EXISTENCE OF WEAK SOLUTIONS TO A CONVECTION–DIFFUSION EQUATION IN A UNIFORMLY LOCAL LEBESGUE SPACE

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Abstract. We consider the local existence and the uniqueness of a weak solution of the initial boundary value problem to a convection–diffusion equation in a uniformly local function space $L^r_{uloc,\rho}(\Omega)$, where the solution is not decaying at $|x| \to \infty$. We show that the local existence and the uniqueness of a solution for the initial data in uniformly local $L^r$ spaces and identify the Fujita-Weissler critical exponent for the local well-posedness found by Escobedo-Zuazua [10] is also valid for the uniformly local function class.

1. Introduction. We consider the Cauchy problem of a convection-diffusion equation in spatially non-decaying function space. Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space with $n \geq 1$, and $\Omega \subset \mathbb{R}^n$ be an unbounded domain with uniform $C^2$ boundary. We consider the Cauchy-Dirichlet problem of a time dependent convection-diffusion equation: For $a \in \mathbb{R}^n$,

$$
\begin{cases}
\partial_t u - \Delta u = a \cdot \nabla(|u|^{p-1}u), & t > 0, x \in \Omega, \\
u(t, x) = 0, & t > 0, x \in \partial \Omega, \\
u(0, x) = u_0(x), & x \in \Omega,
\end{cases}
$$

(1.1)

where $u = u(t, x): \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is the unknown function, and $u_0 = u_0(x): \Omega \to \mathbb{R}$ is a given initial data.
For the case $\Omega = \mathbb{R}^n$, Escobedo and Zuazua [10] showed that for initial data $u_0 \in L^1(\mathbb{R}^n)$, there exists a unique global strong solution $u \in C((0, \infty); L^1(\mathbb{R}^n))$ of (1.1) in

$$u \in C((0, \infty); \mathcal{W}^{2, q}(\mathbb{R}^n)) \cap C^1((0, \infty); L^q(\mathbb{R}^n)),$$

(1.2)

for every $q \in (1, \infty)$. They also described the large time behavior of solutions of (1.1) and showed a decay property when the initial data is in $L^1(\mathbb{R}^n)$. When $p = 1 + \frac{1}{n}$, they proved that the large time behavior of solutions with initial data in $L^1(\mathbb{R}^n)$ is given by a one-parameter family of self-similar solutions. The relevant parameter is the mass of the solution that is conserved for all $t$. When $p > 1 + \frac{1}{n}$, the convection term is too weak and they proved that the large time behavior of solutions is given by the heat kernel.

Note that problem (1.1) has been considered by a number of authors (see e.g. [2, 5], [8]–[14], [16, 21, 27, 28]).

On the other hand, in [3, 4, 7, 19, 20, 22] and [23], the authors make use of spaces of functions which have the property that their elements have some uniform size when it is measured in balls of fixed radius but arbitrary center. These spaces are called as uniformly local spaces. These spaces are natural and useful for finding the solutions of parabolic equations in unbounded domains with non-decaying initial functions. The spaces enjoy suitable inclusion properties and have locally compact embeddings and besides any constant functions belong to them. In particular, when we analyze parabolic equations in unbounded domains, these spaces will allow us to consider a solution with no prescribed behavior at infinity and allowing for local singularities.

**Definition 1.1 (Uniformly local $L^r$ space).** Let $1 \leq r \leq \infty$ and $\rho > 0$. The uniformly local $L^r$ space on $\Omega$ denoted by $L^r_{uloc, \rho}(\Omega)$, is defined by

$$L^r_{uloc, \rho}(\Omega) := \left\{ f \in L^1_{loc}(\Omega) : \|f\|_{L^r_{uloc, \rho}} < \infty \right\},$$

where for $\rho > 0$

$$\|f\|_{L^r_{uloc, \rho}} = \begin{cases} \sup_{x \in \Omega} \left( \int_{B_\rho(x) \cap \Omega} |f(y)|^r dy \right)^{\frac{1}{r}} , & 1 \leq r < \infty, \\ \sup_{x \in \Omega} \sup_{y \in B_\rho(x) \cap \Omega} |f(y)| , & r = \infty. \end{cases} \tag{1.3}$$

Here we identify $L^\infty_{uloc, \rho}(\Omega)$ as $L^\infty(\Omega)$. The space $L^r_{uloc, \rho}(\Omega)$ is a Banach space with the norm defined in (1.3). We define the subspace $L^r_{uloc, \rho}(\Omega)$ as the closure of the space of bounded uniformly continuous functions $\text{BUC}(\Omega)$ in the space $L^r_{uloc, \rho}(\Omega)$, i.e.,

$$L^r_{uloc, \rho}(\Omega) := \overline{\text{BUC}(\Omega)}_{\| \cdot \|_{L^r_{uloc, \rho}}},$$

and define $L^\infty_{uloc, \rho}(\Omega) = \text{BUC}(\Omega)$.

The Sobolev spaces $W^{k, r}_{uloc, \rho}(\Omega)$ for $1 \leq r \leq \infty$, $\rho > 0$ and $k = 1, 2, \ldots$ are analogously introduced. We define by

$$W^{k, r}_{uloc, \rho}(\Omega) := \left\{ f \in L^r_{loc}(\Omega) : \|f\|_{W^{k, r}_{uloc, \rho}} < \infty \right\},$$
where for \( \rho > 0 \),
\[
\|f\|_{W^{k,r}_{uloc,\rho}} = \|f\|_{L^r_{uloc,\rho}} + \sum_{|\alpha| = k} \|\partial^\alpha_x f\|_{L^r_{uloc,\rho}}. \tag{1.4}
\]

We denote \( W^{1,2}_{uloc,\rho}(\Omega) \) as \( H^1_{uloc,\rho}(\Omega) \) for simplicity. Since the spaces \( L^r_{uloc,\rho}(\Omega) \) restrict the local property of function, it holds that \( L^q_{uloc,\rho}(\Omega) \subset L^r_{uloc,\rho}(\Omega) \) for any \( r \leq q \) and for all \( \rho > 0 \). Besides for any \( 0 < \rho < \rho' \), we may regard that \( L^r_{uloc,\rho}(\Omega) \) can be identified with \( L^r_{uloc,\rho'}(\Omega) \) since two norms equipped each spaces are equivalent. Hence when \( \rho = 1 \), we abbreviate the notation such as \( L^r_{uloc}(\Omega) := L^r_{uloc,1}(\Omega) \) and \( L^r_{uloc}(\Omega) := L^r_{uloc,1}(\Omega) \) and we may regard that those spaces are identified with \( L^r_{uloc,\rho}(\Omega) \) and \( L^r_{uloc,\rho}(\Omega) \) for any \( \rho > 0 \), respectively.

We introduce the weak solution to (1.1) in uniformly local space \( L^r_{uloc}(\Omega) \) as follows.

**Definition 1.2** (Weak \( L^r_{uloc}(\Omega) \)-solution). Let \( 1 \leq r < \infty \) and \( \rho > 0 \). For an initial data \( u_0 \in L^r_{uloc,\rho}(\Omega) \) and \( T > 0 \), we say that \( u \) is a weak \( L^r_{uloc}(\Omega) \)-solution of (1.1) in \((0,T) \times \Omega\), if
1. \( u \in C([0,T] : L^r_{uloc,\rho}(\Omega)) \cap L^2(0, T : H^1_{uloc,\rho}(\Omega)) \),
2. \( u(0) = u_0 \) in \( L^r_{uloc,\rho}(\Omega) \),
3. \( u \) satisfies
\[
\int_0^T \int_{\Omega} \left\{ -u \partial_t \phi + \nabla u \cdot \nabla \phi + a|u|^{p-1}u \cdot \nabla \phi \right\} dx dt = 0
\]
for all \( \phi \in C_0^\infty((0,T) \times \Omega) \).

We state our main result for the existence of a weak solution to (1.1) in uniformly local \( L^r \) spaces.

**Theorem 1.3** (Existence of a weak solution). Let \( p > 1 \) and \( 1 \leq r < \infty \) with
\[
\begin{align*}
    r &\geq n(p - 1) \quad \text{if} \quad p > 1 + \frac{1}{n}, \\
    r &> 1 \quad \text{if} \quad p = 1 + \frac{1}{n}, \\
    r &\geq 1 \quad \text{if} \quad 1 < p < 1 + \frac{1}{n}.
\end{align*}
\tag{1.5}
\]

There exists a positive constant \( \gamma_0 \), depending only on \( n \), \( p \) and \( r \), such that, if for any initial data \( u_0 \in L^r_{uloc,\rho}(\Omega) \) satisfies
\[
\rho^{\frac{p-1}{2}} \|u_0\|_{L^r_{uloc,\rho}} \leq \gamma_0 \tag{1.6}
\]
for some \( \rho > 0 \), then there exists a unique weak \( L^r_{uloc}(\Omega) \)-solution \( u \) of (1.1) in \((0,\rho^2) \times \Omega\) such that
\[
\sup_{0 < t < \rho^2} \|u(t)\|_{L^r_{uloc,\rho}} \leq C\|u_0\|_{L^r_{uloc,\rho}}, \tag{1.7}
\]
where \( C \) is independent of \( u \). Besides the solution has a uniform estimate
\[
\|u\|_{L^\infty((0,\rho^2) \times \Omega)} \leq C \left( \int_0^{\rho^2} \|u(t)\|_{L^r_{uloc,\rho}} dt \right)^{\frac{1}{r}} \tag{1.8}
\]
and hence \( u \in L^\infty((0,\rho^2) \times \Omega) \).
In the assumption on the initial data (1.6), the constant $\gamma_0 > 0$ is a constant depending only on $n$, $p$ and $r$. Hence one can regard this condition on the initial data as the restriction on the choice of $\rho > 0$. Since the function class $L^r_{u_{loc}, \rho}(\Omega)$ does not depend on $\rho > 0$, we have a room for the choice of $\rho > 0$ depending on the initial data. This choice is reflecting how long the local solution can be continued.

