A Unified Analytical Design Method of Standard Controllers using Inversion Formulae *

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Abstract

The aim of this paper is to present a comprehensive range of design techniques for the synthesis of the standard compensators (Lead and Lag networks as well as PID controllers) that in the last twenty years have proved to be of great educational value in a vast number of undergraduate and postgraduate courses in Control throughout Italy, but that to-date remain mostly confined within this country. These techniques hinge upon a set of simple closed-form formulae for the computation of the parameters of the controller as functions of the typical specifications introduced in Control courses, i.e., the steady-state performance, the stability margins and the crossover frequencies.

Keywords: Feedback control, Lead and Lag networks, PID controllers, stability margins, steady-state performance, crossover frequencies.

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1 Introduction

The standard compensators presented in every course and textbook on control systems design belong to two important families: Lead/Lag networks and PID controllers. The structures of the controllers of these two big sets of compensators are particularly simple, and this partly justifies the standard practice of using these compensators as prototype examples to illustrate the various control synthesis methods in Control courses. However, the importance of these compensator structures resides also in their relevance in applications. It is often argued that the 90% of the compensators used in industry is made up of PID controllers alone [2, 6].

The common trend in both traditional and modern approaches to Control education is to formulate the feedback control problem as one in which the design specifications are first expressed using time domain parameters of the response (speed of the response, overshoot, undershoot, steady-state accuracy, etc). These requirements are then transformed into frequency domain specifications (DC gain, bandwidth, resonant peak, phase and gain margins, crossover frequencies, etc). Alternatively – but less realistically from a practical perspective – the design specifications can be expressed from the very beginning in the frequency domain.

In both situations, the design is effectively carried out using frequency domain considerations on Bode, Nyquist or – nowadays less frequently – Nichols plots, which constitute different types of graphical representations of the frequency responses involved in the control problem. The tuning techniques introduced in the vast majority of control courses are mainly based on trial-and-error considerations on these diagrams. For example, when a Lead network is employed in a feedback control system with the objective of increasing the phase margin of the loop gain, one can use the well-known analytical formula that provides the frequency at which the maximum phase lead is delivered by the network, and impose that frequency to be equal to the gain crossover frequency of the uncompensated plant. However, this procedure is not exact, as it does not take into account the fact that any Lead network with unity DC gain amplifies the magnitude of the frequency response of the plant at all (finite) frequencies, and hence the gain crossover frequency of the loop gain will necessarily be greater than the one in which the maximum lead is attained. Therefore, the specification on the phase margin is not exactly met. This suggests that the problem of placing this frequency can be solved iteratively using rules of thumb. This is the approach usually taken in the majority of Control courses and textbooks. This renders the synthesis procedure rather clumsy, and less suitable to be employed for educational purposes. This is particularly true within the context of written exercises, not in the least because all the aforementioned plots can only be drawn with pen and paper only in a very approximate fashion, especially nowadays when less and less emphasis is given to the rules for drawing these diagrams as a result of the increasing role that MATLAB® has to the same purpose. It is very difficult to construct a written exercise, or test, of exam, in which the control design problem consists in the exploitation of graphical techniques to compute the parameters of the desired compensator. Another consequence of the clumsiness associated with the classic trial-and-error design method is the fact that this procedure is difficult to automate into an algorithm that can be used for educational purposes (and for the same reason it is also unsuited to be used as a self-tuning strategy).

In this paper, we present an alternative methodology that can be successfully employed both in an educational and in a practical context to carry out the design of a standard compensator given standard control system specifications such as steady-state performance, gain and phase crossover frequencies, phase and gain margins. This method is based on a set of very simple closed-form formulae, known as Inversion Formulae, which deliver the parameters of the compensator as an explicit function of the specifications. These formulae first appeared for generic first-order compensators in [7], and their geometric interpretation in the context of control feedback design was explained in [11]. Surprisingly, the pioneering paper [7], which gives an extremely powerful tool for the design of standard compensators, has never
been cited in the literature, and this in part explains why this method is still relatively unknown to the wider scientific community. As such, its potential as a precise and meaningful tool in Control education is still to be fully examined. A significant exception which is worth mentioning is the Italian Control literature. In the Technical Report [3] the procedure presented in [7] had already been outlined for Lead and Lag networks. This technique also appeared in the Italian control textbook [8]. This undergraduate textbook has been by far the most utilised one in University courses and technical secondary institution (Istituti Tecnici) courses throughout Italy over the past twenty years. Due to its popularity, the same technique has later appeared in other University textbooks in Italy, see e.g. [5] and [4]. In the latter, a hint on how to adapt this technique to PID controllers was also presented. However, the success of this technique for educational purposes has so far remained confined within the Italian Control literature.

The aim of this paper is to present this technique in the most comprehensive way possible, to make its potential in control systems design education clear to the wider scientific community.

The educational value of the method outlined in this paper is motivated by the following facts.

1. The entire synthesis procedure can be carried out by pen, paper and a scientific calculator; it is therefore very suitable to be employed in all forms of written questions and exercises;

2. The synthesis procedure forces the students to follow the classical order of taking into account the steady state specifications first, and then to design the remaining part of the compensator;

3. Even though the synthesis methodology described here can be carried out by pen and paper, this technique has also an important graphical counterpart. In other words, it is shown that the Inversion Formulae enable the control system design problem to be solved analytically with pen and paper, or graphically on Nyquist, Bode or Nichols plots (without necessarily using trial-and-error or iterative procedures), thus retaining important links to other parts of a programme of a course of Control, [13];

4. Unlike the traditional design methodologies, the feasibility of the design procedure can be checked a priori. Furthermore, once the Bode gain of the compensator is computed from the steady-state requirements, very simple considerations can lead students to the selection of the most suitable type of compensator to be employed;

5. The mathematical tools that are needed to explain the method are basic notions of trigonometry and complex numbers. Hence, the use of Inversion Formulae reinforces the use of manipulations of complex numbers which is crucial in control systems education;

6. The situations in which some of the parameters of the compensator turn out to be positive can be fruitfully linked to important considerations on the shape of the Bode plot of the compensator;

7. The method based on the Inversion Formulae can be implemented as an extremely simple algorithm, for example using MATLAB®. An example will be presented in this paper;

8. For the most part, there is a tendency of Control courses and textbooks to neglect the synthesis techniques for richer compensator structures such as the Lead-Lag network. The Inversion Formulae enable these compensators to be addressed without a significant increase in the design complexity. This is an important advantage, because Lead-Lag networks offer additional flexibility with respect to standard Lead and Lag networks, that results in the ability to satisfy further specifications or constraints.
In this paper, we present the design technique based on the Inversion Formulae by first presenting the feedback control problem in the way it is usually introduced in undergraduate and postgraduate Control courses and textbooks. Lead and Lag networks will be the first compensator structures to be considered. It will be shown how simple considerations on the plant transfer function and on the specifications of the problem can guide students to the choice of the correct network to employ. In the second part of the paper, PID controllers will be introduced. The design approach is similar in spirit to the one for Lead-Lag networks, but the way steady-state specifications are accommodated in these two scenarios are slightly different. Indeed, the formulae that deliver the parameters of the PID controller depend on the type of steady-state specification, as also shown in [9]. The understanding of such difference is a crucial aspect in the understanding of the problem of steady-state specifications in a control feedback design problem.

In this paper we give great emphasis to the numerical examples, because these show what we believe is the most important feature of the Inversion Formulae in Control education, i.e., in the possibility of devising simple and at the same time complete and educationally relevant written exercises that have the potential to guide students through all stages of the compensator design process. To stress the potential offered by these formulae in all kinds of written exercises we illustrate the solutions of the numerical problems proposed here in a closed-form. However, we also show that this analytical method also has a fundamental graphical counterpart that adds a further dimension to the learning experience of the synthesis of standard compensators as highlighted in [13].

