An Exact Solution with $f^2 = 1$ and $\Lambda \neq 0$ in the LTB model.

by

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Abstract

The exact solution in the LTB model with $f^2 = 1$, $\Lambda \neq 0$ is studied. The initial conditions for the metrical function and its derivatives generate the solution with complicated structure including the solutions like ”stripping of the shell”, ”collapse” and ”core”, or ”accretion”. In the limit of big time the solution allows the constant Hubble function and the density, depending on time. The transformation to the FRW model is shown.

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1 The Introduction

As it is shown in [3] the LTB model [1] is reduced to the Cauchy problem for the equation

\[ \left( \frac{\partial e^{\omega(\mu,\tau)/2}}{\partial \tau} \right)^2 = f^2(\mu) - 1 + \frac{1}{2} F(\mu) e^{\omega(\mu,\tau)/2} + \frac{\Lambda}{3} e^\omega \]  

(1)

whit initial conditions [3]

\[ \begin{align*}
\omega(\mu, \tau)|_{\tau=0} &= \omega_0(\mu), & \dot{\omega}(\mu, \tau)|_{\tau=0} &= \dot{\omega}_0(\mu), \\
\ddot{\omega}(\mu, \tau)|_{\tau=0} &= \ddot{\omega}_0(\mu)
\end{align*} \]

(2)

and constants

\[ \begin{align*}
\omega(\mu, 0)|_{\mu=0} &= \omega_0(0), & \dot{\omega}(\mu, 0)|_{\mu=0} &= \dot{\omega}_0(0), \\
\ddot{\omega}(\mu, 0)|_{\mu=0} &= \ddot{\omega}_0(0), & \Lambda,
\end{align*} \]

(3)

where \( \mu \) and \( \tau \) are dimensionless co-moving coordinates, corresponding to Lagrangian coordinate and time, \( \omega(\mu, \tau) \) is the metrical function, \( \dot{} = \frac{\partial}{\partial \tau} \), \( \Lambda \) is the cosmological constant,

\[ f^2(\mu) - 1 = e^{\omega_0(\mu)} \left( \frac{\dot{\omega}_0(\mu)}{2} + \frac{3}{4} \ddot{\omega}_0(\mu) - \Lambda \right), \]

(4)

\[ F(\mu) = e^{3\omega_0(\mu)/2} \left( \frac{\dot{\omega}_0^2(\mu)}{2} - \frac{3}{2} \Lambda \right) + 2 e^{\omega_0(\mu)/2} \left[ 1 - f^2(\mu) \right]. \]

(5)

In this article we study the class of the LTB models defined by the following initial conditions:

\[ f^2(\mu) = 1, \quad \Lambda \neq 0. \]

(6)

The Bonnor function [2]

\[ R(\mu, \tau) = e^{\omega(\mu,\tau)/2} \]

(7)

will be also used. For the obtained solution will be studied the Hubble function

\[ h(\mu, \tau) = \frac{\dot{R}'(\mu, \tau)}{R'(\mu, \tau)} = \frac{\partial \ln R'(\mu, \tau)}{\partial \tau} \]

(8)

and the density law [4]

\[ 8\pi \delta(\mu, \tau) = \frac{e^\frac{3}{2}[\omega_0(\mu) - \omega(\mu, \tau)]}{\omega'(\mu, \tau)} \times \]

\[ \left\{ 3 [\omega_0(\mu)]' \left[ -\dot{\omega}_0(\mu) - \frac{1}{2} \ddot{\omega}_0^2(\mu) + \frac{\Lambda}{6} \right] - 2 [\dot{\omega}_0(\mu)]' - 2 \dot{\omega}_0(\mu) [\dot{\omega}_0(\mu)]' \right\} \]

in the limit of \( \tau \to +\infty \).

