RESEARCH ARTICLE

The Exponentiated Exponential Burr XII distribution: Theory and application to lifetime and simulated data

Majdah Badr¹, Muhammad Ijaz²*

¹ Faculty of Science for Girls, Statistics Department, University of Jeddah, Jeddah, Saudi Arabia,
² Department of Statistics, University of Peshawar, Peshawar, KPK, Pakistan

* ijaz.statistics@gmail.com

Abstract

The paper addresses a new four-parameter probability distribution called the Exponentiated Exponential Burr XII or abbreviated as EE-BXII. We derive various statistical properties in addition to the parameter estimation, moments, and asymptotic confidence bounds. We estimate the precision of the maximum likelihood estimators via a simulation study. Furthermore, the utility of the proposed distribution is evaluated by using two lifetime data sets and the results are compared with other existing probability distributions. The results clarify that the proposed distribution provides a better fit to these data sets as compared to the existing probability distributions.

Introduction

These days no one can deny the importance of probability distributions because of their wide applicability in almost all fields of sciences. For example, to compute the system reliability Abdel-Ghaley et al. [1] used the Burr type two distribution. Other applications are found in the lifetime testing experiment [2], for wealth and income data analysis [3,4]. The existing probability distributions are however not suitable for the data sets having the non-monotonic (bath-tub or inverse bath-tub) hazard rate shapes. For example, the Gamma and Exponential distribution fail to model the increasing decreasing failure functions. It is therefore, researchers are working in distribution theory so that to develop such models that can capture not only the monotonic but also the non-monotonic hazard rate shapes of the real phenomena.

To increase the model fitting to applied data, many existing distributions have been improved by the researchers, for example, Gupta et al. [5] studied the general class of Exponentiated distribution. Mudholkar and Srivastava [6] discussed the Exponentiated Weibull distribution. Nadarajah and Kotz [7] defined and studied Exponentiated Fréchet distribution, Nadarajah and Kotz studied the Beta Gumbel distribution [8]. Nadarajah [9] discussed the Exponentiated Gumbel distribution with climate application. Nadarajah and Kotz [10] introduced a Beta Exponentiated distribution. Barreto Souza et al. [11] studied a Beta generalized...
Exponential distribution. Cordeiro and deCastro [12] used the Kumaraswamy distribution. Alzaatreh et al. [13] extended the Weibull Pareto distribution. Recently, Alzaghal et al. [14] defined and studied the Exponentiated T—X family of distribution with applications. Ali et.al [15] defined Alpha-Power Exponentiated Inverse Rayleigh distribution and its applications to real and simulated data. More recently Ijaz et.al [16–18] introduced a new scheme and families for generating the new probability distributions.

Among other probability distributions, the Burr XII distribution is the most commonly used distribution to capture non-monotonic hazard rates. This distribution was first derived by Burr in 19942 [19] and became very famous among practitioners due to broad applications including in reliability analysis [20], finance data modeling [21], environmental data analysis [22] and so on. The Burr distribution has been studied by many researchers, for example, Al-Hussaini [23] studied the Bayesian estimation, Abdel-Ghaly [20] used for modeling the software growth reliability, Mousa [24] studied the statistical inference by using censored data.

This paper focuses on one of the techniques used to generate improved probability distributions by replacing the baseline distribution in the existing family of distributions, for example [25]. In the current paper, the BXII distribution is improved by replacing $F(x)$ and $f(x)$ of BXII in the Exponentiated T-X family of distributions presented by Alazaghal et.al [14] given in Eq (1).

The derived probability distribution covers the monotonic and also the non-monotonic hazard rate shapes. The proposed distribution is not only flexible than BXII distribution but also provides better results than EP, SLL, Kw—GEP, Lomax—PII, McDonald Weibull (McW), Beta Weibull (BW), and Transmuted Marshall-Olkin Frechét (TMOFr) distribution.

The pdf of the Exponentiated T-X family of Alzaghal et.al [14] is defined by

$$ g(x) = c[f(x)]^{(1 - (F(x))^{b})}{b}, \quad x > 0, \; c, \; b > 0. $$

Let $F(x)$ and $f(x)$ denote the cdf and pdf of a random variable $X$ of a Burr XII distribution as follows

$$ F(x) = 1 - (1 + x^{a})^{-k}, \quad x > 0, \; a, \; k > 0 $$

$$ f(x) = k\alpha x^{-a}(1 + x^{a})^{-(k+1)}, \quad x > 0, \; a, \; k > 0. $$

The EE-BXII distribution

By putting (2) and (3) in (1), we obtained the following cdf of the Exponentiated Exponential Burr—XII (EE-BXII) distribution

$$ G(x) = 1 - \{1 - [1 - (1 + x^{a})^{-k}]^{c}\}^{b}, \quad x > 0, \; \alpha, \; c, \; \beta \text{ and } k > 0 $$

where $\alpha$, $c$, $\beta$, and $k$ are shape parameters.

