A Positive Mass Theorem for Manifolds with Boundary

Pengzi Miao* and Sven Hirsch†

*Department of Mathematics, University of Miami, Coral Gables, FL
†Department of Mathematics, Duke University, Durham, NC

Abstract
We derive a positive mass theorem for asymptotically flat manifolds with boundary whose mean curvature satisfies a sharp estimate involving the conformal Green’s function. The theorem also holds if the conformal Green’s function is replaced by the standard Green’s function for the Laplacian operator. As an application, we obtain an inequality relating the mass and harmonic functions that generalizes H. Bray’s mass-capacity inequality in his proof of the Riemannian Penrose conjecture.

1 Introduction
One of the fundamental results in mathematical relativity is the Riemannian positive mass theorem:

Theorem 1.1 (Theorem 5.3 in [30]). Suppose \((M^n, g)\) is an \(n\)-dimensional, \(n \geq 3\), complete, asymptotically flat manifold with non-negative scalar curvature. Then the ADM mass of \((M, g)\) is non-negative and is zero if and only if \((M^n, g)\) is isometric to the Euclidean space \((\mathbb{R}^n, \delta)\).

This result was first proven in low dimensions by R. Schoen and S.-T. Yau [28, 29] using minimal surface methods and later by E. Witten [33] for spin manifolds exploiting the Bochner-Weitzenböck formula for the Dirac operator (see also [27, 3]). Recently, R. Schoen and S.-T. Yau [30] extended their arguments to arbitrary dimensions (see [21, 22] by J. Lohkamp for a different approach).

The minimal surface method applies to the case in which the manifold has nonempty boundary with non-positive mean curvature, i.e. the mean curvature vector vanishes or points toward the infinity (see [28]). If the boundary has positive mean curvature, the positivity of the mass is a more subtle question as the boundary no longer acts as a barrier for minimal surfaces. For instance, consider a manifold \(M\) obtained by cutting out a rotationally symmetric ball in a negative-mass Schwarzschild manifold. In this case, \(M\) has negative mass and a zero area singularity (see [5]) is shielded by the boundary due to the relatively large positive mean curvature of \(\partial M\).

The spinor method also adapts to the setting of manifolds with boundary. If the boundary mean curvature is non-positive, positivity of the mass was shown in [10] (see also [13]). Refining the spinor analysis, M. Herzlich [12] relaxed the requirement of non-positive mean curvature and showed that a condition \(H \leq 4 \sqrt{\frac{\mu}{\text{vol}\partial M}}\) in 3-dimension implies the positivity of the mass. For higher dimensional spin manifolds, M. Herzlich proved a similar result in [14].

In this work, we consider asymptotically flat manifolds of dimensions \(n \geq 3\), with boundary allowed to have positive mean curvature. We show that if the mean curvature satisfies an upper estimate involving the conformal Green’s function, then the manifold has non-negative mass. More precisely, we have
Theorem 1.2. Let \((M^n, g)\) be an \(n\)-dimensional, asymptotically flat manifold with non-negative scalar curvature \(R\), with boundary \(\Sigma\). Let \(H\) be the mean curvature of \(\Sigma\) with respect to the \(\infty\)-pointing unit normal \(\nu\). Let \(u\) be the conformal Green’s function given by

\[
\begin{align*}
\Delta u - \frac{n-2}{4(n-1)} R u &= 0 \text{ in } M \\
u u &\to 0 \text{ at } \infty \\
u u &= 1 \text{ at } \Sigma.
\end{align*}
\]

Then the estimate

\[H(x) \leq - \frac{n-1}{n-2} \nabla_\nu u(x) \text{ for all } x \in \Sigma\] (1.1)

implies positivity of the ADM mass of \((M^n, g)\); moreover, \((M, g)\) has zero mass if and only if \((M^n, g)\) is isometric to \((\mathbb{R}^n, \delta)\) minus a round ball.

By the assumption \(R \geq 0\) and the maximum principle, we have an immediate corollary:

Corollary 1.3. The same result as above holds true if one replaces \(u\) with the standard Green’s function

\[
\begin{align*}
\Delta v &= 0 \text{ in } M \\
v &\to 0 \text{ at } \infty \\
v &= 1 \text{ at } \Sigma.
\end{align*}
\]

and require

\[H(x) \leq - \frac{n-1}{n-2} \nabla_\nu v(x) \text{ for all } x \in \Sigma.\] (1.2)

In \cite{4}, H. Bray proved the following theorem which plays a key role in his proof of the Riemannian Penrose inequality.

Theorem 1.4 (Bray \cite{4}). Let \((M^3, g)\) be a complete, asymptotically flat 3-manifold with nonnegative scalar curvature, with boundary \(\Sigma\) which has zero mean curvature. Let \(\varphi(x)\) be a function on \((M^3, g)\) which satisfies

\[
\begin{align*}
\Delta \varphi &= 0 \text{ in } M \\
\varphi &\to 1 \text{ at } \infty \\
\varphi &= 0 \text{ at } \Sigma.
\end{align*}
\]

Then

\[m \geq C,\] (1.3)

where \(m\) is the ADM mass of \((M^3, g)\) and \(C > 0\) is the constant in the asymptotic expansion

\[\varphi(x) = 1 - \frac{C}{|x|} + o(|x|^{-1}), \text{ as } x \to \infty.\]

Moreover, equality in (1.3) holds if and only if \((M^3, g)\) is isometric to a spatial Schwarzschild manifold outside its horizon.

