SELF-DUAL CODES BETTER THAN THE GILBERT–VARSHAMOV BOUND

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Abstract. We show that every self-orthogonal code over \( \mathbb{F}_q \) of length \( n \) can be extended to a self-dual code, if there exists self-dual codes of length \( n \). Using a family of Galois towers of algebraic function fields we show that over any nonprime field \( \mathbb{F}_q \), with \( q \geq 64 \), except possibly \( q = 125 \), there are self-dual codes better than the asymptotic Gilbert–Varshamov bound.

1. Introduction

Let \( \mathbb{F}_q \) be the finite field of cardinality \( q \) where \( q \) is a power of some prime number \( p \). We mean by a code \( C \) over \( \mathbb{F}_q \) always a linear code; i.e., \( C \) is a linear subspace of the \( n \)-dimensional vector space \( \mathbb{F}_q^n \). The number \( n \) is called the length of \( C \), and the dimension \( k \) of \( C \) as an \( \mathbb{F}_q \)-vector space is called the dimension of \( C \). The weight of an element \( x = (x_1, \ldots, x_n) \in \mathbb{F}_q^n \) is defined as

\[
\text{wt}(x) = |\{i : x_i \neq 0\}|
\]

The minimum distance \( d \) of a code \( C \) is defined as

\[
d = \min_{0 \neq x \in C} \text{wt}(x).
\]

The space \( \mathbb{F}_q^n \) is equipped with the standard symmetric bilinear form \( \langle \cdot, \cdot \rangle \), defined by

\[
\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i
\]

for \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{F}_q^n \). Clearly this bilinear form is non-degenerate; i.e., for every \( 0 \neq x \in \mathbb{F}_q^n \) there is some \( y \in \mathbb{F}_q^n \) such that \( \langle x, y \rangle \neq 0 \). Two vectors \( u, v \in \mathbb{F}_q^n \) are called orthogonal if \( \langle u, v \rangle = 0 \). In this case we also write \( u \perp v \). If \( C \subseteq \mathbb{F}_q^n \) is a code then the set

\[
C^\perp := \{ y \in \mathbb{F}_q^n \mid y \perp x \text{ for all } x \in C \}
\]

A.B. was supported by the BAGEP Award of the Science Academy with funding supplied by Mehves Demiren in memory of Selim Demiren and TÜBİTAK Proj. 112T233. H.S. was supported by TÜBİTAK Proj. 114F432.
is also a linear subspace of $\mathbb{F}_q^n$; it is called the dual code of $C$. It is clear from linear algebra that

$$\dim C + \dim C^\perp = n.$$  \hfill (1.1)

A code $C$ is called self-orthogonal if $C \subseteq C^\perp$. It is called self-dual if $C = C^\perp$. An $[n,k,d]$-code $C$ is a code of length $n$, dimension $k$ and minimum distance $d$. The ratios

$$R(C) := k/n \quad \text{and} \quad \delta(C) := d/n$$

are called the rate and the relative minimum distance of $C$, resp. It is clear from Equation (1.1) that the rate of a self-orthogonal code $C$ satisfies $R(C) \leq 1/2$; a self-dual code has rate $R(C) = 1/2$.

It has been known for a long time that the class of self-dual codes over $\mathbb{F}_q$ is asymptotically good and it attains the Gilbert–Varshamov bound $[2, 3]$. This means: there exists a sequence $(C_i)_{i \geq 0}$ of self-dual codes over $\mathbb{F}_q$ with length $n_i \to \infty$ such that the limit $\delta := \lim_{i \to \infty} \delta(C_i)$ exists and the point $(\delta, R) \in \mathbb{R}^2$ with $R = 1/2$ lies on or above the Gilbert–Varshamov bound

$$R \geq 1 - H_q(\delta).$$  \hfill (1.2)

Here $H_q(\delta)$ denotes the $q$-ary entropy function, defined by $H_q(0) = 0$ and

$$H_q(\delta) = \delta \log_q(q - 1) - \delta \log_q(\delta) - (1 - \delta) \log_q(1 - \delta)$$

for $0 < \delta \leq 1 - q^{-1}$.

