On the bosonic symmetries of the Dirac equation with nonzero mass

Volodimir Simulik
Principal researcher, Institute of Electron Physics, National Academy of Sciences, Uzhgorod 88000, Ukraine
E-mail: vsimulik@gmail.com

Abstract. Bosonic symmetries of the Dirac equation with nonzero mass, which existence is under consideration after our publications in the years 2011–2015, are proved here on the basis of two different methods. The first one appeals to the 64-dimensional gamma matrix representation of the Clifford algebra $C^{\ell}_R(0,6)$ over the field of real numbers and the 28-dimensional gamma matrix representation of the algebra $SO(8)$ (over the field of real numbers as well). The second way of proof is based on the start from the relativistic canonical quantum mechanics of spin $1/2$ particle multiplet and its relationship with the Dirac equation, which is given by the extended Foldy-Wouthuysen transformation suggested by us in 2014–2017. Both the Lorentz and Poincaré bosonic symmetries are considered. The 31-dimensional algebra of invariance is found. The bosonic solutions and conservation laws are found as well. The considered phenomenon is called the Fermion-Boson duality of the Dirac equation according to P. Garbaczewski’s titles suggested in 1986.

1. Introduction

It is well-known that the Dirac equation is invariant with respect to the transformations, which are determined by the spin $s = \frac{1}{2}$ representation of the Poincaré group. On the basis of this fact the conclusion that the Dirac equation describes the spin $s = \frac{1}{2}$ fields and particles (fermions) is formulated. The representations of the proper orthochronous Poincaré group (inhomogeneous Lorentz group) have the principal importance.

Below we are able to demonstrate another hidden half of the Dirac equation possibilities. We consider here briefly the bosonic symmetries, solutions and conservation laws for the Dirac equation with nonzero mass.

Note that the Dirac equation taken itself (as well as every other field equation) does not carry the complete information on the subject what field (particle) is described by this equation. The complete information is given only by the pair of conceptions: equation and the transformation law of the field function. Therefore, the transformations, which are determined not by the 1/2 eigen values of the particle spin operator, have a special importance and physical meaning among the additional transformations, with respect to which the Dirac equation is invariant.

Here two proofs of Bose properties of the Dirac equation with nonzero mass are considered. The first one is based on the new mathematical objects. These objects are the 64-dimensional gamma matrix representation of the Clifford algebra $C^{\ell}_R(0,6)$ over the field of real numbers and the 28-dimensional gamma matrix representation of the algebra $SO(8)$ (over the field of real numbers as well). The additional elements of gamma matrix representations of these algebras...
unable us to prove the bosonic symmetries of the Dirac equation and, as a consequence, to
demonstrate the bosonic solutions and conservation laws.

The second proof is based on the start from the relativistic canonical quantum mechanics of
Bose spin (1,0) particle multiplet and its link with the Dirac equation, which is given by the
extended Foldy–Wouthuysen transformation.

In addition to the Dirac equation the \( \gamma \) matrix representations of the Clifford algebra are
used widely for the multicomponent Dirac-like equations of higher and arbitrary spin such as
the Bhabha, Bargman–Wigner, Rarita–Schwinger (for the field with the spin \( s=3/2 \)), Iwanenko-
Landau-Dirac–Kähler equations, etc. Thus, the application of the Clifford algebra in the
quantum theory is more wide then the 4-component Dirac equation and corresponded spinor
field. It is evident that the program of finding of more wide matrix representations then the
16-dimensional \( C_\ell(1,3) \) and 15-dimensional \( \text{SO}(1,5) \) algebras representations is the interesting
task. The corresponding matrix representations can be useful both in the known quantum field
theory models and in the development of new approaches for higher spin.

Starting from the first steps of quantum mechanics and during the period of its growth many
authors, see, e.g. \[1, 2\] and the list of references in \[3\], have been investigated the Fermi-Bose
duality of the massless spinor field and of the related electromagnetic field in the terms of
field strengths. Such investigations were called the relations between the Dirac and Maxwell
equations, the Maxwell–Dirac isomorphism \[4\], etc. Recent interest to the problem is known as
well \[5, 6\].

For the most simple case of the massless Dirac equation the bosonic symmetries, solutions
and conservation laws were found by us more then 15 years ago, see e.g. \[7, 8\] and the references
therein. The corresponding fermionic properties of the slightly generalized original Maxwell
equations were considered in \[9, 10\]. The investigations of the general case of nonzero mass
lead us to the new mathematical objects \[11, 12\]. Only after introducing of the 64-dimensional
gamma matrix representation of the Clifford algebra \( C_\ell^R(0,6) \) \[13, 14\] over the field of real
numbers (as well as the 28-dimensional gamma matrix representation of the real algebra \( \text{SO}(8) \)
\[15, 16\]) we were successful in extension of the results to the general case, when the mass in the
Dirac equation is nonzero.