Using Corollary 1.3, we obtain the following generalization of Bray’s theorem.
Theorem 1.5. Let \((M^n, g)\) be an \(n\)-dimensional, asymptotically flat manifold, with nonnegative scalar curvature, with boundary \(\Sigma\). Let \(H\) be the mean curvature of \(\Sigma\) with respect to the \(\infty\)-pointing unit normal \(\nu\). Let \(\phi(x)\) be a function on \((M, g)\) which satisfies

\[
\begin{cases}
\Delta \phi = 0 \text{ in } M \\
\phi \to 1 \text{ at } \infty \\
\phi = c \text{ at } \Sigma,
\end{cases}
\]

where \(c > -1\) is a constant and \(c \neq 1\). If

\[
\frac{2c}{1-c^2} \frac{\partial \phi}{\partial \nu} \geq \frac{n-2}{n-1} H,
\]

then

\[
m \geq C,
\]

where \(m\) is the ADM mass of \((M^n, g)\) and \(C\) is the constant in

\[
\phi = 1 - \frac{C}{|x|^{n-2}} + o(|x|^{2-n}), \text{ as } x \to \infty.
\]

Moreover, equality in (1.5) holds if and only if \((M^n, g)\) is isometric to the exterior region outside a rotationally symmetric sphere in an \(n\)-dimensional spatial Schwarzschild manifold, which is

\[
\left(\mathbb{R}^n / \{|x| < r_0\}, \left(1 + \frac{m}{2|x|^{n-2}}\right)^{\frac{n-2}{n-2}} \delta_{ij}\right)
\]

for some constants \(r_0 > 0\).

Remark 1.6. The mass \(m\) of the Schwarzschild manifold in the rigidity statement of Theorem 1.5 can be arbitrary, in particular \(m\) can be negative. Moreover, if \(m > 0\), \(r_0\) can be arbitrary, meaning that \(\Sigma\) can be either outside the Schwarzschild horizon or inside the Schwarzschild horizon.

Remark 1.7. If \(H \leq 0\), one can take \(c = 0\). In this case, Theorem 1.5 reduces to Theorem 1.4.

Remark 1.8. An equivalent formulation of Theorem 1.5 shows that Corollary 1.3 corresponds to a version of Theorem 1.5 if \(c = 1\). See Theorem 5.1 in Section 5 for details.

Remark 1.9. In application, given a harmonic function \(\phi\) that goes to 1 at infinity on an asymptotically flat manifold with nonnegative scalar curvature, one can consider the level sets \(\Sigma_c = \phi^{-1}(c)\). As long as condition (1.4) holds at some \(\Sigma_c\), one will have an estimate of the mass in terms of \(\phi\).

We now outline the proof of Theorem 1.2. The idea is simply to show the manifold \((M, g)\) in Theorem 1.2 admits a legitimate compact fill-in \((\Omega, \tilde{g})\) so that, if \((\Omega, \tilde{g})\) is glued to \((M, g)\), the resulting manifold satisfies the assumptions of the positive mass theorem. To produce such a fill-in, one conformally deforms \((M, g)\) using the given function \(u\) (or \(v\)). The idea of conformally deforming an asymptotically flat manifold with boundary to produce a compact piece appeared in the pioneer work of G. Bunting and A. Masood-ul-Alam [7] and was used by H. Bray [4] in his proof of the Riemannian Penrose inequality. A recent application of this idea to obtain capacity estimates was given by C. Mantoulidis, L.-F. Tam and the first author [24]. Once Theorem 1.2 is proved, Theorem 1.5 follows by considering a conformally deformed metric \(\tilde{g} = \left(\frac{1+\phi}{2}\right)^{\frac{n-2}{n}} g\) on \(M\) and applying Corollary 1.3 to \((M^n, \tilde{g})\).
The rest of this paper is organized as follows. We provide some background material in Section 2. In Section 3, we prove Theorem 1.2 for harmonically flat metrics. The general case of Theorem 1.2 is proved in Section 4. In Section 5, we use Corollary 1.3 to prove Theorem 1.5. In Section 6 we give some examples and discussions.

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2 Prerequisites

We start by recalling the definition of a manifold being asymptotically flat (see [30, 6] for instance). Here we only consider manifolds with one end, and the general case can be treated in the same fashion.

Definition 2.1. Let \( n \geq 3 \). A Riemannian manifold \((M^n, g)\) is asymptotically flat if there is a compact set \( K \subset M \) such that \( M/K \) is diffeomorphic to \( \mathbb{R}^n \) minus a ball and in this coordinate chart, \( g \) satisfies

\[
g = \delta + O(|x|^{-\tau}), \quad \partial g = O(|x|^{-\tau-1}), \quad \partial^2 g = O(|x|^{-\tau-2}),
\]

and \( R = O(|x|^{-q}) \), where \( \tau > \frac{n-2}{2} \) and \( q > n \). Here \( R \) is the scalar curvature of \( g \).

