CONDENSATION OF NON-REVERSIBLE ZERO-RANGE PROCESSES

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Abstract. In this article, we investigate the condensation phenomena for a class of non-reversible zero-range processes on a fixed finite set. By establishing a novel inequality bounding the capacity between two sets, and by developing a robust framework to perform quantitative analysis on the metastability of non-reversible processes, we prove that the condensed site of the corresponding zero-range processes approximately behaves as a Markov chain on the underlying graph whose jump rate is proportional to the capacity with respect to the underlying random walk. The results presented in the current paper complete the generalization of the work of Beltran and Landim [4] on reversible zero-range processes, and that of Landim [22] on totally asymmetric zero-range processes on a one-dimensional discrete torus.

1. Introduction

Metastability is a generic phenomenon that occurs in several of models in probability theory and statistical physics, such as random perturbations of dynamical systems [9, 14, 25], low-temperature ferromagnetic spin systems [6, 11, 12, 28, 29], the stochastic partial differential equations [5], and the system of sticky particles [4, 7, 18, 22]. For an extensive discussion of recent developments in this field, certain monographs [10, 30] can be referred to.

A crucial breakthrough was made by the phenomenal works [8, 9] of Bovier, Eckhoff, Gayrard, and Klein. They connected potential theoretic notions, such as the equilibrium potential and the capacity, and important quantities related to metastable behavior of the system, such as transition time and hitting probability. Based on this connection, they established a robust framework for the quantitative investigation of the metastable behavior of reversible random dynamics. This framework is now called the potential theoretic approach, and has been successfully applied to numerous metastable situations. For a detailed description of this approach, we recommend referring to [10].

Under the presence of multiple metastable valleys, it is natural to describe the successive metastable transition as a limiting Markov chain after suitable time rescaling. Based
on the same level of knowledge of potential theoretic quantities as the potential theoretic approach introduced above, Beltran and Landim [2, 3] developed a framework for obtaining this description, and this approach is now known as the martingale approach. One of the advantages of this approach is that it can be used for successful analyses of condensing phenomena for sticky interacting particle systems such as simple inclusion processes [7, 18] and zero-range processes [4, 22]. The current paper focuses on the latter, i.e., the condensation of sticky zero-range processes. Specifically, we provide a complete generalization of the previous results obtained in [4, 22], based on a novel methodology for non-reversible dynamics.

From the perspectives of either the potential theoretic approach or the martingale approach, investigation of metastability of non-reversible dynamics is far more complicated and challenging than that of reversible dynamics, mainly because of two reasons. First, accurate estimation of the capacity between metastable valleys, which is a crucial step in both approaches, involves taking into consideration the so-called flow structure corresponding to the dynamics. For the reversible case, the estimation of the capacity is carried out via the Dirichlet principle, which expresses the capacity as a minimization problem in a space of functions with a certain boundary condition, and the optimizer is given by the equilibrium potential. For non-reversible processes, the minimization and maximization problems for the capacity have been obtained from [15] and [31], respectively. These principles are now called the Dirichlet–Thomson principles for non-reversible processes. These problems are defined in the space of flows, of which the treatment is far more complicated than that of the space of functions. Furthermore, these problems require all test flows to be divergence-free. Divergence-free flows are delicate objects, and hence this additional restriction is a major source of the technical difficulty of the problem. In spite of this difficulty, several metastability results for non-reversible processes have been provided. In [22], Landim provided a detailed analysis of the condensation of totally asymmetric zero-range process on a one-dimensional discrete torus. This result is the first sharp metastability result for non-reversible dynamics. This work confronts a min–max problem for the capacity directly, instead of relying on the Dirichlet–Thomson principles. For the results based on the Dirichlet–Thomson principles, we recommend referring to [25, 27, 28].

Second, the so-called mean jump rate between metastable valleys is difficult to estimate in the non-reversible case. Mean jump rate is an essential notion in the Markov chain description of metastability, in the spirit of the martingale approach, and thus needs to be estimated precisely. For the reversible case, this mean jump rate is merely obtained based on the capacities between valleys. By contrast, there is no clear relationship between the capacity and mean jump rate in the non-reversible case. In [3], the collapsed chain is introduced as a
Our achievements in the current article can be divided into three parts. First, we provide a generalized version of the Dirichlet and Thomson principles (Theorem 5.3) that is far more convenient to apply in the asymptotic analysis of capacity than the classical principle, as we have removed the divergence-free restriction.

The second achievement is the establishment of a general method to deduce mean jump rate from the capacity estimate. This has been previously addressed in [27], but the methodology described therein also relies on the divergence-free flow. We remove this restriction. Consequently, we can reduce the entire problem to constructions of certain approximating functions and flows. This reduction is thoroughly explained in Section 6 and is believed to be model-independent. The model-dependent part is the construction of approximating objects, and this part requires a deep understanding of the flow structure.

The third achievement is the completion of the dynamical analysis of the condensation of zero-range processes on a fixed finite set. There have been several studies on the condensation of zero-range processes [1, 4, 16, 19, 20, 22, 24]. Condensation is a wide-spread phenomenon and it indicates that a macroscopically significant portion of particles is concentrated on one site with dominating probability. We recommend referring to [13, 22] and the references therein for examples of condensation. Pertaining to the dynamical analysis of condensation of zero-range processes, it has been conjectured within the community that the transition of a condensed site occurs approximately as a Markov chain whose jump rate is proportional to the capacity between two sites for the underlying random walk. This has been confirmed in [4] for the reversible zero-range processes with $\alpha > 2$, and in [22] for the totally asymmetric zero-range processes on the discrete torus with $\alpha > 3$, where $\alpha$ is a parameter that represents the stickiness of constituent particles. In this article, we extend this result for any non-reversible zero-range process on any fixed finite set for all $\alpha > 2$, and finally verify that the conjecture holds for this level of generality. This main result is stated in Theorem 6.3.

Organization of the article. The rest of this paper is organized as follows. We introduce zero-range processes and relevant notations in Section 2 and state the main result regarding the condensation of zero-range processes in Section 3. In Section 4, we introduce adjoint dynamics and prove the sector condition. This sector condition is not a crucial ingredient of the proof, but simplifies the proof remarkably in some scenarios. In Section 5, we review the flow structure and then formulate the generalized Dirichlet–Thomson principle, which is one of the main achievements of the article. In Section 6, we develop the general framework for the quantitative analysis of metastability of non-reversible processes, and prove the main
result stated in Section 3. In Sections 7 and 8 we construct several approximating objects playing a central role in the proof presented in Section 6.

2. Zero-range processes

In this section, the zero-range process and several relevant notions are introduced. Most notations are inspired from [4] and hence similar to those therein. However, different notations for several sets and functions are used.

2.1. Underlying random walk. A zero-range process is a system of interacting particles. Herein, a Markov chain is introduced that describes the underlying movement of the particles. The zero-range interaction mechanism among them is explained in the next subsection.

Let $S$ be a finite set with $|S| = \kappa$, and let $\{X(t) : t \geq 0\}$ be a continuous-time, irreducible Markov chain on $S$, so that the jump rate from a site $x \in S$ to $y \in S$ is given by $r(x, y)$ for some $r : S \times S \to [0, \infty)$. We assume that $r(x, x) = 0$ for all $x \in S$. The invariant measure of Markov chain $X(\cdot)$ is denote by $m(\cdot)$, namely,

$$\sum_{y \in S} m(x) r(x, y) = \sum_{y \in S} m(y) r(y, x) \quad \text{for all } x \in S . \quad (2.1)$$

Let

$$M_* = \max \{m(x) : x \in S\} , \quad S_* = \{x \in S : m(x) = M_*\} \quad \text{and} \quad \kappa_* = |S_*| . \quad (2.2)$$

That is, $S_* \subseteq S$ represents the set of sites with maximum mass, with respect to the invariant measure $m(\cdot)$, and $\kappa_*$ denotes the number of these sites. The normalized mass is defined by

$$m_*(x) = \frac{m(x)}{M_*} \in (0, 1] ; \quad x \in S , \quad (2.3)$$

so that $m_*(x) = 1$ for all $x \in S_*$.

It is assumed that $\kappa_* \geq 2$, so that the zero-range process corresponding to this Markov chain, defined in the next subsection, exhibits the metastable behavior. In particular, in Theorem 3.1, it is observed that under the invariant distribution of the zero-range dynamics, most particles are concentrated at a site of $S_*$, with dominating probability.

For $f : S \to \mathbb{R}$, the generator $L_X$ and the Dirichlet form $D_X(\cdot)$ associated with the Markov chain $X(\cdot)$ can be written as

$$(L_X f)(x) = \sum_{y \in S} r(y, x)(f(y) - f(x)) ; \quad x \in S \quad \text{and}$$

$$D_X(f) = \sum_{x \in S} m(x) f(x)(-L_X f)(x) = \frac{1}{2} \sum_{x, y \in S} m(x) r(x, y) [f(y) - f(x)]^2 ,$$

respectively.
2.2. Zero-range process. A zero-range process is now defined that is an interacting system of $N$ particles, where the particles follow the law of random walk $X(\cdot)$ defined above, but interact through the zero-range interaction.

**Definition of particle systems.** In the study of sticky zero-range process, a parameter $\alpha$ represent the stickyness of constituent particles. In this article, we assume that $\alpha > 2$. Let $a : \mathbb{N} \to \mathbb{R}$ (the convention $\mathbb{N} = \{0, 1, 2, \cdots\}$ is used) be a function defined by

$$a(n) = \begin{cases} 1 & \text{if } n = 0 \\ n^\alpha & \text{if } n \geq 1. \end{cases}$$

Moreover, let $g : \mathbb{N} \to \mathbb{R}$ be a function defined by

$$g(n) = \begin{cases} 0 & \text{if } n = 0 \\ a(n)/a(n - 1) & \text{if } n \geq 1. \end{cases}$$

For $N \in \mathbb{N}$, the set $\mathcal{H}_N \subset \mathbb{N}^S$, representing the set of configuration of $N$ particles on $S$, is defined by

$$\mathcal{H}_N = \{ \eta = (\eta_x)_{x \in S} \in \mathbb{N}^S : \sum_{x \in S} \eta_x = N \}.$$

The zero-range process $\{\eta_N(t) : t \geq 0\}$ is defined as a continuous-time Markov chain on $\mathcal{H}_N$ associated with the generator

$$(\mathcal{L}_N f)(\eta) = \sum_{x,y \in S} g(\eta_x) r(x, y)(f(\sigma^{x,y}\eta) - f(\eta)) ; \eta \in \mathcal{H}_N,$$

for $f : \mathcal{H}_N \to \mathbb{R}$, where $\sigma^{x,y}\eta \in \mathcal{H}_N$ is the configuration obtained from $\eta$ by sending a particle at site $x$ to $y$. More precisely, $\sigma^{x,y}\eta = \eta$ if $\eta_x = 0$, and if $\eta_x \geq 1$, then

$$(\sigma^{x,y}\eta)_z = \begin{cases} \eta_z - 1 & \text{if } z = x \\ \eta_z + 1 & \text{if } z = y \\ \eta_z & \text{otherwise}. \end{cases}$$

For convenience $\sigma^{x,x}$ is regarded as the identity operator. It is not difficult to verify that the zero-range process defined above is irreducible. For $\eta \in \mathcal{H}_N$, let $\mathbb{P}^N_\eta$ be the law of the zero-range process $\eta_N(\cdot)$ starting from $\eta$, and let $\mathbb{E}^N_\eta$ be the corresponding expectation.

