DIVISORS ON BURNIAT SURFACES

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Abstract. In this short note, we extend the results of [Alexeev-Orlov, 2012] about Picard groups of Burniat surfaces with $K^2 = 6$ to the cases of $2 \leq K^2 \leq 5$. We also compute the semigroup of effective divisors on Burniat surfaces with $K^2 = 6$. Finally, we construct an exceptional collection on a nonnormal semistable degeneration of a 1-parameter family of Burniat surfaces with $K^2 = 6$.

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Introduction

This note strengthens and extends several geometric results of the paper [AO12], joint with Dmitri Orlov, in which we constructed exceptional sequences of maximal possible length on Burniat surfaces with $K^2 = 6$. The construction was based on certain results about the Picard group and effective divisors on Burniat surfaces.

Here, we extend the results about Picard group to Burniat surfaces with $2 \leq K^2 \leq 5$. We also establish a complete description of the semigroup of effective $\mathbb{Z}$-divisors on Burniat surfaces with $K_X^2 = 6$. (For the construction of exceptional sequences in [AO12] only a small portion of this description was needed.)

Finally, we construct an exceptional collection on a nonnormal semistable degeneration of a 1-parameter family of Burniat surfaces with $K^2 = 6$.

1. Definition of Burniat surfaces

In this paper, Burniat surfaces will be certain smooth surfaces of general type with $q = p_g = 0$ and $2 \leq K^2 \leq 6$ with big and nef canonical class $K$ which were defined by Peters in [Pet77] following Burniat. They are Galois $\mathbb{Z}_2$-covers of (weak) del Pezzo surfaces with $2 \leq K^2 \leq 6$ ramified in certain special configurations of curves.
Recall from [Par91] that a $\mathbb{Z}_3^2$-cover $\pi: X \to Y$ with smooth and projective $X$ and $Y$ is determined by three branch divisors $A, B, C$ and three invertible sheaves $L_1, L_2, L_3$ on the base $Y$ satisfying fundamental relations $L_2 \otimes L_3 \simeq L_1(A), L_3 \otimes L_1 \simeq L_2(B), L_1 \otimes L_2 \simeq L_3(C)$. These relations imply that $L_i^3 \simeq \mathcal{O}_Y(B + C), L_i^2 \simeq \mathcal{O}_Y(C + A), L_i^3 \simeq \mathcal{O}_Y(A + B)$.

One has $X = \text{Spec}_Y \mathcal{A}$, where the $\mathcal{O}_Y$-algebra $\mathcal{A}$ is $\mathcal{O}_Y \oplus \bigoplus_{i=1}^3 L_i^{-1}$. The multiplication is determined by three sections in $\text{Hom}(L_i^{-1} \otimes L_j^{-1}, L_k^{-1}) = H^0(L_i \otimes L_j \otimes L_k^{-1})$, where $\{i, j, k\}$ is a permutation of $\{1, 2, 3\}$, i.e. by sections of the sheaves $\mathcal{O}_Y(A), \mathcal{O}_Y(B), \mathcal{O}_Y(C)$ vanishing on $A, B, C$.

Burniat surfaces with $K^2 = 6$ are defined by taking $Y$ to be the del Pezzo surface of degree 6, i.e. the blowup of $\mathbb{P}^2$ in three noncollinear points, and the divisors $\bar{A} = \sum_{i=0}^3 A_i, \bar{B} = \sum_{i=0}^3 B_i, \bar{C} = \sum_{i=0}^3 C_i$ to be the ones shown in red, blue, and black in the central picture of Figure 1 below.

The divisors $A_i, B_i, C_i$ for $i = 0, 3$ are the $(-1)$-curves, and those for $i = 1, 2$ are 0-curves, fibers of rulings $Bl_3 \mathbb{P}^2 \to \mathbb{P}^1$. The del Pezzo surface also has two contractions to $\mathbb{P}^2$ related by a quadratic transformation, and the images of the divisors form a special line configuration on either $\mathbb{P}^2$. We denote the fibers of the three rulings $f_1, f_2, f_3$ and the preimages of the hyperplanes from $\mathbb{P}^2$’s by $h_1, h_2$.

**Figure 1.** Burniat configuration on $Bl_3 \mathbb{P}^2$

Burniat surfaces with $K^2 = 6 - k, 1 \leq k \leq 4$ are obtained by considering a special configuration in Figure 1 for which some $k$ triples of curves, one from each group $\{A_1, A_2\}, \{B_1, B_2\}, \{C_1, C_2\}$, meet at common points $P_s$. The corresponding Burniat surface is the $\mathbb{Z}_2^2$-cover of the blowup of $Bl_3 \mathbb{P}^2$ at these points.

Up to symmetry, there are the following cases, see [BC11]:

1. $K^2 = 5$: $P_1 = \bar{A}_1 \bar{B}_1 \bar{C}_1$ (our shortcut notation for $\bar{A}_1 \cap \bar{B}_1 \cap \bar{C}_1$).
2. $K^2 = 4$, nodal case: $P_1 = \bar{A}_1 \bar{B}_1 \bar{C}_1, P_2 = \bar{A}_1 \bar{B}_2 \bar{C}_2$.
3. $K^2 = 4$, non-nodal case: $P_1 = \bar{A}_1 \bar{B}_1 \bar{C}_1, P_2 = \bar{A}_2 \bar{B}_2 \bar{C}_2$.
4. $K^2 = 3$: $P_1 = \bar{A}_1 \bar{B}_1 \bar{C}_2, P_2 = \bar{A}_1 \bar{B}_2 \bar{C}_1, P_3 = \bar{A}_2 \bar{B}_2 \bar{C}_1$.
5. $K^2 = 2$: $P_1 = \bar{A}_1 \bar{B}_1 \bar{C}_1, P_2 = \bar{A}_1 \bar{B}_2 \bar{C}_2, P_3 = \bar{A}_2 \bar{B}_2 \bar{C}_2, P_4 = \bar{A}_2 \bar{B}_1 \bar{C}_1$.

