Tight Bounds for the Price of Anarchy of Simultaneous First Price Auctions

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Abstract

We study the Price of Anarchy of simultaneous First-Price auctions for buyers with submodular and subadditive valuations. The current best upper bounds for the Bayesian Price of Anarchy of these auctions are $e/(e-1)$ [34] and 2 [16], respectively. We provide matching lower bounds for both cases even for the case of the full information and for mixed Nash equilibria. An immediate consequence of our results, is that for both cases, the Price of Anarchy of these auctions stays the same, for mixed, correlated, coarse-correlated, and Bayesian Nash equilibria.

We bring some novel ideas to the theoretical discussion of upper bounding the Price of Anarchy in Bayesian Auctions settings. We suggest an alternative way to bid against price distributions. Using our approach we were able to re-provide the upper bounds of $e/(e-1)$ [34] for XOS bidders. An advantage of our approach, is that it reveals a worst-case price distribution, that is used as a building block for the matching lower bound construction.

Finally, we apply our techniques on Discriminatory Price multi-unit auctions. We complement the results of [13] for the case of subadditive valuations, by providing a matching lower bound of 2. For the case of submodular valuations, we provide a lower bound of 1.109. For the same class of valuations, we were able to reproduce the upper bound of $e/(e-1)$ using our non-smooth approach.

1 Introduction

Combinatorial auctions is a fundamental, well-studied resource allocation problem that involves the interaction of $n$ selfish agents in competition for $m$ indivisible resources/goods. The preferences of each player for different bundles of the items are expressed via a valuation set function. The main challenge is to design a (truthful) mechanism that allocates the items in an efficient way in the equilibrium. That is, to find a partition of the items that maximizes the social welfare, which is the sum of the values of the players. Although it is well-known that this can be achieved optimally by the VCG mechanism [36, 12, 18], unfortunately this might take exponential time in $m$ and $n$ [27, 28].

However, in practice, several simple non-truthful mechanisms have been recruited. The most notable examples are generalized second price (GSP) auctions for internet adwords [14, 35], simultaneous ascending price auctions for wireless spectrum allocation [26], or eBay independent second price auctions. Furthermore, in such auctions, the expressive power of the buyers is heavily restricted by the bidding language and they are not able to represent their complex preferences precisely. In light of the above, Christodoulou, Kovács and Schapira [11] proposed the study of

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simple, non-truthful auctions using the price of anarchy [21] as a measure of inefficiency of those auctions.\footnote{In this setting, the price of anarchy is defined as the worst-case ratio of the optimal social welfare over the social welfare obtained in a (Bayesian) Nash equilibrium.}

**Item bidding.** Of particular, both practical and theoretical, interest are the so-called combinatorial auctions with *item-bidding*. In such an auction, the auctioneer sells each item by running *simultaneously* \( m \) independent single-item auctions. By fixing different types of single-item auctions, two main variants that have been studied are *simultaneous second-price auctions (SPAs)* [11, 3, 16] and *simultaneous first-price auctions (FPAs)* [20, 34, 16]. In both cases, the bidders are asked to submit a bid for each item. Then each item is assigned to the highest bidder. The main difference is that in a SPA a winner is charged an amount equal to the second highest bid while in the latter a winner pays his own bid.

FPAs have been shown to be more efficient than SPAs. For general valuations, Hassidim et al.[20] showed that FPAs are efficient whenever pure equilibria exist, but can be highly inefficient for mixed, and Bayesian Nash equilibria, for settings with complementarities. For two important classes of valuation functions, namely *fractionally subadditive* and *subadditive*\footnote{Fractionally subadditive valuations, are also known as XOS valuations. For definitions of those valuation functions we refer the reader to the technical sections.}, and for mixed and Bayesian Nash equilibria, [20, 34] and [16] showed that FPAs have lower, constant, price of anarchy comparing with the respective bounds obtained in SPAs [11, 3, 16]. The current best upper bounds, applying different techniques, show that for XOS valuations the price of anarchy is at most \( e/(e-1) \) \cite{34}, while for subadditive valuations it is at most 2 [16].

**Upper Bound techniques and the Smoothness Framework.** The *smoothness framework* \cite{31} has been proven a quite successful technique, providing tight bounds for the price of anarchy of selfish routing games \cite{9, 10, 5, 31, 2, 1}. In a routing game\footnote{We refer the reader to \cite{29}, for a detailed discussion of the subject. Here our main objective is to give a brief definition of “smoothness” for routing games, and illustrate the challenges that arise in applying the framework to the combinatorial auctions setting.}, a pure strategy \( s_i \) for a player \( i \) is a path that connects a given source to a given destination, each player cares about minimizing their own latency, while the system’s objective is to minimize total time\footnote{In a nutshell, the smoothness framework works as follows for routing games: One compares any Nash equilibrium strategy profile \( s \), with the optimal pure strategy profile \( s^* \). For every player, one considers the pure deviation that corresponds to their optimal strategies, using the Nash equilibrium inequality to provide a bound on the cost of this deviation \( c_i(s) \leq c_i(s_i^*, s_{-i}) \), where \( c_i \) is the cost function and \( s_{-i} \) is the vector of strategies of all players except \( i \). Having summed up these inequalities, one can show a bound on the quantity \( \sum_i c_i(s_i^*, s_{-i}) \) with respect to the optimal social cost and the cost in the Nash equilibrium.}. The importance of the smoothness framework for routing games stems from the fact, that once an upper bound is shown for pure Nash equilibria, the bound holds for more general equilibrium concepts like mixed, correlated and coarse correlated equilibria. More importantly, in such games such an upper bound is normally coupled by a matching lower bound.

There have been attempts to define analogous frameworks for auction settings and for games with incomplete information \cite{32, 33}. One main nuisance in auctions with complete or incomplete information, is that *strategy profiles* are different than *allocation profiles* as opposed to routing games where these notions coincide. Therefore, a deviation to the optimal set is not defined in a straightforward way. Importantly, a “smooth” deviation to the optimal set needs to be independent of the distribution of the prices (caused by the bids of other players), otherwise the bound for pure equilibria does not extend for more general classes of equilibria.
Using the smoothness framework and pure deviations, Christodoulou, Kovács and Schapira [11], and Bhawalkar and Roughgarden [3] showed upper bounds of 2 and $O(\log n)$, for XOS and subadditive valuations, respectively for SPAs$^5$, while Hassidim et al. [20] showed the BPoA is 4 and $O(\log n)$ for FPAs. Syrgkanis and Tardos [34] used smoothness but applied mixed deviations, and showed an improved upper bound of $e/(e-1)$ for FPAs, for XOS bidders.

Interestingly, in contrast to the case of Routing Games, the PoA of FPAs, that is the case of interest of this work, the PoA does not attain the same value for all classes of equilibria$^6$. Therefore smoothness, cannot provide the tight answer for such games$^7$. Moreover, for subadditive valuations, defining a smooth deviation didn’t succeed in providing constant upper bounds. Feldman et al. [16], came up with an elegant distribution-dependent deviation that is not smooth. They suggested a deviating bid that follows the price distribution, and they were able to show constant upper bounds on the Bayesian PoA of 2 and 4, for FPAs and SPAs, respectively.

**Our Contribution.** Following the work of [20, 16, 34], we study the price of anarchy of FPAs for games with (in)complete information. Our main concern is the development of tools that provide tight bounds for the price of anarchy of these auctions. Our results complement the current knowledge for simultaneous first-price auctions. We provide matching lower bounds to the results by Syrgkanis and Tardos [34] and those by Feldman et al. [16], showing that even for the case of full information and mixed Nash equilibria the PoA is at least $e/(e-1)$ for submodular valuations (and therefore for XOS) and 2 for subadditive valuations.

An immediate consequence of our results, is that the price of anarchy of these auctions stays the same, for mixed, correlated, coarse-correlated, and Bayesian Nash equilibria. Only for pure Nash equilibria it is equal to 1. Our findings suggest that smoothness may provide tight results for certain classes of auctions, using as a base class the class of mixed Nash equilibria, and not that of pure. This is in contrast to what is known for routing games, where the respective base class was that of pure equilibria.

We bring some novel ideas to the theoretical discussion of upper bounding the Price of Anarchy in Bayesian Auctions settings. Two main techniques have been applied; smoothness, and the very recent technique by Feldman et al [16]. We suggest an alternative non-smooth way to bid against price distributions. Using our approach we were able to re-provide the upper bound of $e/(e-1)$ [34] for XOS bidders. An advantage of our approach, is that it reveals the worst-case price distribution, that is used as a building block of the matching lower bound construction. It is more closely related to the technique of Feldman et al. [16], as the deviation we prescribe depends on the price distributions, and in fact leads to tight results also when the distribution is pure. In fact, we carefully select a pure strategy that optimizes a certain function (see discussion at the end of section 3.1). We illustrate the main idea for a single item, and then we show how this extends for XOS valuations.

For buyers with additive valuations, we show that mixed Nash equilibria are efficient, whenever they exist. This suggests an interesting separation on the price of anarchy between the full and the incomplete information cases, as it was already known that this is not the case for Bayesian Nash equilibria [22].

Finally, we apply our techniques on discriminatory price auctions [22]. We complement the results by de Keijzer et al. [13] for the case of subadditive valuations, by providing a matching lower bound of 2, for the standard bidding format. For the case of submodular valuations, we

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$^5$In SPAs, an additional no-overbidding assumption must be used otherwise the PoA can be arbitrarily high. This assumption is not needed for FPAs.

$^6$Pure equilibria are always efficient, but the BPoA is greater than one even for a single item.

$^7$It was also noted in [3, 32] that smoothness does not give tight results for SPAs.
were able to provide a lower bound of 1.109. We were also able to reproduce their upper bound of \(e/(e - 1)\) for submodular bids, using our non-smooth approach. Note that the previous lower bound for such auctions was 1.0004 [13] for Bayesian Nash equilibria. Both our lower bounds hold for the case of mixed Nash equilibria.

**Related Work.** A long line of research aims to design simple auctions with good performance guarantee (see e.g. [19, 8]). The (in)efficiency of first-price price auctions has been studied in economics (cf. [22]) starting from the seminal work by Vickrey [36]. Existence of efficient equilibria of simultaneous sealed bid auctions in full information settings was first studied in [4].

Christodoulou, Kovács and Schapira [11] extended the concept of PoA to the Bayesian setting and applied it to item-bidding auctions. Hassidim et al. [20] showed that, in FPAs, pure Nash equilibria are always efficient for general valuations, whereas, for SPAs, Fu, Kleinberg and Lavi [17] proved that the PoA is at most 2. For Bayesian Nash equilibria, Syrgkanis and Tardos [34] and Feldman et al. [16] showed improved upper bounds on the PoA and BPoA for simultaneous FPAs. Syrgkanis and Tardos came up with a general composability framework of smooth mechanisms, that shown to be quite powerful, as led to upper bounds for several settings.

Few lower-bound results are known for the PoA of simultaneous auctions. Hassidim et al [20] presented an example with PoA= \(\Omega(\sqrt{m})\) for FPA with superadditive valuations. As suggested in [16], similar lower bound can be derived for SPAs, as well. Under the non-overbidding assumption, Bhawalkar and Roughgarden [3] gave a lower bound of 2.013 for SPAs with subadditive bidders and \(\Omega(n^{1/4})\) for correlated bidders. In [16], they show similar results under the weak non-overbidding assumption.

