Sensitivity coefficients for thermal properties measurements using a boundary condition of the 4th kind

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Abstract. In the parameter estimation the sensitivity analysis of the temperature to the unknown parameter (such as thermal conductivity, heat capacity) plays a fundamental role. Considering the experimental configuration for thermal properties measurements of solid materials, the heater in contact with the specimen can be modeled through an high conductivity thin layer at which an heat flux is applied. In such a case the one dimensional finite Cartesian body, representing the sample, is subject to a boundary condition of the 4th kind at the heated boundary. The transient temperature within the finite body, assuming the back side surface insulated, is derived by means of the Green’s Function Solution Equation. Then the scaled sensitivity coefficients are computed in part analytically and in part numerically, for two different location: at the interface between the heater and the sample, and at the sample back side. The results show that the sensitivity coefficients with respect to the thermal conductivity and to the heat capacity of the sample are uncorrelated.

1. Introduction

Sensitivity coefficients have many applications such as parameter estimation, optimal experimental design and uncertainty or error analysis [1]. In parameter estimation, considerable insight is provided by carrying out a sensitivity analysis of the temperature to the unknown parameters, such as thermal conductivity, volumetric heat capacity or thermal diffusivity. In particular the knowledge of sensitivity coefficients not only allows a deep understanding of the phenomenon but it can represent an important design tool. In fact sensitivity coefficients provide useful design information concerning the effect of parameters on the model response [1].

As described in [2], the parameter estimation technique requires measured temperature values. It is essential that the measured response be sensitive to the parameters of interest: the more sensitive the temperature (or large the sensitivity coefficient) is, the more valuable the temperature measurements are [1]. Also, in order to gain much insight and information from the results as possible, it must be that the sensitivity coefficients with respect to the parameters of interest are uncorrelated and large in magnitudes [3]. For these reasons, the experimental apparatus have to be properly designed.

The sensitivity coefficients allow the experimental apparatus to be made closer to the optimal. For example, in an apparatus for which the sensitivity coefficients had not been investigated it is possible that additional materials in the experimental configuration (for instance the thin heater giving up heat to the sample) can have a large impact on the temperature than the material of interest (sample), with direct consequences on the quality of the results [1]. Definitely, on the one hand the sensitivity
coefficients make possible a preliminary evaluation of the goodness of the experimental results (at least from a qualitative point of view), on the other hand are directly involved in the estimation of the parameters when minimizing the ordinary least square norm [4-6].

The focus of the current paper is to provide sensitivity coefficients in the event that the effect of the thin heater in contact with the specimen is simulated through an high-conductivity thin layer to which a jump in the heat flux is applied. The heat conduction problem thus addressed consists in a one dimensional finite Cartesian body, which represents the sample, subject to a boundary condition of the fourth kind [7, Chap. 2] at the heated boundary \( x=0 \), and insulated at the back side \( (x=L) \). The governing equations are solved by means of the Green’s Function Solution Equation (GFSE) [7]. Once the thermal field is obtained, the so-called “scaled” sensitivity coefficients are computed by performing the partial derivative of the temperature with respect to the parameters of interest. Their computation is carried out in part analytically and in part numerically by using a two-point central difference scheme [8]. The coefficients are presented in a graphical form and they are calculated for two different location: at the interface between the heater and the sample, and at the sample back side. The results of the analysis show that the sensitivity coefficients with respect the thermal properties of the sample are uncorrelated, and both the thermal conductivity and the heat capacity can be estimated.

2. Analytical formulation

Figure 1a shows the schematic of an experimental apparatus for the thermal properties measurements of an orthotropic solid specimen: a thin layer heater located at the interface of two samples of the same material and thickness and gives heat up at surface of both samples. For sake of thermal symmetry, the three layer configuration (specimen-heater-specimen) reduces to the simplified configuration depicted in figure 1b. In particular the finite body, initially at uniform temperature \( T_{\text{in}} \), is in perfect thermal contact with a high-conductivity layer at the boundary surface \( x=0 \). The thin layer, which a step change in heat flux is applied, it is at the same initial temperature \( T_{\text{in}} \); its thickness and heat capacity are denoted by \( L_f \) and \( C_f \), respectively. Also the finite body is insulated at the back side \( x=L \), and its properties are considered temperature-independent.

