The AdS/CFT Correspondence Conjecture and Topological Censorship

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Abstract

In [1] it was shown that \((n + 1)\)-dimensional asymptotically anti-de-Sitter spacetimes obeying natural causality conditions exhibit topological censorship. We use this fact in this paper to derive in arbitrary dimension relations between the topology of the timelike boundary-at-infinity, \(\mathcal{I}\), and that of the spacetime interior to this boundary. We prove as a simple corollary of topological censorship that any asymptotically anti-de Sitter spacetime with a disconnected boundary-at-infinity necessarily contains black hole horizons which screen the boundary components from each other. This corollary may be viewed as a Lorentzian analog of the Witten and Yau result [2], but is independent of the scalar curvature of \(\mathcal{I}\). Furthermore, as shown in [1], the topology of \(V'\), the Cauchy surface (as defined for asymptotically anti-de Sitter spacetime with boundary-at-infinity) for regions exterior to event horizons, is constrained by that of \(\mathcal{I}\); the homomorphism \(\Pi_1(\Sigma_0) \to \Pi_1(V')\) induced by the inclusion map is onto where \(\Sigma_0\) is the intersection of \(V'\) with \(\mathcal{I}\). In 3 + 1 dimensions, the homology of \(V'\) can be completely determined from this as shown in [1]. In this paper, we prove in arbitrary dimension that \(H_{n-1}(V; \mathbb{Z}) = \mathbb{Z}^k\) where \(V\) is the closure of \(V'\) and \(k\) is the number of boundaries \(\Sigma_i\) interior to \(\Sigma_0\). As a consequence, \(V\) does not contain any wormholes or other compact, non-simply connected topological structures. Finally, for the case of \(n = 2\), we show that these constraints and the onto homomorphism of the fundamental groups from which they follow are sufficient to limit the topology of interior of \(V\) to either \(B^2\) or \(I \times S^1\).

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I. INTRODUCTION

The global structure and topology of asymptotically anti-de Sitter spacetime is a topic of particular interest due to its relevance to string theory. Spacetimes that are products of an asymptotically anti-de Sitter spacetime in \( n + 1 \) dimensions and a Riemannian manifold arise in the low energy limit of certain D-brane configurations, notably for the cases of \( n = 2 \) and \( n = 4 \) [3]. Furthermore, Maldacena has proposed that supergravity in an asymptotically anti-de Sitter spacetime corresponds to a conformal field theory on the boundary-at-infinity of this spacetime in the large \( N \) limit [4]. This conjecture, the adS/CFT correspondence conjecture, is supported by recent calculations which, for example, show a direct connection between black hole entropy as calculated classically and the number of states of the conformal field theory on the boundary-at-infinity [3]. Thus, the adS/CFT correspondence conjecture provides new insight into the old puzzle of black hole entropy in the context of string theory. Moreover, it is believed that this conjecture, if true, may hold answers to other long-standing puzzles in gravity.

Now it is well known that asymptotically anti-de Sitter spacetimes admit black holes and wormholes of various topologies [3, 10] (for a recent review see [11]). These spaces exhibit a boundary-at-infinity which carries the topology of the event horizons. Furthermore one can show that there exist initial data sets with very general topology that evolve as anti-de Sitter spacetimes [12].

It is therefore natural to ask what implications the topology of an asymptotically anti-
de Sitter spacetime has for the adS/CFT correspondence. That is, if the topology of an asymptotically anti-de Sitter spacetime is arbitrary, can a conformal field theory that only detects the topology of its boundary-at-infinity correctly describe its physics? Recently, Witten and Yau in [2] have addressed part of this issue in the context of a generalization of the adS/CFT correspondence in which this conjecture is formulated in terms of Riemannian manifolds [14]. They show that the topology of a complete Einstein manifold \( M \) of negative curvature and boundary \( N \) admitting positive scalar curvature is constrained. In particular, they show \( H_n(M; Z) = 0 \) and thus \( N \) is connected. A consequence of this result is that Riemannian manifolds satisfying these conditions do not admit wormholes.

However, the interesting results in [2] do not address the relation of the topology of an asymptotically anti-de Sitter space to that of the boundary-at-infinity in the context of Lorentzian spacetimes, the standard arena for the adS/CFT correspondence conjecture. We will address this issue. We will do so by exploiting the fact that asymptotically anti-de Sitter spacetimes are members of a class of spacetimes that exhibit topological censorship; that is any causal curve with initial and final endpoints on the boundary-at-infinity \( I \) can be continuously deformed to a curve that lies in \( I \) itself. Thus causal curves passing through the interior of an asymptotically anti-de Sitter spacetime detect no topological structure not also present in the boundary-at-infinity. This somewhat surprising result was first proven to hold for asymptotically flat spacetimes by Friedman, Schleich and Witt [15]. It has been generalized to apply to a broader class of spacetimes [1,16,17]. Of relevance here is the proof in [1] that topological censorship holds for a class of spacetimes with timelike boundary-at-infinity that includes asymptotically anti-de Sitter spacetimes. This proof, like other proofs of topological censorship, holds in any such \((n+1)\)-dimensional spacetime with \( n \geq 2 \). This fact immediately implies that causal curves passing through the interior of an asymptotically anti-de Sitter spacetime detect only topological structure also present in the boundary-at-infinity in any such dimension.

