Light production metrics of radiation sources

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Abstract
Light production by a radiation source is evaluated and reviewed as an important concept of physics from the black-body point of view. The mechanical equivalent of the lumen, the unit of perceived light, is explained and evaluated using radiation physics arguments. The existence of an upper limit of luminous efficacy is illustrated for various sources and implications are highlighted.

Keywords: visual perception, light sources, light-emitting diode

(Some figures may appear in colour only in the online journal)

1. Introduction

Physics students are exposed to black-body radiation in undergraduate quantum physics or in graduate/undergraduate statistical physics without any clue regarding its significance as to the fundamental role it plays in the development of light source calibration, development and of lighting in general. Perhaps, only students with astrophysics, advanced optics, atmospheric physics or medical physics curriculum will be more aware generally of infra-red and radiation physics on the basis of black-body fundamentals.

Presently, lighting is undergoing a tremendous evolution because of the swift evolution of the light-emitting-diode (LED), which is now gaining larger and larger luminous efficacy to a point such that it is now replacing, at a very impressive pace, our traditional home lighting, LCD-monitors backlights, car headlights, traffic light signalling, street lighting, etc. Additionally, LED colours are becoming more versatile and sharper, both in the case of the traditional inorganic LED or its organic counterpart, the OLED.

The underlying basis of lighting progress is the existence of the Haitz law [1] (illustrated in figure 1), which is similar to Moore’s law of electronics evolution. The Haitz law [1] states that ‘for a given wavelength, the amount of light generated by LEDs increases by a factor of
20 and the cost per lumen (unit of perceived light) decreases by a factor of 10, every period of ten years'.

Light production by a radiation source is described by a luminous efficacy ratio $\eta_L$ as the product $\eta_C \times \eta_P$ where:

- $\eta_C$ is the conversion efficacy ratio that is number of photons produced (having any wavelength) over input energy (usually electrical but it could also be mechanical, thermal or chemical);
- $\eta_P$ is the light perception (or photometric) efficacy ratio (PER). It is the ratio of number of photons perceived by the human eye (photon wavelength in the visible spectrum) to the total number of photons (note that some authors call it $\eta_S$, the spectral efficacy).

This work can be taught as an application chapter in a general course of statistical physics, optics or semiconductor physics at the undergraduate or graduate level since physicists can contribute readily in improving light production through increase of either $\eta_C$ or $\eta_P$ once the basis for luminous efficacy is explained and illustrated along some notions of colorimetry and light calibration.

The notions reviewed in this work are primarily concerned with $\eta_P$ (the perception efficacy) and are clearly important not only in physics and technology but also for energy savings and efficiency, renewable energy and consequently for sustainable development of the planet.

This paper is organized as follows: in section 2, a review of lighting metrology is made, in section 3 luminous efficacy of radiation sources is explained and derived and in section 4, we derive the maximum luminous efficacy on the basis of the colorimetry standard established by the CIE\(^1\). This standard is briefly explained and reviewed in the appendix that comes after section 5 carrying discussions and conclusions.

\(^1\) CIE is *Commission Internationale de l’Eclairage* or International Organization for Lighting based in Vienna (Austria) that sets standards for light and colours and is responsible of the metrology of lighting like the NIST (National Institute of Standards and Technology) in the US.
2. Metrology of lighting

The SI system of units is based on seven entities: the metre, the kilogram, the second, the ampere, the kelvin, the mole and the candela as the unit of luminous intensity (lumen is candela per unit solid angle).

Prior to 1979, the SI system of units defines the candela as follows: ‘a pure sample of platinum at its melting temperature (T = 2042 K) emits exactly 60 candelas cm\(^{-2}\) sr\(^{-1}\) along the normal to the sample (per unit surface, in cm\(^2\) and per unit solid angle, in sr or steradian)’.

The SI system changed the definition during the 16th General Conference on Weights and Measures in October 1979 to: ‘the candela is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 540 \times 10^{12} \text{ Hz} with a radiant intensity, in that direction, of 1/683 Watts per unit solid angle’.

In order to relate these two definitions, despite the obvious equivalence of 555 nm wavelength and 540 \times 10^{12} \text{ Hz} frequency, some reminders about black-body radiation must be given.

