Density perturbations in the brane-world

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In Randall-Sundrum-type brane-world cosmologies, density perturbations generate Weyl curvature in the bulk, which in turn backreacts on the brane via stress-energy perturbations. On large scales, the perturbation equations contain a closed system on the brane, which may be solved without solving for the bulk perturbations. Bulk effects produce a non-adiabatic mode, even when the matter perturbations are adiabatic, and alter the background dynamics. As a consequence, the standard evolution of large-scale fluctuations in general relativity is modified. The metric perturbation on large-scales is not constant during high-energy inflation. It is constant during the radiation era, except at most during the very beginning, if the energy is high enough.

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I. INTRODUCTION

According to string and M-theory, gravity is a higher-dimensional theory, reducing to Einstein’s four-dimensional theory of general relativity at low enough energies. In the brane-world scenario, the standard model matter fields are confined to a 3-brane in 1 + 3 + d dimensions, while the gravitational field can propagate in the bulk, i.e., also in the d extra dimensions, being localized at the brane at low energies. Recent developments show that the d extra space dimensions need not be small, or even compact, thus allowing the intriguing possibility that corrections could occur even at TeV scales.

These exciting theoretical developments may offer a promising route towards a quantum gravity theory. However, as well as theoretical elegance, they must also pass the increasingly stringent tests provided by cosmological observations. Primarily, this involves developing higher-dimensional perturbation theory and then applying it to analyze the generation and evolution of density and tensor perturbations on the brane, leading to a prediction of the CMB (cosmic microwave background) anisotropies and galaxy distribution.

This is an ambitious and difficult program, but initial steps have already been taken, at least in the case of a particular class of models that generalize the Randall-Sundrum models \[4\]. Large-scale adiabatic density perturbations from inflation on the brane have been computed \[5\] (see also \[6\]), using the conservation of the curvature perturbation on uniform-density hypersurfaces. This conservation follows from adiabaticity and the conservation of energy-momentum on the brane, and is independent of the form of the field equations \[4\]. In \[6\], the backreaction effect of metric fluctuations in the fifth dimension was neglected. In the general case, i.e., incorporating also the fluctuations in the nonlocal quantities that carry the bulk influence onto the brane, it has been shown that large-scale scalar perturbations contain a closed system on the brane— and thus can in principle be evaluated purely from initial conditions on the brane, without knowledge of bulk dynamics \[5\]. In this paper, we solve the closed system given in \[5\] to find the evolution of large-scale density perturbations on the brane.

A general perturbation formalism has been developed \[4,5\], encompassing equations on the brane and in the bulk, and in principle able to describe all scales. However, the general equations are extremely complicated. A first application of the equations has been made to large-scale tensor perturbations from inflation on the brane \[8\]. Unlike the scalar case, large-scale tensor perturbations cannot be evaluated without the bulk perturbation equations. We develop the outline argument presented first in \[5\], and analyze large-scale density perturbations and their evolution, from after Hubble-crossing in inflation through the radiation era. This provides the basis for predicting the large-angle scalar anisotropies generated in CMB temperature, and seeing how the bulk effects modify general relativistic predictions. We show that in general, the perturbation \(\Phi\) (a covariant analog of the Bardeen metric perturbation) is no longer constant during high-energy inflation, but grows. However, \(\Phi\) is constant during the radiation era, as in general relativity, except at most in the early radiation era, if the energy density is still high relative to the brane tension. This means that the Sachs-Wolfe plateau in the CMB anisotropies is likely to be preserved \[9\], but the limits which COBE measurements place on inflationary potentials will be changed, and could also

\[\text{The Sachs-Wolfe formula has not yet been calculated in brane-world cosmologies, but we expect that the corrections to the general relativistic result will be very small, since the energy scale at last scattering is much less than the brane tension.}\]
become sensitive to the form of the potential.