Markakis and Telelis [25] studied uniform price *multi-unit* auctions. De Keijzer et al. [13] bounded the BPoA for several formats of multi-unit auctions with first or second pricing rules. Auctions employing greedy algorithms were studied by Lucier and Borodin [23]. A number of works [30, 6, 32] studied the PoA of generalized SPAs in the full information and Bayesian settings and even with correlated bidders [24]. Chawla and Hartline [7] proved that for the generalized FPAs with symmetric bidders, the pure Bayesian Nash equilibria are unique and always efficient.

## 2 Preliminaries

### 2.1 Simultaneous first-price auctions

*Simultaneous first-price auctions* constitute a simple type of combinatorial auctions. In a combinatorial auction with \(n\) players and \(m\) items, every player (or *bidder*) \(i \in [n]\) has a valuation for each subset of items, given by a valuation function \(v_i : 2^{[m]} \to \mathbb{R}_{\geq 0}\), where \(v_i \in V_i\) for some possible set of valuations \(V_i\). A valuation profile for all players is \(v \in \times_i V_i\). The \(v_i\) functions are monotone and normalized, that is, \(S \subseteq T \Rightarrow v_i(S) \leq v_i(T)\), and \(v_i(\emptyset) = 0\).

In the *Bayesian* setting, the valuation of each player \(i\) is drawn from \(V_i\) according to some known distribution \(D_i\). We assume that the \(D_i\) are independent (and possibly different) over the players. In the *full information* setting the valuation \(v_i\) is fixed and known by all other players for all \(i \in [n]\). Note that the latter is a special Bayesian combinatorial auction, in which player \(i\) has valuation \(v_i\) with probability 1.

In a *simultaneous (or item bidding)* auction every player \(i \in [n]\) submits a non-negative bid \(b_i(j)\) for each item \(j \in [m]\). The items are then allocated by independent auctions: for each \(j \in [m]\), the bidder \(i\) with the highest bid \(b_i(j)\) receives the item. We consider the case when the payment for each item is the *first price* payment: a player pays his own bid (the highest bid) for every item.
he receives. Our (upper bound) results hold for arbitrary randomized tie-breaking rules, as long as, for any fixed \( b = (b_1, \ldots, b_n) \), the probabilities for the players to get the item are fixed.

The allocation \( X = (X_1, \ldots, X_n) \) determined this way is thus a partition (allowing empty sets \( X_i \)) of the items. The most common global objective in combinatorial auctions is to maximize the sum of the valuations of the players for their received sets of items, i.e., to maximize the social welfare \( SW(X) \) of the allocation, where \( SW(X) = \sum_{i \in [n]} v_i(X_i) \). Therefore, for an optimal allocation \( O(v) = O = (O_1, \ldots, O_n) \) the value \( SW(O) \) is maximum among all possible allocations. Switching to more precise game theoretic definitions, we define a pure (bidding) strategy \( b_i \) for player \( i \) to be a vector of bids for the \( m \) items \( b_i = (b_i(1), \ldots, b_i(m)) \). As usual, \( b_{-i} \) denotes the strategies of all players except for \( i \). The pure strategy profile of all bidders is then \( b = (b_1, \ldots, b_n) \).

For any set \( S \) of items, we use the notation \( b_i(S) = \sum_{j \in S} b_i(j) \) to denote the sum of the bids of player \( i \) for \( S \). A mixed strategy \( B_i \) of player \( i \) is a probability distribution over pure strategies. \( B = (B_1, \ldots, B_n) \) is a profile of mixed strategies.

Assume that the players submitted bids for the items according to \( b_i \) and the simultaneous first-price auction yields the allocation \( X(b) \). The social welfare of \( b \) is \( SW(b) = SW(X(b)) \).

The utility \( u_i \) of player \( i \) is defined as his valuation for the received set, minus his payments: \( u_i(b) = v_i(X_i(b)) - b_i(X_i(b)) \)

**Types of valuations.** Our results concern different classes of valuation functions, which we define next, in increasing order of inclusion. Let \( v : 2^{[n]} \rightarrow \mathbb{R}_{\geq 0} \) be a valuation function, or (non-negative) set function in general. Then \( v \) is called

- additive, if \( v(S) = \sum_{j \in S} v(j) \);
- submodular, if \( v(S \cup T) + v(S \cap T) \leq v(S) + v(T) \);
- fractionally subadditive or XOS, if \( v \) is determined by a (finite) set of additive valuations \( f_\gamma \) for \( \gamma \in \Gamma \), so that \( v(S) = \max_{\gamma \in \Gamma} f_\gamma(S) \);
- subadditive, if \( v(S \cup T) \leq v(S) + v(T) \);

where the given (in)equalities must hold for arbitrary subsets \( S \) and \( T \) of the items. It is well-known that each one of the above classes is strictly contained in the next class, e.g., an additive set function is always submodular but not vice versa, a submodular is always XOS, etc. [15]. As an equivalent definition, submodular valuations are exactly the valuations with *decreasing marginal bids*, meaning that \( v(\{j \} \cup S) - v(T) \leq v(\{j \} \cup S) - v(S) \) holds for any item \( j \) and any \( S \subseteq T \).

### 2.2 Nash equilibria, and the price of anarchy.

We review five standard equilibrium concepts studied in this paper: pure, mixed, correlated, coarse correlated and Bayesian Nash equilibria. The first four of them is for the *full information* setting and the last one is defined in the *Bayesian* setting. Let \( v = (v_1, \ldots, v_n) \) be the players valuation functions. In the Bayesian setting, \( v_i \) is drawn from \( V_i \) according to some known distribution. Recall that \( B = (B_1, \ldots, B_n) \) denotes the bidding strategies of the players. Then, \( B \) is called a

- pure Nash equilibrium, if \( B_i \) is a pure strategy \( b_i \) and \( u_i(b) \geq u_i(b_i', b_{-i}) \).
- mixed Nash equilibrium, if \( B_i = x_i B_i \) and \( E_{b_i \sim B_i} [u_i(b_i)] \geq E_{b_i \sim B_i} [u_i(b_i', b_{-i})] \).
- correlated Nash equilibrium, if \( E_{b_i \sim B_i} [u_i(b_i)[b_i]] \geq E_{b_i \sim B_i} [u_i(b_i', b_{-i})[b_i]] \).
- coarse correlated Nash equilibrium, if \( E_{b_i \sim B_i} [u_i(b_i)] \geq E_{b_i \sim B_i} [u_i(b_i', b_{-i})] \).
- Bayesian Nash equilibrium, if \( B_i(v) = x_i B_i(v_i) \) and \( E_{v_i \sim B_i(v_i)} [u_i(b_i)] \geq E_{v_i \sim B_i(v_i)} [u_i(b_i', b_{-i})] \)

where the given inequalities hold for all players \( i \) and (pure) deviating bids \( b_i' \). It is well-known that each one of the first four classes is contained in the next class, i.e., pure \( \subseteq \) mixed \( \subseteq \) correlated
where $P_i$ denotes the payment of $i$ under bidding profile $b$.\]
3 Upper bounds

3.1 Single Item Auction

Here we discuss a new approach to obtain bounds for the PoA in simultaneous first price, and discriminatory auctions. While the upper bounds that we derive with the help of this idea, can also be obtained based on the very general smoothness framework [32, 34, 13], the approach we introduce here does not adhere to this framework. The strength of our approach consists in its potential to lead to better (in some cases tight) lower bounds, as we demonstrate in the subsequent sections.

In order to keep the presentation pure, and to focus on the main ingredients, in this section we illustrate this approach on a single item auction with full information. For this case we obtain the bound of $\frac{e}{e-1}$. We show that the same idea can be used for obtaining the same bound for discriminatory auctions, and various generalizations of simultaneous auctions with $m$ items: XOS valuations in Bayesian NE; and XOS valuations in coarse correlated equilibria. It is also trivial (so the proof is omitted) to extend our approach of single item to arbitrary valuations (with coarse correlated equilibria) when the players receive at most one item in the optimal allocation.

**Theorem 3.1.** The PoA of mixed Nash equilibria in first-price single-item auctions is at most $\frac{e}{e-1}$.

**Proof.** Let $v = (v_1, \ldots, v_n)$ be the valuations of the players, and suppose that $v_i = \max_{k \in [n]} v_k$. We consider a mixed Nash equilibrium $B = (B_1, B_2, \ldots, B_n)$. Let $p_i$ denote the highest bid in $b_{-i}$, and $F(x) = F_i(x)$ be the cumulative distribution function (CDF) of $p_i$, that is, $F(x) = P_{b_{-i}}[p_i \leq x]$. The following lemma prepares the ground for the selection of an appropriate deviating bid.

**Lemma 3.2.** For any pure strategy $a$ of player $i$, $E_{b \sim B}[u_i(b)] \geq F(a)(v_i - a)$.

**Proof.** If $F$ is continuous in $a$, then $F(a) = P[p_i \leq a] = P[p_i < a]$, tie-breaking in $a$ does not matter, and $F(a)$ equals also the probability that bidder $i$ gets the item if he bids $a$. Therefore, $F(a)(v_i - a) = E_{b_{-i} \sim B_{-i}}[u_i(a, b_{-i})] \leq E_{b \sim B}[u_i(b)]$, since $B$ is a NE. If $F$ is not continuous in $a$ ($P[p_i = a] > 0$), then, as a CDF, it is at least right-continuous. By the previous argument $E[u_i(b)] \geq F(a + \epsilon)(v_i - a - \epsilon)$ holds for every $x = a + \epsilon$ where $F$ is continuous, and the lemma follows by taking $\epsilon \to 0$. \hfill \Box

Since in a Nash equilibrium the expected utility of every player is non-negative, it also holds that $\sum_{k=1}^{n} E[u_k(b)] \geq F(a)(v_i - a)$. Analogues of the following calculation are standard in the simultaneous auctions literature (the expectations are with regard to $b \sim B$): $E[u_k(b)] = E[u_k(X_k(b)) - b_k(X_k(b))]$. Note that $b_k(X_k(b)) = b_k$ whenever $b_k$ is a winning bid, and $b_k(X_k(b)) = 0$ otherwise. Let $b_{\max}$ be the maximum bid in a given bidding profile $b$. By summing up over all players, and combining with Lemma 3.2, we obtain

$$E[SW(b)] = E\left[ \sum_{k=1}^{n} v_k(X_k(b)) \right] \geq F(a)(v_i - a) + E[b_{\max}] \geq F(a)(v_i - a) + E[p_i], \quad (1)$$

for any (deviating) bid $a$. We choose the bid $a^*$ that maximizes the right hand side of (1), i.e. $a^* = \arg \max_a F(a)(v - a)$ (see Figure 1 for an illustration). Then, we need to find the maximum value of $\lambda$, such that,

*Note that this bound is not tight, see Section 5.*
Next we argue that $a^*$ leads to tight bounds against both pure and mixed strategies. The following lemma serves the latter purpose\textsuperscript{10}.

**Lemma 3.3.** For any non-negative random variable $p$ drawn from $F$, and any fixed number $v$, it is $F(a^*)(v-a^*) + \mathbb{E}[p] \geq (1 - \frac{1}{e}) v$.

