Graded Holonomic D-modules on Monomial Curves

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Abstract In this paper, we study the holonomic $D$-modules when $D$ is the ring of $k$-linear differential operators on $A = k[\Gamma]$, the coordinate ring of an affine monomial curve over the complex numbers $k = \mathbb{C}$. In particular, we consider the graded case, and classify the simple graded $D$-modules and compute their extensions. The classification over the first Weyl algebra $D = A_1(k)$ is obtained as a special case.

Keywords Rings of differential operators · $D$-modules · Representation theory

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Introduction

Let $\Gamma \subseteq \mathbb{N}_0$ be a numerical semigroup, generated by positive integers $a_1, \ldots, a_r$ without common factors, such that $H = \mathbb{N}_0 \setminus \Gamma$ is a finite set. We consider its semigroup algebra $A = k[\Gamma]$ over the field $k = \mathbb{C}$ of complex numbers. Since $X = \text{Spec}(A)$ is the affine monomial curve $X = \{(t^{a_1}, t^{a_2}, \ldots, t^{a_r}) : t \in k\} \subseteq \mathbb{A}^r_k$, we call $A$ a monomial curve.

We described the ring $D = \text{Diff}(A)$ of differential operators on a monomial curve $A = k[\Gamma]$ in Eriksen [8], using the graded structure. For any degree $w$, there is a homogeneous differential operator $P_w$ of minimal order in $D_w$, given by

$$P_w = t^w \prod_{\gamma \in \Omega(w)} (E - \gamma)$$

where $E = t\partial$ is the Euler derivation, and $\Omega(w) = \{\gamma \in \Gamma : \gamma + w \notin \Gamma\}$. Moreover, $E$ together with $\{P_w : |w| \in \{a_1, \ldots, a_r\} \text{ or } |w| \in H\}$ is a set of homogeneous generators for $D$, and $D_w = P_w \cdot k[E]$. We also showed that the associated graded ring $\text{gr} D$ is the finitely generated semigroup algebra $\text{gr} D = k[\Gamma'] \subseteq k[t, \xi]$, with

$$\Gamma' = \{(m, n) \in \mathbb{N}_0^2 : n \geq \sigma(m - n)\} \subseteq \mathbb{N}_0^2$$

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where we write $\sigma(w)$ for the cardinality of $\Omega(w)$, which satisfies $\sigma(w) = d(P_w)$.

In this paper, we use these results to study the holonomic $D$-modules on a monomial curve. Note that $\text{gr} D$ is a positively graded $k$-algebra via the Bernstein filtration, but it is not generated by homogeneous elements of degree one. We use the results of Shukla [11] to prove that for any good filtration of $M$, there is a period $m \geq 1$ and polynomials $P_0, \ldots, P_{m-1} \in \mathbb{Q}[t]$ such that $\dim_k (M_{nm+r}) = P_r(n)$ for $0 \leq r < m$ and for all sufficiently large integers $n$. Moreover, the polynomials

$$P_r(t) = \frac{e}{dt} t^d + \text{terms of lower degree}$$

all have the same leading term. We define $d(M) = d$ and $e(M) = e$ to be the dimension and multiplicity of $M$, and prove the Bernstein inequality $d(M) \geq 1$.

We define a $D$-module to be holonomic if $d(M) = 1$, and use the properties of dimension and multiplicity to study holonomic $D$-modules, generalizing the standard results; see Coutinho [2]. In particular, we show that a $D$-module is holonomic if and only if it has finite length; this means that we can study the length category of holonomic $D$-modules via its simple modules and their extensions. The simplest example of a monomial curve is $A = k[t]$, with the first Weyl algebra $D = A_1(k)$ as its ring of differential operators. The simple modules over the first Weyl algebra were classified in Block [1], and form a very large set. We therefore restrict our attention to the graded case.

The simple graded modules over the graded ring $D$ are the simple objects in the length category $\text{grHol}_D$ of graded holonomic $D$-modules. We classify the simple graded $D$-modules, up to graded isomorphisms and twists:

**Theorem** If $A = k[t]$ is a monomial curve, then the simple graded left modules over the ring $D$ of differential operators on $A$, up to graded isomorphisms and twists, are

$$\{M_0\} \cup \{M_\alpha : \alpha \in J^*\} \cup \{M_\infty\}$$

where $J^* = \{\alpha \in k : 0 \leq \text{Re}(\alpha) < 1, \alpha \neq 0\}$. Moreover, we have that $M_0 = A$ and that $M_\alpha = D/D \cdot (E - \alpha)$ for all $\alpha \in J^*$.

Any ring of differential operators on a monomial curve is Morita equivalent to the first Weyl algebra $A_1(k)$, which is itself the ring of differential operators on the monomial curve $A = k[t]$. We obtain the following special case:

**Corollary** The simple graded left modules over the first Weyl algebra $D = A_1(k)$, up to graded isomorphisms and twists, are given by $M_0 = D/D\partial$, $M_\infty = D/Dt$, and $M_\alpha = D/D(D - \alpha)$ for all $\alpha \in J^*$.

