Generators for K-theoretic Hall algebras of quivers with potential

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Abstract
K-theoretic Hall algebras (KHAs) of quivers with potential \((Q, W)\) are a generalization of preprojective KHAs of quivers, which are conjecturally positive parts of the Okounkov–Smironov quantum affine algebras. In particular, preprojective KHAs are expected to satisfy a Poincaré–Birkhoff–Witt theorem. We construct semi-orthogonal decompositions of categorical Hall algebras using techniques developed by Halpern-Leistner, Ballard–Favero–Katzarkov, and Špenko–Van den Bergh. For a quotient of KHA\((Q, W)_Q\), we refine these decompositions and prove a PBW-type theorem for it. The spaces of generators of KHA\((Q, 0)_Q\) are given by (a version of) intersection K-theory of coarse moduli spaces of representations of \(Q\).

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1 Introduction

1.1 Hall algebras of quivers with potential

Let \(Q\) be a quiver with vertex set \(I\), edge set \(E\), and let \(W\) be a potential of \(Q\). For a dimension vector \(d \in \mathbb{N}^I\), denote by \(\mathcal{X}(d) = R(d)/G(d)\) the moduli stack of representations of \(Q\) of dimension \(d\). The potential induces a regular function

\[ \text{Tr } W : \mathcal{X}(d) \to \mathbb{A}^1_C. \]  

(1)

Kontsevich–Soibelman [8] defined a Cohomological Hall algebra of \((Q, W)\), inspired by the Donaldson–Thomas theory of the Jacobi algebra Jac \((Q, W)\), on the cohomol-
ogy of vanishing cycles of (1):

\[
\text{CoHA} \left( Q, W \right) := \bigoplus_{d \in \mathbb{N}} H^* \left( \mathcal{X}(d), \varphi_{\text{Tr} \, W} \mathbb{Q} \right),
\]

where the multiplication \( m = p^* q^* \) is defined using the maps

\[
\mathcal{X}(d) \times \mathcal{X}(e) \xrightarrow{q} \mathcal{X}(d, e) \xrightarrow{p} \mathcal{X}(d + e)
\]

from the stack \( \mathcal{X}(d, e) \) parametrizing pairs of representations \( A \subset B \) with \( A \) of dimension \( d \) and \( B \) of dimension \( d + e \). In [8, Section 8.1], Kontsevich–Soibelman proposed a categorification of the CoHA using categories of matrix factorizations of the regular function \( \text{Tr} \, W \):

\[
\text{HA} \left( Q, W \right) := \bigoplus_{d \in \mathbb{N}} \text{MF} \left( \mathcal{X}(d), W \right).
\]

The category \( \text{MF} \left( \mathcal{X}(d), W \right) \) is equivalent to the category of singularities \( D_{\text{sg}} \left( \mathcal{X}(d)_0 \right) \) of the zero fiber of \( \text{Tr} \, W \). The category \( \text{HA} \left( Q, W \right) \) is monoidal with respect to \( m \), see [13, Theorem 1.1], and it is called the categorical Hall algebra (HA) of \( (Q, W) \). Its \( K_0 \) is called the K-theoretic Hall algebra (KHA) of \( (Q, W) \). For any edge \( e \in E \), let \( \mathbb{C}^* \) act on \( R(d) \) by scaling the linear map corresponding to \( e \). Let \( (\mathbb{C}^*)^E \) be the product of these multiplicative groups. One can consider a graded version of Hall algebras \( \text{HA}^{gr} \left( Q, W \right) \) for any group \( \mathbb{C}^* \subset (\mathbb{C}^*)^E \) scaling \( \text{Tr} \, W \) with weight 2.

### 1.2 Preprojective Hall algebra

For a quiver \( Q \) and a dimension vector \( d \in \mathbb{N}^I \), let \( \mathfrak{P}(d) \) be the moduli stack of dimension \( d \) representations of the preprojective algebra of \( Q \). Varagnolo–Vasserot [25] studied the categorical preprojective Hall algebra of \( Q \):

\[
\text{HA} \left( Q \right) := \bigoplus_{d \in \mathbb{N}^I} \text{Db} \left( \mathfrak{P}(d) \right).
\]

Its \( K_0 \) is called the preprojective KHA of \( Q \). A \( \mathbb{C}^* \)-equivariant version is conjectured [13, Conjecture 1.2] to be the positive part of the Okounkov–Smirnov quantum affine algebra \( U_q \left( \mathfrak{g}_Q \right) \) [11]. There is also a preprojective CoHA of \( Q \) [18, 26], and a \( \mathbb{C}^* \)-equivariant version is conjectured [3] to be the positive part of the Maulik–Okounkov Yangian \( Y_{\text{MO}} \left( \mathfrak{g}_Q \right) \) [9].

To any quiver \( Q \) one associates a quiver with potential \( (\tilde{Q}, \tilde{W}) \) with a natural grading \( \mathbb{C}^* \subset (\mathbb{C}^*)^E \) such that there is an equivalence of underlying categories

\[
\text{HA}^{gr} \left( \tilde{Q}, \tilde{W} \right) \cong \text{HA} \left( Q \right),
\]

see [13, Subsection 3.2.3] for a comparison of the multiplications of the two Hall algebras.
1.3 The PBW theorem for CoHAs

Davison–Meinhardt [4, Theorem C] proved a Poincaré-Birkhoff-Witt (PBW) theorem beyond the case of \((\tilde{Q}, \tilde{W})\), namely for all pairs \((Q, W)\) where \(Q\) is symmetric. Let \(X(d)\) be the coarse space of \(\mathcal{X}(d)\). The coarse space map

\[
\pi : \mathcal{X}(d) \to X(d)
\]

induces an increasing (perverse) filtration

\[
P^{\leq i} \subset H^\cdot (\mathcal{X}(d), \varphi_{TrWQ})
\]

starting in degree 1. Denote by \(\text{gr} \, \text{CoHA}(Q, W)\) the associated graded of the CoHA \((Q, W)\) with respect to the filtration \(P^{\leq i}\). The Davison–Meinhardt PBW theorem for CoHAs says there is an isomorphism of algebras:

\[
\text{gr} \, \text{CoHA}(Q, W) \cong \text{Sym} \left( P^{\leq 1} \otimes H^\cdot (B\mathbb{C}^*) \right),
\]

where the right hand side is a super-symmetric algebra and the generator of \(H^\cdot (B\mathbb{C}^*)\) is in perverse degree 2. As a corollary of (4), there is a natural Lie algebra structure (called the BPS Lie algebra) on \(P^{\leq 1}\). There is an isomorphism of \(\mathbb{N}^I\)-graded vector spaces:

\[
P^{\leq 1} \cong \bigoplus_{d \in \mathbb{N}^I} H^\cdot (X(d), \varphi_{TrWICX(d)}).
\]

To prove the PBW theorem, Davison–Meinhardt first formulate a version of (4) for the analogous algebra of constructible sheaves on the coarse spaces \(X(d)\). The map \(\pi : \mathcal{X}(d) \to X(d)\) can be approximated by proper maps and thus \(\pi_*\) commutes with vanishing cycles. Therefore the sheaf version of (4) for a general potential follows formally from the one for \(W = 0\). The sheaf version of (4) for \(W = 0\) follows from an explicit description of the summands of \(R\pi_*IC_{\mathcal{X}(d)}\) due to Meinhardt–Reineke [10, Proposition 4.3 and Theorem 4.6].

1.4 Semi-orthogonal decompositions for categorical HA

We are interested in proving versions of (4) for categorical and K-theoretic Hall algebras. The main tools used in proving (4), intersection complexes and the BBDG decomposition theorem [2], do not admit obvious generalizations to the categorical or the K-theoretic setting.

To prove a version of (4) for categorical HAs, we replace the use of the BBDG decomposition theorem by semi-orthogonal decompositions inspired by work of Halpern-Leistner [5], Ballard–Favero–Katzarkov [1], and Špenko–Van den Bergh [19, 20].
For $w \in \mathbb{Z}$, denote by $\text{MF}(\mathcal{X}(d), W)_w$ the category of matrix factorizations on which $z \cdot \text{Id} \subseteq G(d)$ acts with weight $w$. We denote by $d = (d_1, \ldots, d_k)$ a partition of $d$, by $\mathcal{X}(d)$ the stack of representations $0 = R_0 \subset R_1 \subset \cdots \subset R_k$ such that $R_i / R_{i-1}$ has dimension $d_i$ for $1 \leq i \leq k$, and by $p_d$ and $q_d$ the natural maps

$\times_{i=1}^k \mathcal{X}(d_i) \xleftarrow{q_d} \mathcal{X}(d) \xrightarrow{p_d} \mathcal{X}(d)$.

Let $M(d)$ be the weight lattice of a maximal torus of $G(d)$, let $M(d)_\mathbb{R} := M(d) \otimes \mathbb{Z} \mathbb{R}$, let $\mathbb{S}_d$ be the Weyl group of $G(d)$, and let $\delta \in M(d)_{\mathbb{R}}^{\mathbb{S}_d}$. We define a set $S_{w}^d(\delta)$ of partitions $(d_i, w_i)_{i=1}^k$ of $(d, w)$ in Subsection 3.3.1. We define categories of generators $\mathbb{M}(d; \delta)_w \subset D^b(\mathcal{X}(d))_w$ after we state Theorem 1.1. For an ordered partition $A = (d_i, w_i)_{i=1}^k$ in $S_{w}^d(\delta)$, let

$\mathbb{M}_A(\delta) := \otimes_{i=1}^k \mathbb{M}(d_i; \delta_{A_i})_{w_i}$, \hspace{2cm} (6)

where the weights $\delta_{A_i} \in M(d_i)_{\mathbb{R}}^{\mathbb{S}_{d_i}}$ are determined by $A$ and $\delta$. Denote by $\mathbb{S}_A(\delta) := \text{MF}(\mathbb{M}_A(\delta), \oplus_{i=1}^k W_{d_i})$ the category of matrix factorizations with factors in $\mathbb{M}_A(\delta)$.

**Theorem 1.1** Assume $Q$ is symmetric. Let $\delta \in M(d)_{\mathbb{R}}^{\mathbb{S}_d}$. There is a semi-orthogonal decomposition

$\text{MF}(\mathcal{X}(d), W)_w = \left\{ p_d^* q_d^* (\mathbb{S}_A(\delta)) \right\}$,

where the right hand side is after all ordered partitions $A = (d_i, w_i)_{i=1}^k$ in $S_{w}^d(\delta)$. The functor $p_d^* q_d^*$ is fully faithful on the categories $\mathbb{S}_A(\delta)$. The order of the categories is as in Subsection 3.3.4, see also Subsection 2.8.1. There are analogous semi-orthogonal decompositions for $\text{MF}^{\mathbb{R}^r}$.

Similarly to the Davison–Meinhardt approach, the semi-orthogonal decomposition for $W$ arbitrary follows from the semi-orthogonal decomposition for $W = 0$. It is important that we formulate a categorical statement as it is not clear how to make the reduction from $W$ arbitrary to $W = 0$ directly in $K$-theory. The categories $\mathbb{M}(d)_w$ are examples of noncommutative resolutions of singularities of $X(d)$ constructed by Špenko–Van den Bergh [19]. In general, they are not equivalent to $D^b(\mathcal{Y})$ for $\mathcal{Y}$ a stack.

We now explain the definition of $\mathbb{M}(d; \delta)_w$. Denote by $M$ the weight lattice of $G(d)$, by $M_{\mathbb{R}} := M \otimes \mathbb{Z} \mathbb{R}$, by $\nu$ the sum of simple weights of $G(d)$, by $\rho$ half the sum of positive weights of $G(d)$, and by $\mathbb{W}$ the multiset of weights of $R(d)$. Consider the region

$\mathbb{W} := \left( \sum_{\beta \in \mathbb{W}[0, \beta]} \right) \oplus \mathbb{C} \nu \subseteq M_{\mathbb{R}}$, \hspace{2cm} (7)

where the Minkowski sum is over weights $\beta$ in $\mathbb{W}$. Denote by $\partial \mathbb{W}$ the boundary of $\mathbb{W}$. The full subcategory $\mathbb{M}(d; \delta)$ of $D^b(\mathcal{X}(d))$ is generated by the vector bundles
for a dominant weight of $G(d)$ such that

$$\chi + \rho + \delta \in \frac{1}{2} \mathbb{W}.$$  \hspace{1cm} (8)

The category $\mathcal{M}(d; \delta)_w$ is the subcategory of $\mathcal{M}(d; \delta)$ of complexes on which $z \cdot \text{Id}$ acts with weight $w$.

### 1.5 Filtrations on KHA

Inspired by the Davison–Meinhardt PBW theorem (4), we search for filtrations on KHA whose associated graded are $q$-deformed super-symmetric algebras. We replace the perverse filtration $P^{\leq}$ on $H^* (\mathcal{X}(d), \varphi_{Tr_W})$ with the filtration $F^{\leq}$ on $K_0(\text{MF}(\mathcal{X}(d), W))$ induced by the semi-orthogonal decompositions from Theorem 1.1. These filtrations depend on $\delta \in M(\mathcal{X}(d), W)$. For generators $x_{e,v} \in K_0(\mathcal{S}(e; \delta_e)_v)$ and $x_{f,u} \in K_0(\mathcal{S}(f; \delta_f)_u)$, we show that

$$x_{e,v} \cdot x_{f,u} = \left( x_{f,u} q^{\gamma(f,e)} \right) \cdot \left( x_{e,v} q^{-\delta(f,e)} \right)$$  \hspace{1cm} (9)

in the associated graded $\text{gr} K_0(\text{MF}(\mathcal{X}(d), W))$ with respect to the filtration $F^{\leq}$, where the factors $q^{\gamma(f,e)}, q^{-\delta(f,e)} \in K_0^{T(e) \times T(f)}(\text{pt})$ depend only on $e$ and $f$, see Proposition 4.5, part (c) for a precise categorical analogue.

