Thermofield Dynamics for Twisted Poincaré-Invariant Field Theories: Wick Theorem and S-matrix

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Abstract

Poincaré invariant quantum field theories can be formulated on non-commutative planes if the statistics of fields is twisted. This is equivalent to state that the co-product on the Poincaré group is suitably twisted. In the present work we present a twisted Poincaré invariant quantum field theory at finite temperature. For that we use the formalism of Thermofield Dynamics (TFD). This TFD formalism is extend to incorporate interacting fields. This is a non trivial step, since the separation in positive and negative frequency terms is no longer valid in TFD. In particular, we prove the validity of Wick’s theorem for twisted scalar quantum field at finite temperature.
I. INTRODUCTION

Field theory at finite temperature is of paramount importance in describing several aspects of many-body systems and particle physics, involving for instance spontaneous symmetry breaking or the restoration of symmetry, as in super-conductivity and in high energy physics [1, 2]. For this reason, there is a strong interest in the extension of thermal quantum field theories to the realm of fields associated with aspects of nature such as non-commutativity [3, 4].

A central ingredient of a non-commutative quantum field theory on Groenenwold-Moyal plane [5, 6] is the mixing of ultraviolet and infrared divergences (UV-IR mixing) in the perturbation theory [7]. This UV-IR mixing, being a signature of non-commutativity due to the appearance of non-planar diagrams in perturbative expansions, is relevant for applications of non-commutativity in condensed matter systems, including the finite temperature cases, as in the quantum hall effect [8]. In addition, temperature effects are important as a way for testing non-commutativity in space coordinates, considering the limit of high temperature [9]. These results have motivated different studies involving temperature aspects [10–17], using the imaginary formalism [18] or thermofield dynamics (TFD) [1, 2], i.e., a real time formalism that in equilibrium is equivalent to the Keldish-Schwinger formalism [19, 20].

The imaginary time formalism takes the Boltzmann factor, \( \exp(-\beta H) \), where \( \beta \) is the inverse of temperature \( T \) (\( \beta = 1/T \), with \( \kappa_B \equiv 1 \)), under a Wick rotation in the time evolution, such that time \( t \) is mapped into a imaginary time: \( t \to t = i\tau \), with \( 0 \leq \tau \leq \beta \). The theory satisfies the KMS (Kubo, Martin and Schwinger) boundary condition, and then the propagator is written as a Fourier series in the imaginary time, by using the Matsubara frequencies: \( \omega_n = 2\pi n/\beta \) (\( \omega_n = \pi(2n + 1)/\beta \)) for bosons (fermions), corresponding to the period \( \beta = T^{-1} \), with \( T \) being the temperature. As as consequence, in the momentum space, the original theory is reduced to a 3-dimensional Euclidean formalism, involving an infinite summation of the Matsubara frequencies. However imaginary time formalism can be applied just to systems in equilibrium. This problem does not exist in real time formalism.

On the other hand, TFD is fully structured on algebraic methods using the notion of \( \mathbb{C}^* \)-algebras [21, 23] and synthesized in two ingredients: a doubling in the Hilbert space (a consequence of the commutants of the von Neumann algebra used in the definition of
the C*-algebra) and a Bogoliubov transformation, i.e., a rotation involving the two distinct types of linear spaces, leading to the thermalization of the theory. A consequence of such an apparatus is that there is no imaginary time in the theory and the propagator is written in two pieces: one describes the $T = 0$ theory, while the other gives rise to temperature effects. This aspect is useful when effects of temperature are competing with other parameters of a theory as is the case of non-commutative approach. This fact has been explored in the context of non-commutativity, in particular for twisted Poincaré-invariant theories [24].

In quantum field theory, a crucial property is the statistics of particles. That is encapsulated in the problem of how the states of the system behaves under the interchange of two particles. This many particle property has an entangled relation with the spin nature of the identical particles that forms the system. On the other hand, the spin nature of these particles is related to the representation of the Poincaré group. For space-time dimension strictly greater than 3, there are only two possible classes of spin, half-integer and integer, leading to two classes of statistics, fermions and bosons, respectively. This is the celebrated spin-statistics theorem (See for example the book [25]).

Recently, it was shown that if space-time is noncommutative, e.g. $[x^\mu, x^\nu] = i\theta^{\mu\nu}$ with $\theta^{\mu\nu} = -\theta^{\nu\mu}$ and $\theta^{0i} \neq 0$, then it is possible to construct a field theory invariant under Poincaré group, but the spin-statistics theorem is modified [26–28, 28–38]. This type of theory has been called twisted-Poincaré field theories and lies in the fact that in order to have Poincaré invariance, one has to twist the action of the group in the tensor states, i.e. in the many particles states. That is possible by twisting the co-product action of the Poincaré group into the many particles Hilbert space and such a procedure leads to non-trivial results when it is applied, for instance, to some process in quantum chromodynamics. Despite the fact that temperature effects play an important role in non-commutative approaches, only some preliminary aspects of thermal phenomena have been explored in the context of twisted Poincaré-invariant theories [12, 24].

The main goal in the present work is to develop a thermal perturbative quantum-field approach of a many-particle twisted Poincaré system. In order to accomplish such a result, we use TFD, exploring several aspects of its algebraic structure. We first prove the Wick theorem in TFD under general basis. One of the advantages of such demonstration is that it allows to implement interactions in TFD formalism. This result is up to our best knowledge not available in the literature. It is therefore an important result for TFD in its own right.
and its applications. We then extend this formalism to twisted Poincaré invariant quantum field theories at finite temperature.

The present work is organized in the following manner. In the next section, we present an outline of TFD, emphasizing some aspects to be explored later. In Section 3, we demonstrate the validity of Wick theorem in the TFD formalism. In Section 4, after recollecting some facts about twisted Poincaré quantum field theories, we demonstrate the validity of the thermal Wick theorem in this non-commutative context. In Section 5, concluding remarks are presented.

II. THE THERMOFIELD DYNAMICS PROPAGATOR

In this section, we present an outline of thermofield dynamics, in order to fix the notation and to emphasize aspects to be explored later, as the thermal Green’s function. We consider the scalar field, only, for simplicity. Let $\mathcal{O}$ be a physical observable. Its thermal average can be written as

$$\langle\mathcal{O}\rangle_\beta = \frac{1}{Z(\beta)} \text{Tr}(\mathcal{O} \rho(\beta)),$$

where

$$Z(\beta) = \sum_n e^{-\beta E_n},$$

is the canonical partition function, with $E_n$ being the $n$-th eigenvalue of a Hamiltonian $H$, and

$$\rho(\beta) = \frac{1}{Z(\beta)} \sum_n e^{-\beta E_n} |n\rangle\langle n|,$$

is the corresponding density matrix.