The probability density function (pdf) related to (4) is defined by

$$ g(x) = \alpha k\beta x^{a-1}(1 + x^{a})^{-(k+1)}[1 - (1 + x^{a})^{-k}]^{c-1}\{1 - [1 - (1 + x^{a})^{-k}]^{c}\}^{b-1}, $$

$$ x > 0, \; \alpha, \; c, \; \beta \text{ and } k > 0. $$

The reliability function of the EE-BXII is given by

$$ R(x) = \{1 - [1 - (1 + x^{a})^{-k}]^{c}\}^{b}, \quad x > 0, \; \alpha, \; c, \; \beta \text{ and } k > 0. $$
The hazard or the failure rate function \( h(x) \) of the EE—BXII distribution is defined by

\[
h(x) = \frac{\alpha k c x^{\alpha-1} \left[1 - \left(1 + x^a\right)^{\frac{1}{c}}\right]^{c-1}}{1 - \left[1 - \left(1 + x^a\right)^{\frac{1}{c}}\right]^{(k+1)}}.
\]

It is to be noted that EE-BXII reduces to the BXII distribution when \( \beta = c = 1 \). For \( \beta = 1 \), it becomes the Exponentiated Pareto (EP) distribution. For \( \beta = c = k = 1 \) it reduces to the standard Log–Logistic (SLL) distribution which is not known in the literature. For \( \alpha = 1, c = \theta a \), we get the Kumaraswamy—Generalized Exponentiated Pareto (Kw—GEP) distribution. At \( \beta = c = \alpha = 1 \), we get the Lomax (L) distribution or Pareto type II (PII) distribution. The sub-models for EE-BXII are given in Table 1.

We plot the pdf, reliability function, and failure rate function of the EE—BXII distribution for the selected values of the parameters. Fig 1 shows the plot of the density function and it is observed that the pdf can accommodate the real data sets with decreasing, symmetry and right-skewed shapes. Fig 2 evaluates the reliability function of the EE—BXII distribution, which has decreasing behavior for the selected values of parameters. Fig 3 demonstrates

| Model        | Parameters |
|--------------|------------|
| EE-Burr XII  | \( \alpha \) | \( C \) | \( k \) | \( \beta \) |
| BXII         | \( \alpha \) | 1 | \( k \) | 1 |
| EP           | \( \alpha \) | \( C \) | \( k \) | 1 |
| SLL          | \( \alpha \) | 1 | 1 | 1 |
| Kw—GEP       | 1 | \( \alpha \) | \( k \) | \( \beta \) |
| Lomax—PII    | 1 | 1 | \( k \) | 1 |

Table 1. The sub-models of the EE-BXII.
Fig 2. The reliability curves of the EE—BXII distribution with \((\alpha, c, k, \beta)\).

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Fig 3. The hazard curve of the EE—BXII distribution with \((\alpha, c, k, \beta)\).

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various shapes of the hazard function with different parameter values for the EE—BXII distribution. This function is very flexible and provides various shapes—decreasing, J-shaped, and upside-down behaviors.

**Statistical properties of the EE—BXII distribution**

This section elaborates on some of the important statistical properties of the EE—BXII distribution. The properties include the ordinary moments, quantile, median and mode, skewness and kurtosis, the asymptotic variance covariance matrix, and asymptotic confidence intervals.

**$R^{th}$ moments**

The $r^{th}$ moments of the EE—BXII around zero are given by

$$
\mu_r = E(X^r) = \frac{\alpha \beta^r}{\gamma^r} \int_0^\infty x^{r-1} (1 + x^\gamma)^{-(k+1)} \left(1 - \frac{1}{1 + x^\gamma}\right)^{(k+1)-1} \left(1 - \frac{1}{1 + x^\gamma}\right)^{-1} \, dx.
$$

Substituting $r = 1$ in (8), we obtain the mean of the EE—BXII distribution and as follows

$$
\mu_1 = \frac{\alpha \beta^1}{\gamma^1} \int_0^\infty x^{0} (1 + x^\gamma)^{-(k+1)} \left(x^\gamma\right)^{(k+1)-1} \, dx,
$$

(9)

When $r = 2$ in (8), we obtain the second moment for EE—BXII distribution and is given by

$$
\mu_2 = \frac{\alpha \beta^2}{\gamma^2} \int_0^\infty x^{2} (1 + x^\gamma)^{-(k+1)} \left(x^\gamma\right)^{(k+1)-1} \, dx,
$$

(10)

By using Eqs (9) and (10), we get the variance of EE—BXII

$$
\text{var}(x) = \frac{\alpha \beta^2}{\gamma^2} \int_0^\infty (1 + x^\gamma)^{-(k+1)} \left(x^\gamma\right)^{(k+1)-1} \left(1 + \frac{1}{1 + x^\gamma}\right)^2 \, dx
$$

(11)

When we put $r = 3$ in (8), we obtain $\mu_3$ and is given by

$$
\mu_3 = \frac{\alpha \beta^3}{\gamma^3} \int_0^\infty x^{2} (1 + x^\gamma)^{-(k+1)} \left(x^\gamma\right)^{(k+1)-1} \, dx,
$$

(12)

When we replace $r = 4$ in (8), we obtained the following result

$$
\mu_4 = \frac{\alpha \beta^4}{\gamma^4} \int_0^\infty x^{2} (1 + x^\gamma)^{-(k+1)} \left(x^\gamma\right)^{(k+1)-1} \, dx
$$

(13)
Skewness and Kurtosis

The Skewness and Kurtosis of EE—BXII is defined by using (12) and (13) as

\[
SK(x) = c k \beta \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{i+j} \binom{\beta-1}{i} \binom{c(i+1)}{j} \frac{[x^{(i+1)} - 1]}{[x^{(j+1)} - 1]}
\]

\[
\div ck \beta \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{i+j} \binom{\beta-1}{i} \binom{c(i+1)}{j} \left\{ \frac{[x^{(i+1)} - 1]}{[x^{(j+1)} - 1]} - \left( \frac{[x^{(i+1)} - 1]}{[x^{(j+1)} - 1]} \right)^2 \right\} \] (14)

The Kurtosis is defined by

\[
kurt(x) = \frac{\left[ \frac{[x^{(i+1)} - 1]}{[x^{(j+1)} - 1]} \right]}{\left( \frac{[x^{(i+1)} - 1]}{[x^{(j+1)} - 1]} \right)^2} \] (15)

The quantile, median and mode of EE—BXII distribution

The quantile function \( t_q \) of the EE—BXII is the real solution of the inverse cdf of the EE—BXII distribution which is proceeding as follows

\[ F(x) = q \] (16)

where \( q \) is uniformly distributed over the interval 0 to 1.

