Odd Scalar Curvature in Field-Antifield Formalism

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**Abstract**

We consider the possibility of adding a Grassmann-odd function \(\nu\) to the odd Laplacian. Requiring the total \(\Delta\) operator to be nilpotent leads to a differential condition for \(\nu\), which is integrable. It turns out that the odd function \(\nu\) is not an independent geometric object, but is instead completely specified by the antisymplectic structure \(E\) and the density \(\rho\). The main impact of introducing the \(\nu\) term is that it makes compatibility relations between \(E\) and \(\rho\) obsolete. We give a geometric interpretation of \(\nu\) as (minus 1/8 times) the odd scalar curvature of an arbitrary antisymplectic, torsion-free and \(\rho\)-compatible connection. We show that the total \(\Delta\) operator is a \(\rho\)-dressed version of Khudaverdian’s \(\Delta_E\) operator, which takes semidensities to semidensities. We also show that the construction generalizes to the situation where \(\rho\) is replaced by a non-flat line bundle connection \(F\). This generalization is implemented by breaking the nilpotency of \(\Delta\) with an arbitrary Grassmann-even second-order operator source.

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1 Introduction

Conventionally [1, 2, 3, 4] the geometric arena for quantization of Lagrangian theories in the field-antifield formalism [5, 6, 7] is taken to be an antisymplectic manifold \((M; E)\) with a measure density \(\rho\). Each point in the manifold \(M\) with local coordinates \(\Gamma^A\) and Grassmann parity \(\varepsilon_A \equiv \varepsilon(\Gamma^A)\) represents a field-antifield configuration \(\Gamma^A = \{\phi^\alpha; \phi^*\alpha\}\), the antisymplectic structure \(E\) provides the antibracket \((\cdot, \cdot)\), and the density \(\rho\) yields the path integral measure. However, up until recently, it has been necessary to impose a compatibility condition [2, 8] between the two geometric structures \(E\) and \(\rho\) to ensure nilpotency of the odd Laplacian \(\Delta = \Delta^\rho \equiv (−1)^{ε_A} \partial^l A \rho E^{AB} \partial^l B\), \(\partial^l \equiv \partial/\partial Γ^A\).

In this paper, we show that the compatibility condition between \(E\) and \(\rho\) can be omitted if one adds an odd scalar function \(\nu\) to the odd Laplacian \(\Delta^\rho\),

\[
\Delta = \Delta^\rho + \nu
\]

such that the total \(\Delta\) operator is nilpotent

\[
\Delta^2 = 0 .
\]

Nilpotency is important for the field-antifield formalism in many ways, for instance in securing that the physical partition function \(Z\) is independent of gauge-choice, see Appendix A. (More precisely, what is really vital is the nilpotency of the underlying \(\Delta_E\) operator, cf. Sections 8-9.) In physics terms, the addition of the \(\nu\) function to the odd Laplacian \(\Delta^\rho\) implies that the quantum master equation

\[
\Delta e^{\hbar W} = 0
\]

(1.4)

is modified with a \(\nu\) term at the two-loop order \(O(\hbar^2)\):

\[
\frac{1}{2}⟨W, W⟩ = i\hbar \Delta^\rho W + \hbar^2 \nu ,
\]

(1.5)

and \(\Delta^\rho\) is in general no longer a nilpotent operator. It turns out that the zeroth-order \(\nu\) term is uniquely determined from the nilpotency requirement (1.3) apart from an odd constant. One particular solution to the zeroth-order term, which we call \(\nu_\rho\), takes a special form [9]

\[
\nu_\rho \equiv \nu_\rho^{(0)} + \frac{\nu_\rho^{(1)}}{8} - \frac{\nu_\rho^{(2)}}{24} ,
\]

(1.6)

where \(\nu_\rho^{(0)}\), \(\nu_\rho^{(1)}\) and \(\nu_\rho^{(2)}\) are defined as

\[
\nu_\rho^{(0)} \equiv \frac{1}{\sqrt{\rho}}(\Delta_1 \sqrt{\rho}) ,
\]

(1.7)

\[
\nu_\rho^{(1)} \equiv (−1)^{ε_A} (∂_B \partial^l A E^{AB}) ,
\]

(1.8)

\[
\nu_\rho^{(2)} \equiv (−1)^{ε_A ε_C} (∂_D E^{AB}) E_{BC} (∂^l A E^{CD}) = (−1)^{ε_B} (∂_A E_{BC}) E^{CD} (∂^l D E^{BA}) .
\]

(1.9)

Here, \(\Delta_1\) in eq. (1.7) denotes the expression (1.1) for the odd Laplacian \(\Delta^\rho=1\) with \(\rho\) replaced by 1. In particular, the odd scalar \(\nu_\rho\) is a function of \(E\) and \(\rho\), so there is no call for new independent geometric
structures on the manifold $M$. In Sections 2–6 we show that $\Delta_\rho + \nu$ is the only possible $\Delta$ operator within the set of all second-order differential operators. The now obsolete compatibility condition [2, 8] between $E$ and $\rho$ can be recast as $\nu_\rho = \text{odd constant}$, thereby making contact to the previous approach [2], which uses the odd Laplacian $\Delta_\rho$ only. The explicit formula (1.6) for $\nu_\rho$ is proven in Section 7 and Appendix B. The formula (1.6) first appeared in Ref. [9]. That paper was devoted to Khudaverdian’s $\Delta_E$ operator [10, 11, 12, 13], which takes semidensities to semidensities. This is no coincidence: At the bare level of mathematical formulas the construction is intimately related to the $\Delta_E$ operator, as shown in Sections 8–9. However the starting point is different. On one hand, Ref. [9] studied the $\Delta_E$ operator in its minimal and purest setting, which is a manifold with an antisymplectic structure $E$ but without a density $\rho$. On the other hand, the starting point of the current paper is a $\Delta$ operator that takes scalar functions to scalar functions, and this implies that a choice of $\rho$ (or $F$, cf. below) should be made. Later in Sections 10 and 11 we interpret the odd $\nu_\rho$ function as (minus $1/8$ times) the odd scalar curvature $R$ of an arbitrary antisymplectic, torsion-free and $\rho$-compatible connection,

$$\nu_\rho = -\frac{R}{8}. \quad (1.11)$$

One of the main priorities for the current article is to ensure that all arguments are handled in completely general coordinates without resorting to Darboux coordinates at any stage. This is important to give a physical theory a natural, coordinate-independent, geometric status in the antisymplectic phase space. We shall also throughout the paper often address the question of generalizing the density $\rho$ to a non-flat line bundle connection $F$. It is well-known [2] that a density $\rho$ gives rise to a flat line bundle connection

$$F_A = (\partial_A \ln \rho). \quad (1.12)$$

In fact, several mathematical objects, for instance the odd Laplacian $\Delta_\rho$ and the odd scalar $\nu_\rho$, can be formulated entirely using $F$ instead of $\rho$. Surprisingly, many of these objects continue to be well-defined for non-flat $F$’s as well, where the nilpotency (and the ordinary physical description) is broken down. In Section 5 we shall therefore temporarily digress to contemplate a modification of the nilpotency condition that addresses these mathematical observations. Finally, Section 12 contains our conclusions.

General remark about notation. We have two types of grading: A Grassmann grading $\varepsilon$ and an exterior form degree $p$. The sign conventions are such that two exterior forms $\xi$ and $\eta$, of Grassmann parity $\varepsilon_\xi$, $\varepsilon_\eta$ and exterior form degree $p_\xi$, $p_\eta$, respectively, commute in the following graded sense:

$$\eta \wedge \xi = (-1)^{\varepsilon_\xi \varepsilon_\eta + p_\xi p_\eta} \xi \wedge \eta \quad (1.13)$$

inside the exterior algebra. We will often not write the exterior wedges “$\wedge$” explicitly.

## 2 General Second-Order $\Delta$ operator

We here introduce the setting and notation more carefully, and argue that the $\Delta$ operator must be equal to $\Delta_\rho + \nu_\rho$ up to an odd constant. (The undetermined odd constant comes from the fact that the square $\Delta^2 = \frac{1}{2} [\Delta, \Delta]$ does not change if $\Delta$ is shifted by an odd constant.) Consider now an arbitrary Grassmann-odd, second-order, differential operator $\Delta$ that takes scalar functions to scalar functions. In this paper, we shall only discuss the non-degenerate case, where the second-order term in $\Delta$ is of maximal rank, and hence provides for a non-degenerated antibracket $(\cdot, \cdot)$, cf. the Definition (2.6) below. (The non-degeneracy assumption is motivated by the fact that it is satisfied for currently known applications. The degenerate case may be dealt with via for instance the antisymplectic conversion
mechanism [14, 15].) Due to the non-degeneracy assumption, it is always possible to organize $\Delta$ as

$$\Delta = \Delta_F + \nu ,$$  \hfill (2.1)

where $\nu$ is a zeroth-order term and $\Delta_F$ is an operator with terms of second and first order [2]

$$\Delta_F \equiv \frac{(-1)^{\epsilon_A}}{2} (\vec{\partial}^A F_B) E^{AB} \vec{\partial}^B .$$  \hfill (2.2)

Here, $E^{AB} = E^{AB}(\Gamma)$, $F_A = F_A(\Gamma)$ and $\nu = \nu(\Gamma)$ is a $(2,0)$-tensor, a line bundle connection, and a scalar, respectively. We shall sometimes use the slightly longer notation $\Delta_F \equiv \Delta_{F,E}$ to acknowledge that it depends on two inputs: $F$ and $E$. The line bundle connection $F_A$ transforms under general coordinate transformations $\Gamma^A \rightarrow \Gamma'^B$ as

$$F_A = (\vec{\partial}^A \Gamma'^B) F_B' + (\vec{\partial}^A \ln J), \quad J \equiv \text{sdet} \frac{\partial \Gamma'^B}{\partial \Gamma^A} .$$  \hfill (2.3)

These transformation properties guarantee that the expressions (2.1) and (2.2) remain invariant under general coordinate transformations. The Grassmann-parities are

$$\epsilon(E^{AB}) = \epsilon_A + \epsilon_B + 1 , \quad \epsilon(F_A) = \epsilon_A , \quad \epsilon(\nu) = 1 .$$  \hfill (2.4)

One may, without loss of generality, assume that the $(2,0)$-tensor $E^{AB}$ has a Grassmann-graded skewsymmetry

$$E^{AB} = - (-1)^{(\epsilon_A+1)(\epsilon_B+1)} E^{BA} .$$  \hfill (2.5)

