Anomaly Matching and Syzygies
in $N=1$ Gauge Theories

Ph. Brax, C. Grojean and C.A. Savoy

CEA-SACLAY, Service de Physique Théorique
F-91191 Gif-sur-Yvette Cedex, FRANCE

Abstract

We investigate the connection between the moduli space of $N = 1$ supersymmetric gauge theories and the set of polynomial gauge invariants constrained by classical/quantum relations called syzygies. We examine the existence of a superpotential reproducing these syzygies and the link with the ´t Hooft anomaly matching between the fundamental fields at high energy and the gauge invariant degrees of freedom at low energy for the flavour symmetry group. We show that the anomaly matching is equivalent to the vanishing of the flavour anomaly on the normal space to the manifold defined by the syzygies. For normal spaces in a real representation of the flavour group we strengthen the connection between the ´t Hooft anomaly matching and the existence of a superpotential by constructing a flavour invariant polynomial whose gradient vanishes at least on the solutions of the syzygies. This corroborates a recent definition of confining theories. We illustrate our general result by considering two examples based on the $SU(N_c)$ and Spin(7) gauge theories. We also examine the role of syzygies in the context of non-Abelian duality. We emphasize the relevance of non-perturbative effects in the dual magnetic theories in proving the equivalence of the electric and magnetic syzygies.

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e-mail addresses: brax, grojean, savoy@spht.saclay.cea.fr
1 Introduction

Recently there has been a tremendous increase of one’s understanding of supersymmetric field theories in four dimensions. Extended supersymmetric theories with $N = 2, 4$ provide a wealth of information on the non-perturbative nature of quantum field theories. The $N = 1$ theories have also been thoroughly investigated. Nevertheless their description is not as complete as their extended counterparts. This is certainly due to the less restrictive mathematical framework of $N = 1$ theories. Yet, a new insight on the infrared (IR) properties of asymptotically free supersymmetric gauge theories has been provided by the recent work \[1\] on these theories. The most striking result on $N = 1$ theories is the existence of a new type of duality. This duality relates two apparently different theories in the short distance regime that are described by the same effective theory in the IR limit. In the same vein the basic question concerning the issue of colour confinement has been tackled and clarified in these non-perturbative approaches for a large class of supersymmetric theories. The dual descriptions and the strong coupling effects have also received a new treatment and a broader understanding via the study of the D-brane dynamics \[6\].

The key ideas in these studies are the non-renormalization theorems and the holomorphy of the superpotential on the one hand \[8–11\], and the matching of the global symmetry anomalies as advocated by ‘t Hooft \[12\] on the other hand. Effective theories have been written and argued to describe the IR behaviour of some gauge theories in terms of gauge invariant composite chiral superfields \[1, 2, 9\]. The perturbative quantum corrections only affect the Kähler metrics through wave-function renormalization, the non-perturbative quantum effects are fixed to a great extent by the holomorphy and by the global symmetries of the theory. The remaining ambiguities in the IR theory are lifted and further support for their validity is obtained through the use of several arguments: the decoupling of flavours (chiral multiplets) establishing a descent link among a series of theories with the same gauge group and different representations for the chiral supermultiplets, the reduction of the gauge symmetry through the Higgs mechanism.

The relevance of the analytic gauge invariants in the study of supersymmetric theories has been noticed a long time ago. Indeed, the $F$-terms in the scalar potential are components of the gradient of the superpotential, an invariant analytic function. The $D$-terms are Hermitean functions of the scalar fields, but a beautiful result in algebraic geometry \[13\] provides the link with holomorphic invariants, at least for the analysis of the vacua \[14–17\]. The degeneracy of the classical solutions is described in terms of massless fields, the moduli. The fundamental mathematical property of the (classical) moduli space (modulo the group of gauge symmetry) is its isomorphism with an algebraic variety of analytic gauge invariant polynomials in the primordial fields, which we call the chiral ring, following ref. \[1\]. The chiral rings can be classified into two categories, those which have algebraic constraints among the invariants of the

\[1\] For a complete review on first duality examples see \[3–5\] and many other examples constructed later have been reviewed in \[6\].
integrity basis, called syzygies, and those with a free basis. This classification is also related to the possible patterns of gauge and flavour symmetry breaking and to the structure of the commutant, the largest symmetry group commuting with the gauge group. These results are summarized in the section 2 of this paper.

A powerful and necessary criterion for the existence of an effective theory describing the IR regime of an asymptotically free gauge theory was stated by 't Hooft: there should be matching of the (formal) anomalies of the global symmetries calculated with either the UV or the IR massless fermions. This has been extensively exploited to classify confining theories (for a review see [18]). In section 3, we also stress that the 't Hooft matching conditions for the unbroken global (flavour) symmetries have to be satisfied at the origin of the UV and IR field spaces. It is then easy to select the unbroken flavour subgroups that are allowed at low energies. The breaking has to be provided by an appropriate superpotential which is restricted by the global symmetries, including a $R$-symmetry.

One of the main aims of this paper is the analysis of the necessity of a sufficient condition recently obtained [19] from the isomorphism between the moduli space and the chiral ring constrained by the syzygies. With this completion the conjecture reads:

*The 't Hooft anomaly matching conditions are satisfied for supersymmetric gauge theories if and only if the syzygies of the chiral ring derive from a superpotential.*

Rather than proving this statement we shall be interested in making explicit the intimate link between a physical condition, the t'Hooft anomaly matching, and the geometry of the space of gauge invariants. We show that the anomaly matching is equivalent to the vanishing of the flavour anomaly on the normal space to the zeros of the syzygies. In the case where the normal space is real under the action of the residual flavour subgroup, we construct a flavour group invariant polynomial in the gauge invariants whose gradient vanishes at all the solutions of the syzygies. For the proof to be complete it remains to be shown that all of the zeros of the gradient of this superpotential are solutions of the syzygies. Now, as discussed in section 4, if a superpotential exists that reproduces the syzygies, the normal space transforms as a real representation of the flavour group, implying that the correspondence between the anomaly matching condition and the existence of superpotential occurs in the widest possible sense.

The global symmetries of a supersymmetric gauge theory always possesses an Abelian subfactor $U(1)_R$, the $R$ symmetry, which differs notably from the rest of the flavour symmetries. The role of the $R$ symmetry has been crucial in analyzing both confining theories and the $N = 1$ duality. The main difference between the $R$ symmetry and the flavor symmetries stems from the non-trivial action of the $R$ symmetry on the gauginos, the fermions in the gauge vector multiplets, while the flavour symmetries leave the vector multiplets invariant. Another essential property of the $R$ symmetry is the different $R$ charges of the
bosons and fermions in a chiral multiplet. The boson have a $R$ charge which is shifted by one unit from the charges of the fermions, i.e. $R_F = R_B - 1$. This plays a noteworthy role in the analysis of the matching conditions as the fermions are the fields involved in the anomaly calculations.

At low energy we shall only consider the cases where the gauge invariant polynomials in the matter fields are sufficient to satisfy the matching conditions. In the rest of this paper we shall not be interested in the explicit role that the invariants constructed using the field strength chiral superfield $W_\alpha$ can have.

It is well-known that the $R$-symmetry only allows for a polynomial superpotential for chiral supermultiplets in representations of the gauge group with special values of Dynkin index, and the corresponding theories have been extensively analysed [20–24]. Our result corroborates the conjecture that the confining theories are those that admit a polynomial superpotential [22, 25]. These proofs are given in section 4.

The second object of this paper is to present a consistency check on dual theories: we verify that the solutions of the magnetic theories satisfy the syzygies of the electric theory if and only if a non-perturbative superpotential is added to the former. This is done by considering the theories with any number $N_f$ of chiral multiplets in the fundamental representations of the gauge groups $SU(N_c)$ and $Spin(7)$. We give the (unique) superpotentials for any $N_f$ which are consistent with the flavour symmetries and with the successive decoupling of flavours. Then we find the gauge invariants and their syzygies and proceed to test the electric syzygies in terms of the magnetic solutions. These results are displayed in sections 5 and 6.

Finally, in section 7, the geometry of the moduli space of the $SU(N)$ theories with a large number of flavours is determined in an attempt to have access to the (classical) Kähler potential of the dual theory. This is done by finding the non-compact invariance group of the equations of motion. It is shown that the dense part of the moduli space consists of two conjugated orbits of this non-compact group whose closure are the singular orbits. This provides a coset parameterisation of the solutions which, however, is not (explicitly) holomorphic and breaks the Kähler invariance. Therefore it is not possible to obtain the induced Kähler potential for the dual theory in a straightforward way. But some issues suggested by these results are under investigation.