![Figure 1. Schematic of the experimental apparatus for thermal properties measurements (a) and simplified schematic for the addressed problem (b).](image)

The analytical formulation of this transient, linear, Cartesian heat conduction problem (denoted by X42B10T00 according to the numbering system devised in [9]) is defined in dimensionless form as follows.

\[
\frac{\partial^2 \bar{T}}{\partial \bar{x}^2} = \frac{\partial \bar{T}}{\partial \bar{t}} \quad (0 < \bar{x} < 1; \bar{t} > 0)
\]
- \left( \frac{\partial \bar{T}}{\partial \bar{t}} \right)_{\bar{t}=0} + P \frac{\partial \bar{T}}{\partial \bar{t}} = 1 \right)_{\bar{t}=1} = 0 \quad (\bar{t} > 0) \\
\bar{T}_i(\bar{r}) = \bar{T}(0, \bar{r}) \quad (\bar{t} > 0); \quad \bar{T}_i(0) = 0 \\
\bar{T}(\bar{x}, 0) = 0 \quad (0 < \bar{x} < 1) 
\tag{1d}

The dimensionless variables appearing in the above equations are defined as:
\[ \bar{T} = \frac{T - T_{in}}{q_L L / k}, \quad \bar{x} = \frac{x}{L}, \quad \bar{r} = \frac{\alpha t}{L^2}, \quad \bar{T}_i = \frac{T_i - T_{in}}{q_L L / k}, \quad P = \frac{C_i}{CL} \]

3. Temperature solution

The solution to the problem defined by equations (1) is not available in heat conduction literature. In particular the dimensionless governing equations (1a)-(1f) are solved by means of the GFSE method. As the series solution obtained by means of the standard GFSE requires a large number of terms to reach the numerical convergence, an alternative temperature solution is determined.

3.1. Solution with GFSE

The Green’s function for the X42 problem is provided in literature \[7, p 609, \#X42\]. In dimensionless form it results in:
\[ \bar{G}_{X42}(\bar{x}, \bar{r} | \bar{x}', \bar{r}') = \frac{1}{P + 1} + 2 \sum_{m=1}^{\infty} \frac{X_m(\bar{x})X_m(\bar{x}')}{\eta_m} \exp\left(-\beta_m^2(\bar{r} - \bar{r}')\right) \]
\tag{2}

Where \( \beta_m \) is the m-th eigenvalue whose computation is discussed in Subsection 3.3, and
\[ \eta_m = (P \beta_m)^2 + P + 1 \]
\[ X_m(\bar{x}) = \cos(\beta_m \bar{x}) - P \beta_m \sin(\beta_m \bar{x}) \]
\tag{2a}
\tag{2b}

By applying the GFSE to the current problem, considering that the only non-homogeneous term is the boundary condition of the fourth kind, the dimensionless temperature solution \( \bar{T}(\bar{x}, \bar{r}) \) can be obtained performing the following integral:
\[ \bar{T}(\bar{x}, \bar{r}) = \int_{\bar{r}=0}^{\bar{r}} \bar{G}_{X42}(\bar{x}, \bar{r} | 0, \bar{r}') d\bar{r} \]
\tag{3}

Then substituting equation (2) into the above integral yields
\[ \bar{T}(\bar{x}, \bar{r}) = \frac{\bar{r}}{P + 1} + 2 \sum_{m=1}^{\infty} \frac{\cos(\beta_m \bar{x}) - P \beta_m \sin(\beta_m \bar{x})}{\eta_m \beta_m^2} - 2 \sum_{m=1}^{\infty} \frac{\cos(\beta_m \bar{x}) - P \beta_m (\beta_m \bar{x})}{\eta_m \beta_m^2} \exp\left(-\beta_m^2 \bar{r}'\right) \]
\tag{4}

The first summation in equation (4) exhibits an algebraic convergence, that is it requires a large number of terms. For this reason, an alternative solution with better convergence properties is determined in the following subsection.

