MILLER–ABRAHAMS RANDOM RESISTOR NETWORK, MOTT RANDOM WALK AND 2-SCALE HOMOGENIZATION

ALESSANDRA FAGGIONATO

This preprint is and will remain unpublished. The results contained here have been generalized to a very large class of random resistor networks in arXiv:2108.11258 A. Faggionato; Scaling limit of the conductivity of random resistor networks on point processes. Please see arXiv:2108.11258 for results and proofs.

Abstract. The Miller-Abrahams (MA) random resistor network is given by a complete graph on a marked simple point process with edge conductivities depending on the marks and decaying exponentially in the edge length. As Mott random walk, it is an effective model to study Mott variable range hopping in amorphous solids as doped semiconductors. By using 2-scale homogenization we prove that a.s. the infinite volume conductivity of the MA resistor network is given by an effective homogenized matrix $D$ times the mean point density. We also derive homogenization results for the electrical potential. Moreover, $D$ admits a variational characterization and equals the limiting diffusion matrix of Mott random walk. This result clarifies the relation between the two models and it also allows to extend to the MA resistor network the existing bounds on $D$ in agreement with the physical Mott law [16, 17]. The latter concerns the low temperature stretched exponential decay of conductivity in amorphous solids. The techniques developed here can be applied to other models, as e.g. the random conductance model, without ellipticity assumptions.

Keywords: Marked simple point process, Mott variable range hopping, Miller–Abrahams random resistor network, Mott random walk, homogenization, 2-scale convergence.

AMS 2010 Subject Classification: 60G55, 74Q05, 82D30

1. Introduction

The Miller–Abrahams (MA) random resistor network [20] has been introduced in order to study electron transport in amorphous media as doped semiconductors in the regime of strong Anderson localization. These solids present an anomalous conductivity decay at zero temperature, described by Mott law.

Calling $x_i$ the impurity positions in the doped semiconductor, the electron Hamiltonian has exponentially localized quantum eigenstates with localization centers $x_i$ and corresponding energy $E_i$ close to the Fermi level, set equal to zero in what follows. At low temperature phonons induce transitions between the localized eigenstates, the rate of which can be calculated by the Fermi
golden rule [20, 27]. In the simplification of spinless electrons, the resulting rate for an electron to hop from \( x_i \) to the unoccupied site \( x_j \) is then given by (cf. [1, Eq. (3.6)])

\[
\exp \left\{ -\frac{2}{\gamma} |x_i - x_j| - \beta \{E_j - E_i\}_+ \right\}. \tag{1}
\]

In (1) \( \gamma \) is the localization length, \( \beta = 1/kT \) is the inverse temperature and \( \{a\}_+ := \max\{0, a\} \).

The above set \( \{x_i\} \) can be modelled by a random simple point process, marked by random variables \( E_i \) (called energy marks) which can be taken i.i.d. with common distribution \( \nu \). The physically relevant distributions for inorganic media are of the form \( \nu(dE) = c |E|^\alpha dE \) with finite support \([-A, A]\) for some exponent \( \alpha \geq 0 \) [21, 27] (one says that the marked simple point process \( \{(x_i, E_i)\} \) is the \( \nu \)-randomization of \( \{x_i\} \)). Mott law [22, 23, 27] then predicts that, for \( d \geq 2 \), the DC conductivity matrix \( \sigma(\beta) \) of the medium decays to zero as \( \beta \to \infty \) as

\[
\sigma(\beta) \approx A(\beta) \exp\left\{ -\kappa \frac{\alpha+1}{\alpha+d+1} \beta \right\}, \tag{2}
\]

where the prefactor matrix \( A(\beta) \) exhibits a negligible \( \beta \)-dependence (we keep the matrix formalism to cover anisotropic media). Strictly speaking, Mott derived the above asymptotics for \( \alpha = 0 \), while Efros and Shklovskii derived it for \( \alpha = d-1 \). The cases \( \alpha = 0, d-1 \) are the two physically relevant ones. For \( d = 1 \) the DC conductivity presents an Arrhenius–type decay as \( \beta \to \infty \) for all \( \alpha \geq 0 \) [19], i.e.

\[
\sigma(\beta) \approx A(\beta) \exp\left\{ -\kappa \beta \right\}. \tag{3}
\]

Due to localization one can treat the above electron conduction by a hopping process of classical particles (see also [2, 3]), thus leading anyway to a complicated simple exclusion process due to the Pauli blocking. The reversible measure of the exclusion process is the Fermi-Dirac distribution (i.e. the Bernoulli product probability measure such that the probability of having a particle at \( x_i \) is proportional to \( e^{-\beta (E_i - \gamma)} \), \( \gamma \in \mathbb{R} \) being the chemical potential). Effective simplified models in a mean field approximation are given by the MA random resistor network [1, 20, 25, 27] and by Mott random walk [17]. The MA random resistor network has nodes \( x_i \) and, between any pair of nodes \( x_i \neq x_j \), it has an electrical filament of conductivity

\[
c_{x_i, x_j} := \exp \left\{ -\frac{2}{\gamma} |x_i - x_j| - \frac{\beta}{2} (|E_i| + |E_j| + |E_i - E_j|) \right\}. \tag{4}
\]

Mott random walk is the continuous–time random walk with state space \( \{x_i\} \) and probability rate for a jump from \( x_i \) to \( x_j \) given by (4). We point out that the r.h.s. of (4) corresponds to the leading term for \( \beta \) large of (1) multiplied by the probability in the Fermi-Dirac distribution that \( x_i \) and \( x_j \) are, respectively, occupied and unoccupied by an electron (see [1, Eq. (3.7)]).

The original derivation of the laws (2) and (3) is rather heuristic. More robust arguments have been proposed in the physical literature (see [1, 20, 25, 26, 27]). We recall some rigorous results for Mott random walk. They
hold under general conditions (see the references below for the details). We start with \( d \geq 2 \). In [17, Thm. 1] and [9, Thm. 1.2] an invariance principle (respectively annealed and quenched) is stated for Mott random walk, with asymptotic diffusion matrix \( D(\beta) \) admitting a variational characterization [17, Thm. 2]. In addition, lower and upper bounds on \( D(\beta) \) in agreement with Mott law (2) have been obtained respectively in [17, Thm. 1] and [16, Thm. 1]: for \( \beta \) large

\[
\frac{c_1}{\beta^{\frac{\alpha+1}{\alpha+2}}} \leq D(\beta) \leq \frac{c_2}{\beta^{\frac{\alpha+1}{\alpha+2}}},
\]

(5)

for suitable \( \beta \)-independent positive constants \( c_1, c_1', c_2, c_2' \). For \( d = 1 \) annealed and quenched invariance principles have been obtained in [8, Thm. 1.1]. Again \( D(\beta) \) has a variational characterization and satisfies bounds in agreement with (3) (see [8, Thm. 1.2]):

\[
\frac{c_1}{\beta^{\frac{\alpha+1}{\alpha}}} \leq D(\beta) \leq \frac{c_2}{\beta^{\frac{\alpha+1}{\alpha}}},
\]

(6)

for suitable \( \beta \)-independent positive constants \( c_1, c_1', c_2, c_2' \). By invoking Einstein relation (which has been rigorously proved for \( d = 1 \) in [15]) the bounds in (5) and (6) extend to the mobility matrix defined in terms of linear response.

Similar results for the conductivity matrix of the MA resistor network were absent. Our main result (cf. Theorem 1) fills this gap and clarifies the connection between the MA resistor network and Mott random walk. Indeed, for ergodic stationary marked simple point processes \( \{(x_i, E_i)\} \) we prove that the limit of the conductivity of the MA resistor network read on enlarging boxes exists under suitable rescaling (we call this limit the infinite volume conductivity matrix). We also provide a variational characterization of the infinite volume conductivity matrix of the MA resistor network, implying that it equals the asymptotic diffusion matrix \( D(\beta) \) of Mott random walk times the mean point density. As a consequence we get that the infinite volume conductivity matrix satisfies [5] for \( d \geq 2 \) under the assumptions of [17, Thm. 1] and [16, Thm. 1] and satisfies [6] for \( d = 1 \) under the assumptions of [8, Thm. 1.2]. The matrix \( D(\beta) \) equals also the effective homogenized matrix associated to the rescaled Markov generator of Mott random walk (see [12, Thm. 1]). As a consequence, due to [12, Thm. 2], under conditions much weaker than the ones leading to the above quenched/annealed invariance principles, Mott random walk satisfies a weak form of central limit theorem with asymptotic diffusion matrix \( D(\beta) \). Our second main result is given by the homogenization property of the electrical potential in the MA random resistor network (cf. Theorem 2).

We point out that our results do not restrict to Mott variable range hopping (shortly, v.r.h.), i.e. to the MA random resistor network with conductivities (4). Indeed, our Theorems 1 and 2 are stated for more general MA random resistor networks. We also stress that we have followed here the convention used in Physics for the diffusion matrix, which is given by twice the covariance matrix of \( B_t \), where \( (B_t)_{t \geq 0} \) is the Brownian motion emerging in the CLT/invariance principle. This explains the factor 1/2 appearing in Definition 2.1 for \( D \) and not appearing in [17, Thm. 2], [8, Thm. 1.1]. This choice
has the advantage to identify $D(\beta)$ with the effective homogenized matrix (cf. [12]).

We conclude with some comments on the technical aspects. Our proofs are based on homogenization with 2-scale convergence (cf. [28, 29] and references therein). Thinking of $\omega := \{(x_i, E_i)\}$ as a microscopic picture of the medium and introducing the scaling parameter $\epsilon > 0$, the 2-scale convergence allows to explore the ergodicity properties of the medium (cf. Prop. 4.3 below) when averaging on enlarging boxes of size $1/\epsilon$ quantities as $\varphi(\epsilon x) g(\tau x, \omega)$, $\tau x, \omega$ being the environment viewed from site $x_i$. Note that, while $\epsilon x_i$ belongs to the macroscopic world, $\tau x_i, \omega$ refers to the microscopic one (hence the presence of 2 scales).

In [29] the authors have proved homogenization for the Poisson equation $u + Lu = f$ by 2-scale convergence, on $\mathbb{R}^d$ and on bounded domains with mixed boundary conditions, $L$ being the generator of a diffusion in random environments. Analogous results for Mott random walk on $\mathbb{R}^d$, but not on bounded domains, have been obtained in [12]. In [29] Section 7] the above results of [29] on bounded domains have been applied to get that the effective homogenized matrix $D$ equals the infinite volume conductivity in a model related to percolation, under the a priori check that $D > 0$. To get Theorem 1 one could have also thought to adapt the strategy developed for diffusions with random coefficients in [6] to difference operators by using the results of [24], but [24] requires ellipticity assumptions (which are not valid in Mott v.r.h.). We have developed here a direct proof based on 2-scale homogenization, which avoids both the a priori check that $D > 0$ and elliptic assumptions. Our proof of Theorem 1 and 2 is very general and can be applied as well to other resistor networks, as e.g. the conductance model [4] without any ellipticity assumption. For the conductance model the identification between the asymptotic diffusion matrix and the conductivity matrix had already been derived under ellipticity assumptions (see [5] and references therein). We stress that, in addition to the lack of ellipticity, Mott v.r.h. presents further technical difficulties due to long jumps and the absence of an underlying lattice structure (thus leading to the concept of amorphous gradients), not present in the above models.

We conclude mentioning that other rigorous results on the Miller–Abrahams random resistor network have been recently obtained in [13] and [14], where percolation properties of the subnetwork given by filaments with lower bounded conductances have been analyzed. By means of these results and the present Theorem 1 in a forthcoming paper we will show for $d \geq 2$ that one can go beyond the bounds (5) and get the asymptotics of the infinite volume conductivity for $\beta$ large.

Outline of the paper. In the rest we remove the inverse temperature $\beta$ from the notation. In Section 2 we introduce the model and state our main results (cf. Theorem 1 and Theorem 2 for $D_{1,1} > 0$). In Section 3 we analyze the effective diffusive equation. In Section 4 we recall basic facts on marked simple point processes and their Palm distribution. In Section 5 we introduce
the proper Hilbert space to analyze the electrical potential. In Section 6 we prove Theorem 1 when $D_{1,1} = 0$. In Section 7 we consider the space of square integrable forms. In Section 8 we define the family of typical environments. In section 9 we recall the definitions of several types of convergence (including the weak 2-scale convergence). Sections 10 and 11 are devoted to the weak 2-scale limit points of the electrical potential and its gradient. Finally, in Section 12 we conclude the proof of Theorems 1 and 2 when $D_{1,1} > 0$. We collect some minor results in Appendix A.

2. Model and main results

We denote by $\Omega$ the space of locally finite subsets $\omega \subset \mathbb{R}^d \times \mathbb{R}$ such that for each $x \in \mathbb{R}^d$ there exists at most one element $E \in \mathbb{R}$ with $(x, E) \in \omega$. We write a generic element $\omega \in \Omega$ as $\omega = \{(x_i, E_i)\}$ ($E_i$ is called the mark at the point $x_i$) and we set $\hat{\omega} := \{x_i\}$. We will identify the sets $\omega = \{(x_i, E_i)\}$ and $\hat{\omega} = \{x_i\}$ with the counting measures $\sum_i \delta_{(x_i, E_i)}$ and $\sum_i \delta_{x_i}$, respectively. On $\Omega$ one defines in a standard way a metric such that the $\sigma$-algebra $\mathcal{B}(\Omega)$ of Borel sets is generated by the sets $\{\omega(A) = k\}$, with $A$ and $k$ varying respectively among the Borel sets of $\mathbb{R}^d \times \mathbb{R}$ and the nonnegative integers.

We consider a marked simple point process, which is a measurable function from a probability space to the measurable space $(\Omega, \mathcal{B}(\Omega))$. We denote by $\mathcal{P}$ its law and by $\mathbb{E}[:]$ the associated expectation. $\mathcal{P}$ is therefore a probability measure on $\Omega$. We assume that $\mathcal{P}$ is stationary and ergodic w.r.t. translations. More precisely, given $x \in \mathbb{R}^d$ we define the translation $\tau_x : \Omega \to \Omega$ as

$$
\tau_x \omega := \{(x_i - x, E_i)\} \quad \text{if} \quad \omega = \{(x_i, E_i)\}.
$$

Then stationarity means that $\mathcal{P}(\tau_x A) = A$ for any $A \in \mathcal{B}(\Omega)$, while ergodicity means that $\mathcal{P}(A) \in \{0, 1\}$ for any $A \in \mathcal{B}(\Omega)$ such that $\tau_x A = A$ for all $x \in \mathbb{R}^d$. Due to our assumptions stated below, $\mathcal{P}$ has finite positive intensity $m$, i.e.

$$
m := \mathbb{E}[\hat{\omega}([0, 1]^d)] \in (0, +\infty). \quad (7)
$$

As a consequence, the Palm distribution $\mathcal{P}_0$ associated to $\mathcal{P}$ is well defined [10, Chp. 12]. Roughly, $\mathcal{P}_0$ can be thought as $\mathcal{P}$ conditioned to the event $\Omega_0$, where

$$
\Omega_0 := \{\omega \in \Omega : 0 \in \hat{\omega}\}. \quad (8)
$$

We will provide more details on $\mathcal{P}$ and $\mathcal{P}_0$ in Section 11. Below, we write $\mathbb{E}_0[\cdot]$ for the expectation w.r.t. $\mathcal{P}_0$.

In addition to the marked simple point process with law $\mathcal{P}$ we fix a nonnegative Borel function

$$
\mathbb{R}^d \times \mathbb{R}^d \times \Omega \ni (x, y, \omega) \mapsto c_{x,y}(\omega) \in [0, +\infty)
$$

such that $c_{x,x}(\omega) = 0$ for all $x \in \mathbb{R}^d$. The value of $c_{x,y}(\omega)$ will be relevant only when $x, y \in \hat{\omega}$. For later use we define

$$
\lambda_k(\omega) := \int_{\mathbb{R}^d} \hat{\omega}(x)c_{0,x}(\omega)|x|^k, \quad (9)
$$

where $|x|$ denotes the norm of $x \in \mathbb{R}^d$. 

We define the effective diffusion matrix $D$ as the $d \times d$ non-negative symmetric matrix such that
\[
a \cdot Da = \inf_{f \in L^\infty(P_0)} \frac{1}{2} \int dP_0(\omega) \int d\omega(x)c_{0,x}(\omega) (a \cdot x - \nabla f(\omega, x))^2 ,
\]
where $\nabla f(\omega, x) := f(\tau_x \omega) - f(\omega)$.

Above, and in what follows, we will denote by $a \cdot b$ the scalar product of the vectors $a$ and $b$.

**Assumptions.** We make the following assumptions:

(A1) the law $P$ of the marked simple point process is stationary and ergodic w.r.t. spatial translations;

(A2) $P$ has finite positive intensity as stated in (7);

(A3) $P(\omega \in \Omega : \tau_x \omega \neq \tau_y \omega \forall x \neq y \in \hat{\omega}) = 1$;

(A4) the weights $c_{x,y}(\omega)$ are symmetric and covariant, i.e. $c_{x,y}(\omega) = c_{y,x}(\omega)$ $\forall x, y \in \hat{\omega}$ and $c_{x,y}(\omega) = c_{x-a,y-a}(\tau_a \omega)$ $\forall x, y \in \hat{\omega}$ and $\forall a \in \mathbb{R}^d$;

(A5) $\lambda_0, \lambda_2 \in L^1(P_0)$;

(A6) for some $\alpha \in (0, 1)$ it holds
\[
\mathbb{E}_0\left[ \int d\omega(z)c_{0,y}(\omega)^\alpha \right] < +\infty ,
\]
\[
\mathbb{E}_0\left[ \int d\omega(z)c_{0,x}(\omega)^\alpha |z|^2 \right] < +\infty ,
\]
\[
\limsup_{\ell \to +\infty} \sup_{\omega \in \Omega_0} \sup_{x \in \hat{\omega} \cap \Lambda_\ell} c_{0,x}(\omega)^{1-\alpha} < +\infty ;
\]

(A7) $c_{x,y}(\omega) > 0$ for all $x, y \in \hat{\omega}$.

We discuss the above assumptions at the end of this section.

**Warning 2.1.** Since $D$ is a symmetric matrix, at cost of an orthonormal change of coordinates and without loss of generality, we will suppose that $D$ is diagonal. In other words, our results refer to the principal directions of $D$. Note that $a \in \mathbb{R}^d \setminus \{0\}$ is eigenvector of eigenvalue zero if $a \cdot Da = 0$.

In the rest, $\ell$ will be a positive number. We consider the stripe $S_\ell := \mathbb{R} \times (-\ell/2, \ell/2)^{d-1}$ and the box $\Lambda_\ell := (-\ell/2, \ell/2)^d$. We consider the $\ell$–parametrized resistor network $(RN)_\ell^\omega$ on $S_\ell$ with electrical filaments defined as follows. To each unordered pair $\{x, y\}$, such that $x \in \hat{\omega} \cap \Lambda_\ell$ and $y \in \hat{\omega} \cap \Lambda_\ell$, we associate an electrical filament of conductivity $c_{x,y}(\omega)$. We can think of $(RN)_\ell^\omega$ as a weighted unoriented graph with vertex set $\hat{\omega} \cap \Lambda_\ell$, edge set
\[
\mathbb{E}_\ell^\omega := \{ \{x, y\} : x \in \hat{\omega} \cap \Lambda_\ell, y \in \hat{\omega} \cap \Lambda_\ell, x \neq y \}
\]
and weight of the edge $\{x, y\}$ given by the conductivity $c_{x,y}(\omega)$, see Figure 1.
\( \ell \geq \ell_0(\omega) \), it holds
\[
\hat{\omega} \cap \Lambda_\ell \neq \emptyset, \\
\{ x \in \hat{\omega} \cap S_\ell : x_1 \leq -\ell/2 \} \neq \emptyset, \\
\{ x \in \hat{\omega} \cap S_\ell : x_1 \geq \ell/2 \} \neq \emptyset, \\
\sum_{y \in \hat{\omega} \cap S_\ell} c_{x,y}(\omega) < +\infty \quad \forall x \in \hat{\omega} \cap \Lambda_\ell. 
\]
\tag{15}

Indeed, it is enough to apply Proposition 4.3 in Section 4 with suitable test functions \( \varphi \), to bound the series in (15) by \( \sum_{y \in \hat{\omega}} c_{x,y}(\omega) = \lambda_0(\tau_x \omega) \) and use that \( E_0[\lambda_0] < +\infty \).

