Generalized Harmonic Progression Part II

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Abstract

In a previous paper, we saw how to create formulae for the sum of the terms of a harmonic progression of order \( k \), \( HP_k(n) \), with integer parameters, \( a \) and \( b \). In this new paper we make those formulae more general by lifting the restriction that the parameters be integers. These new formulae always hold, except when \( \frac{ib}{a} \in \mathbb{Z} \). This paper employs a slightly modified version of the reasoning used previously. Nonetheless, we make another brief exposition of the principle used to derive them.

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1 Introduction

This article demonstrates how to produce a formula for a harmonic progression of order \( k \) with complex parameters:

\[
HP_k(n) = \sum_{j=1}^{n} \frac{1}{(aij + b)^k},
\]

with \( a \) an integer, \( b \) a complex number and \( i \) the imaginary unit.
The motivation to create such a formula came from the realization that, unlike the sum of $1/(j^2 - 1)$ over $j$, a sum such as $1/(j^2 + 1)$ can’t be derived from the formulae created in [4].

If we can produce a formula for $HP_k(n)$ with integer $a$ and complex $b$, we may be able to produce formulae for most, if not all, sums of the type $\sum_j 1/p(j)$, where $p(j)$ is any polynomial with complex coefficients. Besides, $HP_k(n)$ provides the partial sums of the Hurwitz zeta function at the positive integers, and allows us to easily obtain $\zeta(k,b)$ by taking its limit when $n$ goes to infinity. With a little more effort, it’s even possible to obtain the analytic continuation of the Hurwitz zeta function.

The choice of integer $a$ and complex $b$ is good enough in most of the cases (since one can always make $a$ integer by dividing the parameters by $a$), and when it’s not, we can fall back on the formulae from the previous paper [4], which assume $a$ and $b$ to be integers.

We need to make use again of Faulhaber’s formula for the sum of powers of the first $n$ positive integers, which are given by:

\[
\sum_{j=1}^{n} j^{2i} = \frac{n^{2i}}{2} + \sum_{j=0}^{i} \frac{(2i)!B_{2j}n^{2i+1-2j}}{(2j)!(2i+1-2j)!} \\
\]

\[
\sum_{j=1}^{n} j^{2i+1} = \frac{n^{2i+1}}{2} + \sum_{j=0}^{i} \frac{(2i+1)!B_{2j}n^{2i+2-2j}}{(2j)!(2i+2-2j)!},
\]

where $B_{2j}$ are the non-null Bernoulli numbers.

2 Review of preview results

Before we begin, let’s recap some results from [4] for a bit of context. We’ve seen that the sum of the terms of a generalized harmonic progression with integer parameters, $a$ and $b$, for even and odd powers is given, respectively, by:

\[
\sum_{j=1}^{n} \frac{1}{(aj+b)^{2k}} = \frac{1}{2b^{2k}} + \frac{1}{2(an+b)^{2k}} - \frac{(-1)^k(2\pi)^{2k}}{2} \int_0^1 \sum_{j=0}^{k} B_{2j} \frac{(2 - 2^{2j})(1 - u)^{2k-2j}}{(2j)!(2k-2j)!} (\sin 2\pi(an+b)u - \sin 2\pi bu) \cot \pi au \, du,
\]

\[
\sum_{j=1}^{n} \frac{1}{(aj+b)^{2k+1}} = -\frac{1}{2b^{2k+1}} + \frac{1}{2(an+b)^{2k+1}} - \frac{(-1)^k(2\pi)^{2k+1}}{2} \int_0^1 \sum_{j=0}^{k} B_{2j} \frac{(2 - 2^{2j})(1 - u)^{2k+1-2j}}{(2j)!(2k+1-2j)!} (\cos 2\pi(an+b)u - \cos 2\pi bu) \cot \pi au \, du
\]
Notice that there can be singularities on both sides of these equations if, for example, $aj + b = 0$ for some $j \leq n$ (that is, $a$ divides $b$ and they have different signs). In those cases, if we ignore the singularities wherever they occur the formula still holds.

This means that, for instance, if $b = 0$ then the above formulae reduce to the the generalized harmonic numbers formulae from [3].

