A class of fourth-order parabolic equation with logarithmic nonlinearity

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Abstract
In this paper, we study a class of fourth-order parabolic equation with the logarithmic nonlinearity. By using the potential well method, we obtain the existence of the unique global weak solution. In addition, we also obtain results of decay and blow-up in the finite time for the weak solution.

MSC: Primary 35K35; secondary 35A01; 35B44; 35K55

Keywords: Global existence; Blow-up; Logarithmic source term

1 Introduction
In this paper, we study the following problem:

\[
\begin{align*}
&u_t + \Delta^2 u = |u|^{q-2}u \log |u|, \quad x \in \Omega, t > 0, \\
&u(x, t) = \Delta u(x, t) = 0, \quad x \in \partial \Omega, t > 0, \\
&u(x, 0) = u_0(x), \quad x \in \Omega,
\end{align*}
\]

where \( \Omega \) is a bound domain in \( \mathbb{R}^n \) with smooth boundary, \( 2 < q < 2 + \frac{4}{n} \), \( u_0(x) \in H_0^2(\Omega) \setminus \{0\} \).

Many papers have been devoted to the fourth-order parabolic equation. Qu and Zhou [1] studied the following fourth-order equation:

\[
u_t + \Delta^4 u = |u|^{p-1}u - \frac{1}{|\Omega|} \int_{\Omega} |u|^{p-1} u \, dx.
\]

Using the method of potential wells, they established a threshold result for the global existence and blow-up for the sign-changing weak solutions. Zhou [2] proved new blow-up conditions and the maximum of the blow-up time for Eq. (1.2). Li, Gao and Han [3] considered

\[
\begin{align*}
&u_t + D^4 u - (|u|^p - 2u_x)_x = |u|^{p-1}u - \frac{1}{|\Omega|} \int_{\Omega} |u|^{p-1} u \, dx, \quad x \in \Omega, \\
&D u = D^3 u = 0, \quad (x, t) \in \partial \Omega \times (0, T), \\
&u(x, 0) = u_0(x), \quad x \in \Omega.
\end{align*}
\]

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They obtained the existence, uniqueness and blow-up of solutions. Liu and Liu [4] considered the following equation:

\[
\begin{align*}
    u_t - \Delta^2 u + D\left( a(Du)D^3 u + \frac{a'(Du)}{2} (D^2 u)^2 \right) \\
    = |u|^{p-1} u - \frac{1}{|\Omega|} \int_{\Omega} |u|^{p-1} u \, dx, \\
    (x, t) \in \Omega \times (0, T).
\end{align*}
\]

They combine the potential well method, the classical Galerkin method and the energy method to give a threshold result for the global existence and non-existence of sign-changing weak solutions to the problem. The relevant equations have also been studied in [5, 6].

In this paper, we study a fourth-order parabolic equation with the logarithmic nonlinearity. The second-order parabolic equation with the logarithmic nonlinearity is diffusely studied. Chen, Luo and Liu [7] studied the heat equation with the logarithmic nonlinearity. Ji, Yin and Cao [8] established the existence of positive periodic solutions and discussed the instability of such solutions for the semilinear pseudo-parabolic equation with the logarithmic source. Nahn and Truong [9] studied the following nonlinear equation:

\[
\begin{align*}
    u_t - \Delta u_t - \text{div}(|\nabla u|^{p-2} \nabla u) = |u|^{p-2} u \log(|u|).
\end{align*}
\]

They obtained results as regards the existence or non-existence of global weak solutions. He, Gao and Wang [10] considered the following equation:

\[
\begin{align*}
    u_t - \Delta u_t - \text{div}(|\nabla u|^{p-2} \nabla u) = |u|^{q-2} u \log(|u|),
\end{align*}
\]

where \(2 < p < q < p(1 + \frac{2}{n})\), they proved the decay and the finite time blow-up for weak solutions.

In this paper, we prove the existence of the unique global weak solution of the problem (1.1) based on the potential well method. In addition, we also obtain some properties of the solutions. This paper is organized as follows: in Sect. 2, we introduce some lemmas. In Sect. 3, we mainly introduce the existence of the unique local weak solution of the problem (1.1). In Sect. 4, under some conditions, we obtain the existence of the unique global weak solution of the problem (1.1). Meanwhile, we find that the solution is decaying. In the last section, we prove the blow-up theorem.

### 2 Some lemmas

We first consider the energy functional \(J\) and Nehari functional \(I\) defined on \(H^2_0(\Omega) \setminus \{0\}\) as follows:

\[
\begin{align*}
    J(u) &= \frac{1}{2} \int_{\Omega} |\Delta u|^2 \, dx - \frac{1}{q} \int_{\Omega} |u|^q \log |u| \, dx + \frac{1}{q^2} \int_{\Omega} |u|^q \, dx, \\
    I(u) &= \int_{\Omega} |\Delta u|^2 \, dx - \int_{\Omega} |u|^q \log |u| \, dx.
\end{align*}
\]

We can see that \(J\) and \(I\) are continuous from the Gagliardo–Nirenberg multiplicative embedding inequality (see [11]).
By (2.1) and (2.2), we have

\[ J(u) = \frac{1}{q} I(u) + \left( \frac{1}{2} - \frac{1}{q} \right) \int_{\Omega} |\Delta u|^2 \, dx + \frac{1}{q^2} \int_{\Omega} |u|^q \, dx. \] (2.3)

Let \( \mathcal{N} = \{ u \in H^2_0(\Omega) \setminus \{ 0 \} : I(u) = 0 \} \). Lemma 2.1 indicates \( \mathcal{N} \) is not empty. Thus, we can define

\[ d = \inf_{u \in \mathcal{N}} J(u). \] (2.4)

It is obvious that \( d > 0 \) by (2.3), (2.4), \( 2 < q < 2 + \frac{4}{n} \) and the definition of \( \mathcal{N} \). For a fixed \( u \in H^2_0(\Omega) \setminus \{ 0 \} \), we consider the function \( j : \lambda \to J(\lambda u) \) for \( \lambda > 0 \).

**Lemma 2.1** Let \( u \in H^2_0(\Omega) \setminus \{ 0 \} \). Then the following results hold:

1. \( \lim_{\lambda \to 0^+} j(\lambda) = 0, \lim_{\lambda \to +\infty} j(\lambda) = 0 \);
2. there exists a unique \( \lambda > 0 \) such that \( j(\lambda) = 0 \);
3. \( j(\lambda) \) is increasing on \((0, \lambda)\), decreasing on \((\lambda, +\infty)\) and attains the maximum at \( \lambda \);
4. \( I(\lambda u) > 0 \) for \( 0 < \lambda < \lambda \), \( I(\lambda u) < 0 \) for \( \lambda < \lambda < +\infty \) and \( I(\lambda u) = 0 \).

**Proof** For \( u \in H^2_0(\Omega) \setminus \{ 0 \} \), by the definition of \( j \), we have

\[ j(\lambda) = J(\lambda u) = \frac{1}{2} \int_{\Omega} |\Delta (\lambda u)|^2 \, dx - \frac{1}{q} \int_{\Omega} |\lambda u|^q \log |\lambda u| \, dx + \frac{1}{q^2} \int_{\Omega} |\lambda u|^q \, dx. \]

It is obvious that (1) holds due to \( 2 < q < 2 + \frac{4}{n} \) and \( \int_{\Omega} |u|^q \, dx \neq 0 \). We have

\[ j'(\lambda) = \lambda \int_{\Omega} |\Delta u|^2 \, dx - \frac{\lambda^{q-1}}{q} \int_{\Omega} |u|^q \log |u| \, dx - \frac{\lambda^{q-1}}{q} \log \lambda \int_{\Omega} |u|^q \, dx \]

\[ = \lambda \int_{\Omega} |\Delta u|^2 \, dx - \lambda^{q-1} \int_{\Omega} |u|^q \log |u| \, dx - \lambda^{q-1} \log \lambda \int_{\Omega} |u|^q \, dx. \]

