The associated family of an elliptic surface and 
an application to minimal submanifolds

Marcos Dajczer and Theodoros Vlachos

Abstract

It is well-known that in any codimension a simply connected Euclidean minimal 
surface has an associated one-parameter family of minimal isometric deformations. 
In this paper, we show that this is just a special case of the associated family to any 
simply connected elliptic surface for which all curvature ellipses of a certain order 
are circles. We also provide the conditions under which this associated family is 
trivial, extending the known result for minimal surfaces. As an application, we 
show how the associated family of a minimal Euclidean submanifold of rank two is 
determined by the associated family of an elliptic surface clarifying the geometry 
around the associated family of these higher dimensional submanifolds.

1 Introduction

It is a well-known fact that a simply connected minimal surface in a space form of any 
dimension allows a one-parameter family of isometric minimal deformations, called the 
associated family, and that in Euclidean space this family can be parametrically given 
by means of the generalized Weierstrass representation; see [16]. In this paper, we show 
that this associated family is just a special case of the associated family to an elliptic 
surface for which all ellipses of curvature of a certain order are circles. Minimal surfaces 
can be seen as those elliptic surfaces for which the ellipse of curvature of order zero is a 
circle. Several basic properties of the new associated family are also given, in particular, 
we state when the family is trivial. Our second main result is an application of the result 
on surfaces to minimal submanifolds, which was our initial motivation and is explained 
in the sequel.

Euclidean submanifolds of rank two have been studied in different contexts; see [1], 
[7], [8], [11] and [12]. A submanifold having rank two means that the image of the Gauss 
map is a surface in the corresponding Grassmannian or, equivalently, that the kernel 
of the second fundamental form (relative nullity subspace) has constant codimension 
two. The study of the minimal ones is particularly interesting since they belong to the 
important class of austere submanifolds introduced in [13]. As a special case, one has the
ones that carry a Kaehler structure described in [8] by a Weierstrass type representation in terms of $m$-isotropic surfaces. A minimal surface is called $m$-isotropic if all ellipses of curvature up to order $m$ are circles. In turn, the $m$-isotropic surfaces can be constructed by the use of a Weierstrass type representation given in [10] based on results in [6].

It turns that the normal bundle of an elliptic surface splits as the orthogonal sum of a sequence of plane bundles (except the last one in odd codimension) such that each fiber contains an ellipse of curvature that is then ordered accordingly; see next section for details. Euclidean minimal submanifolds of rank two have been parametrically described in [8] by means of the class of elliptic surfaces for which the curvature ellipses of a certain order are circles.

To some surprise, it was observed in [9] that any simply connected Euclidean minimal submanifold of rank two allows an associated family of submanifolds of the same class. As in the surface case, this family is obtained by rotating the second fundamental form while keeping fixed the normal bundle and the induced normal connection. This fact, together with the representation in [8] discussed above, suggests that an elliptic surface in a space form for which the ellipses of curvature of a certain order are circles should have some kind of associate family preserving that property, and that was the starting point of this paper.

There is an abundance of examples of surfaces with circular ellipses of curvature, specially minimal ones. In particular, there are the surfaces for which all but the last one ellipse of curvature is a circle. These have been studied in the round sphere [3] and in hyperbolic space [17] under the name of superconformal. Other interesting examples are holomorphic curves in the nearly Kaehler sphere $S^6$. The theory of these surfaces started in [5] and was developed in [2], [14] and [15]. The first ellipse of curvature is always a circle but there is a class for which second curvature ellipse is not a circle; see Case 3 of Theorem 6.5 in [14].

For the purpose of this paper, the most important known examples are Lawson’s surfaces. These are minimal surfaces in spheres that decompose as a direct sum of elements in the associated family $h_\theta$, $\theta \in [0, \pi)$, of a minimal surface $h$ in $S^3$. More precisely, we consider surfaces in $S^{4n-1} \subset \mathbb{R}^{4n}$ given as

$$f = a_1 h_{\theta_1} \oplus \ldots \oplus a_n h_{\theta_n}$$

where $0 \leq \theta_1 < \cdots < \theta_n < \pi$, the real numbers $a_1, \ldots, a_n$ satisfy $\sum_{j=1}^n a_j^2 = 1$ and $\oplus$ denotes the orthogonal sum with respect to an orthogonal decomposition of $\mathbb{R}^{4n}$. It has been checked in [22] that all ellipses of curvature of even order are circles while that the ones of odd order generically are not. These surfaces are part of Lawson’s conjecture [19] which asserts that the only non-flat minimal surfaces in spheres that are locally isometric to minimal surfaces in $S^3$ are Lawson’s surfaces.

Most of what is done in this paper for surfaces can be extended to elliptic submanifolds of rank two. But in the final section of the paper, we limit ourselves to show how the associated family of a minimal Euclidean submanifold of rank two is determined.
by the associated family to an elliptic surface with a circular ellipse of curvature. This result completely clarifies the geometry around the associated family of these higher dimensional submanifolds.

Finally, we observe that a key ingredient of our proofs is the classical Burstin-Mayer-Allendoerfer theory as discussed in Vol. IV of Spivak [20]. Similar to the case of curves, this theory shows that certain tensors associated to a set of Frenet type equations are a complete set of invariants for a submanifold of a space form. Among these tensors, one has the higher order fundamental forms some of which are preserved by our associated family. We should point out that isometric deformations of submanifolds that also preserve higher fundamental forms, starting with the second fundamental form, up to a stated order was somehow considered in [4].

2 Preliminaries

In this section we recall from [20] some basic definitions for submanifolds in space forms, and from [8] the notions of elliptic surface, ellipse of curvature and polar surface to an elliptic surface and some of their basic properties, which will be used in the sequel without further reference.

Let \( f: M^n \to \mathbb{Q}^N_c \) be a substantial isometric immersion of a connected \( n \)-dimensional Riemannian manifold into either the Euclidean space \( \mathbb{R}^N (c = 0) \), the round sphere \( \mathbb{S}^N (c > 0) \) or the hyperbolic space \( \mathbb{H}^N (c < 0) \) with vector valued second fundamental form \( \alpha_f \) and induced Riemannian connection \( \nabla \perp \) in the normal bundle \( N_f M \). That \( f \) is substantial (called full in [20]) means that the codimension cannot be reduced.

The \( k \)-th-normal space \( N^f_k(x) \) of \( f \) at \( x \in M^n \) for \( k \geq 1 \) is defined as

\[
N^f_k(x) = \text{span}\{\alpha^{k+1}_f(X_1, \ldots, X_{k+1}) : X_1, \ldots, X_{k+1} \in T_x M\}.
\]

Thus \( \alpha^2_f = \alpha_f \) and for \( s \geq 3 \) the symmetric tensor \( \alpha^s_f: TM \times \cdots \times TM \to N_f M \), called the \( s \)-th-fundamental form, is defined inductively by

\[
\alpha^s_f(X_1, \ldots, X_s) = (\nabla^\perp_{X_1} \ldots \nabla^\perp_{X_{s-1}} \alpha_f(X_2, X_1))^\perp
\]

where \( (\;)^\perp \) denotes taking the projection onto the normal subspace \( (N^f_1 \oplus \ldots \oplus N^f_{s-2})^\perp \).

We always admit that \( f \) is regular (called nicely curved in [20]) which means that all the \( N^f_k \)'s have constant dimension for each \( k \) and thus form normal subbundles. This means "geometrically" that at each point the submanifold bends in the same number of directions. For any submanifold this condition is verified along connected components of an open dense subset of \( M^n \).

A surface \( g: L^2 \to \mathbb{Q}^N_c \) is called elliptic in [8] if there exists a (unique up to a sign) almost complex structure \( J: TL \to TL \) such that the second fundamental form satisfies

\[
\alpha_g(X, X) + \alpha_g(JX, JX) = 0 \quad \text{for all } X \in TL.
\]
Then all the $N_k^g$ have dimension two except the last one that is one-dimensional if the codimension is odd. Therefore, the normal bundle $N_gL$ splits as

$$N_gL = N_1^g \oplus \cdots \oplus N_{\tau_g}^g,$$

where $\tau_g$ (sometimes called the geometric degree of $g$) is the index of the last subbundle. Thus, the induced bundle $g^*T\mathcal{Q}_c^N$ splits as

$$g^*T\mathcal{Q}_c^N = N_0^g \oplus N_1^g \oplus \cdots \oplus N_{\tau_g}^g,$$

where $N_0^g = g_*TL$. Setting

$$\tau_g^o = \begin{cases} 
\tau_g & \text{if } N \text{ is even} \\
\tau_g - 1 & \text{if } N \text{ is odd},
\end{cases}$$

it turns out that the almost complex structure $J$ on $TL$ induces an almost complex structure $J_s$ on each $N_s^g$, $1 \leq s \leq \tau_g^o$, defined by

$$J_{s+1} \pi_s(\nabla^\perp_X \xi) = \pi_s(\nabla^\perp_X J_{s+1} \xi) = \pi_s(\nabla^\perp_X J_s \xi) \quad \text{if } \xi \in N_{s+1}^g$$

(1)

and

$$J_s \pi_{s-1}(\nabla^\perp_X \xi) = \pi_{s-1}(\nabla^\perp_X J_s \xi) = \pi_{s-1}(\nabla^\perp_X J_{s-1} \xi) \quad \text{if } \xi \in N_s^g.$$  

(2)

For any $\varphi \in \mathbb{S}^1 = [0, \pi]$ let $R_{\varphi}^s : N_s^g \to N_s^g$, $0 \leq s \leq \tau_g^o$, denote the map given by

$$R_{\varphi}^s = \cos \varphi I + \sin \varphi J_s.$$  

(3)

It follows from (1) and (2) that

$$R_{\varphi}^{s+1} \pi_{s+1}(\nabla^\perp_X \xi) = \pi_{s+1}(\nabla^\perp_X R_{\varphi}^s \xi) \quad \text{if } \xi \in N_{s+1}^g$$

(4)

and

$$(R_{\varphi}^s)^t \pi_s(\nabla^\perp_X \xi) = \pi_s(\nabla^\perp_X (R_{\varphi}^{s+1})^t \xi) \quad \text{if } \xi \in N_{s+1}^g.$$  

(5)

for any $1 \leq s \leq \tau_g^o - 1$.

