Nonlinear conditions for instability of the free surface of a conducting liquid in an external electric field in a confined axisymmetric geometry

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Abstract. The behavior of the free surface of a perfectly conducting liquid in an external uniform electric field is considered in the framework of the Hamiltonian formalism for the case of bounded axisymmetric geometry of the system (the fluid is bounded by a cylindrical rigid wall). Taking into account the influence of quadratic nonlinearities, we derive an amplitude equation which describes the evolution of the boundary. Using this equation, we find the condition for the hard excitation of boundary instability that leads to an explosive growth of surface perturbations. The differences in the description of the dynamics of axisymmetric perturbations of the boundary from the cases of plane, square, and hexagonal symmetries of the problem are discussed.

1. Introduction
It is well known that the free surface of a conducting liquid (liquid metal) exposed to a sufficiently strong electric field is unstable [1–4]. Interaction of the field with the induced surface charges leads to the formation of a single cone-shaped cusp or a system of such cusps [5–9]. The field enhancement at the cusp apex gives rise to intense field evaporation [10, 11] or field emission [12–15] processes. These phenomena are used in many applications, in particular, in the liquid-metal ion sources [16–18].

The character of instability development of a charged conducting liquid surface is determined by two main factors: the general laws of electrohydrodynamic processes and the system geometry. For an unbounded free surface, the dynamics of instability development is defined by a nonlinear interaction of three plane waves that form a hexagonal structure (their wave vectors are rotated by 2π/3 relative to each other) [19–21].

In the present work, we consider the behavior of a liquid surface in the bounded axisymmetric geometry of the system (this is schematically shown in figure 1) that occurs in applications related to emitters of liquid metal ions. In this case, the spatial scale of instability is determined by geometry of the system (its characteristic size) rather than by the dominant mode of electrohydrodynamic instability, defined by the capillary length, as in the case of the unlimited boundary of a fluid.
We derive an amplitude equation which describes the evolution of the boundary taking into account the influence of nonlinearities in the first nonvanishing order of perturbation theory. In terms of this equation, we demonstrate that even at subcritical electric fields, a disturbance of a sufficient magnitude can break the equilibrium of a planar surface. At that, the growth of surface perturbations has an explosive character. We formulate a nonlinear criterion of a hard excitation of the instability, i.e., a condition that permits, proceeding from some initial data such as the shape of the surface and the distribution of velocities, to answer the question of whether or not the initial perturbation will lead to the loss of the stability of a planar boundary and, at the developed stage of the instability, to the formation of cusp-like structure.

2. Initial equations; Hamiltonian formalism
At first, let us consider the potential motion of a perfectly conducting ideal liquid of infinite depth with an unbounded free surface \( z = \eta(x, y, t) \) in an external uniform electric field \( E \) directed along the \( z \) axis. The velocity potential \( \Phi \) and the electric field potential \( \varphi \) (we have \( \mathbf{v} = \nabla \Phi \) and \( \mathbf{E} = -\nabla \varphi \)) obey the Laplace equations,

\[
\begin{align*}
\nabla^2 \Phi &= 0, & z < \eta, \\
\nabla^2 \varphi &= 0, & z > \eta,
\end{align*}
\]

with the conditions at infinity

\[
\begin{align*}
\Phi &\to 0, & z \to -\infty, \\
\varphi &\to -EZ, & z \to \infty.
\end{align*}
\]
The dynamic and kinematic conditions on the free surface are
\[
\frac{\partial \Phi}{\partial t} + (\nabla \Phi)^2 = \frac{\varepsilon_0}{2\rho} \left[ (\nabla \varphi)^2 - E^2 \right] + \frac{\sigma}{\rho} \nabla \cdot \frac{\nabla \perp}{\sqrt{1 + (\nabla \perp \eta)^2}} - g \eta, \quad z = \eta,
\]
\[
\frac{\partial \eta}{\partial t} = \frac{\partial \Phi}{\partial z} - \nabla \perp \cdot \nabla \perp \sqrt{1 + (\nabla \perp \eta)^2} - g \eta, \quad z = \eta,
\]
and, since the surface of a conducting liquid is equipotential,
\[
\varphi = 0, \quad z = \eta.
\]
Here the operator \( \nabla \perp \) acts in the plane \( \{x, y\} \), \( g \) is the acceleration of gravity, \( \sigma \) is the surface tension, \( \rho \) is the mass density, and \( \varepsilon_0 \) is the electric constant. The terms on the right-hand side of the dynamic boundary condition (nonstationary Bernoulli equation at the liquid boundary) are responsible for electrostatic, capillary, and gravitational forces, respectively.