For $q = \ell^2$ a square, the bound (1.2) was improved in $[9]$. More precisely it was shown that the class of self-dual codes attains the Tsfasman–Vladut–Zink bound; i.e., there is a sequence of self-dual codes $(C_i)_{i \geq 0}$ over $\mathbb{F}_q$ with parameters $[n_i, n_i/2, d_i]$ with $n_i \to \infty$ and

$$\liminf_{i \to \infty} \frac{d_i}{n_i} \geq \frac{1}{2} - \frac{1}{\ell - 1}.$$  \hfill (1.3)

Our aim is to extend (1.3) to all nonprime finite fields and hence to improve the bound (1.2) for almost all nonprime values of $q$ (all, except $q \leq 49$ and $q = 125$).

Our proof relies on the use of specific towers of algebraic function fields having many rational places. These towers will allow us to construct self-orthogonal algebraic geometry codes $C_i$ of increasing length whose dual codes $C_i^\perp$ have a large minimum distance. We will then show that there are self-dual codes $\tilde{C}_i$ with $C_i \subseteq \tilde{C}_i \subseteq C_i^\perp$ whose relative minimum distance satisfies the corresponding Tsfasman-Vladut-Zink bound

$$\liminf_{i \to \infty} \delta(\tilde{C}_i) \geq \frac{1}{2} - \frac{1}{2} \left( \frac{1}{\ell^{\lceil r/2 \rceil} - 1} + \frac{1}{\ell^{\lfloor r/2 \rfloor} - 1} \right)$$

for $q = \ell^r$, $r > 1$ odd.
Taking \( r = 2 \) in (1.4), we recover (1.3). Together, (1.3) and (1.4) give the following result.

**Theorem 1.1.** For any nonprime finite field \( \mathbb{F}_q \) with \( q \geq 64 \), except possibly \( q = 125 \), there are self-dual codes over \( \mathbb{F}_q \) better than the Gilbert–Varshamov bound.

2. Embedding Self-Orthogonal Codes into Self-Dual Codes

In this section we show that every self-orthogonal code \( C \subseteq \mathbb{F}_n^q \) can be extended to a self-dual code \( \tilde{C} \subseteq \mathbb{F}_n^q \), if at least one self-dual code exists in \( \mathbb{F}_n^q \). The results in this section should be classically known in the theory of quadratic spaces over finite fields and finite geometries. In particular, Lemma 2.2 follows immediately from the classification of quadratic spaces over finite fields by dimension and discriminant and can be found in \[5, Theorem 1\], and results along the lines of Theorem 2.1 for \( q = 2 \) can be found in \[2\], among possibly others. Since we could not find a good reference in this form and generality, we give proofs of these results below.

**Theorem 2.1.** Let \( C \subseteq \mathbb{F}_n^q \) be a self-orthogonal code over \( \mathbb{F}_q \) of length \( n \). Assume that

\[ (*) \ n \text{ is even and, in case } q \equiv 3 \pmod{4}, \ n \text{ is a multiple of 4.} \]

Then there exists a self-dual code \( \tilde{C} \subseteq \mathbb{F}_n^q \) such that \( C \subseteq \tilde{C} \).

Condition \( (*) \) above is necessary and sufficient for the existence of a self-dual code of length \( n \) over \( \mathbb{F}_q \). Necessity can be seen easily by using the discriminant of the bilinear form \( \langle \cdot , \cdot \rangle \), but we do not need it here. To show sufficiency (to show that under the above condition \( (*) \) at least one self-dual code over \( \mathbb{F}_q \) of length \( n \) exists) constitutes the first step for the proof of Theorem 2.1.

**Lemma 2.2.** Assume that \( n \) is even and, in case \( q \equiv 3 \pmod{4} \), \( n \) is a multiple of 4. Then there exists a self-dual code \( E \subseteq \mathbb{F}_n^q \).

**Proof.** First we consider case that \( q \) is even or \( q \equiv 1 \pmod{4} \) and \( n = 2m \) is even. If \( q \) is even, let \( \alpha = 1 \). If \( q \equiv 1 \pmod{4} \), since the multiplicative group \( \mathbb{F}_q^\times \) is cyclic of order \( q - 1 \equiv 0 \pmod{4} \), there is an element \( \alpha \in \mathbb{F}_q^\times \) such that \( \alpha^2 = -1 \). We consider the vectors

\[ c_1 = (\alpha, 1, 0, 0, \ldots, 0, 0), \quad c_2 = (0, 0, \alpha, 1, \ldots, 0, 0), \quad \ldots, \quad c_m = (0, 0, \ldots, 0, 0, \alpha, 1). \]

These vectors span an \( m \)-dimensional subspace \( E \subseteq \mathbb{F}_n^q \) which is obviously self-dual.