### 1.6 The PBW theorem for KHAs

The categories $\mathcal{M}(d; \delta)_w$ may contain complexes generated in smaller dimensions, that is, complexes in the image of $p_{d,*}q_d^*$ for $d = (d_i)_{i=1}^k$ a partition of $d$ with $k \geq 2$. These partitions are indexed by a set $T^d_w(\delta)$ defined in Subsection 3.3.1. For a partition $A = (d_i, w_i)_{i=1}^k$ in $T^d_w(\delta)$, we define a coproduct-type map

$$\Delta_A : K_0(\text{MF}(\mathcal{X}(d), W)) \rightarrow K_0(\mathcal{S}(\delta)).$$

The inclusion $i_d : \mathcal{X}(d)_0 \hookrightarrow \mathcal{X}(d)$ induces an algebra morphism

$$i_* : \text{KHA}(Q, W) \rightarrow \text{KHA}(Q, 0),$$  \hspace{1cm} (10)

see [13, Proposition 3.6]. Denote its image by $\text{KHA}(Q, W)'$. It is a $\mathbb{N}^I \times \mathbb{Z}$-graded algebra and denote its $(d, w) \in \mathbb{N}^I \times \mathbb{Z}$-graded part by $\text{KHA}(Q, W)'_{d,w}$. For $d \in \mathbb{N}^I$ and $w \in \mathbb{Z}$, let $P(d; \delta)_w \subset \text{KHA}(Q, W)'_{d,w}$ be the space of primitive-like elements with respect to $\Delta$, see Subsection 5.2 for a precise definition. For $A = (d_i, w_i)_{i=1}^k$ in a set of partitions $U^d_w(\delta)$, define

$$P_A(\delta) := \otimes_{i=1}^k P(d_i; \delta_{Ai})_{w_i},$$
where the weights $\delta_{Aj}$ are determined by $A$ and $\delta$. Any summand $(d', w')$ of a partition of $(d, w)$ in $S_w^d(\delta)$ has a corresponding Weyl invariant weight $\delta'$. Denote by $T_w^d(\delta)$ the set of two terms partitions in $T_w^d(\delta')$ for summands $(d', w')$ as above.

**Theorem 1.2** Let $Q$ be a symmetric quiver with potential $W$. Let $d \in \mathbb{N}^I$, $w \in \mathbb{Z}$, and $\delta_d \in M(d)^{\mathbb{Q}_d}$. Then $\text{KHA}(Q, W)'_{d,w}$ is generated (as a $\mathbb{Q}$-vector space) by $P_A(\delta)$ for $A \in U_w^d(\delta)$. The relations between elements in $P_A(\delta)$ are generated by

$$x_{e,v} \cdot x_{f,u} = (x_{f,u}q^{\gamma(f,e)}) \cdot (x_{e,v}q^{-\delta(f,e)})$$

(11)

for $x_{e,v} \in P(e; \delta_e)_v$ and $x_{f,u} \in P(f; \delta_f)_u$ for a partition $(e, v), (f, u)$ in $T_w^d(\delta)$, and where the factors $q^{\gamma(f,e)}, q^{-\delta(f,e)} \in K^0(T(e) \times T(f)(pt))$ depend only on $e$ and $f$, see Subsection 4.3.3 for their definitions.

It suffices to prove Theorem 1.2 for $W = 0$. In this case, Theorem 1.2 follows from Theorem 1.1 and computations using shuffle formulas from [15, Section 5]. In particular, there is an isomorphism

$$P(d; 0)_0 \cong IK_0(X(d)).$$

The right hand side is the intersection $K$-theory space (with rational coefficients) of $X(d)$ introduced in [15], where a definition of intersection $K$-theory was proposed for a more general class of Artin stacks with good moduli spaces. The spaces $P(d; 0)_0$ can be viewed as twisted by $w$ versions of $IK_0(X(d))$. There is a Chern character map to the space of generators of the CoHA:

$$\text{ch} : \bigoplus_{d \in \mathbb{N}^I} P(d; 0)_0 \rightarrow P^{\leq 1}.$$  

Using the equivalence (3), we obtain versions of Theorems 1.1 and 1.2 for the categorical preprojective Hall algebra $\text{HA}(Q)$.

### 1.7 Past and future work

Using a theorem of Toda [21] describing moduli stacks of sheaves on surfaces via quivers with potentials, we prove an analogue of Theorem 1.1 for categorical Hall algebras of surfaces in [16].

There are versions of the KHA for any stability condition of $Q$. Davison–Meinhardt proved the PBW theorem for a generic stability condition on an arbitrary pair $(Q, W)$; when the stability condition is zero, $Q$ has to be symmetric. There are also $T$-equivariant versions of the KHA for a torus $T \subset (\mathbb{C}^*)^E$ which fixes $W$. It is interesting to prove versions of Theorems 1.1 and 1.2 in these more general cases. In the case of tripled quivers and certain equivariant actions and for a Hall algebra constructed using nilpotent representations of $Q$, the algebra morphism (10) is an inclusion [25, Subsection 2.4.1], see also [18, Remark 4.7] for the analogous statement in cohomology.
and thus an equivariant version of Theorem 1.2 would imply a PBW theorem for the full preprojective KHA.

It is also interesting to study the dependence on $\delta$ of the spaces of generators $M(d; \delta)_w$ and $P(d; \delta)_w$. We plan to return to these problems in future work.

1.8 Structure of the paper

In Sect. 2, we introduce notations and recall the definitions of categories of singularities and matrix factorizations. In Sect. 3, we prove preliminary results about weight spaces of $G(d)$ and partitions of $(d, w)$ for $d \in \mathbb{N}^I$ and $w \in \mathbb{Z}$. In Sect. 4, we prove Theorem 1.1, we discuss a categorical version of the $q$-deformed commutator, and prove a categorical analogue of (9). In Sect. 5, we construct the coproduct-type map $\Delta$ and prove Theorem 1.2.

2 Notations and preliminaries

2.1 Table

We list the most frequent notations used in the paper:

2.2 Notations

All stacks and schemes considered are over $\mathbb{C}$. For a stack $\mathcal{X}$, denote by $\text{Coh}(\mathcal{X})$ the abelian category of sheaves on $\mathcal{X}$, by $D^b(\mathcal{X})$ the derived category of bounded complexes of coherent sheaves on $\mathcal{X}$, and by $\text{Perf}(\mathcal{X})$ the category of perfect complexes on $\mathcal{X}$. The categories considered are dg and we denote by $\otimes$ the product of dg categories [7, Subsections 2.2 and 2.3] (Fig. 1).

The functors considered in the paper are derived unless otherwise stated. For a morphism $f$, denote by $L_f$ its cotangent complex. For $\mathcal{X}$ a stack, denote by $L_{\mathcal{X}}$ the cotangent complex of $\mathcal{X} \to \text{Spec } \mathbb{C}$.

We denote by $K_0$ the Grothendieck group of a triangulated category. For $\mathcal{X}$ a stack, denote by $K_0(\mathcal{X}) := K_0(\text{Perf}(\mathcal{X}))$ and by $G_0(\mathcal{X}) := K_0(\mathcal{D}^b(\mathcal{X}))$. For $M$ a $\mathbb{Z}$-module, denote by $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$.

2.2.1. Let $Q = (I, E)$ be a symmetric quiver with source and target maps $s, t : E \to I$. Let $W$ be a potential. For $d \in \mathbb{N}^I$, let

$$\mathcal{X}(d) := R(d)/G(d)$$

be the stack of representations of $Q$ of dimension $d$. For $d, e \in \mathbb{N}^I$, let

$$\chi(d, e) := \sum_{i \in I} d_i e_i - \sum_{a \in E} d_{s(a)} e_{t(a)}.$$
| Notation | Description | Location defined |
|----------|-------------|------------------|
| $X(d) = R(d)/G(d)$ | moduli of representations of a quiver | Subsection 2.2.1 |
| $Q^p$ | quiver with one vertex and no loops | Subsection 2.2.1 |
| $T(d) \subset G(d)$ | maximal torus | Subsection 2.2.2 |
| $M,(M_{\mathbb{R}},M^+ \text{ etc.)}$ | weight space (real weights, dominant weights etc.) | Subsection 2.2.2 |
| $N$ | coweight lattice | Subsection 2.2.2 |
| $\beta^i_j$ | weights of the standard representation | Subsection 2.2.2 |
| $\rho_d$ | half the sum of positive roots of $G(d)$ | Subsection 2.2.2 |
| $\Lambda_d$ | diagonal cocharacter | Subsection 2.2.2 |
| $\mathcal{S}_d$ | Weyl group of $G(d)$ | Subsection 2.2.2 |
| $\nu_d, \tau_d$ | Weyl invariant weights | Subsection 2.2.2 |
| $\mathcal{W}, \mathcal{W}_c$ | set of weights associated to $R(d)$ | Subsection 2.2.2 |
| $\mathcal{S}(d)$ | determinant one matrices | Subsection 2.2.2 |
| $p_{\lambda}, q_{\lambda}$ | maps used to define the Hall product | Subsection 2.3.2 |
| $n_{\lambda}$ | width of the magic categories | Subsection 2.3.2 |
| $\mu \geq \lambda$ | comparison of cocharacters | Subsection 2.3.5 |
| $\mathcal{S}_d$ | set of partitions | Subsection 2.3.2 |
| $F_r(\lambda)$ | Face of the polytope $\mathcal{W}(d)$ | Subsection 2.5.1 |
| $M(d, \delta)$ | Categories of generators | Subsection 2.7 |
| $S(d, \delta)$ | categories of generators for arbitrary potential | Subsection 2.7 |
| $r$-invariant | quantity used to decompose weights | Subsection 3.1.1 |
| $p$-invariant | quantity used to decompose weights | Subsection 3.1.1 |
| $F_r(\lambda)^{\text{int}}$ | interior of a face of the polytope $\mathcal{W}$ | Proposition 3.2 |
| $A_\chi$ | partition associated to $\chi$ | Subsection 3.3.1 |
| $S_A^{d}(\delta)$ | set of partitions used in SOD | Subsection 3.3.1 |
| $\chi_A$ | weight associated to a partition $A$ | Subsection 3.3.1 |
| $T_\mathcal{W}^{\delta}(\delta)$ | set of partitions | Subsection 3.3.2 |
| $U_\mathcal{W}^{\delta}(\delta)$ | set of partitions | Subsection 3.3.2 |
| $O$ | set used in the order of SODs | Subsection 3.3.4 |
| $M_A(\delta), S_A(\delta)$ | Hall products of categories of generators | Subsection 4.1 |
| $B_{d,w}$ | set of $(r,p)$-invariants | Subsection 4.3.1 |
| $D^p(\mathcal{X}) \leq \Lambda$ | subcategory of $D^p(\mathcal{X})$ | Subsection 4.3.2 |
| $W_{e,f}$ | sets of weights | Subsection 4.3.3 |
| $L_{e,f}, N_{e,f}, \rho_{e,f}$ | weights | Subsection 4.3.3 |
| $q^{\delta(e,f)}, q^{-\delta(e,f)}$ | equivariant monomials | Subsection 4.3.3 |
| $w_{e,f}$ | element of the Weyl group | Subsection 4.3.3 |
| $\Delta_A, \Delta_{AB}$ | coproduct-like maps | Subsection 5.1.1 |
| $K_0(S_A(\delta)')$ | quotient of $K_0(S_A(\delta))$ | Subsection 5.1.1 |
| $m \otimes m$ | twisted Hall product | Subsection 5.1.4 |
| $P(d, \delta)$ | space of primitive elements | Subsection 5.2 |
| $P_A(\delta)$ | space of Hall products of primitive elements | Subsection 5.2 |

**Fig. 1** Notation introduced in the paper
The quiver \( Q \) is symmetric, so \( \chi(d, e) = \chi(e, d) \). Let
\[
K_0^{T(d)}(pt) = \mathbb{Z} \left[ a_{ij}^{\pm 1} \middle| i \in I, 1 \leq j \leq d_i \right].
\] (12)

We denote by \( Q^o \) the quiver with one vertex and no loops.

We may use the notation \( W_d \) when we want to emphasize the dimension for the regular function (1), but we usually write \( W \). We assume that (1) has 0 as the only critical value. We denote by \( X(d)_0 \) the (derived) zero fiber of (1).

2.2.2. Let \( d \in \mathbb{N}^I \). Fix maximal torus and Borel subgroups \( T(d) \subset B(d) \subset G(d) \).

We use the convention that the weights of the Lie algebra of \( B(d) \) are negative; it determines a dominant chamber of weights of \( G(d) \). Denote by \( M \) the weight space of \( G(d) \), let \( M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R} \), and let \( M^+ \subset M \) and \( M^+_{\mathbb{R}} \subset M_{\mathbb{R}} \) be the dominant chambers. When we want to emphasize the dimension vector, we write \( M(d) \) etc. Denote by \( N \) the coweight lattice of \( G(d) \) and by \( N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} \). Let \( (\cdot, \cdot) \) be the natural pairing between \( N_{\mathbb{R}} \) and \( M_{\mathbb{R}} \).

Denote by \( \beta^i_j \) the weights of the standard representation of \( T(d) \) for \( i \in I \) and \( 1 \leq j \leq d_i \), where \( d = (d^i)_{i \in I} \in \mathbb{N}^I \), and by \( \rho_d \) half the sum of positive roots of \( G(d) \). We denote by \( 1_d := z \cdot \text{Id} \) the diagonal cocharacter of \( G(d) \). Consider the real weights
\[
\nu_d := \sum_{i \in I, j \leq d_i} \beta^i_j,
\]
\[
\tau_d := \frac{\nu_d}{\langle 1_d, \nu_d \rangle}.
\]

Denote by \( \mathcal{W} \) the multiset of weights of \( R(d) \) (counted with multiplicities). Let \( \mathcal{S}_d \) be the Weyl group of \( G(d) \) and let \( \chi \) be a weight. Denote by \( w\chi \) the standard Weyl action and by \( w \ast \psi := w(\psi + \rho) - \rho \) the shifted Weyl group action. For a weight \( \chi \) in \( M \), let \( \chi^+ \) be the dominant weight in the shifted Weyl orbit of \( \chi \) if there is such a weight, or zero otherwise. For \( \chi \in M(d)^+ \), let \( \Gamma_{G(d)}(\chi) \) be the representation of \( G(d) \) of highest weight \( \chi \); when there is no danger of confusion, we drop the subscript.

Denote by \( SG(d) := \ker(\det : G(d) \to \mathbb{C}^*) \).

2.3 Cocharacters and the multiplication map

2.3.1. The action of \( 1_d \) on \( R(d) \) is trivial. Let \( D^b(\mathcal{X}(d))_w \) be the category of complexes on which \( 1_d \) acts with weight \( w \). We have an orthogonal decomposition
\[
D^b(\mathcal{X}(d)) \cong \bigoplus_{w \in \mathbb{Z}} D^b(\mathcal{X}(d))_w.
\]
2.3.2. For a cocharacter \( \lambda : \mathbb{C}^* \to SG(d) \), consider the maps of fixed and attracting loci

\[
\mathcal{X}(d)^\lambda \xleftarrow{q_\lambda} \mathcal{X}(d)^{\lambda \geq 0} \xrightarrow{p_\lambda} \mathcal{X}(d).
\]

We say that two cocharacters \( \lambda \) and \( \lambda' \) are equivalent and write \( \lambda \sim \lambda' \) if \( \lambda \) and \( \lambda' \) have the same fixed and attracting stacks as above.