To see why singularities can be removed, let’s look at the formula for the odd powers. From that formula, it’s easy to see why it still holds when we ignore singularities. For example, let’s take the forward difference of $HP(n)$:

$$2\pi \int_0^1 (1 - u)[\cos 2\pi(an + b)u - \cos 2\pi(a(n - 1) + b)u] \cot \pi a(1 - u) \, du = -\frac{1}{an + b} - \frac{1}{a(n - 1) + b}$$

Now, if for instance $an + b = 0$, the left-hand side of the above equation yields $1/a$, even though $1/(an+b)$ is a singularity on the right-hand side. Therefore, by ignoring the singularity, the equation still holds.

### 3 Approach based on exponential

The reasoning to build a formula for the harmonic progression is to use the Taylor series expansion of $e^{2\pi(aij+b)}$, and seize upon the fact that it’s constant for any integer $a$ and complex $b$:

$$e^{2\pi(aij+b)} = e^{2\pi b} \Rightarrow \sum_{i=0}^{\infty} \frac{(2\pi(aij+b))^i}{i!} = e^{2\pi b}, \text{ if } a \text{ is integer}$$

Starting from the above initial equation, we end up with the below after a few steps:

$$\frac{1}{aij + b} (e^{2\pi b} - 1) = 2\pi \sum_{i=0}^{\infty} \left( \frac{(2\pi(aij+b))^{2i}}{(2i+1)!} + \frac{(2\pi(aij+b))^{2i+1}}{(2i+2)!} \right)$$

Note we now have two power series, as opposed to one previously.

At this point we notice that this formula doesn’t work if $e^{2\pi b} = 1$, which happens when $b$ is a pure complex number with integer imaginary part ($\Re(b) = 0$ and $\Im(b) \in \mathbb{Z}$). In most situations, we can work around this issue by making $b$ non-integer, by dividing $b$ by $a$. By the same token, the formula won’t apply if $a$ is not integer ($a \notin \mathbb{Z}$). In that case we can also make $a$ integer by dividing both parameters by $a$:

$$\frac{1}{aij + b} \sum_{j=1}^{n} \frac{1}{aij + b} = \frac{1}{a} \sum_{j=1}^{n} \frac{1}{ij + b/a}$$

However, in both cases this work-around won’t work if $ib/a$ is integer, including 0. In those cases we can resort to the previous formulae.
3.1 Lagrange’s identities

Through Lagrange’s identities, which stem from the sum of the terms of a geometric progression, we can create functions whose power series take similar forms to the power series we obtain when we expand the initial equation from the previous section.

For the complex harmonic progression, these functions are the same as the ones used in \[\text{(1)}\], with only a slight transformation (\(a := a\dot{i}\)). Nonetheless, they are obtained by means of Lagrange’s trigonometric identities, along with the identities \(\cos(x+y) = \cos x \cos y - \sin x \sin y\) or \(\sin(x + y) = \sin x \cos y + \cos x \sin y\):

\[
\sum_{j=1}^{k} \frac{2\pi n(a\dot{i}j + b)}{k} = -\frac{1}{2} \cos \left(\frac{2\pi b n}{k} + \frac{1}{2} \cos 2\pi n \left(a\dot{i} + \frac{b}{k}\right) + \cos \pi n \left(a\dot{i} + \frac{2b}{k}\right) \sin \pi a\dot{i} \cot \frac{\pi a\dot{i}n}{k}\right)
\]

\[
\sum_{j=1}^{k} \frac{2\pi n(a\dot{i}j + b)}{k} = -\frac{1}{2} \sin \left(\frac{2\pi b n}{k} + \frac{1}{2} \sin 2\pi n \left(a\dot{i} + \frac{b}{k}\right) + \sin \pi n \left(a\dot{i} + \frac{2b}{k}\right) \sin \pi a\dot{i} \cot \frac{\pi a\dot{i}n}{k}\right)
\]