The antibracket $(f,g)$ of two functions $f = f(\Gamma)$ and $g = g(\Gamma)$ is defined via a double commutator* with the $\Delta$-operator, acting on the constant unit function 1,

$$(f, g) \equiv \langle -1 \rangle^{\epsilon_f} ([[-\Delta, f], g])_1 \equiv \langle -1 \rangle^{\epsilon_f} (\Delta f g) - (\langle -1 \rangle^{\epsilon_f} \Delta f) g + (\langle -1 \rangle^{\epsilon_f} f g (\Delta 1)

= (f \vec{\partial}^A E^{AB} (\vec{\partial}^B g) = - (\langle -1 \rangle^{\epsilon_f+1}(\epsilon_f+1) (g, f) ,$$  \hfill (2.6)

where use is made of the skewsymmetry (2.5) in the third equality. By the non-degeneracy assumption, there exists an inverse matrix $E_{AB}$ such that

$$E^{AB} E_{BC} = \delta^A_C = E_{CB} E^{BA} .$$  \hfill (2.7)

Since the tensor $E^{AB}$ possesses a graded $A \leftrightarrow B$ skewsymmetry (2.5), the inverse tensor $E_{AB}$ must be skewsymmetric,

$$E_{AB} = - (-1)^{\epsilon_A \epsilon_B} E_{BA} .$$  \hfill (2.8)

In other words, $E_{AB}$ is a two-form

$$E = \frac{1}{2} d\Gamma^A E_{AB} \wedge d\Gamma^B .$$  \hfill (2.9)

The Grassmann parity is

$$\epsilon(E_{AB}) = \epsilon_A + \epsilon_B + 1 .$$  \hfill (2.10)

*Here, and throughout the paper, $[A, B]$ and $\{A, B\}$ denote the graded commutator $[A, B] \equiv AB - (-1)^{\epsilon_A \epsilon_B} BA$ and the graded anticommutator $\{A, B\} \equiv AB + (-1)^{\epsilon_A \epsilon_B} BA$, respectively.
3 Nilpotency Conditions: Part I

The square $\Delta^2 = \frac{1}{2} [\Delta, \Delta]$ of an odd second-order operator (2.1) is generally a third-order differential operator, which we, for simplicity, imagine has been normal ordered, i.e. with all derivatives standing to the right. Nilpotency (1.3) of the $\Delta$ operator leads to conditions on $E^{AB}$, $F_A$ and $\nu$. Let us therefore systematically, over the next four Sections 3–6, discuss order by order the consequences of the nilpotency condition $\Delta^2 = 0$, starting with the highest (third) order terms, and going down until we reach the zeroth order.

The third-order terms of $\Delta^2$ vanish if and only if the Jacobi identity

$$\sum_{\text{cycl. } f,g,h} (-1)^{(\epsilon_f+1)(\epsilon_h+1)} (f, (g, h)) = 0 \quad (3.1)$$

for the antibracket $(\cdot, \cdot)$ holds. We shall always assume this from now on. Equivalently, the two-form $E_{AB}$ is closed,

$$dE = 0 \quad (3.2)$$

In terms of the matrices $E^{AB}$ and $E_{AB}$, the Jacobi identity (3.1) and the closeness condition (3.2) read

$$\sum_{\text{cycl. } A,B,C} (-1)^{(\epsilon_A+1)(\epsilon_C+1)} E^{AD} (\partial^l_D E^{BC}) = 0 \quad , \quad (3.3)$$

$$\sum_{\text{cycl. } A,B,C} (-1)^{\epsilon_A \epsilon_C} (\partial_A^l E^{BC}) = 0 \quad , \quad (3.4)$$

respectively. By definition, a non-degenerate tensor $E_{AB}$ with Grassmann-parity (2.10), skewsymmetry (2.8), and closeness relation (3.4) is called an antisymplectic structure.

Granted the Jacobi identity (3.1), the second-order terms of $\Delta^2$ can be written on the form

$$\frac{1}{4} \mathcal{R}^{AB} \delta_B^l \delta_A^l \quad , \quad (3.5)$$

where $\mathcal{R}^{AB}$ with upper indices is a shorthand for

$$\mathcal{R}^{AD} \equiv E^{AB} \mathcal{R}_{BC} E^{CD} (-1)^{\epsilon_C} \quad , \quad (3.6)$$

and $\mathcal{R}_{AB}$ with lower indices is the curvature tensor for the line bundle connection $F_A$:

$$\mathcal{R}_{AB} \equiv [\partial_A^l + F_A, \partial_B^l + F_B] = (\partial_A^l F_B) - (-1)^{\epsilon_A \epsilon_B} (A \leftrightarrow B) \quad . \quad (3.7)$$

Remarkably, the two tensors $\mathcal{R}_{AB}$ and $\mathcal{R}^{AB}$ carry opposite symmetry:

$$\mathcal{R}_{AB} = -(-1)^{\epsilon_A \epsilon_B} \mathcal{R}_{BA} \quad , \quad (3.8)$$

$$\mathcal{R}^{AB} = (-1)^{\epsilon_A \epsilon_B} \mathcal{R}^{BA} \quad . \quad (3.9)$$

It follows that in the non-degenerate case, the second-order terms of $\Delta^2$ vanish if and only if the line bundle connection $F_A$ has vanishing curvature

$$\mathcal{R}_{AB} = 0 \quad . \quad (3.10)$$
The zero curvature condition (3.10) is an integrability condition for the local existence of a density \( \rho \),

\[
F_A = \left( \partial_A \ln \rho \right) .
\]  

(3.11)

Under the \( F \leftrightarrow \rho \) identification (3.11) the \( \Delta_F \) operator (2.2) just becomes the ordinary odd Laplacian \( \Delta_\rho \) from eq. (1.1),

\[
\Delta_F = \Delta_\rho .
\]  

(3.12)

Conventionally the field-antifield formalism requires the \( F \leftrightarrow \rho \) identification (3.11) to hold globally. Nevertheless, we shall present many of the constructions below using \( F \) rather than \( \rho \), to be as general as possible.

There exists a descriptive characterization: Granted the Jacobi identity (3.1), the second-order terms of \( \Delta^2 \) vanish if and only if there is a Leibniz rule for the interplay of the so-called “one-bracket” \( \Phi_1^1 \equiv \Delta - (\Delta 1) = \Delta_F \) and the “two-bracket” \( (\cdot, \cdot) \)

\[
\Delta_F(f,g) = (\Delta_F f, g) - (-1)^{|f|}(f, \Delta_F g) .
\]  

(3.13)

See Ref. [16, 17] for more details.

4 A Non-Zero \( F \)-Curvature?

In eq. (3.10) of the previous Section 3 we learned that the nilpotency condition (1.3) completely kills the line bundle curvature \( R \). Nevertheless, several constructions continue to be well-defined for non-zero \( R \). For instance, both the important scalars \( \nu_F \) and \( R \) fall into this category, cf. eqs. (7.1) and (11.7) below. Another example, which turns out to be related to our discussion, is the Grassmann-odd 2-cocycle of Khudaverdian and Voronov [8, 11, 18]. It is defined using two (possibly non-flat) line bundle connections \( F^{(1)} \) and \( F^{(2)} \) as follows:

\[
\nu(F^{(1)}; F^{(2)}, E) \equiv \frac{1}{4} \text{div}_{F^{(12)}} X \equiv \frac{(-1)^{|A|}}{4} (\partial_A + \frac{F^{(1)} + F^{(2)}}{2})(E^{AB}(F^{(1)}_B - F^{(2)}_B)) ,
\]  

(4.1)

where the divergence “\( \text{div} \)” is defined in eq. (10.13),

\[
F^{(12)} \equiv \frac{F^{(1)} + F^{(2)}}{2} ,
\]  

(4.2)

and

\[
X^{A}_{(12)} \equiv E^{AB}(F^{(1)}_B - F^{(2)}_B) .
\]  

(4.3)

It is clear from Definition (4.1) that \( \nu(F^{(1)}; F^{(2)}, E) \) behaves as a scalar under general coordinate transformations. This is because the average \( F^{(12)} \) is again a line bundle connection, and \( X \) is a vector field since the difference \( F^{(1)}_B - F^{(2)}_B \) is a co-vector (=one-form), cf. eq. (2.3). That \( \nu(F^{(1)}; F^{(2)}, E) \) is a 2-cocycle

\[
\nu(F^{(1)}; F^{(2)}, E) + \nu(F^{(2)}; F^{(3)}, E) + \nu(F^{(3)}; F^{(1)}, E) = 0
\]  

(4.4)

follows easily by rewriting Definition (4.1) as

\[
\nu(F^{(1)}; F^{(2)}, E) = \nu^{(0)}_{F^{(1)}} - \nu^{(0)}_{F^{(2)}} ,
\]  

(4.5)

where \( \nu^{(0)}_F \) generalizes eq. (1.7):

\[
\nu^{(0)}_F \equiv \frac{(-1)^{|A|}}{4} (\partial_A + \frac{F_A}{2})(E^{AB} F_B) .
\]  

(4.6)
Note that Definitions (4.1) and (4.6) continue to make sense for non-flat $F$’s. We should stress that $\nu^{(0)}_F$ itself is not a scalar, but we shall soon see that it can be replaced in eq. (4.5) by a scalar $\nu_F$, cf. eq. (7.1) below. In other words, $\nu(F^{(1)}; F^{(2)}, E)$ is a 2-coboundary.

The $F$-curvature $R_{AB}$ is also an interesting geometric object in its own right. It can be identified with a Ricci two-form of a tangent bundle connection $\nabla$, cf. eq. (11.4) in Section 11 below. The Ricci two-form

$$ R = \frac{1}{2} d\Gamma^A R_{AB} \wedge d\Gamma^B (-1)^{\varepsilon_B} $$

(4.7)

is closed

$$ dR = 0, $$

(4.8)

due to the Bianchi identity

$$ \sum_{\text{cycl. } A,B,C} (-1)^{\varepsilon_A \varepsilon_C} (\partial^l_A R_{BC}) = 0, $$

(4.9)

so the two-form (4.7) defines a cohomology class.

