Singularity analysis and analytic solutions for the Benney-Gjevik equations

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March 15, 2021

Abstract

We apply the Painlevé Test for the Benney and the Benney-Gjevik equations which describe waves in falling liquids. We prove that these two nonlinear 1 + 1 evolution equations pass the singularity test for the travelling-wave solutions. The algebraic solutions in terms of Laurent expansions are presented.

Keywords: Painlevé Test, Singularity analysis, Liquid film,

1 Introduction

In 1966 in a short article Benney [1] proved the existence of two-dimensional waves with finite amplitude in the case of a laminar flow of a thin liquid film down an inclined plane by studying the nonlinearised system. The total set of differential equations which describe the problem is the equation of motion for the laminar flow and the continuity equation for the liquid film [1–6].

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By using the approach described in [2] and considering nonlinear terms Benney [1] reduced the problem to a third-order nonlinear $1+1$ evolution equation [1], up to a second-order correction, namely
\[ u_t + A(u) u_x + \varepsilon (B(u) u_{xx} + C(u) u_x^2) + \varepsilon^2 (D(u) u_{xxx} + E(u) u_x u_{xx} + F(u) u_x^3) = 0. \] (1)

Functions $A(u), B(u), \ldots$ are presented in Appendix A, which depend also upon the Rayleigh number, $R$. The parameter $\varepsilon$ is taken to be an infinitesimal, i.e. $\varepsilon^2 \to 0$, but for our analysis we consider it as a normal parameter. Finally the dependent variable $u(t,x)$ describes the location of the free surface of the liquid film. For a review on the Benney equation see [7, 8].

While the analysis of Benney [1] includes the linear case of [2, 3], the effect of the surface tension on the whole evolution of the problem has been omitted. However, such an analysis was later performed in 1970 by Gjevik [9].

In a similar way as Benney, Gjevik was able to reduce the problem to the nonlinear $1+1$ evolution equation
\[ u_t + A(u) u_x + \varepsilon (B(u) u_{xx} + C(u) u_x^2) + \varepsilon^2 (\bar{D}(u) u_x u_{xxx} + \bar{E}(u) u_{xxxx}) = 0. \] (2)

In the linear consideration, $\varepsilon^2 \to 0$, equations (1) and (2) are identical However, the main difference between these two evolution equations is that (1) is of third-order while (2) is of fourth-order.

The approach that Benney followed in order to study the existence of wave solutions for equation (1) is based on an approximate solution around a stationary point, $u_0$, and more specifically he considered $u(t,x) = 1 + \omega \eta(t,x)$ and linearised around $\omega$. However, such an analysis fails to provide an answer to whether the Benney equation is integrable. In this work, we study the integrability of the Benney-Gjevik equations (1) and (2) by using the method of singularity analysis for differential equations [10].

Singularity analysis is a powerful tool to study the integrability of differential equations and has a wide range of applications in mathematical physics, for instance see [11–14] and references therein. As far as concerns $1+1$ evolution equations, Glöckle et al in [15] present a detailed application on the Painlevé Test for a wide range of nonlinear equations and they show how the results of the Painlevé Test can be used to find new similarity solutions. A similar analysis was done by Steeb et al in [16] a few years earlier. For other applications of the singularity analysis on diffusion equations we refer the reader to [17–19]. The plan of the paper follows.

In Section 2 we present the basic properties and definitions for the singularity analysis while a demonstrative example is given. Sections 3 and 4 include the main results of our analysis in which we prove the integrability of the Benney and the Benney-Gjevik equations and we present the algebraic solution. Finally, we draw our conclusions in Section 5.

### 2 Preliminaries

The development of the Painlevé Test for the determination of integrability of a given equation or system of equations and its systematization has been succinctly summarized by Ablowitz, Ramani and Segur in the so-called ARS algorithm [31–33]. The main steps of the ARS algorithm are described as: (a) first step is to find the leading-order behaviour, (b) second step is the derivation of the resonances and (c) final step is the consistency test. The determination of the leading-order term provides the movable pole for the differential equation, while the resonances describe the position of the constants of integration and also define the explicit

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1 Any differential equation on falling liquid films which is derived with the approach described in [1] is termed a Benny equation. However, on his original paper Benney provides a third-order derivative in the first nonlinear correction $\varepsilon^2$. 