2 Formulation of the control problem

Consider the classic feedback control architecture in Figure 1 where $G(s)$ is the transfer function of the plant, which is assumed to be stable. In Figure 1 the symbols $R(s)$, $U(s)$ and $Y(s)$ respectively represent the Laplace transforms of the reference signal $r(t)$, of the control input $u(t)$ and of the controlled output $y(t)$. Let $E(s)$ represent the Laplace transform of the tracking error $e(t) \triangleq r(t) - y(t)$. The first and most basic control problem that we consider is the one that aims at satisfying standard specifications on the steady-state performance, on the phase margin and on the gain crossover frequency. To express these specifications mathematically, we define the loop gain transfer function as the product $L(s) \triangleq C(s) G(s)$. When $L(s)$ is strictly proper and the polar plot of $L(j\omega)$ for $\omega \geq 0$ has a single intersection with the unit circle and the negative real semiaxis (except for the trivial intersection at the origin as $\omega \rightarrow \infty$), the gain and phase margins are well defined, and ensure that the polar plot of $L(j\omega)$ does not encircle the critical point $-1$ in view of the simplified version of the Nyquist criterion, [10]. We denote by $\omega_g$ the gain crossover frequency, i.e., the frequency at which the polar plot of $L(j\omega)$ intersects the unit circle. Hence, $\omega_g$ is such that $|L(j\omega_g)| = 1$, and the phase margin is defined as the angle $\text{PM} \triangleq \arg L(j\omega_g) + \pi$. Similarly, we denote by $\omega_p$ the phase crossover frequency, i.e., the frequency at which the polar plot of $L(j\omega)$ intersects the negative real half-axis. As such, $\omega_p$ is such that $\arg L(j\omega_p) = -\pi$, and the gain margin is defined as $\text{GM} \triangleq 1/|L(j\omega_p)|$.
Problem 2.1 Find a controller $C(s)$ such that the steady-state requirements on the tracking error $e(t)$ are satisfied, and such that the gain crossover frequency and the phase margin of the loop gain transfer function $L(s)$ are $\omega_g$ and $PM$, respectively, i.e., such that

$$|L(j\omega_g)| = 1 \quad \text{and} \quad \arg L(j\omega_g) = PM - \pi. \hspace{1cm} (1)$$

An alternative control problem can be formulated by specifying the gain margin and the phase crossover frequency:

Problem 2.2 Find $C(s)$ such that the steady-state requirements on the tracking error are satisfied, and such that the phase crossover frequency and the gain margin of $L(s)$ are $\omega_p$ and $GM$, respectively, i.e., such that

$$|L(j\omega_p)| = GM^{-1} \quad \text{and} \quad \arg L(j\omega_p) = -\pi. \hspace{1cm} (2)$$

In some cases, the compensators with a richer dynamic structure will allow an additional degree of freedom to be exploited to the end of satisfying a further specification. In these cases, the control problem considered here is the one in which, in addition to the steady-state specification, the gain crossover frequency, the phase and the gain margin are imposed. This is the case of Lead-Lag networks and of PID controllers in which the steady-state performance requirements do not lead to a constraint on the Bode gain.

Problem 2.3 Find a controller $C(s)$ that meets the steady-state requirements, and such that the gain crossover frequency, the phase margin and the gain margin of the $L(s)$ are $\omega_g$, $PM$ and $GM$, respectively. In other words, $C(s)$ must guarantee that a frequency $\omega_p > 0$ exists such that (1) and (2) hold.

The first step of the design procedure consists in writing the transfer function $C(s)$ of the compensator as the product of a constant $K$, that is determined by imposing the steady-state requirements, by the transfer function $\bar{C}(s)$ with unity DC gain.

Since the term $K$ is known after the steady-state constraints have been imposed, it can be considered as being part of the plant, see Figure 3. Let us define $\tilde{G}(s) \triangleq KG(s)$. In order to compute the parameters.
of the compensator, we write $\bar{G}(j\omega)$ and $\bar{C}(j\omega)$ in polar form as

$$\bar{G}(j\omega) = |\bar{G}(j\omega)| e^{j \text{arg} \bar{G}(j\omega)}, \quad \bar{C}(j\omega) = M(\omega) e^{j \phi(\omega)}.$$  

The loop gain frequency response can be written as $L(j\omega) = |\bar{G}(j\omega)| M(\omega) e^{j (\text{arg} \bar{G}(j\omega) + \phi(\omega))}$. Consider Problem 2.1. From (1) we find

i-a) \hspace{0.5cm} M_g = 1/|\bar{G}(j\omega_g)|,  

ii-a) \hspace{0.5cm} \varphi_g = PM - \pi - \text{arg} \bar{G}(j\omega_g),

where $M_g \overset{\text{def}}{=} M(\omega_g)$ and $\varphi_g \overset{\text{def}}{=} \phi(\omega_g)$. At this point, since $M_g$ and $\varphi_g$ are known, by solving the equation

$$\bar{C}(j\omega_g) = M_g e^{j \varphi_g} \quad (3)$$

via the so-called Inversion Formulae we find all the remaining parameters of the compensator. In the case of Problem 2.2, the imposition of (2) leads to

i-b) \hspace{0.5cm} M_p = 1/(GM|\bar{G}(j\omega_p)|),  

ii-b) \hspace{0.5cm} \varphi_p = -\pi - \text{arg} \bar{G}(j\omega_p),

and the equation to be solved has the same structure of (3), with $M_p$, $\varphi_p$ and $\omega_p$ instead of $M_g$, $\varphi_g$ and $\omega_g$, respectively.

### 2.1 Standard compensators

The families of compensators that are considered in this paper are the phase-correction networks (Lead, Lag and Lead-Lag) and the PID controllers. These are the two most studied and utilised types of compensators, and are those that are introduced with no exceptions in all undergraduate and postgraduate textbooks of control feedback design:

#### 2.1.1 Phase-Correction Networks

- Lead network: $C_{\text{Lead}}(s) = K \frac{1 + \tau s}{1 + \alpha \tau s}$
- Lag network: $C_{\text{Lag}}(s) = K \frac{1 + \alpha \tau s}{1 + \tau s}$
- Lead-Lag network: $C_{LL}(s) = K \frac{(1 + \tau_1 s)(1 + \tau_2 s)}{(1 + \alpha \tau_1 s)(1 + \frac{\tau_2}{\alpha} s)}$

where $\alpha \in (0, 1)$ and $\tau, \tau_1, \tau_2 > 0$. The transfer function of the Lead-Lag network used in this paper generalises the one given above, because it includes the case of complex conjugate poles and zeros.
• Lead-Lag network with complex poles and zeros:

\[ C'_{LL}(s) = K \frac{s^2 + 2 \xi_1 \omega_n s + \omega_n^2}{s^2 + 2 \xi_2 \omega_n s + \omega_n^2}, \]

with \( \xi_1, \xi_2 > 0 \) and \( \omega_n > 0 \). The Lead-Lag network \( C_{LL}(s) \) can always be written as \( C'_{LL}(s) \) by setting

\[ \xi_1 = \frac{\tau_1 + \tau_2}{2 \sqrt{\tau_1 \tau_2}}, \quad \xi_2 = \frac{\alpha \tau_1 + \frac{\tau_2}{\tau_1}}{2 \sqrt{\xi_1 \xi_2}}, \quad \omega_n = \frac{1}{\sqrt{\tau_1 \tau_2}}. \tag{4} \]

Conversely, \( C'_{LL}(s) \) can be written as in \( C_{LL}(s) \) if and only if \( \xi_1 > 1 \) and \( \xi_2 > 1 \). In this case, by defining \( \hat{\xi}_1^\pm := \xi_1 \pm \sqrt{\xi_1^2 - 1} \) and \( \hat{\xi}_2^\pm := \xi_2 \pm \sqrt{\xi_2^2 - 1} \), \( C'_{LL}(s) \) can be written as in \( C_{LL}(s) \) with:

\[ \alpha = \frac{\hat{\xi}_2^-}{\hat{\xi}_1^-}, \quad \tau_1 = \frac{\hat{\xi}_1^-}{\omega_n}, \quad \tau_2 = \frac{\hat{\xi}_1^+}{\omega_n} \text{ if } \xi_1 < \xi_2; \]

\[ \alpha = \frac{\hat{\xi}_2^-}{\hat{\xi}_1^+}, \quad \tau_1 = \frac{\hat{\xi}_1^+}{\omega_n}, \quad \tau_2 = \frac{\hat{\xi}_1^-}{\omega_n} \text{ if } \xi_1 > \xi_2. \]

### 2.1.2 PID controllers

- **PID controller:** \( C_{PID}(s) = K_p \left( 1 + \frac{1}{T_i s} + T_d s \right) \)
- **PI controller:** \( C_{PI}(s) = K_p \left( 1 + \frac{1}{T_i s} \right) \)
- **PD controller:** \( C_{PD}(s) = K_p \left( 1 + T_d s \right) \)

with \( K_p, T_i, T_d > 0 \). In addition to these controllers, sometimes the proper versions of the PID and PD controllers are also introduced. The second one is basically equivalent to a Lead network. These more complex structures will not be considered here. For details on these structures, see [9].

