SECOND ORDER SOBOLEV EMBEDDINGS
AND FOURTH-ORDER $pq$-LAPLACIANS

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Abstract. Motivated by the study of Sobolev embeddings of second order
on intervals and their approximation numbers, we examine the non-linear
$pq$-biharmonic eigenvalue problem on the unit segment subject to Navier boundary
conditions for general $1 < p, q < \infty$. By establishing existence, uniqueness
and symmetry of periodic solutions, we fully describe the eigenvalues and
eigenfunctions. In the case $q = p'$, these are given explicitly in terms of gene-
ralized trigonometric functions. However, remarkably, this is not so the case
when $q \neq p'$, as the eigenfunctions are related to a different family of functions
which are solution of a coupled system of integral equations. After determin-
ing these, we give an explicit description of the $s$-numbers of the Sobolev
embedding of second order with homogeneous Dirichlet boundary conditions.

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Date: April 2022.
1. Introduction

This paper centers around a fourth order non-linear eigenvalue problem associated to optimal norms and approximation numbers of second order Sobolev embeddings on intervals. The specific mathematical framework is as follows.

Let us begin by recalling the known context of first order Sobolev embeddings. For \( t_0 > 0 \), let \( \mathcal{I} = [0, t_0] \). For convenience we center the lower end of this interval at the origin, but all the results we report below have analogues for \( \mathcal{I} \) any bounded segment. For \( 1 < p < \infty \), let \( W_0^{1,p} \equiv W_0^{1,p}(\mathcal{I}) \) denote the closure of \( C_0^\infty \text{(Int} \mathcal{I}) \) in the real Sobolev space \( W^{1,p}(\mathcal{I}) \). The usual norm of the latter is

\[
\|u\|_{W^{1,p}} = \|u\|_{p,\mathcal{I}} + \|u'|_{p,\mathcal{I}}
\]

but in \( W_0^{1,p} \) the seminorm \( \|u\|_{W_0^{1,p}} = \|u'|_{p,\mathcal{I}} \) is also a norm equivalent to \( \|u\|_{W^{1,p}} \).

Here and elsewhere, we consider that the norm in \( W_0^{1,p} \) is this equivalent norm and the notation \( \|u\|_{p,\mathcal{I}} = \|u\|_{L^p(\mathcal{I})} \) will often come handy. For \( 1 < q < \infty \), consider embeddings

\[
E_1 : W_0^{1,p}(\mathcal{I}) \to L^q(\mathcal{I}),
\]

which are compact operators. As the underlying spaces are reflexive and strictly convex, then there exist non-zeros \( u_D \in W_0^{1,p}(\mathcal{I}) \) such that

\[
\|E_1\| = \sup_{u \in W_0^{1,p}} \frac{\|u\|_{q,\mathcal{I}}}{\|u_D\|_{q,\mathcal{I}}} > 0.
\]

We characterize the optimizers \( u_D \), by following the ideas described in [12 Sect. 3.2]. Indeed, according to the duality map approach described in [12 Propositions 1.10 and 1.11], there exists an optimizer such that \( \|u_D\|_{q,\mathcal{I}} = \|E_1\| \) and \( \|u_D\|_{W_0^{1,p}} = 1 \), which corresponds exactly to a solution of the “infinite-dimensional Lagrange multipliers” equation

\[
(1) \quad \lambda \langle v, \text{grad} \|u\|_{q,\mathcal{I}} \rangle_{L^q} = \left \langle v, \text{grad} \|u\|_{W_0^{1,p}} \right \rangle_{L^p}, \quad v \in C_0^\infty \text{(Int} \mathcal{I})
\]

where the eigenvalue \( \lambda = \lambda_1^q = \|E_1\|^{-q} \) is minimal. In the sense of distributions, the Gâteaux derivatives of the norms are, respectively,

\[
\text{grad} \|u\|_{q,\mathcal{I}} = \|u\|_{q,\mathcal{I}}^{1-q} \text{sgn} (u(t)) |u(t)|^{q-1} \quad u \in L^q(\mathcal{I}) \setminus \{0\}
\]

and

\[
\text{grad} \|u'\|_{p,\mathcal{I}} = -\|u'\|_{p,\mathcal{I}}^{1-p} \left (\text{sgn} (u'(t)) |u'(t)|^{p-1} \right )' \quad u \in W_0^{1,p}(\mathcal{I}) \setminus \{0\}.
\]

Then, (1) yields

\[
\lambda_1^q \int_0^{t_0} v(t) \text{sgn} (u_D(t)) |u_D(t)|^{-q} \, dt = \int_0^{t_0} v'(t) \text{sgn} (u_D'(t)) |u_D'(t)|^{p-1} \, dt
\]

which can be recast into an eigenvalue problem, c.f. [12 Sect. 1.3] (also see [8] and [9]). Indeed \( u_D \) is exactly an eigenfunction of

\[
(2) \quad -(\text{sgn} (u') |u'|^{p-1})' = \lambda \text{sgn} (u) |u|^{q-1}
\]

\[
u(0) = u(t_0) = 0
\]

for \( \lambda = \lambda_1^q > 0 \) the smallest eigenvalue.

The other extremal functions of the Sobolev embedding \( E_1 \) are exactly the other eigenfunctions of (2). But note that, for \( p \neq q \), the differential equation is not
homogeneous, so extra challenges arise when we estimate s-numbers and describe the eigenfunctions. We will elaborate fully on this in the context of second order Sobolev embeddings in Section 6. At this point, we only comment that the eigenfunctions and eigenvalues of (2) are found explicitly in terms of generalized trigonometric functions. Thus, the exact values of the different s-numbers for $E_1$ are given in terms of known special functions. See Section 5 as well as [6, 12] and references therein.

From this well studied framework, the following questions arise.

- Is there an analogue eigenvalue problem for higher order Sobolev embeddings?
- If so, can we expect a full characterization of extremal functions?
- Once the equation is posed, what properties do the eigenpairs have?
- How are the first and the higher order optimizers related?

In the present paper we address these questions for the specific case of the second order Sobolev space

$$W^{2,p}_D = \{ f \in W^{2,p}(I) : f(0) = f(t_0) = 1 \}$$

equipped with the norm

$$\| u\|_{W^{2,p}_D} = \| u''\|_{p,I}$$

and the corresponding compact embedding

$$E_2 : W^{2,p}_D(I) \to L^q(I).$$

Just as in the first order case, we seek for extremal elements $u_D \in W^{2,p}_D(I)$ such that

$$\| E_2 \| = \sup_{u \in W^{2,p}_D(I)} \frac{\| u\|_{q,I}}{\| u''\|_{p,I}} = \frac{\| u_D\|_{q,I}}{\| u''_D\|_{p,I}}.$$ 

To characterize $u_D$, we consider once again the duality maps approach which we now describe in more details, see [8, 9] and [10, Chapter 2].

Let $0 \neq T : X \to Y$ be a general compact map, where $X, Y, X^*$ and $Y^*$ are real and strictly convex spaces. Let $J_X$ and $J_Y$ be given by $J_X(x) = \text{grad} \| x \|_X$ and $J_Y(y) = \text{grad} \| y \|_Y$, where “grad” always denotes the derivative in Gâteaux sense evaluated at the corresponding points. Then, [12, Proposition 1.10], there exists $x_1 \in X$ such that $\| x_1 \|_X = 1$ and $T(x_1) = \| T\|$. Moreover, [12, Proposition 1.11], in the following diagram,

$$
\begin{align*}
X & \xrightarrow{T} Y \\
\downarrow J_X & \quad \downarrow J_Y \\
X^* & \xleftarrow{T^*} Y^*
\end{align*}
$$

$x = x_1$ satisfies the equation

$$T^* J_Y T x = \nu J_X x, \quad \text{with } \nu = \| T\|.$$ 

For the case of first order Sobolev embeddings described above, we take $X = W^{1,p}_0$, $Y = L^q$, $T = E_1$ and (1) is equivalent to (4).

So, now set $X = W^{2,p}_D$, $Y = L^q$ and $T = E_2$. Since $1 < p, q < \infty$, both $X$ and $Y$ are reflexive and strictly convex. Also $Y^*$, being $L^q'$, is strictly convex.
Furthermore, \(\|\cdot\|_X\) is Gâteaux-differentiable on \(X \setminus \{0\}\), hence \(X^*\) is also strictly convex, \([12]\) Proposition 1.8. In the sense of distributions,
\[
\hat{J}_Y(u) = \|u\|_{q,\mathcal{I}}^{1-q} \text{sgn}(u) |u|^{q-1}, \quad u \in L^q \setminus \{0\},
\]
and
\[
\hat{J}_X(u) = \|u''\|_{p,\mathcal{I}}^{1-p} \left(\text{sgn}(u'') |u''|^{p-1}\right)'' , \quad u \in W^{2,p}_D \setminus \{0\}.
\]

The equation (4) takes the form
\[
(5) \quad \|u\|_{q,\mathcal{I}}^{1-q} \int_0^t v \text{sgn}(u) |u|^{q-1} \, dx = \|E_2\| \|u''\|_{p,\mathcal{I}}^{1-p} \int_0^t v'' \text{sgn}(u'') |u''|^{p-1} \, dx,
\]
for all \(v \in C_c^\infty(\text{Int} \mathcal{I})\), and we know that there exists a solution \(u = u_D \in W^{2,p}_D\) with \(\|u_D''\|_{p,\mathcal{I}} = 1\) and \(\|u_D\|_{q,\mathcal{I}} = \|E_2\|\). We now derive an eigenvalue equation from this weak problem with this choice of normalization.

The Poisson equation with Dirichlet boundary conditions is uniquely solvable in \(L^{q'}\). That is, for each \(f \in L^{q'}(\mathcal{I})\) there exists \(g \in W^{2,q'}_D(\mathcal{I})\) such that \(-g'' = f\). In particular, there exists \(\tilde{u} \in W^{2,q'}_D\) such that \(-\tilde{u}'' = \text{sgn}(u_D) |u_D|^{q-1} \in L^{q'}(\mathcal{I})\). In weak form, this amounts to
\[
\int_0^t v \text{sgn}(u_D) |u_D|^{q-1} = -\langle v, \tilde{u}'' \rangle = -\int_0^t v'' \tilde{u}'' \, dx
\]
for all \(v \in C_c^\infty(\text{Int} \mathcal{I})\). Substituting into (5), we then have that
\[
\int_0^t v'' \left(\|E_2\|^q \text{sgn}(u_D'') |u_D''|^{p-1} + \tilde{u}''\right) = 0
\]
for all \(v \in C_c^\infty(\text{Int} \mathcal{I})\). Hence
\[
\tilde{u}'' = -\|E_2\|^q \text{sgn}(u_D'') |u_D''|^{p-1} \in L^{p'} \cap W^{2,q'}_D
\]
and moreover
\[
\|E_2\|^q \left(\text{sgn}(u_D'') |u_D''|^{p-1}\right)'' = \text{sgn}(u_D) |u_D|^{q-1}.
\]
Thus, \(u_D\) is an eigenfunction of the differential equation
\[
\nu \left(\text{sgn}(u'') |u''|^{p-1}\right)'' = \text{sgn}(u) |u|^{q-1}
\]
with boundary conditions
\[
u(0) = u(t_0) = u''(0) = u''(t_0) = 0.
\]

Note that the first two boundary conditions are consequence of the inclusion
\[
u_D \in W^{2,p}_D
\]
and the last two are consequence of the inclusion
\[
u_D'' |^{p-1} \in W^{2,q'}_D.
\]

From this reasoning, it follows that the s-numbers problem for the embedding \(E_2\) with a suitable normalization can be recast as an eigenvalue problem similar to (2), that we write as
\[
(6) \quad (\text{sgn}(u'') |u''|^{p-1})'' = \lambda \text{sgn}(u) |u|^{q-1} \quad 0 \leq t \leq t_0
\]
\[
u(0) = u(t_0) = u''(0) = u''(t_0) = 0
\]
for \( u \neq 0 \) and \( \lambda > 0 \). This equation will be the object of study in the first part of
this paper, covering sections 2. The other crucial part of this paper is the final
Section 6. There, we determine explicit invariants for the optimal approximation
problem associated to \( E_2 \). In order to motivate the general case, in Section 5 we
first give specific details of the special case \( q = p' \). We also include an appendix,
containing several interesting results about the non-periodic solutions of (6).