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form of the Laurent expansion. Finally the consistency test proves that the Laurent expansion solves the original differential equations. More details on the method and various discussions can be found in the review of [10], while a recent discussion on the connection of the singularity analysis with the symmetry approach is given in [22].

The ARS algorithm is somewhat like a recipe which can be difficult to implement at times. These difficulties can be due to the inherent complexity of the equation under consideration or it can be due to results at variance with the prevailing canon for an equation which is patently integrable. This has lead to changes in the paradigms of singularity analysis. Originally the singularity had to be a pole, where the alternate description as the method of polelike expansions. Singularity analysis is valid when the leading-order term has an exponent a fraction number or a negative integer number. On the other hand, for a positive fractional power by inverse the dependent variable it eventually leads to a negative exponent and hence into a pole. As the analysis is undertaken over the complex plane of the independent variable, the presence of fractional exponents means the division of the plane by branch cuts. A practical limitation would be not to have a large denominator thereby leading to many subsets of the complex plane. Another fallacy was that resonances, apart from the generic $-1$, had to be nonnegative. This was demonstrated to be not so with some elementary equations the closed-form solutions of which could be used to valid the point. Finally the occurrence of multiple $-1$ resonances was taken to mean that the solution had to had a logarithmic singularity. This was shown by explicit example not to be so.

It is straightforward to find that the application of Lie’s theory [23] for the two 1+1 PDEs of our consideration provides the two symmetry vectors $\Gamma_1 = \partial_t$ and $\Gamma_2 = \partial_x$ for both equations. The application of the invariants obtained by these symmetries reduces the PDEs to ODEs. Static solutions follow by applying the symmetry vector $\Gamma_1$ and stationary solutions follow by applying the invariants of the symmetry vector $\Gamma_2$. In our case, as we are interested in travelling-wave solutions we apply the invariant functions of the the symmetry vector $\Gamma_1 + c\Gamma_2$ which are $\zeta = x - ct$, $y = u(\zeta)$, in which $c$ is the wave speed.

By assuming $\zeta$ to be the new independent variable and $u(\zeta)$ the new dependent variable, the Benney equation reduces to the third-order ODE

$$ (A (y) - 1) y' + \varepsilon \left( B (y) y'' + C (y) (y')^2 \right) + \varepsilon^2 \left( D (y) y''' + E (y) y'y'' + F (y) (y')^3 \right) = 0. \tag{3} $$

On the other hand, the Bennery-Gjevik equation takes the form of the fourth-order ODE

$$ (A (y) - 1) y' + \varepsilon \left( B (y) y'' + C (y) (y')^2 \right) + \varepsilon^2 \left( D (y) y''' + E (y) y(4) \right) = 0. \tag{4} $$

For more details on the application of symmetries on PDEs and on physical problems we refer the reader in [24–30] and references therein.

We continue with the application of the singularity analysis to these ODEs.

### 3 Integrability of the Benney equation

We commence our discussion by studying the linearised Benney equation

$$ (A (y) - 1) y' + \varepsilon \left( B (y) y'' + C (y) (y')^2 \right) = 0. \tag{5} $$
We substitute \( y(\zeta) = y_0 (\zeta - \zeta_0)^{-\frac{1}{2}} \) into \( (5) \) which is reduced to the algebraic equation

\[
0 = a_1 (y_0, R, a, \varepsilon) (\zeta - \zeta_0)^{8P-1} + a_2 (y_0, R, a, \varepsilon) (\zeta - \zeta_0)^{6P-1} + a_3 (y_0, R, a, \varepsilon) (\zeta - \zeta_0)^{2P-2} + a_4 (y_0, R, a, \varepsilon) (\zeta - \zeta_0)^{5P-2}.
\]

From this we find that the leading-order behaviour \( y(\zeta) = y_0^{-\frac{1}{2}} (\zeta - \zeta_0) \), in which \( y_0 = y_0 (R, a) \), is given by a third-order polynomial equation.