The equations for the surface motion can be written in the Hamiltonian form [19, 22, 23],
\[
\frac{\partial \eta}{\partial t} = -\frac{\delta H}{\delta \psi}, \quad \frac{\partial \psi}{\partial t} = \frac{\delta H}{\delta \eta},
\]
where the elevation \( \eta \) and the value of the velocity potential at the free surface \( \psi \equiv \Phi|_{z=\eta} \) are canonically conjugate quantities.

The Hamiltonian \( H \) coincides with the total energy of the system [24],
\[
H = H_{\text{kin}} + H_{\text{pot}}, \quad H_{\text{kin}} = \iint_{z<\eta} \frac{(\nabla \Phi)^2}{2} \, dx \, dy \, dz, \quad H_{\text{pot}} = -\iint_{z>\eta} \frac{\varepsilon_0}{2\rho} \left[ (\nabla \varphi)^2 - E^2 \right] \, dx \, dy \, dz + \iint \left[ \frac{g \eta^2}{2} + \frac{\sigma}{\rho} \left( \sqrt{1 + (\nabla \perp \eta)^2} - 1 \right) \right] \, dx \, dy.
\]
It is possible to express the Hamiltonian explicitly in terms of the canonical variables. Rewriting \( H \) in the form of a surface integral with the help of Green’s formulas and expanding the integrand in powers series of \( \eta \) and \( \psi \), we obtain (see [25] for details)
\[
H = H^{(2)} + H^{(3)}, \tag{1}
\]
where
\[
H^{(2)} = \iint \left[ \frac{\psi \hat{k} \psi}{2} - \frac{\varepsilon_0 E^2 \eta \hat{k} \eta}{2\rho} + \frac{g \eta^2}{2} + \frac{\sigma}{2\rho} (\nabla \perp \eta)^2 \right] \, dx \, dy, \sinh
\]
\[
H^{(3)} = \frac{\varepsilon_0 E^2}{2\rho} \iint \eta \left[ (\nabla \perp \eta)^2 - (\hat{k} \eta)^2 \right] \, dx \, dy.
\]
Here \( \hat{k} \) is the two-dimensional integral operator with a difference kernel whose Fourier transform is equal to the absolute value of the wave vector \( \hat{k} \exp(i \mathbf{k} r) = |\mathbf{k}| \exp(i \mathbf{k} r) \).

We derived the expression for the Hamiltonian (1) with allowance for quadratic nonlinear terms for the kinetic energy \( H_{\text{kin}} \) and cubic nonlinear terms for the potential energy \( H_{\text{pot}} \). This approximation can be used under the assumption that the surface-slope angles are small, i.e., \( |\nabla \perp \eta| \ll 1 \), and also under the condition that the applied electric field strength is close to the critical value for the instability development (see, for example, [25]). Note that for the cases of plane and square symmetries of the problem, three-wave interactions disappear, and there arises a need to take into account the fourth order terms of the Hamiltonian [26, 27]. Below we will consider the case of axisymmetric perturbations of the boundary, where it is sufficient to take into account cubic nonlinearities of the Hamiltonian (i.e., the above term \( H^{(3)} \)) in the first nonvanishing order.
3. Amplitude equation; stationary solutions

As it was mentioned above, for an unbounded free surface, the hexagonal structure of the surface perturbations is formed as a result of the interaction of three plane waves whose wave vectors make angles of $2\pi/3$ with each other [19, 21, 25]. Here we consider the behavior of the liquid surface in confined axisymmetric geometry of the system as it is shown schematically in figure 1. The fluid free surface is bounded by a rigid wall corresponding to the hollow electrode at the position

$$r \equiv \sqrt{x^2 + y^2} = R,$$

and, consequently, the protrusion on the free surface possesses an axial symmetry. Such a problem geometry is important in many applications, in particular, in the liquid-metal ion sources [11, 16, 17, 28].

