Cohomology classes of admissible normal functions

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Abstract. We study the map associating the cohomology class of an admissible normal function on the product of punctured disks, and give some sufficient conditions for the surjectivity of the map. We also construct some examples such that the map is not surjective.

Introduction

Let $H$ be a polarizable variation of Hodge structure of weight $-1$ on a product of punctured disks $S^* := (\Delta^*)^n$. Let $\text{NF}(S^*, H)_{S}^{\text{ad}}$ be the group of admissible normal functions with respect to $S = \Delta^n$ (see [12]). Let $\text{NF}(S^*, H_{\mathbb{Q}})_{S}^{\text{ad}}$ be its scalar extension by $\mathbb{Z} \to \mathbb{Q}$. This is identified with the group of extension classes of $\mathbb{Q}S^*$ by $H_{\mathbb{Q}}$ as admissible variations of $\mathbb{Q}$-mixed Hodge structures with respect to $S$. Let $j : S^* \to S$, $i_0 : \{0\} \to S$ denote the inclusions. We have a canonical morphism associating the cohomology class of an admissible normal function

$$\text{NF}(S^*, H_{\mathbb{Q}})_{S}^{\text{ad}} \to \text{Hom}_{\text{MHS}}(\mathbb{Q}, H^1i_0^*R^j_*H_{\mathbb{Q}}),$$

where MHS denotes the category of graded-polarizable $\mathbb{Q}$-mixed Hodge structures [6]. Let $j_!^*H_{\mathbb{Q}}$ denote the intermediate direct image [1]. This exists as a (shifted) pure Hodge module [11]. It is known (see e.g. [2] and also [10]) that $H^1i_0^*j_!^*H_{\mathbb{Q}}$ is a subspace of $H^1i_0^*R^j_*H_{\mathbb{Q}}$, and the above morphism is naturally factored by

$$(0.1) \quad \text{NF}(S^*, H_{\mathbb{Q}})_{S}^{\text{ad}} \to \text{Hom}_{\text{MHS}}(\mathbb{Q}, H^1i_0^*j_!^*H_{\mathbb{Q}}).$$

The target of (0.1) does not change by replacing $H$ with the nilpotent orbit associated to $H$. It is rather easy to show that (0.1) is surjective if $n = 1$ (since $H^1i_0^*j_!^*H_{\mathbb{Q}} = 0$ in this case) or if $H$ is a nilpotent orbit or corresponds to a family of Abelian varieties. More generally, we have

**Theorem 1.** Let $\mathcal{V}$ denote the underlying filtered $\mathcal{O}$-module of $H$. Assume $\text{Gr}_{F^2}\mathcal{V} = 0$, or more generally, $F^{-1}\mathcal{V}$ is stable by the action of vector fields (i.e. $F^{-1}\mathcal{V}$ is defined by a local system). Then the morphism (0.1) is surjective.

For a more general statement, see Theorem (1.5) below. In this paper we also show that (0.1) is not necessarily surjective for $n \geq 2$ in general. More precisely, we have the following
Theorem 2. Assume \( H \) is an nilpotent orbit of weight \(-1\) on \( \Delta^* \) such that \( \text{dim} \, \text{Im} \, N = 1 \) and \( \text{Gr}_F^1 V \neq 0 \). Then there is a (non-horizontal) one-parameter family of polarizable variations of Hodge structures \( H^{(\lambda)} \) on \( \Delta^* \) for \( |\lambda| \ll 1 \) (shrinking \( \Delta \) if necessary) such that \( H^{(0)} \) coincides with \( H \), the limit mixed Hodge structure of \( H^{(\lambda)} \) is independent of \( \lambda \), and moreover, if \( H^{(\lambda)} \) denotes also its pull-back by the morphism \( S^* = (\Delta^*)^n \ni (t_1, \ldots, t_n) \mapsto t_1 \cdots t_n \in \Delta^* \) with \( n \geq 2 \), then for \( 0 < |\lambda| \ll 1 \), the morphism (0.1) vanishes although its target does not, where \( H^{1,i_0}_* j_* H \mathbb{Q} \cong \bigoplus_{k=1}^{n-1} Q \) as mixed Hodge structures.

Note that the situation in Theorem 2 is obtained by iterating unnecessary blowing-ups along smooth centers contained in a smooth divisor, and it is quite difficult to generalize it to a more natural situation. So Theorem 2 does not imply any obstructions to a strategy for solving the Hodge conjecture which is recently studied by M. Green, P. Griffiths, R. Thomas, and others, see [2], [7], [15] (and also Remark (2.5) below).

In Section 1 we recall some basics of the cycle classes of admissible normal functions, and prove Theorem 1. In Section 2 we prove Theorem 2.