It is also known that more information about the topology of a spacetime that exhibits topological censorship can be found by algebraic topology. In 3 + 1 dimensions, Chruściel
and Wald \[18\] noted that topological censorship implies that black holes are topological 2-spheres in stationary, asymptotically flat spacetimes. Jacobson and Venkataramani \[19\] generalized this result to show that it holds for a quite general class of (3 + 1)-dimensional asymptotically flat spacetimes, including spacetimes with black hole formation by collapse. For the case of (3 + 1)-dimensional asymptotically anti-de Sitter spacetimes, we showed in \[1\] that the sum of the genera of black holes is bounded above by that of a spatial cut of the boundary-at-infinity. Furthermore, the integral homology of Cauchy surfaces (as defined for asymptotically anti-de Sitter spacetime with boundary-at-infinity) is torsion free and consequently completely determined by the Betti numbers. Moreover, the first Betti number is equal to the sum of the genera of the black holes and the second Betti number is equal to the number of black holes. Thus topological censorship restricts both the topology of black holes and that of the spacetime exterior to them in 3 + 1 dimensions.

Though the proof of topological censorship in \[1\] holds in (n + 1)-dimensional spacetime for \( n \geq 2 \), some of the stronger results in \[1\] have been derived by using certain special properties of the topology of (3 + 1)-dimensional spacetimes. Clearly one can extend some of these arguments using algebraic topology to other dimensions. We will do so in this paper.

We first prove a simple corollary of topological censorship, that any asymptotically anti-de Sitter spacetime with a disconnected boundary-at-infinity necessarily contains black hole horizons which screen the boundary components from each other. This result is independent of the scalar curvature of the boundary-at-infinity. But, in a certain sense, it is a Lorentzian analog of the Witten and Yau result \[2\]. Furthermore the topology of \( V' \), the Cauchy surface (as defined for asymptotically anti-de Sitter spacetime with boundary-at-infinity) for regions exterior to event horizons is constrained by that of the boundary-at-infinity; we show that the homomorphism \( \Pi_1(\Sigma_0) \to \Pi_1(V') \) induced by the inclusion map is onto where \( \Sigma_0 \) is the intersection of \( V' \) with the boundary-at-infinity. We also prove that the integral homology \( H_{n-1}(V; \mathbb{Z}) = \mathbb{Z}^k \) where \( V \) is the closure of \( V' \) and \( k \) is the number of boundaries \( \Sigma_i \) interior to \( \Sigma_0 \). As a consequence \( V \) itself does not contain any wormholes or other compact non-simply topological structures. Furthermore, the integral homology \( H_k(V; \mathbb{Z}) \) is torsion free.
for \( k = n - 2 \). For the case of \( n = 2 \), these constraints and the onto homomorphism of the fundamental groups are sufficient to limit the topology of interior of \( V \) to either \( B^2 \) or \( I \times S^1 \). Therefore, in \( 2 + 1 \) dimensions, the topology of the boundary-at-infinity almost completely characterizes that of the interior, a desirable conclusion for the adS/CFT correspondence conjecture. However, in \( 4 + 1 \) dimensions, these constraints and the onto homomorphism of fundamental groups in and of themselves are not sufficient to completely characterize the topology of the interior. As we will see, they do not suffice to constrain the number of compact simply connected topological structures in the interior.

II. TOPOLOGICAL CENSORSHIP IN \((N + 1)\)-DIMENSIONAL ASYMPOTICALLY ANTI-DE SITTER SPACETIMES

Precisely, we will consider an \((n + 1)\)-dimensional connected spacetime \( \mathcal{M} \), with metric \( g_{ab} \), which can be conformally included into a spacetime-with-boundary \( \mathcal{M}' = \mathcal{M} \cup \mathcal{I} \), with metric \( g'_{ab} \), such that \( \partial \mathcal{M}' = \mathcal{I} \) is timelike (i.e., is an \( n \)-dimensional Lorentzian hypersurface in the induced metric) and \( \mathcal{M} = \mathcal{M}' \setminus \mathcal{I} \). Note that the boundary-at-infinity \( \mathcal{I} \) can have multiple components, that is the cardinality of \( \Pi_0(\mathcal{I}) \) can be greater than one. The conditions on the conformal factor \( \Omega \in C^1(\mathcal{M}') \) are that (a) \( \Omega > 0 \) and \( g'_{ab} = \Omega^2 g_{ab} \) on \( \mathcal{M} \), and (b) \( \Omega = 0 \) and \( d\Omega \neq 0 \) pointwise on \( \mathcal{I} \). These are the standard conditions on \( \Omega \) in a conformal compactification of a spacetime with infinitely extendible null geodesics such as asymptotically anti-de Sitter spacetimes.

The conformal compactification of the universal anti-de Sitter spacetime in \( n + 1 \) dimensions (cf. [13] p. 131) is a canonical example of such a spacetime. Its boundary-at-infinity \( \mathcal{I} \) is \( n \)-dimensional Minkowski spacetime.\(^2\) Group actions on this spacetime generate local adS spacetimes containing black holes and wormholes (see for example [4,5]). In contrast to

\(^2\)Precisely, the boundary-at-infinity is the Einstein static universe \( S^{n-1} \times R \) in which \( n \)-dimensional Minkowski spacetime is itself conformally included.
universal anti-de Sitter spacetime, these spacetimes can have disconnected $\mathcal{I}$.