2 Flat Directions, Analytic Invariants and Syzygies

In this section we recall some important mathematical results that allow a general analysis of supersymmetric gauge theories. In global supersymmetric theories the scalar potential is a sum of $F$– and $D$–terms: the $F$–terms are the norm of the gradient $W_i$ of a holomorphic function, the superpotential $W(z^i)$. For the $D$–terms, the relation with holomorphy is more subtle and appears when we are interested in supersymmetric vacua. In a supersymmetric theory with
scalar fields $z^i$, Kähler potential $\mathcal{K}(z, z^*)$ and gauge group $G$, the $D$–terms are defined as

$$D^A = \frac{\partial \mathcal{K}}{\partial z^i} (T^A z)^i$$

where $T^A$ ($A = 1, 2, \ldots, \text{dim } G$) are the Hermitian generators of $G$ in the representation of the scalar fields.

A sufficient and necessary condition for the vanishing of the $D$–terms is:

For any holomorphic gauge-invariant polynomial $I(z^i)$ in the scalar fields, a solution $\xi$ of the equation

$$\frac{\partial I}{\partial z^i}|_{z^i=\xi} = C \frac{\partial \mathcal{K}}{\partial z^i}|_{z^i=\xi}$$

where $C$ is a complex constant, is a solution of $D^A = 0$ (of course the whole $G$–orbit associated to $\xi$ is a solution). Conversely to any solution of $D^A = 0$ one can associate a holomorphic gauge invariant satisfying (2).

The proof of this result is obtained by studying the closed orbits of the complexified $G_c$ of the gauge group $G$ and the ring of $G_c$–invariant analytic polynomials. This ring is finitely generated: one can find an integrity basis i.e. a set of $G_c$–invariant holomorphic homogeneous polynomials $\{I^a(z)\}_{a=1}^d$ such that every $G_c$–invariant polynomial in $z$ can be written as a polynomial in the $I^a(z)$.

The elements of an integrity basis are not always algebraically independent. In general, there exist algebraic relations (called syzygies) satisfied by the fundamental invariants:

$$S^\alpha (I^a(z)) = 0 \quad \alpha = 1, \ldots, N$$

To each $G_c$–orbit corresponds a vector in $\mathbb{C}^d$ made out of the values taken by the invariants $\{I^a(z)\}$ along this orbit. Conversely, it can be shown that to each vector in $\mathbb{C}^d$ satisfying the syzygies is associated a unique closed $G_c$–orbit. In that sense the algebraic manifold defined by the syzygies is identified with the set of closed $G_c$–orbits. Notice that the origin $\{I^a = 0\}$ is associated with the unique closed $G_c$–orbit of $z^i = 0$.

The existence of the syzygies can be related to the index of the matter field representation, $\mu = \sum_i \mu_i$, where $\mu_i$ is the index of the irreducible representation $R_i$ defined by $\text{tr} (T^a(R_i)T^b(R_i)) = \mu_i \delta^{ab}$. For low indices $\mu < \mu_{\text{adj}}$, where $\mu_{\text{adj}}$ is the index of the adjoint representation, the gauge invariant rings have been classified and it has been shown that there are no syzygies. For indices larger than the index of the adjoint representation the generic situation is that there are syzygies, with a few exceptional cases with no syzygies.

\footnote{A trivial example is provided by the $SU(N_c)$ gauge theory with $N_c$ fields in the fundamental and antifundamental representations; the fundamental invariants are the mesons $M = z\bar{z}$ and the baryons $B = \det z$ and $B = \det \bar{z}$; classically, they are constrained by $\det (M - BB) = 0$. See below for further examples.}
Equations (2) can be seen as a condition for the points of a closed $G^c$-orbit to extremize the Kähler potential, the constant $C^{-1}$ being a Lagrange multiplier. A result of geometric invariant theory [13] states that the points extremizing the Kähler potential on a $G^c$-orbit form a unique $G$-orbit and are solutions of $\{D^A = 0\}$. Identifying the points on a same $G$-orbit, there is a one-to-one correspondence between any two of the following sets:

1. the algebraic manifold defined by the syzygies (3);
2. the closed $G^c$-orbits;
3. the solutions of (3) modulo gauge transformations;
4. the solutions of $D^A = 0$ modulo gauge transformations.

Therefore, there is a homeomorphism between the set of the scalar field space defined by $D^A = 0$, i.e. the moduli space, $\mathcal{M}_z$, and the algebraic variety defined by the fundamental invariants $I^a$ constrained by $S^a (I^a(z)) = 0$, $\mathcal{M}_I$:

$$\mathcal{M}_I \sim \mathcal{M}_z.$$ (4)

This homeomorphism is instrumental in studying the low energy physics of supersymmetric gauge theories.

There is a natural stratification of $\mathcal{M}_z$ according to the unbroken gauge symmetry, i.e., the little group $G_z \subset G$ at each point $z$. A stratum $(G_z)$ is the set of all points which have the same invariance $gG_zg^{-1}$, where $g$ is any element of $G$. The generic or principal stratum has the smallest residual symmetry $G_z$ while the more singular ones correspond to larger numbers of unbroken gauge symmetries. For $\mu < \mu_{adj}$ the gauge group is generically broken to a non-void residual gauge group, while for $\mu > \mu_{adj}$ the gauge group is totally broken in the generic stratum. Because of the isomorphism (4), there is a corresponding stratification of the algebraic manifold defined by the syzygies, $\mathcal{M}_I$.

A powerful property for the study of supersymmetric gauge theories is the following theorem [13, 28]:

The tangent space to the moduli space at a point $\xi$ in a stratum is analytically isomorphic to the tangent space at the point $I^a(\xi)$ to the corresponding stratum of the manifold of gauge invariant polynomials.

In particular, it has an important consequence on anomalies of the flavour group in both theories and the ’t Hooft matching conditions.

### 3 The ’t Hooft Anomaly Matching Conditions

Let us recall that the matching conditions concern the global (unbroken) anomalous symmetries $H$ of a given gauge theory. It specifies that the anomalies $\text{tr } H^3$ and $\text{tr } H$ are equal when evaluated with the high energy content of the gauge theory, i.e. the matter fields $z^i$ in diverse representations of the gauge and flavour groups, and with the low energy degrees of freedom of the theory. The (perturbative) degeneracy of the supersymmetric vacua requires a more careful analysis. In a supersymmetric theory with gauge symmetry group $G$ and superpotential $W(z^i)$ with flavour symmetry $H$, at any vacuum $z$ such that
\( W_i(z) = 0 \) and \( D^A(z, z^*) = 0 \), the mass matrices for the chiral multiplets are \( W_{ij}(z) \) and \( D^A_i(z, z^*) \). If the residual global (flavour) group at \( z \) is \( H_z \subset H \), the mass matrix is \( H_z \)-invariant and the massive states are in a real (vector-like) representation of \( H_z \). Therefore the anomalies \( \text{tr} H_z^3 \) and \( \text{tr} H_z \) can be calculated at \( z \) where the massive states are excluded or at the origin \( z = 0 \) where all states are massless. In particular the ’t Hooft conditions can be checked at the origin of the UV theory on the one hand, and at the origin of the IR theory on the other hand, for each relevant flavour subgroup \( H_z \subset H \).

The role of the superpotentials (both in the UV and the IR theories) is then clear: \textit{they cannot change the anomaly matching, but they could restrict the vacua, hence the flavour invariance of the theories to a subgroup whose anomalies match}, provided that such superpotentials that are consistent with the symmetries of the theory exist. This argument corresponds to examining the anomaly matching at the origin of the moduli space. Given the UV fields, \( z^i \), and the IR composites, \( I^a \), the flavour symmetries consistent with the ’t Hooft conditions are easily checked by considering the chains of subgroups \( H_z \) and calculating their anomalies at the origin.

Since the inclusion of a superpotential is easily taken into account, we assume for simplicity that the UV theory has no superpotential. As seen in the previous section the moduli space is homeomorphic to the variety of gauge invariant polynomials. These gauge invariants are chiral superfields which are good candidates for a low energy description of the gauge theory. The direct relationship between the moduli space and gauge invariants entails a close relationship between the IR and UV anomalies.

Let us decompose \( M_z \) and \( M_I \) into strata and then apply the theorem quoted in the previous section. The relevant point of this theorem is the analyticity of the mapping. It implies that chiral superfields in the tangent space of the \( D^A = 0 \) set at \( z \) are mapped to chiral superfields in the tangent space of the manifold of gauge invariants at \( I^a(z) \). The tangent space within a given stratum corresponds to singlet fields under the residual gauge group. They fall into representations of the residual global symmetry \( H_z \) and are mapped into corresponding representations in the tangent space to the variety of gauge invariants. An immediate consequence is \textit{the equality between the contributions to the anomalies from the fields associated to both tangent spaces}. Our results in the next section rely on this powerful property.