II. BRANE DYNAMICS

We follow the approach and notation of [5]. The 5-dimensional (bulk) field equations are

\[ \bar{G}_{AB} = \kappa^2 \left[ -\Lambda \bar{g}_{AB} + \delta(\chi) \{ -\lambda g_{AB} + T_{AB} \} \right], \]

where tildes denote the bulk generalization of standard general relativity quantities, and \( \kappa^2 = 8\pi/M_p^2 \), where \( M_p \) is the fundamental 5-dimensional Planck mass, which is typically much less than the effective Planck mass on the brane, \( M_p = 1.2 \times 10^{19} \) GeV. The brane is given by \( \epsilon = 0 \), so that a natural choice of coordinates is \( x^\alpha = (x^\mu, \chi) \), where \( x^\mu = (t, x^i) \) are spacetime coordinates on the brane. The brane tension is \( \lambda \), and \( g_{AB} = \bar{g}_{AB} - n_A n_B \) is the induced metric on the brane, with \( n_A \) the spacelike-like unit normal to the brane. Standard-model matter fields confined to the brane make up the brane energy-momentum tensor \( T_{AB} \) (with \( T_{AB} \mathcal{R}^B = 0 \)). The bulk cosmological constant \( \Lambda \) is negative, and is the only 5-dimensional (bulk) field equations are

\[ E_{\mu\nu} = -\frac{6}{\kappa^2 \Lambda} \mathcal{U} \left( u_\mu u_\nu + \frac{1}{3} h_{\mu\nu} \right) + \mathcal{P}_{\mu\nu} + \mathcal{Q}_\mu u_\nu + \mathcal{Q}_\nu u_\mu, \]

where \( h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu \) projects into the comoving rest-space. Here

\[ \mathcal{U} = -\frac{1}{6} \kappa^2 \Lambda \mathcal{E}_{\mu\nu} u^\mu u^\nu \]

is an effective nonlocal energy density on the brane (which need not be positive), arising from the free gravitational field in the bulk. It carries Coulomb-type effects from the bulk onto the brane. There is an effective nonlocal anisotropic stress

\[ \mathcal{P}_{\mu\nu} = -\frac{1}{6} \kappa^2 \Lambda \left[ h_{\mu}^{\alpha} h_{\nu}^{\beta} - \frac{1}{3} h^{\alpha\beta} h_{\mu\nu} \right] \mathcal{E}_{\alpha\beta} \]

on the brane, which carries Coulomb, gravito-magnetic and gravitational wave effects of the free gravitational field in the bulk. The effective nonlocal energy flux on the brane,

\[ \mathcal{Q}_\mu = \frac{1}{6} \kappa^2 \Lambda h_{\mu}^{\alpha} \mathcal{E}_{\alpha\beta} u^\beta, \]

carries Coulomb and gravito-magnetic effects from the free gravitational field in the bulk. (Note that there is no energy flux in the bulk, and thus no transfer of energy between bulk and brane; this situation changes if bulk scalar fields are present [6].)

III. LOCAL AND NONLOCAL CONSERVATION EQUATIONS

The local and nonlocal bulk modifications may be consolidated into an effective total energy-momentum tensor:

\[ G_{\mu\nu} = -\Lambda g_{\mu\nu} + \kappa^2 T_{\mu\nu}^{\text{tot}}, \]

where

\[ \mathcal{E}_{AB} = \bar{C}_{ACBD} n^C n^D, \]

which is symmetric and traceless and without components orthogonal to the brane, so that \( \mathcal{E}_{AB} \mathcal{R}^B = 0 \) and \( \mathcal{E}_{AB} \rightarrow \mathcal{E}_{\mu\nu} g_A \mu g_B \nu \) as \( \chi \rightarrow 0 \).
\[ T_{\mu\nu}^{\text{tot}} = T_{\mu\nu} + \frac{6}{\lambda} S_{\mu\nu} - \frac{1}{\kappa^2} \xi_{\mu\nu}. \]  

The effective total energy density, pressure, anisotropic stress and energy flux are

\[ \rho^{\text{tot}} = \rho \left(1 + \frac{\rho}{2\lambda}\right) + \frac{6\dot{U}}{\kappa^2\lambda}, \]

\[ P^{\text{tot}} = p + \frac{\rho}{2\lambda} (\rho + 2p) + \frac{2\dot{U}}{\kappa^2\lambda}, \]

\[ \pi_{\mu\nu}^{\text{tot}} = \frac{6}{\kappa^4\lambda} \mathcal{P}_{\mu\nu}, \]

\[ q_{\mu}^{\text{tot}} = \frac{6}{\kappa^4\lambda} Q_{\mu}. \]

