UNIQUENESS OF WEAK SOLUTIONS OF THE
THREE-DIMENSIONAL COMPRESSIBLE NAVIER-STOKES
EQUATIONS WITH POTENTIAL FORCE

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Abstract. We prove uniqueness of weak solutions of the three-dimensional
compressible Navier-Stokes equations with potential force. We make use of the
Lagrangean framework in comparing the instantaneous states of corresponding
fluid particles in two different solutions. The present work provides qualitative
results on how the weak solutions depend continuously on initial data and
steady states.

1. Introduction

We are interested in the 3-D compressible Navier-Stokes equations with an ex-
ternal potential force in the whole space $\mathbb{R}^3$ ($j = 1, 2, 3$):

\begin{equation}
\begin{aligned}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u^j)_t + \text{div}(\rho u^j u) + (P)_{x_j} &= \mu \Delta u^j + \lambda (\text{div} u)_{x_j} + \rho f^j.
\end{aligned}
\end{equation}

Here $x \in \mathbb{R}^3$ is the spatial coordinate and $t \geq 0$ stands for the time. The unknown
functions $\rho = \rho(x, t)$ and $u = (u^1, u^2, u^3)(x, t)$ represent the density and velocity
vector in a compressible fluid. The function $P = P(\rho)$ denotes the pressure, $f =
(f^1(x), f^2(x), f^3(x))$ is a prescribed external force and $\mu, \lambda$ are positive viscosity
constants. The system (1.1) is equipped with initial condition

\begin{equation}
(\rho(\cdot, 0) - \rho_s, u(\cdot, 0)) = (\rho_0 - \rho_s, u_0),
\end{equation}

where the non-constant time-independent function $\rho_s = \rho_s(x)$ (known as the steady
state solution to (1.1)) can be obtained formally by taking $u \equiv 0$ in (1.1):

\begin{equation}
\nabla P(\rho_s(x)) = (x) = \rho_s(x)f(x).
\end{equation}

The well-posedness problem of the Navier-Stokes system (1.1) is an important but
challenging research topic in fluid mechanics, and we now give a brief review
on the related results. The local-in-time existence of classical solution to the full
Navier-Stokes equations was proved by Nash [Nas62] and Tani [Tan77], and some
Serrin type blow-up criteria for smooth solutions was recently obtained by Suen
[Sue20d]. Later, Matsumura and Nishida [MN80] obtained the global-in-time exis-
tence of $H^3$ solutions when the initial data was taken to be small with respect to $H^3$
norm, the results were then generalised by Danchin [Dan00] who showed the global
existence of solutions in critical spaces. In the case of large initial data, Lions [Lio98]
obtained the existence of global-in-time finite energy weak solutions, yet the problem of uniqueness for those weak solutions remains completely open. In between the two types of solutions as mentioned above, a type of “intermediate weak” solutions were first suggested by Hoff in [Hof95, Hof02, Hof05, Hof06] and later generalised by Matsumura and Yamagata in [MY01], Sue in [Sue13, Sue14, CS16] and other systems which include compressible magnetohydrodynamics (MHD) [SH12, Sue12, Sue20b], compressible Navier-Stokes-Poisson system [Sue20a] and chemotaxis systems [LS16]. Solutions as obtained in this intermediate class are less regular than those small-smooth type solutions obtained by Matsumura and Nishida [MN80] and Danchin [Dan00], which contain possible codimension-one discontinuities in density, pressure, and velocity gradient. Nevertheless, those intermediate weak solutions would be more regular than the large-weak type solutions developed by Lions [Ljo98], hence the uniqueness and continuous dependence of solutions may be obtained; see [Hof06] and the compressible MHD [Sue20b].

In this present work, we compare solutions of (1.1)-(1.2) from the class of intermediate weak solutions as mentioned above. Our results extend that of Cheung and Sue [CS16], which proved uniqueness of weak solution to (1.1)-(1.2) with H"older continuous density function \( \rho \). Such condition on \( \rho \) is too strong in the sense that this would exclude solutions with codimension-one singularities, which are physically interesting features for the weak solutions; see [Hof02] for a detailed discussion on the propagation of singularities. The main novelties of this current work can be summarised as follows:

- We provide detailed descriptions on the global-in-time existence and regularity of weak solution to (1.1)-(1.2), see Theorem 2.4 in section 2;
- We remove the H"older continuity restriction on density from [CS16] which allows us to include a larger class of weak solutions;
- We give a more precise results on how the weak solutions to (1.1)-(1.2) depend continuously on initial data and steady state solutions, see (1.28) in Theorem 1.1.

We now give a precise formulation of our results. For \( r \in (1, \infty] \), we define the following function spaces:

\[
\begin{align*}
L^r &= L^r(\mathbb{R}^3) = \{ u \in L^1_{loc}(\mathbb{R}^3) : \| \nabla^k u \|_{L^r} < \infty \}, \\
D^{k,r} &= \{ u \in L^1_{loc}(\mathbb{R}^3) : \| \nabla^k u \|_{L^r} < \infty \}, \\
W^{k,r} &= L^r \cap D^{k,r}, \\
H^k &= W^{k,2}.
\end{align*}
\]

We introduce the usual convective derivative \( \frac{d}{dt} \) with respect to a velocity field \( u \) as follows. For a given function \( w : \mathbb{R}^3 \times (0, T) \to \mathbb{R} \), we define

\[
\frac{d}{dt}(w) = \dot{w} := w_t + u \cdot \nabla w,
\]

where \( \nabla w \) is the gradient of \( w \). For \( w : \mathbb{R}^3 \times (0, T) \to \mathbb{R}^3 \), we define

\[
\frac{d}{dt}(w) = \dot{w} := w_t + \nabla w u,
\]

where \( \nabla w \) is the \( 3 \times 3 \) matrix of partial derivatives of \( w \).

We define the system parameters \( P, f, \mu, \lambda \) as follows. For the pressure function \( P = P(\rho) \) and the external force \( f \), we assume that

\[
P(\rho) = a\rho \text{ with } a > 0;
\]

\[
\text{There exists } \psi \in H^2 \text{ such that } f = \nabla \psi \text{ and } \psi(x) \to 0 \text{ as } |x| \to \infty.
\]

(1.4)

\[
\frac{d}{dt}(w) = \dot{w} := w_t + u \cdot \nabla w,
\]

where \( \nabla w \) is the gradient of \( w \). For \( w : \mathbb{R}^3 \times (0, T) \to \mathbb{R}^3 \), we define

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\frac{d}{dt}(w) = \dot{w} := w_t + \nabla w u,
\]

where \( \nabla w \) is the \( 3 \times 3 \) matrix of partial derivatives of \( w \).

