LOCAL WELL-POSEDNESS FOR THE QUADRATIC SCHRÖDINGER EQUATION IN TWO-DIMENSIONAL COMPACT MANIFOLDS WITH BOUNDARY

MARCÉL NOGUEIRA AND MAHENDRA PANTHEE

ABSTRACT. We consider the quadratic NLS posed on a bidimensional compact Riemannian manifold \((M, g)\) with \(\partial M \neq \emptyset\). Using bilinear and gradient bilinear Strichartz estimates for Schrödinger operators in two-dimensional compact manifolds proved by J. Jiang in [10] we deduce a new evolution bilinear estimates. Consequently, using Bourgain’s spaces, we obtain a local well-posedness result for given data \(u_0 \in H^s(M)\) whenever \(s > \frac{2}{3}\) in such manifolds.

1. INTRODUCTION

Let \((M, g)\) be a compact Riemannian manifold with boundary of dimension 2. Denote by \(\Delta_g\) the Beltrami-Laplace operator with respect to metric \(g\) on \(M\). We consider the Dirichlet or Neumann problem for the quadratic Schrödinger equation on \(M\),

\[
\begin{cases}
  i\partial_t u + \Delta_g u = \alpha u^2 + \beta \overline{u}^2 + \gamma |u|^2, & \text{in } [0, \infty) \times (M \setminus \partial M) \\
  u(0, x) = u_0(x), & \text{on } \partial M, \\
  Bu(t, x) = 0, & \text{on } \partial M,
\end{cases}
\]

(1.1)

where \(u = u(t, x)\) is a complex function, \(\alpha, \beta, \gamma \in \mathbb{C}\) are complex constants, \(B\) is the boundary operator, either \(Bf = f \mid_{\partial M}\) in the Dirichlet case or \(Bf = \partial_{\nu} f \mid_{\partial M}\) in the Neumann case, with \(\partial_{\nu}\) denoting the normal derivative\(^1\) along the boundary \(\partial M\).

The study of well-posedness issues to initial value problem (IVP) associated with the nonlinear Schrödinger (NLS) equation with quadratic nonlinearities has attracted attention of several mathematicians in the last decades (see for example [2, 14, 15] and references therein). In these works such issues are addressed considering Euclidean spatial domains. Very few is known when the problem is posed on general manifolds. In this work, we are interested in addressing the well-posedness issues for the IVP (1.1) when \(M\) is a compact Riemannian manifold with boundary.

As in the Euclidean case, Strichartz’s type inequalities play a vital role while dealing with the well-posedness issues for given data with low Sobolev regularity. In the case of general manifold global

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\(^1\)We have \(\partial_{\nu} u = \frac{\partial u}{\partial v} = (\nabla u, \nu)\) (Normal component of \(\nabla u\)), where \(\nu\) is the unit outward-pointing normal to \(\partial M\).
in time Strichartz’s type inequalities are not available. In a pioneer work [6], Bourgain obtained local in time Strichartz’s type estimate for the NLS equation posed on a standard flat 2-torus \( \mathbb{T}^2 \). In recent time, much attention has been drawn to find such estimates with loss of derivative for solutions of the linear Schrödinger equation

\[
i\partial_t u + \Delta_x u = 0; \quad u(x, 0) = f(x),
\]

posed on general manifolds of dimension \( d \geq 2 \). In such spaces, these estimates are given by

\[
\|e^{it\Delta} f\|_{L^p(I; L^q(M))} \lesssim C(d, p, q, \Gamma) \|f\|_{H^{\ell}(M)},
\]

for some \( 0 \leq \ell < \frac{d}{2} \) where \((p, q)\) is a \(d\)-admissible pair, i.e., \( 2/p + d/q = d/2 \) with \( q < \infty \) and \( I \subset \mathbb{R} \) is a finite interval. The number \( \ell \) depends on the geometry of \( M \) and is called the loss of regularity index. In the flat case, \( M = (\mathbb{R}^d, \delta_{ij}) \) we have \( \ell = 0 \) and one can take \( I = \mathbb{R} \). For a complete discussion we refer the readers to [16, 18] and references therein.

For a pioneer work concerning the Strichartz’s type estimate considering non-Euclidean geometries, we refer to [7], where a local in time version of (1.3) with loss of derivative (\( \ell = 1/p \)) for the solution of (1.2) posed on compact manifold without boundary was derived. Such estimates for the NLS equation posed on manifolds with boundary can be found in [4] and [5].

A powerful refinement of the estimate (1.3) is known as bilinear Strichartz’s type estimate

\[
\left( \int_{[0, 1] \times M} |e^{it\Delta} f(x) e^{it\Delta} h(x)|^2 dt dx \right)^{\frac{1}{2}} \lesssim C(\min(\Gamma, \Lambda)) \|f\|_{L^2} \|h\|_{L^2},
\]

that hold for \( s > s_0(M) \) and spectrally localized \( f, h \) in dyadic intervals of order \( \Gamma, \Lambda \) respectively, i.e.,

\[
1_{\varLambda \leq \sqrt{-\Delta} \leq 2\Lambda}(f) = f, \quad 1_{\Gamma \leq \sqrt{-\Delta} \leq 2\Gamma}(h) = h.
\]

The claim that the bilinear version (1.4) is a refinement of (1.3) can be justified with the help of the following remark. If \( d = 2 \), then (4, 4) is a \(2\)-admissible pair. Considering \( h = f \) in (1.4), we obtain

\[
\|e^{it\Delta} f\|_{L^4([0, 1] \times M)} \lesssim \Lambda \|f\|_{L^4(M)},
\]

and consequently

\[
\|e^{it\Delta} f\|_{L^4([0, 1] \times M)} \lesssim \|f\|_{H^{s/2}(M)}.
\]

The estimates (1.4) has proven to be one of the most important tools to obtain the local well-posedness results. More precisely, taking \( M = S^2 \) endowed with its standard metric, using the precise knowledge of its spectrum \( \{\lambda_k = k(k+1)\} \) for \( k \in \mathbb{N} \) and estimates about spectral projectors of the form

\[
\chi_{\lambda} f = \sum_k \chi(\lambda_k - \lambda) P_k f,
\]

acting on functions \( f \) over \( M \), where \( \chi \in C_0^\infty(\mathbb{R}) \), Burq, Gérard and Tzvetkov in [9], showed that (1.4) is true for every \( s > s_0(M) : = \frac{1}{4} \). The authors in [9] also proved the validity of (1.4) for bidimensional Zoll manifolds in same range of \( s \).
In the case of manifolds with boundary, \( \partial M \neq \emptyset \), we do not have the precise knowledge of the eigenvalues as in the cases of flat torus and sphere, where we know eigenvalues of the Laplacean precisely. In these cases it is possible to use the arithmetic property of these eigenvalues. For general manifolds with boundary, our poor knowledge of spectrum does not allow us to use the same technique.

Recently, Anton [1] considered manifolds with boundary, \( M = \mathbb{B}^3 \) (the three dimensional ball) and proved (1.4) and the following estimate involving gradient

\[
\| (\nabla_x (e^{it\Delta} f)) e^{it\Delta} h \|_{L^2([0,1] \times M)} \leq C \Lambda \left( \min(\Lambda, \Gamma) \right) s \| f \|_{L^2(M)} \| h \|_{L^2(M)}
\]

for \( s_0(\mathbb{B}^3) = \frac{1}{2} \). Using these bilinear estimates, the authors in [1], [9], obtained local and well-posedness results for the nonlinear cubic Schrödinger equations for initial data in \( H^s(M) \) for \( s > s_0(M) \) on such manifolds. Observe that the author in [1] proved the local well-posedness result for the cubic nonlinear Schrödinger equation with Dirichlet boundary condition and radial data in \( H^s \) for every \( s > \frac{1}{2} \). Later, Jiang in [10] considered two dimensional compact manifold with boundary and showed validity of the estimates (1.4) and (1.5) for \( s > s_0(M^2) = \frac{2}{3} \) and consequently obtained local well-posedness of the cubic NLS in \( H^s(M) \), \( s > \frac{2}{3} \).

In this work, we use the techniques used in [1] and [10] to get crucial bilinear estimates corresponding to the quadratic NLS (1.1) posed on a two dimensional compact manifold with boundary and prove the following local well-posedness result.

**Theorem 1.** Let \((M, g)\) be a two dimensional compact manifold with boundary. For any \( u_0 \in H^s(M) \), with \( s > s_0 := \frac{2}{3} \), there exist \( T = T(\| u_0 \|_{H^s(M)}) > 0 \) and a unique solution \( u(t) \) of the initial value problem (1.1), on the time interval \([0, T]\), such that

(i) \( u \in X^{s,b}(M) \);

(ii) \( u \in C([0, T], H^s(M)) \);

for suitable \( b \) close to \( \frac{1}{2} + \). Moreover the map \( u_0 \mapsto u(t) \) is locally Lipschitz from \( H^s(M) \) into \( C([0, T], H^s(M)) \).

Having proved the local well-posedness of the IVP (1.1) in Theorem 1, a natural question to ask is about the global well-posedness. Generally, conserved quantities play a vital role to answer such question. Recall that, one of the important properties of solutions of the nonlinear Schrödinger equations with nonlinearity of the form \( N_p(u) := |u|^p - u (p > 1) \) is that mass and energy of the solutions are conserved (at least for smooth solutions) by the flow. However, in the case of (1.1), by multiplying the equation by \( \overline{u} \), integrating and taking the imaginary part, one can easily conclude that the mass is conserved during evolution of system (1.1) if

\[
\text{Im}(\alpha |u|^2 \overline{u} + |u|^2 \overline{u} + \gamma |u|^2 \overline{u}) = 0.
\]

(1.6)

For instance, the condition (1.6) is satisfied if we take \( \alpha = \gamma \in \mathbb{R} \) and \( \beta = 0 \). On the other hand, it is much more difficult to obtain a condition for energy conservation. In fact, by multiplying the equation
by \(\partial_t u\) integrating and taking the real parts we obtain that
\[
- \int_M \partial_t(|\nabla u|^2) dx = \int_M 2\text{Re}[\overline{\partial_t u}(\alpha u^2 + \beta \overline{u}^2 + \gamma |u|^2)] dx. \tag{1.7}
\]
Looking at the RHS of (1.7), we see that it is nontrivial to rewrite it as a derivative of a function involving \(u\). This is one of the main differences between the equation (1.1) and the NLS equation with nonlinearity \(N_p(u)\). This fact constitutes an obstacle in the development of a global well-posedness theory for the equation under investigation in this work. However, in some cases where one has non-trivial perturbations of the NLS equation, it is possible to obtain an \textit{a priori} estimate which leads to global well-posedness results (see for instance [17]).