**Notation 1.1.** We generally denote the divisors upstairs by $D$ and the divisors downstairs by $\bar{D}$ for the reasons which will become clear from Lemmas 2.1, 3.1. We denote $Y = Bl_3 \mathbb{P}^2$ and $\epsilon: Y' \to Y$ is the blowup map at the points $P_s$. The exceptional divisors are denoted by $E_s$.

The curves $\bar{A}_i, \bar{B}_i, \bar{C}_i$ are the curves on $Y$, the curves $\bar{A}_i', \bar{B}_i', \bar{C}_i'$ are their strict preimages under $\epsilon$. (So that $\epsilon^*(\bar{A}_i) = \bar{A}_i' + E_1$ in the case (1), etc.) The divisors $\bar{A}_i', \bar{B}_i', \bar{C}_i', E_s$ are the curves (with reduced structure) which are the preimages of
The latter curves and $ar{E}_s$ under $\pi': X' \to Y'$. The surface $X'$ is the Burniat surface with $K^2 = 6 - k$.

The building data for the $\mathbb{Z}^2$-cover $\pi': X' \to Y'$ consists of three divisors $A' = \sum A'_i$, $B' = \sum B'_i$, $C' = \sum C'_i$. It does not include the exceptional divisors $\bar{E}_s$, they are not in the ramification locus.

One has $\pi''(A'_i) = 2A'_i$, $\pi''(B'_i) = 2B'_i$, $\pi''(C'_i) = 2C'_i$, and $\pi''(\bar{E}_s) = E_s$.

For the canonical class, one has $2K_X = \pi''(-K_Y)$. Indeed, from Hurwitz formula $2K_X = \pi^*(2K_Y + R')$, where $R' = A' + B' + C'$. Therefore, the above identity is equivalent to $R' = -3K_{Y'}$. This holds on $Y = \text{Bl}_3 \mathbb{P}^2$, and

$$R' = \epsilon^* R - 3 \sum \bar{E}_s = \epsilon^*(-3K_Y) - 3 \sum \bar{E}_s = -3K_{Y'}.$$

For the surfaces with $K^2 = 6, 5$ and 4 (non-nodal case), $-K_Y$ and $K_X$ are ample. For the remaining cases, including $K^2 = 2, 3$, the divisors $-K_Y$ and $K_X$ are big, nef, but not ample. Each of the curves $L_j$ (among $A_i, B_i, C_j$) through two of the points $P_j$ is a $(−2)$-curve (a $\mathbb{P}^1$ with square $−2$) on the surface $Y$. (For example, for the nodal case with $K^2 = 4$ $L_1 = A_1$ is such a line). Its preimage, a curve $L_j$ on $X$, is also a $(−2)$-curve. One has $-K_Y L_j = K_X L_j = 0$, and the curve $L_j$ is contracted to a node on the canonical model of $X$.

Note that both of the cases with $K^2 = 2$ and 3 are nodal.

2. Picard group of Burniat surfaces with $K^2 = 6$

In this section, we recall two results of [AO12].

**Lemma 2.1** ([AO12], Lemma 1). The homomorphism $\bar{D} \mapsto \frac{1}{2} \pi^*(\bar{D})$ defines an isomorphism of integral lattices $\frac{1}{2} \pi^*: \text{Pic } Y \to \text{Pic } X / \text{Tors}$. Under this isomorphism, one has $\frac{1}{2} \pi^*(-K_Y) = K_X$.

This lemma allows one to identify $\mathbb{Z}$-divisors $\bar{D}$ on the del Pezzo surface $Y$ with classes of $\mathbb{Z}$-divisors $D$ on $X$ up to torsion, equivalently up to numerical equivalence. This identification preserves the intersection form.

The curves $A_0, B_0, C_0$ are elliptic curves (and so are the curves $A_3 \simeq A_0$, etc.). Moreover, each of them comes with a canonical choice of an origin, denoted $P_{00}$, which is the point of intersection with the other curves which has a distinct color, different from the other three points. (For example, for $A_0$ one has $P_{00} = A_0 \cap B_3$.)

On the elliptic curve $A_0$ one also defines $P_{10} = A_0 \cap C_3$, $P_{01} = A_0 \cap C_1$, $P_{11} = A_0 \cap C_2$. This gives the 4 points in the 2-torsion group $A_0[2]$. We do the same for $B_0, C_0$ cyclically.

**Theorem 2.2.** ([AO12], Theorem 1) One has the following:

1. The homomorphism

$$\phi: \text{Pic } X \to \mathbb{Z} \times \text{Pic } A_0 \times \text{Pic } B_0 \times \text{Pic } C_0\quad L \mapsto (d(L) = L \cdot K_X, L|_{A_0}, L|_{B_0}, L|_{C_0})$$

is injective, and the image is the subgroup of index 3 of

$$\mathbb{Z} \times (\mathbb{Z} P_{00} + A_0[2]) \times (\mathbb{Z} P_{00} + B_0[2]) \times (\mathbb{Z} P_{00} + C_0[2]) \simeq \mathbb{Z}^4 \times \mathbb{Z}_2^6.$$

consisting of the elements with $d + a_0^0 + b_0^0 + c_0^0$ divisible by 3. Here, we denote an element of the group $\mathbb{Z} P_{00} + A_0[2]$ by $(a_0^0, a_0^1 a_0^2)$, etc.
(2) \( \phi \) induces an isomorphism \( \text{Tors}(\text{Pic} \ X) \to A_0[2] \times B_0[2] \times C_0[2] \).

(3) The curves \( A_i, B_i, C_i, 0 \leq i \leq 3 \), generate \( \text{Pic} \ X \).

This theorem provides one with explicit coordinates for the Picard group of a Burniat surface \( X \), convenient for making computations.

3. Picard group of Burniat surfaces with \( 2 \leq K^2 \leq 5 \)

In this section, we extend the results of the previous section to the cases \( 2 \leq K^2 \leq 5 \). First, we show that Lemma 2.1 holds verbatim if \( 3 \leq K^2 \leq 5 \).

Lemma 3.1. Assume \( 3 \leq K^2 \leq 5 \). Then the homomorphism \( \bar{D} \mapsto \frac{1}{2} \pi^*(\bar{D}) \) defines an isomorphism of integral lattices \( \frac{1}{2} \pi'^* : \text{Pic} \ Y' \to \text{Pic} \ X'/\text{Tors} \), and the inverse map is \( \frac{1}{2} \pi' \). Under this isomorphism, one has \( \frac{1}{2} \pi'^*(-K_{Y'}) = K_{X'} \).