As a corollary of Theorem 1.3, we have:

**Corollary 1.1.** Let $p > 1 + \frac{1}{n}$. Then there exists a constant $\gamma$ such that, if $u_0 \in L^{n(p-1)}(\Omega)$ and $\|u_0\|_{L^{n(p-1)}(\Omega)} \leq \gamma$, then problem (1.1) has a global solution.

The local well-posedness problem for the Fujita type nonlinear heat equation was discussed by many authors: For $1 < p < \infty$ and $a \neq 0$,

$$\begin{aligned}
\frac{\partial u}{\partial t} - \Delta u = au^p, & \quad t > 0, x \in \Omega, \\
u(0, x) = u_0(x) > 0, & \quad x \in \Omega.
\end{aligned}$$

In particular, Weissler [26] obtained the sharp well-posedness result in the Lebesgue space: If

$$\begin{aligned}
 r \geq \frac{n}{2}(p - 1) & \quad \text{if } p > 1 + \frac{2}{n}, \\
r > 1 & \quad \text{if } p = 1 + \frac{2}{n}, \\
r \geq 1 & \quad \text{if } 1 < p < 1 + \frac{2}{n},
\end{aligned}$$

then the solution exists and well-posed in the Lebesgue space $L^r(\Omega)$. The exponent appears naturally from the invariant scaling equipped with the equation itself:

$$u_\lambda(t, x) = \lambda^{\frac{n}{p-1}} u(\lambda^2 t, \lambda x),$$

where $u_\lambda$ also solves the equation (1.9). The threshold scaling space appears when the exponent of the coefficient $\lambda^{\frac{n}{p-1}}$ of the scaled function (1.10) coincides the $L^1$ invariant scaling. The corresponding result to the convection-diffusion equation (1.1) also holds for the critical exponent $p = 1 + \frac{1}{n}$ (cf. [10]). In our case, even in the uniformly local spaces, the well-posedness threshold coincides with the usual Lebesgue space case. This stands for that the role of exponent of the function space essentially limited in a local sense and the behavior of the solution at spatial infinity does not give a large difference for the time local well-posedness as far as the function space remains in a uniform sense.

Note that since the above weak solution does not decay at $|x| \to \infty$, it does not satisfy the $L^1$ conservation law anymore. Hence the global existence of the solution is not clear.

2. Preliminaries. In this section, we review some fundamental inequalities which will be used throughout this paper.

**Lemma 2.1.** Let $u \in W^{1, p}(\Omega)$, then $u^+, u^-, |u| \in W^{1, p}(\Omega)$ and

$$\nabla u^+ = \begin{cases} 
\nabla u & \text{if } u > 0, \\
0 & \text{if } u \leq 0,
\end{cases} \quad \nabla u^- = \begin{cases} 
0 & \text{if } u \geq 0, \\
\nabla u & \text{if } u < 0.
\end{cases}$$

Besides it holds that

$$\nabla |u| = \begin{cases} 
\nabla u & \text{if } u > 0, \\
0 & \text{if } u = 0, \\
-\nabla u & \text{if } u < 0.
\end{cases}$$
For the proof see [17].
Here we summarize important properties for functions belonging to the uniformly local $L'$ spaces.

**Proposition 2.1** (Properties of uniformly local spaces). (1) If $1 \leq p \leq q$, then for any $\rho > 0$, we have $L^q_{uloc,\rho}(\Omega) \subset L^p_{uloc,\rho}(\Omega)$.  
(2) Let $1 \leq r < \infty$. If $f \in L^r_{uloc,\rho}(\Omega)$ for some $\rho > 0$, then for any $\rho' > 0$, $f \in L^r_{uloc,\rho'}(\Omega)$ and

$$
\|f\|_{L^r_{uloc,\rho'}} \leq C \|f\|_{L^r_{uloc,\rho}}
$$

(2.11)

for some constant $C$ depending only on $n$, $\rho$ and $\rho'$ if $\rho' > \rho$.

**Remark 1.** The class of compact supported smooth functions; $C^\infty_0(\Omega)$ is not dense in $L^r_{uloc,\rho}(\Omega)$.

**Proof.** Proposition 2.1 (1) The inclusion follows from the Hölder inequality. (2) For the case $\rho' < \rho$, the inequality (2.11) directly follows from the definition of the uniform local norm with the constant $C = 1$. Hence we assume $\rho' > \rho$. Let $A := \bigcup_{y \in \mathbb{R}^n} B_\rho(y)$. Obviously $B_{\rho'}(x) \subset A$. Since $A$ is an open cover of $B_{\rho'}(x)$, there exist a finite open subcover $B$ of $A$ such that for $i = 1, 2, \ldots, N$

$$
B_{\rho'}(x) \subset B = \bigcup_{y_i \in B_{\rho'}(x)} B_{\rho}(y_i) = \bigcup_{i=1}^N B_{\rho}(y_i).
$$

Then for any $x \in \Omega$

$$
\int_{B_{\rho}(x) \cap \Omega} |f(y)|^r \, dy \leq \int_{\bigcup_{i=1}^N B_{\rho}(y_i) \cap \Omega} |f(y)|^r \, dy
$$

$$
\leq N \sup_{x \in \Omega} \int_{B_{\rho}(x) \cap \Omega} |f(y)|^r \, dy \leq C \sup_{x \in \Omega} \int_{B_{\rho}(x) \cap \Omega} |f(y)|^r \, dy.
$$

Since $N$ is only depending on $n$, it yields that

$$
\|f\|_{L^r_{uloc,\rho'}} \leq C \|f\|_{L^r_{uloc,\rho}}.
$$

\[\square\]

**Proposition 2.2.** Let $n \geq 1$, $a \in \Omega$, $\rho > 0$ and $1 \leq p, q, r < \infty$ with

$$
\frac{1}{q} = \frac{1}{p} - \frac{2\theta}{r} \left(\frac{1}{2} - \frac{1}{n}\right).
$$

Then there exists a constant $C > 0$ such that for any function $f$ satisfying $f \in L^p(B_{\rho}(a) \cap \Omega)$ with $|f|^\frac{q}{2} \in H^{1}_{0}(B_{\rho}(a) \cap \Omega)$,

$$
\left(\int_{B_{\rho}(a) \cap \Omega} |f|^q \, dy \right)^{\frac{1}{q}} \leq C \left(\int_{B_{\rho}(a) \cap \Omega} |f|^p \, dy \right)^{\frac{1}{p}} \left(\int_{B_{\rho}(a) \cap \Omega} |\nabla |f|\ast|^{2} \, dy \right)^{\frac{1}{2}}
$$

(2.12)

**Lemma 2.3** (The Gagliardo-Nirenberg inequality). Let $0 < r \leq \infty$, $1 \leq p, q \leq \infty$, and $\theta \in [0, 1]$ satisfying

$$
\frac{1}{q} = (1 - \theta) \frac{1}{p} + \theta \left(\frac{1}{r} - \frac{1}{n}\right).
$$
Then there exists a constant $C_{GN} > 0$, depending only on $p, q, r$ and $n$ such that for any $f \in L^p(\Omega) \cap W^{1,r}_0(\Omega)$,
\[
\|f\|_{L^p} \leq C_{GN} \|f\|_{L^p}^{1-\theta} \|\nabla f\|_{L^r}^{\theta}.
\]
(2.13)

For the proof see ([15], [24]).

**Proof.** Proposition 2.2 It suffices to show the inequality (2.12) for $f \in C^1_0(B_p(a) \cap \Omega)$. From the Gagliardo-Nirenberg inequality (2.13), we see that for $1 \leq \rho, \tilde{q} \leq \infty$, there exists a constant $C > 0$ such that for $g \in L^\rho(B_p(a) \cap \Omega)$ and $\rho > 0$,
\[
\left( \int_{B_p(a) \cap \Omega} |g|^\frac{\rho}{\tilde{q}} \, dy \right)^{\frac{1}{\rho}} \leq C \left( \int_{B_p(a) \cap \Omega} |g|^\rho \, dy \right)^{\frac{1}{\rho}} \left( \int_{B_p(a) \cap \Omega} |\nabla g|^2 \, dy \right)^{\frac{\rho}{\tilde{q}}},
\]
(2.14)

with
\[
\frac{1}{\tilde{q}} = \frac{1-\theta}{\tilde{p}} + \theta \left( \frac{1}{2} - \frac{1}{n} \right).
\]

Substituting $g(x) = |f(x)|^{\frac{\tilde{q}}{2}}$, putting $\frac{\tilde{p}}{2} = q$ and $\frac{\tilde{q}}{2} = p$, we obtain from (2.14) that
\[
\left( \int_{B_p(a) \cap \Omega} |f|^q \, dy \right)^{\frac{1}{q}} \leq C \left( \int_{B_p(a) \cap \Omega} |f|^p \, dy \right)^{\frac{1-p}{r}} \left( \int_{B_p(a) \cap \Omega} |\nabla f|^{\frac{2}{\tilde{r}}} \, dy \right)^{\frac{\tilde{r}}{2}},
\]
(2.15)

where
\[
\frac{1}{\tilde{q}} = \frac{1-\theta}{p} + \frac{2\theta}{r} \left( \frac{1}{2} - \frac{1}{n} \right).
\]
(2.16)

Finally we state the existence of the strong solution to (1.1) in $L^\infty(\Omega)$. Let $BUC(\Omega)$ be a class of set of all bounded uniformly continuous functions on $\Omega$.