**Definition 2.2** (Electrical potential). Suppose that \( \omega, \ell \) satisfy (15). Then we denote by \( V_\ell^\omega \) the electrical potential of the resistor network \( (RN)_\ell^\omega \) with values 0 and 1 on \( \{ x \in \hat{\omega} \cap S_\ell : x_1 \leq -\ell/2 \} \) and \( \{ x \in \hat{\omega} \cap S_\ell : x_1 \geq \ell/2 \} \), respectively. In particular, \( V_\ell^\omega \) is the unique function \( V_\ell^\omega : \hat{\omega} \cap S_\ell \to \mathbb{R} \) such that
\[
\sum_{y \in \hat{\omega} \cap S_\ell} c_{x,y}(\omega) (V_\ell^\omega(y) - V_\ell^\omega(x)) = 0 \quad \forall x \in \hat{\omega} \cap \Lambda_\ell, 
\]
and satisfying the boundary conditions
\[
\begin{cases}
V_\ell^\omega(x) = 0 & \text{if } x \in \hat{\omega} \cap S_\ell, \ x_1 \leq -\ell/2, \\
V_\ell^\omega(x) = 1 & \text{if } x \in \hat{\omega} \cap S_\ell, \ x_1 \geq +\ell/2.
\end{cases} 
\]
\tag{16}

As discussed in Section 5, the above electrical potential exists and is unique (here we use (A7)) and has values in \([0,1]\). We recall that, given \((x,y)\) with \( \{x,y\} \in \mathbb{B}_\ell^\omega \) (cf. (14)),
\[
i_{x,y}(\omega) := c_{x,y}(\omega)(V_\ell^\omega(y) - V_\ell^\omega(x))
\]
\tag{18}
is the current flowing from \( x \) to \( y \) under the electrical potential \( V_\ell^\omega \). For simplicity we have dropped the dependence on \( \ell \) in the notation \( i_{x,y}(\omega) \).
Definition 2.3 (Effective conductivity). Suppose that \( \omega, \ell \) satisfy (15). We call \( \sigma_\ell(\omega) \) the effective conductivity of the resistor network \((RN)_\ell^\omega\) along the first direction under the electrical potential \(V_\ell^\omega\). More precisely, \( \sigma_\ell(\omega) \) is given by

\[
\sigma_\ell(\omega) := \sum_{x \in \omega \cap S_+: y \in \omega \cap \Lambda_\ell \cap \{x \leq -\ell/2\}} i_{x,y}(\omega) = \sum_{x \in \omega \cap S_+: y \in \omega \cap \Lambda_\ell \cap \{x \leq -\ell/2\}} c_{x,y}(\omega) \left(V_\ell^\omega(y) - V_\ell^\omega(x)\right).
\]

(19)

We recall two equivalent characterizations of the conductivity \( \sigma_\ell(\omega) \) (cf. Appendix A). For any \( \gamma \in [\ell/2, \ell/2] \), \( \sigma_\ell(\omega) \) equals the current flowing through the hyperplane \( \{x \in \mathbb{R}^d : x_1 = \gamma\} \):

\[
\sigma_\ell(\omega) = \sum_{x \in \omega \cap S_+: y \in \omega \cap S_+ \cap \{x_1 \leq \gamma\}} \sum_{\{x,y\} \in B_\ell^\omega, y > \gamma} i_{x,y}(\omega).
\]

(20)

Note that (19) corresponds to (20) with \( \gamma = -\ell/2 \). \( \sigma_\ell(\omega) \) also satisfies the identity

\[
\sigma_\ell(\omega) = \sum_{\{x,y\} \in B_\ell^\omega} c_{x,y}(\omega) \left(V_\ell^\omega(x) - V_\ell^\omega(y)\right)^2.
\]

(21)

We can now state our first main result concerning the infinite volume asymptotics of \( \sigma_\ell(\omega) \):

Theorem 1. For \( \mathcal{P}-a.a. \ \omega \) it holds

\[
\lim_{\ell \to +\infty} \ell^{2-d} \sigma_\ell(\omega) = mD_{1,1}.
\]

(22)

To clarify the link with homogenization and state our further results, it is convenient to rescale space in order to deal with fixed stripe and box. More precisely, we set \( \varepsilon := 1/\ell \). Then \( \varepsilon > 0 \) is our scaling parameter. We set

\[
\begin{align*}
S &:= \mathbb{R} \times (-1/2, 1/2)^{d-1}, \\
\Lambda &:= (-1/2, 1/2)^d, \\
S_- &:= \{x \in S : x_1 \leq -1/2\}, \\
S_+ &:= \{x \in S : x_1 \geq 1/2\}.
\end{align*}
\]

(23)

We write \( V_\varepsilon : \varepsilon \hat{\omega} \cap S \to [0,1] \) for the function given by \( V_\varepsilon(\varepsilon x) := V_\ell^\omega(x) \) (note that the dependence on \( \omega \) in \( V_\varepsilon \) is understood, as for other objects below).

We introduce the atomic measures

\[
\mu^\varepsilon_{\omega, \Lambda} := \varepsilon^d \sum_{x \in \varepsilon \omega \cap \Lambda} \delta_x, \quad \nu^\varepsilon_{\omega, \Lambda} := \sum_{(x,y) \in \mathcal{E}_\varepsilon} \varepsilon^d c_{x,y}/\varepsilon(\omega) \delta_{(x,(y-x)/\varepsilon)},
\]

(24)

where \( \mathcal{E}_\varepsilon \) is the set of pairs \( (x,y) \) such that \( x \neq y \) are in \( \varepsilon \hat{\omega} \cap S \) and \( \{x,y\} \) intersect \( \Lambda \). Equivalently, \( \mathcal{E}_\varepsilon := \{(\varepsilon x, \varepsilon y) : \{x,y\} \in B_\ell^\omega\} \).

Given a function \( f : \varepsilon \hat{\omega} \cap S \to \mathbb{R} \), we define the amorphous gradient \( \nabla_\varepsilon f \) on pairs \( (x,z) \) with \( x \in \varepsilon \hat{\omega} \cap S \) and \( x + \varepsilon z \in \varepsilon \hat{\omega} \cap S \) as

\[
\nabla_\varepsilon f(x,z) = \frac{f(x + \varepsilon z) - f(x)}{\varepsilon}.
\]

(25)
Moreover, we define the operator
\[ \mathbb{L}^\varepsilon_{\omega} f(x) := \varepsilon^{-2} \sum_{y \in \varepsilon \wedge S} c_{x/y} \varepsilon \left[ f(y) - f(x) \right], \quad x \in \varepsilon \wedge \Lambda, \]  
whenever the series in the r.h.s. is absolutely convergent.

Since \( \mathbb{E}_0[\lambda_0] < +\infty \), we have \( \mathbb{P}_0(\lambda_0 < \infty) = 1 \). By Lemma 4.1 in Section 4 it follows that \( \mathbb{P}(\Omega_1) = 1 \), where \( \Omega_1 \) is the translation invariant Borel set
\[ \Omega_1 := \{ \omega \in \Omega : \lambda_0(\tau_{x} \omega) < +\infty \ \forall x \in \hat{\omega} \} \cap \Omega' \]  
(see (15) for the definition of \( \Omega' \)). Let \( \omega \in \Omega_1 \) and let \( f : \varepsilon \wedge S \to \mathbb{R} \) be a bounded function. Since \( \lambda_0(\tau_{x} \omega) = \sum_{y \in \omega} c_{x/y}(\omega) \), \( \mathbb{L}^\varepsilon_{\omega} f(x) \) is well defined for all \( x \in \varepsilon \wedge \Lambda \) and the measure \( \nu^\varepsilon_{\omega,\Lambda} \) has finite mass (\( \mu^\varepsilon_{\omega,\Lambda} \) always has finite mass as \( \wedge \omega \) is locally finite). As the amorphous gradient \( \nabla^\varepsilon_{\omega} \) is bounded too, we have that \( \nabla^\varepsilon_{\omega} f \in L^2(\nu^\varepsilon_{\omega,\Lambda}) \). Moreover, if in addition \( f \) is zero outside \( \Lambda \), it holds (cf. Lemma 5.1)
\[ \langle f, -\mathbb{L}^\varepsilon_{\omega} f \rangle_{L^2(\nu^\varepsilon_{\omega,\Lambda})} \equiv \frac{1}{2} \langle \nabla^\varepsilon_{\omega} f, \nabla^\varepsilon_{\omega} f \rangle_{L^2(\nu^\varepsilon_{\omega,\Lambda})} < +\infty. \]  

**Definition 2.4.** Given \( \omega \in \Omega_1 \) we define the Hilbert space
\[ H^1_{0,\omega} \equiv \{ f : \varepsilon \wedge S \to \mathbb{R} \text{ s.t. } f(x) = 0 \ \forall x \in \varepsilon \wedge (S_- \cup S_+) \} \]  
educed with norm \( \| f \|_{H^1_{0,\omega}} \equiv \| f \|_{L^2(\nu^\varepsilon_{\omega,\Lambda})} + \| \nabla^\varepsilon_{\omega} f \|_{L^2(\nu^\varepsilon_{\omega,\Lambda})} \). In addition, we set \( K^\varepsilon_{\omega} \equiv H^1_{0,\omega} + \psi \), where \( \psi : S \to [0, 1] \) is the function
\[ \psi(x) := \begin{cases} x_1 + \frac{1}{2} & \text{if } x \in \Lambda, \\ 0 & \text{if } x \in S_-, \\ 1 & \text{if } x \in S_+. \end{cases} \]  

Note that \( K^\varepsilon_{\omega} \) is given by the functions \( f : \varepsilon \wedge S \to \mathbb{R} \) such that \( f(x) = 0 \) for all \( x \in \varepsilon \wedge \Lambda \) and \( f(x) = 1 \) for all \( x \in \varepsilon \wedge S_+ \).

Given \( \omega \in \Omega_1 \), in Section 5 we will derive that, due to (16) and (17), \( V^\varepsilon \) is the unique function in \( K^\varepsilon_{\omega} \) such that \( \mathbb{L}^\varepsilon_{\omega} V^\varepsilon(x) = 0 \) for all \( x \in \varepsilon \wedge \Lambda \) (cf. Lemma 5.2). We point out that, by (21) and (28), the rescaled conductivity \( \ell^2 - d \sigma_\varepsilon(\omega) \) equals the flow energy associated to \( V^\varepsilon \):
\[ \ell^2 - d \sigma_\varepsilon(\omega) = \langle V^\varepsilon, -\mathbb{L}^\varepsilon_{\omega} V^\varepsilon \rangle_{L^2(\nu^\varepsilon_{\omega,\Lambda})} \equiv \frac{1}{2} \langle \nabla^\varepsilon_{\omega} V^\varepsilon, \nabla^\varepsilon_{\omega} V^\varepsilon \rangle_{L^2(\nu^\varepsilon_{\omega,\Lambda})}. \]  

Theorem 1 can therefore be restated as
\[ \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^2} \langle \nabla^\varepsilon_{\omega} V^\varepsilon, \nabla^\varepsilon_{\omega} V^\varepsilon \rangle_{L^2(\nu^\varepsilon_{\omega,\Lambda})} = mD_{1,1}(\nabla \psi, \nabla \psi)_{L^2(\Lambda, dx)} = mD_{1,1}, \quad \mathbb{P}-a.s. \]  
(32)

Note that the second identity in (32) is immediate as \( \nabla \psi = e_1 \). To prove Theorem 1 we distinguish the cases \( D_{1,1} = 0 \) and \( D_{1,1} > 0 \). The proof for \( D_{1,1} = 0 \) (which is simpler) is given in Section 6 while the proof for \( D_{1,1} > 0 \) will take the rest of our investigation and will be concluded in Section 12. In the case \( D_{1,1} > 0 \) we can say more on the behavior of \( V^\varepsilon \):
Theorem 2. Suppose that $D_{1,1} > 0$. Then there exists a translation invariant Borel set $\Omega_{typ}$ of typical environments with $\Omega_{typ} \subset \Omega_1$ and $P(\Omega_{typ}) = 1$, such that for any $\omega \in \Omega_{typ}$, holds, $V_\varepsilon \in L^2(\mu_{\varepsilon,\Lambda}^\ast)$ converges weakly and 2-scale converges weakly to $\psi \in L^2(\Lambda, dx)$.

The definition of the above types of convergence is recalled in Section 9.

Warning 2.2. Recall that $D$ is diagonal (see Warning 2.1). When $D_{1,1} > 0$, at cost to permute the coordinates and without loss of generality, we assume that $D_{i,i} > 0$ for $1 \leq i \leq d_\ast$ and $D_{i,i} = 0$ for $d_\ast < i \leq d$.

In Section 3 we will characterize $\psi$ as the unique weak solution on $\Lambda$ of the so-called effective equation given by $\nabla_\ast \cdot (D \nabla_\ast \psi) = 0$ with suitable mixed Dirichlet-Neumann conditions, where $\nabla_\ast$ denotes the projection of $\nabla$ on the first $d_\ast$ coordinate (cf. Definition 3.6). Due to Theorem 2, the equation $\nabla_\ast \cdot (D \nabla_\ast \psi) = 0$ represents the effective macroscopic law of the electrical potential $V_\varepsilon$ in the limit $\varepsilon \downarrow 0$, when $D_{1,1} > 0$.

2.1. Comments on Assumptions (A1),...,(A7). If the marked simple point process is the $\nu$–randomization of an ergodic stationary simple point process $\xi$ on $\mathbb{R}^d$ (i.e. under $\mathcal{P}(\cdot | \hat{\omega})$ the marks are i.i.d. with common law $\nu$) and $\nu$ is not degenerate (i.e. $\nu \neq \delta_0$), then condition (A1) is automatically satisfied (see [17] Section 2.1). The point process $\xi$ can be genuinely amorphous as the Poisson point process or can keep some lattice structure as the random set $\xi := U + \xi \subset \mathbb{R}^d$, where $U$ and $\xi$ are independent, $U$ is a random vector with uniform distribution on $[0, 1]^d$ and $\xi$ is given by the vertex set of a site/bond Bernoulli percolation in $\mathbb{Z}^d$.

Always in the case of $\nu$–randomization, if $\nu$ is not degenerate, then (A3) is also fulfilled. In the general case, since the event in (A3) is translation invariant, (A3) is equivalent to the identity $\mathcal{P}(\omega \in \Omega_0 : \tau_x \omega \neq \tau_y \omega \ \forall x \neq y \ in \ \hat{\omega}) = 1$ (cf. e.g. [10], [17, Lemma 1]).

To verify (A5) and (11), (12) in (A6) the following property is very useful: given $n \in \mathbb{N}, x \in \mathbb{R}^d$ and a box $B \subset \mathbb{R}^d$, it holds $\mathbb{E}_0[\hat{\omega}(x + B)^n] \leq C\mathbb{E}[\hat{\omega}(0, 1)^{n+1}]$ for some positive constant $C$ independent from $x$, cf. [17] Lemma 1-(iv). If, as in Mott v.r.h., there exist $C' > 0$ such that $c_{0,x}(\omega) \leq C' f(|k|)$ for any $k \in \mathbb{Z}^d$ and $x \in k + [0, 1]^d$, then one can bound

$$\int \hat{\omega}(z) c_{0,z}(\omega)^\gamma |z|^\chi \leq C(\gamma, \chi) \sum_{k \in \mathbb{Z}^d} f(|k|)^\gamma (1 + |k|)^\chi \hat{\omega}(k + [0, 1]^d).$$

As a consequence, if $\mathbb{E}[\hat{\omega}(0, 1)^2] < +\infty$, we have $\mathbb{E}_0[\int d\hat{\omega}(z) c_{0,z}(\omega)^\gamma |z|^\chi] < +\infty$ if $\sum_{k \in \mathbb{Z}^d} f(|k|)^\gamma (1 + |k|)^\chi < +\infty$. By Campbell’s formula (take $f(x, \omega) := \mathbb{I}(\|x\|_\infty \leq 1/2)\hat{\omega}(-1, 1)^d$ in [15] below), $\mathbb{E}_0[\hat{\omega}([-1, 1]^d)] < +\infty$ implies that $\mathbb{E}[\hat{\omega}(0, 1)^2] < +\infty$. In particular, for Mott v.r.h. Assumption (A5), (11) and (12) are satisfied if and only if $\mathbb{E}[\hat{\omega}(0, 1)^2] < +\infty$.

Condition (13) can be relaxed. For the sake of simplicity, and since (13) is true for Mott v.r.h., we have preferred the present form. Condition (A7) is not strictly necessary. It guarantees the uniqueness of the electrical potential
and it is always satisfied by Mott v.r.h.. Due to the above discussion, for
Mott v.r.h., our assumptions reduce to Assumptions (A1), (A2), (A3) and the
requirement that $\mathbb{E}[\omega([0,1]^d)^2] < +\infty$.

Finally, we point out that the marks $E_i$ could indeed belong to any Polish
space instead of $\mathbb{R}$, results and proofs would not change.

3. Effective equation with mixed boundary conditions

In this section we assume that $D_{1,1} > 0$. Recall the definition of $d_*$ given in
Warning 2.2. We are interested in elliptic operators with mixed (Dirichlet and
Neumann) boundary conditions. We set

$$F_- := \{ x \in \bar{\Lambda} : x_1 = -1/2 \}, \quad F_+ := \{ x \in \bar{\Lambda} : x_1 = 1/2 \}, \quad F := F_- \cup F_+ .$$

Given a domain $A \subset \mathbb{R}^d$, $L^2(A)$ and $H^1(A)$ refer to the Lebesgue measure $dx$.

**Definition 3.1.** We introduce the following three functional spaces:

- We define $H^1(\Lambda, d_*)$ as the Hilbert space given by functions $f \in L^2(\Lambda)$
  with weak derivative $\partial_i f$ in $L^2(\Lambda)$ for any $i = 1, \ldots, d_*$, endowed with
  the norm $\| f \|_{1,*} := \| f \|_{L^2(\Lambda)} + \sum_{i=1}^{d_*} \| \partial_i f \|_{L^2(\Lambda)}$. Moreover, given $f \in
  H^1(\Lambda, d_*)$, we define

$$\nabla_* f := (\partial_1 f, \partial_2 f, \ldots, \partial_{d_*} f, 0, \ldots, 0). \quad (33)$$

- We define $H^1_0(\Lambda, F, d_*)$ as the closure in $H^1(\Lambda, d_*)$ of

$$\left\{ \varphi |_\Lambda : \varphi \in C^\infty_c(\mathbb{R}^d \setminus F) \right\} . \quad (34)$$

- We define the functional set $K$ as (cf. (30))

$$K := \{ \psi |_\Lambda + f : f \in H^1_0(\Lambda, F, d_*) \} . \quad (35)$$

**Remark 3.2.** Let $f \in H^1(\Lambda, d_*)$. Given $1 \leq i \leq d_*$, by integrating $\partial_i f$ times

$$\varphi(x_1, \ldots, x_{d_*}) \phi(x_{d_*+1}, \ldots, x_d)$$

with $\varphi \in C^\infty_c(\mathbb{R}^{d_*})$ and $\phi \in C^\infty_c(\mathbb{R}^{d-d_*})$, one obtains that the function $f(\cdot, y_1, \ldots, y_{n-d_*})$ belongs to $H^1((-1/2, 1/2)^{d_*})$ for

$$\text{a.e. } (y_1, \ldots, y_{n-d_*}) \in (-1/2, 1/2)^{n-d_*} .$$

Being a closed subspace of the Hilbert space $H^1(\Lambda, d_*)$, $H^1_0(\Lambda, F, d_*)$ is a
Hilbert space. We also point out that in the definition of $K$ one could replace
$\psi |_\Lambda$ by any other function $\varphi \in H^1(\Lambda, d_*) \cap C(\bar{\Lambda})$ such that $\varphi \equiv 0$ on $F_-$ and

$$\phi \equiv 1 \text{ on } F_+ ,$$

as follows from the next lemma:

**Lemma 3.3.** Let $u \in H^1(\Lambda, d_*) \cap C(\bar{\Lambda})$ satisfy $u \equiv 0$ on $F$. Then $u \in
H^1_0(\Lambda, F, d_*)$.

**Proof.** We use some idea from the proof of [7, Theorem 9.17]. We set $u_n(x) := G(nu(x))/n$, where $G \in C^1(\mathbb{R})$ satisfies: $|G(t)| \leq |t|$ for all $t \geq 0$, $G(t) = 0$
for $|t| \leq 1$ and $G(t) = t$ for $|t| \geq 2$. Note that $\partial_i u_n(x) = G'(nu(x)) \partial_i u(x)$ for

$$1 \leq i \leq d_* \quad (36)$$

(cf. [7, Prop. 9.5]). Hence, $u_n \rightarrow u$ and $\partial_i u_n \rightarrow 1_{\{u=0\}} \partial_i u = \partial_i u$ a.e.

In the last identity, we have used that $\partial_i u = 0$ a.e. on $\{u = 0\}$ which follows
as a byproduct of Remark 3.2 and Stampacchia's theorem (see thereom 3 and
Remark (ii) to Theorem 4 in [11, Section 6.1.3]). By dominated convergence
one obtains that \( u_n \to u \) in \( H^1(\Lambda, d_\ast) \). Since \( H^1_0(\Lambda, F, d_\ast) \) is a closed subspace of \( H^1(\Lambda, d_\ast) \), it is enough to prove that \( u_n \in H^1_0(\Lambda, F, d_\ast) \). Due to our hypothesis on \( u \) and the definition of \( G \), \( u_n \equiv 0 \) in a neighborhood of \( F \) inside \( \Lambda \). Hence the thesis follows by applying the implication (iii) \( \Rightarrow \) (i) in Proposition 3.4. Equivalently, it is enough to observe that, by adapting [7, Cor. 9.8] or [11, Theorem 1, Sec. 4.4], there exists a sequence of functions \( \varphi_k \in C_c^\infty(\mathbb{R}^d) \) such that \( \varphi_{k|\Lambda} \to u_n \) in \( H^1(\Lambda, d_\ast) \). Since \( u_n \equiv 0 \) in a neighborhood of \( F \), it is easy to find \( \phi \in C_c^\infty(\mathbb{R}^d \setminus F) \) such that \( (\phi \varphi_k)|\Lambda \to u_n \) in \( H^1(\Lambda, d_\ast) \). Hence \( u_n \in H^1_0(\Lambda, F, d_\ast) \).

One can adapt the proof of [7, Prop. 9.18] to get the following criterion assuring that a function belongs to \( H^1_0(\Lambda, F, d_\ast) \):

**Proposition 3.4.** Given a function \( u \in L^2(\Lambda) \), the following properties are equivalent:

(i) \( u \in H^1_0(\Lambda, F, d_\ast) \);

(ii) there exists \( C > 0 \) such that

\[
\left| \int_\Lambda u \partial_i \varphi dx \right| \leq C \| \varphi \|_{L^2(\Lambda)} \quad \forall \varphi \in C_c^\infty(S), \forall i : 1 \leq i \leq d_\ast ; \tag{35}
\]

(iii) the function

\[
\bar{u}(x) := \begin{cases} u(x) & \text{if } x \in \Lambda, \\ 0 & \text{if } x \in S \setminus \Lambda, \end{cases} \tag{36}
\]

belongs to \( H^1(S, d_\ast) \) (which is defined similarly to \( H^1(\Lambda, d_\ast) \)). Moreover, in this case it holds \( \partial_i \bar{u} = \partial_i u \) for \( 1 \leq i \leq d_\ast \), where \( \partial_i u \) is defined similarly to \( \bar{u} \).

**Lemma 3.5** (Poincaré inequality). It holds \( \| f \|_{L^2(\Lambda)} \leq \| \partial_1 f \|_{L^2(\Lambda)} \) for any \( f \in H^1_0(\Lambda, F, d_\ast) \).

