LOCAL WELL-POSEDNESS FOR THE DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION WITH $L^2$-SUBCRITICAL DATA

SHAOMING GUO
Department of Mathematics, University of Wisconsin Madison
Madison, WI, USA

XIANFENG REN AND BAOXIANG WANG∗
LMAM, School of Mathematical Sciences, Peking University
Beijing 100871, China

(Communicated by Chongchun Zeng)

Abstract. Considering the Cauchy problem of the derivative NLS
\[ iu_t + \partial_{xx} u = i\mu \partial_x (|u|^2 u), \quad u(0, x) = u_0(x), \]
we will show its local well-posedness in modulation spaces $M^{1/2}_{2,q}(\mathbb{R})$ ($4 \leq q < \infty$). It is well-known that $H^{1/2}$ is a critical Sobolev space of the derivative NLS. Noticing that $H^{1/2} \subset M^{1/2}_{2,q} \subset B^{1/2}_{2,q}$ ($q \geq 2$) are sharp inclusions, our result contains a class of functions in $L^2 \setminus H^{1/2}$.

1. Introduction. In this paper, we consider the Cauchy problem for the derivative nonlinear Schrödinger equations with a derivative nonlinearity (DNLS)
\[ iu_t + \partial_{xx} u = i\mu \partial_x (|u|^2 u), \quad u(x, 0) = u_0(x), \quad (1.1) \]
where $u$ is a complex valued function of $(x, t) \in \mathbb{R} \times [0, T]$ for some $T > 0$, $\mu \in \mathbb{R}$. The DNLS equation arises in one-dimensional compressible magneto-hydrodynamic in the presence of the Hall effect and the propagation of circular polarized nonlinear Alfvén waves in magnetized plasmas (cf. [37, 38, 19, 36]).

Using the gauge transform
\[ G(u)(x) = \exp \left( -i \int_x^{-\infty} |u(y)|^2 dy \right) u(x), \quad (1.2) \]
Hayashi and Ozawa [25, 27, 28] obtained the global well-posedness of DNLS in $H^1$ with condition $\|u_0\|_2 < \sqrt{2\pi}$ (see also [26, 40]). Takaoka [44] considered the rougher data and he established the local well-posedness in $H^{1/2}$ by using the equivalent equation of $v = Gu$:
\[ iv_t + \partial_{xx} v = -i\mu v^2 \partial_x \overline{v} - \frac{|\mu|^2}{2} |v|^4 v, \quad v(x, 0) = v_0(x). \quad (1.3) \]
On the other hand, Biagioni and Linares [5] showed that DNLS is ill-posed in $H^s$ with $s < 1/2$, i.e., the solution map $u_0 \to u(t)$ in is not uniformly continuous in $H^s$ if $s < 1/2$. So, $H^{1/2}$ is the critical Sobolev space in all $H^s$ so that DNLS is locally well-posed. The global well-posedness in $H^{1/2}$ and Wu [22] with condition $\|u_0\|_2 < \sqrt{4\pi}$. Very recently, Bahouri and Perelman [1] have removed the condition $\|u_0\|_2 < \sqrt{4\pi}$ by using the integrability structure of the DNLS, i.e., DNLS is globally well-posed in $H^{1/2}$.

However, the critical space for DNLS in the scaling sense is $L^2$, i.e., for any solution $u$ of DNLS, the scaling solution $u_\sigma(t, x) := \sigma^{1/2} u(\sigma^2 t, \sigma x)$ has an invariant norm in $L^2$ for any $\sigma > 0$. This fact implies that there is a gap between $L^2$ and $H^{1/2}$ for the well-posedness of DNLS. One can naturally ask what is the reasonable well-posed space with the regularity at the same level with $L^2$. In order to answer this question, Grünrock [18, 20] applied the $H^s$ spaces for which the norm is defined by

$$||u||_{H^s_q} := ||\langle \xi \rangle^s \hat{u}||_{p'} , \ 1/p + 1/p' = 1$$

and he obtained that DNLS is local well-posed in $H^{1/2}_p$ for any $1 < p \leq 2$. Using the scaling argument, $H^{1/2}_p$ $(1 < p \leq 2)$ can be regarded as subcritical spaces. In this paper we consider the initial data in modulation spaces $M^{1/2}_{2,q}$ which are larger than $H^{1/2}_{q'}$ for $2 < q < \infty$.

Now let us recall the definition of modulation spaces by using the short-time Fourier transform (STFT) of a function $f$ with respect to $g \in \mathcal{S}$

$$V_g f(x, \omega) = \int_{\mathbb{R}^n} e^{-it\cdot\omega} g(t-x) f(t) dt,$$

where $g$ is said to be a window function. STFT is a basic tool in time frequency analysis theory, which is closely related to the Wigner transform and the wave packets transform introduced by Córdoba and Fefferman in [12]. We write

$$\|f\|_{M^s_{p,q}} = \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |V_g f(x, \omega)|^p dx \right)^{q/p} \langle \omega \rangle^{sq} d\omega \right)^{1/q},$$

with the usual modifications if $p$ or $q$ is infinite. Let $1 \leq p, q \leq \infty$. The modulation space (or Feichtinger space) $M^s_{p,q}$ is defined as the space of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ for which $\|f\|_{M^s_{p,q}}$ is finite (see Feichtinger [16]). One can refer to [17] for their basic properties.

We write $\Box_k = \mathcal{F}^{-1} \chi_{[k-1/2, k+1/2]} \mathcal{F}$, where $\mathcal{F}$ ($\mathcal{F}^{-1}$) denotes the (inverse) Fourier transform on $\mathbb{R}$, $\chi_E$ denotes the characteristic function on $E$. The modulation space $M^s_{2,q}$ can be equivalently defined in the following way (cf. [51, 49, 48, 50]):

$$\|f\|_{M^{s}_{2,q}(\mathbb{R})} = \left( \sum_{k \in \mathbb{Z}} \langle k \rangle^{sq} \| \Box_k f \|_{L^2(\mathbb{R})}^q \right)^{1/q}, \quad (1.4)$$

where $\langle k \rangle = (1 + |k|^2)^{1/2}$. Let $2 \leq q \leq \infty$, $1/q + 1/q' = 1$. We have $H^s_{q'} \subset M^{s}_{2,q}$ (see Appendix). Combining it with the inclusions between Besov and modulation spaces, we have (cf. [42, 46, 50] and Appendix)

$$\tilde{H}^{1/2}_{q'} \subset M^{1/2}_{2,q} \subset B^{1/q}_{2,q} \subset L^2 , \ q < \infty.$$
and these inclusions are optimal. Hence, $M^{1/2}_{2,q}$ ($2 \leq q < \infty$) can be regarded as subcritical spaces and $M^{1/\infty}_{2,\infty}$ is a critical space for the DNLS (see Appendix for further discussions on the critical spaces for DNLS). Our main result is the following

**Theorem 1.1.** Let $4 \leq q < \infty$, $v_0 \in M^{1/2}_{2,q}$. Then there exists a $T > 0$ such that DNLS (1.3) is locally well posed in $L^\infty(0,T;M^{1/2}_{2,q}) \cap X^{1/2}_{q,\Delta}([0,T])$, where $X^{s,\Delta}_{q,\Delta}$ is defined in (2.3).

The regularity index $1/2$ in $M^{1/2}_{2,q}$ is optimal. In fact, there is an ill-posedness for the DNLS in $M^{s}_{2,q}$ if $s < 1/2$; cf. [47]. Noticing that $M^{s}_{2,q_1} \subset M^{s,\infty}_{2,q_2}$ for any $q_1 \leq q_2$, the interesting case is $q \gg 1$ in Theorem 1.1. In order to have a better understanding to $M^{1/2}_{2,q}$, we observe the following two typical functions:

$$\tilde{v}_0(\xi) = (\xi)^{-1/2-\eta}, \quad \tilde{u}_0(\xi) = \xi^{-1/2+\theta} \chi_{(0,1)}(\xi).$$

We see that $u_0 \in L^2$ if and only if $u_0 \in M^{1/2}_{2,q}$ for some $q > 2$, if and only if $\eta > 0$; $v_0 \in L^2$ if and only if $v_0 \in M^{1/2}_{2,q}$, if and only if $\theta > 0$. One may further ask what happens if the initial data in $M^{1/2}_{2,\infty}$, we conjecture that DNLS is also ill-posed in $M^{1/2}_{2,\infty}$, since $L^2$ and $M^{1/2}_{2,\infty}$ have no inclusion relations.

There are some recent papers which have been devoted to the study of nonlinear PDE with initial data in modulation spaces $M^{s}_{p,1}$, see [2, 3, 10, 11, 13, 14, 15, 29, 30, 31, 32, 41, 43, 47, 52]. An interesting feature is that modulation spaces $M^{s}_{p,1}$ contains a class of initial data out of the critical Sobolev spaces $H^{s_c}$ in the case $s_c > 0$, for which the nonlinear PDE is well-posed for $s > s_c$ and ill-posed for $s < s_c$. Guo [21] considered a class of initial data in $M^{s}_{2,q}$ for the cubic NLS, where the case $q > 2$ was first taken into account by using $U^p$ and $V^p$ spaces. For more recent development related to the study of NLS in modulation spaces, we refer to [8, 9, 39] and the reference therein.

Let $c < 1, C > 1$ denote positive universal constants, which can be different at different places, $a \lesssim b$ stands for $a \leq Cb$, $a \sim b$ means that $a \lesssim b$ and $b \lesssim a$. If $a = b + \ell$, $|\ell| \leq C$, then we write $a \approx b$. $a \gg b$ means that $a > b + C$. We write $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$. We write $p'$ as the dual number of $p \in [1, \infty]$, i.e., $1/p + 1/p' = 1$. We will use Lebesgue spaces $L^p := L^p(\mathbb{R}), \| \cdot \|_p := \| \cdot \|_{L^p}$, Sobolev spaces $H^s = (1 - \Delta)^{-s/2}L^2$. Some properties of these function spaces can be found in [4, 45].

2. $U^p$ and $V^p$ spaces.

2.1. **Definitions.** $U^p$ and $V^p$, as a development of Bourgain’s spaces [6, 7] were first applied by Koch and Tataru in the study of NLS, cf. [33, 34, 35]. Using $U^p$ and $V^p$, Hadac, Herr and Koch [23] obtained the well-posedness and scattering results for the critical KP-II equation. Let $\mathcal{Z}$ be the set of finite partitions $-\infty = t_0 < t_1 < \ldots < t_{K-1} < t_K = \infty$. Let $1 \leq p < \infty$. For any $\{t_k\}_{k=0}^K \subset \mathcal{Z}$ and $\{\phi_k\}_{k=0}^K \subset L^2$ with $\sum_{k=0}^{K-1} \|\phi_k\|_2^p = 1$, $\phi_0 = 0$. A step function $a : \mathbb{R} \rightarrow L^2$ given by

$$a = \sum_{k=1}^K \chi_{[t_{k-1}, t_k)} \phi_k$$
Similarly for the definition of $v$ is finite, where we use the convention that $v$ is a subspace of all $(v)$ (here for which the norm is given by $\|v\|_{U^p} = \inf \left\{ \sum_{j=1}^{\infty} |c_j| : u = \sum_{j=1}^{\infty} c_j a_j, a_j \in \mathcal{A}(U^p), c_j \in \mathbb{C} \right\}$.

We define $V^p$ as the normed space of all functions $v: \mathbb{R} \to L^2$ such that $\lim_{t \to \pm \infty} v(t)$ exist and for which the norm

$$\|v\|_{V^p} = \sup_{\{t_k\} \subseteq \mathbb{Z}} \left( \sum_{k=1}^{K} \|v(t_k) - v(t_{k-1})\|_{L^2}^p \right)^{1/p}$$

is finite, where we use the convention that $v(-\infty) = \lim_{t \to -\infty} v(t)$ and $v(\infty) = 0$ (here $v(\infty)$ and $\lim_{t \to \infty} v(t)$ are different notations). Likewise, we denote by $V^p$ the subspace of all $v \in V^p$ so that $v(-\infty) = 0$. Moreover, we define the closed subspace $V^p_{rc} (V^p_{rc, -\Delta})$ as all of the right continuous functions in $V^p (V^p_{rc})$. We define

$$U^p_{\Delta} := e^{-i\Delta} U^p, \quad \|u\|_{U^p_{\Delta}} = \|e^{-it\Delta} u\|_{U^p},$$

$$V^p_{\Delta} := e^{-i\Delta} V^p, \quad \|u\|_{V^p_{\Delta}} = \|e^{-it\Delta} u\|_{V^p}.$$  

Similarly for the definition of $V^p_{rc, -\Delta}: V^p_{rc, -\Delta}$.

We introduce the frequency-uniform localized $U^p_{\Delta}$-spaces $X^p_q(I)$ and $V^p_{\Delta}$-spaces $Y^p_q(I)$ for which the norms are defined by

$$\|u\|_{X^p_q(I)} = \left( \sum_{\lambda \in I \in \mathbb{Z}} \langle \lambda \rangle^q q \|\square \lambda u\|_{U^p_{\Delta}}^{q} \right)^{1/q}, \quad X^p_q := X^p_q(\mathbb{R}),$$

$$\|u\|_{Y^p_q(I)} = \left( \sum_{\lambda \in I \in \mathbb{Z}} \langle \lambda \rangle^q q \|\square \lambda u\|_{V^p_{\Delta}}^{q} \right)^{1/q}, \quad Y^p_q := Y^p_q(\mathbb{R}),$$

$$\|u\|_{X^p_{\Delta}} := \|e^{-it\Delta} u\|_{X^p_q}, \quad \|v\|_{Y^p_{\Delta}} := \|e^{-it\Delta} v\|_{Y^p_q}.$$  

For the time local version of $X^p_{\Delta}$, the norm can be defined as

$$\|u\|_{X^p_{\Delta}(0,T)} = \inf \{ \|v\|_{X^p_{\Delta}} : v \in X^p_q, v(t) = u(t), \forall t \in [0,T] \}. \quad (2.4)$$

Besov type Bourgain’s spaces $X^{s,b,q}$ are defined by

$$\|u\|_{X^{s,b,q}} := \|\chi_{|\tau+\xi^2| \in [2^{j-1},2^j]} |\xi|^s |\tau + \xi^2|^{b} \hat{u}(\tau, \xi) \|_{L^2_{\tau,\xi}} \|_{L^q_{\xi}}.$$  

2.2. Known results on $U^p$ and $V^p$. We list some known results in $U^p$ and $V^p$ (cf. [33, 34, 35, 23]).

**Proposition 2.1.** (Embedding) Let $1 \leq p < q < \infty$. We have the following results.

(i) $U^p$ and $V^p$, $V^p_{rc}$, $V^p_{rc,-}$ are Banach spaces.

(ii) $U^p \subset V^p_{rc}$, $V^p \subset L^\infty(\mathbb{R}, L^2)$. Every $u \in U^p$ is right continuous on $t \in \mathbb{R}$

(iii) $V^p \subset V^q$, $V^p \subset V^p_{rc}$, $V^p_{rc} \subset V^p_{rc,-}$

(iv) $X^{0,1/2,1} \subset U^p_{\Delta} \subset V^2_{rc,-, \Delta} \subset X^{0,1/2,\infty}$. 

**SHAOMING GUO, XIANFENG REN AND BAOXIANG WANG**
It is known that, for the free solution of the Schrödinger equation \(u(x, t) = e^{i\Delta t}u_0\), \(\hat{u}(\xi, \tau)\) is supported on a curve \(\tau + \xi^2 = 0\), which is said to be the dispersion relation. For the solution \(u\) of DNLS, \(\hat{u}(\xi, \tau)\) can be supported in \((\xi, \tau) \in \mathbb{R}^2\), we need to consider the size of \(|\xi^2 + \tau|\), which is said to be the dispersion modulation. By the second inclusion in (ii) of Proposition 2.1 we have

\[
X^s_{q, \Delta} \subset L^\infty(\mathbb{R}, M^s_{2,q}).
\]

By the last inclusion of (iv) in Proposition 2.1, we see that

**Lemma 2.2** (Dispersion Modulation Decay). Suppose that the dispersion modulation \(|\tau + \xi^2| \gtrsim \mu\) for a function \(u \in L^2_{x,t}\), then we

\[
\|u\|_{L^2_{x,t}} \lesssim \mu^{-1/2}\|u\|_{V^s}.
\]

The next interpolation inequality was obtained by Hadac, Herr and Koch [23].

**Proposition 2.3.** (Interpolation) Let \(1 \leq p < q < \infty\). There exists a positive constant \(c(p, q) > 0\), such that for any \(u \in V^p\) and \(M > 1\), there exists a decomposition \(u = u_1 + u_2\) satisfying

\[
\frac{1}{M}\|u_1\|_{U^p} + e^{cM}\|u_2\|_{U^q} \lesssim \|u\|_{V^p}.
\]

**Proposition 2.4.** (Duality) Let \(1 \leq p < \infty\), \(1/p + 1/p' = 1\). Then \((U^p)^* = V^{p'}\) in the sense that

\[
T : V^{p'} \rightarrow (U^p)^*; \quad T(v) = B(\cdot, v),
\]

is an isometric mapping. The bilinear form \(B : U^p \times V^{p'}\) is defined in the following way: For a partition \(t := \{t_k\}_{k=0}^K \subset \mathbb{Z}\), we define

\[
B_t(u, v) = \sum_{k=1}^K \langle u(t_{k-1}), v(t_k) - v(t_{k-1}) \rangle.
\]

Here \(\langle \cdot, \cdot \rangle\) denotes the inner product on \(L^2\). For any \(u \in U^p, v \in V^{p'}\), there exists a unique number \(B(u, v)\) satisfying the following property. For any \(\varepsilon > 0\), there exists a partition \(t\) such that

\[
|B(u, v) - B_t(u, v)| < \varepsilon, \quad \forall t' \supset t.
\]

Moreover,

\[
|B(u, v)| \leq \|u\|_{U^p}\|v\|_{V^{p'}}.
\]

In particular, let \(u \in V^1\) be absolutely continuous on compact interval, then for any \(v \in V^{p'}\),

\[
B(u, v) = \int \langle u'(t), v(t) \rangle dt.
\]

The duality of \(U^p\) and \(V^{p'}\) obtained by Hadac, Herr and Koch [23] is of importance for us to make nonlinear estimates for the dispersive equations. For its applications to the DNLS, we need further consider its frequency-uniform decomposition version, namely, the duality of \(X^s_q\) and \(Y^{s-\ast}_{q'}\).
2.3. Duality of $X_q^*$ and $Y_{q'}^{-s}$.

Proposition 2.5. (Duality) Let $1 \leq q < \infty$. Then $(X_q^*)^* = Y_{q'}^{-s}$ in the sense that

$$T : Y_{q'}^{-s} \rightarrow (X_q^*)^*; \quad T(v) = B(\cdot, v),$$

is an isometric mapping, where the bilinear form $B(\cdot, \cdot)$ is defined in Proposition 2.4. Moreover, we have

$$|B(u, v)| \leq \|u\|_{X_q^*} \|v\|_{Y_{q'}^{-s}}.$$  \hfill (2.10)

Proof. By the orthogonality, we see that

$$B_k(\square_k u, \square_l v) = 0, \quad k \neq l,$$

which implies that

$$B(\square_k u, \square_l v) = 0, \quad k \neq l.$$  

For any $v \in Y_{q'}^{-s}$, by Proposition 2.4 and Hölder’s inequality, we have

$$|B(u, v)| = \left| \sum_{k \in \mathbb{Z}} B(\square_k u, \square_k v) \right| \leq \sum_{k \in \mathbb{Z}} ||\square_k u||_{U^2} ||\square_k v||_{V^2} \leq \|u\|_{X_q^*} \|v\|_{Y_{q'}^{-s}}.$$  

It follows that $Y_{q'}^{-s} \subset (X_q^*)^*$ and $\|v\|_{(X_q^*)^*} \leq \|v\|_{Y_{q'}^{-s}}$.

Conversely, considering the map

$$X_q^* \ni f \rightarrow \{\square_k f\} \in \ell_q^s(\mathbb{Z}; U^2),$$

where

$$\ell_q^s(\mathbb{Z}; U^2) := \{\{f_k\}_{k \in \mathbb{Z}} : \|f_k\|_{\ell_q^s(\mathbb{Z}; U^2)} := \|\{\langle (k)^s f_k \rangle_{U^2}\} \|_{\ell_q^s} < \infty\},$$

we see that it is an isometric mapping from $X_q^*$ into a subspace of $\ell_q^s(\mathbb{Z}; U^2)$. So, $v \in (X_q^*)^*$ can be regarded as a continuous functional in a subspace of $\ell_q^s(\mathbb{Z}; U^2)$. In view of Hahn-Banach Theorem, it can be extended onto $\ell_q^s(\mathbb{Z}; U^2)$ (the extension is written as $\tilde{v}$) and its norm will be preserved. In view of the well-known duality $(\ell_q^s(\mathbb{Z}; X))^* = \ell_{q'}^{-s}(\mathbb{Z}; X^*)$, we have

$$(\ell_q^s(\mathbb{Z}; U^2))^* = \ell_{q'}^{-s}(\mathbb{Z}; V^2),$$

and there exists $\{v_k\}_{k \in \mathbb{Z}} \in \ell_{q'}^{-s}(\mathbb{Z}; V^2)$ such that

$$\langle \tilde{v}, \{f_k\} \rangle = \sum_{k \in \mathbb{Z}} B(f_k, v_k), \quad \forall \{f_k\} \in \ell_q^s(\mathbb{Z}; U^2).$$

Moreover, $\|v\|_{(X_q^*)^*} = \|\{v_k\}\|_{\ell_{q'}^{-s}(\mathbb{Z}; V^2)}$. Hence, for any $u \in X_q^*$,

$$\langle v, u \rangle = \langle \tilde{v}, \{\square_k u\} \rangle = \sum_{k \in \mathbb{Z}} B(\square_k u, v_k).$$

From $B_k(\square_k u, v) = B_k(u, \square_k v)$ we see that $B(\square_k u, v) = B(u, \square_k v)$. It follows that

$$v = \sum_{k \in \mathbb{Z}} \square_k v_k.$$  

Obviously, we have

$$\|v\|_{Y_{q'}^{-s}} \leq \|\{v_k\}\|_{\ell_{q'}^{-s}(\mathbb{Z}; V^2)} = \|v\|_{(X_q^*)^*}.$$
This proves \((X_q^s)^* \subset Y_q^{-s}\).

Now we apply the duality to the norm calculation to the inhomogeneous part of the solution of DNLS in \(X_q^s\). It is known that (1.1) is equivalent to the following integral equation:

\[
  u(t) = e^{it\Delta} u_0 - A \left( i\mu u^2 \partial_x \Pi + \frac{|\mu|^2}{2} |u|^4 u \right),
\]

where

\[
  A(f) = \int_0^t e^{i(t-s)\Delta} f(s) ds.
\]

By Propositions 2.4 and 2.5, we see that, for supp \(v \subset \mathbb{R} \times [0,T]\),

\[
  \|A(f)\|_{X_{p,q}^{1/2}} = \sup \left\{ \left| B \left( \int_0^t e^{-i\Delta} f(s) ds, v \right) \right| : \|v\|_{Y_{p,q}^{-1/2}} \leq 1 \right\}
\]

\[
  \leq \sup_{\|v\|_{Y_{p,q}^{-1/2}} \leq 1} \left| \int_0^T \langle f(s), e^{is\Delta} v(s) \rangle ds \right|
\]

\[
  \leq \sup_{\|v\|_{Y_{p,q}^{-1/2}} \leq 1} \left| \int_0^T \langle f(s), v(s) \rangle ds \right|. \tag{2.12}
\]

3. **Frequency localized estimates in \(L^4\).** Let \(I \subset \mathbb{R}\) be an interval with finite length. For simply, we denote

\[
  u_\lambda = \Box \lambda u, \quad u_I = \sum_{\lambda \in I \cap \mathbb{Z}} u_\lambda,
\]

**Lemma 3.1.** ([21]) Let \(I \subset \mathbb{R}\) with \(|I| < \infty\). For any \(\theta \in (0, 1), \beta > 0\), we have

\[
  \|u_I\|_{L^4_{x,t}([0,T])} \lesssim (T^{1/8} + T^{(1-\theta)/8}|I|^{\beta+(1-\theta)/4})|u|_{X_q^0(I)}^{\beta}. \tag{3.1}
\]

In particular, if \(1 \leq |I| < \infty, 0 < T < 1\), then for any \(0 < \varepsilon \ll 1\),

\[
  \|u_I\|_{L^4_{x,t}([0,T])} \lesssim T^{\varepsilon/3}|I|^{\varepsilon}|u|_{X_q^0(I)}. \tag{3.2}
\]

**Lemma 3.2.** Let \(I \subset \mathbb{R}\) with \(1 \leq |I| < \infty, 0 < T < 1\), then for any \(4 \leq q < \infty, 0 < \varepsilon \ll 1\), we have

\[
  \|u_I\|_{L^4_{x,t}([0,T])} \lesssim T^{\varepsilon/3}|I|^{1/4-1/q+\varepsilon}\sup_{\lambda \in I} \|\lambda\|_{X_q^{1/2}(I)}, \tag{3.3}
\]

\[
  \|u_I\|_{L^q_{x,t} L^\infty_{x,t} \cap V_q^2} \lesssim |I|^{1/2-1/q}\sup_{\lambda \in I} \|\lambda\|_{X_q^{1/2}(I)}, \tag{3.4}
\]

\[
  \|u_I\|_{L^\infty_{x,t}} \lesssim |I|^{1-1/q}\sup_{\lambda \in I} \|\lambda\|_{X_q^{1/2}(I)}. \tag{3.5}
\]

**Proof.** By (3.2) and Hölder’s inequality, we have (3.3). Using \(M_{2,2}^0 = L^2, V_q^2 \subset L^\infty_{x,t} L^2_x\) and Hölder’s inequality, we have (3.4). In view of Bernstein’s inequality,

\[
  \|u_I\|_{L^\infty_{x,t}} \lesssim |I|^{1/2}\|u_I\|_{L^\infty_{x,t} L^2_x}.
\]

Combining it with (3.4), we have (3.5). \(\square\)
Corollary 3.3. Let $I_j = [a2^j, b2^j]$, $0 < a < b$, $0 < T < 1$, then for any $4 \leq q < \infty$, $0 < \varepsilon \ll 1$, we have
\[
\|u_{I_j}\|_{L^q_{x,t}([0,T])} \lesssim T^{\varepsilon/4}2^{j(-1/4-1/q+\varepsilon)}\|u\|_{X^{1/2}_q(I_j)},
\]
\[
\|u_{I_j}\|_{L^q_{x,t}L^2_{x,t}} \lesssim 2^{-j/q}\|u\|_{X^{1/2}_q(I_j)},
\]
\[
\|u_{I_j}\|_{L^q_{x,t}} \lesssim 2^{j(1/2-1/q)}\|u\|_{X^{1/2}_q(I_j)}.
\]

Corollary 3.4. Let $\lambda_0 \gg 1$, $\lambda \in \mathbb{N}$, $I_j = \lambda_0 - [a2^j, b2^j] \subset [c\lambda_0, \lambda_0]$ for some $0 < a < b$, $0 < T < 1$, then for any $4 \leq q < \infty$, $0 < \varepsilon \ll 1$, we have
\[
\|u_{I_j}\|_{L^q_{x,t}([0,T])} \lesssim T^{\varepsilon/3}(\lambda_0)^{-1/2}2^{j(1/4-1/q+\varepsilon)}\|u\|_{X^{1/2}_q(I_j)},
\]
\[
\|u_{I_j}\|_{L^q_{x,t}L^2_{x,t}} \lesssim (\lambda_0)^{-1/2}2^{j(1/2-1/q)}\|u\|_{X^{1/2}_q(I_j)},
\]
\[
\|u_{I_j}\|_{L^q_{x,t}} \lesssim (\lambda_0)^{-1/2}2^{j(1-1/q)}\|u\|_{X^{1/2}_q(I_j)}.
\]

Corollary 3.5. Let $\lambda_0 \in \mathbb{R}$ with $\lambda_0 \gg 1$, $I \subset [c\lambda_0, \lambda_0]$ or $I \subset [-\lambda_0, -c\lambda_0]$ for some $c \in (0,1)$, $0 < T < 1$, then for any $4 \leq q < \infty$, $0 < \varepsilon \ll 1$, we have
\[
\|u_{I}\|_{L^q_{x,t}([0,T])} \lesssim T^{\varepsilon/3}(\lambda_0)^{-1/4-1/q+\varepsilon}\|u\|_{X^{1/2}_q(I)},
\]
\[
\|u_{I}\|_{L^q_{x,t}L^2_{x,t}} \lesssim (\lambda_0)^{-1/q}\|u\|_{X^{1/2}_q(I)},
\]
\[
\|u_{I}\|_{L^q_{x,t}} \lesssim (\lambda_0)^{1/2-1/q}\|u\|_{X^{1/2}_q(I)}.
\]

4. Bilinear estimates. The following bilinear estimate can be found in [20], [35], [21].

Lemma 4.1 (Bilinear Estimate 1). Let $0 < T \leq 1$. Suppose that $\hat{u}$, $\hat{v}$ are localized in some compact intervals $I_1, I_2$ with $\text{dist}(I_1, I_2) \geq \lambda$. Then for any $0 < \varepsilon \ll 1$, we have
\[
\|u\hat{v}\|_{L^2_{x,t}([0,T])} \lesssim T^{\varepsilon/4}\lambda^{-1/2+\varepsilon}\|u\|_{V^2_2}\|v\|_{V^2_2}.
\]

Similar to Grünrock’s bilinear estimate, we can consider the following bilinear estimate, which is useful for our late purpose.