A spacetime-with-boundary $\mathcal{M}'$ is defined to be globally hyperbolic if $\mathcal{M}'$ is strongly causal and the intersection of the causal future of $p$ with the causal past of $q$, $J^+(p, \mathcal{M}') \cap J^-(q, \mathcal{M}')$, is compact for all $p, q \in \mathcal{M}'$. This definition has exactly the same form as that for the case of a spacetime without boundary. However, observe that the inclusion of the boundary is key for spacetimes with timelike $\mathcal{I}$. In particular, $\mathcal{M}$, is not globally hyperbolic but $\mathcal{M}' = \mathcal{M} \cup \mathcal{I}$ is globally hyperbolic for many spacetimes with timelike $\mathcal{I}$ such as asymptotically anti-de Sitter spacetimes.

For example, it is well known that universal anti-de Sitter spacetime $\mathcal{M}$ itself is not globally hyperbolic (cf. [13] p. 132); as $\mathcal{I}$ is timelike, one can find points $q$ and $p$ such that $J^+(p, \mathcal{M}) \cap J^-(q, \mathcal{M})$ is not compact. Such points are those such that past directed radially outward null curves from $q$ and future directed radially outward null curves from $p$ intersect $\mathcal{I}$ before they intersect each other. As these geodesics leave $\mathcal{M}$ before intersecting, $J^+(p, \mathcal{M}) \cap J^-(q, \mathcal{M})$ is not compact. However, observe that $J^+(p, \mathcal{M}') \cap J^-(q, \mathcal{M}')$ is compact as it includes the appropriate part of $\mathcal{I}$. Intuitively, one is including additional information about the spacetime by including the boundary-at-infinity and this additional information is sufficient to ensure physical predictability.

We will also require that the spacetime satisfy a modified form of the Averaged Null Energy Condition (ANEC): For each point $p$ in $\mathcal{M}$ near $\mathcal{I}$ and any future complete null geodesic $s \to \eta(s)$ in $\mathcal{M}$ starting at $p$ with tangent $X$, $\int_0^\infty \text{Ric}(X,X) \, ds \geq 0$. This condition is necessary to ensure that all radially outward directed null geodesics from a closed outer

3Note that the causal future of a set $S$ relative to $U$, $J^+(S,U)$, is the union of $S \cap U$ with the set of all points that can be reached from $S$ by a future directed non-spacelike curve in $U$. The interchange of the past with future in the previous definition yields $J^-(S,U)$.

4 The term ANEC usually refers to a condition of this form except that the integral is taken over geodesics complete to both past and future [21].
trapped surface focus to a conjugate point in finite affine parameter, ensuring that this surface is not visible to \( I \) (see for example Prop. 9.2.1 in [13], a related proof using the null energy condition).

ANECD is stated in geometric form but can be interpreted physically by invoking the Einstein equations to relate the Ricci tensor to its sources. In particular, if the Einstein equations with cosmological constant hold, \( R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi T_{ab} \), then as \( g_{ab} X^a X^b = 0 \) for any null vector \( X \), \( \text{Ric}(X,X) = R_{ab} X^a X^b = 8\pi T_{ab} X^a X^b = 8\pi T(X,X) \). Clearly, the cosmological constant does not appear in this expression. Consequently, ANEC depends only on the stress energy tensor. ANEC is satisfied by the stress energy tensor of physically reasonable sources of matter. In particular, it is obvious that spacetimes with negative cosmological constant containing no matter will satisfy this condition.

Finally, a spacetime satisfies the **generic condition** if every timelike or null geodesic with tangent vector \( X \) contains a point at which \( X^a X^b X^c R_{d[ab]c} X^f \) is not zero. The generic condition will be satisfied if a spacetime contains matter or gravitational radiation in a non-symmetric configuration.

We begin by reminding readers that proofs of topological censorship, in particular that of [4], hold in \((n + 1)\)-dimensional spacetimes for \( n \geq 2 \). Namely,

**Theorem 1.** Let \( \mathcal{M}' \) be a globally hyperbolic spacetime-with-boundary with timelike boundary \( I \) that satisfies ANEC. Let \( I_0 \) be a connected component of \( I \). Furthermore assume either (i) \( I_0 \) admits a compact spacelike cut or (ii) \( \mathcal{M}' \) satisfies the generic condition. Then every causal curve whose initial and final endpoints belong to \( I_0 \) is fixed endpoint homotopic to a curve on \( I_0 \).

This is an alternate but completely equivalent statement of theorem 2.2 proven in [4]. The proof of theorem 1 uses the result

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\(^5\)All known examples of spacetimes that do not satisfy the generic condition have a high degree of local or global symmetry.
**Theorem 2.** Let \( \mathcal{M}' \) be a globally hyperbolic spacetime-with-boundary with timelike boundary \( \mathcal{I} \) that satisfies ANEC. Let \( \mathcal{I}_0 \) be a connected component of \( \mathcal{I} \) of \( \mathcal{M}' \). Furthermore assume that either (i) \( \mathcal{I}_0 \) admits a compact spacelike cut or (ii) \( \mathcal{M}' \) satisfies the generic condition. Then \( \mathcal{I}_0 \) cannot communicate with any other component of \( \mathcal{I} \), i.e., \( J^+(\mathcal{I}_0) \cap (\mathcal{I} \setminus \mathcal{I}_0) = \emptyset \).

