SELF-ADJOINT CYCLICALLY COMPACT OPERATORS AND THEIR APPLICATIONS

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ABSTRACT. This paper is devoted to self-adjoint cyclically compact operators on Hilbert–Kaplansky modulus over a ring of bounded measurable functions. The spectral theorem for such a class of operators are given. We apply this result to partial integral equations on the space with mixed norm of measurable functions and to compact operators relative to von Neumann algebras. We will give a condition of solvability of partial integral equations with self-adjoint kernel. Moreover, a general form of compact operators relative to a type I von Neumann algebra is given.

Keywords: compact operator; cyclically compact operator; partial integral equation; von Neumann algebra.

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1. INTRODUCTION

The modern structure theory of $AW^*$-modules originated with the articles by I. Kaplansky \cite{9,10} and nowadays this theory has many applications in the operator algebras.

One of the important instrument in the theory of operator algebras is a different form spectral theorem. In \cite{15} it was proved an important spectral theorem for Hilbert–Kaplansky modules. Another important concept is the compactness property. Cyclically compact sets and operators in lattice-normed spaces were introduced by Kusraev (see \cite{12}). In \cite{12} (see also \cite{13}) a general form of cyclically compact operators in Hilbert–Kaplansky module, as well as a variant of Fredholm alternative for cyclically compact operators, are given. In \cite{7} it was proved that every cyclically compact operator acting in Banach–Kantorovich space over a ring measurable functions can be represented as a measurable bundle of compact operators acting in Banach spaces. In \cite{11} it has been shown that the algebra of all locally measurable operators with respect to a type I von Neumann algebra can be represented as an algebra of all bounded module-linear operators acting on a Hilbert–Kaplanskiy module over the ring of measurable function on a measure space. This result played a crucial role in the description of derivations on algebras of locally measurable operators with respect to type I von Neumann algebras and their subalgebras (see for example, \cite{11,13,15}).

It is well-known that one of the important notions in the theory of operator algebras is compact operators relative to von Neumann algebras with faithful semi-finite trace introduced by M. G. Sonis 1971 (see \cite{14}). Later V. Kaftal \cite{8} has shown that Sonis’s definition of compact operator relative to von Neumann algebras with faithful semi-finite trace is viable for general von Neumann algebras too and he obtained most of the classical characterizations of compact operators.
In this paper we shall investigate self-adjoint cyclically compact operators on Hilbert–Kaplansky module over a ring of bounded measurable functions and their applications.

In Section 2 we give preliminaries from the theory of Hilbert–Kaplansky module and give a general form of self-adjoint cyclically compact operators on Hilbert–Kaplansky module over a ring of measurable functions.

In section 3 we apply the above theorem to partial integral equations on the space with mixed norm of measurable functions. We will give a condition of solvability of partial integral equations with self-adjoint kernel.

In section 4 we will give a general form of a self-adjoint compact operator relative to a type I von Neumann algebra.

2. Self-Adjoint Cyclically Compact Operators on Hilbert–Kaplansky Modules

Let us recall some notions and results from the theory of Hilbert–Kaplansky modules (see [13]).

Let \((\Omega, \Sigma, \mu)\) be a measurable space and suppose that the measure \(\mu\) has the direct sum property, i.e. there is a family \(\{\Omega_i\}_{i \in J} \subset \Sigma, 0 < \mu(\Omega_i) < +\infty, i \in J\) such that for any \(A \in \Sigma, \mu(A) < +\infty\) there exist a countable subset \(J_0 \subset J\) and a set \(B\) with zero measure such that \(A = \bigcup_{i \in J_0} (A \cap \Omega_i) \cup B\).

We denote by \(L^\infty(\Omega)\) the algebra of all (equivalence classes of) complex measurable bounded functions on \(\Omega\) and let \(\nabla\) be the set of all idempotents of the algebra \(L^\infty(\Omega)\).

