Embedding a balanced binary tree on a bounded point set

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Abstract. Given an undirected planar graph $G$ with $n$ vertices and a set $S$ of $n$ points inside a simple polygon $P$, a point-set embedding of $G$ on $S$ is a planar drawing of $G$ such that each vertex is mapped to a distinct point of $S$ and the edges are polygonal chains surrounded by $P$. A special case of the embedding problem is that in which $G$ is a balanced binary tree. In this paper, we present a new algorithm for embedding an $n$-vertex balanced binary tree $BBT$ on a set $S$ of $n$ points inside a simple $m$-gon $P$ in $O(m^2 + n \log^2 n + mn)$ time with at most $O(m)$ bends per edge.

Key words. Point-set embedding. Bounded point set. Simple polygon. Balanced binary tree. Straight skeleton.

1. Introduction

Let $G$ be an undirected $n$-vertex planar graph, and $S$ be a set of $n$ points. The point-set embedding problem, also known as the point-set embeddability problem, aims at drawing $G$ with no edges crossing, such that each vertex is mapped to a distinct point of $S$ and the edges are polygonal chains.

The largest subclass of planar graphs that admits a straight-line embedding, i.e. an embedding with no edge-bends, on any point set is the class of outer planar graphs; this was shown by Gritzmann et al. \cite{11} for the first time, later it was rediscovered by Castaneda et al. \cite{7}, and finally Bose \cite{4} presented an efficient algorithm for it. The point-set embeddability problem has been extensively studied \cite{4,5,6,7,8,9,10,11,12,13,14,15,16}, but few researches have considered a bounding polygon for the point set \cite{3}. Note that deciding if there is a straight-line embedding of a general graph on a given set of points $S$ is NP-hard \cite{6}.

The constraint that the edges should be placed inside a surface with prescribed shape can be important for example to design a wired network on a surface with given shape.

Let $T$ be a tree with $n$ vertices and let $S$ be a set of $n$ points inside a polygon $P$ with $k$ reflex vertices, by an immediate implication of Lemma 7 of \cite{9} there exists an $O(n^2 \log n)$ time algorithm that computes a point-set embedding of $T$ on $S$ inside $P$ such that each edge has at most $2 \left\lfloor \frac{k}{2} \right\rfloor$ bends.

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Given a simple polygon $P$ with $m$ vertices, Bagheri et al. \cite{3} has presented an $O(m^2n^{1.2})$ algorithm for planar poly-line drawing of an $n$-vertex complete binary tree on the surface of $P$, such that the total number of the edge bends is bounded by $O(mn^{1.6})$, in the worst case.

Let $BBT$ be an $n$-vertex balanced binary tree, i.e. a binary tree in which the depth of the two subtrees of every node differs by at most one. In this paper, we present a new algorithm for point-set embedding of $BBT$ on a set of $n$ points $S$ surrounded by a simple $m$-gon $P$, in $O(m^2 + n\log^2 n + mn)$ time with at most $O(m)$ bends per edge. We use the partitioning idea of Bagheri et al. \cite{3} in our embedding algorithm.

This paper is organized as follows. Preliminary definitions are in Section 2. In Section 3 we present our new embedding algorithm for embedding a balanced binary tree on a point set surrounded by a simple polygon. In Section 4 we present an example for our new embedding algorithm. Conclusions and open problems are in Section 5.

2. Preliminaries

The skeleton of a simple polygons is a one-dimensional structure that describes the two-dimensional structure of the polygon with lower complexity than the original polygon.

There are two types of skeletons for a simple polygon, the medial axis and the straight skeleton. The medial axis of a simple polygon consists of all interior points of the simple polygon whose closest point on the boundary of it is not unique. The straight skeleton $SS$ is defined as the union of the pieces of angular bisectors traced out by the polygon vertices during a shrinking process \cite{1}. In this shrinking process all edges of $P$ are contracted inwards parallel to themselves and at a constant speed. So, each vertex of $P$ moves along the angular bisector of its incident edges until the boundary does not change topologically. Then, a new node is created and the shrinking process continues in the resulting polygon(s) until the area of all the resulting polygons becomes zero.

In a partition of a polygon into subpolygons, two subpolygons sharing an edge are called neighbours. By Lemma 1, the straight skeleton of a simple polygon $P$ with $m$ vertices partitions the interior of $P$ into a cycle of $m$ monotone neighbor subpolygons, called faces. According to Theorem 1 of \cite{1}, each face of $P$ shares exactly one edge, called the boundary edge, with the polygon $P$.