On an asymptotically flat \((M^n, g)\), the ADM mass [1] \( m \) is given by

\[
m = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r \to \infty} \int_{|x|=r} (g_{ij,j} - g_{jj,i}) \nu^i,
\]

where \( \omega_{n-1} \) is the volume of the unit sphere in \( \mathbb{R}^n \), and the unit normal \( \nu \) and volume integral are with respect to the Euclidean metric. The fact that \( m \) is a geometric invariant of \((M, g)\) was shown by Bartnik [3] and by Chruściel [8] independently.

We next address some regularity questions regarding the positive mass theorem. Originally the result was proved for smooth manifolds, but since then there have been also several singular cases, proven in [26], [31], [19], [25], [20], [32], and [23]. Combined with Theorem 1.1, results and proofs in [26, 25] and [32] in particular give the following theorems:

Theorem 2.2 (Theorem 1 in [26], Theorem 2 in [25]). Let \((M^n, g)\) be an asymptotically flat manifold. Suppose \( \Sigma^{n-1} \subset M^n \) is a closed hypersurface which divides \( M \) into an exterior part \( M_1 \) and an interior part \( M_2 \). Suppose the metric \( g \) is smooth up to \( \Sigma \) on both sides of \( \Sigma \) and has non-negative scalar curvature away from \( \Sigma \). Let \( H_1, H_2 \) be the mean curvature of \( \Sigma \) with respect to the unit normal pointing to infinity in \((M_1, g)\) and \((M_2, g)\), respectively. If

\[
H_1 \leq H_2,
\]

then the ADM mass of \((M^n, g)\) is non-negative and is zero if and only if \((M_1, g)\) is isometric to \((\mathbb{R}^n/\Omega, \delta)\) for a bounded domain \( \Omega \subset \mathbb{R}^n \) with smooth boundary and \((M_2, g)\) is isometric to \((\Omega, \delta)\).

Theorem 2.3 (Theorem 7.2 in [32]). Let \((M^n, g)\) be an asymptotically flat manifold such that \( g \in W^{1,p}_{\text{loc}} \) for some \( p > n \) and is smooth with non-negative scalar curvature away from a point \( q \). Then \( m \geq 0 \) and \( m = 0 \) only if \( M^n \) is diffeomorphic to \( \mathbb{R}^n \) and \( g \) is flat away from \( q \).
3 The Harmonically Flat Case

As in [4], we first prove Theorem 1.2 for the case $g = \xi^{4n-2}\delta$ near $\infty$, where $\xi$ is a Euclidean harmonic function. This property is often known as $(\mathbb{M}^n, g)$ being harmonically flat (see [4]).

Proof. Let $(\tilde{\mathbb{M}}^n, \tilde{g})$ be the one point compactification of $(\mathbb{M}^n, u^{4n-2}g)$ where $u$ is the conformal Green’s function from Theorem 1.2. Note that in particular $\tilde{g}|_{\Sigma} = g|_{\Sigma}$ and by the maximum principle $u > 0$. Also, we are able to extend $\tilde{g}$ smoothly to the point at $\infty$ as H. Bray’s proof of Theorem 1.4, also see the detailed exposition in section 2 of [24]. This argument is based on the removable singularity theorem for harmonic functions and thus crucially requires the manifold to be harmonically flat. Note that in this case $u$ is harmonic near $\infty$. In the general setting this conformal blow down may produce a singular metric which requires to be smoothed which is performed in section 4 following [32] and [24].

The conformally transformed manifold $\tilde{M}$ has zero scalar curvature $\tilde{R}$ due to the well known formula

$$\tilde{R} = u^{-\frac{4}{n-2} - \frac{4(n-1)}{n-2}}u^{-\frac{n}{n-2}}\Delta u. \quad (3.1)$$

Moreover, the mean curvature transforms under conformal transformation and change of unit normal via

$$\tilde{H} = -u^{\frac{2-n}{n-2}}H - \frac{2(n-1)}{n-2}u^{\frac{n}{n-2}}\nabla u. \quad (3.2)$$

Note that we have to change the sign of the unit normal of $\tilde{\Sigma}$ so that it corresponds with the normal of $\Sigma$ after gluing $\tilde{M}$ into $M$ as carried out below. Identity (3.2) leads in combination with $u|_{\Sigma} = 1$ to

$$\tilde{H} = -H - \frac{2(n-1)}{n-2}u \nabla_u u. \quad (3.3)$$

Next we glue $M$ and $\tilde{M}$ together along $\Sigma$ to obtain a manifold $\hat{M}$ with corner $\Sigma$. More precisely we define $(\hat{M}, \hat{g})$ by $\hat{M} = (M \sqcup \tilde{M})/\sim\sim$ and $\hat{g}|_{\hat{M}} = g$, $\hat{g}|_{\tilde{M}} = \tilde{g}$. Then $\hat{M}$ is asymptotically flat, has non-negative scalar curvature and no boundary. Furthermore, we have due to (1.1) and (3.3)

$$\tilde{H} = -\tilde{H} - \frac{2(n-1)}{n-2}\nabla_u u \geq H.$$ 

Thus all the assumptions of the positive mass theorem with corners, Theorem 2.2, are satisfied and we obtain $\hat{m} \geq 0$. Since we did not change the asymptotic behavior of the metric during the construction of $(\hat{M}, \hat{g})$ we have $\hat{m} = \hat{\hat{m}} \geq 0$ as desired.

