Corrigendum to “On Maximizing a Monotone $k$-Submodular Function under a Knapsack Constraint” [Oper. Res. Lett. 50.1 (2022) 28–31]

Zhongzheng Tang$^1$  Chenhao Wang$^{2,3}$  Hau Chan$^4$

1 Beijing University of Posts and Telecommunications  
2 Beijing Normal University-Zhuhai  
3 BNU-HKBU United International College  
4 University of Nebraska-Lincoln  
tangzhongzheng@amss.ac.cn, chenhwang@bnu.edu.cn, hchan3@unl.edu

Abstract

This corrigendum provides corrections for an error in the previously published letter “On Maximizing a Monotone $k$-Submodular Function under a Knapsack Constraint” [Oper. Res. Lett. 50.1 (2022) 28–31].

The aim of this corrigendum is to correct the error in the proof of Theorem 1 of [1]. This error resulted from the incorrect statement of inequality (6). Fortunately, the corrections made in this corrigendum do not affect the statement of Theorem 1. The proposed original algorithm (Algorithm 2) still works, and its approximation ratio can even be improved from $\frac{1}{2} - \frac{1}{2e} \approx 0.316$ to 0.4.

This corrigendum is divided into two parts. We first describe how the error happens, and then present a revised proof of Theorem 1.

1 Error in Inequality (6)

Inequality (6) in [1] states that

\[
    f(S_A) \geq f(S^{t^*}) = f(Y) + g(S^{t^*}) \\
    = f(Y) + g(S^{t^*} \cup \{(a_{t^*+1}, i_{t^*+1})\}) - g(S^{t^*} \cup \{(a_{t^*+1}, i_{t^*+1})\}) + g(S^{t^*}) \\
    = f(Y) + g(S^{t^*} \cup \{(a_{t^*+1}, i_{t^*+1})\}) - f(S^{t^*} \cup \{(a_{t^*+1}, i_{t^*+1})\}) - f(S^{t^*}) \\
    \geq f(Y) + \frac{1}{2}(1 - e^{-1})g(T) - f(Y)/2 \\
    \geq \frac{1}{2}(1 - e^{-1})f(T).
\]

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However, the second last inequality may not hold, and more specifically, the inequality
\[ f(S^{t^*} \cup \{(a_{t^*+1}, i_{t^*+1})\}) - f(S^{t^*}) \leq f(Y)/2 \]
is incorrect. Since the item-index pair \((a_{t^*+1}, i_{t^*+1})\) may be not contained in the optimal solution \(T\) (though \(a_{t^*+1} \in U(T)\), the index may be inconsistent), it does not satisfy the condition of Eq. (2) in [1], and thus we cannot use Eq. (2) to bound the difference \(f(S^{t^*} \cup \{(a_{t^*+1}, i_{t^*+1})\}) - f(S^{t^*})\).

## 2 Revised proof of Theorem 1

In this section, we present a revised proof of Theorem 1 in [1]. In the previous version of Theorem 1 in [1], it is stated that Algorithm 2 has an approximation ratio \(\frac{1}{2} - \frac{1}{2e} \approx 0.316\). In the revised version, the ratio can be proven to be at least 0.4.

**Theorem 2.1.** For maximizing \(f\) under a knapsack constraint, Algorithm 2 has an approximation ratio of 0.4.

**Proof.** Let \(T = \{(e_1, d_1^T), \ldots, (e_{|T|}, d_{|T|}^T)\}\) be an optimal solution. If \(|T| = 1\), our algorithm must find it in Line 1. So we only need to consider \(|T| \geq 2\). First, we assume w.l.o.g. that the items in \(U(T) = \{e_1, \ldots, e_{|T|}\}\) are arranged in decreasing order of maximum possible marginal gain. Precisely, denote \(G_0 = \emptyset\), and for \(j = 1, \ldots, |T|\), denote
\[
G_j = G_{j-1} \cup \{(e_j, d_j)\} \text{ where } (e_j, d_j) \in \arg \max_{(e,d) \in U(T) \setminus U(G_{j-1}) \times [k]} \Delta_{e,d}(G_{j-1}).
\]
That is, \(G_{j-1} = \bigcup_{i=1}^{j-1} \{(e_i, d_i)\}\), and \((e_j, d_j)\) is the item-index pair that brings the largest marginal gain to \(G_{j-1}\). Note that this definition is independent of the cost, and is indeed obtained by a greedy procedure over items in \(U(T)\). Though all items in \(G_j\) comes from \(T\), the dimensions may be different. Further, for \(j = 0, 1, \ldots, |T|\), define
\[
T_j = (T \setminus \{(e_1, d_1^T), \ldots, (e_j, d_j^T)\}) \cup G_j.
\]
It is easy to see that \(T_0 = T, T_{|T|} = G_{|T|}\), and \(T_j\) is obtained by replacing the first \(j\) item-index pairs of \(T\) by \(G_j\).

