Exergetic Port-Hamiltonian Systems: Navier-Stokes-Fourier Fluid

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Abstract: The Exergetic Port-Hamiltonian Systems modeling language combines a graphical syntax inspired by bond graphs with a port-Hamiltonian semantics akin to the GENERIC formalism. The syntax enables the modular and hierarchical specification of the composition pattern of lumped and distributed-parameter models. The semantics reflects the first and second law of thermodynamics as structural properties. Interconnected and hierarchically defined models of multiphysical thermodynamic systems can thus be expressed in a formal language accessible to humans and computers alike. We discuss a composed model of the Navier-Stokes-Fourier fluid on a fixed spatial domain as an example of an open distributed-parameter system. At the top level, the system comprises five subsystems which model kinetic energy storage, internal energy storage, thermal conduction, bulk viscosity, and shear viscosity.

Keywords: port-Hamiltonian systems, geometric fluid mechanics, thermodynamics, exergy, compositionality, bond graphs, GENERIC, exterior calculus

1. INTRODUCTION

Due to the aggravating sustainability crisis, control- and optimization-oriented modeling of multiphysical thermodynamic systems is gaining in importance for the design of healthy energy systems. This is but one reason why port-Hamiltonian models of open thermodynamic systems, and in particular of fluid systems, are of practical interest.

The Navier-Stokes-Fourier fluid extends Euler’s model of an ideal fluid with viscosity and thermal conduction. As a thermodynamic model, it does not resolve all relevant details. Its entropy reflects incomplete information about the chaotic motion of atoms and entropy growth reflects irreversible processes which, in the absence of external forcing, drive the system towards equilibrium. Based on the assumption of local equilibrium, material properties pertinent to internal energy storage are defined by a thermodynamic potential relating intensive quantities (e.g. temperature, pressure) and (volumetric densities of) extensive quantities (e.g. internal energy, entropy, mass).

Van der Schaft and Maschke (2001) show that the Euler equations for an ideal compressible fluid can be written in port-Hamiltonian form using a Stokes-Dirac structure. Rashad et al. (2021a,b) provide a firm theoretical basis by deriving this model via Lie-Poisson reduction, see Sec. 4.1. Califano et al. (2021) extend the model with bulk and shear viscosity. Their Navier-Stokes model however ignores the thermal domain. The simplification hinges on the barotropicity assumption. The goal of the present article is to continue this line of research with a complete thermodynamic model including also thermal conduction.

Port-Hamiltonian theory is often considered insufficient for thermodynamic modeling. This has motivated related modeling frameworks introduced by Ramirez et al. (2013) and Van der Schaft and Maschke (2018). Exergetic Port-Hamiltonian Systems (EPHS) introduced by Lohmayer et al. (2021) are port-Hamiltonian systems which have a thermodynamic structure and interpretation that is akin to the GENERIC formalism of nonequilibrium thermodynamics introduced by Grmela and Öttinger (1997). The EPHS framework is partly inspired by the work of Badlyan et al. (2018) who rewrite a GENERIC model of the Navier-Stokes-Fourier fluid in port-Hamiltonian form using an ‘exergy-like potential’. While they rewrite the GENERIC model into a monolithic port-Hamiltonian system, our model has explicit compositional structure. Based on the graphical EPHS syntax, the exchange of energy between model components and boundary ports is made explicit. Rather than using the fluid’s internal energy as a state variable, we choose entropy since it leads to simpler EPHS models, see Lohmayer et al. (2021). Instead of assuming an Euclidian spatial domain, we work with a general Riemannian manifold and use Cartan’s exterior calculus. Eventually, this shall enable a rather general framework for distributed-parameter port-Hamiltonian systems and their spatial discretization.

In Sec. 2 and 3 we provide relevant background on EPHS and exterior calculus, respectively. In Sec. 4 we discuss the fluid model component by component. In Sec. 5 we conclude the article.

2. EXERGETIC PORT-HAMILTONIAN SYSTEMS

The EPHS framework combines port-Hamilton theory with ideas from the GENERIC formalism and categorical systems theory, which leads to a formal modeling language for multiphysical thermodynamic systems, see Lohmayer.
et al. (2021); Lohmayer and Leyendecker (2022). Using the EPIS syntax, the composition pattern of a system and its subsystems can be expressed in a graphical, yet formal manner. As syntactic primitive, a box represents the typed interface of a system in terms of its boundary parts. A model semantics is defined by a functor which assigns to every box the set of systems with that interface and to every composition pattern a function which sends appropriately typed subsystems to the composite system.

