Unification of All String Models with $c < 1$

S.Kharchev, A.Marshakov, A.Mironov

*P.N.Lebedev Physics Institute*

*Leninsky prospect, 53, Moscow, 117 924*

A.Morozov

*Institute of Theoretical and Experimental Physics,*

*Bol.Cheremushkinskaya st., 25, Moscow, 117 259*

A.Zabrodin

*Institute of Chemical Physics*

*Kosygina st., 117334, Moscow*

**ABSTRACT**

A 1-matrix model is proposed, which nicely interpolates between double-scaling continuum limits of all multimatrix models. The interpolating partition function is always a KP $\tau$-function and always obeys $\mathcal{L}_{-1}$-constraint and string equation. Therefore this model can be considered as a natural unification of all models of 2d-gravity (string models) with $c \leq 1$.

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1E-mail address: theordep@sci.fian.msk.su
The model. The purpose of this letter is to introduce a new theory, which we call Generalized Kontsevich's Model (GKM) and to describe its structure and appealing properties. The partition function of the GKM is defined by the following integral over $N \times N$ Hermitean matrix:

\[ Z_N^{(V)}[M] \equiv \frac{\int e^{U(M,Y)}dY}{\int e^{-U_2(M,Y)}dY} , \]  

where

\[ U(M,Y) = Tr[V(M+Y) - V(M) - V'(M)Y] \]  

and

\[ U_2(M,Y) = \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} U(M, \epsilon Y) , \]

is an $Y^2$-term in $U$. $M$ is also a Hermitean $N \times N$ matrix with eigenvalues $\{\mu_i\}$, $V(\mu)$ is arbitrary analytic function.

Integrable structure. After the shift of variables $X = Y + M$ and integration over angular components of $X$, $Z_N^{(V)}[M]$ acquires the form of

\[ Z_N^{(V)}[M] = \left[ \frac{\text{det } \Phi_i(\mu_j)}{\Delta(M)} \right] , \]

where $\Delta(M) = \prod_{i<j} (\mu_i - \mu_j)$ is the Van-der-Monde determinant, and functions

\[ \tilde{\Phi}_i(\mu) = \left[ V''(\mu) \right]^{1/2} e^{V(\mu)-\mu V'(\mu)} \int e^{-\nu(x)+\nu'(\mu)x^i} dx \]  

The only assumption necessary for the derivation of (4) from (1) is the possibility to represent the potential $V(\mu)$ as a formal series in positive integer powers of $\mu$.

Formula (4) with arbitrary entries $\phi_i(\mu)$ is characteristic for generic KP $\tau$-function $\tau^G(T_n)$ in Miwa’s coordinates

\[ T_n = \frac{1}{n} Tr M^{-n} , \quad n \geq 1 \]  

and the point $G$ of Grassmannian is defined by potential $V$ through the set of basis vectors $\{\phi_i(\mu)\}$. (We remind that a priori definition is $\tau^G(T_n) = < 0 | e^{\sum T_n J_n} G | 0 >$, where $J$
stands for the free-fermion $U(1)$ current and $G$ is an exponent of quadratic combination of free fermion operators.) Therefore

$$Z^{(V)}[M] = \tau^{(V)}(T_n). \quad (7)$$

The case of finite $N$ in this formalism is distinguished by the condition that only $N$ of the parameters $\{\mu_i\}$ are finite. In order to take the limit $N \to \infty$ in the GKM (1) it is enough to bring all the $\mu'_i$s from infinity. In this sense this a smooth limit, in contrast to the singular conventional double-scaling limit, which one needs to take in ordinary (multi)matrix models.

$L_{-1}$-constraint. The set of function $\{\tilde{\Phi}_i(\mu)\}$ in (4) is, however, not arbitrary. They are all expressed through a single function — potential $V(\mu)$, — and are in fact recurrently related: if we denote the integral in (5) through $F_i(V'(\mu))$, then

$$F_i(\lambda) = (\partial/\partial \lambda)^{i-1} F_1(\lambda). \quad (8)$$

This relation is enough to prove, that

$$\frac{\partial}{\partial T_1} \log Z_N^{(V)} = -Tr M + Tr \frac{\partial}{\partial \Lambda_{tr}} \log \det F_i(\lambda_j) \quad (9)$$

whenever potential $V(\mu)$ grows faster than $\mu$ as $\mu \to \infty$.