**Proof.** Set $A = F(a^*)(v-a^*)$. Then

$$F(a^*)(v-a^*) + \mathbb{E}[p] \geq A + \int_{0}^{v-A} (1 - F(x)) dx$$

$$= v - \int_{0}^{v-A} F(x) dx \geq v - \int_{0}^{v-A} \frac{A}{v-x} dx$$

$$= v + A \ln \left( \frac{A}{v} \right) \geq v + \frac{v}{e} \ln \left( \frac{1}{e} \right) = \left( 1 - \frac{1}{e} \right) v,$$

where the last inequality is due to the fact that $A \ln \left( \frac{A}{v} \right)$ is minimized for $A = \frac{v}{e}$.

The theorem follows by applying Lemma 3.3 to (1). \hfill \Box

### Worst-case price distribution.

The CDF that makes all the inequalities of (the proof of) Lemma 3.3 tight (see Figure 2), is

$$\hat{F}(x) = \begin{cases} \frac{v}{e(v-x)} & , \text{for } x \leq \left( 1 - \frac{1}{e} \right) v \\ 1 & , \text{for } x > \left( 1 - \frac{1}{e} \right) v \end{cases}$$

Observe that for $x \leq \left( 1 - \frac{1}{e} \right) v$, $\hat{F}(x)(v-x) = \frac{v}{e}$ and for $x > \left( 1 - \frac{1}{e} \right) v$, $\hat{F}(x)(v-x) = v - x < v - (1 - \frac{1}{e})v = \frac{v}{e}$. So, the bid that maximizes the quantity $\hat{F}(a)(v-a)$ is any value $a \in [0, \left( 1 - \frac{1}{e} \right) v]$. The given distribution for $p_i$ makes inequality (2) tight. In order to construct a (tight) lower bound for the PoA, we also need to tighten the inequalities in (1). Note that the inequality of Lemma 3.2 is tight for all $a \in [0, \left( 1 - \frac{1}{e} \right) v]$. Intuitively, we need to construct a Nash equilibrium, where $F(p_i)$ is equal to $\hat{F}(x)$ and $b_i$ doesn’t exceed $p_i$. We present the construction in Section 4.1.

\textsuperscript{10}Note that if $F$ is pure, then it easy to verify that the $\lambda$ that corresponds to $a^*$ is equal to 1, leading to a PoA=1.
Figure 2: The CDF $\hat{F}(x)$ makes all the inequalities of Lemma 3.3 tight, i.e. for every $x \in [0, (1 - \frac{1}{e}) v]$, $F(x)(v - x) = A = \frac{v}{e}$.

**Remark 1.** Here we discuss our technique and the smoothness technique that achieves the same upper bound [34]. In [34], a particular mixed bidding strategy $A_0$ was defined for each player $i$, such that for every price $p = \max_{i' \neq i} b_{i'}$, $E_{A_0}[u_i(A_0, p)] + p \geq v(1 - 1/e)$. If we denote $g(A, F) = E_{A,F}[u_i(a, p) + p]$, it can be implied that $\max_A \min_f g(A, p) \geq v(1 - 1/e)$. In Lemma 3.3 we show that $\min_f \max_a g(a, F) = v(1 - 1/e)$.

3.2 Upper bound for coarse correlated equilibria with XOS valuations

In this section, we generalize the proof of section 3.1 to prove that for all XOS (fractionally subadditive) valuation functions, the PoA is at most $\frac{e}{e-1}$ even in coarse correlated equilibria.

**Theorem 3.4.** The PoA for simultaneous first-price auctions with XOS valuations for coarse correlated equilibria is at most $\frac{e}{e-1} \approx 1.58$.

**Lemma 3.5.** Let $S$ be a set of items, and $f_i$ be a maximizing additive function of $S$ for player $i$ with XOS valuation function $v_i$. Then for any strategy profile $b$, $u_i(b) \geq \sum_{j \in S} \mathbb{P}[j \in X_i(b)](f_i(j) - b_i(j))$.

**Proof.** By the definition of XOS valuations, we have $v_i(T) \geq f_i(T)$, for every $T \subseteq S$.

$$u_i(b) \geq \sum_{T \subseteq S} \mathbb{P}[X_i(b) = T](f_i(T) - b_i(T))$$

$$= \sum_{T \subseteq S} \sum_{j \in T} \mathbb{P}[X_i(b) = T] (f_i(j) - b_i(j))$$

$$= \sum_{j \in S} \sum_{T \subseteq S, j \in T} \mathbb{P}[X_i(b) = T] (f_i(j) - b_i(j))$$

$$= \sum_{j \in S} \mathbb{P}[j \in X_i(b)](f_i(j) - b_i(j))$$
Let $p_{ij}$ be the random variable indicating the highest bid on item $j$ among other players than $i$ and $F_{ij}(x) = \Pr[p_{ij} \leq x]$ be the CDF of $p_{ij}$.

**Lemma 3.6.** For any pure strategy $b'_i$ of player $i$ and any set of items $S$,

$$\mathbb{E}_{b \sim B}(SW(b)) = \sum_{i=1}^{n} \left( \mathbb{E}_{b \sim B}(u_i(b)) + \mathbb{E}_{b \sim B}(b_i(X_i(b))) \right)$$

$$\geq \sum_{i=1}^{n} \left( \sum_{j \in O_i} F_{ij}(a_i(j))(f_i(j) - a_i(j)) + \sum_{j \in O_i} \mathbb{E}[p_{ij}] \right)$$

$$\geq \sum_{i=1}^{n} \sum_{j \in O_i} \left( 1 - \frac{1}{e} \right) f_i(j) = \left( 1 - \frac{1}{e} \right) \sum_{i=1}^{n} v_i(O_i) = \left( 1 - \frac{1}{e} \right) SW(O).$$

For inequality (3), the second term is derived based on inequality

$$\sum_{i=1}^{n} b_i(X_i(b)) = \sum_{i=1}^{n} \max_k b_k(O_i) \geq \sum_{i=1}^{n} \max_{k \neq i} b_k(O_i).$$

So, the PoA is at most $\frac{e}{e-1} \approx 1.58$.

### 3.3 Upper bound for Bayesian equilibria with XOS valuations

For any Bayesian bidding strategy $B$, player $i$ and his valuation $v_i \sim D_i$, we define $p_{ij}(v_i)$ to be the random variable indicating the maximum bid on item $j$ of players other than $i$, i.e., $p_{ij}(v_i) = \max_{k \neq i} b_k(j)$ where $b_{-i} \sim B_{-i}(v_{-i})$ and $v_{-i} \sim D_{-i}$. Let $F_{ij}^{v_i}(x)$ be the CDF of $p_{ij}(v_i)$. Given any valuation $v$, let $f_i^S$ be the maximizing additive function for player $i$ on set $S$. Similarly to Lemmas 3.2 and 3.6, we can prove the following.

**Lemma 3.7.** Let $S$ be any set of items and $B$ be some Bayesian Nash equilibrium $B$. For any player $i$ with valuation $v_i$, let $a_i$ be any pure strategy of player $i$. Then,

$$\mathbb{E}_{b \sim B(v)}(u_i(b)) \geq \sum_{j \in S} F_{ij}^{v_i}(a_i(j))(f_i^S(j) - a_i(j)).$$

**Theorem 3.8.** The BPoA of simultaneous first price auctions is at most $\frac{e}{e-1}$.
Proof. Suppose \( B \) is a Bayesian Nash equilibrium.

\[
\sum_{i=1}^{n} \mathbb{E}_{b \sim B(v)} \left[ u_i^n(b) \right] = \sum_{i=1}^{n} \mathbb{E}_{v, w} \left[ u_i^n(b) \right] \\
\geq \sum_{i=1}^{n} \sum_{j \in O_i(v, w_{-i})} F_{ij}^{v_i}(a_i(j)) \left( f_{v_i}(v_{-i}, w_{-i})(j) - a_i(j) \right) \\
\geq \sum_{i=1}^{n} \sum_{j \in O_i(w)} F_{ij}^{w_i}(a_i(j)) \left( f_{v_i}(v_{-i})(j) - a_i(j) \right) \\
\geq \sum_{i=1}^{n} \sum_{j \in O_i(w)} \left( 1 - 1/e \right) f_{v_i}(v_{-i})(j) - \mathbb{E}_{b \sim B(v)} \left[ p_{ij}(w) \right] 
\]

The first inequality is due to Lemma 3.7 for \( S = O_i(v, w_{-i}) \), and the last inequality follows by Lemma 3.3. So, we get the following inequality by replacing \( w \) with \( v \).

\[
\sum_{i=1}^{n} \mathbb{E}_{b \sim B(v)} \left[ u_i(b) + \sum_{j \in O_i(v)} p_{ij}(v_i) \right] \geq (1 - 1/e) \cdot \mathbb{E}_{v} \left[ SW(O(v)) \right] 
\]

Furthermore,

\[
\mathbb{E}_{b \sim B(v)} \left[ SW(b) \right] = \sum_{i=1}^{n} \mathbb{E}_{b \sim B(v)} \left[ u_i(b) \right] + \mathbb{E}_{v} \left[ \sum_{i=1}^{n} b_i(X_i(b)) \right] \\
\geq \sum_{i=1}^{n} \mathbb{E}_{b \sim B(v)} \left[ u_i(b) + \sum_{j \in O_i(v)} p_{ij}(v_i) \right] \\
\geq (1 - 1/e) \cdot \mathbb{E}_{v} \left[ SW(O(v)) \right] 
\]

The first inequality is derived based on inequality (4). So, the BPoA is at most \( \frac{1}{1 - \frac{1}{e}} \approx 1.58 \). 

4 Lower bounds

4.1 Submodular valuations

Here we present a lower bound of \( \frac{e}{e-1} \) for the price of anarchy in simultaneous first price auctions with submodular valuations. This implies a lower bound for fractionally subadditive valuations. In the appendix A we show a class of instances with inefficient mixed equilibria that in the limit converge to the construction presented in this section.

**Theorem 4.1.** The price of anarchy of simultaneous first price auctions with full information and submodular valuations is at least \( \frac{e}{e-1} \approx 1.58 \).

**Proof.** We construct an instance with \( n + 1 \) players and \( n^n \) items (in the Appendix A we present a more generalized instance with \( n^4 \) items). Let \( [n] \) denote the set of integers \( \{1, \ldots, n\} \). We define the set of items as \( M = [n]^n \), that is, they correspond to all the different vectors \( w = (w_1, w_2, \ldots, w_n) \).
with \( w_i \in [n] \). Intuitively, they can be thought of as the nodes of an \( n \) dimensional grid, with coordinates in \([n]\) in each dimension.

We associate each real player \( i \in [n] \) with one of the dimensions (directions) of the grid. In particular, for any fixed player \( i \), his valuation for a subset of items \( S \subseteq M \) is the size (number of elements) in the \( n-1 \)-dimensional projection of \( S \) in direction \( i \). Formally,

\[
v_i(S) = |\{ w_{-i} | \exists w_i \text{ s.t. } (w_i, w_{-i}) \in S \}|.
\]

It is straightforward to check that \( v_i \) has decreasing marginal utilities, and is therefore submodular.