**Proposition 2.4** (Existence of a strong solution). Let $n \geq 1$. Then for any $u_0 \in BUC(\Omega)$ with $u_0 = 0$ on $\partial \Omega$, there exist $T = T(\|u_0\|_{L^\infty(\Omega)}) > 0$ and a unique mild solution $u \in C([0,T); BUC(\Omega))$ to (1.1), i.e.,
\[
u(t) = e^{t\Delta_0}u_0 + \int_0^t e^{(t-s)\Delta_0} a \cdot \nabla(|u(s)|^{p-1}u(s)) \, ds
\]
(2.17)
in $C([0,T); BUC(\Omega))$. Here $e^{t\Delta_0}u_0$ is the heat semigroup with the Dirichlet Laplacian $-\Delta_0$ in $W^{0,\infty}_0(\Omega)$ with suitable $\alpha > 0$. Besides the solution satisfies
\[
u \in C([0,T); BUC(\Omega) \cap C^\alpha(\Omega)).
\]

**Proof.** Proposition 2.4 For $M > 0$ and $T > 0$, we let
\[
X_T = \left\{ f \in C([0,T); BUC(\Omega)), \|f\|_{X_T} \right\}
\]
\[
\|f\|_{X_T} = \sup_{t \in [0,T]} \|f(t)\|_{L^\infty(\Omega)} + \sup_{t \in [0,T]} t^{\frac{\tilde{r}}{2}} \|\nabla|\alpha f(t)\|_{L^\infty(\Omega)} \leq M
\]
where $M$ and $T$ are constants depending on $\|u_0\|_{L^\infty(\Omega)}$ and $p$ and $n$ determined later. $X_T$ is a complete metric space with the metric; for any $f$ and $g \in X_T$,
\[
d_{X_T}(f,g) = \|f-g\|_{X_T} = \sup_{0 < t < T} \|f(t) - g(t)\|_{L^\infty(\Omega)}.
\]
Then we consider a map
\[
\Phi : u \in X_T \rightarrow \Phi[u] = e^{t \Delta_0} u_0 + \int_0^t e^{(t-s) \Delta_0} a \cdot \nabla (|u(s)|^{p-1} u(s)) \, ds
\] (2.18)
and show that \( \Phi \) is a contraction mapping from \( X_T \) to itself. This implies an existence of the fixed point for the map \( \Phi \) on \( X_T \) and it becomes a solution to the corresponding integral equation (2.17) has a unique fixed point and it becomes an \( L^r \)-mild solution.

The first term related to the initial data \( u_0 \) in the right hand side of (2.18) is directly estimated by the dissipative estimate as follows:
\[
\| e^{t \Delta_0} u_0 \|_{L^\infty(\Omega)} \leq \| u_0 \|_{L^\infty(\Omega)}. \tag{2.19}
\]
While for the second part of the norm \( \| \cdot \| \),
\[
\| \nabla e^{(t-s) \Delta_0} f \|_{L^\infty(\Omega)} \leq C (t-s)^{-\frac{1}{2}} \| f \|_{L^\infty(\Omega)}. \tag{2.20}
\]
Hence combining (2.19) and (2.20), we obtain
\[
\| e^{t \Delta_0} u_0 \|_{X_T} = \sup_{0 < t < T} \| e^{t \Delta_0} u_0 \|_{L^\infty(\Omega)} \leq C_0 \| u_0 \|_{L^\infty(\Omega)}. \tag{2.21}
\]

We show for \( T > 0 \), the following two estimates to be verified:

**Lemma 2.5.** If \( T > 0 \) is sufficiently small depending on \( n, p \) and \( \| u_0 \|_{\infty} \), then
\[
\| \Phi[u] \|_{X_T} \leq M, \tag{2.22}
\]
\[
\| \Phi[u] - \Phi[v] \|_{X_T} \leq \frac{1}{2} \| u - v \|_{X_T}, \tag{2.23}
\]
for any \( u \) and \( v \in X_T \).

**Proof.** Lemma 2.5 For some \( T > 0 \), let \( I = (0, T) \) and \( M = 4 \max(1, C_\alpha) \| u_0 \|_{\infty} \). To see the estimate (2.22) hold, we invoke the \( L^p-L^q \) type dissipative estimate
\[
\sup_{t \in I} \| \Phi[u(t)] \|_{L^\infty(\Omega)} \leq \| u_0 \|_{L^\infty(\Omega)} + |a| \left\| \int_0^t \nabla e^{(t-s) \Delta_0} (|u(s)|^{p-1} u(s)) \, ds \right\|_{L^\infty(I; L^\infty(\Omega))}
\leq \| u_0 \|_{L^\infty(\Omega)} + C \left\| \int_0^t (t-s)^{-\frac{1}{2}} \| u(s) \|_{\infty}^p \, ds \right\|_{L^\infty(I; L^\infty(\Omega))}
\leq \| u_0 \|_{L^\infty(\Omega)} + C \sup_{0 < t < T} \| u(t) \|_{\infty}^p \int_0^t (t-s)^{-\frac{1}{2}} \, ds
\leq \frac{1}{4} M + CT^\frac{1}{2} \sup_{0 < t < T} \| u(t) \|_{L^\infty(\Omega)}^p. \tag{2.24}
\]
Hence by choosing \( T \) small enough such that
\[
CT^\frac{1}{2} M^{p-1} \leq \frac{1}{4} \tag{2.25}
\]
then we obtain
\[
\| \Phi[u] \|_{L^\infty(I; L^\infty(\Omega))} \leq \frac{1}{4} M + \frac{1}{4} M = \frac{1}{2} M. \tag{2.26}
\]
Similar to (2.24), we see that
\[
\|\nabla^\alpha \Phi[u(t)]\|_{L^\infty(\Omega)} \leq C_{\alpha} t^{-\frac{\alpha}{2}} \|u_0\|_{L^\infty(\Omega)}
\]
\[
+ |a| \int_0^t \|\nabla^\alpha e^{(t-s)\Delta_0} (|u(s)|^{p-1}u(s))ds\|_{L^\infty(\Omega)}
\]
\[
\leq C_{\alpha} t^{-\frac{\alpha}{2}} \|u_0\|_{L^\infty(\Omega)} + C \sup_{0 < t < T} \|u(t)\|_{L^p(\Omega)}^p \int_0^t |t-s|^{-\frac{\alpha}{2} - \frac{1}{2}} ds
\]
\[
\leq C_{\alpha} t^{-\frac{\alpha}{2}} \|u_0\|_{L^\infty(\Omega)} + Ct^{\frac{1}{4} - \frac{\alpha}{2}} \sup_{0 < t < T} \|u(t)\|_{L^\infty(\Omega)}.
\]  
(2.27)

Hence it follows that
\[
\sup_{t \in I} t^{\frac{1}{4}} \|\nabla^\alpha \Phi[u(t)]\|_{L^\infty(\Omega)} \leq C_{\alpha} \|u_0\|_{L^\infty(\Omega)} + CT^{\frac{1}{4}} \sup_{0 < t < T} \|u(t)\|_{L^\infty(\Omega)}^p
\leq \frac{1}{4} M + CT^{\frac{1}{4}} M^p \leq \frac{1}{2} M
\]  
(2.28)

under the condition (2.25). Combining (2.26) and (2.28), we obtain the estimate (2.22).

The estimate (2.23) is obtained in an almost similar way to (2.22). Since the first term of the right hand side of (2.18) is common, we have for any \(u \) and \(v \in X_T\), that
\[
\|\Phi[u] - \Phi[v]\|_{L^\infty(\Omega)} \leq \sup_{t \in I} \int_0^t \|u(t) - v(t)\|_{L^\infty(\Omega)} ds
\]
\[
\leq CT^{\frac{1}{4}} \sup_{t \in I} \max(\|u(t)|^{p-1}, |v(t)|^{p-1}) \cdot \|u(t) - v(t)\|_{L^\infty(\Omega)}
\leq CT^{\frac{1}{4}} M^{p-1} \sup_{t \in I} \|u(t) - v(t)\|_{L^\infty(\Omega)}.
\]  
(2.29)

Hence by setting
\[
CT^{\frac{1}{4}} M^{p-1} \leq \frac{1}{4},
\]  
(2.30)

we conclude from (2.29) that
\[
\|\Phi[u] - \Phi[v]\|_{L^\infty(\Omega)} \leq \frac{1}{4} \sup_{t \in I} \|u(t) - v(t)\|_{L^\infty(\Omega)}.
\]  
(2.31)

Similar to the bound estimate, we proceed
\[
\|\nabla^\alpha (\Phi[u] - \Phi[v])\|_{L^\infty} \leq \sup_{t \in I} \int_0^t \|u(t) - v(t)\|_{L^\infty(\Omega)} ds
\]
\[
\leq CT^{\frac{1}{4} - \frac{\alpha}{2}} M^{p-1} \sup_{t \in I} \|u(t) - v(t)\|_{L^\infty(\Omega)}.
\]  
(2.32)

Hence by (2.30), we conclude from (2.32) that
\[
\sup_{t \in I} t^{\frac{1}{4}} \|\Phi[u(t)] - \Phi[v(t)]\|_{L^\infty} \leq \frac{1}{4} \sup_{t \in I} \|u(t) - v(t)\|_{L^\infty}.
\]  
(2.33)

Combining (2.31) and (2.33), we obtain the desired estimate (2.23).
Now we see from Lemma 2.5 that the map $\Phi$ is contraction from $X_T$ to $X_T$ and by virtue of the Banach fixed point theorem, there exists a unique fixed point of $\Phi$ in $X_T$. By the definition, this fixed point satisfies the integral equation (2.17) and besides, $u(t) \to u_0$ as $t \to 0$. Hence $u$ is the $L^\infty$-mild solution to (1.1). This shows the existence of solution. The uniqueness and the continuous dependence of the initial data is obtained by very similar estimate (2.29).

