Dynamical mapping method in nonrelativistic models of quantum field theory

A.N.Vall, S.E.Korenblit, V.M.Leviant, A.B.Tanaev.
Irkitsk State University, 664003, Gagarin bl-d, 20, Irkutsk, Russia.

Abstract

The solutions of Heisenberg equations and two-particles eigenvalue problems for nonrelativistic models of current-current fermion interaction and $N, \Theta$ model are obtained in the frameworks of dynamical mapping method. The equivalence of different types of dynamical mapping is shown. The connection between renormalization procedure and theory of selfadjoint extensions is elucidated.

1. General consideration

The main problem of QFT follows from the fact that any solutions of Heisenberg equations (HE) are the operator distributions which products, always appearing in that equations, are ill-defined.

\[
(i\partial_t - \mathcal{E}(\mathbf{P})) \Psi_\alpha(\mathbf{x}, t) = [\Psi_\alpha(\mathbf{x}, t), H_I\{\Psi\}] = \Psi(\mathbf{x}, t), \quad \text{for} \quad H\{\Psi\} = H_0\{\Psi\} + H_I\{\Psi\}
\]

So, the correct definition of field equations (and Hamiltonian itself) implies some knowledge about qualitative properties of its solutions which in their turn depend on the form of these equations by a very singular manner. The usual way to go out from this closed circle is connected with perturbation theory. It is based on the assumption that product of Heisenberg fields (HF) may be defined as well as for the free ones and solution of HE may be obtained by perturbation in the Fock space of renormalized free fields. However, it is impossible on such a way to work with nonrenormalizable theory and to understand the origin of the bound states. We consider another possibility which is based on the idea of dynamical mapping and reduce the product of HF to the normal ordering for the product of the physical fields. It is originated from the works of R.Haag [1], O.Greenberg [2], H.Umezawa [3], and L.D.Faddeev [4] M.I.Shirokov [5] (see also [6]).

In this approach the problem of making a sense for formal expression of HF:

\[
\Psi(\mathbf{x}, t) = e^{iH(t-t_0)} \Psi(\mathbf{x}, t_0) e^{-iH(t-t_0)} \rightarrow \mathcal{F}^t[\Psi(\mathbf{x}, t_0)],
\]

for Hamiltonian given as a functional $H = H[\Psi(\mathbf{x}, t)]$, is divided on two parts. The first one is the construction of the following operator realization of the initial fields $\Psi(\mathbf{x}, t_0) = \Psi[\psi]$ via physical fields $\psi(\mathbf{x}, t) \equiv \psi\{A_\alpha(\mathbf{k})\}$, which, on the one hand, should be consistent with CCR (CAR) $(\alpha = 1, 2)$

\[
\{\Psi_\alpha(\mathbf{x}, t), \Psi_\beta(\mathbf{y}, t)\} = 0 = \{\psi_\alpha(\mathbf{x}, t), \psi_\beta(\mathbf{y}, t)\},
\]

\[
\{\Psi_\alpha(\mathbf{x}, t), \Psi_\beta^\dagger(\mathbf{y}, t)\} = \delta_3(\mathbf{x} - \mathbf{y}) \delta_{\alpha\beta} = \{\psi_\alpha(\mathbf{x}, t), \psi_\beta^\dagger(\mathbf{y}, t)\},
\]

\[
\{A_\alpha(\mathbf{k}), A_\beta(\mathbf{q})\} = 0; \quad \{A_\alpha(\mathbf{k}), A_\beta^\dagger(\mathbf{q})\} = \delta_3(\mathbf{k} - \mathbf{q}) \delta_{\alpha\beta},
\]

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E-mail VALL@physdep.irkutsk.su, KORENB@math.isu.rannet.ru
and on the other hand leads to unique stable vacuum \( |0\rangle \) and one-particle state \( |1\vec{k},\alpha\rangle \) with
definite spectrum \( E(\vec{k}) \):

\[
H | 0 \rangle = V w_0 | 0 \rangle; \quad A_\alpha(\vec{k}) | 0 \rangle = 0; \quad V \text{ is space volume}; \quad (4)
\]

\[
[H, A_\alpha^\dagger(\vec{k})] | 0 \rangle = E(k) A_\alpha^\dagger(\vec{k}) | 0 \rangle = E(k) | 1\vec{k},\alpha\rangle = (H - V w_0) | 1\vec{k},\alpha\rangle. \quad (5)
\]

Moreover, let us suppose that for such operator realization the reduced (time-independent)
Hamiltonian for the definite moment \( t = t_0 \) does not contain "fluctuation" terms \( \mathbb{I} \) up to
fourth order and looks like :

\[
H \equiv H\{A\} = V w_0 + \hat{H}\{A\} =: H\{\Psi[\lambda]\} := V w_0 + \hat{H}_0\{A\} + \hat{H}_r\{A\};
\]

\[
\hat{H}_0\{A\} \overset{\text{def}}{=} \int d^3k E(k) A_\alpha^\dagger(\vec{k}) A_\alpha(\vec{k}); \quad [\hat{H}_0, A_\alpha^\dagger(\vec{k})] \equiv E(k) A_\alpha^\dagger(\vec{k});
\]

\[
\hat{H}_r\{A\} = \int d^3q \int d^3p \int d^3\kappa \int d^3l \delta_3(\vec{q} + \vec{p} - \vec{\kappa} - \vec{l}) K_{22}^{q+p} \left( \frac{\vec{q} - \vec{p}}{2}, \frac{\vec{\kappa} - \vec{l}}{2} \right).
\]

\[
\cdot A_\alpha^\dagger(\vec{r}) A_\beta^\dagger(\vec{l}) A_\beta(\vec{p}) A_\alpha(\vec{q}) + \sum (A_\alpha^\dagger)^m (A)^n; \quad m, n \geq 3.
\]

\[
K_{22}^{P}(\vec{r},\vec{s}) = K_{22}^{P}(-\vec{r},-\vec{s}); \quad K_{22}^{*}(\vec{s},\vec{r}) = K_{22}^{*}(\vec{r},\vec{s});
\]

The general consideration of the existence of the operator realizations of such kind is a subject
of our another work. They always exist for the Lee models considered below.

This work will be concentrated on the second part of the problem which is the construction
of the corresponding dynamical mapping (Haag expansion) \( \mathcal{F}^t[\Psi(\vec{x},t_0)] \) as a series of normal
ordered products of physical fields \( \Psi(\vec{x},t_0) \) or \( \psi(\vec{x}) \), \( A(\vec{k}) \):

\[
e^{iH(t-t_0)} A_\alpha(\vec{k}) e^{-iH(t-t_0)} \equiv e^{-iE(k)(t-t_0)} a_\alpha(\vec{k},t) = e^{-iE(k)(t-t_0)} A_{t_0}^\dagger [A_\alpha(\vec{k})], \quad (7)
\]

\[
a_\alpha(\vec{k},t) = A_\alpha(\vec{k}) + \int d^3l \int d^3p A_\beta^\dagger(\vec{l} + \vec{p}) A_\beta(\vec{k} + \vec{p}) A_\alpha(\vec{l}) F_A^{(1)} (t; \vec{p} | \vec{l}, \vec{k}) + \ldots \quad (8)
\]

\[
\Rightarrow \quad (\text{for } m = n) \Rightarrow A_{t_0}^\dagger [A_\alpha(\vec{k})] = A_\alpha(\vec{k}) + \sum_{n=1}^{\infty} \int d^3l \prod_{j=1}^{n} \left\{ \int d^3q_j \int d^3p_j \right\} \cdot \prod_{j=1}^{n} \left\{ A_\beta^\dagger(\vec{q}_j) \right\} \prod_{j=n+1} \left\{ A_\beta(\vec{p}_j) \right\} A_\alpha(\vec{l}) Y_A^{(n)} (t; \vec{k}; \{ \vec{q}_j \}_{1}^{n}; \{ \vec{p}_j \}_{n+1}^{1}, \vec{l}), \quad (9)
\]

and the usage of these coefficient functions for eigenvalue problem. The condition \( m = n \) in
the last eq.\( \mathbb{I} \) means the absence of "fluctuation terms" (with \( m \neq n \)) in reduced Hamiltonian
\( \mathbb{I} \), which commutes with particle number operator for that case.

