On the Equitable Vertex Arboricity of Graphs

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Abstract

The equitable coloring problem, introduced by Meyer in 1973, has received considerable attention and research. Recently, Wu, Zhang and Li introduced the concept of equitable \((t, k)\)-tree-coloring, which can be regarded as a generalization of proper equitable \(t\)-coloring. The \textit{strong equitable vertex \(k\)-arboricity} of \(G\), denoted by \(v_{\text{eq}}^k(G)\), is the smallest integer \(t\) such that \(G\) has an equitable \((t', k)\)-tree-coloring for every \(t' \geq t\). The exact value of strong equitable vertex \(k\)-arboricity of complete equipartition bipartite graph \(K_{n,n}\) was studied by Wu, Zhang and Li. In this paper, we first get a sharp upper bound of strong equitable vertex arboricity of complete bipartite graph \(K_{n,n+\ell}\) \((1 \leq \ell \leq n)\), that is, \(v_{\text{eq}}^k(K_{n,n+\ell}) \leq 2\left\lfloor \frac{n+\ell+1}{3} \right\rfloor\). Next, we obtain a sufficient and necessary condition on an equitable \((q, \infty)\)-tree coloring of a complete equipartition tripartite graph, and study the strong equitable vertex arboricity of forests. For a simple graph \(G\) of order \(n\), we show that \(1 \leq v_{\text{eq}}^k(G) \leq \left\lfloor \frac{n}{2} \right\rfloor\). Furthermore, graphs with \(v_{\text{eq}}^k(G) = 1, 3, 5\) and \(6\) are characterized, respectively. In the end, we obtain the Nordhaus-Gaddum type results of strong equitable vertex \(k\)-arboricity for general \(k\).

\textbf{Keywords:} Equitable coloring, vertex \(k\)-arboricity, \(k\)-tree-coloring, complete multipartite graph, Nordhaus-Gaddum type result.

\textsuperscript{*}Research supported by the National Science Foundation of China (Nos. 11551001, 61164005, 11101232, 11461054 and 11161037), the National Basic Research Program of China (No. 2010CB334708) and the Program for Changjiang Scholars and Innovative Research Team in Universities (No. IRT1068), the Research Fund for the Chunhui Program of Ministry of Education of China (No. Z2014022) and the Nature Science Foundation from Qinghai Province (Nos. 2012-Z-943, 2014-ZJ-721 and 2014-ZJ-907).

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AMS subject classification 2010: 05C05; 05C12; 05C35.

1 Introduction

All graphs considered in this paper are finite and simple. For a real number \( x \), \( \lceil x \rceil \) is the least integer not less than \( x \) and \( \lfloor x \rfloor \) is the largest integer not larger than \( x \). We use \( V(G) \), \( E(G) \), \( \delta(G) \) and \( \Delta(G) \) to denote the vertex set, edge set, minimum degree and maximum degree of \( G \), respectively. For a vertex \( v \in V(G) \), let \( N_G(v) \) denote the set of neighbors of \( v \) in \( G \) and \( d_G(v) = |N_G(v)| \) denote the degree of \( v \). We often use \( d(v) \) for \( d_G(v) \) and refer the reader to [4] for undefined terms and notation.

A mapping \( f : V(G) \to \{1, 2, \ldots, t\} \) is a \( t \)-coloring of a graph \( G \). A \( t \)-coloring of \( G \) is proper if any two adjacent vertices have different colors. For \( 1 \leq i \leq t \), let \( V_i = \{v | f(v) = i\} \). A \( t \)-coloring of a graph \( G \) is said to be equitable if \( ||V_i| - |V_j|| \leq 1 \) for all \( i \) and \( j \), that is, every color class has size \( \lceil|V(G)|/t\rceil \) or \( \lfloor|V(G)|/t\rfloor \). A graph \( G \) is said to be properly equitably \( t \)-colorable if \( G \) has a proper equitable \( t \)-coloring. The smallest number \( t \) for which \( G \) is properly equitably \( t \)-colorable is called the equitable chromatic number of \( G \), denoted by \( \chi_e(G) \).

The equitable coloring problem, introduced by Meyer [10], is motivated by a practical application to municipal garbage collection [13]. In this context, the vertices of the graph represent garbage collection routes. A pair of vertices share an edge if the corresponding routes should not be run on the same day. It is desirable that the number of routes ran on each day be approximately the same. Therefore, the problem of assigning one of the six weekly working days to each route reduces to finding a proper equitable 6-coloring. For more applications such as scheduling, constructing timetables and load balance in parallel memory systems, we refer to [2, 3, 5, 8, 9, 11].

Note that a properly equitably \( t \)-colorable graph may admit no proper equitable \( t' \)-colorings for some \( t' > t \). For example, the complete bipartite graph \( H := K_{2m+1,2m+1} \) has no proper equitable \( (2m + 1) \)-colorings, although it satisfies \( \chi_e(H) = 2 \). This fact motivates another interesting parameter for proper equitable coloring. The equitable chromatic threshold of \( G \), denoted by \( \chi_e^=(G) \), is the smallest integer \( t \) such that \( G \) has a proper equitable \( t' \)-coloring for all \( t' \geq t \). This notion was first introduced by Fan et al. in [6].

For a graph \( G \), a \( d \)-relaxed \( k \)-coloring, also known as a \( d \)-defective coloring, of \( G \) is a function \( f \) from \( V(G) \) to \( \{1, 2, \ldots, k\} \) such that each subgraph \( G[V_i] \) is a graph of maximum degree at most \( d \). A \( d \)-relaxed equitable \( k \)-coloring is a \( d \)-relaxed \( k \)-coloring that is equitable. In [6], Fan, Kierstead, Liu, Molla, Wu and Zhang first considered relaxed equitable coloring of graphs. They proved that every graph has a proper equitable \( \Delta(G) \)-coloring such that each color class induces a forest with maximum degree at most one.
On the basis of this research, Wu, Zhang and Li \cite{14} introduced the notion of equitable $(t, k)$-tree-coloring, which can be viewed as a generalization of proper equitable $t$-coloring.

A $(t, k)$-tree-coloring is a $t$-coloring $f$ of a graph $G$ such that each component of $G[V_i]$ is a tree of maximum degree at most $k$. A $(t, \infty)$-tree-coloring is called a $t$-tree-coloring for short. An equitable $(t, k)$-tree-coloring is a $(t, k)$-tree-coloring that is equitable. The equitable vertex $k$-arboricity of a graph $G$, denoted by $va_k^=(G)$, is the smallest integer $t$ such that $G$ has an equitable $(t, k)$-tree-coloring. The strong equitable vertex $k$-arboricity of $G$, denoted by $va_k^\equiv(G)$, is the smallest integer $t$ such that $G$ has an equitable $(t', k)$-tree-coloring for every $t' \geq t$. It is clear that $va_0^=(G) = \chi^=(G)$ and $va_0^\equiv(G) = \chi^\equiv(G)$ for every graph $G$, and $va_k^=(G)$ and $va_k^\equiv(G)$ may vary a lot.