The Krull-Schmidt Theorem holds in $\text{grHol}_D$ since it is a length category. In Eriksen [9], we describe a constructive method for finding the indecomposable objects in $\text{grHol}_D$ when the simple objects and their extensions are given. We have therefore computed the extensions of the simple graded $D$-modules:

**Proposition** If $D$ is the ring of differential operators on a monomial curve, then the extensions of $M_\beta$ by $M_\alpha$ in $\text{grHol}_D$ are given by

$$\text{Ext}^1_D(M_\alpha, M_\beta) = \begin{cases} k^\times \cong k, & (\alpha, \beta) = (0, \infty), (\infty, 0) \\ k^\times \cong k, & \alpha = \beta \in J^* \\ 0, & \text{otherwise} \end{cases}$$

for all simple graded $D$-modules $M_\alpha, M_\beta$ with $\alpha, \beta \in J^* \cup \{0, \infty\}$.
For example, when \( D = A_1(k) \) is the first Weyl algebra, the proposition shows that \( \text{Ext}^1_{D_1}(M_{\infty}, M_0)_0 = k\xi \cong k \) for a non-split graded extension \( \xi \). We represent \( \xi \) by the graded extension

\[
0 \to D_t/D\partial t \to D/D\partial t \to D/Dt \to 0
\]

where \( D/Dt = M_{\infty} \) and \( D/D\partial t \cong M_0[-1] \). Hence \( M = D/D\partial t \) is the unique indecomposable graded holonomic \( D \)-module of length two with composition series \( M \supseteq M_0[-1] \supseteq 0 \) and simple factors \( M_{\infty} \) and \( M_0[-1] \).

It turns out that this example can be generalized. In fact, we use the results of this paper to classify all graded holonomic \( D \)-modules in Eriksen [4], where we prove the following result:

**Theorem** Let \( D = A_1(k) \) be the first Weyl algebra. The indecomposable \( D \)-modules in \( \text{grHol}_D \) are, up to graded isomorphisms and twists, given by

\[
M(\alpha, n) = D/D (E - \alpha)^n, \quad M(\beta, n) = D/D w(\beta, n)
\]

where \( n \geq 1, \alpha \in J^* = \{ \alpha \in k : 0 \leq \text{Re}(\alpha) < 1, \alpha \neq 0 \}, \beta \in \{0, \infty\}, \) and \( w(\beta, n) \) is the alternating word on \( n \) letters in \( t \) and \( \partial \), ending with \( \partial \) if \( \beta = 0 \), and ending in \( t \) if \( \beta = \infty \).

We remark that the assumption that \( k = \mathbb{C} \) is one of convenience; the results would hold over any algebraically closed field \( k \) of characteristic 0, and the methods would be applicable if \( k \) is any field of characteristic 0.

**1 Differential operators on affine monomial curves**

Let \( \Gamma \subseteq \mathbb{N}_0 \) be a numerical semigroup, and let \( \{a_1, a_2, \ldots, a_r\} \) be a minimal set of generators of \( \Gamma \). Without loss of generality, we may assume that these generators are without common factors, such that \( H = \mathbb{N}_0 \setminus \Gamma \) is a finite set. We consider the semigroup algebra \( A = k[\Gamma] \cong k[t^{a_1}, t^{a_2}, \ldots, t^{a_r}] \subseteq k[t] \) over the field \( k = \mathbb{C} \) of complex numbers, and the algebra \( D = \text{Diff}(A) \) of \( k \)-linear differential operators on \( A \). Notice that \( A = k[\Gamma] \) is the coordinate ring of an affine monomial curve \( X \subseteq \mathbb{A}^r \). By abuse of language, we shall call \( A = k[\Gamma] \) a monomial curve.

We notice that \( A = k[\Gamma] \) is a positively graded ring. Let \( S = \{t^n : n \in \Gamma\} \subseteq A \) be the multiplicatively closed subset consisting of the homogeneous elements in \( A \), and consider the graded localization \( A \subseteq S^{-1}A = k[t, t^{-1}] = T \). By general localization results for rings of differential operators, there is a localization

\[
D = \text{Diff}(A) \to S^{-1}D = S^{-1}A \otimes_A D = k[t, t^{-1}] \otimes_A D = \text{Diff}(T)
\]

and we may identify \( D \) with the subring \( \{P \in \text{Diff}(T) : P(A) \subseteq A \} \subseteq \text{Diff}(T) \); see for instance Smith, Stafford [12].