**Proof.** Given \( f \in C_c^\infty(\mathbb{R}^d \setminus F) \), by Schwarz inequality, for any \( (x_1, x') \in \Lambda \) we have

\[
f(x_1, x')^2 = \left( \int_{-1/2}^{x_1} \partial_1 f(s, x') ds \right)^2 \leq \int_{-1/2}^{1/2} \partial_1 f(s, x')^2 ds .
\]

By integrating over \( \Lambda \) we get the desired estimate for \( f \in C_c^\infty(\mathbb{R}^d \setminus F) \). Since \( C_c^\infty(\mathbb{R}^d \setminus F) \) is dense in \( H^1_0(\Lambda, F, d_\ast) \), we get the thesis.

**Definition 3.6.** We say that \( v \) is a weak solution of the equation

\[
\nabla_\ast \cdot (D \nabla_\ast v) = 0 \tag{37}
\]

on \( \Lambda \) with boundary conditions

\[
\begin{align*}
v(x) &= 0 & \text{if } x & \in F_- , \\
v(x) &= 1 & \text{if } x & \in F_+ , \\
D \nabla_\ast v(x) \cdot n(x) &= 0 & \text{if } x & \in \partial \Lambda \setminus F ,
\end{align*} \tag{38}
\]

if \( v \in K \) (cf. [34]) and if \( \int_\Lambda \nabla_\ast u \cdot D \nabla_\ast v \ dx = 0 \) for all \( u \in H^1_0(\Lambda, F, d_\ast) \).

Above \( n \) denotes the outward unit normal vector to the boundary in \( \partial \Lambda \) (which is well defined on \( \partial \Lambda \setminus F \)).
Remark 3.7. In the above definition it would be enough to require that \( \int_{\Lambda} \nabla_s u \cdot D \nabla_s v dx = 0 \) for all \( u \in C^c_c(\mathbb{R}^d \setminus F) \) since the functional \( H^1_0(\Lambda, F, d_s) \ni u \mapsto \int_{\Lambda} \nabla_s u \cdot D \nabla_s v dx \in \mathbb{R} \) is continuous.

We shortly motivate the above definition. To simplify the notation we take \( d_s = d \). We recall Green’s formula for a Lipschitz domain \( B \):

\[
\int_B (\partial_i f) g dx = -\int_B f (\partial_i g) dx + \int_{\partial B} f g (n \cdot e_i) dS, \quad \forall f, g \in C^1(\bar{B}),
\]

(39) where \( n \) denotes the outward unit normal vector to the boundary \( \partial B \) and \( dS \) is the surface measure on \( \partial B \). By taking \( f = \partial_j v \) and \( g = u \) in (39) we get

\[
\int_B u \nabla \cdot (D \nabla v) dx = -\int_B \nabla u \cdot (D \nabla v) dx + \int_{\partial B} u (\nabla v \cdot (D n)) dS,
\]

(40) for all \( v \in C^2(\bar{B}) \) and \( u \in C^1(\bar{B}) \). By taking (40) with \( B = \Lambda \) we see that \( v \in C^2(\bar{\Lambda}) \) satisfies \( \nabla \cdot (D \nabla v) = 0 \) on \( \Lambda \) and \( \nabla v \cdot (D n) \equiv 0 \) on \( \partial \Lambda \setminus F \) if and only if \( \int_\Lambda \nabla u \cdot (D \nabla v) dx = 0 \) for any \( u \in C^1(\bar{\Lambda}) \) with \( u \equiv 0 \) on \( F \). Such a set \( C \) of functions \( u \) is dense in \( H^1(\Lambda, F, d) \). Indeed \( C \subset H^1_0(\Lambda, F, d) \) by Lemma 3.3 while \( C^\infty_c(\mathbb{R}^d \setminus F) \subset C \). Hence, we conclude that \( v \in C^2(\bar{\Lambda}) \) satisfies \( \nabla \cdot (D \nabla v) = 0 \) on \( \Lambda \) and \( \nabla v \cdot (D n) \equiv 0 \) on \( \partial \Lambda \setminus F \) if and only if \( \int_\Lambda \nabla u \cdot (D \nabla v) dx = 0 \) for any \( u \in H^1_0(\Lambda, F, d) \). We have therefore proved that \( v \in C^2(\bar{\Lambda}) \) is a classical solution of (37) and (38) if and only if it is a weak solution in the sense of Definition 3.6.

Lemma 3.8. There exists a unique weak solution \( u \in K \) of the equation \( \nabla_s \cdot (D \nabla_s u) = 0 \) with boundary conditions (38). Furthermore, \( u \) is the unique minimizer of

\[
\inf_{v \in K} \int \nabla_s v \cdot D \nabla_s v dx.
\]

(41)

Proof. To simplify the notation, in what follows we write \( \psi \) instead of \( \psi_{|\Lambda} \).

We define the bilinear form \( a(f, g) := \int_\Lambda \nabla_s f \cdot D \nabla_s g dx \) on the Hilbert space \( H^1_0(\Lambda, F, d_s) \). The bilinear form \( a(\cdot, \cdot) \) is symmetric and continuous (since \( D \) is symmetric). Due to the Poincaré inequality (cf. Lemma 3.5) and since \( D_{1,1} > 0 \), \( a(\cdot, \cdot) \) is also coercive.

By definition we have that \( u \in K \) is a weak solution of equation \( \nabla_s \cdot (D \nabla_s u) = 0 \) with b.c. (38) if and only if, setting \( f := u - \psi \), \( f \in H^1_0(\Lambda, F, d_s) \) and \( f \) satisfies

\[
\int \nabla_s f \cdot D \nabla_s v dx = -\int \nabla_s \psi \cdot D \nabla_s v dx \quad \forall v \in H^1_0(\Lambda, F, d_s).
\]

(42)

Note that the r.h.s. is a continuous functional in \( v \in H^1_0(\Lambda, F, d_s) \). Due to the above observations and by Lax–Milgram theorem we conclude that there exists a unique such function \( f \), hence there is a unique weak solution \( u \) of equation \( \nabla_s \cdot (D \nabla_s u) = 0 \) with b.c. (38). Moreover \( f \) satisfies

\[
\frac{1}{2} a(f, f) + \int \nabla_s \psi \cdot D \nabla_s f dx = \inf_{g \in H^1_0(\Lambda, F, d_s)} \left\{ \frac{1}{2} a(g, g) + \int \nabla_s \psi \cdot D \nabla_s g dx \right\}.
\]

(43)
By adding to both sides \( \frac{1}{2} \int \nabla_x \psi \cdot D\nabla_x \psi \, dx \), we get that \( \frac{1}{2} \int \nabla_x u \cdot D\nabla_x u \, dx \) = inf_{v \in K} \frac{1}{2} \int \nabla_x v \cdot D\nabla_x v \, dx. \)

From the above lemma we immediately get:

**Corollary 3.9.** The function \( \psi|_A \) (cf. (30)) is the unique weak solution \( u \in K \) of the equation \( \nabla_x \cdot (D\nabla_x u) = 0 \) with boundary conditions (38).

4. Preliminary facts on \( \Omega, \mathcal{P} \) and \( \mathcal{P}_0 \)

In this section we recall some basic facts on the space \( \Omega \) and on the Palm distribution \( \mathcal{P}_0 \) associated to \( \mathcal{P} \).

The space \( \Omega \) of realizations of marked point processes is endowed with a Prohorov-like metric \( d \) such that the following facts are equivalent: (i) a sequence \( (\omega_n) \) converges to \( \omega \) in \( (\Omega, d) \), (ii) \( \lim_{n \to \infty} \int_{\mathbb{R}^d \times \mathbb{R}} f(x,s) d\omega_n(x,s) = \int_{\mathbb{R}^d \times \mathbb{R}} f(x,s) d\omega(x,s) \), for any bounded continuous function \( f : \mathbb{R}^d \times \Omega \to \mathbb{R} \) vanishing outside a bounded set and (iii) \( \lim_{n \to \infty} \omega_n(A) = \omega(A) \) for any bounded Borel set \( A \subset \mathbb{R}^d \times \mathbb{R} \) with \( \omega(\partial A) = 0 \) (see [10, App. A2.6 and Sect. 7.1]). In addition, \( (\Omega, d) \) is a separable metric space. Indeed, the above distance \( d \) is defined on the larger space \( \mathcal{N} \) of counting measures \( \mu = \sum k_i \delta_{(x_i, E_i)} \), where \( k_i \in \mathbb{N} \) and \( \{(x_i, E_i)\} \) is a locally finite subset of \( \mathbb{R}^d \times \mathbb{R} \), and one can prove that \( (\mathcal{N}, d) \) is a Polish space having \( \Omega \) as Borel subset [10, Cor. 7.1.IV, App. A2.6.I].

We recall some properties of the Palm distribution \( \mathcal{P}_0 \) associated to the measure \( \mathcal{P} \) on \( \Omega \). \( \mathcal{P}_0 \) is a probability measure with support inside \( \Omega_0 \) and it can be characterized by the identity

\[
\mathcal{P}_0(A) = \frac{1}{m} \int_{\Omega} \mathcal{P}(d\omega) \int_{[0,1]^d} d\hat{\omega}(x) 1_A(\tau_x \omega), \quad \forall A \subset \Omega_0 \text{ Borel}. \tag{44}
\]

The above identity (44) is a special case of the so-called Campbell’s formula (cf. [10, Eq. (12.2.4)]); for any nonnegative Borel function \( f : \mathbb{R}^d \times \Omega \to [0, \infty) \) it holds (recall (7))

\[
\int_{\mathbb{R}^d} dx \int_{\Omega_0} \mathcal{P}_0(d\omega) f(x, \omega) = \frac{1}{m} \int_{\Omega} \mathcal{P}(d\omega) \int_{\mathbb{R}^d} d\hat{\omega}(x) f(x, \tau_x \omega). \tag{45}
\]

An alternative characterization of \( \mathcal{P}_0 \) is described in [29, Section 1.2].

A fact frequently used in the rest is the following (see [17, Lemma 1]): given a translation invariant Borel subset \( A \subset \Omega \), it holds \( \mathcal{P}(A) = 1 \) if and only if \( \mathcal{P}_0(A) = 1 \).

We recall some basic technical facts discussed in [12]:

**Lemma 4.1.** [12] Lemma 4.1. Given a Borel subset \( A \subset \Omega_0 \), the following facts are equivalent:

(i) \( \mathcal{P}_0(A) = 1 \);

(ii) \( \mathcal{P}(\omega \in \Omega : \tau_x \omega \in A \forall x \in \hat{\omega}) = 1 \);

(iii) \( \mathcal{P}_0(\omega \in \Omega_0 : \tau_x \omega \in A \forall x \in \hat{\omega}) = 1 \).
Lemma 4.2. [17 Lemma 1–(i)] [12 Lemma 4.3] Let $k : \Omega_0 \times \Omega_0 \to \mathbb{R}$ be a Borel function such that (i) at least one of the functions $\int d\hat{\omega}(x)|k(\omega, \tau_x \omega)|$ and $\int d\hat{\omega}(x)|k(\tau_x \omega, \omega)|$ is in $L^1(\mathcal{P}_0)$, or (ii) $k(\omega, \omega) \geq 0$. Then
\[
\int d\mathcal{P}_0(\omega) \int d\hat{\omega}(x)k(\omega, \tau_x \omega) = \int d\mathcal{P}_0(\omega) \int d\hat{\omega}(x)k(\tau_x \omega, \omega). \tag{46}
\]

We conclude by focusing on ergodicity. Since by Assumption (A1) $\mathcal{P}$ is ergodic, we have the following result (cf. [10 Prop. 12.2.6I]): given a nonnegative Borel function $g : \Omega_0 \to [0, \infty)$ it holds
\[
\lim_{n \to \infty} \frac{1}{(2n)^d} \int_{[-n,n]^d} d\hat{\omega}(x) g(\tau_x \omega) = m \mathbb{E}_0[g] \quad \mathcal{P}-\text{a.s.} \tag{47}
\]
One can indeed refine the above result. To this aim we define $\mu^\varepsilon$ as the atomic measure on $\mathbb{R}^d$ given by $\mu^\varepsilon := \varepsilon^d \sum_{x \in \hat{\omega}} \delta_{x \varepsilon}$. Then it holds:

Proposition 4.3. [12 Prop. 3.1] Let $g : \Omega_0 \to \mathbb{R}$ be a Borel function with $\|g\|_{L^1(\mathcal{P}_0)} < +\infty$. Then there exists a translation invariant Borel subset $\mathcal{A}[g] \subset \Omega$ such that $\mathcal{P}(\mathcal{A}[g]) = 1$ and such that, for any $\omega \in \mathcal{A}[g]$ and any $\varphi \in C_c(\mathbb{R}^d)$, it holds
\[
\lim_{\varepsilon \to 0} \int d\mu^\varepsilon(x) \varphi(x) g(\tau_x \varepsilon \omega) = \int dx \mu(\varepsilon \omega) \cdot \mathbb{E}_0[g]. \tag{48}
\]

The above proposition (which is the analogous e.g. of [29 Theorem 1.1]) is at the core of 2-scale convergence. It corresponds to a refined version of ergodicity. The variable $x$ appears in the l.h.s. of (48) at the macroscopic scale in $\varphi(x)$ and at the microscopic scale in $g(\tau_x \varepsilon \omega)$.

Definition 4.4. Given a function $g : \Omega_0 \to [0, +\infty]$ such that $\mathbb{E}_0[g] < +\infty$, we define $\mathcal{A}[g]$ as $\mathcal{A}[g_*]$ (cf. Proposition 4.3), where $g_* : \Omega_0 \to \mathbb{R}$ is defined as $g$ on $\{g < +\infty\}$ and as $0$ on $\{g = +\infty\}$.

5. The Hilbert space $H^{1,\varepsilon}_{\omega,0}$ and the amorphous gradient $\nabla_\varepsilon f$

In this section we come back to the Hilbert space $H^{1,\varepsilon}_{\omega,0}$ introduced in Section 2 proving some properties used there and extending the discussion. In addition, in Subsection 5.1 we collect some basic properties of the amorphous gradient $\nabla_\varepsilon$, which will be frequently used in the proof of Theorem 2.

Let $\omega \in \Omega_1$ (cf. [27]). Recall Definition 2.4 of $H^{1,\varepsilon}_{\omega,0}$ and $K^{\varepsilon}_{\omega}$. As discussed in Section 2 if $f : \varepsilon \omega \cap S \to \mathbb{R}$ is bounded, then $f \in L^2(\mu^\varepsilon_{\omega,\Lambda})$, $\nabla_\varepsilon f \in L^2(\nu^\varepsilon_{\omega,\Lambda})$ and $L^\varepsilon f \in L^2(\mu^\varepsilon_{\omega,\Lambda})$. By definition of $\nu^\varepsilon_{\omega,\Lambda}$, given bounded functions $f, g : \varepsilon \omega \cap S \to \mathbb{R}$, we have
\[
\langle \nabla_\varepsilon f, \nabla_\varepsilon g \rangle_{L^2(\nu^\varepsilon_{\omega,\Lambda})} = \varepsilon^{d-2} \sum_{(x,y) \in E_\varepsilon} c_{\varepsilon^2,\varepsilon^2}(\omega)(f(y) - f(x))(g(y) - g(x)). \tag{49}
\]

Lemma 5.1. Let $\omega \in \Omega_1$. Given $f, g : \varepsilon \omega \cap S \to \mathbb{R}$ with $f \in H^{1,\varepsilon}_{\omega,0}$ and $g$ bounded, it holds
\[
\langle f, -L^\varepsilon_\omega g \rangle_{L^2(\mu^\varepsilon_{\omega,\Lambda})} = \frac{1}{2} \langle \nabla_\varepsilon f, \nabla_\varepsilon g \rangle_{L^2(\nu^\varepsilon_{\omega,\Lambda})}. \tag{50}
\]
By the Lax–Milgram theorem we conclude that there exists a unique function
\[ f \] such that
\[ a(f, g) = 0 \] for all \( g \in V \). Moreover, by Assumption (A7) and (15), it holds
\[ \inf \{ \langle \nabla_v \psi, \nabla_v \rangle_{L^2(\nu_{\omega}, \Lambda)} \mid v \in K^\varepsilon_{\omega} \} > 0 \] due to Lemma 5.1.

\[ \omega \] satisfies (55), thus implying Item (i). Since \( a(f, f) = \frac{1}{2} \langle \nabla f, \nabla f \rangle_{L^2(\nu_{\omega}, \Lambda)} \),
the uniqueness of the solution \( f \) of (55) corresponds to Item (ii). Moreover,
always by the Lax–Milgram theorem, $f_\varepsilon$ is the unique minimizer of the functional $H^{1,\varepsilon}_{0,\omega} \ni v \mapsto \frac{1}{2}a(v,v) + \frac{1}{2} \langle \nabla_\varepsilon \psi, \nabla_\varepsilon v \rangle_{L^2(\nu_{\varepsilon,\lambda}^\varepsilon)}$, and therefore of the functional $H^{1,\varepsilon}_{0,\omega} \ni v \mapsto \frac{1}{4} \langle \nabla_\varepsilon (v + \psi), \nabla_\varepsilon (v + \psi) \rangle_{L^2(\nu_{\varepsilon,\lambda}^\varepsilon)}$. This proves Item (iii). □

**Remark 5.3.** As $V_\varepsilon$ is “harmonic” on $\varepsilon \hat{\omega} \cap \Lambda$ (cf. Lemma 3.2 (i)) and $\omega \in \Omega_1$, $V_\varepsilon$ has values in $[0, 1]$.

**Lemma 5.4.** There exists a translation invariant Borel subset $\Omega_2 \subset \Omega_1$ such that $\mathcal{P}(\Omega_2) = 1$ and, for all $\omega \in \Omega_2$,

$$\limsup_{\varepsilon \downarrow 0} \|\psi\|_{L^2(\mu_{\varepsilon,\lambda}^\varepsilon)} < +\infty, \quad \limsup_{\varepsilon \downarrow 0} \|\nabla_\varepsilon \psi\|_{L^2(\nu_{\varepsilon,\lambda}^\varepsilon)} < +\infty, \quad (56)$$

$$\limsup_{\varepsilon \downarrow 0} \|V_\varepsilon\|_{L^2(\nu_{\varepsilon,\lambda}^\varepsilon)} < +\infty, \quad \limsup_{\varepsilon \downarrow 0} \|\nabla_\varepsilon V_\varepsilon\|_{L^2(\nu_{\varepsilon,\lambda}^\varepsilon)} < +\infty. \quad (57)$$

**Proof.** By Proposition 4.3 applied with suitable test functions $\varphi$, there exists a translation invariant Borel set $\Omega_2 \subset \Omega_1$ such that $\lim_{\varepsilon \downarrow 0} \mu_{\varepsilon}(\Lambda) = m$ and $\lim_{\varepsilon \downarrow 0} \int_{\Lambda} \mu_{\varepsilon}(dx) \lambda(\tau_\varepsilon x, \omega) = E_0[\lambda_2]$ for any $\omega \in \Omega_2$.

Let us take $\omega \in \Omega_2$. Since $\psi, V_\varepsilon$ have value in $[0, 1]$ and $\mu_{\varepsilon,\lambda}^\varepsilon$ has mass $\mu_{\varepsilon}(\Lambda) \to m$, we get the first bounds in (56) and (57).

Let us prove that $\limsup_{\varepsilon \downarrow 0} \|\nabla_\varepsilon \psi\|_{L^2(\nu_{\varepsilon,\lambda}^\varepsilon)} < +\infty$. We have (recall (49))

$$\|\nabla_\varepsilon \psi\|_{L^2(\nu_{\varepsilon,\lambda}^\varepsilon)}^2 = \varepsilon^{d-2} \sum_{(x,y) \in E_\varepsilon} c_{x/\varepsilon,y/\varepsilon}(\omega) (\psi(y) - \psi(x))^2$$

$$\leq \varepsilon^{d-2} \sum_{(x,y) \in E_\varepsilon} c_{x/\varepsilon,y/\varepsilon}(\omega) (y_1 - x_1)^2$$

$$\leq 2 \varepsilon^{d-2} \sum_{x \in \varepsilon \hat{\omega} \cap \Lambda} \sum_{y \in \varepsilon \hat{\omega} \cap S} c_{x/\varepsilon,y/\varepsilon}(\omega) (y_1 - x_1)^2 \quad (58).$$

We can rewrite the last expression as

$$2 \varepsilon^d \sum_{x \in \varepsilon \hat{\omega} \cap (\varepsilon^{-1} A)} \sum_{y \in \varepsilon \hat{\omega} \cap (\varepsilon^{-1} S)} c_{x,y}(\omega) (y_1 - x_1)^2,$$

which is upper bounded by $2 \varepsilon^d \sum_{x \in \varepsilon \hat{\omega} \cap (\varepsilon^{-1} A)} \lambda(\tau_\varepsilon x, \omega) = 2 \int_{\Lambda} \mu_{\varepsilon}(dx) \lambda(\tau_\varepsilon x, \omega)$. The last integral converges to $2E_0[\lambda_2] < +\infty$ as $\omega \in \Omega_2$. This concludes the proof that $\limsup_{\varepsilon \downarrow 0} \|\nabla_\varepsilon \psi\|_{L^2(\nu_{\varepsilon,\lambda}^\varepsilon)} < +\infty$.

Since $V_\varepsilon$ minimizes (53), we have $\|\nabla_\varepsilon V_\varepsilon\|_{L^2(\nu_{\varepsilon,\lambda}^\varepsilon)} \leq \|\nabla_\varepsilon \psi\|_{L^2(\nu_{\varepsilon,\lambda}^\varepsilon)}$. Hence $\limsup_{\varepsilon \downarrow 0} \|\nabla_\varepsilon V_\varepsilon\|_{L^2(\nu_{\varepsilon,\lambda}^\varepsilon)} < +\infty$ by the second bound in (56). □

**5.1. Some properties of the amorphous gradient $\nabla_\varepsilon$.** In Section 2 we have defined $\nabla_\varepsilon f$ for functions $f : \varepsilon \hat{\omega} \cap S \to \mathbb{R}$. The definition can be extended by replacing $S$ with any set $A \subset \mathbb{R}^d$. Given $f, g : \varepsilon \hat{\omega} \to \mathbb{R}$, it is simple to check the following Leibniz rule:

$$\nabla_\varepsilon (fg)(x, z) = \nabla_\varepsilon f(x, z)g(x) + f(x + \varepsilon z)\nabla_\varepsilon g(x, z).\quad (59)$$

Let $\varphi \in C^1_c(\mathbb{R}^d)$. Let $\ell$ be such that $\varphi(x) = 0$ if $|x| \geq \ell$. Fix $\phi \in C^1_c(\mathbb{R}^d)$ with values in $[0, 1]$, such that $\phi(x) = 1$ for $|x| \leq \ell$ and $\phi(x) = 0$ for $|x| \geq \ell + 1$. 