Proof: The proof is similar to that of Lemma 2.1. The map \( \frac{1}{2} \pi^* \) establishes an isomorphism of \( \mathbb{Q} \)-vector spaces (\( \text{Pic} \ Y' \) \( \otimes \mathbb{Q} \) and (\( \text{Pic} \ X' \) \( \otimes \mathbb{Q} \) together with the intersection product because:

1. Since \( h^i(\mathcal{O}_{X'}) = h^i(\mathcal{O}_{Y'}) = 0 \) for \( i = 1, 2 \) and \( K_{X'}^2 = K_{Y'}^2 \), by Noether's formula the two vector spaces have the same dimension.
2. \( \frac{1}{2} \pi'^* D_1 \cdot \frac{1}{2} \pi'^* D_2 = \frac{1}{4} \pi'^* (D_1 \cdot D_2) = D_1 D_2 \).

A crucial observation is that \( \frac{1}{2} \pi'^* \) sends \( \text{Pic} \ Y' \) to integral classes. To see this, it is sufficient to observe that \( \text{Pic} \ Y' \) is generated by divisors \( \bar{D} \) which are in the ramification locus and thus for which \( D = \frac{1}{2} \pi'^* (\bar{D}) \) is integral.

Consider for example the case of \( K^2 = 5 \). One has \( \text{Pic} \ Y' = \epsilon^*(\text{Pic} \ Y) \oplus \mathbb{Z} E \). The group \( \epsilon^*(\text{Pic} \ Y) \) is generated by \( A'_0, B'_0, C'_0, A'_1, B'_1, C'_1 \). Since \( \epsilon^*(A_1) = A'_1 + E_1 \), the divisor class \( E_1 \) lies in group spanned by \( A'_1 \) and \( \epsilon^*(\text{Pic} \ Y) \). So we are done.

In the nodal case \( K^2 = 4 \), \( E_1 \) is spanned by \( B'_1 \) and \( \epsilon^*(\text{Pic} \ Y) \). \( E_2 \) by \( B'_2 \) and \( \epsilon^*(\text{Pic} \ Y) \); exactly the same for the non-nodal case. In the case \( K^2 = 3 \), \( E_1 \) is spanned by \( C'_1 \) and \( \epsilon^*(\text{Pic} \ Y) \), \( E_2 \) by \( B'_2 \) and \( \epsilon^*(\text{Pic} \ Y) \), \( E_3 \) by \( A'_2 \) and \( \epsilon^*(\text{Pic} \ Y) \).

Therefore, \( \frac{1}{2} \pi'^* (\text{Pic} \ Y') \) is a sublattice of finite index in \( \text{Pic} \ X'/\text{Tors} \). Since the former lattice is unimodular, they must be equal.

One has \( \frac{1}{2} \pi' \circ \frac{1}{2} \pi'^* (\bar{D}) = \bar{D} \), so the inverse map is \( \frac{1}{2} \pi' \).

Remark 3.2. I thank Stephen Coughlan for pointing out that the above proof that \( \text{Pic} \ Y' \) is generated by the divisors in the ramification locus does not work in the \( K^2 = 2 \) case. In this case, each of the lines \( A_i, B_i, C_i, i = 1, 2 \) contains exactly two of the points \( P_1, P_2, P_3 \). What we can see easily is the following: there exists a free abelian group \( H \cong \mathbb{Z}^8 \) which can be identified with a subgroup of index 2 in \( \text{Pic} \ Y' \) and a subgroup of index 2 in \( \text{Pic} \ X'/\text{Tors} \).

Consider a \( \mathbb{Z} \)-divisor (not a divisor class) on \( Y' \)

\[
\bar{D} = a_0 A'_0 + \ldots + c_3 C'_3 + \sum_s e_s E_s
\]

such that the coefficients \( e_s \) of \( E_s \) are even. Then we can define a canonical lift

\[
D = a_0 A_0 + \ldots + c_3 C_3 + \sum_s \frac{1}{2} e_s E_s,
\]
which is a divisor on $X'$, and numerically one has $D = \frac{1}{2} \pi^*(\bar{D})$. Note that $\bar{D}$ is linearly equivalent to 0 iff $D$ is a torsion.

By Theorem 2.2, for a Burniat surface with $K^2 = 6$, we have an identification

$$V := \text{Tors Pic}(X) = A_0[2] \times B_0[2] \times C_0[2] = \mathbb{Z}_2^2 \times \mathbb{Z}_2^2 \times \mathbb{Z}_2^2.$$ 

It is known (see [BC11]) that for Burniat surfaces with $2 \leq K^2 \leq 6$ one has $\text{Tors Pic}(X) \cong \mathbb{Z}_2^6$ with the exception of the case $K^2 = 2$ where $\text{Tors Pic}(X) \cong \mathbb{Z}_2^3$. We would like to establish a convenient presentation for the Picard group and its torsion for these cases which would be similar to the above.

**Definition 3.3.** We define the following vectors, forming a basis in the $\mathbb{Z}_2$-vector space $V$: $\vec{A}_1 = 00 10 00$, $\vec{A}_2 = 00 11 00$, $\vec{B}_1 = 00 00 10$, $\vec{B}_2 = 00 00 11$, $\vec{C}_1 = 10 00 00$, $\vec{C}_2 = 11 00 00$.

Further, for each point $P_s = A_i B_j C_k$ we define a vector $\vec{P}_s = \vec{A}_i + \vec{B}_j + \vec{C}_k$.

**Definition 3.4.** We also define the standard bilinear form $V \times V \to \mathbb{Z}_2$: $(x_1, \ldots, x_6) \cdot (y_1, \ldots, y_6) = \sum_{i=1}^{6} x_i y_i$.

**Lemma 3.5.** The restriction map $\rho : \text{Tors Pic}(X') \to A_0[2] \times B_0[2] \times C_0[2]$ is injective, and the image is identified with the orthogonal complement of the subspace generated by the vectors $\vec{P}_s$.