From the expressions \( \mathbb{I} \),\( \mathbb{I} \) it’s follows that vacuum and one-particle states remain stable
for all \( t \):

\[
a_\alpha(\vec{k},t) | 0 \rangle \longrightarrow A_\alpha(\vec{k}) | 0 \rangle \equiv 0; \quad | 1\vec{k},\alpha \rangle = a_\alpha^\dagger(\vec{k},t) | 0 \rangle \longrightarrow A_\alpha^\dagger(\vec{k}) | 0 \rangle;
\]

\[
[H, a_\alpha^\dagger(\vec{k},t)] | 0 \rangle = E(k) A_\alpha^\dagger(\vec{k}) | 0 \rangle, \quad (10)
\]
what allows one to define the normal ordering directly for HF and the normal ordered Hamiltonian \[ H \] now correctly defines the nonlinear terms in reduced HE \[ \{ H \} \]:

\[
(i\partial_t - E(P)) \Psi_\alpha(x, t) = \left[ \Psi_\alpha(x, t), \hat{H}_I(\Psi) \right].
\] (11)

\[
i\partial_t a_\alpha(\vec{k}, t) = \left[ a_\alpha(\vec{k}, t), \hat{H}_I(\{ \alpha \}) \right] \Rightarrow \int d^3l Q_{(\alpha)}(\vec{l}, t) a_\alpha(\vec{l}, t);
\] (12)

\[
Q_{(\alpha)}(\vec{k}, \vec{l}; t) = \int d^3q \int d^3\rho \ a_\beta^\dagger(\vec{q}, t) a_\beta(\vec{\rho}, t) \ e^{it[E(k) + E(q) - E(p) - E(\vec{l})]}.
\]

\[
\delta_3(\vec{k} + \vec{q} - \vec{p} - \vec{l}) \frac{2}{i} K_{22}^{\vec{l}+\vec{p}} \left( \frac{\vec{l} - \vec{p}}{2}, \frac{\vec{k} - \vec{q}}{2} \right) = \int d^3\rho \ a_\beta^\dagger(\vec{l} + \vec{\rho}, t) a_\beta(\vec{k} + \vec{\rho}, t) e^{it[E(\vec{l} + \vec{\rho}) + E(\vec{k} + \vec{\rho}) - E(\vec{l})]}.
\]

\[
\frac{2}{i} K_{22}^{\vec{l}+\vec{p}} \left( \frac{\vec{l} - \vec{k} - \vec{p}}{2}, \frac{\vec{k} - \vec{q}}{2} \right).
\] (13)

The case \( \{ H \} \) means, moreover, the stability and absence of any polarization not only for vacuum and one-particle states but also for arbitrary \( N \)-particle ones. So, for arbitrary \( N \) one can reduce the product:

\[
\langle 0 \mid \prod_{i=1}^{N} a_\alpha(\vec{k}_i, t) \rangle \rightarrow_{t_0} \langle 0 \mid \sum_{i=1}^{N} d^3s_i A_\alpha(s_i). \] (14)

However, if fluctuation terms appear with min \( (m, n) \geq N_0 \), then such reduction \( \{ 14 \} \) is possible only for \( N < N_0 \).

There exist two essentially different choices for the initial moment \( t_0 \) leading to corresponding different choices of physical fields:

- \( t_0 \rightarrow -\infty \), (Greenberg, Umezawa)
  - nonoperator initial condition
  - \( w \lim_{t \rightarrow -\infty} \langle f_{in} \mid \Psi(\vec{x}, t) - \psi_{in}(\vec{x}, t) \mid i_{in} \rangle = 0 \)
  - \( \{ \psi_{in}[A_{in}] \} \rightarrow \) incomplete Fock space
  - new fields \( V_{in} \) for every bound state
  - \( \hat{H} \rightarrow \hat{H}_0 \{ A_{in} \} + \hat{H}_0 \{ V_{in} \} + \ldots \)

- \( t_0 = 0 \), (Faddeev, Shirokov)
  - operator initial condition
  - \( s \lim_{t \rightarrow 0} \Psi(\vec{x}, t) = \Psi [\psi(\vec{x}, 0)] \)
  - \( \{ \psi[A] \} \rightarrow \) complete Fock space
  - no any new fields for bound states
  - \( \hat{H} = \hat{H}_0 \{ A \} + \hat{H}_I \{ A \} \)

The second choice \( t_0 = 0 \) is used here. It seems more economical, and both bound and scattering eigenstates look equal in rights for that choice.

One can check by direct substitution that solutions of both scattering and bound state two-particles eigenvalue problems

\[
\hat{H} \mid R_{\alpha\beta}^\pm(P, q) \rangle = E_2(P, q) \mid R_{\alpha\beta}^\pm(P, q) \rangle; \quad E_2(P, q) \equiv E(P/2 + q) + E(P/2 - q);
\]

\[
\hat{H} \mid B_{\alpha\beta}^P \rangle = M_2(P) \mid B_{\alpha\beta}^P \rangle;
\] (15)

exist in the Fock eigenspace of kinetic part of reduced Hamiltonian \( \{ \hat{H} \} \)

\[
\hat{H}_0 \{ A \} A_{\alpha_1}^\dagger(\vec{k}_1) \cdots A_{\alpha_n}^\dagger(\vec{k}_n) \mid 0 \rangle = \left( \sum_{j=1}^{n} E(k_j) \right) A_{\alpha_1}^\dagger(\vec{k}_1) \cdots A_{\alpha_n}^\dagger(\vec{k}_n) \mid 0 \rangle,
\] (16)
in the following form:

\[ | R_{\alpha\beta}^{1}(\mathcal{P}, \bar{q}) \rangle = \int d^3 \kappa \ | R_{\alpha\beta}^{0}(\mathcal{P}, \bar{r}) \rangle \Phi^{1}_{\hat{P}}(\kappa); \quad | B_{\alpha\beta}^{\mathcal{P}}(\kappa) \rangle = \int d^3 \kappa \ | R_{\alpha\beta}^{0}(\mathcal{P}, \bar{r}) \rangle B_{\alpha\beta}^{\mathcal{P}}(\kappa); \]

where corresponding wavefunctions satisfy to the usual Lippman-Schwinger equations:

\[ \Phi^{1}_{\hat{P}}(\kappa) = \delta_{3}(\kappa - \bar{q}) + \frac{1}{E_{2}(\mathcal{P}, q) - E_{2}(\mathcal{P}, \kappa) \pm i\delta} \cdot \int d^3 r \ \Phi^{1}_{\hat{P}}(\vec{r}) \ 2K_{22}^{\mathcal{P}}(\vec{r}, \kappa), \]

\[ B_{\alpha\beta}^{\mathcal{P}}(\kappa) = \frac{1}{M(\mathcal{P}) - E_{2}(\mathcal{P}, \kappa)} \cdot \int d^3 r \ B_{\alpha\beta}^{\mathcal{P}}(\vec{r}) \ 2K_{22}^{\mathcal{P}}(\vec{r}, \kappa). \] (18)