## 3 Lead, Lag and Lead-Lag networks

We begin by considering compensators which belong to the family of Lead and Lag networks. The first step consists in the computation of the DC gain \( K \) of the phase-correction network, using the steady-state specifications. For phase-correction networks we can isolate the static gain by writing

\[ C_{Lead}(s) = K \tilde{C}_{Lead}(s), \quad C_{Lag}(s) = K \tilde{C}_{Lag}(s) \text{ and } C_{LL}(s) = K \tilde{C}_{Lead-Lag}(s), \]

where \( \tilde{C}_{Lead}(s), \tilde{C}_{Lag}(s) \) and \( \tilde{C}_{LL}(s) \) have unity DC gain. As is well known from classical control theory, in the unity feedback case, steady-state specifications that lead to the sharp assignment of the steady-state error to a given non-zero constant are such that the DC gain of the network is fixed. More precisely, if we assign the position error \( e_p \) for type-0 plants, the velocity error \( e_v \) for type-1 plants, or the acceleration error \( e_a \) for type-2 plants, to a given non-zero constant, the DC gain of the network is determined.

When our aim is to solve Problems 2.1.1 using a Lead, Lag or Lead-Lag network (that from now on will be considered with unity DC gain), simple standard considerations on the frequency response of these compensators suggest that the choice of the type of compensator to be employed can be made according to the following table.

The considerations that emerge from this table result from any of the graphical and analytical methods to characterise the frequency response of the compensator. For example, the Bode plot of the magnitude of a Lead network shows that such compensator amplifies the magnitude of the plant at all finite frequencies. Therefore, the gain crossover frequency of the loop gain to be selected in order for a Lead network
to solve the problem must necessarily be greater than the gain crossover frequency of the uncompensated system. That is, the magnitude of the uncompensated system at the frequency that we want to select as the gain crossover frequency of the loop gain must be smaller than 1, so that its inverse (which is $M_g$) must be greater than 1. Similar considerations on the Bode plot (or on the Nyquist or Nichols plots) can be used to justify the other entries in the table above, and can guide students (and engineers) towards the choice of the correct type of compensator.

Notice that Table 1 also gives a reason why usually Lead-Lag networks are defined only in the case in which $\tau_1 > \tau_2$. In fact, when $\tau_2 > \tau_1$, if $\varphi_g \in (0, \frac{\pi}{2})$ Problem 2.1 can be solved using simply a Lead network, while if $\varphi_g \in (-\frac{\pi}{2}, 0)$ a proportional controller (with gain equal to $M_g$) can even provide a phase margin greater than PM. However, because of the way Problem 2.1 has been formulated, it is more natural to also consider the case $\tau_2 > \tau_1$. Moreover, even if in some situations a Lead or a Lag network can be used instead of a Lead-Lag network to satisfy specifications on the phase margin and gain crossover frequency, a Lead-Lag network still presents the advantage of allowing a further parameter (such as the gain margin) to be assigned as well.

In order to solve Problems 2.1 and 2.2 for all types of phase-correction networks, we use the following simple lemma.

**Lemma 3.1** Let $P, Q \in \mathbb{R}$, $M \in \mathbb{R}_+$ and $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Consider the following equation

$$\frac{1 + jP}{1 + jQ} = Me^{j\varphi}. \quad (5)$$

Solving (5) with respect to $P$ and $Q$ yields

$$P = \frac{M - \cos \varphi}{\sin \varphi} \quad Q = \frac{M \cos \varphi - 1}{M \sin \varphi}$$

The proof of Lemma 3.1 follows straightforwardly by equating the real and imaginary parts of (5) once $Me^{j\varphi}$ is expressed as $M(\cos \varphi + j \sin \varphi)$, see also [7]. A geometric proof of the same result can be found in [11].

Lemma 3.1 is the result that allows the parameters of the phase-correction network to be computed in closed form.

**Lead network:** Equation (3) with $\tilde{C}(j\omega) = \tilde{C}_{\text{Lead}}(j\omega)$ is solvable in $\alpha \in (0, 1)$ and $\tau > 0$ if and only if

$$0 < \varphi_g < \frac{\pi}{2} \quad \text{and} \quad M_g > \frac{1}{\cos \varphi_g}. \quad (6)$$

| $\varphi_g \in (-\frac{\pi}{2}, 0)$ | $\tau_2 > \tau_1$ | $\tau_1 > \tau_2$ |
| $\varphi_g \in (0, \frac{\pi}{2})$ | Lead-Lag ($\tau_2 > \tau_1$) | Lag or Lead-Lag ($\tau_1 > \tau_2$) |

| $\varphi_g \in (0, \frac{\pi}{2})$ | Lead or Lead-Lag ($\tau_2 > \tau_1$) | Lead-Lag ($\tau_1 > \tau_2$) |

Table 1: Use of phase-correction networks
If (6) is satisfied, the solution of (3) with \( \tilde{C}(j\omega_p) = \tilde{C}_{\text{Lead}}(j\omega_p) \) is given by

\[
\alpha = \frac{M_g \cos \varphi_g - 1}{M_g (M_g - \cos \varphi_g)} \quad \text{and} \quad \tau = \frac{M_g - \cos \varphi_g}{\omega_g \sin \varphi_g}.
\]  

(7)

Eqs. (7) are called *Inversion Formulae for the Lead network*. This result is a consequence of Lemma 3.1 with \( P = \tau \omega_g \) and \( Q = \alpha \tau \omega_g \). Conditions (6) ensure that \( \tau > 0 \) and \( \alpha \in (0, 1) \). These conditions can be also written as \( M_g > 1 \) and \( 0 < \varphi_g < \arccos(1/M_g) \). It is of significant educational value to show the link between the solvability of (6) and the dynamic characteristics of the Lead network: the Bode plot of the magnitude of a Lead network shows that the effect of this network on the polar plot of the plant is to amplify the magnitude for all non-zero frequencies: this means that the gain crossover frequency of the loop gain transfer function must be greater than that of the plant alone. Therefore, it is essential that the gain crossover frequency to be chosen for the loop gain must be greater than the one of \( \tilde{G}(s) \): in case it is not, a Lead network achieving that goal does not exist. This fact is in line with the fact that \( M_g > 1 \).

In the case of Problem 2.2, the solution of (3) with \( \tilde{C}(j\omega_p) = \tilde{C}_{\text{Lead}}(j\omega_p) \) is given by the same equations written above with \( M_p, \varphi_p \) and \( \omega_p \) instead of \( M_g, \varphi_g \) and \( \omega_g \).

**Lag network:** Equation (3) with \( \tilde{C}(j\omega_p) = \tilde{C}_{\text{Lag}}(j\omega_p) \) is solvable in \( \alpha \in (0, 1) \) and \( \tau > 0 \) if and only if

\[
-\frac{\pi}{2} < \varphi_g < 0 \quad \text{and} \quad M_g < \cos \varphi_g.
\]  

(8)

If (8) are satisfied, the solution of (3) with \( \tilde{C}(j\omega_p) = \tilde{C}_{\text{Lag}}(j\omega_p) \) is given by

\[
\alpha = \frac{M_g (\cos \varphi_g - M_g)}{1 - M_g \cos \varphi_g} \quad \text{and} \quad \tau = \frac{M_g \cos \varphi_g - 1}{\omega_g M_g \sin \varphi_g}.
\]  

(9)

which are called *Inversion Formulae for the Lag network*. This result follows from Lemma 3.1 with \( P = \alpha \tau \omega_g \) and \( Q = \tau \omega_g \). Conditions (8) can be also written as \( M_g < 1 \) and \( -\arccos M_g < \varphi_g < 0 \).