Lemma 4.2 (Bilinear Estimate 2). Let $0 < T \leq 1$. Suppose that $\hat{u}, \hat{v}$ are localized in some compact intervals $I_1, I_2$ with $\text{dist}(I_1, I_2) \geq \lambda$. Then for any $0 < \varepsilon \ll 1$, we have
\[
\|uv\|_{L^2_{x,t}([0,T])} \lesssim T^{\varepsilon/4}\lambda^{-1/2+\varepsilon}\|u\|_{V^2_2}\|v\|_{V^2_2}.
\]

Proof. First, we show that if $\hat{u}_0, \hat{v}_0$ are localized in $I_1, I_2$ with $\text{dist}(I_1, I_2) \geq \lambda$, then
\[
\|e^{it\Delta}u_0e^{it\Delta}v_0\|_{L^2_{x,t}([0,T])} \lesssim \lambda^{-1/2}\|u_0\|_2\|v_0\|_2.
\]

We have
\[
\mathcal{F}_x(e^{it\Delta}u_0e^{it\Delta}v_0) = \int e^{-it(\xi^2-2\xi_1+2\xi_1^2)}\hat{u}_0(\xi - \xi_1)\hat{v}_0(\xi_1)d\xi_1.
\]

It follows that
\[
\mathcal{F}_{x,t}(e^{it\Delta}u_0e^{it\Delta}v_0) = \int \delta(2\xi_1^2-2\xi_1+\xi^2+\tau)\hat{u}_0(\xi - \xi_1)\hat{v}_0(\xi_1)d\xi_1.
\]
Denote
\[ g(\xi_1) = 2\xi_1^2 - 2\xi_1 + \xi^2 + \tau, \]
we see that
\[ g'(\xi_1) = 4\xi_1 - 2\xi, \quad g(\xi_1^\pm) = 0, \quad \xi^\pm = \frac{\xi}{2} \pm \sqrt{\frac{\xi^2}{4} - \frac{\xi^2 + \tau}{2}} := \frac{\xi}{2} \pm y. \]
Recall that \( \delta(g(\xi_1)) = \delta(\xi_1 - \xi^+)/|g'(\xi^+)| - \delta(\xi_1 - \xi^-)/|g'(\xi^-)| = \delta(\xi_1 - \xi^+)/4y - \delta(\xi_1 - \xi^-)/4y, \) we have
\[ \mathcal{F}_{x,t}(e^{it\Delta}u_0 e^{it\Delta}v_0) = \frac{1}{4y} \tilde{u}_0 \left( \frac{\xi}{2} - y \right) \tilde{v}_0 \left( \frac{\xi}{2} + y \right) - \frac{1}{4y} \tilde{u}_0 \left( \frac{\xi}{2} + y \right) \tilde{v}_0 \left( \frac{\xi}{2} - y \right). \quad (4.6) \]
By symmetry, it suffices to estimate the first term in (4.6). Changing of variables \( y = \sqrt{\frac{\xi^2}{4} - \frac{\xi^2 + \tau}{2}}, \) we see that
\[
\left\| e^{it\Delta}u_0 e^{it\Delta}v_0 \right\|^2_{L^2_x} \leq \int_{\mathbb{R}^2} \frac{c}{|y|} \left| \tilde{u}_0 \left( \frac{\xi}{2} - y \right) \right|^2 \left| \tilde{v}_0 \left( \frac{\xi}{2} + y \right) \right|^2 dyd\xi
\]
\[
\lesssim \int_{\mathbb{R}^2} \frac{1}{|\xi_1 - \xi^2|} |\tilde{u}_0(\xi_1)|^2 |\tilde{v}_0(\xi_2)|^2 d\xi_1 d\xi_2
\]
\[
\lesssim \lambda^{-1} \int_{\mathbb{R}^2} |\tilde{u}_0(\xi_1)|^2 |\tilde{v}_0(\xi_2)|^2 d\xi_1 d\xi_2 = C\lambda^{-1} \|u_0\|_{L^\infty}^2 \|v_0\|_{L^\infty}^2, \quad (4.7) \]
where in the last inequality, we have applied \( \text{dist}(I_1, I_2) > \lambda. \) By testing atoms in \( U^2, \) then applying the interpolation in Proposition 2.3, we have the Bilinear Estimate 2.

5. Trilinear estimates. We need to have a bound of the second term of the integral equation (2.11) in \( \dot{X}^{1/2}_{p,\Delta}. \) More precisely, we want to show that
\[
\left\| \int_0^t e^{(t-s)\Delta} (u^2 \partial_x \bar{w}) (s) ds \right\|_{\dot{X}^{1/2}_{p,\Delta}} \lesssim T^\epsilon \|u\|^3_{\dot{X}^{1/2}_{p,\Delta}}, \quad (5.1)
\]
\[
\left\| \int_0^t e^{(t-s)\Delta} (uv \partial_x \bar{w}) (s) ds \right\|_{\dot{X}^{1/2}_{p,\Delta}} \lesssim T^\epsilon \|u\|_{X^{1/2}_{p,\Delta}} \|v\|_{X^{1/2}_{p,\Delta}} \|u\|_{X^{1/2}_{p,\Delta}} \quad (5.2)
\]

Proof of (5.1). In view of (2.12), it suffices to show that
\[
\left| \int_{\mathbb{R} \times [0,T]} vu^2 \partial_x \bar{w} dx dt \right| \lesssim T^\epsilon \|u\|^3_{\dot{X}^{1/2}_{p,\Delta}} \|v\|_{\dot{Y}^{-1/2}_{p',\Delta}}, \quad (5.3)
\]
We perform a uniform decomposition with \( u, v \) in the left hand side of (5.3), it suffice to prove that
\[
\left| \sum_{\lambda_0, \ldots, \lambda_3} \langle \lambda_0 \rangle^{1/2} \int_{[0,T] \times \mathbb{R}} \bar{v}_{\lambda_0} u_{\lambda_1} u_{\lambda_2} \partial_x \bar{v}_{\lambda_3} dx dt \right| \lesssim T^\epsilon \|u\|^3_{\dot{X}^{1/2}_{p,\Delta}} \|v\|_{\dot{Y}^{-1/2}_{p',\Delta}}. \quad (5.4)
\]
In order to keep the left hand side of (5.4) nonzero, we have the frequency constraint condition (FCC)
\[
\lambda_1 + \lambda_2 - \lambda_3 - \lambda_0 \approx 0 \quad (5.5)
\]
and dispersion modulation constraint condition (DMCC)
\[
\max_{0 \leq k \leq 3} |\xi_k^2 + \tau_k| \gtrsim |(\xi_0 - \xi_1)(\xi_0 - \xi_2)|, \quad (5.6)
\]
where we assume that $\hat{v}_{\lambda_0}, \hat{u}_{\lambda_1}, \ldots, \hat{u}_{\lambda_3}$ in (5.4) are functions of $(\xi_0, \tau_0), \ldots, (\xi_3, \tau_3)$, respectively. It suffices to consider the cases that $\lambda_0$ is minimal or secondly minimal number in $\lambda_0, \ldots, \lambda_3$ (in the opposite case, one can instead $\lambda_0, \ldots, \lambda_3$ by $-\lambda_0, \ldots, -\lambda_3$).

**Step 1.** We assume that $\lambda_0 = \min_{0 \leq k \leq 3} \lambda_k$. We separate the proof into two subcase $\lambda_0 \geq 0$ and $\lambda_0 < 0$.

**Step 1.1.** We consider the case $\lambda_0 \geq 0$. Let us denote $I_0 = [0, 1)$, $I_j = [2^{j-1}, 2^j)$, $j \geq 1$. We decompose $\lambda_0 + [0, \infty)$ by dyadic decomposition, i.e.,

$$\lambda_k \in \lambda_0 + [0, \infty) = \bigcup_{j_k \geq 0} (\lambda_0 + I_{j_k}), \quad k = 1, 2, 3.$$ 

Recall that

$$u_{\lambda_0 + I_{j_k}} = \sum_{\lambda_k \in \lambda_0 + I_{j_k}} u_{\lambda_k}, \quad k = 1, 2, 3.$$ 

By the symmetry we can assume that $j_1 \leq j_2$. Moreover, in view of FCC (5.5), we see that $j_2 \approx j_3$. It follows that

$$\mathcal{L}^+ (u, v) := \sum_{0 \leq \lambda_0 \leq \lambda_k, \ k = 1, 2, 3} (\lambda_0)^{1/2} \int_{[0, T] \times \mathbb{R}} \nabla \lambda_0 u_{\lambda_1} u_{\lambda_2} \partial_{x} \pi_{\lambda_3} \, dx \, dt$$

$$= \sum_{\lambda_0 \geq 0, \ j_1 \leq j_2, \ j_3 \approx j_2} (\lambda_0)^{1/2} \int_{[0, T] \times \mathbb{R}} \nabla \lambda_0 u_{\lambda_0 + I_{j_1}} u_{\lambda_0 + I_{j_2}} \partial_{x} \pi_{\lambda_0 + I_{j_3}} \, dx \, dt$$

$$\lesssim \sum_{\lambda_0 \geq 0, \ j_1 \leq j_2 \approx j_3 \leq 1} \lambda_0 + j_1 \leq 10, \ j_2 \approx j_3 \gg 1 \sum_{\lambda_0 \geq 0, \ 10 < j_1 \leq j_2 \approx j_3} (\lambda_0)^{1/2}$$

$$\times \int_{[0, T] \times \mathbb{R}} \nabla \lambda_0 u_{\lambda_0 + I_{j_1}} u_{\lambda_0 + I_{j_2}} \partial_{x} \pi_{\lambda_0 + I_{j_3}} \, dx \, dt$$

$$:= \mathcal{L}^+_m (u, v) + \mathcal{L}^+_m (u, v) + \mathcal{L}^+_h (u, v). \quad (5.7)$$

It is easy to see that in $\mathcal{L}^+_m (u, v)$, the frequency of $u_{\lambda_0 + I_{j_k}}$ ($k = 1, 2, 3$) are all localized in a neighbourhood of $\lambda_0$. So, by Hölder’s inequality, $\|\square_k u\|_4 \leq \|\square_k u\|_2$ and $U^2_\Delta \subset V^2_\Delta \subset L^2_\Delta$, we have

$$|\mathcal{L}^+_m (u, v)| \lesssim \sum_{\lambda_0 \approx \lambda_1 \approx \lambda_2 \approx \lambda_3} (\lambda_0)^{3/2} \|v_{\lambda_0}\|_{L^4 \Delta L^4 \Delta} \|u_{\lambda_1}\|_{L^4 \Delta L^4 \Delta} \|u_{\lambda_2}\|_{L^4 \Delta L^4 \Delta} \|u_{\lambda_3}\|_{L^4 \Delta L^4 \Delta}$$

$$\lesssim T \|u\|_{X^{1/2}_{p, \Delta}} \|v\|_{Y^{1/2}_{p, \Delta}}. \quad (5.8)$$

In $\mathcal{L}^+_h (u, v)$, we easily see that the frequency of $v_{\lambda_0}$ and $u_{\lambda_0 + I_{j_k}}$ are localized near $\lambda_0$, which are much less than those of $u_{\lambda_0 + I_{j_2}}$ and $u_{\lambda_0 + I_{j_3}}$. So, we can use bilinear estimate (4.1) and Lemma 3.2 to obtain that

$$|\mathcal{L}^+_m (u, v)|$$

$$\lesssim \sum_{\lambda_0 \geq 0, \ j_1 \leq 10, \ j_2 \approx j_3 \geq 1} (\lambda_0)^{1/2} \|u_{\lambda_0 + I_{j_1}} \partial_{x} \pi_{\lambda_0 + I_{j_2}}\|_{L^2_{x,t}} \|\nabla \lambda_0 u_{\lambda_0 + I_{j_2}}\|_{L^2_{x,t}}$$

$$\lesssim T^{<2/2} \sum_{\lambda_0 \geq 0, \ j_1 \leq 10, \ j_2 \approx j_3 \geq 1} (\lambda_0)^{1/2} \|u_{\lambda_0 + I_{j_1}} \nabla \lambda_0 u_{\lambda_0 + I_{j_2}}\|_{V_\Delta^2} \|u_{\lambda_0 + I_{j_3}}\|_{V_\Delta^2}$$

$$\lesssim T^{<2/2} \sum_{\lambda_0 \geq 0, j \geq 1, |\xi_1| \|\xi_2| \leq 1} (\lambda_0)^{1/2} \|u_{\lambda_0 + I_{j_1}} \nabla \lambda_0 u_{\lambda_0 + I_{j_2}}\|_{V_\Delta^2} \|u_{\lambda_0 + I_{j_3}}\|_{V_\Delta^2}$$
Now we estimate
\[\sum_{j=0}^\infty 2^{(-2/p+2c)} \|v_{j,0}\|_{V_{2,\Delta}}^2 \|u_{j,0}+\ell\|_{V_{2,\Delta}}^2 \lesssim T^{c/2} \quad (5.9)\]
Noticing that \(0 < \varepsilon < 1/p\), we have from Hölder’s inequality that
\[|\mathcal{L}_m^+(u, v)| \lesssim T^{c/2} \|u\|_{X_{p,\Delta}^{1/2}}^2 \sum_{\lambda_0 \geq 0, \|\ell\| \leq 1} \|v_{\lambda_0}\|_{V_{2,\Delta}^2} (\lambda_0)^{1/2} \|u_{\lambda_0}+\ell\|_{V_{2,\Delta}^2}^2 \lesssim T^{c/2} \|v\|_{X_{p,\Delta}^{1/2}}^3 \|v\|_{X_0^{0,\Delta}}^2. \quad (5.10)\]
Now we estimate \(\mathcal{L}_h^+(u, v)\).
\textbf{Case 1.} \(v_{\lambda_0}\) has the highest dispersion modulation in the right hand side of \(\mathcal{L}_h^+(u, v)\) in (5.7). In view of DMCC (5.6), we have
\[|\xi_0^2 + \tau_0| \gtrsim 2^{j_1+j_2}.\]
It follows from the dispersion modulation decay (2.6) that
\[\|v_{\lambda_0}\|_{L_{x,t}^\infty} \lesssim 2^{-2(\lambda_0+1)/2} \|v_{\lambda_0}\|_{V_{2,\Delta}^2}. \quad (5.11)\]
By Hölder’s inequality, we have
\[|\mathcal{L}_h^+(u, v)| \lesssim \sum_{\lambda_0 \geq 0, 10<j_1 \leq j_2 \approx j_3} (\lambda_0)^{1/2} \|v_{\lambda_0}\|_{L_7^2 L_\infty^\infty} \|u_{\lambda_0}+I_{j_1}\|_{L_7^2 L_\infty^\infty} \times \|u_{\lambda_0}+I_{j_2}\|_{L_7^1 L_4^4} \|\partial_x u_{\lambda_0} + I_{j_3}\|_{L_7^1 L_4^4} \approx \sum_{\lambda_0 \geq 0, 10<j_1 \leq j_2 \approx j_3} (\lambda_0)^{1/2} \|v_{\lambda_0}\|_{V_{2,\Delta}^2} (\lambda_0+2^{j_1})^{-1/2} \|\partial_x u_{\lambda_0} + I_{j_3}\|_{L_7^1 L_4^4} \approx \sum_{\lambda_0 \geq 0, 10<j_1 \leq j_2 \approx j_3} (\lambda_0)^{1/2} \|v_{\lambda_0}\|_{V_{2,\Delta}^2} (\lambda_0+2^{j_1})^{-1/2} \|\partial_x u_{\lambda_0} + I_{j_3}\|_{L_7^1 L_4^4} \approx \sum_{\lambda_0 \geq 0, 10<j_1 \leq j_2 \approx j_3} (\lambda_0)^{1/2} \|v_{\lambda_0}\|_{V_{2,\Delta}^2} (\lambda_0+2^{j_1})^{-1/2} \|\partial_x u_{\lambda_0} + I_{j_3}\|_{L_7^1 L_4^4} \approx \sum_{\lambda_0 \geq 0, 10<j_1 \leq j_2 \approx j_3} (\lambda_0)^{1/2} \|v_{\lambda_0}\|_{V_{2,\Delta}^2} (\lambda_0+2^{j_1})^{-1/2} \|\partial_x u_{\lambda_0} + I_{j_3}\|_{L_7^1 L_4^4} \approx \sum_{\lambda_0 \geq 0, 10<j_1 \leq j_2 \approx j_3} (\lambda_0)^{1/2} \|v_{\lambda_0}\|_{V_{2,\Delta}^2} (\lambda_0+2^{j_1})^{-1/2} \|\partial_x u_{\lambda_0} + I_{j_3}\|_{L_7^1 L_4^4}.\]
Making the summation on \(j_2, j_3\), applying Hölder’s inequality on \(\lambda_0\) and finally summing over all \(j_1\), we have for \(0 < \varepsilon < 1/2p\),
\[|\mathcal{L}_h^+(u, v)| \lesssim T^{c/2} \sum_{\lambda_0 \geq 0, 10<j_1 \leq j_2 \approx j_3} (\lambda_0)^{1/2} \|v_{\lambda_0}\|_{V_{2,\Delta}^2} (\lambda_0+2^{j_1})^{-1/2} \|\partial_x u_{\lambda_0} + I_{j_3}\|_{L_7^1 L_4^4} \approx \sum_{\lambda_0 \geq 0, 10<j_1 \leq j_2 \approx j_3} (\lambda_0)^{1/2} \|v_{\lambda_0}\|_{V_{2,\Delta}^2} (\lambda_0+2^{j_1})^{-1/2} \|\partial_x u_{\lambda_0} + I_{j_3}\|_{L_7^1 L_4^4}.\]
\textbf{Case 2.} \(u_{\lambda_0}+I_{j_1}\) has the highest dispersion modulation in the right hand side of \(\mathcal{L}_h^+(u, v)\) in (5.7). In order to use DMCC (5.6), \(u_{\lambda_0}+I_{j_1}\) takes \(L_7^2 L_\infty^\infty\) norm, \(u_{\lambda_0}+I_{j_2}\) and \(u_{\lambda_0}+I_{j_3}\) take \(L_7^4 L_\infty^\infty\) norms. Indeed, we have
\[|\mathcal{L}_h^+(u, v)| \lesssim \sum_{\lambda_0 \geq 0, 10<j_1 \leq j_2 \approx j_3} (\lambda_0)^{1/2} \|v_{\lambda_0}\|_{L_7^\infty L_\infty^\infty} \|u_{\lambda_0}+I_{j_1}\|_{L_7^2 L_\infty^\infty} \|u_{\lambda_0}+I_{j_2}\|_{L_7^4 L_\infty^\infty} \|u_{\lambda_0}+I_{j_3}\|_{L_7^4 L_\infty^\infty} \|\partial_x u_{\lambda_0} + I_{j_3}\|_{L_7^1 L_4^4} \approx \sum_{\lambda_0 \geq 0, 10<j_1 \leq j_2 \approx j_3} (\lambda_0)^{1/2} \|v_{\lambda_0}\|_{L_7^\infty L_\infty^\infty} \|u_{\lambda_0}+I_{j_1}\|_{L_7^2 L_\infty^\infty} \|u_{\lambda_0}+I_{j_2}\|_{L_7^4 L_\infty^\infty} \|u_{\lambda_0}+I_{j_3}\|_{L_7^4 L_\infty^\infty} \|\partial_x u_{\lambda_0} + I_{j_3}\|_{L_7^1 L_4^4} \approx \sum_{\lambda_0 \geq 0, 10<j_1 \leq j_2 \approx j_3} (\lambda_0)^{1/2} \|v_{\lambda_0}\|_{L_7^\infty L_\infty^\infty} \|u_{\lambda_0}+I_{j_1}\|_{L_7^2 L_\infty^\infty} \|u_{\lambda_0}+I_{j_2}\|_{L_7^4 L_\infty^\infty} \|u_{\lambda_0}+I_{j_3}\|_{L_7^4 L_\infty^\infty} \|\partial_x u_{\lambda_0} + I_{j_3}\|_{L_7^1 L_4^4} \approx \sum_{\lambda_0 \geq 0, 10<j_1 \leq j_2 \approx j_3} (\lambda_0)^{1/2} \|v_{\lambda_0}\|_{L_7^\infty L_\infty^\infty} \|u_{\lambda_0}+I_{j_1}\|_{L_7^2 L_\infty^\infty} \|u_{\lambda_0}+I_{j_2}\|_{L_7^4 L_\infty^\infty} \|u_{\lambda_0}+I_{j_3}\|_{L_7^4 L_\infty^\infty} \|\partial_x u_{\lambda_0} + I_{j_3}\|_{L_7^1 L_4^4}.\]
Using the fact \(\|v_{\lambda_0}\|_{L_7^\infty L_\infty^\infty} \lesssim \|v_{\lambda_0}\|_{L_7^\infty L_\infty^\infty} \lesssim \|v_{\lambda_0}\|_{V_{2,\Delta}^2}\), the dispersion modulation decay estimate (5.11) and Lemma 3.2, this case reduces to the same estimate as in Case 1, see (5.12).
Applying the bilinear estimate (4.1), DMCC (5.6) and Lemma 3.2, we have

\[ |L^+_h(u, v)| \leq \sum_{\lambda_0 \geq 0, \ 10 < j_1, j \approx j_3} \langle \lambda_0 \rangle^{1/2} \| v_{\lambda_0} \|_{L_h^{\infty}} \| u_{\lambda_0 + j_1} \|_{L^2_x} \| u_{\lambda_0 + i_2} \|_{L^2_{x,t}[0, T]} \cdot \]

Applying the bilinear estimate (4.1), DMCC (5.6) and Lemma 3.2, we have

\[ |L^+_h(u, v)| \leq \sum_{\lambda_0 \geq 0, \ 10 < j_1, j \approx j_3} \langle \lambda_0 \rangle^{1/2} \| v_{\lambda_0} \|_{L_h^{\infty}} T^{\varepsilon/4} 2^{j_3 (1/2 - \varepsilon)} \| u_{\lambda_0 + j_1} \|_{V^2_\Delta} \| u_{\lambda_0 + i_2} \|_{V^2_\Delta} \]

\[ \leq T^{\varepsilon/4} \sum_{\lambda_0 \geq 0, \ 10 < j_1, j \approx j_3} 2^{-(j_1 + j_2 + j_3) / p + j_3} \| v_{\lambda_0} \|_{V^2_\Delta} \| u_{\lambda_0 + j_1} \|_{V^2_\Delta} \| u_{\lambda_0 + i_2} \|_{V^2_\Delta} \]

Making the summation on \( j_2, j_3 \), we have

\[ |L^+_h(u, v)| \leq T^{\varepsilon/4} \sum_{\lambda_0 \geq 0, \ j_1 > 10} 2^{j_1 (3 - p + \varepsilon)} \| v_{\lambda_0} \|_{V^2_\Delta} \| u_{\lambda_0} \|_{X^{j_1/2}_{p, \Delta} (\lambda_0 + j_1)} \| u_{\lambda_0} \|_{X^{j_1/2}_{p, \Delta}} \]

This reduces to the same estimate as in the first inequality of (5.12).

**Case 3.** \( u_{\lambda_0 + j_3} \) has the highest dispersion modulation in the right hand side of \( L^+_h(u, v) \) in (5.7). If \( j_1 \approx j_3 \) in the right hand side of \( L^+_h(u, v) \) in (5.7), then we can repeat the proof as in Case 2 to obtain the desired estimates. So, it suffices to consider the case \( j_1 \ll j_3 \). \( j_1 \ll j_3 \) implies that the frequency of \( u_{\lambda_0 + j_3} \) is much higher than that of \( u_{\lambda_0 + j_1} \), and we can use the bilinear estimate (4.2). By Hölder’s inequality,

\[ |L^+_h(u, v)| \leq \sum_{\lambda_0 \geq 0, \ 10 < j_1, j \approx j_3} \langle \lambda_0 \rangle^{1/2} \| v_{\lambda_0} \|_{L_h^{\infty}} \| u_{\lambda_0 + j_1} \|_{L^2_x} \| u_{\lambda_0 + i_2} \|_{L^2_{x,t}[0, T]} \cdot \]

**Step 1.2.** We consider the case \( \lambda_0 < 0 \). We can assume that \( \lambda_0 \ll 0 \). For short, we denote by \( \lambda \in h_- \) and \( \lambda \in l_- \) the fact \( \lambda \in [\lambda_0, 3\lambda_0/4) \) and \( \lambda \in [3\lambda_0/4, 0) \) respectively. We need to consider the following three cases \( \lambda_1 \in h_- , \lambda_2 \in l_- \) and \( \lambda_3 \in [0, \infty) \) separately.

**Case A.** \( \lambda_1 \in h_- \). We consider the following four subcases as in Table 1.

| Case          | \( \lambda_1 \)       | \( \lambda_2 \)       | \( \lambda_3 \)       |
|---------------|------------------------|------------------------|------------------------|
| \( h_- h_- a \) | \( [\lambda_0, 3\lambda_0/4) \) | \( [\lambda_0, 3\lambda_0/4) \) | \( [\lambda_0, \infty) \) |
| \( h_- l_- a_- \) | \( [\lambda_0, 3\lambda_0/4) \) | \( [3\lambda_0/4, 0) \) | \( [\lambda_0, 0) \) |
| \( h_- l_- a_+ \) | \( [\lambda_0, 3\lambda_0/4) \) | \( [3\lambda_0/4, 0) \) | \( (0, \infty) \) |
| \( h_- a_+ a \) | \( [\lambda_0, 3\lambda_0/4) \) | \( [0, \infty) \) | \( [\lambda_0, \infty) \) |

Table 1. In \( \lambda_1, \ldots, \lambda_3 \), there is at least one frequency near \( \lambda_0 \)

**Case \( h_- h_- a \).** By FCC (5.5), we see that \( \lambda_3 \in [\lambda_0, \lambda_0/2 + C) \). Hence, \( \lambda_3 \) is also near \( \lambda_0 \). Using the same way as in the proof of Step 1, we can get the result and the details are omitted.