Let us write \( B = k[t] \) for the normalization of \( A \), and \( \text{Diff}(B) \) for its algebra of \( k \)-linear differential operators. We remark that \( B \) is itself the coordinate ring of an affine monomial curve, the affine line \( \mathbb{A}^1 \), and that \( \text{Diff}(B) = A_1(k) = k[t, \partial] \) is the first Weyl algebra, generated by \( t \) and \( \partial = d/dt \) and with relation \( [\partial, t] = 1 \). In particular, it follows that

\[
\text{Diff}(T) \cong k[t, t^{-1}] \otimes_{k[t]} A_1(k) \cong k[t, t^{-1}]/(\partial)
\]
There is a grading on \( \text{Diff}(T) \) induced by the grading on \( T \), such that \( P \in \text{Diff}(T) \) is homogeneous of degree \( w \) if and only if \( P(T_i) \subseteq T_{i+w} \) for all integers \( i \). In concrete terms, \( t^n \partial^m \in \text{Diff}(T) \) is homogeneous of degree \( n-m \). Moreover, \( D = \text{Diff}(A) \) is the graded subalgebra

\[
D = \{ P \in \text{Diff}(T) : P(A) \subseteq A \} \subseteq k[t, t^{-1}][\partial]
\]

We write \( D = \bigoplus_w D_w \), where \( D_w \) is the linear subspace of differential operators of degree \( w \).

Based on these results, we gave an explicit description of the algebra \( D \) of differential operators on an affine monomial curve in Eriksen [8]: For any degree \( w \in \mathbb{Z} \), we have that \( D_w = P_w \cdot k[E] \), where \( E = t \partial \) is the Euler derivation of degree zero, and \( P_w \) is the homogeneous differential operator of degree \( w \) given by

\[
P_w = t^w \prod_{\gamma \in D(w)} (E - \gamma)
\]

determined by the set \( D(w) = \{ \gamma \in \Gamma : \gamma + w \notin \Gamma \} \). In particular, we proved the following result:

**Theorem 1** If \( A = k[\Gamma] \) is a monomial curve, then the \( k \)-algebra \( D \) of differential operators on \( A \) is generated by the Euler derivation \( E \) and the differential operators \( P_w \) for all degrees \( w \) such that \( |w| \in \{a_1, a_2, \ldots, a_r\} \) or \( |w| \in H \).

The following structural results on the algebra \( D \) of differential operators on \( A \) is a consequence of the work of Smith, Stafford [12]:

**Theorem 2** If \( A = k[\Gamma] \) is a monomial curve, then the ring \( D \) of differential operators on \( A \) is Morita equivalent with the first Weyl algebra \( \text{Diff}(B) = A_1(k) \), and \( D \) has the following properties:

1. \( D \) is a simple Noetherian ring
2. \( A \) is a simple left \( D \)-module
3. \( D \) has Krull dimension 1 and Gelfand-Kirillov dimension 2
4. \( D \) is a hereditary ring

**Proof** Since the normalization of \( X = \text{Spec}(A) \) is the affine line \( \text{Spec}(B) = \mathbb{A}^1 \), and the normalization map \( \mathbb{A}^1 \to X \) is injective, it follows from Proposition 3.3, Theorem 3.4, Theorem 3.7 and Proposition 4.2 in Smith, Stafford [12] that \( D \) is Morita equivalent with the Weyl algebra \( A_1(k) \), that \( D \) is a simple ring, and that \( A \) is a simple left \( D \)-module. The rest follows from the fact that the properties are preserved under Morita equivalence, and hold for \( A_1(k) \). □

Let \( D^p \subseteq D \) be the set of differential operators in \( D \) of order at most \( p \). We call \( \{D^p\} \) the order filtration of \( D \), and consider the associated graded ring

\[
gr D = \bigoplus_{p \geq 0} D^p / D^{p-1}
\]

It is well-known that \( gr D \) is a commutative ring, and that \( gr \text{Diff}(B) = k[t, \xi] \), where \( \xi \) is the image of \( \partial \). In Eriksen [8], we proved that, for \( \Gamma \neq \mathbb{N}_0 \), the ring \( gr D = k[\Gamma^*] \subseteq k[t, \xi] \) is a semigroup algebra with minimal set of generators

\[
\{ t^{\sigma(-w)} \xi^{\sigma(w)} : |w| \in \{a_1, \ldots, a_r\} \text{ or } |w| \in H \} \cup \{t\xi\}
\]
In the proof, we use the fact that $P_w$ has leading term $t^{w+\sigma(w)}\partial^{\sigma(w)}$, where $\sigma(w)$ is the cardinality of $\Omega(w)$, and that $\sigma(-w) = \sigma(w)+w$ for all integers $w$. In particular, $\text{gr } D = k[F']$ is a finitely generated $k$-algebra of Krull dimension two, and $\mathbb{N}_0^2 \setminus F'$ is a finite set.

Let us write $D^p_w = D^p \cap D_w \subseteq D$ for the linear subspace of differential operators of degree $w$ and order at most $p$ in $D$, and define

$$B^n = \bigoplus_{2p+w \leq n} D^p_w$$

for all integers $n$. Then $\{B^n\}$ is a filtration of $D$, which we call the Bernstein filtration. Note that any differential operator $P \in B^n$ is of the form

$$P = \sum_{i+j \leq n} c_{ij} t^i \partial^j$$

Clearly, the associated graded ring $\oplus_n B^n/B^{n-1}$ is naturally isomorphic to $\text{gr } D$. Moreover, since $\text{gr } D \subseteq k[t, \xi]$, it follows that $B^0 = k$, that $B^n = 0$ for $n < 0$, and that $\dim_k B^n$ is finite for all integers $n$.