## 5 Breaking the Nilpotency

Due to the above mathematical reasons we shall digress in this Section 5 to contemplate how a non-zero $F$-curvature could arise in antisymplectic geometry, although we should stress that it remains unclear if it is useful in physics. Nevertheless, the strategy that we shall adapt here is to append a general Grassmann-even (possibly degenerate) second-order operator source $\frac{1}{2} \Delta_R$ to the right-hand side of the nilpotency condition (1.3):

$$ \Delta^2 = \frac{1}{2} \Delta_R. $$

(5.1)

A covariant and general way of realizing the second-order $\Delta_R$ operator is to write

$$ \Delta_R \equiv \Delta_{F,R} + V_R + n_R, $$

(5.2)

where

$$ \Delta_{F,R} \equiv \frac{(-1)^{\varepsilon_A}}{2} (\partial^l_A + F_A) R^{AB} \partial^l_B, $$

(5.3)

is an Grassmann-even Laplacian based on $F_A$ and $R^{AB}$. We have included a Grassmann-even vector field

$$ V_R \equiv V^A_R \partial^l_A $$

(5.4)

and a scalar function $n_R$ to give a systematic treatment. Note that the vector field $V_R$ is the difference of the subleading connection terms inside $\Delta_R$ and $\Delta_{F,R}$. We shall show below that the $n_R$ term is completely determined by consistency, while $V_R$ in principle can be any locally Hamiltonian vector field subjected to the following restriction: Both $V^A_R$ and $n_R$ should be proportional to the $R$-source (or its derivatives) in order to restore nilpotency (1.3) in the limit $R \to 0$.

The new condition (5.1) still imposes the Jacobi identity (3.1) for the antibracket $(\cdot, \cdot)$ at the third order, since the modification is just of second order. (We mention, for later, that the Jacobi identity alone guarantees the existence of a nilpotent $\Delta_F$ operator and its quantization scheme, cf. Sections 8-9, regardless of how the nilpotency (5.1) of $\Delta$ is broken at lower orders.) The second-order terms in eq. (5.1) implies that the $F$-curvature $R^{AB}$ defined in eq. (3.7) should be identified with the principal
symbol $\mathcal{R}^{AB}$ appearing inside the $\Delta_{F;\mathcal{R}}$ operator (5.3), thereby justifying the notation. Note that the Leibniz rule (3.13) is no longer valid. To see this, it is useful to define an even $\mathcal{R}$-bracket [19]

$$
(f, g)_{\mathcal{R}} \equiv [\Delta_{\mathcal{R}}, f], g]_1 \equiv \Delta_{\mathcal{R}}(fg) - (\Delta_{\mathcal{R}} f)g - f(\Delta_{\mathcal{R}} g) + fg(\Delta_{\mathcal{R}} 1) = (f\partial_A^B)\mathcal{R}^{AB}(\partial_B^g) = (-1)^{\epsilon_f\epsilon_g}(g, f)_{\mathcal{R}}.
$$

(5.5)

It turns out that the $\mathcal{R}$-bracket $(\cdot, \cdot)_{\mathcal{R}}$ measures the failure of the Leibniz rule:

$$
\frac{1}{2}(f, g)_{\mathcal{R}} = (-1)^{\epsilon_f}\Delta_f(f, g) - (-1)^{\epsilon_f}(\Delta_FF, g) + (f, \Delta_Fg) .
$$

(5.6)

Note that this $\mathcal{R}$-bracket $(\cdot, \cdot)_{\mathcal{R}}$ does not satisfy a Jacobi identity. (In fact, we shall see that the closeness relation (4.8) for $\mathcal{R}_{AB}$ will instead lead to a compatibility relation (5.8) below.) Since $\Delta_F^2 - \frac{1}{2}\Delta_{F;\mathcal{R}}$ is a first-order operator, cf. eqs. (2.1) and (5.1), the commutator

$$
\frac{1}{2}[\Delta_{F;\mathcal{R}}, \Delta_F] = [\Delta_F, \Delta_F^2 - \frac{1}{2}\Delta_{F;\mathcal{R}}]
$$

(5.7)

becomes a second-order operator at most. (We shall improve this estimate in Lemma 5.1 below.) This fact already implies that the two brackets $(\cdot, \cdot)$ and $(\cdot, \cdot)_{\mathcal{R}}$ are compatible in the sense that

$$
\sum_{\text{cycl. } f, g, h} (-1)^{\epsilon_f(\epsilon_{h+1})}((f, g), h)_{\mathcal{R}} = \sum_{\text{cycl. } f, g, h} (-1)^{\epsilon_f(\epsilon_{h+1})+\epsilon_g}((f, g)_{\mathcal{R}}, h) .
$$

(5.8)

Phrased differently, one may define a one-parameter family of antisymplectic two-forms

$$
E(\theta) = E + \theta R \equiv E + R\theta = \frac{1}{2}d\Gamma^A E_{AB}(\theta) \wedge d\Gamma^B , \quad dE(\theta) = 0 ,
$$

(5.9)

which depends on a Grassmann-odd parameter $\theta$. In components it reads

$$
E_{AB}(\theta) = E_{AB} + \mathcal{R}_{AB}\theta ,
$$

(5.10)

$$
E^{AB}(\theta) = E^{AB} + (-1)^{\epsilon_A}\mathcal{R}_{AB} = E^{AB} + \mathcal{R}_{AB}(\theta(-1)^{\epsilon_B} .
$$

(5.11)

There exists locally an antisymplectic one-form potential

$$
U(\theta) \equiv U_A(\theta)d\Gamma^A , \quad U_A(\theta) \equiv U_A + F_A\theta ,
$$

(5.12)

dU(\theta) = E(\theta) , \quad \partial_A U_B(\theta) - (-1)^{\epsilon_A}\epsilon_B(A \leftrightarrow B) = E_{AB}(\theta) .
$$

We will now improve the estimate from eq. (5.7):

**Lemma 5.1** The commutator $[\Delta_F, \Delta_{F;\mathcal{R}}]$ is always a first-order operator at most.

**Proof of Lemma 5.1:** Note that the commutator $[\Delta_F, \Delta_{F;\mathcal{R}}]$ appears inside the square

$$
(\Delta_F(\theta))^2 = \Delta_F^2 + \theta[\Delta_{F;\mathcal{R}}, \Delta_F] = \Delta_F^2 + [\Delta_F, \Delta_{F;\mathcal{R}}]\theta
$$

(5.13)

of the Grassmann-odd second-order operator

$$
\Delta_F(\theta) \equiv \Delta_F + \theta\Delta_{F;\mathcal{R}} \equiv \Delta_F + \Delta_{F;\mathcal{R}}\theta = \frac{(1)^{\epsilon_A}}{2}(\partial_A^F + F_A)E^{AB}(\theta)\partial_B^I .
$$

(5.14)
One knows from the general discussion in the previous Section 3 that the third-order terms in the square (5.13) vanish because $E^{AB}(\theta)$ satisfies the Jacobi identity (3.3). Moreover, the second-order terms in the square (5.13) are of the form
\begin{equation}
\frac{(-1)^{\varepsilon_c}}{4} E^{AB}(\theta) \mathcal{R}_{BC} E^{CD}(\theta) \partial^A_D \partial^B_A = \frac{1}{4} \mathcal{R}^{AB} \partial^A_B \partial^A_A,
\end{equation}
cf. eqs. (3.5) and (3.6). It is easy to see that the two $\theta$-dependent terms inside the left-hand side of eq. (5.15) cancel against each other. In fact, each of the two terms vanish separately due to skewsymmetry:
\begin{equation}
(-1)^{\varepsilon_c+\varepsilon_f} E^{AB} \mathcal{R}_{BC} E^{CD} \mathcal{R}_{DF} E^{FG} = \mathcal{R}^{AC} E_{CD} \mathcal{R}^{DG} = (-1)^{(\varepsilon_A+1)(\varepsilon_G+1)}(A \leftrightarrow G).
\end{equation}
Therefore, the $\theta$-dependent part of the square (5.13) must be of first order at most.

(One may also give a proof of Lemma 5.1 based on Lemma B.1 in Appendix B.) Lemma 5.1 implies (for instance via the technology of Ref. [16]) that
\begin{equation}
\Delta_{F,R}(f,g) - (\Delta_{F,R} f,g) - (f, \Delta_{F,R} g) = (-1)^{\varepsilon_f} \Delta_f(f,g)_R - (-1)^{\varepsilon_f}(\Delta_f f, g)_R
\end{equation}
\begin{equation}
- (f, \Delta_f g)_R,
\end{equation}
\begin{equation}
\Delta^2 + \frac{1}{2} \Delta_{F,R} (f,g) = ((\Delta^2 - \frac{1}{2} \Delta_{F,R}) f, g) + (f, (\Delta^2 - \frac{1}{2} \Delta_{F,R}) g).
\end{equation}
More generally, there exists a superformulation
\begin{equation}
\Delta(\theta) \equiv \Delta + \theta \Delta_R \equiv \Delta + \Delta_R \theta = \frac{(-1)^{\varepsilon_A}}{2} (\partial^A_A + F_A(\theta)) E^{AB}(\theta) \partial^B_B + \nu(\theta),
\end{equation}
where
\begin{equation}
\nu(\theta) \equiv \nu + \theta n_R \equiv \nu + n_R \theta,
\end{equation}
and
\begin{equation}
F_A(\theta) \equiv F_A + 2 E_{AB} V_R^B \theta \equiv F_A - 2 V_R^B E_{BA} \theta.
\end{equation}
The nilpotency condition
\begin{equation}
\left(\Delta(\theta) - \frac{1}{2} \partial^A_A\right)^2 = 0
\end{equation}
precisely encodes the deformed condition (5.1) and its consistency relation
\begin{equation}
0 = [\Delta, [\Delta, \Delta]] = [\Delta, \Delta_R] = [\Delta_\nu, \Delta_{F,R} + V_R + n_R] = [\Delta_{F,R} + V_R, \Delta_R + n_R - V_R, \nu].
\end{equation}
Note in the last line of eq. (5.23) that the first term $[\Delta_{F,R}]$ and the two last terms $[\Delta_{F,R} + V_R, \nu]$ are all of first order. Hence, the second term $[\Delta_{F,R}, V_R]$ must be of first order as well. This in turn implies that $V_R$ should be a generating vector field for an anticanonical transformation:
\begin{equation}
V_R(f,g) = (V_R(f), g) + (f, V_R(g))
\end{equation}
Since the antibracket is non-degenerated, it follows that $V_R$ must be a locally Hamiltonian vector field, which we, for simplicity, will assume is a globally Hamiltonian vector field
\begin{equation}
V_R = - 2(\nu_R, \cdot),
\end{equation}
with some Fermionic globally defined Hamiltonian $\nu_R$. The factor "$-2" in eq. (5.25) is chosen for later convenience. The Hamiltonian $\nu_R$ in eq. (5.25) should be considered as an additional geometric
input, which labels the different ways (5.1) of breaking the nilpotency of $\Delta$. It is a priori only defined in eq. (5.25) up to an odd constant. We fix this constant by requiring that

$$\nu_R \to 0 \quad \text{for} \quad R \to 0. \quad (5.26)$$

Altogether, the Hamiltonian $\nu_R$ does not contribute to the curvature

$$\overrightarrow{\partial}_A^B \theta - (-1)^{e_A e_B} (A \leftrightarrow B) = \mathcal{R}_{AB} \quad (5.27)$$

of the line bundle connection

$$F_A(\theta) = F_A + 4(\overrightarrow{\partial}_A \nu_R) \theta. \quad (5.28)$$

