No-regret Learning in Cournot Games

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Abstract

In this work, we study the interaction of strategic players in continuous action Cournot games with limited information feedback. Cournot game is the essential model for many socio-economic systems where players learn and compete. In addition, in many practical settings these players do not have full knowledge of the system or of each other. In this limited information setting, it becomes important to understand the dynamics and limiting behavior of the players. Specifically, we assume players follow strategies such that in hindsight their payoffs are not exceeded by any single deviating action. Given this no-regret guarantee, we prove that under standard assumptions, the players’ joint action (both in the sense of time average and final iteration convergence) converges to the unique Nash equilibrium. In addition, our results naturally extend the existing regret analysis on time average convergence to obtain final iteration convergence rates. Together, our work presents significantly sharper and generalized convergence results, and shows how exploiting the game information feedback can influence the convergence rates.

1 Introduction

Game-theoretic models have been used to describe the cooperative and competitive behaviors of a group of players in a wide range of systems in robotics, distributed control and resource allocation Buşoniu et al. (2010); Stone and Veloso (2000); Olfati-Saber and Murray (2007). In this paper, we study the interaction of strategic players in Cournot games Cournot (1838), which is one of the best studied models for competition between self-interested agents in a system Johari and Tsitsiklis (2005). The Cournot competition is the essential market model for many socio-economic systems such as energy Pipattanasomporn et al. (2009), transportation Guériau et al. (2015) and healthcare systems Isern and Moreno (2016). This competition can be thought as multiple agents competing to satisfy an elastic demand by changing their production levels. For example, most of the US electricity market is built upon a Cournot competition model Kirschen and Strbac (2004), where energy producers bid to serve the loads in the grid, and a market price is cleared based on the total supply and demand. Each producer’s payoff is based on the market price multiplied by its share of the supply. The goal of producers is then to maximize their individual payoffs by strategically choosing the production levels.

Previous works in Cournot games mostly focused on analyzing the equilibrium behavior, especially the Nash equilibria. However, this raises the question of how the players would reach an equilibrium when they do not start at one. This question was actually considered by Cournot in the original 1838 paper Cournot (1838) in the case of two players. Since then, a rich set of results have generalized the conditions under which the players can get to a Nash equilibrium. For example, Milgrom and Roberts (1990, 1991) proposed a dynamic and adaptive behavior rule to reach a Nash equilibrium for an arbitrary number of players and encompasses both best-response dynamics and Bayesian learning. However, most of the known results require full information of other players’ actions and the exact market price function, which are not usually available in practice. For many applications of interest, providing full feedback is either impractical (e.g., distributed control Lian et al. (2002)) or explicitly
disallowed due to privacy and market power concerns (e.g., energy markets [Quinn (2009)]). Therefore in this work, we move away from both the static view of assuming players are at a Nash equilibrium and the assumption that they have full knowledge of the system. Instead, we analyze the long-run outcome of the learning dynamics under limited information feedback, and ask the following two fundamental questions:

1. Will strategic learning agents reach an equilibrium in Cournot games?
2. If so, how quickly do they converge to the equilibrium?

Reasoning about these questions requires specifying the dynamics, which describes how players act before achieving the equilibrium. In particular, we consider player dynamics induced by no-regret learning algorithms that attempt to find the best strategies according to some objective. In terms of the information structure, we consider bandit feedbacks. In the Cournot game, it means that the players only receive the price (a single number) from the system and nothing else. Therefore they must make decisions based only on the observed history of the price and their own actions. This is an important feature of many socio-economic games where each player only has access to local information (i.e., the realized payoffs) and are not informed about the attributes of the other participants.

Studying the behavior of players under limited feedback is a challenging problem which has started to receive some recent attention. Most existing works focus on no-regret algorithms because of their inherent robustness. In particular, a player’s no-regret dynamics definition could be directly translated to the coarse correlated equilibrium condition [Young (2004)]. However, coarse correlated equilibria (CCE) is a loose notion of equilibrium and may contain actions that are manifestly suboptimal for players [Nisan et al. (2007)]. Therefore, showing convergence to CCE, although important [Syrgkanis et al. (2015); Foster et al. (2016)], may not be satisfactory.

Since CCE is about the time averaged or expected behavior, a more refined notion is the final iteration convergence (i.e., the actual sequence of play). [Bravo et al. (2018); Bervoets et al. (2018); Zhou et al. (2018)] showed that joint action generated by online mirror descent converges to the Nash equilibrium in the class of variationally stable games. Paper [Cohen et al. (2017)] showed that the joint action produced by multiplicative weights converges to the Nash equilibrium in potential games. However, each of the aforementioned papers focused on a specific class of algorithms and do not generalize to all no-regret dynamics. In addition, the standard Cournot game definition does not guarantee the variational stability and existence of potential functions. Consequently, establishing convergence to finer notion of equilibria (e.g., Nash) for a broader class of no-regret algorithms remains open.

In the most relevant result to our work, [Nadav and Piliouras (2010)] proved each player’s average and actual action profiles converge to his Nash strategy in Cournot games with linear price and convex cost functions. However, as we will discuss in Section 2.3, Cournot games are monotone games under these assumptions and the analysis is greatly simplified. Our work is a strict generalization of their model and results. In fact, to the best of our knowledge, our work is one of the few that obtain positive convergence result without the monotocity or stronger assumptions.

1.1 Our Contributions

In this work, we study the dynamics of no-regret learning algorithms in Cournot games, and our major contributions are in the following two aspects.

First, we proved that in repeated Cournot game with concave, decreasing inverse demand function, and increasing, convex cost functions, when every player experiences no-regret, both the time-average and outcome action profile converge to the Nash strategy. This result holds for all no-regret algorithms. This is a much sharper result compared to the standard convergence result of the time-averaged action converging to a coarse correlated equilibrium.