We can then derive power series for the left-hand side of the above equations with the employment of \(\text{(1)}\) and \(\text{(2)}\), and come up with the following power series for each function on the right-hand side:

\[
\sum_{i=0}^{\infty} (-1)^i \left(\frac{2\pi bn}{k}\right)^{2i} \left(\sum_{j=0}^{i} \frac{(a\dot{i}k/b)^{2j}}{(2i)! (2j - 2j)!} + \sum_{j=0}^{i-1} \frac{(a\dot{i}k/b)^{2j+1}}{(2j + 1)! (2i - 2j - 1)!}\right) = \cos 2\pi n \left(a\dot{i} + \frac{b}{k}\right)
\]

\[
\sum_{i=0}^{\infty} (-1)^i \left(\frac{2\pi bn}{k}\right)^{2i+1} \left(\sum_{j=0}^{i} \frac{(a\dot{i}k/b)^{2j}}{(2j)! (2i + 1 - 2j)!} + \sum_{j=0}^{i-1} \frac{(a\dot{i}k/b)^{2j+1}}{(2j + 1)! (2i - 2j)!}\right) = \sin 2\pi n \left(a\dot{i} + \frac{b}{k}\right)
\]

\[
\sum_{i=0}^{\infty} (-1)^i \left(\frac{2\pi bn}{k}\right)^{2i} \left(\sum_{j=0}^{i} \frac{(a\dot{i}/b)^{2j}}{(2i - 2j)!} \sum_{p=0}^{j} \frac{B_{2p} k^{2j+1-2p}}{(2j + 1 - 2p)!(2p)!} + \sum_{j=0}^{i-1} \frac{(a\dot{i}/b)^{2j+1}}{2j - 2j - 1)!} \sum_{p=0}^{j} \frac{B_{2p} k^{2j+2-2p}}{(2j + 2 - 2p)!(2p)!}\right)
\]

\[
= \cos \pi n \left(a\dot{i} + \frac{2b}{k}\right) \sin \pi a\dot{i} \cot \frac{\pi a\dot{i}n}{k}
\]

\[
\sum_{i=0}^{\infty} (-1)^i \left(\frac{2\pi bn}{k}\right)^{2i+1} \left(\sum_{j=0}^{i} \frac{(a\dot{i}/b)^{2j}}{(2i + 1 - 2j)!} \sum_{p=0}^{j} \frac{B_{2p} k^{2j+1-2p}}{(2j + 1 - 2p)!(2p)!} + \sum_{j=0}^{i-1} \frac{(a\dot{i}/b)^{2j+1}}{2j - 2j - 1)!} \sum_{p=0}^{j} \frac{B_{2p} k^{2j+2-2p}}{(2j + 2 - 2p)!(2p)!}\right)
\]

\[
= \sin \pi n \left(a\dot{i} + \frac{2b}{k}\right) \sin \pi a\dot{i} \cot \frac{\pi a\dot{i}n}{k}
\]
3.2 Harmonic progression

Going back to equation (3), we need to sum it over \( j \), expand \((aij + b)^{2i}\) and \((aij + b)^{2i+1}\) using the binomial theorem, and replace the sums of \(j^{2q}\) and \(j^{2q+1}\) over \( j \) with their respective Faulhaber’s formulae. After we make all the possible simplifications, we end up with a few power series.

First we have the independent term, given by:

\[
-\pi \sum_{i=0}^{\infty} \left( \frac{(2\pi b)^{2i}}{(2i+1)!} + \frac{(2\pi b)^{2i+1}}{(2i+2)!} \right) = -\frac{e^{2\pi b} - 1}{2b}
\]

For the other, more complicated power series, we can obtain closed-forms\(^1\) with the aid of their correlated functions from (3.1):

\[
\pi \sum_{i=0}^{\infty} \frac{(2\pi b)^{2i}}{2i+1} \left( \sum_{j=0}^{i} \frac{(ai/n)^{2j}}{(2j)!((2i-2j)!)} + \sum_{j=0}^{i-1} \frac{(ai/n)^{2j+1}}{(2j+1)!(2i-2j-1)!} \right) = -\frac{i\pi}{n} \int_{0}^{\infty} \cos 2\pi x \left( ai + \frac{b}{n} \right) dx
\]

\[
\pi \sum_{i=0}^{\infty} \frac{(2\pi b)^{2i+1}}{2i+2} \left( \sum_{j=0}^{i} \frac{(ai/n)^{2j}}{(2j)!((2i+1-2j)!)} + \sum_{j=0}^{i} \frac{(ai/n)^{2j+1}}{(2j+1)!(2i-2j)!} \right) = -\frac{\pi}{n} \int_{0}^{\infty} \sin 2\pi x \left( ai + \frac{b}{n} \right) dx
\]