2 The Input Equation and Previous Analysis

The initial conditions [3] produce the equation which we are interested in:

\[ \frac{3}{\Lambda} r(\mu, \tau) \left( \frac{\partial r(\mu, \tau)}{\partial \tau} \right)^2 = k(\mu) + r^3(\mu, \tau), \]

(10)
where
\[ r(\mu, \tau) = \frac{R(\mu, \tau)}{R_0(\mu)}, \quad R_0(\mu) = R(\mu, \tau)|_{\tau=0}, \quad (11) \]
\[ k(\mu) = \frac{3}{4} \frac{\ddot{\omega}_0(\mu)}{\Lambda} - 1. \quad (12) \]
The initial condition for the equation (10) is
\[ r(\mu, 0) = 1. \quad (13) \]
In accordance with (14), the general number of the initial conditions (2) is decreased by one. (11) binds two initial conditions \( \omega_0(\mu) \) and \( \dot{\omega}_0(\mu) \) as well:
\[ \ddot{\omega}_0(\mu) + \frac{3}{4} \dot{\omega}_0^2(\mu) - \Lambda = 0. \quad (14) \]
The definition of the initial value of the velocity follows from (10) and (13):
\[ \frac{3}{\Lambda} \left( \frac{\partial r(\mu, \tau)}{\partial \tau} \right)^2 \bigg|_{\tau=0} = k(\mu) + 1, \quad (15) \]
and, so, the restriction for function \( k(\mu) \) has the form:
\[ k(\mu) \geq -1. \quad (16) \]
The boundary value \( k(\mu) = -1 \) is reached in two cases: 1) \( \dot{\omega}_0(\mu) = 0 \), in this case the initial velocity is equal to zero:
\[ \frac{\partial r(\mu, \tau)}{\partial \tau} \bigg|_{\tau=0} = 0; \quad (17) \]
or 2) under the approximation ”\( \Lambda \to +\infty \)” and the initial velocity being an arbitrary finite function.
It follows also from (10) that the minimal distance from the centre and the particle is
\[ r_{\min}(\mu, \tau) = \frac{|k(\mu)|}{1/3}. \quad (18) \]
Before solving the equation (10) we will study two limit cases of this equation. They appear in the process of the competition of two summands at the right part of the equation (10): \( k(\mu) \) and \( r^3(\mu, \tau) \). Equating them, we obtain the definition of the boundary, presented by the equation (18). First limit case is that of small \( r(\mu, \tau) \):
\[ 0 < r(\mu, \tau) \ll |k(\mu)|^{1/3}. \quad (19) \]
In this case the main contribution to the exact solution, satisfying the initial condition (13), is given by the formula
\[ r_{\ll}^{\text{main}}(\mu, \tau) = \left( 1 \pm \frac{2}{\sqrt{3\Lambda}} \right) \frac{\tau}{\tau_0} \quad (20) \]
where
\[ \tau_0 = \frac{2}{\sqrt{3\Lambda}}. \quad (21) \]
This solution has the form of the Bonnor’s solution [2].

The second limit case is that of big \( r(\mu, \tau) \):
\[ r(\mu, \tau) \gg |k(\mu)|^{1/3} \quad (22) \]
The main contribution to the exact solution in this case is given by the formula
\[ r_{\gg}^{\text{main}}(\mu, \tau) = e^{\pm \frac{4}{3} \frac{\tau}{\tau_0}}. \quad (23) \]
The initial conditions are "forgotten" in this limit and the dynamics is defined only by $\Lambda$.

Let us study the input equation in the limit $\Lambda \to 0$. The equation (10) reads
\[ 3 r(\mu, \tau) \left( \frac{\partial r(\mu, \tau)}{\partial \tau} \right)^2 = \frac{3}{4} \dot{\omega}_0^2(\mu) + \Lambda \left( r^3(\mu, \tau) - 1 \right) \] (24)
and reduced in this limit to the equation
\[ \frac{\partial r^{3/2}(\mu, \tau)}{\partial \tau} = \pm \frac{3}{4} \dot{\omega}_0(\mu). \] (25)

This equation is studied in [4]. It is shown there that for $\dot{\omega}_0(\mu) = \text{const}$ the equation (25) reduced to the FRW model. In case $\dot{\omega}_0(\mu) \neq \text{const}$ the equation (25) includes the FRW model as the main part of the exact solution in the limit of big time: $\tau \gg \frac{3}{\dot{\omega}_0(\mu)}$.