2. Gamma matrix representations of the \( C_\ell^R(0,6), C_\ell^R(4,2) \) Clifford and the \( \text{SO}(8) \)
Lie algebras

One of the principal objects of the relativistic quantum mechanics and field theory is the Dirac
equation, see, e.g. \[17\]. This equation describes the particle-antiparticle doublet with spins
\( s=(1/2,1/2) \) or, in other words, spin 1/2 fermion-antifermion doublet. The presentation of
Dirac equation operator \( D \equiv i\gamma^\mu \partial_\mu - m \), as well as a list of other operators of quantum spinor
field, in the terms of \( \gamma \) matrices gives a possibility to use the anti-commutation relations between
the Clifford algebra operators directly for finding the symmetries, solutions, conservation laws,
fulfilling the canonical quantization and calculating the interaction processes in the quantum
field models. The important fact is that application of the Clifford algebra essentially simplifies
the calculations. We use the definition of Clifford algebra given in \[18, 19\].

Thus, for the case of free non-interacting spinor field the Dirac equation has the form

\[
(i\gamma^\mu \partial_\mu - m)\psi(x) = 0,
\]

where

\[
x \in \text{M}(1,3), \quad \partial_\mu \equiv \partial/\partial x^\mu, \quad \mu = 0,3, \quad j = 1, 2, 3,
\]

\[
\text{M}(1,3) = \{ x \equiv (x^\mu) = (x^0 = t, x^j) \equiv (x^j) \} \text{ is the Minkowski space-time and 4-component}
\]

function \( \psi(x) \) belongs to rigged Hilbert space \( S^{3,4}_J \subset \mathbb{H}^{3,4} \subset \mathbb{S}^{3,4} \). Due to a special role of
the time variable \( x^0 = t \in (x^\mu) \) (in obvious analogy with nonrelativistic theory), in general
consideration one can use the quantum-mechanical rigged Hilbert space. Here the Schwartz test function space \( S^{3,4} \) is dense in the Schwartz generalized function space \( S^{3,4*} \) and \( H^{3,4} \) is the quantum-mechanical Hilbert space of 4-component functions over coordinate space \( \mathbb{R}^3 \subset M(1,3) \). Here the system of units \( h = c = 1 \) is chosen, the metric tensor in Minkowski space-time \( M(1,3) \) is given by
\[
g^{\mu\nu} = g_{\mu\nu} = g^0_0, \quad (g^\mu_\nu) = \text{diag} (1, -1, -1, -1); \quad x_{\mu} = g_{\mu\nu} x^\nu,
\]
and summation over the twice repeated indices is implied.

The Dirac \( \gamma \) matrices are taken in the standard Dirac-Pauli representation
\[
\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^\ell = \begin{pmatrix} 0 & \sigma^\ell \\ -\sigma^\ell & 0 \end{pmatrix}, \quad \ell = 1, 2, 3,
\]
where the Pauli matrices are given by
\[
\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^1 \sigma^2 = i\sigma^3, 123! - \text{circle.}
\]

Four operators (5) satisfy the anti-commutation relations of the Clifford algebra
\[
\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}, \quad \gamma_{\mu} = g_{\mu\nu}\gamma^{\nu}, \quad \gamma^{1\ell} = -\gamma^{\ell}, \quad \gamma^{10} = \gamma^0,
\]
and realize the 16-dimensional \((2^4 = 16)\) matrix representation of the Clifford algebra \( Cl^C(1,3) \) over the field of complex numbers.

For our purposes we introduce the additional matrix
\[
\gamma^4 = \gamma^0\gamma^1\gamma^2\gamma^3 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Such matrix satisfies the anti-commutation relations of the Clifford algebra as well \( \gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}, \quad \gamma_{\mu} = g_{\mu\nu}\gamma^{\nu}, \quad \gamma^{1\ell} = -\gamma^{\ell}, \quad \gamma^{10} = \gamma^0 \),

and does not contribute to the definition of this algebra as 16-dimensional \((2^4 = 16)\) \( Cl^C(1,3) \).

Here and in our publications (see, e. g. the articles [11–16]) we use the anti-Hermitian \( \gamma^{4} = \gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3} \) matrix instead of the \( \gamma^{5} \) matrix of other authors [17]. Our \( \gamma^{4} \) is equal to \( \gamma_{\text{standard}}^{5} \). Notation \( \gamma^{5} \) is used in [11–16] and below for a completely different matrix \( \gamma^{5} \equiv \gamma^{1}\gamma^{3}\hat{C} \).