**Proof.** The restrictions of the following divisors to $V$ give the subset $B_0[2]$:

$$0, \quad A_1 - A_2 = 00 10 00, \quad A_1 - A_3 - C_0 = 00 11 00, \quad A_2 - A_3 - C_0 = 00 01 00.$$ 

Among these, the divisors containing $A_1$ are precisely those for which the vector $v \in B_0[2] \subset V$ satisfies $v \cdot \vec{A}_1 = 1$. Repeating this verbatim, one has the same results for the divisors $A_2, \ldots, C_2$ and vectors $\vec{A}_2, \ldots, \vec{C}_2$.

Let $\bar{D}$ be a linear combination of the divisors $\vec{A}_1 - \vec{A}_2$, $\vec{A}_1 - \vec{A}_3 - \vec{C}_0$, $\vec{A}_2 - \vec{A}_3 - \vec{C}_0$, and the corresponding divisors for $C_0[2], A_0[2]$. Define the vector $\nu(D) \in V$ to be the sum of the corresponding divisors $\vec{A}_1 - \vec{A}_2 \in V$, etc.

Now assume that the vector $\nu(D)$ satisfies the condition $\nu(D) \cdot \vec{P}_s = 0$ for all the points $P_s$. Then the coefficients of the exceptional divisors $\vec{E}_s$ in the divisor $\epsilon^*(\bar{D})$ on $X'$ are even (and one can also easily arrange them to be zero since the important part is working modulo 2). Therefore, a lift of $\epsilon^*(\bar{D})$ to $X'$ is well defined and is a torsion in $\text{Pic}(X')$.

This shows that the image of the homomorphism $\rho : \text{Tors Pic}(X') \to V$ contains the space $(\vec{P}_s)^\perp$. But this space already has the correct dimension. Indeed, for $3 \leq K^2 \leq 5$ the vectors $\vec{P}_s$ are linearly independent, and for $K^2 = 2$ the vectors $\vec{P}_1 = \vec{A}_1 + \vec{B}_1 + \vec{C}_1$, $\vec{P}_2 = \vec{A}_1 + \vec{B}_2 + \vec{C}_2$, $\vec{P}_3 = \vec{A}_2 + \vec{B}_1 + \vec{C}_2$, $\vec{P}_4 = \vec{A}_2 + \vec{B}_2 + \vec{C}_1$ are linearly dependent (their sum is zero) and span a subspace of dimension 3; thus the orthogonal complement has dimension 3 as well. Therefore, $\rho$ is a bijection of $\text{Tors Pic}(X')$ onto $(\vec{P}_s)^\perp$. 

**Theorem 3.6.** One has the following:

1. The homomorphism

$$\phi : \text{Pic}(X') \to \mathbb{Z}_2^{1+k} \times \text{Pic}(A'_0 \times \text{Pic}(B'_0 \times \text{Pic}(C'_0)

$$

$$L \mapsto (d(L) = L \cdot K_{X'}, L \cdot \frac{1}{2} E_s, L|_{A'_0}, L|_{B'_0}, L|_{C'_0})$$

where $k = \dim \mathbb{Z}_2^{1+k}$.
is injective, and the image is the subgroup of index $3 \cdot 2^n$ in $\mathbb{Z}^{4+k} \times A'_0[2] \times B'_0[2] \times C'_0[2]$, where $n = 6 - K^2$ for $3 \leq K^2 \leq 6$ and $n = 3$ for $K^2 = 2$.

2. $\phi$ induces an isomorphism $\text{Tors}(\text{Pic} X') \cong (\mathbb{P}_2^1)^{+} \subset A'_0[2] \times B'_0[2] \times C'_0[2]$.

3. The curves $A'_i, B'_i, C'_i$, $0 \leq i \leq 3$, generate $\text{Pic} X'$.

Proof. (2) is (3.5) and (1) follows from it. For (3), note that $\text{Pic} X'/\text{Tors} = \text{Pic} Y'$ is generated by the divisors $A'_i, B'_i, C'_i$ and that the proof of the previous theorem shows that $\text{Tors} \text{Pic} X'$ is generated by certain linear combinations of these divisors.

\section{Effective divisors on Burniat surfaces with $K^2 = 6$}

Since $\frac{1}{2}\pi^*$ and $\frac{1}{3}\pi_*$ provide isomorphisms between the $\mathbb{Q}$-vector spaces $(\text{Pic} Y) \otimes \mathbb{Q}$ and $(\text{Pic} X) \otimes \mathbb{Q}$, it is obvious that the cones of effective $\mathbb{Q}$- or $\mathbb{R}$-divisors on $X$ and $Y$ are naturally identified. In this section, we would like to prove the following description of the semigroup of effective $\mathbb{Z}$-divisors:

\begin{theorem}
The curves $A_i, B_i, C_i$, $0 \leq i \leq 3$, generate the semigroup of effective $\mathbb{Z}$-divisors on Burniat surface $X$.
\end{theorem}

We start with several preparatory lemmas.

\begin{lemma}
The semigroup of effective $\mathbb{Z}$-divisors on $Y$ is generated by the $(-1)$-curves $A_0, B_0, C_0, A_3, B_3, C_3$.
\end{lemma}

Proof. Since $-KY$ is ample, the Mori-Kleiman cone $NE_1(Y)$ of effective curves in $(\text{Pic} Y) \otimes \mathbb{Q}$ is generated by extremal rays, i.e. the $(-1)$-curves $A_0, B_0, C_0, A_3, B_3, C_3$. We claim that moreover the semigroup of integral points in $NE_1(Y)$ is generated by these points, i.e. the polytope $Q = NE_1(Y) \cap \{ C \mid -KYC = 1 \}$ is totally generating. The vertices of this polytope in $\mathbb{R}^3$ are $(-1,0,0)$, $(0, -1, 0)$, $(0, 0, -1)$, $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$, and the lattice $\text{Pic} Y = \mathbb{Z}^4$ is generated by them. It is a prism over a triangular base, and it is totally generating because it can be split into 3 elementary simplices.

\begin{lemma}
The semigroup of nef $\mathbb{Z}$-divisors on $Y$ is generated by $f_1, f_2, f_3, h_1,$ and $h_2$.
\end{lemma}

Proof. Again, for the $\mathbb{Q}$-divisors this is obvious by MMP: a divisor $D$ is nef iff $\tilde{D} \tilde{F} \geq 0$ for $\tilde{F} \in \{ \tilde{A}_0, \tilde{B}_0, \tilde{C}_0, \tilde{A}_3, \tilde{B}_3, \tilde{C}_3 \}$, and the extremal nef $D$ divisors correspond to contractions $Y \to Y'$ with $\text{rk} \text{Pic} Y' = 1$. Another proof: the extremal nef divisors correspond to the faces of the triangular prism from the proof of Lemma 4.2, and there are 5 of them: 3 sides, top, and the bottom.