The following claim is firstly noticed by Ward and Živný (implicitly in Theorem 5.1 [2]) and formalized by Xiao et al. [3].

**Claim 2.2.** \(f(T) \leq f(G_j) + f(T_j) \leq 2f(G_j) + \sum_{(a,d) \in T_j \setminus G_j} \Delta_{a,d}(G_j)\).

Let \(Y = G_2\). For any item \(e \in U(T) \setminus U(Y)\), any dimension \(d \in [k]\) and any set \(Z \subseteq V \setminus U(Y) \setminus \{e\} \times [k]\), we have
\[
f(Y \cup Z \cup \{(e,d)\}) - f(Y \cup Z) \leq f(G_1) - f(\emptyset) = f(G_1),
\]
where the inequality comes from the orthant submodularity and the definition of \(G_1\). Similarly, since \(G_1 \subseteq Y\), we have
\[
f(Y \cup Z \cup \{(e,d)\}) - f(Y \cup Z) \leq f(G_1 \cup \{(e,d)\}) - f(G_1) \leq f(G_2) - f(G_1).
\]
It follows from the summation of the above two inequalities that
\[ f(Y \cup Z \cup \{(e, d)\}) - f(Y \cup Z) \leq \frac{f(G_2)}{2} = \frac{f(Y)}{2}. \]  
(2)

Eq. (2) provides an upper bound on the marginal gain of bringing any item in \( U(T) \setminus U(Y) \) to any solution that contains \( Y \).

Now, we consider the iteration in which Algorithm 2 chooses fixed set \( Y = G_2 \) at the beginning of the greedy procedure, i.e. \( S^0 = Y \).

Let \( t^* + 1 \) be the first step in which the algorithm considered an item in \( U(T) \) but does not add it due to the budget constraint (implying that \( c(S_{t^*}) + c(a_{t^*+1}) > B \) and \( S_{t^*+1} = S_{t^*} \)). We can further assume that \( t^* + 1 \) is the first step \( t \) for which \( S_t = S_{t-1} \). This assumption is without loss of generality, because if it happens earlier for some \( t' < t^* + 1 \), then \( a_{t'} \) does not belong to the optimal solution \( T \), nor the approximate solution we are interested in; thus, we can remove \( a_{t'} \) from the ground set \( V \), without affecting the analysis, the optimal solution \( T \), and the approximate solution obtained in the iteration with \( S^0 = Y \).

Let \( e^* \in \arg \max_{e \in U(T_2 \setminus Y)} c(e) \) be the item of largest cost in \( T \setminus Y \), and \( (e^*, d^*) \in T \). Let \( R = T_2 \setminus Y \setminus \{(e^*, d^*)\} \) be the remainder, that is, \( T_2 = Y \cup \{(e^*, d^*)\} \cup R \). It is not hard to see that \( c(S_{t^*}) \geq c(R) + c(Y) \), because adding the item \( a_{t^*+1} \) (see Line 5 of Algorithm 2) to \( S_{t^*} \) would exceed the budget, that is, \( c(S_{t^*}) + c(a_{t^*+1}) > B \geq c(T) = c(R) + c(Y) + c(e^*) \).

