Ascribing quantum system to Schwarzschild spacetime with naked singularity

Andrzej Góźdź,†,∗ Aleksandra Pędrań,‡ and Włodzimierz Piechocki‡,

1Institute of Physics, Maria Curie-Skłodowska University, pl. Marii Curie-Skłodowskiej 1, 20-031 Lublin, Poland
2Department of Fundamental Research, National Centre for Nuclear Research, Pasteura 7, 02-093 Warszawa, Poland
(Dated: April 18, 2022)

Abstract

We quantize the Schwarzschild spacetime with naked singularity using the affine coherent states quantization method. The novelty of our approach is quantization of both temporal and spatial coordinates. Quantization smears the gravitational singularity indicated by the Kretschmann invariant avoiding its localization in the configuration space. This way we resolve the singularity problem of considered spacetime at quantum level.

∗andrzej.gozdz@umcs.lublin.pl
†aleksandra.pedrak@ncbj.gov.pl
‡wlodzimierz.piechocki@ncbj.gov.pl
I. INTRODUCTION

One of the motivations of this paper is constructing the tools to be used in the quantization of the Lemaître-Tolman-Bondi model of spacetime. Another one is testing the idea of quantization of both temporal and spatial variables of simple gravitational system to be used later in the case of more sophisticated gravitational models.

The system we consider to be quantized is the celebrated Schwarzschild spacetime \[^1, 2\]. We ascribe to this gravitational system a quantum system by making use of the affine coherent states (ACS) approach that we have recently used for the quantization of the Belinski-Khalatnikov-Lifshitz scenario with generic cosmological singularity \[^3, 4\].

To this end, we quantize not only spatial but also temporal coordinates. Instead of phase space used in Hamiltonian formulations, we introduce the notion of an extended configuration space including the time variable. This space is used to quantize both elementary and composite observables.

As far as we are aware, our paper is the first one which proposes the quantization of the temporal and spatial variables in general relativity. Quite general rationale for such dealing is the following: the distinction between space and time violates relativity; in particular, the general covariance of arbitrary transformations of temporal and spatial coordinates.

By resolving the gravitational singularity problem, we mean showing that quantization smears the singularity indicated by the Kretschmann scalar avoiding its localization in the configuration space.

Recently, we have found that the ACS quantization depends on the choice of the parametrization of the affine group \[^5\]. In this paper we present another “parameter” of the ACS method, unknown before, that is connected with the freedom in the choice of the center of the affine group.

There are at least three goals of this paper: (i) presenting a powerful quantization method especially suitable for quantization of gravitational systems, (ii) applying successfully this method to the resolution of gravitational singularity of an isolated object, and (iii) showing that treating temporal and spatial coordinates on the same footing, supporting the covariance of general relativity, enables the construction of consistent quantum theory.

The paper is organized as follows: In Sec. II we recall the known properties of the Schwarzschild spacetime. Sec. III is devoted to the quantum theory. We
recall the formalism of the affine coherent states quantization method. Then, we quantize the temporal and spatial coordinates which are elementary observables. Quantization of the main observable, the Kretschmann scalar, is carried out in Sec. IV. It includes examination of the expectation value of Kretschmann’s operator. We conclude in Sec. V. Appendixes include some practical rules concerning calculations of special expressions, eigensolutions for elementary observables, expectation value of the Kretschmann operator within some basis of the carrier space, and determination of some parameters used in the paper.

In the following we choose $G = c = 1$ except where otherwise noted.

II. CLASSICAL MODEL

One of the simplest vacuum solutions to Einstein’s equations, representing the spherically symmetric black hole is the Schwarzschild spacetime. The Schwarzschild metric in the so-called Schwarzschild coordinates $(t, r, \theta, \phi) \in \mathbb{R} \times (0, \infty) \times S^2$ reads [6, 7]:

$$ds^2 = -(1 - \frac{r_s}{r}) dt^2 + (1 - \frac{r_s}{r})^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$  \hspace{1cm} (1)

where $t$ is the time coordinate measured by a stationary clock located infinitely far from black hole, $r$ is the radial coordinate measured as the circumference (divided by $2\pi$) of a sphere centered around the black hole, $\theta$ and $\phi$ are angle coordinates of the sphere $S^2$, $r_s = 2M$ denotes the Schwarzschild radius defining the event horizon, and $M$ is the mass parameter of the black hole. It is commonly known that $r = r_s$ defines not gravitational, but a coordinate singularity.

The event horizon divides the Schwarzschild spacetime into the interior and exterior regions of that black hole. The exterior metric, defined by (1), is static. In the interior region, the exterior spatial radial and temporal coordinates exchange their character so that the metric coefficients become time dependent [7]. There exists the isometry of the interior of the Schwarzschild black hole with the vacuum Kantowski-Sachs spacetime (see, e.g. [8]) which can be used for the quantization of the former. We make some remarks on that quantization in the concluding section.

In this paper we ascribe a quantum system to the Schwarzschild spacetime devoid of the event horizon. Such gravitational model is defined by the metric (1) with $M < 0$, which is static for any $r > 0$ (see, e.g. [7]). This way we avoid the problem of bearing of the horizon on the quantization which simplifies the latter.

To identify the curvature singularity, we cannot use the Ricci scalar and tensor as these are vanishing for the vacuum solution. However, another curvature invariant,
the Kretschmann scalar is non-zero and reads [6, 7]:

$$\mathcal{K} := R^{\alpha \beta \gamma \delta} R_{\alpha \beta \gamma \delta} = \frac{48 M^2}{r^6}, \quad (2)$$

so that it exhibits the gravitational singularity as \( r \to 0 \).

The Kretschmann invariant is the main observable to be examined at quantum level.

### III. QUANTUM DESCRIPTION

The classical description of the model presented in the previous section includes two elementary observables: time and radial coordinates. The former is timelike and the latter is spacelike. In the standard quantization procedure, the time variable may play the role of an evolution parameter as in the Schrödinger equation. In what follows, we quantize both the temporal and spatial coordinates. For both variables the time \( t \) and the radial coordinates \( r \), we construct their quantum counterparts. Both quantum observables (operators) we treat on the same footing. It means that time is no longer a parameter, but similarly as the radial coordinate, a quantum observable represented by an appropriate operator obtained by a quantization procedure. In the following, as it was mentioned earlier, we are using the affine coherent states quantization (ACS). As we will see later the ACS quantization leads to the operators \( \hat{t} \) and \( \hat{r} \) which, in general, do not commute. Due to the Heisenberg uncertainty principle they cannot be considered as a compatible pair of quantum observables. In addition, because of this property, one cannot construct the common eigenstates of both observables, which would represent spacetime position states. In such case, the most appropriate candidates for the spacetime position states are the coherent states. The coherent states furnish a set of non–orthogonal states. It means that, in general, the spacetime position states are always connected by a non–zero transition amplitudes. They cannot be considered as a set of independent alternatives as it is in the case of common eigenstates of commuting self-adjoint operators.

In this paper we want to check if introducing of time as quantum observables can help to resolve at the quantum level the main problem of general relativity, which is the existence of solutions with gravitational singularities. To begin with, we address the singularity problem of the simplest solution to Einstein’s gravity, but we plan to apply this approach to more advanced singular solutions within general relativity.
A. **Affine coherent states quantization**

The covariance of general relativity requires to treat both variables \( t \) and \( r \) on the same footing in both the classical and quantum descriptions. To fulfill this condition we begin with introducing the notion of the extended configuration space \( T \) of our system by including time as next coordinate variable required in description of this quantum system. It is defined as follows

\[
T = \{(t, r) \mid (t, r) \in \mathbb{R} \times \mathbb{R}_+\}, \quad \mathbb{R}_+ = (0, +\infty),
\]

where \( t \) and \( r \) are the time and the radial coordinates, respectively, which occur in the line element (1). The corresponding operators \( \hat{t} \) and \( \hat{r} \) are constructed in the subsection B by the ACS quantization procedure.

As usually in quantum mechanics, to define a quantum observable one needs to determine an operational procedure which connects this observable with its quantum description. In the case of \( t \) and \( r \) one needs to measure time and spacial distance. To relate the values of measured time and radial variables to our states and operators we introduce the consistency conditions (26) and (27). They represent compatibility of expectation values of the time and position operators, within the coherent states, with measured values.

The other space variables \( \theta \) and \( \phi \) of (1), used to implement the spherical symmetry of considered spacetime, do not enter the definition of \( T \) as the main observable to be quantized, the Kretschmann scalar, does not depend on these variables. In the following we sketch the basic facts about affine quantization required in further considerations. The most important formula in this subsection is the expression (23) for quantization of any arbitrary classical mechanics function defined on the configuration space of a given physical system.

Since the configuration space is a half-plane, every point \( (t, r) \in T \) can be uniquely identified with the corresponding element \( g(\chi_1(t, r), \chi_2(t, r)) \) of the affine group \( \text{Aff}(\mathbb{R}) \), where \( \chi(t, r) = (\chi_1(t, r), \chi_2(t, r)) \) is a one-to-one mapping between \( T \) and any arbitrary chosen fixed parametrization of \( \text{Aff}(\mathbb{R}) \).