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1. Cohomology classes

1.1. Calculation of cohomology classes. Let \( S = \Delta^n \), \( S^* = (\Delta^*)^n \), and \( H \) be a polarizable variation of Hodge structure of weight \(-1\) on \( S^* \). Let \( H \) be the limit mixed Hodge structure of \( H \) with \( \mathbb{Q} \)-coefficients, see [13]. We assume that the monodromies \( T_i \) are unipotent. Set \( N_i = \frac{1}{2\pi i} \log T_i \). Let \( j : S^* \to S \), \( i_0 : \{0\} \to S \) denote the inclusions. Let \( j_* H \mathbb{Q} \) be the intermediate direct image \([1]\). Then \( i^*_0 j_* H \mathbb{Q} \) is calculated by the complex

\[
I^*(H; N_1, \ldots, N_n) := [0 \to H \overset{\oplus_i N_i}{\to} \bigoplus_i \text{Im} \, N_i \to \bigoplus_{i \neq j} \text{Im} \, N_i N_j \to \cdots],
\]

where \( H \) is put at the degree 0, see e.g. [5]. It is a subcomplex of the Koszul complex

\[
K^*(H; N_1, \ldots, N_n) := [0 \to H \overset{\oplus_i N_i}{\to} \bigoplus_i \text{H}(-1) \to \bigoplus_{i \neq j} \text{H}(-2) \to \cdots],
\]

calculating \( i^*_0 R_j j_* H \mathbb{Q} \). (Here \( \text{Im} \, N_i \subset \text{H}(-1) \), \( \text{Im} \, N_i N_j \subset \text{H}(-2) \), see [6], 2.1.13 for Tate twist.) This is obtained by iterating the restriction functors \( i^*_j = C(\psi_{t_j} \to \varphi_{t_j}) \), where the \( i_j \) denote the inclusion of \( \{ t_j = 0 \} \) with \( t_j \) the coordinates of the polydisk, see [11].

We have a canonical morphism

\[
\text{NF}(S^*, H)_{ad}^\text{ad} \to \text{Hom}_{\text{MHS}}(\mathbb{Q}, H^{1,i^*_0 R_j j_* H \mathbb{Q}}),
\]

associating the cohomology class of an admissible normal function on \( S^* \) with respect to \( S \). By definition [12], \( \nu \in \text{NF}(S^*, H)_{ad}^\text{ad} \) corresponds (using [3]) to a short exact sequence of admissible variations of Hodge structures ([8], [14])

\[
0 \to H \to H' \to Z_{S^*} \to 0.
\]
Passing to the limit, we get a short exact sequence of mixed $\mathbb{Q}$-Hodge structures endowed with the action of the logarithm of the monodromies

$$0 \to (H; N_1, \ldots, N_n) \to (H'; N'_1, \ldots, N'_n) \to (Q; 0, \ldots, 0) \to 0.$$ 

Let $\sigma : Q \to H'$ be a splitting of the surjection $H' \to Q$ as $\mathbb{Q}$-vector spaces. Then we have for any $i$

$$N'_i \sigma(1) \in \text{Im } N_i,$$

restricting over curves transversal to each divisor $\{t_i = 0\}$, see e.g. [12], 2.5.4. Hence

$$(1.5) \quad (N'_i \sigma(1)) \in (\bigoplus_i \text{Im } N_i)^0 := \text{Ker}(\bigoplus_i \text{Im } N_i \to \bigoplus_{i \neq j} \text{Im } N_i N_j),$$

with

$$(\bigoplus_i \text{Im } N_i)^0 / \text{Im}(\bigoplus_i N_i) = H^1 I^*(H; N_1, \ldots, N_n) = H^1 i^*_0 j^*_s H^* Q.$$

1.2. Proposition. The cohomology class of an admissible normal function $\nu$ (i.e. the image of $\nu$ by (1.3)) is given by

$$(1.2.1) \quad (N'_i \sigma(1)) \in H^1 I^*(H; N_1, \ldots, N_n) = H^1 i^*_0 j^*_s H^* Q \subset H^1 i^*_0 R^* j^*_s H^* Q.$$

Proof. Applying $H^* i^*_0 R^* j^*_s$ to (1.4), we get a long exact sequence

$$(1.2.2) \quad H^0 i^*_0 R^* j^*_s Q \to H^1 i^*_0 R^* j^*_s H^* Q \to H^1 i^*_0 R^* j^*_s H^* Q \to,$$

and the cohomological class is given by the image of $1 \in Q = H^0 i^*_0 R^* j^*_s Q$ by $\partial$ (which is a morphism of mixed Hodge structures). Moreover, (1.2.2) is induced by the short exact sequence of complexes of mixed Hodge structures

$$0 \to K^*(H; N_1, \ldots, N_n) \to K^*(H'; N'_1, \ldots, N'_n) \to K^*(Q; 0, \ldots, 0) \to 0,$$

where $K^*(\ast)$ denotes the Koszul complex as in (1.2). So the assertion follows.

1.3. Mixed nilpotent orbits. We say that $((H, W'); N_1, \ldots, N_n)\) is a mixed nilpotent orbit if $H$ is a mixed $\mathbb{Q}$-Hodge structure endowed with a finite increasing filtration $W'$ and $N_i : H \to H(-1)$ are nilpotent morphisms preserving $W'$ such that the following two conditions are satisfied:

(i) The relative monodromy filtration for $N_i$ with respect to $W'$ exists for any $i$.

(ii) Each $(\text{Gr}_{W'} N_1, \ldots, N_n)$ is a pure nilpotent orbit of weight $k$ for any $k$.