For \( \lambda \) a cocharacter of \( SG(d) \), we introduce the weights

\[
L^{\lambda > 0} := \det \mathcal{X}(d)^{\lambda > 0},
\]

\[
N^{\lambda > 0} := \det \mathcal{R}(d)^{\lambda > 0},
\]

and their analogues for \( \lambda \geq 0, \lambda < 0, \) and \( \lambda \leq 0 \).

For a cocharacter \( \lambda \) of \( SG(d) \), let \( n_\lambda = \langle \lambda, L^{\lambda > 0} \rangle \).

2.3.3. Let \( d \in \mathbb{N}^I \). We call \( d := (d_i)_{i=1}^k \) a partition of \( d \) if \( d_i \in \mathbb{N}^I \) are all non-zero and \( \sum_{i=1}^k d_i = d \). We define similarly partitions of \( (d, w) \in \mathbb{N}^I \times \mathbb{Z} \). For a cocharacter \( \lambda \) of \( SG(d) \), there is an associated partition \((d_i)_{i=1}^k\) such that

\[
\mathcal{X}(d)^{\lambda \geq 0} \cong \mathcal{X}(d) \xrightarrow{q} \mathcal{X}(d)^{\lambda} \cong \mathcal{X}(d_i),
\]

where recall the stack \( \mathcal{X}(d) \) from Subsection 1.4. Define the length \( \ell(\lambda) := k \).

Equivalence classes of antidominant cocharacters are in bijection with ordered partitions \((d_i)_{i=1}^k\) of \( d \). For an ordered partition \((d_i)_{i=1}^k\) of \( d \), fix a corresponding antidominant cocharacter \( \lambda_d \) of \( SG(d) \) which induces the maps

\[
\mathcal{X}(d)^{\lambda} \cong \mathcal{X}(d_i) \xleftarrow{q} \mathcal{X}(d)^{\lambda \geq 0} \cong \mathcal{X}(d) \xrightarrow{p} \mathcal{X}.
\]

The multiplication in the Hall algebra is induced by the functor \( p_{\lambda, \mu} q_{\lambda}^* \). We may drop the subscript \( \lambda \) in the functors \( p_{\lambda} \) and \( q_{\lambda}^* \) when the cocharacter \( \lambda \) is clear.

2.3.4. Let \( (d_i)_{i=1}^k \) be a partition of \( d \). There is an identification

\[
\bigoplus_{i=1}^k M(d_i) \cong M(d),
\]

where the simple roots \( \beta_j^i \) in \( M(d_i) \) correspond to the first \( d_i \) simple roots \( \beta_j^i \) of \( d \) etc.

2.3.5. For cocharacters \( \lambda, \mu : \mathbb{C}^* \to SG(d) \), we write \( \mu \geq \lambda \) if for every weight \( \beta \) in \( M \) with \( \langle \lambda, \beta \rangle > 0 \), we have that \( \langle \mu, \beta \rangle > 0 \).
2.4 Partitions

2.4.1. Let $e = (e_i)_{i=1}^l$ and $d = (d_i)_{i=1}^k$ be two partitions of $d \in \mathbb{N}^I$. We write $e \geq d$ if there exist integers $a_0 = 0 < a_1 < \cdots < a_{k-1} \leq a_k = l$ such that for any $0 \leq j \leq k - 1$, we have

$$\sum_{a_j+1}^{a_{j+1}} e_i = d_{j+1}.$$

There is a similarly defined order on pairs $(d, w) \in \mathbb{N}^I \times \mathbb{Z}$.

If $\lambda$ and $\mu$ are cocharacters with associated partitions $d$ and $e$ such that $\mu \geq \lambda$, then $e \geq d$.

2.4.2. We define $T$ to be the unique (oriented) tree such that:

1. each vertex is indexed by a partition $(d_1, \ldots, d_k)$ of some $d \in \mathbb{N}^I$,
2. for each $d \in \mathbb{N}^I$, there is a unique vertex indexed by the partition $(d)$ of size one,
3. if $\bullet$ is a vertex indexed by $(d_1, \ldots, d_k)$ and $d_{m} = (e_1, \ldots, e_s)$ is a partition of $d_m$ for some $1 \leq m \leq k$, then there is a unique vertex $\bullet'$ indexed by $(d_1, \ldots, d_{m-1}, e_1, \ldots, e_s, d_{m+1}, \ldots, d_k)$ and with an edge from $\bullet$ to $\bullet'$, and
4. all edges in $T$ are as in (3).

Note that each partition $(d_1, \ldots, d_k)$ of some $d \in \mathbb{N}^I$ gives an index of some (not necessary unique) vertex. A subtree $T \subset T$ is called a path of partitions if it is connected, contains a vertex indexed by $(d)$ for some $d \in \mathbb{N}^I$ and a unique end vertex $\bullet$. The partition $(d_1, \ldots, d_k)$ at the end vertex $\bullet$ is called the associated partition of $T$.

We define the Levi group associated to $T$ to be

$$L(T) := \times_{i=1}^k G(d_i).$$

Note that each edge $\ell \in T$ corresponds to a partition of some $\mathbb{N}^I$: if $\ell$ connects $(d_1, \ldots, d_k)$ and $(d_1, \ldots, d_{m-1}, e_1, \ldots, e_s, d_{m+1}, \ldots, d_k)$ as in (3), then $\ell$ corresponds to the partition $(e_1, \ldots, e_s)$ of $d_m$ and thus there is an associated cocharacter $\lambda_\ell$ as in Subsection 2.3.3.

2.5 Regions in weight spaces

2.5.1. The polytope $\mathbb{W}$ is defined as follows:

$$\mathbb{W} := \left( \sum_{\beta \in \mathcal{W} \circ [0, \beta]} \right) \oplus \mathbb{R} \tau_d \subset M_{\mathbb{R}}.$$

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where the Minkowski sum is over weights $\beta \in \mathcal{W}$. Denote by $\mathcal{W}_c \subset \mathcal{W}$ the hyperplane where the coefficient of $\tau_d$ is $c$. Note that

$$\mathcal{W} = \mathcal{W}_0 \oplus \mathbb{R}\tau_d.$$ 

Let $\mathcal{W}^o := \mathcal{W} \setminus \partial \mathcal{W}$.

Let $\lambda$ be a cocharacter of $SG(d)$ and let $r \geq 0$. Define the hyperplane $H_r(\lambda)$ as the locus of weights $\psi \in M_\mathbb{R}$ such that

$$\langle \lambda, \psi \rangle + \langle \lambda, rN^{\lambda > 0} \rangle = 0.$$

The boundary of the polytope $r\mathcal{W}$ is contained in the subspaces $H_r(\lambda)$ for $\lambda$ a cocharacter of $SG(d)$. Let $F_r(\lambda) := H_r(\lambda) \cap r\mathcal{W}$. When $r = \frac{1}{2}$, we drop the subscript and write $F(\lambda)$.

2.5.2. Consider a partition $(d_i)_{i=1}^k$ of $d \in \mathbb{N}^I$ with associated Levi group $L = \times_{i=1}^k G(d_i)$. Using the identification in Subsection 2.3.4, define the region

$$\mathcal{W}(L) := \left( \bigoplus_{1 \leq i \leq k} \mathcal{W}(d_i) \right)_0 \oplus \mathbb{R}\tau_d \subset \bigoplus_{1 \leq i \leq k} M(d_i) \cong M(d) \mathbb{R}.$$

2.6 Categories of singularities and matrix factorizations

2.6.1. Let $Y$ be a (quasi-smooth) scheme with an action of a reductive group $G$. Consider the quotient stack $\mathcal{Y} = Y/G$. The category of singularities of $\mathcal{Y}$ is a triangulated category defined as the quotient of triangulated categories

$$D_{sg}(\mathcal{Y}) := D^b(\mathcal{Y})/\text{Perf}(\mathcal{Y}).$$

If $\mathcal{Y}$ is smooth, then $D_{sg}(\mathcal{Y})$ is trivial. There is an exact sequence

$$K_0(\mathcal{Y}) \to G_0(\mathcal{Y}) \to K_0 \left( D_{sg}(\mathcal{Y}) \right) \to 0. \quad (14)$$

2.6.2. A reference for the next two subsections is [22, Section 2.2]. Let $\mathcal{X} = X/G$ be a quotient stack with $X$ a smooth affine scheme and consider a regular function

$$f : \mathcal{X} \to \mathbb{A}^1_{\mathbb{C}}.$$

Consider the category of matrix factorizations $\text{MF}(\mathcal{X}, f)$. It has objects $(\mathbb{Z}/2\mathbb{Z}) \times G$-equivariant factorizations $(P, d_P)$, where $P$ is a $G$-equivariant coherent sheaf, $(1)$ is the twist corresponding to a non-trivial $\mathbb{Z}/2\mathbb{Z}$-character on $X$, and

$$d_P : P \to P(1)$$
with \( d_P \circ d_P = f \). Alternatively, the objects of \( \text{MF}(\mathcal{X}, f) \) are tuplets

\[
(\alpha : F_1 \rightarrow F_2 : \beta),
\]

where \( F_1 \) and \( G \) are \( F_2 \)-equivariant coherent sheaves, and \( \alpha \) and \( \beta \) are \( G \)-equivariant morphisms such that \( \alpha \circ \beta \) and \( \beta \circ \alpha \) are multiplication by \( f \). Let \( \mathcal{X}_0 := 0 \times_{\mathbb{A}^1} \mathcal{X} \). The stack \( \mathcal{X}_0 \) is quasi-smooth. By a theorem of Orlov [12], there is an equivalence

\[
D_{sg}(\mathcal{X}_0) \cong \text{MF}(\mathcal{X}, f).
\]

For a triangulated subcategory \( \mathcal{A} \) of \( D^b(\mathcal{X}) \), define \( \text{MF}(A, f) \) as the full subcategory of \( \text{MF}(\mathcal{X}, f) \) with objects pairs \((P, d_P)\) with \( P \) in \( \mathcal{A} \).

2.6.3. Assume there exists an extra action of \( \mathbb{C}^* \) on \( X \) which commutes with the action of \( G \) on \( X \) and which scales \( f \) with weight 2. Denote by (1) the twist by the character \( \text{pr}_2 : G \times \mathbb{C}^* \rightarrow \mathbb{C}^* \).

Consider the category of graded matrix factorizations \( \text{MF}^{gr}(\mathcal{X}, f) \). It has objects pairs \((P, d_P)\) with \( P \) an equivariant \( G \times \mathbb{C}^* \)-sheaf on \( X \) and \( d_P : P \rightarrow P(1) \) a \( G \times \mathbb{C}^* \)-equivariant morphism. For a triangulated subcategory \( \mathcal{B} \) of \( D^b_{\mathbb{C}^*}(\mathcal{X}) \), define \( \text{MF}^{gr}(\mathcal{B}, f) \) as the full subcategory of \( \text{MF}^{gr}(\mathcal{X}, f) \) with objects pairs \((P, d_P)\) with \( P \) in \( \mathcal{B} \). The gradings used in this paper are induced by groups \( \mathbb{C}^* \) as described in Subsection 1.1. For \( f \) zero and the trivial \( \mathbb{C}^* \)-action on \( \mathcal{X} \), there is an equivalence

\[
\text{MF}^{gr}(\mathcal{X}, 0) \cong D^b(\mathcal{X}),
\]

see [22, Remark 2.3.7].

2.7 Categories of generators

Let \( \delta \in M^{\Theta, d}_{\mathbb{R}} \). The category

\[
\mathcal{M}(d; \delta)
\]

is generated by the vector bundles \( \mathcal{O}_{X(d)} \otimes \Gamma_{G(d)}(\chi) \) for \( \chi \) a dominant weight of \( G(d) \) such that

\[
\chi + \rho + \delta \in \frac{1}{2} \mathcal{W}.
\]

The category \( \mathcal{M}(d; \delta)_w \subset \mathcal{M}(d; \delta) \) is the subcategory of complexes of weight \( w \) with respect to the diagonal cocharacter. Then there is an orthogonal decomposition:

\[
\mathcal{M}(d; \delta) \cong \bigoplus_{w \in \mathbb{Z}} \mathcal{M}(d; \delta)_w
\]
The category $\mathbb{M}(d; \delta)$ can be also described as the subcategory of $D^b(\mathcal{X}(d))$ of complexes $\mathcal{F}$ such that

$$\langle \lambda, \mathcal{F}|_0 \rangle \subset \left[ \frac{-n_\lambda}{2} - \langle \lambda, \delta \rangle, \frac{n_\lambda}{2} - \langle \lambda, \delta \rangle \right],$$

for all cocharacters $\lambda$ of $SG(d)$, where, by abuse of notation, $\langle \lambda, \mathcal{F}|_0 \rangle$ is the set of weights of $\lambda$ on $\mathcal{F}|_0$, the restriction of $\mathcal{F}$ to the origin 0 in $R(d)$, see [6, Section 2].

Let $W$ be a potential for $Q$. Denote by $\mathbb{S}(d; \delta)_w := MF(\mathbb{M}(d; \delta)_w, W)$. For a given grading, denote by $\mathbb{S}^{gr}(d; \delta)_w := MF^{gr}(\mathbb{M}(d; \delta)_w, W)$.

### 2.8 Semi-orthogonal decomposition

2.8.1. Let $I$ be a set. Assume there is a set $O \subset I \times I$ such that for any $i, j \in I$ we have that $(i, j) \in O$, or $(j, i) \in O$, or both $(i, j) \in O$ and $(j, i) \in O$. An element $o \in I$ is minimal if $I \times o \subset O$. An element $m \in I$ is maximal if $m \times I \subset O$.

Let $\mathbb{T}$ be a triangulated category. We will construct semi-orthogonal decompositions

$$\mathbb{T} = \langle A_i \rangle,$$

by subcategories $A_i$ with $i \in I$ such that for any $i, j \in I$ with $(i, j) \in O$ and objects $A_i \in A_i, A_j \in A_j$, we have that:

$$\text{Ext}^a_T(A_j, A_i) = 0$$

for any $a \in \mathbb{Z}$.

If $o$ is a minimal element in $I$, then the inclusion $A_o \hookrightarrow \mathbb{T}$ admits a right adjoint $\mathbb{T} \to A_o$. If $m$ is a maximal element in $I$, then the inclusion $A_m \hookrightarrow \mathbb{T}$ admits a left adjoint $R : \mathbb{T} \to A_m$.

