\[ O_\infty \] Realized on Bose Fock Space

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Abstract

We study the semigroup of Bogoliubov endomorphisms of the canonical commutation relations which give rise to representations of the Cuntz algebra \[ O_\infty \] on Fock space and describe the corresponding Cuntz algebra generators in detail.

1 Introduction

The appearance of the Cuntz algebras \[ O_d \] is a generic feature of quantum field theory. This fact has been discovered, within the algebraic approach [2], by Doplicher and Roberts [3] who associated with each localized morphism \( \varrho \) of dimension \( d \) and obeying permutation group statistics a multiplet \( \Psi_1, \ldots, \Psi_d \) of local field operators, acting on a Hilbert space which contains each superselection sector with some multiplicity, such that Cuntz’ relations hold

\[
\Psi_j^* \Psi_k = \delta_{jk} 1, \quad \sum_j \Psi_j^* \Psi_j = 1 \tag{1.1}
\]

and such that, for any local observable \( A \),

\[
\varrho(A) = \sum_j \Psi_j A \Psi_j^*. \tag{1.2}
\]

Regarding the \( \Psi_j \) as an orthonormal basis for the closure \( H(\varrho) \) of their linear span, one says that ‘\( \varrho \) is implemented by the Hilbert space \( H(\varrho) \) of isometries’.

These observations motivated a study of endomorphisms of the CAR algebra which can be implemented by Hilbert spaces of isometries on Fock space [4]. Here we present a similar analysis for endomorphisms of the canonical commutation relations (CCR) or, more precisely, of the Weyl relations. As in [4], we find it convenient to use Araki’s ‘selfdual’ formulation [5, 6] which is briefly introduced in Sect. 2. We discuss a class of natural endomorphisms of the CCR algebra (‘Bogoliubov endomorphisms’) in Sect. 3. As a generalization of Shale’s

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condition for automorphisms [7], we state a necessary and sufficient condition for
Bogoliubov endomorphisms to be implementable in a fixed Fock representation.
The derivation of this result is based on known criteria for quasi-equivalence of
quasi-free states [5, 8, 6] and can, at one point, be reduced to the CAR case,
by using an inequality due to Araki and Yamagami [9].

The topological semigroup of implementable endomorphisms is the subject
of Sect. 4. It can be written as a product of a subgroup consisting of automor-
phisms which are close to the identity, and the sub-semigroup of endomorphisms
which leave the given Fock state invariant. This decomposition enables us to
determine the connected components of the semigroup. It also plays a role
in Sect. 5 which is concerned with the construction of orthonormal bases for
the Hilbert spaces $H(\omega)$. These Hilbert spaces themselves carry the structure
of symmetric Fock spaces and thus are, for genuine endomorphisms, infinite-
dimensional. The C*-algebra generated by a single $H(\omega)$ is $O_{\infty}$. Implementers
are constructed by an adaptation of Ruijsenaars’ formulas for unitary imple-
menters of automorphisms [10]. They can be written as products of certain
isometries belonging to the commutant of the range of $\omega$ times a Wick ordered
exponential of an expression which is bilinear in creation and annihilation opera-
tors. The connection with the aforementioned product decomposition is that,
roughly speaking, the first factor carries the exponential term, whereas the sec-
ond is responsible for the additional isometries. The proof of completeness of
implementers can thereby be reduced to the case of endomorphisms which leave
the given Fock state invariant.

2 Basic Notions

Let $\mathcal{K}^0$ be an infinite-dimensional complex linear space, equipped with a non-
degenerate hermitian sesquilinear form $\gamma$ and an antilinear involution $f \mapsto f^*$, such that
$$\gamma(f^*, g^*) = -\gamma(g, f), \quad f, g \in \mathcal{K}^0.$$  
(The reader who is unfamiliar with Araki’s approach [2, 3] should think of $\mathcal{K}^0$ as
being the complexification of the real linear space $\text{Re} \mathcal{K}^0 \equiv \{f \in \mathcal{K}^0 \mid f^* = f\}$, together with its canonical conjugation.
$-i\gamma$ should be viewed as the sesquilinear extension of a nondegenerate symplectic form on $\text{Re} \mathcal{K}^0$. Hopefully, the reader
will not be confused in the following by the appearance of too many stars with
different meanings.) The selfdual CCR algebra $\mathcal{C}(\mathcal{K}^0, \gamma)$ [4, 5] over $(\mathcal{K}^0, \gamma)$ is the simple *-algebra
which is generated by $1$ and elements $f \in \mathcal{K}^0$, subject to the commutation relation
$$f^* g - gf^* = \gamma(f, g)1, \quad f, g \in \mathcal{K}^0. \quad (2.1)$$
We henceforth assume the existence of a distinguished Fock state $\omega_{P_1}$. Here $P_1$
is a basis projection of $\mathcal{K}^0$, i.e. a linear operator, defined on the whole of $\mathcal{K}^0$,
which satisfies
$$P_1^2 = P_1, \quad \gamma(f, P_1 g) = \gamma(P_1 f, g),$$
$$P_1 f + P_1 (f^*)^* = f, \quad \gamma(f, P_1 f) > 0 \text{ if } P_1 f \neq 0 \quad (2.2)$$
for $f, g \in \mathcal{K}^0$. Let
$$P_2 \equiv 1 - P_1, \quad C \equiv P_1 - P_2, \quad \langle f, g \rangle \equiv \gamma(f, Cg).$$
The positive definite inner product $\langle \ , \ \rangle$ turns $\mathcal{K}^0$ into a pre-Hilbert space. We assume its completion $\mathcal{K}$ to be separable. By continuity, the involution $*$ extends to a conjugation on $\mathcal{K}$, $P_1$ and $P_2$ to orthogonal projections, $C$ to a self-adjoint unitary, and $\gamma$ to a nondegenerate hermitian form. These extensions will be denoted by the same symbols. Setting

$$\mathcal{K}_n \equiv P_n(\mathcal{K}), \quad n = 1, 2,$$

we get a direct sum decomposition $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$ which is orthogonal with respect to both $\gamma$ and $\langle \ , \ \rangle$. The following notations will frequently be used for $A \in \mathfrak{B}(\mathcal{K})$, where $\mathfrak{B}(\mathcal{K})$ is the algebra of bounded linear operators on $\mathcal{K}$:

$$A_{mn} \equiv P_m A P_n, \quad m, n = 1, 2,$$

$$A^\dagger \equiv CA^*C,$$

$$\overline{A}f \equiv A(f^*)^*, \quad f \in \mathcal{K}.$$

The components $A_{mn}$ of $A$ are regarded as operators from $\mathcal{K}_n$ to $\mathcal{K}_m$, and $A$ will sometimes be written as a matrix $\left( \begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right)$. $A^\dagger$ is the adjoint of $A$ relative to $\gamma$, whereas $A^*$ is the Hilbert space adjoint. $\overline{A}$ may be viewed as the complex conjugate of $A$. Thus one has relations like

$$\overline{P_2} = P_1 = P_1^\dagger = P_1^*, \quad \overline{C} = -C, \quad A_{12}^\dagger = A_{21}^\dagger = -A_{12}^*, \quad \overline{A_{11}} = \overline{A_{22}} \quad \text{etc.}$$

The Fock state $\omega_{P_1}$ is the unique state which is annihilated by all $f \in \text{ran} \ P_2$:

$$\omega_{P_1}(f^*f) = 0 \quad \text{if} \quad P_1f = 0.$$

(In the conventional setting mentioned above, $\omega_{P_1}$ is the Fock state corresponding to the complex structure $iC$ on $\text{Re} \mathcal{K}$.) Let $\mathcal{F}_s(\mathcal{K}_1)$ be the symmetric Fock space over $\mathcal{K}_1$ and let $D$ be the dense subspace of algebraic tensors. A GNS representation $\pi_{P_1}$ for $\omega_{P_1}$ is provided by

$$\pi_{P_1}(f) = a^*(P_1f) + a(P_1(f^*)), \quad f \in \mathcal{K}$$

where $a^*(g)$ and $a(g)$, $g \in \mathcal{K}_1$, are the usual creation and annihilation operators on $D$. The cyclic vector inducing the state $\omega_{P_1}$ is $\Omega_{P_1}$, the Fock vacuum. The operators $\pi_{P_1}(a)$, $a \in \mathfrak{C}(\mathcal{K}, \gamma)$, have invariant domain $D$, are closable, and $\pi_{P_1}(a^*) \subset \pi_{P_1}(a)^*$. In particular, if $f \in \text{Re} \mathcal{K}$, then $\pi_{P_1}(f)$ is essentially self-adjoint on $D$, and the unitary Weyl operator $w(f)$ is defined as the exponential of the closure of $i\pi_{P_1}(f)$. Its vacuum expectation value is

$$\omega_{P_1}(w(f)) = \langle \Omega_{P_1}, w(f)\Omega_{P_1} \rangle = e^{-\frac{1}{2}\|f\|^2},$$

and the Weyl relations hold

$$w(f)w(g) = e^{-\frac{i}{2}\gamma(f,g)}w(f + g), \quad f, g \in \text{Re} \mathcal{K}.$$

The Weyl operators generate a simple C*-algebra $\mathcal{W}(\mathcal{K}, \gamma)$ which acts irreducibly on $\mathcal{F}_s(\mathcal{K}_1)$. If $\mathcal{H}$ is a subspace of $\mathcal{K}$ with $\mathcal{H} = \mathcal{K}^\gamma$, then the C*-algebra generated by all $w(f)$ with $f \in \text{Re} \mathcal{H}$ is denoted by $\mathcal{W}(\mathcal{H})$. If $\mathcal{H}_0$ is the orthogonal complement of $\mathcal{H}$ with respect to $\gamma$, then duality holds

$$\mathcal{W}(\mathcal{H})' = \mathcal{W}(\mathcal{H}_0)'$$

(a prime denotes the commutant).

\footnote{A state $\omega$ over $\mathfrak{C}(\mathcal{K}, \gamma)$ is a linear functional with $\omega(1) = 1$ and $\omega(A^*A) \geq 0$, $A \in \mathfrak{C}(\mathcal{K}, \gamma)$.}
Lemma 2.1. For $f \in \mathcal{K}$, let $\mathcal{H}_f$ be the subspace spanned by $f$ and $f^*$. Then the closure of $\pi_{P_1}(f)$ is affiliated with $\mathcal{W}(\mathcal{H}_f)'$.