Now let $\tilde{D}$ be a nonnegative linear combination $\tilde{D} = \sum a_i f_j + b_j h_j$ and let us assume that $a_1 > 0$ (resp. $b_1 > 0$). Since the intersections of $f_1$ (resp. $h_1$) with the curves $F$ above are 0 or 1, it follows that $\tilde{D} - f_1$ (resp. $\tilde{D} - h_1$) is also nef. We finish by induction.

We write the divisors $\tilde{D}$ in $\text{Pic} Y$ using the symmetric coordinates

$$(d; a_0^0, b_0^0, c_0^0, a_3^0, b_3^0, c_3^0),$$

where $d = D(-KY)$, $a_0^0 = D\tilde{A}_0$, $\ldots$, $c_3^0 = D\tilde{C}_3$.

Note that, as in Theorem 2.2, Pic $Y$ can be described either as the subgroup of $\mathbb{Z}^4$ with coordinates $(d; a_0^0, b_0^0, c_0^0)$ satisfying the congruence $3|(d + a_0^0 + b_0^0 + c_0^0)$,
The function $\text{Definition 4.7.}$ We say that an effective divisor $h$ on $Y$ is strictly positive, with the exception of the following divisors, up to symmetry:

$$D = n_1 a_1 + n_2 b_2$$

Indeed, since $2p(a) = \chi(D + K_Y) > 0$, we have $p_a(D) = 0$. Therefore, we can write $D = \sum n_i a_i + m_j b_j$ with $n_i, m_j \in \mathbb{Z}_{\geq 0}$. Let us say $n_1 > 0$. If $D = n_1 a_1$, then $p_a(D) = -(n_1 - 1)$. Otherwise, $n_1 a_1 + g \leq D$, where $g = f_j, j \neq 1$, or $g = h_j$. Then using the elementary formula $p_a(D_1 + D_2) = p_a(D_1) + p_a(D_2) + D_1 D_2 - 1$, we see that $p_a(n_1 a_1 + g) = 0$. Continuing this by induction and adding more $f_j$'s and $h_j$'s, one easily obtains that $p_a(D) > 0$ with the only exceptions listed above. Starting with $m_1 h_1$ instead of $n_1 a_1$ works the same.

**Lemma 4.4.** The function $p_a(D) = \frac{D(D + K_Y)}{2} + 1$ on the set of nef $\mathbb{Z}$-divisors on $Y$ is strictly positive, with the exception of the following divisors, up to symmetry:

1. $(2n; n, 0, 0; n, 0, 0)$ for $n \geq 1$, one has $p_a = -(n - 1)$
2. $(2n; n - 1, 1, 0; n - 1, 1, 0)$ for $n > 1$, one has $p_a = 0$.
3. $(2n + 1; n, 1, 1; n - 1, 0, 0)$ and $(2n + 1; n - 1, 0, 0; n, 1, 1)$ for $n \geq 1$, $p_a = 0$.
4. $(6; 2, 2, 0, 0, 0)$ and $(6; 0, 0, 0; 2, 2, 2)$, $p_a = 0$.

The divisors in (1) are in the linear system $|nf_1|$, where $f_1$ is a fiber of one of the three rulings $\overline{Y} \to \mathbb{P}^1$. The divisors in (2) and (3) are obtained from these by adding a section. The divisors in (4) belong to the linear systems $|2h_1|$ and $|2h_2|$.

**Proof.** Let $\overline{D}$ be a nef $\mathbb{Z}$-divisor. By Lemma 4.3, we can write $\overline{D} = \sum n_i f_i + m_j h_j$ with $n_i, m_j \in \mathbb{Z}_{\geq 0}$. Let us say $n_1 > 0$. If $\overline{D} = n_1 f_1$, then $p_a(\overline{D}) = -(n_1 - 1)$. Otherwise, $n_1 f_1 + g \leq \overline{D}$, where $g = f_j, j \neq 1$, or $g = h_j$. Then using the elementary formula $p_a(D_1 + D_2) = p_a(D_1) + p_a(D_2) + D_1 D_2 - 1$, we see that $p_a(n_1 f_1 + g) = 0$. Continuing this by induction and adding more $f_j$’s and $h_j$’s, one easily obtains that $p_a(\overline{D}) > 0$ with the only exceptions listed above. Starting with $m_1 h_1$ instead of $n_1 f_1$ works the same.

**Corollary 4.5.** The function $\chi(D) = \frac{D(D - K_X)}{2} + 1$ on the set of nef $\mathbb{Z}$-divisors on $Y$ is strictly positive, with the same exceptions as above.

**Proof.** Indeed, since $\chi(\mathcal{O}_X) = 1$, one has $\chi(D) = p_a(D)$.

**Lemma 4.6.** Assume that $\overline{D} \neq 0$ is a nef divisor on $X$ with $p_a(\overline{D}) > 0$. Then the divisor $\overline{D} + K_Y$ is effective.

**Proof.** One has $\chi(\overline{D} + K_Y) = \frac{(\overline{D} + K_Y)\overline{D}}{2} + 1 = p_a(\overline{D}) > 0$. Since $h^2(\overline{D} + K_Y) = h^0(-\overline{D}) = 0$, this implies that $h^0(\overline{D}) = 0$.

**Definition 4.7.** We say that an effective divisor $D$ on $X$ is in minimal form if $DF \geq 0$ for the elliptic curves $F \in \{A_0, B_0, C_0, A_3, B_3, C_3\}$, and for the curves among those that satisfy $DF = 0$, one has $D|F \neq 0$ in $F[2]$.