**Case \( h_- l_- a_- \).** By FCC (5.5), we see that \( \lambda_3 \geq 3\lambda_0/4 - C \). We decompose \( \lambda_k \) in the following way\(^1\):

\[ \lambda_1 \in [\lambda_0 + [0, \lambda_0/4]) = \bigcup_{j_1 \geq 0} (\lambda_0 + I_{j_1}); \quad \lambda_2 \in [3\lambda_0/4, 0) = \bigcup_{j_2 \geq 0} -I_{j_2}; \]

\(^1\)We can assume that \( I_{j_1} = [2^{j_1 - 1}, 2^j] \cap [0, -\lambda_0/4) \), it is convenient to use such a kind of notations below.
\[ \lambda_3 \in \left[ \frac{3\lambda_0}{4} - C, 0 \right] = \bigcup_{j_3 \geq 0} -I_{j_3}. \]

Again, in view of (5.5) we see that

\[(j_1 \lor j_3) \approx j_2.\]

We have the following three subcases:

\[ j_1 \ll j_3, \quad j_2 \approx j_3; \quad \text{or} \quad j_3 \ll j_1, \quad j_1 \approx j_2; \quad \text{or} \quad j_1 \approx j_2 \approx j_3. \]

It follows that

\[
\sum_{\lambda_0 < 0} (\lambda_0)^{1/2} \int_{[0,T] \times \mathbb{R}} |\nabla_{\lambda_0} u|_{\lambda_0, 3\lambda_0/4} u_{[3\lambda_0/4, 0]} \partial_x \bar{u}_{[3\lambda_0/4 - C, 0]} | \, dx \, dt \\
\lesssim \sum_{\lambda_0 < 0, \, (j_1 \lor j_3) \approx j_2 \leq \ln(\lambda_0)} (\lambda_0)^{1/2} \int_{[0,T] \times \mathbb{R}} |\nabla_{\lambda_0} u_{\lambda_0 + j_1} u_{-j_2} \partial_x \bar{u}_{-j_3} | \, dx \, dt.
\]

We consider the case \( j_1 \ll j_3, \, j_2 \approx j_3 \). We have

\[
\mathcal{L}_{h_{-1-1-1}}^- (u, v) := \sum_{\lambda_0 < 0, \, j_1 \ll j_2 \approx j_3 \leq \ln(\lambda_0)} (\lambda_0)^{1/2} \int_{[0,T] \times \mathbb{R}} |\nabla_{\lambda_0} u_{\lambda_0 + j_1} u_{-j_2} \partial_x \bar{u}_{-j_3} | \, dx \, dt \\
\lesssim \left( \sum_{\lambda_0 < 0, \, j_1 \ll 10 \ll j_2 \approx j_3 \leq \ln(\lambda_0)} (\lambda_0)^{1/2} \right) \times \int_{[0,T] \times \mathbb{R}} |\nabla_{\lambda_0} u_{\lambda_0 + j_1} u_{-j_2} \partial_x \bar{u}_{-j_3} | \, dx \, dt \\
:= \mathcal{L}_{h_{-1-1-1}}^{-m} (u, v) + \mathcal{L}_{h_{-1-1-1}}^{-h} (u, v).
\]

In order to estimate \( \mathcal{L}_{h_{-1-1-1}}^{-m} (u, v) \), we follow the same ideas as in (5.9). We may assume that \( 2^{10} \leq -\lambda_0/16 \). It follows that \( \lambda_0 + 2^{j_1} + 2^{j_2} \leq \lambda_0/16 \). Now we can use the bilinear estimate (4.1), Corollary 3.3. For \( 0 \leq \varepsilon < 1/2p \), we have

\[
\mathcal{L}_{h_{-1-1-1}}^{-m} (u, v) \\
\lesssim \sum_{\lambda_0 < 0, \, j_1 \ll 10 \ll j_2 \approx j_3 \leq \ln(\lambda_0)} (\lambda_0)^{-1/2 + 2\varepsilon} 2^{j_1} \|u_{\lambda_0} \|_{V^2_{\Delta}} \|u_{-j_2} \|_{V^2_{\Delta}} \|u_{\lambda_0 + \epsilon} \|_{V^2_{\Delta}} \|u_{-j_2 - j_3} \|_{V^2_{\Delta}} \\
\lesssim T^{\varepsilon/2} \sum_{\lambda_0 < 0, \, |\xi|, |\xi_1| \leq 1, \, 1 \ll j_3 \leq \ln(\lambda_0)} (\lambda_0)^{-1/2 + 2\varepsilon} 2^{j_1} \|u_{\lambda_0} \|_{V^2_{\Delta}} \|u_{\lambda_0 + \epsilon} \|_{V^2_{\Delta}} \|u_{-j_2 - j_3} \|_{V^2_{\Delta}} \\
\lesssim T^{\varepsilon/2} \|v\|_{Y^{\varepsilon}_{p, \Delta}} \|u\|^3_{\tilde{X}^{1/2}_{p, \Delta}},
\]

(5.14)

For the estimate of \( \mathcal{L}_{h_{-1-1-1}}^{-h} (u, v) \), we need to use DMCC (5.6), we have

\[
\max_{0 \leq k \leq 1} |\xi_2 + r_k| \geq |(\xi_0 - \xi_1)(\xi_0 - \xi_2)| \geq 2^{j_1} \langle \lambda_0 \rangle.
\]

(5.15)

If \( v_{\lambda_0} \) has the highest dispersion modulation, we have

\[
\mathcal{L}_{h_{-1-1-1}}^{-h} (u, v)
\]
\[
\lesssim \sum_{\lambda_0 < 0, \ 10 < j_1, j_2 \approx j_3 \leq \ln(\lambda_0)} \langle \lambda_0 \rangle^{1/2} 2^{j_3} \| v_{\lambda_0} \|_{L_t^\infty L_x^{\infty}} \| u_{\lambda_0 + I_{j_1}} \|_{L_t^\infty L_x^{\infty}} \| u-I_{j_2} \|_{L_t^{4} L_x^{4}} \| u-I_{j_3} \|_{L_t^{4} L_x^{4}}.
\]

(5.16)

In view of (2.6), Corollary 3.3, we have
\[
\mathcal{L}_{h-1}^{\gamma} (u, v) \lesssim T^{2\varepsilon/3} \sum_{\lambda_{0} < 0, \ 10 < j_3 \approx j_3 \leq \ln(\lambda_0)} \langle \lambda_0 \rangle^{1/2} 2^{j_3} 2^{-j_1/2} \langle \lambda_0 \rangle^{-1/2} \| v_{\lambda_0} \|_{V_{2}^3} \times \langle \lambda_0 \rangle^{-1/2} 2^{j_3/2} (1/2 - 1/p) \| u \|_{X_{p, \Delta}^{1/2}(\lambda_0 + I_{j_1})} 2^{(j_2 + j_3)(-1/4 - 1/p + \varepsilon)} \| u \|_{X_{p, \Delta}^{1/2}}^2
\]
\[
\lesssim T^{2\varepsilon/3} \sum_{\lambda_{0} < 0, \ 10 < j_1, j_2 \approx j_3 \leq \ln(\lambda_0)} 2^{-j_1/2} 2^{j_3/2} 2^{(j_2 + j_3)(-1/4 - 1/p + \varepsilon)} \| v_{\lambda_0} \|_{V_{2}^3} \times \| u \|_{X_{p, \Delta}^{1/2}(\lambda_0 + I_{j_1})} \| u \|_{X_{p, \Delta}^{1/2}}^2.
\]

(5.17)

Making the summation on \( j_3 \), then using the same way as in (5.12), one has that for \( 0 < \varepsilon < 1 < 2p \),
\[
\mathcal{L}_{h-1}^{\gamma} (u, v) \lesssim T^{2\varepsilon/3} \sum_{\lambda_{0} < 0, \ 10 < j_1} 2^{(-3/p + 2\varepsilon) j_1} \| v_{\lambda_0} \|_{V_{2}^3} \| u \|_{X_{p, \Delta}^{1/2}(\lambda_0 + I_{j_1})} \| u \|_{X_{p, \Delta}^{1/2}}^2.
\]
\[
\lesssim T^{\varepsilon/2} \| u \|_{X_{p, \Delta}^{1/2}}^3 \| v \|_{V_{p, \Delta}^{3}}.
\]

(5.18)

If \( u_{\lambda_0 + I_{j_1}} \) has the highest dispersion modulation, we have
\[
\mathcal{L}_{h-1}^{\gamma} (u, v) \lesssim \sum_{\lambda_{0} < 0, \ 10 < j_1} \langle \lambda_0 \rangle^{1/2} 2^{j_1} \| v_{\lambda_0} \|_{L_t^{\infty} L_x^{\infty}} \| u_{\lambda_0 + I_{j_1}} \|_{L_t^{\infty} L_x^{\infty}} \| u-I_{j_2} \|_{L_t^{4} L_x^{4}} \| u-I_{j_3} \|_{L_t^{4} L_x^{4}}.
\]

(5.19)

Applying the dispersion modulation decay estimate (2.6) to \( u_{\lambda_0 + I_{j_1}} \), and \( \| v_{\lambda_0} \|_{L_t^{\infty} L_x^{\infty}} \lesssim \| v_{\lambda_0} \|_{V_{2}^3} \), we can reduce the estimate of (5.19) to the case as in (5.17) and (5.18), the details are omitted.

Now we consider the case that \( u-I_{j_2} \) has the highest modulation. In \( \mathcal{L}_{h-1}^{\gamma} (u, v) \), it is easy to see that \( \lambda_0 \approx \ln(\lambda_0) \), which implies that \( \lambda_0 + 2^{j_1} + 2^{j_3} \approx \lambda_0/16 \). Using the bilinear estimate (4.1) to \( \partial_x \pi_{-I_{j_3}} u_{\lambda_0 + I_{j_1}}, \) and the dispersion modulation decay estimate (2.6), one has that
\[
\mathcal{L}_{h-1}^{\gamma} (u, v) \lesssim \sum_{\lambda_{0} < 0, \ 10 < j_1, j_2 \approx j_3 \leq \ln(\lambda_0)} \langle \lambda_0 \rangle^{1/2} \| v_{\lambda_0} \|_{L_t^{\infty} L_x^{\infty}} \| u_{\lambda_0 + I_{j_1}} \|_{L_t^{\infty} L_x^{\infty}} \| u-I_{j_2} \|_{L_t^{4} L_x^{4}} \| u-I_{j_3} \|_{L_t^{4} L_x^{4}}
\]
\[
\times 2^{-j_1/2} \langle \lambda_0 \rangle^{-1/2} \| u-I_{j_2} \|_{V_{2}^3}.
\]

(5.20)

It follows from Lemma 3.2 and (5.20) that
\[
\mathcal{L}_{h-1}^{\gamma} (u, v) \lesssim T^{\varepsilon/4} \sum_{\lambda_{0} < 0, \ 10 < j_1} \langle \lambda_0 \rangle^{-1/2} 2^{-j_1/2} 2^{j_3(1/2 - 1/p)} \| u \|_{X_{p, \Delta}^{1/2}(\lambda_0 + I_{j_1})} \| u \|_{X_{p, \Delta}^{1/2}}^2.
\]

(5.21)
Making the summation on \( j_3 \) and using the same way as in (5.12), we have for \( 0 < \varepsilon < 1/2p \),
\[
\mathcal{H}_{h-L-L-}^{\varepsilon}(u,v) \lesssim T^{\varepsilon/4} \sum_{\lambda_0 \in (0,J_1]} \| u_{\lambda_0} \|_{X_p^2}^{2^{j_1/p}} \| u \|_{X_p^2(\lambda_0 + I_{j_3})} \| u \|_{X_p^2(\lambda_0 + I_{j_3})}^2
\lesssim T^{\varepsilon/2} \| u \|_{X_p^2} \| v \|_{Y_{p',\Delta}^r}.
\]
(5.22)

We consider the case \( j_3 \ll j_1 \approx j_2 \). We denote
\[
\mathcal{H}_{h-L-L-}^{-\varepsilon}(u,v) := \sum_{\lambda_0 \in (0,J_1]} \langle \lambda_0 \rangle^{1/2} \int_{[0,T] \times \mathbb{R}} \| u_{\lambda_0 + I_{j_1}} u_{-I_{j_2}} \partial_x u_{-I_{j_3}} \| dt.
\]
(5.23)

In \( \mathcal{H}_{h-L-L-}^{-\varepsilon}(u,v) \), observing DMCC (5.6), we have
\[
\max_{0 \leq k \leq 1} (\xi_k^2 + \tau_k \geq |(\xi_0 - \xi_1)(\xi_0 - \xi_2)| \approx 2^{j_1} \langle \lambda_0 \rangle.
\]
(5.24)

In view of (5.24) we see that the lower bound of the highest dispersion modulation is the same as that of the case \( j_1 \ll j_3 \approx j_2 \). Moreover, \( \partial_x u_{-I_{j_3}} \) has the lower frequency, which leads to that the derivative in front of \( u_{-I_{j_3}} \) becomes easier to handle. So, we omit the details of the proof in this case.

For the case \( j_1 \approx j_2 \approx j_3 \), we divide the proof into two subcases: \( j_3 \ll 1 \) and \( j_3 \gg 1 \). The first case is very easy, since both \( u_{-I_{j_2}} \) and \( u_{-I_{j_3}} \) have the low frequency in a neighbourhood of 0. One can directly use Hölder's inequality to get the desired estimate. For the case \( j_3 \gg 1 \), we consider that the highest dispersion modulation (larger than \( 2^{j_1} \langle \lambda_0 \rangle \)) is due to \( u_{\lambda_0} \), \( u_{\lambda_0 + I_{j_1}} \), \( u_{-I_{j_2}} \) and \( u_{-I_{j_3}} \), separately. Denote
\[
\mathcal{H}_{h-L-L-}^{-\varepsilon}(u,v) \lesssim \sum_{\lambda_0 \in (0,J_1]} \langle \lambda_0 \rangle^{1/2} \int_{[0,T] \times \mathbb{R}} \| u_{\lambda_0 + I_{j_1}} u_{-I_{j_2}} \partial_x u_{-I_{j_3}} \| dt.
\]

If \( u_{\lambda_0} \) has the highest dispersion modulation, we have
\[
\mathcal{H}_{h-L-L-}^{-\varepsilon}(u,v) \lesssim \sum_{\lambda_0 \in (0,J_1]} \langle \lambda_0 \rangle^{1/2} 2^{j_3} \| u_{\lambda_0} \|_{L_\infty^2 L_\infty^2} \| u_{\lambda_0 + I_{j_1}} \|_{L_\infty^2 L_\infty^2} \| u_{-I_{j_2}} \|_{L_\infty^2} \| u_{-I_{j_3}} \|_{L_\infty^2}.
\]

We can follow the same way as in dealing with (5.16) to have the desired estimate. If \( u_{\lambda_0 + I_{j_1}} \) has the highest dispersion modulation, we have
\[
\mathcal{H}_{h-L-L-}^{-\varepsilon}(u,v) \lesssim \sum_{\lambda_0 \in (0,J_1]} \langle \lambda_0 \rangle^{1/2} 2^{j_3} \| u_{\lambda_0} \|_{L_\infty^2} \| u_{\lambda_0 + I_{j_1}} \|_{L_\infty^2} \| u_{-I_{j_2}} \|_{L_\infty^2} \| u_{-I_{j_3}} \|_{L_\infty^2}.
\]

If \( u_{-I_{j_2}} \) has the highest dispersion modulation, we have
\[
\mathcal{H}_{h-L-L-}^{-\varepsilon}(u,v) \lesssim \sum_{\lambda_0 \in (0,J_1]} \langle \lambda_0 \rangle^{1/2} 2^{j_3} \| u_{\lambda_0} \|_{L_\infty^2} \| u_{\lambda_0 + I_{j_1}} \|_{L_\infty^2} \| u_{-I_{j_2}} \|_{L_\infty^2} \| u_{-I_{j_3}} \|_{L_\infty^2}.
\]

Then, using (2.6), Lemma 3.2 and similar estimates as in the above, we can get the result, as desired.
Case \( h \perp L_a \). By FCC (5.5), we see that \( \lambda_3 < -\lambda_0/4 + C \). We decompose \( \lambda_k \):

\[
\lambda_1 \in \lambda_0 + [0, -\frac{\lambda_0}{4}) = \bigcup_{j_1 \geq 0} (\lambda_0 + I_{j_1}); \quad \lambda_2 \in [\frac{3\lambda_0}{4}, 0) = \bigcup_{j_2 \geq 0} -I_{j_2}; \quad \lambda_3 \in [0, -\frac{\lambda_0}{4} + C) = \bigcup_{j_3 \geq 0} I_{j_3}.
\]

Again, in view of FCC (5.5) we see that

\[
(\lambda_2 \vee \lambda_3) \approx \lambda_1, \quad \lambda_1 \leq \ln(\lambda_0) + C.
\]

It follows that

\[
j_2 \ll j_3 \approx j_1; \quad \text{or} \quad j_3 \ll j_2 \approx j_1; \quad \text{or} \quad j_1 \approx j_2 \approx j_3.
\]

We have

\[
\sum_{\lambda_0 < 0} \langle \lambda_0 \rangle^{1/2} \int_{[0,T]} |v_{\lambda_0}u_{\lambda_0}, 3\lambda_0/4) u|3\lambda_0/4, 0| \partial_x \Pi_{[0, -\lambda_0/4 + C]} dxdt \leq \sum_{\lambda_0 < 0, (j_2 \vee j_3) \approx j \leq \ln(\lambda_0)} \langle \lambda_0 \rangle^{1/2} \int_{[0,T]} |v_{\lambda_0}u_{\lambda_0 + I_{j_1}} u_{-I_{j_2}} \partial_x \Pi_{I_{j_3}}| dxdt.
\]

From the proof of the Case \( h \perp L \), it suffices to consider the case \( j_2 \ll j_3 \approx j_1 \). We estimate

\[
L_{h \perp L_a}^{-1}(u, v) := \sum_{\lambda_0 < 0, j_2 \ll j_3 \approx j_1 \leq \ln(\lambda_0)} \langle \lambda_0 \rangle^{1/2} \int_{[0,T]} |v_{\lambda_0}u_{\lambda_0 + I_{j_1}} u_{-I_{j_2}} \partial_x \Pi_{I_{j_3}}| dxdt.
\]

In the right hand side of \( L_{h \perp L_a}^{-1}(u, v) \), by DMCC (5.6),

\[
\max_{0 \leq k \leq 3} \xi_k^2 + \tau_k \geq 2^{j_1} \langle \lambda_0 \rangle.
\]

If \( v_{\lambda_0} \) has the highest dispersion modulation, in a similar way as in (5.16)–(5.18), we have for \( 0 < \varepsilon < 1/2p \),

\[
L_{h \perp L_a}^{-1}(u, v) \lesssim \sum_{\lambda_0 < 0, j_2 \ll j_3 \approx j_1 \leq \ln(\lambda_0)} \langle \lambda_0 \rangle^{1/2} 2^{j_3} \| v_{\lambda_0} \|_{L_2^2 L_\infty} \| u_{\lambda_0 + I_{j_1}} \|_{L_4^4 L_\infty} \| u_{-I_{j_2}} \|_{L_p^p L_\infty} \| u_{I_{j_3}} \|_{L_4^4} \lesssim T^{2e/3} \sum_{\lambda_0 < 0, j_2 \ll j_3 \approx j_1 \leq \ln(\lambda_0)} 2^{(-2/p + 2e)j_1} \| v_{\lambda_0} \|_{X^{1/2}_{p, \Delta}} \| u \|_{X^{1/2}_{p, \Delta}} \lesssim T^e/2 \| u \|_{X^{3/2}_{p, \Delta}}^3 \| v \|_{Y_{p, \Delta}}.
\]

Now we consider the case that \( u_{-I_{j_2}} \) has the highest dispersion modulation.

\[
L_{h \perp L_a}^{-1}(u, v) \lesssim \sum_{\lambda_0 < 0, j_2 \ll j_3 \approx j_1 \leq \ln(\lambda_0)} \langle \lambda_0 \rangle^{1/2} 2^{j_3} \| v_{\lambda_0} \|_{L_2^2 L_\infty} \| u_{\lambda_0 + I_{j_1}} \|_{L_4^4 L_\infty} \| u_{-I_{j_2}} \|_{L_p^p L_\infty} \| u_{I_{j_3}} \|_{L_4^4}.
\]

Applying the dispersion modulation decay estimate (2.6) to \( u_{-I_{j_2}} \), we can reduce the estimate of (5.28) to the case as in (5.27), the details are omitted.
Assume that \( u_{\lambda_0+I_1} \) has the highest dispersion modulation. Using the bilinear estimate (4.1) to \( u_{-I_2} \partial_x \overline{u}_{I_3} \), and the dispersion modulation decay estimate (2.6), then applying Lemma 3.2, one has that for \( 0 < \varepsilon < 1/2p \),

\[
\mathcal{L}_{h_{-1},T}^-(u,v) \lesssim \sum_{\lambda_0 < 0, \ j_2 < j_3 \approx j_1 \leq \ln(\lambda_0)} |\langle \lambda_0 \rangle|^{1/2} \| v_{\lambda_0} \|_{L^p_{x,t}} \| u_{\lambda_0+I_1} \|_{L^2_{x,t}} \| u_{-I_2} \partial_x \overline{u}_{I_3} \|_{L^2_{x,t}} \lesssim T^{\varepsilon/4} \sum_{\lambda_0 < 0, \ j_1 \leq \ln(\lambda_0)} \| v_{\lambda_0} \| \| v_{\lambda_0} \|^{2j_1(-2/p+\varepsilon)} \| u \|_{X^{1/2}_{p,\Delta}(\lambda_0+I_1)} \| u \|_{X^{1/2}_{p,\Delta}} \lesssim T^{\varepsilon/4} \| u \|^3_{X^{1/2}_{p,\Delta}} \| v \|_{Y_{p',\Delta}}. \tag{5.29}
\]

If \( u_{I_3} \) has the highest dispersion modulation, we can use the same way as in the case that \( u_{\lambda_0+I_1} \) has the highest dispersion modulation to obtain the result. The details are omitted.

Case \( h_{-a+a} \). First, using FCC (5.5), we have \( \lambda_3 \gg -C \). Moreover, if \( \lambda_3 \leq C \), we see that \( \lambda_2 \leq C \). We have

\[
\sum_{\lambda_0 < 0} |\langle \lambda_0 \rangle|^{1/2} \int_{[0,T] \times \mathbb{R}} \overline{v}_{\lambda_0} u_{[\lambda_0, 3\lambda_0/4]} u_{[0, \infty]} \partial_x \overline{u}_{[\lambda_0, \infty]} \, dx dt \leq \sum_{\lambda_0 < 0} |\langle \lambda_0 \rangle|^{1/2} \int_{[0,T] \times \mathbb{R}} \overline{v}_{\lambda_0} u_{[\lambda_0, 3\lambda_0/4]} u_{[0, C]} \partial_x \overline{u}_{[-C, C]} \, dx dt + \sum_{\lambda_0 < 0} |\langle \lambda_0 \rangle|^{1/2} \int_{[0,T] \times \mathbb{R}} \overline{v}_{\lambda_0} u_{[\lambda_0, 3\lambda_0/4]} u_{[0, \infty]} \partial_x \overline{u}_{[C, \infty]} \, dx dt := L_{h_{-a+a},l}^-(u,v) + L_{h_{-a+a},h}^-(u,v). \tag{5.30}
\]

In \( L_{h_{-a+a},l}^-(u,v) \), there are two lower frequency in a neighbourhood of the origin and two higher frequency near \( \lambda_0 \). Hence, we can use the bilinear estimate (4.1) to obtain that

\[
\mathcal{L}_{h_{-a+a},l}^-(u,v) \lesssim T^{\varepsilon/2} \| u \|_{X^{1/2}_{p,\Delta}}^3 \| v \|_{Y_{p',\Delta}}. \tag{5.31}
\]

We estimate \( L_{h_{-a+a},h}^-(u,v) \). Considering the dyadic version of \( \lambda_k \):

\[
\lambda_1 \in \lambda_0 + [0, -\frac{\lambda_0}{4}) = \bigcup_{j_1 \geq 0} (\lambda_0 + I_{j_1}); \ \lambda_2 \in [0, \infty) = \bigcup_{j_2 \geq 0} I_{j_2}; \ \lambda_3 \in [C, \infty) = \bigcup_{j_3 \geq 1} I_{j_3}.
\]

Again, in view of FCC (5.5) we see that

\[
|\langle j_1 \cup j_2 | \approx j_3, \ j_1 \leq \ln(\lambda_0) + C.
\]

We have the following three subcases:

\[
j_1 \ll j_2 \approx j_3; \text{ or } j_2 \ll j_1 \approx j_3; \text{ or } j_1 \approx j_2 \approx j_3.
\]

Using the dyadic version, we have

\[
\mathcal{L}_{h_{-a+a},h}^-(u,v) := \left( \sum_{\lambda_0 < 0, \ j_2 \ll j_1 \ll j_3} + \sum_{\lambda_0 < 0, \ j_2 \ll j_1 \ll j_3} + \sum_{\lambda_0 < 0, \ j_1 \approx j_2 \approx j_3} \right) |\langle \lambda_0 \rangle|^{1/2} \times \int_{[0,T] \times \mathbb{R}} |\overline{v}_{\lambda_0} u_{\lambda_0+I_1} \partial_x \overline{u}_{I_3} | \, dx dt := I + II + III. \tag{5.32}
\]
We mainly estimate $I$. Using the bilinear estimate (4.2), Corollary 3.3, Hölder’s inequality, we have

$$I_1 := \sum_{\lambda_0 \neq 0, \ j_1 \leq 10, j_2 \equiv j_3} \langle \lambda_0 \rangle^{1/2} \int_{[0,T] \times \mathbb{R}} |\nabla_{\lambda_0} u_{\lambda_0 + I_{j_1}} u_{I_{j_2}} \partial_x \pi_{I_{j_3}}| \ dx \ dt$$

$$\leq \sum_{\lambda_0 \neq 0, \ j_1 \leq 10, j_2 \equiv j_3} \langle \lambda_0 \rangle^{1/2} \|\nabla_{\lambda_0} \partial_x \pi_{I_{j_3}} \|_{L^2_{x,t}} \|u_{\lambda_0 + I_{j_1}} u_{I_{j_2}}\|_{L^2_{x,t}}$$

$$\leq T^{\varepsilon/2} \sum_{\lambda_0 \neq 0, \ j_1 \leq 10, j_2 \equiv j_3} \langle \lambda_0 \rangle^{1/2} (\langle \lambda_0 \rangle + 2^{j_2})^{-1+2\varepsilon} 2^{j_2} \|v_{\lambda_0}\|_{V^2_\Delta} \|u_{I_{j_3}}\|_{V^2_\Delta}$$

$$\leq T^{\varepsilon/2} \sum_{\lambda_0 \neq 0, \ j_1 \leq 10, j_2 \equiv j_3} \langle \lambda_0 \rangle^{1/2} (\langle \lambda_0 \rangle + 2^{j_2})^{-1/2} \|v_{\lambda_0}\|_{V^2_\Delta} \|u_{\lambda_0 + I_{j_1}} u_{I_{j_3}}\|_{V^2_\Delta}$$

$$\lesssim T^{\varepsilon/2} \|u\|_{X^3_{t,\Delta}}^3 \|v\|_{Y^p_{t,\Delta}}.$$

(5.33)

We estimate

$$I_h := \sum_{\lambda_0 \neq 0, \ 10 \leq j_1 \leq j_2 \equiv j_3} \langle \lambda_0 \rangle^{1/2} \int_{[0,T] \times \mathbb{R}} |\nabla_{\lambda_0} u_{\lambda_0 + I_{j_1}} u_{I_{j_2}} \partial_x \pi_{I_{j_3}}| \ dx \ dt. \quad (5.34)$$

In the right hand side of $I_h$, in view of DMCC (5.6), the highest dispersion modulation

$$\max_{0 \leq k \leq 3} \|\xi_k^2 + \tau_k\| \gtrsim 2^{j_1} (\langle \lambda_0 \rangle + 2^{j_2}). \quad (5.35)$$