The following results are immediate.

**Observation 1.1** If $H$ is spanning subgraph of $G$, then $va_k^\equiv(H) \leq va_k^\equiv(G)$.

**Observation 1.2** Let $G$ be a graph of order $n$. Then $G$ has an equitable $(q, k)$-tree-coloring for every $\left\lceil \frac{n}{2} \right\rceil \leq q \leq n$.

The exact value of strong equitable vertex $1$-arboricity of complete equipartition bipartite graph $K_{n,n}$ where $n \equiv 2(\text{mod } 3)$ was studied by Wu, Zhang and Li \cite{14}. In Section 2, we first get a sharp upper bound of strong equitable vertex arboricity of complete bipartite graph $K_{n,n+\ell}$ ($1 \leq \ell \leq n$), that is, $va_2^\equiv(K_{n,n+\ell}) \leq 2 \left\lfloor \frac{n+\ell+1}{2} \right\rfloor$. We next investigate the strong equitable vertex $k$-arboricity of forests. Wu, Zhang and Li \cite{14} got a sufficient and necessary condition on an equitable $(q, \infty)$-tree coloring of a complete equipartition bipartite graph. At the end of Section 2, we obtain a sufficient and necessary condition on an equitable $(q, \infty)$-tree coloring of a complete equipartition tripartite graph.

In Section 3, we show that $1 \leq va_k^\equiv(G) \leq \left\lceil \frac{n}{2} \right\rceil$ for a simple graph $G$ of order $n$. Furthermore, graphs with $va_k^\equiv(G) \in \{1, \left\lceil \frac{n}{2} \right\rceil, \left\lceil \frac{n}{2} \right\rceil - 1\}$ are respectively characterized.

Let $G(n)$ denote the class of simple graphs of order $n$ ($n \geq 2$). For $G \in G(n)$, $\bar{G}$ denotes the complement of $G$. Give a graph parameter $f(G)$ and a positive integer $n$, the Nordhaus-Gaddum Problem is to determine sharp bounds for (1) $f(G) + f(\bar{G})$ and (2) $f(G) \cdot f(\bar{G})$, as $G$ ranges over the class $G(n)$, and characterize the extremal graphs, i.e., graphs that achieve the bounds. The Nordhaus-Gaddum type relations have received wide attention; see a survey paper \cite{1} by Aouchiche and Hansen in 2013. In Section 4, we obtain the Nordhaus-Gaddum type results of strong equitable vertex $k$-arboricity for general $k$. 

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2 Results for some specific graphs

In this section, we give the strong equitable vertex $k$-arboricity of a complete bipartite graph, a forest and a complete tripartite graph.

2.1 Results for complete bipartite graphs

Wu, Zhang and Li [14] obtain the exact value of strong equitable vertex 1-arboricity of complete equipartition bipartite graph $K_{n,n}$ where $n \equiv 2 (\text{mod } 3)$. In this subsection, we prove a sharp upper bound for the strong equitable vertex 2-arboricity of a complete bipartite graph.

**Theorem 2.1** Let $K_{n,n+\ell}$ ($1 \leq \ell \leq n$) be a complete bipartite graph. Then

$$va_2(K_{n,n+\ell}) \leq 2 \left\lfloor \frac{n + \ell + 1}{3} \right\rfloor.$$  

Moreover, the bound is sharp.

**Proof.** By definition, to show $va_2(K_{n,n+\ell}) \leq 2 \left\lfloor \frac{n + \ell + 1}{3} \right\rfloor$, it suffices to show that $K_{n,n+\ell}$ has an equitable $(q, 2)$-tree-coloring for every $q$ such that $q \geq 2 \left\lfloor \frac{n + \ell + 1}{3} \right\rfloor + 1$. Note that

$$3q - 2n \geq 6 \left\lfloor \frac{n + \ell + 1}{3} \right\rfloor + 3 - 2n \geq 6 \left( \frac{n + \ell - 1}{3} \right) + 3 - 2n = 2\ell + 1.$$  

Then $q \geq \frac{2n+2\ell+1}{3}$. Furthermore, $\frac{2n+2\ell+1}{3} \leq q \leq 2n + \ell$.

Let $X, Y$ be the partite sets of $K_{n,n+\ell}$ with $|X| = n$ and $|Y| = n + \ell$. Let $e = xy$ be an edge of $K_{n,n+\ell}$ with $x \in X$ and $y \in Y$. Color $x$ and $y$ with 1, and divide each of $X \setminus \{x\}$ and $Y \setminus \{y\}$ into $\frac{q-1}{2}$ classes equitably and color the vertices of each class with a color in $\{2, 3, \cdots, q\}$. Note that $|X \setminus \{x\}| = n - 1$ and $|Y \setminus \{y\}| = n + \ell - 1$.

If $\frac{2n+2\ell+1}{3} \leq q \leq n$, then

$$2 \leq \frac{n - 1}{\frac{q-1}{2}} \leq \frac{n + \ell - 1}{\frac{q-1}{2}} \leq 3,$$

and hence each color class contains two or three vertices. Therefore, the resulting coloring is an equitable $(q, 2)$-tree-coloring of $K_{n,n+\ell}$.

If $n + \ell \leq q \leq 2n - 1$, then

$$1 \leq \frac{n - 1}{\frac{q-1}{2}} \leq \frac{n + \ell - 1}{\frac{q-1}{2}} \leq 2,$$

and hence each color class contains one or two vertices. Thus, the resulting coloring is an equitable $(q, 2)$-tree-coloring of $K_{n,n+\ell}$, as desired.
If \( n \leq q \leq n + \frac{\ell}{2} \), then
\[
2 = \frac{2n + \ell}{n + \frac{\ell}{2}} \leq \frac{2n + \ell}{q} \leq \frac{2n + \ell}{n} = 2 + \frac{\ell}{n} \leq 3.
\]
Hence, each color class contains one or two vertices. Thus, the resulting coloring is an equitable \((q, 2)\)-tree-coloring of \(K_{n,n+\ell} \), as desired.