We define the system parameters \( P, f, \mu, \lambda \) as follows. For the pressure function \( P = P(\rho) \) and the external force \( f \), we assume that

(1.6)

\[
P(\rho) = a\rho \text{ with } a > 0;
\]

(1.7)

\[
\text{There exists } \psi \in H^2 \text{ such that } f = \nabla \psi \text{ and } \psi(x) \to 0 \text{ as } |x| \to \infty.
\]
The viscosity coefficients $\mu$ and $\lambda$ are assumed to satisfy
\begin{equation}
\lambda \geq 0, \quad \mu > 0.
\end{equation}

Next, we define $\rho_s$ as mentioned at the beginning of this section. Given a constant density $\rho_\infty > 0$, we say $(\rho_s,0)$ is a steady state solution to (1.1) if $\rho_s \in C^2(\mathbb{R}^3)$ and the following holds
\begin{equation}
\begin{cases}
\nabla P(\rho_s(x)) = \rho_s(x) \nabla \psi(x), \\
\lim_{|x| \to \infty} \rho_s(x) = \rho_\infty.
\end{cases}
\end{equation}

By solving (1.9), $\rho_s$ can be expressed explicitly as follows:
\begin{equation}
\rho_s(x) = \rho_\infty \exp\left(\frac{1}{a} \psi(x)\right).
\end{equation}

From now on, we choose $\rho_\infty \equiv 1$ and take $\rho_s$ which satisfies (1.10). We also write $P_s = P(\rho_s)$ for simplicity.

We further introduce two important functions, namely the effective viscous flux $F$ and vorticity $\omega$, which are defined by
\begin{equation}
\rho_s F = (\mu + \lambda) \text{div} u - (P(\rho) - P(\rho_s)) , \quad \omega = \omega^{j,k} = u_{x_k}^j - u_{x_j}^k.
\end{equation}

By the definitions of $F$ and $\omega$, and together with (1.14), $F$ and $\omega$ satisfy the elliptic equations
\begin{align}
\Delta (\rho_s F) &= \text{div}(\rho \dot{u} - \rho f + \nabla P(\rho_s)), \\
\mu \Delta \omega &= \nabla \times (\rho \dot{u} - \rho f + \nabla P(\rho_s)).
\end{align}

The functions $F$ and $\omega$ play essential roles for studying intermediate weak solutions to compressible flows, see [Hof95, Hof02, SH12] for more detailed discussions.

Weak solutions to the system (1.1)-(1.2) can be defined as follows. We say that $(\rho, u, f, \rho_s)$ on $\mathbb{R}^3 \times [0,T]$ is a weak solution of (1.1)-(1.2) if the following conditions hold:
\begin{align}
\rho_s &= \text{a steady state solution to (1.9)} \quad \text{which satisfies (1.10)}; \tag{1.14} \\
\rho - \rho_s &= \text{a bounded map from } [0,T] \text{ into } L^1_{\text{loc}} \cap H^{-1} \text{ and } \rho \geq 0 \text{ a.e.}; \tag{1.15} \\
\rho_0 u_0 &= L^2; \rho_0, P_s, \nabla u_0, \rho f \in L^2(\mathbb{R}^3 \times (0,T)); \rho |u|^2 \in L^1(\mathbb{R}^3 \times (0,T)); \tag{1.16}
\end{align}

For all $t_2 \geq t_1 \geq 0$ and $C^1$ test functions $\varphi$ which are Lipschitz on $\mathbb{R}^3 \times [t_1,t_2]$ with $\text{supp } \varphi(\cdot, t) \subset K$, $t \in [t_1,t_2]$, where $K$ is compact and
\begin{equation}
\int_{\mathbb{R}^3} \rho(x,\cdot) \varphi(x,\cdot) dx \bigg|_{t_1}^{t_2} = \int_{t_1}^{t_2} \int_{\mathbb{R}^3} (\rho \varphi_t + \rho u \cdot \nabla \varphi) dx dt;
\end{equation}

The weak form of the momentum equation
\begin{align}
\int_{\mathbb{R}^3} (\rho u^j)(x,\cdot) \varphi(x,\cdot) dx \bigg|_{t_1}^{t_2} &= \int_{t_1}^{t_2} \int_{\mathbb{R}^3} \left[ \rho u^j \varphi_t + \rho u^j u \cdot \nabla \varphi + (P - P_s) \varphi_{x_j} \right] dx dt \\
&- \int_{t_1}^{t_2} \int_{\mathbb{R}^3} \left[ \mu \nabla u^j \cdot \nabla \varphi + (\mu - \xi)(\text{div } u) \varphi_{x_j} \right] dx dt \\
+ \int_{t_1}^{t_2} \int_{\mathbb{R}^3} (\rho f - \nabla P_s) \cdot \varphi dx dt. \tag{1.18}
\end{align}
holds for test functions \( \varphi \) which are locally Lipschitz on \( \mathbb{R}^3 \times [0, T] \) and for which 
\( \varphi, \varphi_t, \nabla \varphi \in L^2(\mathbb{R}^3 \times (0, T)), \nabla \varphi \in L^\infty(\mathbb{R}^3 \times (0, T)) \), and \( \varphi(\cdot, T) = 0 \).

For the two solutions \((\rho, u, f, \rho_s)\) and \((\bar{\rho}, \bar{u}, \bar{f}, \bar{\rho}_s)\) we compare, they will be assumed to satisfy
\[
\text{(1.19)} \quad u, \bar{u} \in C(\mathbb{R}^3 \times (0, T]) \cap L^1((0, T); W^{1, \infty}) \cap L^{\infty}_{loc}((0, T]; L^\infty);
\]
\[
\text{(1.20)} \quad \rho - \rho_s, \bar{\rho} - \bar{\rho}_s, u, \bar{u}, f, \bar{f} \in L^2(\mathbb{R}^2 \times (0, T)).
\]

One of the solutions \((\rho, u, f, \rho_s)\) will have to satisfy
\[
\text{(1.21)} \quad ||f||_{L^\infty} < \infty,
\]
\[
\text{(1.22)} \quad \rho, \rho^{-1} \in L^\infty(\mathbb{R}^3 \times (0, T)),
\]
and
\[
\text{(1.23)} \quad \int_0^T \int_{\mathbb{R}^3} |u|^r dx dt < \infty
\]
for some \( r > 3 \), and the other solution \((\bar{\rho}, \bar{u}, \bar{f}, \bar{\rho}_s)\) will have to satisfy
\[
\text{(1.24)} \quad \|\bar{\rho}_s\|_{L^\infty} + \|\nabla \bar{\rho}_s\|_{L^\infty} < \infty,
\]
\[
\int_0^T \left[ \|\nabla \bar{F}(\cdot, t)\|_{L^2}^2 + t\|\nabla \bar{\omega}(\cdot, t)\|_{L^2}^2 + t^\alpha\|\nabla \bar{F}(\cdot, t)\|_{L^3}^2 + t^\alpha\|\nabla \bar{\omega}(\cdot, t)\|_{L^3}^2 \right] dt
\]
\[
+ \int_0^T \left[ \|\bar{u}(\cdot, t)\|_{L^\infty}^2 + t\|\nabla \bar{u}(\cdot, t)\|_{L^\infty}^2 \right] dt < \infty,
\]
where \( \bar{F} \) and \( \bar{\omega} \) are as in (1.12)–(1.13) and \( \alpha = \frac{1}{2} \); and
\[
\text{(1.26)} \quad \bar{f} \in L^{2q},
\]
for some \( q \in [1, \infty) \). Finally, we assume that
\[
\text{(1.27)} \quad \rho_0 - \bar{\rho}_0 \in L^2 \cap L^{2p},
\]
where \( p \) is the H"older conjugate of \( q \).