Black-body radiation was noticed for the first time by Kirchhoff [3, 4] who used to watch the colour change of the cavities present in the heated metals worked by blacksmiths in his neighbourhood. He noticed a systematic colour change as the metal is heated which followed a typical sequence from red to orange, yellow, white and finally to blue. Cavity colour change did not depend on the nature of the metal but on the size of the cavity.

In quantum physics language, we have a photon gas in the cavity obeying the Planck radiation law [3] with the (thermal) average spectral energy density\(^2\) \(E(\lambda)\) of the Bose–Einstein form (the chemical potential is zero since the number of photons is not fixed):

\[
E(\lambda) = \frac{8\pi hc}{\lambda^5} \left[\exp\left(\frac{hc}{\lambda k_BT}\right) - 1\right]
\]

\(\lambda\) is the wavelength, \(h\) is Planck constant, \(c\) is velocity of light in vacuum, \(k_B\) is Boltzmann constant and \(T\) the temperature. This law is derived in many statistical physics books and is usually enough for a standard physics curriculum. It represents the average spectral energy density (see footnote 2) of a black-body at temperature \(T\).

Since a black-body (absorbs and) emits radiation, we need a Planck law equivalent for the emission spectral power density (see footnote 2). Henceforth, the latter is called the emission power spectrum (EPS).

By analogy with electrical charge current density \(J\) (\(J = \rho v\) with \(\rho\) the charge density and \(v\) the velocity), we multiply the average spectral energy density by the velocity of light \(c\) and divide by \(4\pi\), the whole space solid angle, in order to obtain a power current density. This results in the Planck distribution for the EPS (see footnote 2) function at wavelength \(\lambda\), temperature \(T\) and unit solid angle:

\[
P_B(\lambda) = \frac{2hc^2}{\lambda^5} \left[\exp\left(\frac{hc}{\lambda k_BT}\right) - 1\right].
\]

Light sensed by the human eye is given by averaging over wavelength, \(P_B(\lambda)\) with the eye sensitivity function called \(V(\lambda)\) by the CIE (see footnote 1). This function, depicted in figure 2, peaks at a 555 nm (yellow–green) wavelength, the maximum sensitivity of the human eye. The analytical expressions [5] for daylight and dark sensitivities \(V(\lambda), V'(\lambda)\) differ

\(\lambda\) in \(\text{nm}\), \(E(\lambda)\) the photon average spectral energy density has units of energy/volume/wavelength, whereas EPS functions \(P(\lambda)\) and \(P_B(\lambda)\) (black-body) have units of power/surface/wavelength since they correspond to photons emitted at the source surface. The term ‘spectral density’ originates from the fact we are working per unit wavelength.
Figure 2. $V(\lambda)$ function giving the eye sensitivity versus wavelength in daylight (photopic) and $V'(\lambda)$ in the dark (scotopic). They are shifted one with respect to the other by 45 nm, the Purkinje shift.

slightly from the tabulated values provided by the CIE (see footnote 1). Daylight sensitivity is $V(\lambda) = 1.019 \exp(-285 \left(\frac{\lambda}{1000} - 0.559\right)^2)$ (also called photopic) with $\lambda$ expressed in nm.

Dark sensitivity $V'(\lambda)$ (scotopic) is shifted with respect to $V(\lambda)$ by 45 nm to shorter wavelengths and peaks at 510 nm (Purkinje shift). Analytically [5] $V'(\lambda) = 0.992 \exp(-321.9 \left(\frac{\lambda}{1000} - 0.503\right)^2)$. Both analytic expressions are obtained [5] from a nonlinear least-square fit to Gaussian functions. They are useful in the case of continuous spectral distribution and show less than 1% error with black-body sources from 1500 to 20 000 K. Nevertheless, they must be used with caution in the case of narrow-band or line sources [5].