For the rigidity part we proceed in two steps: First, we apply the rigidity statement of the positive mass theorem with corners [26], [25] to deduce that in the case of $\hat{m} = 0$, $(\hat{M}^n, g)$ is isometric to $(\mathbb{R}^n/\Omega, \delta)$ where $\Omega$ is some smooth, open set and $\delta$ the standard metric.

Second, we follow section 5 of [24] to show that $\Omega$ is a round ball. More precisely, let $B$ be the smallest ball which contains and touches $\Sigma = \partial\Omega$ at some point $p$. Without loss of generality $B$ is centered at the origin and has radius 1.

Next, we consider the fundamental solution to the Laplace equation $\varphi(r) = \frac{1}{r^{n-2}}$ and define $\psi := \varphi - u$. Assume $\Sigma$ is not a sphere which implies $\psi \neq 0$ and more precisely $\psi > 0$ in $\mathbb{R}^n \setminus B$ by the strong maximum principle. Applying Hopf’s Lemma we obtain $\nabla u < \nabla \psi$. 

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Observe, that by construction the mean curvature \( H \) of \( \Sigma \) is bounded from below by the mean curvature \( H(S) \) of \( S = \partial B \). By virtue of condition (1.1), we have at \( p \)

\[-\nabla_\nu u \geq \frac{n-1}{n-2} H \geq \frac{n-1}{n-2} H(S) = -\nabla_\nu \psi \]

which is a contradiction. \( \square \)

4 The General Case

In this section, we prove Theorem 1.2 in the general case. Let \( u \) be the conformal Green’s function from Theorem 1.2, i.e. \( L u = 0 \) where

\[ L = \Delta - \frac{n-2}{4(n-1)} R \]

is the conformal Laplacian with respect to \( g \). As in Appendix A of [24] we have the following asymptotic behavior of \( u \) at \( \infty \):

**Lemma 4.1.** There exists a constant \( D \) such that

\[ u(x) = \frac{D}{|x|^{n-2}} + \mathcal{O}(|x|^{-\gamma}) \]

where \( \gamma := \min(q-2, n+\tau-2, n-1) > n-2 \).

**Proof.** Let \( \mathbb{R}^n/B_{R_1}(0) \) be the asymptotically flat end. We start by construction a barrier to the conformal Green’s function. For this purpose let \( \psi = ar^{-n+2} - r^{-n+2-\epsilon} \) with \( a, \epsilon > 0 \). Then we have as in the proof of Lemma A2 in [24]

\[ L \psi = (-n + 2 - \epsilon)(-n + 1 - \epsilon)r^{-n-\epsilon} + \mathcal{O}(|x|^{-\kappa}) \],

where \( \kappa := \min(n + \tau, q) > n \). Choosing \( \epsilon < \min(\tau, n-q) \) there is an \( R_2 \geq R_1 \) independent of \( a \) such that \( \Delta \psi \leq 0 \) on \( \mathbb{R}^3/B_{R_2}(0) \). Moreover, we set \( a \gg 1 \) such that \( \psi > u \) on the complement of \( \mathbb{R}^n/B_{R_0}(0) \) in \( M \). Hence we have by the maximum principle \( u \leq \psi \leq ar^{-n+2} \) and similarly \( u \geq -ar^{-n+2} \).

By the Schauder Estimates, Theorem 6.2 in [11], we have

\[ |u| \leq C|x|^{-\kappa}, \quad |\nabla u| \leq C|x|^{-\kappa-1}, \quad |\nabla^2 u| \leq C|x|^{-\kappa-2} \]

where \( C \) denotes some constant which may change in the following computations from line to line.

We extend \( u \) smoothly onto \( \mathbb{R}^n \) such that \( u \leq C \) in \( B_{R_0}(0) \) with \( R_0 \leq R_1 \). Let \( f = \Delta u = \mathcal{O}(|x|^{-\kappa}) \), where \( \Delta \) is the Euclidean Laplacian, and define

\[ w(x) := -\frac{1}{n(n-2)\alpha_n} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} f(y) dy \]

where \( \alpha_n \) is the volume of the unit ball in \( \mathbb{R}^n \). This is well defined due due to the decay properties of \( f = \Delta u \). Also, observe that \( \Delta w = f \). Then we compute as in the proof of Lemma A2 in the Appendix A of [24] denoting \( |x| = r \):
Let Lemma 4.2. Combining 4.1-4.5, we obtain

\[ \int_{B_r(x)/B_r(0)} \frac{1}{|x-y|^{n-2}} f(y) dy \leq C r^{-\kappa} \int_{B_r(x)/B_r(0)} \frac{1}{|x-y|^{n-2}} dy \leq C r^{-\kappa+2}, \]

(4.1)

\[ \int_{\mathbb{R}^n/(B_r(0) \cup B_r(x))} \frac{1}{|x-y|^{n-2}} f(y) dy \leq C r^{-n+2} \int_{\mathbb{R}^3/(B_r(0) \cup B_r(x))} |y|^{-n+\tau} \leq C r^{-\kappa+2}. \]