We construct a function \( \phi(\lambda) = \lambda^{-1} j'(\lambda) \), thus we obtain

\[ \phi'(\lambda) = -(q - 2) \lambda^{q-3} \int_{\Omega} |u|^q \log |u| \, dx. \]
\[-(q - 2)\lambda^{q - 3}\log\lambda \int_{\Omega} |u|^q \, dx - \lambda^{q - 3} \int_{\Omega} |u|^q \, dx \]
\[= -\lambda^{q - 3} \left((q - 2) \int_{\Omega} |u|^q \log |u| \, dx + (q - 2) \log \lambda \int_{\Omega} |u|^q \, dx + \int_{\Omega} |u|^q \, dx \right),\]

which implies that there exists a \(\lambda_1 > 0\) such that \(\psi(\lambda)\) is increasing on \((0, \lambda_1)\), decreasing on \((\lambda_1, +\infty)\). Using the Poincaré inequality and \(u \in H_0^2(\Omega) \setminus \{0\}\), we have \(0 < \int_{\Omega} |u|^2 \, dx \leq C_1 \int_{\Omega} |Du|^2 \, dx \leq C_1 C_2 \int_{\Omega} |\Delta u|^2 \, dx\), where \(C_1, C_2\) are the Poincaré constants. Since \(\lim_{\lambda \to 0^+} \psi(\lambda) = \int_{\Omega} |\Delta u|^2 \, dx > 0\), \(\lim_{\lambda \to +\infty} \psi(\lambda) = -\infty\), there exists a unique \(\tilde{\lambda} > 0\) such that \(\psi(\tilde{\lambda}) = 0\), i.e. \(j(\tilde{\lambda}) = 0\). (So (2) holds. Thus \(j'(\lambda) = \lambda \psi(\lambda) > 0\) for \(0 < \lambda < \tilde{\lambda}\) and \(j'(\lambda) < 0\) for \(\tilde{\lambda} < \lambda < +\infty\), which indicates \(j(\lambda)\) is increasing on \((0, \tilde{\lambda})\), decreasing on \((\tilde{\lambda}, +\infty)\) and attains the maximum at \(\tilde{\lambda}\). So (3) holds. From (2.2), we have

\[I(\lambda u) = \int_{\Omega} |\Delta (\lambda u)|^2 \, dx - \int_{\Omega} |\lambda u|^q \log |\lambda u| \, dx\]
\[= \lambda^2 \int_{\Omega} |\Delta u|^2 \, dx - \lambda^2 \int_{\Omega} |u|^q \log |u| \, dx - \lambda^q \log \lambda \int_{\Omega} |u|^q \, dx\]
\[= \lambda \left(\lambda \int_{\Omega} |\Delta u|^2 \, dx - \lambda^{q - 1} \int_{\Omega} |u|^q \log |u| \, dx - \lambda^{q - 1} \log \lambda \int_{\Omega} |u|^q \, dx\right)\]
\[= \lambda j'(\lambda)\].

Thus, \(I(\lambda u) > 0\) for \(0 < \lambda < \tilde{\lambda}\), \(I(\lambda u) < 0\) for \(\tilde{\lambda} < \lambda < +\infty\) and \(I(\tilde{\lambda} u) = 0\). So (4) holds. \(\square\)

**Lemma 2.2** There exists a \(u > 0\) with \(u \in \mathcal{M}\) such that \(J(u) = d\).

**Proof** By (2.4), we suppose \(\{u_k\}_{k=1}^{\infty} \subset \mathcal{M}\) is a minimizing sequence of \(J\). Since \(\{|u_k|\}_{k=1}^{\infty} \subset \mathcal{M}\) is also a minimizing sequence of \(J\), we consider the case where \(u_k > 0\) a.e. in \(\Omega\), \(k \in \mathbb{N}\) without loss of generality. Thus, \(\lim_{k \to \infty} J(u_k) = d\), (2.5)

which implies that \(\{J(u_k)\}_{k=1}^{\infty}\) is bounded, i.e. there exists a constant \(C_3 > 0\) such that \(|J(u_k)| \leq C_3\). Using (2.3), \(J(u_k) = 0\) and \(|J(u_k)| \leq C_3\), we have

\[\left(\frac{1}{2} - \frac{1}{q}\right) \int_{\Omega} |\Delta u_k|^2 \, dx + \frac{1}{q} \int_{\Omega} |u_k|^q \, dx \leq C_3.\]

(2.6)

Combining \(2 < q < 2 + \frac{4}{n}\) with (2.6), we have

\[\int_{\Omega} |\Delta u_k|^2 \, dx \leq \left(\frac{1}{2} - \frac{1}{q}\right)^{-1} C_3.\]

(2.7)

Using (2.7) and the Poincaré inequality, we have

\[\int_{\Omega} |u_k|^2 \, dx \leq C_4 \int_{\Omega} |Du_k|^2 \, dx \leq C_4 C_5 \int_{\Omega} |\Delta u_k|^2 \, dx \leq \left(\frac{1}{2} - \frac{1}{q}\right)^{-1} C_3 C_4 C_5,\]

where \(C_4, C_5\) are the Poincaré constants. The above inequality implies that \(\{u_k\}_{k=1}^{\infty}\) is bounded in \(H_0^2(\Omega)\). Let \(\mu_1 > 0\) be sufficiently small such that \(q + \mu_1 < \frac{2n}{n - 2}\). Since \(H_0^2(\Omega) \hookrightarrow\)
$L^{q+\mu_1}(\Omega)$ is a compact embedding, there exist a function $u$ and a subsequence of $\{u_k\}_{k=1}^\infty$, still denoted by $\{u_k\}_{k=1}^\infty$, such that

$$u_k \to u, \quad \text{weakly in } H^1_0(\Omega),$$

$$u_k \to u, \quad \text{strongly in } L^{q+\mu_1}(\Omega),$$

$$u_k \to u(x), \quad \text{a.e. in } \Omega.$$

Then we have $u \geq 0$ a.e. in $\Omega$. First, we prove $u \neq 0$. Using the dominated convergence theorem, we obtain

$$\int_\Omega |u|^q \log |u| \, dx = \lim_{k \to \infty} \int_\Omega |u_k|^q \log |u_k| \, dx, \quad (2.8)$$

$$\int_\Omega |u|^q \, dx = \lim_{k \to \infty} \int_\Omega |u_k|^q \, dx. \quad (2.9)$$

It follows from the weak lower semicontinuity of the $L^2$ norm that

$$\int_\Omega |\Delta u|^2 \, dx \leq \liminf_{k \to \infty} \int_\Omega |\Delta u_k|^2 \, dx. \quad (2.10)$$

Using (2.1), (2.5), (2.8), (2.9) and (2.10), we have

$$J(u) = \frac{1}{2} \int_\Omega |\Delta u|^2 \, dx - \frac{1}{q} \int_\Omega |u|^q \log |u| \, dx + \frac{1}{q^2} \int_\Omega |u|^q \, dx$$

$$\leq \liminf_{k \to \infty} \frac{1}{2} \int_\Omega |\Delta u_k|^2 \, dx - \frac{1}{q} \int_\Omega |u_k|^q \log |u_k| \, dx + \frac{1}{q^2} \int_\Omega |u_k|^q \, dx$$

$$= \liminf_{k \to \infty} \left( \frac{1}{2} \int_\Omega |\Delta u_k|^2 \, dx - \frac{1}{q} \int_\Omega |u_k|^q \log |u_k| \, dx + \frac{1}{q^2} \int_\Omega |u_k|^q \, dx \right)$$

$$= \liminf_{k \to \infty} J(u_k) = d. \quad (2.11)$$

Using (2.2), (2.8), (2.10) and $u_k \in \mathcal{N}$, we have

$$I(u) = \int_\Omega |\Delta u|^2 \, dx - \int_\Omega |u|^q \log |u| \, dx$$

$$\leq \liminf_{k \to \infty} \int_\Omega |\Delta u_k|^2 \, dx - \liminf_{k \to \infty} \int_\Omega |u_k|^q \log |u_k| \, dx$$

$$= \liminf_{k \to \infty} \left( \int_\Omega |\Delta u_k|^2 \, dx - \int_\Omega |u_k|^q \log |u_k| \, dx \right)$$

$$= \liminf_{k \to \infty} I(u_k) = 0. \quad (2.12)$$

By $u_k \in \mathcal{N}$ and using the Sobolev embedding inequality and the Poincaré inequality, we have

$$\int_\Omega |\Delta u_k|^2 \, dx = \int_\Omega |u_k|^q \log |u_k| \, dx$$

$$\leq \frac{e^{-1}}{\mu_1^2} \int_\Omega |u_k|^{q+\mu_1} \, dx$$
\[
\int_\Omega |u_k|^q \log |u_k| \, dx = \int_\Omega |\Delta u_k|^2 \, dx \geq \frac{C_8}{q}.
\]