By using (2), Eq (16) takes the following form

\[ t_q = \left( 1 - \left[ 1 - \{ 1 - q \}^\frac{1}{\beta} \right]^\frac{1}{\beta} \right) - 1. \] (17)

The median of EE—BXII distribution can be easily obtained by using \( q = 0.5 \) in (17).

The median then takes the following form

\[ t_{0.5} = \left( 1 - \left[ 1 - \{ 0.5 \}^\frac{1}{\beta} \right]^\frac{1}{\beta} \right) - 1. \] (18)

Mode

The mode of the EE—BXII distribution is that point by which the function \( f(x) \) reached to the maximum probability. The mode of EE—BXII can be easily obtained by solving the following equation

\[
g'(x) = z k c \beta \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{i+j} \binom{\beta-1}{i} \binom{c(i+1)}{j} x^{a-2} \]

\[
\times \{ x(1 - k(j+1)x^a + (a - 1)(1 + x^a)^{-k(j+1)-1} \} = 0. \] (19)

The above equation is not in explicit form and hence one can estimate the values of the mode by using real data sets.

Table 2 describes some of the numerical results for mean, median, mode, Standard deviation (SD), skewness and kurtosis by using different values of parameters. Table 2 concludes the following results...
Table 2. The mean, SD, mode, median, skewness, and kurtosis.

| α  | c  | k  | β  | Mean  | SD  | Mode  | Median | Skewness | Kurtosis |
|----|----|----|----|-------|-----|-------|--------|----------|----------|
| 2  | 1  | 5  | 1  | 0.7761 | 0.3767 | -3.002×10⁻⁸ | 0.14867 | 17.5122 | 81.5019 |
| 2  | 1  | 5  | 2  | 0.2165 | 0.2599 | -2.512×10⁻⁸ | 0.0718 | 3.9569 | 10.4821 |
| 2  | 1  | 5  | 3  | 0.1143 | 0.1846 | -2.133×10⁻⁸ | 0.0473 | 3.4897 | 9.7860 |
| 4  | 1  | 5  | 1  | 0.3157 | 0.3419 | 7.009×10⁻⁹ | 0.0718 | 3.8679 | 8.3785 |
| 5  | 1  | 5  | 1  | 0.3047 | 0.3284 | 6.213×10⁻⁹ | 0.0087 | 3.6432 | 7.6053 |
| 2  | 0.5 | 5  | 1  | 0.4294 | 0.4549 | -1.255×10⁻⁸ | 0.0592 | 5.0754 | 19.2669 |
| 2  | 0.8 | 5  | 1  | 0.6447 | 0.4371 | -3.303×10⁻⁸ | 0.1153 | 9.0246 | 35.9706 |
| 2  | 1  | 5  | 1  | 0.7761 | 0.3767 | -3.003×10⁻⁸ | 0.1287 | 17.5122 | 81.5019 |
| 2  | 1  | 6  | 1  | 0.5615 | 0.3436 | -6.925×10⁻⁸ | 0.1225 | 10.1328 | 34.6881 |
| 2  | 1  | 7  | 1  | 0.4472 | 0.3114 | 5.697×10⁻⁸ | 0.1041 | 7.8049 | 23.685 |
| 4  | 1  | 7  | 1  | 0.5891 | 0.3165 | 6.959×10⁻⁸ | 0.1040 | 11.2318 | 29.5862 |
| 5  | 1  | 8  | 1  | 0.5613 | 0.3342 | 1.846×10⁻⁸ | 0.0905 | 8.9515 | 21.5714 |
| 6  | 1  | 9  | 1  | 0.5441 | 0.3495 | 2.697×10⁻⁸ | 0.0801 | 7.6818 | 17.5692 |
| 2  | 1  | 6  | 2  | 0.1794 | 0.2262 | -1.615×10⁻⁸ | 0.0595 | 3.8117 | 10.0123 |
| 2  | 1  | 7  | 3  | 0.0868 | 0.1468 | -1.626×10⁻⁸ | 0.0336 | 3.4538 | 9.7523 |
| 1  | 2  | 6  | 3  | 0.3409 | 0.2786 | -1.745×10⁻¹⁶ | 0.1062 | 5.2950 | 11.6339 |
| 4  | 3  | 7  | 4  | 0.4427 | 0.2886 | 5.790×10⁻⁸ | 0.1179 | 7.4387 | 16.7381 |
| 5  | 4  | 8  | 5  | 0.5216 | 0.2889 | 1.803×10⁻⁸ | 0.1213 | 10.2123 | 24.4879 |

1. The mean, SD, median, and kurtosis are decreasing when α, c, and k is constant but β varies and also the same results were found when we fixed the values of c, k, β but α varies.

2. The mean, median, and kurtosis are increasing at the fixed values of α, k, β but c varies however the SD is decreasing.