### 3 Admissible sets

For $Q$ a quiver and $d \in \mathbb{N}^I$, denote by $Q^d$ the subquiver of $Q$ with vertices $i \in I$ with $d_i \neq 0$. Recall that $Q^0$ is the quiver with one vertex and no loops.

#### 3.1 The $r$-invariant

3.1.1. Assume that $Q = Q^d$ is connected and that $Q$ is not $Q^0$. Denote by $\mathbb{R}^{\leq 0} := (-\infty, 0] \subset \mathbb{R}$. Then

$$\mathbb{R}r_d \oplus \bigoplus_{\beta \in \mathcal{W}} \mathbb{R}^{\leq 0} \beta = M_{\mathbb{R}}.$$
Indeed, $Q$ is symmetric, so it suffices to show that all the weights $\beta^*_a - \beta^*_b$ are in the $\mathbb{R}$-span of $\beta \in \mathcal{W}$ for $i, j \in I$, $1 \leq a \leq d_i$, $1 \leq b \leq d_j$. This is true as there is a path between $i$ and $j$.

For $\chi$ a weight in $M_{\mathbb{R}}$, define its $r$-invariant to be the smallest nonnegative real number $r$ such that $\chi \in r \partial \mathcal{W}$. For $\chi$ a real weight in $M_{\mathbb{R}}$ such that $\chi \in r \partial \mathcal{W}$, we have that:

$$ r = \max \frac{\langle \lambda, \chi \rangle}{\langle \lambda, N^\lambda \rangle} = -\min \frac{\langle \lambda, \chi \rangle}{\langle \lambda, N^\lambda \rangle}, $$

where the min and max are taken over all $SG(d)$ cocharacters $\lambda$, see [6, Lemma 2.9] applied to the $T(d)$-representation $R(d)$, for $\varepsilon = 0$, and for the (slightly) larger polytope $\mathcal{W}$ considered in this paper. When $\chi \in M_{\mathbb{R}}^+$, we have that:

$$ r = -\min \frac{\langle \lambda, \chi \rangle}{\langle \lambda, N^\lambda \rangle}, $$

where the min is taken over all $SG(d)$ antidominant cocharacters $\lambda$.

Recall that $\mathcal{W}$ is the set of weights of $R(d)$ counted with multiplicity. If $\chi \in M_{\mathbb{R}}$ has $r(\chi) = r$, then there are coefficients $-r \leq c_{\beta} \leq 0$ for $\beta \in \mathcal{W}$ and $c \in \mathbb{R}$ such that

$$ \chi = \sum_{\beta \in \mathcal{W}} c_{\beta} \beta + c \tau_d. \quad (15) $$

The $p$-invariant for $\chi$ with $r(\chi) = r$ is defined as the smallest number of coefficients $c_{\beta}$ for $\beta \in \mathcal{W}$ equal to $r$ in a formula (15).

**Proposition 3.1** (a) Two weights of dimension $d$ that differ by a multiple of $\tau_d$ have the same $(r, p)$-invariant.

(b) Let $\chi$ be a weight, $w \in \mathcal{G}_d$, and $\delta \in M_{\mathbb{R}}^{\mathcal{G}_d}$. Then

$$ (r, p)(\chi + \rho + \delta) = (r, p)(w * \chi + \rho + \delta). $$

**Proof** The proof of (a) is clear. To prove (b), observe that $\mathcal{W} \subset M_{\mathbb{R}}$ is Weyl invariant and that $w * \chi + \rho + \delta = w(\chi + \rho) - \rho + \rho + \delta = w(\chi + \rho + \delta)$.

The following is a stronger form of [6, Lemma 3.12].

**Proposition 3.2** Let $\chi \in M_{\mathbb{R}}$ be a real weight with $r(\chi) = r$ and let $\lambda$ be a cocharacter of $SG(d)$ with $\chi \in F_r(\lambda)$. Then there are coefficients $c \in \mathbb{R}$ and

$$ c_{\beta} = \begin{cases} 0 & \text{if } \langle \lambda, \beta \rangle < 0, \\ -r & \text{if } \langle \lambda, \beta \rangle > 0, \\ \text{in } [-r, 0] & \text{if } \langle \lambda, \beta \rangle = 0 \end{cases} $$
such that

\[ \chi = \sum_{\beta \in \mathcal{W}} c_{\beta} \beta + c_{\tau} d. \]  

(16)

Conversely, every \( \chi \in r\mathcal{W} \) of the form (16) is on \( F_r(\lambda) \). Thus if \( \mu \) is a cocharacter with \( F_r(\mu) \subset F_r(\lambda) \), then \( \mu \geq \lambda \). Define

\[ F_r(\lambda)^{\text{int}} := F_r(\lambda) \setminus \bigcup_{\mu > \lambda} F_r(\mu). \]

If \( \chi \in F_r(\lambda)^{\text{int}} \), then we can choose the coefficients \( c_{\beta} \in (-r, 0] \) in (16) for \( \langle \lambda, \beta \rangle = 0 \).

**Proof** Let \( \alpha \in M_{\mathbb{R}} \) be a small weight with \( \langle \lambda, \alpha \rangle > 0 \). Then \( \chi - \alpha \) will not be in \( r\mathcal{W} \) and thus it will have \( r(\chi - \alpha) > r(\chi) \). Further, \( \chi + \alpha \) will still be inside \( r\mathcal{W} \), and thus \( r(\chi + \alpha) \leq r(\chi) \). In the above expression

\[ \chi = \sum_{\beta \in \mathcal{W}} c_{\beta} \beta + c_{\tau} d, \]

assume that there exists a weight \( \langle \lambda, \beta \rangle > 0 \) such that its coefficients \( c_{\beta} > -r \). But then for small \( \varepsilon > 0 \), we will have that

\[ \chi - \varepsilon \beta = \sum_{\beta \in \mathcal{W}} c'_{\beta} \beta + c_{\tau} d \]

with all coefficients \( -r \leq c'_{\beta} \leq 0 \), so \( \chi + \varepsilon \beta \in r\mathcal{W} \), which contradicts the above observation. Thus \( c_{\beta} = -r \) for all \( \beta \in \mathcal{W} \) with \( \langle \lambda, \beta \rangle > 0 \). The argument for why \( c_{\beta} = 0 \) when \( \langle \lambda, \beta \rangle < 0 \) is similar. The converse statement is clear. If \( F_r(\mu) \subset F_r(\lambda) \), then \( \{ \beta | \langle \lambda, \beta \rangle > 0 \} \subset \{ \beta | \langle \mu, \beta \rangle > 0 \} \), so \( \mu \geq \lambda \).

Assume next that \( \chi \in F_r(\lambda)^{\text{int}} \). We show that one can choose the coefficients \( 0 \geq c_{\beta} > -r \) for weights \( \beta \) such that \( \langle \lambda, \beta \rangle = 0 \). Let \( \psi := \chi + rN^\lambda > 0 \). Then

\[ \psi = \sum_{\beta \in \mathcal{W}^\lambda} c_{\beta} \beta + c_{\tau} d, \]

where \( \mathcal{W}^\lambda \) is the multiset \( \{ \beta | \langle \lambda, \beta \rangle = 0 \} \) and \( -r \leq c_{\beta} \leq 0 \). If \( r(\psi) = r \), then by the above argument there exists an antidominant cocharacter \( \mu \) such that

\[ \psi = \sum_{\beta \in \mathcal{W}^\lambda} c_{\beta} \beta + c_{\tau} d, \]

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and the coefficients are
\[ c_\beta = \begin{cases} 
0 & \text{if } \langle \mu, \beta \rangle < 0, \\
-r & \text{if } \langle \mu, \beta \rangle > 0, \\
\text{in } [-r, 0] & \text{if } \langle \mu, \beta \rangle = 0.
\end{cases} \]

Say that \( \lambda \) has associated Levi group \( L(\lambda) \subset G(d) \) and that \( \mu \) has associated Levi group \( L(\mu) \subset L(\lambda) \). Let \( \mu' \) be a cocharacter of \( SG(d) \) with associated Levi \( L(\mu) \subset G(d) \). We then have that
\[ \langle \mu', \chi \rangle + r \langle \mu', N^{\mu',>0} \rangle = 0. \]

This means that \( \chi \in F_r(\mu') \). We have \( \mu' \geq \lambda \) and \( \chi \in F_r(\lambda)^{\text{int}} \), so \( \lambda = \mu' \), which implies that \( r(\psi) < r \). We can thus choose all the coefficients \( c_\beta \) to be \( -r < c_\beta \leq 0 \) for weights \( \beta \) with \( \langle \lambda, \beta \rangle = 0 \).

**Proposition 3.3** Let \( \chi \) be a weight in \( M_\mathbb{R} \) with \( r(\chi) = r \geq \frac{1}{2} \). Assume that there is a set \( J \subset \mathcal{W} \) and \( \psi \) a sum of weights in \( \mathcal{W} \setminus J \) with positive coefficients < \( r \) such that
\[ \chi = -r \sum J \beta - \psi. \]

Assume that \( \chi \in F_r(\lambda)^{\text{int}} \). Then \( \{ \beta | \langle \lambda, \beta \rangle > 0 \} \subset J \).

**Proof** Denote by \( \alpha_+ \) a sum of weights in \( \{ \beta | \langle \lambda, \beta \rangle > 0 \} \) with positive coefficients, by \( \alpha_0 \) a sum of weights in \( \{ \beta | \langle \lambda, \beta \rangle = 0 \} \) with positive coefficients, and by \( \alpha_- \) a sum of weights in \( \{ \beta | \langle \lambda, \beta \rangle < 0 \} \) with positive coefficients. Using Proposition 3.2, there is a weight \( \phi_0 \) with \( r(\phi_0) < r \) such that
\[ \chi = -r N^{\lambda,>0} + \phi_0. \]

Let \( J_+ = J \cap \{ \beta | \langle \lambda, \beta \rangle > 0 \} \) etc. Choose \( \psi_+, \psi_0, \psi_- \) such that
\[ \chi = -r \sum J \beta - \psi_+ - \psi_0 - \psi_. \]

We can thus write
\[ r N^{\lambda,>0} - r \sum J_+ \beta - \psi_+ = r \sum J_0 \beta + r \sum J_- \beta + \psi_0 - \phi_0 + \psi_. \]

The right hand side has \( \lambda \)-weight \( \leq 0 \), while the left hand side has \( \lambda \)-weight \( \geq 0 \). The \( \lambda \)-weight of the left hand side is thus zero and this implies that \( J_+ = \{ \beta | \langle \lambda, \beta \rangle > 0 \} \).

3.1.2. The following is an immediate corollary of Proposition 3.2:
Corollary 3.4 Let $\chi \in M_R^+ \subset M^+ \subset R$ with $r(\chi) = r$ and let $\lambda$ be the antidominant cocharacter of $SG(d)$ with Levi group $L$ such that $\chi \in F_r(\lambda)^{int}$. Then
\[ \chi = -r N_{\lambda > 0} + \psi, \]
where $r(\psi) = s < r$, $\psi$ is $L$-dominant, and $\psi \in sW(L)$.

Recall the tree $T$ from Subsection 2.4.2. Applying Corollary 3.4 repeatedly, we obtain a decomposition of $\chi$, see Proposition 3.5. Before we state it, we introduce some notation.

For $d_a$ a summand of a partition of $d$, denote by $M(d_a) \subset M(d)$ the subspace as in the decomposition from Subsection 2.3.4 and let $A \subset \{1, \ldots, d\}$ be the set of indices of weights of standard representation corresponding to $M(d_a) \subset M(d)$. Assume that $j$ is a partition of a dimension $d_a \in \mathbb{N}$, alternatively, $j$ is an edge of the tree $T$ introduced in Subsection 2.4.2. Let $\lambda_j$ be the corresponding antidominant cocharacter of $T(d_a)$, see Subsection 2.4.2. Let $W_j \subset W$ be the multiset of weights with $\langle \lambda_j, \beta \rangle > 0$. Define
\[ N_j := \sum_{\beta \in W_j} \beta. \]
If $R(d) = g(d)$, we use the notation
\[ g_j := N_j. \quad (18) \]

Proposition 3.5 Let $\chi$ be a weight in $M(d)^+ \subset R$. There exists:

1. a path of partitions $T$, see Subsection 2.4.2,
2. coefficients $r_j$ for $j \in T$ such that $r_j > 1/2$ if $j$ corresponds to a partition with length $> 1$, and $r_j = 0$ otherwise; further, if $j$, $j' \in T$ are vertices corresponding to partitions with length $> 1$, and with a path from $j$ to $j'$, then $r_j > r_j' > 1/2$, and
3. weight $\psi \in W(d)$ such that:
\[ \chi = -\sum_{j \in T} r_j N_j + \psi. \quad (19) \]

We call the expansion (19) the standard form of $\chi \in M_R^+$. For future reference, we note the following:

Proposition 3.6 In the setting of Proposition 3.5, let $L$ be the Levi group corresponding to the path of partitions $T$ and let $\mathfrak{S}_L$ be the Weyl group of $L$. Then $\sum_{j \in T} r_j N_j \in M(d)^{\mathfrak{S}_L} \subset R$. 
Proof It suffices to show that, in the setting of Corollary 3.4, we have that $N^\lambda > 0 \in M(d)_{\mathbb{R}^L}$. The weight $N^\lambda > 0$ is a linear combination after the edges of $Q$:

$$N^\lambda > 0 = \sum_{e \in E} \text{Hom}(\mathbb{C}^{d_{i(e)}}, \mathbb{C}^{d_{i(e)}})^\lambda > 0 \in M(d).$$

It thus suffices to check the claim when $Q$ is an edge between vertices $i$ and $j$. Assume for simplicity that $\lambda$ corresponds to a length 2 partition $(e, f)$ of $d$. Then

$$N^\lambda > 0 = \text{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j})^\lambda > 0 = e_i \left( \sum_{a > e_j} \beta_a^j \right) - f_j \left( \sum_{a \leq e_i} \beta_a^i \right),$$

which is indeed invariant under the action of $S_e \times S_f$. \hfill \Box

3.2 A corollary of the Borel–Weyl–Bott theorem

For future reference, we state [6, Section 3.2]. For $I$ a set of weights, we denote by $\sigma_I$ the sum of the weights in $I$. Let $\chi$ be a weight in $M$. Let $\chi^+$ be the dominant Weyl-shifted conjugate of $\chi$ if it exists, and zero otherwise. Let $w$ be the element of the Weyl group such that $w^*(\chi - \sigma_I)$ is dominant or zero. It has length $\ell(w) =: \ell(I)$.