Next we consider the case \( q \equiv 3 \pmod{4} \) and \( n = 4k \). Since \( \mathbb{F}_q^\times \) has order \( q - 1 \equiv 2 \pmod{4} \), the element \( -1 \in \mathbb{F}_q \) is not a square. So the set

\[ A := \{1 + \alpha^2 \mid \alpha \in \mathbb{F}_q\} \subseteq \mathbb{F}_q^\times \]
Then \( \phi \) exists a self-dual code \( E \) where
\[
\phi \left( \begin{array}{c} \alpha, \beta, 0, 0, 0, 0, 0, \ldots, 0, 0, 0, 0, 0, 0, 0 \end{array} \right),
\]
and
\[
\phi \left( \begin{array}{c} 0, 0, 0, 0, 0, 0, 0, \ldots, 0, 0, 0, 0, 0, 0, 0, 0 \end{array} \right).
\]

Now the vectors \( c_1, d_1, \ldots, c_k, d_k \) span a self-dual code \( E \subseteq \mathbb{F}_q^n \).

In order to prove Theorem 2.1 we use Witt’s Theorem which holds in a more general setting as follows. Let \( K \) be an arbitrary field and let \( V \) be a vector space over \( K \). Let \( s : V \times V \rightarrow K \) be a symmetric bilinear form, and let \( W \subseteq V \) be a subspace of \( V \). An injective linear map \( \varphi : W \rightarrow V \) is called an isometry from \( W \) to \( V \) if \( \langle \varphi(w_1), \varphi(w_2) \rangle = \langle w_1, w_2 \rangle \) holds for all \( w_1, w_2 \in W \). Now we can state Witt’s Theorem.

**Theorem 2.3** (Witt [6]). Let \( V \) be a vector space over \( K \) with \( \dim V = n < \infty \), where \( K \) is a field of characteristic \( \text{char } K \neq 2 \). Let \( s \) be a non-degenerate symmetric bilinear form on \( V \) and let \( W \subseteq V \) be a subspace of \( V \). Assume that \( \varphi : W \rightarrow V \) is an isometry. Then \( \varphi \) can be extended to an isometry \( \tilde{\varphi} : V \rightarrow V \); i.e., \( \varphi \) is the restriction of \( \tilde{\varphi} \) to \( W \).

Pless has shown that under certain conditions, an analog of Witt’s Theorem holds in characteristic 2. In the particular case, where \( K \) a finite field of characteristic 2, it gives the following:

**Theorem 2.4** (Pless [4]). Let \( 1 = (1, 1, \ldots, 1) \). Consider the setting of Theorem 2.3, where \( K \) is a finite field of characteristic 2. Assume moreover that the following holds: If \( 1 \in W \), then \( \varphi(1) = 1 \). Otherwise, if \( 1 \notin W \), then \( 1 \notin \varphi(W) \). Then the conclusion of Theorem 2.3 holds.

**Proof of Theorem 2.1** We are given a self-orthogonal code \( C \subseteq \mathbb{F}_q^n \). By Lemma 2.2 there exists a self-dual code \( E \subseteq \mathbb{F}_q^n \). As \( \dim C \leq \dim E \), there is an injective linear map \( \varphi : C \rightarrow E \) (if \( q \) is even, choose \( \varphi \) so that the condition in Theorem 2.1 is satisfied). Since both codes \( C \) and \( E \) are self-orthogonal, it follows that \( \varphi \) is in fact an isometry, and we can extend \( \varphi \) to an isometry \( \tilde{\varphi} : \mathbb{F}_q^n \rightarrow \mathbb{F}_q^n \) by Witt’s Theorem (or Pless' Theorem if \( q \) is even). Then the space \( \tilde{C} := \tilde{\varphi}^{-1}(E) \) is a self-dual code (as \( \tilde{\varphi}^{-1} \) is an isometry) and it contains \( C \).
3. Self-dual algebraic geometry codes