**Lemma 1.** \cite{1} The straight skeleton of a simple polygon $P$ with $m$ vertices consists of exactly $m$ connected faces, $m - 2$ dummy vertices, and $2m - 3$ dummy edges.

**Lemma 2.** \cite{1} Each face $f(e)$ is monotone in direction of its boundary edge $e$.

We denote the edge of $P$ that is adjacent to the face $f$ by $e(f)$, and the corresponding face of the boundary edge $e$ by $f(e)$. The bisector pieces of the straight skeleton of $P$ are called dummy edges, and their endpoints that are not vertices of $P$ are called dummy vertices. We call the dummy edge that is connected to a boundary edge the boundary connected dummy edge or $bcd$ edge. The dummy edges that are not $bcd$ edges are called the internal dummy edges.

So, each face has one boundary edge and two $bcd$ edges. Also, it may have some internal dummy edges. Figure \cite{1}(a) shows the straight skeleton of a simple polygon, in which the line segments $ab$ and $de$ are the two $bcd$ edges of the $f(ae)$, and $bc$ and $cd$ are the internal dummy edges. A corner $v$ of a polygon in the plane is said to be a reflex corner if the
angle at $v$ inside the polygon is greater than 180 degrees. In Figure 1(b) the dashed line splits the reflex dummy vertex into two convex corners.

The neighbor vertices of each vertex of a straight skeleton $SS$ are the vertices of $SS$ that are adjacent to it. The set of all the faces that are adjacent to a vertex of $SS$ are the neighbor faces of it. In Figure 1(a), the vertices $b$ and $d$ are the neighbor vertices of $c$, and the face $abcde$ is the neighbor face of the vertices $a$, $b$, $c$, $d$, and $e$. For each edge $(i, j)$ of $SS$ we define its weight by subtracting sum of the areas of adjacent faces of $i$ from the sum of the areas of adjacent faces of $j$. The middle edge of $SS$ is the edge that has the minimum positive weight among all the other edges of $SS$. The middle point of $SS$ is the middle point of its middle edge, if the middle edge is unique, otherwise the common point of the middle edges.

Now, we review the drawing algorithm presented by Bagheri et al. Let $CBT$ be an $n$-vertex complete binary tree and $P$ be a simple polygon with $m$ vertices, their algorithm draws $CBT$ on the surface of $P$ in $O(m^2n^{2.2})$ time with $O(mn^{1.6})$ total edge bends. This algorithm partitions the surface of the given simple $m$-gon into $n+1$ equal area subpolygons (where $n$ is the number of the nodes of the input complete binary tree) as follows.

The bisecting chain is a polygonal chain partitioning the input simple polygon into two equal area subpolygons $R$ and $L$. Their algorithm first computes a bisecting chain $C$ for the input simple polygon by a nonrecursive bisecting algorithm using the straight skeleton of $P$. The root of the input complete binary tree $CBT$ is laid on $C$. Then, the bisecting chains of the subpolygons $R$ and $L$ are computed such that each bisecting chain has one end point on $C$. This process is recursively repeated for drawing the left and right subtrees of $CBT$ inside $L$ and $R$, respectively. Bisecting the new subpolygons continues until getting $\frac{n+1}{2}$ new subpolygons. In fact, they lie the edges and internal nodes of the $CBT$ on the partitioning line segments and each leaf on the surface of one of the subpolygons.

3. The new embedding algorithm

In this section, we describe our new embedding algorithm which embeds an $n$-vertex balanced binary tree $BBT$ on a point set $S$ surrounded by a simple polygon $P$ with
$m$ vertices. We use the idea of Bagheri et al. [3]: using the straight skeleton of $P$ we decompose the surface of $P$ into convex regions such that each region contains zero or one single point of $S$ and the remaining points lie on the partitioning line segments. We use the partitioning chains as the edges.

Fig. 2. Splitting a reflex vertex of a face into two convex vertices.

Fig. 3. Case I: $\hat{xyw} > 180$ degrees.