2 Holonomic $D$-modules on monomial curves

Let $M$ be a left $D$-module. A filtration of $M$ is a chain $M_0 \subseteq M_1 \subseteq \ldots$ of $k$-linear subspaces of $M$ such that $\cup_n M_n = M$ and $B^m \cdot M_n \subseteq M_{n+m}$ for all integers $m, n \geq 0$. By convention, we let $M_n = 0$ for $n < 0$. We say that the filtration is good if the associated graded module $\text{gr } M = \oplus_n M_n/M_{n-1}$ is a finitely generated module over $\text{gr } D$.

We recall that there is a good filtration of $M$ if and only if $M$ is a finitely generated $D$-module. Moreover, if $\{M_n\}$ and $\{M'_n\}$ are two good filtrations of $M$, then there is an integer $N$ such that

$$M_{n-N} \subseteq M'_n \subseteq M_{n+N}$$

for all integers $n$. This is a standard result; see for instance Chapter 1 in Björk [1].

We fix a good filtration $\{M_n\}$ of a left $D$-module $M$, and remark that $\dim_k M_n$ is finite for all integers $n \geq 0$. In fact, we have that $(\text{gr } M)_n = M_n/M_{n-1}$ is finitely generated over $B_0 = k$, since $\text{gr } M$ is a finitely generated module over $\text{gr } D$, and $\text{gr } D$ is positively graded with $(\text{gr } D)_0 = B_0 = k$. The Hilbert function of $\text{gr } M$ is the function $H(\text{gr } M, -)$ given by

$$H(\text{gr } M, n) = \dim_k (M_n/M_{n-1}) = \dim_k M_n - \dim_k M_{n-1}$$

and $H^1(\text{gr } M, n) = \dim_k M_n$ is the first iterated Hilbert function of $\text{gr } M$.

**Proposition 3** Let $M$ be a non-zero, finitely generated left $D$-module and let $\{M_n\}$ be a good filtration of $M$. Then there is a positive integer $m \geq 1$ and polynomials $P_r(t) \in \mathbb{Q}[t]$ for $0 \leq r < m$, such that $\dim_k (M_{nm+r}) = P_r(n)$ for all sufficiently large integers $n$. Moreover, we have that

$$P_r(t) = \frac{e}{d!} t^d + \text{terms of lower degree}$$

for $0 \leq r < d$, where $d$ is the Krull dimension of $\text{gr } M$, and $e > 0$ is a positive integer.
Proof To simplify notation, we write $R = \text{gr } D$, and $N = \text{gr } M$. We consider the Hilbert series $H_N(t) = \sum_n H(N,n) \cdot t^n$ of $N$ in $\mathbb{Z}[t]$, which can be written as a rational function

$$H_N(t) = \frac{Q(t)}{(1 - t)^d}$$

where $Q(t) \in \mathbb{Z}[t]$ with $Q(1) > 0$, and $d \geq 1$ is a positive integer. This follows from Proposition 4.4.1 and the following remarks in Bruns, Herzog [3], since there is a homogeneous system of parameters for $N$ of common degree $l > 0$. Let

$$m = \min\{l \geq 1 : H_N(t) \cdot (1 - t)^d \in \mathbb{Z}[t]\}$$

and let $a(t) = H_N(t) \cdot (1 - t)^d \in \mathbb{Z}[t]$. Then $a(1) > 0$, and by Proposition 2.3 and Remark 2.4 in Campbell et al. [4], it follows that there are polynomials $p_r(t) \in \mathbb{Q}[t]$ for $0 \leq r < m$ such that $\deg p_r(t) \leq d - 1$ and $p_r(n) = H(N,nm + r)$ for all $n$ large enough. Furthermore, since $a(1) > 0$, it follows that $p_r(t)$ has degree $d - 1$ for at least one integer $r$ with $0 \leq r < m$. Let us write $a_r(t)$ for the polynomial

$$a_r(t) = \sum_{n \geq 0} a_{nm + r} \cdot t^n$$

in $\mathbb{Z}[t]$, where the coefficients $a_i$ for $i \geq 0$ are given by $a(t) = \sum a_it^i$. We obtain polynomials $a_0(t), \ldots, a_{m - 1}(t) \in \mathbb{Z}[t]$ with $a(1) = a_0(1) + \cdots + a_{m - 1}(1)$. Furthermore, the polynomial $p_r(t)$ has the form

$$p_r(t) = \frac{a_r(1)}{(d - 1)!} t^d + \text{terms of lower degree}$$

for $0 \leq r < m$, where $a_r(1) \geq 0$ for all $r$. To compute the first iterated Hilbert function, we choose a positive integer $n_0$ such that $H(N,nm + r) = p_r(n)$ for $n > n_0$ and $0 \leq r < m$. If we let $C_r = H(N,nm + r + 1) + \cdots + H(N,nm + m - 1)$, then we have, for $n > n_0$, that