In its turn, at \( m, n \geq 3 \) for the first coefficient function of (8)

\[ Y_{A}^{(1)}(t; \bar{k}; \bar{q} | \bar{p}; \bar{l}) \equiv \delta_{3}(\bar{k} + \bar{q} - \bar{p} - \bar{l}) \ F_{A}^{(1)}(t; \bar{p} - \bar{k} | \bar{l}, \bar{k}) \] (19)

from reduced HE (12) follows an integral equation with the same kernel:

\[ F_{A}^{(1)}(t; -\bar{k} - \bar{q} | \frac{\mathcal{P}}{2} + \bar{k}, \frac{\mathcal{P}}{2} + \bar{q}) \equiv F_{A}^{(1)}(t; \bar{k} + \bar{q} | \frac{\mathcal{P}}{2} - \bar{k}, \frac{\mathcal{P}}{2} - \bar{q}) = \]

\[ = \int_{0}^{t} d\eta e^{i\eta[E_{2}(\mathcal{P}, q) - E_{2}(\mathcal{P}, \kappa)]} \left[ 2 \frac{1}{i} K_{22}^{\mathcal{P}}(\bar{k}, \bar{q}) + \int d^3 r e^{i\eta[E_{2}(\mathcal{P}, \kappa) - E_{2}(\mathcal{P}, r)]} \right] \]

\[ \cdot \left[ \frac{1}{2} K_{22}^{\mathcal{P}}(\bar{r}, \bar{q}) \ F_{A}^{(1)}(\eta; \bar{k} + \bar{r} | \frac{\mathcal{P}}{2} - \bar{k}, \frac{\mathcal{P}}{2} - \bar{r}) \right]. \] (20)

It contains all information about two-particle sector, directly determining the scattering wave function for \( E_{2}(\mathcal{P}, q) = E_{2}(\mathcal{P}, \kappa) \pm i\delta \):

\[ \Phi^{\pm}_{\hat{P}}(\kappa) = \delta_{3}(\kappa - \bar{q}) + F_{A}^{(1)*}(t = \mp\infty; -\bar{k} - \bar{q} | \frac{\mathcal{P}}{2} + \bar{k}, \frac{\mathcal{P}}{2} + \bar{q}), \] (21)

where the simply derived expression for scattering state was used:

\[ | R_{\alpha\beta}^{0}(\mathcal{P}, \bar{q}) \rangle = | R_{\alpha\beta}^{0}(\mathcal{P}, \bar{q}) \rangle + \int_{0}^{\mp\infty} dt \ e^{-i(t(E_{2}(\mathcal{P}, q) - E_{2}(\mathcal{P}, \kappa) \pm i\delta))} a_{\alpha}(\frac{\mathcal{P}}{2} + \bar{k}, t) a_{\beta}^{\dagger}(\frac{\mathcal{P}}{2} - \bar{k}, t) | 0), \]

which follows directly from (7) and definition of scattering state:

\[ | R_{\alpha\beta}^{\mp}(\mathcal{P}, \bar{q}) \rangle = | R_{\alpha\beta}^{0}(\mathcal{P}, \bar{q}) \rangle + \frac{1}{E_{2}(\mathcal{P}, q) \pm i\delta - \hat{H}} \hat{H} | R_{\alpha\beta}^{0}(\mathcal{P}, \bar{q}) \rangle. \] (23)

It is a simple matter to see that all integral equations above are exactly solvable for degenerate kernels: \( K_{22}^{\mathcal{P}}(\vec{r}, \vec{s}) = \sum_{n} V_{n}(\vec{r}) U_{n}(\vec{s}). \)
The dynamical mapping formulae (7), (8) give two forms of instantaneous Bethe-Salpeter matrix element \( \langle 0 \mid a_{\alpha} (\vec{P}/2 + \vec{k}, t) a_{\beta} (\vec{P}/2 - \vec{k}, t) \mid 2, \mathcal{P} \rangle \), leading to the following identities when \( |2, \mathcal{P}\rangle \) stands for \( |R_{\alpha \beta}^\pm (\vec{P}, \vec{q})\rangle \) or \( |\mathcal{B}_{\alpha \beta}\rangle \):

\[
\Phi_{\mathcal{P} q}^\pm (\vec{k}) \left[ e^{it(\mathcal{E}_2 (\mathcal{P}, \mathcal{K}) - \mathcal{E}_2 (\mathcal{P}, \mathcal{K}) + \delta)} - 1 \right] = \int d^3 r \left\{ \Phi_{\mathcal{P} q}^\pm (\vec{r}) \right\},
\]

\[
B_{\mathcal{P}} (\vec{k}) \left[ e^{it(\mathcal{E}_2 (\mathcal{P}, \mathcal{K}) - \mathcal{M}_2 (\mathcal{P}))} - 1 \right] \cdot F_A^{(1)} \left( t; \vec{k} + \vec{r} \mid \frac{\mathcal{P}}{2} - \vec{k}, \frac{\mathcal{P}}{2} - \vec{r} \right),
\]

which have a sense of off-shell extension of unitarity relation.

2. Four-fermion models

As a first example let us consider the contact four-fermion model. Defining

\[
x = (\vec{x}, t); \quad t = t_0; \quad \mathbf{P}_x = -i \vec{\nabla}_x; \quad \epsilon_{\alpha \beta} = -\epsilon_{\beta \alpha}; \quad \chi_{(\Psi)} (x) = \epsilon_{\alpha \beta} \Psi_{\alpha} (x) \Psi_{\beta} (x);
\]

\[
S_{(\Psi)} (x) = \Psi_{\alpha}^\dagger (x) \Psi_{\alpha} (x), \quad \vec{J}_{(\Psi)} (x) = \Psi_{\alpha}^\dagger (x) \mathbf{P}_x \Psi_{\alpha} (x);
\]

\[
\{ \Psi_{\alpha} (x), \Psi_{\beta}^\dagger (y) \} \bigg|_{x = y_0} = 0; \quad \text{with the convention:}
\]

\[
\{ \Psi_{\alpha} (x), \Psi_{\beta}^\dagger (y) \} \bigg|_{x = y_0} = \delta_{\alpha \beta} \delta_3 (\vec{x} - \vec{y}) \Rightarrow \bigg|_{\vec{x} = \vec{y}} \frac{1}{V^3},
\]

let us consider the local 4-fermion interactions with the following densities:

\[
\mathcal{H}_1 (x) = \Psi_{\alpha}^\dagger (x) \mathcal{E} (\mathbf{P}) \Psi_{\alpha} (x) - \frac{\lambda}{8} \chi_{(\Psi)} (x) \chi_{(\Psi)} (x);
\]

\[
\mathcal{H}_2 (x) = \Psi_{\alpha}^\dagger (x) \mathcal{E} (\mathbf{P}) \Psi_{\alpha} (x) - \frac{\lambda}{4} S_{(\Psi)}^2 (x) + \frac{\mu}{4} \vec{J}_{(\Psi)}^2 (x) \Rightarrow \mathcal{H}_2 (x) := w_0 + \Psi_{\alpha}^\dagger (x) \mathcal{E} (\mathbf{P}) \Psi_{\alpha} (x) - \frac{\lambda}{8} \chi_{(\Psi)} (x) \chi_{(\Psi)} (x) + \frac{\mu}{4} \Psi_{\alpha}^\dagger (x) \left( \vec{J}_{(\Psi)} (y) \cdot \mathbf{P}_x \right) \Psi_{\alpha} (x) \bigg|_{x = y},
\]

which give the HE (11), (12) for HF:

\[
\Psi_{\alpha} (x) = \int \frac{d^3 k}{(2 \pi)^{3/2}} e^{i (\vec{k} \cdot \vec{x} - i \mathcal{E} (\mathbf{k}) t)} a_{\alpha} (\vec{k}, t),
\]

\[
(i \partial_t - \mathcal{E} (\mathbf{P})) \Psi_{\alpha} (x) = \hat{\mathcal{V}}_{(\Psi)} (t; \vec{x}, \mathbf{P}) \Psi_{\alpha} (x) \equiv \mathcal{J}_{\alpha} (x),
\]