Before leaving this section, we record some notations that will be used throughout this work. We write \(A \sim B\) if there are constants \(c_1, c_2 > 0\) such that \(A \leq c_1 B\) and \(B \leq c_2 A\). Throughout this work we will denote dyadic numbers \(2^m\) for \(m \in \mathbb{N}\) by capital letters, e.g. \(N = 2^n, L = 2^l, \ldots\). The letter \(C\) will be used to denote a positive constant that may vary from line to line. We use \(\| \cdot \|_{L^p}\) to denote \(L^p(M)\) norm and \((\cdot, \cdot)_{L^2}\) to denote the inner product in \(L^2(M)\). Moreover, we denote by \(\langle x \rangle := \sqrt{1+x^2}\).

2. \textbf{Function Spaces and Preliminary Results}

2.1. \textbf{Spectral properties of the Laplace-Beltrami operator.} Consider the following eigenvalues problems when \(M\) is compact:

\begin{enumerate}[-]
    \item Closed problem \(-\Delta_g f = \lambda f\) in \(M\); \(\partial M = \emptyset\);
    \item Dirichlet problem \(-\Delta_g f = \lambda f\) in \(M\); \(f|_{\partial M} = 0\);
    \item Neumann problem \(-\Delta_g f = \lambda f\) in \(M\); \(\partial_v f|_{\partial M} = 0\).
\end{enumerate}

In our case, we will deal with (ii) and (iii). We have the following standard result about the spectrum.

\textbf{Theorem 2.} Let \(M\) be a compact manifold with boundary \(\partial M\) (eventually empty), and consider one of the above mentioned eigenvalue problems. Then:

\begin{enumerate}[-]
    \item The set of eigenvalue consists of an infinite sequence\footnote{Sometimes we denote the eigenvalues by \(\lambda_l = \mu_l^2\).} \(\lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_l \leq \lambda_{l+1} \leq \ldots \to +\infty\),
    \item Each eigenvalue has finite multiplicity and the eigenspaces \(\mathcal{E}_j := \{u \mid -\Delta_g u = \lambda_j u\}\) corresponding to distinct eigenvalues are \(L^2(M)\)-orthogonal;
    \item The direct sum of the eigenspaces \(\mathcal{E}_j\) is dense in \(L^2(M)\) for the \(L^2\)-norm topology. Furthermore, each eigenfunction is \(C^\infty\)-smooth and analytic.
\end{enumerate}

\textbf{Proof.} See [3], p.53. \qed
From now on, we will list the eigenvalues of the problems (ii) and (iii) as

\[(0 \leq) \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \to +\infty,\]

with each eigenvalue repeat a number of times equal to its multiplicity. The third assertion in Theorem 2 shows that the sequence \( \{e_j\}_{j \in \mathbb{N}} \) is an orthonormal basis of \( L^2(M) \). For any \( f \in L^2(M) \), one can write \( f = \sum_j (f \mid e_j)_{L^2} e_j \) in \( L^2 \)-sense. We finish this subsection by introducing a spectral projector operator. For a dyadic number \( k \), we use \( \mathbf{1}_{\Lambda \leq \sqrt{-\Delta} \leq 2\Lambda} \) to denote the spectral projector

\[
\sum_{\Lambda \leq \mu_j \leq 2\Lambda} P_j f = \sum_{\Lambda \leq \mu_j \leq 2\Lambda} e_j \int_M f \hat{e_j} dx,
\]

where \( \mu_j^2 = \lambda_j \) are eigenvalues corresponding to eigenvectors \( e_j \) of \( -\Delta = -\Delta_g \).

### 2.2. Function spaces.

**Definition 3.** Let \((M,g)\) be a compact Riemannian manifold, and consider the Laplace-Beltrami operator \(-\Delta := -\Delta_g\) on \(M\). Let \( (e_k) \) be an \( L^2 \) orthonormal basis formed by the eigenfunctions of \(-\Delta\), with eigenvalues \( \lambda_k := \mu_k^2 \). Let \( P_k \) be the orthogonal projector along \( e_k \). For \( s \geq 0 \) we define the natural Sobolev space generated by \((1 - \Delta)^{s/2} H^s(M)\), equipped with the following norm

\[
\|u\|_{H^s(M)}^2 := \| (1 - \Delta)^{s/2} u \|_{L^2(M)}^2 = \sum_k \langle \mu_k \rangle^s \|P_k u\|_{L^2(M)}^2.
\]  

(2.1)

We define the Hilbert spaces \( X^{s,b}(\mathbb{R} \times M) \) as the completion of \( C_0^\infty(\mathbb{R} \times M) \) with respect to the norm

\[
\|u\|_{X^{s,b}(\mathbb{R} \times M)}^2 = \sum_k \| \tau + \mu_k \|^b \langle \mu_k \rangle^{s/2} \| \hat{P_k} u(\tau) \|_{L^2(\mathbb{R}, L^2(M))}^2 = \| \langle -\tau \rangle u(t,\cdot) \|_{H^b(\mathbb{R}; L^2(M))}^2,
\]  

(2.2)

where \( \hat{P_k} u(\tau) \) denotes the Fourier transform of \( P_k u \) with respect to the time variable.

**Proposition 4.** The following properties are valid

(i) For \( s_1 \leq s_2 \) and \( b_1 \leq b_2 \), one has \( X^{s_2,b_2}(\mathbb{R} \times M) \hookrightarrow X^{s_1,b_1}(\mathbb{R} \times M) \).

(ii) \( X^{0,\frac{1}{2}}(\mathbb{R} \times M) \hookrightarrow L^3(\mathbb{R}, L^2(M)) \).

(iii) If \( b > \frac{1}{2} \), then the inclusion \( X^{s,b}(\mathbb{R} \times M) \hookrightarrow C(\mathbb{R}, H^s(M)) \) holds.

**Proof.** The part (i) follows directly from the definition of the \( X^{s,b} \)-norm in 2.2. The part (ii) follows from the fact that \( u \in X^{s,b}(\mathbb{R} \times M) \), if and only if, \( S(-\tau) u(t,\cdot) \in H^b(\mathbb{R}, H^s(M)) \) and from the immersion \( H^{1/6}(\mathbb{R}) \hookrightarrow L^3(\mathbb{R}) \). The proof of (iii), is also a consequence of (2.2). \( \square \)

In order to use a contraction mapping argument to obtain local existence, we need to define local in time version of \( X^{s,b} \).

\footnote{For \( s < 0 \) we define \( H^s(M) \) as the closure of \( L^2(M) \) under the norm (2.1).}
Definition 5. For every compact interval \( I \subset \mathbb{R} \), we define the restriction space \( X^{s,b}(I \times M) \) as the space of functions \( u \) on \( I \times M \) that admit extensions to \( \mathbb{R} \times M \) in \( X^{s,b}(\mathbb{R} \times M) \). The space \( X^{s,b}(I \times M) \) is equipped with the restriction norm

\[
\|u\|_{X^{s,b}(I \times M)} = \inf_{w \in X^{s,b}(\mathbb{R} \times M)} \{ \|w\|_{X^{s,b}(\mathbb{R} \times M)} \mid w = u \text{ on } I \}.
\]

Another property we are going to use frequently refers to the dyadic decompositions and their relation to the norm of the \( X^{s,b} \) spaces. More explicitly, considering \( u \), we can decompose with respect to the space variables as

\[ u = \sum_{N} u_N = \sum_{N} 1_{\sqrt{-\Delta} \sim N}(u) \]

where \( N \) denotes the sequence of dyadic integers. Using the definition of the operator \( 1_{\sqrt{-\Delta} \sim N} \) we can establish the norm equivalence relation

\[
\|u\|_{X^{s,b}}^2 \sim \sum_{N} N^{2s} \|u_N\|_{X^{0,b}}^2 \sim \sum_{N} \|u_N\|_{X^{s,b}}^2.
\]  

(2.3)

In an analogous manner, we can decompose \( u \) with respect to the “time-space” variable

\[ u = \sum_{L} u_L = \sum_{L} 1_{(\tau + \mu_k) \sim L}(u) \]

where \( L \) denotes the sequence of dyadic integers. Also, using the definition of the operator \( 1_{(\tau + \mu_k) \sim L} \) we can establish the following norm equivalence

\[
\|u\|_{X^{0,b}}^2 \sim \sum_{L} L^{2b} \|u_L\|_{L^2(\mathbb{R} \times M)}^2 \sim \sum_{L} \|u_L\|_{X^{0,b}}^2.
\]  

(2.4)

2.2.1. Linear estimates in the function spaces.

Proposition 6. (Linear estimates in the \( X^{s,b} \) spaces). Let \( b, s > 0 \) and \( u_0 \in H^s(M) \). Then

(i)

\[
\|S(t)u_0\|_{X^{s,b}(\mathbb{R} \times M)} \lesssim T^{1-b} \|u_0\|_{H^s(M)}
\]  

(2.5)

(ii) Let \( 0 < b' < \frac{1}{2} \) and \( 0 < b < 1 - b' \). Then for all \( F \in X^{s,-b'}_T(M) \),

\[
\left\| \int_0^t S(t-t')F(t')dt' \right\|_{X^{s,b}(\mathbb{R} \times M)} \lesssim T^{1-b-b'} \|F\|_{X^{s,-b'}_T(\mathbb{R} \times M)}.
\]  

(2.6)

provided \( 0 < T \leq 1 \).

Proof. For the proof of this proposition we refer to [8], [10] or [12]. \( \square \)

From (2.5) we know that \( \|S(t)u_0\|_{X^{s,b}_1(\mathbb{R} \times M)} \leq C \|u_0\|_{H^s(M)} \), for some \( C > 0 \). From the definition of \( X^{s,b}_T \) spaces we know that \( T_1 < T_2 \) implies \( X^{s,b}_{T_1} \subset X^{s,b}_{T_2} \). Therefore for \( T \leq 1 \),

\[
\|S(t)u_0\|_{X^{s,b}_T(\mathbb{R} \times M)} \leq C \|u_0\|_{H^s(M)}.
\]  

(2.7)
2.3. **Bilinear Strichartz estimates and applications.** In this subsection we record some estimates obtained in [10] while working on the well-posedness of the cubic NLS equation

\[ i \partial_t u + \Delta_g u = 0; \quad u(x, 0) = f(x), \]

posed on bi-dimensional compact manifolds with boundary. We start with the following lemma.