Now let us continue the investigation of the deformed condition (5.1). The first-order terms of eq. (5.1) cancel if and only if

$$\Delta^2_F - \frac{1}{2} \Delta_{F,R} = (\nu - \nu_R, \cdot). \quad (5.29)$$

This is a differential equation for the function $\nu = \nu(\Gamma)$, or, equivalently, for the difference $\nu - \nu_R$. It now becomes clear that the $\nu_R$ function provides an auxiliary curvature background for the $\nu$ function. Since we assume that $\nu_R$ is given, we will now focus on the difference $\nu - \nu_R$ rather than on $\nu$ itself. The Frobenius integrability condition for eq. (5.29) comes from the fact that the operator $\Delta^2_F - \frac{1}{2} \Delta_{F,R}$ differentiates the antibracket, cf. eq. (5.18). This implies that the difference $\nu - \nu_R$ can be written as a contour integral

$$(\nu - \nu_R)(\Gamma) = (\nu - \nu_R)(\Gamma_0) + \int_{\Gamma_0}^{\Gamma} ((\Delta^2_F - \frac{1}{2} \Delta_{F,R})^{A_B} E_{A_B}) \bigg|_{\Gamma = \Gamma'} d\Gamma' \quad (5.30)$$

that is independent of the curve (aside from the two endpoints). It only depends on $E, F,$ and an odd integration constant $(\nu - \nu_R)(\Gamma_0)$. In particular, we conclude that the difference $\nu - \nu_R$ does not introduce any new geometric structures. The first-order commutator from Lemma 5.1 can now be expressed in terms of the difference $\nu - \nu_R$ as follows:

$$\frac{1}{2} [\Delta_{F,R}; \Delta_F] = [\Delta_F; \Delta^2_F - \frac{1}{2} \Delta_{F,R}] = \Delta_F (\nu - \nu_R, \cdot) - (\nu - \nu_R, \Delta_F (\cdot))$$

$$= (\Delta_F (\nu - \nu_R, \cdot) - \frac{1}{2} (\nu - \nu_R, \cdot)_{\mathcal{R}}. \quad (5.31)$$

Here, eq. (5.29) is used in the second equality and the deformed Leibniz rule (5.6) is used in the third (=last) equality.

Finally, the zeroth-order terms of eq. (5.1) cancel if and only if

$$n_R = 2(\Delta_F \nu), \quad (5.32)$$

so this fixes completely the Grassmann-even function $n_R$. One can show that if the Hamiltonian vector field $V^A_R$ vanishes in the flat limit $\mathcal{R} \to 0$, then the $n_R$ function, defined via eq. (5.32), automatically does the same, cf. eq. (6.2) below. The nilpotency-breaking operator $\Delta_R$ will therefore vanish for $\mathcal{R} \to 0$, as it should.

### 6 Nilpotency Conditions: Part II

After this digression into non-zero $\mathcal{R}$ curvature, let us now return to the nilpotent (and ordinary physical) situation $\Delta^2 = 0$, where $\mathcal{R}, V^A_R$ and $n_R$ are all zero. Not much changes for the condition
(5.29) for the first-order terms other than one should remove the \(\nu_R\) function and the \(\Delta_{F,R}\) operator from the Frobenius integrability condition (5.18), the differential eq. (5.29), and the contour integral (5.30). (Of course, now the Frobenius integrability condition is just an easy consequence of the Leibniz rule (3.13) applied twice.) The condition (5.32) for the zeroth-order terms becomes

\[
(\Delta_F \nu) = 0 .
\]

Equation (6.1) is not an independent condition but it follows instead automatically from the previous requirements. \(\text{Proof:}\)

\[
-(\Delta_F \nu) = \frac{(-1)^{\varepsilon_A}}{2} (\partial_A + F_A)(\nu, \Gamma^A) + \frac{(-1)^{\varepsilon_A}}{2} (\partial_A + F_A) \Delta^2_F \Gamma^A
\]

\[
= \frac{(-1)^{\varepsilon_A+\varepsilon_B}}{4} (\partial_A + F_A)(\partial_B + F_B)(\Gamma^B, \Delta_F \Gamma^A)
\]

\[
= \frac{(-1)^{\varepsilon_A}}{8} (\partial_A + F_A)(\partial_B + F_B) \Delta_F (\Gamma^B, \Gamma^A)
\]

\[
= \frac{(-1)^{\varepsilon_A+\varepsilon_C}}{16} (\partial_A + F_A)(\partial_B + F_B)(\partial_C + F_C)(\Gamma^C, (\Gamma^B, \Gamma^A))(-1)^{\varepsilon_A+1}(\varepsilon_A+1) = 0 .
\]

Here, the \(\nu\) eq. (5.29) is used in the second equality, the Leibniz rule (3.13) in the fourth equality, the Jacobi identity (3.1) in the sixth (=last) equality, and the zero curvature condition (3.10) in the second, fourth and sixth equality.

\(\square\)

7 An Explicit Solution \(\nu_F\)

Remarkably, the integral (5.30) can be performed.

**Proposition 7.1** The odd quantity

\[
\nu_F \equiv \nu_F^{(0)} + \frac{\nu^{(1)}}{8} - \frac{\nu^{(2)}}{24}
\]

is a solution to the differential eq. (5.29) for the difference \(\nu - \nu_R\), even if the line bundle connection \(F\) is not flat.

Here, \(\nu_F^{(0)}\), \(\nu^{(1)}\) and \(\nu^{(2)}\) are given by eqs. (4.6), (1.8) and (1.9), respectively. Proposition 7.1 is proven in Appendix B by repeated use of the Jacobi identity (3.3) and the closeness relation (3.4). Notice that under the \(F \leftrightarrow \rho\) identification (3.11), the \(F\)-dependent Definitions (4.6) and (7.1) reduce to their \(\rho\) counterparts (1.6) and (1.7),

\[
\nu_F = \nu_{\rho} , \quad \nu_F^{(0)} = \nu_{\rho}^{(0)} .
\]

**Notation:** \(\nu_F\) or \(\nu_{\rho}\) with subscript “\(F\)” or “\(\rho\)” denotes one particular solution (7.1) or (1.6) to the difference \(\nu - \nu_R\) in eq. (5.29), respectively.

**Proposition 7.2** The \(\nu_F\) quantity (7.1) is invariant under general coordinate transformations, i.e. it is a scalar, even if the line bundle connection \(F\) is not flat.
Proof of Proposition 7.2: Under an arbitrary infinitesimal coordinate transformation $\delta \Gamma^A = X^A$, one calculates [9]

$$\delta \nu_F^{(0)} = -\frac{1}{2} \Delta_1 \text{div}_1 X,$$

(7.3)

$$\delta \nu_F^{(1)} = 4\Delta_1 \text{div}_1 X + (-1)^{\varepsilon_A} (\delta_C E^{AB}) (\delta_B \delta_A X^C),$$

(7.4)

$$\delta \nu_F^{(2)} = 3(-1)^{\varepsilon_A} (\delta_C E^{AB}) (\delta_B \delta_A X^C),$$

(7.5)

where $\Delta_1$ and $\text{div}_1$ denote the expressions (1.1) and (10.14) for the odd Laplacian $\Delta_{\rho=1}$ and the divergence $\text{div}_{\rho=1}$ with $\rho$ replaced by 1. One easily sees that while the three constituents $\nu_F^{(0)}$, $\nu_F^{(1)}$ and $\nu_F^{(2)}$ separately have non-trivial transformation properties, the linear combination $\nu_F$ in eq. (7.1) is indeed a scalar. Proposition 7.2 also follows from the identification of $\nu_F$ as an odd scalar curvature, cf. eq. (11.8) below.

The difference $\nu - \nu_R$ is only determined up to an odd integration constant because the defining relation (5.29) is a differential relation. The explicit solution $\nu_F$ in (7.1) provides us with an opportunity to fix this odd integration constant once and for all. Out of all the solutions to the difference $\nu - \nu_R$, we choose the $\nu_F$ solution (7.1), i.e. we identify from now on

$$\nu \equiv \nu_F + \nu_R.$$

(7.6)

We do this for two reasons. Firstly, any odd constants inside the $\nu_F$ expression (7.1) can only arise implicitly through $E$ and $F$, which means that if $E$ and $F$ do not carry any odd constants, then the $\nu_F$ solution (7.1) will be free of odd constants as well. Similarly, the $\nu_R$ part does not contain odd constants because of the boundary condition (5.26). Secondly, the expression $\nu_F$ is the only solution that has an interpretation as an odd scalar curvature, cf. eq. (11.8) below. This completes the reduction of a general second-order $\Delta$ operator to

$$\Delta = \Delta_F + \nu = \Delta_F + \nu_F + \nu_R \rightarrow \Delta_\rho + \nu_\rho \quad \text{for} \quad R \to 0.$$

(7.7)