The conversion from radiometry (covering all wavelengths, i.e. $\lambda \in [0, \infty]$) to photometry (covering the visible interval, i.e. $\lambda \in [380 \text{ nm}, 780 \text{ nm}]$) is carried out by multiplying the total power per unit surface (see footnote 2) (in Watts $m^{-2}$) perceived by eye:

$$\int_{0}^{\infty} P_{B}(\lambda) \times V(\lambda) \, d\lambda$$

(3)

by a conversion factor $K_m$ called the ‘mechanical equivalent of the lumen’ such that:

$$\text{Perceived light} = K_m \int_{0}^{\infty} P_{B}(\lambda)V(\lambda) \, d\lambda.$$  

(4)

It is important to notice that we are carrying the integration over all positive frequencies and not the visible spectrum (wavelength interval $[380 \text{ nm}, 780 \text{ nm}]$) since we are dealing with radiation energy ($\lambda \in [0, \infty]$) and counting on $V(\lambda)$ to quantify perception.

In order to evaluate $K_m$, we use the old definition of the candela and get:

$$K_m = \frac{60 \, \text{cd}/\text{cm}^2/\text{sr}}{\int_{0}^{\infty} P_{B}(\lambda) V(\lambda) \, d\lambda}.$$  

(5)

Transforming to SI units, the numerator becomes $6 \times 10^5$ lumens $m^{-2}$ (since lumen = cd sr$^{-1}$, i.e. candela per unit solid angle). The numerical evaluation of the denominator integral
Figure 3. $\eta_P$ versus temperature of the black-body. Notice that the maximum is 95 lm Watt$^{-1}$ and that the temperature is about 7000 K.

gives 883.6 Watt m$^{-2}$ resulting in $K_m = 679$ lm Watt$^{-1}$ (lumens Watt$^{-1}$) which is close to 683 lm Watt$^{-1}$, the standard value adopted by the SI$^3$ system.

3. Luminous efficacy of radiation sources

3.1. Luminous efficacy of black-body radiation sources

Having determined $K_m$ we are now in the position of determining the luminous efficacy of any radiation source.

The value of $\eta_P$ for any radiation source characterized by an EPS function $P(\lambda)$ (akin to the black-body $P_B(\lambda)$) is given by:

$$\eta_P = K_m \frac{\int_{\lambda} P(\lambda)V(\lambda) \, d\lambda}{\int_{\lambda} P(\lambda) \, d\lambda}. \quad (6)$$

In view of the perceived light definition equation 4, the wavelength interval $\mathcal{D}_\lambda$ should be $[0, \infty]$, however we can redefine it in order to suit the radiation source spectrum.

In the particular case of a source of the thermal black-body type, we apply the above formula (6) with $P(\lambda) = P_B(\lambda)$ and $\mathcal{D}_\lambda = [0, \infty]$:

$$\eta_P = K_m \frac{\int_0^{\infty} P_B(\lambda)V(\lambda) \, d\lambda}{\int_0^{\infty} P_B(\lambda) \, d\lambda}. \quad (7)$$

The result is the temperature-dependent PER curve depicted in figure 3.

PER has lm Watt$^{-1}$ dimensions, thus we define efficiency as a dimensionless ratio (in %) yielding the fraction with respect to the ideal efficacy of 683 lm Watt$^{-1}$. It shows that the Sun efficacy is about 93 lm Watt$^{-1}$ (temperature about 6000 K) or an efficiency of 93/683 = 13.6% and that an ordinary tungsten light bulb based on the incandescence phenomenon [6] is about 15 lm Watt$^{-1}$ (temperature about 3000 K) or 2% only. For a candle considered as a black-body at $T = 1800$ K, we get 0.6 lm Watt$^{-1}$, which corresponds to an efficiency of 0.6/683 or about 0.1%.

Report of the 21st meeting (23–24 February 2012) of the Consultative Committee for Photometry and Radiometry (CCPR), Bureau International des Poids et Mesures (2012).
The consequence in terms of lighting is that when one acquires a 60 Watts bulb of the tungsten type, the light produced by the bulb is $60 \times 15 \text{ lm Watt}^{-1} = 900$ lumens in total and that has tremendous consequences for the quality and cost of the lighting desired.

3.2. Luminous efficacy of white radiation sources

A white source is considered as having a flat EPS function $P(\lambda)$ defined over the entire visible interval, nevertheless in practice the interval is limited and one has to define precisely the wavelength interval over which this flatness is observed.

Two cases are encountered in lighting systems.