The 16 elements of the \( \gamma \) matrix representation of the \( Cl^C(1,3) \) algebra are demonstrated by the Table 1 in [20]. In the papers [21, 22] 16 elements of the Clifford-Dirac algebra are linked to the 15 elements of the SO(3,3) Lie algebra. In our papers [14–16] we were able to give the link between the \( \gamma \) matrix representation of the \( Cl^C(1,3) \) algebra and the 15 elements of the SO(1,5) Lie algebra:
\[
s^{\mu\nu} = \left\{ \begin{array}{l}
     s^{\mu\nu} = \frac{1}{4} \left[ \gamma^\mu, \gamma^\nu \right], \quad s^{\mu\nu} = 2s^{\mu\nu}, \quad \mu, \nu = 0, 1, 2, 3 \\
     s^{\mu\nu} = 0, 5, \quad \mu, \nu = 0, 4.
\end{array} \right.
\]

The generators (3.4) satisfy the commutation relations of the SO(1,5) Lie algebra:
\[
\left[ s^{\mu\nu}, s^{\rho\sigma} \right] = -s^{\rho\sigma} s^{\mu\nu} - s^{\sigma\mu} s^{\rho\nu} - s^{\nu\rho} s^{\mu\sigma} - s^{\mu\rho} s^{\nu\sigma} - g^{\mu\rho} g^{\sigma\nu} - g^{\sigma\rho} g^{\nu\mu} - g^{\nu\rho} g^{\mu\sigma} - g^{\mu\sigma} g^{\nu\rho}, \quad (g^{\mu\nu}) = \text{diag}(+1, -1, -1, -1, -1).
\]

The 15 elements of the \( \gamma \) matrix representation of the SO(1,5) algebra are demonstrated by the Table 2 in [20]. Thus, it is proved that one and the same 15 not unit elements determine both the \( \gamma \) matrix representation of the \( Cl^C(1,3) \) Clifford algebra and the \( \gamma \) matrix representation of the SO(1,5) Lie algebra (similarly for SO(6) algebra). Note that set of elements of the SO(1,5)
representation (8) contains the spin operators \((s^{23}, s^{31}, s^{12})\) of the spinor field and Dirac theory. Note that here in (8) the anti-Hermitian realization of the SO(1,5) operators is chosen, for the reasons see, e.g., [11–16], for the possibility and mathematical correction to choose the anti-Hermitian operators see, e.g., [23, 24].

Comparison of the Tables 1 and 2 in [20] shows evidently that \(\mathcal{C}^C(1,3)\) and SO(1,5) are completely different algebras. Moreover, the algebra SO(1,5) can not be the subalgebra of the \(\mathcal{C}^C(1,3)\) algebra. Indeed, the first one is Clifford and the second is Lie algebra, which have different structure. In particular, Lie algebras do not contain the unit element as \(I_4 = -\gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^4\) from \(\mathcal{C}^C(1,3)\).

The new matrix representations of the Clifford and SO(8) algebras have been found in order to prove the known Bose symmetries of the massless Dirac equation for the general case of nonzero mass. We have started from the case of massless Dirac equation and the spinor field of zero mass. Indeed, at first the Bose symmetries of the massless Dirac equation have been found. We have used essentially so-called Pauli-Gürsey-Ibragimov symmetry [25–27] of the Dirac equation with zero mass, i.e. the fact that this equation is invariant with respect to the transformations generating by the eight operators

\[
\{\gamma^2 \hat{C}, \gamma^3 \hat{C}, \gamma^4 \hat{C}, \gamma^4 \gamma^1 \hat{C}, \gamma^1 \gamma^2 \gamma^3 \gamma^4 \hat{C}, \gamma^4 \hat{C}, i \gamma^4 \hat{C}, i, I\},
\]

where imaginary unit \(i = \sqrt{-1}\) is considered as the operator, \(\hat{C}\) is the operator of complex conjugation, \(\hat{C} \psi = \psi^*\) (the operator of involution in \(H^{3,4}\)), i.e. these operators are here the nontrivial orts of the algebra. Of course, the Pauli-Gürsey-Ibragimov algebra is defined over the field of real numbers.

At first we have proved that six of the operators from (10)

\[
s^0_{\text{PGI}} = -\frac{i}{2} \gamma^2 \hat{C}, \quad s^1_{\text{PGI}} = -\frac{i}{2} \gamma^3 \hat{C}, \quad s^2_{\text{PGI}} = -\frac{i}{2} \gamma^4 \hat{C}, \quad s^3_{\text{PGI}} = -\frac{i}{2} \gamma^4 \gamma^1 \hat{C}, \quad s^4_{\text{PGI}} = -\frac{i}{2} \gamma^4 \gamma^1 \gamma^2 \gamma^3 \gamma^4 \hat{C}, \quad s^5_{\text{PGI}} = -\frac{i}{2} \gamma^4 \gamma^1 \gamma^2 \gamma^3 \gamma^4 \hat{C},
\]

realizes the additional \(D(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)\) representation [28, 29] (and [7, 8]) of the Lie algebra of universal covering \(\mathcal{L} = SL(2, C)\) of the proper orthochronous Lorentz group \(L^+_\mathcal{L} = SO(1,3) = \{A = (A_{\mu}^\nu)\}\). Moreover, six operators (11) commute with all operators from (8).