We also introduce a dummy player 0 having valuation 0 for any subset of items.

Given these valuations, we describe a mixed Nash equilibrium \( B = (B_1, \ldots, B_n) \) having a PoA arbitrarily close to \( e/(e-1) \), for large enough \( n \). The dummy player bids 0 for every item, and receives the item if all of the real players bid 0 for it. The utility and welfare of the dummy player is always 0. For real players the mixed strategy \( B_i \) is the following. Every player \( i \) picks a number \( \ell \in [n] \) uniformly at random, and an \( x \) according to the distribution with CDF

\[
G(x) = (n-1) \left( \frac{1}{(1-x)^{\frac{1}{n-1}}} - 1 \right),
\]

where \( x \in \left[0, 1 - \left(\frac{n-1}{n}\right)^{n-1}\right] \). Subsequently, he bids \( x \) for every item \( w = (\ell, w_{-i}) \), with \( w_i = \ell \) as \( i \)th coordinate, and bids 0 for the rest of the items, see Figure 3 for the cases of \( n = d = 2 \) and \( n = d = 3 \). That is, in any \( b_i \) in the support of \( B_i \), the player bids a positive \( x \) only for a \( n-1 \)-dimensional slice of the items. Observe that \( G(\cdot) \) has no mass points, so tie-breaking matters only in case of 0 valuations for an item, in which case player 0 gets the item.

Let \( F(x) \) denote the probability that bidder \( i \) gets a fixed item \( j \), given that he bids \( b_i(j) = x \) for this item, and the bids in \( b_{-i} \) are drawn from \( B_{-i} \) (due to symmetry, this probability is the same for all items \( w = (\ell, w_{-i}) \)). For every other player \( k \), the probability that he bids 0 for item \( j \) is \( (n-1)/n \), and the probability that \( j \) is in his selected slice but he bids lower than \( x \) is \( G(x)/n \). Multiplying over the \( n-1 \) other players, we obtain

\[
F(x) = \left( \frac{G(x)}{n} + \frac{n-1}{n} \right)^{n-1} = \frac{(1 - \frac{1}{n})^{n-1}}{1-x}.
\]
Notice that $v_i$ is an additive valuation restricted to the slice of items that player $i$ bids for in a particular $b_i$. Therefore the expected utility of $i$ when he bids $x$ in $b_i$ is $F(x)(1-x)$ for one of these items, and comprising all items $E[v_i(b_i)]=n^{-1}F(x)(1-x)=n^{-1}(1-1/n)^{n-1}$.

Next we show that $B$ is a Nash equilibrium. In particular, the bids $b_i$ in the support of $B_i$ maximize the expected utility of a fixed player $i$.

First, we fix an arbitrary $w_{-i}$, and focus on the set of items $C:=\{(\ell,w_{-i})|\ell \in [n]\}$, which we call a column for player $i$. Recall that $i$ is interested in getting only one item within $C$, on the other hand his valuation is additive over items from different columns. Moreover, in a fixed $b_{-i}$, every other player $k$ submits the same bid for all items in $C$, because either the whole $C$ is in the current slice of $k$, and he bids the same value $x$, or no item from the column is in the slice and he bids 0. Consider first a deviating bid, in which $i$ bids a positive value for more than one items in $C$, say (at least) the values $x \geq x' > 0$ where $x$ is his highest bid in $C$. Then his expected utility for this column is strictly less than $F(x)(1-x)$, because his value is $F(x)\cdot 1$, but he might have to pay $x+x'$, in case he gets both items. Consequently, bidding $x$ for only one item in $C$ and 0 for the rest of $C$ is more profitable.

Second, observe that restricted to a fixed column, submitting any bid $x \in [0,1-(n-1)^{n-1}]$ for one arbitrary item results in the constant expected utility of $(1-1/n)^{n-1}$, whereas a bid higher than $1-1/n)$ guarantees the item but pays more so the utility becomes strictly less than $(1-1/n)^{n-1}$ for this column. In summary, bidding for exactly one item from each column, an arbitrary (possibly different) bid $x \in [0,1-(n-1)^{n-1}]$ is a best response for $i$ yielding the above expected utility, which concludes the proof that $B$ is a Nash equilibrium.

It remains to calculate the expected social welfare of $B$, and the optimal social welfare. We define a random variable w.r.t. the distribution $B$. Let $Z_j = 1$ if one of the real players 1, $n$ gets item $j$, and $Z_j = 0$ if player 0 gets the item. The expected social welfare is

$$E_{b \sim B}[SW(b)] = \sum_j E[Z_j] \cdot 1 = n^n (1 - Pr(\text{no real player bids for } j)) = n^n \left(1 - \left(1 - \frac{1}{n}\right)^n\right).$$

Finally, we show that the optimum social welfare is $n^n$. An optimal allocation can be constructed as follows: For each item $(w_1, w_2, \ldots, w_n)$ compute $r = (\sum_{i=1}^n w_i \mod n)$. Allocate this item to the player $r+1$. It is easy to see that this way each player is allocated $n^{n-1}$ items; Figure 3(c) shows the optimum allocation for $n=3$. We claim that any two items allocated to the same player differ in at least two coordinates. This proves that they belong to different columns of this player, and all contribute 1 to the valuation of the player, which is therefore $n^{n-1}$. Since this valuation is maximum possible for every player, the obtained social welfare of $n^n$ is optimal.

We prove the claim next. Assume for the sake of contradiction that there is a player who has been allocated two items that differ in exactly one coordinate. Let them be $(w_1, \ldots, w_{j-1}, w_j, w_{j+1}, \ldots, w_n)$ and $(w_1, \ldots, w_{j-1}, w'_j, w_{j+1}, \ldots, w_n)$. Since both items have been allocated to the same player, the following should hold: $\left(\sum_{i=1}^n w_i \mod n\right) = \left(\sum_{i=1, i \neq j}^n w_i + w'_j \mod n\right)$ and so, $w_j \mod n = \left(w'_j \mod n\right)$. But since $1 \leq w_j, w'_j \leq n$, we can derive that $w_j = w'_j$, a contradiction.

Thus, the Price of Anarchy is $\frac{1}{1-(1-\frac{1}{n})^{n-1}}$, and for large $n$ it converges to $\frac{1}{1-\frac{2}{e}} \approx 1.58$. □

### 4.2 Subadditive valuations

**Theorem 4.2.** The price of anarchy of simultaneous first price of mixed Nash equilibria and subadditive valuations is at least 2.
Proof. Consider two players and $m$ items with the following valuations: player 1 is a unit-demand player with valuation $v < 1$ if she gets at least one item; player 2 has valuation 1 for getting less than $m$ items (but at least one) and 2 if she gets all the items. Inspired by [20], we use the following distribution functions:

$$G(x) = \frac{(m-1)x}{1-x} \quad x \in (0, 1/m]; \quad F(y) = \frac{v-1/m}{v-y} y \in (0, 1/m].$$

Player 1 picks one of the $m$ items uniformly at random, and bids $x$ for this item and 0 for all other items. Player 2 bids $y$ for each of the $m$ items. $x$ and $y$ are drawn from $F(x)$ and $G(y)$, respectively. In case of a tie, the item is always allocated to player 2.

Let $B$ denote this mixed bidding profile. We are going to prove that $B$ is a mixed Nash equilibrium for every $v > 1/m$.

If player 1 bids any $x$ in the range $(0, 1/m]$ for the one item, she gets the item with probability $F(x)$, since a tie appears with zero probability. Her expected utility is $F(x)(v-x) = v-1/m$. So, for every $x \in (0, 1/m]$ her utility is $v-1/m$. If player 1 picks $x$ according to $G(x)$, her utility is still $v-1/m$, since she bids 0 with zero probability. Bidding something greater than $1/m$ results in a utility less than $v-1/m$. Regarding player 1, it remains to show that her utility while bidding only for one item is at least her utility while bidding for more items. Suppose player 1 bids $x_i$ for item $i, 1 \leq i \leq m$. W.l.o.g., assume $x_i \geq x_{i+1}$, for $1 \leq i \leq m-1$. Player 1 doesn’t get any item if and only if $y \geq x_1$. So, with probability $F(x_1)$, she gets at least one item and she pays at least $x_1$. Therefore, her expected utility is at most $F(x_1)(v-x_1) = v-1/m$, but would be strictly less if she had nonzero payments for other items with positive probability. This means that bidding only $x_1$ for one item and zero for the rest of them dominates the strategy we have assumed.

If player 2 bids $y$ for all items, where $y \in [0, 1/m]$, she gets $m$ items with probability $G(y)$ and $m-1$ items with probability $1-G(y)$. Her expected utility is $G(y)(2-my)+(1-G(y))(1-(m-1)y) = G(y)(1-y) + 1 - (m-1)y = 1$. Bidding something greater than $1/m$ results in utility less than 1. Suppose now that player 2 bids $y_i$ for item $i$, for $1 \leq i \leq m$. Player 1 bids for item $i$ (according to $G(x)$), with probability $1/m$. So, the expected utility of player 2 is

$$\frac{1}{m} \sum_{i=1}^{m} \left( G(y_i) \left( 2 - \sum_{j=1}^{m} y_j \right) + (1 - G(y_i)) \left( 1 - \sum_{j=1}^{m} y_j \right) \right)$$

$$= \frac{1}{m} \sum_{i=1}^{m} \left( G(y_i)(1-y_i) + 1 - \sum_{j=1}^{m} y_j \right)$$

$$= \frac{1}{m} \sum_{i=1}^{m} \left( my_i + 1 - \sum_{j=1}^{m} y_j \right)$$

$$= \frac{1}{m} \left( m \sum_{i=1}^{m} y_i + m - m \sum_{j=1}^{m} y_j \right) = 1.$$

Overall, we proved that $B$ is a mixed Nash equilibrium. It is easy to see that the social welfare in the optimum allocation is 2. In this Nash equilibrium, player 2 bids 0 with probability $1 - \frac{1}{mv}$, so, with at least this probability, player 1 gets one item.
$$SW(B) \leq \left(1 - \frac{1}{mv}\right)(v + 1) + \frac{1}{mv}2 = 1 + v + \frac{1}{mv} - \frac{1}{m}$$

If we set $v = 1/\sqrt{m}$, then $SW(B) \leq 1 + \frac{2}{\sqrt{m}} - \frac{1}{m}$. So, $PoA \geq \frac{2}{1+\frac{2}{\sqrt{m}} - \frac{1}{m}}$ which, for large $m$, converges to 2.

\[\square\]

5 A separation result for additive valuations

For additive valuations, we show that the mixed Nash equilibria are efficient, whenever they exist. This implies an interesting separation between mixed equilibria with full information and Bayesian equilibria, that are known to be inefficient [22]. For the sake of completeness, we present a lower bound of 1.06 for the Bayesian case of a single item, in Appendix B.

5.1 The PoA with single item is 1

We consider a first-price, single-item auction, where the valuations of the players for the item are given by $(v_1, v_2, \ldots, v_n)$. We show that the PoA in mixed strategies is 1. For any mixed Nash equilibrium of strategies $B = (B_1, B_2, \ldots, B_n)$, let $B_i$ denote the probability measure of the distribution of bid $b_i$; in particular, $B_i(I) = \mathbb{P}[b_i \in I]$ for any real interval $I$. The corresponding cumulative distribution function (CDF) of $b_i$ is denoted by $G_i$ (i.e., $G_i(x) = B_i(( - \infty, x)]$).