The brane energy-momentum tensor separately satisfies the conservation equations, \( \nabla^\mu T_{\mu\nu} = 0 \), giving

\[ \dot{\rho} + \Theta (\rho + p) = 0, \]

\[ D_{\mu} \rho + (\rho + p) A_{\mu} = 0, \]

where a dot denotes \( u^\mu \nabla_\mu, \Theta = D^\mu u_\mu \) is the volume expansion rate of the \( u^\mu \) congruence, \( A_\mu = \dot{u}_\mu \) is its 4-acceleration, and \( D_\mu \) is the projected covariant spatial derivative. The Bianchi identities on the brane imply that the projected Weyl tensor obeys the constraint

\[ \nabla^\mu \xi_{\mu\nu} = \frac{6\kappa^2}{\lambda} \nabla^\mu S_{\mu\nu}. \]

This shows how nonlocal bulk effects are sourced by local bulk effects, which include spatial gradients and time derivatives: evolution and inhomogeneity in the matter fields can generate nonlocal gravitational effects in the bulk, which backreact on the brane. The brane energy-momentum tensor and the consolidated effective energy-momentum tensor are both conserved separately. Conservation of \( T_{\mu\nu}^{\text{tot}} \) gives, upon using Eqs. (1)–(14), propagation equations for the nonlocal energy density \( \mathcal{U} \) and energy flux \( Q_\mu \). In linearized form, these are

\[ \dot{\mathcal{U}} + \frac{4}{3} \dot{\Theta} \mathcal{U} + D^\mu Q_\mu = 0, \]

\[ \dot{Q}_\mu + 4H Q_\mu + \frac{1}{3} D_\mu \mathcal{U} + \frac{2}{3} H A_\mu + D^\nu \mathcal{P}_{\mu\nu} = -\frac{1}{2} \kappa^4 (\rho + p) D_\mu \rho, \]

where \( H = \dot{a}/a \) (\( = \frac{1}{4} \dot{\Theta} \)) is the Hubble rate in the background. The nonlocal tensor mode, which satisfies \( D^\nu \mathcal{P}_{\mu\nu} = 0 \neq \mathcal{P}_{\mu\nu}, \) does not enter the nonlocal conservation equations. Furthermore, there is no evolution equation at all for \( \mathcal{P}_{\mu\nu}, \) reflecting the fact that in general the equations do not close on the brane, and one needs bulk equations to determine brane dynamics. There are bulk degrees of freedom whose impact on the brane cannot be predicted by brane observers. The evolution of the nonlocal energy density and flux, which carry scalar and vector modes of the bulk gravitational field, is determined on the brane, while the evolution of the nonlocal anisotropic stress, which carries scalar, vector and tensor modes of the bulk field, is not.

The generalized Raychaudhuri equation on the brane in linearized form is

\[ \dot{\Theta} + \frac{1}{3} \Theta^2 - D^\mu A_\mu + \frac{\kappa^2}{2} (\rho + 3p) - \Lambda = -\frac{1}{2} \kappa^2 (2\rho + 3p) \rho - \frac{6\dot{U}}{\kappa^2\lambda}, \]

where the general relativistic case is recovered when the right-hand side is set to zero. In the background, this gives

\[ \dot{H} = -H^2 - \frac{\kappa^2}{6} \left[ \rho + 3p + \frac{\rho}{2\lambda} (2\rho + 3p) \right], \]

\[ + \frac{1}{3} \Lambda - \frac{2}{\kappa^2\lambda} \mathcal{U} \left( \frac{a_o}{a} \right)^4, \]

where the solution for \( \mathcal{U} \) follows from Eq. (15), \( a_o \) is the initial scale factor and \( \mathcal{U} = \mathcal{U}(a_o). \) The first integral of this equation is the generalized Friedmann equation on the brane:

\[ H^2 = \frac{\kappa^2}{3} \rho \left(1 + \frac{\rho}{2\lambda}\right) + \frac{1}{3} \Lambda - \frac{K}{a^2} + \frac{2}{\kappa^2\lambda} \mathcal{U} \left( \frac{a_o}{a} \right)^4, \]

where \( K = 0, \pm 1. \) Local bulk effects modify the background dynamics. In particular, inflation at high energies \( (\rho \gtrsim \lambda) \) proceeds at a higher rate than the corresponding rate in general relativity. This introduces important changes to the dynamics of the early universe [3][8][9], and accounts for an increase in the amplitude of scalar and tensor fluctuations at Hubble-crossing [3][8].

