Equilibrium and dynamics of porous and cracked media

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Abstract. We established equilibrium and motion equations for media with internal structure, characterized by integral geometry parameters. These equations are linear differential equations of the infinite order. Besides usual elastic waves, there are many waves with sufficiently lower velocities without bottom limit. Corresponding dispersion equations have both real and imaginary roots. Complex roots represent parametric resonances (catastrophes). This model of structured continuum describes intermediate states between statics and dynamics. It means that the equilibrium equations are valid for large scales, while the dynamic equations are valid in small ones.

1. Construction of the operator for continuous image of the real body

The classic continuous model blindly supposes that close points have close physical properties like stresses, strains electrical resistivity. Each point equally represents real classic continuum. This hypothesis contradicts real micro-inhomogeneous media, where physical properties of skeleton and fluid distinguish in orders of magnitude. Thus, juxtaposition of skeleton and fluid points does not mean similarity of their physical properties. The solution of this contradiction is to build a new continuum model with a structure described by the parameters of integral geometry.

The main idea of constructing a model is not to postulate the continuity of a real medium, but to construct a continuous image of a real microstructure of finite dimensions. There is known result of integral geometry that the specific surface of pores and cracks

\[ \sigma_0 l_0 = 4(1 - f), \]

where \( f \) is the porosity of rock sample. One-dimensional operator of field translation from point \( x \) into point \( x \pm l_0 \) given by this expression [3]:

\[ u(x \pm l_0) = u(x)e^{\pm l_0 D_x}; D_x = \frac{\partial}{\partial x}; P(D_x; l_0) = e^{\pm l_0 D_x}. \tag{1} \]

In (1) \( P(D_x; l_0) \) is one-dimensional operator of field translation. Using these operators, we construct continuous image of real body, saturated by field in any point, including pores and cracks. The representative of media is some finite volume, not a point. In three-dimensional case due to averaging over spherical angles \( \theta, \varphi \) the translating operator takes a form [2]:
\[ P(D_x, D_y, D_z; l_0) = \frac{1}{4\pi} \int_0^\infty \int_0^\pi \exp \left[ l_0 (D_x \sin \theta \cos \phi + D_y \sin \theta \sin \phi + D_z \cos \theta) \right] \sin \theta d\theta d\phi \]

\[ = \frac{\sinh(l_0 \sqrt{\Delta})}{l_0 \sqrt{\Delta}} = E + \frac{l_0^2}{3!} \Delta + \frac{l_0^4}{5!} \Delta \Delta + \ldots, \]  

(2)

here \( E \) is the unit operator and \( \Delta \) is the Laplace operator. Hypotheses Cauchy and Poisson have simple mathematical expression \( P = E \), i.e., the nature itself creates continuous body from real rocks. In case of finite sizes microstructures, the equation of motion expresses not real fast changing forces but their continuous image, given by operator \( P(D_x, D_y, D_z; l_0) \), i.e.:

\[ \frac{\partial}{\partial x_k} [P(\sigma_{ik})] = \rho \ddot{u}_i = \frac{\partial}{\partial x_k} \left[ \left( E + \frac{l_0^2}{3!} \Delta + \frac{l_0^4}{5!} \Delta \Delta + \ldots \right) (\sigma_{ik}) \right]. \]  

(3)

For one-dimensional stationary vibrations, instead of (3), we have more simple equation:

\[ u_{xx} \left( E + \frac{l_0^2}{3!} \Delta + \frac{l_0^4}{5!} \Delta \Delta + \ldots \right) + k_x^2 u = 0. \]

Here \( k_x^2 \) is a square of ordinary wave number for longitudinal or shear waves. Searching the solution in form \( u = A \exp(i k x) \) we obtain the dispersion equation for unknown wavenumber \( k \):

\[ \frac{\sin (k l_0)}{k l_0} = \frac{k_x^2}{k^2}. \]  

(4)