Second, we provided the convergence rate of different no-regret algorithms, and linked it to the regret analysis of algorithm. In general Cournot games, we showed that convergence rate of zeroth-order no-regret algorithm is $O(T^{-1/4})$ and that of first-order no-regret algorithm is $O(T^{-1/2})$. For game with strongly monotocity structure, these rates could be improved to $O(T^{-1/3})$ and $O(T^{-1})$, respectively. These results provide new and quantitative insights for game mechanism design, i.e., the price of not having the gradient information is $O(T^{-1/4})$ and the benefits of using linear price functions could be as high as $O(T^{-1/2})$ in terms of the players’ convergence rate.
2 Problem Setup and Preliminaries

2.1 Model of Cournot competition

In this section, we first review the Cournot game setup and then provide two motivating examples of its applications in infrastructure and social systems. Assumptions on the inverse demand function and individual cost models are stated at the end of this section.

**Definition 1 (Cournot game Cournot (1838)).** Consider $N$ players with homogeneous products in a limited market, where the strategy space of player $i$ is the production level $x_i \geq 0$. The utility function of player $i$ is denoted as $\pi_i(x_1, \ldots, x_N) = p(\sum_{j=1}^{N} x_j)x_i - C_i(x_i)$, where $p$ is the market clearing price (inverse demand) function that maps the total production quantity to a price in $\mathbb{R}$ and $C_i(\cdot)$ is the cost function of player $i$.

The goal of each player $i$ in the Cournot game is to choose the best production quantity $x_i$ such that maximizes his own utility $\pi_i$. An important concept in game theory is the Nash equilibrium, at which state no player can increase his expected payoff via a unilateral deviation. A pure strategy Nash equilibrium of the Cournot game defined by $(\pi_1, \ldots, \pi_N)$ is a vector $\mathbf{x}^* \geq 0$ such that for all $i$:

$$\pi_i(x_i^*, \mathbf{x}_{-i}^*) \geq \pi_i(\tilde{x}_i, \mathbf{x}_{-i}^*), \quad \text{for all } \tilde{x}_i,$$

(1)

where $\mathbf{x}_{-i}$ denotes the actions of all players except $i$. Below, we briefly discuss two Cournot game examples in socio-economic systems.

**Example 1 (Wholesale Electricity Market)** The Cournot model is the most widely adopted framework for electricity market design Kirschen and Strbac (2004). Suppose there are $N$ electricity producers, each supplying the market with $x_i$ units of energy. In an uncongested grid, the electricity is priced as a decreasing function $p(y)$ of the total generated energy $y = \sum_{i=1}^{N} x_i$. In practice, linear price and production cost functions are commonly adopted. The profit of generator $i$ can be written as: $\pi_i(x_i; \mathbf{x}_{-i}) = x_i(a - b \sum_{j=1}^{N} x_j) - c_i x_i$, where $c_i \geq 0$ is the marginal production cost of $i$.

**Example 2 (Lotteries)** Lotteries are becoming an increasing important mechanism to allocated limited resources in social contexts, with examples in housing Friedman and Weinberg (2014), parking Zhang et al. (2015) and buying limited goods Zhong et al. (2012). These lotteries typically allocates each player with a number of “coupons” and a player’s chance of winning depends on how many coupons is played in round. Suppose $x_1, \ldots, x_N$ are the coupons used by the players, then a decreasing price function $p(\sum_{i=1}^{N} x_i)$ can be used to model the fact that each player is less likely to win as others spend more coupons. The profit of player $i$ is given by $\pi_i(x_i; \mathbf{x}_{-i}) = p(\sum_{j=1}^{N} x_j)x_i - x_i$, where $-x_i$ represent the cost of spending the coupons.

In this paper, we restrict our attentions to Cournot games satisfying the following assumptions:

**Assumption 1.** We assume the price function and individual cost functions:

1) The price function $p$ is concave, strictly decreasing, and twice differentiable on $[0, y_{\text{max}}]$, where $y_{\text{max}}$ is the first point where $p$ becomes 0. For $y > y_{\text{max}}, p(y) = 0$. In addition, $p(0) > 0$. \hspace{1cm} (A1)

2) The individual cost function $C_i(x_i)$ is convex, strictly increasing, and twice differentiable, with $p(0) > C_i(0)$, for all $i$. \hspace{1cm} (A2)

These assumptions are standard in the literature (e.g., see Johari and Tsitsiklis (2005) and the references within). The assumption $p(0) > C_i(0)$ is to avoid the triviality of a player never participating in the game. Paper Szidarovszky and Yakowitz (1977) first proved the following proposition, that Cournot games satisfying assumptions (A1) and (A2) have unique equilibrium.

**Proposition 1.** Cournot game satisfying assumption (A1) and (A2) has exactly one Nash Equilibrium.

Detailed proof of Proposition 1 is deferred to Appendix B.1.

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1 In congested grid, all electricity producers still compete in a Cournot game manner (i.e., bidding quantities), while the independent system operator (ISO) transmits electricity and sets congestion prices to maximize social surplus of the entire system Yao et al. (2005).
2.2 Review of No-Regret Algorithms

In this section, we review learning methods that players could employ to increase their individual rewards in an online manner. In particular, we consider a family of online learning algorithms with worst-case performance guarantees, namely no-regret algorithms.

An online algorithm $A$ is called no-regret (or no-external regret) if the difference between the total payoff it receives and that of the best fixed decision in hindsight is sublinear as a function of time [Hazan, 2016]. Formally, at iteration $t$, the online player chooses $x_t \in \mathcal{X}$. After the player has committed to this choice, a convex cost function $f_i \in \mathcal{F} : \mathcal{X} \to \mathbb{R}$ is revealed. Here $\mathcal{F}$ is the bounded family of cost functions available to the adversary. The cost incurred by the online player is $f_i(x_t)$, the value of the cost function for the choice $x_t$. Let $T$ denote the total number of game iterations. Let $A$ be an algorithm for decision mapping, which maps a certain game history to a decision for the next round. We formally define the regret of $A$ after $T$ iterations as:

$$R_T(A) = \sup_{\{f_1, f_2, \ldots, f_T\} \subseteq \mathcal{F}} \left\{ \sum_{t=1}^{T} f_i(x_t) - \min_{x \in \mathcal{X}} \sum_{t=1}^{T} f_i(x) \right\}, \quad (2)$$

An algorithm $A$ is said to have no regret, if its regret is sublinear as a function of $T$, i.e., $R_T(A) = o(T)$, since this implies that on average the algorithm performs as well as the best fixed strategy in hindsight. In the context of learning in multi-agent games, such worst-case guarantee property would be desirable since no player wants to realize that the decision policy they employed was strictly inferior to a plain policy prescribing the same action throughout, under the fog of war.