We will study now the exact solution of the equation (10). The equation depends on the sign of $k(\mu)$ and its solution is reduced to the calculation of the following integral:
\[ \frac{1}{\sqrt{|k(\mu)|}} \int \frac{d \left( r^{3/2} \right)}{\sqrt{\frac{1}{|k(\mu)|} r^3 + \text{sign} k(\mu)}} = F(\mu) \pm \frac{\tau}{\tau_0}, \] (26)
the function $F(\mu)$ being an undetermined function will be defined by the initial conditions as follows:
\[ F(\mu) = \frac{1}{\sqrt{|k(\mu)|}} \left. \int \frac{d \left( r^{3/2} \right)}{\sqrt{\frac{1}{|k(\mu)|} r^3 + \text{sign} k(\mu)}} \right|_{\tau=0}. \] (27)

The upper sign in the equation (26) corresponds to the expansion from the initial state into the infinity and the lower sign corresponds to the fall to the its centre. We will name the solution with sign + the "expanding" solution and the solution with − the "falling" solution.

### 3 The "pure" solutions (function $k(\mu)$ has a constant sign)

We will study three cases: sign $k(\mu) = -1$, sign $k(\mu) = 0$, sign $k(\mu) = 1$.

Case 1: sign $k(\mu) = -1$. The equation (26) reads:
\[ \frac{1}{\sqrt{|k(\mu)|}} \int \frac{d \left( r^{3/2} \right)}{\sqrt{\frac{1}{|k(\mu)|} r^3 - 1}} = F_1(\mu) \pm \tau, \] (28)
and after the calculation of the integral has the form
\[ r^{3/2}(\mu, \tau) = \sqrt{|k(\mu)|} \text{ch} \left( F_1(\mu) \pm \frac{\tau}{\tau_0} \right), \] (29)
where
\[ F_1(\mu) = \text{Arch} \frac{1}{\sqrt{|k(\mu)|}} = \text{Arch} \left( \frac{2}{3} \frac{\tau_0}{\sqrt{|k(\mu)|}} \dot{r}(\mu, 0) \right), \] (30)

First we point out the special case $k(\mu) = -1$. Its peculiarity is the following: at the moment of time $\tau = 0$ all particles have the velocity equal to zero: $F_1(\mu) = 0$. The particles spread out to infinity from the point $r(\mu, 0) = 1$.

We will study now the falling solution more detail. All particles start from the initial position $r(\mu, 0) = 1$. In accordance with the solution (24) the velocity of the particle $\frac{\partial r(\mu, \tau)}{\partial \tau}$ is equal to zero at time
\[ \bar{\tau}(\mu) = \tau_0 F_1(\mu). \] (31)
The distance from the centre to the particle at this moment is

$$r(\mu, \bar{\tau}) = |k(\mu)|^{1/3}. \quad (32)$$

There are no particles which are able to fall onto the centre $r = 0$ Because in this case $|k(\mu)| \neq 0$. So, all particles with Lagrangian coordinates from the set

$$0 < \dot{\omega}_0^2(\mu) < \frac{4}{3} \Lambda \quad (33)$$

fall to the centre at the period of time $0 < \tau < \bar{\tau}(\mu)$, have the zero velocity at time $\tau = \bar{\tau}$ and go to the infinity after $\bar{\tau}$. Because of this Case 1 named "stripping of the shell".

Case 2: $k(\mu) = 0$. The equation (10) reads:

$$\frac{dr}{d\tau} = \pm \sqrt{\frac{\Lambda}{3}} r = \pm \frac{2}{3} \frac{\tau}{\tau_0} \quad (34)$$

and has the solution

$$r(\mu, \tau) = e^{\pm \frac{2}{3} \frac{\tau}{\tau_0}}, \quad F_2(\mu) = 0. \quad (35)$$

The expanded solution has no peculiarity. The falling solution shows that the particles with Lagrangian coordinate

$$\dot{\omega}_0^2(\mu) = \frac{4}{3} \Lambda \quad (36)$$

come to the centre for infinite time with zero velocity. Because of this Case 2 named "collapse".

This case coincides with the limit (22) of the equation (40).