We are ready to state the following main results which are given in Theorem 1.1

**Theorem 1.1.** Given \( \alpha > 0 \), let \( P, f, \lambda, \mu \) be the system parameters in (1.1) satisfying (1.6)–(1.8). Given \( M, T \) and \( r > 3 \), there is a positive constant \( C \) depending on \( \alpha \), \( M, T \) and \( r \) such that if \((\rho, u, f, \rho_s)\) and \((\bar{\rho}, \bar{u}, \bar{f}, \bar{\rho}_s)\) are weak solutions of (1.1) satisfying (1.19)–(1.20) with \((\rho, u, f, \rho_s)\) satisfying (1.21)–(1.23) and \((\bar{\rho}, \bar{u}, \bar{f}, \bar{\rho}_s)\) satisfying (1.24)–(1.26), if (1.27) holds, and if all the norms occurring in the above conditions are bounded by \( M \), then
\[
\left( \int_0^T |u - \bar{u}|^2 dx dt \right)^{\frac{1}{2}} + \sup_{0 \leq T \leq T} \|\rho - \bar{\rho}\|_{H^{-1}}\]
\[
\leq C \left[ \|\rho_0 - \bar{\rho}_0\|_{L^{2r}} + \|\rho_0 u_0 - \bar{\rho}_0 \bar{u}_0\|_{L^2} \right]
\]
\[
\text{(1.28)} \quad + C \left[ \left( \int_{\mathbb{R}^3} |\rho_s - \bar{\rho}_s|^2 \right)^{\frac{1}{2}} + \left( \int_{\mathbb{R}^3} |\bar{f} - \bar{f} \circ S|^2 dx dt \right)^{\frac{1}{2}} \right],
\]
where \( S \) is defined in (2.12) later. If we further have \( \int_0^T t \|\nabla \bar{f}(\cdot, t)\|_{L^\infty} \leq M \), then \( \bar{f} \circ S \) may be replaced by \( \bar{f} \) in (1.28).
Lemma 2.2. For terms of hold:

\[ (2.1) \]

one can still obtain the same conclusion \((1.28)\) from Theorem \((1.1)\).

Remark 1.2. Similar to the case as in Hoff \cite{Hof06}, under a more general condition on \(P\), namely

\[ (1.29) \]

one can still obtain the same conclusion \((1.28)\) from Theorem \((1.1)\).

The rest of the paper is organised as follows. In section 2, we recall some known facts and useful estimates, and we further discuss the global-in-time existence and regularity of weak solution to \((1.1)-(1.2)\). In section 3 we address the uniqueness of weak solutions given in Theorem \((1.1)\) by making use of the Lagrangean framework and bounds on the weak solutions.

2. Existence and regularity of weak solution

In this section, we state some known facts and estimates which will be useful for later analysis. We also discuss the global-in-time existence and regularity of weak solution to \((1.1)-(1.2)\) which will be summarised in Theorem 2.3.

To begin with, we state the following Gagliardo-Nirenberg type inequalities and the proof can be found in Ziemer \cite{Zie89}:

Proposition 2.1. For \(p \in [2,6], q \in (1,\infty)\) and \(r \in (3,\infty)\), there exists some generic constant \(C > 0\) such that for any \(f \in H^1\) and \(g \in L^q \cap D^{1,r}\), we have

\[ (2.1) \]

\[ (2.2) \]

Next, we recall the following lemma which gives some useful estimates on \(u\) in terms of \(F\) and \(\omega\).

Lemma 2.2. For \(r_1, r_2 \in (1,\infty)\) and \(t > 0\), there exists a universal constant \(C\) which depends only on \(r_1, r_2, \mu, \lambda, a, \gamma\) and \(\rho_s\) such that, the following estimates hold:

\[ (2.3) \]

\[ (2.4) \]

Proof. In view of the Poisson equations \((1.12)\) and \((1.13)\), we can apply standard \(L^p\)-estimate on \(F\) and \(\omega\) to obtain \((2.3)-(2.4)\); see \cite{Sue13, Sue14, CS16} for more details.

Before we discuss the existence and some further properties of weak solution to \((1.1)-(1.2)\), we introduce the notion of piecewise Hölder continuous as follows (also refer to Hoff \cite{Hof02} for more details):

Definition 2.3. We say that a function \(\phi(\cdot, t)\) is piecewise \(C^{\beta(t)}\) if it has simple discontinuities across a \(C^{1+\beta(t)}\) curve \(C(t) = \{y(s, t) : s \in I \subset \mathbb{R}\}\), where \(\beta(t) > 0\) is a function in \(t\), \(I\) is an open interval and the curve \(C(t)\) is the \(u\)-transport of \(C(0)\) given by:

\[ y(s, t) = y(s, 0) + \int_0^t u(y(s, \tau), \tau) d\tau. \]
Theorem 2.4. Given regularity of weak solution \((\rho, u, \rho_s)\) to \((1.1)-(1.2)\):  
\[ \begin{array}{l}
\rho_0 \geq 0 \text{ a.e., } \\
\int_{\mathbb{R}^3} \rho_0 |u_0|^2 dx < \infty, \\
\|\rho_0 - \hat{\rho}\|_{L^2} + \|\rho_0 \frac{\partial u}{\partial t}\|_{L^2} \ll 1,
\end{array} \]  
where \(q > 6\). The solution can be shown to satisfy conditions \((1.1)-(1.20)\) and if \(\inf \rho_0 > 0\), then the solution further satisfies \((1.22)-(1.23)\) and the energy estimates: for \(u_0 \in H^s\) for some \(s \in [0, 1]\), then it holds  
\[ \sup_{0 \leq \tau \leq T} \int_{\mathbb{R}^3} |\rho u|^2 + |\rho - \rho_s|^2 + \tau^{1-s} |\nabla u|^2 + \tau^s |\dot{u}|^2 dx \]  
\[ + \int_0^T \int_{\mathbb{R}^3} (|\nabla u|^2 + \tau^{1-s} |\dot{u}|^2 + \tau^s |\nabla \dot{u}|^2) dx d\tau \leq C(T), \]  
where \(\sigma = \max\{2-s, 3-3s\}\) and \(C(T)\) is a generic positive constant which depends on \(T\). Moreover, if \(\rho_0\) is piecewise \(C^{\beta_0}\) for some \(\beta_0 > 0\) in the sense of Definition 2.3, then for each positive time \(T\) and \(t \in [0, T]\), there exists function \(\beta(t) \in (0, \beta_0]\) such that \(\rho\) is piecewise \(C^{\beta(t)}\) on \([0, T]\).