In order to find the resonances, i.e. the position of the other constant of integration, we substitute

\[
y(\zeta) = y_0 (\zeta - \zeta_0)^{-\frac{1}{2}} + m (\zeta - \zeta_0)^{-\frac{1}{2} + s}
\]

into \( (5) \) and we linearize around the parameter \( m \). There again the leading-order terms provide the polynomial equation

\[
(S + 1) (3S + 1) = 0
\]

the zeros of which give the resonances; they are \( S = -1 \) and \( S = -\frac{1}{3} \). As we discussed above, the existence of the resonance, \( S = -1 \), is important in order that the singularity exist and be movable. In particular it is related with the constant of integration, \( \zeta_0 \).

From the second resonance we extract the information that the algebraic solution of equation \( (5) \) is given by a Left Painlevé Series, with a step \( \frac{1}{3} \), i.e. the algebraic solution is

\[
y(\zeta) = y_0 (\zeta - \zeta_0)^{-\frac{1}{2}} + y_1 (\zeta - \zeta_0)^{-\frac{2}{3}} + \sum_{I=2}^{\infty} y_I (\zeta - \zeta_0)^{-\frac{1+4I}{3}}
\]

in which \( y_1 \) is the second constant of integration and \( y_I = y_I (y_1, R, a) \).

Now in the case of the full Benney equation, when the nonlinear term \( \varepsilon^2 \) is included, the situation is different because the new terms contribute to the existence of the singularity and a substitution \( y(\zeta) = y_0 (\zeta - \zeta_0)^{-\frac{1}{2}} \) into \( (5) \) fails to provide a singular leading-order behaviour. In order to surpass this difficulty we perform the change of variables \( y(\zeta) = \frac{1}{\zeta (\zeta)} \) and in the new equation we substitute \( Y(\zeta) = Y_0 (\zeta - \zeta_0)^P \) from which we obtain the algebraic equation

\[
0 = \beta_1 (Y_0, R, a, \varepsilon) (\zeta - \zeta_0)^{13P-1} + \beta_2 (Y_0, R, a, \varepsilon) (\zeta - \zeta_0)^{11P-1} + \beta_3 (Y_0, R, a, \varepsilon) (\zeta - \zeta_0)^{7P-2} + \beta_4 (Y_0, R, a, \varepsilon) (\zeta - \zeta_0)^{10P-2} + \beta_5 (Y_0, R, a, \varepsilon) (\zeta - \zeta_0)^{9P-3} + \beta_6 (Y_0, R, a, \varepsilon) (\zeta - \zeta_0)^{6P-3} + \beta_7 (Y_0, R, a, \varepsilon) (\zeta - \zeta_0)^{3P-1}.
\]

From this system we obtain the leading-order term

\[
Y(\zeta) = Y_0 (\zeta - \zeta_0)^{-\frac{1}{2}} \quad \text{(13)}
\]

with corresponding resonances

\[
s = -1 \quad \text{,} \quad s = 1 \quad \text{,} \quad s = -\frac{17}{6}.
\]

Consequently the algebraic solution is expressed by a full Painlevé Series with step \( \frac{1}{3} \), that is

\[
Y(\zeta) = \sum_{J=-\infty}^{\infty} Y_J (\zeta - \zeta_0)^{-\frac{1+4J}{3}} + Y_{-1} (\zeta - \zeta_0)^{-\frac{2}{3}} + Y_0 (\zeta - \zeta_0)^{-\frac{1}{2}} + Y_1 (\zeta - \zeta_0)^{-\frac{1+4}{3}} + \sum_{I=2}^{\infty} Y_I (\zeta - \zeta_0)^{-\frac{1+4I}{3}}.
\]

The latter Laurent expansion is replaced in equation \( (5) \) such that to perform the consistency test. We find that expression \( (13) \) is a solution for the Benney equation.

At this point it is important to mention that the imaginary coefficient of the leading-order term \( Y_0 \) is directly related with periodic solutions. We continue our study with the Benney-Gjevik equation.
4 Integrability of the Benney-Gjevik Equation

For the Benney-Gjevik equation we apply the same procedure as for the full Benney equation. Indeed, we replace \( y(ζ) = \frac{1}{ζ} \) and we perform the singularity analysis for the new fourth-order ODE expressed in terms of \( Y(ζ) \).

