HOMOTOPY TYPES OF $\text{Spin}^c(n)$-GAUGE GROUPS OVER $S^4$

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ABSTRACT. The gauge group of a principal $G$-bundle $P$ over a space $X$ is the group of $G$-equivariant homeomorphisms of $P$ that cover the identity on $X$. We consider the gauge groups of bundles over $S^4$ with $\text{Spin}^c(n)$, the complex spin group, as structure group and show how the study of their homotopy types reduces to that of $\text{Spin}(n)$-gauge groups over $S^4$. We then advance on what is known by providing a partial classification for $\text{Spin}(7)$- and $\text{Spin}(8)$-gauge groups over $S^4$.

KEYWORDS: Gauge groups, Homotopy types, Spin groups

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1. Introduction

Let $G$ be a topological group and $X$ a space. The gauge group $\mathcal{G}(P)$ of a principal $G$-bundle $P$ over $X$ is defined as the group of $G$-equivariant bundle automorphisms of $P$ which cover the identity on $X$. A detailed introduction to the topology of gauge groups of bundles can be found in [25, 40]. The study of gauge groups is important for the classification of principal bundles, as well as understanding moduli spaces of connections on principal bundles [7, 48, 50]. Gauge groups also play a key role in theoretical physics, where they are used to describe the parallel transport of point particles by means of connections on bundles. Famously, Donaldson [12] discovered a deep link between the gauge groups of certain $\text{SU}(2)$-bundles and the differential topology of 4-manifolds.

Key properties of gauge groups are invariant under continuous deformation and so studying their homotopy theory is important. Having fixed a topological group $G$ and a space $X$, an interesting problem is that of classifying the possible homotopy types of the gauge groups $\mathcal{G}(P)$ of principal $G$-bundles $P$ over $X$.

Crabb and Sutherland showed [8 Theorem 1.1] that if $G$ is a compact, connected, Lie group and $X$ is a connected, finite CW-complex, then the number of distinct homotopy types of $\mathcal{G}(P)$, as $P \to X$ ranges over all principal $G$-bundles over $X$, is finite. In fact, since isomorphic $G$-bundles give rise to homeomorphic gauge groups, it will suffice to the let $P \to X$ range over the set of isomorphism classes of principal $G$-bundles over $X$.

Explicit classification results have been obtained, especially for the case of gauge groups of bundles with low rank, compact, Lie groups as structure groups.
and \( X = S^4 \) as base space. In particular, the first such result was obtained by Kono [30] in 1991. Using the fact that isomorphism classes of principal SU(2)-bundles over \( S^4 \) are classified by \( k \in \mathbb{Z} \cong \pi_1(\text{SU}(2)) \) and denoting by \( \mathcal{G}_k \) the gauge group of the principal SU(2)-bundle \( P_k \rightarrow S^4 \) corresponding to the integer \( k \), Kono showed that there is a homotopy equivalence \( \mathcal{G}_k \cong \mathcal{G}_l \) if, and only if, \((12, k) = (12, l)\), where \((m, n)\) denotes the greatest common divisor of \( m \) and \( n \). Since 12 has six divisors, it follows that there are precisely six homotopy types of SU(2)-gauge groups over \( S^4 \).

Results formally similar to that of Kono have been obtained for principal bundles over \( S^4 \) with different structure groups, among others, by: Hamanaka and Kono [17] for SU(3)-gauge groups; Theriault [51, 52] for SU(5)-gauge groups; and Deninger for Spin(7)-gauge groups, as well as [46] for Sp(2)-gauge groups; Cutler [9, 10] for Sp(3)-gauge groups and U(n)-gauge groups; Kishimoto, Theriault and Tsutaya for G2-gauge groups; Kamiyama, Kishimoto, Kono and Tsukuda for SO(3)-gauge groups; Hasui, Kishimoto, Kono and Sato [21] for PU(3)- and PSp(2)-gauge groups; and Hasui, Kishimoto, So and Theriault [22] for bundles with exceptional Lie groups as structure groups. There are also several classification results for gauge groups of principal bundles with base spaces other than \( S^4 \) [6, 16, 18, 21, 23, 24, 31–34, 37–39, 41, 43, 44, 49, 53, 55].

The complex spin group Spin\(^c\)(n) was first introduced in 1964 in a paper of Atiyah, Bott and Shapiro [33]. There has been an increasing interest in the Spin\(^c\)(n) groups ever since the publication of the Seiberg-Witten equations for 4-manifolds [56], whose formulation requires the existence of Spin\(^c\)(n)-structures, and more recently for the role they play in string theory [5, 13, 42].

In this paper we examine Spin\(^c\)(n)-gauge groups over \( S^4 \). We begin by recalling some basic properties of the complex spin group Spin\(^c\)(n) and showing that, provided \( n \geq 3 \), it can be expressed as a product of a circle and the real spin group Spin\(^c\)(n).

**Theorem 1.1.** For \( n \geq 6 \) and any \( k \in \mathbb{Z} \), we have

\[
\mathcal{G}_k(\text{Spin}^c(n)) \cong S^1 \times \mathcal{G}_k(\text{Spin}(n)).
\]

The homotopy theory of Spin\(^c\)(n)-gauge groups over \( S^4 \) therefore reduced to that of the corresponding Spin(n)-gauge groups. We advance on what is known on Spin(n)-gauge groups by providing a partial classification for Spin(7)- and Spin(8)-gauge groups over \( S^4 \).

**Theorem 1.2.** (a) If \((168, k) = (168, l)\), there is a homotopy equivalence

\[
\mathcal{G}_k(\text{Spin}(7)) \cong \mathcal{G}_l(\text{Spin}(7))
\]

after localising rationally or at any prime;

(b) If \( \mathcal{G}_k(\text{Spin}(7)) \cong \mathcal{G}_l(\text{Spin}(7)) \) then \((84, k) = (84, l)\).
We note that the discrepancy by a factor of 2 between parts (a) and (b) is due to the same discrepancy for $G_2$-gauge groups.

**Theorem 1.3.** (a) If $(168, k) = (168, l)$, there is a homotopy equivalence

$$\mathcal{G}_k(\text{Spin}(8)) \simeq \mathcal{G}_l(\text{Spin}(8))$$

after localising rationally or at any prime;

(b) If $\mathcal{G}_k(\text{Spin}(8)) \simeq \mathcal{G}_l(\text{Spin}(8))$ then $(28, k) = (28, l)$. Furthermore, if $k$ and $l$ are multiples of 3, then $(3, k) = (3, l)$.