Define a function \( g(S) = f(S \cup Y) - f(Y) \) for all \( S \subseteq V \setminus U(Y) \), which is also a monotone submodular function. \( g \) satisfies the restricted submodularity property: for any \( k \)-submodular function \( f \), any instance with an optimal solution \( OPT \), and any partial greedy solution \( S \), the function value \( f'(S) \) is at least \( \frac{1}{2}(1 - e^{-2 \frac{c(S)}{c(OPT)}}) \cdot f'(OPT) \). Naturally applying this conclusion to function \( g \) and solution \( R \), we obtain
\[ g(S_{t^*} \setminus Y) \geq \frac{1}{2}(1 - e^{-\frac{2c(S_{t^*} \setminus Y)}{c(R)}}) g(R) \geq \frac{1}{2}(1 - e^{-2}) g(R). \]  
(3)

Moreover, we have
\[ \frac{3}{2} f(Y) = f(Y) + \frac{1}{2} f(Y) \geq f(Y) + \left( f(Y \cup R \cup \{(e^*, d^*)\}) - f(Y \cup R) \right) \]  
(4)
\[ = f(T_2) - g(R) \]  
(5)
\[ \geq f(T) - f(Y) - g(R), \]  
(6)
where (4) comes from Eq. (2), (5) comes from the facts that \( T_2 = Y \cup R \cup \{(e^*, d^*)\} \) and \( g(R) = f(R \cup Y) - f(Y) \), and (6) comes from Claim 2.2. It immediately implies that
\[ \frac{5}{2} f(Y) \geq f(T) - g(R). \]  
(7)

Finally, combining (3) and (7), we obtain a lower bound on the output \( f(S_A) \) of our algorithm:
\[ f(S_A) \geq f(S_{t^*}) = f(Y) + g(S_{t^*} \setminus Y) \]  
\[ \geq f(Y) + \frac{1}{2}(1 - e^{-2}) g(R) \]  
(8)
\[ \geq \frac{2}{5} f(T) - g(R) + \frac{1}{2}(1 - e^{-2}) g(R) \]  
(9)
\[ \geq \frac{2}{5} \cdot f(T). \]  
(10)
Eq. (8) follows from Eq. (3), Eq. (9) follows from Eq. (7), and Eq. (10) follows from the fact that \( \frac{1}{2}(1 - e^{-2}) > \frac{2}{5} \). Therefore, the outcome \( S_A \) of Algorithm 2 is at least 0.4-approximation of the optimum.

References

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On Maximizing a Monotone \( k \)-Submodular Function under a Knapsack Constraint

Zhongzheng Tang\(^1\) Chenhao Wang\(^2,3\) Hau Chan\(^4\)

1 Beijing University of Posts and Telecommunications
2 Beijing Normal University-Zhuhai
3 BNU-HKBU United International College
4 University of Nebraska-Lincoln

tangzhongzheng@amss.ac.cn, chenhwang@bnu.edu.cn, hchan3@unl.edu

Abstract

We study the problem of maximizing a non-negative monotone \( k \)-submodular function \( f \) under a knapsack constraint, where a \( k \)-submodular function is a natural generalization of a submodular function to \( k \) dimensions. We present a deterministic \((1 - \frac{1}{2e}) \approx 0.316\)-approximation algorithm that evaluates \( f \) \( O(n^4k^3) \) times, based on the result of Sviridenko (2004) on submodular knapsack maximization.\(^1\)

1 Introduction

A \( k \)-submodular function is a generalization of submodular function, where the input consists of \( k \) disjoint subsets of the domain, instead of a single subset. The \( k \)-submodular maximization problem has been studied in the unconstrained setting \(^{15}\), under cardinality constraints \(^{10}\), and under matroid constraints \(^{12}\), because it appears in a broad range of applications (e.g., influence maximization with \( k \) kinds of topics, and sensor placement with \( k \) kinds of sensors \(^{10}\)).

Let \( V \) be a finite set. Let \((k + 1)V := \{(X_1, \ldots, X_k) \mid X_i \subseteq V \forall i \in [k], X_i \cap X_j = \emptyset \forall i \neq j\}\) be the family of \( k \) disjoint sets, where \([k] := \{1, \ldots, k\}\). A function \( f : (k + 1)V \rightarrow \mathbb{R} \) is called \( k \)-submodular \(^{7}\), if for any \( x = (X_1, \ldots, X_k) \) and \( y = (Y_1, \ldots, Y_k) \) in \((k + 1)V\), we have

\[
    f(x) + f(y) \geq f(x \sqcup y) + f(x \sqcap y),
\]

where

\[
    x \sqcup y := \left( X_1 \cup Y_1 \setminus \left( \bigcup_{i \neq 1} X_i \cup Y_i \right), \ldots, X_k \cup Y_k \setminus \left( \bigcup_{i \neq k} X_i \cup Y_i \right) \right),
\]

\[
    x \sqcap y := (X_1 \cap Y_1, \ldots, X_k \cap Y_k).
\]