If $v_{\lambda_0}$ has the highest dispersion modulation, in a similar way as in (5.16)–(5.18), we have

$$I_h \lesssim \sum_{\lambda_0 \neq 0, \ 10 \leq j_1 \leq j_2 \equiv j_3} \langle \lambda_0 \rangle^{1/2} 2^{j_1/2} \|v_{\lambda_0}\|_{L^2_t L^\infty_x} \|u_{\lambda_0 + I_{j_1}} \|_{L^\infty_t L^2_x} \|u_{I_{j_2}}\|_{L^4_x} \|u_{I_{j_3}}\|_{L^4_x},$$

$$\lesssim \sum_{\lambda_0 \neq 0, \ 10 \leq j_1 \leq j_2 \equiv j_3} \langle \lambda_0 \rangle^{1/2} 2^{j_1/2} (\langle \lambda_0 \rangle + 2^{j_2})^{-1/2} \|v_{\lambda_0}\|_{V^2_\Delta} \|u_{\lambda_0 + I_{j_1}} \|_{L^4_x} \|u_{I_{j_2}}\|_{L^4_x} \|u_{I_{j_3}}\|_{L^4_x},$$

$$\lesssim T^{2\varepsilon/3} \sum_{\lambda_0 \neq 0, \ 10 \leq j_1 \leq j_2 \equiv j_3} \langle \lambda_0 \rangle^{1/2} 2^{j_1/2} (\langle \lambda_0 \rangle + 2^{j_2})^{-1/2} \|v_{\lambda_0}\|_{V^2_\Delta}$$

$$\times (\langle \lambda_0 \rangle + 2^{j_2})^{-1/2-2/p+2\varepsilon} \|u\|_{X^3_{t,\Delta}}^2 \|v\|_{Y^p_{t,\Delta}}.$$

(5.36)

Summing over all $j_2$, then using Hölder’s on $\lambda_0$, we obtain that for $0 < \varepsilon < 1/2p$,

$$I_h \lesssim T^{\varepsilon/2} \|u\|_{X^3_{t,\Delta}}^3 \|v\|_{Y^p_{t,\Delta}}.$$

(5.37)

If $u_{\lambda_0 + I_{j_1}}$ has the highest dispersion modulation, in a similar way as in (5.36) and (5.37), we have for $0 < \varepsilon < 1/2p$,

$$I_h \lesssim \sum_{\lambda_0 \neq 0, \ 10 < j_1 \leq j_2 \equiv j_3} \langle \lambda_0 \rangle^{1/2} 2^{j_1} \|v_{\lambda_0}\|_{L^\infty_t L^\infty_x} \|u_{\lambda_0 + I_{j_1}} \|_{L^\infty_t L^2_x} \|u_{I_{j_2}}\|_{L^4_x} \|u_{I_{j_3}}\|_{L^4_x},$$

$$\lesssim T^{\varepsilon/2} \|u\|_{X^3_{t,\Delta}}^3 \|v\|_{Y^p_{t,\Delta}}.$$

(5.38)

If $u_{I_{j_3}}$ has the highest dispersion modulation, we have

$$I_h \lesssim \sum_{\lambda_0 \neq 0, \ 10 < j_1 \leq j_2 \equiv j_3} \langle \lambda_0 \rangle^{1/2} 2^{j_3} \|v_{\lambda_0}\|_{L^\infty_t L^\infty_x} \|u_{\lambda_0 + I_{j_1}} \|_{L^\infty_t L^2_x} \|u_{I_{j_2}}\|_{L^4_x} \|u_{I_{j_3}}\|_{L^4_x},$$

(5.39)
Using the bilinear estimate (4.2), and the dispersion modulation decay (5.35) and (2.6), then applying Lemma 3.2,
\[ I_h \lesssim T^{\varepsilon/4} \sum_{\lambda_0 \leq 0, j_1 > 0} 2^{(-3/p+\varepsilon)j_1} \|v_{\lambda_0}\|_{V_{p,\Delta}}^2 \|u\|_{X_{p,\Delta}^1(\lambda_0+I_{j_1})}^2 \|u\|_{X_{p,\Delta}^1}^2. \]
\[ \lesssim T^{\varepsilon/4} \|u\|_{X_{p,\Delta}^1}^{\lambda_0/2} \|v\|_{Y_{p,\Delta}^\varepsilon}. \]

(5.40)

If \( u_{I_{j_2}} \) has the highest dispersion modulation, we have
\[ I_h \lesssim \sum_{\lambda_0 \leq 0, 10 < j_1 \leq j_2} \langle \lambda_0 \rangle^{1/2} \|v_{\lambda_0}\|_{L_{x,t}^\infty} \|u_{I_{j_2}}\|_{L_{x,t}^2} \|u_{\lambda_0+I_{j_1}}\|_{H_{x,t}^{1/2}} \|\partial_x \mathfrak{P}_{I_{j_2}}\|_{L_{x,t}^2}^2. \]

(5.41)

One sees that \( \text{dist}(\lambda_0 + I_{j_1}, I_{j_3}) \gtrsim \langle \lambda_0 \rangle + 2^{j_3} \). Using the bilinear estimate (4.1) and the dispersion modulation decay (2.6) and (5.35),
\[ I_h \lesssim T^{\varepsilon/4} \sum_{\lambda_0 \leq 0, 10 < j_1 \leq j_2} \langle \lambda_0 \rangle^{1/2} \|v_{\lambda_0}\|_{V_{p,\Delta}} \|u_{\lambda_0+I_{j_1}}\|_{H_{x,t}^{1/2}} \|u_{I_{j_2}}\|_{V_{p,\Delta}}^2 \times 2^{j_3} \|u_{\lambda_0+I_{j_1}}\|_{V_{p,\Delta}} \|u_{I_{j_2}}\|_{V_{p,\Delta}}^2. \]

(5.42)

Applying Corollaries 3.3, 3.4 and making the summations in turn to \( j_3 \) and \( j_2, \lambda_0, j_1 \), we have
\[ I_h \lesssim T^{\varepsilon/4} \sum_{\lambda_0 \leq 0, j_2 > j_1 > 10} 2^{j_2(-3/p+\varepsilon)2^{-j_2/2}} \langle \lambda_0 \rangle^{1/2} \|v_{\lambda_0}\|_{V_{p,\Delta}} \|u\|_{X_{p,\Delta}^1(\lambda_0+I_{j_1})} \|u\|_{X_{p,\Delta}^1(\lambda_0+I_{j_1})}^2 \|u\|_{X_{p,\Delta}^1}^2 \|L_{x,t}^{-}\|_{V_{p,\Delta}^{\varepsilon}}. \]

(5.43)

Case B. \( \lambda_1 \in L_\nu \). Since \( \lambda_1 \) and \( \lambda_2 \) are symmetry, it suffices to consider the following four subcases as shown in Table 2.

| Case | \( \lambda_1 \in \) | \( \lambda_2 \in \) | \( \lambda_3 \in \) |
|------|-----------------|-----------------|-----------------|
| \( L_{x,t}^{-}(u,v) \) | \( (3\lambda_0/4,0) \) | \( (3\lambda_0/4,0) \) | \( (-\lambda_0/16,0) \) |
| \( L_{x,t}^{-}(u,v) \) | \( (3\lambda_0/4,0) \) | \( (3\lambda_0/4,0) \) | \( (-\lambda_0/16,0) \) |
| \( L_{x,t}^{-}(u,v) \) | \( (3\lambda_0/4,0) \) | \( (0,\infty) \) | \( (-\lambda_0,0) \) |
| \( L_{x,t}^{-}(u,v) \) | \( (3\lambda_0/4,0) \) | \( (0,\infty) \) | \( (-\lambda_0,0) \) |

Table 2. \( \lambda_1, \lambda_2, \lambda_3 \) far away from \( \lambda_0 \).

Case \( L_{x,t}^{-}(u,v) \). From FCC (5.5) it follows that \( \lambda_3 \geq \lambda_0/2 - C \). Moreover, we see that \( \lambda_1, \lambda_2 < \lambda_0/8 \). So, it suffices to estimate
\[ \mathcal{L}_{\xi_0}^{-}(u,v) \]
\[ := \sum_{\lambda_0 \leq 0} \langle \lambda_0 \rangle^{1/2} \left| \int_{[0,T] \times \mathbb{R}} \pi_{\lambda_0} u_{(3\lambda_0/4,\lambda_0/8)} u_{(3\lambda_0/4,\lambda_0/8)} \partial_x \mathfrak{P}_{(\lambda_0/2-C,\lambda_0/16)} dxdt \right|. \]

(5.44)

Using the dyadic decomposition and Corollary 3.3, we see that
\[ \|\partial_x \mathfrak{P}_{(\lambda_0/2-C,0)}\|_{L_{x,t}^{\infty}} \lesssim \sum_{j \in \mathbb{N}(\lambda_0)} \|\partial_x u_{I_j}\|_{L_{x,t}^{\infty}} \lesssim \langle \lambda_0 \rangle^{1-1/p} \|u\|_{X_{p,\Delta}^{1/2}}. \]

(5.45)

\[ \|\partial_x \mathfrak{P}_{(\lambda_0/2-C,0)}\|_{L_{x,t}^{L_{x,t}}} \lesssim T^{\varepsilon/3} \sum_{j \in \mathbb{N}(\lambda_0)} \|\partial_x u_{I_j}\|_{L_{x,t}^{L_{x,t}}} \lesssim \langle \lambda_0 \rangle^{3/4-1/p+\varepsilon} \|u\|_{X_{p,\Delta}^{1/2}}. \]

(5.46)
Similarly, \( \partial_x \bar{\pi}[0, -\lambda_0/16] \) has the same estimates as in (5.45) and (5.46). By DMCC (5.6), the highest dispersion modulation satisfies
\[
\sqrt{3} \| \xi_k^2 + \tau_k \| \geq (\lambda_0)^2. 
\]
(5.47)

If \( v\lambda_0 \) has the highest dispersion modulation, we have
\[
L_{-1,-1}^{-1}(u, v) \leq \sum_{\lambda_0 \in 0} (\lambda_0)^{-3/2} \| v\lambda_0 \|_2^2 \| u(3\lambda_0/4, \lambda_0/8) \|_L^2 \| \partial_x \pi[\lambda_0/2-C,-\lambda_0/16] \|_L^\infty \|
\]
(5.48)

Applying the highest dispersion modulation decay estimate (2.6), Corollary 3.5, (5.45) and (5.46), we have
\[
L_{-1,-1}^{-1}(u, v) \leq T^{3/2} \sum_{\lambda_0 \in 0} (\lambda_0)^{-3/2} \| v\lambda_0 \|_2^2 \| u \|_3 \leq T^{3/2} \| v \|_1 \| u \|_3 \leq 2^{1/2} \| v \|_1 \| u \|_3. 
\]
(5.49)

If \( u(\lambda_0/2-C,-\lambda_0/16) \) gains the highest dispersion modulation, we can use an analogous way to get the result. In fact, taking \( L_\infty, L_1, L_2, \) and \( L_4 \), norms for \( v\lambda_0 \), \( u(3\lambda_0/4, \lambda_0/8) \) and \( \partial_x u(\lambda_0/2-C,-\lambda_0/16) \), respectively, then using the dispersion modulation decay estimate (2.6), Corollary 3.5, (5.45), we also have (5.49). Finally, in the case \( u(3\lambda_0/4, \lambda_0/8) \) having the highest dispersion modulation, taking \( L_\infty, L_1, L_2, \) and \( L_4 \), and \( L_4 \), norms for \( v\lambda_0 \), \( u(3\lambda_0/4, \lambda_0/8) \), \( u(3\lambda_0/4, \lambda_0/8) \) and \( \partial_x u(\lambda_0/2-C,-\lambda_0/16) \), respectively, then using the dispersion modulation decay estimate (2.6), Corollary 3.5, (5.45) and (5.48), we also have (5.49).

Case \( I_{-1,-1}^{-1} \). We easily see that \( \lambda_3 \leq -\lambda_0 + C \). We collect \( \lambda_k \) in the following dyadic version:
\[
\lambda_k \in \{3\lambda_0/4, 0\} = \bigcup_{j_k} I_{j_k}, \quad k = 1, 2;
\]
\[
\lambda_3 \in \{-\lambda_0/16, -\lambda_0 + C\} = \bigcup_{j_3 \neq -1} (-\lambda_0 - I_{j_3}), \quad I_{-1} = [-C, 0). 
\]

By FCC (5.5), we see that \( j_3 \approx (j_1 \lor j_2) \leq \ln(\lambda_0) \). We need to estimate
\[
L_{-1,-1}^{-1}(u, v) := \sum_{\lambda_0 \in 0} (\lambda_0)^{-1/2} \int_{[0, T] \times \mathbb{R}} |\nabla_{x,t} u(3\lambda_0/4, \lambda_0/4) \partial_x \pi[-\lambda_0/16, -\lambda_0 + C] | \ dx \ dt 
\]
(5.50)

In the right hand side of (5.50), from DMCC (5.6) it follows that
\[
\max_{0 \leq k \leq 3} (\xi_k^2 + \tau_k) \geq (\lambda_0)^2. 
\]
(5.51)
If $v_{\lambda_0}$ attains the highest dispersion modulation, using a similar way as in (5.16)–(5.18), we have for $0 < \varepsilon < 1/2p$,
\[
\mathcal{L}_{L_\infty}^{-1}(u, v) \leq \sum_{\lambda_0 \in 0, j_1 \ll j_2 \approx j_3} \langle \lambda_0 \rangle \frac{1}{2}\|v_{\lambda_0}\|_{L^n_\infty} \left\|u_{-I_{j_1}}\right\|_{L^2_{x,t}} \left\|u_{-I_{j_2}}\right\|_{L^2_{x,t}} \left\|\partial_x u_{-\lambda_0-I_{j_3}}\right\|_{L^4_{x,t}}
\leq \sum_{\lambda_0 \in 0, j_1 \ll j_2 \approx j_3} \langle \lambda_0 \rangle^{-1/2}\|v_{\lambda_0}\|_{V_2^2} \left\|u_{-I_{j_1}}\right\|_{V_2^2} \left\|u_{-I_{j_2}}\right\|_{L^4_{x,t}} \left\|\partial_x u_{-\lambda_0-I_{j_3}}\right\|_{L^4_{x,t}}
\leq T^{2\varepsilon/3} \sum_{j_1 \ll j_3, \lambda_0 \in 0} 2^{-j_1/p_2} (-2/p+2\varepsilon) j_3 \|v_{\lambda_0}\|_{V_2^2} \|u\|_{X_{p,\Delta}^{1/2}(-\lambda_0-I_{j_3})} \|u\|_{X_{p,\Delta}^{1/2}}^2
\leq T^{2\varepsilon/3} \sum_{j_3, \lambda_0 \in 0} 2^{-j_3/p_2} (-2/p+2\varepsilon) j_3 \|v_{\lambda_0}\|_{V_2^2} \|u\|_{X_{p,\Delta}^{1/2}(-\lambda_0-I_{j_3})} \|u\|_{X_{p,\Delta}^{1/2}}^2
\leq T^{\varepsilon/2} \|u\|_{X_{p,\Delta}^{1/2}}^3 \|v\|_{Y_{p,\Delta}^\ast}.
\tag{5.52}
\]
In the case $u_{-I_{j_1}}$ attaining the highest dispersion modulation, we have from (2.6) and (5.51) that
\[
\mathcal{L}_{L_\infty}^{-1}(u, v) \leq \sum_{\lambda_0 \in 0, j_1 \ll j_2 \approx j_3} \langle \lambda_0 \rangle \frac{1}{2}\|v_{\lambda_0}\|_{L^n_\infty} \left\|u_{-I_{j_1}}\right\|_{L^2_{x,t}} \left\|u_{-I_{j_2}}\right\|_{L^2_{x,t}} \left\|\partial_x u_{-\lambda_0-I_{j_3}}\right\|_{L^4_{x,t}}
\leq \sum_{\lambda_0 \in 0, j_1 \ll j_2 \approx j_3} \langle \lambda_0 \rangle^{-1/2}\|v_{\lambda_0}\|_{V_2^2} \left\|u_{-I_{j_1}}\right\|_{V_2^2} \left\|u_{-I_{j_2}}\right\|_{L^4_{x,t}} \left\|\partial_x u_{-\lambda_0-I_{j_3}}\right\|_{L^4_{x,t}},
\tag{5.53}
\]
which is the same as in the right hand side of the second inequality as in (5.52).
If $u_{-I_{j_2}}$ has the highest modulation, we have
\[
\mathcal{L}_{L_\infty}^{-1}(u, v) \leq \sum_{\lambda_0 \in 0, j_1 \ll j_2 \approx j_3} \langle \lambda_0 \rangle \frac{1}{2}\|v_{\lambda_0}\|_{L^n_\infty} \left\|u_{-I_{j_1}}\right\|_{L^2_{x,t}} \left\|u_{-I_{j_2}}\right\|_{L^2_{x,t}} \left\|\partial_x u_{-\lambda_0-I_{j_3}}\right\|_{L^4_{x,t}}.
\tag{5.54}
\]
Noticing that for any $\lambda_1 \in -I_{j_1}$, $\lambda_3 \in -\lambda_0-I_{j_3}$, if $j_1 \ll j_3$, we have $|\lambda_3| - |\lambda_1| \gtrsim |\lambda_0|$. Using (5.15), the bilinear estimate (4.1), Corollaries 3.3 and 3.4, and noticing that $j_3 \lesssim \ln(\lambda_0)$, we have
\[
\mathcal{L}_{L_\infty}^{-1}(u, v) \leq T^{2\varepsilon/4} \sum_{\lambda_0 \in 0, j_1 \ll j_2 \approx j_3} 2^{-j_1/p_2} (-2/p+2\varepsilon) j_3 \|v_{\lambda_0}\|_{V_2^2} \|u\|_{X_{p,\Delta}^{1/2}(-\lambda_0-I_{j_3})} \|u\|_{X_{p,\Delta}^{1/2}}^2
\leq T^{2\varepsilon/4} \sum_{\lambda_0 \in 0, j_1 \ll j_2 \approx j_3} 2^{-j_1/p_2} (-2/p+2\varepsilon) j_3 \|v_{\lambda_0}\|_{V_2^2} \|u\|_{X_{p,\Delta}^{1/2}(-\lambda_0-I_{j_3})} \|u\|_{X_{p,\Delta}^{1/2}}^2.
\tag{5.55}
\]
$\mathcal{L}_{L_\infty}^{-2}(u, v)$ and $\mathcal{L}_{L_\infty}^{-3}(u, v)$ can be estimated in an analogous way as above, we omit the details.

**Case $l-a+a_0$**. In view of the frequency constraint condition we see that $\lambda_2 \in [0, -3\lambda_0/4 + C)$ and $\lambda_3 \in [-\lambda_0/4 - C, -\lambda_0)$. We consider the dyadic collections of $\lambda_k$:
\[
\lambda_1 \in \left[\frac{3\lambda_0}{4}, 0\right] \cup_{j_1} -I_{j_1}, \quad \lambda_2 \in \left[0, \frac{3\lambda_0}{4} + C\right] \cup_{j_2} I_{j_2}, \quad \lambda_3 \in \left[-\frac{\lambda_0}{4} - C, -\lambda_0\right] \cup_{j_3 \geq 0} (-\lambda_0-I_{j_3}).
\]
It is easy to see that $j_1 \approx (j_2 \lor j_3) \lesssim \ln(\lambda_0)$. We need to estimate
\[
\mathcal{L}^{-}_{a_+ h_+} (u, v) := \sum_{\lambda_0 \ll 0} \langle \lambda_0 \rangle^{1/2} \left| \int_{[0,T] \times \mathbb{R}} \tau_{\lambda_0} u(3\lambda_0/4) u|_{[0, -3\lambda_0/4 + C]} \partial_x \vec{\pi}|_{-\lambda_0/4 - C, -\lambda_0} \, dxdt \right|
\]
\[
\lesssim \left( \sum_{\lambda_0 \ll 0, \, j_1 \ll j_2 \approx j_3} + \sum_{\lambda_0 \ll 0, \, j_2 \ll j_1 \approx j_3} + \sum_{\lambda_0 \ll 0, \, j_1 \ll j_2 \approx j_3} \right) \langle \lambda_0 \rangle^{1/2} \times \int_{[0,T] \times \mathbb{R}} \left| \tau_{\lambda_0} u_{-I_{j_1}} u_{I_{j_2}} \partial_x \vec{\pi} \right|_{-\lambda_0 - I_{j_3}} \, dxdt.
\]  
(5.56)

Using a similar way as in the estimate of (5.50), we can get the desired result and we omit the details.

Case $I_{-a_+ h_+}$. We consider the dyadic collections of $\lambda_k$:
\[
\lambda_1 \in \left[ \frac{3\lambda_0}{4}, 0 \right) = \bigcup_{j_1} -I_{j_1}, \lambda_2 \in [0, \infty) = \bigcup_{j_2 \geq 0} I_{j_2}, \lambda_3 \in [-\lambda_0, \infty) = \bigcup_{j_3 \geq 0} (-\lambda_0 + I_{j_3}).
\]

From FCC (5.5) it follows that $j_2 \approx (j_1 \lor j_3)$. We need to estimate
\[
\mathcal{L}^{-}_{-a_+ h_+} (u, v) := \sum_{\lambda_0 \ll 0} \langle \lambda_0 \rangle^{1/2} \left| \int_{[0,T] \times \mathbb{R}} \tau_{\lambda_0} u(3\lambda_0/4) u|_{[0, \infty)} \partial_x \vec{\pi}|_{-\lambda_0, \infty} \, dxdt \right|
\]
\[
\lesssim \left( \sum_{\lambda_0 \ll 0, \, j_1 \ll j_2 \approx j_3} + \sum_{\lambda_0 \ll 0, \, j_2 \ll j_1 \approx j_3} + \sum_{\lambda_0 \ll 0, \, j_1 \ll j_2 \approx j_3} \right) \langle \lambda_0 \rangle^{1/2} \times \int_{[0,T] \times \mathbb{R}} \left| \tau_{\lambda_0} u_{-I_{j_1}} u_{I_{j_2}} \partial_x \vec{\pi} \right|_{0} \, dxdt := \Gamma_1 + \Gamma_2 + \Gamma_3.
\]  
(5.57)

In the right hand side of (5.57), by DMCC (5.6) the highest dispersion modulation satisfies
\[
\max_{0 \leq k \leq 3} \left| \xi_k^2 + \tau_k \right| \gtrsim \langle \lambda_0 \rangle (\langle \lambda_0 \rangle + 2^{j_2}).
\]  
(5.58)

If $v_{\lambda_0}$ has the highest dispersion modulation, using a similar way as in (5.16)–(5.18), we have
\[
\Gamma_1 \lesssim \sum_{\lambda_0 \ll 0, \, j_1 \ll j_2 \approx j_3} \langle \lambda_0 \rangle^{1/2} \left\| v_{\lambda_0} \right\|_{L^2_t L^\infty_x} \left\| u_{-I_{j_1}} \right\|_{L^2_t L^\infty_x} \left\| u_{I_{j_2}} \right\|_{L^4_t} \left\| \partial_x u_{-\lambda_0 + I_{j_3}} \right\|_{L^4_t}
\]
\[
\lesssim \sum_{\lambda_0 \ll 0, \, j_1 \ll j_2 \approx j_3} (\langle \lambda_0 \rangle + 2^{j_2})^{-1/2} \left\| v_{\lambda_0} \right\|_{V^2} \left\| u_{-I_{j_1}} \right\|_{V^2} \left\| u_{I_{j_2}} \right\|_{L^4_t} \left\| \partial_x u_{-\lambda_0 + I_{j_3}} \right\|_{L^4_t}
\]
\[
\lesssim T^{2/3} \sum_{j_1 \ll j_3, \, \lambda_0 \ll 0} 2^{-j_1/p} 2^{-2/p + 2/3 j_3} \left\| v_{\lambda_0} \right\|_{L^2_x} \left\| u \right\|_{X^1_{p, \Delta}} (-\lambda_0 + I_{j_3}) \left\| u \right\|_{X^{1/2}_{p, \Delta}}^2
\]
\[
\lesssim T^{-2/3} \left\| u \right\|_{Y^{1/2}_{p, \Delta}}^3 \left\| v \right\|_{Y^{1/2}_{p, \Delta}}.
\]  
(5.59)

If $u_{-I_{j_1}}$ has the highest dispersion modulation, the estimate of $\Gamma_1$ is similar to (5.59) by considering
\[
\Gamma_1 \lesssim \sum_{\lambda_0 \ll 0, \, j_1 \ll j_2 \approx j_3} \langle \lambda_0 \rangle^{1/2} \left\| v_{\lambda_0} \right\|_{L^2_t L^\infty_x} \left\| u_{-I_{j_1}} \right\|_{L^2_t L^\infty_x} \left\| u_{I_{j_2}} \right\|_{L^4_t} \left\| \partial_x u_{-\lambda_0 + I_{j_3}} \right\|_{L^4_t}
\]
\[
\sum_{\lambda_0 \leq 0, \ j_1 \ll j_2 \approx j_3} \left( \langle \lambda_0 \rangle + 2^{4j} \right)^{-1/2} \| v_{\lambda_0} \|_{V_{3,2}^2} \| u_{-I_1} \|_{V_{3,2}^2} \| u_{I_2} \|_{L_{x,t}^6} \| \partial_x u_{-\lambda_0 + I_3} \|_{L_{x,t}^4},
\]

(5.60)

for which the right hand side of (5.60) is the same one as the second inequality of (5.59).

If \( u_{I_2} \) gains the highest dispersion modulation, using (5.58), the bilinear estimate (4.1) and Corollaries 3.3, 3.4, we have

\[
\Gamma_1 \lesssim \sum_{\lambda_0 \leq 0, \ j_1 \ll j_2 \approx j_3} \left( \langle \lambda_0 \rangle + 2^{4j} \right)^{1/2} \| v_{\lambda_0} \|_{L_{x,t}^6} \| u_{I_2} \|_{L_{x,t}^2} \| u_{-I_1} \|_{L_{x,t}^2} \| \partial_x \pi_{-\lambda_0 + I_3} \|_{L_{x,t}^2},
\]

\[
\lesssim \sum_{\lambda_0 \leq 0, \ j_1 \ll j_2 \approx j_3} \left( \langle \lambda_0 \rangle + 2^{4j} \right)^{1/2} \| v_{\lambda_0} \|_{V_{3,2}^2} \| u_{-I_1} \|_{V_{3,2}^2} \| u_{I_2} \|_{V_{3,2}^2} \| u_{-\lambda_0 + I_3} \|_{V_{3,2}^2}.
\]

\[
\lesssim T^{e/4} \sum_{j_1 \ll j_3, \ \lambda_0 \leq 0} 2^{-j_1/p} \left( -2/p+\epsilon \right)^{j_1} \| v_{\lambda_0} \|_{V_{3,2}^2} \| u \|_{X_{p,\Delta}^{1/2}(-\lambda_0 + I_3)} \| u \|_{X_{p,\Delta}^{3/2}}.
\]

(5.61)

If \( u_{-\lambda_0 + I_3} \) has the highest dispersion modulation, one can use similar way as in (5.61) to get the same estimate and we omit the details.

Case \( a_+a_+h \). Finally, we consider the following case as in Table 3.

| Case | \( \lambda_1 \in [0, \infty) \) | \( \lambda_2 \in [0, \infty) \) | \( \lambda_3 \in [\lambda_0, \infty) \) |
|------|-----------------|-----------------|-----------------|
| \( a_+a_+h \) | \( [0, \infty) \) | \( [0, \infty) \) | \( [\lambda_0, \infty) \) |

Table 3. \( \lambda_1, \lambda_2, \lambda_3 \geq 0 \).

In view of FCC (5.5) we see that \( \lambda_3 \in [-\lambda_0 - C, \infty) \). We consider the dyadic collections of \( \lambda_k \):

\( \lambda_k \in [0, \infty) = \bigcup_{j_k} I_{j_k}, \ k = 1, 2; \ \lambda_3 \in [-\lambda_0 - C, \infty) = \bigcup_{j_3} (-\lambda_0 + I_{j_3}) \), \( I_{-1} = [-C, 0] \).