Further, the application of the simplest linear combinations of the generators \(s^{\mu\nu}\) (11) and (8) gives the possibility [7, 8, 28, 29] to find Bose representations of the Lorentz group \(\mathcal{L}\) and the Poincaré group \(\mathcal{P} \supset \mathcal{L} = SL(2, C)\) (here \(\mathcal{P}\) is the universal covering of the proper orthochronous Poincaré group \(P^+_\mathcal{P} = T(4) \times L^+_\mathcal{L} \supset L^+_\mathcal{L}\)), with respect to which the massless Dirac equation is invariant. Thus, we have found the Bose \(D(1,0) \oplus (0,0)\) and \(D(\frac{1}{2}, \frac{1}{2})\) representations of the Lie algebra of the Lorentz group \(\mathcal{L}\) together with the tensor-scalar of the spin \(s=(1,0)\) and vector representations of the Lie algebra of the Poincaré group \(\mathcal{P}\), with respect to which the Dirac equation with \(m=0\) is invariant.

After that the goal of further investigations has been formulated as follows. To find the similar symmetries of the Dirac equation with nonzero mass. Our first idea in this direction was to find the complete set of combinations of the Pauli-Gürsey-Ibragimov operators (10) and the elements of the matrix representation of the \(\mathcal{C}^C(1,3)\) algebra demonstrated in the Table 1 in [20].

Below the results are presented in modern mathematical notations of Clifford algebras. Consider the fact that seven \(\gamma\) matrices

\[
\gamma^1, \gamma^2, \gamma^3, \gamma^4 = \gamma^0 \gamma^1 \gamma^2 \gamma^3, \gamma^5 = \gamma^1 \gamma^3 \hat{C}, \gamma^6 = i \gamma^1 \gamma^3 \hat{C}, \gamma^7 = i \gamma^0,
\]

where \(\gamma^\mu\) matrices are given in (4) and operator \(\hat{C}\) is defined after the formula (10), satisfy the anti-commutation relations

\[
\gamma^A \gamma^B + \gamma^B \gamma^A = -2 \delta^{AB}, \quad A, B = 1, 7,
\]

(13)
of the Clifford algebra generators over the field of real numbers. Due to the evident fact that only six operators of (12) are linearly independent, $\gamma^4 = -i\gamma^7\gamma^1\gamma^2\gamma^3$, it is the representation of the Clifford algebra $\mathcal{C}(0,6)$ of the dimension $2^6 = 64$.

The first 16 operators are given in the Table 1 in [20], the next 16 are found from them with the help of the multiplication by imaginary unit $i = \sqrt{-1}$. Last 32 are found from first 32 with the help of multiplication by operator $\hat{C}$ of complex conjugation. Thus, if to introduce the notation "stand CD" ("stand" and CD are taken from standard Clifford–Dirac) for the set of 16 with the help of multiplication by operator $\hat{C}$ of complex conjugation. Thus, if to introduce the notation "stand CD" ("stand" and CD are taken from standard Clifford–Dirac) for the set of 16 matrices from the Table 1 in [20], then the set of 64 elements of $\mathcal{C}(0,6)$ algebra representation will be given by

$$\left\{ (\text{stand CD}) \cup i \cdot (\text{stand CD}) \cup \hat{C} \cdot (\text{stand CD}) \cup i \hat{C} \cdot (\text{stand CD}) \right\}. \quad (14)$$

As the consequences of the equalities $\gamma^4 \equiv \prod_{\mu=0}^{3} \gamma^\mu \rightarrow \prod_{\bar{\mu}=0}^{4} \gamma^{\bar{\mu}} = -I$, known from the standard Clifford–Dirac algebra $\mathcal{C}(1,3)$, and the anti-commutation relations (13), in $\mathcal{C}(0,6)$ algebra for the matrices $\gamma^A$ (12) the following extended equalities are valid: $\gamma^7 \equiv -\prod_{\Delta=1}^{6} \gamma^\Delta \rightarrow \prod_{A=1}^{7} \gamma^A = I, \quad \gamma^5 \gamma^6 = i$.

Another realization $\mathcal{C}(4,2)$ can be formed directly by the four $\gamma^\mu$ generators of $\mathcal{C}(1,3)$ taking together with the independent generators from (10) of the Pauli-Gürsey-Ibragimov algebra [25–27].

The main structure elements of such joint set are given by the matrices $(\gamma^0, \gamma^1, \gamma^2, \gamma^3, i, \hat{C}I_4)$, where $\gamma^\mu$ are $4 \times 4$ Dirac matrices in standard representation. It is easy to see that simplest set of the Clifford–Dirac algebra generators can be constructed from these elements in the following form $(i\gamma^0, i\gamma^1, i\gamma^2, i\gamma^3, \hat{C}I_4, i\hat{C}I_4)$. Therefore, the $4 \times 4$ matrix generators of the corresponding Clifford–Dirac algebra over the field of real numbers can be found by the simple redefinition

$$\hat{\gamma}^1 \equiv i\gamma^1, \quad \hat{\gamma}^2 \equiv i\gamma^3, \quad \hat{\gamma}^3 \equiv \hat{C}I_4, \quad \hat{\gamma}^4 \equiv i\hat{C}I_4, \quad \hat{\gamma}^5 \equiv i\gamma^0, \quad \hat{\gamma}^6 \equiv -\gamma^2, \quad (15)$$

of the matrices $(i\gamma^0, i\gamma^1, \gamma^2, i\gamma^3, 4 \times 4 \hat{C}I_4, \gamma^2, i\hat{C}I_4)$, where $\gamma^\mu$ are given in (4), (5).