If \( l_0 \to 0 \), then \( k \to k_x \). At fixed frequency it means, that we have ordinary velocities of longitudinal and shear waves. At finite \( l_0 \), these velocities are decreasing down to zero, and when \( k l_0 > \pi \), the roots of equation (4) are complex, corresponding to unstable solutions [1]. Numerical investigation of real roots of (4) shows that the group velocity grows with increase in specific surface, and the ratio of shear velocity to longitudinal one rises too. It means that the Poisson ratio is decreasing up to negative values. Evidently, negative values of Poisson ratio for oil-gas collectors appear due to dispersion effects [4]. The complex roots correspond to the energy distribution of the Gutenberg-Richter seismological law, since the specific surface of the cracks is proportional to the deficit of the potential energy of the medium. The slope angles of these graphs practically coincide [5].

2. The random sizes of microstructures

If distances between cracks (pores) are random values, then \( l = l_0 \xi \), where \( \xi \) is a random value with gamma distribution and \( l_0 \) is constant, the \( P \) operator takes a form

\[ < P(D_x, D_y, D_z; \alpha, l_0 \xi) >= \frac{1}{4\pi} \int\int_0^\infty \int_0^\infty \mathcal{L} \{\exp(-\alpha \xi)\} \exp[l_0 \xi (D_i n_i)] d\xi. \]  

(5)

In (5) \( \alpha = \frac{1}{\sigma^2} \) is the inverse dispersion of random distances between crack. The integration gives an expression for imaging operator:

\[ P(D_x, D_y, D_z; \alpha, l_0) = \frac{\alpha}{\alpha - 1} \left( 1 - \sqrt{\Delta} \frac{l_0}{\alpha} \right)^{1-\alpha} - \left( 1 + \sqrt{\Delta} \frac{l_0}{\alpha} \right)^{1-\alpha}. \]  

(6)

At \( \alpha \to \infty \) an expression (6) takes a form (2). At \( \alpha \to 1 \) an image operator tends to the limit:

\[ P(D_x, D_y, D_z; \alpha, l_0) \to \frac{\arctan(l_0 \sqrt{\Delta})}{l_0 \sqrt{\Delta}} = E - \frac{l_0^2 \Delta}{3} + \frac{l_0^4 \Delta \Delta}{5} + \ldots l_0 \sqrt{\Delta} < \alpha. \]
The roots of (6) are real, and there are no catastrophes.

3. The intermediate states between statics and dynamics

The difference of operators $P - E$ is zero for continuous media, and the chain of Laplace operators for micro-inhomogeneous media. Therefore, an equilibrium state takes place on a large scale, but on a small scale it is disturbed. Simplest equation for such intermediate is:

$$P \left( \frac{\partial \sigma_{ik}}{\partial x_k} \right) = (P - E) \rho \ddot{u}_i. \quad (7)$$

The discrete set of dispersion equations roots means that the wave fields in porous and cracked media have quantum character. Every root on the complex plane is a state. Transition from one state to another requires a finite part of energy. For small distances dispersion between pores and cracks (large values of $\alpha$ (figure 1)) there are area with no roots (classic equilibrium at $k l_0 < \pi$). Then, there is area of real roots (blue color), corresponding to usual stationary vibrations. In complex roots area from the equality of vertical ($\gamma = k l_0$) and horizontal ($x$) coordinates we obtain that the velocities of nonstationary waves are close to the velocities of usual shear waves.

![Figure 1. Real (axis X) and imaginary (axis Y) parts of dispersion equation complex roots that correspond to equation (7). The large value of $\alpha = 300$ corresponds to small dispersion of distances between cracks (pores). Wave velocity is equal to ratio of horizontal coordinate X (for any discrete point) to vertical. Their equality means ordinary shear wave velocity. Red and blue colors correspond to catastrophes and damping solutions. Catastrophes are moving with velocities, closed to Rayleigh waves.](image-url)
Figure 2. Notations the same like on the figure 1. Nevertheless a small value $\alpha = 5$ corresponds to sufficiently large dispersion. Numbers on the vertical axe less about two orders than in the case of small dispersion. It means, that the catastrophes velocity in 85 times less than shear wave velocity.