\(^1\)This manuscript is published in Operations Research Letters, but there is an error in the proof of Theorem 1. We provide a corrigendum in the end of this manuscript.
Denote $x \preceq y$, if $x = (X_1, \ldots, X_k)$ and $y = (Y_1, \ldots, Y_k)$ with $X_i \subseteq Y_i$ for each $i \in [k]$. Define the marginal gain when adding item $a$ to the $i$-th dimension of $x$ to be

$$\Delta_{a,i}(x) := f(X_1, \ldots, X_{i-1}, X_i \cup \{a\}, X_{i+1}, \ldots, X_k) - f(x).$$

It is not hard to see that, a $k$-submodular function $f$ satisfies the orthant submodularity, that is,

$$\Delta_{a,i}f(x) \geq \Delta_{a,i}f(y), \quad \text{for any } x, y \in (k+1)^V \text{ with } x \preceq y, a \notin \bigcup_{j \in [k]} Y_j, i \in [k].$$

A function $f : (k + 1)^V \to \mathbb{R}$ is called monotone, if $f(x) \leq f(y)$ for any $x \preceq y$. Ward and Živný [15] shows that when monotonicity holds, $f$ is $k$-submodular if and only if it is orthant submodular.

In this note, we study the maximization problem of a non-negative monotone $k$-submodular function under a knapsack constraint, and give a deterministic $(1 - \frac{1}{e})$-approximation algorithm (see Theorem 3.1). It is an adaption to Sviridenko’s $(1 - \frac{1}{e})$-approximation algorithm for submodular knapsack maximization [14].

**Related works.** It is well known that the diminishing return property characterizes the submodular function. This property often appears in practice, and various problems can be formulated as submodular function maximization, under different constraints. Unfortunately, submodular function maximization is generally known to be NP-hard [2]. Therefore, approximation algorithms that can run in polynomial time have been extensively studied.

**Submodular knapsack.** For monotone submodular maximization under a knapsack constraint, Sviridenko [14] presents a greedy $(1 - \frac{1}{e})$-approximation algorithm with $O(n^5)$ queries, which enumerates all feasible sets of size no more than than 3 and then expands each set of size 3 greedily by the marginal density. This is the best possible approximation ratio among polynomial-time algorithms. Faster algorithms with $(1 - \frac{1}{e} - \epsilon)$-approximation exist [11], but the time is exponential to $\frac{1}{\epsilon}$.

Yaroslavtsev et al. [17] presented a Greedy+Max algorithm that is a $\frac{1}{2}$-approximation with query complexity $O(Kn)$, where $K$ is an upper bound on the number of elements in any feasible solution. Huang et al. [8] considered this problem in a streaming setting.

**$k$-submodular maximization.** One decade ago, Huber and Kolmogorov [7] proposed $k$-submodular functions, which express the submodularity on choosing $k$ disjoint sets of elements instead of a single set, and recently become a popular subject of research [3, 4, 9, 13]. For unconstrained non-monotone $k$-submodular maximization, Ward and Živný [15] proposed a max($\frac{1}{2}, \frac{1}{1+\epsilon}$)-approximation algorithm with $a = \max\{1, \sqrt{\frac{k-1}{4}}\}$. Later, Iwata et al. [8] improved the approximation ratio to $\frac{1}{2}$, which is improved to $\frac{k^2+1}{2k^2+1}$ by Oshima [11] more recently. For unconstrained monotone $k$-submodular maximization, Ward and Živný [15] proved that a greedy algorithm is $\frac{1}{2}$-approximation, and later, Iwata et al. [8] proposed a randomized $\frac{1}{2k-1}$-approximation algorithm, which is asymptotically tight.

