A SPECTRAL THEORY OF POLYNOMIALLY BOUNDED SEQUENCES AND APPLICATIONS TO THE ASYMPTOTIC BEHAVIOR OF DISCRETE SYSTEMS

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Abstract. In this paper using a transform defined by the translation operator we introduce the concept of spectrum of sequences that are bounded by \( n^\nu \), where \( \nu \) is a natural number. We apply this spectral theory to study the asymptotic behavior of solutions of fractional difference equations of the form \( \Delta^\alpha x(n) = Tx(n) + y(n), \ n \in \mathbb{N}, \) where \( 0 < \alpha \leq 1 \). One of the obtained results is an extension of a famous Katznelson-Tzafriri Theorem, saying that if the \( \alpha \)-resolvent operator \( S_\alpha \) satisfies \( \sup_{n \in \mathbb{N}} \|S_\alpha(n)\|/n^{\nu} < \infty \) and the set of \( z_0 \in \mathbb{C} \) such that \((z - k^\nu(z)T)^{-1}\) exists, and together with \( k^\nu(z) \), is holomorphic in a neighborhood of \( z_0 \) consists of at most 1, where \( k^\nu(z) \) is the Z-transform of \( k^\nu(n) := \Gamma(\alpha + n)/\Gamma(\alpha)\Gamma(n + 1) \), then

\[
\lim_{n \to \infty} \frac{1}{n^{\nu}} \sum_{k=0}^{n+1} \frac{(\nu + 1)!}{k!(\nu + 1 - k)!}( -1)^{\nu+1+k}S_\alpha(n + k) = 0.
\]

1. Introduction

Let us consider difference equations of the form

\[
(1.1) \quad x(n + 1) = Tx(n) + y(n), \quad n \in \mathbb{N},
\]

where \( T \) is a bounded operator in a Banach space \( X \), \( \{x(n)\}_{n=1}^{\infty} \) and \( \{y(n)\}_{n=1}^{\infty} \) are sequences in \( X \). The asymptotic behavior of solutions of the above mentioned equations is a central topic in Analysis and Dynamical Systems. There are numerous methods for this study of this topic. The reader is referred to \( \mathbb{N} \) and its references for information on classical methods of Dynamical Systems in the finite dimensional case. On the other hand, in the infinite dimensional case, by Harmonic Analysis and Operator Theory, many results on the asymptotic behavior of solutions of Eq. (1.1) have been obtained, see e.g. \[1\, 3\, 4\, 5\, 6\, 7\, 8\, 11\, 14\, 15\, 16\]. Among many interesting results in this direction is a famous theorem due to Katznelson-Tzafriri (see \[9\]) saying that if \( T \) is a bounded operator in a Banach space \( X \) such that

\[
(1.2) \quad \sup_{n \in \mathbb{N}} \|T^n\| < \infty,
\]

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and $\sigma(T) \subset \{1\}$, then
\begin{equation}
\lim_{n \to \infty} (T - I)T^n = 0.
\end{equation}

There are a lot of extensions and improvements of this result as well as simple proofs of it, see e.g. [1, 3, 5, 7, 9, 11, 14, 15, 16] and the references therein.

As shown in [15] the above mentioned Katznelson-Tzafriri Theorem is equivalent to its weaker version for individual orbits. Namely, the following statement: Let $T$ be a bounded operator in a Banach space $X$ such that (1.2) holds and $\sigma(T) \subset \{1\}$. Then, for each $x \in X$
\begin{equation}
\lim_{n \to \infty} (T - I)T^n x = 0.
\end{equation}

In [11] a simple proof of this weaker version is given, based on a transform associated with the translation operator of sequences.

The main concern of this paper is to extend the above mentioned Katznelson-Tzafriri Theorem to fractional difference equations of the form
\begin{equation}
\Delta^\alpha x(n) = Tx(n) + y(n), \quad n \in \mathbb{N},
\end{equation}
where $0 < \alpha \leq 1$, the operator $\Delta^\alpha$ (the fractional difference operator in the sense of Riemann-Liouville) and other operators are defined as follows (see [11] and its references for more details): for each $n \in \mathbb{N}$,
\begin{align*}
(\Delta^\alpha)f(n) &= \Delta^1 \circ \Delta^{-(1-\alpha)}f(n), \\
(\Delta^1)f(n) &= f(n + 1) - f(n), \\
(\Delta^{-\alpha})f(n) &= \sum_{k=0}^{n} k^\alpha(n - k)f(k),
\end{align*}
where $\Gamma(\cdot)$ is the Gamma function defined below. Our method relies on a spectral theory of polynomially bounded sequences that will be presented in the next sections, and that would be of independent interest. The obtained results will be illustrated in simple cases of ordinary difference equations and then stated for fractional difference equations. Our main result is Theorem 4.16. To our best knowledge, it is a new extension of the Katznelson-Tzafriri Theorem to fractional difference equations.

2. Preliminaries and Notations

2.1. Notations. Throughout this paper we will denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ the set of natural numbers, integers, real numbers and the complex plane, respectively. For $z \in \mathbb{C}$, $\Re z$ stands for its real part. The gamma function $\Gamma(z)$ is defined to be
\begin{equation}
\Gamma(z) = \int_{0}^{\infty} x^{z-1}e^{-x}dx, \quad \Re z > 0.
\end{equation}

For a Banach space $X$, $L(X)$ denotes the space of all bounded linear operators from $X$ to itself. We also use the following standard notations: $\rho(T)$ denotes the resolvent set of a given operator $T$, that is, $\rho(T) := \{\lambda \in \mathbb{C} : (\lambda - T)^{-1}\text{exists}\}$, and $\sigma(T) := \mathbb{C}\setminus\rho(T)$. For each $\lambda \in \rho(T)$ we denote $R(\lambda, T) := (\lambda - T)^{-1}$. Moreover, we will denote by $\Gamma$ the unit circle in the complex plane.
For a given nonnegative integer \( \nu \), we denote by \( l^\nu_\infty(X) \) the space of all sequences in a Banach space \( X \) such that
\[
\sup_{n \in \mathbb{N}} \| x(n) \|^{\nu} < \infty.
\]
It is easy to see that \( l^\nu_\infty(X) \) is a Banach space with norm
\[
\| x \|_\nu := \sup_{n \in \mathbb{N}} \| x(n) \|^{\nu}
\]
for each \( x = \{ x(n) \}_{n \in \mathbb{N}} \). We will denote by \( c^\nu_0(X) \) the subspace of \( l^\nu_\infty(X) \) consisting of all sequences \( \{ x(n) \}_{n \in \mathbb{N}} \) such that
\[
\lim_{n \to \infty} \frac{\| x(n) \|}{n^{\nu}} = 0.
\]
We can check that \( c^\nu_0(X) \) is a complete subspace of \( l^\nu_\infty(X) \), so the quotient space
\[
Y := l^\nu_\infty(X)/c^\nu_0(X)
\]
is well defined as a Banach space. If \( f \in l^\nu_\infty(X) \) we will denote its equivalence class by \( \bar{f} \). In the space \( l^\nu_\infty(X) \) let us consider the translation operator \( S \) defined as
\[
[Sx](n) = x(n + 1), \quad n \in \mathbb{N}, \quad x \in l^\nu_\infty(X).
\]
This is a bounded operator. Moreover, this operator leaves \( c^\nu_0(X) \) invariant. Hence, it induces an operator \( \bar{S} \) on \( Y \).

