In the Hartle–Hawking ‘no boundary’ approach to quantum cosmology, the Universe is described by a Euclidean path integral on an $n$-dimensional manifold $M$ with a single-boundary component $\Sigma$ representing the ‘present’ (see figure 1). The path integral depends on the induced metric $h_{ij}$ on $\Sigma$ and the boundary values of any matter fields $\psi$ on $\Sigma$, thus yielding a ‘wavefunction of the Universe’ $\Psi[h_{ij}, \psi|\Sigma]$. Whether one should also sum over topologies of $M$ is an open question; such a sum can qualitatively change the locations of the peaks of the wavefunction [3, 4], but does not affect the main thrust of this paper.

In the absence of a full-fledged quantum theory of gravity, of course, such a path integral is not very well defined. The hope is that minisuperspace models and semiclassical saddle-point approximations might still give useful information about, for example, inflation [5]. If $\Sigma$ is to be spacelike, one cannot ordinarily find a saddle point with a globally Lorentzian metric—such topology-changing geometries are forbidden by fairly mild energy conditions [6]. It may be that the dominant saddle points are complex [7], with consequent ambiguities in the integration contour. If we restrict ourselves to real metrics, though, we are naturally led to ‘real tunnelling geometries’ [8], geometries in which an initial Riemannian metric is joined to a Lorentzian metric along a hypersurface $\Sigma_0$, as shown in figure 1.

For the resulting geometry to be smooth, with no ‘boundary layer’ stress–energy tensor at $\Sigma_0$, the induced metrics $h_{ij}$ must match across $\Sigma_0$ and the extrinsic curvature $K_{ij}[\Sigma_0]$ of the signature-changing hypersurface must vanish. A classical solution of the field equations with such a geometry may or may not exist, depending on the topology of $M$ and the sign of the
cosmological constant. Some of the known restrictions on the topology of $M$ are described in [8, 9].

For the special case of a three-manifold $M$ with a negative cosmological constant, the question of which topologies admit real tunnelling geometries is almost completely solved. Any three-dimensional Einstein metric with $\Lambda < 0$ is hyperbolic (that is, has a constant negative curvature). Thurston has shown that a compact three manifold with a nontrivial boundary admits a hyperbolic metric if and only if it is prime, homotopically atoroidal, and not homeomorphic to a certain twisted product of a two torus and an interval [10]. If in addition $M$ is acylindrical, it admits a hyperbolic metric for which the extrinsic curvature of the boundary vanishes, and thus allows a real tunnelling geometry.

Real tunnelling geometries lead to an elegant classical picture of a universe born from 'nothing'. Quantum mechanically, though, the picture is less clear. The induced metric and extrinsic curvature are conjugate variables, and in a quantum theory, they should not be specified simultaneously. In the saddle-point approximation, in particular, the requirement that $K_{ij}[\Sigma_0] = 0$ determines the boundary metric $h_{ij}$ nearly uniquely. Hence real tunnelling geometries do not determine a wavefunction $\Psi[h_{ij}]$, but merely a contribution to the wavefunction at a few particular values of $h_{ij}$.

One alternative, proposed in [11], is to forget for a moment about the requirement that $K_{ij}[\Sigma_0] = 0$, and consider the wavefunction $\Psi[h_{ij}]$ as a functional of the spatial metric on $\Sigma_0$. We can then ask whether a real tunnelling geometry is probable—that is, whether configurations with $K_{ij} = 0$ occur at or near the peaks of the wavefunction. This is still a bit tricky, since the spatial metric contains information about time as well as spatial geometry [12]; we should really ask whether a transition to Lorentzian signature is probable at a fixed time. Following York [13], we can use the mean curvature $K = h^{ij}K_{ij}$ as an 'extrinsic time' coordinate. The wavefunction will then be a functional of $K$ and the conformal metric $\tilde{h}_{ij}$, that is, the spatial metric modulo Weyl transformations [14, 15].

The question is then whether configurations with $K_{ij} = 0$ are probable at time $K = 0$. To answer this in the semiclassical approximation, we must first find a form of the Einstein–Hilbert action that fixes $K$ and $\tilde{h}_{ij}$ at the boundary. (For simplicity, I will omit matter terms.) Recall first [1, 11] that the Euclidean action

$$I_E[g] = -\frac{1}{16\pi G} \int_M d^n x \sqrt{g} (R - 2\Lambda) - \frac{1}{8\pi G} \int_{\Sigma_0} d^{n-1} x \sqrt{\tilde{h}} K$$

(1)

is appropriate for a fixed boundary metric $h_{ij}$:

$$\delta I_E = \text{(equations of motion)} - \frac{1}{16\pi G} \int_{\Sigma_0} d^{n-1} x \sqrt{\tilde{h}} (K_{ij} - h_{ij} K) \delta h^{ij}$$

(2)
so no boundary contributions appear in the variation when \( \delta h^{ij} \) vanishes. Adding a term

\[
\frac{1}{8\pi G} \int_{\Sigma_0} d^{n-1}x \sqrt{h} K
\]

to (1), we obtain a new ‘York time’ action

\[
I_Y[g] = -\frac{1}{16\pi G} \int_M d^n x \sqrt{g} \left( R[g] - 2\Lambda \right) - \frac{1}{8\pi G(n-1)} \int_{\Sigma_0} d^{n-1}x \sqrt{h} K,
\]

