Perfect Gauge Actions on Anisotropic Lattices* †
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On anisotropic lattices, where generally the lattice is rather coarse in spatial directions, a parametrized classically perfect action could help reducing lattice artifacts considerably. We investigate the possibility of constructing such actions for SU(3) gauge theory. We present two different methods to do so, either repeating the procedure used to create our newly parametrized isotropic FP action, or performing one single step starting with the isotropic result. The anisotropic action is parametrized using an ansatz including anisotropically APE-like smeared (“fat”) links. The parametrized classically perfect action with anisotropy $\xi = a_s/a_t = 2$ is constructed and the renormalized anisotropy is measured using the torelon dispersion relation. It turns out that the renormalization is small.

1. INTRODUCTION

Measuring heavy states (e.g. glueballs) on moderately large lattices is feasible using anisotropic lattices whose resolution in temporal direction is finer than the one in spatial directions [1–7]. Having rather coarse (spatial) lattices calls for the use of improved actions; a radical choice there is the classically perfect FP action [8] which has been constructed for (isotropic) SU(3) gauge theory [9–12]. Our new parametrization [13,14] can serve as a basis for the construction of the corresponding anisotropic action as will be shown later. Anisotropic classically perfect gauge actions are an independent approach for measuring heavy states in pure gauge theory and thus probe universality.

2. CREATING PERFECT ACTIONS

In this paper we consider SU(N) pure gauge theory in $d = 4$ Euclidean space on a periodic lattice, the numerical work is done for SU(3).

For asymptotically free theories, the classically perfect FP action is determined by a saddle point equation [8]:

$$\mathcal{A}^{FP}(V) = \min_{\{U\}} \{\mathcal{A}^{FP}(U) + T(U,V)\}, \quad (1)$$

where $T(U,V)$ is the blocking kernel that defines the block transformation connecting the initial (fine) configuration $U$ and the blocked (coarse) configuration $V$.

In order to get a FP action for coarse configurations occuring in MC simulations, eq. (1) is used as a recursion relation connecting an action on a coarse configuration $V$ to one on a finer configuration $U$. The recursion step is iterated $3 \sim 4$ times using some appropriate simple starting action $\mathcal{A}(U)$ on the smoothest configuration. In principle the final FP action contains infinitely many couplings and thus has to be parametrized such that it may be used in a numerical simulation.

3. GETTING ANISOTROPIC

We pursue two different possibilities of modifying the method sketched in the previous section to obtain an anisotropic action. Firstly, the procedure may be repeated with an anisotropic starting action for smooth configurations, which may be a simple anisotropic discretization of the continuum action. Due to the isotropic nature of the blocking the anisotropy is relayed up to the level of coarse configurations.

Secondly, we can start with the parametrized isotropic FP action on coarse configurations and perform one or several additional purely spa-
tial blocking steps each doubling the anisotropy \( \xi = a_s/a_t \) (if one is working with scale-2 blocking steps) such that one obtains actions with anisotropies \( \xi = 2, 4, 8, \ldots \). These actions are not Fixed Point actions in the narrower sense as the action on the l.h.s. of eq. (1) has got an anisotropy twice as large as the one on the r.h.s. However, the actions are still classically perfect. In particular, they possess exactly scale invariant lattice instanton solutions.

4. SCALAR FIELDS, PERTURBATIVE TESTS

For scalar fields it can be proven analytically that both methods mentioned in the last section are completely equivalent and yield the same result. Furthermore it can be shown that the couplings for anisotropic actions on scalar fields can be made short-ranged in all directions, this has already been studied earlier in [15].

In addition, we study gauge fields in the quadratic approximation, where the action is

\[
A = \frac{1}{2N_c} \sum_{n,r} \rho_{\mu\nu}(r) \text{Tr}[A_\mu(n + r) A_\nu(n)]
\]

neglecting terms of order 3 and higher in the gauge potential. Fig. 1 displays the nice spectra obtained for different anisotropies. Also, in the perturbative potential the anisotropic classically perfect actions perform as well as the isotropic FP action.

5. ANISOTROPIC STARTING ACTION

Repeating the isotropic FP action program with an anisotropic starting action has got the advantage that the recursion steps are exactly the same as for the isotropic action, i.e. exactly the same RG transformation may be used. Furthermore, one may construct perfect actions of any desired anisotropy \( \xi \). However, the procedure is costly as one has to perform again several RGT and fitting steps for each value of \( \xi \). Besides, in each step it has to be carefully checked that the action parametrizations used are accurate on the respective fluctuations and that the renormalization of the anisotropy is small. That is why we prefer to perform a small number of purely spatial blocking steps starting with the isotropic action parametrized in [13,14].

6. SPATIAL BLOCKING STEP

To perform a purely spatial blocking step we modify the symmetric blocking kernel [9]:

\[
T(U, V) = -\frac{\kappa}{N_c} \sum_{nB,\mu} \left( \text{Re} \text{Tr}[V_\mu(nB)Q_\mu^\dagger(nB)] - N_\mu^B \right),
\]

where \( Q_\mu(nB) \) is a mean of products of fine links that connect two coarse lattice sites (we use \( Q_\mu \) of type III blocking, defined in [12]). Thus, the fine links are first smeared and subsequently blocked to the coarse lattice. The normalization term \( N_\mu^B \) assures that the partition function remains unchanged under the corresponding RG transformation. The parameter \( \kappa \) determines the weight of the blocking kernel relative to the action value \( A(U) \) and thus the freedom of the fine link products to fluctuate with respect to the coarse links.