Our exposition is almost entirely self-contained. However, we mention that the
literature contains previous investigations of the eigenvalue problem (6) in the ho-

mogeneous case \( p = q \). We highlight that, by means of techniques different from
those we apply below, Drábek and Ōtani reported in [7] that the first eigenfunc-
tion exists and that it is symmetric with respect to \( t_0/2 \). Remarkably, they also
showed that the higher eigenfunctions are generated from this first eigenfunction
by re-scaling.

2. The eigenvalue problem

If there is an eigenpair \( (u, \lambda) \) for (6), integrating by parts twice and using the
boundary conditions, gives

\[
\lambda = \frac{\|u''\|^p_{p,x}}{\|u\|^q_{q,x}} > 0.
\]

The boundary conditions prevent \( u \) from being a linear function, other than \( u = 0 \). Hence there is \( u \neq 0 \) satisfying (6), only for \( \lambda \) is positive.

The following “converse” of this elementary observation is one of our main re-

sults. The proof will be given in sections 3 and 4.

**Theorem 2.1.** Let \( t_0 > 0 \) be fixed. If \( p \neq q \), then for all \( \lambda > 0 \) there exists a
unique solution \( u(t) \) positive on \((0, t_0)\) satisfying (6). If \( p = q \), then there exists
a unique \( \lambda \equiv \lambda(t_0) > 0 \) such that a solution \( u(t) \) positive on \((0, t_0)\) satisfying (6)
exists. Moreover this solution is unique up to multiplication by constants.

From now on we assume that \( \lambda > 0 \). Writing

\[
(8) \quad u_1(t) = u(t), \quad u_2(t) = u'(t),
\]

we get the equivalent systems of differential equations

\[
(u_1'(t) = u_2(t), \quad u_2'(t) = -\sgn(u''(t))|u''(t)|^{p-1} \quad \text{and} \quad w_2(t) = -(\sgn(u''(t))|u''(t)|^{p-1}),
\]

or integral equations

\[
(u_1(t) = \int_0^t u_2(s)ds, \quad u_2(t) = \alpha - \int_0^t \sgn(u_1(s))|u_1(s)|^{p'-1} ds
\]

\[
(w_1(t) = \int_0^t w_2(s)ds, \quad w_2(t) = \beta - \lambda \int_0^t \sgn(u_1(s))|u_1(s)|^{q-1} ds,
\]

both subject to the initial and final conditions

\[
(9) \quad u_1(0) = u_1(t_0) = w_1(0) = w_1(t_0) = 0.
\]
In (8), $\alpha$ and $\beta$ are to be determined for the final condition to be satisfied. Without further mention, when referring below to the systems (\star) we mean both (8D) and (8I).

It is useful to fix $\lambda$ and understand (6) in the context of (8I) as a dynamical system seeking for the trajectory

$$
\varphi(t) = \begin{bmatrix}
u_1(t) \\
u_2(t) \\
w_1(t) \\
w_2(t)
\end{bmatrix}
$$

starting from an initial state at $t = 0$

$$
\varphi(0) = \zeta = \begin{bmatrix} 0 \\ \alpha \\ 0 \\ \beta \end{bmatrix}.
$$

As the equation is local in its variable, it is readily seen that if such a trajectory returns to the initial state at $t = 2t_0$, $\varphi(2t_0) = \varphi(0)$, then it is $2t_0$-periodic. Below we will determine conditions for the existence of periodic trajectories, for the suitable combination of $\alpha$ and $\beta$. We will subsequently show that these periodic trajectories are symmetric with respect to half the period, with

$$
\varphi(t_0) = \begin{bmatrix} 0 \\ \mp \alpha \\ 0 \\ \mp \beta \end{bmatrix}
$$

depending on whether the components are even or odd with respect to a quarter period.

We can also write (8I) as

(10) \hspace{1cm} \varphi'(t) = \zeta + \int_0^t F(\lambda; \varphi(s)) \, ds

for

$$
F(\lambda; x, y, z, w) = \begin{bmatrix}
y \\
-\text{sgn}(z) |z|^{p-1} \\
-\lambda \text{sgn}(x) |x|^{q-1} \\
w
\end{bmatrix}.
$$

Note that $F(\lambda; \cdot) : \mathbb{R}^4 \to \mathbb{R}^4$ is continuous and Lipschitz (for any fixed $\lambda, p, q$) because $p, q > 1$. Therefore, by Picard-Lindelöf or Cauchy-Peano, for arbitrary $\psi \in \mathbb{R}^4$ and $t_1 \in \mathbb{R}$, there exists $\delta > 0$ such that there is a unique solution satisfying (10) and the initial condition $\varphi(t_1) = \psi$ for $t \in [t_1, t_1 + \delta]$.

If for $\lambda > 0$ we have a solution of (10), then

$$
u_2(0) = \nu'(0) = \alpha \quad \text{and} \quad w_2(0) = -(\text{sgn}(\nu''(0)) |\nu''(0)|^{p-1})' = \beta
$$

must satisfy $\alpha \beta > 0$. Indeed, if we assume $\alpha > 0$ and $\beta \leq 0$ (the other case being similar), then $u_1$ would be increasing with increasing derivative (convex) and $w_1$ will be decreasing with decreasing derivative (concave), therefore the boundary condition at $t_0$ could not be satisfied.

This observation renders the next lemma and Lemma A.1 in the appendix includes a more general setting.
Lemma 2.1. Consider the system of integral equations \((\ast I)\) with \(\lambda > 0\) and initial conditions \(u_1(0) = w_1(0) = 0\).

i) If \(\alpha > 0\) and \(\beta < 0\) then \(u_1\) is strictly increasing and \(w_1\) is strictly decreasing for \(t > 0\).

ii) If \(\alpha < 0\) and \(\beta > 0\) then \(u_1\) is strictly decreasing and \(w_1\) is strictly increasing for \(t > 0\).

We will see in the appendix that, for some pairs \((p,q)\), the solution develops a singularity at finite time and for some it exists and is unique for all \(t \in (0, \infty)\).

We now introduce the notation defining the maximal interval of existence and uniqueness, and describe general behaviour of the solution in that interval. From now on we will write the point of singularity of the solution by \(t_\infty \equiv t_\infty(\lambda, p, q, \zeta)\), defined as

\[
(11) \quad t_\infty(\lambda, p, q, \zeta) = \sup \left\{ t_1 > 0 : \sup_{t \in (0, t_1)} \left( \sum_{v \in \{u_k, w_k\}_{k=1}^2} |v(t)| \right) < \infty \right\}.
\]

That is, all components of \(\varphi(t)\) are bounded for \(0 \leq t < t_\infty\) and

\[
\limsup_{t \to t_\infty} \left( \sum_{v \in \{u_k, w_k\}_{k=1}^2} |v(t)| \right) = \infty.
\]

The next lemma implies that \(t_\infty\) is well defined and justifies identifying it.

Lemma 2.2. Given \(\lambda, p, q, \alpha\) and \(\beta\) fixed, there exists a unique solution to \((\ast I)\) for all \(t \in (0, t_\infty)\).

Proof. Let \(\varphi^1\) and \(\varphi^2\) be two solutions for the same data at \(t = 0\). Let \(0 < t_1 \leq t_\infty\) be given by

\[
t_1 = \sup \{ 0 < t_2 \leq t_\infty : \varphi^1(t) = \varphi^2(t) \forall 0 < t < t_2 \}.
\]

If \(t_1 < t_\infty\), then all entries of the trajectories are bounded up to \(t_1\), so

\[
\varphi^1(t_1) = \varphi^2(t_1) = \lim_{t \to t_1} \varphi^j(t)
\]

exists. By Picard-Lindelöf or Cauchy-Peano, starting with \(\psi = \varphi^j(t_1)\) we then have

\[
\varphi^1(t) = \varphi^2(t) \quad \forall t \in [t_1, t_1 + \delta].
\]

This contradicts the definition of \(t_1\), unless \(t_1 = t_\infty\). \(\square\)

We now establish a series of statements, describing the behaviour of the entries of the solution vector near \(t_\infty\). Their validity reflects the fact that the components of \(\varphi(t)\) depend cyclically on one another. Below and elsewhere, we write the word “lim” to mean that the “lim inf” and the “lim sup” coincide.

Lemma 2.3. Let \(\lambda, p, q, \alpha = u_2(0)\) and \(\beta = w_2(0)\) be fixed. Assume that \(t_\infty = \infty\) in \((11)\). If one of the components of the solution vector \(\varphi(t)\) of the systems \((\ast)\) is uniformly bounded in \([0, \infty)\), then all the components of this solution vector have infinitely many zeros.
Proof. Without loss of generality, we assume it is $u_1(t)$ the one component that is bounded. Firstly note that one of the components of $x(t)$ has infinitely many zeros if and only if all the components have infinitely many zeros. So, aiming at applying \textit{reductio ad absurdum}, let us begin by assuming that all the components have finitely many zeros. If

$$\liminf_{t \to \infty} |u_1(t)| > 0,$$

then $|w_2(t)| \to \infty$. Hence, arguing cyclically, also $|u_1(t)| \to \infty$ which is impossible, so necessarily

$$\liminf_{t \to \infty} |u_1(t)| = 0.$$

Now, if

$$\limsup_{t \to \infty} |u_1(t)| > 0,$$

then $u_2(t)$ has infinitely many zeros which is not what we are assuming, hence

$$\lim_{t \to \infty} |u_1(t)| = 0.$$

Thus, as all have finitely many zeros, also

$$\lim_{t \to \infty} |u_2(t)| = \lim_{t \to \infty} |w_k(t)| = 0.$$

This being the case, if one of the components of $x(t)$ vanishes at $t_1$ for some $t_1 \geq 0$, then the exists $t_2 > t_1$ such that the next component in the vector (counting them cyclically) also vanishes. But this is a contradiction, unless all components of the solution vector have infinitely many zeros. This confirms the claim. \hfill $\square$

We will show below a much stronger result than this lemma for the solution of the eigenvalue equation (6). Namely, a solution vector such that $u_1(t_1) = w_1(t_1) = 0$ for some $t_1 > 0$ is always uniformly bounded, periodic and symmetric. Before that, we continue with our plan to settle the behaviour of the solution near the singular point. The case where $t_\infty$ is finite is now in place.

\textbf{Lemma 2.4.} Let $\lambda, p, q, \zeta$ be fixed. Assume that the point of singularity for the system (11), as defined in (11), is $t_\infty < \infty$. Then,

$$\left| \liminf_{t \to t_\infty} v(t) \right| = \limsup_{t \to t_\infty} v(t) = \infty. \quad \text{for all } v \in \{u_1, u_2, w_1, w_2\}.\quad (12)$$

\textbf{Proof.} Without loss of generality we assume that,

$$\limsup_{t \to t_\infty} |u_1(t)| = \infty.$$

There are two possibilities to consider.

One possibility is that

$$0 \leq \liminf_{t \to t_\infty} u_1(t) \leq \infty.$$

In that case, according to (11), near $t_\infty$ we should have $w_2(t)$ monotonic decreasing and negative, hence $w_1(t)$ monotonic decreasing and negative and $u_2(t)$ is monotonic increasing and positive. Then, $u_1(t)$ is monotonic increasing and positive too. Therefore in fact

$$\lim_{t \to t_\infty} u_1(t) = \infty.$$
But then, from the formulation (\ref{eq:D}), it follows that
\[
\lim_{t \to t_{\infty}} u_2(t) = \infty \quad \text{and also that} \quad \lim_{t \to t_{\infty}} w_k(t) = -\infty.
\]
This is (12).

