The pde2path add-on toolbox p2poc for solving optimal control problems

– Quickstart Guide –

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Abstract

p2poc is an add–on toolbox to the Matlab package pde2path. It is aimed at the numerical solution of infinite time horizon optimal control (OC) problems for parabolic systems of PDE over 1D or 2D spatial domains. The basic idea is to treat the OC problem via the associated canonical system in two steps. First we use pde2path to find branches of stationary solutions of the canonical system, also called canonical steady states (CSS). In a second step we use the results and the spatial discretization of the first step to calculate the objective values of time-dependent canonical paths ending at a CSS with the so called saddle point property. This is a (typically very high dimensional) boundary value problem (BVP) in time, which we solve by combining a modification of the BVP solver TOM with a continuation algorithm in the initial states. We explain the design and usage of the package via two example problems, namely the optimal management of a distributed shallow lake model, and of a semi-arid grazing system. Both show interesting bifurcations of so called patterned CSS, and in particular the latter also a variety of patterned optimal steady states. The package (library and demos) can be downloaded at www.staff.uni-oldenburg.de/hannes.uecker/pde2path.

1 Introduction

Denoting the state variable (vector) by \( v = v(t,x) \in \mathbb{R}^N \), and the control by \( k = k(t,x) \in \mathbb{R} \), we consider optimal control problems of the form

\[
V(v_0(\cdot)) := \max_{k(\cdot)} J(v_0(\cdot), k(\cdot, \cdot)), \quad J(v_0(\cdot), k(\cdot, \cdot)) := \int_0^\infty e^{-\rho t} J_{ci}(v(t), k(t)) \, dt, \tag{1a}
\]

where \( J_{ci}(v(\cdot,t), k(\cdot,t)) = \int_\Omega J_c(v(x,t), k(x,t)) \, dx \) \tag{1b}

is the spatially integrated current value objective function, with the local current value \( J_c : \mathbb{R}^{N+1} \rightarrow \mathbb{R} \) a given function; \( \rho > 0 \) is the discount rate, and \( v : [0, \infty) \times \Omega \rightarrow \mathbb{R}^N \) fulfills a PDE of the form

\[
\partial_t v = -G_1(v, k) := D\Delta v + g_1(v, k), \quad v|_{t=0} = v_0. \tag{1c}
\]

Here, \( D \in \mathbb{R}^{N \times N} \) is a diffusion matrix, \( \Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_d^2} \) is the Laplacian, and \( \frac{\partial}{\partial t} \) holds in a bounded domain \( \Omega \subset \mathbb{R}^d \) with suitable boundary conditions (BC), where for simplicity we restrict to homogeneous Neumann \( \partial_{\nu} v = 0 \), \( \nu \) the outer normal. In applications, \( J_c \) and \( G_1 \) of course often also depend on a number of parameters, which however for simplicity we do not display here.\(^1\)

\(^1\)\(G_1\) in (1) can in fact be of a much more general form, but for simplicity here we stick to (1c). The convention that \( \partial_t v = -G_1(v) \) instead of \( \partial_t v = G_1(v) \) with \( G_1(v) = D\Delta v + g_1(v) \) is inherited via pde2path from the Matlab pdetoolbox, which assembles \(-\Delta\) into the stiffness matrix \( K\).
Using Pontryagin’s Maximum Principle, an optimal solution \((v, \lambda) \in \mathbb{R}^{2N}\) (or equivalently \((v, k) \in \mathbb{R}^{N+1}\)) has to satisfy the canonical system

\[
\begin{align*}
\partial_t v &= \partial_{\lambda} \mathcal{H} = D \Delta v + g_1(v, k), \quad v|_{t=0} = v_0, \\
\partial_t \lambda &= \rho \lambda - \partial_v \mathcal{H} = \rho \lambda + g_2(v, k) - D \Delta \lambda.
\end{align*}
\] (2a)

Here \(\lambda = \lambda(t, x) \in \mathbb{R}^N\) are called the co–states,

\[
\mathcal{H} = \mathcal{H}(v, \lambda, k) = J_c(v, k) + \langle \lambda, D \Delta v + g_1(v, k) \rangle
\] (2c)

is called the (local) current value Hamiltonian, \(\langle a, b \rangle := \sum_{i=1}^N a_i b_i\), \(k\) is obtained from solving

\[
\partial_k H = 0
\] (2d)

and derivatives like \(\partial_{\lambda} \mathcal{H}\) are taken in the variational sense, i.e., the functional to be maximized is

\[
\int_0^\infty e^{-\rho t} \int_{\Omega} \mathcal{H}(v(x, t), \lambda(x, t), k(x, t)) \, dx \, dt.
\] (3)

In principle we want to solve (2) for \(t \in [0, \infty)\), but in (2a) we have initial data for only half the variables, and in (2b) we have anti–diffusion, such that (2) is ill–posed as an initial value problem. For convenience we set

\[
u(t, \cdot) := \left( \begin{array}{c} v(t, \cdot) \\ \lambda(t, \cdot) \end{array} \right) : \Omega \to \mathbb{R}^{2N},
\] (4)

and write (2) as

\[
\partial_t u = -G(u, \eta) := D \Delta u + f(u), \quad D = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix}, \quad f(u) = \begin{pmatrix} g_1(u) \\ g_2(u) \end{pmatrix},
\] (5a)

and where \(\eta \in \mathbb{R}^p\) stands for parameters present, which for instance also includes the discount rate \(\rho\). Besides the boundary condition \(\partial_{\nu} u = 0\) and the initial condition

\[
v|_{t=0} = v_0,\] (5b)

we then impose the transversality condition

\[
\lim_{t \to \infty} e^{-\rho t} u(t) = 0.
\] (5c)

A solution \(u\) of the canonical system (5) is called a canonical path, and an equilibrium of (5a) (which automatically fulfils (5c)) is called canonical steady state (CSS). With a slight abuse of notation we also call \((v, k)\) with \(k\) given by (2d) a canonical path. See also, e.g., [GCF+08, GU15] for more formal definitions, for references to background literature, and further comments on the notions of optimal systems, and, e.g., the significance of the transversality condition (5c). Since the canonical system (2) only gives necessary conditions for optimal solutions, we proceed in two steps:

(a) CSS branches. We compute (approximate) CSS of (3), i.e., solutions \(\hat{u}\) of

\[
G(u, \eta) = 0,
\] (6)

together with the BC. For this use the package pde2path [LWR13, DRW14] to set up a FEM discretization of (6) as a continuation/bifurcation problem in one of the parameters, which we call \(\eta\) again. This gives branches \(\eta \mapsto \hat{u}(\eta)\) of solutions, which is in particular useful to possibly find several solutions \(\hat{u}^{(l)}(\eta), j = l, \ldots, m\) at fixed \(\eta\). By computing the associated \(J_{ci}(\hat{u}^{(l)}, k^{(l)})\) we we can identify which of these is optimal amongst the CSS. Given a CSS \(\hat{u}\), for simplicity we also write \(J_{ci}(\hat{u}) := J_{ci}(\hat{u}^{(l)}, k^{(l)})\), and moreover, have