Proposition 3.7 Let $\lambda$ be a cocharacter of $SG(d)$. Recall the maps $p_\lambda$ and $q_\lambda$ from (13). Let $\chi \in M^+$. Then there is a spectral sequence converging to $p_{\lambda^*}q_{\lambda^*}^*(\mathcal{O}_{\chi(d)} \otimes \Gamma_{G(d)^+}(\chi))$ with terms $\mathcal{O}_{\chi(d)} \otimes \Gamma_{G(d)}((\chi - \sigma_I)^+)$ in degree $|I| - \ell(I)$ corresponding to subsets $I \subset \{ \beta | \langle \lambda, \beta \rangle < 0 \}$.

We will be using Proposition 3.7 in conjunction with the following computation.

Proposition 3.8 Assume that $Q = Q^d$ is connected and that $Q$ is not $Q^o$. Let $\chi \in M^+_R$ with $r(\chi) = r > \frac{1}{2}$ and let $\lambda$ be the antidominant cocharacter of $SG(d)$ with $\chi \in F_r(\lambda)^{\text{int}}$. Let $I$ be a non-empty subset of $\{ \beta | \langle \lambda, \beta \rangle < 0 \}$. Then $\chi - \sigma_I$ is in $r\mathbb{W}$. If $\chi - \sigma_I$ lies on $F_r(\mu)^{\text{int}}$, then $\mu < \lambda$. In particular, we have that

$$(r, p)(\chi - \sigma_I) < (r, p)(\chi).$$

Proof By Proposition 3.2, there are coefficients $c \in \mathbb{R}$ and

$$c_{\beta} = \begin{cases} 0 & \text{if } \langle \lambda, \beta \rangle < 0, \\ -r & \text{if } \langle \lambda, \beta \rangle > 0, \\ \text{in } (-r, 0] & \text{if } \langle \lambda, \beta \rangle = 0 \end{cases}$$

such that

$$\chi = \sum_{\beta \in \mathbb{W}} c_{\beta} \beta + c_{\tau_d}.$$
The representation $R(d)$ is symmetric. Consider the set

$$\tilde{I} := \{-\beta | \beta \in I \} \subset \{\beta | \langle \lambda, \beta \rangle > 0\}.$$ 

Then $-\sigma_I = \sum_{\tilde{I}} \beta$. This means that

$$\chi - \sigma_I = \sum \tilde{I} \beta.$$ 

where

$$c'_{\beta} = \begin{cases} c_{\beta} + 1 = -r + 1 & \text{if } \beta \in \tilde{I}, \\ c_{\beta} & \text{otherwise} \end{cases}.$$ 

If $-r + 1 > 0$, replace the weight $(-r + 1)\beta$ with the weight $(r - 1)(-\beta)$. All the coefficients in $\chi - \sigma_I$ are thus between $-r$ and 0, so $\chi - \sigma_I \in r\mathbb{W}$. Because $r > \frac{1}{2}$, the set of weights $\beta$ with $c'_{\beta} = r$ is included in $\{\beta | \langle \lambda, \beta \rangle > 0\}$; when $I$ is non-empty, this inclusion is strict. By Proposition 3.3, if $\chi - \sigma_I$ lies on $F_r(\mu)^{\text{int}}$, then $\mu < \lambda$. □

### 3.3 Admissible sets

#### 3.3.1. In this subsection, we explain how to write a dominant weight as the sum of a “large part” $\chi_A$ (which appears in setting the summands of the semi-orthogonal decomposition in Theorem 1.1) and a “small part” (which is in $\frac{1}{2}\mathbb{W}$).

We assume that $Q = Q^d$ is connected and that $Q$ is not $Q^o$. Fix $\delta \in M^B_d$. Let $\chi \in M^+$ and consider the standard form (19):

$$\chi + \rho + \delta = -\sum_{j \in T} r_j N_j + \psi,$$

for $\psi \in \frac{1}{2}\mathbb{W}$. Let $L \cong x_{i=1}^k G(d_i)$. Write $\chi = \sum_{i=1}^k \chi_i$ with $\chi_i \in M(d_i)^+$ for $1 \leq i \leq k$ and consider the associated partition

$$A = A_{\chi} := (d_i, w_i)_{i=1}^k,$$

where $w_i = \langle 1_{d_i}, \chi_i \rangle$. Let $S^d_w(\delta)$ be the set of partitions $A$ for which there exists $\chi \in M^+$ with $\langle 1_{d}, \chi \rangle = w$ and such that $A_{\chi} = A$.

Let $\chi' \in M^+$ be a different weight from $\chi$ with $A_{\chi'} = A$. Consider the standard form

$$\chi' + \rho + \delta = -\sum_{j \in T'} r'_j N'_j + \psi.$$ 

Then $T \cong T'$, and under this identification, $\lambda_j = \lambda'_j$ and $r_j = r'_j$. Indeed, the order of the cocharacters $\lambda_j$ and the coefficients $r_j$ are computed only using the weights $w_i$.
of $\chi_i$ and $\chi_i'$ for $1 \leq i \leq k$. To any such partition $A$, associate the weight

$$
\chi_A := -\sum_{j \in T} r_j N_j - \rho^{\lambda_d < 0} - \delta.
$$

(20)

For $1 \leq i \leq k$, consider the weights $\delta_{Ai} \in M(d_i)_{\mathbb{R}}$ defined by:

$$
- \chi_A = \sum_{i=1}^{k} \delta_{Ai} \text{ in } M(d)_{\mathbb{R}} \cong \bigoplus_{i=1}^{k} M(d_i)_{\mathbb{R}}.
$$

(21)

To see the weights $\delta_{Ai}$ are Weyl invariant, note that $\delta$ is $S_{d_i}$-invariant and $\rho^{\lambda_d < 0}$ is a linear combination of $\mathfrak{gl}(d_i)$, and thus also $\times_{i=1}^{k} S_{d_i}$-invariant. The claim for $\sum_{j \in T} r_j N_j$ follows from Proposition 3.6.

The sets $S^d_w(\delta)$ and the weights $\chi_A$ are used in formulating of the semi-orthogonal decomposition from Theorem 1.1.

**Remark 3.9** In general, it is not clear how to characterize the weights $\chi_A$. Given an explicit example of a quiver, one may attempt to describe the weights $\chi_A$ and to compute combinatorially the set $S^d_w(\delta)$ in order to make Theorem 1.1 more explicit.

The example of $Q$ the Jordan quiver (the quiver with one vertex and one loop) is discussed in [24], see also Subsection 4.2. Let $\chi \in M(d)^+$ with $\langle 1_d, \chi \rangle = w$ and let $\delta = -w \tau_d$. Recall the notation (18). The decomposition (19) can be written as

$$
\chi + \rho - w \tau_d = -\sum_{j \in T} r_j g_j + \sum_{i=1}^{k} (\psi_i + \rho_i),
$$

(22)

for a partition $(d_i)_{i=1}^{k}$ of $d$, where $\psi_i \in M(d_i)^+$ with $\psi_i \in \frac{1}{2}\mathfrak{h}(d_i)_0$, and $\rho_i$ is half the sum of positive roots of $\mathfrak{gl}(d_i)$.

Note that the only possibility for $\psi_i = 0$, see [24, Lemma 3.2]. Then $\chi_A = \chi$. So, in this case, there is a bijection between $S^d_w(\delta)$ and the integral dominant weights $\chi \in M(d)^+$ with $\langle 1_d, \chi \rangle = w$.

The example of $Q$ the quiver with one vertex and three loops has been considered in [14], [17]. Let $\chi \in M(d)^+$ with $\langle 1_d, \chi \rangle = w$ and let $\delta = -w \tau_d$. The analogue of the decomposition (22) is:

$$
\chi + \rho - w \tau_d = -\sum_{j \in T} 3r_j g_j + \sum_{i=1}^{k} (\psi_i + \rho_i).
$$

The weight $\chi_A$ is computed as

$$
\chi_A = w \tau_d - \sum_{j \in T} \left(3r_j - \frac{1}{2}\right) g_j.
$$
It is combinatorially more convenient (as explained in loc. cit.) to consider \( \tilde{\chi}_A := \chi_A - 2\rho \lambda_d \). Then \( \tilde{\chi}_A = w \tau_d - \sum_{j \in T} 3(r_j - \frac{1}{2}) \mathfrak{g}_j \) is a dominant weight. Because \( \tilde{\chi}_A \) is \( \times_{i=1}^k \mathfrak{S}_{d_i} \)-invariant, it is a linear combination \( \tilde{\chi}_A = \sum_{i=1}^k v_i \tau_{d_i} \). Using this, one obtains that the set \( S^d_w(\delta) \) is in bijection (see the statement of [16, Theorem 1.1], [17, Theorem 1.1]), with the set of tuplets \( (d_i, v_i)_{i=1}^k \in (\mathbb{N} \times \mathbb{Z})^k \) with sum \( (d, w) \) such that

\[
\frac{v_1}{d_1} < \ldots < \frac{v_k}{d_k}.
\]

3.3.2. We assume that \( Q = Q^d \) is connected and that \( Q \) is not \( Q^o \). Let \( \chi \in M^+ \) and assume that \( \chi + \rho + \delta \in \frac{1}{2} \mathcal{D} \mathcal{W} \). Using Corollary 3.4, write

\[
\chi + \rho + \delta = -\frac{1}{2} N^{\mathcal{W}^0} + \psi,
\]

for \( \psi \in \frac{1}{2} \mathcal{W}^0 \) and a partition \( d = (d_i)_{i=1}^k \). Write \( \chi = \sum_{i=1}^k \chi_i \) with \( \chi_i \in M(d_i)^+ \) for \( 1 \leq i \leq k \) and consider the associated partition

\[
A = A_\chi := (d_i, w_i)_{i=1}^k,
\]

where \( w_i = \langle 1_{d_i}, \chi_i \rangle \). Let \( T^d_w(\delta) \) be the set of partitions \( A \) for which there exists \( \chi \in M^+ \) with \( \langle 1_d, \chi \rangle = w \) such that \( A_\chi = A \). We define \( \chi_A \) as in the previous Subsection.

Let \( U^d_w(\delta) \) be the set of partitions \( A = (e_i, v_i)_{i=1}^l \) for which there exists a partition \( B = (d_i, w_i)_{i=1}^k \in S^d_w(\delta) \), integers

\[
a_0 = 0 < a_1 < \ldots < a_{k-1} \leq a_k = l
\]

such that for any \( 1 \leq j \leq k \), \( (e_i, v_i)_{i=a_j-1+1}^{a_j} \) is a partition in \( T^d_{w_j}(\delta_{B_j}) \).

**Remark 3.10** We continue the discussion from Remark 3.9.

If \( Q \) is the Jordan quiver, then \( T^d_w(\delta) \) is empty.

If \( Q \) is the quiver with one vertex and three loops, then \( T^d_w(\delta) \) is in bijection with the set of tuplets \( (d_i, v_i)_{i=1}^k \in (\mathbb{N} \times \mathbb{Z})^k \) with sum \( (d, w) \) such that

\[
\frac{v_1}{d_1} = \ldots = \frac{v_k}{d_k}.
\]

3.3.3. We assume that \( Q = Q^d \).

Assume first that \( Q = Q^o \). Then \( \delta \) is a multiple of \( \tau_d \). Let \( U^d_w(\delta) = S^d_w(\delta) \) be the set of partitions \( (1, w_i)_{i=1}^d \) of \( (d, w) \in \mathbb{N} \times \mathbb{Z} \) with \( w_1 \geq \ldots \geq w_d \). Let \( T^d_w(\delta) \) be empty for \( d > 1 \) and \( T^d_1(\delta) = \{(1, w)\} \).

Assume that \( Q \) is a disconnected quiver and let \( \delta \in M(d)\mathfrak{S}_d^d \). If \( Q \) is a disjoint union of connected quivers \( Q_j \) for \( j \in J \), write \( d_j \) and \( \delta_j \) for the corresponding dimension

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vector and Weyl invariant weight of $Q_j$ for $j \in J$. Let $C$ be the set of partitions $(w_j)_{j \in J}$ of $w$. Let

\[ S^d_w(\delta) := \bigsqcup_C \left( \times_{j \in J} S^d_{w_j}(\delta_j) \right), \]

\[ T^d_w(\delta) := \bigsqcup_C \left( \times_{j \in J} T^d_{w_j}(\delta_j) \right). \]

Then $U^d_w(\delta) \cong \bigsqcup_C \left( \times_{j \in J} U^d_{w_j}(\delta_j) \right)$.

3.3.4. Assume first that $Q = Q^d$ is connected and that $Q$ is not $Q^o$. Consider two partitions $A = (d_i, w_i)_{i=1}^k$ and $B = (e_i, v_i)_{i=1}^l$ in $S^d_w(\delta)$. Let $\chi_A$ and $\chi_B$ be two weights with associated sets $A$ and $B$:

\[ \chi_A + \rho + \delta := - \sum_{j \in T_A} r_{A,k} N_{A,k} + \psi_A, \]

\[ \chi_B + \rho + \delta := - \sum_{j \in T_B} r_{B,k} N_{B,k} + \psi_B. \]

The set $R \subset S^d_w(\delta) \times S^d_w(\delta)$ contains pairs $(A, B)$ for which there exists $c \geq 1$ such that $r_{A,c} > r_{B,c}$ and $r_{A,i} = r_{B,i}$ for $i < c$, or for which there exists $c \geq 1$ such that $r_{A,i} = r_{B,i}$ for $i \leq c$, $\lambda_{A,i} = \lambda_{B,i}$ for $i < c$, and $\lambda_{A,c} > \lambda_{B,c}$, or with $A = B$.

Let $O := S^d_w(\delta) \times S^d_w(\delta) \setminus R$.

For $Q^o$, let $R = \{(A, A) | A \in S^d_w(\delta)\}$, and let $O := S^d_w(\delta) \times S^d_w(\delta) \setminus R$.

Assume $Q$ is a disconnected quiver. We continue with the notation from Subsection 3.3.3. Consider the set $R_j \subset S^d_{w_j}(\delta_j) \times S^d_{w_j}(\delta_j)$ for the quiver $Q_j$, let

\[ R := \bigsqcup_C \left( \times_{j \in J} R_j \right) \subset S^d_w(\delta) \times S^d_w(\delta), \]

and let $O := S^d_w(\delta) \times S^d_w(\delta) \setminus R$.