Let us first fix some notation. For background on the theory of algebraic function fields, we refer to [8]. We will consider function fields $F/\mathbb{F}_q$ where $\mathbb{F}_q$ is the full constant field of $F$. We will denote by

- $g(F)$ the genus of $F$,
- $(x)$ the principal divisor of $0 \neq x \in F$,
- $x(P)$ the value of the function $x \in F$ at the place $P \in \mathbb{P}_F$,
- $\mathbb{P}_F$ the set of places of $F/\mathbb{F}_q$,
- $v_P$ the normalized discrete valuation of $F/\mathbb{F}_q$ associated with the place $P \in \mathbb{P}_F$,
- $N(F)$ the number of places of degree one (rational places) of $F/\mathbb{F}_q$,
- $\text{supp} A$ the support of the divisor $A$ of $F/\mathbb{F}_q$,
- $(\omega)$ the divisor of the differential $\omega \neq 0$,
- $\text{res}_P(\omega)$ the residue of the differential $\omega$ at the place $P \in \mathbb{P}_F$.

For a divisor $A$ of $F/\mathbb{F}_q$ we define the Riemann-Roch space

$$L(A) := \{ x \in F^\times | (x) + A \geq 0 \} \cup \{ 0 \}.$$ 

For a finite separable extension $E$ of $F$ we will denote by

- $\text{Con}_{E/F}(A)$ the conorm of the divisor $A$ of $F$ in $E/F$,
- $\text{Cotr}_{E/F}(\omega)$ the cotrace of the differential $\omega$ of $F$ in $E/F$,
- $\text{Diff}(E/F)$ the different of the extension $E/F$.

A rational place $P \in \mathbb{P}_F$ is said to split completely in the extension $E/F$ if there are $[E:F]$ distinct places of $E$ above $P$.

Let $F/\mathbb{F}_q$ be an algebraic function field of genus $g$ over the finite field $\mathbb{F}_q$. Let $P_1, P_2, \ldots, P_n$ be pairwise different rational places of $F/\mathbb{F}_q$. Put $D = P_1 + P_2 + \ldots + P_n$, and let $G$ be a divisor of $F/\mathbb{F}_q$, such that $\text{supp} G \cap \text{supp} D = \emptyset$. We consider the algebraic geometry code $C_L(G, D)$, which is as usual defined as follows:

$$C_L(G, D) := \{(f(P_1), f(P_2), \ldots, f(P_n)) | f \in L(G)\}.$$ 

It is well known that this is a linear code of length $n$ and minimum distance $d$, with

$$d \geq n - \deg G \quad \text{(if } C_L(G, D) \neq 0).$$

In [7], sufficient criteria for self-duality of algebraic geometry codes are given. In particular, we have the following description of the dual $C_L(G, D)^\perp$ of the code $C_L(G, D)$:

**Theorem 3.1.** Suppose $\omega$ is a differential such that

1. $v_{P_i}(\omega) = -1$, for $i = 1, 2, \ldots, n$,
2. $\text{res}_{P_i}(\omega) = \text{res}_{P_j}(\omega)$ for $1 \leq i, j \leq n$. 


Then we have

\[ \mathcal{C}_L(G, D) = \mathcal{C}_L(D + (\omega) - G, D). \]

**Proof.** See [7]. \(\square\)

**Corollary 3.2.** (in the setting as above) If \(D + (\omega) \geq 2G\) then \(\mathcal{C}_L(G, D) \subseteq \mathcal{C}_L(G, D)^\perp\).

**Proof.**

\[ G \leq D + (\omega) - G \Rightarrow L(G) \subseteq L(D + (\omega) - G) \]
\[ \Rightarrow \mathcal{C}_L(G, D) \subseteq \mathcal{C}_L(D + (\omega) - G, D) = \mathcal{C}_L(G, D)^\perp. \]

\(\square\)

**Corollary 3.3.** (in the setting as above) If \(D + (\omega) = 2G\) then \(\mathcal{C}_L(G, D) = \mathcal{C}_L(G, D)^\perp\); i.e., the code \(\mathcal{C}_L(G, D)\) is self-dual.