\[
J(\hat{u}) = J_{ci}(\hat{u}) / \rho.
\] (7)
(b) Canonical paths. In a second step (b), we calculate canonical paths ending at a CSS \( \hat{u} \) (and
often starting at the state values \( v_0 \) of a different CSS \( \hat{u}_0 \)), and the objective values of the canonical
paths. For this we choose a truncation time \( T \) and modify \( (5) \) to the condition that \( u(T) \in W_s(\hat{u}) \)
and near \( \hat{u} \), where \( W_s(\hat{u}) \) denotes the stable manifold of \( \hat{u} \). In practice, we approximate \( W_s(\hat{u}) \) by
the stable eigenspace \( E_s(\hat{u}) \), and thus consider the BVP
\[
\begin{align*}
\partial_t u &= -G(u), \\
v|_{t=0} &= v_0, \\
u(T) &\in E_s(\hat{u}) \quad (\text{and } \|u(T) - \hat{u}\| \text{ small}).
\end{align*}
\]
Using the spatial FEM discretizations, the implementation of \( G \), and the results from the first step (a), if the mesh in the FEM consists of \( n \) nodes, then \( u(t) \in \mathbb{R}^{2Nn} \), and \( (8) \) yields a system of \( 2Nn \) ODEs in the form (with a slight abuse of notation)
\[
M \frac{d}{dt} u = -G(u),
\]
while the initial and transversality conditions become
\[
\begin{align*}
v|_{t=0} &= v_0, \\
\Psi(u(T) - \hat{u}) &= 0 \quad (\text{and } \|u(T) - \hat{u}\| \text{ small}).
\end{align*}
\]
Here \( M \in \mathbb{R}^{2Nn \times 2Nn} \) is the mass matrix of the FEM mesh, and \( \Psi \in \mathbb{R}^{Nn \times 2Nn} \) defines the projection
onto \( E_s(\hat{u}) \). Moreover, \( (9) \) consists of \( Nn \) initial conditions for the states, while the costates \( \lambda \)
(and hence the control \( k \)) are free. Thus, to have \( 2Nn \) BC altogether we need \( \dim E_s(\hat{u}) = Nn \). On
the other hand, we always have \( \dim E_s(\hat{u}) \leq Nn \), see \[GU15, Appendix A\]. We define the defect
\[
d(\hat{u}) := \dim E_s(\hat{u}) - Nn
\]
and call a CSS \( \hat{u} \) with \( d(\hat{u}) = 0 \) a CSS with the saddle–point property (SPP). At first sight it may appear
that \( d(\hat{u}) \) depends on the spatial discretization, i.e., on the number of \( n \) of nodes. However, \( d(\hat{u}) \)
remains constant for finer and finer meshes, see \[GU15, Appendix A\] for further comments.

For \( \hat{u} = (\hat{v}, \lambda) \) with the SPP, and \( \|v_0 - \hat{v}\| \) sufficiently small, we may expect the existence of a
solution \( u \) of \( (9) \), which moreover can be found from a Newton loop for \( (9) \) with initial guess
\( u(t) \equiv \hat{u} \). On the other hand, for larger \( \|v_0 - \hat{v}\| \) a solution of \( (9) \) may not exist, or a good initial
guess may be hard to find, and therefore we use a continuation process also for \( (9) \). In the simplest setting, assume that for some \( \alpha \in [0, 1] \) we have a solution \( u_\alpha \) of \( (9) \) with \( (9b) \) replaced by
\[
v(0) = \alpha v_0 + (1 - \alpha) \hat{v},
\]
(e.g., \( \alpha = 0 \) and \( u \equiv \hat{u} \)). We then increase \( \alpha \) by some stepsize \( \delta_\alpha \) and use \( u_\alpha \) as initial guess for
\( (9a) \), \( (9c) \) and \( (11) \), ultimately aiming at \( \alpha = 1 \).

To actually solve \( (9a) \), \( (9c) \) and \( (11) \) we use TOM \[MS02, MT04, MST09\] in a version \texttt{mtom}
which accounts for the mass matrix \( M \) on the lhs of \( (9a) \).\footnote{mtom is an ad-hoc modification of TOM, and will be replaced by an official version of TOM which handles mass matrices once that becomes available. In the following, when discussing, e.g., the behaviour and usage of \texttt{mtom}, we note that almost all of this is derived from TOM.} \footnote{see also \url{www.dm.uniba.it/~mazzia/mazzia/?page_id=433}} This predictor \( u_\alpha \) – corrector (\texttt{mtom}
for \( \alpha + \delta_\alpha \)) continuation method corresponds to the “natural” parametrization of the continuation
by \( \alpha \), and is thus implemented in \texttt{p2poc} as \texttt{iscnat} (Initial State Continuation NATural). We also
give the option to use a secant predictor
\[
u^j(t) = u^{j-1}(t) + \delta_\alpha \tau(t), \quad \tau(t) = \left( u^{(j-1)}(t) - u^{(j-2)}(t) \right) / \left( u^{(j-1)}(\cdot) - u^{(j-2)}(\cdot) \right) \|_2,
\]
\[\text{(12)}\]
where \( u^{j-2} \) and \( u^{j-1} \) are the two previous steps. However, the corrector still works at fixed \( \alpha \), in contrast to the arclength predictor–corrector iscarc described next.

It may happen that no solution of (14), (15) and (11) is found for \( \alpha > \alpha_0 \) for some \( \alpha_0 < 1 \), i.e., that the continuation to the intended initial states fails. In that case, often the BVP shows a fold in \( \alpha \), and we use a modified continuation process, letting \( \alpha \) be a free parameter and using a pseudo–arclength parametrization by \( \sigma \) in the BC at \( t = 0 \). Since mtom does not allow free parameters we add the dummy ODE \( \dot{\alpha} = 0 \), and BCs at continuation step \( j \),

\[
\langle s, (u(0) - u^{(j-1)}(0)) + s_\alpha(\alpha - \alpha^{(j-1)}) = \sigma, \quad (13)
\]

with \( u^{(j-1)}(\cdot) \) the solution from the previous continuation step \( j - 1 \), and \( (s, s_\alpha) \in \mathbb{R}^{2N} \times \mathbb{R} \) appropriately chosen with \( \| (s, s_\alpha) \|_s = 1 \), where \( \cdot \) \( \cdot \) is a suitable norm in \( \mathbb{R}^{2N+1} \), which may contain different weights of \( v \) and \( v_\alpha \). For \( s = 0 \) and \( s_\alpha = 1 \) we find iscnat with stepsize \( \delta_\alpha = \sigma \) again. To get around folds we may use the secant

\[
s := \xi\left(u^{(j-1)}(0) - u^{(j-2)}(0)\right)/\|u^{(j-1)}(0) - u^{(j-2)}(0)\|_2 \text{ and } s_\alpha = 1 - \xi
\]

with small \( \xi \), and also a secant predictor

\[
(u^{j}, \alpha^{j})^{\text{pred}} = (u^{j-1}, \alpha^{j-1}) + \sigma \tau
\]

for \( t \mapsto u^{j}(t) \) with

\[
\tau = \xi\left(u^{(j-1)}(\cdot) - u^{(j-2)}(\cdot)\right)/\|u^{(j-1)}(\cdot) - u^{(j-2)}(\cdot)\|_2 \text{ and } \tau_\alpha = 1 - \xi.
\]

This essentially follows [GCF+08 §7.2], and is implemented in a routine iscarc (Initial State Continuation ARClength).