**Lead-Lag network:** As already observed, in this case the compensator has an additional parameter that can be exploited to satisfy a further requirement other than the gain crossover frequency and the phase margin. One can, for example, assign the value of a parameter (a damping ratio or the natural frequency) and then solve for the other two. However, a more interesting problem to be solved in this case is Problem 2.3. Let us first consider the Lead-Lag network \( \tilde{C}_{\text{LL}}^v(s) \) with complex poles and zeros and with unity DC gain. Its frequency response can be written for \( \omega \neq \omega_n \) as

\[
\tilde{C}_{\text{LL}}^v(j\omega) = \frac{1 + jP(\omega)}{1 + jQ(\omega)}, \quad P(\omega) = \frac{2\varsigma_1 \omega_n \omega_n}{\omega_n^2 - \omega^2}, \quad Q(\omega) = \frac{2\varsigma_2 \omega \omega_n}{\omega_n^2 - \omega^2}.
\]

As such, when we want to assign the gain crossover frequency and the phase margin, we need to solve (5) with \( P = P_g = P(\omega_g), Q = Q_g = Q(\omega_g), M = M_g \) and \( \varphi = \varphi_g \). Similarly, when we want to assign the phase crossover frequency and the phase margin, we need to solve (5) with \( P = P(\omega_p), Q = Q(\omega_p), M = M_p \) and \( \varphi = \varphi_p \). As such, in order to solve Problem 2.3 we must solve

\[
P_g = \frac{2\varsigma_1 \omega_n \omega_n}{\omega_n^2 - \omega_g^2}, \quad Q_g = \frac{2\varsigma_2 \omega \omega_n}{\omega_n^2 - \omega_g^2},
\]

\[
P_p = \frac{2\varsigma_1 \omega_p \omega_n}{\omega_n^2 - \omega_p^2}, \quad Q_p = \frac{2\varsigma_2 \omega_p \omega_n}{\omega_n^2 - \omega_p^2}.
\]  

(10)
in which \( P_g \) and \( Q_g \) are completely assigned by the specifications, whereas \( P_p \) and \( Q_p \) are functions of \( \omega_p \), which is not assigned. From \( \frac{\zeta_1}{\zeta_2} = \frac{P_g}{P_p} = \frac{Q_g}{Q_p} \) we obtain

\[
\frac{M_g - \cos \varphi_g}{\cos \varphi_g - M_g} = \frac{M_p - \cos \varphi_p}{\cos \varphi_p - M_p},
\]

which is an equation in the unknown \( \omega_p \). If \( G(s) \) is a rational function of \( s \in \mathbb{C} \), it is a simple exercise of trigonometry to verify that (11) is a polynomial equation in \( \omega_p \), and therefore it is easy to derive all its solutions in closed form, whenever the degree is lower or equal to 5, or numerically. Using (10) we find the parameters

\[
\zeta_1 = \frac{\omega_g^2 - \omega_p^2}{2 \Phi_2} \sqrt{\frac{\Phi_2}{\omega_g \omega_p \Phi_1}},
\]

\[
\zeta_2 = \frac{\omega_g^2 - \omega_p^2}{2 \Psi_2} \sqrt{\frac{\Psi_2}{\omega_g \omega_p \Psi_1}},
\]

\[
\omega_n = \sqrt{\omega_g \omega_p \frac{\Phi_1}{\Phi_2}} = \sqrt{\omega_g \omega_p \frac{\Psi_1}{\Psi_2}},
\]

where \( \Phi_1 = \omega_g P_p^{-1} - \omega_p P_g^{-1} \), \( \Phi_2 = \omega_p P_p^{-1} - \omega_g P_g^{-1} \), \( \Psi_1 = \omega_g Q_p^{-1} - \omega_p Q_g^{-1} \) and \( \Psi_2 = \omega_p Q_p^{-1} - \omega_g Q_g^{-1} \).

However, it is easily seen that for some solutions \( \omega_p \), some of these parameters may be negative. A simple argument based on elementary inequalities gives the following result.

**Proposition 3.1** Problem 2.3 admits solutions with a Lead-Lag network with complex poles/zeros if and only if a solution \( \omega_p \) of (11) exists such that \( \Phi_1, \Phi_2, \Psi_1 \) and \( \Psi_2 \) all have the same sign, and

- are all positive if \( \omega_p < \omega_g \);
- are all negative if \( \omega_p > \omega_g \).

Moreover, Problem 2.3 admits solutions with a Lead-Lag network with real poles/zeros if and only if a solution \( \omega_p \) of (11) exists such that

\[
\max\{\Phi_1 \cdot \Phi_2, \Psi_1 \cdot \Psi_2\} < \frac{\omega_g \omega_p (\omega_g^2 - \omega_p^2)}{4}.
\]

**Proof:** From the expressions of \( \zeta_1 \), \( \zeta_2 \) and \( \omega_n \), we see that we must have \( \Phi_1, \Phi_2, \Psi_1 \) and \( \Psi_2 \) all positive when \( \omega_p < \omega_g \) and all negative when \( \omega_p > \omega_g \). The second statement follows by imposing \( \zeta_1 > 1 \) and \( \zeta_2 > 1 \).  

### 3.1 Design examples using Phase-Correction Networks

In this section our aim is to show how the simple method outlined in the previous sections can be easily employed to solve a range of problems that can be fruitfully used as written exercises of a basic Control course. First, we notice that when the aim is to assign the gain crossover frequency and the phase margin, we can complement the results in Table 1 with the considerations on the feasibility of the networks presented in the previous section, see Table 2. A graphical representation – that complements the use of
Consider the control scheme in Figure 5.

### Algorithm 1 Solution of Question 1 in MATLAB®

1. s=tf('s');
2. G=0.5*(s+10)/(s*(s^2+2*s+10));
3. wg=3;
4. PM=pi/4;
5. C=evalfr(G,j*wg);
6. M=1/abs(C);
7. phi=PM-(pi+angle(C));
8. if (sin(phi)<0)|(cos(phi)<0)|M<1/cos(phi),
   disp('No solutions with a Lead network');
9. end
10. alpha=(M*cos(phi)-1)/(M*(M-cos(phi)));
11. tau=(M-cos(phi))/(wg*sin(phi));

### Table 2: of the points $M_g e^{j\phi_g}$ of the Nyquist plane for which the problem admits solutions with a Lead, Lag or Lead-Lag network is given in Figure 4.

This figure provides a useful mean to gain insight into the design procedure presented here. The desired gain crossover frequency $\omega_g$ defines a point $A$ on the Nyquist plot of the plant, i.e., $A = \bar{G}(j\omega_g)$. The specification on the phase margin defines a point $B$ on the unit circle that the loop gain has to cross at exactly the same frequency, i.e., $B = e^{j(\pi + PM)} = L(j\omega_g)$. As such, the design reduces to finding the compensator structure such that $C(j\omega_g)A = B$. Loosely speaking, we may say that the network brings point $A$ into point $B$ at the frequency $\omega_g$. The solution of this problem is exactly the one given by the inversion formulae. The feasibility of each type of network imposes a constraint on $M_g$ and $\phi_g$, i.e., on the position that point $A$ must have with respect to point $B$ in order for a network with positive parameters to exist. These feasibility constraints are represented graphically by the shaded regions in Figure 4.
Figure 5: Unity feedback control scheme.

**Question 1.** Design a phase-correction network that satisfies the following static and dynamic specifications:

- velocity constant equal to 0.5;
- phase margin equal to 45°;
- gain crossover frequency equal to 3 rad/sec.

Find also the range of phase margins that are achievable at the crossover frequency 3 rad/sec with this phase-correction network. Also, determine the range of phase margins that at this gain crossover frequency ensures closed-loop stability.