Thus, $Z^{(V)}$ satisfies a simple identity:

$$\frac{1}{Z^{(V)}} L^{(V)}_{-1} Z_N^{(V)} = \frac{\partial}{\partial T_1} \log Z_N^{(V)} + Tr M - Tr \frac{\partial}{\partial \Lambda_{tr}} \log \det F_i(\lambda_j) = 0 \quad (10)$$

where operator $L^{(V)}_{-1}$ is defined to be

$$L^{(V)}_{-1} = \sum_{n \geq 1} Tr \left[ \frac{1}{V''(M)n+1} \frac{\partial}{\partial T_n} \right] + \frac{1}{2} \sum_{i,j} \frac{1}{V''(\mu_i)V''(\mu_j)} \frac{\partial}{\partial T_1} \quad (11)$$

(the items with $i = j$ are included into the sum). The reason why this operator is denoted by $L_{-1}$ will be clear after reductions of GKM will be discussed. From eqs.(9),(10) it follows, that partition function of GKM usually satisfies the constraint
\( \mathcal{L}^{(V)}_{-1} \tau^{(V)} = 0. \) \hspace{1cm} (12)

**Reductions.** The integral \( \mathcal{F}^{(V)}[\Lambda], \Lambda \equiv \mathcal{V}'(M) \), in the numerator of (1) satisfies the Ward identity

\[
\text{Tr} \left\{ \epsilon(\Lambda) \left[ \mathcal{V}'(\frac{\partial}{\partial \Lambda_{tr}}) - \Lambda \right] \right\} \mathcal{F}^{(V)}_N = 0 \hspace{1cm} (13)
\]

(as result of invariance under any shift of integration variable \( X \to X + \epsilon(M) \)). If potential \( \mathcal{V}(\mu) \) is restricted to be a polynomial of degree \( K + 1 \), this identity implies, that the functions (8) obey additional relations:

\[
F_{m+Kn}(\lambda) = \lambda^n F_m(\lambda) + \sum_{i=1}^{m+Kn-1} s_i F_i(\lambda). \hspace{1cm} (14)
\]

Since the sum at the r.h.s. does not contribute to determinant (5), we can say that all the functions \( F_n \) are expressed through the first \( K \) functions \( F_1...F_K \) by multiplication by powers of \( \lambda = \mathcal{V}'(\mu) \). Such situation (when the basis vectors \( \phi_i \), defining the point of Grassmannian are proportional to the first \( K \) ones) corresponds to reduction of KP-hierarchy. This reduction depends on the form of \( \mathcal{V}'(\mu) \) and in the case of \( \mathcal{V}(\mu) = \mathcal{V}_K(\mu) = \text{const} \cdot \mu^{K+1} \) coincides with the well-known \( K \)-reduction of the KP-hierarchy (KdV as \( K = 2 \), Boussinesq as \( K = 3 \) etc.). Thus in such cases partition function of GKM becomes \( \tau^{(K)} \)-function of the corresponding hierarchy. Generic \( \tau^{(K)} \) possesses an important property: it is almost independent of all time-variables \( T_{nK} \). To be exact,

\[
\partial \log \tau^{(K)} / \partial T_{nK} = a_n = \text{const} \hspace{1cm} (15)
\]

If \( \mathcal{V} = \mathcal{V}_K \), the generic expression (12) for the \( \mathcal{L}_{-1} \)-operator turns into

\[
\mathcal{L}^{(K)}_{-1} = \frac{1}{K} \sum_{n>K} nT_n \partial / \partial T_{n-K} + \frac{1}{2K} \sum_{a+b=K \atop a,b>0} aT_a bT_b + \partial / \partial T_1 \hspace{1cm} (16)
\]