### 2.2. Vector-valued holomorphic functions.

In this paper we say that a function \( f(z) \) defined for all \( z \in \Omega \subset \mathbb{C} \) with values in a complex Banach space \( X \) is holomorphic (or analytic) for \( z \in \Omega \) if for each \( z_0 \in \Omega \)
\[
f'(z_0) := \lim_{h \to 0, h \neq 0} \frac{f(z_0 + h) - f(z_0)}{h}
\]
exists. A family of continuous functionals \( W \subset X^* \) is said to be separating if \( x \in X \) and \( \langle x, \phi \rangle = 0 \) for all \( \phi \in W \), then \( x = 0 \). We will need the following whose proof can be found in \([2, \text{Theorem 3.1}]\), or \([3, \text{Theorem A.7}]\):

**Theorem 2.1.** Let \( \Omega \subset \mathbb{C} \) be open and connected, and let \( f : \Omega \to X \) be bounded on every compact subset of \( \Omega \). Assume further that \( W \subset X^* \) is separating subset such that \( x^* \circ f \) is holomorphic for all \( x^* \in W \). Then \( f \) is holomorphic.

We will need an auxiliary result that is a special kind of maximum principle for holomorphic functions (for the proof see e.g. \([3, \text{Lemma 4.6.6}]\)):

**Lemma 2.2.** Let \( U \) be an open neighborhood of \( i\eta \) such that \( U \) contains the closed disk \( \bar{B}(i\eta, 2r) = \{ z \in \mathbb{C} : |z - i\eta| \leq 2r \} \). Let \( h : U \to X \) be holomorphic and \( c \geq 0 \), \( k \in \mathbb{N} \) such that
\[
\| h(z) \| \leq \frac{c}{|\Re z|^k}, \quad \text{if} \quad |z - i\eta| = 2r, \quad \Re z \neq 0.
\]
Then
\[
\| h(z) \| \leq \left( \frac{4}{3} \right)^k \frac{c}{r^k}, \quad \text{for all} \quad z \in \bar{B}(i\eta, r).
\]
3. Spectrum of a polynomially bounded sequence

The following lemma is the key for us to set up a spectral theory for polynomially bounded sequences.

**Lemma 3.1.** Assume that $\tilde{S}$ is the operator induced by the translation $S$ in the quotient space $l^\infty_\nu(\mathbb{K})/c_0^\nu(\mathbb{K})$. Then

$$\sigma(\tilde{S}) \subset \Gamma.$$ 

Moreover, for each $|\lambda| \neq 1$ with $|\lambda| < 2$ and $f \in l^\nu_\infty(\mathbb{K})$, the following estimate is valid:

$$\|R(\lambda, \tilde{S})f\| \leq \frac{C}{|\lambda| - 1} \|f\|_\nu,$$

where $C$ is a certain positive number, independent of $f$.

**Proof.** We will prove that if $|\lambda| \neq 1$, then $\lambda \in \rho(\tilde{S})$. In other words, $\sigma(\tilde{S}) \subset \Gamma$. And after that, we will give estimates of the resolvent $R(\lambda, \tilde{S})$ of a given sequence $f \in l^\nu_\infty(\mathbb{K})$. To study the invertibility of the operator $(\lambda - \tilde{S})$, we consider the non-homogeneous linear difference equation

$$x(n+1) - \lambda x(n) = f(n), \quad n \in \mathbb{N}.$$

To prove that $\lambda \in \rho(\tilde{S})$ for each $|\lambda| \neq 1$ we will show that this equation \(3.2\) has a unique solution $x \in l^\nu_\infty(\mathbb{K})$ modulo $c_0^\nu(\mathbb{K})$ given $f \in l^\nu_\infty(\mathbb{K})$.

We first consider the case $|\lambda| < 1$. In this case, we will use the Variation of Constants Formula

$$x(n) = \lambda^{n-1}x(1) + \sum_{k=1}^{n-1} \lambda^{n-1-k}f(k), \quad n \in \mathbb{N}.$$ 

Since the sequence $f$ grows polynomially, the series $\sum_{k=1}^\infty \lambda^{n-1-k}f(k)$ is absolutely convergent. Also, by $|\lambda| < 1$ the sequence $\{\lambda^{n-1}x(1)\}_{n \in \mathbb{N}}$ is in $c_0^\nu(\mathbb{K})$. Therefore, Eq. \(3.2\) has a unique solution

$$\left\{ x_f(n) := \sum_{k=1}^{n-1} \lambda^{n-1-k}f(k) \right\}_{n \in \mathbb{N}} \text{ modulo } c_0^\nu(\mathbb{K}).$$

Now suppose that $g$ is any element in the class $\tilde{f}$. We will show that $\bar{x}_g = \bar{x}_f$. Or equivalently, we have to show that whenever $h \in c_0^\nu(\mathbb{K})$, the sequence

$$\{x_h(n)\}_{n \in \mathbb{N}} = \left\{ \sum_{k=1}^{n-1} \lambda^{n-1-k}h(k) \right\}_{n \in \mathbb{N}} \in c_0^\nu(\mathbb{K}).$$

In fact, as $h \in c_0^\nu(\mathbb{K})$, given $\varepsilon > 0$ there exists a natural number $M$ such that for all $k \geq M$,

$$\frac{|h(k)|}{k^\nu} < \frac{1 - |\lambda|}{2} \varepsilon. $$

Therefore, for all $n \geq M + 1$,

$$\frac{\|x_h(n)\|}{n^\nu} \leq \sum_{k=1}^{M-1} |\lambda|^{n-1-k} \|h(k)\| + \sum_{k=M}^{n-1} |\lambda|^{n-1-k} \frac{n^\nu}{n^\nu} \|h(k)\|.$$
This means \( g \) is any representative of the class \( \bar{\lambda} \) such that for all \( n \geq K \),

\[
\frac{|\lambda|^n}{n^\nu} \sum_{k=1}^{M-1} |\lambda|^{-1-k} \|h(k)\| \leq \frac{\varepsilon}{2}.
\]

Consequently, given any \( \varepsilon > 0 \) there exists a number \( K \) such that for all \( n \geq K \),

\[
\frac{\|x_h(n)\|}{n^\nu} \leq \varepsilon.
\]

This means

\[
\lim_{n \to \infty} \frac{\|x_h(n)\|}{n^\nu} = 0.
\]