Proof. The existence result follows mainly from Suen [Sue13, Sue14, CS16], which show that the global-in-time weak solution \((\rho, u, \rho_s)\) satisfies \((1.1)-(1.20)\). The bounds \((1.22)-(1.23)\) and the energy estimates \((2.6)\) for \(s = 0\) follow by [CS16 Theorem 1.1], and the general cases for \(s \in [0, 1]\) can be proved by the interpolation techniques of Hoff [Hof02] or Suen [Sue20b] provided that \(\inf \rho_0 > 0\).

To prove that \(\rho\) is piecewise \(C^{\beta(t)}\) on \([0, T]\) for the case when \(\rho_0\) is piecewise \(C^{\beta_0}\), it involves an argument which is based on the observation of “enhanced regularity” gained by the effective viscous flux \(F\). Details of the proof can be found in [Hof02 Sue20b Sue20c] and we only give a sketch here. We introduce a decomposition of \(u\) which is given by \(u = u_F + u_P\), where \(u_F, u_P\) satisfy  
\[ \begin{array}{l}
(\mu + \lambda) \Delta (u_F)_j = (\rho_s F)_{x_j} + (\mu + \lambda)(\omega)_{x_j} \\
(\mu + \lambda) \Delta (u_P)_j = (P - P_s)_{x_j},
\end{array} \]  
Using the estimates \((2.3)\) on \(F\) and \(\omega\), and together with \((2.6)\), we readily have  
\[ \int_0^T \|\nabla u_F(\cdot, \tau)\|_{L^\infty} d\tau \leq C(T). \]  
On the other hand, in order to control \(u_P\), by applying the results from Bahouri-Chemin [BC94] on Newtonian potential, we can make use of the pointwise bounds \((1.22)\) on \(\rho\) to show that \(u_P\) is, in fact, log-Lipschitz with bounded log-Lipschitz...
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This is sufficient to guarantee that the integral curve \( x(y, t) \) as defined by
\[
\begin{cases}
  \dot{x}(t) = u(x(t), t) \\
  x(0) = y,
\end{cases}
\]
is Hölder-continuous in \( y \). Upon integrating the mass equation along integral curves \( x(t, y) \) and \( x(t, z) \), subtracting and recalling the definition (1.11) of \( F \), we arrive at
\[
\log \rho(x(T, y), T) - \log \rho(x(T, z), T) = \log \rho_0(y) - \log \rho_0(z) + \int_0^T [P(\rho(x(\tau, y), \tau) - P(\rho(x(\tau, z), \tau))]d\tau
\]
\[+ \int_0^t [F(x(\tau, y), \tau) - F(x(\tau, z), \tau)]d\tau. \tag{2.9}\]

Since \( P \) is increasing, the second term on the right side of the above can be dropped out. Moreover, with the help of the estimate (2.3) on \( F \) and the Hölder-continuity of \( x(y, t) \), the third term can be bounded by \( M \). Hence we can conclude from (2.9) that \( \rho(\cdot, t) \) is \( C^{\beta(t)} \) on \([0, T] \) for some \( \beta(t) \in (0, \beta_0] \) with bounded modulus. \( \square \)

Remark 2.5. Under the assumption that the initial density \( \rho_0 \) is piecewise Hölder continuous, by Theorem 2.4, we can see that \( \rho \) is piecewise Hölder continuous. As a consequence, it further implies that \( \nabla u \in L^1((0, T); W^{1, \infty}) \). To see how it works, we make use of the Poisson equation (2.7) again and apply properties of Newtonian potentials to conclude that the \( C^{1+\beta(t)}(R^3) \) norm of \( u_P \) remains finite in finite time, hence the following bound holds for \( \nabla u_P \) as well:
\[
\int_0^T ||\nabla u_P(\cdot, \tau)||_\infty d\tau \leq C(T). \tag{2.10}\]

Together with the bound (2.8) on \( u_F \), we conclude that condition (1.19) holds for the weak solution to (1.1)-(1.2) with piecewise Hölder continuous initial density. The results of Theorem 1.1 therefore do apply to this class of weak solutions, which includes solutions with Riemann-like initial data.

We end this section by stating some results on Lagrangean structure which will be used in this paper. As suggested by Hoff in [Hof06], weak solutions with minimal regularity are best compared in a Lagrangian framework. In other words, we try to compare the instantaneous states of corresponding fluid particles in two different solutions. To achieve our goal, we employ some delicate estimates on particle trajectories. More precisely, for \( T > 0 \), the bound (2.8) and (2.10) guarantee the existence and uniqueness of the mapping \( X(y, t, t') \in C(R^3 \times [0, T]^2) \) satisfying
\[
\begin{cases}
  \frac{\partial X}{\partial t}(y, t, t') = u(X(y, t, t'), t) \\
  X(y, t, t') = y
\end{cases}
\]
where \( (\rho, u, B) \) is a weak solution to (1.1)-(1.2). Moreover, the mapping \( X(\cdot, t, t') \) is Lipschitz on \( R^3 \) for \((t, t') \in [0, T]^2 \). The results are given in the following proposition and the proof can be found in Hoff [Hof06].

Proposition 2.1. Let \( T > 0 \) and \( u \) satisfy (1.19). Then there is a unique function \( X \in C(R^3 \times [0, T]^2) \) satisfying (2.11). In particular, \( X(\cdot, t, t') \) is Lipschitz on \( R^3 \) for
\((t, t') \in [0, T]^2\), and there is a constant \(C\) such that 
\[
\left\| \frac{\partial X}{\partial y}(\cdot, t, t') \right\|_{L^\infty} \leq C, \quad (t, t') \in [0, T]^2.
\]

With respect to velocities \(u\) and \(\bar{u}\), for \(y \in \mathbb{R}^3\), we let \(X, \bar{X}\) be two integral curves given by 
\[
\begin{cases}
\frac{\partial X}{\partial t}(y, t, t') = u(X(y, t, t'), t) \\
X(y, t', t) = y
\end{cases}
\]
and 
\[
\begin{cases}
\frac{\partial \bar{X}}{\partial t}(y, t, t') = \bar{u}(\bar{X}(y, t, t'), t) \\
\bar{X}(y, t', t) = y.
\end{cases}
\]
We then define \(S(x, t)\), \(S^{-1}(x, t)\) by 
\[
S(x, t) = \bar{X}(X(x, 0, t), t, 0),
\]
and 
\[
S^{-1}(x, t) = X(\bar{X}(x, 0, t), t, 0).
\]

The following proposition provides some properties of \(S\) and \(S^{-1}\), which will be crucial for later analysis.