The DC gain of the phase-correction network $K$ must be selected so as to satisfy the specification on the velocity constant:

$$K_v = \lim_{s \to 0} s C(s) G(s) = \lim_{s \to 0} s K G(s) = K,$$

so that $K = 0.5$. The gain $K$ is now considered to be part of the plant, i.e., we define $\tilde{G}(s) = K G(s)$. In order to select the right compensation structure, we compute $M_g$ and $\varphi_g$:

$$M_g = \frac{1}{|\tilde{G}(3j)|} = 6 \sqrt{\frac{37}{109}} \approx 3.4957 > 1,$$

$$\varphi_g = \text{PM} - (\pi + \arctan(3/10))$$

$$= \frac{7}{4} \pi - \arctan(3/10) + \arctan 6 \simeq 18.84^\circ.$$

As such, using Table 1 we see that a Lead network may be used. Since the conditions (6) are both satisfied, we expect the problem to be solvable. A simple computation, that can even be carried out in closed-form with pen and paper, shows that
\[ \alpha = \frac{3 \cdot 85 \sqrt{2} - 109}{36 \cdot 37 - 3 \cdot 85 \sqrt{2}} \approx 0.2590, \]
\[ \tau = \frac{12 \cdot 37 - 85 \sqrt{2}}{3 \cdot 29 \sqrt{2}} \approx 2.6317 \text{ sec}. \]

The corresponding MATLAB\textsuperscript{®} instructions are shown in Algorithm 1, and the required compensator that satisfies all the specifications is given by
\[ C_{\text{Lead}}(s) = 0.5 \frac{1 + 2.6317 s}{1 + 0.6817 s}. \]

A graphical plot on the Nyquist plane of the frequency response \( \overline{G}(j\omega) \) is shown with the black line in Fig. 6, where \( A \) denotes the point of \( \overline{G}(j\omega) \) at frequency \( \omega_g = 3 \) rad/sec. The compensator \( C_{\text{Lead}}(s) \) has been designed such that \( L(j\omega) \) shown with red line passes through point \( B = e^{j(\text{PM} + \pi)} \) at frequency \( \omega_g \). Intuitively, we can say that the point \( A \) is brought to point \( B \) by multiplication with the compensator frequency response at \( \omega = \omega_g \). The gray area in Fig. 6 denotes the set of all the points that can be brought to the desired point \( B \) using a Lead network.

The smallest phase margin achievable with a Lead network at the gain crossover frequency \( \omega_g = 3 \) rad/sec is
\[ \text{PM}_{\text{min}} = \pi + \text{arg} \overline{G}(j\omega_g) = \frac{\pi}{2} + \arctan \frac{3}{10} - \arctan 6 \simeq 26.1616^\circ, \]
and the largest phase margin is
\[ \text{PM}_{\text{max}} = \pi + \text{arg} \overline{G}(j\omega_g) + \arccos(|\overline{G}(j\omega_g)|) \]
\[ = \frac{\pi}{2} + \arctan \frac{3}{10} - \arctan 6 + \arccos \frac{1}{6} \sqrt{\frac{109}{37}} \simeq 99.54^\circ. \]

**Question 2.** Design a phase-correction network that satisfies the following specifications:

- velocity error equal to 0.1;
- phase margin equal to 60°;
- gain crossover frequency equal to 1 rad/sec.

Find also the range of phase margins that are achievable at this crossover frequency with this phase-correction network.

Since the velocity error is equal to \( e_v = 1/K_v \) and \( K_v = K \) as shown in Question 1, it is found that \( K = 10 \). We define \( \tilde{G}(s) = KG(s) \). In order to select the right compensation structure, we compute \( M_g \) and \( \phi_g \):
\[ M_g = \frac{1}{10} \sqrt{\frac{85}{101}} \simeq 0.0917 < 1, \]
\[ \phi_g = -\frac{\pi}{6} - \arctan(1/10) + \arctan \frac{2}{9} \simeq -23.18^\circ. \]
Therefore, we can use a Lag network. Since $-\arccos(M_g) < \varphi_g < 0$, a Lag network solving the problem exists, and is characterised by the parameters $\alpha = 0.0829$ and $\tau = 25.3559$ sec by simply replacing $M_g$ and $\varphi_g$ thus found into the Inversion Formulae (9) for the Lag network.

The smallest phase margin achievable with a Lag network at the gain crossover frequency $\omega_g = 1$ rad/sec is

$$PM_{\text{min}} = \pi + \arg \tilde{G}(j \omega_g) - \arccos \frac{1}{|\tilde{G}(j \omega_g)|} \simeq -1.55^\circ,$$

and the largest phase margin is

$$PM_{\text{max}} = \pi + \arg \tilde{G}(j \omega_g) \simeq 83.18^\circ.$$

Intuitively, the designed Lag network brings the point $A = \tilde{G}(j \omega_g)$ to the desired point $B = e^{j240^\circ}$ as shown in Fig. 7. The gray area denotes the set of points that can be brought to $B$ by a Lag network. If $A = \tilde{G}(j \omega_g)$ is not within this area, the problem cannot be solved with this type of network.

**Question 3.** Let $K = 10$. Find the interval of gain margins achieved using a Lag network at the phase crossover frequency $\omega_p = 4$ rad/sec that guarantee asymptotic stability of the closed loop.

A simple computation shows that

$$M_p = \frac{2}{\text{GM}\sqrt{29}},$$

$$\varphi_p = -\frac{\pi}{2} + \arctan \frac{2}{5} + \arctan \frac{4}{3},$$

from which it follows that

$$\alpha = \frac{52 - \frac{20}{\text{GM}}}{145 \text{GM} - 52}, \quad \tau = \frac{145 \text{GM} - 52}{56}.$$

It follows that

$$C(s) = \frac{56 + \left(52 - \frac{20}{\text{GM}}\right)s}{56 + (145 \text{GM} - 52)s}.$$

The characteristic polynomial is

$$(145 \text{GM} - 52)s^4 + (290 \text{GM} - 48)s^3$$

$$+ (112 + 1450\text{GM} - \frac{200}{\text{GM}})s^2 + (6320 - \frac{2000}{\text{GM}})s + 5600 = 0.$$
The asymptotic stability of the closed loop can at this point be studied using the Routh criterion on this polynomial. Such study will lead to a set of intervals for GM that guarantee asymptotic stability of the closed-loop.

**Question 4.** Design a Lead-Lag network that meets the following specifications:

- velocity constant equal to 0.1;
- phase margin equal to 45°;
- gain margin equal to 3;
- gain crossover frequency equal to 1 rad/sec.

![Design of Lead-Lag network on the Nyquist plane to meet the specifications of Question 4.](image)

First, notice that only a Lead-Lag network can simultaneously meet all the specifications. The DC gain of the Lead-Lag network $K$ must be selected so as to satisfy the specification on the velocity constant:

$$K_v = \lim_{s \to 0} s C_{LL}(s) G(s) = \lim_{s \to 0} s K G(s) = K,$$

so that $K = 0.1$. Now, we consider this gain to be part of the plant, and define $\tilde{G}(s) = K G(s)$. The frequency response of the plant at $\omega = \omega_g$ is

$$G(j \omega_g) = \frac{10 + j}{j(2j + 9)},$$

which leads to

$$M_g = \frac{1}{K} \sqrt{\frac{85}{101}},$$

$$\varphi_g = \frac{\pi}{4} - \pi - \left( \arctan \frac{1}{10} - \frac{\pi}{2} - \arctan \frac{2}{9} \right)$$

$$= \frac{3}{4} \pi - \arctan \frac{1}{10} + \arctan \frac{2}{9}.$$

A simple goniometric calculation shows that

$$\cos \varphi_g = \frac{103}{2} \sqrt{\frac{2}{85 \cdot 101}}.$$
which leads to
\[
\gamma = \frac{M_g - \cos \phi_g}{\cos \phi_g - M_g^{-1}} = \frac{170 - 103 \sqrt{2} K}{103 \sqrt{2} K - 202 K^2}.
\]

We express \(M_p\) and \(\phi_p\) as a function of the phase crossover frequency \(\omega_p\), which is still unknown:
\[
M_p = \frac{\omega_p \sqrt{4 \omega_p^2 + (10 - \omega_p^2)^2}}{GM \cdot K \sqrt{\omega_p^2 + 100}}
\]
\[
\cos \phi_p = \frac{\omega_p (\omega_p^2 + 10)}{\sqrt{(100 + \omega_p^2)((10 - \omega_p^2)^2 + 4 \omega_p^2)}}
\]

Plugging these expressions into (11) yields the polynomial equation
\[
\omega^6 - (16 + H + \gamma H) \omega^4 + 10 (10 - H - \gamma H) \omega^2 + 100 \gamma H^2 = 0
\]
where \(H = GM \cdot K\). This biquadratic equation has two positive solutions: \(\omega'_p = 3.9591\) and \(\omega''_p = 2.3686\). Using the first solution, we obtain positive values for \(\Phi_1, \Phi_2, \Psi_1\) and \(\Psi_2\). Therefore, the condition of Proposition 3.1 are not satisfied. If we use \(\omega_p = \omega''_p = 2.3686\), those values are all negative, and therefore they satisfy the condition of Proposition 3.1. The corresponding values of the parameters of the Lead-Lag network are \(\zeta_1 = 20.7474\), \(\zeta_2 = 1.6747\) and \(\omega_n = 0.2980\). Since \(\zeta_1\) and \(\zeta_2\) are both greater than 1, the solution can be also given in terms of a Lead-Lag network with real poles/zeros, with \(\alpha = 0.0728\), \(\tau_1 = 1.1120\) sec and \(\tau_2 = 139.1776\) sec.