3. The mean, SD, median, and kurtosis are decreasing when α, c, β are kept constant but k varies.

4. The mean, median and kurtosis are decreasing when α, k varies but c, β are kept constant however the SD is increasing.

5. The mean, SD, median and kurtosis are decreasing when α, c is constant but k, β varies.

6. The mean, SD, median and kurtosis are increasing when α, c, and β are not kept constant.

7. In general, it is observed that the proposed probability distribution is unimodal and positively skewed.

Fig 4 presents the plot of skewness and kurtosis for the EE—BXII distribution versus the parameter values of c. Fig 5 reflects the plot of skewness and kurtosis against the parameter values of β. Fig 6 explored the plot of skewness and kurtosis with different values of the parameter α. Fig 7 demonstrates the plot of skewness and kurtosis with various values of k.

Maximum likelihood estimates of the parameters

Let X₁, X₂, . . . , Xₙ be a random sample of size n from EE—BXII distribution with parameters (α, c, β). The likelihood function is defined by L(θ|x) = ∏ₙ f(xᵢ) Where, f(.) is defined in (2) and θ = (α, c, k, β). By taking the log of the likelihood function, we have

\[ l = \ln L = n \ln[kcβ] + (α - 1) \sum_{i=0}^{n} \ln xᵢ - (k + 1) \sum_{i=0}^{n} \ln[1 + xᵢ^α] + (c - 1) \sum_{i=0}^{n} \ln[1 - (1 + xᵢ^α)^{-k}] + (β - 1) \sum_{i=0}^{n} \ln[1 - (1 - (1 + xᵢ^α)^{-k})] \]  

(20)
Differentiate (20) with respect to $\alpha$, $c$, $\beta$, and $k$, respectively to have the following results:

\[
\begin{align*}
\frac{\partial l_1}{\partial \alpha} &= \frac{n}{\alpha} + \sum_{i=0}^{\alpha} \ln [x_i] - (k+1) \sum_{i=0}^{\alpha} \frac{x_i^a \ln [x_i]}{1+x_i^a} \\
&\quad - (1-c) \sum_{i=0}^{\alpha} kx_i^a \ln [x_i] \frac{(1+x_i^a)^{-(k+1)}}{1-(1+x_i^a)^{-k}} \\
&\quad + (1-\beta) \sum_{i=0}^{\alpha} c\beta x_i^c \ln [x_i] \frac{(1+x_i^a)^{-(k+1)}}{1-(1+x_i^a)^{-k}},
\end{align*}
\]

(21)

\[
\begin{align*}
\frac{\partial l_2}{\partial c} &= \frac{n}{c} + \sum_{i=0}^{\alpha} \ln [1-(1+x_i^a)^{-k}] \\
&\quad - (\beta-1) \sum_{i=0}^{\alpha} \frac{(1-(1+x_i^a)^{-k}) \ln [1-(1+x_i^a)^{-k}]}{1-(1+x_i^a)^{-k}},
\end{align*}
\]

(22)

Fig 4. The plot of skewness and kurtosis for selected values of the EE—BXII distribution versus the parameter $c$.

Fig 5. The plot of skewness and kurtosis for selected values of the EE—BXII distribution versus the parameter $\beta$.
Equate the above Eqs from (21) to (24) to zero and then solve these equations simultaneously for \( x \) will yield the maximum likelihood estimates \( (\hat{\alpha}, \hat{c}, \hat{k}, \hat{\beta}) \) of parameters \( \alpha, c, k, \beta \).
Since these nonlinear equations cannot be solved analytically but the numerical methods i-e Newton Raphson and some others can be easily used to obtain the estimates of these parameters. In the current study, Mathematica 9.0 package was used to evaluate the numerical values for these parameters.

The asymptotic variance—Covariance matrix

The asymptotic variance-covariance of \( \dot{a}, \dot{c}, \dot{k} \) and \( \dot{b} \) are obtained by using the second derivatives of the log-likelihood function. In this section, we derive the asymptotic variance-covariance matrix by using \( I_j(\theta) = -\frac{\partial^2 \ln L}{\partial \theta_i \partial \theta_j} \) that contains variances and covariances of estimates. Unfortunately, the exact mathematical expressions for the above expectation are complicated to obtain. Accordingly, we have the following approximate variance-covariance matrix.

\[
F = [I_j(\theta)]^{-1}, i, j = 1, 2, 3, 4, 5, 6 \text{ and } \theta = (a, c, k, b).
\]

The second partial derivative of parameters of the EE—BXII are given by

\[
\frac{\partial^2 I}{\partial x^2} = -\frac{n}{\sigma^2} (1 + k) \sum_{i=1}^n \left( \frac{-\ln|x_i^2 x_i^{2k}}{(1 + x_i^2)^2} + \frac{\ln|x_i^2 x_i^{2k}}{1 + x_i^2} \right) + (-1)
\]

\[
+ \sum_{i=1}^n \left( -\frac{k^2 \ln|x_i^2 x_i^{2k}(1 + x_i^2)^{-2k}}{(1 - (1 + x_i^2)^{-k})^2} + \frac{(1 - k)k \ln|x_i^2 x_i^{2k}(1 + x_i^2)^{-2k}}{1 - (1 + x_i^2)^{-k}} + \frac{k \ln|x_i^2 x_i^{2k}(1 + x_i^2)^{-k}}{1 - (1 + x_i^2)^{-k}} \right)
\]

\[
+ (-1 + \beta) \sum_{i=1}^n \left( -\frac{c^2 k \ln|x_i^2 x_i^{2k}(1 + x_i^2)^{-2k}}{(1 - (1 + x_i^2)^{-k})^2} + \frac{(1 - k)k \ln|x_i^2 x_i^{2k}(1 + x_i^2)^{-2k}}{1 - (1 + x_i^2)^{-k}} + \frac{k \ln|x_i^2 x_i^{2k}(1 + x_i^2)^{-k}}{1 - (1 + x_i^2)^{-k}} \right)
\]

\[
(25)
\]