The other possibility is that
\[
\liminf_{t \to t_{\infty}} u_1(t) < 0.
\]
From the formulation (\ref{eq:D}) it then follows that
\[
\liminf_{t \to t_{\infty}} u_2(t) = -\infty \quad \text{and} \quad \limsup_{t \to t_{\infty}} u_2(t) = +\infty,
\]
as \(u_1(t)\) becomes highly oscillatory at \(t_{\infty} < \infty\) with negative minima and positive maxima. Likewise, and for similar reasons, also
\[
\liminf_{t \to t_{\infty}} w_k(t) = -\infty \quad \text{and} \quad \limsup_{t \to t_{\infty}} w_k(t) = +\infty.
\]
and then
\[
\liminf_{t \to t_{\infty}} u_1(t) = -\infty \quad \text{and} \quad \limsup_{t \to t_{\infty}} u_1(t) = +\infty.
\]
Therefore, once again, we have (12). \(\Box\)

**Remark 2.1.** Irrespective of whether \(t_{\infty}(\lambda, p, q, \zeta)\) is finite or not, we have the following assertion that will be used repeatedly below. If one of the \(v(t)\) from (11) does not have a zero in \(t \in (t_1, t_{\infty})\) for some \(0 < t_1 < t_{\infty}\), then all the components of the solution vector are monotonic for \(t \in (t_2, t_{\infty})\) where \(t_2 \geq t_1\) is large enough. Moreover, in that case
\[
\lim_{t \to t_{\infty}} u_k(t) = -\lim_{t \to t_{\infty}} w_k(t).
\]
The proof of this is very similar to the line of arguments in the first possibility in the proof of Lemma 2.4.

We show in the appendix, that the solution does not become oscillatory near \(t_{\infty}\) for a range of parameters \((p, q)\). Namely
\[
\lim_{t \to t_{\infty}} |u_k(t)| = \lim_{t \to t_{\infty}} |w_k(t)| = \infty
\]
and (13) holds true. In fact we conjecture that the latter is the case for all \(p > 1\) and \(q > 1\), but we are not currently able to complete the proof of this claim. We will not need this fact in the following, but only that \(t_{\infty}\) is defined as in (11) and that the components of the solution vector satisfy what we established in the previous lemmas and the remark.

The next re-scaling property will be repeatedly invoked in the forthcoming sections. Let \((\lambda, u)\) be a solution to (6) for \(t_0 > 0\) and that \(a, b > 0\). Then, \((\tilde{\lambda}, \tilde{u})\) where
\[
\tilde{u}(t) = au(bt), \quad \tilde{\lambda} = \lambda a^{p-q} b^{2p} \quad \text{and} \quad \tilde{t}_0 = \frac{t_0}{b},
\]

\[
(14) \quad \tilde{u}(t) = au(bt), \quad \tilde{\lambda} = \lambda a^{p-q} b^{2p} \quad \text{and} \quad \tilde{t}_0 = \frac{t_0}{b},
\]
is a solution to (6) as well for $0 < t < t_0$. In obvious notation
\begin{equation}
\hat{\varphi}(t) = \begin{bmatrix} \hat{u}_1(t) \\ \hat{u}_2(t) \\ \hat{w}_1(t) \\ \hat{w}_2(t) \end{bmatrix} = \begin{bmatrix} aw_1(bt) \\ abw_2(bt) \\ a^{p-1}b^{2p-2}w_1(bt) \\ a^{p-1}b^{2p-1}w_2(bt) \end{bmatrix}
\end{equation}
is the corresponding solution to (*) so that
\begin{equation}
\hat{\varphi}(0) = \begin{bmatrix} 0 \\ ab\alpha \\ 0 \\ a^{p-1}b^{2p-1}\beta \end{bmatrix} = \begin{bmatrix} \alpha \\ \hat{\alpha} \\ \hat{\beta} \end{bmatrix}.
\end{equation}

### 3. Stability of Solutions and Proof of Existence

The following statement is one of the crucial ingredients in the proof of existence in Theorem 2.1.

**Lemma 3.1.** Let $\lambda, \ p, \ q$ be fixed. Consider the evolution systems (*). Let $\alpha_2 \leq \alpha_1$ and $\beta_1 \leq \beta_2$. Let $t_1 > 0$ be any point such that all the quantities \( u_k^2(t) \) and \( w_k^2(t) \) are finite for $0 \leq t < t_1$. Then,
\begin{equation}
(17) \quad u_k^2(t) \leq u_k^1(t) \quad \text{and} \quad w_k^1(t) \leq w_k^2(t) \quad \forall k = 1, 2 \quad t \in (0, t_1).
\end{equation}
Moreover,
\begin{equation}
(18) \quad u_1^1(t) - u_1^2(t) \geq (\alpha_1 - \alpha_2) t \quad \text{and} \quad w_1^2(t) - w_1^1(t) \geq (\beta_2 - \beta_1) t \quad \forall t \in (0, t_1).
\end{equation}

In fact, if one of the inequalities involving $\alpha_j$ or $\beta_j$ is strict, then $u_1^2(t) < u_1^1(t)$ and $w_1^1(t) < w_1^2(t)$ for $0 < t \leq t_1$.

**Proof.** Without loss of generality we assume that one of the inequalities in the hypothesis is strict. By this hypothesis $u_2^0(0) \leq u_1^1(0)$ and $w_2^0(0) \leq w_1^1(0)$, one of these inequalities being strict. Then, using the formulation (17), there exists $\varepsilon > 0$ such that (17) holds true with all four inequalities strict for all $t \in (0, \varepsilon)$. Hence, for some $c > 0$,
\begin{equation}
w_1^2(t) - w_1^1(t) = \int_0^t (u_2^2(t) - u_2^1(t)) ds > ct
\end{equation}
for all $t \in (0, \varepsilon)$.

Now, by virtue of the latter,
\begin{equation}
u_2^1(t) - u_2^2(t) = \alpha_1 - \alpha_2 + \int_0^t (\text{sgn} (w_1^2(s)) |w_1^2(s)|^{p'-1} - \text{sgn} (w_1^1(s)) |w_1^1(s)|^{p'-1}) ds
\end{equation}
is increasing in $t \in (0, \varepsilon)$ and so
\begin{equation}
u_2^1(t) - u_2^2(t) > \alpha_1 - \alpha_2
\end{equation}
for all such $t$. Thus
\begin{equation}
u_1^1(t) - u_1^2(t) = \int_0^t (u_2^1(s) - u_2^2(s)) ds > (\alpha_2 - \alpha_1) t
\end{equation}

\footnote{Here and everywhere below, the indices $j$ (on top) refer to corresponding sub-indices of $\alpha$ or $\beta$, in context.}
for all $t \in (0, \varepsilon)$. Moreover,

$$w_2^2(t) - w_2^1(t) = \beta_2 - \beta_1 + \lambda \int_0^t (\text{sgn} (u_1^2(s)) |u_1^2(s)|^{q-1} - \text{sgn} (u_1^2(s)) |u_1^2(s)|^{q-1}) \, ds$$

is therefore also increasing in $t \in (0, \varepsilon)$ and so

$$w_2^2(t) - w_2^1(t) > \beta_2 - \beta_1$$

for all such $t$. The latter ensures

$$w_1^2(t) - w_1^1(t) > (\beta_2 - \beta_1)t$$

(i.e. we can take $c = \beta_2 - \beta_1$ in the estimate above for the $w_1^1$).

As a conclusion of the previous paragraph, so far we now know that (17) and (18) hold true for all $t \in (0, \varepsilon)$. But suppose that there exists $0 < t_2 < t_1$ such that $u_1^2(t_2) = w_2^2(t_2)$ or $u_1^2(t_2) = w_2^2(t_2)$. Because of the other quantities are still increasing, this is impossible. Therefore (17) and (18) hold true for all $t \in (0, t_1)$.

**Lemma 3.2.** Let $p$, $q$ and $\lambda$ be fixed. There exists $\alpha > 0$ and $\beta > 0$, such that both $u_1^1(t)$ and $u_1^2(t)$ have at least one zero each in $(0, t_\infty)$. In fact, we can find a pair $(\alpha, \beta)$, such that one of these two functions has at least two zeros, $0 < t_1 < t_2$, the other has one zero $0 < r_1 < t_2$ and these are the first zeros of the respective functions in $(0, t_\infty)$.

**Proof.** Firstly we show the first claim. By virtue of Remark 2.1 if $\alpha > 0$ and $\beta > 0$ are such that one of the components, $u_1^1(t) \equiv u_1(t)$ or $w_1^1(t) \equiv w_1(t)$, is positive and does not vanish in $(0, t_\infty)$, then necessarily that component is eventually monotonic increasing and the other component has a zero and it is eventually monotonic decreasing. Therefore, in the rest of the proof of the first claim, we assume that it is $u_1^1(t)$ that is positive and does not vanish, aiming at contradictions.

Let $t_1 > 0$ be such that $u_1^1(t_1) = 0$ and $u_1^1(t) > 0$ for all $t \in (0, t_1)$. Then $(u_2^1)'(t_1) = 0$ and $(u_2^1)'(t) < 0$ for all $t \in (0, t_1)$. Hence $u_1^1(t)$ is concave for all $t \in (0, t_1)$. Let

$$0 < \alpha_2 = \frac{\alpha_1 t_1 - u_1^1(t_1)}{t_1}.$$

Then $\alpha_2 > 0$ (because of concavity of $u_1^1$) and also $\alpha_2 < \alpha_1$. From Lemma 3.1 it follows that

$$u_1^1(t_1) - u_1^2(t_1) \geq (\alpha_1 - \alpha_2) t_1 = u_1^1(t_1).$$

Hence $u_2^1(t_1) \leq 0$, so it must have a zero below or on $t_1$, and four possibilities arise.

One is that either $u_2^1(t)$ or $w_2^2(t)$ are bounded in $(0, t_\infty)$. According to Lemma 2.3 this is impossible whenever $t_\infty < \infty$. Then $t_\infty = \infty$ and in this case, the conclusion follows from Lemma 2.3. Another is that all $u_2^1(t)$ and $w_2^2(t)$ become unbounded oscillatory near $t_\infty$, and once again the conclusion follows. See Remark 2.1. A third possibility is that the $u_2^1(t)$ point upwards (the limits at $t_\infty$ is $+\infty$), $w_2^2(t)$ point downwards and once again the conclusion follows with $(\alpha, \beta) = (\alpha_2, \beta_1)$.

The fourth possibility is that for the pair $(\alpha_2, \beta_1)$, $u_2^1(t)$ point downwards and $w_2^2(t)$ upwards at $t_\infty$. If this is the case, by Lemma 3.1 and by continuity of the solutions in $(\alpha, \beta)$ at all fix $t$, two sub-cases arise. Either, there exists a critical $\alpha = \alpha_3$, $\alpha_2 < \alpha_3 < \alpha_1$, such that

$$u_k(t_\infty) = \begin{cases} -\infty & \text{when } \alpha_2 \leq \alpha < \alpha_3 \\ +\infty & \text{when } \alpha_3 < \alpha \leq \alpha_1 \end{cases} \quad \text{and} \quad w_k(t_\infty) = \begin{cases} +\infty & \text{when } \alpha_2 \leq \alpha < \alpha_3 \\ -\infty & \text{when } \alpha_3 < \alpha \leq \alpha_1. \end{cases}$$
But then, for $\alpha_4 = \frac{\alpha + \alpha_2}{2}$, the behaviour of the solution falls into the previous possibility and hence the conclusion follows. Or

$$u_k(t) = -\infty \text{ and } w_k(t) = +\infty \quad \text{for all } \alpha_2 \leq \alpha < \alpha_1.$$  

In that case, let $r_1 > 0$ be such that $w_1^1(r_1) < 0$. Then, for $\alpha_2 < \alpha_5 < \alpha_1$ where $\alpha_5$ is sufficiently close to $\alpha_1$, we have $w_k^1(r_1) < 0$. Since both $u_1^0(t)$ and $w_1^1(t)$ have each positive zeros, once again the conclusion follows. This completes the proof of the first claimed statement.

For the final statement, assume without loss of generality that $u_k(t) = +\infty$, that $w_k(t) = -\infty$, that $u_1(t)$ has at least two zeros (possibly double) and that $w_1(t)$ has at least one zero. That we can reduce to this assumption follows from the choice of initial conditions. Let $t_1$ be the first zero and $t_2 \geq t_1$ be the second zero of $u_1(t)$. If $t_1 = t_2$ (i.e. if we have a double zero) we decrease slightly $\alpha$ to get $t_1 < t_2$. So we can suppose the latter. We know that $u_2(t)$ has at least two zeros, one $s_1 \in (0, t_1)$ and another $s_2 \in (t_1, t_2)$. Hence $w_2(t)$ has certainly one zero $r_1 \in (s_1, s_2) \subset (s_1, t_2)$. Now, either $r_1(t)$ is the first zero of $w_2$ or not. But, either case, the final statement claimed in the lemma holds true.