Then the relative monodromy filtration for $\bigoplus_{i \in I} N_i$ with respect to $W'$ exists for any subset $I$ of $\{1, \ldots, n\}$, see [8]. The category of mixed nilpotent orbits is an abelian category such that any morphisms are strictly compatible with $F$ and $W'$, see loc. cit. A mixed nilpotent orbit defines an admissible variation of mixed Hodge structure on $S^* = (\Delta^*)^n$ with respect to $S = \Delta^n$ using the coordinates $t_i$ of $\Delta^n$. Here we use the correspondence between the multivalued horizontal sections of $\mathcal{V}$ and the holomorphic sections of the Deligne extension.
\( \tilde{V} \) of \( V \) annihilated by \((t_k \frac{\partial}{\partial t_k})^{\dim H}\) for any \( k \). It is defined by assigning to a horizontal section \( v \)

\[
(1.3.1) \quad \tilde{v} = \exp(-\sum_{k=1}^{n}(\log t_k)N_k)v.
\]

In the case \( S^* = (\Delta^*)^n, S = \Delta^n \), and \( H \) is a nilpotent orbit on \( S^* \), we will denote by

\[
(1.3.2) \quad \text{NF}(S^*, H_Q)^{\text{mno}}_S,
\]

the subgroup of \( \text{NF}(S^*, H_Q)^{\text{ad}}_S (= \text{NF}(S^*, H)^{\text{ad}}_S \otimes \mathbb{Q}) \) consisting of admissible normal functions (tensored by \( \mathbb{Q} \)) corresponding to extension classes in the category of mixed nilpotent orbits.

The following would be known to specialists.

\textbf{1.4. Proposition.} Assume \( H \) is a nilpotent orbit. Then (0.1) is surjective. More precisely, (1.1.3) induces a surjective morphism

\[
(1.4.1) \quad \text{NF}(S^*, H_Q)^{\text{mno}}_S \to \text{Hom}_{\text{MHS}}(\mathbb{Q}, H^{1^n_0 j^i_0} H_Q).
\]

**Proof.** It is enough to show the surjectivity of (1.4.1). In the notation of (1.1.5), take \( \alpha \in \text{Hom}_{\text{MHS}}(\mathbb{Q}, (\bigoplus_i \text{Im} N_i)^0/\text{Im}((\bigoplus_i N_i)). \)

This is identified with an element of \((\bigoplus_i \text{Im} N_i)^0/\text{Im}((\bigoplus_i N_i))\) considering the image of 1 in \( \mathbb{Q} \). We have an exact sequence

\[
0 \to \text{Im}((\bigoplus_i N_i) \to (\bigoplus_i \text{Im} N_i)^0 \to (\bigoplus_i \text{Im} N_i)^0/\text{Im}((\bigoplus_i N_i) \to 0.
\]

Take lifts \( \alpha'_Q \) and \( \alpha'_{\mathbb{C}} \) of \( \alpha \) to \((\bigoplus_i \text{Im} N_i)^0\) and \( F^0(\bigoplus_i \text{Im} N_i)^0_\mathbb{C}\) respectively. There is \( \beta \in H^i_{\mathbb{C}} \) such that

\[
(1.4.2) \quad (N_i(\beta)) = \alpha'_{\mathbb{C}} - \alpha'_Q \text{ in } (\bigoplus_i \text{Im} N_i)^0_\mathbb{C}.
\]

We will construct an extension \( H' \) of \( \mathbb{Q} \) by \( H \) such that the image of the extension class by (1.4.1) corresponds to \( \alpha \) as follows.

As a \( \mathbb{Q} \)-vector spaces we have

\[
H'_Q = H_Q \oplus \mathbb{Q}.
\]

The action of \( N_i' \) on \( H_Q' \) is defined by

\[
(1.4.3) \quad N'_i(a, b) = (N_i a + b(\alpha'_Q)_i, 0) \text{ for } a \in H_Q, b \in \mathbb{Q},
\]

where \((\alpha'_Q)_i \in (\text{Im} N_i)_{\mathbb{Q}}\) is the \( i \)-th component of \( \alpha'_Q \) in \((\bigoplus_i \text{Im} N_i)^0_{\mathbb{C}}\). The weight filtration \( W' \) is defined by \( \text{Gr}_{-1}^{W'} H' = H \) and \( \text{Gr}_0^{W'} H' = \mathbb{Q} \). The Hodge filtration \( F \) is defined by

\[
(1.4.4) \quad F^p H'_\mathbb{C} = \begin{cases} 
F^p H_\mathbb{C} & \text{if } p > 0, \\
F^p H_\mathbb{C} + \mathbb{C}(\beta, 1) & \text{if } p \leq 0,
\end{cases}
\]
where $F^pH_C$ is identified with a subspace of $H'_C$.

We have to show that $H'$ satisfies the conditions of mixed nilpotent orbits. By [8] it is enough to show that the relative monodromy filtration exists for each $N'_i$, and the Griffiths transversality $N'_i F^pH'_C \subset F^{p-1}H'_C$ is satisfied. The first condition is trivially satisfied since $(\alpha'_Q)_i \in (\text{Im } N_i)_Q$. The second condition is reduced to

$$N'_i(\beta, 1) = N_i(\beta) + (\alpha'_Q)_i \in F^{-1}H_C,$$

and follows from (1.4.2), i.e. $N_i(\beta) + (\alpha'_Q)_i = (\alpha'_P)_i$. (Note that the Hodge filtration $F$ on $\text{Im } N_i \subset H_i(-1)$ is shifted by 1 so that $F^0(H(-1))_C = F^{-1}H_C$.)