(1) White source as a truncated black-body source. This is a black-body source taken at a temperature $T = 5800$ K with a spectrum limited by definition to $\lambda_{\text{min}} = 400$ nm and $\lambda_{\text{max}} = 700$ nm. The PER is obtained from:

$$\eta_P = \frac{\int_{\lambda_{\text{min}}}^{\lambda_{\text{max}}} P(\lambda) V(\lambda) \, d\lambda}{\int_{\lambda_{\text{min}}}^{\lambda_{\text{max}}} P(\lambda) \, d\lambda}.$$ (8)

Using $P(\lambda) = P_B(\lambda)$ we get a PER of about $250 \text{ lm Watt}^{-1}$.

(2) Equal energy white source.

For instance, an ‘equal energy white source’ possesses by definition a flat EPS over the entire visible interval. Mathematically $P(\lambda) = W$ for $\lambda \in [380 \text{ nm}, 780 \text{ nm}]$ i.e. $\lambda_{\text{min}} = 380$ nm, whereas $\lambda_{\text{max}} = 780$ nm. Thus we get:

$$\eta_P = \frac{\int_{\lambda_{\text{min}}}^{\lambda_{\text{max}}} W V(\lambda) \, d\lambda}{\int_{\lambda_{\text{min}}}^{\lambda_{\text{max}}} W \, d\lambda}.$$ (9)

This yields about $179 \text{ lm Watt}^{-1}$ and leads to the conclusion that flatness is not enough to increase PER. We need a compromise between flatness and wavelength interval length.

3.3. Luminous efficacy of fluorescent sources

Fluorescent light sources are known as cold sources (‘eco light’ bulbs that are sold in Europe are also compact fluorescent light sources) as opposed to thermal (black-body like) or incandescent sources. They need a special circuit called a ballast to stabilize current and accelerate electrons in order to make them collide inelastically with a gas mixture of heavy atoms (typically mercury, terbium and argon) producing radiation.

An example EPS of the three-band type is displayed in figure 4. It shows several peaks over a finite wavelength interval in sharp contrast with the black-body spectrum, which is smooth and continuous extending over an infinite wavelength interval.

Emission by a fluorescent lamp extends over a finite interval $[\lambda_{\text{min}}, \lambda_{\text{max}}]$ with $\lambda_{\text{min}} = 380$ nm and $\lambda_{\text{max}} = 700$ nm for this case. The evaluation of the PER is done by spline interpolating the data displayed in figure 4. Using the general definition equation (8) we obtain a PER of $343 \text{ lm Watt}^{-1}$ which represents an efficiency of 50%. Usually, in fluorescent lamps, $\eta_C$ is about 20% which makes the overall value $\eta_L = 68.6 \text{ lm Watt}^{-1}$ and the total efficiency at 10%. Such efficiency is quite interesting, however the problem with fluorescent light is that it suffers from flicker (fluctuating light intensity) due to the ballast and random collision phenomena, besides it relies on mercury which is a highly environment polluting source. Some ballasts also produce an annoying type of low frequency noise$^4$.

$^4$ In ‘flicker-free’ fluorescent lamps, electronic ballasts reduce flicker substantially by converting power supply frequency to a much higher frequency such that the human eye cannot detect any fluctuation in the light intensity. Moreover, electronic ballasts produce less hum than other types.
3.4. Luminous efficacy of lasers and LEDs

White light, black-body and fluorescent radiators are considered as broadband emitters since their radiation spans (at least) the entire visible spectrum. However this is not the case of LEDs and lasers since they are somehow closer to monochromatic (narrow-band) sources.

The new CIE definition of the mechanical equivalent of the lumen is that a monochromatic source that is an EPS peaking at $\lambda_0 = 555$ nm (at the maximum sensitivity of the eye) with power of 1 W produces exactly 683 lumens.

This can be understood readily from the general PER definition:

$$\eta_P = \frac{\int_{\lambda_{\min}}^{\lambda_{\max}} \delta(\lambda - \lambda_0)V(\lambda) \, d\lambda}{\int_{\lambda_{\min}}^{\lambda_{\max}} \delta(\lambda - \lambda_0) \, d\lambda}$$

where we used: $P(\lambda) = a\delta(\lambda - \lambda_0)$ with $a$ a constant and the condition that $\lambda_0 \in [\lambda_{\min}, \lambda_{\max}]$. The integral gives the result: PER = $K_m V(\lambda_0) = K_m$ since $V(\lambda_0 = 555 \text{ nm}) = 1$.