We substitute \( Y(ζ) = Y₀ζ^p \) which gives us the polynomial equation

\[
0 = γ₁(Y₀, R, a, ε, W)(ζ - ζ₀)^8p−1 + γ₂(Y₀, R, a, ε, W)(ζ - ζ₀)^6p−1 + \]
\[
+ γ₃(Y₀, R, a, ε, W)(ζ - ζ₀)^5p−2 + γ₄(Y₀, R, a, ε, W)(ζ - ζ₀)^2p−2 + \]
\[
+ γ₅(Y₀, R, a, ε, W)(ζ - ζ₀)^5p−4 \quad (14)
\]

from which we extract the leading-order behaviour to be

\[
Y(ζ) = Y₀(ζ - ζ₀)^{-\frac{2}{3}}, \quad Y₀ = Y₀(R, ε, W). \quad (15)
\]

in which \( Y₀ \) is given by a fourth-order polynomial.

Moreover the resonances are calculated to be

\[
s = -1, \quad s = \frac{8}{3}, \quad s = \frac{10}{3}, \quad s = \frac{17}{3} \quad (16)
\]

which indicates that the solution is expressed in terms of a Right Painlevé Series with step \( \frac{1}{3} \), i.e.

\[
Y(ζ) = Y₀(ζ - ζ₀)^{-\frac{2}{3}} + Y₁(ζ - ζ₀) + \sum_{l=2}^{∞} Y_l(ζ - ζ₀)^{\frac{2}{3}l}. \quad (17)
\]

In addition, because the three resonances are positive, this indicates that the leading-order behaviour is an attractor for the differential equation. Finally, the Laurent expansion (17) satisfy the consistency test which means that expression (17) is an analytic solution for the Benney-Gjevik equation. The coefficients of (17) are \( Y₁ = 0, \quad Y₆ = -\frac{25}{18}Y₀(R, ε, W) \), \( Y₈ = \text{arbitrary} \), \( Y₉ = 0 \), \( Y₉ = \text{arbitrary} \) etc. The integration constants are the coefficients \( Y₈ \), \( Y₁₀ \) and \( Y₁₇ \).

5 Conclusions

In this work we studied the existence of travelling-wave solutions for the Benney and the Benney-Gjevik equations by using the singularity analysis. In particular we applied the invariants of the Lie symmetry vector \( \partial_t + c\partial_x \) and reduced the 1 + 1 evolution equations (1), (2) to a third-order and a fourth-order ODE, respectively. We proved that theses equations pass the Painlevé Test and consequently are integrable.

For completeness in our analysis we have considered also the case where in the Benney equation (1) parameter \( ε^2 \rightarrow 0 \) which is explicitly the case studied in [2,3]. However, what is worth mentioning is that all the leading-order terms can have complex coefficients which means that the solutions can be periodic.

The importance of our analysis is that we were able to prove the integrability for the Benney and the Benney-Gjevik equations while also to present for the first time the algebraic expressions for the travelling-wave solutions in terms of Laurent expansions.

We conclude this work by mentioning that the scopus of this work was to prove the existence of analytic solutions for the evolution equations (1), (2) which in particular are singular perturbative equations. Until now, according to our knowledge of the literature, travelling-wave solutions have been found as approximate...
solutions at the limits of the behaviour for these two equations by using perturbation theory. In our analysis we considered the problem without assuming any perturbative term and we were able to prove the existence of real solutions (travelling-wave solutions) for arbitrary initial conditions, without resorting to the Cauchy theorem for local existence.

Acknowledgements

AP and GL were funded by Comisión Nacional de Investigación Científica y Tecnológica (CONICYT) through FONDECYT Iniciación 11180126. GL thanks to Department of Mathematics and to Vicerrectoría de Investigación y Desarrollo Tecnológico at Universidad Católica del Norte for financial support.

A Coefficients for equations (1) and (2)

The coefficients of equation (1) are as follows [1]

\[
\begin{align*}
A(u) &= 2u^2 \\
B(u) &= -\frac{8}{15}Ru^6 + \frac{2}{3}u^3 \cot a \\
C(u) &= -\frac{16}{5}Ru^5 + 2u^2 \cot a \\
D(u) &= -2u^4 - \frac{32}{63}R^2u^{10} + \frac{40}{63}Ru^7 \cot a \\
E(u) &= -\frac{52}{3}u^3 + \frac{433}{63}R^2u^9 + \frac{392}{45}Ru^6 \cot a \\
F(u) &= -14u^2 - 29R^2u^8 + \frac{64}{5}Ru^5 \cot a,
\end{align*}
\]

where \( R \) denotes the Rayleigh’s number, \( a \) is the inclination of the plane on which the motion occurs, while the dependent variable \( u(t, x) \) describes the surface of the flow. While for equation (2) coefficients \( \bar{D}(u) \) and \( \bar{E}(u) \)

\[
\bar{D}(u) = 3Wu^2, \quad \bar{E}(u) = 3Wu^3
\]

in which \( W \) is a parameter of order \( a^{-2} \) [9].