Lemma 3. Let $a$ be a straight line that passes through the line segment $e(f)$. Each line perpendicular to the straight line $a$ passing through a reflex vertex of $f(c)$, splits the reflex vertex into two convex vertices.
Proof. In Figure 2 we pass through the reflex vertex \( \overline{xyz} \) a straight line \( b \) perpendicular to the line \( a \), where \( x, y \) and \( z \) are three consecutive vertices of \( f(v_i, v_{i+1}) \). Assume that the lemma is false. Let \( b \) be a line passing through the reflex vertex \( \overline{xyz} \) and perpendicular to the straight line \( a \), and let \( w \) be the intersection point of the lines \( a \) and \( b \). The line \( b \) splits \( \overline{xyz} \) vertex into the vertices \( \overline{xyw} \) and \( \overline{wyz} \). Two cases arise: \( \overline{xyw} > 180 \) degree and \( \overline{wyz} > 180 \) degree. Figure 3 shows the first case, \( \overline{xyw} > 180 \). In Figure 3 there exists a line \( c \) perpendicular to the straight line \( a \) that intersects \( f(v_i, v_{i+1}) \) in more than two points and this contradicts Lemma 2. Figure 4 shows the second case, \( \overline{wyz} > 180 \) degrees. In Figure 4 there is a line \( c \) perpendicular to the straight line \( a \) that intersects \( f(v_i, v_{i+1}) \) in more than two points and this contradicts Lemma 2.

In this paper, the functions \( \text{Num}(BBT) \) returns the number of the nodes of the balanced binary tree \( BBT \). Also, the function \( \text{Num}(P) \) returns the number of the points inside the simple polygon \( P \). Let \( BBT_1 \) and \( BBT_2 \) denote the left and right subtrees of \( BBT \), respectively. Our algorithm consists of two general steps, the recursive and the non recursive step (see Algorithm 1). In the first step, the non recursive step, we compute a partitioning chain \( C \) that divides the input simple polygon into two subpolygons \( R \) and \( L \), such that the number of the points in \( L \) and \( R \) are equal to \( \text{Num}(BBT_1) \) and \( \text{Num}(BBT_2) \), respectively and the remaining single point lies on \( C \). Then in the second step, we recursively partition the new subpolygons \( L \) and \( R \) into other subpolygons until we get convex regions with zero or one single point in each region. If a point lie on a straight skeleton edge or on a decomposing edge, we assume that it is contained into an arbitrary face that is incident to it. In the following, we explain these steps.

**Step 1. The non recursive step.**

In this step, the partitioning algorithm \( \text{PartitioningAlg} \) is called (line 3 of Algorithm 1). It partitions the surface of \( P \) into two subpolygons \( R \) and \( L \) using the straight skeleton of \( P \), such that \( \text{Num}(BBT_1) = \text{Num}(L) \), \( \text{Num}(BBT_2) = \text{Num}(R) \), and the remaining single point lies on the partitioning chain.
Algorithm 1 The Embedding Algorithm

Input: A simple polygon $P$, a balanced binary tree $BBT$, a point set $S$ inside $P$

Output: A point-set embedding of $BBT$ on $S$ such that the edges are polygonal chains inside $P$

1: if $BBT$ is null then
2: Stop
3: $(R, L, SCR, SCL, q, SSSR, SSSL) =$ PARTITIONINGALG$(P, S, BBT)$
4: RECEMBEDDING$(L, SSSL, SCL, q, BBT_1, l)$
5: RECEMBEDDING$(R, SSSR, SCR, q, BBT_2, r)$
6: Map the root of $BBT$ on the point $q$

We first partition $P$ into a sequence of the neighbor convex subpolygons using the straight skeleton of $P$ as follows. We find the straight skeleton $SS$ of $P$. Then, we split each reflex vertex of the faces into the convex vertices. By Lemma 3, we can split each reflex vertex of a face $f(e)$ into two convex vertices using a straight line passing through the reflex vertex and perpendicular to the straight line that contains the edge $e$. We find the middle point of $SS$, called $ms$, lying on the boundary of one or more faces of $SS$. We select one of the incident faces of $ms$ as the starting face $sf$. Let $\partial P$ be the boundary of $P$, considered as a counter clockwise path $CCW$, starting from and ending at the boundary edge of the starting face $sf$. Starting from $sf$, we process each face, one by one, in the order its boundary edge appears in $CCW$, and add the splitting line segments to the reflex vertices. So we pass a straight line segment, called splitting line segment, through each reflex vertex of the faces of $P$. The union of $SS$ and the splitting line segments is a structure called split straight skeleton or $SSS$. After adding the splitting line segments, all subfaces of $SSS$ are convex; in Figure 1(b) the dashed line shows the splitting line segment. The function $Split(P)$ adds the splitting line segments to $SS$. Each face of $P$ is incident to one edge of $P$, so $SSS$ partitions the surface of $P$ into a cycle of the neighbor convex subfaces.