Since $\nabla_{x} \varphi(x, z) = 0$ if $|x| \geq \ell$ and $|x + \varepsilon z| \geq \ell$, by the mean value theorem we conclude that
\[ |\nabla_{x} \varphi(x, z)| \leq \|\nabla \varphi\|_{\infty} |z| (\varphi(x) + \varphi(x + \varepsilon z)). \] (60)
If in addition $\varphi \in C^{2}(\mathbb{R}^{d})$, by Taylor expansion $|\nabla_{x} \varphi(x, z) - \nabla \varphi(x) \cdot z| \leq \varepsilon C(\varphi)|z|^{2}$ for some constant $C(\varphi)$ depending only on $\varphi$. Note that $\nabla_{x} \varphi(x, z) - \nabla \varphi(x) \cdot z = 0$ if $|x| \geq \ell$ and $|x + \varepsilon z| \geq \ell$. Hence we get that
\[ |\nabla_{x} \varphi(x, z) - \nabla \varphi(x) \cdot z| \leq \varepsilon C(\varphi)|z|^{2}(\varphi(x) + \varphi(x + \varepsilon z)). \] (61)

6. PROOF OF THEOREM 1 WHEN $D_{1,1} = 0$

We need to prove (32), i.e. that $\mathcal{P}$–a.s. $\lim_{\varepsilon \downarrow 0} \langle \nabla_{x} V_{\varepsilon}, \nabla_{x} V_{\varepsilon} \rangle_{L^{2}(\nu_{\varepsilon, \lambda})} = 0$. As $D_{1,1} = 0$ and by (10), given $\delta > 0$ we can fix $f \in L^{\infty}(P_{0})$ such that
\[ E_{0} \left[ \int d\omega(x)c_{0,x}(\omega) \left( x_{1} - \nabla f(\omega, x) \right)^{2} \right] \leq \delta. \] (62)
Given $\varepsilon > 0$ we define the function $v_{\varepsilon}: \varepsilon \hat{\omega} \cap S \rightarrow \mathbb{R}$ as
\[ v_{\varepsilon}(x) := \begin{cases} \psi(x) + \varepsilon f(\tau_{x/\varepsilon}) & \text{if } x \in \Lambda, \\ 0 & \text{if } x \in S_{-}, \\ 1 & \text{if } x \in S_{+}. \end{cases} \] (63)
By Lemma 5.2(iii) it is enough to prove that $\lim_{\varepsilon \downarrow 0} \langle \nabla_{x} v_{\varepsilon}, \nabla_{x} v_{\varepsilon} \rangle_{L^{2}(\nu_{\varepsilon, \lambda})} = 0 \mathcal{P}$–a.s.. We write
\[ \frac{1}{2} \langle \nabla_{x} v_{\varepsilon}, \nabla_{x} v_{\varepsilon} \rangle_{L^{2}(\nu_{\varepsilon, \lambda})} \leq \varepsilon^{d-2} \sum_{x \in \hat{\omega} \cap \varepsilon^{-1} \Lambda} \sum_{y \in \hat{\omega} \cap \varepsilon^{-1} S} c_{x,y}(\omega) \left( v_{\varepsilon}(x) - v_{\varepsilon}(y) \right)^{2}. \] (64)
We split the sum in the r.h.s. into three contributions $C(\varepsilon)$, $C_{-}(\varepsilon)$ and $C_{+}(\varepsilon)$, corresponding respectively to the cases $y \in \hat{\omega} \cap \varepsilon^{-1} \Lambda$, $y \in \hat{\omega} \cap \varepsilon^{-1} S_{-}$ and $y \in \hat{\omega} \cap \varepsilon^{-1} S_{+}$, while in all the above contributions $x$ varies among $\hat{\omega} \cap \varepsilon^{-1} \Lambda$.
If $x, y \in \hat{\omega} \cap \varepsilon^{-1} \Lambda$, then $v_{\varepsilon}(y) - v_{\varepsilon}(x) = \varepsilon(y_{1} - x_{1} - \nabla f(\tau_{x} \omega, y - x))$. Hence, we can bound
\[ C(\varepsilon) \leq \varepsilon^{d} \sum_{x \in \hat{\omega} \cap \varepsilon^{-1} \Lambda} \sum_{y \in \hat{\omega}} c_{x,y}(y)(y_{1} - x_{1} - \nabla f(\tau_{x} \omega, y - x))^{2}. \] (65)
By ergodicity (cf. (47), Proposition 4.3) the r.h.s. converges $\mathcal{P}$–a.s. to the l.h.s of (62), and therefore it is bounded by $\delta \mathcal{P}$–a.s.. Hence, $\lim_{\varepsilon \downarrow 0} C(\varepsilon) \leq \delta$.
We now consider $C_{-}(\varepsilon)$ and prove that $\lim_{\varepsilon \downarrow 0} C_{-}(\varepsilon) = 0$. If $x \in \hat{\omega} \cap \varepsilon^{-1} \Lambda$ and $y \in \hat{\omega} \cap \varepsilon^{-1} S_{-}$, then $v_{\varepsilon}(x) - v_{\varepsilon}(y) = \varepsilon^{2}(x_{1} - f(\tau_{x} \omega))^{2} \leq 2\varepsilon^{2} x_{1}^{2} + 2\varepsilon^{2} \|f\|_{\infty}^{2} \leq 2\varepsilon^{2}(x_{1} - y_{1})^{2} + 2\varepsilon^{2} \|f\|_{\infty}^{2}$. Hence it remains to show that
\[ \varepsilon^{d} \sum_{x \in \hat{\omega} \cap \varepsilon^{-1} \Lambda} \sum_{y \in \hat{\omega} \cap \varepsilon^{-1} S_{-}} c_{x,y}(\omega)(x_{1} - y_{1})^{2} + 1 \] (66)
goes to zero as $\varepsilon \downarrow 0$. Given $\rho \in (0, 1/2)$ we set $\Lambda_{\rho} := (-\rho, \rho)^{d}$. We denote by $A_{1}(\rho, \varepsilon)$ the sum in (66) restricted to $x \in \hat{\omega} \cap \varepsilon^{-1} \Lambda_{\rho}$ and $y \in \hat{\omega} \cap \varepsilon^{-1} S_{-}$. We denote by $A_{2}(\rho, \varepsilon)$ the sum coming from the remaining addenda so that (66) equals $A_{1}(\rho, \varepsilon) + A_{2}(\rho, \varepsilon)$. Given $x, y$ as in $A_{1}(\rho, \varepsilon)$, it holds $x_{1} - y_{1} \geq
(1/2 − ρ)/ε ≥ 1 for ε small enough. In this case, we can bound \( c_{x,y}(\omega) [(x_1 − y_1)^2 + 1] \leq C c_{x,y}(\omega)^a \), for some universal positive constant \( C \). Indeed, due to [13], \( \lim_{\epsilon \to +\infty} \epsilon^2 \rho(\ell) < +\infty \) where \( \rho(\ell) := \sup_{\omega \in \Omega_0} \sup_{\ell \in \omega} c_{0,\ell}(\omega)^{1−a} \). Due to the above observations,

\[
A_1(\rho, \epsilon) \leq C(\omega) \epsilon^d \sum_{x \in \omega \cap \epsilon^{-1} \Lambda} \sum_{y \in \omega} c_{x,y}(\omega)^a \mathbf{1}(|x − y| \geq \rho/\epsilon).
\]

By the ergodic theorem and [11], we get that \( \lim_{\epsilon \downarrow 0} A_1(\rho, \epsilon) = 0 \) \( \mathcal{P} \)-a.s. We move to \( A_2(\rho, \epsilon) \). By Proposition 4.3 with suitable test functions, we get that (68) converges as \( \epsilon \downarrow 0 \) to \( \mathbb{E}_{\Omega_0} |\lambda_2^0 + \lambda_0^0(\ell(\Lambda \setminus \Lambda_\rho)) \), where here \( \ell(\cdot) \) denotes the Lebesgue measure. To conclude the proof that \( \lim_{\epsilon \downarrow 0} C_-(\epsilon) = 0 \), it is therefore enough to take the limit \( \rho \uparrow 1/2 \).

By the same arguments used for \( C_-(\epsilon) \), one proves that \( \lim_{\epsilon \downarrow 0} C_+(\epsilon) = 0 \).

7. Square integrable forms and effective diffusion matrix

**Warning 7.1.** *From this section, until Section 12 included, we assume that \( D_{1,1} > 0 \). In particular, \( d_\ast \geq 1 \) is defined according to Warning 2.2.*

As typical in homogenization theory [18], the variational formula (10) defining the effective diffusion matrix \( D \) admits a geometrical interpretation in the Hilbert space of square integrable forms. We recall here this interpretation. We also collect some facts taken from [12]. They are mainly an adaptation to the present contest of very general facts (see e.g. [18, 29]) and can be easily checked (all proofs have been provided in [12]).

7.1. Square integrable forms. We define \( \nu \) as the Radon measure on \( \Omega \times \mathbb{R}^d \) such that

\[
\int d\nu(\omega, z) g(\omega, z) = \int d\mathcal{P}_0(\omega) \int d\tilde{\omega}(z) c_{0,\tilde{z}}(\omega) g(\omega, z)
\]

for any nonnegative Borel function \( g(\omega, z) \). We point out that \( \nu \) has finite total mass since \( \nu(\Omega \times \mathbb{R}^d) = \mathbb{E}_{\Omega_0} |\lambda_0^0| < +\infty \). Elements of \( L^2(\nu) \) are called *square integrable forms*.

Given a function \( u : \Omega_0 \to \mathbb{R} \), its gradient \( \nabla u : \Omega \times \mathbb{R}^d \to \mathbb{R} \) is defined as

\[
\nabla u(\omega, z) := u(\tau_z \omega) − u(\omega).
\]

If \( u \) is defined \( \mathcal{P}_0 \)-a.s., then \( \nabla u \) is well defined \( \nu \)-a.s. by Lemma 4.1. If \( u \) is bounded and measurable, then \( \nabla u \in L^2(\nu) \). The subspace of potential forms \( L^2_{\text{pot}}(\nu) \) is defined as the following closure in \( L^2(\nu) \):

\[
L^2_{\text{pot}}(\nu) := \{ \nabla u : u \text{ is bounded and measurable} \}.
\]

The subspace of solenoidal forms \( L^2_{\text{sol}}(\nu) \) is defined as the orthogonal complement of \( L^2_{\text{pot}}(\nu) \) in \( L^2(\nu) \).
Definition 7.1. Given a square integrable form \( v \in L^2(\nu) \) we define its divergence \( \text{div} \, v \in L^1(\mathcal{P}_0) \) as
\[
\text{div} \, v(\omega) = \int d\omega(\varepsilon) c_{0,\varepsilon}(\omega)(v(\omega, z) - v(\tau_z \omega, -z)).
\] (71)

The r.h.s. of (71) is well defined since it corresponds to an absolutely convergent series by Lemma 4.2.

For any \( v \in L^2(\nu) \) and any bounded and measurable function \( u : \Omega \to \mathbb{R} \), it holds (cf. [12, Lemma 5.4])
\[
\int d\mathcal{P}_0(\omega) \text{div} \, v(\omega) u(\omega) = -\int d\nu(\omega, z) v(\omega, z) \nabla u(\omega, z).
\] (72)

As a consequence we have that, given \( v \in L^2(\nu), \, v \in L^2_{\text{sol}}(\nu) \) if and only if \( \text{div} \, v = 0 \, \mathcal{P}_0\text{-a.s.} \) (cf. [12, Cor. 5.5]). We also have (cf. [12, Lemma 5.8]):

Lemma 7.2. The functions \( g \in L^2(\mathcal{P}_0) \) of the form \( g = \text{div} \, v \) with \( v \in L^2(\nu) \) are dense in \( \{ w \in L^2(\mathcal{P}_0) : \mathbb{E}_0[w] = 0 \} \).

7.2. Diffusion matrix. As \( \lambda_2 \in L^1(\mathcal{P}_0), \) given \( a \in \mathbb{R}^d \) the form
\[
u_a(\omega, z) := a \cdot z
\] (73)
is square integrable, i.e. it belongs to \( L^2(\nu) \). We note that the symmetric diffusion matrix \( D \) defined in (10) satisfies, for any \( a \in \mathbb{R}^d \),
\[
q(a) := a \cdot Da = \inf_{v \in L^2_{\text{sol}}(\nu)} \frac{1}{2} \int d\nu(\omega, x) (u_a(x) + v(\omega, x))^2 = \inf_{v \in L^2_{\text{sol}}(\nu)} \frac{1}{2} \| u_a + v^a \|^2_{L^2(\nu)},
\] (74)
where \( v^a = -\Pi u_a \) and \( \Pi : L^2(\nu) \to L^2_{\text{sol}}(\nu) \) denotes the orthogonal projection of \( L^2(\nu) \) on \( L^2_{\text{sol}}(\nu) \). It follows easily that \( v^a \) is characterized by the properties
\[
v^a \in L^2_{\text{sol}}(\nu), \quad v^a + u_a \in L^2_{\text{sol}}(\nu).
\] (75)
Moreover it holds (cf. [12, Section 6]):
\[
Da = \frac{1}{2} \int d\nu(\omega, z) z (a \cdot z + v^a(\omega, z)) \quad \forall a \in \mathbb{R}^d.
\] (76)

By (74) the kernel \( \text{Ker}(q) \) of the quadratic form \( q \) is given by
\[
\text{Ker}(q) := \{ a \in \mathbb{R}^d : q(a) = 0 \} = \{ a \in \mathbb{R}^d : u_a \in L^2_{\text{sol}}(\nu) \}.
\] (77)

The following result is the analogous of [29, Lemma 5.1]:

Lemma 7.3. [12, Lemma 6.1] It holds
\[
\text{Ker}(q)^\perp = \left\{ \int d\nu(\omega, z) b(\omega, z) z : b \in L^2_{\text{sol}}(\nu) \right\}.
\] (78)

It is simple to check that Warning 2.2 and Lemma 7.3 imply the following:

Corollary 7.4. \( \text{Span}\{e_1, e_2, \ldots, e_d\} = \left\{ \int d\nu(\omega, z) b(\omega, z) z : b \in L^2_{\text{sol}}(\nu) \right\} \).
7.3. The contraction \( b(\omega, z) \mapsto \hat{b}(\omega) \) and the set \( A_1[b] \).

**Definition 7.5.** Let \( b(\omega, z) : \Omega_0 \times \mathbb{R}^d \to \mathbb{R} \) be a Borel function with \( \|b\|_{L^1(\nu)} < +\infty \). We define the Borel function \( c_b : \Omega_0 \to [0, +\infty] \) as

\[
c_b(\omega) := \int d\hat{\omega}(z)c_{0,z}(\omega)|b(\omega, z)|,
\]

the Borel function \( \hat{b} : \Omega_0 \to \mathbb{R} \) as

\[
\hat{b}(\omega) := \begin{cases} 
\int d\hat{\omega}(z)c_{0,z}(\omega)b(\omega, z) & \text{if } c_b(\omega) < +\infty, \\
0 & \text{if } c_b(\omega) = +\infty,
\end{cases}
\]

and the Borel set \( A_1[b] := \{\omega \in \Omega : c_b(\tau_z \omega) < +\infty \forall z \in \hat{\omega}\} \).

We consider the atomic measures (\( \mu^\varepsilon_x \) was introduced in Section 4)

\[
\mu^\varepsilon_x := \varepsilon^d \sum_{x \in \omega} \delta_x, \quad \nu^\varepsilon_x := \sum_{x \in \omega} \sum_{y \in \omega} \varepsilon^d c_{z,y}(\omega) \delta_{(x, \frac{y-z}{\varepsilon})}.
\]

**Lemma 7.6.** [12, Lemma 7.2] Let \( b(\omega, z) : \Omega_0 \times \mathbb{R}^d \to \mathbb{R} \) be a Borel function with \( \|b\|_{L^1(\nu)} < +\infty \). Then

(i) \( \|\hat{b}\|_{L^1(\mathcal{P}_0)} \leq \|b\|_{L^1(\nu)} = \|c_b\|_{L^1(\mathcal{P}_0)} \) and \( \mathbb{E}_0[\hat{b}] = \nu(b); \)

(ii) given \( \omega \in \mathcal{A}_1[b] \) and \( \varphi \in C_c(\mathbb{R}^d) \), it holds

\[
\int d\mu^\varepsilon_x(\omega)\varphi(x)\hat{b}(\tau_x/\varepsilon \omega) = \int d\nu^\varepsilon_x(\omega)\varphi(x)b(\tau_x/\varepsilon, z)
\]

(82)

(the series in the l.h.s. and in the r.h.s. are absolutely convergent);

(iii) \( \mathcal{P}(\mathcal{A}_1[b]) = \mathcal{P}_0(\mathcal{A}_1[b]) = 1 \) and \( \mathcal{A}_1[b] \) is translation invariant.

7.4. The transformation \( b(\omega, z) \mapsto \hat{b}(\omega, z) \).

**Definition 7.7.** Given a Borel function \( b : \Omega_0 \times \mathbb{R}^d \to \mathbb{R} \) we set

\[
\tilde{b}(\omega, z) := \begin{cases} 
b(\tau_z \omega, -z) & \text{if } z \in \hat{\omega}, \\
0 & \text{otherwise}.
\end{cases}
\]

By applying Lemma 4.1 and using Assumption (A3), one gets:

**Lemma 7.8.** [12, Lemma 8.2] Given a Borel function \( b : \Omega_0 \times \mathbb{R}^d \to \mathbb{R} \), it holds \( \tilde{b}(\omega, z) = b(\omega, z) \) if \( z \in \hat{\omega} \). If \( b \in L^1(\nu) \), then \( \|b\|_{L^1(\nu)} = \|\hat{b}\|_{L^1(\nu)} \). If \( b \in L^2(\nu) \), then \( \|b\|_{L^2(\nu)} = \|\hat{b}\|_{L^2(\nu)} \) and \( \text{div } \hat{b} = -\text{div } b \).

**Definition 7.9.** Let \( b : \Omega_0 \times \mathbb{R}^d \to \mathbb{R} \) be a Borel function with \( \|b\|_{L^1(\nu)} < +\infty \). If \( \omega \in \mathcal{A}_1[b] \cap \mathcal{A}_1[\hat{b}] \cap \Omega_0 \), we set \( \text{div}_b(\omega) := \hat{b}(\omega) - \tilde{b}(\omega) \in \mathbb{R} \).

**Lemma 7.10.** [12, Lemma 8.5] Let \( b : \Omega_0 \times \mathbb{R}^d \to \mathbb{R} \) be a Borel function with \( \|b\|_{L^2(\nu)} < +\infty \). Then \( \mathcal{P}_0(\mathcal{A}_1[b] \cap \mathcal{A}_1[\hat{b}]) = 1 \) and \( \text{div}_b = \text{div } b \) in \( L^1(\mathcal{P}_0) \).
Lemma 7.11. [12, Lemma 8.6] Let $b : \Omega_0 \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a Borel function with $\|b\|_{L^2(\nu)} < +\infty$ and such that its class of equivalence in $L^2(\nu)$ belongs to $L^2_{\text{sol}}(\nu)$. Let
\[ \mathcal{A}_d[b] := \{ \omega \in \mathcal{A}_1[b] \cap \mathcal{A}_1[\hat{b}] : \text{div}_b(\tau_{\omega}\omega) = 0 \ \forall \omega \in \hat{\omega} \}. \tag{84} \]
Then $\mathcal{P}(\mathcal{A}_d[b]) = 1$ and $\mathcal{A}_d[b]$ is translation invariant.

Lemma 7.12. [12, Lemma 8.7] Suppose that $b : \Omega_0 \times \mathbb{R}^d \rightarrow \mathbb{R}$ is a Borel function with $\|b\|_{L^2(\nu)} < +\infty$. Take $\omega \in \mathcal{A}_1[b] \cap \mathcal{A}_1[\hat{b}]$. Then for any $\varepsilon > 0$ and any $u : \mathbb{R}^d \rightarrow \mathbb{R}$ with compact support it holds
\[ \int d\mu^\omega_{\varepsilon}(x) u(x) \text{div}_b(\tau_{\omega/\varepsilon}\omega) = -\varepsilon \int d\nu^\omega_{\varepsilon}(x,z) \nabla_z u(x,z) b(\tau_{\omega/\varepsilon}\omega, z). \tag{85} \]

Lemma 7.13. [12, Lemma 8.3]
(i) Let $b : \Omega_0 \times \mathbb{R}^d \rightarrow [0, +\infty]$ and $\varphi, \psi : \mathbb{R}^d \rightarrow [0, +\infty]$ be Borel functions. Then, for each $\omega \in \Omega$, it holds
\[ \int d\nu^\omega_{\varepsilon}(x,z) \varphi(x) \psi(x + \varepsilon z) b(\tau_{\omega/\varepsilon}\omega, z) = \int d\nu^\omega_{\varepsilon}(x,z) \varphi(x) \psi(x + \varepsilon z) \tilde{b}(\tau_{\omega/\varepsilon}\omega, z). \tag{86} \]

(ii) Let $b : \Omega_0 \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a Borel function with $\|b\|_{L^1(\nu)} < +\infty$ and take $\omega \in \mathcal{A}_1[b] \cap \mathcal{A}_1[\hat{b}]$. Given functions $\varphi, \psi : \mathbb{R}^d \rightarrow \mathbb{R}$ such that at least one between $\varphi, \psi$ has compact support and the other is bounded, identity [86] is still valid.

