The new approach to the microscopic description of the phase transitions starting from the only first principles was developed on an example of the transition normal metal-superconductor. This means mathematically, that the free energy is calculated in the range of temperatures, which includes a point of phase transition, without introducing any artificial parameters similar to an order parameter, but only starting from microscopic parameters of Hamiltonian. Moreover the theorems about connection of a vacuum amplitude with thermodynamics potentials are realized. The functional of a superconductor’s free energy in a magnetic field was obtained with help the developed method. The obtained functional is generalization of Ginzburg-Landau functional for the case of arbitrary value of a gap, arbitrary spatial inhomogeneities and nonlocal magnetic response. The explicit expressions for the extremals of this functional were obtained in the low-temperature limit and the high-temperature limit at the condition of slowness of gap’s changes.

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I. FORMULATION OF THE PROBLEM

Exact microscopic description of a phase transition is opened problem of modern theoretical physics. The problem can be solved for several simplest models only, but different phenomenological approaches exist for the rest cases. The essence of the matter lies in the following. Basic problem of statistical mechanics is calculation of a partition function (of grand canonical ensemble in a total case) $Z$ or calculation of a density matrix $\hat{\rho}$:

$$Z = Sp\left(e^{-\beta(\hat{H} - \mu \hat{N})}\right) = e^{-\beta \Omega}, \quad \hat{\rho} = \frac{1}{Z} e^{-\beta(\hat{H} - \mu \hat{N})},$$

where $\hat{H} = \hat{H}_0 + \hat{V}$ is full Hamiltonian of a system, $\hat{N}$ is the particle operator, $\mu$ is the chemical potential, $\Omega$ is the grand thermodynamics potential. Replacement of Hamiltonian $\hat{H}$ in canonical ensemble by Hamiltonian $\hat{H} - \mu \hat{N}$ in grand canonical ensemble leads to the shift of reference of particle’s energy from zero to Fermi surface: $\varepsilon(k_F) = 0$. Hence, the potential $\Omega$ plays a part of Helmholtz free energy with a reference of particle’s energy from Fermi surface. Therefore we shall call the grand thermodynamics potential by free energy for brevity. Let’s transform the partition function (1) to a form:

$$Z = Sp\left(e^{-\beta(\hat{H}_0 - \mu \hat{N})} e^{\beta(\hat{H}_0 - \mu \hat{N})} e^{-\beta(\hat{H} - \mu \hat{N})}\right) = Sp\left(Z_0 \hat{\rho}_0 \bar{U}(\beta)\right) = Z_0 R(\beta),$$

where $Z_0$ is the partition function for a system of noninteracting particles,

$$\bar{U}(\beta) = e^{+\beta(\hat{H}_0 - \mu \hat{N})} e^{-\beta(\hat{H} - \mu \hat{N})} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^\beta d\tau_1 \ldots \int_0^\beta d\tau_n \hat{T} \left\{ \hat{H}_1(\tau_1) \ldots \hat{H}_1(\tau_n) \right\}$$

is the evolution operator in the interaction representation (it describes evolution of the system in imaginary time $it \rightarrow \tau$, $\hat{T}$ is the ordering operator in time), $\hat{H}_1(\beta) = e^{+\beta(\hat{H}_0 - \mu \hat{N})} \hat{V} e^{-\beta(\hat{H}_0 - \mu \hat{N})}$ is the interaction operator of particles in interaction representation,

$$R(\beta) = \langle \bar{U}(\beta) \rangle_0 = Sp\left(\hat{\rho}_0 \bar{U}(\beta)\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^\beta d\tau_1 \ldots \int_0^\beta d\tau_n Sp\left(\hat{\rho}_0 \hat{T} \left\{ \hat{H}_1(\tau_1) \ldots \hat{H}_1(\tau_n) \right\} \right)$$
is the vacuum amplitude of the system. The averaging $\langle \Phi_0 | \Psi_0 \rangle \equiv Sp(\hat{\rho}_0 \ldots)$ is done over ensemble of noninteracting particles. It is necessary to note, that the vacuum amplitude describes dynamics of the system in the multiparticle state $\Phi_0$ under the influence of the internal interaction, unlike Green function describing dynamics of a particle in one-particle state $\phi_k$ under the influence of its interaction with other particles.

The partition function $Z_0$ can be found exactly for any system. If particles interact, then the situation becomes complicated essentially. The only several models can be solved exactly [2, 3], but in the rest cases the values [4] can not be found exactly and it is necessary to develop a perturbation theory. Solution of the basic problem of the statistical mechanics reduces to the calculation of a transition amplitude "vacuum-vacuum" [4]. In order to formulate our problem let’s consider the case of zero temperature $T = 0$. In this case the time is real, and the vacuum amplitude is determined by the following:

$$ R(t) = \langle \Phi_0 | U(t - t_0) | \Phi_0 \rangle_{t_0 = 0} = \langle \Phi_0 | U(t) | \Phi_0 \rangle e^{iW_0t} = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_0^t dt_1 \ldots \int_0^t dt_n \langle \Phi_0 | \hat{T} \left\{ \hat{H}_I(t_1) \ldots \hat{H}_I(t_n) \right\} | \Phi_0 \rangle, $$

(5)

where $\hat{H}_I(t) = e^{+it(\hat{H}_0 - \mu \hat{N})} e^{-it(\hat{H}_0 - \mu \hat{N})}$ is operator of particles’ interaction in the interaction representation, $W_0$ is ground state energy of a system without interaction. The averaging $\langle \Phi_0 | \Psi_0 \rangle \equiv \langle \Phi_0 | \ldots | \Phi_0 \rangle$ is done over the ensemble of noninteracting particles. Knowing a vacuum amplitude we can calculate the ground state energy of the system of the interacting particles $E_0$ using the theorem [1]:

$$ E_0 = W_0 + \lim_{t \to \infty(1-i\delta)} i \frac{d}{dt} \ln R(t), $$

(6)

where $\delta$ is infinitely small value, but $\infty \cdot \delta \to \infty$. The theorem is correct if the limit transition

$$ \left[ \frac{d}{dt} \ln R(t) \right]_{\infty(1-i\delta)} = iW_0 + \frac{\langle \Phi_0 | \Psi_0 \rangle^2 (2(-iE_0) e^{-iE_0 \delta} e^{-E_0 \delta} \cdot \infty)}{\langle \Phi_0 | \Psi_0 \rangle^2 e^{-iE_0 \delta} e^{-E_0 \delta} \cdot \infty} = iW_0 - iE_0 $$

(7)

is possible. This transition is possible at the condition when the symmetries of ground state of the system with interaction $|\Psi_0\rangle$ and without interaction $|\Phi_0\rangle$ are identical:

$$ \langle \Phi_0 | \Psi_0 \rangle \neq 0. $$

(8)

The expressions (7) and (8) mean: 1) potential of interaction is being switched slowly in the system in the ground state without interaction, 2) the ground state of the system with interaction is being obtained by continuous way from the ground state without interaction while the switching of the interaction (adiabatic hypothesis). For nonzero temperature the analog of the theorem (6) has a form:

$$ U = - \frac{\partial}{\partial \beta} \ln Z_0 - \frac{\partial}{\partial \beta} \ln R(\beta), $$

(9)

where $U$ is the internal energy of a system. Moreover, we can calculate the free energy:

$$ \Omega = - \frac{1}{\beta} \ln Z_0 - \frac{1}{\beta} \ln R(\beta). $$

(10)

If the wave functions are orthogonal $\langle \Phi_0 | \Psi_0 \rangle = 0$ - the adiabatic hypothesis is not valid, then the symmetries of the system with interaction and without one is different. This means, that an initial system without interaction suffers phase transition stipulated by the interaction. A nonfulfillment of the adiabatic hypothesis means nonfulfillment of the theorem (6). For the system with the broken symmetry we can calculate a vacuum amplitude on the free propagators $G_0(k, t)$ and use the formula (6). However we shall find a wrong ground state energy $E_0$. This means that a state exists with more low energy than the found value. Moreover, in consequence of the breakdown of the condition (8) a system becomes unstable: $\Gamma$-matrix, which determines vacuum amplitude $R(t)$, one-particle propagator $G(k, t)$ and two-particle propagator $K(k_1, k_2, t)$ in stair approximation, has the form:

$$ \Gamma(t) = c e^{-\alpha t} + c' e^{+\alpha t}. $$

(11)

The value of $\Gamma$ is increasing infinitely at $t \to \infty$ [1, 4], it means an instability of the system. Similar instability was observed experimentally as the process of formation of a charge density wave in TbTe$_3$ [4].
From the aforesaid we can see, that perturbation theory making possible to calculate the vacuum amplitude (thermodynamics function $U$, $\Omega$ and so on) in ranges including a point of phase transition (for example, temperature $T_C$) doesn’t exist. In other words, it is not possible to find $R(\beta)$ in the system with condensed phase starting from the first principles. However the phenomenological approach exists for calculation of $\Omega$. It assumes, that at the temperature $T < T_C$ the condensed phase is characterized by some order parameter $\eta$ exists. Near a transition point for type II phase transition or near a point of overcooling for type I phase transition the parameter $\eta$ is small, and the free energy can be represented in a form of Landau expansion:

$$\Omega(T, V, \mu, h) = \Omega_0 + \int d^3r (a\eta^2 + b\eta^4 + g(\nabla\eta)^2 - \eta h), \tag{12}$$

where $h$ is external field. The equilibrium value of order parameter $\eta$ is determined by an extremal of the functional

$$\frac{\delta \Omega}{\delta \eta} = 0. \tag{12}$$

However, the expansion (12) is correct in the range $(T_C - T)/T_C \ll 1$ only, hence $\eta$ can be found in this range only.

In order to obtain $\eta$ at any temperature the concept of quasi-averages (anomalous averages) is introduced. In the paper [3] generalization of the method of quasi-averages has been represented in terms of Green function - Nambu-Gor’kov formalism [7, 8]. It is postulated, that in the condensed phase, in addition to normal propagators $G(k, \sigma, t' - t) = -i\langle \Psi_0 | T \{ C_{k\sigma} (t') C_{k\sigma}^\dagger (t) \} | \Psi_0 \rangle$, anomalous propagators $F$ exist, for example:

- **Ferromagnetic**: $F_{fer} = -i\langle \Psi_0 | T \{ C_{k\uparrow} (t') C_{k\uparrow}^\dagger (t) \} | \Psi_0 \rangle$

- **Solid, liquid**: $F_{sol} = -i\langle \Psi_0 | T \{ C_{k\uparrow} (t') C_{k\uparrow}^\dagger (t) \} | \Psi_0 \rangle$

- **Superconductor with singlet pairing**: $F_{sup} = -i\langle \Psi_0 | T \{ C_{-k\uparrow} (t') C_{k\uparrow} (t) \} | \Psi_0 \rangle$

They are proportional to the according order parameters. The anomalous propagators $F$ can not be obtained by summation of diagrams consisting of normal propagators $G$ only. This fact is result of different symmetries of a perturbed state and an unperturbed state.

For construction of the self-consistent perturbation theory the "source term" $\hat{H}_S$ is introduced in Hamiltonian instead of the interaction operator $\hat{V}$. The source term is induced by the order parameter (for ferromagnetic - magnetic field orientating spins, for superconductor - a source of Cooper pairs). The source term inducts specification structure and changes symmetry of a system: $\Phi_0 \rightarrow \Phi'_0$, such that $\langle \Phi'_0 | \Psi_0 \rangle \neq 0$. This means that a system has an internal long range field $\lambda$, generated by the source $\hat{H}_S$, and according order parameter $\eta$, which is function of $H$ and depends on the interaction constant $\lambda$ and temperature $T$: $\eta = \eta_{AX}(H)$. In turn, $H$ is function of $\eta$: $H = H_{AX}(\eta)$. Hence, these two equations can be combined:

$$\eta = \eta_{AX}(H_{AX}(\eta)) \tag{14}$$

and they can be solved in $\eta$. This means, that order parameter is determined in self consistent way. In Green function formalism this fact has the following form: the free matrix propagator $\Phi_0$ (anomalous part is zero $F = 0$) and the dressed propagator $\Phi$ (with normal $G$ part and anomalous $F$ parts) obey Dyson equation [3, 8]:

$$\Gamma^{-1} = \Phi^{-1} - \Sigma(G), \tag{15}$$

and what’s more the mass operator $\Sigma(G)$ is determined by self consistent way. This means, that elements of diagrams for the mass operator are dressed propagators $\Phi$, which contain the sought mass operator. Artificiality of the introduction of the source $\hat{H}_S$ lies in the fact that the expression for a mass operator is **postulated**. Moreover, $\Sigma$ can not be obtained by summation of the diagram consisting of free propagators $\Phi_0$ only, that is $\Sigma(\Phi_0) = 0$. For example, $\Sigma$ has the form for a superconductor:

$$\Sigma(\omega, p) = [1 - Z(\omega, p)] |\omega| \mathbf{1} + \chi(\omega, p) \tau_3 + \varphi(\omega, p) \tau_1 + \tilde{\varphi}(\omega, p) \tau_2, \tag{16}$$

where $Z$ is some coefficient, the field $\chi$ determines a shift of chemical potential $\mu$ at the transition to the superconductive state, the fields $\varphi$ and $\tilde{\varphi}$ play a part of order parameter in superconductivity - a gap, and specter of excitations is represented by them $E^2(p) = \varepsilon^2(p) + \varphi^2(p) + \tilde{\varphi}^2(p)$; $\mathbf{1}$, $\tau_i$ are unit matrix and Pauli matrices. The dependence of all fields on frequency $\omega$ takes into account a delay and a damping of quasi-particles. In Hartree-Fock approximation we have $Z = 1$ and $\Sigma$ doesn’t depend on $\omega$. We can suppose $\tilde{\varphi} = 0$, $\chi = 0$ and denote $\varphi(p) \equiv \Delta(p)$, then

$$\Delta(T = 0) = \frac{\lambda}{V} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \int \frac{d^3p}{(2\pi)^3} iF(\omega, p, \Delta), \quad \Delta(T) = \frac{\lambda}{V} T \sum_{n=-\infty}^{n=\infty} \int \frac{d^3p}{(2\pi)^3} iF(\omega_n, p, \Delta), \tag{17}$$
where \( \lambda < 0 \) is the interaction constant, and the summation is done over Fermi frequencies \( \omega_n = (2n + 1)\pi T \). We can see, that the order parameter is determined by the anomalous propagator \( F \), and the equations (17) are equations of self consistency for order parameter \( \Delta \) as specific case of the total equation (14).

The described approach is phenomenological too, because the anomalous propagators and corresponding order parameters are introduced to the theory from elsewhere. This means that we select the required states from all possible state artificially. In this sense this approach likes Landau approach (12). The source of Cooper pairs for superconductor or the magnetic field \( H = \frac{J_0}{g \mu_B} \langle S_z \rangle \) for ferromagnetic \( (J_0 \text{ is the exchange integral, } g \text{ is the gyromagnetic relation, } \mu_B \text{ is Bohr magneton, } \langle S_z \rangle \text{ is the average projection of spin onto axis } z) \) can not be interpreted as real field. So, for iron the internal magnetic field is \( \sim 10^6 \) oersted, but for the ordering of spins it is necessary the effective magnetic field \( H \sim 10^9 \) oersted. Thus, self magnetization has nonmagnetic nature evidently.

As it has been noted in [9], in spite of all progresses reached in description of phase transitions and in calculation of main characteristics of a system in critical region, the basic problem of phase transitions is not solved: calculation explicit expressions for thermodynamical functions of a system in ranges, which include a point of phase transition as function of temperature, external fields and microscopic parameters of Hamiltonian. It is necessary to have a description of phase transitions on microscopic level. This presupposes a direct calculation of free energy \( \Omega(T, V, h) \) from first principles, but we must not construct it.

At the present moment the two approaches can be separated for solution of the formulated problem. In the papers (9)–(11) a partition function of Ising model is calculated by the method of collective variables, which are oscillation modes of spin moment. Then a partition function can be written via these variables as a functional. Investigation of Euler–Lagrange equations shows, that among a set of collective variables the variable connected with order parameter \( \Delta \) exists. In the paper [12] a partition function is calculated by the saddle point method for classical system with short range attraction and repulsion between particles. It has been shown, that the free energy can be represented by the form which is analogous to Landau expansion where the saddle point plays a role the order parameter. The saddle point method has the sense as method of separation of states giving largest contribution in a partition function and it is equivalent to the mean field method. With the help of the saddle point method the processes of cluster formation can be investigated. The cluster formation can be considered as a phase transition from spatially homogeneous distribution to spatially inhomogeneous distribution. In the papers [13]–[16] both short range potentials and long range potentials were considered. Critical temperature and critical concentration, when clusters form in a system, dependence of size of a cluster on temperature have been obtained.

One of the important applications of the microscopic theory of phase transitions is description of thermodynamics and electrodynamics of superconductors. It is necessary to know the functional of free energy \( \Omega(\beta, \Delta, A) \), where \( A \) is potential of magnetic field. Then the equations \( \frac{\delta \Omega}{\delta \Delta} = 0, \frac{\delta \Omega}{\delta A} = 0 \) will describe equilibrium states of the condensed phase and normal phase. Two basic methods for obtaining of the sought equations exist. First of them is joint solution of Gor’kov equations and the equation of self consistency (17). At the temperature \( T \rightarrow T_C \), when \( \Delta/T_C \rightarrow 0 \), the solution of these equation can be represented in the form of series in degrees of \( \Delta \). Moreover, magnetic penetration depth is bigger then Pippard coherent length \( l_0 \), hence the potential \( A \) changes a little on a coherent length. As a result we have the well-known Ginzburg-Landau equation.