In the particle dynamics defined above, each particle interacts only with the particles at the same site through the function $g(\cdot)$. Thus, it is called zero-range process. Moreover, in this model $g(n)$ is a decreasing function for $n \geq 1$, and hence the movement of particles is slowed down as the number of particle at the same site becomes larger. This observation
heuristically explains that the particles are sticky, and this sticky behavior eventually causes their condensation.

**Invariant measure and partition function.** For $\zeta \in \mathbb{N}^S_0$, $S_0 \subseteq S$, let

$$m_\zeta = \prod_{x \in S_0} m_* (x)^{\zeta_x} \quad \text{and} \quad a(\zeta) = \prod_{x \in \mathbb{N}} a(\zeta_x) \ .$$

Then, the unique invariant measure $\mu_N(\cdot)$ on $\mathcal{H}_N$ of the zero-range process defined above is given by

$$\mu_N(\eta) = \frac{N^\alpha \ m_\eta}{Z_N a(\eta)} ; \ \eta \in \mathcal{H}_N ,$$

where $Z_N$ represents the partition function that turns $\mu_N$ into a probability measure, that is,

$$Z_N = N^\alpha \sum_{\eta \in \mathcal{H}_N} \frac{m_\eta}{a(\eta)} .$$

Let

$$\Gamma_x = \sum_{j=0}^{\infty} \frac{m_*(x)^j}{a(j)}$$

for $x \in S$, and

$$\Gamma(\alpha) = \sum_{j=0}^{\infty} \frac{1}{a(j)} ,$$

so that $\Gamma_x = \Gamma(\alpha)$ for $x \in S_*$. The series converge because $\alpha > 2$. Let now a constant be defined by

$$Z = \kappa_* \Gamma(\alpha)^{\kappa_* - 1} \prod_{x \in S \setminus S_*} \Gamma_x .$$

The asymptotic result for the partition function is obtained as follows.

**Proposition 2.1.** We have that

$$\lim_{N \to \infty} Z_N = Z .$$

**Proof.** See [4, Proposition 2.1]. \qed

**Dirichlet form.** For $f, g : \mathcal{H}_N \to \mathbb{R}$, the inner product $\langle f, g \rangle_{\mu_N}$ is defined by

$$\langle f, g \rangle_{\mu_N} = \sum_{\eta \in \mathcal{H}_N} f(\eta) g(\eta) \mu_N(\eta) .$$

Then, for $f : \mathcal{H}_N \to \mathbb{R}$, the Dirichlet form associated with the process $\eta_N(\cdot)$ is defined by

$$\mathcal{D}_N(f) = \langle f, -\mathcal{L}_N f \rangle_{\mu_N} .$$

By summation by parts, the Dirichlet form can be rewritten as

$$\mathcal{D}_N(f) = \frac{1}{2} \sum_{x, y \in S} \mu(\eta) g(\eta_x) r(x, y) [f(\sigma^{x,y} \eta) - f(\eta)]^2 .$$

**2.3. Equilibrium potential and capacity.** Two potential theoretic notions, namely, equilibrium potential and capacity, related to the underlying random walk and the associated
zero-range process introduced above will now be explained. Denote by \( \tau_A \) the hitting times \( \tau_A = \inf \{ t : X(t) \in A \} \).

The hitting time \( \tau_A \) of a set \( A \subset S \) is defined analogously. It should be noted that standard Roman fonts are used for representing subsets or elements of \( S \), and calligraphic fonts for representing subsets of \( H_N \). The configurations in \( H_N \) are denoted by Greek letters.

Let \( P_x, x \in S \), denote the law of the underlying Markov chain \( X(\cdot) \) starting from a site \( x \).

For two disjoint and non-empty sets \( A, B \subset S \), the equilibrium potential \( h_{A,B} : S \rightarrow [0,1] \) for the process \( X(\cdot) \) is defined by

\[
h_{A,B}(x) = P_x[\tau_A < \tau_B] ; \quad x \in S.
\]

It is well known that the equilibrium potential \( h_{A,B} \) satisfies

\[
h_{A,B} \equiv 1 \text{ on } A, \quad h_{A,B} \equiv 0 \text{ on } B, \quad \text{and } L_X h_{A,B} \equiv 0 \text{ on } (A \cup B)^c.
\]

Then, the capacity between \( A \) and \( B \) with respect to the process \( X(\cdot) \) is defined by

\[
cap_X(A, B) = D_X(h_{A,B}).
\]

Thus, by (2.6), the following alternative representation of capacity is obtained:

\[
cap_X(A, B) = -\sum_{x \in A} m(x)(L_X h_{A,B})(x) = \sum_{x \in B} m(x)(L_X h_{A,B})(x).
\]

One can define the equilibrium potential and the capacity for zero-range processes as well. For two disjoint and non-empty sets \( A, B \subset H_N \), the equilibrium potential is defined by

\[
h_{A,B}(\eta) = h_{A,B}^N(\eta) := P_N^\eta[\tau_A < \tau_B] ; \quad \eta \in H_N.
\]

Here and in the following the notation is simplified by dropping the dependency on \( N \). As in (2.6), the equilibrium potential \( h_{A,B} \) satisfies

\[
h_{A,B} \equiv 1 \text{ on } A, \quad h_{A,B} \equiv 0 \text{ on } B, \quad \text{and } L_N h_{A,B} \equiv 0 \text{ on } (A \cup B)^c.
\]

The capacity between \( A \) and \( B \) is defined by

\[
cap_N(A, B) = D_N(h_{A,B}) = -\sum_{\eta \in A} \mu_N(\eta)(L_N h_{A,B})(\eta) = \sum_{\eta \in B} \mu_N(\eta)(L_N h_{A,B})(\eta),
\]

where the last two equalities follow from (2.8).

Finally, it should be remarked that, if a set in the definitions above is a singleton, then the curly brackets will be dropped. For instance, if \( A = \{a\} \) and \( B = \{b\} \), then the notation \( h_{a,b} \) will be used instead of \( h_{\{a\},\{b\}} \).
3. Main result

In this section, the main result for the condensation of non-reversible zero-range processes is presented. This phenomenon can be understood as metastable behavior; hence, metastable valleys around the condensed configurations are first defined in Section 3.1. Then, in Section 3.2, the main result describing the metastable behavior as a limiting Markov chain is presented.

3.1. Metastable valleys.

Auxiliary sequences. Several auxiliary sequences are introduced to concretely describe metastability. For two sequences \((a_N)_{N \in \mathbb{N}}, (b_N)_{N \in \mathbb{N}}\) of positive real numbers, the notation \(a_N \ll b_N\) implies that

\[
\lim_{N \to \infty} \frac{b_N}{a_N} = \infty.
\]

A sequence of positive integers \((\pi_N)_{N \in \mathbb{N}}\) is defined by

\[
\pi_N = \left\lfloor N^{\frac{1}{\alpha} + \frac{1}{2}} \right\rfloor \ll N,
\]

where \([x]\) denotes the largest integer not larger than \(x\).

Let now \((\ell_N)_{N \in \mathbb{N}}\) and \((b_N(z))_{N \in \mathbb{N}}, z \in S \setminus S_\star\), be sequences of positive integer such that

\[
1 \ll \ell_N \ll \pi_N, \quad 1 \ll b_N(z) \quad \text{for all } z \in S \setminus S_\star,
\]

and

\[
\lim_{N \to \infty} \frac{\ell_N^{1+\alpha(\kappa-1)}}{N^{1+\alpha}} \prod_{z \in S \setminus S_\star} m_\star(z)^{-b_N(z)} = 0.
\]  \hfill (3.1)

For instance,

\[
\ell_N = \left\lfloor N^{\frac{1}{\pi_N^{\kappa-1}}} \right\rfloor \quad \text{and} \quad b_N(z) = \left\lfloor \frac{\log N}{-2\kappa \log m_\star(z)} \right\rfloor \quad \text{for } z \in S \setminus S_\star,
\]

satisfy all the assumptions above.

Metastable valleys. For each \(x \in S_\star\), the metastable valley representing the set of configurations such that most particles are condensed at the vertex \(x\) is defined by

\[
\mathcal{E}_N^x = \{ \eta \in \mathcal{H}_N : \eta_x \geq N - \ell_N \text{ and } \eta_z \leq b_N(z) \text{ for all } z \in S \setminus S_\star \}.
\]

For a non-empty set \(A \subseteq S_\star\), define

\[
\mathcal{E}_N(A) = \bigcup_{x \in A} \mathcal{E}_N^x,
\]

and let

\[
\mathcal{E}_N = \mathcal{E}_N(S_\star) \quad \text{and} \quad \Delta_N = \mathcal{H}_N \setminus \mathcal{E}_N.
\]
It should be emphasized that the definitions of the invariant measure as well as the metastable valleys are identical to the reversible zero-range process considered in [4]. Hence the following result on invariant measure is immediate from [4, display (3.2)].

**Theorem 3.1.** The invariant measure $\mu_N(\cdot)$ is concentrated on the valleys defined above, in the sense that

$$\lim_{N \to \infty} \mu_N(\mathcal{E}_N^x) = \frac{1}{\kappa_*} \text{ for all } x \in S_* \text{ and } \lim_{N \to \infty} \mu_N(\Delta_N) = 0.$$  

This theorem explains the static condensation of zero-range process. The main concern here is the dynamical analysis of this condensation behavior. Suppose that almost all particles are condensed at a certain site of $S_*$. Then, after a sufficiently long time, the particles are moved and condensed at another site, and this will be sequentially repeated. This is a type of metastable behavior; hence, its analysis lies on the framework of Beltran and Landim [2, 3].

### 3.2. Condensation of zero-range processes.

The standard method for expressing theing behavior in terms of the convergence to a Markov chain under the presence of multiple metastable valleys is the martingale approach developed in [2, 3] and enhanced in [24]. The main result of this study is explained in the spirit of this approach to metastability. To this end, a projection function $\Psi : \mathcal{H}_N \to S_* \cup \{0\}$ is first defined by

$$\Psi(\eta) = \begin{cases} 
  x & \text{if } x \in \mathcal{E}_N^x, \\
  0 & \text{if } x \in \Delta_N.
\end{cases}$$

A projection of the zero-range process $\eta_N(\cdot)$ is then defined by

$$Y_N(t) = \Psi(\eta_N(t)).$$

It should be noticed that the (non-Markov) process $Y_N(\cdot)$ on $S_* \cup \{0\}$ and represents the valley at which the zero-range process is staying at time $t$. The null state $0$ indicates the state at which the zero-range process does not exhibit condensation. Then, metastability can be represented in terms of the convergence of $Y_N(\cdot)$ to a Markov chain $Y(\cdot)$ on $S_* \cup \{0\}$ defined below.

**Limiting Markov chain describing metastable behavior.** Define a constant by

$$I_\alpha = \int_0^1 u^\alpha (1 - u)^\alpha du. \quad (3.2)$$

The following remark on notation, which is valid throughout the paper, is now in order.

**Notation 3.2.** The notation $u, v \in T$ or $\{u, v\} \subset T$ for some set $T$ automatically implies that $u$ and $v$ are different elements of the set $T$.  

Let \( \{Y(t): t \geq 0\} \) be a Markov chain on \( S \cup \{0\} \), whose jump rate is given by
\[
a(x, y) = \frac{1}{M \Gamma(\alpha)} \cap(x, y); \quad x, y \in S^*,
\]
and \( a(x, y) = 0 \) otherwise. As the capacity is symmetric, it is easy to verify that the invariant measure \( \mu(\cdot) \) of this process is given by
\[
\mu(x) = \begin{cases} 
1/\kappa & \text{if } x \in S^*, \\
0 & \text{if } x = 0.
\end{cases}
\]
The Markov chain \( Y(\cdot) \) is a long-range process in the sense that \( a(x, y) > 0 \) for all \( x, y \in S^* \).