From FCC (5.5) it follows that \( j_3 \approx (j_1 \lor j_2) \). We need to estimate

\[
\mathcal{L}_{a_+a_+}^-(u, v)
\]

\[
:= \sum_{\lambda_0 \leq 0} \langle \lambda_0 \rangle^{1/2} \left| \int_{[0,T] \times \mathbb{R}} \pi_{\lambda_0} u_{[0,\infty)} u_{[0,\infty)} \partial_x \pi_{[-\lambda_0 - C, \infty)} \ dx dt \right|
\]

\[
\lesssim \left( \sum_{\lambda_0 \leq 0, \ j_1 \ll j_2 \approx j_3} + \sum_{\lambda_0 \leq 0, \ j_2 \ll j_1 \approx j_3} \sum_{\lambda_0 \leq 0, \ j_1 \approx j_2 \approx j_3} \right) \langle \lambda_0 \rangle^{1/2} \times \int_{[0,T] \times \mathbb{R}} |\pi_{\lambda_0} u_{I_1} u_{I_2} \partial_x \pi_{-\lambda_0 + I_3} \ dx dt := \Upsilon_1 + \Upsilon_2 + \Upsilon_3.
\]

(5.62)

It suffices to estimate \( \Upsilon_1 \). The highest dispersion modulation in the right hand side of \( \Upsilon_1 \) satisfies

\[
\max_{0 \leq k \leq 3} |\xi_k^2 + \tau_k| \gtrsim \langle \lambda_0 \rangle + 2^{j_1} \langle \lambda_0 \rangle + 2^{j_2} \rangle.
\]

(5.63)

Now we compare the highest dispersion modulation between (5.58) and (5.63), we see that the the highest modulation in \( \Upsilon_1 \) is larger than that of \( \Gamma_1 \). If \( v_{\lambda_0} \) has the highest dispersion modulation, using the same estimates as in (5.59) for
In the right hand side of $L_{I_{j_1}}$, $u_{I_{j_2}}$, $\partial_x \bar{\pi}_{-\lambda_0 + I_{j_3}}$ in the spaces $V^2_\Delta$, $L^4_{2, \tau}$, we can obtain that $\Upsilon_1$ has the same upper bound as that of $\Gamma_1$ in (5.59). The other cases are also similar to those estimates in (5.60) and (5.61).

**Step 2.** We consider the case that $\lambda_0$ is the secondly minimal integer in $\lambda_0, \ldots, \lambda_3$. Namely, there is a bijection $\pi : \{1, 2, 3\} \rightarrow \{1, 2, 3\}$ such that

$$\lambda_{\pi(1)} \leq \lambda_0 \leq \lambda_{\pi(2)} \wedge \lambda_{\pi(3)}.$$  

(5.64)

**Step 2.1.** We assume that $\lambda_0 \gg 0$ and $\lambda_1 = \lambda_{\pi(1)}$. By (5.64), we need to estimate

$$\mathcal{L}(u, v) := \sum_{\lambda_0 \gg 0} \langle \lambda_0 \rangle^{1/2} \int_{[0,T] \times \mathbb{R}} \bar{\pi}_{\lambda_0} u(-\infty, \lambda_0) u(\lambda_0, \infty) \partial_x \bar{\pi}(\lambda_0, \infty) \, dx \, dt$$

$$= \sum_{\lambda_0 \gg 0} \langle \lambda_0 \rangle^{1/2} \int_{[0,T] \times \mathbb{R}} \bar{\pi}_{\lambda_0} u(-\infty, -\lambda_0/2) u(\lambda_0, \infty) \partial_x \bar{\pi}(\lambda_0, \infty) \, dx \, dt$$

$$+ \sum_{\lambda_0 \gg 0} \langle \lambda_0 \rangle^{1/2} \int_{[0,T] \times \mathbb{R}} \bar{\pi}_{\lambda_0} u(-\lambda_0/2, \lambda_0/2) u(\lambda_0, \infty) \partial_x \bar{\pi}(\lambda_0, \infty) \, dx \, dt$$

$$+ \sum_{\lambda_0 \gg 0} \langle \lambda_0 \rangle^{1/2} \int_{[0,T] \times \mathbb{R}} \bar{\pi}_{\lambda_0} u(\lambda_0/2, \lambda_0) u(\lambda_0, \infty) \partial_x \bar{\pi}(\lambda_0, \infty) \, dx \, dt$$

$$:= \mathcal{L}^-(u, v) + \mathcal{L}^0(u, v) + \mathcal{L}^+(u, v).$$

(5.65)

First, we estimate $\mathcal{L}^0(u, v)$. From FCC (5.5) it follows that

$$\mathcal{L}^0(u, v) = \sum_{\lambda_0 \gg 0} \langle \lambda_0 \rangle^{1/2} \int_{[0,T] \times \mathbb{R}} \bar{\pi}_{\lambda_0} u(-\lambda_0/2, 0) u(2\lambda_0 - C, \infty) \partial_x \bar{\pi}(\lambda_0, \infty) \, dx \, dt$$

$$+ \sum_{\lambda_0 \gg 0} \langle \lambda_0 \rangle^{1/2} \int_{[0,T] \times \mathbb{R}} \bar{\pi}_{\lambda_0} u(0, \lambda_0/2) u(3\lambda_0 - 2C, \infty) \partial_x \bar{\pi}(\lambda_0, \infty) \, dx \, dt$$

$$:= \mathcal{L}^0^-(u, v) + \mathcal{L}^0^+(u, v).$$

(5.66)

We further divide $\mathcal{L}^0^+(u, v)$ into two parts

$$\mathcal{L}^0^+(u, v) = \sum_{\lambda_0 \gg 0} \langle \lambda_0 \rangle^{1/2} \int_{[0,T] \times \mathbb{R}} \bar{\pi}_{\lambda_0} u(0, \lambda_0/2) u(3\lambda_0 - 2C, 2\lambda_0) \partial_x \bar{\pi}(\lambda_0, \infty) \, dx \, dt$$

$$+ \sum_{\lambda_0 \gg 0} \langle \lambda_0 \rangle^{1/2} \int_{[0,T] \times \mathbb{R}} \bar{\pi}_{\lambda_0} u(0, \lambda_0/2) u(2\lambda_0, \infty) \partial_x \bar{\pi}(\lambda_0, \infty) \, dx \, dt$$

$$:= \mathcal{L}^0_1^+(u, v) + \mathcal{L}^0_2^+(u, v).$$

(5.67)

Again, in view of FCC (5.5) we have

$$\mathcal{L}^0_1^+(u, v) = \sum_{\lambda_0 \gg 0} \langle \lambda_0 \rangle^{1/2} \int_{[0,T] \times \mathbb{R}} \bar{\pi}_{\lambda_0} u(0, \lambda_0/2) u(3\lambda_0 - 2C, 2\lambda_0) \partial_x \bar{\pi}(\lambda_0, 3\lambda_0/2 + C) \, dx \, dt.$$  

(5.68)

In the right hand side of $\mathcal{L}^0_1^+(u, v)$, by DMCC (5.6) we have

$$\sum_{k=0}^{3} |\xi_k^2 + \tau_k| \gg (\lambda_0)^2.$$
By Corollary 3.3 we have
\[
\|u(t,\lambda_0/2)\|_{L_t^\infty L_{x,\Delta}^2} \lesssim \sum_{j \leq \log_2^{\lambda_0} + 1} \|u_j\|_{V_{2,\Delta}^2} \lesssim \|u\|_{X_{p,\Delta}^{1/2}}. \tag{5.69}
\]
If \(v_{\lambda_0}\) gains the highest dispersion modulation, by Hölder’s inequality, the dispersion modulation decay estimate \((2.6), (5.69)\) and Lemma 3.5, we have for \(0 < \epsilon < 1/4p\),
\[
|\mathcal{L}^{0,+}_t(u, v)| \lesssim \sum_{\lambda_0 > 0} \langle \lambda_0 \rangle^{-1/2} \|v_{\lambda_0}\|_{V_{2,\Delta}^2} \|u(0,\lambda_0/2)\|_{L_t^\infty L_{x,\Delta}^2} \|u(3\lambda_0/2-C,2\lambda_0)\|_{L_{x,t}^4} \|\partial_x \overline{\Pi}_0(\lambda_0,3\lambda_0/2+C)\|_{L_{x,t}^4}.
\]
If \(u(0,\lambda_0/2)\) the highest dispersion modulation, taking \(L_t^\infty, L_{x,t}^2, L_{x,\Delta}^4, L_{x,t}^4\) norms to \(v_{\lambda_0}, u(0,\lambda_0/2), u(3\lambda_0/2-C,2\lambda_0), \partial_x \overline{\Pi}_0(\lambda_0,3\lambda_0/2+C)\), then applying \((2.6)\), we have
\[
|\mathcal{L}^{0,+}_t(u, v)| \lesssim T^{3/2} \sum_{\lambda_0 > 0} \langle \lambda_0 \rangle^{-2/2+2\epsilon} \|v_{\lambda_0}\|_{V_{2,\Delta}^2} \|u(0,\lambda_0/2)\|_{V_{2,\Delta}^2} \|u(3\lambda_0/2-C,2\lambda_0)\|_{L_{x,t}^4} \|\partial_x \overline{\Pi}_0(\lambda_0,3\lambda_0/2+C)\|_{L_{x,\Delta}^4},
\]
which reduces the same estimate as the first inequality in \((5.70)\).

Let \(u(3\lambda_0/2-C,2\lambda_0)\) have the highest dispersion modulation. Taking \(L_t^\infty, L_{x,t}^2, L_{x,\Delta}^4\) norms to \(v_{\lambda_0}, u(3\lambda_0/2-C,2\lambda_0), u(0,\lambda_0/2)\partial_x \overline{\Pi}_0(\lambda_0,3\lambda_0/2+C)\), then applying the dispersion modulation decay \((2.6)\) to \(u(3\lambda_0/2-C,2\lambda_0)\) and the bilinear estimate \((4.1)\) to \(u(0,\lambda_0/2)\partial_x \overline{\Pi}_0(\lambda_0,3\lambda_0/2+C)\), one obtains that \(\mathcal{L}^{0,+}_t(u, v)\) has the desired estimate. When \(\partial_x u(0,\lambda_0/2+C)\) has the highest dispersion modulation, the argument is similar.

Now we estimate \(\mathcal{L}^{0,+}_h(u, v)\). We adopt the following decompositions:
\[
\lambda_1 \in (0, \lambda_0/2) = \bigcup_{j_1 > 0} I_{j_1}, \quad \lambda_2 \in [2\lambda_0, \infty) = \bigcup_{j_1 > 0} (2\lambda_0 + I_{j_2}),
\]
\[
\lambda_3 \in [\lambda_0, \infty) = \bigcup_{j_2 > 0} (\lambda_0 + I_{j_3}).
\]
By FCC \((5.5)\), we have
\[
j_3 \approx j_1 \vee j_2, \quad j_1 \leq \log_2^{\lambda_0} + 1.
\]
\[
|\mathcal{L}_h^{0,+}(u, v)| \lesssim \sum_{\lambda_0 > 0} \langle \lambda_0 \rangle^{1/2} \int_{[0,T] \times \mathbb{R}} |\varphi_{\lambda_0} u_{I_{j_1}} u_{2\lambda_0 + I_{j_2}} \partial_x \overline{\Pi}_{\lambda_0 + I_{j_3}}| \, dx dt
\]
\[
\lesssim \left( \sum_{\lambda_0 > 0} \sum_{1 \leq j_2 < j_3 = j_2} \sum_{1 \leq j_1} \sum_{\lambda_0 > 0, 1 \leq j_2 = j_3} \right) \langle \lambda_0 \rangle^{1/2}
\]
\[
\times \int_{[0,T] \times \mathbb{R}} |\varphi_{\lambda_0} u_{I_{j_1}} u_{2\lambda_0 + I_{j_2}} \partial_x \overline{\Pi}_{\lambda_0 + I_{j_3}}| \, dx dt = I + II + III. \tag{5.71}
\]
In the right hand side of \((5.71)\), by DMCC \((5.6)\) we have
\[
\sum_{k=0}^3 |\xi_k^2 + \tau_k| \gtrsim \langle \lambda_0 \rangle \langle \lambda_0 \rangle + 2j^2. \tag{5.72}
\]
Noticing that \((5.72)\) is the same as \((5.58)\), we can follow the same ideas in the estimates of \(\Gamma_1\) to get the bound of \(I\), see \((5.59)\)–\((5.61)\).
Let us observe that \( j_1, j_2, j_3 \lesssim \log \lambda_0 \) in II and III. It follows that the estimates of II and III are easier than that of I. We omit the details.

\( \mathcal{L}^{0-}(u, v) \) can be handled in a similar way as that of \( \mathcal{L}^{0+}(u, v) \) and we omit the details.

Now we estimate \( \mathcal{L}^+(u, v) \). We use the dyadic decomposition

\[
\lambda_1 \in (\lambda_0/2, \lambda_0) = \bigcup_{j_1 > 0} (\lambda_0 - I_{j_1}), \quad \lambda_k \in [\lambda_0, \infty) = \bigcup_{j_k \geq 0} (\lambda_0 + I_{j_k}), \quad k = 2, 3.
\]

By FCC (5.5), we have

\[
|\mathcal{L}^+(u, v)| \leq \sum_{\lambda_0 > 0, j_2 = j_1 \lor j_3} (\lambda_0)^{1/2} \int_{[0,T] \times \mathbb{R}} |\nabla_{\lambda_0} u_{\lambda_0 - I_{j_1}} u_{\lambda_0 + I_{j_2}} \partial_x \nabla_{\lambda_0 + I_{j_3}}| \, dx dt. \tag{5.73}
\]

Comparing (5.73) with (5.7), we see that \( \mathcal{L}^+(u, v) \) in (5.73) is rather similar to (5.7) in Step 1.1. We omit the details of the proof.

We estimate \( \mathcal{L}^-(u, v) \). In view of FCC (5.5) we see that

\[
\mathcal{L}^-(u, v) = \sum_{\lambda_0 > 0} (\lambda_0)^{1/2} \int_{[0,T] \times \mathbb{R}} \nabla_{\lambda_0} u_{(-\infty, -\lambda_0/2]} u_{[5\lambda_0/2 - C, \infty)} \partial_x \nabla_{\lambda_0} \, dx dt. \tag{5.74}
\]

Decompose \( \lambda_k \) in the following dyadic way:

\[
\lambda_1 \in (-\infty, -\lambda_0/2] = \bigcup_{j_1 \geq 0} (-\lambda_0/2 - I_{j_1}), \quad \lambda_3 \in [\lambda_0, \infty) = \bigcup_{j_k \geq 0} (\lambda_0 + I_{j_k})
\]

\[
\lambda_2 \in [5\lambda_0/2 - C, \infty) = \bigcup_{j_2 \geq -1} (5\lambda_0/2 + I_{j_2}), \quad j_3 = [-C, 0).
\]

It follows that \( j_2 \approx j_1 \lor j_3 \).

\[
|\mathcal{L}^-(u, v)| \leq \sum_{\lambda_0 > 0, j_2 \approx j_1 \lor j_3} (\lambda_0)^{1/2} \int_{[0,T] \times \mathbb{R}} |\nabla_{\lambda_0} u_{-\lambda_0/2 - I_{j_1}} u_{5\lambda_0/2 + I_{j_2}} \partial_x \nabla_{\lambda_0 + I_{j_3}}| \, dx dt. \tag{5.75}
\]

By DMCC (5.6) we have

\[
\max_{0 \leq k \leq 3} |\xi_k^2 + \tau_k| \gtrsim ((\lambda_0) + 2^{j_1})((\lambda_0) + 2^{j_2}). \tag{5.76}
\]

By (5.76), we see that the dispersion modulation estimate (2.6) gives better decay. So, the estimate of \( \mathcal{L}^-(u, v) \) is easier than that of the above cases and we will not perform the details.

**Step 2.2.** We consider the case \( \lambda_0 \ll 0, \lambda_1 \approx \lambda_0 \approx \lambda_2 \approx \lambda_3 \). According to the size of \( \lambda_2 \), we divide the proof into four subcases \( \lambda_2 \in \tilde{h}_-, \lambda_2 \in l-, \lambda_2 \in (0, -3\lambda_0/4) \) and \( \lambda_2 \in [-3\lambda_0/4, \infty) \). Moreover, in view of FCC (5.5), in order to keep the left hand side of (5.4) nonzero, it suffices to consider the following four subcases in Table 4.

**Case 1.** Let us observe that all \( \lambda_k \ (k = 1, 2, 3) \) are localized in a neighbourhood of \( \lambda_0 \), which is essentially the same as in Case 1.2 as in Step 1.2, we omit the details of the proof.

**Case 2.** Similar to Case 1. as in Step 1.2 and we omit its proof.

**Case 3.** By separating \( \lambda_3 \in [\lambda_0, -3\lambda_0/4 + C) = [\lambda_0, 0) \cup (0, -3\lambda_0/4 + C) \), we see that this case is quite similar to Case 2.
Case 4. Observing that $\lambda_1$ and $\lambda_2$ are far away from 0, but $\lambda_3 \in [\lambda_0, \infty)$ containing 0, we need to further split $[\lambda_0, \infty)$ into $[\lambda_0, 0]$ and $[0, \infty)$. We consider the following two subcases of Case 4, see Table 5.

| Case | $\lambda_1 \in$ | $\lambda_2 \in$ | $\lambda_3 \in$ |
|------|-----------------|-----------------|-----------------|
| 1    | $(5\lambda_0/4 - C, \lambda_0)$ | $(\lambda_0, 3\lambda_0/4)$ | $(\lambda_0, 3\lambda_0/4 + C)$ |
| 2    | $(2\lambda_0 - C, \lambda_0)$ | $(3\lambda_0/4, 0)$ | $(\lambda_0, C)$ |
| 3    | $(11\lambda_0/4 - C, \lambda_0)$ | $(0, -3\lambda_0/4)$ | $(\lambda_0, -3\lambda_0/4 + C)$ |
| 4    | $(-\infty, \lambda_0)$ | $[-3\lambda_0/4, \infty]$ | $(\lambda_0, \infty)$ |

Table 4. $\lambda_0 < 0$, $\lambda_1 < \lambda_0 \leq \lambda_2 \wedge \lambda_3$.

Case 4.1. By FCC (5.5), we have $\lambda_1 \in (-\infty, 7\lambda_0/4 + C)$. So, we need to estimate

$$\mathcal{L}^{2, \infty}(u, v) := \sum_{\lambda_0 < 0} \langle \lambda_0 \rangle^{1/2} \int_{[0,T] \times \mathbb{R}} \overline{v}_{\lambda_0} u_{(-\infty, 7\lambda_0/4 + C) u_{[-3\lambda_0/4, \infty]} \partial_x \pi_{[\lambda_0, 0]} dx dt.}$$

(5.77)

We can decompose $\lambda_k$ in the following way:

$$\lambda_1 \in (-\infty, 7\lambda_0/4 + C) = \bigcup_{j_1 \geq -1} (7\lambda_0/4 - I_{j_1}), I_{-1} = [-C, 0),$$

$$\lambda_2 \in (-3\lambda_0/4, \infty) = \bigcup_{j_3 \geq 0} (-3\lambda_0/4 + I_{j_3}), \lambda_3 \in [\lambda_0, 0) = \bigcup_{j_3 \geq 0} -I_{j_3}.$$

It follows that $j_1 \approx j_2 \vee j_3, j_3 \leq 1 + \log_2 |\lambda_0|$

$$|\mathcal{L}^{2, \infty}(u, v)| \leq \sum_{\lambda_0 < 0, j_1 \approx j_2 \vee j_3} \langle \lambda_0 \rangle^{1/2} \int_{[0,T] \times \mathbb{R}} |\overline{v}_{\lambda_0} u_{7\lambda_0/4 - I_{j_1}} u_{-3\lambda_0/4 + I_{j_2}} \partial_x \pi_{-I_{j_3}}| dx dt. \quad (5.78)$$

By DMCC (5.6) we have

$$\max_{0 \leq k \leq 3} |\xi_k^2 + \tau_k| \gtrsim (\langle \lambda_0 \rangle + 2^{j_1})(\langle \lambda_0 \rangle + 2^{j_2}).$$

(5.79)

One can imitate the procedure as in Case $l_{-a\xi}h_+$ in Step 1.2 to obtain the result, as desired.

Case 4.2. We need to estimate

$$\mathcal{L}^{3, \infty}(u, v) := \sum_{\lambda_0 < 0} \langle \lambda_0 \rangle^{1/2} \int_{[0,T] \times \mathbb{R}} \overline{v}_{\lambda_0} u_{(-\infty, \lambda_0)} u_{[-3\lambda_0/4, \infty]} \partial_x \pi_{[0, \infty)} dx dt. \quad (5.80)$$

We can decompose $\lambda_k$ in the following way:

$$\lambda_1 \in (-\infty, \lambda_0) = \bigcup_{j_1 > 0} (\lambda_0 - I_{j_1}),$$

$$\lambda_2 \in \bigcup_{j_3 > 0} -I_{j_3},$$

$$\lambda_3 \in \bigcup_{j_3 > 0} [-\lambda_0, -3\lambda_0/4 + C].$$

Table 5. Two subcases of Case 4.
\[ \lambda_2 \in (-3\lambda_0/4, \infty) = \bigcup_{j_2} I_{j_2}, \quad \lambda_3 \in [0, \infty) = \bigcup_{j_3 \geq 0} I_{j_3}. \]

It follows that \( j_2 \approx j_1 \lor j_1, j_2 \geq \log_2 |\lambda_0| - C \)

\[ |\mathcal{L}^{3, \infty}(u, v)| \leq \sum_{\lambda_0 < 0, \ j_1 \lor j_2 \geq \log_2 |\lambda_0|} (\langle \lambda_0 \rangle^{1/2}) \int_{[0, T] \times \mathbb{R}} |\nabla_{x_0} u_{\lambda_0} - I_{j_1} u_{I_{j_2}} \partial_x \nabla_{I_{j_3}}| \, dxdt. \]

(5.81)

By DMCC (5.6) we have for \( j_1 \geq 10 \),

\[ \max_{0 \leq k \leq 3} |\xi_k^2 + \tau_k| \geq 2^{j_1} (\langle \lambda_0 \rangle + 2^{j_2}). \]

(5.82)

Comparing (5.82) with (5.35), (5.81) and (5.32), one can imitate the procedures as in the Case \( h_{-a, \pm a} \) to obtain the result, as desired. The details are omitted.

**Step 2.3.** We consider the remaining cases. If \( \pi(1) = 2 \), namely \( \lambda_2 \) is the smallest one in all \( \lambda_k \), by symmetry we see that it is the same as the case \( \pi(1) = 1 \). If \( \pi(1) = 3 \), it follows from FCC (5.5) we have \( \lambda_0 \approx \lambda_1 \approx \lambda_2 \approx \lambda_3 \). So, the summation in the left hand side of (5.4) is essentially on \( \lambda_0 \), the other summations are finitely many. By Hölder’s inequality we have the result, as desired.

**Step 3.** We consider the case \( |\lambda_0| \lesssim 1 \). It follows that \( \langle \lambda_0 \rangle^{1/2} \sim 1 \) in the left hand side of (5.4), which means that we have gained half order derivative. So, this case becomes easier to handle and the details of the proof are omitted.

6. **Quintic linear estimates.** For the sake of convenience, we denote for \( w = (w^{(0)}, \ldots, w^{(5)}) \),

\[ \mathcal{L}(w) = \sum_{\lambda_0, \ldots, \lambda_5 \in \mathbb{Z}} \langle \lambda_0 \rangle^{1/2} \int_{\mathbb{R} \times [0, T]} \prod_{k=0}^{5} w_{\lambda_k}^{(k)} (x, t) \, dxdt, \]

(6.1)

In the following we always assume that

\[ w_{\lambda_k}^{(k)} = u_{\lambda_k}, \quad k = 1, 3, 5; \quad w_{\lambda_0}^{(0)} = \nabla_{x_0}, \quad w_{\lambda_k}^{(k)} = \nabla_{\lambda_k}, \quad k = 2, 4, \]

(6.2)

which means that for \( w = (\bar{u}, u, \bar{u}, u, \bar{u}, u) \),

\[ \mathcal{L}(w) = \sum_{\lambda_0, \ldots, \lambda_5 \in \mathbb{Z}} \langle \lambda_0 \rangle^{1/2} \int_{\mathbb{R} \times [0, T]} \nabla_{\lambda_0} u_{\lambda_1} \nabla_{\lambda_2} u_{\lambda_3} \nabla_{\lambda_4} u_{\lambda_5} (x, t) \, dxdt = \sum_{\lambda_0 \in \mathbb{Z}} \langle \lambda_0 \rangle^{1/2} \int_{\mathbb{R} \times [0, T]} |\nabla_{\lambda_0} u|^4 u(x, t) \, dxdt. \]

(6.3)

Our goal in this section is to show that

**Lemma 6.1.** Let \( p \in [4, \infty), \ 0 < T < 1, \ 0 < \varepsilon \ll 1 \). Let \( \mathcal{L}(w) \) be as in (6.1) and (6.2). We have

\[ |\mathcal{L}(w)| \lesssim T^\varepsilon \|v\|_{Y^{\theta_{p, \Delta}}} \|u\|_{X^{1/2}_{p, \Delta}}^{5/2} \]

(6.4)

and by duality,

\[ \|u|^4 u\|_{X^{-1/2}_{p, \Delta}} \lesssim T^\varepsilon \|u\|_{X^{1/2}_{p, \Delta}}^{5}. \]

(6.5)
We divide the proof of Lemma 6.1 into a few steps according to the size of \( \lambda_0 \). We can assume that \( \lambda_0 > 0 \), since in the opposite case one can substitute \( \lambda_0, \ldots, \lambda_5 \) by \( -\lambda_0, \ldots, -\lambda_5 \).

**Step 1.** Let us assume that \( \lambda_0, \ldots, \lambda_5 \) satisfy
\[
\lambda_0 = \max_{0 \leq k \leq 5} |\lambda_k|.
\] (6.6)

For short, considering the higher, lower and all frequency of \( \lambda_k \), we use the following notations:
\[
\lambda_k \in h \Leftrightarrow \lambda_k \in [c \lambda_0, \lambda_0]
\]
\[
\lambda_k \in h_- \Leftrightarrow \lambda_k \in [-\lambda_0, -c \lambda_0]
\]
\[
\lambda_k \in l \Leftrightarrow \lambda_k \in [0, c \lambda_0]
\]
\[
\lambda_k \in l_- \Leftrightarrow \lambda_k \in [-c \lambda_0, 0]
\]
\[
\lambda_k \in a \Leftrightarrow \lambda_k \in [-\lambda_0, \lambda_0]
\]
\[
\lambda_k \in a_+ \Leftrightarrow \lambda_k \in [0, \lambda_0]
\]
\[
\lambda_k \in a_- \Leftrightarrow \lambda_k \in [-\lambda_0, 0]
\]
for some \( c > 0 \). First, we consider the case that there are two higher frequency in \( \lambda_1, \ldots, \lambda_5 \), say, \( \lambda_1, \lambda_3 \) belong to higher frequency intervals. We denote by \( (\lambda_k) \in hhaaa \) that all \( \lambda_0, \ldots, \lambda_5 \) satisfy conditions (6.6) and
\[
\lambda_1, \lambda_3 \in h, \quad \lambda_2, \lambda_4, \lambda_5 \in a. \quad (6.7)
\]

For \( w = (\bar{v}, u, \bar{u}, u, \bar{u}) \), we write
\[
\mathcal{L}_{hhaaa}(w) = \sum_{(\lambda_k) \in hhaaa} (\lambda_0)^{1/2} \int_{\mathbb{R} \times [0,T]} \prod_{k=0}^{5} w^{(k)}_{\lambda_k}(x,t) dx dt,
\] (6.8)
we will always use the notation
\[
\mathcal{L}_{bcdef}(w) = \sum_{(\lambda_k) \in bcdef} (\lambda_0)^{1/2} \int_{\mathbb{R} \times [0,T]} \prod_{k=0}^{5} w^{(k)}_{\lambda_k}(x,t) dx dt,
\] (6.9)
where \( b, c, d, e, f \in \{a, a_+, a_- \}, h, h_-, l, l_- \} \).

**Lemma 6.2.** Let \( p \in [4, \infty) \), \( 0 < T < 1 \), \( 0 < \varepsilon \ll 1 \). We have
\[
|\mathcal{L}_{bcdef}(w)| \lesssim T^2 \|v\|_{y^0} \|u\|_{\Lambda^0_{p', \Delta}}^5
\] (6.10)
if at least two ones in “\( b, c, d, e, f \)” belong to \( \{h, h_-\} \).