Matrices (15) together with the matrix $\hat{\gamma}^7 \equiv \hat{\gamma}^1\hat{\gamma}^2\hat{\gamma}^3\hat{\gamma}^4\hat{\gamma}^5\hat{\gamma}^6 = \gamma^4$ satisfy the anti-commutation relations of the Clifford–Dirac algebra generators in the form

$$\hat{\gamma}^A\hat{\gamma}^B + \hat{\gamma}^B\hat{\gamma}^A = 2g^{AB}; \quad A, B = 1, 7, (g^{AB}) = (+ + + + + + + +) \quad (16)$$

As well as in (12) among the generators of (16) only the 4+2+6 matrices (15) are independent and form the basis of the algebra. Therefore, the found above algebra over the field of real numbers is defined as $\mathcal{C}(4,2)$ and the dimension of this algebra is $2^6 = 64$.

Operators (12) generate also the 28 matrices

$$s^{\bar{A}B} = \{ s^{AB} = \frac{1}{4}[\gamma^A, \gamma^B], s^{A\bar{A}} = -s^{\bar{A}A} = \frac{1}{2}\gamma^A \}, \quad \bar{A}, \bar{B} = 1, 8, \quad (17)$$

which satisfy the commutation relations of the Lie algebra SO(8)

$$[s^{\bar{A}B}, s^{\bar{C}D}] = \delta^{\bar{A}C}s^{\bar{B}D} + \delta^{\bar{C}B}s^{\bar{A}D} + \delta^{\bar{D}B}s^{\bar{A}C} + \delta^{\bar{A}D}s^{\bar{C}B}. \quad (18)$$

It is evident that here we have the algebra over the field of real numbers as well. Furthermore, it is evident that 28 elements (17) of SO(8) do not form any Clifford algebra and do not form any subalgebra of the Clifford algebra. It is independent from the Clifford algebra mathematical object. Note that here (as in (12)) the anti-Hermitian realization of the SO(8) operators is chosen, for the reasons see, e.g., [14–16] and [23, 24]. The explicit form of the 28 elements of the $\gamma$ matrix representation of the SO(8) algebra is given in the Table 3 in [20].
Consider next interesting representations of corresponding algebras. The fundamental assertions are as follows. (i) The subalgebra SO(6) of the algebra SO(8), which is determined by the operators
\[
\{s_{\vec{A}\vec{B}}\} = \{s_{\vec{A}\vec{B}} \equiv \frac{1}{4}[\gamma^{\vec{A}}, \gamma^{\vec{B}}]\}, \quad \vec{A}, \vec{B} = \vec{1}, \vec{6},
\]  

(19)
is the algebra of invariance of the Dirac equation in the Foldy–Wouthuysen representation [30, 31] (in (19) the six matrices \{\gamma^{\vec{A}}\} = \{\gamma^1, \gamma^2, \gamma^3, \gamma^4, \gamma^5, \gamma^6\} are known from (4), (12)). (ii) On the basis of SO(6) (19) the 31-dimensional Lie algebra SO(6)⊕iγ^0SO(6)⊕iγ^0 has been found [14–16], which is formed by the elements from C_\mathbb{R}(0,6) and is the maximal pure matrix algebra of invariance of the Dirac equation in the Foldy–Wouthuysen representation:
\[
(i\partial_{\mu} - \gamma^0\omega)V_{\mu}(x) = 0; \quad \omega \equiv \sqrt{-\Delta + m^2}, \quad x \in \mathbb{M}(1,3), \quad \phi \in \{S^{3,4} \subset H^{3,4} \subset S^{3,4*}\}. \tag{20}
\]

For the Dirac equation only the part of this algebra is pure matrix, other elements contain the operator \(\omega \equiv \sqrt{-\Delta + m^2}\), see, e.g., [14–16].

The start of these investigations has been given in [11, 12]. The exact interpretation of algebraic objects was given for the first time in [20].

3. Bose symmetries of the Dirac equation
The new additional operators (in comparison of 8 Pauli-Gürsey-Ibragimov generators), which the matrix representations of C_\mathbb{R}(0,6) and SO(8) algebras include, give the possibility to prove the Bose symmetry not only for the case of massless Dirac equation, but for the general case of the Dirac equation with nonzero mass as well.