There are a collection of learning algorithms satisfy such no-regret property, e.g., EXP3 [Auer et al., 2002], online mirror descend [Shalev-Shwartz and Singer, 2007b], following the regularized leader [Shalev-Shwartz and Singer, 2007a]. Based on the available information to the decision maker, these algorithms can be grouped into two classes: first-order algorithms and zeroth-order algorithms.

**First-order no-regret algorithms** The common assumption for first-order methods is that decision makers are able to obtain the payoff gradient $\nabla f_i(x_t)$. Therefore, they can adjust actions by taking a step from their previous action in the direction of the gradient. This step may result in an action outside of the feasible set, and the algorithm will project back to the feasible set. Online mirror decent is the most widely adopted first-order no-regret algorithms [Mertikopoulos and Zhou, 2019], along with its variants such as follow the regularized leader and dual averaging [Xiao, 2010]. A pseudocode implementation of online mirror descend is provided in Algorithm 2 in Appendix A.

**Zeroth-order no-regret algorithms** As opposed to the first-order algorithms, in which the decision maker has access to the gradient information $\nabla f_i(x_t)$, in zeroth-order no-regret algorithm setting, the incurred loss $f_i(x_t)$ is the only feedback available to the online player at iteration $t$. This is the so-called bandit learning framework. In the game-theoretic setting (especially non-cooperative games), such bandit feedback assumption is more realistic, because players usually have only local information while the payoff gradients computation depends on the actions of all players.

When a function can be queried at multiple point, there are efficient ways to estimate the function gradient via directional sampling method. However, in the Cournot game setting, the multiple-point estimation techniques do not work. When a player attempts to get a second query of their payoff function, this function may have already changed due to the change of other players' action, i.e., instead of sampling $\pi_i(\cdot, x_{-i})$, the $i$-th player would be sampling $\pi_i(\cdot, x'_{-i})$ for some $x'_{-i} \neq x_{-i}$. FKM [Flaxman et al., 2005] algorithm is a well-known no-regret learning algorithm under the bandit feedback setup, which is also known as “gradient descent without a gradient”. A pseudocode implementation of FKM is provided in Algorithm 2 in Appendix A.

2.3 Existing Convergence Results

Existing learning in games literature obtaining convergence results mostly focus on the following two classes of games: monotone game and variational stable game.

**Definition 2 (Monotone game).** Define the game gradient as:

$$g(x) = [\nabla_1 \pi_1(x), \nabla_2 \pi_2(x), \ldots, \nabla_N \pi_N(x)]^T.$$
The game is monotone (diagonally strictly concave) if it satisfies, \( \forall x, x' \in \mathcal{X} \)
\[
\langle g(x) - g(x'), x - x' \rangle \leq 0,
\]
with equality if and only if \( x = x' \).

Rosen [1965] showed that every concave N-person game satisfying this monotonicity condition (also known as the diagonal strict concavity condition), has a unique equilibrium point. It also proved that, if the game does not start off from the equilibrium and each player attempt to change his strategy following his payoff gradient, the system of multi-agent learning process is globally asymptotically stable. Furthermore, starting from any feasible point in the joint strategy set, the system will always converge to the unique Nash Equilibrium point of the original N-person concave game. Actually, this diagonal strict concavity condition of Rosen is a common assumption and cornerstone for many learning in games literatures and convergence result [Bervoets et al. (2018); Bravo et al. (2018)].

**Definition 3** (Variational stable game). Given a concave N-player game \( (\mathcal{N}, \mathcal{X} = \prod_{i=1}^{\mathcal{N}} \mathcal{X}_i, \{u_i\}_{i=1}^{\mathcal{N}}) \), the game is called globally variational stable, if
\[
\langle g(x), x - x^* \rangle = \sum_{i=1}^{\mathcal{N}} \langle g_i(x), x_i - x_i^* \rangle \leq 0, \forall x \in \mathcal{X},
\]
where \( x^* \) is the Nash equilibrium, and the equality holds if \( x = x^* \).

The variational stable condition is a weaker condition than Rosen’s monotonicity condition. Recent papers [Zhou et al. (2018); Mertikopoulos and Zhou (2019)] show the convergence of online mirror descend algorithm in variational stable games. It allows us to treat convergence questions in general games with continuous action spaces, without having to restrict ourselves to a specific monotone subclass (such as potential or common interest games). The following table summarizes the conditions required for Nash equilibrium, variational stability and monotonicity.

| Monotonicity, Variational stability, Concave Game and Nash Equilibrium | First-order requirement | Second-order test |
|---|---|---|
| Nash equilibrium | \( \langle g(x^*), x - x^* \rangle \leq 0 \) | N/A |
| Variational stability | \( \langle g(x), x - x^* \rangle \leq 0 \) | \( H^G(x^*) < 0 \) |
| Monotonicity | \( \langle g(x^*), x - x^* \rangle \leq 0 \) | \( H^G(x) \prec 0 \) |

In the above table, \( H^G \) defines the Hessian of a game \( G \) and \( H^G(x) = (H^G_{ij}(x))_{i,j \in N} \). Each block \( H^G_{ij}(x) = \frac{\partial^2 u_i(x)}{\partial x_i \partial x_j} \). We want to emphasize that Cournot games with (A1) and (A2) may neither be a monotone game (Definition 2) nor a variational stable game (Definition 3). See the example below.