Let \(X\) be a unitary \(L^\infty(\Omega)\)-module. The mapping \(\langle \cdot, \cdot \rangle: X \times X \to L^\infty(\Omega)\) is a \(L^\infty(\Omega)\)-valued inner product, if for all \(\xi, \eta, \zeta \in X\) and \(a \in L^\infty(\Omega)\) the following are satisfied:

1. \(\langle \xi, \xi \rangle \geq 0, \langle \xi, \xi \rangle = 0 \iff \xi = 0\);
2. \(\langle \xi, \eta \rangle = \langle \eta, \xi \rangle\);
3. \(\langle a\xi, \eta \rangle = a\langle \xi, \eta \rangle\);
4. \(\langle \xi + \eta, \zeta \rangle = \langle \xi, \zeta \rangle + \langle \eta, \zeta \rangle\).

Using a \(L^\infty(\Omega)\)-valued inner product, we may introduce the norm in \(X\) by the formula

\[ \|\xi\|_\infty = \sqrt{\|\langle \xi, \xi \rangle\|_{L^\infty(\Omega)}}, \]

and the vector norm

\[ \|\xi\| = \sqrt{\langle \xi, \xi \rangle}. \]

A Hilbert–Kaplansky module over \(L^\infty(\Omega)\) is a unitary module over \(L^\infty(\Omega)\) such that it is complete with respect the norm \(\|\cdot\|_\infty\) and the following two properties are true:

1. let \(\xi\) be an arbitrary element in \(X\), and let \(\{\pi_i\}_{i \in I}\) be a partition of unity in \(\nabla\) with \(\pi_i\xi = 0\) for all \(i \in I\), then \(\xi = 0\);
2. let \(\\{\xi_i\}_{i \in I}\) be a norm-bounded family in \(X\), and let \(\{\pi_i\}_{i \in I}\) be a partition of unity in \(\nabla\), then there exists an element \(\xi \in X\) such that \(\pi_i\xi = \pi_i\xi_i\) for all \(i \in I\).

An orthogonal basis in a Hilbert–Kaplansky module \(X\) over \(L^\infty(\Omega)\) is a orthogonal set whose orthogonal complement is \(\{0\}\). A Hilbert–Kaplansky module \(X\) is said to be \(\alpha\)-homogeneous, if \(\alpha\) is a cardinal and \(X\) has a basis of cardinality \(\alpha\).

A Hilbert–Kaplansky module \(X\) is said to be \(\sigma\)-finite-generated if there exists a partition of unity \(\{\pi_n\}_{n \in F}\) in \(\nabla\), where \(F \subseteq \mathbb{N}\), such that \(\pi_nX\) is a \(n\)-homogeneous module over \(\pi_nL^\infty(\Omega)\) for all \(n \in F\).
Let $C$ be a subset in $X$. Denote by $\text{mix}(C)$ the set of all vectors $\xi$ from $X$ for which there is a partition of unity $\{\pi_i\}_{i \in I}$ in $\nabla$ such that $\pi_i\xi \in C$ for all $i \in I$, i.e.

$$\text{mix}(C) = \left\{ \xi \in X : \exists \pi_i \in \nabla, \pi_i\pi_j = 0, i \neq j, \bigvee_{i \in I} \pi_i = 1, \pi_i\xi \in C, i \in I \right\}.$$ 

In other words $\text{mix}(C)$ is the set of all mixings obtained by families $\{\xi_i\}_{i \in I}$ taken from $C$.

A subset $C$ is said to be cyclic if $C = \text{mix}(C)$.

For a nonempty set $A$ by $\nabla(A)$ denotes the set of all partitions of unity in $\nabla$ with the index set $A$, i.e.,

$$\nabla(A) = \left\{ \nu : A \to \nabla : (\forall \alpha, \beta \in A)(\alpha \neq \beta \to \nu(\alpha) \land \nu(\beta) = 0) \land \bigvee_{\alpha \in A} \nu(\alpha) = 1 \right\}.$$ 

If $A$ is a partially ordered set then we can order the set $\nabla(A)$ as:

$$\nu \leq \mu \iff (\forall \alpha, \beta \in A)(\nu(\alpha) \land \nu(\beta) \neq 0 \to \alpha \leq \beta) \ (\nu, \mu \in \nabla(A)).$$

Then this relation is a partial order in $\nabla(A)$, in particular, if $A$ is directed upward or downward, then so does $\nabla(A)$.