Now, let $\mathcal{O} = A(t') B(t)$, where $A(t')$ and $B(t)$ are any two operators in instants $t'$ and $t$, respectively, given in the Heisenberg picture by

$$A(t') = e^{it' H} A e^{-it' H} \quad B(t) = e^{it H} B e^{-it H}.$$

The KMS (Kubo-Martin-Schwinger) condition reads

$$\langle A(t') B(t) \rangle_\beta = \langle B(t - i\beta) A(t') \rangle_\beta = \langle B(t) A(t' - i\beta) \rangle_\beta.$$

In the Euclidean quantum field theory at finite temperature, i.e. $t \to i\tau$, the KMS condition implies that correlation functions are periodic, with period $\beta$.  

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TFD is introduced by a doubling in the degrees of freedom of thermofield theory \[2\]:

\[
(A_i A_j) = \tilde{A}_i \tilde{A}_j,
\]

\[
(c A_i + A_j) = c^* \tilde{A}_i + \tilde{A}_j,
\]

\[
(A_i^\dagger) = (\tilde{A}_i)^\dagger,
\]

\[
(\tilde{A}_i) = -\xi A_i,
\]

with \(\xi = -1\) for bosons and \(\xi = +1\) for fermions; such that the physical variables are described by non-tilde operators. The tilde variables are defined in the commutant of the von Neumann algebra of quantum operators and are associated with generators of the modular group given by \(\hat{A} = A - \tilde{A}\). With these elements, reducible representations of Lie-groups can be studied, in particular, kinematical symmetries as the Poincaré group. The other basic ingredient of TFD is a Bogoliubov transformation, \(U(\alpha)\), introducing a rotation in the tilde and non-tilde variables, such that thermal effects emerge from a condensate state. The rotation parameter \(\alpha\) is associated with temperature, and this procedure leads to the usual statistical thermal average.

Consider \(\phi(x)\) a real scalar field. The TFD prescription leads us then to the doubling: \(\phi(x) \rightarrow \phi(x) \otimes \tilde{\phi}(x)\). These operator fields are expanded in modes as

\[
\phi(x) = \int d\mu(p) \left( c(p)e^{ipx} + c(p)^\dagger e^{-ipx} \right), \quad (1)
\]

\[
\tilde{\phi}(x) = \int d\mu(p) \left( \tilde{c}(p)e^{-ipx} + \tilde{c}(p)^\dagger e^{ipx} \right), \quad (2)
\]

where \(d\mu(p)\) represents an invariant measure in momentum space (for the case of continuous field it can be written as usual like \(d\mu(p) = d^3p/(2\pi)^32p_0\)) and \(c(p)^\dagger, \tilde{c}(p)^\dagger, c(p), \tilde{c}(p)\) are the creation and annihilation operators that act on the Fock space of many-particle states.

We write Eqs. (1) and (2) in a compact matrix form

\[
\phi(x)_1 = \int d\mu(p) \left( c(p)_1 e_p + c(p)_2 e_{-p} \right) = \phi^+(x)_1 + \phi^-(x)_1,
\]

where \(e_p = e^{ipx}\), and

\[
\phi^+(x)_1 = \int d\mu(p) c(p)e_p, \quad \phi^-(x)_2 = \int d\mu(p)\tilde{c}(p)e_{-p}
\]

\[
\phi^-(x)_1 = \int d\mu(p) c^\dagger(p)e_{-p}, \quad \phi^+(x)_2 = \int d\mu(p)\tilde{c}(p)^\dagger e_p.
\]
Here the superscripts $+$ and $-$ denote the positive and negatives frequencies, respectively, frequencies of the Fourier representation of the fields.

The Bogoliubov transformation is introduced by the canonical operator

$$U_\beta = e^{-iG_\beta},$$

where

$$G_\beta = i \int d\mu(p) f_\beta(p) \left( c(p)\tilde{c}(p)^\dagger - c(p)\tilde{c}(p) \right),$$

with $f(\beta)$ being defined intrinsically by the equation

$$\tanh(f_\beta(p)) = e^{-\frac{E_p}{2\beta}}, \quad (3)$$

with $E_p$ being the dispersion relation for bosons. The origin of this relation lies in the Bose-Einstein distribution. Applying the Bogoliubov transformation in the usual Poincaré-invariant vacuum state $|0 \rangle$, one obtains a thermal Poincaré-invariant vacuum state

$$|0; \beta \rangle \equiv U_\beta|0 \rangle. \quad (4)$$

This vacuum state is annihilated by the thermal annihilation operator $c_\beta(p) \equiv U_\beta c(p) U_\beta^{-1}$, i.e.

$$c_\beta(p)|0; \beta \rangle = 0. \quad (5)$$

Thermal particle states are created by thermal creation operator

$$c_\beta(p)^\dagger \equiv U_\beta c(p)^\dagger U_\beta^{-1} \quad (6)$$

acting on the thermal vacuum, Eq. (4).

We write now the doubled operators in terms of thermal operators. For that we use the Bogoliubov transformation, such that

$$c(p)_J = (U_\beta^{-1})_{JK} c_\beta(p)_K,$$

with

$$c(p)_J = \begin{pmatrix} c(p) \\ \tilde{c}(p)^\dagger \end{pmatrix}, \quad c_\beta(p)_J = \begin{pmatrix} c_\beta(p) \\ \tilde{c}_\beta(p)^\dagger \end{pmatrix},$$

and

$$(U_\beta)_{JK} = \begin{pmatrix} u & v \\ v & u \end{pmatrix}$$
is the matrix representation of Bogoliubov transformation, where
\[ u \equiv u(f_\beta(p)) = \frac{e^{\beta \epsilon}}{\sqrt{e^{2\beta \epsilon} - 1}}, \quad v \equiv v(f_\beta(p)) = \frac{1}{\sqrt{e^{2\beta \epsilon} - 1}}. \]
Then we write \( \phi(x)_A \), the representation of the doubled field in terms of thermal operators, as
\[ \phi(x)_J = \phi^+_\beta(x)_J + \phi^-_\beta(x)_J. \] (7)
For \( A = 1 \)
\[ \phi^+_\beta(x)_1 = \int d\mu(p)(ue_p c_\beta(p) + ve_{-p} \tilde{c}_\beta(p)), \]
and
\[ \phi^-_\beta(x)_1 = \int d\mu(p)(ue_{-p} c_\beta(p) + ve_p \tilde{c}_\beta(p)). \]
Now the fields \( \phi^+ \) and \( \phi^- \) describe the annihilation and creation, respectively, parts of \( \phi(x)_A \).
In compact notation
\[ \phi(x)_J = \phi^+_\beta(x)_J + \phi^-_\beta(x)_J = \int d\mu(p)Q_\beta(x, p)_JK C_\beta(p)_K \]
\[ + \int d\mu(p)\bar{Q}_\beta(x, p)_JK C_\beta(p)_K^+, \]
where \( C_\beta(p)_J = (c_\beta(p), \tilde{c}_\beta(p)) \),
\[ Q_\beta(x, p)_JK = \begin{pmatrix} ue_p & ve_{-p} \\ ve_p & ue_{-p} \end{pmatrix}. \] (8)
This notation for the fields will be useful in the demonstration of the Wick theorem. For the field at zero temperature, the super-index +(-) is associated with positive (negative) frequencies. For the case of finite temperature, this notion is lost, since there is a mixing of the frequencies. However, we still can use this separation in components given in Eq. (7), as long as one understands that it is not refereeing to the separation in positive-negative frequencies. The causal thermal propagator is defined by
\[ \Delta(x - y; \beta)_JK = \langle 0; \beta | T \phi(x)_J \phi(y)_K | 0; \beta \rangle \]
\[ = \Theta(x_0 - y_0)\Delta_>(x, y; \beta)_JK + \Theta(y_0 - x_0)\Delta<_<(x, y; \beta)_JK, \]
where \( \Theta(x) \) is a Heaviside step function (\( \Theta(x) = 1 \), if \( x \geq 0 \), and 0 otherwise), \( \Delta_> \) and \( \Delta_< \) denote the advanced and retarded propagators, respectively.
In this section, the formalism of TFD for interacting fields is developed in order to demonstrate the Wick theorem for thermal theories under algebraic basis, achieving so in a very general result. We deal only with real scalar field, although the generalization for charged field and spinor fields is straightforward. In the following we set $\beta = 1/T$ and chemical potential $\mu = 0$.