**Proposition 2.2.** Let \(S\) and \(S^{-1}\) be as given in (2.12) - (2.13). Then we have

- \(S^{\pm 1}\) is continuous on \(R^3 \times [0, T]\) and Lipschitz continuous on \(R^3 \times [\tau, T]\) for all \(\tau > 0\), and there is a constant \(C\) such that 
  \[
  \left\| \nabla S^{\pm 1}(\cdot, t) \right\|_{L^\infty} \leq C, \quad t \in [0, T];
  \]
- \((S_t + \nabla Su)(x, t) = \bar{u}(S(x, t), t)\) a.e. in \(R^3 \times (0, T)\); 
- \(\rho(S(x, t), t) \rho_0(X(x, 0, t)) \det \nabla S(x, t) = \rho(x, t) \rho_0(X(x, 0, t))\) a.e. in \(R^3 \times (0, T)\); 
- If \(u, u \in L^2(R^3 \times (0, T))\), then for all \(t \in (0, T)\), 
  \[
  \int_{R^3} |x - S(x, t)|^2 dx \leq C t \int_0^t \int_{R^3} |u(x, \tau) - \bar{u}(S(x, \tau), \tau)|^2 dx d\tau,
  \]
  \[
  \int_{R^3} |x - S^{-1}(x, t)|^2 dx \leq C t \int_0^t \int_{R^3} |u(S^{-1}(x, \tau), \tau) - \bar{u}(x, \tau)|^2 dx d\tau.
  \]

**Proof.** The proof can be found on page 1752 in [Hog06]. \(\square\)

### 3. Proof of Theorem 1.1

We are now ready to give the proof of Theorem 1.1. Throughout this section, \(C\) always denotes a generic positive constant which depends on the parameters \(P, f, \lambda, \mu, T, a, r\) as described in Theorem 1.1. First, we let \(\psi : R^3 \times [0, T] \to R^3\) be a
test function satisfying
\[
- \int_{\mathbb{R}^3} \rho_0(x)u_0(x)\psi(x,0)dx = \int^T_0 \int_{\mathbb{R}^3} \begin{bmatrix} \rho u \cdot (\psi_t + \nabla \psi u) + (P(\rho) - P_s) \text{div}(\psi) - \mu \nabla u^j \cdot \nabla \psi^j \\
- \lambda \text{div}(u)\psi - \lambda(\text{div}(u)\psi + (\rho - \rho_s)f \cdot \psi) \end{bmatrix} dx d\tau,
\]
(3.1)

where we used the expressions \((1.7)\) and \((1.9)\) for \(\nabla P_s\). Define \(\tilde{\psi} = \psi \circ S^{-1}\). Then we have
\[
\int_{\mathbb{R}^3} \tilde{\rho}_0(x)\tilde{u}_0(x)\tilde{\psi}(x,0)dx = \int^T_0 \int_{\mathbb{R}^3} \begin{bmatrix} \tilde{\rho} \tilde{u} \cdot (\psi_t + \nabla \psi \tilde{u}) + (\tilde{P}(\tilde{\rho}) - \tilde{P}_s) \text{div}(\tilde{\psi}) - \mu \nabla \tilde{u}^j \cdot \nabla \tilde{\psi}^j \\
- \lambda \text{div}(\tilde{u})\tilde{\psi} - \lambda(\text{div}(\tilde{u})\tilde{\psi} + (\tilde{\rho} - \tilde{\rho}_s)\tilde{f} \cdot \tilde{\psi}) \end{bmatrix} dx d\tau,
\]
(3.2)

with \(\tilde{P}_s = P(\tilde{\rho}_s)\). Notice that using the definition of \(\tilde{F}\) and \(\omega\) from \((1.11)\) (replacing \(F\) by \(F\), \(u\) by \(\tilde{u}\), etc.),
\[
\int^T_0 \int_{\mathbb{R}^3} \begin{bmatrix} (\tilde{P}_s - \tilde{P}) \text{div}(\tilde{\psi}) + \mu \nabla \tilde{u}^j \cdot \nabla \tilde{\psi}^j + \lambda \text{div}(\tilde{u})\text{div}(\tilde{\psi}) \end{bmatrix} \\
= \int^T_0 \int_{\mathbb{R}^3} \begin{bmatrix} (\mu + \lambda) \text{div}(\tilde{u}) - \tilde{P} + \tilde{P}_s \end{bmatrix} \text{div}(\tilde{\psi}) + \int^T_0 \int_{\mathbb{R}^3} \mu(\tilde{u}^j_{x_k} - \tilde{u}^j_{x_j})\tilde{\psi}^j_{x_k} \\
= - \int^T_0 \int_{\mathbb{R}^3} \begin{bmatrix} \nabla(\tilde{\rho}_s \tilde{F}) \cdot \psi + \mu \tilde{\omega}^{j,k}_{x_k} \psi^j \end{bmatrix} \\
+ \int^T_0 \int_{\mathbb{R}^3} \begin{bmatrix} \nabla(\tilde{\rho}_s \tilde{F}) \cdot (\psi - \psi \circ S^{-1}) + \mu \tilde{\omega}^{j,k}_{x_k} (\psi^j - \psi^j \circ S^{-1}) \end{bmatrix} \\
= \int^T_0 \int_{\mathbb{R}^3} \begin{bmatrix} (P_s - P) \text{div}(\psi) + \mu \nabla \tilde{u}^j \cdot \nabla \psi^j + \lambda \text{div}(\tilde{u})\text{div}(\psi) \end{bmatrix} \\
+ \int^T_0 \int_{\mathbb{R}^3} \begin{bmatrix} \nabla(\tilde{\rho}_s \tilde{F}) \cdot (\psi - \psi \circ S^{-1}) + \mu \tilde{\omega}^{j,k}_{x_k} (\psi^j - \psi^j \circ S^{-1}) \end{bmatrix}.
\]

Moreover, we have
\[
\int_{\mathbb{R}^3} \tilde{\rho} \tilde{u} \cdot (\psi_t + \nabla \psi \tilde{u})dx = \int_{\mathbb{R}^3} \tilde{\rho}(S)\tilde{u}(S) \cdot (\psi_t(S) + \nabla \tilde{\psi} \tilde{u}(S))|\det(S)|dx \\
= \int_{\mathbb{R}^3} A_0 \tilde{\rho}(S)(\psi_t + \nabla \psi u)dx,
\]
where we used the fact that \( A_0 \rho = (\bar{\rho} \circ S) | \det(S) | \) from Proposition \([2,2]\). Hence by taking the difference between (3.1) and (3.2), for all \( \psi \), we have

\[
\int_{\mathbb{R}^3} (\bar{\rho} u_0 - \rho_0 u_0) \cdot \psi(x, 0) dx
\]