**Cournot game example** Suppose a two-player Cournot game with assumptions (A1) and (A2), where the price function is \( P(Q) = a - b(x_1 + x_2)^3 \) and the payoff function for both player are \( C_i(x_i) = x_i^4 \). The utility function for both players are:
\[
\pi_i(x_1, x_2) = [a - b(x_1 + x_2)^3]x_i - x_i^4 = (a - bx_1^3 - bx_2^3 - 3bx_1x_2^2 - 3bx_2x_1^2)x_i - x_i^4,
\]
(5)
The unique Nash equilibrium is \( x_1^* = x_2^* = \sqrt[3]{\frac{a}{20b+4}} \). We want to find \( a, b \) such that, \( \exists x_1, x_2 \geq 0 \) violating the (MC) and (VS) conditions. Specifically, we want to find \( a, b \) such that,
\[
\exists x_1, x_2 \geq 0, [x_1 - x_1^* \quad x_2 - x_2^*] \begin{bmatrix}
    a & -4bx_1^3 - 9bx_2^2x_2 - 6bx_1x_2^2 - bx_2^3 - 4x_2^3 \\
    a & -4bx_2^3 - 9bx_1^2x_2 - 6bx_1x_2^2 - bx_1^3 - 4x_1^3
\end{bmatrix} > 0,
\]
(6)
By simulation, there exists many such examples. For instance, when setting \( a = 10, b = 1 \) and the price function equals \( p(x_1 + x_2) = 10 - (x_1 + x_2)^3 \). The NE equilibrium is \( x_1^* = x_2^* = \sqrt[3]{\frac{10}{22}} \approx 0.7469 \). Pick \( x_1 = 0.25 \) and \( x_2 = 0.5 \), by simple calculation, we find that:
\[
[x_1 - x_1^* \quad x_2 - x_2^*] \begin{bmatrix}
    a & -4bx_1^3 - 9bx_2^2x_2 - 6bx_1x_2^2 - bx_2^3 - 4x_2^3 \\
    a & -4bx_2^3 - 9bx_1^2x_2 - 6bx_1x_2^2 - bx_1^3 - 4x_1^3
\end{bmatrix} \approx 0.142 > 0
\]
which contradicts both the global variational stability condition and the monotone concavity condition.

As a Cournot game with (A1) and (A2) may neither be a monotone game nor a globally variational stable game, previous convergence results do not apply.
3 Convergence of No-regret Algorithms in Cournot Games

Before we proceed to discuss about the main convergence results of this work. First, we formally define the notion of convergence. Then, we prove the convergence result in two steps, by first showing the convergence of players’ payoff (in both time-average and last iteration manner), then showing the convergence of players’ actions. At the end of this section, we discussed the convergence rate for different no-regret algorithms,

Definition 4 (Convergence in measure). Let μ be a measure on $\mathbb{N}$ (e.g., the counting measure). We say that a sequence $a_t$ converges in measure to a if $\forall \epsilon > 0, \lim_{t \to \infty} \mu(\{|a_t - a| > \epsilon\}) = 0$.

The reason we need to work with convergence in measure rather than the standard notion of convergence (i.e., $\lim_{t \to \infty} a_t = a$) is because the latter condition is too stringent for no-regret algorithms. Consider a no-regret algorithm $\mathcal{A}$. We can construct another algorithm $\mathcal{A}'$ in the following manner. Let $M$ be some positive integer larger than 1. Then the actions produced by $\mathcal{A}'$ is the same as $\mathcal{A}$ except for times $M, M^2, M^3, \ldots$. At these times, $\mathcal{A}'$ takes on the action 0 (or any other arbitrary action). Both $\mathcal{A}$ and $\mathcal{A}'$ would be no-regret algorithms, since $\mathcal{A}'$ only deviates at a set of vanishing small fraction of points. On the other hand, for $\mathcal{A}'$, its actions cannot converge in the standard sense. Therefore, given only the regret bound, the best final time convergence result we can hope for is convergence in measure as defined in Definition 4.

3.1 Payoff Convergence

In this part, we prove that each player’s payoff in the game converge to the payoff at the Nash equilibrium point. Theorem 1 shows the time-averaged convergence and Theorem 2 sharps the result by proving the final iteration convergence.

Theorem 1 (Time average convergence). Suppose that after $T$ iterations, every player has expected regret $o(T)$. As $T \to \infty$, every player’s time-averaged payoff $\frac{1}{T} \sum_{t=1}^{T} \pi_i(x_t)$, converges to the payoff at the Nash equilibrium $\pi_i(x^*)$, for all $i$.

Intuitively, the proof flows naturally from the definition of no-regret algorithm. Given that every player has a sublinear regret compared to any fixed action in hindsight, we can obtain a lower bound of the players’ payoff by their best response substracting the regret term $o(T)$. In addition, the players’ payoff is upper bounded by their best response. By the squeeze theorem in calculus, we are able to show the limit of the individual payoff converges to the payoff at the Nash equilibrium, as time approaches infinity. Details about the proof could be found in Appendix B.2.

Theorem 2 (Actual payoff convergence). Suppose that after $T$ iterations, every player has expected regret $o(T)$. As $T \to \infty$, every player’s realized payoff $\pi_i(x_t), \forall i$, converges to the payoff at the Nash equilibrium $\pi_i(x^*)$ in measure:

$$\forall \epsilon > 0, |\pi_i(x_t) - \pi_i(x^*)| < \epsilon,$$

for all but $o(T)$ periods $t \in [1, T]$.

