A CRYSTAL EMBEDDING INTO LUSZTIG DATA OF TYPE A

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Abstract. Let $i$ be a reduced expression of the longest element in the Weyl group of type $A$, which is adapted to a Dynkin quiver with a single sink. We present a simple description of the crystal embedding of Young tableaux of arbitrary shape into $i$-Lusztig data, which also gives an algorithm for the transition matrix between Lusztig data associated to reduced expressions adapted to quivers with a single sink.

1. Introduction

Let $U_q(g)$ be the quantized enveloping algebra associated to a symmetrizable Kac-Moody algebra $g$. The negative part of $U_q(g)$ has a basis called a canonical basis [16] or lower global crystal basis [7], which has many fundamental properties. The canonical basis forms a colored oriented graph $B(\infty)$, called a crystal, with respect to Kashiwara operators. The crystal $B(\infty)$ plays an important role in the study of combinatorial aspects of $U_q(g)$-modules together with its subgraph $B(\lambda)$ associated to any integrable highest weight module $V(\lambda)$ with highest weight $\lambda$.

Suppose that $g$ is a finite-dimensional semisimple Lie algebra with the index set $I$ of simple roots. Let $i = (i_1, \ldots, i_N)$ be a sequence of indices in $I$ corresponding to a reduced expression of the longest element in the Weyl group of $g$. A PBW basis associated to $i$ is a basis of the negative part of $U_q(g)$ [17], which is parametrized by the set $B_i$ of $N$-tuple of non-negative integers. One can identify $B(\infty)$ with $B_i$ since the associated PBW basis coincides with the canonical basis at $q = 0$ [20]. We call an element in $B_i$ an $i$-Lusztig datum or Lusztig parametrization associated to $i$.

Consider the map

\[
\psi^i_\lambda : B(\lambda) \otimes T_{-\lambda} \rightarrow B_i,
\]

given by the $i$-Lusztig datum of $b \in B(\lambda)$ under the embedding of $B(\lambda) \otimes T_{-\lambda}$ into $B(\infty)$, where $T_{-\lambda} = \{ t_{-\lambda} \}$ is an abstract crystal with $\text{wt}(t_{-\lambda}) = -\lambda$ and $\varphi_i(t_{-\lambda}) = -\infty$ for $i \in I$. In this paper, we give a simple combinatorial description of (1.1) when

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\( g = \mathfrak{g}l_n \) and \( i \) is a reduced expression adapted to a Dynkin quiver of type \( A_{n-1} \) with a single sink (Theorem 5.4). It is well-known that when \( i \) is adapted to a quiver with one direction, for example \( i = i_0 = (1,2,1,3,2,1,\ldots,n-1,\ldots,1) \), the \( i_0 \)-Lusztig datum of a Young tableaux is simply given by counting the number of occurrences of each entry in each row. But the \( i \)-Lusztig datum for arbitrary \( i \) is not easy to describe in general, and one may apply a sequence of Lusztig’s transformations \([17]\) or the formula for a transition map \( R^i_{i_0} : B_{i_0} \to B_i \) by Berenstein-Fomin-Zelevinsky \([2]\). We remark that our algorithm for computing \( \psi^i_{\lambda} \) is completely different from the known methods, and hence provides an alternative description of \( R^i_{i_0} \).

Let us explain the basic ideas in our description of \( \psi^i_{\lambda} \). Suppose that \( \Omega \) is a quiver of type \( A_{n-1} \) with a single sink and \( i \) is adapted to \( \Omega \). Let \( J \subset I \) be a maximal subset such that each connected component of the corresponding quiver \( \Omega J \subset \Omega \) has only one direction. Let \( g_J \) be the maximal Levi subalgebra and \( u_J \) the nilradical associated to \( J \), respectively.

The first step is to prove a tensor product decomposition \( B_i \cong B^J(\infty) \otimes B_J(\infty) \), as a crystal, where \( B_J(\infty) \) is the crystal of the negative part of \( U_q(g_J) \) and \( B^J(\infty) \) is the crystal of the quantum nilpotent subalgebra \( U_q(u_J) \). The isomorphism is just given by restricting the Lusztig datum to each part, and it is a special case of the bijection introduced in \([1, 20]\) using crystal reflections. Here we show that it is indeed a morphism of crystals by using Reineke’s description of \( B(\infty) \) in terms of representations of \( \Omega \) \([19]\). We refer the reader to a recent work by Salibury-Schultze-Tingley \([23]\) on \( i \)-Lusztig data, which also implies the combinatorial description of Kashiwara operators on \( B(\infty) \) used in this paper.

The next step is to construct an embedding of \( B(\lambda) \otimes T_{-\lambda} \) into \( B^J(\infty) \otimes B_J(\infty) \) using a crystal theoretic interpretation of Sagan and Stanley’s skew RSK algorithm \([22]\), which was observed in the author’s previous work \([12]\) (see also \([13, 14]\)), and using the embedding \([11] \) in case of \( i \) adapted to a quiver with one direction. Hence we obtain an \( i \)-Lusztig datum of a Young tableau for any \( i \) adapted to \( \Omega \). One may consider the image of the embedding by using a combinatorial description of \(*\)-crystal structure on \( B_i \) in \([19]\), but we do not discuss it here.

Our description of the embedding \( \psi^i_{\lambda} \) also provides an algorithm for a transition map \( R^i_{i_0} : B_{i_0} \to B_i \) together with its inverse \( R^i_{i_0} \) since \( \psi^i_{\lambda} \) naturally extends to an isomorphism from another realization of \( B(\infty) \) given by the set of large tableaux \([3, 5]\). Therefore we obtain an algorithm for a transition map \( R^i_{i'} = R^i_{i_0} \circ R^i_{i_0} \) for any \( i \) and \( i' \) which are adapted to quivers with a single sink. Roughly speaking, \( R^i_{i'} \) is given by a composition of skew RSK and its inverse algorithms with respect to various maximal Levi subalgebras depending on \( i \) and \( i' \).
The paper is organized as follows: In Sections 2 and 3, we review necessary background on crystals and related materials. In Section 4, we give an explicit description of the crystal $B_i$ when $i$ is adapted to a Dynkin quiver of type $A_{n-1}$ with a single sink, and then prove the decomposition of $B_i$ as a tensor product of two subcrystals. Finally in Section 5, we construct an embedding of the crystal of Young tableaux of arbitrary shape $\lambda$ into $B_i$.

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2. Review on crystals

2.1. Let us give a brief review on crystals (see [1, 7, 9] for more details). We denote by $Z_+$ the set of non-negative integers. Fix a positive integer $n$ greater than 1. Throughout the paper, $\mathfrak{g}$ denotes the general linear Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ which is spanned by the elementary matrices $e_{ij}$ for $1 \leq i, j \leq n$. Let $P^\vee = \bigoplus_{i=1}^{n} \mathbb{Z}e_{ii}$ be the dual weight lattice and $P = \text{Hom}_{\mathbb{Z}}(P^\vee, \mathbb{Z}) = \bigoplus_{i=1}^{n} \mathbb{Z}e_i$ be the weight lattice of $\mathfrak{g}$ with $(e_i, e_{jj}) = \delta_{ij}$ for $i, j$. Define a symmetric bilinear form $(\cdot | \cdot)$ on $P$ such that $(e_i | e_j) = \delta_{ij}$ for $i, j$. Set $I = \{1, \ldots, n-1\}$. Then $\{ \alpha_i := e_i - e_{i+1} \mid i \in I \}$ is the set of simple roots and $\{ h_i := e_{ii} - e_{i+1i+1} \mid i \in I \}$ is the set of simple coroots of $\mathfrak{g}$. Let $\Phi^+ = \{ \epsilon_i - \epsilon_j \mid 1 \leq i < j \leq n \}$ denote the set of positive roots of $\mathfrak{g}$.

Let $W \cong S_n$ be the Weyl group of $\mathfrak{g}$, which is generated by simple reflections $s_i$ for $i \in I$. Let $w_0$ be the longest element in $W$, which is of length $N := n(n-1)/2$, and let $R(w_0) = \{ (i_1, \ldots, i_N) \mid w_0 = s_{i_1} \ldots s_{i_N} \}$ be the set of reduced expressions of $w_0$.

For $J \subset I$, let $\mathfrak{g}_J$ be the subalgebra of $\mathfrak{g}$ generated by $e_{ii}$ for $1 \leq i \leq n$ and the root vectors associated to $\pm \alpha_j$ for $j \in J$. Let $\Phi^+_J$ be the set of positive roots of $\mathfrak{g}_J$ and $\Phi^+_J(J) = \Phi^+ \setminus \Phi^+_J$.

A $\mathfrak{g}$-crystal is a set $B$ together with the maps $\text{wt} : B \to P$, $\varepsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{-\infty\}$ and $\tilde{e}_i, \tilde{f}_i : B \to \mathbb{Z} \cup \{0\}$ for $i \in I$ satisfying the following conditions: for $b \in B$ and $i \in I$,

1. $\varphi_i(b) = \langle \text{wt}(b), h_i \rangle + \varepsilon_i(b)$,
2. $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$, $\text{wt}(\tilde{e}_i b) = \text{wt}(b) + \alpha_i$ if $\tilde{e}_i b \in B$,
3. $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$, $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$, $\text{wt}(\tilde{f}_i b) = \text{wt}(b) - \alpha_i$ if $\tilde{f}_i b \in B$,
4. $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$ for $b' \in B$,
5. $\tilde{e}_i b = \tilde{f}_i b = 0$ when $\varphi_i(b) = -\infty$.