8 The $\Delta_E$ operator

Let us briefly outline the connection to Khudaverdian’s $\Delta_E$ operator [10, 11, 12, 13], which takes semidensities to semidensities. The $\Delta_E$ operator was defined in Ref. [9] as

$$\Delta_E = \Delta_1 + \nu_{(1)}^{(1)} - \nu_{(2)}^{(2)},$$

(8.1)

where $\Delta_1$ denotes the expression (1.1) for the odd Laplacian $\Delta_{\rho=1}$ with $\rho$ replaced by 1. Some of the strengths of Definition (8.1) are that it works in any coordinate system and that it is manifestly independent of $\rho$ or $F$. However, it is a rather lengthy calculation to demonstrate in a $\rho$-less or $F$-less environment that $\Delta_E$ has the pertinent transformation property under general coordinate transformations, and that it is nilpotent

$$\Delta_E^2 = 0,$$

(8.2)

cf. Ref. [9]. Once we are given a density $\rho$, the situation simplifies considerably. Then, the $\Delta_E$ operator becomes just the operator $\Delta \equiv \Delta_\rho + \nu_\rho$ conjugated with the square root of $\rho$:

$$\Delta_E = \sqrt{\rho} \Delta \frac{1}{\sqrt{\rho}}.$$

(8.3)
Proof of eq. (8.3): Let \( \sigma \) denote an arbitrary semidensity. Then, it follows from the explicit \( \nu_{\rho} \) formula (1.6) that

\[
(\Delta_E \sigma) = (\Delta_1 \sigma) + \left( \frac{\nu^{(1)}}{8} - \frac{\nu^{(2)}}{24} \right) \sigma = (\Delta_1 \sigma) - \frac{\sigma}{\sqrt{\rho}} + \nu_{\rho} \sigma
\]

\[
= \sqrt{\rho} (\Delta_1 \frac{\sigma}{\sqrt{\rho}}) + \nu_{\rho} \sigma = \sqrt{\rho} (\Delta_1 \frac{\sigma}{\sqrt{\rho}}) + \frac{\sigma}{\sqrt{\rho}}.
\]

(8.4)

It is remarkable that the \( \sqrt{\rho} \)-conjugated \( \Delta \) operator \( \sqrt{\rho} \Delta \sqrt{\rho} \) does not depend on \( \rho \) at all! On the other hand, it is obvious that the operator \( \sqrt{\rho} \Delta \sqrt{\rho} \) is nilpotent and that it satisfies the required transformation law under general coordinate transformations, i.e. that it takes semidensities to semidensities. This is because the \( \Delta \) operator itself is a nilpotent operator and \( \Delta \) takes scalar functions to scalar functions. Let us also mention that

\[
\nu_{\rho} = (\Delta 1) = \frac{1}{\sqrt{\rho}} (\Delta_E \sqrt{\rho}) \cdot
\]

(8.5)

The right-hand side of eq. (8.5) served as a definition of the odd scalar \( \nu_{\rho} \) in Ref. [9].

More generally, the operators \( \Delta_E \) and \( \Delta = \Delta_F + \nu_F + \nu_R \) are linked via

\[
\Delta_E = \Delta - \frac{(-1)^{\varepsilon_A}}{2} F_A (\Gamma^A, \cdot) - \nu_F^{(0)} - \nu_R.
\]

(8.6)

Equation (8.6) may be viewed as a generalization of eq. (8.3) to non-flat \( F \)'s, or, equivalently, to non-nilpotent \( \Delta \)'s, cf. eq. (3.10). It might be worth emphasizing that \( \Delta_E \) is nilpotent even in this situation, since \( \Delta_E \) only depends on \( E \).

9 \hspace{1cm} F-Independent Formalism

There exists [9, 15] a manifestly \( F \)-independent quantization scheme based on the \( \Delta_E \) operator. Since we will demand that the quantization is covariant with respect to the antisymplectic phase space, it will be necessary to use first-level formalism or one of its higher-level generalizations [2, 20]. See Ref. [8] for a review of the multi-level formalism. It turns out to be most efficient to use the second-level formalism in order not to deal directly with weak quantum master equations [21]. Let \( \Gamma^A \) denote all the zeroth- and first-level fields and antifields, and let \( \lambda^\alpha \) denote the second-level Lagrange multipliers for the first-level gauge-fixing constraints. Assume also that there is no dependence on the corresponding second-level antifields \( \lambda^\alpha \). The second-level partition function

\[
Z = \int [d\Gamma] [d\lambda] e^{\frac{i}{\hbar} (W_E + X_E)}
\]

(9.1)

contains two Boltzmann semidensities: a gauge-generating semidensity \( e^{\frac{i}{\hbar} W_E} \) and a gauge-fixing semidensity \( e^{\frac{i}{\hbar} X_E} \), where \( W_E \) and \( X_E \) denote the corresponding quantum actions. The two Boltzmann semidensities are both required to satisfy strong quantum master equations

\[
\Delta_E e^{\frac{i}{\hbar} W_E} = 0, \quad \Delta_E e^{\frac{i}{\hbar} X_E} = 0,
\]

(9.2)

or equivalently,

\[
\frac{1}{2} (W_E, W_E) = i\hbar \Delta_1 W_E + \hbar^2 \Delta_E 1, \quad \frac{1}{2} (X_E, X_E) = i\hbar \Delta_1 X_E + \hbar^2 \Delta_E 1,
\]

(9.3)
where
\[
\Delta E^1 = \frac{\nu^{(1)}}{8} - \frac{\nu^{(2)}}{24}.
\] (9.4)

The caveat is that the quantum actions \( W_E \) and \( X_E \) are not scalars. They obey non-trivial transformation laws under general coordinate transformations, since they are logarithms of semidensities. It is shown in Appendix A that the partition function (9.1) is independent of the gauge choice \( X_E \).

If we are given a density \( \rho \), we may introduce a nilpotent \( \Delta \) operator (8.3) and Boltzmann scalars \( e^{i\bar{h}W} \) and \( e^{i\bar{h}X} \) by dressing appropriately with square roots of \( \rho \):
\[
\sqrt{\rho} \Delta = \Delta E \sqrt{\rho}, \quad e^{i\bar{h}W_E} = \sqrt{\rho} e^{i\bar{h}W}, \quad e^{i\bar{h}X_E} = \sqrt{\rho} e^{i\bar{h}X}.
\] (9.5)

Then \( \Delta = \Delta + \nu F \) and the two scalar actions \( W \) and \( X \) will satisfy the strong quantum master eq. (1.4) from the Introduction, which in non-exponential form reads
\[
\frac{1}{2}(W, W) = i\hbar \Delta W + \hbar^2 \nu, \quad \frac{1}{2}(X, X) = i\hbar \Delta X + \hbar^2 \nu.
\] (9.6)

The partition function (9.1) then reduces to the familiar \( W \)-\( X \) form:
\[
Z = \int \rho[\text{d}\Gamma][\text{d}\lambda] e^{i(W + X)}. \] (9.7)

Conversely, since the partition function (9.7) via the above identifications (9.5) can be written in the manifestly \( \rho \)-independent form (9.1), one may state that in this sense the partition function (9.7) does not depend on \( \rho \). The point is that the well-known ambiguity in the choice of measure that exists in the field-antifield formalism has been fully transcribed into an ambiguity in the choice of the Boltzmann semidensity \( e^{i\bar{h}W_E} \). Put differently, if one splits the Boltzmann semidensity \( e^{i\bar{h}W_E} \) into a Boltzmann scalar \( e^{i\bar{h}W} \) and a density \( \rho \) as done in eq. (9.5), the measure ambiguity sits inside the scalar \( e^{i\bar{h}W} \), not in \( \rho \), as \( \rho \) actually drops out of \( Z \).

More generally, imagine that we are given a non-nilpotent operator \( \Delta \equiv \Delta_E + \nu_F + \nu_R \) with a non-flat line bundle connection \( F \) that satisfies the deformed nilpotency condition (5.1). We can still define the partition function in this situation via the above quantization scheme (9.1) based on the nilpotent \( \Delta_E \) operator. Such an approach will of course be manifestly \( F \)-independent by construction.

10 Connection

We now introduce a connection \( \nabla : TM \times TM \to TM \). See Ref. [19, 22] for related discussions. The left covariant derivative \( (\nabla_A X)^B \) of a left vector field \( X^A \) is defined as [19]
\[
(\nabla_A X)^B \equiv (\partial^B_A X^B) + (-1)^{\varepsilon(X^B)}(\varepsilon_A + \varepsilon_X) \Gamma^B_A C X^C, \quad \varepsilon(X^A) = \varepsilon_X + \varepsilon_A. \] (10.1)

The word “left” implies that \( X^A \) and \( (\nabla_A X)^B \) transform with left derivatives
\[
X'^B = X^A(\partial^B_A \Gamma'^B), \quad (\partial^B_A \Gamma'^B)(\nabla_B X)^C = (\nabla_A X)^B(\partial^B_A \Gamma'^C), \] (10.2)
under general coordinate transformations \( \Gamma^A \to \Gamma'^A \). It is convenient to introduce a reordered Christoffel symbol
\[
\Gamma^A_{BC} \equiv (-1)^{\varepsilon_A \varepsilon_B} \Gamma^A_{BC}. \] (10.3)
to minimize the appearances of sign factors. On an antisymplectic manifold \((M; E)\), it is furthermore possible to define a Christoffel symbol with three lower indices

\[ \Gamma_{ABC} \equiv E_{AD} \Gamma^D_{BC}(-1)^\epsilon_b . \]  

(10.4)

Let us also define

\[ \gamma_{ABC} \equiv \Gamma_{ABC} - \frac{1}{3}(E_{A(B} \partial_{C)} + E_{AC} \partial_B - (-1)^\epsilon_b \epsilon^C_{(B \leftrightarrow C)} . \]  

(10.5)

\(\gamma_{ABC}\) is not a tensor but it still has some useful properties, see eqs. (10.8) and (10.11) below. One can think of \(\gamma_{ABC}\) as parametrizing all the possible connections \(\nabla\) on \((M; E)\).