By this we have proved that \( \bar{x}_f = \bar{x}_g \) whenever \( \bar{f} = \bar{g} \). Namely, we have showed that if \( |\lambda| < 1 \), then \( \bar{x}_f = (\lambda - \bar{S})^{-1} \bar{f} \). In other words, \( \lambda \in \rho(S) \). Moreover, for any representative \( g \) of the class \( \bar{f} \)

\[
\|R(\lambda, \bar{S})\bar{f}\|_\nu = \|\bar{x}_f\|_\nu = \inf_{g \in \bar{f}} \|x_g\|_\nu \leq \|x_g\|_\nu = \sup_{n \in \mathbb{N}} \left\| \sum_{k=1}^{n-1} \lambda^{n-1-k} g(k) \right\|_{n^\nu}
\]

\[
\leq \sup_{n \in \mathbb{N}} \sum_{k=1}^{n-1} |\lambda|^{n-1-k} \left\| g(k) \right\|_{k^\nu}
\]

\[
\leq \left( \sup_{n \in \mathbb{N}} \sum_{k=1}^{n-1} |\lambda|^{n-1-k} \right) \|g\|_{\nu}
\]

\[
\leq \frac{\|g\|_{\nu}}{1 - |\lambda|}.
\]

Finally, as \( g \) is any representative of the class \( \bar{f} \), we have

\[
\|R(\lambda, \bar{S})\bar{f}\| \leq \inf_{g \in \bar{f}} \frac{\|g\|_{\nu}}{1 - |\lambda|} = \frac{\|\bar{f}\|_{\nu}}{1 - |\lambda|}.
\]

Next, we consider the case \( |\lambda| > 1 \). We can verify that the formula

\[
x(n) = \lambda^{n-1} x(1) - \sum_{k=n}^{\infty} \lambda^{n-k-1} f(k), \quad n \in \mathbb{N},
\]

(3.3)

\[
x(n + 1) = \lambda^n x(1) - \sum_{k=n+1}^{\infty} \lambda^{n-k} f(k)
\]
gives the general solution to Eq. (3.2). In fact, since \( |\lambda| > 1 \) and \( f \) grows polynomially the series \( \sum_{k=n}^{\infty} \lambda^{n-k-1} f(k) \) is absolutely convergent for each \( n \in \mathbb{N} \). Moreover, by (3.3), for each \( n \in \mathbb{N} \),

\[
x(n + 1) = \lambda^n x(1) - \sum_{k=n+1}^{\infty} \lambda^{n-k} f(k)
\]
\[ x_f := \left\{ -\sum_{k=n}^{\infty} \lambda^{n-k} f(k) \right\}_{n \in \mathbb{N}}. \]

Indeed,

\[
\|x_f\|_\nu \leq \sup_{n \in \mathbb{N}} \sum_{k=n}^{\infty} \frac{|\lambda|^{n-k-1} \|f(k)\|}{n^\nu} = \sup_{n \in \mathbb{N}} \sum_{k=n}^{\infty} \frac{|\lambda|^{n-k-1} k^\nu}{n^\nu} \|f(k)\|_\nu \\
\leq \sup_{n \in \mathbb{N}} \sum_{k=n}^{\infty} \frac{k^\nu}{n^\nu} \|f\|_\nu = \sup_{n \in \mathbb{N}} \sum_{j=1}^{\infty} |\lambda|^{-j} \left( 1 + \frac{j-1}{n} \right)^\nu \|f\|_\nu \\
\leq \sum_{j=1}^{\infty} |\lambda|^{-j} j^\nu \|f\|_\nu.
\]

We are interested in the behavior of \( \sum_{j=1}^{\infty} |\lambda|^{-j} j^\nu \) as \(|\lambda|\) gets closer and closer to 1 (and \(\infty\), respectively). To this end, we note that for each \(j \in \mathbb{N}\),

\[
|\lambda|^{-j} j^\nu \leq \int_j^{j+1} |\lambda|^{-t} t^\nu dt.
\]

Therefore,

\[
\sum_{j=1}^{\infty} |\lambda|^{-j} j^\nu \leq |\lambda| \int_0^{\infty} |\lambda|^{-t} t^\nu dt = |\lambda| \int_0^{\infty} e^{-t \ln(|\lambda|)} t^\nu dt = \frac{|\lambda| |\nu|}{|\ln(|\lambda|)|^{\nu+1}}.
\]

Consequently, since \(\ln(|\lambda|)\) is equivalent to \(|\lambda| - 1\) as \(|\lambda|\) is close to 1, there exists a number \(C\) independent of \(\lambda\) such that for \(1 < |\lambda| < 2\)

\[
(3.5) \quad \|x_f\|_\nu \leq \frac{|\lambda| |\nu|}{|\ln(|\lambda|)|^{\nu+1}} \|f\|_\nu \leq \frac{C}{|\lambda| - 1^{\nu+1}} \|f\|_\nu.
\]

Similarly as in the previous case where \(|\lambda| < 1\), we will prove that \(\bar{x}_f = \bar{x}_g\) whenever \(\bar{f} = \bar{g}\). Namely, if \(\bar{h} = 0\), then \(\bar{x}_h = 0\). In fact, for a given \(\epsilon > 0\), there exists a natural number \(N\) such that for all \(k \geq N\), \(\|h(k)\|/k^\nu < \epsilon\). Therefore, for all \(n \geq N\),

\[
\frac{\|x_k(n)\|}{n^\nu} \leq \sum_{k=n}^{\infty} \frac{|\lambda|^{n-k-1}}{n^\nu} \|h(k)\| \\
\leq \sum_{k=n}^{\infty} \frac{|\lambda|^{n-k-1}}{n^\nu} \epsilon k^\nu \\
\leq \epsilon |\lambda|^{-1} \sum_{j=0}^{\infty} |\lambda|^{-j} \left( 1 + \frac{j}{n} \right)^\nu.
\]
Since \(|\lambda| > 1|\) is fixed, the series \(\sum_{j=0}^{\infty} |\lambda|^{-j}(1 + j/n)^{\nu}\) is convergent, so this shows that

\[
\lim_{n \to \infty} \frac{\|x_h(n)\|}{n^{\nu}} = 0.
\]

That is, \(\bar{x}_h = 0\). This yields that \(\lambda \in \rho(\bar{S})\) and \(\bar{x}_f = (\lambda - \bar{S})^{-1}f\). Finally, with \(\Box\) the proof of the lemma is complete.

**Definition 3.2.** Let \(f \in \ell^\nu_{\infty}(X)\) be a given sequence in \(X\). Then its spectrum is defined to be the set of all complex \(\xi_0 \in \Gamma\) such that the complex function \(R(\lambda, \bar{S})\bar{f}\) has no analytic extension to any neighborhood of \(\xi_0\). The spectrum of a sequence \(f \in \ell^\nu_{\infty}(X)\) will be denoted by \(\sigma_\nu(f)\).

Before we proceed we introduce some notations: \(D_{|z|>1} := \{z \in \mathbb{C} : |z| > 1\}\), \(B(\xi_0, \delta) := \{z \in \mathbb{C} : |z - \xi_0| < \delta\}\).

**Lemma 3.3.** Let \(f \in \ell^\nu_{\infty}(X)\). Then, \(\xi_0 \in \Gamma\) is in \(\sigma_\nu(f)\) if and only if the function \(g : D_{|z|>1} \ni \lambda \mapsto R(\lambda, \bar{S})\bar{f} \in \ell^\nu_{\infty}(X)\) cannot be extended to an analytic function in any neighborhood of \(\xi_0\).