We first discuss an example before embarking upon the general cases. For simplicity we describe the case of the \( SU(N_c) \) gauge theory with \( N_f < N_c \) flavours of quarks and antiquarks \([29]\). This theory possesses two global axial symmetries that we shall denote by \( SU(N_f)_A \times U(1)_R \). The charges of both the quarks and the antiquarks are \((N_f, 1 - N_c/N_f)\). The chiral ring is generated by the (meson) invariants \( M_{ij} \) transforming under the axial flavour group as \((0, 2(1 - N_c/N_f))\). The anomalies of the various possible \( SU(N_f)_A \times U(1)_R \) subgroups calculated (at the origins) for the quark and antiquarks and for the mesons do not match, so that the low energy theory cannot preserve any flavour symmetry. At the generic stratum of the \( D^A = 0 \) manifold, the gauge group
is broken to $SU(N_c - N_f)$, and there is no residual global symmetries. At
the singular strata, the gauge group is broken to $SU(N_c - r)$, $r < N_f$, and
the axial flavour group is broken to $U(N_f - r)$. The tangent spaces to the
strata are, respectively, all the quarks but $(N_f - r)$ families in the fundamental
representations of $SU(N_c - r)$, and all the mesons $M_{ij}$ except those for which
$i > r$ and $j > r$. The global anomalies of $U(N_f - r)$ are easily checked to
coincide. However, by taking into account the other mesonic fields, which are
not in the tangent space this matching is destroyed. Therefore one has to look
for a theory where besides the $SU(N_c - r)$ gauge theory with $(N_f - r)$ flavours,
the IR physics is defined in terms of this restricted set of mesons.

The only non-perturbative superpotential consistent with the flavour sym-
metries is

$$W = (N_c - N_f) \left( \frac{\det M_D}{\Lambda^{3N_c - N_f}} \right)^{1/(N_f - N_c)}$$

(5)

implying $\det M \neq 0$ and restricting $M$ to the generic stratum. The singular
strata are excluded as they must be because of the anomaly mismatch. Ac-
tually, the only way to introduce a supersymmetric theory with the mesonic
fields restricted to the coset $U(N_f)/U(N_f - r)$ would be through a non-linear
realization of the global symmetries, namely, through a non-trivial Kähler poten-
tial. Of course, this is a difficult task in the absence of important constraints
such as holomorphy and perturbative non-renormalization which allow for the
determination of the superpotential. Anyway, as such, the superpotential (5)
destabilizes the theory which has no ground state.

On more general grounds, at a non-trivial vacuum, the anomalies receive two
contributions, one from the charged fields under the residual gauge group, $G_z$,
and another one from the singlet fields. Neither all singlet fields nor all chiral
invariants belong to the lesser dimensional tangent spaces to the singular strata
which the theorems of the previous section apply to, yet they may contribute
to the anomalies which may not match as in the previous example. As already
stressed, the anomaly matching for the corresponding flavour subgroups are
easily checked at the origins.

Although the flavour symmetries that constrain the non-perturbatively gen-
erated potential are model dependent, a general structure can be already estab-
lished from the general form of the non-anomalous $R$-symmetry. The charges
of the matter fields $z_i$ under the $U(1)_R$ can be chosen as

$$R_i = 1 - \frac{\mu_{adj}}{n\mu_i}$$

(6)

where $n$ is the number of matter field representations. The charges of the
fermions in the same chiral multiplets are $R_i - 1 = -\frac{\mu_{adj}}{n\mu_i}$. The low energy
superpotential is a combination of terms having $R = 2$ which can be expressed
in terms of the matter field content of the gauge invariants as

$$W \sim (\prod_i z_i^{\mu_i})^{2/(\mu - \mu_{adj})}$$

(7)
The special cases where $W = 0$ have been classified and studied in Refs. [21, 22]. When $\mu < \mu_{adj}$ the superpotential has a runaway behaviour with a minimum at infinity. In a theory with confinement, this general form has to be reproduced in terms of various combinations of the chiral invariants, and there can be several. Its determination then relies on different arguments involving decoupling and higgsing [1, 2, 22, 30]. We give some examples of (5) in section 3.

For low indices $\mu < \mu_{adj}$ the gauge invariant rings have been classified [20, 23] and it has been shown that there are no syzygies. There were several prior examples of confining theories of this kind in the literature [18–21]. The case study of all these theories has not been carried out. Nevertheless the studied cases reveal the pattern emerging from the 't Hooft matching, either the anomalies do not match and there is a superpotential, or the anomalies do match and there is a branch of the low energy theory with a vanishing superpotential. We refer to these detailed studies and concentrate on the $\mu \geq \mu_{adj}$ in the following.

4 Anomaly Matching and Syzygies

The gauge theories with $\mu \geq \mu_{adj}$ have syzygies generically. There are a few classified cases where the basic invariants are not constrained. Barring these examples which have been extensively studied [14, 21, 23, 24] in the literature, in the following we shall be interested exclusively in theories with non-void syzygies, i.e. there exists relations between the basic invariants of the theory. We shall now examine the connection between the anomaly matching and the existence of a superpotential which reproduces the syzygies.

Let us recall that the moduli space $\mathcal{M}_z$ is homeomorphic to the manifold $\mathcal{M}_I$ defined in terms of the composite fields $I^a$ by the syzygies (3). The gauge group is generically broken, i.e. the gauge group is completely broken on the generic stratum, a dense open set. In this section we concentrate on the principal stratum. For the singular ones, where there remains a residual gauge group, the analysis is more involved as already discussed in the previous section. Since the gauge group is completely broken, the anomaly calculated from the chiral fields in the tangent space to the moduli space at a point $z$ of the generic stratum is the same as the anomaly (formally) calculated from the gauge invariants in the tangent space to the corresponding point of the chiral ring with the syzygies (3). Then [20, 46]

The 't Hooft anomaly matching conditions are satisfied for supersymmetric gauge theories if the syzygies $S^a(I^a) = 0$ of the gauge invariant ring $\mathbb{C}[I^a]/\{S^a = 0\}$ derive from a superpotential $W(I)$ i.e. $S^a = \frac{\partial W}{\partial I^a} = 0$.

Let us show this result. Assume that there exists a superpotential $W(I^a)$ such that the syzygies are the $F$–terms

$$S^a = \frac{\partial W}{\partial I^a}$$

\[3\] See the remark at the end of this section concerning the case $\mu = \mu_{adj}$ and the quantum modifications.
The tangent space to the variety defined by the syzygies is given by the zero eigenvectors of the matrix
\[ S_b^a = \frac{\partial S^a}{\partial I^b} (I^a) = \frac{\partial^2 W}{\partial I^a \partial I^b}, \] (9)
namely, by the massless composite superfields. The anomalies in the IR supersymmetric theory defined by the superpotential \( W(I^a) \) are the same as the formal one defined within the tangent space to \( S_a^a = 0 \). As already stated, these coincide with the anomalies calculated within the tangent space to the set of solutions of \( D^A = 0 \), defined by the zero eigenstates of the matrix \( D^A_\alpha \), namely, with the massless states of the UV supersymmetric theory. Of course, the mass matrices are invariant under the residual global (flavour) group \( H_z \subset H \), and the massive states are in a real (vector-like) representation of \( H_z \). These anomalies of \( H_z \) coincide also when calculated at the origins, \( z^i = 0 \) and \( I^a = 0 \), respectively. Therefore the ’t Hooft matching conditions are satisfied.

Let us examine the statement:

The ’t Hooft anomaly matching implies the existence of a flavour invariant polynomial whose gradient vanishes on the zeros of the syzygies.

The ’t Hooft conditions imply the matching of the anomalies of \( H_z \) calculated in the fundamental theory with all the fields \( z^i \) and in the low energy theory, with all the composite fields \( I^a \). The former also match with the anomalies calculated with the zero eigenstates of the mass matrix \( D^A_\alpha \), namely on \( M_z \), which, in turn, match with the anomalies calculated with the zero eigenstates of \( S^a_\alpha (I) \), namely on \( M_I \). Therefore the \( N \times d \) matrix \( S^a_\alpha \) \( (N < d) \) projects onto the normal space to \( M_I \) which transforms as an anomaly free representation of \( H_z \). There are two distinct possibilities concerning the representations of \( H_z \) forming the normal space. When the normal space is an anomaly free chiral representation of the little group there cannot exist a superpotential reproducing the syzygies. This follows from our earlier analysis showing that the existence of a superpotential implies that the normal space is a real representation of the little group. In the following we only focus on the case where the normal space is a real representation of \( H_z \) and show how this allows to construct a \( H \)-invariant polynomial whose gradient is zero when the syzygies are satisfied. It is important to notice that the restoration of the \( H \)-invariance of the constructed polynomial concerns the flavour symmetries acting on the fermions associated to the invariants. This implies the invariance of the polynomial under the \((R-1)\) Abelian symmetry.