For the Spin$(8)$ case, in addition to the same 2-primary indeterminacy appearing in the Spin$(7)$ case, there are also known $[28, 47]$ difficulties at the prime 3 due to the non-vanishing of $\pi_{10}(\text{Spin}(8))_3$.

2. Spin$^c$ $(n)$ groups

For $n \geq 1$, the complex spin group Spin$^c$ $(n)$ is defined as the quotient

$$\frac{\text{Spin}(n) \times U(1)}{\mathbb{Z}/2\mathbb{Z}}$$

where $\mathbb{Z}/2\mathbb{Z} \cong \{(1, 1), (-1, -1)\} \subseteq \text{Spin}(n) \times U(1)$ denotes the central subgroup of order 2. The group Spin$^c$ $(n)$ is special case of the more general notion of Spin$^k$ $(n)$ group introduced in [1].

The first low rank Spin$^c$ $(n)$ groups can be identified as follows:

- Spin$^c$ $(1) \cong U(1) \cong S^1$;
- Spin$^c$ $(2) \cong U(1) \times U(1) \cong S^1 \times S^1$;
- Spin$^c$ $(3) \cong U(2) \cong S^1 \times S^3$;
- Spin$^c$ $(4) \cong \{(A, B) \in U(2) \times U(2) \mid \det A = \det B\}$.

The group Spin$^c$ $(n)$ fits into a commutative diagram

$$
\begin{array}{ccc}
\{\pm 1\} & \xleftarrow{pr_1} & \{(1, 1), (-1, -1)\} & \xrightarrow{pr_2} & \{\pm 1\} \\
\downarrow & & \downarrow & & \downarrow \\
\text{Spin}(n) & \xleftarrow{pr_1} & \text{Spin}(n) \times S^1 & \xrightarrow{pr_2} & S^1 \\
\downarrow \lambda & & \downarrow q & & \downarrow 2 \\
\text{SO}(n) & \xleftarrow{pr_1} & \text{Spin}^c(n) & \xrightarrow{pr_2} & S^1, \\
\end{array}
$$

where $q$ is the quotient map, $\lambda: \text{Spin}(n) \to \text{SO}(n)$ denotes the double covering map of the group $\text{SO}(n)$ by $\text{Spin}(n)$ and $2: S^1 \to S^1$ denotes the degree 2 map. Furthermore, we observe that the map

$$\lambda \times 2: \text{Spin}^c(n) \to \text{SO}(n) \times S^1$$

is a double covering of $\text{SO}(n) \times S^1$ by Spin$^c$ $(n)$.
3. Method of classification

A principal bundle isomorphism determines a homeomorphism of gauge groups induced by conjugation \[^40\]. We therefore begin by considering isomorphism classes of principal Spin\(^c\)\((n)\)-bundles over \(S^4\). These are classified by the free homotopy classes of maps \(S^4 \rightarrow \text{BSpin}^c(n)\). Since \(\text{Spin}^c(n)\) is connected, \(\text{BSpin}^c(n)\) is simply-connected and hence there are isomorphisms

\[
[S^4, \text{BSpin}^c(n)]_{\text{free}} \cong \pi_3(\text{Spin}^c(n)) \cong \pi_3(\text{SO}(n)) \cong \begin{cases}
0 & n = 1, 2 \\
\mathbb{Z}^2 & n = 4 \\
\mathbb{Z} & n = 3, n \geq 5.
\end{cases}
\]

**Remark 3.1.** Note that for \(n = 3\) we have \(\text{Spin}^c(3) \cong \text{U}(2)\), and the homotopy types of \(\text{U}(2)\)-gauge groups over \(S^4\) have been studied by Cutler in \[^10\].

For \(n \geq 5\), let \(\mathcal{G}_k\) denote the gauge group of the \(\text{Spin}^c(n)\)-bundle \(P_k \rightarrow S^4\) classified by \(k \in \mathbb{Z}\). By \[^2, 15\], there is a homotopy equivalence

\[
\text{B}\mathcal{G}_k \cong \text{Map}_k(S^4, \text{BSpin}^c(n)),
\]

the latter space being the \(k\)-th component of \(\text{Map}(S^4, \text{BSpin}^c(n))\), meaning the connected component containing the map classifying \(P_k \rightarrow S^4\).

There is an evaluation fibration

\[
\text{Map}_k^*(S^4, \text{BSpin}^c(n)) \longrightarrow \text{Map}_k^*(S^4, \text{BSpin}^c(n)) \xrightarrow{\text{ev}} \text{BSpin}^c(n),
\]

where \(\text{ev}\) evaluates a map at the basepoint of \(S^4\) and the fibre is the \(k\)-th component of the pointed mapping space \(\text{Map}^*(S^4, \text{BSpin}^c(n))\). This fibration extends to a homotopy fibration sequence

\[
\mathcal{G}_k \longrightarrow \text{Spin}^c(n) \longrightarrow \text{Map}_k^*(S^4, \text{BSpin}^c(n)) \longrightarrow \text{B}\mathcal{G}_k \longrightarrow \text{BSpin}^c(n).
\]

Furthermore, by \[^45\] there is, for each \(k \in \mathbb{Z}\), a homotopy equivalence

\[
\text{Map}_k^*(S^4, \text{BSpin}^c(n)) \cong \text{Map}_0^*(S^4, \text{BSpin}^c(n)).
\]

The space on the right-hand side is homotopy equivalent to \(\text{Map}_0^*(S^3, \text{Spin}^c(n))\) by the exponential law, and is more commonly denoted as \(\Omega_0^3\text{Spin}^c(n)\). We therefore have the following homotopy fibration sequence

\[
\mathcal{G}_k \longrightarrow \text{Spin}^c(n) \xrightarrow{\partial_k} \Omega_0^3\text{Spin}^c(n) \longrightarrow \text{B}\mathcal{G}_k \longrightarrow \text{BSpin}^c(n),
\]

which exhibits the gauge group \(\mathcal{G}_k\) as the homotopy fibre of the map \(\partial_k\). This is a key observation, as it implies that the homotopy theory of the gauge groups \(\mathcal{G}_k\) depends on the maps \(\partial_k\).