We now apply this scheme to interacting fields. The $n$-point correlation function for an interacting field is defined as

$$\Delta(x_1, ..., x_n; \beta)_{J_1 ... J_n} = \langle \Omega; \beta \mid T \Phi(x_1)_{J_1} ... \Phi(x_n)_{J_n} \mid \Omega; \beta \rangle,$$

where $\Phi(x)_{J_i}$ is an interacting field defined in Heisenberg picture as

$$\Phi(x)_{J_i} = e^{it\hat{H}} \phi(0, \vec{x}_i)_{J_i} e^{-it\hat{H}},$$

with

$$\hat{H} = H_0 - \tilde{H}_0 + H_{\text{int}} - \tilde{H}_{\text{int}} \equiv \hat{H}_0 + \tilde{H}_{\text{int}}$$

being a duplicated Hamiltonian operator, and

$$\Phi(x) = \Phi(x)^+ + \Phi(x)^-, \quad \Phi(x)^\dagger = \Phi(x)^+ + \Phi(x)^-, \quad \Phi(x)^- = \Phi(x)^+ + \Phi(x)^-, \quad \Phi(x)^\dagger = \Phi(x)^+ + \Phi(x)^-.$$

Using the matrix notation, introduced in Eq.s (7-8), we have

$$\Phi(x)_{J_i} = \int d\mu(p) \ (Q_{\beta}(x, p)_{JK} C_{\beta}(p)_K + \tilde{Q}_{\beta}(x, p)_{JK} C_{\beta}(p)_K),$$

In the following we obtain the Gellmann-Low formula, defining a thermal interacting vacuum. In interacting picture, let us introduce thermal fields as

$$\varphi(x)_{J_i} = e^{it\hat{H}_0} \phi(0, \vec{x}_i)_{J_i} e^{-it\hat{H}_0}.$$

The thermal evolution operator from an instant $t_1$ to instant $t_2$ is defined as

$$\hat{U}(t_1, t_2) = U(t_1, t_2) \tilde{U}(t_1, t_2) = T \exp \left( -i \int_{t_1}^{t_2} dt \ \hat{H}_I \right),$$

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such that
\[ \hat{H}_I = e^{it\hat{H}_0} \hat{H} e^{-it\hat{H}_0}. \]

Recall that the Heisenberg and interaction pictures coincide with each other at \( t = 0 \).

The free thermal-field vacuum is defined as
\[ |0; \beta \rangle = Z_0(\beta)^{-1/2} e^{-\beta \hat{H}_0} |0 \rangle, \quad \text{with} \quad Z_0(\beta) = \text{Tr}(e^{-\beta \hat{H}_0}). \]

Interacting thermal vacuum is defined similarly by
\[ |\Omega; \beta \rangle = Z(\beta)^{-1/2} e^{-\beta \hat{H}} |0 \rangle, \quad \text{with} \quad Z(\beta) = \text{Tr}(e^{-\beta \hat{H}}). \]

Both vacua are related to one another by
\[ |\Omega; \beta \rangle = \left( \frac{Z_0(\beta)}{Z(\beta)} \right)^{1/2} \hat{U}(0, i\beta^2) |0; \beta \rangle \]
\[ \langle \Omega; \beta | = \langle \Omega; \beta | \hat{U}(-i\beta^2, 0) \left( \frac{Z_0(\beta)}{Z(\beta)} \right)^{1/2}. \]

We use these relations to obtain the Gellman-Low formula. Indeed, by using Eqs. (10) and (11) into the expression of \( n \)-point function, Eq. (9), using the definition of time-ordered operator together with some algebraic manipulation and the definition of normalization factor \( Z_0(\beta)/Z(\beta) \), we thus obtain
\[ \Delta(x_1, \ldots, x_n; \beta)_{J_1 \ldots J_n} = \frac{\langle 0; \beta | T \hat{U}(-\infty, \infty) \varphi(x_1)_{J_1} \cdots \varphi(x_n)_{J_n} |0; \beta \rangle}{\langle 0; \beta | T \hat{U}(-\infty, \infty) |0; \beta \rangle}. \]

Therefore, we describe the thermal \( n \)-point function for an interacting field theory in terms of free thermal fields, as long as we write \( \hat{U}(-\infty, \infty) \) in terms of free thermal fields. This happens for weak coupling constants, which is proper for the development of perturbation methods.

The Gellmann-low formula allows us to evaluate \( n \)-point functions by perturbation methods. To derive such a formula, we expand the expression
\[ \hat{U}(-\infty, \infty) = T \exp \left( -i \int_{-\infty}^{\infty} dt \hat{H}_I(t) \right) = T \exp \left( -i \int dt \hat{H}_I(x) \right), \]
where \( \hat{H}_I(x) \) is a Hamiltonian density. In order to prove the Wick theorem, it is usually assumed that \( \hat{H}_I(x) \) is written as a sum of local products of fields \( \varphi(x)_{J} \), e.g. \( g_n \varphi(x)^{J_n} \). Then all we have to compute is a term like
\[ \langle 0; \beta | T \varphi(x_1)_{J_1} \cdots \varphi(x_n)_{J_n} |0; \beta \rangle. \]
We hereafter perform this task for the case of $n = 2$ in order to understand the necessary steps. The generalization for an arbitrary $n$ will be straightforwardly. The start point is then the expression

$$\langle 0; \beta | T \varphi(x_1) J_1 \varphi(x_2) J_2 | 0; \beta \rangle.$$  \hspace{1cm} (13)

We observe in hindsight that the key step in this proof is to consider a separation of fields in positive and negative frequencies, considering a proper interpretation in case of thermal fields.