Take any net $(\xi_\alpha)_{\alpha \in A}$ in $X$. For each $\nu \in \nabla(A)$ put $\xi_\nu = \text{mix}_{\alpha \in A}\nu(\alpha)\xi_\alpha$. If all the mixings exist then we have a net $(\xi_\nu)_{\nu \in \nabla(A)}$ in $X$. Every subnet of the net $(\xi_\nu)_{\nu \in \nabla(A)}$ is called a cyclical subnet of the original net $(\xi_\alpha)_{\alpha \in A}$.

Recall [13] that a subset $C \subset X$ is said to be cyclically compact if $C$ is cyclically and any sequence in $C$ has a cyclic subsequence that norm converges to some element of $C$. A subset in $X$ is called relatively cyclically compact if it is contained in a cyclically compact set.

An operator $T$ on $X$ is called $L^\infty(\Omega)$-linear if $T(a\xi + b\eta) = aT(\xi) + bT(\eta)$ for all $a, b \in L^\infty(\Omega)$, $\xi, \eta \in X$.

A $L^\infty(\Omega)$-linear operator $T$ on $X$ is called cyclically compact if the image $T(C)$ of any bounded subset $C \subset X$ is relatively cyclically compact.

For every $\xi, \eta \in X$ we define an operator $\xi \otimes \eta$ on $X$ by the rule

$$(\xi \otimes \eta)(\zeta) = \langle \zeta, \eta \rangle \xi, \zeta \in X.$$ 

It is well-known [13, Theorem 8.5.6] that if $T$ is a cyclically compact operator on $X$ then there exists a partition of unity $\{\pi_0, \pi_1, \ldots, \pi_k, \ldots, \pi_\infty\}$ in $\nabla$ and orthonormal system $\{\xi_{k,n}\}_{n=1}^k$, $\{\eta_{k,n}\}_{n=1}^k$ in $\pi_kX$ and a families $\{f_{k,n}\}_{n=1}^k$ in $\pi_k L^\infty(\Omega)$, where $k = 1, \ldots, n, \ldots, \infty$, such that the followings are true:

1. $\pi_0 T = 0$;
2. $\pi_\infty \sum_{n=1}^\infty f_{\infty,n} \xi_{\infty,n} \otimes \eta_{\infty,n} + \sum_{k=1}^\infty \pi_k \sum_{n=1}^k f_{k,n} \xi_{k,n} \otimes \eta_{k,n}.$
3. the representation is valid

$$T = \pi_\infty \sum_{n=1}^\infty f_{\infty,n} \xi_{\infty,n} \otimes \eta_{\infty,n} + \sum_{k=1}^\infty \pi_k \sum_{n=1}^k f_{k,n} \xi_{k,n} \otimes \eta_{k,n}.$$ 

The following is the main result of this section.

**Theorem 2.1.** If $T$ is a self-adjoint cyclically compact operator on $X$ then there are partition of unity $\{\pi_0, \pi_1, \ldots, \pi_k, \ldots, \pi_\infty\}$ in $\nabla$ and orthonormal families $\{\xi_{k,n}\}_{n=1}^k$ in $\pi_k X$ and a families $\{f_{k,n}\}_{n=1}^k$ in $\pi_k L^\infty(\Omega)$, where $k = 1, \ldots, n, \ldots, \infty$, such that the following hold:
(1) $\pi_0 T = 0$;
(2) $\pi_\infty |f_{\infty,n}| \downarrow 0$;
(3) the representation is valid

\begin{equation}
T = \pi^\infty \sum_{n=1}^\infty f_{\infty,n} \xi_{\infty,n} \otimes \xi_{\infty,n} + \sum_{k=1}^\infty \pi_k \sum_{n=1}^k f_{k,n} \xi_{k,n} \otimes \xi_{k,n}.
\end{equation}

Proof. Let $T$ be an operator of the form (2.1). Without lost of generality we can assume that $\pi_\infty = 1$, and therefore the operator $T$ has the form $T = T^* = \sum f_n \eta_n \otimes \xi_n$.