As the standard parametrization of the affine group (see [5] for more details) we assume the parametrization \( (p, q) \in \mathbb{R} \times \mathbb{R}_+ \) which obey the following multiplication law

\[
g(p_1, q_1) \cdot g(p_2, q_2) := g(p_1 + q_1 p_2, q_1 q_2) \in \text{Aff}(\mathbb{R}),
\]

and the left invariant measure on this group is defined as

\[
d\mu(p, q) = dp \frac{dq}{q^2}.
\]
The corresponding left invariant integration over the affine group is given by

\[ \int_{\text{Aff}(\mathbb{R})} d\mu(p, q) := \frac{1}{2\pi} \int_{-\infty}^{\infty} dp \int_{0}^{\infty} dq/q^2. \] (6)

It enables defining the Hilbert space of functions on the affine group \( \mathcal{H}_g := L^2(\text{Aff}(\mathbb{R}), d\mu(g)) \), where \( g = g(p, q) \in \text{Aff}(\mathbb{R}) \).

Because \( (p, q) = \chi(t, r) \) is a one-to-one function, the coordinates \( (t, r) \in T \) also parameterize the affine group \( \text{Aff}(\mathbb{R}) \), i.e., we have the mapping \( (t, r) \rightarrow g(\chi_1(t, r), \chi_2(t, r)) \). The corresponding measure (not necessarily invariant) in the \( (t, r) \) parametrization reads

\[ d\mu(p, q) = \sigma(t, r)dtdr = \left| \frac{\partial\chi_1}{\partial t} \frac{\partial\chi_2}{\partial r} \right| dtdr =: d\lambda(t, r). \] (7)

It is known (see [5] for more details) that the affine group has two (inequivalent) irreducible unitary representations defined in the Hilbert space \( \mathcal{H}_x := L^2(\mathbb{R}^+, d\nu(x)) \), where \( d\nu(x) := dx/x \).

We choose the one defined as follows

\[ U(p, q)\Psi(x) := e^{ipx}\Psi(qx) \], (8)

where \( \Psi(x) \in \mathcal{H}_x \).

The coherent states in the standard parametrization of the affine group, \( \langle x|g(p, q)\rangle \in \mathcal{H}_x \), are defined as follows

\[ \langle x|g(p, q)\rangle = U(p, q)\Phi_0(x) = e^{ipx}\Phi_0(qx), \] (10)

where \( \Phi_0(x) \in \mathcal{H}_x \) is the so-called fiducial vector. It is a sort of a free “parameter” of the ACS quantization. First of all, it should be normalized so that we should have

\[ \langle \Phi_0|\Phi_0 \rangle := \int_0^{\infty} d\nu(x)\langle \Phi_0|x\rangle\langle x|\Phi_0 \rangle = \int_0^{\infty} d\nu(x)|\Phi_0(x)|^2 = 1, \] (11)

where we have used the formula [5]

\[ \int_0^{\infty} d\nu(x)|x\rangle\langle x| = \hat{1}, \] (12)
which applies to \( \mathcal{H}_x \).

The resolutions of the identity \( \hat{1} \) in the Hilbert space \( \mathcal{H}_x \), in terms of the coherent states, reads [5]

\[
\frac{1}{A_{\Phi_0}} \int_{\text{Aff}(\mathbb{R})} d\mu(p, q) \ |g(p, q)\rangle\langle g(p, q)| = \hat{1},
\]

(13)

where

\[
A_{\Phi_0} := \int_0^\infty \frac{dx}{x^2} |\Phi_0(x)|^2 < \infty,
\]

(14)

which defines another condition to be imposed on the fiducial vector \( \Phi_0(x) \).

Using (13) we can (formally) map any observable \( f : T \to \mathbb{R} \) into a symmetric operator \( \hat{f} : \mathcal{H}_x \to \mathcal{H}_x \) as follows (see, App. (A) and [3, 5] for more details)

\[
\hat{f} := \frac{1}{A_{\Phi_0}} \int_{\text{Aff}(\mathbb{R})} d\mu(p, q) |g(p, q)\rangle f(p, q) \langle g(p, q)|.
\]

(15)

However, as it was shown in the paper [5], the affine quantization is dependent on the parametrization of the affine group \( \text{Aff}(\mathbb{R}) \) and it has to be considered also as a kind of a free “parameter” in the ACS quantization.

A fundamental expression in the ACS quantization is a non-orthogonal decomposition of unity constructed from the coherent states \( |h(t, r)\rangle = |g(\chi(t, r))\rangle \):

\[
\frac{1}{A_{\Phi_0}} \int_{\text{Aff}(\mathbb{R})} d\lambda(t, r) |h(t, r)\rangle\langle h(t, r)| = \frac{1}{A_{\Phi_0}} \int_{\text{Aff}(\mathbb{R})} d\mu(p, q) \ |g(p, q)\rangle\langle g(p, q)| = \hat{1}.
\]

(16)

The affine group manifold itself is a homogenous space. All points in this manifold are equivalent to each other. This means that from the physical point of view we have an additional freedom in mapping of the configuration space onto the group manifold \( g(\chi(\cdot)) : T \to \text{Aff}(\mathbb{R}) \). More precisely, using the ACS quantization it is usually assumed that the element \( (t_0 = 0, r_0 = 1) \in T \) of the configuration space is mapped onto the unit element \( g(0, 1) \) of the affine group \( \text{Aff}(\mathbb{R}) \). Because of the homogeneity of the group manifold, this assignment is in fact arbitrary. Every choice of the mapping \( g(\chi(t, r)) \in \text{Aff}(\mathbb{R}) \) fixes in some way a relative position between configuration space and the affine group manifold by the relation \( T \ni (t_0 = 0, r_0 = 1) \to g(a, b) = g(\chi(0, 1)) \). It defines the point \( g(a, b) \in \text{Aff}(\mathbb{R}) \) which we call the “center” of the group manifold associated to this configuration space. In the standard parametrization, where the transformation \( \chi \) is the identity transformation, the center is identified with the unity \( g(0, 1) \) of the affine group.

To use this freedom one can check that the resolution of unity is invariant with respect to any arbitrary left shift operation of the affine group manifold:

\[
\frac{1}{A_{\Phi_0}} \int_{\text{Aff}(\mathbb{R})} d\mu(p, q) \ |g(a, b) \cdot g(p, q)\rangle\langle g(a, b) \cdot g(p, q)| = \hat{1},
\]

(17)

7
however, it is not invariant with respect to the right shift operation:

$$\frac{1}{A_{\Phi_0}} \int_{\text{Aff}(\mathbb{R})} d\mu(p,q) \ |g(p,q) \cdot g(a,b)\rangle \langle g(p,q) \cdot g(a,b)| = \Delta(g(a,b)^{-1})\hat{1}. \quad (18)$$

In general, the function $\Delta(g)$ is the Haar modulus of the Lie group $G$ defined as

$$\int_G d\mu(g) f(g \cdot h) =: \Delta(h^{-1}) \int_G d\mu(g) f(g), \quad (19)$$

where $d\mu(g)$ denotes the left invariant measure on $G$. Note that the right shift of the unity resolution is still proportional to resolution of unity. The right shift translates the “center” of the affine group manifold to the new point $g(a,b)$.

Summing up, in the ACS quantization, which is a deformation of the resolution of the unit operator, we have three free “parameters”: choice of a fiducial vector, choice of an affine group parametrization, and choice of a center of the group manifold.

In fact, any choice of the center can be done by an appropriate choice of the mapping $(p,q) = \chi(t,r)$. However, from technical point of view it is useful to distinguish both operations: choice of the group parametrization and a choice of the appropriate center, because the left $g' = g \cdot \tilde{g}$ and right $g'' = \tilde{g} \cdot g$ shift of the element $g \in \text{Aff}(\mathbb{R})$ on the group manifold commutes, i.e., both operations are independent. This is useful property in calculations with invariant measure.

Using this freedom, the quantization process (15) can be now generalized to a deformation of the resolution of unity rewritten in a general parametrization with the additional right shift which fixes the center of the mapping between the configuration space and $\text{Aff}(\mathbb{R})$. Again introducing the shortcut $|h(t,r)\rangle = |g(\chi(t,r))\rangle$ the required resolution of unity read

$$\frac{\Delta(h(a',b'))}{A_{\Phi_0}} \int_{\text{Aff}(\mathbb{R})} d\lambda(t,r) |h(t,r) \cdot h(a',b')\rangle \langle h(t,r) \cdot h(a',b')| = \hat{1}. \quad (20)$$

Now, the ACS quantization of any function $f(t,r)$ on the configuration space is defined as

$$\hat{f} = \frac{\Delta(h(a',b'))}{A_{\Phi_0}} \int_{\text{Aff}(\mathbb{R})} d\lambda(t,r) |h(t,r) \cdot h(a',b')\rangle f(t,r) \langle h(t,r) \cdot h(a',b')|, \quad (21)$$

where the shift of the group manifold center is given by $h(a',b') = g(\chi(a',b')) =: g(a,b) \in \text{Aff}(\mathbb{R})$.