The $s^{th}$-order curvature ellipse $\mathcal{E}_s^g(x) \subset N_s^g(x)$ of $g$ at $x \in L^2$ for $0 \leq s \leq \tau_g^o$ is

$$\mathcal{E}_s^g(x) = \{ \alpha_{s+1}^g(Z_\psi, \ldots, Z_\psi) : Z_\psi = \cos \psi Z + \sin \psi JZ \quad \text{and} \quad \psi \in [0, \pi) \},$$

where we understand that $\alpha^1 = g$, and assume that $Z \in T_xL$ has unit length and satisfies $\langle Z, JZ \rangle = 0$. It follows from the ellipticity condition that such a $Z$ always exists and that $\mathcal{E}_s^g(x)$ is indeed an ellipse.
We point out that $E^g_1$ given by the above definition coincides with the standard definition only if the mean curvature vanishes, in which case the higher order ellipses also coincide.

By $E^g_\ell$ being a circle we mean that the curvature ellipse $E^g_\ell(x)$ is a circle for any $x \in L^2$. A fundamental fact in this paper is that $E^g_\ell(x)$ is a circle if and only if $J_s(x)$ is orthogonal. Notice that $E^g_0$ is a circle if and only if $g$ is a minimal surface.

A polar surface to an elliptic surface $g: L^2 \to \mathbb{Q}^N_{c-\kappa} \subset \mathbb{R}^N$ defined as follows:

(i) If $N-c$ is odd, then the polar surface $h: L^2 \to S^{N-1}$ is the spherical image of a unit normal field spanning the last one-dimensional normal bundle.

(ii) If $N-c$ is even, then the polar surface $h: L^2 \to \mathbb{R}^N$ is any surface such that $T_{h(x)}L = N^g_{\tau_\theta}(x)$ up to parallel identification in $\mathbb{R}^N$.

It is known that in case (ii) any elliptic surface admits locally many polar surfaces. It turns out that a polar surface to an elliptic surface is necessarily elliptic. Moreover, if the elliptic surface has a circular ellipse of curvature then its polar surface has the same property at the “corresponding” normal bundle. In particular, for the polar surface to an $m$-isotropic surface the last $m+1$ ellipses of curvature are circles. Notice that in this case the polar surface is not necessarily minimal.

3 The results for surfaces

In this section, we state our results on the associated family to an elliptic surface with circular ellipses of curvature. We assert the existence of the associated family and discuss when the family is trivial. Then, we state a general result that shows that the families associated to two consecutive circular ellipses coincide.

**Theorem 1.** Let $g: L^2 \to \mathbb{Q}^N_c$, $N \geq 6$, be a simply connected substantial elliptic surface with $E^g_\ell$ a circle for some $1 \leq \ell \leq \tau^o_g - 1$. For each $\theta \in S^1$ there exists an elliptic isometric immersion $g_\theta: L^2 \to \mathbb{Q}^N_c$ with respect to the same almost complex structure and a vector bundle isometry $\psi_\theta: N_g L \to N_{g_\theta} L$ that preserves the fundamental forms $\alpha^{k}_{g_\theta} = \psi_\theta \alpha^k_g$ for $2 \leq k \leq \ell + 1$ as well as the normal curvature tensor.

In fact, the property of an ellipse of curvature being a circle remains for the elements in the associated family defined next.

**Definition 2.** The associated family $G_\ell$ to an elliptic surface $g: L^2 \to \mathbb{Q}^N_c$ with $E^g_\ell$ a circle for some $0 \leq \ell \leq \tau^o_g - 1$ is the set of elliptic isometric immersions

$$G_\ell = \{g_\theta: L^2 \to \mathbb{Q}^N_c : \theta \in S^1 = [0, \pi)\}$$
given by Theorem 1 for \( \ell \geq 1 \) and is the standard associated family to a minimal surface for \( \ell = 0 \).

By the associate family being trivial we mean that \( G_{\ell} \) only contains one element, that is, any \( g_0 \) is congruent to \( g \) in the ambient space.

**Theorem 3.** Let \( g : L^2 \to \mathbb{Q}_c^N \), \( N \geq 6 \), be a simply connected substantial elliptic surface such that \( \mathcal{E}_\ell^0 \) is a circle for some \( 0 \leq \ell \leq \tau_g - 1 \). If a pair of surfaces \( g_\theta, g_\tilde{\theta} \in G_\ell \) for \( \theta \neq \tilde{\theta} \) are congruent, then \( N \) is even and the \( \mathcal{E}_s^g \)'s are circles for \( \ell \leq s \leq \tau_g \). Conversely, if \( N \) be even and the \( \mathcal{E}_s^g \)'s are circles for \( \ell \leq s \leq \tau_g \), then the associated family \( G_\ell \) is trivial.

In odd codimension the one-parameter associated family is never trivial. For minimal surfaces in a special case this was already observed in [18] and [21].

The following is a consequence of the above result and basic properties of polar surfaces of elliptic surfaces.

**Corollary 4.** Let \( g : L^2 \to \mathbb{Q}_c^N \), \( c = 0, 1 \), be a simply connected substantial elliptic surface where \( N \) is even and \( \mathcal{E}_\ell^0 \) is a circle for some \( 0 \leq \ell \leq \tau_g - 1 \). Then, the associated family \( G_\ell \) is trivial if and only if \( N \) is even and \( g \) is (locally) a polar surface to an \( m \)-isotropic surface for \( m = \tau_g - \ell \).

Finally, we prove the following result which shows that if two ellipses of consecutive order are circles then the associated families coincide.

**Theorem 5.** Let \( : L^2 \to \mathbb{Q}_c^N \) be a simply connected elliptic surface such that the ellipses \( \mathcal{E}_\ell^0, \mathcal{E}_{\ell+r}^0 \) are circles for some \( 0 \leq \ell < \ell + r \). Then, the following facts are equivalent:

(i) \( G_\ell \cap G_{\ell+r} \neq \emptyset \).

(ii) The ellipses \( \mathcal{E}_j^g \) are circles for \( \ell \leq j \leq \ell + r \).

(iii) \( G_\ell = G_{\ell+r} \).

For the case of minimal surfaces we thus have the following.

**Corollary 6.** Let \( g : L^2 \to \mathbb{Q}_c^N \), \( N \geq 6 \), be a simply connected substantial minimal surface with \( \mathcal{E}_\ell^0 \) a circle for some \( 1 \leq \ell \leq \tau_g - 1 \). Then \( G_\ell = G_0 \) if and only if \( g \) is \( \ell \)-isotropic.
3.1 The compatibility equations

A key ingredient in the proofs are the basic equations from the classical Burstin-Mayer-Allendoerfer theory discussed in Vol. IV of [20]. They naturally extend the situation for curves under similar regularity conditions. The main result is that for a regular submanifold of a space form the tensors determined by the Frenet equations are a complete set of invariants.