The transformations \( \{ h \} \) of \( H \) define the orbit \( \{ hI \} \) of each point \( I \in \mathbb{C}[I^a] \). The covariance of the syzygies defines a representation \( \{ \tilde{h} \} \) such that
\[ S^\alpha (hI) = \tilde{h}^\alpha \beta S^\beta (I). \] (10)

At any point \( I_0 \in M_I \) the gradients \( S^\alpha_\beta (I_0) \) are invariant under the action of the little group of \( I_0, H_{I_0} \). It is then possible to construct a basis of the syzygies and a basis of the invariants associated to the point \( I_0 \), with the indices \( \alpha \) and
\[ a \text{ decomposed into } \alpha = \{i, m\} \text{ and } a = \{i, r\} \text{ such that:} \]

\[
\frac{\partial S^a}{\partial I^a}(I_0) = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}
\]

where \( D \) is a non-singular diagonal matrix. The set \( \{I^i\} \) spans \( \mathcal{N}_{I_0} \), the normal space to \( \mathcal{M}_I \) at the point \( I_0 \), and \( \{I^r\} \), the tangent space. Under the flavour group \( H \),

\[
I_0 \in \mathcal{M}_I \rightarrow hI_0 \in \mathcal{M}_I, \quad H_{I_0} \rightarrow hH_{I_0}h^{-1}, \quad \mathcal{N}_{I_0} \rightarrow h\mathcal{N}_{I_0}.
\]

The reality of \( \mathcal{N}_{I_0} \) under \( H_{I_0} \) implies the existence of an unitary matrix \( \eta \) such that for any element \( h_N \) of the representation of \( H_{I_0} \) on \( \mathcal{N}_{I_0} \): \( h_N \ast \eta h_N \eta^{-1} \). Now, if \( I_0 \) is the projection of \( I \) on \( \mathcal{M}_I \) (the projection is always defined at least in a neighborhood of \( \mathcal{M}_I \)), we define \( W(I) \) as the integral along the normal direction \( (I - I_0) \):

\[
W(I) = \int_0^1 dt \, S^{i_1}(I_0 + t(I - I_0))(I - I_0)^{i_2}\eta_{i_1i_2}
\]

where the basis associated to \( I_0 \) has been chosen. By construction, this function is \( H_{I_0} \)-invariant. To see its \( H \)-invariance, we have to relate the basis at different points. Consider any element \( h \) of \( H \). The unitarity of \( H \) ensures that \( hI_0 \) is the projection of \( hI \) (note that the covariance of the syzygies guarantees that \( hI_0 \) belongs to \( \mathcal{M}_I \)). It is easy to verify that \( h^{-1}S \) and \( h^{-1}I \) are the bases associated to \( hI_0 \). The coordinates of the point \( hI \) in the new basis are the same as those of \( I \) in the old basis. Moreover \( \eta \) is independent of the basis. Thus

\[
W(hI) = \int_0^1 dt \, (h^{-1})^{i_1i_2}S^{i_3}(h(I_0 + t(I - I_0))(I - I_0)^{i_3}\eta_{i_1i_3}
\]

\[= W(I) \]

which proves the invariance of \( W \) under the action of flavour group \( H \). Note that the reality of \( \mathcal{N}_{I_0} \), which we assume to follow from the 't Hooft matching condition, is necessary in our construction of \( W \); the invariance under \( H \) then follows from covariance. The \( H \)-invariance of the polynomial \( W \) concerns the flavour symmetries acting on the fermions associated to the invariants. When \( \mu \neq \mu_{adj} \) the \( R \) symmetry is broken at low energy. The restored symmetry acting on \( W \) is the \( (R - 1) \) symmetry of the fermions. Assuming that the directions \( (I - I_0)^i \) have a well-defined \( R \) charge, this leads to the invariance condition \( (R_S - 1) + (R_I - 1) = 0 \). The \( R \) charge of the polynomial \( W \) is explicitly given by \( R_W \equiv R_S + R_I = 2 \). The \( R \) symmetry of the polynomial \( W \) is then two as required for a superpotential.

The gradients of \( W(I) \) along the tangent directions to \( \mathcal{M}_I \),

\[
\partial_i W(I) = \int_0^1 dt S^{i_1}(I_0 + t(I - I_0))(I - I_0)^{i_2}\eta_{i_1i_2}
\]

10
obviously vanishes when $I = I_0$. Along the normal directions the gradient becomes

$$\partial_i W(I) = S_i(I) + \int_0^1 dt (S_{ii}(I_0 + t(I - I_0)) - S_{ii}(I_0 + t(I - I_0)))(I - I_0)^{ii} \quad (16)$$

where the indices are lowered using $\eta$. This vanishes for $I = I_0$. This guarantees that the polynomial $W(I)$ has a vanishing gradient on the zeros of the syzygies. The previous formulæ imply that the gradient of $W$ is equal to the set of syzygies when the matrix $S_a^\alpha$ is symmetric. This requires a connection between the $H$-representations of the syzygies and of the invariants which seems difficult to establish in general.

Up to now our discussion has been restricted to the normal space to one particular point $I_0$ on $\mathcal{M}_I$. We have constructed explicitly a polynomial $W$ on the normal space at $I_0$ which has a dependence on the point $I_0$. We shall now extend our analysis to non-singular strata on the non-singular manifold $\hat{\mathcal{M}}_I$. By a standard result of differential geometry \cite{32}, we can choose a tubular neighbourhood of this manifold $\hat{\mathcal{M}}_I$ which is diffeomorphic to the normal bundle $N$. This is an open neighbourhood of the manifold $\hat{\mathcal{M}}_I$. Locally in an open neighbourhood the coordinates describing the tubular neighbourhood are $(I_0, E)$ where $I_0 \in \hat{\mathcal{M}}_I$ and the vector $E \in N_{I_0}$ in the normal space at $I_0$. This implies that every point in the tubular neighbourhood is represented by the vector $I = I_0 + E$.

We can extend the definition of $W$ to the stratum of $I_0$ on the manifold $\hat{\mathcal{M}}_I$. We choose $I_0$ such that its stratum possesses a little group $H_{I_0} \neq 0$ with the smallest dimension. Let us now introduce the slice $\Sigma_{I_0}$ passing through $I_0$ in the stratum of $I_0$. It is such that it parametrizes the orbits and is in a sense normal to them. In particular the tangent space to the slice $\Sigma_{I_0}$ at $I_0$ comprises only singlets of the little group $H_{I_0}$. Consider another point $I'_0 \in \Sigma_{I_0} \cap \mathcal{M}_I$ then the little groups $H_{I_0} \equiv H_{I'_0}$ and one can easily verify that the normal spaces coincide.

Choose a point $I'$ in the tubular neighbourhood such that $I' - I'_0 = I - I_0$ identified with $E$ in the normal space. By the line integral similar to (13) starting from $I'_0$ one can define a polynomial $W'(I')$ with an explicit dependence on $I'_0$.

One can also extend the polynomial $W(I)$ defined in the normal space at $I_0$ to $W(I')$ using the Taylor polynomial in $\delta I_0 = I'_0 - I_0$. A calculation shows that the two polynomials coincide

$$W'(I') \equiv W(I') \quad (17)$$

This allows to extend the definition of the polynomial $W$ to the whole tubular neighbourhood of the slice $\Sigma_{I_0}$ and therefore by the $H$-invariance to the tubular neighbourhood to the stratum of $I_0$ in $\mathcal{M}_I$. We have then defined a polynomial $W$ on each of the tubular neighbourhoods of the non-singular strata of the manifold $\mathcal{M}_I$.

Although this provides a working definition of a superpotential $W$ there are still two points which should be clarified in order to complete the proof of the
existence of a superpotential reproducing the syzygies. First of all we have not proved that the polynomials $W_i, i = 1 \ldots p$, corresponding to $p$ different non-singular strata coincide on the singular sets where the closures of the strata meet. Assuming that this is the case allows to extend the polynomial $W$ to the singular strata by analytic continuation. Secondly we have not shown that the zeros of the gradient of the polynomial $W$, when extended to the whole set of gauge invariants from the tubular neighbourhood of $\mathcal{M}_I$ by analytic continuation, coincide with $\mathcal{M}_I$. These two points require some global analysis which goes beyond the scope of the present paper.

In general, one can a write a IR superpotential $W(I)$ which is invariant under the global symmetries of the UV theory. The ambiguities in its construction can be resolved by means of the decoupling technique. Two examples are discussed in the next section.