If \( n + \frac{\ell}{2} + 1 \leq q \leq n + \ell \), then
\[
1 \leq 1 + \frac{n}{n + \ell} = \frac{2n + \ell}{2n + \ell} \leq \frac{2n + \ell}{q} \leq \frac{2n + \ell}{n + \frac{\ell}{2} + 1} = 2 - \frac{4}{2n + \ell - 2} \leq 2.
\]
Hence, each color class contains one or two vertices. Thus, the resulting coloring is an equitable \((q, 2)\)-tree-coloring of \(K_{n,n+\ell} \), as desired.

If \( 2n \leq q \leq 2n + \ell \), then
\[
1 = \frac{2n + \ell}{2n + \ell} \leq \frac{2n + \ell}{q} \leq \frac{2n + \ell}{2n} = 1 + \frac{\ell}{2n} \leq 2.
\]
Hence, each color class contains one or two vertices. Thus, the resulting coloring is an equitable \((q, 2)\)-tree-coloring of \(K_{n,n+\ell} \), as desired.

**Lemma 2.1** If the cardinality of color classes is at least 4 in any equitable \((t, 2)\)-tree-coloring of \(K_{n,n+\ell} \), then all vertices of each color class must belong to the same partite set.

**Proof.** Assume, to the contrary, that all the vertices in some color class belong to different partite sets of \(K_{n,n+\ell} \). Then as this color class has at least 4 vertices, the subgraph induced by all vertices of this color class either has a cycle, contrary to that this subgraph should be a tree; or has maximum degree at least 3. Contrary to the definition of equitable \((t, 2)\)-tree-colorings. So, the conclusion holds.

**Proposition 2.1** Let \( n = 3t \) \((t \leq 2)\). Then \(va_2^=(K_{n,n+1}) = 2 \left\lceil \frac{n+2}{3} \right\rceil \).

**Proof.** Suppose \( n = 3t \) \((t \leq 2)\) and \( \ell = 1 \). By Theorem 2.1, we have \(va_2^=(K_{n,n+1}) \leq 2 \left\lceil \frac{n+2}{3} \right\rceil \). Next, in order to prove that \(va_2^=(K_{n,n+1}) \geq 2 \left\lceil \frac{n+2}{3} \right\rceil \), we need to prove that \(K_{n,n+1} \) has no equitable \((2t - 1, 2)\)-tree-coloring. Assume, to the contrary, that \(K_{n,n+1} \) has an equitable \((2t - 1, 2)\)-tree-coloring. Then cardinality of each color class is at least 4 because \( \left\lceil \frac{2n+1}{2t-1} \right\rceil = \left\lceil \frac{6t+1}{2t-1} \right\rceil = 4 \) for \( t \leq 2 \). By Lemma 2.1, we have that all the vertices of every color class belong to some partite set of \(K_{n,n+1} \). However, for \( t \leq 2 \), \( n \) and \( n + 1 \) both are divisible by 4. a contradiction. So, \(va_2^=(K_{n,n+1}) = 2 \left\lceil \frac{n+2}{3} \right\rceil \).

From Proposition 2.1 we can see that this bound of Theorem 2.1 is sharp.
2.2 Results for forests

We first give the exact value of strong equitable vertex \( k \)-arboricity of a wheel, which also be used later.

**Lemma 2.2** Let \( W_n \) be a wheel of order \( n \). Then

\[
va_k^\equiv (W_n) \leq \left\lceil \frac{n}{k} \right\rceil.
\]

Moreover, the bound is sharp.

**Proof.** From Observation 1.2, we only need to prove that \( W_n \) has an equitable \((q, k)\)-tree-coloring for each \( q \) with \( \left\lceil \frac{n}{k} \right\rceil \leq q \leq \left\lceil \frac{n}{2} \right\rceil \). Set \( V(W_n) = \{v_1, v_2, \ldots, v_n\} \). Without loss of generality, let \( v_1 \) be the center of \( W_n \). For \( 1 \leq i \leq q \), we set \( V_i = \{v_{jq+i} \in V(W_n) | 0 \leq j \leq k\} \). We give a vertex-coloring \( c \) of \( V(W_n) \) with \( q \) colors such that all the vertices of \( V_i \) receive the color \( i \) \( (1 \leq i \leq q) \). Clearly, the maximum degree of the induced subgraph of each \( V_i \) is at most \( k \). So \( va_k^\equiv (W_n) \leq \left\lceil \frac{n}{k} \right\rceil \), as desired.

**Proposition 2.2** Let \( K_{1,n-1} \) be a star of order \( n \). Then

\[
1 + \left\lceil \frac{n-k-1}{k+2} \right\rceil \leq va_k^\equiv (K_{1,n-1}) \leq \left\lceil \frac{n}{k} \right\rceil.
\]

Moreover, the bound is sharp.

**Proof.** From Lemma 2.2 and Observation 1.1, \( va_k^\equiv (K_{1,n-1}) \leq \left\lceil \frac{n}{k} \right\rceil \). We now show that \( va_k^\equiv (K_{1,n-1}) \geq 1 + \left\lceil \frac{n-k-1}{k+2} \right\rceil \). In order to show this, we need to prove that \( K_{1,n-1} \) has no equitable \( \left(\left\lceil \frac{n-k-1}{k+2} \right\rceil, k\right)\)-tree-coloring. Assume, to the contrary, that \( K_{1,n-1} \) has an equitable \( \left(\left\lceil \frac{n-k-1}{k+2} \right\rceil, k\right)\)-tree-coloring. Since

\[
\frac{n}{\left\lfloor \frac{n-k-1}{k+2} \right\rfloor} \geq \frac{n}{k+2} = k + 2,
\]

it follows that each color class contains at least \( k + 2 \) vertices. Then the color class containing the center of \( K_{1,n-1} \) also contains at least \( k + 2 \) vertices, and hence the maximum degree of the subgraph induced by all the vertices of this color class is at least \( k + 1 \), a contradiction. So \( va_k^\equiv (K_{1,n-1}) \geq 1 + \left\lceil \frac{n-k-1}{k+2} \right\rceil \).

**Proposition 2.3** Let \( G \) be a forest with maximum degree \( \Delta \). If \( k \geq \Delta \), then \( va_k^\equiv (G) = 1 \).

