NEW LOWER BOUNDS ON $\chi(\mathbb{R}^d)$ FOR 
$d = 8, \ldots, 12$

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1 Introduction

We study the infinite graph whose vertices correspond to points in $d$-dimensional Euclidean space and where two vertices $p, q$ are considered adjacent $p \sim q$ if $d(p, q) = 1$. We denote this graph by $\mathbb{R}^d$.

Recall that the chromatic number $\chi(G)$ of a graph $G$ is the smallest number of colors needed for a proper coloring, i.e. the smallest $k$ so that there exists a function $f : V(G) \rightarrow \{1, 2, \ldots, k\}$ such that whenever $p \sim q$, we have $f(p) \neq f(q)$.

Determining the chromatic number of the plane $\chi(\mathbb{R}^2)$ is a well known problem in combinatorics. For several decades the bounds have held fast at

$$4 \leq \chi(\mathbb{R}^2) \leq 7.$$ 

See Soifer’s “The mathematical coloring book” [7] for an encyclopedic overview of this problem and its history.

The problem has also been studied in higher dimensions. See [7] and Kupavskii–Raigorodskii’s paper [4]. With the aid of extensive calculations in the free and open-source software Sage [8], we obtain new lower bounds for $\chi(\mathbb{R}^d)$ in dimensions $d = 8 \ldots 12$. We verified all of the computations in Maple 17.
Our main result is the following.

**Theorem 1.1.** We have that $\chi(\mathbb{R}^8) \geq 19$, $\chi(\mathbb{R}^9) \geq 21$, $\chi(\mathbb{R}^{10}) \geq 26$, $\chi(\mathbb{R}^{11}) \geq 32$, and $\chi(\mathbb{R}^{12}) \geq 32$.

As far as we are aware, the best previously published lower bounds were, respectively:

- $\chi(\mathbb{R}^8) \geq 16$ by Larman and Rogers in 1972 [5], and $\chi(\mathbb{R}^9) \geq 21$,
- $\chi(\mathbb{R}^{10}) \geq 23$, $\chi(\mathbb{R}^{11}) \geq 23$, and $\chi(\mathbb{R}^{12}) \geq 25$ by Kupavskii and Raigorodskii in 2009 [4].

The rest of the paper is organized as follows. We prove the bounds $\chi(\mathbb{R}^{10}) \geq 26$, $\chi(\mathbb{R}^{11}) \geq 32$, and $\chi(\mathbb{R}^{12}) \geq 32$ in Section 2; the bound $\chi(\mathbb{R}^9) \geq 21$ in Section 3; and the bound $\chi(\mathbb{R}^8) \geq 19$ in Section 4.

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## 2 Hypercube graphs

Graphs constructed on vertices of the $d$-dimensional cube $\{0, 1\}^d$ provide important examples in geometric graph theory. Frankl and Wilson’s proof that

$$\chi(\mathbb{R}^d) \geq \exp(Cd)$$

for some constant $C > 0$, for example, uses such graphs [2]. See also Kahn and Kalai’s subsequent counterexample to Borsuk’s conjecture [3].

Define the hypercube graph $C(d, u)$ to have vertices $V = \{0, 1\}^d$, with two vertices adjacent if their Hamming distance is $u$. Note that Hamming distance $u$ corresponds to Euclidean distance $\sqrt{u}$. So by dilating Euclidean space by a factor of $1/\sqrt{u}$, we see that $C(d, u)$ is a unit-distance graph in $\mathbb{R}^d$.

For $d$ odd $C(d, u)$ is bipartite, so for our purposes these are not very interesting. For $d$ even $C(d, u)$ has two isomorphic connected components — we denote one of these “half-cube” connected components on
For example $C(5, 2)$ is a graph on 32 vertices, regular of degree 10, and the half-cube $H(5, 2)$ is a graph on 16 vertices.

Recall that for a graph $G$ with $|V(G)|$ vertices and independence number $\alpha(G)$, we have that

$$\chi(G) \geq \frac{|V(G)|}{\alpha(G)}.$$

One checks that the independence number $\alpha(H(5, 2)) = 2$. Then the independence-number bound gives that $\chi(H) \geq 8$, and since $H$ is a unit-distance graph in $\mathbb{R}^5$, this also gives that $\chi(\mathbb{R}^5) \geq 8$. This half-cube example is well known; see for example [5].

In Figure 1 we summarize our results on chromatic numbers of hypercube graphs for small $d$ and $u$. In some cases we were able to compute $\chi(C(d, u))$ exactly. In some other cases we were not, but we were still able to compute the independence number $\alpha(C(d, u))$, giving a lower bound.