**Definition 5.1.** For a given $B$, for any bidder $i \in [n]$, let $F_i(b_i)$ denote the CDF of $\max_{j \neq i} b_j$.

Let $\varphi_i(b_i)$ denote the probability that $i$ gets the item with bid $b_i$. Then $\varphi(b_i) \leq F_i(b_i)$, due to a possible tie at $b_i$. Therefore, if he bids $b_i$, then for his expected utility holds that

$$\mathbb{E}[u_i(b_i)] = \mathbb{E} \left[ u_i(b_i) | b_i \sim B_i \right] = \varphi_i(b_i)(v_i - b_i) \leq F_i(b_i)(v_i - b_i).$$

Let $\mathbb{E}[u_i] = \mathbb{E}_{b_i \sim B_i}[u_i]$ denote his overall expected utility, that is defined by the (Lebesgue) integral $\mathbb{E}[u_i] = \int_{(-\infty, \infty)} \varphi_i(x)(v_i - x)dB_i$. Furthermore, assuming $\mathbb{P}[b_i \in I] > 0$ for some interval $I$, let $\mathbb{E}[u_i | b_i \in I] = \mathbb{E}_{b_i \sim B_i}[u_i | b_i \in I]$ be the expected utility of $i$, on condition that his bid is in $I$. By definition

$$\mathbb{E}[u_i | b_i \in I] = \frac{\int_I \varphi_i(x)(v_i - x)dB_i}{\mathbb{P}[b_i \in I]} \leq \frac{\int_I F_i(x)(v_i - x)dB_i}{\mathbb{P}[b_i \in I]}.$$

**Lemma 5.2.** In any mixed Nash equilibrium $B$, for every player $i$ holds that if $\mathbb{P}[b_i \in I] > 0$, then $\mathbb{E}[u_i | b_i \in I] = \mathbb{E}[u_i]$.

**Proof.** Assume that $\mathbb{E}[u_i | b_i \in I] > \mathbb{E}[u_i]$. Then, the player would be better off by submitting bids only in the interval $I$ (according to the distribution $B'_i(I') = B_i(I')/B_i(I)$ for all $I' \subset I$). If $\mathbb{E}[u_i | b_i \in I] < \mathbb{E}[u_i]$, the proof is analogous: in this case the player would be better off by bidding outside the interval. Both cases would contradict $B$ being a Nash equilibrium.

\[\square\]

\[11\] Our result holds for arbitrary (randomized) tie-breaking rules, as long as, for any fixed $b = (b_1, \ldots, b_n)$, the probabilities for the players to get the item are fixed.
In the next lemma we show, that for any two players with positive utility, the infimum of their bids must be equal, and they both bid higher than this value with probability 1.

**Lemma 5.3.** Assume that in a mixed Nash equilibrium \( B \) there are bidders \( i \) and \( j \), with positive utilities \( \mathbb{E}[u_i] > 0 \), and \( \mathbb{E}[u_j] > 0 \). Let \( q_i = \inf_x \{ G_i(x) > 0 \} \) and \( q_j = \inf_x \{ G_j(x) > 0 \} \). Then \( q_i = q_j = q \), and \( G_i(q) = G_j(q) = 0 \), consequently \( F_i(q) = F_j(q) = 0 \).

**Proof.** Assume w.l.o.g. that \( q_i > q_j \). Note that, by the definition of \( q_j \), player \( j \) bids with positive probability in the interval \( I = [q_j, q_i) \).

On the other hand, \( F_j(x) = 0 \) over interval \( I \), since (at least) player \( i \) bids higher than \( x \) with probability 1. This implies \( \mathbb{E}[u_j|b_i \in I] = 0 \). Using Lemma 5.2 we obtain \( \mathbb{E}[u_i] = \mathbb{E}[u_i|b_i \in I] = 0 \), contradicting our assumptions. This proves \( q_i = q_j = q \).

Next we show \( G_i(q) = G_j(q) = 0 \). Observe first, that because of \( \mathbb{E}[u_j] > 0 \), \( v_j > q \) must hold, since \( q \) is the smallest possible bid of \( j \), and similarly, \( v_i > q \). Assume now that \( G_i(q) > 0 \) and \( G_j(q) = 0 \). Then, \( \mathbb{P}[b_i = q] > 0 \), but \( \mathbb{E}[u_i|b_i = q] = 0 \), since \( j \) bids higher. This contradicts again Lemma 5.2 for the interval \([q, q]\).

Second, assume that \( G_i(q) > 0 \) and \( G_j(q) > 0 \). In case \( b_i = b_j = q \), bidder \( i \) or bidder \( j \) receives the item with probability smaller than 1. W.l.o.g., we assume it is player \( i \). In this case bidder \( i \) is better off by bidding \( q + \epsilon \) for a small enough \( \epsilon \) instead of bidding \( q \), since in case of bids \( b_i = q + \epsilon \), and \( b_j = q \), he gets the item for sure. This contradicts \( B \) being a Nash equilibrium, and altogether we conclude \( G_i(q) = G_j(q) = 0 \).

Finally, this immediately implies \( F_i(q) = F_j(q) = 0 \), since for both \( i \) and \( j \), (at least) the other one bids higher than \( q \) with probability 1. \( \square \)

**Theorem 5.4.** In a single-item auction the PoA of mixed Nash equilibria is 1.

**Proof.** Let \( v_j \) be the maximum valuation in the single item auction with full-information. Assume for the sake of contradiction that a mixed Nash equilibrium \( B \) has \( \mathbb{E}_{b \in B}[SW(b)] < SW(OPT) = v_i \). Then, there is a nonempty set of bidders \( J \subset [n] \setminus \{i\} \), who all get the item with positive probability in \( B \), moreover \( v_k < v_i \) holds for all \( k \in J \). Let \( j \in J \) denote the player with maximum valuation \( v_j < v_i \) among players in \( J \).

We show that \( \mathbb{E}[u_i] > 0 \), and \( \mathbb{E}[u_j] > 0 \). Let us first consider the distribution \( F_i(x) \) of the maximum bid in \( B_{-i} \). If \( F_i(v_i - \delta) = 0 \) for all \( \delta > 0 \), then the highest bid in \( B_{-i} \), and thus the payment of player \( j \) is at least \( v_i > v_j \) whenever \( j \) wins the item. Thus for his utility \( \mathbb{E}[u_j] < 0 \), contradicting that \( B \) is a Nash equilibrium. Therefore, there exists a small \( \delta \), such that \( F_i(v_i - \delta) > 0 \). This implies \( \mathbb{E}[u_i] > 0 \), otherwise, by bidding \( v_i - \delta/2 \) only, player \( i \) would have higher than 0 utility.

Now assume for the sake of contradiction that \( \mathbb{E}[u_j] = 0 \) (\( \mathbb{E}[u_j] < 0 \) is impossible in an equilibrium). Note that \( F_j(v_j) > 0 \), otherwise \( j \) would get the item with positive probability, and always for a price higher than \( v_j \). On the other hand, if there were a small \( \delta' \) such that \( F_j(v_j - \delta') > 0 \), then \( j \) could improve his 0 utility by bidding \( v_j - \delta'/2 \) only. The latter implies, that \( j \) can get the item (with positive probability) only with bids \( v_j \) or higher, so he never bids higher so as to avoid negative expected utility. Moreover, \( F_j(v_j - \delta') = 0 \) for all \( \delta' > 0 \) implies that the minimum bid of at least one player \( k \) is at least \( v_j \) (\( \inf_x \{ G_k(x) > 0 \} \geq v_j \)). This implies that the winning bids of player \( i \) are also at least \( v_j \) (both when \( i = k \), and when \( i \neq k \)). But then \( i \) could improve his utility by bidding exactly \( v_j + \epsilon \) (instead of \( \leq v_j + \epsilon \)) with probability \( G_i(v_j + \epsilon) \) for a small enough \( \epsilon \). With this bid, the additional utility of \( i \) would (at least) get arbitrarily close to \( \varphi_j(v_j)(v_i - v_j) > 0 \), where \( \varphi_j(v_j) \) is the probability of \( j \) winning the item in \( B \).

Thus, we established the existence of players \( i \) and \( j \), with different valuations \( v_j < v_i \), and both with strictly positive expected utility in \( B \). According to Lemma 5.3, for the infimum of these
two players' bids $q_i = q_j = q$, and $F_i(q) = F_j(q) = 0$ hold. Furthermore, $q < v_j < v_i$, otherwise the utility of $j$ could not be positive. By the definition of $q_i = q$, for any $\epsilon > 0$ it holds that $\mathbb{P}[q \leq b_i < q + \epsilon] > 0$. Therefore, by Lemma 5.2, and by the definition of conditional expectation, for the interval $I = [q, q + \epsilon)$ we have

\[
\mathbb{E}[u_i] = \mathbb{E}[u_i | b_i \in I] \leq \frac{\int_I F_i(x) (v_i - x) dB_i}{\mathbb{P}[b_i \in I]} < \frac{\int_I F_i(q + \epsilon) (v_i - q) dB_i}{\mathbb{P}[b_i \in I]} = F_i(q + \epsilon) (v_i - q).
\]

Rearranging terms, this yields $F_i(q + \epsilon) > \mathbb{E}[u_i] / (v_i - q) > 0$ for arbitrary $\epsilon > 0$. This contradicts the fact that $F_i(q) = 0$, and that $F(x)$ as a cumulative distribution function is right-continuous in every point. \hfill \square

### 5.2 Upper bound for additive valuations

We extend the above proof for additive valuations.

**Theorem 5.5.** For simultaneous first-price auctions with additive valuations the PoA of mixed Nash equilibria is 1.

**Proof.** Let $B$ be a mixed Nash equilibrium in the $m$ item case. We argue first that for any fixed bidder $i$, it is without loss of generality to assume that in $B_i$ his bids for each item are drawn from independent distributions. If this were not the case, we could determine the distribution $B_i'$ of $b_i(j)$ for any item to have the same CDF $G_i$ as the distribution of bids for this item in $B_i$. Then we would replace $B_i$ by the product distribution for the items $B_i' = \times B_i'$. Since both the expected valuation and the expected payment for item $j$ would remain the same in this new strategy, and the valuation and utility of the player are the sum of valuations and utilities over the items, none of these amounts would be affected. Furthermore the same additivity holds for any other player $k$, whose 'price function' $F_k^j(\cdot)$ for item $j$ would also not be influenced. Thus, with $B_i$ replaced by the strategy $B_i'$, the mixed profile $B' = (B_i', B_{-i})$ would remain a mixed Nash with the same expected social welfare as $B$.