**Proof.** Suppose that \( G \) is a forest with the maximum degree \( \Delta \) and \( k \geq \Delta \). The graph induced by all vertices is a forest of maximum degree at most \( \Delta \). From the definition of the strong equitable vertex \( k \)-arboricity of \( G \), we have \( va_\Delta^\equiv (G) = 1 \).
2.3 Results for complete tripartite graphs

In this section, we will prove a sufficient and necessary condition for a partial $q$-coloring of $K_{n,n,n}$ to be an equitable $(q, \infty)$-tree coloring. Let $K_{n,n,n}$ be a complete tripartite graph with three partite sets $X$, $Y$ and $Z$. For a partial $q$-coloring $c$ of $K_{n,n,n}$ (not need to be proper), we let $V_i$ ($1 \leq i \leq q$) be its color classes and $a = \lfloor \frac{3n}{q} \rfloor$. Set

- $X_1 = \{ V_i | V_i \cap X = a + 1, |V_i \cap Y| = 0, |V_i \cap Z| = 0 \}$;
- $X'_1 = \{ V_i | V_i \cap X = a, |V_i \cap Y| = 1, |V_i \cap Z| = 0 \}$;
- $X''_1 = \{ V_i | V_i \cap X = a, |V_i \cap Y| = 0, |V_i \cap Z| = 1 \}$;
- $X_2 = \{ V_i | V_i \cap X = a, |V_i \cap Y| = 0, |V_i \cap Z| = 0 \}$;
- $X'_2 = \{ V_i | V_i \cap X = a - 1, |V_i \cap Y| = 1, |V_i \cap Z| = 0 \}$;
- $X''_2 = \{ V_i | V_i \cap X = a - 1, |V_i \cap Y| = 0, |V_i \cap Z| = 0 \}$;
- $X_1 = \{ V_i | V_i \cap Y = a + 1, |V_i \cap X| = 0, |V_i \cap Z| = 0 \}$;
- $X'_1 = \{ V_i | V_i \cap Y = a, |V_i \cap X| = 0, |V_i \cap Z| = 1 \}$;
- $X''_1 = \{ V_i | V_i \cap Y = a, |V_i \cap X| = 0, |V_i \cap Z| = 1 \}$;
- $Y_2 = \{ V_i | V_i \cap Y = 0, |V_i \cap Z| = 0 \}$;
- $Y'_2 = \{ V_i | V_i \cap Y = 0, |V_i \cap Z| = 0 \}$;
- $Y''_2 = \{ V_i | V_i \cap Y = 0, |V_i \cap Z| = 1 \}$;
- $Z_1 = \{ V_i | V_i \cap Z = a + 1, |V_i \cap X| = 0, |V_i \cap Y| = 0 \}$;
- $Z'_1 = \{ V_i | V_i \cap Z = a, |V_i \cap X| = 0, |V_i \cap Y| = 0 \}$;
- $Z''_1 = \{ V_i | V_i \cap Z = a, |V_i \cap X| = 0, |V_i \cap Y| = 0 \}$;
- $Z_2 = \{ V_i | V_i \cap Z = 0, |V_i \cap Y| = 0 \}$;
- $Z'_2 = \{ V_i | V_i \cap Z = 0, |V_i \cap Y| = 0 \}$;
- $Z''_2 = \{ V_i | V_i \cap Z = 0, |V_i \cap Y| = 1 \}$.

We now in a position to give our main result.

**Theorem 2.2** If $K_{n,n,n}$ is a complete tripartite graph with three partite sets $X$, $Y$ and $Z$, where $3n = aq + r$ and $0 \leq r \leq q - 1$, and $c$ is a partial $q$-coloring of $K_{n,n,n}$, then $c$ is an equitable $(q, \infty)$-tree coloring of $K_{n,n,n}$ if and only if

\[(a+1)|X_1| + a|X_2| + a|X'_1| + a|X''_1| + (a-1)|X'_2| + (a-1)|X''_2| + |Y'_1| + |Y'_2| + |Z'_1| + |Z'_2| = n \quad (1)\]
Furthermore, we also have
\begin{align}
(a+1)|Y_1|+a|Y_2|+a|Y'_1|+a|Y'_1|+(a-1)|Y'_2|+(a-1)|Y'_2|+|X'_1|+|X'_2|+|Z'_1|+|Z'_2| &= n \quad (2) \\
(a+1)|Z_1|+a|Z_2|+a|Z'_1|+a|Z'_1|+(a-1)|Z'_2|+(a-1)|Z'_2|+|X'_1|+|X'_2|+|Y'_1|+|Y'_2| &= n \quad (3)
\end{align}

**Proof.** Let \( V_i \) (\( 1 \leq i \leq q \)) be the color classes of \( c \). Firstly, we suppose that \( c \) is an equitable \((q, \infty)\)-tree coloring of \( K_{n,n,n} \). Since \( 3n = aq + r \) and \( 0 \leq r \leq q - 1 \), the size of each color class of \( c \) is either \( a \) or \( a+1 \). It is obvious that \( \min\{|V_i \cap X|, |V_i \cap Y|, |V_i \cap Z|\} \leq 1 \) for every \( 1 \leq i \leq q \), because otherwise we would find a cycle in some color class \( V_i \), which contradicts to the definition of the equitable \((t, \infty)\)-tree-coloring of \( K_{n,n,n} \). Thus,
\[
\bigcup_{j=1}^{2}(X_j \cup X'_j \cup Y_j \cup Y'_j \cup Z_j \cup Z'_j) = \bigcup_{i=1}^{q} V_i,
\]
and Equality (1) \( \sim \) (3) hold accordingly. On the other hand, if Equality (1) \( \sim \) (3) hold, then \( c \) is a \( q \)-coloring of \( K_{n,n,n} \) and the size of each color class of \( c \) is either \( a \) or \( a+1 \). Furthermore, we also have \( \min\{|V_i \cap X|, |V_i \cap Y|, |V_i \cap Z|\} \leq 1 \) for every \( 1 \leq i \leq q \). Hence \( c \) is an equitable \((t, \infty)\)-tree-coloring of \( K_{n,n,n} \).
\[ \square \]

## 3 Graphs with given strong equitable vertex \( k \)-arboricity

In this section, we give the lower and upper bounds for the strong equitable vertex \( k \)-arboricity of simple graphs of order \( n \).

**Proposition 3.1** Let \( G \) be a simple graph of order \( n \). Then
\[ 1 \leq va_k^\equiv(G) \leq \lceil n/2 \rceil. \]

**Proof.** It is clear that \( va_k^\equiv(G) \geq 1 \). In order to show \( va_k^\equiv(G) \leq \lceil n/2 \rceil \), we only need to prove that \( G \) has an equitable \((q,k)\)-tree-coloring for every \( q \) satisfying \( \lceil n/2 \rceil \leq q \leq n \). If \( q = \lceil n/2 \rceil \), then each color class of the resulting equitable tree-coloring of \( G \) contains 1 or 2 vertices. Suppose \( \lceil n/2 \rceil < q \leq n \). We can easily construct an equitable \((q,k)\)-tree-coloring of \( G \) by coloring the color class of 2 vertices with two different colors. Hence, we have \( va_k^\equiv(G) \leq \lceil n/2 \rceil \).
\[ \square \]

Graphs with the strong equitable vertex \( k \)-arboricity can be 1 and \( \lceil n/2 \rceil \) are characterized, respectively.