It is clear that the real electrode geometry for liquid-metal ion sources (see, for instance, [7, 13]) is much more complicated than presented in figure 1, which makes it almost impossible to describe analytically the fluid motion in an electric field. To consider the effect of confined geometry on the fluid dynamics, it is necessary to introduce some simplifications. As an example, for plane problems it is possible to use periodic boundary conditions. In the considered case of axial symmetry, by analogy with these conditions, we take for the velocity and electric field potentials

$$\Phi_r|_{r=0} = \Phi_r|_{r=R} = 0,$$  
$$\varphi_r|_{r=0} = \varphi_r|_{r=R} = 0,$$

i.e., the fluid velocity and the field strength do not have radial components at the symmetry axis $r = 0$ and on the cylindrical surface $r = R$ (in the region above the liquid, it is shown by the dotted line in figure 1). The condition that the electric field is directed vertically above the electrode boundary $r = R$ is a rather rough idealization, however, it does not contradict the general picture of the field distribution in space. Figure 1 assumes the alignment of fluid and electrode levels at the boundary $r = R$ (otherwise, a singularity in the electric field distribution would occur) that, for real systems (see figures in [7, 13]), is driven by wetting effects. The use of condition (3) allows to find the electric field distribution in the region $r < R$ independently of its distribution at the periphery $r > R$ and, thereby, to significantly simplify the problem. In particular, it becomes possible to apply the Hamiltonian formalism described in the previous section.

For the axisymmetric flow of a fluid, in terms of the canonical functions $\eta(r, t)$ and $\psi(r, t)$, the additional conditions (2) and (3) reduce to

$$\psi_r|_{r=0} = \psi_r|_{r=R} = 0,$$  
$$\eta_r|_{r=0} = \eta_r|_{r=R} = 0,$$

Then, in the Hamiltonian (1), it is sufficient to integrate over the range $0 < r < R$, and its components $H^{(2)}$ and $H^{(3)}$ take the form

$$H^{(2)} = \pi \int_0^R \left[ \frac{\varepsilon_0 E^2}{\rho} \eta \dot{k} \eta + \frac{\sigma}{\rho} \eta^2 \right] r \, dr,$$  
$$H^{(3)} = \frac{\pi \varepsilon_0 E^2}{\rho} \int_0^R \eta \left[ \eta^2 - (\hat{k} \eta)^2 \right] r \, dr,$$

where we assume that the condition $R \ll \sqrt{\sigma/\rho g}$ is satisfied, and hence the gravitational forces can be neglected as compared to the capillary ones.

We will seek a solution of the motion equations in the form

$$\psi(r, t) = B(t) J_0(\mu_1 r), \quad \eta(r, t) = A(t) J_0(\mu_1 r),$$

where $B(t)$ and $A(t)$ are time-dependent functions.
where \( A \) and \( B \) are the corresponding amplitudes, \( J_0 \) is the zero-order Bessel function, \( \mu_1 = m_1/R \), and \( m_1 \approx 3.83 \) is the first zero of the first-order Bessel function \( J_1 \) (i.e., \( J_1(\mu_1 R) = 0 \)) that ensures the conditions (4) and (5). The possibility of using such a presentation for the Laplacian in the cylindrical geometry. Further we will use the following auxiliary formulas for the elevation amplitude, canonical functions \( \eta \) that ensures the conditions (4) and (5). The possibility of using such a presentation for the capillary forces),

\[
A > 0 \text{ accelerates the development of linear instability for surface } [2,3,25]. \text{ One can see from the structure of the right-hand side of (7) that the nonlinearity}
\]

\[
\text{Excluding } B \text{ from these expressions, we obtain the key nonlinear ordinary differential equation for the elevation amplitude,}
\]

\[
\frac{d^2 A}{dt^2} = \frac{\mu_1^2}{\rho} \left( \varepsilon_0 E^2 - \mu_1 \sigma \right) A + \frac{3c_1 \mu_1^2 \varepsilon_0 E^2}{4\rho} A^2. \tag{7}
\]