Here $0$ is a formal symbol and $-\infty$ is the smallest element in $\mathbb{Z} \cup \{-\infty\}$ such that $-\infty + n = -\infty$ for all $n \in \mathbb{Z}$. Unless otherwise specified, a crystal means a $\mathfrak{g}$-crystal throughout the paper for simplicity.
Let $B_1$ and $B_2$ be crystals. A tensor product $B_1 \otimes B_2$ is a crystal, which is defined to be $B_1 \times B_2$ as a set with elements denoted by $b_1 \otimes b_2$, where

$$wt(b_1 \otimes b_2) = wt(b_1) + wt(b_2),$$
$$\varepsilon_i(b_1 \otimes b_2) = \max\{\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle wt(b_1), h_i \rangle\},$$
$$\varphi_i(b_1 \otimes b_2) = \max\{\varphi_i(b_1) + \langle wt(b_2), h_i \rangle, \varphi_i(b_2)\},$$

where $i \in I$. Here we assume that $0 \otimes b_2 = b_1 \otimes 0 = 0$.

A morphism $\psi : B_1 \to B_2$ is a map from $B_1 \cup \{0\}$ to $B_2 \cup \{0\}$ such that

1. $\psi(0) = 0$,
2. $wt(\psi(b)) = wt(b)$, $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$, and $\varphi_i(\psi(b)) = \varphi_i(b)$ when $\psi(b) \neq 0$,
3. $\psi(\overline{e}_ib) = \overline{e}_i\psi(b)$ when $\psi(b) \neq 0$ and $\psi(\overline{e}_ib) \neq 0$,
4. $\psi(\overline{f}_ib) = \overline{f}_i\psi(b)$ when $\psi(b) \neq 0$ and $\psi(\overline{f}_ib) \neq 0$,

for $b \in B_1$ and $i \in I$. We call $\psi$ an embedding and $B_1$ a subcrystal of $B_2$ when $\psi$ is injective.

The dual crystal $B^\vee$ of a crystal $B$ is defined to be the set $\{b^\vee | b \in B\}$ with $wt(b^\vee) = -wt(b)$, $\varepsilon_i(b^\vee) = \varphi_i(b)$, $\varphi_i(b^\vee) = \varepsilon_i(b)$, $\overline{e}_i(b^\vee) = \overline{f}_i(b)^\vee$, and $\overline{f}_i(b^\vee) = \overline{e}_i(b)^\vee$ for $b \in B$ and $i \in I$. We assume that $0^\vee = 0$.

For $\mu \in P$, let $T_\mu = \{t_\mu\}$ be a crystal, where $wt(t_\mu) = \mu$, $\overline{e}_it_\mu = \overline{f}_i t_\mu = 0$, and $\varepsilon_i(t_\mu) = \varphi_i(t_\mu) = -\infty$ for all $i \in I$.

2.2. Let $q$ be an indeterminate. Let $U = U_q(\mathfrak{g})$ be the quantized enveloping algebra of $\mathfrak{g}$, which is an associative $\mathbb{Q}(q)$-algebra with 1 generated by $e_i, f_i$, and $q^h$ for $i \in I$ and $h \in P^\vee$. Let $U^\minus{} = U_q^\minus{}(\mathfrak{g})$ be the negative part of $U$, the subalgebra generated by $f_i$ for $i \in I$. We put $[m] = \frac{q^m - q^{-m}}{q - q^{-1}}$ and $[m]! = [1][2] \cdots [m]$ for $m \in \mathbb{N}$. Let $t_i = q^{h_i}$, $e_i^{(m)} = e_i^m/[m]!$, and $f_i^{(m)} = f_i^m/[m]!$ for $m \in \mathbb{N}$ and $i \in I$. Let $A_0$ denote the subring of $\mathbb{Q}(q)$ consisting of rational functions regular at $q = 0$.

For $i \in I$, let $T_i$ be the $\mathbb{Q}(q)$-algebra automorphism of $U$ given by

$$T_i(t_j) = t_j t_i^{-a_{ij}},$$
$$T_i(e_j) = \begin{cases} -f_i t_i, & \text{if } j = i, \\ \sum_{k+l = -a_{ij}} (-1)^k q^{-l} e_i^{(k)} e_j e_i^{(l)}, & \text{if } j \neq i, \end{cases}$$
\[ T_i(f_j) = \begin{cases} -t_i^{-1}e_i, & \text{if } j = i, \\ \sum_{k+l=-a_{ij}}(-1)^k q^k f_i^{(k)} f_j f_i^{(l)}, & \text{if } j \neq i, \end{cases} \]

for \( j \in I \), where \( a_{ij} = \langle \alpha_j, h_i \rangle \). Note that \( T_i \) is denoted by \( T_i^{\nu} \) in [17] (see also [20]).

For \( i = (i_1, \ldots, i_N) \in R(w_0) \) and \( c = (c_1, \ldots, c_N) \in \mathbb{Z}_+^N \), consider the vectors of the following form:

\[ (2.2) \quad b_i(c) = f_i^{(c_1)} T_i T_{i_2} \cdots T_{i_{N-1}} f_i^{(c_N)}. \]

The set \( B_i := \{ b_i(c) \mid c \in \mathbb{Z}_+^N \} \) is a \( \mathbb{Q}(q) \)-basis of \( U^- \), which is often referred to as a PBW basis [17].

The \( A_0 \)-lattice of \( U^- \) generated by \( B_i \) is independent of the choice of \( i \), which we denote by \( L(\infty) \). If \( \pi : L(\infty) \to L(\infty)/qL(\infty) \) is the canonical projection, then \( \pi(B_i) \) is a \( \mathbb{Q} \)-basis of \( L(\infty)/qL(\infty) \) and also independent of the choice of \( i \), which we denote by \( B(\infty) \). Indeed the pair \( (L(\infty), B(\infty)) \) coincides with the Kashiwara’s crystal base of \( U^- \) [7], that is, \( L(\infty) \) is invariant under \( \tilde{e}_i, \tilde{f}_i \), and \( \tilde{e}_i B(\infty) \subset B(\infty) \cup \{0\} \), \( \tilde{f}_i B(\infty) \subset B(\infty) \cup \{0\} \) for \( i \in I \), where \( \tilde{e}_i \) and \( \tilde{f}_i \) denote the modified Kashiwara operators on \( U^- \) given by

\[ \tilde{e}_i x = \sum_{k \geq 1} f_i^{(k-1)} x_k, \quad \tilde{f}_i x = \sum_{k \geq 0} f_i^{(k+1)} x_k, \]

for \( x = \sum_{k \geq 0} f_i^{(k)} x_k \in U^- \), where \( x_k \in T_i(U^-) \cap U^- \) for \( k \geq 0 \) (see [18, 20]). The set \( B(\infty) \) equipped with the induced operators \( \tilde{e}_i \) and \( \tilde{f}_i \) becomes a crystal, where \( \varepsilon_i(b) = \max\{ k \mid \tilde{e}_i^k b \neq 0 \} \) for \( i \in I \) and \( b \in B(\infty) \).

Let \( P^+ = \{ \lambda \in P \mid \langle \lambda, h_i \rangle \geq 0 \text{ for } i \in I \} \) be the set of dominant integral weights. For \( \lambda \in P^+ \), let \( V(\lambda) \) be the irreducible highest weight \( U^- \)-module with highest weight \( \lambda \), which is given by \( U^-/\sum_{i \in I} U^- f_i^{(\langle \lambda, h_i \rangle + 1)} \cdot 1 \) as a left \( U^- \)-module. If \( \pi_\lambda : U^- \to V(\lambda) \) is the canonical projection, then \( L(\lambda) := \pi_\lambda(L(\infty)) \) is an \( A_0 \)-lattice of \( V(\lambda) \) and \( B(\lambda) := \pi_\lambda(B(\infty)) \setminus \{0\} \) is a \( \mathbb{Q} \)-basis of \( L(\lambda)/qL(\lambda) \). The pair \( (L(\lambda), B(\lambda)) \) is called the crystal base of \( V(\lambda) \). The set \( B(\lambda) \) becomes a crystal with respect to \( \tilde{e}_i \) and \( \tilde{f}_i \) induced from those on \( B(\infty) \), where \( \varepsilon_i(b) = \max\{ k \mid \tilde{e}_i^k b \neq 0 \} \) and \( \varphi_i(b) = \max\{ k \mid \tilde{f}_i^k b \neq 0 \} \) for \( i \in I \) and \( b \in B(\lambda) \) [7].