Case 1. Let $T \geq 0$. Since $T = |T| = \sqrt{T^* T} = \sqrt{T T^*}$ and

$$TT^* = \sum_{n=1}^\infty f_n \xi_n \otimes \eta_n \sum_{m=1}^\infty f_m \eta_m \otimes \xi_m = \sum_{n,m=1}^\infty f_n f_m \langle \eta_m, \eta_n \rangle \xi_m \otimes \xi_n = \sum_{n=1}^\infty f_n^2 \xi_n \otimes \xi_n$$

we get

$$T = \sum_{n=1}^\infty f_n \xi_n \otimes \xi_n.$$

Case 2. Let $T$ be an arbitrary self-adjoint cyclically compact operator. Consider its representation in the form $T = T_+ - T_-$, where $T_+, T_- \geq 0$ and $T_+ T_- = 0$. By case 1 there are orthonormal families $\{\xi^+_n\}_{n \in \mathbb{N}}, \{\xi^-_n\}_{n \in \mathbb{N}}$ in $X$ and families $\{f^+_n\}_{n \in \mathbb{N}}, \{f^-_n\}_{n \in \mathbb{N}}$ in $L^\infty(\Omega)$ such that $f^+_n \downarrow 0, f^-_n \downarrow 0$ and the following representations are valid

$$T_+ = \sum_{n=1}^\infty f^+_n \xi^+_n \otimes \xi^+_n,$$

$$T_- = \sum_{n=1}^\infty f^-_n \xi^-_n \otimes \xi^-_n.$$

Since $T_+ T_- = 0$ we get that the elements $\xi^+_n$ and $\xi^-_n$ are mutually orthogonal for all $i, j \in \mathbb{N}$.

Now consider the following sequence in $\nabla$:

$$z_n = c((f^-_1 - f^+_1)_+) \downarrow 1,$$

where $f_+$ is the positive part of self-adjoint element $f$ from $L^\infty(\Omega)$ and $c(f)$ it’s support, i.e. $c(f)$ is the indicator function of the set $\{\omega \in \Omega : f(\omega) \neq 0\}$. Set

$$f^{(1)}_1 = -z_1 f^-_1 + z^+_1 f^+_1,$$

$$f^{(1)}_n = (z_n - z_{n-1})f^-_1 + z^+_n f^+_n, n \geq 2,$$

$$\xi^{(1)}_1 = z_1 \xi^-_1 + z^+_1 \xi^+_1,$$

$$\xi^{(1)}_n = (z_n - z_{n-1})\xi^-_1 + z^+_n \xi^+_n, n \geq 2.$$

Let us check that $|f^{(1)}_n| \downarrow 0$. Firstly we shall show that $|f^{(1)}_1| \geq |f^{(1)}_2|$. Indeed,

$$|z_1 f^{(1)}_1| = z_1 f^-_1 \geq z_1 f^+_1 = |z_1 f^{(1)}_2|.$$
and 
\[ z_n^+ f_1^{(1)} = z_1^+ f_1^+ = (z_2 - z_1) f_1^+ \geq (z_2 - z_1) f_1^- + z_2^+ f_2^- = z_1^+ f_2^{(1)}. \]