Let \( Q_x \) denote the law of the Markov chain \( Y(\cdot) \) starting at \( x \in S^* \).

**Notation 3.3.** Let \( \{\hat{Y}(t): t \geq 0\} \) denote the Markov chain on \( S^* \) with jump rate \( a(\cdot, \cdot) \). This process is obtained by neglecting the null-state \( 0 \) in \( Y(\cdot) \). Then, one can verify by simple algebra that \( \hat{Y}(\cdot) \) is an irreducible Markov chain and is reversible with respect to the invariant measure \( \mu(\cdot) \) conditioned on \( S^* \).

**Main result.** The main result of this study is the following theorem, which describes the metastable transition of the condensation of the zero-range processes in a precise manner.

For \( t \geq 0 \), define
\[
W_N(t) = Y_N(N^{1+\alpha}t).
\]

**Theorem 3.4.** For all \( x \in S^* \) and for all \( (\eta_N)_{N \in \mathbb{N}} \) such that \( \eta_N \in \mathcal{E}_N^x \) for all \( N \), the finite dimensional distributions of the process \( W_N(\cdot) \) under \( \mathbb{P}_{\eta_N}^N \) converges to that of the law \( Q_x \), as \( N \) tends to infinity.

The proof of this theorem is given in Section 6. It should be stressed that, in the theorem above, the convergence of the finite dimensional distributions can be replaced with that in the soft topology [21].

### 4. Adjoint dynamics and sector condition

For the investigation of the metastability of non-reversible processes, numerous computations are involved with both the original and the adjoint dynamics simultaneously. In particular, the Dirichlet and Thomson principles stated in Theorem 5.2 highlight this fact. Accordingly, the notations related to the adjoint chains and the symmetrized chains of the non-reversible zero-range processes are introduced in Section 4.1. The sector condition for zero-range processes is proved in Section 4.2.
4.1. Adjoint dynamics and symmetrized dynamics. For two sites $x, y \in S$, let

$$r^*(x, y) = r(y, x) m(y)/m(x).$$

The adjoint generators of $L_X$ with respect to $L^2(m)$ is define by, for all $f : S \to \mathbb{R}$,

$$(L^*_X f)(x) = \sum_{y \in S} r^*(x, y) (f(y) - f(x)) ; x \in S.$$  

Analogously, the adjoint generator of $L_N$ with respect to $L^2(\mu_N)$ is defined by, for all $f : H_N \to \mathbb{R}$,

$$(L^*_N f)(\eta) = \sum_{x, y \in S} g(\eta_x, \eta_y) r^*(x, y) (f(\sigma^{x,y} \eta) - f(\eta)) ; \eta \in H_N.$$  

Then, the processes generated by $L^*_X$ and $L^*_N$ are denoted by \{\text{X}^*(t) : t \geq 0\} and \{\text{N}^*(t) : t \geq 0\}, respectively, and are called the adjoint dynamics. It should be noted that the original and adjoint dynamics share the invariant measure and the Dirichlet form.

The equilibrium potentials $h^*_{A,B}, h^*_{A}, B$, and the capacities $\text{cap}^*_X(A, B), \text{cap}^*_N(A, B)$ for these adjoint dynamics are defined as before. It is known from [15, display (2.4)] that although the equilibrium potentials for the original dynamics and the adjoint dynamics are quite different, the corresponding capacities are the same, i.e., $\text{cap}^*_X(A, B) = \text{cap}^*_X(A, B)$ and $\text{cap}^*_N(A, B) = \text{cap}^*_N(A, B)$ for all $A, B \subset S$ and $A, B \subset H_N$.

Another process of interest is the symmetrized zero-range process \{\text{N}^*(t) : t \geq 0\} on $H_N$ with generator $L^*_N = (1/2)(L_N + L^*_N)$. One can verify that this process is reversible with respect to the invariant measure $\mu_N(\cdot)$. Hence, the process $\eta^*_N(\cdot)$ is that considered in [4]. Let $\text{cap}^*_N(\cdot, \cdot)$ denote the capacity with respect to the process $\eta^*_N(\cdot)$.

4.2. Sector condition for the zero-range processes. In [4], several estimates were obtained in the context of reversible zero-range processes, and can be employed in this study using the so-called sector condition for the zero-range process $\eta_N(\cdot)$, which is proved in Proposition 4.12 below. In particular, Corollary 4.3 provides rough estimates of the capacity $\text{cap}_N(\cdot, \cdot)$ via the estimates in [4] for the symmetrized capacity $\text{cap}^*_N(\cdot, \cdot)$. It should be emphasized that for sharp estimates, an entirely new idea is required.

For $u \in S$, let $\omega^u = (\omega^u_x)_{x \in S} \in H_1$ be the configuration with one particle at site $x$, namely,

$$\omega^u_x = \begin{cases} 1 & \text{if } x = u \\ 0 & \text{otherwise.} \end{cases}$$

For $u \in S$ and $\eta \in H_N$, let $\eta + \omega^u \in H_{N+1}$ be the configuration obtained from $\eta$ by adding a particle at site $u$. The configuration $\eta - \omega^u \in H_{N-1}$ can be defined similarly, provided that
\( \eta_u \geq 1 \). Remark that, for \( u \in S \) and \( \eta \in \mathcal{H}_N \) such that \( \eta_u \geq 1 \), we have

\[
\mu_N(\eta) g(\eta_u) = a_N \mu_{N-1}(\eta - \omega^u) m(u),
\]

where \( a_N \) is defined by

\[
a_N = \frac{N^\alpha Z_{N-1}}{(N-1)^\alpha Z_N M^*}.
\]

By Proposition 2.1, it is immediate that

\[
\lim_{N \to \infty} a_N = M^*^{-1}.
\]

**Remark 4.1.** Henceforth, all constants are assumed to depend on the set \( S \), the underlying random walk \( X(\cdot) \), and the parameter \( \alpha \). Later on, dependency on a new parameter \( \epsilon \) will be additionally allowed, and this will be explicitly stated.

**Proposition 4.2** (Sector condition for zero-range processes). There exists a constant \( C_0 > 0 \) such that for all \( f, g : \mathcal{H}_N \to \mathbb{R} \), we have

\[
\langle g, -\mathcal{L}_N f \rangle_{\mu_N} \leq C_0 \mathcal{D}_N(f) \mathcal{D}_N(g).
\]

**Proof.** By (4.1) and the change of variable \( \eta - \omega^x = \zeta \), we have

\[
\langle g, -\mathcal{L}_N f \rangle_{\mu_N} = \sum_{\eta \in \mathcal{H}_N} \sum_{x, y \in S} \mu_N(\eta) g(\eta_u) r(x, y) [f(\eta) - f(\sigma^{x,y}\eta)] g(\eta)
\]

\[
= a_N \sum_{\zeta \in \mathcal{H}_N} \mu_{N-1}(\zeta) A(f, g; \zeta),
\]

where

\[
A(f, g; \zeta) = \sum_{x, y \in S} m(x) r(x, y) [f(\zeta + \omega^x) - f(\zeta + \omega^y)] g(\zeta + \omega^x).
\]

By (2.1),

\[
A(f, f; \zeta) = \frac{1}{2} \sum_{x, y \in S} m(x) r(x, y) [f(\zeta + \omega^x) - f(\zeta + \omega^y)]^2.
\]

Therefore, the Dirichlet form can be rewritten as

\[
\mathcal{D}_N(f) = \frac{a_N}{2} \sum_{\zeta \in \mathcal{H}_{N-1}} \sum_{x, y \in S} \mu_{N-1}(\zeta) m(x) r(x, y) [f(\zeta + \omega^x) - f(\zeta + \omega^y)]^2.
\]

For \( \zeta \in \mathcal{H}_{N-1} \), let

\[
\overline{g}(\zeta) = \frac{1}{\kappa} \sum_{z \in S} g(\zeta + \omega^z),
\]

where \( \kappa = |S| \). By (2.1), it holds that

\[
\sum_{x, y \in S} m(x) r(x, y) [f(\zeta + \omega^x) - f(\zeta + \omega^y)] = 0.
\]
From this identity, it follows that, for all $\lambda > 0$,
\[
|A(f, g; \zeta)| = \left| \sum_{x, y \in S} m(x) r(x, y) [f(\zeta + \omega^x) - f(\zeta + \omega^y)] [g(\zeta + \omega^x) - g(\zeta)] \right|
\]
\[
\leq \frac{\lambda}{2} \sum_{x, y \in S} m(x) r(x, y) [f(\zeta + \omega^x) - f(\zeta + \omega^y)]^2
\]
\[
+ \frac{1}{2\lambda} \sum_{x, y \in S} m(x) r(x, y) [g(\zeta + \omega^x) - g(\zeta)]^2. \tag{4.5}
\]

Let $E \subseteq S \times S$ be defined by $E = \{(x, y) : r(x, y) > 0\}$, and let
\[
C_1 = \min_{(x, y) \in E} m(x) r(x, y), \quad C_2 = \max_{(x, y) \in E} m(x) r(x, y).
\]

To each $u, v \in S$, a canonical path
\[
u = z_1(u, v), z_2(u, v), \ldots, z_{k(u,v)}(u, v) = v
\]
is assigned such that
\[
(z_i(u, v), z_{i+1}(u, v)) \in E \text{ for all } 1 \leq i \leq k(u, v) - 1.
\]

Here, it can be assumed that all $z_i(u, v), 1 \leq i \leq k(u, v)$, are different; hence, $k(u, v) \leq \kappa$. The existence of such a path is ensured by the irreducibility of $X(\cdot)$. Then, the last summation of (4.5) can be bounded above by
\[
C_2(\kappa - 1) \sum_{x \in S} |g(\zeta + \omega^x) - \bar{g}(\zeta)|^2 = \frac{C_2(\kappa - 1)}{\kappa} \sum_{u, v \in S} |g(\zeta + \omega^u) - g(\zeta + \omega^v)|^2. \tag{4.6}
\]

By the Cauchy–Schwarz inequality, and the fact that $k(u, v) \leq \kappa$, the last summation can be bounded above by
\[
(\kappa - 1) \sum_{u, v \in S} \sum_{i=1}^{k(u,v)-1} |g(\zeta + \omega^{z_i(u,v)}) - g(\zeta + \omega^{z_{i+1}(u,v)})|^2
\]
\[
\leq \kappa^2(\kappa - 1) \sum_{(x, y) \in E} |g(\zeta + \omega^x) - g(\zeta + \omega^y)|^2 \tag{4.7}
\]
\[
\leq \frac{\kappa^3}{C_1} \sum_{(x, y) \in E} m(x) r(x, y) [g(\zeta + \omega^x) - g(\zeta + \omega^y)]^2.
\]

By (4.6) and (4.7), there exists a constant $C > 0$ such that
\[
\sum_{x, y \in S} m(x) r(x, y) [g(\zeta + \omega^x) - \bar{g}(\zeta)]^2
\]
\[
\leq C \sum_{x, y \in S} m(x) r(x, y) [g(\zeta + \omega^x) - g(\zeta + \omega^y)]^2. \tag{4.8}
\]
By (4.3), (4.5), and (4.8),
\[
|\langle \mathbf{g}, -\mathcal{L}_N \mathbf{f} \rangle_{\mu_N}| \leq \lambda \mathcal{D}_N(\mathbf{f}) + \frac{C}{2\lambda} \mathcal{D}_N(\mathbf{g}).
\]

The proof can be completed by optimizing over \( \lambda > 0 \).

Henceforth, the constant \( C_0 \) will always be used to denote the constant appearing in Proposition 4.2. The following corollary is immediate from [15, Lemmata 2.5 and 2.6].