Using (2.8) and (2.14), we have

\[
e^{-1} \int_\Omega |u|^q \, dx \geq \int_\Omega |u|^q \log |u| \, dx = \lim_{k \to \infty} \int_\Omega |u_k|^q \log |u_k| \, dx \geq C_8,
\]

which indicates \( u \neq 0 \). Next, we will study \( I(u) = 0 \). If \( I(u) < 0 \), by Lemma 2.1, there exists a \( \lambda_1 \) such that \( I(\lambda_1 u) = 0 \) and \( 0 < \lambda_1 < 1 \). Thus, \( \lambda_1 u \in \mathcal{N} \). By (2.3), (2.4), (2.9) and (2.10), we have

\[
d \leq J(\lambda_1 u) = \frac{1}{q} I(\lambda_1 u) + \left( \frac{1}{2} - \frac{1}{q} \right) \int_\Omega |\Delta (\lambda_1 u)|^2 \, dx + \frac{1}{q^2} \int_\Omega |\lambda_1 u|^q \, dx
\]

\[
= \left( \frac{1}{2} - \frac{1}{q} \right) \int_\Omega |\Delta (\lambda_1 u)|^2 \, dx + \frac{1}{q^2} \int_\Omega |\lambda_1 u|^q \, dx
\]

\[
= \left( \frac{1}{2} - \frac{1}{q} \right) \lambda_1^2 \int_\Omega |\Delta u|^2 \, dx + \frac{\lambda_1^q}{q^2} \int_\Omega |u|^q \, dx
\]

\[
\leq \left( \frac{1}{2} - \frac{1}{q} \right) \lambda_1^2 \int_\Omega |\Delta u|^2 \, dx + \frac{\lambda_1^q}{q^2} \int_\Omega |u|^q \, dx
\]

\[
= \lambda_1^2 \left[ \left( \frac{1}{2} - \frac{1}{q} \right) \int_\Omega |\Delta u|^2 \, dx + \frac{1}{q^2} \int_\Omega |u|^q \, dx \right]
\]

\[
\leq \lambda_1^2 \liminf_{k \to \infty} \left[ \left( \frac{1}{2} - \frac{1}{q} \right) \int_\Omega |\Delta u_k|^2 \, dx + \frac{1}{q^2} \int_\Omega |u_k|^q \, dx \right]
\]

\[
= \lambda_1^2 \liminf_{k \to \infty} J(u_k)
\]

\[
= \lambda_1^2 d,
\]

which indicates \( \lambda_1 \geq 1 \) by \( d > 0 \). It contradicts \( 0 < \lambda_1 < 1 \). Then, by (2.12), we have \( I(u) = 0 \). Therefore, \( u \in \mathcal{N} \). By (2.4), we have \( J(u) \geq d \). By (2.11), we have \( J(u) \leq d \). So, \( J(u) = d \). \( \square \)

**Lemma 2.3** ([9]) For any \( u \in W^{1,p}_0(\Omega), \) \( p \geq 1 \), and \( r \geq 1 \), the inequality

\[
\|u\|_q \leq C\|Du\|_p^\theta \|u\|_r^{1-\theta},
\]

is valid, where

\[
\theta = \left( \frac{1}{r} - \frac{1}{q} \right) \left( \frac{1}{n} - \frac{1}{p} + \frac{1}{r} \right)^{-1},
\]
and for $p \geq n = 1, r \leq q \leq \infty$; for $n > 1$ and $p < n, q \in [r, p^*)$ if $r \leq p^*$ and $q \in [p^*, r]$ if $r \geq p^*$; for $p = n > 1, r \leq q \leq \infty$; for $p > n > 1, r \leq q \leq \infty$.

Here, the constant $C$ depends on $n, p, q, a, n, d, r.$

Lemma 2.4 ([12]) Let $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a nonincreasing function. Assume that there is a constant $A > 0$ such that

$$\int_{s}^{+\infty} h(t) dt \leq Ah(s), \quad 0 \leq s < +\infty.$$ 

Then $h(t) \leq h(0)e^{-\frac{t}{A}}$, for all $t > 0$.

3 Local existence and uniqueness

Definition 3.1 (Weak solution) A function $u$ is a solution of problem (1.1) over $[0, T]$ if $u \in L^\infty(0, T; H^2_0(\Omega))$ with $u_t \in L^2(0, T; L^2(\Omega))$, satisfies the initial condition $u(0) = u_0(x) \in H^2_0(\Omega) \setminus \{0\}$, and

$$\int_{\Omega} u_t w dx + \int_{\Omega} \Delta u \Delta w dx = \int_{\Omega} |u|^{q-2} u \log |u| w dx, \quad (3.1)$$

for any $w \in H^2_0(\Omega)$, and for a.e. $t \in [0, T]$.

Theorem 3.1 (Local existence) Let $u_0 \in H^2_0(\Omega) \setminus \{0\}$. Then there exists a positive constant $T_0$ such that the problem (1.1) has a unique weak solution $u(x, t)$ on $\Omega \times (0, T_0)$. Furthermore, $u(x, t)$ satisfies the energy inequality

$$\int_{0}^{t} \int_{\Omega} u^2 dx ds + f(u(t)) \leq f(u_0), \quad t \in [0, T_0]. \quad (3.2)$$

Proof In the space of $H^2_0(\Omega)$, we take a basis $\{w_j\}_{j=1}^\infty$ and define the finite dimensional space

$$V_m = \text{span}\{w_1, w_2, \ldots, w_m\}.$$ 

Let $u_{0m}$ be an element of $V_m$ such that

$$u_{0m} = \sum_{j=1}^{m} a_{mj} w_j \rightarrow u_0 \quad \text{strongly in } H^2_0(\Omega), \quad (3.3)$$

as $m \rightarrow \infty$. We can find the approximate solution $u_m(x, t)$ of the problem (1.1) in the form

$$u_m(x, t) = \sum_{j=1}^{m} \alpha_{mj}(t) w_j(x), \quad (3.4)$$

where $\alpha_{mj}$ ($1 \leq j \leq m$) satisfy the ordinary differential equations

$$\int_{\Omega} u_m w_j dx + \int_{\Omega} \Delta u_m \Delta w_j dx = \int_{\Omega} |u_m|^{q-2} u_m \log |u_m| w_j dx, \quad (3.5)$$
for $i \in \{1, 2, \ldots, m\}$, with

$$\alpha_{mj}(0) = a_{mj}, \quad i \in \{1, 2, \ldots, m\}. \quad (3.6)$$

We find from Peano’s theorem that (3.5)–(3.6) has a local solution $\alpha_{mj}$, and there exists a positive $T_m > 0$ such that $\alpha_{mj} \in C^1([0, T_m])$, therefore $u_m \in C^1([0, T_m]; H_0^2(\Omega))$. Multiplying the $i$th equation in (3.5) by $\alpha_{mi}$, summing over $i$ from 1 to $m$, we have

$$\int_{\Omega} u_m u_m \, dx + \int_{\Omega} |\Delta u_m|^2 \, dx = \int_{\Omega} |u_m|^q \log |u_m| \, dx. \quad (3.7)$$

Integrating the above formula with respect to $s$ over $(0, t)$, we have

$$y_m(t) = y_m(0) + \int_0^t \int_{\Omega} |u_m|^q \log |u_m| \, dx \, ds, \quad (3.8)$$

where

$$y_m(t) = \frac{1}{2} \int_{\Omega} |u_m|^2 \, dx + \int_0^t \int_{\Omega} |\Delta u_m|^2 \, dx \, ds. \quad (3.9)$$