An antisymplectic connection \(\Gamma^B_A\) satisfies by definition [19]

\[ 0 = (\nabla A E)^{BC} \equiv (\partial^B_A E^{BC}) + \left( \Gamma^B_A \Gamma^D_{BC} - (-1)^{(\epsilon_b+1)(\epsilon_c+1)}(B \leftrightarrow C) \right) , \]  

so that the antisymplectic metric \(E^{AB}\) is covariantly preserved. In terms of the two-form \(E_{AB}\), the antisymplectic condition reads

\[ 0 = (\nabla A E)_{BC} \equiv (\partial^B_A E_{BC}) - ((-1)^{\epsilon_b A} b \Gamma_{BAC} - (-1)^{\epsilon_b c} (B \leftrightarrow C)) . \]  

(10.7)

Written in terms of the \(\gamma_{ABC}\) symbol, the antisymplectic condition (10.7) becomes a purely algebraic equation, due to the closeness relation (3.4):

\[ \gamma_{ABC} = (-1)^{\epsilon_b A \epsilon_b B + \epsilon_b B \epsilon_c C + \epsilon_b C \epsilon_a A} \gamma_{CBA} . \]  

(10.8)

A torsion-free connection has the following symmetry in the lower indices:

\[ \Gamma^A_{BC} = -(-1)^{(\epsilon_b + 1)(\epsilon_c + 1)} \Gamma^A_{CB} , \]  

(10.9)

\[ \Gamma_{ABC} = (-1)^{\epsilon_b c \gamma_{ABC}} , \]  

(10.10)

\[ \gamma_{ABC} = (-1)^{\epsilon_b \epsilon_c \gamma_{ABC}} . \]  

(10.11)

Note that \((-1)^{\epsilon_b A \epsilon_b B} \gamma_{BAC} = \gamma_{ABC} = (-1)^{\epsilon_b b \epsilon_c \gamma_{ACB}}\) is totally symmetric for an antisymplectic torsion-free connection. (Similar results hold for even symplectic structures.)

A connection \(\nabla\) can be used to define a divergence of a Bosonic vector field \(X^A\) as

\[ \text{str}(\nabla X) \equiv (-1)^{\epsilon_A (\nabla A X)^A} = ((-1)^{\epsilon_A A} \partial^B_A + \Gamma^B_{BA}) X^A , \]  

(10.12)

\[ \varepsilon_X = 0 . \]

On the other hand, the divergence is defined in terms of \(F\) or \(\rho\) as

\[ \text{div}_F X \equiv (-1)^{\epsilon_A} (\partial^B_A + F_A) X^A , \]  

\[ \text{div}_\rho X \equiv (-1)^{\epsilon_A} \rho \partial^B_A (\rho X^A) . \]  

(10.13)

(10.14)

See Ref. [23] for a mathematical exposition of divergence operators on supermanifolds. Under the \(F \leftrightarrow \rho\) identification (3.11), the last two Definitions (10.13) and (10.14) agree:

\[ \text{div}_F X = \text{div}_\rho X . \]  

(10.15)
In order to have a unique divergence operator (and hence a unique notion of volume), it is necessary to impose the following compatibility condition between \( F_A \) and the Christoffel symbols \( \Gamma^{A}_{BC} \):

\[
\Gamma^{B}_{BA} = (-1)^{\varepsilon_A}F_A .
\]  

(10.16)

We shall only consider antisymplectic, torsion-free, and \( F \)-compatible connections \( \nabla \), i.e., connections that satisfy the three conditions (10.6), (10.9) and (10.16). The first and third condition ensure the compatibility with \( E \) and \( F \), respectively. The second (the torsion-free condition) guarantees compatibility with the closeness relation (3.4). It can be demonstrated that connections satisfying these three conditions exist locally for \( N > 1 \), where \( 2N \) denotes the number of antisymplectic variables \( \Gamma^A, A = 1, \ldots, 2N \). (There are counterexamples for \( N=1 \) where \( \nabla \) need not exist.) For connections satisfying the three conditions, the \( \Delta_F \) operator can be written on a manifestly covariant form

\[
\Delta_F = \frac{(-1)^{\varepsilon_A}}{2}\nabla_A E^{AB}\nabla_B = \frac{(-1)^{\varepsilon_B}}{2} E^{BA}\nabla_A\nabla_B .
\]  

(10.17)

### 11 Curvature

The Riemann curvature tensor \( R^{C}_{ABD} \) is defined as the commutator of the \( \nabla \) connection

\[
( [\nabla_A, \nabla_B]X)^C = R^{C}_{ABD}X^D(-1)^{\varepsilon_C}X^{(\varepsilon_C + \varepsilon_D)} ,
\]  

(11.1)

so that

\[
R^{C}_{ABD} = (\partial^l_A\Gamma^{C}_{BD}) + (-1)^{\varepsilon_B\varepsilon_C}\Gamma^{E}_{CAB}\Gamma^{E}_{BD} - (-1)^{\varepsilon_A\varepsilon_B}(A \leftrightarrow B) .
\]  

(11.2)

It is useful to define a reordered Riemann curvature tensor \( R^{A}_{BCD} \) as

\[
R^{A}_{BCD} \equiv (-1)^{\varepsilon_A}R^{C}_{ABD} = (-1)^{\varepsilon_A\varepsilon_B}(\partial^l_A\Gamma^{C}_{BD}) + \Gamma^{E}_{ABE}\Gamma^{E}_{CD} - (-1)^{\varepsilon_B\varepsilon_C}(B \leftrightarrow C) .
\]  

(11.3)

It is interesting to consider the various contractions of the Riemann curvature tensor. There are two possibilities. Firstly, there is the Ricci two-form

\[
\mathcal{R}_{AB} \equiv R^{C}_{AB}(-1)^{\varepsilon_C} = (\partial^j_AF_B) - (-1)^{\varepsilon_A\varepsilon_B}(A \leftrightarrow B) .
\]  

(11.4)

However, the Ricci two-form \( \mathcal{R}_{AB} \) typically vanishes, cf. eq. (3.10), and even if it does not vanish, its antisymmetry (3.8) means that \( \mathcal{R}_{AB} \) cannot successfully be contracted with the antisymplectic metric \( E^{AB} \) to yield a non-zero scalar curvature, cf. eq. (2.5). Secondly, there is the Ricci tensor

\[
R_{AB} \equiv R^{C}_{CAB} = (-1)^{\varepsilon_C}(\partial^j_C + F_C)\Gamma^{C}_{AB} - (\partial^j_AF_B)(-1)^{\varepsilon_B} \Gamma^C_{AD}\Gamma^D_{CB} .
\]  

(11.5)

Note that when the torsion tensor and Ricci two-form vanish, the Ricci tensor \( R_{AB} \) possesses exactly the same \( A \leftrightarrow B \) symmetry (2.5) as the antisymplectic metric \( E^{AB} \) with upper indices

\[
R_{AB} = -(-1)^{(\varepsilon_A+1)(\varepsilon_B+1)}R^{BA} .
\]  

(11.6)

The \textit{odd scalar curvature} \( R \) is therefore defined in antisymplectic geometry as the contraction of the Ricci tensor \( R_{AB} \) and the antisymplectic metric \( E^{BA} \),

\[
R \equiv R_{AB}E^{BA} = E^{AB}R_{BA} .
\]  

(11.7)

**Proposition 11.1** For an arbitrary, antisymplectic, torsion-free, and \( F \)-compatible connections \( \nabla \), the scalar curvature \( R \) does only depend on \( E \) and \( F \) through the odd scalar \( \nu_F \),

\[
R = -8\nu_F ,
\]  

(11.8)

even if the line bundle connection \( F \) is not flat.
Proposition 11.1 is shown in Appendix C. In particular, one concludes that the scalar curvature $R$ does not depend on the connection $\Gamma^{A}_{BC}$ used.

One can perform various consistency checks on the formalism. Here, let us just mention one. For an antisymplectic connection $\nabla$, one has

$$0 = [\nabla_{A}, \nabla_{B}]E^{CD} = R_{AB}^{\ C}F^{EF}D - (-1)^{(\varepsilon_{C}+1)(\varepsilon_{D}+1)}(C \leftrightarrow D),$$

or, equivalently,

$$R^{C}_{\ ABF}E^{FD} = - (-1)^{\varepsilon_{A}\varepsilon_{B}+(\varepsilon_{C}+1)(\varepsilon_{D}+1)+(\varepsilon_{A}+\varepsilon_{B})(\varepsilon_{C}+\varepsilon_{D})}R^{D}_{\ BAF}E^{FC}.$$  (11.10)

Contracting the $A \leftrightarrow C$ and $B \leftrightarrow D$ indices in eq. (11.10) indeed produces the identity $R = R$. Had the signs turn out differently, the odd scalar curvature (11.7) would have been stillborn, i.e. always zero.

## 12 Conclusions

In this paper, we have first of all analyzed a general non-degenerate, second-order $\Delta$ operator, and found that nilpotency determines the $\Delta$ operator uniquely (after dismissing an odd constant). The result is that $\Delta$ has to be $\Delta_{\rho} + \nu_{\rho}$, where $\Delta_{\rho}$ is the odd Laplacian, and $\nu_{\rho}$ is an odd scalar function (=zeroth-order operator) that only depends on the density $\rho$ and the antisymplectic structure $E$.

Secondly, we have shown that several constructions in antisymplectic geometry can be extended to a non-flat line bundle connection $F$, which replaces $\rho$. We did this by breaking the nilpotency $\Delta^{2} = \frac{1}{2}\Delta_{\mathcal{R}}$ by a general second-order operator $\Delta_{\mathcal{R}}$, which acts as a source for the $F$-curvature $\mathcal{R}$. In this more general case, the $\Delta$ operator takes the form $\Delta_{F} + \nu_{F} + \nu_{\mathcal{R}}$, where $\Delta_{F}$ and $\nu_{F}$ are generalizations of the odd Laplacian $\Delta_{\rho}$ and the odd scalar $\nu_{\rho}$, respectively. The $\nu_{\mathcal{R}}$ term is an auxiliary curvature background encoded in the $\Delta_{\mathcal{R}}$ operator. Thirdly, we have identified the $\nu_{F}$ function with (minus $1/8$ times) the odd scalar curvature $R$ of an arbitrary antisymplectic, torsion-free, and $F$-compatible connection.

One may summarize by saying that two notions of curvature play an important rôle in this paper: 1) a line bundle curvature $\mathcal{R}_{AB}$ defined in eq. (3.7) and 2) an odd scalar curvature $R$ defined in eq. (11.7). The former provides a natural framework for several mathematical constructions, but it remains currently unclear if it would be useful in physics. On the other hand, the field-antifield formalism naturally embraces the latter type of curvature both physically and mathematically. Concretely, we saw that the odd scalar curvature $R$ manifests itself via a zeroth-order term $\nu_{F}$ in the $\Delta$ operator, which could potentially be used in a physical application some day. Altogether, the odd scalar curvature $R$ and $\nu_{F}$ represent an important milestone in our understanding of the symmetries and the supergeometric structures behind the powerful field-antifield formalism.

