On the Physical Properties of Spherically Symmetric Self-Similar Solutions

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Abstract
In this paper, we are exploring some of the properties of the self-similar solutions of the first kind. In particular, we shall discuss the kinematic properties and also check the singularities of these solutions. We discuss these properties both in co-moving and also in non co-moving (only in the radial direction) coordinates. Some interesting features of these solutions turn up.

Keywords: Self-Similar Solutions.

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1 Introduction

A set of field equations remains invariant under a scale transformation if we assume appropriate matter field. This implies the existence of scale invariant solutions to the field equations. Such solutions are called self-similar solutions. The special feature of self-similar solutions is that, by a suitable coordinate transformations, the number of independent variables can be reduced by one and hence reduces the field equations. In other words, self-similarity refers to an invariance which simply allows the reduction of a system of partial differential equations to ordinary differential equations.

Similarity solutions were first studied in General Relativity (GR) by Cahill and Taub [1]. They studied these solutions in the cosmological context and under the assumption of spherically symmetric distribution of a self-gravitating perfect fluid. They assumed that the solution was such that the dependent variables are essentially functions of a single independent variable constructed as a dimensionless combination of the independent variables and that the model contains no other dimensional constants. They showed that the existence of a similarity of the first kind in this situation could be invariantly formulated in terms of the existence of a Homothetic vector (HV).

In GR, self-similarity is defined by the existence of a HV field. Such similarity is called the first kind (or homothety). There exists a natural generalization of homothety called Kinematic self-similarity, which is defined by the existence of a kinematic self-similar (KSS) vector field. The basic condition characterizing a manifold vector field $\xi$ as a self-similar generator [2] is given by

$$\mathcal{L}_\xi A = \lambda A,$$

where $\lambda$ is constant and $A$ is independent physical field. This field can be scalar (e.g. $\mu$), vector (e.g. $u_a$), or tensor (e.g. $g_{ab}$). In GR, the gravitational field is represented by the metric tensor $g_{ab}$, and an appropriate definition of geometrical self-similarity is necessary.

A kinematic self-similarity satisfies the condition

$$\mathcal{L}_\xi u_a = \alpha u_a,$$

with

$$\mathcal{L}_\xi h_{ab} = 2\delta h_{ab},$$

where $\alpha$ and $\delta$ are constants and $h_{ab} = g_{ab} - u_a u_b$ is the projection tensor.
KSS perfect fluid solutions have been explored by several authors. Carter and Henriksen [3] defined the other kinds of self-similarity namely second, zeroth and infinite kind. In the context of kinematic self-similarity, homothety is considered as the first kind.

The only barotropic equation of state which is compatible with self-similarity of first kind is \( p = k \mu \). Carr [4] has classified the self-similar perfect fluid solutions of first kind with this equation of state for the dust case \((k = 0)\) and the case \(0 < k < 1\) has been studied by Carr and Coley [5]. Coley [6] has shown that the Friedmann Robertson Walker solution is the only spherically symmetric homothetic perfect fluid solution in the parallel case. McIntosh [7] has discussed that a stiff fluid \((k = 1)\) is the only compatible perfect fluid with the homothety in the orthogonal case.

Benoit and Coley [8] have studied spherically symmetric spacetimes which admit a KSS vector of the second and zeroth kind. Sintes et al. [9] have considered spacetimes which admit a KSS vector of infinite kind. In all these papers the equation of state has not been specified.

In recent papers, Maeda et al. [10,11] investigated the KSS vector of the second kind in the tilted case. They assumed the perfect fluid spacetime obeying a relativistic polytropic equation of state. Further, they assumed two kinds of polytropic equation of state in GR and showed that such spacetimes must be vacuum in both cases. They studied the case in which a KSS vector is not only tilted to the fluid flow but also parallel or orthogonal.

Daud and Ziad [12] have found the homotheties of spherically symmetric spacetimes admitting maximal isometry groups larger than SO(3) along with their metrics. They have used the homothety equations without imposing any restriction on the stress-energy tensor. It would be worth interesting to explore the physical properties of these solutions. In this paper, we would evaluate the kinematical properties of these self-similar solutions of the first kind. We shall discuss the physical properties, such as acceleration, rotation, expansion, shear, shear invariant and expansion rate both in co-moving and non co-moving (only in radial direction) coordinates. Further, we would look for the singularities of these solutions.