This theorem is a restated form of theorem 2.1 proven in [1]. The maximally extended Schwarzschild and Schwarzschild-anti-de Sitter solutions provide simple examples of space-times satisfying the conditions of this theorem. They both have two disconnected components of \( \mathcal{I} \). Causal curves originating from one component of \( \mathcal{I} \) cannot end on the other; instead they end on the black hole singularity.

It is useful to mention that theorem 1 follows from theorem 2 by constructing a covering space of \( \mathcal{M}' \) in which all non-contractible curves not homotopic to curves on \( \mathcal{I}_0 \) are unwound. Any causal curve with endpoints on \( \mathcal{I}_0 \) not fixed endpoint homotopic to a causal curve in \( \mathcal{I}_0 \) will begin on a different component of \( \mathcal{I} \) in this covering space. However, this covering space is itself a globally hyperbolic spacetime-with-boundary satisfying the conditions of theorem 2. Thus such a curve cannot exist. Hence the result.

Ref. [1] provides a natural restatement of theorem 1 in terms of the region of spacetime that can communicate with a given component of the boundary-at-infinity. This region, the domain of outer communications \( \mathcal{D} = I^-(\mathcal{I}_0) \cap I^+(\mathcal{I}_0) \), is the subset of \( \mathcal{M} \) that is in causal contact with \( \mathcal{I}_0 \). As we shall see later, one can think of the domain of outer communications as the region of \( \mathcal{M} \) which is exterior to event horizons. Clearly, \( \mathcal{D} \) is also the interior of an \((n+1)\)-dimensional spacetime-with-boundary \( \mathcal{D}' = \mathcal{D} \cup \mathcal{I}_0 \). Now theorem 1 can be conveniently restated in terms of the fundamental group of the domain of outer

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6Note that the timelike future of a set \( S \) relative to \( U \), \( I^+(S, U) \), is the set of all points that can be reached from \( S \) by a future directed timelike curve in \( U \). The interchange of the past with future in the previous definition yields \( I^-(S, U) \).
communications. Observe that the inclusion map \( i : \mathcal{I}_0 \to \mathcal{D}' \) induces a homomorphism of fundamental groups \( i_* : \Pi_1(\mathcal{I}_0) \to \Pi_1(\mathcal{D}') \). Then

**Theorem 3.** If \( \mathcal{M}' \) is a globally hyperbolic spacetime-with-boundary that satisfies the conditions given in theorem 1, then the group homomorphism \( i_* : \Pi_1(\mathcal{I}_0) \to \Pi_1(\mathcal{D}') \) induced by inclusion is surjective.

Theorem 3 says roughly that every loop in \( \mathcal{D} \) is deformable to a loop in \( \mathcal{I} \). Moreover, it implies that \( \Pi_1(\mathcal{D}) (= \Pi_1(\mathcal{D}')) \) is isomorphic to the factor group \( \Pi_1(\mathcal{I})/\ker i_* \). In particular, if \( \mathcal{I} \) is simply connected then so is \( \mathcal{D} \), thus generalizing the result of [20].

**III. CAUSAL DISCONNECTEDNESS OF DISJOINT COMPONENTS OF THE BOUNDARY-AT-INFINITY**

The boundary of the region of spacetime visible to observers at \( \mathcal{I} \) by past directed causal curves is referred to as the future event horizon. This horizon is a set of one or more null surfaces, also called black hole horizons, generated by null geodesics that have no future endpoints but possibly have past endpoints. Precisely these horizons are characterized as the boundary of the causal past of \( \mathcal{I} \), \( \mathcal{J}^- (\mathcal{I}) \). A past event horizon is similarly defined; \( \mathcal{J}^+(\mathcal{I}) \). The past event horizon also can consist of one or null surfaces known as white hole horizons.

Theorems 1, 2 and 3 provide a partial characterization of the topology of the region of spacetime exterior to the event horizons in \( (n + 1) \)-dimensions. In particular, they demonstrate that no causal curve links with event horizons in a manner such that it cannot be deformed to a curve on the boundary-at-infinity. Rather causal curves in the spacetime will only carry information about the non-triviality of curves on \( \mathcal{I} \). Thus the topology of event horizons in spacetimes that exhibit topological censorship is constrained.

An immediate result of theorem 2 is that spacetimes with disconnected \( \mathcal{I} \) contain black hole horizons which screen the boundary components from each other.

**Corollary.** Let \( \mathcal{M}' \) satisfy the conditions given in theorem 1. If \( \mathcal{I} \) is disconnected, then the
spacetime contains black hole horizons, namely $\hat{\mathcal{J}}^-(\mathcal{I}_0) \neq \emptyset$.