Recalling that $\varphi(x_i) J_i$ is a free field at zero temperature, the expectation value we want to compute is with respect to thermal vacuum. Thus, we write these free fields in terms of thermal creation and annihilation operators as

$$\phi(x_i) J_i = \phi(x_i) J_i^+ \phi(x_i) J_i^- \equiv \int d\mu(p) \left( Q_{\beta}(x_i, p) J_i K_i C_{\beta}(p) K_i + \bar{Q}_{\beta}(x_i, p) J_i K_i \bar{C}_{\beta}(p) K_i \right).$$

We stress once again that the above decomposition is not in terms of positive and negative frequencies, once both frequencies mixed up inside each $\phi(x_i) J_i^+$ and $\phi(x_i) J_i^-$. 

In order to compute the time-ordered product appearing inside 2-point function given in Eq. (13), we introduce one more ingredient: a normal ordering for thermal operators. This is defined by carrying all thermal creation operators to the left of all thermal annihilation operators. For instance,

$$c_{\beta}(p) \bar{c}_{\beta}(q) c_{\beta}(k) \equiv c_{\beta}(k) c_{\beta}(p) \bar{c}_{\beta}(q),$$

where $: :$ is used as the usual symbol for normal ordering (although normal ordering is here being performed for thermal creation and annihilation operators, and not the non-thermal creation and annihilation operators). In addition, observe that tilde operators commute with non-tilde operators. Using this result, we compute the time-ordered product appearing inside 2-point function, Eq. (13). By simple algebraic manipulations, one may bring the time-ordered product into a sum of normal ordered products plus commutators of thermal fields. Then we have

$$T \phi(x_1) J_1 \phi(x_2) J_2 = \theta(x_1^0 - x_2^0) \left( \phi(x_1) J_1 \phi(x_2) J_2 : + \left[ \phi(x_1) J_1^+, \phi(x_2) J_2^- \right] \right)$$

$$+ \theta(x_2^0 - x_1^0) \left( \phi(x_2) J_2 \phi(x_1) J_1 : + \left[ \phi(x_2) J_2^+, \phi(x_1) J_1^- \right] \right).$$

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Since this time-ordered product is acting on thermal vacuum, thermal normal-ordering product terms vanish. Therefore, we obtain

\[ \langle 0; \beta | T \varphi(x_1) J_1 \varphi(x_2) J_2 | 0; \beta \rangle = \theta(x_1^0 - x_2^0) \left[ \phi(x_1)^+ J_1, \phi(x_2)^- J_2 \right] + \theta(x_2^0 - x_1^0) \left[ \phi(x_2)^+ J_2, \phi(x_1)^- J_1 \right] \]
\[ \equiv \Delta_0(x_1 - x_2; \beta) J_1 J_2, \]

which is Feynman propagator for thermal free fields [2].

The generalization to the \( n \)-point functions is straightforward. The procedure is essentially the same we have done so far. Thus, the Wick theorem for thermal scalar fields in general is written as

\[ \langle 0; \beta | T \varphi(x_1) J_1 ... \varphi(x_n) J_n | 0; \beta \rangle = \sum_{\pi} \prod_{i,j} \Delta_0(x_{\pi(i)} - x_{\pi(j)}; \beta) J_{\pi(i)} J_{\pi(i)} \]

i.e. the \( n \)-point function for thermal fields may be written as a product of Feynman propagators for thermal free fields.

IV. WICK THEOREM FOR TWISTED POINCARÉ TFD

In this section, we demonstrate the validity of the Wick theorem for the twisted Poincaré quantum field theory at finite temperature using TFD formalism. We begin with by recollecting some aspects of the twisted Poincaré quantum field theory.

Twisted Quantization for Scalar Fields

The main idea of twisted Poincaré quantum field theory is to construct a Poincaré-invariant quantum field on the Groenewold-Moyal (GM) plane. The lesson is that we can perform the usual quantization for one-particle states, i.e., obtaining the usual unitary irreps of the Poincaré group. However, when we deal with a many-particle system, we have to twist statistics of fields. This is suitably seen as a twisting of the action of the co-product of the Poincaré group. Notice that a co-product is the algebraic structure that encodes the action of the underlying symmetry group on tensor products. Therefore, it is the proper structure to be considered when dealing with symmetries and statistics of many-body systems.
Let $\mathcal{A}_\theta(\mathbb{R}^D)$ be the algebra of smooth functions on $\mathbb{R}^D$ with moyal product defined by the map
\[ m_\theta : \mathcal{A}_\theta(\mathbb{R}^D) \otimes \mathcal{A}_\theta(\mathbb{R}^D) \to \mathcal{A}_\theta(\mathbb{R}^D) \]
\[ f_1 \otimes f_2 \mapsto f_1 \ast f_2 \equiv m_\theta(f_1 \otimes f_2), \]
such that
\[ m_\theta(f_1 \otimes f_2) = m_0(F_\theta f_1 \otimes f_2), \]
where $m_0$ is the point-wise product, i.e. $m_0(f_1 \otimes f_2) \equiv f_1(x) \cdot f_2(x)$, and
\[ F_\theta = e^{\frac{\theta}{2} \theta^{\mu \nu} \partial_\mu \otimes \partial_\nu} \]
is a *twist element*, where $\theta^{\mu \nu} = -\theta^{\nu \mu}$ is a constant matrix.

With this twisted product, we obtain the action of the co-product of the Poincaré group $\mathcal{P}(\mathbb{R}^D)$ by solving the compatibility equation
\[ m_\theta(\Delta_\theta(g) f_1 \otimes f_2) = g m_\theta(f_1 \otimes f_2), \] (14)
for any $f_1, f_2 \in \mathcal{A}_\theta(\mathbb{R}^D)$, with $g$ representing an element of the Poincaré group, and $\Delta_\theta(g)$ being the action of twisted co-product of $g$. In terms of the usual co-product $\Delta_0(g) = g \otimes g$, the twisted co-product that solves the compatibility condition given in Eq. (14) is
\[ \Delta_\theta g = F_\theta \Delta_\theta g \tau_0 \neq \tau_0 \Delta_\theta g \quad \text{for all } g \in \mathcal{P}(\mathbb{R}^D). \]

In order to study statistics, one usually considers a relation between the flip operator
\[ \tau_0(f_1 \otimes f_2) = f_2 \otimes f_2, \] (15)
and the co-product of the isometry group. Since the co-product has been twisted, it no longer commutes with the flip operator for all elements of the Poincaré group, i.e.
\[ \Delta_\theta(g)\tau_0 \neq \tau_0 \Delta_\theta(g) \quad \text{for all } g \in \mathcal{P}(\mathbb{R}^D). \]
However, if the flip operator is also twisted as
\[ \tau_\theta = F_\theta^{-1} \tau_0 F_\theta, \]
then
\[ \Delta_\theta(g)\tau_\theta = \tau_\theta \Delta_\theta(g) \quad \text{for all } g \in \mathcal{P}(\mathbb{R}^D). \]
But this twisting of the flip operator is equivalent to the twist of the statistics of the fields, since we can define the following projectors

\[ P^\pm_\theta = \frac{1 \pm \tau_\theta}{2}, \]

where the super-index + or − here means projectors for boson or fermion subspaces, respectively. In what follows, we consider bosons only.