As an application, consider a laser emitting at $\lambda_0 = 570$ nm with a power of 50 mW. It has a PER = $K_m V(\lambda_0)$ that produces 30 lumens.

Turning to lighting with LEDs, one of the main advantages of LED is that its lifetime is extremely long (on the order of 100 000 hours) because it is a rugged solid state device. Moreover it does not rely on a ballast or mercury, which makes it safer than fluorescent lamps whose lifetime is on the order of several 1000 hours.

In the LED case, the EPS function is usually approximated by a Gaussian or a superposition of several Gaussian functions. In the single Gaussian approximation $P(\lambda) = \exp\left[-\frac{(\lambda - \lambda_0)^2}{2\Delta^2}\right]$ can be used to evaluate $\eta_P$ from equation (8).

The LED is characterized by an average wavelength $\lambda_0$ and a standard deviation $\Delta \lambda$. As an example, we consider a blue LED with $\lambda_0 = 450$ nm and $\Delta \lambda = 20$ nm. The PER obtained from equation (8) is about 39.7 lm Watt$^{-1}$ and therefore an efficiency of 6%. The small efficiency is due to small overlap between the blue LED spectrum and the eye sensibility curve $V(\lambda)$, moreover that number is further reduced after multiplication by $\eta_C$ which is typically anywhere between 20% to 50% yielding a total efficiency of 1.2% to 3%.

Figure 4. EPS of a three-band type fluorescent bulb compared to eye sensitivity curve (in green). The mercury peak is around 450 nm (data adapted from Hoffmann [7]).
This is to be contrasted with the present status of white LEDs, which has a relatively large PER as illustrated by the Haitz law in figure 1. That might be due to the fact that a flat EPS enhances the PER as previously seen with white sources however the question might be asked more generally in specific terms as explained in the next section.

4. Maximum luminous efficacy of radiation sources

An important question can now be asked: for a given colour (chromaticity), is there a maximum PER that can be realized with any radiation source?

Mathematically this can be answered with the following set of assumptions. Given a source endowed with a normalized EPS function

$$P(\lambda)$$

$$\int_{\lambda_{\text{min}}}^{\lambda_{\text{max}}} P(\lambda) \, d\lambda = 1.$$  \hspace{1cm} (11)

Is it possible to find the best $$P(\lambda)$$ such that the PER given by (using equation 6):

$$\eta_P = K_m \int_{\lambda_{\text{min}}}^{\lambda_{\text{max}}} P(\lambda)V(\lambda) \, d\lambda.$$  \hspace{1cm} (12)

for some given colour represented by chromaticity coordinates $$x_c, y_c$$ (see the appendix) is maximized. Note that the above integral can be written as:

$$\eta_P = K_m \int_{\lambda_{\text{min}}}^{\lambda_{\text{max}}} P(\lambda) \bar{y}(\lambda) \, d\lambda.$$  \hspace{1cm} (13)

This stems from the identification of the eye sensitivity function $$V(\lambda)$$ by the CIE colour matching function $$\bar{y}(\lambda)$$ operating in the yellow–green part of the visible spectrum as explained in the appendix.

Thus we have an optimization problem for an unknown function $$P(\lambda)$$ subject to three constraints: normalization equation (11) and fixed colour $$(x_c, y_c)$$ constraints equation (A.1).

Following Ohta et al’s suggestion [8], we transform the problem into its discrete version by dividing the wavelength interval $$[\lambda_{\text{min}}, \lambda_{\text{max}}]$$ into $$N$$ values $$\lambda_i$$ with a step $$\Delta \lambda$$ such that the objective function to be optimized is:

$$\max \int P(\lambda) \bar{y}(\lambda) \, d\lambda \rightarrow \max \sum_{i=1}^{N} P_i \bar{y}_i.$$  \hspace{1cm} (14)

Discrete values $$P_i, \bar{x}_i, \bar{y}_i, \bar{z}_i$$ correspond to $$P(\lambda), \bar{x}(\lambda), \bar{y}(\lambda), \bar{z}(\lambda)$$ taken at $$\lambda = \lambda_i$$.