By Proposition (1.2) together with (1.4.3) for $(a, b) = (0, 1)$, the image by (1.4.1) of this extension class is given by $\alpha'_Q$. So Proposition (1.4) follows.

The above argument can be extended as follows. Let $\hat{\mathcal{V}}$ denote the Deligne extension of the underlying filtered $\mathcal{O}$-module $\mathcal{V}$ of $H$. Then we have an isomorphism

$$H_C \cong \bigcap_j \text{Ker } (t_j \frac{\partial}{\partial t_j})^{\dim H} \subset \hat{\mathcal{V}}_0,$$

using the coordinates of $S$, and $H_C$ is also identified with a quotient $\mathcal{V}_0/m_0 \mathcal{V}_0$ of $\mathcal{V}_0$ where $m_0$ denotes the maximal ideal of $\mathcal{O}_{S, 0}$. Analyzing the proof of Proposition (1.4), we get

**1.5. Theorem.** Assume the filtration induced by the inclusion $H_C \hookrightarrow \hat{\mathcal{V}}_0$ coincides with the Hodge filtration of $H_C$ which is by definition the quotient filtration by $\hat{\mathcal{V}}_0 \rightarrow \hat{\mathcal{V}}_0/m_0 = H_C$, i.e. $F^{-1}\hat{\mathcal{V}}_0$ is generated over $\mathcal{O}_{S, 0}$ by its intersection with $\bigcap_j \text{Ker } (t_j \frac{\partial}{\partial t_j})^{\dim H} \subset \hat{\mathcal{V}}_0$, or equivalently, $F^{-1}\mathcal{V}$ is generated over $\mathcal{O}_{S^*}$ by the sections $\tilde{v}_1, \ldots, \tilde{v}_m$ of $\mathcal{V}$ corresponding to some horizontal sections $v_1, \ldots, v_m$ as in (1.3.1). Then (0.1) is surjective.

**Proof.** This follows from the same argument as in the proof of Proposition (1.4) if we replace (1.4.4) by

$$F^p\hat{\mathcal{V}}' = \begin{cases} F^p\hat{\mathcal{V}} & \text{if } p > 0, \\ F^p\hat{\mathcal{V}} + \mathcal{O}_S(\beta, 1) & \text{if } p \leq 0, \end{cases}$$

where $\mathcal{V}'$ is the underlying $\mathcal{O}$-module of the extension $H'$, and $\hat{\mathcal{V}}'$ is its Deligne extension. Here $H'_C$ is identified with a subspace of $\hat{\mathcal{V}}'$ using $\bigcap_j \text{Ker } (t_j \frac{\partial}{\partial t_j})^{\dim H'}$ as above. The hypothesis implies that the image of $(\alpha'_P)_i \in H_C$ in $\hat{\mathcal{V}}_0$ belongs to $F^{-1}\hat{\mathcal{V}}_0$ in the notation of the proof of Proposition (1.4), and hence the Griffiths transversality is satisfied. So Theorem (1.5) follows.

**2. Deformation of nilpotent orbits**

We will construct a family of variations of Hodge structures $H^{(\lambda)}$ on $\Delta^*$ for $\lambda \in \mathbb{C}^*$ with $|\lambda| \ll 1$ by modifying the Hodge filtration $F$ of $H$ so that the limit mixed Hodge structure does not change. Here we may forget the rational structure and consider the real structure instead, since we have the rational structure which does not change by the deformation.
We will identify nilpotent orbit with the associated limit mixed Hodge structure $H$ endowed with the action of $N$. Let $\langle *, * \rangle$ be a skew-symmetric form on $H_C$ giving a polarization of a nilpotent orbit as in [5].

2.1. Proposition. With the assumption of Theorem 2, there is a 4-dimensional $\mathbb{R}$-nilpotent orbit $H_1$ generated by $u_1, \ldots, u_4 \in H_C$ satisfying the following conditions.

(i) $u_1 \in F^1 W_{-1} H_C$, $u_2 \in W_0 H_R$, $u_3 \in W_{-2} H_C$, $u_4 \in F^{-2} W_{-1} H_C$.

(ii) $\bar{u}_1 = u_4$, $\bar{u}_2 = u_2$, $\bar{u}_3 = -u_3$, $Nu_2 = u_3$, $Nu_j = 0$ ($j \neq 2$), where $N = \frac{1}{2\pi i} \log T$.

(iii) $[u_1], [u_4] \in Gr_{1}^W H_C$ have type $(1, -2)$ and $(-2, 1)$ respectively.

(iv) $u_2 + au_3 \in F^0 W_0 H_C$ for some $a \in \mathbb{C}$.

(v) $\langle u_i, u_j \rangle = 0$ unless $\{i, j\} = \{1, 4\}$ or $\{2, 3\}$.