A component for exergy storage is defined by a state space \( X \ni x \) and a Hamiltonian function \( H : X \to \mathbb{R} \) of the form
\[
H(x) = E(x) - E(x_0) - \theta_0(S(x) - S(x_0)) + \pi_0(V(x) - V(x_0)) - \mu_0(N(x) - N(x_0)).
\]

\( E, S, V, N \) are the total energy, entropy, volume and mass. \( \theta_0, \pi_0, \mu_0 \) are the reference temperature, pressure and chemical potential that define the system’s environment. \( H \) is equal to the amount of work \( E(x) - E(x_0) \) which may be extracted from the system in the reversible limit, implying \( S(x) = S(x_0) \), before the overall system including its reference environment reaches the equilibrium state \( x_0 \). Hence, \( \theta_0, \pi_0, \) and \( \mu_0 \) may be regarded as Lagrange multipliers which are constant because the environment is considered to be infinitely large. \( H \) is a Lyapunov/storage function for the stability of \( x_0 \). As the entropy \( S \) grows, the exergy \( H \) diminishes. For an isolated EPIS, \( -S \) alone is not a Lyapunov function because it is unbounded in the absence of constraints which state that \( E, V, N \) are conserved while \( S \) may grow, see Sec. 3.3 of Lohmayer et al. (2021) for more details. Since \( H \) is a potential, we may ignore the constant shifts related to \( x_0 \).

The Dirac structure of an EPIS preserves exergetic power, and thus energy and entropy. It represents reversible exchange of exergy among components, akin to ideal wires in electrical circuits. The resistive structure is defined such that energy is conserved while the net exergetic power supplied to a dissipative process is non-negative. Together, this implies the first and second law of thermodynamics.

3. EXTERIOR CALCULUS

The spatial domain of the fluid is a smooth, oriented and compact \( n \)-dimensional Riemannian manifold \( Z \) with a possibly non-empty boundary \( \partial Z \) and inclusion map \( i : \partial Z \rightarrow Z \). The \( C^\infty(Z) \)-module \( \Omega^k(Z) = \Gamma(T^*Z)^k \) of differential \( k \)-forms contains all the anti-symmetric covariant tensors of rank \( k \). \( 0 \)-forms are merely smooth functions, i.e. \( \Omega^0(Z) = C^\infty(Z) \). Differential forms of degrees \( 0 \leq k \leq n \) form the exterior algebra whose \( \Omega^k(Z) \times \Omega^l(Z) \rightarrow \Omega^{k+l}(Z) \) essentially is the anti-symmetricized tensor product \( \otimes \). A \( k \)-form can be integrated on a \( k \)-dimensional submanifold. The exterior derivative \( d : \Omega^k(Z) \rightarrow \Omega^{k+1}(Z) \) appears in the Stokes-Cartan theorem generalizing the fundamental theorem of calculus (\( k = 0 \)), the classical Stokes theorem (\( k = 1 \)), and the divergence theorem (\( k = 2 \)). It expresses a duality between differentiation and restriction to the boundary, i.e. pullback along the inclusion map \( i^* \). An integration by parts formula follows: \( \forall \alpha \in \Omega^k(Z) \) and \( \forall \beta \in \Omega^{k-1}(Z) \)
\[
\int_Z d(\alpha \wedge \beta) = \int_{\partial Z} i^*(\alpha \wedge \beta) = \int_{\partial Z} i^* \alpha \wedge i^* \beta
\]
\[
= \int_Z d \alpha \wedge \beta + (-1)^k \int_Z \alpha \wedge d \beta
\]

(2)

The spatial domain \( Z \) carries a Riemannian metric, i.e. a symmetric positive-definite tensor \( g \in \Gamma(T^*Z \otimes T^*Z) \) which defines an inner product on every tangent space \( g : \Omega^k(Z) \times \Omega^l(Z) \rightarrow \mathbb{R} \) (\( k + l \)-forms). This group and a vector field \( \mathbf{X} \) for an isolated EPIS, \( \Gamma \rightarrow \Omega \rightarrow \Omega^{k-1}(Z) \) fixes a vector field \( X \in \Gamma(TX) \) as the first argument of a \( k \)-form. It can be expressed via the Hodge star, see Hirani (2003):
\[
\iota_{X} : \Omega^k(Z) \rightarrow \Omega^{k-1}(Z)
\]
\[
(v, \alpha) \mapsto \iota_{X} \alpha = (-1)^{n-k} \star_{X} \star_{X}(v \wedge \alpha)
\]

(3)

Cartan’s magic formula \( L_{\mathbf{X}} \times \mathbf{X} = d(\mathbf{X} \times \mathbf{X}) + \iota_{\mathbf{X}} (d \mathbf{X}) \) gives the Lie derivative of a form \( \alpha \) and thereby shows a duality between exterior derivative and interior product. We refer to Abraham and Marsden (1978) for a relatively detailed introduction to these concepts. We define \( \sigma = (-1)^{n-1} \).