The normalization constraint equation (11) becomes:

$$\Delta \lambda \sum_{i=1}^{N} P_i = 1$$  \hspace{1cm} (15)

whereas the chromaticity constraints (see the appendix) become:

$$x_c = \frac{\sum_{i=1}^{N} P_i \bar{y}_i}{\sum_{i=1}^{N} P_i (\bar{x}_i + \bar{y}_i + \bar{z}_i)}$$

$$y_c = \frac{\sum_{i=1}^{N} P_i \bar{y}_i}{\sum_{i=1}^{N} P_i (\bar{x}_i + \bar{y}_i + \bar{z}_i)}.$$  \hspace{1cm} (16)

This can be transformed into:

$$\sum_{i=1}^{N} P_i [x_c (\bar{x}_i + \bar{y}_i + \bar{z}_i) - \bar{x}_i] = 0,$$

$$\sum_{i=1}^{N} P_i [y_c (\bar{x}_i + \bar{y}_i + \bar{z}_i) - \bar{y}_i] = 0.$$  \hspace{1cm} (17)
The problem now is expressed in the standard simplex form (see numerical recipes [9], chapter 10): one ought to find a set of \( N \) values \( P_i \) such that equation (14) is maximized under three constraints given by equations (15) and (17).

The simplex method [9] results are displayed in figure 5. The constant PER curves or iso-PER curves represented in the CIE diagram (see appendix) get closer to the CIE contour as the PER is increased.

Low values of PER are in the blue region of the CIE diagram which explains the result obtained for the blue diode, whereas larger PER values occur as we move towards the yellow part of the CIE diagram.

We find the largest value of 679 lm Watt\(^{-1}\), as before, and the corresponding iso-PER curve in the vicinity of the 555 nm region, the area of highest sensibility of the eye, confirming the candela standard once again and the SI metrological data (see footnote 3).

5. Discussion and conclusion

Some lighting metrics have been introduced and perspectives for future development regarding the increase of lighting intensity and quality were presented.

Despite the fact white LEDs are presently showing tremendous potential in terms of quality and PER increase, the diagram presented in figure 5 indicates that a white source maximum is between 350 and 400 lm Watt\(^{-1}\) which is yet to be reached by white LEDs. Presently, a white LED called LUXEON\(^{TM}\)4014 produces 150 lm Watt\(^{-1}\) with a 30 mA drive current.

The performance enhancement of white LEDs is tricky since they are generally made of a blue LED and a phosphor that converts, partially, blue light into yellow. The superposition of yellow and blue produces a broadband white light whereas the superposition of red, green and blue LEDs would produce a lower quality narrow-band white light.
Consequently, in order to increase performance of a white LED, one has to work on optimizing both blue LEDs; light production, drive current, etc, and simultaneously improve the geometrical design and chemical composition of the phosphor.

This has many important implications for, e.g., very bright light sources. For instance, car headlights employ traditionally halogen lamps (typically 25 lm Watt\(^{-1}\)) or high intensity discharge (HID) lamps (around 100 lm Watt\(^{-1}\)), also called xenon lamps.

White LED based headlights with roughly 50 to 60 lm Watt\(^{-1}\) are serious competitors to HID lamps since optimally designed optical systems with lower output loss can be easily fabricated.

On the other hand, yellow–green light sources may achieve a large increase since their maximum PER is around the theoretical standard of 683 lm Watt\(^{-1}\) adopted by both CIE and SI organizations.

The conclusion is that work must be targeted towards increasing rather the conversion efficacy \(\eta_C\) in the white sources case and in particular the white LED case.

### Appendix. CIE chromaticity coordinates

In 1931, the CIE undertook a series of historical measurements called colour matching experiments in order to calibrate colorimetry and human colour perception. A number of 'standard' observers had to make a comparison between a colour of a given wavelength \(\lambda\) and a superposition of three selected wavelengths called RGB primaries [10]. The weight of each of the three colours to perform the match was recorded. The observations were done at a fixed distance of 50 cm with two possibilities for the eye angle opening (defined by the observation diameter value) of \(2^\circ\) and \(10^\circ\).