**Proof.** It suffices to show that if \(g\) can be extended to an analytic function in a neighborhood of \(\xi_0\), then \(\xi_0 \notin \sigma_\nu(f)\). Suppose that \(g(\lambda) = h(\lambda)\) for all \(\lambda \in D_{|z|>1} \cap B(\xi_0, \delta)\) where \(h\) is an analytic function in a small disk \(B(\xi_0, \delta)\). Then, the function \((\lambda - \bar{S})h(\lambda)\) is analytic in \(B(\xi_0, \delta)\). We observe that, for \(\lambda \in D_{|z|>1} \cap B(\xi_0, \delta)\)

\[
(\lambda - \bar{S})h(\lambda) = (\lambda - \bar{S})g(\lambda) = (\lambda - \bar{S})R(\lambda, \bar{S})\bar{f} = \bar{f}.
\]

That is, the function \((\lambda - \bar{S})h(\lambda)\) is a constant in an open and connected subset \(D_{|z|>1} \cap B(\xi_0, \delta)\) of the disk \(B(\xi_0, \delta)\). Hence, \((\lambda - \bar{S})h(\lambda) = \bar{f}\) for all \(\lambda\) in \(B(\xi_0, \delta)\). In particular, when \(|\lambda| < 1\) and \(\lambda \in B(\xi_0, \delta)\), \(h = R(\lambda, \bar{S})\bar{f}\). That means, \(h(\lambda)\) is an analytic extension of the function \(R(\lambda, \bar{S})\bar{f}\) as a complex function on \(\{z \in \mathbb{C} : |z| \neq 1\}\) to a neighborhood of \(\xi_0\). \(\Box\)

**Proposition 3.4.** Let \(f \in \ell^\nu_{\infty}(X)\) be a given sequence in \(X\). Then the following assertions are valid:

i) \(\sigma_\nu(f)\) is a closed subset of \(\Gamma\);

ii) The sequence \(f\) is in \(c_0^\nu(X)\) if and only if \(\sigma_\nu(f) = \emptyset\);

iii) If \(\xi_0\) is an isolated element of \(\sigma_\nu(f)\), then the point \(\xi_0\) is a pole of the complex function \(R(\lambda, \bar{S})\bar{f}\) of order up to \(\nu + 1\).

**Proof.** Part (i) is obvious from the definition of the spectrum of \(x\).

Part (ii): Clearly, if \(f \in c_0^\nu(X)\), then \(\sigma_\nu(f) = \emptyset\). Conversely, if \(\sigma_\nu(f) = \emptyset\), then the complex function \(\bar{f}(\lambda) := R(\lambda, \bar{S})\bar{f}\) is an entire function. Moreover, it is bounded. In fact, from \([3,4]\) for large \(|\lambda| > 2\),

\[
\|\bar{f}(\lambda)\|_\nu \leq \|\bar{x}_f\| \leq \sum_{j=1}^{\infty} |\lambda|^{-j}j^{\nu}\|f\|_\nu
\]

\[
= |\lambda|^{-1} \sum_{k=0}^{\infty} |\lambda|^{-k}(k + 1)^{\nu}\|f\|_\nu
\]

\[
\leq |\lambda|^{-1} \sum_{k=0}^{\infty} \frac{(k + 1)^{\nu}}{2^k} \|f\|_\nu.
\]
Since the series \( \sum_{k=0}^{\infty} (k+1)^{\nu}/2^k \) is convergent, we have
\[
\lim_{|\lambda| \to \infty} \| \hat{f}(\lambda) \|^\nu = 0.
\]

By the Liouville Theorem, this complex function \( \hat{f}(\lambda) := R(\lambda, \tilde{S}) \) is the zero function, so \( \hat{f} = 0 \) since \( R(\lambda, \tilde{S}) \) is injective for each large \( |\lambda| \). That means \( f \in c_0^\nu(\mathbb{H}) \).

Part (iii): Without loss of generality we may assume that \( \xi_0 = 1 \). Consider \( \lambda \) in a small neighborhood of 1 in the complex plane. We will express \( \lambda = e^z \) with \( |z| < \delta_0 \). Choose a small \( \delta_0 > 0 \) such that if \( |z| < \delta_0 \), then
\[
\frac{1}{|1 - |\lambda||} \leq \frac{2}{|\Re z|}.
\]

It follows from Lemma 3.1 that for \( 0 < |\Re z| < \delta_0 \),
\[
\| R(\lambda, \tilde{S}) \hat{x} \| \leq \frac{C}{1 - |\lambda|^\nu+1} \| \hat{x} \| \leq \frac{C 2^{\nu+1}}{|\Re z|^{\nu+1}} \| \hat{x} \|.
\]

Set \( f(z) = R(e^z, \tilde{S}) \hat{x} \) with \( |z| < \delta_0 \). Since 1 is a singular point of \( \| R(\lambda, \tilde{S}) \hat{x} \| \), 0 is a singular point of \( f(z) \) in \( \{ |z| < \delta_0 \} \). For each \( n \in \mathbb{Z} \) and \( 0 < r < \delta_0 \), we have
\[
\left\| \frac{1}{2\pi i} \int_{|z|=r} \left( 1 + \frac{z^2}{r^2} \right)^{\nu+1} f(z) \, dz \right\| \leq \frac{1}{2\pi} \int_{|z|=r} \left| 1 + \frac{z^2}{r^2} \right|^{\nu+1} \| f(z) \| \, |dz|.
\]

If \( z = re^{i\varphi} \), where \( \varphi \) is real, one has
\[
\left| 1 + \frac{z^2}{r^2} \right|^{\nu+1} = \left| 1 + e^{2i\varphi} \right|^{\nu+1} = |e^{-i\varphi} + e^{i\varphi}|^{\nu+1} = (2|\cos \varphi|)^{\nu+1} = 2^{\nu+1} r^{-\nu-1} |\Re z|^{\nu+1}.
\]

Therefore,
\[
\left\| \frac{1}{2\pi i} \int_{|z|=r} \left( 1 + \frac{z^2}{r^2} \right)^{\nu+1} \frac{f(z)}{z^{\nu+1}} \, dz \right\| \leq \frac{1}{2\pi} \int_{|z|=r} 2^{\nu+1} r^{-n-\nu-2} |\Re z|^{\nu+1} \frac{C 2^{\nu+1}}{|\Re z|^{\nu+1}} \| \hat{x} \| \, |dz| = \frac{C 4^{\nu+1} r^{-n-\nu-2}}{2\pi} \int_{|z|=r} |dz| \| \hat{x} \|. \quad (3.6)
\]

Consider the Laurent series of \( f(z) \) at \( z = 0 \),
\[
f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,
\]
where
\[
a_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{n+1}} \, dz, \quad n \in \mathbb{Z}.
\]
It follows that for each \( n \in \mathbb{Z} \),
\[
\frac{1}{2\pi i} \int_{|z|=r} \left( 1 + \frac{z^2}{r^2} \right)^{\nu+1} \frac{f(z)}{z^{n+1}} dz = \frac{1}{2\pi i} \int_{|z|=r} \sum_{k=0}^{\nu+1} \frac{(\nu+1)!}{k!(\nu+1-k)!} \frac{1}{2\pi i} \frac{f(z)}{z^{n+1}} dz \cdot \frac{1}{r^{2k}} \frac{z^{-2k}}{z^{n+1}} dz
\]
\[
= \sum_{k=0}^{\nu+1} \frac{(\nu+1)!}{k!(\nu+1-k)!} \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{n+1}} dz \cdot \frac{1}{r^{2k}} \frac{z^{-2k}}{z^{n+1}} dz
\]
\[
= \sum_{k=0}^{\nu+1} \frac{(\nu+1)!}{k!(\nu+1-k)!} r^{-2k} a_{n-2k}
\]
This, together with (3.6), shows
\[
\left| \sum_{k=0}^{\nu+1} \frac{(\nu+1)!}{k!(\nu+1-k)!} r^{-2k} a_{n-2k} \right| \leq C 4^{\nu+1} r^{-\nu-1} \|\bar{x}\|.
\]