Finally, given \( \hat{u} \), to calculate \( \Psi \), at startup we solve the generalized adjoint eigenvalue problem

\[
\partial_\alpha G(\hat{u})^T \Phi = \Lambda M \Phi
\]

for the eigenvalues \( \Lambda \) and (adjoint) eigenvectors \( \Phi \), which also gives the defect \( d(\hat{u}) \) by counting the negative eigenvalues in \( \Lambda \). If \( d(\hat{u}) = 0 \), then from \( \Phi \in \mathbb{C}^{2Nn \times 2Nn} \) we generate a real base of \( E_u(\hat{u}) \) which we sort into the matrix \( \Psi \in \mathbb{R}^{Nn \times 2Nn} \).

Acknowledgement. I thank D. Grass, ORCOS Wien, for introducing me to the field of optimal control over infinite time horizons, and for clarifying (and posing) many questions regarding the aim of software based on Pontryagin’s Maximum Principle for spatially distributed OC problems. In 1D, an alternative to p2poc is a finite differences module by Dieter for OCMat, [http://orcos.tuwien.ac.at/research/ocmat_software/], see [Gra15].

2 Examples and implementation details

2.1 The SLOC model

Following [BX08], in [GU15] we consider a model for phosphorus \( P = P(t,x) \) in a shallow lake, and phosphate load \( k = k(x,t) \) as a control, which in 0D, i.e., in the ODE setting, has analyzed in detail for instance in [KWT10]. Here we explain how we set up the spatial so called Shallow Lake Optimal Control (SLOC) problem in p2poc, and refer to [GU15] for details about the modelling and the interpretation of results. The model reads

\[
V(P_0(\cdot)) := \max_{k(\cdot)} J(P_0(\cdot), k(\cdot, \cdot)), \quad J(P_0(\cdot), k(\cdot, \cdot)) := \int_0^\infty e^{-\rho t} J_{el}(P(t), k(t)), dt
\]

(17a)
where \( J_c(P,k) = \ln(k) - \gamma P^2 \) is the local current value objective function,

\[
J_{ci}(P(t),k(t)) = \int_{\Omega} J_c(P(x,t),k(x,t)) \, dx
\]

(17b)
is the spatially integrated current value objective function, and \( P \) fulfills the PDE

\[
\partial_t P(x,t) = k(x,t) - bP(x,t) + \frac{P(x,t)^2}{1 + P(x,t)^2} + D\Delta P(x,t),
\]

\[
\partial_n P(x,t)|_{\partial \Omega} = 0, \quad P(x,t)|_{t=0} = P_0(x), \quad x \in \Omega \subset \mathbb{R}^d.
\]

(17c)

The parameter \( b > 0 \) is the phosphor degradation rate, and \( \gamma > 0 \) are ecological costs of the phosphor contamination \( P \). One wants a low \( P \) for ecological reasons, but for economical reasons a high phosphate load \( k \), for instance from fertilizers used by farmers. Thus, the objective function consists of the concave increasing function \( \ln(k) \), and the concave decreasing function \( -\gamma P^2 \). We consider two scenarios, namely

Scenario 1: \( D = 0.5, \rho = 0.03, \gamma = 0.5, b \in (0.5, 0.8) \) (primary bif. param.),

Scenario 2: \( D = 0.5, \rho = 0.3, b = 0.55, \gamma \in (2.5, 3.7) \) (primary bif. param.).

(18)

(19)

With the co-state \( q \) and local current value Hamiltonian

\[
\mathcal{H}(P,q,\lambda) = J_c(P,k) + q[k - bP + \frac{P^2}{1 + P^2} + D\Delta P],
\]

(20)

the canonical system for (17) becomes, with \( k(x,t) = -\frac{1}{q(x,t)} \),

\[
\partial_t P(x,t) = k(x,t) - bP(x,t) + \frac{P(x,t)^2}{1 + P(x,t)^2} + D\Delta P(x,t),
\]

\[
\partial_t q(x,t) = 2\gamma P(x,t) + q(x,t) \left( \rho + b - \frac{2P(x,t)}{(1 + P(x,t)^2)^2} \right) - D\Delta q(x,t),
\]

\[
\partial_n P(x,t)|_{\partial \Omega} = 0, \quad \partial_n q(x,t)|_{\partial \Omega} = 0, \quad P(x,t)|_{t=0} = P_0(x), \quad x \in \Omega.
\]

(21a)

(21b)

(21c)

We now explain how to use \texttt{p2poc} to calculate CSS and canonical paths for (21). For this we discuss files from the demo directory \texttt{slocdemo} (except for obvious library files), assuming that \texttt{slocdemo} is in the same directory as the libraries \texttt{p2plib}, \texttt{p2poclib} and \texttt{tom}.

### 2.1.1 Basics of \texttt{pde2path}, and the setup for CSS

We very briefly review the data structures of \texttt{pde2path}, and refer to [UWR14, DRUW14] for more details and the underlying algorithms. The basic structure is a Matlab struct, henceforth called \texttt{p} like problem, which has a (large) number of fields (and subfields), as indicated in Table 1. However, most of these can be set to standard values by calling \texttt{p=stanparam(p)}. At least in simple problems, the user only has to provide:

1. The geometry of the domain \( \Omega \) and the boundary conditions.
2. Function handles (in the semilinear setting of interest here) \texttt{sG} and, for speedup, \texttt{sGjac}, implementing \( G \), and its Jacobian.
3. An initial guess for a solution \( u \) of \( G(u) = 0 \), i.e., an initial guess for a CSS.
Table 1: Selection (with focus on the semilinear case \texttt{p.sw.sfem=1}) of fields in the structure \texttt{p} describing a \texttt{pde2path} problem; see \texttt{stanparam.m} in \texttt{p2plib} for detailed information on the contents of these fields and the standard settings.

| field   | purpose                                                                 |
|---------|-------------------------------------------------------------------------|
| fuha    | struct of function handles: in particular the function handles \texttt{p.fuha.sG}, \texttt{p.fuha.sGjac}, \texttt{p.fuha.bc}, \texttt{p.fuha.bcjac} defining and Jacobians. |
| nc, sw  | numerical controls and switches such as \texttt{p.sw.bifcheck},...       |
| u,np,nu | the solution \texttt{u} (including all parameters/auxiliary variables in \texttt{u(p.nu+1:end)}), the number of nodes \texttt{p.np} in the mesh, and the number of nodal values \texttt{p.nu} of PDE-variables |
| tau,branch | tangent \texttt{tau(1:p.mu+p.nc.nq+1)}, and the branch, filled via \texttt{bradat.m} and \texttt{p.fuha.outfu}. |
| sol     | other values/fields calculated at runtime, e.g.: \texttt{ds} (stepsize), \texttt{res} (residual), ... |
| usrlam  | vector of user set target values for the primary parameter, default \texttt{usrlam=[]}; |
| eqn,mesh | the tensors \texttt{c,a,b} for the semilinear FEM setup, and the geometry data and mesh. |
| plot, file | switches (and, e.g., figure numbers and directory name) for plotting and file output |
| mat     | problem matrices, e.g., mass/stiffness matrices \texttt{M}, \texttt{K} for the semilinear FEM setting. |

Typically, the steps 1-3 are put into an init routine, here \texttt{p=slinit(p,lx,ly,nx,ny,sw,rho)}, where \texttt{lx,ly,nx,ny} are parameters to describe the domain size and discretization, and \texttt{sw} is used to set up different initial guesses, see Table 2. The only additions/modifications to the standard \texttt{pde2path} setting for CSS problems are as follows: (the additional function handle) \texttt{p.fuha.jc} should be set to the local current value objective function, here \texttt{p.fuha.jc=@sljcf}, and \texttt{p.fuha.outfu} to \texttt{ocrbra}, i.e., \texttt{p.fuha.outfu=@ocrbra}. This automatically puts

\[
J_{\text{con}}(u) := \frac{1}{|\Omega|} \int_{\Omega} J_c(u(x)) \, dx
\]

at position 4 of the calculated output–branch. Here we generally use the \textit{averaged} current objective function since typically we want to normalize \(J_{\text{c}}\) by the domain size for simple comparison between different domains. Finally, it is useful to set \texttt{p.fuha.con=@slcon}, where \texttt{k=slcon(p,u)} extracts the control \texttt{k} from the states \texttt{v}, costates \texttt{\lambda} and parameters \texttt{\eta}, all contained in the vector \texttt{u}.