The last item at the r.h.s. may be eliminated by the shift of time-variables:

\[
T_n \to \tilde{T}_n^{(K)} = T_n + \frac{K}{n} \delta_{n,K+1}. \hspace{1cm} (17)
\]
This shift is, however, $K$-dependent and does not seem to have too much sense. However, only expressed in terms of these $\hat{T}$'s the constraint (12) acquires the form of

$$\mathcal{L}_{-1}^{(K)} \tau^{(K)} = \left\{ \frac{1}{K} \sum_{n > K \atop n \neq 0 \text{mod } K} n \hat{T}_n \partial \partial \hat{T}_{n-K} + \frac{1}{2K} \sum_{a+b=K \atop a, b > 0} a \hat{T}_a \partial \partial b \hat{T}_b \right\} \tau^{(K)} =$$

$$= \sum_n a_n (n+1) \hat{T}_{(n+1)K} \tau^{(K)} . \quad (18)$$

with the l.h.s. familiar from [1]. The sum at the r.h.s. of (18) does not contribute to the “string equation”

$$\frac{\partial}{\partial T_1} \frac{\mathcal{L}_{-1}^{(K)} \tau^{(K)}}{\tau^{(K)}} = 0 . \quad (19)$$

Moreover, in variance with generic $\tau^{(K)}$ the partition function $Z^{(K)}$ of GKM is expected to obey (15) and (18) with all $a_n = 0$.

**Universal string equation.** Generalization of (19) to the case of arbitrary potential

$$\frac{\partial}{\partial T_1} \frac{\mathcal{L}_{-1}^{(V)} \tau^{(V)}}{\tau^{(V)}} = 0 . \quad (20)$$

may be transformed to the following form

$$\sum_{n \geq -1} \mathcal{T}_n \frac{\partial^2 \log \tau}{\partial T_1 \partial T_n} = u , \quad (21)$$

where

$$\mathcal{T}_n \equiv \text{Tr} \frac{1}{V^n(M)} M^{n+1} , \quad (22)$$

$$u \equiv \frac{\partial^2 \log \tau}{\partial T_1^2} , \quad \frac{\partial \log \tau}{\partial T_0} \equiv 0 , \quad \frac{\partial \log \tau}{\partial T_{-1}} \equiv T_1 .$$

If Baker-Akhiezer are introduced:

$$\Psi_{\pm}(z|T_k) = e^{\sum \mathcal{T}_k z^k} \frac{\tau(T_n \pm \frac{z^n}{n})}{\tau(T_n)} , \quad (23)$$

string equation (22) can be rewritten in the form of bilinear relation
\[
\sum_i \frac{\Psi_+(\mu_i)\Psi_-(\mu_i)}{\mu_i} = u. \tag{24}
\]

\textit{\mathcal{W}-constraints.} According to the arguments of refs.\[1\] the constraint

\[
\mathcal{L}^{\{K\}}_{-1} \tau^{\{K\}} = 0 \tag{25}
\]

(i.e. (18) with the vanishing r.h.s., as it is in fact the case if we deal with the model (1)) implies the entire tower of \(\mathcal{W}\)-constraints

\[
\mathcal{W}^{(k)}_{K\mu} \tau^{\{K\}} = 0, \quad k = 2, 3, \ldots, K; \quad n \geq 1 - k \tag{26}
\]

imposed on \(\tau^{\{K\}}\). Here \(\mathcal{W}^{(p)}_{K\mu}\) is the \(n - th\) harmonics of the \(p - th\) generator of Zamolodchikov’s \(W_K\)-algebra (the proper notation would be \(\mathcal{W}^{(p)\{K\}}_{n}\), but it is a bit too complicated). There is a Virasoro Lie sub-algebra, generated by \(\mathcal{W}^{(2)}_{K\mu} = \mathcal{L}^{\{K\}}_{n}\), and the particular \(\mathcal{L}^{\{K\}}_{-1}\) is just the operator (16). This is the origin of our notation \(\mathcal{L}^{\{V\}}_{-1}\) in the generic situation (where the entire Virasoro subalgebra of \(W_\infty\) was not explicitly specified).