In the constrained setting, Ohsaka and Yoshida [10] proposed a $\frac{1}{2}$-approximation algorithm for nonnegative monotone $k$-submodular maximization with a total size constraint (i.e., $\bigcup_{i \in [k]} |X_i| \leq B$ for an integer $B$) and a $\frac{1}{3}$-approximation algorithm for that with individual size constraints (i.e., $|X_i| \leq B_i$, $\forall i \in [k]$ with integers $B_i$). Sakaue [12] proposed a $\frac{1}{2}$-approximation algorithm for nonnegative monotone $k$-submodular maximization with a matroid constraint on the union of the sets. Thus, our work completes the picture by studying a knapsack constraint.
ALGORITHM 1: Greedy (without constraint)

**Input:** Set $V = \{1, 2, \ldots, n\}$, monotone $k$-submodular function $f$

**Output:** A solution $S \in \mathcal{S}$

1. $S \leftarrow \emptyset$
2. for $a = 1$ to $n$ do
3. \quad $i_a \leftarrow \arg \max_{i \in [k]} \Delta_{a,i}(S)$
4. \quad $S \leftarrow S \cup \{(a, i_a)\}$
5. end for
6. return $S$

## 2 Preliminaries

For notational ease, we identify $(k + 1)^V$ with the set

$$\mathcal{S} = \{\bigcup_{j=1}^f \{(a_j, i_j)\} \mid \forall t \in [|V|], a_j \in V \ i_j \in [k] \ \forall j \in [t], a_j \neq a_{j'} \ \forall j \neq j'\} \cup \emptyset$$

that is, any $k$-disjoint set $x = (X_1, \ldots, X_k) \in (k + 1)^V$ uniquely corresponds to an item-index pairs set $S \in \mathcal{S}$, such that $(a_j, i_j) \in S$ if and only if $a_j \in X_{i_j}$. From now on, we rewrite $f(x)$ as $f(S)$ with some abuse of notations, and thus $\Delta_{a,i}(S)$ means the marginal gain $f(S \cup \{(a, i)\}) - f(S)$. For any $S \in \mathcal{S}$, we define $U(S) := \{a \in V \mid \exists i \in [k] \text{ s.t. } (a, i) \in S\}$ to be the set of items included, and the size of $S$ is $|S| = |U(S)|$. In the remainder of this note, let $f$ be an arbitrary non-negative, monotone, $k$-submodular function. We further assume that $f(\emptyset) = 0$, which is without loss of generality because otherwise we can redefine $f(S) := f(S) - f(\emptyset)$ for all $S \in \mathcal{S}$.

We first introduce an important lemma.

**Lemma 2.1.** For any $S, S' \in \mathcal{S}$ with $S \subseteq S'$, we have

$$f(S') - f(S) \leq \sum_{(a, i) \in S \setminus S'} \Delta_{a,i}(S).$$

**Proof.** Let $t = |S'| - |S|$. Define arbitrary manner sets $\{S_j\}_{j=0}^t$ such that (1) $S_0 = S$; (2) $|S_j \setminus S_{j-1}| = 1$ for $j \in [t]$; (3) $S_t = S'$. Let $\{(a_j, i_j)\} := S_j \setminus S_{j-1}$ for $j \in [t]$. Then we have

$$f(S') - f(S) = \sum_{j=1}^t \Delta_{a_j,i_j}(S_{j-1}) \leq \sum_{j=1}^t \Delta_{a_j,i_j}(S),$$

where the inequality follows from the orthant submodularity.

The following proposition from Ward and Živný [15] says that Greedy (see Algorithm 1) is $\frac{1}{2}$-approximation for maximizing $f$ without constraint (note that Theorem 5.1 of [15] states a more general conclusion which holds for a large class of $k$-set functions). Greedy considers items in an arbitrary order, and assign each item the best index that brings the largest marginal gain.

**Proposition 2.2** ([15]). Let $T \in \mathcal{S}$ be a solution that maximizes $f$ in the unconstrained setting, and $S \in \mathcal{S}$ be the solution returned by Greedy. Then $f(T) \leq 2 \cdot f(S)$.