Given now $\varphi$ with compact support and $\psi$ bounded, it holds
\[ \int d\nu^\omega_{\varepsilon}(x,z) \nabla_z \varphi(x,z) \psi(x + \varepsilon z) b(\tau_{\omega/\varepsilon}\omega, z) \]
\[ = -\int d\nu^\omega_{\varepsilon}(x,z) \nabla_z \varphi(x,z) \psi(x) \tilde{b}(\tau_{\omega/\varepsilon}\omega, z). \tag{87} \]
Moreover, the above integrals in [86], [87] (under the hypothesis of this Item (ii)) correspond to absolutely convergent series and are therefore well defined.

Recall the set $\mathcal{A}[g]$ introduced in Prop. 4.3 and Definition 4.4.

Lemma 7.14. Suppose that $\omega$ belongs to the sets $\mathcal{A}_1[1]$, $\mathcal{A}[\lambda_0]$, $\mathcal{A}_1[|z|^21_{\{|z|\geq\ell\}}]$ and $\mathcal{A}[\int d\omega(z) c_{0,\varepsilon}(\omega)|z|^21_{\{|z|\geq\ell\}}]$ for all $\ell \in \mathbb{N}$. Then $\forall \varphi \in C_c^2(\mathbb{R}^d)$ we have
\[ \lim_{\varepsilon \downarrow 0} \int d\nu^\omega_{\varepsilon}(x,z) \left[ \nabla_{\varepsilon} \varphi(x,z) - \nabla \varphi(x) \cdot z \right]^2 = 0. \tag{88} \]

The above lemma is related to [12, Lemma 15.2]. We give the proof, since we need to isolate the conditions leading to [88] (which in [12] are assured by the property that $\omega$ belongs to the space $\Omega_{\text{typ}}$ in [12]).

Proof. Let $\ell, \phi$ be defined as done before [60]. The upper bound given by [60] with $\nabla_{\varepsilon} \varphi(x,z)$ replaced by $\nabla \varphi(x) \cdot z$ is also true. We will apply the above bounds for $|z| \geq \ell$. On the other hand, we apply [61] for $|z| < \ell$. As a result, we can bound
\[ \int d\nu^\omega_{\varepsilon}(x,z) \left[ \nabla_{\varepsilon} \varphi(x,z) - \nabla \varphi(x) \cdot z \right]^2 \leq C(\varphi)[A(\varepsilon, \ell) + B(\varepsilon, \ell)], \tag{89} \]
where (cf. (86))

\[ A(\varepsilon, \ell) := \int d\nu^\varepsilon_b(x, z)|z|^2(\phi(x) + \phi(x + \varepsilon z))1_{\{z \geq \ell\}} \]

\[ = 2 \int d\nu^\varepsilon_b(x, z)|z|^2\phi(x)1_{\{z \geq \ell\}} = 2 \int d\mu^\varepsilon_b(x)\phi(x)h_\ell(\tau_{x/\varepsilon} \omega), \]

\[ h_\ell(\omega) := \int d\hat{\omega}(z)c_{0,z}(\omega)|z|^21_{\{z \geq \ell\}}, \]

\[ B(\varepsilon, \ell) := \varepsilon^2\ell^4 \int d\nu^\varepsilon_b(x, z)(\phi(x) + \phi(x + \varepsilon z)) \]

\[ = 2\varepsilon^2\ell^4 \int d\nu^\varepsilon_b(x, z)\phi(x) = 2\varepsilon^2\ell^4 \int d\mu^\varepsilon_b(x)\phi(x)\lambda_0(\tau_{x/\varepsilon} \omega). \]

We now apply Prop. 4.3. As \( \omega \in A_1[||z|^21_{\{z \geq \ell\}}] \cap A[h_\ell] \), we conclude that

\[ \lim_{\varepsilon, \ell \downarrow 0} \int d\mu^\varepsilon_b(x)\phi(x)h_\ell(\tau_{x/\varepsilon} \omega) = \int dx m\phi(x)E_0[h_\ell]. \]

Hence \( \lim_{\ell \rightarrow \infty, \varepsilon \downarrow 0} A(\varepsilon, \ell) = 0 \) by dominated convergence as \( E_0[\lambda_2] < +\infty \). As \( \omega \in A_1[1] \cap A[\lambda_0] \) the integral \( \int d\mu^\varepsilon_b(x)\phi(x)\lambda_0(\tau_{x/\varepsilon} \omega) \) converges to \( \int dx m\phi(x)E_0[\lambda_0] \) as \( \varepsilon \downarrow 0 \). As a consequence, \( \lim_{\varepsilon, \ell \downarrow 0} B(\varepsilon, \ell) = 0 \). Coming back to (89) we finally get (88). \( \square \)

8. The set \( \Omega_{\text{typ}} \) of typical environments

Recall the definitions of the set \( A[g] \) (cf. Proposition 4.3 and Definition 4.4) and of the set \( A[g] \) (cf. Definition 7.5).

In the construction of the sets below, we will use the separability of \( L^2(\nu) \) and \( L^2(\mathcal{P}_0) \). Since \( (\mathcal{N}, d) \) is a separable metric space (cf. Section 4), the same holds for \( (\Omega, d) \) and \( (\Omega_0, d) \). By [12, Theorem 4.13] we then get that the space \( L^p(\mathcal{P}_0) \) is separable for \( 1 \leq p < +\infty \). The separability of \( L^2(\nu) \) is proved in [12, Lemma 9.2].

- **The functional sets** \( \mathcal{G}_1, \mathcal{H}_1 \). We fix a countable set \( \mathcal{H}_1 \) of Borel functions \( b : \Omega_0 \times \mathbb{R}^d \rightarrow \mathbb{R} \) such that \( ||b||_{L^2(\nu)} < +\infty \) for any \( b \in \mathcal{H}_1 \) and such that \( \{\text{div } b : b \in \mathcal{H}_1\} \) is a dense subset of \( \{w \in L^2(\mathcal{P}_0) : E_0[w] = 0\} \) when thought of as set of \( L^2 \)-functions (recall Lemma 7.2). For each \( b \in \mathcal{H}_1 \) we define the Borel function \( g_b : \Omega_0 \rightarrow \mathbb{R} \) as (cf. Definition 7.9)

\[ g_b(\omega) := \begin{cases} \text{div}_x b(\omega) & \text{if } \omega \in A_1[b] \cap A_1[\hat{b}], \\ 0 & \text{otherwise}. \end{cases} \] (90)

Note that by Lemma 7.10 \( g_b = \text{div } b, \mathcal{P}_0 \)-a.s. Finally we set \( \mathcal{G}_1 := \{g_b : b \in \mathcal{H}_1\} \).

- **The functional sets** \( \mathcal{G}_2, \mathcal{H}_2 \). We fix a countable set \( \mathcal{G}_2 \) of bounded Borel functions \( g : \Omega_0 \rightarrow \mathbb{R} \) such that the set \( \{\nabla g : g \in \mathcal{G}_2\} \), thought in \( L^2(\nu) \), is dense in \( L^2_{\text{pot}}(\nu) \) (this is possible by the definition of \( L^2_{\text{pot}}(\nu) \)). We define \( \mathcal{H}_2 \) as the set of Borel functions \( h : \Omega_0 \times \mathbb{R}^d \rightarrow \mathbb{R} \) such that \( h = \nabla g \) for some \( g \in \mathcal{G}_2 \).

- **The functional set** \( \mathcal{W} \). We fix a countable set \( \mathcal{W} \) of Borel functions \( b : \Omega_0 \times \mathbb{R}^d \rightarrow \mathbb{R} \) such that, thought of as subset of \( L^2(\nu) \), \( \mathcal{W} \) is dense in
In what follows, \( \Delta \) equals \( S \). By Lemma 7.8, \( \bar{b} \in L^2_{sol}(\nu) \) for any \( b \in L^2_{sol}(\nu) \). Hence, at cost to enlarge \( \mathcal{W} \), we assume that \( \bar{b} \in \mathcal{W} \) for any \( b \in \mathcal{W} \) (recall Definition 7.7).

**Definition 8.1** (Definition of the functional set \( \mathcal{G} \)). We define \( \mathcal{G} \) as the union of the following countable sets of Borel functions on \( \Omega_0 \), which are \( \mathcal{P}_0 \)-square integrable: \( \{1\} \), \( \mathcal{G}_1 \), \( \mathcal{G}_2 \) and \( \{u_{b,i}\1(|u_{b,i}| \leq M)\} \) with \( b \in \mathcal{W} \), \( i \in \{1, \ldots, d\} \), \( M \in \mathbb{N} \) and \( u_{b,i}(\omega) := \int d\bar{\omega}(z)c_{0,z}(\omega)zib(\omega, z). \)

**Definition 8.2** (Definition of the functional set \( \mathcal{H} \)). We define \( \mathcal{H} \) as the union of the following countable sets of Borel functions on \( \Omega_0 \times \mathbb{R}^d \), which are \( \nu \)-square integrable: \( \mathcal{H}_1 \), \( \mathcal{H}_2 \), \( \mathcal{W} \), \( \{(\omega, z) \mapsto z_i : 1 \leq i \leq d\} \).

Recall the transformation \( b \mapsto \bar{b} \) given in Definition 7.5 and the parameter \( \alpha \in (0, 1) \) appearing in Assumption (A6).

**Definition 8.3.** The set \( \Omega_{typ} \subset \Omega \) of typical environments is the intersection of the following sets:

- \( \mathcal{A}[gg'] \) for all \( g, g' \in \mathcal{G} \) (recall that \( 1 \in \mathcal{G} \));
- \( \mathcal{A}[bb'] \cap \mathcal{A}[\bar{b}\bar{b}'] \) as \( b, b' \in \mathcal{H} \);
- \( \Omega_2 \) (cf. Lemma 3.4);
- \( \mathcal{A}[[z]^k] \cap \mathcal{A}[\lambda_k] \) for \( k = 0, 2 \);
- \( \mathcal{A}[\int d\bar{\omega}(z)c_{0,z}(\omega)|z|^2\1_{|z|\geq n}] \) for all \( n \in \mathbb{N} \);
- \( \mathcal{A}[c_{0,z}(\omega)\alpha] \cap \mathcal{A}[\int d\bar{\omega}(z)c_{0,z}(\omega)\alpha^2\1_{|z|\geq n}] \) for all \( n \in \mathbb{N} \);
- \( \mathcal{A}[b] \cap \mathcal{A}[\bar{b}] \cap \mathcal{A}[\bar{b}^2] \) for all \( b \in \mathcal{H} \);
- \( \mathcal{A}[\bar{b}^2] \cap \mathcal{A}[\bar{b}^2] \cap \mathcal{A}[\bar{b}] \cap \mathcal{A}[\bar{b}] \) for all \( b \in \mathcal{H} \);
- \( \mathcal{A}[b, z_i] \) for \( 1 \leq i \leq d \) for all \( b \in \mathcal{W} \);
- \( \mathcal{A}[u_{b,i,M}] \) for all \( b \in \mathcal{W} \), \( 1 \leq i \leq d \) and \( M \in \mathbb{N} \), where \( u_{b,i,M} := |u_{b,i}|\1(|u_{b,i}| \geq M) \) and \( u_{b,i}(\omega) := \int d\bar{\omega}(z)c_{0,z}(\omega)zib(\omega, z) \) (see definition of \( \mathcal{G} \));
- \( \mathcal{A}[c_{0,z}(\omega)^2z_i] \cap \mathcal{A}[\int d\bar{\omega}(z)c_{0,z}(\omega)^2z_i] \);
- \( \mathcal{A}[\bar{b}^2] \) for all \( b \in \mathcal{W} \) (recall (84)).

As \( \lambda_0, \lambda_1 \in L^1(\mathcal{P}_0) \), due to (11), (12) and our definition of \( \mathcal{G}, \mathcal{H}, \mathcal{W}, \) the sets listed in Definition 8.3 are well defined (recall in particular Lemmata 7.6, 7.8, 7.10). As these sets are translation invariant with full \( \mathcal{P} \)-measure (see Proposition 4.3, Lemma 7.6 and Lemma 7.11), the same holds for \( \Omega_{typ} \).

**9. Weak/strong convergence and 2-scale convergence**

Recall \( \mu_{\omega,\lambda}^x \) and \( \nu_{\omega,\lambda}^x \) given in (24). Recall \( \mu^x \) and \( \nu^x \) given in (84). We also define

\[
\mu^x_{\omega,S} := \varepsilon^d \sum_{x \in \Omega_{sol}^S} \delta_x, \quad \nu^x_{\omega,S} := \sum_{x \in \Omega_{sol}^S} \sum_{y \in \Omega_{sol}^S} \varepsilon^d c_{x,y}^x(\omega)\delta_{(x, y)}(\omega). \quad (91)
\]

In what follows, \( \Delta \) equals \( S \) or \( \Lambda \).
9.1. Weak/strong convergence.

**Definition 9.1.** Fix $\omega \in \Omega$ and a family of $\epsilon$-parametrized functions $v_\epsilon \in L^2(\mu_{\omega,\Delta})$.

- We say that the family $\{v_\epsilon\}$ converges weakly to the function $v \in L^2(\Delta, mdx)$, and write $v_\epsilon \rightharpoonup v$, if the family $\{v_\epsilon\}$ is bounded (i.e. $\limsup_{\epsilon \downarrow 0} \|v_\epsilon\|_{L^2(\mu_{\omega,\Delta})} < +\infty$) and
  \[
  \lim_{\epsilon \downarrow 0} \int d\mu_{\omega,\Delta}(x) v_\epsilon(x) \varphi(x) = \int_{\Delta} dx m(v)(x) \varphi(x)
  \]  
  for all $\varphi \in C_c(\Delta)$.

- We say that the family $\{v_\epsilon\}$ converges strongly to $v \in L^2(\Delta, mdx)$, and write $v_\epsilon \rightarrow v$, if $\{v_\epsilon\}$ is bounded and it holds
  \[
  \lim_{\epsilon \downarrow 0} \int d\mu_{\omega,\Delta}(x) v_\epsilon(x) g(x) = \int_{\Delta} dx m(v)(x) g(x),
  \]  
  for any family of functions $g_\epsilon \in L^2(\mu_{\omega,\Delta})$ weakly converging to $g \in L^2(\Delta, mdx)$.

Trivially, strong convergence implies weak convergence.

**Remark 9.2.** Given $v_\epsilon$ and $v$ as in Definition 9.1, we have that $v_\epsilon \rightarrow v$ if $v_\epsilon \rightharpoonup v$ and $\lim_{\epsilon \downarrow 0} \|v_\epsilon\|_{L^2(\mu_{\omega,\Delta})} = \|v\|_{L^2(\Omega, mdx)}$ (cf. the proof of [28, Prop. 1.1]).

9.2. Weak 2-scale convergence.

**Definition 9.3.** Fix $\tilde{\omega} \in \Omega_{\text{typ}}$, an $\epsilon$-parametrized family of functions $v_\epsilon \in L^2(\mu_{\tilde{\omega},\Delta})$ and a function $v \in L^2(\Delta \times \Omega, mdx \times P_0)$. We say that $v_\epsilon$ is weakly 2-scale convergent to $v$, and write $v_\epsilon \rightharpoonup 2 v$, if the family $\{v_\epsilon\}$ is bounded, i.e.

\[
\lim_{\epsilon \downarrow 0} \int d\mu_{\tilde{\omega},\Delta}(x) v_\epsilon(x) \varphi(x) g(\tau_{x/\epsilon} \tilde{\omega}) = \int dP_0(\omega) \int_{\Delta} dx m(v)(x, \omega) \varphi(x) g(\omega),
\]

for any $\varphi \in C_c(\Delta)$ and any $g \in G$.

One can define also the strong 2-scale convergence, but we will not need it in what follows. As $\tilde{\omega} \in \Omega_{\text{typ}} \subset A[g]$ for all $g \in G$, by Proposition 1.3 one gets that $v_\epsilon \rightarrow 2 v$ where $v_\epsilon := \varphi \in L^2(\mu_{\tilde{\omega},\Delta})$ and $v := \varphi \in L^2(\Delta, mdx)$ for any $\varphi \in C_c(\Delta)$.

It is standard to prove the following fact by using the first item in Definition 8.3 (cf. [28, Prop. 2.2], [29, Lemma 5.1] and in particular [12, Lemma 10.5]):

**Lemma 9.4.** Let $\tilde{\omega} \in \Omega_{\text{typ}}$. Then, given a bounded family of functions $v_\epsilon \in L^2(\mu_{\tilde{\omega},\Delta})$, there exists a subsequence $\{v_{\epsilon_k}\}$ such that $v_{\epsilon_k} \rightharpoonup 2 v$ for some $v \in L^2(\Delta \times \Omega, mdx \times P_0)$ with $\|v\|_{L^2(\Omega, \Omega, mdx \times P_0)} \leq \limsup_{\epsilon \downarrow 0} \|v_\epsilon\|_{L^2(\mu_{\tilde{\omega},\Delta})}$.

Recall the definition of the measure $\nu$ given in (69).
Definition 9.5. Given \( \hat{\omega} \in \Omega_{\text{typ}} \), an \( \varepsilon \)-parametrized family of functions \( w_{\varepsilon} \in L^2(\nu_{\hat{\omega},\Delta}) \) and a function \( w \in L^2(\Delta \times \Omega \times \mathbb{R}^d, mdx \times dv) \), we say that \( w_{\varepsilon} \) is weakly \( 2 \)-scale convergent to \( w \), and write \( w_{\varepsilon} \xrightarrow{\varepsilon} w \), if \( \{w_{\varepsilon}\} \) is bounded in \( L^2(\nu_{\hat{\omega},\Delta}) \), i.e. \( \limsup_{\varepsilon \downarrow 0} \|w_{\varepsilon}\|_{L^2(\nu_{\hat{\omega},\Delta})} < +\infty \), and

\[
\lim_{\varepsilon \downarrow 0} \int \frac{d\nu_{\hat{\omega},\Delta}(x,z)w_{\varepsilon}(x,z)\varphi(x)\mu(z,\tau_{\varepsilon/\varepsilon} \hat{\omega}, z)}{b} = \int_{\Delta} dx \int \frac{dv(\omega, z)w(x,\omega, z)\varphi(x)\mu(\omega, z)}{b(\omega, z)}, \tag{95}
\]

for any \( \varphi \in C_c(\Delta) \) and any \( b \in \mathcal{H} \).

It is standard to prove the following fact by using the second item in Definition 8.3 (cf. [12, Lemma 10.7]):

Lemma 9.6. Let \( \hat{\omega} \in \Omega_{\text{typ}} \). Then, given a bounded family of functions \( w_{\varepsilon} \in L^2(\nu_{\hat{\omega},\Delta}) \), there exists a subsequence \( \{w_{\varepsilon_k}\} \) such that \( w_{\varepsilon_k} \xrightarrow{\varepsilon} w \) for some \( w \in L^2(\Delta \times \Omega \times \mathbb{R}^d, mdx \times dv) \) with \( \|w\|_{L^2(\Delta \times \Omega \times \mathbb{R}^d, mdx \times dv)} \leq \limsup_{\varepsilon \downarrow 0} \|w_{\varepsilon}\|_{L^2(\nu_{\hat{\omega},\Delta})} \).

10. 2-scale limits of uniformly bounded functions

We fix \( \hat{\omega} \in \Omega_{\text{typ}} \). The domain \( \Delta \) below can be \( \Lambda, S \). We consider a family of functions \( \{f_{\varepsilon}\} \) with \( f_{\varepsilon} : \hat{\varepsilon} \hat{\omega} \cap S \rightarrow \mathbb{R} \) such that

\[
\limsup_{\varepsilon \downarrow 0} \|f_{\varepsilon}\|_{\infty} < +\infty, \tag{96}
\]

\[
\limsup_{\varepsilon \downarrow 0} \|f_{\varepsilon}\|_{L^2(\nu_{\hat{\omega},\Delta})} < +\infty, \tag{97}
\]

\[
\limsup_{\varepsilon \downarrow 0} \|
abla_{\varepsilon} f_{\varepsilon}\|_{L^2(\nu_{\hat{\omega},\Delta})} < +\infty. \tag{98}
\]

Due to Lemmata 9.4 and 9.6 along a subsequence \( \{\varepsilon_k\} \) we have

\[
L^2(\mu_{\hat{\omega},\Delta}) \ni f_{\varepsilon_k} \xrightarrow{\varepsilon_k} v \in L^2(\Delta \times \Omega, mdx \times \mathcal{P}_0), \tag{99}
\]

\[
L^2(\nu_{\hat{\omega},\Delta}) \ni \nabla_{\varepsilon_k} f_{\varepsilon_k} \xrightarrow{\varepsilon_k} w \in L^2(\Delta \times \Omega \times \mathbb{R}^d, mdx \times dv), \tag{100}
\]

for suitable functions \( v, w \).

Warning 10.1. In this section (with exception of Lemma 10.1 and Claim 10.4), when taking the limit \( \varepsilon \downarrow 0 \), we understood that \( \varepsilon \) varies along the subsequence \( \{\varepsilon_k\} \) satisfying (99) and (100). We set \( f_{\varepsilon}(x) := 0 \) for \( x \in \hat{\varepsilon} \hat{\omega} \setminus S \).

The structural results presented below (cf. Propositions 10.2 and 10.3) correspond to a general strategy in homogenization by 2-scale convergence (see Propositions 12.1 and 14.1 in [12], Lemmata 5.3 and 5.4 in [29], Theorems 4.1 and 4.2 in [28]). Condition (96) would not be strictly necessary, but it allows important technical simplifications, and in particular it allows to avoid the cut-off procedures developed in [12 Sections 11,13] in order to deal with the long jumps in the Markov generator (26). We will apply Propositions 10.2 and
only to the following cases: $\Delta = \Lambda$ and $f_\varepsilon = V_\varepsilon$; $\Delta = S$ and $f_\varepsilon = V_\varepsilon - \psi$. In both cases (96), (97) and (98) are satisfied by Remark 5.3 and Lemma 5.4.