Multiplying both sides by \( r^{2\nu} \) gives
\[
\left| \sum_{k=0}^{\nu+1} \frac{(\nu+1)!}{k!(\nu+1-k)!} r^{2\nu-2k} a_{n-2k} \right| \leq C 4^{\nu+1} r^{\nu-1} \|\bar{x}\|.
\]

Observe that in the left side is a polynomial in terms of \( r \) whose zero power term is \( a_{n-2\nu} \). Therefore, when \( \nu - n - 1 \geq 1 \) if we let \( r \) to get closer and closer to zero, then \( a_{n-2\nu} \) must be zero. That is for all \( \nu \geq n + 2 \), the coefficients \( a_{n-2\nu} = 0 \). This yields that for all \( j \leq -\nu - 2 \), \( a_j = 0 \). In other words, \( z = 0 \), or \( \lambda = 1 \) is a pole of the complex function \( \hat{f}(\lambda) := R(\lambda, \bar{S}) \hat{f} \) with order up to \( \nu + 1 \). □

Before proceeding we introduce a notation: Let \( 0 \neq z \in \mathbb{C} \) such that \( z = re^{i\varphi} \) with reals \( r, \varphi \), and \( F(z) \) be any complex function. Then we define
\[
(3.7) \quad \lim_{\lambda \sim z} F(\lambda) := \lim_{s \sim z} F(se^{i\varphi}).
\]

**Corollary 3.5.** Let \( f \in l''(\mathbb{X}) \), and \( \xi_0 \in \Gamma \) be an isolated point in \( \sigma_\nu(f) \). Moreover, assume that
\[
(3.8) \quad \lim_{\lambda \sim \xi_0} (\lambda - \xi_0) R(\lambda, \bar{S}) \hat{f} = 0.
\]

Then the singular point \( \xi_0 \) of \( R(\lambda, \bar{S}) \hat{f} \) is removable and the complex function \( R(\lambda, \bar{S}) \hat{f} \) is zero in the connected open subset of its domain that contains \( \xi_0 \).

**Proof.** By Proposition 3.4, \( \xi_0 \) is a pole of order up to \( \nu + 1 \). Consider the Laurent series of \( R(\lambda, \bar{S}) \hat{f} \) in a neighborhood of \( \xi_0 \) we have
\[
R(\lambda, \bar{S}) \hat{f} = \sum_{j=-\nu-1}^{\infty} \frac{a_j}{(\lambda - \xi_0)^{j+1}}.
\]
If (3.8) is satisfied, then for any \( k \geq 1 \) the following is also valid:
\[
\lim_{\lambda \sim z} (\lambda - \xi_0)^k R(\lambda, \bar{S}) \hat{f} = 0.
\]
If we let \( k \) take on the values \( 1, 2, \ldots, \nu + 1 \), then we see that \( a_j = 0 \) for all \( j = -\nu - 1, -\nu, \ldots \). That is, the function \( R(\lambda, \bar{S}) \hat{f} \) is zero (so analytic) in a neighborhood of \( \xi_0 \). From the properties of analytic functions this function must be zero in the connected open subset of its domain as well. □
4. Applications

4.1. Asymptotic behavior of polynomially bounded solutions of difference equations. In this subsection we will apply the results obtained in the previous section to study the polynomially bounded solutions of difference equations of the form

\[ x(n + 1) = T x(n) + F(n), \quad n \in \mathbb{N}, \]

where \( T \) is a bounded linear operator in \( X \) and \( F \in c_0^\nu(X) \).

**Definition 4.1.** A bounded operator \( T \) from a Banach space \( X \) to itself is said to be \( \nu \)-polynomially power bounded, if

\[ \sup_{n \in \mathbb{N}} \| T^n \|^{k \nu} < \infty, \]

where \( \nu \) is a nonnegative integer.

**Lemma 4.2.** Let \( x \in l_\nu^\infty(X) \) be a solution of (4.1). Then

\[ \sigma_\nu(x) \subset \sigma(T) \cap \Gamma. \] (4.2)

Moreover, for \( \lambda \in \rho(\bar{T}) \cap \rho(T) \),

\[ R(\lambda, \bar{T})\bar{x} = R(\lambda, \bar{S})\bar{x}. \] (4.3)

**Proof.** Consider the operator of multiplication by \( T \) in the spaces \( l_\nu^\infty(X) \). It is easy to see that the operator is bounded and preserves \( c_0^\nu(X) \), so it induces an operator \( \bar{T} \) in the quotient space \( l_\nu^\infty(X)/c_0^\nu(X) \). Moreover, \( \sigma(\bar{T}) \subset \sigma(T) \). Since \( x \) is a solution of (4.1), for each \( |\lambda| \neq 1 \) we have

\[ R(\lambda, \bar{S})\bar{x} = R(\lambda, \bar{S})\bar{T} \bar{x} + R(\lambda, \bar{S})\bar{F} = \bar{T} R(\lambda, \bar{S})\bar{x}, \]

This, together with the identity \( \lambda R(\lambda, \bar{S})\bar{x} - \bar{x} = R(\lambda, \bar{S})\bar{S} \bar{x} \), shows

\[ \lambda R(\lambda, \bar{S})\bar{x} - \bar{x} = \bar{T} R(\lambda, \bar{S})\bar{x}. \]

Therefore,

\[ \bar{x} = \lambda R(\lambda, \bar{S})\bar{x} - \bar{T} R(\lambda, \bar{S})\bar{x} = (\lambda - \bar{T}) R(\lambda, \bar{S})\bar{x}. \]

If \( \xi_0 \in \Gamma \) and \( \xi_0 \notin \sigma(T) \), then there exists a neighborhood of \( \xi_0 \) (in \( \mathbb{C} \)) such that for any \( \lambda \in U \) and \( |\lambda| \neq 1 \),

\[ R(\lambda, \bar{T})\bar{x} = R(\lambda, \bar{S})\bar{x}. \] (4.4)

As the left hand side function is an analytic extension in a neighborhood \( U \) of \( \xi_0 \), by [4.1], the complex function \( R(\lambda, \bar{S})\bar{x} \) has an analytic extension to the neighborhood \( U \) of \( \xi_0 \), that is \( \xi_0 \notin \sigma_\nu(x) \). In other words, \( \sigma_\nu(x) \subset \sigma(T) \cap \Gamma \). Moreover, [4.3] is proved. \( \square \)

We will prove the following that extends the famous Katznelson-Tzafriri to the case of \( \nu \)-polynomially bounded operator.