The condition for inflation becomes \[ 3 \]

\[ w < -\frac{1}{3} \left( \frac{2\rho + \lambda}{\rho + \lambda} \right), \]

where \( w = p/\rho. \) As \( \rho/\lambda \to \infty, \) we have \( w < -\frac{2}{3}; \) while the general relativity condition \( w < -\frac{1}{3} \) is recovered as \( \rho/\lambda \to 0. \)

If \( \mathcal{U} = 0, \) i.e., if the background bulk is conformally flat, then Eqs. (16) and (17) show that the effective equation of state index for the total energy-momentum tensor is

\[ w^{\text{tot}} = \frac{\rho^{\text{tot}}}{\mathcal{U}} \equiv \frac{w + (1 + 2w)\rho/2\lambda}{1 + \rho/2\lambda} \approx 1 + 2w, \]

where the last equality holds at very high energies \( (\rho \gg \lambda). \) Thus for slow-roll inflation, \( w^{\text{tot}} \) and \( w \) are both close to \( -1. \) The high-energy inflation condition \( w < -\frac{2}{3} \) is \( w^{\text{tot}} < -\frac{1}{3}. \) During high-energy reheating with \( w \approx 0 \) on average, we have \( w^{\text{tot}} \approx 1, \) so that the effective equation of state is stiff, while high-energy radiation-domination \( (w = \frac{1}{3}) \) has \( w^{\text{tot}} \approx \frac{3}{2}, \) i.e., an ultra-stiff effective equation of state. The effective sound speed at very high energies is also altered:

\[ (c_s^2)^{\text{tot}} = \frac{\mathcal{U}}{\rho^{\text{tot}}} \approx c_s^2 + w + 1, \]

where \( c_s^2 = \rho/\mathcal{U}. \)
IV. SCALAR PERTURBATIONS ON THE BRANE

Scalar perturbations are covariantly (as well as gauge-invariantly and locally) characterized as the case when all perturbed quantities are expressible as spatial gradients of scalars. In particular, the nonlocal perturbed bulk effects are described by $D_\nu U$ and $F_{\alpha\beta}$:

$$Q_\mu = D_\mu Q, \quad P_{\mu\nu} = \left[h_{\mu\alpha}h_{\nu}\beta - \frac{1}{2}h_{\alpha\beta}h_{\mu\nu}\right]D_\alpha D_\beta P.$$  \hfill (24)

Note that there is no transverse traceless mode from the bulk, since the nonlocal traceless mode has nonzero spatial divergence $F_{\alpha\beta}$:

$$D^\nu P_{\mu\nu} = \frac{3}{2}D^2(D_{\mu}P).$$

The bulk gravitational field affects scalar perturbations via scalar Coulomb modes, given by the spatial gradients of the ‘potentials’ $U$, $Q$ and $P$.

For adiabatic matter perturbations the 4-acceleration is

$$A_\mu = -\frac{c_s^2}{\rho(1+w)}D_\mu \rho.$$  \hfill (25)

The gradients

$$\Delta_\mu = \frac{a}{\rho}D_\mu \rho, \quad Z_\mu = aD_\rho \Theta,$$  \hfill (26)

describe inhomogeneities in the matter and expansion [4], and the dimensionless gradients describing inhomogeneity in the nonlocal quantities are $F_{\alpha\beta}$:

$$U_\mu = \frac{a}{\rho}D_\mu U, \quad Q_\mu = \frac{1}{\rho}D_\mu Q, \quad P_\mu = \frac{1}{a\rho}D_\mu P.$$  \hfill (27)