3. Crystal of Young tableaux

3.1. Let us recall some necessary background on semistandard tableaux and related combinatorics following [3]. Let \( \mathcal{P} \) be the set of partitions. We identify \( \lambda = (\lambda_i)_{i \geq 1} \in \mathcal{P} \) with a Young diagram. Let \( \lambda/\mu \) denote a skew Young diagram associated to \( \lambda, \mu \in \mathcal{P} \) with \( \lambda \supset \mu \), and let \( (\lambda/\mu)^\circ \) denote the skew Young diagram obtained by \( 180^\circ \)-rotation of \( \lambda/\mu \).
Let \( \Lambda \) be a linearly ordered set. For a skew Young diagram \( \lambda/\mu \), let \( \text{SST}_\Lambda(\lambda/\mu) \) be the set of all semistandard tableaux of shape \( \lambda/\mu \) with entries in \( \Lambda \). Let \( \mathcal{W}_\Lambda \) be the set of finite words in \( \Lambda \). For \( T \in \text{SST}_\Lambda(\lambda/\mu) \), let \( \text{sh}(T) \) denote the shape of \( T \), and let \( w(T) \) be a word in \( \mathcal{W}_\Lambda \) obtained by reading the entries of \( T \) row by row from top to bottom, and from right to left in each row.

Let \( T \in \text{SST}_\Lambda(\lambda^\pi) \) be given for \( \lambda \in \mathcal{P} \). For \( a \in \Lambda \), we define \( T \leftarrow a \) to be the tableau obtained by applying the Schensted's column insertion of \( a \) into \( T \) in a reverse way starting from the rightmost column of \( T \) so that \( \text{sh}(T \leftarrow a) = \mu^\pi \) for some \( \mu \supset \lambda \) obtained by adding a box in a corner of \( \lambda \). We also denote by \( T^\pi \) the unique tableau in \( \text{SST}_\Lambda(\lambda) \), which is Knuth equivalent to \( T \). Note that the map \( T \mapsto T^\pi \) gives a bijection from \( \text{SST}_\Lambda(\lambda^\pi) \) to \( \text{SST}_\Lambda(\lambda) \), where the inverse map is given by \((\cdots (a_r \leftarrow a_{r-1}) \leftarrow \cdots) \leftarrow a_1\) for \( S \in \text{SST}_\Lambda(\lambda) \) with \( w(S) = a_1 \cdots a_r \).

Let \( B \) be another linearly ordered set, and let

\[
\mathcal{M}_{\Lambda \times B} = \left\{ M = (m_{ab})_{a \in \Lambda, b \in B} \left| m_{ab} \in \mathbb{Z}_+, \sum_{a,b} m_{ab} < \infty \right. \right\}.
\]

Let \( J_{\Lambda \times B} \) be the set of biwords \((a, b) \in \mathcal{W}_\Lambda \times \mathcal{W}_B\) such that (1) \( a = a_1 \cdots a_r \) and \( b = b_1 \cdots b_r \) for some \( r \geq 0 \), (2) \((a_1, b_1) \leq \cdots \leq (a_r, b_r)\), where for \((a, b)\) and \((c, d)\) \( a < d \) implies \((a, b) < (c, d)\). There is a bijection

\[
\begin{align*}
J_{\Lambda \times B} & \rightarrow \mathcal{M}_{\Lambda \times B} \\
(a, b) & \mapsto M(a, b)
\end{align*}
\]

where \( M(a, b) = (m_{ab}) \) with \( m_{ab} = |\{k \mid (a_k, b_k) = (a, b)\}| \) and the pair of empty words \((\emptyset, \emptyset)\) corresponds to the zero matrix \( O \).

For \((a, b) \in J_{\Lambda \times B}\), we write \( M[b, a] = M(a, b)^t \in \mathcal{M}_{B \times \Lambda} \), where \( M^t \) denotes the transpose of \( M \in \mathcal{M}_{B \times \Lambda} \). For \((a, b) \in J_{\Lambda \times B}\), there exist unique \( a^r \in \mathcal{W}_\Lambda \) and \( b^r \in \mathcal{W}_B \), which are rearrangements of \( a \) and \( b \), respectively, satisfying \( M(b^r, a^r) = M(a, b)^t \in \mathcal{M}_{B \times \Lambda} \) with \((b^r, a^r) \in J_{B \times \Lambda}\), or equivalently

\[
M[a^r, b^r] = M(a, b) \in \mathcal{M}_{\Lambda \times B}.
\]

Fix \( \lambda \in \mathcal{P} \). Let \( T \in \text{SST}_\Lambda(\lambda^\pi) \) and \( M \in \mathcal{M}_{\Lambda \times B} \) be given, where \( M = M(a, b) \) for some \((a, b) \in J_{\Lambda \times B}\). Suppose that \( a^r = a_1^r \cdots a_r^r \) and \( b^r = b_1^r \cdots b_r^r \). We define the pair of tableaux \( P(T \leftarrow M) \) and \( Q(T \leftarrow M) \) inductively as follows: For \( 1 \leq i \leq r \), put \( P^{(i)} = (P^{(i-1)} \leftarrow a_{r-i+1}^r) \), and \( \lambda^{(i)} = \text{sh}(P^{(i)})^\pi \) with \( P^{(0)} = T \) and \( \lambda^{(0)} = \lambda \). Define \( P(T \leftarrow M) = P^r \) and \( \mu = \text{sh}(P^r)^\pi \), and define \( Q(T \leftarrow M) \) to be the tableau of shape \((\mu/\lambda)^\pi\), where \((\lambda^{(i)}/\lambda^{(i-1)})^\pi\) is filled with \( b_{r-i+1}^r \) for \( 1 \leq i \leq r \).
Then the map

\[
\kappa : \text{SST}_k(\lambda^\pi) \times \mathcal{M}_{\lambda \times \lambda} \to \bigcup_{\mu \supset \lambda} \text{SST}_k(\mu^\pi) \times \text{SST}_\mu((\mu/\lambda)^\pi)
\]

is a bijection, which is a skew analogue of the usual RSK correspondence \cite{22}.

3.2. Let \([n] = \{1 < \cdots < n\}\) and \([\overline{n}] = \{\overline{n} < \cdots < \overline{1}\}\) be linearly ordered sets. We regard \([n]\) as a crystal \(B(\epsilon_1)\), where \(\text{wt}(k) = \epsilon_k\) for \(k \in [n]\), and \([\overline{n}]\) as the dual crystal \([n]^\vee\), where \(\overline{k} = k^\vee\) for \(k \in [n]\). Then \(\mathcal{W}_n\) and \(\mathcal{W}_{\overline{n}}\) are crystals, where we identify \(w = w_1 \cdots w_r\) with \(w_1 \otimes \cdots \otimes w_r\). The crystal structure on \(\mathcal{W}_n\) is easily described by so-called signature rule (cf. \cite{10} Section 2.1).

Let \(\mathcal{P}_n\) be the set of partitions \(\lambda = (\lambda_1, \ldots, \lambda_n)\) of length less than or equal to \(n\). For \(\lambda \in \mathcal{P}_n\), \(\text{SST}_{[n]}(\lambda)\) is a crystal under the identification of \(T\) with \(w(T) \in \mathcal{W}_n\), and it is isomorphic to \(B(\lambda)\), where we regard \(\lambda\) as \(\sum_{i=1}^n \lambda_i \epsilon_i \in P^+\) \cite{10}, while \(\text{SST}_{[\overline{n}]}(\lambda)\) is isomorphic to \(B(-w_0 \lambda)\). One can define a crystal structure on \(\text{SST}_{[n]}(\mu/\nu)\) for a skew Young diagram \(\mu/\nu\) in a similar way. Note that \(\text{SST}_{[n]}(\lambda)^\vee \cong \text{SST}_{[\overline{n}]}(\lambda^\pi)\), where the isomorphism is given by taking the 180°-rotation and replacing the entry \(i\) with \(n - i + 1\) for \(i \in [n]\).

For \(0 \leq t \leq n\), let \(\sigma^{-1} : \text{SST}_{[n]}(1^t) \to \text{SST}_{[\overline{n}]}(1^{n-t})\) be a bijection, where \(\sigma^{-1}(T)\) is the tableau with entries \([\overline{n}] \setminus \{\overline{k_1}, \ldots, \overline{k_t}\}\) for \(T\) with entries \(k_1 < \cdots < k_t\). For \(d \geq \lambda_1\), define

\[
\sigma^{-d} : \text{SST}_{[n]}(\lambda) \to \text{SST}_{[\overline{n}]}(\sigma^{-d}(\lambda)), \tag{3.4}
\]

where \(\sigma^{-d}(\lambda) = (d^n)/\lambda\), and the \(i\)th column of \(\sigma^{-d}(T)\) from the left is obtained by applying \(\sigma^{-1}\) to the \(i\)th column of \(T \in \text{SST}_{[n]}(\lambda)\) (which is assumed to be empty if \(i > \lambda_1\)). Then \(\sigma^{-d}\) commutes with \(\tilde{e}_i\) and \(\tilde{f}_i\) for \(i \in I\), where \(\text{wt}(\sigma^{-d}(T)) = \text{wt}(T) - d(\epsilon_1 + \cdots + \epsilon_n)\). We have an isomorphism of crystals

\[
\text{SST}_{[n]}(\lambda) \otimes T_\xi \to \text{SST}_{[\overline{n}]}(\sigma^{-d}(\lambda)), \tag{3.5}
\]

where \(\xi = -d(\epsilon_1 + \cdots + \epsilon_n)\). Also (3.4) and (3.5) hold when \([n]\) and \([\overline{n}]\) are exchanged (assuming that \(\overline{k} = k\) for \(k \in [n]\)).