Now let \( n > 1 \). Then
\[ z_{n-1} f_n^{(1)} = z_{n-1}^+ f_n^+ \geq z_{n-1} f_n^+ = z_{n-1} f_n^+ = z_{n-1} f_{n+1}^{(1)}, \]
\[ (z_n - z_{n-1}) f_n^{(1)} = (z_n - z_{n-1}) f_1^- \geq (z_n - z_{n-1}) f_n^+ = (z_n - z_{n-1}) f_{n+1}^{(1)} \]
and
\[ z_n^+ f_n^{(1)} = z_n^+ f_n^+ = (z_n + 1) f_n^+ + z_1^+ f_1^- \geq (z_n + 1) f_1^- + z_1^+ f_1^+ + z_1^+ f_{n+1}^{(1)} = z_n^+ f_{n+1}^{(1)}. \]

So \( |f_n^{(1)}| \geq |f_{n+1}^{(1)}| \). Since \( f_n^+ \downarrow 0 \) and \( z_n^+ \downarrow 0 \) we get \( |f_n^{(1)}| \downarrow 0 \).

Direct computations show that
\[ \langle \xi_1^{(1)}, \xi_1^{(1)} \rangle = z_1 + z_1^+ = 1, \]
\[ \langle \xi_n^{(1)}, \xi_n^{(1)} \rangle = z_n + (z_n - z_{n-1}) + z_n^+ = 1, \quad n > 1, \]
\[ \langle \xi_n^{(1)}, \xi_m^{(1)} \rangle = 0, \quad n \neq m. \]

This means that \( \{ \xi_n^{(1)} \} \) is an orthonormal system.

Set
\[ T_1 = \sum_{n=1}^{\infty} f_n^{(1)} \xi_n^{(1)} \otimes \xi_n^{(1)} - \sum_{n=2}^{\infty} f_n^- \xi_n^- \otimes \xi_n^- . \]

Let us show that \( T = T_1 \). It clear that \( T(\xi_k^-) = T_1(\xi_k^-) \) for all \( k > 1 \). Further
\[ T_1(\xi_1^-) = z_1 T(\xi_1^-) + \sum_{n=2}^{\infty} (z_n - z_{n-1}) T(\xi_1^-) = T(\xi_1^-), \]
\[ T_1(\xi_k^+) = z_k T(\xi_k^+) + z_k^+ T(\xi_k^+) = T(\xi_k^+) . \]

This means that \( T = T_1 \).

Continuing by this way we obtain
\[ T = \sum_{n=1}^{\infty} f_n \xi_n \otimes \xi_n . \]

The proof is complete. \( \square \)

3. Applications to Partial Integral Equations

In this section we shall apply the main result of the previous section to partial integral equations on the space with mixed norm of measurable functions. For more information about the partial integral operators and equations can be found in the monograph [2].

Let \( (S, \Xi, \nu) \) be a measure space and let \( L^{2,\infty}(S \times \Omega) \) be the set of all complex-valued measurable functions \( f \) on \( S \times \Omega \) such that
\[ \int_S |f(s, \omega)|^2 \, d\mu(s) \in L^\infty(\Omega). \]
Then $L^{2,\infty}(S \times \Omega)$ is a Hilbert–Kaplansky module over $L^\infty(\Omega)$ with respect to inner product defined by

$$\langle f, g \rangle = \int_S f(s, \omega) \overline{g(s, \omega)} \, d\mu(s), \quad f, g \in L^{2,\infty}(S \times \Omega).$$

Let us take a complex-valued measurable function $k(t, s, \omega)$ on $S^2 \times \Omega$ such that

$$\int_S \int_S |k(t, s, \omega)|^2 \, d\mu(t) \, d\mu(s) \in L^\infty(\Omega)$$

and define an operator $T : L^{2,\infty}(S \times \Omega) \to L^{2,\infty}(S \times \Omega)$ by the rule

$$T(f)(t, \omega) = \int_S k(t, s, \omega) f(s, \omega) \, d\mu(s), \quad f \in L^{2,\infty}(S \times \Omega).$$

For any $\omega \in \Omega$ we put $k_\omega(t, s) = k(t, s, \omega)$.