It can be easily seen from equation (7) that the flat surface of the fluid (it corresponds to zero amplitude, \( A = 0 \)) is a stationary solution. It is stable when the coefficient of the linear term in the right-hand side is negative. If this coefficient is positive then the solution becomes unstable. From here it is possible to find a threshold value of the external field. The instability will develop if the electric field exceeds the value \( E_c \) (the destabilizing electrostatic forces will dominate over the capillary forces),

\[
E_c^2 = \frac{\mu_1 \sigma}{\varepsilon_0} = \frac{m_1 \sigma}{\varepsilon_0 R}.
\]

This value differs from the threshold value arising from the stability analysis of an unbounded flat surface [2,3,25]. One can see from the structure of the right-hand side of (7) that the nonlinearity accelerates the development of linear instability for \( A > 0 \), and retards it for \( A < 0 \). The first corresponds to a tendency to the formation of cusps on the surface observed in experiments [7,13] and numerical calculations [9,29].

In addition to the trivial stationary solution, equation (7) admits one more, nontrivial solution (see also the paper [30] where the plane symmetric solution has been obtained). We find it assuming that the right-hand side of (7) is equal to zero,

\[
A_c = \frac{4}{3c_1 \mu_1} \left( \frac{\mu_1 \sigma}{\varepsilon_0 E^2} - 1 \right) = \frac{4}{3c_1 \mu_1} \left( \frac{E_c^2}{E^2} - 1 \right). \tag{8}
\]
The boundary shape corresponding to this solution is described by the simple expression (it is depicted in figure 1):

$$\eta = A_c J_0(\mu_1 r) + O(A_c^2).$$  \hfill (9)

Obviously, $A_c < 0$ at $E > E_c$ and $A_c > 0$ at $E < E_c$. It is clear from general considerations that the solution is unstable in the first case (it adjoins with the stable trivial solution) and it is stable in the second case (the stability can be broken if the azimuthal perturbations of the free boundary are considered; see, for instance, [31, 32]).

4. Nonlinear instability criterion

The boundary instability of the conducting liquid can develop even at subcritical electric fields, $E < E_c$, when the flat surface of the liquid is stable in the linear approximation. In this case, the hard regime of instability onset occurs due to the nonlinearity effect.

Let us formulate a criterion for the development of the instability in this situation. Equation (7) can be rewritten in the form of Newton’s second law:

$$\frac{d^2 A}{dt^2} = -\frac{\partial P(A)}{\partial A},$$ \hfill (10)

$$P(A) = -\frac{\mu_1^2}{2\rho} (\varepsilon_0 E^2 - \mu_1 \sigma) A^2 - \frac{c_1 \mu_1^3 \varepsilon_0 E^2}{4\rho} A^3,$$ \hfill (11)

where $A$ can be considered as a coordinate of some “particle” and the function $P(A)$ plays the role of the potential.

The integral of motion for (10) is the total energy of the particle (the sum of its kinetic and potential energies)

$$U = \frac{1}{2} \left( \frac{dA}{dt} \right)^2 + P(A),$$

related to the Hamiltonian of the considered system by simple expression

$$U = \frac{\mu_1}{\varepsilon_0 \pi R^2} H.$$

Consider the most important case of subcritical field, $E < E_c$. For clarity, we rewrite the potential (11) in terms of the threshold field,

$$P(A) = \frac{\mu_1^2 \varepsilon_0 E^2}{2\rho} \left[ \left( \frac{E_c^2}{E^2} - 1 \right) A^2 - \frac{c_1 \mu_1}{2} A^3 \right].$$

Typical dependence of the potential $P$ on $A$ is shown in figure 2. It is seen that this dependence has two extrema. One extremum (minimum) corresponds to the trivial stationary solution $A = 0$. Near this point, the particle with relatively small energy $U$ is in the potential well; its motion is bounded. For the considered problem concerning the behavior of the conducting liquid in an electric field, this corresponds to the free surface oscillations around the equilibrium position (the plane $z = 0$).