Let us now consider the cases where the $R$ symmetry belongs to the residual flavour symmetry. This is feasible only in the case $\mu = \mu_{\text{adj}}$ when the $R$ charges of the scalars $z_i$ vanish. Consequently the $R$ charges of the invariants and of the syzygies vanish. The $(R-1)$ charges of the syzygies and of the invariants are all $(-1)$ implying that the normal space cannot be real. The matching condition for the $R$ symmetry is violated. Let us focus on one particular invariant syzygy, one can associate to this syzygy a new invariant $X$ whose role is to cancel the anomaly. Introducing the $X$ field corresponds to an extension of the space of invariants. This field is such that its $(R-1)$ symmetry is unity in order to render the normal space real. Having paired the syzygy with the field $X$ in a real normal space one can define the invariant polynomial (13) by integration along the $X$ direction. This leads to $W = XS$. This superpotential reproduces the classical moduli space $\mathcal{M}_I$ augmented with the $X$ direction at the origin. In general this superpotential is deformed by quantum effects which forbids the unwanted $X$ direction.

As the superpotential that we have constructed is a polynomial we can deduce conditions on $\mu$. Replacing the $I^a(z^i)$ polynomials, the IR superpotential $W$ is a polynomial in the fields $z^i$. Now by comparing with the general structure (7) required by the $R$-symmetry, one obtains the following condition:

\begin{equation}
\mu = \mu_{\text{adj}} + k, \quad k = 0, 1, 2.
\end{equation}

This is the confinement condition\footnote{Strictly speaking this only applies to gauge theories without a UV superpotential. In the presence of a UV superpotential, the results stand for the unbroken gauge and flavour symmetries and the massless states even though the anomaly matching, as previously noticed, can be analysed at the origin of the moduli space. Indeed the massive states decoupled below the symmetry breaking scale are in a real representation of the residual flavour group.} for theories with $\mu \geq \mu_{\text{adj}}$. This condition has been already introduced in the literature\footnote{By requiring a polynomial structure in (7). Here, it follows from the necessary 't Hooft matching conditions on anomalies. In particular this excludes all higher order Dynkin index theories where the possible superpotential would have a branch cut at the origin.} by requiring a polynomial structure in (7). Here, it follows from the necessary 't Hooft matching conditions on anomalies. In particular this excludes all higher order Dynkin index theories where the possible superpotential would have a branch cut at the origin.
Notice that we are including in the series (18) the theories with quantum modified moduli spaces [1, 13, 24, 33], where the (quantum modified) syzygies are usually introduced as constraints rather than as field equations. We have shown how the case \( \mu = \mu_{\text{adj}} \) can be included by introducing a new field \( X \) when the syzygies are invariant under the flavour group. With this in mind, we find it more appropriate to include these theories among the other confining theories in (18).

Another important remark concerning the quantum modified moduli spaces is the following. By writing the general superpotential in terms of the different invariants there are ambiguities due to the presence of different combinations of invariants with the right quantum numbers. When the superpotential is obtained through decoupling in the descent procedure, the constraint corresponding to the equations of motion turns out to be the quantum modified one [1]. The syzygies follow if one discards the dimensionful scale, which is obviously absent in the classical syzygy. Of course, the deformation of the manifold \( \mathcal{M}_I \) when taking into account the quantum modified syzygy must preserve the isomorphism (4). However, we do not have a general proof of this conjecture.

5 A Case Study: \( SU(N_c) \) and \( Spin(7) \) Gauge Theories

Let us illustrate these general results by a study of the well-known case of the supersymmetric \( SU(N_c) \) gauge group with \( N_f \) flavours of quarks and antiquarks, denoted by \( Q^i_{\alpha} \) and \( \tilde{Q}^{\dot{\alpha}}_{i} \), respectively. The global symmetries are \( SU(N_f) \times SU(N_f) \times U(1)_B \times U(1)_R \) with \( Q^i_{\alpha} \) transforming as \((N_f, 1, 1, 0)\) and \( \tilde{Q}^{\dot{\alpha}}_{i} \) as \((1, N_f, 1, 0)\). The chiral ring is generated by meson and baryon composite fields (with obvious contractions of the colour indices):

\[
M^{ij} = Q^i \tilde{Q}^j, \\
B_{i_1...i_{N_f-N_c}} = \epsilon_{i_1...i_{N_f}} Q^{i_{N_f-N_c+1}} \ldots Q^{i_{N_f}}, \\
B_{i_1...i_{N_f-N_c}} = \epsilon_{i_1...i_{N_f}} \tilde{Q}^{i_{N_f-N_c+1}} \ldots \tilde{Q}^{i_{N_f}},
\]

constrained by the (overdetermined) set of syzygies:

\[
B_{i_1...i_{N_f-N_c}} M^{i_1j} = 0, \\
M^{i_1j} \bar{B}_{j_1...j_{N_f-N_c}} = 0, \\
B_{i_1...i_{N_f-N_c}} \bar{B}_{j_1...j_{N_f-N_c}} = \epsilon_{i_1...i_{N_f}} \epsilon_{j_1...j_{N_f}} M^{i_{N_f-N_c+1}j_{N_f-N_c+1}} \ldots M^{i_{N_f}j_{N_f}},
\]

if \( N_f \geq N_c \). In this case one can easily solve the syzygies. The solutions are up to a \( U(N_f) \times U(N_f) \) transformation given by

\[
M = \text{diag}(M_1 \ldots M_{N_c}, 0 \ldots 0), \\
B_{N_{c+1}...N_f} = B, \\
\bar{B}_{N_{c+1}...N_f} = \bar{B},
\]
with all other components of \( B \) and \( \hat{B} \) vanishing. The syzygies \([21]\) are satisfied by \([21]\) provided:

\[
BB = \prod_{i=1}^{N_c} M_i .
\]  

(22)

The global symmetry preserved by the solution of the syzygies \([21]\) is \( H_z = SU(N_f - N_c)^2 \times U(1)^{N_c - 1} \) for \( N_f \geq N_c \). The quark representation decomposes as \( N_c \) copies of the fundamental representation of \( SU(N_f - N_c) \). The tangent space to the moduli space defined by the action of \( U(N_f) \times U(N_f) \) on \([21]\) is defined by the action of the generators \( \left( \frac{U(N_f)}{U(N_f - N_c) \times U(N_c)} \right)^2 \) on \( M \). This gives two \( N_c \times (N_f - N_c) \) matrices, transforming as \( N_c \) copies of the fundamental representations of each \( SU(N_f - N_c) \) factor in \( H_z \). This proves that the \( SU(N_f - N_c) \) anomalies match. Similarly the \( U(1)'s \) are vector-like and the anomalies match. This is a consequence of isomorphism between the moduli spaces defined for \( Q^i, \tilde{Q}^i \) and the \( M's, B's \) and \( B'\)s that fulfill \([21]\).

From the global symmetries, one deduces the superpotential

\[
W = (N_f - N_c) \left( \frac{\det M - tr \left( B \| M^{N_f - N_c} \| \hat{B} \right)}{M^{3N_c - N_f}} \right)^{1/(N_f - N_c)}
\]  

(23)

where \( \| M^{N_f - N_c} \|^{i_1 \cdots i_{N_f - N_c}} \cdots ^{j_{N_f - N_c}} \) is the minor of rank \( N_f - N_c \) of the meson matrix \( M \) specified by the appropriate rows and columns. This superpotential has the required charge \( R = 2 \). Starting from \( N_f > N_c + 1 \), and adding a source to the \( N_f \) flavour \( mM_{N_f,N_f} \) one can check that it correctly satisfies the holomorphic decoupling condition yielding the same superpotential with one less flavour, and \( mM_{N_f} = \Lambda^{3N_c - N_f} = \Lambda_{N_f - 1}^{3N_c - N_f} \). In particular, going down to \( N_f = N_c + 1 \), one gets \( W = (\det M - B^2 \hat{B})/\Lambda_{N_c + 1}^{2N_c} \) and decoupling the \( N_c + 1 \) flavour the superpotential yields the quantum constraint of the \( N_f = N_c \) theory, \( W = X (\det M - B^2 \hat{B} - \Lambda_{N_c}^{2N_c}) \) with the help of a Lagrange multiplier \( X = M_{N_f,N_f}/\Lambda_{N_c + 1}^{2N_c} \). For \( N_f = N_c + 1 \) the equations for the extrema of the superpotential reproduce the syzygies. When \( N_f > N_c + 1 \), the superpotential is not polynomial and it does not yield the syzygies.