In the later proofs we will use the following inequality from Wolsey [16].
Lemma 2.3 (16). Let $P$ and $D$ be arbitrary positive integers, and $(\rho_i)_{i=1}^P$ be arbitrary nonnegative real values with $\rho_1 > 0$. Then

$$\frac{\sum_{i=1}^P \rho_i}{\min_{t=1,\ldots,P}(\sum_{i=1}^{t-1} \rho_i + D\rho_t)} \geq 1 - (1 - \frac{1}{D})^P \geq 1 - e^{-P/D}.$$  

3 Knapsack constraint

Given a set $V = \{1, \ldots, n\}$, nonnegative integers $c_a \in \mathbb{N}$ for all $a \in V$, and budget $B \in \mathbb{R}$, we consider the following maximization problem with a knapsack constraint,

$$\max_{S \in \mathcal{S}} \left\{ f(S) : \sum_{a \in U(S)} c_a \leq B \right\}. \quad (1)$$

Sviridenko [14] considers the special case of $k = 1$ (i.e., submodular maximization with a knapsack constraint), and presents a greedy $(1 - \frac{1}{e})$-approximation algorithm with $O(n^5)$ queries, which enumerates all feasible sets of size at most 3 and then expands each set of size 3 greedily by the marginal density. We adapted it to Algorithm 2 for problem (1) by enumerating all feasible sets of size at most 2 and then expanding each set of size 2 greedily, and prove an approximation ratio of $\frac{1}{2} - \frac{1}{2e}$. For any solution $S \in \mathcal{S}$, define $c(S)$ to be the total cost of all items in $S$.

In Line 1 of Algorithm 2 it enumerates all feasible singleton solutions, and store the currently best solution as $S_A$; it takes $O(nk)$ oracle queries. Then it considers all feasible sets of size two, and completes each such set greedily with respect to the density, subject to the knapsack constraint. There are $O(n^2k^2)$ such sets, and for each set it takes $O(n^2k)$ queries. Thus, the time complexity is $O(n^4k^3)$.

**Algorithm 2: Greedy for (1)**

1. Let $S_A \in \arg \max_{S \in \mathcal{S}} f(S)$ be a singleton solution giving the largest value.
2. for every $I \in \mathcal{S}$ of size 2 do
3.   $S^0 \leftarrow I$, $V^0 \leftarrow V \setminus U(I)$
4.   for $t$ from 1 to $n$ do
5.     Let $\theta_t = \max_{a \in V^{t-1}, i \in [k]} \frac{\Delta_a(i,(S^{t-1}))}{c_a}$, and assume that the maximum is attained on $(a_t, i_t)$
6.     if $c(S^{t-1}) + c_{a_t} \leq B$ then
7.         $S^t = S^{t-1} \cup \{(a_t, i_t)\}$
8.     else
9.         $S^t = S^{t-1}$
10.    end if
11. $V^t = V^{t-1} \setminus \{a_t\}$
12. end for
13. $S_A \leftarrow S^n$ if $f(S^n) > f(S_A)$
14. end for
15. return $S_A$

Theorem 3.1. For maximizing $f$ under a knapsack constraint, Algorithm 2 has an approximation ratio of $\frac{1}{2} - \frac{1}{2e}$, and evaluates $f$ $O(n^4k^3)$ times.
Proof. Let \( T = \{(a_1^*, i_1^*), \ldots, (a_{|T|}^*, i_{|T|}^*)\} \) be an optimal solution. If \(|T| = 1\), our algorithm must find it in Line 1. So we only need to consider \(|T| \geq 2\). We order the set \( T \) so that for any \( t = 1, \ldots, |T| \),

\[
f(T^t) = \max_{(a,i) \in T \setminus T^{t-1}} f(T^{t-1} \cup \{(a,i)\}),
\]

where \( T^t = \{(a_1^*, i_1^*), \ldots, (a_t^*, i_t^*)\} \), and \( T^0 = \emptyset \).

Let \( Y = T^2 \) be the set that consists of the first two items of \( T \). For any item \((a_j^*, i_j^*) \in T, j \geq 3\), and any set \( Z \subseteq V \setminus \{a_1^*, a_2^*, a_3^*\} \times [k] \), by the ordering of the sets in \( T \), we have

\[
\begin{align*}
  f(Y \cup Z \cup \{(a_j^*, i_j^*)\}) - f(Y \cup Z) &\leq f(T^1) - f(\emptyset) \leq f(T^1); \\
  f(Y \cup Z \cup \{(a_j^*, i_j^*)\}) - f(Y \cup Z) &\leq f(T^1 \cup \{(a_j^*, i_j^*)\}) - f(T^1) \leq f(T^2) - f(T^1).
\end{align*}
\]