In the case of large dispersions (figure 2) the vertical coordinate is almost two orders than analogous coordinate on the figure 1. It means that with large dispersions catastrophes move approximately 85 times slower than shear waves. Decreasing $\alpha = 1$ causes degeneration of gamma distribution into exponential distribution and eliminates the catastrophes.

4. Singularities in the gamma distribution. Singing sands
In the case $\alpha < 1$, the gamma distribution has the integrating singularity. Very large dispersions, such as $(\frac{1}{\alpha} - 1)k l_0 \gg 1$, give us another representation of image operator:

$$P(D_x, D_y, D_z; \alpha, l_0) \approx E + \frac{\exp(2\pi i m \alpha) - 1}{2} = \frac{\exp(2\pi i m \alpha) + 1}{2}. \quad (8)$$

If $\alpha$ is rational fracture the set of integer numbers $m$ is finite, but if $\alpha$ is irrational number the set of $m$ is infinite. As a result, for large numbers $x$ and small $\alpha$ corresponding dispersion equation takes the form of attractor:

$$k^2 = \frac{\exp(2\pi i m \alpha) - 1}{\exp(2\pi i m \alpha) + 1} k_P^2. \quad (9)$$

At $\alpha \to 0$ one solution of (9) ($k = 0$) is usual state of equilibrium. Nevertheless, at any sufficiently small $\alpha$ there are many or infinite number of complex solutions, corresponding to catastrophes of different scales. The modulus of expression (8) tends to unite, but the operator does not tend to unite. The dispersion equation roots are located on the unite circle. All of them are complex except one, corresponding to stable state.

5. Experiment
In experiment, we used tube with sand. Average diameter of particles was $1 \pm 2$ mm. The pressure about 50 MPa is applied for 10s. Experiment reviled 125 acts of seismic emission, much larger than unite [6]. It means that we had the case of many scenarios. The characteristic time of pressing (10 s) and particle destroy frequency (1 MHz) do not correspond to characteristic frequencies of seismic emission (30÷1000 Hz). Therefore, emissions relate to collective particles acts. After experiment we found ultra-small particles that verify the large particle sizes dispersion [6].
Figure 3. The common view of micro-seismic vibrations in pressed sand. Time of observation is 10s.

6. Conclusion
The micro-inhomogeneous media have many degrees of freedom (theoretically infinite). Thus, the equations of motion for structured media are differential equations of the infinite order. Stable solutions, in addition to ordinary waves, describe many slow waves, not limited below in speed, as well as catastrophes of different scales. Unstable solutions describe catastrophes and damping processes. At very small size of microstructure these equations degenerates to usual elastic equation of motion.

Small dispersion of micro-structures sizes stabilizes structured bodies. Catastrophes move with small velocities and decreasing amplitudes. Large dispersion destabilizes structured bodies again, resulting in unpredictable number of scenarios and details of catastrophes.

We obtained equation of intermediate states between statics and dynamics, when equilibrium in a large scale accompany the dynamic processes in smaller scales. These states generate micro-seismic emission with many elementary acts. In macro-scale it creates sound vibration in wide range of frequencies (30–1000 Hz).

Acknowledgments
The authors gratefully acknowledge PETROBRAS/CENPES for the financial support of the GEOMEC research and educational project.

References
[1] Sibiryakov B P and Sibiryakov E B 2016 J. Min. Sci. 52 1090–9
[2] Sibiryakov B P, Prilous B I and Kopeykin A V 2013 Phys. Mesomech. 16 141–51
[3] Maslov V P 1973 Operator Theory (Moscow: Nauka)
[4] Gregory A R 1976 Geophysics 41 895–921
[5] Riznichenko Y V 1985 Seismology Challenges (Moscow: Nauka)
[6] Sibiryakov B P and Bobrov B A 2008 Phys. Mesomech. 11 80–4