If either of these conditions fails then $D - F$ must also be effective since $F$ is then in the base locus of $|D|$. A minimal form is obtained by repeating this procedure until it stops or one obtains a divisor of negative degree, in which case $D$ obviously was not effective. We do not claim that a minimal form is unique.

**Proof of Thm. 4.1.** Let $D$ be an effective divisor on $X$. We have to show that it belongs to the semigroup $S = \{A_i, B_i, C_i, \ 0 \leq i \leq 3\}$.

**Step 1:** One can assume that $D$ is in minimal form. Obviously.

**Step 2:** The statement is true for $d \leq 6$. There are finitely many cases here to check. We checked them using a computer script. For each of the divisors, putting it in minimal form makes it obvious that it is either in $S$ or it is not effective because it has negative degree, with the exception of the following three divisors, in the notations of Theorem 2.2: $(3; 1 10 1 10 1 10), (3; 0 0 0 0 0 0 0), (3; 1 0 0 1 0 0 1 0 0)$. The first two divisors are not effective by [AO12, Lemma 5]. The third one is not effective because it is $K_X$ and $h^0(K_X) = p_g(X) = 0$. 


Step 3: The statement is true for nef divisors of degree \( d \geq 7 \) which are not the exceptions listed in Lemma 4.4.

One has \( K_X(K_X - D) < 0 \), so \( h^0(K_X - D) = 0 \) and the condition \( \chi(D) > 0 \) implies that \( D \) is effective. We are going to show that \( D \) is in the semigroup \( S \).

Consider the divisor \( D - K_X \) which modulo torsion is identified with the divisor \( \bar{D} + K_Y \) on \( Y \). By Lemmas 4.6 and 4.2, \( \bar{D} + K_Y \) is a positive \( \mathbb{Z} \)-combination of \( \bar{A}_0, \bar{B}_0, \bar{C}_0, \bar{A}_3, \bar{B}_3, \bar{C}_3 \). This means that

\[ D = K_X + (\text{a positive combination of } A_0, B_0, C_0, A_3, B_3, C_3) + \text{(torsion } \nu) \]

A direct computer check shows that for any torsion \( \nu \) the divisor \( K_X + F + \nu \) is in \( S \) for a single curve \( F \in \{ A_0, B_0, C_0, A_3, B_3, C_3 \} \). (In fact, for any \( \nu \neq 0 \) the divisor \( K_X + \nu \) is already in \( S \).) Thus,

\[ D - (\text{a nonnegative combination of } A_0, B_0, C_0, A_3, B_3, C_3) \in S \implies D \in S. \]

Step 4: The statement is true for nef divisors in minimal form of degree \( d \geq 7 \) which are the exceptions listed in Lemma 4.4.

We claim that any such divisor is in \( S \), and in particular is effective. For \( d = 7,8 \) this is again a direct computer check. For \( d \geq 9 \), the claim is true by induction, as follows: If \( D \) is of exceptional type (1,2, or 3) of Lemma 4.4 then \( D - C_1 \) has degree \( d' = d - 2 \) and is of the same exceptional type. This concludes the proof. \( \square \)

Remark 4.8. Note that we proved a little more than what Theorem 4.1 says. We also proved that every divisor \( D \) in minimal form and of degree \( \geq 7 \) is effective and is in the semigroup \( S \).

Remark 4.9. For Burniat surfaces with \( 2 \leq K^2 X \leq 5 \), a natural question to ask is whether the semigroup of effective \( \mathbb{Z} \)-divisors is generated by the preimages of the \((-1)\)- and \((-2)\) curves on \( Y' \). These include the divisors \( \bar{A}_i', \bar{B}_i', \bar{C}_i' \) and \( E_s \) but in some cases there are other curves, too.

5. Exceptional collections on degenerate Burniat surfaces

Degenerations of Burniat surfaces with \( K^2 X = 6 \) were described in [AP09]. Here, we will concentrate on one particularly nice degeneration depicted in Figure 2.

![Figure 2. One-parameter degeneration of Burniat surfaces](image)
in the central fiber $f^{-1}(0)$ to a configuration shown in the left panel. The surface $\mathcal{Y}$ is obtained from $Y \times \mathbb{A}^1$ by two blowups in the central fiber, along the smooth centers $A_0$ and then (the strict preimage of) $C_3$. The resulting 3-fold $\mathcal{Y}$ is smooth, the central fiber $\mathcal{Y}_0 = \text{Bl}_3 \mathbb{P}^2 \cup \text{Bl}_2 \mathbb{P}^2 \cup (\mathbb{P}^1 \times \mathbb{P}^1)$ is reduced and has normal crossings. This central fiber is shown in the third panel.

The log canonical divisor $K_{\mathcal{Y}} + \frac{1}{2} \sum_{i=0}^3 (A_i + \bar{B}_i + \bar{C}_i)$ is relatively big and nef over $\mathbb{A}^1$. It is a relatively minimal model. The relative canonical model $\mathcal{Y}_{\text{can}}$ is obtained from $\mathcal{Y}$ by contracting three curves. The 3-fold $\mathcal{Y}_{\text{can}}$ is singular at three points and not $\mathbb{Q}$-factorial. Its central fiber $\mathcal{Y}_{0\text{can}}$ is shown in the last, fourth panel.

The 3-folds $\pi : \mathcal{X} \to \mathcal{Y}$ and $\pi_{\text{can}} : \mathcal{X}_{\text{can}} \to \mathcal{Y}_{\text{can}}$ are the corresponding $\mathbb{Z}_2^3$-Galois covers. The 3-fold $\mathcal{X}$ is smooth, and its central fiber $X_0$ is reduced and has normal crossings. It is a relatively minimal model: $K_X$ is relatively big and nef.

The 3-fold $\mathcal{X}_{\text{can}}$ is obtained from $\mathcal{X}$ by contracting three curves. Its canonical divisor $K_{\mathcal{X}_{\text{can}}}$ is relatively ample. It is a relative canonical model. We note that $\mathcal{X}$ is one of the 6 relative minimal models $\mathcal{X}^{(k)}$, $k = 1, \ldots, 6$, that are related by flops.