Since the solution is in $X_T$, it is bounded uniformly continuous in $x$ variable for all $t < T$. Besides the solution is belonging to $W^{\alpha,\infty}(\Omega)$ and hence it is Hölder continuous in $x$. Then the mild solution satisfies the equation (1.1) with the initial data pointwisely and this completes the proof of Proposition 2.4. 

3. A priori estimates. In this section, we give some a priori estimates for a weak solution to (1.1). All the estimates holds for the weak solution to (1.1) if we assume that the solution exists. In what follows, we denote $B_p(x) \cap \Omega$ for $x \in \Omega$, $\rho > 0$ by simply $B_p(x)$ unless otherwise specified.

**Proposition 3.1** (A priori estimate). Let $r$ satisfy (1.5) and $r > 1$. Let $u_0 \in L^r_{uloc,\rho}(\Omega)$ and $u$ be a $L^r_{uloc}(\Omega)$-solution of (1.1) in $(0, T) \times \Omega$, where $T > 0$. Then there exists a positive constant $\gamma_1$ such that, if
\[
\rho^{\frac{n-2}{2}} \sup_{0 < s \leq t} \|u(s)\|_{L^r_{uloc, \rho}} \leq \gamma_1
\]
for some $\rho > 0$, then there exists a constant $\mu > 0$ depending only on $p$, $r$, $n$ and $\gamma_1$ such that
\[
\sup_{0 < s \leq t} \|u(s)\|_{L^r_{uloc, \rho}} \leq \mu \|u_0\|_{L^r_{uloc, \rho}}
\]
for $0 < t \leq \min\{\mu \rho^2, T\}$, where $\mu$ is a positive constant depending only on $n$, $p$ and $r$.

**Proof.** Proposition 3.1 Let $x \in \Omega$ and $\zeta$ be a smooth function in $C_0^\infty(\Omega)$ such that
\[
\begin{align*}
0 \leq \zeta \leq 1 \quad \text{and} \quad |\nabla \zeta| \leq 2\rho^{-1} \quad \text{in} \ \Omega, \\
\zeta = 1 \quad \text{on} \quad B_p(x), \quad \zeta = 0 \quad \text{in} \quad \Omega \setminus B_{2p}(x).
\end{align*}
\]
In what follows, for any ball $B_p(x) \cap \Omega \neq \emptyset$, we denote $B_p(x) \cap \Omega$ by $B_p(x)$ for simplicity. For any $0 < \tau < t \leq T$, multiplying (1.1) by $(\sgn u)|u|^{r-1}\zeta^k$ and integrating it in $(0, \tau) \times \Omega$, we have
\[
\begin{align*}
\frac{1}{r} \int_{B_{2\rho}(x)} |u(\tau, y)|^{r-1}\zeta(y)^k \, dy - \frac{1}{r} \int_{B_p(x)} |u(0, y)|^{r-1}\zeta(y)^k \, dy \\
+ \int_0^\tau \int_{\Omega} \nabla u(s, y) \cdot \nabla((\sgn u(s, y))|u(s, y)|^{r-1}\zeta^k) \, dy \, ds \\
= \int_0^\tau \int_{\Omega} a \cdot \nabla(|u(s, y)|^{p-1} u(s, y))(\sgn u(s, y))|u(s, y)|^{r-1}\zeta^k \, dy \, ds.
\end{align*}
\]
Since
\[
\nabla u \cdot \nabla((\sgn u)|u|^{r-1}\zeta^k) \geq (r-1)|u|^{r-2}|\nabla u|^2\zeta^k - k(|\sgn u||u|^{r-1}\zeta^{k-1}\nabla u \cdot \nabla \zeta|
\[
\geq \{ (r-1) - k \varepsilon \}|u|^{r-2}|\nabla u|^2\zeta^k - \frac{k}{4\varepsilon} |u|^{r-1}\zeta^{k-2}|\nabla \zeta|^2
\]
and
\[
\left| \nabla(|u|^\frac{2}{r}\zeta^k) \right|^2 \leq \frac{r^2}{2}|u|^{r-2}|\nabla u|^2\zeta^k + \frac{k^2}{2}|u|^{r-1}\zeta^{k-2}|\nabla \zeta|^2,
\]

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\]
we see from (3.5) and (3.6) that
\[
\nabla u \cdot \nabla ((\text{sgn} u) |u|^{r-1} \zeta^k)
\geq \{r(1) - \varepsilon k\} \left\{ \frac{2}{r^2} \| \nabla (|u|^\frac{\varepsilon}{2} \zeta^\frac{k}{2}) \|^2 - \frac{k^2}{r^2} |u|^r \zeta^{k-2} |\nabla \zeta| \|^2 \right\} - \frac{k}{4r^2} |u|^r \zeta^{k-2} |\nabla \zeta|^2
\]
\[
= C_1 \left| \nabla (|u|^\frac{\varepsilon}{2} \zeta^\frac{k}{2}) \right|^2 - C_2 |u|^r \zeta^{k-2} |\nabla \zeta|^2.
\]
By the Young inequality, we see that
\[
\int_0^\tau \int_{B_{2r}(x)} \frac{p}{p + r - 1} a \cdot \nabla (|u(s, y)|^{p + r - 1}) \zeta(y)^k dy ds
\leq \frac{kp}{p + r - 1} \int_0^\tau \int_{B_{2r}(x)} a \cdot |u(s, y)|^{p + r - 1} \zeta^{k-1} |\nabla \zeta(y)| dy ds
\leq C_2 \int_0^\tau \int_{B_{2r}(x)} |u(s, y)|^r \zeta(y)^{k-2} |\nabla \zeta(y)|^2 dy ds
+ C_3 \int_0^\tau \int_{B_{2r}(x)} |u(s, y)|^{2p + r - 2} \zeta(y)^k dy ds.
\] (3.8)
We obtain from (3.4), (3.7) and (3.8),
\[
\frac{1}{r} \int_{B_{2r}(x)} |u(\tau, y)|^r \zeta(y)^k dy - \frac{1}{r} \int_{B_{2r}(x)} |u(0, y)|^r \zeta(y)^k dy
+ C_1 \int_0^\tau \int_{B_{2r}(x)} \left| \nabla (|u(s, y)|^\frac{\varepsilon}{2} \zeta(y)^\frac{k}{2}) \right|^2 dy ds
\leq C_2 \int_0^\tau \int_{B_{2r}(x)} |u(s, y)|^r \zeta(y)^{k-2} |\nabla \zeta(y)|^2 dy ds
+ C_3 \int_0^\tau \int_{B_{2r}(x)} |u(s, y)|^{2p + r - 2} \zeta(y)^k dy ds.
\] (3.9)
Now we estimate the last term of the right hand side of (3.9) using the Gagliardo-Nirenberg inequality (Proposition 2.2). In particular, choosing \( \tilde{q} = \frac{4}{r}(p - 1) + 2 \) and \( \tilde{p} = 1 \), and setting \( g(s, y) := |u(s, y)| \zeta(y)^{\frac{r}{2p + r - 2}} \), we obtain by the Hölder inequality that for \( r \geq n(p - 1) \)
\[
\int_0^\tau \int_{B_{2r}(x)} |u(s, y)|^{2p + r - 2} \zeta(y)^k dy ds
\leq C \sup_{\rho < s < \tau} \left( \int_{B_{2\rho}(x)} |g(s, y)|^{n(p - 1)} dy \right)^{\frac{2}{n}} \int_0^\tau \int_{B_{2r}(x)} |\nabla |g(s, y)|^\frac{k}{2} |^2 dy ds
\leq C \sup_{\rho < s < \tau} \left( \rho^{p - 1} \int_{B_{2\rho}(x)} |u(s, y)|^r dy \right)^{\frac{2(p - 1)}{2(p - 1)}}
\times \int_0^\tau \left( \int_{B_{2r}(x)} \left| \nabla (|u(s, y)|^\frac{k}{2}) \right|^2 dy + \rho^{-2} \int_{B_{2\rho}(x)} |u(s, y)|^r dy \right) ds.
\] (3.10)
Hence by Proposition 2.1, we obtain from (3.10)