\[
\hat{\mathcal{V}}_{(\Psi)} (t; \vec{x}, \mathbf{P}) = -\frac{\lambda}{2} S_{(\Psi)}^2 (x) + \mu \left( \mathcal{J}_{(\Psi)} (x) \cdot \mathbf{P}_x \right) - \frac{i}{2} \left( \vec{\nabla}_x \cdot \mathcal{J}_{(\Psi)} (x) \right);
\]

\[
\frac{2}{i K_{22}^P} \left( \vec{r} - \frac{\mathcal{P}}{2}, \vec{k} - \frac{\mathcal{P}}{2} \right),
\]
and for the following simple operator realizations via physical fields

\[ \Psi_\alpha(\vec{x}, 0) = \Psi_\alpha \left[ \psi_\alpha \{ A(\vec{k}) \} \right] \Rightarrow \begin{cases} -\epsilon_{\alpha\beta} \psi_\beta^i(\vec{x}, 0); & (\text{+}), \quad E(k) \Rightarrow E^+(k) \\ \psi_\alpha(\vec{x}, 0); & (-), \quad E(k) \Rightarrow E^-(k) \end{cases} \]


\[ \psi_\alpha(x) = \int \frac{d^3 k}{(2\pi)^3/2} e^{i(\vec{k} \cdot \vec{x}) - iE(k)t} A_\alpha(\vec{k}) = e^{-iE(P)} \psi_\alpha(\vec{x}, 0), \quad (30) \]

lead to above reduced Hamiltonian (11) \( H \{ A \} =: H_2 \{ A \} : \) (note, that \( H_1(x) \) looks like normal form of \( H_2(x) \) for the case \( \mu = 0 \)), with the following degenerate kernel and parameters:

\[ \frac{2}{i} K_{22}^P(\vec{r}, \vec{s}) \Rightarrow \frac{i}{2(2\pi)^3} \left\{ L + \mu \left( \vec{r} + \vec{s} \right)^2 \right\}; \quad L \equiv \lambda - \mu P^2; \quad (31) \]

\[ E(k) \Rightarrow E^\pm(k) = \frac{\mu}{4V^*} \left( k^2 + \langle k^2 \rangle \right) + \frac{\lambda}{4V^*} \mp \left( \mathcal{E}(k) - \frac{\lambda}{2V^*} \right); \quad (32) \]

\[ w_0 \Rightarrow w_0^\pm = \frac{1 \pm 1}{V^*} \left( \langle \mathcal{E}(k) \rangle - \frac{\lambda}{2V^*} \right); \quad \langle \mathcal{E}(k) \rangle \equiv V^* \int \frac{d^3 k}{(2\pi)^3} \mathcal{E}(k). \]

Here \( \mathcal{E}(k) \) is arbitrary ”bare” one-particle spectrum and \( V^* \) has a sense of the volume of excitation.

Now the solution of (20) for the first coefficient function reads:

\[ F^{(1)} \left( t; \vec{k} + \vec{q} \mid \frac{P}{2} - \frac{\vec{k}}{2} - \frac{\vec{q}}{2} \right) = \int_0^t dt e^{i\eta \epsilon(t)} \left( \frac{E(t)}{2\pi i} \right)^{i\omega + \Delta \cdot \epsilon(t)} \int_{-i\omega + \Delta \cdot \epsilon(t)} \frac{d\sigma e^{\sigma}}{\sigma + iE(\sigma, \kappa)}. \]

\[ \left\{ \frac{D_{\alpha 1}^P(\sigma) + q^2 D_{\alpha 2}^P(\sigma)}{D_0^P(\sigma)} \right\} + 2\mu q^j \left[ \frac{\epsilon_\mu(\sigma)}{1 - 2\mu J_0^P(\sigma)} + \frac{\epsilon_\mu(\sigma)}{1 - 2\mu J_D^P(\sigma)} \right] \kappa^i \right\}, \quad (33) \]

where \( \epsilon(\eta) = \epsilon(t) \equiv \text{sgn}(t), \Delta > 0, \) and

\[ \left\{ \frac{J_{\alpha j}^P(\sigma)}{J_{\alpha j}^P(\sigma)} \right\} = \frac{1}{2} \int \frac{d^3 r}{(2\pi)^3} \cdot \frac{1}{E_2(\sigma, r) - i\sigma} \left( \frac{(r^2)^n}{(\vec{r}, \vec{P})^2} \right); \quad \left( \begin{array}{c} \eta \\ \langle \vec{r}, \vec{P} \rangle \end{array} \right); \quad (34) \]

\[ J_{\alpha j}^P(\sigma) = \delta_{\alpha j} J_0^P(\sigma) + \frac{\epsilon_{\alpha j}^P}{P^2} J_0^P(\sigma), \quad J_0^P(\sigma) = \frac{1}{2} \left[ J_0^P(\sigma) - J_0^P(\sigma) \right]; \]

\[ D_0^P(\sigma) = \left\{ \left[ \mu J_0^P(\sigma) - 1 \right]^2 - \mu^2 J_0^P(\sigma) J_0^P(\sigma) - L J_0^P(\sigma). \right\} \]

\[ D_{\alpha j}^P(\sigma) = \left\{ L + \mu \kappa^2 + \mu^2 \left[ J_0^P(\sigma) - \kappa^2 J_0^P(\sigma) \right] \right\}. \quad (35) \]

Setting \( J_{\{\alpha j\}}(q) \equiv J_{\{\alpha j\}}(\pm \delta - iE_2(\sigma, q)), \quad D_{\{\alpha j\}}(q) = D_{\{\alpha j\}}(\pm \delta - iE_2(\sigma, q)), \) from eq. (18) or from (21) for the scattering eigenfunctions with fixed spin \( J=0,1, \) defined in symmetrical basis \( (\sigma_j, j = 1, 2, 3) -\text{usual Pauli matrices}): \]

\[ (\delta_{\alpha j}, (\sigma_j)_{\alpha j}) \rightarrow (\epsilon_{\alpha j}, (\tau_j)_{\alpha j}), \quad (\epsilon_{\alpha j} = i(\sigma_j)_{\alpha j}, \quad (\tau_j)_{\alpha j} = \tau_j)_{\alpha j} = i(\sigma_2)_{\alpha j}, \]
As: \( \phi_{\overline{P}q}^{\pm}(\vec{k})_{\alpha\beta} = \epsilon_{\alpha\beta} \Phi_{\overline{P}q}^{\pm}(\vec{k}) \); \( \Phi_{\overline{P}q}^{\pm(1,m)}(\vec{k})_{\alpha\beta} = \tau_{\alpha\beta}^{(m)} \Phi_{\overline{P}q}^{\pm(1)}(\vec{k}) \); (36)

\( \tau_{\alpha\beta}^{(0)} = i \tau_{\alpha\beta}^{(1)} \); \( \tau_{\alpha\beta}^{(1)} = \frac{\mp i}{\sqrt{2}} \left( \tau_{\alpha\beta}^{(1)} \pm i \tau_{\alpha\beta}^{(2)} \right) \); \( \Phi_{\overline{P}q}^{\pm(1,m)}(\vec{k})_{\alpha\beta} = -\Phi_{\overline{P}q}^{\pm(1,m)}(-\vec{k})_{\beta\alpha} \);

\[ J, m; \mathcal{P}, q = \int d^3 k \phi_{\overline{P}q}^{\pm(1,m)}(\vec{k})_{\alpha\beta} | R_{\alpha\beta}^0(\mathcal{P}, \vec{k}) \];

one has the following expressions:

\[ \Phi_{\overline{P}q}^{\pm(1)}(\vec{k}) = \frac{1}{2} \left[ \Phi_{\overline{P}q}^{\pm}(\vec{k}) + (-1)^j \Phi_{\overline{P}q}^{\pm}(-\vec{k}) \right] \]; (37)