\[
2\pi \sum_{i=0}^{\infty} \frac{(2\pi b)^{2i}}{2i+1} \left( \sum_{j=0}^{i} \frac{(ai/n)^{2j}}{(2i-2j)!} \sum_{p=0}^{j} \frac{B_{2p}n^{2j+1-2p}}{(2j+1-2p)!(2p)!} + \sum_{j=0}^{i-1} \frac{(ai/n)^{2j+1}}{(2j+1)!} \sum_{p=0}^{j} \frac{B_{2p}n^{2j+2-2p}}{(2j+2-2p)!}(2p)! \right) 
\]

\[= -\frac{2\pi i}{n} \int_{0}^{\infty} \cos \pi x \left( ai + \frac{2b}{n} \right) \sin (\pi ax) \cot \frac{\pi ax}{n} dx
\]

\[
2\pi \sum_{i=0}^{\infty} \frac{(2\pi b)^{2i+1}}{2i+2} \left( \sum_{j=0}^{i} \frac{(ai/n)^{2j}}{(2i+1-2j)!} \sum_{p=0}^{j} \frac{B_{2p}n^{2j+1-2p}}{(2j+1-2p)!(2p)!} + \sum_{j=0}^{i} \frac{(ai/n)^{2j+1}}{(2j+2-2p)!}(2p)! \right) 
\]

\[= -\frac{2\pi}{n} \int_{0}^{\infty} \sin \pi x \left( ai + \frac{2b}{n} \right) \sin (\pi ax) \cot \frac{\pi ax}{n} dx
\]

So, by putting it all together and making a change of variables on the integral, one finds the below closed-form, which works as long as \( a \) is integer and \( i b \) is not integer:

\[
\sum_{j=1}^{n} \frac{1}{aij + b} = -\frac{1}{2b} + \frac{1}{2(ai + b)} + \frac{2\pi}{e^{2\pi b} - 1} \int_{0}^{1} e^{\pi(ai + 2b)u} \sin \pi anu \cot \pi au \, du
\]

In this formula, the independent term came from the first power series, the second term

\(^1\)In a stricter sense, a closed-form doesn’t include integrals.
from the next two power series and the third term from the last two power series, which is somewhat evident.

As discussed before, provided that $ib/a \notin \mathbb{Z}$, we can lift the restriction that $a$ be integer by transforming the formula, for example, as follows:

$$\frac{1}{a} \sum_{j=1}^{n} \frac{1}{ij + b/a} = -\frac{1}{2b} + \frac{1}{2(ai n + b)} + \frac{2\pi}{a (e^{2\pi b/a} - 1)} \int_{0}^{1} e^{\pi (in + 2b/a)u} \sin \pi nu \cot \pi u du,$$

which holds for any $a, b \in \mathbb{C}$. There are endless ways of doing that, the example is just one of the possibilities.

### 3.3 Generalization

To generalize the previous formula we got, first we note that we get a recursion at each new step. For example, for the third power we have:

$$\left(e^{2\pi b} - 1\right) \sum_{j=1}^{n} \frac{1}{(ai n + b)^3} = \sum_{j=1}^{n} \left(\frac{(2\pi)^2}{2!} \frac{1}{ai j + b} + \frac{2\pi}{1! (ai j + b)^2} + \frac{1}{2i + 3)!} + \frac{2\pi (aij + b)^{2i + 1}}{(2i + 4)!} \right)$$

That means that we only need to worry about the last sum in the above equation. As before, this recurrence is such that in the final formula the terms that go outside of the integral reduce to very simple forms, the integral being the only challenging part.

Let $p_k(u)$ be the polynomial in $u$ that goes within the integral. Then for integer $a$ and non-integer $ib/a$:

$$\sum_{j=1}^{n} \frac{1}{(ai j + b)^k} = -\frac{1}{2b^k} + \frac{1}{2(ai n + b)^k} + (2\pi)^k \int_{0}^{1} p_k(u) e^{\pi (in + 2b/a)u} \sin \pi anu \cot \pi au du,$$

where $p_k(u)$ is given by the recurrence equation:

$$(e^{2\pi b} - 1) p_k(u) = \begin{cases} 1, & \text{if } k = 1 \\ \frac{(1-u)^{k-1}}{(k-1)!} + \sum_{j=1}^{k-1} \frac{p_j(u)}{(k-j)!}, & \text{if integer } k > 1 \end{cases}$$

and where term $(1 - u)^{k-1}/(k-1)!$ came from the observation of the patterns.