Intuitively, point \(A = \hat{G}(j \omega_g)\) is brought to \(B = e^{j225^\circ} = L(j \omega_g)\) and point \(C = \hat{G}(j \omega''_p)\) is brought to \(D = \frac{e^{j(180^\circ)}}{GM}\). The gray area in Fig. 8 represents the set of all points that can be brought to \(B\) using a Lead-Lag network.

This example has shown the effectiveness of this design method from another perspective. In fact, even in the case of a plant with a reasonably rich dynamical structure, the design problem is found to admit a closed-form solution. Indeed, the 6th degree polynomial equation in \(\omega_p\) is biquadratic, and is therefore solvable in finite terms. This leads to a “mathematically exact” solution of this design problem. To the best of the authors’ knowledge, this is the first time an exact explicit solution for this problem is given for a Lead-Lag network.

### 4 PID controllers

Consider the example in Section 3.1 where system \(G(s)\) is of type 1. Lead and Lag networks are not suitable compensators for a design problem involving steady-state specifications that require zero velocity error (i.e., that the resulting control system tracks a ramp reference signal with zero steady-state error). In this case, a compensator that meets the desired steady-state specification is a PID controller, because of the presence of a pole at the origin in its transfer function. If the steady-state specification simply consists in achieving zero velocity error, the use alone of a compensator with a pole at the origin is sufficient to guarantee that this requirement is satisfied. In this case, the steady-state specification does not lead to constraints on the parameters of the controller. However, there are steady-state requirements that lead to such constraints. For example, if the controller is required to guarantee that the acceleration error be
Figure 9: Graphical representation on the Nyquist plane of admissible values of $M_g$ and $\phi_g$ for PID, PD and PI compensators.

no greater than, say, 0.2, we obtain a constraint on the ratio $K_p/T_i$, which is known also as integration constant of the PID controller. In fact,

$$e_a = \lim_{s \to 0} \frac{1}{s^2 C(s)G(s)} = \frac{T_i}{K_p} \leq 0.2$$

leads to $K_p/T_i \geq 5$. Hence, in this case the ratio $K_p/T_i$ is assigned by the steady-state requirements. These two situations must be addressed separately, because of the significant differences arising in the design phase.

The graphical representation of the frequency response of PID controllers on the Nyquist plane, usually omitted in undergraduate textbooks, is helpful to understand the physical meaning of the regulator synthesis, see Fig. 9. The frequency response of a PID controller is a vertical line passing through point $(K_p, 0)$. Variations of parameters determine the gray area of admissible values of $M_g$ and $\phi_g$ useful in the synthesis procedure of the compensator.

First, we consider the case where the steady-state specifications do not lead to a constraint on the integral constant of the PID controller.

In order to find the parameters of the controller such that $i-a)$ and $ii-a)$ are met, equation (3), that in the case $\tilde{C}'(s) = C(s)$ becomes

$$M_g e^{j\phi_g} = K_p \frac{1 + j \omega_g T_i - \omega_g^2 T_i T_d}{j \omega_g T_i},$$

which must be solved in $K_p, T_i, T_d > 0$. By equating real and imaginary parts of both sides of (12) we get

$$\omega_g M_g T_i \cos \phi_g = \omega_g K_p T_i,$$

$$-M_g \omega_g T_i \sin \phi_g = K_p - K_p \omega_g^2 T_i T_d,$$

in the three unknowns $K_p, T_i$ and $T_d$. A possibility to carry out the design at this point is to freely assign one of the unknowns and to solve (13-14) for the other two.\[1\]

Another possibility is to exploit the remaining degree of freedom so as to satisfy some further time or frequency domain requirements. Here, we consider two important situations: the first is the one where the ratio $T_d/T_i$ is chosen to ensure that the zeros of the PID controller are real; the second is the one where a gain margin constraint is to be satisfied.

\[1\] From (13) it is easily seen that $K_p$ cannot be chosen arbitrarily. If we choose $T_i$, (13) gives $K_p = M_g \cos \phi_g$, and (14) leads
4.0.1 Imposition of the ratio $T_d/T_i$

The ratio $\sigma \triangleq T_d/T_i$ is an important parameter. When $\sigma^{-1} \geq 4$, the zeros of the PID controller are real, and they are complex conjugate when $\sigma^{-1} < 4$. In the following theorem, necessary and sufficient conditions are given for the solvability of (3) in the case of a standard PID controller when the ratio $\sigma$ is given, [1].

**Theorem 4.1** ([1] Ch. 4, pp. 140–141). Let $\sigma = T_d/T_i$ be assigned. Equation (3) with $\bar{C}(s) = C_{PID}(s)$ admits solutions in $K_p, T_i, T_d > 0$ if and only if $\varphi_s \in (-\pi/2, \pi/2)$. If this condition is satisfied, the solution of (12) is given by

\[
K_p = \frac{M_g \cos \varphi_g}{\tan \varphi_g + \sqrt{\tan^2 \varphi_g + 4 \sigma}} \quad (15)
\]

\[
T_i = \frac{\omega_s^2 \sigma T_i^2 - \omega_s T_i \tan \varphi_g - 1 = 0}{2 \omega_s \sigma} \quad (16)
\]

\[
T_d = T_i \sigma \quad (17)
\]

**Proof:** (Only if). As already observed, equating real part to real part and imaginary part to imaginary part in (12) results in (13) and (14). Since $K_p$ must be positive, from (13) – which can be written as $K_p = M_g \cos \varphi_g$ – we get that $\varphi_g$ must satisfy $-\pi/2 < \varphi_g < \pi/2$. If this inequality is satisfied, it is also easy to see that (14) always admits a positive solution. In fact, (14) can be written as

\[
\omega_s^2 \sigma T_i^2 - \omega_s T_i \tan \varphi_g - 1 = 0, \quad (18)
\]

in $T_i$, that always admits two real solutions, one positive and one negative.

(If). From (15), it follows that (13) is satisfied. Moreover, since as aforementioned (14) can be written as (18) and $\sqrt{\tan^2 \varphi + 4 \sigma} > |\tan \varphi|$, the positive solution is given by (16). \[\blacksquare\]

4.0.2 Imposition of the Gain Margin

Another possibility in the solution of the control problem in the case of unconstrained $K_i$ is to fix the gain margin to a certain value $GM$. From $\arg L(j\omega_p) = -\pi$ and $GM = |L(j\omega_p)|^{-1}$ we obtain $M_p = 1/(GM|\bar{G}(j\omega_p)|)$ and $\varphi_p = -\pi - \arg \bar{G}(j\omega_p)$. Therefore, now the parameters $K_p, T_i, T_d > 0$ of the PID controller must be determined so that (12) and

\[
M_p e^{j\varphi_p} = K_p \frac{1 + j \omega_p T_i - \omega_p^2 T_i T_d}{j \omega_p T_i} \quad (19)
\]

to

\[
T_d = \frac{1 + \omega_s T_i \tan \varphi_g}{\omega_s^2 T_d - \omega_s \tan \varphi_g} \quad (20)
\]

However, in order to guarantee $K_p > 0$ and $T_d > 0$, the angle $\varphi_p$ must be such that $\cos \varphi_p < 0$ and $T_i$ must be chosen to be smaller than $-1/(\omega_s \tan \varphi_s)$. These two conditions can be simultaneously satisfied only when $\varphi_s \in (-\pi/2, 0)$. If we choose $T_d$, we get $K_p = M_g \cos \varphi_g$ and

\[
T_i = \frac{1}{\omega_s^2 T_d - \omega_s \tan \varphi_s} \quad (21)
\]

which implies that in order to ensure $T_i > 0$ we must choose $T_d$ to be greater than $\tan \varphi_s / \omega_s$. Therefore, $T_d$ is arbitrary when $\varphi_s \in (-\pi/2, 0)$, while when $\varphi_s \in (0, \pi/2)$, we must choose $T_d > \tan \varphi_s / \omega_s$. 