**Proposition 3.2** Let \( G \) be a graph with maximum degree \( \Delta \). Then \( va_k^\equiv(G) = 1 \) if and only if \( k \geq \Delta \) and \( G \) is a forest with the maximum degree \( \Delta \).

**Proof.** Suppose \( va_k^\equiv(G) = 1 \). By the definition of the strong equitable vertex \( k \)-arboricity, graph \( G \) induced by all vertices is a forest such that \( \Delta \leq k \). Conversely, assume that \( G \) is a forest with the maximum degree \( \Delta \) and \( k \geq \Delta \). From Proposition 2,3 we have \( va_\Delta^\equiv(G) = 1 \).
\[ \square \]
Theorem 3.1 Let $G$ be a simple graph of order $n$. Then $\nu_k^\equiv(G) = \left\lceil \frac{n}{2} \right\rceil$ if and only if either $n \geq 3$ is odd and $G = K_n$, or $n$ is even and $G$ satisfies one of the following conditions.

1. For $n = 2$, $G = K_2$ or $G = 2K_1$.
2. For $n = 4$, $\bar{G}$ does not contain $P_4$ as its subgraph.
3. For $n \geq 6$, every maximum matching $M$ in $\overline{G}$ satisfies $|M| \leq m - 1$, where $n = 3m + 2r$ and $m + r = \lceil n/2 \rceil - 1$.

We give the proof of Theorem 3.1 by the following lemmas.

Lemma 3.1 Let $G$ be a simple graph of order $n$, and let $n \geq 3$ be an odd integer. Then $\nu_k^\equiv(G) = \left\lceil \frac{n}{2} \right\rceil$ if and only if $G = K_n$.

Proof. We first consider the case that $n$ is odd. Suppose $\nu_k^\equiv(G) = \left\lceil \frac{n}{2} \right\rceil$. We claim that $G = K_n$. Assume, to the contrary, that $G \neq K_n$. From Observation 1.1 we only need to show $\nu_k^\equiv(G) \leq \left\lceil \frac{n}{2} \right\rceil - 1$ for $G = K_n \setminus e$, where $e \in E(G)$. Observe that there exists a color class with 3 vertices, say $C_1$. Let $n = 2\ell + 1$. Then $\left\lceil n/2 \right\rceil = \ell + 1$ and $\frac{n-3}{2} = \frac{2\ell - 2}{2} = \ell - 1 = \left\lceil n/2 \right\rceil - 2$. From this argument, there are $\left\lceil n/2 \right\rceil - 2$ color classes with 2 vertices and one color class with 3 vertices, and hence $G$ has an equitable $([\left\lceil n/2 \right\rceil - 1, k])$-tree-coloring, which contradicts to the fact that $\nu_k^\equiv(G) = \left\lceil \frac{n}{2} \right\rceil$. So $G = K_n$.

Conversely, we assume that $G = K_n$. From Proposition 3.1, we have $\nu_k^\equiv(G) \leq \left\lceil \frac{n}{2} \right\rceil$. It suffices to show that $\nu_k^\equiv(G) \geq \left\lceil \frac{n}{2} \right\rceil$. Assume, to the contrary, that $G$ has an equitable $([\left\lceil n/2 \right\rceil - 1, k])$-tree-coloring. Let $n = 2\ell + 1$ ($\ell \geq 1$). Then $\left\lceil n/2 \right\rceil - 1 = \ell$, and

$$\frac{n}{\left\lceil n/2 \right\rceil - 1} = \frac{2\ell + 1}{\ell} = 2 + \frac{1}{\ell}.$$ 

Clearly, we have

$$2 < \frac{n}{\left\lceil n/2 \right\rceil - 1} \leq 3.$$

Then there exists a color class with 3 vertices, and the subgraph induced by the vertices in $C_1$ contains a cycle, a contradiction. So $\nu_k^\equiv(G) = \left\lceil \frac{n}{2} \right\rceil$. 

Lemma 3.2 Let $G$ be a simple graph of order $n$, let $n \geq 3$ be an even integer. Then $\nu_k^\equiv(G) = \left\lfloor \frac{n}{2} \right\rfloor$ if and only if $G$ satisfies one of the following conditions.

1. For $n = 2$, $G = K_2$ or $G = 2K_1$.
2. For $n = 4$, $\bar{G}$ does not contain $P_4$ as its subgraph.
3. For $n \geq 6$, every maximum matching $M$ in $\overline{G}$ satisfies $1 \leq |M| \leq m - 1$, where $n = 3m + 2r$ and $m + r = \lceil n/2 \rceil - 1$. 

Proof. Suppose \( v_{a_k}^\equiv(G) = \left[ \frac{n}{2} \right] \). Set \( n = 2\ell \). We claim that \( 1 \leq |M| \leq m - 1 \) for any maximum matching \( M \) in \( \overline{G} \). Assume, to the contrary, that there exists a maximum matching such that \( |M| \geq m \), where \( n = 3m + 2r \) and \( m + r = \left[ \frac{n}{2} \right] - 1 \). Set \( M = \{v_1w_1, v_2w_2, \ldots, v_rw_r\} \), where \( r \geq m \). Choose \( M' = \{v_1w_1, v_2w_2, \ldots, v_mw_m\} \). Set \( V(G) \setminus (\{v_1, v_2, \ldots, v_m\} \cup \{w_1, w_2, \ldots, w_m\}) = \{u_{2m+1}, u_{2m+2}, \ldots, u_n\} \). Choose \( S_1 = \{v_i, w_i, u_{2m+i}\} \), where \( 1 \leq i \leq m \), and choose \( S_{m+j} = \{u_{3m+2j-1}, u_{3m+2j}\} \), where \( 1 \leq j \leq \frac{n-3m}{2} \). Then \( S_1, S_2, \ldots, S_{n-3m} \) are \( \left[ \frac{n}{2} \right] - 1 \) color classes such that the subgraph induced by each \( S_i \) is a forest of order 2 or 3. So \( G \) has an equitable \((\left[ \frac{n}{2} \right] - 1, k)\)-tree-coloring, which contradicts to the fact that \( v_{a_k}^\equiv(G) = \left[ \frac{n}{2} \right] \).