**Lemma 3.2** (Lang \[^26\] Theorem 2.6). The adjoint of \(\partial_k : \text{Spin}^c(n) \rightarrow \Omega_0^3\text{Spin}^c(n)\) is homotopic to the Samelson product \(\langle k\epsilon, 1 \rangle : S^3 \wedge \text{Spin}^c(n) \rightarrow \text{Spin}^c(n)\), where \(\epsilon \in \pi_3(\text{Spin}^c(n))\) is a generator and \(1\) denotes the identity map on \(\text{Spin}^c(n)\). \(\square\)
As the Samelson product is bilinear, we have \( \langle k\epsilon, 1 \rangle \approx k\langle \epsilon, 1 \rangle \), and hence, taking adjoints once more, \( \partial_k \approx k\partial_1 \).

**Lemma 3.3** (Theriault [46, Lemma 3.1]). Let \( X \) be a connected CW-complex and let \( Y \) be an \( H \)-space with a homotopy inverse. Suppose that \( f \in [X, Y] \) has finite order and let \( m \in \mathbb{N} \) be such that \( mf = \ast \). Then, for any integers \( k, l \in \mathbb{Z} \) such that \( (m, k) = (m, l) \), the homotopy fibres of \( kf \) and \( lf \) are homotopy equivalent when localised rationally or at any prime. \( \square \)

**Remark 3.4.** The lemma of Theriault is the local analogue of a lemma used by Hamanaka and Kono in their study [17] of \( SU(3) \)-gauge groups over \( S^4 \).

Part (a) of Theorems 1.2 and 1.3 will follow as applications of Lemma 3.3. For parts (b) we will need to determine suitable homotopy invariants of the gauge groups.

### 4. Spin\( ^c(n) \)-gauge groups

We begin with a decomposition of \( \text{Spin}^c(n) \) as a product of spaces which will be reflected in an analogous decomposition of \( \text{Spin}^c(n) \)-gauge groups.

**Lemma 4.1.** For \( n \geq 3 \), we have \( \text{Spin}^c(n) \approx S^1 \times \widetilde{\text{Spin}}^c(n) \), where \( \widetilde{\text{Spin}}^c(n) \) denotes the universal cover of \( \text{Spin}^c(n) \).

**Proof.** We have \( \pi_1(\text{Spin}^c(n)) \approx \mathbb{Z} \) for \( n \geq 3 \) (see, e.g. [27]). By the Hurewicz and the universal coefficient theorems, we have isomorphisms
\[
\mathbb{Z} \approx \pi_1(\text{Spin}^c(n)) \cong H_1(\text{Spin}^c(n); \mathbb{Z}) \cong H^1(\text{Spin}^c(n); \mathbb{Z}).
\]
Therefore, we have maps \( S^1 \rightarrow \text{Spin}^c(n) \) and \( \text{Spin}^c(n) \rightarrow K(\mathbb{Z}, 1) \approx S^1 \) representing generators of \( \pi_1(\text{Spin}^c(n)) \) and of \( H^1(\text{Spin}^c(n); \mathbb{Z}) \), respectively, such that the composite induces an isomorphism in \( \pi_1 \). Therefore, the homotopy fibration
\[
\widetilde{\text{Spin}}^c(n) \longrightarrow \text{Spin}^c(n) \longrightarrow K(\mathbb{Z}, 1) \approx S^1
\]
defining the universal cover of \( \text{Spin}^c(n) \) admits a right homotopy splitting and hence, as \( \text{Spin}^c(n) \) is a group, we have
\[
\text{Spin}^c(n) \approx S^1 \times \widetilde{\text{Spin}}^c(n). \quad \square
\]

Note that as \( \text{Spin}^c(n) \) is a Lie group, we can equip \( \widetilde{\text{Spin}}^c(n) \) with a group structure for which the covering map
\[
\varphi: \widetilde{\text{Spin}}^c(n) \rightarrow \text{Spin}^c(n)
\]
is a group homomorphism.

**Lemma 4.2.** For \( n \geq 3 \), we have \( \widetilde{\text{Spin}}^c(n) \approx \text{Spin}(n) \).
Proof. Let $f : \text{Spin}^c(n) \to K(\mathbb{Z}/2\mathbb{Z}, 1)$ be the map in the fibration sequence

\[
\mathbb{Z}/2\mathbb{Z} \to S^1 \times \text{Spin}(n) \xrightarrow{q} \text{Spin}^c(n) \xrightarrow{f} K(\mathbb{Z}/2\mathbb{Z}, 1)
\]

arising from the double covering of $\text{Spin}^c(n)$ and let $g : \text{Spin}^c(n) \to K(\mathbb{Z}, 1)$ be the generator of $H^1(\text{Spin}^c(n); \mathbb{Z})$ realising the splitting of $\text{Spin}^c(n)$ in Lemma 4.1. We claim that there is a homotopy commutative diagram

\[
\begin{array}{ccc}
\text{Spin}^c(n) & \xrightarrow{f} & K(\mathbb{Z}/2\mathbb{Z}, 1) \\
\downarrow{g} & & \downarrow{=} \\
K(\mathbb{Z}, 1) & \xrightarrow{\rho} & K(\mathbb{Z}/2\mathbb{Z}, 1)
\end{array}
\]

where $\rho$ is the map induced by the mod 2 reduction $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$. Since the target space is an Eilenberg-MacLane space, it will be sufficient to check cohomology.

Indeed, on the one hand, since $g$ represents the generator of $H^1(\text{Spin}^c(n); \mathbb{Z})$, the composite $\rho \circ g$ represents the unique class in $H^1(\text{Spin}^c(n); \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ (cf. Harada and Kono [19] for the mod 2 cohomology of $\text{Spin}^c(n)$).