**Theorem 4.3.** Let \( T \in L(X) \) be \( \nu \)-polynomially bounded such that \( \sigma(T) \cap \Gamma \subset \{1\} \), where \( \nu \) is a nonnegative integer. Then

\[ \lim_{n \to \infty} \frac{1}{n^\nu} (T - I)^{\nu + 1} T^n = 0. \] (4.5)
Theorem 4.5. Let polynomially bounded operator $T$ obtain the above mentioned Katznelson-Tzafriri Theorem. An elementary proof of this theorem is given in [11]. In Theorem 4.3, when $\nu = 0$, we obtain the above mentioned Katznelson-Tzafriri Theorem.

Below is an individual version of Katznelson-Tzafriri Theorem for possibly non-$\nu$ polynomially bounded operator $T$.

**Theorem 4.5.** Let $T \in L(\mathbb{X})$ satisfy $\sigma(T) \cap \Gamma \subset \{1\}$. Then, for each $x_0 \in \mathbb{X}$,

$$\lim_{n \to \infty} \frac{1}{n^\nu} (T-I)^{\nu+1} T^n x_0 = 0,$$

provided that

$$\sup_{n \in \mathbb{N}} \frac{\|T^n x_0\|}{n^\nu} < \infty.$$

**Proof.** The proof is similar to that of Theorem 4.3. In analogy to the sequence $x$ in the proof of Theorem 4.3, we can use the sequence $(T^n x_0)_{n=1}^\infty$.

Let us consider homogeneous linear difference equations of the form

$$x(n + 1) = T x(n), \quad n \in \mathbb{N},$$

where $T \in L(\mathbb{X})$. Each solution $\{x(n)\}_{n=1}^\infty$ of this equation (4.7) is of the form $x(n) = T^{n-1} x_0$, $n \in \mathbb{N}$, for some $x_0 \in \mathbb{X}$.
Theorem 4.6. Let $T \in L(X)$, and let $x$ be a $\nu$ polynomially bounded solution of Eq. (4.1). Assume further that the following conditions are satisfied:

1) $\sigma(T) \cap \Gamma$ is countable
2) For each $\xi_0 = e^{i\phi_0} \in \sigma(T) \cap \Gamma$

\begin{equation}
\{z = re^{i\phi_0}, r > 1\} \subset \rho(T); \tag{4.8}
\end{equation}

\begin{equation}
\lim_{\lambda \downarrow z} (\lambda - \xi_0)R(\lambda, \bar{T})\bar{x} = 0. \tag{4.9}
\end{equation}

Then

\begin{equation}
\lim_{n \to \infty} \frac{x(n)}{n^\nu} = 0. \tag{4.10}
\end{equation}

Proof. Since $\sigma_\nu(f) \subset \sigma(T) \cap \Gamma$, if $\sigma(T) \cap \Gamma$ is empty, then the claim of the theorem is clear. Next, if it is not, then from the countability of $\sigma_\nu(f)$ as a closed subset of $\Gamma$ there must be an isolated point, say $\xi_0$ of $\sigma_\nu(f)$. However, by condition (4.9) and Corollary 3.5 the set of non-removable singular points of the complex function $R(\lambda, S)\bar{x} = R(\lambda, \bar{T})\bar{x}$ cannot have an isolated point. That means, $\sigma_\nu(x)$ must be empty set, so by Proposition 3.4, the sequence $x = \{x(n)\}_{n=1}^\infty$ must be in $c_0^\nu(X)$, that is (4.10). \qed

The following result gives a sufficient condition for the stability of polynomially bounded solutions that is well known as Arendt-Batty-Ljubich-Vu Theorem (see [3]):

Corollary 4.7. Let $T \in L(X)$ be $\nu$-polynomially power bounded. Assume further that

1) $\sigma(T) \cap \Gamma$ is countable,
2) For each $\xi_0$ of $\sigma(T) \cap \Gamma$, and each $x \in X$,

\begin{equation}
\lim_{\lambda \downarrow z} (\lambda - \xi_0)R(\lambda, T)x = 0. \tag{4.11}
\end{equation}

Then, for each $x \in X$,

\begin{equation}
\lim_{n \to \infty} \frac{T^nx}{n^\nu} = 0. \tag{4.12}
\end{equation}

Proof. It is clear that $x(n) = T^nx$ is a solution of Eq. (4.1). By the Spectral Radius Theorem the spectral radius $r_\sigma(T)$ of $T$ must satisfy $r_\sigma(T) \leq 1$ because of the polynomial boundedness of $T$, so (4.8) is satisfied. By Theorem 4.6 we only need to check condition (4.9). We have

\begin{equation}
0 \leq \lim_{\lambda \downarrow z} (\lambda - \xi_0)R(\lambda, \bar{T})\bar{x} \nu \leq \limsup_{\lambda \downarrow z, n \in N} \frac{\|R(\lambda, T)\| T^n x}{n^\nu}
\end{equation}

\begin{equation}
\leq \limsup_{\lambda \downarrow z, n \in N} \frac{\|T^n\| (\lambda - \xi_0)R(\lambda, T)x}{n^\nu}
\end{equation}

\begin{equation}
= \sup_{n \in N} \frac{\|T^n\|}{n^\nu} \lim_{\lambda \downarrow z} (\lambda - \xi_0)R(\lambda, T)x. \tag{4.13}
\end{equation}

Since $T$ is $\nu$-polynomially power bounded $\sup_{n \in N}(\|T^n\|/n^\nu)$ is finite, so (4.13) yields that condition (4.9) is satisfied. \qed
4.2. Asymptotic behavior of solutions of fractional difference equations. Consider fractional difference equations of the form
\begin{equation}
\Delta^\alpha x(n) = Tx(n) + y(n), \quad n \in \mathbb{N},
\end{equation}
where $0 < \alpha \leq 1$, $T \in L(\mathbb{X})$ and $y \in c_0'(\mathbb{X})$.

**Definition 4.8.** ([10] Definition 3.1) Let $T$ be a bounded operator defined on a Banach space $\mathbb{X}$ and $\alpha > 0$. We call $T$ the *generator of an $\alpha$-resolvent sequence* if there exists a sequence of bounded and linear operator $\{S_\alpha(n)\}_{n \in \mathbb{N}} \subset L(\mathbb{X})$ that satisfies the following properties
\begin{itemize}
  \item[i)] $S_\alpha(0) = I$;
  \item[ii)] $S_\alpha(n + 1) = k^n(n + 1)I + T \sum_{j=0}^{n} k^{n-j} S_\alpha(j)$, for all $n \in \mathbb{N}$.
\end{itemize}

As shown in [10] Theorem 3.4 and a note before it, $S_\alpha$ is determined by one of the following formulas:

**Theorem 4.9.** Let $\alpha > 0$ and $T$ be a bounded operator defined on a Banach space $\mathbb{X}$. The following properties are equivalent:
\begin{itemize}
  \item[i)] $T$ is the generator of an $\alpha$-resolvent sequence $\{S_\alpha(n)\}_{n \in \mathbb{N}}$;
  \item[ii)] $S_\alpha(n) = \sum_{j=0}^{n} \frac{\Gamma(n-j+(j+1)\alpha)}{\Gamma(n-j+1)\Gamma(j+\alpha)} T^j$;
  \item[iii)] $S_\alpha(n) = \frac{1}{2\pi i} \int_{C} z^n((z-1)^{\alpha}z^{1-\alpha} - T)^{-1} \, dz$,
\end{itemize}
where $C$ is a circle, centered at the origin of the complex plane, that encloses all spectral values of $(z-1)^{\alpha}z^{1-\alpha} - T$.