Choose $\mu_2$ such that $0 < \mu_2 < 2 + \frac{4}{n} - q$. Using Lemma 2.3, the Poincaré inequality and the Young inequality, we have

$$\int_{\Omega} |u_m|^q \log |u_m| \, dx \le \frac{e^{-1}}{\mu_2} \int_{\Omega} |u_m|^{q+\mu_2} \, dx \le \frac{e^{-1}}{\mu_2} C_9^q \mu_2 \|D u_m\|_{L^2} \|u_m\|^{(1-\theta)(q+\mu_2)} \le \frac{e^{-1}}{\mu_2} C_9^q \mu_2 C_{10}^{\frac{(q+\mu_2)}{2}} \|\Delta u_m\|_2 \|u_m\|^{(1-\theta)(q+\mu_2)} \le \varepsilon \|\Delta u_m\|_2^2 + \left( \frac{e^{-1}}{\mu_2} C_9^q \mu_2 C_{10}^{\frac{(q+\mu_2)}{2}} \right)^{\frac{2(1-\theta)(q+\mu_2)}{2q+\mu_2}} \|u_m\|_2^{2(1-\theta)(q+\mu_2)}, \quad (3.10)$$

where $C_9$ is the constant of Lemma 2.3, $C_{10}$ is the Poincaré constant, $0 < \varepsilon < 1$, and $\theta = n(\frac{1}{2} - \frac{1}{q+\mu_2})$. Let $\gamma = \frac{(1-\theta)(q+\mu_2)}{2q+\mu_2}$ and $C_{11} = \left( \frac{e^{-1}}{\mu_2} C_9^q \mu_2 C_{10}^{\frac{(q+\mu_2)}{2}} \right)^{\frac{2(1-\theta)(q+\mu_2)}{2q+\mu_2}}$, thus (3.10) becomes

$$\int_{\Omega} |u_m|^q \log |u_m| \, dx \le \varepsilon \int_{\Omega} |\Delta u_m|^2 \, dx + C_{11} \left( \int_{\Omega} |u_m|^2 \, dx \right)^\gamma. \quad (3.11)$$

It is easy to check $\gamma > 1$ according to $2 < q < 2 + \frac{4}{n}$. Using (3.3), (3.8), (3.9) and (3.11), we have

$$y_m(t) = y_m(0) + \int_0^t \int_{\Omega} |u_m|^q \log |u_m| \, dx \, ds \le y_m(0) + \int_0^t \left[ \varepsilon \int_{\Omega} |\Delta u_m|^2 \, dx + C_{11} \left( \int_{\Omega} |u_m|^2 \, dx \right)^\gamma \right] \, ds.$$
\[
\begin{align*}
&= \frac{1}{2} \int_{\Omega} |u_m(0)|^2 \, dx + \int_0^t \int_{\Omega} |\Delta u_m(0)|^2 \, dx \, ds \\
&\quad + \int_0^t \left[ \int_{\Omega} |\Delta u_m|^2 \, dx + C_{11} \left( \int_{\Omega} |u_m|^2 \, dx \right)^\gamma \right] ds \\
&\leq C_{12} + \int_0^t \left[ \int_{\Omega} |\Delta u_m|^2 \, dx + C_{11} \left( \int_{\Omega} |u_m|^2 \, dx \right)^\gamma \right] ds \\
&\leq C_{12} + \frac{\epsilon}{2} \int_{\Omega} |u_m|^2 \, dx + \int_0^t \int_{\Omega} |\Delta u_m|^2 \, dx \, ds \\
&\quad + C_{11} 2^{\gamma'} \int_0^t \left( \int_0^s \int_{\Omega} |\Delta u_m|^2 \, dx \, dy \right)^\gamma ds \\
&\quad + C_{11} 2^{\gamma'} \int_0^t \left( \frac{1}{2} \int_{\Omega} |u_m|^2 \, dx \right)^\gamma ds \\
&\leq C_{12} + \epsilon y_m(t) + C_{11} 2^{\gamma'} \int_0^t y_m(s)^{\gamma'} \, ds.
\end{align*}
\] (3.12)

Using \(0 < \epsilon < 1\) and (3.12),
\[
y_m(t) \leq \frac{C_{12}}{1 - \epsilon} + \frac{C_{11} 2^{\gamma'}}{1 - \epsilon} \int_0^t y_m(s)^{\gamma'} \, ds.
\] (3.13)

Using the integral inequality of Gronwall–Bellman–Bihari type and combining with (3.13), there exists \(T_0\) such that
\[
y_m(t) \leq C_{13}(T_0), \quad t \in [0, T_0],
\] (3.14)

where \(C_{13}(T_0)\) is a positive constant dependent on \(T_0\). Multiplying equation (3.5) by \(a_m^i\), summing over \(i\) from 1 to \(m\) and integrating with respect to time variable on \([0, t]\), we have
\[
\int_0^t \int_{\Omega} u_m^2 \, dx \, ds + J(u_m(t)) = J(u_m(0)), \quad \text{for all} \ t \in [0, T_0].
\] (3.15)

We find from (3.3) and the continuity of the \(f\) that there exists a constant \(C_{14} > 0\) such that
\[
J(u_m(0)) \leq C_{14}, \quad \text{for all} \ m.
\] (3.16)

Using (2.1), (3.9), (3.11), (3.14), (3.15) and (3.16), we have
\[
C_{14} \geq J(u_m(t))
\]
\[
\begin{align*}
&= \frac{1}{2} \int_{\Omega} |\Delta u_m(t)|^2 \, dx - \frac{1}{q} \int_{\Omega} |u_m(t)|^q \log |u_m(t)| \, dx + \frac{1}{q^2} \int_{\Omega} |u_m(t)|^q \, dx \\
&\geq \frac{1}{2} \int_{\Omega} |\Delta u_m(t)|^2 \, dx - \frac{1}{q} \int_{\Omega} |u_m(t)|^q \log |u_m(t)| \, dx \\
&\geq \frac{1}{2} \int_{\Omega} |\Delta u_m(t)|^2 \, dx - \frac{\epsilon}{q} \int_{\Omega} |\Delta u_m(t)|^2 \, dx - \frac{C_{11}}{q} \left( \int_{\Omega} |u_m(t)|^2 \, dx \right)^\gamma.
\end{align*}
\]
\[
\int_\Omega |\Delta u_m(t)|^2 \, dx \leq \left( \frac{1}{2} - \frac{\varepsilon}{q} \right)^{-1} \left[ C_{14} + \frac{C_{11}}{q} (2C_{13}(T_0))^\gamma \right].
\]  

which implies that

\[
\int_\Omega |\Delta u_m(t)|^2 \, dx \leq \left( \frac{1}{2} - \frac{\varepsilon}{q} \right)^{-1} \left[ C_{14} + \frac{C_{11}}{q} (2C_{13}(T_0))^\gamma \right].
\]  

By the Poincaré inequality and (3.18), we obtain

\[
\int_\Omega |u_m(t)|^2 \, dx \leq C_{15} \int_\Omega |D u_m(t)|^2 \, dx \leq C_{15} C_{16} \int_\Omega |\Delta u_m(t)|^2 \, dx
\]

\[
\leq C_{15} C_{16} \left( \frac{1}{2} - \frac{\varepsilon}{q} \right)^{-1} \left[ C_{14} + \frac{C_{11}}{q} (2C_{13}(T_0))^\gamma \right],
\]

where \(C_{15}, C_{16}\) are the Poincaré constants. We can easily obtain from the above inequality

\[
\|u_m\|_{L^\infty(0, T_0; H^2_0(\Omega))} \leq C_{17}(T_0),
\]

where \(C_{17}(T_0)\) is a positive constant dependent on \(T_0\). Using (3.15)–(3.17), we have

\[
\left( \frac{1}{2} - \frac{\varepsilon}{q} \right) \int_\Omega |\Delta u_m(t)|^2 \, dx - \frac{C_{11}}{q} (2C_{13}(T_0))^\gamma + \int_0^T \int_\Omega u_{mt}^2 \, dx \, ds \leq C_{14},
\]

which implies that

\[
\|u_m\|_{L^2(0, T_0; L^2(\Omega))} \leq C_{18}(T_0),
\]

where \(C_{18}(T_0)\) is a positive constant dependent on \(T_0\). It follows from (3.20) and (3.22) that there exist a function \(u\) and a subsequence of \(\{u_m\}_{m=1}^\infty\) still denoted \(\{u_m\}_{m=1}^\infty\) such that

\[
u_m \to u \quad \text{weakly star in } L^\infty(0, T_0; H^2_0(\Omega)),
\]

\[
u_{mt} \to u_t \quad \text{weakly in } L^2(0, T_0; L^2(\Omega)).
\]

We obtain from the Aubin–Lions–Simon lemma (see [13]) together with (3.23) and (3.24)

\[
u_m \to u \quad \text{strongly in } C(0, T_0; L^2(\Omega)).
\]

So, \(u_m \to u\) a.e. \((x, t) \in \Omega \times (0, T_0)\). This implies that

\[
|\nu_m|^{q-2}\nu_m \log |\nu_m| \to |u|^{q-2}u \log |u| \quad \text{a.e. } (x, t) \in \Omega \times (0, T_0).
\]

According to \(2 < q < 2 + \frac{4}{n}\), we can choose \(\mu_3\) such that \(1 < \frac{q(q-1+\mu_3)}{q-1} < \frac{2n}{n-2}\). Then, using the Sobolev embedding inequality and combining (3.19), we have

\[
\int_\Omega |\nu_m|^{q-2}\nu_m \log |\nu_m| \, dx
\]
satisfies the initial condition $u(0) = 0$, by (3.23), (3.24) and (3.28), we see that $u$ satisfies the initial condition $u(0) = 0$ and

\[
\int_{\Omega} u_t(t)w_0 dx + \int_{\Omega} \Delta u(t)\Delta w_0 dx = \int_{\Omega} |u(t)|^{q-2}u(t)\log|u(t)|w_0 dx,
\]

for all $w_0 \in H_0^2(\Omega)$, and for a.e. $t \in [0, T_0]$. So, $u$ is a desired solution of problem (1.1).