(4.2)

Moreover, we additionally split the integral over \( B_r(0) \) into \( B_r(0)/B_{R_0}(0) \cup B_{R_0}(0) \):

\[ \int_{B_r(0)/B_{R_0}(0)} \left( \frac{1}{|x-y|^{n-2}} - \frac{1}{|x|^{n-2}} \right) f(y) dy 
= \int_{B_{R_0}(0)/B_r(0)} \frac{|x|^{2n-4} - |x-y|^{2n-4}}{|x|^{n-2} - |x-y|^{n-2}} f(y) dy 
\leq \sum_{j=1}^{2n-4} \int_{B_r(0)/B_{R_0}(0)} C \frac{|y|^{-\kappa+j}}{|x|^{n-2+j}} \leq C r^{-n+1}. \]

(4.3)

Similarly, we have

\[ \int_{B_{R_0}(0)} \left( \frac{1}{|x-y|^{n-2}} - \frac{1}{|x|^{n-2}} \right) f(y) dy 
= \int_{B_{R_0}(0)} \frac{|x|^{2n-4} - |x-y|^{2n-4}}{|x|^{n-2} - |x-y|^{n-2}} f(y) dy 
\leq \sum_{j=1}^{2n-4} \int_{B_{R_0}(0)} C \frac{|y|^{2n-4}}{|x|^{n-2+2n}} \leq C r^{-n+1}. \]

(4.4)

Lastly, we note

\[ \frac{1}{|x|^{n-2}} \int_{\mathbb{R}^n/B_r(0)} f(y) dy \leq C r^{-\kappa+2}. \]

(4.5)

Combining 4.1-4.5, we obtain

\[ -n(n-2)\alpha_n w = \int_{\mathbb{R}^n} \frac{\Delta u}{r^{n-2}} + O(|x|^{-\min(q-2,n+\tau-2,n-1)}). \]

In particular, we have by the maximum principle \( w = u \). As in [23], Schauder theory, Theorem 6.3 in [11], gives the higher order estimates. \( \blacksquare \)

Without loss of generality we may assume \( R_1 = 1 \), i.e. the asymptotically flat end is diffeomorphic to \( \mathbb{R}^n/B_1(0) \). Next, we introduce the Kelvin transform \( K \) and prove the following elementary lemma:

**Lemma 4.2.** Let \((M^n,g)\) be asymptotically flat, i.e. \( g_{ij} = \delta_{ij} + \sigma_{ij} \) where \( \sigma_{ij} = O_2(|x|^{-\tau}) = O_2(|y|^{-\tau}) \) and \( K(x) := \frac{x}{|x|^2} \). Then we have for \( y = K(x) \) and \( h_{ij} = g(\partial_i y, \partial_j y) \)

\[ h_{ij} = |y|^{-4} \delta_{ij} + O(|y|^\tau-4). \]

Moreover, we have

\[ \partial_k h_{ij} = -4|y|^{-5} \partial_k |y| \delta_{ij} + O(|y|^\tau-5). \]
Proof. We compute
\begin{align*}
    h_{ij} &= g\left(\frac{\partial_i x}{x^2} - 2\frac{(\partial_i x, x)x}{|x|^4}, \frac{\partial_j x}{x^2} - 2\frac{(\partial_j x, x)x}{|x|^4}\right)
    = |y|^{-4}[\delta_{ij} + \sigma_{ij} + |y|^{-4}(4y_iy_jy_ky_l\sigma^{kl} - 2|y|^2(y_iy^k\sigma_{jk} + y_jy^k\sigma_{ik}))]
    = |y|^{-4}\delta_{ij} + O(|y|^\gamma).
\end{align*}

The second statement follows analogously.

Next, we study the metric \( \tilde{g} = u^{\frac{4}{n-2}}g. \) A priori \( \tilde{g} \) is defined on \( \mathbb{R}^n/B_1(0) \) but by inverting the coordinates via the Kelvin transform \( K \) we may view \( \tilde{g} \) as metric on \( B_1(0) \). Thereby the point \( \infty \) of the one point compactification corresponds to the origin under the Kelvin transform and we wish to extend \( \tilde{g} \) there. This is done in the following lemma which is based on Lemma 6.1 in [24]. Moreover, we also obtain some regularity for \( \tilde{g} \) which is necessary in order to perform the smoothing procedure below.

**Lemma 4.3.** The metric \( \tilde{g} \) extends continuously across the origin to a \( W^{1,p} \) metric for some \( p > n \).

**Proof.** We compute in the coordinates \( y = K(x) \) for \( h_{ij} \)
\begin{align*}
    u^{\frac{4}{n-2}}h_{ij} = & D\delta_{ij} + O(|y|^\gamma) + (Dy)|y| + O(|y|^\gamma) + O(|y|^{\gamma+1})
    = O(|y|^\gamma).
\end{align*}

Hence \( h \) is continuous in the origin. Next, we compute in a similar fashion
\begin{align*}
    \partial_k(u^{\frac{4}{n-2}}h_{ij}) &= 4D|y|^{n-7} + O(|y|^{\gamma-5})D|y|^{n-2} + O(|y|^\gamma)\frac{\delta_{ij}}{|y|}
    + (Dy)|y|^{n-7} + O(|y|^{\gamma-5})D|y|^{n-2} + O(|y|^\gamma)\frac{\delta_{ij}}{|y|}
    = O(|y|^{\gamma+n+1}).
\end{align*}

Thus \( u^4h_{ij} \in W^{1,p}(B_1(0)) \) for \( p = \frac{n}{\min(q-n-1,\gamma-1)} > n \).