The paper has been organised as follows. In the next section, we shall write all such solutions. In section 3, we shall discuss the physical properties of these solutions of the first kind both in co-moving and non co-moving coordinates. In section 4, singularities of these solutions will be explored. Finally, in the last section, we shall conclude the results.
2 Self-Similar Spherically Symmetric Solutions

The line element of general spherically symmetric spacetime is given by [13]
\[ ds^2 = e^{\nu(t,r)} dt^2 - e^{\lambda(t,r)} dr^2 - e^{x(t,r)} d\Omega^2, \]  
where \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \).

Daud and Ziad [12] have solved homothetic equations and found the different self-similar solutions of the first kind. There are two classes of such solutions, one of which admit 5 HVs and the second class admit 7 HVs. The first class of self-similar solutions which admit 5 HVs is given by the following three different metrics

\[ ds^2 = r^{2(1-\alpha)} dt^2 - dr^2 - r^2 d\Omega^2, \quad (5) \]
where \( \alpha \) is an arbitrary constant but cannot be 1.

\[ ds^2 = dt^2 - t^{2(1-\alpha)} dr^2 - t^2 d\Omega^2, \quad (6) \]
\[ ds^2 = dt^2 - dr^2 - \beta^2 (t + \alpha r)^2 d\Omega^2, \quad (7) \]
where \( \beta \) is an arbitrary constant. The second class of self-similar solutions which admit 7 HVs is given by the line element

\[ ds^2 = dt^2 - e^{x(t)} [dr^2 + \Sigma^2 (m, r) d\Omega^2], \quad (8) \]
where \( \Sigma = a \sinh(r/a) \), \( r \) and \( \sin(r/a) \) for \( m = 1/a^2 \), 0, \( -1/a^2 \) (a being an arbitrary constant) respectively subject to the following constraint

\[ 2me^{-x} + \dot{x} \neq 0, \]

(dot denotes differentiation with respect to \( t \)) and

\[ e^x = 2(t\alpha + \gamma)^2, \quad \alpha^2 \neq \frac{1}{a^2}, \]
\[ = \delta(t - \gamma)^\alpha, \quad \alpha \neq 0, \]
\[ = (t\alpha + \gamma)^2, \quad \alpha^2 \neq -\frac{1}{a^2}, \quad (9) \]
where \( \gamma \) is an arbitrary constant. Another self-similar solution which also admits 7 HVs is given by the following metric

\[ ds^2 = e^{y(r)} [dt^2 - a^2 \cosh(\frac{t}{a}) d\Omega^2] - dr^2 \quad (10) \]
subject to the constraint
\[ 2e^{-y} + a^2y'' \neq 0, \]
(prime indicates derivative with respect to \( r \)) and
\[ e^y = (r\alpha + \beta)^2, \quad a^2 \neq \frac{1}{a^2}. \] (11)

3 Kinematics of the Velocity Field

In this section, we shall discuss some of the kinematical properties of the self-similar solutions given by Eqs.(5)-(8) and (10) both in co-moving and non co-moving coordinates. The kinematical properties [13] can be listed as follows. The acceleration is defined by
\[ \dot{u}_a = u_{a:b}u^b. \] (12)

The rotation is given by
\[ \omega_{ab} = u_{[a;b]} + \dot{u}_{[a}u_{b]}. \] (13)

The expansion scalar, which determines the volume behaviour of the fluid, is defined by
\[ \Theta = u^a_i_a. \] (14)

The shear tensor, which provides the distortion arising in fluid flow leaving the volume invariant, can be found by
\[ \sigma_{ab} = u_{(a;b)} + \dot{u}_{(a}u_{b)} - \frac{1}{3}\Theta h_{ab}, \] (15)

The shear invariant is given by
\[ \sigma = \sigma_{ab}\sigma^{ab}. \] (16)

The rate of change of expansion with respect to proper time is given by Raychaudhuri’s equation [14]
\[ \frac{d\Theta}{d\tau} = -\frac{1}{3}\Theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}u^a_i u^b_b - R_{ab}u^a_i u^b_b. \] (17)

Now we discuss the properties of these solutions both in co-moving and also in non co-moving coordinates.
3.1 Kinematic Properties in Co-Moving Coordinates

First we evaluate the kinematical properties of the self-similar solutions in the co-moving coordinates.