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A Independence of Gauge-Fixing in the $F$-Independent Formalism

In this Appendix A, we prove in two different ways that the partition function (9.1) is independent of gauge-fixing. Let us introduce the following shorthand notation

$$w \equiv e^{\hat{w}E}, \quad x \equiv e^{\hat{x}E},$$

(A.1)

for the two Boltzmann semidensities, so that the partition function (9.1) simply becomes

$$Z = \int \! [d\Gamma][d\lambda] \, w \, x.$$  \hspace{1cm}  \text{(A.2)}

The Boltzmann semidensities $w$ and $x$ are $\Delta_{E}$-closed because of the two quantum master eqs. (9.2). Since the $\Delta_{E}$ operator is nilpotent, one may argue on general grounds that an arbitrary infinitesimal variation of $x$ should be $\Delta_{E}$-exact, which may be written as

$$\delta x = [\Delta_{E}, \delta \Psi] x \equiv \Delta_{E}(\delta \Psi \, x) + \delta \Psi(\Delta_{E}x),$$

(A.3)

if one assumes that $x$ is invertible and satisfies the quantum master eq. (9.2). Phrased equivalently, the variation $\delta X_{E}$ of the quantum action is BRST-exact,

$$\delta X_{E} = (X_{E}, \delta \Psi) + \frac{\hbar}{i} \Delta_{1}(\delta \Psi) = \sigma_{X_{E}}(\delta \Psi),$$

(A.4)

where $\sigma_{X_{E}}=(X_{E}, \cdot) + \frac{\hbar}{i} \Delta_{1}$ is a quantum BRST-operator. One may now proceed in at least two ways. One axiomatic way [24] uses that the $\Delta_{E}$ operator (8.1) is symmetric,

$$\Delta_{T}E = \Delta_{E},$$

(A.5)

i.e. stable under integration by part. Then, an infinitesimal variation (A.3) of the gauge-fixing Boltzmann semidensity $x$ changes the partition function as

$$\delta Z = \int \! [d\Gamma][d\lambda] \, w \, \delta x = \int \! [d\Gamma][d\lambda] \, w \, [\Delta_{E}, \delta \Psi] x$$

$$= \int \! [d\Gamma][d\lambda] \, [(\Delta_{E}w) \, \delta \Psi \, x + w \, \delta \Psi \, (\Delta_{E}x)] = 0,$$

(A.6)

where the symmetry property (A.5) is used in the third equality and the two quantum master equations (9.2) in the fourth (= last) equality. Notice how this proof requires very little knowledge of the detailed form of $\Delta_{E}$. Another proof [2, 5, 21] uses an intrinsic infinitesimal redefinition of the integration variables,

$$\delta \Gamma^{A} = \frac{i}{2\hbar} (\Gamma^{A}, X_{E}-W_{E}) \delta \Psi + \frac{1}{2} (\Gamma^{A}, \delta \Psi) = \frac{w}{2x} (\Gamma^{A}, x \, \delta \Psi), \quad \delta \lambda^{\alpha} = 0,$$

(A.7)

to induce the allowed variation (A.3) of $x$. Now it is instructive to write the path integral integrand as a volume form $\Omega \equiv wx[d\Gamma][d\lambda]$ with measure density $wx$. The Lie-derivative is

$$\delta \Omega = (\text{div}_{wx}\delta \Gamma)\Omega.$$  \hspace{1cm}  \text{(A.8)}

In detail, the field-antifield redefinition (A.7) yields the following logarithmic variation of $\Omega$:

$$\text{div}_{wx}\delta \Gamma = \frac{(-1)^{\nu_{(1)}}}{wx} \delta_{A}(wx \, \delta \Gamma^{A}) = \frac{(-1)^{\nu_{(2)}}}{2wx} \delta_{A}^{x}w^{2}(\Gamma^{A}, x \, \delta \Psi) = \frac{w}{x} \Delta_{w^{2}}x \, \delta \Psi$$

$$= \frac{1}{x} \Delta_{1}(x \, \delta \Psi) - (\Delta_{1}w) \frac{\delta \Psi}{w} = \frac{1}{x} \Delta_{1}(x \, \delta \Psi) + (\frac{\nu_{(1)}}{8} - \frac{\nu_{(2)}}{24}) \delta \Psi = \frac{1}{x} \Delta_{E}(x \, \delta \Psi)$$

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\[ \frac{1}{x}[\Delta_E, \delta \Psi] x = \delta \ln x. \]  

(A.9)

Here, a non-trivial property of the odd Laplacian (1.1) is used in the fourth equality, the two quantum master equations (9.2) are used in the fifth and seventh equality, and the formula (A.3) for the allowed variation of \( x \) is used in the eighth (=last) equality. If one reads the above eq. (A.9) in the opposite direction, one sees that all allowed variations (A.3) of the gauge-fixing Boltzmann semidensity \( x \) can be reproduced by an intrinsic field-antifield redefinition (A.7),

\[ \delta Z = \int [\delta \Gamma][d\lambda] \omega \delta x = \int \Omega \delta \ln x = \int \Omega \, \text{div} \omega \delta \Gamma = \int \delta \Omega = 0. \]  

(A.10)

One concludes that the partition function \( Z = \int \Omega \) must be independent of the gauge-fixing \( x \) part since an intrinsic redefinition of dummy integration variables cannot change the value of the path integral.

**B Proof of Proposition 7.1**

In this Appendix B, we show that the \( \nu_F \) expression (7.1) satisfies the differential eq. (5.29) for the difference \( \nu - \nu_R \). We start by recalling that the \( \Delta_F \) operator (2.2) is

\[ \Delta_F \equiv \Delta_1 + V, \]  

(B.1)

where \( \Delta_1 \) denotes the expression (1.1) for the odd Laplacian \( \Delta_{\rho=1} \) with \( \rho \) replaced by 1, and where we, for convenience, have defined

\[ V \equiv \frac{(-1)^{\varepsilon_A}}{2} F_A(\Gamma^A, \cdot). \]  

(B.2)

**Lemma B.1** The square of the \( \Delta_F \) operator is

\[ \Delta_F^2 \equiv \Delta_1^2 + [\Delta_1, V] + V^2 = \Delta_1^2 + \frac{1}{2} \Delta_{F,R} + (\nu_F^{(0)}, \cdot). \]  

(B.3)

**Proof of Lemma B.1**: One finds by straightforward calculations that

\[ 4V^2 = (-1)^{\varepsilon_A + \varepsilon_B} F_A(\Gamma^A, F_B(\Gamma^B, \cdot)) \]

\[ = \frac{(-1)^{\varepsilon_A}}{2} F_B F_A((\Gamma^A, \Gamma^B), \cdot) + (-1)^{\varepsilon_A} F_A E^{AC}(\partial_C F_B)(\Gamma^B, \cdot) \]

\[ = \frac{(-1)^{\varepsilon_A}}{2} F_B F_A((\Gamma^A, \Gamma^B), \cdot) + (-1)^{\varepsilon_A} F_A E^{AC}[F_C \partial_B + \mathcal{R}_{CB}(-1)^{\varepsilon_B}](\Gamma^B, \cdot) \]

\[ = \frac{(-1)^{\varepsilon_A}}{2} (F_A E^{AB} F_B, \cdot) + (-1)^{\varepsilon_A + \varepsilon_C} F_A E^{AB} \mathcal{R}_{BC}(\Gamma^C, \cdot), \]  

(B.4)

and

\[ 2[\Delta_1, V] = (-1)^{\varepsilon_A} \Delta_1 F_A(\Gamma^A, \cdot) + (-1)^{\varepsilon_A} F_A(\Gamma^A, \Delta_1(\cdot)) \]

\[ = (-1)^{\varepsilon_A} \Delta_1 F_A(\Gamma^A, \cdot) + (F_A(\Gamma^A, \cdot) + F_A(\Gamma^A, \cdot) + (-1)^{\varepsilon_A} F_A(\Gamma^A, \Delta_1(\cdot))) \]

\[ = \frac{(-1)^{\varepsilon_A + \varepsilon_B}}{2} (\partial_B E^{BC} \partial_C F_A)(\Gamma^A, \cdot) + (F_A \partial_B)(\Gamma^B, \Gamma^A, \cdot) + F_A(\Delta_1 \Gamma^A, \cdot) \]

\[ = \frac{(-1)^{\varepsilon_B}}{2} (\partial_B E^{BC} \partial_C F_A + \mathcal{R}_{CA}(-1)^{\varepsilon_A} F_A(\Gamma^A, \cdot) + \frac{1}{2} (F_A \partial_B + (-1)^{\varepsilon_B} \partial_A F_B + (-1)^{\varepsilon_A + \varepsilon_B} \mathcal{R}_{BA})(\Gamma^B, \Gamma^A, \cdot) + F_A(\Delta_1 \Gamma^A, \cdot)) \]
Proof of Lemma B.2:
Combine

\[
\begin{align*}
&= \frac{(-1)^{\varepsilon_C}}{2} E^{CB} (\partial_B F_C, \cdot) + (\Delta_1 \Gamma^C)(F_C, \cdot) + \frac{(-1)^{\varepsilon_A + \varepsilon_B}}{2} (\partial_B E^{BC} R_{CA})(\Gamma^A, \cdot) \\
&\quad + \frac{(-1)^{\varepsilon_B}}{2} (\partial_A F_B)((\Gamma^B, \Gamma^A), \cdot) - \frac{(-1)^{(\varepsilon_A + \varepsilon_B + \varepsilon_C)}}{2} E^{CB} R_{BA} \partial_B \Gamma^C(\Gamma^A, \cdot) + F_A(\Delta_1 \Gamma^A, \cdot) \\
&\quad = \frac{(-1)^{\varepsilon_B}}{2} (E^{BA} \partial_A F_B, \cdot) + (F_A \Delta_1 \Gamma^A, \cdot) + \frac{(-1)^{\varepsilon_A + \varepsilon_C}}{2} \partial_A E^{AB} R_{BC}(\Gamma^A, \cdot) \\
&\quad = \frac{(-1)^{\varepsilon_A}}{2} (\partial_A (E^{AB} F_B), \cdot) + \frac{(-1)^{\varepsilon_A + \varepsilon_C}}{2} \partial_A E^{AB} R_{BC}(\Gamma^C, \cdot),
\end{align*}
\]

where the Jacobi identity (3.1) has been applied in the third and fifth equality of eqs. (B.4) and (B.5), respectively.

(As an aside, we mention that Lemma B.1 can be used to prove Lemma 5.1 in Section 5.) When one compares Lemma B.1 with the \(\nu\) differential eq. (5.29), one sees the first clue that the \(\nu_F\) expression (7.1) is a solution. More precisely, Lemma B.1 has extracted the \(\nu_F^{(0)}\) part for us. Next task is to uncover the \(\nu^{(1)}\) term (1.8).