\[ \Phi_{\overline{P}q}^{\pm(1)}(\vec{k}) = \frac{1}{2} \left[ \delta_3(\vec{k} - \vec{q}) + (-1)^j \delta_3(\vec{k} + \vec{q}) + \frac{T_{\mathcal{P}}^{(1)}(q; \vec{\kappa})}{E_2(\mathcal{P}, \vec{\kappa}) - E_2(\mathcal{P}, \vec{q}) \mp i0} \right] \];

\[ T_{\mathcal{P}}^{(0)}(q; k) = C_1(q) + k^2 C_2(q) = \frac{D_{q1}^{(\pm)}(q) + k^2 D_{q2}^{(\pm)}(q)}{(2\pi)^2 D_0^{(\pm)}(q)} \], (38)

\[ T_{\mathcal{P}}^{(1)}(q; k) = \left( \vec{k} \cdot C_3^{(\pm)}(q) \right) = \frac{2\mu}{(2\pi)^3} k^2 \left\{ \frac{\Pi_{\mathcal{P}}^{(1)}(q)}{1 - 2\mu I_\delta(q)} + \frac{\Pi_{\mathcal{P}}^{(2)}(q)}{1 - 2\mu I_\delta(q)} \right\} q \];

where projectors are \( \Pi_{\perp}^{(1)}(\mathcal{P}) = \left( \delta_{ij} - \frac{p_i p_j}{p^2} \right) \); \( \Pi_{\parallel}^{(1)}(\mathcal{P}) = \frac{p_i p_j}{p^2} \). (39)

The bound state wave functions look like simple residues of scattering one \( \Phi_{\overline{P}q}^{\pm}(\vec{k}) \) for corresponding poles for \( E_2(\mathcal{P}, q) \pm i\sigma \Rightarrow i\sigma \Rightarrow M_2^{(0)}(\mathcal{P}) \);

\[ D_0^{(\pm)}(\sigma) = 0; \quad 1 - 2\mu I_\delta(q) = 0; \quad 1 - 2\mu I_\delta(q) = 0; \quad C_n(\mathcal{P}) \sim C_n(q) \]; (40)

\( \Phi_{\overline{P}q}^{\pm}(\vec{k}) \Rightarrow B^{(0)}(\vec{k}) = Z^{(0)}(\mathcal{P}, \vec{k}) \left[ E_2(\mathcal{P}, q) - M_2^{(0)}(\mathcal{P}) \right]^{-1} \);

\( Z^{(0)}(\mathcal{P}, \vec{k}) = C_1(\mathcal{P}) + k^2 C_2(\mathcal{P}); \quad Z^{(1)}_{\perp}(\mathcal{P}, k) = \left( \vec{k} \cdot \tilde{C}_3^{(1)}(\mathcal{P}) \right) \);

For the case \( \mu \rightarrow 0 \); \( C_1(q) \Rightarrow \frac{\lambda(2\pi)^{-3}}{\lambda I_{0}^{(\pm)}(q) - 1}; \quad C_2^{(\pm)}(q) \Rightarrow 0. \) (41)

The obtained solutions directly satisfy to extended unitarity relation \([24]\).

3. Case \( \mu = 0 \) and linearisation of HE

Returning to HE, let us consider the conserved charge densities \( S_{\psi}(x) \):

\[ i\partial_t S_{\psi}(x) = \Psi_\alpha^\dagger(x) \left[ E(\hat{\mathcal{P}}) - E(\hat{\mathcal{P}}) \right] \Psi_\alpha(x) - i\mu \vec{V} \cdot \left( \Psi_\alpha^\dagger(x) J_{\psi}(x) \Psi_\alpha(x) \right) \];

\[ Q(t) = \int d^3 x \mathcal{S}_{\psi}(\vec{x}, t); \quad \partial_t Q(t) = -\mu \int d^3 \vec{r} \cdot \left( \Psi_\alpha^\dagger(x) J_{\psi}(x) \Psi_\alpha(x) \right) R_{\psi} \rightarrow 0. \]

It is clear that for \( \mu = 0 \) the HE for \( S_{\psi}(x) \) contains only kinetic term. Then the initial conditions lead to the simple form of dynamical mapping for this operator:

\[ S_{\psi}(\vec{x}, t) \equiv e^{iHt} S_{\psi}(\vec{x}, 0) e^{-iHt} \Rightarrow e^{iHt} S_{\psi}(\vec{x}, 0) e^{-iHt} \Rightarrow e^{iHt} S_{\psi}(\vec{x}, 0) e^{-iHt} \equiv S_{\psi}(\vec{x}, t). \) (42)
So, HE \((29)\) become linear equation with respect to HF \(\Psi_\alpha(\vec{x}, t)\):

\[
(i\partial_t - E(P)) \Psi_\alpha(\vec{x}, t) = -\frac{\lambda}{2} S(\psi)(\vec{x}, t) \Psi_\alpha(\vec{x}, t) \Rightarrow -\frac{\lambda}{2} S(\psi)(\vec{x}, t) \Psi_\alpha(\vec{x}, t),
\]

and its solution gives the closed expression of HF in terms of the physical one:

\[
\Psi_\alpha(\vec{x}, t) = T \exp \left\{ i \int_0^t d\eta \left[ \frac{\lambda}{2} S(\psi)(\vec{x}, \eta) - E(P) \right] \right\} \psi_\alpha(\vec{x}, 0) = e^{-iE(P)T} \exp \left\{ i \int_0^t d\eta S(\psi) (\vec{x} + \eta \vec{v}(P), \eta) \right\} \psi_\alpha(\vec{x}, 0),
\]

where \(\vec{v}(P) = \vec{\nabla}_P E(P)\) is corresponding group velocity. Dynamical mapping is given by normal ordering of this formal solution. It seems difficult to obtain such kind of solution in terms of in-fields.

4. \(N, \Theta\) model

This model is determined by the following Hamiltonian, CCR, and HE:

\[
H = \int d^3x \left\{ N^\dagger(x) \mu(\nabla^2) N(x) + \Theta^\dagger(x) w(\nabla^2) \Theta(x) + \lambda \int d^4y \int d^4z \alpha(x - y) \alpha(x - z) N^\dagger(x) N(x) \Theta^\dagger(y) \Theta(z) \right\},
\]

\[
\alpha(x - y) = \alpha(x - y) \delta(t_x - t_y); \quad \mu(\nabla^2) e^{ikx} = m(k)e^{ikx}; \quad w(\nabla^2) e^{ikx} = \omega(k)e^{ikx}.
\]

\[
\{ N(x), N^\dagger(y) \} = \delta(t_x - t_y), \quad [\Theta(x), \Theta^\dagger(y)] = \delta(t_x - t_y), \quad \{ \Theta(x), N^\dagger(y) \} = \delta(t_x - t_y),
\]

\[
(i\partial_t - \mu(\nabla^2)) N(x) = \lambda \int d^4y \int d^4z \alpha(x - y) \alpha(x - z) \Theta^\dagger(y) N(x) \Theta(z),
\]

\[
(i\partial_t - w(\nabla^2)) \Theta(x) = \lambda \int d^4y \int d^4z \alpha(y - x) \alpha(y - z) N^\dagger(y) N(y) \Theta(z).
\]

All others (anti) commutators vanish. It is seen from this HE that as in the previous case \((12)\) the HE for the operator \(N^\dagger(x) N(x)\) contains only kinetic term:

\[
(i\partial_t (N^\dagger(x) N(x)) = N^\dagger(x) \left[ \mu(\nabla^2) - \mu(\nabla^2) \right] N(x).
\]