The pair $(\alpha, \beta)$ ensuring the last statement of the above lemma renders four obvious possibilities.

1. $u_1(t)$ is the one with two zeros, $t_1 < t_2$, $w_1(t)$ the one with at least one zero $r_1 < t_2$, and
   - a) either $r_1 \geq t_1$
   - b) or $r_1 < t_1$.

2. (Roles of $u_1$ and $w_1$ swapped), $w_1(t)$ is the one with two zeros, $r_1 < r_2$, $u_1(t)$ the one with at least one zero $t_1 < r_2$, and
   - a) either $t_1 \geq r_1$
   - b) or $t_1 < r_1$.

**Corollary 3.1.** Let $p$, $q$ and $\lambda$ be fixed. There exists $\alpha > 0$ and $\beta > 0$, such that both $u_1(t)$ and $w_1(t)$ have a simple zero at the same point $t_1 > 0$ and both are strictly positive for $t \in (0, t_1)$.

**Proof.** Apply Lemma 3.2 and get a pair $(\alpha_0, \beta_0)$ rendering one of the possibilities above. Suppose that $[I]$ occurs. Fix

$$d = \left| \min_{t_1 \leq t \leq t_2} u_1^0(t) \right| \quad \text{and} \quad \alpha_1 \geq \frac{d}{t_1} + \alpha_0.$$  

If the parameter $\alpha$ in the equation (11) is varied continuously from $\alpha_0$ to $\alpha_1$, the component $u_1(t)$ of the solution increases, the difference between the two zeros $t_1 - t_2 \rightarrow 0$, $w_1(t)$ decreases and $r_1$ decreases being trapped between $t_1$ and $t_2$. At $\alpha_1$, according to Lemma 3.2

$$u_1^1(t) - u_1^0(t) \geq (\alpha_1 - \alpha_0)t \geq (\alpha_1 - \alpha^0)t_1 \geq d$$  

for all $t_1 < t < t_2$. That is $u_1^1(t) > 0$ for all such $t$. Therefore, there is an intermediate value of $\alpha$ such that either $t_1 = r_1$ or $t_2 = r_1$.

Now assume the latter, i.e. $t_2 = r_1$. If $t_1 \neq r_1$, from the possible shapes of $u_1(t)$, it becomes clear that $u_2(t)$ should have a zero below $t_1$ and another zero between $t_1$ and $t_2$, generating a contradiction. If, on the other hand, $t_1 = r_1 = t_2$ (that is the zero of $u_1$ is double), $w_1(t)$ is forced to have a zero below $r_1$ generating also a contradiction. So the only possibility for the case $[I]$ to occur is that eventually,
in the continuous deformation described above, \( t_1 = r_1 < t_2 \). From this, the conclusion follows.

Suppose the possibility \([1\text{D}]\) occurs. Let \( d = |w_1^0(t_1)| = -w_1^0(t_1) \). If \( \beta \) increases from \( \beta_0 \), then \( r_1 \) increases, and \( t_1 \) and \( t_2 \) decrease. Let \( \beta_1 \) be such that
\[
(\beta_1 - \beta_0)t_1 \geq d.
\]
Then, by Lemma 3.1,
\[
w_1^1(t_1) - w_1^0(t_1) \geq (\beta_1 - \beta_0)t_1 \geq d.
\]
Hence \( w_1^1(t_1) > 0 \). Therefore there should have been a point in the continuous deformation where \( \beta \) was such that \( r_1 = t_1 \). As in the previous case, \( u_1(t) \) cannot have a double zero at that point, so the conclusion follows again.

Finally, if the cases \([1\text{J}]\) occurs, we argue by swapping the roles of the components \( u_1(t) \) and \( w_1(t) \).

We highlight that, in the context of this corollary, \( t_1 \) depends on \( p \), \( q \) and \( \lambda \). Therefore, it directly implies the existence of a positive solution for \([1\text{J}]\), given these parameters, only when \( t_0 = t_1 \). However, when combined with the rescaling \([1\text{E}]\), it yields the existence part of Theorem 2.1 as we shall see next.

**Proof of existence in Theorem 2.1.** For the first statement, let \( p \neq q \) and \( \lambda > 0 \). According to Corollary 3.1, there exists \( t_1 > 0 \) such that the corresponding first component of the solution vector \( u_1(t) \) is positive in \((0, t_1)\). For given \( t_0 > 0 \) take from Theorem 2.1
\[
b = \frac{t_0}{t_1} \quad \text{and} \quad a = \left( \frac{t_1}{t_0} \right)^{\frac{2p}{p-q}}
\]
in \([1\text{J}]\)–\([1\text{E}]\). Then \( u(t) = au_1(bt) \) is the solution sought after. This ensures the first statement.

For the second statement, let \( p = q \). By Corollary 3.1 for eigenvalue parameter \( \lambda = 1 \), there exists \( t_1 > 0 \) such that the corresponding \( u_1(t) > 0 \) for all \( t \in (0, t_1) \). Take
\[
b = \frac{t_0}{t_1} \quad \text{and} \quad a = 1.
\]
Then \( u(t) = \tilde{u}_1(t) \) in \([1\text{J}]\)–\([1\text{E}]\), is a positive solution to \([1\text{G}]\) for \( t \in (0, t_0) \) and
\[
\lambda \equiv \lambda(t_0) = \left( \frac{t_0}{t_1} \right)^{2p}.
\]

We now highlight a crucial observation.

**Remark 3.1.** *In the context of the initial value problem \([1\text{D}]\), Theorem 2.1 has the following consequence. Let \( u_1(t) \), the first component of a solution, be such that \( u_1(0) = u_1(t_0) = 0 \) and \( u_1(t) > 0 \) for all \( 0 < t < t_0 \). If the corresponding \( w_1 \) satisfies \( w_1(0) = w_1(t_0) = 0 \), then \( w_1(t) > 0 \) for all \( 0 < t < t_0 \). Indeed, argue by contradiction. Assume that \( w(r_1) = 0 \) for some \( r_1 \in (0, t_0) \). As \( w_1 = w_1' \), then there exists \( 0 < s_1 < s_2 < t_0 \) such that \( w_1(s_k) = 0 \). Hence, for similar reasons, there must exist \( t_1 \in (0, t_0) \) such that \( u_1(t_1) = 0 \) creating a contradiction. Note that in this argumentation, the roles of \( u_1 \) and \( w_1 \) can be exchanged.*
From Lemma 2.3, it follows that if the entries of the solution vector are uniformly bounded, then \( u_k, w_k \) have infinitely many zeros. If, additionally, the zeros of \( u_1 \) and \( w_1 \) coincide, we will see below that they are periodic.

4. Symmetries, periodicity and uniqueness of the solution

In this section we will show that any solution of (6) on \([0, t_0]\) is either even or odd with respect to \( t \) and that it can be extended periodically to a function on \( \mathbb{R} \). Although they are rather straightforward, the next statements are crucial in the context of equation (6).

Lemma 4.1. Let \( u(t) \) be a solution of (6). Then, \( u(t) \) vanishes \( n \) times in the segment \((0, t_0)\) if and only if \( u''(t) \) also vanish \( n \) times in the segment \((0, t_0)\).

Proof. Suppose that \( u(t) \) vanishes \( n \) times in \((0, t_0)\). Since \( \alpha = u'(0) \neq 0 \), it follows that \( u'(t) \) vanishes at least \( n + 1 \) times in \((0, t_0)\). Then, as \( u''(t) \) vanishes at the end points of the segment, \( u''(t) \) must vanish at least \( n \) times in \((0, t_0)\). So, in the notation of the systems (\(*\)), \( w_1(t) \) has the same property.

Now, if \( w_1(t) \) vanishes more than \( n \) times, arguing cyclically like in the last paragraph, we would have that \( u(t) = u_1(t) \) vanishes more than \( n \) times. So this is impossible. In turns, \( u''(t) \) must vanish exactly \( n \) times.

The proof of the converse is identical. \( \square \)

Lemma 4.2. The solutions of (6) on \((0, t_0)\) are related as follows. If \( u(t) \) is a solution of (6) for eigenvalue \( \lambda > 0 \) such that \( u(t) \) has \( n \) zeros in \((0, t_0)\), then all solutions of (6) with \( n \) zeros in \((0, t_0)\) are of the form \( \tilde{u}(t) = au(t) \) for corresponding eigenvalue \( \tilde{\lambda} = \lambda a^{p-n} \) where \( a > 0 \).

Proof. Firstly, let \( p \neq q \) be fixed. We proceed by reductio ad absurdum. By invoking the dilation (14) with \( b = 1 \), the negation of the conclusion is equivalent to assuming that, for the same fixed \( \lambda > 0 \), there exists two different solutions of (6) on \((0, t_0)\) with \( n \) zeros. We therefore suppose the latter.

Call these solutions \( u^*(t) \neq u^2(t) \), and write \( \alpha_k = u_k^2(0) > 0 \) and \( \beta_k = u_k^2(0) > 0 \), in the notation of the systems (\(*\)). According to Lemma 2.2, either \( \alpha_1 \neq \alpha_2 \) or \( \beta_1 \neq \beta_2 \). Without loss of generality, we can further assume that \( \alpha_1 \neq \alpha_2 \). The other case is similar and leads to the same conclusion.

Let \( a, b > 0 \) be such that

\[
a^{p-q}b^{2p} = 1 \quad \text{and} \quad ab = \frac{\alpha_1}{\alpha_2}.
\]

Then necessarily \( b \neq 1 \). Let

\[
\tilde{\alpha} = ab\alpha_2 = \alpha_1 \quad \text{and} \quad \tilde{\beta} = a^{p-1}b^{2p-1}\beta_2.
\]

Hence, \( \tilde{u}(t) = au^2(bt) \) is a solution of (6) on \((0, \tilde{t}_0)\) for the same eigenvalue \( \lambda > 0 \) but end point \( \tilde{t}_0 = \frac{1}{b} \tilde{t}_0 \neq t_0 \). Moreover, \( \tilde{u}(t) \) and \( \tilde{u}''(t) \) have \( n \) zeros in \((0, \tilde{t}_0)\). The corresponding solution vector \( \tilde{\phi}(t) \) is a solution of the system (15) with \( \tilde{\alpha}_2(t) = \alpha_2 \) and \( \tilde{w}_2(t) = \tilde{\beta} \), for which \( \tilde{u}_1(t) \) and \( \tilde{w}_1(t) \) have \( n \) zeros in \((0, \tilde{t}_0)\). Additionally they are such that

\[
\tilde{u}_1(0) = \tilde{w}_1(0) = \tilde{u}_1(\tilde{t}_0) = \tilde{w}_1(t_0) = 0.
\]

Now, two possibilities arise. Either \( \beta_2 < \tilde{\beta} \) or \( \beta_2 > \tilde{\beta} \). But these two possibilities are incompatible with Lemma 4.1. Indeed, the first possibility renders \( \tilde{u}_1(t) < u_1^2(t) \) for any function \( u_1 \).
and \( w^2(t) < \tilde{w}_1(t) \) for all \( t > 0 \) whereas the second possibility renders \( \tilde{u}_1(t) > w^2(t) \) and \( w^2(t) > \tilde{w}_1(t) \) for all \( t > 0 \). Either case, we have that either \( \tilde{u}_1(t_0) \neq 0 \) or \( \tilde{u}_1(t_0) \neq 0 \) or that one of these functions does not have exactly \( n \) zeros in \((0, t_0)\). Any of this, contradicts what we know already about \( \tilde{u}(t) \). Hence we must have \( b = 1 \) and so \( u^2(t) = \tilde{u}(t) = u^1(t) \). The conclusion of the lemma for \( p \neq q \) follows.