If \(v_P((\omega) + D)\) is even for all \(P \in \text{supp}((\omega) + D)\), we can obtain self-dual algebraic geometry codes by taking \(G = \frac{D+(\omega)}{2}\). Otherwise, we can construct self-dual codes using Theorem 2.1 as follows:

We will call a divisor \(A\) even if \(v_p(A)\) is even for all \(P \in \mathbb{P}_F\). For a divisor \(A\) we will denote by \(\lfloor A \rfloor\) (respectively \(\lceil A \rceil\)) the largest (respectively smallest) even divisor \(B\) with \(B \leq A\) (respectively \(B \geq A\)). Clearly \(2A = \lfloor A \rfloor + \lceil A \rceil\).

**Theorem 3.4.** Let \(n\) be an even integer, that is also a multiple of 4 in case \(q \equiv 3 \pmod{4}\). Let \(P_1, P_2, \ldots, P_n\) be pairwise different rational places of \(F/\mathbb{F}_q\), let \(D = P_1 + P_2 + \ldots + P_n\) and let \(\omega\) be a differential, such that

1. \(v_{P_i}(\omega) = -1\), for \(i = 1, 2, \ldots, n\),
2. \(\text{res}_{P_i}(\omega) = \text{res}_{P_j}(\omega)\) for \(1 \leq i, j \leq n\).

Then there exists a self-dual code of length \(n\) and minimum distance \(d\) satisfying

\[ d \geq \frac{\deg(|D + (\omega)|)}{2} - \deg(\omega). \]

**Proof.** Let \(G = \frac{|(\omega) + D|}{2}\). Since \(2G = |(\omega) + D| \leq (\omega) + D\), it follows from Corollary 3.2 that the code \(\mathcal{C}_L(G, D)\) is self-orthogonal. By Theorem 3.1 we have

\[ \mathcal{C}_L(G, D)^\perp = \mathcal{C}_L(D + (\omega) - G, D) = \mathcal{C}_L(\frac{|D + (\omega)|}{2}, D). \]
Hence for the minimum distance of $C_L(G, D)^\perp$ we obtain the estimate
\[
d(C_L(G, D)^\perp) = d(C_L(\frac{[D + (\omega)]}{2}, D)) \geq n - \deg(\frac{[D + (\omega)]}{2})
\]
\[
= \deg(\frac{[D + (\omega)]}{2}) - \deg(\omega)
\]
\[
= \frac{\deg([D + (\omega)])}{2} - \deg(\omega).
\]

By Theorem 2.1 we see that there is a self-dual code $\tilde{C}$ with $C_L(G, D) \subseteq \tilde{C} \subseteq C_L(G, D)^\perp$. From this inclusion we obtain for the minimum distance $d(\tilde{C})$ of $\tilde{C}$
\[
d(\tilde{C}) \geq d(C_L(G, D)^\perp) \geq \frac{\deg([D + (\omega)])}{2} - \deg(\omega).
\]

\[
4. \text{ ASYMPTOTICALLY GOOD SELF-DUAL CODES}
\]

The real strength of algebraic geometry codes becomes apparent when considering asymptotic questions, i.e., families of codes of increasing length. The length of an algebraic geometry code is limited by the number of rational places $N(F)$ of the function field $F$. Hence to consider codes of increasing length one is naturally led to work with function fields with many rational places, which will necessarily have large genera. Thus let us briefly recall the notion of a tower of function fields.

A tower $\mathcal{F}$ of function fields over $\mathbb{F}_q$ is an infinite sequence $\mathcal{F} = (F_0, F_1, F_2, \ldots)$ of function fields $F_i/\mathbb{F}_q$, with the following properties:

(1) $F_0 \subseteq F_1 \subseteq F_2 \subseteq \ldots$

(2) The field $\mathbb{F}_q$ is the full constant field of $F_i$, for $i = 0, 1, 2, \ldots$.

(3) For each $i \geq 1$, the extension $F_i/F_{i-1}$ is finite and separable.

(4) $g(F_i) \to \infty$ as $i \to \infty$.