Case 3: $\text{sign } k(\mu) = 1$. The equation (26) reads:

$$\frac{1}{\sqrt{k(\mu)}} \int \frac{d\left(r^{3/2}\right)}{\sqrt{\frac{1}{k(\mu)} r^3 + 1}} = F_3(\mu) \pm \tau, \quad (37)$$

and after calculation of the integral has the form

$$r^{3/2}(\mu, \tau) = \sqrt{k(\mu)} sh \left( F_3(\mu) \pm \frac{\tau}{\tau_0} \right), \quad (38)$$

where

$$F_3(\mu) = Arsh \frac{1}{\sqrt{k(\mu)}}. \quad (39)$$

In accordance with the solution (38) the velocity $\frac{dr(\mu, \tau)}{d\tau}$ of the particle with Lagrangian coordinates satisfied the condition

$$\dot{\omega}_0^2(\mu) > \frac{4}{3} \Lambda \quad (40)$$

is never equal to zero and these particles reach the centre at the moment of time

$$\bar{\tau}(\mu) = \tau_0 F_3(\mu). \quad (41)$$

Because of this Case 3 case named "accretion".

We will study now the condition of application of the Bonnor’s solution (20) which takes a place in the limit (22). First, we rewrite the solution (27) in the form:

$$r^{3/2}(\mu, \tau) = ch \frac{\tau}{\tau_0} \pm \sqrt{k_3(\mu) + 1} sh \frac{\tau}{\tau_0}. \quad (42)$$

We will transform this solution, taking in ... the fact that

$$k_3(\mu) \gg 1. \quad (43)$$
It is follows from (42) in this case

\[ \text{Arsh} \left( \frac{1}{\sqrt{k_3(\mu)}} \right) = F_3(\mu) = \frac{\bar{\tau}(\mu)}{\tau_0} < 1. \]  

(44)

For

\[ \tau < \bar{\tau} < \tau_0 \]  

(45)

it is follows from (42) - (44) that

\[ r^{3/2}(\mu, \tau) \approx 1 \pm \sqrt{k_3(\mu)} \frac{\tau}{\tau_0}, \]  

(46)

what equivalent to the equation (20). All particles with Lagrangian coordinates satisfied the condition (43), form the core with equation of motion containing the Bonnor's solution as the main part. So, the Case 3 contains the Bonnor's core and nonbonnor's part.

4 The Asymptotic Propertys of the Solution

In this section the density law (9) and the Hubble function (8) will be studied for the expanding solution with in the limit

\[ \tau(\mu) \to +\infty. \]  

(47)

Let us use the expanding solution of the Case 3. In accordance with the definition of Hubble function (8) we calculate first \( R'(\mu, \tau) \):

\[ R'(\mu, \tau) = \left( \frac{R'_0 k + \frac{1}{3} R_0 k'}{k^{3/3} \text{sh}^{1/3} \alpha} \right) \text{sh} \alpha + \frac{2}{3} R_0 k F' \text{ch} \alpha \]  

(48)

where

\[ \alpha = F_1(\mu) + \frac{\bar{\tau}}{\tau_0} \]  

(49)

In the limit of big time the Hubble function is

\[ h(\mu, \tau) = \pm \frac{2}{3} \frac{\tau}{\tau_0} \]  

(50)

and does not depend on time and initial conditions, but depends only on \( \Lambda \). The density

\[ 4\pi \delta(\mu, \tau) = \frac{1}{R^2 R'} \]  

(51)

in the limit (47) depends on time as well:

\[ \delta(\mu, \tau \to \infty) = \Phi(\mu) e^{-2\bar{\tau}/\tau_0}, \]  

(52)

where the function \( \Phi(\mu) \) represents the initial conditions.

5 The Structure of the Solution for Alternating Sign Function \( k(\mu) \)

The solutions has been built in sections (2) - (4) describe the three different "pure" cases of the evolution of the initial profile of the density. These cases are differ by the form of solution but have the common propriety: the sign of the function \( k(\mu) \) does not depend on \( \mu \). The general case of the initial conditions (2) and (3) supposes the function \( k(\mu) \) with alternating sign. This case will be studied here.
The main idea of building the mixed solution is to satisfy the condition of the nonintersection of layers of the particles (Lagrangian coordinate is not dependent on time). In accordance with this idea the mixed falling solution has the following structure. The central part of the gaze is described by the solution "Case 3", where the particles fall during the time $(41)$ with nonzero velocity. The shell falls to the centre during the time $(31)$, stops at time $\bar{\tau}$ and then goes to the infinity (Case 1). Case 1 and Case 3 are bounded by the case 2. After this qualitative analysis the building of the mixed solution is reduced to the study of the condition of nonintersection of the layers of particles.