(3.3) \quad = \int_0^T \int_{\mathbb{R}^3} [\rho (u - \bar{u} \circ S)(\psi_t + \nabla \psi u) + (1 - A_0) \rho (\bar{u} \circ S)(\psi_t + \nabla \psi u)]

\[+ \int_0^T \int_{\mathbb{R}^3} [(P_s - P) \text{div}(\psi) + \mu \nabla \bar{u}^j \cdot \nabla \psi^j + \lambda \text{div}(\bar{u}) \text{div}(\psi)]
\]

\[+ \int_0^T \int_{\mathbb{R}^3} [(1 - A_0) \rho (\bar{u} \circ S)(\psi_t + \nabla \psi u)]
\]

\[+ \int_0^T \int_{\mathbb{R}^3} \nabla (\bar{\rho}_s F) \cdot (\psi - \psi \circ S^{-1}) + \mu \bar{\omega}_{jk} ^i (\psi^j - \psi \circ S^{-1})]
\]

\[+ \int_0^T \int_{\mathbb{R}^3} (\bar{u} \circ S - \bar{u})(\mu \Delta \psi + \lambda \nabla \text{div}(\psi))
\]

\[+ \int_0^T \int_{\mathbb{R}^3} (P - \bar{P}) \text{div}(\psi) + \int_0^T \int_{\mathbb{R}^3} (\bar{P}_s - P_s) \text{div}(\psi)
\]

\[+ \int_0^T \int_{\mathbb{R}^3} (\bar{\rho}_s (\bar{f} \circ S) \cdot \psi - \rho_s f \cdot \psi).
\]

Next we extend \( \rho, u \) to be constant in \( t \) outside \([0, T]\) and let \( \rho^\varepsilon \) and \( u^\varepsilon \) be the corresponding smooth approximation obtained by mollifying in both \( x \) and \( t \). Then we define \( \psi^\varepsilon : \mathbb{R}^3 \times [0, T] \rightarrow \mathbb{R}^3 \) to be the solutions satisfying

\[
\begin{cases}
\rho^\varepsilon (\psi^\varepsilon_t + u^\varepsilon \cdot \nabla \psi^\varepsilon) + \mu \Delta \psi^\varepsilon + \lambda \nabla \text{div}(\psi^\varepsilon) = G \\
\psi^\varepsilon(\cdot, T) = 0.
\end{cases}
\]

By simple estimates (or refer to \([Hof06, \text{Lemma 3.1}]\), \( \psi^\varepsilon \) satisfies the following bounds in terms of \( G \)

\[
\sup_{0 \leq \tau \leq T} \int_{\mathbb{R}^3} [\psi^\varepsilon(x, \tau)]^2 + [\nabla \psi^\varepsilon(x, \tau)]^2 \leq \int_0^T \int_{\mathbb{R}^3} [\psi^\varepsilon_t + \nabla \psi^\varepsilon u^\varepsilon]^2 + [D_x^2 \psi^\varepsilon]^2
\]

(3.4) \quad \leq C \int_0^T \int_{\mathbb{R}^3} |G|^2,

(3.5) \quad \sup_{0 \leq \tau \leq T} \| \psi^\varepsilon(\cdot, \tau) \|_{L^\infty} + \int_0^T \int_{\mathbb{R}^3} |\psi^\varepsilon| \leq C(G),

for some positive constant \( C(G) \) which depends on \( G \). We now take \( \psi = \psi^\varepsilon \) in (3.3) to obtain

\[
\int_{\mathbb{R}^3} (\bar{\rho} u_0 - \rho_0 u_0) \cdot \psi^\varepsilon(x, 0) dx = \int_0^T \int_{\mathbb{R}^3} z \cdot G + \sum_{i=1}^{7} R_i,
\]
where $z = u - \bar{u} \circ S$ and $\mathcal{R}_1, \ldots, \mathcal{R}_7$ are given by:

\[
\mathcal{R}_1 = \int_0^T \int_{\mathbb{R}^3} \left[ \nabla (\bar{\rho}_s \tilde{F}) \cdot (\psi^\varepsilon - \psi^\varepsilon \circ S^{-1}) + \mu \tilde{\omega}^{j,k} (\psi^\varepsilon - \psi^\varepsilon \circ S^{-1}) \right], \\
\mathcal{R}_2 = \int_0^T \int_{\mathbb{R}^3} \left[ \rho (f - \bar{f} \circ D) \cdot \psi^\varepsilon + (1 - A_0) \rho (\tilde{f} \circ S) \cdot \psi^\varepsilon \right], \\
\mathcal{R}_3 = \int_0^T \int_{\mathbb{R}^3} (\bar{u} \circ S - \bar{u}) \cdot (\mu \Delta \psi^\varepsilon + \lambda \text{div}(\psi^\varepsilon)), \\
\mathcal{R}_4 = \int_0^T \int_{\mathbb{R}^3} \left[ (\rho - \rho^x) \psi^\varepsilon_t + \nabla \psi^\varepsilon (\rho u - \rho^x u^x) \right], \\
\mathcal{R}_5 = \int_0^T \int_{\mathbb{R}^3} (1 - A_0) \rho (\bar{u} \circ S) \cdot (\psi^\varepsilon_t + \nabla \psi^\varepsilon u), \\
\mathcal{R}_6 = \int_0^T \int_{\mathbb{R}^3} (P - \bar{P}) \text{div}(\psi^\varepsilon) + \int_0^T \int_{\mathbb{R}^3} (\bar{P}_s - P_s) \text{div}(\psi^\varepsilon) \\
\mathcal{R}_7 = \int_0^T \int_{\mathbb{R}^3} (\bar{\rho}_s (\tilde{f} \circ S) \cdot \psi^\varepsilon - \rho_s f \cdot \psi^\varepsilon). \\
\]

The left side of (3.6) can be readily bounded by

\[
\left| \int_{\mathbb{R}^3} (\bar{\rho}_0 - \rho_0 u_0) \cdot \psi^\varepsilon (x, 0) dx \right| \leq \|\rho_0 u_0 - \bar{\rho}_0 \bar{u}_0\|_{L^2} \left( \int_0^T \int_{\mathbb{R}^3} |G|^2 \right)^{\frac{1}{2}}.
\]