(3)

whose variation is

\[
\delta I_Y = \text{(equations of motion)}
\]

\[
-\frac{1}{16\pi G} \int_{\Sigma_0} d^{n-1}x \sqrt{h} \left( K_{ij} - \frac{1}{n-1} h_{ij} K \right) \delta h^{ij} + \frac{1}{8\pi G(n-1)} \int_{\Sigma_0} d^{n-1}x \sqrt{h} \delta K.
\]

(4)

For a fixed conformal geometry on \( \Sigma_0 \), the only allowed metric variations are of the form \( \delta h^{ij} = \delta \phi h^{ij} \), so the first boundary term in (4) vanishes; for \( K \) fixed, the second term vanishes as well.

Saddle-point contributions to the path integral come from extrema of (3), that is, classical solutions of the Einstein field equations with prescribed mean curvature and conformal geometry at \( \Sigma_0 \). For a solution \( \bar{g}_{ab} \) with \( K[\Sigma_0] = 0 \), the action (3) is

\[
\bar{I}_Y[\bar{g}] = -\frac{\Lambda}{4\pi G(n-2)} Vol_{\bar{g}}(M),
\]

(5)

and the saddle-point contribution to the path integral is

\[
\Psi[h_{ij}, K = 0] \sim \Delta_{\bar{g}} \exp \left\{ -\frac{\Lambda}{4\pi G(n-2)} Vol_{\bar{g}}(M) \right\},
\]

(6)

where the Van Vleck–Morette determinant \( \Delta_{\bar{g}} \) is a combination of determinants coming from quadratic terms in the action and from gauge fixing.

It is immediately evident from (4) that the extrema of the classical action \( \bar{I}_Y \) occur at \( K_{ij}[\Sigma_0] = 0 \): when one varies the boundary metric\(^1\) while keeping \( K[\Sigma_0] = 0 \), only the second term on the right-hand side of (4) contributes to \( \delta \bar{I}_Y \). The question of whether these extrema are actually minima was first addressed in [11]. In more than three spacetime dimensions, the answer is not known: the second variation of the action includes a term proportional to the Weyl tensor, whose contribution I do not know how to control.

In three spacetime dimensions, though, the Weyl tensor term is absent, and the problem is more tractable. It was shown in [11] that for \( \Lambda > 0 \), extrema with vanishing extrinsic curvature at \( \Sigma_0 \) are local minima of \( \bar{I}_Y \), and thus—assuming that the Van Vleck–Morette contribution is small—local maxima of the wavefunction (6). For this case, then, real tunnelling geometries are ‘probable’.

The main aim of this paper is to extend this analysis to geometries for which \( \Lambda < 0 \), a case left unresolved in [11]. This extension is made possible by a new result of Agol, Storm and Thurston [19], who prove—assuming the correctness of Perelman’s recent work on the geometrization theorem [20]—the following:

Let \((M, g)\) be a compact hyperbolic three manifold with a minimal surface (i.e., \( K = 0 \)) boundary. If \( M \) is acylindrical, it admits a hyperbolic metric \( \nu \) with a totally geodesic (i.e., \( K_{ij} = 0 \)) boundary. Then \( Vol(M, g) \geq Vol(M, \nu) \).

\(^1\) Note that I am considering only variations among classical solutions. Variations of the conformal factor off the space of solutions can make the action arbitrarily negative [1]. There is evidence that such variations are unimportant in the full path integral [16–18], but the question is not yet settled.
In other words, given a topological restriction on $M$ (acylindricity) that guarantees the existence of a hyperbolic metric for which $K_{ij}[\Sigma_0] = 0$, the geometry for which $K_{ij}[\Sigma_0] = 0$ has a minimal volume among all hyperbolic geometries for which $K[\Sigma_0] = 0$. But all vacuum solutions of the Einstein field equations in three dimensions with $\Lambda < 0$ are hyperbolic, and when $\Lambda < 0$, the smallest volumes give the largest contributions to the wavefunction (6). We thus conclude—again assuming that the Van Vleck–Morette contribution is small—that real tunnelling geometries are ‘probable’ for $\Lambda < 0$ as well.

Thus far, I have assumed that the determinant $\Delta_3$ in (6) is unimportant in determining the peaks of the wavefunction. In three spacetime dimensions, this factor is essentially the Ray-Singer torsion [21], which can be computed in certain cases (see, for example, the appendix of [3]). For a manifold with boundary, the dependence on the extrinsic curvature—or, roughly equivalently, on the boundary spin connection in the Chern–Simons formalism—is analysed carefully in [22]. In principle, it should be possible to use this result to test the assumption that $\Delta_3$ can be neglected here.

To extend these results to more than three dimensions requires control of the Weyl curvature term discussed in [11]. But the success in three dimensions at least makes the Hartle–Hawking description of quantum tunnelling from Riemannian to Lorentzian signature more plausible.

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