3.2. Alternative solution

An alternative solution to the problem defined by equations (1) may be derived by means of the Alternative GFSE (AGFSE) \[7, \text{Chap. 3}\]. In this case the temperature solution is considered as the sum of two terms: the former, denoted by \( \bar{T}'(\bar{x}, \bar{r}) \), which satisfies only the boundary conditions, and the latter which represents the transient solution, denoted by \( \bar{T}(\bar{x}, \bar{r}) \).
\[ \overline{T}(\bar{x},\bar{t}) = \overline{T}'(\bar{x},\bar{t}) + \overline{T}''(\bar{x},\bar{t}) \] (5)

Then, by substituting equation (5) into the governing equations (1), the original problem can be split up into a set of two simpler problems:

- a non-homogeneous problem defined for \( \overline{T}'(\bar{x},\bar{t}) \), as:
  \[
  -\frac{\partial^2 \overline{T}'}{\partial \bar{x}^2} + \frac{\partial \overline{T}'}{\partial \bar{t}} = 0 \quad \text{at} \quad \bar{t} = 0 \]
  \[
  -\frac{\partial \overline{T}'}{\partial \bar{x}} \bigg|_{\bar{x}=0} + P \frac{\partial \overline{T}'}{\partial \bar{t}} \bigg|_{\bar{x}=1} = 1
  \]
(6a)

(6b)

- and a transient problem defined for \( \overline{T}''(\bar{x},\bar{t}) \), with homogeneous boundary conditions:
  \[
  -\frac{\partial^2 \overline{T}''}{\partial \bar{x}^2} + \frac{\partial \overline{T}''}{\partial \bar{t}} = 0 \quad \text{at} \quad \bar{t} = 0
  \]
  \[
  -\frac{\partial \overline{T}''}{\partial \bar{x}} \bigg|_{\bar{x}=0} + P \frac{\partial \overline{T}''}{\partial \bar{t}} \bigg|_{\bar{x}=1} = 0
  \]
(7a)

(7b)

\[ \overline{T}''(\bar{x},0) = -\overline{T}'(\bar{x},0) \] (7c)

As the temperature solution, equation (4), contains a time dependent term which dominates temperature for large times, that is \( \bar{t} \approx (P+1) \), the \( \overline{T}'(\bar{x},\bar{t}) \) solution should be:

\[ \overline{T}'(\bar{x},\bar{t}) = f(\bar{x}) + \frac{\bar{t}}{P+1} \] (8)

Where the \( f(\bar{x}) \) function can be obtained by substituting equation (8) in equations (6). It results in:

\[ f(\bar{x}) = \frac{\bar{x}^2}{2(P+1)} - \frac{\bar{x}}{(P+1)} \] (9)

Instead the problem defined by equations (7) is conveniently solved by means of the GFSE: in this case the only non-homogeneous term is the initial condition. The solution \( \overline{T}''(\bar{x},\bar{t}) \) can be obtained by the following equation:

\[ \overline{T}''(\bar{x},\bar{t}) = -\frac{1}{\bar{x}^2} \int_{\bar{x}=0}^{\bar{x}} \overline{G}_{x_{x_2}}(\bar{x},\bar{t} | \bar{x}',0) f(\bar{x}') d\bar{x}' - P \overline{G}_{x_{x_2}}(\bar{x},\bar{t} | 0,0) f(0) \] (10)

In the above equation we have considered that \( \overline{T}''(\bar{x},0) = f(\bar{x}) \), and the Green’s function is defined by equation (2). Then by performing the integral in equation (10) one can obtain the \( \overline{T}'(\bar{x},\bar{t}) \) solution:

\[ \overline{T}'(\bar{x},\bar{t}) = \frac{1}{3(P+1)^2} - 2 \sum_{m=1}^{\infty} \frac{\cos(\beta_m \bar{x}) - P \beta_m \sin(\beta_m \bar{x})}{\eta_m \beta_m^2} \exp(-\beta_m^2 \bar{t}) \] (11)