4. FLUID MODEL

In this section we present the fluid model by discussing its five subsystems. Interconnection of the kinetic energy subsystem defined in Sec. 4.1 and the internal energy subsystem defined in Sec. 4.2 yields a model of an ideal fluid. The subsystems defined in Sec. 4.3 to 4.5 model irreversible processes and can be added to the overall system independently of each other.

4.1 Kinetic Energy Subsystem

The kinetic energy subsystem is central to fluid motion. A model of an isolated, ideal, compressible fluid can be derived via Lie-Poisson reduction of Hamiltonian systems on semidirect product Lie groups, see Marsden et al. (1984b,a). The point of departure is a Lagrangian/material description of fluid motion formalized as a curve on the Lie group of automorphisms of \( Z \). This group and a vector space of advected quantities (e.g. mass) together form a semidirect product Lie group because automorphisms act on advected quantities. The reduction procedure takes the Lagrangian description on the semidirect product Lie group to an Eulerian/spatial description on the dual of its semidirect product Lie algebra. A Lie algebra isomorphism between the reduced phase space and a space of differential forms enables the straightforward generalization of the Lie-Poisson structure to a Stokes-Dirac structure and thus also the generalization of the Hamiltonian model of ideal compressible fluid motion to a port-Hamiltonian model, see Rashad et al. (2021a). This permits the consideration of spatial domains with permeable boundaries and it enables the extension of the model through interconnection with other port-Hamiltonian systems.

The composition pattern of the subsystem is given by the following expression in the EPIS syntax:
First, we discuss the semantic data associated to box $H_k$ representing storage of kinetic energy. Linear momentum and advected mass are the canonical phase space variables. The momentum $p$ is a 1-form since its change is equal to the force 1-form that is (energetically) dual to the velocity vector field. The integral over $Z$ of the mass $n$-form $m$ yields the total fluid mass. Rather than momentum, the model uses the velocity (or specific momentum) 1-form $v = p/m$ as a state variable, since it leads to simpler equations and presumably has advantages for numerical computation. Thus, the energy variables are $(v, m) \in \mathcal{X} = \Omega^1(Z) \times \Omega^0(Z).$ (4)

The Hamiltonian $H(v, m) = \int_Z \frac{1}{2} m g(v^s, v^t) = \int_Z \frac{1}{2} m \cdot \delta v$ (5)

The partial variational derivative $\delta_v H$ e.g. is defined by $H(v + \epsilon \delta v, m) = H(v, m) + \epsilon \int_Z \delta_v H \wedge \delta v + \mathcal{O}(\epsilon^2)$ for $\epsilon \to 0$. We obtain the following partial derivatives:

$$\delta_v H = \sigma \star m \star v \quad (\delta_m H = \frac{1}{2} \star (v \wedge \star v))$$

(6a) (6b)

The box $H_k$ has ports $u$ and $m$ related to $v$ and $m$, respectively. Each port has a flow variable $f$ and an effort variable $e$. The port variables are defined by

$$(u, f, m, f, u, e, m, e) = (v, m, \delta_v H, \delta_m H).$$

(7)

We now discuss box $D_1$ which represents the Stokes-Dirac structure generalizing the original Lie-Poisson structure. Ports $u$ and $m$ are related to $v$ and $m$, while $b_k$ is related to energy exchange across $\partial Z$. The port variables are constrained by the following relation:

$$\begin{bmatrix} u, f \\ m, f \\ b_k, f \end{bmatrix} = \begin{bmatrix} S(\cdot) & d(\cdot) & 0 \\ \sigma d(\cdot) & 0 & 0 \\ -\sigma \star (\cdot) & 0 & \star (\cdot) \end{bmatrix} \begin{bmatrix} u, e \\ m, e \\ b_k, e \end{bmatrix}$$

(8a) (8b)

where $S(\cdot) = 1/m \cdot \delta v$ (3) $\star (\cdot) = -\frac{1}{2} \star (\cdot)$

(8c)

The Dirac structure implies a power-balance equation:

$$\int_Z (u, e \wedge u, f + m, e \wedge m, f) + \int_{\partial Z} b_k, e \wedge b_k, f = 0$$