Finally let \( p = q \). Just as before, suppose that for the same \( \lambda > 0 \) we have two positive solutions of (6) with \( n \) zeros, \( u^1(t) \) and \( u^2(t) \). Let \( c = \frac{a_2}{a_1} \), where \( a_k \) are the second components of the corresponding solution vectors. Set \( \tilde{u}(t) = cu^1(t) \). As the equation is homogeneous when \( p = q \), then \( \tilde{u}(t) \) is also a solution for the same eigenvalue \( \lambda \).

After this we can proceed in similar way as the case \( p \neq q \). We have \( \tilde{u}_2(0) = \alpha_2 = u^2_2(0) \) and, by virtue of Lemma 3.1, the only possibility not leading to a contradiction is that also \( \tilde{u}_2(0) = \beta_2 = w^2_2(0) \). But then, the uniqueness statement Lemma 2.2 ensures that

\[
 u^2(t) = \tilde{u}(t) = cu^1(t),
\]

as claimed. \( \square \)

We continue by completing the proof of Theorem 2.1

**Proof of uniqueness in Theorem 2.1** The case \( p \neq q \) is a direct consequence of Lemma 4.2.

Let \( p = q \) instead. By virtue of Lemma 4.2 we only need to show that \( \lambda \) is unique. Suppose that we have two \( \lambda_1 \neq \lambda_2 \) with corresponding eigenvectors \( u^1(t) \) and \( u^2(t) \), positive on \((0, t_0)\) and vanishing at the end points alongside their second derivatives. Set

\[
 b = \left( \frac{\lambda_2}{\lambda_1} \right)^{\frac{1}{2p}},
\]

and consider \( u^3(t) = u^1(bt) \). Then \( \varphi^2(t) \) and \( \varphi^3(t) \) are both solutions vectors of the systems (*) for the same \( \lambda \equiv \lambda_2 = \lambda_1 b^{2p} \).

Now, let

\[
 a = \frac{u^3_2(0)}{u^2_2(0)}.
\]

Then, \( au^2_3(0) = u^3_2(0) = \alpha_3 \), that is, both \( au^2(t) \) and \( u^3(t) \) match their derivatives at \( t = 0 \). Now the third derivative satisfies the following three possibilities. If \( \beta_2 = \beta_3 \), then \( u^2(t) = u^3(t) \) and so \( b = 1 \) and \( \lambda_1 = \lambda_2 \) contradicting the original assumption. If \( \beta_2 < \beta_3 \), according to Lemma 3.1, then \( w^2_1(t) < w^3_1(t) \) and \( u^2_1(t) > u^3_1(t) \) for all \( t > 0 \). But this is impossible too, as then the first zeros of \( u^1_1(t) \) and \( w^1_1(t) \) cannot coincide, contradicting the fact that \( u^3(t) \) is a dilation of \( u^1(t) \). Finally, if \( \beta_2 > \beta_3 \) we reach the same conclusion by analogous arguments. This completes the proof of uniqueness for the eigenvalue when \( p = q \), and also the full proof of Theorem 2.1. \( \square \)

The rest of this section is devoted to describing the symmetries of the eigenfunctions. The next statement is a direct consequence of the uniqueness part of Theorem 2.1

**Theorem 4.1** Let \( u(t) \) be a positive solution of (6) on \((0, t_0)\). Then, \( u(t) = u(t_0 - t) \) for all \( 0 < t < \frac{t_0}{2} \). Moreover, \( u(t) \) can be extended to a 2\( t_0 \)-periodic
function $u_* \in C^1(\mathbb{R})$ satisfying
\[(\text{sgn} (u_*''(t)) |u_*''(t)|^{p-1})'' = \lambda \text{sgn} (u_*'(t)) |u_*'(t)|^{q-1} \quad \forall t \in \mathbb{R}.
\]

**Proof.** Start with a solution $u(t)$. Since the equation (19) is invariant under translations and the change $t \mapsto -t$, then $\tilde{u}(t) = u(t_0 - t)$ is also a solution of (19). That is, for the same $\lambda > 0$ and satisfying the same boundary conditions. But we have uniqueness in Theorem 2.1. For $p \neq q$, this implies directly that $u(t) = \tilde{u}(t)$. For $p = q$, it implies that $u(t) = cu(t)$. Then
\[u \left( \frac{t_0}{2} \right) = cu \left( \frac{t_0}{2} \right) = cu \left( \frac{t_0}{2} \right),
\]
so $c = 1$ and once again $u(t) = \tilde{u}(t)$.

In order to achieve the second conclusion, note that $u \in C^1(0, t_0)$ and that the first conclusion implies that $u'(0) = -u'(t_0)$.

**Corollary 4.1.** Let $u(t)$ be a solution of (19) with exactly $n - 1$ zeros in $(0, t_0)$. Then, these zeros are all simple and located at
\[t_j = \frac{j t_0}{n} \quad j = 1, \ldots, n - 1.
\]
Moreover, $u''(t)$ also vanish exactly at the points $t_j$.

**Proof.** Let $u^1(t)$ be a positive solution of (19) on the segment $(0, \frac{t_0}{4})$ for the same eigenvalue $\lambda > 0$. From Theorem 4.1, it follows that $u^1(t)$ is a solution of the same equation on $(0, t_0)$ with exactly $n - 1$ (simple) zeros in the interior of the segment. If $p = q$, as all eigenfunctions are multiple of one another, then the claimed conclusion follows. If $p \neq q$, by Lemma 4.2 we obtain $u^1(t) = u(t)$ and the claimed conclusions follow. \qed

5. The case $q = p'$

Let $B(a, b)$ denote the Beta function and $B_x(a, b)$ denote the incomplete Beta function [19] 8.17.1. For $1 < r, s < \infty$, let
\[(20) \quad \pi_{r, s} := 2 \int_0^1 \frac{dt}{(1 - t^r)^{\frac{s}{r}}} = \frac{2}{s} B \left( \frac{1}{s}, \frac{1}{r} \right)
\]
and let $\sin_{r,s} : \mathbb{R} \rightarrow [-1, 1]$ be the $2\pi_{r,s}$-periodic odd function whose inverse in $[0, \pi_{r,s}]$ is
\[F_{r,s}(y) = \int_0^y \frac{dt}{(1 - t^r)^{\frac{s}{r}}} = \frac{1}{2} B_y \left( \frac{1}{s}, \frac{1}{r} \right) \quad \forall y \in [0, 1]
\]
and is even with respect to $\pi_{r,s}$. Then $\sin_{2,2}(x) = \sin(x)$ and, except possibly at the points $x = \frac{(2k+1) \pi_{r,s}}{2}$ for $k \in \mathbb{Z}$, the functions $\sin_{r,s}(x)$ are $C^\infty$. Note that (20) and the fact that the Beta function is symmetric, yield the relation
\[(21) \quad s \pi_{r,s} = r' \pi_{s',r'}.
\]

In this section we show that, for $q = p'$, the systems $(*)$ are exactly solvable in terms of $\sin_{r,s}$ and therefore the value of $\|E_2\|$ can be found explicitly in terms of $B(a, b)$. The next two statements summarize our main findings about $E_2$ for $q = p'$. 


Theorem 5.1. Let $1 < p < \infty$ and fix $I = [0, \pi]$. Then

\begin{align}
\|E_2\| &\leq \sup_{f \in W^{1, p}_0(I)} \|f\|_{p, I} \cdot \|f''\|_{p, I} = \left( \frac{2}{p} \right)^\frac{1}{p'} \left( \frac{1}{2} \left( \frac{1}{p} + 1 \right) \right)^{\frac{1}{p'} - \frac{1}{p}}
\end{align}

and the extremal functions are of the form $f(x) = c \sin_{2, p'}(x)$ where $c \in \mathbb{R}$ is a non-zero constant.

For $1 < p < \infty$, let us re-write the equation (1) with the substitution $q = p'$. We seek for $u \neq 0$ and $\lambda > 0$ such that

\begin{align}
\begin{aligned}
\left(\text{sgn} \left( u'' \right) |u''|^{p-1} \right)'' &\equiv \lambda \text{sgn} \left( u \right) |u|^{p'-1} \quad 0 \leq t \leq t_0 \\
u(0) &= u(t_0) = u''(0) = u''(t_0) = 0.
\end{aligned}
\end{align}

Theorem 5.2. The eigenvalues and eigenfunctions of (23) are fully characterized as follows. For any given constant $c > 0$ and $n \in \mathbb{N}$, $\lambda = \lambda_n(c)$ is of the form

\begin{align}
\lambda_n(c) = \left( \frac{\pi_{2, p'} \pi_{p, 2n^2}}{t_0^p} \right)^p c^{p - p'}
\end{align}

with corresponding $u(x) = f_{n, c}(x)$ of the form

\begin{align}
\begin{aligned}
f_{n, c}(x) &\equiv c \sin_{2, p'} \left( \frac{\pi_{2, p'} \pi_{p, 2n^2}}{t_0} \right)
\end{aligned}
\end{align}

For $p \neq 2$, this eigenpair is the unique solution such that the eigenfunction has positive derivative at $x = 0$ and changes sign exactly $n - 1$ times on $(0, t_0)$.

We prove the validity of these two statements below. Let us begin by recalling properties of the generalized trigonometric functions and their role in the solution of the equation (2), associated to first order Sobolev embeddings $E_1$. The eigenpairs $(u, \lambda)$ of (2), have a close expression in terms of $\pi_{p, q}$ and $\sin_{p, q}$. Concretely, the full set of solutions of the Dirichlet problem (2) is given by

\begin{align}
\begin{aligned}
u(t) &\equiv u_{n, \alpha}(t) = \frac{\alpha t_0}{n \pi_{p, q}} \sin_{p, q} \left( \frac{n \pi_{p, q}}{t_0} t \right)
\end{aligned}
\end{align}

for corresponding

\begin{align}
\begin{aligned}
\lambda &\equiv \lambda_{n, \alpha} = \left( \frac{n \pi_{p, q}}{t_0} \right)^q \left( \frac{\alpha |p - q| (1 - p)}{p} \right)
\end{aligned}
\end{align}

where $\alpha \neq 0$ is a real parameter and $n \in \mathbb{N}$. Here we are interested in the case $q = p'$, which is independent of $\alpha$ only for $p = q = 2$.

The generalised cosine, $\cos_{p, q} : \mathbb{R} \to [-1, 1]$, is defined as

\begin{align}
\cos_{p, q} = \frac{d}{dx} \sin_{p, q}(x), \quad x \in \mathbb{R}.
\end{align}

From the properties of $\sin_{p, q}(x)$ it follows that $\cos_{p, q}(x)$ is an even, $2\pi_{p, q}$ periodic function, decreasing on $[0, \pi_{p, q}/2]$. If $x \in [0, \pi_{p, q}/2]$, then

\begin{align}
\cos_{p, q}(x) = (1 - (\sin_{p, q} x)^q)^{1/p}
\end{align}

and

\begin{align}
|\sin_{p, q} x|^q + |\cos_{p, q} x|^p = 1 \quad \forall x \in \mathbb{R}.
\end{align}

The functions $\sin_{p, q}$ and $\cos_{p, q}$ have a long history that can be traced back to about 40 years ago. Indeed, these and other analogue functions were examined by Schmid [22], Lindqvist [15, 16], Elbert [13], and Otani [20], in connection with
Let us compute the exact values of these constants. It has also been discovered that these play a significant role in describing optimal Sobolev embeddings and related integral operators [12].

The proof of Theorem 5.1 relies of the decomposition $E_2 = I_1I_2$ where the embeddings $I_j$ are to be understood in the context of the following diagram,

$$
X := W^{2,p}_D(I) \xrightarrow{E_2} Y := L^{p'}(I) \xrightarrow{I_1} I_1 \xrightarrow{I_2} Z := W^{1,2}_0(I)
$$

(26)

We will see that $u(x) = \sin_{p',r}(\pi x/t_0)$ is the extremal function for embedding $I_2$ and $u'(x)$ is the extremal functions for $I_1$. Hence, $u(x)$ is extremal for $E_2$ and eventually that would lead to $\|E_2\| = \|I_1\|\|I_2\|$.