Proof. Let $T$ be the closure of $\pi_{P_1}(f)$, with domain $D(T)$. We have to show that, for any $A \in \mathcal{W}(\mathcal{H}_f)'$

$$A(D(T)) \subset D(T), \quad AT = TA \text{ on } D(T).$$

Now by virtue of the CCR (2.1),

$$||T\phi||^2 = ||T^*\phi||^2 + \gamma(f,f)||\phi||^2$$

for $\phi \in \mathcal{D}$. Hence, for a given Cauchy sequence $\phi_n \in \mathcal{D}$, $T\phi_n$ converges if and only if $T^*\phi_n$ does. This implies that

$$D(T) = D(T^*).$$

Let $f^{\pm} \in \text{Re}\mathcal{H}_f$ be defined as $f^+ \equiv \frac{1}{2}(f + f^*)$, $f^- \equiv \frac{1}{2}(f - f^*)$, and let $T^\pm$ be the (self–adjoint) closure of $\pi_{P_1}(f^{\pm})$. We claim that

$$D(T) = D(T^+) \cap D(T^-), \quad T = T^+ - iT^- \text{ on } D(T).$$

For if $\phi \in D(T)$, then there exists a sequence $\phi_n \in \mathcal{D}$ converging to $\phi$ such that $\pi_{P_1}(f)\phi_n$ and $\pi_{P_1}(f^*)\phi_n$ converge. Thus $\phi$ belongs to the domain of the closure of $\pi_{P_1}(f^{\pm})$. Conversely, if $\phi \in D(T^+) \cap D(T^-)$, then there exists a sequence $\phi_n \in \mathcal{D}$ converging to $\phi$ such that both $\pi_{P_1}(f + f^*)\phi_n$ and $\pi_{P_1}(f - f^*)\phi_n$ converge (cf. [10]). Therefore $\pi_{P_1}(f)\phi_n$ is also convergent, i.e. $\phi$ is contained in $D(T)$, and $T\phi = (T^+ - iT^-)\phi$.

Now if $A \in \mathcal{W}(\mathcal{H}_f)'$, then $A$ commutes with the one–parameter unitary groups $w(tf^{\pm}) = \exp(itT^{\pm})$. As a consequence, $A$ leaves $D(T^\pm)$ invariant and commutes with $T^\pm$ on $D(T^\pm)$. It follows that $A(D(T)) \subset D(T)$ and $AT = TA$ on $D(T)$ as was to be shown. \qed

3 Implementability of Endomorphisms

Bogoliubov endomorphisms are the unital *–endomorphisms of $\mathcal{C}(\mathcal{K},\gamma)$ which map $\mathcal{K}$, viewed as a subspace of $\mathcal{C}(\mathcal{K},\gamma)$, into itself. They are completely determined by their restrictions to $\mathcal{K}$ which are called Bogoliubov operators. Hence $V \in \mathcal{B}(\mathcal{K})$ is a Bogoliubov operator if and only if it commutes with complex conjugation and preserves the hermitian form $\gamma$. Bogoliubov operators form a unital semigroup denoted by

$$\mathcal{S}(\mathcal{K},\gamma) \equiv \{V \in \mathcal{B}(\mathcal{K}) \mid \overline{V} = V, \ V^\dagger V = 1\}.$$

Each $V \in \mathcal{S}(\mathcal{K},\gamma)$ extends to a unique Bogoliubov endomorphism of $\mathcal{C}(\mathcal{K},\gamma)$ and to a unique *–endomorphism of $\mathcal{W}(\mathcal{K},\gamma)$. By abuse of notation, both endomorphisms are denoted by $\varrho_V$, so that $\varrho_V(f) = Vf$, $f \in \mathcal{K}$, and $\varrho_V(w(g)) = w(Vg)$, $g \in \text{Re}\mathcal{K}$.

The condition $V^\dagger V = 1$ entails that $V$ is injective and $V^*$ surjective; hence ran $V$ is closed, and $V$ is a semi–Fredholm operator [12]. We claim that the Fredholm index $-\text{ind } V = \dim \ker V^\dagger$ cannot be odd, in contrast to the CAR case [1]. For let $f \in \ker V^\dagger$ such that $0 = \gamma(f,g) = \langle f, Cg \rangle \forall g \in \ker V^\dagger$.

\footnote{We may disregard unbounded Bogoliubov operators $V$ (defined on $\mathcal{K}^0$) since the topologies induced by the corresponding states $\omega_{P_1} \circ \varrho_V$ on $\mathcal{K}^0$ differ from the one induced by $\omega_{P_1}$. Hence these states cannot be quasi–equivalent to $\omega_{P_1}$ (cf. [1, 14]), and $\varrho_V$ cannot be implemented.}
Then \( f \in (C \ker V^\dagger)^\perp = (\ker V^*)^\perp = \text{ran} \, V \), but \( \text{ran} \, V \cap \ker V^\dagger = \{0\} \) due to \( V^\dagger V = 1 \), so \( f \) has to vanish. This shows that the restriction of \( \gamma \) to \( \ker V^\dagger \) stays nondegenerate. It follows that \( \dim \ker V^\dagger \) cannot be odd (there is no nondegenerate symplectic form on an odd–dimensional space).

On the other hand, each even number (and \( \infty \)) occurs as \( \dim \ker V^\dagger \) for some \( V \). Hence we have an epimorphism of semigroups

\[
S(\mathcal{K}, \gamma) \to \mathbb{N} \cup \{\infty\}, \quad V \mapsto -\frac{1}{2} \text{ind} \, V = \frac{1}{2} \dim \ker V^\dagger
\]

\((0 \in \mathbb{N} \text{ by convention}). \)

Let

\[
S^n(\mathcal{K}, \gamma) \equiv \{V \in S(\mathcal{K}, \gamma) \mid \text{ind} \, V = -2n\}, \quad n \in \mathbb{N} \cup \{\infty\}.
\]

\(S^0(\mathcal{K}, \gamma)\) is the group of Bogoliubov automorphisms (isomorphic to the symplectic group of \( \Re \mathcal{K} \)). It acts on \( S(\mathcal{K}, \gamma) \) by left multiplication. Analogous to the CAR case, the orbits under this action are the subsets \( S^n(\mathcal{K}, \gamma) \), and the stabilizer of \( V \in S^n(\mathcal{K}, \gamma) \) is isomorphic to the symplectic group \( \text{Sp}(n) \).

We are interested in endomorphisms \( g_V \) which can be implemented by Hilbert spaces of isometries on \( \mathcal{F}_s(\mathcal{K}_1) \). This means that there exist isometries \( \Psi_j \) on \( \mathcal{F}_s(\mathcal{K}_1) \) which fulfill the Cuntz algebra relations \((1.1)\) and implement \( g_V \), according to \((1.2)\)

\[
g_V(w(f)) = \sum_j \Psi_j(w(f))\Psi_j^*, \quad f \in \Re \mathcal{K}.
\]

As explained in \([4]\), such isometries exist if and only if \( g_V \), viewed as a representation of \( W(\mathcal{K}, \gamma) \) on \( \mathcal{F}_s(\mathcal{K}_1) \), is quasi–equivalent to the defining (Fock) representation.

To study \( g_V \) as a representation, for fixed \( V \in S(\mathcal{K}, \gamma) \), let us decompose it into cyclic subrepresentations. Let \( e_1, e_2, \ldots \) be an orthonormal basis in \( \mathcal{K}_1 \cap \ker V^\dagger \) and let \( \alpha = (\alpha_1, \ldots, \alpha_l) \) be a multi–index with \( \alpha_j \leq \alpha_{j+1} \). Such \( \alpha \) has the form

\[
\alpha = (\alpha'_1, \ldots, \alpha'_1, \alpha'_2, \ldots, \alpha'_r, \ldots, \alpha'_r)
\]

with \( \alpha'_1 < \alpha'_2 < \cdots < \alpha'_r \) and \( l_1 + \cdots + l_r = l \). Let

\[
\phi_\alpha \equiv (l_1! \cdots l_r!)^{-\frac{1}{2}} a^*(e_{\alpha_1}) \cdots a^*(e_{\alpha_l}) \Omega_{P_1},
\]

\[
\mathcal{F}_\alpha \equiv \mathit{W}(\text{ran} \, V)\phi_\alpha,
\]

\[
\pi_\alpha \equiv \rho |_{\mathcal{F}_\alpha}.
\]

**Lemma 3.1.** \( g_V = \oplus_\alpha \pi_\alpha \), where the sum extends over all multi–indices \( \alpha \) as above, including \( \alpha = 0 \) (\( \phi_0 \equiv \Omega_{P_1} \)). Each \((\pi_\alpha, \mathcal{F}_\alpha, \phi_\alpha)\) is a GNS representation for \( \omega_{P_1} \circ g_V \) (regarded as a state over \( W(\mathcal{K}, \gamma) \)).

**Proof.** By definition, the \( \phi_\alpha \) constitute an orthonormal basis for \( \mathcal{F}_s(\mathcal{K}_1 \cap \ker V^\dagger) \), and \((\pi_\alpha, \mathcal{F}_\alpha, \phi_\alpha)\) is a cyclic representation of \( W(\mathcal{K}, \gamma) \). Since the closures of
\[ a^*(e_j) \text{ and } a(e_j) \text{ are affiliated with } W(\ker V^\dagger)'' = W(\ker V') \] (see Lemma 2.1 and (2.3)), there holds for \( f \in \mathbb{R} \), with \( N_\alpha \equiv (l_1! \cdots l_r)!^{-1} \)

\[ \langle \phi_\alpha, \pi_\alpha(w(f))\phi_\alpha \rangle = N_\alpha(a^*(e_{a_1}) \cdots a^*(e_{a_l})\Omega_{P_1}, w(V f)a^*(e_{a_1}) \cdots a^*(e_{a_l})\Omega_{P_1}) \\
= N_\alpha(\Omega_{P_1}, w(V f)\underbrace{a(e_{a_1})a^*(e_{a_1}) \cdots a(e_{a_l})a^*(e_{a_l})}_{N_\alpha^{-1}\Omega_{P_1}}\Omega_{P_1}) \\
= \langle \Omega_{P_1}, w(V f)\Omega_{P_1} \rangle. \]

This proves that \( \langle \phi_\alpha, \pi_\alpha, \phi_\alpha \rangle \) is a GNS representation for \( \omega_{P_1} \circ \phi_V \). Similarly, one finds that \( \langle \phi_\alpha, w(V f)\phi_\alpha \rangle = 0 \) for \( \alpha \neq \alpha' \), so the \( \pi_\alpha \) are mutually orthogonal.