For the first solution given by Eq. (5), the acceleration and expansion are zero but one of the rotation components exists given by

$$\omega_{01} = \frac{1 - \alpha}{r^\alpha}. \quad (18)$$

The only non-zero component of the shear is

$$\sigma_{01} = -\omega_{01} \quad (19)$$

and consequently the shear invariant becomes

$$\sigma = -\frac{(1 - \alpha)^2}{r^2}. \quad (20)$$

By using Raychaudhuri equation, the rate of change of expansion turns out to be

$$\frac{d\Theta}{d\tau} = \frac{\alpha - 1}{r^2}. \quad (21)$$

For the second self-similar solution given by Eq. (6), the acceleration, rotation and expansion are zero. The non-zero shear components are

$$\sigma_{11} = 2(\alpha - 1)t^{(1 - 2\alpha)}, \quad \sigma_{22} = -2t, \quad \sigma_{33} = \sigma_{22} \sin^2 \theta. \quad (22)$$

The shear invariant becomes

$$\sigma = \frac{4}{t^2}[(\alpha - 1)^2 + 2]. \quad (23)$$

The expansion rate is given by Raychaudhuri equation as follows

$$\frac{d\Theta}{d\tau} = -\frac{1}{t^2}(5\alpha^2 - 9\alpha + 12). \quad (24)$$

Now we evaluate the above quantities for the self-similar solution given by Eq. (7). It is easy to see that the acceleration, rotation and expansion turn out to be zero. The non-zero components of the shear are

$$\sigma_{22} = -2\beta^2(t + \alpha r), \quad \sigma_{33} = \sigma_{22} \sin^2 \theta. \quad (25)$$
The shear invariant takes the following form
\[ \sigma = \frac{8}{(t + \alpha r)^2}. \]  
(26)

The rate of change of expansion is
\[ \frac{d\Theta}{d\tau} = -\sigma. \]  
(27)

Now we calculate the above kinematical quantities for another self-similar solutions given by Eq.(8). For this class, the acceleration, rotation and expansion are zero. The non-zero components of the shear take the form
\[ \begin{align*}
\sigma_{11} & = -x_t e^x, \\
\sigma_{22} & = -x_t e^x \Sigma^2(m, r), \\
\sigma_{33} & = \sigma_{22} \sin^2 \theta.
\end{align*} \]  
(28)

The shear invariant becomes
\[ \sigma = 3x_t^2. \]  
(29)

The expansion rate turns out to be
\[ \frac{d\Theta}{d\tau} = \frac{3}{4}(2x_{tt} - 3x_t^2). \]  
(30)

Finally, we explore the above quantities for the self-similar solution given by Eq.(10). For this solution, the expansion is zero while the acceleration and rotation are given by respectively
\[ \begin{align*}
\dot{u}_1 & = -\frac{yr}{2}, \\
\omega_{01} & = yr \hat{e}_x.
\end{align*} \]  
(31)

The non-zero components of the shear are
\[ \begin{align*}
\sigma_{01} & = -\omega_{01}, \\
\sigma_{22} & = -2a \cosh\left(\frac{t}{a}\right) \sinh\left(\frac{t}{a}\right) e^x, \\
\sigma_{33} & = \sigma_{22} \sin^2 \theta.
\end{align*} \]  
(33)
The shear invariant becomes
\[ \sigma = -y_r^2 + 8 \frac{\tanh(\frac{t}{a})}{a^2 e^y}. \] (34)

The rate of change of expansion is given by
\[ \frac{d\Theta}{d\tau} = -\sigma + \frac{2a^2 y_{rr} e^y + 3a^2 y_r^2 e^y - 8}{4a^2 e^y}. \] (35)

### 3.2 Kinematic Properties in Non Co-Moving Coordinates

Here we discuss the kinematical properties of the self-similar solutions in the non co-moving coordinates only in radial direction.