Next, we will study uniqueness of the solution. We obtain from (3.29) for any $v \in L^2(0, T_0; H_0^2(\Omega))$

\[
\int_{\Omega} u_t(t)v dx + \int_{\Omega} \Delta u(t)\Delta v dx = \int_{\Omega} |u(t)|^{q-2}u(t)\log|u(t)|v dx.
\]

We suppose there are two solutions $u_1$ and $u_2$. Let $w = u_1 - u_2$, thus we have $w(0) = 0$, $w \in L^2(0, T_0; H_0^2(\Omega))$ and $w_t \in L^2(0, T_0; L^2(\Omega))$. We set

\[
v(s) = \begin{cases} 
 u_1(s) - u_2(s), & s \in [0, t], \\
 0, & s \in [t, T_0].
\end{cases}
\]

From (3.30), it follows that

\[
\int_0^t \int_{\Omega} w_s w_t dx ds + \int_0^t \int_{\Omega} |\Delta w|^2 dx ds = \int_0^t \int_{\Omega} (|u_1|^{q-2}u_1 \log|u_1| - |u_2|^{q-2}u_2 \log|u_2|)w dx ds.
\]

According to $0 \leq \int_0^t \int_{\Omega} |\Delta w|^2 dx ds$, (3.31) becomes

\[
\int_0^t \int_{\Omega} w_s w_t dx ds \leq \int_0^t \int_{\Omega} (|u_1|^{q-2}u_1 \log|u_1| - |u_2|^{q-2}u_2 \log|u_2|)w dx ds.
\]

where $C_{19}$ is the embedding constant. Using (3.26), (3.27) and Lion’s lemma (see [13]), we obtain

\[
|u_m|^{q-2}u_m \log|u_m| \rightarrow |u|^{q-2}u \log|u| \quad \text{weakly* in } L^\infty(0, T_0; L^{\frac{q}{q-1}}(\Omega)).
\]

Passing to the limit in (3.5) and (3.6) as $m \to \infty$, by (3.23), (3.24) and (3.28), we see that $u$ satisfies the initial condition $u(0) = 0$ and

\[
\int_{\Omega} u_t(t)w_0 dx + \int_{\Omega} \Delta u(t)\Delta w_0 dx = \int_{\Omega} |u(t)|^{q-2}u(t)\log|u(t)|w_0 dx,
\]
We construct a function \( f: \mathbb{R}^* \to \mathbb{R} \), \( f(s) = |s|^{q-2} s \log |s| \). Thus, we find that there exists a positive constant \( C_{20} \) such that

\[ |u_1|^{q-2} u_1 \log |u_1| - |u_2|^{q-2} u_2 \log |u_2| \leq C_{20}|w|, \tag{3.33} \]

By (3.32) and (3.33),

\[ \int_0^t \int_{\Omega} w_s w \, dx \, ds \leq C_{20} \int_0^t \int_{\Omega} w^2 \, dx \, ds, \]

i.e.,

\[ \frac{1}{2} \int_{\Omega} w^2 \, dx \leq \frac{1}{2} \int_{\Omega} w(0)^2 \, dx + C_{20} \int_0^t \int_{\Omega} w^2 \, dx \, ds \leq C_{20} \int_0^t \int_{\Omega} w^2 \, dx \, ds. \tag{3.34} \]

Using Gronwall’s inequality and combining with (3.34), we have

\[ \int_{\Omega} w^2 \, dx \leq 0. \]

So, the uniqueness is derived.

Finally, we will study (3.2). Let \( \phi(t) \) is a nonnegative function which belongs to \( C([0, T_0]) \).

From (3.15), we can obtain

\[ \int_0^{T_0} \phi(t) \, dt \int_0^t \int_{\Omega} u_{m}^2 \, dx \, ds + \int_0^{T_0} J(u_m(t)) \phi(t) \, dt = \int_0^{T_0} \phi(u_m(0)) \phi(t) \, dt. \tag{3.35} \]

As \( m \to \infty \),

\[ \int_0^{T_0} \phi(u_m(0)) \phi(t) \, dt \to \int_0^{T_0} \phi(u_0) \phi(t) \, dt \]

and

\[ \int_0^{T_0} \phi(t) \, dt \int_0^t \int_{\Omega} u_{m}^2 \, dx \, ds \to \int_0^{T_0} \phi(t) \, dt \int_0^t \int_{\Omega} u_s^2 \, dx \, ds \]

hold. Since \( \int_0^{T_0} J(u_m(t)) \phi(t) \, dt \) is lower semi-continuous with respect to the weak topology of \( L^2(0, T_0; H^2_0(\Omega)) \), we know that

\[ \int_0^{T_0} J(u(t)) \phi(t) \, dt \leq \liminf_{m \to \infty} \int_0^{T_0} J(u_m(t)) \phi(t) \, dt. \]

Hence, by (3.35), it follows that

\[ \int_0^{T_0} \phi(t) \, dt \int_0^t \int_{\Omega} u_s^2 \, dx \, ds + \int_0^{T_0} J(u(t)) \phi(t) \, dt \leq \int_0^{T_0} J(u_0) \phi(t) \, dt, \]
as $m \to \infty$. $\phi(t)$ is arbitrary nonnegative function, so we have

$$
\int_0^t \int_{\Omega} u_x^2 \, dx \, ds + J(u(t)) \leq J(u_0), \quad t \in [0, T_0].
$$

4 Global existence and decay estimates

Now as in [9], we introduce the following sets: $\mathcal{W}_1 = \{ u \in H_0^2(\Omega) \setminus \{0\} : J(u) < d \}$, $\mathcal{W}_2 = \{ u \in H_0^2(\Omega) \setminus \{0\} : J(u) = d \}$, $\mathcal{W}^- = \{ u \in \mathcal{W}_2 : J(u) < 0 \}$, $\mathcal{W}^+ = \mathcal{W}_1 \cup \mathcal{W}_2$, $\mathcal{W}^+_1 = \{ u \in \mathcal{W}_1 : J(u) < 0 \}$, $\mathcal{W}^+_2 = \{ u \in \mathcal{W}_2 : J(u) < 0 \}$, $\mathcal{W}^-_1 = \mathcal{W}^-_1 \cup \mathcal{W}^-_2$.

**Definition 4.1** (Maximal existence time) Let $u(t)$ be a solution of problem (1.1). We define the maximal existence time $T_{\text{max}}$ as follows:

$$
T_{\text{max}} = \sup \{ T > 0 : u(t) \text{ exists on } [0, T] \}.
$$

Then:

(i) if $T_{\text{max}} < +\infty$, we say that $u(t)$ blows up in finite time and $T_{\text{max}}$ is the blow-up time;

(ii) if $T_{\text{max}} = +\infty$, we say that $u(t)$ is global.

**Theorem 4.1** Let $u_0 \in \mathcal{W}^+$. Then the problem of (1.1) admits a unique global weak solution such that

$$
u(t) \in \mathcal{W}^+, \quad t \in [0, \infty),$$

and

$$
\int_0^t \int_{\Omega} u_x^2 \, dx \, ds + J(u(t)) \leq J(u_0), \quad t \in [0, \infty). \quad (4.1)
$$

Furthermore, if $u_0 \in \mathcal{W}^+_1$, the solution $u(t)$ decays exponentially.