A free scalar field invariant under the twisted Poincaré group is written as

\[ \phi(x) = \int d\mu(p) \left[ a(p)e^{ipx} + a(p)^\dagger e^{-ipx} \right], \]

where the twisted creation and annihilation operators \( a(p) \) and \( a(p)^\dagger \), respectively, satisfy

\[
\begin{align*}
    a(p)a(q) &= e^{i\theta_{\mu\nu} p_\mu q_\nu} a(q)a(p), \\
    a(p)^\dagger a(q)^\dagger &= e^{i\theta_{\mu\nu} p_\mu q_\nu} a(q)^\dagger a(p)^\dagger, \\
    a(p)a(q)^\dagger &= e^{i\theta_{\mu\nu} p_\mu q_\nu} a(p)^\dagger a(q) + 2p_0 \delta^{(4)}(p - q).
\end{align*}
\]

An interacting Hamiltonian, in interaction representation, for these fields may be written as

\[ H_I(t) = g_n \int d^4x : \phi^* n(x) := g_n \int d^4x : \phi(x) \times \ldots \times \phi(x) :, \]

where

\[ e^{ipx} \times e^{iqx} = e^{-i\frac{1}{2}p_\mu \theta_{\mu\nu} q_\nu} e^{i(p+q)x} . \] (16)

In the following, we thermalize such a twisted theory exploring the algebraic structure of TFD. This will represent an easy in the procedure, if we attempt to perform the same calculation using the imaginary-time approach, in particular to address the case of interacting fields.

**Twisted Thermofield Dynamics**

In order to construct a twisted thermal field theory using TFD, we observe that during such a development, the order that we use each formalism does not matter. Indeed, one can first construct a TFD for scalar field, by doubling of the Hilbert space and using the Bogoliubov transformations, and then Poincaré twist on top of resulting Hilbert space; or likewise, one can first Poincaré twist the Hilbert space and then applies the TFD procedures on top of the twisted Hilbert space.
Let us consider the doubled scalar field as given in Eqs. (1) and (2). The generator of time translation is given by a Hamiltonian in the form $\hat{H} = H - \tilde{H}$, such that

$$\hat{H} = \int d\mu(p) \omega(p) (c(p)c(p)\dagger - \tilde{c}(p)\tilde{c}(p)\dagger).$$

A total momentum operator is given by

$$\hat{P}_\nu = P_\nu - \tilde{P}_\nu = \int d\mu(p) p_\nu(c(p)\dagger - \tilde{c}(p)\dagger),$$

with

$$\begin{align*}
[\hat{P}_\nu, c(p)] &= -p_\nu c(p), & [\hat{P}_\nu, c(p)\dagger] &= p_\nu c(p)\dagger, \\
[\hat{P}_\nu, \tilde{c}(p)] &= -\tilde{p}_\nu \tilde{c}(p), & [\hat{P}_\nu, \tilde{c}(p)\dagger] &= \tilde{p}_\nu \tilde{c}(p)\dagger.
\end{align*}$$

We use $\hat{P}$ to construct a co-product, acting on $\hat{H}$. For that, we introduce the twisted annihilation and creation operators $a(p), a(p)\dagger$ in terms of usual annihilation and creation operators $c(p), c(p)\dagger$ as

$$\begin{align*}
a(p) &= c(p)e^{\frac{i}{2}p_\mu \theta^{\mu\nu} P_\nu}, \\
a(p)\dagger &= e^{-\frac{i}{2}p_\mu \theta^{\mu\nu} P_\nu} c(p)\dagger, \\
\tilde{a}(p) &= \tilde{c}(p)e^{\frac{i}{2}p_\mu \theta^{\mu\nu} P_\nu}, \\
\tilde{a}(p)\dagger &= e^{-\frac{i}{2}p_\mu \theta^{\mu\nu} P_\nu} \tilde{c}(p)\dagger,
\end{align*}$$

where we have used $(\hat{P}_\nu)\dagger = -\hat{P}_\nu$. It is important to observe that, using the twisted doubled operators, the momentum operator is written as

$$\hat{P}_\nu = \int d\mu(p) p_\nu (a(p)\dagger a(p) - \tilde{a}(p)\dagger \tilde{a}(p))$$

The Bogoliubov transformation is now defined by

$$U_\beta = e^{-iG_\beta}$$

with

$$G_\beta = i \int d\mu(p) f_\beta(p) (a(p)\dagger \tilde{a}(p) - a(p)\tilde{a}(p)), $$

where $f_\beta(p)$ is given in Eq. (3). The twisted thermal fields are thus given by

$$\begin{align*}
\phi_\beta(x) &= \int d\mu(p) (a_\beta(p)e^{ipx} + a_\beta(p)\dagger e^{-ipx}), \\
\tilde{\phi}_\beta(x) &= \int d\mu(p) (\tilde{a}_\beta(p)e^{ipx} + \tilde{a}_\beta(p)\dagger e^{-ipx}),
\end{align*}$$

(17)
where the twisted thermal annihilation and creation operators are
\[
a_\beta(p) = U_\beta a(p) U_\beta^{-1},
\]
\[
a_\beta(p)\dagger = U_\beta a(p)\dagger U_\beta^{-1}.
\]
In terms of untwisted operators, the above operators may be written as
\[
a_\beta(p) = U_\beta c(p)e^{i\hat{p}_\mu\theta_\mu\nu\hat{P}_\nu} U_\beta^{-1},
\]
\[
a_\beta(p)\dagger = U_\beta c(p)\dagger e^{-i\hat{p}_\mu\theta_\mu\nu\hat{P}_\nu} U_\beta^{-1}.
\]
Since
\[
[\hat{P}, G_\beta] = 0,
\]
which can be easily checked by direct inspection, then we write
\[
a_\beta(p) = c_\beta(p)e^{i\hat{p}_\mu\theta_\mu\nu\hat{P}_\nu},
\]
\[
a_\beta(p)\dagger = e^{-i\hat{p}_\mu\theta_\mu\nu\hat{P}_\nu} c_\beta(p),
\]
\[
\tilde{a}_\beta(p) = \tilde{c}_\beta(p)e^{i\hat{p}_\mu\theta_\mu\nu\hat{P}_\nu},
\]
\[
\tilde{a}_\beta(p)\dagger = e^{-i\hat{p}_\mu\theta_\mu\nu\hat{P}_\nu} \tilde{c}_\beta(p).
\]
Let us now discuss S-matrices for interacting twisted thermal fields. We first recall that in the usual TFD, the free-field 2-point function may be written as
\[
\langle 0; \beta | T \phi(x) \phi(y) | 0; \beta \rangle = \langle 0, 0 | T \phi_\beta(x) \phi_\beta(y) | 0, 0 \rangle.
\]
Then we may define the twisted thermal free-field 2-point function as
\[
\Delta^\beta_\phi(x - y; \beta) = \langle 0; \beta | T \phi(x) * \phi(y) | 0; \beta \rangle,
\]
where the noncommutative product * is defined in Eq.(16).