\[
\int_0^\tau \int_{B_2(x)} |u(s, y)|^{2p+\gamma-2} \zeta(y)^k dy ds \\
\leq C \left( \rho^{\frac{-\gamma}{2}} \sup_{0 < s < \tau} \sup_{x \in \Omega} \int_{B_r(x)} |u(s, y)|^\gamma dy \right)^{2(p-1)} \\
\times \left( \sup_{x \in \Omega} \int_0^\tau \int_{B_r(x)} |\nabla (|u(s, y)|^\frac{\gamma}{2}) \zeta(y)^{\frac{k}{2}}|^2 dy ds + \rho^{-2} \sup_{x \in \Omega} \int_0^\tau \int_{B_r(x)} |u(s, y)|^\gamma dy ds \right) \\
\leq C \left( \rho^{\frac{-\gamma}{2}} \sup_{0 < s < \tau} \|u(s)\|_{L^\gamma_{uloc,p}} \right)^{2(p-1)} \\
\times \left( \sup_{x \in \Omega} \int_0^\tau \int_{B_r(x)} |\nabla (|u(s, y)|^\frac{\gamma}{2}) \zeta(y)^{\frac{k}{2}}|^2 dy ds + \tau \rho^{-2} \sup_{0 < s < \tau} \|u(s)\|_{L^\gamma_{uloc,p}} \right) \\
\]  

for all \(0 < \tau < t \leq T\). From (3.11) and (3.9),

\[
\frac{1}{\tau} \int_{B_2(x)} |u(\tau, y)|^\gamma \zeta(y)^k dy - \frac{1}{\tau} \int_{B_2(x)} |u(0, y)|^\gamma \zeta(y)^k dy \\
+ C_1 \int_0^\tau \int_{B_r(x)} |\nabla (|u(s, y)|^\frac{\gamma}{2}) \zeta(y)^{\frac{k}{2}}|^2 dy ds \\
\leq C_2 \rho^{-2} \sup_{x \in \Omega} \int_0^\tau \int_{B_r(x)} |u(s, y)|^\gamma dy \\
+ C_3 \left( \rho^{\frac{-\gamma}{2}} \sup_{0 < s < \tau} \|u(s)\|_{L^\gamma_{uloc,p}} \right)^{2(p-1)} \\
\times \left( \sup_{x \in \Omega} \int_0^\tau \int_{B_r(x)} |\nabla (|u(s, y)|^\frac{\gamma}{2}) \zeta(y)^{\frac{k}{2}}|^2 dy ds + \tau \rho^{-2} \sup_{0 < s < \tau} \|u(s)\|_{L^\gamma_{uloc,p}} \right) \\
\]  

By taking the supremum for \(\tau \in (0, t)\) in the right hand side of (3.12), we obtain

\[
\sup_{x \in \Omega} \int_0^\tau \int_{B_r(x)} |u(\tau, y)|^\gamma dy + C_1 \sup_{x \in \Omega} \int_0^\tau \int_{B_r(x)} |\nabla (|u(s, y)|^\frac{\gamma}{2}) \zeta(y)^{\frac{k}{2}}|^2 dy ds \\
\leq C_2 \rho^{-2} \sup_{0 < s < t} \int_{B_r(x)} |u(s, y)|^\gamma dy + \sup_{x \in \Omega} \int_{B_r(x)} |u(0, y)|^\gamma dy \\
+ C_3 \left( \sup_{x \in \Omega} \int_0^t \int_{B_r(x)} |\nabla (|u(s, y)|^\frac{\gamma}{2}) \zeta(y)^{\frac{k}{2}}|^2 dy ds + t \rho^{-2} \sup_{0 < s < t} \|u(s)\|_{L^\gamma_{uloc,p}} \right) \\
\]  

for \(0 < \tau < t \leq T\). Taking a sufficiently small \(\gamma_1\) and \(t \rho^{-2}\) if necessary, and by taking the supremum for \(\tau \in (0, t)\), we deduce from (3.13) that

\[
\sup_{0 < \tau < t} \sup_{x \in \Omega} \int_{B_r(x)} |u(\tau, y)|^\gamma dy \\
\leq Ct \rho^{-2} \sup_{0 < s < t} \int_{B_r(x)} |u(s, y)|^\gamma dy + \sup_{x \in \Omega} \int_{B_r(x)} |u(0, y)|^\gamma dy \\
\]  

(3.14)
for $0 < \tau < t \leq T$. Hence from (3.14), we obtain
\[
\sup_{0 \leq s \leq t} \|u(s)\|_{L^r_{\text{loc}, \rho}} \leq C\|u(0)\|_{L^r_{\text{loc}, \rho}},
\] (3.15)
for $0 < t \leq \min\{\mu \rho^2, T\}$.

**Proposition 3.2** (Difference estimate). Let $r$ satisfy (1.5), $r > 1$ and $T > 0$. Let $u_0$ and $v_0 \in L^r_{\text{loc}, \rho}(\Omega)$ be two initial data and suppose that $u$ and $v$ be a corresponding $L^r_{\text{loc}}(\Omega)$-solution of (1.1) in $(0, T) \times \Omega$, respectively. Then there exists a positive constant $\gamma_2$ such that, if
\[
\rho \frac{1}{r} \frac{d}{dt} \int_\Omega |w(s)|^r \zeta^k \, dy + \int_\Omega \nabla w(s) \cdot \nabla (|w(s)|^{r-1} (\text{sgn } w(s)) \zeta^k) \, dy
\]
\[= - \int_\Omega (|u|^{p-1} u - |v|^{p-1} v) (a \cdot \nabla (\text{sgn } w(s)) |w(s)|^{r-1}) \zeta^k \, dy \tag{3.18}
\]
\[= \int_\Omega (|u|^{p-1} u - |v|^{p-1} v) (\text{sgn } w(s)) |w(s)|^{r-1} a \cdot \nabla \zeta^k \, dy.
\]
Observing that
\[
\nabla w \cdot \nabla (|w|^{r-1} (\text{sgn } w) \zeta^k) \geq C_1 \left| \nabla (|w|^{\frac{r}{2} \zeta^{\frac{k}{2}}}) \right|^2 - C_2 |w|^{r-2} |\nabla \zeta|^2. \tag{3.19}
\]
By the mean value theorem
\[
|u|^{p-1} u - |v|^{p-1} v = \left| \int_0^1 \frac{d}{d\theta} (v + \theta(u - v))^{p-1} (v + \theta(u - v)) \, d\theta \right|
\]
\[\leq p |u - v| \int_0^1 |v + \theta(u - v)|^{p-1} \, d\theta \tag{3.20}
\]
\[\leq p |u - v| (\max(|u|, |v|))^{p-1}.
\]
Therefore, by (3.19) and (3.20) we obtain from (3.18)

\[
\frac{1}{r} \frac{d}{dt} \int_{B_{2r}(x)} |w(s)|^r \zeta^k dy + C \int_{B_{2r}(x)} \left| \nabla (|w(s)|^\frac{r}{2} \zeta^\frac{k}{2}) \right|^2 dy \\
\leq C \int_{B_{2r}(x)} \left( \max(|u(s)|, |v(s)|) \right)^{r-1} |\nabla |w(s)|^r| \zeta^k dy \\
+ C \int_{B_{2r}(x)} \left( \max(|u(s)|, |v(s)|) \right)^{r-1} |w(s)|^r |\nabla \zeta|^k dy \\
+ C \int_{B_{2r}(x)} |w(s)|^r \zeta^{k-2} |\nabla \zeta|^2 dy. 
\]  

(3.21)

Now we estimate the first and last term of the right hand side of (3.21) using the Young and the Hölder inequalities. The first term of the right hand side of (3.21) follows:

Let \( U(s) = \max(|u(s)|, |v(s)|) \), then

\[
\int_{B_{2r}(x)} \left( \max(|u(s)|, |v(s)|) \right)^{r-1} |\nabla |w(s)|^r| \zeta^k dy \\
= C \int_{B_{2r}(x)} U(s)^{r-1} |w(s)|^\frac{r}{2} |\nabla |w(s)|^\frac{r}{2}| \zeta^k dy \\
\leq C \int_{B_{2r}(x)} U(s)^{2r-2} |w(s)|^r \zeta^k dy + C \int_{B_{2r}(x)} |\nabla |w(s)|^\frac{r}{2}| \zeta^k dy. 
\]  

(3.22)

Now we estimate the first term of the right hand side of (3.22) using the Hölder and the Sobolev inequalities and obtain that

\[
\int_0^\tau \int_{B_{2r}(x)} |U(s, y)|^{2p-2} |w(s, y)|^r \zeta(y)^k dy ds \\
\leq C \int_0^\tau \left( \int_{B_{2r}(x)} |U(s, y)|^{n(p-1)} dy \right)^\frac{2}{n} \left( \int_{B_{2r}(x)} |w(s, y)|^\frac{2(p-1)}{p-2} \zeta(y)^\frac{k}{p-2} dy \right)^\frac{p-2}{n} ds \\
\leq C \int_0^\tau \left( \int_{B_{2r}(x)} |U(s, y)|^{n(p-1)} dy \right)^\frac{2}{n} \left( \int_{B_{2r}(x)} |\nabla \left( |w(s, y)|^\frac{r}{2} \zeta(y)^\frac{k}{2} \right)|^2 dy \right) ds \\
\leq C \sup_{0 < s < \tau} \left( \rho^{\frac{n}{r} - \frac{2}{2p-2}} \int_{B_{2r}(x)} |U(s, y)|^r dy \right) \\
\times \int_0^\tau \left( \int_{B_{2r}(x)} |\nabla (|w(s, y)|^\frac{r}{2})|^2 dy + \rho^{\frac{2}{r} - 2} \int_{B_{2r}(x)} |w(s, y)|^r dy \right) ds. 
\]  

(3.23)