It remains to show that \( \oplus_\alpha \pi_\alpha = \pi_{(X_1)} \). We claim that \( \mathcal{F}_0 \) equals \( \mathcal{F}_{s}(\ker P_1^V) \), the symmetric Fock space over the closure of \( \ker P_1^V \). The inclusion \( \mathcal{F}_0 \subset \mathcal{F}_{s}(\ker P_1^V) \) holds because vectors of the form \( w(V f)\Omega_{P_1} = \exp i(a^*(P_1 V f) + a(P_1 V f))\Omega_{P_1} \in \mathcal{F}_{s}(\ker P_1^V) \) are total in \( \mathcal{F}_0 \). The converse inclusion may be proved inductively. Assume that \( a^*(g_1) \cdots a^*(g_m)\Omega_{P_1} \) is contained in \( \mathcal{F}_0 \) for all \( m \leq n, g_1, \ldots, g_m \in \ker P_1^V \). Then, for \( f \in \ker P_1^V \) and \( g_1, \ldots, g_n \in \ker P_1^V \), \( \frac{1}{n!}a^*(g_1) \cdots a^*(g_n)\Omega_{P_1} \) has a limit \( a^*(P_1 f)a^*(g_1) \cdots a^*(g_n)\Omega_{P_1} + a(P_1 f)a^*(g_1) \cdots a^*(g_n)\Omega_{P_1} \) in \( \mathcal{F}_0 \) as \( t \searrow 0 \).

By assumption, the second term lies in \( \mathcal{F}_0 \), and so does the first. Since each \( g \in \ker P_1^V \) is a linear combination of such \( P_1 f \), it follows that \( a^*(g_1) \cdots a^*(g_{n+1})\Omega_{P_1} \) is contained in \( \mathcal{F}_0 \) for arbitrary \( g_j \in \ker P_1^V \), and, by induction, for arbitrary \( n \in \mathbb{N} \). But such vectors span a dense subspace in \( \mathcal{F}_{s}(\ker P_1^V) \), so \( \mathcal{F}_0 = \mathcal{F}_{s}(\ker P_1^V) \) as claimed.

Finally, \( X_1 \cap \ker V^\dagger = \ker V^*P_1 \), where \( V^*P_1 \) is regarded as an operator from \( X_1 \) to \( X \). Thus we have \( \mathcal{K}_1 = \ker P_1^V \oplus (X_1 \cap \ker V^\dagger) \) and \( \pi_{(X_1)} \cong \pi_0 \otimes \pi_{(X_1 \cap \ker V^\dagger)} \). Under this isomorphism, \( \pi_\alpha \) is identified with \( \pi_0 \otimes (\mathbb{C}\phi_\alpha) \). Since the \( \phi_\alpha \) form an orthonormal basis for \( \mathcal{F}_{s}(X_1 \cap \ker V^\dagger) \), the desired result \( \oplus_\alpha \pi_\alpha = \pi_{(X_1)} \) follows.

As a consequence, the representation \( \phi_V \) is quasi-equivalent to the GNS representation associated with the quasi-free state \( \omega_{P_1} \circ \phi_V \). So \( \phi_V \) is implementable if and only if \( \omega_{P_1} \circ \phi_V \) and \( \omega_{P_1} \) (i.e. their GNS representations) are quasi-equivalent. Now the two-point function of \( \omega_{P_1} \circ \phi_V \) (as a state over \( C(X, \gamma) \)) is given by

\[ \omega_{P_1} \circ \phi_V(f^* g) = \gamma(f, S g) = \langle f, \tilde{S} g \rangle, \quad f, g \in X, \]

with

\[ S \equiv V^\dagger P_1 V, \quad \tilde{S} \equiv V^* P_1 V. \]

The latter operators contain valuable information about \( \omega_{P_1} \circ \phi_V \). For example, it can be shown (cf. [13]) that \( \omega_{P_1} \circ \phi_V \) is a pure state over \( W(X, \gamma) \) if and only if \( S \) is a basis projection, that is, if and only if \( S \) is idempotent (the remaining conditions in (2.2) are automatically fulfilled). This is further equivalent to \( [P_1, V^\dagger V] = 0 \), by the following chain of equivalences:

\[ S^2 = S \iff 0 = SS^\dagger \iff 0 = V^*P_1VCV^*P_2V \iff 0 = P_1VCV^*P_2 \iff 0 = P_1V^\dagger P_2 \iff 0 = [P_1, V^\dagger]. \]
On the other hand, the criterion for quasi–equivalence of quasi–free states, in the form given by Araki and Yamagami [3], states that \( \omega_{P_1} \circ \varrho_V \) is quasi–equivalent to \( \omega_{P_1} \) if and only if \( P_1 - \tilde{S}P \) is a Hilbert–Schmidt operator on \( \mathcal{K} \). This condition can be simplified in the present context, as the following result shows.

**Theorem 3.2.** Let a Bogoliubov operator \( V \in S(\mathcal{K}, \gamma) \) be given. Then there exists a Hilbert space of isometries \( H(\varrho_V) \) which implements the endomorphism \( \varrho_V \) in the Fock representation determined by the basis projection \( P_1 \) if and only if \( [P_1, V] \) (or, equivalently, \( V_{12} \)) is a Hilbert–Schmidt operator. The dimension of \( H(\varrho_V) \) is 1 if \( \text{ind} V = 0 \), otherwise \( \infty \).

**Proof.** First note that \( [P_1, V] = V_{12} - V_{21} = V_{12} - \overline{V_{12}} \) is Hilbert–Schmidt (HS) if and only if \( V_{12} \) is HS.

By the preceding discussion, \( \varrho_V \) is implementable if and only if \( P_1 - \tilde{S}P \) is HS. In this case, \( P_2(P_1 - \tilde{S}P)^2P_2 = P_2\tilde{S}P = V_{12}V_{12} \) is of trace class, hence \( V_{12} \) is HS.

Conversely, assume \( V_{12} \) to be HS. Let \( V = V'V \) be the polar decomposition of \( V \). Then \( [V] = [V'] \) is a bounded bijection with a bounded inverse, and \( |V| - 1 = (|V|^2 - 1)(|V| + 1)^{-1} = (V^* - V)V(|V| + 1)^{-1} = 2(V_{12}^* + V_{21}^*)V(|V| + 1)^{-1} \) is HS. Thus, by a corollary of an inequality of Araki and Yamagami [9], \( (|V|A|V|)^\sharp - A^\sharp \) is HS for any positive \( A \in \mathfrak{B}(\mathcal{K}) \). Applying this to \( A = V^*P_1V \), we get that

\[
\tilde{S}P - (V^*P_1V')^\sharp \text{ is HS. (3.2)}
\]

Now \( V' \) is an isometry with \( \overline{V'} = V' \), i.e., a CAR Bogoliubov operator [1].

Since \( [P_1, V] \) and \( [P_1, V^{-1}] = [V^{-1}] [V', P_1] [V]^{-1} = [V]^{-1} [V - 1, P_1] [V]^{-1} \) are HS, the same holds true for \( [P_1, V'] = [P_1, V]^{-1} \). So \( V' \) fulfills the implementability condition for CAR Bogoliubov operators derived in [1], and, as shown there, this forces \( P_1 - (V^*P_1V')^\sharp \) to be HS. This, together with (3.2), implies that \( P_1 - \tilde{S} \) is HS as claimed.

It remains to prove the statement about \( \dim H(\varrho_V) \). Let \( \tilde{\varrho}_V \) be the normal extension of \( \varrho_V \) to \( \mathfrak{B}(\mathfrak{F}_s(\mathcal{K}_1)) \). Then \( \mathfrak{B}(H(\varrho_V)) \cong \tilde{\varrho}_V(\mathfrak{B}(\mathfrak{F}_s(\mathcal{K}_1)))^\prime = \varrho_V(\mathfrak{W}(\mathcal{K}, \gamma))^\prime = \mathfrak{W}(\text{ran} V^\dagger)^\prime = \mathfrak{W}(\ker V^\dagger)^\prime \). The latter (and hence \( H(\varrho_V) \)) is one–dimensional if \( \ker V^\dagger = \{0\} \) and infinite–dimensional if \( \ker V^\dagger \neq \{0\} \).

**Remark.** Shale’s original result [6] asserts that a Bogoliubov automorphism \( \varrho_V \), \( V \in S^0(\mathcal{K}, \gamma) \), is implementable if and only if \( |V| - 1 \) is HS. This condition is equivalent to \( [P_1, V] \) being HS, not only for \( V \in S^0(\mathcal{K}, \gamma) \), but for all \( V \in S(\mathcal{K}, \gamma) \) with \( -\text{ind} V < \infty \). However, the two conditions are not equivalent for \( V \in S^\infty(\mathcal{K}, \gamma) \), as the following example shows. Let \( \mathcal{K}_1 = \mathcal{K} \oplus \mathcal{K}' \) be a decomposition into infinite–dimensional subspaces. Choose an operator \( V_{12} \) from \( \mathcal{K}_2 \) to \( \mathcal{K} \) with \( \text{tr} |V_{12}|^2 < \infty \), but \( \text{tr} |V_{12}|^4 = \infty \). Let \( V_{21} = \overline{V_{12}} \) and \( |V_{11}| = (P_1 + |V_{21}|^2)^\sharp \). Choose an isometry \( v_{11} \) from \( \mathcal{K}_1 \) to \( \mathcal{K}' \) and set \( v_{11} = v_{11}|V_{11}|, V_{22} = \overline{V_{11}} \). These components define a Bogoliubov operator \( V \in S^\infty(\mathcal{K}, \gamma) \) (cf. (3.4a)–(3.4c) below) which violates the condition of Theorem 3.2. But it fulfills Shale’s condition since \( |V|^2 - 1 = 2(|V_{12}|^2 + |V_{21}|^2) \) is HS and since \( |V| - 1 = (|V|^2 - 1)(|V| + 1)^{-1} \).