By calling \texttt{p=cont(p)}, \texttt{pde2path} then first uses a Newton–loop to converge to a (numerical) solution, and afterwards attempts to continue in the given parameter. If \texttt{p.sw.bifcheck>0}, then \texttt{pde2path} detects, localizes and saves to disk bifurcation points on the branch. Afterwards, the bifurcating branches can be computed by calling \texttt{swibra} and \texttt{cont} again. These (and other) \texttt{pde2path} commands (continuation, branch switching, and plotting) are typically put into a script file, here \texttt{bdcmds.m}, see Table 2.

Naturally, there are some modifications to the standard \texttt{pde2path} plotting commands, see, e.g., \texttt{plot1D.m}. These work as usual by overloading the respective \texttt{pde2path} functions by putting the adapted file in the current directory. See Fig. 1 for example results of running \texttt{bdcmds}.

### 2.1.2 Canonical paths

The goal is to calculate canonical paths from some starting state \texttt{v(0)} to a CSS \texttt{\hat{u}_1} with the SPP. For this we use one of the continuation algorithm \texttt{iscnat} or \texttt{iscarc} which in turn call \texttt{mtom}, based on TOM. Since we only wanted minimal modifications of TOM we found it convenient (though somewhat dangerous) to pass a number of parameters to the functions called by \texttt{mtom} via global variables. Thus, at the start of the canonical path scripts (here \texttt{cpdemo.m}) we define a number of global variables, see Table 4.

\footnote{Note that we do not use \texttt{slcon} in \texttt{s1sG}. However, putting this function for the control into \texttt{p} has the advantage that for instance plotting and extracting the value of the control can easily be done by calling some convenience functions of \texttt{p2poc}.}
Table 2: The init routine `slinit.m`, the rhs `slsG.m`, the objective function `sljcf.m`, and the function `slcon.m`. See also, e.g., the source code of `slsGjac` for the implementation of $G_u$.

```matlab
function p=slinit(p,lx,ly,nx,ny,sw) % init-routine
  p=stanparam(p); % set generic parameters to standard, if needed reset below..
  p.nc.neq=2; p.fuha.sG=@slsG; p.fuha.sGjac=@slsGjac; %rhs
  p.fuha.outfu=@ocbra; p.fuha.jcf=@sljcf; p.fuha.con=@slcon; % current-val-obj
  p.usrlam=[0.55 0.6 0.65 0.7 0.75]; % target-values for bif-param lam
  [p.mesh.geo,bc]=recnbc2(lx,ly); p.vol=4*lx*ly; % geometry, and volume of dom
  p.fuha.bc=@(p,u) bc; p.fuha.bcjac=@(p,u) bc; % standard Neumann BC
  p.xplot=lx; p.sw.spcalc=0; p.sw.jac=1; p.file.smod=100; % some more switches
  par=[0.03;0.55;0.5;0.5]; p.nc.ilam=2; % startup param values, and index of main param
  % r=par(1); bp=par(2); cp=par(3); D=par(4);
  p.nc.dsmin=1e-6; p.nc.dsmax=0.5; p.nc.lammax=0.8; p.nc.lammin=0.549; p.sol.ds=0.1;
  p=stanmesh(p,nx,ny);p=setbmesh(p); p.sol.xi=1/p.np; % mesh
  p.eqn.c=[1;0;0;0;1]; p.eqn.a=0; p.eqn.b=0; % diffusion tensor and a,b
  switch sw % choose initial guess arcoording to switch
    case 1; u=0.3*ones(p.np,1); v=-13*ones(p.np,1); u0=[u v]; p.u=u0(:); % FSC
    case 2; u=2*ones(p.np,1); v=-4*ones(p.np,1); u0=[u v]; p.u=u0(:); % FSM
    case 3; u=3*ones(p.np,1); v=-4*ones(p.np,1); u0=[u v]; p.u=u0(:); par=[0.3;0.55;2.5;0.5]; p.nc.lammax=3.7; p.nc.lammin=2.5; p.usrlam=3.25;
    end
  p.u=[p.u; par]; p.sw.sfem=1; p=setfemops(p); % semilin. setting
  [p.u,res]=nloop(p,p.u); fprintf('first res=%g
',res); plotsol(p,1,1,1);
end

function r=slsG(p,u) % CS for SLOC
  % p_t=D*p/1-q-b*p+p^2/(1+p^2)
  % q_t=-D lap q+2cp*p+q*(rho+bp-2*p/(1+p^2)^2;
  par=u(p.nu+1:end); r=par(1); bp=par(2); cp=par(3); D=par(4);
  P=u(1:p.np); q=u(p.np+1:p.nu); f1=-1./q-bp*P+P.^2./(1+P.^2); f2=2*cp*P+q.*(r+bp-2*P./(1+P.^2).^2);
  r=[f1;f2]; f=D*p.mat.K*u(1:p.nu)-p.mat.M*f;
end

function jc=sljcf(p,u) % current value J
  cp=u(p.nu+3:end); pv=u(1:p.np); kv=-1./u(p.np+1:p.u); jc=log(kv)-cp*pv.^2;
end

function k=slcon(p,u) % extract control from states/costates
  k=-1./u(p.np+1:p.u);
end
```

The usage of `p2poc` to compute canonical paths is best understood by running and inspecting the demo file `cdemo.m`. However, we start with a brief overview of the involved `p2poclib` functions, with the line-numbers refering to Table 6.

```matlab
[alv,vv,sol,udat,tlv,tv,uv]=iscnat(alvin,sol,usec,opt,fn); %line 9
```

Input: `alvin` as the vector of desired $\alpha$ values, for instance `alvin=[0.25 0.5 1]`. `sol,usec` can be empty (typically on first call), but on subsequent calls should contain the last solution and the last secant (if `opt.msw=1`). `fn` is a structure containing the filenames for the start CSS and end CSS, and `opt` is an options structure containing TOM options and some more,

---

5Taking $v$ from a CSS is basically for convenience: Of course, the initial states $v_0$ can be arbitrary, and there are no initial conditions for the co-states (and in particular those of the “start CSS” are not used). However, the construction is also motivated by the fact that one of the most interesting questions is if given $v$ from some CSS $\hat{u}_0$ there exists a canonical path to an end CSS $\hat{u}_1$, and whether this yields a higher value. Then, it is of course also interesting if one can also go the other way round, and for this we provide the `flip` parameter in `setfnflip`, see below.