The Frenet equations for a regular isometric immersion \( f: M^n \to \mathbb{Q}_c^N \) are given by

\[
\tilde{\nabla}_X \xi = -A_s^s X + D_s^s \xi + S_s^s \xi \quad \text{if} \quad \xi \in N_s^f \quad \text{and} \quad X \in T_x M, \quad s \geq 1,
\]

in term of the linear maps

\[
A^s: TM \times N_s^f \to N_{s-1}^f \quad \text{defined by} \quad A^s_x X = -\pi_{s-1}(\tilde{\nabla}_X \xi),
\]

\[
D^s: TM \times N_s^f \to N_s^f \quad \text{defined by} \quad D^s_x \xi = \pi_s(\nabla^\perp_X \xi),
\]

\[
S^s: TM \times N_s^f \to N_{s+1}^f \quad \text{defined by} \quad S^s_x \xi = \pi_{s+1}(\nabla^\perp_X \xi),
\]

where \( \tilde{\nabla} \) is the connection in the induced bundle \( f^*(T\mathbb{Q}_c^N) = N_0^f \oplus N_f M \) and \( \pi_0 \) is the projection onto \( N_0^f = f^*(TM) \). Notice that \( A_1^s \) is the standard Weingarten operator and that \( D^s \) is a connection in \( N_s^f \) compatible with the metric. An important fact is that the tensors \( A^s \) and \( S^s \) are completely determined by the higher fundamental forms since

\[
S^s_X(\alpha_{s+1}^+(X_1, \ldots, X_{s+1})) = \alpha_{s+2}^+(X, X_1, \ldots, X_{s+1})
\]

and

\[
\langle A^s_x X, \eta \rangle = \langle \xi, S^{s-1}_X \eta \rangle \quad \text{for} \quad \xi \in N_s^f \quad \text{and} \quad \eta \in N_{s-1}^f.
\]

We briefly summarize the basic results of the theory: Let \( f, \tilde{f}: M^n \to \mathbb{Q}_c^N \) be two regular isometric immersions. If there are vector bundle isometries \( \phi_k: N_k^I \to N_k^I \) for all \( k \geq 1 \), which preserve the fundamental forms \( \alpha^{k+1} \) and the induced normal connections \( D^k \), then there is an isometry \( \tau \) of \( \mathbb{Q}_c^N \) such that \( \tilde{f} = \tau \circ f \) and \( \phi_k = \tau_s|N_k^f \). Moreover, there is a set of equations given below, namely, the Generalized Gauss and Codazzi equations, that relate the higher fundamental forms and the induced connections. It turns out that the set of connections \( D^k \) in \( N_k^f \) is the unique set for which the higher order fundamental forms satisfy the Codazzi equations. Furthermore, the Generalized Gauss and Codazzi equations are the integrability conditions that assure the existence of an isometric immersion provided all data involved has been provided.

The Generalized Gauss equation.

\[
A_{S^s_{X_1} \xi}^{s+1} X - A_{S^s_{X_1} \xi}^{s+1} Y = D^s_X D^s_Y \xi - D^s_Y D^s_X \xi - S^s_{X_1} A^s_{X_1} Y + S^s_{Y_1} A^s_{Y_1} X - D^s_{[X,Y]} \xi \quad (7)
\]
for all $X, Y \in TM$ and $\xi \in N^f_s$.

The Generalized Codazzi equation

$$D_{s+1}^{s+1}(S^s_X \xi) - D_{s}^{s+1}(S^s_X \xi) + S^s_X D^s_Y \xi - S^s_Y D^s_X \xi - S^s_{[X,Y]} \xi = 0 \quad (8)$$

for all $X, Y \in TM$ and $\xi \in N^f_s$.

Using (8) we have that (8) has the equivalent form

$$D^s_{s+1} X A^s_{s+1} Y - D^s_{s} Y A^s_{s+1} X + A^s_{s+1} D^s_{s+1} Y \xi - A^s_{s} D^s_{s+1} X \xi - A^s_{[X,Y]} = 0 \quad (9)$$

for all $X, Y \in TM$ and $\xi \in N^f_{s+1}$.

We conclude with some useful symmetric equations.

**Proposition 7.** It holds that

$$S_{s+1}^s S^s_X \xi = S^s_{X s+1} S^s_\xi \text{ or, equivalently, that } A^s_{s s+1} X = A^s_{s s+1} Y \quad (10)$$

for any $\xi \in N^f_s$ and $X, Y \in TM$.

**Proof:** To prove the first equation take $\xi = \alpha^s_{s+1}(X_1, \ldots, X_{s+1})$ and use the symmetry of the higher fundamental forms. For the proof of the equivalent second equation take $\xi \in N^f_s$, $\eta \in N^f_{s-1}$ and use (8) twice to obtain

$$\langle A^s_{s s+1} Y - A^s_{s s+1} X, \eta \rangle = \langle \xi, S^s_{s s+1} S^s_{s-2} \eta - S^s_{s} S^s_{s-2} \eta \rangle = 0,$$

and this concludes the proof. ■

### 3.2 The proofs

For a substantial elliptic surface $g: L^2 \to Q^N_c$ with a circular ellipse of curvature in a space form we first define a one-parameter family of compatible connections. Hereafter, we assume that $E^g_0$ is a circle for given $0 \leq \ell \leq \tau^0_g - 1$, that is, the almost complex structure $J_\ell$ ($J_0 = J$) is a vector bundle isometry. Notice that $J_\ell$ is parallel with respect to the induced connection on $N^g_{\ell}$ by dimension reasons. Thus, for any $\varphi \in S^1 = [0, \pi)$ the map $R^\varphi_{\ell}: N^g_{\ell} \to N^g_{\ell}$ defined by (3) is also a parallel isometry, i.e.,

$$\pi_{\ell}(\nabla^1_X R^\varphi_{\ell} \xi) = R^\varphi_{\ell} \pi_{\ell}(\nabla^1_X \xi). \quad (11)$$

Let $\tilde{\nabla}^g: TL \times g^*TQ^N_c \to g^*TQ^N_c$ for each $\theta \in S^1$ be the map defined by modifying the induced connection $\nabla$ of $g^*TQ^N_c$ as follows:

$$\begin{cases} 
\pi_{\ell+1}(\tilde{\nabla}^g_X \xi) = \pi_{\ell+1}(\nabla_X R^\theta \xi) \quad \text{if } \xi \in N^g_{\ell} \\
\pi_{\ell}(\tilde{\nabla}^g_X \eta) = R^\theta_{-\ell} \pi_{\ell}(\nabla_X \eta) \quad \text{if } \eta \in N^g_{\ell+1},
\end{cases}$$
and \( \tilde{\nabla}^g = \tilde{\nabla} \) in all other cases. We also define the map \( \nabla^g : TL \times N_gL \rightarrow N_gL \) by
\[
\nabla^g_X \xi = \tilde{\nabla}^g_X \xi - \pi_0(\tilde{\nabla}^g_X \xi).
\]

For \( \ell = 0 \), we have the map \( \alpha_\theta : TL \times TL \rightarrow N_gL \) given by
\[
a_\theta(X,Y) = \pi_1(\tilde{\nabla}^g_X g_\ast Y).
\]

Thus \( a_\theta(X,Y) = \alpha_g(J_\theta X,Y) \) where \( J_\theta = \cos \theta I + \sin \theta J \). Also \( \nabla^g = \nabla^\perp \). Then, the triple \( (\alpha_\theta, \langle , \rangle, \nabla^\perp) \) satisfies the Gauss, Codazzi and Ricci equations. Therefore, if \( L \) is simply connected if follows from the Fundamental theorem of submanifolds that there exists an isometric minimal surface \( g_\theta : L \rightarrow \mathbb{Q}^n_c \) and a parallel vector bundle isometry \( \phi_\theta : (N^g\nabla L, \nabla^\perp) \rightarrow (N^g\nabla g_\theta, \nabla^\perp(g_\theta)) \) such that \( \alpha_{g_\theta} = \phi_\theta \alpha_\theta \). Of course, the family \( g_\theta \) with \( \theta \in \mathbb{S}^1 \) is just the standard associated family of the minimal surface \( g \).

For \( \ell \geq 1 \), the map \( \nabla^g \) is obtained modifying the normal connection of \( g \) as follows:
\[
\begin{align*}
\pi_{\ell+1}(\nabla^g_X \xi) & = \pi_{\ell+1}(\nabla^g_X R_\theta^\ell \xi) \quad \text{if} \quad \xi \in N^g_{\ell} \\
\pi_\ell(\nabla^g_X \eta) & = R_\perp^\ell \pi_\ell(\nabla^g_X \eta) \quad \text{if} \quad \eta \in N^g_{\ell+1},
\end{align*}
\]
and \( \nabla^g = \nabla^\perp \) in all other cases.

**Lemma 8.** For \( \ell \geq 1 \) the map \( \nabla^g \) is a Riemannian connection whose curvature tensor satisfies \( R^\theta = R^\perp \).

**Proof:** Take \( \xi \in N^g_{\ell} \) and \( \eta \in N^g_{\ell+1} \). Then,
\[
\nabla^g_X f \xi = (\pi_{\ell-1} + \pi_\ell)(\nabla^g_X f \xi) + \pi_{\ell+1}(\nabla^g_X f R_\theta^\ell \xi) = X(f) \xi + f \nabla^g_X \xi
\]
and
\[
\nabla^g_X f \eta = (\pi_{\ell+1} + \pi_{\ell+2})(\nabla^g_X f \eta) + R_\perp^\ell \pi_\ell(\nabla^g_X f \eta) = X(f) \eta + f \nabla^g_X \eta.
\]
Moreover, we obtain using (11) that
\[
\langle \nabla^g_X \xi, \eta \rangle + \langle \xi, \nabla^g_X \eta \rangle = \langle \nabla^g_X R_\theta^\ell \xi, \eta \rangle + \langle \xi, R_\perp^\ell \nabla^g_X \eta \rangle = \langle R_\theta^\ell \nabla^g_X \xi, \eta \rangle + \langle \xi, R_\perp^\ell \nabla^g_X \eta \rangle = 0,
\]
and that the connection is Riemannian follows easily from (13), (14) and (15).