**Theorem 4.10.** ([10] Theorem 3.7) Let $0 < \alpha < 1$ and $\{y(n)\}_{n \in \mathbb{N}}$ is given. The unique solution of Eq. (4.13) with initial condition $u(0) = x$ can be represented by
\begin{equation}
\hat{x}(z) := \sum_{j=0}^{\infty} x(j)z^{-j}.
\end{equation}

Let us denote $D_{|z|>1} := \{z \in \mathbb{C} : |z| > 1\}$, and $D_{|z|<1} := \{z \in \mathbb{C} : |z| < 1\}$. For each $\{x(n)\}_{n \in \mathbb{N}} \in l^\infty(\mathbb{X})$ we will set $x(0) = 0$, so some properties of the Z-transform of sequences can be stated in the following:

**Proposition 4.11.** Let $\{x(n)\}_{n \in \mathbb{N}}$ and $\{y(n)\}_{n \in \mathbb{N}}$ be in $l^\infty(\mathbb{X})$. Then
\begin{itemize}
  \item[i)] $\hat{x}(z)$ is a complex function in $z \in D_{|z|>1}$;
  \item[ii)] $\hat{S}x(z) = z\hat{x}(z) - z\hat{x}(0)$;
  \item[iii)] $x \hat{y}(z) = \hat{x}(z) \hat{y}(z)$.
\end{itemize}

**Proof.** For the proof see e.g. [8] Chapter 6.

To study fractional difference equations we will need the following analog of [12] Lemma 3.3:
Lemma 4.12. Let \( \{x(n)\}_{n \in \mathbb{N}} \in l^\infty_\nu(X) \). If the Z-transform \( \hat{x}(z) \) of the sequence \( x \) has a holomorphic extension to a neighborhood of \( z_0 \in \Gamma \), then \( z_0 \notin \sigma_\nu(x) \).

**Proof.** Assume that \( \hat{x}(z) \) (with \( |z| > 1 \)) can be extended to a holomorphic function \( g_0(z) \) in \( B(z_0, \delta) \) with a sufficiently small positive \( \delta \). We will show that \( R(z, S)x \) (with \( |z| > 1 \)) has a holomorphic extension in a neighborhood of \( z_0 \). By setting \( x(0) = 0 \) we define a sequence \( \{g_k(z)\}_{k=1}^\infty \) as follows:

\[
g_k(z) := z^{k-1}\hat{x}(z) - \sum_{j=0}^{k-1} z^{k-1-j}x(j), \quad k \in \mathbb{N}.
\]

(4.15)

We are going to prove that this defines a bounded function \( g(z) \) with \( z \) in a small disk \( B(z_0, \delta) : \{z \in \mathbb{C} : |z - z_0| < \delta\} \), and then, applying a necessary and sufficient condition for a locally bounded function to be holomorphic to prove that \( R(z, S)x \) is holomorphic. To prove the boundedness of \( g(z) \) in a small disk \( B(z_0, \delta) \) we will use a special maximum principle as in [12]. We have

\[
R(z, S)x := (z - S)^{-1}x = z^{-1}(I - z^{-1}S)^{-1}x
\]

\[
= z^{-1}\sum_{n=0}^\infty z^{-n}S(n)x = \sum_{n=0}^\infty z^{-n-1}S(n)x.
\]

Therefore, for \( z \in B(z_0, \delta) \cap D_{|z|>1} \) and for each \( k \in \mathbb{N} \),

\[
[R(z, S)x](k) = \sum_{n=0}^\infty z^{-n-1}x(n + k) = z^{-1}\left(z^k\hat{x}(z) - \sum_{j=0}^{k-1} z^{k-j}x(j)\right) = g_k(z).
\]

By (3.1) and (3.3), for \( z \in B(z_0, \delta) \cap D_{|z|>1} \), there is a certain number \( C \) such that

\[
\sup_{k \in \mathbb{N}} \frac{\|g_k(z)\|_{k^\nu}}{k^\nu} = \|g(z)\|_{\nu} = \left\|\left(\sum_{n=0}^\infty z^{-n-1}x(n + k)\right)\right\|_{k=1}^\infty \leq \frac{C}{(|z| - 1)^\nu + 1}\|z\|_{\nu}.
\]

(4.16)

On the other hand, for \( z \in B(z_0, \delta) \cap D_{|z|<1} \) we have for all \( k \in \mathbb{N} \),

\[
\|g_k(z)\| \leq |z|^{k-1}\|g_0(z)\| + \sum_{j=0}^{k-1} |z|^{k-1-j}\|x(j)\| \leq |z|^{k-1}\|g_0(z)\| + \sum_{j=0}^{k-1} |z|^{k-1-j}\|x\|_{\nu} \leq \left(\sup_{z \in B(z_0, \delta)} \|g_0(z)\| + \|x\|_{\nu}\right)\sum_{j=0}^{k-1} |z|^{k-1-j}\|x\|_{\nu} = M\sum_{j=0}^{k-1} |z|^{k-1-j}\|x\|_{\nu}
\]

where \( M := \sup_{z \in B(z_0, \delta)} \|g_0(z)\| + \|x\|_{\nu} \). Hence, for all \( k \in \mathbb{N} \),

\[
\frac{\|g_k(z)\|_{k^\nu}}{k^\nu} \leq M\sum_{j=0}^{k-1} |z|^{k-1-j}\left(\frac{j}{k}\right)^\nu \leq M\sum_{j=0}^{k-1} |z|^{k-1-j} \leq \frac{M}{1 - |z|}.
\]

(4.17)
By (4.16) and (4.17) we have proved that there is a positive number $K$ such that for $z \in B(z_0, \delta)$ and for each $k \in \mathbb{N}$, this estimate is valid:

\[
\frac{\|g_k(z)\|}{k^\nu} \leq \frac{K}{|z - 1|^{\nu + 1}}.
\]

Applying the maximum principle Lemma 2.2 as in [12] to the function $g_k(z)/k^\nu$ gives the boundedness of $g_k(z)/k^\nu$ in $B(z_0, \delta/2)$. In fact, it is clear that for each $k \in \mathbb{N}$ the function $g_k(z)/k^\nu$ is holomorphic in $z \in B(z_0, \delta)$. Therefore, $g_k(z)/k^\nu$ is bounded by a number independent of $k$, so $g(z)$ is bounded in $B(z_0, \delta/2)$. We are now ready to apply Theorem 2.1, a criterion for a locally bounded function to be holomorphic.

In fact, since the family $W := \{x^* \circ p_k, \ x^* \in X^*, \ p_k : \{x_n\} \mapsto x_k, \ k \in \mathbb{N}\}$ is separating and $x^* \circ p_k(g(\cdot)) = x^*(g_k(\cdot))$ is holomorphic, the complex function $g(z)$ is holomorphic for $z \in B(z_0, \delta/2)$.