Conversely, we suppose that if any maximum matching \( M \) in \( \overline{G} \), then \( 1 \leq |M| \leq m - 1 \), where \( n = 3m + 2r \) and \( m + r = \left[ \frac{n}{2} \right] - 1 \). From Proposition 3.1 we have \( v_{a_k}^\equiv(G) \leq \left[ \frac{n}{2} \right] \). It suffices to show that \( v_{a_k}^\equiv(G) \geq \left[ \frac{n}{2} \right] \). Assume, to the contrary, that \( G \) has an equitable \((\left[ \frac{n}{2} \right] - 1, k)\)-tree-coloring. Set \( n = 2\ell (\ell \geq 3) \). Then \( \left[ \frac{n}{2} \right] - 1 = \ell - 1 \), and hence

\[
\frac{n}{\ell} - 1 = \frac{2\ell}{\ell - 1} = 2 + \frac{2}{\ell - 1}.
\]

Since \( \ell \geq 3 \), it follows that \( \frac{n}{\ell} - 1 \leq 3 \). Let \( m \) denote the number of those color classes such that each color class contains exactly 3 vertices, and let \( r \) denote the number of those color classes such that each color class contains exactly 2 vertices. Then \( n = 3m + 2r \) and \( m + r = \left[ \frac{n}{2} \right] - 1 \). Since \( |M| \leq m - 1 \) for any maximum matching \( M \) in \( \overline{G} \), there exists a color class with 3 vertices, say \( C_1 \), such that \( M \cap E(\overline{G}[C_1]) = \emptyset \). We claim that \( E(\overline{G}[C_1]) = \emptyset \). Assume, to the contrary, that \( E(\overline{G}[C_1]) \neq \emptyset \). Let \( e \in E(\overline{G}[C_1]) \). Then \( M \cup \{e\} \) is a matching in \( \overline{G} \), which contradicts to the fact that \( M \) is maximum matching in \( \overline{G} \). Therefore, \( E(\overline{G}[C_1]) = \emptyset \), and hence \( G[C_1] \) is a cycle, a contradiction. So \( v_{a_k}^\equiv(G) = \left[ \frac{n}{2} \right] \).

Graphs with \( v_{a_k}^\equiv(G) = \left[ \frac{n}{2} \right] - 1 \) can be also characterized.

**Theorem 3.2** Let \( G \) be a simple graph of order \( n (n \geq 9, n \neq 10) \). Then \( v_{a_k}^\equiv(G) = \left[ \frac{n}{2} \right] - 1 \) if and only if \( G \) satisfies one of the following conditions. (1) \( n \geq 12 \) is even, and every maximum matching \( M \) in \( \overline{G} \) satisfies \( 2 \leq |M| \leq m - 1 \), where \( n = 3m + 2r \) and \( m + r = \left[ \frac{n}{2} \right] - 2 \). (2) \( n \geq 9 \) is odd, and every maximum matching \( M \) in \( \overline{G} \) satisfies \( 1 \leq |M| \leq m - 1 \), where \( n = 3m + 2r \) and \( m + r = \left[ \frac{n}{2} \right] - 2 \).

**Proof.** We distinguish the following two cases to show our proof.

**Case 1.** \( n = 2\ell \)

Suppose \( v_{a_k}^\equiv(G) = \left[ \frac{n}{2} \right] - 1 \). Then \( G \) has an equitable \((q,k)\)-tree-coloring for \( q \geq \left[ \frac{n}{2} \right] - 1 \). For \( q = \left[ \frac{n}{2} \right] - 1 \), since

\[
2 < \frac{n}{\left[ \frac{n}{2} \right] - 1} = \frac{2\ell}{\ell - 1} \leq 3,
\]
it follows that there is at least one color class of order 3 in \( G \). We claim that \( 2 \leq |M| \leq m - 1 \) for any maximum matching \( M \) in \( \overline{G} \). Assume, to the contrary, that there exists a maximum matching such that \( |M| = 1 \) or \( |M| \geq m \), where \( n = 3m + 2r \) and \( m + r = \lceil n/2 \rceil - 2 = \ell - 2 \).

Suppose \( |M| = 1 \). Recall that there is at least one color class of order 3 in \( G \). We furthermore claim that there is only one color class of order 3 in \( G \). Assume, to the contrary, that there are two color classes of order 3 in \( G \), say \( C_1, C_2 \). From the definition, there is an edge of \( C_1 \) (\( i = 1, 2 \)) in \( G \), say \( e_i \). Clearly, \( \{e_1, e_2\} \) is a matching in \( \overline{G} \), which contradicts the fact that \( M \) is a maximum matching \( \overline{G} \) and \( |M| = 1 \). So there is only one color class of order 3 in \( G \). Then each of the remaining classes contains 2 vertices in \( G \). This contradicts to the fact that \( n \) is even.

Suppose that \( |M| \geq m \), where \( n = 3m + 2r \) and \( m + r = \lceil n/2 \rceil - 2 = \ell - 2 \). Set \( M = \{v_1w_1, v_2w_2, \ldots, v_rw_r\} \), where \( r \geq m \). Choose \( M' = \{v_1w_1, v_2w_2, \ldots, v_rw_r\} \). Set \( V(G) \setminus (\{v_1, v_2, \ldots, v_m\} \cup \{w_1, w_2, \ldots, w_r\}) = \{u_{2m+1}, u_{2m+2}, \ldots, u_n\} \). Choose \( S_i = \{v_i, w_i, u_{2m+i}\} \), where \( 1 \leq i \leq m \), and choose \( S_{m+j} = \{u_{3m+2j}, u_{3m+2j+1}, u_{3m+2j+2}\} \), where \( 1 \leq j \leq \frac{n-3m}{2} \). Then \( S_1, S_2, \ldots, S_{\frac{n-m}{2}} \) are \( \lceil n/2 \rceil - 2 \) color classes such that the subgraph induced by each \( S_i \) is a forest of order 2 or 3. So \( G \) has an equitable \((\lceil n/2 \rceil - 2, k)\)-tree-coloring, which contradicts to the fact that \( v_a_k(G) = \lceil \frac{n}{2} \rceil - 1 \). So \( |M| \leq m - 1 \).