Now we only need to solve this recurrence. Let $p(x)$ be the generating function of $p_k(u)$. Then, looking at the recurrence, we conclude that:

$$(e^{2\pi b} - 1) p(x) - (-1 + e^x) p(x) = x e^{(1-u)x} \Rightarrow p(x) = -\frac{x e^{(1-u)x}}{e^x - e^{2\pi b}}$$

The solution of this equation is not trivial, that is, the general term of the power series of $p(x)$ is not simple, unfortunately. Nonetheless, it gives us the following generalization, after
we drop the requirement that \( a \) be integer.

Let \( \delta_{ij} \) be the Kronecker delta (\( \delta_{ij} = 1 \), if \( i = j \), \( \delta_{ij} = 0 \) otherwise), and \( \text{Li}_k(z) \) the polylogarithm function, a Dirichlet series given by:

\[
\text{Li}_k(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^k}
\]

Then, if \( ib/a \notin \mathbb{Z} \):

\[
\sum_{j=1}^{n} \frac{1}{(a + ib)^k} = -\frac{1}{2b^k} + \frac{1}{2(a + ib)^k} + e^{-2\pi b/a} \left( \frac{2\pi}{a} \right)^k \int_0^1 \sum_{j=1}^{k} \frac{\delta_{1j} + \text{Li}_{j+1} \left( e^{-2\pi ib/a} \right)}{(j-1)!(k-j)!} e^{\pi u(i n + 2b/a)} \sin \pi n u \cot \pi u \, du
\]

One can also transform this formula and derive the following identity. If \( b \notin \mathbb{Z} \):

\[
\sum_{j=1}^{n} \frac{1}{(j + ib)^k} = -\frac{1}{2b^k} + \frac{1}{2(n + b)^k} + (2\pi i)^k e^{-2\pi ib} \int_0^1 \sum_{j=1}^{k} \frac{\delta_{1j} + \text{Li}_{j+1} \left( e^{-2\pi ib} \right)}{(j-1)!(k-j)!} e^{\pi i u(n + 2b)} \sin \pi n u \cot \pi u \, du,
\]

whose generating function is:

\[
f(x) = -\frac{ixe^{ix(1-u)}}{e^{ix} - e^{2\pi ib}} \Rightarrow \frac{f^{(k)}(0)}{k!} = i^k e^{-2\pi ib} \sum_{j=1}^{k} \frac{\delta_{1j} + \text{Li}_{j+1} \left( e^{-2\pi ib} \right)}{(j-1)!(k-j)!}
\]

### 3.4 Application examples

Let’s see two examples on how we can apply the formula we just got to obtain sums that weren’t possible with the formulae we created in the previous paper.

The first example is the sum of \( 1/(j^2 + 1) \) over \( j \). The first step to derive this formula is to decompose \( 1/(j^2 + 1) \) into a linear combination:

\[
\frac{1}{j^2 + 1} = \frac{1}{2(ij + 1)} - \frac{1}{2(ij - 1)}
\]

Thus we get the below closed-form:

\[
\sum_{j=1}^{n} \frac{1}{j^2 + 1} = -\frac{1}{2} + \frac{1}{2(n^2 + 1)} + \frac{4\pi}{e^{4\pi} - 1} \int_0^1 e^{4\pi u} \cos 2\pi n(1 - u) \sin 2\pi n u \cot 2\pi u \, du
\]
Generally, if we want to know what the formula for the sum of \(1/(j^{2k} + 1)\) looks like, we need to find all the \(2k\) complex roots of \(x^{2k} = -1\) (say they are \(x_j\)), and a linear combination such that \(\sum_{j=1}^{2k} c_j/(x - x_j) = 1/(x^{2k} + 1)\).