Similarly, we can see

\[
\frac{\partial^2 I}{\partial x \partial c} = \sum_{i=1}^n \left( \frac{k \ln|x_i^2 x_i^{2k}(1 + x_i^2)^{-1-k}}{1 - (1 + x_i^2)^{-k}} \right)
\]

\[
+ (-1 + \beta) \sum_{i=1}^n \left( -\frac{c k \ln|x_i^2 x_i^{2k}(1 + x_i^2)^{-1-k}}{1 - (1 + x_i^2)^{-k}} \right)
\]

\[
+ \frac{k \ln|x_i^2 x_i^{2k}(1 + x_i^2)^{-1-k}}{1 - (1 + x_i^2)^{-k}} \right)
\]

\[
(26)
\]
\[
\frac{\partial^2 I}{\partial \alpha \partial \theta} = -\sum_{i=1}^{n} \frac{\text{Ln}[x_i] x_i^\theta}{1 + x_i^\theta} + (-1) \\
+ \sum_{i=1}^{n} \left( k \text{Ln}[x_i] \text{Ln}[1 + x_i^\theta] x_i^\theta \left( 1 + x_i^\theta \right)^{-1-2k} + \frac{\text{Ln}[x_i] x_i^\theta (1 + x_i^\theta)^{-1-k}}{1 - (1 + x_i^\theta)^{-k}} - \frac{k \text{Ln}[x_i] \text{Ln}[1 + x_i^\theta] x_i^\theta (1 + x_i^\theta)^{-1-k}}{1 - (1 + x_i^\theta)^{-k}} \right) \\
+ \frac{(-1 + \beta) \sum_{i=1}^{n} \left( -c^2 k \text{Ln}[x_i] \text{Ln}[1 + x_i^\theta] x_i^\theta (1 + x_i^\theta)^{-1-2k} (1 - (1 + x_i^\theta)^{-k})^{-2-2k} \\
+ \frac{c k \text{Ln}[x_i] \text{Ln}[1 + x_i^\theta] x_i^\theta (1 + x_i^\theta)^{-1-2k} (1 - (1 + x_i^\theta)^{-k})^{-2-2k}}{(1 - (1 + x_i^\theta)^{-k})^2} \right)}{1 - (1 + x_i^\theta)^{-k}} \\
- \frac{c k \text{Ln}[x_i] \text{Ln}[1 + x_i^\theta] x_i^\theta (1 + x_i^\theta)^{-1-k} (1 - (1 + x_i^\theta)^{-k})^{-1+c}}{1 - (1 + x_i^\theta)^{-k}} + \frac{c k \text{Ln}[x_i] \text{Ln}[1 + x_i^\theta] x_i^\theta (1 + x_i^\theta)^{-1-k} (1 - (1 + x_i^\theta)^{-k})^{-1+c}}{1 - (1 + x_i^\theta)^{-k}} \right) 
\]

(27)

\[
\frac{\partial^2 I}{\partial \alpha \partial \beta} = \frac{\partial^2 I}{\partial \beta \partial \alpha} = \sum_{i=1}^{n} \left( \frac{c k \text{Ln}[x_i] \text{Ln}[1 + x_i^\theta] x_i^\theta (1 + x_i^\theta)^{-1-k} (1 - (1 + x_i^\theta)^{-k})^{-1+c}}{1 - (1 + x_i^\theta)^{-k}} \right) 
\]

(28)

\[
\frac{\partial^2 I}{\partial \alpha^2} = -\sum_{i=1}^{n} \left( \frac{c \text{Ln}[1 - (1 + x_i^\theta)^{-k}]^2 (1 - (1 + x_i^\theta)^{-k})^{-k} \left( 1 - (1 + x_i^\theta)^{-k} \right)^2 + \frac{c \text{Ln}[1 - (1 + x_i^\theta)^{-k}]^2 (1 - (1 + x_i^\theta)^{-k})^{-k} \left( 1 - (1 + x_i^\theta)^{-k} \right)^2}{(1 - (1 + x_i^\theta)^{-k})^2} \right)}{1 - (1 + x_i^\theta)^{-k}} \\
+ \frac{c \text{Ln}[1 - (1 + x_i^\theta)^{-k}]^2 (1 - (1 + x_i^\theta)^{-k})^{-k} \left( 1 - (1 + x_i^\theta)^{-k} \right)^2}{1 - (1 + x_i^\theta)^{-k}} \\
+ \frac{c \text{Ln}[1 - (1 + x_i^\theta)^{-k}]^2 (1 - (1 + x_i^\theta)^{-k})^{-k} \left( 1 - (1 + x_i^\theta)^{-k} \right)^2}{1 - (1 + x_i^\theta)^{-k}} \right) 
\]