For the first solution, the acceleration components turn out to be
\[ \dot{u}_0 = \frac{1 - \alpha}{r^\alpha}, \quad \dot{u}_1 = \frac{\alpha - 1}{r}. \] (36)

The rotation component is given by
\[ \omega_{01} = \dot{u}_0 \] (37)
and the expansion becomes
\[ \Theta = \frac{\alpha - 3}{r}. \] (38)

The non-zero components of the shear are
\[ \sigma_{00} = 4(1 - \alpha)r^{(1-2\alpha)}, \quad \sigma_{11} = -\frac{4\alpha}{3r}, \]
\[ \sigma_{22} = \frac{(\alpha - 9)r}{3}, \quad \sigma_{33} = \sigma_{22}\sin^2 \theta, \quad \sigma_{01} = \frac{2(4\alpha - 3)}{3r^\alpha} \] (39)

and the shear invariant is
\[ \sigma = \frac{2(45 + 13\alpha^2 - 38\alpha)}{3r^2}. \] (40)

The expansion rate turns out to be
\[ \frac{d\Theta}{d\tau} = -\frac{1}{3r^2}(27\alpha^2 - 76\alpha + 93) + \frac{\alpha - 1}{r}. \] (41)
For the second solution, the acceleration components become
\[ \dot{u}_0 = \frac{\alpha - 1}{t}, \quad \dot{u}_1 = \frac{(1 - \alpha)}{t^\alpha}. \] (42)

The rotation component is given by
\[ \omega_{01} = -\dot{u}_1 \] (43)

and the expansion becomes
\[ \Theta = \frac{(3 - \alpha)}{t}. \] (44)

The non-zero components of the shear are
\[ \sigma_{00} = 2\dot{u}_0, \quad \sigma_{11} = \frac{2}{3}(5\alpha - 3)t^{(1-2\alpha)}, \quad \sigma_{22} = -2t, \quad \sigma_{33} = \sigma_{22}\sin^2\theta, \quad \sigma_{01} = -\frac{2\alpha}{t^\alpha}. \] (45)

The shear invariant becomes
\[ \sigma = \frac{4}{t^2}(\frac{25}{9}\alpha^2 - \frac{16}{3}\alpha + 4). \] (46)

The rate of change of expansion will become
\[ \frac{d\Theta}{d\tau} = -\frac{1}{9t^2}(112\alpha^2 - 219\alpha + 171) + \frac{(1 - \alpha)}{t} - \frac{(1 - \alpha)(\alpha - 2)}{t^\alpha}. \] (47)

For the self-similar solution given by Eq.(7), the acceleration, rotation and expansion are zero. The non-zero components of the shear are
\[ \sigma_{22} = 2\beta^2(t + \alpha r)(\alpha - 1), \quad \sigma_{33} = \sigma_{22}\sin^2\theta. \] (48)

The shear invariant becomes
\[ \sigma = \frac{8(1 - \alpha)^2}{(t + \alpha r)^2}. \] (49)

The expansion rate can be found by using Raychaudhuri equation as given below
\[ \frac{d\Theta}{d\tau} = -\sigma. \] (50)
Now we consider the metric given by Eq. (8). In this case, the acceleration components turn out to be

\[
\dot{u}_0 = -\frac{x_t}{2}, \quad \dot{u}_1 = \frac{x_t}{2}e^{\frac{x}{r}}.
\]

(51)

The only non-zero rotation component is given by

\[
\omega_{01} = -\dot{u}_1
\]

(52)

and the expansion becomes

\[
\Theta = \frac{3x_t}{2} - 2e^{\frac{x}{r}}\frac{\Sigma_r}{\Sigma}.
\]

(53)

The non-zero components of the shear are

\[
\sigma_{00} = -x_t, \quad \sigma_{11} = -2(e^x + \frac{2\Sigma_r}{3\Sigma}e^{\frac{x}{r}}), \quad \sigma_{22} = -\frac{5}{3}e^{\frac{x}{r}}\Sigma\Sigma_r,
\]

\[
\sigma_{33} = \sigma_{22} \sin^2 \theta, \quad \sigma_{01} = x_t e^{\frac{x}{r}} + \frac{2\Sigma_r}{3\Sigma}
\]

(54)

and consequently the shear invariant becomes

\[
\sigma = x_t^2 + \frac{4}{e^{2x}}(e^x + \frac{2\Sigma_r}{3\Sigma}e^{\frac{x}{r}})^2 - \frac{1}{e^x}(x_t e^{\frac{x}{r}} + \frac{2\Sigma_r}{3\Sigma})^2 + \frac{50\Sigma_r^2}{9e^{2x}}.
\]