Let \( V \in S(\mathcal{K}, \gamma) \) with \( V_{12} \) compact. Due to stability under compact perturba-
tions \( \mathcal{V}_{11} \) and \( \mathcal{V}_{22} = \overline{\mathcal{V}_{11}} \) are semi-Fredholm with
\[
\text{ind} \mathcal{V}_{11} = \text{ind} \mathcal{V}_{22} = \frac{1}{2} \text{ind} \mathcal{V}.
\]
(3.3)
We will occasionally use the relation \( \mathcal{V}^\dagger \mathcal{V} = 1 \) componentwise:
\[
\begin{align*}
\mathcal{V}_{11}^* \mathcal{V}_{11} - \mathcal{V}_{21}^* \mathcal{V}_{21} &= P_1, \\
\mathcal{V}_{22}^* \mathcal{V}_{22} - \mathcal{V}_{12}^* \mathcal{V}_{12} &= P_2, \\
\mathcal{V}_{11}^* \mathcal{V}_{12} - \mathcal{V}_{21}^* \mathcal{V}_{22} &= 0, \\
\mathcal{V}_{22}^* \mathcal{V}_{21} - \mathcal{V}_{12}^* \mathcal{V}_{11} &= 0.
\end{align*}
\]
(3.4a, 3.4b, 3.4c, 3.4d)
Since \( \mathcal{V}_{11} \) is injective by (3.4a) and has closed range, we may define a bounded operator \( \mathcal{V}_{11}^{-1} \) as the inverse of \( \mathcal{V}_{11} \) on \( \text{ran} \mathcal{V}_{11} \) and as zero on \( \ker \mathcal{V}_{11} \). These operators will be needed later. Note that \( \dim \ker \mathcal{V}_{11}^* = \frac{1}{2} \text{ind} \mathcal{V} \).

4 On the Semigroup of Implementable Endomorphisms

According to Theorem 3.2, the semigroup of implementable Bogoliubov endomorphisms is isomorphic to the following semigroup of Bogoliubov operators:
\[
\mathcal{S}_{P_1}^{\mathcal{K}} (\mathcal{K}, \gamma) \equiv \{ \mathcal{V} \in \mathcal{S} (\mathcal{K}, \gamma) \mid \mathcal{V}_{12} \text{ is Hilbert–Schmidt} \}.
\]
\( \mathcal{S}_{P_1}^{\mathcal{K}} (\mathcal{K}, \gamma) \) is a topological semigroup with respect to the metric \( d_{P_1} (\mathcal{V}, \mathcal{V}') = \| \mathcal{V} - \mathcal{V}' \| + \| \mathcal{V}_{12} - \mathcal{V}'_{12} \|_{\text{HS}} \), where \( \| \cdot \|_{\text{HS}} \) denotes Hilbert–Schmidt norm. It contains the closed sub-semigroup of diagonal Bogoliubov operators
\[
\mathcal{S}_{\text{diag}}^{\mathcal{K}} (\mathcal{K}, \gamma) = \{ \mathcal{V} \in \mathcal{S} (\mathcal{K}, \gamma) \mid [\mathcal{P}_1, \mathcal{V}] = 0 \}
\]
which is isomorphic to the semigroup of isometries of the Hilbert space \( \mathcal{K}_1 \), via the map \( \mathcal{V} \mapsto \mathcal{V}_{11} \). The Fredholm index yields a decomposition
\[
\mathcal{S}_{P_1}^{\mathcal{K}} (\mathcal{K}, \gamma) = \bigcup_{n \in \mathbb{N} \cup \{ \infty \}} \mathcal{S}_{P_1}^{n} (\mathcal{K}, \gamma), \quad \mathcal{S}_{P_1}^{n} (\mathcal{K}, \gamma) = \mathcal{S}_{P_1}^{\mathcal{K}} (\mathcal{K}, \gamma) \cap \mathcal{S}^{n} (\mathcal{K}, \gamma).
\]
The group \( \mathcal{S}_{P_1}^{0} (\mathcal{K}, \gamma) \) is usually called the restricted symplectic group \( \mathcal{S}_{P_1}^{0} (\mathcal{K}, \gamma) \). It has a natural normal subgroup
\[
\mathcal{S}_{\text{HS}}^{\mathcal{K}} (\mathcal{K}, \gamma) \equiv \{ \mathcal{V} \in \mathcal{S} (\mathcal{K}, \gamma) \mid \mathcal{V} - 1 \text{ is Hilbert–Schmidt} \} \subset \mathcal{S}_{P_1}^{0} (\mathcal{K}, \gamma).
\]
We will eventually show that each \( \mathcal{V} \in \mathcal{S}_{P_1} (\mathcal{K}, \gamma) \) can be written as a product \( \mathcal{V} = \mathcal{U} \mathcal{W} \) with \( \mathcal{U} \in \mathcal{S}_{\text{HS}}^{\mathcal{K}} (\mathcal{K}, \gamma) \) and \( \mathcal{W} \in \mathcal{S}_{\text{diag}}^{\mathcal{K}} (\mathcal{K}, \gamma) \). Assume that such \( \mathcal{U} \) and \( \mathcal{W} \) exist. Then \( \mathcal{P}_V \equiv \mathcal{U} \mathcal{P}_1 \mathcal{U}^\dagger \) is a basis projection such that
\[
\mathcal{P}_1 - \mathcal{P}_V \text{ is Hilbert–Schmidt,} \quad \mathcal{V}^\dagger \mathcal{P}_V \mathcal{V} = \mathcal{P}_1.
\]
(4.1)
so the corresponding Fock state \( \omega_{\mathcal{P}_V} \) is unitarily equivalent to \( \omega_{\mathcal{P}_1} \) and fulfills \( \omega_{\mathcal{P}_V} \circ \varrho_{\mathcal{V}} = \omega_{\mathcal{P}_1} \). In order to construct such basis projections, let us investigate
the set $\mathcal{P}_{P_1}$ of basis projections of $\mathcal{K}$ which differ from $P_1$ only by a Hilbert–Schmidt operator. Let $\mathcal{E}_{P_1}$ be the infinite-dimensional analogue of the open unit disk $\mathbb{D}$, consisting of all Hilbert–Schmidt operators $Z$ from $\mathcal{K}_1$ to $\mathcal{K}_2$ which are symmetric in the sense that

$$Z = \overline{Z}^*$$

and have norm less than 1 (the latter condition is equivalent to $P_1 + Z^\dagger Z$ being positive definite on $\mathcal{K}_1$, since $Z^\dagger = -Z^*$ and $Z$ is compact). Then the following is more or less well–known (cf. [15]).

**Proposition 4.1.** $P \mapsto P_{21} P_{11}^{-1}$ defines a bijection from $\mathcal{P}_{P_1}$ onto $\mathcal{E}_{P_1}$, with inverse given by

$$Z \mapsto P_Z \equiv (P_1 + Z)(P_1 + Z^\dagger Z)^{-1}(P_1 + Z^\dagger).$$

The restricted symplectic group $\mathcal{S}_0^\theta(\mathcal{K}, \gamma)$ acts transitively on either set, in a way compatible with the above bijection, through the formulas

$$P \mapsto UPU^\dagger$$

$$Z \mapsto (U_{21} + U_{22} Z)(U_{11} + U_{12} Z)^{-1}.$$  

The restrictions of these actions to the subgroup $\mathcal{S}_{HS}(\mathcal{K}, \gamma)$ remain transitive, as follows from the fact that, for $Z \in \mathcal{E}_{P_1}$,

$$U_Z \equiv (P_1 + Z)(P_1 + Z^\dagger Z)^{-1/2} + (P_2 - Z)(P_2 + Z Z^\dagger)^{-1/2}$$

lies in $\mathcal{S}_{HS}(\mathcal{K}, \gamma)$ and fulfills $U_Z P_1 U_Z^\dagger = P_Z$ (equivalently, under the action (4.4), $U_Z$ takes $0 \in \mathcal{E}_{P_1}$ to $Z$).

**Proof.** Having made $\mathcal{K}$ into a Hilbert space, the conditions on $P$ to be a basis projection (2.2) may be rewritten as

$$P = P^\dagger = 1 - \overrightarrow{P} = P^2; \quad CP \text{ is positive definite on } \text{ran} \ P;$$

or, in components:

$$P_{11} = P_{11}^* = P_1 - P_{22},$$

$$P_{22} = P_{22}^* = P_2 - P_{11},$$

$$P_{21} = P_{21}^* = -P_{12}^*,$$

$$P_{11}^2 - P_{11} = P_{21}^* P_{21},$$

$$P_{22}^2 - P_{22} = P_{12}^* P_{12},$$

$$(P_1 - P_{11}) P_{12} = P_{12} P_{22},$$

$$(P_2 - P_{22}) P_{21} = P_{21} P_{11},$$

$$\begin{pmatrix} P_{11} & P_{12} \\ -P_{21} & -P_{22} \end{pmatrix}$$

is positive definite on $\text{ran} \ P$.  

(4.7h)
Moreover, $P_1 - P$ is Hilbert–Schmidt if and only if $P_2P$ is.

Now let $P \in \mathcal{F}_{P_1}$. Then $P_{22} \leq 0$ by (4.7b), hence, by (4.7a),

$$P_{11} = P_1 - P_{22} \geq P_1$$

has a bounded inverse. Thus $Z = P_{21}P_{11}^{-1}$ is a well-defined Hilbert–Schmidt operator. By (4.7a)–(4.7c) and (4.7g),

$$Z - Z^* = P_{21}P_{11}^{-1} - P_{11}^{-1}P_{21}^*$$

$$= P_{11}^{-1}(P_2 - P_{22})P_{21} - P_{21}P_{11})P_{11}^{-1}$$

$$= 0,$$

so $Z$ is symmetric in the sense of (4.2). Furthermore, by (4.7d),

$$P_1 - Z^*Z = P_1 - P_{11}^{-1}P_{21}^*P_{21}P_{11}^{-1}$$

$$= P_1 - P_{11}^{-1}(P_{11}^2 - P_{11})P_{11}^{-1}$$

$$= P_{11}^{-1}$$

is positive definite on $\mathcal{H}_1$, which proves $Z \in \mathcal{E}_{P_1}$.