In what follows we consider an interacting Hamiltonian in interaction picture given in polynomial form as
\[
\hat{H}_I = H_I - \tilde{H}_I = \lambda \int d^d x ( : \phi^m(x) : - : \tilde{\phi}^m(x) : ),
\]
where : : denotes normal ordering of the creation and annihilation operators over the doubled Hilbert space.
The $S$-matrix is defined as
\[
\hat{S}_\theta = T \exp \left( -i \int dx^0 \hat{H}_I x^0 \right) = T \exp \left( -i \int dx^{d+1} : \phi^* n(x) : - : \tilde{\phi}^* n(x) : \right).
\]

In [27], it was proven that $\hat{S}_\theta = \hat{S}_0$, in a non-thermal formalism. This result is straightforwardly generalized to a thermal formalism. Indeed, notice that twisted thermal fields can be written in the form
\[
\phi_\beta(x) = \phi_{\beta 0} e^{\frac{i}{2} \theta_{\mu} \theta_{\nu} \hat{P}_\nu},
\]
where $\phi_{\beta 0}$ is an untwisted thermal field constructed with untwisted operators $c_\beta(p)$ and $\hat{c}_\beta(p)$. Now, let us consider the product of two plane wave vectors $e_p = e^{ipx}$ of positive frequencies
\[
e_p * e_q = e^{-\frac{i}{2} p_\mu \theta_{\mu \nu} q_\nu} e_{p+q}.
\]
Note that we consider $a(p) = a(-p)$. Thus a twisted thermal field may be written as
\[
\phi_\beta(x) = \int d\mu(p) \left( a_\beta(p)e_p + a_\beta(-p)e_{-p} \right).
\]
For $n = 2$ in $\hat{H}$, the first term of the $\hat{S}_\theta$ expansion is
\[
\hat{S}_\theta^{(1)} = -i \lambda \int dx^{d+1} : \phi \phi : - : \tilde{\phi} \tilde{\phi} :.
\]
As a consequence, we have terms of the type
\[
a_\beta(p)a_\beta(q)e_p * e_q = a_\beta(p)a_\beta(q)e^{\frac{i}{2} p_\mu \theta_{\mu \nu} q_\nu} e_{p+q}.
\]
With the expression, we derive
\[
a_\beta(p)a_\beta(q)e_p * e_q = c_\beta(p)e_{\frac{i}{2} p_\mu \theta_{\mu \nu} \hat{P}_\nu} c_\beta(q)e_{\frac{i}{2} q_\mu \theta_{\mu \nu} \hat{P}_\nu} e^{-\frac{i}{2} p_\mu \theta_{\mu \nu} q_\nu} e_{p+q} \quad (22)
\]
\[
= c(p)c(q)e_{(p+q)_\mu \theta_{\mu \nu} \hat{P}_\nu} \quad (23)
\]
Writing $\partial_\mu e_{p+q} = i(p + q)_\mu e_{p+q}$, we have
\[
e_{p+q} e_{\frac{i}{2} (p+q)_\mu \theta_{\mu \nu} \hat{P}_\nu} = e_{p+q} e_{\frac{i}{2} \theta_{\mu} \theta_{\nu} \hat{P}_\nu},
\]
and
\[
a_\beta(p)a_\beta(q)e_p * e_q = a_\beta(p)a_\beta(q)e^{\frac{i}{2} \theta_{\mu} \theta_{\nu} \hat{P}_\nu}.
\]
Thus the $S$-matrix term of first order in $\lambda$ is

$$
\hat{S}_{\theta}^{(1)} = -i\lambda \int d^{d+1}x (:\phi^2 : - :\tilde{\phi}^2 :) e^{\frac{i}{2} \tilde{\theta} \tilde{\Phi}_{\mu} \theta^{\mu\nu} \tilde{P}_\nu}.
$$

Considering that the interaction does not involve long range forces, we expand the exponential in the integrand and discard surface terms. We finally obtain

$$
\hat{S}_{\theta}^{(1)} = \hat{S}_{\theta}^{(1)0}.
$$

The same procedure goes through in second order in $\lambda$

$$
\hat{S}_{\theta}^{(2)} = \frac{(-i\lambda^2)}{2!} \int d^{d+1}x_1 d^{d+1}x_2 [\Theta(x_1^0 - x_2^0)] (:\phi \phi)(x_1) :: (\phi \phi)(x_2) :

+ \Theta(x_2^0 - x_1^0) (:\phi \phi)(x_2) :: (\phi \phi)(x_1) :

- \Theta(x_1^0 - x_2^0) (:\tilde{\phi} \tilde{\phi})(x_1) :: (\tilde{\phi} \tilde{\phi})(x_2) :

- \Theta(x_2^0 - x_1^0) (:\tilde{\phi} \tilde{\phi})(x_2) :: (\tilde{\phi} \tilde{\phi})(x_1) :].
$$

In terms of twisted thermal annihilation and creation operators, the above expression displays terms into the form

$$
T^*_\beta = \Theta(x_1^0 - x_2^0) a_\beta(p_1) a_\beta(q_1) :: a_\beta(p_2) a_\beta(q_2) :

\times (e_{p_1} e_{q_1})(x_1) * (e_{p_2} e_{q_2})(x_2).
$$

Using Eq. (23), we have

$$
T^*_\beta = \Theta(x_1^0 - x_2^0) \left[ c_1(p_1) c_1 :: c_1(p_2) c_1 : e^{-\frac{i}{2} \tilde{\theta}(p_1 + q_1) \mu \theta^{\mu\nu}(p_2 + q_2 \nu)}

\times \left( e_{p_1 + q_1}(x_1) e_{p_2 + q_2}(x_2) e^{\frac{i}{2} \tilde{\theta} \tilde{\Phi}_{\mu} \theta^{\mu\nu} \tilde{P}_\nu} \right) \right].
$$

The momentum conservation is expressed as

$$
e^{-\frac{i}{2} \tilde{\theta}(p_1 + q_1) \mu \theta^{\mu\nu}(p_2 + q_2 \nu)} = 1.
$$

Then we show that the $S$-matrix, to all orders of $\lambda$, satisfies $\hat{S}_{\theta}^{(2)} = \hat{S}_{\theta}^{(2)0}$. Therefore, for $n = 2$ we have

$$
\hat{S}_{\theta} = \hat{S}_{0}.
$$

Let us now obtain a Wick theorem for the twisted TFD. First, we calculate expressions like $\langle 0| T \phi_{\beta J_1} * \phi_{\beta J_2} * \cdots * \phi_{\beta J_n} |0 \rangle$, or, equivalently

$$
\langle 0; \beta | T \phi_{J_1} * \phi_{J_2} * \cdots * \phi_{J_n} |0; \beta \rangle.
$$