**Lemma 7.** Let \((M, g)\) be a two dimensional compact manifold with boundary. If for any \(f, h \in L^2(M)\) we have

\[ 1 \leq \sqrt{-\Delta} \leq 2 \Lambda (f) = f, \quad 1 \leq \sqrt{-\Delta} \leq 2 \Gamma (h) = h, \]

where \(\Lambda\) and \(\Gamma\) are dyadic integers, then for any \(s > s_0 = \frac{2}{3}\), there exists \(C > 0\) such that

\[ \| e^{it\Delta} f e^{it\Delta} h \|_{L^2([0,1] \times M)} \leq C (\min(\Gamma, \Lambda))^{s} \| f \|_{L^2(M)} \| h \|_{L^2(M)}. \]

**Proof.** See [10], p.85. 

**Lemma 8.** Let \(s > s_0 = \frac{2}{3}\), and \(\Gamma, \Lambda\) be dyadic integers. The following statements are equivalent:

1. For any \(f, h \in L^2(M)\) satisfying

\[ 1 \leq \sqrt{-\Delta} \leq 2 \Lambda (f) = f, \quad 1 \leq \sqrt{-\Delta} \leq 2 \Gamma (h) = h \]

one has

\[ \| e^{it\Delta} f e^{it\Delta} h \|_{L^2([0,1] \times M)} \leq C (\min(\Gamma, \Lambda))^{s} \| f \|_{L^2(M)} \| h \|_{L^2(M)}. \]

2. For any \(b > \frac{1}{2}\) and any \(f, h \in X^0, b(\mathbb{R} \times M)\) satisfying

\[ 1 \leq \sqrt{-\Delta} \leq 2 \Lambda (f) = f, \quad 1 \leq \sqrt{-\Delta} \leq 2 \Gamma (h) = h \]

one has

\[ \| fh \|_{L^2(\mathbb{R} \times M)} \leq C (\min(\Gamma, \Lambda))^{s} \| f \|_{X^0, b(\mathbb{R} \times M)} \| h \|_{X^0, b(\mathbb{R} \times M)}. \]

**Proof.** See [10], p.99. 

**Lemma 9.** Let \((M, g)\) be a two dimensional compact manifold with boundary. If for any \(f, h \in L^2(M)\) we have

\[ 1 \leq \sqrt{-\Delta} \leq 2 \Lambda (f) = f, \quad 1 \leq \sqrt{-\Delta} \leq 2 \Gamma (h) = h. \]

Then for any \(s > s_0 = \frac{2}{3}\), there exists \(C > 0\) such that

\[ \| (\nabla_x (e^{it\Delta} f)) e^{it\Delta} h \|_{L^2([0,1] \times M)} \leq C \Lambda (\min(\Lambda, \Gamma))^{s} \| f \|_{L^2(M)} \| h \|_{L^2(M)}. \]

**Proof.** See [10], p.86. 

**Lemma 10.** Let \(s > s_0 = \frac{2}{3}\), and \(\Gamma, \Lambda\) be dyadic integers. The following statements are equivalent:
(1) For any \( f, h \in L^2(M) \) satisfying
\[
I_{\lambda \leq \sqrt{-\Lambda} \leq 2\lambda}(f) = f, \quad I_{\Gamma \leq \sqrt{-\Delta} \leq 2\Gamma}(h) = h
\]
one has
\[
\| (\nabla_x e^{i\Delta} f) e^{i\Delta} h \|_{L^2([0,1] \times M)} \leq C\Lambda (\min(\Lambda, \Gamma))^s \| f \|_{L^2(M)} \| h \|_{L^2(M)}.
\]
(2) Let \( b > \frac{1}{2} \). Then, for any \( f, h \in X^{0, b}([R \times M) \) satisfying
\[
I_{\lambda \leq \sqrt{-\Lambda} \leq 2\lambda}(f) = f, \quad I_{\Gamma \leq \sqrt{-\Delta} \leq 2\Gamma}(h) = h
\]
one has
\[
\| (\nabla_x f) h \|_{L^2([R \times M)} \leq C\Lambda (\min(\Lambda, \Gamma))^s \| f \|_{X^{0, b}([R \times M)} \| h \|_{X^{0, b}([R \times M)}.
\]

Proof. See [10], p.100. \( \square \)

3. BILINEAR ESTIMATES

It is well known that in the framework of Bourgain’s spaces the local well-posedness results can usually be reduced to the proof of adequate \( k \)-linear estimates. In our case, due to the non-linear structure of (1.1) it is necessary to prove the bilinear estimates. These bilinear estimates are crucial to perform the contraction argument as we shall see in the next section.

In this section, we use the linear and the nonlinear estimates stated in the previous section to prove the following crucial bilinear estimates.

**Proposition 11.** Let \( s_0 < s \), \( b' = 1 - \frac{s}{2} \). Then there exists \( C > 0 \), such that the bilinear estimate
\[
\| u_1 u_2 \|_{X^{s, b'}} \leq C \| u_1 \|_{X^{s, b}} \| u_2 \|_{X^{s, b}}
\]
holds provided \( b = \frac{1}{2} + \varepsilon_1 \), for an adequate selection of the parameters \( 0 < \varepsilon_1 << 1 \).

**Proposition 12.** Let \( s_0 < s \). Then the bilinear inequality
\[
\| u_1 \overline{u_2} \|_{X^{s, -b}} \leq C \| u_1 \|_{X^{s, b}} \| u_2 \|_{X^{s, b}}
\]
holds true for some \( C > 0 \), provided \( b = \frac{1}{2} - \varepsilon, b_2 = \frac{1}{2} + \varepsilon_2 \), for an adequate selection of the parameters \( 0 < \varepsilon, \varepsilon_2 << 1 \).

Before providing proofs for Propositions [11] and [12], we record some auxiliary results. We begin with an interpolation lemma.

**Lemma 13.** For every \( s' > s_0 = \frac{\theta}{2} \), there exist \((b, b')\) satisfying respectively, \( 0 < b' < \frac{1}{2} < b \), \( b + b' < 1 \); and \( 0 < \varepsilon < 1 \), such that
\[
s' > \frac{3\theta}{2} + (s_0 + \delta)(1 - \theta),
\]
\[
b' > \frac{\theta}{6} + \frac{1}{2} + \varepsilon(1 - \theta).
\]
Proof. Let \( \frac{2}{3} < s' < 2 \) and write \( s' = s_0 + 2\delta \) for some \( (\delta > 0) \). Choose \( \theta \in (0, 1) \) such that

\[
\frac{3\theta}{2} + (s_0 + \delta)(1 - \theta) < s',
\]

then \( (\frac{5}{6} - \delta)\theta < s' - (s_0 + \delta) \), (any choice works if \( \frac{5}{6} \leq \delta \), otherwise \( \theta \) has to be close enough to 0).

Next, we choose \( \epsilon \) such that \( b' > \frac{\theta}{6} + (1 - \theta)b \) with \( b' = \frac{1}{2} - 2\epsilon, b = \frac{1}{2} + \epsilon \). That happens when

\[
(\frac{1}{2} - 2\epsilon) > \frac{\theta}{6} + (1 - \theta)(\frac{1}{2} + \epsilon) \iff \epsilon < \frac{\theta}{9 - 3\theta}.
\]

\[\square\]

To deal with dyadic summations, the following lemma proved in [8], page 282, will be very useful.

Lemma 14. For every \( \gamma > 0 \), every \( \theta > 0 \) there exists \( C > 0 \) such that if \( (c_N) \) and \( (d_N) \) are two sequences of non-negative numbers indexed by the dyadic integers, then,

\[
\sum_{N \leq \gamma N'} \left( \frac{N}{N'} \right)^{\theta} c_N d_N' \leq C \left( \sum_{N} c_N^2 \right)^{\frac{1}{2}} \left( \sum_{N'} d_N'^2 \right)^{\frac{1}{2}}.
\] (3.3)

Now, we provide proof of the bilinear estimate stated in Proposition 11.

Proof of Proposition 11. Using the duality relation between \( X^{s,-b} \) and \( (X^{s,-b})^* \approx X^{-s,b'} \) (see appendix for the proof of this fact) to prove (3.1), it suffices to establish the following inequality

\[
| \int_{\mathbb{R} \times M} u_1 u_2 \overline{u_0} dx dt | \leq C \| u_1 \|_{X^{s,b_1}} \| u_2 \|_{X^{s,b_2}} \| u_0 \|_{X^{-s,b'}}.
\] (3.4)

for all \( u_0 \in X^{-s,b'} \). We start inserting the dyadic decompositions on the spatial frequencies of

\[
u_j = \sum_j u_{jN_j}, \quad (j = 0, 1, 2)
\]
in the left hand side of (3.4), where

\[
u_{jN_j} := \sum_{k:N_j \leq k \leq 2N_j} P_k u.
\]

Related to this dyadic decomposition, we have the following equivalences of norms

\[
\| u_j \|_{X^{s,b}}^2 \sim \sum_{N_j} \| u_{jN_j} \|_{X^{s,b}}^2 \sim \sum_{N_j} N_j^{s+b} \| u_{jN_j} \|_{X^{0,b}}^2,
\] (3.5)

for \( j = 0, 1, 2 \), which will be very useful in the next steps of the proof.

Let

\[
I := \int_{\mathbb{R} \times M} u_1 u_2 \overline{u_0} dx dt.
\] (3.6)

Using the triangle inequality, we obtain

\[
| I | \leq \sum_{N_0,N_1,N_2} \left| \int_{\mathbb{R} \times M} u_{1N_1} u_{2N_2} \overline{u_{0N_0}} dx dt \right|.
\] (3.7)

Observe that the summation in (3.7) is over all triples of dyadic numbers \((N_0,N_1,N_2)\). To simplify the notation, we write \( N = (N_0,N_1,N_2) \) and
Under these considerations, we split the summation \( \sum_N |I(N)| \) in the following manner
\[
\sum_N |I(N)| \leq \sum_{N: N_2 \leq N_1} |I(N)| + \sum_{N: N_1 < N_2} |I(N)| =: \Sigma_1 + \Sigma_2.
\] (3.9)

Moreover, we split the summation \( \sum_{N: N_2 \leq N_1} |I(N)| \) in two frequency regimes
\[
\sum_{N: N_2 \leq N_1} |I(N)| \leq \sum_{N: N_2 \leq N_1, N_0 \leq CN_1} |I(N)| + \sum_{N: N_2 \leq N_1, N_0 > CN_1} |I(N)| =: \Sigma_3 + \Sigma_4.
\] (3.10)

Similarly, the summation \( \sum_{N: N_1 < N_2} |I(N)| \) is also split in two frequency regimes
\[
\sum_{N: N_1 < N_2} |I(N)| \leq \sum_{N: N_1 < N_2, N_0 \leq CN_2} |I(N)| + \sum_{N: N_1 < N_2, N_0 > CN_2} |I(N)| =: \Sigma_3 + \Sigma_4.
\] (3.11)

Therefore, combining (3.7), (3.9), (3.10) and (3.11) we arrive at
\[
|I| \leq \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4.
\] (3.12)

In what follows, we estimate \( \Sigma_1, \Sigma_2 \) and \( \Sigma_3, \Sigma_4 \) in the four frequency regimes that appear in (3.10) and (3.11) respectively. To simplify the notations further, in sequel, for \( j = 1, 2 \) we define \( u_{N_j}^{(j)} := u_{jN_j} \) and \( u_{N_0}^{(0)} := u_{0N_0} \).

First we will estimate \( \Sigma_1 \) and \( \Sigma_2 \) considering the frequency regimes with condition \( N_2 \leq N_1 \). Using symmetry, estimates for the terms \( \Sigma_3 \) and \( \Sigma_4 \) in the frequency regimes with condition \( N_2 > N_1 \) follow with analogous arguments. For this purpose, we need the following lemma.