Then \(N^\dagger(x) N(x) = N_0^\dagger(x) N_0(x)\). Defining the HF \(N(x)\) and physical field \(N_0(x)\) and HE in momentum representation

\[
N(x) \Theta(x) = \int \frac{d^3k}{(2\pi)^3/2} \left\{ n(k, t) e^{ikx - im(k)t} \right\}; \quad \tilde{\alpha}(p) = \int \frac{d^3x}{(2\pi)^3/2} e^{ipx} \alpha(x);
\]

\[
i\partial_t n(l, t) = \lambda \int d^3p \int d^3q \tilde{\alpha}(q) \tilde{\alpha}(-p) e^{i(E_{l+q}^{+1} - E_p^{-1})t} n(1, t) n(p, t) n(l + q - p, t),
\]

\[
i\partial_t o(l, t) = \lambda \tilde{\alpha}(l) \int d^3p \int d^3q \tilde{\alpha}(p - q - l) e^{i(E_{l+q-p}^{+1} - E_p^{-1})t} n(1, t) n(p, t) o(l + q - p, t),
\]
where $E_{q}^{p+q} = \omega(q) + m(p)$; and $N_{o}(x), \Theta_{o}(x)$ are defined by the same identities (48) with $n(k, t) \rightarrow N(k), o(k, t) \rightarrow \Theta(k)$, one has, as above, the linear HE for operator $o(k, t)$

$$o(k, t) = \Theta(k) + \int_{0}^{t} d\eta \int d^{3}l \ K_{N}(k, l; \eta) o(l, \eta),$$

$$K_{N}(k, l; t) = -i\lambda\tilde{\alpha}(k)\tilde{\alpha}(-l) \int d^{3}q \ e^{i(E_{q}^{p+q} - E_{q}^{p+k})t} N^{\dagger}(q) N(q + k - l),$$

with initial conditions $o(k, 0) = \Theta(k)$, $n(k, 0) = N(k)$ which has the similar formal solution

$$o(k, t) = \Theta(k) + \int d^{3}l \ R_{N}(k, l; t)\Theta(l) = T \left\{ \exp \left[ \int_{0}^{t} d\eta \hat{K}_{N}(\eta) \right] \Theta \right\}(k), \quad (51)$$

where $\hat{K}_{N}(\eta)$ is integral operator with the kernel $K_{N}(k, l; \eta)$. Note, that for a given $o(k, t)$ the equation (19) for $n(k, t)$ is also linear.

For the reduced Hamiltonian

$$H_{t} = \lambda \int d^{3}p \int d^{3}k \int d^{3}q \tilde{\alpha}(-q)\tilde{\alpha}(k + q - p) N^{\dagger}(p)N(k)\Theta^{\dagger}(k + q - p)\Theta(q),$$

$$H_{0} = \int d^{3}p \ \left[ m(p) N^{\dagger}(p)N(p) + \omega(p) \Theta^{\dagger}(p)\Theta(p) \right], \quad H = H_{0} + H_{t}, \quad (52)$$

as for the previous case, it’s possible to find coefficient functions of dynamical mapping and two-particles bound and scattering eigenstates. The dynamical mapping up to third order now reads:

$$o(p, t) = O_{0}[\Theta(k), N(k)] = \Theta(p) + \int d^{3}q \int d^{3}k N^{\dagger}(q)N(p + q - k)\Theta(k) \cdot F(t; q, p; p + q - k, k) + \ldots$$

$$n(p, t) = N_{0}[\Theta(k), N(k)] = N(p) + \int d^{3}q \int d^{3}k \Theta^{\dagger}(q)N(p + q - k)\Theta(k) \cdot F(t; p, q; p + q - k, k) + \ldots, \quad (53)$$

where the first coefficient function may be found as:

$$F(t; l, q; l + q - p, p) = -i\lambda\tilde{\alpha}(q)\tilde{\alpha}(-p) \int_{0}^{t} d\xi e^{i\xi E_{q}^{p+q}z(t)} \frac{1}{2\pi i}. \quad (54)$$

$$\int_{-i\infty + \Delta \varepsilon(t)}^{i\infty + \Delta \varepsilon(t)} d\sigma \frac{e^{\sigma \xi}}{(\sigma + iE_{p}^{l+q}) [1 + I^{l+q}(\sigma)]}, \quad I^{l+q}(\sigma) = \lambda \int d^{3}k \frac{|\tilde{\alpha}(k)|^{2}}{E_{k}^{l+q} - i\sigma}.$$  

The familiar solutions of eq. (18) for bound and scattering eigenstates of Hamiltonian (52) (2) (3) :

$$H \ | R^{\pm}\{N(p - q)\Theta(q)\} \rangle = E_{q}^{p} | R^{\pm}\{N(p - q)\Theta(q)\} \rangle \quad H \ | B^{P} \rangle = M(P) | B^{P} \rangle;$$

$$| R^{\pm}\{N(p - k)\Theta(k)\} \rangle = \int d^{3}q R_{0}^{p, k}(q) N^{\dagger}(p - q)\Theta^{\dagger}(q) | 0 \rangle, \quad (55)$$
satisfy to the orthogonality conditions:

\[
R_{\pm}^p (q) = \delta_3 (k - q) + \frac{\lambda \tilde{\alpha} (q) \tilde{\alpha} (-k)}{E_k^p - E_q^p \pm i \delta} Q_k^{p(\pm)}; \quad Q_k^{p(\pm)} = [1 + i P (\pm \delta - i E_k^p)]^{-1};
\]

\[
\begin{align*}
| B^p \rangle &= \int d^3 q B^p (q) N^\dagger (p - q) \Theta^\dagger (q) | 0 \rangle; \quad 1 + i P (E_k^p p) = 0. \quad (56) \\
B^p (q) &= \frac{\lambda \tilde{\alpha} (q)}{M (p) - E_q^p}; \quad J^p (\sigma) = \int d^3 k \frac{| \tilde{\alpha} (k) |^2}{(E_k^p - i \sigma) (E_k^p - M (p))};
\end{align*}
\]

\[
\int d^3 k \int B^p (k) B^p (k) = | Z^p |^2 J^p (-i M (p)) = 1, \quad (57)
\]

satisfy to the orthogonality conditions:

\[
\langle R^\pm \{ N (p - k) \Theta (k) \} | R^\mp \{ N (p_1 - k_1) \Theta (k_1) \} \rangle = \\
= \delta_3 (p - p_1) \int d^3 q \; R_{\pm}^p (q) R_{\mp}^{p, k_1} (q) = \delta_3 (p - p_1) \delta_3 (k - k_1); \quad (58)
\]

\[
\langle B^p | R^\mp \{ N (p - k) \Theta (k) \} \rangle = \delta_3 (p_1 - p) \int d^3 q B^{\dagger p} (q) R_{\pm}^{p, k} (q) = 0.
\]

By definition, the S-matrix reads:

\[
\begin{align*}
\langle N_{\text{in}} (p - k) \Theta_{\text{in}} (k) | \hat{S}^{\pm 1} | N_{\text{in}} (p_1 - k_1) \Theta_{\text{in}} (k_1) \rangle & \overset{\text{def}}{=} \langle R^\pm \{ N (p - k) \Theta (k) \} | R^\mp \{ N (p_1 - k_1) \Theta (k_1) \} \rangle = \\
&= \delta_3 (p - p_1) \int d^3 q \; R_{\pm}^p (q) R_{\mp}^{p, k_1} (q) = \delta_3 (p - p_1) \; S_{\pm}^p (k, k_1); \quad (59)
\end{align*}
\]

\[
S_{\pm}^p (k, k_1) = \{ \delta_3 (k - k_1) \mp 2 \pi i \delta (E_{k1}^p - E_k^p) \lambda \tilde{\alpha} (-k) \tilde{\alpha} (k) Q_k^{p(\pm)} \}.
\]