$$H^2(N,nm + r) = \sum_{s=0}^{m-1} \sum_{t=0}^{n} H(N,tm + s) - C_r$$

$$= \sum_{s=0}^{m-1} \sum_{t=n_0+1}^{n} p_s(t) - C_r + H^2(N,n_0 m + m - 1)$$

$$= \sum_{s=0}^{m-1} \sum_{t=n_0+1}^{n} \frac{a_s(1)}{(d - 1)!} t^d + \text{terms of lower degree}$$

$$= \sum_{t=n_0+1}^{n} \frac{a(1)}{(d - 1)!} t^d + \text{terms of lower degree}$$

since $C_r$ and $H^1(N,n_0 m + m - 1)$ are constants. Hence, there are polynomials $P_r(t) \in \mathbb{Q}[t]$ for $0 \leq r < m$ of the form

$$P_r(t) = \frac{e}{d!} t^d + \text{terms of lower degree}$$

such that $H^2(nm + r) = P_r(n)$ for all $n > n_0$, where $d$ is the Krull dimension of $N$ and $e = a(1) > 0$ is a positive integer. \qed
We remark that \( d \) and \( e \) in Proposition 3 are independent of the chosen good filtration \( \{ M_n \} \) of \( M \). We define the dimension of \( M \) to be \( d(M) = d \) and the multiplicity of \( M \) to be \( e(M) = e \) for any finitely generated left \( D \)-module \( M \neq 0 \). These invariants have the following properties: If \( 0 \to M' \to M \to M'' \to 0 \) is a short exact sequence of \( D \)-modules, then \( d(M) = \max\{d(M'), d(M'')\} \). If moreover \( d(M') = d(M) = d(M'') \), then \( e(M) = e(M') + e(M'') \). This follows from the fact that a good filtration of \( M \) induces good filtrations of \( M' \) and \( M'' \).

**Proposition 4 (Bernstein’s inequality)** For any finitely generated left \( D \)-module \( M \neq 0 \), we have \( d(M) \geq 1 \).

**Proof** From Theorem 2, it follows that \( D \) is a simple ring, and the algebra homomorphism \( D \to \text{End}_k(M) \) is therefore injective. This implies that \( M \) cannot be finite dimensional over \( k \), hence \( d(M) \geq 1 \).

We say that a finitely generated left \( D \)-module \( M \) is holonomic if \( M \neq 0 \) and \( d(M) = 1 \). By convention, \( M = 0 \) is also considered to be holonomic. If \( M \) is holonomic, then it has finite length. In fact, if \( M' \subseteq M \) is a non-zero submodule of \( M \), then \( M' \) is holonomic and \( e(M') < e(M) \) if \( M' \neq M \). Hence the length of \( M \) is at most \( e(M) \).

**Proposition 5** Let \( M \) be a finitely generated left \( D \)-module. Then the following conditions are equivalent:

1. \( M \) is holonomic
2. \( M \) has finite length
3. \( M \) is cyclic and not isomorphic to \( D \)

**Proof** If \( M \) is holonomic, then it has finite length by the comment above. If \( M \) has finite length, then \( M \) is cyclic by Theorem 1.8.18 in Björk [1], since \( D \) is a simple ring by Theorem 2. Moreover, \( M \) is not isomorphic to \( D \) since \( d(M) = 1 \) and \( d(D) = 2 \). In fact, the Bernstein filtration is a good filtration of \( D \), and we have

\[
\dim_k B^n = 1 + 2 + \cdots + (n+1) - s = \frac{(n+1)(n+2)}{2} - s = \frac{1}{2}n^2 + \text{ terms of lower degree}
\]

for \( n \) sufficiently large, where \( \text{gr} D = k[I'] \subseteq k[t, \xi] \) and \( s \) is the cardinality of \( \mathbb{N}^2 \setminus I' \). Finally, we show that if \( M \) is cyclic and not isomorphic to \( D \), then it is holonomic. In this case, we may assume that \( M = D/I \) with \( I \neq 0 \), and there is a non-zero element \( P \in I \) and a principal left ideal \( J = D \cdot P \subseteq I \). The short exact sequence

\[
0 \to D \xrightarrow{P} D \to D/J \to 0
\]

shows that \( d(D/J) = 1 \). In fact, if \( d(D/J) = 2 \), then \( e(D) = 1 \) and \( e(D/J) = 0 \), which is a contradiction since \( e(D) = 1 \) from the computation above. The short exact sequence \( 0 \to I/J \to D/J \to D/I \to 0 \) gives that \( d(M) = d(D/I) \leq d(D/J) = 1 \), and this implies that \( M \) is holonomic. \( \square \)
3 Graded holonomic D-modules

Let \( D \) be the algebra of differential operators on a monomial curve \( A = k[G] \), with the \( \mathbb{Z} \)-grading \( D = \oplus_w D_w \) described in Section 1, and consider the category \( \text{grMod}_D \) of \( \mathbb{Z} \)-graded left \( D \)-modules. An object of \( \text{grMod}_D \) is a left \( D \)-module \( M \) with a grading
\[
M = \oplus_w M_w
\]
such that \( D_w \cdot M_w \subseteq M_{w+w} \), and a morphism \( \phi : M \to N \) in \( \text{grMod}_D \) is a \( D \)-module homomorphism which is homogeneous of degree zero, such that \( \phi(M_w) \subseteq N_w \). For any graded \( D \)-module \( M \) in \( \text{grMod}_D \) and any integer \( n \), we denote by \( M[n] \) the \( n \)th twisted \( D \)-module of \( M \), with grading given by \( M[n]_i = M_{n+i} \).