On the other hand, applying $\pi_1$ to the fibration sequence (1) we obtain an exact sequence

\[
\pi_1(\mathbb{Z}/2\mathbb{Z}) \to \pi_1(S^1 \times \text{Spin}(n)) \xrightarrow{q_*} \pi_1(\text{Spin}^c(n)) \xrightarrow{f_*} \pi_1(K(\mathbb{Z}/2\mathbb{Z}, 1)).
\]

Recalling from Section 2 that $q_*$ induces multiplication by 2 on the fundamental groups, we therefore have

\[
0 \to \mathbb{Z} \xrightarrow{q_* = 2} \mathbb{Z} \xrightarrow{f_*} \mathbb{Z}/2\mathbb{Z} \to 0
\]

since $\pi_1(\text{BS}^1 \times \text{BSpin}(n)) \cong 0$. Hence $f_*$ is reduction mod 2 on $\pi_1$. Applying the Hurewicz theorem and changing coefficients to $\mathbb{Z}/2\mathbb{Z}$ then gives a commutative diagram

\[
\begin{array}{ccc}
\pi_1(\text{Spin}^c(n)) & \xrightarrow{\pi_1(f)} & \pi_1(K(\mathbb{Z}/2\mathbb{Z}, 1)) \\
\xrightarrow{= h_1} & & \xrightarrow{= h_1} \\
H_1(\text{Spin}^c(n); \mathbb{Z}) & \xrightarrow{H_1(f)} & H_1(K(\mathbb{Z}/2\mathbb{Z}, 1); \mathbb{Z}) \\
\downarrow & & \downarrow \cong \\
H_1(\text{Spin}^c(n); \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{f_*} & H_1(K(\mathbb{Z}/2\mathbb{Z}, 1); \mathbb{Z}/2\mathbb{Z}).
\end{array}
\]

Since $\pi_1(f)$ and the composite in the left column are both reduction mod 2, the diagram implies that $H_1(f)$ is also reduction mod 2. Hence $f_*$ is an isomorphism in mod 2 homology. Finally, by the universal coefficient theorem for cohomology with field coefficients, we see that

\[
f^* : H^1(K(\mathbb{Z}/2\mathbb{Z}, 1); \mathbb{Z}/2\mathbb{Z}) \to H^1(\text{Spin}^c(n); \mathbb{Z}/2\mathbb{Z})
\]

is an isomorphism. Therefore $f : \text{Spin}^c(n) \to K(\mathbb{Z}/2\mathbb{Z}, 1)$ also represents the unique class in $H^1(\text{Spin}^c(n); \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and hence we have $f \cong \rho \circ g$. Taking
fibres, we obtain a diagram of homotopy fibrations

\[
\begin{array}{ccc}
\tilde{\text{Spin}}^c(n) & \rightarrow & \text{Spin}^c(n) \\
\downarrow \psi & & \downarrow \phi \\
S^1 \times \text{Spin}(n) & \rightarrow & \tilde{\text{Spin}}^c(n) \\
\downarrow & & \downarrow f \\
S^1 & \simeq K(\mathbb{Z}, 1) & \rightarrow & K(\mathbb{Z}/2\mathbb{Z}, 1)
\end{array}
\]

which defines a map \(\psi : \tilde{\text{Spin}}^c(n) \rightarrow S^1 \times \text{Spin}(n)\). In particular, the fibration in the leftmost column induces an exact sequence

\[
\pi_m(\Omega S^1) \rightarrow \pi_m(\tilde{\text{Spin}}^c(n)) \rightarrow \pi_m(S^1 \times \text{Spin}(n)) \rightarrow \pi_m(S^1)
\]

for each \(m > 1\). Given that the projection \(\text{pr}_2 : S^1 \times \text{Spin}(n) \rightarrow \text{Spin}(n)\) induces isomorphisms \(\pi_m(S^1 \times \text{Spin}(n)) \cong \pi_m(\text{Spin}(n))\) for \(m > 1\) and that the groups \(\tilde{\text{Spin}}^c(n)\) and \(\text{Spin}(n)\) are both simply-connected, the composite \(\text{pr}_2 \circ \psi\) induces isomorphisms on all homotopy groups and is therefore a homotopy equivalence by Whitehead’s theorem. Hence \(\tilde{\text{Spin}}^c(n) \cong \text{Spin}(n)\). \(\square\)

We are now ready to show that the decomposition

\[
\mathcal{G}_k(\text{Spin}^c(n)) \simeq S^1 \times \mathcal{G}_k(\text{Spin}(n))
\]

for \(n \geq 6\) holds as stated in Theorem 1.1.

Proof of Theorem 1.1 Identifying the universal cover of \(\text{Spin}^c(n)\) as \(\text{Spin}(n)\) as in Lemma 4.2 there is a covering fibration

\[
\text{Spin}(n) \xrightarrow{g} \text{Spin}^c(n) \xrightarrow{g} S^1
\]

where \(g\) is a group homomorphism. Let \(s : S^1 \rightarrow \text{Spin}^c(n)\) be a right homotopy inverse of \(g\), which exists by Lemma 4.1.

As \(\pi_3(\text{Spin}^c(n)) \cong 0\) for \(n \geq 6\), there is a lift in the diagram

\[
\begin{array}{ccc}
S^1 & \rightarrow & \text{Spin}(n) \\
\downarrow s & & \downarrow a \\
\mathcal{G}_k(\text{Spin}^c(n)) & \rightarrow & \text{Spin}^c(n) \\
\end{array}
\]

Define the map \(b\) to be the composite

\[
\mathcal{G}_k(\text{Spin}^c(n)) \rightarrow \text{Spin}^c(n) \xrightarrow{g} S^1.
\]

Since \(s\) is a right homotopy inverse for \(g\), the map \(a\) is a right homotopy inverse for \(b\). Therefore we have \(\mathcal{G}_k(\text{Spin}^c(n)) \simeq S^1 \times F_b\), where \(F_b\) denotes the homotopy fibre of \(b\).
As the covering map \( \varphi : \text{Spin}(n) \to \text{Spin}^c(n) \) is a group homomorphism, it classifies to a map
\[
B\varphi : B\text{Spin}(n) \to B\text{Spin}^c(n).
\]
Since \( \varphi \) induces an isomorphism in \( \pi_3 \), it respects path-components in \( \text{Map}_k(S^4, -) \) and \( \text{Map}_k^*(S^4, -) \) for any \( k \in \mathbb{Z} \). We therefore have a diagram of fibration sequences
\[
\cdots \to \text{Map}_k^*(S^4, B\text{Spin}(n)) \to \text{Map}_k(S^4, B\text{Spin}(n)) \to B\text{Spin}(n)
\]
\[
\cdots \to \text{Map}_k^*(S^4, B\text{Spin}^c(n)) \to \text{Map}_k(S^4, B\text{Spin}^c(n)) \to B\text{Spin}^c(n).
\]

Furthermore, observe that for all \( k \in \mathbb{Z} \) we have
\[
\pi_m(\text{Map}_k^*(S^4, B\text{Spin}(n))) \cong \pi_m(\Omega_0^3\text{Spin}(n)) \cong \pi_{m+3}(\text{Spin}(n))
\]
and, similarly, \( \pi_m(\text{Map}_k^*(S^4, B\text{Spin}^c(n))) \cong \pi_{m+3}(\text{Spin}^c(n)) \). Since \( \varphi \) induces isomorphisms on \( \pi_m \) for \( m \geq 2 \), it follows that \( (B\varphi)_* \) induces isomorphisms
\[
\pi_m((B\varphi)_*): \pi_m(\text{Map}_k^*(S^4, B\text{Spin}(n))) \xrightarrow{\cong} \pi_m(\text{Map}_k^*(S^4, B\text{Spin}^c(n)))
\]
for all \( m \) and is therefore a homotopy equivalence by Whitehead’s theorem.