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are simultaneously satisfied. By equating real and imaginary part of \((12)\) and \((19)\) we obtain \((13), (14)\) in addition to
\[
\begin{align*}
\omega_p M_p T_i \cos \varphi_p &= \omega_p K_p T_i, \\
-M_p \omega_p T_i \sin \varphi_p &= K_p - K_p \omega_p^2 T_i T_d.
\end{align*}
\]
From \((13)\) and \((20)\), we obtain
\[
M_g \cos \varphi_g = M_p \cos \varphi_p
\] (22)
in the unknown \(\omega_p\). For the control problem to be solvable, it is required that \((22)\) admits at least one strictly positive solution. For the given \(G(s)\) the solution of \((22)\) can be found by solving a polynomial equation in \(\omega_p\), and therefore all its solutions can be determined either in closed form when the system order is not too high or numerically with arbitrary precision. Indeed, as for Lead-Lag networks, it is a simple exercise of trigonometry to see that if \(G(s) = \hat{G}(s) e^{-t_0 s}\), equation \((22)\) is not polynomial in \(\omega_p\) and it needs to be solved numerically.

**Theorem 4.2** Consider Problem 2.3 with the additional specification on the gain margin \(GM\). Equations \((12)\) and \((19)\) admit solutions in \(K_p, T_i, T_d > 0\) if and only if \(\varphi_g \in (-\pi/2, \pi/2)\) and \((22)\) admits a positive solution \(\omega_p\) such that
\[
\begin{align*}
\begin{cases}
\omega_p < \omega_g \\
\omega_k \tan \varphi_g > \omega_p \tan \varphi_p \\
\omega_p \tan \varphi_g > \omega_g \tan \varphi_p
\end{cases}
\quad \text{or} \quad
\begin{cases}
\omega_p > \omega_g \\
\omega_k \tan \varphi_g < \omega_p \tan \varphi_p \\
\omega_p \tan \varphi_g < \omega_g \tan \varphi_p
\end{cases}
\end{align*}
\]
If \(\varphi_g \in (-\pi/2, \pi/2)\) and \((23)\) is satisfied, the problem admits solutions with
\[
\begin{align*}
K_p &= M_g \cos \varphi_g \quad (24) \\
T_i &= \frac{(\omega_k - \omega_p^2)}{\omega_k \omega_p (\omega_p \tan \varphi_g - \omega_k \tan \varphi_p)} \quad (25) \\
T_d &= \frac{\omega_k \tan \varphi_g - \omega_p \tan \varphi_p}{\omega_g^2 - \omega_p^2} \quad (26)
\end{align*}
\]
**Proof:** *(Only if)*. As already seen, a necessary condition for the problem to admit solutions is that \(\omega_p\) is a solution of \((22)\). From \((13)\) and \((14)\), and from \((20)\) and \((21)\), we obtain
\[
\begin{align*}
- \omega_k T_i \tan \varphi_g &= 1 - \omega_k^2 T_i T_d, \\
- \omega_p T_i \tan \varphi_p &= 1 - \omega_p^2 T_i T_d.
\end{align*}
\]
By solving \((27)\) and \((28)\) in \(T_i\) and \(T_d\), we obtain \((24),(26)\). For \(K_p\) to be positive, it is necessary that \(\varphi_g \in (-\pi/2, \pi/2)\). Moreover, the time constants \(T_i\) and \(T_d\) are positive if \(\omega_g\) and \(\omega_p\) satisfy \((23)\).

(If). It is a matter of straightforward substitution of \((24),(26)\) into \((13), (14), (20)\) and \((21)\). \(\square\)

Now we consider the case in which the steady-state requirements lead to constraints on \(K_i = K_p / T_i\). Hence, now the integration constant \(K_i\) is determined via the imposition of the steady-state requirements; for example, for type-0 (resp. type-1) plants, \(K_i\) is computed via the imposition of the velocity error (resp.
acceleration error).

As such, the factor $K_i/s$ can be separated from $\bar{C}_{\text{PID}}(s) = 1 + T_i s + T_d s^2$, and viewed as part of the plant. In this way, the part of the controller to be designed is $\bar{C}_{\text{PID}}(s)$, and the feedback scheme reduces to that of Figure I0. Let $\tilde{G}(s) \equiv \frac{K_0}{T_i s} G(s)$, so that the loop gain transfer function can be written as $L(s) = \bar{C}_{\text{PID}}(s) \tilde{G}(s)$. Write $\tilde{G}(j\omega)$ and $\bar{C}_{\text{PID}}(j\omega)$ in polar form as

$$\tilde{G}(j\omega) = |\tilde{G}(j\omega)| e^{j \arg\tilde{G}(j\omega)}; \quad \bar{C}_{\text{PID}}(j\omega) = M(\omega) e^{j \varphi(\omega)},$$

so that the loop gain frequency response can be written as $L(j\omega) = |\tilde{G}(j\omega)| M(\omega) e^{j (\arg\tilde{G}(j\omega) + \varphi(\omega))}$. If the crossover frequency $\omega_c$ and the phase margin PM of the loop gain transfer function $L(s)$ are assigned, the equations $|L(j\omega_c)| = 1$ and PM = $\pi + \arg L(j\omega_c)$ must be satisfied, and as already observed these can be written as $M_g = 1/|\tilde{G}(j\omega_c)|$ and $\varphi_g = \text{PM} - \pi - \arg \tilde{G}(j\omega_c)$. Alternatively, $M_g$ and $\varphi_g$ can be computed as functions of the frequency response of $G(s)$ at $\omega = \omega_c$:

$$M_g = \left| \frac{K_p}{T_i j \omega_c} \tilde{G}(j\omega_c) \right|^{-1} = \frac{\omega_c}{|K_i| \tilde{G}(j\omega_c)} \quad (29)$$

$$\varphi_g = \text{PM} - \pi - \arg \left[ \frac{K_p}{T_i j \omega_c} \tilde{G}(j\omega_c) \right] = \text{PM} - \frac{\pi}{2} - \arg \tilde{G}(j\omega_c), \quad (30)$$

since $K_p, T_i > 0$. In order to find the parameters of the controller such that i-a) and ii-a) are met, equation

$$M_g e^{j \varphi_g} = 1 + j \omega_c T_i - T_d \omega_c^2 \quad (31)$$

must be solved in $T_i > 0$ and $T_d > 0$. The closed-form solution to this problem is given in the following theorem.

**Theorem 4.3**  *Equation (31) admits solutions in $T_i > 0$ and $T_d > 0$ if and only if*

$$0 < \varphi_g < \pi \quad \text{and} \quad M_g \cos \varphi_g < 1. \quad (32)$$

*If (32) are satisfied, the solution of (31) is given by*

$$K_p = \frac{K_i}{\omega_c} M_g \sin \varphi_g, \quad (33)$$

$$T_i = \frac{1}{\omega_c} M_g \sin \varphi_g, \quad (34)$$

$$T_d = \frac{1 - M_g \cos \varphi_g}{\omega_c M_g \sin \varphi_g}. \quad (35)$$
The two conditions (32) can be alternatively written as
\[
\phi_g \in \left( \arccos \frac{1}{M_g}, \pi \right) \quad \text{if } M_g > 1,
\]
\[
\phi_g \in (0, \pi) \quad \text{if } M_g < 1.
\]
In fact, when \( \phi_g \in (0, \pi/2) \), condition \( \cos \phi_g < 1/M_g \) is always satisfied when \( M_g < 1 \), and is satisfied when \( \phi_g > \arccos(1/M_g) \) when \( M_g > 1 \). When \( \phi_g \in (\pi/2, \pi) \), the condition \( \cos \phi_g < 1/M_g \) is always satisfied since \( \cos \phi_g < 0 \) and \( (1/M_g) > 0 \).

### 4.1 Design examples using PID controllers

Consider the unity feedback control scheme in Figure 5.

**Question 1.** Design a compensator that meets the following specifications:

- zero velocity error;
- phase margin equal to 45°;
- gain crossover frequency equal to 3 rad/sec.