**Proof:** Let $\mathcal{I}_1$ be any component of $\mathcal{I}$ not connected to $\mathcal{I}_0$. Theorem 2 shows that there is no causal curve connecting $\mathcal{I}_0$ and $\mathcal{I}_1$. Thus the causal past of $\mathcal{I}_0$ is disjoint from the causal future of $\mathcal{I}_1$, $\mathcal{J}^-(\mathcal{I}_0) \cap \mathcal{J}^+(\mathcal{I}_1) = \emptyset$. Now as both are subsets of $\mathcal{M}'$, clearly $\mathcal{J}^-(\mathcal{I}_0)$ is not itself $\mathcal{M}'$. Thus $\hat{\mathcal{J}}^-(\mathcal{I}_0) \neq \emptyset$.

Observe that a similar argument shows that the spacetime contains a past event horizon, $\hat{\mathcal{J}}^+(\mathcal{I})$; as $\mathcal{J}^+(\mathcal{I}_1)$ is also not itself $\mathcal{M}'$, $\hat{\mathcal{J}}^+(\mathcal{I}_1) \neq \emptyset$. Note that $\hat{\mathcal{J}}^-(\mathcal{I}_0)$ and $\hat{\mathcal{J}}^+(\mathcal{I}_1)$ may coincide as is the case in the maximally extended Schwarzschild and Schwarzschild-anti-de Sitter spacetimes.

In simple terms, these results show that black hole spacetimes formed from the collapse of topological structures must always have both black hole and white hole horizons. This behavior is quantitatively different than that of spacetimes containing black holes formed by collapse of matter which may, but need not exhibit a white hole horizon. Therefore, white holes are an essential feature of black hole spacetimes formed from collapse of topology.

The implications of the corollary for adS/CFT correspondence are immediate. In an asymptotically de-Sitter spacetime satisfying reasonable physical conditions, any component of the boundary-at-infinity cannot causally communicate with any other disjoint component of the boundary-at-infinity. Thus a field operator on one component of the boundary-at-infinity cannot causally interact with another field operator on any other disjoint component. Thus a field operator on one component of $\mathcal{I}$ will commute with any other field operator on any disjoint component $\mathcal{I}$. Thus conformal field theories defined on disjoint components of the boundary-at-infinity do not interact dynamically.

Clearly however, one can set up correlations in the initial vacuum states of the conformal field theories. In fact, the necessary appearance of white hole horizons may yield a natural way to do so. However, any such correlations are not dynamic.
IV. THE TOPOLOGY OF REGIONS EXTERIOR TO BLACK HOLE HORIZONS IN $N + 1$ DIMENSIONS

One can obtain further information about regions exterior to black hole horizons if one considers the topology of the intersections of certain spacelike hypersurfaces with the horizons; those for which this intersection is a set of closed spacelike $(n - 1)$-manifolds (good cuts of the horizons). A characterization of these regions can then be given by the analysis of the topology of these spacelike hypersurfaces, specifically Cauchy surfaces as defined for asymptotically anti-de Sitter spacetimes, in terms of their homology. One can show that in $n + 1$ dimensions, spacetimes that obey topological censorship must have spacelike surfaces, which contain no wormholes or other non-simply connected compact topological structures. These results are the generalization of the previous results for $3 + 1$ dimension reported in [1] to arbitrary dimension.

Precisely, let $M'$ be as described in theorem 1, and $D'$ be the domain of outer communications of a component $I_0$ of its timelike boundary. A Cauchy surface (for asymptotically anti-de Sitter spacetime with boundary-at-infinity) for $D'$ is defined to be a subset $V' \subset D'$ which is met once and only once by each inextendible causal curve in $D'$. Then $V'$ will be a spacelike hypersurface which, as a manifold-with-boundary, has boundary on $I$. This definition of Cauchy surface for asymptotically anti-de Sitter spacetimes with boundary-at-infinity parallels that for spacetimes without boundary. For brevity, we will call these Cauchy surfaces for asymptotically anti-de Sitter spacetimes with boundary-at-infinity simply Cauchy surfaces in the remainder of the paper.

It can be shown, as in the standard case of spacetime without boundary, that a spacetime-with-timelike-boundary $M'$ which is globally hyperbolic admits a Cauchy surface $V'$ for $D'$. Furthermore, $D'$ is homeomorphic to $R \times V'$. (This can be shown by directly modifying the proof of Prop. 6.6.8 in [13].) Examples of spacetimes whose domains of outer communications admit such Cauchy surfaces are the locally anti-de Sitter spacetimes and related models given in [3–10].
If a globally hyperbolic $\mathcal{D}'$ has topology $R \times V'$ (e.g. if $V'$ is a Cauchy surface for $\mathcal{D}'$) then $\mathcal{D}'$ can be continuously deformed to $V'$ so that $\mathcal{I}$ gets deformed to $\Sigma_0$. This process preserves fundamental groups and hence allows the application of algebraic topology to further characterize the the spatial hypersurface $V'$. Precisely,

**Theorem 4.** Assume $\mathcal{D}' (= \mathcal{D} \cup \mathcal{I})$ is a globally hyperbolic spacetime that satisfies the conditions given in theorem 1. Suppose $V'$ is a Cauchy surface for $\mathcal{D}'$ such that its closure $V = \overline{V}$ in $\mathcal{M}'$ is a compact topological $n$-manifold-with-boundary whose boundary $\partial V$ (corresponding to the edge of $V'$ in $\mathcal{M}'$) consists of a disjoint union of compact $(n-1)$-manifolds, $\partial V = \sqcup_{i=0}^{k} \Sigma_i$ where $\Sigma_0$ is on $\mathcal{I}$ and the $\Sigma_i$, $i = 1, \ldots, k$, are on the event horizon. Then the group homomorphism $i_* : \Pi_1(\Sigma_0) \rightarrow \Pi_1(V)$ induced by inclusion $i : \Sigma_0 \rightarrow V$ is onto.