In what follows we will use the following control on long filaments (recall (81)):

**Lemma 10.1.** Given $\tilde{\omega} \in \Omega_{\text{typ}}$, $\ell > 0$ and $\varphi \in C_c(\mathbb{R}^d)$, it holds

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-2} \int d\nu_{\tilde{\omega}}^\varepsilon (x, z) |\varphi(x)| \mathbf{1}(|z| \geq \ell/\varepsilon) = 0. \quad (101)$$

**Proof.** Let $\alpha \in (0, 1)$ be as in (A6). We set $\kappa(t) := \sup_{t \in [0, |z|]} |c_0.(\omega)\alpha|$ and $h_{\alpha,n}(\omega) := \int d\tilde{\omega}(z) c_{0,z}(\omega)\alpha \mathbf{1}(|z| \geq n)$ for $n \in \mathbb{N}$. For $\ell/\varepsilon \geq n$, we can bound the l.h.s. of (101) by

$$\varepsilon^{-2} \int d\mu_{\tilde{\omega}}(x) |\varphi(x)| \int d\tau_{x/\varepsilon}\tilde{\omega}(z) c_{0,z}(\tau_{x/\varepsilon}\tilde{\omega}) \mathbf{1}(|z| \geq \ell/\varepsilon)$$

$$\leq \varepsilon^{-2}\kappa(\ell/\varepsilon) \int d\mu_{\tilde{\omega}}(x) |\varphi(x)| \int d\tau_{x/\varepsilon}\tilde{\omega}(z) c_{0,z}(\tau_{x/\varepsilon}\tilde{\omega}) \alpha \mathbf{1}(|z| \geq n)$$

$$= \varepsilon^{-2}\kappa(\ell/\varepsilon) \int d\mu_{\tilde{\omega}}(x) |\varphi(x)| h_{\alpha,n}(\tau_{x/\varepsilon}\tilde{\omega}). \quad (102)$$

By (13) we have $\lim_{\varepsilon \downarrow 0} \varepsilon^{-2}\kappa(\ell/\varepsilon) < +\infty$. Since $\tilde{\omega} \in \Omega_{\text{typ}} \subset A_t^{c_0}(\omega) \alpha \cap A_{h_{\alpha,n}}$, we have $\int d\mu_{\tilde{\omega}}(x) |\varphi(x)| h_{\alpha}(\tau_{x/\varepsilon}\tilde{\omega}) \rightarrow \int dx m |\varphi(x)| E_t[h_{\alpha,n}]$ as $\varepsilon \downarrow 0$. By taking the limit $n \rightarrow \infty$ we get (101) due to (11). \qed

**Proposition 10.2.** For $dx$–a.e. $x \in \Delta$, the map $v(x, \omega)$ given in (99) does not depend on $\omega$.

**Proof.** Recall the definition of the functional sets $\mathcal{G}_1, \mathcal{H}_1$ given in Section 8. We claim that $\forall \varphi \in C^1_c(\Delta)$ and $\forall \psi \in \mathcal{G}_1$ it holds

$$\int_\Delta dx m \int dP_0(\omega) v(x, \omega) \varphi(\omega) \psi(\omega) = 0. \quad (103)$$

Before proving our claim, let us explain how it leads to the thesis. Since $\varphi$ varies among $C^1_c(\Delta)$ while $\psi$ varies in a countable set, (103) implies that, $dx$–a.e. on $\Delta$, $\int dP_0(\omega) v(x, \omega) \varphi(\omega) = 0$ for any $\psi \in \mathcal{G}_1$. We conclude that, $dx$–a.e. on $\Delta$, $v(x, \cdot)$ is orthogonal in $L^2(P_0)$ to $\{w \in L^2(P_0) : E_0[w] = 0\}$ (due to the density of $G_1$), which is equivalent to the fact that $v(x, \omega) = E_0[v(x, \cdot)]$ for $P_0$–a.e. $\omega$.

It now remains to prove (103). We first note that, by (94), (99) and since $\tilde{\omega} \in \Omega_{\text{typ}}$ and $\psi \in \mathcal{G}_1 \subset \mathcal{G}$,

$$\text{l.h.s. of (103)} = \lim_{\varepsilon \downarrow 0} \int d\mu_{\tilde{\omega},\Delta}(x) f_\varepsilon(x) \varphi(x) \psi(\tau_{x/\varepsilon}\tilde{\omega}). \quad (104)$$
Let us take \( \psi = g_b \) with \( b \in H_1 \) as in \( (90) \). By Lemma 7.12 and since \( \tilde{\omega} \in \Omega_{\text{typ}} \subset A_1[b] \cap A_1[\hat{b}] \), we have

\[
\int d\mu_{\tilde{\omega}, \Delta}^\varepsilon(x_0) f_\varepsilon(x) \varphi(x) \psi(\tau_{x/\varepsilon} \tilde{\omega}) = \int d\mu_{\tilde{\omega}}^\varepsilon(x_0) f_\varepsilon(x) \varphi(x) \psi(\tau_{x/\varepsilon} \tilde{\omega})
= -\varepsilon \int d\nu_{\tilde{\omega}}^\varepsilon(x, z) \nabla_\varepsilon (f_\varepsilon \varphi)(x, z) b(\tau_{x/\varepsilon} \tilde{\omega}, z).\
\]

As usual, we think \( C_\varepsilon(\Delta) \subset C_\varepsilon(\mathbb{R}^d) \) and we keep the same notation for \( \varphi \) thought in \( C_\varepsilon(\mathbb{R}^d) \). By \( (59) \) we have

\[
- \varepsilon \int d\nu_{\tilde{\omega}}^\varepsilon(x, z) \nabla_\varepsilon (f_\varepsilon \varphi)(x, z) b(\tau_{x/\varepsilon} \tilde{\omega}, z) = -\varepsilon C_1(\varepsilon) + \varepsilon C_2(\varepsilon),
\]

where

\[
C_1(\varepsilon) := \int d\nu_{\tilde{\omega}}^\varepsilon(x, z) \nabla_\varepsilon f_\varepsilon(x, z) \varphi(x) b(\tau_{x/\varepsilon} \tilde{\omega}, z),
\]

\[
C_2(\varepsilon) := \int d\nu_{\tilde{\omega}}^\varepsilon(x, z) f_\varepsilon(x + \varepsilon z) \nabla_\varepsilon \varphi(x, z) b(\tau_{x/\varepsilon} \tilde{\omega}, z).
\]

Due to \( (104), (105), \) and \( (106) \), to get \( (103) \) we only need to show that \( \lim_{\varepsilon \downarrow 0} \varepsilon C_1(\varepsilon) = 0 \) and \( \lim_{\varepsilon \downarrow 0} \varepsilon C_2(\varepsilon) = 0 \).

We start with \( C_1(\varepsilon) \). By Schwarz inequality and since \( \tilde{\omega} \in \Omega_{\text{typ}} \subset A_1[b^2] \)

\[
|C_1(\varepsilon)| \leq \left[ \int d\nu_{\tilde{\omega}}^\varepsilon(x, z) \varphi(x) \right]^{1/2} \left[ \int d\nu_{\tilde{\omega}}^\varepsilon(x, z) \nabla_\varepsilon f_\varepsilon(x, z)^2 \right]^{1/2}.
\]

Since \( \tilde{\omega} \in \Omega_{\text{typ}} \subset A_1[b^2] \cap A_1[\hat{b}^2] \), the last integral in the r.h.s. converges to a finite constant as \( \varepsilon \downarrow 0 \). It remains to prove that \( \int d\nu_{\tilde{\omega}}^\varepsilon(x, z) \varphi(x) \nabla_\varepsilon f_\varepsilon(x, z)^2 \) remains bounded from above as \( \varepsilon \downarrow 0 \). We call \( \ell \) the distance between the support of \( \varphi \) (which is contained in \( \Delta \) as \( \varphi \in C_\varepsilon^1(\Delta) \)) and \( \partial \Delta \). Then, between the pairs \( (x, z) \) with \( x + \varepsilon z \notin S \) contributing to the above integral, only the pairs \( (x, z) \) such that \( x \in \Delta \) and \( |z| \geq \ell/\varepsilon \) can give a nonzero contribution. In both cases \( \Delta = \Lambda \) and \( \Delta = S \) we can estimate

\[
\int d\nu_{\tilde{\omega}}^\varepsilon(x, z) \varphi(x) \nabla_\varepsilon f_\varepsilon(x, z)^2 \leq \int d\nu_{\tilde{\omega}, \Delta}^\varepsilon(x, z) \varphi(x) \nabla_\varepsilon f_\varepsilon(x, z)^2
+ \int d\nu_{\tilde{\omega}}^\varepsilon(x, z) \varphi(x) \nabla_\varepsilon f_\varepsilon(x, z)^2 [1(|z| \geq \ell/\varepsilon)].
\]

The first addendum in the r.h.s. of \( (107) \) is bounded due to \( (98) \). The second addendum goes to zero due to \( (96) \) (implying that \( |\nabla f| \leq C/\varepsilon \) for small \( \varepsilon \)) and Lemma 10.1. Hence the l.h.s. of \( (107) \) remains bounded as \( \varepsilon \downarrow 0 \). This completes the proof that \( \lim_{\varepsilon \downarrow 0} \varepsilon C_1(\varepsilon) = 0 \).

We move to \( C_2(\varepsilon) \). Let \( \phi \) be as in \( (60) \). Using \( (60) \) and \( (96) \), and afterwards Lemma 7.13 (i), for some \( \varepsilon \)-independent constants \( C \)’s (which can change from
line to line), for \( \varepsilon \) small we can bound
\[
|C_2(\varepsilon)| \leq C \int d\nu_{\omega,\varepsilon}(x, z) |\nabla_\varepsilon \varphi(x, z) b(\tau_{x/\varepsilon} \omega, z)|
\leq C \int d\nu_{\omega,\varepsilon}(x, z) |z| |b(\tau_{x/\varepsilon} \omega, z)| (\phi(x) + \phi(x + \varepsilon z))
\leq C \int d\nu_{\omega,\varepsilon}(x, z) \phi(x)|z|(|b| + \tilde{b})(\tau_{x/\varepsilon} \omega, z)
\leq C \left[ \int d\nu_{\omega,\varepsilon}(x, z) \phi(x)|z|^2 \right]^{1/2} \left[ 2 \int d\nu_{\omega,\varepsilon}(x, z) \phi(x)(b^2 + \tilde{b}^2)(\tau_{x/\varepsilon} \omega, z) \right]^{1/2}. \tag{108}
\]

The first integral in the last line of (108) equals \( \int d\mu_{\omega,\varepsilon}(x) \phi(x) \lambda_2(\tau_{x/\varepsilon} \omega) \). Since \( \tilde{\omega} \in \Omega_{\text{typ}} \subset A_1[|z|^2] \cap A[\lambda_2] \), this integral converges to a finite constant as \( \varepsilon \downarrow 0 \).

The second integral in the last line of (108) equals
\[
\int d\mu_{\omega,\varepsilon}(x) \phi(x)(\tilde{b}^2 + \tilde{b}^2)(\tau_{x/\varepsilon} \omega) \tag{109}
\]
as \( \tilde{\omega} \in \Omega_{\text{typ}} \subset A_1[b^2] \cap A[\tilde{b}^2] \). Since \( \tilde{\omega} \in \Omega_{\text{typ}} \subset A[\tilde{b}^2] \cap A[\tilde{b}^2] \), the integral (109) converges to a finite constant. This implies that \( \lim_{\varepsilon \downarrow 0} C_2(\varepsilon) = 0 \).

Due to Proposition 10.2, we can write \( v(x) \) instead of \( v(x, \omega) \), where \( v \) is given by (99). Recall the index \( d_* \) introduced in Warning 2.2 and recall (33).

**Proposition 10.3.** Let \( v \) and \( w \) be as in (99) and (100). Then it holds:

(i) \( v \) has weak derivatives \( \partial_j v \in L^2(\Delta, dx) \) for \( 1 \leq j \leq d_* \);
(ii) \( w(x, \omega, z) = \nabla_* v(x) \cdot z + v_1(x, \omega, z) \), where \( v_1 \in L^2(\Delta, dx; L^2_{\text{pot}}(\nu)) \).

We stress that \( L^2(\Delta, dx; L^2_{\text{pot}}(\nu)) \) denotes the space of square integrable maps \( f : \Delta \rightarrow L^2_{\text{pot}}(\nu) \), where \( \Delta \) is endowed with the Lebesgue measure.

**Proof.** Given a square integrable form \( b \), we define \( \eta_b := \int d\nu(\omega, z) z b(\omega, z) \).

Note that \( \eta_b \) is well defined since both \( b \) and the map \((\omega, z) \mapsto z\) are in \( L^2(\nu) \) (for the latter use that \( \mathbb{E}_0[\lambda_2] < +\infty \)). We observe that \( \eta_b = -\eta_{b'} \) by Lemma 4.2 with \( k(\omega, \omega') := z \xi_{\omega, z}(\omega) b(\omega, z) \) if \( \omega' \) can be written as \( \tau_{\omega} \omega \) with \( z \in \tilde{\omega} \) and \( k(\omega, \omega') := 0 \) otherwise (the function \( k \) is well defined \( \mathcal{P}_0\)-a.s. due to Assumption (A3)). We claim that for each solenoidal form \( b \in L^2_{\text{sol}}(\nu) \) and each function \( \varphi \in C_c^2(\Delta) \), it holds
\[
\int_\Delta dx \ m \varphi(x) \int d\nu(\omega, z) w(x, \omega, z) b(\omega, z) = - \int_\Delta dx \ m v(x) \nabla \varphi(x) \cdot \eta_b. \tag{110}
\]

Before proving (110) we show how to conclude the proof of Proposition 10.3. We start with Item (i). Due to Corollary 7.4 there are solenoidal forms \( b_1, b_2, \ldots, b_d \), such that \( \eta_{b_1}, \eta_{b_2}, \ldots, \eta_{b_d} \) equals \( e_1, e_2, \ldots, e_d \). Given \( 1 \leq i \leq d_* \) consider the measurable function
\[
g_i(x) := \int d\nu(\omega, z) w(x, \omega, z) b_i(\omega, z), \quad x \in \Delta. \tag{111}
\]
We have that $g_i \in L^2(\Delta, dx)$. Indeed, by Schwarz inequality and since $w \in L^2(\Delta \times \Omega \times \mathbb{R}^d, dx \times \nu)$, we can bound

$$\int_\Delta g_i(x)^2 \, dx = \int_\Delta dx \left[ \int d\nu(\omega, z) w(x, \omega, z) b_i(\omega, z) \right]^2 \leq \| b_i \|_{L^2(\nu)}^2 \int_\Delta dx \int d\nu(\omega, z) w(x, \omega, z)^2 < \infty. \quad (112)$$

Moreover, we have that $\int_\Delta dx \varphi(x) g_i(x) = -\int_\Delta dx v(\partial_i \varphi(x))$ by (110) and since $\eta_b = \epsilon_i$. This proves that $\partial_i v(x) = -g_i(x) \in L^2(\Delta, dx)$, $\partial_i v$ being the weak derivative of $v$ w.r.t. the $i$–th coordinate. This concludes the proof of Item (i).

We move to Item (ii) (always assuming (110)). By Item (i) and Corollary 11.4 we can replace the r.h.s. of (110) by $\int_\Delta dx (\nabla_s v(x) \cdot \eta_b) \varphi(x)$. Hence (110) can be rewritten as

$$\int_\Delta dx \varphi(x) \int d\nu(\omega, z) [w(x, \omega, z) - \nabla_s v(x) \cdot z] b(\omega, z) = 0. \quad (113)$$

By the arbitrariness of $\varphi$ we conclude that $dx$–a.s. on $\Delta$

$$\int d\nu(\omega, z) [w(x, \omega, z) - \nabla_s v(x) \cdot z] b(\omega, z) = 0, \quad \forall b \in L^2_{\text{solv}}(\nu). \quad (114)$$

Let us now show that the map $w(x, \omega, z) - \nabla_s v(x) \cdot z$ belongs to $L^2(\Delta, dx; L^2(\nu))$. Indeed, we have $\int_\Delta dx \| w(x, \cdot, \cdot) \|_{L^2(\nu)}^2 = \| w \|_{L^2(\Delta \times \Omega, dx \times \nu)}^2 < +\infty$ and also

$$\int_\Delta dx \| \nabla_s v(x) \cdot z \|_{L^2(\nu)}^2 \leq \int_\Delta dx |\nabla_s v(x)|^2 \int d\nu(\omega, z) |z|^2 < \infty, \quad (115)$$

where the last bound follows from the fact that $\nabla_s v \in L^2(\Delta, dx)$ (see Item (i)) and that $\mathbb{E}_0[\lambda]_2 < +\infty$.

As the map $w(x, \omega, z) - \nabla_s v(x) \cdot z$ belongs to $L^2(\Delta, dx; L^2(\nu))$, for $dx$–a.e. $x$ in $\Delta$ we have that the map $(\omega, z) \mapsto w(x, \omega, z) - \nabla_s v(x) \cdot z$ belongs to $L^2(\nu)$ and therefore, by (114), to $L^2_{\text{pov}}(\nu)$. This concludes the proof of Item (ii).

It remains to prove (110). Since both sides of (110) are continuous as functions of $b \in L^2_{\text{solv}}(\nu)$, it is enough to prove it for $b \in \mathcal{W}$. Since $\tilde{\omega} \in \Omega_{\text{typ}}$, along $\{\epsilon_k\}$ it holds $\nabla_\epsilon f_\epsilon \xrightarrow{\mathcal{A}_d} w$ as in (100) and since $b \in \mathcal{W} \subset \mathcal{H}$ (cf. (95)) we can write

l.h.s. of (110) \begin{align*}
= \lim_{\epsilon \to 0} \int d\nu^\epsilon_{\tilde{\omega}}(x, z) \nabla_\epsilon f_\epsilon(x, z) \varphi(x) b(\tau_{x/\epsilon}\tilde{\omega}, z) \\
&= \lim_{\epsilon \to 0} \int d\nu^\epsilon_{\tilde{\omega}}(x, z) \nabla_\epsilon (f_\epsilon \varphi)(x, z) b(\tau_{x/\epsilon}\tilde{\omega}, z). \quad (116)
\end{align*}

Since $b \in \mathcal{W} \subset L^2_{\text{solv}}(\nu)$ and $\tilde{\omega} \in \Omega_{\text{typ}} \subset \mathcal{A}_d[b]$, from Lemma 7.12 we get

$$\int d\nu^\epsilon_{\tilde{\omega}}(x, z) \nabla_\epsilon (f_\epsilon \varphi)(x, z) b(\tau_{x/\epsilon}\tilde{\omega}, z) = 0.$$
Above we used the natural inclusion $C_c(\Delta) \subset C_c(\mathbb{R}^d)$. Using the above identity and \([59]\), we get
\[
\int d\nu^\phi_\varepsilon(x,z) \nabla_\varepsilon f_\varepsilon(x,z) \varphi(x)b(\tau_{x/\varepsilon}\hat{\omega}, z) \quad (117)
\]
\[= - \int d\nu^\phi_\varepsilon(x,z) \tilde{f}_\varepsilon(x, z + \varepsilon z) \nabla_\varepsilon \varphi(x, z)b(\tau_{x/\varepsilon}\hat{\omega}, z). \]
As a byproduct of \((117)\) and \((87)\) in Lemma 7.13–(ii), we get
\[
\int d\nu^\phi_\varepsilon(x,z) \nabla_\varepsilon f_\varepsilon(x,z) \varphi(x)b(\tau_{x/\varepsilon}\hat{\omega}, z) = \int d\nu^\phi_\varepsilon(x,z) \tilde{f}_\varepsilon(x) \nabla_\varepsilon \varphi(x, z)\tilde{b}(\tau_{x/\varepsilon}\hat{\omega}, z). \quad (118)
\]
By combining \((116)\) and \((118)\) we therefore have that
\[
\text{l.h.s. of } (123) \text{ such that } |z| \geq \ell/\varepsilon.
\]
We claim that $\lim_{\varepsilon \downarrow 0} R_1(\varepsilon) = 0$. We call $\ell$ the distance between the support $\Delta_\varphi \subset \Delta$ of $\varphi$ and $\partial \Delta$. Then in $R_1(\varepsilon)$ the contribution comes only from pairs $(x,z)$ such that $x \in \Delta_\varphi$ and $x + \varepsilon z \not\in S$ and therefore from pairs $(x,z)$ such that $x \in \Delta$ and $|z| \geq \ell/\varepsilon$:
\[
R_1(\varepsilon) = \int d\nu^\phi_\varepsilon(x,z) \nabla_\varepsilon f_\varepsilon(x,z) \varphi(x)b(\tau_{x/\varepsilon}\hat{\omega}, z) \mathbf{1}(x \in \Delta, |z| \geq \ell/\varepsilon). \quad (120)
\]
By Schwarz inequality we have therefore that $R_1(\varepsilon)^2 \leq I_1(\varepsilon)I_2(\varepsilon)$, where
\[
I_1(\varepsilon) := \int d\nu^\phi_\varepsilon(x,z) \nabla_\varepsilon f_\varepsilon(x,z)^2 \varphi(x)\mathbf{1}(|z| \geq \ell/\varepsilon), \quad (121)
\]
\[
I_2(\varepsilon) := \int d\nu^\phi_\varepsilon(x,z) \varphi(x)b(\tau_{x/\varepsilon}\hat{\omega}, z)^2 = \int d\mu^\phi_\varepsilon(x) \varphi(x)\mathbf{b}^2(\tau_{x/\varepsilon}\hat{\omega}). \quad (122)
\]
Note that the last identity concerning $I_2(\varepsilon)$ uses that $\hat{\omega} \in \Omega_{\text{typ}} \subset \mathcal{A}[b^2]$. Then $\lim_{\varepsilon \downarrow 0} I_1(\varepsilon) = 0$ due to Lemma \([10.1]\) while $I_2(\varepsilon)$ converges to a bounded constant when $\varepsilon \downarrow 0$ since $\hat{\omega} \in \Omega_{\text{typ}} \subset \mathcal{A}[b^2]$. This proves that $R_1(\varepsilon) \to 0$.

We now move to $R_2(\varepsilon)$.