(55)

The rate of change of expansion will be

\[
\frac{d\Theta}{d\tau} = -\frac{3}{2}x_t^2 (1 + \frac{e^{\frac{x}{r}}}{2}) - \frac{x_t}{2} (3 + e^{\frac{x}{r}}) + \frac{2\Sigma_r}{3\Sigma}(5e^{-\frac{x}{r}}x_t - 8e^{-x})
\]

\[
-\frac{74\Sigma_r^2}{9e^{2x}\Sigma^2} + 2e^{-\frac{x}{r}}\frac{\Sigma_{rr}}{\Sigma} + \frac{x_t}{2} - 4.
\]

(56)

Finally, we evaluate the kinematical properties of the self-similar solution given Eq.(10). For this solution, the acceleration components become

\[
\dot{u}_0 = \frac{y_r}{2}e^{\frac{y}{r}}, \quad \dot{u}_1 = -\frac{y_r}{2}.
\]

(57)

The non-zero rotation component will be

\[
\omega_{01} = \dot{u}_0
\]

(58)
and the expansion is given by

\[ \Theta = \frac{2}{ae^y} \tanh \left( \frac{t}{a} \right) - \frac{3}{2} y_r. \]  \hfill (59)

The non-zero shear components are

\[
\sigma_{00} = 2y_r e^{y}, \quad \sigma_{11} = \frac{4 \tanh \left( \frac{t}{a} \right)}{3ae^y}, \\
\sigma_{22} = -a \cosh \left( \frac{t}{a} \right) \left[ e^y \left( 2 \sinh \left( \frac{t}{a} \right) + y_r e^{y} a \cosh \left( \frac{t}{a} \right) \right) \right. \\
\quad + \frac{a}{3} \cosh \left( \frac{t}{a} \right) \left( \frac{2}{ae^y} \tanh \left( \frac{t}{a} \right) - \frac{3}{2} y_r \right)], \\
\sigma_{33} = \sigma_{22} \sin^2 \theta, \\
\sigma_{01} = -y_r e^{y} - \frac{2}{3a} \tanh \left( \frac{t}{a} \right). \hfill (60)
\]

The shear invariant is

\[
\sigma = 3y_r^2 - \frac{4}{3a} y_r e^{-y} \tanh \left( \frac{t}{a} \right) - \frac{4}{3a^2 e^y} \tanh^2 \left( \frac{t}{a} \right) \\
+ \frac{2e^{-y}}{a^2 \cosh^2 \left( \frac{t}{a} \right)} \left( 2 \sinh \left( \frac{t}{a} \right) + y_r e^{y} a \cosh \left( \frac{t}{a} \right) \right)^2 \\
+ \frac{2e^{-2y}}{9} \frac{2}{a^2 e^y} \tanh \left( \frac{t}{a} \right) - \frac{3}{2} y_r^2 \right)^2 + \frac{4e^{-y}}{3a \cosh \left( \frac{t}{a} \right)} \left( \frac{2}{ae^y} \tanh \left( \frac{t}{a} \right) \\
- \frac{3}{2} y_r \right) \left( 2 \sinh \left( \frac{t}{a} \right) + y_r e^{y} a \cosh \left( \frac{t}{a} \right) \right). \hfill (61)
\]

The expansion rate is given by

\[
\frac{d\Theta}{d\tau} = -\frac{1}{3} \left( \frac{2}{ae^y} \tanh \left( \frac{t}{a} \right) - \frac{3y_r}{2} \right)^2 + \frac{3}{2} y_{rr} + \frac{3}{4} y_r^2 \\
- \frac{1}{4a^2 e^y} \left( 2y_{rr} a^2 e^y + 3y_r^2 e^y a^2 - 8 \right) - \frac{y_r}{2} - \sigma. \hfill (62)
\]

4 **Singularities**

In this section, we shall discuss the singularities of the self-similar solutions. The Kretschmann scalar is defined by

\[ K = R_{abcd} R^{abcd}, \hfill (63) \]
where $R_{abcd}$ is the Riemann tensor. For the first solution given by Eq.(5), it reduces to

$$K = \frac{2}{r^4}(1 - \alpha)(1 - \alpha)\alpha^2 + 1. \quad (64)$$

It is obvious that $K$ diverges at the point $r = 0$. Thus the solution is singular at $r = 0$.