Proof. Since $\dim \text{Im} N = 1$, we have $N^2 = 0$ and the weight filtration $W$ on $H_R$ is given by

\[ W_{-3} H_R = 0, \quad W_{-2} H_R = \text{Im} N, \quad W_{-1} H_R = \text{Ker} N, \quad W_0 H_R = H_R. \]

Moreover, we have an isomorphism compatible with $F$

\[ N : \text{Gr}_0^W H_R (= \mathbb{R}) \xrightarrow{\sim} \text{Gr}_2^W H_R (-1) (= (\mathbb{R}(1))(-1)). \]

By the second assumption of Theorem 2, there are

\[ u_1 \in F^1 W_{-1} H_C, \quad u_4 \in F^{-2} W_{-1} H_C, \]

such that $\bar{u}_1 = u_4$ and condition (iii) is satisfied. By the first assumption of Theorem 2, there is

\[ u_3 \in F^{-1} W_{-2} H_C, \]

which is purely imaginary, i.e. $\bar{u}_3 = -u_3$. By (2.1.1) there is

\[ u_2 \in W_0 H_R \text{ such that } Nu_2 = u_3. \]

Note that

\[ (2.1.2) \quad \text{Gr}_0^W H_R = \mathbb{R}[u_2], \quad \text{Gr}_2^W H_R = \mathbb{R}[iu_3]. \]

To show condition (iv), note that $\text{Gr}_0^W H_C$ is the direct sum of $F^0$ and $F^0$. Since $i[v - \bar{v}] \in \text{Gr}_1^W H_C$ is real, it is written as $[w] + [\bar{w}]$ for some $w \in F^0 W_{-1} H_C$. Thus we may assume $v - \bar{v} \in W_{-2} H_C$ replacing $v$ with $v + iw$. Then $v - au_3$ is real for some $a \in \mathbb{C}$, and we may replace $u_2$ with $v - au_3$.

As for condition (v), it is satisfied modifying $u_1, u_4$ by a multiple of $u_3$ if necessary, since $\langle u_2, u_3 \rangle \neq 0$ by (2.1.2) and $\langle u_j, u_3 \rangle = 0$ for $j = 1, 4$ (because $\langle W_{-1}, W_{-2} \rangle = 0$). This completes the proof of Proposition (2.1).

2.2. Construction. With the notation of Proposition (2.1), let $H_1$ be the mixed $\mathbb{R}$-Hodge structure spanned by $u_1, \ldots, u_4$ in $H_C$, and $H_2$ be its orthogonal complement in $H_C$. We
have the decomposition as mixed $\mathbf{R}$-Hodge structures endowed with a pairing and the action of $N$
\[ H_{\mathbf{R}} = H_1 \oplus H_2. \]
For the proof of Theorem 2, we may then assume
\[ H_{\mathbf{R}} = H_1, \quad H_2 = 0, \]
so that $u_1, \ldots, u_4$ is a $\mathbf{C}$-basis of $H_{\mathbf{C}}$. Note that $H_{\mathbf{C}}$ is identified with the vector space of horizontal sections of $\mathcal{V}$ and also with that of holomorphic sections of the Deligne extension $\hat{\mathcal{V}}$ annihilated by $(t_k \frac{\partial}{\partial t_k})^{\dim H}$ for any $k$ (using (1.3.1)).

Let $z = \log t$ with $t$ the coordinate of $\Delta$. The $u_j$ ($j = 1, \ldots, 4$) induce a basis of the Deligne extension $\hat{\mathcal{V}}$ as in (1.3.1), i.e.
\[ \tilde{u}_2 = u_2 - zu_3, \quad \tilde{u}_j = u_j \ (j \neq 2). \]

Setting $\xi = t \frac{d}{dt}$, we have
\[ \xi \tilde{u}_2 = -\tilde{u}_3, \quad \xi \tilde{u}_j = 0 \ (j \neq 2). \]

Let $a, C \in \mathbf{C}$. For $\lambda \in \mathbf{C}$ with $|\lambda|$ sufficiently small, define
\begin{align*}
(2.2.1) \quad w_1 &= \tilde{u}_1 + C\lambda t (\tilde{u}_2 + a\tilde{u}_3 + \frac{1}{2}\lambda t \tilde{u}_4), \\
 w_2 &= \tilde{u}_2 + (a - 1)\tilde{u}_3 + \lambda t \tilde{u}_4, \quad w_3 = -\tilde{u}_3 + \lambda t \tilde{u}_4, \quad w_4 = \tilde{u}_4.
\end{align*}

Let $\hat{\mathcal{V}}^{(\lambda)}$ denote the Deligne extension of the underlying $\mathcal{O}$-module $\mathcal{V}^{(\lambda)}$ of $H^{(\lambda)}$. Define the Hodge filtration $F$ on $\hat{\mathcal{V}}^{(\lambda)}$ by
\[ F^p = \sum_{i=1}^{2-p} \mathcal{O}_S w_i. \]

For $\lambda = 0$, the Hodge filtration on $H^{(0)}$ coincides with that of $H$ choosing $a$ appropriately. The Griffiths transversality holds for $\xi = t \frac{d}{dt}$ since
\begin{align*}
(2.2.2) \quad \xi w_1 &= C\lambda t w_2, \quad \xi w_2 = w_3, \quad \xi w_3 = \lambda t w_4, \quad \xi w_4 = 0.
\end{align*}