We can extend the fibration diagram (2) to the left as
\[
\begin{array}{c}
\mathcal{G}_k(\text{Spin}(n)) \to \text{Spin}(n) \xrightarrow{\varphi^k} \text{Map}_k^*(S^4, B\text{Spin}(n)) \to \cdots \\
\downarrow{\mathcal{G}_k(\varphi)} \quad \downarrow{\varphi} \quad = (B\varphi)_* \\
\mathcal{G}_k(\text{Spin}^c(n)) \to \text{Spin}^c(n) \xrightarrow{\partial^c_k} \text{Map}_k^*(S^4, B\text{Spin}^c(n)) \to \cdots 
\end{array}
\]
where \( \partial^c_k \) denotes the boundary map associated to \( \text{Spin}(n) \)-gauge groups over \( S^4 \).

Since \( (B\varphi)_* \) is a homotopy equivalence, the leftmost square is a homotopy pullback. Since we know that there is a fibration
\[
\text{Spin}(n) \xrightarrow{\varphi} \text{Spin}^c(n) \xrightarrow{g} S^1,
\]
it follows that we also have a fibration
\[
\mathcal{G}_k(\text{Spin}(n)) \xrightarrow{\mathcal{G}_k(\varphi)} \mathcal{G}_k(\text{Spin}^c(n)) \xrightarrow{b} S^1.
\]
In particular, the space \( \mathcal{G}_k(\text{Spin}(n)) \) is seen to be the homotopy fibre \( F_b \) of the map \( b: \mathcal{G}_k(\text{Spin}^c(n)) \to S^1 \) and hence we have
\[
\mathcal{G}_k(\text{Spin}^c(n)) \cong S^1 \times \mathcal{G}_k(\text{Spin}(n)). \quad \Box
\]

In light of Theorem \ref{thm:homotopy-fibres}, the homotopy theory of \( \text{Spin}^c(n) \)-gauge groups over \( S^4 \) for \( n \geq 6 \) is completely determined by that of \( \text{Spin}(n) \)-gauge groups over \( S^4 \).
Remark 4.3. By a result of Cutler [10], there is a decomposition
\[ G_k(U(2)) \simeq S^1 \times G_k(SU(2)) \]
of U(2)-gauge groups over \( S^4 \) whenever \( k \) is even. Given that \( \text{Spin}^c(3) \cong U(2) \) and \( \text{Spin}(3) \cong SU(2) \), the statement of Theorem [4.1] still holds true when \( n = 2 \) provided that \( k \) is even. Cutler also shows that \( G_k(U(2)) \simeq S^1 \times G_k(\text{PU}(2)) \) for odd \( k \), so Theorem [4.1] does not hold for \( n = 2 \).

5. \( \text{Spin}(n) \)-gauge groups

We now shift our focus to principal \( \text{Spin}(n) \)-bundles over \( S^4 \) and the classification of their gauge groups. In the interest of completeness, we recall that, for \( n \leq 6 \), the following exceptional isomorphisms hold.

| \( n \) | \( \text{Spin}(n) \) |
|---|---|
| 1 | O(1) |
| 2 | U(1) |
| 3 | SU(2) |
| 4 | SU(2) \times SU(2) |
| 5 | Sp(2) |
| 6 | SU(4) |

Table 1. The exceptional isomorphisms.

The cases \( n = 1, 2 \) are trivial. Indeed, as \( \pi_3(O(1)) \cong \pi_3(U(1)) \equiv 0 \), there is only one isomorphism class of \( O(1) \)- and \( U(1) \)-bundles over \( S^4 \) (namely, that of the trivial bundle), and hence there is only one possible homotopy type for the corresponding gauge groups. The case \( n = 3 \) was studied by Kono in [30]. The case \( n = 4 \) can be reduced to the \( n = 3 \) case by [4, Theorem 5]. The case \( n = 5 \) was studied by Theriault in [46]. Finally, the case \( n = 6 \) was studied by Cutler and Theriault in [11].

We shall now explore the \( n = 7 \) case. Recall that we have a fibration sequence
\[ G_k(\text{Spin}(7)) \to \text{Spin}(7) \xrightarrow{k^h} \Omega^3_{0} \text{Spin}(7). \]

**Lemma 5.1.** Localised away from the prime 2, the boundary map
\[ \text{Spin}(7) \xrightarrow{h} \Omega^3_{0} \text{Spin}(7) \]
has order 21.

**Proof.** Harris [20] showed that \( \text{Spin}(2m + 1) \cong (p) \text{ Sp}(m) \) for odd primes \( p \). This result was later improved by Friedlander [14] to a \( p \)-local homotopy equivalence
of the corresponding classifying spaces. Then, in particular, localising at an odd prime \( p \), we have a commutative diagram

\[
\begin{array}{cccccc}
\text{Spin}(7) & \xrightarrow{\partial_1} & \Omega_0^3\text{Spin}(7) & \xrightarrow{=} & \text{Map}_1(S^4, \text{BSpin}(7)) & \xrightarrow{=} & \text{BSpin}(7) \\
\downarrow{=} & & \downarrow{=} & & \downarrow{=} & & \downarrow{=} \\
\text{Sp}(3) & \xrightarrow{\partial_1'} & \Omega_0^3\text{Sp}(3) & \xrightarrow{=} & \text{Map}_1(S^4, \text{BSp}(3)) & \xrightarrow{=} & \text{BSp}(3)
\end{array}
\]

where \( \partial_1' : \text{Sp}(3) \to \Omega_0^3\text{Sp}(3) \) denotes the boundary map associated to \( \text{Sp}(3) \)-gauge groups over \( S^4 \) studied in [9]. Hence the result follows from the calculation in [9 Theorem 1.2] where it is shown that \( \partial_1' \) has order 21 after localising away from the prime 2.

\[\square\]

**Lemma 5.2.** Let \( F \to X \to Y \) be a homotopy fibration, where \( F \) is an \( H \)-space, and let \( \partial : \Omega Y \to F \) be the homotopy fibration connecting map. Let \( \alpha : A \to \Omega Y \) and \( \beta : B \to \Omega Y \) be maps such that

1. \( \mu \circ (\alpha \times \beta) : A \times B \to \Omega Y \) is a homotopy equivalence, where \( \mu \) is the loop multiplication on \( \Omega Y \);
2. \( \partial \circ \beta : B \to \Omega Y \) is nullhomotopic.