---

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Table 3: Selected commands from the script file bdcmds.m. See the source code for more details, and, e.g., bdcmds2D.m for the quite similar commands in 2D. Note that these files are in Matlab cell-modes, and the cells should be executed one by one. In the last command, p.fuha.con must be set.

%% stat. BD for sloc, main script-file. First set matlab paths:
path('./p2plib','path');path('./p2poclib','path');

%% ---- Scenario 1, FSC/FSI branch
close all; p=[];lx=2*pi/0.44; ly=0.1; nx=50; ny=1; sw=1; p=slinit(p,lx,ly,nx,ny,sw);
p=setfn(p,'f1'); screenlayout(p); p=cont(p,100);

%% ---- FSM branch
sw=2; p=slinit(p,lx,ly,nx,ny,sw); p=setfn(p,'f2'); p.nc.dsmax=0.2; p=cont(p,15);

%% ---- bif from f1 (set bpt* and p* and repeat as necessary)
p=swibra('f1','bpt1','p1',-0.05); p.nc.dsmax=0.3; p.nc.neigdet=50; p=cont(p,150);

%% ---- plotting of BD, L2 and J_{ca}, and solution plots
clf(3); pcmp=3; plotbraf('f1','bpt1',3,pcmp,'lab',12,'cl','k'); % FSC (+ other branches)
clf(3); pcmp=4; plotbraf('f1','bpt1',3,pcmp,'lab',12,'cl','k'); % FSC (+ other branches)
plot1Df('p1','pt16',1,1,1,2); plot1Df('p1','pt71',2,1,1,2); % solution plots ...

stancssvalf('p1','pt16'); % extract values <P>, <k>, J_{c,a} from solution in p1/pt16;

Table 4: Global variables for the computation of canonical paths, i.e., mainly for interfacing the driver scripts with the functions called by TOM.

| name     | purpose                                      |
|----------|----------------------------------------------|
| s0,s1    | pde2path structs containing the boundary values at $t = 0$ (s0) and at $t = T$ (s1) |
| psi      | the matrix $\Psi$ to encode the BC at $t = T$ |
| u0, u1   | vectors containing the current values of $u$ at the boundaries |
| par      | the parameter values from s1 (only for convenience) |
| um1, um2 | solutions at continuation steps $j - 1$ and $j - 2$ (to calculate secant predictors and used in extended system in iscarc) |
| sig      | current (arc length) stepsize in iscarc       |

see Table 5 and Remark 2.1

Output: alv as the $\alpha$ vector of successful solves; vv as the canonical path values $J_{\alpha}$ of the successful solves; sol as the (last) canonical path; ydat contains the 2 last steps and the last secant (useful for repeated calls), and for using iscnat as startup for iscarc. Finally, if opt.retsw=1, then tlv, tv, uv contain data of all successful solves, namely: for the $j$-th step, $j=1:length(alv)$, $tlv(j)$, contains the meshsize (in $t$), $tv(j,1:tlv(j))$ the mesh, and $uv(j,1:n,1:tlv(j))$ the solution. Thus, via $sol.x=xv(j,1:tlv(j)); sol.y=squeeze(uv(j,1:n,1:tlv(j)))$; the solution of the $j$-th step is recovered. This is useful for posteriori inspecting some solution from the continuation (see line 24, and skiba.m in vegdemo). Note that uv can be large, and might give memory problems. If opt.retsw=0, then tlv, tv, uv are empty.

[alv,vv,usec,esol,tlv,uv,uv]=iscarc(esol,usec,opt,fn); (line 15)

Input: as for iscnat (without alvin), but with esol containing the extended solution $(u_\alpha, \alpha)$, and similar for the secant usec; as in iscnat, if opt.start=1, then esol, usec can be empty.

Output: alv, val $\in \mathbb{R}^{1 \times m}$ as the vectors of achieved $\alpha$ and $J_\alpha(\alpha)$; usec, esol as the last secant/solution; tlv, tv, uv as in iscnat, but all in the sense of extended solutions, i.e. $(u_\alpha, \alpha)$.

Remark 2.1. Concerning the original TOM options we remark that typically we run iscnat and iscarc with weak error requirements and what appears to be the fastest monitor and order options, i.e., opt.Monitor=3; opt.order=2; Once continuation is successful (or also if it fails at some $\alpha$), we can always postprocess by calling mtom again with a higher order, stronger error requirements,
Figure 1: Example bifurcation diagrams and solution plots from running `bdcmds.m`. For $b < b_{\mathrm{fold}} \approx 0.73$ there are three branches of FCSS, here called FSC (Flat State Clean, low $P$), FSI (Flat State Intermediate), and FSM (Flat State Muddy, high $P$). On FSC∪FSI there are a number of bifurcations to patterned CSS branches. The BD in $J_{\mathrm{ca}}$ in (b) shows that at, e.g., $b = 0.65$ we have $J(\text{FSI}) < J(\text{FMS}) < J(p1/p16) < J(p1/p71)$, suggesting the FSC to be an optimal steady state. However, this does not exclude p1/p16 or p1/p71 to be optimal steady states, and moreover, there might be a path from the FSC to some other CSS with the SPP, dominating the FSC as a CSS. See [GU15] for further discussion.

Table 5: Switches/controls in `opt` besides the TOM options. Note that `iscarc` only has a very elementary stepsize control via `opt.sigmin,opt.sigmax`.

| name        | purpose                       | name        | purpose                       |
|-------------|-------------------------------|-------------|-------------------------------|
| start       | 1 for startup, 0 else         | M,lu,vsw    | mass matrix, lu-switch and    |
| retsw       | 1 to return full continuation data | (extra) verbosity for `mtom` |
| rhoi        | index of $\rho$ in `par`      | t1          | truncation time $T$           |
| nti         | # of points in startup $t$–mesh | tv          | current $t$–mesh              |
| nsteps      | number of steps for `iscarc`  | sigmin,sigmax | min&max stepsize for `iscarc` |

and different monitor options, e.g., mesh–refinement based on condition rather than error alone. See the original TOM documentation.

The functions `iscnat` and `iscarc` are the two main user interface functions for the canonical path numerics. There are a number of additional functions for internal use, and some convenience functions, which we briefly review as follows:

`[\Psi,\mu,d,t]=getPsi(s1)`; compute $\Psi$, the eigenvalues $\mu$, the defect $d$, and a suggestion for $T$. Note that this becomes expensive with large $2nN$ (i.e., the total number of DoF).

`[sol,info]=mtom(\text{ODE,BC,solinit,opt,varargin})`; the ad–hoc modification of TOM, which allows for $M$ in (9). Extra arguments $M$ and $\text{lu,vsw}$ in `opt`. If `opt.lu=0`, then \_ is used for solving linear systems instead of an LU–decomposition, which becomes too slow when $nN \times m$ becomes too large. See the TOM documentation for all other arguments including `opt`, and note that the modifications in `mtom` can be identified by searching “HU” in `mtom.m`. Of course `mtom` (as any other function) can also be called directly (line 26), which for instance is useful to postprocess the output of some continuation by changing parameters by hand.