Therefore, by (3.22), (3.23), we obtain from (3.21)
\[
\frac{1}{r} \int_{B_{2r}(x)} |w(\tau, y)|^r \zeta(y)^k \, dy - \frac{1}{r} \int_{B_{2r}(x)} |w(0, y)|^r \zeta(y)^k \, dy \\
+ C \int_0^\tau \int_{B_{2r}(x)} |\nabla(|w(s, y)|^2 \zeta(y)^{\frac{k}{2}})|^2 \, ds \, dy \\
\leq C \int_0^\tau \int_{B_{2r}(x)} |w(s, y)|^r \zeta(y)^k \nabla \zeta(y)^2 \, dy \\
+ C \sup_{0 < s < \tau} \left( \rho^{\frac{r-1}{r-n}} \int_{B_{2r}(x)} |\max(|u|, |v|)|^r \, dy \right)^{\frac{2(p-1)}{r-p}} \\
\times \left( \int_0^\tau \left( \int_{B_{2r}(x)} |\nabla(|w(s, y)|^2 \zeta)|^2 \, dy \right) \, ds \right)
\]

By the Gagliardo-Nirenberg inequality, we obtain from (3.24)
\[
\frac{1}{r} \int_{B_{2r}(x)} |w(\tau, y)|^r \zeta(y)^k \, dy - \frac{1}{r} \int_{B_{2r}(x)} |w(0, y)|^r \zeta(y)^k \, dy \\
+ C \int_0^\tau \int_{B_{2r}(x)} |\nabla(|w(s, y)|^2 \zeta(y)^{\frac{k}{2}})|^2 \, ds \, dy \\
\leq C \rho^{-2} \sup_{x \in \Omega} \int_0^\tau \int_{B_{r}(x)} |w(s, y)|^r \, dy \\
+ C \left( \rho^{\frac{r-1}{r-n}} \sup_{0 < s < \tau} \|u(s)\|_{L^{r}_{uloc, \rho}}^r + \rho^{\frac{r-1}{r-n}} \sup_{0 < s < \tau} \|v(s)\|_{L^{r}_{uloc, \rho}}^r \right)^{\frac{2(p-1)}{r-p}} \\
\times \left( \sup_{x \in \Omega} \int_0^\tau \left( \int_{B_{r}(x)} |\nabla(|w(s, y)|^2 \zeta)|^2 \, dy \right) \, ds \right)
\]

for all \(0 < \tau < t \leq T\).

By taking the supremum for \(\tau \in (0, t)\) in the right hand side of (3.25), we obtain
\[
\sup_{x \in \Omega} \int_{B_{r}(x)} |w(\tau, y)|^r \, dy + C \sup_{x \in \Omega} \int_0^\tau \int_{B_{r}(x)} |\nabla(|w(s, y)|^2 \zeta)|^2 \, ds \, dy \\
\leq C \rho^{-2} \sup_{0 < s < \tau} \int_{B_{r}(x)} |w(s, y)|^r \, dy + \sup_{x \in \Omega} \int_{B_{r}(x)} |w(0, y)|^r \, dy \\
+ C \gamma_2 \left( \sup_{x \in \Omega} \int_0^t \left( \int_{B_{r}(x)} |\nabla(|w(s, y)|^2 \zeta)|^2 \, dy \right) \, ds \right)
\]

for \(0 < \tau < t \leq T\). Taking a sufficiently small \(\gamma_2\) and \(t \rho^{-2}\) if necessary, and by taking the supremum for \(\tau \in (0, t)\), we deduce from (3.26) that
\[
\sup_{0 < \tau < t} \sup_{x \in \Omega} \int_{B_{r}(x)} |w(\tau, y)|^r \, dy \\
\leq C \rho^{-2} \sup_{0 < \tau < t} \sup_{x \in \Omega} \int_{B_{r}(x)} |w(s, y)|^r \, dy + \sup_{x \in \Omega} \int_{B_{r}(x)} |w(0, y)|^r \, dy
\]
for \(0 < \tau < t \leq T\). Hence from (3.27), we obtain

\[
\sup_{0 < s < t} \|w(s)\|_{L_{\text{uloc}, r}^p} \leq C \|w(0)\|_{L_{\text{uloc}, r}^p}
\]

(3.28)

for \(0 < t \leq \min\{\mu \rho^2, T\}\).

To obtain the critical existence of the weak solution, the \(L^\infty\) a priori estimate for the weak solution is essential. For related results, see([1, 19]).

**Proposition 3.3** (\(L^\infty\)-a priori estimate). Let \(u\) be a \(L^\infty_{\text{uloc}}(\Omega)\)-solution of (1.1) in \((0, T) \times \Omega\), where \(0 < T < \infty\) and \(r \geq 1\). For some positive constant \(\gamma_3\), if

\[
\rho^{\frac{1}{p-r} - \frac{1}{p}} \sup_{0 < s \leq T} \|u(s)\|_{L_{\text{uloc}, r}^p} \leq \gamma_3
\]

(3.29)

for some \(\rho > 0\), then there exist a constant \(C > 0\) such that

\[
\|u\|_{L^\infty((t_1, t) \times B_{R_1}(x))} \leq CD^{\frac{n+2}{2}} \left( \int_{t_2}^{t_1} \int_{B_{R_2}(x)} |u|^r \, dy \, ds \right)^{\frac{1}{r}},
\]

(3.30)

\[
\int_{t_1}^{t} \int_{B_{R_1}(x)} |\nabla u|^2 \, dy \, ds \leq CD \int_{t_2}^{t_1} \int_{B_{R_2}(x)} |u|^2 \, dy \, ds,
\]

(3.31)

for all \(x \in \Omega, 0 < R_1 < R_2\) and \(0 < t_2 < t_1 \leq T\), where

\[
D = C_4(R_2 - R_1)^{-2} + (t_1 - t_2)^{-1}.
\]

**Proof.** Proposition 3.3 Let \(x \in \Omega, 0 < R_1 < R_2, 0 < t_2 < t_1 < t \leq T\). For \(j = 0, 1, 2, ..., \) set

\[
r_j := R_1 + (R_2 - R_1)2^{-j}, \quad \tau_j := t_1 - (t_1 - t_2)2^{-2j}, \quad Q_j = (\tau_j, t) \times B_{r_j}(x).
\]

Let \(\zeta_j\) be a piecewise smooth function in \(Q_j\) satisfying

\[
\begin{align*}
0 \leq \zeta_j(t, x) &
\leq 1 \quad \text{in } \Omega, \\
\zeta_j(t, x) &
\equiv 1 \quad \text{on } Q_{j+1}, \\
\zeta_j &
\equiv 0 \quad \text{near } [\tau_j, t] \times \partial B_{r_j}(x) \cup \{\tau_j\} \times B_{r_j}(x), \\
|\nabla \zeta_j| &
\leq \frac{2^{j+1}}{R_2 - R_1}, \quad \text{in } Q_j \\
0 \leq \partial_t \zeta_j &
\leq \frac{2^{2(j+1)}}{t_2 - t_1}, \quad \text{in } Q_j.
\end{align*}
\]

(3.32)

Multiplying (1.1) by \(|u(t, y)|^{\beta - 2} u(t, y) \zeta_j(t, y)|^k| u(t, y)|^k dy\) and integrating it in \(\Omega\), we obtain

\[
\frac{1}{\beta} \frac{d}{dt} \left( \int_{B_{r_j}(x)} |u(t, y)|^{\beta} \zeta_j(t, y)^k \, dy \right) + \frac{2(2\beta - k - 2)}{\beta^2} \int_{B_{r_j}(x)} |\nabla u(t, y)|^2 \zeta_j(t, y)^k \, dy \\
\leq \frac{p k|a|^2}{2(p + \beta - 1)} \int_{B_{r_j}(x)} |u(t, y)|^{2p+\beta-2} \zeta_j(t, y)^k \, dy \\
+ \frac{k}{2} \left( \frac{p}{p + \beta - 1} + 1 \right) \int_{B_{r_j}(x)} |u(t, y)|^2 \zeta_j(t, y)^k - 2 |\nabla \zeta_j(t, y)|^2 \, dy \\
+ \frac{k}{\beta} \int_{B_{r_j}(x)} \zeta_j(t, y)^{k-1} |u(t, y)|^{\beta} \partial_t \zeta_j(t, y) \, dy.
\]

(3.33)
For the highest order term, using the Hölder and the Sobolev inequalities, we obtain
\[
\int_{B_{r_j}(x)} |u(t,y)|^{2(p-1)} |u(t,y)|^\beta \zeta^k \, dy \\
\leq \left( \int_{B_{r_j}(x)} |u(t,y)|^{n(p-1)} \, dy \right)^{\frac{2}{n}} \left( \int_{B_{r_j}(x)} \left( |u(t,y)|^\beta \zeta^k \right)^{\frac{n}{n-2}} \, dy \right)^{\frac{n-2}{n}} \\
\leq C_s^2 \left( \int_{B_{r_j}(x)} |u(t,y)|^{n(p-1)} \, dy \right)^{\frac{2}{n}} \left( \int_{B_{r_j}(x)} |(u(t,y)|\zeta_j(t,y)^\frac{k}{r_j} |^2 \, dy \right).
\]