Another method, proposed in [17], is the direct calculation of a vacuum amplitude \( R(\beta) \). The concept lies in the fact that we consider electrons in a normal metal propagating in random "field" of thermodynamic fluctuations of order parameter \( \Delta q \), where \( q \) is small wave-vector. The operator of the interaction of electrons with the fluctuations can be written as:

\[
\hat{H}_{\text{int}} = \sum_{\mathbf{p}} \left[ \Delta q \hat{G}_\mathbf{p}^+ \hat{c}_-^\mathbf{p} + \Delta q \hat{c}_-^\mathbf{p} \hat{G}_\mathbf{p}^+ \right],
\]

where \( \mathbf{p} = \mathbf{p} \pm \mathbf{q}/2 \). A correction to the thermodynamics potential from any interaction is represented via the vacuum amplitude \( R \):

\[
\Delta \Omega = -T \ln R(\beta) \approx -T[R(\beta) - 1].
\]

Then using Wick theorem we can represent (18) via the propagators. For the correction of second order we have:

\[
\Delta \Omega = -\frac{T}{2} \int_0^{1/T} d\tau_1 \int_0^{1/T} d\tau_2 \langle \hat{H}_{\text{int}}(\tau_1) \hat{H}_{\text{int}}(\tau_2) \rangle
\]

\[
= -T \int_0^{1/T} d\tau_1 \int_0^{1/T} d\tau_2 |\Delta q|^2 \sum_{\mathbf{p}} G_0(p_+, \tau_1 - \tau_2) G_0(-p_-, \tau_1 - \tau_2).
\]
The correction $\Delta \Omega$ is represented via the free propagators $G_0$ of normal state only - we consider normal metal at $T > T_c$, where the fluctuation source of Cooper pair $|S|$ acts. As a result we have Landau expansion:

$$\Omega_s - \Omega_n = \sum_q \left[ \alpha(T)|\Delta q|^2 + \frac{b}{2}|\Delta q|^4 + \gamma q^2|\Delta q|^2 \right],$$

(21)

where $\alpha(T) \propto (T - T)$, $b, \gamma$ are expansion coefficients.

In our opinion, this approach is not successively microscopic, because the artificial element is used - the external source of Cooper pairs $|S|$, that implies some seed order parameter. As for calculation of the correction $\Delta \Omega$ the free propagators $G_0$ of normal phase is used only, the condensed phase is considered as fluctuations against the background of normal phase. This means, that we can obtain the limit $\Omega(T \rightarrow T)$ only.

Ginzburg-Landau equations are correct for description of thermodynamics and electrodynamics of a superconductor at arbitrary temperatures, spatial inhomogeneities, magnetic fields and currents, moreover with spin density, crystallization and so on). Further we can apply the developed method in order to calculate free energy be described [5], then our method can be generalized to the rest transitions (ferromagnetism, waves of charge and so on). Since in Nambu-Gor’kov formalism (the method of anomalous propagators) any phase transition can as an example. The correction $\Delta \Omega$ is represented via the free propagators $G_0$ of normal state only - we consider normal metal at $T > T_c$. This means, that the equations are correct in the range $T \rightarrow T_C$ or in the range $H \rightarrow H_{C2}$ (intensity of magnetic field is near the second critical magnetic field $H_{C2}$).

1. The gap is much less than critical temperature. Then the parameter $\Delta(r, T)/T_C \ll 1$ can be expansion parameter. This means, that the equations are correct in the range $T \rightarrow T_C$ or in the range $H \rightarrow H_{C2}$ (intensity of magnetic field is near the second critical magnetic field $H_{C2}$).

2. $\Delta(r, T)$ changes slowly on the coherent length $l(T)$, which is size of a Cooper pair.

3. Magnetic field $H(r) = \text{rot}\mathbf{A}(r)$ changes slowly on the coherent length, that is the magnetic penetration depth is $\lambda(T) \gg l(0)$. This means, that electrodynamics of a superconductor is local.

In the papers [18, 19] the equations has been proposed, where the first restriction is absent. These equations were obtained from Gor'kov equations and they are the generalization of Ginzburg-Landau equations for the case of arbitrary value of $\Delta(r, T)/T$. However spatial inhomogeneities are slow and electrodynamics is local.

Our aim is the description of phase transitions on microscopic level starting from the first principles only. Mathematically this means to develop a method of calculation of the partition function $\Pi$ (the free energy $\Omega$) in ranges of temperatures and parameters of interaction, which include a point of phase transition, without introducing any artificial parameters of type of order parameter $\eta$ and sources of ordering $\tilde{H}_S$, but starting from microscopic parameters of Hamiltonian $H = \tilde{H}_0 + \tilde{V}$ only. The theorems (6,9,10) about connection of a vacuum amplitude with thermodynamics potentials must be realized. Thus, in microscopic theory of phase transitions the equation of self-consistency must be postulated.

To solve the problem means to develop the perturbation theory for the vacuum amplitudes $\eta^\pm$ of normal phase only - we consider normal metal at $T > T_c$, where the fluctuation source of Cooper pair $|S|$ acts. As a result we have Landau expansion: $\Delta \Omega = \sum q \left[ \frac{\alpha(T)}{2}|\Delta q|^2 + \frac{b}{4}|\Delta q|^4 + \gamma q^2|\Delta q|^2 \right]$.

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Ginzburg-Landau equations are correct for description of thermodynamics and electrodynamics of a superconductor at arbitrary temperatures, spatial inhomogeneities, magnetic fields and currents, moreover with nonlocal magnetic response. The obtained expression will be the generalization of Ginzburg-Landau functional in above mentioned sense.

In the section [11] we consider the instability of normal Fermi system at the switching of attraction between particles. Mathematically this is expressed in the fact that a two-particle propagator $K$, calculated on free one-particle propagators in stair approximation, has a pole $\alpha$ which doesn’t belong to a free propagator $K_0$ and situated on complex axis. This means a presence of bound states of particles in the system (with the binding energy $|\alpha|$) and evolution of the system in time as $\sim e^{\alpha t}$. On the other hand, a two-particle propagator has a pole at the energy of the bound state if the bound state of two isolated particles exists. The residue in this pole is product of Bethe-Solpiter amplitudes $\eta^\pm$ - the amplitudes of pairing.

In the section [11] we generalizes the result of two-particles problem to a many-particle system. It follows from the identity principle, that amplitudes of pairing are determined by dynamics of all particles of the system, and observed values of the amplitudes of pairing are the result of the averaging over a system. Thus, the collective (condensate) of pairs exists. In order to find one-particle propagator $G_2$ and to generalize the two-particle problem to a many-particle case we proposed the method of an uncoupling of correlations. The method considers interaction of a additional fermion with fluctuations of pairing (formation and decay of the pairs). As a result of such interaction, the law of quasi-particles’ dispersion changes as $\varepsilon(k) \rightarrow \sqrt{\varepsilon^2(k) + \Delta^2}$, where $\Delta$ and $\Delta^+$ are amplitudes of pairing playing a role of a gap and they are analog of Bethe-Solpiter amplitudes in two-particle problem. Gor’kov equations and the existence of anomalous propagators $F$ and $F^+$ follow from Dyson equation for the described above process. This fact means breakdown of a global gauge symmetry, namely number of particles is not conserved in the course of a presence of a pairs’ condensate. After calculation of particles’ interaction with fluctuations of pairing all characteristics of a system must be calculated over the new vacuum with broken symmetry.
For calculation of observed values of the amplitude of pairing \( \Delta \) and \( \Delta^+ \) it is necessary to know a vacuum amplitude \( R(t) \) of a system. The vacuum amplitude must be calculated over the new vacuum with broken symmetry. This gives possibility to use the theorem about connection of a vacuum amplitude with ground state energy of a system. In the section [V] the method of uncoupling of correlations is proposed. The method allows to represent a vacuum amplitude via anomalous propagators \( F \) and \( F^+ \). Thus we obtain a functional of ground state energy over the fields \( \Delta \) and \( \Delta^+ \). An extremal of the obtained functional is the equation of self consistency for the parameter \( \Delta \) in Nambu-Gor’kov formalism. This means, that order parameter is averaged Bethe-Solpiter amplitude over all system. In the section [VI] we generalize the results of two previous sections for the case of nonzero temperature. Using the method of uncoupling of correlations we calculate a vacuum amplitude \( R \) and a functional of free energy \( \Omega(\Delta, T) \) over the fields \( \Delta \) and \( \Delta^+ \). In a high-temperature limit \( T \to T_C \) the expansion of free energy in powers of \( |\Delta| \) has a form of Landau expansion. This fact proves, that the averaged over a system the amplitudes of pairing \( \Delta \) and \( \Delta^+ \) have properties, which are analogous to the properties of an order parameter in a phenomenological theory. In the section [VI] we consider the case of a pairing with nonzero momentum of a pair’s center of mass.

In the sections [VII] and [VIII] using the developed method of microscopic description of a phase transition we obtained the functional of free energy of a spatial inhomogeneous superconductor in magnetic field. The functional generalizes Ginzburg-Landau functional in cases of arbitrary temperatures, arbitrary spatial inhomogeneities and nonlocality of magnetic response. The free energy has a form of Ginzburg-Landau functional in high-temperature limit at the condition of slowness of gap’s change in space. Corresponding equations of superconductor’s state demonstrate the nonlinear connection between the current and the field. In the low-temperature limit in the case of weak field \( H \ll H_C \) at the condition of slowness of gap’s change the nonlocal connection between the current and the field appears, which is the long-wave limit of Pippard law. The last fact proves the nonlocality of the obtained functional of free energy.

II. THE INSTABILITY OF NORMAL STATE AND TWO-PARTICLE DYNAMICS

Let we have a system from \( N \) noninteracting fermions in volume \( V \). In ideal Fermi gas propagation of a particle with momentum \( k \), energy \( \varepsilon \approx v_F(\varepsilon - k) \) counted off Fermi surface (we are using system of units, where \( \hbar = k_B = 1 \)) and spin \( \sigma \) is described by the free propagator:

\[
G_0(k, t) = \begin{cases} 
-\frac{i}{\pi} \int \Phi_0 |C_{k, \sigma}(t)C_{k, \sigma}^+(0)|\Phi_0, & t > 0 \\
\frac{i}{\pi} \int \Phi_0 |C_{k, \sigma}^+(0)C_{k, \sigma}(t)|\Phi_0, & t \leq 0
\end{cases}
\]

\[
G(k, t) = \frac{d\omega}{2\pi} G(k, \omega) e^{-i\omega t} \Rightarrow G_0(k, \omega) = \frac{1}{\omega - \varepsilon(k)} = \frac{A_0}{\omega - \varepsilon(k)} + \frac{B_0}{\omega + \varepsilon(k)},
\]

where

\[
A_0 = \frac{1}{2} \left( 1 + \frac{\varepsilon}{|\varepsilon|} \right), \quad B_0 = \frac{1}{2} \left( 1 - \frac{\varepsilon}{|\varepsilon|} \right), \quad \theta_t = \begin{cases} 1, & t > 0 \\
0, & t < 0
\end{cases},
\]

(23)

\( C_{k, \sigma}(t) \) and \( C_{k, \sigma}^+(t) \) are creation and annihilation operators in Heisenberg representation. Now let attractive force acts between particles. The force is described by matrix element of interaction potential:

\[
|l, -l, \uparrow\downarrow | \Phi |k, -k\rangle = \lambda w_k w_k < 0, \quad w_k = \begin{cases} 1, & \varepsilon(k) < \omega_D \\
0, & \varepsilon(k) > \omega_D
\end{cases},
\]

(24)

moreover interacting particles have opposite spins. This model potential is correct if a range of interaction \( r_0 \) is much smaller than average distance between particles: \( r_0 \sqrt{N/V} \sim r_0 k_F \ll 1 \), that means "slowness" of collisions. In turn, account of Fermi statisity, in the limit of "slow" collisions particles can scatter with opposite spins only.

The mass operator \( \Sigma(G_0) \) in stair approximation is determined by so called \( \Gamma \)-matrix (amplitude of scattering), which is solution of the equation represented by the diagram in Fig[II]. For the case \( q = 0 \) \( \Gamma \)-matrix has a form:

\[
\Gamma = \frac{\lambda w_k w_k}{1 - i\lambda \int \frac{d^3q}{(2\pi)^3} \frac{d\omega}{2\pi} w_k G_0(k, \alpha - \omega), G_0(k, \omega)} = \frac{\lambda w_k w_k}{1 - |\lambda|^{mk_F} \ln \frac{4\pi^2}{\alpha^2} - 1}.
\]

(25)

This expression has a pole in the point \( \alpha_0 \). In its neighborhood the expression has a view:

\[
\Gamma(\alpha \to \alpha_0) = -\frac{2\pi^2}{mk_F} \frac{\pm i|\alpha_0|}{\alpha \pm i|\alpha_0|}, \quad \alpha^2_0(\lambda \to 0) = -4\omega_D^2 \exp \left\{ -\frac{4\pi^2}{mk_F|\lambda|} \right\}.
\]

(26)
If with the help of Fourier transformation to pass from \( \omega \)-representation to \( t \)-representation, then we shall have the expression \([11]\), which increases infinitely at \( t \to \infty \). Hence, according to the first diagram in Fig. 1 we have the same bad behavior of the mass operator \( \Sigma \). The presence of a pole in imaginary axis means instability of a system. In our opinion, this result can be interpreted as follows: in a system because of the interaction \([24]\) the strong fluctuations exists, but the free propagator \( G_0 \) doesn’t consider these perturbations. In turn, it leads to the instable solution of Dyson equation. This means, that besides the interaction between particles \( V(k) \) the interaction of particles with aforesaid fluctuations exists too. For stability of the solution, dressed propagators \( G_s \), which considers scattering on the fluctuations, must be in the equation for the mass operator \( \Sigma \) and scattering amplitude \( \Gamma \) in Fig. 1 instead the free propagators \( G_0 \) .

Let’s determine a two-particle propagator by the expression:

\[
K(x_1, x_2; x_3, x_4) = \langle \Psi_0 | T \hat{C}(x_1) \hat{C}(x_2) \hat{C}^+(x_3) \hat{C}^+(x_4) | \Psi_0 \rangle
\]

\[
K_0(x_1, x_2; x_3, x_4) = \langle \Phi_0 | T \hat{C}(x_1) \hat{C}(x_2) \hat{C}^+(x_3) \hat{C}^+(x_4) | \Phi_0 \rangle = G_0(x_1, x_3)G_0(x_2, x_4),
\]

where \( K \) is a propagator in a system with interaction, \( K_0 \) is a free two-particle propagator and it is equal to a product of one-particle propagators \( G_0 \), \( x \equiv (\xi, t) \). The free and dressed two-particle propagators are connected by Bethe-Solpiter equation (in an operator view):

\[
K = K_0 + K_0 ivK = K_0 + K_0 \Gamma K_0 \Rightarrow \Gamma = v + ivK_0 \Gamma,
\]

where \( v(x_1, x_2; x_3, x_4) = V(r_1 - r_2)\delta(x_1 - x_2)\delta(x_2 - x_4)\delta(t_1 - t_2) \). The last equation in \([28]\) for the scattering amplitude \( \Gamma \) is represented graphically in Fig. 1. We can see from the equation \([28]\), that a presence of the pole in \( \Gamma \) means a presence the same pole in the two-particle propagator \( K \). As it is well known \([20, 22, 27]\), a two-particle propagator has a pole structure at the values of energy corresponding to a bound state. The pole \( |\alpha_0| \) \([20]\) doesn’t belong to the free propagator \( K_0 \), but it appears as a result of the attraction \([24]\) \( \lambda < 0 \). Hence, the pole means a presence of bound states of two particles in a system with the binding energy \( E_s \approx |\alpha_0| \).

For investigation of the bound state and calculation of of particles’ interaction with above described fluctuations let’s consider the problem of two particles at first. Previously we considered dynamics of two selected particles being in a field of the rest particles of a system. Therefore the propagator \( K \) determined dynamics of all particles of a system in such approximation. Now we shall consider the system consisting from two particles only being in the state \( \Phi_s \) with the energy \( E_s \). Interaction between them is described by the potential \( V(k) \equiv V_{12} \):

\[
(H_1 + H_2 + V_{12})\Phi_s = E_s\Phi_s.
\]

Let’s determine a propagator for two particles as kernel of the integral operator finding \( \Phi(t) \) known \( \Phi_1(t') \):

\[
\Phi(\xi_1, \xi_2, t, t') = -\int K(\xi_1, \xi_2; t; \xi'_1, \xi'_2, t')\Phi_1(\xi'_1, \xi'_2, t')d\xi'_1d\xi'_2.
\]

Then the two-particle propagator can be written in Fourier representation as \([20]\):

\[
-K(\xi_1, \xi_2; \xi'_1, \xi'_2; E) = i \sum_s \frac{\Phi_s(\xi_1, \xi_2)\Phi_s^*(\xi'_1, \xi'_2)}{E - E_s + i\gamma}.
\]

Figure 1: The mass operator \( \Sigma \) expressed via \( \Gamma(q, \alpha) \)-matrix. Bethe-Solpiter equation for \( \Gamma(q, \alpha) \)-matrix.
If the bound state is among states $s$, then the pole of the function $K(\xi_1, \xi_2, E)$ corresponds to the bound state $s$ at a real $E$, where $E$ is equal to energy of the bound state $E_s$.

The residue in a pole $E = E_s$ is $\Phi_s(\xi_1, \xi_2)\Phi_s^*(\xi'_1, \xi'_2)$. As $K_0$ hasn’t a pole in $E = E_s$, where $E_s$ is the energy of the bound state, then a pole of $K$ means a presence of a pole at $\Gamma$ according to the equation (28). For the function $K(x_1, x_2; x'_1, x'_2)$ with different times the expression can be written:

$$iK(\xi_1, \xi_2, \tau_1; \xi'_1, \xi'_2, \tau'_1; E) = \sum_s \Pi_s(\xi_1, \xi_2, \tau_1; \xi'_1, \xi'_2, \tau'_1) \frac{\Pi_s(\xi_1, \xi_2, \tau_1; \xi'_1, \xi'_2, \tau'_1)}{E - E_s + i\gamma},$$

where we denoted that

$$t_1 - t_2 = \tau'_1, \quad t_1 - t'_2 = \tau'_1, \quad t_1 + t_2 = 2t, \quad t'_1 + t'_2 = 2t'.$$

This expression is the formula (31) at $\tau_1 = \tau'_1 = 0$.

The residue $\Pi_s(\xi_1, \xi_2, \tau_1; \xi'_1, \xi'_2, \tau'_1)$ for the bound state can be written in multiplicative form as in the case $\tau_1 = \tau'_1 = 0$:

$$\Pi_s = \eta_s(\xi_1, \xi_2, \tau_1)\eta^+_s(\xi'_1, \xi'_2, \tau'_1).$$

(33)

The values $\eta$ and $\eta^+$ are Bethe-Solpiter amplitudes [22–25]. They are connected with the wave functions $\Phi_s$ as (in momentum representation $\xi \equiv k$): $\Phi_s(k_1, k_2) = \int \eta_s(k_1, k_2, \epsilon) \frac{d\epsilon}{2\pi}$. In the equation for $K$ (28), $K_0$ can be neglected near a pole corresponding to a bound state. Resulting homogeneous equation has the solution (32), moreover the function $\eta_s(\xi_1, \xi_2, \tau_1)$ satisfies the equation

$$\eta = iK_0\eta \iff \eta_s(k_1, k_2, \epsilon) = iG_0(k_1, E_s/2 - \epsilon)G_0(k_2, E_s/2 + \epsilon) \int V(k)\eta_s(k_1 + q, k_2 - q, \epsilon') \frac{d\epsilon'}{2\pi} \frac{dk}{(2\pi)^3}.$$

(34)

The term in the sum (32), corresponding to the bound state $E_s$, can be represented by the diagram in Fig 2 where

![Diagram](image)

Figure 2: The two-particle propagator for isolated pair of particles in neighborhood of a pole corresponding to the bound state $E_s$.

the dotted line means the multiplier $\frac{1}{E - E_s + i\gamma}$ - propagation of two particles in bound state, and the blocks are $\eta_s(k_1, k_2, \epsilon)$ and $\eta^+_s(k_1 + q, k_2 - q, \epsilon)$ - transition amplitudes in bound state and back. The diagram in Fig 2 can be interpreted by the following way. Two particles with momentums $k_1$ and $k_2$ form a bound state with energy $E_s$ with amplitude $\eta_s(k_1, k_2)$. Further, the two particles propagate together. Then the bound state can decay with amplitude $\eta^+_s(k_1 + q, k_2 - q)$. As a result the two free particles appear with momentums $k_1 + q$ and $k_2 - q$.