This model was considered also by Umezawa, Matsumoto, Tachiki \cite{3} in the framework of dynamical mapping method using the "in" physical fields. The dynamical mapping for this case looks like this:

\[
n (l, t) \equiv \mathcal{N}_{-\infty}^t \{ N_{\text{in}} (k), \Theta_{\text{in}} (k) \} = N_{\text{in}} (l) + \\
+ \int d^3 q \; \int d^3 k \left[ R_{\pm}^{l+q, k} (q) - \delta_3 (k - q) \right] e^{i (E_{q1}^p - E_{k1}^p) t} \Theta_{\text{in}}^\dagger (l + q - k) \Theta_{\text{in}} (k) + \\
+ \int d^3 k \; B_{l+q, k} (k) e^{i (E_{q1}^p - M (l + k)) t} \Theta_{\text{in}}^\dagger (l + k) V_{\text{in}} (l + k) + \ldots ;
\]

\[
e^{-it E_{q1}^p} o (q, t) n (p - q, t) = e^{-it M (p)} B^p (q) V_{\text{in}} (p) + \\
+ \int d^3 k \; e^{-it E_{q1}^p} R_{\pm}^{p, k} (k) \Theta_{\text{in}} (k) V_{\text{in}} (p - k) + \ldots.
\]

Unfortunately, the second term in the last equation was omitted in \cite{3}. With this correction, one can compare their results with the our approach using the uniqueness of HF and making the dynamical mapping onto the "in" field by two steps:

\[
n (k, t) = \mathcal{N}_{-\infty}^t \{ N_{\text{in}} (k) \} = \mathcal{N}_{0}^t \{ N (k) \} = \mathcal{N}_{0}^t \{ \mathcal{N}_{-\infty}^0 \{ N_{\text{in}} (k) \} \}.
\]

One can see that obtained consistency conditions have the same form as above mentioned off-shell extended unitarity relations \cite{24}:

\[
B_{l+q, k} (q) \left\{ e^{i (E_{q1}^p - M (l + k)) t} - 1 \right\} = \int d^3 p \; B_{l+q, k} (p) F (t; l, q; l + q - p, p), \quad (62)
\]

\[
R_{\pm}^{l+q, k} (q) \left\{ e^{i (E_{q1}^p - E_{k1}^p) t} - 1 \right\} = \int d^3 p \; R_{\pm}^{l+q, k} (p) F (t; l, q; l + q - p, p).
\]
and hold identically for solutions \[^{54-57}\]. Moreover, the scattering and bound states for this two different approaches are connected correspondingly as:

\[
\langle B^p | = \langle 0 | \int d^3q \ B^{p*}(q) N(p - q) \Theta(q) =
\]

\[
= \langle 0 | \int d^3q \ B^{p*}(q) \left\{ B^p(q) V_{in}(p) + \int d^3k \ R^{p,k}_+(q) \Theta_{in}(k) N_{in}(p - k) + \ldots \right\} =
\]

\[
= \langle 0 | V_{in}(p); | R^\pm \{N(p - k) \Theta(k)\}) = \int d^3q R^{p,k}_+(q) N^\dagger(p - q) \Theta^\dagger(q) | 0 \rangle =
\]

\[
= \int d^3q R^{p,k}_+(q) \left\{ \tilde{B}^p_+(q) V_{in}^\dagger(p) + \int d^3l \ R^{p,l}_+(q) N_{in}^\dagger(p - l) \Theta_{in}(l) + \ldots \right\} | 0 \rangle =
\]

\[
= \int d^3l N_{in}^\dagger(p - l) \Theta_{in}^\dagger(l) | 0 \rangle \left\{ \delta_3(l - k), \begin{array}{c} (+) \\ \lambda^2 E^3 + B \end{array}, \begin{array}{c} (+) \\ (-) \end{array} \right\}
\]

So, as expected, the bound state \( | B^p \rangle \), obtained by selfconsistency method of \[^8\] with the help of new bound state operator \( V_{in}(p) \) coincides with the one obtained from direct solution of eigenvalue problem in terms of constituent fields, and in their turn the scattering eigenstates are nothing but two-particles in- and out- states from \[^8\].

5. Divergencies and selfadjoint extensions

Now some remarks about divergency problem are needed. It’s absent for \( N, \Theta \) model due to \( \hat{\alpha}(p) \). However for four-fermion models with quadratic ”bare” spectra \( \mathcal{E}(k) \) the two-particle eigenvalue problem \[^8\] may be reduced in configuration space to the Schroedinger equation with singular delta-like potential, considered in \[^8\]. A simple cut-off procedure for integrals \[^8\] with forthcoming incorporation of cut-off parameter and ”bare” coupling constant into the binding energy \( M_2(0) \) via subtraction procedure leads to the same answer where arbitrary binding energy serves as a parameter of selfadjoint extension of free Hamiltonian \((-\nabla^2_\pi)\) in two-particle sector \[^8\].

The example of such dimensional transmutation of cut-off parameter and ”bare” coupling constant into the binding energy for 2D Schroedinger equation was given in \[^1\]. So doing, we obtain from \[^37\], \[^38\], \[^31\], for Hamiltonian \( \mathcal{H}_1 \) \[^29\] with \( \mathcal{E}(k) = k^2/2m + \mathcal{E}_0, \lambda_0 = \lambda m/2, \) and \( M_2(P) = P^2/4m + 2\mathcal{E}_0 - b^2/m \) the following renormalized eigenfunctions:

\[
\Phi^\pm_{Rq}(k) = \delta_3(k - q - q) - \frac{[2\pi^2(b \pm iq)]^{-1}}{(k^2 - q^2 + i0)}; \quad B(k) = \frac{\sqrt{8\pi b}}{k^2 + b^2};
\]

where the bound state equation \[^31\] leads to transmutation condition:

\[
1 = \lambda \ J_0^P (-iM_2(P)) \Rightarrow \lambda_0 \int_{k < \Lambda} \frac{d^3k}{(2\pi)^3} \cdot \frac{1}{k^2 + b^2} \Rightarrow \frac{\lambda_0}{4\pi} \left( \frac{2\Lambda}{\pi} - b \right); \quad b = \frac{2}{\pi} \Lambda - \frac{4\pi}{\lambda_0}.
\]

The Fourier-images \( \phi(\vec{x}) \) of wave functions \[^{64}\] satisfy to Schroedinger equation with singular delta-potential:

\[
(-\nabla^2 + b^2) \phi_B(\vec{x}) = \lambda_0 \delta_3(\vec{x}) \phi_B(\vec{x}).
\]
So, the existence of selfadjoint extension of free Hamiltonian corresponding to such singular one implies definite behavior of "bare" coupling constant as well as in [8]:

$$\lambda_0(\Lambda) = 4\pi \left[ \frac{2}{\pi} \Lambda - b \right]^{-1} \simeq \frac{2\pi^2}{\Lambda} \left( 1 + \frac{\pi b}{2\Lambda} + \ldots \right).$$  

(65)

The case of Hamiltonian \([27]\) becomes less trivial because it requires renormalization of mass and "gap" of one-particle spectrum \([32]\) as well:

$$E^\pm(k) = \frac{k^2}{2\mathcal{M}^\pm} + E_0^\pm; \quad E_0^\pm = g \left( \frac{\langle k^2 \rangle}{(2mc)^2} + 1 \right) \pm (2g - \mathcal{E}_0);$$

where we put: $\mu \Rightarrow \frac{\lambda}{(2mc)^2}; \quad g = \frac{\lambda}{4V^*}; \quad \mathcal{M}^\pm = m \left( \frac{g}{2mc^2} \mp 1 \right)^{-1}. $  

(66)