Conversely, we suppose that every maximum matching \( M \) in \( \overline{G} \) satisfies \( 2 \leq |M| \leq m - 1 \), where \( 2 \leq |M| \leq m - 1 \), where \( n = 3m + 2r \) and \( m + r = \lceil n/2 \rceil - 2 \). From Theorem 3.1 and Proposition 3.1 we have \( v_{a_k}(G) \leq \lceil \frac{n}{2} \rceil - 1 \). It suffices to show that \( v_{a_k}(G) \geq \lceil \frac{n}{2} \rceil - 1 \). Assume, to the contrary, that \( G \) has an equitable \((\lceil n/2 \rceil - 2, k)\)-tree-coloring. Set \( n = 2\ell \) \((\ell \geq 6)\). Then \( \lceil \frac{n}{2} \rceil - 2 = \ell - 2 \), and hence

\[
\frac{n}{\lceil \frac{n}{2} \rceil - 2} = \frac{2\ell}{\ell - 2} = 2 + \frac{4}{\ell - 2}.
\]

Since \( \ell \geq 6 \), it follows that \( \frac{n}{\lceil \frac{n}{2} \rceil - 2} \leq 3 \). Let \( m \) denote the number of those color classes such that each color class contains exactly 3 vertices, and let \( r \) denote the number of those color classes such that each color class contains exactly 2 vertices. Then \( n = 3m + 2r \) and \( m + r = \lceil n/2 \rceil - 2 \). Since \( |M| \leq m - 1 \) for any maximum matching \( M \) in \( \overline{G} \), there exists a color class with 3 vertices, say \( C_1 \), such that \( M \cap E(\overline{G}[C_1]) = \emptyset \). We claim that \( E(\overline{G}[C_1]) = \emptyset \). Assume, to the contrary, that \( E(\overline{G}[C_1]) \neq \emptyset \). Let \( e \in E(\overline{G}[C_1]) \). Then \( M \cup \{e\} \) is a matching in \( \overline{G} \), which contradicts to the fact that \( M \) is a maximum matching in \( G \). Therefore, \( E(\overline{G}[C_1]) = \emptyset \), and hence \( G[C_1] \) is a cycle, a contradiction. So \( v_{a_k}(G) = \lceil \frac{n}{2} \rceil - 1 \).

**Case 2.** \( n = 2\ell + 1 \)

We claim that \( 1 \leq |M| \leq m - 1 \) for any maximum matching \( M \) in \( \overline{G} \). Assume, to the contrary, that there exists a maximum matching such that \( |M| \geq m \), where \( n = 3m + 2r \) and \( m + r = \lceil n/2 \rceil - 2 = \ell - 2 \). Set \( M = \{v_1w_1, v_2w_2, \ldots, v_rw_r\} \), where \( r \geq m \).
Choose $M' = \{v_1w_1, v_2w_2, \ldots, v_mw_m\}$. Set $V(G) \setminus (\{v_1, v_2, \ldots, v_m\} \cup \{w_1, w_2, \ldots, w_m\}) = \{u_{2m+1}, u_{2m+2}, \ldots, u_n\}$. Choose $S_i = \{v_i, w_i, u_{2m+i}\}$, where $1 \leq i \leq m$, and choose $S_{m+j} = \{u_{3m+2j-1}, u_{3m+2j}\}$, where $1 \leq j \leq \frac{n-3m}{2}$. Then $S_1, S_2, \ldots, S_{\frac{n-3m}{2}}$ are $\lceil n/2 \rceil - 2$ color classes such that the subgraph induced by each $S_i$ is a forest of order 2 or 3. So $G$ has an equitable $(\lceil n/2 \rceil - 2, k)$-tree-coloring, which contradicts the fact that $va_k^=(G) = \lceil \frac{n}{2} \rceil - 1$. So $1 \leq |M| \leq m - 1$.

Conversely, we suppose that if any maximum matching $M$ in $\overline{G}$ satisfies $1 \leq |M| \leq m - 1$, where $n = 3m + 2r$ and $m + r = \lceil n/2 \rceil - 2$. From Theorem 3.1 and Proposition 3.1 we have $va_k^=(G) \leq \lceil \frac{n}{2} \rceil - 1$. It suffices to show that $va_k^=(G) \geq \lceil \frac{n}{2} \rceil - 1$. Assume, to the contrary, that $G$ has an equitable $(\lceil n/2 \rceil - 2, k)$-tree-coloring. Set $n = 2\ell + 1$ ($\ell \geq 4$). Then $\lceil \frac{n}{2} \rceil - 2 = \ell - 1$, and hence

$$\frac{n}{\lceil \frac{n}{2} \rceil - 2} = \frac{2\ell + 1}{\ell - 1} = 2 + \frac{3}{\ell - 1}.$$ 

Since $\ell \geq 4$, it follows that $\frac{n}{\lceil \frac{n}{2} \rceil - 2} \leq 3$. Let $m$ denote the number of those color classes such that each color class contains exactly 3 vertices, and let $r$ denote the number of those color classes such that each color class contains exactly 2 vertices. Then $n = 3m + 2r$ and $m + r = \lceil n/2 \rceil - 2$. Since $|M| \leq m - 1$ for any maximum matching $M$ in $\overline{G}$, there exists a color class with 3 vertices, say $C_1$, such that $M \cap E(\overline{G}[C_1]) = \emptyset$. We claim that $E(\overline{G}[C_1]) = \emptyset$. Assume, to the contrary, that $E(\overline{G}[C_1]) \neq \emptyset$. Let $e \in E(\overline{G}[C_1])$. Then $M \cup \{e\}$ is a matching in $\overline{G}$, which contradicts to the fact that $M$ is maximum matching in $\overline{G}$. Therefore, $E(\overline{G}[C_1]) = \emptyset$, and hence $G[C_1]$ is a cycle, a contradiction. So $va_k^=(G) = \lceil \frac{n}{2} \rceil - 1$.

\section{Nordhaus-Gaddum-type results}

In this section, we investigate the Nordhaus-Gaddum-type problem on the strong equitable vertex $k$-arboricity of graphs.

\begin{proposition}
For any $G \in \mathcal{G}(n)$, if $n \geq 2$ is even, then
\begin{enumerate}
\item $2 \leq va_k^=(G) + va_k^=(\overline{G}) \leq 2\lceil \frac{n}{2} \rceil$;
\item $1 \leq va_k^=(G) \cdot va_k^=(\overline{G}) \leq \lceil \frac{n}{2} \rceil^2$.
\end{enumerate}

Moreover, the bounds are sharp.
\end{proposition}

\begin{proof}
(1) From Proposition 3.1, $va_k^=(G) \geq 1$ and $va_k^=(\overline{G}) \geq 1$, and hence $va_k^=(G) + va_k^=(\overline{G}) \geq 2$. From Proposition 3.1, $va_k^=(G) \leq \lceil \frac{n}{2} \rceil$. Furthermore, we have $va_k^=(G) \leq \lceil \frac{n}{2} \rceil$ for $G$. Hence, $va_k^=(G) + va_k^=(\overline{G}) \leq 2\lceil \frac{n}{2} \rceil$, as desired.