4. Crystal of Lusztig data

4.1. Let \(i = (i_1, \ldots, i_N) \in R(w_0)\) be given. We have

\[
\Phi^+ = \{ \beta_1 := \alpha_{i_1}, \beta_2 := s_{i_1}(\alpha_{i_2}), \ldots, \beta_N := s_{i_1} \cdots s_{i_{N-1}}(\alpha_{i_N}) \}.
\]
Since \( \pi(B_1) = B(\infty) \) and \( b_1 : \mathbb{Z}_+^N \to B_1 \) is a bijection by (2.2), one can define a crystal structure on \( \mathbb{Z}_+^N \) by

\[
\tilde{f}_i c = c' \quad \text{if and only if} \quad \tilde{f}_i b_1(c) \equiv b_1(c') \pmod{qL(\infty)} \quad \text{for} \quad c, c' \in \mathbb{Z}_+^N \quad \text{and} \quad i \in I,
\]

with \( \text{wt}(c) = -(c_1\beta_1 + \cdots + c_N\beta_N) \), for \( c = (c_k) \in \mathbb{Z}_+^N \). We call the crystal \( \mathbb{Z}_+^N \) the crystal of \( i \)-Lusztig data, and denote it by \( B_i \). Recall that (4.1)

\[
\tilde{f}_i c = (c_1 + 1, c_2, \ldots, c_N), \quad \text{for} \quad c = (c_k) \in B_i.
\]

Let \( \Omega \) be a Dynkin quiver of type \( A_{n-1} \). We call a vertex \( i \in I \) a sink (resp. source) of \( \Omega \) if there is no arrow going out of \( i \) (resp. coming into \( i \)). For \( i \in I \), let \( s_i \Omega \) be the quiver given by reversing the arrows which end or start at \( i \). We say that \( i = (i_1, \ldots, i_N) \in R(w_0) \) is adapted to \( \Omega \) if \( i_1 \) is a sink of \( \Omega \), and \( i_k \) is a sink of \( s_{i_{k-1}} \cdots s_{i_1} \Omega \) for \( 2 \leq k \leq N \).

Let \( B_{\Omega} \) be the crystal \( B_i \) for \( i \in R(w_0) \) which is adapted to \( \Omega \). Note that \( B_{\Omega} \) is independent of the choice of \( i \) [15]. For \( c = (c_k) \in B_i \), we write \( c_{ij} = c_k \) if \( \beta_k = \epsilon_i - \epsilon_j \) for \( 1 \leq i < j \leq N \). For \( c = (c_{ij}) \) and \( c' = (c'_{ij}) \in B_{\Omega} \), put \( c \pm c' = (c_{ij} \pm c'_{ij}) \). For \( 1 \leq k < l \leq n \), let \( 1_{kl} = (c_{ij}^{kl}) \in B_{\Omega} \) be such that \( c_{ij}^{kl} = \delta_{ik}\delta_{jl} \).

In the next subsections, we consider some special cases of \( \Omega \), which give simple descriptions of the crystal \( B_{\Omega} \).}

4.2. We first consider the quiver \( \Omega \) where all the arrows are of the same direction. Suppose that \( \Omega = \Omega^+ \), where

\[
\Omega^+ : \quad \bullet \quad \cdots \quad \bullet \\
1 \quad 2 \quad \cdots \quad n-1
\]

For example, \( i = (1, 2, 1, 3, 2, 1, \ldots, n - 1, n - 2, \ldots, 2, 1) \) is adapted to \( \Omega^+ \). We assume that \( A = B = [n] \) and define an injective map

\[
(4.3) \quad B_{\Omega^+} \hookrightarrow M_{A \times B}, \quad c \quad \mapsto \quad M^+(c)
\]

where \( M^+(c) = (m_{ij}^+) \) is a strictly upper triangular matrix given by \( m_{ij}^+ = c_{ij} \) when \( 1 \leq i < j \leq n \) and \( 0 \) otherwise, for \( c = (c_{ij}) \in B_{\Omega^+} \). For \( M \in M_{A \times B} \), let \( M^+ = (m_{ij}^+) \) be the projection of \( M = (m_{ij}) \) onto the image of \( B_{\Omega^+} \) under (4.3), that is, \( m_{ij}^+ = m_{ij} \) for \( 1 \leq i < j \leq n \), and \( 0 \) otherwise.

Let us define \( \tilde{f}_i \) and \( \tilde{e}_i \) for \( i \in I \) on the image of \( B_{\Omega^+} \) in \( M_{A \times B} \) under (4.3). Given \( c \in B_{\Omega^+} \), suppose that \( M^+(c) = M(a, b) \) for some \( (a, b) \in J_{A \times B} \) under (3.1). Recall
that $b$ is an element in a crystal $W_{[n]}$. For $i \in I$, we define

$$
\bar{e}_iM^+(c) = \begin{cases} 
M(a, \bar{e}_ib)^+, & \text{if } \bar{e}_ib \neq 0, \\
0, & \text{if } \bar{e}_ib = 0,
\end{cases}
$$

$$
(4.4)
$$

$$
\bar{f}_iM^+(c) = \begin{cases} 
M(a, \bar{f}_ib), & \text{if } \bar{f}_ib \neq 0, \\
M(a, b) + E_{i+1}, & \text{if } \bar{f}_ib = 0,
\end{cases}
$$

where $E_{i+1}$ is an elementary matrix in $M_{A \times B}$. Note that $0$ is a formal symbol, not the zero matrix $O$.

Next suppose that $\Omega = \Omega^-$, where

$$
\begin{array}{c}
\Omega^- : \\
1 & 2 & \cdots & n-1
\end{array}
$$

In this case, we assume that $A = [n]$ and $B = [m]$, and define an injective map

$$
(4.5)
$$

$$
\begin{array}{c}
\mathcal{B}_{\Omega^-} & \hookrightarrow & M_{A \times B} \\
c & \mapsto & M^-(c)
\end{array}
$$

where $M^-(c) = (m^-_{ab})$ is a strictly upper triangular matrix given by $m^-_{n-j+1,i} = c_{ij}$ when $1 \leq i < j \leq n$, and 0 otherwise, for $c = (c_{ij}) \in \mathcal{B}_{\Omega^-}$. For $M = (m_{ij}) \in M_{A \times B}$, let $M^-(c)$ be the projection of $M$ onto the image of $\mathcal{B}_{\Omega^-}$ under (4.5), that is, $m^-_{ij} = m_{ij}$ for $i + j \leq n$, and 0 otherwise.

Let us define $\bar{e}_i$ and $\bar{f}_i$ for $i \in I$ on the image of $\mathcal{B}_{\Omega^-}$ in $M_{A \times B}$ under (4.5). Given $c = (c_{ij}) \in \mathcal{B}_{\Omega^-}$, suppose that $M^-(c) = M(a, b)$ for some $(a, b) \in J_{A \times B}$. For $i \in I$, we define

$$
\bar{e}_iM^-(c) = \begin{cases} 
M(a, \bar{e}_ib)^-, & \text{if } \bar{e}_ib \neq 0, \\
0, & \text{if } \bar{e}_ib = 0,
\end{cases}
$$

$$
(4.6)
$$

$$
\bar{f}_iM^-(c) = \begin{cases} 
M(a, \bar{f}_ib), & \text{if } \bar{f}_ib \neq 0, \\
M(a, b) + E_{n-i}, & \text{if } \bar{f}_ib = 0,
\end{cases}
$$

where $E_{n-i}$ is an elementary matrix in $M_{A \times B}$.

**Proposition 4.1.** Suppose that $\Omega$ is either $\Omega^+$ or $\Omega^-$. The operators $\bar{e}_i$ and $\bar{f}_i$ for $i \in I$ on $\mathcal{B}_{\Omega}$ induced from (4.4) and (4.6) coincide with those in (1.1), that is,

$$
\bar{e}_iM^\pm(c) = M^\pm(\bar{e}_ic), \quad \bar{f}_iM^\pm(c) = M^\pm(\bar{f}_ic),
$$

for $i \in I$ and $c \in \mathcal{B}_{\Omega}$. Here we assume that $M^\pm(0) = 0$. 
Proof. We will consider only the case when $\Omega = \Omega^+$, since the proof for the case when $\Omega = \Omega^-$ is similar. Let us first recall the description of (4.1) on $B_\Omega$ in [19, Theorem 7.1] (see also [21, Section 4.1]). Let $c = (c_{ij}) \in B_\Omega$ be given. For $i \in I$, put

$$c_k^{(i)} = \sum_{s=1}^{k} (c_{si+1} - c_{s-1,i}) \quad (1 \leq k \leq i),$$

$$c^{(i)} = \max\{c_k^{(i)} | 1 \leq k \leq i\},$$

$$k_0 = \min\{1 \leq k \leq i | c_k^{(i)} = c^{(i)}\},$$

$$k_1 = \max\{1 \leq k \leq i | c_k^{(i)} = c^{(i)}\},$$

where we assume that $c_0 = 0$. Then one can compute from [19, Theorem 7.1]

$$\bar{e}_i c = \begin{cases} c + 1_{k_0} - 1_{k_0 + 1}, & \text{if } c^{(i)} > 0 \text{ and } k_0 < i, \\ c - 1_{i+1}, & \text{if } c^{(i)} > 0 \text{ and } k_0 = i, \\ 0, & \text{if } c^{(i)} = 0, \end{cases}$$

(4.7)

$$\bar{f}_i c = \begin{cases} c - 1_{k_1} + 1_{k_1 + 1}, & \text{if } k_1 < i, \\ c + 1_{i+1}, & \text{if } k_1 = i. \end{cases}$$