**Claim 10.4.** We have
\[
\lim_{\varepsilon \downarrow 0} \int d\nu^\phi_\varepsilon(x,z) f_\varepsilon(x)\left[ \nabla_\varepsilon \varphi(x, z) - \nabla \varphi(x) \cdot z \right] \tilde{b}(\tau_{x/\varepsilon}\hat{\omega}, z) = 0. \quad (123)
\]

**Proof.** Given $\ell \in \mathbb{N}$ we write the integral in \((123)\) as $A_{\ell}(\varepsilon) + B_{\ell}(\varepsilon)$, where $A_{\ell}(\varepsilon)$ is the contribution coming from $z$ with $|z| \leq \ell$ and $B_{\ell}(\varepsilon)$ is the contribution
coming from z with $|z| > \ell$. Due to (61) and (96) we can bound

$$A_\ell(\varepsilon) \leq C\ell^2 \varepsilon \int d\nu_0^\ell(x, z)\left(\phi(x) + \phi(x + \varepsilon z)\right)\left|\tilde{b}(\tau_{x/\varepsilon\hat{\omega}}, z)\right|.$$  

(124)

Hence, using now (86) in Lemma 7.13 we can bound

$$A_\ell(\varepsilon) \leq C\ell^2 \varepsilon \int d\nu_0^\ell(x, z)\phi(x)\left(|b| + |\tilde{b}|\right)(\tau_{x/\varepsilon\hat{\omega}}, z).$$  

(125)

Since $\omega \in \Omega_{\text{typ}} \subset A_1[b] = A_1[|b|]$ (recall that $\tilde{b} \in W$ for all $b \in W$), the r.h.s. of (125) can be written as

$$C\ell^2 \varepsilon \int d\mu_0^\ell(x)\phi(x)\left(\hat{|b|} + \hat{|\tilde{b}|}\right)(\tau_{x/\varepsilon\hat{\omega}}, z).$$  

(126)

Since $\omega \in \Omega_{\text{typ}} \subset A[|b|] \cap A[|\tilde{b}|]$ (recall that $\tilde{b} \in W$ for all $b \in W$), the integral in (126) converges to a finite constant as $\varepsilon \downarrow 0$. Hence, coming back to (125), $\lim_{\ell \to 0} A_\ell(\varepsilon) = 0$.

It remains to prove that $\lim_{\ell \to \infty} \limsup_{\varepsilon \downarrow 0} B_\ell(\varepsilon) = 0$. We reason as above but now we apply (60) and a similar bound for $\nabla \varphi(x) \cdot z$. Due to (96), (60) and (86) in Lemma 7.13 we can bound

$$B_\ell(\varepsilon) \leq C \int d\nu_0^\ell(x, z)\phi(x)\left(|b| + |\tilde{b}|\right)(\tau_{x/\varepsilon\hat{\omega}}, z)|z|1(|z| \geq \ell).$$  

(127)

By Schwarz inequality

$$B_\ell(\varepsilon) \leq C C_\ell(\varepsilon)^{1/2} D_\ell(\varepsilon)^{1/2}$$  

(128)

where

$$C_\ell(\varepsilon) := 2 \int d\nu_0^\ell(x, z)\phi(x)(|b|^2 + |	ilde{b}|^2)(\tau_{x/\varepsilon\hat{\omega}}, z)$$

$$= 2 \int d\mu_0^\ell(x)\phi(x)\left(|\hat{b}|^2 + |\hat{\tilde{b}}|^2\right)(\tau_{x/\varepsilon\hat{\omega}})$$

$$D_\ell(\varepsilon) := \int d\nu_0^\ell(x, z)\phi(x)|z|^21(|z| \geq \ell) = \int d\mu_0^\ell(x)\phi(x)\tilde{h}_\ell(\tau_{x/\varepsilon\hat{\omega}})$$

where $h_\ell(\omega, z) := |z|^21(|z| \geq \ell)$. Note that in the identities concerning $C_\ell(\varepsilon)$ and $D_\ell(\varepsilon)$ we have used that $\omega \in \Omega_{\text{typ}} \subset A_1[|b^2|] \cap A_1[|\tilde{b}|^2]$ and $\hat{\omega} \in \Omega_{\text{typ}} \subset A_1[|z|^2] \subset A_1[|h_\ell|]$. As $\hat{\omega} \in \Omega_{\text{typ}}$, which is included in the sets $A_1[|b^2|], A_1[|\tilde{b}|^2], A[|\tilde{b}|^2], A[|b^2|], A_1[h_\ell]$ and $A[\hat{h}_\ell]$, we get

$$\limsup_{\varepsilon \downarrow 0} B_\ell(\varepsilon) \leq C \left[\int dx \, m(\phi(x))\mathbb{E}_0(\hat{|b|^2} + |\hat{\tilde{b}}|^2)\right]^{1/2} \mathbb{E}_0[h_\ell]^{1/2},$$  

(129)

and the r.h.s. goes to zero as $\ell \to \infty$. □

We come back to (110). By combining (119), (123) and the limit $R_1(\varepsilon) \to 0$, we conclude that

$$\text{l.h.s. of (110)} = \lim_{\varepsilon \downarrow 0} \int d\nu_0^\ell(x, z)f_(x)\partial \varphi(x) \cdot \tilde{z}\tilde{b}(\tau_{x/\varepsilon\hat{\omega}}, z).$$  

(130)
Due to (130) and since $\eta_b = -\eta_b$, to prove (110) we only need to show that
\[
\lim_{\varepsilon \downarrow 0} \int d\nu_{\omega}(x, z) \tilde{f}_\varepsilon(x) \nabla \varphi(x) \cdot z \tilde{b}(\tau_{x/\varepsilon}, z) = \int dx \, m v(x) \nabla \varphi(x) \cdot \eta_b. \tag{131}
\]
To this aim we observe that
\[
\int d\nu_{\omega}(x, z) \tilde{f}_\varepsilon(x) \partial_i \varphi(x) z_i \tilde{b}(\tau_{x/\varepsilon}, \omega) = \int d\mu_{\omega}(x) \tilde{f}_\varepsilon(x) \partial_i \varphi(x) u_{b, i}(\tau_{x/\varepsilon}, \omega), \tag{132}
\]
where $u_{b, i}(\omega) := \int d\tilde{w}(z) c_{a, z}(\omega) z_i \tilde{b}(\omega, z)$ (recall that $\tilde{\omega} \in \Omega_{typ} \subset A_1[\tilde{b}(\omega, z)]$). We claim that
\[
\lim_{\varepsilon \downarrow 0} \int d\mu_{\omega}(x) \tilde{f}_\varepsilon(x) \partial_i \varphi(x) u_{b, i}(\tau_{x/\varepsilon}, \omega) = \int dx \, m v(x) \partial_i \varphi(x) E_0[u_{b, i}]. \tag{133}
\]
Since the r.h.s. equals $\int_{\Delta} dx \, m v(x) \partial_i \varphi(x)(\eta_b \cdot e_i)$, our target (131) then would follow as a byproduct of (132) and (133). It remains therefore to prove (133).

Given $M \in \mathbb{N}$ let $u_{b, i, M} := |u_{b, i}| \mathbb{1}[|u_{b, i}| \geq M]$. Due to Prop. 4.3 (recall that $\tilde{b} \in W$ for any $b \in W$ and that $\tilde{\omega} \in \Omega_{typ} \subset A[u_{b, i, M}]$ for all $b \in W$)
\[
\lim_{\varepsilon \downarrow 0} \int d\mu_{\omega}(x) |\partial_i \varphi(x)| u_{b, i, M}(\tau_{x/\varepsilon}) = \int dx \, m |\partial_i \varphi(x)| E_0[u_{b, i, M}].
\]
As $u_{b, i} \in L^1(P_\omega)$ we then get that
\[
\lim_{M \uparrow \infty} \lim_{\varepsilon \downarrow 0} \int d\mu_{\omega}(x) |\partial_i \varphi(x)| u_{b, i, M}(\tau_{x/\varepsilon}) = \lim_{M \uparrow \infty} \int dx \, m |\partial_i \varphi(x)| E_0[u_{b, i, M}] = 0. \tag{134}
\]
Due to (96) and (134), to get (133) it is enough to show that
\[
\lim_{M \uparrow \infty} \lim_{\varepsilon \downarrow 0} \int d\mu_{\omega}(x) \tilde{f}_\varepsilon(x) \partial_i \varphi(x) u_{b, i}(\tau_{x/\varepsilon}) \mathbb{1}[|u_{b, i}(\tau_{x/\varepsilon})| \leq M] = \int_{\Delta} dx \, m v(x) \partial_i \varphi(x) E[u_{b, i}]. \tag{135}
\]
Note that in (135) we can replace $d\mu_{\omega}(x) \tilde{f}_\varepsilon(x) \partial_i \varphi(x)$ by $d\mu_{\omega, \Delta}(x) \tilde{f}_\varepsilon(x) \partial_i \varphi(x)$. Due to (96) and since $u_{b, i} \mathbb{1}[|u_{b, i}| \leq M] \in G$, by (94) we have
\[
\lim_{\varepsilon \downarrow 0} \int d\mu_{\omega, \Delta}(x) \tilde{f}_\varepsilon(x) \partial_i \varphi(x) u_{b, i}(\tau_{x/\varepsilon}) \mathbb{1}[|u_{b, i}(\tau_{x/\varepsilon})| \leq M] = \int_{\Delta} dx \, m v(x) \partial_i \varphi(x) E[u_{b, i}] \mathbb{1}[|u_{b, i}| \leq M]. \tag{136}
\]
By dominated convergence, we get (135) from (136). \hfill \Box

11. 2-scale limit points of $V_\varepsilon$ and $\nabla_\varepsilon V_\varepsilon$

In this section $\tilde{\omega}$ is a fixed configuration in $\Omega_{typ}$. Due to Lemmas 5.4, 9.4, and 9.6 along a subsequence $\varepsilon_k$ we have that
\[
L^2(\mu_{\tilde{\omega}, \Lambda}) \ni V_\varepsilon \overset{2}{\to} v \in L^2(\Lambda \times \Omega, m dx \times \mathcal{P}_0), \tag{137}
\]
\[
L^2(\nu_{\tilde{\omega}, \Lambda}) \ni \nabla_\varepsilon V_\varepsilon \overset{2}{\to} w \in L^2(\Lambda \times \Omega \times \mathbb{R}^d, m dx \times \nu), \tag{138}
\]
for suitable functions \( v \) and \( w \). In the rest of this section, when considering the limit \( \varepsilon \downarrow 0 \), we understand that \( \varepsilon \) varies in the sequence \( \{\varepsilon_k\} \).

**Proposition 11.1.** Let \( v \) be as in (137). Then \( v - \psi_\Lambda \in H^1_0(\Lambda, F, d_\nu) \).

**Proof.** We apply the results of Section 10 to the case \( \Delta = S \) and \( f_\varepsilon := V_\varepsilon - \psi \). Since \( f_\varepsilon \) is zero on \( S \setminus \Lambda \) and takes values in \([-1, 1]\) on \( \Lambda \), conditions (96) and (97) are satisfied. In addition, we have \( \nabla_\varepsilon f_\varepsilon(x, z) = 0 \) if \( \{x, x + \varepsilon z\} \) does not intersect \( \Lambda \) and therefore \( \|f_\varepsilon\|_{L^2(\nu^\varepsilon_{\omega,s})} = \|f_\varepsilon\|_{L^2(\nu^\varepsilon_{\omega,\Lambda})} \). By Lemma 5.4 we therefore conclude that also (98) is satisfied.

At cost to refine the subsequence \( \{\varepsilon_k\} \), without loss of generality we can assume that along \( \{\varepsilon_k\} \) itself we have

\[
L^2(\mu^\varepsilon_{\omega,S}) \ni f_\varepsilon \xrightarrow{\varepsilon} \hat{v} \in L^2(S \times \Omega, m \, dx \times \mathcal{P}_0),
\]

\[
L^2(\nu^\varepsilon_{\omega,S}) \ni \nabla_\varepsilon f_\varepsilon \xrightarrow{\varepsilon} \hat{w} \in L^2(S \times \Omega \times \mathbb{R}^d, m \, dx \times \nu),
\]

for suitable functions \( \hat{v}, \hat{w} \). By Proposition 10.2 we have \( \hat{v} = \hat{v}(x) \). We recall that in the proof of Proposition 10.3 we have in particular derived (110): for each solenoidal form \( b \in L^2(\nu^\varepsilon_{\omega,S}) \) and each function \( \varphi \in C^2_\varepsilon(S) \), it holds

\[
\int_S \varphi(x) \int_\nu \omega(z) \hat{w}(x, \omega, z) b(\omega, z) = - \int_S \hat{v}(x) \nabla \varphi(x) \cdot \eta_b. \tag{141}
\]

Since \( f_\varepsilon \equiv 0 \) on \( S \setminus \Lambda \), it is simple to derive from the definition of 2–scale convergence that \( \hat{v}(x) = 0 \) \( dx \)–a.e. on \( S \setminus \Lambda \) and that \( \hat{w}(x, \cdot, \cdot) \equiv 0 \) \( dx \)–a.e. on \( S \setminus \Lambda \). Therefore (141) implies that

\[
\left| \int_\Lambda \hat{v}(x) \nabla \varphi(x) \cdot \eta_b \right| = \left| \int_\Lambda \varphi(x) \int_\nu \omega(z) \hat{w}(x, \omega, z) b(\omega, z) \right|. \tag{142}
\]

By Schwarz inequality we can bound

\[
C^2 := \int_\Lambda \left[ \int_\nu \omega(z) \hat{w}(x, \omega, z) b(\omega, z) \right]^2 \leq \int_\Lambda \left[ \int_\nu \omega(z) \hat{w}(x, \omega, z)^2 \right] \left[ \int_\nu \omega(z) b(\omega, z)^2 \right] = \|\hat{w}\|^2_{L^2(\Lambda \times \Omega, dx \times \nu)} \|b\|^2_{L^2(\nu)} < +\infty. \tag{143}
\]

By applying now Schwarz inequality to (142) we conclude that

\[
\left| \int_\Lambda \hat{v}(x) \nabla \varphi(x) \cdot \eta_b \right| \leq C \|\varphi\|_{L^2(\Lambda, dx)}. \tag{144}
\]

The above bound, Proposition 3.4 and Corollary 7.4 imply that \( \hat{v} \in H^1_0(\Lambda, F, d_\nu) \). To get the thesis it remains to observe that \( \hat{v} = v - \psi_\Lambda \) \( dx \)–a.e. on \( \Lambda \), which follows from the definition of 2–scale convergence, (137) and since \( L^2(\mu^\varepsilon_{\omega,\Lambda}) \ni \psi_\Lambda \xrightarrow{\varepsilon} \psi_\Lambda \in L^2(\Lambda, dx) \).

**Proposition 11.2.** Let \( w \) be as in (138). For \( dx \)–a.e. \( x \in \Lambda \), the map \( (\omega, z) \mapsto w(x, \omega, z) \) belongs to \( L^2_{\text{sol}}(\nu) \).
Proof. We use that \( \langle \nabla \epsilon u, \nabla \epsilon V \rangle_{L^2(\nu_{\omega,\lambda})} = 0 \) for any \( u \in H^1_{\omega,0} \) (cf. Lemma 5.2 (iii)). We take \( u(x) := \epsilon \phi(x) g(\tau_{x/\epsilon} \omega) \), where \( \phi \in C_c(\Lambda) \) and \( g \in G_2 \) (cf. Section 8). Due to (59) we have
\[
\nabla \epsilon u(x, z) = \epsilon \nabla \epsilon \phi(x, z) g(\tau_{x+\epsilon z/\epsilon} \omega) + \phi(x) \nabla g(\tau_{x/\epsilon} \omega, z),
\]
where \( \nabla g(\omega, z) = g(\tau_{z} \omega) - g(\omega) \). Due to (145), the identity \( \langle \nabla \epsilon u, \nabla \epsilon V \rangle_{L^2(\nu_{\omega,\lambda})} = 0 \) can be rewritten as
\[
\epsilon \int d\nu^\epsilon_{\omega,\lambda}(x, z) \nabla \epsilon \phi(x, z) g(\tau_{x+\epsilon z/\epsilon} \omega) \nabla \epsilon V(x, z) + \int d\nu^\epsilon_{\omega,\lambda}(x, z) \phi(x) \nabla g(\tau_{x/\epsilon} \omega, z) \nabla \epsilon V(x, z) = 0.
\]
We first show that
\[
\limsup_{\epsilon \downarrow 0} \left| \int d\nu^\epsilon_{\omega,\lambda}(x, z) \nabla \epsilon \phi(x, z) g(\tau_{x+\epsilon z/\epsilon} \omega) \nabla \epsilon V(x, z) \right| < +\infty. \tag{147}
\]
By applying Schwarz inequality, using that \( g \) is bounded as \( g \in G_2 \) and that \( \limsup_{\epsilon \downarrow 0} \| \nabla \epsilon V \|_{L^2(\nu_{\omega,\lambda})} < +\infty \) due to (57), and since \( \omega \in \Omega_{\typ} \subset \Omega_2 \), to get (147) it is enough to show that \( \limsup_{\epsilon \downarrow 0} \| \nabla \epsilon \phi \|_{L^2(\nu_{\omega,\lambda})} < +\infty \). As \( \omega \in \Omega_{\typ} \), by Lemma 7.14 it remains to prove that \( \limsup_{\epsilon \downarrow 0} \| \nabla \epsilon \phi \cdot z \|_{L^2(\nu_{\omega,\lambda})} < +\infty \).

To conclude we observe that, since \( \omega \in \Omega_{\typ} \subset \mathcal{A}_1[|z|^2] \cap \mathcal{A}[\lambda_2] \),
\[
\int d\nu^\epsilon_{\omega,\lambda}(x, z) |\nabla \phi(x)|^2 z^2 = \int d\mu^\epsilon_{\omega}(x) |\nabla \phi(x)|^2 \lambda_2(\tau_{x/\epsilon} \omega) \rightarrow \int dx m|\nabla \phi(x)|^2 \mathbb{E}_0[\lambda_2] < +\infty. \tag{148}
\]
This completes the proof of (147).

Coming back to (146), using (147) to treat the first addendum and applying the 2-scale convergence \( \nabla \epsilon V \xrightarrow{\text{w}} w \) in (138) to treat the second addendum, we conclude that
\[
\int \Lambda dx \int d\nu(\omega, z) \phi(x) \nabla g(\omega, z) w(x, \omega, z) = 0 \quad \forall g \in G_2. \tag{149}
\]
Note that above we have applied (95) as \( \nabla g \in \mathcal{H}_2 \subset \mathcal{H} \). Since \( \{ \nabla g : g \in G_2 \} \) is dense in \( L^2_{\text{pot}}(\nu) \), the above identity implies that, for \( dx \)-a.e. \( x \in \Lambda \), the map \( (\omega, z) \mapsto w(x, \omega, z) \) belongs to \( L^2_{\text{stal}}(\nu) \).

12. 2-scale limit of \( V_\epsilon \): proof of Theorem 2 for \( D_{1,1} > 0 \)

In this section we give the proof of Theorem 2 assuming that \( D_{1,1} > 0 \). In particular, we will get (32).
12.1. Convergence of \( V_ε \) to \( \psi \). We fix \( \tilde{\omega} \in Ω_{\text{typ}} \) and prove the convergences in Theorem 2 for \( \tilde{\omega} \) instead of \( \omega \) there. Due to Lemmas 9.4 and 9.6 along a subsequence \( \{ε_k\} \) we have that \( \mathcal{L}^2(\mu_{\tilde{\omega},A}) \ni V_ε \xrightarrow{ε \to 0} v \in \mathcal{L}^2(\Lambda \times Ω, mdx \times P_0) \) and \( \mathcal{L}^2(\nu_{\tilde{\omega},A}) \ni \nabla_ε V_ε \xrightarrow{ε \to 0} w \in \mathcal{L}^2(\Lambda \times Ω \times \mathbb{R}^d, mdx \times ν) \) (cf. (137) and (138)). We claim that for \( dx \)-a.e. \( x \in Λ \) it holds

\[
\int dν(ω, z)w(x, ω, z)z = 2D\nabla_v v(x). \tag{150}
\]

By Proposition 11.2 for \( dx \)-a.e. \( x \in Λ \), the map \( (ω, z) \mapsto w(x, ω, z) \) belongs to \( \mathcal{L}^2_{\text{sol}}(ν) \). On the other hand, by Proposition 10.3 we know that \( w(x, ω, z) = \nabla_v v(x) \cdot z + v_1(x, ω, z) \), where \( v_1 \in \mathcal{L}^2(Λ, \mathcal{L}^2_{\text{pot}}(ν)) \). Hence, by (75), for \( dx \)-a.e. \( x \in Λ \) we have that \( v_1(x, \cdot, \cdot) = v^a \), where \( a := \nabla_v v(x) \). As a consequence (using also (76)), for \( dx \)-a.e. \( x \in Λ \), we have

\[
\int dν(ω, z)w(x, ω, z)z = \int dν(ω, z)[ν(x) \cdot z + \nabla_v v(x)(ω, z)] = 2D\nabla_v v(x),
\]

thus proving (150).