We wish to study and classify the graded holonomic \( D \)-modules, up to graded isomorphisms in \( \text{grMod}_D \) and twists. The full subcategory \( \text{grHol}_D \subseteq \text{grMod}_D \) of graded holonomic \( D \)-modules consists of graded \( D \)-modules \( M \) of finite length, with composition series
\[
M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_{n-1} \supseteq M_n = 0
\]
in \( \text{grMod}_D \) of length \( n \leq e(M) \); see Proposition [5]. We are interested in the simple objects in \( \text{grMod}_D \), since these are the simple factors \( M_i/M_{i+1} \) in the composition series.

**Lemma 6** A graded left \( D \)-module \( M \) is a simple object in \( \text{grMod}_D \) if and only if it is simple considered as a left \( D \)-module.

**Proof** It is clear that if \( M \) is simple as a left \( D \)-module, then it is simple in \( \text{grMod}_D \). To prove the converse, assume that \( M \) is simple in \( \text{grMod}_D \). Then it follows from Theorem II.7.5 in Năstăescu, van Oystaeyen [10] that it is either simple or 1-critical considered as a left \( D \)-module. We claim that it cannot be 1-critical. In fact, we may choose a homogeneous element \( m \neq 0 \) of degree \( w \) in \( M \), and consider the short exact sequence
\[
0 \to I \to D(-w) \xrightarrow{m} M \to 0
\]
where \( D(-w) \) is a twist of \( D \) and \( I = \{ P \in D(-w) : P \cdot m = 0 \} \) is the annihilator of \( m \). Since \( I \neq 0 \), it follows from Proposition [5] that \( D/I \) is a holonomic \( D \)-module, and therefore of finite length. This implies that \( M \) is Artinian, of Krull dimension zero, and it is therefore not 1-critical. It follows that any simple object in \( \text{grMod}_D \) is simple considered as a left \( D \)-module.

The simple modules over the Weyl algebra \( A_1(k) \) were classified in Block [2]. We shall classify the simple objects in \( \text{grMod}_D \) by adapting Block’s results to the graded situation. For any graded left \( D \)-module \( M \), we define
\[
T_S(M) = \{ m \in M : sm = 0 \text{ for an element } s \in S \} \subseteq M
\]
It follows from the fact that \( S \) is an Ore set for \( D \) that \( T_S(M) \) is a left \( D \)-module. We say that \( M \) is an \( S \)-torsion module if \( T_S(M) = M \), and that it is an \( S \)-torsion free module if \( T_S(M) = 0 \). Notice that any simple left \( D \)-module is either an \( S \)-torsion module or an \( S \)-torsion free module, and that it is an \( S \)-torsion module if and only if \( S^{-1}M = 0 \).
There is a graded division algorithm in the ring Diff(T) in the following sense: For any homogeneous differential operators $P, Q \in \text{Diff}(T)$ with $Q \neq 0$, we have that

$$P = L \cdot Q + R$$

for unique homogeneous differential operators $L, R \in \text{Diff}(T)$, where the order $d(R) < d(Q)$. Therefore, any homogeneous left ideal $I \subset \text{Diff}(T)$ is principal. In fact, if $P \neq 0$ is an element in $I$ with minimal order, then $I = \text{Diff}(T) \cdot P$. This implies that a graded left Diff(T)-module is simple considered as a Diff(T)-module if and only if it is a simple object in the category of graded left Diff(T)-modules, and we can talk about simple graded left Diff(T)-modules without ambiguity.

**Proposition 7** The assignment $M \mapsto S^{-1}M$ defines a bijective correspondence

$$S^{-1} : \text{grSimp}_{D}[S\text{-torsion free}] \to \text{grSimp}_{\text{Diff}(T)}$$

from the set of isomorphism classes of simple graded left $D$-modules that are $S$-torsion free, and the set of isomorphism classes of simple graded left Diff(T)-modules.