**Proof** We will consider the following two cases.

First we address the case of the initial data $u_0 \in \mathcal{W}^+_1$.

Let $\{ w_j \}_{j=1}^{\infty}$, $\{ u_{0m} \}_{m=1}^{\infty}$, and $\{ u_m \}_{m=1}^{\infty}$ be the same as those stated in the proof of the local existence in the second section. Multiplying the (3.5) by $\alpha'(m)$, summing over $i$ from 1 to $m$ and integrating with respect to variable on $[0, t]$, we have

$$
\int_0^t \int_{\Omega} u_{mx}^2 \, dx \, ds + J(u_m(t)) = J(u_0(0)), \quad t \in [0, T_{\text{max}}), \quad (4.2)
$$

where $T_{\text{max}}$ is the maximal existence time of solution $u_m(x, t)$. We will prove that $T_{\text{max}} = \infty$. By (3.3), (3.6) and the continuity of $J$, we have

$$
J(u_m(0)) \to J(u_0) \quad \text{as } m \to \infty. \quad (4.3)
$$

Using (4.2) and (4.3) and combining with $J(u_0) < d$, we have

$$
\int_0^t \int_{\Omega} u_{mx}^2 \, dx \, ds + J(u_m(t)) < d, \quad t \in [0, T_{\text{max}}), \quad (4.4)
$$
for sufficiently large $m$. Next, we will study

$$u_m(t) \in H^1_1, \quad t \in [0, T_{\max}),$$

(4.5)

for sufficiently large $m$. We assume that (4.5) does not hold and think that there exists a smallest time $t_0$ such that $u_m(t_0) \notin H^1_1$. Then, we have $u_m(t_0) \in \partial H^1_1$. So, we have

$$J(u_m(t_0)) = d,$$

(4.6)

or

$$I(u_m(t_0)) = 0.$$  

(4.7)

(4.6) contradicts with (4.4). If (4.7) holds, from (2.4) we can obtain

$$J(u_m(t_0)) \geq \inf_{u \in \mathcal{N}} J(u) = d,$$

which contradicts with (4.4). Hence, we have (4.5), i.e., $J(u_m(t)) < d$, and $I(u_m(t)) > 0$, for any $t \in [0, T_{\max})$, for sufficiently large $m$. Then, by (2.3), we have

$$d > J(u_m(t)) = \frac{1}{q} J(u_m(t)) + \left( \frac{1}{2} - \frac{1}{q} \right) \int_{\Omega} |\Delta u_m(t)|^2 \, dx + \frac{1}{q} \int_{\Omega} |u_m(t)|^q \, dx$$

$$\geq \left( \frac{1}{2} - \frac{1}{q} \right) \int_{\Omega} |\Delta u_m(t)|^2 \, dx \quad \text{(4.8)}$$

Using (4.8) and combining with the Poincaré inequality, we have

$$\int_{\Omega} |u_m(t)|^2 \, dx \leq C_{21} \int_{\Omega} |D u_m(t)|^2 \, dx \leq C_{21} C_{22} \int_{\Omega} |\Delta u_m(t)|^2 \, dx$$

$$\leq \left( \frac{1}{2} - \frac{1}{q} \right)^{-1} C_{21} C_{22} d,$$

(4.9)

where $C_{21}$ and $C_{22}$ are the Poincaré constants. By (4.4) and (4.8), we have

$$\int_{0}^{t} \int_{\Omega} u_{m}^2 \, dx \, ds + \left( \frac{1}{2} - \frac{1}{q} \right) \int_{\Omega} |\Delta u_m(t)|^2 \, dx < d.$$  

(4.10)

Equations (4.9) and (4.10) imply that $T_{\max} = \infty$. Then the rest is similar to the proof of the local existence, and we see that there exists a unique global weak solution $u(t) \in H^1_1$ of the problem (1.1), and

$$\int_{0}^{t} \int_{\Omega} u_{m}^2 \, dx \, ds + J(u(t)) \leq J(u_0), \quad t \in [0, \infty).$$

Now we address the case of the initial data $u_0 \in H^1_2$. 

First, we can choose a sequence \( \{\rho_m\}_{m=1}^\infty \subset (0, 1) \) and \( \lim_{m \to \infty} \rho_m = 1 \). Next, we consider the following problem:

\[
\begin{cases}
  u_t + \Delta^2 u = |u|^{q-2}u \log |u|, & x \in \Omega, t > 0, \\
  u(x, t) = \Delta u(x, t) = 0, & x \in \partial \Omega, t > 0, \\
  u(x, 0) = u_{0m}(x), & x \in \Omega,
\end{cases}
\]

(4.11)

where \( u_{0m} = \rho_m u_0 \). By \( I(u_0) > 0 \) and Lemma 2.1, we see that there exists a \( \overline{\lambda}_2 > 1 \). Hence, \( I(\rho_m u_0) > 0 \) and \( I(\rho_m u_0) = I(u_0) = \mu \) hold. So, we have \( u_{0m} \in \mathcal{H}_t^* \). Similar to the previous case, we see that the problem (4.11) admits that, for any positive \( I \), there exists a unique global \( u_m \) which satisfies \( u_m \in L^\infty(0, \infty; H^2_0(\Omega)) \), \( u_{m+} \in L^\infty(0, \infty; L^2(\Omega)) \), \( u_m(0) = u_{0m} = \rho_m u_0 \rightarrow u_0 \) strongly in \( H^2_0(\Omega) \), and

\[
\int_{\Omega} u_{m+} w \, dx + \int_{\Omega} \Delta u_m \Delta w \, dx = \int_{\Omega} |u_m|^{q-2} u_m \log |u_m| w \, dx,
\]

(4.12)

for any \( w \in H^2_0(\Omega) \), and for a.e. \( t \in [0, \infty) \). Moreover, we have

\[
u_m(t) \in \mathcal{H}_t^*, \quad t \in [0, \infty)
\]

and

\[
\int_0^t \int_{\Omega} u_{m+}^2 \, dx \, ds + f(u_{m+}(t)) \leq f(u_{0m}(t)) < d, \quad t \in [0, \infty).
\]

The remainder of the proof can be processed as the previous case.

Finally, we discuss the decay results.

Since \( u_0 \in \mathcal{H}_t^* \), similar to the first case, we obtain \( u(t) \in \mathcal{H}_t^* \) for any \( t \in [0, \infty) \). By (2.3) and (4.1), we obtain

\[
J(u_0) > J(u(t)) = \frac{1}{q} I(u(t)) + \left( \frac{1}{2} - \frac{1}{q} \right) \int_{\Omega} |\Delta u(t)|^2 \, dx + \frac{1}{q^2} \int_{\Omega} |u(t)|^q \, dx \\
\geq \left( \frac{1}{2} - \frac{1}{q} \right) \int_{\Omega} |\Delta u(t)|^2 \, dx + \frac{1}{q^2} \int_{\Omega} |u(t)|^q \, dx.
\]

(4.13)

By \( I(u(t)) > 0 \), (2.4) and Lemma 2.1, there exists a \( \overline{\lambda}_3 > 1 \) such that \( I(\overline{\lambda}_3 u(t)) = 0 \). We have

\[
d \leq J(\overline{\lambda}_3 u(t)) \\
= \frac{1}{q} I(\overline{\lambda}_3 u(t)) + \left( \frac{1}{2} - \frac{1}{q} \right) \int_{\Omega} |\Delta (\overline{\lambda}_3 u(t))|^2 \, dx + \frac{1}{q^2} \int_{\Omega} |\overline{\lambda}_3 u(t)|^q \, dx \\
= \left( \frac{1}{2} - \frac{1}{q} \right) \int_{\Omega} |\Delta (\overline{\lambda}_3 u(t))|^2 \, dx + \frac{1}{q^2} \int_{\Omega} |\overline{\lambda}_3 u(t)|^q \, dx \\
= \frac{1}{2} \overline{\lambda}_3^2 \int_{\Omega} |\Delta u(t)|^2 \, dx + \frac{1}{q^2} \overline{\lambda}_3^q \int_{\Omega} |u(t)|^q \, dx \\
= \overline{\lambda}_3^q \left( \frac{1}{2} - \frac{1}{q} \right) \overline{\lambda}_3^{2-q} \int_{\Omega} |\Delta u(t)|^2 \, dx + \frac{1}{q^2} \int_{\Omega} |u(t)|^q \, dx.
\]
\[
\leq \bar{\lambda}_3^q \left( \frac{1}{2} - \frac{1}{q} \right) \int_\Omega |\Delta u(t)|^2 \, dx + \frac{1}{q^2} \int_\Omega |u(t)|^q \, dx. \tag{4.14}
\]