### III. THE UNCOUPLING OF CORRELATIONS AND A MULTIPARTICLE DYNAMICS.

In the previous section we considered dynamics of two isolated particles. Now we have to generalize the obtained results to the multi-particle case - propagation of two interacting particles in a system of identical fermions. This situation differs from the previous case by the following conditions:

1. Each pair of fermions is in field of all the rest particles.

2. All particles of a system are identical. Moreover, the average size of a pair $l_0 \sim 1/\sqrt{|m|2m} \gg \sqrt{V/N}$ is more big than average distant between particles, that means the wave packages of pairs overlap strongly.
Mathematically this means, that the amplitudes $\eta$ and $\eta^+$ are not solution of the equation (34), which is correct for isolated pair only. Now the amplitudes are determined by dynamics of all particles of the system, and their observed value is result of an averaging over a system. Pairs in such system are effective, namely two fermions having formed a bound state with an amplitude $\eta_s(k_1, k_2, \epsilon)$ (Fig.2) are not fixed pair: one from partners in a pair can leave the bound state with a fermion from another pair with amplitude $\eta^+_s(k_1 + q, k_2 - q, \epsilon)$. Thus, the collective of pairs (condensate) exists.

Let’s consider the two-particle propagator $K_{E \to E_s}$ represented in Fig.2. As it has been noted earlier, fermions with opposite momentums and opposite spins form a pair. Let’s suppose, that corresponding amplitudes of pairing don’t depend on time. In order to obtain an one-particle propagator $G^0$ we shall use the method of uncoupling of correlations considered in the Appendix A and we shall be acting analogously to Fig.11. The procedure of an uncoupling is represented in Fig.3. We connect the entering line and the outgoing line corresponding to particles with opposite momentums and opposite spins form a pair. Let’s suppose, that corresponding amplitudes of pairing are not fixed in consequence of the identity principle and strong intersection of wave packages of pairs.

Some pair of fermions decays in components of correlations considered in the Appendix A, and we shall be acting analogously to Fig.11. The procedure of an uncoupling of correlations for a two-particle propagator of a pair being in a field of all rest fermion is represented in Fig.3. We connect the entering line and the outgoing line corresponding to particles with opposite momentums and energy parameter $-\omega$ each. As a result we have the intermediate propagator $G^0$. Since partners in each pair are not fixed in consequence of the identity principle and strong intersection of wave packages of pairs, that is a condensate of pairs exists, then it is necessary to cut the dotted line - the propagator of a pair $g_{E \to E_s, \pm q}$. Then the points of a joining of the dotted lines correspond to interaction with the effective field (in accordance with the rules of diagram technics in the Appendix A). The fluctuation of pairing play a role of the above mentioned effective field. The amplitudes of such interaction we denote as $-i\Delta(k, -\omega)$ and $i\Delta^+(k, \omega)$. These values correspond to the amplitudes of the two-particle problem $\eta_s(k, -\omega)$ and $\eta^+_s(k, \omega)$ to the extent that their observed values is result of averaging over a system $\langle \eta_s(k, -\omega) \rangle \sim \Delta$ and $\langle \eta^+_s(k, \omega) \rangle \sim \Delta^+$ in consequence of statistical correlations between pairs and they are determined by dynamics of all system’s particles.

![Figure 3: The procedure of uncoupling of correlations for a two-particle propagator of a pair being in a field of all rest fermion of a system. The result of the uncoupling is the dressed one-particle propagator $G^0$, as a consequence of interaction of a free fermion with fluctuations of pairing.](image)

The result of the procedure of uncoupling of correlations means the follows. Let an additional particle with momentum $k, \omega$ propagates through a system of identical fermions. In the process of propagation a particle can form bound states with other fermions according to the following mechanism. Some pair of fermions decays in components with momentums $-k, -\omega$ and $k, \omega$ with amplitude $i\Delta^+$. Second particle of the decayed pair is in state of the additional pair $(k, \omega)$ and it is identical to the additional particle. The second particle propagates through a system further.

First particle of decayed pair forms bound state with the initial additional particle with amplitude $-i\Delta$. Anew formed pair replenishes the condensate of pairs in a system. Thus, the dressed propagator $G^0$ takes into account interaction of a particle, initially described by free propagator $G^0$, with fluctuations of pairing. Intensity of the interaction is the amplitudes $-i\Delta$ and $i\Delta^+$.

Starting from the aforesaid, we can write the mass operator for such process (Fig.11) as

$$-i\Sigma = -i\Delta iG^0(-k, -\omega)i\Delta^+ \Rightarrow \Sigma = \frac{\Delta\Delta^+}{\omega + \epsilon(k)},$$  

(35)
Hamiltonian of a system of free quasi-particles has a view:

Then, using the definition (38), we can find:

where

Helmholtz free energy in a superconductive state. The states chemical potential of a quasi-particles’ system equals to zero. Therefore the grand potential Ω coincides with (36). For this mass operator we postulated (as the anomalous averages). From Dyson equation (A6) we can obtain the dressed one-particle propagator:

is dispersion law of dressed particles (quasi-particles). The amplitude Δ is named by gap, because minimal work for one-particle propagator:

Figure 4: The diagram for the mass operator Σ describing interaction of a fermion with fluctuations of pairing.

This mass operator has been proposed in [20, 21], however an existence of the amplitudes Δ, Δ+ and the equation of self-consistency were postulated (as the anomalous averages). From Dyson equation (A6) we can obtain the dressed one-particle propagator:

where

is dispersion law of dressed particles (quasi-particles). The amplitude Δ is named by gap, because minimal work for creation of one-particle excitations is 2Δ. In accordance with the definition of one-particle propagator we can write:

where the system is placed in another ground state Ψ₀. In the state Ψ₀ interaction of particles with fluctuation of pairing (existence of condensate of pairs) is taken into account, moreover Ψ₀(Δ = 0) = Φ₀, Gₛ(Δ = 0) = G₀. A propagator defines occupations number of quasi-particles \( n_k \) by the following way:

Hence, we can suppose, that

where \( |Ψ₀, 1_p^+⟩ \) and \( |Ψ₀, 1_k^+⟩ \) are states with one added particle and one removed particle with momentum \( k \) accordingly. Hamiltonian of a system of free quasi-particle has a view:

Chemical potential of a quasi-particles’ system equals to zero. Therefore the grand potential Ω coincides with Helmholtz free energy in a superconductive state. The states \( |Ψ₀, 1_p^+⟩ \) and \( |Ψ₀, 1_k^+⟩ \) are eigenvectors of the Hamiltonian (41):

Then, using the definition (38), we can find:

that coincides with (36). For \( t ≤ 0 \) the proof is analogous.
Dyson equation can be represented in other form. Let’s use the definition \((\omega - \varepsilon(k))G_0 = 1\). On the other hand we have \(G_0 = G_S/(1 + G_S\Sigma)\). Moreover, let’s introduce the notations

\[-G_S\Sigma = \Delta F^+, \quad -G_S\Sigma = \Delta^+ F^+\]  

(44)

Then, we can obtain the set of equations:

\[
(\omega - \varepsilon(k))G_S(k, \omega) + \Delta F^+(k, \omega) = 1
\]

(45)

\[
(\omega + \varepsilon(k))F^+(k, \omega) + \Delta^+ G_S(k, \omega) = 0.
\]

(46)

These equations are Gor’kov equations in momentum representation. However, unlike phenomenological approach (where existence of the anomalous propagator \(F\) and the equation for order parameter are postulated) these equations are obtained by microscopic way with help of the procedure of uncoupling of correlations. From the equations, we can find, that

\[
F^+ = \frac{-\Delta^+}{\omega^2 - E^2}, \quad F = \frac{-\Delta}{\omega^2 - E^2}.
\]

(47)

The anomalous propagators describe creation of two fermions from the condensate of pairs - \(F^+\), formation a pair by two particles with leaving to the condensate - \(F\). Moreover, \(F\) and \(F^+\) are the infinity sum of the serial processes of creation and annihilation of pairs described by amplitudes \(\Delta\) and \(\Delta^+\). Mathematically this is expressed in the fact that

\[
F_{\alpha\beta}^+(k, t) = \Delta^+ \sqrt{\Delta^+ + \Delta} A_S B_S e^{-iEt} + i\theta - i^\Delta \sqrt{\Delta^+ + \Delta} A_S B_S e^{iEt},
\]

(48)

\[
F_{\alpha\beta}(k, t) = \Delta \sqrt{\Delta^+ + \Delta} A_S B_S e^{-iEt} + i\theta - i^\Delta \sqrt{\Delta^+ + \Delta} A_S B_S e^{iEt},
\]

(49)

Let’s prove the formulas (48) (49). For this let’s consider the expression:

\[
\langle \Psi_0 | C_{-k, \beta}^+(t) C_{k, \alpha}^+(0) | \Psi_0 \rangle = \langle \Psi_0 | e^{i\hat{H}_t} C_{-k, \beta}^+ e^{-i\hat{H}_t} C_{k, \alpha}^+ | \Psi_0 \rangle
\]

\[
= \langle \Psi_0, 1_{-k, \beta}^h | e^{i\Delta t} e^{-i(\hat{H}_0 + E(k)t)} \sqrt{A_S} | \Psi_0, 1_{k, \alpha}^p \rangle = \sqrt{A_S B_S} e^{-iE(k)t} \langle \Psi_0, 1_{-k, \beta}^h | \Psi_0, 1_{k, \alpha}^p \rangle
\]

(50)

that corresponds to (48). In the last equality the fact has been used, that the states, created by addition of a particle to a state \(k, \alpha\) or removing of a particle from a state \(-k, \beta\) at \(\alpha \neq \beta\), are identical in the course of existence of the pair condensate. The rest cases is proved analogously. It is not difficult to see, that \(F(\Delta = 0) = F^+(\Delta = 0) = 0\), because \(A_0(k)B_0(k) = 0\). From (45) we can see, that the existence of nonzero anomalous propagators \(F\) and \(F^+\) means breakdown of global gauge symmetry in a system, that is number of particles is not conserved in the course of existence of a pair condensate. Hence the states \(\Phi_0\) and \(\Psi_0\) have different symmetries:

\[
\langle \Phi_0 | \Psi_0 \rangle = 0.
\]

(51)

However distribution function over \(N\) has a maximum at the average number of particles \(\langle N \rangle\) determined by the expression:

\[
\langle N \rangle = 2 \sum_k n_k = -2i \int \frac{d^3k}{(2\pi)^2} \lim_{t \to 0} G_S(k, t) = 2 \int \frac{d^3k}{(2\pi)^2} B_S(k) \approx N = 2 \int \frac{d^3k}{(2\pi)^2} B_0(k).
\]

(52)

Now let’s return to the left part of the expression (25) for \(\Gamma\)-matrix. After considering of particles’ interaction with fluctuations of pairing we have to substitute dressed propagators \(G_S\) instead free propagators \(G_0\) in the formula (25). It is not difficult to verify, that \(\Gamma\) hasn’t poles at any \(\alpha\) and \(\Delta\). This means, that the problem of instability of a system
is removed, and we can use dressed propagator for further calculations confidently. Moreover, the absence of poles means the absence of bound states, because we have taken into account them in the specter of quasi-particles $E(k)$.

It is necessary to note, that the mass operator $\varepsilon$ and Gor’kov equation haven’t parameter of interaction between particles. This means, that the amplitudes of pairing $\Delta$ and $\Delta^+$ exists regardless of interaction between particles and its type. However, as we shall see below, the interaction determines the average value of the amplitudes, that is observed in experiment. This average value is not zero in the case of attraction between particle only.

IV. GROUND STATE ENERGY.

A. Summary kinetic energy of particles.

In order to calculate ground state energy it is necessary to know kinetic energy of particles and energy of their interaction. The operator of kinetic energy of all particles of a system has a form:

$$\hat{\mathcal{W}} = \sum_{k, \alpha} v_F (k - k_F) C_{k, \alpha}^+ C_{k, \alpha} = 2 \sum_k \varepsilon(k) C_{k, \alpha}^+ C_{k, \alpha}. \quad (53)$$

Then the corresponding average value is

$$\langle W \rangle = -2i \sum_k G(k, t \to 0^+) \varepsilon(k) = -2i \lim_{t \to 0} \sum_k \int \frac{d\omega}{2\pi} G(k, \omega) e^{-i\omega t} \varepsilon(k)$$

$$= 2 \sum_k B(k) \varepsilon(k) = V \nu_F \int_{-\nu_F k_F}^{\infty} B(\varepsilon) \varepsilon d\varepsilon. \quad (54)$$

Here $\nu_F = \frac{k_F^2}{2\pi^2}$ is density of states on Fermi surface. Since the interaction $V_{\mathrm{I}, \mathrm{lk} \to \mathrm{kn}}$ exists in the layer $-\omega_D < \varepsilon(k) < \omega_D$ only, we can suppose that

$$G = \left[ \begin{array}{c} G_0; \ \varepsilon(k) > \omega_D \\ G_S; \ \varepsilon(k) < \omega_D \end{array} \right], \quad A(k) = \left[ \begin{array}{c} A_0(k); \ \varepsilon(k) > \omega_D \\ A_S(k); \ \varepsilon(k) < \omega_D \end{array} \right], \quad B(k) = \left[ \begin{array}{c} B_0(k); \ \varepsilon(k) > \omega_D \\ B_S(k); \ \varepsilon(k) < \omega_D \end{array} \right]. \quad (55)$$

Then we can separate a normal part and a superconductive part of the kinetic energy:

$$\langle W \rangle = V \nu_F \int_{-\nu_F k_F}^{-\omega_D} B_0 \varepsilon d\varepsilon + V \nu_F \int_{-\omega_D}^{\omega_D} B_S \varepsilon d\varepsilon + V \nu_F \int_{\omega_D}^{\infty} B_0 \varepsilon d\varepsilon$$

$$= W_n + V \nu_F \int_{-\omega_D}^{\omega_D} B_S \varepsilon d\varepsilon - V \nu_F \int_{-\omega_D}^{\omega_D} B_0 \varepsilon d\varepsilon = W_n - V \nu_F 2 \int_{-\omega_D}^{\omega_D} \left( \varepsilon^2 - \frac{\varepsilon^2}{|\varepsilon|} \right)$$

$$= W_n + V \nu_F 2 \left( \omega_D^2 - \omega_D \sqrt{\omega_D^2 + \Delta^2} + \Delta^2 \arcsinh \frac{\omega_D}{\Delta} \right). \quad (56)$$

We can see, that the pairing leads to a loss in the kinetic energy.

B. Vacuum amplitude.

Since we took account of interaction of particles with fluctuations of pairing and we discovered that the ground state of a system $|\Psi_0\rangle$ has other symmetry as compared with the initial state $|\Phi_0\rangle$, hence the vacuum amplitude of a system can be written as:

$$R(t) = \langle \Psi_0 | U(t - t_0) | \Psi_0 \rangle |_{t_0 = 0} = \langle \Psi_0 | U(t) | \Psi_0 \rangle e^{i \mathcal{W} t}$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_0^t \cdots \int_0^t \int_0^t \langle \Psi_0 | T \left\{ \hat{H}_f(t_1) \cdots \hat{H}_f(t_n) \right\} | \Psi_0 \rangle, \quad (57)$$
\[
\hat{H}_0 + \hat{V} = \sum_{\alpha} \sum_{\mathbf{k}} E(k) C_{\mathbf{k},\alpha}^+ C_{\mathbf{k},\alpha} + \frac{1}{2V} \sum_{\alpha,\beta,\gamma,\delta} \sum_{\mathbf{k},\mathbf{l},\mathbf{m},\mathbf{n}} V_{\mathbf{klmn}} C_{\mathbf{l},\beta,\gamma}^+ C_{\mathbf{k},\alpha}^+ C_{\mathbf{m},\gamma} C_{\mathbf{n},\delta},
\]

where momentum is conserved \( \mathbf{k} + \mathbf{l} = \mathbf{m} + \mathbf{n} \) and spin is conserved \( \alpha + \beta = \gamma + \delta \). \( V \) is volume of a system. The sequence order of indices of matrix elements and of creation and annihilation operators is important.

The expressions for several first orders \( (n = 0, 1, 2, ...) \) in the expansion of vacuum amplitude are (let us suppose \( t_2 > t_1 \) for definiteness):

\[
R_0(t) = \langle \Psi_0 | \Psi_0 \rangle = 1
\]

\[
R_1(t) = \frac{1}{1!} \frac{1}{V} \int_0^t dt_1 \sum_{\alpha,\beta} \sum_{\mathbf{k},\mathbf{l},\mathbf{m},\mathbf{n}} \left( -\frac{i}{2} V_{\mathbf{klmn}} \right) \langle \Psi_0 | C_{\mathbf{l},\beta}^+ (t_1) C_{\mathbf{k},\alpha} (t_1) C_{\mathbf{m},\gamma} (t_1) C_{\mathbf{n},\delta} (t_1) | \Psi_0 \rangle
\]

\[
R_2(t) = \frac{1}{2!} \frac{1}{V^2} \int_0^t dt_1 \int_0^{t_1} dt_2 \sum_{\alpha,\beta,\gamma,\delta} \sum_{\mathbf{k},\mathbf{l},\mathbf{m},\mathbf{n}} \left( -\frac{i}{2} V_{\mathbf{klmn}} \right) \sum_{\alpha',\gamma',\delta'} \sum_{\mathbf{k}',\mathbf{l}',\mathbf{m}',\mathbf{n}'} \left( -\frac{i}{2} V_{\mathbf{k}'\mathbf{l}'\mathbf{m}'\mathbf{n}'} \right)
\times \langle \Psi_0 | C_{\mathbf{l}',\gamma'}^+ (t_2) C_{\mathbf{k},\alpha'}^+ (t_2) C_{\mathbf{m}',\gamma'} (t_2) C_{\mathbf{n}',\delta'} (t_2) C_{\mathbf{l},\beta}^+ (t_1) C_{\mathbf{k},\alpha}^+ (t_1) C_{\mathbf{m},\gamma} (t_1) C_{\mathbf{n},\delta} (t_1) | \Psi_0 \rangle + ... 
\]

where \( C^+(t) = e^{i\hat{H}_0 t} C e^{-i\hat{H}_0 t} \) \( C(t) = e^{i\hat{H}_0 t} C e^{-i\hat{H}_0 t} \) are operators of creation and annihilation in Heisenberg representation, that coincides with interaction representation for an ensemble of noninteracting particles.