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where $\phi_J(x) = (\phi(x), \tilde{\phi}(x))$. The Fourier representation for these twisted doubled fields is

$$\phi(x)_J = \int d\mu(p) \left( a(p)e_p + a(p)\dagger e_{-p} \right) = \phi^+(x)_J + \phi^-(x)_J,$$

and

$$\phi^+(x)_1 = \int d\mu(p)a(p)e_p, \quad \phi^-(x)_2 = \int d\mu(p)\tilde{a}(p)e_{-p}$$

and

$$\phi^-(x)_1 = \int d\mu(p)a^\dagger(p)e_{-p}, \quad \phi^+(x)_2 = \int d\mu(p)\tilde{a}(p)^\dagger e_p.$$

We write now the twisted operators in terms of thermal operators. For that we use the Bogoliubov transformation, such that

$$a(p)_J = (U^{-1}_\beta)_{JK} a_\beta(p)_K,$$

with

$$a(p)_J = \left( \begin{array}{c} a(p) \\ \tilde{a}(p)^\dagger \end{array} \right), \quad a_\beta(p)_J = \left( \begin{array}{c} a_\beta(p) \\ \tilde{a}_\beta(p)^\dagger \end{array} \right).$$

Then we write $\phi_\beta(x)_J$, the representation of the doubled field in terms of thermal operators, as

$$\phi(x)_J = \phi^+_\beta(x)_J + \phi^-_\beta(x)_J.$$

For $J = 1$

$$\phi^+_\beta(x)_1 = \int d\mu(p)(ue_p a_\beta(p) + ve_{-p} \tilde{a}_\beta(p)),$$

and

$$\phi^-_\beta(x)_1 = \int d\mu(p)(ue_{-p} a_\beta(p)^\dagger + ve_p \tilde{a}_\beta(p)^\dagger).$$

Now the fields $\phi^+$ and $\phi^-$ describe the annihilation and creation, respectively, parts of $\phi(x)$. In compact notation

$$\phi(x)_J = \phi^+_\beta(x)_J + \phi^-_\beta(x)_J$$

$$= \int d\mu(p)Q_\beta(x,p)_{JK} A_\beta(p)_K$$

$$+ \int d\mu(p)\tilde{Q}_\beta(x,p)_{JK} A_\beta(p)^\dagger_K,$$

where $A_\beta(p)_J = (a_\beta(p), \tilde{a}_\beta(p))$, and

$$Q_\beta(x,p)_{JK} = \begin{pmatrix} ue_p & ve_{-p} \\ ve_p & ue_{-p} \end{pmatrix}.$$
Writing the twisted operators in terms of the untwisted creation and annihilation operators we obtain

\[ A_\beta(p)_J = \left( c_\beta(p) e^{\frac{i}{2} p_\mu \theta^{\mu \nu} P_{\nu}} \right) = \left( c_\beta(p) \right) e^{\frac{i}{2} p_\mu \theta^{\mu \nu} P_{\nu}} \]

and

\[ A_\beta(p)_J^{\dagger} = \left( c_\beta(p)^{\dagger} \right) e^{-\frac{i}{2} p_\mu \theta^{\mu \nu} P_{\nu}}, \]

we have the compact notation

\[ A_\beta(p)_J = C_\beta(p)_J e^{\frac{i}{2} p_\mu \theta^{\mu \nu} P_{\nu}}, \]

\[ A_\beta(p)_J^{\dagger} = C_\beta(p)_J^{\dagger} e^{-\frac{i}{2} p_\mu \theta^{\mu \nu} P_{\nu}}. \]  

(25)

(26)

Now we are able to calculate the products \( \phi_\beta^+(x)_J \ast \phi_\beta^+(y)_K \) in

\[ \phi_\beta(x)_J \ast \phi_\beta(y)_K = \int d\mu(p) d\mu(q) [Q_\beta(x, p)_J J_r A_\beta(p)_J \bar{Q}_\beta(q, y)_K K K^\dagger \bar{A}_\beta(q)_K^\dagger, \]

\[ + Q_\beta(x, p)_J J_r A_\beta(p)_J \bar{Q}_\beta(q, y)_K K K^\dagger A_\beta(q)_K^\dagger, \]

\[ + Q_\beta(x, p)_J J_r A_\beta(p)_J^{\dagger} A_\beta(q)_K^\dagger, Q_\beta(y, q)_K K K^\dagger A_\beta(q)_K^\dagger, \]

\[ + Q_\beta(x, p)_J J_r A_\beta(p)_J^{\dagger} Q_\beta(y, q)_K K K^\dagger A_\beta(q)_K^\dagger]. \]

The first term involves

\[ \phi_\beta^+(x)_1 \ast \phi_\beta^+(y)_1 = A_\beta(p)_1 A_\beta(q)_1 Q_\beta(x, p)_11 \ast Q_\beta(y, q)_11 \]

\[ + A_\beta(p)_1 A_\beta(q)_2 Q_\beta(x, p)_11 \ast Q_\beta(y, q)_12 \]

\[ + A_\beta(p)_2 A_\beta(q)_1 Q_\beta(x, p)_12 \ast Q_\beta(y, q)_11 \]

\[ + A_\beta(p)_2 A_\beta(q)_2 Q_\beta(x, p)_12 \ast Q_\beta(y, q)_12 \]

Notice that

\[ Q_\beta(x, p)_J K = U_{JK}^{-1} E(x, p)_J K \]

with

\[ E(x, p)_J K = \begin{pmatrix} e_p & e_{-p} \\ e_p & e_{-p} \end{pmatrix}. \]

Then the products of the Q-matrix are given by

\[ Q_\beta(x, p)_J K \ast Q_\beta(y, q)_L M = Q_\beta(x, p)_J K Q_\beta(y, q)_L M e^{\frac{i}{2} p_\mu \theta^{\mu \nu} P_{\nu}}, \]

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with
\[ p_\mu = \begin{cases} +p_\mu & \text{if } j = 1; \\ -p_\mu & \text{if } j = 2. \end{cases} \]

Using Eqs. (23) and (26), we obtain
\[ \phi_+^+(x_j) \ast \phi_+^+(y) = \phi_+^+(x_j) \phi_+^+(y) e^{\phi_+^+(p+\overline{p})_{\theta_{\mu}} \phi_+^+(p_{\mu})}. \tag{27} \]

With the twisted co-product, given in Eq. (27), we write the time ordering in terms of the normal ordering
\[ D_{JK}(x_1 - x_2) = T\phi(x_1) \ast \phi(x_2) \]
\[ = \theta(x^0_1 - x^0_2)D_{JK}(x_1 - x_2) + \theta(x^0_2 - x^0_1)D_{KJ}(x_1 - x_2) \]
where
\[ D_{JK}(x_1 - x_2) = (\phi_+^+(x_1) + \phi_-^+(x_1)) \ast (\phi_+^+(x_2) + \phi_-^+(x_2)) \] and
\[ D_{KJ}(x_1 - x_2) = \theta(x^0_2 - x^0_1)(\phi_+^+(x_2) + \phi_-^+(x_2)) \ast (\phi_+^+(x_1) + \phi_-^+(x_1)). \]