We prove Theorem 2 by contradiction. In particular, suppose that $\exists \epsilon > 0$, such that more than a sub-linear fraction of $t \in \{1, 2, \ldots, T\}$ satisfies that: $|\pi_i(x_t) - \pi_i(x^*)| \geq \epsilon$. It will lead to contradiction of the sub-linear regret property of no-regret algorithms. One can interpret Theorem 2 from two angles. Firstly, given any $\epsilon > 0$ (fix the error bound), the set of time that realized payoff significantly deviates from the Nash equilibrium equals $O(\frac{R}{\sqrt{T}})$, which depends on the used algorithm. For algorithms with tighter regret bound, the set of time vanishes faster. On the other hand, for any set with more than sublinear of points $T_1 \subseteq T$ (fix time periods), we have $|\pi_i(x_t) - \pi_i(x^*)| < O(\frac{R}{\sqrt{T_1}})$, for all $t \in T_1$ but a set of containing sublinear number of points. Thus, algorithms with tighter regret bound have smaller error bound.

3.2 Action Convergence

Now we turn our attention to prove the action convergence. The following two propositions are needed for the proof of Theorem 3.
Proposition 2 (Inverse function theorem). Consider function $f: \mathbb{R}^n \to \mathbb{R}^n$, and $f(x_0) = y_0$. Let $J = \frac{df}{dx}|_{x=x_0}$ as the Jacobian of function $f$. If $J$ evaluated at $x_0$ is invertible, then there exists a continuous and differentiable function $g$ such that,

$$g(f(x)) = x,$$

for $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ where $\mathcal{X}$ is some set around $x_0$ and $\mathcal{Y}$ is some set around $y_0$.

Proposition 3 (Lipschitz continuity). Suppose function $f: \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable on $\mathcal{X}$. Then it satisfies the Lipschitz condition that, there exists a constant $L \in \mathbb{R}^+$ such that,

$$||f(x_1) - f(x_2)|| \leq L||x_1 - x_2||, \forall x_1, x_2 \in \mathcal{X},$$

Theorem 3 (Convergence in action). Let $x^*$ denote the Nash Equilibrium, suppose that $x$ satisfies:

$$||\pi(x) - \pi(x^*)|| \leq \epsilon, \text{(closeness in payoff)},$$

then it implies that,

$$||x - x^*|| \leq L \cdot \epsilon, \text{(closeness in action),}$$

where $\pi(x) = [\pi_1(x), \pi_2(x), ..., \pi_N(x)]^T$ is the payoff vector of all players, $\pi_i(x) = \langle \sum_{j=1}^N x_j \rangle x_i - C_i(x_i)$ is the individual payoff function of player $i$, and $L \in \mathbb{R}^+$ is a constant.

Proof of Theorem 3 is deferred to Appendix B.4.

3.3 Convergence Rate Discussion

The previous sections proved the convergence of time-averaged and final iteration payoffs and action to the Nash equilibrium. However, the proofs do not explicitly provide us the convergence rate. Recall that the payoff convergence is based on the following equation,

$$\frac{1}{T} \sum_{t=1}^T \pi_i(x_{t,i}, x_{t, -i}) - \frac{R_T(A)}{T} \leq \frac{1}{T} \sum_{t=1}^T \pi_i(x_{t,i}, x_{t, -i}) \leq \frac{1}{T} \sum_{t=1}^T \pi_i(x_{t,i}, x_{t, -i}),$$

where $R_T(A)$ is the regret of algorithm $A$ after $T$ iterations. Thus, the rate of convergence rate naturally connects to the regret of the algorithm.

Zeroth-order algorithm. Paper [Flaxman et al. 2005] gives that the expected regret of FKM algorithm is $R_T(A) = O(T^{-\frac{1}{2}})$. By Theorem 2, we have that,

$$||\pi_i(x_t) - \pi_i(x^*)|| \leq O\left(\frac{R_T(A)}{T}\right) = O(T^{-\frac{1}{2}}),$$

for all but a set of measure zero as $T$ goes to infinity. Therefore, by Theorem 3, the action is also bounded by $||x_t - x^*||_2 \leq O(T^{-\frac{1}{2}})$ for all but a set of measure zero as $T$ goes to infinity.

First-order algorithm. The regret bound for online mirror descend(OMD) and follow-the-regularized-leaders(FTRL) are both $R_T(A) = O(T^{-\frac{1}{2}})$ [Hazan 2016]. By Theorem 2, we have that,

$$||\pi_{t,i}(x_t) - \pi_{t,i}(x^*)|| \leq O\left(\frac{R_T(A)}{T}\right) = O(T^{-\frac{1}{2}}),$$

for all but a set of measure zero as $T$ goes to infinity. By Theorem 3, the action is also bounded by, $||x_t - x^*||_2 \leq O(T^{-\frac{1}{2}})$ for all but a set of measure zero as $T$ goes to infinity.

Game property and convergence rate. In section 2.3, we gave an example showing that Cournot game with (A1) and (A2) may neither be a monotone game nor a variational stable game. But what if we restrict the price and individual cost function to certain class such that Cournot game becomes monotone (or strongly monotone)? Will it lead to different convergence rate?

For example, consider a linear Cournot game where the price model $P(\sum_{i=1}^N x_i) = a - b \sum_{i=1}^N x_i$ is linear ($a$ is a positive constant and $b_i > 0$) and the individual cost function $C_i(x_i) = 0, \forall i$. A game is strongly monotone if it satisfies the following stronger variant of monotocity:

$$\sum_{i \in N} \lambda_i \langle g_i(x') - g_i(x), x' - x \rangle \leq -\frac{\beta}{2} ||x' - x||^2,$$  \hspace{1cm} (7)
where $\lambda, \beta > 0$. By simple calculation, we see that linear Cournot games with zero marginal generation cost are always strongly convex. Paper [Bravo et al. (2018)] proves that for $\beta$-strongly monotone games, zeroth-order no-regret algorithms can achieve $O(T^{-\frac{3}{4}})$ convergence rate; and first-order no-regret algorithms can achieve $O(T^{-1})$. The detailed proof could be found in Theorem 5.2 in their paper for readers’ reference. Compared with our convergence results in general Cournot games, that is $O(T^{-\frac{1}{4}})$ for zeroth-order algorithms and $O(T^{-\frac{1}{2}})$ for first-order algorithms, the gain for having strongly monotone game structure can accelerate players’ equilibration rate.