Finally by substituting equations (8) and (11), bearing in mind equation (9), into equation (5), an alternative form for the temperature solution is obtained:

\[ \overline{T}(\bar{x},\bar{t}) = \frac{\bar{t}}{P+1} + \frac{\bar{x}^2}{2(P+1)} - \frac{\bar{x}}{P+1} + \frac{1}{3(P+1)^2} - 2 \sum_{m=1}^{\infty} \frac{\cos(\beta_m \bar{x}) - P \beta_m \sin(\beta_m \bar{x})}{\eta_m \beta_m^2} \exp(-\beta_m^2 \bar{t}) \] (12)
3.3. Computation of the eigenvalues
The eigenvalues appearing in the temperature solutions, equations (4) and (12), are computed as roots of the eigencondition (transcendental equation) defined as [7],

$$\beta_n \cot(\beta_n) = -P^{-1}$$ (13)

Its roots may be computed by using explicit approximate relations based on the second-order modified Newton method [10]. These relations provide an approximate value of the exact eigenvalue with high accuracy (8-decimal place after one iteration, and 15-decimal place after two iterations) for the $P$ range $[0, \infty)$.

3.4. Computational solution
The infinite series appearing in the alternative temperature solution, equation (12), exhibits an exponential convergence at any time except near $\tilde{t} = 0$. However as we cannot take into account infinite terms a convergence criterion for this series is needed. Following the procedure given in [11], the maximum number of required terms $m_{\text{max}}$ to get a truncation error of $10^{-4}$ ($A=2,3,..15$) in the series solution may be taken as:

$$m_{\text{max}} = \lceil \frac{1}{2} + \frac{1}{\pi} \left[ \frac{A \ln(10)}{\tilde{t}} \right]^{1/2} \rceil$$ (14)

In particular, the above equation has been obtained by using $(m-1/2)\pi$ as a conservative estimate for $\beta_m$. In addition, according to equation (14) a large number of terms are required for early times. However for times less than the so-called deviation time $\tilde{t}_d$, the temperature solution can be replaced by a semi-infinite transient solution (X40B1T00) with an error less than $10^{-4}$ ($A=2,3,..15$) [9, 12]. The solution for the X40B1T00 case is a well-established exact analytical solution available in the heat conduction literature [13, p 306, #12] and discussed in [14]:

$$\tilde{T}(\tilde{x}, \tilde{t}) = \sqrt{\frac{4\tilde{t}}{\pi}} \exp \left( -\frac{\tilde{x}^2}{4\tilde{t}} \right) - \tilde{x} \text{erfc} \left( \frac{\tilde{x}}{\sqrt{4\tilde{t}}} \right) - P \left[ \text{erfc} \left( \frac{\tilde{x}}{\sqrt{4\tilde{t}}} \right) - \text{Am}(\tilde{x}, \tilde{t}, P) \right] \quad 0 \leq \tilde{t} \leq \tilde{t}_d = \frac{(2-\tilde{x})^2}{10A}$$ (15)

where $\text{Am}(\tilde{x}, \tilde{t}, P)$ is the so-called Amos function defined as

$$\text{Am}(\tilde{x}, \tilde{t}, P) = \exp \left( \frac{\tilde{x}}{P} + \frac{\tilde{t}}{P^2} \right) \text{erfc} \left( \frac{\tilde{x}}{\sqrt{4\tilde{t}}} + \frac{\sqrt{\tilde{t}}}{P} \right) = \exp \left( -\frac{\tilde{x}^2}{4\tilde{t}} \right) \text{erf} \left( \frac{\tilde{x}}{\sqrt{4\tilde{t}}} + \frac{\sqrt{\tilde{t}}}{P} \right)$$ (15a)

A computer code in Matlab ambient for calculating the solution of the current problem is provided in [10]. Also, in this reference, a more detailed derivation of the solution is given, and a numerical intrinsic verification of the above code is carried out.

For sake of completeness, a plot of the dimensionless temperature as a function of time for different $P$ values is given in figure 2.