A tower $\mathcal{F} = (F_0, F_1, F_2, \ldots)$ is called a Galois tower, if all extensions $F_i/F_0$ are Galois. For a Galois tower $\mathcal{F} = (F_0, F_1, F_2, \ldots)$, a place $P \in \mathbb{P}_{F_0}$ and a place $Q \in \mathbb{P}_{F_i}$, we will denote by $e_i(P)$ the ramification index $e(Q|P)$ of $Q|P$. Note that since all extensions $F_i/F_0$ are Galois, $e_i(P)$ is well-defined; i.e., does not depend on the chosen place $Q$ of $F_i$ lying over $P$.

We define the genus $\gamma(\mathcal{F}/F_0)$ of $\mathcal{F}$ over $F_0$
\[
\gamma(\mathcal{F}) := \lim_{i \to \infty} \frac{g(F_i)}{[F_i : F_0]}.
\]

It can be shown, that this limit exists (it can be $\infty$).
A place $P$ of $F_0$ is said to be ramified in the tower $\mathcal{F} = (F_0, F_1, F_2, \ldots)$, if the place $P$ is ramified in the extension $F_i/F_0$ for some $i \geq 1$. The set

$$V(\mathcal{F}/F_0) := \{ P \in \mathbb{P}(F_0)|P \text{ is ramified in } F \}$$

is called the ramification locus of $\mathcal{F}$ over $F_0$.

A rational place $P$ of $F_0$ is said to split completely in the tower $\mathcal{F} = (F_0, F_1, F_2, \ldots)$, if the place $P$ splits completely in all extensions $F_i/F_0$.

**Theorem 4.1.** Suppose there exists a tower $\mathcal{F} = (F_0, F_1, \ldots)$ of function fields $F_i/F_q$ satisfying the following conditions:

1. The extension $F_i/F_0$ is Galois for every $i \geq 1$.
2. The ramification locus $V(\mathcal{F}/F_0)$ of the tower is finite. Moreover for any place $P \in V(\mathcal{F}/F_0)$, we have $\lim_{i \to \infty} e_i(P) = \infty$, where $e_i(P)$ denotes the ramification index of the place $P$ in the extension $F_i/F_0$.
3. There exists a differential $\omega$ of $F_0$, such that:

   - $\supp((\omega)) = \{ R_0, R_1, \ldots, R_k \} \cup \{ S_0, S_1, \ldots, S_m \} \subseteq \mathbb{P}(F_0)$,
   - $m > 0$, the places $S_0, S_1, \ldots, S_m$ are rational and split completely in the tower, moreover we have $v_{S_j}(\omega) = -1$ and $\text{res}_{S_j}(\omega) = 1$ for $0 \leq j \leq m$,
   - the places $R_0, R_1, \ldots, R_k$ are ramified in the tower,
4. if $q \equiv 3 \mod 4$ then $4[ F_r : F_0 ] \cdot m$ for some $r \geq 1$
5. if $q$ is even or $q \equiv 1 \mod 4$ then $2[ F_r : F_0 ] \cdot m$ for some $r \geq 1$

Then there exists a sequence $(C_i)_{i \geq 0}$ of self-dual codes over $\mathbb{F}_q$, such that

$$n(C_i) \to \infty \quad \text{and} \quad \liminf_{i \to \infty} \frac{d_i}{n_i} \geq \frac{1}{2} - \frac{\gamma(\mathcal{F}/F_0)}{m},$$

where

$$\gamma(\mathcal{F}/F_0) = \lim_{i \to \infty} \frac{g(F_i)}{[ F_i : F_0 ]}$$

denotes the genus of the tower $\mathcal{F}$.

**Proof.** For $i \geq 0$ consider the differential $\omega_i = \text{Cotr}_{F_i/F_0}(\omega)$ of $F_i$. Let $D_i = \text{Con}_{F_i/F_0}(S_0 + S_1 + \ldots + S_m)$. Since for $0 \leq j \leq m$ the place $S_j$ splits completely in the tower and since $v_{S_j}(\omega) = -1$ and $\text{res}_{S_j}(\omega) = 1$, we have $v_Q(\omega_i) = -1$ and $\text{res}_Q(\omega_i) = 1$ for any $Q \in \supp(D_i)$. Without loss of generality we can assume that $i \geq r$. Since $|\supp(D_i)| = m \cdot [ F_i : F_0 ]$ it follows hence by Theorem 3.4 that there exists a self-dual code $C_i$ of length $n_i = m \cdot [ F_i : F_0 ]$ and minimum distance $d_i$ satisfying

$$d_i \geq \frac{\deg([D_i + (\omega_i)])}{2} - \deg(\omega_i).$$
Next we want to estimate \( \deg([D_i + (\omega_i)]\). Let \( T := \text{supp}(D_i + (\omega_i)) \). Clearly we have

\[
\deg([D_i + (\omega_i)]) \geq \deg(D_i + (\omega_i)) - \sum_{Q \in T} \deg Q.
\]