In Appendix B we propose the method of uncoupling of correlations for approximate calculation of a vacuum amplitude \( R(t) \). As an example Hartree-Fock normal processes have been considered there. We shall generalize this method for anomalous processes here. In the case the particles interact by the potential \[ V_{-\mathbf{1},\mathbf{-k},-\mathbf{k}} \]. Hence, the vacuum amplitude has a form:

\[
R(t) = 1 + \frac{1}{1!} \frac{1}{V} \int_0^t dt_1 \sum_{\alpha,\beta} \sum_{\mathbf{k},\mathbf{l}} \left( -\frac{i}{2} V_{\mathbf{l},-\mathbf{1},-\mathbf{k},-\mathbf{k}} \right) \langle \Psi_0 | C_{\mathbf{l},\beta}^+ (t_1) C_{\mathbf{k},\alpha}^+ (t_1) C_{\mathbf{-1},\alpha} (t_1) C_{\mathbf{-1},\beta} (t_1) | \Psi_0 \rangle
\]

\[
+ \frac{1}{2!} \frac{1}{V^2} \int_0^t dt_2 \int_0^{t_1} dt_1 \sum_{\alpha,\beta} \sum_{\mathbf{k},\mathbf{l}} \sum_{\alpha',\gamma'} \sum_{\mathbf{k}',\mathbf{l}'} \left( -\frac{i}{2} V_{\mathbf{l}',-\mathbf{1},-\mathbf{k}',-\mathbf{k}'} \right) \times \langle \Psi_0 | C_{\mathbf{l}',\gamma'}^+ (t_2) C_{\mathbf{k},\alpha'}^+ (t_2) C_{\mathbf{k}',\alpha'} (t_2) C_{\mathbf{-1},\beta}^+ (t_1) C_{\mathbf{k},\alpha}^+ (t_1) C_{\mathbf{-1},\beta} (t_1) | \Psi_0 \rangle + ...
\]

We can uncouple correlations by following way taking into account anticommutation of the operators \( C \) and \( C^+ \):

\[
R(t) \approx 1 + \frac{1}{1!} \frac{1}{V} \int_0^t dt_1 \sum_{\alpha,\beta} \sum_{\mathbf{k},\mathbf{l}} \left( -\frac{i}{2} V_{\mathbf{l},-\mathbf{1},-\mathbf{k},-\mathbf{k}} \right) \langle \Psi_0 | C_{\mathbf{k},\alpha}^+ (t_1) C_{\mathbf{k},\beta} (t_1) | \Psi_0 \rangle \langle \Psi_0 | C_{\mathbf{-1},\alpha} (t_1) C_{\mathbf{-1},\beta} (t_1) | \Psi_0 \rangle
\]

\[
+ \frac{1}{2!} \frac{1}{V^2} \int_0^t dt_2 \int_0^{t_1} dt_1 \sum_{\alpha,\beta} \sum_{\mathbf{k},\mathbf{l}} \sum_{\alpha',\gamma'} \sum_{\mathbf{k}',\mathbf{l}'} \left( -\frac{i}{2} V_{\mathbf{l}',-\mathbf{1},-\mathbf{k}',-\mathbf{k}'} \right) \times \langle \Psi_0 | C_{\mathbf{k},\alpha}^+ (t_2) C_{\mathbf{k},\beta} (t_2) | \Psi_0 \rangle \langle \Psi_0 | C_{\mathbf{-1},\alpha}^+ (t_1) C_{\mathbf{-1},\beta}^+ (t_1) | \Psi_0 \rangle \langle \Psi_0 | C_{\mathbf{1},\alpha} (t_1) C_{\mathbf{1},\beta} (t_1) | \Psi_0 \rangle
\]

\[
+ \ldots = 1 + R_1 + \frac{1}{2!} R_2^2 + ... = \exp(R_1)
\]

As a result of the uncoupling of correlations we can see, that anomalous processes \( CC \) and \( C^+ C^+ \) give contribution to the vacuum amplitude of a system with the interaction \[ V_{-\mathbf{1},\mathbf{-k},-\mathbf{k}} \] only. Another combinations of the uncoupling with an obtaining of normal propagator \( C^+ C \) don’t conserve a momentum. Such representation of the vacuum amplitude by uncoupled correlations is analogous to Fock approximation for normal processes, and it means a neglect of dynamic correlation between pairs. Then \( R(t) \) can be written as

\[
\ln R(t) = R_1(t) = \frac{1}{V} \int_0^t dt_1 \sum_{\alpha,\beta} \sum_{\mathbf{k},\mathbf{l}} \left( -\frac{i}{2} V_{-\mathbf{1},-\mathbf{k},-\mathbf{k}} \right) \frac{\Delta^{+\Delta} \Delta^{+\Delta}}{\Delta^{+\Delta}} \langle \Psi_0 | F_{\alpha\beta}^+(1, t_1 - t_1) \frac{\Delta^{+\Delta}}{\Delta^{+\Delta}} (-i) F_{\alpha\beta} (k, t_1 - t_1)
\]
transferring the state $\Psi_0$ is correct in the case of phase transition even.

The transition amplitude "vacuum-vacuum". We use the theorem because interaction of particle with pairing fluctuation, changing a symmetry of a system: $\Phi_0 \rightarrow \Phi_0, \lambda \rightarrow \lambda$. Then the interaction tries to reconstruct a system so, that the gap $\Delta$ is as much as possible. We can use the theorem because interaction of particle with pairing fluctuation, changing a symmetry of a system: $\Phi_0 \rightarrow \Phi_0, \lambda \rightarrow \lambda$. Then the interaction tries to reconstruct a system so, that the gap $\Delta$ is as much as possible. We can use the theorem because interaction of particle with pairing fluctuation, changing a symmetry of a system: $\Phi_0 \rightarrow \Phi_0, \lambda \rightarrow \lambda$. Then the interaction tries to reconstruct a system so, that the gap $\Delta$ is as much as possible. We can use the theorem because interaction of particle with pairing fluctuation, changing a symmetry of a system: $\Phi_0 \rightarrow \Phi_0, \lambda \rightarrow \lambda$. Then the interaction tries to reconstruct a system so, that the gap $\Delta$ is as much as possible. We can use the theorem because interaction of particle with pairing fluctuation, changing a symmetry of a system: $\Phi_0 \rightarrow \Phi_0, \lambda \rightarrow \lambda$. Then the interaction tries to reconstruct a system so, that the gap $\Delta$ is as much as possible.

Using the formulas (48) and (49) we can rewrite $R(t)$ in the following form:

$$\ln R(t) = -\frac{i\lambda}{V} \left( \frac{\nu F}{2} \right)^2 t \int_{-\omega_D}^{\omega_D} \sqrt{A(\varepsilon)B(\varepsilon)} \, d\varepsilon \int_{-\omega_D}^{\omega_D} \sqrt{A(\varepsilon)B(\varepsilon)} \, d\varepsilon = -i\lambda V \left( \frac{\nu F}{2} \right)^2 \Delta^2 \text{arcsinh} \left( \frac{\omega_D}{\Delta} \right) t. \quad (63)$$

This formula was obtained from such arguments. Since the area of action of the potential is limited by the layer $2\omega_D$ in a neighborhood of Fermi surface, then the amplitudes $\Delta$ and $\Delta^+$ is not equal to zero in this area only. As it was pointed before, presented manner of uncoupling of correlations is analogous to Fock exchange interaction for normal processes. In Hartree-Fock approximation a decay of quasi-particles is absent [1]. This means, that the amplitude of pairing is real: $\Delta = \Delta^+$ in a momentum space.

For calculation of the contribution of interaction to internal energy it is necessary to use the theorem (49), which connect a vacuum amplitude with a ground state energy:

$$\Omega_\lambda = \frac{d}{dt} \ln R(t) = -\frac{\lambda}{V} \sum_l \int \frac{d\omega}{2\pi} w_l F^+(1, \omega) \sum_k \int \frac{d\omega}{2\pi} w_k F(k, \omega)$$

$$= \lambda V \left( \frac{\nu F}{2} \right)^2 \int_{-\omega_D}^{\omega_D} \frac{\Delta}{2E} \, d\varepsilon \int_{-\omega_D}^{\omega_D} \frac{\Delta}{2E} \, d\varepsilon = \lambda V \left( \frac{\nu F}{2} \right)^2 \Delta^2 \text{arcsinh} \left( \frac{\omega_D}{\Delta} \right). \quad (64)$$

Since $\lambda < 0$, then the interaction tries to reconstruct a system so, that the gap $\Delta$ is as much as possible. We can use the theorem because interaction of particle with pairing fluctuation, changing a symmetry of a system: $\Phi_0 \rightarrow \Phi_0, \lambda \rightarrow \lambda$. Switching of the interaction $V_{1,-1k,-k}$, transferring the state $\Psi_0$ in some other $\Psi'_0$, doesn’t change symmetry of a system: $\langle \Psi_0 | \Psi'_0 \rangle \neq 0$. This means that the adiabatic hypothesis is correct in the case of phase transition even.
C. Internal energy.

On the basis of above obtained results we can write expression for internal energy of a system (at the temperature $T = 0$ internal energy coincides with free energy):

\[
\Omega = \langle W \rangle + \Omega_n = -2i \lim_{t \to 0} \sum_{k} \frac{d\omega}{2\pi} G(k, \omega) e^{-it\varepsilon(k)} - \frac{\lambda}{V} \sum_{k} \int \frac{d\omega}{2\pi} w_k F^+(k, \omega) \sum_{k} \int \frac{d\omega}{2\pi} w_k F(k, \omega). \tag{65}
\]

As it was pointed before, the gap is real $\Delta = \Delta^+$, then $F(k, \omega) = F^+(k, \omega)$. We can see, that the energy depends on the unknown amplitude of pairing $\Delta$, which corresponds to Bethe-Salpeter amplitude $\eta$ in the two-particle problem. The amplitude $\Delta$ is determined by dynamics of all particles of a system, and its observed value is a result of averaging over a system. The procedure of averaging means mathematically, that the observer value of $\Delta$ minimizes the internal energy:

\[
\frac{d\Omega}{d\Delta} = 0 \implies (-i)\Delta = \frac{\lambda}{V} \sum_{k} \int \frac{d\omega}{2\pi} w_k F(k, \omega), \tag{66}
\]

that coincides with (17). It means that the self-consistency equation for order parameter in Nambu-Gor’kov formalism is an extremal of the obtained free energy functional (65), and the order parameter is the averaged Bethe-Salpeter amplitude over a system.

The functional (65) can be written in an explicit form in quadratures:

\[
\Omega = \Omega_n - V \nu_F \frac{1}{2} \int_{-\omega_D}^{\omega_D} \left( \frac{\varepsilon^2}{E} - \frac{\varepsilon^2}{|\varepsilon|} \right) + V \nu_F \frac{1}{2} \int_{-\omega_D}^{\omega_D} \frac{\Delta}{2E} \frac{d\varepsilon}{dE}, \tag{67}
\]

where $\Omega_n$ is the energy of a normal phase, $g = \frac{\nu_F}{\nu}$ is the effective interaction constant. Value of the energy on the extremal (66) is

\[
\Omega_{\text{min}} = \Omega_n + V \nu_F \frac{1}{2} (\omega_D^2 - \omega_D) \sqrt{\omega_D^2 + \Delta^2}. \tag{68}
\]

Thus, we solved the basic problem of statistical mechanics (at zero temperature): the calculation of a partition function (free energy) and, in particular, of a vacuum amplitude in a system of interacting particles for case, when the interaction causes a phase transition, that is symmetry of a system changes at a switching of the interaction. Unlike other methods, this result was obtained from first principles without introducing any artificial parameters of type of order parameter, but starting from parameters of the Hamiltonian only.

V. NONZERO TEMPERATURES.

A. Normal and anomalous propagators.

In the sections III and IV we have described microscopic mechanism of formation of long-range order in a system at zero temperature. In this section we shall formulate the approach for case of nonzero temperatures. Let we have a system from $N$ noninteracting fermions being in volume $V$ at temperature $T$. Then we must use Matsubara propagators, where time $t$ is complex: $t \to -i\tau$, $\tau \in [0, \beta]$. In ideal Fermi gas propagation of a particle with momentum $k$, energy $\varepsilon \approx v_F(|k| - k_F)$ and spin $\sigma$ is described by the free propagator:

\[
G_0(k, \tau) = \begin{cases} 
- \frac{i}{\omega_n - \varepsilon(k)} & \text{for } \tau > 0 \\
\frac{i}{\omega_n + \varepsilon(k)} & \text{for } \tau < 0 
\end{cases},
\]

\[
G(k, \tau) = \frac{1}{\beta} \sum_{n=-\infty}^{n=+\infty} G(k, \omega_n) e^{-i\omega_n \tau}, \quad G(k, \omega_n) = \frac{1}{2} \int_{-\beta}^{\beta} G(k, \tau) e^{i\omega_n \tau} d\tau
\]

\[
G_0(k, \omega_n) = \frac{i}{i\omega_n - \varepsilon(k)} = \frac{i}{i\omega_n + \varepsilon(k)} = \frac{i}{\omega_n - |\varepsilon|} + \frac{i}{\omega_n + |\varepsilon|}, \tag{69}
\]
where
\[ g_0^+ = \frac{1}{e^{-|c|\beta} + 1}, \quad g_0^- = \frac{1}{e^{c|\beta} + 1}, \quad \omega_n = \frac{(2n+1)\pi}{\beta}, \] (70)

\( C_{k,\sigma}(\tau) \) and \( C_{k,\sigma}^+(\tau) \) are operators of creation and annihilation in Heisenberg representation. \( \hat{\rho}_0 \) is density matrix of noninteracting particles:
\[ \hat{\rho}_0 = \exp \left\{ \Omega - \hat{H}_0 + \mu \hat{N} \right\} = \exp \left\{ \frac{\Omega - \sum_{k,\sigma} \frac{k^2}{2m} C_{k,\sigma} C_{k,\sigma}^+ + \mu \hat{N}}{T} \right\} = \exp \left\{ \frac{\Omega - \sum_{k,\sigma} \varepsilon(k) C_{k,\sigma} C_{k,\sigma}^+}{T} \right\}, \] (71)

where \( \varepsilon(k) = \frac{k^2}{2m} - \mu \approx v_F (k - k_F) \) is kinetic energy of particles counted off from Fermi surface.

Now let an attracting force acts between particles. The force is described by the matrix element of interaction (24). In this case the instability of a system appears again with regard to a pairing (as in the section II) and \( \Gamma \)-matrix has a pole structure:
\[ \Gamma(0, 0) = \frac{\lambda}{1 + \frac{mk_F}{\pi} \ln \frac{2\omega_D}{\omega}} \approx -\frac{2\pi^2}{mk_F} \frac{T_C}{T - T_C}, \quad T_C = 2\gamma \frac{\omega_D}{\pi} \left( 1 - \frac{1}{2|\lambda|/v_F} \right), \] (72)

where \( \nu_F = \frac{mk_F}{\pi} \) is density of states on Fermi surface. We can see that bound states exist in a system while temperature is not higher than critical temperature \( T_C \) - at higher temperatures particles have large kinetic energy, so that an attraction between them leads to a scattering only.

Generalization of the two particle problem on the multiparticle case is done analogously to the section III. In the case of nonzero temperature the mass operator has a form:
\[ -\Sigma(k, \omega_n) = (-\Delta) iG_0^+ (-k, \omega_n)(-\Delta^+) = \frac{-\Delta \Delta^+}{i\omega_n + \varepsilon(k)}. \] (73)

Then it follows from Dyson equation, that a dressed propagator has a view:
\[ \frac{1}{G_0} = \frac{1}{G_S} - i\Sigma \Rightarrow G_S(k, \omega_n) = \frac{i}{\omega_n - \varepsilon(k) - \Sigma(k, \omega_n)} = \frac{i}{i\omega_n + \varepsilon} \left( \frac{A_S}{i\omega_n - |\varepsilon|} + \frac{B_S}{i\omega_n + |\varepsilon|} \right). \] (74)

It can be written with help of a total definition of Green function in \((k, t)\)-space:
\[ G_S(k, \tau) = \tau_2 - \tau_1 = \begin{cases} -i\text{Sp} \left\{ \hat{\varrho}_0 C_{k,\sigma}(\tau_2) C_{k,\sigma}^+(\tau_1) \right\}, & \tau > 0 \\ i\text{Sp} \left\{ \hat{\varrho}_0 C_{k,\sigma}^+(\tau_1) C_{k,\sigma}(\tau_2) \right\}, & \tau \leq 0 \end{cases} \]
\[ = -i\theta_\tau (g_S^+ A_S e^{-E\tau} + g_S B_S e^{E\tau}) + i\theta_\tau (g_S A_S e^{-E\tau} + g_S^+ B_S e^{E\tau}), \] (75)

where
\[ g_S^+ = \frac{1}{e^{-E\beta} + 1}, \quad g_S^- = \frac{1}{e^{E\beta} + 1}. \] (76)
\( \hat{\varrho}_0 \) is the density matrix of noninteracting quasi-particles:
\[ \hat{\varrho}_0 = \exp \left\{ \frac{\Omega - \sum_{k,\sigma} E(k) C_{k,\sigma} C_{k,\sigma}^+}{T} \right\}. \] (77)

It should be noted, that the state described by the density matrix \( \rho_0 \) has another symmetry in comparison with the initial state \( \rho_0 \). Occupation numbers \( n(k) \) is determined by the following manner:
\[ n(k) = -i \lim_{\tau \to 0^-} G(k, \tau) = g^- A + g^+ B, \quad \lim_{T \to 0} n(k) = B(k). \] (78)

Let’s introduce the designations:
\[ -G\Sigma = \Delta F^+, \quad -G^+\Sigma^+ = \Delta^+ F.. \] (79)
Then Dyson equation can be rewritten in a form of Gor’kov equations:

\[(i\omega_n - \varepsilon)G + \Delta F^+ = i\]
\[(i\omega_n + \varepsilon)F^+ + G\Delta = 0.\]

The expressions for anomalous propagators follow from Gor’kov equations:

\[F^+(k, \omega_n) = \frac{-i\Delta^+}{(i\omega_n)^2 - E^2(k)}, \quad F(k, \omega_n) = (F^+(k, \omega_n))^+ = \frac{i\Delta}{(i\omega_n)^2 - E^2(k)}.\]

These expressions are analogous to the expressions (47). We can write the anomalous propagators in a form of a vacuum average of creation and annihilation operators:

\[F_{\alpha\beta}(k, \tau) = \frac{\Delta^+}{\sqrt{\Delta^+\Delta}} \left\{ \begin{array}{ll}
i\text{Sp}\{\hat{g}_0C_{-k,\beta}^{\dagger}(\tau_2)C_{k,\alpha}^{\dagger}(\tau_1)\}, & \tau > 0 \\
i\text{Sp}\{\hat{g}_0C_k^{\dagger}(\tau_1)C_{-k,\beta}^{\dagger}(\tau_2)\}, & \tau \leq 0 \end{array} \right\} \]
\[= ig_{\alpha\beta}\frac{\Delta^+}{\sqrt{\Delta^+\Delta}}\sqrt{A_SBS} \left\{ \begin{array}{ll}
(g^+_S e^{-E\tau} - g^-_Se^{E\tau})\theta_\tau - (g^+_Se^{E\tau} - g^-_Se^{-E\tau})\theta_{-\tau}\right\}, \quad \tau > 0\]
\[F_{\alpha\beta}(k, \tau) = \frac{\Delta}{\sqrt{\Delta^+\Delta}} \left\{ \begin{array}{ll}
-i\text{Sp}\{\hat{g}_0C_{k,\alpha}^{\dagger}(\tau_2)C_{-k,\beta}^{\dagger}(\tau_1)\}, & \tau > 0 \\
-i\text{Sp}\{\hat{g}_0C_{-k,\alpha}^{\dagger}(\tau_1)C_{k,\beta}^{\dagger}(\tau_2)\}, & \tau \leq 0 \end{array} \right\} \]
\[= ig_{\alpha\beta}\frac{\Delta}{\sqrt{\Delta^+\Delta}}\sqrt{A_SBS} \left\{ -(g^+_S e^{-E\tau} - g^-_Se^{E\tau})\theta_\tau + (g^+_Se^{E\tau} - g^-_Se^{-E\tau})\theta_{-\tau}\right\}, \quad \tau \leq 0\]

that is analogous to the expressions (48) and (49).