The first application of the matrix representations of the algebras C_\mathbb{R}(0,6) and SO(8) is the symmetry analysis (the search of groups and algebras with respect to which the equation is invariant) of the Dirac equation with nonzero mass. It is easy to understand that the Foldy–Wouthuysen representation [30, 31] is preferable for such analysis. Indeed, in this representation one must calculate the commutation relations of possible pure matrix symmetry operators from (14) (or from Table 3 in [20]) only with two elements of the Foldy–Wouthuysen equation (20) operator: they are \(s^0\) and \(i\). After the determining of the symmetries of the Foldy–Wouthuysen equation one can find the symmetries of the Dirac equation on the basis of the inverse Foldy–Wouthuysen transformation. Note that after such transformation only the small part of symmetry operators will be pure matrix, the main part of operators will contain the nonlocal operator \(\omega \equiv \sqrt{-\Delta + m^2}\) and the functions of it.

Furthermore, in the Foldy–Wouthuysen representation two subsets \(s^{31}, s^{12}\) and \(s^{45}, s^{64}, s^{56}\) of operators \(s_{\vec{A}\vec{B}}\) from the Table 3 in [20] (i) determine two different sets of SU(2) spin 1/2 generators, (ii) commute between each other and (iii) commute with the operator of the Dirac equation in the Foldy–Wouthuysen representation. Therefore, we can use here the methods developed in [7–10] for the case \(m = 0\) (the brief consideration is given in Section 2 above). The start of such investigations for \(m \neq 0\) has been presented in [14–16], where on this bases the Bose-symmetries, Bose-solutions and Bose-conservation laws for the Dirac equation with nonzero mass have been found. Among the Bose-symmetries the important Lorentz and Poincaré algebras of invariance of the Dirac equation with nonzero mass were found.

The following assertions have been proved. Now we can determine two different realizations of the \(D(0,1) \oplus (\frac{1}{2},0)\) representation of the Lie algebra of universal covering \(\mathcal{L} = \text{SL}(2,\mathbb{C})\) of the proper ortochronous Lorentz group \(L_+ = \text{SO}(1,3) = \{\Lambda = (\Lambda_\mu^\nu)\}\) with respect to which the equation (20) is invariant:
\[
s_{\mu\nu} = \{s_{0k} = \frac{i}{2}\gamma^k\gamma^4, \quad s_{km} = \frac{1}{4}[\gamma^k, \gamma^m]\}, \quad \gamma^4 \equiv \gamma^0\gamma^1\gamma^2\gamma^3, \quad (k, m = \vec{1}, \vec{3}), \tag{21}
\]
by one and the same equation both the spin 1/2 particle-antiparticle doublet and spin s=(1,0) multiplet are described formally in [32-37]. The extended Foldy-Wouthuysen operator, which transforms such model into the canonical quantum mechanics for the particles of arbitrary mass and spin has been formulated, is not the quantum mechanical one (see, e.g., the consideration in [30, 31]). The relativistic from the relativistic canonical quantum mechanics is presented. Note that the Dirac equation here the second way of proof of the bosonic properties of the Dirac equation is given. The start of this consideration in [14-16], where on this bases the Bose Poincaré symmetries, Bose solutions and Bose conservation laws for the Dirac equation (1) have been proved.

\[ s_{II}^{\mu \nu} = \{ s_{I}^{01} = -\frac{i}{2} \gamma^{2} \tilde{C}, s_{I}^{02} = -\frac{1}{2} \gamma^{2} \tilde{C}, s_{I}^{03} = \frac{1}{2} \gamma^{0}, s_{I}^{23} = -\frac{1}{2} \gamma^{0} \gamma^{2} \tilde{C}, s_{I}^{31} = \frac{i}{2} \gamma^{0} \gamma^{2} \tilde{C}, s_{I}^{12} = -\frac{i}{2} \} , \]  

(22)

On the basis of operators (21), (22) the generators of bosonic representations are constructed as follows

\[ s_{\mu \nu}^{TS} = \{ s_{I}^{0k} = s_{I}^{0k}, s_{I}^{mn} = s_{I}^{mn} + s_{I}^{mn} \}, \  s_{\nu}^{\mu} = \{ s_{I}^{0k} = -s_{I}^{0k} + s_{I}^{0k}, s_{I}^{mn} = s_{I}^{mn} \} , \]  

(23)

where \( s_{\mu \nu}^{TS} \) and \( s_{\mu \nu}^{\nu} \) are the generators of the tensor-scalar \( D(1,0) \oplus (0,0) \) and irreducible vector \( D(1,0) \) representations of the Lie algebra SO(1,3) of the Lorentz group \( L \) respectively, with respect to which the Foldy–Wouthuysen equation (20) is invariant.