4 Semi-orthogonal decompositions and relations in the categorical Hall algebra

4.1 Semi-orthogonal decompositions

In this Section, we prove Theorem 1.1. Let

\[ M^l_A(\delta) := \bigotimes_{i=1}^k M^l (d_i; \delta_{Ai})_{w_i}, \]

\[ S_A(\delta) := \text{MF} \left( M^l_A(\delta), \bigoplus_{i=1}^k W_{d_i} \right). \]
Given a grading, recall that \( S_A^{gr}(\delta) := MF^{gr}\left(\bigoplus_{i=1}^k W_{d_i}\right) \). We first discuss the generation statement.

**Proposition 4.1** The categories \( p_d q_d^* (M_A(\delta)) \) for partitions \( A \in S_w^d(\delta) \) generate \( D^b(\mathcal{X}(d)) \).

**Proof** We may assume that \( Q = Q^d \) is connected.

For \( Q^\alpha \), let \( A = (1, w_i)_{i=1}^d \) with \( w_1 \geq \cdots \geq w_d \). Let \( \chi_A := \sum_{i=1}^d w_i \beta_i \). We have that

\[
\mathcal{M}(1)_w \cong \mathcal{O}_{\mathcal{X}(1)}(w)
\]

in \( D^b Coh(\mathcal{X}(1)) \cong D^b Rep(C^s) \). Then \( p_d q_d^*(M_A(\delta)) \) is the category generated by \( \Gamma_{GL(d)}(\chi) \). These vector bundles generate \( Coh(\mathcal{X}(d)) \cong Rep(GL(d)) \) and thus \( D^b(\mathcal{X}(d)) \).

We next assume that \( Q \) is different from \( Q^\alpha \). Let \( \chi \) be a dominant weight for \( G(d) := G \) and let \( w = (1_d, \chi) \). It is enough to show that the sheaf \( \mathcal{O}_{\mathcal{X}(d)}(\chi) \cong \Gamma_{GL(d)}(\chi) \) is generated by the categories \( p_d q_d^*(M_A(\delta)) \) for partitions \( A \in S_w^d(\delta) \). If \( r(\chi + \rho + \delta) \leq 1 \), then \( \mathcal{O}_{\mathcal{X}(d)}(\chi) \cong \Gamma_{GL(d)}(\chi) \) is in \( \mathcal{M}(1)_w \). Assume that \( r(\chi + \rho + \delta) > 1 \). The set of \( (r, p) \)-invariants less than a fixed pair is finite. We use induction on \( (r, p)(\chi + \rho + \delta) \). By Proposition 3.5, we have

\[
\chi + \rho + \delta = - \sum_{j \in T} r_j N_j + \psi,
\]

where the sum is taken over a path of partitions \( T \) of antidominant cocharacters, \( r_j > \frac{1}{2}, \psi \in \frac{1}{2} \mathbb{W}(L) \), and \( L \cong \times_{i=1}^k G(d_i) \). Let \( \lambda = \lambda_d \). The weight \( \psi \) is \( L \)-dominant.

Let \((1_d, \psi) := v \), then \( v = w + (1_d, \delta_d) \).

Write \( \chi = \sum_{i=1}^k \chi_i \) in \( M(d)_\mathbb{R} \cong \bigoplus_{i=1}^k M(d_i)_\mathbb{R} \). The weight \( \chi \) is in \( M(d)_+ \), so \( \chi_i \in M(d_i)_+ \) for every \( 1 \leq i \leq k \). Let \( w_i = (1_d, \chi_i) \). Write \( \tau_d = \sum_{i=1}^k \tau_i \), then \( \tau_i \) is a multiple of \( \tau_d \). Then \( \psi = \sum_{i=1}^k (\psi_i + v \tau_i) \), where \( \psi_i \in \frac{1}{2} \mathbb{W}(d_i)_0 \). Consider weights \( \delta_i \in M(d_i)_\mathbb{R} \) by

\[
\rho^{\lambda < 0} + \delta + \sum_{j \in T} r_j N_j = \sum_{i=1}^k \delta_i \text{ in } M(d)_\mathbb{R} \cong \bigoplus_{i=1}^k M(d_i)_\mathbb{R}.
\]

We claim that actually \( \delta_i \in M(d_i)_\mathbb{R}^{\mathfrak{g} d_i} \). It suffices to check that the weight \( \rho^{\lambda < 0} + \delta + \sum_{j \in T} r_j N_j \) is \( \times_{i=1}^k \mathfrak{g} d_i \)-invariant. Note that \( \delta \) is \( \mathfrak{g} d \)-invariant and \( \rho^{\lambda < 0} \) is a linear combination of \( \mathfrak{g} l(d_i) \), and thus also \( \times_{i=1}^k \mathfrak{g} d_i \)-invariant. The claim for \( \sum_{j \in T} r_j N_j \) follows from Proposition 3.6.

For \( 1 \leq i \leq k \), we have that

\[
\chi_i + \rho_i + \delta_i = \psi_i + v \tau_i \in \frac{1}{2} \mathbb{W}(d_i).
\]
Then \( A = A_\chi := (d_i, w_i)_{i=1}^k \) is in \( S^d_u(\delta) \), the weights \( \delta_{Ai} \) are \( \delta_i \) for \( 1 \leq i \leq k \), and so \( \mathcal{O}_{\chi(d_i)} \otimes \Gamma_G(d_i)(\chi_i) \in \mathbb{M}(d_i; \delta_i)_{w_i} \).

Consider the complex \( \mathcal{O}_{\chi(d)} \otimes \Gamma_L(\chi) \) in \( \mathbb{M}_A \). By Proposition 3.7, there is a spectral sequence with terms \( \mathcal{O}_{\chi(d)} \otimes \Gamma_G((\chi - \sigma_I)^+) \) converging to

\[
p_d q^*_\chi (\mathcal{O}_{\chi(d)} \otimes \Gamma_L(\chi)),
\]

where \( \sigma_I \) is a sum of weights in \( I \subset \{ \beta | \langle \lambda, \beta \rangle < 0 \} \). By Propositions 3.1 and 3.8, we have that

\[
(r, p)((\chi + \rho - \sigma_I + \delta)^+) = (r, p)((\chi + \rho - \sigma_I + \delta) < (r, p)((\chi + \rho + \delta)
\]

for \( I \) a non-empty set. The conclusion thus follows.

Before we discuss the orthogonality statement, we note a preliminary computation.

**Proposition 4.2** Let \( G \) be a reductive group, \( V \) a \( G \)-representation with origin \( 0, \lambda \) a cocharacter, and \( w \in \mathbb{Z} \). Let \( X = V/G \) be the quotient stack. Assume that \( F \) and \( E \) are in \( D^b(X^\lambda \geq 0) \) such that \( F|_0 \) has \( \lambda \)-weights \( > w \) and \( E|_0 \) has \( \lambda \)-weights \( \leq w \).

Then

\[
\text{Ext}^a_{X^\lambda \geq 0} (F, E) = 0
\]

for any \( a \in \mathbb{Z} \).

**Proof** This statement is contained in [5, Amplification 3.18]. We briefly explain a proof. We can assume that \( F \) and \( E \) are locally free sheaves. The vanishing (23) is then clear for \( a \neq 0 \).

Let \( Z = V^{\lambda \geq 0}/\mathbb{C}^* \). For \( a = 0 \), it suffices to check the statement for locally free sheaves on \( Z \). We can assume that \( F = \mathcal{O}_Z(u) \) and \( E = \mathcal{O}_Z(v) \) with \( v \leq w < u \). We have that:

\[
\text{Hom}_Z (\mathcal{O}_Z(u), \mathcal{O}_Z(v)) \cong \left( \mathbb{C} \left[ (V^{\lambda \geq 0})^\vee \right] (v - u) \right)^{\mathbb{C}^*} = 0.
\]

We next discuss the orthogonality statement.

**Proposition 4.3** Recall the set \( O \subset U^d_w(\delta) \times U^d_w(\delta) \) from Subsection 3.3.4. Consider partitions \( A = (d_i, w_i)_{i=1}^k, B = (e_i, v_i)_{i=1}^l \) in \( S^d_w(\delta) \) with \( (A, B) \in O \) and with associated antidominant cocharacters \( \lambda_A \) and \( \lambda_B \). Denote by \( \chi^A = \chi^A(d)^{\lambda_A} \) and \( \chi^B = \chi^B(d)^{\lambda_B} \). Let \( \mathcal{F}_A, \mathcal{F}'_A \in \mathbb{M}_A, \mathcal{F}_B, \mathcal{F}'_B \in \mathbb{M}_B \), and \( a \in \mathbb{Z} \). Then

\[
\text{Ext}^a_{\chi^A(d)} (p_d q^*_\chi \mathcal{F}_A, p_d q^*_\chi \mathcal{F}'_A) = 0,
\]

\[
\text{Ext}^a_{\chi^A(d)} (p_d q^*_\chi \mathcal{F}_A, p_d q^*_\chi \mathcal{F}'_A) \cong \text{Ext}^a_{\chi^A} (\mathcal{F}_A, \mathcal{F}'_A).
\]
Proof We may assume that \( Q = Q^d \) is connected.

If \( Q = Q^o \), then \( d_i = e_i = 1 \). Let \( \chi_A := \sum_{i=1}^k w_i \beta_i \) and \( \chi_B := \sum_{i=1}^k v_i \beta_i \). The categories \( \text{Coh}(\mathcal{X}(d)) \cong \text{Rep}(G(d)) \) and \( \text{Coh}(\mathcal{X}(1)^{\times d}) \cong \text{Rep}(T(d)) \) are semisimple, so we can assume that \( a = 0 \). Let \( G := G(d) \) and \( T := T(d) \). We can also assume that \( \mathcal{F}_A \cong \mathcal{F}_A' \cong \Gamma(\chi_A) \) and \( \mathcal{F}_B \cong \Gamma(\chi_B) \). Then

\[
\text{Hom}_\text{Rep}(G)(\Gamma_G(\chi_A), \Gamma_G(\chi_B)) \cong 0,
\]
\[
\text{Hom}_\text{Rep}(G)(\Gamma_G(\chi_A), \Gamma_G(\chi_A)) \cong \text{Hom}_\text{Rep}(T)(\Gamma_T(\chi_A), \Gamma_T(\chi_A)) \cong \mathbb{C}.
\]

We assume next that \( Q \) is not \( Q^o \). Denote the Levi group of \( \lambda_A \) by \( L \), the Levi group of \( \lambda_B \) by \( H \), and let \( G := G(d) \). It is enough to prove the result for generators of the above categories:

\[
\mathcal{F}_A = \Gamma_L(\chi_A) \otimes \mathcal{O}_{\chi_A},
\]
\[
\mathcal{F}_A' = \Gamma_L(\chi_A') \otimes \mathcal{O}_{\chi_A},
\]
\[
\mathcal{F}_B = \Gamma_H(\chi_B) \otimes \mathcal{O}_{\chi_B}.
\]

Let \( \mu_A \) and \( \mu_B \) be antidominant cocharacters such that

\[
\chi_A + \rho + \delta, \chi_A' + \rho + \delta \in F_r(\mu_A)^{\text{int}} \subset F_r(\lambda_A),
\]
\[
\chi_B + \rho + \delta \in F_s(\mu_B)^{\text{int}} \subset F_r(\lambda_A).
\]

Note that \( \mu_A \geq \lambda_A \) and \( \mu_B \geq \lambda_B \) in the notation of Subsection 2.3.5, see Proposition 3.2. In particular, if \( \beta \) is a weight such that \( \langle \lambda_A, \beta \rangle > 0 \), then \( \langle \mu_A, \beta \rangle > 0 \) and similarly for \( B \).

Assume for simplicity that \( s > r \) or \( s = r \) and \( \mu_A \) is not \( \mu_B \), which corresponds to the case \( c = 1 \) in the definition of the comparison set \( O \) from Subsection 3.3.4. The general case follows similarly.

For the first statement, by adjunction one needs to prove

\[
\text{Ext}_A^q(\chi_A)_{\lambda_B} \geq 0 \left( p_\varepsilon^* p_d^* q_d^* \mathcal{F}_A, q_\varepsilon^* \mathcal{F}_B \right) = 0. \tag{24}
\]

We have that \( \chi_B + \rho + \delta \in F_s(\mu)^{\text{int}} \). By Propositions 3.7 and 3.8, there is a spectral sequence with vector bundles \( \mathcal{O}_{\chi_A} \otimes \Gamma_G((\chi_A - \sigma_I)^+) \) converging to \( p_d^* q_d^* \mathcal{F}_A \) with \( \chi_A + \rho - \sigma_I + \delta \in r \mathbb{N} \), where \( \sigma_I \) is the sum of weights in \( I \subset \beta \ | (\lambda_A, \beta) < 0 \). Further, if \( (\chi_A + \rho - \sigma_I + \delta)^+ \in F_r(\omega)^{\text{int}} \), then \( \omega \leq \lambda_A \). For any such set \( I \), we have that either \( s > r \) or \( s = r \) and \( \omega \) is not \( \mu_B \), and so

\[
\langle \mu_B, (\chi_A - \sigma_I)^+ \rangle > \langle \mu_B, \chi_B \rangle.
\]

The statement in (24) follows from Proposition 4.2.

For the second statement, we need to show that

\[
\text{Ext}_A^q(\chi_A)_{\lambda_B} \geq 0 \left( p_d^* p_d^* q_d^* \mathcal{F}_A, q_d^* \mathcal{F}_A' \right) \cong \text{Ext}_A^q(\mathcal{F}_A, \mathcal{F}_A'), \tag{25}
\]

\[\otimes \]

\[\circ\]

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We have that \( \chi_A' + \rho + \delta \in F_r(\mu_A) \) \( \text{int} \). We use the notations from the previous paragraph. By Proposition 3.8, if \( \chi + \rho - \sigma_I + \delta \) \( \in F_r(\omega) \) \( \text{int} \), then \( \omega \leq \lambda_A \) and equality holds only for \( I \) empty. For any non-empty \( I \), we have that

\[
\langle \mu_A, (\chi_A - \sigma_I)^+ \rangle > \langle \mu_A, \chi_A' \rangle,
\]

and the statement in (25) follows from Proposition 4.2.

\[ \square \]

**Proof of Theorem 1.1** Assume \( W = 0 \). Then the semi-orthogonal decomposition follows from Propositions 4.1 and 4.3.