### B. Kinetic energy and entropy.

In order to calculate a free energy it is necessary to know kinetic energy of particles of a system, energy of their interaction and entropy of collective excitations. Average kinetic energy of all particles of a system is

\[\langle W \rangle = -2i \sum_k G(k, \tau \rightarrow 0^-)\varepsilon(k) = -\frac{2i}{\beta} \lim_{\tau \rightarrow 0^-} \sum_{k \in [-\omega_D, \omega_D]} \sum_{n=-\infty}^{n=+\infty} G(k, \omega_n)e^{-i\omega_n t}\varepsilon(k) \]
\[= 2 \sum_k (g^- A + g^+ B)\varepsilon(k) = V\nu_F \int_{-\nu_F k_F}^{\nu_F k_F} (g^- A + g^+ B)\varepsilon d\varepsilon.\]

Since the interaction $V_{1-1k-k}$ exists only in the layer $-\omega_D < \varepsilon(k) < \omega_D$, then we can suppose that

\[g^- = \left[ g^-_0; \varepsilon(k) > \omega_D \right] \quad g^+ = \left[ g^+_0; \varepsilon(k) > \omega_D \right].\]

Hence, one may write expression for kinetic energy separating normal and superconductive parts by analogy (56):

\[\langle W \rangle = W_n - V\nu_F \int_{-\omega_D}^{\omega_D} (g^-_0 - g^+_0)\frac{\varepsilon^2}{|\varepsilon|} d\varepsilon + V\nu_F \int_{-\omega_D}^{\omega_D} (g^-_0 - g^-_0)\frac{\varepsilon^2}{|\varepsilon|} d\varepsilon \]
\[= W_n + V\nu_F \int_{-\omega_D}^{\omega_D} \tanh\left(\frac{\beta|\varepsilon|}{2}\right)\frac{\varepsilon^2}{|\varepsilon|} d\varepsilon - V\nu_F \int_{-\omega_D}^{\omega_D} \tanh\left(\frac{\beta|\varepsilon|}{2}\right)\frac{\varepsilon^2}{|\varepsilon|} d\varepsilon. \]

In the limit of low temperatures $\beta \rightarrow \infty$ this expression reduces to the expression (56). If we suppose that $\Delta = 0$, then we shall have $W = W_n$.

At temperature $T \neq 0$ a gas of collective excitation exists - boholons with the spectrum $E = \sqrt{\varepsilon^2(k) + \Delta^2}$. Since boholons are product of decay of Cooper pairs on fermions, hence occupation numbers of states by boholons are

\[f_S(k) = \frac{1}{e^{\beta E} + 1} = \frac{1}{2} \left( 1 - \tanh\left(\frac{\beta E}{2}\right) \right).\]
Then entropy of a system is
\[
S = -2 \sum_k [f(k) \ln f(k) + (1 - f(k)) \ln (1 - f(k))]
\]
\[
= S_0 - 2V \frac{\nu F}{2} \int_{-\omega_D}^{\omega_D} [f_S \ln f_S + (1 - f_S) \ln (1 - f_S)] \, dz + 2V \frac{\nu F}{2} \int_{-\omega_D}^{\omega_D} [f_0 \ln f_0 + (1 - f_0) \ln (1 - f_0)] \, dz. \quad (89)
\]

Here we separated the normal part again, where \( f_0 = (e^{\beta |\epsilon|} + 1)^{-1} \), so that \( S = S_n \) at \( \Delta = 0 \). The multiplier "2" appeared as result of summation over spin states.

### C. Vacuum amplitude.

In the previous subsections we considered the interaction of particles with fluctuations of pairing, and we found, that the state of a system described by the density matrix \( \hat{\rho}_0 \) has another symmetry in comparison with the initial state \( \hat{\rho}_0 \). Hence vacuum amplitude of a system can be written in a form:

\[
R(\beta) = \langle \tilde{U}(\beta) \rangle_0 = S_p \left( \hat{\rho}_0 \tilde{U}(\beta) \right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^\beta \! d\tau_1 \cdots \int_0^\beta \! d\tau_n \! S_p \left( \hat{\rho}_0 T \left\{ \hat{H}_1(\tau_1) \cdots \hat{H}_1(\tau_n) \right\} \right), \quad (90)
\]

where \( \hat{H}_1(\tau) = e^{+\tau \hat{h}_0} \tilde{V} e^{-\tau \hat{h}_0} \) is the interaction operator of particles in interaction representation. The averaging \( S_p \left( \hat{\rho}_0 U(\beta) \right) \) is made over ensemble of noninteracting quasi-particles. The potential \( V_{-1,k,-k} \) acts between particles \( \alpha \). Hence we can write (by analogy with (60)) expended expression for vacuum amplitude:

\[
R(\beta) = 1 + \frac{1}{11V} \int_0^\beta \! d\tau_1 \sum_{\alpha,\gamma,k,l} \left( -\frac{1}{2} V_{-1,k,-k} \right) \! S_p \left( \hat{\rho}_0 C_{-1,\gamma}(\tau_1) C^+_{1,\alpha}(\tau_1) C_{k,\alpha}(\tau_1) C_{-k,\gamma}(\tau_1) \right)
\]
\[
+ \frac{1}{21V^2} \int_0^\beta \! d\tau_2 \int_0^\beta \! d\tau_1 \sum_{\alpha,\gamma,k,l} \left( -\frac{1}{2} V_{-1,k,-k} \right) \sum_{\alpha',\gamma',k',l'} \left( -\frac{1}{2} V_{-1,k',-k'} \right)
\]
\[
\times S_p \left( \hat{\rho}_0 C_{-1,\gamma'}(\tau_2) C^+_{1,\alpha'}(\tau_2) C_{k',\alpha'}(\tau_2) C_{-k',\gamma'}(\tau_2) C_{-1,\gamma}(\tau_1) C^+_{1,\alpha}(\tau_1) C_{k,\alpha}(\tau_1) C_{-k,\gamma}(\tau_1) \right)
\]
\[
+ \ldots = 1 + R_1 + \frac{1}{2!} R_2 + \ldots = \exp(R_1) \quad (91)
\]

where we took into account that \( S_p \{ \hat{\rho}_0 \} = 1 \). Then, by analogy with the equation (89), we can uncouple correlations by the following way taking into account anticommutation of operators \( C \) and \( C^+ \):

\[
R(\beta) \approx 1 + \frac{1}{11V} \int_0^\beta \! d\tau_1 \sum_{\alpha,\gamma,k,l} \left( -\frac{1}{2} V_{-1,k,-k} \right) \! S_p \left( \hat{\rho}_0 C^+_{1,\alpha}(\tau_1) C_{-1,\gamma}(\tau_1) \right) S_p \left( \hat{\rho}_0 C_{-k,\gamma}(\tau_1) C_{k,\alpha}(\tau_1) \right)
\]
\[
+ \frac{1}{21V^2} \int_0^\beta \! d\tau_2 \int_0^\beta \! d\tau_1 \sum_{\alpha,\gamma,k,l} \left( -\frac{1}{2} V_{-1,k,-k} \right) \sum_{\alpha',\gamma',k',l'} \left( -\frac{1}{2} V_{-1,k',-k'} \right)
\]
\[
\times S_p \left( \hat{\rho}_0 C^+_{1,\alpha'}(\tau_2) C_{-1,\gamma'}(\tau_2) \right) S_p \left( \hat{\rho}_0 C_{-k',\gamma'}(\tau_2) C_{k',\alpha'}(\tau_2) \right) S_p \left( \hat{\rho}_0 C_{-k,\gamma}(\tau_1) C^+_{1,\alpha}(\tau_1) \right) S_p \left( \hat{\rho}_0 C_{k,\alpha}(\tau_1) C_{k,\alpha}(\tau_1) \right)
\]
\[
+ \ldots = 1 + R_1 + \frac{1}{2!} R_2^2 + \ldots = \exp(R_1) \quad (92)
\]

Let’s take into account that our approximation is analogous to Fock approximation for normal processes. Then we can suppose \( \Delta = \Delta^+ \), hence \( F = -F^+ \). Then \( R(t) \) can be written as

\[
\ln R(\beta) = R_1(\beta) = \frac{1}{V} \int_0^\beta \! d\tau_1 \sum_{\alpha,\gamma,k,l} \left( -\frac{1}{2} V_{-1,k,-k} \right) i F_{\alpha\gamma}(1, \tau_1 - \tau_1) i F_{\alpha\beta}(k, \tau_1 - \tau_1)
\]
\[
= \frac{2}{V} \sum_{k,l} \left( -\frac{1}{2} V_{-1,k,-k} \right) F(1, \tau \to 0^{-}) F(k, \tau \to 0^{-}) \beta
\]
\[
= \frac{\beta \lambda}{V} \sum_{k} w_k \frac{1}{\beta} \sum_{n=-\infty}^{n=+\infty} F(k, \omega_n) \sum_{k} w_k \frac{1}{\beta} \sum_{n=-\infty}^{n=+\infty} F(k, \omega_n)
\]
\[
= -\beta \lambda V \left( \frac{\nu_F}{2} \right)^2 \int_{-\omega_D}^{\omega_D} \tanh \left( \frac{\beta E}{2} \right) \frac{\Delta E}{2E} d\varepsilon \int_{-\omega_D}^{\omega_D} \tanh \left( \frac{\beta E}{2} \right) \frac{\Delta E}{2E} d\varepsilon.
\] (93)

This equation is represented graphically as well as in Fig. 6. In order to calculate a contribution of interaction in free energy we can use the formula (10):

\[
\Omega_{\lambda} = -\frac{1}{\beta} \ln R(\beta) = -\frac{\lambda}{V} \sum_{k} w_k \frac{1}{\beta} \sum_{n=-\infty}^{n=+\infty} F(k, \omega_n) \sum_{k} w_k \frac{1}{\beta} \sum_{n=-\infty}^{n=+\infty} F(k, \omega_n)
\]
\[
= \lambda V \left( \frac{\nu_F}{2} \right)^2 \int_{-\omega_D}^{\omega_D} \tanh \left( \frac{\beta E}{2} \right) \frac{\Delta E}{2E} d\varepsilon \int_{-\omega_D}^{\omega_D} \tanh \left( \frac{\beta E}{2} \right) \frac{\Delta E}{2E} d\varepsilon.
\] (94)

In the limit of low temperatures \( \beta \to \infty \) this equation transforms to the equation (14). If we suppose that \( \Delta = 0 \), then we shall have \( \Omega_{\lambda} = 0 \).

\[\text{D. Free energy.}\]

Starting from the above found results we can write the expression a for free energy of a system:

\[
\Omega = \langle W \rangle - \frac{1}{\beta} S + \Omega_{\lambda} = -\frac{2i}{\beta} \lim_{\tau \to 0} \sum_{k} \sum_{n=-\infty}^{n=+\infty} G(k, \omega_n) e^{-i\omega_n \tau} \varepsilon(k)
\]
\[+ \frac{2}{\beta} \sum_{k} [f(k) \ln f(k) + (1 - f(k)) \ln(1 - f(k))] - \frac{\lambda}{V} \sum_{k} w_k \frac{1}{\beta} \sum_{n=-\infty}^{n=+\infty} F(k, \omega_n) \sum_{k} w_k \frac{1}{\beta} \sum_{n=-\infty}^{n=+\infty} F(k, \omega_n)
\] (95)

We can see, that the energy depends on the unknown amplitude of pairing \( \Delta \), which corresponds to Bethe-Salpeter amplitude \( \eta \) in two-particle problem. The observed value of \( \Delta \) must minimize the free energy:

\[
\frac{d\Omega}{d\Delta} = 0 \Rightarrow (-i) \Delta = \frac{\lambda}{V^\beta} \sum_{k} w_k F(k, \omega_n),
\] (96)

that coincides with (17). As in the previous section the equation of a self-consistence for order parameter in Nambu-Gor’kov formalism is the extremal of the functional of free energy (95), and the order parameter is averaged Bethe-Salpeter amplitude over a system.

The functional (95) can be written in an explicit form in quadratures:

\[
\Omega = \Omega_n + V \frac{\nu_F}{2} \int_{-\omega_D}^{\omega_D} \left[ \tanh \left( \frac{\beta |\varepsilon|}{2} \right) \frac{\varepsilon^2}{|\varepsilon|} - \tanh \left( \frac{\beta E}{2} \right) \frac{\varepsilon^2}{E} \right] d\varepsilon
\]
\[+ \frac{2V}{\beta} \frac{\nu_F}{2} \int_{-\omega_D}^{\omega_D} [f_s \ln f_s + (1 - f_s) \ln(1 - f_s)] - f_0 \ln f_0 - (1 - f_0) \ln(1 - f_0)] d\varepsilon
\]
\[+ V \frac{\nu_F}{2} g \int_{-\omega_D}^{\omega_D} \tanh \left( \frac{\beta E}{2} \right) \frac{\Delta E}{2E} d\varepsilon \int_{-\omega_D}^{\omega_D} \tanh \left( \frac{\beta E}{2} \right) \frac{\Delta E}{2E} d\varepsilon,
\] (97)

where \( \Omega_n \) is the energy of a normal phase, \( g = \frac{\lambda V^2}{\beta} \) is the effective interaction constant, \( V \) is volume of a system. \( g \)
can be expressed via critical temperature \( \beta_C \) with help of the equation \( \Delta(\beta_C) = 0 \) as following:

\[1 = -g \int_{-\omega_D}^{\omega_D} \tanh \left( \frac{\beta_C |\varepsilon|}{2} \right) \frac{1}{2|\varepsilon|} d\varepsilon.
\] (98)

If to suppose \( \Delta = 0 \), then we shall have \( \Omega = \Omega_n \). The equilibrium value of \( \Delta \) is determined by balance of kinetic energy, entropy and energy of interaction, that corresponds to a minimum of the free energy.

Let’s consider a low-temperature limit of the free energy (96): \( \Delta \beta \gg 1 \) at \( T \to 0 \). This means, that the value \( \Delta - \Delta_0 \) can be parameter of expansion, where \( \Delta_0 = \Delta(T = 0) \) is equilibrium value of the gap (amplitude of pairing) at
Then expressions (100) are simplified:

The corresponding mass operator describing an interaction of particles with fluctuations of pairing is:

\[ \Omega = \Omega_n + V (\alpha_0(T) + b_0(T)\Delta + d_0\Delta^2), \tag{99} \]

where coefficients of the expansion are

\[
\alpha_0(T) = -\frac{\nu_F}{4}\Delta_0^2 + 2\nu_F T^2 - \frac{\nu_F}{2}\sqrt{8\pi\Delta_0} T^3 e^{-\frac{\Delta_0}{T}} + \frac{\nu_F}{2}(g+1)\Delta_0^2 - \frac{\nu_F}{2}(g+1)\sqrt{8\pi\Delta_0^3} T e^{-\frac{\Delta_0}{T}}
\]

\[
b_0(T) = -\frac{\nu_F}{2}(g+1)\Delta_0 + \frac{\nu_F}{2}(g+1)\sqrt{8\pi\Delta_0} T e^{-\frac{\Delta_0}{T}}, \quad d_0 = \frac{\nu_F}{2}2(g+1)
\tag{100}
\]

Since in the expression \(\frac{b_0(T)}{2d_0}\) the multipliers \(g+1\) is cancelled and, as a rule, \(|g| \ll 1\), we can suppose that \(g+1 \approx 1\). Then expressions (100) are simplified:

\[
\alpha_0(T) = \frac{\nu_F}{4}\Delta_0^2 + 2\nu_F T^2 - \frac{\nu_F}{2}\sqrt{8\pi\Delta_0} T^3 e^{-\frac{\Delta_0}{T}} + \frac{\nu_F}{2}\sqrt{8\pi\Delta_0^3} T e^{-\frac{\Delta_0}{T}}
\]

\[
b_0(T) = -\frac{\nu_F}{2}2\Delta_0 + \frac{\nu_F}{2}\sqrt{8\pi\Delta_0} T e^{-\frac{\Delta_0}{T}}, \quad d_0 = \nu_F
\tag{101}
\]

Let’s consider a high-temperature limit of the free energy: \(\Delta \beta_c \ll 1\) at \(T \to T_c\). Expansion in powers of \(\Delta\) gives:

\[ \Omega = \Omega_n + V \left( \alpha(T)\Delta^2 + \frac{1}{2}b\Delta^4 + \frac{1}{3}d\Delta^6 \right), \tag{102} \]

where the coefficients of the expansion are

\[
\alpha(T) = \frac{\nu_F}{2} \frac{T - T_c}{T_c}
\]

\[
b = \frac{\nu_F}{2} \frac{7\zeta(3)}{8\pi^2 T_c^2}
\]

\[
d = \frac{\nu_F}{2} \left( \frac{52.31\zeta(5)}{\pi^4} + 4.83 \right) \frac{1}{4T_c^4}
\tag{103}
\]

This expansion has a form of Landau expansion of free energy in powers of order parameter.