$f=mrhs(t,u,k)$; $J=fjac(t,u)$; and $f=mrhse(t,u,k)$; $J=fjace(t,u)$; the rhs and its Jacobian to be called within `mtom`. These are just wrappers which calculate $f$ and $J$ by calling the resp. functions in the `pde2path–struct s1`, which were already set up and used to calculate the CSS. $\text{s1}$ is passed as a global variable. Similar remarks apply to `mrhse` and `fjace` for the extended setting in `iscarc`. 

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Table 6: Selected commands from the script file `cpdemo.m`. See the source code for more details and, e.g., more fancy plotting.

1. `% driver script for Shallow Lake Optimal Control, first set paths and globals
   path('../tom',path); path('../p2poclib',path);
2. close all; clear all; global s0 s1 u0 u1 Psi par xi ym1 ym2 sig;
3. `%% Preparations: put filenames into fn, set some bvp parameters
   sd0='f1'; sp0='pt12'; sd1='p3'; sp1='pt19'; flip=1; % p3->FSC
   fn=setfnflip(sd0,sp0,sd1,sp1,flip); opt=[]; opt=ocstanopt(opt);
4. `%% the solve and continue call, and some plots
   sol=[]; alvin=[0.1 0.25 0.5 0.75 1]; v=[15,30];
5. `[alv,vv,sol,ydat,tlv,xv,yv]=iscnat(alvin,sol,[],opt,fn); slsolplot(sol,v);
6. `%% ---- A fold in alpha, here iscarc needed. Prep. and initial iscarc call
   sd0='f1'; sp0='pt12'; sd1='p1'; sp1='pt71'; flip=1; fn=setfnflip(sd0,sp0,sd1,sp1,flip);
   esol=[]; ysec=[]; opt.nsteps=3; opt.alvin=[0.2 0.25]; sig=0.1; opt.nti=10; opt.tv=[];
7. `[alv,vv,ysec1,esol1,tlv,xv,yv]=iscarc(esol,ysec,opt,fn); opt.start=0;
8. `%% subsequent iccarc-calls (repeat this cell)
   opt.nsteps=20; ysec=ysec1; esol=esol1; % new input (for repeated calls)
   [alv1,vv1,ysec1,esol1,tlv1,xv1,yv1]=iscarc(esol,ysec,opt,fn);
9. `%% Postprocess sol from iscarc, first a simple plot of J over alpha
   alv0=alv; vv0=vv; xv0=xv; yv0=yv; tlv0=tlv; % save results for skibademo.m
   figure(6); clf; plot(alv0(1,:),vv0(1,:),'-*'); xlabel('\alpha'); ylabel('J_a');
10. `%% fix al from iscarc to some given value and compute CPs, first j=22, then j=34
    j=22; tlv=tlv(j); n=s1.nu; sol.x=xv(j,1:tlv(j));sol.y=squeeze(yv(j,1:n,1:tlv(j)));
    al=0.6; u0=al*s0.u(1:n)+(1-al)*s1.u(1:n); u1=s1.u(1:s1.nu);
    opt.M=s1.mat.M; sol=mtom(@mrhs,@cbcf,sol,opt); v=[100,30]; slsolplot(sol,v);

bc=cbcf(ya,yb);[ja,jb]=cbcjac(ya,yb); bc=cbcf(ya,yb);[ja,jb]=cbcjace(ya,yb): The
boundary conditions (in time) for (9) and the associated Jacobians. Implemented by passing
u_0, \hat{u}_1, \Psi and similar globally. The *e (as in extended) versions are for iscarc.
jcaval=jcai(s1,sol,rho) and djca=isjca(s1,sol,rho); The normalized objective value
\[ J_a(u) = \int_0^T e^{-\rho t} J_c(a(v(t,\cdot),k(t,\cdot))) \, dt \] (23)
of the solution u in sol (with J_c taken from s1.fuha.jcf), and similarly the normalized
discounted value of a CSS contained in sol.y(:,end).

fn=setfnflip(sd0,sp0,sd1,sp1,flip); generate the filename struct fn from sd0, sp0 (sd0/sp0.mat
contains IC u_0) and sd1,sp1 (contains u); if flip=1, then interchange *0 and *1.
psol3D(p,sol,wnr,cmp,v,tit); x-t plots of canonical paths; plot component cmp of a canonical
path sol to figure wnr, with view v and title tit. If cmp=0, then plot the control k, extracted
from sol via p.fuha.con.

Thus, after having set up p as in §2.1.1 for the CSS, including G and p.fuha.jcf, the user does
not need to set up any additional functions to calculate canonical paths and their values. However,
typically there are some functions which should be adapted to the given problem, e.g., for plotting,
for instance

slsolplot(sol,v); (line 10) which calls:

zdia=zldiagn(sol,wnr); Plot some norms on a canonical path as functions of t to figure(wnr).

This is for instance useful to check the convergence behaviour of the canonical path as t \to T,
cef. (d),(f) in Fig. 2.
Some results of running \texttt{cpdemo} are shown in Fig. 2 (a) shows the "easy" case of a canonical path from p3/pt19 to FSC (up to line 11), while (b) to (f) illustrate the case of a fold in $\alpha$ when trying to get a canonical path from p1/pt71 to the FSC. (c)-(e) show the two canonical paths obtained from picking two canonical paths from the output of \texttt{iscarc} and a posteriori correcting to $\alpha = 0.6$ (lines 24-26, only for the "upper" canonical path). Line 21 from \texttt{slcpdemo} prepares the calculation of Skiba paths, explained next.

### 2.1.3 A patterned Skiba point

In ODE OC applications, if there are several locally stable OSS, then often an important issue is to identify their domains of attractions. These are separated by so-called threshold or Skiba–points (if $N = 1$) or Skiba–manifolds (if $N > 1$). Roughly speaking, these are initial states from which there are several optimal paths with the same value but leading to different CSS. In PDE applications, even under spatial discretization with moderate $nN$, Skiba manifolds should be expected to become very complicated objects. Thus, here we just give one example how to compute a patterned Skiba point between FSC and FSM.

In Line 17-19 of \texttt{cpdemo.m} we attempt to find a path from $P_{FS}$ given by p1/pt71 to $(P, q)_{FSC}$ given by FSC/pt12; this fails due to the fold in $\alpha$. However, for given $\alpha$ we can also try to find...
a path from the initial state $P_α(0) := αP_S + (1 − α)P_{FSC}$ to the FSM, and compare to the path to the FSC. For this, in line 21 of cpdemo.m, we stored the $α$ and $J_{α}$ values into alv0, vv0, and also the path data into tlv0, tv0. See skibademo.m in Table 7 (in particular line 11 and the following) how to put the values $uv0(j, :, 1)$ into $s0$ and subsequently find the paths to the FSM, and Fig. 3 for illustration.