Assume \( W \) is arbitrary. Let \( (d_i, w_i)_{i=1}^k \) be a partition of \( (d, w) \). Let \( D \) be a subcategory of \( \otimes_{i=1}^k D^b(\mathcal{X}(d_i))_{w_i} \) on which \( p_d^*q_d^* \) is fully faithful. Then

\[
\text{MF} \left( p_d^*q_d^* D, W_d \right) \cong p_d^* \text{MF} \left( D, p_d^* W_d \right) \cong p_d^*q_d^* \text{MF} \left( \otimes_{i=1}^k W_{d_i} \right).
\]

The semi-orthogonal decomposition for \( \text{MF}(\mathcal{X}(d), W) \) follows from the semi-orthogonal decomposition for \( W = 0 \) and [13, Proposition 2.1]. The argument for \( \text{MF}_{gr}(\mathcal{X}(d), W) \) is similar and follows from [13, Proposition 2.2].

\[ \square \]

### 4.2 The Jordan quiver

The Jordan quiver is the quiver with one vertex and one loop. Consider the Hall algebra for the Jordan quiver \( J \) and zero potential:

\[
\text{HA}(J, 0) := \bigoplus_{d \in \mathbb{N}} D^b(\mathcal{X}(d)).
\]

The categories \( \mathbb{M}(d, \delta_d)_w \) are independent of \( \delta_d \), so we drop it from the notation. The category \( \mathbb{M}(d)_{d^v} \subset D^b(\mathcal{X}(d))_{d^v} \) is generated by the line bundles \( \mathcal{O}_{\mathcal{X}(d)}(v) \) for \( v \in \mathbb{Z} \). In particular, \( \mathbb{M}(d)_{d^v} \cong D^b(\text{pt}) \). The semi-orthogonal decomposition in Theorem 1.1 is

\[
D^b(\mathcal{X}(d))_w = \left\{ \otimes_{i=1}^k \mathbb{M}(d_i)_{d^v} \right\},
\]

where the right hand side is after all partitions \( A = (d_i, d^v_i)_{i=1}^k \) of \( (d, w) \) with \( v_i > v_j \) for every \( 1 \leq i < j \leq k \). The corresponding weight of \( A \) is

\[
\chi_A = \sum_{i=1}^k v_i v_{d_i} \in M(d)^+.
\]

In \( K_0(\mathcal{X}(d)) \), we have that:

\[
\left[ \mathcal{O}_{\mathcal{X}(1)}(v) \right]^d = d! \left[ \mathcal{O}_{\mathcal{X}(d)}(v) \right].
\]
In KHA($J, 0)_Q$, the generators are $K_0(\mathcal{X}(1))_Q$.

For a discussion of Theorem 1.1 for preprojective Hall algebras and for examples in those cases, see [16].

4.3 Relations in the Hall algebra

4.3.1. Let $(d, w) \in \mathbb{N}^I \times \mathbb{Z}$ and $\delta \in M_{\mathbb{R}}^{S_d}$. Consider the set

$$B_{d,w} = \left\{ \frac{1}{2}, (r, p)(A) \mid A \in S^d_w(\delta) \right\}.$$

Let $\beta : B_{d,w} \to \mathbb{N} \setminus \{0\}$ be the order preserving bijection. Define a filtration $F_{d,w}^{\leq i}$ on $D^b(\mathcal{X}(d))_w$ containing $\mathbb{M}_A$ for $\beta((r, p)(A)) \leq i$. Then $F_{d,w}^{\leq 1} := \mathbb{M}(d; \delta)_w$. The filtration depends on $\delta$. It is not clear whether we can choose $\delta \in M_{\mathbb{R}}^{S_d}$ for all $d \in \mathbb{N}^I$ such that these filtrations are compatible with multiplication. However, once we fix a partition $A = (d_i, w_i)_{i=1}^k$ in $S^d_w(\delta)$, the filtrations on $D^b(\mathcal{X}(d_i))_{w_i}$ for the weights $\delta_{A_i}$ are compatible in the following sense. Consider integers $\ell_j \geq 1$ for $1 \leq j \leq k$ with sum $\ell$ such that $\ell_a \geq 2$ for some $1 \leq a \leq k$. Let $\alpha$ be an integer such that

$$F_{d_1,w_1}^{\leq \ell_1} \cdots F_{d_a,w_a}^{\leq \ell_a} \cdots F_{d_k,w_k}^{\leq \ell_k} \subset F_{d,w}^{\leq \ell + \alpha}.$$

Then

$$F_{d_1,w_1}^{\leq \ell_1} \cdots F_{d_a,w_a}^{\leq \ell_a - 1} \cdots F_{d_k,w_k}^{\leq \ell_k} \subset F_{d,w}^{\leq \ell - 1 + \alpha}.$$

We define the analogous filtrations for MF and MF_{gr} with the same property as above.

4.3.2. Let $A \in S^d_w(\delta)$. Consider the subcategory $D^b(\mathcal{X}(d))_{<A}$ of $D^b(\mathcal{X}(d))$ generated by the categories $\mathbb{M}_B(\delta)$ for $(r, p)(B) < (r, p)(A)$. Let $D^b(\mathcal{X}(d))_{<A}$ be the subcategory of $D^b(\mathcal{X}(d))$ generated by $\mathbb{M}_A(\delta)$ and $D^b(\mathcal{X}(d))_{<A}$. By Theorem 1.1 for $W = 0$, there is a semi-orthogonal decomposition

$$D^b(\mathcal{X}(d))_{<A} = \{ \mathbb{M}_A(\delta), D^b(\mathcal{X}(d))_{<A} \}.$$

We denote by $\Phi_A : D^b(\mathcal{X}(d))_{<A} \to \mathbb{M}_A(\delta)$ the left adjoint to the natural inclusion. Denote by MF($\mathcal{X}(d), W)_{<A}$ and MF_{gr}($\mathcal{X}(d), W)_{<A}$ the analogous categories for a general potential.

4.3.3. For $e, f \in \mathbb{N}^I$, let $\lambda_{e,f}$ be an antidominant cocharacter corresponding to the partition $(e, f)$ of $e + f$, see Subsection 2.3.3. Consider the set $\mathcal{W}_{e,f} = \{ \beta \mid \langle \lambda_{e,f}, \beta \rangle < 0 \}$. We use the notations $L_{e,f} := L^{\lambda_{e,f}}$, $N_{e,f} := N^{\lambda_{e,f}}$. Let $\rho_{e,f} := \rho^{\lambda_{e,f}} < 0$ be half the sum of positive roots pairing negatively with $\lambda_{e,f}$. Then

$$L_{e,f} = N_{e,f} - 2\rho_{e,f}.$$
Recall the notations from (12). Define the equivariant monomials
\[ q^{\gamma(e,f)} \in K_0(BG(e)) \quad \text{and} \quad q^{-\delta(e,f)} \in K_0(BG(f)) \]
by the equality:
\[ L_{e,f} = q^{\gamma(e,f)} q^{-\delta(e,f)} \in K_0(BG(e) \times BG(f)). \]

Denote also by \( N_{e,f} \) and \( L_{e,f} \) the corresponding \( G(e) \times G(f) \)-equivariant line bundles. Let \( w_{e,f} \) be the Weyl group element
\[ w_{e,f} = (w^a)_{a \in I} \in \mathfrak{S} \cong \prod_{a \in I} \mathfrak{S}_{e^a + f^a}, \]
such that for a vertex \( a \in I \):
\[ w^a(i) = \begin{cases} i + f, & \text{if } 1 \leq i \leq e, \\ i - e, & \text{if } e < i \leq e + f. \end{cases} \]

Then \( w_{f,e} = w_{e,f}^{-1} \).

**Proposition 4.4** (a) We have that \( w_{e,f} \cdot \rho - \rho = -2\rho_{f,e} \).
(b) Multiplication by \(-w_{f,e}\) gives a bijection of sets \((-w_{f,e}) \cdot \mathcal{W}_{f,e} \cong \mathcal{W}_{e,f}\).

**Proof** To simplify notation for (a), assume that there is only one vertex. The weights contributing to \( w_{e,f} \cdot \rho \) which are not positive are of the form \( \beta_{i+f} - \beta_{j-e} \) with \( 1 \leq i \leq e < j \leq e + f \). Then
\[ w_{e,f} \cdot \rho - \rho = \sum_{i \leq e < j} (\beta_{i+f} - \beta_{j-e}) = -2\rho_{f,e}. \]

For (b), the weights in \( \mathcal{W}_{f,e} \) are of the form \( \beta_i^a - \beta_j^b \) where \( a \) and \( b \) be vertices with an edge between them, \( 1 \leq i \leq f^a \), and \( f^b + 1 \leq j \leq e^b + f^b \). Then
\[ (-w_{f,e}) \cdot (\beta_i^a - \beta_j^b) = \beta_j^b - \beta_{j-f^b}^b - \beta_i^{a+e^a}. \]
The weight \( \beta_j^b - \beta_{j-f^b}^b - \beta_i^{a+e^a} \) is in \( \mathcal{W}_{e,f} \) because there is an edge between \( b \) and \( a \) as \( Q \) is symmetric, \( 1 \leq j - f^b \leq e^b \), and \( e^a + 1 \leq i + e^a \leq e^a + f^a \).

4.3.4. In this subsection, we denote by \( A \) a partition in \( \Delta_w^d(\delta_d) \) with parts \((e, v), (f, u)\) and associated weights \( \delta_e \) and \( \delta_f \), see (21).

**Proposition 4.5** Consider vector bundles \( E = \mathcal{O}(\chi(e)) \otimes \Gamma_{G(e)}(\chi_e) \in M(e; \delta_e)_v \) and \( F = \mathcal{O}(\chi(f)) \otimes \Gamma_{G(f)}(\chi_f) \in M(f; \delta_f)_u \).
(a) We have that
\[ p_{f,e} q_{f,e}^* (\mathcal{F} \boxtimes \mathcal{E} \otimes L_{f,e}[\chi(e, f)]) \in D^b(\mathcal{X})_{\leq A}. \]

(b) There is a natural isomorphism
\[ \Phi_A \left( p_{f,e} q_{f,e}^* (\mathcal{F} \boxtimes \mathcal{E} \otimes L_{f,e}[\chi(e, f)]) \right) \cong \mathcal{E} \boxtimes \mathcal{F}. \]

(c) There is a natural map
\[ p_{f,e} q_{f,e}^* (\mathcal{F} \boxtimes \mathcal{E} \otimes L_{f,e}[\chi(e, f)]) \to p_{e,f} q_{e,f}^* (\mathcal{E} \boxtimes \mathcal{F}) \]
and its cone is in \( D^b(\mathcal{X})_{< A} \).

**Proof** We may assume that \( Q = Q^d \) is connected.

If \( Q = Q^o \), then \( d = e = 1 \) and the statements follow from the Borel–Weyl–Bott theorem.

We assume next that \( Q \) is not \( Q^o \).

(a) We may assume that
\[ \mathcal{E} = \mathcal{O}_{\mathcal{X}(e)} \otimes \Gamma_G(e)(\chi_e), \]
\[ \mathcal{F} = \mathcal{O}_{\mathcal{X}(f)} \otimes \Gamma_G(f)(\chi_f). \]

Let \( H := G(f) \times G(e) \). By Proposition 3.7, there exists a spectral sequence with terms
\[ \mathcal{O}_{\mathcal{X}(d)} \otimes \Gamma_G(d) \left( (\chi_f + \chi_e + L_{f,e} - \sigma_I)^+ \right) \]
in degree \( |I| - \ell(I) \) converging to \( p_{f,e} q_{f,e}^* \left( \Gamma_H(\chi_f + \chi_e) \otimes L_{f,e} \otimes \mathcal{O}_{\mathcal{X}(e) \times \mathcal{X}(f)} \right) \),
where \( I \subset \mathcal{W}_{f,e} \). Recall that \( L_{f,e} = N_{f,e} - 2\rho_{f,e} \). Using Corollary 3.4, write
\[ \chi_e + \chi_f + \rho + \delta_d = r N_{e,f} + \psi \]
for \( r > \frac{1}{2} \) and \( r(\psi) < r \). Using Proposition 4.4, we have that
\[ \chi_f + \chi_e + N_{f,e} - 2\rho_{f,e} + \rho + \delta_d = (1 - r) N_{f,e} + w_{e,f} \psi, \]
and thus for \( I \subset \mathcal{W}_{f,e} \) we have that
\[ \chi_f + \chi_e + N_{f,e} - \sigma_I - 2\rho_{f,e} + \rho + \delta_d = (1 - r) N_{f,e} - \sigma_I + w_{e,f} \psi. \]

Denote the left hand side by \( \theta \). Using Proposition 3.3, we have \( r(\theta) \leq r \) and if \( \theta \in r \partial \mathcal{W} \), then \( I = \mathcal{W}_{f,e} \).
To show (b), observe first that the corresponding highest weight for the partial sum \( I = \mathcal{W}_{f,e} \) is

\[
w_{f,e} \ast (\chi_f + \chi_e + N_{f,e} - 2\rho_{f,e} - N_{f,e}) = \chi_e + \chi_f.
\]

The corresponding shift is \(|\mathcal{W}_{f,e}| - \ell (\mathcal{W}_{f,e}) = -\chi(e, f)\). There are thus natural isomorphisms

\[
\Phi_A \left( p_{f,e} q_{f,e}^* \left( \mathcal{F} \boxtimes \mathcal{E} \otimes L_{f,e} [\chi(e, f)] \right) \right) \sim \mathcal{E} \boxtimes \mathcal{F}
\]

for \( \mathcal{E} = \mathcal{O}_{\mathcal{X}(e)} \otimes \Gamma_G(e)(\chi_e) \) and \( \mathcal{F} = \mathcal{O}_{\mathcal{X}(f)} \otimes \Gamma_G(f)(\chi_f) \). Part (c) follows from (b) using the adjoint pair of functors \( \Phi_A : D^b(\mathcal{X}(d)) \leq_A \boxright \mathbb{M}_A(\delta) : p_\delta q^* \).

\[\square\]

5 The PBW theorem for KHAs

5.1 Preliminaries

In this Subsection, we assume that \( Q = Q^d \).