The proof of this result is given in [1]. Clearly, theorem 4 implies that if $\Sigma_0$ is simply connected, then so is $V$.

One may gain further insight into the consequences of topological censorship by asking how can one modify the topology of $V$ yet still satisfy the restriction $\Pi_1(\Sigma_0) \rightarrow \Pi_1(V)$? A standard method of constructing topological spaces is by connected sum: one takes $V$ and sews in a closed $n$-manifold $N$ by removing one $n$-ball from the interior of $V$ and one $n$-ball from $N$ then identifying the resulting $(n-1)$-sphere boundaries with each other to form $\tilde{V} = V \# N$. For $N = S^{n-1} \times S^1$, this procedure adds a $n$-handle or wormhole to the space. One can similarly add another compact connected topological structure to the space by choosing $N$ to be any other closed manifold besides $S^n$.

Clearly this procedure can add factors that will modify $V$ such that it will no longer satisfy $i_* : \Pi_1(\Sigma_0) \rightarrow \Pi_1(V)$ being onto. For example, the addition of a wormhole will produce a new generator of $\Pi_1(\tilde{V})$; a curve passing through the new handle will not be homotopic to any curve in $V$. Similarly, adding another compact connected topological structure which has nontrivial $\Pi_1(N)$ will also introduce new generators to $\Pi_1(\tilde{V})$. However, connected sums involving compact simply connected topological structures will not change
the fundamental group and thus the addition of such structures is not constrained by this argument.

An alternate view of this effect is provided in terms of the homology of $V$. For example, taking the connected sum of $V$ with a handle $N$ introduces a $(n-1)$-sphere that does not bound an $n$-ball. Therefore, the rank of $H_{n-1}(V\#N;\mathbb{Z})$ is greater than that of $H_{n-1}(V;\mathbb{Z})$. It is clear on an intuitive level that if connected sums with wormholes and other compact connected topological structures can change the topology of $V$, then the information provided in theorem 4 must constrain the number of wormholes and other compact non-simply connected topological structures.

A further characterization of the spacetime is given by the analysis of the topology of $V$ in terms of its homology. In the following we will assume that $\Sigma_0$ is orientable, the generalization to the non-orientable case being straightforward.

**Theorem 5.** If $\Sigma_0$ is orientable and $i_*:\Pi_1(\Sigma_0)\to\Pi_1(V)$ is onto, then the natural homomorphism $H_1(\Sigma_0;\mathbb{Z})\to H_1(V;\mathbb{Z})$ is onto. The integral homology $H_k(V;\mathbb{Z})$ is torsion free for $k = 0, n-2, n-1, \text{and } n$. Furthermore, $H_{n-1}(V;\mathbb{Z}) = \mathbb{Z}^k$ where $k$ is the number of boundaries $\Sigma_i$ interior to $\Sigma_0$.

**Proof:** We use the fact that the first integral homology group of a space is isomorphic to the fundamental group modded out by its commutator subgroup. Hence, modding out by the commutator subgroups of $\Pi_1(\Sigma_0)$ and $\Pi_1(V)$, respectively, induces from $i_*$ a surjective homomorphism from $H_1(\Sigma_0;\mathbb{Z})$ to $H_1(V;\mathbb{Z})$.

We next prove the torsion free claims. The assumption on fundamental groups and the orientability of $\Sigma_0$ imply that $V$ is orientable. Then, since $V$ has boundary, $H_n(V;\mathbb{Z}) = 0$.

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7 Theorem 2 contradicts the possibility of a nonorientable $D'$ satisfying the conditions of theorem 2 with orientable $I$. The orientable double cover of $D'$ would also satisfy the conditions of theorem 2 and would contain two copies of $I$ connected by causal curves. As this cannot happen, $D'$ must be orientable for orientable $I$. As orientable $\Sigma_0$ implies orientable $I$, it follows that $V$ is orientable.
Also $H_0(V; Z) = Z$ as $V$ is connected. Further, it is a standard fact that $H_{n-1}(V; Z)$ is free (cf. [22], p. 379). This follows from Poincaré duality for manifolds-with-boundary and the fact that the relative first cohomology group is in general free. Thus we need to show that $H_{n-2}(V; Z)$ is free. The arguments we use for this we also use to show $H_{n-1}(V; Z) = Z^k$.