We write \( D_{JK}(x_1 - x_2) \) as
\[ D_{JK}(x_1 - x_2) = (\phi_+^+(x_1) + \phi_-^+(x_1)) \ast (\phi_+^+(x_2) + \phi_-^+(x_2)) \]
\[ = \phi_+^+(x_1) \ast \phi_+^+(x_2) + \phi_+^+(x_1) \ast \phi_-^+(x_2) + \phi_-^+(x_1) \ast \phi_+^+(x_2) + \phi_-^+(x_1) \ast \phi_-^+(x_2), \]

where the underlined terms are normal ordered with respect to the thermal operators. Using the deformed product commutator \( [\phi_+^+(x_1), \phi_-^+(x_2)]^* \), we have
\[ D_{JK}(x_1 - x_2) = \phi_+^+(x_1) \ast \phi_+^+(x_2) + \phi_+^+(x_1) \ast \phi_-^+(x_2) + \phi_-^+(x_1) \ast \phi_+^+(x_2) + \phi_-^+(x_1) \ast \phi_-^+(x_2) \]
\[ + [\phi_+^+(x_1), \phi_-^+(x_2)]^*. \]

Relatively to thermal operators the normal ordering is denoted as
\[ N_\theta a_\beta(p) a_\beta(q)^\dagger = a_\beta(q)^\dagger a_\beta(p) \]
\[ N_\theta a_\beta(p)^\dagger a_\beta(q) = a_\beta(p)^\dagger a_\beta(p) \]
\[ N_\theta a_\beta(p_1) \tilde{a}_\beta(q_1)^\dagger \tilde{a}_\beta(p_2) a_\beta(q_2)^\dagger = \tilde{a}_\beta(q_1)^\dagger a_\beta(q_2)^\dagger \tilde{a}_\beta(p_2) a_\beta(p_1) \]
\[ \vdots \]
Thus we obtain

\[ T\phi(x_1) \ast \phi(x_2) = N_0 \phi(x_1) \ast \phi(x_2) \]

\[ + \theta(x_1^0 - x_2^0) \left[ \phi_\beta^+(x_1), \phi_\beta^-(x_2) \right]^* \]

\[ + \theta(x_2^0 - x_1^0) \left[ \phi_\beta^+(x_2), \phi_\beta^-(x_1) \right]^*. \]

Defining the contraction of the fields as

\[
\phi(x_1) \ast \phi(x_2) = + \theta(x_1^0 - x_2^0) \left[ \phi_\beta^+(x_1), \phi_\beta^-(x_2) \right]^* \\
+ \theta(x_2^0 - x_1^0) \left[ \phi_\beta^+(x_2), \phi_\beta^-(x_1) \right]^*,
\]

we obtain

\[ T\phi(x_1) \ast \phi(x_2) = N_0 \phi(x_1) \ast \phi(x_2) + \phi(x_1) \ast \phi(x_2) \]

Therefore

\[ \langle 0; \beta | T\phi(x_1) \ast \phi(x_2) | 0; \beta \rangle = \langle 0; \beta | N_0 \phi(x_1) \ast \phi(x_2) | 0; \beta \rangle \]

\[ + \langle 0; \beta | \phi(x_1) \ast \phi(x_2) | 0; \beta \rangle \]

The last expression is our guide for defining the Feynman propagator for the twisted TFD, i.e.

\[ \Delta_0^N(x_1 - x_2; \beta)_{JK} = \langle 0; \beta | \phi(x_1) \ast \phi(x_2) | 0; \beta \rangle \]

\[ = \langle 0; \beta | \phi(x_1) \phi(x_2)_K e^{\frac{i}{2} [\vec{\nabla} + \nabla] \mu \nu \epsilon \nu} | 0; \beta \rangle \]

\[ = \Delta_0(x_1 - x_2; \beta)_{JK} \]

where we expanded the exponential and used the fact that \( \hat{P}_\mu |0; \beta\rangle = 0 \).

For any number of fields we are interesting in products of the type

\[
\Pi_i^\pm \phi(x_i), J_i = \phi(x_1), J_1 \ast \phi(x_2), J_2 \ast \cdots \ast \phi(x_n), J_n \\
= (\phi_\beta^+(x_1), J_1 \ast \phi_\beta^-(x_1), J_2 \ast \cdots \ast \phi_\beta^+(x_n), J_n + \phi_\beta^-(x_n), J_n)
\]

We have then \( 2^n \) products of the form

\[ \Pi_i^\pm \phi(x_i), J_i = \phi_\beta^+(x_1), J_1 \ast \cdots \ast \phi_\beta^+(x_n), J_n. \] (28)

With each of the fields in terms of untwisted operators we have

\[ \Pi_i^\pm \phi(x_i), J_i = \phi_\beta^+(x_1), J_1 \cdots \phi_\beta^+(x_n), J_n e^{\frac{i}{2} \sum_{j=1}^n (p_j) \mu \theta^{\mu \nu} \epsilon \nu}. \]
This implies, together with $\hat{P}_\mu|0; \beta\rangle = 0$, that

$$\langle 0; \beta|T\phi(x_1)J_1 \ast \cdots \ast \phi(x_n)J_n|0; \beta\rangle = \langle 0; \beta|T\phi(x_1)J_1 \cdots \phi(x_n)J_n|0; \beta\rangle.$$ 

Therefore, the Wick theorem for twisted TFD is

$$T\prod_{i=1}^{n}\phi(x_i)J_i = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \sum_{\{\pi\}} N_\theta \prod_{j=1}^{n-2k}\phi(x_{\pi_j})J_{\pi_j} \prod_{\{(\pi l', \pi l)\}} G_0(x_{\pi l'} - x_{\pi l}; \beta) J_{\pi l'}J_{\pi l}$$

where $\pi = (\pi_1, \cdots, \pi_n)$ is a permutation $(1, \cdots, n)$, with $\pi_j$ an element of the combination $(1, \cdots, n)$ token $n - 2k$ a $n - 2k$, for $j = 1, \ldots, n - 2k$. $\{(\pi l, \pi l')\}$ is the set of all pairs inside $\pi l', \pi l$ for $n - 2k \leq l, l' \leq n$, with $l \neq l'$. This result shows that the TFD Wick theorem remains unchanged.

V. CONCLUDING REMARKS

In this work we have developed a perturbative approach for twisted Poincaré invariant theories at finite temperature. We have used the algebraic formalism thermofield dynamics in order to derive the thermal Wick theorem for this type of non-commutative theory. During the demonstration, we rederive the Wick theorem for TFD, but under very general basis, a result that is not available in the literature, at least at our best knowledge.

Our results can be readily extended to other fields, as fermion and gauge fields. This aspect will be addressed in detail in another place.

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