From the all considered above we can see, that averaged over a system Bethe-Salpeter amplitude \(\eta\) and \(\eta^*\) - the amplitude of pairing \(\Delta\) and \(\Delta^+\) have the properties, which are analogous to the properties of a order parameter. Thus, we have solved the basic problem of statistical mechanics: the calculation of a partition function (free energy) and, in particular, of a vacuum amplitude in a system of interacting particles for case, when the interaction causes a phase transition, that is symmetry of a system changes at a switching of the interaction.

**VI. THE PAIRING WITH NONZERO MOMENTUM OF CENTER OF MASS OF A PAIR.**

Let fermions with momentums \(k + \frac{q}{2}\) and \(-k + \frac{q}{2}\) pair up, so that the momentum of a center of mass of a pair is \(q\). The free propagators, corresponding to these states are

\[
G_0 \left( k + \frac{q}{2}, \omega \right) = \frac{1}{\omega - \varepsilon \left( k + \frac{q}{2} \right)} = \frac{1}{\omega - \varepsilon_+}
\]

\[
G_0 \left( -k + \frac{q}{2}, -\omega \right) = \frac{1}{-\omega - \varepsilon \left( -k + \frac{q}{2} \right)} = \frac{1}{-\omega - \varepsilon_-}
\tag{104}
\]

We can suppose that

\[
\varepsilon_+ = \frac{1}{2m} \left( k + \frac{q}{2} \right)^2 - \mu \approx \varepsilon + \frac{2kq}{2m}, \quad \varepsilon_- = \frac{1}{2m} \left( -k + \frac{q}{2} \right)^2 - \mu \approx \varepsilon - \frac{2kq}{2m}
\tag{105}
\]

The corresponding mass operator describing an interaction of particles with fluctuations of pairing is:

\[
(-i)\Sigma_q = -i\Delta iG_0 \left( -k + \frac{q}{2}, -\omega \right) i\Delta^+ = (-i) \frac{\Delta \Delta^+}{\omega + \varepsilon_-}
\tag{106}
\]
Then can find the dressed propagator from Dyson equation:
\[ G_{S} = \frac{1}{G_{0}^{-1} - \Sigma_{q}} = \frac{\omega + \varepsilon_{-}}{(\omega - E_{+})(\omega - E_{-})}, \]

(107)

where the specters of quasi-particles are
\[
E_{+} = \frac{\varepsilon_{+} - \varepsilon_{-}}{2} + \sqrt{\left(\frac{\varepsilon_{+} + \varepsilon_{-}}{2}\right)^{2} + |\Delta|^{2}} \approx \frac{kq}{2m} + \sqrt{\varepsilon^{2} + |\Delta|^{2}}
\]

\[
E_{-} = \frac{\varepsilon_{+} - \varepsilon_{-}}{2} - \sqrt{\left(\frac{\varepsilon_{+} + \varepsilon_{-}}{2}\right)^{2} + |\Delta|^{2}} \approx \frac{kq}{2m} - \sqrt{\varepsilon^{2} + |\Delta|^{2}}.
\]

(108)

We can see, that if to assume \(q = 0\), then the specter turn into the usual specter of boholons: \(E = \pm \sqrt{\varepsilon^{2} + |\Delta|^{2}}\). The critical momentum \(q_{cr}\) exists when a minimum of the specter touches Fermi surface. Then for excitation of a system it is necessary infinitely small energy. This means, that superfluidity of Fermi gas is absent. The critical momentum is:
\[
E_{+}(q = q_{cr}, \varepsilon = 0) = E_{-}(q = q_{cr}, \varepsilon = 0) = 0 \Rightarrow q_{cr} = \frac{2}{v_{F}}|\Delta|.
\]

(109)

Similar pairing can take place in high-temperature superconductors (cuprates) with a mirror nesting \(\varepsilon \left( \mathbf{k} + \frac{\mathbf{q}}{2} \right) = \varepsilon \left( -\mathbf{k} + \frac{\mathbf{q}}{2} \right)\) of regions of Fermi contour \([26, 27] \).

**VII. FREE ENERGY IN A CASE OF SLOW SPATIAL INHOMOGENEITY.**

In the previous sections we supposed, that amplitudes of pairing \(\Delta\) and \(\Delta^{+}\) don’t depend on spatial coordinates. This takes place in interminable, homogeneous, isotropic and isolated from external fields superconductor. However in a total case these conditions are not realized. For example, in a sufficiently strong magnetic field the inclusions of normal phase can exist in volume of a superconductor. Another example - a contact of a superconductor and a normal metal. In this case the order parameter is suppressed in a boundary layer of a superconductor, however it appears in boundary layer of a normal metal.

Let’s consider some region of a superconductor, where a distribution of \(\Delta\) is inhomogeneous and the amplitude of pairing can be both smaller and larger than its equilibrium value \(\Delta_{0}\) - Fig[4] When a quasi-particle propagates along the axis Ox its energy is constant \(E = \sqrt{\varepsilon(k)^{2} + |\Delta|^{2}}\), but its momentum changes. If a pair moves into region, where \(|\Delta| < |\Delta_{0}|\), then forces appear tearing the pair. Each element of the pair gets some increment of momentum \(q\): \((\mathbf{k}, -\mathbf{k}) \rightarrow (\mathbf{k} + \mathbf{q}, -\mathbf{k} - \mathbf{q})\). If the pair moves into region, where \(|\Delta| > |\Delta_{0}|\), then a forces appear increasing a bound energy of the pair. Each element of the pair gets some increment of momentum \(q\) too. Since the amplitude of pairing is function of coordinates \(\Delta(\mathbf{r})\), moreover we suppose that order parameter is real \(\Delta = \Delta^{+}\) in momentum space and it has an identical dimension in q-space and in r-space, then we can write the Fourier-transformations:
\[
\Delta(\mathbf{r}) = \sum_{\mathbf{q}} \Delta(q)e^{i\mathbf{q}\cdot\mathbf{r}} = \frac{V}{(2\pi)^{3}} \int \Delta(q)e^{i\mathbf{q}\cdot\mathbf{r}}d^{3}q,
\]

\[
\Delta^{+}(\mathbf{r}) = \sum_{\mathbf{q}} \Delta(q)e^{-i\mathbf{q}\cdot\mathbf{r}} = \frac{V}{(2\pi)^{3}} \int \Delta(q)e^{-i\mathbf{q}\cdot\mathbf{r}}d^{3}q,
\]

\[
\Delta(q) = \frac{1}{V} \int \Delta(\mathbf{r})e^{-i\mathbf{q}\cdot\mathbf{r}}d^{3}r = \frac{1}{V} \int \Delta^{+}(\mathbf{r})e^{i\mathbf{q}\cdot\mathbf{r}}d^{3}r.
\]

(110)

The mass operator for above-mentioned process is shown in Fig[7] In analytical representation it has a view:
\[
-\Sigma_{q}(\mathbf{k}, \omega_{n}) = (-\Delta_{q})iG_{0}^{+}(-\mathbf{k} - \mathbf{q}, \omega_{n})(-\Delta_{q}^{+}) = \frac{-\Delta_{q}\Delta_{q}^{+}}{i\omega_{n} + \varepsilon(\mathbf{k} + \mathbf{q})},
\]

(111)

where the free propagator \(G_{0}\) is
\[
G_{0} = \frac{1}{i\omega_{n} - \varepsilon(\mathbf{k} + \mathbf{q})} = \frac{i\omega_{n} + \varepsilon_{q}(\mathbf{k})}{(i\omega_{n})^{2} - \varepsilon^{2}_{q}(\mathbf{k})}.
\]

(112)
The anomalous propagators are

\[ F_{\alpha\beta}(k+q, \tau) = ig_{\alpha\beta} \frac{\Delta_k}{2E_q} \left[ \left( g_k^+ e^{-E_k\tau} - g_k^- e^{E_k\tau} \right) \theta_\tau - \left( g_q^+ e^{-E_q\tau} - g_q^- e^{E_q\tau} \right) \theta_{-\tau} \right], \]

\[ F_{\alpha\beta}(k+q, \tau) = ig_{\alpha\beta} \frac{\Delta_q}{2E_q} \left[ - \left( g_k^+ e^{-E_k\tau} - g_k^- e^{E_k\tau} \right) \theta_\tau + \left( g_q^+ e^{E_q\tau} - g_q^- e^{-E_q\tau} \right) \theta_{-\tau} \right]. \]
We can see, that in the approximation of slowness of changes of \( \Delta(r) \) the anomalous propagators depend on \( q \) by means of \( \Delta(q) \) only.

Kinetic energy of a system is determined by the following way:

\[
\langle W \rangle = -2i \sum_k \varepsilon(k + q) G(k + q, \tau \to 0) = 2 \sum_k \varepsilon(k + q) \left( g_q^- A(k + q) + g_q^+ B(k + q) \right)
\]

\[
= 2 \sum_k \varepsilon(k) \left( g_q^- A_q + g_q^+ B_q \right) + 2 \sum_k \left( g_q^- - g_q^+ \right) \left( \frac{1}{E_q} \right) \frac{(kq)^2}{m^2}
\]

\[
= W_n + V \frac{\nu_F}{2} \int_{\omega_D}^{\infty} \left[ \tanh \left( \frac{\beta |\varepsilon|}{2} \right) \frac{\varepsilon^2}{|\varepsilon|} - \tanh \left( \frac{\beta E_q}{2} \right) \frac{\varepsilon^2}{E_q} \right] d\varepsilon
\]

\[
+ V \frac{\nu_F}{2} \frac{1}{3} \frac{\nu_F^2 q^2}{2} \int_{-\omega_D}^{\omega_D} \left[ \tanh \left( \frac{\beta |\varepsilon|}{2} \right) \frac{1}{|\varepsilon|} - \tanh \left( \frac{\beta E_q}{2} \right) \frac{1}{E_q} \right] d\varepsilon.
\]

(119)

We can see, that the term, which is proportional to \( q^2 \), is added to the kinetic energy \( \text{(87)} \) (with the replacement \( E \to E_q = \sqrt{\varepsilon^2(k) + |\Delta(q)|^2} \)). In the approximation of slowness of changes the expressions for entropy and vacuum amplitude coincide with the expressions \( \text{(89)} \) and \( \text{(94)} \) accordingly, however it should be written \( \Delta(q) \) instead of \( \Delta = \text{const} \). Then we can write the free energy:

\[
\Omega(q) = \Omega_n(q) + \Omega(\Delta_q) + V \frac{\nu_F}{2} \int_{\omega_D}^{\infty} \left[ \tanh \left( \frac{\beta |\varepsilon|}{2} \right) \frac{1}{|\varepsilon|} - \tanh \left( \frac{\beta E_q}{2} \right) \frac{1}{E_q} \right] d\varepsilon,
\]

(120)

where \( \Omega(\Delta_q) \) coincides with the expression \( \text{(96)} \), where the replacement \( \Delta \to \Delta(q) \) was done.

Expanding the free energy \( \text{(120)} \) in powers of \( \Delta \) we can obtain the expression:

\[
\Omega(q) = \Omega_n(q) + V \left( \alpha(T) \Delta^2_q + \frac{1}{2} b \Delta^4_q + \gamma q^2 \Delta^2_q \right),
\]

(121)

where the coefficient \( \gamma \) is

\[
\gamma = \frac{\nu_F}{2} \frac{7 \zeta(3) \nu_F^2}{24 \pi^2 T^3} = \frac{\nu_F}{2} l_0^3,
\]

(122)

where \( l_0 \) is a coherence length at \( T = 0 \) (Pippard length). The expansion \( \text{(121)} \) has a form of Landau expansion of free energy in powers of order parameter at the condition \( q l_0 \ll 1 \). We can see, that a spatial inhomogeneity increases the free energy of a superconductor. Hence in most cases we can be restricted by the term \( \sim q^2 \) in the expansion, because more fast changes of \( \Delta \) increase the free energy essentially.

The full free energy of a system in a spatial inhomogeneous case can be obtained by summation of the expression \( \text{(120)} \) over all possible \( q \):

\[
\Omega = \Omega_n + \sum_q \Omega_q(q) = \Omega_n + \frac{V}{(2\pi)^3} \int \sigma_q(q) d^3q = \Omega_n + \frac{V^2}{(2\pi)^3} \int \left( \alpha(T) \Delta^2_q + \frac{1}{2} b \Delta^4_q + \gamma q^2 \Delta^2_q \right) d^3q
\]

(123)

Let’s pass from momentum space to real space using the expressions \( \text{(111)} \):

\[
\int \Delta_q \Delta_q d^3q = \int \Delta_q \left[ \frac{1}{V} \int \Delta(r) e^{-iqr} d^3r \right] d^3q = \frac{1}{V} \int \Delta(r) \left[ \int \Delta(q) e^{-iqr} d^3q \right] d^3r = \frac{(2\pi)^3}{V^2} \int \Delta(r) \Delta^+(r) d^3r
\]

(124)

\[
\int q \Delta_q \Delta_q d^3q = \int q \Delta_q \left[ \frac{1}{V} \int e^{-iqr} (-i) \frac{\partial}{\partial r} \Delta(r) d^3r \right] d^3q
\]

\[
= -i \int \left[ \int q \Delta(q) e^{-iqr} d^3q \right] \frac{\partial}{\partial r} \Delta(r) d^3r = \frac{(2\pi)^3}{V^2} \int \left[ \frac{\partial}{\partial r} \Delta(r) \right] \frac{\partial}{\partial r} \Delta^+(r) d^3r.
\]

(125)

For the term \( \Delta^4 \) and terms with more high powers the situation is more difficult. This is because a square of a Fourier transform is not equal to a Fourier transform of a square: \( \left( \frac{1}{V} \int \Delta(r) e^{-iqr} d^3r \right)^2 \neq \frac{1}{V} \int \Delta^2(r) e^{-iqr} d^3r \). Apparently this fact results to some nonlocality of a superconductor’s state in zero magnetic field described in \( \text{(28)} \), where a value of gap in a point is determined by a distribution of gap in some neighborhood: \( \Delta(r) = \int d^3r' Q(r, r') \Delta'(r) \). However
in first approximation this correlation can be neglected and we can write the expansion of free energy in powers of $\Delta \Delta^+$ in real space:

$$\Omega = \Omega_n + \int \left[ \alpha(T)|\Delta(r)|^2 + \frac{b}{2} |\Delta(r)|^4 + \gamma \left| \frac{\partial}{\partial r} \Delta(r) \right|^2 \right] d^3r.$$  \hfill (126)

This expansion coincides with Ginzburg-Landau expansion in zero magnetic field.

VIII. FREE ENERGY OF A SUPERCONDUCTOR IN MAGNETIC FIELD.

In this section we shall generalize the previous results for the case, when a superconductor is placed in a magnetic field $\mathbf{H}(r) = \mathbf{rot} \mathbf{A}(r)$. Our aim is to obtain the functional of free energy $\Omega \left( \Delta(r), \frac{\partial}{\partial T} \Delta(r), \mathbf{A}(r) \right)$, which is correct for an arbitrary value of the relation $\Delta(T)/T$, for an arbitrary scale of a change of $\Delta(r)$ in comparison with a coherent length $l(T)$, and for an arbitrary value of a magnetic penetration depth $\lambda(T)$ in comparison with a coherent length $l_0$ (nonlocal electromagnetic response). Thus, the all three restriction on Ginzburg-Landau functional, described in section II are excluded.

Let the microscopic magnetic field exists in a superconductor with a potential $\mathbf{A}$ and an intensity $\mathbf{H}$:

$$\mathbf{A}(r) = \sum_q a(q)e^{iqr} \Rightarrow \mathbf{H}(r) = \mathbf{rot} \mathbf{A}(r) = \sum_q \mathbf{q} \times a(q)e^{iqr}. \hfill (127)$$

Then the magnetic field inducts a current:

$$\mathbf{J}(r) = \frac{c}{4\pi} \mathbf{rot} \mathbf{H}(r) = -\frac{c}{4\pi} \sum_q \mathbf{q} \times a(q)e^{iqr} = -\frac{c}{4\pi} \sum_q \left( q(qa) - aq^2 \right)e^{iqr} \equiv \sum_q j(q)e^{iqr}. \hfill (128)$$

Energy of the magnetic field is

$$W_f = \frac{1}{8\pi} \int |\mathbf{H}(r)|^2 d^3r = \frac{V}{8\pi} \sum_q \left( q^2 a_q^2 - (qa_q)^2 \right). \hfill (129)$$

A magnetic field affects on a superconductor essentially. In the first place, a distribution of order parameter becomes inhomogeneous. As it was shown in the section VII an inhomogeneity leads to same growth of momentum of each element of a pair: $\mathbf{k} \rightarrow \mathbf{k} + \mathbf{q}, \mathbf{-k} \rightarrow -\mathbf{k} - \mathbf{q}$, moreover the order parameter depends on the momentum $\Delta = \Delta(q)$. In the second place, an ordinary momentum must be replaced by a canonical momentum: $\mathbf{k} + \mathbf{q} \rightarrow \mathbf{k} + \mathbf{q} - \frac{e}{\hbar} \mathbf{a}_q$, moreover the order parameter depends on the momentum $\Delta = \Delta(q - \frac{e}{\hbar} \mathbf{a}_q)$.

Figure 8: The diagram for the mass operator $\Sigma$ describing the interaction of a charged fermion with fluctuations of pairing in a spatially inhomogeneous system situated in magnetic field with a potential $a(q)$.