At this point we have shown that $g(z)$ is holomorphic for $z \in B(z_0, \delta/2)$, and as $g(z) = R(z, S)x$ for $|z| > 1$. This yields that $R(z, S)x$ has a holomorphic extension $g(z)$ to a neighborhood of $z_0$. This completes the proof of the lemma. □

**Definition 4.13.** We denote by $\sigma_{Z, \nu}(x)$ the set of all points $\xi_0$ on $\Gamma$ such that the $Z$-transform of a sequence $x := \{x(n)\}_{n \in \mathbb{N}} \in l_\nu^\infty(X)$ cannot be extended holomorphically to any neighborhood of $\xi_0$, and call this set the $Z$-spectrum of the sequence $x$.

In the simplest case where $\nu = 0$, $\sigma_\nu(x)$ may be different from $\sigma_{Z, \nu}(x)$. In fact, the following numerical sequence $x := \{x(n)\}_{n \in \mathbb{N}} \in l_0^\infty(\mathbb{R})$, where

\[
x(n) := \begin{cases} 0, & n = 0, \\ 1/n, & n \in \mathbb{N}
\end{cases}
\]

is in $c_0(\mathbb{R})$. Obviously, $\tilde{x} = 0$, so $\sigma(x) = \emptyset$. However, $1 \in \sigma_{Z, \nu}(x)$ because $\tilde{x}(z) = \sum_{j=1}^{\infty} z^{-j}/j$ cannot be extended holomorphically to a neighborhood of 1. In general, we only have the following inclusion.

**Corollary 4.14.** For each $x := \{x(n)\}_{n \in \mathbb{N}} \in l_\nu^\infty(X)$,

\[
\sigma_\nu(x) \subset \sigma_{Z, \nu}(x).
\]

**Proof.** The corollary is an immediate consequence of Lemma 4.12 and the definitions of the spectra mentioned in the statement. □

Before we proceed, we introduce a notation

\[
\Sigma_0 := \{z_0 \in \Gamma \subset \mathbb{C} : (z - \tilde{k}^\alpha(z)T)^{-1} \text{ exists and } (z - \tilde{k}^\alpha(z)T)^{-1} \text{ and } \tilde{k}^\alpha(z)
\]

are holomorphic in a neighborhood of $z_0}\}

and $\Sigma = \Gamma \setminus \Sigma_0$.

**Lemma 4.15.** Let $\alpha > 0$ and $S_\alpha := \{S_\alpha(n)\}_{n \in \mathbb{N}} \subset L(X)$ be the resolvent of Eq. (4.13) that satisfies

\[
\sup_{n \in \mathbb{N}} \frac{\|S_\alpha(n)\|}{n^\nu} < \infty.
\]

Then

\[
\sigma_\nu(S_\alpha) \subset \Sigma.
\]
In other words, Theorem 4.16.

Proof. It suffices to show that if \( z_0 \in \Sigma_0 \), then \( z_0 \notin \sigma_\nu(S_\alpha) \). Taking the \( \mathbb{Z} \)-transform of \( S_\alpha \) from the equation in Definition 4.8 gives

\[
z\tilde{S}_\alpha(z) - zS_\alpha(0) = \tilde{S}\overline{S}_\alpha(z) = z\tilde{k}_\alpha(z)I - zk_\alpha(0)I + \tilde{k}_\alpha(z) \cdot T\tilde{S}_\alpha(z).
\]

Therefore, for \( z \in D_{\{z\geq 1\}} \),

\[
(z - \tilde{k}_\alpha(z)T)\tilde{S}_\alpha(z) = zS_\alpha(0) + z\tilde{k}_\alpha(z)I - zk_\alpha(0)I.
\]

Let \( z_0 \in \Sigma_0 \). Then \( (z - \tilde{k}_\alpha(z)T)^{-1} \) exists. Hence, \( \tilde{S}_\alpha(z) = (z - \tilde{k}_\alpha(z)T)^{-1}(zS_\alpha(0) + z\tilde{k}_\alpha(z)I - zk_\alpha(0)I) \).

And, it is clear that \( \tilde{S}_\alpha(z) \) has a holomorphic extension to a neighborhood of \( z_0 \) because both \( \tilde{k}_\alpha(z) \) and \( (z - \tilde{k}_\alpha(z)T)^{-1} \) are holomorphic in a neighborhood of \( z_0 \), so \( z_0 \notin \sigma_\nu(S_\alpha) \). By Corollary 4.14 this yields \( z_0 \notin \sigma_\nu(S_\alpha) \). This completes the proof of the lemma.

**Theorem 4.16.** Let \( 0 < \alpha \leq 1 \) and \( \Sigma \subset \{1\} \). Assume further that the \( \alpha \)-resolvent \( S_\alpha \) of Eq. (4.13) satisfies

\[
\sup_{n \in \mathbb{N}} \frac{||S_\alpha(n)||}{n^\nu} < \infty.
\]

Then

\[
\lim_{n \to \infty} \frac{1}{n^\nu} \sum_{k=0}^{\nu+1} \frac{(\nu + 1)!}{k!(\nu + 1 - k)!} (\nu + 1 + k)! S_\alpha(n + k) = 0.
\]

**Proof.** As in the proof of Theorem 4.3 we can show that

\[
\lambda \mapsto R(\lambda, \tilde{S})(\tilde{S} - I)^{\nu+1}\tilde{S}_\alpha
\]

has a holomorphic extension to a neighborhood of 1 in the complex plane. Moreover, since \( \sigma_\nu(S_\alpha) \subset \Sigma \) this function has a holomorphic extension to a neighborhood of all points of \( \Gamma \). Namely, \( \sigma_\nu((\tilde{S} - I)^{\nu+1}\tilde{S}_\alpha) = \emptyset \). Therefore, \( (\tilde{S} - I)^{\nu+1}S_\alpha \in c_0(\mathbb{X}) \).

In other words,

\[
\lim_{n \to \infty} \frac{1}{n^\nu} ((\tilde{S} - I)^{\nu+1}S_\alpha)(n) = 0.
\]

Then it follows that

\[
(\tilde{S} - I)^{\nu+1} = (\nu + 1)^{\nu+1} (I - S)^{\nu+1} = (-1)^{\nu+1} \sum_{k=0}^{\nu+1} \frac{(\nu + 1)!}{k!(\nu + 1 - k)!} (\nu + 1 + k)! S_\alpha(k).
\]

This, together with (4.20), yields (4.19). The theorem is proved.

**Remark 4.17.** When \( \alpha = 1 \) Eq. (4.13) becomes

\[
x(n + 1) = (I + T)x(n) + y(n), \quad n \in \mathbb{N}.
\]

As shown in [10] \( S_\alpha(n) = (I + T)^n, \ n \in \mathbb{N} \). With this formula, (4.19) becomes

\[
\lim_{n \to \infty} \frac{1}{n^\nu} T^{\nu+1}(I + T)^n.
\]

Hence, Theorem 4.16 coincides with Theorem 4.3 when \( \alpha = 1 \). In other words, Theorem 4.16 is an extension of the Katnelson-Tzafriri Theorem for fractional difference equations (4.13).
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