Theorem 3.1. There are partition $\{\Omega_0, \Omega_1, \ldots, \Omega_k, \ldots, \Omega_\infty\}$ of $\Omega$ and orthonormal families $\{g_{k,n}\}_{n=1}^k$ in $L^{2,\infty}(S \times \Omega)$ and families $\{\lambda_{k,n}\}_{n=1}^k$ in $L^\infty(\Omega)$, where $k = 1, \ldots, n, \ldots, \infty$, such that the following hold:

1. $\chi_{\Omega_0}T = 0$;
2. $\chi_{\Omega_\infty} |\lambda_{\infty,n}| \downarrow 0$;
3. the representation is valid

$$T(f) = \chi_{\Omega_\infty} \sum_{n=1}^\infty \lambda_{\infty,n} \langle f, g_{\infty,n} \rangle g_{\infty,n} + \sum_{k=1}^\infty \chi_{\Omega_k} \sum_{n=1}^k \lambda_{k,n} \langle f, g_{k,n} \rangle g_{k,n}.$$

Let us consider the following partial integral equation

$$\int_S k(t, s, \omega) f(s, \omega) \, d\mu(s) = \lambda(\omega) f(t, \omega),$$

where $\lambda(\omega) \in L^\infty(\Omega)$, $f \in L^{2,\infty}(S \times \Omega)$. 
Theorem 3.2 implies the following condition of solvability of partial integral equation with self-adjoint kernel.

**Corollary 3.2.** Suppose that \( k(t, s, \omega) = k(s, t, \omega) \). Then partial integral equation (3.3) is solvable if and only if there exists a non zero \( \pi \in \nabla \) and \( \lambda_{k,n} \) such that \( \pi \lambda = \pi \lambda_{k,n} \).

**Remark 3.3.** Suppose that \((\Omega, \Sigma, \mu)\) is an atomless measure space. Then the operator \( T \) defined by (3.2) is compact if and only if \( k(t, s, \omega) \equiv 0 \). Indeed, suppose that \( k(t, s, \omega) \neq 0 \). Since \( T \) is a compact operator, it follows that there exists an eigenvalue \( \lambda \) of \( T \). Note that the subspace of eigenvectors corresponding to \( \lambda \) is finite dimensional.

On the other hand, since the operator \( T \) is \( L^\infty(\Omega) \)-linear, we conclude that the subspace of eigenvectors corresponding to \( \lambda \) is a non trivial \( L^\infty(\Omega) \)-module, in particular, it is an infinite dimensional, because \( L^\infty(\Omega) \) is an infinite dimensional. This contradiction implies that \( k(t, s, \omega) \equiv 0 \).

4. Self-adjoint compact operators in type I von Neumann algebras

In this section we shall apply the main result of the section 2 to compact operators relative von Neumann algebras of type I. For more information about the compact operators relative to von Neumann algebras can be found in [8].

An operator \( x \in M \) is compact relative to \( M \), if it is the limit in the norm of finite operators in \( M \), i.e. of operators for which the left support

\[
l(y) = \inf \{ p \in P(M) : py = y \}
\]

is finite.

**Theorem 4.1.** Let \( M \) be a type I von Neumann algebra and let \( x \) be a compact operator relative to \( M \). If \( x \) is self-adjoint then there are a sequence of mutually orthogonal central projections \( \{ z_0, z_1, \ldots, z_k, \ldots, z_\infty \} \) in \( M \) and the families of mutually orthogonal abelian projections \( \{ p_{k,n} \}_{n=1}^\infty \) in \( z_k M \) and a families of central elements \( \{ f_{k,n} \}_{k,n=1}^\infty \) in \( z_k M \), where \( k = 1, \ldots, n, \ldots, \infty \), such that the following hold:

1. \( z_0 x = 0 \);
2. \( z_\infty |f_{\infty,n}| \downarrow 0 \);
3. the representation is valid

\[
x = z_\infty \sum_{n=1}^\infty f_{\infty,n} p_{\infty,n} + \sum_{k=1}^\infty z_k \sum_{n=1}^k f_{k,n} p_{k,n}.
\]

For the proof of this theorem we need several lemmata.