The second claim amounts to show that the tensor defined by
\[
R(X,Y) \xi = R^\theta(X,Y) \xi - R^\perp(X,Y) \xi
\]
vanishes. In the sequel, some of the arguments will just be sketched to avoid writing rather long but straightforward computations.
We divide the proof in several cases:

Case 1. The case $\xi \in N_{\ell}^0 \oplus \cdots \oplus N_{\ell-2}^0 \oplus N_{\ell+3}^0 \oplus \cdots \oplus N_{\ell}^0$ is trivial.

Case 2. Take $\xi \in N_{\ell+2}^0$. Then,

$$\mathcal{R}(X,Y) \xi = (R_{-\theta}^\ell - I) \left( A_{A_{\xi}^{\ell+1}X,Y}^{\ell+1} - A_{A_{\xi}^{\ell+2}X,Y}^{\ell+1} \right),$$

and the claim follows from (10).

Case 3. Take $\xi \in N_{\ell+1}^\ell$. Then,

$$\mathcal{R}(X,Y) \xi = B_\theta(X,Y) - B_\theta(Y,X) - B_0(X,Y) + B_0(Y,X) + (I - R_{-\theta}^\ell) \pi_\ell(\nabla_{X,Y}^\perp)$$

where

$$B_\theta(X,Y) = R_{-\theta}^\ell \pi_\ell(\nabla_{X}^\perp \pi_{\ell+1}(\nabla_{Y}^\perp)) + \pi_\ell(\nabla_{Y}^\perp R_{-\theta}^\ell \pi_\ell(\nabla_{X}^\perp)) + \pi_{\ell-1}(\nabla_{X}^\perp R_{-\theta}^\ell \pi_\ell(\nabla_{Y}^\perp)).$$

Observe that (11) can be written as

$$D_X^\ell R_{\theta}^\ell \xi = R_{\theta}^\ell D_X^\ell \xi.$$  \hspace{1cm} (16)

We obtain using (10) and (13) that

$$\mathcal{R}(X,Y) \xi = (I - R_{-\theta}^\ell) \left( D_X^\ell A_{\xi}^{\ell+1}Y - D_Y^\ell A_{\xi}^{\ell+1}X + A_{D_{\xi}^{\ell+1}X}^{\ell+1}X - A_{D_{\xi}^{\ell+1}Y}^{\ell+1}Y - A_{r_{\xi}^{\ell+1}X,Y}^{\ell+1} \right)$$

$$+ A_{R_{-\theta}^\ell A_{\xi}^{\ell+1}Y}^\ell X - A_{R_{-\theta}^\ell A_{\xi}^{\ell+1}X}^\ell Y.$$  \hspace{1cm} (17)

For $\eta \in N_{\ell-1}^{\ell-1}$, we have using (12) that

$$\langle A_{r_{\xi}^{\ell+1}Y}^\ell X, \eta \rangle = \langle A_{r_{\xi}^{\ell+1}Y}^{\ell+1}X, \eta \rangle = \langle \xi, S_{Y}^\ell R_{\theta}^\ell S_{X}^{\ell-1} \eta \rangle.$$  \hspace{1cm} (18)

Since $R_{\theta}^\ell S_{\xi}^{\ell-1} = S_{\xi}^{\ell-1} R_{\theta}^\ell \xi$ from (14), we obtain from (10) that

$$S_{Y}^\ell R_{\theta}^\ell S_{X}^{\ell-1} \xi = S_{X}^\ell R_{\theta}^\ell S_{Y}^{\ell-1} \xi.$$  \hspace{1cm} (19)

Now the claim follows from (9), (17) and (18).

Case 4. Take $\xi \in N_{\ell}^\ell$. First assume $\ell \geq 2$. Using (7), (8), (10) and (16) we obtain

$$R_{\theta}(X,Y) \xi = S_{Y}^{\ell-1} A_{\xi}^\ell X - S_{X}^{\ell-1} A_{\xi}^\ell Y - R_{-\theta}(S_{Y}^{\ell-1} A_{r_{\xi}^{\ell}Y}^\ell X - S_{X}^{\ell-1} A_{r_{\xi}^{\ell}Y}^\ell Y).$$

On the other hand, it holds that

$$R_{\theta}^\ell(S_{Y}^{\ell-1} A_{\xi}^\ell X - S_{X}^{\ell-1} A_{\xi}^\ell Y) = S_{Y}^{\ell-1} A_{R_{\theta}^\ell \xi}^\ell X - S_{X}^{\ell-1} A_{R_{\theta}^\ell \xi}^\ell Y.$$  \hspace{1cm} (20)
In fact, it follows using (6) that
\[
\langle R^\ell_f(S^\ell-1_X^f A^\ell_X S^\ell-1_X^f A^\ell_Y^f), R^\ell_f(\delta) \rangle = \langle A^\ell_A Y, A^\ell_X X \rangle - \langle A^\ell_X X, A^\ell_Y Y \rangle
\]
and
\[
\langle S^\ell-1_{R^\ell_f X} A^\ell_{R^\ell_f X} S^\ell-1_{X} A^\ell_{R^\ell_f X} Y, R^\ell_f(\delta) \rangle = \langle A^\ell_{R^\ell_f \delta} Y, A^\ell_{R^\ell_f \delta} X \rangle - \langle A^\ell_{R^\ell_f \delta} X, A^\ell_{R^\ell_f \delta} Y \rangle.
\]
Since \( N^g_\ell = \text{span}\{\xi, J_\ell \xi\} \), to obtain (20) is suffices to compute the right hand side of both equations for \( \delta = J_\ell \xi \) and observe that they coincide.

We now have from (19) and (20) that \( R^\theta(X, Y)\xi = 0 \), and this proves the claim since also \( R^\perp(X, Y)\xi = 0 \) from the Ricci equation.

For \( \ell = 1 \), we have that
\[
R^\theta(X, Y)\xi = R^\ell_{-\theta} \left( \alpha_g(X, A_{R^\ell_f \xi} Y) - \alpha_g(Y, A_{R^\ell_f \xi} X) \right).
\]
Since \( A_{R^\ell_f \xi} = R_\theta A_{\xi} \), we obtain from the Ricci equation that \( R^\theta(X, Y)\xi = R^\perp(X, Y)\xi \) and the claim also follows in this case.

**Case 5.** Take \( \xi \in N^g_{\ell-1} \). Then,
\[
\mathcal{R}(X, Y)\xi = B_\theta(X, Y) - B_\theta(Y, X) - B_\theta(X, Y) + B_\theta(Y, X)
\]
where
\[
B_\theta(X, Y) = \pi_\ell(\nabla^X_\perp \pi_{\ell-1}(\nabla^X_\perp \xi)) + \pi_\ell(\nabla^X_\perp \pi_\ell(\nabla^X_\perp \xi)) + \pi_{\ell+1}(\nabla^X_\perp R^\theta_{\pi_\ell}(\nabla^X_\perp \xi)).
\]
It follows that
\[
\mathcal{R}(X, Y)\xi = (R^\ell_{-\theta} - I) \left( D^\ell_X (S^\ell-1_X^f S^\ell-1_X^f) - D^\ell_Y (S^\ell-1_X^f S^\ell-1_X^f) + S^\ell-1_X D^\ell-1_X^f - S^\ell-1_X D^\ell-1_X^f - S^\ell-1_{X,Y} \right)
\]
\[
+ S^\ell_X R^\theta_{\pi_\ell} S^\ell_Y S^\ell_X S^\ell_Y \xi - \pi_\ell R^\theta_{\pi_\ell} S^\ell_X S^\ell_Y \xi,
\]
and the claim follows from (8) and (18).

To conclude the proof, we observe that the case \( \tau^g = \tau_g - 1 \) and \( \xi \in N^g_{\tau_g} \) is included in the above cases.

**Proof of Theorem** \( \square \) We argue that the triple \( (\alpha_g, \langle , \rangle, \nabla^\theta) \) satisfies the Gauss, Codazzi and Ricci equations. Then, according to the Fundamental theorem of submanifolds there exists an isometric immersion \( g_\theta: L^2 \to Q_\ell^N \) and a parallel vector bundle isometry
\[
\phi_\theta: (N_g L, \nabla^\theta) \to (N_{g_\theta} L, \nabla^\perp(g_\theta))
\]
(21)
such that \( \alpha_{g_\theta} = \phi_\theta \alpha_g \).
The Gauss equation holds since the second fundamental form remains the same. The Codazzi equation

\[(\nabla_X^\theta \alpha_g)(Y, Z) = (\nabla_Y^\theta \alpha_g)(X, Z)\]

is trivially satisfied for \(\ell \geq 3\) since \(\nabla^\theta \alpha_g = \nabla^\perp \alpha_g\). For \(\ell = 1\), we obtain

\[(\nabla_X^\theta \alpha_g)(Y, Z) = \pi_1((\nabla_X^\perp \alpha_g)(Y, Z)) + \pi_2((\nabla_X^\perp \alpha_g)(Y, R_\theta Z))\]

while for \(\ell = 2\), we have

\[(\nabla_X^\theta \alpha_g)(Y, Z) = (\pi_1 + \pi_2)((\nabla_X^\perp \alpha_g)(Y, Z)),\]

and again the Codazzi equation follows.