**Corollary 4.3.** For any two disjoint, non-empty subsets \( \mathcal{A}, \mathcal{B} \) of \( \mathcal{H}_N \), it holds that
\[
\text{cap}_N^s(\mathcal{A}, \mathcal{B}) \leq \text{cap}_N(\mathcal{A}, \mathcal{B}) \leq C_0 \text{cap}_N^s(\mathcal{A}, \mathcal{B}).
\]

### 5. Generalized Dirichlet-Thomson Principles

The major technical difficulty in the quantitative analysis of metastability, in the spirit of the potential theoretic analysis of Bovier, Eckhoff, Gayrard, and Klein [8, 9] or the martingale approach of Beltran and Landim [2, 3], is the sharp estimates of the capacities between metastable valleys. For reversible processes, the notable observation is that the Dirichlet principle expresses the capacity as the infimum of a variational formula for a class of functions whose minimum is achieved by the equilibrium potential between valleys. Hence, the sharp upper bound of the capacity can be immediately obtained if we are able to find a test function of the Dirichlet principle that accurately approximates the equilibrium potential between valleys. Moreover, the lower bound of capacity can be usually obtained by the dimension reduction technique [8, 9] or by the Thomson principle for reversible Markov chains [26].

The reader is referred to the recent monograph [10] for a comprehensive discussion on this matter.

Recently, a sharp analysis of the capacity for several non-reversible dynamics has also been obtained in [25, 27, 28], based on the Dirichlet principle [15] and Thomson principle [31] for non-reversible dynamics. These principles are stated in Theorem 5.2. As they express the capacity as infimum and supremum, respectively, of certain variational formulas, and the optimizers for these formulas are explicitly known, the same strategy as in the reversible case can be used. However, this program is notoriously complicated because these principles require the divergence-free flow as a test flow. Obtaining a divergence-free flow approximating the optimal flow requires deep intuition about the underlying processes as well as highly complicated computations. In view of this difficulty, one of the main achievement of this study is Theorem 5.3 that removes the divergence-free restriction for the test flow, and in turn allows the use of any flow to bound the capacity from below and above. To explain this new result, the flow structure for the zero-range process is first explained in
Section 5.1. Then, in Section 5.2, the Dirichlet and Thomson principles for non-reversible Markov chains are reviewed and the proposed generalization is developed.

5.1. Flow structure. Herein, the flow structure is interpreted in terms of the non-reversible zero-range processes considered in this study. The reader is referred to [15, 31, 27] for a summary of the general theory in the context of Markov chains and to [25] for diffusion processes.

The flow structure is constructed on a directed graph whose vertex set is \( \mathcal{H}_N \). Two configurations \( \eta, \zeta \in \mathcal{H}_N \) are called adjacent and denoted as \( \eta \sim \zeta \), if \( \zeta \) can be obtained by a legitimate jump of a particle in the configuration \( \eta \) or vice versa. That is, \( \eta \sim \zeta \) if there exists \( \xi \in \mathcal{H}_{N-1} \) and \( x, y \in S \) satisfying \( r(x, y) + r(y, x) > 0 \) such that \( \eta = \xi + \omega^x \) and \( \zeta = \xi + \omega^y \). It should be noted that \( \eta \sim \zeta \) if and only if \( \zeta \sim \eta \). Finally, the set of directed edges is defined by

\[
\mathcal{H}^\circ_N = \{ (\eta, \zeta) \in \mathcal{H}_N \times \mathcal{H}_N : \eta \sim \zeta \} .
\]

It should be remarked that \( (\eta, \zeta) \in \mathcal{H}^\circ_N \) if and only if \( (\zeta, \eta) \in \mathcal{H}^\circ_N \); however, these two elements must be distinguished.

The conductance, adjoint conductance, and symmetrized conductance between \( \eta \) and \( \zeta = \sigma^x:y \eta \) for some \( x, y \) satisfying \( r(x, y) + r(y, x) > 0 \) are defined by

\[
\begin{align*}
c_N(\eta, \zeta) &= \mu_N(\eta) g(\eta_x) r(x, y) , \\
c^*_N(\eta, \zeta) &= \mu_N(\eta) g(\eta_x) r^*(x, y) , \\
c^*_N(\eta, \zeta) &= (1/2) [c_N(\eta, \zeta) + c^*_N(\eta, \zeta)] ,
\end{align*}
\]

respectively. If \( \eta = \xi + \omega^x \) and \( \zeta = \xi + \omega^y \) for some \( \xi \in \mathcal{H}_{N-1} \), then, by (4.1), these conductances can be written as

\[
\begin{align*}
c_N(\eta, \zeta) &= a_N \mu_{N-1}(\xi) m(x) r(x, y) , \\
c^*_N(\eta, \zeta) &= a_N \mu_{N-1}(\xi) m(y) r(y, x) .
\end{align*}
\]

From these expressions, it is apparent that \( c^*_N(\eta, \zeta) = c_N(\zeta, \eta) \) and \( c^*_N(\eta, \zeta) = c^*_N(\zeta, \eta) > 0 \). Thus, \( c^*_N(\cdot, \cdot) \) is a symmetric, positive function on \( \mathcal{H}^\circ_N \).

An anti-symmetric real-valued function on \( \mathcal{H}^\circ_N \) is called a flow, i.e., \( \phi : \mathcal{H}^\circ_N \rightarrow \mathbb{R} \) is a flow if and only if \( \phi(\eta, \zeta) = -\phi(\zeta, \eta) \) for all \( (\eta, \zeta) \in \mathcal{H}^\circ_N \). Let \( \mathcal{F}_N \) denote the set of flows on \( \mathcal{H}^\circ_N \). On this set, an inner product is defined by,

\[
\langle \phi, \psi \rangle = \langle \phi, \psi \rangle_{\mathcal{F}_N} = \frac{1}{2} \sum_{(\eta, \zeta) \in \mathcal{H}^\circ_N} \frac{\phi(\eta, \zeta) \psi(\eta, \zeta)}{c^*_N(\eta, \zeta)} ; \phi, \psi \in \mathcal{F}_N .
\]

The flow norm is defined by \( \| \phi \|^2 = \| \phi \|^2_{\mathcal{F}_N} := \langle \phi, \phi \rangle \) for \( \phi \in \mathcal{F}_N \).
Another important notion related to flows is *divergence*. For each $\eta \in H_N$ and $\phi \in \mathfrak{F}_N$, the divergence of the flow $\phi$ at $\eta$ is defined by

$$(\text{div } \phi)(\eta) := \sum_{\zeta : \eta \sim \zeta} \phi(\eta, \zeta).$$

The divergence of $\phi$ on a set $A \subseteq H_N$ is defined by

$$(\text{div } \phi)(A) = \sum_{\eta \in A} (\text{div } \phi)(\eta).$$

A flow $\phi \in \mathfrak{F}_N$ is called divergence-free at $\eta$ if $(\text{div } \phi)(\eta) = 0$. It is called divergence-free on $A \subseteq H_N$ if $(\text{div } \phi)(\eta) = 0$ for all $\eta \in A$.

For $f : H_N \to \mathbb{R}$ and for $(\eta, \zeta) \in H_N^\otimes$, the objects $\Phi_f = \Phi^N_f$, $\Phi^{*N}_f$, and $\Psi_f = \Psi^N_f$ are defined by

$$\begin{align*}
\Phi_f(\eta, \zeta) &= f(\eta)c_N(\eta, \zeta) - f(\zeta)c_N(\zeta, \eta), \\
\Phi^{*N}_f(\eta, \zeta) &= f(\eta)c_N(\zeta, \eta) - f(\zeta)c_N(\eta, \zeta), \\
\Psi_f(\eta, \zeta) &= c_N(\eta, \zeta) [f(\eta) - f(\zeta)] = (1/2)(\Phi_f + \Phi^{*N}_f)(\eta, \zeta).
\end{align*}$$

(5.4)

It is elementary to verify that these objects, as functions on $H_N^\otimes$, are anti-symmetric; hence, they are flows. The following properties for these flows are well known and will be frequently used later.

**Proposition 5.1.** With notations as above, the followings hold.

1. For all $f : H_N \to \mathbb{R}$ and $\eta \in H_N$,

$$(\text{div } \Phi_f)(\eta) = -\mu_N(\eta)(L_N^* f)(\eta) \quad \text{and} \quad (\text{div } \Phi^{*N}_f)(\eta) = -\mu_N(\eta)(L_N f)(\eta).$$

Therefore, for disjoint non-empty subsets $A, B$ of $H_N$, the flows $\Phi_{h_A, B}$ and $\Phi^{*N}_{h_A, B}$ are divergence-free on $(A \cup B)^c$.

2. For all $f, g : H_N \to \mathbb{R}$,

$$\langle \Psi_f, \Phi_g \rangle = \langle -L_N f, g \rangle_{\mu_N} \quad \text{and} \quad \langle \Psi_f, \Phi^{*N}_g \rangle = \langle -L_N^* f, g \rangle_{\mu_N}.$$

3. For all $f : H_N \to \mathbb{R}$ and $\phi \in \mathfrak{F}_N$,

$$\langle \Psi_f, \phi \rangle = \sum_{\eta \in H_N} f(\eta)(\text{div } \phi)(\eta).$$

4. For all $f : H_N \to \mathbb{R}$, it holds that $||\Psi_f||^2 = \mathcal{D}_N(f)$. Therefore,

$$||\Psi_{h_A, B}||^2 = ||\Psi^{*N}_{h_A, B}||^2 = \text{cap}_N(A, B).$$

**Proof.** The proof follows by elementary algebra. The reader is referred to [31] for the proof. \qed
5.2. **Generalization of the Dirichlet–Thomson principles.** Several classes of functions and flows are defined to explain the Dirichlet and the Thomson principles for non-reversible Markov chains and their generalizations. Fix two disjoint non-empty subsets \( A, B \) of \( \mathcal{H}_N \) and \( a, b \in \mathbb{R} \). In the definitions below, the dependency on \( N \) will be neglected as there is no risk of confusion.

- Let \( C_{a,b}(A, B) \) be the class of real-valued functions \( f \) on \( \mathcal{H}_N \) satisfying \( f|_A \equiv a \) and \( f|_B \equiv b \), i.e.,
  \[
  C_{a,b}(A, B) = \{ f : \mathcal{H}_N \to \mathbb{R} : f(\eta) = a, \ \forall \eta \in A \text{ and } f(\eta) = b, \ \forall \eta \in B \} .
  \]

- Let \( \mathcal{G}_a(A, B) \subset \mathcal{F}_N \) be the set of flows whose divergence on \( A \) is \( a \), i.e.,
  \[
  \mathcal{G}_a(A, B) = \{ \phi \in \mathcal{F}_N : (\text{div } \phi)(A) = a \}
  \]

- Let \( \mathcal{D}\mathcal{G}_a(A, B) \subset \mathcal{G}_a(A, B) \) be the set of divergence-free flows from \( A \) to \( B \) of strength \( a \), i.e.,
  \[
  \mathcal{D}\mathcal{G}_a(A, B) = \{ \phi \in \mathcal{G}_a(A, B) : (\text{div } \phi)(\eta) = 0 \text{ for all } \eta \in (A \cup B)^c \}
  \]

It should be noticed that \( \phi \in \mathcal{D}\mathcal{G}_a(A, B) \) implies that \( (\text{div } \phi)(B) = -a \).

In the following theorem, the Dirichlet and the Thomson principles are stated for non-reversible Markov chains.