(29)

\[
\frac{\partial^2 I}{\partial \beta^2} = -\sum_{i=1}^{n} \left( \frac{c \text{Ln}[1 - (1 + x_i^\theta)^{-k}]^2 (1 - (1 + x_i^\theta)^{-k})^{-k} \left( 1 - (1 + x_i^\theta)^{-k} \right)^2 + \frac{c \text{Ln}[1 - (1 + x_i^\theta)^{-k}]^2 (1 - (1 + x_i^\theta)^{-k})^{-k} \left( 1 - (1 + x_i^\theta)^{-k} \right)^2}{(1 - (1 + x_i^\theta)^{-k})^2} \right)}{1 - (1 + x_i^\theta)^{-k}} \\
+ \frac{c \text{Ln}[1 - (1 + x_i^\theta)^{-k}]^2 (1 - (1 + x_i^\theta)^{-k})^{-k} \left( 1 - (1 + x_i^\theta)^{-k} \right)^2}{1 - (1 + x_i^\theta)^{-k}} \\
+ \frac{c \text{Ln}[1 - (1 + x_i^\theta)^{-k}]^2 (1 - (1 + x_i^\theta)^{-k})^{-k} \left( 1 - (1 + x_i^\theta)^{-k} \right)^2}{1 - (1 + x_i^\theta)^{-k}} \right) 
\]

(30)
\[ \frac{\partial^2 I}{\partial \alpha \partial \beta} = \sum_{i=1}^{n} \ln \left( \frac{1 + x_i^c}{1 + x_i^c} \right) \] 
\[ + \beta \sum_{i=1}^{n} \left( \frac{-c \ln (1 + x_i^c) \ln (1 - (1 + x_i^c)^{-k} (1 + x_i^c)^{-k}) (1 - (1 + x_i^c)^{-k})^{-1+c}}{(1 - (1 + x_i^c)^{-k})^2} \right) \] 
\[ - \frac{\ln (1 + x_i^c) \ln (1 - (1 + x_i^c)^{-k} (1 + x_i^c)^{-k}) (1 - (1 + x_i^c)^{-k})^{-1+c}}{1 - (1 + x_i^c)^{-k}} \] 
\[ \frac{\partial^2 I}{\partial k^2} = \sum_{i=1}^{n} \frac{\ln (1 - (1 + x_i^c)^{-k} (1 + x_i^c)^{-k})}{1 - (1 + x_i^c)^{-k}} \] 
\[ \frac{\partial^2 I}{\partial \alpha \partial k} = \sum_{i=1}^{n} \frac{-ck \ln (1 + x_i^c) (1 + x_i^c)^{-k} (1 - (1 + x_i^c)^{-k})^{-1+c}}{1 - (1 + x_i^c)^{-k}} \] 
\[ \frac{\partial^2 I}{\partial \beta \partial k} = \sum_{i=1}^{n} \frac{-ck \ln (1 + x_i^c) (1 + x_i^c)^{-k} (1 - (1 + x_i^c)^{-k})^{-1+c}}{1 - (1 + x_i^c)^{-k}} \] 
\[ \frac{\partial^2 I}{\partial \beta^2} = -\frac{n}{\widehat{\beta}^2} \] 

The asymptotic confidence interval

For a large sample size, the ML estimators under appropriate regularity conditions are consistent and asymptotically unbiased as well as asymptotically normally distributed. Therefore, two-sided approximate $100(1 - \tau)$% confidence intervals for the ML estimator say $\hat{\omega}$ of a population value $\omega$ can be obtained by

\[ P \left( -z \leq \frac{\hat{\omega} - \omega}{\sigma_{\hat{\omega}}} \leq z \right) = 1 - \tau \]

where $z$ is the $100(1 - \frac{\tau}{2})$th standard normal percentile. The two-sided approximate $100(1 - \tau)$% confidence intervals for $\omega$ will be given as follows

\[ L_{\omega} = \hat{\omega} - z_{\tau} \frac{\sigma_{\hat{\omega}}}{\sigma_{\hat{\omega}}} \quad \text{and} \quad U_{\omega} = \hat{\omega} + z_{\tau} \frac{\sigma_{\hat{\omega}}}{\sigma_{\hat{\omega}}} \]

where $\sigma_{\hat{\omega}}$ is the standard deviation of each estimator and $\hat{\omega}$ is the estimated value of $\alpha$, $c$, $k$, and $\beta$ respectively. Furthermore, $L_{\omega}$ is the lower limit and $U_{\omega}$ is the upper limit of the interval estimates.

Simulation study

A Simulation study has been performed using Mathematica 9.0 so that to illustrate the theoretical results of the estimation problem. The performance of the resulting estimators of the parameters have been considered in terms of Bias and mean square error (MSE). The Bias and
MSE are defined by

$$\text{Bais} = \frac{1}{1000} \sum_{i=1}^{1000} \hat{\theta}_i - \theta_0,$$

$$\text{MSE} = \frac{1}{1000} \sum_{i=1}^{1000} (\hat{\theta}_i - \theta_0)^2$$

The following algorithm is considered to conduct the simulation study, to evaluate the asymptotic variances and covariance of each estimator and also to derive confidence bounds.

**Step1.** A random sample of size $n = 10, 20, 30, 40, 50, 100, 150, 200$ each with 1000 replications were generated from the EE-BXII distribution. The true parameter values were selected as $a = 1.2$, $c = 0.6$, $k = 3$, $\beta = 0.2$.