Table 7: The script file skibademo.m. See text for comments.

```matlab
1 % Skiba example, continues cpdemo.m
2 % find paths from the yv0 initial states from cpdemo.m to FSM
3 js=10; je=30; jl=js-je+1; % alpha-range; now set the target and Psi to FSM:
4 s1=loadp('f2','pt11'); u1=s1.u(1:s1.nu); [Psi,muv,d,t1]=getPsi(s1);
5 a0l=length(alv0); tva=zeros(jl,opt.Nmax+1); % some prep. and fields to hold paths
6 uva=zeros(jl,n+1,opt.Nmax+1); opt.start=0; alva=[]; vva=[]; tavl=[]; sol=[];
7 alvin=[0.1 0.25 0.5 0.75 1]; % we run from uv0(j,:) to FSM with iscnat
8 tv=linspace(0,opt.t1,opt.nti); se=2; opt.tv=tv.^se./opt.t1^(se-1); doplot=1;
9 opt.msw=0; opt.Stats_step='off'; v=[50,8]; % switch off stats
10 for j=js:je;
11     fprintf('j=%i, al=%g
', j, alv0(j)); s0.u(1:n)=uv0(j,1:n,1)';
12     [alv,vv,sol,udat]=iscnat(alvin,[],[],opt,fn);
13     if alv(end)==1; Jd=vv0(j)-vv(end); fprintf('J1-J2=%g
',Jd); % contin. successful
14     alva=[alva alv0(j)]; vva=[vva vv(end)]; tlv0(j,1:tlv0(j))=sol.x; tv0(j,1:tlv0(j))=sol.y;
15     if abs(Jd)<0.05; doplot=asknu('plot path?',doplot); % Skiba point(s) found
16     if doplot==1; sol0=[]; alp=alv0(j); % plot the paths to FSC and FSM
17     sol0.x=tv0(j,1:tlv0(j));sol0.y=squeeze(uv0(j,1:n,1:tlv0(j)));
18     psol3Dm(s1,sol0,sol,1,1,[]); view(v); zlabel('P'); pause
19     end
20 end
21 end
22 end
23
% plot value diagram
figure(6); plot(alv0(js:jep),vv0(js:jep),’-*b’);hold on;plot(alva,vva,’-*r’);
xlabel(‘\alpha’,’FontSize’,s1.plot.fs); ylabel(‘J_{a}’,’FontSize’,s1.plot.fs);
```

2.1.4 Further comments

To keep the demos simple, here we do not include versions of iscnat and iscarc that use the truncation time $T$ as an additional free parameter [GU15, Fig.5]. However, the files bdcmds.m

(a) A Skiba at $α = 0.454$  
(b) Paths to FSC (blue) and FSM (red)
and \texttt{cpdemo.m} also contain some of the commands used to study Scenario 2, see \cite{GU15} §2.3 for the results, and the directory \texttt{slodemo} contains the script files \texttt{bdcmds2D.m} and \texttt{cpdemo2D.m}, used to compute CSS and canonical paths for \cite{GJ14} over the 2D domain $\Omega = (-L,L) \times (-\frac{L}{2}, \frac{L}{2})$ (based on exactly the same init file \texttt{slinit.m}), some auxiliary plotting functions \texttt{plotsoflf.m} and \texttt{plotsoflfu.m}, and the function \texttt{sol2mov.m} used to generate movies of canonical paths. See, e.g., \cite[Fig. 7,8]{GU15} for some results, while some movies can be downloaded at the \texttt{pde2path} homepage.

### 2.2 The vegOC model

Our second example, from \cite{Uec15}, concerns the optimal control of a reaction diffusion system used to model grazing in a semi arid system for biomass (vegetation) $v$ and soil water $w$, following \cite{BX10}. Here, semi arid means that there is enough water to support some vegetation, but not enough water for a dense homogeneous vegetation. This is an important problem as it is estimated that semi arid areas cover about 40% of the world’s land area and support about two billion people, often by grazing livestock, \url{www.allcountries.org/maps/world_climate_maps.html}. In semi arid areas, often overgrazing is a serious threat as it may lead to irreversible desertification, see, e.g., \cite{SBB+09}, and the references therein.

Denoting the harvesting (grazing) effort as the control by $E$, we consider

\begin{align}
V(v_0, w_0) &= \max_{E(\cdot)} J(v_0, w_0, E),
\intertext{and current value objective function $J_c = J_c(v, E) = pH - cE$, which thus depends on the price $p$, the costs $c$ for harvesting/grazing, and $v, E$ in a classical Cobb–Douglas form with elasticity parameter $0 < \alpha < 1$. For the modeling, and the meaning and values of the parameters $(g, \eta, d, \delta, \beta, \xi, r_u, r_w, d_{1,2})$ we refer to \cite{BX10, Uec15} and the references therein (see also Table 8, line 9 for the parameter values), and here only remark that $\rho = 0.03, p = 1.1, \alpha = 0.3, c = 1$ are the economic parameters, and we take the rainfall $R$ as the main bifurcation parameter. Furthermore, we have the BC and IC}
\begin{align}
\partial_t v &= \partial_v w = 0 \text{ on } \partial \Omega, \quad (v, w)|_{t=0} = (v_0, w_0).
\end{align}

Denoting the co-states by $(\lambda, \mu)$ we have the local current value Hamiltonian

\begin{align}
\mathcal{H}(v, w, \lambda, \mu, E) &= J_c(v, E) + \lambda [d_1 \Delta v + (gw^p - d(1 + \delta v) - H) v]
\intertext{and obtain the canonical system}
\begin{align}
\partial_t v &= \mathcal{H}_\lambda = d_1 \Delta v + (gw^p - d(1 + \delta v) - H) v,
\partial_t w &= \mathcal{H}_\mu = d_2 \Delta w + R(\beta + \xi v) - (r_u v + r_w) w,
\partial_t \lambda &= \rho \lambda - \mathcal{H}_v = \rho \lambda - \rho \lambda - pcv^{\alpha-1}E^{1-\alpha} - \lambda [g(\eta + 1)w v^p - 2d\delta v - d - \alpha v^{\alpha-1}E^{1-\alpha}]
\intertext{and}
\partial_t \mu &= \rho \mu - \mathcal{H}_w = \rho \mu - \lambda gw v^{\alpha+1} + \mu (r_u v + r_w) - d_2 \Delta \mu,
\end{align}

where $E$ is obtained from solving $\partial_E \mathcal{H} = 0$, giving

\begin{align}
E = \left( \frac{c}{(p-\lambda)(1-\alpha)} \right)^{-1/\alpha} v.
\end{align}
With the notation $u = (v, w, \lambda, \mu)$, the IC, the BC, and the transversality condition are

$$
(v, w)|_{t=0} = (v_0, w_0), \quad \partial_t u = 0 \text{ on } \partial \Omega, \quad \lim_{t \to \infty} e^{-\rho t} u(t) = 0. \tag{26f}
$$

To study (26), we write it as $\partial_t u = -G(u)$ and basically need to set up $G$ and the BC. This follows the general `pde2path` settings with the OC related modifications already explained in §2.1 and thus we only give the following remarks, first concerning `veginit.m`, see Table 8.

In line 2 we only set up `p.fuha.sG` since in this demo we use `p.sw.jac=0` (numerical Jacobians), and hence do not need to set `p.fuha.sGjac`.

Lines 4-7 set the Neumann BC and diffusion tensor for the 4 component system (see `gnbc.m` and `isoc.m` for documentation).

Lines 8-10 set the desired $R$ values for output of CSS to disk, the parameter values, and the main bifurcation parameter. Of course, one could also hard-code all parameters except $R$, but we generally recommend to treat parameters as parameters since this is needed if later a continuation in some other parameter is desired, and since it usually makes the code more readable.