On the other hand, suppose that $M^+(c) = M(a, b)$ for some $(a, b) \in J_{\mathbb{A} \times \mathbb{B}}$ under (3.1), where $b = b_1 \ldots b_r$. By definition of $(a, b)$, the subword of $b$ consisting of $i$ and $i + 1$ is

$$\overbrace{i + 1 \ldots i + 1}^{c_{1i+1}} \underbrace{i \ldots i}_{c_1} \overbrace{i + 1 \ldots i + 1}^{c_{2i+1}} \underbrace{i \ldots i}_{c_2} \ldots \overbrace{i + 1 \ldots i + 1}^{c_{ki+1}} \underbrace{i \ldots i}_{c_k}.$$

By the tensor product rule of crystals [21] (cf. [10, Proposition 2.1.1]), it is straightforward to see that $c^{(i)} > 0$ if and only if $\varepsilon_i(b) > 0$ and $\bar{e}_i b = b_1 \cdots (\bar{e}_i b_s) \cdots b_r$ for some $1 \leq s \leq r$ with $b_s = i + 1$, where $b_s$ is the leftmost $i + 1$ in $i + 1 \cdots i + 1$. This implies that $\bar{e}_i M^+(c) = M^+(\bar{e}_i c)$. Similarly, we see that

1. $k_1 < i$ if and only if $\varphi_i(b) > 0$ and $\bar{f}_i b = b_1 \cdots (\bar{f}_i b_s) \cdots b_r$ with $b_s = i$ for some $1 \leq s \leq r$, where $b_s$ is the rightmost $i$ in $i \cdots i$.

2. $k_1 = i$ if and only if $\varphi_i(b) = 0$ or $\bar{f}_i b = 0$,

which implies that $\bar{f}_i M^+(c) = M^+(\bar{f}_i c)$. \qed

4.3. Now we suppose that $\Omega$ is a quiver with a single sink, that is,

$$\bullet \cdots \bullet \bullet \cdots \bullet,$$

1 $r$ $n - 1
for some $r \in I$. Note that we have $\Omega = \Omega^+$ if $r = 1$ and $\Omega = \Omega^-$ when $r = n - 1$. So we assume that $r \in I \setminus \{1, n - 1\}$. Put
\[
J = I \setminus \{r\}, \quad J_1 = \{ j \in J \mid j < r \}, \quad J_2 = \{ j \in J \mid j > r \},
\]
where $J = J_1 \cup J_2$. Then we have $\Phi^+ = \Phi^+(J) \cup \Phi^+_{J_1} \cup \Phi^+_{J_2}$ where
\[
\Phi^+_{J_1} = \{ \epsilon_i - \epsilon_j \mid 1 \leq i < j \leq r \},
\]
\[
\Phi^+_{J_2} = \{ \epsilon_i - \epsilon_j \mid r < i < j \leq n \},
\]
\[
\Phi^+(J) = \{ \epsilon_i - \epsilon_j \mid 1 \leq i \leq r < j \leq n \}.
\]
We set
\[
\mathcal{B}^J_\Omega = \{ c = (c_{ij}) \in \mathcal{B}_\Omega \mid c_{ij} = 0 \text{ unless } \epsilon_i - \epsilon_j \in \Phi^+(J) \},
\]
which is a subcrystal of $\mathcal{B}_\Omega$. Note that we have $f_i c = 0$ on $\mathcal{B}^J_\Omega$ if $f_i c \not\in \mathcal{B}^J_\Omega$ for $i \in I$ and $c \in \mathcal{B}^J_\Omega$. Let $\Omega_{J_k}$ be the quiver corresponding to the vertices $J_k$ ($k = 1, 2$) in $\Omega$. Then $\mathcal{B}_{\Omega_{J_k}}$ is the crystal of the negative part of the quantum group $U_q(\mathfrak{g}_{J_k})$, whose crystal structure is described in (4.3) and (4.6), respectively. We identify $\mathcal{B}_{\Omega_{J_k}}$ with the subset of $\mathcal{B}_\Omega$ consisting of $c = (c_{ij})$ with $c_{ij} = 0$ for $\epsilon_i - \epsilon_j \not\in \Phi^+_{J_k}$, and then regard it as a subcrystal of $\mathcal{B}_\Omega$ where $\tilde{e}_i c = f_i c = 0$ with $\varphi_i(c) = \varphi_i(c) = -\infty$ for $i \in J \setminus J_k$.

We define a bijection
(4.8) \[
\begin{array}{ccc}
\mathcal{B}^J_\Omega & \longrightarrow & M_{[r] \times [n] \setminus \{r\}} \\
c & \longmapsto & M(c)
\end{array}
\]
where $M(c) = (m_{ab})$ is given by $m_{r+j} = c_{ij}$ for $c = (c_{ij}) \in \mathcal{B}^J_\Omega$ ($1 \leq i \leq r < j \leq n$).

Given $c \in \mathcal{B}^J_\Omega$, suppose that $M(c) = M(a, b) = M[\bar{a}^r, \bar{b}^r] = (m_{ab})$ for some $(a, b) \in \mathcal{J}_k \times \mathcal{B}$ (see (3.2)). For $i \in I$, we define
\[
\tilde{e}_i M(c) = \begin{cases} 
M[\bar{e}_i \bar{a}^r, \bar{b}^r], & \text{if } i \in J_1 \text{ and } \bar{e}_i \bar{a}^r \neq 0, \\
M(a, \bar{e}_i b), & \text{if } i \in J_2 \text{ and } \bar{e}_i b \neq 0, \\
M(a, b) - E_{\tau_{r+1}}, & \text{if } i = r \text{ and } m_{\tau_{r+1}} > 0, \\
0, & \text{otherwise},
\end{cases}
\]
(4.9) \[
\begin{cases} 
M \left[ \bar{f}_i \bar{a}^r, \bar{b}^r \right], & \text{if } i \in J_1 \text{ and } \bar{f}_i \bar{a}^r \neq 0, \\
M \left( a, \bar{f}_i b \right), & \text{if } i \in J_2 \text{ and } \bar{f}_i b \neq 0, \\
M(a, b) + E_{\tau_{r+1}}, & \text{if } i = r, \\
0, & \text{otherwise}.
\end{cases}
\]
For \( c \in \mathcal{B}_{\Omega} \), let \( c^I \) and \( c_{J_k} \) be the restrictions of \( c \) to \( \mathcal{B}^I_{\Omega} \) and \( \mathcal{B}_{\Omega_{J_k}} \) \((k = 1, 2)\), respectively. Then we have the following decomposition of \( \mathcal{B}_{\Omega} \) as a tensor product of its subcrystals.

**Theorem 4.2.** The map

\[
\begin{array}{c}
\mathcal{B}_{\Omega} \\
\xrightarrow{c}
\end{array} \mathcal{B}^I_{\Omega} \otimes \mathcal{B}_{\Omega_{J_1}} \otimes \mathcal{B}_{\Omega_{J_2}}
\]

is an isomorphism of crystals. Moreover, for \( i \in I \) and \( c \in \mathcal{B}^I_{\Omega} \) such that \( \bar{e}_i c \in \mathcal{B}^I_{\Omega} \) and \( \bar{f}_i c \in \mathcal{B}^I_{\Omega} \), the operators \( \bar{e}_i \) and \( \bar{f}_i \) on \( \mathcal{B}^I_{\Omega} \) induced from (4.9) coincide with those in (4.11) respectively, that is,

\[
\bar{e}_i M (c) = M (\bar{e}_i c), \quad \bar{f}_i M (c) = M (\bar{f}_i c).
\]

**Proof.** As in Proposition 4.1, it is done by comparing with the description of (4.11) on \( \mathcal{B}_{\Omega} \) using [19] Theorem 7.1.

For \( c \in \mathcal{B}_{\Omega} \), let \( \psi(c) = c'^I \otimes c_{J_1} \otimes c_{J_2} \). It is clear that \( \psi \) is a bijection. So it remains to show that \( \psi \) commutes with \( \bar{e}_i \) and \( \bar{f}_i \) for \( i \in I \).

Suppose that \( c = (c_{ij}) \in \mathcal{B}_{\Omega} \) is given. First, we have by (4.2)

\[
\bar{e}_r c = \begin{cases} 
  c - 1_{r+1}, & \text{if } c_{r+1} > 0, \\
  0, & \text{if } c_{r+1} = 0,
\end{cases} \quad \bar{f}_r c = c + 1_{r+1},
\]

which immediately implies that

\[
\bar{e}_r M (c') = M (\bar{e}_r c'), \quad \bar{f}_r M (c') = M (\bar{f}_r c'),
\]

assuming that \( M(0) = 0 \), and hence \( \psi \) commutes with \( \bar{e}_r \) and \( \bar{f}_r \).