The steady-state specification is automatically satisfied by using a PID controller or a PI controller. Let us consider the case of a PID controller. The extra freedom in this case can be used to select the ratio \( T_i/T_d \). Let us choose for example \( T_i/T_d = 8 \), so that \( \sigma = 1/8 \) guarantees that the zeros of the PID controller are real. Then, we compute \( M_g \) and \( \phi_g \):

\[
M_g = \frac{1}{|G(3j)|} = 3\sqrt{\frac{37}{109}} \approx 1.7479,
\]
\[
\phi_g = \text{PM} - (\pi + \arccos(1/M_g))
\]
\[
= 7/4 \pi - \arctan(3/10) + \arctan(6) \approx 18.84°.
\]

Since \( \phi_g \in (-\pi/2, \pi/2) \), the problem admits a solution with a PID controller. Using (15-17) we find \( K_p = 1.6542, T_i = 1.5017 \) sec and \( T_d = 0.1877 \) sec. This choice guarantees that the controller has real zeros, which in this case are \( -4.5471 \) and \( -0.7802 \).
Figure 12: Design of PID controller on the Nyquist plane to meet the specifications of Question 2.

Let us attempt to solve the same problem with a PI controller. To this end, we compute

$$\phi_g = \text{PM} - \frac{\pi}{2} - \arg(G(j\omega_g)) \simeq 108.8384^\circ,$$

so that a PI controller solving the problem does not exist.

The graphical representation of the problem solution with PID regulator is shown in Fig. 11. The designed PID brings the point $A = G(j\omega_g)$ to $B = e^{j(\text{PM}+\pi)}$. The gray area corresponds to the set of admissible points that can be brought to $B$ by a PID controller.

**Question 2.** Design a compensator that meets the following specifications:

- zero velocity error and acceleration error not greater than 0.2;
- phase margin equal to 45°;
- gain crossover frequency equal to 3 rad/sec.

The correct compensator structure to be employed in this case is the PID controller. As already observed, the steady-state requirement in this case imposes the ratio $K_i = K_p/T_i$. In particular, in this case we need $K_i \geq 5$. Let us choose $K_i = 5$. Hence,

$$M_g = \frac{\omega_g}{K_i |G(j\omega_g)|} \simeq 1.0487,$$

$$\phi_g = \text{PM} - \frac{\pi}{2} - \arg(G(j\omega_g)) \simeq 108.8384^\circ.$$

The conditions $0 < \phi_g < \pi$ and $M_g \cos \phi_g < 1$ are both satisfied, so that the problem admits solutions. The parameters of the controller in this case are

$$T_i = \frac{M_g \sin \phi_g}{\omega_g} \simeq 0.3308 \text{ sec},$$

$$T_d = \frac{1 - M_g \cos \phi_g}{\omega_g M_g \sin \phi_g} \simeq 0.4496 \text{ sec},$$

$$K_p = K_i T_i \simeq 1.6542.$$
As such, the PID controller
\[
C_{\text{PID}} = 1.6542 \left(1 + \frac{1}{0.3308s} + 0.4496s \right)
\]
solves the control problem. However, since in this case \(T_i < 4T_d\), the zeros of the compensator are complex conjugate and equal to \(-1.1122 \pm 2.3423j\). The Nyquist plot of \(\hat{G}(j\omega)\) is shown in Fig. [12]. It can be shown that \(\hat{G}(j\omega)\) is a rotation and an amplification of \(G(j\omega)\) and the area of points that can be brought to \(B\) by the controller is shown in gray.

5 Conclusion

In this paper we presented a design method for all types of standard compensators that are ubiquitously addressed in Control subjects, and which represent the very vast majority of compensators used in industry. This method, based on the so-called Inversion Formulae, enables the synthesis to be carried out precisely and just with the aid of a pen and a piece of paper. This represents the most remarkable value and potential of this method in Control education. In fact, these techniques – that have been employed in several Universities in Italy for several years – do not rely on iterative procedures to be performed on Bode or Nyquist plots, and appear therefore to be very suitable for numerical exercises that can test students’ skills in every single aspect of the compensator design process. In this paper we tried to focus our attention on the most important educational aspects of this technique, emphasizing the links that can be established with the classic diagrams of the frequency response, because we firmly believe that this is a key aspect for a deep understanding of Control synthesis techniques. However, the value of this method lies also in the fact that can be easily adapted to different design scenarios:

- Here for the sake of brevity we restricted our attention to standard PID controllers. However, these techniques are easily adapted to PI and PD controllers, as well as to PID controllers with an additional pole introduced in the derivative action for physical implementability. [9];

- The approach based on the Inversion Formulae can easily be adapted to the discrete-time case as shown in [12];

- Even if in the examples proposed in this paper the transfer function was rational, it is very easy to see that this method can be readily applied to systems with finite delays as well.

The relevance of this method for written exercises has been demonstrated in this paper with a number of Question examples that are extremely difficult to tackle with the standard approaches, and that shed some light on some aspects of the control design that would otherwise remain neglected. In particular, we have proposed several different design exercises aimed at designing the parameters of the compensator (even in closed-form as a further evidence of the simplicity of the method) in the presence of standard specifications on the steady-state performance and stability margins and crossover frequencies. This method can even provide an \emph{a priori} answer to the question if the desired stability margin guarantees asymptotic stability of the closed-loop, by combining the Inversion Formulae with the Routh criterion, while often it is with an \emph{a posteriori} check that in Control courses students are encouraged to ensure a positive margin indeed leads to asymptotic stability.
Appendix A: Relationship between the two Lead-Lag network transfer functions

The relationship between $C_{LL}(s)$ and $C'_{LL}(s)$ can be proved by writing $C_{LL}(s)$ as

$$C_{LL}(s) = \frac{s^2 + \frac{\tau_1 + \tau_2}{\tau_1 \tau_2} s + \frac{1}{\tau_1 \tau_2}}{s^2 + \frac{\alpha^2 \tau_1 + \tau_2}{\alpha \tau_1 \tau_2} s + \frac{1}{\tau_1 \tau_2}}$$

and by comparing this expression with $C'_{LL}(s)$. Expressions (4) immediately follow. This shows that $C_{LL}(s)$ can always be written as $C'_{LL}(s)$. To prove the opposite implication, we solve (4) in $\alpha$, $\tau_1$ and $\tau_2$. Solving the third of (4) gives

$$\tau_1 = 1/(\tau_2 \omega_n^2).$$

Plugging this into the first of (4) gives

$$\zeta_1 = \frac{1 + \tau_2^2 \omega_n^2}{2 \tau_2 \omega_n},$$

which leads to the equation in $\tau_2$:

$$\tau_2^2 \omega_n^2 - 2 \zeta_1 \tau_2 \omega_n + 1 = 0,$$

whose solutions are $\tau_2^\pm = \frac{\zeta_1^\pm}{\omega_n}$. To find the first set of solutions, let us first consider $\tau_2 = \tau_2^+$, and we plug it into (36) to get $\tau_1 = \frac{\zeta_1^1}{\omega_n}$. We plug these expressions in the second of (4) and we get

$$\alpha^2 \zeta_1^2 - 2 \zeta_2 \alpha + \zeta_1^+ = 0$$

which gives $\alpha = \frac{\zeta_2^+}{\zeta_1^1}$. As such, the first solution yields $\tau_2 = \frac{\zeta_1^1}{\omega_n}$, $\tau_1 = \frac{\zeta_1^1}{\omega_n}$ and $\alpha = \frac{\zeta_2^+}{\zeta_1^1}$. These two solutions both lead to $\tau_1 > 0$ and $\tau_2 > 0$. However, in the case $\zeta_2 > \zeta_1 > 1$, the only solution that gives $\alpha \in (0, 1)$ is $\tau_2 = \frac{\zeta_1^1}{\omega_n}$, $\tau_1 = \frac{\zeta_1^1}{\omega_n}$ and $\alpha = \frac{\zeta_2^+}{\zeta_1^1}$. As one can see with simple irrational inequalities. When $\zeta_1 > \zeta_2 > 1$, both solutions lead to $\alpha \in (0, 1)$, and are therefore both feasible. The second set of solutions is found by picking $\tau_2 = \tau_2^-$ . By following the same steps, we get $\tau_2 = \frac{\zeta_1^-}{\omega_n}$, $\tau_1 = \frac{\zeta_1^-}{\omega_n}$ and $\alpha = \frac{\zeta_2^+}{\zeta_1^-}$. When $\zeta_1 > \zeta_2 > 1$, $\alpha$ is always greater than 1. When $\zeta_2 > \zeta_1 > 1$, the only solution that yields $\alpha \in (0, 1)$ is $\tau_2 = \frac{\zeta_1^1}{\omega_n}$, $\tau_1 = \frac{\zeta_1^-}{\omega_n}$ and $\alpha = \frac{\zeta_2^-}{\zeta_1^-}$. 

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