However, this does not hold for $\lambda \frac{\partial}{\partial \lambda}$, and $\{H^{(\lambda)}\}$ is a non-horizontal family. Note that $Nu_2 = u_3, Nu_j = 0$ ($j \neq 0$), and
\begin{align*}
(2.2.3) \quad w_1 &= u_1 + C\lambda t (u_2 - (z - a)u_3 + \frac{1}{2}\lambda t u_4), \\
 w_2 &= u_2 - (z - a + 1)u_3 + \lambda t u_4, \quad w_3 = -u_3 + \lambda t u_4, \quad w_4 = u_4, \\
 \bar{w}_1 &= u_4 + \bar{C}\bar{\lambda} \bar{t} (u_2 + (\bar{z} - \bar{a})u_3 + \frac{1}{2}\bar{\lambda} \bar{t} u_1), \\
 \bar{w}_2 &= u_2 + (\bar{z} - \bar{a} + 1)u_3 + \bar{\lambda} \bar{t} u_1, \quad \bar{w}_3 = u_3 + \bar{\lambda} \bar{t} u_1, \quad \bar{w}_4 = u_1.
\end{align*}

Assume $C \in \mathbf{C}$ satisfies the condition
\[ \langle u_1, u_4 \rangle = C \langle u_2, u_3 \rangle. \]
Then we have the orthogonal relation
\begin{equation}
\langle F^p \mathcal{V}^{(\lambda)}, F^{-p} \mathcal{V}^{(\lambda)} \rangle = 0 \quad \text{for any } p.
\end{equation}
Indeed, the pairing is skew-symmetric and the above condition on $C$ implies
\[ \langle w_1, w_2 \rangle = \langle w_1, w_3 \rangle = 0. \]

So it remains to show that $F^p \cap \overline{F}^{-1-1}$ is 1-dimensional at each point of $\Delta^*$, and for a generator $\eta_{2-p}$ of $F^p \cap \overline{F}^{-1-1}$, we have the positivity (as in [6], 2.1.15):
\[ (2\pi i)^{-1} \langle \eta_{2-p}, i^{-2p-1} \bar{\eta}_{2-p} \rangle > 0 \quad (p = -2, -1, 0, 1). \]

It is enough to show these for $p = 1$ and 0, using the complex conjugation. For $p = 1$ the assertion is easy (using $|\lambda| \ll 1$) since $F^1$ is generated by $w_1$. For $p = 0$, we first see that $F^0 \cap \overline{F}^{-1}$ is at most 1-dimensional. Indeed, otherwise it must coincide with $F^0$ and hence $\overline{F}^{-1}$ must contain $w_1$. However, $\bar{w}_1, w_2, \bar{w}_3, w_1$, or equivalently, $w_1, w_2, w_3, \bar{w}_1$ are linearly independent at each point of $\Delta^*$ if $|\lambda| \ll 1$. The last assertion follows from (2.2.3) by calculating the following determinant:
\[
\begin{pmatrix}
1 & C\lambda t & -C\lambda t(z-a) & \frac{1}{2}C\lambda^2 t^2 \\
0 & 1 & -z+a-1 & \lambda t \\
\frac{1}{2}C\lambda^2 t^2 & C\bar{\lambda} t & \bar{C}\bar{\lambda} t(z-\bar{a}) & 1 \\
\end{pmatrix}
\]

Here we assume $\text{Im} \ z \in [0, 2\pi)$ so that $|tz| = |t\log t|$ is bounded on $\Delta^*$. Thus we get $\dim F^0 \cap \overline{F}^{-1} \leq 1$. From (2.2.3) we also deduce
\[
\begin{align*}
w_1 - C\lambda t w_2 &= u_1 + C\lambda t(u_3 - \frac{1}{2}C\lambda t(u_4)) = C\lambda t \bar{w}_3 + (1 - C\lambda \bar{\lambda} t)u_1 - \frac{1}{2}C\lambda^2 t^2 u_4, \\
\bar{w}_1 - C\bar{\lambda} t \bar{w}_2 &= u_4 + C\bar{\lambda} t(-u_3 - \frac{1}{2}C\bar{\lambda} t u_1) = -C\bar{\lambda} t \bar{w}_3 + \frac{1}{2}C\bar{\lambda}^2 t^2 u_1 + u_4.
\end{align*}
\]

This implies that $u_1, u_4$ are linear combinations of $w_1, w_2, \bar{w}_1, \bar{w}_2, \bar{w}_3$ with coefficients in $Q[\lambda, \bar{\lambda}, t, \bar{t}, P^{-1}]$ where $P = 1 - C\lambda \bar{\lambda} t + \frac{1}{4}C\bar{\lambda}^2 t^2 \bar{\lambda}^2 \bar{t}^2$. (Here $\lambda$ is viewed as a variable, but $a, c$ are constant.) Substitute these to the following equality which also follows from (2.2.3):
\[
\bar{w}_2 - w_2 = (z + \bar{z} - a - \bar{a} + 2)u_3 + \bar{\lambda} t u_1 - \lambda t u_4
\]
\[
= (z + \bar{z} - a - \bar{a} + 2)\bar{w}_3 - (z + \bar{z} - a - \bar{a} + 1)\bar{\lambda} t u_1 - \lambda t u_4.
\]

Then we get
\[ \lambda f_1 w_1 + (\lambda f_2 + 1)w_2 = g_1 \bar{w}_1 + g_2 \bar{w}_2 + g_3 \bar{w}_3, \]
where $f_1, f_2 \in Q[\lambda, \bar{\lambda}, t, \bar{t}, z, \bar{z}, P^{-1}]$ and $g_1, g_2, g_3 \in Q[\lambda, \bar{\lambda}, t, \bar{t}, z, \bar{z}, P^{-1}]$. Let $\eta_2$ denote the left-hand side of the above equality so that
\[ \eta_2 \in F^0 \cap \overline{F}^{-1}, \quad \eta_2 - w_2 = \lambda(f_1 w_1 + f_2 w_2). \]
Since the pairing gives a polarization for \( \lambda = 0 \) and \( |tz| = |t \log t| \) is bounded on \( \Delta^* \) (assuming \( \text{Im} z \in [0, 2\pi) \)), the assertion follows.