Since \((\omega_i) = \text{Con}_{F_i/F_0}(\omega) + \text{Diff}(F_i/F_0)\), it follows that every place in the support of \(D_i + (\omega_i)\) lies over a place in the ramification locus \(V(F/F_0)\) of \(F\) (which is finite!). Hence

\[
\sum_{Q \in T} \deg Q \leq \sum_{P \in V(F/F_0)} \frac{[F_i : F_0]}{e_i(P)} \deg P.
\]

Hence

\[
d_i \geq \frac{\deg([D_i + (\omega_i)])}{2} - \deg(\omega_i)
\]

\[
\geq \frac{\deg(D_i + (\omega_i)) - \sum_{Q \in T} \deg Q}{2} - \deg(\omega_i)
\]

\[
\geq \left( \deg(D_i) - \deg(\omega_i) - \sum_{P \in V(F/F_0)} \frac{[F_i : F_0]}{e_i(P)} \deg P \right)/2.
\]

Dividing by \(n_i = m \cdot [F_i : F_0] = \deg(D_i)\) and using \(\deg(\omega_i) = 2g(F_i) - 2\), we obtain

\[
\frac{d_i}{n_i} \geq \frac{1}{2} - \frac{g(F_i)}{m \cdot [F_i : F_0]} + \frac{1}{m \cdot [F_i : F_0]} - \left( \sum_{P \in V(F/F_0)} \frac{1}{e_i(P)} \deg P \right)/(2m).
\]

Letting \(i \to \infty\) and noting that for all \(P \in V(F/F_0)\) we have \(\lim_{i \to \infty} e_i(P) = \infty\), we obtain the desired result. \(\square\)

Galois towers over all non-prime finite fields were constructed in [1]. In particular in [1, Theorem 1.1] it is shown that for \(\ell\) a prime power, \(q = \ell^r\) with \(r > 1\) odd there exists a tower \(F = (F_0, F_1, F_2, \ldots)\) of functions fields over \(\mathbb{F}_q\), satisfying the conditions in Theorem 4.1 with \(m = 1\) and

\[
\gamma(F/F_0) \geq \frac{1}{2} \left( \frac{1}{\ell^{(r-1)/2} - 1} + \frac{1}{\ell^{(r+1)/2} - 1} \right).
\]

Note that in the corresponding tower the field \(F_0 = \mathbb{F}_q(z)\) is a rational function field and we take

\[
\omega := \frac{dz}{z - 1}.
\]

Then all conditions of Theorem 4.1 are easily verified.

**Remark 4.2.** Let \(q\) be a prime power, that is not a prime, with \(q \geq 64\) and \(q \neq 125\). Let \(\delta_0\) be such that \(1 - H_q(\delta_0) = 1/2\). Then there is a prime power \(\ell\) and an integer \(r > 1\) such that \(q = \ell^r\) and

\[
\delta_0 < \frac{1}{2} - \frac{1}{2} \left( \frac{1}{\ell^{[r/2]} - 1} + \frac{1}{\ell^{[r/2]} - 1} \right) := \delta_1.
\]
Proof of Remark 4.2. Since $1 - H_q(\delta)$ is a strictly decreasing function, it is sufficient to show that

(4.2) $1 - H_q(\delta_1) < 1/2$.