We now take a function \( φ \in C^2_c(\mathbb{R}^d) \) which is zero on \( S \setminus Λ \) (note that we are not taking \( φ \in C^2_c(S) \)). By Lemma 5.2 (ii) we have the identity \( \langle \nabla_ε φ, \nabla_ε V_ε \rangle_{L^2(\nu_{\tilde{\omega},A})} = 0 \). The above identity and Lemma 7.14 (use that \( \tilde{\omega} \in Ω_{\text{typ}} \)) imply that

\[
0 = \langle \nabla_ε φ, \nabla_ε V_ε \rangle_{L^2(\nu_{\tilde{\omega},A})} = \int dν_{\tilde{\omega},A}(x, z)\nabla_ε φ(x) \cdot z \nabla_ε V_ε(x, z) + o(1). \tag{151}
\]

Hence

\[
0 = \lim_{ε \to 0} \int dν_{\tilde{\omega},A}(x, z)\nabla_ε φ(x) \cdot z \nabla_ε V_ε(x, z). \tag{152}
\]

For each \( n \geq 3 \) let \( A_n := [-1/2 + 1/n, 1/2 - 1/n]^d \) and let \( φ_n \in C_c(Λ) \) be a function with values in \([0, 1]\) such that \( φ_n \equiv 1 \) on \( A_n \). By Schwarz inequality

\[
\left| \int dν_{\tilde{\omega},A}(x, z)(φ_n(x) - 1)\nabla_ε φ(x) \cdot z \nabla_ε V_ε(x, z) \right| ≤ \|\nabla_ε V_ε\|_{L^2(\nu_{\tilde{\omega},A})} \left[ \int_{Λ \setminus A_n} dμ_{\tilde{\omega},A}(x)\lambda_2(τ_{x/ε}\tilde{ω}) \right]^{1/2}. \tag{153}
\]

By ergodicity (equivalently by applying Prop. 4.3 to suitable functions \( φ, φ' \in C_c(\mathbb{R}^d) \) with \( φ \leq ι_{Λ \setminus A_n} \leq φ' \) and using that \( \tilde{ω} \in Ω_{\text{typ}} \subset A_{[λ_2]} \)) we have \( \lim_{ε \to 0} \int_{Λ \setminus A_n} dμ_{\tilde{\omega},A}(x)\λ_2(τ_{x/ε}\tilde{ω}) = ℓ(Λ \setminus A_n)\mathcal{E}_0[λ_2] \). As a byproduct with Lemma 5.4 we conclude that

\[
\lim_{n \to \infty} \limsup_{ε \to 0} \text{l.h.s. of } (153) = 0. \tag{154}
\]

Using (152) we get

\[
\lim_{n \to \infty} \limsup_{ε \to 0} \int dν_{\tilde{\omega},A}(x, z)φ_n(x)\nabla_ε φ(x) \cdot z \nabla_ε V_ε(x, z) = 0. \tag{155}
\]
On the other hand, due to (138) and since \( \tilde{\omega} \in \Omega_{\text{typ}} \) (recall that the form \((\omega, z) \mapsto z\) belongs to \(\mathcal{H}\), recall that \(\phi_n \in C_c(\Lambda)\) and apply (95)), we can rewrite (155) as

\[
\lim_{n \to \infty} \int_{\Lambda} dx \int dv(\omega, z) \phi_n(x) \nabla \varphi(x) \cdot zw(x, \omega, z) = 0. \tag{156}
\]

Reasoning as in (153) we get

\[
0 = \int_{\Lambda} dx \int dv(\omega, z) \nabla \varphi(x) \cdot zw(x, \omega, z). \tag{157}
\]

As a byproduct of (156) and (157) we conclude that \(0 = \int_{\Lambda} dx \nabla \varphi(x) \cdot D \nabla v(x) = \int_{\Lambda} dx \nabla \varphi(x) \cdot D \nabla v(x)\) for any \(\varphi \in C_0^2(\mathbb{R}^d)\) with \(\varphi \equiv 0\) on \(S \setminus \Lambda\) (we write \(\varphi \in C\)). If we take \(\varphi \in C_0^\infty(\mathbb{R}^d \setminus F)\), then \(\varphi|_{\Lambda}\) can be approximated in the space \(H^1(\Lambda)\) by functions \(\tilde{\varphi}|_{\Lambda}\) with \(\tilde{\varphi} \in C\). Hence by density we conclude that \(0 = \int_{\Lambda} dx \nabla \varphi(x) \cdot D \nabla v(x)\) for any \(\varphi \in H^1(\Lambda, F, d\alpha)\). Due to Proposition 11.1 we also have that \(v \in K\) (cf. (34) in Definition 3.1). Hence, by Definition 3.6 and Lemma 3.8 \(v\) is the unique weak solution of the equation \(\nabla \cdot (D \nabla v) = 0\) with boundary conditions (38). By Corollary 3.9 we conclude that \(v = \psi|_{\Lambda}\). Since the limit point is always \(\psi|_{\Lambda}\) whatever the subsequence \(\{\varepsilon\}\), we get the \(V_\varepsilon \in L^2(\mu_{\overline{\varepsilon}, \Lambda}^\varepsilon)\) weakly 2-scale converges to \(\psi|_{\Lambda} \in L^2(\Lambda \times \Omega, m dx \times P_0)\) as \(\varepsilon \downarrow 0\), and not only along some subsequence. As \(\psi|_{\Lambda}\) does not depend from \(\omega\) and since \(1 \in \mathcal{G}\), we derive from (94) that \(L^2(\mu_{\overline{\varepsilon}, \Lambda}^\varepsilon) \ni V_\varepsilon \to \psi \in L^2(\Lambda, m dx)\) according to Definition 9.1.

12.2. Convergence of the energy flow. Let us show that, given \(\tilde{\omega} \in \Omega_{\text{typ}}\), it holds \(\lim_{\varepsilon \to 0} \frac{\varepsilon}{2} \langle \nabla \varepsilon V_\varepsilon, \nabla \varepsilon V_\varepsilon \rangle_{L^2(\nu_{\overline{\varepsilon}, \Lambda}^\varepsilon)} = m D_{1,1}\). To this aim we apply Lemma 5.2(ii) with \(u := V_\varepsilon - \psi\), which belongs to \(H^1_{0,\omega}\). Then we have \(\langle \nabla \varepsilon (V_\varepsilon - \psi), \nabla \varepsilon V_\varepsilon \rangle_{L^2(\nu_{\overline{\varepsilon}, \Lambda}^\varepsilon)} = 0\). This implies that

\[
\langle \nabla \varepsilon V_\varepsilon, \nabla \varepsilon V_\varepsilon \rangle_{L^2(\nu_{\overline{\varepsilon}, \Lambda}^\varepsilon)} = \langle \nabla \varepsilon \psi, \nabla \varepsilon V_\varepsilon \rangle_{L^2(\nu_{\overline{\varepsilon}, \Lambda}^\varepsilon)}; \tag{158}
\]

Claim 12.1. It holds \(\lim_{\varepsilon \to 0} \int d\nu_{\overline{\varepsilon}, \Lambda}^\varepsilon(x, z) |\nabla \varepsilon \psi(x, z) - z_1|^2 = 0\).

**Proof.** If \(x, x + \varepsilon z \in \Lambda\), then \(\nabla \varepsilon \psi(x, z) = z_1\). We have only 4 relevant alternative cases: (a) \(x \in \Lambda, x + \varepsilon z \in S_+\); (b) \(x \in S_-, x + \varepsilon z \in \Lambda\); (c) \(x \in \Lambda, x + \varepsilon z \in S_-\); (d) \(x \in S_-, x + \varepsilon z \in \Lambda\). Below we treat only case (a), since the other cases can be treated similarly. Hence we assume (a) to hold. Then \(x_1 + \frac{1}{2} = \psi(x) \leq \psi(x + \varepsilon z) \leq x_1 + \varepsilon z_1 + \frac{1}{2}\) and therefore \(0 \leq \nabla \varepsilon \psi(x, z) \leq z_1\). This implies that \(|\nabla \varepsilon \psi(x, z) - z_1|^2 \leq z_1^2\). Fix \(\delta \in (0, 1/2)\)
and set \( \Lambda_{\delta} = (-1/2 + \delta, 1/2 - \delta)^d \). We can bound

\[
\int dv_{\Lambda, \delta}^e(x, z) |\nabla_x \psi(x, z) - z_1|^2 \mathbb{I}(x \in \Lambda_{\delta}, x + \varepsilon z \in S_+)
\]

\[
\leq \int dv_{\Lambda}^e(x, z) z_1^2 \mathbb{I}(x \in \Lambda_{\delta}, z_1 \geq \delta/\varepsilon)
\]

\[
\leq \int_{\Lambda_{\delta}} d\mu_{\omega}^e(x) \int d\tilde{\tau}_{x/\varepsilon} \tilde{\omega}(z)c_{0, z}(\tau_{x/\varepsilon} \tilde{\omega}) z_1^2 \mathbb{I}(|z| \geq \delta/\varepsilon)
\]

\[
\leq \kappa(\delta/\varepsilon) \int_{\Lambda_{\delta}} d\mu_{\omega}^e(x) \int d\tilde{\tau}_{x/\varepsilon} \tilde{\omega}(z)c_{0, z}(\tau_{x/\varepsilon} \tilde{\omega})^n z_1^2 \leq \kappa(\delta/\varepsilon) \int_{\Lambda_{\delta}} d\mu_{\omega}^e(h(\tau_{x/\varepsilon} \tilde{\omega})) \tag{159}
\]

where \( \kappa(\ell) := \sup_{\omega \in \Omega_{\delta, |z| \geq \ell}} c_{0, z}(\omega)^{1-\alpha} \) and \( h(\omega) := \int d\tilde{\omega}(z)c_{0, z}(\omega)^n z_1^2 \). We have that \( \lim_{\varepsilon \downarrow 0} \kappa(\delta/\varepsilon) = 0 \) by (13). Since \( \omega \in \Omega_{\text{typ}} \subset A_1[c_{0, z}(\omega)^n z_1^2] \cap A[h] \), the last integral in (159) converges to a finite constant as \( \varepsilon \downarrow 0 \). This concludes the proof that the l.h.s. of (159) converges to zero as \( \varepsilon \downarrow 0 \).

We can bound

\[
\int dv_{\Lambda, \delta}^e(x, z) |\nabla_x \psi(x, z) - z_1|^2 \mathbb{I}(x \in \Lambda \setminus \Lambda_{\delta}, x + \varepsilon z \in S_+)
\]

\[
\leq \int dv_{\Lambda}^e(x, z) z_1^2 \mathbb{I}(x \in \Lambda \setminus \Lambda_{\delta}) \leq \int_{\Lambda \setminus \Lambda_{\delta}} d\mu_{\omega}^e(x) \lambda_2(\tau_{x/\varepsilon} \tilde{\omega}). \tag{160}
\]

By Prop. 4.3 and since \( \tilde{\omega} \in \Omega_{\text{typ}} \subset A_1[z_1^2] \cap A[\lambda_2] \), \( \lim_{\varepsilon \downarrow 0} \int_{\Lambda \setminus \Lambda_{\delta}} d\mu_{\omega}^e(x) \lambda_2(\tau_{x/\varepsilon} \tilde{\omega}) = \ell(\Lambda \setminus \Lambda_{\delta}) \mathcal{E}_0[\lambda_2] \). It then follows that the l.h.s. of (159) converges to zero as \( \varepsilon \downarrow 0 \) and afterwards \( \delta \downarrow 0 \).

As a byproduct of Claim 12.1 and (158), we get

\[
\lim_{\varepsilon \downarrow 0} \langle \nabla_v V_\varepsilon, \nabla_v \psi \rangle_{L^2(v_{\omega, \Lambda}^\varepsilon)} = \lim_{\varepsilon \downarrow 0} \int dv_{\Lambda, \delta}^e(x, z) z_1 \nabla_v V_\varepsilon(x, z). \tag{161}
\]

By applying Schwarz inequality as in (153), we get that

\[
\lim_{\varepsilon \downarrow 0} \int dv_{\Lambda, \delta}^e(x, z) z_1 \nabla_v V_\varepsilon(x, z) = \lim_{n \uparrow \infty} \lim_{\varepsilon \downarrow 0} \int dv_{\Lambda, \delta}^e(x, z) \phi_n(x) z_1 \nabla_v V_\varepsilon(x, z). \tag{162}
\]

By Lemma 9.6 from any vanishing sequence \( \{\varepsilon_k\} \) we can extract a sub-subsequence \( \{\varepsilon_{k_n}\} \) such that \( \nabla_v V_\varepsilon \rightharpoonup w \) along the sub-subsequence as in (138). Since \( \phi_n \in C_c(\Lambda) \), as a byproduct of (161) and (162) we obtain that

\[
\lim_{\varepsilon \downarrow 0} \langle \nabla_v V_\varepsilon, \nabla_v \psi \rangle_{L^2(v_{\omega, \Lambda}^\varepsilon)} = \lim_{n \uparrow \infty} \int_\Lambda dx m \phi_n(x) \int dv(x, z) z_1 w(x, \omega, z)
\]

\[
= \int_\Lambda dx m \int dv(x, z) z_1 w(x, \omega, z) \tag{163}
\]

along \( \{\varepsilon_{k_n}\} \). Due to (150) the last term equals \( m \int_\Lambda 2(D\nabla v(x)) \cdot e_1 dx \). Since \( v = \psi_{1\Lambda} \) as derived in the first part of the proof, we get that \( \nabla v(x) = e_1 \). As a
consequence, the last term of (163) equals $2mD_{11}$, thus allowing to conclude the proof.

APPENDIX A. PROOF OF EQUATIONS (20) AND (21)

For simplicity of notation we write $i_{x,y}$ instead of $i_{x,y}(\omega)$. It is also convenient to set $A_0 := \hat{\omega} \cap A_\ell$, $A_{-1} := \{ x \in \hat{\omega} \cap S_\ell : x_1 \leq -\ell/2 \}$ and $A_1 := \{ x \in \hat{\omega} \cap S_\ell : x_1 \geq \ell/2 \}$.

We start proving (20). Due to definition (19) of $\sigma_\ell(\omega)$ we can write the r.h.s. of (20) as

$$\sigma_\ell(\omega) - \sum_{x \in A_{-1}} \sum_{y \in A_0 : y_1 \leq \gamma} i_{x,y} + \sum_{x \in A_0 : y \in A_0 \cup A_1 : x_1 \leq \gamma} \sum_{y_1 \leq \gamma} i_{x,y}.$$  

By antisymmetry $-\sum_{x \in A_{-1}} \sum_{y \in A_0 : i_{x,y}} = \sum_{x \in A_{-1}} \sum_{y \in A_0 : i_{y,x}}$. Hence, (164) can be rewritten as

$$\sigma_\ell(\omega) + \sum_{x \in A_0 : y \in A_{-1}} \sum_{x_1 \leq \gamma} i_{x,y} + \sum_{x \in A_0 : y \in A_0 \cup A_1 : x_1 \leq \gamma} \sum_{y_1 \leq \gamma} i_{x,y}.$$  

By antisymmetry $\sum_{x \in A_0 : y \in A_0 : i_{x,y}} = 0$. By adding this zero sum to (164) we get $\sigma_\ell(\omega) + \sum_{x \in A_0 : (\text{div } i)_x} (\text{div } i)_x$ being the divergence of the current field at $x$ given by $(\text{div } i)_x := \sum_{y \in \hat{\omega} \cap S_\ell} i_{x,y}$. To conclude the proof of (20) we observe that $(\text{div } i)_x = 0$ for any $x \in A_0$ by (16).

We move to the proof of (21). Due to (18) we can write the r.h.s. of (21) as

$$2^{-1} \sum_{(x,y) : (x,y) \in \mathcal{B}_\ell^\omega} c_{x,y}(\omega)(V_{\ell}^{\omega}(x) - V_{\ell}^{\omega}(y))^2 = 2^{-1}C_1 - 2^{-1}C_2,$$  

where $C_1 := \sum_{(x,y)} i_{x,y}V_{\ell}^{\omega}(y)$ and $C_2 := \sum_{(x,y)} i_{x,y}V_{\ell}^{\omega}(x)$. We analyze the two contributions $C_1$ and $C_2$ separately. As $V \equiv 0$ on $A_{-1}$ and $V \equiv 1$ on $A_1$ we can write

$$C_1 = \sum_{x \in A_{-1}, y \in A_0} i_{x,y}V_{\ell}^{\omega}(y) + \sum_{x \in A_0, y \in A_0} i_{x,y}V_{\ell}^{\omega}(y) + \sum_{x \in A_0, y \in A_1} i_{x,y} + \sum_{x \in A_1, y \in A_0} i_{x,y}V_{\ell}^{\omega}(y).$$  

(167)

Note that, by antisymmetry of the current, we can rewrite (167) as

$$C_1 = \sum_{x \in A_0, y \in A_1} i_{x,y} - \sum_{y \in A_0} V_{\ell}^{\omega}(y) \sum_{x \in A_0 \cup A_{-1} \cup A_1} i_{y,x} = \sum_{x \in A_0, y \in A_1} i_{x,y},$$  

(168)

where the last identity follows from the fact that $(\text{div } i)_x = 0$ for any $x \in A_0$.

We now move to $C_2$. Always by the above zero divergence property, in $C_2$ we can remove the contribution from $x \in A_0$. Hence, using also (17), we get

$$C_2 = \sum_{x \in A_{-1}, y \in A_0} i_{x,y}V_{\ell}^{\omega}(x) + \sum_{x \in A_1, y \in A_0} i_{x,y}V_{\ell}^{\omega}(x) = \sum_{x \in A_1, y \in A_0} i_{x,y}.$$  

(169)
By combining (166), (168) and (169) we conclude that the r.h.s. of (21) equals \( \sum_{x \in A_0, y \in A_1} i_{x,y} \). This last term equals \( \sigma_\ell(\omega) \) due to (20) with \( \gamma \) very near to \( \ell/2 \) (as \( \tilde{\omega} \) is a locally finite set).

**Acknowledgements.** I thank Andrey Piatnitski for useful discussions. I thank Annibale Faggionato and Bruna Tecchio for their warm hospitality in Codroipo, where part of this work has been completed.

**References**

[1] V. Ambegoakar, B.I. Halperin, J.S. Langer; *Hopping conductivity in disordered systems*. Phys. Rev. B 4, 2612-2620 (1971).

[2] G. Androulakis, J. Bellissard, C. Sadel; *Dissipative Dynamics in Semiconductors at Low Temperature*. J. Stat. Phys. 147, Issue 2, 448–486 (2012).

[3] J. Bellissard, R. Rebolledo, D. Spehner, W. Von Waldenfels; *The Quantum Flow Of Electronic Transport I: The Finite Volume Case*. Unpublished. Available online.

[4] M. Biskup; *Recent progress on the random conductance model*. Probability Surveys, Vol. 8, 294-373 (2011).

[5] M. Biskup, M. Salvi, T. Wolff; *A central limit theorem for the effective conductance: linear boundary data and small ellipticity contrasts*. Commun. Math. Phys. 328, 701-731 (2014).

[6] A. Bourgeat, A. Piatnitski; *Approximations of effective coefficients in stochastic homogenization*. Ann. I. H. Poincaré 40, 153–165 (2004).

[7] H. Brezis; *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. New York, Springer Verlag, 2010.

[8] P. Caputo, A. Faggionato; *Diffusivity of 1-dimensional generalized Mott variable range hopping*. Ann. Appl. Probab. 19, 1459–1494 (2009).

[9] P. Caputo, A. Faggionato, T. Prescott; *Invariance principle for Mott variable range hopping and other walks on point processes*. Ann. Inst. H. Poincaré Probab. Statist. 49, 654–697 (2013).

[10] D.J. Daley, D. Vere-Jones; *An Introduction to the Theory of Point Processes*. New York, Springer Verlag, 1988.

[11] L.C. Evans, R.F. Gariepy; *Measure theory and fine properties of functions*. Boca Raton, CRC, 1992.

[12] A. Faggionato; *Stochastic homogenization in amorphous media and applications to exclusion processes*. Preprint [arXiv:1903.07311] (2019).

[13] A. Faggionato, H.A. Mimun; *Connection probabilities in the Poisson Miller–Abrahams random resistor network and other Poisson random graphs with bounded edges*. ALEA, Lat. Am. J. Probab. Math. Stat. 16, 463-486 (2019).

[14] A. Faggionato, H.A. Mimun; *Left-right crossings in the Miller-Abrahams random resistor network on a Poisson point process*. [arXiv:1912.07482] (2019).

[15] A. Faggionato, N. Gantert, M. Salvi; *Einstein relation and linear response in one-dimensional Mott variable-range hopping*. Ann. Inst. H. Poincaré Probab. Statist. 55, 1477–1508 (2019).

[16] A. Faggionato, P. Mathieu; *Mott law as upper bound for a random walk in a random environment*. Commun. Math. Phys. 281, 263–286 (2008).

[17] A. Faggionato, H. Schulz-Baldes, D. Spehner; *Mott law as lower bound for a random walk in a random environment*. Commun. Math. Phys., 263, 21–64 (2006).

[18] V.V. Jikov, S.M. Kozlov, O.A. Oleinik *Homogenization of differential operators and integral functionals*. Berlin, Springer Verlag, 1994.

[19] J. Kurkijärvi; *Hopping conductivity in one dimension*. Phys. Rev. B 8, 922–924. (1973).
[20] A. Miller, E. Abrahams; Impurity Conduction at Low Concentrations. Phys. Rev. 120, 745–755 (1960).
[21] N. Minami; Local fluctuation of the spectrum of a multidimensional Anderson tight binding model. Commun. Math. Phys. 177, 709–725 (1996).
[22] N.F. Mott: J. Non-Crystal. Solids 1, 1 (1968); N. F. Mott, Phil. Mag 19, 835 (1969).
[23] N.F. Mott, E.A. Davis; Electronic processes in non-crystalline materials. Oxford Classic Texts in the Physical Sciences, OUP Oxford, Oxford, 2012.
[24] A. Piatnitski, E Remy; Homogenization of elliptic difference operators. SIAM J. Math. Anal. 33, 53-83 (2001).
[25] M. Pollak, M. Ortuño, A. Frydman; The electron glass. Cambridge University Press, United Kingdom, 2013.
[26] M. Sahimi; Applications of percolation theory. Taylor & Francis, CRC Press, 1994.
[27] S. Shklovskii, A.L. Efros; Electronic Properties of Doped Semiconductors. Springer Verlag, Berlin, 1984.
[28] V.V. Zhikov; On an extension of the method of two-scale convergence and its applications. (Russian) Mat. Sb. 191, no. 7, 31–72 (2000); translation in Sb. Math. 191, no. 7-8, 973–1014 (2000).
[29] V.V. Zhikov, A.L. Pyatnitskii; Homogenization of random singular structures and random measures. (Russian) Izv. Ross. Akad. Nauk Ser. Mat. 70, no. 1, 23–74 (2006); translation in Izv. Math. 70, no. 1, 19–67 (2006).

Alessandra Faggionato. Dipartimento di Matematica, Università di Roma ‘La Sapienza’ P.le Aldo Moro 2, 00185 Roma, Italy
Email address: faggiona@mat.uniroma1.it