The second example is the sum of \(1/(j^2 + 2j + 2)\). It can be linearly decomposed as:

\[
\frac{1}{j^2 + 2j + 2} = \frac{1}{2(ij + 1 + i)} - \frac{1}{2(ij - 1 + i)},
\]

which implies the below closed-form:

\[
\sum_{j=1}^{n} \frac{1}{j^2 + 2j + 2} = \frac{1}{4} + \frac{1}{2(n^2 + 2n + 2)} + \frac{2\pi}{e^{4\pi} - 1} \int_0^1 \left(e^{4\pi(1-u)} + e^{4\pi u}\right) \cos 2\pi(n + 2)u \sin 2\pi n(1 - u) \cot 2\pi(1 - u) \, du
\]

So, it turns out the real problem is finding out the linear combination that yields the polynomial that one is interested in.

## 4 Approaches based on cosine and sine

Though the exponential-based approach is probably more useful, if we use an \(\cos 2\pi(j + b) = \cos 2\pi b\) and \(\sin 2\pi(j + b) = \sin 2\pi b\) as initial equations, we can obtain different formulae.

For cosine, we start from:

\[
\cos 2\pi(j + b) = \cos 2\pi b = \sum_{i=0}^{\infty} \frac{(-1)^i(2\pi(j + b))^{2i}}{(2i)!},
\]

which produces the recurrence equations:

\[
\begin{cases}
\frac{2\sin(\pi b)^2}{j+b} = 2\pi \sum_{i=0}^{\infty} \frac{(-1)^i(2\pi(j+b))^{2i+1}}{(2i+2)!} \\
\frac{2\sin(\pi b)^2}{(j+b)^2} = (2\pi)^2 \sum_{i=0}^{\infty} \frac{(-1)^i(2\pi(j+b))^{2i}}{(2i+2)!} \\
\frac{2\sin(\pi b)^2}{(j+b)^4} = \frac{(2\pi)^2}{2}(j+b)^2 - (2\pi)^3 \sum_{i=0}^{\infty} \frac{(-1)^i(2\pi(j+b))^{2i+1}}{(2i+4)!} \\
\frac{2\sin(\pi b)^2}{(j+b)^6} = \frac{(2\pi)^4}{24}(j+b)^4 - (2\pi)^5 \sum_{i=0}^{\infty} \frac{(-1)^i(2\pi(j+b))^{2i+1}}{(2i+6)!} \\
\frac{2\sin(\pi b)^2}{(j+b)^8} = \frac{(2\pi)^6}{168}(j+b)^6 - (2\pi)^7 \sum_{i=0}^{\infty} \frac{(-1)^i(2\pi(j+b))^{2i+1}}{(2i+8)!} \\
\end{cases}
\]

Likewise, from:

\[
\sin 2\pi(j + b) = \sin 2\pi b = \sum_{i=0}^{\infty} \frac{(-1)^i(2\pi(j + b))^{2i+1}}{(2i + 1)!},
\]
we obtain the below recurrence equations:

\[
\begin{align*}
\sin \frac{2\pi b}{j+b} &= 2\pi \sum_{i=0}^{\infty} \frac{(-1)^i (2\pi (j+b))^{2i}}{(2i+1)!} \\
\sin \frac{2\pi b}{(j+b)^2} &= \frac{2\pi}{j+b} - (2\pi)^2 \sum_{i=0}^{\infty} \frac{(-1)^i (2\pi (j+b))^{2i+1}}{(2i+3)!} \\
\sin \frac{2\pi b}{(j+b)^3} &= \frac{2\pi}{(j+b)^2} - (2\pi)^3 \sum_{i=0}^{\infty} \frac{(-1)^i (2\pi (j+b))^{2i+2}}{(2i+5)!} \\
\sin \frac{2\pi b}{(j+b)^4} &= \frac{2\pi}{(j+b)^3} - (2\pi)^4 \sum_{i=0}^{\infty} \frac{(-1)^i (2\pi (j+b))^{2i+3}}{(2i+7)!} \\
&\vdots
\end{align*}
\]

The formulae derived with these approaches usually break for integer and half-integer \( b \).

### 4.1 Cosine approach

Using the two systems of recurrence equations, we have more than one way to derive formulae for \( HP_k(n) \). Unless we combine them, we have a very complicated recursion to determine the formula for \( HP_k(n) \). Let’s go over the simplest way.