4. Sensitivity coefficients
The scaled sensitivity coefficients are defined as partial derivatives of the temperature with respect to the model parameter $\gamma$ of interest (e.g. $q_{10}^*, k, C, C_i$) multiplied by the parameter itself:

$$X_\gamma = \gamma \frac{\partial T}{\partial \gamma}$$ (16)
The relative sensitivity coefficient can be expressed as follows:

\[ X_f = \frac{X_f}{q'_{f0} L/k} = \frac{\partial T}{\partial f} \]  

4.1. Heat flux as a parameter

If the parameter of interest is the heat flux applied to the boundary surface \( \bar{x} = 0 \), the corresponding dimensional scaled sensitivity coefficient is:

\[ X_{q'_{f0}} = q'_{f0} \frac{\partial T}{\partial q'_{f0}} = q'_{f0} \frac{\partial}{\partial q'_{f0}} \left( \frac{q'_{f0} L}{k} \right) = q'_{f0} \left[ \frac{q'_{f0} L}{k} \frac{\partial T}{\partial q'_{f0}} \right] = q'_{f0} \frac{L}{k} \bar{T} \]  

In dimensionless form it results in \( \bar{X}_{q'_{f0}} = \bar{T} \), where \( \bar{T} \) is defined by equation (12). Thus figure 2 also shows the sensitivity coefficient to the applied heat flux.

4.2. Thermal conductivity and volumetric heat capacities as parameters

As \( \bar{T} \) depends on thermal conductivity of the specimen \( k \), and also it depends on the volumetric heat capacities of both specimen and heater, \( C \) and \( C_f \) respectively, three coefficients should be considered.

When the parameter of interest is the thermal conductivity, the relative sensitivity coefficient can be expressed as follows

\[ X_k = k \frac{\partial T}{\partial k} = k \frac{\partial}{\partial k} \left( \frac{q'_{f0} L}{k} \right) = k \left[ \frac{q'_{f0} L}{k} \frac{\partial \bar{T}}{\partial k} + \bar{T} \frac{\partial}{\partial k} \left( \frac{q'_{f0} L}{k} \right) \right] = k \left[ \frac{q'_{f0} L}{k} \frac{\partial \bar{T}}{\partial k} - \frac{q'_{f0} L}{k^2} \bar{T} \right] \]  

where \( \bar{T} = \bar{T}[\bar{t}, P, \beta_0(P), \bar{\alpha}(k, C), P(C, C_f)] \). Thus, by using the chain rule:

\[ \frac{\partial \bar{T}}{\partial k} = \frac{\partial \bar{T}}{\partial \bar{t}} \frac{\partial \bar{t}}{\partial \alpha} \frac{\partial \alpha}{\partial k} \]  

By substituting equation (20) in equation (19), one can obtain the scaled sensitivity coefficient with respect to the thermal conductivity. In dimensionless form it results in:

Figure 2. Dimensionless temperature as a function of the dimensionless time, for different \( P \) values, at \( \bar{x} = 0 \) (a) and at \( \bar{x} = 1 \) (b).
\[
\tilde{X}_k = \tilde{r} \frac{\partial \tilde{T}}{\partial \tilde{t}} - \tilde{T}
\]  

(21)

Following the same procedure described above, the scaled sensitivity coefficient with respect to the volumetric heat capacity \( C \) is obtained.

\[
\tilde{X}_C = C \frac{\partial \tilde{T}}{\partial C} = -\tilde{r} \frac{\partial \tilde{T}}{\partial \tilde{t}} - \tilde{P} \frac{\partial \tilde{T}}{\partial \tilde{P}} - \tilde{P} \sum_{m=1}^{\infty} \frac{\partial \tilde{T}}{\partial \beta_m} \frac{d\beta_m}{d\tilde{P}}
\]  

(22)

In similar manner the scaled sensitivity coefficient with respect to the volumetric heat capacity of the heater \( C_i \) in dimensionless form, results in:

\[
\tilde{X}_{C_i} = C_i \frac{\partial \tilde{T}}{\partial C_i} = \tilde{P} \frac{\partial \tilde{T}}{\partial \tilde{P}} + \tilde{P} \sum_{m=1}^{\infty} \frac{\partial \tilde{T}}{\partial \beta_m} \frac{d\beta_m}{d\tilde{P}}
\]  

(23)

It is worth to note that all scaled sensitivity coefficients discussed above sum to zero for all values of time and position.

\[
\tilde{X}_{q_{10}} + \tilde{X}_k + \tilde{X}_C + \tilde{X}_{C_i} = 0
\]  