Consider a Hilbert space \( H \). A mapping \( s : \Omega \to H \) is called simple, if \( s(\omega) = \sum_{k=1}^n \chi_{A_k}(\omega)c_k \), where \( A_k \in \Sigma, A_i \cap A_j = \emptyset, i \neq j, c_k \in H, k = 1, \ldots, n, n \in \mathbb{N} \). A mapping \( u : \Omega \to H \) is said to be measurable, if for each \( A \in \Sigma \) with \( \mu(A) < \infty \) there is a sequence \( \{ s_n \} \) of simple maps such that \( \| s_n(\omega) - u(\omega) \| \to 0 \) almost everywhere on \( A \).

Denote by \( B(\Omega, H) \) the set of all bounded measurable mappings from \( \Omega \) into \( H \), and let \( L^\infty(\Omega, H) \) denote the space of all equivalence classes with respect to the equality almost everywhere. The equivalence class from \( L^\infty(\Omega, H) \) which contains the measurable map \( \xi \in B(\Omega, H) \) denotes as \( \hat{\xi} \). We shall identify
the element $\xi \in B(\Omega, H)$ and the class $\hat{\xi}$. It is clear that the function $\omega \to \|\xi(\omega)\|$ is measurable for all $\xi \in B(\Omega, H)$. Denote by $\|\hat{\xi}\|$ the equivalence class containing the function $\|\xi(\omega)\|$. The algebraic operations on $L^\infty(\Omega, H)$ defined by usual way: $\hat{\xi} + \hat{\eta} = \hat{\xi + \eta}, a\hat{\xi} = \hat{a\xi}$ for all $\hat{\xi}, \hat{\eta} \in L^\infty(\Omega, H), a \in L^\infty(\Omega)$. Let us consider $L^\infty(\Omega)$-valued inner product

$$\langle \xi, \eta \rangle = \langle \xi(\omega), \eta(\omega) \rangle_H,$$

where $\langle \cdot, \cdot \rangle_H$ is the inner product in $H$. Then $L^\infty(\Omega, H)$ is a Hilbert–Kaplansky module over $L^\infty(\Omega)$.

It is known [1] that $\alpha$-dimensional Hilbert space $H$ the Hilbert–Kaplansky module $L^\infty(\Omega, H)$ is $\alpha$-homogeneous.

Let $B(L^\infty(\Omega, H))$ be the algebra of all bounded $L^\infty(\Omega)$-linear operators on $L^\infty(\Omega, H)$. Taking into account that $L^\infty(\Omega, H)$ is a Hilbert–Kaplansky module over $L^\infty(\Omega)$ we get that $B(L^\infty(\Omega, H))$ is an $AW^*$-algebra of type I with the center is $*$-isomorphic to $L^\infty(\Omega)$. Suppose that $\dim H = \alpha$. Then $L^\infty(\Omega, H)$ is $\alpha$-homogeneous and by [9] Theorem 7 the algebra $B(L^\infty(\Omega, H))$ has the type $I_\alpha$. The center $Z(B(L^\infty(\Omega, H)))$ of this $AW^*$-algebra isomorphic with the algebra $L^\infty(\Omega)$ which is a von Neumann algebra, and thus by [10] Theorem 2 $B(L^\infty(\Omega, H))$ is also a von Neumann algebra. Consequently, if $\dim H = \alpha$ then $B(L^\infty(\Omega, H))$ is a of type $I_\alpha$ von Neumann algebra.

Now let us consider an arbitrary of type $I_\alpha$ homogeneous von Neumann algebra $M$ with the center isomorphic to $L^\infty(\Omega)$. Taking into account that two von Neumann algebras of the same type $I_\alpha$ with isomorphic centers are mutually $*$-isomorphic, we conclude that the algebra $M$ is $*$-isomorphic to the algebra $B(L^\infty(\Omega, H))$, where $\dim H = \alpha$.