The spatial gradient of the conservation equations (13), (14) and (17), and the generalized Raychaudhuri equation (18), leads to a system of equations for these gradient quantities $F_{\alpha\beta}$. The gradients define scalars via their comoving divergences:

$$F \equiv aD^\alpha F_\alpha, \quad \text{with} \quad F = \Delta, Z, U, Q, P,$$  \hfill (28)

where $\Delta$ is a covariant analog of the Bardeen density perturbation $\epsilon_m$ (see [33]). Then the system of equations governing scalar perturbations on the brane follows from the gradient system given in [33] as

$$\Delta = 3wH\Delta - (1 + w)Z,$$  \hfill (29)

$$\dot{Z} = -2HZ - \left(\frac{c_s^2}{1 + w}\right)D^2 \Delta - \left(\frac{6\rho}{\kappa^2 \lambda}\right)U$$

$$- \frac{1}{2}k^2 \rho \left[1 + (4 + 3w)\frac{\rho}{\kappa^2 \lambda} - \frac{2c_s^2}{1 + w} \frac{6\rho}{\kappa^4 \lambda} \Delta\right],$$  \hfill (30)

$$\dot{U} = (3w - 1)HU - \left(\frac{4c_s^2}{1 + w}\right)\left(\frac{U}{\rho}\right)H \Delta$$

$$- \left(\frac{4U}{3\rho}\right)Z - aD^2Q,$$  \hfill (31)

$$\dot{Q} = (1 - 3w)HQ - \frac{1}{3a}U - \frac{2a}{3}\dot{a}D^2P$$

$$+ \frac{1}{6a}\left[\left(\frac{8c_s^2}{1 + w}\right)\frac{U}{\rho} - \kappa^2 \rho(1 + w)\right]H \Delta.$$  \hfill (32)

In general relativity, only the first two equations apply, with $\lambda^{-1}$ set to zero in Eq. (30). In this case we can decouple the density perturbations via a second-order equation for $\Delta$, whose independent solutions are adiabatic growing and decaying modes. Local bulk effects modify the background dynamics, while nonlocal bulk effects introduce new fluctuations. This leads to fundamental changes to the simple general relativity picture. There is no equation for $P$, so that in general, scalar perturbations on the brane cannot be predicted by brane observers without additional information from the unobservable bulk. Thus in general, one must solve also the scalar perturbations in the bulk in order to determine the perturbation evolution on the brane.

However, there is a crucially important exception to this, arising from the fact that $P$ only occurs in Eqs. (24)–(25) via the Laplacian term $D^2P$. We can use the shear propagation equation on the brane $F_{\alpha\beta}$ to provide an order-of-magnitude comparison of $P$ with $\Delta$:

$$\sigma_{\mu\nu} + 2H\sigma_{\mu\nu} + E_{\mu\nu} + D_\mu A_{\nu} = \frac{3}{\kappa^2 \lambda}P_{\mu\nu},$$

where $E_{\mu\nu}$ is the electric Weyl tensor on the brane. This equation, together with Eqs. (24) and (25) shows that

$$\frac{1}{\kappa^2 \lambda}|D^2P| \sim \frac{1}{\rho}|D^2\rho|,$$

and then Eqs. (26)–(28) imply

$$|P| \sim \frac{\kappa^2 \lambda}{a^2 \rho}|\Delta|.$$

Then

$$|aD^2P| \sim \frac{k^2}{a^2 H^2} \frac{\kappa^4 \rho}{a}|\Delta|,$$

on using the high-energy Friedmann equation ($H^2 \sim \kappa^2 \rho^2 / \lambda$). Thus, for $k \ll aH$, i.e. on large scales, well beyond the Hubble horizon, we can neglect the $D^2P$ term in Eq. (33) relative to the $\Delta$ term. Thus on large scales, the system closes on the brane, and brane observers can predict scalar perturbations from initial conditions intrinsic to the brane, without the need to solve the bulk perturbation equations. Note that the $D^2Q$ term in Eq. (31) may also be neglected relative to the $U$ term. This follows from the shear constraint equation $F_{\alpha\beta}$

$$D^\nu \sigma_{\mu\nu} - \frac{3}{2}D_\mu \Theta = -\frac{6}{\kappa^2 \lambda}Q_{\mu},$$

which gives, on taking the divergence,
\[ |Q| \sim \frac{\kappa^2 \lambda}{a \rho} |Z| \sim \frac{1}{a H |U|}, \]