Following the same method given in [Hof06] and with the help of the bounds (3.4)-(3.5), the terms $\mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4$ and $\mathcal{R}_5$ can be estimated as follows:

\[
|\mathcal{R}_2| \leq C \left[ \left( \int_0^T \int_{\mathbb{R}^3} |f - \bar{f} \circ S|^2 \right)^{\frac{1}{2}} + \|\rho_0 - \bar{\rho}_0\|_{L^2} \right] \left( \int_0^T \int_{\mathbb{R}^3} |G|^2 \right)^{\frac{1}{2}},
\]

\[
|\mathcal{R}_3| \leq C \left( \int_0^T \int_{\mathbb{R}^3} |z|^2 \right)^{\frac{1}{2}} \left( \int_0^T \int_{\mathbb{R}^3} |G|^2 \right)^{\frac{1}{2}},
\]

\[
\lim_{\varepsilon \to 0} \mathcal{R}_4 = 0,
\]

and

\[
|\mathcal{R}_5| \leq C \left[ \|\rho_0 - \bar{\rho}_0\|_{L^2} + \left( \int_0^T \int_{\mathbb{R}^3} |z|^2 \right)^{\frac{1}{2}} \right] \left( \int_0^T \int_{\mathbb{R}^3} |G|^2 \right)^{\frac{1}{2}}.
\]
It remains to estimate the terms $\mathcal{R}_1$, $\mathcal{R}_6$ and $\mathcal{R}_7$. For $\mathcal{R}_1$, using Hölder inequality, we have

$$|\mathcal{R}_1| \leq C \left( \int_0^T \int_{\mathbb{R}^3} |z|^2 \right)^{\frac{1}{3}} \int_0^T t^{\frac{1}{2}} \| \nabla (\tilde{\rho}, \tilde{F}) (\cdot, t) \|_{L^4} \| \nabla \psi (\cdot, t) \|_{L^4} dt$$

$$+ C \left( \int_0^T \int_{\mathbb{R}^3} |z|^2 \right)^{\frac{1}{3}} \int_0^T t^{\frac{1}{2}} \| \nabla \tilde{\omega} (\cdot, t) \|_{L^4} \| \nabla \psi (\cdot, t) \|_{L^4} dt$$

$$\leq C \left( \int_0^T \int_{\mathbb{R}^3} |z|^2 \right)^{\frac{1}{3}} \left( \int_0^T \int_{\mathbb{R}^3} |G|^2 \right)^{\frac{1}{3}} \left( \int_0^T \int_{\mathbb{R}^3} |D^2 \psi|^2 \right)^{\frac{1}{3}}$$

$$\times \left( \int_0^T t^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} |\tilde{u}|^4 \right)^{\frac{1}{3}} + C \right).$$

Using (2.3) and the boundedness assumption (1.24) on $\tilde{\rho}$, the term involving $\tilde{F}$ and $\tilde{\omega}$ can be bounded by

$$\int_{\mathbb{R}^3} t^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} |\tilde{u}|^4 \right)^{\frac{1}{3}} + C$$

and with the help of (2.3) and the energy estimates (2.6), we further have

$$\int_0^T t^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} |\tilde{u}|^4 \right)^{\frac{1}{3}}$$

$$\leq C \left( \int_0^T \int_{\mathbb{R}^3} |\tilde{u}|^2 \right)^{\frac{1}{3}} \left( \int_{\mathbb{R}^3} |\nabla \tilde{u}|^2 \right)^{\frac{1}{3}}$$

$$\leq C \left( \int_0^T t^{4s-3} \right)^{\frac{1}{3}} \left( \int_0^T t^{1-s} \int_{\mathbb{R}^3} |\tilde{u}|^2 \right)^{\frac{1}{3}} \left( \int_0^T \int_{\mathbb{R}^3} |\tilde{u}|^2 \right)^{\frac{1}{3}} \leq C T^{\frac{2s-2}{3}}.$$

Hence we conclude

$$|\mathcal{R}_1| \leq C T^{\frac{2s-2}{3}} \left( \int_0^T \int_{\mathbb{R}^3} |z|^2 \right)^{\frac{1}{3}} \left( \int_0^T \int_{\mathbb{R}^3} |G|^2 \right)^{\frac{1}{3}}.$$

In particular, for $[t_1, t_2] \subseteq [0, T]$, if we define

$$\mathcal{R}_1(t_1, t_2) = \int_{t_1}^{t_2} \int_{\mathbb{R}^3} \left[ \nabla (\tilde{\rho}, \tilde{F}) \cdot (\psi - \psi \circ S^{-1}) + \mu \tilde{\omega}^{jk} \left( \psi - \psi \circ S^{-1} \right) \right],$$

then we also have

$$|\mathcal{R}_1(t_1, t_2)| \leq C |t_2 - t_1|^{\frac{2s-1}{3}} \left( \int_0^{t_2} \int_{\mathbb{R}^3} |z|^2 \right)^{\frac{1}{3}} \left( \int_0^{t_2} \int_{\mathbb{R}^3} |G|^2 \right)^{\frac{1}{3}}.$$

For $\mathcal{R}_6$, using the assumption (1.6) on $P$, we have

$$\| (P - \tilde{P})(\cdot, t) \|_{H^{-1}} \leq C \| (\rho - \tilde{\rho})(\cdot, t) \|_{H^{-1}}.$$