Before establishing the proofs of theorems 5.1 and 5.2 we recall four additional known formulas connecting properties of the generalized trigonometric functions and their derivatives. See [11] Lemma 2.2, Props. 3.1 and 3.2. Let $r, s \in (1, \infty)$. A direct calculation gives

$$
\cos_{r, s} x = -\frac{r}{s} (\cos_{r, s} x)^{2-r} (\sin_{r, s} x)^{s-1} \quad \text{and} \quad [(\sin_{r, s} x)^{r-1}]' = (r-1) (\sin_{r, s} x)^{r-2} \cos_{r, s} x.
$$

Thus, as in [11], we get the general formula

$$
[\cos_{r, s}(\pi x/t/2)]^r = [\sin_{s', r'}(\pi x/r' (1-t)/2)]^r,
$$

(27)

which we will employ mostly in the case $r = 2$ and $s = p'$. Finally, let $u(x) = \sin_{2,p'}(x)$. Applying (27), then (24), yields

$$
u''(x) = -\operatorname{sgn}(u(x)) |u(x)|^{p'-1} \frac{\pi_{p, 2}}{\pi_{2, p'}} = -\operatorname{sgn}(u(x)) |u(x)|^{p'-1} \frac{p'}{2}
$$

(28)

Proof of Theorem 5.1. As $W^{2,p}_D(I) = W^{1,-p}_0(I) \cap W^{2,p}(I)$, it is readily seen that

$$
W^{2,p}_D(I) = \left\{ f \in W^{2,p}(I) : f(0) = 0 \text{ and } \int_I f' = 0 \right\}.
$$

Moreover, $W^{2,p}_D(I) \subset W^{1,q}_0(I)$ for any $1 < q < \infty$ and in particular for $q = 2$. We will use these facts below.

We begin by finding an upper bound for $\|E_2\|$. Note that

$$
\|E_2\| = \sup_{f \in W^{2,p}_D(I)} \frac{\|f\|_{p', \mathcal{X}}}{\|f'\|_{p, \mathcal{X}}} = \sup_{f \in W^{2,p}_D(I)} \frac{\|f\|_{p', \mathcal{X}}}{\|f'\|_{2, \mathcal{X}}} = \frac{\sup_{f \in W^{2,p}_D(I)} \|f\|_{p', \mathcal{X}}}{\sup_{f \in W^{2,p}_D(I)} \|f'\|_{2, \mathcal{X}}} = N_1 N_2.
$$

Let us compute the exact values of these constants.
On the one hand, we claim that

\[ N_1 = \frac{\pi_2^{1/p'} (2 + p')^\frac{1}{2} - 1}{2^{1/p' - 1}} = (2 + p')^\frac{1}{2} - \frac{1}{p'} B\left(\frac{1}{2}, \frac{1}{p'}\right) 2^{\frac{1}{2} - \frac{1}{p'}} (p')^{\frac{1}{2}}. \]

The second equality follows from (21). To show the first equality, recall the following classical result of Talenti [23, page 357] (see also [18, (45.4)]). For all \( 1 < r, s < \infty \),

\[ \sup_{f \in W_r^{1,p}} \frac{\|f\|_{L^s(I)}}{\|f\|_{L^r(I)}} = \frac{t_0^{1/p'} (r' + s)\frac{1}{2} - (r')\frac{1}{2}s^{1/p'}}{2B\left(\frac{1}{2}, \frac{1}{p'}\right)} \]

and the extremals of this are any non-zero multiple of \( \sin_{r,s}(\frac{\pi r-x}{t_0}) \). Substituting \( t_0 = \pi_{2,p'} \), \( r = 2 \) and \( s = p' \), it follows that the middle expression of (30) and the right hand side of (31) coincide. Then, since the optimizer \( \sin_{2,p'} \in W_r^{2,p}(I) \), we have that

\[ N_1 = \frac{\|\sin_{2,p'}\|_{L^p(I)}}{\|\cos_{2,p'}\|_{L^2(I)}} \leq \sup_{f \in W_r^{1,p}} \frac{\|f\|_{L^p(I)}}{\|f\|_{L^2(I)}} \leq \sup_{f \in W_r^{1,p}} \frac{\|f\|_{L^p(I)}}{\|f\|_{L^2(I)}} = N_1. \]

This confirms the first equality of (29).

On the other hand, we find the value of \( N_2 \) as follows. By substituting \( f' = u \),

\[ N_2 = \sup_{f \in W_r^{1,p}} \frac{\|f'\|_{L^2(I)}}{\|f''\|_{L^p(I)}} = \sup_{u \in S} \frac{\|u\|_{L^2(I)}}{\|u''\|_{L^p(I)}} \]

where \( S \subseteq W_r^{1,2}(I) : = \{ u \in AC(I) : f_u = 0 \} \) is the subspace

\[ S = \{ u \in AC(I) : u = f' \text{ for some } f \in W_r^{2,p}(I) \}. \]

Now, let us show that

\[ \sup_{u \in W_r^{1,2}(I)} \frac{\|u\|_{L^2(I)}}{\|u''\|_{L^p(I)}} = \frac{t_0^{1/p'} (p' + 2)\frac{1}{2} - (p')\frac{1}{2} s^{1/p'}}{2B\left(\frac{1}{2}, \frac{1}{p'}\right)} \]

where the extremals are non-zero multiples of \( \cos_{2,p'}(\frac{\pi r-x}{t_0}) \). Indeed, recall that the optimizer of the supremum on the left hand side is odd with respect to \( t_0/2 \) (see [14] or [5]). Then,

\[ \sup_{u \in W_r^{1,2}(I)} \frac{\|u\|_{L^2(I)}}{\|u''\|_{L^p(I)}} = \sup_{v \in L_p(0,t_0/2)} \frac{\|Hv\|_{L^2(0,t_0/2)}}{\|v\|_{L_p(0,t_0/2)}}, \]

where \( H : L_p(0,t_0/2) \rightarrow L_2(0,t_0/2) \) is the Hardy operator. According to [12, Theorem 4.6], the supremum on the right hand side is attained whenever \( u \) is any non-zero multiple of \( \cos_{2,p'}(\frac{\pi r-x}{t_0}) \). Hence, from (30) with \( r = p \) and \( s = 2 \) together with the fact that

\[ H(\cos_{2,p'}(\frac{\pi r-x}{t_0})) = \frac{t_0 \sin_{2,p'}(\frac{\pi r-x}{t_0})}{\pi_{2,p'}}, \]

we obtain (31) with the extremals as claimed. Now, for \( t_0 = \pi_{2,p'}, \cos_{2,p'}(\frac{\pi r-x}{t_0}) \in S \). Hence,

\[ \frac{\|\sin_{2,p'}(\frac{\pi r-x}{t_0})\|_{L^2(I)}}{\|\cos_{2,p'}(\frac{\pi r-x}{t_0})\|_{L^2(I)}} \leq N_2 \leq \sup_{u \in W_r^{1,2}(I)} \frac{\|u\|_{L^2(I)}}{\|u''\|_{L^p(I)}} = \frac{\|\sin_{2,p'}(\frac{\pi r-x}{t_0})\|_{L^2(I)}}{\|\cos_{2,p'}(\frac{\pi r-x}{t_0})\|_{L^2(I)}} \]
Thus
\[ N_2 = \frac{\pi^{\frac{1}{p'} + \frac{1}{2}} (p' + 2)^{\frac{1}{2}} (p')^{\frac{1}{4}}}{2^{\frac{3}{2}}} B\left(\frac{1}{2}, \frac{1}{p'}\right) = (2 + p')^{\frac{3}{4}} B\left(\frac{1}{2}, \frac{1}{p'}\right) (p')^{\frac{1}{4}} - p' \cdot \frac{p'}{2}. \]

Therefore, we get
\[ (32) \quad N_1 N_2 = \frac{2}{p'} B\left(\frac{1}{2}, \frac{p' + 1}{p'}\right) \sin^{\frac{1}{2}}. \]

So we have an upper bound for \( \|E_2\| \). Let us now show that there is equality. Set \( u_1(x) = \sin_{2,p'}(x) \). By using (28), we obtain
\[ \frac{\|u_1\|_{p, I}}{\|u_1^2\|_{p, I}} = \frac{2}{p'} \|u_1\|_{p, I}^{1 - \frac{1}{p'}}. \]

Now,
\[ \int_0^{\pi_{2,p'/2}} (\sin_{2,p'}(x))^p dx = \int_0^{\pi_{2,p'/2}} \frac{z^{p'}}{(1 - z^2)^{1/2}} dz = \frac{1}{p'} B\left(\frac{1}{2}, \frac{p' + 1}{p'}\right), \]

which can be obtain by substituting \( z = \sin_{2,p'}(x), dz = \cos_{2,p'}(x) \) and invoking identity (23). Thus,
\[ \frac{\|u_1\|_{p, I}}{\|u_1^2\|_{p, I}} = \left(\frac{2}{p'}\right)^{1 - \frac{1}{2}} \left(\frac{\frac{1}{2} + \frac{1}{p'} - \frac{1}{p'} - 1}{p'}\right)^{1/p' - 1/p} \]

which gives exactly the expression for \( \|E_2\| \).

The uniqueness of the extremal function follows from the uniqueness of the extremal functions in the above arguments.

**Remark 5.1.** Evidently, Theorem 5.1 follows from Theorem 5.2 as the extremal function of \( E_2 \) is the first eigenfunction of (23). However, the proof we include above has the advantage of clearly distinguishing the connection between first order embeddings and second order embeddings in the general case for \( p \) and \( q \). It shows that only for the case \( q = p' \) the extremal functions coincide.

**Proof of Theorem 5.2** Let \( t_0 = \pi_{2,p'} \). From (28), applied twice, we obtain that for \( u(x) = \sin_{2,p'}(x) \) we have
\[ ((u''(x))^{p-1})'' = -\left(\frac{p'}{2}\right)^{p-1} u''(x) = \left(\frac{p'}{2}\right)^p u^{p-1}(x), \]

for all \( 0 < x < \pi_{2,p'} \). As \( u''(x) = -\frac{p'}{2} u^{p-1}(x) \), then also \( u''(0) = u''(\pi_{2,p'}) = 0 \). Hence, \( \sin_{2,p'}(x) \) is a positive eigenfunction for the problem (23) on \( I = [0, \pi_{2,p'}] \) with \( \lambda = (p'/2)^p \).

For the general case \( t_0 > 0 \), observe that
\[ (33) \quad f_n(x) = c \sin_{2,p'}(\pi_{2,p'}nx/t_0), \]
satisfies (23) on \([0, t_0]\) with eigenvalue
\[ \lambda_n(c) = \frac{(\pi_{2,p'} t_{0}^{p/p})^p}{t_{0}^{p/p}} e^{p-p'}. \]
Note that $\lambda_n(c)$ is the $n$-th eigenvalue in the spectrum, for fixed $c > 0$ and that $f_{n,c}$ has exactly $n - 1$ zeros in $(0, t_0)$. Finally, uniqueness follows directly from Lemma 4.2.

6. Approximation of Sobolev embedding

In this final section we derive a precise connection between the $s$-numbers of $E_2$ and the eigenpairs of (6). For this purpose, it is convenient to fix the norm of the eigenfunctions. We will call an eigenpair $(f, \lambda)$ a spectral couple of (6), if $\|f''\|_{p,\mathcal{I}} = 1$ and $f'(0) > 0$. Below we refer to such $f$ as a spectral function and to the corresponding eigenvalue $\lambda > 0$ as a spectral number.

In earlier publications, the choice $\|f\|_{q,\mathcal{I}} = 1$ is used. To distinguish the connection with our choice of normalization, we write $(f, \lambda)$ for the corresponding spectral couple and call it of second kind.