The dependence of "bare" quantities on cut-off parameter $\Lambda$ is now fixed as:

$$\frac{1}{V^*} = \frac{\Lambda^3}{6\pi^2}; \quad \langle k^2 \rangle = \frac{3}{5}\Lambda^2; \quad g = \Lambda^2 G(\Lambda); \quad (2mc)^2 = \Lambda^2 \nu(\Lambda); \quad \mathcal{E}_0 = \Lambda^2 \epsilon(\Lambda);$$

$$2\mathcal{M}^\pm \equiv 2\mathcal{M}^\pm(\Lambda) = \frac{\nu(\Lambda) \gamma^\pm(\Lambda)}{G(\Lambda)}; \quad \gamma^\pm(\Lambda) \equiv \frac{\mu \mathcal{M}^\pm}{2V^*} = \left[ 1 \mp \frac{c\sqrt{\nu(\Lambda)}}{\Lambda G(\Lambda)} \right]^{-1};$$

$$E_0^\pm(\Lambda) = \Lambda^2 \left\{ G(\Lambda) \left[ \frac{3}{5\nu(\Lambda)} + (1 \pm 2) \right] \mp \epsilon(\Lambda) \right\}; \quad \text{where:}$$

$$\nu(\Lambda) = \nu_0 + \frac{\nu_1}{\Lambda} + \frac{\nu_2}{\Lambda^2} + \ldots; \quad \text{and the same for } \mathcal{M}^\pm(\Lambda), \epsilon(\Lambda), \gamma^\pm(\Lambda), G(\Lambda).$$

For $G_0, \nu_0 \neq 0$ the finiteness of the quantities follows:

$$\gamma_0 = 1; \quad \gamma_1^\pm = \pm \frac{c\sqrt{\nu_0}}{G_0}; \quad \mathcal{M}_0 = \frac{\nu_0}{2G_0}; \quad \mathcal{M}_1^\pm = \frac{1}{2G_0} \left[ \nu_1 - 2\mathcal{M}_0(G_1 \mp c\sqrt{\nu_0}) \right];$$

$$E_0^\pm = \Lambda^2 \left\{ G_0 \left( \frac{3}{5\nu_0} + (1 \pm 2) \right) \mp \epsilon_0 \right\} + \Lambda \left\{ G_1 \left( \frac{3}{5\nu_0} + (1 \pm 2) \right) \mp \epsilon_1 - \frac{3G_0\nu_1}{5\nu_0^2} \right\} + \left[ G_2 \left( \frac{3}{5\nu_0} + (1 \pm 2) \right) \mp \epsilon_2 - \frac{3}{5} \left( \frac{G_0\nu_2 + G_1\nu_1}{\nu_0} - \frac{G_0\nu_1}{\nu_0^2} \right) \right].$$

(69)

Thus, the renormalization conditions imply that first and second square brackets in \([63]\) vanish. As well as in the previous case, the bound state equation \([40]\) for $J=0$ serves as a transmutation condition and looks like:

$$D_0^P (-iM_2(\mathcal{P})) = (\gamma - 1)^2 - \mu J_0^P (-iM_2(\mathcal{P})) \left[ \frac{\Lambda}{\mu} + \gamma < k^2 > - \mathcal{P}^2 - (2 - \gamma)b^2 \right] \Rightarrow$$

$$\Rightarrow (\gamma - 1)^2 - \frac{\gamma}{\Lambda^3} \beta_0 \left( 3\xi \Lambda - b \right) \left[ \Lambda^2 \left( \nu + \frac{3}{5}\gamma \right) - \mathcal{P}^2 - (2 - \gamma)b^2 \right] = 0;$$

where: $M_2(\mathcal{P}) \equiv E_2(\mathcal{P}, q = ib) \Rightarrow \frac{\mathcal{P}^2}{4M^\pm} + 2E_0^\pm - \frac{b^2}{M^\pm},$ and for:

$$\xi = \frac{2}{3\pi}; \quad Y = \frac{3}{2} \gamma + \tilde{\nu}; \quad \Gamma = \frac{(\gamma - 1)^2}{\gamma(2 - \gamma)}; \quad \tilde{\nu}(\Lambda) = \nu(\Lambda) - \frac{\mathcal{P}^2}{\Lambda^2}; \quad b = \Lambda z(\Lambda);$$

$$z(\Lambda) = z_0 + \frac{z_1}{\Lambda} + \ldots,$$  

and the same for $Y(\Lambda), \Gamma(\Lambda),$  

(71)
it leads to cubic equation: \( z^3 - 3ζ z^2 - Y z + ξ(3Y - Γ) = 0 \). Now the finiteness of \( b \) implies the condition \( z_0 = 0 \) which together with (38), (39) means that:

\[
3Y_0 = Γ_0 = 0; \quad ν_0 = -\frac{3}{5} γ_0 = -\frac{3}{5}; \quad M_0 = -\frac{0.3}{G_0}; \quad ε_0 = 2G_0; \quad (so ν_0, G_0 < 0);
\]

\[
γ_1^± = ±i \frac{e}{G_0} \sqrt{\frac{3}{5}}; \quad ε_1 = 2G_1 ± \frac{5}{3} G_0 ν_1; \quad Y_1 = \frac{3}{5} γ_1 + ν_1; \quad \text{if in addition,}
\]

\[
ε_2 = 2G_2, \quad \text{then: } E^+(k) \Rightarrow E^-(k) = \frac{k^2}{2M_0} - \frac{5}{3} \left( G_0 ν_2 + G_1 ν_1 + \frac{5}{3} G_0 ν_2^2 \right),
\]

and if besides \( ν_1 = 0 \), then the "bare" spectrum becomes also unique.

The renormalized T-matrix for \( J=0 \) (38) after subtraction

\[
\mathcal{D}_0^{(0)}(q) \Rightarrow \mathcal{D}_0^{(0)}(q) - \mathcal{D}_0^{(0)}(-iM_2(\mathcal{P})), \quad \text{looks like:} \quad \mathcal{T}_p^{(0)}(q; k) = \frac{\Lambda^2 \left( ν + \frac{3}{5} \right) - \mathcal{P}^2 + q^2 + k^2(1-γ)}{\mathcal{M} \left[ \Lambda^2 \left( ν + \frac{3}{5} \right) - \mathcal{P}^2 \right] (b ± iq) + (2-γ) \left[ \frac{2}{π} \mathcal{M}_1(b^2 + q^2) - (b^2 ± (iq)^3) \right]},
\]

and under the conditions (72) takes the form:

\[
\mathcal{T}_p^{(0)}(q; k) \mid_{Λ→∞} = -\frac{8π}{\mathcal{M}_0} \cdot \frac{Y_1}{(Y_1 + b ± iq)(b ± iq)}. \quad (74)
\]

Corresponding boundstate wave function with \( J=0 \) has the same form as for the previous case (14). Note that for the last case the Galilean invariance which means independence on \( \mathcal{P} \) of both scattering and boundstate eigenfunctions is restored manifestly only due to the applied renormalization procedure. For \( J=1 \) the T-matrix (39) becomes:

\[
\mathcal{T}_p^{(1)}(q; k) = \frac{2μ}{(2π)^3} \cdot \frac{(\vec{k} \cdot \vec{q})}{1 - \frac{2}{3} μI_1^{(0)}(q)}, \quad \text{and for } γ_0 = 1 \text{ it tends to zero like } Λ^{-3} \text{ with } Λ → ∞. \quad \text{So, there are no any scattering and bound states with } J=1 \text{ for such selfadjoint extension, defined by (38), (72).}
\]

To find another extension let us apply the above described renormalization procedure to the case \( J=1 \). The denominator of (72) with \( q \Rightarrow ib_1, b_1 = Λ y \), gives the following transmutation conditions: \( y^3 - 3ζ y^2 - ξ(3/(2γ) - 1) = 0, \ y_0 = 0 \), which mean that \( γ_0 = 3/2 \). Then the finiteness of \( \mathcal{M}_0 \) leads to the requirements: \( G_0 = G_1 = ν_0 = ν_1 = ε_0 = ε_1 = 0 \). However, for this case both T-matrices \( \mathcal{T}_p^{(0)}(q; k), \mathcal{T}_p^{(1)}(q; k) \) tend to zero like \( Λ^{-1} \). So, such extension is completely equivalent to the free Hamiltonian.

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