Now we are in a position to verify the following Proposition, which is known to people who are familiar with the work in [20, 31, 10, 25, 32, 23, 24].

**Proposition 4.4.** Suppose \( (M^n, g) \) has corner singularity along a hypersurface \( \Sigma \) with \( H_1 \leq H_2 \) as in Theorem 2.2. Also, suppose there is a point singularity \( q \in M_2 \) where \( g \) is in \( W^{1,p} \) for some \( p > n \) near \( q \). If \( g \) has non-negative scalar curvature away from \( \Sigma \) and \( \{q\} \), then \( m \geq 0 \) with equality if and only if \( (M_1, g) \) is isometric to \( (\mathbb{R}^n/\Omega, \delta) \) for a bounded domain \( \Omega \) with smooth boundary.

**Proof.** Due to the assumption \( H_1 \leq H_2 \) at \( \Sigma \), we may exactly follow Proposition 3.1 in [26] to approximate \( g \) by a family of smooth metrics \( g_\delta \) such that \( g_\delta(x) = g(x) \) for \( \text{dist}(\Sigma, x) \geq \delta \) and the scalar curvature \( R_\delta \) satisfies
\begin{align*}
    R_\delta(x) &\geq -C 
\end{align*}

for \( \text{dist}(\Sigma, x) < \delta \). (4.6) in particular shows that the integral of the negative part of the scalar curvature can be made arbitrarily small during the approximation process.

Near the point singularity \( q \), we approximate \( g \) as in [32] Lemma 4.1 and [24] Lemma 3.6 to obtain smooth metrics \( \{g_\epsilon\} \) so that \( g_\epsilon = g \) outside \( B_\epsilon(\{q\}) \), \( \|g_\epsilon\|_{W^{1,p}(B_\epsilon(\{q\}))} \leq C \) where \( C \) is independent of \( \epsilon \). By Lemma 3.7 in [24], the uniform \( W^{1,p} \) bound on \( g_\epsilon \) implies that the integral of the negative
part of $R_g$ over $B_\varepsilon(\{q\})$ becomes arbitrarily small. (We note that Lemma 3.7 in [24] is stated in a slightly more general version and for our purpose the result already follows from the estimate on the term $I_B$ in Lemma 3.7 in [24].)

Since both approximations $g_\varepsilon$ and $g_\delta$ are local, we can perform them simultaneously to approximate $g$ by a smooth metric with $\int_M R^-$ becoming arbitrarily small. Here $R^-$ denotes the negative part of the scalar curvature.

Now we proceed as usual and apply conformally transform $M$ such that $R \geq 0$ everywhere, see [29] and section 4 in [26]. Thereby the mass converges as in [26] and we may deduce the mass is non-negative by the standard positive mass theorem.

If the mass is zero, we need to apply the argument from [25]. This is due to [25] relying on Ricci flow which has the advantage that the mass stays constant during the smoothing process so we can apply the rigidity statement of the positive mass theorem to the flow solution with initial data $g$ (see also section 7 of [32]). This shows the rigidity of the Proposition.

**Proof of Theorem 1.2** Due to Lemma 1.3, the conformally filled in manifold $(\hat{M}, \hat{g})$ constructed in Section 3 satisfies the conditions of Proposition 1.4. Hence Theorem 1.2 follows from Proposition 1.4 in the same way that the harmonically flat case is proved in Section 3.

**5 Application**

In this section, we prove Theorem 1.5 which generalizes H. Bray’s result in Theorem 1.4.

**Proof of Theorem 1.5** By the maximum principle, $\phi > -1$ on $M$. Define

$$w = \frac{2}{1 + \phi} \quad \text{and} \quad \tilde{g} = w^{-\frac{4}{n-2}} g.$$  

$(M^n, \tilde{g})$ is asymptotically flat with nonnegative scalar curvature. The fact $\Delta w^{-1} = 0$ implies

$$\tilde{\Delta} w = 0.$$  

Moreover, $w \to 1$ at $\infty$ and $w = \frac{2}{1+c}$ at $\Sigma$. Next, define

$$v = \frac{1 + c}{1 - c} (w - 1).$$  

Then

$$\tilde{\Delta} v = 0, \quad v \to 0 \text{ at } \infty, \quad \text{and} \quad v = 1 \text{ at } \Sigma.$$  

Let $\tilde{H}$ be the mean curvature of $\Sigma$ in $(M, \tilde{g})$ with respect to the $\infty$-pointing unit normal $\tilde{\nu}$. By (3.2), it follows that

$$\tilde{H} = w^{\frac{2}{n-2}} \left[ H + \frac{2(n-1)}{(n-2)} w \nabla_\nu w^{-1} \right],$$  