For the Dirac equation in the space of Dirac spinors \( \{ \psi \} \) (i.e. in the Pauli–Dirac representation) the form the generators of the tensor-scalar \( D(1,0) \oplus (0,0) \) and irreducible vector \( D(1,0) \) representations of the Lie algebra SO(1,3) of the Lorentz group \( L \) is similar to (23) (with (21), (22)) but the gamma operators are in this case too much complicated and are given by

\[ \tilde{\gamma} = \gamma - \frac{\gamma^{1}}{\omega} \cdot \nabla + m \]  

(24)

\[ \tilde{\gamma}^{5} = \gamma^{1} \tilde{\gamma} \tilde{C}, \  \tilde{\gamma}^{0} = \gamma^{1} \tilde{\gamma} \tilde{C}, \  \tilde{\gamma}^{7} = \gamma^{0} \tilde{\gamma}, \  \tilde{\gamma}^{3} = \gamma^{0} \tilde{\gamma} \]  

(25)

where \( \tilde{C} = (1 + 2\gamma^{0} \gamma^{2} + \gamma^{1}) \tilde{C} \) and \( \omega \equiv \sqrt{-\Delta + m^{2}} \).

In [15, 16] we used also the bosonic representation of (23), in which the Casimir operators are diagonal and the proof of Bose properties is most convenient. One can easy find the continuation of this consideration in [14–16], where on this bases the Bose Poincaré symmetries, Bose solutions and Bose conservation laws for the Dirac equation (1) have been proved.

4. Derivation from the first principles of relativistic canonical quantum mechanics

Here the second way of proof of the bosonic properties of the Dirac equation is given. The start from the relativistic canonical quantum mechanics is presented. Note that the Dirac equation is not the quantum mechanical one (see, e.g., the consideration in [30, 31]). The relativistic canonical quantum mechanics for the particles of arbitrary mass and spin has been formulated in [32–37]. The extended Foldy–Wouthuysen operator, which transforms such model into the corresponding manifestly covariant field theory, has been suggested. In such quantum mechanics both the spin 1/2 particle-antiparticle doublet and spin s=(1,0) multiplet are described formally by one and the same equation

\[ (i\partial_{0} - \omega) f(x) = 0, \  f = \text{column} | f^{1}, f^{2}, f^{3}, f^{4} | , \]  

(26)

for the 4-component wave function and one and the same extended Foldy–Wouthuysen operator

\[ V = \frac{i\gamma^{4} \partial_{k} + \omega + m}{\sqrt{2\omega(\omega + m)}} \left[ \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & C \end{array} \right], \  V^{-1} = \frac{-i\gamma^{4} \partial_{k} + \omega + m}{\sqrt{2\omega(\omega + m)}} \left[ \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & C \end{array} \right], \]  

(27)

transforms (26) into the Dirac equation (in the case of anti-Hermitian operators [14–16, 23, 24]). Here \( C f^{3} = f^{3*} \) and the s=0 contribution is taken as the \( f^{4} \) component of the column.
The equation (26) should be considered in the space of states $S^{3,4} \subset H^{3,4} \subset S^{3,4*}$. The general solution of the Schrödinger–Foldy equation (26) is given by

$$f(x) = \frac{1}{(2\pi)^{3/2}} \int d^3k e^{-ikx} \left[ c^1(k) d_1 + c^2(k) d_2 + c^3(k) d_3 + c^4(k) d_4 \right],$$

(28)

where $\{d_n\}$ are the orts of the 4-dimensional Cartesian basis. The solution (28) can be used for both for the spin $1/2$ particle-antiparticle doublet and for spin $s=(1,0)$ Bose multiplet.

The operator (27) consists of two operators [32–37]. The first one, which contains the complex conjugation $C$, transforms the quantum mechanical equation (26) into the Foldy–Wouthuysen equation (20) and the solution (28) transforms into the solution of equation (20), which contains the positive and negative frequency parts. The second one is the Foldy–Wouthuysen transformation [30, 31]. For this point the description of spin $1/2$ particle-antiparticle doublet and spin $s=(1,0)$ multiplet is similar.

Below the specification for spin $s=(1,0)$ Bose multiplet (not for the spin $1/2$ Fermi particle-antiparticle doublet) is demonstrated. The generators of the corresponding SU(2)-spin that satisfy the commutation relations $[s^j, s^k] = i\epsilon^{jkn}s^n$ of the SU(2) algebra are given by

$$s^1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad s^2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad s^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$  

(29)

It is easy to verify that the commutation relations $[s^j, s^k] = i\epsilon^{jkn}s^n$ are valid. The Casimir operator for this reducible representation of the SU(2)-algebra is given by $\tilde{\mathbf{s}^2} = 2 \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix} = 1(1+1) \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$, where $I_3$ is a $3 \times 3$ unit matrix. The stationary complete set of operators is given by $\tilde{\mathbf{p}^2}$, $s^3 = s_z$.

The equations on the eigenvalues for these operators have the form

$$\tilde{\mathbf{p}^2} e^{-ikx} d_\alpha = \frac{\mathbf{k}}{\mathbf{p}} e^{-ikx} d_\alpha; \quad s^3 d_1 = d_1, \quad s^3 d_2 = 0, \quad s^3 d_3 = -d_3, \quad s^3 d_4 = 0.$$  

(30)

Interpretation of the amplitudes $c^n(\mathbf{k})$ in the solution (28) follows from equations (30). The functions $c^1(\mathbf{k})$, $c^2(\mathbf{k})$, $c^3(\mathbf{k})$, are the quantum-mechanical momentum-spin amplitudes of the boson with the spin $s=1$ and the eigenvalues of the spin projection $1$, $0$, $-1$, respectively. The function $c^4(\mathbf{k})$ is the amplitude of the spinless boson.