It is useful to rewrite this formula in the form of our standard affine group parametrization

$$\hat{f} = \frac{\Delta(h(a',b'))}{A_{\Phi_0}} \int_{\text{Aff}(\mathbb{R})} d\lambda(t,r) |g(\chi(t,r)) \cdot g(a,b)\rangle f(t,r) \langle g(\chi(t,r)) \cdot g(a,b)|. \quad (22)$$
After change of variables under integral $p = \chi_1(t, r)$ and $q = \chi_2(t, r)$, and performing the right shift operation (19) in the coherent states, one gets the final expression for quantization of the function $f(t, r)$:

$$\hat{f} = \frac{1}{A_{\Phi_0}} \int_{\text{Aff}(\mathbb{R})} d\mu(p, q) |g(p, q)\rangle f\left(\chi^{-1}_1\left(p - \frac{a}{b}q, \frac{q}{b}\right)\right) \langle g(p, q)|. \quad (23)$$

**B. Quantization of elementary observables**

The formula (23) allows to quantize almost any real function on the configuration space, $T$, giving the corresponding operator. The most elementary observables are time and radial coordinates

$$\hat{t} = \frac{1}{A_{\Phi}} \int_{\text{Aff}(\mathbb{R})} d\mu(p, q) |g(p, q)\rangle \chi^{-1}_1\left(p - \frac{a}{b}q, \frac{q}{b}\right) \langle g(p, q)|, \quad (24)$$

$$\hat{r} = \frac{1}{A_{\Phi}} \int_{\text{Aff}(\mathbb{R})} d\mu(p, q) |g(p, q)\rangle \chi^{-2}_1\left(p - \frac{a}{b}q, \frac{q}{b}\right) \langle g(p, q)|. \quad (25)$$

They are required for description of the Schwarzschild spacetime.

As it was mentioned earlier in the subsection A, we have to relate the values of measured time and coordinate with our quantum description. For this purpose we have to choose the group parametrization and the group manifold center $g(a, b)$ to fulfil the following consistency conditions:

$$\langle \hat{t}; h(t, r) \rangle = t, \quad (26)$$

$$\langle \hat{r}; h(t, r) \rangle = r, \quad (27)$$

where $\langle \hat{A}; \psi \rangle := \langle \psi | \hat{A} | \psi \rangle$ denotes expectation value of the observable $\hat{A}$ in the state labelled by $\psi$. These condition relates the measured time and radial coordinate to the corresponding quantum observables and states.

It turns out that we do not need to reparameterize our group to fulfil required conditions (26) and (27). We only have to choose properly the group manifold center parameters $g(a, b)$ in the standard parametrization. In general, these parameters are dependent on the choice of the fiducial vector.

In the following we get two useful expressions for expectation values within the coherent states $|g(t, r)\rangle$, which can be easily obtained by applying invariance of the Haar measure.
For any arbitrary operator (23) quantized by means of the affine group we get

\[
\langle \hat{f}; g(t, r) \rangle = \frac{1}{A_{\Phi}} \int_{\text{Aff}(\mathbb{R})} d\mu(p, q) \langle g(t, r) | g(p, q) \rangle f \left( p - \frac{a}{b} q, \frac{q}{b} \right) \langle g(p, q) | g(t, r) \rangle \\
= \frac{1}{A_{\Phi}} \int_{\text{Aff}(\mathbb{R})} d\mu(p, q) |\langle g(0, 1) | g(p, q) \rangle|^2 f \left( t + \left( p - \frac{a}{b} q \right) r, \frac{q}{b} r \right),
\]

(28)

and for any product of two such operators we obtain

\[
\langle \hat{f}_1 \hat{f}_2; g(t, r) \rangle = \frac{1}{A_{\Phi}} \int_{\text{Aff}(\mathbb{R})} d\mu(p_1, q_1) \frac{1}{A_{\Phi}} \int_{\text{Aff}(\mathbb{R})} d\mu(p_2, q_2) \\
f_1 \left( t + (p_1 - \frac{a}{b} q_1) r, \frac{q_1}{b} r \right) f_2 \left( t + (p_2 - \frac{a}{b} q_2) r, \frac{q_2}{b} r \right) \\
\langle g(0, 1) | g(p_1, q_1) \rangle \langle g(p_1, q_1) | g(p_2, q_2) \rangle \langle g(p_2, q_2) | g(0, 1) \rangle.
\]

(29)

Using the formula (28), the expectation value for the time observable can be written in the following form

\[
\langle \hat{t}; g(t, r) \rangle = \frac{1}{A_{\Phi}} \int_{\text{Aff}(\mathbb{R})} d\mu(p, q) |\langle g(0, 1) | g(p, q) \rangle|^2 \left( t + r \left( p - \frac{a}{b} q \right) \right).
\]

(30)

After integration over \( p \) and \( q \) one gets

\[
\langle \hat{t}; g(t, r) \rangle = t + \left( \langle \hat{p} \rangle_0 - \frac{a}{b} \langle \hat{q} \rangle_0 \right) r,
\]

(31)

where we introduce the abbreviations:

\[
\tilde{f} := \frac{1}{A_{\Phi}} \int_{\text{Aff}(\mathbb{R})} d\mu(p, q) |g(p, q)\rangle f(p, q) \langle g(p, q)|,
\]

(32)

and

\[
\langle \tilde{f} \rangle_0 := \frac{1}{A_{\Phi}} \int_{\text{Aff}(\mathbb{R})} d\mu(p, q) |\langle g(0, 1) | g(p, q) \rangle|^2 f(p, q) = \langle g(0, 1) | \tilde{f} | g(0, 1) \rangle.
\]

(33)

Thus, \( \langle \tilde{f} \rangle_0 \) denotes the expectation value of the operator \( \tilde{f} \) in the fixed coherent state \( |g(0, 1)\rangle \) corresponding to the unity of the affine group. In the case of more complicated expressions like \( f_1^n f_2^m \) instead of the notation (32), where the check symbol is over the expression, we write \( (f_1^n f_2^m)^{\check{\cdot}} \).

Similarly, we obtain

\[
\langle \hat{r}; g(t, r) \rangle = \frac{\langle \hat{q} \rangle_0}{b} r.
\]

(34)
Assuming $\langle \hat{p} \rangle_0 - \frac{a}{b} \langle \hat{q} \rangle_0 = 0$ and $b = \langle \hat{q} \rangle_0$, i.e. $a = \langle \hat{p} \rangle_0$, the self consistency conditions (26) and (27) become fulfilled.

An important property of any quantum observable $\hat{A}$ is its variance. The variance determines the value of smearing of a quantum observable. This influences behaviour of a given physical system substantially. In the quantum state labelled by $\psi$ the variance is defined as follows

$$\text{var}(\hat{A}; \psi) := \langle (\hat{A} - \langle \hat{A}; \psi \rangle)^2; \psi \rangle = \langle \hat{A}^2; \psi \rangle - \langle \hat{A}; \psi \rangle^2. \quad (35)$$

Formally, the variance is the stochastic deviation from the expectation value of the observable $\hat{A}$.

Suppose the operator $\hat{A}$ is essentially self-adjoint on some dense subspace $S$ of the Hilbert space $H$. For every quantum state $\psi \in S$ of a physical system which belongs to the domain of the operator $A$ one can check that

$$\left( \text{var}(\hat{A}; \psi) = 0 \right) \iff \left( \hat{A}\psi = \lambda\psi, \quad \lambda \in \mathbb{R} \right), \quad (36)$$

i.e., the variance of the operator $\hat{A}$ is equal to 0, if and only if, the quantum system is in an eigenstate of the operator $\hat{A}$. Then the corresponding observable is not smeared.

The statement (36) is implied by properties of the scalar product, norm and the operator itself:

$$\text{var}(\hat{A}; \psi) = \langle (\hat{A} - \langle \hat{A}; \psi \rangle)\psi | (\hat{A} - \langle \hat{A}; \psi \rangle)\psi \rangle = \| (\hat{A} - \langle \hat{A}; \psi \rangle)\psi \|^2.$$

Thus,

$$\left( \text{var}(\hat{A}; \psi) = 0 \right) \Rightarrow \left( (\hat{A} - \langle \hat{A}; \psi \rangle)\psi = 0 \right) \Rightarrow \left( \hat{A}\psi = \langle \hat{A}; \psi \rangle\psi \right).$$

The latter equality means that $\langle \hat{A}; \psi \rangle$ is the eigenvalue of $\hat{A}$ corresponding to the eigenstate $\psi$.

On the other hand, if $\hat{A}\psi = \lambda\psi$, we have

$$\text{var}(\hat{A}; \psi) = \langle \hat{A}^2; \psi \rangle - \langle \hat{A}; \psi \rangle^2 = \lambda^2\langle \psi | \psi \rangle - \lambda^2\langle \psi | \psi \rangle^2 = 0,$$

as $\psi$ is a normalized vector. This completes the verification of the validity of (36).