Then the orders of \( \partial \) and \( \partial \circ \alpha \) coincide.

**Proof.** Let \( \theta : \Omega Y \times F \to F \) denote the canonical homotopy action of the loopspace \( \Omega Y \) onto the homotopy fibre \( F \), and let \( e = \mu \circ (\alpha \times \beta) \). Consider the diagram

\[
\begin{array}{ccccccc}
& & A \times B & & \\
& \downarrow{pr_1} & \downarrow{\alpha \times \beta} & \downarrow{e} & \\
A & \xrightarrow{\alpha} & \Omega Y \times \Omega Y & \xrightarrow{\mu} & \Omega Y & \\
\downarrow{\sigma} & & \downarrow{id \times \vartheta} & & \downarrow{\vartheta} & & \downarrow{\theta} & & \downarrow{F} \to \\
\Omega Y & \xleftarrow{\Omega Y \times F} & \Omega Y \times F & \xrightarrow{\theta} & F.
\end{array}
\]

The left portion of the diagram commutes by the assumption that \( \partial \circ \beta \simeq * \), while the right and bottom portions commute by properties of the canonical action \( \theta \). Therefore

\[\partial \simeq \partial \circ \alpha \circ pr_1 \circ e^{-1},\]

and hence the orders of \( \partial \) and \( \partial \circ \alpha \) coincide. \(\square\)

**Lemma 5.3.** Localised at the prime 2, the order of the boundary map

\[
\text{Spin}(7) \xrightarrow{\partial_1} \Omega_0^3\text{Spin}(7)
\]

is at most 8.
Proof. The strategy here will be to show that $\partial_8$ is nullhomotopic. This will suffice as we have $\partial_8 \simeq 8\partial_1$ by Lemma 5.2.

By a result of Mimura [35, Proposition 9.1], the fibration

$$G_2 \xrightarrow{\alpha} \text{Spin}(7) \longrightarrow S^7$$

splits at the prime 2. Let $\beta : S^7 \to \text{Spin}(7)$ denote a right homotopy inverse for $\text{Spin}(7) \to S^7$. Then the composite

$$G_2 \times S^7 \xrightarrow{\alpha \times \beta} \text{Spin}(7) \times \text{Spin}(7) \xrightarrow{\mu} \text{Spin}(7)$$

is a 2-local homotopy equivalence.

Observe that we have $\partial_8 \circ \beta \simeq *$ since $\pi_{10}(\text{Spin}(7)) \cong \mathbb{Z}/8\mathbb{Z}$ and $\partial_8 \circ \beta \simeq 8\partial_1 \circ \beta$. Therefore, by Lemma 5.2 the order of $\partial_8$ equals the order of $\partial_8 \circ \alpha$. As $\alpha$ is a group homomorphism, there is a diagram of evaluation fibrations

$$G_2 \xrightarrow{\alpha} \Omega^3 G_2 \xrightarrow{\partial_1} \Omega^3 \text{Spin}(7).$$

Since $\partial_8' \simeq 8\partial_1' \simeq *$ by [29, Theorem 1.1], we must have $\partial_8 \simeq *$. □

Proof of Theorem 1.2 (a). Lemmas 5.1 and 5.3 imply that $168\partial_1 \simeq *$, so the result follows from Lemma 5.3. □

We now move on to consider Spin(8)-gauge groups.

**Lemma 5.4.** Localised at the prime 2 (resp. 3), the order of the boundary map

$$\text{Spin}(8) \xrightarrow{\partial_1} \Omega^3 \text{Spin}(8)$$

is at most 8 (resp. 3).

_Proof._ There is a fibration

$$\text{Spin}(7) \longrightarrow \text{Spin}(8) \longrightarrow S^7$$

which splits after localisation at any prime. Therefore, we have a homotopy equivalence $\text{Spin}(8) \simeq \text{Spin}(7) \times S^7$ realised by maps $\alpha : \text{Spin}(7) \to \text{Spin}(8)$ and $\beta : S^7 \to \text{Spin}(8)$, where $\alpha$ is a group homomorphism. Integrally, we have $\pi_{10}(\text{Spin}(8)) \cong \mathbb{Z}/24\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$ (see, e.g. the table in [36]). Hence the same argument presented in the proof of Lemma 5.3 shows that $8\partial_1 \simeq *$ and $3\partial_1 \simeq *$ after localising at $p = 2$ and $p = 3$, respectively. □

**Lemma 5.5.** Let $p \neq 3$ be an odd prime. Then the $p$-primary orders of the maps $\partial_1 : \text{Spin}(7) \to \Omega^3 \text{Spin}(7)$ and $\partial_1 : \text{Spin}(8) \to \Omega^3 \text{Spin}(8)$ coincide.
Lemma 6.2 shown to be as follows. moto, Theriault and Tsutaya constructed a space in mod 2 cohomology in dimensions 1 through 6. The cohomology of is shown that, integrally, hence the result now follows from the calculations in [9, Theorem 1.1] where it follows from Lemma 5.3. □

6. Homotopy invariants of Spin(\(n\))-gauge groups

Lemma 6.1. If \(G_k(\text{Spin}(7)) \cong G_l(\text{Spin}(7))\), then \((21, k) = (21, l)\).