Next, we fix \( i \in J_1 \). Let

\[
c_k^{(i)} = \begin{cases} 
  c_{i+1}, & \text{if } k = 1, \\
  c_{i+1} + \sum_{s=2}^{k} (c_{i+s} - c_{i+1+s-1}), & \text{if } 2 \leq k \leq n - r, \\
  c_{i+s} + (c_{i+s} - c_{i+1+s-1}), & \text{if } k = n - r + 1, \\
  c_{n-r+1} + \sum_{s=1}^{k-n+r-1} (c_{i-r-s} - c_{i+1-r-s+1}), & \text{if } n - r + 2 \leq k \leq n - i,
\end{cases}
\]

and

\[
c^{(i)} = \max \{ c_k^{(i)} \mid 1 \leq k \leq n - i \},
\]

\[
k_0 = \min \{ 1 \leq k \leq n - i \mid c_k^{(i)} = c^{(i)} \},
\]

and

\[
k_1 = \max \{ 1 \leq k \leq n - i \mid c_k^{(i)} = c^{(i)} \}.
\]

(4.10)
Note that if $c^{(i)} > 0$, then we have $c_{i+k_0+r} > 0$ when $k_0 \leq n - r$, and $c_{i+n-k_0+1} > 0$ when $k > n - r$. Also if $k_0 > n - r$, then we necessarily have $c^{(i)} > 0$. By Theorem 7.1, one can compute directly that

$\tilde{e}_i c = \begin{cases} c - 1_i k_{0+r} + 1_{i+1 k_{0+r}}, & \text{if } c^{(i)} > 0 \text{ and } k_0 \leq n - r, \\ c - 1_i n-k_0+1 + 1_{i+1 n-k_0+1}, & \text{if } n - r + 1 \leq k_0 \leq n - i - 1, \\ c - 1_{i+1}, & \text{if } k_0 = n - i, \\ 0, & \text{if } c^{(i)} = 0, \end{cases}$

(4.11)

$\tilde{f}_i c = \begin{cases} c + 1_i k_{1+r} - 1_{i+1 k_{1+r}}, & \text{if } k_1 \leq n - r, \\ c + 1_i n-k_1+1 - 1_{i+1 n-k_1+1}, & \text{if } n - r + 1 \leq k_1 \leq n - i - 1, \\ c + 1_{i+1}, & \text{if } k_1 = n - i, \end{cases}$

(see for example, the Auslander-Reiten quiver in Example 5.6, which might be helpful to see which $c_{ij}$'s are involved for $\tilde{e}_i$ and $\tilde{f}_i$, and how they are arranged).

Case 1. Suppose that $c = c^J \in B_{1j}^J$, that is, $c_{ij} = 0$ unless $c_i - c_j \in \Phi^+(J)$. We have

$c^{(i)}_{n-r} \geq c^{(i)}_{n-r+1} = \cdots = c^{(i)}_{n-i},$

which implies that $k_0 \leq n - r$. Note that if $k_1 > n - r$, then we have $c_{i+1 n} = 0$ and hence $k_1 = n - i$.

Let $(a, b) \in J_{[I]} \times ([I] \setminus [J])$ be such that $M(c) = [a^\tau, b^\tau]$. Note that the subword of $a^\tau$ consisting of $i$ and $i+1$ is

$\begin{array}{cccccccc}
\underbrace{i \cdots i} & \underbrace{i+1 \cdots i+1} & \cdots & \underbrace{i \cdots i} & \underbrace{i+1 \cdots i+1} \\
\tilde{e}_{i+r+1} & \tilde{e}_{i+1 r+1} & & \tilde{e}_n & \tilde{e}_{i+1 n}
\end{array}$

By the tensor product rule (2.1), we have $\varepsilon_i (a^\tau) = c^{(i)}$ and

$M(\tilde{e}_i c) = M(\tilde{e}_i a^\tau, b^\tau) = \tilde{e}_i M(c),$

If $k_1 \leq n - r$, then $\tilde{f}_i a^\tau \neq 0$ and

$M \left( \tilde{f}_i c \right) = M(\tilde{f}_i a^\tau, b^\tau) = \tilde{f}_i M(c).$

If $k_1 > n - r$, then $\tilde{f}_i a^\tau = 0$ and $\tilde{f}_i c \notin B_{1j}^J$, which implies that $\tilde{f}_i M(c) = 0$ and $\tilde{f}_i c = 0$ in $B_{1j}^J$, respectively. We have $\tilde{f}_i M(c) = M(\tilde{f}_i c) = 0$.

Case 2. Suppose that $c \in B_{\Omega}$ is arbitrary. We assume that

$M_1 := M(c^J) = [a^\tau, b^\tau], \quad M_2 := M^-(c_{J_1}) = [a', b'],$

for some $(a, b) \in J_{[I]} \times ([I] \setminus [J])$ and $(a', b') \in J_{[I]} \times [J]$. By Proposition 4.1 and the arguments in Case 1, we see that

$\varepsilon_i (M_1) = \varepsilon_i (a^\tau), \quad \varepsilon_i (M_2) = \varepsilon_i (b').$

(4.12)
Since \( \langle \text{wt}(M_1), h_i \rangle = \langle \text{wt}(a^\tau), h_i \rangle \) and \( \langle \text{wt}(M_2), h_i \rangle = \langle \text{wt}(b^\tau), h_i \rangle \), we also have
\[
\varphi_i(M_1) = \varphi_i(a^\tau), \quad \varphi_i(M_2) = \varphi_i(b^\tau).
\]
By (4.10) and (4.11), we have
\[
\psi(\bar{e}_i c) = \left( \bar{e}_i e_i \right) \otimes c_{J_1} \otimes c_{J_2} \iff k_0 \leq n - r
\]
\[
\iff \bar{e}_i(a^\tau \otimes b') = (\bar{e}_i a^\tau) \otimes b'
\]
\[
\iff \varphi_i(a^\tau) \geq \varepsilon_i(b'),
\]
\[
\psi(\bar{e}_i c) = c^J \otimes (\bar{e}_i c_{J_1}) \otimes c_{J_2} \iff k_0 > n - r
\]
\[
\iff \bar{e}_i(a^\tau \otimes b') = a^\tau \otimes (\bar{e}_i b')
\]
\[
\iff \varphi_i(a^\tau) < \varepsilon_i(b').
\]
Therefore, we have by (4.12) and (4.13)
\[
\psi(\bar{e}_i c) = \begin{cases} 
(\bar{e}_i c^J) \otimes c_{J_1} \otimes c_{J_2}, & \text{if } \varphi_i(M_1) \geq \varepsilon_i(M_2), \\
c^J \otimes (\bar{e}_i c_{J_1}) \otimes c_{J_2}, & \text{if } \varphi_i(M_1) < \varepsilon_i(M_2).
\end{cases}
\]
Similarly, we have
\[
\psi(\bar{f}_i c) = \begin{cases} 
(\bar{f}_i c^J) \otimes c_{J_1} \otimes c_{J_2}, & \text{if } \varphi_i(M_1) > \varepsilon_i(M_2), \\
c^J \otimes (\bar{f}_i c_{J_1}) \otimes c_{J_2}, & \text{if } \varphi_i(M_1) \leq \varepsilon_i(M_2).
\end{cases}
\]
It follows that \( \psi \) commutes with \( \bar{e}_i \) and \( \bar{f}_i \) for \( i \in J_1 \).

By the same arguments, we can show that \( \psi \) commutes with \( \bar{e}_i \) and \( \bar{f}_i \) for \( i \in J_2 \).

This completes the proof. \( \square \)

Let \( B_J(\infty) \) denote the \( \mathfrak{g}_J \)-crystal of the negative part of \( U_q(\mathfrak{g}_J) \), and extend it to a \( \mathfrak{g} \)-crystal with \( \bar{e}_i b = \bar{f}_i b = 0 \) and \( \varepsilon_i(b) = \varphi_i(b) = -\infty \) for \( i \in I \setminus J \) and \( b \in B_J(\infty) \).

Let \( W_J \) be the Weyl group of \( \mathfrak{g}_J \) generated by \( s_j \) for \( j \in J \), and let \( w^J \) be the longest element in the set of coset representatives of minimal length in \( W/W_J \).

Consider \( i \in R(w_0) \) corresponding to \( w_0 = w^J w_J \), where \( w_J \) is the longest element in \( W_J \). Let \( U^-(J) \) be the \( \mathbb{Q}(q) \)-subspace of \( U^- \) spanned by \( b_1(c) \in B_1 \) for \( c \in \mathbb{Z}_+^N \) such that \( c = c^J \). Then \( U^-(J) \) is independent of the choice of \( i \), and forms a subalgebra of \( U^- \) called the \textit{quantum nilpotent subalgebra} associated to \( w^J \) \footnote{For more details and its generalization to the case of a symmetrizable Kac-Moody algebra.}. By using a PBW basis, we see that the multiplication in \( U^- \) gives an isomorphism of a \( \mathbb{Q}(q) \)-vector space
\[
U^- \cong U^-(J) \otimes U^-_J
\]
(see \footnote{For more details and its generalization to the case of a symmetrizable Kac-Moody algebra.}) for more details and its generalization to the case of a symmetrizable Kac-Moody algebra). The image of a PBW basis of \( U^-(J) \) under the canonical projection \( \pi \) forms a subcrystal of \( \pi(B_1) = B(\infty) \) in \( L(\infty)/qL(\infty) \), which we denote
by $B^J(\infty)$. Then we have the following tensor product decomposition of $B(\infty)$, which is a crystal version of (1.14).