A projection $p \in B(L^\infty(\Omega, H))$ is called $\sigma$-finite-generated if $p(L^\infty(\Omega, H))$ is a $\sigma$-finite-generated module.

**Lemma 4.2.** A projection $p \in M \cong B(L^\infty(\Omega, H))$ is finite if and only if it is $\sigma$-finitely-generated.

**Proof.** "if" part. Let $p \in M$ be a finite projection.

Case 1. Let $p$ be a projection such that $pMp$ is a $n$-homogeneous von Neumann algebra. Then $p \in pMp \equiv B(p(L^\infty(\Omega, H)))$ is also $n$-homogeneous algebra. This implies that $p(L^\infty(\Omega, H))$ is a $n$-homogeneous Hilbert–Kaplansky module. This means that $p$ is a $n$-homogeneous projection. In particular, $p$ is a $\sigma$-finite-generated projection.

Case 2. Let $p$ be an arbitrary finite projection. Then there exists a system of mutually orthogonal central projections $(q_n)_{n \in F} \subset P(pMp)$, where $F \subseteq \mathbb{N}$, with $\sum_{n \in F} q_n = p$ such that $q_n pMp$ is a homogeneous von Neumann algebra of type $I_n$. By case 1 we get that $q_n p$ is $n$-homogeneous for all $n \in F$. Thus $p = \sum_{n \in F} q_n p$ is $\sigma$-finite-generated.

"only if" part. Let $p$ be a $\sigma$-finite-generated projection. Then there exists a partition of unity $\{z_n\}_{n \in F}$ in $\nabla$, where $F \subseteq \mathbb{N}$, such that $z_n p(L^\infty(\Omega, H))$ is a $n$-homogeneous module over $z_n L^\infty(\Omega)$ for all $n \in F$. Therefore $z_n pMp \cong B(z_n p(L^\infty(\Omega, H)))$ is a homogeneous von Neumann algebra of type $I_n$ for all $n \in F$. Thus $p = \sum_{n \in F} q_n p$ is finite. The proof is complete.

**Lemma 4.3.** Let $M \equiv B(L^\infty(\Omega, H))$. If $x \in M$ is a compact operator relative $M$ then it is cyclically compact.
Proof. Let $x \in M$ be a compact operator relative $M$. By [8, Theorem 1.3] for every $n \in \mathbb{N}$ there exists a projection $p_n \in M$ such that $\|xp_n\| < 1/n$ and $1 - p_n$ is finite. By Lemma 4.2, $1 - p_n$ is $\sigma$-finite-generated. Therefore $1 - p_n$ is a cyclically compact operator on $L^\infty(\Omega, H)$. Thus $x(1 - p_n)$ is also a cyclically compact operator on $L^\infty(\Omega, H)$. Since $\|x - x(1 - p_n)\| = \|xp_n\| < 1/n$ we obtain that $x$ is also a cyclically compact operator on $L^\infty(\Omega, H)$. The proof is complete.

Proof of Theorem 4.1. Firstly, we will consider a case homogeneous von Neumann algebra. In this case by Lemma 4.3, $x$ is a cyclically compact operator. Therefore by Theorem 2.2 we can assume that without lost of generality it has the following form

$$x = \sum_{k=1}^{\infty} f_k \xi_k \otimes \xi_k.$$  

According to [9, Lemma 13] we obtain that $p_k = \xi_k \otimes \xi_k$ is an abelian projection for all $k$. So

$$x = \sum_{k=1}^{\infty} f_k p_k.$$  

Now if $M$ is an arbitrary von Neumann algebra of type I then we can consider the decomposition of the algebra $M$ to homogeneous summands and apply the above assertion. The proof is complete.

Remark 4.4. If $M$ is a type I$_n$, $n < \infty$, von Neumann algebra which represented as $n \times n$-matrix algebra over its center, then Theorem 4.1 gives us that any self-adjoint element from $M$ can be represented as diagonal matrix (cf. [6]).

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