This led to the existence of colour matching functions (CMF) \(\bar{r}(\lambda), \bar{g}(\lambda), \bar{b}(\lambda)\) corresponding to the red, green, blue weight coefficients that matched the colour \(\lambda\).

The study showed that not all matching weights are positive and that some values were negative. The algebraic values of the coefficients meaning that CMF functions took positive and negative values originating from the overlap versus wavelength between human cone sensitivities.

This pushed the CIE to perform a linear transformation over \(\bar{r}(\lambda), \bar{g}(\lambda), \bar{b}(\lambda)\) in order to define three strictly positive CMF functions \(\bar{x}(\lambda), \bar{y}(\lambda), \bar{z}(\lambda)\) displayed in figure A1.

The linear transformation is based on equal area of \(\bar{x}(\lambda), \bar{y}(\lambda), \bar{z}(\lambda)\) over the visible spectrum and the choice for the middle spectrum function \(\bar{y}(\lambda)\) to be taken equal to \(V(\lambda)\), the photopic eye sensitivity depicted in figure 2.

If we have a radiation source characterized by an EPS function \(P(\lambda)\) its tristimulus coordinates are given by:

\[
X = K_m \int_{\lambda_{\text{min}}}^{\lambda_{\text{max}}} P(\lambda)\bar{x}(\lambda) \, d\lambda,
\]

\[
Y = K_m \int_{\lambda_{\text{min}}}^{\lambda_{\text{max}}} P(\lambda)\bar{y}(\lambda) \, d\lambda,
\]

\[
Z = K_m \int_{\lambda_{\text{min}}}^{\lambda_{\text{max}}} P(\lambda)\bar{z}(\lambda) \, d\lambda.
\]  

(A.1)

Thus, \(X, Y, Z\) are respectively akin to a PER that is proper to the corresponding colour matching function, with \(Y\) corresponding to eye sensitivity since \(\bar{y}(\lambda) = V(\lambda)\).

The colour of the \(P(\lambda)\) source is given by a point with coordinates \((x, y)\) in the CIE diagram displayed in figure A2. \((x, y)\) are called chromaticity coordinates with values
Figure A1. Colour matching functions of the CIE for eye opening of 2°. Functions $\bar{x}$ (in red), $\bar{y}$ (in green) and $\bar{z}$ (in blue) cover approximately the corresponding RGB colour zones. $\bar{y}(\lambda) = V(\lambda)$ as decided by the CIE.

Figure A2. CIE diagram displaying colour of points with coordinates $(x, y)$ and the black-body radiation colour path as a function of absolute temperature. The various symbols $D_T$ correspond to daylight type source (illuminant or synthetic source) at a given temperature $T/100$. For instance $D_{65}$ is for $T = 6500$ K (adapted from Hoffmann [7]).

explicitly given by:

\[
x = \frac{X}{X + Y + Z}, \quad y = \frac{Y}{X + Y + Z}.
\]

\[\text{(A.2)}\]
The CIE diagram (called tongue or horseshoe diagram) shown in figure A2 displays several interesting characteristics.

1. The contour contains pure colours (completely saturated or free of any white content) having wavelengths indicated on the borderline in nanometres. The corresponding wavelengths are called dominant since they control the colour from pure (on the border) to white point at the centre with coordinates $x = \frac{1}{3}$, $y = \frac{1}{3}$.

2. Colours within the horseshoe diagram are unsaturated and as we move forward towards the white point they become pastel like. This stems from the increase of white content as we proceed towards the white point.

3. Black-body colour appears on a path as a function of absolute temperature. It follows the red, orange, yellow, white and finally blue sequence as temperature is increased. This describes heated metals and agrees with Kirchhoff’s observations [3].

4. Illuminants (artificial daylight sources) indicated by $D_{50}$, $D_{65}$ and $D_{93}$ appear at their corresponding colour with index (50,65,93) equal to absolute temperature divided by 100. Black-body sources with temperatures of 5000 K, 6500 K and 9300 K have colours close to the white point.

5. The CIE contour is closed from below by a straight line (also called the purple line) that does not carry any dominant wavelength. It means that most purple colours cannot be obtained by altering the white content of some main (dominant) colour as done before. This is another consequence of the cone overlap that resulted in negative CMF weights.

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