The mass operator for a process of interaction of a fermion (with charge $e$) with a fluctuation of pairing is shown in Fig.8. In analytical representation this diagram has the form:

$$-\Sigma \left( \mathbf{k} + \mathbf{q} - \frac{e}{\hbar} \mathbf{a}_q, \omega_n \right) = -\Delta \left( \mathbf{q} - \frac{e}{\hbar} \mathbf{a}_q \right) iG^+_0 \left( -\mathbf{k} - \mathbf{q} + \frac{e}{\hbar} \mathbf{a}_q, \omega_n \right) \left( -\Delta^+ \left( \mathbf{q} - \frac{e}{\hbar} \mathbf{a}_q \right) \right) \hfill (129)$$

$$= -\Delta \left( \mathbf{q} - \frac{e}{\hbar} \mathbf{a}_q \right) \Delta^+ \left( \mathbf{q} - \frac{e}{\hbar} \mathbf{a}_q \right) i\omega_n + \varepsilon \left( \mathbf{k} + \mathbf{q} - \frac{e}{\hbar} \mathbf{a}_q \right). \hfill (130)$$
where the free propagator $G_0$ is

$$G_0 = \frac{1}{i\omega_n - \varepsilon(k + q - \frac{e}{c}a_q)} = i\frac{i\omega_n + \varepsilon(k + q - \frac{e}{c}a_q)}{(i\omega_n)^2 - \varepsilon^2(k + q - \frac{e}{c}a_q)^2}.$$  (131)

Then from Dyson equation we can obtain the dressed propagator:

$$\frac{1}{G_0} = \frac{1}{G_S} - i\Sigma_q \Rightarrow G_S = i\frac{i\omega_n + \varepsilon(k + q - \frac{e}{c}a_q)}{(i\omega_n)^2 - E^2(k + q - \frac{e}{c}a_q)^2},$$  (132)

where $E$ is the specter of quasi-particles in a inhomogeneous system situated in magnetic field:

$$E^2(k + q - \frac{e}{c}a_q) = \varepsilon^2(k + q - \frac{e}{c}a_q) + |\Delta(k - \frac{e}{c}a_q)|^2.$$

where we have introduced the notations $E_{q,a}$ and $\varepsilon_{q,a}$ for convenience. Then Dyson equation can be represented in the form of Gor'kov equations. From these equations the expressions for anomalous propagators follow:

$$(i\omega_n - \varepsilon_{q,a})G + \Delta_{q,a}F^+ = i$$  \[F^+(k + q - \frac{e}{c}a_q, \omega_n) = \frac{-i\Delta_{q,a}}{(i\omega_n - \varepsilon_{q,a})E_{q,a}}.\]  (134)

If to suppose $q = 0, a = 0$, then we shall have the expressions (80-82).

In the space $(k, t)$ the normal propagator has the form:

$$G(k + \frac{e}{c}a_q, \tau) = \frac{1}{2} \left( 1 + \frac{\varepsilon_{q,a}}{E_{q,a}} \right) \theta_{\tau} - \frac{\varepsilon_{q,a}}{E_{q,a}} \theta_{\tau},$$

$$A \left( k + q - \frac{e}{c}a_q \right) = \frac{1}{2} \left( 1 + \frac{\varepsilon_{q,a}}{E_{q,a}} \right), \quad B \left( k + q - \frac{e}{c}a_q \right) = \frac{1}{2} \left( 1 - \frac{\varepsilon_{q,a}}{E_{q,a}} \right).$$  (136)

The anomalous propagators are

$$F_{\alpha,\beta}^+(k + q - \frac{e}{c}a_q, \tau) = ig_{\alpha,\beta} \frac{\Delta_{q,a}}{2E_{q,a}} \left[ \left( g_{\alpha,\beta}e^{-E_{q,a}} - g_{\alpha,\beta}e^{E_{q,a}} \right) \theta_{\tau} - \left( g_{\alpha,\beta}e^{E_{q,a}} - g_{\alpha,\beta}e^{-E_{q,a}} \right) \theta_{\tau} \right],$$

$$F_{\alpha,\beta}(k + q - \frac{e}{c}a_q, \tau) = ig_{\alpha,\beta} \frac{\Delta_{q,a}}{2E_{q,a}} \left[ \left( g_{\alpha,\beta}e^{-E_{q,a}} - g_{\alpha,\beta}e^{E_{q,a}} \right) \theta_{\tau} + \left( g_{\alpha,\beta}e^{E_{q,a}} - g_{\alpha,\beta}e^{-E_{q,a}} \right) \theta_{\tau} \right].$$  (137)

where $g_{\alpha,\beta}^+$ and $g_{\alpha,\beta}^-$ are statistical multipliers:

$$g^-(k + q - \frac{e}{c}a_q) = \frac{1}{e^{\beta E_{q,a}} + 1}, \quad g^+(k + q - \frac{e}{c}a_q) = \frac{1}{e^{-\beta E_{q,a}} + 1}.\]  (138)

We can see, that the normal and anomalous propagators have a complicated dependence on vector $q$, amplitudes of pairing $\Delta, \Delta^\dagger$ and magnetic field $a(q)$.

Kinetic energy of a system is determined in the following way:

$$\langle W \rangle = -2i \sum_k \varepsilon \left( k + q - \frac{e}{c}a_q \right) G \left( k + q - \frac{e}{c}a_q, \tau \rightarrow 0^+ \right) = 2 \sum_k \varepsilon_{q,a} \left( g_{q,a}^A q,a + g_{q,a}^+ B q,a \right)$$

$$= \sum_k \varepsilon_{q,a} \left( 1 - \frac{\varepsilon_{q,a}}{E_{q,a}} \tanh \frac{\beta E_{q,a}}{2} \right)$$

$$= W_n + \sum_{k, (|\varepsilon(k)| < \omega_d, |k| > k_F)} \varepsilon_{q,a} \left( \frac{\varepsilon_{q,a}}{E_{q,a}} \tanh \frac{\beta |\varepsilon_{q,a}|}{2} - \frac{\varepsilon_{q,a}}{E_{q,a}} \tanh \frac{\beta E_{q,a}}{2} \right) \equiv W_n + W_S.\]  (139)
We can see, that the kinetic energy depends on the vectors \( \mathbf{q} \) and \( \mathbf{a}(\mathbf{q}) \) in a complicated way. If we suppose \( a = 0 \) and \( q \) is small, then we shall obtain the expression \[ (119) \]

Free energy of a superconductor is the sum of the following terms:

\[
\Omega = \Omega_n + W_S - \frac{1}{\beta} S_S + \Omega_\lambda + W_{\text{field}}(\mathbf{a}),
\]

where \( W_S \) is the kinetic energy of fermions of a system in superconductive phase, \( S_S \) is the entropy of boholons, \( \Omega_\lambda = -\frac{1}{\beta} \ln R(\beta) \) is the energy corresponding to an interaction, \( W_{\text{field}}(\mathbf{a}) \) is the energy of the magnetic field \( \Omega(\mathbf{q}, \mathbf{a}_q) = \sum_{k} \left[ \mathbf{q}_a(k) \mathbf{q}_{a}(k) \frac{\beta E_{\mathbf{q}}}{2} \left( \frac{\beta E_{\mathbf{q}}}{2} \right) \right] d\varepsilon
\]

\[
+ \frac{1}{\beta} \sum_{k} \left[ f_{\mathbf{q}}(k) \ln f_{\mathbf{q}}(k) + (1 - f_{\mathbf{q}}(k)) \right] - f_{\mathbf{q}}(k) \ln f_{\mathbf{q}}(k) - (1 - f_{\mathbf{q}}(k)) \ln(1 - f_{\mathbf{q}}(k))
\]

where

\[
f_{\mathbf{q}}^S(k) = \frac{1}{e^{\beta E_{\mathbf{q}}(k)} + 1}, \quad f_{\mathbf{q}}^0(k) = \frac{1}{e^{\beta E_{\mathbf{q}}(k)} + 1}
\]

are occupation numbers of states by boholons, moreover with help of the term \( f_{\mathbf{q}}^0(k) \), a normal part of entropy is separated, such that \( \Omega_S(\Delta = 0) = 0 \). The full free energy is sum of the expression \[ (131) \] over all possible \( \mathbf{q} \):

\[
\Omega = \Omega_n + \sum_{\mathbf{q}} \left\{ \Omega_S(\mathbf{q}, \mathbf{a}_q) + w_{\text{field}}(\mathbf{q}, \mathbf{a}_q) \right\}.
\]

Unlike Ginzburg-Landau functional the obtained functional of free energy \[ (133) \] is correct for an arbitrary value of the relation \( \Delta(T)/T \), for an arbitrary scale of a change of \( \Delta(r) \) in comparison with a coherent length \( l(T) \), for an arbitrary value of a magnetic penetration depth \( \lambda(T) \) in comparison with a coherent length \( l_0 \) - it describes a nonlocal response to magnetic field. However the obtained expression is complicated for analyze. For its simplification let’s suppose that \( \Delta(r) \) changes in space slowly. Then it is necessary to expand the expression \[ (144) \] in degrees of \( \mathbf{q} - \mathbf{a}_q \) keeping terms which are proportional to the vector in second degree only. Then supposing \( E_{\mathbf{q}, \mathbf{a}} \approx \sqrt{\varepsilon^2(k) + \Delta_{\mathbf{q}, \mathbf{a}}^2} \), we have:

\[
\Omega(\mathbf{q}, \mathbf{a}_q) = \Omega_n(\mathbf{q}, \mathbf{a}_q) + V \sum_{\mathbf{q}} \left\{ a(T) \Delta_{\mathbf{q}, \mathbf{a}}^2 + \frac{1}{2} b \Delta_{\mathbf{q}, \mathbf{a}}^4 + \frac{1}{3} d \Delta_{\mathbf{q}, \mathbf{a}}^6 + \gamma \left( \mathbf{q} - \mathbf{e}_q \right)^2 \Delta_{\mathbf{q}, \mathbf{a}}^2 \right\} + \frac{V}{8\pi} \sum_{\mathbf{q}} \left( q^2 a_q^2 - (\mathbf{q} a_q)^2 \right),
\]
Figure 9: The screening of the external magnetic field \( H_{\text{ext}} \), impressed in parallel to a superconductive cylinder, by inducted closed currents. The inducted currents are directed so as to compensate the external field. The resultant current goes around a lateral area of cylinder, but strives to zero inside of the cylinder. The area where the resultant current is not equal to zero is surface layer with thickness \( \sim \lambda \). The magnetic field penetrates in a superconductor in the depth \( \sim \lambda \) too.

where the coefficients \( \alpha(T), b, d \) are determined by the formulas (103), and the coefficient \( \gamma \) is determined by the formula (122). The expansion (145) has a form of Ginzburg-Landau expansion of free energy in degrees of order parameter. Observed configuration of the order parameter \( \Delta_{\mathbf{q}} \mathbf{a} \) and the magnetic field \( \mathbf{a}(\mathbf{q}) \) minimizes the free energy:

\[
\frac{\delta \Omega}{\delta \Delta} = 0 \Rightarrow \alpha(T)\Delta_{\mathbf{q}} \mathbf{a} + b\Delta^3_{\mathbf{q}} \mathbf{a} + d\Delta^5_{\mathbf{q}} \mathbf{a} + \gamma \left( \mathbf{q} - \frac{e}{c} \mathbf{a}_q \right)^2 \Delta_{\mathbf{q}} \mathbf{a} = 0
\]

(146)

\[
\frac{\delta \Omega}{\delta \mathbf{a}} = 0 \Rightarrow \mathbf{j}(\mathbf{q}) = 2e\gamma \mathbf{q} \Delta^2_{\mathbf{q}} \mathbf{a} - 2\gamma \frac{e^2}{c} \Delta^2_{\mathbf{q}} \mathbf{a} \mathbf{q}(\mathbf{q}),
\]

(147)

where \( \mathbf{j}(\mathbf{q}) \) is Fourier component of a current:

\[
\mathbf{j}(\mathbf{q}) = -\frac{c}{4\pi} \mathbf{q} \times \mathbf{q} \times \mathbf{a}(\mathbf{q}) = -\frac{c}{4\pi} \left( \mathbf{q}(\mathbf{qa}) - \mathbf{a}(\mathbf{q}) \right).
\]

(148)

These equations take more simple form in a transverse gauge \( \mathbf{q} \cdot \mathbf{a}(\mathbf{q}) = 0 \). This gauge gives a condition of closure of a current (special case of conservation of charge):

\[
\mathbf{q} \cdot \mathbf{a}(\mathbf{q}) = 0 \Rightarrow \mathbf{j}(\mathbf{q}) = -\frac{c}{4\pi} \mathbf{a}(\mathbf{q}) \mathbf{q}^2 \Rightarrow \mathbf{q} \cdot \mathbf{j}(\mathbf{q}) = 0 \Leftrightarrow \text{div} \mathbf{J}(\mathbf{r}) = 0.
\]

(149)

The closed currents (149) screen a magnetic field in a superconductor - Fig[9]. The currents is analogy to molecular currents of Ampere, their resulting gives rise to observed magnetic effects. In transverse gauge the functional of free energy has a form:

\[
\Omega = \Omega_n + V \sum_{\mathbf{q}} \left( \alpha(T)\Delta^2_{\mathbf{q}} \mathbf{a} + \frac{1}{2}b\Delta^4_{\mathbf{q}} \mathbf{a} + \frac{1}{3}d\Delta^6_{\mathbf{q}} \mathbf{a} + \gamma \left( \mathbf{q}^2 + \frac{e^2}{c^2} \mathbf{a}_q^2 \right) \Delta^2_{\mathbf{q}} \mathbf{a} \right) + \frac{V}{8\pi} \sum_{\mathbf{q}} q^2 a_q^2.
\]

(150)
The equations of extremals are

\[ \alpha(T) \Delta_q a + b \Delta^3_q a + d \Delta^5_q a + \gamma \left( q^2 + \frac{c^2}{e^2 a_y^2} \right) \Delta_q a = 0 \]  

\[ j(q) = -2\gamma \frac{e^2}{c} \Delta^2_q a a(q). \]  

From (152) one can see, that the value \( Q = -2\gamma \frac{e^2}{c} \Delta^2_q a \) is a Fourier transform of a kernel in the integral law of a magnetic response (Pippard law). The order parameter is function of \( a \) and \( a(q) \):

\[ \Delta^2_q a(T) = \frac{|\alpha(T)|}{b} \left( 1 - \frac{\gamma}{|\alpha(T)|} \left( q^2 + \frac{c^2}{e^2 a_y^2} \right) \right) = \frac{|\alpha(T)|}{b} \left( 1 - \lambda^2(T) \left( q^2 + \frac{c^2}{e^2 a_y^2} \right) \right). \]  

where smallness of \( 1/q \) in comparison with the coherent length \( l(T) \): \( ql(T) \ll 1 \) is supposed, and we assumed that the coefficient \( d = 0 \) for simplification. From the formula (153) one can see, that the kernel of the magnetic response \( Q \) is function of magnetic field. Hence the electrodynamics of a superconductor is nonlinear. If to suppose \( \Delta = \text{const} \) at given temperature, then we shall obtain London equation:

\[ j(q) = -2\gamma \frac{e^2}{c} \Delta^2(T) a(q) \equiv -\frac{c}{4\pi \lambda^2(T)} a(q) \Rightarrow \lambda^2(T) = \frac{c^2}{8\pi e^2} \frac{b}{|\alpha(T)| \gamma}. \]  

where \( \lambda^2(T) \) is the magnetic penetration depth in a superconductor. It is necessary to note, that \( \lambda \propto \frac{m}{e^2 n_s} = \frac{2m}{(2e)^2 n_s/2} \). This means that the exchange of mass, charge and concentration of superconductive electrons to corresponding values of Cooper pairs doesn’t change the observed values.

However it is necessary to note, that in a high-temperature limit a gap (and a kernel \( Q \)) depends on vector \( q \) and field \( a(q) \) strongly. From the formula (153) one can see, that the gap decreases at an increase of \( q \), hence a magnetic penetration depth \( \lambda \) increases. Moreover with a rise of temperature this dependence becomes stronger (at \( T = T_C \) we have \( \lambda = \infty \)). Besides at temperature \( T = T_C \) the critical magnetic field is zero \( H_C = 0 \). This means, that in the limit \( T \rightarrow T_C \) any magnetic field \( H(q) \) cannot be considered as weak. Therefore it suppresses order parameter essentially and penetrates in a superconductor deeply (in macroscopic distant even). For example, the penetration of magnetic field along a core of Abrikosov vortex in a type II superconductor. Such structure exists in infinitely weak magnetic field at \( T \rightarrow T_C \).

For research of the nonlocal characteristics of the functional of free energy (14,143) let’s consider a low-temperature limit \( \Delta \beta \gg 1 \) at \( T \rightarrow 0 \). A value of gap is close to the value at zero temperature \( \Delta(T) \leq \Delta_0 \). Moreover, magnetic field is weak, such that it changes a value of gap lightly, that is the magnetic field is much smaller than critical field \( H \ll H_C \). Either as above, we assume that a change of a gap in space is slow. Starting from aforesaid and using the expansion (99) we obtain the free energy:

\[ \Omega = \Omega_n + V \sum_q \left( \alpha_0(T) + b_0(T) \Delta_q a + d_0 \Delta^2_q a + \gamma \left( q - \frac{e}{c} a_y \right)^2 \Delta^2_q a \right) + \frac{V}{8\pi} \sum_q \left( q^2 a_y^2 - (qa_y)^2 \right), \]  

where coefficients \( \alpha_0(T), b_0(T), d_0 \) are determined by the formulas (101), and coefficient \( \gamma \) is determined by the formula (122). The observed configurations of order parameter \( \Delta_q a \) and magnetic field \( a(q) \) minimized free energy:

\[ \frac{\delta \Omega}{\delta \Delta} = 0 \Rightarrow b_0(T) + 2d_0 \Delta_q a + \gamma \left( q - \frac{e}{c} a_y \right)^2 \Delta_q a = 0 \]  

\[ \frac{\delta \Omega}{\delta a} = 0 \Rightarrow j(q) = 2e\gamma q \Delta^2_q a - 2\gamma \frac{e^2}{c} \Delta^2_q a a(q). \]  

In the transverse gauge \( q \cdot a(q) = 0 \) the functional of free energy and the equations for the extremals have a form:

\[ \Omega = \Omega_n + V \sum_q \left( \alpha_0(T) + b_0(T) \Delta_q a + d_0 \Delta^2_q a + \gamma \left( q^2 + \frac{c^2}{e^2 a_y^2} \right) \Delta^2 q a \right) + \frac{V}{8\pi} \sum_q q^2 a_y^2 \]  

\[ b_0(T) + 2d_0 \Delta_q a + 2\gamma \left( q^2 + \frac{c^2}{e^2 a_y^2} \right) \Delta_q a = 0 \]  

\[ j(q) = -2\gamma \frac{e^2}{c} \Delta^2 q a a(q). \]
If in the equation (160) to assume $\Delta = \text{const}$ at given temperature, then we shall have London equation again:

$$j(q) = -2\gamma \frac{e^2}{c} \Delta^2(T) a(q) \equiv -\frac{c}{4\pi \lambda^2(T)} a(q) \Rightarrow \lambda^2(T) = \frac{c^2}{2\pi e^2} \frac{d^2}{b_0^2(T) \gamma}. \quad (161)$$

The set of equations (159,160) allows to generalize London equation. From the equation (159) we can find value of $a$ parameter is Bethe-Solpiter amplitude averaged over a system due to statistical and dynamical correlations. Since in

where we took into account a slowness of changes of a gap in space: $l_0 q \ll 1$. $2\frac{\Delta^2}{\gamma_0} = 2\Delta^2 = l_0^2$ is a coherent length at temperature $T = 0$. Thus we have the nonlocal kernel $Q(q)$, where radius of a nonlocality is equal to the coherent length $l_0$. This result corresponds to nonlocal Pippard electrodynamics (long wavelength limit). This fact proves nonlocality of the obtained functional of free energy of a superconductor (141) (143). For generalization in case of a large value $l_0 q$ it is necessary to expand the free energy (141) in degrees of $q$. Then we can obtain a short wavelength limit of $Q$: $Q \sim 1/ql_0$. Starting from correctness of the asymptotics (145) and (155) of the functional (141) (143) we can make a conclusion about its correctness for description of a superconductive phase.