**Proof** If $M$ is a simple graded left $D$-module that is $S$-torsion free, then $S^{-1}M$ is a simple graded left Diff(T)-module, since $S$ consists of homogeneous elements and localization is exact; see also Lemma 2.2.1 in Block 2. This defines the map $S^{-1}$, and that it is injective follows from the proof of Lemma 2.2.1 in Block 2. To show that $S^{-1}$ is surjective, let us consider a graded simple left Diff(T)-module $N$. We choose a non-zero homogeneous element $n \in N$ of degree $w$, which gives a short exact sequence

$$0 \to I \to \text{Diff}(T)[-w] \to N \to 0$$

where $I = \{P \in \text{Diff}(T) : P \cdot n = 0\} = \text{ann}(n)$. Let $J \equiv I \cap D[-w] \subset D[-w]$. This is a non-zero homogeneous ideal in $D[-w]$, and $M = D[-w]/J \subset \text{Diff}(T)[-w]/I \cong N$ is a graded $D$-submodule. Since $J \neq 0$, $M$ is a graded $D$-module of finite length by Proposition 5, hence it contains a graded simple left $D$-module $K$. Then it follows from Lemma 2.2.1 in Block 2 and its proof that $N \cong S^{-1}K$ since $K \subset M \subset N$. \qed

Next, we classify the simple graded left Diff(T)-modules, up to graded isomorphisms and twists. Any simple graded left Diff(T)-module $N$ must be of the form $N \cong \text{Diff}(T)[-w]/I$, where $I = \text{Diff}(T) \cdot P$ is a homogeneous principal left ideal that is maximal. Hence, we must have that $P = t^{\omega}(E - \alpha)$ for $\alpha \in k$, and $t^{\omega}$ is a unit in Diff(T). We obtain the cases $\alpha = 0$, which gives the graded simple module

$$N_0 = \text{Diff}(T)/\text{Diff}(T) \cdot E = \text{Diff}(T)/\text{Diff}(T) \cdot \partial \cong T$$

and $\alpha \neq 0$, which gives the graded simple module

$$N_\alpha = \text{Diff}(T)/\text{Diff}(T) \cdot (E - \alpha)$$

It is not difficult to see that $N_\alpha \cong T[\alpha]$ if $\alpha \in \mathbb{Z}$, with isomorphism given by $1 \mapsto t^{\alpha}$. Similarly, we have that $N_\alpha \cong N_\beta[\alpha - \beta]$ when $\alpha - \beta \in \mathbb{Z}$, with isomorphism given by $1 \mapsto t^{\alpha - \beta}$.

**Lemma 8** Let $J = \{\alpha \in k : 0 \leq \Re(\alpha) < 1\}$. Then the set of simple graded Diff(T)-modules, up to graded isomorphisms and twists, are given by $\{N_\alpha : \alpha \in J\}$. 


Proof We claim that if $\alpha - \beta \not\in \mathbb{Z}$, then $N_\alpha \not\cong N_\beta$ as left $\text{Diff}(T)$-modules. In light of the comments above, this is enough to prove the lemma, since $J$ is a fundamental domain for $k/\mathbb{Z}$. To prove the claim, we consider the Weyl algebra $A_1(k)$ as a special case of $D = \text{Diff}(A)$ with $A = k[T]$ and $T = \mathbb{N}_0$. By Proposition 7 we have that $M_\alpha = A_1(k)/A_1(k) \cdot (E - \alpha)$ corresponds to $N_\alpha = S^{-1}M_\alpha$ under localization when $\alpha \not\in \mathbb{Z}$, and $M_0 = A_1(k)/A_1(k) \cdot \partial$ corresponds to $N_0$. By results of Dixmier on the Weyl algebra, we have that $M_\alpha \not\cong M_\beta$ when $\alpha - \beta \not\in \mathbb{Z}$; see Lemma 24 in Dixmier [6] and Proposition 4.4 in Dixmier [7], and this proves the claim. □

Let $A = k[T]$ be a monomial curve, let $D$ be the ring of differential operators on $A$, and let $A_1(k)$ be the ring of differential operators on $B = k[t]$. Then $D$ is Morita equivalent with the Weyl algebra $A_1(k)$, and by Section 3.14 of Smith, Stafford [12], the equivalence $\text{Mod}_D \rightarrow \text{Mod}_{A_1(k)}$ of module categories is given by $M \mapsto D(A,B) \otimes_D M$, where $D(A,B)$ is the $A_1(k)$-$D$ bimodule

$$D(A,B) = \{ P \in \text{Diff}(T) : P(A) \subseteq B \} \subseteq \text{Diff}(T).$$

Notice that $D(A,B)$ is a graded $A_1(k)$-$D$ bimodule, with $S^{-1}D(A,B) = \text{Diff}(T)$. Hence, there is an induced equivalence $F : \text{grMod}_D \rightarrow \text{grMod}_{A_1(k)}$ of categories of graded modules, and it commutes with localization:

$$\text{grMod}_D \xrightarrow{F} \text{grMod}_{A_1(k)}$$

We know that $\text{grSimp}_{\text{Diff}(T)}[\text{S-torsion free}] \cong \text{grSimp}_{\text{Diff}(T)}$, and we write $M_\alpha$ for the unique simple graded left $D$-module $M$ such that $S^{-1}M = N_\alpha$ for $\alpha \in J$. In fact, it follows from the proof of Proposition 7 that $M_\alpha$ is the unique simple submodule of $D/D \cdot (E - \alpha)$ if $\alpha \neq 0$, and that $M_0$ is the unique simple submodule of $D/D \cdot \partial = A$. We claim that $M_0 = A$ if $\alpha = 0$, and that $M_\alpha = D/D \cdot (E - \alpha)$ if $\alpha \neq 0$. If $\alpha = 0$, this is clear since $A$ is a simple left $D$-module. If $\alpha \neq 0$, then the claim follows from the following result, since simple modules are preserved by Morita equivalence:

Lemma 9 If $A = k[t]$, then $M_\alpha = D/D \cdot (E - \alpha)$ for all $\alpha \in J$ with $\alpha \neq 0$.