Box $D_2$ represents a Dirac structure which transforms the force acting on the fluid via port $f$ to a change in velocity:

$$\begin{bmatrix} u, f \\ f, e \end{bmatrix} = \begin{bmatrix} 0 & \star (\cdot) \\ \sigma/m & 0 \end{bmatrix} \begin{bmatrix} u, e \\ f, f \end{bmatrix}$$

(9)

The composition pattern states the interconnection of components $H_k, D_1, D_2$ and boundary ports $f, b_k$. Here, ‘boundary’ refers to the system boundary represented by the outer box of the expression. This is orthogonal to the concept of distributed versus boundary ports in the sense of $\Omega$ vs $\partial \Omega$ since different systems may be defined on the same spatial domain and the definition of a single system may involve different spatial domains. Ports are connected to 0-junctions where flow variables sum to zero and effort variables are equal. E.g. $H_k, u, f + D_1, u, f + D_2, u, f = 0$ and $H_k, u, e = D_1, u, e = D_2, u, e$. Because naming is hard, we use the same labels for ports of different boxes connected to the same junction. This shall also apply to the interconnection of the five subsystems. By applying the composition pattern to the semantic data associated to the three inner boxes, we obtain the semantic data associated to the composite subsystem filling the outer box:

$$\dot{v} = -\tau_\nu (dv) - \mu (\frac{1}{2} \star (u \wedge \star v)) + \star m \cdot f$$

(10a)

$$\dot{m} = -d (s \cdot m \cdot v) = -L \cdot s m$$

(10b)

$$f \cdot e = \star v$$

(10c)

$$b_k \cdot f = -\tau (s \cdot m \cdot v)$$

(10d)

$$b_k \cdot e = \tau (\star (v \wedge \star v)/2)$$

(10e)

The Lie derivative appears in (10b) since $m$ is advected. While $f, e$ is the force acting on a fluid particle, $f, e$ is the velocity (or volume flux). While $b_k \cdot f$ is the mass influx, $b_k \cdot e$ is the kinetic energy per unit of mass.

4.2 Internal Energy Subsystem

While kinetic energy comes from macroscopic motion, internal energy corresponds to microscopic motion, which at the macroscopic scale manifests as temperature and pressure. The composition pattern of the internal energy subsystem is given by the following expression:

$$H(s, m) = \int_Z (s \cdot U(s, m) - \theta_0 s - \mu_0 m)$$

(12)
The intensive variables temperature $\theta$, chemical potential $\mu$ and pressure $\pi$ take values in $\Omega^0(\mathcal{Z})$ and are defined by
\[
\begin{align*}
\theta &= \frac{\partial U(\star s, \star m)}{\partial (\star s)} & (13a) \\
\mu &= \frac{\partial U(\star s, \star m)}{\partial (\star m)} & (13b) \\
\pi &= \theta \star s + \mu \star m - U(\star s, \star m). & (13c)
\end{align*}
\]
Eq. (13c) follows from 1-homogeneity of $U$. The partial variational derivatives are consequently given by
\[
\delta s H = \theta - \theta_0 \in \Omega^0(\mathcal{Z}) & (14a) \\
\delta m H = \mu - \mu_0 \in \Omega^0(\mathcal{Z}) & (14b)
\]
and the port variables of $H_1$ are defined by
\[
(s, f, m, s, e, m, e) = (\hat{s}, \hat{m}, \delta s H, \delta m H). & (15)
\]
Box $D_1$ represents the reversible coupling of fluid motion and internal energy storage. Since the boundary ports labeled $f$ of the kinetic and internal energy subsystems are connected, $f, e$ is the fluid velocity that advects $s, m$ and hence also the internal energy, while $f, f$ is the force which results from the internal energy and affects fluid motion. Ports $b_s$ and $b_e$ are related to advection of internal energy across $\partial \mathcal{Z}$. The port variables of $D_2$ are constrained by the following relation defining a Stokes-Dirac structure:
\[
\begin{pmatrix} s, f \\ m, f \\ f, f \end{pmatrix} = \begin{pmatrix} 0 & 0 & d(\star s) \\ 0 & 0 & d(\star m) \\ \sigma \star s d(\cdot) & \sigma \star m d(\cdot) & 0 \end{pmatrix} \begin{pmatrix} s, e \\ m, e \\ f, e \end{pmatrix} & (16a)
\]
\begin{align*}
[bs, f] &= \begin{pmatrix} 0 & 0 & -i^*(\star s) \\ i^*(\cdot) & 0 & 0 \\ 0 & 0 & -i^*(\star m) \\ 0 & i^*(\cdot) & 0 \end{pmatrix} \begin{pmatrix} s, e \\ m, e \\ f, e \end{pmatrix} & (16b)
\end{align*}