The proof that the curvature tensor \(R^\theta\) of \(\nabla^\theta\) satisfies the Ricci equation

\[R^\theta(X, Y)\xi = \alpha_g(X, A_\xi Y) - \alpha_g(Y, A_\xi X)\]

is a consequence of Lemma 8 above. Finally, the statements on the fundamental forms is part of Proposition 9 given next.

The following result provides the expressions for the higher fundamental forms of the associated family \(g_\theta\) in terms of the ones corresponding to \(g\). We observe that this can be used to give an alternative (but more complicate) definition for the associated family by means of the version in [20] of the Fundamental theorem of submanifolds as part of the Burstin-Mayer-Allendoerfer theory.

**Proposition 9.** Let \(g: L^2 \to \mathbb{Q}_c^N\) be a simply connected elliptic surface with \(\mathcal{E}_\ell^g\) a circle for some \(1 \leq \ell \leq \tau_g - 1\). Then, up to identification, the higher fundamental forms of \(g_\theta\) are given by

\[\alpha_{g_\theta}^s(X_1, \ldots, X_s) = \begin{cases} \alpha_g^s(X_1, \ldots, X_s) & \text{if } 2 \leq s \leq \ell + 1, \\ R_{\theta}^{s-1} \alpha_g^s(X_1, \ldots, X_s) = \alpha_g^s(R_\theta X_1, \ldots, X_s) & \text{if } \ell + 2 \leq s \leq \tau_g^o + 1. \end{cases}\]

Moreover, if \(\tau_g^o < \tau_g\), then

\[\alpha_{g_\theta}^{\tau_g + 1}(X_1, \ldots, X_{\tau_g + 1}) = \alpha_g^{\tau_g + 1}(R_\theta X_1, \ldots, X_{\tau_g + 1}).\]

**Proof:** Since \(\alpha_{\theta}^2 = \alpha_g\), the case \(2 \leq s \leq \ell\) follows easily from the definitions. For the other cases, we have

\[\alpha_{g_\theta}^{\ell+1}(X_1, \ldots, X_{\ell+1}) = \pi_\ell(\nabla_{X_{\ell+1}}^\theta \alpha_{g_\theta}^{\ell}(X_1, \ldots, X_\ell)) = \pi_\ell(\nabla_{X_{\ell+1}}^\theta \alpha_g^{\ell}(X_1, \ldots, X_\ell))\]

and

\[\alpha_{g_\theta}^{\ell+1}(X_1, \ldots, X_{\ell+1}) = \alpha_g^{\ell+1}(X_1, \ldots, X_{\ell+1})\]

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and

\[ \alpha^{\ell+2}_{g\theta}(X_1, \ldots, X_{\ell+2}) = \pi_{\ell+1}(\nabla^\theta_{X_{\ell+2}} \alpha^{\ell+1}_{g\theta}(X_1, \ldots, X_{\ell+1})) = \pi_{\ell+1}(\nabla^\theta_{X_{\ell+2}} \alpha^{\ell+1}_g(X_1, \ldots, X_{\ell+1})) = \pi_{\ell+1}(\nabla^\theta_{X_{\ell+2}} R^\ell g \alpha^{\ell+1}_g(X_1, \ldots, X_{\ell+1})) = \pi_{\ell+1}(\nabla^\theta_{X_{\ell+2}} \alpha^{\ell+1}_g(R_\theta X_1, \ldots, X_{\ell+1})) = \alpha^{\ell+2}_g(R_\theta X_1, \ldots, X_{\ell+2}) = R_\theta^{\ell+1} \alpha^{\ell+2}_g(X_1, \ldots, X_{\ell+2}), \]

and the remaining of the proof is immediate. □

**Proof of Theorem** Since the case of \( \ell = 0 \) is well-known we argue for \( \ell \geq 1 \). Suppose first that the surfaces \( g_\theta \) and \( g_\vartheta \) in \( G_t \) are congruent. Without loss of generality, we may assume that \( \theta = 0 \). Then, there exists a parallel vector bundle isometry \( \psi: N_\theta L \rightarrow N_{g\theta} L \) such that \( \psi(N_\theta^s) = N_{g\theta}^s \) and \( \alpha^{s}_{g\theta} = \psi \alpha^s_\theta \) for any \( 2 \leq s \leq \tau^\theta_\vartheta \). Proposition 9 yields

\[ \psi \alpha^s_\theta = \begin{cases} \alpha^s_g & \text{if } 2 \leq s \leq \ell + 1, \\ R^{s-1}_\theta \alpha^s_g & \text{if } \ell + 2 \leq s \leq \tau^\theta_\vartheta + 1, \end{cases} \]

and if \( \tau^\theta_\vartheta < \tau_\theta \) then

\[ \psi \alpha^{s-1}_{g\theta}(X_1, \ldots, X_{\tau_\theta + 1}) = \alpha^{s-1}_g(R_\theta X_1, \ldots, X_{\tau_\theta + 1}). \]

Therefore, we have that \( N \) is even and

\[ \psi = I \text{ on } N^\theta_1 \oplus \ldots \oplus N^\theta_\ell \text{ and } \psi = R^\theta_\vartheta \text{ on } N^\vartheta_\ell \text{ if } s \geq \ell + 1. \]

(22)

We conclude that \( R^\theta_\vartheta \) is an isometry for \( s \geq \ell + 1 \) and hence \( J_\vartheta \) is an isometry for \( s \geq \ell \).

Conversely, suppose that \( N \) is even and any \( E_s(g) \) is a circle for all \( s \geq \ell \), or equivalently, that \( R^s_\theta \) is an isometry for \( s \geq \ell \). Then \( \psi: N_\theta L \rightarrow N_{g\theta} L \) given by (22) is a vector bundle isometry that preserves the second fundamental form. To conclude the proof it remains to show that \( \psi \) is parallel, i.e., \( \nabla^\theta_X \psi \xi = \nabla^\vartheta_X \psi \xi \). For that we distinguish several cases:

**Case 1.** The case \( \xi \in N^\theta_1 \oplus \ldots \oplus N^\theta_{\ell-1} \) is trivial.

**Case 2.** Assume that \( \xi \in N^\theta_{\ell-1} \). We have using (11) that

\[ \nabla^\theta_X \psi \xi = \nabla^\theta_X \xi = (\pi_{\ell-1} + \pi_\ell)(\nabla^\ell_X \xi) + \pi_{\ell+1}(\nabla^\perp_X R^\ell_\vartheta \xi) = (\pi_{\ell-1} + \pi_\ell)(\nabla^\perp_X \xi) + R^\ell R^{\ell+1}_\vartheta \pi_{\ell+1}(\nabla^\perp_X \xi) = \psi \nabla^\perp_X \xi. \]

**Case 3.** Assume that \( \xi \in N^\theta_{\ell+1} \). We have,

\[ \nabla^\theta_X \psi \xi = \nabla^\theta_X R^{\ell+1}_\vartheta \xi = R^{\ell-1}_g \pi_\ell(\nabla^\perp_X R^{\ell+1}_\vartheta \xi) + (\pi_{\ell+1} + \pi_{\ell+2})(\nabla^\perp_X R^{\ell+1}_\vartheta \xi). \]
and
\[
\psi \nabla^\perp_X \xi = \pi_\ell(\nabla^\perp_X \xi) + R_\theta^\ell \pi_{\ell+1}(\nabla^\perp_X \xi) + R_\theta^\ell \pi_{\ell+2}(\nabla^\perp_X \xi).
\]
To obtain equality we observe that \((5)\) yields
\[
R_\theta^\ell \pi_{\ell}(\nabla^\perp_X \xi) = \pi_\ell(\nabla^\perp_X \xi)
\]
and that the \(N^g_{\ell+1}\) and \(N^g_{\ell+2}\) components are equal due to \((11)\) and \((4)\), respectively.