Then, we pass through $ms$ a straight line segment, called $sl_1$ that is perpendicular to $e(f(ms))$ (Figure 5). By Lemma 2 this line splits one of the subfaces of $f$ into at most two simple subfaces. We call the subface that is in the same direction with $e(f)$, $f(t)$ and the other subface, if exists, $f(m)$.

Fig. 5. Two new subfaces $f(m)$ and $f(t)$. 
Let \( SC \) be a sequence of the consecutive subfaces of \( SSS \), where the faces are in CCW order and the subfaces of each face \( f \) are ordered with increasing \( e(f) \) direction such that \( SC.first = f(t) \) and \( SC.last = f(m) \) (see Figure 6). Starting from \( f(t) \) we process the subfaces, one by one, in the order that they appear in \( SC \). Let \( S_1 = Num(BBT_1) \) denote the number of the nodes of the left subtree of \( BBT \). Let \( SUM \) denote the sum of the points surrounded by the processed subfaces. We process the subfaces until \( SUM \) be greater than \( S_1 \) and then stop. Then, we split the current subface \( cf \) into two subfaces \( cf(m) \) and \( cf(t) \) using a straight line segment, called the dividing line segment. Note that \( cf(m) \) is the subface that is not in the same direction with \( e(cf) \) and the other subface is \( cf(t) \). Moreover, the dividing line segment must intersect a point of \( cf \) such that the sum of the points surrounded by the processed subfaces, i.e. \( SUM \), plus \( Num(cf(m)) \) be equal to \( S_1 \). Each subface may have four types of the edges: a boundary edge, two boundary connected edges, the splitting line segments, and some dummy edges. We select a point on one of the dummy edges of the current subface, called center point. If the current subface \( cf \) has no dummy edges and so is a triangle, we select the intersection point of the two \( bcd \) edges of it. In fact, we use the dividing line segments used in proof of Lemma 6 by Gritzmann et al. [11]. Therefore, we order all the points surrounded by \( cf \) radially around the center point, find \((S_1 - SUM + 1)th\) point inside \( cf \), and index it by \( q \). A dividing line segment \( sl_2 \) is a line segment passing through the center point and \( q \). The dividing line segments exist since all subpolygons are convex. So we partition the simple polygon \( P \) into two new subpolygons \( R \) and \( L \), such that \( L \) is the subpolygon between \( sl_1 \) and \( sl_2 \) that contains \( f(t) \) and \( R \) is the subpolygon between \( sl_1 \) and \( sl_2 \) that contains \( f(m) \) (Figure 7). In fact, the subfaces form \( f(t) \) to \( cf(m) \) in the CCW direction constitute \( L \). And \( R \) consists of the subfaces form \( f(m) \) to \( cf(t) \) in the clockwise direction.

Consider the boundary of the subpolygon \( L \) as a clockwise path \( path_1 \) and the boundary of \( R \) as a counter clockwise path \( path_2 \) starting from and ending at \( sl_2 \). Then, \( SCL \) and \( SCR \) are two lists of consecutive subfaces in \( path_1 \) and \( path_2 \) order, respectively. Figure 8 shows \( SCL = f_1 \rightarrow f_2 \rightarrow f_3 \rightarrow f_4 \rightarrow f_5 \) (the subfaces of \( L \)) and \( SCR = f'_1 \rightarrow f'_2 \rightarrow f'_3 \rightarrow f'_4 \rightarrow f'_5 \) (the subfaces of \( R \)). Let \( SSS(R) \) and \( SSS(L) \) be functions that return the portions of the \( SSS \) that corresponds to \( R \) and \( L \), respectively. The pseudo-code of the non recursive algorithm is given in the Figure ??.

![Fig. 6. A sequence of the neighbor faces.](image)

**Step 2. The recursive step.**
Given the point \( q \) and the new subpolygons \( R \) and \( L \) computed in the previous step, we call the recursive embedding algorithm, called \( \text{RecEmbeddingAlg} \), for embedding the left and right subtrees of \( BBT \) on the points inside \( L \) and \( R \), respectively (line 4 and 5 of Algorithm 1).

The recursive embedding algorithm \( \text{RecEmbeddingAlg} \) takes a simple polygon \( P \) which consists of a sequence of the neighbor subfaces \( SC \), the skeleton like structure \( SLS \) of \( P \), a point \( q \in S \) on the border of the first subface of \( SC \), and a balanced binary tree \( BBT \). Moreover, the last input \( sid \) indicates whether the input subpolygon is \( R \) or \( L \). The output of \( \text{RecEmbeddingAlg} \) is a point set embedding of \( BBT \) on \( S \) inside \( P \) (see Algorithm 3).