The variances of the operators $\hat{t}$ and $\hat{r}$ in the coherent states $|g(t, r)\rangle$ can be directly calculated. They describe the smearing of both observables. The behaviour of variances and expectation values for time and radial coordinate allows to determine if they behave similarly to their classical counterparts or not.
Because of the self-consistency condition the only unknown components are \( \langle \hat{r}^2; g(t, r) \rangle \) and \( \langle \hat{r}^2; g(t, r) \rangle \). Using the formula (29)

\[
\langle \hat{t}^2; g(t, r) \rangle = t^2 + 2 \left( \frac{\langle \hat{p} \rangle_0}{\langle \hat{q} \rangle_0} \right)_0 tr + \sigma_t r^2 ,
\]

where

\[
\sigma_t := \left( \left\{ \hat{p}^2 - \frac{\langle \hat{p} \rangle_0}{\langle \hat{q} \rangle_0} (\hat{p} \hat{q} + \hat{q} \hat{p}) + \left( \frac{\langle \hat{p} \rangle_0}{\langle \hat{q} \rangle_0} \right)^2 \hat{q}^2 \right\} + \sigma_t r^2 ,
\]

In Eq. (37) the second term vanishes and the variance of the time coordinate operator reads

\[
\text{var}(\hat{t}; g(t, r)) = \sigma_t r^2 .
\]

Similarly, for the radial coordinate operator \( \hat{r} \) we get

\[
\langle \hat{r}^2; g(t, r) \rangle = \left( \frac{\langle \hat{q}^2 \rangle_0}{\langle \hat{q} \rangle_0} \right)_0 r^2 ,
\]

so that the variance of \( \hat{r} \) becomes

\[
\text{var}(\hat{r}; g(t, r)) = \sigma_r r^2 ,
\]

where

\[
\sigma_r := \left( \frac{\langle \hat{q}^2 \rangle_0 - \langle \hat{q} \rangle_0^2}{\langle \hat{q} \rangle_0^2} \right)_0 .
\]

In both cases the standard deviation from the expectation value (square root of the variance) is proportional to the radius \( r \). The coefficients in (38) and (42) depend only on the fiducial vector \( \Phi_0(x) \). One can see that while approaching to classical singularity \( r \to 0 \), the quantum radial observable behaves as the classical one because its expectation value goes to zero and its variance also goes to zero. However, the ratio of the standard deviation from the expectation value to the expectation value of \( \hat{r} \) is constant. This suggests an existence of non-zero relative fluctuations of the radial coordinate even at singularity. Such fluctuations can be a germ which leads to larger fluctuations of other quantum observables, like spacetime invariants, and finally to avoiding the singularity in the Schwarzschild spacetime.

To find the lowest bound of the product \( \sigma_t \sigma_r \) one can use the Heisenberg type uncertainty principle in the form proposed by Robertson [10]. In this case we get

\[
\text{var}(\hat{t}; g(t, r)) \text{var}(\hat{r}; g(t, r)) \geq \frac{\langle i \hat{p}, \hat{q} \rangle_0^2}{4\langle \hat{q} \rangle_0^2} r^4 ,
\]

12
which gives the required lowest bound for product of both smearing coefficients

\[ \sigma_t \sigma_r \geq \frac{\langle i[\hat{p}, \hat{q}] \rangle_0^2}{4\langle \hat{q} \rangle_0^2}. \]  \hspace{1cm} (44)

As an example we give values of the above constants for some particular fiducial vectors. Let us take

\[ \Phi_0(x) = \frac{1}{\sqrt{(2n - 1)!}} x^n e^{-\frac{x}{2}}, \]  \hspace{1cm} (45)

where \( n > 1 \) is a natural number selected to ensure the convergence properties.

One easily gets

\[ \sigma_t = \frac{2n - 1}{2n - 2}, \]  \hspace{1cm} (46)
\[ \sigma_r = \frac{1}{2n - 2}. \]  \hspace{1cm} (47)

Thus, the inequality (44) reads

\[ \sigma_t \sigma_r \geq \frac{1}{4(2n - 1)^2}. \]  \hspace{1cm} (48)

IV. QUANTIZATION OF THE KRETSCHMANN SCALAR

Observables which characterize the behaviour of the spacetime at a given space-time point are the curvature invariants. In our case the most important is the Kretschmann scalar (2). The classical Kretschmann invariant diverges as \( r \to 0 \). Does this singularity survive quantization? Is the expectation value of the Kretschmann operator \( \hat{C} \) regular across the configuration space \( T \)? What is the quantum smearing of \( \hat{C} \)? These are the issues to be addressed in this section.

Using our quantization rules (23), the quantum Kretschmann observable can be written as

\[ \hat{C} = 48M^2\langle \hat{q} \rangle_0^6 \frac{1}{A_{\Phi_0}} \int_{\text{Aff}(\mathbb{R})} d\mu(p, q)|g(p, q)\rangle \frac{1}{Q^6} \langle g(p, q)|. \]  \hspace{1cm} (49)

A. Eigenproblem for \( \hat{C} \) operator

As the first step, let us consider the eigenproblem of the operator \( \hat{C} \) which allows to establish eigenfunctions (or rather generalized eigenfunctions) and spectrum of the
Kretschmann operator

\[ \int_{\mathbb{R}^+} d\nu(y) \, K_K(x, y) \, \psi_k^{(K)}(y) = k \psi_k^{(K)}(x), \]  
\[ \tag{50} \]

written in terms of the integral kernel

\[ K_K(x, y) = \langle x | \hat{K} | y \rangle = \frac{48M^2}{q^6} \left[ \int_{\mathbb{R}^+} \frac{dq}{q^8} \Phi_0(q)^2 \right] \delta(x - y)x^7 = A \delta(x - y)x^7, \]
\[ \tag{51} \]

where the coefficient \( A = \frac{48M^2}{\mathcal{Aq}_0} \left[ \int_{\mathbb{R}^+} \frac{dq}{q^8} \Phi_0(q)^2 \right] \). It must be noticed that the condition \( A < \infty \) requires an appropriate behavior of the fiducial vector at \( x \) equal to zero and infinity.

Direct calculations lead to the following generalized eigenfunctions

\[ \psi_k^{(K)}(x) = \delta \left( x^6 - \frac{k}{\mathcal{A}} \right), \quad 0 < k < \infty, \]
\[ \tag{52} \]

and the positive spectrum \( 0 < k < \infty \) of the Kretschmann operator.

For further interpretation it is useful to calculate a form of these solutions as functions of the affine group elements. In the standard parametrization \((p, q)\) the above states can be written as

\[ \psi_k^{(K)}(p, q) = \frac{1}{6} \left( \frac{A}{k} \right)^{\frac{5}{6}} \exp \left[ i \frac{k}{\mathcal{A}} \Phi_0^*(q \sqrt{\frac{q}{A}}) \right]. \]
\[ \tag{53} \]

It is obtained due to the useful transformation formula

\[ \langle g(p, q) | f \rangle = \int_{\mathbb{R}^+} d\nu(x) \, e^{-ipx} \Phi_0^*(q \sqrt{\frac{x}{\mathcal{A}}}) f(x). \]
\[ \tag{54} \]

According to general quantum rules one can expect that \( |\psi_k^{(K)}(t, r)|^2 \) is related to density probability (in this case it cannot be normalized) of finding the Schwarzschild spacetime in the Kretschmann observable eigenstate if this physical system is in the coherent state. As one can see this density probability is independent of \( t \) and depends only on the explicit form of the fiducial vector.

An important information implied by the eigenproblem solution of \( \hat{K} \) is that the quantum Kretschmann scalar can be potentially infinite because its spectrum is not bounded from above.
B. Expectation value for the $\hat{K}$ operator

Expectation values which give a link between quantum theory and observed values of quantum observables are state dependent. This feature is related to an important question about quantum states of our physical system. As we mentioned earlier, the fundamental observables $\hat{t}$ and $\hat{r}$ do not commute, but classically they are good observables of our quantum system so that the Schwarzschild spacetime cannot be in any common eigenstate of $\hat{t}$ and $\hat{r}$. In fact, it is a consequence of the Heisenberg uncertainty principle. In this context we need to check if the expectation values of the operator $\hat{K}$, determined within the coherent states representing elementary states of the spacetime, behave like the classical Kretschmann scalar.

Using the formula (A16) from the appendix A, one gets simple general expression for the expectation value of the Kretschmann operator

$$\langle \Psi | \hat{K} | \Psi \rangle \equiv \langle \hat{K} ; \Psi \rangle = \mathcal{A} \int_{\mathbb{R}^4} dx \, x^6 |\Psi(x)|^2.$$ \hspace{1cm} (55)

It turns out that the classical form of the Kretschmann scalar is proportional to the expectation value of the Kretschmann operator calculated within the coherent states $|g(t, r)\rangle$ fulfilling the consistency conditions

$$\langle \hat{K} ; g(t, r) \rangle = 48 M^2 \langle (q^{-6})^{-} \rangle_0 \frac{1}{r^6}.$$ \hspace{1cm} (56)

Therefore, the mean value $\langle \hat{K} ; g(t, r) \rangle$ has formally the singularity at $r = 0$, as in the classical case.