Using (4.13) and (4.14), we have
\[
d \leq \bar{\lambda}_3^q J(u_0),
\]
which implies that
\[
\bar{\lambda}_3 \geq \left( \frac{d}{J(u_0)} \right)^{\frac{1}{q}}. \tag{4.15}
\]

It follows from (2.2) that
\[
0 = I(\bar{\lambda}_3 u(t)) = \int_\Omega |\Delta (\bar{\lambda}_3 u(t))|^2 \, dx - \int_\Omega |\bar{\lambda}_3 u(t)|^q \log |\bar{\lambda}_3 u(t)| \, dx
\]
\[= \bar{\lambda}_3^2 \int_\Omega |\Delta u(t)|^2 \, dx - \bar{\lambda}_3^q \log \bar{\lambda}_3 \int_\Omega |u(t)|^q \, dx
\]
\[-\bar{\lambda}_3^q \int_\Omega |u(t)|^q \log |u(t)| \, dx
\]
\[= \bar{\lambda}_3^q I(u(t)) + \bar{\lambda}_3^2 \int_\Omega |\Delta u(t)|^2 \, dx - \bar{\lambda}_3^q \int_\Omega |\Delta u(t)|^2 \, dx
\]
\[-\bar{\lambda}_3^q \log \bar{\lambda}_3 \int_\Omega |u(t)|^q \, dx
\]
\[= \bar{\lambda}_3^q I(u(t)) + (\bar{\lambda}_3^q - \bar{\lambda}_3^2) \int_\Omega |\Delta u(t)|^2 \, dx
\]
\[-\bar{\lambda}_3^q \log \bar{\lambda}_3 \int_\Omega |u(t)|^q \, dx. \tag{4.16}
\]

Using (4.15) and (4.16), we have
\[
\bar{\lambda}_3^q I(u(t)) \geq (\bar{\lambda}_3^q - \bar{\lambda}_3^2) \int_\Omega |\Delta u(t)|^2 \, dx + \bar{\lambda}_3^q \log \bar{\lambda}_3 \int_\Omega |u(t)|^q \, dx
\]
\[\geq (\bar{\lambda}_3^q - \bar{\lambda}_3^2) \int_\Omega |\Delta u(t)|^2 \, dx,
\]
which implies that
\[
I(u(t)) \geq (1 - \bar{\lambda}_3^{2-q}) \int_\Omega |\Delta u(t)|^2 \, dx. \tag{4.17}
\]

It follows from (4.15) and (4.17) that
\[
I(u(t)) \geq (1 - \bar{\lambda}_3^{2-q}) \int_\Omega |\Delta u(t)|^2 \, dx
\]
\[\geq \left[ 1 - \left( \frac{d}{J(u_0)} \right)^{\frac{1}{q^2 - 1}} \right] \int_\Omega |\Delta u(t)|^2 \, dx
\]
\[\geq C_{23} \left[ 1 - \left( \frac{d}{J(u_0)} \right)^{\frac{1}{q^2 - 1}} \right] \int_\Omega |u(t)|^q \, dx
\]
\begin{align}
\geq C_{25}^{-1} C_{24}^{-1} \left[ 1 - \left( \frac{d}{J(u_0)} \right)^{\frac{3}{2} - 1} \right] \int_{\Omega} |u(t)|^2 \, dx,
\end{align}

where $C_{23}$ and $C_{24}$ are the Poincaré inequality constants. Hence, by (4.18), we obtain

\begin{align}
I(u(t)) \geq & \frac{1}{3} \left[ 1 - \left( \frac{d}{J(u_0)} \right)^{\frac{3}{2} - 1} \right] \int_{\Omega} \Delta u(t)^2 \, dx \\
+ & \frac{1}{3} C_{23}^{-1} \left[ 1 - \left( \frac{d}{J(u_0)} \right)^{\frac{3}{2} - 1} \right] \int_{\Omega} |Du(t)|^2 \, dx \\
+ & \frac{1}{3} C_{23}^{-1} C_{24}^{-1} \left[ 1 - \left( \frac{d}{J(u_0)} \right)^{\frac{3}{2} - 1} \right] \int_{\Omega} |u(t)|^2 \, dx \\
\geq & C_{25} \left( \int_{\Omega} |\Delta u(t)|^2 \, dx + \int_{\Omega} |Du(t)|^2 \, dx + \int_{\Omega} |u(t)|^2 \, dx \right) \\
= & C_{25} \|u(t)\|^2_{H_0^2(\Omega)},
\end{align}

where

\begin{align*}
C_{25} = \max \left\{ \frac{1}{3} \left[ 1 - \left( \frac{d}{J(u_0)} \right)^{\frac{3}{2} - 1} \right], C_{21}^{-1} \left[ 1 - \left( \frac{d}{J(u_0)} \right)^{\frac{3}{2} - 1} \right], \frac{C_{21}^{-1} C_{22}^{-1}}{3} \left[ 1 - \left( \frac{d}{J(u_0)} \right)^{\frac{3}{2} - 1} \right] \right\}.
\end{align*}

Integrating the $I(u(s))$ with respect to $s$ over $(t, T)$ and using the embedding $H_0^2(\Omega) \hookrightarrow L^2(\Omega)$, we obtain

\begin{align}
\int_t^T I(u(s)) \, ds &= - \int_t^T \int_{\Omega} u(s)u(s) \, dx \, ds = - \frac{1}{2} \int_{\Omega} u(T)^2 \, dx + \frac{1}{2} \int_{\Omega} u(t)^2 \, dx \\
\leq & \frac{1}{2} \int_{\Omega} u(t)^2 \, dx \\
\leq & \frac{1}{2} C_{26} \|u(t)\|^2_{H_0^2(\Omega)},
\end{align}

where $C_{26}$ is the embedding constant. From (4.19) and (4.20), we have

\begin{align}
\int_t^T C_{25} \|u(t)\|^2_{H_0^2(\Omega)} \, ds \leq & \frac{1}{2} C_{26} \|u(t)\|^2_{H_0^2(\Omega)}, \quad \text{for all } t \in [0, T].
\end{align}

Let $T \to \infty$ in (4.21), we can get

\begin{align}
\int_t^\infty \|u(t)\|^2_{H_0^2(\Omega)} \, ds \leq & \frac{1}{2} C_{25}^{-1} C_{26} \|u(t)\|^2_{H_0^2(\Omega)}.
\end{align}

From Lemma 2.4, we have

\begin{align*}
\|u(t)\|^2_{H_0^2(\Omega)} \leq \|u(0)\|^2_{H_0^2(\Omega)} e^{1 - \frac{2t}{C_{25} C_{26}}}, \quad t \in [0, \infty).
\end{align*}

The above inequality implies that the solution $u(t)$ decays exponentially. \(\square\)
5 Blow up

Theorem 5.1 If $u_0 \in \mathcal{W}_1^-$, the unique local weak solution $u(t)$ of the problem (1.1) blows up in finite time, i.e., there exists a $T_*>0$ such that

$$\lim_{t \to T_*} \int_{\Omega} |u(t)|^2 \, dx = \infty.$$ 

Proof Since $u_0 \in \mathcal{W}_1^-$, it follows from the local existence that there exists a unique local weak solution $u(t)$ of the problem (1.1) such that

$$\int_0^t \int_{\Omega} u^2 \, dx \, ds + J(u(t)) \leq J(u_0) < d, \quad t \in [0, T_{\max}]. \quad (5.1)$$