The composition pattern states the interconnection of components $H_1, D_1$, and boundary ports $f, s, b_s, b_e$. Its application to the semantic data associated to $H_1$ and $D_1$ yields semantic data associated to the entire subsystem:
\[
\begin{align*}
\dot{s} &= -d(\star s \star f, e) + s, f & (17a) \\
\dot{m} &= -d(\star m \star f, e) & (17b) \\
f, f &= \sigma \partial_{\pi} & (17c) \\
s, e &= \theta - \theta_0 & (17d) \\
b_s, f &= -i^*(\star s \star f, e) & (17e) \\
b_e, e &= i^*(\star m \star f, e) & (17f) \\
b_s, e &= i^*(\mu - \mu_0) & (17g) \\
b_e, e &= i^*(\mu - \mu_0) & (17h)
\end{align*}
\]
Terms with $f, e = \star \nu$ in (17a) and (17b) are Lie derivatives since $s, m$ are advected quantities. Eq. (17c) follows from
\[
d\pi = \star s d\theta + \star m d\mu & (18)
\]
which in turn is a consequence of (13). While $f, e$ is the fluid velocity, $f, f$ is the force resulting from thermodynamic pressure. Port $s$ is relevant for adding irreversible processes. While $s, f$ is the local change in entropy caused by irreversible processes, $s, e$ is the relative temperature. While $b_s, f$ is the entropy influx, $b_e, e$ is the relative temperature at the boundary. Similarly, $b_s, f$ is the mass influx and $b_e, e$ is the relative chemical potential at the boundary.

### 4.3 Thermal Conduction

Thermal conduction is a relaxation process counteracting nonuniform temperature with a diffusive heat flux. The composition pattern of the thermal conduction subsystem is given by the following identity expression:

\[
\begin{align*}
[s, f] &= \begin{pmatrix} -1 & \frac{1}{\theta_0} \frac{\partial}{\partial \theta} \frac{1}{\theta} d \left( \frac{1}{\theta} \right) \end{pmatrix} \begin{pmatrix} s, e \\ f, e \end{pmatrix} & (19a) \\
[b_t, f] &= \begin{pmatrix} 0 & i^* \frac{1}{\theta} \frac{\partial}{\partial \theta} \frac{1}{\theta_0} d \left( \frac{1}{\theta_0} \right) \end{pmatrix} \begin{pmatrix} s, e \\ e, e \end{pmatrix} & (19b)
\end{align*}
\]

Ports $s$ and $b_t$ are related to entropy flux due to thermal conduction within $\mathcal{Z}$ and across $\partial \mathcal{Z}$, respectively. Box $R_t$ represents the irreversible process. Its port variables are constrained by the following relation defining a resistive structure:

\[
\begin{align*}
\int_\mathcal{Z} \dot{u} &= \int_{\partial \mathcal{Z}} i^*(-\phi) &= \int_\mathcal{Z} d(-\phi) & (19a) \\
\text{The minus sign comes from the induced Stokes orientation on } \partial \mathcal{Z} \text{ which makes } \phi \text{ the outgoing heat flux. We hence have the local energy balance } \dot{u} = d(-\phi). \text{ Fourier’s law gives } \phi = -\kappa s d\theta. \text{ The thermal conductivity } \kappa \geq 0 \text{ is constant or depends on } \theta. \text{ Using } \dot{u} = \frac{\partial U(\star s)}{\partial (\star s)} \wedge \hat{s} \hat{s}, \text{ we obtain } \dot{s} = \frac{1}{\theta} u = \frac{1}{\theta} d(\theta - \theta_0) = \frac{1}{\theta} d(\kappa \phi). \text{ We get the same local energy balance from (19a)} \text{ and the interconnection of } H_1, s \text{ and } R_t, s \text{ yielding } \dot{s} + s, f = 0 \text{ and } \theta - \theta_0 = s, e. \text{ To see this, we identify the heat flux in (19a)}:
\end{align*}
\]

\[
\begin{align*}
\phi &= \star \left( \kappa \frac{1}{\theta_0} \frac{\partial}{\partial \theta} \frac{1}{\theta} d \left( \frac{1}{\theta} \right) \right) = \star \left( \kappa \frac{\partial^2}{\partial \theta^2} \frac{1}{\theta} \right) = \star (\kappa d\theta) = \kappa s d\theta
\end{align*}
\]