We obtain new lower bounds on $\chi(\mathbb{R}^d)$ for $d = 10$ and $d = 11$ this way, in both cases with Hamming distance $u = 4$. According to a calculation in Sage, the halfcube $H(10, 4)$ has independence number $\alpha(H(10, 4)) = 20$. There are 512 vertices, so then

$$\chi(H(10, 4)) \geq \frac{512}{20} = 25.6,$$

and since the chromatic number is an integer,

$$\chi(\mathbb{R}^{10}) \geq 26.$$

According to another calculation in Sage, the half-cube $H(11, 4)$ has independence number $\alpha(H(10, 4)) = 32$. There are 1024 vertices, and then we have

$$\chi(\mathbb{R}^{11}) \geq \frac{1024}{32} = 32.$$

This last example also gives a new record lower bound $\chi(\mathbb{R}^{12}) \geq 32$.

Since Hamming distance 4 corresponds to Euclidean distance 2, these graphs still have rational coordinates when rescaled to unit-distance graphs.
Figure 1: Chromatic numbers of some hypercube graphs $\chi(C(d,u))$ for some small $d$ and $u$. We restrict to $u$ even, since $C(d,u)$ is bipartite for $u$ odd. In some cases we compute chromatic number exactly, and have an independence-number lower bound. For $(d,u) = (10,2)$ and $(11,2)$ the lower bound is by monotonicity of rows.

### 3 Hyperplane slices of hypercube graphs

Let $C(d,u,s)$ denote the intersection of $C(d,u)$ with the hyperplane at height $s$

$$x_1 + \cdots + x_d = s.$$  

Clearly, $C(d,u,s)$ is a unit distance graph in $\mathbb{R}^{d-1}$, so

$$\chi(R^{d-1}) \geq \chi(C(d,u,s)).$$

For example, $C(10,4)$ has $2^{10} = 1024$ vertices and is regular of degree $\binom{10}{4}$, where the subgraph $C(10,4,5)$ has $\binom{10}{5} = 252$ vertices and is regular of degree 100.

We found with a computation in Sage that

$$\alpha(C(10,4,5)) = 12,$$

so

$$\chi(R^9) \geq \frac{252}{12} = 21.$$  

This gives an alternate proof for the currently best known bound of Kupavskii and Raigorodskii [4].
4 A unit-distance graph in \( \mathbb{R}^8 \)

Let \( G_0 \) be a graph whose vertices are the 240 shortest vectors in the \( E8 \) lattice. These may also be described as the vertices of an 8-dimensional Gosset polytope.

Concretely, there are 112 vertices with integer entries obtained from arbitrary permutations of the vectors

\[(\pm 2, \pm 2, 0, 0, 0, 0, 0),\]

and 128 vertices with integer entries obtained from all vectors

\[(\pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1)\]

with an even number of minus signs.

Adjacency is with respect to Euclidean distance 4. So for example, \((2, 2, 0, 0, 0, 0, 0, 0)\) is adjacent to \((0, 0, 2, 2, 0, 0, 0, 0)\).

The graph \( G_0 \) lives on a 7-dimensional sphere of radius \( 2\sqrt{2} \). A pair of vertices on this sphere is adjacent if and only if the corresponding vectors are orthogonal.

Now define \( P \) to be the set of integer points within distance 4 of the origin.

Our algorithm is as follows.

1. Initialize \( G_0 \) as above.
2. Choose a vertex \( x \in P \) outside of the current graph \( G_i \).
3. If the independence number remains unchanged on adding vertex \( x \) to the graph \( G_i \) then add \( X \), i.e. if \( \alpha(G_i + x) = \alpha(G_i) \), then set \( V(G_{i+1}) := V(G_i) + x \).
4. If there are any more points in \( P \) outside of the graph \( V(G_i) \), go to step (2).

After experimentation, we found that at least 49 points (Figure 2) could be added without increasing the independence number, which then gives a unit-distance graph \( G \) with

\[\chi(G) \geq \frac{289}{16} = 18.0625,\]
Figure 2: List of 49 additional points added to the vertices of the 8-dimensional Gosset polytope to obtain a unit-distance graph with chromatic number at least 19.
so $\chi(\mathbb{R}^8) \geq 19$. Our graph $G$ can be rescaled to have rational coordinates and unit distance, so this also gives the new lower bound $\chi(\mathbb{Q}^8) \geq 19$.

Mann used extensive computer-aided calculations to hunt for unit-distance graphs with large chromatic number in 2003 — he established the lower bounds $\chi(\mathbb{Q}^6) \geq 10$, $\chi(\mathbb{Q}^7) \geq 13$, and $\chi(\mathbb{Q}^8) \geq 16$.

Cibulka studied similar constructions in 2008. He also added points to Gosset polytopes one at a time, and used computer-aided calculations to establish the bounds $\chi(\mathbb{Q}^5) \geq 8$ and $\chi(\mathbb{Q}^7) \geq 15$.

Unit-distance graphs built on Gosset polytopes go back at least to the landmark paper of Larman and Rogers in 1972 — they in turn thanked McMullen for suggesting the idea. Larman and Rogers exhibited a configuration of 64 points, a “spindle” over the 7-dimensional Gosset polytope, to establish the previous record lower bound $\chi(\mathbb{R}^8) \geq 16$.

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