Next let $Z \in \mathcal{E}_{P_1}$ and let $P_Z$ be given by (4.3). We associate with $Z$ an operator

$$Y \equiv (P_1 + Z^\dagger Z)^{-1} = (P_1 - Z^*Z)^{-1}$$

(4.9)

which is bounded by assumption. Then $P_Z = P_Z^\dagger = P_Z^*$ since $(P_1 + Z^\dagger)(P_1 + Z) = Y^{-1}$. To prove that $P_Z + \overline{P_Z} = 1$ holds, note that $ZY^{-1} = Y^{-1}Z$ and therefore $\overline{Y}Z = ZY$, $YZ^\dagger = Z^\dagger Y$. It follows that

$$P_Z + \overline{P_Z} = (P_1 + Z)Y(P_1 + Z^\dagger) + (P_2 - Z^\dagger)\overline{Y}(P_2 - Z)$$

$$= Y + YZ^\dagger + ZZ^\dagger Y + \overline{Y} - YZ^\dagger - ZY + Z^\dagger ZY$$

$$= Y^{-1}Y + Y^{-1}\overline{Y}$$

$$= P_1 + P_2$$

$$= 1.$$

Since $P_2P_Z$ is clearly HS and since

$$CP_Z = (P_1 - Z)Y(P_1 - Z^\dagger)$$

(4.10)

is positive definite on ran $P_Z = \text{ran}(P_1 + Z)$, we get that $P_Z \in \mathcal{F}_{P_1}$ as desired.

To show that these two maps are mutually inverse, let first $Z \in \mathcal{E}_{P_1}$. Then $(P_Z)_{21}(P_Z)_{11}^{-1} = ZYY^{-1} = Z$. Conversely, let $P \in \mathcal{F}_{P_1}$ be given and set $Z \equiv P_{21}P_{11}^{-1}$. Then $ZP_{11} = P_{21}$ and $P_{11}Z^\dagger = P_{21}^\dagger = P_{12}$. By (4.8) and (4.9),

$Y = P_{11}$, hence $P_{11}Z^\dagger = Z^\dagger P_{11}$. Thus we get

$$P - P_Z = P - (P_1 + Z)P_{11}(P_1 + Z^\dagger)$$

$$= P - P_{11} - ZP_{11}^\dagger - P_{11}Z^\dagger - ZP_{11}Z^\dagger$$

$$= P - P_{11} - P_{21} - P_{22} - ZZ^\dagger P_{11}^\dagger$$

$$= P_{22} - ZZ^\dagger P_{11}^\dagger$$

$$= P_2 - (P_2 + ZZ^\dagger P_{11}^\dagger) (\text{by (4.7b)})$$

$$= 0.$$
It remains to prove the statements about the group actions. It is fairly obvious that $S^0_{P_1}(\mathcal{K}, \gamma)$ acts on $P_1$ via (4.4). The proof that $U_Z$ is a Bogoliubov operator which takes $P_1$ to $P_2$ is also straightforward. To show that $U_Z \in S_{\text{HS}}(\mathcal{K}, \gamma)$, let $Y$ be given by (19). Then

$$Y^\dagger - P_1 = Y^\dagger (P_1 - Y^{-1})(P_1 + Y^{-\dagger})^{-1} = Y^\dagger Z^* Z (P_1 + Y^{-\dagger})^{-1}$$

is of trace class. Therefore $(U_Z - 1)P_1 = (P_1 + Z)Y^\dagger - P_1 = Y^\dagger - P_1 + ZY^\dagger$ is HS, which implies $U_Z \in S_{\text{HS}}(\mathcal{K}, \gamma)$.

Finally we have to show that the action (4.4) on $P_{P_1}$ carries over to an action (14) on $\mathcal{E}_{P_1}$. Thus, for given $Z \in \mathcal{E}_{P_1}$ and $U \in S^0_{P_1}(\mathcal{K}, \gamma)$, we have to compute the operator $Z' = P_{P_1}^{-1}P'_{P_1}$ which corresponds to $P' = UP_ZU^\dagger$. By definition,

$$P'_{P_1} = (U_{21} + U_{22}Z)Y(U_{11} + U_{12}Z)^*,$$

$$P_{P_1} = (U_{11} + U_{12}Z)Y(U_{11} + U_{12}Z)^*.$$ (4.11)

Suppose that $(U_{11} + U_{12}Z)f = 0$ for some $f \in \mathcal{K}_I$. Then $\|f\| = \|U_{11}^{-1}U_{12}Zf\|$. Since $\|U_{11}^{-1}U_{12}\|^2 = \|U_{11}^*U_{11}^{-1}U_{12}^{-1}U_{12}\| = \|U_{12}^*(P_1 + U_{12}U_{12}^*)^{-1}U_{12}\| = \|U_{12}\|^2/(1 + \|U_{12}\|^2) < 1$ and $\|Z\| < 1$, it follows that $f = 0$. Hence $U_{11} + U_{12}Z$ is injective, and, as a Fredholm operator with vanishing index (3.3), it has a bounded inverse. So we get from (4.11) that $Z' = P_{P_1}P'_{P_1}^{-1} = (U_{21} + U_{22}Z)^{-1}P_{P_1}$ as claimed.

The following construction will enable us to assign, in an unambiguous way, to each Bogoliubov operator $V \in S_{P_1}(\mathcal{K}, \gamma)$ a basis projection $P_V$ such that (4.4) holds.

**Lemma 4.2.** Let $\mathcal{H} \subset \mathcal{K}$ be a closed *--invariant subspace such that $\gamma|_{\mathcal{H} \times \mathcal{H}}$ is nondegenerate and such that $[P_1, E]$ is Hilbert–Schmidt where $E$ is the orthogonal projection onto $\mathcal{H}$. Let $A \equiv ECE$ be the self–adjoint operator, invertible on $\mathcal{H}$, such that $\gamma(f, g) = (f, Ag)$, $f, g \in \mathcal{H}$, and let $A_{\pm}$ be the unique positive operators such that $A = A_+ - A_-$ and $A_{\pm}A_{\mp} = 0$. Further let $A^{-1}$ be defined as the inverse of $A$ on $\mathcal{H}$ and as zero on $\mathcal{H}^\perp$, and similarly for $A_{\pm}^{-1}$. Then $A^{-1}C$ is the $\gamma$–orthogonal projection onto $\mathcal{H}$, $P_+ \equiv A_+^{-1}C$ is a basis projection of $\mathcal{H}$, and $P_+P_{P_1}$ is Hilbert–Schmidt. Moreover, $P_+ = P_1E$ if and only if $[P_1, E] = 0$.

**Proof.** Let $E' = 1 - E$. Since $ECE'$ and $ECE'$ are compact by assumption, $C - ECE' - ECE = A + ECE'$ is a Fredholm operator on $\mathcal{K}$ with vanishing index. Hence $A$ is Fredholm on $\mathcal{H}$ with ind $A = 0$. $A$ is injective since $\gamma$ is nondegenerate on $\mathcal{H}$. It is therefore a bounded bijection on $\mathcal{H}$ with a bounded inverse (the same true for $A_\pm$ as operators on ran $A_\pm$). Thus $Q \equiv A^{-1}C$ is well–defined. It fulfills $Q^2 = A^{-1}(ECE)A^{-1}C = Q$ and $Q^\dagger = C(CA^{-1}C)^\dagger = Q$. So $Q$ is a projection, self–adjoint with respect to $\gamma$. Since its range equals ran $A^{-1} = \mathcal{H}$, it is the $\gamma$–orthogonal projection onto $\mathcal{H}$.

By a similar argument, $P_+$ is also a $\gamma$–orthogonal projection. It is straightforward to see that $P_+ = P_1E$ if and only if $[P_1, E] = 0$. To show that $P_+$ is actually a basis projection of $\mathcal{H}$ (cf. (4.6)), note that $\overline{A_+} = A_-$ because of $A = -A$ (and uniqueness of $A_{\pm}$). This implies $P_+ + P_{P_1} = A_+^{-1}C - A_+^{-1}C = A_+^{-1}C = 1_{\mathcal{H}}$. Positive definiteness of $CP_+$ on ran $P_+$ follows from $\langle f, CP_+f \rangle = \|A_+^{-1/2}Cf\|^2$. 

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To prove that $P_2P_+$ is HS, let $D = EP_1E - A_+$. Since $EP_1E - EP_2E = A = A_+ - A_-$, we have $D = \bar{D}$. We claim that $D$ is of trace class. Since $ECE'$ is HS,

$$ECE'CE = E(1 - E)CE$$

$$= E - (ECE)^2$$

$$= E - A^2$$

$$= (E + |A|)(E - |A|)$$

is of trace class. Since $E + |A|$ has a bounded inverse (as an operator on $\mathcal{H}$) and since $|A| = A_+ + A_-$, it follows that $E - |A| = EP_1E + EP_2E - A_+ - A_- = D + \bar{D} = 2D$ is of trace class as claimed. As a consequence, $A_+P_2 = (EP_1E - D)P_2$ is HS ($P_1EP_2$ is HS by assumption). By boundedness of $A_+^{-1}$, $P_+P_2 = -A_+^{-2}(A_+P_2)$ and $P_2P_+ = (P_+P_2)\dagger$ are also HS. This completes the proof.

Now let $V \in \mathcal{S}_{P_1}(\mathcal{K}, \gamma)$. We already showed in Section 3 that the restriction of $\gamma$ to $\ker V^\dagger$ is nondegenerate. We also showed in the proof of Theorem 4.3 that $[P_1, V']$ is Hilbert–Schmidt where $V'$ is the isometry arising from polar decomposition of $V$. Hence $[P_1, E]$ is Hilbert–Schmidt where $E = C(1 - V'V'^\dagger)C$ is the orthogonal projection onto $\ker V^\dagger$. Thus Lemma 4.2 applies to $\mathcal{H} = \ker V^\dagger$.

**Definition 4.3.** For $V \in \mathcal{S}_{P_1}(\mathcal{K}, \gamma)$, let $P_{V+}$ be the basis projection of $\ker V^\dagger$ given by Lemma 4.2, and set

$$P_V \equiv VP_1V^\dagger + P_{V+} \in \mathcal{P}_{P_1},$$

$$Z_V \equiv (P_V)^{21}(P_V)^{-1}\dagger11 \in \mathcal{E}_{P_1}$$

(cf. Proposition 4.3). Further let $U_V \in \mathcal{S}_{\text{HS}}(\mathcal{K}, \gamma)$ be the Bogoliubov operator associated with $Z_V$ according to (4.5), and define $W_V \equiv U_V^\dagger V \in \mathcal{S}_{\text{diag}}(\mathcal{K}, \gamma)$.

$P_V$ clearly is a basis projection which satisfies (4.1). Actually, any basis projection $P$ fulfilling $V^\dagger PV = P_1$ or, equivalently, $PV = VP_1$, is of the form $P = VP_1V^\dagger + P'$ where $P'$ is some basis projection of $\ker V^\dagger$. What had to be proved above is that $P'$ can be chosen such that $P_2P'$ is Hilbert–Schmidt, in the case $\dim \ker V^\dagger = \infty$. In fact, any such choice would suffice for what follows.