References

[1] D.J. Benney, Studies in Applied Mathematics 45, 150 (1966)
[2] T.B. Benjamin, J. Fluid Mech. 2, 554 (1957)
[3] C.S. Yih, Physics of Fluids, 6, 321 (1963)
[4] C.C. Mei, J. Math. Phys. 45, 266 (1966)
[5] A.D.D Craik, Wave Interactions and Fluid Flows, Cambridge University Press, Cambridge (1988)
[6] S. Kalliadasis, C. Ruyer-Quil, B. Scheid and M.G. Velarde, Falling Liquid Films, Springer-Verlag, London, (2012)
[7] R.V. Craster, O.K. Matar, J. Fluid. Mech. 553, 96 (2006)
[8] A. Oron, S.H. Davis, S.G. Bankoff, Rev. Mod. Phys. 69, 931 (1997)
[9] B. Gjevik, The Physics of Fluids 13, 1915 (1970)
[10] A. Ramani, B. Grammaticos and T. Bountis, Phys. Rept. 180, 159 (1989)
[11] S. Cotsakis and P.G.L. Leach, Proceedings of Institute of Mathematics of NAS of Ukraine, eConf, 43, 128 (2002)
[12] H.I. Levine, J. Math. Phys. 34, 4781 (1993)
[13] S. Cotsakis and P.G.L. Leach, J. Phys. A: Math. Gen. 27, 1625 (1994)
[14] A. Paliathanasis, P.G.L. Leach and T. Taves, EPJC 77, 909 (2017)
[15] W.G. Glöckle, G. Baumann and T.F. Nonnenmacher, J. Math. Phys. 33, 2456 (1992)
[16] W-H. Steeb, A. Granel, M. Kloke and B.M. Spieker, Phys. Scr. 31, 5 (1984)
[17] F. Li and X. Zheng, Applied Math. Lett. 25, 2179 (2012)
[18] B. Fuchssteiner and S. Carillo, Physica A: Statistical Mechanics and its Applications 154, 467 (1989)
[19] O. Costin and S. Tanveer, Communications in Partial Differential Equations 31, 593 (2006)
[20] B. Abraham-Shrauner, J. Math. Phys. 34, 4809 (1993)
[21] F. M. Mahomed and P. G. L. Leach, Quaestiones Mathematicae 8, 241 (1985)
[22] A. Paliathanasis and P.G.L. Leach, Int. J. Geom. Meth. Mod. Phys. 13, 11630009 (2016)
[23] G.W. Bluman and S. Kumei, Symmetries and Differential Equations, Springer, New York, (1989)
[24] U. Obaidullah and S. Jamal, J. Appl. Math. Comp. 65, 541 (2021)
[25] S. Jamal, Mathematica, 7, 574 (2019)
[26] S. Jamal, Gen. Relat. Gravit. 49, 88 (2017)
[27] S. Jamal and A. G. Johnpillai, Math. Mod. Anal. 25, 10115 (2020)
[28] S. Jamal, J. Differential Equations 266, 4018 (2019)
[29] S. Jamal and N. Mnguni, Quaestiones Mathematicae doi.org/10.2989/16073606.2020.1790438
[30] A. Mathebula and S. Jamal, Indian J. Phys. doi.org/10.1007/s12648-020-01810-7
[31] M.J. Ablowitz, A. Ramani and H. Segur, Lettere al Nuovo Cimento 23, 333 (1978)
[32] M.J. Ablowitz, A. Ramani and H. Segur, J. Math. Phys. 21, 715 (1980)
[33] M.J. Ablowitz, A. Ramani and H. Segur, J. Math. Phys. 21, 1006 (1980)