The other extremum (maximum) corresponds to the nontrivial stationary solution $A_c$. The value of the potential at this point is equal to

$$P_{\text{max}} = P(A_c) = \frac{8 \varepsilon_0 E^2}{27 c_1^2 \rho} \left( \frac{E_c^2}{E^2} - 1 \right)^3.$$

As it is clear from figure 2, the motion of the “particle” is unbounded if its energy $U$ is sufficiently high, and the particle leaves the potential well near the point $A = 0$, passing over the potential barrier at the point $A = A_c$. This occurs when the condition $U > P_{\text{max}}$ is satisfied. It is clear
Figure 2. Dependence of the auxiliary potential $P$ upon the amplitude $A$ for $E = 0.9E_c$. This potential is normalized to its extremum value $P_c$, and the amplitude is normalized to the radius $R$. The energy of a “particle” for its bounded and unbounded movement is shown by horizontal lines.

that such unbounded motion corresponds to the excitation of the boundary instability of the conducting liquid (compare with [33]).

In terms of the total energy $H$ of the system (conducting liquid with the free surface in an external uniform electric field), we gain that the condition for the instability development in the situation where the flat surface is linearly stable (i.e., when the condition $E < E_c$ is fulfilled) is

$$H > H_c = \frac{8\epsilon_0 \pi R^2 \varepsilon_0 E^2}{27 \epsilon_0^2 \mu_1 \rho} \left( \frac{E_c}{E^2} - 1 \right)^3. \quad (12)$$

This nonlinear criterion suggests that, for the excitation of the instability, the system at the initial moment of time must have sufficient energy (in terms of the Newtonian particle) to leave the potential well (figure 2). This requires either an initial deformation of the free surface of the liquid or a kinetic energy store at the initial moment of time (the system should get an impulse).

5. Concluding remarks

Thus, we have shown that if the condition (12) is valid, i.e., the total energy of the system $H$ exceeds the threshold value $H_c$, then nonlinear interaction between surface waves leads to the hard loss of stability of a charged liquid surface. The growth of the boundary perturbations occurs even if the electric field value is subcritical, $E < E_c$, and the flat boundary of the fluid is stable in the linear approximation. According to the derived amplitude equation (7), the instability development has an explosive character: the surface perturbation amplitude $A$ grows without limit with asymptotics

$$A \propto \frac{1}{(t_c - t)^2}, \quad t \to t_c,$$

where $t_c$ corresponds to the moment of “explosion”.

Thus, even the simplest model, where the nonlinearity is taken into account only in the first nonvanishing order, reveals a tendency to the development of singularities in solutions of the electrohydrodynamic equations. Notice that both the experimental data [7, 13, 34] and the
results of the numerical calculations [9, 29, 35, 36] indicate that the higher-order nonlinearities have the destabilizing influence. The particular self-similar solutions of the motion equations found previously [8,37] (they describe the dynamics of the conical cusps formation on the liquid boundary in an electric field) also testify that the tendency to the formation of singularities persists when considering high-order nonlinearities. A similar tendency is indicated by exact solutions for the equilibrium shape of the boundary of a conducting fluid in an electric field. These solutions describe a loss of simple connectivity of the system, i.e., the formation of droplets [30,38,39].

It should be noted that the solution for the stationary configuration of the boundary (8) and (9) is exact in the leading order of the expansion with respect to the supercriticality parameter $(E^2 - E^2_c)/E^2_c$. The specific feature of the cylindrical geometry of the considered problem is the complexity of constructing the next order of the expansion. When considering four-wave interactions, which become essential in the next approximation, we should take into account an infinite number of terms in the expansion of $\eta$ in Bessel functions. Instead of expressions (6) and, respectively, (9), there arises a need to use the following series:

$$\eta(r, t) = A(t) J_0 (m_1 r/R) + \sum_{n=2}^{\infty} A_n(t) J_0 (m_n r/R),$$

where $m_n$ denotes $n$-th zero of the Bessel function of the first order ($n = 2, 3, 4, \ldots$) and $A_n$ are the amplitudes of corresponding harmonics. It is valid for these amplitudes $A_n \propto A^2$, i.e., they have the same order of smallness. This fact significantly complicates the analysis of influence of the higher-order nonlinearities as compared to the cases of the plane, square, and hexagonal symmetries, where the number of harmonics which need to be considered is finite.

The approach developed in the present paper can be applied to the analysis of instabilities of the free surface [3,40] or interface [41,42] of dielectric fluids in an applied electric field.

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