However, to determine its behaviour in quantum case fully we have to calculate its variance. Applying (29) to the operator (49) gives

$$\langle \hat{K}^2 ; g(t, r) \rangle = (48 M^2)^2 \langle (q^{-6})^{-} \rangle_0^2 \frac{1}{r^{12}}.$$ \hspace{1cm} (57)

Combining the expressions (56) and (57) we get the required variance of the Kretschmann operator within the coherent states

$$\text{var}(\hat{K} ; g(t, r)) = (48 M^2)^2 \left( \langle (q^{-6})^{-} \rangle_0^2 - \langle (q^{-6})^{-} \rangle_0^2 \right) \frac{1}{r^{12}}.$$ \hspace{1cm} (58)

The variance (58) tends also to infinity as $r$ approaches zero. However, the ratio of the expectation value $\langle \hat{K} ; g(t, r) \rangle$ and the standard deviation $\sqrt{\text{var}(\hat{K} ; g(t, r))}$ is independent on $r$ and $t$

$$s = \frac{\langle (q^{-6})^{-} \rangle_0}{\sqrt{\langle (q^{-6})^{-} \rangle_0^2 - \langle (q^{-6})^{-} \rangle_0^2}},$$ \hspace{1cm} (59)
Figure 1: The $1/r$ dependence of the expectation value of the Kretschmann operator $\langle \hat{K}; g(t, r) \rangle$ defined by (56). The blue area defines the points for which distance from expected value is smaller than $\sqrt{\text{var}(\hat{K}; g(t, r))}$ (the distance is counted along fixed $1/r$ line). The fiducial vector is taken as in (45) with $n = 25$.

i.e., both, the expectation value of $\hat{K}$ and its standard deviation are proportional. This behavior of the variance protects the mean value of the quantum Kretschmann observable within the coherent states to be singular. The operator $\hat{K}$ represents a well behaving smeared observable which is completely undetermined at the classical singularity $r = 0$, see Fig.1. Fluctuations of the Kretschmann quantum observable grow to infinity.

This is a novel mechanism which allows to omit singularity after quantization of classical variables.

Above, our new mechanism was checked only on the fundamental set of states, i.e., for the affine coherent states. In the appendix C we show that the expectation values and variances of the Kretschmann operator within the dense set of states

$$
\Psi_n(x) = N x^n \exp \left[ i\gamma_0 x - \frac{\gamma^2 x^2}{2} \right],
$$

(60)
where \( N^2 = 2\gamma^n/(n - 1)! \) and \( n = 1, 2, \ldots \), behaves exactly in the same way as it was obtained for the affine coherent states.

V. CONCLUSIONS

The extension of the configuration space to include temporal variable at the same footing as spatial variables is the novelty in the programme of quantization of gravity. In this paper we have used this idea to address the issue of the fate of the naked gravitational singularity of the Schwarzschild spacetime at the quantum level. Quantization of the time variable has enabled resolving the singularity problem. The above idea seems to be fruitful and worth of being applied to more realistic models of spacetime with naked singularities like the ones considered in a series of papers by Pankaj Joshi and his collaborators (see, e.g., [16–23] and references therein).

If isolated objects with naked singularities do occur in the real world, their examinations may bring highly valuable data to be used in the construction of quantum gravity. It is so because the isolated objects with covered singularities, i.e. black holes, may have screened some essential quantum gravity data due to the presence of horizons.

The solution of the eigenproblem for the Kretschmann operator shows that the spectrum is bounded from below and unbounded from above. The latter seems to lead to an embarrassment, but further examinations in the context of expectation value and the variance of the Kretschmann operator indicate the resolution of this difficulty.

Making use of the affine coherent states quantization, we have found that the expectation value of the Kretschmann operator \( \hat{\mathcal{K}} \) is singular and behaves like \( 1/r^6 \) as in the classical case. However, its variance behaves like \( 1/r^{12} \). One can say that quantization smears the singularity, avoiding its localization in the region of the configuration space including the singularity. In addition, since the variance not only does not vanish but diverges as \( r \to 0 \), the state corresponding to \( r = 0 \) cannot be any eigenstate of the operator \( \hat{\mathcal{K}} \), which is suggested by the property (36). Thus, the system cannot occupy the state corresponding to the gravitational singularity. One can say that probability of finding our system in the singular state is equal to zero.

The above result, carried out for the affine coherent states, has been confirmed in App. C for any vector of the carrier space \( L^2(\mathbb{R}^+, d\nu(x)) \). This proves the generality of our singularity avoiding mechanism. Our conclusion seems to be true for any quantum state of the system under consideration.

The issue of possible resolution of the singularity problem of the Schwarzschild black hole (\( M > 0 \)) at quantum level, has been addressed in several papers (see,
e.g., [11–13] and references therein). It is based on the isometry of the interior of the black hole with the vacuum Kantowski-Sachs spacetime. An interesting approach is presented in [11]. The corrections to the Raychaudhuri equation in the interior of the Schwarzschild black hole derived from loop quantum gravity (LQG) has been examined. The resulting effective equation implies the defocusing of geodesics which prevents the formation of conjugate points so that leads to the resolution of the singularity problem. In [12] the Kruskal-Szekeres coordinates [7] are applied. Quantum corrections of LQG are used to resolve the singularity problem, and the resulting quantum extension of spacetime has interesting features. An effective LQG model of the Schwarzschild black hole interior based on Thiemann’s identities is proposed in [13]. The effective dynamics leads to the resolution of the classical singularity. A spherically symmetric vacuum gravity is quantized using LQG techniques in [14]. Dirac’s quantization procedure leads to the resolution of the singularity of the classical theory inside black holes. The loop quantization of the model of Schwarzschild interior coupled to a massless scalar field has been studied [15]. Obtained results indicates the existence of a non-vanishing minimal mass of that black hole, which implies the existence of some black hole remnants after the Hawking evaporation.

An extension of the present paper to the case of the Schwarzschild black hole is straightforward. It will be considered in the context of quantization of the Lemaître-Tolman-Bondi model of isolated object, with naked or covered singularity, in the near future [24].

ACKNOWLEDGMENTS

We would like to thank Jan Ostrowski for helpful discussions.

Appendix A: Some remarks about calculations

According to methodology of integral quantization, one can see that for every classical observable \( f(t, r) \) the corresponding quantized operator \( \hat{f} \) (see (23)) is symmetric because its quadratic form

\[
\langle \Psi | \hat{f} | \Psi \rangle = \frac{1}{A_{\Phi_0}} \int_{\text{Aff}(\mathbb{R})} d\mu(p, q) \langle g(p, q) | \Psi \rangle^2 f \left( \chi^{-1} \left( p - \frac{a}{b} q, \frac{q}{b} \right) \right) \tag{A1}
\]
is real for \( \Psi \) belonging to the domain of the operator \( \hat{f} \). The operator \( \hat{f} \) can be bounded by the following expression

\[
\| \hat{f} | h \| ^2 = \left| \frac{1}{A_{\Phi_0}^{2}} \int_{\text{Aff}(\mathbb{R})} d\mu(p_1, q_1) \int_{\text{Aff}(\mathbb{R})} d\mu(p_2, q_2) f \left( \chi^{-1} \left( p_1 - \frac{a}{b} q_1, \frac{q_1}{b} \right) \right) f \left( \chi^{-1} \left( p_2 - \frac{a}{b} q_2, \frac{q_2}{b} \right) \right) \langle h | g(p_1, q_1) \rangle \langle g(p_1, q_1) | g(p_2, q_2) \rangle \langle g(p_2, q_2) | h \rangle \right| \\
\leq \left| \frac{1}{A_{\Phi_0}^{2}} \int_{\text{Aff}(\mathbb{R})} d\mu(p_1, q_1) \int_{\text{Aff}(\mathbb{R})} d\mu(p_2, q_2) f \left( \chi^{-1} \left( p_1 - \frac{a}{b} q_1, \frac{q_1}{b} \right) \right) \langle h | g(p_1, q_1) \rangle \langle g(p_1, q_1) | g(p_2, q_2) \rangle \langle g(p_2, q_2) | h \rangle \right| \\
\leq \left[ \int_{\text{Aff}(\mathbb{R})} d\mu(p_1, q_1) \int_{\text{Aff}(\mathbb{R})} d\mu(p_2, q_2) f \left( \chi^{-1} \left( p_1 - \frac{a}{b} q_1, \frac{q_1}{b} \right) \right) \langle g(p_1, q_1) | g(p_2, q_2) \rangle \right] \| | h \| ^2 ,
\]

(A2)

for all \( | h \rangle \) in the domain of the operator \( \hat{f} \). The last step is obtained by making use of the Schwartz inequality \( | \langle g(p, q) | h \rangle | \leq \| | h \| \|. \) If the above integral contained in square bracket is finite the operator \( \hat{f} \) is continuous in \( L^2(\mathbb{R}_+, d\nu(x)) \), i.e., \( \hat{f} \) is a self-adjoint operator.