The previous example dealt with a vector-like theory with the matter fields in a real representation of the gauge group. In the following we will discuss the case of \( Spin(7) \) with \( N_f \) spinors in the 8 representation \([35]\). In order to describe the low energy dynamics of the theory let us study the global symmetries. There is a non-Abelian flavour symmetry \( SU(N_f) \) and also a single anomaly free \( U(1)_R \) symmetry which is non-anomalous. The spinors have charges \( (N_f, 1 - \frac{5}{N_f}) \). The gauge invariants are the mesons \( M^{ij} = q^i q^j \) which is a symmetric matrix and the baryons \( B^{i_1 i_2 i_3 i_4} = q^{i_1} q^{i_2} q^{i_3} q^{i_4} \). It is also convenient to define the baryons as the Hodge dual of \( B \), i.e. as an \((N_f - 4)\) antisymmetric tensor.
There are two different syzygies. The first one is formally
\[ \varepsilon_{i_1 \cdots i_{N_f}} \varepsilon^{j_1 \cdots j_{N_f}} M_{i_1 j_1} \cdots M_{i_{N_f} j_{N_f}} = M_{i j} B^{i_{N_f} \cdots i_{N_f}} B^{j_{N_f} \cdots j_{N_f}}. \]  
(24)

The second one denoted by \( M^2 B + B^2 = 0 \) reads
\[ M_{i_1 j_1} M_{i_2 j_2} B^{i_1 i_2 j_3 \cdots j_{N_f} - 4} + \varepsilon^{a b c d} j_3 \cdots j_{N_f} - 4 B^{c f} a_b B_{e f c d} = 0. \]  
(25)

Let us define the gauge invariant polynomial
\[
\Omega_{N_f} = \det M + a_{N_f} B^{i_1 \cdots i_{N_f} - 4} B^{j_1 \cdots j_{N_f} - 4} M_{i_1 j_1} \cdots M_{i_{N_f} j_{N_f} - 4} \\
+ b_{N_f} \varepsilon^{i_1 \cdots i_{N_f} j_{N_f} - 6} B_{i_1 i_2 k_1 \cdots k_{N_f} - 6} B_{k_5 \cdots k_{N_f}} M_{j_1 k_1} \cdots M_{j_{N_f} - 6 k_{N_f} - 6}
\]  
(26)

where \( a_{n-1} = (n - 4)a_n, \ b_{n-1} = (n - 4)(n - 6)b_n \) and the initial values \( a_6 = -1, \ b_6 = -1 \). There is a unique superpotential
\[ W_{N_f} = (5 - N_f) \left( \frac{\Omega_{N_f}}{\Lambda_{N_f}^{15 - N_f}} \right)^{1/(N_f - 5)}. \]  
(27)

which has a \( R \)-charge of two and satisfies the decoupling requirements. It reproduces the syzygies when \( N_f = 6 \). This corresponds to the anomaly matching case and \( \mu = \mu_{\text{adj}} + 1 \) as \( \mu_{\text{spinor}} = 1, \ \mu_{\text{adj}} = 5 \). For \( N_f \), the anomalies match when including the Lagrange multiplier charged under \( U(1)_R \). For \( 6 < N_f < 15 \) the superpotential is not polynomial and therefore there is no matching.

We have thus given two examples where the possibilities \( \mu = \mu_{\text{adj}} + k, \ k = 1, 2 \) for the index of the matter fields have been illustrated.

### 6 Syzygies and Duality

As shown in the previous section the gauge theories with \( \mu > \mu_{\text{adj}} + 2 \) are far more difficult to analyse than their counterparts with either no syzygies or a superpotential describing the low energy physics. Let us first recall the conventional lore \(^3\) about gauge theories with higher indices and their relation with syzygies.

It is generally assumed that theories with \( 2 + \mu_{\text{adj}} < \mu < 3 \mu_{\text{adj}} \) have an infrared fixed point \(^3\) where they are described by a superconformal theory. This fixed point describes either an interacting theory or it is a free theory in magnetic phase. Using this conjecture one can describe the set of chiral primary fields \(^5\) of the superconformal theories. The primary chiral superfields form a chiral ring under the operator product defined by the short distance expansion. This ring is identified with the ring of gauge invariant polynomials. The syzygies

---

\(^5\) Let us recall that chiral fields are defined by \( D \Phi = 0 \) where \( D \) is the supersymmetric fermionic derivative. Primary fields are such that \( \Phi \neq D \chi \) for a given superfield \( \chi \).
are therefore exact quantum relations between the chiral primary fields at the superconformal fixed point.

At these fixed points, the dimension $d$ of the chiral primary field is related to the $R$-charge by $d = \frac{3}{2} |R|$. Unitary conditions restrict these dimensions to be greater than one, the bound being saturated for a free field. So the description in terms of infrared fixed points breaks down when the dimensions of certain operators becomes formally less than one. For instance for vector-like theories with quarks and antiquarks this happens for the mesons when $\mu < \frac{3}{4}\mu_{adj}$. In that case the theory is supposed to possess a dual description where the dual mesons become a free field, leading to a "free magnetic phase". Unfortunately there is no prescription for constructing such dual models. Nevertheless the known examples have to satisfy stringent consistency checks.

In this section, we present, for two well known examples, a new check. Whereas in "electric theories" the gauge invariants are composite fields and so subject to syzygies, in the magnetic dual theories, some of them appear as elementary fields and do not have a priori to satisfy the syzygies. As suggested in [3, 36], these “magnetic syzygies” should appear as non-perturbative effects, which we will show explicitly by an appropriate regularization of the superpotentials generated non perturbatively in the magnetic theories.

Let us first describe the duality of the $SU(N_c)$ with $N_f > N_c + 2$ quarks and antiquarks. The dual gauge group [2] is $SU(N_f - N_c)$ with $N_f$ matter fields $q, \bar{q}$ in the fundamental and antifundamental representations respectively. The dual theory also possesses dual mesons $M_D$, invariant under $SU(N_f - N_c)$ and that couple to $q$ and $\bar{q}$ through a superpotential:

$$W = \frac{1}{\mu} \text{tr} \left( M_D q \bar{q}^t \right),$$

where $\mu$ is a scale parameter. This superpotential is not sufficient to show the equivalence between the original theory and its dual, and in particular is not sufficient to identify the dual mesons $M_D$ with the electric ones $Q\bar{Q}$. This requires non perturbative modification of the magnetic moduli space. Along a flat direction with $\langle M_D \rangle \neq 0$, the matter fields $q$ and $\bar{q}$ become massive and decouple from the low energy theory. The pure gauge theory then undergoes gaugino condensation which generates an extra contribution to the superpotential [36]

$$W_{n.p.} = (N_c - N_f) \left( \frac{\det M_D}{\Lambda^{N_c - N_f}} \right)^{1/(N_f - N_c)}.$$  \hspace{1cm} (29)

The analysis of the vacua of the dual theory necessitates to introduce source terms $W_{\text{reg.}} = \text{tr} (mM_D)$ in the superpotential. The vacua are obtained by taking the limit of zero sources. From the total superpotential $W + W_{n.p.} + W_{\text{reg.}}$, the equations of motion for the mesons lead to a vacuum satisfying

$$M_D = \left( \Lambda^{N_c - N_f} \det \left( \frac{m + \frac{qq^t}{\mu}}{\mu} \right) \right)^{1/N_c} \left( m + \frac{qq^t}{\mu} \right)^{-1}. \hspace{1cm} (30)$$
As in the electric theory, a flavour and gauge transformation diagonalize the matter fields

\[(q^i, \tilde{q}^i) = \begin{cases} (a_i, \tilde{a}_i) \delta^i_\alpha & \text{for } i \leq N_f - N_c, \\ 0 & \text{for } i > N_f - N_c. \end{cases} \]  

(31)

The matrix parameter \(m\) regularizes the matrices on the right hand side of (30). In the limit \(m \to 0\), the vacuum (30) becomes

\[\Lambda^{3N_c - N_f} \prod_{i=1}^{N_f - N_c} \frac{a_i \tilde{a}_i}{\mu} \left( \begin{array}{cc} 0 & 0 \\ 0 & m_D \end{array} \right), \]  

(32)

where \(m_D\) is a \(N_c \times N_c\) matrix of determinant one. The claim is that this relation exactly reproduces the syzygies of the electric theory. Indeed, according to ref. [3], the identification between the electric and magnetic gauge invariants is

\[M \to M_D, \]  

\[(B, \tilde{B})^{i_1 \ldots i_{N_f - N_c}} \to \sqrt{\Lambda^{3N_c - N_f} \mu^{N_c - N_f}} (b, \tilde{b})^{i_1 \ldots i_{N_f - N_c}},\]  

(33)

where \(b^{i_1 \ldots i_{N_f - N_c}}\) is defined by the totally antisymmetric gauge invariant combination \(q^{[i_1 \ldots q^{i_{N_f - N_c}]}\). Using the electric variables, the relation (32) reads

\[BB = || M^{N_c} ||, \]  

(34)

whereas the two other sets of electric syzygies are also trivially satisfied. In that sense, the magnetic theory with an appropriate regularization satisfies the same set of syzygies as the electric theory.