**Step2.** The Newton Raphson method is used for solving Eqs (21)–(24) for $\alpha$, $c$, $k$, and $\beta$ respectively.

**Step3.** For each sample, the parameters of the distribution are estimated based on the complete sample data.

**Step4.** The Bias and MSE of estimators are calculated and tabulated.

**Step5.** For a large sample of size $n = 100, 200, 300$, we obtained asymptotic variances and covariances of each estimator.

**Step6.** Finally, the approximate confidence limits at 95% confidence level for each estimator is computed.

Table 3 presents the Bias and MSEs of the parameters $\alpha$, $c$, $k$, $\beta$. It has been noted that both the Bias and MSEs are decreased as the sample of size $n$ increases. Table 4 describes the

| $n$ | $\alpha$ | Bias | MSE | $c$ | Bias | MSE | $k$ | Bias | MSE | $\beta$ | Bias | MSE |
|-----|---------|------|-----|-----|------|-----|-----|------|-----|---------|------|-----|
| 10  | -0.2053 | 0.7879 |      | 15.363 | 23.024 | 4.9158 | 24.1648 | 0.6528 | 0.2244 |
| 20  | -0.2103 | 0.4422 |      | 0.4025 | 0.1654 | -1.1660 | 1.3596 | 0.4579 | 0.2098 |
| 30  | -0.3442 | 0.0285 |      | 0.1826 | 0.0233 | -1.2164 | 1.2795 | 0.0346 | 0.0198 |
| 40  | -0.3456 | 0.0212 |      | 0.1744 | 0.0218 | -1.3541 | 0.4161 | 0.0254 | 0.0128 |
| 50  | -0.5337 | 0.0202 |      | 0.1623 | 0.0205 | -2.1514 | 0.4086 | 0.0226 | 0.0124 |
| 100 | -0.5642 | 0.0196 |      | 0.1596 | 0.0194 | -3.2542 | 0.3962 | 0.0221 | 0.0120 |
| 150 | -1.0681 | 0.0123 |      | 0.1324 | 0.0162 | -4.5211 | 0.3054 | 0.0213 | 0.0115 |
| 200 | -1.0542 | 0.0112 |      | 0.1301 | 0.0155 | -4.8743 | 0.2981 | 0.0193 | 0.0102 |

Table 4. Confidence bounds of the estimators with a confidence level of 0.95.

| $n$ | Parameters | Estimated mean | Lower boundary | Upper boundary | Width |
|-----|------------|----------------|----------------|----------------|-------|
| 100 | $\alpha$  | 1.07894        | 1.03668        | 2.19456       | 1.15788 |
|     | $c$       | 1.21323        | 1.06162        | 1.40324       | 0.29783 |
|     | $k$       | 1.14898        | 1.10541        | 1.32792       | 0.26628 |
|     | $\beta$   | 1.21986        | 1.19593        | 1.32283       | 0.12690 |
| 200 | $\alpha$  | 0.991207       | 0.843477       | 1.82589       | 0.98241 |
|     | $c$       | 1.19913        | 1.47256        | 1.73885       | 0.26629 |
|     | $k$       | 1.13091        | 1.08682        | 1.34863       | 0.26181 |
|     | $\beta$   | 1.21807        | 1.17309        | 1.23985       | 0.06676 |
| 300 | $\alpha$  | 0.984851       | 0.100079       | 0.99987       | 0.89908 |
|     | $c$       | 1.18572        | 1.05752        | 1.32385       | 0.26533 |
|     | $k$       | 1.20332        | 1.15126        | 1.25792       | 0.10666 |
|     | $\beta$   | 1.21016        | 1.16734        | 1.23214       | 0.06482 |

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confidence limits for the parameters $\alpha$, $c$, $k$, $\beta$ and it is noted that the width of the confidence limits are decreases as the sample of size $n$ increases. From Table 5, it is evident that the variances of estimators are increasing with the increase in sample size expect a parameter $\beta$.

Applications to real data

We consider an application to the failure times for a particular windshield device obtained by Murthy et al. [26] for authentication of the flexibility, utility and potentiality of the EE-BXII model. We compare the EE-BXII distribution with models such as BXII, EP, SLL, Kw—GEP, Lomax—PII, McDonald Weibull (McW) [22], Beta Weibull (BW) [27] and transmuted Marshall-Olkin Fréchet (TMOFr) [28]. For the selection of the optimum distribution, we compute “Cramer-von Mises ($W^2$), Anderson Darling ($A^2$), Akaike information criterion (AIC), corrected Akaike information criterion (CAIC) and Hannan-Quinn information criterion (HQIC) for all competing and sub distributions. We compute the MLEs, their standard errors (in parentheses) and goodness of fit statistics (GOFs) values for the BXII, EP, SLL, Kw—GEP, Lomax—PII, McW, BW and TMOFr models.

Data set 1. Aircraft windshield data. The data set reflects the failure times of 84 Aircraft Windshield and been taken from [12]. The data set values are as follows.

| Data set 1. Aircraft windshield data. The data set reflects the failure times of 84 Aircraft Windshield and been taken from [12]. The data set values are as follows. |
|---|
| 0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779, 1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.82, 3, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663. |
Table 6. Estimates and SE (in parentheses) for aircraft windshield data.