$\tilde{\nu} = w^{\frac{2}{n-2}} \nu,$

and

$$\nabla_{\tilde{\nu}} v = w^{\frac{2}{n-2}} \nabla_\nu v$$  

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Hence, at $\Sigma$, by (1.4),
\[
\tilde{H} + \frac{n-1}{n-2} \nabla_{\nu} v
= w^{\frac{n-2}{n-1}} \left[ H + \frac{(n-1)}{(n-2)} \left( 2w \nabla_{\nu} w^{-1} + \nabla_{\nu} v \right) \right]
= w^{\frac{n-2}{n-1}} \left[ H - \frac{(n-1)}{(n-2)} \frac{2c}{(1-c^2)} \nabla_{\nu} v \right]
\leq 0.
\]
Thus, by Corollary 1.3
\[\tilde{m} \geq 0 \quad (5.1)\]
where $\tilde{m}$ is the ADM mass of $(M^n, \tilde{g})$. Since by the definition of mass $\tilde{m}$ and $m$ are related by $\tilde{m} = m - C$, we conclude from (5.1) that
\[m \geq C.\]
If $m = C$, then $\tilde{m} = 0$. By Corollary 1.3 $(M^n, \tilde{g})$ is isometric to $(\mathbb{R}^n/\{|x| < r_0\}, \delta)$ for some constant $r_0 > 0$. In this case, $w$ is a Euclidean harmonic function that goes to 1 at $\infty$ and equals a positive constant $\frac{2}{1+c}$ at $\Sigma = \{|x| = r_0\}$. Hence,
\[w = 1 + \frac{m}{2|x|^{n-2}},\]
where $m$ is a constant satisfying $m = \frac{(1-c)}{1+c} 2r_0^{n-2}$. This completes the proof. 

It is worth of pointing out the following equivalent form of Theorem 1.5.

Theorem 5.1. Let $(M^n, g)$ be an $n$-dimensional, asymptotically flat manifold, with non-negative scalar curvature, with boundary $\Sigma$. Let $\varphi(x)$ be a function on $(M, g)$ which satisfies
\[
\begin{cases}
\Delta \varphi &= 0 \text{ in } M \\
\varphi &\to 1 \text{ at } \infty \\
\varphi &= 0 \text{ at } \Sigma.
\end{cases}
\]
Let $H$ be the mean curvature of $\Sigma$ in $(M^n, g)$ with respect to the $\infty$-pointing normal $\nu$. If there is a constant $c > -1$ such that
\[\frac{2c}{1+c} \nabla_{\nu} \varphi \geq \frac{n-2}{n-1} H, \quad (5.2)\]
then
\[m \geq (1-c)C, \quad (5.3)\]
where $m$ is the ADM mass of $(M^n, g)$ and $C$ is the constant in $\varphi = 1 - \frac{C}{|x|^{n-2}} + o(|x|^{2-n})$, as $x \to \infty$.

Moreover, equality in (5.3) holds if and only if $(M^n, g)$ is isometric to an $n$-dimensional spatial Schwarzschild manifold outside a rotationally symmetric sphere, i.e.
\[
\left( \mathbb{R}^n/\{|x| < r_0\}, \left( 1 + \frac{m}{2|x|^{n-2}} \right)^{\frac{1}{n-2}} \delta_{ij} \right) \quad \text{for some constants } r_0 > 0.
\]
Proof. If \( c \neq 1 \), the theorem follows directly from Theorem 1.5 by letting \( \varphi = \frac{1}{1-c}(\phi - c) \). If \( c = 1 \), the theorem reduces to Corollary 1.3. \( \square \)

Remark 5.2. For a constant \( c > -1 \) satisfying (5.2) to exist, one needs to have

\[
2\nabla_{\nu}\varphi > \frac{n-2}{n-1} H.
\]

If \( n = 3 \), this coincides with the condition used in [24, Theorem 1.5].

6 Examples and Discussions

Example 6.1 (Boundary with \( H \leq 0 \)). Every manifold \((M^n, g)\) whose boundary \( \partial M \) has non-positive mean curvature satisfies condition (1.1) and thus has positive mass \( m \). More precisely, \( m \geq C > 0 \) by Theorem 1.5.

Example 6.2 (Regions in Schwarzschild manifold). Let \((M^n, g) = (\mathbb{R}^n / B_{r_0}(0), (1 + \frac{m}{2r^n-1})^{\frac{n}{n-2}} \delta)\) be a spacelike slice in Schwarzschild spacetime of mass \( m \). Note, \( m \) is allowed to be negative. We begin by computing the mean curvature of the boundary \( S_{r_0}(0) \) using (3.2):

\[
H = (2\frac{n}{n-2} r_0^{n-1} - 2\frac{2}{n-2} mr_0) \frac{n-1}{(2r_0^{n-2} + m)^{\frac{n-2}{n}}}.
\]

Next, we observe that the Green’s function is given by

\[
u(r) = \frac{2r_0^{n-2} + m}{2r_0^{n-2} + m}.
\]

Combining this with the observation \(|\partial_r|_g = (1 + \frac{m}{2r^n-1})^{\frac{n}{n-2}}\), we have at \( S_{r_0} \)

\[
-n\frac{n-1}{n-2} \nabla_{\nu}u = 2\frac{n}{n-2} r_0^{n-1} \frac{n-1}{(2r_0^{n-2} + m)^{\frac{n-2}{n}}}.
\]

Thus we have as predicted by the rigidity statement of Theorem 1.5

\[
H = -\frac{2c(n-1)}{(1+c)(n-2)} \nabla_{\nu}u
\]

for

\[
c = \frac{2r_0^{n-2}}{2r_0^{n-2} + m}.
\]

Moreover, \( \frac{2c}{c+1} \) can approach 1 while maintaining negative mass which shows that Theorem 1.2 is sharp in this sense.