**Lemma 15.** Let, \( N_2 \leq N_1 \) and \( I(N) \) be as defined in (3.3). If (2.10) and (2.11) hold for \( s > s_0 \), then for all \( s' > s_0 \) there exist \( 0 < b', b_1 < \frac{1}{2} \) and \( C > 0 \) such that the following estimates hold
\[
|I(N)| \leq CN_2^{s'} \|u_{N_0}^{(0)}\|_{X^{0,b_1}} \|u_{N_1}^{(1)}\|_{X^{0,b_1}} \|u_{N_2}^{(2)}\|_{X^{0,b_1}},
\] (3.13)
\[
|I(N)| \leq C \left( \frac{N_1}{N_0} \right)^2 N_2^{s'} \|u_{N_1}^{(1)}\|_{X^{0,b'}} \|u_{N_0}^{(0)}\|_{X^{0,b'}} \|u_{N_2}^{(2)}\|_{X^{0,b'}}.
\] (3.14)

**Proof.** Applying Hölder’s inequality in (3.8), we obtain
\[
|I(N)| \leq \int_{R \times M} \prod_{j=0}^2 |u_{N_j}^{(j)}| \, dx \, dt \leq \|u_{N_2}^{(2)}\|_{L^1(R \times L^2)} \|u_{N_0}^{(0)}\|_{L^1(R \times L^2)} \|u_{N_1}^{(1)}\|_{L^1(R \times L^2)}.
\] (3.15)

Using the Sobolev embedding \( H^{s'}(M) \rightarrow L^{s''}(M) \), (3.15) yields
\[
|I(N)| \leq CN_2^s \prod_{j=0}^2 \|u_{N_j}^{(j)}\|_{L^1(R \times L^2)},
\] (3.16)

Now, using the property (ii) of Proposition \( \[ \) we obtain
\[
|I(N)| \leq C N_2^{3/2} \prod_{j=0}^2 \|u_{N_j}^{(j)}\|_{X^{0,b'}}.
\] (3.17)
On the other hand, applying the Cauchy-Schwarz inequality in (3.8), we obtain

$$|I(N)| \leq \|u^{(0)}_{N_0}u^{(2)}_{N_2}\|_{L^2(\mathbb{R} \times M)}\|u^{(1)}_{N_1}\|_{L^2(\mathbb{R} \times M)}.$$  (3.18)

Now, applying the estimate (2.10) in Lemma 8 in the estimate (3.18), we find that for $b > \frac{1}{2}$,

$$|I(N)| \leq C(\min(N_0,N_2))^{s_0+\delta}\|u^{(0)}_{N_0}\|_{X^{0,b}(\mathbb{R} \times M)}\|u^{(2)}_{N_2}\|_{X^{0,b}(\mathbb{R} \times M)}\|u^{(1)}_{N_1}\|_{L^2(\mathbb{R} \times M)},$$  (3.19)

where $s_0 = \frac{2}{3}$ and $0 < \delta \ll 1$.

Hence, from (3.19) we obtain

$$|I(N)| \leq CN_2^{s_0+\delta}\prod_{j=0}^{2}\|u^{(j)}_{N_j}\|_{X^{0,b}(\mathbb{R} \times M)}.$$  (3.20)

Now, we further decompose each $u^{(j)}_{N_j}$ (for $j = 0, 1, 2$) according to the sum of $u^{(j)}_{N_j}$ localized in each spacetime frequency interval of the form $L_j \leq (\tau + \mu_k) < 2L_j$ for $j = 0, 1, 2$. More explicitly, for $j = 0, 1, 2$, we decompose

$$u^{(j)}_{N_j} = \sum_{L_j}u^{(j)}_{N_j,L_j}, \quad \text{with} \quad u^{(j)}_{N_j,L_j} = 1_{L_j \leq (\tau + \mu_k) < 2L_j}(u^{(j)}_{N_j,L_j}),$$  (3.21)

where $L_j$ denote the dyadic integers. Using (2.4) we obtain the following norm equivalences for a generic function $u$

$$\|u\|^2_{X^{0,b}} \sim \sum_{L_j}L_j^{2b}\|u_{L_j}\|^2_{L^2(\mathbb{R} \times M)} \sim \sum_{L_j}\|u_{L_j}\|^2_{X^{0,b}}.$$  (3.22)

Observe that, denoting $L = (L_0, L_1, L_2)$, we have

$$|I(N)| = \sum_{L}\sum_{L}I(N,L) \leq \sum_{L}\sum_{L}|I(N,L)|,$$  (3.23)

where

$$I(N,L) := \int_{\mathbb{R} \times M}u^{(0)}_{N_0,L_0}u^{(1)}_{N_1,L_1}u^{(2)}_{N_2,L_2}dxdt.$$  (3.24)

Next, using the estimates (3.17) and (3.20), which are also true if we replace the functions $u^{(j)}_{N_j}$ by the functions $u^{(j)}_{N_j,L_j}$, and then, using (3.22) with $u^{(j)}_{N_j}$ instead of $u$, we obtain

$$|I(N,L)| \leq CN_2^{3/2}(L_0L_1L_2)^{\frac{3}{2}}\prod_{j=0}^{2}\|u^{(j)}_{N_j,L_j}\|_{L^2(\mathbb{R} \times M)},$$  (3.25)

and

$$|I(N,L)| \leq CN_2^{s_0+\delta}(L_0L_1L_2)^{b}\prod_{j=0}^{2}\|u^{(j)}_{N_j,L_j}\|_{L^2(\mathbb{R} \times M)}.$$  (3.26)

Interpolating the inequalities (3.25) and (3.26), we have for every $\theta \in (0, 1)$ that

$$|I(N,L)| \leq CN_2^{\theta+(1-\theta)(s_0+\delta)}(L_0L_1L_2)^{\theta+(1-\theta)b}\prod_{j=0}^{2}\|u^{(j)}_{N_j,L_j}\|_{L^2(\mathbb{R} \times M)}.$$  (3.27)
To continue the interpolation argument we use Lemma 13. It follows from Lemma 13 that for every $s_0 < s'$ there exists $b' < 1/2$ such that
\[
|I(N,L)| \leq CN_2^3 (L_0 L_1 L_2)^b' \prod_{j=0}^2 \|u_{N_j L_j}^{(j)}\|_{L^2(\mathbb{R} \times M)}.
\] (3.28)

Using (3.28) in (3.23), we obtain that
\[
|I(N)| \leq \sum_L |I(N,L)| \leq \sum_{L_0,L_1,L_2} CN_2^3 (L_0 L_1 L_2)^b' \prod_{j=0}^2 \|u_{N_j L_j}^{(j)}\|_{L^2(\mathbb{R} \times M)}.
\]

Now, choosing $b' < b_1$, where $b_1 \in (b', \frac{1}{2})$ and using the norm equivalence (3.22), we have
\[
|I(N)| \leq CN_2^3 \sum_{L_0,L_1,L_2} (L_0 L_1 L_2)^b' \|u_{N_0 L_0}^{(0)}\|_{X^{0,b_1}} \|u_{N_1 L_1}^{(1)}\|_{X^{0,b_1}} \|u_{N_2 L_2}^{(2)}\|_{X^{0,b_1}}
\] (3.29)

An application of the Cauchy-Schwarz inequality in the summation involving $L_2$ in (3.29) yields
\[
|I(N)| \leq CN_2^3 \sum_{L_0,L_1,L_2} \left( \prod_{j=0}^1 L_j^{b'-b_1} \|u_{N_j L_j}^{(j)}\|_{X^{0,b_1}} \right) \left( \sum_{L_2} L_2^{2(b' - b_1)} \right)^{1/2} \left( \sum_{L_2} \|u_{N_2 L_2}^{(2)}\|_{X^{0,b_1}}^2 \right)^{1/2}
\] (3.30)

In a similar manner, using the Cauchy-Schwarz inequality in the summations in $L_0$ and $L_1$ (as was done in the summation involving $L_2$) in (3.30) and applying (3.22), we obtain the desired estimate (3.13)
\[
|I(N)| \leq CN_2^3 \|u_{N_0}^{(0)}\|_{X^{0,b_1}} \|u_{N_1}^{(1)}\|_{X^{0,b_1}} \|u_{N_2}^{(2)}\|_{X^{0,b_1}}.
\]

Now, we move to prove the estimate (3.14). For this, we need to establish a new estimate for $I(N)$. We start noting that, as the functions $u_{N_j}^{(j)}$ are localized at frequency $\sim N_j$, we have
\[
u_{N_j}^{(j)} = \sum_{\mu_k \sim N_j} c_k e_k,
\]
where $\mu_k \sim N_j$ means that $\mu_k \in [N_j, 2N_j]$ and $c_k := (u_{N_k}^{(j)}, e_k)_{L^2}$. Of course, $e_k$ are the eigenfunctions with eigenvalues $\mu_k^2$ that is, $-\Delta e_k = \mu_k^2 e_k$. For these functions, we define the operators $T$ and $V$ as follows
\[
T(u_{N_j}^{(j)}) = \sum_{k: \mu_k \sim N_j} c_k \left( \frac{N_j}{\mu_k} \right)^2 e_k, \quad \text{and} \quad V(u_{N_j}^{(j)}) = \sum_{k: \mu_k \sim N_j} c_k \left( \frac{\mu_k}{N_j} \right)^2 e_k.
\] (3.31)

Observe that, for these operators one has the following norm equivalences
\[
\|T(u_{N_j}^{(j)})\|_X \sim \|u_{N_j}^{(j)}\|_X \sim \|V(u_{N_j}^{(j)})\|_X,
\] (3.32)
where $X = L^2$ or $X^{0,b}$. 


Therefore, if
\[ j \]
where \( \partial \nu \) denotes the normal derivative on the boundary and \( d \sigma \) is the induced measure on \( \partial M \). Since we are either assuming Dirichlet or Neumann boundary conditions, all boundary integrals vanish.

\[ \frac{1}{N_j^2} \Delta_g \left( \sum_{\mu \sim N_j} c_k \left( \frac{N_j}{\mu_k} \right)^2 e_k \right) = \frac{1}{N_j^2} \Delta_g (T(u_{N_j}^{(j)})). \]

In particular,
\[ I(N) = \int_{\mathbb{R} \times M} \sum_{j=0}^{2} u_{N_j}^{(j)} d\sigma \]
\[ \frac{1}{N_0^2} \int_{\mathbb{R} \times M} \Delta_g (T(u_{N_0}^{(0)})) \sum_{j=1}^{2} u_{N_j}^{(j)} d\sigma. \]

Next, we apply a general formula\[ \text{[3.33]} \]
\[ I(N) = -\frac{1}{N_0^2} \int_{\mathbb{R} \times M} T(u_{N_0}^{(0)}) \Delta_g \left( u_{N_1}^{(1)} u_{N_2}^{(2)} \right) d\sigma. \]

in the identity $[3.33]$, to get
\[ I(N) = I_1(N) + I_2(N) + I_3(N), \]
where
\[ \begin{cases} I_1(N) := -\frac{1}{N_0^2} \int_{\mathbb{R} \times M} T(u_{N_0}^{(0)}) u_{N_1}^{(1)} \Delta_g (u_{N_2}^{(2)}) d\sigma, \\ I_2(N) := -\frac{1}{N_0^2} \int_{\mathbb{R} \times M} T(u_{N_0}^{(0)}) u_{N_2}^{(2)} \Delta_g (u_{N_1}^{(1)}) d\sigma, \\ I_3(N) := -\frac{2}{N_0^2} \int_{\mathbb{R} \times M} T(u_{N_0}^{(0)}) (\nabla u_{N_1}^{(1)}, \nabla u_{N_2}^{(2)})_g d\sigma. \end{cases} \]

In what follows, we estimate the terms $I_1(N)$, $I_2(N)$ and $I_3(N)$ separately.