The same analysis can be applied to the Spin(7) theory. Its dual theory is a chiral \(SU(N_f - 4)\) gauge theory with \(N_f\) matter fields \(q^i\) in the antifundamental representation and one field \(S\) in the symmetric representation. There is also a symmetric meson matrix \(M_D\) which is a singlet under the gauge group. These fields are coupled by the superpotential

\[W = \frac{1}{\mu^2} \text{tr} (M_DqS \bar{q}) + \frac{1}{\mu^{N_f - 7}} \text{det} S. \]  

(35)

This tree-level superpotential needs to be completed by a non-perturbative part and source terms. As before, we consider a flat direction where the symmetric field is proportional to the identity and \(M_D \neq 0\). The matter fields become massive leading to gaugino condensation of the pure gauge theory. This leads to the following term in the superpotential

\[W_{n.p.} = (5 - N_f) \left( \frac{\text{det} M_D}{\Lambda^{15 - N_f}} \right)^{1/(N_f - 5)} \]  

\[W_{\text{reg.}} = \text{tr} (mM_D) + \text{tr} (mS), \]  

(36)
where sources $m$ and $m_s$ have been introduced for regularization purposes. Firstly, the $D$-terms have to vanish, i.e.
\[ 2S^1S - q^\dagger q = \lambda 1, \]
with a suitable real number $\lambda$. Then once again, gauge and flavour transformations can simultaneously diagonalize $q$ and $S$. The solutions of equations of motion derived from the total superpotential are,
\[ S = \left( -\mu^{N_f-7} \det \left( m_s + \frac{M_D qq^\dagger}{\mu^2} \right) \right)^{1/(N_f-5)} \left( m_s + \frac{M_D qq^\dagger}{\mu^2} \right)^{-1}, \]  
and
\[ M_D = \left( \Lambda^{15-N_f} \det \left( m + \frac{qS q^\dagger}{\mu^2} \right) \right)^{1/5} \left( m + \frac{qS q^\dagger}{\mu^2} \right)^{-1}. \]
The physical vacua correspond to the limit $m \to 0$ and $m_s \to 0$ where (38) and (39) become
\[ S = \left( -\mu^{N_f-7} \frac{M_{N_f-4} q_{N_f-4}^2}{\mu^2} \right)^{1/(N_f-5)} \left( \begin{array}{c} s \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right), \]  
$s$ being a $(N_f - 5) \times (N_f - 5)$ matrix of determinant one, and
\[ M_D = \left( \Lambda^{15-N_f} \prod_{i=1}^{N_f-4} \frac{q_i^2 S_i}{\mu^2} \right)^{1/5} \left( \begin{array}{c} 0 \\ 0 \\ 0 \\ m_D \end{array} \right), \]  
$m_D$ being a $5 \times 5$ matrix of determinant one. Quarks and mesons have only one non-vanishing flavour index in common that is taken here to be the $(N_f - 4)$th one. Equations (40) and (41) assures that only one $5 \times 5$ minor of $M_D$ is non vanishing and it is given by
\[ \| M_D^5 \| = -\Lambda^{15-N_f} \mu^{1-N_f} M_{N_f-4} bb, \]  
where $b$ is the only non vanishing dual baryon $b = \prod_{i=1}^{N_f-4} q_i$. Providing the following identification between electric and magnetic gauge invariants
\[ M \rightarrow M_d, \]
\[ B^{i_1\ldots i_{N_f-4}} \rightarrow \sqrt{-\Lambda^{15-N_f} \mu^{1-N_f}} b^{i_1\ldots i_{N_f-4}}, \]
the relation (42) is the only non trivial electric syzygy expressed in terms of magnetic variables.

So we have seen in the two examples explicitly studied how the syzygies of the electric theory are fulfilled in the magnetic theory thanks to non-perturbative effects that eliminate magnetic vacua without counterparts in the electric theory.
7 Geometry of the Moduli Spaces

In supersymmetric theories, the Kähler potential is sensitive to perturbative quantum effects, non-perturbative ones, threshold and decoupling corrections. For this reason it is not obvious how to extend the analysis of confinement and duality to this sector of the theory. Nevertheless, the Kähler potential encodes important information on the dynamics. It gives the σ-model geometry of the complex scalar manifolds, with the quantum modifications that includes a scale dependence. Therefore, one would like to understand the relationship between the Kähler structures of dual pairs of theories, for instance. Of course, a classical approach would be incomplete, but it could be of some usefulness if only the perturbative corrections are the most relevant, which is consistent with the fact that we do not expect quantum modifications of the moduli spaces in dual theories.

In this section, we present an approach to the geometry of the moduli space which is based on a study of the hidden symmetries of the scalar potential. This provides a parameterisation in terms of IR degrees of freedom. We show that the moduli space consists of one orbit of a non-compact group $F$ (the so-called commutant as it commutes with the gauge symmetry) and its closure, corresponding to the singular orbits. The commutant group is a non-trivial non-compact extension of the compact flavour symmetry. Therefore the geometry of the moduli space is that of a Kähler coset space $F/H$, where $H$ is the little group of the orbit.

Of course this structure is to be identified in both the dual theories, providing IR parameterisations along the lines of the usual approach to spontaneously broken symmetries. This would give access to a comparison of the classical Kähler potential of the dual theories, but there is a major obstacle in the fact that the parameterisations are not holomorphic. In spite of this failure, the possibility of identifying (in both senses) the geometry of the moduli space of dual theories is worthwhile presenting.

We shall carry out our analysis in the explicit context of the $SU(N_c)$ gauge theory with $N_f$ flavours of quarks and antiquarks. The classical moduli space is not modified non-perturbatively when $N_f > N_c$. The construction of the duality between the electric and the magnetic theories has been obtained in the conformal range $\frac{2}{3}N_c < N_f < 3N_c$ in which the electric and magnetic theories are both asymptotically free. We denote by $Q^i_\alpha$ and $\tilde{Q}^i_{\tilde{\alpha}}$ the corresponding scalar fields. The Kähler potential is
\[
K(Q, Q^\dagger, \tilde{Q}, \tilde{Q}^\dagger) = \text{tr} (Q^\dagger Q + \tilde{Q}^\dagger \tilde{Q}).
\] (44)

In order to identify this manifold, one has to determine the symmetries of the flat potential condition $V = 0$, namely the simultaneous zeros of the $D^A$'s and $F_i$'s. In the electric theory, due to the absence of superpotential, the vacua are only restricted to the $D$-flatness equations which are equivalent to the single

\[{}^6\text{For different attempts see refs. [38, 39].}\]
relation:

\[(Q_i^\alpha)^* Q_\beta^\beta - \tilde{Q}_\alpha^\alpha (\tilde{Q}_\beta^\beta)^* = \lambda \delta_{\alpha\beta},\]  

(45)

where \(\lambda\) is a real number. Equation (45) explicitly exhibits an \(SU(N_f, N_f) \times U(1)_B\) flavour invariance [14], acting on the \(2N_f\) component vectors \((Q, \tilde{Q}^*)\), including a Cartan generator corresponding to the \(U(1)_R\). Moreover there is an obvious invariance under dilation. Finally the symmetry of the moduli space is

\[\mathcal{F}_e = U(N_f, N_f) \times D.\]  

(46)

Notice that holomorphy is not preserved by the action of \(\mathcal{F}_e\). This is the main drawback of this approach where the supersymmetry is not explicit.

The moduli space corresponds to the following orbits of \(\mathcal{F}_e\). There are two conjugated baryonic orbits corresponding to \(\lambda > 0\) and \(\lambda < 0\). They are called baryonic orbits as there is always a non-zero baryon on these branches of the moduli space. Each of them forms a single orbit under the symmetry group \(\mathcal{F}_e\). The case \(\lambda < 0\) is obtained from \(\lambda > 0\) by exchanging the roles of \(Q\) and \(\tilde{Q}^*\). Their common boundary \(\lambda = 0\) is the mesonic orbit.

Therefore, the vacua manifold has been identified as a finite set of \(SU(N_f, N_f) \times U(1)_B \times SU(N_c) \times D\) orbits. Each of these orbits can be represented by the quotient of the symmetry group (acting transitively on the orbit) by the stabilizer (or little group) of one point. Thus we have to identify the stabilizer of each representative point considered earlier to characterize each orbit.

For the baryonic orbit, the little group associated to the generic vacuum takes a simple structure of direct product

\[\mathcal{H}_e = SU(N_f - N_c, N_f) \times U(1) \times SU(N_c)_D,\]  

(47)

where \(SU(N_c)_D\) is the diagonal combination of the gauge \(SU(N_c)_G\) and an \(SU(N_c)\) subgroup of \(U(N_f, N_f)\). The \(U(1)\) is a combination of the \(U(1)_B\) and an element of the Cartan subalgebra of \(SU(N_f, N_f)\). Here the gauge group is completely broken; then after eliminating the spurious massless scalars associated to the Higgs mechanism, the real dimension of the coset \(\mathcal{F}_e/\mathcal{H}_e\) is \(4N_f N_c - 2N_c^2 + 2\).