We have computed the straight skeleton in the previous step, while in this step we compute the straight skeleton like structure \( SLS \) of the input simple polygon as follows. Given the sequence of the consecutive neighbor subfaces \( SC \) and the point \( q \in S \) on the border of the first subface of \( SC \), we insert a dummy node at the middle point of the common border of each two neighbor subfaces of \( SC \) called the backbone point. Moreover, we insert a dummy node at the middle point of

\[
\text{Perimeter}(SC.\text{last}) - \text{Commonedge}(SC.\text{last}, SC.\text{last.prev}),
\]
Algorithm 2 The Partitioning Algorithm

**Input:** A simple polygon $P$, a point set $S$ inside $P$, a balanced binary tree $BBT$

**Output:** Two simple polygons $R$ and $L$, two lists $SCR$ and $SCL$, the point $q \in S$ on the partitioning polygonal chain, two skeleton like structures $SLS(R)$ and $SLS(L)$

1. Set $SS = StraightSkel(P)$, $ms = MidPoint(SS)$, $SSS = Split(SS)$, $f = an\ incident\ face$ to $ms$, $SUM = 0$, $BBT_1 = LeftSubTree(BBT)$
2. Split $f$ into $f(m)$ and $f(t)$ using a line passing through $ms$
3. Create the list of the subfaces of $SSS$ called $SC$
4. $S_1 = Num(BBT_1)$
5. $f = SC.first$
6. while $SUM + Num(f) \leq S_1$ do
7. $SUM = SUM + Num(f)$
8. $f = f.next$
9. $cf = f$
10. Divide $cf$ into $cf(m)$ and $cf(t)$ using a line passing through the point $q$ inside $cf$
11. Construct the list of the left new subpolygon $L$ subfaces called $SCL$
12. Construct the list of the right new subpolygon $R$ subfaces called $SCR$
13. Return two new subpolygons $R$ and $L$, $SCR$ and $SCL$, the point $q$, $SSS(R)$ and $SSS(L)$.

Consider the backbone points of the neighbor subfaces of $SC$, the starting point $q$, and the point $q'$ (Figure 8 depicts these points on the subfaces of $L$ and $R$). We connect these points to each other, using straight line segments, called backbone line segments. Consider the backbone as a directed path $qq'$ that partitions the consecutive subfaces into two sequences of the subfaces, the sequence of the subfaces that are in the left side of the backbone, called $SCL$ and the ones in the right side of it, called $SCR$. We slightly perturb the backbone line segments such that all new subfaces be convex, since finding a dividing line segment requires a convex simple polygon.

Now, we construct a new list by concatenating the lists $SCR$ and $SCL$. Two cases arise; $sid = r$ and $sid = l$. In the first case, that is the case in which the input subpolygon is $R$, let $SC = Concatenation(SCR, SCL)$ where $SC.first = SCR.first$ and $SC.last = SCL.last$. Now, assume that the input subpolygon is $L$, in this situation $SC = Concatenation(SCL, SCR)$ where $SC.first = SCL.first$ and $SC.last = SCR.last$.

Then, we process the subfaces of $SC$, starting from $SC.first$, one by one (Figures 9 and 10 depict the consecutive subfaces of $L$ and $R$). As the non recursive step, when the sum of the points surrounded by the processed subfaces, $SUM$, is greater than the number of the nodes of the left child of $BBT$, $Num(BBT_1)$, the process stops. Then, we select a point of the backbone that is incident to the current subface $cf$, called the point $t'$, order all the points surrounded by $cf$ radially around $t'$, and find $(Num(BBT_1) - SUM + 1)th$ point inside $cf$, called $t$. Then, we divide the current subface by the dividing line segment passing through $t'$ and $t$. We map the root of $BBT$ on $t$, and connect $t$ to $q$ using $tt'$ and the portion of the backbone between $q$ and $t'$. And finally, we recursively call the recursive embedding algorithm for subtrees of $BBT$, and the two newly created subpolygons (see Figure 13).

**Theorem 1.** Given an $n$-vertex balanced binary tree $BBT$ and a set of $n$ points $S$ bounded by a simple $m$-gon $P$, we can embed $BBT$ on $S$ inside $P$ using our embedding algorithm.
Algorithm 3 The Recursive Embedding Algorithm

**Input:** A simple polygon \( P \), its skeleton like structure SLS, a sequence of the consecutive subfaces \( SC \), a balanced binary tree \( BBT \), a point set \( S \) inside \( P \), an starting point \( q \in S \) which is incident to \( SC.first \), the direction \( sid \) that in which the input subpolygon is placed.