**Estimate for the Term $I_1(N)$**. From $[3.31]$, we can deduce that
\[ \Delta_g (u_{N_j}^{(j)}) = \sum_{\mu \sim N_j} c_k \Delta_g (e_k) = -N_j^2 \sum_{\mu \sim N_j} c_k \left( \frac{\mu_k}{N_j} \right)^2 e_k = -N_j^2 V(u_{N_j}^{(j)}). \]

Therefore, if $j = 2$ in $[3.37]$, we get
\[ \Delta_g (u_{N_2}^{(2)}) = -N_2^2 V(u_{N_2}^{(2)}). \]

\[ \text{[3.37]} \]

4The Green’s theorem states that
\[ \int_M (u \Delta v - v \Delta u) \, dx = \int_M (\frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu}) \, d\sigma \]

where \( \partial / \partial \nu \) denotes the normal derivative on the boundary and \( d\sigma \) is the induced measure on \( \partial M \). Since we are either assuming Dirichlet or Neumann boundary conditions, all boundary integrals vanish.

5We have that \((\nabla_g f, \nabla_g h)_g\) is the pointwise scalar product with respect to the metric g of \( \nabla_g f \) and \( \nabla_g h \).
Now, considering the definition of the term $I_1$ in (3.36), using (3.38) and applying the Hölder’s inequality, we find

$$|I_1(N)| \leq \left( \frac{N_2}{N_0} \right)^2 \int_{\mathbb{R}} \|T(u_{N_0}^{(0)})\|_{L^2} \|V(u_{N_2}^{(2)})\|_{L^2} \|u_{N_1}^{(1)}\|_{L^2} \ dt. \tag{3.39}$$

Using the Sobolev embedding $H^\frac{3}{2}(M) \hookrightarrow L^\infty(M)$ and norm equivalence (3.32) with $X = L^2$, we deduce from (3.39) that

$$|I_1(N)| \leq \left( \frac{N_2}{N_0} \right)^2 \int_{\mathbb{R}} \|T(u_{N_0}^{(0)})\|_{L^2} \|V(u_{N_2}^{(2)})\|_{L^2} \|u_{N_1}^{(1)}\|_{L^2} \ dt. \tag{3.40}$$

Thus, applying Hölder’s inequality in (3.40) and using the property (ii) in Proposition 4, we get

$$|I_1(N)| \leq \left( \frac{N_1}{N_0} \right)^2 \int_{\mathbb{R}} \|u_{N_0}^{(0)}\|_{X^{0,0}} \|u_{N_1}^{(1)}\|_{X^{0,0}} \|u_{N_2}^{(2)}\|_{X^{0,0}}. \tag{3.41}$$

On the other hand, using (3.38) and the Cauchy-Schwarz inequality in the term $I_1$ in (3.36), we obtain

$$|I_1(N)| \leq C \left( \frac{N_2}{N_0} \right)^2 \int_{\mathbb{R}} \|u_{N_1}^{(1)}\|_{L^2(M)} \|T(u_{N_0}^{(0)})V(u_{N_2}^{(2)})\|_{L^2(M)} \ dt \tag{3.42} \leq C \left( \frac{N_2}{N_0} \right)^2 \|u_{N_1}^{(1)}\|_{L^2(\mathbb{R} \times M)} \|T(u_{N_0}^{(0)})V(u_{N_2}^{(2)})\|_{L^2(\mathbb{R} \times M)}.$$

Next, using Lemma 8 and the norm equivalence (3.32) with $X = X^{0,0}$, the estimate (3.42) yields

$$|I_1(N)| \leq C \left( \frac{N_2}{N_0} \right)^2 \left( \min(N_0, N_2) \right)^{s_0 + \delta} \|u_{N_1}^{(1)}\|_{X^{0,0}} \|T(u_{N_0}^{(0)})\|_{X^{0,0}} \|V(u_{N_2}^{(2)})\|_{X^{0,0}} \tag{3.43} \leq C \left( \frac{N_1}{N_0} \right)^2 N_2^{s_0 + \delta} \|u_{N_1}^{(1)}\|_{X^{0,0}} \|u_{N_0}^{(0)}\|_{X^{0,0}} \|u_{N_2}^{(2)}\|_{X^{0,0}}.$$

Now, we interpolate (3.41) and (3.43) as in (3.27), and use Lemma 13 to obtain

$$|I_1(N)| \leq C \left( \frac{N_1}{N_0} \right)^2 \left( N_2^{s_0 + \delta} \right) \|u_{N_1}^{(1)}\|_{X^{0,0}} \|u_{N_0}^{(0)}\|_{X^{0,0}} \|u_{N_2}^{(2)}\|_{X^{0,0}} \tag{3.44}$$

where we have $s_0 < s'$ and $b' < \frac{1}{2}$.

**Estimate for the Term $I_2(N)$**. As in the estimate of the term $I_1(N)$, we use (3.37) with $j = 1$ that is, $\Delta_g(u_{N_1}^{(1)}) = -N_1^2 V(u_{N_1}^{(1)})$ and apply Hölder’s inequality in the term $I_2$ in (3.36), to obtain

$$|I_2(N)| \leq \left( \frac{N_1}{N_0} \right)^2 \int_{\mathbb{R}} \|T(u_{N_0}^{(0)})\|_{L^2} \|V(u_{N_1}^{(1)})\|_{L^2} \|u_{N_2}^{(2)}\|_{L^2} \ dt. \tag{3.45}$$

Now, using the Sobolev embedding $H^\frac{3}{2}(M) \hookrightarrow L^\infty(M)$ and norm equivalence (3.32) with $X = L^2$, we deduce from (3.45) that

$$|I_2(N)| \leq \left( \frac{N_1}{N_0} \right)^2 \int_{\mathbb{R}} \|T(u_{N_0}^{(0)})\|_{L^2} \|V(u_{N_1}^{(1)})\|_{L^2} \|u_{N_2}^{(2)}\|_{L^2} \ dt \tag{3.46} \leq \left( \frac{N_1}{N_0} \right)^2 \int_{\mathbb{R}} \|u_{N_0}^{(0)}\|_{L^2} \|u_{N_1}^{(1)}\|_{L^2} \|u_{N_2}^{(2)}\|_{L^2} \ dt.$$
Thus, applying Hölder’s inequality in (3.46) and using the property (ii) of Proposition 4, we get
\[
|I_2(N)| \leq \left( \frac{N_1}{N_0} \right)^2 N_2^3 \| u_N(0) \|_{X^{s_0, b}} \| u_N^{(2)} \|_{X^{s_0, b}} \| u_N^{(1)} \|_{X^{s_0, b}}. \tag{3.47}
\]

On the other hand, using Cauchy-Schwarz inequality in the term \( I_2 \) in (3.36) we obtain that
\[
|I_2(N)| \leq C \left( \frac{N_1}{N_0} \right)^2 \int_{\mathbb{R}} \| T(u_{N_0}^{(0)}) u_{N_2}^{(2)} \|_{L^2(M)} \| V(u_{N_1}^{(1)}) \|_{L^2(M)} dt \leq C \left( \frac{N_1}{N_0} \right)^2 \| T(u_{N_0}^{(0)}) u_{N_2}^{(2)} \|_{L^2(\mathbb{R} \times M)} \| V(u_{N_1}^{(1)}) \|_{L^2(\mathbb{R} \times M)}. \tag{3.48}
\]

Next, using Lemma 8 in the estimate (3.48) and the relation (3.32) with \( X = X^{0, b} \), we get
\[
|I_2(N)| \leq C \left( \frac{N_1}{N_0} \right)^2 N_2 \| u_{N_0}^{(0)} \|_{X^{0, b}} \| u_{N_2}^{(2)} \|_{X^{0, b}} \| u_{N_1}^{(1)} \|_{X^{0, b}}. \tag{3.49}
\]

Now, we can interpolate (3.47) with (3.49) as in (3.27), and use the Lemma 13 to obtain
\[
|I_2(N)| \leq C \left( \frac{N_1}{N_0} \right)^2 N_2 |N_0^{s_0 + \delta} \| u_{N_0}^{(0)} \|_{X^{0, b}} \| u_{N_2}^{(2)} \|_{X^{0, b}} \| u_{N_1}^{(1)} \|_{X^{0, b}}, \tag{3.50}
\]

where \( s_0 < s' \) and \( b' < b \).

**Estimate for the Term** \( I_3(N) \). To estimate the term \( I_3(N) \), we use the inequality
\[
|\langle \nabla g f, \nabla g h \rangle| \leq |\nabla g f||\nabla g h|, \tag{3.51}
\]

with \( f = u_{N_1}^{(1)} \) and \( h = u_{N_2}^{(2)} \), and Hölder’s inequality in the identity defining \( I_3 \) in (3.36), to get
\[
|I_3(N)| \leq C \frac{N_0^2}{N_0^3} \int_{\mathbb{R}} \| T(u_{N_0}^{(0)}) \|_{L^2} \| \nabla u_{N_1}^{(1)} \|_{L^2} \| \nabla u_{N_2}^{(2)} \|_{L^2} dt. \tag{3.52}
\]

Now, using Sobolev embedding \( H^\frac{3}{2}(M) \hookrightarrow L^\infty(M) \) and the norm equivalence (3.32) with \( X = L^2 \), we obtain from (3.52)
\[
|I_3(N)| \leq C \frac{N_0^2}{N_0^3} \int_{\mathbb{R}} \| u_{N_0}^{(0)} \|_{L^2} \| \nabla u_{N_1}^{(1)} \|_{L^2} \| \nabla u_{N_2}^{(2)} \|_{L^2} dt. \tag{3.53}
\]

Using the inequality \( \| \nabla u \|_{L^2} \leq C \| u \|_{X^s} \) we deduce from (3.53) that
\[
|I_3(N)| \leq C \frac{N_0^2 N_1^3 + 1}{N_0^3} \int_{\mathbb{R}} \| u_{N_0}^{(0)} \|_{L^2} \| u_{N_1}^{(1)} \|_{L^2} \| u_{N_2}^{(2)} \|_{L^2} dt. \tag{3.54}
\]