**Case 4.** Assume that \(\xi \in N^g_s\) for \(s \geq \ell + 2\). We have,
\[
\nabla^\perp_X \psi \xi = \nabla^\perp_X R_\theta^s \xi = (\pi_{s-1} + \pi_s + \pi_{s+1})(\nabla^\perp_X R_\theta^s \xi),
\]
\[
\psi \nabla^\perp_X \xi = R^s_\theta \pi_{s-1}(\nabla^\perp_X \xi) + R^s_\theta \pi_s(\nabla^\perp_X \xi) + R^s_{\ell+1} \pi_{s+1}(\nabla^\perp_X \xi),
\]
and equality follows from \((4), (5)\) and \((11)\). □

**Proof of Corollary 4.** The proof follows from Theorem 3 and the fact proved in \([8]\) that an elliptic surface has circular curvature ellipses from some order on if and only if any polar surface has circular curvature ellipses up to that order. □

**Proof of Theorem 5.** To see that \((i) \Rightarrow (ii)\) suppose that a surface \(g_\theta\) in \(G_\ell\) is congruent to \(\tilde{g}_\omega\) in \(G_{\ell+r}\). First assume that \(\ell \geq 1\), and let
\[
\phi_\theta: (N_g^L, \nabla^\perp) \to (N_{g_\theta}^L, \nabla^\perp(g_\theta)), \quad \sigma_\omega: (N_g^L, \nabla^\perp) \to (N_{\tilde{g}_\omega}^L, \nabla^\perp(\tilde{g}_\omega))
\]
be the parallel vector bundle isometries given by \((21)\). By assumption, there exists a parallel bundle isometry
\[
\psi: (N_{g_{\theta}}^L, \nabla^\perp(g_\theta)) \to (N_{\tilde{g}_{\omega}}^L, \nabla^\perp(\tilde{g}_{\omega})),
\]
such that \(\alpha_{{g_\theta}}^{s+1} = \psi \alpha_{{g_\omega}}^{s+1}\) for any \(1 \leq s \leq \tau_\theta - 1\). Proposition \(9\) yields
\[
\alpha_{{g_\omega}}^{s+1} = \sigma_\omega \alpha_{{g_\theta}}^{s+1}
\]
for any \(1 \leq s \leq \ell + r\) and
\[
\alpha_{{g_\theta}}^{s+1} = \phi_\theta R_\theta^s \alpha_{{g_\theta}}^{s+1}
\]
for any \(\ell + 1 \leq s \leq \ell + r\). We obtain that \(\sigma_\omega \alpha_{{g_\theta}}^{s+1} = \psi \circ \phi_\theta \circ R_\theta^s \alpha_{{g_\theta}}^{s+1}\), that is,
\[
R_\theta^s = (\psi \circ \phi_\theta)^{-1} \circ \sigma_\omega
\]
on \(N^g_s\) for \(\ell + 1 \leq s \leq \ell + r\). Since \(R_\theta^s\) is an isometry for any \(\ell + 1 \leq s \leq \ell + r\), then all the \(E_j^g, \ell \leq j \leq \ell + r\), are circles.

Now assume \(\ell = 0\) and let \(\phi_\theta: (N_g^L, \nabla^\perp) \to (N_{g_{\theta}}^L, \nabla^\perp(g_\theta))\) be the parallel bundle isometry such that
\[
\alpha_{{g_\theta}}^{s+1} = \phi_\theta \circ R_\theta^s \alpha_{{g_\theta}}^{s+1}
\]
for any $1 \leq s \leq \tau_g^t$. Proposition \ref{prop:alpha} yields $\alpha_{g_\omega}^{s+1} = \sigma_\omega \alpha_g^{s+1}$ for $1 \leq s \leq \ell + r$. Then,

$$\alpha_{g_\omega}^{s+1} = \sigma_\omega \circ (\phi_\theta \circ R_\theta^s)^{-1} \alpha_{g_\omega}^{s+1}.$$

Since $\alpha_{g_\omega}^{s+1} = \psi \alpha_g^{s+1}$ we find that $R_\theta^s = (\psi \circ \phi_\theta)^{-1} \circ \sigma_\omega$ on $N_g^s$ for any $1 \leq s \leq \ell + r$. Hence, all the $E_j^g, 0 \leq j \leq \ell + r$, are circles.

We show that $(ii) \Rightarrow (iii)$. At first, we assume $\ell = 0$ and prove that $G_0 = G_r$. According to Theorem \ref{thm:fundamental} the second fundamental form of $g_\theta \in G_r$ is given by $\alpha_{g_\theta} = \psi_\theta \alpha_g$. The second fundamental form of $h_\theta \in G_0$ is given by

$$\alpha_{h_\theta} = \phi_\theta \circ R_\theta^1 \alpha_g,$$

where $\phi_\theta: (N_g L, \nabla^\bot) \to (N_{h_\theta} L, \nabla^\bot(h_\theta))$ is a parallel vector bundle isometry. Hence $\alpha_{h_\theta} = T_\theta \alpha_{g_\theta}$ where $T_\theta: N_g L \to N_{h_\theta} L$ is the bundle isometry given by

$$T_\theta = \phi_\theta \circ R_\theta^1 \circ \psi_\theta^{-1} \text{ on } N_{g_\theta} \text{ if } 1 \leq s \leq r \text{ and } T_\theta = \phi_\theta \circ \psi_\theta^{-1} \text{ on } N_{r+1}^\theta \oplus \cdots \oplus N_{\tau_g}^\theta.$$

We show next that $T_\theta$ is parallel, i.e.,

$$\nabla_X^{\perp} T_\theta \xi_\theta = T_\theta(\nabla_X^{\perp} \xi_\theta) \tag{23}$$

where $\xi_\theta = \psi_\theta \xi$ and $\xi \in N_{g_\theta}^t, 1 \leq s \leq \tau_g$. We need to distinguish several cases:

**Case 1.** Assume that $1 \leq s \leq r - 1$. We have that

$$\nabla_X^{\perp} T_\theta \xi_\theta = \nabla_X^{\perp} \phi_\theta R_\theta^s \xi_\theta = \phi_\theta(\nabla_X^{\perp} R_\theta^s \xi_\theta).$$

Using \ref{eq:theta}, \ref{eq:psi}, \ref{eq:bundle} and \ref{eq:isometry} we obtain

\begin{align*}
T_\theta(\nabla_X^{\perp} \xi_\theta) &= T_\theta \circ \psi_\theta(\nabla_X^{\perp} \xi_\theta) = T_\theta \circ \psi_\theta(\pi_{s-1}(\nabla_X^{\perp} \xi_\theta) + \pi_s(\nabla_X^{\perp} \xi_\theta) + \pi_{s+1}(\nabla_X^{\perp} \xi_\theta)) \\
&= T_\theta \circ \psi_\theta(\pi_{s-1}(\nabla_X^{\perp} \xi_\theta) + \pi_s(\nabla_X^{\perp} \xi_\theta) + \pi_{s+1}(\nabla_X^{\perp} \xi_\theta)) \\
&= \phi_\theta \circ R_\theta^{s-1}(\pi_{s-1}(\nabla_X^{\perp} \xi_\theta)) + \phi_\theta \circ R_\theta^s(\pi_s(\nabla_X^{\perp} \xi_\theta)) + \phi_\theta \circ R_\theta^{s+1}(\pi_{s+1}(\nabla_X^{\perp} \xi_\theta)) \\
&= \phi_\theta(\pi_{s-1}(\nabla_X^{\perp} R_\theta^s \xi_\theta) + \pi_s(\nabla_X^{\perp} R_\theta^s \xi_\theta) + \pi_{s+1}(\nabla_X^{\perp} R_\theta^s \xi_\theta)) \\
&= \phi_\theta(\pi_{s-1}(\nabla_X^{\perp} R_\theta^s \xi_\theta) + \pi_s(\nabla_X^{\perp} R_\theta^s \xi_\theta) + \pi_{s+1}(\nabla_X^{\perp} R_\theta^s \xi_\theta)),
\end{align*}

and this proves \ref{eq:23}.

**Case 2.** Assume that $s = r$. As before, we have

$$\nabla_X^{\perp} T_\theta \xi_\theta = \phi_\theta(\nabla_X^{\perp} R_\theta^s \xi_\theta).$$

Using \ref{eq:theta}, \ref{eq:psi}, \ref{eq:bundle} and \ref{eq:isometry} it follows that

\begin{align*}
T_\theta(\nabla_X^{\perp} \xi_\theta) &= T_\theta \circ \psi_\theta(\nabla_X^{\perp} \xi_\theta) = T_\theta \circ \psi_\theta(\pi_{r-1}(\nabla_X^{\perp} \xi_\theta) + \pi_r(\nabla_X^{\perp} \xi_\theta) + \pi_{r+1}(\nabla_X^{\perp} \xi_\theta)) \\
&= T_\theta \circ \psi_\theta(\pi_{r-1}(\nabla_X^{\perp} \xi_\theta) + \pi_r(\nabla_X^{\perp} \xi_\theta) + \pi_{r+1}(\nabla_X^{\perp} R_\theta^s \xi_\theta)) \\
&= \phi_\theta \circ R_\theta^{r-1}(\pi_{r-1}(\nabla_X^{\perp} \xi_\theta)) + \phi_\theta \circ R_\theta^s(\pi_r(\nabla_X^{\perp} \xi_\theta)) + \phi_\theta(\pi_{r+1}(\nabla_X^{\perp} R_\theta^s \xi_\theta)) \\
&= \phi_\theta(\pi_{r-1}(\nabla_X^{\perp} R_\theta^s \xi_\theta) + \pi_r(\nabla_X^{\perp} R_\theta^s \xi_\theta) + \pi_{r+1}(\nabla_X^{\perp} R_\theta^s \xi_\theta)) \\
&= \phi_\theta(\pi_{r-1}(\nabla_X^{\perp} R_\theta^s \xi_\theta) + \pi_r(\nabla_X^{\perp} R_\theta^s \xi_\theta) + \pi_{r+1}(\nabla_X^{\perp} R_\theta^s \xi_\theta)),
\end{align*}
and this proves (23).