where the last relation follows from Eq. (34). The system Eqs. (29)–(32) then reduces to 3 coupled equations in \( \Delta, Z \) and \( U \), plus a decoupled equation for \( Q \), which determines \( Q \) once the other 3 quantities are solved for. Thus there are in general 3 modes of large-scale density perturbations: a non-adiabatic mode is introduced by bulk effects. This mode is carried by fluctuations \( \Phi \) and Eq. (37) reduces to \( \Phi = C(\frac{\rho}{\lambda}) \). This is a covariant analog of the Bardeen metric potential \( \Phi_H \), and the covariant local curvature perturbation

\[ C = aD\mu C_\mu, \quad C_\mu = a^3D_\mu R, \]

where \( R \) is the Ricci curvature of the surfaces orthogonal to \( u^\mu \). (Note that these surfaces are in general shearing, and non-uniform in \( \rho, \Theta, U \) and \( R \).

Then the coupled system for density perturbations, Eqs. (29)–(31), can be rewritten on large scales as

\[ \dot{\Phi} = -H \left[ 1 + \frac{(1 + w)\kappa^2 \rho}{2H^2} \left( 1 + \frac{\rho}{\lambda} \right) \right] \Phi + \frac{(1 + w)\kappa^2 \rho}{4H} C - \frac{3(1 + w)\kappa^2 \rho^2}{\lambda H} U, \]

\[ \dot{C} = -\frac{72c_s^2 HU}{U \lambda^2 \rho^2} \Phi, \]

\[ \dot{U} = H \left[ 3w - 1 - \frac{4U}{3a^2 H^2} \right] U + \left( \frac{U}{3a^2 H^2} \right) C - \frac{2U}{3a^2 H^2} \left[ 1 + \frac{\rho}{\lambda} + \frac{6c_s^2H^2}{(1 + w)\kappa^2 \rho} \right] \Phi. \]

The general relativistic case is recovered when we set \( \lambda^{-1} = 0 \) and \( \dot{C} = 0 \); in this case, Eq. (38) falls away, and Eq. (37) reduces to

\[ C = C_\rho, \quad \dot{C}_\rho = 0, \]

which expresses conservation of the covariant curvature perturbation along each fundamental world-line. The value of \( C_\rho \) will in general vary from world-line to world-line, so that its conservation is local, and is not an indicator of purely adiabatic perturbations. (In general relative, \( C = 0 \) on large scales for a flat background even when there are non-adiabatic perturbations.) Bulk effects destroy the local conservation of \( C \) in general, by Eq. (37). Bulk effects destroy the local conservation of \( C \) in general, by Eq. (37).

However, there is an important special case when local conservation is regained: when the nonlocal energy density \( U \) vanishes in the background. This does not mean that fluctuations in the nonlocal energy density zero, i.e., we still have \( U \neq 0 \) in general. It can be argued that vanishing \( U \) in the background is more natural, if one believes that the bulk background should be conformally flat, and thus strictly anti-de Sitter. (Quantum effects may nucleate a black hole in the bulk [5], in which case the Schwarzschild-anti de Sitter bulk, with \( U_0 \) proportional to the black hole mass, would be a natural background.) From now on, we will assume a conformally flat background bulk and a spatially flat brane background; thus \( U_0 = 0 = K \) in the background generalized Friedmann equation (4). Equation (33) shows that the non-adiabatic total pressure perturbation is then proportional to \( (\rho/\lambda)U \), which will be enhanced at high energies and suppressed at low energies.

V. LARGE-SCALE SCALAR PERTURBATIONS

We rewrite the coupled system for large-scale perturbations by introducing two useful new quantities. We define, following [14],

\[ \Phi = \kappa^2 \rho a^2 \Delta, \]

where the last relation follows from Eq. (34). The system Eqs. (29)–(32) then reduces to 3 coupled equations in \( \Delta, Z \) and \( U \), plus a decoupled equation for \( Q \), which determines \( Q \) once the other 3 quantities are solved for. Thus there are in general 3 modes of large-scale density perturbations: a non-adiabatic mode is introduced by bulk effects. This mode is carried by fluctuations \( \Phi \) and Eq. (37) reduces to \( \Phi = C(\frac{\rho}{\lambda}) \). This is a covariant analog of the Bardeen metric potential \( \Phi_H \), and the covariant local curvature perturbation

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The general relativistic case is recovered when we set \( \lambda^{-1} = 0 \) and \( \dot{C} = 0 \); in this case, Eq. (38) falls away, and Eq. (37) reduces to

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When $\mathcal{U} = 0$ in the background, Eq. (39) holds, and Eq. (88) gives

$$U = U_0 f, \quad \dot{U}_0 = 0, \quad f = \exp \int_{a_o}^a (3w - 1) d \ln a.$$  \hspace{1cm} (40)

This shows that $U$ rapidly redshifts away during inflation, so that non-adiabatic effects from nonlocal bulk influence are small. By contrast, the modifications to the background dynamics from local bulk effects are strong during inflation at high energy.