**Lemma B.2**
\[
8(\Delta_1^2 \Gamma^A) = (\nu^{(1)}, \Gamma^A) - (-1)^{\varepsilon_C} (\partial_B E^{CD})(\partial_D \partial_C E^{BA}).
\]  

**Proof of Lemma B.2:** Combine

\[
(\partial_B \Delta_1 E^{BA}) - 2(\Delta_1^2 \Gamma^A) = \partial_B \Delta_1 E^{BA} = \frac{1}{2} (-1)^{\varepsilon_C} (\partial_B E^{CD})(\partial_D \partial_C E^{BA}) + (\partial_B \Delta_1 \Gamma^C) \partial_C E^{BA},
\]

and

\[
(\partial_B \Delta_1 E^{BA}) = \partial_B \Delta_1 (\Gamma^B, \Gamma^A) = \partial_B (\Delta_1 \Gamma^B, \Gamma^A) - (-1)^{\varepsilon_B} \partial_B (\Gamma^B, \Delta_1 \Gamma^A) \\
= \frac{1}{2} (\nu^{(1)}, \Gamma^A) + (\partial_C \Delta_1 \Gamma^B) \partial_B E^{CA} - 2(\Delta_1^2 \Gamma^A).
\]

So far, we have reproduced the \(\nu_F^{(0)}\) and the \(\nu^{(1)}\) part of the \(\nu_F\) solution to the \(\nu\) differential eq. (5.29). Finally, we should extract the \(\nu^{(2)}\) term (1.9). The prefactor 1/24 in the \(\nu_F\) formula (7.1) hints that such a calculation is going be lengthy. Rewrite first Lemma B.2 as

\[
8(\Delta_1^2 \Gamma^B) E_{BA} = (\partial_B \nu^{(1)}) - \nu_A^I,
\]

where

\[
\begin{align*}
\nu_A^I &\equiv (-1)^{\varepsilon_D} (\partial_C E^{DF})(\partial_D E^{CB}) E_{BA} = \nu_A^I + \nu_A^{III}, \\
\nu_A^{II} &\equiv (-1)^{\varepsilon_D} (\partial_D E^{BC})(\partial_C E^{DF}) E_{BA} = - \nu_A^{II} - \nu_A^{IV}, \\
\nu_A^{III} &\equiv (-1)^{\varepsilon_D} (\partial_C E^{DF}) \partial_D ((\partial_D E^{CB}) E_{BA})
\end{align*}
\]
\[
\nu_A^{IV} = (1)^{c-c_F} \nu \vec{D}_C E^{CD} E_D F \vec{D}_E F \vec{E} \vec{B}_E BA = \nu_A^{IV} + \nu_A^Y , \tag{B.12}
\]
\[
\nu_A^{IV} \equiv (1)^{c-c_F} \nu \vec{D}_C E^{CD} E_D F \vec{D}_E F \vec{E} \vec{B}_E BA = \nu_A^{IV} + \nu_A^Y , \tag{B.13}
\]
\[
\nu_A^{V} = (1)^{c-c_F} \nu \vec{D}_C E^{CD} E_D F \vec{D}_E F \vec{E} \vec{B}_E BA = -2 \nu_A^{IV} , \tag{B.14}
\]
\[
\nu_A^{VI} = (1)^{c-c_F} \nu \vec{D}_C E^{CD} E_D F \vec{D}_E F \vec{E} \vec{B}_E BA = \nu_A^{V} + \nu_A^{VII} , \tag{B.15}
\]
\[
\nu_A^{VIII} = (1)^{c} \nu \vec{D}_C E^{CD} E_D F \vec{D}_E F \vec{E} \vec{B}_E BA = \nu_A^{Y} . \tag{B.16}
\]

Here, the Jacobi identity (3.3) is used in the second equality of eq. (B.14), and the closeness relation (3.4) is used in the second equalities of eqs. (B.11) and (B.15). Altogether eqs. (B.10)–(B.16) yield

\[
\nu_A^{IV} = \nu_A^{IV} + \nu_A^{VIII} = 2 \nu_A^{IV} + \nu_A^Y = - \nu_A^{IV} + \nu_A^Y = - \nu_A^{IV} - \frac{2}{3} \nu_A^{VIII} . \tag{B.17}
\]

Ultimately, we would like to show that \( \nu_A^{IV} \) is equal to \((\vec{D}_A \nu^{(2)})/3\). The achievement in eq. (B.17) is more modest: The free “A” index on the \( \nu_A^{IV} \) expression has been moved to a derivative \( \vec{D}_A \) in \( \nu_A^{IV} \) and \( \nu_A^{VIII} \). On the other hand, differentiation with respect to \( \Gamma^A \) of the two expressions (1.9) and (1.10) for the \( \nu^{(2)} \) quantity (1.9) yields two more relations

\[
\nu_A^{IV} + 2 \nu_A^{VIII} = (\vec{D}_A \nu^{(2)}) = \nu_A^{IV} - \nu_A^{VIII} - \nu_A^X , \tag{B.18}
\]

where

\[
\nu_A^{VIII} = (1)^{c} \nu \vec{D}_C E^{CD} E_D F \vec{D}_E F \vec{E} \vec{B}_E BA = \nu_A^{IV} + \nu_A^Y , \tag{B.19}
\]
\[
\nu_A^{X} = (1)^{c} \nu \vec{D}_C E^{CD} E_D F \vec{D}_E F \vec{E} \vec{B}_E BA = (1)^{c} \nu \vec{D}_C E^{CD} E_D F \vec{D}_E F \vec{E} \vec{B}_E BA = \nu_A^{IV} + \nu_A^Y , \tag{B.20}
\]
\[
\nu_A^{X} = (1)^{c} \nu \vec{D}_C E^{CD} E_D F \vec{D}_E F \vec{E} \vec{B}_E BA = (1)^{c} \nu \vec{D}_C E^{CD} E_D F \vec{D}_E F \vec{E} \vec{B}_E BA = \nu_A^{IV} + \nu_A^Y , \tag{B.21}
\]

Here, the Jacobi identity (3.1) is used in the fourth equality of eq. (B.20). Remarkably, the \( \nu_A^{X} \) term vanishes due to an antisymmetry under the index permutation \( FDC \leftrightarrow BHG \). Altogether, \( \nu_A^{X} = \nu_A^{IV} \) and

\[
\nu_A^{X} = - \nu_A^{IV} - \frac{2}{3} \nu_A^{VIII} = - \nu_A^{IV} - \frac{2}{3} (\nu_A^{IV} - \nu_A^{VIII} - \nu_A^X) = \frac{1}{3} (\vec{D}_A \nu^{(2)}) . \tag{B.22}
\]

Combining eqs. (B.3), (B.9) and (B.22) shows that the \( \nu_F \) expression (7.1) satisfies the \( \nu \) differential eq. (5.29).

## C Proof of Proposition 11.1

In this Appendix C, we prove that the odd scalar curvature \( R \) is minus eight times the odd scalar \( \nu_F \). The odd scalar curvature

\[
R \equiv R_{AB} E^{BA} = R_I + R_{II} - R_{III} - R_{IV} \tag{C.1}
\]
inherits four terms \( R, R_{II}, R_{III} \) and \( R_{IV} \) from the expression (11.5) for the Ricci tensor \( R_{AB} \). They are defined as

\[
R_I \equiv (\varepsilon_A (\partial_A \Gamma^A_{BC})E^{CB} = R_V - R_{VI}, \quad (C.2)
\]

\[
R_{II} \equiv (\varepsilon_A F_A \Gamma^A_{BC}E^{CB} = - (\varepsilon_B F_A (\partial_B + F_B)E_{BA}, \quad (C.3)
\]

\[
R_{III} \equiv (\varepsilon_B E^{BA} (\partial_B F_B), \quad (C.4)
\]

\[
R_{IV} \equiv \Gamma_A^C \Gamma^D_{CB}E^{BA} = - R_{IV} - R_{VI}, \quad (C.5)
\]

\[
R_V \equiv (\varepsilon_A \partial_A (\Gamma^A_{BC}E^{CB} = - (\varepsilon_B \partial_A (\partial_B + F_B)E^{BA}
\]

\[
= - \nu^{(1)} - (\varepsilon_A \Gamma^A_{BC}E^{CB} = - \nu^{(1)} - (\varepsilon_A \partial_A (E^{AB}F_B), \quad (C.6)
\]

Here, the antisymplectic and the torsion-free conditions (10.6) and (10.9) are used in the second equality of eq. (C.5), and a contracted version of the antisymplectic condition (10.6)

\[
(-1)^{\varepsilon_B (\partial_B + F_B)E^{BA} + (-1)^{\varepsilon_A \Gamma^A_{BC}E^{CB} = 0 \quad (C.8)
\]

is used in the second equalities of eqs. (C.3) and (C.6). Inserting back in eq. (C.1), one finds that

\[
R = - 8\nu^{(0)} - \nu^{(1)} - \frac{1}{2}R_{VI}, \quad (C.9)
\]

where \( \nu^{(0)} \) and \( \nu^{(1)} \) are given in eqs. (4.6) and (1.8). Now it remains to eliminate \( R_{VI} \) from eq. (C.9). Note that \( R_{VI} \) only depends on the torsion-free part of the connection \( \Gamma^A_{BC} \), so one does in principle not need the torsion-free condition (10.9) from now on. One calculates that

\[
\frac{1}{2}R_{VI} = \frac{1}{2}(-1)^{\varepsilon_A (\varepsilon_B + 1)} \Gamma_B^A \varepsilon^{CD} (\partial_A E_{DF})E^{FB} = (-1)^{\varepsilon_A \Gamma_B^A \varepsilon^{CD} (\partial_D E_{AF})E^{FB}
\]

\[
= \Gamma_B^A E^{CD} (\partial_D E_{BF})E_{FA} = - \nu^{(2)} - R_{VI} \quad (C.10)
\]

Here, the closeness relation (3.4) is used in the second equality and the antisymplectic condition (10.6) in the fourth equality. In other words,

\[
R_{VI} = - \frac{2}{3} \nu^{(2)} \quad (C.11)
\]

Combining eqs. (C.9) and (C.11) yields the main result of Proposition 11.1:

\[
R = - 8\nu_F \quad (C.12)
\]

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