The connection between any eigenpair, spectral couples and spectral couples of second kind becomes evident via re-scaling. Indeed, let $(\tilde{f}, \tilde{\lambda})$ be any eigenpair of (6). Then,

$$\tilde{\lambda} = \frac{\|\tilde{f}''\|_{p,\mathcal{I}}}{\|\tilde{f}\|_{q,\mathcal{I}}}.$$

Let

$$\alpha = \operatorname{sgn}(f'(0))$$

and

$$\alpha = \frac{1}{\|f\|_{q,\mathcal{I}}}.$$

Then,

$$\left( f(t) = \alpha \tilde{f}(t), \quad \lambda = \left( \frac{\|\tilde{f}''\|_{p,\mathcal{I}}}{\|\tilde{f}\|_{q,\mathcal{I}}} \right)^q \right)$$

is a spectral couple and

$$\left( f(t) = \alpha \tilde{f}(t), \quad \lambda = \left( \frac{\|\tilde{f}''\|_{p,\mathcal{I}}}{\|\tilde{f}\|_{q,\mathcal{I}}} \right)^p \right)$$

is a spectral couple of second kind. Our discussion below only refers to spectral couples.

The next lemma shows that the spectral functions form a unique chain, linked by re-scaling and generating a corresponding chain of spectral numbers. In turn, the latter form an increasing sequence of positive numbers accumulating at $+\infty$.

**Lemma 6.1.** Let $1 < p, q < \infty$ and $n \in \mathbb{N}$.

i) There is a unique spectral couple $(f, \lambda)$ such that $f$ has $n - 1$ distinct zeros in $\operatorname{Int}(\mathcal{I})$.

ii) Let $(f_1, \lambda_1)$ be the spectral couple on $\mathcal{I} = [0, 1]$ where $\lambda_1$ is the first spectral number. Let $f_{1*} : \mathbb{R} \to \mathbb{R}$ be the 2-periodic odd function, such that $f_{1*}(t) = f_1(t)$ for all $t \in [0, 1]$. Then,

$$\left( f_n(t) = \frac{f_{1*}(nt)}{n^2}, \quad \lambda_n = n^{2q} \lambda_1 \right)$$

is the spectral couple on $[0, 1]$ associated to the $n$-th spectral number.
iii) Let \((f_n, \lambda_n)\) be the \(n\)-th spectral couple of the previous item. Then,
\[
SN_n(t) = t^{2-1/p}f_n(t/t_0), \quad sn_n = n^{q/p - 1 - 2q} \lambda_n
\]
is the spectral couple on \(I = [0, t_0]\) which has \(n - 1\) distinct zeros in \((0, t_0)\).
In particular
\[
SN_n(t) = t^{2-1/p}f_1(nt/t_0), \quad \text{and} \quad sn_n = n^{2q\lambda_1^{p - 1 - 2q}}
\]
are the \(n\)-th spectral function and the \(n\)-th spectral number on \(I\), respectively.

**Proof.** The first item follows directly from Theorem [11] and Corollary [11]. The other statements follow from applying the substitutions [11], then conducting the corresponding computations. \(\square\)

Below we will adhere disambiguously to the notation of this lemma. The following direct consequence of it, summarizes the original purpose of this paper. That is, the calculation of the norm of the second order Sobolev embedding.

**Theorem 6.1.** For all \(1 < p, q < \infty\), the second order embedding
\[
E_2 : W^{2,p}_D(I) \to L^q(I)
\]
has norm
\[
\|E_2\| = \sup_{u \in W^{2,p}_D(I)} \frac{\|u\|_{q,I}}{\|u''\|_{p,I}} = \|SN_1\|_{q,I} = sn_1^{-1/q} = |I|^{1/q - 1/p} \lambda_1^{-1/q},
\]
where \(\lambda_1\) is the first spectral number of the unit interval \([0, 1]\).

It is natural now to describe the connection between the different s-numbers of \(E_2\) and the spectral couples. We begin by surveying the classical definitions.

**Definition 6.1.** Let \(s : T \mapsto \{s_n(T)\} \in \ell_\infty(\mathbb{N})\) be a rule which assigns to every bounded linear operator \(T \in B(X, Y)\) on every pair of Banach spaces \(X\) and \(Y\), a sequence of non-negative numbers satisfying the following properties.

\(\begin{array}{l}
(S1) \|T\| = s_1(T) \geq s_2(T) \geq \ldots. \\
(S2) s_n(S + T) \leq s_n(S) + \|T\| \text{ for } S, T \in B(X, Y) \text{ and } n \in \mathbb{N}. \\
(S3) s_n(BTA) \leq \|B\|s_n(T)\|A\| \text{ whenever } A \in B(X_0, X), T \in B(X, Y), B \in B(X, Y_0), \text{ and } n \in \mathbb{N}. \\
(S4) s_n(Id : \mathbb{R}^n \to \mathbb{R}^n) = 1 \text{ for } n \in \mathbb{N}. \\
(S5) s_n(T) = 0 \text{ when } \text{rank}(T) < n. \\
\end{array}\)

We call \(s_n(T)\) \((\text{or } s_n(T : X \to Y))\) an \(n\)-th s-number of \(T\). Moreover, when \((S4)\) is replaced by

\(\begin{array}{l}
(S6) s_n(Id : E \to E) = 1 \text{ for every Banach space } E \text{ with } \dim(E) \geq n, \\
\end{array}\)
we say that \(s_n(T)\) is the \(n\)-th s-number of \(T\) in the strict sense.

Many standard s-numbers of approximation theory are defined in relation to the moduli of injectivity and surjectivity, which we will recall.

**Definition 6.2.** Let \(T \in B(X, Y)\). The modulus of injectivity of \(T\) is
\[
j(T) = \sup\{\rho \geq 0 : \|Tx\|_Y \geq \rho\|x\|_X \text{ for all } x \in X\}.
\]
The modulus of surjectivity of \(T\) is
\[
q(T) = \sup\{\rho \geq 0 : T(B_X) \supset \rho B_Y\}.
\]
Below we denote the embedding of a closed linear subspace \( M \subset X \) into \( X \) by \( J_M^X \) and the canonical map of \( X \) onto the quotient space \( X/M \) by \( Q_M^X \). The standard \( n \)-th \( s \)-numbers and their terminology is as follows.

**Definition 6.3.** Let \( T \in B(X,Y) \) and \( n \in \mathbb{N} \).

- **Approximation numbers** of \( T \)
  \[ a_n(T) = \inf \{ \| T - F \| : F \in B(X,Y), \text{rank}(F) < n \} \]
- **Isomorphism numbers** of \( T \)
  \[ i_n(T) = \sup \{ \| A^{-1} B^{-1} \| \} \]
  the supremum taken over all possible Banach spaces \( G \) with \( \dim(G) \geq n \) and maps \( A \in B(Y,G) \), \( B \in B(G,X) \) such that \( ATB \) is the identity on \( G \).
- **Gelfand numbers** of \( T \)
  \[ c_n(T) = \inf \{ \| T J_M^X \| : \text{codim}(M) < n \} \]
- **Bernstein numbers** of \( T \)
  \[ b_n(T) = \sup \{ j(T J_M^X) : \dim(M) \geq n \} \]
- **Kolmogorov numbers** of \( T \)
  \[ d_n(T) = \inf \{ \| Q_M^X T \| : \dim(N) < n \} \]
- **Mityagin numbers** of \( T \)
  \[ m_n(T) = \sup \{ q(Q_M^X T) : \text{codim}(N) \geq n \} \]

In the context of this definition, recall the fundamental relation \[ a_n(T) \geq \max[c_n(T),d_n(T)] \geq \min[c_n(T),d_n(T)] \]
\[ \geq \max[b_n(T),m_n(T)] \geq \min[b_n(T),m_n(T)] \geq i_n(T). \]

Moreover, the approximation numbers are the largest \( s \)-numbers and the isomorphism numbers are the smallest strict \( s \)-numbers. In the next theorem we will use notation \( S_n \) and \( s_n \) from Lemma 6.1.

**Theorem 6.2.** For all \( 1 < p,q < \infty \), the second order embedding
\[ E_2 : W^{2,p}_D(I) \to L^q(I) \]
has \( s \)-numbers obeying the following relations.

i) If \( p < q \), then
\[ i_n(E_2) \geq s_n^{-1/q} = \frac{|I|^{1/q + 2 - 1/p}}{n^2 \lambda_1^{1/q}}. \]

ii) If \( p \geq q \), then
\[ a_n(E_2) \leq s_n^{-1/q} = \frac{|I|^{1/q + 2 - 1/p}}{n^2 \lambda_1^{1/q}}. \]

**Proof.** Fix \( 1 < p \leq q < \infty \).

Proof of i). We show that
\[ i_n(E_2) \geq s_n^{-1/q}. \]

Set \( 0 = a_0 < a_1 < \ldots < a_n = t_0 \) where \( a_i - a_{i-1} = t_0/n \) and \( I_i = (a_{i-1},a_i) \). Then, \( S_n(a_i) = 0 = S_n''(a_i) \) for \( i = 0,...,n \). Let \( g_i = S_n(x)I_i \) for \( 1 \leq i \leq n \).
Then $M_n = \text{span}\{g_1, \ldots, g_n\}$ has dimension $n$. Also, $\|g_i''\|_{p, \mathcal{I}_i} = \|g_i''\|_{p, \mathcal{I}_j}$ for every $1 \leq i \leq n$.

Now, let $\mu = \text{sn}_1$ for $\mathcal{I}_1$. By virtue of Theorem 6.1 applied on the interval $\mathcal{I}_i$,

$$\text{SN}_1(t) = \frac{\text{sgn}(g'_i(0))}{\|g_i''\|_p} g_i(t) \quad \forall t \in \mathcal{I}_i.$$ 

Hence,

$$\|g_i\|_q = \mu^{-1} \|g_i''\|_p^q$$

for all $i \in \{1, \ldots, n\}$.

Recall the next identity valid for $p \leq q$. [12 Lemma 8.14],

$$\inf_{\alpha \in \mathbb{R}^n} \frac{\left(\sum_{i=1}^n |\alpha_i|^q\right)^{1/q}}{\left(\sum_{i=1}^n (\frac{1}{|\alpha_i|^p})\right)^{1/p}} = n^{1/q - 1/p},$$

(35)

and the infimum is attained when $|\alpha_i| = c$, $i = 1, \ldots, n$. Since the supports of the $g_i$ are disjoint, we have,

$$\inf_{0 \neq u \in M_n} \frac{\|u\|_q}{\|u''\|_p, \mathcal{I}} = \inf_{\alpha \in \mathbb{R}^n \setminus \{0\}} \frac{\left(\sum_{i=1}^n |\alpha_i g_i|_q, \mathcal{I}\right)^{1/q}}{\left(\sum_{i=1}^n |\alpha_i (g_i'')_{p, \mathcal{I}}\right)^{1/p}} = \inf_{\alpha \in \mathbb{R}^n \setminus \{0\}} \frac{\left(\sum_{i=1}^n |\alpha_i|^q\right)^{1/q}}{\left(\sum_{i=1}^n |\alpha_i|^p\right)^{1/p}} \mu^{-1/q} = n^{1/q - 1/p} \mu^{-1/q} = \text{sn}_n^{-1/q}.$$ 

The last equality follows from Lemma 6.1.

This allows us to complete the proof of i) as follows. In the definition of the isomorphism number, replace $G = M_n$ with norm $\| \cdot \|_{q, \mathcal{I}}$, $X = W^{2,p}_{D}$, $Y = L^q$ and $T = E_2$. Now set $B : G \rightarrow X$ the operator $B(u) = u$ and $A : Y \rightarrow G$ the projection obtained by completing $\{g_i\}$ to a basis of $L^q(\mathcal{I})$. Then $\|A\| = 1$ and

$$\|B\|^{-1} = \left(\sup_{0 \neq u \in M_n} \frac{\|u''\|_{p, \mathcal{I}}}{\|u\|_{q, \mathcal{I}}}\right)^{-1} = \inf_{0 \neq u \in M_n} \frac{\|u\|_{q, \mathcal{I}}}{\|u''\|_{p, \mathcal{I}}} = \text{sn}_n^{-1/q}.$$ 

So, indeed $i_n(E_2) \geq \text{sn}_n^{-1/q}$.