To see that energy is conserved at the resistive structure, we note that $s, e = \theta - \theta_0$ where $\theta$ stems from the energy $E$ and the shift $-\theta_0$ stems from the entropy $S$, see the first and second line of (1). By omitting the shift, i.e. by letting $s, e = \theta$, we assert that $E$ is a metric Casimir, meaning that the irreversible process conserves energy. The exergy balance of the system consisting of $H_1$ and $R_t$ reads

\[
\begin{align*}
\dot{H} &= \int_\mathcal{Z} \delta s H \wedge \hat{s} = \int_\mathcal{Z} s, e \wedge (-s, f) = \int_\mathcal{Z} \frac{\theta - \theta_0}{\theta} d(-\phi) \\
&= \int_{\partial \mathcal{Z}} i^*(\theta - \theta_0) \wedge i^* \left( \frac{1}{\theta} \phi \right) \\
&= \theta_0 \int_\mathcal{Z} \frac{1}{\theta_0} d \left( \frac{1}{\theta} s, e \right) \wedge \left( \kappa \frac{\partial^2}{\partial \theta^2} \frac{1}{\theta_0} d \left( \frac{1}{\theta} s, e \right) \right).
\end{align*}
\]

Eq. (20) says that the stored exergetic power is equal to the supplied exergetic power minus the lost exergetic power.
In accordance with the second law, the last line shows that the lost exergetic power is non-negative. The first line shows that the stored exergetic power is equal to the Carnot efficiency multiplied by the net incoming heat flux. While the first and second line are related to the strong form of the evolution equation together with boundary conditions, the last line is related to the weak form which implicitly includes boundary conditions. The weak form shows that the resistive structure is defined by a symmetric operator in accordance with Onsager’s reciprocal relations. By analyzing the structure of this symmetric operator we can identify the thermodynamic force and flux. The term \( \frac{1}{\theta_0} d(\tau f.e) = -d(\phi) \) is called the thermodynamic force since its integral yields the rate of local volume change. We have in this case is the heat flux, the differential (i.e. variational derivative) of the exergetic energy is left invariant. The operator \( \star(k_{\theta^2}) \) turns the thermodynamic force into the thermodynamic flux which in this case is the heat flux \(-\phi\). The integrand is the product of force and flux, which is equal to the entropy production rate. Multiplication with the leading factor \( \theta_0 \) finally yields the exergy destruction rate.

4.4 Bulk Viscosity

Bulk viscosity is a relaxation process counteracting local changes in volume and thus in particular leads to damping of acoustic waves. The composition pattern of the bulk viscosity subsystem is given by the following expression:

Port \( f \) is related to the viscous force counteracting volume change, port \( s \) is related to the corresponding increase in entropy, and port \( b_1 \) is related to the viscous force acting across \( \partial Z \). The port variables of \( R_b \) are constrained by the following relation defining a resistive structure:

\[
\begin{pmatrix}
  f.f \\
  s.f \\
\end{pmatrix}
= \begin{pmatrix}
  -\sigma d(\mu_b \star d(\cdot)) + \frac{1}{\theta_0} \sigma d(\mu_b \star d(s.e)) \\
  -\mu_b \star d(\cdot) + \frac{1}{\theta_0} \mu_b \star d(s.e) \\
\end{pmatrix}
\begin{pmatrix}
  f.e \\
  s.e \\
\end{pmatrix}
\]

(21a)

\[
\begin{pmatrix}
  b_1.f \\
  b_1.e \\
\end{pmatrix}
= \begin{pmatrix}
  \frac{1}{\theta_0} \mu_b \star d(f.e) - \mu_b \star d(s.e) \\
  \frac{1}{\theta_0} \mu_b \star d(s.e) \\
\end{pmatrix}
\]

(21b)

To gain insight about (21), we note that \( \star 1 \in \Omega^0(Z) \) is called Riemannian volume form since its integral yields the volume of the manifold. Its Lie derivative \( L_v(\star 1) \) gives the rate of local volume change. We have

\[
L_v(\star 1) = d(\iota_v(\star 1)) \quad (\star 0) \quad d(\star (v \wedge \star 1)) = d \star v
\]