**Theorem 5.2.** Let \( A \) and \( B \) be two disjoint and non-empty subsets of \( \mathcal{H}_N \). Then, the capacity between \( A \) and \( B \) satisfies the following variational formulas:

\[
\text{cap}_N(A, B) = \inf_{f \in C_{1,0}(A, B), \phi \in \mathcal{D}\mathcal{G}_0(A, B)} ||\Phi_f - \phi||^2 \tag{5.5}
\]

\[
\text{cap}_N(A, B) = \sup_{g \in C_{0,0}(A, B), \psi \in \mathcal{G}_1(A, B)} \frac{1}{||\Phi_g - \psi||^2} . \tag{5.6}
\]

Furthermore, the unique optimizer of the first variational formula is

\[
(f_0, \phi_0) = \left( \frac{h_{A,B} + h^*_{A,B}}{2}, \frac{\Phi h^*_{A,B} - \Phi h^*_{\mathcal{H},B}}{2} \right) ,
\]

and of the second variational formula is

\[
(g_0, \psi_0) = \left( \frac{h^*_{A,B} - h_{A,B}}{2 \text{cap}_N(A, B)}, \frac{\Phi h^*_{A,B} + \Phi h^*_{\mathcal{H},B}}{2 \text{cap}_N(A, B)} \right) .
\]

In the previous theorem, the first variational formula (5.5) was established in [15] and is called the Dirichlet principle. The second formula (5.6) was developed in [31] and is known as the Thomson principle.
To use Dirichlet principle, a test function $f \in C_{1,0}(A, B)$ and a test flow $\phi \in \mathcal{D}\Phi_0(A, B)$ should be suitably chosen so that the upper bound $\text{cap}_B(A, B) \leq \|\Phi_f - \phi\|^2$ is obtained. The sharpness of this bound is closely related to the fact that $(f, \phi)$ approximates $(f_0, \phi_0)$. Indeed, determining $f \in C_{1,0}(A, B)$ that approximates the optimizer $f_0$ especially when $A$ and $B$ are metastable valleys is not usually a difficult task. As mentioned earlier, the technical obstacle appears in the construction of a test flow $\phi \in \mathcal{D}\Phi_0(A, B)$ properly approximating the optimizing flow $\phi_0$. The requirement that $(\text{div} \, \phi)(\eta) = 0$ for all $\eta \in (A \cup B)^c$ is a severe restriction in usual applications. In particular, the underlying graph for the flow structure, which is $H^*_N$ for the present model, is complicated, and the problem is thus particularly difficult. The following Theorem eliminates this divergence-free restriction and hence widens the range of potential applications.

**Theorem 5.3.** Let $A$ and $B$ be two disjoint and non-empty subsets of $H_N$, and let $\varepsilon$ be any real number.

1. For $f \in C_{1,0}(A, B)$ and $\phi \in \mathcal{G}_{\varepsilon}(A, B)$, we have that
   \[
   \text{cap}_B(A, B) \leq \|\Phi_f - \phi\|^2 + 2\varepsilon + 2 \sum_{\eta \in (A \cup B)^c} h_{A, B}(\eta) (\text{div} \, \phi)(\eta). \tag{5.7}
   \]

2. For $g \in C_{0,0}(A, B)$ and $\psi \in \mathcal{G}_{1+\varepsilon}(A, B)$, we have that
   \[
   \text{cap}_B(A, B) \geq \frac{1}{\|\Phi_f - \phi\|^2} \left[ 1 - 2\varepsilon - 2 \sum_{\eta \in (A \cup B)^c} h_{A, B}(\eta) (\text{div} \, \psi)(\eta) \right]. \tag{5.8}
   \]

**Proof.** By (2) of Proposition 5.1 and the fact that $\mathcal{L}_N h_{A, B} \equiv 0$ on $(A \cup B)^c$, we have
   \[
   \left\langle \Psi_{h_{A, B}}, \Phi_f \right\rangle = \left\langle -\mathcal{L}_N h_{A, B}, f \right\rangle_{\mu_N} = - \sum_{\eta \in A \cup B} \mu_N(\eta) (\mathcal{L}_N h_{A, B})(\eta) f(\eta). \tag{5.9}
   \]

Hence, by (2.9),
   \[
   \left\langle \Psi_{h_{A, B}}, \Phi_f \right\rangle = a \text{cap}_B(A, B) \text{ for all } f \in C_{a,0}(A, B). \tag{5.10}
   \]

Moreover, by (3) of Proposition 5.1
   \[
   \left\langle \Psi_{h_{A, B}}, \phi \right\rangle = b + \sum_{\eta \in (A \cup B)^c} h_{A, B}(\eta) (\text{div} \, \phi)(\eta) \text{ for all } \phi \in \mathcal{G}_b(A, B). \tag{5.11}
   \]

For part (1), let $f \in C_{1,0}(A, B)$ and $\phi \in \mathcal{G}_{\varepsilon}(A, B)$. Then, by (5.9) and (5.10),
   \[
   \left\langle \Phi_f - \phi, \Psi_{h_{A, B}} \right\rangle = \text{cap}_B(A, B) - \left[ \varepsilon + \sum_{\eta \in (A \cup B)^c} h_{A, B}(\eta) (\text{div} \, \phi)(\eta) \right]. \tag{5.11}
   \]

Furthermore, by the Cauchy-Schwarz inequality and (4) of Proposition 5.1
   \[
   \left\langle \Phi_f - \phi, \Psi_{h_{A, B}} \right\rangle \leq \|\Phi_f - \phi\|^2 \left\|\Psi_{h_{A, B}}\right\|^2 = \|\Phi_f - \phi\|^2 \text{cap}_B(A, B). \tag{5.12}
   \]
By (5.11) and (5.12),
\[ \| \Phi_{f} - \phi \|^2 \text{cap}_{N}(A, B) \geq \text{cap}_{N}(A, B)^2 - 2 \text{cap}_{N}(A, B) \left[ \varepsilon + \sum_{\eta \in (A \cup B)^c} h_{A, B}(\eta) \left( \text{div} \phi(\eta) \right) \right]. \]

Thus, part (1) is proved. The proof of part (2) is similar. For \( g \in C_{0,0}(A, B) \) and \( \psi \in \mathcal{S}_{1+\varepsilon}(A, B) \), again by (5.9), (5.10), we have
\[ \langle \Phi_{g} - \psi, \Psi_{h_{A, B}} \rangle = 1 - \varepsilon - \sum_{\eta \in (A \cup B)^c} h_{A, B}(\eta) \left( \text{div} \psi(\eta) \right). \]

Hence, by computations as before, the proof of part (2) is completed. \( \square \)

Based on this theorem, one can prove that \( \text{cap}_{N}(A, B) \approx a_{N} \) for some sequence \( (a_{N})_{N \in \mathbb{N}} \) as follows. The essential part is to determine \( f \in C_{1,0}(A, B) \) and \( \phi \in \mathcal{S}_{\varepsilon N}(A, B) \) where \( \varepsilon_{N} \ll a_{N} \), such that
\[ \sum_{\eta \in (A \cup B)^c} h_{A, B}(\eta) \left( \text{div} \phi(\eta) \right) \ll a_{N} \text{ and } \| \Phi_{f} - \phi \|^2 \approx a_{N}. \]  

(5.13)

Then, by (1) of Theorem 5.3 we obtain \( \text{cap}_{N}(A, B) \approx a_{N} \). In a similar manner, (5.8) is used to obtain \( \text{cap}_{N}(A, B) \geq a_{N} \) and completes the estimate. Moreover, the first condition of (5.13) is valid if
\[ \sum_{\eta \in (A \cup B)^c} |(\text{div} \phi)(\eta)| \ll a_{N}. \]  

(5.14)

This condition usually holds if \( A \) and \( B \) contain all the valleys. Otherwise, (5.14) is not easy to verify, and the summation in (5.13) involving equilibrium potentials should be handled directly. This can be achieved using a general argument presented in Lemma 8.5.

6. Metastability of non-reversible zero-range processes

In this section Theorem 6.3 is proved. Most arguments presented here are not model-dependent; the special feature of zero-range dynamics hardly plays a role. The model-dependent part, which is the construction processes for approximating objects, is postponed to Sections 7 and 8.

6.1. Brief review of the martingale approach to metastability. A summary of the general results obtained in [2, 3, 24] regarding the metastability is first presented. In [2, 3], Beltran and Landim demonstrated that, up to several technical estimates, obtaining the sharp asymptotics for the so-called mean jump rates between metastable valleys is crucial and sufficient for describing the metastable or ing behavior in terms of the convergence to the Markov chain, after a suitable time rescaling. This approach is called the martingale approach to metastability. The mode of convergence for this original work is the soft topology developed in [21]. Recently, Landim, Loulakis and Mourragui in [24] showed that the finite
dimensional convergence can be proved by establishing an additional estimate. For the present model, this estimate corresponds to (6.4) below. In this subsection, these results are briefly summarized in terms of non-reversible zero-range processes.

Theorem 6.1. The following theorem is proved in [3, Theorem 2.1] and in [24, Proposition 1.1].

The trace chain of the zero-range process \( \eta_N(\cdot) \) on the set \( \mathcal{E}_N \) is first defined. For \( t \geq 0 \), let
\[
T^{\mathcal{E}_N}(t) = \int_0^t \mathbf{1}\{\eta_N(s) \in \mathcal{E}_N\} \, ds ,
\]
which represents the amount of time for which the zero-range process stays in one of the valleys up to time \( t \). Let \( S^{\mathcal{E}_N}(t) \) be the generalized inverse of \( T^{\mathcal{E}_N}(t) \), i.e.,
\[
S^{\mathcal{E}_N}(t) = \sup\{s \geq 0 : T^{\mathcal{E}_N}(s) \leq t\} .
\]
The trace chain of \( \eta_N(\cdot) \) on \( \mathcal{E}_N \) is defined by \( \eta^{\mathcal{E}_N}_N(t) = \eta_N(S^{\mathcal{E}_N}(t)) \), \( t \geq 0 \). Then, it is known that \( \eta^{\mathcal{E}_N}_N(\cdot) \) is a Markov chain on \( \mathcal{E}_N \) with stationary measure \( \mu_N(\cdot)/\mu_N(\mathcal{E}_N) \). For two configurations \( \eta, \zeta \in \mathcal{E}_N \), let \( j_N(\eta, \zeta) \) be the jump rate between \( \eta \) and \( \zeta \) for the chain \( \eta^{\mathcal{E}_N}_N(\cdot) \). Finally, for \( x, y \in S_\ast \), the mean jump rate between two valleys \( \mathcal{E}_N^x \) and \( \mathcal{E}_N^y \) is defined by
\[
r_N(x, y) = \frac{1}{\mu_N(\mathcal{E}_N^x)} \sum_{\eta \in \mathcal{E}_N^x} \sum_{\zeta \in \mathcal{E}_N^y} \mu_N(\eta) j_N(\eta, \zeta) .
\]

For each \( x \in S_\ast \), let \( \xi_N^x \in \mathcal{H}_N \) be the configuration such that all particles are concentrated at site \( x \). In addition, define
\[
\tilde{\mathcal{E}}_N^x = \mathcal{E}_N \setminus \mathcal{E}_N^x = \mathcal{E}_N(S_\ast \setminus \{x\}) ; \ x \in S_\ast \\
\tilde{\mathcal{E}}_N^{x,y} = \mathcal{E}_N \setminus (\mathcal{E}_N^x \cup \mathcal{E}_N^y) = \mathcal{E}_N(S_\ast \setminus \{x, y\}) ; \ x, y \in S_\ast .
\]

The following theorem is proved in [3, Theorem 2.1] and in [24, Proposition 1.1].