**Lemma.** $H_{n-2}(V; Z)$ is free.

To prove the lemma we first consider the relative homology sequence for the pair $V \supset \Sigma_0$,

$$
\cdots \to H_1(\Sigma_0) \overset{\alpha}{\to} H_1(V) \overset{\beta}{\to} H_1(V, \Sigma_0) \overset{\partial}{\to} \tilde{H}_0(\Sigma_0) = 0 \quad (4.1)
$$

where we have assumed in the above sequence and from now on in the relative homology arguments that the coefficients are over $Z$. Here $\tilde{H}_0(\Sigma_0)$ is the reduced zeroth-dimensional homology group. Since, as discussed previously, $\alpha$ is onto, we have $\ker \beta = \im \alpha = H_1(V)$ which implies $\beta \equiv 0$. Hence $\ker \partial = \im \beta = 0$, and thus $\partial$ is injective. This implies that $H_1(V, \Sigma_0) = 0$.

Now consider the relative homology sequence for the triple $V \supset \partial V \supset \Sigma_0$,

$$
\cdots \to H_1(\partial V, \Sigma_0) \to H_1(V, \Sigma_0) = 0 \to H_1(V, \partial V) \overset{\partial}{\to} \tilde{H}_0(\partial V, \Sigma_0) \to \cdots \quad (4.2)
$$

Since $H_0(\partial V, \Sigma_0)$ is torsion free and $\partial$ is injective, $H_1(V, \partial V)$ is torsion free. Next, Poincaré-Lefschetz duality gives $H^{n-1}(V) \cong H_1(V, \partial V)$. Hence $H^{n-1}(V)$ is torsion free. The universal coefficient theorem implies that

$$
H^{n-1}(V) \cong \Hom(H_{n-1}(V), Z) \oplus \Ext(H_{n-2}(V), Z) \quad (4.3)
$$

The functor $\Ext(-,-)$ is bilinear in the first argument with respect to direct sums and $\Ext(Z^k, Z) = Z^k$. Hence $H^{n-1}(V)$ cannot be torsion free unless $H_{n-2}(V)$ is. This completes the proof of the lemma.

The boundary surfaces $\Sigma_1, \Sigma_2, \ldots, \Sigma_k$ clearly determine $k$ linearly independent $(n-1)$-cycles in $V$, and hence $b_{n-1} \geq k$.

It remains to show that $b_{n-1} \leq k$. Since both $H_{n-1}(V)$ and $H^{n-1}(V)$ are finitely generated and torsion free, we have $H_{n-1}(V) \cong H^{n-1}(V) \cong H_1(V, \partial V)$, where we have
again made use of Poincaré-Lefschetz duality. Hence, $b_{n-1} = \text{rank } H_1(V, \partial V)$. To show that $\text{rank } H_1(V, \partial V) \leq k$, we refer again to the long exact sequence (4.2). By excision, $H_0(\partial V, \Sigma_0) \cong H_0(\partial V \setminus \Sigma_0, \emptyset) = H_0(\partial V \setminus \Sigma_0)$. Hence, by the injectivity of $\partial$, $\text{rank } H_1(V, \partial V) \leq \text{rank } H_0(\partial V, \Sigma_0) = \text{the number of components of } \partial V \setminus \Sigma_0 = k$. This completes the proof of theorem 5.

An easy consequence of theorem 5 is that $b_1(\Sigma_0) \geq b_1(V)$. As observed earlier, the addition of a wormhole changes not only $\Pi_1(V)$ but also $H_{n-1}(V)$. Also note that rank $H_{n-1}(V; Z) = k$, the number of boundaries of $V$ interior to the cut of the boundary-at-infinity $\Sigma_0$. Therefore, there is no element of $H_{n-1}(V; Z)$ associated with a structure in the interior of $V$. That is, there are no wormholes or other compact non-simply connected topological structures in $V$.

**Corollary.** Given $V$ satisfying the conditions of theorem 5, then there exists no closed manifold $N$ with $b_1(N) > 0$ such that $V = U \# N$.

**Proof:** Observe that as $N$ is a closed manifold, that $U$ has the same boundaries as $V$; it follows that $b_{n-1}(U) \geq k$. The Mayer-Vietoris sequence yields

$$0 \to H_n(V) \to H_{n-1}(S^{n-1}) \to H_{n-1}(U - B^n) \oplus H_{n-1}(N - B^n) \to H_{n-1}(V) \to H_{n-2}(S^{n-1}) \cdots .$$

Now the above sequence is exact as $H_{n-2}(S^{n-1}) = 0$. The alternating sum of the ranks must vanish, thus $1 - (b_{n-1}(U) + 1) - b_1(N) + k = 0$ using $b_n(V) = 0$, $b_{n-1}(U - B^n) = b_{n-1}(U) + 1$ as $U$ and $V$ are manifolds with boundary. Clearly this implies $b_{n-1}(U) + b_{n-1}(N) = k$. But this is a contradiction to $b_1(N) > 0$. ■