Proof It is sufficient, in light of the comments above, to show that $D/D \cdot (E - \alpha)$ is a simple module over $D = A_1(k)$. This follows from Lemma 24 in Dixmier [6]. □

Finally, we classify the simple graded $D$-modules that are $S$-torsion modules. We first consider the case $A = k[t]$, and the $S'$-torsion modules over the Weyl algebra $D = A_1(k)$, where $S' = k[t]^*$. In this case, it follows from Proposition 4.1 and Corollary 4.1 in Block [2] that the simple $S'$-torsion $D$-modules are given by

$$V(\beta) = D \otimes_{A_1} A/(t - \beta) \cong D/D \cdot (t - \beta)$$

for $\beta \in k$. Moreover, $V(\beta)$ is a graded left $D$-module, or an $S$-torsion module, if and only if $\beta = 0$. Hence, $V(0) = D/D \cdot t$ is the unique simple graded left module over the Weyl algebra $D = A_1(k)$ that is $S$-torsion.

Let $D$ be the ring of differential operators on $A$ when $A = k[T]$ is any monomial curve. Since $D$ is Morita equivalent to the Weyl algebra and the property of being
S-torsion is preserved under Morita equivalence, there is a unique simple graded left $D$-module $M_\infty$ that is S-torsion. Moreover, we have that

$$M_\infty = D(B, A) \otimes_{A_1(k)} V(0)$$

where $D(B, A) = \{ P \in \text{Diff}(T) : P(B) \subseteq A \} \subseteq \text{Diff}(T)$ by Section 3.14 in Smith, Stafford [12]. We summarize these results as follows:

**Theorem 10** If $A = k[\Gamma]$ is a monomial curve, then the simple graded left modules over the ring $D$ of differential operators on $A$, up to graded isomorphisms and twists, are given by

$$\{ M_0 \} \cup \{ M_\alpha : \alpha \in J^* \} \cup \{ M_\infty \}$$

where $J^* = \{ \alpha \in k : 0 \leq \text{Re}(\alpha) < 1, \alpha \neq 0 \}$. Moreover, we have that $M_0 = A$ and that $M_\alpha = D/D \cdot (E - \alpha)$ for all $\alpha \in J^*$.

**Corollary 11** The simple graded left modules over the first Weyl algebra $D = A_1(k)$, up to graded isomorphisms and twists, are given by $M_0 = D/D \partial$, $M_\infty = D/Dt$, and $M_\alpha = D/D \cdot (E - \alpha)$ for all $\alpha \in J^*$.

Let $M_\alpha, M_\beta$ be simple graded $D$-modules in $\text{grHol}_D$, with $\alpha, \beta \in J \cup \{ \infty \}$. Their extensions in $\text{grHol}_D$ are given by $\text{Ext}_D^1(M_\alpha, M_\beta)$.

**Proposition 12** If $D$ is the ring of differential operators on a monomial curve, then the extensions of $M_\beta$ by $M_\alpha$ in $\text{grHol}_D$ are given by

$$\text{Ext}_D^1(M_\alpha, M_\beta) = \begin{cases} k\xi \cong k, & (\alpha, \beta) = (0, \infty), (\infty, 0) \\ k\xi \cong k, & \alpha = \beta \in J^* \\ 0, & \text{otherwise} \end{cases}$$

for all simple graded $D$-modules $M_\alpha, M_\beta$ with $\alpha, \beta \in J^* \cup \{ 0, \infty \}$.

**Proof** By the comments above, we may assume that $D = A_1(k)$ is the Weyl algebra. We show the computation of the extensions in the case $\alpha = \infty$ and $\beta = 0$; all other cases can be done in a similar way. The $D$-module $M_\infty = D/Dt$ over the Weyl algebra $D = A_1(k)$ has the free resolution

$$0 \leftarrow M_\infty \leftarrow D \leftarrow D \leftarrow 0$$

When we apply $\text{Hom}_D(-, M_0)$ to this exact sequence, we obtain

$$\text{Hom}_D(D, M_0) \cong M_0 \xrightarrow{i} \text{Hom}_D(D, M_0) \cong M_0 \rightarrow 0$$

and $\text{Ext}_D^1(M_\infty, M_0) \cong M_0/tM_0 \cong A/tA = k[t]/(t) \cong k$, which is concentrated in degree zero. \[\square\]
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