Example 6.3 (Conformal minimal boundary). Suppose that \( g = \phi^{n-2} \tilde{g} \) such that \( \Sigma \) is a minimal surface with respect to \( \tilde{g} \) where \( \phi \) solves

\[
\begin{cases}
\Delta \phi &= 0 \text{ in } M \\
\phi &\to 2 \text{ at } \infty \\
\phi &= 1 \text{ at } \Sigma.
\end{cases}
\]
Observe that $\tilde{R}$ has by (3.1) also non-negative scalar curvature. Note that $v\phi$ is a harmonic function on $(M^n, \tilde{g})$ satisfying $(\phi v)|_\Sigma = 1$ and $(\phi v)(x) \to 0$ for $|x| \to \infty$. Precisely the same properties hold true for the function $(2 - \phi)$ and thus we deduce by the uniqueness property for harmonic functions $\phi v = 2 - \phi$. Therefore, we have at $\Sigma$

$$\nabla_\nu v = -\frac{2}{v^2} \nabla_\nu \phi = -2 \nabla_\nu \phi.$$ 

Next, we compute using (3.2)

$$H = \tilde{H} \phi^{-\frac{1}{n-2}} + 2(n-1)\phi^{-\frac{n}{n-2}} \nabla_\nu \phi = 2(n-1)\frac{n}{n-2} \nabla_\nu \phi = -\frac{n-1}{n-2} \nabla_\nu v.$$ 

Hence condition (1.2) is satisfied with equality and we have $m \geq 0$. Conversely, suppose we have $H \equiv -\frac{4}{n-2} \nabla_\nu v$. Then, we define $\phi := \frac{2}{1+v}$ and $\tilde{g} = \phi^{\frac{-1}{n-2}} g$. Applying again (3.2), we obtain

$$\tilde{H} = H \phi^{\frac{1}{n-2}} + 2(n-1)\phi^{\frac{n}{n-2}} \nabla_\nu (\phi^{-1}) = H + \frac{n-1}{n-2} \nabla_\nu v = 0.$$ 

Thus $\Sigma$ is a minimal surface with respect to $\tilde{g}$. Indeed, this calculation shows that condition (1.2) holds if and only if $\tilde{H} \leq 0$ under the above conformal transformation. In particular, this suggests Bray’s result and proof of Theorem 1.4 in [4] can also be applied to derive Corollary 1.3 and Theorem 1.5 in the harmonically flat case.

Next, we want to relate Theorem 1.2, Theorem 1.5 and Example 6.3 to the Riemannian Penrose conjecture. This conjecture was shown by G. Huisken and T. Ilmanen [18] (one black hole) and by H. Bray [4] (multiple black holes). The latter proof has been extended up to dimension 7 by H. Bray and D. Lee [6].

One of the key steps in H. Bray’s proof is to prove the mass-capacity estimate, Theorem 1.4. There an asymptotically flat manifold with non-negative scalar curvature is first reflected across its horizon, then conformally transformed via the Green’s function so the positive mass theorem may be applied. In view of the positive mass theorem for manifolds with boundary, the reflection argument can now be dropped. More precisely, the asymptotically flat manifold can be directly conformally transformed via the Green’s function as presented in Example 6.3.

As it turns out Theorem 1.4 is the reason the proof of the Penrose inequality breaks down in higher dimension as discovered by H. Bray. This is due to minimal hypersurfaces being no longer regular which then in particular complicates the reflection argument. Therefore, showing a positive mass theorem for manifolds with singular boundary may be considered a strategy for proving the higher dimensional Penrose inequality.

We also recall that M. Herzlich [12, 14] showed a positive mass theorem for asymptotically flat manifolds with boundary $\partial M$ whose mean curvature satisfies an upper bound which depends on the area of $\partial M$ in 3-dimension and depends on the Yamabe invariant $\mathcal{Y}$ of $\partial M$ in higher dimensions. More precisely, the latter case assumes

$$H \leq |\Sigma|^{-\frac{1}{n-1}} \sqrt{\frac{n-1}{n-2} \mathcal{Y}(\Sigma)}.$$ 

The proofs in [12, 14] make use of estimates of the first eigenvalue of the Dirac operator by C. Bär [2] and by O. Hijazi [15, 16]. Results in [12, 14] have been extended to the asymptotically hyperbolic setting by O. Hijazi, S. Montiel and S. Raulot [17]. It is plausible that Theorem 1.2 or Corollary 1.3 may have an analogue in the asymptotically hyperbolic setting.
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