Proof of ii). Now we claim that for $q \leq p$,

$$a_n(E_2) \leq \text{sn}_n^{-1/q}.$$ 

Using the partition of the interval into $n$ sub-intervals as in the previous part, we now let

$$(Tu)(x) = \chi_{\mathcal{I}_i}(x) \left( u(a_{i-1}) + \frac{u(a_i) - u(a_{i-1})}{a_i - a_{i-1}}(x - a_{i-1}) \right)$$

and $T = \sum_{i=1}^n T_i$. Note that $T$ defined on $W^{2,p}(\mathcal{I})$ is an operator of rank $n - 1$. Then,

$$a_n(E_2) \leq \sup_{0 \neq f \in W^{2,p}_D(\mathcal{I})} \frac{\|f - Tf\|_{q, \mathcal{I}}}{\|f''\|_{p, \mathcal{I}}} \leq \sup_{0 \neq u \in W^{2,p}(\mathcal{I})} \frac{\|u - Tu\|_{q, \mathcal{I}}}{\|u''\|_{p, \mathcal{I}}}.$$ 

Now, for any $0 \neq u \in W^{2,p}(\mathcal{I})$,

$$\|u - Tu\|_{q, \mathcal{I}} = \left( \sum_{i=1}^n \|u - T_i u\|_{q, \mathcal{I}}^q \right)^{1/q}.$$
Also, for all \( x \in \mathcal{I}_i \), \( u''(x) = (u - Tu)''(x) \). Hence,
\[
\|u''\|_{p,\mathcal{I}} = \left(\sum_{i=1}^{n} \|u_{i} - T_{i}u_{i}\|_{q,\mathcal{I}_i}^{p}\right)^{1/p}.
\]

Let \( u_{i} = \chi_{\mathcal{I}_i} u \). Then, for any \( u_{i} \in W^{2,p} (\mathcal{I}_i) \) we have \((u_{i} - T_{i}u_{i})(a_{i}) = (u - T_{i}u_{i})(a_{i-1}) = 0 \) and \((u_{i} - T_{i}u_{i})'' = u''_{i} \). Thus,
\[
a_{n}(E_{2}) \leq \sup_{u \in W^{2,p}(\mathcal{I})} \left(\frac{\sum_{i=1}^{n} \|u_{i} - T_{i}u_{i}\|_{q,\mathcal{I}_i}^{q}}{\sum_{i=1}^{n} \|u''_{i}\|_{p,\mathcal{I}_i}^{p}}\right)^{1/q}
\]
\[
\leq \sup_{u_{i} \in W^{2,p}(\mathcal{I}_i)} \left(\frac{\sum_{i=1}^{n} \|u_{i} - T_{i}u_{i}\|_{q,\mathcal{I}_i}^{q}}{\sum_{i=1}^{n} \|u''_{i}\|_{p,\mathcal{I}_i}^{p}}\right)^{1/q}
\]
\[
= \frac{\sum_{i=1}^{n} \|f_{i}''\|_{p,\mathcal{I}_i}^{p}}{\sum_{i=1}^{n} \|f_{i}'\|_{q,\mathcal{I}_i}^{q}}.
\]

By virtue of Theorem 6.1 applied on the sub-segments \( \mathcal{I}_{i} \), which are of equal length, we then have
\[
\sup_{f_{i} \in W^{2,p}(\mathcal{I}_{i})} \left(\frac{\sum_{i=1}^{n} \|f_{i}'\|_{q,\mathcal{I}_i}^{q}}{\sum_{i=1}^{n} \|f_{i}''\|_{p,\mathcal{I}_i}^{p}}\right)^{1/q}
\]
\[
\leq \frac{n^{-1/q+2-1/p}}{\lambda_{1}^{1/q+2-1/p}} \frac{\sum_{i=1}^{n} \|\alpha_{i}'\|_{q,\mathcal{I}_i}^{q}}{\sum_{i=1}^{n} \|\alpha_{i}''\|_{p,\mathcal{I}_i}^{q}}
\]
where \( |\alpha_{i}| = \|f_{i}''\|_{p,\mathcal{I}_i} \).

Invoke now the identity [12] Lemma 8.23], valid for \( q \leq p \),
\[
(37) \quad \sup_{\alpha \in \mathbb{R}^{n}} \left(\frac{\sum_{i=1}^{n} |\alpha_{i}'|^{q}}{\sum_{i=1}^{n} |\alpha_{i}|^{p}}\right)^{1/q} = n^{1/q-1/p},
\]
where the supremum being attained when \( |\alpha_{i}| = c \), \( i = 1, \ldots, n \). Hence, we finally get
\[
a_{n}(E_{2}) \leq \frac{\sum_{i=1}^{n} |\alpha_{i}'|^{q}}{\sum_{i=1}^{n} |\alpha_{i}|^{p}} = \frac{n^{-1/q}}{\lambda_{1}^{1/q}}.
\]
as claimed.

The second statement in the previous theorem is an improvement on the results obtained in [3] and [4].

The next statement encompasses the other original main purpose of this paper, but we omit details of its proof, which is beyond the current scope. It follows from Lemma 6.1 and Theorem 6.2 alongside with results settled in [3] and [4]. Basic
Theorem]. An interesting related question, is whether we can replace \( d_n(E_2) \) by \( c_n(E_2) \) and \( b_n(E_2) \) by \( m_n(E_2) \) below. Our omission of the proof is motivated by the fact that we aim at reporting on a full investigation of this question in future.

**Theorem 6.3.** Let \( 1 < p, q < \infty \).

i) If \( p \leq q \), then
\[
i_n(E_2) = b_n(E_2) = sn_n^{-1/q} = \frac{|I|^{1/q+2-1/p}}{n^2 \lambda_1^{1/q}}.
\]

ii) If \( q \leq p \), then
\[
a_n(E_2) = d_n(E_2) = sn_n^{-1/q} = \frac{|I|^{1/q+2-1/p}}{n^2 \lambda_1^{1/q}}.
\]

**Appendix A. The singularity of the solutions**

In this appendix we provide more details about the behaviour of solutions of the systems (*) near the point of singularity \( t_\infty \). The results presented in the main body of the paper are independent of the ones below. Here \( \gamma \in \mathbb{R}^4 \). Consider the general system
\[
\begin{align*}
&\phi_1(t) = u_2(t) \\
&\phi_2(t) = \lambda |u_1(t)|^{q-1}
\end{align*}
\]
for \( t \geq t_1 \) subject to
\[
\begin{align*}
&u_1(t_1) = \gamma_1, u_2(t_1) = \gamma_2, \ w_1(t_1) = \gamma_3, \ w_2(t_1) = \gamma_4.
\end{align*}
\]
This is equivalent to the system of integral equations
\[
\begin{align*}
u_1(t) &= \gamma_1 + \int_{t_1}^{t} u_2(s) ds & \quad & u_2(t) = \gamma_2 - \int_{t_1}^{t} [w_1(s)]^{p' - 1} ds \\
&\quad & w_1(t) &= \gamma_3 + \int_{t_1}^{t} w_2(s) ds & \quad & w_2(t) = \gamma_4 - \lambda \int_{t_1}^{t} |u_1(s)|^{q - 1} ds.
\end{align*}
\]
Adapting the definition (11), we now write
\[
t_\infty(\lambda, p, q, \gamma) = \sup \left\{ t_2 > t_1 : \sup_{t \in (t_1, t_2)} |\phi(t)| < \infty \right\}.
\]
That is, all components of \( \varphi(t) \) are finite for \( t_1 \leq t < t_\infty \) and
\[
\lim_{t \to t_\infty} |\varphi(t)| = \infty.
\]

The proof of the following lemma is straightforward.

**Lemma A.1.** Let \( \lambda > 0 \) and \( p, q > 1 \) be fixed.

- If \( \gamma_1 \geq 0, \gamma_2 > 0, \gamma_3 \leq 0 \) and \( \gamma_4 < 0 \), then \( u_j(t) \) are increasing and \( w_k(t) \) are decreasing for all \( t \in (t_1, t_\infty) \). Moreover, \( u_j(t) \uparrow \infty \) and \( w_j(t) \downarrow -\infty \) as \( t \to t_\infty \).
- If \( \gamma_1 \leq 0, \gamma_2 < 0, \gamma_3 \geq 0 \) and \( \gamma_4 > 0 \), then \( u_j(t) \) are decreasing and \( w_k(t) \) are increasing for all \( t \in (t_1, t_\infty) \). Moreover, \( u_j(t) \downarrow -\infty \) and \( w_j(t) \uparrow \infty \) as \( t \to t_\infty \).

We include full proof of the next lemma, as it is interesting in the context of the property (12).
Lemma A.2. Let $\lambda > 0$. Let $p_1 \geq p_2$, $q_1 \leq q_2$.

- If $\gamma_1 > 1$, $\gamma_2 > 0$, $\gamma_3 < -1$ and $\gamma_4 < 0$, then $u_1^j(t) \leq u_2^j(t)$ and $w_2^j(t) \leq w_1^j(t)$ for all $t \in (t_1, \infty)$.
- If $\gamma_1 < -1$, $\gamma_2 < 0$, $\gamma_3 > 1$ and $\gamma_4 > 0$, then $u_2^j(t) \leq u_1^j(t)$ and $w_1^j(t) \leq w_2^j(t)$ for all $t \in (t_1, \infty)$.

Proof. Note that the previous lemma implies monotonicity of the solutions. Also note that $p_1' \leq p_2'$. We only consider the proof of the first statement. Since

$$\frac{d}{dt}(u_2^j(t) - u_1^j(t))|_{t=t_1} = -(\gamma_3^{p_2'} - \gamma_3^{p_1'}) = |\gamma_3|^{p_2'} - |\gamma_2|^{p_1'} > 0,$$

then, for $t > t_1$ (small enough first and then generic), we have all the following. To begin with, $u_1^j(t) \geq u_1^j(t) > 1$. Hence, $|u_1^j(t)|^{q_2} - |u_1^j(t)|^{q_1} > 1$ so that

$$-\int_{t_1}^{t} |u_1^j(s)|^{q_2-1} ds \leq -\int_{t_1}^{t} |u_1^j(t)|^{q_1-1} ds < 0.$$

Thus, $w_2^j(t) \leq w_1^j(t) < 0$ and so $w_1^j(t) \leq w_1^j(t) < -1$. This implies that

$$-[u_1^j(t)]^{p_2'} = |w_2^j(t)|^{p_2'} - |w_1^j(t)|^{p_2'} \geq |u_1^j(t)|^{p_1'} - [w_1^j(t)]^{p_1'} > 0.$$

And so $w_2^j(t) \geq w_2^j(t)$, which completes the proof of the first statement. \hfill $\square$

The proof of the next consequence of this lemma by contradicion is straightforward.

Corollary A.1. Let $p_1 \geq p_2$ and $q_1 \geq q_2$. Suppose that either of the two same conditions for the entries of $\gamma$ as in Lemma A.2 hold true. Then $t_\infty(\lambda, p_2, q_2, \gamma) \geq t_\infty(\lambda, p_1, q_1, \gamma)$.

Corollary A.2. Let $p \geq 2$ and $q \leq 2$. Then $t_\infty = \infty$ and \hfill $\square$ holds true.

Proof. Let $t_1 > 0$ be such that (without loss of generality)

$$u_1(t_1) > 1, \quad w_2(t_1) > 0, \quad w_1(t_1) < -1, \quad \text{and} \quad w_2(t_1) < 0.$$

The result follows from the previous corollary taking $p_1 = p$, $q_1 = q$, $p_2 = 2$ and $q_2 = 2$, combined with the fact that

$$t_\infty(\lambda, 2, 2, \gamma) = \infty$$

for any initial $\gamma \in \mathbb{R}^4$. \hfill $\square$

We conjecture that $t_\infty < \infty$ for $p > q$ and $t_\infty = \infty$ for $p \leq q$. Also note that $t_\infty < \infty$ can occur. For example, observe that for $p = 4/3$ and $q = 3$, $\beta = -\alpha = -1$ and $\lambda = 1$, \hfill $\square$ reduces to

$$u''(t) = u^2(t)$$

$$u(0) = 0, \quad u'(0) = 1.$$

This differential equation has as solutions Jacobi Elliptic functions, which indeed have poles on the real line.

Acknowledgments

This research was funded by the UK’s Royal Society International Exchange Grant “James Orthogonality and Higher order Sobolev Embeddings”. We are also grateful to our colleagues at the Czech Technical University in Prague for hosting many of the discussions that eventually lead to this paper.
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