Setting \( a = 1 \) without loss of generality, and intertwining the two systems of equations, the cosine-based \( HP_k(n) \) formula is:

\[
\sum_{j=1}^{n} \frac{1}{(j+b)^k} = -\frac{1}{2b^k} + \frac{1}{2(n+b)^k} - \frac{(2\pi)^k}{2} \int_{0}^{1} p_k(u) \left( \cos 2\pi (n+b)u - \cos 2\pi bu \right) \cot \pi u \, du,
\]

and below is the simplified recurrence equation on \( p_k(u) \):

\[
p_k(u) = \frac{(-1)^{(k+1)/2}}{2 \sin \pi b} \left( \frac{1 - u}{(k-1)!} \right)^{k-1} \sum_{j=1}^{\lfloor k/2 \rfloor} \frac{(-1)^j p_{2j-1}(u)}{(k-2j+1)!}. \begin{cases} -\frac{1}{\sin \pi b}, & \text{if } k \text{ is odd} \\ \frac{1}{\cos \pi b}, & \text{if } k \text{ is even} \end{cases}
\]

To solve this equation, we need to separate odd from even. For the odd we can make \( q_k(u) = p_{2k+1}(u) \), and hence for \( k \geq 0 \):

\[
q_k(u) = \frac{(-1)^k}{2(\sin \pi b)^2} \left( \frac{(1 - u)^{2k}}{(2k)!} - \sum_{j=0}^{k-1} \frac{(-1)^j q_j(u)}{(2k - 2j)!} \right)
\]

The recurrence for even is very similar and is a function of \( q_k(u) \). The generating functions of odd and even \( p_k(u) \) are therefore:

\[
f(x) = \frac{x \cos x (1 - u)}{\cos x - \cos 2\pi b} \quad \text{and} \quad g(x) = \sin x \frac{x \cos x (1 - u)}{\cos x - \cos 2\pi b},
\]

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and the final formulae can be expressed as:

\[
\sum_{j=1}^{n} \frac{1}{(j + b)^{2k+1}} = -\frac{1}{2b^{2k+1}} + \frac{1}{2(n + b)^{2k+1}} \\
- \frac{(2\pi)^{2k+1}}{2} \int_{0}^{1} \frac{f^{(2k+1)}(0)}{(2k+1)!} \left( \cos 2\pi(n + b)u - \cos 2\pi bu \right) \cot \pi u \, du, \text{ and}
\]

\[
\sum_{j=1}^{n} \frac{1}{(j + b)^{2k}} = -\frac{1}{2b^{2k}} + \frac{1}{2(n + b)^{2k}} \\
- \frac{(2\pi)^{2k}}{2 \sin 2\pi b} \int_{0}^{1} \left( \frac{(-1)^{k}(1 - u)^{2k-1}}{(2k-1)!} + \frac{g^{(2k)}(0)}{(2k)!} \right) \left( \cos 2\pi(n + b)u - \cos 2\pi bu \right) \cot \pi u \, du
\]

4.2 Sine approach

Since the process for sine is analogous, we only show the final formulae. Let \( f(x) \) and \( g(x) \) be the functions:

\[
f(x) = \frac{x \sin x(1 - u)}{\cos x - \cos 2\pi b} \quad \text{and} \quad g(x) = \frac{x \sin x(1 - u)}{\cos x - \cos 2\pi b}
\]

The even and odd \( HP_{k}(n) \) formulae can be expressed as a function of their derivatives by:

\[
\sum_{j=1}^{n} \frac{1}{(j + b)^{2k}} = -\frac{1}{2b^{2k}} + \frac{1}{2(n + b)^{2k}} \\
\quad + \frac{(2\pi)^{2k}}{2} \int_{0}^{1} \frac{f^{(2k)}(0)}{(2k)!} \left( \sin 2\pi(n + b)u - \sin 2\pi bu \right) \cot \pi u \, du, \text{ and}
\]

\[
\sum_{j=1}^{n} \frac{1}{(j + b)^{2k+1}} = -\frac{1}{2b^{2k+1}} + \frac{1}{2(n + b)^{2k+1}} \\
\quad + \frac{(2\pi)^{2k+1}}{2 \sin 2\pi b} \int_{0}^{1} \left( \frac{(-1)^{k}(1 - u)^{2k}}{(2k)!} + \frac{g^{(2k+1)}(0)}{(2k+1)!} \right) \left( \sin 2\pi(n + b)u - \sin 2\pi bu \right) \cot \pi u \, du
\]
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