Next, we prove $u(t) \in \mathcal{W}_1^-$ for $t \in [0, T_{\max}]$. We assume $u(t)$ leaves $\mathcal{W}_1^+$ at time $t = t_1$, then there exists a sequence $\{t_n\}$ such that $I(u(t_n)) \leq 0$ as $t_n \to t_1$. It follows from lower semi-continuity of $L^2$ norm that

$$I(u(t_1)) \leq \liminf_{n \to \infty} I(u(t_n)) \leq 0. \quad (5.2)$$

We have $I(u(t_1)) = 0$ according to $u(t_1) \notin \mathcal{W}_1^+$. By (2.4) and (5.1), we have

$$d = \inf_{u \in \mathcal{W}} J(u) < J(u(t_1)),$$

which is a contradiction. So, $u(t) \in \mathcal{W}_1^-$ for $t \in [0, T_{\max}]$. Next, we will study that $u(t)$ blows up in finite time by contradiction. Thus, we assume $u(t)$ is global. We contract a function $\Phi : [0, \infty) \to \mathbb{R}^+$, and

$$\Phi(t) = \int_0^t \int_{\Omega} u^2 \, dx \, ds. \quad (5.3)$$

We can easily obtain

$$\Phi'(t) = \int_{\Omega} u^2 \, dx. \quad (5.4)$$

By (2.2) and (5.4), we have

$$\Phi''(t) = 2 \int_{\Omega} uu_t \, dx = 2 \int_{\Omega} |u|^{q-2} u \log |u| \, dx - 2 \int_{\Omega} |\Delta u|^2 \, dx = -2I(u). \quad (5.5)$$

From $u(t) \in \mathcal{W}_1^-$ and (5.5), we can obtain

$$\Phi''(t) > 0. \quad (5.6)$$

Thus, it follows from $u_0 \in \mathcal{W}_1^-$ and (5.4) that

$$\Phi'(t) \geq \Phi'(0) = \int_{\Omega} u_0^2 \, dx > 0. \quad (5.7)$$
Using the Hölder inequality and combining (5.5), we have

\[
\frac{1}{4} (\Phi'(t) - \Phi'(0))^2 = \frac{1}{4} \left( \int_0^t \Phi''(s) ds \right)^2 = \left( \int_0^t \int_{\Omega} uu_s dx ds \right)^2 \\
\leq \int_0^t \int_{\Omega} u^2 dx ds \int_0^t \int_{\Omega} u_s^2 dx ds.
\]

(5.8)

By (2.3) and (5.5), we have

\[
\Phi''(t) = -2qJ(u) + 2q \left( \frac{1}{2} - \frac{1}{q} \right) \int_{\Omega} |\Delta u|^2 dx + \frac{2}{q} \int_{\Omega} |u|^q dx \\
= -2qJ(u(t)) + 2q \int_0^t \int_{\Omega} u_s dx ds \\
+ 2q \left( \frac{1}{2} - \frac{1}{q} \right) \int_{\Omega} |\Delta u|^2 dx + \frac{2}{q} \int_{\Omega} |u|^q dx.
\]

(5.9)

Since \( u(t) \in H^{-1} \), \( I(u(t)) < 0 \). By Lemma 2.1, there exists a \( \overline{\lambda}_4 \), \( 0 < \overline{\lambda}_4 < 1 \) such that \( I(\overline{\lambda}_4 u(t)) = 0 \). It follows from (2.3) and (2.4) that

\[
d = \inf_{u \in \mathcal{H}} f(u) \leq f(\overline{\lambda}_4 u(t)) \\
= \frac{1}{q} \int_{\Omega} |\Delta u|^2 dx + \frac{1}{q^2} \int_{\Omega} |\overline{\lambda}_4 u(t)|^q dx \\
= \left( \frac{1}{2} - \frac{1}{q} \right) \int_{\Omega} |\Delta u(t)|^2 dx + \frac{1}{q^2} \int_{\Omega} |u(t)|^q dx \\
\leq \left( \frac{1}{2} - \frac{1}{q} \right) \int_{\Omega} |\Delta u(t)|^2 dx + \frac{1}{q^2} \int_{\Omega} |u(t)|^q dx.
\]

(5.10)

Combining (5.9) with (5.10), we have

\[
\Phi''(t) = -2qJ(u_0) + 2q \int_0^t \int_{\Omega} u_s^2 dx ds + 2q \left( \frac{1}{2} - \frac{1}{q} \right) \int_{\Omega} |\Delta u|^2 dx + \frac{2}{q} \int_{\Omega} |u|^q dx \\
= -2qJ(u_0) + 2q \int_0^t \int_{\Omega} u_s^2 dx ds \\
+ 2q \left[ \left( \frac{1}{2} - \frac{1}{q} \right) \int_{\Omega} |\Delta u(t)|^2 dx + \frac{1}{q^2} \int_{\Omega} |u(t)|^q dx \right] \\
\geq 2q(d - J(u_0)) + 2q \int_0^t \int_{\Omega} u_s^2 dx ds.
\]

(5.11)

Using (5.3), (5.8) and (5.11), we have

\[
\Phi(t)\Phi''(t) = \int_0^t \int_{\Omega} u_s^2 dx ds \Phi''(t) \\
\geq \int_0^t \int_{\Omega} u_s^2 dx ds \left[ 2q(d - J(u_0)) + 2q \int_0^t \int_{\Omega} u_s^2 dx ds \right]
\]
\[ \geq \Phi(t)2q(d - J(u_0)) + \frac{q}{2} (\Phi'(t) - \Phi'(0))^2. \] (5.12)

We fix a \( t_2 > 0 \). It follows from (5.7) that we have

\[ \Phi(t) \geq \Phi(t_2) = \int_0^{t_2} \int_{\Omega} u^2 \, dx \, ds \geq t_2 \int_{\Omega} u_0^2 \, dx > 0, \quad \text{for } t \in [t_2, \infty). \] (5.13)

Hence, by (5.12) and (5.13), we have

\[ \Phi(t)\Phi''(t) - \frac{q}{2} (\Phi'(t) - \Phi'(0))^2 \geq \Phi(t)2q(d - J(u_0)) \]
\[ \geq t_2 \int_{\Omega} u_0^2 \, dx > 0, \quad \text{for } t \in [t_2, \infty). \] (5.14)

We choose \( T > t_2 \) sufficiently large and contract a function \( \Psi(t) \) as follows:

\[ \Psi(t) = \Phi(t) + (T - t) \int_{\Omega} u_0^2 \, dx, \quad t \in [t_2, T]. \] (5.15)

From (5.13) and (5.15), we can easily see that for any \( t \in [t_2, T] \), \( \Psi(t) \geq \Phi(t) > 0 \) holds. It follows from (5.4) and (5.14) that, for any \( t \in [t_2, T] \), \( \Psi'(t) = \Phi'(t) - \Phi'(0) \) holds, thus we also have \( \Psi''(t) = \Phi''(t) > 0 \) from (5.6). Thus, we can obtain from (5.14)

\[ \Psi(t)\Psi''(t) - \frac{q}{2} \Psi'(t)^2 \geq \Phi(t)\Phi''(t) + \frac{q}{2} (\Phi'(t) - \Phi'(0))^2 \]
\[ \geq \Phi(t)2q(d - J(u_0)) \]
\[ \geq t_2 \int_{\Omega} u_0^2 \, dx > 0, \] (5.16)

for \( t \in [t_2, T] \). Let \( \chi(t) = \Psi(t)^{-\frac{q+2}{2}} \). Thus,

\[ \chi'(t) = -\frac{q-2}{2} \Psi(t)^{-\frac{q+2}{2}} \Psi'(t). \] (5.17)

From (5.16) and (5.17), we have

\[ \chi''(t) = \frac{q(q-2)}{4} \Psi(t)^{-\frac{q+2}{2}} \Psi'(t)^2 - \frac{q-2}{2} \Psi(t)^{-\frac{q+2}{2}} \Psi''(t) \]
\[ = \frac{q-2}{2} \Psi(t)^{-\frac{q+2}{2}} \left[ \frac{q}{2} \Psi'(t)^2 - \Psi'(t)\Psi''(t) \right] < 0, \] (5.18)

for \( t \in [t_2, T] \). This shows that, for any sufficiently large \( T > t_2 \), \( \chi(t) \) is a concave function in \([t_2, T]\). \( \chi(t_2) > 0 \) and \( \chi''(t_2) < 0 \), so there exists a finite time \( T_\ast > t_2 > 0 \) such that

\[ \lim_{t \to T_\ast} \chi(t) = 0, \]

which implies

\[ \lim_{t \to T_\ast} \Psi(t) = \infty. \]
Hence, we have
\[
\lim_{t \to T^-} \int_0^t \int_{\Omega} u^2 \, dx \, ds = \infty,
\]
i.e.,
\[
\lim_{t \to T^-} \int_{\Omega} u^2 \, dx = \infty.
\]
This is a contradiction to our assumption. □

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