Using Hölder inequality in (3.54) and applying the property (ii) in Proposition 4, we get
\[
|I_3(N)| \leq C \left( \frac{N_1}{N_0} \right)^2 N_2 \| u_{N_0}^{(0)} \|_{X^{0, b}} \| u_{N_1}^{(1)} \|_{X^{0, b}} \| u_{N_2}^{(2)} \|_{X^{0, b}}. \tag{3.55}
\]

On the other hand, using (3.51) and the Cauchy-Schwarz inequality in the term \( I_3 \) of (3.36), we have
\[
|I_3(N)| \leq C \frac{1}{N_0^3} \| \nabla u_{N_1}^{(1)} \|_{L^2(\mathbb{R} \times M)} \| (\nabla u_{N_0}^{(2)}) T(u_{N_0}^{(0)}) \|_{L^2(\mathbb{R} \times M)}. \tag{3.56}
\]
Next, using the inequality $\|\nabla u_N\|_{L^2} \leq CN\|u_N\|_{L^2}$ and the bilinear estimate (2.10) in (3.56) we obtain
\[
|I_3(N)| \leq C\frac{N_1}{N_0}N_2(\min(N_0,N_2))^{s_0+\delta}\|u_N(1)\|_{L^2([0,T])}\|u_N(2)\|_{X^{0,\delta}}\|T(u_N(0))\|_{X^{0,0}}. \tag{3.57}
\]
Thus, using (3.32) we conclude from (3.57) that
\[
|I_3(N)| \leq C\left(\frac{N_1}{N_0}\right)^2 N_2^{s_0+\delta}\|u_N(1)\|_{X^{0,\delta}}\|u_N(2)\|_{X^{0,\delta}}\|u_N(0)\|_{X^{0,\delta}}. \tag{3.58}
\]
Now, we can interpolate (3.55) and (3.58) as in (3.27), and use Lemma (13) to obtain
\[
|I_3(N)| \leq C\left(\frac{N_1}{N_0}\right)^2 N_2^{s_0}\|u_N(1)\|_{X^{0,0'}}\|u_N(0)\|_{X^{0,0'}}\|u_N(2)\|_{X^{0,0'}}, \tag{3.59}
\]
where we have $s_0 < s'$ and $b' < \frac{1}{2}$.

Finally, combining the estimates obtained for $I_j(N)$ ($j = 1, 2, 3$), in (3.44), (3.50) and (3.59), we obtain the required estimate (3.14) as follows
\[
|I(N)| \leq |I_1(N)| + |I_2(N)| + |I_3(N)| \leq C\left(\frac{N_1}{N_0}\right)^2 N_2^{s_0}\|u_N(1)\|_{X^{0,0'}}\|u_N(0)\|_{X^{0,0'}}\|u_N(2)\|_{X^{0,0'}},
\]
where $s_0 < s'$ and $b' < \frac{1}{2}$. \qed

Now, we come back to estimate the terms $\Sigma_1$ and $\Sigma_2$ using Lemma [13].

**Estimate for $\Sigma_1$.** We saw in Lemma [15] that, for a fixed $s > s_0$, one can find $s'$ with $s_0 < s' < s$ such that (3.13) holds true. Hence, one has
\[
\Sigma_1 = \sum_{N_0N_0 \leq CN_1} |I(N)| \leq C\sum_{N_0N_0 \leq CN_1} N_2^{s_0}\|u_N(1)\|_{X^{0,0'}}\|u_N(0)\|_{X^{0,0'}}\|u_N(2)\|_{X^{0,0'}}. \tag{3.60}
\]
Using the norm equivalence (3.3), we obtain from (3.60) that
\[
\Sigma_1 \leq C\sum_{N_0N_0 \leq CN_1} \left(\frac{N_0}{N_1}\right)^s N_2^{s_0-s}\|u_N(0)\|_{X^{0,0'}}\|u_N(1)\|_{X^{0,0'}}\|u_N(2)\|_{X^{0,0'}}
= C\sum_{N_0N_0 \leq CN_1} \left(\frac{N_0}{N_1}\right)^s \|u_N(0)\|_{X^{0,0'}}\|u_N(1)\|_{X^{0,0'}} \left(\sum_{N_2} \|u_N(2)\|_{X^{0,0'}} N_2^{s_0-s}\right). \tag{3.61}
\]
Now, using Cauchy-Schwarz inequality in (3.61) and the norm equivalence we find
\[
\Sigma_1 \leq C\|u_2\|_{X^{0,0'}} \sum_{N_0N_0 \leq CN_1} \left(\frac{N_0}{N_1}\right)^s \|u_N(0)\|_{X^{0,0'}}\|u_N(1)\|_{X^{0,0'}.} \tag{3.62}
\]
Thus, using (3.3) in Lemma [14] about dyadic summations with $N = N_0$ and $N' = N_1$ in (3.62) we conclude that
\[
\Sigma_1 \leq C\|u_2\|_{X^{0,0'}} \|u_1\|_{X^{0,0'}}\|u_0\|_{X^{0,0'}}. \tag{3.63}
\]
Estimate for $\Sigma_2$. We use the estimate (3.14) and the norm equivalence (3.3), to get
\[
\Sigma_2 = \sum_{N:N_0>C_{N_1},N_2\leq N_1} |I(N)| \leq C \sum_{N:N_0>C_{N_1}} \left( \frac{N_1}{N_0} \right)^2 \left[ u_{N_0}^{(0)} \right]_{X^{0,b}} \left[ u_{N_1}^{(1)} \right]_{X^{0,b}} \left[ u_{N_2}^{(2)} \right]_{X^{0,b}} \\
\leq C \sum_{N:N_0>C_{N_1}} \left( \frac{N_1}{N_0} \right)^{2-s} \left[ u_{N_0}^{(0)} \right]_{X^{-s,b}} \left[ u_{N_1}^{(1)} \right]_{X^{s,b}} \left[ u_{N_2}^{(2)} \right]_{X^{s,b}} \\
= C \sum_{N_0,N_1,N_0>C_{N_1}} \left( \frac{N_1}{N_0} \right)^{2-s} \left[ u_{N_0}^{(0)} \right]_{X^{-s,b}} \left[ u_{N_1}^{(1)} \right]_{X^{s,b}} \left( \sum_{N_2} \left[ u_{N_2}^{(2)} \right]_{X^{s,b}} N_2^{-s} \right).
\]
(3.64)

Applying the Cauchy-Schwarz inequality, we obtain from (3.64)
\[
\Sigma_2 \leq C \left\| u_2 \right\|_{X^{-s,b}} \sum_{N_0,N_1,N_0>C_{N_1}} \left( \frac{N_1}{N_0} \right)^{2-s} \left[ u_{N_0}^{(0)} \right]_{X^{-s,b}} \left[ u_{N_1}^{(1)} \right]_{X^{s,b}}.
\]
Finally, using Lemma 14 similarly to (3.63) we conclude that
\[
\Sigma_2 \leq C \left\| u_2 \right\|_{X^{-s,b}} \left\| u_1 \right\|_{X^{s,b}} \left\| u_0 \right\|_{X^{-s,b}}.
\]
(3.65)

Estimate for $\Sigma_3$ and $\Sigma_4$. Using symmetry, analogously to $\Sigma_1$ and $\Sigma_2$, one can obtain similar estimates for the terms $\Sigma_3$ and $\Sigma_4$.

Gathering estimates for $\Sigma_1$, $\Sigma_2$, $\Sigma_3$ and $\Sigma_4$ in (3.12) we conclude the proof of the proposition. □

Now, we move to prove the second crucial bilinear estimate stated in Proposition 12.

Proof of Proposition 12: By a duality argument, to prove (3.2), it suffices to establish the following inequality
\[
\left| \int_{\mathbb{R} \times M} \overline{u_1} u_2 u_0 \right| \leq C \left\| u_1 \right\|_{X^{-s,b}} \left\| u_2 \right\|_{X^{s,b}} \left\| u_0 \right\|_{X^{-s,b}}.
\]
(3.66)
for all $u_0 \in X^{-s,b}$. As in the proof of (3.1) we use the dyadic decompositions $u_j = \sum_j u_{jN_j}$ ($j = 0, 1, 2$) in the left hand side of (3.66).

Let
\[
J := \int_{\mathbb{R} \times M} \overline{u_1} u_2 u_0 dx dt,
\]
(3.67)
and use triangle inequality to get
\[
|J| \leq \sum_{N_0,N_1,N_2} \left| \int_{\mathbb{R} \times M} \overline{u_{1N_1}} u_{2N_2} u_{0N_0} \right|.
\]
(3.68)

Observe that the summation in (3.68) is taken over all triples of dyadic numbers. For abbreviation, let $N = (N_0,N_1,N_2)$ and
\[
J(N) := \int_{\mathbb{R} \times M} \overline{u_{1N_1}} u_{2N_2} u_{0N_0} dx dt.
\]
(3.69)

With these notations we write $\Sigma_N |J(N)|$ in the following manner
\[
\sum_N |J(N)| \leq \sum_{N:N_0 \leq N_1} |J(N)| + \sum_{N:N_1 < N_2} |J(N)|.
\]
(3.70)
Here too, we split the sums in four frequency regimes as we did in the proof of (3.1). More precisely, we write

$$\sum_{N: N_2 \leq N_1} |J(N)| \leq \sum_{N: N_2 \leq N_1, N_0 \leq CN_1} |J(N)| + \sum_{N: N_2 \leq N_1, N_0 > CN_1} |J(N)| =: \tilde{\Sigma}_1 + \tilde{\Sigma}_2,$$

(3.71)

and

$$\sum_{N: N_1 < N_2} |J(N)| \leq \sum_{N: N_1 < N_2, N_0 \leq CN_2} |J(N)| + \sum_{N: N_1 < N_2, N_0 > CN_2} |J(N)| =: \tilde{\Sigma}_3 + \tilde{\Sigma}_4.$$

(3.72)

Therefore, combining (3.68), (3.70), (3.71) and (3.72) we arrive at

$$|J| \leq \tilde{\Sigma}_1 + \tilde{\Sigma}_2 + \tilde{\Sigma}_3 + \tilde{\Sigma}_4.$$  

(3.73)

In this way, we reduced the proof to estimating the each term $\tilde{\Sigma}_j$ ($j = 1, 2, 3, 4$). To simplify the exposition, let $u^{(j)}_{N_j} := u_{J N_j}$, ($j = 0, 2$) and $u^{(1)}_{N_1} := \overline{u_{J N_1}}$. We start estimating $\tilde{\Sigma}_1$.