(24)

### 4.3. Computation of the scaled sensitivity coefficients \( \tilde{X}_k, \tilde{X}_C \) and \( \tilde{X}_{C_i} \)

The sensitivity coefficient \( \tilde{X}_k \) may be computed analytically through equation (12) and its partial derivative with respect to time. As regards the coefficients \( \tilde{X}_C \) and \( \tilde{X}_{C_i} \), they cannot be evaluated analytically as the derivative \( \frac{d\beta_m}{d\tilde{P}} \), appearing in equations (22) and (23), cannot be given in an algebraic form (see equation (13)). For this reason, they will be computed numerically by using a two point central difference (CD) scheme [8, p 440, #12-6c] for the derivatives \( \frac{\partial \tilde{T}}{\partial \tilde{C}} \) and \( \frac{\partial \tilde{T}}{\partial \tilde{C}_i} \), as suggested by the following equation.

\[
\frac{\partial \tilde{T}}{\partial \gamma} = \tilde{T} \left[ \gamma, \tilde{T}(\gamma + \Delta \gamma), \tilde{P}(\gamma + \Delta \gamma), \beta_m(\gamma + \Delta \gamma) \right] - \tilde{T} \left[ \gamma, \tilde{T}(\gamma - \Delta \gamma), \tilde{P}(\gamma - \Delta \gamma), \beta_m(\gamma - \Delta \gamma) \right] + O(\Delta \gamma^2)
\]

(25)

Bearing in mind that \( \gamma = C \) and \( \gamma = C_i \).

Also for early times (less than the deviation time \( \tilde{t}_0 \)) the temperature field within the specimen may be evaluated accurately by using equation (15). Therefore the sensitivity coefficients can be computed analytically by performing the derivatives of equation (15) with respect to \( \tilde{t} \) and to \( \tilde{P} \). While the coefficient \( \tilde{X}_{q_{10}} \) is obtained immediately, the sensitivity coefficients \( \tilde{X}_k, \tilde{X}_C \) and \( \tilde{X}_{C_i} \) can be evaluated through the following algebraic expressions:

\[
\tilde{X}_k = \tilde{r} \frac{\partial \tilde{T}}{\partial \tilde{t}} - \tilde{T} = (P + \tilde{x}) \text{erfc} \left( \frac{\tilde{x}}{\sqrt{4\tilde{t}}} \right) + \left( \frac{\tilde{T}}{P} - P \right) \text{Am}(\tilde{x}, \tilde{T}, P) - 2 \left( \frac{\tilde{T}}{P} \right) \text{exp} \left( \frac{-\tilde{x}^2}{4\tilde{t}} \right)
\]

(26)

\[
\tilde{X}_C = -\tilde{r} \frac{\partial \tilde{T}}{\partial \tilde{t}} - \tilde{P} \frac{\partial \tilde{T}}{\partial \tilde{P}} = P \text{erfc} \left( \frac{\tilde{x}}{\sqrt{4\tilde{t}}} \right) - 2 \left( \frac{\tilde{T}}{P} \right) \text{exp} \left( \frac{-\tilde{x}^2}{4\tilde{t}} \right) + \left( \tilde{x} + P - \tilde{T} \right) \text{Am}(\tilde{x}, \tilde{T}, P)
\]

(27)

\[
\tilde{X}_{C_i} = P \frac{\partial \tilde{T}}{\partial \tilde{P}} = 2 \left( \frac{\tilde{T}}{P} \right) \text{exp} \left( \frac{-\tilde{x}^2}{4\tilde{t}} \right) - P \text{erfc} \left( \frac{\tilde{x}}{\sqrt{4\tilde{t}}} \right) + \left( P - \tilde{x} - \frac{2\tilde{T}}{P} \right) \text{Am}(\tilde{x}, \tilde{T}, P)
\]

(28)
5. Results

Figures 3-5 show the dimensionless sensitivity coefficients $\widetilde{k}$, $\widetilde{C}$, and $\widetilde{f}$ as a function of time for different $P$ values respectively. Two different locations ($\bar{x} = 0$ and $\bar{x} = 1$) are also considered. In these figures a curve for $P = 0$ is plotted as a reference case; in fact, in such a case, the thin heater is completely neglected and the solution matches to that of the X22B10T0 problem. For sake of brevity, this solution is not given here but it is available in [7, 16].