**Theorem 6.1.** Suppose that
\[
\lim_{N \to \infty} N^{1+\alpha} r_N(x, y) = a(x, y) \quad \text{for all } x, y \in S_\ast ,
\]
(6.1)
\[
\lim_{N \to \infty} \sup_{\eta \in \mathcal{E}_N, \eta \neq \xi_N^x} \frac{\operatorname{cap}_{\mathcal{E}_N}(\tilde{\mathcal{E}}_N^x, \tilde{\mathcal{E}}_N^\eta)}{\operatorname{cap}_{\mathcal{E}_N}(\eta, \xi_N^x)} = 0 \quad \text{for all } x \in S_\ast , \text{ and}
\]
(6.2)
\[
\lim_{N \to \infty} \frac{\mu_N(\Delta_N)}{\mu_N(\mathcal{E}_N^\eta)} = 0 \quad \text{for all } x \in S_\ast .
\]
(6.3)

Then, for all \( x \in S_\ast \) and for all sequences \( (\eta_N)_{N=1}^\infty \) such that \( \eta_N \in \mathcal{E}_N^x \) for all \( N \), the process \( W_N(\cdot) \) under \( \mathbb{P}_N^\eta \) converges to \( Q_x \) with respect to the soft topology developed in [21]. In addition, suppose that
\[
\lim_{\delta \to 0} \lim_{N \to \infty} \sup_{\eta \in \mathcal{E}_N} \sup_{2\delta \leq s \leq 3\delta} \mathbb{P}_N^\eta \left[ \eta_N(N^{1+\alpha} s) \in \Delta_N \right] = 0 \quad \text{for all } x \in S_\ast .
\]
(6.4)
Then, the finite dimensional distributions of the process $W_N(\cdot)$ under $\mathbb{P}_N^\eta$ converge to the those of $Q_x$.

It should be remarked that the conditions (6.1), (6.2) and (6.3) are called (H0), (H1) and (H2), respectively, in [2, 3]. By Theorem 3.1, the condition (6.3) is immediate. The condition (6.2) has been verified in [4, Section 6] for reversible zero-range process, namely,

$$
\lim_{N \to \infty} \sup_{\eta \in E^N_x, \eta \neq x_N} \text{cap}_N^s(E_N^x, \xi_N^x) = 0.
$$

(6.5)

Hence, (6.2) is an immediate consequence of this result and the sector condition, i.e., Corollary 4.3.

The condition (6.4) will now be investigated. The proof is similar to that in [24, Example 4.2].

**Proposition 6.2.** The condition (6.4) holds.

**Proof.** By [24, Lemma 2.4], it suffices to verify that for all $x \in S_*$, we have

$$
\lim_{N \to \infty} \sup_{\eta \in E^N_x, \eta \neq x_N} \mathbb{P}_N^\eta \left[ \tau_{\xi_N^x} > N^{1+\alpha} \delta \right] = 0 \quad \text{for all } \delta > 0 \quad \text{(6.6)}
$$

$$
\lim_{\delta \to 0} \limsup_{N \to \infty} \sup_{\delta < t < 3\delta} \mathbb{P}_N^\xi \left[ \eta_N(N^{1+\alpha} t) \in \Delta_N \right] = 0. \quad \text{(6.7)}
$$

For (6.6), by the Markov inequality and [3, Proposition 6.2] with $g \equiv 1$, we have

$$
\mathbb{P}_N^\eta \left[ \tau_{\xi_N^x} > N^{1+\alpha} \delta \right] \leq \frac{1}{N^{1+\alpha} \delta} \mathbb{E}_N^\eta \left[ \tau_{\xi_N^x} \right] \leq \frac{1}{N^{1+\alpha} \delta} \text{cap}_N^s(\eta, \xi_N^x). \quad \text{(6.8)}
$$

It follows from [1, Theorem 2.2] that

$$
\text{cap}_N^s(E_N^x, \xi_N^x) \geq C N^{-(1+\alpha)}. \quad \text{(6.9)}
$$

By (6.8), (6.9) and Corollary 4.3, it can be concluded that

$$
\mathbb{P}_N^\eta \left[ \tau_{\xi_N^x} > N^{1+\alpha} \delta \right] \leq \frac{C \text{cap}_N^s(E_N^x, \xi_N^x)}{\delta} \text{cap}_N^s(\eta, \xi_N^x). \quad \text{(6.10)}
$$

Hence, the estimate of (6.6) follows from (6.5).

The second requirement (6.7) is now considered. Remark from the definition of $\mu_N$ that we have $\mu_N(\xi_N^x) = Z_N^{-1}$. Hence, for $t > 0$,

$$
\mathbb{P}_N^{\xi_N^x} \left[ \eta_N(N^{1+\alpha} t) \in \Delta_N \right] \leq \frac{\mathbb{P}_N^{\eta} \left[ \eta_N(N^{1+\alpha} t) \in \Delta_N \right]}{\mu_N(\xi_N^x)} = \frac{\mu_N(\Delta_N)}{\mu_N(\xi_N^x)} = Z_N \mu_N(\Delta_N).
$$

Hence, (6.7) is obtained by Proposition 2.1 and Theorem 3.1. \qed
Therefore, the proof of Theorem 3.4 is reduced to the asymptotic estimate (6.1), and this estimate is the core of the entire problem. In particular, this estimate is reduced to the estimate of the capacity between metastable valleys in the reversible case, owing to the following identity for reversible Markov chains:

\[
r_N(x, y) = \frac{1}{2} \left[ \text{cap}_N(E^x_N, \tilde{E}^x_N) + \text{cap}_N(E^y_N, \tilde{E}^y_N) - \text{cap}_N(E^x_N \cup E^y_N, \tilde{E}^{x,y}_N) \right]; \quad x, y \in S_*. \tag{6.10}
\]

Unfortunately, this relation is no longer valid in the non-reversible case; hence, the estimation of \( r_N(x, y) \) becomes a more delicate task. The general strategy developed in [27] can be summarized as follows.

1. Let the mean holding rate be defined by

\[
\lambda_N(x) = \sum_{y \in S \setminus \{x\}} r_N(x, y).
\]

Then, the estimate of \( \lambda_N(x) \) follows from the capacity estimate. More precisely, it is known from [3, display (A.8)] that

\[
\lambda_N(x) = \frac{\text{cap}_N(E^x_N, \tilde{E}^x_N)}{\mu_N(E^x_N)}.
\tag{6.11}
\]

Then, by Theorem 3.1 it suffices to estimate \( \text{cap}_N(E^x_N, \tilde{E}^x_N) \) to obtain the sharp asymptotics of \( \lambda_N(x) \). This estimate of the capacity between valleys is obtained in Corollary 6.4 below.

2. The second step is to compute the sharp asymptotics of \( \frac{r_N(x, y)}{\lambda_N(x)} \) using the so-called collapsed chain introduced in [15]. This can be briefly explained as follows. Fix a point \( x \in S_* \). Then, the collapsed chain is obtained from the original chain \( \eta_N(\cdot) \) by carefully collapsing the set \( E^x_N \) into a single point \( \sigma \). The precise definition and basic properties of the collapsed chain are presented in Section 6.4. If \( \mathbb{P}_\sigma^N \) denotes the law of this collapsed chain starting from \( \sigma \), then it has been proven in [3, Proposition 4.2] that

\[
\frac{r_N(x, y)}{\lambda_N(x)} = \mathbb{P}_\sigma^N \left[ \tau^y_N < \tau^{x,y}_N \right]. \tag{6.12}
\]

The right-hand side of the previous equality can be regarded as the value of the equilibrium potential between \( E^y_N \) and \( \tilde{E}^{x,y}_N \) at the collapsed state \( \sigma \), with respect to the collapsed chain. The estimate of this value is based on the capacity estimate for the collapsed chain, the sector condition of the collapsed chain, and a careful investigation of the relation between the original and the collapsed chain. This argument is explained in detail in Sections 6.4, 6.5, and 6.6.

The proof of Theorem 3.4 based on this strategy is also given in Section 6.6.
6.2. **Capacity estimates.** Herein, the main capacity estimates are provided. To this end, certain potential-theoretic notations for the Markov chain $\hat{Y}(\cdot)$ on $S_*$ of Notation 3.3 should be first introduced. Notice here that the Markov chain $\hat{Y}(\cdot)$ describes the limiting metastable behavior of the present model.

**Limiting Markov chain $\hat{Y}(\cdot)$**. For $f : S_* \to \mathbb{R}$, the generator of the Markov chain $\hat{Y}(\cdot)$ on $S_*$ can be written as

$$
(\mathcal{L}_Y f)(x) = \sum_{y \in S_* \setminus \{x\}} \frac{\text{cap}_X(x, y)}{M_* \Gamma(\alpha) I_\alpha} [f(y) - f(x)] ; \ x \in S_* .
$$

(6.13)

The invariant measure for $\hat{Y}(\cdot)$ is the uniform measure $\mu(\cdot)$ on $S_*$, i.e.,

$$
\mu(x) = 1/\kappa_* \ \text{for all} \ x \in S_* .
$$

Thus, for $f : S_* \to \mathbb{R}$, the Dirichlet form can be written as

$$
\mathcal{D}_Y(f) = \sum_{x \in S_*} \mu(x) f(x) [- (\mathcal{L}_Y f)(x)] = \frac{1}{2} \sum_{x, y \in S_*} \frac{\text{cap}_X(x, y)}{M_* \Gamma(\alpha) I_\alpha} \kappa_* [f(y) - f(x)]^2
$$

Let $\hat{Q}_x$ denote the law of chain $\hat{Y}(\cdot)$ starting from $x \in S_*$. Then, for two disjoint non-empty sets $A, B \subseteq S_*$, the equilibrium potential and capacity between them with respect to the chain $\hat{Y}(\cdot)$ are defined by

$$
\mathfrak{h}_{A,B}(x) = \hat{Q}_x(\tau_A < \tau_B) \ \text{for} \ x \in S_* \ \text{and} \ \text{cap}_Y(A, B) = \mathcal{D}_Y(\mathfrak{h}_{A,B}) ,
$$

(6.14)

respectively.

**Main capacity estimates.** The main capacity estimates are now stated.

**Theorem 6.3.** For disjoint, non-empty subsets $A, B$ of $S_*$, we have that

$$
\lim_{N \to \infty} N^{1+\alpha} \text{cap}_N(\mathcal{E}_N(A), \mathcal{E}_N(B)) = \text{cap}_Y(A, B) .
$$

The proof of this result is given in the next subsection. In addition, if $(A, B)$ is a partition of $S_*$, i.e., $A \cup B = S_*$, the equilibrium potential $\mathfrak{h}_{A,B}$ for $\hat{Y}(\cdot)$ becomes the indicator function on $A$; hence, the following corollary is obtained.

**Corollary 6.4.** Suppose that two disjoint, non-empty subsets $A, B$ of $S_*$ satisfy $A \cup B = S_*$. Then,

$$
\lim_{N \to \infty} N^{1+\alpha} \text{cap}_N(\mathcal{E}_N(A), \mathcal{E}_N(B)) = \frac{1}{M_* \kappa_* \Gamma(\alpha) I_\alpha} \sum_{x \in A, y \in B} \text{cap}_X(x, y) .
$$
6.3. **Approximation of optimal flows and proof of Theorem 6.3.** Herein, the proof of Theorem 6.3 is provided based on the generalized Dirichlet and Thomson principles of Theorem 5.3. Several technical details about the construction of approximations of equilibrium potentials and optimal flows are postponed to Sections 7 and 8.

Another parameter $\epsilon > 0$ that denotes a sufficiently small number is now introduced. In particular, we shall assume that $\epsilon \in (0, \epsilon_0)$ where $\epsilon_0$ is a sufficiently small number to be introduced in Lemma 7.6.