The condition $V^\dagger P_1V = P_1$ translates into the condition

$$Z_VV_{11} = V_{21}$$

(4.12)

for $Z_V$. Again, each $Z \in \mathcal{E}_{P_1}$ fulfilling (4.12) would do, but we prefer to have a definite choice. It follows from symmetry (4.2) that any $Z$ which solves (4.12) must have the form

$$Z = V_{21}V_{11}^{-1} + V_{22}^{-1}V_{12}P_{\ker V_{11}} + Z'$$

(4.13)

where $P_{3\mathcal{K}}$ denotes the orthogonal projection onto some closed subspace $\mathcal{K} \subset \mathcal{K}$, $V_{11}^{-1}$ and $V_{22}^{-1}$ have been defined below (4.4), and $Z'$ is a symmetric Hilbert–Schmidt operator from $\ker V_{11}$ to $\ker V_{22}$. The freedom in the choice of $Z'$ corresponds to the freedom in the choice of $P'$. Note that $Z$ can be written, with
respect to the decompositions \( \mathcal{K}_1 = \text{ran} V_{11} \oplus \ker V_{11}^* \), \( \mathcal{K}_2 = \text{ran} V_{22} \oplus \ker V_{22}^* \), as

\[
Z = \begin{pmatrix} \text{ran} V_{22} V_{21} V_{11}^{-1} & V_{22}^{-1} V_{12}^* P_{\ker V_{11}^*} \\ P_{\ker V_{22} \cdot V_{21} V_{11}^{-1}} & Z' \end{pmatrix}.
\]

The Hilbert–Schmidt norm of \( Z \) is minimized by choosing \( Z' = 0 \), but there are examples in which this choice violates the condition \( \|Z\| < 1 \), i.e. it does not always define an element of \( \mathcal{E}_{P_1} \). This is in contrast to the CAR case where the choice analogous to \( Z' = 0 \) appears to be natural. As we shall see in Section 4, \( Z_V \) describes the values of implementers on the Fock vacuum.

The operators \( U_V \) and \( W_V \) constitute the product decomposition of \( V \) that was announced earlier. \( W_V \) is diagonal because \( P_1 W_V = P_1 U_V^V V = U_V^V V P_1 = W_V P_1 \). Explicitly, one computes that

\[
W_V = \begin{pmatrix} (P_1 + Z_V^\dagger Z_V^\dagger V_{11} & 0 \\ 0 & (P_2 + Z_V Z_V^\dagger)^\dagger V_{22} \end{pmatrix}
\]

with respect to the decomposition \( \mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2 \). Let us summarize the properties of these operators.

**Proposition 4.4.** Definition 4.3 establishes a decomposition of \( V \in S_{P_1}(\mathcal{K}, \gamma) \)

\[
V = U_V W_V
\]

where \( U_V \in S_{\text{HS}}(\mathcal{K}, \gamma) \) and \( W_V \in S_{\text{diag}}(\mathcal{K}, \gamma) \) have the properties

\[
\begin{align*}
\text{ind} U_V &= 0, & Z_{U_V} &= Z_V, & P_{U_V} &= P_V; \\
\text{ind} W_V &= \text{ind} V, & Z_{W_V} &= 0, & P_{W_V} &= P_1.
\end{align*}
\]

In particular, if \( V \in S^0_{P_1}(\mathcal{K}, \gamma) \), then

\[
U_V = \begin{pmatrix} |V_{11}| & V_{12}v_{22}^* \\ V_{21}v_{11}^* & |V_{22}| \end{pmatrix}, \quad W_V = \begin{pmatrix} v_{11} & 0 \\ 0 & v_{22} \end{pmatrix}
\]

where \( v_{11} \equiv V_{11}|V_{11}|^{-1} \) and \( v_{22} = \frac{v_{11}^\dagger}{v_{11}^\dagger} \) are the unitary parts of \( V_{11} \) and \( V_{22} \); whereas if \( V \in S_{\text{diag}}(\mathcal{K}, \gamma) \), then \( U_V = \mathbf{1} \) and \( W_V = V \).

**Remark.** The product decomposition described above is the generalization to the infinite-dimensional case of a construction given by Maaß [10]. The exact analogue of the construction given in [4] in the fermionic case would be to define \( W' \in S_{\text{diag}}(\mathcal{K}, \gamma) \) through \( W'_{11} \equiv V_{11}|V_{11}|^{-1} \) (the isometric part of \( V_{11} \)), to choose a Bogoliubov operator \( u' \) from \( \ker W'^\dagger \) to \( \ker V^\dagger \) such that \( u' P_1 = P_V u' \), and to set \( U' \equiv W' V'^\dagger + u' \in S^0_{P_1}(\mathcal{K}, \gamma) \). Then \( U' \) and \( W' \) would also have the properties listed in Proposition 4.4, with the exception that \( U' - \mathbf{1} \) is not necessarily Hilbert–Schmidt. On the other hand, this choice has the merit that the definition of \( W' \) is completely canonical (independent of the choice of \( Z \)).

Though it was not shown in [4], it holds true also in the CAR case that each implementable Bogoliubov operator can be written as a product of two factors where the first differs from \( \mathbf{1} \) only by a Hilbert–Schmidt part, and the second is diagonal.

**Corollary 4.5.** \( S_{P_1}(\mathcal{K}, \gamma) = S_{\text{HS}}(\mathcal{K}, \gamma) \cdot S_{\text{diag}}(\mathcal{K}, \gamma) \). The orbits of the action of \( S^0_{P_1}(\mathcal{K}, \gamma) \) on \( S_{P_1}(\mathcal{K}, \gamma) \) are the subsets \( S^n_{P_1}(\mathcal{K}, \gamma), \, n \in \mathbb{N} \cup \{\infty\} \). They coincide with the connected components of \( S_{P_1}(\mathcal{K}, \gamma) \).
5 Normal Form of Cuntz Algebra Generators

The first step in the construction of implementers consists in a generalization of the definition of ‘bilinear Hamiltonians’ from the finite rank case to the case of bounded operators. If $H$ is a finite rank operator on $\mathcal{K}$ such that $H^* = \overline{H} = -H$, then $e^{HC}$ belongs to $\mathcal{S}_{18}(\mathcal{K}, \gamma)$. Expanding $H = \sum f_j (g_j, \cdot)$, one obtains a skew–adjoint element $b_0(H) \equiv \sum f_j g_j^*$ of $\mathcal{S}(\mathcal{K}, \gamma)$ which is a linear function of $H$, independent of the choice of $f_j, g_j \in \mathcal{K}$. Then $\pi_{P_1}(b_0(H))$ is essentially skew–adjoint on $\mathcal{D}$, and, if $b(H)$ denotes its closure, $\exp(\frac{1}{2}b(H))$ is a unitary which implements the automorphism induced by $e^{HC}$.

Using Wick ordering, the definition of bilinear Hamiltonians can be extended to arbitrary bounded operators $H$ which are symmetric in the sense of

$$
H_{11} = \overline{H_{22}}, \quad H_{12} = \overline{H_{12}}, \quad H_{21} = \overline{H_{21}}.
$$

Without loss of generality, we henceforth assume that $\mathcal{K}_1 = L^2(\mathbb{R}^d)$. Then let $\mathcal{S} \subset \mathcal{F}_s(\mathcal{K}_1)$ be the dense subspace consisting of finite particle vectors $\phi$ with $n$–particle wave functions $\phi^{(n)}$ in the Schwartz space $\mathcal{S}(\mathbb{R}^{dn})$. The unsmeared annihilation operator $a(p)$ with (invariant) domain $\mathcal{S}$ is defined as usual

$$(a(p)\phi)^{(n)}(p_1, \ldots, p_n) \equiv \sqrt{n + 1} \phi^{(n+1)}(p, p_1, \ldots, p_n).$$

Let $a^*(p)$ be its quadratic form adjoint on $\mathcal{S} \times \mathcal{S}$. Then Wick ordered monomials $a^*(q_1) \cdots a^*(q_m)a(p_1) \cdots a(p_n)$ make sense as quadratic forms on $\mathcal{S} \times \mathcal{S}$ and, for $\phi, \phi' \in \mathcal{S}$,

$$\langle \phi, a^*(q_1) \cdots a^*(q_m)a(p_1) \cdots a(p_n)\phi' \rangle \equiv \langle a(q_1) \cdots a(q_m)\phi, a(p_1) \cdots a(p_n)\phi' \rangle$$

is a Schwartz function to which tempered distributions may be applied. In particular, the distributions $H_{jk}(p, q)$, $j, k = 1, 2$, given by

$$
\langle f, H_{11}g \rangle = \int \overline{f(p)}H_{11}(p, q)g(q) \, dp \, dq,
$$

$$
\langle f, H_{12}g^* \rangle = \int \overline{f(p)}H_{12}(p, q)\overline{g(q)} \, dp \, dq,
$$

$$
\langle f^*, H_{21}g \rangle = \int f(p)H_{21}(p, q)g(q) \, dp \, dq,
$$

$$
\langle f^*, H_{22}g^* \rangle = \int f(p)H_{22}(p, q)\overline{g(q)} \, dp \, dq
$$

for $f, g \in \mathcal{S}(\mathbb{R}^d) \subset \mathcal{K}_1$, give rise to the following quadratic forms on $\mathcal{S} \times \mathcal{S}$:

$$
H_{11}a^* a \equiv \int a(p)^* H_{11}(p, q)a(q) \, dp \, dq,
$$

$$
H_{12}a^* a \equiv \int a(p)^* H_{12}(p, q)a(q)^* \, dp \, dq,
$$

$$
H_{21}aa \equiv \int a(p) H_{21}(p, q)a(q) \, dp \, dq
$$

$$
H_{22}aa^* \equiv \int a(q)^* H_{22}(p, q)a(p) \, dp \, dq = H_{11}a^* a.
$$

$^3$The bilinear Hamiltonian corresponding to an antisymmetric operator $H = -\overline{H^*}$ vanishes.
Wick ordering of $H_{22}aa^*$ is necessary to make this expression well-defined. The last equality follows from symmetry of $H$:

$$H_{11}(p, q) = H_{22}(q, p), \quad H_{12}(p, q) = H_{12}(q, p), \quad H_{21}(p, q) = H_{21}(q, p).$$