Since
\[
\frac{1}{2} \left| \nabla (u \zeta_j^k) \right|^2 - \frac{k^2}{4} u \zeta_j^{k-2} |\nabla \zeta_j|^2 \leq \left| \nabla u \right|^2 \zeta_j^k
\]
and (3.34), integrating (3.33) over \( t \in I_j \), we obtain
\[
\sup_{t \in I_j} \int_{B_{r_j}(x)} |u(t,y)|^\beta \zeta_j(t,y)^k \, dy + \frac{2\beta - k - 2}{\beta} \int_{I_j} \int_{B_{r_j}(x)} \left| \nabla (|u(t,y)|\zeta_j(t,y)^\frac{k}{r_j}) \right|^2 \, dy \, ds \\
\leq \frac{pk|a|^{2\beta}}{2(p + \beta - 1)} C_s^2 \int_{I_j} \left( \int_{B_{r_j}(x)} |u(t,y)|^{n(p-1)} \, dy \right)^{\frac{2}{n}} \left( \int_{B_{r_j}(x)} \left| \nabla (|u(t,y)|\zeta_j(t,y)^\frac{k}{r_j}) \right|^2 \, dy \right) ds \\
+ \frac{k}{2} \left( \frac{p\beta}{p + \beta - 1} + \beta + \frac{k(2\beta - k - 2)}{\beta} \right) \int_{I_j} \int_{B_{r_j}(x)} |u(t,y)|^\beta \zeta_j(t,y)^{k-2} |\nabla \zeta_j(t,y)|^2 \, dy \, ds \\
+ k \int_{I_j} \int_{B_{r_j}(x)} |u(t,y)|^\beta \zeta_j^{k-1} (t,y) \partial_t \zeta_j(t,y) \, dy.
\]

(3.35)

Let \( \gamma_3 > 0 \) be taken as
\[
\frac{pk|a|^{2\beta}}{2(p + \beta - 1)} C_s^2 \gamma_3^{\frac{2}{n}} \leq 1 - \frac{k + 2}{n(p - 1)}.
\]

Then under the assumption (3.29), we estimate the first term of the right hand side of (3.35) and it cancels by the second term of the right hand side. Thus from (3.35) and using the estimate for the derivatives \( \zeta_j \) in (3.32), we obtain that
\[
\sup_{t \in I_j} \int_{B_{r_j}(x)} u(t)^\beta \zeta_j(t)^k \, dy + \int_{I_j} \int_{B_{r_j}(x)} \left| \nabla (u \zeta_j^\frac{k}{r_j}) \right|^2 \, dy \, ds \\
\leq 2k \left( \frac{p\beta}{p + \beta - 1} + \beta + \frac{k(2\beta - k - 2)}{\beta} \right) \int_{I_j} \int_{B_{r_j}(x)} |u(t,y)|^\beta \, dy \, ds \\
= C2^j \left( \frac{\beta}{(R_2 - R_1)^2} + \frac{1}{t_1 - t_2} \right) \int_{I_j} \int_{B_{r_j}(x)} |u(t,y)|^\beta \, dy \, ds,
\]

(3.36)

for any \( j = 0, 1, 2, \ldots \) and \( \beta > r \). Now applying the Gagliardo-Nirenberg inequality, Proposition 2.2, for any function \( f \in C_0^1(B_{r_j}(x)) \) and \( \theta \in (0, 1) \) with choosing \( r = 2 + \frac{4}{n} = 2(1 + \frac{2}{n}) \), \( p = 2 \), \( q = 2 \). We obtain for letting \( \gamma = 1 + \frac{2}{n} \)
\[
\int_{B_{r_j}(x)} |f|^{2\gamma} \, dy \leq C2^j \left( \int_{B_{r_j}(x)} |f|^2 \, dy \right)^{\frac{\gamma}{2}} \int_{B_{r_j}(x)} |\nabla f|^2 \, dy.
\]

(3.37)
Integrating (3.37) with respect to time \( t \in I_j \),
\[
\int_{I_j} \int_{B_{r_j}(x)} |u|^{\beta_j} dy ds \leq C^{2\gamma} \left( \sup_{t \in I_j} \int_{B_{r_j}(x)} |u|^{\beta_j} dy \right)^{\frac{2}{\gamma}} \int_{I_j} \int_{B_{r_j}(x)} |\nabla (u^{\frac{k-1}{j}})|^2 dy ds.
\]
Hence we obtain the reversed Hölder estimate:
\[
\left( \int_{Q_{j+1}} |u(t,y)|^{\beta_j} dy ds \right)^{\frac{1}{\beta_j}} \leq C^{2j} \left\{ \frac{\beta_j}{(R_2 - R_1)^2} + \frac{1}{t_1 - t_2} \right\} \int_{Q_j} |u(t,y)|^{\beta_j} dy ds,
\]
where \( Q_j = I_j \times B_{r_j}(x) = (\tau_j, t) \times B_{r_j}(x) \) and \( \zeta_j = 1 \) on \( Q_{j+1} \). Furthermore, by (3.36) with \( \beta = 2 \) and \( k = 2 \) we have (3.31). We use the estimate (3.39) iteratively with choosing \( \beta = \beta_j = r\gamma_j \), where \( \gamma = 1 + \frac{2}{n} \) and \( j = 1, 2, \ldots \). Since it holds
\[
\left( \int_{Q_{j+1}} |u(t,y)|^{\beta_j} dy ds \right)^{\frac{1}{\beta_j}} \leq \left( \int_{Q_{j+1}} |u(t,y)|^{\beta_j} dy ds \right)^{\frac{1}{\beta_j}} \leq \left( \int_{Q_{j+1}} |u(t,y)|^{\beta_j} dy ds \right)^{\frac{1}{\beta_j}} \leq \left( \int_{Q_{j+1}} |u(t,y)|^{\beta_j} dy ds \right)^{\frac{1}{\beta_j}} \leq \left( \int_{Q_{j+1}} |u(t,y)|^{\beta_j} dy ds \right)^{\frac{1}{\beta_j}} \leq \left( \int_{Q_{j+1}} |u(t,y)|^{\beta_j} dy ds \right)^{\frac{1}{\beta_j}} \leq \left( \int_{Q_{j+1}} |u(t,y)|^{\beta_j} dy ds \right)^{\frac{1}{\beta_j}} \leq \left( \int_{Q_{j+1}} |u(t,y)|^{\beta_j} dy ds \right)^{\frac{1}{\beta_j}}
\]
we see that
\[
M_{j+1} \leq (C^{2j})^{\frac{1}{\gamma_j}} \left( \frac{r\gamma_j}{(R_2 - R_1)^2} + \frac{1}{t_1 - t_2} \right)^{\frac{1}{\gamma_j}} M_j
\]
(3.40)
where
\[
D = C_1 (R_2 - R_1)^{-2} + (t_1 - t_2)^{-1}.
\]
The inequality (3.41) implies that
\[
M_{j+1} \leq M_0 \prod_{k=0}^{j} (CD)^{\frac{1}{\gamma_j}} M_j
\]
and
\[
\lim_{j \to \infty} M_j \leq C \sum_{j=0}^{\infty} \frac{1}{\beta_j} (CD)^{\frac{1}{\gamma_j}} M_0.
\]
It follows that
\[
\lim_{j \to \infty} M_j \leq C \sum_{j=0}^{\infty} \frac{1}{\beta_j} (CD)^{\frac{1}{\gamma_j}} M_0.
\]
Since \( \gamma = 1 + \frac{2}{n} \),
\[
\sum_{j=0}^{\infty} \frac{1}{\beta_j} = \sum_{j=0}^{\infty} \frac{1}{r\gamma_j} = \frac{\gamma}{r(\gamma - 1)} = \frac{n + 2}{2r}
\]
and
\[
\sum_{j=0}^{\infty} \frac{j}{\beta_j} < \infty.
\]
We obtain from (3.42) that
\[
\|u\|_{L^\infty(Q_\infty)} \leq CD^{\frac{n+2}{2r}} \|u\|_{L^r(Q_0)}.
\]
(3.43)
Hence
\[ \|u\|_{L^∞((t_1, t) \times B_{R_1}(x))} \leq CD^{\frac{n+2}{2\nu}} \left( \int_{t_2}^{t} \int_{B_{R_2}(x)} |u|^r \, dy \, ds \right)^{\frac{1}{r}}. \] (3.44)

4. **Proof of Theorem.** In this section, we will give the proof of our main theorem.