The remaining argument is similar. Now the distribution of bids ($B_i'$), for any particular item $j$ corresponds to a mixed Nash equilibrium of the single item auction for this item. Otherwise a player could improve his utility for $j$, and consequently the sum of his utilities for all items. In turn, this implies that the social welfare for each item $j$ is optimal, a player (or players) with maximum valuation receive the item, which concludes the proof. \hfill \square

### 6 Discriminatory auction

#### 6.1 Upper bounds

Consider a discriminatory auction with submodular valuations, with $n$ players and $m$ items. Recall that $v_i(j)$ denotes the valuation of player $i$ for $j$ copies of the item. For any player $i$, we define $v_{ij} = \frac{v_i(j)}{j}$. It is easy to see that for submodular functions, $v_{ij} \geq v_{i(j+1)}$ for all $j \in [m - 1]$. Let $\beta_j(b)$ be the $j^{th}$ lowest bid among the winning bids under the strategy profile $b$ (in case $b$ is not a pure strategy, $\beta_j(b)$ is a random variable depending on $b$). Assume that the strategy profile
\( B = (B_1, \ldots, B_n) \) is an equilibrium (pure Nash, mixed Nash, correlated, coarse correlated). We define the following functions:

\[
F_{ij}(x) = \Pr[\beta_j(B_{-i}) \leq x] \quad \text{for } 1 \leq j \leq m,
\]

\[
G_{ij}(x) = \Pr[\beta_j(B_{-i}) \leq x < \beta_{j+1}(B_{-i})] = F_{ij}(x) - F_{i(j+1)}(x) \quad \text{for } 1 \leq j \leq m - 1.
\]

We define separately \( G_{im}(x) = \Pr[\beta_m(B_{-i}) \leq x] = F_{im}(x) \). Notice that

\[
F_{ij}(x) = \sum_{k=j}^{m} G_{ik}(x),
\]

\[
\sum_{j=1}^{m} F_{ij}(x) = \sum_{j=1}^{m} jG_{ij}(x).
\]

For any pure or mixed Bayesian Nash equilibrium \( B(v) = (B_1(v), \ldots, B_n(v_n)) \) and for any player \( i \) with valuation \( v_i \), we define \( F_{ij}^v(x) \) and \( G_{ij}^v(x) \) as above, with respect to \( b_{-i} \sim B_{-i}(v_{-i}) \) and \( v_{-i} \sim D_{-i} \).

### 6.1.1 Upper bound for coarse correlated equilibria with submodular valuations

**Theorem 6.1.** The PoA for discriminatory auctions with submodular valuations for coarse correlated equilibria is at most \( \frac{e}{e-1} \approx 1.58 \).

**Proof.** Let \( a_i = (a_{i1}, \ldots, a_{im}, 0, \ldots, 0) \) be a strategy profile for player \( i \). If her valuation is \( v_i = (v_{i1}, \ldots, v_{im}) \) (based on the definition of \( v_{ij} \)), similarly to Lemma 3.2, we get

\[
\mathbb{E}_{b \sim B} [u_i(b)] \geq \frac{o_i}{m-o_i} \sum_{j=1}^{o_i} jG_{ij}(a_i)(v_{ij} - a_i),
\]

where \( G_{ij}(a_i) \) is actually the probability of \( a_i \) being the \( j \)th lowest bid among winning bids under \( B \). Therefore, since player \( i \) bids according to \( a_i \), \( G_{ij}(a_i) \) is the probability of player \( i \) getting exactly \( j \) items. Summing up over all players

\[
\sum_{i=1}^{n} \mathbb{E}_{b \sim B} [u_i(b)] \geq \sum_{i=1}^{n} \frac{o_i}{m-o_i} \sum_{j=1}^{o_i} jG_{ij}(a_i)(v_{ij} - a_i)
\]

\[
\mathbb{E}_{b \sim B} [SW(b)] \geq \sum_{i=1}^{n} \sum_{j=1}^{o_i} jG_{ij}(a_i)(v_{ij} - a_i) + \mathbb{E}_{b \sim B} \left[ \sum_{j=1}^{m} \beta_j(b) \right].
\]

Similarly to inequality (4), it holds that

\[
\sum_{j=1}^{m} \beta_j(b) \geq \sum_{i=1}^{n} \sum_{j=1}^{o_i} \beta_j(b) \geq \sum_{i=1}^{n} \sum_{j=1}^{o_i} \beta_j(b_{-i}).
\]

So, we finally have
We define $F^{av}_i(x) = \frac{1}{\alpha_i} \sum_{j=1}^{o_i} F_{ij}(x)$, and let $\beta^{av}_i$ be a random variable with $F^{av}_i(x)$ as CDF. $F^{av}_i(x)$ is a cumulative distribution defined on $\mathbb{R}^+$, since $F^{av}_i(0) = 0$, $\lim_{x \to +\infty} (F^{av}_i(x) = 1$ and $F^{av}_i(x)$ is the average of non-decreasing functions, so it is itself a non-decreasing function. Moreover, $\mathbb{E}[\beta^{av}_i] = \int_{0}^{\infty} (1 - F^{av}_i(x))dx = \int_{0}^{\infty} (1 - \frac{1}{\alpha_i} \sum_{j=1}^{o_i} F_{ij}(x))dx = \frac{1}{\alpha_i} \sum_{j=1}^{o_i} \int_{0}^{\infty} (1 - F_{ij}(x))dx = \frac{1}{\alpha_i} \mathbb{E}_{b_i \sim B_i} \left[ \sum_{j=1}^{o_i} \beta_j(b_{-i}) \right]$. If we use $F^{av}_i(x)$ and $\beta^{av}_i$ in the above expression, we get

$$
\mathbb{E}_{b \sim B}[SW(b)] \geq \sum_{i=1}^{n} \left( \sum_{j=1}^{o_i} jG_{ij}(a_i)(v_{ij} - a_i) + \mathbb{E}_{b_i \sim B_i} \left[ \sum_{j=1}^{o_i} \beta_j(b_{-i}) \right] \right)
$$

$$
\geq \sum_{i=1}^{n} \left( (v_{io_i} - a_i) \sum_{j=1}^{o_i} F_{ij}(a_i) + \mathbb{E}_{b_i \sim B_i} \left[ \sum_{j=1}^{o_i} \beta_j(b_{-i}) \right] \right).
$$

For the second inequality we apply Lemma 3.3. So, the PoA is at most $\frac{e}{e-1} \approx 1.58$.

**6.1.2 Upper bound for Bayesian equilibria with submodular valuations**

**Theorem 6.2.** The BPoA of discriminatory auctions with submodular valuations is at most $\frac{e}{e-1}$.

**Proof.** The proof follows the same line as the corresponding proof for simultaneous first price auctions. So, we only highlight the most important steps. Let $a^v_i = (a_{i1}, \ldots, a_{im}, 0, \ldots, 0)$ be a strategy profile for player $i$ for valuations $v$.

$$
\mathbb{E}_{b \sim B(v)}[SW(b)] = \sum_{i=1}^{n} \left( \mathbb{E}_{v_i \sim \phi(v_i)} \left[ u^{v_i}(b) \right] + \mathbb{E}_{v_i \sim \phi(v_i)} \left[ \sum_{j=1}^{o_i} \beta_j(b_{-i}) \right] \right)
$$

$$
\geq \sum_{i=1}^{n} \left( (v_{io_i}(v_i) - a_i) \sum_{j=1}^{o_i} F^{vi}_{ij}(a_i) + \mathbb{E}_{b_i \sim B_{-i}(v)} \left[ \sum_{j=1}^{o_i} \beta_j(b_{-i}) \right] \right).
$$

We define $F^{av}_i(v_i) = \frac{1}{\alpha_i(v_i)} \sum_{j=1}^{o_i(v_i)} F^{vi}_{ij}(x)$ and $\beta^{av}_i(v_i)$ to be a random variable that follows $F^{av}_i(x)$. By using them in the above expression we get

$$
\mathbb{E}_{b \sim B(v)}[SW(b)] \geq \sum_{i=1}^{n} \mathbb{E}_{v_i \sim \phi(v_i)} \left[ o_i(v_i) (v_{io_i}(v_i) - a_i) F^{av}_i(a_i) + \mathbb{E}[\beta^{av}_i(v_i)] \right] \geq \sum_{i=1}^{n} \mathbb{E}_{v_i \sim \phi(v_i)} \left[ o_i(v_i) \left( 1 - \frac{1}{e} \right) v_{io_i}(v_i) \right]
$$

$$
= \left( 1 - \frac{1}{e} \right) \mathbb{E}_{v_i \sim \phi(v_i)} \left[ \sum_{i=1}^{n} v_i(o_i(v_i)) \right] = \left( 1 - \frac{1}{e} \right) \mathbb{E}_{v_i \sim \phi(v_i)}[SW(O(v))].
$$
For the second inequality we apply Lemma 3.3. So, the BPoA is at most $\frac{e}{e-1}$. \hfill \Box

6.2 Lower bounds

6.2.1 Submodular valuations

**Theorem 6.3.** The price of anarchy for submodular discriminatory auctions is at least 1.099.

*Proof.* We present an example for a discriminatory auction with submodular valuations and show that the PoA of the mixed Nash equilibria is at least 1.099.

We design a game with two players and two identical items. Player 1 has valuation $v$ if she gets one or more items, i.e. $(v, v)$, whereas player 2 is additive with value 1 for each item, i.e. $(1, 2)$.

We use the following distribution functions defined by Hassidim et al [20]:

\[
G(x) = \frac{x}{1 - x} \quad x \in [0, 1/2]; \quad F(y) = \frac{v - 1/2}{v - y} \quad y \in [0, 1/2].
\]

Consider the following mixed strategy profile. Player 1 bids $(x, 0)$ and player 2 bids $(y, y)$, where $x$ and $y$ are drawn from $G(x)$ and $F(y)$, respectively. Noting that player 2 bids 0 with probability $F(0) = 1 - 1/2v$, we need a tie-breaking rule for the case of bidding 0, in which player 2 always gets the item. We claim that this mixed strategy profile is a Nash equilibrium.

First we prove that playing $(x, 0)$ for player 1 is a best response for every $x \in [0, 1/2]$. Notice that $(x, x')$ with $x' \leq x$, is dominated by $(x, 0)$, since if player 1 gets at least one item, she should pay at least $x$ and getting both items doesn’t add to her utility.

\[
u_1(x, 0) = F(x) \cdot (v - x) = v - 1/2.
\]

Notice that bidding higher than 1/2 guarantees the item but pays more. Now we need to show that $(y, y)$ is a best response for player 2, for every $y \in [0, 1/2]$. Consider any strategy $(y, y')$ with $y, y' \in [0, 1/2]$ and $y \geq y'$.

\[
u_2(y, y') = \mathbb{P}[x \leq y'](2 - y - y') + \mathbb{P}[x > y'](1 - y)
\]

\[= G(y')(2 - y - y') + (1 - G(y'))(1 - y) = G(y')(1 - y') + 1 - y = 1 + y' - y \leq 1
\]

Notice that $u_2(y, y) = 1$. Bidding strictly higher than 1/2 for both items is not profitable, since then her utility becomes strictly less than 1. Now we calculate the expected social welfare of this Nash equilibrium.

\[
\mathbb{E}[SW] = \mathbb{P}[y \geq x]2 + \mathbb{P}[x > y](1 + v)
\]

\[= 2 - (1 - v) \int_{0}^{1/2} F(x) dG(x)
\]

This expression is maximized for $v = 0.643$. For this value of $v$, $\mathbb{E}[SW] = 1.818$. Since $SW(O) = 2$, we get $\text{PoA} = 1.099$.

\[12\text{This can be actually improved to 1.109 by a similar construction with 3 items. For large } m \text{ it goes to one, so we don’t really believe that this construction is tight. That is the reason that we prefer to put the simplest version of 2 items.}\]
6.2.2 Subadditive valuations

We provide a tight lower bound of 2 for subadditive valuations in discriminatory auctions which is similar to the lower bound of Section 4.2, adjusted to discriminatory auctions.