**IX. CONCLUSION.**

In this paper on the example of superconductivity we described the type II phase transition on a microscopic level, namely starting from first principles. This means, that the method of calculation of a free energy $\Omega(T,N/V)$ has been developed in a range of temperatures, which includes a point of pase transition, without introducing any artificial parameters of type of order parameter and sources of ordering, but starting from microscopic parameters of Hamiltonian only. Moreover, the theorems about connection of a vacuum amplitude with thermodynamics potentials are realized.

Microscopic picture of a phase transition lies in the following. At switching of attraction between particles of a Fermy system the instability relatively formation of bound states of two fermions rises. The given states are characterized by Bethe-Solpiter amplitudes - amplitudes of pairing, which can be found exactly in the case of an isolated pair. However, in consequence of statistical correlations between pairs the amplitudes are determined by dynamics of all particles of a system. Their observing value is result of averaging over the system. Thus, a collective (condensate) of pairs exists. A particle propagating through a system interacts with fluctuations of pairing. As a result of such interaction a dispersion law of quasi-particles is changed and anomalous propagators appear. This means, that a spontaneous symmetry breakdown takes place. After consideration of interaction of particles with fluctuations of pairing all characteristics of a system must be calculated over the new vacuum with broken symmetry. So, calculation of a vacuum amplitude over new ground state gives a possibility to use the theorem about connection of a vacuum amplitude with a ground state energy (with free energy at nonzero temperature). As a result, the free energy is function of amplitudes $\Omega = \Omega(\Delta,\Delta')$, and their observed value minimizes the free energy. Analyze of the obtained functional of free energy shows, that the amplitude of pairing plays a part of an order parameter. Namely, the order parameter is Bethe-Solpiter amplitude averaged over a system due to statistical and dynamical correlations. Since in Nambu-Gor’kov formalism (the method of anomalous propagators) any phase transition can be described by another method can be generalized to the rest transitions (ferromagnetism and antiferromagnetism, waves of charge and spin density, ferroelectricity and so on).
The functional of a superconductor’s free energy \([141][143]\) has been obtained in this paper using the developed method of microscopic description of phase transitions and generalizing its in the cases of spatial inhomogeneity and presence of magnetic field. The functional generalizes Ginzburg-Landau functional for cases of arbitrary temperatures, arbitrary spatial inhomogeneities and a nonlocality of a magnetic response. The equations of superconductor’s state are extremals of the functional obtained by variation over the gap \(\Delta\) and the magnetic field \(a\). The equations determined equilibrium configurations of a gap and a magnetic field at given conditions.

Appendix A: The method of uncoupling of correlations and Dyson equation.

As it was shown in \([20]\) for propagators \(G\) (one-particle), \(K_2\) (two-particle), \(K_3\) (tree-particle) ... \(K_n\) (n-particle) the set of coupling equations can be written. The set of equations is analogous to BBGKY hierarchy for a s-particle probability density function. For example, for \(G\) and \(K_2\) these equations have a view:

\[
G(1,1') = G_0(1,1') + i \int G_0(1,2)U(2,3)K_2(2,3;1',3^+)dx_3dx_4.
\]

\[
K_2(1,2;1',2') = G(2,2')G_0(1,1') - G(2,1')G_0(1,2') + i \int G_0(1,2)U(2,3)K_3(2,3,2;1',2',4^+)dx_3dx_4,
\]

where \(3^+ \equiv (\xi_3, t_3 + 0), U(2,3) \equiv V(\xi_2, \xi_3)\delta(t_2 - t_3), dx \equiv d\xi dt\). The cross term \(G(2,2')G_0(1,1') - G(2,1')G_0(1,2')\) appeared as result of calculation of Fermi symmetry of particles (it would be "+" for bosons). The equation for a three-particle propagator \(K_3\) will be determined by four-particle propagator \(K_4\) and so on: \(K_n = f(K_{n+1})\).

It is obviously that this set of equations can not be solved. However, in most cases for description of a system it is enough to know functions \(G\) and \(K_2\) (less). Then the method of uncoupling of correlations is used. The function \(K_2\) in a zero approximation is

\[
K_2^{(0)} = G_0(1,1')G_0(2,2') - G_0(1,2')G_0(2,1').
\]

We can see, that in an absence of interaction the two-particle propagator is represented in a multiplicative form by the one-particle free propagators \(G_0\). Statistical correlation exists only in the course of Pauli exclusion principle. Then in first approximation let’s use the free propagator \(K_2^{(0)}\) instead of the dressed propagators \(K_2\) in the formula \([A1]\). Hence the correction for the dressed one-particle propagator \(G\) is

\[
G^{(1)}(1,1') = i \int G_0(1,2)U(2,3)G_0(3,3^+)G_0(2,1')dx_3dx_4 - i \int G_0(1,2)U(2,3)G_0(3,3')G_0(2,3)dx_2dx_3.
\]

The correction \([A4]\) is represented graphically in Fig.10 Integration is carried over coordinates of the internal lines. The first term corresponds to direct Hartree interaction, the second term corresponds to exchange Fock interaction. In order to obtain the next approximation for \(G\) it is necessary to find \(K_2^{(1)}\). In a symbolic representation we have

![Figure 10: The correction of first order \(G_1\) obtained by uncoupling of correlations in the equation \([A1]\).](image)

the equations:

\[
K_2^{(1)} = G^{(1)}G_0 - G_0G^{(1)} + iG_0U^{(0)}K_3^{(0)}
\]

\[
G^{(2)} = iG_0UK_2^{(1)}.
\]

The procedure of uncoupling of correlations can be represented in another way. Let a correction for two-particle propagator \(K^{(1)}\) is determined by the matrix elements: \(V_{kkl}\) is a direct interaction and \(V_{kilk}\) is an exchange interaction.
A two-particle propagator has two entering momentums (represented by lines with corresponding indexes) and two outgoing momentums. An one-particle propagator has one entering momentum and one outgoing momentum. The procedure of uncoupling of correlations consist in the fact, that we connect two lines in $K$ taking into account conservation of momentum and spin: Fig.11. The connection means integration over intermediate momentums and energy parameters (in a momentum-energy representation $\xi, t \rightarrow k, \omega$) in the formula (A4). As a result, we have the same diagrams for $G$ as in Fig.10.

The procedure can be generalized to higher corrections (with two and more lines of interaction). In general case the rules of diagram technique are:

1. The multiplier $iG(k, \omega)$ is associated with each bold line, the multiplier $iG_0(k, \omega)$ is associated with each thin line.
2. The multiplier $-iV_{kl}\omega$ is associated with each dashed line of interaction between particles, and the multiplier $-iV_{kl}$ is associated with line of interaction of a particle with an external field.
3. The multiplier $-1$ associated with each fermion loop and we make summation over all possible spin configurations.
4. Momentum $k + 1 = m + n$, energy parameter and spin are reserved in every vertex.
5. Integration is made over each intermediate momentum and summation is made over each intermediate energy parameter: $\sum_k \rightarrow \frac{1}{V} \int \frac{d^3 k}{(2\pi)^3}$ ($V$ is volume of a system) and $\int \frac{d\omega}{2\pi}$.

The diagrams of the type Fig.10 are summarized with help of the mass operator $\Sigma$ - any diagram without external lines. Hence a dressed propagator $G$ can be found from Dyson equation:

$$iG = iG_0 + iG_0(-i)\Sigma iG \Rightarrow G = \frac{1}{G_0^{-1} - \Sigma} \quad (A6)$$

The mass operator $\Sigma$ has the sense of a mean field of all particles of a system acting on a test (marked) article. In this fact the sense of the procedure of uncoupling of correlations is: interaction and propagation of all particle of a system is reduced to propagation of each particle in the mean field of all rest particles.

Other approach exists (more widely represented in literature) for obtaining of the diagram expansion for $G$. In this approach a one-particle propagator is determined as

$$G(k_2, k_1, t_2 - t_1) = \lim_{T_1 \rightarrow -\infty(1-i\delta)}^{T_2 \rightarrow +\infty(1-i\delta)} \frac{-i\langle \Phi_0 | T[\hat{U}(T_2, T_1)\hat{C}^+_k(t_2)\hat{C}^+_k(t_2)]|\Phi_0 \rangle}{\langle \Phi_0 | \hat{U}(T_2, T_1)|\Phi_0 \rangle}, \quad (A7)$$

where $\hat{C}^+(t), \hat{C}(t)$ is creation and annihilation operators in interaction representation, $\hat{U}$ is evolution operator in interaction representation, $\Phi_0$ is ground state of a system of noninteracting fermions. This definition of a propagator is equivalent to the definition

$$G(k, t_2 - t_1) = -i\langle \Psi_0 | T[C_{k,\sigma}(t_2)C^+_{k,\sigma}(t_1)]|\Psi_0 \rangle, \quad (A8)$$

where $\hat{C}^+(t), \hat{C}(t)$ is creation and annihilation operators in Heisenberg representation, $\Psi_0$ is ground state of a system of interacting fermions, and expansion of $G$ in series of perturbation theory is possible if the condition of adiabaticity is realized (N):

$$\langle \Phi_0 | \Psi_0 \rangle \neq 0. \quad (A9)$$
In the method of uncoupling of correlations we didn’t use the condition and Wick theorem unlike the standard formulation of the perturbation theory based on . Thus the advantage of stated above method of uncoupling of correlations consists in that we can formulate a perturbation theory without using of the adiabatic hypothesis.

Appendix B: The method of uncoupling of correlations for a vacuum amplitude in the case of normal processes.

Let us consider the processes of direct and exchange interaction of first order, which is described by the matrix elements \( V_{jkl} \) and \( V_{ijkl} \), moreover the interaction doesn’t act to spins of particles. Contribution to vacuum amplitude of such processes is (let’s suppose \( t_2 > t_1 \) for definiteness):

\[
R(t) = 1 + \frac{1}{1!} \int_0^t dt_1 \sum_{k,l} \left( \frac{-i}{2} V_{kkl} \right) \langle \Phi_0 | C_{1,a}^+ (t_1) C_{1,b}^+ (t_1) C_{k,a}^+ (t_1) C_{k,b}^+ (t_1) | \Phi_0 \rangle \\
+ \frac{1}{2!} \int_0^t dt_1 \sum_{k,l} \left( \frac{-i}{2} V_{kkl} \right) \sum_{\alpha, \beta} \langle \Phi_0 | C_{1,a}^+ (t_1) C_{1,b}^+ (t_1) C_{k,a}^+ (t_1) C_{k,b}^+ (t_1) | \Phi_0 \rangle \\
+ \frac{1}{2!} \int_0^t dt_1 \sum_{k,l} \left( \frac{-i}{2} V_{kkl} \right) \sum_{\alpha, \beta} \langle \Phi_0 | C_{1,a}^+ (t_1) C_{1,b}^+ (t_1) C_{k,a}^+ (t_1) C_{k,b}^+ (t_1) | \Phi_0 \rangle \times \langle \Phi_0 | C_{1,a}^+ (t_1) C_{1,b}^+ (t_1) C_{k,a}^+ (t_1) C_{k,b}^+ (t_1) | \Phi_0 \rangle \\
+ \frac{1}{2!} \int_0^t dt_1 \sum_{k,l} \left( \frac{-i}{2} V_{kkl} \right) \sum_{\alpha, \beta} \langle \Phi_0 | C_{1,a}^+ (t_1) C_{1,b}^+ (t_1) C_{k,a}^+ (t_1) C_{k,b}^+ (t_1) | \Phi_0 \rangle \times \langle \Phi_0 | C_{1,a}^+ (t_1) C_{1,b}^+ (t_1) C_{k,a}^+ (t_1) C_{k,b}^+ (t_1) | \Phi_0 \rangle \\
+ \cdots \quad (B1)
\]

For approximate calculation \( R(t) \) we shall use the method of uncoupling of correlations, which lies in the fact that an average of four creation and annihilation operators is represented by a product of averages of pairs of the operators: \( \langle C^+ C^+ C C \rangle \rightarrow \langle C^+ C \rangle \langle C^+ C \rangle \). The averages correspond to propagators of particles with initial and final states corresponding to the matrix element of interaction \( V_{klmn} \) taking into account conservation of momentum and spin. It is achieved by preliminary transposition of the operators before the uncoupling taking into account Fermi commutation. For Hartree and Fock processes the procedure corresponds to the diagrams in Fig. In each process of scattering we connect incoming and outgoing lines taking into account the laws of conservation. The obtained diagram must not have free ends. Then we have the amplitude of transition ”vacuum-vacuum”. Analytically it will be so:

\[
R(t) = 1 + (-1)^2 \frac{1}{1!} \int_0^t dt_1 \sum_{k,l} \sum_{\alpha, \beta} \left( \frac{-i}{2} V_{kkl} \right) \langle \Phi_0 | C_{1,a}^+ (t_1) C_{1,b}^+ (t_1) C_{k,a}^+ (t_1) C_{k,b}^+ (t_1) | \Phi_0 \rangle \\
+ (-1)^2 \frac{1}{2!} \int_0^t dt_1 \sum_{k,l} \sum_{\alpha, \beta} \left( \frac{-i}{2} V_{kkl} \right) \sum_{\alpha', \beta'} \langle \Phi_0 | C_{1,a}^+ (t_1) C_{k,a}^+ (t_1) | \Phi_0 \rangle \langle \Phi_0 | C_{1,b}^+ (t_1) C_{k,b}^+ (t_1) | \Phi_0 \rangle \\
+ (-1)^4 \frac{1}{2!} \int_0^t dt_1 \sum_{k,l} \sum_{\alpha, \beta} \left( \frac{-i}{2} V_{kkl} \right) \sum_{\alpha', \beta'} \langle \Phi_0 | C_{1,a}^+ (t_1) C_{k,a}^+ (t_1) | \Phi_0 \rangle \langle \Phi_0 | C_{1,b}^+ (t_1) C_{k,b}^+ (t_1) | \Phi_0 \rangle \times \langle \Phi_0 | C_{1,a}^+ (t_1) C_{1,b}^+ (t_1) C_{k,a}^+ (t_1) C_{k,b}^+ (t_1) | \Phi_0 \rangle \\
+ (-1)^2 \frac{1}{2!} \int_0^t dt_1 \sum_{k,l} \sum_{\alpha, \beta} \left( \frac{-i}{2} V_{kkl} \right) \sum_{\alpha', \beta'} \langle \Phi_0 | C_{1,a}^+ (t_1) C_{k,a}^+ (t_1) | \Phi_0 \rangle \langle \Phi_0 | C_{1,b}^+ (t_1) C_{k,b}^+ (t_1) | \Phi_0 \rangle \times \langle \Phi_0 | C_{1,a}^+ (t_1) C_{1,b}^+ (t_1) C_{k,a}^+ (t_1) C_{k,b}^+ (t_1) | \Phi_0 \rangle \\
+ \cdots
\]
\[
\frac{1}{2i}(\mathcal{L}^\dagger - \mathcal{L})^2 \int_0^t dt_1 \int_0^t dt_2 \sum_{\alpha, \beta} \sum_{k, l} \left( -\frac{i}{2} V_{k\ell kl} \right) \sum_{\alpha'} \sum_{k', l'} \left( \frac{i}{2} V_{k'\ell' k'l'} \right) \\
\times \langle \Phi_0 | C^+_\mathcal{L}_\beta(t_1) C^+_\ell_\beta(t_1) | \Phi_0 \rangle \langle \Phi_0 | C^+_k \alpha(t_1) C^+_k \alpha(t_1) | \Phi_0 \rangle \langle \Phi_0 | C^+_k \alpha'(t_2) C^+_k \alpha'(t_2) \rangle | \Phi_0 \rangle \\
+ \ldots = 1 + (R_1^{\text{Hartree}} + R_1^{\text{Fock}}) + \frac{1}{2i}(R_1^{\text{Hartree}} + R_1^{\text{Fock}})^2 + \ldots = \exp(R_1^{\text{Hartree}} + R_1^{\text{Fock}})
\]  

(B2)

Figure 12: The procedure of uncoupling of correlations in vacuum amplitude for a correction of first order. In the average of matrix element of interaction operator $V(C^+C^+CC)$ we connect the lines taking into account conservation of momentum and spin. As a result, we obtain the correction of first order $R_1$ for a vacuum amplitude of a view $V(C^+C)(C^+C)$.

The uncoupling lets to write expression for $R_1(t)$ via normal propagators:

\[
R_1^{\text{Hartree}} = (2s + 1)^2 \int_0^t dt \sum_{k, l} \left( -\frac{i}{2} V_{k\ell kl} \right) iG_0(l, t - t)iG_0(k, t - t)
\]
\[
= (-2i) \sum_{k, l} V_{k\ell kl}B_0(l)B_0(k)t  
\]  

(B3)

\[
R_1^{\text{Fock}} = (2s + 1)(-1) \int_0^t dt \sum_{k, l} \left( \frac{i}{2} V_{k\ell kl} \right) iG_0(l, t - t)iG_0(k, t - t)
\]
\[
= i \sum_{k, l} V_{k\ell kl}B_0(l)B_0(k)t  
\]  

(B4)

The multiplier $(2s + 1)$ is result of summation over spin states (number of spin configurations), $s = 1/2$. We can see, that the method of uncoupling of correlations lets to calculate vacuum amplitude simply, selecting contributions of processes of each type. The method can be generalized to the processes of higher order. We can see, that the proposed method of obtaining of diagram expansion for a vacuum amplitude by uncoupling of correlations doesn’t demand of an use of Wick theorem and the adiabatic hypothesis.

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