It follows from the summation of the above two inequalities that

\[
f(Y \cup Z \cup \{(a_2^*, i_2^*)\}) - f(Y \cup Z) \leq f(Y)/2. \tag{2}
\]

Now, we consider the iteration in which the algorithm chooses set \( Y \) at the beginning of the greedy procedure, i.e. \( S^0 = Y \). Define a function \( g(S) = f(S) - f(Y) \) for all \( S \supseteq Y \), which is also a monotone \( k \)-submodular function.

Let \( \hat{t} + 1 \) be the first step that the algorithm does not add item \( a_{\hat{t}+1} \in U(T) \) to the current set \( U(S^{\hat{t}}) \) because its addition would exceed the budget. Thus \( S^{\hat{t}+1} = S^{\hat{t}} \). We can further assume that \( \hat{t} + 1 \) is the first step \( t \) for which \( S^t = S^{t-1} \). This assumption is without loss of generality, because if it happens earlier for some \( t' < \hat{t} + 1 \), then \( a_{t'} \) does not belong to the optimal solution \( T \), nor the approximate solution we are interested in; thus, we can remove \( a_{t'} \) from the ground set \( V \), without affecting the analysis, the optimal solution \( T \), and the approximate solution obtained in the iteration with \( S^0 = Y \).

Note that \( Y \subseteq T \cap S^t \), for any \( t = 0, \ldots, \hat{t} \). Define \( OPT_g(V') \) to be the optimal value of function \( g \) over items \( V' \subseteq V \) without constraint. We greedily construct a set \( \tilde{S} \in \mathcal{F} \) over items \( U(T) \cup U(S^t) \): starting with \( Y \subseteq \tilde{S} \), consider every item in \( U(S^t \setminus Y) \) in the same order as it is added to \( U(S^t) \) in Algorithm 2, and then consider every item in \( U(T) \setminus U(S^t) \) in an arbitrary order; when considering each item, assign the best index that brings the largest marginal gain. Clearly \( S^t \subseteq \tilde{S} \), as the indices in \( S^t \) are assigned greedily. For any \( t = 0, \ldots, \hat{t} \), we have

\[
g(T) \leq OPT_g(U(T) \cup U(S^t)) \leq 2 \cdot g(\tilde{S}) \leq 2 \left( g(S^t) + \sum_{(a,i) \in \tilde{S} \setminus S^t} (g(S^t \cup \{(a,i)\}) - g(S^t)) \right) \tag{3}
\]

\[
= 2 \left( g(S^t) + \sum_{(a,i) \in \tilde{S} \setminus S^t} (f(S^t \cup \{(a,i)\}) - f(S^t)) \right)
\]

\[
\leq 2 \left( g(S^t) + (B - c(Y)) \theta_{\hat{t}+1} \right).
\]

The second inequality follows from the fact that \( \tilde{S} \) is obtained greedily and thus achieves a 2-approximation by Proposition 2.2. The third inequality is because of Lemma 2.1. The last inequality follows from \( f(S^t \cup \{(a,i)\}) - f(S^t) \leq c_a \cdot \theta_{\hat{t}+1} \) and \( \sum_{(a,i) \in \tilde{S} \setminus S^t} c_a \leq B - c(Y) \).
Let $B_t = \sum_{\tau=1}^{t} c_\tau$ and $B_0 = 0$. Define $B' = B_{t+1}$ and $B'' = B - c(Y)$. By the definition of the item $a_{i+1}$, we have $B' > B \geq B''$. For $j = 1, \ldots, B'$, we define $\rho_j = \theta_j$ if $j = B_{t+1} + 1, \ldots, B_t$ (that is, $\rho_1 = \cdots = \rho_{B_1} = \theta_1$, $\rho_{B_1+1} = \cdots = \rho_{B_2} = \theta_2$, $\ldots$, $\rho_{B_t+1} = \cdots = \rho_{B''} = \theta_{t+1}$). Using this definition, we obtain $g(S^t) = \sum_{\tau=1}^{s-1} c_\tau \theta_\tau = \sum_{j=1}^{B_\tau} \rho_j$ for $t = 1, \ldots, \hat{t}$ and $g(S^t \cup \{a_{i+1}, i_{i+1}\}) = \sum_{\tau=1}^{\hat{t}+1} c_\tau \theta_\tau = \sum_{j=1}^{B'} \rho_j$. Then we have equalities