4 Numerical Experiments

In this section, we provide two Cournot game examples and visualize the performance of different no-learning algorithms. We hope these toy examples can help readers quickly grasp the key theoretic results in three perspectives: 1) the convergence behavior; 2) convergence rate comparison between different classes of no-regret algorithms; and 3) the impact of game structure on convergence rate.

Setup We consider two two-player Cournot games with different price and cost functions. **G1:** a general Cournot game that is neither monotone nor variational stable. We take the example in Section 2.3 that $p(x) = 10 - (x_1 + x_2)^3$ with cost $C_i(x_i) = x_i^4$, $i = 1, 2$. The Nash equilibrium of this game is $x_1^* = x_2^* \approx 0.7469$.

**G2:** a strongly concave Cournot game where $p(x) = 1 - (x_1 + x_2)$ and $C_i(x_i) = 0$. Nash equilibrium of this game is $x_1^* = x_2^* = \frac{1}{3}$. The game is repeated for multiple times with all players either using the FKM algorithm (a zeroth-order algorithm) or OMD (a first-order algorithm). Detailed algorithm implementation is given in Appendix A.

![Figure 1](image)

Fig. 1 shows that all players’ actions converge to the Nash equilibrium, for both FKM and OMD. However, the convergence rate of OMD (column 2) is much faster than FKM (column 1), which demonstrates the benefits of having access to gradient information. This is aligned with the theoretical results in Section 3. In addition, viewed the convergence rate difference between G1 and G2, readers could have a more intuitive understanding on how the game structure affects the convergence rate.

5 Conclusion

In this paper, we study the interaction of strategic players in Cournot games with concave inverse demand function and convex individual cost functions. In particular, we consider all players are risk adverse and use the class of no-regret algorithms. We prove the convergence of all no-regret algorithms.
learning algorithms in Cournot games. In addition, we show the convergence rate difference under different information feedbacks and game structures. It demonstrate that by taking advantages of more information (first-order algorithm v.s. zeroth-order algorithm) and game structure (strongly monotone v.s. general Cournot game), one could achieve much faster convergence rate, compared with the original convergence rate of bandit optimization algorithms.

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Appendix A

A1. Review of FKM

Algorithm 1 FKM

1: Input: decision set $\mathcal{X}$, parameters $\delta, \eta$.
2: Pick $y_1 = 0$ (or arbitrarily).
3: for $t = 1, 2, ..., T$ do $\triangleright$ Number of time slots
4: Draw $u_t \in S_d$ uniformly at random, and set $x_t = y_t + \delta u_t$.
5: Play $x_t$ suffer loss $f_t(x_t)$.
6: Calculate $g_t = n\delta f_t(x_t)u_t$.
7: Update $y_{t+1} = \prod_{X}[y_t - \eta g_t]$.
8: end for

The FKM algorithm is an instantiation of the generic reduction from bandit convex optimization to online convex optimization with spherical gradient estimators over the set $\mathcal{X}_\delta$ [Hazan (2016)]. It iteratively projects onto $\mathcal{X}_\delta$, in order to have enough space for spherical gradient estimation. In particular, in our experiment, we have $\mathcal{X} = \mathbb{R}_{\geq 0}, \mathcal{X}_\delta = \{x \in \mathbb{R} : x \geq \delta\}$. We set $\eta = \frac{1}{20\sqrt{T}}, \delta = \frac{1}{T_1/3}$ as suggested in [Hazan (2016)] Theorem 6.9, and pick $y_1 \geq \delta$ randomly from $N(0, 1)$ (repeat drawing until the condition $y_1 \geq \delta$ holds) for each player.

A2. Review of Online Mirror Descent

Algorithm 2 Online Mirror Descent with Quadratic Regularization

1: Input: decision set $\mathcal{X}$, parameter $\eta > 0$, regularization function $R(x) = \frac{1}{2}||x||^2$ which are strongly convex and smooth.
2: Pick $y_1 = 0$ (or arbitrarily) and $x_1 = \arg \min_{x \in \mathcal{X}} ||y_1 - x||^2$.
3: for $t = 1, 2, ..., T$ do $\triangleright$ Number of time slots
4: Play $x_t$.
5: Observe the payoff function gradient $\nabla f_t(x_t)$.
6: Update $y_t$ according to the rule:
   
   [Lazy version] $y_{t+1} = y_t - \eta \nabla f_t(x_t)$
   
   [Agile version] $y_{t+1} = x_t - \eta \nabla f_t(x_t)$
7: Project to feasible set:
   
   $x_{t+1} = \arg \min_{x \in \mathcal{X}} ||y_{t+1} - x||^2$

8: end for

In particular, in our experiment, each player’s action is one dimensional and non-negative. Thus, we have $R(x) = \frac{1}{2}x^2$ and $\mathcal{X} = \mathbb{R}_{\geq 0}$. We set $\eta = \frac{1}{2\sqrt{T}}$ as suggested in [Hazan (2016)] Theorem 5.6 and pick $y_1 \sim N[0, 1]$ randomly for each player.

The above hyper-parameter choices could be used in general Cournot games. To achieve the acceleration in strongly monotone games, we follow the step size in paper [Bravo et al. (2018)] for G2 in Section 4. In particular, we set $\eta_t = \frac{m_0}{t}, \eta_0 = 0.5, \delta_t = \frac{\delta_0}{t^{1/3}}, \delta_0 = 1$ for FKM and $\eta_t = \frac{m_0}{t}, \eta_0 = \frac{1}{2}$ for OMD.