The Schrödinger–Foldy equation (26) and the set $\{f\}$ of its solutions (28) are invariant with respect to the reducible unitary bosonic representation of the Poincaré group $\mathcal{P}$. The corresponding $4 \times 4$ matrix-differential generators of the Lie algebra of the Poincaré group $\mathcal{P}$ are given by

$$\hat{\mathbf{p}_0} = \omega \equiv \sqrt{-\Delta + m^2}, \quad \hat{\mathbf{p}_t} = i\partial_t, \quad \hat{\mathbf{j}_n} = x_t \hat{\mathbf{p}_n} - x_n \hat{\mathbf{p}_t} + s_n \equiv \hat{\mathbf{m}_n} + s \hat{\mathbf{e}_n},$$  

(31)

$$\hat{\mathbf{j}_0} = \hat{\mathbf{j}_t} = \hat{\mathbf{p}_t} - \frac{1}{2} \{x_t, \hat{\omega}\} - \left( \frac{s \hat{\mathbf{p}_n}}{\hat{\omega} + m} \equiv \hat{\mathbf{s}_n} \right),$$  

(32)

where the spin $s=(1,0)$ SU(2) generators $\hat{\mathbf{s}} = (s \hat{\mathbf{e}_n})$ are given in (29). Here we consider Lie algebra, which is determined by non-local non-Lie type of generators.

The validity of this assertion is verified by the following three steps. (i) The calculation that the $\mathcal{P}$-generators (31), (32) commute with the operator $i\partial_0 - \hat{\omega}$ of the Schrödinger–Foldy equation
(26). (ii) The verification that the $P$-generators (31), (32) satisfy the commutation relations of the Lie algebra of the Poincaré group $P$. (iii) The proof that generators (31), (32) realize the spin $s=(1,0)$ representation of this group. Therefore, the Bargman–Wigner classification on the basis of the corresponding Casimir operators calculation should be given. These three steps can be made by direct and non-cumbersome calculations. The Casimir operators of this reducible bosonic representation of the group $P$ have the form

$$p^2 = \hat{p}^\mu \hat{p}_\mu = m^2 I_4, \quad W = \hat{w}^\mu \hat{w}_\mu = m^2 \hat{s}^2 = m^2 \begin{pmatrix} 1 & (1+1) I_3 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

(33)

where $I_3$ and $I_4$ are the $3 \times 3$ and $4 \times 4$ unit matrices, respectively.

In order to find the images of the operators (29) in the Dirac model it is preferable to rewrite them at first in the unitary equivalent form as follows

$$\hat{s}^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & i & 0 \\ -1 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{s}^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 1 & 0 \\ -i & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{s}^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

(34)

and, furthermore, to rewrite generators (34) in anti-Hermitian form (for the reasons of using the anti-Hermitian operators see [14–16]). Only after that the application of transition operator (27) is possible.

The transition from above given quantum mechanical model to the Dirac equation in the space of Dirac spinors $\{\psi\}$ (i.e. to the Pauli–Dirac representation) is fulfilled by the operator (27). All operators must be taken in anti-Hermitian forms. In the field-theoretical Pauli–Dirac representation the explicit forms of operators (31), (32) are given in [11, 14].

The beginning of specification for the spin 1/2 Fermi particle-antiparticle doublet is in the choice of SU(2) spin operators in the form

$$s^1 = \frac{1}{2} \begin{pmatrix} \sigma^1 & 0 \\ 0 & -\sigma^1 \end{pmatrix}, \quad s^2 = \frac{1}{2} \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}, \quad s^3 = \frac{1}{2} \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix},$$

(35)

where $(\sigma^\ell)$ are given in (5). Further, generators (35) take the places in (31), (32) and in all other points of above given scheme. The finish is in well-known Fermi properties of the Dirac equation (1).

Note that second variant of given in this section derivation of the Bose properties of the Dirac equation can be realized on the basis of the start from the quantum mechanical 8-component spin $s=(1,0,1,0)$ particle-antiparticle multiplet. Such derivation is presented in [32]. In this variant there is no necessity in such steps as transition from (29) to (34). However, in the finish the reduction from 8-component Dirac equation and corresponding theory to the ordinary 4-component Dirac model must be fulfilled.

5. Conclusions

Being the long time hidden, the bosonic properties of the Dirac equation were found recently in [14] and were considered in more details in [15, 16]. Nevertheless, such properties look somewhat unexpected. The author has a hope that after presentation here two different and independent ways of the proof of such properties the veracity of these investigations will be evident. The Fermi–Bose duality of the Dirac equation is proved here as well.
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