(2) From Proposition 3.1, $va_k^=(G) \geq 1$ and $va_k^=(\overline{G}) \geq 1$, and hence $va_k^=(G) \cdot va_k^=(\overline{G}) \geq 1$. From Proposition 3.1, $va_k^=(G) \leq \lceil \frac{n}{2} \rceil$. For $G$, we have $va_k^=(G) \leq \lceil \frac{n}{2} \rceil$. So, $va_k^=(G) \cdot va_k^=(\overline{G}) \leq \lceil \frac{n}{2} \rceil^2$.
\end{proof}
Proposition 4.2  For any $G \in \mathcal{G}(n)$, if $n \geq 5$ is odd, then

(1) $2 \leq va_k^\equiv(G) + va_k^\equiv(\bar{G}) \leq 2\lceil \frac{n}{2} \rceil - 2$;

(2) $1 \leq va_k^\equiv(G) \cdot va_k^\equiv(\bar{G}) \leq (\lceil \frac{n}{2} \rceil - 1)^2$.

Proof.  (1) From Proposition 3.1, $va_k^\equiv(G) \geq 1$ and $va_k^\equiv(\bar{G}) \geq 1$, and hence $va_k^\equiv(G) + va_k^\equiv(\bar{G}) \geq 2$. From Proposition 3.1, $va_k^\equiv(G) + va_k^\equiv(\bar{G}) \leq 2\lceil \frac{n}{2} \rceil$. Suppose that $va_k^\equiv(G) + va_k^\equiv(\bar{G}) = 2\lceil \frac{n}{2} \rceil$. Then $va_k^\equiv(G) = \lceil \frac{n}{2} \rceil$ and $va_k^\equiv(\bar{G}) = \lceil \frac{n}{2} \rceil$. From Theorem 3.1, both $G$ and $\bar{G}$ are all complete graph of order $n$, a contradiction. Suppose that $va_k^\equiv(G) + va_k^\equiv(\bar{G}) = 2\lceil \frac{n}{2} \rceil - 1$. Then without loss of generality, let $va_k^\equiv(G) = \lceil \frac{n}{2} \rceil$ and $va_k^\equiv(\bar{G}) = \lceil \frac{n}{2} \rceil - 1$. From Theorem 3.1, $G$ is a complete graph of order $n$, and hence $\bar{G} = nK_1$. Clearly, $va_k^\equiv(\bar{G}) = 1$. From this together with $va_k^\equiv(G) = \lceil \frac{n}{2} \rceil - 1$, we have $\lceil \frac{n}{2} \rceil - 1 = 1$ and $n = 4$ or $n = 3$, a contradiction. So $va_k^\equiv(G) + va_k^\equiv(\bar{G}) \leq 2\lceil \frac{n}{2} \rceil - 2$.

(2) From Proposition 3.1, $va_k^\equiv(G) \geq 1$ and $va_k^\equiv(\bar{G}) \geq 1$, and hence $va_k^\equiv(G) \cdot va_k^\equiv(\bar{G}) \geq 1$. From Proposition 3.1, $va_k^\equiv(G) \cdot va_k^\equiv(\bar{G}) \leq \lceil \frac{n}{2} \rceil^2$. Suppose that $va_k^\equiv(G) \cdot va_k^\equiv(\bar{G}) = \lceil \frac{n}{2} \rceil^2$. Then $va_k^\equiv(G) = \lceil \frac{n}{2} \rceil$ and $va_k^\equiv(\bar{G}) = \lceil \frac{n}{2} \rceil$. From Theorem 3.1, both $G$ and $\bar{G}$ are all complete graph of order $n$, a contradiction. Suppose that $va_k^\equiv(G) \cdot va_k^\equiv(\bar{G}) = \lceil \frac{n}{2} \rceil (\lceil \frac{n}{2} \rceil - 1)$. Similarly to the proof of (1), we can get a contradiction. So $va_k^\equiv(G) \cdot va_k^\equiv(\bar{G}) \leq (\lceil \frac{n}{2} \rceil - 1)^2$.

To show the sharpness of the upper bounds of the above theorems, we consider the following example.

**Example 1.** If $n$ is even, then we let $G = P_4$. Then $\bar{G} = P_4$. For $k = 1$, $va_1^\equiv(G) = 2$ and $va_1^\equiv(\bar{G}) = 2$. Then $va_1^\equiv(G) + va_1^\equiv(\bar{G}) = 4 = 2\lceil \frac{n}{2} \rceil$ and $va_1^\equiv(G) \cdot va_1^\equiv(\bar{G}) = 4 = \lceil \frac{n}{2} \rceil^2$.

If $n$ is odd, then we let $G = C_5$. Then $\bar{G} = C_5$. For $k \geq 1$, $va_1^\equiv(G) = 2$ and $va_1^\equiv(\bar{G}) = 2$. Then $va_1^\equiv(G) + va_1^\equiv(\bar{G}) = 4 = 2\lceil \frac{n}{2} \rceil - 2$ and $va_1^\equiv(G) \cdot va_1^\equiv(\bar{G}) = 4 = (\lceil \frac{n}{2} \rceil - 1)^2$.

Graphs attaining the lower bounds of above theorems can be characterized.

**Proposition 4.3** Let $G \in \mathcal{G}(n)$, and let $G$ and $\bar{G}$ are both connected. Then $va_k^\equiv(G) + va_k^\equiv(\bar{G}) = 2$ or $va_k^\equiv(G) \cdot va_k^\equiv(\bar{G}) = 1$ if and only if $G = P_4$ for $k \geq 2$, or $G = P_3$ for $k \geq 2$, or $G = P_2 \cup K_1$ for $k \geq 2$, or $G = P_2$ for $k \geq 1$, or $G = 2K_1$ for $k \geq 1$.

Proof.  If $va_k^\equiv(G) + va_k^\equiv(\bar{G}) = 2$ or $va_k^\equiv(G) \cdot va_k^\equiv(\bar{G}) = 1$, then $va_k^\equiv(G) = 1$ and $va_k^\equiv(\bar{G}) = 1$. From Proposition 3.2, we have $G$ and $\bar{G}$ are both a forest and $k \geq \max\{\Delta(G), \Delta(\bar{G})\}$. Therefore, we have $2(n - 1) \geq E(G) + E(\bar{G}) = E(G \cup \bar{G}) = E(K_n) = \frac{n(n-1)}{2}$, and hence $n \leq 4$. So, $G = P_4$ for $k \geq 2$, or $G = P_3$ for $k \geq 2$, or $G = P_2 \cup K_1$ for $k \geq 2$, or $G = P_2$ for $k \geq 1$, or $G = 2K_1$ for $k \geq 1$.

Conversely, suppose $G$ satisfies the condition of this theorem. One can easily check that $va_k^\equiv(G) = 1$ and $va_k^\equiv(\bar{G}) = 1$. Hence, $va_k^\equiv(G) + va_k^\equiv(\bar{G}) = 2$ and $va_k^\equiv(G) \cdot va_k^\equiv(\bar{G}) = 1$, as desired.
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