We next define $\langle b(H) \rangle$: and its Wick ordered powers as quadratic forms on $\mathcal{S} \times \mathcal{S}$:

$$\langle b(H) \rangle : = H_{12}a^*a^* + 2H_{11}a^*a + H_{21}aa,$$

$$\langle b(H) \rangle^l : = \prod_{l_1,l_2,l_3=0}^l \frac{2^{l_2}}{l_1!l_2l_3!}H_{l_1,l_2,l_3}, \quad l \in \mathbb{N},$$

with $H_{l_1,l_2,l_3} \equiv \int H_{12}(p_1, q_1) \cdots H_{12}(p_{l_1}, q_{l_1})H_{11}(p'_{l_1}, q'_{l_1}) \cdots H_{11}(p'_{l_2}, q'_{l_2})$

$$\cdot H_{21}(p''_{l_2}, q''_{l_2}) \cdots H_{21}(p''_{l_3}, q''_{l_3})a^*(p_1) \cdots a^*(p_{l_1})a^*(q_1) \cdots a^*(q_{l_1})$$

$$\cdot a^*(p'_{l_1}) \cdots a^*(p'_{l_2})a(q'_{l_1}) \cdots a(q'_{l_2})a(p''_{l_1}) \cdots a(p''_{l_2})a(q''_{l_1}) \cdots a(q''_{l_2})$$

$$\cdot dp_1 dq_1 \cdots dp_{l_1} dq_{l_1} dp'_{l_1} dq'_{l_1} \cdots dp'_{l_2} dq'_{l_2} dp''_{l_1} dq''_{l_1} \cdots dp''_{l_2} dq''_{l_2}.$$ 

The Wick ordered exponential of $\frac{1}{2}b(H)$ is also well-defined on $\mathcal{S} \times \mathcal{S}$, since only a finite number of terms contributes when applied to vectors from $\mathcal{S}$:

$$\exp\left(\frac{1}{2}b(H)\right) : = \sum_{l=0}^{\infty} \frac{1}{l!2^l} \langle b(H) \rangle^l : .$$

The important point is that these quadratic forms are actually the forms of uniquely determined linear operators, defined on the dense subspace $\mathcal{D}$ and mapping $\mathcal{D}$ into the domain of (the closure of) any creation or annihilation operator, provided that $\|H_{12}\| < 1$, $\quad H_{12}$ is Hilbert–Schmidt. (5.2)

These operators will be denoted by the same symbols as the quadratic forms.

**Lemma 5.1.** Let $H \in \mathfrak{F}(\mathfrak{X})$ satisfy (5.1) and (5.2). Then the following commutation relations hold on $\mathcal{D}$, for $f \in \mathfrak{X}_1$:

$$[H_{l_1,l_2,l_3}, a(f)^*] = l_2a(H_{11}f)^*H_{l_1,l_2-1,l_3} + 2l_3H_{l_1,l_2,l_3-1}a((H_{21}f)^*),$$

$$[a(f), H_{l_1,l_2,l_3}] = 2l_1a(H_{12}f^*)^*H_{l_1-1,l_2,l_3} + l_2H_{l_1,l_2-1,l_3}a(H_{11}^*f),$$

implying that

$$\left[ : \exp\left(\frac{1}{2}b(H)\right) : , a(f)^* \right]$$

$$= a(H_{11}f)^* : \exp\left(\frac{1}{2}b(H)\right) : + : \exp\left(\frac{1}{2}b(H)\right) : a((H_{21}f)^*),$$

$$\left[ a(f), : \exp\left(\frac{1}{2}b(H)\right) : \right]$$

$$= a(H_{12}f^*)^* : \exp\left(\frac{1}{2}b(H)\right) : + : \exp\left(\frac{1}{2}b(H)\right) : a(H_{11}^*f).$$
Proof. Compute as in \cite{4,3}.

For given $V \in \mathcal{S}_{\rho}(\mathcal{K}, \gamma)$, we are now looking for symmetric bounded operators $H$ which satisfy (3.2) and the following intertwiner relation on $\mathcal{D}$

$$
: \exp \left( \frac{1}{2} b(H) \right) : \pi_p(f) = \pi_p(V f) : \exp \left( \frac{1}{2} b(H) \right) :, \quad f \in \mathcal{K}
$$

(5.3)

(taking the closure of $\pi_p(V f)$ is tacitly assumed here). This problem turns out to be equivalent to the determination of the operators $Z$ done in (4.12), (4.13).

**Lemma 5.2.** Each $Z \in \mathcal{E}_p$, fulfilling (4.12) gives rise to a unique solution $H$ of the above problem through the formula

$$
H = \begin{pmatrix}
V_{11} - P_1 + Z^\dagger V_{21} & Z^\dagger \\
(V_{22}^* + V_{12}^* Z^\dagger) V_{21} & V_{22}^* - P_2 + V_{12}^* Z^\dagger
\end{pmatrix},
$$

and each solution arises in this way.

Proof. Let us abbreviate $\eta_H = : \exp(\frac{1}{2} b(H)) :$. Choosing $f \in \mathcal{K}_2$ resp. $f \in \mathcal{K}_1$ and inserting the definition of $\pi_p$, one finds that (5.3) is equivalent to

$$
\eta_H a(g) = (a(V_{11} g) + a^*(V_{12} g^*)) \eta_H, \quad \eta_H a^*(g) = (a^*(V_{11} g) + a(V_{12} g^*)) \eta_H
$$

for $g \in \mathcal{K}_1$. Using the commutation relations from Lemma 5.1, these equations may be brought into Wick ordered form:

$$
0 = a^* (V_{12} + H_{12} V_{22}) g^* \eta_H + \eta_H a \left( \left( (P_1 + H_{11}^*) V_{11} - P_1 \right) g \right),
$$

$$
0 = a^* \left( \left( (P_1 + H_{11} - V_{11} - H_{12} V_{21}) g \right) \eta_H + \eta_H a \left( \left( H_{21} - (P_1 + H_{11}^*) V_{12} \right) g^* \right).$$

As in the CAR case \cite{4}, these equations hold for all $g \in \mathcal{K}_1$ if and only if

$$
0 = V_{12} + H_{12} V_{22}, \quad 0 = P_1 + H_{11} - V_{11} - H_{12} V_{21}, \quad 0 = H_{21} - (P_2 + H_{22}) V_{21}, \quad 0 = P_2 - (P_2 + H_{22}) V_{22}
$$

(5.4a)

(5.4b)

(5.4c)

(5.4d)

(we applied complex conjugation and used $H_{11}^\dagger = H_{22}$).

Now assume that $H$ solves the above problem. It is then obvious from (5.1), (5.2) and (5.4a) that $Z = H_{12}^\dagger$ belongs to $\mathcal{E}_p$ and fulfills (4.12).

Conversely, let $Z \in \mathcal{E}_p$, satisfy (4.12). If there exists a solution $H$ with $H_{12} = Z^\dagger$, then $H_{11}$ is fixed by (5.4d). $H_{22}$ must equal $H_{11}^\star$, and $H_{21}$ is determined by (5.4c). Thus there can be at most one solution corresponding to $Z$, and it is necessarily of the form stated in the proposition.

It remains to prove that the so-defined $H$ has all desired properties, i.e. that $H_{21}$ is symmetric and that (5.4c) holds, the rest being clear by construction. The first claim follows from (3.4c):

$$
H_{21} - H_{21}^\star = (V_{22}^* + V_{12}^* Z^\dagger) V_{21} - V_{12}^* (V_{11} + Z^\dagger V_{21}) = 0,
$$

and the second from (4.12) and (3.4d):

$$
(P_2 + H_{22}) V_{22} = (V_{22}^* - V_{12}^* Z) V_{22} = V_{22}^* V_{22} - V_{12}^* V_{12} = P_2.
$$

\qed
Then define operators \( \Psi \) and, for any element \( \alpha \),

\[
\begin{align*}
H_{11} &= V_{11}^{-1} - P_1 - P_{\ker V_{11}^*} V_{11}^* V_{12}^{-1} V_{21}^* + Z_{11}^* V_{12}^{-1} V_{21}^* \, Z_{12}^* V_{21}^* \, Z_{12}^*, \\
H_{12} &= -V_{12} V_{22}^{-1} - V_{11}^{-1} V_{21}^* P_{\ker V_{22}^*} + Z_{12}^*, \\
H_{21} &= (V_{22}^{-1} - V_{12}^* V_{11}^{-1} V_{21}^* P_{\ker V_{22}^*}) V_{21} + V_{12}^* Z_{12}^* V_{21}^* V_{21}^* + V_{12}^* Z_{12}^*, \\
H_{22} &= V_{22}^{-1} - P_2 - V_{12}^* V_{11}^{-1} V_{21}^* P_{\ker V_{22}^*} + V_{12}^* Z_{12}^*.
\end{align*}
\]

\( H \) corresponds to Ruijsenaars’ operator \( \Lambda \). If one compares the above formula for \( H \) with Ruijsenaars’ formula for \( \Lambda \) in the case of automorphisms (\( \ker V_j^* = \{0\}, \ j = 1, 2, Z' = 0 \), one finds that the off–diagonal components carry opposite signs. This is due to the fact that Ruijsenaars actually constructs implementers for the transformation induced by CVC rather than \( V \), cf. \( (3.27) \) and \( (3.29) \) in [10].