**Proof.** Case \( hh+a+a_+ \). We denote by \( (\lambda_k) \in hh+a+a_+ \) the case that \( \lambda_0, \ldots, \lambda_5 \) satisfy (6.6) and
\[
\lambda_k \in h, \quad k = 1, 3, \quad \lambda_2, \lambda_4, \lambda_5 \in a_+. \quad (6.11)
\]
We decompose \( \lambda_2, \lambda_4, \lambda_5 \) in a dyadic way:
\[
\lambda_k \in [0, \lambda_0] = \bigcup_{j_k \geq 0} f_j, \quad k = 2, 4, 5; \quad 0 \leq j_2, j_4, j_5 \leq \log_2 \lambda_0.
\] (6.12)
We can assume that \( \lambda_0 \gg 1 \). Assume that \( j_{\max} = j_2 \lor j_4 \lor j_5, j_{\min} = j_2 \land j_4 \land j_5 \) and \( j_{\text{med}} \) is the secondly larger one in \( \{j_2, j_4, j_5\} \) in (6.12). By Hölder’s inequality,
\[
|\mathcal{L}_{hh+a+a_+}(w)| \lesssim \sum_{\lambda_0; \ j_{\min}, j_{\text{med}}, j_{\max} \in [\log_2 \lambda_0]} (\lambda_0)^{1/2} \|v_{\lambda_0}\|_{L^\infty_{x,t}} \prod_{k=1,3} \|w^{(k)}_{\lambda_0}\|_{L^4_{x,t,\in\mathbb{R}}},
\]
\[
\times \|u^{j_{\max}}\|_{L^4_{x,t,\in\mathbb{R}}} \|u^{j_{\text{med}}}\|_{L^4_{x,t,\in\mathbb{R}}} \|u^{j_{\min}}\|_{L^4_{x,t,\in\mathbb{R}}},
\] (6.13)
In view of $V_3^2 \subset L_t^\infty L_x^2$ and $\|v\lambda_0\|_{L_t^\infty L_x^2} \gtrsim \|v\lambda_0\|_{L_t^\infty L_x^2}$, by Corollaries 3.3 and 3.5, one has that

$$\left| \mathcal{Z}_{hha_{a_+a_+}}(w) \right| \lesssim T^4e/3 \sum_{\lambda_0: j_{\min}, j_{\med}, j_{\max} \leq \log_2 \lambda_0} \langle \lambda_0 \rangle^{1/2} \|v\lambda_0\|_{V_3^2} \langle \lambda_0 \rangle^{2(-1/4-1/p+\varepsilon)} \|u\|_{X_{p, \Delta}^{1/2, 2}}^2 \times 2^{(j_{\max}+j_{\med})(-1/4-1/p+\varepsilon)} \|u\|_{X_{p, \Delta}^{1/2, 2}}^{2j_{\min}(1/2-1/p)} \|u\|_{X_{p, \Delta}^{1/2}}. \quad (6.14)$$

Choosing $0 < \varepsilon < 1/2p$, and making the summation on $j_{\min}, j_{\med}$ and $j_{\max}$ in order, one obtain that

$$\left| \mathcal{Z}_{hha_{a_+a_+}}(w) \right| \lesssim T^4e/3 \sum_{\lambda_0} \langle \lambda_0 \rangle^{-2/p+2\varepsilon} \|v\lambda_0\|_{V_3^2} \|u\|_{X_{p, \Delta}^{1/2}}^5. \quad (6.15)$$

Noticing that $\{\langle \lambda_0 \rangle^{-2/p+2\varepsilon}\} \in \ell^p$, by Hölder’s inequality we have

$$\left| \mathcal{Z}_{hha_{a_+a_+}}(w) \right| \lesssim T^e \|v\|_{Y_{p, \Delta}^0} \|u\|_{X_{p, \Delta}^{1/2}}^5. \quad (0 < \varepsilon \leq 1/2p) \quad (6.16)$$

**Case $hha_{a_+a_-}$**. We denote by $(\lambda_k) \in hha_{a_+a_-}$ the case that $\lambda_0, ..., \lambda_5$ satisfy (6.6) and

$$\lambda_k \in [c\lambda_0, \lambda_0], \quad k = 1, 3, \quad \lambda_2, \lambda_4 \in [0, \lambda_0], \quad \lambda_5 \in [-\lambda_0, 0]. \quad (6.17)$$

We decompose $\lambda_2, \lambda_4, \lambda_5$ by

$$\lambda_k \in [0, \lambda_0] = \bigcup_{j_k \geq 0} I_{j_k}, \quad k = 2, 4; \quad \lambda_5 \in [-\lambda_0, 0] = \bigcup_{j_5 \geq 0} -I_{j_5}, \quad j_2, j_4, j_5 \leq \log_2 \lambda_0. \quad (6.18)$$

Repeating the procedures as in Case $hha_{a_+a_+}$, we can show that

$$\left| \mathcal{Z}_{hha_{a_+a_-}}(w) \right| \lesssim T^e \|v\|_{Y_{p, \Delta}^0} \|u\|_{X_{p, \Delta}^{1/2}}^5. \quad (6.19)$$

Recall that in the proof above, the condition that two $\lambda_k$ is localized in higher frequency, say $\lambda_1, \lambda_3 \in [c\lambda_0, \lambda_0]$ can guarantee that $\{\langle \lambda_0 \rangle^{-2/p+2\varepsilon}\}$ is convergent in $\ell^p$. Hence, applying the same way as in the above, we can obtain the result for the other cases which contain two higher frequency and three all frequency.

If some $\lambda_k$ is only in lower frequency $[0, c\lambda_0]$, the proof is almost the same as in the above. If all $\lambda_k$ have three or more higher frequency $[c\lambda_0, \lambda_0]$, the proof is easier than the Case $hha_{a_+a_+}$, since the dyadic decomposition starting at 0 for the third higher frequency has at most finite dyadic intervals.

In (6.3), for the sake of symmetry, we can assume that

(H1) $\lambda_1 \geq \lambda_3 \geq \lambda_5$, $\lambda_2 \geq \lambda_4$

One easily sees that $\lambda_0, ..., \lambda_5$ satisfy the following frequency constrained condition:

(H2) $\lambda_0 + \lambda_2 + \lambda_4 = \lambda_1 + \lambda_3 + \lambda_5 + l$, $|l| \leq 10$.

The non-trivial case is that

(H3) $\lambda_0 = \max_{0 \leq k \leq 5} |\lambda_k| \gg 1$.

The case $\lambda_0 = \max_{0 \leq k \leq 5} |\lambda_k| \leq 1$ implies that the summation in (6.3) has at most finite terms. So, in view of (H1), (H2) and (H3), we see that the orders of $\lambda_0, ..., \lambda_5$ have the following 10 cases:

$$\lambda_0 \geq \lambda_2 \geq \lambda_1 \geq \lambda_3 \geq \lambda_5 \geq \lambda_4, \quad (\text{Ord}1)$$
By condition (Ord4) we see that
\[ \lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_5 \geq \lambda_4, \quad \text{(Ord2)} \]
\[ \lambda_0 \geq \lambda_1 \geq \lambda_3 \geq \lambda_5 \geq \lambda_2 \geq \lambda_4, \quad \text{(Ord3)} \]
\[ \lambda_0 \geq \lambda_1 \geq \lambda_3 \geq \lambda_2 \geq \lambda_4 \geq \lambda_5, \quad \text{(Ord4)} \]
\[ \lambda_0 \geq \lambda_1 \geq \lambda_3 \geq \lambda_2 \geq \lambda_5 \geq \lambda_4, \quad \text{(Ord5)} \]
\[ \lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \lambda_5, \quad \text{(Ord6)} \]
\[ \lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \lambda_4 \geq \lambda_3 \geq \lambda_5, \quad \text{(Ord7)} \]
\[ \lambda_0 \geq \lambda_2 \geq \lambda_1 \geq \lambda_3 \geq \lambda_4 \geq \lambda_5, \quad \text{(Ord8)} \]
\[ \lambda_0 \geq \lambda_2 \geq \lambda_1 \geq \lambda_4 \geq \lambda_3 \geq \lambda_5, \quad \text{(Ord9)} \]
\[ \lambda_0 \geq \lambda_2 \geq \lambda_1 \geq \lambda_4 \geq \lambda_3 \geq \lambda_5. \quad \text{(Ord10)} \]

If there are at least two higher frequencies in \( \lambda_1, \ldots, \lambda_5 \) which are localized in \([c\lambda_0, \lambda_0] \cup [-\lambda_0, -c\lambda_0]\), in the proof of Lemma 6.2 we neither consider the orders of \( \lambda_1, \ldots, \lambda_5 \) nor use the constraint condition (H2). Now we study the case that there is only one higher frequency in \( \lambda_1, \ldots, \lambda_5 \). For \( \lambda_1, \ldots, \lambda_5 \), if there is only one higher frequency which is localized in \([c\lambda_0, \lambda_0] \cup [-\lambda_0, -c\lambda_0]\), we see that it must be the biggest one localized in \([c\lambda_0, \lambda_0]\) or the smallest one localized in \([-\lambda_0, -c\lambda_0]\), the other frequency is localized in \([-c\lambda_0, c\lambda_0]\).

First, let us consider (Ord4) case according to the high-low frequency. The case \( \lambda_5 \in \mathcal{H}_- \) and \( \lambda_1, \ldots, \lambda_4 \in \mathcal{I} \) or \( \mathcal{I}_- \) never happens for small \( c > 0 \). So, it suffices to consider the case \( \lambda_1 \in \mathcal{H} \). we divide the proof into a few cases, see Table 6.

| Case (Ord4) | \( \lambda_1 \in \mathcal{H} \) | \( \lambda_2 \in \mathcal{I} \) | \( \lambda_3 \in \mathcal{I} \) | \( \lambda_4 \in \mathcal{I}_- \) | \( \lambda_5 \in \mathcal{I}_- \) |
|------------|-------------------------------|-------------------|-------------------|-------------------|-------------------|
| hllll      | \( h \)                       | \( l \)          | \( l \)          | \( l \)          | \( l \)          |
| hllll_    | \( h \)                       | \( l \)          | \( l \)          | \( l \)          | \( l_\)         |
| hllll_    | \( h \)                       | \( l \)          | \( l \)          | \( l_\)         | \( l_\)         |
| hlll_l_   | \( h \)                       | \( l_\)         | \( l_\)         | \( l_\)         | \( l_\)         |
| hlll_l_l_ | \( h \)                       | \( l_\)         | \( l_\)         | \( l_\)         | \( l_\)         |

Table 6. \( \lambda_0 \geq \lambda_1 \geq \lambda_3 \geq \lambda_2 \geq \lambda_4 \geq \lambda_5 \), only one higher frequency in \( \lambda_1, \ldots, \lambda_5 \).

**Case hllll.** We denote by \((\lambda_k) \in hllll\) the case that \( \lambda_0, \ldots, \lambda_5 \) satisfy (6.6), (Ord4) and
\[ \lambda_1 \in \mathcal{H}, \quad \lambda_k \in \mathcal{I}, \quad k = 2, 3, 4, 5. \quad (6.20) \]

We decompose \( \lambda_1, \ldots, \lambda_5 \) in a dyadic way:
\[ \lambda_1 \in [c\lambda_0, \lambda_0] = \bigcup_{j_1 \geq 0} (\lambda_0 - I_{j_1}), \quad \lambda_k \in [0, \lambda_0] = \bigcup_{j_k \geq 0} I_{j_k}, \quad k = 2, \ldots, 5. \quad (6.21) \]

By condition (Ord4) we see that
\[ 0 \leq j_5 \leq j_4 \leq j_2 \leq j_3 \leq \log^{-1} \lambda_0. \quad (6.22) \]

One has that
\[
\mathcal{L}_{hllll}(w) \leq \sum_{\lambda_0: j_1: j_5 \leq j_4 \leq j_2 \leq j_3 \leq \log^{-1} \lambda_0} \langle \lambda_0 \rangle^{1/2} \int_{[0, T]} |\nabla \lambda_0 w_{\lambda_0 - I_{j_1}} \prod_{k=2}^5 w_{I_{j_k}}^{(k)}(x, t)| dx dt,
\]
\[
(6.23)
\]
Noticing that \( 0 < \varepsilon \), we see that \( j_1 > j_3 + 10 \) implies that \( \mathcal{L}_{hlll}(w) = 0 \). So, we have \( j_1 \leq j_3 + 10 \) in (6.23). By Hölder’s inequality, we have

\[
|\mathcal{L}_{hlll}(w)| \lesssim \sum_{\lambda_0: j_1; j_5 \leq j_1 \leq j_3 \leq j_5 \leq \log_2 \lambda_0} \langle \lambda_0 \rangle^{1/2} \| \nabla \lambda_0 \|_{L^2_{x,t}} \| u_{\lambda_0 - I_{j_1}} \|_{L^4_{x,t}[0,T]} \times \| u_{I_{j_3}} \|_{L^4_{x,t}[0,T]} \| u_{I_{j_4}} \|_{L^4_{x,t}[0,T]} \| u_{I_{j_5}} \|_{L^\infty_{x,t}}.
\]

(6.24)

By Corollaries 3.3 and 3.4, we have

\[
|\mathcal{L}_{hlll}(w)| \lesssim T^{4\varepsilon/3} \sum_{\lambda_0: j_1 \leq j_3 + 10; j_5 \leq j_3 \leq j_5} \langle \lambda_0 \rangle^{1/2} \| v_{\lambda_0} \|_{V^2_\lambda} \langle \lambda_0 \rangle^{-1/2} 2^{j_1(1/4 - 1/p + \varepsilon)} \| u \|_{X^{1/2}_{p,\Delta}(\lambda_0 - I_{j_1})} \times 2^{j_2 + j_3 + j_4} (1/4 - 1/p + \varepsilon) \| u \|_{X^{1/2}_{p,\Delta}}.
\]

(6.25)

Taking \( 0 < \varepsilon \leq 1/4p \) and summarizing over all \( j_5, j_4, j_2 \) and \( j_3 \) in order, we obtain that

\[
|\mathcal{L}_{hlll}(w)| \lesssim T^\varepsilon \sum_{\lambda_0: j_1 \leq j_3 + 10} \| v_{\lambda_0} \|_{V^2_\lambda} \| u \|_{X^{1/2}_{p,\Delta}(\lambda_0 - I_{j_1})} \| u \|_{X^{1/2}_{p,\Delta}}
\]

(6.26)

Noticing that \( 0 < \varepsilon \leq 1/4p \), by Hölder’s inequality, one has that

\[
|\mathcal{L}_{hlll}(w)| \lesssim T^\varepsilon \| v \|_{Y^0_{p',\Delta}} \| u \|_{X^0_{p,\Delta}} \left( \sum_{\lambda_0, \lambda \in \lambda_0 - I_{j_1}} \langle \lambda \rangle^{p/2} \| u_\lambda \|_{Y^0_{p',\Delta}}^{p} \right)^{1/p}
\]

\[
\lesssim T^\varepsilon \| v \|_{Y^0_{p',\Delta}} \| u \|_{X^0_{p,\Delta}} \sum_{j_1} 2^{-3j_1/2} \left( \sum_{\lambda_0, \lambda \in \lambda_0 - I_{j_1}} \langle \lambda \rangle^{p/2} \| u_\lambda \|_{Y^0_{p',\Delta}}^{p} \right)^{1/p}
\]

(6.27)

Case \( hlll_\_ \). We denote by \( (\lambda_k) \in hlll_\_ \) the case that \( \lambda_0, \ldots, \lambda_5 \) satisfy (6.6), (Ord4) and

\[
\lambda_1 \in h; \quad \lambda_k \in I, \quad k = 2, 3, 4; \quad \lambda_5 \in l_.
\]

(6.28)

We decompose \( \lambda_1, \ldots, \lambda_5 \) in a dyadic way:

\[
\lambda_1 \in [c\lambda_0, \lambda_0) = \bigcup_{j_1 \geq 0} (\lambda_0 - I_{j_1}); \quad \lambda_k \in [0, \lambda_0) = \bigcup_{j_k \geq 0} I_{j_k}, \quad k = 2, 3, 4;
\]

\[
\lambda_5 \in [-c\lambda_0, 0) = \bigcup_{j_5 \geq 0} -I_{j_5}.
\]

(6.29)

By condition (Ord4) we see that

\[
0 \leq j_1 \leq j_2 \leq j_3 \leq \log_2 \lambda_0 - C.
\]

(6.30)

One has that

\[
\mathcal{L}_{hlll_\_}(w) \leq \sum_{\lambda_0; j_1; j_5; j_4 \leq j_2 \leq j_3 \leq \log_2 \lambda_0} \langle \lambda_0 \rangle^{1/2} \int_{R \times [0,T]} |\nabla \lambda_0 u_{\lambda_0 - I_{j_1}} u_{-I_{j_5}} \prod_{k=2}^4 w_{I_{j_k}}^{(k)}(x,t) dx dt.
\]

(6.31)
Using the frequency constraint condition (H2), we see that
\[ j_1 \leq j_3 + C. \] (6.32)

Denote
\[ \bar{j}_{\text{max}} = j_2 \lor j_5, \quad \bar{j}_{\text{min}} = j_4 \land j_5, \quad \bar{j}_{\text{med}} \in \{j_2, j_4, j_5\} \setminus \{\bar{j}_{\text{max}}, \bar{j}_{\text{min}}\}. \]

Following the ideas as in the estimate of Case \text{hillll}, by Hölder’s inequality and Corollaries 3.3, 3.4, we have
\[ |\mathcal{L}_{\text{hillll}}(w)| \lesssim T^\epsilon \sum_{\lambda_0: \ j_1 \leq j_3 + C, \ \bar{j}_{\text{max}}, \bar{j}_{\text{med}}, \bar{j}_{\text{min}}} \langle \lambda_0 \rangle^{1/2} \|v_{\lambda_0}\|_{L^2} \langle \lambda_0 \rangle^{-1/2} 2^{j_1(1/4-1/p+\epsilon)} \|u\|_{X^{1/2}_p(\lambda_0 - I_{j_1})} \times 2^{(j_2 + j_3 + j_{\text{med}})(-1/4-1/p+\epsilon)} 2^{j_{\text{min}}(1/2-1/p)} \|u\|_{X^{1/2}_p}^4. \] (6.33)

Summarizing over all \( \bar{j}_{\text{min}}, \bar{j}_{\text{med}}, \bar{j}_{\text{max}} \) in orders, from (6.33) one can control \( |\mathcal{L}_{\text{hillll}}(w)| \) by the right hand side of (6.26). So, we have
\[ |\mathcal{L}_{\text{hillll}}(w)| \lesssim T^\epsilon \|v\|_{Y^0_{p', \Delta}} \|u\|_{X^{1/2}_p}^5. \] (6.34)

Case \text{hillll}. We denote by \( (\lambda_k) \in \text{hillll} \) the case that \( \lambda_0, \ldots, \lambda_5 \) satisfy (6.6), (Ord4) and
\[ \lambda_1 \in h; \quad \lambda_2, \lambda_3 \in l; \quad \lambda_4, \lambda_5 \in l-. \] (6.35)

We decompose \( \lambda_1, \ldots, \lambda_5 \) by
\[ \lambda_1 \in [c\lambda_0, \lambda_0] = \bigcup_{j_1 \geq 0} (\lambda_0 - I_{j_1}); \quad \lambda_2 \in [0, \lambda_0] = \bigcup_{j_k \geq 0} I_{j_k}, \quad k = 2, 3; \]
\[ \lambda_k \in [-c\lambda_0, 0] = \bigcup_{j_k \geq 0} -I_{j_k}, \quad k = 4, 5. \] (6.36)

By condition (Ord4) we see that
\[ 0 \leq j_2 \leq j_3 \leq \log^{\lambda_0}_2 - C, \quad 0 \leq j_4 \leq j_5 \leq \log^{\lambda_0}_2 - C. \] (6.37)

One has that
\[ \mathcal{L}_{\text{hillll}}(w) \leq T^\epsilon \sum_{\lambda_0: \ j_1 \leq j_3; \ j_2 \leq j_3 \leq \log^{\lambda_0}_2} \langle \lambda_0 \rangle^{1/2} \int_{\mathbb{R} \times [0, T]} |v_{\lambda_0} u_{\lambda_0 - I_{j_1}} \bar{\Pi}_{j_2} u_{I_{j_3}} \bar{\Pi}_{-I_{j_4}} u_{-I_{j_5}}(x, t)| dx dt. \] (6.38)

In view if the frequency constraint condition (H2), we see that
\[ I_{j_1} \cap (I_{j_3} + I_{j_4} - I_{j_2} - I_{j_5}) \neq \emptyset. \]

Otherwise \( \mathcal{L}_{\text{hillll}}(w) = 0 \). It follows that
(1) If \( j_4 < j_5 \), then \( j_1 < j_3 + 10 \).
(2) If \( j_4 = j_5 \), then \( j_1 < j_3 \lor j_4 + 10 \).

In the case \( j_4 < j_5 \), we can use a similar way as in Case \text{hillll}. Indeed, putting
\[ \bar{j}_{\text{max}} = j_2 \lor j_5, \quad \bar{j}_{\text{min}} = j_4 \land j_2, \quad \bar{j}_{\text{med}} \in \{j_2, j_4, j_5\} \setminus \{\bar{j}_{\text{max}}, \bar{j}_{\text{min}}\}, \]
then we can repeat the procedure as in Case \text{hillll} to have the estimate
\[ |\mathcal{L}_{\text{hillll}}(w)| \lesssim T^\epsilon \|v\|_{Y^0_{p', \Delta}} \|u\|_{X^{1/2}_p}^5. \] (6.39)
In the case $j_4 = j_5$, we need to separate the proof into the case $j_3 \geq j_4$ and $j_3 < j_4$, respectively. We have

(2a) If $j_3 \leq j_4$, then $j_2 \leq j_3 \leq j_4 = j_5$.

(2b) If $j_3 \geq j_4$, then $j_3 \geq j_4 = j_5 \geq j_2$, or $j_3 \geq j_2 \geq j_4 = j_5$.

In the case (2b), we can use the same way as in Case hllll to show that (6.39) holds. In the case (2a), we have from Corollaries 3.3 and 3.4 that

$$|\mathcal{L}_{hllll}(w)| \lesssim \sum_{\lambda_0; j_3=\min\{j_3\}}^{\lambda_0; j_3<j_4+C} \sum_{j_2<j_3<j_5=j_4} \frac{1}{2^{j_2/3}} \sum_{j_3=\min\{j_3\}}^{j_4} (\lambda_0)^{1/2} \|v_{\lambda_0}\|_{Y^\Delta_2}^2 (\lambda_0)^{-1/4} 2^{j_2 (1/4 - 1/p + \varepsilon)} 2^{j_2 (1/2 - 1/p)} \|u\|_{X^{1/2}_p, \Delta}^4. \quad (6.40)$$

Making the summation in turn to $j_2, j_3$ and $j_4$, we can get (6.39).

**Case hllll.** We denote by $(\lambda_k) \in hllll$ the case that $\lambda_0, ..., \lambda_5$ satisfy (6.6), (Ord4) and

$$\lambda_1 \in h; \ \lambda_3 \in l; \ \lambda_2, \lambda_4, \lambda_5 \in l. \quad (6.41)$$

We decompose $\lambda_1, ..., \lambda_5$ by

$$\lambda_1 \in [c\lambda_0, \lambda_0] = \bigcup_{j_1 \geq 0} (\lambda_0 - I_{j_1}); \ \lambda_3 \in [0, \lambda_0] = \bigcup_{j_3 \geq 0} I_{j_3};$$

$$\lambda_k \in [-c\lambda_0, 0] = \bigcup_{j_k \geq 0} -I_{j_k}, \ k = 2, 4, 5. \quad (6.42)$$

By condition (Ord4) we see that

$$0 \leq j_2 \leq j_4 \leq j_5 \leq \log_2 \lambda_0 - C. \quad (6.43)$$

Using the frequency constraint condition (H2), we see that

(i) If $j_3 \geq j_5$, then $j_1 \leq j_3 + 10$.

(ii) If $j_3 < j_5$, then $j_1 < j_3 + 10$.

In the case (i), we can follow the same way as in Case hllll to obtain the estimate

$$|\mathcal{L}_{hllll}(w)| \lesssim T^{s/3} \|v\|_{Y^\Delta_2}^2 \|u\|_{X^{1/2}_p, \Delta}^{5/2}. \quad (6.44)$$

In the case (ii), we need to further analyze $j_2, j_3, j_4$ and denote

$$j_{\text{max}} = j_3 \lor j_4; \ j_{\text{min}} = j_2 \land j_3; \ j_{\text{med}} \in \{j_2, j_3, j_4\} \setminus \{j_{\text{max}}, j_{\text{min}}\}.$$ 

Then we can repeat the procedures as in Case hllll to obtain (6.44).

**Case hllll.** We denote by $(\lambda_k) \in hllll$ the case that $\lambda_0, ..., \lambda_5$ satisfy (6.6), (Ord4) and

$$\lambda_1 \in h; \ \lambda_2, ..., \lambda_5 \in l. \quad (6.45)$$

We decompose $\lambda_1, ..., \lambda_5$ by

$$\lambda_1 \in [c\lambda_0, \lambda_0] = \bigcup_{j_1 \geq 0} (\lambda_0 - I_{j_1}); \ \lambda_k \in [-c\lambda_0, 0] = \bigcup_{j_k \geq 0} -I_{j_k}, \ k = 2, ..., 5. \quad (6.46)$$

By condition (Ord4) we see that

$$0 \leq j_3 \leq j_2 \leq j_4 \leq j_5 \leq \log_2 \lambda_0 - C. \quad (6.47)$$
Using the frequency constraint condition (H2), we see that
\[ j_5 \leq j_4 + 2, \quad j_1 \leq j_4 + 10. \]
Noticing that \( j_3 \leq j_2 \leq j_4 \leq j_5 \leq j_4 + 2 \), using a similar way as in Case \( \text{hllll} \) we can obtain the estimate
\[ |\mathcal{L}_{hhll} (w) | \lesssim T^c \| v \|_{p, \Delta} \| u \|_{X_{p, \Delta}^{1/2}}^5. \quad (6.48) \]
Next, we consider (Ord1) case according to the high-low frequency. The case \( \lambda_2 \in h \) and \( \lambda_1, \lambda_3, \lambda_4, \lambda_5 \in l \) or \( l_- \) never happens for small \( c > 0 \). So, it suffices to consider the case \( \lambda_4 \in h_- \). We divide the proof into a few cases, see Table 7.

| Case (Ord1) | \( \lambda_2 \in \) | \( \lambda_1 \in \) | \( \lambda_3 \in \) | \( \lambda_5 \in \) | \( \lambda_4 \in \) |
|-------------|------------|------------|------------|------------|------------|
| \( lllhh_- \) | \( l \) | \( l \) | \( l \) | \( l \) | \( h_- \) |
| \( lllh_-h_- \) | \( l \) | \( l \) | \( l \) | \( l_- \) | \( h_- \) |
| \( lhhlll_h_- \) | \( l \) | \( l \) | \( l_- \) | \( l_- \) | \( h_- \) |
| \( lhhlll_h_- \) | \( l \) | \( l_- \) | \( l_- \) | \( l_- \) | \( h_- \) |
| \( llll_h_- \) | \( l_- \) | \( l_- \) | \( l_- \) | \( l_- \) | \( h_- \) |

Table 7. \( \lambda_0 \geq \lambda_2 \geq \lambda_1 \geq \lambda_3 \geq \lambda_5 \geq \lambda_4 \), only one higher frequency in \( \lambda_1, \ldots, \lambda_5 \).

We consider the decomposition of \( \lambda_k \):
\[ \lambda_4 \in [-\lambda_0, -c\lambda_0] = \bigcup_{j_k \geq 0} (-\lambda_0 + I_{j_k}), \quad \lambda_k \in [0, c\lambda_0] = \bigcup_{j_k \geq 0} I_{j_k}; \]
\[ \lambda_1 \in [-c\lambda_0, 0] = \bigcup_{j_k \geq 0} -I_{j_k} \]
for any \( k, l \in \{1, 2, 3, 5 \} \). For short, we denote by \( \lambda_k \) \( \in \text{llllh}_- \) the case that \( \lambda_0, ..., \lambda_5 \) satisfy (6.6), (Ord1) and \( \lambda_1, ..., \lambda_3, \lambda_5 \in l \), \( \lambda_4 \in h_- \). Similarly, we will use the notations \( \lambda_k \) \( \in \text{llll}_-, \ldots, \text{llll}_-h_- \) (Ord1).