For the second solution given by Eq.(6), the Kretschmann scalar turns out to be

$$K = \frac{1}{t^4}[(\alpha^2 + 2)(1 - \alpha)^2 + 4]. \quad (65)$$

and hence the metric is scalar polynomial singular along $t = 0$.

The Kretschmann scalar for the third solution takes the form

$$K = \frac{1}{\beta^4(t + r\alpha)^8}[2t^2 + 2\alpha t r + \alpha^2 r^2 + 2\alpha^2 \beta^2 t r + \alpha^2 \beta^2 r^2 - \alpha^2 \beta^2 t^2 - 2\alpha^3 \beta^2 t r - \alpha^4 \beta^2 r^2]. \quad (66)$$

We see that $K$ diverges only when $t = 0$, $r = 0$ and consequently this solution is singular at these points simultaneously.

For the class of solutions given by Eq.(9), $K$ becomes

$$K = 2\left[\frac{3}{4}(x_{tt} + \frac{1}{2}x_t^2)^2 + \frac{2}{e^{2x}}\Sigma^2(\Sigma_{rr} - \frac{1}{4}x_t^2 e^x \Sigma)^2\right] + \frac{1}{e^{2x}}(1 + \frac{1}{4}x_t^2 e^x \Sigma^2 - \Sigma_r^2)^2. \quad (67)$$

This is singular only when $\Sigma = 0$, i.e., when $r = 0$. Finally, for the last solution, the Kretschmann scalar reduces to

$$K = 2\left[\frac{3}{4}(y_{rr} + \frac{1}{2}y_r^2)^2 + \frac{2}{a^4 e^{2y}}(1 - \frac{1}{4}y_r^2 a^2 e^y)^2\right] + \frac{1}{a^4 e^{2y} \cosh^4(t/a)}(1 + \sinh^2(t/a) - \frac{1}{4}y_r^2 a^2 e^y \cosh^2(t/a))^2. \quad (68)$$

This does not provide any singularity and hence the last solution is singularity free.

\section{Conclusion}

In GR, the term self-similarity can be used in two different ways. One is for the properties of spacetimes and the other is for the properties of matter...
fields. In general, these are not equivalent. Although homothetic solutions can contain many interesting matter fields, but those compatible with homothety are not so common. In this paper, we have considered some self-similar solutions of the first kind and have evaluated some physical properties of these solutions both in co-moving and also in non co-moving coordinates. We also have checked singularities of each solution (if any).

First we discuss the physical properties in co-moving coordinates. For the first self-similar solution, the acceleration and the expansion turn out to be zero while the remaining physical quantities exist and are finite. It can be seen from Eqs.(18)-(20) that these quantities cannot be made zero as $\alpha \neq 1$. Further all these quantities become infinite at $r = 0$ and also the expansion rate can be positive or negative according to the choice of $\alpha$.

For the second solution, the acceleration, rotation and expansion are zero while the rest of the quantities do not vanish as $\alpha \neq 1$. We see from Eq.(24) that the expansion rate does provide positive/negative value according to the choice of $\alpha$. Also, we see that shear invariant and expansion rate turn out to be infinite for $t = 0$. The third solution also gives acceleration, rotation and expansion to be zero. In this case the expansion rate is always negative and become infinite only at $t = 0$, $r = 0$. For the fourth solution again acceleration, rotation and expansion turn out to be zero. Further, the expansion rate is zero only if $x = constant$ otherwise it is finite. For the last self-similar solution, only expansion is zero while the remaining quantities are non-zero and finite.

In non co-moving coordinates, the kinematical quantities turn out to be non-zero for all the solutions except for the solution given by Eq.(7). The reason is that we are also considering velocity in radial direction in addition to the temporal component. Also, we see that the results are more complicated in the non co-moving coordinates as compared to the co-moving coordinates.

Finally, we discuss the singularities of these solutions. We see that the first solution becomes singular at $r = 0$ while the second solution is singular at $t = 0$. The third solution turns out to be singular only if $t = 0$, $r = 0$ whereas the fourth solution is singular at $r = 0$. The last solution turns out to be singularity free solution.
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