**2.3. Theorem.** Let \( H^{(\lambda)} \) denote also the pull-back of the variation of Hodge structure \( H^{(\lambda)} \) in \((2.2)\) by the morphism \( S^* := (\Delta^*)^n \ni (t_1, \ldots, t_n) \mapsto t_1 \cdots t_n \in \Delta^* \). Set \( S = \Delta^n \).

Assume \( \lambda \neq 0 \) and \( |\lambda| \ll 1 \). Then the morphism \( (0.1) \) for \( H^{(\lambda)} \) vanishes although its target does not, where \( H^{1}_i j^* H^{(\lambda)}_R \cong \bigoplus_{k=1}^{n-1} R \) as mixed Hodge structures.

**Proof.** By Proposition \((1.4)\) it is enough to show the vanishing of \((0.1)\). Since the assertion for the rational coefficients follows from that for the real coefficients, we may assume that \( H = H_1 \) as in \((2.2)\). We denote the pull-backs of \( u_j, w_j \) \((j = 1, \ldots, 4)\) in \((2.2)\) also by the same symbols. Then \( N_k u_j = 0 \) \((j \neq 2)\) and \( N_k u_2 = u_3 \) for \( k = 1, \ldots, n \), where \( N_k = \frac{1}{2\pi i} \log T_k \). Since \( N_k = N_{k'} \) for any \( k, k' \), the \( N_k \) will be denoted by \( N \) (which can be viewed as the pull-back of \( N \) on \( \Delta^* \)). Let \( z_k = \log t_k \). The basis of the Deligne extension is given as in \((1.3.1)\), i.e.

\[
(2.3.1) \quad \tilde{u}_2 = u_2 - \sum_{k=1}^{n} z_k u_3, \quad \tilde{u}_j = u_j \ (j \neq 2).
\]

Setting \( \xi_k = t_k \partial_{\xi_k} \), we have \( \xi_k \tilde{u}_2 = -\tilde{u}_3, \xi_k \tilde{u}_j = 0 \ (j \neq 2) \), and

\[
(2.3.2) \quad \xi_k w_1 = C \lambda t_1 \cdots t_n w_2, \quad \xi_k w_2 = w_3, \quad \xi_k w_3 = \lambda t_1 \cdots t_n w_4, \quad \xi_k w_4 = 0.
\]

Note that

\[
\begin{align*}
  w_1 &= u_1 + C \lambda t_1 \cdots t_n (\tilde{u}_2 + au_3 + \frac{1}{2} \lambda t u_4), \\
  w_2 &= \tilde{u}_2 + (a - 1)u_3 + \lambda t_1 \cdots t_n u_4, \\
  w_3 &= -u_3 + \lambda t_1 \cdots t_n u_4, \\
  w_4 &= u_4.
\end{align*}
\]

By the calculation in \((1.1)\), the morphism \((0.1)\) with real coefficients is expressed as

\[
(2.3.3) \quad \text{NF}(S^*, H^{(\lambda)}_R)_{\text{ad}} \rightarrow H^{1}_i j^* H^{(\lambda)}_R = (\bigoplus_{k=1}^{n} \text{Im} \ N)/\text{Im} \ N,
\]

since \( \text{Im} \ N \cong R \) as mixed Hodge structures. (Indeed, \( H \) and \( \text{Im} \ N \subset H(-1) \) can be identified with those associated to the variation on \( \Delta^* \).) Here \( \text{Im} \ N \hookrightarrow \bigoplus_{k=1}^{n} \text{Im} \ N \) is the diagonal. Thus the target of \((2.3.3)\) is nonzero. We have to show that \((2.3.3)\) vanishes.

Assume there is an admissible normal function \( \nu \) such that its image by \((2.3.3)\) does not vanish. Consider the corresponding short exact sequence of admissible variations of \( R \)-mixed Hodge structures

\[
0 \rightarrow H^{(\lambda)} \rightarrow H^{(\lambda)'} \rightarrow R_{S^*} \rightarrow 0.
\]

There is a basis \( u'_j \) \((j = 0, \ldots, 4)\) of complex-valued horizontal multivalued sections of \( V^{(\lambda)'} \) such that \( u'_j = u_j \ (j \neq 0) \), \( u'_0 \) is defined over \( R \), and the image of \( u'_0 \) in \( C \) is 1. By \((1.1)\) we have

\[
N_k u'_0 = c_k u_3 \quad \text{with} \quad c_k \in R \ (k = 1, \ldots, n),
\]

and the image of the normal function \( \nu \) by \((2.3.3)\) is given by \((c_1, \ldots, c_n)\), see \((1.2.1)\). So the above hypothesis on the image of \( \nu \) by \((2.3.3)\) is equivalent to

\[
(2.3.4) \quad c_k \neq c_k' \quad \text{for some} \ k, k'.
\]
Let $\tilde{u}^i_j$ be associated with $u^i_j$ as in (2.3.1). Then $\tilde{u}^i_j = \tilde{u}_j$ ($j \neq 0$) and $$\tilde{u}_0 = u_0 - \sum_{k=1}^n z_k c_k u_3.$$ So we get for any $k$

(2.3.5) \quad \xi_k \tilde{u}_0 = -c_k \tilde{u}_3.