In order to further characterize the geometry of the baryonic branch of moduli space, we first extract a flat subspace associated to the diagonal \(GL(1, \mathbb{C})\) factor in the coset and then introduce as coordinates the \(N_c \times (2N_f - N_c)\) complex matrix \(z\) and define its transformation under \(U(N_f, N_f)\) as

\[z \rightarrow (Az + B)(Cz + D)^{-1}.\]  

(48)

The elements of the coset are then parameterized by the exponentials \(e^{t(z)}\)

\[\begin{pmatrix} Q \\ \tilde{Q}^* \end{pmatrix}|_{D^{A=0}} = e^{t(z)} \begin{pmatrix} 1_{N_c} \\ 0 \end{pmatrix},\]  

(49)

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where
\[ t(z) = \begin{pmatrix} 0 & z \\ -\eta z^\dagger & 0 \end{pmatrix}. \tag{50} \]
and \( \eta \) is the \((N_f - N_c, N_f)\) signature. There is only one \( U(N_f, N_f)\)-invariant Kähler potential up to a Kähler transformation, namely: \( K = \text{tr} \ln(1 + z \eta z^\dagger) \). Nevertheless as \( U(N_f, N_f) \) is not a symmetry of the theory (it does not preserve the kinetic terms), this is not the Kähler potential of the theory on the moduli space.

Let us now check the correspondence between the massless scalar fields and the moduli fields. As already discussed previously, since there is no superpotential and the whole gauge group is Higgsed around the vacuum, all the scalars are moduli with the exception of the \( N_c^2 - 1 \) complex scalars given by \( T_A^{i\beta} Q_i^{\beta} \), associated to the \( SU(N_c) \) massive vector multiplets. All the \( 4N_f N_c - 2N_c^2 + 2 \) remaining scalar fields are massless, and their number coincides with the real dimension of \( G_c/H_c \).

For the mesonic orbit, the pattern is more complicated since the little group associated to the generic vacuum now has a structure of semi-direct product, \( H_r \times SU(N_c - r)_G \), with:
\[ H_r = SU(N_f - r, N_f - r) \times U(1)^2 \times SU(r)_D \rtimes \tilde{H}, \tag{51} \]
where \( SU(r)_D \) is the diagonal combination of the gauge and flavour \( SU(r) \) subgroups, the two \( U(1)\)'s are combinations of the \( U(1)_B \), an element of the Cartan subalgebra of \( SU(N_f, N_f) \) and an element of the Cartan subalgebra of \( SU(N_c)_G \). The semi-direct factor \( \tilde{H} \) is a nilpotent subgroup \(^7\) with generators transforming as \((2(N_f - r), \tau) \oplus (2(N_f - r), r)\) under \( U(N_f - r, N_f - r) \times SU(r)_D \) and the Abelian subalgebra defined by their commutators, which transform as \((1, \text{Adj} \oplus 1)\).

The mesonic orbits are stratified by the index \( r \). The stratum corresponding to \( r = N_c \) is such that the gauge group is completely broken. In fact the mesonic orbits correspond to the “infinitely boosted” baryonic orbit. Hence the moduli space is the closure of the baryonic orbit, \( i.e. \) by applying appropriate boosts and a global dilation to the baryonic orbit one can converge to all the strata of the mesonic orbits. In the same way, by applying infinite boosts, we can go from a mesonic orbit with a stratification index \( r \) to a more singular one with lower index. Geometrically, these boosts correspond to the shrinking of some circles in the moduli space. From a physical point of view, the stratification index is related to the number of massless singlets of the theory. As the stratification index goes from \( r \) to \( r - 1 \), the corresponding orbits differ by a dimension \( 4(r - N_f) - 2 \). In terms of the solutions of the syzygies, the baryonic orbit is equivalent to the open set characterized by \( b \neq \tilde{b} \) while the mesonic orbits correspond to \( b = \tilde{b} \). One retrieves the fact that the mesonic orbits form the natural boundary of the baryonic orbit.

\(^7\) Just as for the Lorentz group, the little groups of singular orbits of non compact groups have semi direct products with nilpotent subgroups.
We now turn to the analysis of the dual theory, or magnetic theory, described in the previous section. Notice in particular the non-perturbative superpotential (29), which restricts the solutions so that the meson field matrix has rank less than $N_c$. This is crucial in identifying the dual moduli spaces. The magnetic theory possesses the same anomaly-free global symmetries as the electric one $SU(N_f) \times SU(N_f) \times U(1)_B \times U(1)_R$, which transform the dual quark superfields as $(\bar{N}_f, 1, N_c/(N_f - N_c), N_c/N_f)$, the antiquark ones as $(1, N_f, -N_c/(N_f - N_c), N_c/N_f)$ and the gauge-singlet superfields $M$ as $(N_f, \bar{N}_f, 0, 2(N_f - N_c)/N_f)$.

The classical moduli space of the magnetic theory is identified with the solutions of the $F$- and $D$-terms. A thorough analysis of these equations and their relationship with the electric moduli space has been given in [37]. In summary

$$q = e^{t(u)} q_0 ;$$
$$\tilde{q} = 0 ;$$
$$M = (0 M_0) e^{t(u)^t}$$

where

$$t(u) = \begin{pmatrix}
0 & u \\
-u^t & 0
\end{pmatrix},$$

with $u$ a $(N_f - N_c) \times N_c$ matrix which parameterises the coset $U(N_f) / U(N_f - N_c) \times U(N_c)$. The following $N_c \times (2N_f - N_c)$ complex matrix can be constructed:

$$z_m = (u^T M_0^T).$$

Consider the homographic action of the group $U(N_f, N_f)$ on $z_m$, analogous to (18). The magnetic moduli space is invariant under this non-linear action of $U(N_f, N_f)$. Consider the image of the origin $z_m = (0, 0)$. As the action on $z_m$ is the same as (18) on the electric baryonic orbit we know that the image of the origin is the whole baryonic orbit. Therefore the baryonic branches of the electric and magnetic moduli spaces are isomorphic as non-compact complex manifolds. The dualising map reads simply $z \leftrightarrow z_m$. This isomorphism is valid at the level of the classical moduli spaces. The non-perturbative part of the superpotential is necessary to study another branch of the dual theory. In the case $q = \tilde{q} = 0$ the mesons $M$ is restricted by the superpotential (29) to have rank $N_c$.

The Kähler potential of the magnetic theory is not known. If the quark kinetic term is expected to become canonical in the UV region, there is no reason for the Kähler geometry of the meson fields to be trivial. One can try to use the isomorphism between the baryonic orbits of the dual theories in order to deduce the Kähler potential of the dual meson field in the UV region. However, we cannot use this twin parameterisation of the electric and magnetic fields in the classical part of the moduli space to deduce the magnetic Kähler potential from the electric one (44). Indeed, the induced metric cannot be straightforwardly
calculated from (49) and (50) because this is not an analytic transformation of variables consistent with the Kählerian geometry. Still, it provides a link that we hope to be able to exploit in the future.

8 Conclusions

We have argued that the 't Hooft matching of the anomalies, when the normal space to the manifold defined by the syzygies is a real representation of the flavour unbroken subgroup, is equivalent to the existence of a polynomial superpotential that has been put forward to characterize the confinement in $N = 1$ supersymmetric gauge theories. This allows a complete classification of the confining theories for $\mu \geq \mu_{\text{adj}}$. Non-confining theories, with $3\mu_{\text{adj}} > \mu > \mu_{\text{adj}} + 2$, are expected to have dual(s). Some of the properties concerning both confinement and duality discussed in this paper have been illustrated by the analysis of two series of theories, $SU(N_c)$ and $Spin(7)$, with a descent relation between the successive decoupling of flavours.

For all confining theories the IR superpotentials for the gauge invariant composites are completely fixed by the flavour symmetries. We have shown, in theories characterized by non-trivial syzygies, that the following additional requirements are strongly related: (i) the matching of the anomalies, (ii) polynomiality of the superpotential, and (iii) the equations of motion leading to the syzygies. We have stressed the equivalence of (i) and (ii) by constructing a superpotential in the case of a normal space in a real representation of the residual flavour group, and we have partially proved the equivalence with (iii) (we were unable to prove that the syzygies are the only solutions of the equations of motion). Of course, one would like to have also a proof of the conjecture that the normal space to the zeros of the syzygies are not in an anomaly free complex representation of the unbroken flavour subgroup.

In the case of dual theories, we have checked, with the aid of explicit examples, the consistency between duality and syzygies. This property is verified only if non-perturbative superpotentials are included.

Finally we have discussed how a non-compact hidden symmetry of the vacua characterizes the classical moduli space geometry, but the non-analyticity of the action of the non-compact hidden group makes it difficult to gain further insight into the Kähler geometry of the dual theories.

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