**Corollary 4.3.** As a $\mathfrak{g}$-crystal, we have

$$B(\infty) \cong B^J(\infty) \otimes B_J(\infty).$$

**Proof.** We have $B_J(\infty) \cong B_{J_1}(\infty) \otimes B_{J_2}(\infty) \cong B_{\Omega J_1} \otimes B_{\Omega J_2}$ and $B^J(\infty) \cong B^J_{\Omega}$. Hence it follows from Theorem 4.2. □

**Remark 4.4.** The isomorphism in Theorem 4.2 is a special case of the bijection $\Omega_w$ for $w \in W$ in [1, Proposition 5.25] when $w = w^d$ (see also [11, Proposition 3.14]). We should remark that $\Omega_w$ is not in general a crystal isomorphism for arbitrary $w \in W$. For example, suppose that $n = 3$ and $i = (1, 2, 1)$, that is, $\beta_1 = \alpha_1$, $\beta_2 = \alpha_1 + \alpha_2$, $\beta_3 = \alpha_2$. If $w = s_1$, then $\Omega_w$ is given by sending $c$ to $(c_{\leq w}, c_{> w})$ where $c_{\leq w} = (c_1, 0, 0)$ and $c_{> w} = (0, c_2, c_3)$ for $c = (c_1, c_2, c_3)$. But the mapping $c \mapsto c_{\leq w} \otimes c_{> w}$ does not define a morphism of crystals in this case (see Proposition 4.1). It would be interesting to characterize $w \in W$ such that the map $\Omega_w : c \mapsto c_{\leq w} \otimes c_{> w}$ is an isomorphism of crystals by using the result in [23], where a connection between crystal structure of Lusztig data and signature rule in tensor product is studied.

5. **Crystal embedding of Young tableaux into Lusztig data**

5.1. Let $\lambda \in \mathcal{P}_n$ be given. For $S \in \text{SST}_{[n]}(\lambda)$, we define

$$c^+(S) = (c_{ij}) \in \mathcal{B}_{\Omega^+},$$

where $c_{ij}$ is given by the number of $j$'s appearing in the $i$th row of $S$ for $1 \leq i < j \leq n$. Then we have the following, which is already well-known to experts in this area and which the author learned from Y. Saito.

**Proposition 5.1.** For $\lambda \in \mathcal{P}_n$, the map

$$\text{SST}_{[n]}(\lambda) \otimes T_{-\lambda} \longrightarrow \mathcal{B}_{\Omega^+},$$

$$S \otimes t_{-\lambda} \longmapsto c^+(S)$$

is an embedding of crystals.

**Proof.** It follows immediately from comparing the crystal structures on $\text{SST}_{[n]}(\lambda)$ and $\mathcal{B}_{\Omega^+}$ described in Proposition 4.1. □

We also have an embedding into $\mathcal{B}_{\Omega^-}$. For $T \in \text{SST}_{[\lambda]}(\lambda)$, we define

$$c^-(T) = (c_{ij}) \in \mathcal{B}_{\Omega^-},$$

(5.2)
where \( c_{ij} \) is given by the number of \( i \)'s appearing in the \((n - j + 1)\)th row of \( T \) for \( 1 \leq i < j \leq n \). Similarly, for \( S \in SST_{[n]}(\lambda) \), we define
\[
(5.3) \quad c_-(S) = c^-(\sigma^{-d}(S)) \in \mathcal{B}_\Omega^-,
\]
for some \( d \geq \lambda_1 \). Note that \( c_-(S) \) does not depend on the choice of \( d \).

**Proposition 5.2.** For \( \lambda \in \mathcal{P}_n \), the maps
\[
SST_{[n]}(\lambda) \otimes T_{w_0\lambda} \rightarrow \mathcal{B}_\Omega^- \quad \text{and} \quad \mathcal{B}_\Omega^- \rightarrow \mathcal{B}_\Omega^-,
\]
are embeddings of crystals.

**Proof.** It follows from (3.4) and Proposition 4.1. \( \square \)

**Example 5.3.** Suppose that \( n = 6 \) and let
\[
S = \begin{pmatrix}
1 & 1 & 1 & 2 & 2 & 3 \\
2 & 3 & 3 & 5 & 6 \\
4 & 4 & 4 \\
5 & 5 & 6 \\
6 & 6
\end{pmatrix} \in SST_{[6]}(6, 5, 3, 3, 2).
\]

Then we have by (5.1)
\[
c^+(S) = \begin{bmatrix}
c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\
c_{23} & c_{24} & c_{25} & c_{26} \\
c_{34} & c_{35} & c_{36} \\
c_{45} & c_{46} \\
c_{56}
\end{bmatrix} = \begin{bmatrix}
2 & 1 & 0 & 0 & 0 \\
2 & 0 & 1 & 1 \\
3 & 0 & 0 \\
2 & 1 \\
2
\end{bmatrix} \in \mathcal{B}_\Omega^+.
\]

On the other hand,
\[
\sigma^{-6}(S) = \begin{pmatrix}
6 \\
5 \\
4 \\
3 \\
2 \\
1 \\
0
\end{pmatrix}, \quad \sigma^{-6}(S)^\wedge = \begin{pmatrix}
6 & 6 & 5 & 4 & 1 & 1 & 1 & 1 \\
5 & 4 & 4 & 3 & 2 \\
4 & 3 & 2 & 1 & 1 \\
3 & 2 & 2 & 1 & 1 \\
2 & 2 & 2 & 2 & 1 \\
1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

Hence by (5.3), we have
\[
c_-(S) = \begin{bmatrix}
c_{56} & c_{46} & c_{36} & c_{26} & c_{16} \\
c_{45} & c_{35} & c_{25} & c_{15} \\
c_{34} & c_{24} & c_{14} \\
c_{23} & c_{13} \\
c_{12}
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 0 & 0 & 2 \\
1 & 0 & 1 & 0 \\
2 & 0 & 0 \\
1 & 1 \\
0
\end{bmatrix} \in \mathcal{B}_\Omega^-.
5.2. Let $\Omega$ be a quiver with a single sink

$$
\bullet \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet \leftarrow \bullet \leftarrow \cdots
$$

for some $r \in I \setminus \{1, n - 1\}$. We keep the notations in Section 4.3.

Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathcal{P}_n$ be given. Choose $d \geq \lambda_1$ and put

$$
\eta = (d - \lambda_r, \ldots, d - \lambda_1) \in \mathcal{P}_r, \quad \zeta = (\lambda_{r+1}, \ldots, \lambda_n) \in \mathcal{P}_{n-r}.
$$

We define a map

$$
(5.5) \quad \text{SST}_{[n]}(\lambda) \rightarrow \text{SST}_{[n][\setminus r]}(\zeta) \times \text{SST}_{[r]}(\eta) \times \mathcal{M}_{[r] \times ([n] \setminus [r])}
$$

where $(S^+, S^-, M)$ is determined by the following steps:

(i) let $S^+ \in \text{SST}_{[n][\setminus r]}(\zeta)$ be given by removing the first $r$ rows in $S$,

(ii) let $S \setminus S^+$ denote the subtableau of $S$ obtained by removing $S^+$, and put

$$
P' = \text{the subtableau of } S \setminus S^+ \text{ with entries in } [r],
$$

$$
Q = \text{the subtableau of } S \setminus S^+ \text{ with entries in } [n] \setminus [r],
$$

(iii) putting $P = \sigma^{-d}(P')$ (see (3.4)), we have for some $\nu \in \mathcal{P}_r$ with $\eta \subset \nu$

$$
(P, Q) \in \text{SST}_{[r]}(\nu^\pi) \times \text{SST}_{[n][\setminus r]}((\nu/\eta)^\pi),
$$

(iv) applying $\kappa^{-1}$ in (3.3), we get

$$
(T, M) = \kappa^{-1}(P, Q) \in \text{SST}_{[r]}(\eta^\pi) \times \mathcal{M}_{[r] \times ([n] \setminus [r])},
$$

(v) let $S^- = T^\wedge \in \text{SST}_{[r]}(\eta)$.

It can be summarized as follows:

$$
(5.6) \quad S \xrightarrow{(\ i\ )} (S^+, S \setminus S^+) \xrightarrow{(\ ii\ )} (S^+, P', Q) \xrightarrow{(\ iii\ )} (S^+, S^-, M) \xrightarrow{(\ iv\ )} (S^+, T, M) \xrightarrow{(\ v\ )} (S^+, P, Q)
$$

Note that the step (i) is injective, and the other steps (ii), (iii), (iv), (v) are bijective by definition. Hence the map (5.5) is injective, where

$$
\text{wt}(S^+) + \text{wt}(S^-) + \text{wt}(M) = \text{wt}(S) - d(\epsilon_1 + \cdots + \epsilon_r).
$$

Now for $S \in \text{SST}_{[n]}(\lambda)$ which is mapped to $(S^+, S^-, M)$ under (5.5), we define

$$
(5.7) \quad c(S) \in \mathcal{B}_\Omega,
$$
to be the unique \( c \in \mathcal{B}_\Omega \) such that
\[
(1) \quad M(c^{l}) = M \quad \text{under (4.8)},
\]
\[
(2) \quad c_{J_1} = c^{-}(S^-) \quad \text{and} \quad c_{J_2} = c^{+}(S^+) \quad \text{under (5.2) and (5.1)},
\]
Note that \( S^- \) depends on \( d \), but \( c^{-}(S^-) \) does not. Then we have the following, which is the main result in this paper.