However, in practice, even the elementary observables \( \hat{t} \) and \( \hat{r} \) are unbounded operators and require more careful procedures of extension their domains. Matrix elements of operators are crucial expressions required in quantum calculations. They can be used to extend such operators by symmetrization of their matrix elements

\[
\langle \psi_2 | \hat{A} | \psi_1 \rangle_{\text{sym}} := \frac{1}{2} \left( \langle \psi_2 | \hat{A} | \psi_1 \rangle + \langle \psi_1 | \hat{A} | \psi_2 \rangle^* \right) .
\]

(A3)

For any symmetric operator \( \hat{A} \) the following identity hold

\[
\langle \psi_2 | \hat{A} | \psi_1 \rangle_{\text{sym}} = \langle \psi_2 | \hat{A} | \psi_1 \rangle ,
\]

(A4)

for \( \psi_1 \) and \( \psi_2 \) in the domain of \( \hat{A} \), however, in other cases this equality can be broken and then the symmetrization (A3) is useful.

Let us assume that \( A(p', q') \), where \( p' = p'(p, q) \) and \( q' = q'(p, q) \) are real functions. A typical matrix elements are of the following form

\[
\langle \psi_2 | \hat{A} | \psi_1 \rangle = \frac{1}{A_{\Phi_0}} \int_{\text{Aff}(\mathbb{R})} d\mu(p, q) \langle \psi_2 | g(p, q) \rangle A(p', q') \langle g(p, q) | \psi_1 \rangle .
\]

(A5)
Calculating in the space $L^2(\mathbb{R}_+, d\nu(x))$ we allow for changing of integration order

$$\langle \psi_2 | \hat{A} | \psi_1 \rangle = \int_{\mathbb{R}_+} d\nu(x_2) \int_{\mathbb{R}_+} d\nu(x_1)$$

$$\psi_2(x_2)^* \left\{ \frac{1}{A\phi_0} \int_{\text{Aff}(\mathbb{R})} d\mu(p, q) e^{ip(x_2-x_1)} A(p', q') \Phi_0(qx_2) \Phi_0(qx_1)^* \right\} \psi_1(x_1)$$

$$= \int_{\mathbb{R}} dx_2 \int_{\mathbb{R}} dx_1 \theta(x_2) \frac{1}{x_2} \psi_2(x_2)^*$$

$$\left\{ \frac{1}{A\phi_0} \int_{\text{Aff}(\mathbb{R})} d\mu(p, q) e^{ip(x_2-x_1)} A(p', q') \Phi_0(qx_2) \Phi_0(qx_1)^* \right\} \theta(x_1) \frac{1}{x_1} \psi_1(x_1) \quad (A6)$$

and we extend integration over $x$ variables on the whole real axis adding the Heaviside function $\theta(x)$. This is useful for regularization of integrals, if needed, in the spirit of distribution theory.

As an example, let us consider the operator and its matrix element between the position eigenstate $|x\rangle$ and any arbitrary vector in the $L^2(\mathbb{R}_+, d\nu(x))$ space

$$\langle x | \hat{p} | \psi \rangle = \theta(x) \frac{1}{2\pi A\phi} \int_{\mathbb{R}} dp \int_{\mathbb{R}_+} dq \langle x | g(p, q) \rangle p \langle g(p, q) | \psi \rangle . \quad (A7)$$

Using the explicit form of the scalar products we get

$$\langle x | \hat{p} | \psi \rangle = \frac{1}{A\phi} \theta(x) \int_{\mathbb{R}_+} dq \int_{\mathbb{R}} dy \left( \int_{\mathbb{R}} dp p e^{ip(x-y)} \right) \Phi_0(qx) \Phi_0(qy)^* \theta(y) \frac{1}{y} \psi(y) . \quad (A8)$$

To regularize the integral over $p$, we use the known formula

$$\int_{\mathbb{R}} dp p e^{ip(x-y)} = -i2\pi\delta'(x-y) , \quad (A9)$$

where prime denotes distributional derivative of the Dirac delta. After using this expression and definition of $\delta'$ one gets

$$\langle x | \hat{p} | \psi \rangle = -i \theta(x) \int_{\mathbb{R}_+} dq \frac{\Phi_0(qx)}{q^2} \frac{\partial}{\partial x} \left[ \theta(x) \Phi_0(qx)^* \frac{1}{x} \psi(x) \right]$$

$$= -i \theta(x) \delta(x) \psi(x) + \theta(x) \left( -i \frac{\partial}{\partial x} + \frac{i}{2x} \right) \psi(x) . \quad (A10)$$

Note, that in this case the position state $|x\rangle$ does not belong to the domain of the operator $\hat{p}$ and using it can require symmetrization.
This formula allows to write more general matrix element

\[ \langle \psi_2 | \hat{p} | \psi_1 \rangle = -i \theta(0) \lim_{x \to 0^+} \frac{\psi_1(x)^* \psi_2(x)}{x} \]

\[ + \int_{\mathbb{R}_+} d\nu(x) \psi_2(x)^* \left( -i \frac{\partial}{\partial x} + \frac{i}{2x} \right) \psi_1(x), \quad (A11) \]

which after symmetrization can be rewritten as

\[ \langle \psi_2 | \hat{p} | \psi_1 \rangle_{\text{sym}} = \theta(0) \lim_{x \to 0^+} \frac{\text{Im}(\psi_2(x)^* \psi_1(x))}{x} \]

\[ + \left( -\frac{i}{2} \right) \int_{\mathbb{R}_+} d\nu(x) \left( \psi_2(x)^* \frac{\partial \psi_1}{\partial x} - \frac{\partial \psi_2^*}{\partial x} \psi_1(x) \right). \quad (A12) \]

Note that for the real functions $\psi$ the expectation value $\langle \psi | \hat{p} | \psi \rangle_{\text{sym}} = 0$. Even more general, the expectation value $\langle \psi | \hat{p} | \psi \rangle_{\text{sym}} = 0$ for $\text{Im} \left( \psi(x)^* \frac{\partial \psi}{\partial x} \right) = 0$.

An interesting quantity is the expectation value of the $\hat{p}$ operator in the gaussian wave packet $\Psi^{(t)}(x)$ constructed from generalized eigenstates $\psi^{(t)}(x) = \sqrt{x} e^{i \tau x}$ of the $\hat{p}$ operator

\[ \Psi^{(t)}(x) = N_t \int_{\mathbb{R}} d\tau f^{(t)}(\tau) \sqrt{x} e^{i \tau x} = N_t \sqrt{x} \exp \left( i \tau_0 x \right) \exp \left[ -\frac{1}{2} \gamma_t^2 x^2 \right], \quad (A13) \]

where

\[ f^{(t)}(\tau) = \frac{1}{\sqrt{2\pi \gamma_t}} \exp \left[ -\frac{(\tau - \tau_0)^2}{2 \gamma_t^2} \right]. \quad (A14) \]

The normalization coefficient is equal to $N_t^2 = 2\gamma_t/\sqrt{\pi}$. Making use of the formula (A11) and $\theta(0) = 1/2$ the required average value is

\[ \langle \Psi^{(t)}(x) | \hat{p} | \Psi^{(t)}(x) \rangle = \tau_0, \quad (A15) \]

as it is expected. The same result one obtains from (A12).

Using the methods of this section, a rather general form of matrix elements can be obtained if the classical observable is dependent only on $q = r$ variable. In this case we need to quantize the function $f(r)$

\[ \langle \Psi | \hat{f} | \Psi \rangle = \frac{1}{A_{\Phi_0}} \int_{\text{Aff}(\mathbb{R})} d\mu(p, g) f(q) |\langle g(p, q) | \Psi \rangle|^2 = \frac{1}{A_{\Phi_0}} \int_{\mathbb{R}_+} dx \frac{|\Psi(x)|^2}{x^2} \int_{\mathbb{R}_+} dq \frac{dq}{q^2} f(q) |\Phi_0(qx)|^2 \quad (A16) \]
Appendix B: Eigensolutions for elementary operators

Because of the importance of the operators $\hat{t}$ and $\hat{r}$, it is interesting to find the eigensolutions of these operators. We will compute them without assuming any special form of fiducial vectors, the only assumption is that a fiducial vector is a real function.

In this case, the constants $a$ and $b$ are as follows

\[ a = \langle \hat{p} \rangle_0 = 0 , \]

\[ b = \langle \hat{q} \rangle_0 = 1 . \]  

They are calculated in App. D.