Figure 3 shows the temperature sensitivity to the thermal conductivity $k$. In particular, for $\bar{x} = 0$, it reduces when the heat capacity for unit area ratio $P$ increases; while for $\bar{x} = 1$ this behavior cannot be observed. It's interesting to note that the sensitivity to $k$ at the backside ($\bar{x} = 1$) is less affected by the $P$ parameter than the sensitivity at $\bar{x} = 0$, when large times are considered. This aspect suggests that for estimating the thermal conductivity, in the event that the heat capacities ratio ($P$) is unknown, the temperature sensor should be placed at $\bar{x} = 1$. In addition, assuming that the $P$ parameter is known, according to the figure 3 if $P < 0.5$ the sensitivity to the thermal conductivity is higher at $\bar{x} = 0$; on the contrary if $P > 0.5$ the highest sensitivity occurs at $\bar{x} = 1$.

The sensitivity coefficient $\widetilde{C}$ is depicted in figure 4. By observing the figure it is possible to note an appreciable difference between the sensitivity at $\bar{x} = 0$ and $\bar{x} = 1$ (not at early times) only for large $P$ values: in this case the sensitivity is higher at the sample backside.

The temperature sensitivity to the heat capacity of the heater is shown in figure 5. The higher sensitivity occurs at the interface between the heater and the sample ($\bar{x} = 0$). By comparing figures 4 and 5 it is possible to state that the temperature of the sample is more sensitive to its heat capacity than to the heat capacity of the heater.

Finally, figure 6 compares the sensitivity coefficients $\widetilde{k}$, $\widetilde{C}$, and $\widetilde{f}$ computed for $P = 0.1$ and for the locations $\bar{x} = 0$ and $\bar{x} = 1$. For time less than the deviation time (defined by equation (15)) the coefficients $\widetilde{k}$ and $\widetilde{C}$ respond in the same way: in particular they are equal for $\bar{x} = 0$ (see equations (26) and (27)). For larger times the temperature sensitivity to the thermal conductivity $k$ approaches to a constant value; while the sensitivity to the heat capacity of the sample $C$ increases linearly with the time. Then the coefficients $\widetilde{k}$ and $\widetilde{C}$ are uncorrelated, and both the thermal properties, $k$ and $C$, can be estimated.

![Figure 3](image1.png)  

Figure 3. Dimensionless scaled sensitivity coefficient to the thermal conductivity as a function of the dimensionless time, for different $P$ values, at $\bar{x} = 0$ (a) and at $\bar{x} = 1$ (b).
Figure 4. Dimensionless scaled sensitivity coefficient to the heat capacity of the sample as a function of the dimensionless time, for different $P$ values, at $\bar{x} = 0$ (a) and at $\bar{x} = 1$ (b).

Figure 5. Dimensionless scaled sensitivity coefficient to the heat capacity of the heater as a function of the dimensionless time, for different $P$ values, at $\bar{x} = 0$ (a) and at $\bar{x} = 1$ (b).

Figure 6. Comparison among dimensionless scaled sensitivity coefficients for $P = 0.1$ at $\bar{x} = 0$ (a) and at $\bar{x} = 1$ (b).
6. Conclusions
The temperature solution for the X42B10T00 problem has been derived by means of the GFSE. Then the sensitivity coefficients with respect to the thermal properties (thermal conductivity and heat capacity of the specimen, and heat capacity of the heater) have been studied in order to establish the best location for the temperature sensor.

The analysis shows that the sensitivity coefficients to the thermal conductivity and to the heat capacity of the sample are uncorrelated and both these thermal properties can be estimated. In particular, when the parameter to estimate is the thermal conductivity the optimal location of the temperature sensor is determined by the heat capacities for unit area ratio ($P$). However when $P$ is unknown the temperature sensor should be placed at the back side of the sample. This is also the best sensor location for estimating the heat capacity of the sample. Moreover, at the back side, the measured temperature is less affected by the heat capacity of the heater.

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