The key equation (36) becomes

$$\frac{d\Phi}{dN} = \left[ 1 + \frac{(1 + w)\kappa^2 \rho}{2H^2} \right] \left[ \frac{(1 + w)\kappa^2 \rho}{4H^2} \right] C_o - \frac{3(1 + w)C_o}{\lambda H^2} e^{2N} fU_o.$$ \hspace{1cm} (41)

where $N = \ln(a/a_o)$ is the number of e-folds. We have thus reduced the coupled system to one simple inhomogeneous linear equation, which may be integrated along the fundamental world-lines. Along each world-line, the constancy of $C_o$ and $U_o$ allows us to track the change in $\Phi$ as $w$ changes, from inflationary behavior through to radiation- and matter-domination.

We can perform a qualitative analysis of the evolution of $\Phi$ as follows. For high-energy slow-roll inflation, $w$ and $\rho$ are nearly constant, and $V \approx \rho \gg \lambda$, where $V(\varphi)$ is the inflaton potential. Then Eq. (11) implies

\text{high-energy inflation: } \Phi \approx \frac{3}{2}(1 + w)C_o \frac{\lambda}{\rho}. \hspace{1cm} (42)

By contrast, the general relativity solution is

$$\Phi_{gr} \approx \frac{3}{4}(1 + w)C_o.$$ \hspace{1cm} (43)

In general relativity, $\Phi$ remains constant on large scales during slow-roll inflation, independent of the form of the inflaton potential. In the brane-world, $\Phi$ is \textit{slowly increasing during high-energy slow-roll inflation}, since $\Phi \sim \rho^{-1}$ and $\rho$ is slowly decreasing. This qualitative analysis is confirmed by the numerical integration of a simple phenomenological model shown in Fig. 1. For more realistic models, i.e., where $V(\varphi)$ is specified, the evolution of $\Phi$ may be more complicated than shown in Fig. 1.

During reheating, in periods when $w$ is approximately constant on average (for example, $w \approx 0$ for $V = \frac{1}{2}m^2\varphi^2$), Eqs. (13) and (11) imply

\text{high-energy $w \approx$ constant reheating: } \Phi \approx \frac{3(1 + w)}{2(7 + 6w)} \frac{\lambda}{\rho_0} C_o e^{3(1+w)N} + \text{const}. \hspace{1cm} (44)

Thus \textit{high-energy $w \approx$ constant reheating on the brane produces amplification of $\Phi$}, unlike general relativity, where $\Phi$ remains constant on large scales during $w \approx$ constant reheating:

$$\Phi_{gr} \approx \frac{3(1 + w)}{2(5 + 3w)} C_o.$$ \hspace{1cm} (45)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{The evolution of $\Phi$ along a fundamental world-line for a mode that is well beyond the Hubble horizon at $N = 0$, about 50 e-folds before inflation ends, and remains super-Hubble through the radiation era. We have modelled a smooth transition from inflation to radiation by $w = \frac{1}{2}(2 - \alpha) \tanh(N - 50) - (1 - \alpha)$, where $\alpha$ is a small positive parameter (chosen as $\alpha = 0.1$ in the plot). Labels on the curves indicate the value of $\rho_0/\lambda$, so that the general relativistic solution is the dashed curve ($\rho_0/\lambda = 0$). For $\rho_0/\lambda \gg 1$, Eq. (24) shows that inflation ends at $N = 50 - 2 \ln((1 - 2\alpha)/3) \approx 47.4$, and at $N = 50$ in general relativity. Only the lowest curve still has $\rho/\lambda \gg 1$ at the start of radiation-domination ($N$ greater than about 53), and one can see that $\Phi$ is still growing, as confirmed by Eq. (44).}
\end{figure}