Comparing to thermal conduction, \( d(\nu) \) is analogous to \( d(\theta) \). Akin to how \( \nu \star (\cdot) \) acts on \( d(\theta) = d(\Re_1.f.e) \) to yield the heat flux, \( \mu_b \star (\cdot) \) acts on \( d \star v = d(\Re_1.f.e) \) to yield a pressure-like quantity counteracting volume change. The 0-form \( \star d \star v \) is called divergence of \( v^\nu \) and \( \mu_b \geq 0 \) is a bulk viscosity coefficient. Akin to how \( d(\cdot) \) acts on the heat flux to yield the net heat flux which leads to a change of internal energy, \( d(\cdot) \) acts on the pressure-like quantity to yield the net viscous force which leads to a change of momentum. This pattern common to many models of relaxation processes is quintessential for linear irreversible thermodynamics (LIT). It must however not lead to linear models, since coefficients may be functions of quantities like temperature. The product of the pressure-like quantity and the rate of volume change \( \mu_b \star (d \star v) d \star v \) is the rate at which kinetic energy irreversibly turns into heat. Consequently, the entropy production rate is given by \( \frac{1}{\theta_0} \mu_b \star (d \star v) d \star v \). Considering the interconnection of the subsystems for kinetic energy, internal energy, and bulk viscosity, we see that (21a) leads to the following contributions to the evolution equations:

\[
\dot{v} = \frac{1}{\theta_0} d(\mu_b \star d \star v) \\
\dot{s} = \frac{1}{\theta_0} \mu_b \star (d \star v) d \star v
\]

Noting that \( f.e, s.e = (\kappa, \theta - \theta_0) \) where \( (\kappa, \theta) \) is the energetic part and \((0, -\theta_0)\) is the entropic part, we assert that the energetic contribution lies in the kernel of (21a) and hence energy is conserved by the irreversible process. By invoking the integration by parts formula (2), we obtain

\[
\int_Z (f.e \wedge f.f + s.e \wedge s.f) - \int_{\partial Z} b_1.e \wedge b_1.f = \int_Z [d(f.e) \wedge \left( \frac{1}{\theta_0} \theta \mu_b \star d(f.e) \right) + d(s.e) \wedge \left( \frac{1}{\theta_0} \mu_b \star (d \star v) \star s.e \right)]
\]

(22)

which shows that (21) defines a symmetric operator, etc.

4.5 Shear Viscosity

We understand shear viscosity as a relaxation process counteracting general deformation of the fluid. Since this includes contraction and expansion, there is overlap with bulk viscosity. On Euclidian spaces, this can be avoided via isotropic-deviatoric splitting of the stress tensor. We carry on, acknowledging that shear viscosity essentially adds more bulk viscosity. While pressure and bulk viscous stress are conveniently expressed as 0-forms, the non-isotropic nature of shear stress requires a stress tensor of type \( \Omega^1(Z) \otimes \Omega^{n-1}(Z) \), where the first leg represents the traction force acting across a surface element paired with the second leg, see Kanso et al. (2007). Pairing the first leg with the velocity vector field \( v^\nu \) and integrating the remaining second leg over a surface yields the power exchanged across the surface. This motivates the product \( \wedge : (\Gamma(TZ) \otimes \Omega^1(Z)) \times (\Omega^1(Z) \otimes \Omega^{n-k}(Z)) \rightarrow \Omega^0(Z) \) which pairs the first legs and uses \( \wedge \) on the second legs. We implicitly use the isomorphism \( \Gamma(TZ) \cong \Gamma(TZ) \otimes \Omega^0(Z) \). Analogous to how \( d(\star \nu) \in \Omega^n(Z) \) is the rate of volume change, \( \nabla(v^\nu) \in \Gamma(TZ) \otimes \Omega^1(Z) \) is the strain rate. The covariant derivative \( \nabla \) measures how \( v^\nu \in \Gamma(TZ) \) changes in the direction given by a vector paired with the second leg. Similar to how \( \mu_b \star (d \star v) \in \Omega^1(Z) \) is the bulk viscous
stress, $T = \mu_s + \mathbf{s} (\text{sym}(\partial_1(\nabla(v^2)))) \in \Omega^1(Z) \otimes \Omega^{n-1}(Z)$ is the shear stress. Subscripts 1 and 2 indicate the leg on which $\partial$ and $\ast$ act and $\text{sym}$ symmetrizes the two legs. Akin to how $d(\mu_b \ast d(\mathbf{s} \mathbf{v})) \in \Omega^1(Z)$ is the net bulk viscous stress, $d\nabla_T \in \Omega^1(Z) \otimes \Omega^n(Z)$ is the net shear stress acting on a volume element paired with the second leg. The covariant exterior derivative $d\nabla_T$ is the formal adjoint of $-\nabla$ shown by the integration by parts formula
\[
\int_Z d(v^2 \wedge T) = \int_Z v^2 \wedge d\nabla_T + \int_Z \nabla(v^2) \wedge T = \int_{\partial Z} i^2_2(v^2 \wedge T) = \int_{\partial Z} i^2_z(v^2) \wedge \nabla(T),
\]
where the boundary terms are two-point tensors since only the second legs can be pulled back to $\partial Z$. The composition pattern is given by the following expression:
\[
\begin{align*}
\int_{\partial Z} i^2_2(v^2 \wedge T) &= \int_{\partial Z} i^2_z(v^2) \wedge \nabla(T),
\end{align*}
\]