First of all we note that it is follows from the formulas (33), (36) and (41) that the function $\dot{\omega}_0^2(\mu)$ should monotonically decrease. Let us write the "pure" solutions in the form

$$R_i(\mu, \tau) = (R_0)_i(\mu) r_i(\mu, \tau),$$

where the index $i$ points the number of the "pure" solution: Case 1, Case 2 or Case 3. The condition of the nonintersection has the following form: if at the moment of time $\tau = 0$ the Euler coordinates of three particles satisfied the inequality

$$R_{01}(\mu) < R_{02}(\mu) < R_{03}(\mu),$$

then this inequality holds for $\tau > 0$:

$$R_{01}(\mu) < R_{02}(\mu) < R_{03}(\mu).$$

We obtain the following inequality expressing the conditions of the nonintersection:

$$R_{03}(\mu) |k_3(\mu)|^{1/3} sh^{2/3} \left( F_3(\mu) \pm \frac{\tau}{70} \right) < R_{02}(\mu) e^{\frac{4 + \dot{\omega}_0}{5}},$$

$$R_{01}(\mu) k_1^{1/3}(\mu) ch^{2/3} \left( F_1(\mu) \pm \frac{\tau}{70} \right).$$

In the limit $\tau \to +\infty$ this inequality has the form

$$R_{03}(\mu) |k_3(\mu)|^{1/3} e^{\frac{4 + \dot{\omega}_0}{5}} < 2^{2/3} R_0^2(\mu) < R_{01}(\mu) k_1^{1/3}(\mu) e^{\frac{4 + \dot{\omega}_0}{5}}$$

and does not depend on time, depending only on the initial conditions. We emphasize here that all functions $R_{01}(\mu), R_{01}(\mu), R_{01}(\mu), k_1(\mu)$ and $k_3(\mu)$ are the fragments of the continuous functions $R(\mu)$ and $k(\mu)$.

6 Discation and Conclusion

This article studies the exact solution in the LTB model, produced by the initial conditions (2) - (3). It is shown that:

- the problem under consideration is defined for initial conditions (2) satisfied the restriction (16);
- in the limit of small time the falling solution is equal in the main part the Bonnor solution [2] and defined by the initial conditions;
- in the limit of big time the solution 'forgets' the initial conditions and has the asymptotic form (23);
- in the limit of $\Lambda \to 0$ the input equation is reduced to the equation, has been studied in [4] where it is shown the following properties of this solution: when $\Lambda = 0$ and $\dot{\omega}_0^2(\mu) = 0$ the input equation is equal to FRW model; when $\dot{\omega}_0^2(\mu) \neq 0$ the FRW model is the main part of the solution in the limit of big time;
• the input equation is strongly depends on the sign of the function \( k(\mu) \). There are three solutions defined by the request of constant sign of the function \( k(\mu) \): "stipping of the shell" for \(-1 < k(\mu) < 0\), "collapse" for \( \text{sign} k(\mu) = 0 \) and "core" or "accretion" for \( \text{sign} k(\mu) > 0 \);

• the structure of the mixed solution defined by the request of the nonintersecting of layers of particles and present in the figure 2;

• in the limit of big time the Hubble function is constant but the density is dependent on the time as \( \exp \left( -2\frac{\tau}{\tau_0} \right) \);

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References

[1] R.C. Tolman, Proc. Nat. Acad. Sci. (Wash), 20, (1934).

[2] W.B. Bonnor, MNRAS 159, 261 - 268 (1972)

[3] A. Gromov, gr-qc/9606068

[4] A. Gromov, gr-qc/9606071

[5] L.D. Landau, E.M. Lifshits, "The Field Theory", Moscow, "Nauka", (1973).

[6] L.E. Gurevich and A.D. Chernin, "The Introduction into Cosmology", Moscow, "Mir", (1978).

[7] A. Krasinski, "Physics in an inhomogeneous Universe" (a revie), Warszawa, 1993.

[8] H.B. Dwight, "Tables of Integrals", NY, (1961).

[9] G.A. Corn, T.M. Corn, "Mathematical Handbook", McGraw-Hill Company, NY, (1968).