$$\min_{s=1, \ldots, B'} \left\{ \sum_{j=1}^{s-1} \rho_j + B'' \rho_s \right\} = \min_{t=0, \ldots, \hat{t}} \left\{ \sum_{j=1}^{B_t} \rho_j + B'' \rho_{B_{t+1}} \right\} = \min_{t=0, \ldots, \hat{t}} \left\{ g(S^t) + B'' \theta_{t+1} \right\}. \quad (4)$$

Combining (4) with (3) and Lemma 2.3 we obtain

$$\frac{g(S^t \cup \{a_{i+1}, i_{i+1}\})}{g(T)} = \frac{\sum_{j=1}^{B'} \rho_j}{g(T)} \geq \frac{\sum_{j=1}^{B'} \rho_j}{2 \cdot \min_{s=1, \ldots, B'} \left\{ \sum_{j=1}^{s-1} \rho_j + B'' \rho_s \right\}} \geq \frac{1}{2} (1 - e^{-B' / B''}) > \frac{1}{2} (1 - e^{-1}). \quad (5)$$

Finally, combining (2) and (5), we obtain a lower bound on the output $f(S_A)$ of our algorithm:

$$f(S_A) \geq f(S^\hat{t}) = f(Y) + g(S^\hat{t})$$

$$= f(Y) + g(S^\hat{t} \cup \{a_{i+1}, i_{i+1}\}) - g(S^\hat{t} \cup \{a_{i+1}, i_{i+1}\}) + g(S^\hat{t})$$

$$= f(Y) + g(S^\hat{t} \cup \{a_{i+1}, i_{i+1}\}) - (f(S^\hat{t} \cup \{a_{i+1}, i_{i+1}\}) - f(S^\hat{t}))$$

$$\geq f(Y) + \frac{1}{2} (1 - e^{-1}) g(T) - f(Y) / 2$$

$$\geq \frac{1}{2} (1 - e^{-1}) f(T). \quad (6)$$

\[\square\]

4 Discussions

We remark that the proof idea of Theorem 1 generally follows the proof of the $(1 - \frac{1}{e})$-approximation algorithm for submodular knapsack maximization in [14]. That is, first upper bound the marginal value brought by a single item in the optimal solution on the basis of $Y$ (as in (2)), then upper bound the optimal value $g(T)$ (as in (5)), and finally derive a lower bound on our solution $f(S_A)$. There are two main differences. First, we reduce the enumeration in the algorithm from subsets of size three in [14] to two. The reason we can do it is that, if the starting set $Y$ is of size $s$, then the RHS of Eq. (2) is $\frac{f(Y)}{s}$; this will be used to derive Eq. (6), which only requires $1 - \frac{1}{e} \geq \frac{1}{2} (1 - e^{-1})$. Thus, a size $s = 2$ is enough for the analysis. Second, the additional difficulty that arises in our problem is that, we can no longer obtain an upper bound on $g(T)$ straightforwardly from Lemma 2.1 because the items in $U(T) \cap U(S^t)$ may have different indices in $T$ and $S^t$. To overcome this,
we construct an intermediary $\tilde{S}$ over the items in $U(T) \cap U(S^t)$ by the Greedy algorithm, such that $g(T) \leq 2 \cdot g(\tilde{S})$ (see Eq. (3)), and then use Lemma 2.1 to upper bound the value $g(\tilde{S})$. Thus, we indirectly obtain an upper bound on $g(T)$.

One may be surprised by the fact that we get a $\frac{1}{2}(1 - \frac{1}{e})$-approximation for any $k$, whereas there is a $(1 - \frac{1}{e})$-approximation when $k = 1$ [14]. The reason for such a jump is that, the approximation ratio of the Greedy (which is used to construct $\tilde{S}$) is trivially 1 for $k = 1$, but jumps to 2 for any $k \geq 2$ (and the analysis is tight). It would be an interesting direction to get an approximation ratio that degrades smoothly as a function of $k$.

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