**Notation 6.5.** Henceforth the constant term $C$ will be allowed to depend on this new parameter $\epsilon$. Furthermore, $o_{N}(1)$ is used for representing a term that vanishes as $N$ tends to $\infty$, and for $o_{\epsilon}(1)$ expressing a term that vanishes as $\epsilon$ tends to $0$. It should be noted that the term $o_{N}(1)$ may depend on $\epsilon$, whereas the term $o_{\epsilon}(1)$ does not depend on $N$. These dependencies of the constant term $C$ and the error term $o_{N}(1)$ on the parameter $\epsilon$ do not incur any technical problem, as we always take $N \to \infty$ first and then $\epsilon \to 0$.

Throughout this subsection, let two disjoint non-empty subsets $A$, $B$ of $S_*$ be fixed. In Section 7 for $\epsilon \in (0, \epsilon_0)$ and sufficiently large $N \in \mathbb{N}$, two real-valued functions $V_{A,B} = V_{A,B}^{N,\epsilon}$ and $V_{A,B}^* = V_{A,B}^{*,N,\epsilon}$ on $\mathcal{H}_N$ are constructed that approximate the equilibrium potentials $h_{E^N(A),E^N(B)}$ and $h_{E^N(A),E^N(B)}^*$, respectively. Furthermore, in Section 7.5 the following properties of these approximating objects are verified.

**Proposition 6.6.** For $\epsilon \in (0, \epsilon_0)$ and sufficiently large $N \in \mathbb{N}$, there are two functions $V_{A,B}$ and $V_{A,B}^*$ satisfying the following properties:

1. For all $x \in S_*$ and $\eta \in E_{x}^n$, it holds that $V_{A,B}(\eta) = V_{A,B}^*(\eta) = h_{A,B}(x)$. That is, these functions are constant and equal to $h_{A,B}(x)$ on each valley $E_{x}^n$, $x \in S_*$.

2. It holds that

$$N^{1+\alpha} \mathcal{D}_N(V_{A,B}) \leq (1 + o_N(1) + o_\epsilon(1)) \operatorname{cap}_Y(A, B).$$

The same inequality holds for $\mathcal{D}_N(V_{A,B}^*)$ as well.

The next step is to construct test flow to approximate $\Phi_{h_{E^N(A),E^N(B)}}^*$ and $\Phi_{h_{E^N(A),E^N(B)}^*}$. Of course, the natural candidates are $\Phi_{V_{A,B}}^*$ and $\Phi_{V_{A,B}^*}$. One may expect that Theorem 5.3 may be used to estimate the capacity based on these objects. Unfortunately, a technical issue arises around the saddle tube defined in Section 8 at which the divergence of these flows is not negligible. This problem is resolved by a systematic correction procedure developed in Section 8 which cleans out the non-negligible flow and in turn allows the application of Theorem 5.3. The consequences of this correction procedure can be summarized as follows.

**Proposition 6.7.** For $\epsilon \in (0, \epsilon_0)$ and sufficiently large $N \in \mathbb{N}$, there exists a flow $\Phi_{A,B} = \Phi_{A,B}^{N,\epsilon} \in \mathcal{F}_N$ satisfying the following properties.
(1) The flow $\Phi_{A,B}$ approximates $\Phi_{V_{A,B}}^{*}$ in the sense that

$$\left\| \Phi_{A,B} - \Phi_{V_{A,B}}^{*} \right\|^{2} = (o_{N}(1) + o_{\epsilon}(1)) N^{-(1+\alpha)} .$$

(2) The divergence of $\Phi_{A,B}$ is negligible on $\Delta_{N}$ in the sense that

$$\sum_{\eta \in \Delta_{N}} \left| (\text{div} \, \Phi_{A,B})(\eta) \right| = o_{N}(1) N^{-(1+\alpha)} .$$

(3) The divergence of $\Phi_{A,B}$ is negligible on $E_{x}^{*}, x \in S_{*} \setminus (A \cup B)$, in the sense that

$$(\text{div} \, \Phi_{A,B})(E_{N}^{*}(A)) = (1 + o_{N}(1)) N^{-(1+\alpha)} \text{cap}_{Y}(A, B) \text{ and}$$

$$(\text{div} \, \Phi_{A,B})(E_{N}^{*}(B)) = - (1 + o_{N}(1)) N^{-(1+\alpha)} \text{cap}_{Y}(A, B) .$$

There also exists $\Phi_{A,B}^{*} = \Phi_{A,B}^{*} \in F_{N}$ that approximates $\Phi_{V_{A,B}}^{*}$ and satisfies the four properties above.

In particular, by (2) and (3) of the previous proposition, we have the following estimate that enables the application of the generalized Dirichlet and Thomson principles.

**Lemma 6.8.** We have that

$$\sum_{\eta \in (E_{N}^{*}(A \cup B))^{c}} h_{E_{N}^{*}(A), E_{N}^{*}(B)}(\eta) (\text{div} \, \Phi_{A,B})(\eta) = o_{N}(1) N^{-(1+\alpha)} .$$

**Proof.** The summation on the left-hand side can be divided as

$$\sum_{\eta \in \Delta_{N}} + \sum_{x \notin A \cup B} \sum_{\eta \in E_{N}^{*}} .$$

As $|h_{E_{N}^{*}(A), E_{N}^{*}(B)}| \leq 1$, the absolute value of the first summation is $o_{N}(1) N^{-(1+\alpha)}$ by (2) of Proposition 6.7. The second summation is $o_{N}(1) N^{-(1+\alpha)}$ by the second estimate of (3) of Proposition 6.7. \qed

Theorem 6.3 may now be proved.

**Proof of Theorem 6.3.** The upper bound of the capacity is first considered. Let

$$f = \frac{V_{A,B} + V_{A,B}^{*}}{2} \in C_{1,0}(E_{N}^{*}(A), E_{N}^{*}(B)) \text{ and}$$

$$\phi = \frac{\Phi_{A,B} - \Phi_{A,B}}{2} \in \mathcal{E}_{\alpha_{N}}(E_{N}^{*}(A), E_{N}^{*}(B)) \text{ for some } \alpha_{N} = o_{N}(1) N^{-(1+\alpha)} .$$

(6.15)
Note that $\alpha = o_N(1)N^{-(1+\alpha)}$ follows from part (4) (for $\Phi_{A,B}$ and $\Phi^*_{A,B}$) of Proposition 6.7.

Then, by part (1) of Theorem 5.3 and Lemma 6.8, we have

$$\text{cap}_N(\mathcal{E}_N(A), \mathcal{E}_N(B)) \leq \|\Phi_f - \phi\|^2 + o_N(1)N^{-(1+\alpha)}.$$ (6.16)

Let

$$\Phi_{A,B} = \Phi^*_{V_{A,B}} + \Theta_N \quad \text{and} \quad \Phi^*_{A,B} = \Phi^*_{V_{A,B}} + \Theta^*_N.$$ (6.17)

Then, we have

$$\Phi_f - \phi = \Phi(\mathcal{V}_{A,B} + \mathcal{V}^*_{A,B})/2 - \frac{\Phi^*_{V_{A,B}} - \Phi^*_{V_{A,B}}}{2} + \frac{\Theta_N - \Theta^*_N}{2} = \Psi_{V_{A,B}} + \frac{\Theta_N - \Theta^*_N}{2}.$$ (6.18)

By (1) and (2) of Proposition 6.6,

$$\|\Psi_{V_{A,B}}\|^2 = D_N(\mathcal{V}_{A,B}) \leq (1 + o_N(1) + o_\epsilon(1))N^{-(1+\alpha)}\text{cap}_Y(A, B),$$

$$\left\|\frac{\Theta_N - \Theta^*_N}{2}\right\|^2 = (o_N(1) + o_\epsilon(1))N^{-(1+\alpha)}.$$ (6.19)

Therefore, by (6.18), (6.19) and the triangle inequality,

$$\|\Phi_f - \phi\|^2 \leq (1 + o_N(1) + o_\epsilon(1))N^{-(1+\alpha)}\text{cap}_Y(A, B).$$ (6.20)

By (6.16) and (6.20), we obtain the upper bound

$$\text{cap}_N(\mathcal{E}_N(A), \mathcal{E}_N(B)) \leq (1 + o_N(1) + o_\epsilon(1))N^{-(1+\alpha)}\text{cap}_Y(A, B).$$ (6.21)

To obtain the lower bound, part (2) of Theorem 5.3 is used. To this end, let

$$g = \frac{\mathcal{V}^*_{A,B} - \mathcal{V}_{A,B}}{2N^{-(1+\alpha)}\text{cap}_Y(A, B)} \in C_{0,0}(\mathcal{E}_N(A), \mathcal{E}_N(B)), \quad \text{and}$$

$$\psi = \frac{\Phi^*_{A,B} + \Phi_{A,B}}{2N^{-(1+\alpha)}\text{cap}_Y(A, B)} \in G_{1+o_N(1)}(\mathcal{E}_N(A), \mathcal{E}_N(B)).$$ (6.22)

Then, by Theorem 5.3 and Lemma 6.8, we have

$$\text{cap}_N(\mathcal{E}_N(A), \mathcal{E}_N(B)) \geq \frac{1}{\|\Phi_g - \psi\|^2} (1 + o_N(1) + o_\epsilon(1)).$$ (6.23)

As we can write

$$\Phi_g - \psi = -\frac{1}{N^{-(1+\alpha)}\text{cap}_Y(A, B)} \left[ \Psi_{V_{A,B}} - \frac{\Theta_N + \Theta^*_N}{2} \right],$$

by similar computations as in the upper bound, we obtain

$$\|\Phi_g - \psi\|^2 \leq (1 + o_N(1) + o_\epsilon(1))\frac{1}{N^{-(1+\alpha)}\text{cap}_Y(A, B)}.$$ (6.24)
Combining (6.23) and (6.24), we have
\[ \text{cap}_N(\mathcal{E}_N(A), \mathcal{E}_N(B)) \geq (1 + o_N(1) + o_\epsilon(1)) N^{-1+\alpha} \text{cap}_Y(A, B). \] (6.25)

By the upper bound (6.21) and the lower bound (6.25), the proof is completed. \hfill \Box

By a careful reading of the previous proof, the estimate obtained in Proposition 6.6 can be strengthened as follows.

**Corollary 6.9.** We have that
\[ \mathcal{D}_N(V_{A,B}) = (1 + o_N(1) + o_\epsilon(1)) N^{-1+\alpha} \text{cap}_Y(A, B). \]

### 6.4. Collapsed chain

The importance of the collapsed chain in the context of metastability has been noticed in [15, 3, 27]. The collapsed chain can be regarded as a special case of lumped Markov chain (cf. [10, Section 9.3]). In [15] the collapsed chain was used for establishing the Dirichlet principle for non-reversible Markov chains on countable spaces. In [3], the relation (6.12) between the mean jump rate and the collapsed chain was obtained, opening up the possibility of rigorous investigation of metastability of non-reversible processes. In [27, Section 8], a method of estimating the right-hand side of (6.12) was obtained and applied to a cyclic random walk in a potential field. In this method, the construction of divergence-free flows was assumed, which is not true in the present case. Accordingly, in this study the method is properly modified to obtain the sharp asymptotics of the mean jump rate. To explain this process, certain well known results on the collapsed chains are presented in this section, in the context of the zero-range processes. All the proofs are elementary and given in [27, Section 8.2].