Let

$$\epsilon = \frac{1}{2} \left( \frac{1}{\ell \lceil r/2 \rceil - 1} + \frac{1}{\ell \lfloor r/2 \rfloor - 1} \right).$$

Using the Taylor series expansion of $\log_q(1 + x)$, we see that

$$\log_q(q - 1) > 1 - \frac{1}{\ln(q)} \cdot \frac{1}{q - 1}, \quad \log_q(\frac{1}{2} - \epsilon) < \frac{1}{\ln(q)} \cdot \left( -\frac{1}{2} - \epsilon \right), \quad \log_q(1 - (\frac{1}{2} - \epsilon)) < \frac{1}{\ln(q)} \cdot \left( -\frac{1}{2} + \epsilon \right).$$

Hence to show Inequality (4.2) it is sufficient to show that

$$\left( \frac{1}{2} - \epsilon \right) \left[ 1 - \frac{1}{\ln(q)} \cdot \frac{1}{q - 1} + \frac{1}{\ln(q)} \cdot (1 + 2\epsilon) \right] > \frac{1}{2}.$$

Noting that $1/\epsilon$ is the harmonic mean of $\ell \lceil r/2 \rceil - 1$ and $\ell \lfloor r/2 \rfloor - 1$ and therefore $2\epsilon \geq 1/(q-1)$, it is enough to show that

$$\frac{1}{\epsilon} > 2 + 2 \ln(q).$$

For the same reason, $1/\epsilon \geq \ell \lceil r/2 \rceil - 1$, so it suffices to show $\ell \lceil r/2 \rceil > 3 + 2 \ln(l^r)$. This inequality is easily checked for $\ell \geq 23$ or $r > 7$. Direct calculation in the finitely many remaining cases shows that Inequality (4.2) holds except for $\ell^r \leq 49$, $(l, r) = (5, 3)$ and $(l, r) = (4, 3)$ (it does however hold for $(l, r) = (2, 6)$). The result follows. \hfill \Box

Inequalities (1.3) and (4.1) and Remark 4.2 together yield the following result over all nonprime finite fields:

**Theorem 4.3.** Let $q = \ell^r$ with $r > 1$. There exists a sequence $(C_i)_{i \geq 0}$ of self-dual codes over $\mathbb{F}_q$ having parameters $[n_i, n_i/2, d_i]$ with $n_i \rightarrow \infty$ and

$$\lim \inf_{i \rightarrow \infty} d_i/n_i \geq \frac{1}{2} - \frac{1}{2} \left( \frac{1}{\ell \lceil r/2 \rceil - 1} + \frac{1}{\ell \lfloor r/2 \rfloor - 1} \right).$$

Hence for all nonprime $q$ with $q \geq 64$ except $q = 125$ there are self-dual codes better than the Gilbert–Varshamov bound.

**Remark 4.4.** This result was obtained for quadratic finite fields in [9]. Although over the field $\mathbb{F}_{49}$ the Tsfasman–Vladut–Zink bound is better than the Gilbert–Varshamov bound on a non-empty interval, this interval does not include codes with $R = 1/2$ (for $q = 49$). Hence our proof works only for $q \geq 64$. 
References

[1] Bassa, A., Beelen, P., Garcia, A., Stichtenoth, H., *Galois Towers over Non-prime Finite Fields*, Acta Arithmetica 164 (2014), 163–179.

[2] F. J. MacWilliams, N. J. A. Sloane, J. G. Thompson, *Good self-dual codes exist*, Discrete Math. vol. 3, 1972, 153–162.

[3] Pless, V., Pierce, J. N., *Self-dual codes over GF(q) satisfy a modified Varshamov-Gilbert bound*, Information and Control 23, 1973, 35–40.

[4] Pless, V., *On Witt’s theorem for nonalternating symmetric bilinear forms over a field of characteristic 2*, Proc. Amer. Math. Soc. 15 1964 979–983.

[5] Pless, V., *On the uniqueness of the Golay codes*, J. Combinatorial Theory 5 1968 215–228.

[6] Serre, J.-P., *A course in arithmetic*, Springer Verlag, New York-Heidelberg, 1973.

[7] Stichtenoth, H., *Self-dual Goppa codes*, J. Pure Appl. Algebra 55, No. 1-2, 1988, 199-211.

[8] Stichtenoth, H., *Algebraic Function Fields and Codes*, Graduate Texts in Mathematics 254, Springer-Verlag, Berlin, 2009.

[9] Stichtenoth, H., *Transitive and self-dual codes attaining the Tsfasman–Vlăduț–Zink bound*, IEEE Trans. Inform. Theory 52, No. 5, 2006, 2218-2224.

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