Besides being a corollary of (24), the constraints (25) can be directly deduced from the Ward identity (13). For the case of \(K = 2\) (which is original Kontsevich’s model \[2\]) this derivation was given in ref.\[3\] (see also \[4,5\] for alternative proofs). Unfortunately, for \(K \geq 3\) the direct corollary of (13) is not just (25), but peculiar linear combinations of these constraints, e.g. for \(K = 3\) they look like

\[
\left\{ \sum_{k \geq 1} (3k - 1) \hat{T}_{3k-1} \mathcal{W}_{3k+3n}^{(2)} + \sum_{a+b=3n} \frac{\partial}{\partial T_{3a+2}} \mathcal{W}_{3b-3}^{(2)} \right\} Z_{\infty}^{(3)} = 0, \quad a, b \geq 0, \quad n \geq -2;
\]

\[
\left\{ \sum_{k \geq 1 + \delta_{n+3,0}} (3k - 2) \hat{T}_{3k-2} \mathcal{W}_{3k+3n}^{(2)} + \sum_{a+b=3n} \frac{\partial}{\partial T_{3a+1}} \mathcal{W}_{3b-3}^{(2)} \right\} Z_{\infty}^{(3)} = 0, \quad a, b \geq 0, \quad n \geq -3. \tag{27}
\]

For identification of (26) with (25) one can argue, that both sets of constraints possess unique, and thus coinciding, solutions.

\textit{Multimatrix models.} While detailed investigation of the properties of multimatrix models in the double-scaling limit (the analogue of ref.\[6\] in the case of conventional
Hermitean 1-matrix model) is still lacking, it has been suggested in [1] that the square roots of their partition functions, \( \sqrt{\Gamma_{ds}^{(K-1)}} \) (\( K - 1 \) is the number of matrices, index \( ds \) means, that partition function is considered in the double scaling limit), possess the following properties:

\[
W_{K^n}^{(k)} \sqrt{\Gamma_{ds}^{(K-1)}} = 0, \quad k = 2, 3, ..., K; \quad n \geq 1 - k.
\]  

(28)

Comparing these properties to the above information about GKM, we obtain:

\[
Z_{\infty}^{(K)} = \sqrt{\Gamma_{ds}^{(K-1)}}
\]

(29)

**Conclusion.** To conclude, we presented a brief description of the properties of the GKM, defined by eq.(1). Its partition function may be considered as a functional of two different variables: potential \( \mathcal{V}(\mu) \) and the infinite-dimensional Hermitean matrix \( M \) with eigenvalues \( \{\mu_i\} \). Partition function \( Z_N^{(\mathcal{V})} \) is an \( N \)-independent KP \( \tau \)-function, considered as a function of time-variables \( T_n = \frac{1}{n} Tr M^{-n} \) and the point of Grassmannian is specified by the choice of potential. The \( N \)-dependence enters only through the argument \( M \) : we return to finite-dimensional matrices if only \( N \) eigenvalues of \( M \) are finite. In this sense the “continuum” limit of \( N \to \infty \) is smooth.

The GKM is associated with a subset of Grassmannian, specified by additional \( \mathcal{L}_{-1} \)-constraint (12). For particularly adjusted potentials \( \mathcal{V}(\mu) = const \cdot \mu^{K+1} \), the corresponding points in Grassmannian lies in the subvarieties, associated with \( K \)-reductions of KP-hierarchy, \( Z_N^{(\mathcal{V})} \) becomes independent of all the time-variables \( T_{K^n} \), and the \( \mathcal{L}_{-1} \)-constraint implies the whole tower of \( W_K \)-algebra constraints on the reduced \( \tau \)-function. These properties are exactly the same as suggested for double scaling limit of the \( K - 1 \)-matrix model, and in fact there is an identification (29).

All this means, that GKM provides an interpolation between double-scaling continuum limits of all multimatrix models and thus between all string models with \( c \leq 1 \). Moreover, this is a reasonable interpolation, because both integrable and “string-equation” structures are preserved. This is why we advertise GKM as a plausible (on-shell) prototype of a unified theory of 2d gravity. All the proofs will be presented in ref.[7].
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References

1. M.Fukuma, H.Kawai, R.Nakayama Int.J.Mod.Phys. A6 (1991) 1385
2. M.Kontsevich Funk.Anal. i Priloz. 25 (1991) 50
3. A.Marshakov, A.Mironov, A.Morozov preprint HU-TFT-91-44, ITEP-M-4/91, FIAN/TD/04-91
4. Yu.Makeenko, G.Semenoff ITEP/UBC preprint, July 1991
5. E.Witten in talk at NYC conference, June 1991
6. Yu.Makeenko et.al. Nucl.Phys. B356(1991) 574
7. S.Kharchev et al. Preprint ITEP-M-9/91 — FIAN/TD-10/91