Note that :\( \exp \left( \frac{1}{2} b(H) \right) \Omega_{P_1} = \exp \left( \frac{1}{2} H_{12} a^* a \right) \Omega_{P_1} \). By Ruijsenaars’ computation [10] (see also [15]), the norm of such vectors is

\[
\left\| \exp \left( \frac{1}{2} b(H) \right) \right\| \Omega_{P_1} = \left( \det(P_1 + H_{12} H_{12}^*) \right)^{-1/4}.
\]

**Definition 5.3.** Let \( V \in S_{P_1}(\mathcal{K}, \gamma) \), and let \( P_V, Z_V \) and \( H_V \) be the operators associated with \( V \) according to Definition 5.3 and Lemma 5.2. Choose a \( \gamma \)-orthonormal basis \( f_1, f_2, \ldots \in P_V(\ker V^* \vDash) \), i.e. a basis such that \( \gamma(f_j, f_k) = \delta_{jk} \) (this is possible because the restriction of \( \gamma \) to \( P_V(\ker V^* \vDash) \) is positive definite). Let \( \psi_j \) be the isometry obtained by polar decomposition of the closure of \( \pi_{P_j}(f_j) \). Then define operators \( \Psi_\alpha(V) \) on \( \mathcal{D} \), for any multi–index \( \alpha = (\alpha_1, \ldots, \alpha_l) \) with \( \alpha_j \leq \alpha_{j+1} \) (or \( \alpha = 0 \)) as in (3.1), as

\[
\Psi_\alpha(V) = \left( \det(P_1 + Z_V^* Z_V) \right)^{\frac{1}{2}} \psi_{\alpha_1} \cdots \psi_{\alpha_l} : \exp \left( \frac{1}{2} b(H_V) \right) : \; .
\]

**Theorem 5.4.** The \( \Psi_\alpha(V) \) extend continuously to isometries (denoted by the same symbols) on the symmetric Fock space \( F_s(\mathcal{K}_1) \) such that

\[
\Psi_\alpha(V)^* \Psi_\beta(V) = \delta_{\alpha \beta} 1, \quad \sum_\alpha \Psi_\alpha(V) \Psi_\alpha(V)^* = 1 \quad (5.6)
\]

and, for any element \( w \) of the Weyl algebra \( W(\mathcal{K}, \gamma) \),

\[
g_V(w) = \sum_\alpha \Psi_\alpha(V) w \Psi_\alpha(V)^*. \quad (5.7)
\]

**Proof.** By (2.20) we have \( \pi_{P_j}(f_j)^* \pi_{P_j}(f_j) = 1 + \pi_{P_j}(f_j) \pi_{P_j}(f_j)^* \) on \( \mathcal{D} \), so the closure of \( \pi_{P_j}(f_j) \) is injective, and \( \psi_j \) is isometric. It is also easy to see, using (5.3), the CCR and \( \| \Psi_\alpha(V) \Omega_{P_1} \| = 1 \), that

\[
\langle \Psi_\alpha(V) \pi_{P_j}(g_1 \cdots g_m) \Omega_{P_1}, \Psi_\alpha(V) \pi_{P_j}(h_1 \cdots h_n) \Omega_{P_1} \rangle
\]

\[
= \langle \pi_{P_j}(g_1 \cdots g_m) \Omega_{P_1}, \pi_{P_j}(h_1 \cdots h_n) \Omega_{P_1} \rangle.
\]

Hence \( \Psi_\alpha(V) \) is isometric on \( \mathcal{D} \) and has a continuous extension to an isometry on \( F_s(\mathcal{K}_1) \).
Let $\mathcal{H}_j \equiv \text{span}(f_j, f_j^*)$, so that $\psi_j \in \mathcal{W}(\mathcal{H}_j)^\ast$ by virtue of Lemma 2.1. Since $\mathcal{H}_j \subset \ker V^\dagger$, there holds $\mathcal{W}(\mathcal{H}_j) \subset \mathcal{W}(\text{ran } V)'$ by duality (2.3). Now let $f \in \text{Re } \mathcal{K}$ and $\phi \in \mathcal{D}$. Since $\phi$ is an entire analytic vector for $\pi_{P_1}(f)$, since $\mathcal{D}$ is invariant under $\pi_{P_1}(f)$, and since $\pi_{P_1}(Vf)$ is affiliated with $\mathcal{W}(\text{ran } V)$ by Lemma 2.1 (the bar denotes closure), it follows from (5.3) that

$$
\Psi_{\alpha}(V)w(f)\phi = \sum_{n=0}^{\infty} \frac{i^n}{n!} \Psi_{\alpha}(V)(\pi_{P_1}(f))^n\phi
$$

By continuity, this entails

$$
\Psi_{\alpha}(V)w = \varrho_V(w)\Psi_{\alpha}(V), \quad w \in \mathcal{W}(\mathcal{K}, \gamma).
$$

We next claim that

$$
\psi_{\alpha}^\ast \Psi_{0}(V) = 0
$$

or, equivalently, that $\pi_{P_1}(f_j)^\ast \Psi_{0}(V) = 0$. To see this, apply Lemma 5.1 and write $\pi_{P_1}(f_j)^\ast \Psi_{0}(V)$ in Wick ordered form:

$$
\pi_{P_1}(f_j)^\ast \Psi_{0}(V) = a((P_1 + H_{12})f_j)^\ast \Psi_{0}(V) + \Psi_{0}(V)a((P_1 + H_{11})f_j)
$$

on $\mathcal{D}$, with $H = H_V$. Then (5.9) holds if and only if

$$
(P_1 + H_{12})f_j^\ast = 0, \quad (P_1 + H_{11})f_j = 0.
$$

Now $f_j \in \text{ran } P_V$ is equivalent to $f_j^\ast \in \ker P_V = \ker CP_V = \ker (P_1 + H_{12})$ (we used (1.16)). This proves the first equation in (5.9). It also shows that $H_{12}f_j = -f_j$. Hence by Lemma 5.2,

$$
(P_1 + H_{11})f_j = (V_{11}^\ast + V_{21}^\ast H_{12})f_j = (V_{11}^\ast - V_{21}^\ast)f_j = P_1 V^\dagger f_j = 0
$$

which proves the second equation in (5.9) and therefore (5.3).

The orthogonality relation $\Psi_{\alpha}(V)^\ast \Psi_{\beta}(V) = 0$ ($\alpha \neq \beta$) now follows from (5.9) and from $\mathcal{W}(\mathcal{H}_1) \subset \mathcal{W}(\mathcal{H}_k)^\ast$ ($j \neq k$) which in turn is a consequence of $\gamma(\mathcal{H}_j, \mathcal{H}_k) = 0$ and (2.3).

The proof of the completeness relation $\sum \Psi_{\alpha}(V)\Psi_{\alpha}(V)^\ast = 1$ is facilitated by invoking the product decomposition $V = U_{V} W_{V}$ from Proposition 1.4. Set $e_j \equiv U_{V}^\dagger f_j$ to obtain a $\gamma$-orthonormal basis $e_1, e_2, \ldots$ in $P_1(\ker W_{V}^\dagger) = \mathcal{K}_1 \cap \ker W_{V}^\dagger$. Let $\phi_{\alpha}^\ast$ be the isometric part of $a(e_j)^\ast$. An application of Definition 5.3 to $W_{V}$ yields implementers $\Psi_{\alpha}(W_{V}) = \psi_{\alpha_1}^\ast \cdots \psi_{\alpha_n}^\ast \Omega_{P_1}$ for $W_{V}$. $Z_{W_{V}} = 0$ entails that

$$
\Psi_{\alpha}(W_{V})\Omega_{P_1} = \psi_{\alpha_1}^\ast \cdots \psi_{\alpha_n}^\ast \Omega_{P_1}.
$$

One computes, using the CCR, that $\psi_{\alpha_1}^\ast \cdots \psi_{\alpha_n}^\ast \Omega_{P_1} = \phi_{\alpha}$, where the $\phi_{\alpha}$ are the cyclic vectors associated with the pure state $\omega_{P_1} \circ \varrho_{W_{V}} = \omega_{P_1}$ as in Lemma 2.1.
Let \( \mathcal{F}'_\alpha \) be the closure of \( \mathcal{W}(\text{ran } W) \phi^\prime_\alpha \). Since the \( \mathcal{F}'_\alpha \) are irreducible subspaces for \( \mathcal{W}(\text{ran } W) \) by Lemma 6.1, they must coincide with the irreducible subspaces \( \text{ran } \Psi_\alpha(W) \). \( \oplus \mathcal{F}'_\alpha = \mathcal{F}_s(\mathcal{K}_1) \) then implies completeness of the \( \Psi_\alpha(W) \).

The proof will be completed by showing that

\[
\Psi_\alpha(V) = \Psi(U_V) \Psi_\alpha(W_V)
\]

holds where \( \Psi(U_V) \) is the unitary implementer for \( U_V \) given by Definition 5.3. It suffices to show that (5.10) holds on \( \Omega_{P_i} \) since any bounded operator fulfilling (5.8) is already determined by its value on \( \Omega_{P_i} \). Because of \( Z_{U_V} = Z_V \) we have

\[
\Psi_0(V) \Omega_{P_1} = \Psi(U_V) \Omega_{P_1},
\]

so it remains to show that \( \psi_{\alpha_1} \cdots \psi_{\alpha_l} \Psi(U_V) \Omega_{P_1} = \Psi(U_V) \psi_{\alpha'_1} \cdots \psi_{\alpha'_l} \Omega_{P_1} \). We claim that

\[
\psi_j \Psi(U_V) = \Psi(U_V) \psi'_j.
\]

For let \( T \) (resp. \( T' \)) be the closure of \( \pi_{P_1}(f_j) \) (resp. \( \pi_{P_1}(e_j) \)), and let \( T^{\pm} \) (resp. \( T'^{\pm} \)) be the corresponding self–adjoint operators as in the proof of Lemma 2.1, so that

\[
D(T) = D(T^+) \cap D(T^-), \quad T = T^+ - iT^-,
\]

and similar for \( T' \). Then there holds

\[
\Psi(U_V) \exp(itT^{\pm}) \Psi(U_V)^* = \exp(itT^{\pm}), \quad t \in \mathbb{R}.
\]

Therefore \( \Psi(U_V) \) maps \( D(T'^{\pm}) \) onto \( D(T^{\pm}) \), and one has \( \Psi(U_V)T^{\pm}\Psi(U_V)^* = T^{\pm} \). Consequently, \( \Psi(U_V)(D(T')) = D(T) \) and \( \Psi(U_V)T'\Psi(U_V)^* = T' \). This implies that \( \psi_j \Psi(U_V)\psi'_j \Psi(U_V)^* = \psi_j \) as claimed. The proof is complete since (5.6) and (5.8) together imply (5.7).

**Corollary 5.5.** There is a unitary isomorphism from \( H(\varphi_V) \), the Hilbert space generated by \( \Psi_\alpha(V) \), onto the symmetric Fock space \( \mathcal{F}_s(P_V(\ker V^\dagger)) \) over \( P_V(\ker V^\dagger) \), which maps \( \Psi_\alpha(V) \) to \( (l_1! \cdots l_r!)^{1/2} a^*(f_{\alpha_1}) \cdots a^*(f_{\alpha_r}) \Omega \), where the notation is as in (3.1), and \( a^*(f_j) \) and \( \Omega \) are now creation operators and the Fock vacuum in \( \mathcal{F}_s(P_V(\ker V^\dagger)) \).

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