Appendix B

B1. Proof of Proposition 1

Proof. For each player $i$, and each $s \geq 0$, define

$$x_i(s) = \begin{cases} x, & \text{such that } x \geq 0 \text{ and } p(s) = C_i'(x) - xp'(s) \\ 0, & \text{if no such exists.} \end{cases}$$
Note that \( x_i(s) \) is monotone decreasing in \( s \) and \( x_i(s) \) is continuous in \( s \). It is now shown that there is a unique non-negative number \( s^* \) such that,

\[
\sum x_i(s^*) = s^*,
\]

(8)

For \( \sum x_i(0) \geq 0 \) and by the positivity of \( C_i \), \(-p', \sum x_i(\xi) = 0 < \xi \) for any \( \xi \) such that \( f(\xi) = 0 \).

As \( x(s) = \sum_{i=1}^N x_i(s) \) is continuous and strictly decreasing for any \( s \) such that \( x(s) > 0 \), there must be exactly one \( s^* \) for which \( x(s^*) = s^* \). By the definition of \( x_i(s) \), each \( x_i(s) \) maximizes \( \pi_i(x_1, \ldots, x_N) = p(\sum_{j=1}^N x_j) x_i - C_i(x_i) \); therefore, \( x(s^*) = (x_1(s^*), \ldots, x_N(s^*)) \) is an equilibrium point of the model, and no other point can be an equilibrium point. The original version of this proof could be found in Szidarovszky and Yakowitz (1977).

**B2. Proof of Theorem 1**

**Proof.** Consider the \( i \)-th player. In each game iteration \( t \), let \((x_{t,i}, x_{t,-i})\) be the moves played by all the players, where we use \( X_{t,-i} \) to denote the actions played by all players except player \( i \). From player \( i \)'s point of view, the payoff he obtains at time \( t \) is the following:

\[
\forall \xi \in X_i, \pi_i(\xi) = \pi_i^{(t)}(\xi, x_{t,-i}).
\]

Note that this payoff function is concave by assumption. Then we have,

By the definition of regret,

\[
R_i(T) = \max_{\hat{x}_i \in X_i} \sum_{t=1}^T \pi_i^{(t)}(\hat{x}_i) - \sum_{t=1}^T \pi_i^{(t)}(x_{t,i}).
\]

Rewriting this in terms of the original utility function, and scaling by the number of iterations we get,

\[
\frac{1}{T} \sum_{t=1}^T \pi_i(x_{t,i}, x_{t,-i}) \geq \frac{1}{T} \sum_{t=1}^T \pi_i(\hat{x}_i, x_{t,-i}) - \frac{1}{T} R_i(T), \forall \hat{x}_i \in X_i.
\]

(9)

Let consider the best response for player \( i \) given all other players’ actions \( x_{t,-i} \) as \((x^*_{t,i}) = \arg \max_{\xi} \pi_i(\xi, x_{t,-i})\). Obviously,

\[
\frac{1}{T} \sum_{t=1}^T \pi_i(x^*_{t,i}, x_{t,-i}) \geq \frac{1}{T} \sum_{t=1}^T \pi_i(x_{t,i}, x_{t,-i}),
\]

(10)

In addition, since \( \pi_i \) is concave with respect to \( x_i \), it follows:

\[
\frac{1}{T} \sum_{t=1}^T \pi_i(x^*_{t,i}, x_{t,-i}) \leq \frac{1}{T} \sum_{t=1}^T \pi_i(\hat{x}_i, x_{t,-i}) = \frac{1}{T} \sum_{t=1}^T \pi_i(\hat{x}_i, x_{t,-i})
\]

(11)

where \( \hat{x}_i = \frac{\sum_{t=1}^T x^*_{t,i}}{T} \).

Combining Eq. (9) and Eq. (11) we have,

\[
\frac{1}{T} \sum_{t=1}^T \pi_i(x_{t,i}, x_{t,-i}) \geq \frac{1}{T} \sum_{t=1}^T \pi_i(x^*_{t,i}, x_{t,-i}) - \frac{1}{T} R_i(T), \forall \hat{x}_i \in X_i.
\]

(12)

Combining Eq. (12) and Eq. (10) leads to,

\[
\frac{1}{T} \sum_{t=1}^T \pi_i(x^*_{t,i}, x_{t,-i}) - \frac{1}{T} R_i(T) \leq \frac{1}{T} \sum_{t=1}^T \pi_i(x_{t,i}, x_{t,-i}) \leq \frac{1}{T} \sum_{t=1}^T \pi_i(x^*_{t,i}, x_{t,-i}),
\]

(13)

By Squeeze Theorem, since \( R_i(T) = o(T) \), as \( T \to \infty \),

\[
\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^T \pi_i(x_{t,i}, x_{t,-i}) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^T \pi_i(x^*_{t,i}, x_{t,-i}),
\]

(14)
which holds for all players. Therefore, as $T \to \infty$, the average payoff of each players’ payoff converges to the payoff at the Nash Equilibrium,

$$\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \pi_i(x_{t,i}, x_{t,-i}) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \pi_i(x^*) = \pi_i(x^*), \forall i$$

\[ \square \]

**B3. Proof of Theorem 2**

*Proof:* Following the Eq (13) in the proof of Theorem 1, we have,

$$\frac{1}{T} \sum_{t=1}^{T} \pi_i(x^*, x_{t,-i}) = \frac{1}{T} R_i(T) \leq \frac{1}{T} \sum_{t=1}^{T} \pi_i(x_{t,i}, x_{t,-i}) \leq \frac{1}{T} \sum_{t=1}^{T} \pi_i(x^*, x_{t,-i}),$$