| Distributions  | Estimates and Standard error (SE) |
|----------------|-----------------------------------|
| EE-Burr XII(α, c, k, β) | 0.483 (0.245) 10.812 (8.34) 0.947 (0.75) 246.626 (189.193) |
| BXII(α, k) | 3.171 (0.449) 0.352 (0.06) |
| EP(α, c, k) | 1.563 (0.345) 3.411 (1.128) 1.182 (0.349) |
| SLL(α) | 1.571 (0.137) |
| Kw—GEP(c, k, β) | 4.373 (0.519) 0.26 (0.085) 220.976 (191.998) |
| Lomax—PII(k) | 0.824 (0.0081) |
| McW (a, b, α, β, c) | 17.686 (6.222) 33.639 (19.994) 1.940 (1.011) 0.306 (0.045) 16.721 (9.622) |
| BW (a, b, α, β) | 34.180 (14.838) 11.496 (6.730) 1.360 (1.002) 0.298 (0.060) |
| TMOFr (α, β, σ, λ) | 200.747 (87.275) 1.952 (0.125) 0.102 (0.017) −0.869 (0.101) |

Table 7. Goodness of fit tests for aircraft windshield data.

| Distribution  | AIC  | CAIC | HQIC | A*   | W*   |
|---------------|------|------|------|------|------|
| EE-Burr XII   | 281.937 | 282.443 | 285.845 | 1.54655 | 0.19141 |
| BXII          | 342.854 | 343.003 | 344.809 | 4.51619 | 0.65524 |
| EP            | 324.777 | 325.077 | 327.709 | 4.25024 | 0.63884 |
| SLL           | 387.548 | 387.597 | 388.525 | 5.01924 | 0.53438 |
| Kw—GEP       | 283.12 | 33.639 (19.994) 1.940 (1.011) 0.306 (0.045) 16.721 (9.622) |
| Lomax—PII    | 406.442 | 406.491 | 407.419 | 21.27859 | 1.44182 |
| McW           | 387.548 | 387.597 | 388.525 | 5.01924 | 0.53438 |
| BW            | 34.180 (14.838) | 11.496 (6.730) | 1.360 (1.002) | 0.298 (0.060) |
| TMOFr         | 200.747 (87.275) | 1.952 (0.125) | 0.102 (0.017) | −0.869 (0.101) |

Fig 8 defines the Empirical and Theoretical pdf and cdf of EE-BXII using windshield data.

Fig 9 presents the Q-Q and PP plot of the proposed distribution using data set 1.

Data set 2. Lifetime of 50 components. The second data presents the lifetime of 50 components which have been cited by Silva et.al [19]. The data set values are as follows.
Table 8 defines the parameter estimates and their standard errors (SE) are enclosed in brackets. The dotted line implies that the corresponding distribution does not contain this parameter. Table 9 reflects the goodness of fit test and we see that the values of AIC, CAIC,
Table 9. Goodness of fit tests for the lifetime of 50 components.

| Distribution   | AIC    | CAIC   | HQIC   | A⁺    | W⁻    |
|----------------|--------|--------|--------|-------|-------|
| EE-Burr XII    | 514.2936 | 515.1632 | 517.2464 | 4.40189 | 0.7708399 |
| BXII           | 548.0896 | 548.3396 | 549.566  | 5.830071 | 1.093033  |
| EP             | 535.6001 | 535.8501 | 537.0766 | 5.688587 | 1.056053  |
| SLL            | 583.6239 | 583.7055 | 584.3621 | 4.878686 | 0.875671  |
| Kw—GEP        | 531.5318 | 532.4013 | 534.4846 | 5.295506 | 0.9659385 |
| Lomax—PII(k)  | 547.0878 | 547.1694 | 547.826  | 5.59071  | 1.034232  |

HQIC, A⁺, and W⁻ for the proposed model are fewer than the values of other distributions. Hence it is evident that the proposed model leads to a better fit using this data as compared to other probability distributions.

Fig 10 defines the Empirical and Theoretical pdf and cdf of EE-BXII using windshield data. Fig 11 presents the Q-Q and PP plot of the proposed distribution using data set 2.

**Conclusion**

The paper presents a novel probability distribution called EE-BXII distribution. It has been observed that the hazard rate function of the EE-BXII can model the real data with decreasing and increasing bathtub shapes. We derived certain mathematical and statistical properties such as quantiles, sub-models, moments, and reliability function. We address the maximum likelihood estimation of the EE-BXII parameters. We consider an application of the proposed density to aircraft windshield data and failure time of components to elucidate the flexibility, utility and potentiality as compared to other probability models. The numerical results show that the EE-BXII distribution is a best fitted model to these data sets as compared to other probability distributions.

The proposed distribution can be further improved by using the Transmutation techniques or some others. Furthermore, future researchers may study the estimation of parameters by other approaches; one can study the proposed distribution with a Bayesian approach.
Fig 11. Q-Q and PP plot of the EE-BXII distribution using data set2.

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Supporting information

S1 Table. The sub models of the EE-BXII.
(TIF)

S2 Table. The mean, SD, mode, median, skewness and kurtosis.
(TIF)

S3 Table. The Bias and MSE of the parameters $\theta = (\alpha, \gamma, k, \beta)$.
(TIF)

S4 Table. Confidence bounds of the estimators with a confidence level of 0.95.
(TIF)

S5 Table. Asymptotic variance and covariance of the estimated matrix.
(TIF)

S6 Table. Estimates and SE (in parentheses) for aircraft windshield data.
(TIF)
Author Contributions
Conceptualization: Majdah Badr.
Formal analysis: Majdah Badr.
Investigation: Majdah Badr.
Methodology: Majdah Badr.
Software: Majdah Badr.
Visualization: Muhammad Ijaz.
Writing – original draft: Majdah Badr, Muhammad Ijaz.
Writing – review & editing: Muhammad Ijaz.

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