**Proof.** Theorem 1.3 Let \( u_0 \in L^r_{uloc,ρ}(Ω) \). Then there exists a sequence \( \{u_{k,0}\} \) in \( BUC(Ω) \) such that
\[ u_{k,0} \longrightarrow u_0 \quad \text{in} \quad L^r_{uloc,ρ}(Ω). \]

For each \( k \), \( u_{k,0} \) in \( BUC(Ω) \) as an initial data, we obtain a unique \( L^∞(Ω) \)-strong solution, \( u_k(t) = u_k(t, x) \in C([0, T); BUC(Ω)) \) for the Cauchy problem (1.1) by Proposition 2.4. Since \( L^∞(Ω) \subseteq L^r_{uloc,ρ}(Ω) \) it follows for \( 0 < T' < T \) that \( C([0, T'); L^∞(Ω)) \subseteq C([0, T'); L^r_{uloc,ρ}(Ω)) \). Hence \( u_k \in C([0, T'); L^r_{uloc,ρ}(Ω)) \). Secondly \( u_k \in L^2(0, T'; H^1_{uloc,ρ}(Ω)) \) by taking \( r = 2 \) in Proposition 3.1. Then the weak form of the equation (1.1) is satisfied therefore \( u_k(t) \) is a \( L^r_{uloc} \)-weak solution to (1.1). We then claim that \( \{u_k(t)\}_k \) satisfies the assumption (3.1). Indeed, since \( u_{k,0} \rightarrow u_0 \) in \( L^r_{uloc,ρ}(Ω) \) as \( k \rightarrow \infty \), we regard, by taking \( k_0 \) sufficiently large if necessary, that
\[ \|u_{k,0}\|_{L^r_{uloc,ρ}} \leq 2\|u_0\|_{L^r_{uloc,ρ}} \] (4.1)

for all \( k \geq k_0 \). Let \( u_k(t) \) be the corresponding strong solution in \( L^∞(Ω) \) to \( u_{k,0} \) and choose \( γ_0 \) such that
\[ γ_0 < \min \left\{ \frac{1}{2} \left( \frac{1}{2C_*} \right)^\frac{1}{2} \gamma_4, \quad γ_4 = \min \{γ_1, γ_2, γ_3\} \right\} \] (4.2)

and \( γ_1, γ_2 \) and \( γ_3 \) are the constants appeared in Proposition 3.1, Proposition 3.2 and Proposition 3.3. By the assumption (1.6) on the data \( u_0 \):
\[ \rho^{\frac{1}{p-1} - \frac{2}{p}} \|u_0\|_{L^r_{uloc,ρ}} \leq γ_0 \]
(4.2) and (4.1), it follows that
\[ \rho^{\frac{1}{p-1} - \frac{2}{p}} \|u_{k,0}\|_{L^r_{uloc,ρ}} \leq 2\rho^{\frac{1}{p-1} - \frac{2}{p}} \|u_0\|_{L^r_{uloc,ρ}} \leq γ_4 \]

for all \( k \geq k_0 \). Since the strong solution \( u_k \in C([0, T'); L^∞(Ω)) \subseteq C([0, T'); L^r_{uloc,ρ}(Ω)) \), one can find a time \( 0 < \tilde{T}_k \leq T' \) such that
\[ \rho^{\frac{1}{p-1} - \frac{2}{p}} \sup_{0 \leq s \leq \tilde{T}_k} \|u_k(s)\|_{L^r_{uloc,ρ}} \leq γ_4. \]

According to Proposition 3.1 and (4.1), we see that
\[ \sup_{0 \leq s \leq \tilde{T}_k} \|u_k(s)\|_{L^r_{uloc,ρ}} \leq C_* \|u_{k,0}\|_{L^r_{uloc,ρ}} \leq 2C_* \|u_0\|_{L^r_{uloc,ρ}} \] (4.3)
for all \( k \geq k_0 \). Therefore, for each fixed solution \( u_k(t) \), we obtain
\[ \rho^{\frac{1}{p-1} - \frac{2}{p}} \sup_{0 \leq s \leq \tilde{T}_k} \|u_k(s)\|_{L^r_{uloc,ρ}} \leq 2C_* \rho^{\frac{1}{p-1} - \frac{2}{p}} \|u_0\|_{L^r_{uloc,ρ}} \leq γ_4 \] (4.4)
for all \( k \geq k_0 \). Therefore, we can obtain a uniform bounds \( \tilde{T}_k \geq \mu ρ^2 \).

Applying Proposition 3.2, for any \( m \) and \( ℓ \in N \) with \( m > ℓ \geq 1 \) it follows that
\[ \sup_{0 \leq s \leq \mu ρ^2} \|u_m(s) - u_ℓ(s)\|_{L^r_{uloc,ρ}} \leq C \|u_{m,0} - u_ℓ,0\|_{L^r_{uloc,ρ}}. \] (4.5)

This estimate (4.5) shows that \( \{u_k(t)\}_k \) is a Cauchy sequence in
C([0, \mu \rho^2]; L^r_{uloc, \rho}(\Omega))$ since $\{u_k, 0\}_k$ is the Cauchy sequence in $L^r_{uloc, \rho}(\Omega)$. Noticing
the fact that $L^\infty([0, T'); L^r_{uloc, \rho}(\Omega))$ is complete and $u_k \in C([0, \mu \rho^2]; L^r_{uloc, \rho}(\Omega))$,
there exists a limit function
\[ u \in BUC([0, \mu \rho^2]; L^r_{uloc, \rho}(\Omega)) \]
such that
\[ u_k \to u \in C([0, \mu \rho^2]; L^r_{uloc, \rho}(\Omega)) \quad \text{as} \ k \to \infty. \quad (4.6) \]
Besides $u_k$ satisfies the equation in the weak sense, Proposition 3.3 yields that
$\{u_k\}_k$ is uniformly bounded under the condition (3.29). Hence by taking subsequence
if necessary, we see that
\[ u \in BUC([0, \mu \rho^2]; L^r_{uloc, \rho}(\Omega)) \cap L^\infty((0, \mu \rho^2) \times \Omega) \]
and
\[ u_k \to u \text{ weak* in } L^\infty((0, \mu \rho^2) \times \Omega) \quad \text{as} \ k \to \infty. \quad (4.7) \]
Since $u_k$ is a $L^r_{uloc}$-weak solution, it satisfies the equation (1.1) in the weak form.
Namely, for each $\phi \in C_0^\infty((0, T) \times \Omega)$ with $T \leq \mu \rho^2$,
\[ \int_0^T \int_\Omega \{ -u_k \partial_t \phi + \nabla u_k \cdot \nabla \phi + a|u_k|^{p-1}u_k \cdot \nabla \phi \} \, dx \, dt = 0. \]
By (4.6) and using Proposition 2.1 finitely many time depending on the support of
the test function $\phi$,\[
\left| \int_0^T \int_\Omega u_k \partial_t \phi \, dx \, dt - \int_0^T \int_\Omega u \partial_t \phi \, dx \, dt \right| 
\leq C(\phi) \int_0^T \left\| u_k(t) - u(t) \right\|_{L^r_{uloc, \rho}} \left\| \partial_t \phi \right\|_{L^r_{uloc, \rho}} \, dt \to 0,
\]
and we obtain that
\[ \int_0^T \int_\Omega u_k \partial_t \phi \, dx \, dt \to \int_0^T \int_\Omega u \partial_t \phi \, dx \, dt \quad (4.8) \]
as $k \to \infty$. Analogously using (3.31) we have
\[ \int_0^T \int_\Omega \nabla u_k \cdot \nabla \phi \, dx \, dt = - \int_0^T \int_\Omega \nabla \phi \, dx \, dt - \int_0^T \int_\Omega \nabla \phi \, dx \, dt = \int_0^T \int_\Omega \nabla \phi \, dx \, dt = 0. \quad (4.9) \]
Furthermore, by applying (3.20) and Proposition 3.3, we see that
\[ \left| \int_0^T \int_\Omega |u_k(t)|^{p-1}u_k(t) a \cdot \nabla \phi(t) \, dx \, dt - \int_0^T \int_\Omega |u(t)|^{p-1}u(t) a \cdot \nabla \phi(t) \, dx \, dt \right| 
\leq |a| \int_0^T \int_\Omega |u_k(t) - u(t)| \left( \max(|u_k(t)|, |u(t)|) \right)^{p-1} |\nabla \phi(t)| \, dx \, dt 
\leq \sup_{0 < t < T} \left\| \nabla \phi(t) \right\|_{L^r_{uloc, \rho}} \sup_{0 < t < T} \left\| u_k(t) - u(t) \right\|_{L^r_{uloc, \rho}} 
\leq \sup_{0 < t < T} \left\| u_k(t) - u(t) \right\|_{L^r_{uloc, \rho}}, \]
where $K = \text{supp} \phi$ and $\rho > 0$ is taken such that $K \subset B_{\rho}(x)$ for some $x \in \Omega$. Hence
\[
\int_0^T \int_\Omega a|u_k|^{p-1} u_k \cdot \nabla \phi dx dt \to \int_0^T \int_\Omega a|u|^{p-1} u \cdot \nabla \phi dx dt
\] (4.10)
as $k \to \infty$. Passing $k \to \infty$, we obtain from (4.8)-(4.10) that
\[
\int_0^T \int_\Omega \{ -u_0 \partial_t \phi + \nabla u \cdot \nabla \phi + a|u|^{p-1} u \cdot \nabla \phi \} dx dt = 0.
\]
This proves the existence of an $L^r_{uloc,\rho}$-weak solution for $u_0 \in L^r_{uloc,\rho}(\Omega)$.

To see the uniqueness of weak solution, let $u$ and $v$ be two $L^r_{uloc,\rho}(\Omega)$-weak solutions of (1.1) with the same initial data $u_0 \in L^r_{uloc,\rho}(\Omega)$ satisfying the condition (1.6). Then it holds in a similar observation that both $u$ and $v$ satisfies the condition (3.16). Then Proposition 3.2 now implies $u = v$ in $C([0,T'); L^r_{uloc,\rho}(\Omega))$. Finally the solution $u$ is approximated uniformly bounded continuous function $u_k$ uniformly in $t$, it belongs to the class $C([0,T'); L^r_{uloc,\rho}(\Omega))$.

This completes the proof of Theorem 1.3.

Proof. Corollary 1.1 Let $p > 1 + \frac{1}{n}$ and $u_0 \in L^{n(p-1)}(\Omega)$. If $\|u_0\|_{L^{n(p-1)}(\Omega)} \leq \gamma$, then $\|u_0\|_{L^{n(p-1)}(\Omega)} \leq \gamma$ for any $\rho > 0$. Then assertion (1.7) of Theorem 1.3 holds for any $\rho > 0$. This implies Corollary 1.1.

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