Table 8: Selected commands from the init-routine `veginit.m`. See the source code for more details.

```plaintext
1 function p=veginit(p,lx,ly,nx,ny,sw,rho) % init-routine for vegOC
2 p=stanparam(p); p.nc.neq=4; p.fuha.sG=@vegsG; p.fuha.jcf=@vegjcf; p.fuha.outfu=@ocbra;
3 p.mesh.geo=rec(lx,ly); p=stanmesh(p,nx,ny); p.sol.xi=0.005/p.np; % generate mesh
4 q=zeros(p.nc.neq); g=zeros(p.nc.neq,1); % setting up Neumann BC for 4 components
5 bc=gnbc(p.nc.neq,4,q,g); p.fuha.bc=@(p,u) bc; p.fuha.bcjac=@(p,u) bc;
6 p.d1=0.05; p.d2=10; p.eqn.a=0; p.eqn.b=0; % setting up $K$ for 4 components
7 c=diag([p.d1, p.d2, -p.d1, -p.d2]); p.eqn.c=isoc(c,4,1);
8 p.usrlam=[4 10 26 28]; % desired $R$ values for output of CSS
9 p.nc.ilam=8; % choose the active par, here Rainfall $R$
10 % now continue with setting a few more param and the initial guess ... see veginit.m
```

Table 9 shows the complete codes for setting up $G$ and the current value $J_c$. Both use the auxiliary function `efu`, which is also used in, e.g., `valf.m` to tabulate characteristic values of CSS.

Figure 4 shows a basic bifurcation diagram of CSS in 1D with $\Omega = (-L, L)$, $L = 5$, from the script file `vegbd1d.m`, which follows the same principles as the one for the SLOC demo. The blue branch in (a) represents the primary bifurcation of PCSS, which for certain $R$ have the SPP, and, moreover, are POSS. See also [Uec15] for more plots, including a comparison with the uncontrolled case of so called “private optimization”, and 2D results for $\Omega = (-L, L) \times (-\sqrt{3}L/2, \sqrt{3}L/2)$ yielding various POSS, including hexagonal patterns.

The script files `vegcpdemo.m` for canonical paths, and `vegskiba.m` for a Skiba point between the flat optimal steady state FSS/pt13 and the POSS p1/pt34, again follow the same principles as in the the SLOC demo. See Figures 5, and 6 for example outputs, and [Uec15] for a detailed discussion. In a nutshell, we find that:

(a) For large $R$ the FCSS is the unique CSS of (26), and is optimal, hence a globally stable FOSS (Flat Optimal Steady State).

(b) For smaller $R$ there are branches of (locally stable) POSS (Patterned Optimal Steady States), which moreover dominate all other CSS.

(c) For the uncontrolled problem, Flat Steady States (FSS) only exist for much larger $R$ than the FCSS under control.

(d) At equal $R$, the profit $J$ (or equivalently the discounted value $J_c/\rho$) of the uncontrolled FSS is much lower than the value of the FCSS under control.
Table 9: Implementation of the rhs $G$ and $J_c$ for (26), and the aux function efu.

```matlab
function r=vegsG(p,u) % rhs for vegOC problem
par=u(p.nu+1:end); rho=par(1); g=par(2); eta=par(3); % extract param
d=par(4); del=par(5); beta=par(6); xi=par(7); rp=par(8); up=par(9); rw=par(10);
cp=par(11); pp=par(12); al=par(13); [e,h,J]=efu(p,u); % calculate H
v=u(1:p.np); w=u(p.np+1:2*p.np); % extract soln-components, states
l1=u(2*p.np+1:3*p.np); l2=u(3*p.np+1:4*p.np); % co-states
f1=(g*w.*v.^eta-d*(1+del*v)).*v-h; f2=rp*(beta+xi*v)-(up*v+rw).*w; % f1,f2
f3=rho*l1-pp*al*h./v-l1.*(g*(eta+1)*w.*v.^eta-2*d*del*v-d-al*h./v)-l2.*(rp*xi-up*w);
f4=rho*l2-l1.*(g*v.^(eta+1))-l2.*(-up*v-rw); f=[f1;f2;f3;f4];
r=p.mat.K*u(1:p.nu)-p.mat.M*f; % the residual

function jc=vegjc(p,u); [e,h,jc]=efu(p,u); % J_c for vegOC, here just an interface

function [e,h,J]=efu(p,varargin) % extract [e,h,J] from p or u
if nargin>1 u=varargin{1}; else u=p.u; end
par=p.u(p.nu+1:end); cp=par(11); pp=par(12); al=par(13);
v=u(1:p.np); l1=u(2*p.np+1:3*p.np);
gas=((pp-l1)*(1-al)./cp).^(1/al); e=gas.*v; h=v.^al.*e.^(1-al);
J=pp.*v."al.*e."-(1-al)-cp*e;
```

3 Summary and outlook

With p2poc we provide a toolbox to study OC problems of class (1) in a simple and convenient way, in 1D and 2D. The class (1) is quite general, and with the pde2path machinery we have a rather powerful tool to study the bifurcations of CSS. The computation of canonical paths is comparatively more involved. We found it to be reasonably fast for up to 4000 degrees of freedom of $u$ at fixed time, e.g., 1000 spatial discretization points and 4 components, and up to 200 temporal discretization points, i.e., up to these values a continuation step in the calculation of a canonical path takes up to a few minutes on a desktop computer.

Of course, there is a rather large number of issues we do not address (yet). One of these are inequality constraints that frequently occur in OC problems, for instance non-negativity of $P$ and $k$ in the SLOC model, and similarly of $v, w$ and $E$ in the vegOC model. In our examples we simply checked these a posteriori and found them to be always fulfilled, i.e., inactive. If such constraints become active one typically encounters multi-point BVPs, see, e.g., [CF+08 Chapter 7]. Some extensions in this direction will be added as required by examples.

As currently p2pOC is based on pde2path, it works most efficiently for spatial 2D problems, while 1D problems are treated as very narrow quasi 1D strips. Presently, pde2path is extended to efficiently treat also 1D and 3D problems, based on the package OOPDE [www.mathe.tu-freiberg.de/nmo/mitarbeiter/uwe-pruefert/software]. Thus, p2pOC will soon be optimized for 1D problems as well.

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Figure 4: Outputs of `bdiddemo.m`. (a),(b) bifurcation diagrams of CSS in 1D; (c) example solutions.

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(a) $R = 26$, the canonical path from the lower PCSS($p1/pt11$) to the FCSS (FSS/pt17)

(b) $R = 26$, the canonical path from the FCSS to the upper PCSS ($p1/pt38$)

Figure 5: Example output of `vegcpcm.m`: Two canonical paths. The leftmost panels indicate the convergence behaviour, the current value profits, and obtained objective values. The middle and right panels show the strategy $E$ and the corresponding behaviour of $(v, w)$. See [Uec15] for comments and more details.

(a) A Skiba at $\alpha \approx 0.9$ (b) Paths of (almost) equal values to the FCSS and the upper PCSS.

Figure 6: $R = 28$, example outputs from `skibademo.m`. In (a), the blue line gives $J$ for the canonical path $t \mapsto u(t)$ from $(v, w)_\alpha(0) := \alpha(v, w)_\text{PS} + (1 - \alpha)(v, w)_\text{FSS}$, where FSS denotes FSS/pt13, and PS denotes $p1/pt16$. The red line gives $J$ for the CP $t \mapsto \tilde{u}(t)$ from $P_\alpha(0)$ to the upper PCSS $p1/pt34$. Similarly, the white surfaces in (b) are for $u$ and the colored ones for $\tilde{u}$. $R = 28$. 

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