In the radiation era, the energy density redshifts rapidly, so that $\rho$ quickly falls below the brane tension $\lambda$. If the energy density at the end of reheating is high enough, then at the start of radiation-domination we have $\rho \gg \lambda$, and we find that $\Phi$ is \textit{amplified during high-energy radiation domination}:

\text{high-energy radiation: } \Phi \approx \frac{2}{9} \frac{\lambda}{\rho_0} C_o e^{4N} + \text{const}. \hspace{1cm} (46)

At low energies on the brane, or in general relativity, we find that $\Phi$ is constant:

\text{low-energy radiation: } \Phi \approx \Phi_{gr} \approx \frac{1}{3} C_o. \hspace{1cm} (47)
This qualitative result is confirmed in Fig. 1. After the radiation era, the energy scale has fallen well below the brane tension, so that in the matter era, we recover the general relativity result:

\[
\phi_{\text{matter}}: \Phi \approx \Phi_{\text{gr}} \approx \frac{3}{10} C_\alpha. \tag{48}
\]

In general relativity, the constancy of \( \Phi \) during slow-roll inflation and radiation- and matter-domination allows one to estimate the amplification in \( \Phi \). CMB large-angle anisotropies as measured by COBE place limits on the inflationary potential, since the potential determines the initial value of \( \Phi \). This simple picture is complicated by high-energy effects in the brane-world. We have given a rough estimate for the slow-roll inflationary evolution in Eq. (42). In particular, \( \Phi \) grows during inflation, so that placing limits on inflationary parameters is more complicated. The evolution of \( \Phi \) is also sensitive to the form of the potential \( V(\varphi) \), although slow-roll conditions will reduce this sensitivity.

**VI. CONCLUSION**

Using the covariant local formalism developed in [3], we have analyzed the evolution of large-scale density perturbations on the brane. Density inhomogeneity on the brane generates Weyl curvature in the bulk, which in turn backreacts on the brane, in the form of a nonlocal energy-momentum tensor. Fluctuations in the nonlocal energy density induce a non-adiabatic mode in large-scale density perturbations. The fluctuations in the nonlocal energy flux are decoupled from the density perturbations, while the nonlocal anisotropic stress plays no role on large scales. This latter feature is what closes the system of brane density perturbation equations, allowing brane observers to evaluate the perturbations on the brane without solving for the bulk perturbations.

We showed that the local and nonlocal bulk effects arising during high-energy inflation, and any high-energy start to the radiation era, modify the simple picture of general relativity. The local covariant version of the metric perturbation, i.e., \( \Phi \), is no longer constant on large scales during these regimes. Computing the constraints on inflationary potentials that are imposed by CMB large-angle anisotropies is therefore more complicated, and more model-dependent. We gave a rough estimate for slow-roll inflation in Eq. (42); together with Eqs. (47) and (48), this indicates how COBE limits can be used to impose constraints on the inflationary potential. A more accurate determination of the observational constraints on brane-world inflation may require numerical integration for the specified form of \( V(\varphi) \), even for large scales. Numerical integration will also be required for the more complicated case, not investigated here, when the background bulk is not conformally flat, i.e., when the nonlocal energy density \( U \) does not vanish in the background. In this case, the coupled system of equations (47) - (48) can no longer be reduced to one equation, since \( C \) and \( U \) are no longer locally conserved.

The formalism we have used is restricted to large scales. When a mode approaches the Hubble radius, the gradient terms can no longer be neglected, and the presence of these terms means that the system of equations no longer closes on the brane. A fuller investigation requires a formalism that can handle all scales, and which necessarily involves the evolution of perturbations in the bulk. A covariant formalism for bulk perturbations has not been developed, but a metric-based formalism has been developed [4,5]. The equations of this formalism are very complicated, and considerable work remains to be done before smaller scale structure can be predicted and compared with observations of the acoustic peaks in the CMB anisotropies. Our results provide a useful initial step for further developments by showing what happens on very large scales.

Although it should be possible to reproduce the Sachs-Wolfe plateau and satisfy COBE limits by suitable restrictions on inflationary parameters, it remains to be seen whether this can be done consistently with the observed small-scale features of CMB anisotropies and with the observed matter distribution. This is the basis on which to confront brane-world theories with cosmological observations.

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