5.1.1. Assume first that \( Q = Q^d \) is connected. Consider a partition \( A \in T^d_w(\delta) \) with terms \((e, v)\) and \((f, u)\) and associated weights \( \delta_e \) and \( \delta_f \) such that

\[-\chi_A = \delta_e + \delta_f,
\]

see (21). Let \( \lambda = \lambda_{e,f} \) and \( m := -n_\lambda/2 - (\lambda, \delta) \). By the definition of the category \( \mathbb{M}(d; \delta) \) and by [5, Corollary 3.28], the functor \( p^*_\lambda \) has image in

\[p^*_\lambda : \mathbb{M}(d; \delta) \rightarrow \left\{ q^*_\lambda \left( D^b(\mathcal{X}(e) \times \mathcal{X}(f))_w \right) \right\},
\]

where the right hand side contains categories for all weights \( w \geq m \). Denote the right hand side by \( \mathbb{E} \). For all summands of \( \mathbb{E} \), the functor \( q^*_\lambda \) induces an equivalence \( D^b(\mathcal{X}(e) \times \mathcal{X}(f))_w \cong q^*_\lambda \left( D^b(\mathcal{X}(e) \times \mathcal{X}(f))_w \right) \). Consider the adjoint to the natural inclusion \( q^*_\lambda : D^b(\mathcal{X}(e) \times \mathcal{X}(f))_w \leq \mathbb{E} \):

\[
\beta_m : \mathbb{E} \rightarrow D^b(\mathcal{X}(e) \times \mathcal{X}(f))_m.
\]

The composition \( \beta_m p^*_\lambda \) induces a functor

\[
\Delta_A := \beta_m p^*_\lambda : \mathbb{M}(d; \delta)_w \rightarrow \mathbb{M}(e; \delta_e)_v \otimes \mathbb{M}(f; \delta_f)_u =: \mathbb{M}_A.
\]

It induces a functor

\[
\Delta_A : \mathbb{S}(d; \delta)_w \rightarrow \mathbb{S}_A(\delta).
\]
Recall the order on the set $T_d^d w(\delta)$ defined as in Subsection 2.4.1. Let $B > A$ be in $T_d^d w(\delta)$. The above construction also provides a functor

$$\Delta_{AB} : S_A(\delta) \rightarrow S_B(\delta).$$

If $Q$ is not connected, define the functors $\Delta$ in the natural way using the decompositions in Subsection 3.3.3.

When $Q = Q^0$, the constructions are non-zero only for $A$ and $B$ the length 1 partitions $(1, w)$ and $\Delta = \text{id}$.

5.1.2. Recall the equivalence $D_{sg}(X(d)_0) \cong \text{MF}(X(d), W)$ and the exact sequence (14). By [13, Proposition 3.6], the pushforward $i_* : X(d)_0 \rightarrow X(d)$ induces an algebra morphism

$$i_* : \text{KHA}(Q, W) \rightarrow \text{KHA}(Q, 0).$$

It can be also described using matrix factorizations

$$i_* : K_0(\text{MF}(X(d), W)) \rightarrow K_0(X(d))$$

$$[(\alpha : F \rightleftharpoons G : \beta)] \mapsto [F] - [G].$$

It also induces a map

$$i_* : K_0(S_A(\delta)) \rightarrow K_0(M_A(\delta)). \tag{26}$$

Denote the image of (26) by $K_0(S_A(\delta))'$.

**Proposition 5.1** Consider partitions $A, B \in T_d^d w(\delta)$ with $B > A$. The following diagram commutes:

\[
\begin{array}{ccc}
K_0(S_A(\delta)) & \xrightarrow{i_*} & K_0(M_A(\delta)) \\
\downarrow \Delta_{AB} & & \downarrow \Delta_{AB} \\
K_0(S_B(\delta)) & \xrightarrow{i_*} & K_0(M_B(\delta)).
\end{array}
\]

**Proof** The maps $i_*$ and $p_*^x$ commute. The functors $\beta_m$ are inverses of functors $q_{e,f}^*$, and the compatibility between $i_*$ and $d_{e,f}^*$ is treated in [13, Proof of Proposition 3.6]. \hfill \Box

5.1.3. Assume that $Q$ is connected. Consider a partition $A$ of $(d, w)$ with terms $(e, v)$ and $(f, u)$. Recall that $N_{e,f} := N_{e,f}^{i,e,f < 0}$, $\rho_{d,e} := \rho_{d,e}^{i,e,f < 0}$, and $L_{e,f}$ from Subsection 4.3.3.

Let

$$\chi_A := r N_{e,f} - \rho_{e,f} - \delta.$$
Let $A'$ be the partition of $(d, w)$ with parts $(f, u')$ and $(e, v')$ corresponding to the weight

$$\chi_{A'} := w_{e,f} \chi_A + L_{f,e}.$$  

Using Proposition 4.4, we see that

$$\chi_{A'} = (1 - r) N_{f,e} - \rho_{f,e} - \delta,$$  \hspace{1cm} (27)

so $\chi_{A'}$ corresponds to a partition $A'$ with terms $(f, u')$, $(e, v')$. Denote the transformation

$$(e, v), (f, u) \mapsto (f, u'), (e, v').$$  \hspace{1cm} (28)

by $A \mapsto A'$. If $A \in T^d_w(\delta)$, then $A' \in T^d_w(\delta)$ by the computation in (27). Define

$$sw : K_0(\mathcal{S}(e; \delta_e)_v) \boxtimes K_0(\mathcal{S}(f; \delta_f)_u) \to K_0(\mathcal{S}(f; \delta'_f)_u') \boxtimes K_0(\mathcal{S}(e; \delta'_e)_v')$$

$$y \mapsto (-1)^{\chi(e,f)} w_{e,f}(y) q^{L_{f,e}}.$$

Then $sw \circ sw = id$.

More generally, consider a partition $A = (d_i, w_i)_{i=1}^k$ of $(d, w)$. Using the transformation (28) for all pairs $(d_i, d_{i+1})$ or $1 \leq i \leq k - 1$, we obtain an action of $\mathfrak{S}_k$ on partitions $A$ of $(d, w)$ of cardinal $k$ which we denote by $A \mapsto \sigma(A)$. For $A$ and $B$ conjugate under $\mathfrak{S}_{|A|}$, we write $A \sim B$. If $A \in T^d_w(\delta)$, then all its conjugates are in $T^d_w(\delta)$. For $\sigma \in \mathfrak{S}_k$, there is a corresponding map

$$sw_{\sigma} : K_0(\mathcal{S}_A(\delta)) \to K_0(\mathcal{S}_{\sigma(A)}(\delta)).$$

If $Q = Q^\circ$, then $T^d_w$ is zero for $d > 1$ and has one element $(1, w)$ for $d = 1$. The swap morphisms are all the identity.

If $Q$ is not connected, extend the definition of $sw_{\sigma}$ using the decompositions from Subsection 3.3.3.

5.1.4. We now state the main result of this Subsection. First, consider pairs $(b, t)$, $(c, s)$, $(e, v)$, $(f, u)$, and $(d, w)$ in $\mathbb{N}^I \times \mathbb{Z}$ such that

$$(e, v) + (f, u) = (b, t) + (c, s) = (d, w).$$

We denote by $A$ the two term partition $(e, v)$, $(f, u)$, and by $B$ the two term partition $(b, t)$, $(c, s)$. Assume that $A$ and $B$ are in $T^d_w(\delta)$. Let $\mathcal{S}$ be the set of partitions $C$ of $T^d_w(\delta)$ with terms $(a_i, \alpha_i)$ for $1 \leq i \leq 4$, some of them possibly zero, such that

$$(a_1, \alpha_1) + (a_2, \alpha_2) = (e, v),$$

$$(a_3, \alpha_3) + (a_4, \alpha_4) = (f, u),$$

$$(a_1, \alpha_1) + (a_3, \alpha_3') = (b, t),$$  \hspace{1cm} (29)
\[(a_2, \alpha'_2) + (a_4, \alpha_4) = (c, s), \tag{30}\]

where \(\alpha'_2, \alpha'_3\) are defined via transformation (28). The weight \(\delta \in M^{\mathfrak{S}_d}_R\) induces corresponding weights for all the other pairs involved. For such \(C \in S\), consider the swap morphism for the transposition \((23) \in \mathfrak{S}_4\) from Subsection 5.1.3. Define

\[
\tilde{m} \boxtimes m := (m \boxtimes m) \text{sw}_{(23)} : K_0(\mathfrak{S}_C(\delta)) \to K_0(\mathfrak{S}_B(\delta)).
\]

Note that the image of \(K_0(\mathfrak{S}_C(\delta))\) under \(\text{sw}_{23}\) is in \(K_0(\mathfrak{S}_{C'}(\delta))\), where \(C'\) is the partition with terms \((a_1, \alpha_1), (a_3, \alpha'_3), (a_2, \alpha'_2),\) and \((a_4, \alpha_4)\), and thus the image of \(K_0(\mathfrak{S}_{C'}(\delta))\) under \(m \boxtimes m\) is in \(K_0(\mathfrak{S}_B(\delta))\) from the relation (29). Recall the definition of \(\text{KHA}(Q, W)'\) and \(\text{KHA}(Q, W)'_{d,w}\) from Subsection 1.6 and of \(K_0(\mathfrak{S}_A(\delta))'\) from (26).

**Theorem 5.2** The following diagram commutes:

\[
\begin{array}{ccc}
K_0(\mathfrak{S}_A(\delta))' & \xrightarrow{m} & K_0(\mathfrak{S}(d; \delta)_w)' \\
\downarrow^{\Delta_{AC}} & & \downarrow^{\Delta_{B}} \\
\bigoplus_{C \in S} K_0(\mathfrak{S}_C(\delta))' & \xrightarrow{\text{m} \boxtimes \text{m}} & K_0(\mathfrak{S}_B(\delta))'.
\end{array}
\]

**Proof of Theorem 5.2** We may assume that \(Q\) is connected.

If \(Q = Q^0\), then \(d = 1\), and the only possibility for \(A\) and \(B\) are the identity partition \((1, w)\). Thus the set \(S = \{(1, w)\}\), and the statement is immediate.

Assume that \(Q\) is not \(Q^0\). Recall the algebra morphism \(i_* : \text{KHA}(Q, W) \to \text{KHA}(Q, 0)\). By Proposition 5.1, \(i_*\) commutes with \(\Delta\). It suffices to check the statement in the zero potential case. The statement now follows from [15, Theorem 5.2]; the statement in loc. cit. for the stack \(R(d)/PGL(d)\) implies Theorem 5.2 for \(w = 0\) and \(\delta = 0\). The proof of [15, Theorem 5.2] is based on explicit shuffle formula for \(m\) and \(\delta\), and the same computation works for all weights \(w \in \mathbb{Z}\) and \(\delta \in M_{\mathfrak{S}_d}^R\). \(\Box\)

**Proposition 5.3** Let \(x_{e,v} \in \text{KHA}(Q, W)_{e,v}'\) and \(x_{f,u} \in \text{KHA}(Q, W)_{f,u}'\). Then

\[
x_{e,v} \cdot x_{f,u} = (x_{f,u} q^{\gamma(f,e)}) \cdot (x_{e,v} q^{-\delta(f,e)}).
\]

**Proof** It suffices to check the statement in the zero potential case, when it follows from a direct computation using shuffle formulas, see [15, Proposition 5.5]. \(\Box\)
5.2 Spaces of generators

One can show using induction and Theorem 1.1 that there is a Künneth-type isomorphism

\[
\bigotimes_{i=1}^k K_0 \left( S(d_i; \delta_i)_{w_i} \right)'_Q \cong K_0 \left( S_A(\delta) \right)'_Q
\]

for \( A = (d_i, w_i)_{i=1}^k \) in \( S^d_w(\delta) \).

We define inductively on \((d, w)\) a (split) subspace

\[
\ell_{d, w} : P(d; \delta)_w \hookrightarrow K_0 \left( S(d; \delta)_{w} \right)'_Q
\]

with a surjection

\[
\pi_{d, w} : K_0 \left( S(d; \delta)_{w} \right)'_Q \twoheadrightarrow P(d; \delta)_w
\]

such that \( \pi_{d, w} \ell_{d, w} = \text{id} \). For any \( A \in T^d_w(\delta) \), denote by \( |A| \) the number of parts of \( A \). For \( d \in \mathbb{N}^I \) with \( \sum_{i \in I} d^i = 1 \), let

\[
P(d; \delta)_w := K_0 \left( S(d; \delta)_{w} \right)'_Q.
\]

Let \( A = (d_i, w_i)_{i=1}^k \in T^d_w(\delta) \) with \( A > (d, w) \), or equivalently with \( k \geq 2 \). Let \( P_A(\delta) := \bigotimes_{i=1}^k P(d_i; \delta_i)_{w_i} \) and let \( \pi_A \) be the natural projection:

\[
\pi_A := \bigotimes_{i=1}^k \pi_{d_i, w_i} : K_0(S_A(\delta))'_Q \twoheadrightarrow P_A(\delta).
\]

Let \( K_A \) be the kernel of the map

\[
\left( \bigoplus_{\sigma \in S_{|A|}} \pi_{\sigma(A)} \right) \left( \bigoplus_{\sigma \in S_{|A|}} S_{\omega_\sigma} \right) \Delta_A : K_0(S(d; \delta)_w)'_Q \twoheadrightarrow \bigoplus_{\sigma \in S_{|A|}} P_{\sigma(A)}(\delta).
\]

Define

\[
P(d; \delta)_w := \bigcap_{A > (d, w)} K_A \hookrightarrow K_0 \left( S(d; \delta)_{w} \right)'_Q.
\]

Let \( O \subset T^d_w(\delta) \setminus \{(d, w)\} \) be a set which contains exactly one set in any \( \mathcal{G} \)-orbit. The following is proved as in [15, Theorem 5.13], the loc. cit. treats the case of stacks \( R(d)/PGL(d) \) and applies directly to the case when \( w = 0 \) and \( \delta = 0 \).
Proposition 5.4  
There is a decomposition
\[
P(d; \delta)_w \oplus \bigoplus_{A \in \mathcal{O}} \left( \bigoplus_{\sigma \in \mathcal{G}_|A|} P_{\sigma}(\delta) \right) \sim \mathcal{S}_w(d; \delta)_{Q'}. \]

In particular, there is a natural projection map
\[
\pi_{d,w} : K_0(\mathcal{S}(d; \delta)_w)' \twoheadrightarrow P(d; \delta)_w
\]
such that \( \pi_{d,w} \ell_{d,w} = \text{id}. \)

Proof of Theorem 1.2  
The space \( K_0(MF(X(d), W)_d) \) is generated by \( P_A(\delta) \) for \( A \in U^d_w(\delta) \) by Theorem 1.1 and Proposition 5.4. There are no relations between the generators \( y_A \) of \( P_A(\delta) \) for \( A = (d_i, w_i) \in S^d_w(\delta) \) by Theorem 1.1. The relations (11) are satisfied by Proposition 5.3. There are no other relations between generators of \( P_A(\delta) \) for \( A \in \mathcal{T}^d_w(\delta) \) by Proposition 5.4.

\[\square\]

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