Finally, one might be worried that Cauchy surfaces satisfying theorem 4 only occur in stationary spacetimes, that is where no black hole formation occurs. However, this is not the case as first pointed out by Jacobson and Venkataramani [K]; one can construct surfaces that characterize cuts of black hole horizons that occur via collapse in quite general asymptotically flat spacetimes. What one does is construct a globally hyperbolic spacetime that consists of a subset of the original spacetime. One can carry out a similar construction
for the asymptotically anti-de Sitter spacetimes. Precisely, let $K$ be a cut of $\mathcal{I}$, and let $\mathcal{I}_K$ be the portion of $\mathcal{I}$ to the future of $K$, $\mathcal{I}_K = \mathcal{I} \cap I^+(K)$. Let $\mathcal{D}_K$ be the domain of outer of communications with respect to $\mathcal{I}_K$, $\mathcal{D}_K = I^+(\mathcal{I}_K) \cap I^-(\mathcal{I}_K) = I^+(K) \cap I^-(\mathcal{I})$. One chooses $\mathcal{D}_K' = \mathcal{D}_K \cup \mathcal{I}$ such that it is globally hyperbolic and such that the closure of its Cauchy surface in $\mathcal{M}'$ has a good intersection with the black hole horizons, i.e. the intersections are $(n-1)$-manifolds. This new spacetime satisfies the conditions of theorem 1 and will contain a surface $V$ as required in theorem 4. Therefore the conclusions of theorem 5 hold for such surfaces as well. Thus, though topological censorship does not determine the topology of arbitrary embedded hypersurfaces, it does do so for hypersurfaces homeomorphic to Cauchy surfaces for the domain of outer communications that make good cuts of the horizons. Details regarding this procedure are discussed further in [1].

V. FURTHER RESULTS IN 2 + 1 AND 4 + 1 DIMENSIONS

Clearly, by using special properties of manifolds in a given dimension $n$, the results obtained here may be strengthened. This is particularly true in low dimension. Of special relevance to the adS/CFT correspondence conjecture are results on asymptotically anti-de Sitter spacetimes in 2 + 1 and 4 + 1 dimensions.

In three dimensions one can show that

**Theorem 6.** Assume $\mathcal{D}'$ is a globally hyperbolic spacetime-with-boundary that satisfies the conditions of theorem 4. Then the 2-dimensional hypersurface $V$ is either $B^2$ or $I \times S^1$.

**Proof:** As all 1-manifolds are orientable, $V$ is orientable. Theorem 5 implies that the rank of the free part of $H_1(V;Z)$ cannot be greater than that of $H_1(\Sigma_0;Z)$, i.e., $b_1(V) \leq b_1(\Sigma_0)$. In the case $n = 2$, $\Sigma_0$ is a 1-manifold so $b_1(\Sigma_0) \leq 1$ thus $b_1(V) \leq 1$. Now $V$ is a closed 2-manifold minus a disjoint union of discs. From the classification of 2-manifolds, $V$ must be a closed 2-manifold minus a disjoint union of disks. The first betti number of such manifolds is $b_1 = 2g + k$ where $g$ is the genus and $k + 1$ the number of disjoint disks; it follows that $g = 0$. Since $V$ must have at least one boundary, the only possible topologies for $V$ are $B^2$
Theorem 6 has very interesting consequences for the topology of (2 + 1)-dimensional spacetimes. If $I$ is disconnected, then $V'$ for the domain of outer communications of each disconnected component of $I$ will have product topology. Thus topological censorship gives a topological rigidity theorem in (2+1)-dimensional gravity. As one has directly characterized the topology of the domain of outer communications for these spacetimes, it follows, by arguments similar to that used to characterize the topology of good cuts of black hole horizons in the (3 + 1)-dimensional case given in [1], that the topology of a good cut of a black hole horizon in (2 + 1)-dimensional spacetime is always $S^1$.

The case of (2 + 1)-dimensional asymptotically flat spacetimes can be similarly treated to produce the same conclusions as theorem 6. It follows that there are no asymptotically flat geons in three dimensions.

In the case of (4+1)-dimensional spacetimes, theorem 5 yields that the integral homology $H_k(V; Z)$ is torsion free except for $k = 1$. However, theorem 5 and the onto homomorphism of the fundamental groups is not enough to even partially fix the topology of $V$. To demonstrate this, it is useful to first study the restricted case for which $\Sigma_0$ is simply connected. It follows that $V$ is a simply connected manifold with boundary. This is a fairly significant restriction; however one will have an infinite number of such manifolds. One obtains these simply by taking the connected sum of $V$ with any closed simply connected 4-manifold. One can readily show that the connected sum of two such manifolds leaves $H_k$ unchanged except for $H_2$. There are an infinite number of closed simply connected 4-manifolds characterized by their Hirzebruch signature and Euler characteristic.

Furthermore the restriction that $V$ is simply connected is not enough to deduce the topology of the boundaries $\Sigma_i$ even in this simple case. It is well known that all closed 3-manifolds are cobordant to $S^3$. In fact one can construct a cobordism with trivial fundamental group [23]. Therefore, one cannot conclude any restriction on the topology of the cuts of black hole horizons in (4 + 1)-dimensional spacetimes from the simple arguments given above.

Finally, it is clear that similar conclusions follow in the case of non-simply connected $\Sigma_0$.  

or $I \times S^1$. 

\[\text{\dag}\]
In particular, one will have an infinite number of manifolds with the same fundamental group and $H_{n-1}(V; Z)$ obtained by taking the connected sum of $V$ with any closed simply connected 4-manifold. Thus the topology of the interior of a $(4 + 1)$-dimensional asymptotically de Sitter spacetime is constrained but not completely characterized by the topology of the boundary-at-infinity.

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