1We point out that the bound (3.14) also holds under the more general condition (1.20) on the pressure; see [Hof02] for a more detailed proof.
Hence together with the bound (3.5) on $\psi$, it implies that
\begin{equation}
\left\| \int_0^T \int_{\mathbb{R}^3} (P - \bar{P}) \text{div}(\psi^\epsilon) \right\| \leq \int_0^T \left( \int_{\mathbb{R}^3} \left| (P - \bar{P})(\cdot, t) \right| \right) \| \text{div}(\psi^\epsilon) \|_{H^1} \, dt
\end{equation}
Following the argument given in [Hof06], the term $\sup_{0 \leq t \leq T} \| (\rho - \bar{\rho})(\cdot, t) \|_{H^{-1}}$ can be bounded by
\begin{equation}
\sup_{0 \leq t \leq T} \| (\rho - \bar{\rho})(\cdot, t) \|_{H^{-1}} \leq C \left[ \| \rho_0 - \bar{\rho}_0 \|_{L^2} + T^{\frac{1}{2}} \left( \int_0^T \int_{\mathbb{R}^3} |z|^2 \right)^{\frac{1}{2}} \right],
\end{equation}
and we conclude from (3.15) that
\begin{equation}
\left\| \int_0^T \int_{\mathbb{R}^3} (P - \bar{P}) \text{div}(\psi^\epsilon) \right\| \leq C \left[ \| \rho_0 - \bar{\rho}_0 \|_{L^2} + \left( \int_0^T \int_{\mathbb{R}^3} |z|^2 \right)^{\frac{1}{2}} \right] \left( \int_0^T \int_{\mathbb{R}^3} |G|^2 \right)^{\frac{1}{2}}.
\end{equation}
The term $\int_0^T \int_{\mathbb{R}^3} (\bar{P}_s - P_s) \text{div}(\psi^\epsilon)$ can be readily bounded by
\begin{equation}
\left\| \int_0^T \int_{\mathbb{R}^3} (\bar{P}_s - P_s) \text{div}(\psi^\epsilon) \right\| \leq C \left( \int_{\mathbb{R}^3} |\rho_s - \bar{\rho}_s|^2 \right)^{\frac{1}{2}},
\end{equation}
and therefore the bounds (3.16) and (3.17) give
\begin{equation}
|\mathcal{R}_6| \leq C \left[ \| \rho_0 - \bar{\rho}_0 \|_{L^2} + \left( \int_0^T \int_{\mathbb{R}^3} |z|^2 \right)^{\frac{1}{2}} \right] \left( \int_0^T \int_{\mathbb{R}^3} |G|^2 \right)^{\frac{1}{2}} + \left( \int_{\mathbb{R}^3} |\rho_s - \bar{\rho}_s|^2 \right)^{\frac{1}{2}}.
\end{equation}
Finally, for the term $\mathcal{R}_7$, we can rewrite it as follows.
\begin{equation}
\mathcal{R}_7 = \int_0^T \int_{\mathbb{R}^3} (\bar{P}_s - P_s) f \cdot \psi^\epsilon + \int_0^T \int_{\mathbb{R}^3} \bar{P}_s (\bar{f} \circ S - f) \cdot \psi^\epsilon
\end{equation}
The term $\int_0^T \int_{\mathbb{R}^3} (\bar{P}_s - P_s) f \cdot \psi^\epsilon$ can be bounded by
\begin{equation}
\left| \int_0^T \int_{\mathbb{R}^3} (\bar{P}_s - P_s) f \cdot \psi^\epsilon \right| \leq C \| f \|_{L^\infty} \left( \int_{\mathbb{R}^3} |\rho_s - \bar{\rho}_s|^2 \right)^{\frac{1}{2}} \left( \int_0^T \int_{\mathbb{R}^3} |\psi^\epsilon|^2 \right)^{\frac{1}{2}} \leq C \| f \|_{L^\infty} \left( \int_{\mathbb{R}^3} |\rho_s - \bar{\rho}_s|^2 \right)^{\frac{1}{2}} \left( \int_0^T \int_{\mathbb{R}^3} |G|^2 \right)^{\frac{1}{2}},
\end{equation}
and similarly, $\int_0^T \int_{\mathbb{R}^3} \bar{P}_s (\bar{f} \circ S - f) \cdot \psi^\epsilon$ can be bounded by
\begin{equation}
\int_0^T \int_{\mathbb{R}^3} \bar{P}_s (\bar{f} \circ S - f) \cdot \psi^\epsilon \leq C \| \bar{P}_s \|_{L^\infty} \left( \int_{\mathbb{R}^3} |\bar{f} \circ S - f|^2 \right)^{\frac{1}{2}} \left( \int_0^T \int_{\mathbb{R}^3} |G|^2 \right)^{\frac{1}{2}}.
\end{equation}
Recalling the assumptions (1.21) and (1.24), we therefore obtain

\[ |R_7| \leq C \left( \int_{\mathbb{R}^3} |\rho_s - \bar{\rho}_s|^2 \right)^{\frac{1}{2}} \left( \int_0^T \int_{\mathbb{R}^3} |G|^2 \right)^{\frac{1}{2}} \]

\[ + C \left( \int_{\mathbb{R}^3} |\tilde{f} \circ S - f|^2 \right)^{\frac{1}{2}} \left( \int_0^T \int_{\mathbb{R}^3} |G|^2 \right)^{\frac{1}{2}}. \]

(3.19)

Combining the estimates (3.7), (3.8), (3.9), (3.10), (3.11), (3.13), (3.18) and (3.19), we arrive at

\[ \left| \int_0^T \int_{\mathbb{R}^3} z \cdot G \right| \leq C \left[ M_0 \left( \int_0^T \int_{\mathbb{R}^3} |G|^2 \right)^{\frac{1}{2}} + |R_1(0, T)| \right], \]

where \( M_0 \) is given by

\[ M_0 = \|\rho_0 - \bar{\rho}_0\|_{L^2} + \|\rho_0 u_0 - \bar{\rho}_0 \bar{u}_0\|_{L^2} + T^\delta \left( \int_0^T \int_{\mathbb{R}^3} |z|^2 \right)^{\frac{1}{2}} \]

\[ + \left( \int_{\mathbb{R}^3} |\rho_s - \bar{\rho}_s|^2 \right)^{\frac{1}{2}} + \left( \int_{\mathbb{R}^3} |\tilde{f} - f \circ S|^2 \right)^{\frac{1}{2}} \]

for some \( \delta > 0 \), and \( C > 0 \) is now fixed. Following the analysis given on page 1758-1759 in Hoff [Hof06], there exists a small time \( \tilde{\tau} > 0 \) such that

\[ \left( \int_{\tilde{\tau}}^T \int_{\mathbb{R}^3} |z|^2 \right)^{\frac{1}{2}} \leq 2CM_0, \]

and consequently

\[ |R_1(0, \tilde{\tau})| \leq M_0 \left( \int_{\tilde{\tau}}^T \int_{\mathbb{R}^3} |G|^2 \right)^{\frac{1}{2}}. \]

By applying (3.20) with \( T \) replaced by \( 2\tilde{\tau} \), we get

\[ \left( \int_0^{2\tilde{\tau}} \int_{\mathbb{R}^3} |z|^2 \right)^{\frac{1}{2}} \leq 4CM_0. \]

Since \( \tilde{\tau} > 0 \) is fixed, we can exhaust the interval \([0, T]\) in finitely many steps to obtain that

\[ \left( \int_0^T \int_{\mathbb{R}^3} |z|^2 \right)^{\frac{1}{2}} \leq CM_0, \]

for some new constant \( C > 0 \). Hence the term \( T^\delta \left( \int_0^T \int_{\mathbb{R}^3} |z|^2 \right)^{\frac{1}{2}} \) can be eliminated from the definition of \( M_0 \) by a Gronwall-type argument. Therefore we conclude that

\[ \left( \int_0^T \int_{\mathbb{R}^3} |z|^2 \right)^{\frac{1}{2}} \leq CM_0, \]

(3.21)

Since the bound (3.21) holds for any \( G \in H^\infty(\mathbb{R}^3 \times [0, T]) \), it shows that \( \|z\|_{L^2([0,T] \times \mathbb{R}^3)} \) can be bounded by \( M_0 \). Finally, using the bound (1.19) on the time integral on \( \|\nabla \bar{u}\|_{L^\infty} \),

\[ \int_0^T \int_{\mathbb{R}^3} |\bar{u} - \bar{u} \circ S|^2 \leq \int_0^T \|\nabla \bar{u}(\cdot, t)\|_{L^\infty} \int_{\mathbb{R}^3} |x - S(x, t)|^2 \]

\[ \leq C \int_0^T \int_{\mathbb{R}^3} |z|^2, \]
and hence $\text{(1.28)}$ follows. This finishes the proof of Theorem 1.1.

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