We split the sum in \( T(U, V) \) into a spatial and a
temporal part with different $Q_\mu(n_B)$ and values of $\kappa$. Whereas $Q_i(n_B)$ is kept the same for the spatial directions $i = 1, 2, 3$, $Q_4(n_B)$ now connects two coarse lattice sites whose (temporal) lattice distance is the same as on the initial fine lattice, i.e. the links are only smeared but not blocked. The parameters $\kappa_t$ and $\kappa_s$ are independent of each other and can be chosen differently to make the resulting action accurately parametrizable with a small number of parameters.

7. **ANISOTROPIC PARAMETRIZATION**

To parametrize the anisotropic actions we generalize our “fat link” parametrization described in [13]:

$$A[U] = \frac{1}{N_c} \sum_x \sum_{\mu<\nu} \sum_{k,l} p_{kl} u_{\mu\nu}(x)^k w_{\mu\nu}(x)^l,$$

(4)

$$u_{\mu\nu} = \text{Re Tr}(1 - U_{\mu\nu}^{pl}), \quad w_{\mu\nu} = \text{Re Tr}(1 - W_{\mu\nu}^{pl}),$$

(5)

where $U_{\mu\nu}^{pl}$, $W_{\mu\nu}^{pl}$ denote the plaquettes built of standard and smeared gauge links, respectively.

Firstly, the coefficients $p_{kl}$ are chosen differently for plaquette orientations parallel ($p_{kl}^{tm}$) and perpendicular ($p_{kl}^{sp}$) to the temporal direction. Secondly, the construction of the smeared plaquettes is altered: There are three different kinds of smeared links contributing to a smeared plaquette $W_{\mu\nu}^{pl}$:

- temporal links (always contributing to a temporal plaquette),
- spatial links that contribute to a spatial plaquette and spatial links that contribute to a temporal plaquette.

In addition, there are four kinds of staples contributing to a smeared link which are “asymmetric” either because they lie in the same plane as the plaquette that is going to be built or because they go in time direction:

Because of this we introduce four different asymmetry parameters $\eta$ (weights of the staples) and for each iteration level $i$ three different parameters $c_i$ (specifying the expansion of the smeared links) [13]. These additional parameters do not influence the overhead of the action significantly, however the time used for parametrizing increases considerably.

Tests with reduced sets of actions (smaller number of non-zero $\eta$ parameters or only one $c_i$ value for all plaquette orientations) show that in order to fit well the perfect action the sophisticated parametrization is indeed needed.

8. **THE $\xi = 2$ ACTION**

The quadratic approximation suggests a ratio of $\kappa_t/\kappa_s = \xi^2$ for the $\kappa$-parameters in the anisotropic blocking kernel. As the blocking is unchanged in spatial direction, $\kappa_s$ is kept at the optimal value of $\kappa_s = 8.8$ obtained for the isotropic case; thus $\kappa_t = 35.2$. Fits on 40 configurations are performed and yield an action with 3 levels of $c_i$, $0 < k + l \leq 4$ for $p_{kl}^{sp}$ and $0 < k + l \leq 3$ for $p_{kl}^{tm}$. The fit is stable as the result does not change considerably when adding further independent configurations, and extending the parameter set by increasing the number of $c$-levels or the number of linear parameters does not further decrease $\chi^2$ significantly.

9. **THE RENORMALIZATION OF THE ANISOTROPY**

The anisotropy of the above action is measured using the torelon dispersion relation as described in [16,17]. On a $12^2 \times 6 \times 30$ lattice at $\beta = 3.3$ with periodic boundary conditions we measure the Polyakov line around the short spatial direction on APE-smeared configurations of smearing levels 3, 6, 9, 12, 15 with smearing parameter $\lambda_s = 0.1$ (see [13]). 23600 alternating Metropolis and over-relaxation sweeps and 4720 measure-
ments are performed. A fully correlated fit using variational techniques yields for the renormalized anisotropy $\xi_R = 1.949(28)$; the renormalization of the input anisotropy $\xi_0 = 2$ is thus very small. A rough estimate of the scale yields $a_s = 0.153(8)$ fm and thus $a_t = 0.079(5)$ fm. Fig. 2 shows the lower part of the dispersion relation including the result of the correlated fit in the range $p^2 = 1..4$. Taking into account larger fitting ranges up to $p^2 = 0..8$, we obtain results that are all consistent with the quoted value of $\xi_R$ within statistical errors.

![Graph](image)

Figure 2. Torelon dispersion relation. The straight line is the correlated fit to $E^2(p) = m_T^2 + p^2$ in the range $p^2 = 1..4$.

10. CONCLUSIONS AND PROSPECTS

The $\xi = 2$ classically perfect gauge action has been constructed using one spatial blocking step starting with the isotropic FP action. It shows a very small renormalization of the anisotropy at $\beta = 3.3$ corresponding to $a_s \approx 0.15$ fm. Scaling tests will be performed on this action measuring the static quark–antiquark potential (including off–axis separations to probe rotational invariance) and glueball masses. Furthermore, the spatial blocking procedure sketched above will be repeated using the $\xi = 2$ parametrized classically perfect action to build a perfect $\xi = 4$ action.

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