Using the frequency constraint condition (H2), we see that
\[ j_4 \leq j_1 + 10 \text{ if } (\lambda_k) \in \text{llllh}_- \cup \text{llllh}_- \cup \text{llll}_-h_-; \]
\[ j_1 \lor \ldots \lor j_5 \lesssim 1, \text{ if } (\lambda_k) \in \text{llll}_-l_- \cup \text{llll}_-l_-h_- \]
The cases \( \text{llll}_-l_-h_- \) and \( \text{llll}_-l_-l_-h_- \) are trivial and we omit the discussions. For the cases \( \text{llllh}_-, \text{llll}_-h_- \) and \( \text{llll}_-h_- \), we need to further analyze \( j_2, j_3, j_5 \) and write
\[ j_{\text{max}} = j_2 \lor j_3 \lor j_5, \quad j_{\text{min}} = j_2 \land j_3 \land j_5, \quad j_{\text{med}} \in \{ j_2, j_3, j_5 \} \setminus \{ j_{\text{max}}, j_{\text{min}} \}. \]
Similar to (6.33), we have
\[ |\mathcal{L}_{llhh} (w) | + |\mathcal{L}_{llhh} (w) | + |\mathcal{L}_{llll} (w) | \]
\[ \lesssim T^c \sum_{\lambda_0; j_4 \leq j_4 + C, j_{\text{max}} j_{\text{med}} j_{\text{min}}} \langle \lambda_0 \rangle^{1/2} \| v \|_{p, \Delta} \langle \lambda_0 \rangle^{-1/2} 2j_4 (1/4 - 1/p + \varepsilon) \| u \|_{X_{p, \Delta}^{1/2}} \]
\[ \times 2^{j_4 + j_{\text{max}} + j_{\text{med}} - (1/4 - 1/p + \varepsilon)} 2^{j_5 (1/2 - 1/p)} \| u \|_{X_{p, \Delta}^{1/2}}. \quad (6.49) \]
Using the same way as above and making the summation in turn to \( j_{\text{min}}, j_{\text{med}}, j_{\text{max}} \) and then to \( \lambda_0 \) and \( j_4 \), we have
\[
|\mathcal{Z}_{lll..}(w)| + |\mathcal{Z}_{lll..h_..}(w)| + |\mathcal{Z}_{lll..h_..}(w)| \lesssim T^c ||v||_{y_{p',\Delta}}^0 ||u||_{X_{p,\Delta}}^5. \tag{6.50}
\]

If \( \lambda_0, ..., \lambda_5 \) have the orders as in (Ord2) and only one of them is lying in higher frequency, there are two possible cases \( \lambda_1 \in h \) and \( \lambda_4 \in h_.. \). We discuss these two cases separately.

| Case (Ord2): \( \lambda_1 \in h \) |
|------------------|---|---|---|---|
| \( hllll \)      | \( h \) | \( l \) | \( l \) | \( l \) |
| \( hllll_.. \)    | \( h \) | \( l \) | \( l \) | \( l \) | \( l \) |
| \( hlll_.._.. \)  | \( h \) | \( l \) | \( l \) | \( l \) | \( l \) |
| \( hlll_..l_.._.. \) | \( h \) | \( l \) | \( l \) | \( l \) | \( l \) |
| \( hlll_..l_..l_.._.. \) | \( h \) | \( l \) | \( l \) | \( l \) | \( l \) |

Table 8. \( \lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_5 \geq \lambda_4 \), only one higher frequency in \( \lambda_1, ..., \lambda_5 \).

As in Table 8, we denote by \( (\lambda_k) \in hllll \) that \( \lambda_0, ..., \lambda_5 \) satisfy (6.6), (Ord2) and \( \lambda_1 \in h, \lambda_2, ..., \lambda_5 \in l \). Similarly for the notations \( (\lambda_k) \in hllll_.., hlll_..l_.._.. \). We consider the decomposition of \( \lambda_k \):
\[
\lambda_1 \in [c\lambda_0, \lambda_0] = \bigcup_{j_k \geq 0} (\lambda_0 - I_{j_k}), \quad \lambda_k \in [0, c\lambda_0] = \bigcup_{j_k \geq 0} I_{j_k}, \quad \lambda_1 \in [-c\lambda_0, 0] = \bigcup_{j_k \geq 0} -I_{j_k}
\]
for any \( k, l \in \{2, 3, 4, 5\} \). We have the following constraint conditions on \( j_1, ..., j_5 \):
\[
\begin{align*}
&j_1 \leq j_3 + C, \quad j_2 \geq j_3 \geq j_5 \geq j_4 \quad \text{if} \quad (\lambda_k) \in hllll, \\
&j_1 \leq j_3 \lor j_4 \geq C, \quad j_2 \geq j_3 \geq j_5 \quad \text{if} \quad (\lambda_k) \in hllll_.., \\
&j_1 \leq j_3 \lor j_4 \geq C, \quad j_2 \geq j_3 \geq j_4 \geq j_5 \quad \text{if} \quad (\lambda_k) \in hlll_.._, \\
&j_1 \leq j_4, \quad j_3 \leq j_5 \leq j_4 \quad \text{if} \quad (\lambda_k) \in hlll_..l_.._, \\
&j_1 \leq j_4 + C, \quad j_2 \leq j_3 \leq j_5 \leq j_4 \quad \text{if} \quad (\lambda_k) \in hlll_..l_..l_.._.. \end{align*}
\]

Using the above constraint conditions, we can follow the same ideas as in (Ord1) case to have the estimate
\[
|\mathcal{L}_{bcdef}(w)| \lesssim T^c ||v||_{y_{p',\Delta}}^0 ||u||_{X_{p,\Delta}}^5. \tag{6.51}
\]

where \( bcdef \in \{hllll, hllll_.., hlll_..l_.._, hlll_..l_..l_.._.., hlll_..l_..l_..l_.._..\} \).

As in Table 9, we denote by \( (\lambda_k) \in lllh_- \) that \( \lambda_0, ..., \lambda_5 \) satisfy (6.6), (Ord2) and \( \lambda_1 \in h, \lambda_2, ..., \lambda_3 \in l \). Similarly for the notations \( (\lambda_k) \in lllh_.._, llll_..l_..l_.._.., hlll_..l_..l_..l_.._.._.. \). We consider the decomposition of \( \lambda_k \):
\[
\begin{align*}
&\lambda_4 \in [-\lambda_0, -c\lambda_0] = \bigcup_{j_k \geq 0} (-\lambda_0 + I_{j_k}), \quad \lambda_k \in [0, c\lambda_0] = \bigcup_{j_k \geq 0} I_{j_k}, \\
&\lambda_l \in [-c\lambda_0, 0] = \bigcup_{j_k \geq 0} -I_{j_k}
\end{align*}
\]
for any \( k, l \in \{1, 2, 3, 5\} \). We have the following constraint conditions on \( j_1, ..., j_5 \):
\[
\begin{align*}
&j_4 \leq j_1 + C, \quad j_4 \geq j_2 \geq j_3 \geq j_5 \quad \text{if} \quad (\lambda_k) \in lllh_-,
\end{align*}
\]
Case (Ord2): $\lambda_4 \in h_-$

| Case | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_5$ | $\lambda_4$ |
|------|-------------|-------------|-------------|-------------|-------------|
| $l_{lll}h_-$ | $l$ | $l$ | $l$ | $l$ | $h_-$ |
| $l_{ll}l_{ll}h_-$ | $l$ | $l$ | $l_-$ | $l_-$ | $h_-$ |
| $l_{ll}l_{ll}l_{ll}h_-$ | $l_-$ | $l_-$ | $l_-$ | $l_-$ | $h_-$ |
| $l_{ll}l_{ll}l_{ll}l_{ll}h_-$ | $l_-$ | $l_-$ | $l_-$ | $l_-$ | $h_-$ |

Table 9. $\lambda_0 \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_5 \geqslant \lambda_4$, only one higher frequency in $\lambda_1, ..., \lambda_5$.

$j_4 \leq j_1 + C, j_1 \geq j_2 \geq j_3, j_5 \leq j_1 + C$ if $(\lambda_k) \in l_{lll}h_-$,

$j_4 \leq j_1 + C, j_1 \leq j_2, j_3 \leq j_5 \leq j_1 + C$ if $(\lambda_k) \in l_{ll}l_{ll}h_-$,

$j_4 \leq j_1 + C, j_2 \leq j_3 \leq j_5 \leq j_1 + C$ if $(\lambda_k) \in l_{ll}l_{ll}l_{ll}h_-$,

$j_4 \leq j_1, j_1 \leq j_2 \leq j_3 \leq j_5$ if $(\lambda_k) \in l_{ll}l_{ll}l_{ll}l_{ll}h$.

Using the above constraint conditions, we can follow the same ideas as in (Ord4) case to have the estimate

$$
|\mathcal{L}_{bcdef}(w)| \lesssim T^5 v y_0^{5/4} ||u||^{5}_{L^1/2},
$$

(6.52)

where $bcdef \in \{l_{llll}h_-, l_{llll}l_{ll}h_-, l_{llll}l_{ll}l_{ll}h_-, l_{llll}l_{ll}l_{ll}l_{ll}h_-, l_{llll}l_{ll}l_{ll}l_{ll}l_{ll}h\}$ as in Table 9.

If $\lambda_0, ..., \lambda_5$ has the orders as in (Ord3) and only one of them is in higher frequency, then there are two possible cases $\lambda_1 \in h$ and $\lambda_4 \in h_-$, see Tables 10 and 11.

Case (Ord3): $\lambda_1 \in h$

| Case | $\lambda_1$ | $\lambda_3$ | $\lambda_5$ | $\lambda_2$ | $\lambda_4$ |
|------|-------------|-------------|-------------|-------------|-------------|
| $llll$ | $h$ | $l$ | $l$ | $l$ | $l$ |
| $llll_-$ | $h$ | $l$ | $l$ | $l_-$ | $l_-$ |
| $llll_{ll}l$ | $h$ | $l$ | $l_-$ | $l_-$ | $l_-$ |
| $llll_{ll}l_{ll}l$ | $h_-$ | $l_-$ | $l_-$ | $l_-$ | $l_-$ |

Table 10. $\lambda_0 \geqslant \lambda_1 \geqslant \lambda_3 \geqslant \lambda_5 \geqslant \lambda_2 \geqslant \lambda_4$, only one higher frequency in $\lambda_1, ..., \lambda_5$.

As in Table 10, we denote by $(\lambda_k) \in h_{llll}$ that $\lambda_0, ..., \lambda_5$ satisfy (6.6), (Ord3) and $\lambda_1 \in h$, $\lambda_2, ..., \lambda_5 \in l$. Similarly for the notations $(\lambda_k) \in h_{llll}_-, ..., h_{ll}l_{ll}l_{ll}l_-$. We consider the decomposition of $\lambda_k$:

$$
\lambda_1 \in [c\lambda_0, \lambda_0] = \bigcup_{j_k \geq 0} (\lambda_0 - I_{j_k}); \quad \lambda_k \in [0, c\lambda_0] = \bigcup_{j_k \geq 0} I_{j_k}, \quad \lambda_1 \in [-c\lambda_0, 0] = \bigcup_{j_k \geq 0} -I_{j_k}
$$

for any $k, l \in \{2, 3, 4, 5\}$. We have the following constraint conditions on $j_1, ..., j_5$:

$j_1 \leq j_3 + C, j_3 \geq j_5 \geq j_2 \geq j_4$ if $(\lambda_k) \in h_{llll},$

$j_1 \leq j_3 \vee j_4 + C, j_3 \geq j_5 \geq j_2$ if $(\lambda_k) \in h_{llll}_-$,

$j_1 \leq j_3 \vee j_4 + C, j_3 \geq j_5, j_2 \leq j_4$ if $(\lambda_k) \in h_{llll}_{ll},$

$j_1 \leq j_3 \vee j_4, j_3 \leq j_5 \leq j_2 \leq j_4$ if $(\lambda_k) \in h_{llll}_{ll}l,$

$j_1 \leq j_4 + C, j_3 \leq j_5 \leq j_2 \leq j_4$ if $(\lambda_k) \in h_{llll}_{ll}l_{ll}.$
Using the above constraint conditions, we can follow the same ideas as in (Ord1) case to have the estimate

$$|\mathcal{L}_{bcdef}(w)| \lesssim T^\varepsilon ||v||_{Y^0_{p', \Delta}} ||u||_{X^{1/2}_{p', \Delta}}^-,$$

(6.53)

where \( bcdef \in \{ hllll_h, hllll_hll, \ldots, hllll_iil, hl_iil_iil, \ldots, hl_iil_iil_iil \} \) in Table 10.

| Case \((Ord3): \lambda_4 \in h_\ - | \lambda_1 \in \lambda_2 \in \lambda_3 \in \lambda_5 \in l \ | \lambda_4 \in \lambda_5 \in l |
|---|---|---|---|---|
| hllll_h_ \_ | l | l | l | l | h_\ - |
| hllll_iil_h_ \_ | l | l | l | l | h_\ - |
| hllll_iil_iil_h_ \_ | l | l | l | l | h_\ - |
| hllll_iil_iil_iil_h_ \_ | l | l | l | l | h_\ - |

**Table 11.** \( \lambda_0 \geq \lambda_1 \geq \lambda_3 \geq \lambda_5 \geq \lambda_2 \geq \lambda_4 \), only one higher frequency in \( \lambda_1, \ldots, \lambda_5 \).

As in Table 11, we denote by \((\lambda_k) \in lllll_h_\ - \) that \( \lambda_0, \ldots, \lambda_5 \) satisfy \((6.6)\), \((Ord3)\) and \( \lambda_1 \in h, \lambda_2, \ldots, \lambda_5 \in l \). Similarly for the notations \((\lambda_k) \in lllll_h_\ - , lllll_iil_h_ \ - \).

We consider the decomposition of \( \lambda_k \):

\[
\lambda_4 \in [-\lambda_0, -c\lambda_0] = \bigcup_{j_4 \geq 0} (-\lambda_0 + I_{j_4}); \quad \lambda_k \in [0, c\lambda_0] = \bigcup_{j_k \geq 0} J_{j_k}, \\
\lambda_l \in [-c\lambda_0, 0] = \bigcup_{j_l \geq 0} -I_{j_l}
\]

for any \( k, l \in \{1, 2, 3, 5\} \). We have the following constraint conditions on \( j_1, \ldots, j_5 \):

\[
\begin{align*}
& j_4 \leq j_1 + C, \quad j_1 \geq j_3 \geq j_5 \geq j_2 \quad \text{if} \quad (\lambda_k) \in lllll_h_\ - , \\
& j_4 \leq j_1 \vee j_2 + C, \quad j_1 \geq j_3 \geq j_5 \quad \text{if} \quad (\lambda_k) \in lllll_iil_h_ \ - , \\
& j_4 \leq j_1 \vee j_2 + C, \quad j_1 \geq j_3, \quad j_5 \leq j_2 \quad \text{if} \quad (\lambda_k) \in lllll_iil_iil_h_ \ - , \\
& j_4 \leq j_1 \vee j_2 + C, \quad j_3 \leq j_5 \leq j_2 \quad \text{if} \quad (\lambda_k) \in lllll_iil_iil_iil_h_ \ - , \\
& j_4 \leq j_1 + C, \quad j_1 \leq j_5 \leq j_2 \quad \text{if} \quad (\lambda_k) \in lllll_iil_iil_iil_iil_h_ \ -.
\end{align*}
\]

Using the above constraint conditions, we can follow the same ideas as in (Ord4) case to have the estimate

$$|\mathcal{L}_{bcdef}(w)| \lesssim T^\varepsilon ||v||_{Y^0_{p', \Delta}} ||u||_{X^{1/2}_{p', \Delta}}^-,$$

(6.54)

where \( bcdef \in \{ hllll_h, hllll_iil, \ldots, hllll_iil_iil, hl_iil_iil_iil_iil \} \) as in Table 11.

If \( \lambda_0, \ldots, \lambda_5 \) has the orders as in (Ord5) and only one of them is in higher frequency, then two possible cases \( \lambda_1 \in h \) and \( \lambda_4 \in h_\ - \) will be happened, see Tables 12 and 13.

As in Table 12, we denote by \((\lambda_k) \in hllll \) that \( \lambda_0, \ldots, \lambda_5 \) satisfy \((6.6)\), \((Ord5)\) and \( \lambda_1 \in h, \lambda_2, \ldots, \lambda_5 \in l \). Similarly for the notations \((\lambda_k) \in hllll_hlll, \ldots, hl_iil_iil_iil_iil \). We consider the decomposition of \( \lambda_k \):

\[
\lambda_1 \in [c\lambda_0, 0] = \bigcup_{j_1 \geq 0} (\lambda_0 - I_{j_1}); \quad \lambda_k \in [0, c\lambda_0] = \bigcup_{j_k \geq 0} I_{j_k}, \quad \lambda_l \in [-c\lambda_0, 0] = \bigcup_{j_l \geq 0} -I_{j_l}
\]

for any \( k, l \in \{2, 3, 4, 5\} \). We have the following constraint conditions on \( j_1, \ldots, j_5 \):

\[
\begin{align*}
& j_1 \leq j_3 + C, \quad j_3 \geq j_2 \geq j_5 \geq j_4 \quad \text{if} \quad (\lambda_k) \in hllll,
\end{align*}
\]
We consider the decomposition of $k,l$ for any case to have the estimate

$\lambda \in 1 \in.$

Table 12. $\lambda_0 \geq \lambda_1 \geq \lambda_3 \geq \lambda_2 \geq \lambda_5 \geq \lambda_4$, only one higher frequency in $\lambda_1, \ldots, \lambda_5$.

Using the above constraint conditions, we can follow the same ideas as in (Ord1) case to have the estimate

$$|\mathcal{L}_{bcdef}(w)| \lesssim T^c||v||_{y_{p,\Delta}}^0 \|u\|_{L^p_{p,\Delta}}^5,$$

(6.55)

where $bcdef \in \{hhll, hlll, hlll, hlll, hlll, hlll\}$ in Table 12.

Table 13. $\lambda_0 \geq \lambda_1 \geq \lambda_3 \geq \lambda_2 \geq \lambda_5 \geq \lambda_4$, only one higher frequency in $\lambda_1, \ldots, \lambda_5$.

As in Table 13, we denote by $(\lambda_k) \in lllll$ that $\lambda_0, \ldots, \lambda_5$ satisfy (6.6), (Ord5) and $\lambda_1 \in h$, $\lambda_2, \ldots, \lambda_5 \in l$. Similarly for the notations $(\lambda_k) \in lllll, \ldots, lllllllllll$.

We consider the decomposition of $\lambda_k$:

$$\lambda_4 \in [-\lambda_0, -c\lambda_0] = \bigcup_{j_4 \geq 0} (-\lambda_0 + I_{j_4}); \lambda_k \in [0, c\lambda_0] = \bigcup_{j_k \geq 0} I_{j_k},$$

$$\lambda_l \in [-c\lambda_0, 0] = \bigcup_{j_l \geq 0} -I_{j_l}$$

for any $k,l \in \{1, 2, 3, 5\}$. We have the following constraint conditions on $j_1, \ldots, j_5$:

- $j_4 \leq j_1 + C$, $j_1 \geq j_3 \geq j_2 \geq j_5$ if $(\lambda_k) \in lllll$
- $j_4 \leq j_1 + C$, $j_1 \geq j_3 \geq j_2$, if $(\lambda_k) \in lllll$
- $j_4 \leq j_1 \vee j_2 + C$, $j_1 \geq j_3$, $j_5 \geq j_2$, if $(\lambda_k) \in ll1lll$
- $j_4 \leq j_1 \vee j_2 + C$, $j_3 \leq j_2 \leq j_5$, if $(\lambda_k) \in ll1lll$.
Using the above constraint conditions, we can follow the same ideas as in (Ord4) case to have the estimate

\[
|\mathcal{L}_{bcdef}(w)| \lesssim T^{5} \|v\|_{Y_{p',\Delta}^{0}}^{5} \|u\|_{X_{p,\Delta}^{1/2}}^{5},
\]

(6.56)

where \( bcdef \in \{111111\ldots, 111111\ldots, 111111\ldots \} \) as in Table 8.

If \( \lambda_{0}, \ldots, \lambda_{5} \) satisfy any case of (Ord6)-(Ord10), we easily see that

\[
\lambda_{0} \geq \lambda_{1}, \quad \lambda_{2} \geq \lambda_{3}, \quad \lambda_{4} \geq \lambda_{5}.
\]

By the frequency constraint condition (H2), one can conclude that

\[
\lambda_{0} \leq \lambda_{1} + 20, \quad \lambda_{2} \leq \lambda_{3} + 20, \quad \lambda_{4} \leq \lambda_{5} + 20.
\]

Hence, we have

\[
\lambda_{0} \approx \lambda_{1}, \quad \lambda_{2} \approx \lambda_{3}, \quad \lambda_{4} \approx \lambda_{5}.
\]

We further have for the cases (Ord8) and (Ord10),

\[
\lambda_{0} \approx \lambda_{1} \approx \lambda_{2} \approx \lambda_{3} \approx \lambda_{4} \approx \lambda_{5}.
\]

For the case (Ord7),

\[
\lambda_{2} \approx \lambda_{3} \approx \lambda_{4} \approx \lambda_{5}.
\]

For the case (Ord9),

\[
\lambda_{0} \approx \lambda_{1} \approx \lambda_{2} \approx \lambda_{3}.
\]

If \( \lambda_{0} \approx \lambda_{1} \), the summations on both \( \lambda_{0} \) and \( \lambda_{1} \) are the summation on \( \lambda_{0} \) together with a finite summation on \( \lambda_{1} \). So, the proof in the case (Ord6) is easier than that of the case (Ord1). The details of the proof are omitted. Up to now, we have finished the proof of Step 1.

**Step 2.** We assume that \( |\lambda_{0}| \) is the second largest one in \( |\lambda_{0}|, \ldots, |\lambda_{5}| \). We can assume, without loss of generality that \( \lambda_{0} \gg 1 \). There exists \( i \in \{1, \ldots, 5\} \) such that \( |\lambda_{i}| = \max_{0 \leq k \leq 5} |\lambda_{k}| \). First, we point that this case is quite similar to that of \( \lambda_{0} \) to be the largest one as in Step 1. Similar to Lemma 6.2, we have

**Lemma 6.3.** Let \( 0 < c_{k} < C_{k}, \ k = 1, 2 \). For \( w = (\bar{v}, u, \bar{u}, u, u) \), we write

\[
\mathcal{L}_{2k}(w) = \sum_{\lambda_{0}>0} (\lambda_{0})^{1/2} \int_{[0,T]} \int_{\mathbb{R}} v_{\lambda_{0}} \prod_{k=1}^{2} w_{[c_{k}\lambda_{0}, C_{k}\lambda_{0}]}^{(k)} \prod_{k=3}^{5} w_{[-C_{\lambda_{0}}, C_{\lambda_{0}}]}^{(k)}(x,t)dxdt,
\]

(6.57)

Let \( p \in [4, \infty) \), \( 0 < T < 1 \), \( 0 < \varepsilon \ll 1 \). We have

\[
|\mathcal{L}_{2k}(w)| \lesssim T^{5} \|v\|_{Y_{p',\Delta}^{0}}^{5} \|u\|_{X_{p,\Delta}^{1/2}}^{5}.
\]

(6.58)

Substituting \( w_{[c_{k}\lambda_{0}, C_{k}\lambda_{0}]}^{(k)} \) by \( w_{[-C_{k}\lambda_{0}, -c_{k}\lambda_{0}]}^{(k)} \) for \( k = 1 \) or \( k = 2 \) in (6.57), (6.58) also holds.

The proof of Lemma 6.3 is almost the same as that of Lemma 6.2 and we omit it. If \( |\lambda_{i}| \) is the largest one in \( |\lambda_{0}|, \ldots, |\lambda_{5}| \), we have from the frequency constraint condition

\[
\lambda_{0} \leq |\lambda_{i}| \leq 5\lambda_{0} + 10 \leq 20\lambda_{0}.
\]

By Lemma 6.3 we can assume that \( \lambda_{k} \in [-c_{k}\lambda_{0}, C_{k}\lambda_{0}] \) if \( k \in \{1, \ldots, 5\} \setminus \{i\} \). Namely, it suffice to consider the case that \( \lambda_{i} \) is higher frequency and the other \( \lambda_{k} \ (k \neq i) \) are lower frequency. By symmetry we can further assume that

\[
\lambda_{2} \geq \lambda_{4}, \quad \lambda_{1} \geq \lambda_{3} \geq \lambda_{5}.
\]
Case 1. We consider the case $|\lambda_1| \vee |\lambda_3| \vee |\lambda_5| = \max_{0 \leq k \leq 5} |\lambda_k|$. It follows that $|\lambda_1| \vee |\lambda_5| = \max_{0 \leq k \leq 5} |\lambda_k|$. We further claim that $|\lambda_1| = \max_{0 \leq k \leq 5} |\lambda_k|$. If not, then $-\lambda_5 = \max_{0 \leq k \leq 5} |\lambda_k|$, which contradicts the frequency constraint condition. It is easy to see that $\lambda_1 > 0$. Hence, we have

$$\lambda_1 \geq \lambda_0 \geq \lambda_k, \quad k = 2, ..., 5.$$  

The possible cases are the following

$$\begin{align*}
\lambda_1 &\geq \lambda_0 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \lambda_5, & (2\text{Ord1}) \\
\lambda_1 &\geq \lambda_0 \geq \lambda_2 \geq \lambda_3 \geq \lambda_5 \geq \lambda_4, & (2\text{Ord2}) \\
\lambda_1 &\geq \lambda_0 \geq \lambda_2 \geq \lambda_4 \geq \lambda_3 \geq \lambda_5, & (2\text{Ord3}) \\
\lambda_1 &\geq \lambda_0 \geq \lambda_3 \geq \lambda_2 \geq \lambda_4 \geq \lambda_5, & (2\text{Ord4}) \\
\lambda_1 &\geq \lambda_0 \geq \lambda_3 \geq \lambda_2 \geq \lambda_5 \geq \lambda_4, & (2\text{Ord5}) \\
\lambda_1 &\geq \lambda_0 \geq \lambda_3 \geq \lambda_5 \geq \lambda_2 \geq \lambda_4. & (2\text{Ord6}) 
\end{align*}$$

We consider Case (2Ord1). Similar to the Step 1, we have the 5 subcases as in Table 14. As in Table 14, we denote by $(\lambda_k) \in 2\text{llll}$ that $\lambda_0, ..., \lambda_5$ satisfy (2Ord1)

| Case 2Ord1 : $\lambda_1$ is maximal |
|------------------|--------------|-----------------|-------------------|---------------------|
| $\lambda_1 \in$  | $\lambda_2 \in$ | $\lambda_3 \in$ | $\lambda_4 \in$ | $\lambda_5 \in$ |
| $2\text{llll}$   | $[\lambda_0, 2\lambda_0]$ | $l$             | $l$               | $l$                 |
| $2\text{llll}_-$ | $[\lambda_0, 2\lambda_0]$ | $l$             | $l$               | $l$                 |
| $2\text{llll}_+$  | $[\lambda_0, 2\lambda_0]$ | $l$             | $l$               | $l$                 |
| $2\text{llll}_-+$ | $[\lambda_0, 2\lambda_0]$ | $l$             | $l$               | $l$                 |
| $2\text{llll}_--+$ | $[\lambda_0, 2\lambda_0]$ | $l$             | $l$               | $l$                 |

Table 14. $\lambda_1 \geq \lambda_0 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \lambda_5$, only one higher frequency in $\lambda_1, ..., \lambda_5$.

and $\lambda_1 \in [\lambda_0, 2\lambda_0]$, $\lambda_2, ..., \lambda_5 \in l$. We consider the decomposition of $\lambda_k$:

$$\lambda_1 \in [\lambda_0, 2\lambda_0] = \bigcup_{j_1 \geq 0} (\lambda_0 + I_{j_1}); \quad \lambda_k \in [0, c\lambda_0] = \bigcup_{j_k \geq 0} I_{j_k}, \quad k = 2, ..., 5.$$  

We have the following constraint conditions on $j_1, ..., j_5$:

$$j_1 \leq j_3 + C, \quad j_2 \geq j_3 \geq j_4 \geq j_5.$$  

Let us write

$$L_{2\text{llll}}(w) = \sum_{(\lambda_k) \in 2\text{llll}} \langle \lambda_0 \rangle^{1/2} \int_{\mathbb{R} \times [0,T]} \prod_{k=0}^5 w^{(k)}_{\lambda_k}(x,t) dx dt,$$  

(6.59)

Applying the dyadic decomposition above and Hölder’s inequality,

$$|L_{2\text{llll}}(w)| \lesssim \sum_{\lambda_0: \ j_1 \leq j_2 + C; \ j_3 \leq j_4 \leq j_5 \leq j_2} \langle \lambda_0 \rangle^{1/2} \|w_{\lambda_0} \|_{L^\infty_t} \|u_{\lambda_0 + I_{j_1}} \|_{L^4_t} \times \|u_{I_{j_3}} \|_{L^2_t} \|u_{I_{j_4}} \|_{L^4_t} \|u_{I_{j_5}} \|_{L^\infty_t},$$  

(6.60)

This estimate reduces to (6.24). Using the same way as in the estimates of (6.24), we have

$$|L_{2\text{llll}}(w)| \lesssim T^s \|v\|_{Y^0_{p', \Delta}} \|w\|_{X^1_{p, \Delta} \frac{1}{2}},$$  

(6.61)
We can use similar way as in Step 1 and case (2Ord1) to handle the cases (2Ord2)-(2Ord6) and we omit the details.