Theorem 6.4. The price of anarchy of discriminatory auctions of mixed Nash equilibria and subadditive valuations is at least 2.

Proof. Consider two players and \( m \) items with the following valuations: player 1 is a unit-demand player with valuation \( v < 1 \) if she gets at least one item; player 2 has valuation 1 for getting less than \( m \) items (but at least one) and 2 if she gets all the items. Inspired by [20], we use the following distribution functions:

\[
G(x) = \frac{(m-1)x}{1-x} \quad x \in [0,1/m]; \quad F(y) = \frac{v-1/m}{v-y} \quad y \in [0,1/m].
\]

Player 1 bids \( b_1 = (x, 0, \ldots, 0) \) and player 2 bids \( b_2 = (y, \ldots, y) \). \( x \) and \( y \) are drawn from \( G(x) \) and \( F(y) \), respectively. In case of a tie, the item is always allocated to player 2.

Let \( B = (b_1, b_2) \) denote this mixed bidding profile. We are going to prove that \( B \) is a mixed Nash equilibrium for every \( v > 1/m \).

If player 1 bids any \( x \) in the range \( (0,1/m] \) for the one item, she gets the item with probability \( F(x) \), since a tie appears with zero probability. Her expected utility is \( F(x)(v-x) = v - 1/m \). So, for every \( x \in (0,1/m] \) her utility is \( v - 1/m \). If player 1 picks \( x \) according to \( G(x) \), her utility is still \( v - 1/m \), since she bids 0 with zero probability. Bidding something greater than \( 1/m \) results in a utility less than \( v - 1/m \). Regarding player 1, it remains to show that her utility while bidding for more items. Suppose player 1 bids \((x_1, \ldots, x_m)\), where \( x_i \geq x_{i+1} \), for \( 1 \leq i \leq m - 1 \). Player 1 doesn’t get any item if and only if \( y \geq x_1 \). So, with probability \( F(x_1) \), she gets at least one item and she pays at least \( x_1 \). Therefore, her expected utility is at most \( F(x_1)(v-x_1) = v - 1/m \), but would be strictly less if she had nonzero payments for other items with positive probability. This means that bidding only \( x_1 \) for one item and zero for the rest of them dominates the strategy we have assumed.

If player 2 bids \( y \) for all items, where \( y \in [0,1/m] \), she gets \( m \) items with probability \( G(y) \) and \( m-1 \) items with probability \( 1-G(y) \). Her expected utility is \( G(y)(2-my)+(1-G(y))(1-(m-1)y) = G(y)(1-y) + 1 - (m-1)y = 1 \). Bidding something greater than \( 1/m \) results in utility less than 1. Suppose now that player 2 bids \((y_1, \ldots, y_m)\), where \( y_i \geq y_{i+1} \) for \( 1 \leq i \leq m - 1 \). If \( x \leq y_m \), player 2 gets all the items; otherwise she gets \( m - 1 \) items and she pays her \( m - 1 \) highest bids. So, her utility is

\[
G(y_m) \left( 2 - \sum_{i=1}^{m} y_i \right) + (1 - G(y_m)) \left( 1 - \sum_{i=1}^{m-1} y_i \right)
= G(y_m)(1 - y_m) + 1 - \sum_{i=1}^{m-1} y_i
= my_m + 1 - \sum_{i=1}^{m} y_i
\leq my_m + 1 - \sum_{i=1}^{m} y_m = 1.
\]
Overall, we proved that \( B \) is a mixed Nash equilibrium. It is easy to see that the social welfare in the optimum allocation is 2. In this Nash equilibrium, player 2 bids 0 with probability \( 1 - \frac{1}{mv} \), so, with at least this probability, player 1 gets one item.

\[
\text{SW}(B) \leq \left(1 - \frac{1}{mv}\right)(v + 1) + \frac{1}{mv}2 = 1 + v + \frac{1}{mv} - \frac{1}{m}
\]

If we set \( v = 1/\sqrt{m} \), then \( \text{SW}(B) \leq 1 + \frac{2}{\sqrt{m}} - \frac{1}{m} \). So, \( \text{PoA} \geq \frac{2}{1 + \frac{2}{\sqrt{m}} - \frac{1}{m}} \) which, for large \( m \), converges to 2.

\[\qed\]

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A Mixed Nash equilibria with submodular valuations

Here, we construct a group of instances with submodular valuations, which have at least one mixed Nash equilibrium. These instances are a generalization of the lower bound of Section 4.1. We believe that this construction is interesting in its own right.

Consider an instance with \( n + 1 \) players and \( n^d \) items, where \( d \) divides \( n \). We will refer to the first \( n \) players as the real players and to the last one as the dummy player. Let \([n]\) denote the set of integers \( \{1, \ldots, n\} \). We define the set of items as \( M = [n]^d \), that is, they correspond to all the different vectors \( w = (w_1, w_2, \ldots, w_d) \) with \( w_i \in [n] \). Intuitively, they can be thought of as the nodes of an \( d \) dimensional grid, with coordinates in \([n]\) in each dimension.

We divide the real players into \( d \) groups of \( n/d \) players. Let \( g(i) \) denote the group that player \( i \) belongs to. We associate each group with one of the dimensions (directions) of the grid. In particular, for any fixed player \( i \), his valuation for a subset of items \( S \subseteq M \) is the size (number of elements) in the \( d - 1 \)-dimensional projection of \( S \) in direction \( g(i) \), times \( v \). Formally,

\[
v_i(S) = v|\{w_{-g(i)} \mid \exists w_{g(i)} \text{ s.t. } (w_{g(i)}, w_{-g(i)}) \in S\}|.
\]

It is straightforward to check that \( v \) has decreasing marginal valuations, and is therefore submodular. The valuation of the dummy player for any subset of items \( S \subseteq M \) is \((v - 1)|S|\).

Given these valuations, we describe a mixed Nash equilibrium \( B = (B_1, \ldots, B_n) \) having a PoA \( \left(1 - (1 - \frac{1}{n})^{n+\frac{2}{d}}\right)^{-1} \) which is arbitrarily close to \( \frac{1}{1 - e^{-1}} \), for large enough \( n \). The dummy player bids \( v - 1 \) for every item, and receives the item if all of the real players bid at most \( v - 1 \) for it. The utility of the dummy player is always 0. For real players the mixed strategy \( B_i \) is the following. Every player \( i \) picks a number \( \ell \in [n] \) uniformly at random, and an \( x \) according to the distribution with CDF

\[
G(x) = (n - 1) \left(\frac{1}{(v - x)^\frac{1}{n-1}} - 1\right),
\]

where \( x \in \left[v - 1, v - (\frac{n-1}{n})^{n-1}\right] \). Subsequently, he bids \( x \) for every item \( w = (\ell, w_{-g(i)}) \), with \( w_{g(i)} = \ell \) as \( g(i) \)th coordinate, and bids \( v - 1 \) for the rest of the items. That is, in any \( b_i \) in the support of \( B_i \), the player bids a positive \( x \) only for a \( d - 1 \) dimensional slice of the items. Observe that \( G(x) \) has no mass points, so tie-breaking matters only in case of \( v - 1 \) bids for an item, in which case the dummy player gets the item.

Let \( F(x) \) denote the probability that any bidder \( i \) gets a fixed item \( j \), given that he bids \( b_i(j) > v - 1 \) for this item, and the bids in \( b_{-i} \) are drawn from \( B_{-i} \) (due to symmetry, this probability is the same for all items \( w = (\ell, w_{-g(i)}) \)). For every other player \( k \), the probability that he bids \( v - 1 \) for item \( j \) is \((n - 1)/n\), and the probability that \( j \) is in his selected slice but he bids lower than \( x \) is \( G(x)/n \). Multiplying over the \( n - 1 \) other players, we obtain

\[
F(x) = \frac{G(x)}{n} + \frac{n-1}{n} = \left(\frac{1}{1 - \frac{1}{n}}\right)^{n-1}.
\]

Notice that \( v_i \) is an additive valuation restricted to the slice of items that player \( i \) bids for in a particular \( b_i \). Therefore the expected utility of \( i \) when he bids \( x \) in \( b_i \) is \( F(x)(v-x) \) for one of these items, and comprising all items \( \mathbb{E}[u_i(b_i)] = n^{d-1}F(x)(v-x) = n^{d-1}(1 - 1/n)^{n-1} \).

Next we show that, for \( v \geq (1 - 1/n)^{-\frac{2}{d}+1} \), \( B \) is a Nash equilibrium. In particular, the bids \( b_i \) in the support of \( B_i \) maximize the expected utility of a fixed player \( i \).
First, we fix an arbitrary \( w_{-g(i)} \), and focus on the set of items \( C := \{(\ell, w_{-g(i)}) | \ell \in [n]\} \), which we call a column for player \( i \). Recall that \( i \) is interested in getting only one item within \( C \), on the other hand his valuation is additive over items from different columns. Observe that restricted to a fixed column, submitting any bid \( x \in [v - 1, v - (\frac{n-1}{n})^{n-1}] \) for one arbitrary item results in the constant expected utility of \((1 - \frac{1}{n})^{n-1}\), whereas a bid higher than \( v - (\frac{n-1}{n})^{n-1} \) guarantees the item but pays more so the utility becomes strictly less than \((1 - \frac{1}{n})^{n-1}\) for this column.

We introduce two functions, \( F_1(x) \) and \( F_2(x) \),

\[
F_1(x) = \left( \frac{G(x)}{n} + \frac{n-1}{n} \right) \frac{d-1}{d} n = \left( \frac{1 - \frac{1}{n}}{(v-x)^{\frac{d-1}{d}}} \right)^{\frac{d-1}{d} n},
\]

\[
F_2(x) = \left( \frac{G(x)}{n} + \frac{n-1}{n} \right)^{\frac{n}{d} - 1} = \left( \frac{1 - \frac{1}{n}}{(v-x)^{\frac{1}{d}}} \right)^{\frac{n}{d} - 1}.
\]

\( F_1(x) \) denotes the probability that any bidder \( i \) gets a fixed item \( j \), while the rest of the players in \( g(i) \) bid 0, given that player \( i \) bids \( b_i(j) = x \) for this item, and the rest bids in \( b_{-i} \) are drawn from \( B_{-i} \). Similarly, \( F_2(x) \) denotes the same probability while the players of all the other groups, apart from \( g(i) \), bid 0. Notice that \( F(x) = F_1(x)F_2(x) \). In any fixed \( b_{-i} \), every other player of group \( k \neq g(i) \) submits the same bid for all items in \( C \), because either the whole \( C \) is in the current slice of \( k \), and he bids the same value \( x \), or no item from the column is in the slice and he bids \( v-1 \). Therefore, \( F_1(x) \) for items in \( C \) are fully dependent distributions, whereas \( F_2(x) \) for items in \( C \) are independent distributions.