Case 3. Assume that \( s = r + 1 \). We have that
\[
\nabla_X T_\theta \xi_\theta = \phi_\theta(\nabla_X \xi).
\]
On the other hand,
\[
T_\theta(\nabla_X \xi_\theta) = T_\theta \circ \psi_\theta(\nabla^\theta \xi) = T_\theta \circ \psi_\theta(\pi_r(\nabla^\theta \xi + \pi_{r+1}(\nabla^\theta \xi + \pi_{r+2}(\nabla^\theta \xi)))
\]
\[
= T_\theta \circ \psi_\theta(R_{\gamma}^\theta \pi_r(\nabla^\theta \xi + \pi_{r+1}(\nabla^\theta \xi + \pi_{r+2}(\nabla^\theta \xi))
\]
\[
= \phi_\theta(\pi_r(\nabla^\theta \xi + \pi_{r+1}(\nabla^\theta \xi + \pi_{r+2}(\nabla^\theta \xi))
\]
and (23) holds true.

Case 4. Assume that \( s \geq r + 2 \). Then, we have
\[
\nabla_X T_\theta \xi_\theta = \phi_\theta(\nabla_X \xi)
\]
and
\[
T_\theta(\nabla_X \xi_\theta) = T_\theta \circ \psi_\theta(\nabla^\theta \xi) = T_\theta \circ \psi_\theta(\pi_s(\nabla^\theta \xi + \pi_{s+1}(\nabla^\theta \xi + \pi_{s+2}(\nabla^\theta \xi)))
\]
\[
= T_\theta \circ \psi_\theta(\pi_s(\nabla^\theta \xi + \pi_{s+1}(\nabla^\theta \xi + \pi_{s+2}(\nabla^\theta \xi))
\]
Thus \( T_\theta \) is parallel and \( g_\theta \in G_0 \). Hence \( G_r = G_0 \).

Now assume \( \ell \geq 1 \). Take \( g_\theta \in G_\ell \), \( h_\theta \in G_{\ell+r} \) and let
\[
\phi_\theta: (N_{g_\theta} L, \nabla^{\theta, \ell}) \to (N_{g_\theta}, \nabla^\perp(g_\theta)) \quad \psi_\theta: (N_{g_\theta} L, \nabla^{\theta, \ell+r}) \to (N_{h_\theta}, \nabla^\perp(h_\theta))
\]
be the corresponding parallel isometries given by (21). Then, we have \( \alpha_{h_\theta} = S \alpha_{g_\theta} \) where \( S: (N_{g_\theta}, \nabla^\perp(g_\theta)) \to (N_{h_\theta}, \nabla^\perp(h_\theta)) \) is the bundle isometry given by
\[
S = \psi_\theta \circ \phi_\theta^{-1} \text{ on } N_{g_\theta}^s \text{ if } 1 \leq s \leq \ell \text{ or } s \geq \ell + r + 1
\]
and
\[
S = \psi_\theta \circ R_{\gamma}^\theta \circ \phi_\theta^{-1} \text{ on } N_{g_\theta}^s \text{ if } \ell + 1 \leq s \leq \ell + r.
\]
We show next that \( S \) is parallel, i.e.,
\[
\nabla_X S \xi_\theta = S(\nabla_X \xi_\theta)
\] (24)
where \( \xi_\theta = \phi_\theta \xi \) and \( \xi \in N_g^s, \leq s \leq \tau_g \). We need to distinguish several cases:

Case 1. Assume that \( 1 \leq s \leq \ell - 1 \). Then, we have
\[
\nabla_X S \xi_\theta = \nabla_X \psi_\theta \xi_\theta = \psi_\theta(\nabla_X^{\theta, \ell+r} \xi) = \psi_\theta(\nabla_X \xi).
\] (25)
We obtain

\[ S(\nabla_{X}^\perp \xi_{\theta}) = S \circ \phi_{\theta}(\nabla_{X}^\perp \xi) = S \circ \phi_{\theta}(\pi_{s-1}(\nabla_{X}^\perp \xi) + \pi_{s}(\nabla_{X}^\perp \xi) + \pi_{s+1}(\nabla_{X}^\perp \xi)) \]
\[ = S \circ \phi_{\theta}(\pi_{s-1}(\nabla_{X}^\perp \xi) + \pi_{s}(\nabla_{X}^\perp \xi) + \pi_{s+1}(\nabla_{X}^\perp \xi)) \]
\[ = \phi_{\theta}(\pi_{s-1}(\nabla_{X}^\perp \xi) + \pi_{s}(\nabla_{X}^\perp \xi) + S \circ \phi_{\theta} \circ \pi_{s+1}(\nabla_{X}^\perp \xi) \]
\[ = \psi_{\theta}(\nabla_{X}^\perp \xi), \]

and this proves (24).

**Case 4.** Assume that (25) still holds. Using (1), (5), (11) and (12) we have

\[ S(\nabla_{X}^\perp \xi_{\theta}) = S \circ \phi_{\theta}(\nabla_{X}^\perp \xi) = S \circ \phi_{\theta}(\pi_{s-1}(\nabla_{X}^\perp \xi) + \pi_{s}(\nabla_{X}^\perp \xi) + \pi_{s+1}(\nabla_{X}^\perp \xi)) \]
\[ = \psi_{\theta}(\pi_{s-1}(\nabla_{X}^\perp \xi) + \pi_{s}(\nabla_{X}^\perp \xi) + \pi_{s+1}(\nabla_{X}^\perp \xi)) \]
\[ = \psi_{\theta}(\nabla_{X}^\perp \xi), \]

and this proves (24).

**Case 2.** For \( s = \ell \) we see that (25) still holds. Using (1), (5), (11) and (12) we have

\[ S(\nabla_{X}^\perp \xi_{\theta}) = S \circ \phi_{\theta}(\nabla_{X}^\perp \xi) = S \circ \phi_{\theta}(\pi_{s-1}(\nabla_{X}^\perp \xi) + \pi_{s}(\nabla_{X}^\perp \xi) + \pi_{s+1}(\nabla_{X}^\perp \xi)) \]
\[ = \psi_{\theta}(\pi_{s-1}(\nabla_{X}^\perp \xi) + \pi_{s}(\nabla_{X}^\perp \xi) + \pi_{s+1}(\nabla_{X}^\perp \xi)) \]
\[ = \psi_{\theta}(\nabla_{X}^\perp \xi), \]

and this proves (24).

**Case 3.** Assume that \( s = \ell + 1 \). Then, we have

\[ \nabla_{X}^\perp S_{\xi_{\theta}} = \nabla_{X}^\perp (\psi_{\theta} R^{\ell+1}_{\theta} \xi) = \psi_{\theta}(\nabla_{X}^{\ell+r} R^{\ell+1}_{\theta} \xi) \]
\[ = \psi_{\theta}(\pi_{\ell}(\nabla_{X}^\perp R^{\ell+1}_{\theta} \xi) + \pi_{s+1}(\nabla_{X}^\perp R^{\ell+1}_{\theta} \xi)) \]

Using (1), (5), (11) and (12) we obtain

\[ S(\nabla_{X}^\perp \xi_{\theta}) = S \circ \phi_{\theta}(\nabla_{X}^\perp \xi) = S \circ \phi_{\theta}(\pi_{s-1}(\nabla_{X}^\perp \xi) + \pi_{s}(\nabla_{X}^\perp \xi) + \pi_{s+1}(\nabla_{X}^\perp \xi)) \]
\[ = \psi_{\theta}(\pi_{s-1}(\nabla_{X}^\perp \xi) + \pi_{s}(\nabla_{X}^\perp \xi) + \pi_{s+1}(\nabla_{X}^\perp \xi)) \]
\[ = \psi_{\theta}(\nabla_{X}^\perp \xi), \]

and this proves (24).