**Estimate for the Term $\tilde{\Sigma}_1$.** In this case we have $N_2 \leq N_1$, $N_0 \leq CN_1$. Consider the expression for $J(N)$ in (3.69). As was done to get (3.17), we use the Hölder’s inequality followed by the property (ii) in Proposition 4 to obtain

$$|J(N)| \leq CN_2^{3/2} \|u^{(0)}_{N_2}\|_{X^{0,b}} \|u^{(1)}_{N_1}\|_{X^{1,b}} \|u^{(2)}_{N_2}\|_{X^{0,b}}.$$  

(3.74)

Next, as was done to obtain (3.20), we use the Cauchy-Schwarz inequality and Lemma 8 to find

$$|J(N)| \leq CN_2^{\delta_{0} + \delta} \|u^{(0)}_{N_2}\|_{X^{0,b}} \|u^{(1)}_{N_1}\|_{X^{1,b}} \|u^{(2)}_{N_2}\|_{X^{0,b}}.$$  

(3.75)

Now, decomposing each function $u^{(j)}_{N_j}$ in (3.69) with respect to the time variable, we can consider

$$J(N) = \sum_{L} J(N, L),$$

(3.76)

where

$$J(N, L) := \int_{\mathbb{R} \times M} u^{(0)}_{N_0} u^{(1)}_{N_1} u^{(2)}_{N_2} dx dt,$$

(3.77)

and the sum is taken over all dyadic integers $L = (L_0, L_1, L_2)$.

Observe that, the estimates (3.74) and (3.75) also hold if one replaces $u^{(j)}_{N_j}$ by $u^{(j)}_{N_j/L_j}$. Now, using the norm equivalences $\|u_{L_j}\|_{X^{0,b}} \simeq L_j^{\frac{b}{2}} \|u_{L_j}\|_{L^2(\mathbb{R} \times M)}$, we obtain from (3.74) and (3.75) with $u^{(j)}_{N_j/L_j}$ replacing $u^{(j)}_{N_j}$ that

$$|J(N, L)| \leq CN_2^{3/2} (L_0 L_1 L_2)^{\frac{1}{2}} \prod_{j=0}^{2} \|u^{(j)}_{N_j/L_j}\|_{L^2(\mathbb{R} \times M)},$$

(3.78)

and

$$|J(N, L)| \leq CN_2^{\delta_{0} + \delta} (L_0 L_1 L_2)^{b} \prod_{j=0}^{2} \|u^{(j)}_{N_j/L_j}\|_{L^2(\mathbb{R} \times M)}.$$  

(3.79)

Interpolating the estimates (3.78) and (3.79), we obtain for $0 < \theta < 1$

$$|J(N, L)| \leq CN_2^{\frac{3}{2} \theta + (1-\theta)(\delta_{0} + \delta)} (L_0 L_1 L_2)^{\frac{1}{2} + (1-\theta)b} \prod_{j=0}^{2} \|u^{(j)}_{N_j/L_j}\|_{L^2(\mathbb{R} \times M)},$$

(3.80)
Consequently, using the Lemma\textsuperscript{13} we get
\[
|J(N, L)| \leq CN_2^{s'} (L_0L_1L_2)^{b'} \prod_{j=0}^{2} \|u_{N_jL_j}^{(j)}\|_{L^2(\mathbb{R} \times \mathbb{M})},
\] (3.81)
where \(s_0 < s'\) and \(b' < \frac{1}{2}\).

Hence, summing over all dyadic triples \(L = (L_0, L_1, L_2)\) and choosing \(b'\) such that \(b' < b_1 < \frac{1}{2}\), we get from (3.81) that
\[
|J(N)| \leq \sum_L |J(N, L)| \leq CN_2^{s'} \sum_L (L_0L_1L_2)^{b'} \prod_{j=0}^{2} \|u_{N_jL_j}^{(j)}\|_{L^2(\mathbb{R} \times \mathbb{M})}.
\] (3.82)

Now, using the norm equivalence (3.22), we get
\[
|J(N)| = CN_2^{s'} \sum_{L_1, L_2} (L_0L_1L_2)^{b'-b_1} \|u_{N_0L_0}^{(0)}\|_{X^{0,b_1}} \prod_{j=1}^{2} \|u_{N_jL_j}^{(j)}\|_{X^{0,b_1}}
\] (3.83)

Therefore, applying the Cauchy-Schwarz inequality successively in the summations involving \(L_0, L_1, L_2\), applying the norm equivalences (3.22), we obtain that for \(s_0 < s'\) there are numbers \(b_1 < \frac{1}{2}\) such that
\[
|J(N)| \leq CN_2^{s'} \|u_{N_0}^{(0)}\|_{X^{0,b_1}} \prod_{j=1}^{2} \|u_{N_j}^{(j)}\|_{X^{0,b_1}}.
\] (3.84)

Now, summing according the regime \(N_2 \leq N_1, N_0 \leq CN_1\), with \(s' < s\), we have from (3.84) that
\[
\tilde{\Sigma}_1 \leq C \sum_{N_0, N_1 \leq CN_1} N_0^s N_2^{s'-s} \|u_{N_0}^{(0)}\|_{X^{s-b_1}} \|u_{N_1}^{(1)}\|_{X^{s-b_1}} \|u_{N_2}^{(2)}\|_{X^{s-b_1}}
\] (3.85)

Thus, applying the Cauchy-Schwarz inequality in the summation in \(N_2\), we obtain
\[
\tilde{\Sigma}_1 \leq C \sum_{N_0, N_1 \leq CN_1} \left( \frac{N_0}{N_1} \right)^s \|u_{N_0}^{(0)}\|_{X^{s-b_1}} \|u_{N_1}^{(1)}\|_{X^{s-b_1}} \left( \sum_{N_2} N_2^{s'-s} \|u_{N_2}^{(2)}\|_{X^{s-b_1}} \right)
\] (3.86)

Finally, in view of Lemma\textsuperscript{14}, we obtain from (3.86) that
\[
\tilde{\Sigma}_1 \leq C \|u_0\|_{X^{s-b_1}} \|u_1\|_{X^{s-b_1}} \|u_2\|_{X^{s-b_1}}.
\]

Estimate for the Term \(\tilde{\Sigma}_2\). In this case we have \(N_2 \leq N_1; N_0 > CN_1\).

In the same way we did in the proof of (3.1), (see Lemma\textsuperscript{13}) we split the integral \(J\) into three terms and analyze each one of them separately. More precisely, we write
\[
J(N) = J_1(N) + J_2(N) + J_3(N),
\]
where
\[
\begin{align*}
J_1(N) & := -\frac{1}{N_0^2} \int_{\mathbb{R}^M} T(u_{N_0}^{(0)}) u_{N_1}^{(1)}(u_{N_2}^{(2)}) dx dt, \\
J_2(N) & := -\frac{1}{N_0^2} \int_{\mathbb{R}^M} T(u_{N_0}^{(0)}) u_{N_2}^{(2)}(u_{N_1}^{(1)}) dx dt, \\
J_3(N) & := -\frac{2}{N_0^2} \int_{\mathbb{R}^M} T(u_{N_0}^{(0)})(\nabla u_{N_1}^{(1)}, \nabla u_{N_2}^{(2)})_g dx dt.
\end{align*}
\] (3.87)

We can obtain the following estimates for \( J_k(N) \).
\[
|J_k(N)| \leq \left( \frac{N_1}{N_0} \right)^2 N_2^2 \prod_{j=0}^{2} \| u_j^{(0)} \|_{X_{s_0/3}^{b}},
\] (3.88)
and
\[
|J_k(N)| \leq \left( \frac{N_1}{N_0} \right)^2 N_2^{2+\delta} \prod_{j=0}^{2} \| u_j^{(0)} \|_{X_{s_0/3}^{b}},
\] (3.89)

where \( k = 1, 2, 3 \). We can use the same estimates and considerations as we did to estimate the terms \( I_k, k = 1, 2, 3 \), in Lemma [15]. More explicitly:

- For \( k = 1 \) we use the same arguments that were used in the estimates (3.41) and (3.43).
- For \( k = 2 \) we use the same arguments that were used in the estimates (3.47) and (3.49).
- For \( k = 3 \) we use the same arguments that were used in the estimates (3.55) and (3.58).

Hence, interpolating (3.88) and (3.89), for each \( k = 1, 2, 3 \) we see that for \( s_0 < s' \) and \( b' < \frac{1}{2} \) one has
\[
|J(N)| \leq \sum_{k=1}^{3} |J_k(N)| \leq C \left( \frac{N_1}{N_0} \right)^2 N_2^{2-s} \frac{N_3^s}{N_1} \| u_{N_0}^{(0)} \|_{X_{-s}^{b'}} \| u_{N_1}^{(1)} \|_{X_{s'}^{b'}} \| u_{N_2}^{(2)} \|_{X_{s'}^{b'}}.
\] (3.90)

Hence, summing in \( N = (N_0, N_1, N_2) \), we get
\[
\overline{\Sigma}_2 \leq C \sum_{N: N_0 = CN_1} \left( \frac{N_1}{N_0} \right)^{2-s} N_2^{2-s} \| u_{N_0}^{(0)} \|_{X_{-s}^{b'}} \| u_{N_1}^{(1)} \|_{X_{s'}^{b'}} \| u_{N_2}^{(2)} \|_{X_{s'}^{b'}}.
\] (3.91)

Estimate for the terms \( \overline{\Sigma}_3 \) and \( \overline{\Sigma}_4 \). By a symmetry argument we can prove the same estimates for \( \overline{\Sigma}_3 : N_1 < N_2; N_0 \leq CN_2 \) and \( \overline{\Sigma}_4 : N_1 < N_2; N_0 > CN_2 \).

Finally, collecting the estimates established in (3.85), (3.91) and for the summations \( \overline{\Sigma}_j \) \( j = 3, 4 \) we obtain the required estimate
\[
|J| \leq \sum_N |J(N)| \leq \overline{\Sigma}_1 + \overline{\Sigma}_2 + \overline{\Sigma}_3 + \overline{\Sigma}_4
\leq C \| u_0 \|_{X_{-s}^{b}} \| u_1 \|_{X_{s'}^{b_2}} \| u_2 \|_{X_{s'}^{b_2}},
\]
where \( b_2 > b_1, b' \) are chosen in a suitable manner. \[\Box\]
4. Proof the Main Result

In this section we use the estimates obtained in the previous section to prove the local well-
posedness stated in Theorem 1 for the IVP (1.1).

Proof of Theorem 1. Let $s > \frac{2}{3}$ and $u_0 \in H^s(M)$. Applying Duhamel’s formula, we can rewrite
the IVP (1.1) in the following equivalent integral equation (with Dirichlet or Neumann conditions)

$$u(t) = e^{it\Delta}u_0 - i \int_0^t e^{i(t-\tau)\Delta} Q(u(\tau), \bar{u}(\tau)) d\tau,$$

(4.1)

where $e^{it\Delta}$ denotes the evolution of the linear Schrödinger equation defined using Dirichlet or Neu-
mann spectral resolution and $Q(u, \bar{u}) := \alpha u^2 + \beta |u|^2 + \gamma |u|^2$.

We define an application

$$\Phi(u)(t) := e^{it\Delta}u_0 - i \int_0^t e^{i(t-\tau)\Delta} Q(u(\tau), \bar{u}(\tau)) d\tau,$$

(4.2)

and use the contraction mapping principle to find a fixed point $u$ that solves the equation (4.1).