Set $w^i_j = w_j$ for $j = 1, \ldots, 4$. There are $h_1, \ldots, h_4 \in O_S$ such that $$w^i_0 := \tilde{u}_0 + \sum_{j=1}^4 h_j w_j \in F^0\hat{V}(\lambda),$$

(shrinking $S$ if necessary). Then the Hodge filtration $F^p$ on the Deligne extension $\hat{V}(\lambda)$ is generated over $O_S$ by $w^i_j$ with $1 \leq i \leq 2 - p$ if $p > 0$, and by $w^i_0$ and $w^i_j$ with $1 \leq i \leq 2 - p$ if $p \leq 0$. By the Griffiths transversality we have for any $k$

$$\xi_k w^i_0 \in F^{-1}\hat{V}(\lambda) = \sum_{j=0}^3 O_S w^i_j.$$ Using (2.3.2), (2.3.5) and the relation $\tilde{u}_3 = \lambda t_1 \cdots t_n w_4 \mod O_S w_3$, we get for any $k$

$$\begin{align*}
\lambda t_1 \cdots t_n (h_3 - c_k) + \xi_k h_4 \in F^{-1}\hat{V}(\lambda),
\iff \lambda t_1 \cdots t_n (h_3 - c_k) + \xi_k h_4 = 0.
\end{align*}$$

This contradicts the hypothesis $c_k \neq c_{k'}$ in (2.3.4), looking at the coefficient of $t_1 \cdots t_n$ in the power series expansion of $h_4$. So the assertion follows.

2.4. Remarks. (i) According to G. Pearlstein, it is possible to describe the examples as above using the theory of period maps as in [9], Th. (6.16) or [4] Th. (2.7).

(ii) The arguments in (2.1–3) cannot be extended to arbitrary cases. For example, we have the surjectivity of (0.1) if the hypothesis of Theorem 1 or Theorem (1.5) is satisfied. We have the vanishing of the target of (0.1) if

$$F^{-1}\text{Gr}^W_{-2} H_{\mathbb{Q}} \cap \text{Ker} \text{Gr}^W N = 0.$$ Indeed, this is shown by taking $\text{Gr}^W$ of the differential

$$d : \bigoplus_i \text{Im} N_i \to \bigoplus_{i \neq j} \text{Im} N_i N_j$$

of the complex $I^*(H; N_1, \ldots, N_n)$ in (1.1.1). Here $N_i = N$ for any $i$.

(iii) In order to extend the arguments in (2.1–3), it may be convenient to assume the following conditions (which are stronger than the conditions coming from Remark (ii) above)

(2.4.1) \quad \text{Gr}^F_{-2}(\text{Ker} N)_{\mathbb{C}} \neq 0, \quad F^{-1} W_{-2}(\text{Ker} N)_{\mathbb{Q}} \neq 0,
where $\text{Ker} \, N \subset H$ denotes a mixed Hodge structure. (Indeed, if we do not assume the condition related to $\text{Ker} \, N$, then it does not seem easy to construct the Hodge filtration $F$ satisfying the Griffiths transversality (2.2.2) and the orthogonality (2.2.4).) It may be more convenient to replace the first condition of (2.4.1) by a stronger one

$$\text{Gr}_F^{-2} \text{Gr}_W^{-1}(\text{Ker} \, N)_C \neq 0.$$ 

2.5. Remark. As a consequence of recent work of R. Thomas (see [14]), M. Green and P. Griffiths (see [7]) and P. Brosnan, G. Pearlstein et al. (see [2]), it is known that the Hodge conjecture is reduced to the non-vanishing of the image by (0.1) of the normal function associated to a Hodge class. Theorem 2 does not imply any obstructions to this strategy since the hypothesis of Theorem 2 is too strong. It is very difficult to generalize the construction in (2.2) to the situation appearing in [7] since the hypothesis $\dim \text{Gr}_{-2}^W H_Q = \dim \text{Gr}_0^W H_Q = 1$ is quite essential in the argument of (2.2). (This condition is essentially equivalent to that the divisor with normal crossings is obtained by iterating unnecessary blowing-ups along smooth centers contained in a smooth divisor.)

Assume, for simplicity, $n = 3$ and

$$\text{Gr}_{-2}^W H_Q = \mathbb{Q}(1) \oplus \mathbb{Q}(1), \quad \text{Gr}_0^W H_Q = \mathbb{Q} \oplus \mathbb{Q},$$

and moreover $N_1, N_2, N_3$ are respectively expressed by

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then it is quite difficult to construct a deformation of the nilpotent orbit such that the image of (0.1) vanishes even in this simple case.

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