**Case 2.** \( |\lambda_2| \vee |\lambda_4| = \max_{0 \leq k \leq 5} |\lambda_k| \). By Lemma 6.3 we can assume that only one frequency \( \lambda_2 \) or \( \lambda_4 \) is localized in higher frequency intervals \([-20\lambda_0, -\lambda_0] \cup [\lambda_0, 20\lambda_0]\).

In view of the frequency constraint condition we see that \(-\lambda_4 = \max_{0 \leq k \leq 5} |\lambda_k| \) and \( \lambda_4 \in \left[ -20\lambda_0, -\lambda_0 \right] \). Hence, we have

\[
\lambda_0 \geq \lambda_k \geq \lambda_4, \quad k = 1, 2, 3, 5.
\]

The possible cases are the following

\[
\begin{align*}
\lambda_0 &\geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_5 \geq \lambda_4, & (3Ord1) \\
\lambda_0 &\geq \lambda_1 \geq \lambda_3 \geq \lambda_2 \geq \lambda_5 \geq \lambda_4, & (3Ord2) \\
\lambda_0 &\geq \lambda_1 \geq \lambda_3 \geq \lambda_5 \geq \lambda_2 \geq \lambda_4, & (3Ord3) \\
\lambda_0 &\geq \lambda_2 \geq \lambda_1 \geq \lambda_3 \geq \lambda_5 \geq \lambda_4, & (3Ord4)
\end{align*}
\]

For example, we consider the case (3Ord1). As in Table 15, we have 5 subcases.

These cases are almost the same as those in Case (Ord2): \( \lambda_4 \in h_- \) and we do not perform the details of the proof.

**Step 3.** We assume that \( |\lambda_0| \) is the third largest one in \( |\lambda_0|, ..., |\lambda_5| \). We can assume, without loss of generality that \( \lambda_0 \geq 0 \). There exists a rearrangement of \( 1, ..., 5 \), which is denoted by \( \pi^1, ..., \pi^5 \) such that

\[
|\lambda_{\pi(1)}| \wedge |\lambda_{\pi(2)}| \geq \lambda_0 \geq |\lambda_{\pi(3)}| \vee |\lambda_{\pi(4)}| \vee |\lambda_{\pi(5)}|
\]

For \( \lambda_{\pi(1)}, \lambda_{\pi(2)} \), we apply the dyadic decomposition starting at \( \pm \lambda_0 \),

\[
either \lambda_{\pi(k)} \in [\lambda_0, \infty) = \bigcup_{j_{\pi(k)} \geq 0} (\lambda_0 + I_{j_{\pi(k)}}),\]

\[
or \lambda_{\pi(k)} \in (-\infty, -\lambda_0] = \bigcup_{j_{\pi(k)} \geq 0} (-\lambda_0 - I_{j_{\pi(k)}}), \quad (6.62)
\]

For \( \lambda_{\pi(3)}, \lambda_{\pi(4)}, \lambda_{\pi(5)} \), we adopt the dyadic decomposition starting at 0:

\[
\lambda_{\pi(k)} \in [0, \lambda_0] = \bigcup_{j_{\pi(k)} \geq 0} I_{j_{\pi(k)}}, \quad or \lambda_{\pi(k)} \in [-\lambda_0, 0] = \bigcup_{j_{\pi(k)} \geq 0} -I_{j_{\pi(k)}}, \quad (6.63)
\]

For example, we consider the following case which is written as \( \langle \lambda_k \rangle \in \mathbb{Z}^+_0 \).

\[
\lambda_0 > 0, \quad \lambda_{\pi(1)}, \lambda_{\pi(2)} \in [\lambda_0, \infty), \quad \lambda_{\pi(3)}, \lambda_{\pi(4)}, \lambda_{\pi(5)} \in [0, \lambda_0].
\]
Let us write
\[
\mathcal{L}^3(w) = \sum_{(\lambda_0) \in \mathbb{Z}^6_{+;3}} (\lambda_0)^{1/2} \int_{R \times [0, T]} \prod_{k=0}^5 u_{\lambda_0}^{(k)}(x,t)dxdt, \tag{6.64}
\]
Denote \( j_{\text{max}} = j_\pi(3) \vee j_\pi(4) \vee j_\pi(5), \) \( j_{\text{min}} = j_\pi(3) \wedge j_\pi(4) \wedge j_\pi(5), \) \( j_{\text{mod}} \in \{ j_\pi(3), j_\pi(4), j_\pi(5) \} \setminus \{ j_{\text{max}}, j_{\text{min}} \}. \) Applying H"older's inequality, we have
\[
|\mathcal{L}^3(w)| \lesssim \sum_{\lambda_0; j_\pi(1), j_\pi(2) \in [0, T]} (\lambda_0)^{1/2} \|v_{\lambda_0}\|_2 \langle \lambda_0 + 2^{j_\pi(1)} \rangle^{-1/2} \langle \lambda_0 + 2^{j_\pi(2)} \rangle^{-1/2} \nonumber
\]
\[
\times \|u_{j_{\text{max}}}^{(4)}\|_2 \|u_{j_{\text{mod}}}^{(5)}\|_2 \|u_{j_{\text{min}}}\|_2 \tag{6.65}
\]
In view of \( V_2^3 \subset L_1^\infty L_2^2, \) \( \|v_{\lambda_0}\|_L^\infty \lesssim \|v_{\lambda_0}\|_L^2, \) by Corollary 3.3 and Lemma 3.2, one has that
\[
|\mathcal{L}^3(w)| \lesssim T^{4\varepsilon/3} \sum_{\lambda_0; j_\pi(1), j_\pi(2) \in [0, T]} (\lambda_0)^{1/2} \|v_{\lambda_0}\|_2 \langle \lambda_0 + 2^{j_\pi(1)} \rangle^{-1/2} \langle \lambda_0 + 2^{j_\pi(2)} \rangle^{-1/2} \nonumber
\]
\[
\times 2^{j_\pi(1) + j_\pi(2)} (1/4-1/p+\varepsilon) 2^{j_{\text{max}} + j_{\text{mod}}}(1/4-1/p+\varepsilon) 2^{j_{\text{min}}}(1/2-1/p) \|u\|^5_{\mathcal{X}_{p,\Delta}^{1/2}}. \tag{6.66}
\]
Choosing \( 0 < \varepsilon \leq 1/4p, \) and making the summation on \( j_{\text{min}}, j_{\text{mod}} \) and \( j_{\text{max}} \) in order, one obtain that
\[
|\mathcal{L}^3(w)| \lesssim T^{4\varepsilon/3} \sum_{\lambda_0; j_\pi(1), j_\pi(2) \in [0, T]} (\lambda_0)^{1/2} \|v_{\lambda_0}\|_2 \langle \lambda_0 + 2^{j_\pi(1)} \rangle^{-1/2} \langle \lambda_0 + 2^{j_\pi(2)} \rangle^{-1/2} \nonumber
\]
\[
\times 2^{j_\pi(1) + j_\pi(2)} (1/4-1/p+\varepsilon) \|u\|^5_{\mathcal{X}_{p,\Delta}^{1/2}}. \tag{6.67}
\]
By splitting the summation \( \sum_{j_\pi(1)} \) into two parts \( \sum_{j_\pi(1) \leq \log_2 \lambda_0} \) and \( \sum_{j_\pi(1) > \log_2 \lambda_0}, \) one sees that
\[
|\mathcal{L}^3(w)| \lesssim T^{4\varepsilon/3} \sum_{\lambda_0} (\lambda_0)^{-2/p+2\varepsilon} \|v_{\lambda_0}\|_2 \|u\|^5_{\mathcal{X}_{p,\Delta}^{1/2}}. \tag{6.68}
\]
Noticing that \( \{ \lambda_0 \}^{-2/p+2\varepsilon} \in L^p, \) by H"older's inequality we have
\[
|\mathcal{L}^3(w)| \lesssim T^\varepsilon \|v\|^p_{Y_{p,\Delta}^{0.5}} \|u\|^5_{\mathcal{X}_{p,\Delta}^{1/2}}. \quad (0 < \varepsilon \leq 1/4p) \tag{6.69}
\]
The proof in the above is also adapted to the other cases, say \( (\lambda_0) \in \mathbb{Z}^6_{+;3}: \)
\[ \lambda_0 > 0, \quad \lambda_\pi(1) \in \{ \lambda_0, \infty \}, \quad \lambda_\pi(2) \in (\infty, -\lambda_0], \quad \lambda_\pi(3) \in [-\lambda_0, 0], \quad \lambda_\pi(4), \lambda_\pi(5) \in [0, \lambda_0]. \]
Since we did not use the frequency constraint condition, there is no essential difference between the cases \( (\lambda_0) \in \mathbb{Z}^6_{+;3} \) and \( (\lambda_0) \in \mathbb{Z}^6_{+;3}. \) We omit the details of the proof for the other cases. When \( \lambda_0 = 0, \) the proof is much easier than the case \( \lambda_0 > 0. \)

**Step 4.** We assume that \( |\lambda_0| \) is the fourth, or five largest one, or the minimal one in \( |\lambda_0|, ..., |\lambda_5|. \) We can assume, without loss of generality that \( \lambda_0 \geq 0. \) Let us consider the case \( \lambda_0 \gg 1 \) be the fourth largest one in \( |\lambda_0|, ..., |\lambda_5|. \) There exists a rearrangement of \( 1, ..., 5, \) which is denoted by \( \pi(1), ..., \pi(5) \) such that
\[
|\lambda_{\pi(1)}| \wedge |\lambda_{\pi(2)}| \wedge |\lambda_{\pi(3)}| \geq \lambda_0 \geq |\lambda_{\pi(4)}| \vee |\lambda_{\pi(5)}|
\]
For \( \lambda_{\pi(1)}, ..., \lambda_{\pi(3)}, \) we apply the dyadic decomposition (6.62) starting at \( \pm \lambda_0. \) For \( \lambda_{\pi(4)}, \lambda_{\pi(5)}, \) we adopt the dyadic decomposition (6.63) starting at 0. Then we can
use a similar way as in Step 3 to obtain the result, as desired. The other cases can be shown along this line. Up to now we have finished the proof of Lemma 6.1.

**Remark on Lemma 6.1.** Since our main goal is to show the subcritical cases $p \gg 2$ which is near the critical case $p = \infty$, the case $2 < p < 4$ is not involved in Lemma 6.1. However, it is easier than that of $p \gg 2$. Indeed, for instance, taking $p = 4$ in (6.15), we have

$$|\mathcal{L}_{h_{a_+,a_+}}(u)| \lesssim T^{4\varepsilon/3} \sum_{\lambda_0} \langle \lambda_0 \rangle^{-1/2+2\varepsilon} \|v_{\lambda_0}\|_{Y_{2+\Delta}^0} \|u\|_{X_{1+\Delta}^{5}}^5.$$  

Using the embedding $X_{p,\Delta}^1 \subset X_{4,\Delta}^{1/2}$ for $p < 4$, we have the desired estimates.

**7. Proof of Theorem 1.1.** We consider the mapping

$$\mathcal{F}: u(t) \to e^{i\|u\|} u_0 - A \left( i u \partial_x \bar{u} + \frac{|\mu|^2}{2} |u|^4 u \right)$$

in the space

$$\mathcal{D} = \left\{ u \in X_{q,\Delta}^{1/2}([0, T]) : \|u\|_{X_{q,\Delta}^{1/2}([0, T])} \leq M \right\}.$$  

We can assume without loss of generality that $2 \ll q < \infty$. By the definition of $X_{q,\Delta}^{1/2}([0, T])$, the trilinear and the quintic linear estimates (5.1) and (6.5), we have

$$\|\mathcal{F} u\|_{X_{q,\Delta}^{1/2}([0, T])} \lesssim \|u_0\|_{M_{1,q}^{1/2}} + T^c \|u\|_{X_{q,\Delta}^{1/2}([0, T])}^3 + T^c \|u\|_{X_{q,\Delta}^{1/2}([0, T])}^5.$$  

(7.3)

By choosing $M = 2C\|u_0\|_{M_{1,q}^{1/2}}$ and $T \ll 1$, we can show that $\mathcal{F}: \mathcal{D} \to \mathcal{D}$ is a contraction mapping, which has a fixed point in $\mathcal{D}$ to solve (2.11). The left part is standard and we omit the details.

**Appendix. 1. On the critical spaces.** We have the embeddings

$$H_{q,\Delta}^{1/2} \subset M_{2,q}^{1/2}. $$

(7.4)

**Proof.** First, we prove the first inclusion. By definition,

$$\|f\|_{M_{2,q}^{1/2}} \equiv \left( \sum_{k \in \mathbb{Z}} \langle k \rangle^{aq} \|\chi_{|k-1/2,k+1/2]} \hat{f}\|_{q}^q \right)^{1/q}.$$

(7.5)

By Hölder’s inequality, we have

$$\|\chi_{|k-1/2,k+1/2]} \hat{f}\|_{2} \lesssim \|\chi_{|k-1/2,k+1/2]} \hat{f}\|_{q}$$

(7.6)

So, we have

$$\sum_{k \in \mathbb{Z}} \langle k \rangle^{aq} \|\chi_{|k-1/2,k+1/2]} \hat{f}\|_{2}^q \leq \sum_{k \in \mathbb{Z}} \langle k \rangle^{aq} \|\chi_{|k-1/2,k+1/2]} \hat{f}\|_{q}^q \leq \sum_{k \in \mathbb{Z}} \int_{|k-1/2,k+1/2]} \langle \xi \rangle^{aq} |\hat{f}(\xi)|^q \ d\xi \leq \sum_{k \in \mathbb{Z}} \int_{|k-1/2,k+1/2]} \langle \xi \rangle^{aq} |\hat{f}(\xi)|^q \ d\xi = \int_{\mathbb{R}} \langle \xi \rangle^{aq} |\hat{f}(\xi)|^q \ d\xi = \|f\|_{M_{2,q}^{1/2}}^q.$$  

(7.7)

We have (7.4), as desired.

□
2. On the scaling. Let \( u_\lambda = \lambda^{1/2} u(\lambda^2 t, \lambda x) \). On the other hand, from the scaling argument we see that for the scaling solution \( u_\lambda \) (cf. Sugimoto and Tomita [42], and Han and Wang [24]),

\[
\| u_\lambda(t, \cdot) \|_{M^{1/2}_{2,q}} \lesssim (1 \vee \lambda^{1/2})\| u \|_{M^{1/2}_{2,q}}, \quad q \geq 2.
\] (7.8)

At first glance, one may think that \( \dot{H}^{1/2}_y \subset M^{1/2}_{2,q} \) contradicts with (7.8) as \( \lambda \gg 1 \).

Indeed, for simply we only consider \( q = \infty \), from \( \dot{H}^{1/2}_1 \subset M^{1/2}_{2,\infty} \) and the scaling of \( \dot{H}^{1/2}_1 \) it follows that

\[
\| f(\varphi \cdot) \|_{M^{1/2}_{2,\infty}} \leq C\| f(\varphi \cdot) \|_{\dot{H}^{1/2}_1} \leq C\lambda^{-1/2}\| f \|_{C\lambda^{1/2}}, \quad \lambda \gg 1
\] (7.9)

If we can find some \( f \) with

\[
\| f(\varphi \cdot) \|_{M^{1/2}_{2,\infty}} \sim \| f \|_{M^{1/2}_{2,\infty}}
\] (7.10)

then we have

\[
\| f \|_{M^{1/2}_{2,\infty}} \lesssim \lambda^{-1/2}\| f \|_{\dot{H}^{1/2}_1}.
\] (7.11)

It seems that (7.11) may have a contradiction by letting \( \lambda \to \infty \). However, \( f \) in (7.10) is depending on \( \lambda \), i.e., \( f \) realizing (7.11) is also changing as \( \lambda \) becomes larger. The following example is helpful to understand (7.10) and (7.11):

\[
f = e^{ix} \mathcal{F}^{-1}(\varphi'(\varphi \cdot)), \quad \text{supp} \varphi \subset B(0, 1).
\]

We can easily check that

\[
\| f \|_{M^{1/2}_{2,\infty}} \sim \| f(\varphi \cdot) \|_{M^{1/2}_{2,\infty}} \sim \| f(\varphi \cdot) \|_{\dot{H}^{1/2}_1} \sim \lambda^{-1/2}, \quad \| f \|_{\dot{H}^{1/2}_1} \sim 1.
\] (7.12)

Strictly speaking, \( \dot{H}^{1/2}_1 \) equipped with the semi-norm \( \| \xi^{1/2} \mathcal{F} \cdot \|_{\infty} \) has the scaling invariance for the DNLS. Its inhomogeneous version \( \dot{H}^{1/2}_1 \) has no scaling property if \( \lambda \ll 1 \). To understand \( M^{1/2}_{2,\infty} \) as a critical space by the scaling argument, we can get that, for \( \lambda < 1 \), \( M^{1/2}_{2,\infty} \) preserves scaling property for DNLS. This is not surprising, since the lower frequency part of \( M^{1/2}_{2,\infty} \) is just the lower frequency part of \( L^2 \). For the higher frequency part, \( M^{1/2}_{2,\infty} \) is between \( B^0_{2,\infty} \) and \( B^1_{2,\infty} \), the scaling is a bit complicated and there are no simple scalings. The upper bound of scalings obeys the rule of \( B^1_{2,\infty} \) (even though the functions in \( M^{1/2}_{2,\infty} \) keeping the scaling of \( B^1_{2,\infty} \) are very special).

Acknowledgments. The work is supported in part by an NSFC grant 11771024. B. Wang is grateful to Professor Ozawa T. for his discussions on the gauge transform for the derivative NLS. The authors are grateful to the reviewer for his/her suggestions and comments.

REFERENCES

[1] H. Bahouri and G. Perelman, Global well-posedness for the derivative nonlinear Schrödinger equation, Preprint, arXiv:2012.01923.
[2] Á. Bényi and K. A. Okoudjou, Local well-posedness of nonlinear dispersive equations on modulation spaces, Bull. Lond. Math. Soc., 41 (2009), 549–558.
[3] Á. Bényi, K. Gröchenig, K. A. Okoudjou and L. G. Rogers, Unimodular Fourier multipliers for modulation spaces, J. Funct. Anal., 246 (2007), 366–384.
4252 SHAOMING GUO, XIANFENG REN AND BAOXIANG WANG

[4] J. Bergh and J. Löfström, *Interpolation Spaces, An Introduction*, Springer-Verlag, 1976.

[5] H. A. Biagioni and F. Linares, Ill-posedness for the derivative Schrödinger and generalized Benjamin-Ono equations, *Trans. Amer. Math. Soc.*, 353 (2001), 3649–3659.

[6] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations, *Geom. Funct. Anal.*, 3 (1993), 107–156.

[7] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. II. The KdV-equation, *Geom. Funct. Anal.*, 3 (1993), 209–262.

[8] L. Chaichenets, D. Hundertmark, P. Kunstmann and N. Pattakos, On the existence of global solutions of the one-dimensional cubic NLS for initial data in the modulation space $M^{p,q} ({\mathbb R})$, *J. Differential Equations*, 263 (2017), 4429–4441.

[9] L. Chaichenets, D. Hundertmark, P. Kunstmann and N. Pattakos, Nonlinear Schrödinger equation, differentiation by parts and modulation spaces, *J. Evol. Equ.*, 19 (2019), 803–843.

[10] J. Chen, D. Fan and L. Sun, Asymptotic estimates for unimodular Fourier multipliers on modulation spaces, *Discrete Contin. Dyn. Syst.*, 32 (2012), 467–485.

[11] M. J. Chen, B. X. Wang, S. X. Wang and M. W. Wong, On dissipative nonlinear evolutional pseudo-differential equations, *Appl. Comput. Harmon. Anal.*, 48 (2020), 182–217.

[12] A. Córdoba and C. Fefferman, Wave packets and Fourier integral operators, *Commun. Partial Differ. Equations*, 3 (1978), 979–1005.

[13] E. Cordero and F. Nicola, Metaplectic representation on Wiener amalgam spaces and applications to the Schrödinger equation, *J. Funct. Anal.*, 254 (2008), 506–534.

[14] E. Cordero and F. Nicola, Some new Strichartz estimates for the Schrödinger equation, *J. Differential Equations*, 245 (2008), 1945–1974.

[15] E. Cordero and F. Nicola, Remarks on Fourier multipliers and applications to the wave equation, *J. Math. Anal. Appl.*, 353 (2009), 583–591.

[16] H. G. Feichtinger, *Modulation Spaces on Locally Compact Abelian Group*, Technical Report, University of Vienna, 1983.

[17] K. Gröchenig, *Foundations of Time-Frequency Analysis*, Birkhäuser, 2001.

[18] A. Grünrock, An improved local well-posedness result for the modified KdV equation, *Int. Math. Res. Not.*, 2004 (2004), 3287–3308.

[19] B. Guo and Y. P. Wu, Orbital stability of solitary waves for the nonlinear derivative Schrödinger equation, *Journal of Differential Equations*, 123 (1994), 35–55.

[20] A. Grünrock, Bi- and trilinear Schrödinger estimates in one space dimension with applications to cubic NLS and DNLS, *Int. Math. Res. Not.*, 2005 (2005), 2525–2558.

[21] S. M. Guo, On the 1D cubic nonlinear Schrödinger equation in an almost critical space, *J. Fourier Anal. Appl.*, 23 (2017), 91–124.

[22] Z. Guo and Y. Wu, Global well-posedness for the derivative nonlinear Schrödinger equation in $H^{1/2}$, *Discrete and Continuous Dynamical Systems*, 37 (2017), 257–264.

[23] M. Hadac, S. Herr and H. Koch, Well-posedness and scattering for the KP-II equation in a critical space, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26 (2009), 917–941.

[24] J. S. Han and B. X. Wang, $\alpha$-Modulation spaces (I) scaling, embedding and algebraic properties, *J. Math. Soc. Japan*, 66 (2014), 1315–1373.

[25] N. Hayashi, The initial value problem for the derivative nonlinear Schrödinger equation in the energy space, *Nonlinear Anal.*, 20 (1993), 823–833.

[26] N. Hayashi and T. Ozawa, On the derivative nonlinear Schrödinger equation, *Phys. D*, 55 (1992), 14–36.

[27] N. Hayashi and T. Ozawa, Finite energy solutions of nonlinear Schrödinger equations of derivative type, *SIAM J. Math. Anal.*, 25 (1994), 1488–1503.

[28] N. Hayashi and T. Ozawa, Remarks on nonlinear Schrödinger equations in one space dimension, *Differential Integral Equations*, 7 (1994), 453–461.

[29] T. Iwabuchi, Navier-Stokes equations and nonlinear heat equations in modulation spaces with negative derivative indices, *J. Differential Equations*, 248 (2010), 1972–2002.

[30] K. Kato, M. Kobayashi and S. Ito, Representation on Schrödinger operator of a free partical via short time Fourier transform and its applications, *Tohoku Math. J.*, 64 (2012), 223–231.

[31] K. Kato, M. Kobayashi and S. Ito, Estimates on modulation spaces for Schrödinger evolution operators with quadratic and sub-quadratic potentials, *J. Funct. Anal.*, 266 (2014), 733–753.

[32] T. Kato, The global Cauchy problems for the nonlinear dispersive equations on modulation spaces, *J. Math. Anal. Appl.*, 413 (2014), 821–840.
[33] H. Koch and D. Tataru, Dispersive estimates for principally normal pseudo-differential operators, *Comm. Pure Appl. Math.*, **58** (2005), 217–284.

[34] H. Koch and D. Tataru, A priori bounds for the 1D cubic NLS in negative Sobolev spaces, *Int. Math. Res. Not.*, **2007** (2007), Art. ID rnm053, 36 pp.

[35] H. Koch and D. Tataru, Energy and local energy bounds for the 1D cubic NLS equation in $H^{1/4}$, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **29** (2012), 955–988.

[36] S. Kwon and Y. Wu, Orbital stability of solitary waves for derivative nonlinear Schrödinger equation, *Journal d’Analyse Mathématique*, **135** (2018), 473–486.

[37] W. Mio, T. Ogino, K. Minami and S. Takeda, Modified nonlinear Schrödinger for Alfvén waves propagating along the magnetic field in cold plasma, *Journal of the Physical Society of Japan*, **41** (1976), 265–271.

[38] E. Mjolhus, On the modulational instability of hydromagnetic waves parallel to the magnetic field, *Journal of Plasma Physics*, **16** (1976), 321–334.

[39] T. Oh and Y. Wang, Global well-posedness of the one-dimensional cubic nonlinear Schrödinger equation in almost critical spaces, *J. Differential Equations*, **269** (2020), 612–640.

[40] T. Ozawa and Y. Tsutsumi, Space-time estimates for null gauge forms and nonlinear Schrödinger equations, *Differential Integral Equations*, **11** (1998), 201–222.

[41] M. Ruzhansky, B. X. Wang and H. Zhang, Global well-posedness and scattering for the fourth order nonlinear Schrödinger equations with small data in modulation and Sobolev spaces, *J. Math. Pures Appl.*, **105** (2016), 31–65.

[42] M. Sugimoto and N. Tomita, The dilation property of modulation spaces and their inclusion relation with Besov spaces, *J. Funct. Anal.*, **248** (2007), 79–106.

[43] B. X. Wang, Globally well and ill posedness for non-elliptic derivative Schrödinger equations with small rough data, *J. Funct. Anal.*, **265** (2013), 3009–3052.

[44] B. X. Wang, L. J. Han and C. Y. Huang, Global smooth effects and well-posedness for the derivative nonlinear Schrödinger equation with small rough data, *Ann. Inst H. Poincare, AN*, **26** (2009), 2253–2281.

[45] B. X. Wang and C. Y. Huang, Frequency-uniform decomposition method for the generalized BO, KdV and NLS equations, *J. Differential Equations*, **239** (2007), 213–250.

[46] B. X. Wang, Z. H. Huo, C. C. Hao and Z. H. Guo, Harmonic Analysis Method for Nonlinear Evolution Equations, I, World Scientific Publishing Co., Pte. Ltd., Hackensack, NJ, 2011.

[47] B. X. Wang and H. Hudzik, The global Cauchy problem for the NLS and NLKG with small rough data, *J. Diff. Eqns.*, **232** (2007), 36–73.

[48] B. X. Wang, L. F. Zhao and B. L. Guo, Isometric decomposition operators, function spaces $E^s_{p,q}$ and applications to nonlinear evolution operators, *J. Funct. Anal.*, **233** (2006), 1–39.

Received August 2020; revised January 2021.

E-mail address: shaomingguo2018@gmail.com
E-mail address: xianfengren@pku.edu.cn
E-mail address: wbx@math.pku.edu.cn