Let $U \subset \mathbb{A}^1$ be the open subset containing 0 and all $t$ for which the fiber $X_t$ is smooth, and let $X_U = \mathcal{X} \times_{\mathbb{A}^1} U$. The aim of this section is to prove the following:

**Theorem 5.1.** Then there exists a sequence of line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_6$ on $X_U$ whose restrictions to any fiber (including the nonnormal semistable fiber $X_0$) form and exceptional collection of line bundles.

**Remark 5.2.** It seems to be considerably harder to construct an exceptional collection on the surface $X^{\text{can}}_0$, the special fiber in a singular 3-fold $\mathcal{X}^{\text{can}}$. And perhaps looking for one is not the right thing to do. A familiar result is that different smooth minimal models $\mathcal{X}^{(k)}$ related by flops have equivalent derived categories. In the same vein, in our situation the central fibers $X^{(k)}_0$, which are reduced reducible semistable varieties, should have the same derived categories. The collection we construct works the same way for any of them.

**Notation 5.3.** On the surface $X_0$, we have 12 Cartier divisors $A_i, B_i, C_i$, $i = 0, 1, 2, 3$. The “internal” divisors $A_i, B_i, C_i$, $i = 1, 2$ have two irreducible components each. Of the 6 “boundary” divisors, $A_0, A_3, C_0$ are irreducible, and $B_0 = B_0' + B_0'', B_3 = B_3' + B_3''$, $C_3 = C_3' + C_3''$ are reducible.

Our notation for the latter divisors is as follows: the curve $C_0'$ is a smooth elliptic curve (on the bottom surface $\mathcal{Y}_0$ the corresponding curve has 4 ramification points), and the curve $C_3'$ is isomorphic to $\mathbb{P}^1$ (on the bottom surface the corresponding curve has 2 ramification points).

For consistency of notation, we also set $A_0' = A_0, A_3' = A_3, C_0' = C_0$.

**Definition 5.4.** Let $\psi = \psi_{C_3} : C_3 \to C_3'$ be the projection which is an isomorphism on the component $C_3'$ and collapses the component $C_0'$ to a point.

We have natural norm map $\psi_* = (\psi_{C_3})_* : \text{Pic} C_3 \to \text{Pic} C_3'$. Indeed, every line bundle on the reducible curve $C_3$ can be represented as a Cartier divisor $\mathcal{O}_{C_3} \left( \sum n_i P_i \right)$, where $P_i$ are nonsingular points. Then we define $\psi_* \left( \mathcal{O}_{C_3} \left( \sum n_i P_i \right) \right) = \mathcal{O}_{C_3'} \left( \sum n_i \psi(P_i) \right)$.

Since the dual graph of the curve $C_3$ is a tree, one has $\text{Pic}^0 C_3 = \text{Pic}^0 C_3'$ and $\text{Pic} C_3 = \text{Pic}^0 C_3' \oplus \mathbb{Z}^2$.

We also have similar morphisms $\psi_{B_0}, \psi_{B_3}$ and norm maps for the other two reducible curves.
**Definition 5.5.** We define a map \( \phi_{C_3} : \text{Pic} X_0 \rightarrow \text{Pic} C_3' \) as the composition of the restriction to \( C_3 \) and the norm map \( \psi_* : C_3 \rightarrow C_3' \). We also have similar morphisms \( \phi_{B_i}, \phi_{B_0} \) for the other two reducible curves. For the irreducible curves \( A_0, A_3, C_0 \) the corresponding maps are simply the restriction maps on Picard groups.

For the following Lemma, compare Theorem 2.2 above.

**Lemma 5.6.** Consider the map

\[
\phi_0 : \text{Pic} X_0 \rightarrow \mathbb{Z} \oplus \text{Pic} A_0' \oplus \text{Pic} B_0' \oplus \text{Pic} C_0'
\]

defined as \( D \mapsto \deg(\text{DK}_{X_0}) \) in the first component and the maps \( \phi_{A_0}, \phi_{B_0}, \phi_{C_0} \) in the other components. Then the images of the Cartier divisors \( A_i, B_i, C_i, i = 0,1,2,3 \) are exactly the same as for a smooth Burniat surface \( X_i \), \( t \neq 0 \).

**Proof.** Immediate check. \( \square \)

**Definition 5.7.** We will denote this image by \( \text{im} \phi_0 \). One has \( \text{im} \phi_0 \simeq \mathbb{Z} \oplus \mathbb{Z}_2 \). We emphasize that \( \text{im} \phi_0 = \text{im} \phi_t = \text{Pic} X_t \), where \( X_t \) is a smooth Burniat surface.

**Lemma 5.8.** Let \( D \) be an effective Cartier divisor \( D \) on the surface \( X_0 \). Suppose that \( D.A_i < 0 \) for \( i = 0,3 \). Then the Cartier divisor \( D - A_i \) is also effective. (Similarly for \( B_i, C_i \).)

**Proof.** For an irreducible divisor this is immediate, so let us do it for the divisor \( C_3 = C_3' + C_3'' \) which spans two irreducible components, say \( X', X'' \) of the surface \( X_0 = X' \cup X'' \cup X''' \). Let \( D' = D|_{X'}, D'' = D|_{X''}, D''' = D|_{X'''} \). Then

\[
D.C_3 = (D'.C_3')_{X'} + (D''.C_3'')_{X''},
\]

where the right-hand intersections are computed on the smooth irreducible surfaces. One has \( (C_3'')^2_{X''} = 0 \) and \( (C_3'')^2_{X''} = -1 \). Therefore, \( (D'.C_3')_{X'} \geq 0 \). Thus, \( D.C_3 < 0 \) implies that \( (D''.C_3'')_{X''} < 0 \). Then \( C_3'' \) must be in the base locus of the linear system \( |D''| \) on the smooth surface \( X'' \). Let \( n > 0 \) be the multiplicity of \( C_3'' \) in \( D'' \). Then the divisor \( D'' - nC_3'' \) is effective and does not contain \( C_3'' \).