Denote $a_t = \pi_i(x^*, x_{t,-i})$ (best response for player $i$ given others’ action), and $b_t = \pi_i(x_t)$, thus we have

$$0 \leq b_t \leq a_t,$$

**Claim:** given any $\epsilon > 0$, there exists at most a sub-linear fraction of time such that,

$$|b_t - a_t| > \epsilon, \forall t \in \{1, 2, ..., T\}$$

**Proof by contradiction:** Suppose the claim does not hold, that is, $\exists \epsilon > 0$, s.t. more than a sub-linear fraction of $t \in \{1, 2, ..., T\}$ satisfies,

$$|b_t - a_t| > \epsilon,$$

Re-arrange the time steps such that all the points that satisfies $|b_t - a_t| > \epsilon$ show up in the front. Say there are $T_1$ such points, then

$$\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} (b_t - a_t) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} |b_t - a_t|(b_t \leq a_t) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} |b_t - a_t| + \frac{1}{T} \sum_{t=T_1}^{T} |b_t - a_t| \geq \lim_{T \to \infty} \frac{T_1}{T} \epsilon + \frac{1}{T} \sum_{t=T_1}^{T} |b_t - a_t|,$$

(15)

Assume that $\frac{T_1}{T} \to 0$ as $T \to \infty$, by Eq. (15), we have

$$\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} |b_t - a_t| \geq \lim_{T \to \infty} \frac{T_1}{T} \epsilon + \frac{1}{T} \sum_{t=T_1}^{T} |b_t - a_t| \geq \lim_{T \to \infty} \frac{T_1}{T} \epsilon > 0$$

(16)

which contradicts the definition of no-regret algorithm.

Hence, the claim is proved that: given any $\epsilon > 0$, as $T \to \infty$ there exists a measure zero set of time ($\lim_{T \to \infty} \frac{o(T)}{T} = 0$) such that $|b_t - a_t| > \epsilon$. Otherwise, $|b_t - a_t| \leq \epsilon, \forall \epsilon > 0$. Equivalently, $\lim_{t \to \infty} \pi_i(x_t) = \arg \max_{x_t} \pi_i(x_t, x_{t,-i})$ for all but a measure zero set of time.

Since this holds for all players, we have,

$$\forall i, \lim_{T \to \infty} \pi_i(x_t) = \pi_i(x^*), \forall t \in \{1, ..., T\}$$

(17)

for all but a measure zero set of time.

\[ \square \]

**B4. Proof of Theorem 3**

*Proof:* Define $\pi = (\pi_1, \pi_2, ..., \pi_N)$ as a function $\mathbb{R}^n \to \mathbb{R}^n$, and $\pi_i(x) = p(\sum_{j=1}^{N} x_j)x_i - C_i(x_i)$ by definition.
Claim: Let $J = \frac{\partial \pi}{\partial x}$, and $x^*$ denotes the N.E. of the Cournot game. Then $J|_{x=x^*}$ is non-singular.

Suppose the claim is true, then $\pi$ is invertible around $\pi^*$ by Lemma 1. Therefore, there exists a continuously differentiable function $g$ (as the inverse function of $\pi$) such that,

$$g(\pi) = x, \forall \pi \in \{ x \in \mathbb{R}^n : ||\pi - \pi^*|| \leq \epsilon \},$$

(18)

By Lipschitz continuity, we have,

$$||x^* - x|| = ||g(\pi^*) - g(\pi)|| \leq L||\pi^* - \pi|| \leq L \cdot \epsilon$$

(19)

So the actions are close. What remains is to prove the claim that $J|_{x=x^*}$ is non-singular.

Proof of the Claim.

$$J|_{x=x^*} = \begin{bmatrix}
0 & p'(\sum_{j=1}^{N} x_j^*)x_1^* & \cdots & p'(\sum_{j=1}^{N} x_j^*)x_N^* \\
p'(\sum_{j=1}^{N} x_j^*)x_2^* & 0 & \cdots & p'(\sum_{j=1}^{N} x_j^*)x_N^* \\
\vdots & \vdots & \ddots & \vdots \\
p'(\sum_{j=1}^{N} x_j^*)x_N^* & p'(\sum_{j=1}^{N} x_j^*)x_2^* & \cdots & 0
\end{bmatrix}$$

(20)

Let define $P^* = p'(\sum_{j=1}^{N} x_j^*) \in \mathbb{R}$, then $J|_{x=x^*}$ could be written as,

$$J|_{x=x^*} = P^*(x^* \cdot 1^T) - P^* \cdot diag(x_1^*, x_2^*, \ldots, x_N^*),$$

Since $P^* > 0 \implies J|_{x=x^*}$ is invertible iff, $(x^* \cdot 1^T - diag(x_1^*, x_2^*, \ldots, x_N^*))$ is invertible.

Suppose $x_i^* \neq 0, \forall i$ (game admits no trivial solutions). Then suppose $v = [v_1, v_2, \ldots, v_N]^T$ solves the following equation,

$$(x^* \cdot 1^T - diag(x_1^*, x_2^*, \ldots, x_N^*))v = 0,$$

(21)

Equivalently,

$$(1^Tv)x_1^* - v_1x_1^* = 0$$

$$(1^Tv)x_2^* - v_2x_2^* = 0$$

$$\vdots$$

$$(1^Tv)x_N^* - v_Nx_N^* = 0$$

Since by assumption $\forall i, x_i^* \neq 0$, thus we have $(1^Tv)1 - v = 0$, and $v$ solves,

$$\begin{bmatrix}
0 & 1 & 1 & \cdots & 1 \\
1 & 0 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & \cdots & 0
\end{bmatrix}v = 0$$

(22)

Since matrix

$$\begin{bmatrix}
0 & 1 & 1 & \cdots & 1 \\
1 & 0 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & \cdots & 0
\end{bmatrix}$$

is non-singular, thus (21) holds iff $v = 0$.

Therefore, $J|_{x=x^*}$ is invertible. \(\square\)

\footnote{Suppose at the Nash Equilibrium point, we have $\sum_i x_i^* \geq y_{max}$ where $p(y_{max}) = 0$. Then at least one of the $x_i^*$ is positive. But then firm $i$ is strictly better off if it reduces $x_i^*$, which contradicts the definition of NE. Therefore, at the NE, $\sum_i x_i^* < y_{max}$ and $P^* > 0$.}