For this, let $T > 0$ and $R > 0$ to be chosen later and consider a ball

$$B_T^R := \{ u \in X^{s,b}_T; \|u\|_{X^{s,b}_T} \leq R \}$$

in the space $X^{s,b}_T$. We will show that for sufficiently small $T > 0$ and an appropriate positive constant
$R > 0$, the application $\Phi$ defined in (4.2) is a contraction map. In fact, applying the linear estimates
(2.5) and (2.6) from Proposition 6 in (4.2), we obtain for $T \leq 1$

$$\|\Phi(u)\|_{X^{s,b}_T} \leq c_0 \|u_0\|_{H^s(M)} + c_1 T^{1-b-b'} \|Q(u, \bar{u})\|_{X^{s-b'}_T}. (4.3)$$

Using the bilinear estimates (3.1) and (3.2), we get from (4.3)

$$\|\Phi(u)\|_{X^{s,b}_T} \leq c_0 \|u_0\|_{H^s(M)} + c_1 T^{1-b-b'} \|u\|_{X^{s,b}_T}^2. (4.4)$$

Set $\theta_1 := 1 - b - b' > 0$ and $R := 2c_0 \|u_0\|_{H^s(M)}$. Therefore, for $u \in B_T^R$ the estimate (4.4) yields

$$\|\Phi(u)\|_{X^{s,b}_T} \leq \frac{R}{2} + c_1 T^{\theta_1} R^2. (4.5)$$

Hence, for a suitable $0 < T \leq 1$ such that $c_1 T^{\theta_1} R < \frac{1}{2}$, one can conclude that $\Phi$ maps $B_T^R$ onto itself.

Let $u, \tilde{u} \in B_T^R$ be solutions with the same initial data $u_0$. In an analogous manner as we did above, it
is easy to get

$$\|\Phi(u) - \Phi(\tilde{u})\|_{X^{s,b}_T} \leq CT^{1-b-b'} \|Q(u, \bar{u}) - Q(\tilde{u}, \bar{u})\|_{X^{s-b'}_T}. (4.6)$$

Observe that, we can write

$$\begin{aligned}
\bar{u}^2 - \bar{\tilde{u}}^2 &= (u - \tilde{u})\bar{u} + (u - \tilde{u})\tilde{u}, \\
u^2 - \tilde{u}^2 &= (u - \tilde{u})u + (u - \tilde{u})\tilde{u}, \\
|u|^2 - |\tilde{u}|^2 &= u(u - \tilde{u}) + (u - \tilde{u})\tilde{u}.
\end{aligned} (4.7)$$

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Now, using (4.7) in (4.6) and then applying the bilinear estimates (3.1) and (3.2), we obtain
\[
\|\Phi(u) - \Phi(\tilde{u})\|_{X^s_{T'}} \leq CT^{1-b-b'}(\|u\|_{X^s_{T'}} + \|\tilde{u}\|_{X^s_{T'}})\|u - \tilde{u}\|_{X^s_{T'}} \leq CT^{\theta_1} R\|u - \tilde{u}\|_{X^s_{T'}}. \tag{4.8}
\]

If we choose \(0 < T \leq 1\) such that \(\max\{c_1 T^{\theta_2} R, CT^{\theta_1} R\} \leq \frac{1}{2}\), it follows from (4.8) that \(\Phi\) is a contraction on the ball \(B^R\). The Lipschitz property is obtained with a similar idea, so the details are omitted. \(\square\)

Remark 16. Let \(B := B^3\) the unit ball in \(\mathbb{R}^3\), denote the Laplacean in \(B\) by \(\Delta := \Delta_B\) and consider the linear Schrödinger group \(S(t) = e^{it\Delta}\). Anton in [1] considered radial data \(u_0, v_0\) spectrally localized at frequency \(\Gamma, \Lambda\) respectively to prove
\[
\|S(t)u_0S(t)v_0\|_{L^2((0,1) \times B)} \leq C(\min(\Gamma, \Lambda))^s\|u_0\|_{L^2(B)}\|v_0\|_{L^2(B)},
\]
and
\[
\|(\nabla S(t)u_0)S(t)v_0\|_{L^2((0,1) \times B)} \leq CT(\min(\Gamma, \Lambda))^s\|u_0\|_{L^2(B)}\|v_0\|_{L^2(B)},
\]
for any \(s > \frac{1}{2}\).

With these estimates at hand, we can obtain the bilinear estimates established in Propositions [1] and [2] for the quadratic NLS [1, 1] posed on \(\mathbb{R}^3\) as well. Consequently, as in [1], we can also establish a local well-posedness result for the quadratic NLS equation [1, 1] with radial data in \(H^s(\mathbb{R}^3)\) for \(s > 1/2\).

5. APPENDIX

In this appendix, we will prove the crucial duality argument used in (3.4). The aim is to prove that there exists an isometric isomorphism \(\Phi : X^{-s,-b}(\mathbb{R} \times M) \rightarrow (X^{s,b}(\mathbb{R} \times M))^*\) such that \(\|\Phi(f)\| = \|f\|_{X^{-s,-b}}\). For this, we need to introduce some definitions and notations. To begin, let us define (for \(f \in C_0^\infty(\mathbb{R} \times M)\))
\[
J^s f = \sum_k \langle \mu_k \rangle^{s/2} P_k f,
\]
and via Fourier transform,
\[
\Lambda^b_k f(t) = \langle (t + \mu_k) b \tilde{f} \rangle (t).
\]

In this manner, we can define the operator
\[
J^s \Lambda^b f := \sum_k \langle \mu_k \rangle^{s/2} P_k [\Lambda^b_k f(t)].
\]

Using this definition and considering \(u \in X^{s,b} \cap L^2_{tx}, v \in X^{-s,-b} \cap L^2_{tx}\), we have
\[
\langle J^s \Lambda^b_v, J^{-s} \Lambda^{-b} u \rangle_{L^2_{tx}} = \langle v(t), u(t) \rangle_{L^2_{tx}}.
\]
In fact, using $L^2$-orthogonality and Plancherel’s theorem, we obtain

$$\langle J^s A^b v, J^{-s} A^{-b} u \rangle_{L^2_{tx}} = \sum_k \int_{\mathbb{R} \times M} P_k [\Lambda^b_k v(t)] \overline{P_k [\Lambda^{-b}_k u(t)]} \, dg dt$$

$$= \sum_k \int_{\mathbb{R} \times M} P_k [(\tau + \mu_k)^b \hat{v}(\tau)] (t) P_k [(\tau + \mu_k)^{-b} \hat{u}(\tau)] (t) \, dg dt$$

$$= \sum_k \int_{\mathbb{R} \times M} P_k [(\tau + \mu_k)^b \hat{v}(\tau)] P_k [(\tau + \mu_k)^{-b} \hat{u}(\tau)] \, dg \, d\tau$$

$$= \sum_k \int_{\mathbb{R} \times M} P_k \hat{v}(\tau) \ P_k \hat{u}(\tau) \, dg \, d\tau.$$  \hfill (5.1)

Now, using that $\hat{\phi} \hat{f} = P_k \hat{f}$, we conclude

$$\langle J^s A^b v, J^{-s} A^{-b} u \rangle_{L^2_{tx}} = \sum_k \int_{\mathbb{R} \times M} \hat{P}_k \hat{v}(\tau) \ \overline{\hat{P}_k \hat{u}(\tau)} \, dg d\tau$$

$$= \sum_k \int_{\mathbb{R} \times M} \hat{P}_k \hat{v}(\tau) \ \overline{\hat{P}_k \hat{u}(-\tau)} \, dg d\tau$$

$$= \sum_k \int_M \int_{\mathbb{R}} P_k v(t) \overline{P_k u(t)} \, dg \, dt$$

$$= \sum_k \int_{\mathbb{R}} \langle P_k v(t) P_k u(t) \rangle_{L^2(M)} \, dt$$

$$= \langle v(t), u(t) \rangle_{L^2(\mathbb{R} \times M)}.  \hfill (5.2)$$

Now, we can state the following lemma.

**Lemma 17.** Let $\langle \cdot, \cdot \rangle$ denotes an inner product in $L^2_{tx}$. Let $\Phi : X^{-s,-b}(\mathbb{R} \times M) \rightarrow (X^{s,b}(\mathbb{R} \times M))^*$ be defined by

$$\Phi_h(f) = \langle J^s A^b f, J^{-s} A^{-b} h \rangle$$

Then $\Phi$ is an isometric isomorphism and we have $\Phi_h(f) = (f, h)$, whenever $f \in X^{s,b} \cap L^2_{tx}$ and $h \in X^{-s,-b} \cap L^2_{tx}$.

**Proof.** For $f \in X^{s,b}$ and $h \in X^{-s,-b}$, we have

$$|\Phi_h(f)| = |\langle J^s A^b f, J^{-s} A^{-b} h \rangle|$$

$$\leq ||J^s A^b f||_{L^2} ||J^{-s} A^{-b} h||_{L^2}$$

$$= ||f||_{X^{s,b}} ||h||_{X^{-s,-b}}.  \hfill (5.3)$$
Hence $\Phi_h \in (X^{s,b})^*$ with $\|\Phi_h\| \leq \|h\|_{X^{-s,-b}}$. Moreover,

$$
\Phi_h = \sup_{\|f\|_{X^{s,b}} \leq 1} |\langle J^s \Lambda^b f, J^{-s} \Lambda^{-b} h \rangle | = \sup_{\|\ell\|_{X^{s,b}} \leq 1} |\langle \ell, J^{-s} \Lambda^{-b} h \rangle | = \|J^{-s} \Lambda^{-b} h\|_{L^2_{tx}} = \|h\|_{X^{-s,-b}}.
$$

(5.4)

It remains to show that $\Phi$ is onto. Let $y$ be a bounded linear functional on $X^{s,b}$. Then

$$
\tilde{z} = y \circ J^{-s} \Lambda^{-b}
$$

is a bounded linear functional on $L^2_{tx}$ and by the Riesz’s representation theorem there exists $\tilde{h} \in L^2_{tx}$ with $\tilde{z}(\tilde{f}) = \langle \tilde{f}, \tilde{h} \rangle$ for all $\tilde{f} \in L^2_{tx}$. Now, note that $h := J^s \Lambda^b \tilde{h}$ belongs to $X^{-s,-b}$ and it is easy to show that $y(f) = \Phi_h(f)$ for all $f \in X^{s,b}$. Finally, let $f \in X^{s,b} \cap L^2_{tx}$ and $h \in X^{-s,-b} \cap L^2_{tx}$. From the above computations we have

$$
\langle f, h \rangle = \langle J^s \Lambda^b f, J^{-s} \Lambda^{-b} h \rangle,
$$

and the proof is completed. □

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DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF CAMPINAS, 13083-859, CAMPINAS, SP, BRAZIL
E-mail address: marcelonogueira19@gmail.com

DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF CAMPINAS, 13083-859, CAMPINAS, SP, BRAZIL
E-mail address: mpanthee@ime.unicamp.br