We first show that if bidder \( i \) bids more than \( v-1 \) for at least two items in \( C \), bidding \( v-1 + \varepsilon \) for all of these items, for a significantly small \( \varepsilon > 0 \), is a dominate strategy. Suppose that bidder \( i \) bids \( x \), which is \( x_j > v-1 \) for every item \( j \in R \subseteq C \) and \( v-1 \) for the rest, where \( k \) is the cardinality of \( R \). Reorder the items in \( R \) in a way that the bids are in non-increasing order. We will use mathematical induction over \( k \). Let \( u_i(x, k) \) be the utility of player \( i \) for bidding \( x \), when the number of bids strictly greater than \( v-1 \) is \( k \).

\[
E[u_i(x, 2)] = F(x_2)F_2(x_1)(v - x_1 - x_2) + F(x_2)(1 - F_2(x_1))(v - x_2) + F_2(x_1)(F_1(x_1) - F(x_2))(v - x_1)
= F(x_2)(v - x_2) + F(x_1)(v - x_1) - F(x_2)F_2(x_1)v
= 2\left(1 - \frac{1}{n}\right)^{n-1} - F(x_2)F_2(x_1)v
\]

\( E[u_i(x, 2)] \) is maximized when both \( x_1 \) and \( x_2 \) are minimized, so for \( x_1 = x_2 = v - 1 + \varepsilon \). Let \( R_{-1} \) denote the set of all \( k \) items apart from the first one (for which bidder \( i \) bids the most). Assume that \( E[u_i(x, k - 1)] \) is maximized when \( x_j = v - 1 + \varepsilon \), for all \( j \in R_{-1} \). Moreover, let \( \mathbb{P}(S \neq \emptyset | S \subseteq R_{-1}) \) and \( \mathbb{P}(S) \) be the probabilities that he gets at least one item from \( R_{-1} \) and gets \( S \), respectively. It is easy to see that \( \mathbb{P}(S \neq \emptyset | S \subseteq R_{-1}) = \sum_{S \subseteq R_{-1}, S \neq \emptyset} \mathbb{P}(S) \).

\[
E[u_i(x, k)] = F_2(x_1)(F_1(x_1) - \mathbb{P}(S \neq \emptyset | S \subseteq R_{-1}))(v - x_1)
+ F_2(x_1)\sum_{S \subseteq R_{-1}, S \neq \emptyset} \mathbb{P}(S)(v - \sum_{j \in S} x_j - x_1) + (1 - F_2(x_1))E[u_i(x_{-1}, k - 1)]
= F(x_1)(v - x_1) - F_2(x_1)\mathbb{P}(S \neq \emptyset | S \subseteq R_{-1})(v - x_1)
+ F_2(x_1)E[u_i(x_{-1}, k - 1)] - F_2(x_1)x_1\sum_{S \subseteq R_{-1}, S \neq \emptyset} \mathbb{P}(S) + (1 - F_2(x_1))E[u_i(x_{-1}, k - 1)]
= (1 - \frac{1}{n})^{n-1} + E[u_i(x_{-1}, k - 1)] - F_2(x_1)\mathbb{P}(S \neq \emptyset | S \subseteq R_{-1})(v - x_1)
- F_2(x_1)x_1\mathbb{P}(S \neq \emptyset | S \subseteq R_{-1})
= (1 - \frac{1}{n})^{n-1} + E[u_i(x_{-1}, k - 1)] - F_2(x_1)\mathbb{P}(S \neq \emptyset | S \subseteq R_{-1})v
\]
$F_2(x_1)$ is minimized when $x_1 = v - 1 + \varepsilon$, for a significantly small $\varepsilon > 0$, $E[u_i(x_{-1}, k-1)]$ is maximized when $x_j = v - 1 + \varepsilon$, for all $j \in R_{-1}$ and $P(S \neq \emptyset | S \subseteq R_{-1})$ is minimized for the same values. So, $E[u_i(x, k)]$ is maximized when $x_j = v - 1 + \varepsilon$, for all $j \in R$.

We next prove that for $v \geq (1 - \frac{1}{n})^{-\frac{3}{2} + \varepsilon}$, bidding $x > v - 1$ only for one item dominates the strategy of bidding $x$ for more than one items.

**Lemma A.1.** For any integer $k \geq 1$,

$$\sum_{r=1}^{k} \binom{k}{r} x^r (1-x)^{k-r} = 1 - (1-x)^k$$

and

$$\sum_{r=1}^{k} \binom{k}{r} x^r (1-x)^{k-r} = k x.$$

**Proof.**

$$\sum_{r=1}^{k} \binom{k}{r} x^r (1-x)^{k-r} = \sum_{r=0}^{k} \binom{k}{r} x^r (1-x)^{k-r} - \binom{k}{0} x^0 (1-x)^{k-0} = (x+1-x)^k - (1-x)^k = 1 - (1-x)^k.$$

$$\sum_{r=1}^{k} \binom{k}{r} x^r (1-x)^{k-r} = \sum_{r=1}^{k} \frac{k!}{r!(k-r)!} x^r (1-x)^{k-r}$$

$$= k \sum_{r=1}^{k} \binom{k-1}{r-1} x^r (1-x)^{k-r}$$

$$= k x \sum_{r=0}^{k-1} \binom{k-1}{r} x^r (1-x)^{k-1-r}$$

$$= k x (x+1-x)^{k-1} = k x.$$

By using Lemma A.1, the utility of bidding $x$ for $k$ items is,

$$E[u_i(x, k)] = F_1(x) \sum_{r=1}^{k} \binom{k}{r} F_2(x)^r (1-F_2(x))^{k-r} (v - r x)$$

$$= F_1(x) \left( \left( 1 - (1 - F_2(x))^k \right) v - k F_2(x) x \right).$$

We are going to bound the value of $v$, so that the utility decreases as $k$ increases. So, we would like the following to hold.

$$E[u_i(x, k + 1)] - E[u_i(x, k)] \leq 0$$

$$F_1(x) \left( (1 - (1 - F_2(x))^{k+1}) v - (k+1) F_2(x)x \right) - F_1(x) \left( (1 - (1 - F_2(x))^k) v - k F_2(x) x \right) \leq 0$$

$$(1 - F_2(x))^k v - (1 - F_2(x))^{k+1} v - F_2(x)x \leq 0$$

$$F_2(x) (1 - F_2(x))^k v - F_2(x)x \leq 0$$

$$(1 - F_2(x))^k v - x \leq 0$$

The quantity $(1 - F_2(x))^k v - x$ is maximized when $x$ is minimized. Therefore,

$$(1 - F_2(x))^k v - x \leq (1 - F_2(v-1))^k v - v + 1$$

$$\leq (1 - F_2(v-1))v - v + 1$$

$$= 1 - F_2(v-1)v.$$
Let $B_r$ gets item $j$, then define a random variable w.r.t. the distribution $\mathcal{B}$. We have

$$\text{Anarchy is } 1 = 0 \text{ with probability } 1 \text{ for large } n\text{, and for large } n \text{ it converges to } 1.$$ 

We can compute the distribution of $\mathcal{B}$.

Finally, we show that the optimum social welfare is $n^d v$. An optimal allocation can be constructed as follows: For each item $(w_1, w_2, ..., w_n)$, compute $r = (\sum_{i=1}^n w_i \mod n)$. Allocate this item to the player $r$. Similar to the Section 4.1, each player is allocated $n^d$ items.

Therefore, the price of anarchy is $v \frac{(1 - \frac{1}{n})^{-\frac{d}{n}}}{1 - e^{-1 - \frac{1}{n}}}$. For $v = (1 - \frac{1}{n})^{-\frac{d}{n} + 1}$, the price of anarchy becomes

$$\frac{(1 - \frac{1}{n})^{-\frac{d}{n} + 1}}{(1 - \frac{1}{n})^{-\frac{d}{n} + 1} - (1 - \frac{1}{n})^n} \approx 1.58.$$

It is easy to see that the case of $d = n$, is the special case of Section 4.1, for which the Price of Anarchy is $\frac{1}{(1 - (\frac{1}{n})^n)}$, and for large $n$ it converges to $\frac{1}{(1 - \frac{1}{n})} \approx 1.58$.

## B A lower bound example for the single item Bayesian PoA.

Bayesian equilibria, were known to be inefficient [22]. Here, for the sake of completeness, we present a lower bound example for the Bayesian price of anarchy, with two players and only one item.

**Theorem B.1.** For single-item auctions the PoA in Bayesian Nash equilibria is at least 1.06.

**Proof.** In the lower-bound instance we have two bidders and only one item. The valuation of bidder 1 is always 1. Let $l = 1 - 2/e$, and $r = 1 - 1/e$. The valuation of bidder 2 is distributed according to the cumulative distribution function $H$:

$$H = \begin{cases} \frac{1}{e(1-x)}, & x \in [0,l] \\ \frac{2}{e(1-x)+2}, & x \in [l,1]. \end{cases}$$

Observe that $v_2 = 0$ with probability 1/2, and is distributed over $[l,1]$ otherwise. Consider the following bidding strategy $\mathcal{B} = (B_1, B_2)$: $B_1$ has a uniform distribution on $[l,r]$ with CDF $G(x) = \frac{x-l}{r-l} = ex - e + 2$ on $[l,r]$; whereas the distribution $B_2$ is determined by the distribution of $v_2$:

$$b_2(v_2) = \begin{cases} 0, & v_2 = 0 \\ \frac{v_2+1}{2}, & v_2 \in [l,1]. \end{cases}$$

Let $F(x)$ denote the CDF of $b_2$. We can compute the distribution of $b_2$ as follows. For $x \in [0,l]$, we have $F(x) = \Pr(b_2 \leq x) = \frac{1}{2}$; for $x \in [l,r]$ we have $F(x) = \Pr(b_2 \leq x) = \Pr(2x+1 \leq x) = \Pr(v_2 \leq 2x - l) = \frac{1}{e(1-x)}$. Finally, $F(x) = 1$ for $x \geq r$. In summary, $b_2 = 0$ with probability 1/2, and is distributed over $[l,r]$ otherwise. On the other hand, $b_1$ is uniformly distributed over $[l,r]$. We do not need to bother about tie-breaking, since there are no mass points in $[l,r]$.
We prove next, that $B$ is a Bayesian NE. Consider first player 2. If $v_2 = 0$, his utility is clearly maximized. If $v_2 \in [l, 1]$, then $\mathbb{E}[v_2(b_2)] = G(b_2)(v_2 - b_2)$. By straightforward calculation we obtain that over $[l, r]$ the function $G(x)(v_2 - x) = \frac{(x-l)(v_2-x)}{r-l}$ is maximized in $b_2 = (v_2 + l)/2$, so $b_2(v_2)$ is best response for bidder 2.

Consider now player 1. Given that over $[l, r]$ the distribution of $b_2$ is $F(x) = \frac{1}{e(1-x)}$, every bid $b_1 \in [l, r]$ is best response for player 1, since his utility $F(b_1)(1 - b_1) = 1/e$ is constant. Now we are ready to compute the social welfare of this Nash equilibrium.

$$SW(B) = Pr[v_2 \leq l] \cdot 1 + \int_{l}^{1} (v_2 \cdot G(b_2(v_2)) + 1 - G(b_2(v_2))) \cdot h(v_2) dv_2 \leq 0.942$$

So the PoA is at least 1.06.