1. Eigenproblem for $\hat{t}$ operator

It is easy to show that eigenfunctions of the differential part of the operator (A10)

\[ \left( -i \frac{\partial}{\partial x} + \frac{i}{2x} \right) \psi^{(t)}_\tau(x) = \tau \psi^{(t)}_\tau(x) \]  

are equal to

\[ \psi^{(t)}_\tau(x) = \sqrt{x} e^{i \tau x} . \]

This implies the following matrix elements of the operator $\hat{t}$

\[ \langle \psi | \hat{t} | \psi \rangle^{(t)} = -i \theta(0) \lim_{x \to 0^+} \frac{\psi^*(x)}{\sqrt{x}} + \tau \langle \psi | \psi \rangle^{(t)} . \]

Because every function $\psi \in L^2(\mathbb{R}_+, dv(x))$ has to converge to 0 as fast as $x$, or faster, when $x$ is going to $0^+$, the condition $\lim_{x \to 0^+} \psi^*(x)/\sqrt{x} = 0$ is fulfilled for such functions and the solutions (B4) are generalized eigensolutions (weak solutions) of the operator $\hat{t}$.

It is also possible to solve the eigenequation for $\hat{t}$ by using integral kernel as it was done for the operator $\hat{K}$. In this case the integral kernel is equal to

\[ K_t(x, y) = \langle x | \hat{t} | y \rangle = \frac{1}{A \Phi_0} \int_{\text{Aff}(\mathbb{R})} d\mu(pq) \langle x | g(p, q) \rangle \left( p - \frac{a}{b} q \right) \langle g(p, q) | y \rangle = \]

\[ = \frac{(-i)}{A \Phi_0} \delta'(x - y) \int_{\mathbb{R}_+} \frac{dq}{q^2} \Phi_0(xq) \Phi_0^*(qy) . \]

Using this method one has to extend the integration over $y$ to the whole real axis by introducing under integral the Heaviside function, as it is shown in the appendix A.
It is interesting to show a form of $\psi_r(x)$ as a function of $(p, q)$ variables. For this purpose one needs to use an explicit form of a fiducial vector. For example, let us assume $\Phi_0(x) = \frac{1}{\sqrt{(2n-1)!}} x^n e^{-\frac{x^2}{2}}$ and then

$$\psi_r(p, q) = \int_{\mathbb{R}^+} d\nu(x) \langle g(p, q)|x\rangle \psi_r(x) = \frac{\Gamma \left( n + \frac{1}{2} \right)}{\sqrt{(2n-1)!} \left( \frac{q}{2} + i(p + \tau) \right)^{n+\frac{1}{2}}}.$$ \hfill (B7)

Obviously, $\psi_r$ does not belong to $\mathcal{H}_x$ because it is not a square integrable function with the measure $d\nu(x)$ on $\mathbb{R}^+.$

2. Eigenproblem for $\hat{r}$ operator

Now, let us examine the eigensolutions of the operator $\hat{r}.$ In this case the assumption about reality of the fiducial vector is not needed.

$$\int_{\mathbb{R}^+} d\nu(y) \mathbf{K}_r(x, y) \psi_s(y) = s \psi_s(x), \quad \text{where} \quad \mathbf{K}_r(x, y) = \frac{1}{A \Phi_0} \int_{\text{Aff}(\mathbb{R})} d\mu(p, q) \langle x|g(p, q)\rangle \frac{q}{b} \langle g(p, q)|y \rangle = \frac{1}{A \Phi_0} \delta(x - y). \quad \text{(B9)}$$

Eigenfunctions of $\hat{r}$ are Dirac delta type functions

$$\psi_s(x) = \delta \left( x - \frac{1}{A \Phi_0} s \right). \quad \text{(B10)}$$

The form of these solutions as a functions of $(p, q)$ reads

$$\psi_s(p, q) = \int_0^\infty d\nu(x) \langle g(p, q)|x\rangle \psi_s(x) = A \Phi_0 \ s \exp \left[ -\frac{i p}{A \Phi_0} \right] \Phi_0 \left( \frac{q}{A \Phi_0} \right). \quad \text{(B11)}$$

Obviously, $\psi_s$ does not belong to $\mathcal{H}_x$ because it is not a square integrable function with the measure $d\nu(x)$ on $\mathbb{R}^+.$
Appendix C: Expectation values of the Kretschmann operator within a basis in the Hilbert space $L^2(\mathbb{R}_+, d\nu(x))$

In this appendix we present a derivation of expectation values and variances of the operators $\hat{t}$, $\hat{r}$ and $\hat{K}$ within a class of quantum states furnishing a basis in the Hilbert space $L^2(\mathbb{R}_+, d\nu(x))$.

Let us consider quantum states similar to the wave packets $\Psi^{(t)}(x)$ defined by (A13), where the only modification is in $x$ dependence of the eigenfunctions $\psi^{(t)}_{\tau}$:

$$\Psi_n(x) = N \int_{\mathbb{R}} d\tau f^{(t)}(\tau) x^n e^{i\tau x} = N x^n \exp \left[ i\tau_0 x - \frac{\gamma^2 x^2}{2} \right], \quad (C1)$$

where $n = 1, 2, \ldots$, the $f^{(t)}(\tau)$ is a Gaussian distribution, and where $N^2 = 2\gamma^n/(n-1)!$.

The expectation values of the operators $\hat{t}$, $\hat{r}$, $\hat{K}$ in the states $\Psi_n$ are as follows

$$\langle \hat{t}; \Psi_n \rangle = \tau_0, \quad (C2)$$

$$\langle \hat{r}; \Psi_n \rangle = A_A \frac{\Gamma \left( n - \frac{1}{2} \right)}{(n-1)!} \gamma, \quad (C3)$$

$$\langle \hat{K}; \Psi_n \rangle = A_A \frac{(n+2)!}{(n+2)!} \frac{1}{(n-1)!} \gamma^6, \quad (C4)$$

and the corresponding variances are

$$\text{var}(\hat{t}; \Psi_n) = \frac{4n-3}{4(n-1)} \gamma^2, \quad (C5)$$

$$\text{var}(\hat{r}; \Psi_n) = A_A^2 \left( \frac{1}{n-1} - \frac{\Gamma \left( n - \frac{1}{2} \right)^2}{(n-1)!^2} \right) \gamma^2, \quad n \geq 2 \quad (C6)$$

$$\text{var}(\hat{K}; \Psi_n) = A_A^2 \frac{(n+5)!}{(n-1)!} - \frac{(n+2)!}{(n-1)!^2} \frac{1}{\gamma^{12}}. \quad (C7)$$

Average value and variance of $\hat{K}$ and $\hat{r}$ are the same as in subsection IV B. The last statement is important because we prove below that the set of the functions $\Psi_n$ is dense in the Hilbert space $L^2(\mathbb{R}_+, d\nu(x))$.

Namely, let us consider the subset of functions $\Psi_n$ where index $n$ is odd. By using linear combination of these functions one can build the set of functions in the following form

$$l_k(x) := \sqrt{2}\gamma x L_k(\gamma^2 x^2) \exp \left( i\tau_0 x - \frac{\gamma^2 x^2}{2} \right), \quad (C8)$$

24
where \( L_k(x) \) are the Laguerre polynomials. Calculating the scalar product, after changing the variables \( y = \gamma^2 x^2 \), one gets

\[
\langle l_k | l_m \rangle = \int_0^\infty d\nu(x) L_k(x)^* l_m(x) = \int_0^\infty dy L_k(y) L_m(y) e^{-y} = \delta_{km}.
\] (C9)

It means, the set of functions (C8) form an orthonormal basis in the Hilbert space \( L^2(\mathbb{R}_+, d\nu) \). Therefore, the set of functions \( \Psi_n(x) \) must be dense in \( L^2(\mathbb{R}_+, d\nu(x)) \) and every function belonging to our Hilbert space can be expressed as a combination of the functions \( \Psi_n(x) \).