Proof. As in the proof of Lemma 5.1, localising at an odd prime, we have an equivalence BSpin(7) \(\cong_{(p)} BSp(3)\). We therefore have a diagram of homotopy fibrations

\[
\begin{array}{cccc}
\text{Spin}(7) & \xrightarrow{\partial_k} & \Omega^3\text{Spin}(7) & \rightarrow & B\mathcal{G}_k(\text{Spin}(7)) & \rightarrow & B\text{Spin}(7) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{Sp}(3) & \xrightarrow{\partial'_k} & \Omega^3\text{Sp}(3) & \rightarrow & B\mathcal{G}_k(\text{Sp}(3)) & \rightarrow & B\text{Sp}(3)
\end{array}
\]

where \(\partial'_k : \text{Sp}(3) \rightarrow \Omega^3\text{Sp}(3)\) denotes the boundary map studied in [9]. Thus, by the five lemma, we have

\[\pi_{11}(B\mathcal{G}_k(\text{Spin}(7))) \cong \pi_{11}(B\mathcal{G}_k(\text{Sp}(3))).\]

Hence the result now follows from the calculations in [9, Theorem 1.1] where it is shown that, integrally,

\[\pi_{11}(B\mathcal{G}_k(\text{Sp}(3))) \cong \mathbb{Z}/(84, k)\mathbb{Z}.\]

□

In their study of the homotopy types of G2-gauge groups over \(S^4\) in [29], Kishimoto, Theriault and Tsutaya constructed a space \(C_k\) for which

\[H^*(C_k) \cong H^*(\mathcal{G}_k(G_2))\]

in mod 2 cohomology in dimensions 1 through 6. The cohomology of \(C_k\) is then shown to be as follows.

Lemma 6.2 ([29, Lemma 8.3]). We have

- if \((4, k) = 1\) then \(C_k \cong S^3\), so \(H^*(C_k) \cong H^*(S^3)\);
- if \((4, k) = 2\) or \((4, k) = 4\) then \(H^*(C_k) \cong H^*(S^3) \oplus H^*(P^5(2)) \oplus H^*(P^6(2))\), where \(P^n(p)\) denotes the nth dimensional mod \(p\) Moore space;
- if \((4, k) = 2\) then \(\text{Sq}^2\) is non-trivial on the degree 4 generator in \(H^*(C_k)\);
- if \((4, k) = 4\) then \(\text{Sq}^2\) is trivial on the degree 4 generator in \(H^*(C_k)\). □

We make use of the same spaces \(C_k\) as follows.

Lemma 6.3. If \(G_k(\text{Spin}(7)) \cong G_l(\text{Spin}(7))\), then we have \((4, k) = (4, l)\).
Proof. As in the proof of Lemma 5.3, recall that we have a 2-local homotopy equivalence

\[ G_2 \times S^7 \xrightarrow{\alpha \times \beta} \text{Spin}(7) \times \text{Spin}(7) \xrightarrow{\mu} \text{Spin}(7). \]

Since the map \( \alpha : G_2 \to \text{Spin}(7) \) is a homomorphism, we have a commutative diagram

\[
\begin{array}{ccc}
G_2 & \xrightarrow{\sigma'_1} & \Omega^3 G_2 \\
\downarrow{\alpha} & & \downarrow{\Omega^3 \alpha} \\
\text{Spin}(7) & \xrightarrow{\delta_1} & \Omega^3 \text{Spin}(7).
\end{array}
\]

Furthermore, as \( \pi_7(\Omega^3 G_2) \cong \pi_{10}(G_2) \cong 0 \), we have

\[ \pi_7(\Omega^3 \text{Spin}(7)) \cong \pi_7(\Omega^3 G_2) \oplus \pi_7(\Omega^3 S^7) \cong \pi_7(\Omega^3 S^7), \]

and thus there is a commutative diagram

\[
\begin{array}{ccc}
S^7 & \xrightarrow{\gamma} & \Omega^3 S^7 \\
\downarrow{\beta} & & \downarrow{\Omega^3 \beta} \\
\text{Spin}(7) & \xrightarrow{\delta_1} & \Omega^3 \text{Spin}(7)
\end{array}
\]

for some \( \gamma \) representing a class in \( \pi_7(\Omega^3 S^7) \cong \pi_{10}(S^7) \cong \mathbb{Z}/8\mathbb{Z} \).

We therefore have a commutative diagram

\[
\begin{array}{ccc}
G_2 \vee S^7 & \xrightarrow{k \sigma'_1 \vee k \gamma} & \Omega^3 G_2 \times \Omega^3 S^7 \\
\downarrow{\alpha \vee \beta} & & \downarrow{\Omega^3 \alpha \times \Omega^3 \beta} \\
\text{Spin}(7) & \xrightarrow{k \delta_1} & \Omega^3 \text{Spin}(7)
\end{array}
\]

which induces a map of fibres \( \phi : M \to \mathcal{G}_k(\text{Spin}(7)) \), where \( M \) denotes the homotopy fibre of the map \( k \sigma'_1 \vee k \gamma \).

Since the lowest dimensional cell in \( G_2 \times S^7 / (G_2 \vee S^7) \) appears in dimension 10, the canonical map \( G_2 \vee S^7 \to G_2 \times S^7 \) is a homotopy equivalence in dimensions less than 9. It thus follows that \( M \) is homotopy equivalent to the homotopy fibre of \( k \sigma'_1 \times k \gamma \) in dimensions up to 8. Since the homotopy fibre of \( k \sigma'_1 \times k \gamma \) is just the product \( \mathcal{G}_k(G_2) \times F_k \), the composite

\[ C_k \times F_k \to \mathcal{G}_k(G_2) \times F_k \to M \xrightarrow{\phi} \mathcal{G}_k(\text{Spin}(7)) \]

induces an isomorphism in mod-2 cohomology in dimensions 1 through 6, and therefore we have

\[ H^*(\mathcal{G}_k(\text{Spin}(7))) \cong H^*(C_k) \otimes H^*(F_k), \quad * \leq 6. \]

From the fibration sequence

\[ \Omega^4 S^7 \longrightarrow F_k \longrightarrow S^7 \]
we see that \( H^* (F_k) \cong H^* (\Omega^4 S^7) \) in dimensions 1 through 6 for dimensional reasons, and hence we have
\[
H^* (F_k) \cong \mathbb{Z} / 2 \mathbb{Z} [y_i] , \quad * \leq 6,
\]
where \(|y_i| = i\), which, in turn, yields
\[
H^* (G_k (\text{Spin}(7))) \cong H^* (C_k) \otimes \mathbb{Z} / 2 \mathbb{Z} [y_3, y_6], \quad * \leq 6.
\]
Since \( H^* (F_k) \) does not contribute any generators in degree 4 to \( H^* (G_k (\text{Spin}(7))) \), the result now follows from Lemma 6.2. Indeed, the presence of a degree 4 generator allows us to distinguish between the \((4, k) = 1 \) case and the \( 2|k \) cases, whereas the vanishing of the Steenrod square \( Sq^2 \) on the degree 4 generator in \( H^* (G_k (\text{Spin}(7))) \) coming from \( H^* (C_k) \) can be used to distinguish between the \((4, k) = 2 \) and \((4, k) = 4 \) cases.
\[ \square \]