**Case 4.** Assume that \( \ell + 2 \leq s \leq \ell + r + 1 \). Then, we have

\[ \nabla_{X}^\perp S_{\xi_{\theta}} = \nabla_{X}^\perp (\psi_{\theta} R^{s}_{\theta} \xi) = \psi_{\theta}(\nabla_{X}^{\ell+r} R^{s}_{\theta} \xi) \]
\[ = \psi_{\theta}(\pi_{s-1}(\nabla_{X}^\perp R^{s}_{\theta} \xi) + \pi_{s}(\nabla_{X}^\perp R^{s}_{\theta} \xi) + \pi_{s+1}(\nabla_{X}^\perp R^{s}_{\theta} \xi)) \]

Using (1), (5), (11) and (12) we deduce that

\[ S(\nabla_{X}^\perp \xi_{\theta}) = S \circ \phi_{\theta}(\nabla_{X}^\perp \xi) = S \circ \phi_{\theta}(\pi_{s-1}(\nabla_{X}^\perp \xi) + \pi_{s}(\nabla_{X}^\perp \xi) + \pi_{s+1}(\nabla_{X}^\perp \xi)) \]
\[ = \psi_{\theta}(\pi_{s-1}(\nabla_{X}^\perp \xi) + \pi_{s}(\nabla_{X}^\perp \xi) + \pi_{s+1}(\nabla_{X}^\perp \xi)) \]
\[ = \psi_{\theta}(\nabla_{X}^\perp \xi), \]

and this proves (24).
Case 5. Assume that $s = \ell + r$. Then, we have
\[
\nabla_X^\perp S\xi_\theta = \nabla_X^\perp (\psi_\theta R^\ell_{-\theta} R^r_{-\theta} \xi)
= \psi_\theta(R^\ell_{-\theta} R^r_{-\theta} - \pi_{\ell+r-1}(\nabla_X^\perp \xi) + \pi_{\ell+r}(\nabla_X^\perp R^\ell_{-\theta} R^r_{-\theta} \xi) + \pi_{\ell+r+1}(\nabla_X^\perp R^r_{-\theta} R^r_{-\theta} \xi)).
\]
Using (4), (5), (11) and (12) we obtain
\[
S(\nabla_X^\perp \xi_\theta) = S \circ \phi_\theta(\nabla_X^\perp \xi_\theta) = S \circ \phi_\theta(\pi_{\ell+r-1}(\nabla_X^\perp \xi_\theta) + \pi_{\ell+r}(\nabla_X^\perp R^\ell_{-\theta} R^r_{-\theta} \xi_\theta) + \pi_{\ell+r+1}(\nabla_X^\perp R^r_{-\theta} R^r_{-\theta} \xi_\theta)),
\]
and this proves (24).

Case 6. Assume that $s = \ell + r + 1$. Then, we have
\[
\nabla_X^\perp S\xi_\theta = \nabla_X^\perp \psi_\theta \xi_\theta = \psi_\theta(\nabla_X^\perp \xi_\theta)
= \psi_\theta(R^\ell_{-\theta} R^r_{-\theta} - \pi_{\ell+r-1}(\nabla_X^\perp \xi_\theta) + \pi_{\ell+r+1}(\nabla_X^\perp \xi_\theta) + \pi_{\ell+r+2}(\nabla_X^\perp \xi_\theta)).
\]
Using (4), (5), (11) and (12) we obtain
\[
S(\nabla_X^\perp \xi_\theta) = S \circ \phi_\theta(\nabla_X^\perp \xi_\theta) = S \circ \phi_\theta(\pi_{\ell+r-1}(\nabla_X^\perp \xi_\theta) + \pi_{\ell+r}(\nabla_X^\perp R^\ell_{-\theta} R^r_{-\theta} \xi_\theta) + \pi_{\ell+r+2}(\nabla_X^\perp R^r_{-\theta} R^r_{-\theta} \xi_\theta)),
\]
and this proves (24).

Case 7. Assume that $s \geq \ell + r + 2$. Then, we have
\[
\nabla_X^\perp S\xi_\theta = \nabla_X^\perp \psi_\theta \xi_\theta = \psi_\theta(\nabla_X^\perp \xi_\theta)
= \psi_\theta(\pi_{s-1}(\nabla_X^\perp \xi_\theta) + \pi_s(\nabla_X^\perp \xi_\theta) + \pi_{s+1}(\nabla_X^\perp \xi_\theta)).
\]
Using (4), (5), (11) and (12) we obtain
\[
S(\nabla_X^\perp \xi_\theta) = S \circ \phi_\theta(\nabla_X^\perp \xi_\theta) = S \circ \phi_\theta(\pi_{s-1}(\nabla_X^\perp \xi_\theta) + \pi_s(\nabla_X^\perp R^\ell_{-\theta} R^r_{-\theta} \xi_\theta) + \pi_{s+1}(\nabla_X^\perp R^r_{-\theta} R^r_{-\theta} \xi_\theta)),
\]
and this proves (24). Thus $S$ is parallel, and the result follows.

4 Minimal submanifolds of rank 2

Let $f : M^n \to \mathbb{R}^N$, $n \geq 3$, be a submanifold of rank two. This means that the relative nullity subspaces $\Delta(x) \subset T_x M$ defined by
\[
\Delta(x) = \{X \in T_x M : \alpha_f(X, Y) = 0 \text{ for all } Y \in T_x M\}
\]
form a codimension two subbundle of the tangent bundle. The submanifold is called \textit{elliptic} if there exists an almost complex structure \( J : \Delta^\perp \to \Delta^\perp \) such that
\[
\alpha_f(X, X) + \alpha_f(JX, JX) = 0 \quad \text{for all} \quad X \in \Delta^\perp.
\]
Hence \( f \) is minimal if and only if \( J \) is orthogonal. As in the case of elliptic surfaces, the normal bundle splits as
\[
N_f M = N^{f}_1 \oplus \cdots \oplus N^{f}_{\tau_f}.
\]
It was shown in [8] that everything explained in this paper about polar surfaces to elliptic surfaces extends to this case. In particular, any elliptic submanifold in case (ii) admits locally many polar surfaces which turn out to be elliptic surfaces.

Hereafter, we assume that \( f : M^n \to \mathbb{R}^N \) is minimal and simply connected of rank two. For any \( \phi \in S^1 \) consider the tensor field \( R\phi \) that is the identity on \( \Delta \) and the rotation through \( \phi \) in \( \Delta^\perp \). It was observed in [9] that the normal valued tensor field given by
\[
\alpha \phi(X, Y) = \alpha_f(R\phi X, Y),
\]
satisfies the Gauss, Codazzi and Ricci equations with respect to the normal connection of \( f \). Hence, for each \( \phi \in S^1 \) there exists a minimal submanifold \( f_{\phi} : M^n \to \mathbb{R}^N \) of rank two that forms the associated family of \( f \).

According to the polar parametrization given in [8, Thm. 10], minimal submanifolds of rank two can be described parametrically along a subbundle of the normal bundle of an elliptic surface whose curvature ellipse of a specific order is circular. More precisely, given an elliptic surface \( g : L^2 \to \mathbb{Q}^N_{c}, \ c = 0, 1, \) with \( E^g_{\ell} \) for some \( 1 \leq \ell \leq \tau_g^0 - 1 \) a circle, consider the map \( f : \Lambda_{\ell} \to \mathbb{R}^N \) defined by
\[
f(\delta) = h(x) + \delta, \quad \delta \in \Lambda_{\ell}(x),
\]
where \( \Lambda_{\ell} = N^{g}_{\ell+1} \oplus \cdots \oplus N^{g}_{\tau_g} \) and \( h \) is any \( \ell \)-cross section to \( g \), is at regular points a minimal submanifold of rank two with polar surface \( g \). Conversely, any minimal submanifold of rank two admits locally such a parametrization with \( g \) a polar map.

The recursive procedure for the construction of the cross sections [8, Prop. 6] yields
\[
h = c\omega g + \text{grad} \omega + \gamma_0 + \gamma_1 + \cdots + \gamma_{\ell},
\]
where \( \omega \) is a solution of the linear elliptic differential equation
\[
\Delta u + \langle X, \text{grad} u \rangle + c\lambda u = 0
\]
for suitable \( X \in TL \) and \( \lambda \in C^\infty(L) \), \( \gamma_0 \) is any section in \( \Lambda_{\ell}, \gamma_1 \in N^g_{1} \) is the unique solution of \( A_{\gamma_1} = \text{Hess}_{\omega} + c\omega I \) and \( \gamma_j \in N^g_{j}, 2 \leq j \leq \ell \), where \( L^2 \) is endowed with the metric which makes \( J \) orthogonal.

Take \( g_{\theta} \in G_{\ell} \) and the corresponding vector bundle isometry \( \phi_{\theta} : N_{g_{\theta}} L \to N_{g_{\theta}} L \). Then,
\[
h_{\theta} = c\omega g_{\theta} + \text{grad} \omega + \phi_{\theta} \gamma_0 + \phi_{\theta} \gamma_1 + \cdots + \phi_{\theta} \gamma_{\ell}
\]
is an \( \ell \)-cross section to \( g_{\theta} \). With these elements we have the following result.
Theorem 10. A submanifold in the associated family of a minimal submanifold of rank two \( f: M^n \to \mathbb{R}^N \) can be locally parametrized as
\[
f_\theta(\delta) = h_\theta(x) + \phi_\theta \delta, \quad \delta \in \Lambda(x).
\]

Proof: It is easy to check that \( f_\theta \) is isometric to \( f \), has the same normal connection and its second fundamental form is given by
\[
\alpha_{f_\theta}(X,Y) = \alpha_f(R_{-\theta}X,Y).
\]

In particular, \( f_\theta \) is minimal and thus belongs to the associated family of \( f \).

The above discussion allows us to give an answer to the question of which minimal submanifolds of rank two have trivial associated family. In fact, this is the case if and only if the associated family of its polar surfaces is trivial, which is equivalent to the fact that a (local) bipolar surface to \( f \), i.e., any polar surface to its polar surface, is \( m \)-isotropic.

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IMPA – Estrada Dona Castorina, 110 22460-320 – Rio de Janeiro – Brazil E-mail: marcos@impa.br

Univ. of Ioannina – Math. Dept. 45110 Ioannina – Greece E-mail: tvlachos@uoi.gr