Port $\mathbf{f}$ is related to the viscous force, $\mathbf{s}$ is related to entropy production, and $\mathbf{b}_2$ is related to viscous stress acting across $\partial Z$. While $\mathbf{R}_2$ represents the resistive structure defining the irreversible process, $\mathbf{D}_p$ represents the Dirac structure which translates between stress $T$ at port $\mathbf{f}_p$ and the resulting force $f \in \Omega^1(Z)$ at port $\mathbf{f}$. Since we have
\[
\int_Z \mathbf{f} \wedge f = \int_Z \mathbf{f} \wedge f(\mathbf{s}) = \int_Z v^2 \wedge f \wedge (\mathbf{s} \wedge 1) = \int_Z v^2 \wedge T,
\]
we can conclude that the following relation defines the Dirac structure associated to $\mathbf{D}_p$:
\[
\begin{bmatrix}
\mathbf{f}, \mathbf{f} \\
\mathbf{s}, \mathbf{f}
\end{bmatrix}
= \begin{bmatrix}
0 & \sigma_2(\cdot) \\
\sigma_2(\mathbf{s}(\cdot)) & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{f}, \mathbf{e} \\
\mathbf{s}, \mathbf{e}
\end{bmatrix}
\] (24)

The resistive structure associated to $\mathbf{R}_2$ is defined by the following relation:
\[
\begin{align*}
\begin{bmatrix}
\mathbf{f}_p, \mathbf{f} \\
\mathbf{s}, \mathbf{f}
\end{bmatrix}
&= \frac{1}{\theta_0} \begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
\mathbf{f}_p, \mathbf{e} \\
\mathbf{s}, \mathbf{e}
\end{bmatrix}
\end{align*}
\]
\[
\begin{align*}
A &= -d\nabla (\mu_s \ast \mathbf{s} \ast \text{sym}(\partial_1(\nabla(v^2))) \mathbf{\theta}) \\
B &= d\nabla (\mu_s \ast \mathbf{s} \ast \text{sym}(\partial_1(\nabla(v^2))) (\cdot)) \\
C &= -\mu_s \nabla(\cdot) \ast \mathbf{s} \ast \text{sym}(\partial_1(\nabla(v^2))) \\
D &= \mu_s \nabla(v^2) \ast \mathbf{s} \ast \text{sym}(\partial_1(\nabla(v^2))) (\cdot)
\end{align*}
\] (25a)
\[
\begin{align*}
b_{2, \mathbf{f}} &= \frac{1}{\theta_0} i^2(\mathbf{b}_1 \ast \mathbf{s} \ast \text{sym}(\partial_1(\mathbf{f}_p, \mathbf{e}))) \mathbf{\theta} \\
&\quad - \mu_s \mathbf{b}_1(\mathbf{s} \ast \text{sym}(\partial_1(\mathbf{f}_p, \mathbf{e}))) (\mathbf{s}, \mathbf{e})
\end{align*}
\] (25b)
\[
\begin{align*}
b_{2, \mathbf{e}} &= i^2(\mathbf{f}_p, \mathbf{e})
\end{align*}
\] (25c)

5. CONCLUSION

Using the Exergetic Port-Hamiltonian Systems (EPHS) framework, we present a structured model of the Navier-Stokes-Fourier fluid. The model has five subsystems for kinetic energy storage, internal energy storage, thermal conduction, bulk viscosity, and shear viscosity. The composed model comprises nine primitive components, namely two for exergy storage, four representing Dirac structures, and three representing resistive structures. The power-preserving interconnection of subsystems, components, and boundary ports is defined explicitly by composition patterns stated in the EPHS syntax. The first and the second law of thermodynamics is implied by the EPHS semantics. The distributed-parameter model is expressed using Cartan’s exterior calculus on Riemannian manifolds. Future work should aim to define model discretization as a natural transformation between functors assigning distributed-parameter and lumped-parameter semantics, since this would imply preservation of the compositional and thermodynamic structure. Once this is achieved, not much would be missing before one could apply the model as a building block in larger systems.

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