**Output:** A point set embedding of \( BBT \) on \( S \) inside \( P \)

1. if \( BBT \) is null then
2. Stop
3. \( SUM = 0 \)
4. Set \( BBT_1 = LeftSubTree(BBT) \), and \( BBT_2 = RightSubTree(BBT) \)
5. Construct the backbone of \( SC \)
6. Construct two new lists of the subfaces \( SCR \) and \( SCL \)
7. if \( sid = r \) then
8. \( SC = Concatenation(SCR,SCL) \)
9. else
10. \( SC = Concatenation(SCL,SCR) \)
11. \( f = SC.first. \)
12. while \( SUM + Num(f) \leq S_1 \) do
13. \( SUM = SUM + Num(f) \)
14. \( f = f.next \)
15. Set \( cf = f \)
16. Divide \( cf \) into \( cf(m) \) and \( cf(t) \) using a line passing through the point \( t \) inside \( f \)
17. Construct the subface list of the left new subpolygon \( L \) called \( SCL \)
18. Construct the subface list of the right new subpolygon \( R \) called \( SCR \)
19. Map the root of \( BBT \) on the point \( t \)
20. Connect \( q \) to \( t \)
21. \( \text{RecEmbeddingAlg}(R,SLS(R),SCR,t,BBT_1,r) \)
22. \( \text{RecEmbeddingAlg}(L,SLS(L),SCL,t,BBT_2,l) \)

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**Fig. 9.** The direction of processing the new subfaces.

\( \text{in } O(m^2 + n \log^2 n + mn) \) time, such that the bend number of each edge is bounded by \( O(m) \) in the worst case.

**Proof.** In Step 1, we first compute the straight skeleton of the input simple \( m \)-gon using the algorithm of [10] in \( O(m^2) \) time, and then add the splitting line segments in \( O(m) \)
time. Processing the points inside the subfaces and finding the dividing line segment take $O(n+m)$ and $O(n \log n)$ time, respectively. Hence, Step 1 runs in $O(m^2 + n \log n)$ time.

In Step 2, we recursively solve two subproblems, each of size $O(n/2)$, which contributes $2T(n/2)$ to the running time. The time complexity of the recursive embedding algorithm, $T(n)$, is bounded by $O(n \log^2 n + mn)$ by solving the recursive equation

$$T(n) = 2T\left(\frac{n}{2}\right) + n \log n + O(m).$$

Where $n \log n$ is the time needed for sorting $n$ points inside the simple polygon and finding the dividing line segment. Moreover, $O(m)$ is the time of constructing the new skeleton like structures of new subpolygons in each recursion. So, the time complexity of the embedding algorithm is $O(m^2 + n \log^2 n + mn)$.

The bend number of each edge is bounded by $O(m)$ in the worst case, since each edge of the input balanced binary tree is a portion of one of the straight skeleton like structures SLS. Recall that the SLS structures are constructed by connecting the middle points of the $bcd$ edges and the splitting edges, whose total number is bounded by $O(m)$, to each other.

4. An Example

In this section, we present an example for point-set embedding of a 15-vertex complete binary tree on the point set of Figure 11 using our new algorithm. In Figures 12, 14, and 15 the vectors depict the paths used as the edges from the circled nodes to their children. Figure 16 represents the output of our embedding algorithm after removing the additional line segments.

5. Concluding Remarks

In this paper, we introduce a new algorithm for point set embedding of an $n$-vertex balanced binary tree $BBT$ on a set of $n$ points $S$ bounded by a simple polygon $P$ with $m$ vertices. Our new embedding algorithm computes the output in $O(m^2 + n \log^2 n + mn)$ time with at most $O(m)$ bends per edge. As a future work, we can concentrate on the other restricted types of the graphs and polygons.
Fig. 11. A point set surrounded by a simple polygon.

Fig. 12. Connecting the root of the 15-vertex complete binary tree to its children.

Fig. 13. Two new subpolygons of the left subpolygon of Figure 12.
Fig. 14. The edges connecting the nodes of the second level to their children.

Fig. 15. The edges that connect the leaves to their parents.

Fig. 16. The embedded 15-vertex complete binary tree on the point set of Figure 11 by our embedding algorithm.
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