**Theorem 5.4.** For \( \lambda \in \mathcal{P}_n \), the map
\[
\begin{align*}
SST[n](\lambda) \otimes T_{-\lambda} & \longrightarrow \mathcal{B}_\Omega \\
S \otimes t_{-\lambda} & \longrightarrow c(S)
\end{align*}
\]
is an embedding of crystals, where \( c(S) \) is given in (5.7).

**Proof.** Put \( M = M_{[\pi \times [(n)\backslash [r]]]} \). Note that \( M_{[\pi \times [(n)\backslash [r]]]} \) has a crystal structure isomorphic to \( \mathcal{B}_{\Omega}^J \) induced from the bijection (4.8), which can be described as in (4.9) by Theorem 4.2.

Choose \( d \geq \lambda_1 \) and let \( \eta \) and \( \zeta \) be as in (5.4). Define a \( \mathfrak{g} \)-crystal
\[
M_{\lambda} = M \otimes SST[\pi](\eta) \otimes SST[n][r](\zeta),
\]
where we extend a \( \mathfrak{g}_{J_1} \)-crystal \( SST[\pi](\eta) \) and a \( \mathfrak{g}_{J_2} \)-crystal \( SST[n][r](\zeta) \) to \( \mathfrak{g} \)-crystals in a trivial way. Put \( \xi = -d(\epsilon_1 + \cdots + \epsilon_r) \).

By (4.9), we see that the crystal structure on \( M_{\lambda} \) coincides with the one given in [12, Section 4.2]. Moreover, if we put \( H_{\lambda} = O \otimes b_1 \otimes b_2 \) where \( O \) is the zero matrix, \( b_1 \) (resp. \( b_2 \)) is the highest weight element in \( SST[\pi](\eta) \) (resp. \( SST[n][r](\zeta) \)) with weight \( -\eta_r \epsilon_1 - \cdots - \eta_1 \epsilon_r \) (resp. \( \zeta_1 \epsilon_{r+1} + \cdots + \zeta_{n-r} \epsilon_n \)), then \( H_{\lambda} \) is the highest weight element with weight \( \lambda + \xi \) and
\[
M_{\lambda} = \{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} H_{\lambda} \mid r \geq 0, i_1, \ldots, i_r \in I \},
\]
[12, Proposition 4.5]. By [12, Proposition 4.6], the map
\[
(5.8) \quad \begin{align*}
SST[n](\lambda) \otimes T_\xi & \longrightarrow M_{\lambda} \\
S \otimes t_\xi & \longrightarrow M \otimes S^- \otimes S^+
\end{align*}
\]
is an embedding of \( \mathfrak{g} \)-crystals, where \((S^+, S^-, M)\) is the triple associated to \( S \) in (5.5). Taking tensor product by \( T_{-\lambda-\xi} \) and then applying (4.8), Propositions 5.1 and 5.2, we have an embedding
\[
(5.9) \quad \begin{align*}
M_{\lambda} \otimes T_{-\xi-\lambda} & \longrightarrow \mathcal{B}_\Omega^J \otimes \mathcal{B}_{\Omega_{J_1}} \otimes \mathcal{B}_{\Omega_{J_2}} \\
M \otimes S^- \otimes S^+ \otimes t_{-\xi-\lambda} & \longrightarrow c^J \otimes c_{J_1} \otimes c_{J_2}
\end{align*}
\]
where \( c = c(S) \) is given in (5.7). Finally composing (5.8), (5.9), and then the inverse of the map in Theorem 4.2, we obtain the required embedding. \qed
Remark 5.5. Suppose that $\Omega$ is a quiver with a single source. Let $B^*_\Omega$ be the set $B_\Omega$ with the $*$-crystal structure [8]. By similar methods as in Theorem 5.4, we can construct an embedding of $SST_{[n]}(\lambda)$ into $B^*_\Omega$ for $\lambda \in \mathcal{P}_n$.

Example 5.6. Suppose that $\Omega$ is given by

\[
\begin{array}{c}
\bullet \\
1 \\
\rightarrow \\
2 \\
\rightarrow \\
3 \\
\rightarrow \\
4 \\
\rightarrow \\
5 \\
\end{array}
\]

Recall that the Auslander-Reiten quiver of representations of $\Omega$ is

which might be helpful for the reader to see (4.11) from Reineke’s description of $B(\infty)$ [19]. Here the vertex “$ij$” denotes the indecomposable representation of $\Omega$ corresponding to the positive root $\epsilon_i - \epsilon_j \in \Phi^+$ for $1 \leq i < j \leq 6$, the solid arrows denote the morphisms between them, and the dotted arrows denote the Auslander-Reiten translation functor denoted by $\tau$ in [19].

Let $S$ be as in Example 5.3. Let us apply the map (5.5) to $S$ following the steps in (5.6). First, we have

\[
S^+ = \begin{pmatrix} 5 & 6 \\ 6 & 6 \end{pmatrix}, \quad S \setminus S^+ = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 \\ 3 & 3 & 5 & 6 \\ 4 & 4 & 4 \end{pmatrix}.
\]

Separating $S \setminus S^+$ into subtableaux with entries in $\{1, 2, 3\}$ and $\{4, 5, 6\}$, we get

\[
P' = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 \\ 2 & 3 & 3 \\ \ldots \\ \ldots \end{pmatrix}, \quad Q = \begin{pmatrix} \ldots & \ldots & \ldots & 5 & 6 \\ 4 & 4 & 4 \end{pmatrix},
\]

and

\[
P = \sigma^{-6}(P') = \begin{pmatrix} 5 & 3 & 2 \\ 3 & 2 & 2 & 1 & 1 \end{pmatrix}.
\]

Applying $\kappa^{-1}$ to the pair
\[(P, Q) = \left( \begin{array}{ccc} 3 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{array}, \quad \begin{array}{ccc} 5 & 0 \\ 4 & 4 \\ \ldots \end{array} \right) \]

where \( \text{sh}(P) = (6, 3)^\pi \) and \( \text{sh}(Q) = ((6, 3)/(3, 1))^\pi \), we have \((T, M)\) where

\[T = \left( \begin{array}{ccc} 3 \\ 2 \\ 2 \end{array} \right) \in \text{SST}_{[3]}((3, 1)^\pi), \quad M = \left[ \begin{array}{ccc} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{array} \right] \in \text{M}_{[3] \times ((6)\setminus[3])}, \]

with

\[S^- = T^\wedge = \left( \begin{array}{ccc} 3 & 2 & 1 \\ 2 & \end{array} \right). \]

Therefore, we have a triple \((S^+, S^-, M)\) associated to \(S\):

\[(S^+, S^-, M) = \left( \begin{array}{ccc} 5 & 5 & 6 \\ 6 & 6 \end{array}, \quad \begin{array}{ccc} 3 & 2 & 1 \\ 2 \end{array}, \quad \left[ \begin{array}{ccc} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{array} \right] \right) \]

Finally, the corresponding \(c(S) = (c^J, c_{J_1}, c_{J_2}) \in \mathcal{B}_\Omega \) in \((5.7)\) is given by

\[c^J = \left[ \begin{array}{ccc} c_{34} & c_{35} & c_{36} \\ c_{24} & c_{25} & c_{26} \\ c_{14} & c_{15} & c_{16} \end{array} \right] = \left[ \begin{array}{ccc} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{array} \right], \]

and

\[c_{J_1} = \left[ \begin{array}{ccc} c_{23} & c_{13} \\ c_{12} \end{array} \right] = \left[ \begin{array}{ccc} 1 & 1 \\ 0 \end{array} \right], \quad c_{J_2} = \left[ \begin{array}{ccc} c_{45} & c_{46} \\ c_{56} \end{array} \right] = \left[ \begin{array}{ccc} 2 & 1 \\ 2 \end{array} \right]. \]

**Remark 5.7.** Let \(\Omega'\) be another quiver of type \(A_{n-1}\) with a single sink. Using Theorem 5.4 one can describe the transition map \(R_{\Omega'}^\Omega: \mathcal{B}_\Omega \to \mathcal{B}_{\Omega'}\) as follows.

Let \(c \in \mathcal{B}_\Omega\) be given. There exist a pair of Young tableaux \((S^+, S^-)\) (but not necessarily unique) such that \(c_{J_1} = c^-(S^-)\) and \(c_{J_2} = c^+(S^+)\). We can apply the inverse algorithm of \((5.6)\) to \((S^+, S^-, c^J)\) to obtain \(S \in \text{SST}_{[n]}(\lambda)\) for some \(\lambda \in \mathcal{P}_n\) such that each \(\lambda_i - \lambda_{i+1}\) is sufficiently large. In fact, we obtain a unique (marginally) large tableau (see \([3, 5]\)) corresponding to \(c\). Let \(c'\) be the Lusztig datum of \(S\) with respect to \(\Omega'\), which is also obtained by the algorithm \((5.6)\). Then we have \(c' = R_{\Omega}^{\Omega'}(c)\).

Note that if either one of \(\Omega\) and \(\Omega'\) is \(\Omega^\pm\), then one may apply only Propositions 5.1 and 5.2 to have \(R_{\Omega}^{\Omega'}\). We also refer the reader to [2, Section 4] for a closed-form
formula for $R^{\Omega}_0$, which is a tropicalization of a subtraction-free rational function connecting two parametrizations of a totally positive variety. It would be interesting to compare these two algorithms.

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