**Proof of Theorem 1.2 (b).** Combine Lemmas 6.1 and 6.3
\[ \square \]

**Lemma 6.4.** If \( G_k (\text{Spin}(8)) \cong G_l (\text{Spin}(8)) \), then \((4, k) = (4, l)\).

**Proof.** As in the proof of Lemma 5.3 the splitting of \( G_2 \to \text{Spin}(7) \to S^7 \) at the prime 2 implies that there is a 2-local homotopy equivalence
\[
\mu \circ (\alpha \times \beta) : G_2 \times S^7 \longrightarrow \text{Spin}(7).
\]
Since the fibration \( \text{Spin}(7) \to \text{Spin}(8) \to S^7 \) also splits after localising at any prime, there is a decomposition
\[
\mu \circ ((i \circ \alpha) \times (i \circ \beta) \times y) : G_2 \times S^7 \times S^7 \longrightarrow \text{Spin}(8),
\]
where \( i : \text{Spin}(7) \to \text{Spin}(8) \) is the inclusion homomorphism and \( y \) is a homotopy inverse for the map \( \text{Spin}(8) \to S^7 \).

Since the map \( i \circ \alpha \) is a homomorphism, we have a commutative diagram
\[
\begin{array}{ccc}
G_2 & \xrightarrow{\delta_i} & \Omega^3 G_2 \\
\downarrow_{i \circ \alpha} & & \downarrow_{\Omega^3(i \circ \alpha)} \\
\text{Spin}(8) & \xrightarrow{\partial_i} & \Omega^3 \text{Spin}(8).
\end{array}
\]
Furthermore, as \( \pi_7 (\Omega^3 G_2) \cong \pi_{10} (G_2) \cong 0 \), we have
\[
\pi_7 (\Omega^3 \text{Spin}(8)) \cong \pi_7 (\Omega^3 S^7) \oplus \pi_7 (\Omega^3 S^7),
\]
and thus there are commutative diagrams
\[
\begin{array}{ccc}
S^7 & \xrightarrow{\delta} & \Omega^3 S^7 \times \Omega^3 S^7 \\
\downarrow_{i \circ \beta} & & \downarrow_{\Omega^3(i \circ \beta) \times \Omega^3 \gamma} \\
\text{Spin}(8) & \xrightarrow{\partial_i} & \Omega^3 \text{Spin}(8)
\end{array} \quad \begin{array}{ccc}
S^7 & \xrightarrow{\delta^*} & \Omega^3 S^7 \times \Omega^3 S^7 \\
\downarrow_{\gamma} & & \downarrow_{\Omega^3(i \circ \beta) \times \Omega^3 \gamma} \\
\text{Spin}(8) & \xrightarrow{\partial_i} & \Omega^3 \text{Spin}(8)
\end{array}
\]
for some $\delta, \delta'$ representing classes in $\pi_7(\Omega^3S^7 \times \Omega^3S^7) \cong (\mathbb{Z}/8\mathbb{Z})^2$. We therefore have a commutative diagram

$$
\begin{array}{ccc}
G_2 \vee (S^7 \vee S^7) & \xrightarrow{k\delta' \vee k(\delta \vee \delta')} & \Omega^3_2G_2 \times (\Omega^3S^7 \times \Omega^3S^7) \\
\downarrow^{a\vee (1 \beta \vee \gamma)} & & \downarrow^{\Omega^3\alpha \times (\Omega^3\beta \times \Omega^3\gamma)} \\
\text{Spin}(8) & \xrightarrow{k \partial_1} & \Omega^3_2\text{Spin}(8).
\end{array}
$$

Arguing as in the proof of Lemma 6.3 we conclude that

$$H^*(G_k(\text{Spin}(7))) \cong H^*(G_k(\text{Spin}(8))), \quad * \leq 6,$$

hence the statement follows from Lemma 6.2.

**Lemma 6.5.** If $G_k(\text{Spin}(8)) \simeq G_l(\text{Spin}(8))$, then $(7, k) = (7, l)$.

**Proof.** Localising at $p = 7$, we have

$$\text{Spin}(8) \simeq \text{Spin}(7) \times S^7 \simeq G_2 \times S^7 \times S^7.$$

Applying the functor $\pi_{11}$ and noting that

$$\pi_{10}(S^7) \cong \pi_{11}(S^7) \cong \pi_{14}(S^7) \cong 0,$$

(see, e.g. [54]) we find that the evaluation fibration

$$\text{Spin}(8) \xrightarrow{d_k} \Omega^3_2\text{Spin}(8) \to B\text{G}_k(\text{Spin}(8)) \to B\text{Spin}(8)$$

reduces to the exact sequence

$$\pi_{11}(G_2) \to \pi_{11}(\Omega^3 G_2) \to \pi_{11}(B\text{G}_k(\text{Spin}(8))) \to 0.$$

Hence the result follows from [29].

**Lemma 6.6.** If $G_k(\text{Spin}(8)) \simeq G_l(\text{Spin}(8))$ and $k$ and $l$ are multiples of 3, then $(3, k) = (3, l)$.

**Proof.** By [47] or [28], when $k$ is a multiple of 3, there is a 3-local homotopy equivalence

$$G_k(\text{Spin}(8)) \simeq S^7 \times \Omega^4S^7 \times G_k(\text{Spin}(7)).$$

Recalling the argument in the proof of Lemma 6.1 we have

$$\pi_{10}(G_k(\text{Spin}(8))) \cong \pi_{10}(S^7) \oplus \pi_{10}(\Omega^4S^7) \oplus \pi_{10}(G_k(\text{Spin}(7)))$$

$$\cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \pi_{10}(G_k(\text{Sp}(3)))$$

$$\cong (\mathbb{Z}/3\mathbb{Z})^2 \oplus \mathbb{Z}/(3, k)\mathbb{Z}.$$  

Hence, if $G_k(\text{Spin}(8)) \simeq G_l(\text{Spin}(8))$ then $\mathbb{Z}/(3, k)\mathbb{Z} \cong \mathbb{Z}/(3, l)\mathbb{Z}$ and thus it must be that $(3, k) = (3, l)$.

**Proof of Theorem 1.3 (b).** Combine Lemmas 6.4, 6.5 and 6.6.
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