By what we just proved, \( D \) must contain \( nC_3'' \). Thus, it passes through the point \( P = C_3' \cap C_3'' \) and the multiplicity of the curve \( (D')_{X'} \) at \( P \) is \( \geq n \), since \( D \) is a Cartier divisor. Suppose that \( D \) does not contain the curve \( C_3' \). Then \( (D'.C_3')_{X'} \geq n \), and

\[
D.C_3 = (D'.C_3')_{X'} + (D''.C_3'')_{X''} \geq n + (-n) = 0,
\]

which provides a contradiction. We conclude that \( D \) contains \( C_3' \) as well, and so \( D - C_3 \) is effective. \( \square \)

**Lemma 5.9.** Let \( D \) be an effective Cartier divisor \( D \) on the surface \( X_0 \). Suppose that \( D.A_i = 0 \) for \( i = 0,3 \) but \( \phi_{A_1}(D) \neq 0 \) in \( \text{Pic} A_i \). Then the Cartier divisor \( D - A_i \) is also effective. (Similarly for \( B_i, C_i \).)

**Proof.** We use the same notations as in the proof of the previous lemma. Since \( D' \) is effective, one has \( (D'.C_3')_{X'} \geq 0 \).

If \( (D''.C_3'')_{X''} < 0 \) then, as in the above proof one must have \( D'' = nC_3'' \) and \( D' \) intersect \( C_3' \) only at the unique point \( P = C_3' \cap C_3'' \) and \( (D'.C_3')_{X'} = n \). But then \( \phi_{C_3}(D) = 0 \) in \( \text{Pic} C_3 \), a contradiction.

If \( (D''.C_3'')_{X''} = 0 \) but \( D'' - nC_3'' \) is effective for some \( n > 0 \), the same argument gives \( DC_3 > 0 \), so an even easier contradiction.
Finally, assume that \((D', C_3')_{X'} = (D'', C_3'')_{X''} = 0\) and \(D''\) does not contain \(C_3''\). By assumption, we have \(D'C_3' = 0\) but \(D'|C_3' \neq 0\) in \(\text{Pic} C_3'\). This implies that \(D' - C_3'\) is effective and that \(D\) contains the point \(P = C_3' \cap C_3''\). But then \((D'', C_3'')_{X''} > 0\). Contradiction. □

The following lemma is the precise analogue of [AO12, Lemma 5] (Lemma 4.5 in the arXiv version).

**Lemma 5.10.** Let \(F \in \text{Pic} X_0\) be an invertible sheaf such that

\[
\text{im } \phi_0(F) = (3; 1 10, 1 10, 1 10) \in \mathbb{Z} \oplus \text{Pic} A_0 \oplus \text{Pic} B_0 \oplus C_0
\]

Then \(h^0(X_0, F) = 0\).

**Proof.** The proof of [AO12, Lemma 5], used verbatim together with the above Lemmas 5.8, 5.9 works. Crucially, the three “corners” \(A_0 \cap C_3, B_0 \cap A_3, C_0 \cap B_3\) are smooth points on \(X_0\). □

**Proof of Thm. 5.1.** We define the sheaves \(L_1, \ldots, L_6\) by the same linear combinations of the Cartier divisors \(A_i, B_i, C_i\) as in the smooth case [AO12, Rem.2] (Remark 4.4 in the arXiv version), namely:

\[
\begin{align*}
L_1 &= \mathcal{O}_X(A_3 + B_3 + C_0 + A_1 - A_2), \\
L_2 &= \mathcal{O}_X(A_0 + B_3 + C_3 + C_2 - A_1), \\
L_3 &= \mathcal{O}_X(C_2 + A_2 - C_0 - A_3), \\
L_4 &= \mathcal{O}_X(B_2 + C_2 - B_0 - C_3), \\
L_5 &= \mathcal{O}_X(A_2 + B_2 - A_0 - B_3), \\
L_6 &= \mathcal{O}_X.
\end{align*}
\]

By [AO12], for every \(t \neq 0\) they restrict to the invertible sheaves \(L_1, \ldots, L_6 \in \text{im } \phi_t = \text{Pic} X_t\) on a smooth Burniat surface which form an exceptional sequence. By Lemma 5.6, the images of \(L_{i|X_0} \in \text{Pic} X_0\) under the map

\[
\phi_0 : \text{Pic} X_0 \to \text{im } \phi_0 = \text{im } \phi_t = \text{Pic} X_t, \quad t \neq 0.
\]

are also \(L_1, \ldots, L_6\). We claim that \(L_{i|X_0}\) also form an exceptional collection.

Indeed, the proof in [AO12] of the fact that \(L_1, \ldots, L_6\) is an exceptional collection on a smooth Burniat surface \(X_t\) \((t \neq 0)\) consists of showing that for \(i < j\) one has

\[
\begin{align*}
(1) \quad &\chi(L_i \otimes L_j^{-1}) = 0, \\
(2) \quad &h^0(L_i \otimes L_j) = 0, \text{ and} \\
(3) \quad &h^0(K_{X_t} \otimes L_i^{-1} \otimes L_j) = 0.
\end{align*}
\]

The properties (2) and (3) are checked by repeatedly applying (the analogues of) Lemmas 5.8, 5.9, 5.10 until \(D.K_{X_t} < 0\) (in which case \(D\) is obviously not effective).

In our case, one has \(\chi(X_0, L_{i|X_0} \otimes L_j^{-1}|_{X_0}) = \chi(X_t, L_{i|X_t} \otimes L_j^{-1}|_{X_t})^{-1} = 0\) by flatness. Since we proved that Lemmas 5.8, 5.9, 5.10 hold for the surface \(X_0\), and since the Cartier divisor \(K_{X_0}\) is nef, the same exact proof for vanishing of \(h^0\) goes through unchanged. □

**Remark 5.11.** The semiorthogonal complement \(A_t\) of the full triangulated category generated by the sheaves \((L_1, \ldots, L_6)|_{X_t}\) is the quite mysterious “quasiphantom”. A viable way to understand it could be to understand the degenerate quasiphantom \(A_0 = (L_1, \ldots, L_6)|_{X_0}\) on the semistable degeneration \(X_0\) first. The irreducible components of \(X_0\) are three bielliptic surfaces and they are glued nicely. Then one could try to understand \(A_t\) as a deformation of \(A_0\).
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