Appendix D: Calculations of \( \langle \hat{p} \rangle_0, \langle \hat{q} \rangle_0, \langle \hat{p}^2 \rangle_0, \) and \( \langle \hat{q}^2 \rangle_0 \)

In the following we present the calculation of component parts which are needed for \( \sigma_t \) and \( \sigma_r \). In the computation we use method described in the Appendix A. For shortness we use the following notation

\[
\langle g(p_1, q_1) | g(p_2, q_2) \rangle = \langle p_1, q_1 | p_2, q_2 \rangle. \]

Now, we calculate

\[
\langle \hat{p} \rangle_0 = \frac{1}{A_{\Phi_0}} \int_{\text{Aff}(\mathbb{R})} d\mu(p, q) \langle 0, 1 | p, q \rangle p \langle p, q | 0, 1 \rangle = \] (D1)

\[
= \frac{i}{A_{\Phi_0}} \left[ \int_{\mathbb{R}_+} dq \int_{\mathbb{R}_+} dy \left( \frac{\Phi_0^*(y)}{y} \right) \Phi_0(qy) \Phi_0^*(0) \Phi_0(y) + \right. \] (D2)

\[
\left. + \int_{\mathbb{R}_+} dq \int_{\mathbb{R}_+} dy \frac{\Phi_0^*(y)}{y} \Phi_0(qy) \Phi_0^*(0) \Phi_0(y) + \right. \] (D3)

\[
\int_{\mathbb{R}_+} dq \int_{\mathbb{R}_+} dy \delta(y) |\Phi_0(y)|^2 |\Phi_0(qy)|^2 \right]. \] (D4)

The second of these integrals can be turned to the form \( -A_{\Phi_0} \int_{\mathbb{R}_+} dy \Phi_0(y)^* \Phi_0'(y)/y \). The third one gives the same formula with the opposite sign. The last one is equal to \( \lim_{x \to 0^+} |\Phi_0(y)|^2/y \). Choosing a fiducial vector for which this limit is equal to zero, one gets

\[
\langle \hat{p} \rangle_0 = 0. \] (D5)

Now, we calculate further required coefficients of type \( \langle \hat{A} \rangle_0 \):

\[
\langle \hat{q} \rangle_0 = \frac{1}{A_{\Phi_0}} \int_{\text{Aff}(\mathbb{R})} d\mu(p, q) \langle 0, 1 | p, q \rangle q \langle p, q | 0, 1 \rangle = \] (D6)

\[
= \frac{1}{A_{\Phi_0}} \int_{\mathbb{R}_+} dq \int_{\mathbb{R}_+} dy \Phi_0^*(y) \Phi_0(qy) \Phi_0^*(qy) \Phi_0(y) = 1. \] (D7)
For variances we need average values of squares of the operators \( \hat{p} \) and \( \hat{q} \):

\[
\langle \hat{p}^2 \rangle_0 = \frac{1}{A_{\Phi_0}^2} \int_{\text{Aff}(\mathbb{R})} d\mu(p_1, q_1) \int_{\text{Aff}(\mathbb{R})} d\mu(p_2, q_2) \langle 0, 1 | p_1, q_1 \rangle p_1 \langle p_1, q_1 | p_2, q_2 \rangle \tag{D8}
\]

\[
p_2 \langle p_2, q_2 | 0, 1 \rangle = (-1)^2 \frac{A_{\Phi_0}^2}{q_1^2} \int_{\mathbb{R}_+} dx \frac{\Phi_0^*(x) \Phi_0(q_1 x)}{x} \int_{\mathbb{R}_+} dy \frac{\Phi_0^*(q_1 y)}{y} \delta'(x - y) \tag{D9}
\]

\[
\left[ \int_{\mathbb{R}_+} \frac{dq_2}{q_2} \left( \int_{\mathbb{R}_+} dz \delta'(z - y) \frac{\Phi_0^*(q_2 z) \Phi_0(z)}{z} \right) \Phi_0^*(y q_2) \right]. \tag{D10}
\]

The integral in the square bracket is equal to:

\[
A_{\Phi_0} y \left( \frac{\Phi_0(y)}{y} \right)' + \frac{\Phi_0(y)}{y} B + A_{\Phi_0} \delta(y) \Phi_0(y), \tag{D12}
\]

where \( B = \int_{\mathbb{R}_+} \frac{dq}{q} \Phi_0(q) \Phi_0^*(q) \). If \( \Phi_0 \) is a real function, \( B = A_{\Phi_0}/2 \).

If one choose the fiducial vector in such a way that the limit \( \lim_{y \to 0^+} \frac{\Phi_0(q_1 y)}{y} \) is equal to zero for every \( q_1 \), then the integration of the part which includes \( \delta'(y) \) is equal to zero. The similar situation one gets after integration over \( dx \) in the original integral. If one take the fiducial vector fulfilling the conditions \( \lim_{y \to 0^+} \frac{\Phi_0(y)}{y} = 0 \) and \( \lim_{y \to 0^+} \left( \frac{\Phi_0(y)}{y} \right)' \leq \infty \) one gets:

\[
\langle \hat{p}^2 \rangle_0 = \int_{\mathbb{R}_+} dy \left| y \left( \frac{\Phi_0(y)}{y} \right)' + \frac{B}{A_{\Phi}} \frac{\Phi_0(y)}{y} \right|^2. \tag{D13}
\]

The last component reads

\[
\langle \hat{q}^2 \rangle_0 = \frac{1}{A_{\Phi_0}^2} \int_{\text{Aff}(\mathbb{R})} d\mu(p_1, q_1) \int_{\text{Aff}(\mathbb{R})} d\mu(p_2, q_2) \langle 0, 1 | p_1, q_1 \rangle q_1 \langle p_1, q_1 | p_2, q_2 \rangle \tag{D14}
\]

\[
q_2 \langle p_2, q_2 | 0, 1 \rangle = \frac{1}{A_{\Phi_0}^2} \int_{\mathbb{R}_+} \frac{dz}{z^2} |\Phi_0(z)|^2. \tag{D15}
\]

[1] K. Schwarzschild, “On the gravitational field of a mass point according to Einstein’s theory”, arXiv:physics/9905030 [physics.hist-ph], translation and foreword by S. Antoci and A. Loinger. 1
[2] J. Droste, “The field of a single centre in Einstein’s theory of gravitation and the motion of a particle in that field”, K. Ned. Akad. Wet. Proc. 19, 197 (1917).
[3] A. Góźdź, W. Piechocki, and G. Plewa, “Quantum Belinski-Khalatnikov-Lifshitz scenario”, Eur. Phys. J. C 79, 45 (2019).
[4] A. Góźdź and W. Piechocki, “Robustnes of the BKL scenario”, Eur. Phys. J. C 80, 142 (2020).
[5] A. Góźdź, W. Piechocki, and T. Schmitz, “Dependence of the affine coherent states quantization on the parametrization of the affine group”, Eur. Phys. J. Plus 136, 18 (2021).
[6] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman and Company, San Francisco, 1973).
[7] P. Chruściel, *Geometry of Black Holes* (International Series of Monographs on Physics, 2020).
[8] M. de Cesare, S. S. Seahra, and E. Wilson-Ewing, “The singularity in mimetic Kantowski-Sachs cosmology”, JCAP 07 (2020) 018.
[9] J. P. Gazeau and R. Murenzi, “Covariant affine integral quantization(s)”, J. Math. Phys. 57 052102 (2016).
[10] H. P. Robertson, “The Uncertainty Principle”, Phys. Rev. 34, 163 (1929).
[11] K. Blanchette, S. Das, S. Hergott, and S. Rastgoo, “Black hole singularity resolution via the modified Raychaudhuri equation in loop quantum gravity”, Phys. Rev. D 103, 084038 (2021).
[12] A. Ashtekar, J. Olmedo, and P. Singh, “Quantum extension of the Kruskal spacetime”, Phys. Rev. D 98, 126003 (2018).
[13] M. Assanioussi and L. Mickel, “Loop effective model for Schwarzschild black hole: A modified \( \bar{\mu} \) dynamics”, Phys. Rev. D 103, 124008 (2021).
[14] R. Gambini and J. Pullin, “Loop Quantization of the Schwarzschild Black Hole”, Phys. Rev. Lett. 110, 211301 (2013).
[15] C. Zhang, Y. Ma, S. Song, and X. Zhang, “Loop quantum deparametrized Schwarzschild interior and discrete black hole mass”, arXiv:2107.10579v1 [gr-qc].
[16] K. Mosani, D. Dey, P. S. Joshi, G. C. Samanta, H. Menon, V. D. Patel, “On the visibility of singularities in general relativity and modified gravity theories”, arXiv:2106.01773 [gr-qc].
[17] K. Mosani, D. Dey, P. S. Joshi, “Globally visible singularity in an astrophysical setup”, arXiv:2103.07179 [gr-qc].
[18] J.-Q. Guo, L. Zhang, Y. Chen, P. S. Joshi, H. Zhang, “Strength of the naked singularity in critical collapse”, Eur. Phys. J. C 80, 924 (2020).
[19] J.-Q. Guo, P. S. Joshi, R. Narayan, L. Zhang, “Accretion disks around naked singularities”, Class. Quantum Grav. 38, 035012 (2021).
[20] K. Mosani, D. Dey, P. S. Joshi, “Global visibility of a strong curvature singularity in nonmarginally bound dust collapse”, Phys. Rev. D 102, 044037 (2020).

[21] K. Mosani, D. Dey, P. S. Joshi, “Strong curvature naked singularities in spherically symmetric perfect fluid collapse”, Phys. Rev. D 101, 044052 (2020).

[22] R. Shaikh, P. S. Joshi, “Can we distinguish black holes from naked singularities by the images of their accretion disks?”, JCAP 10, 064 (2019).

[23] P. Bambhaniya, A. B. Joshi, D. Dey, P. S. Joshi, “Timelike geodesics in Naked Singularity and Black Hole Spacetimes”, Phys. Rev. D 100, 124020 (2019).

[24] A. Góźdź, J. J. Ostrowski, A. Pędrak, and W. Piechocki, “Quantum dynamics of Lemaître-Tolman-Bondi model of massive star”, in progress.