**Definition of collapsed chain.** Let \( x \in S_* \) and \( \overline{H}_N = \left( H_N \setminus \mathcal{E}_N^x \right) \cup \{ o \} \), where \( o \) is a new single point. We can regard \( \overline{H}_N \) as the set obtained from \( H_N \) by collapsing the set \( \mathcal{E}_N^x \) into a single point \( o \). Let \( R_N(\cdot, \cdot) \) be the jump rate of the chain \( \eta_N(\cdot) \), i.e.,
\[ R_N(\eta, \zeta) = \begin{cases} g(\eta_x)r(x, y) & \text{if } \zeta = \sigma^{x,y}\eta \text{ for some } x, y \in S, \\ 0 & \text{otherwise}. \end{cases} \]

The corresponding jump rate on \( \overline{H}_N \) is defined by
\[ \overline{R}_N(\eta, \zeta) = R_N(\eta, \zeta) \text{ for all } \eta, \zeta \in H_N \setminus \mathcal{E}_N^x, \]
\[ \overline{R}_N(\eta, o) = \sum_{\zeta \in \mathcal{E}_N^x} R_N(\eta, \zeta) \text{ for all } \eta \in H_N \setminus \mathcal{E}_N^x, \text{ and} \]
\[ \overline{R}_N(o, \zeta) = \frac{1}{\mu_N(\mathcal{E}_N^x)} \sum_{\eta \in \mathcal{E}_N^x} \mu_N(\eta) R_N(\eta, \zeta) \text{ for all } \zeta \in H_N \setminus \mathcal{E}_N^x. \]
Then, the collapsed chain \( \{ \eta_N(t) : t \geq 0 \} \) is defined as the Markov chain on \( \mathcal{H}_N \) whose jump rate is \( \mathcal{R}_N(\cdot, \cdot) \). Let \( \mathcal{D}_N \) denote the generator of the collapsed chain \( \pi_N(\cdot) \), and let \( \mathcal{D}^s_N \) and \( \mathcal{D}_s^s_N \) denote the generators of the adjoint chain and the symmetrized chain of the collapsed chain, respectively. Let \( \mathcal{D}_N' \) be the Dirichlet form associated with these generators. Denote by \( \mathbb{P}_N^\eta \), \( \eta \in \mathcal{H}_N \), the law of chain \( \pi_N(\cdot) \) starting from \( \eta \).

**Lemma 6.10.** The Markov chain \( \pi_N(\cdot) \) is irreducible on \( \mathcal{H}_N \), and the unique invariant measure \( \mu_N(\cdot) \) is given by

\[
\mu_N(\eta) = \mu_N(\eta) \quad \text{for} \quad \eta \in \mathcal{H}_N \setminus \mathcal{E}_N^x, \quad \text{and} \quad \mu_N(\emptyset) = \mu_N(\mathcal{E}_N^x) .
\]

The proof is based on elementary computations and is left to the reader.

**Flow structure of collapsed chains.** As \( \pi_N(\cdot) \) is a Markov chain on \( \mathcal{H}_N \), the flow structure can be induced, and then potential theory can be developed in the same manner as in Section 5.1. The flow structure of the Markov chain \( \pi_N(\cdot) \) as well as its relation to that of the original chain are summarized now.

The conductance between \( \eta, \zeta \in \mathcal{H}_N \) of the collapsed chain is defined by

\[
c_N(\eta, \zeta) = \mu_N(\eta) \mathcal{R}_N(\eta, \zeta) .
\]

It can be verified that \( c_N(\eta, \zeta) = c_N(\zeta, \eta) \) if \( \eta, \zeta \neq \emptyset \), and that

\[
c_N(\eta, \emptyset) = \sum_{\zeta \in \mathcal{E}_N^x} c_N(\eta, \zeta) \quad \text{and} \quad c_N(\emptyset, \zeta) = \sum_{\eta \in \mathcal{E}_N^x} c_N(\eta, \zeta) .
\]

This linearity is fundamental in the relation between the original and collapsed chain. Define the symmetrized conductance by

\[
c^s_N(\eta, \zeta) = (1/2) \left( c_N(\eta, \zeta) + c_N(\zeta, \eta) \right) ; \eta, \zeta \in \mathcal{H}_N ,
\]

and the edge set by

\[
\mathcal{H}_N^o = \left\{ (\eta, \zeta) \in \mathcal{H}_N \times \mathcal{H}_N : c^s_N(\eta, \zeta) > 0 \right\} .
\]

Then, flows are defined by an anti-symmetric real-valued function on \( \mathcal{H}_N^o \), and the set of flows is denoted by \( \mathcal{F}_N^o \). The inner product and the norm on the flow structure can be defined in the same manner as before and are denoted by \( \langle \cdot, \cdot \rangle_c \) and \( ||\cdot||_c \), respectively. The divergence of a flow is also defined similarly.

**Collapsed objects: flows, functions, equilibrium potential, and capacity.** For \( \phi \in \mathcal{F}_N \), the collapsed flow \( \phi \in \mathcal{F}_N \) is defined by

\[
\phi(\eta, \zeta) = \phi(\eta, \zeta) \quad \forall \eta, \zeta \neq \emptyset , \quad \phi(\eta, \emptyset) = \sum_{\zeta \in \mathcal{E}_N^x} \phi(\eta, \zeta) , \text{and} \quad \phi(\emptyset, \zeta) = \sum_{\eta \in \mathcal{E}_N^x} \phi(\eta, \zeta) .
\]
Then, the following results are known.

**Lemma 6.11.** For \( \phi \in \mathfrak{F}_N \), the flow norm of \( \phi \) satisfies \( ||\phi||_C \leq ||\phi|| \), and the equality holds if and only if,

\[
\phi(\eta, \zeta) = 0 \text{ for all } (\eta, \zeta) \in \mathcal{H}_N^\oplus \text{ such that } \eta, \zeta \in \mathcal{E}_N^x \quad (6.27)
\]

\[
\phi(\eta, \zeta) = \frac{\phi(\eta, \zeta')}{c_N^x(\eta, \zeta')} \quad \text{for all } (\eta, \zeta), (\eta', \zeta') \in \mathcal{H}_N^\oplus \text{ such that } \zeta, \zeta' \in \mathcal{E}_N^x. \quad (6.28)
\]

**Proof.** See [27, Lemma 8.2]. □

**Lemma 6.12.** For \( \phi \in \mathfrak{F}_N \), the divergence of \( \phi \) satisfies

\[
(\text{div } \phi)(\eta) = \begin{cases} 
(\text{div } \phi)(\eta) & \text{if } \eta \neq \varnothing, \\
(\text{div } \phi)(\mathcal{E}_N^x) & \text{if } \eta = \varnothing.
\end{cases}
\]

**Proof.** See [27, display (8.7)]. □

The collapse of functions is now considered. Suppose that a function \( f : \mathcal{H}_N \to \mathbb{R} \) satisfies \( f(\eta) = a \) for all \( \eta \in \mathcal{E}_N^x \), for some \( a \in \mathbb{R} \). Then, the collapsed function \( \overline{f} : \overline{\mathcal{H}}_N \to \mathbb{R} \) is defined by

\[
\overline{f}(\eta) = \begin{cases} 
f(\eta) & \text{if } \eta \neq \varnothing, \\
a & \text{if } \eta = \varnothing.
\end{cases}
\]

As in (5.4), for \( g : \overline{\mathcal{H}}_N \to \mathbb{R} \), three flows can be defined by

\[
\overline{\Phi}_g(\eta, \zeta) = g(\eta) \tau_N(\eta, \zeta) - g(\zeta) \tau_N(\zeta, \eta),
\]

\[
\overline{\Phi}^*_g(\eta, \zeta) = g(\eta) \tau_N(\zeta, \eta) - g(\zeta) \tau_N(\eta, \zeta),
\]

\[
\overline{\Psi}_g(\eta, \zeta) = \tau_N^*(\eta, \zeta) [g(\eta) - g(\zeta)].
\]

**Lemma 6.13.** Suppose that a function \( f : \mathcal{H}_N \to \mathbb{R} \) is constant on \( \mathcal{E}_N^x \) so that the collapsed function \( \overline{f} \) can be defined. Then,

\[
\overline{\Phi}_f = \overline{\Phi}^*_f, \quad \overline{\Phi}_f^* = \overline{\Phi}^*_f, \quad \text{and } \overline{\Psi}_f = \overline{\Psi}^*_f,
\]

where \( \overline{\Phi}_f, \overline{\Phi}_f^* \) and \( \overline{\Psi}_f \) represent the collapsed flows of \( \Phi_f, \Phi_f^* \), and \( \Psi_f \), in the sense of (6.26), respectively.

**Proof.** See [27, Lemma 8.3]. □
Remark 6.14. If $A, B \subset H \setminus \mathcal{E}_N^x$, then the equilibrium potential $h_{A,B}$ with respect to the original dynamics can be considered. In this case, $h_{A,B}$ conditioned on $\mathcal{E}_N^x$ may not be a constant function; hence, the collapsed function of $h_{A,B}$ may not be defined. Thus, we should not regard $\overline{h}_{A,B}$ as a collapse of function $h_{A,B}$ in the sense explained above.

Let $\overline{\text{cap}}^s_N(\cdot, \cdot)$ denote the capacity corresponding to the dynamics associated with the generator $\mathcal{L}^s_N$. It is known that the sector condition of the collapsed chain is inherited from the original chain; therefore, Corollary 4.3 is valid for the collapsed dynamics as well.

**Lemma 6.15.** For two disjoint non-empty subsets $A, B$ of $H \setminus \mathcal{E}_N^x$, we have

$$\overline{\text{cap}}^s_N(A, B) \leq \overline{\text{cap}}^s_N(A, \{o\}) \leq C_0 \text{cap}^s_N(A, B).$$

**Proof.** See [27, Lemma 8.6]. \qed

6.5. **Capacity estimates for collapsed chains.** The capacity estimates for the collapsed chain are an essential ingredient for estimating the mean jump rate. It should be first noted that the capacity for the collapsed process is easy to compute when one of the sets involved is $\{o\}$.

**Lemma 6.16.** For all non-empty subsets $A$ of $H \setminus \mathcal{E}_N^x$,

$$\overline{\text{cap}}^s_N(A, \{o\}) = \text{cap}^s_N(A, \mathcal{E}_N^x).$$

**Proof.** See [13 display (3.10)] or [10] Theorem 9.7]. \qed

A similar result holds for $\overline{\text{cap}}^s_N(A, B)$ as well when either $A$ or $B$ contains $o$. However, to the best of the author’s knowledge, there is no trivial method for comparing $\text{cap}^s_N(A, B)$ and $\overline{\text{cap}}^s_N(A, B)$ when $o \notin A \cup B$. In view of this remark, the following estimate is a non-trivial result.

**Proposition 6.17.** For two disjoint and non-empty subsets $A$ and $B$ of $S_* \setminus \{x\}$ that satisfy $A \cup B = S_* \setminus \{x\}$, it holds that

$$\text{cap}^s_N(\mathcal{E}_N^x(A), \mathcal{E}_N^x(B)) = (1 + o_N(1) + o(1)) \alpha^{1-\omega} \text{cap}^s_N(A, B).$$

**Remark 6.18.** In fact, the condition $A \cup B = S_* \setminus \{x\}$ in the previous proposition is redundant. However, the more general result without this condition is not required in the present study. Moreover, its proof is more complicated and is thus omitted here.

Proposition 6.17 is proved at the end of this subsection. Throughout this subsection, we fix two sets $A, B$ satisfying the condition of Proposition 6.17. Recall the functions $V_{A,B}$ and $V^*_{A,B}$ from Proposition 6.6 and the flows $\Phi_{A,B}$ and $\Phi^*_{A,B}$ from Proposition 6.7. It should be noted that $V_{A,B}(\eta) = h_{A,B}(x)$ for all $\eta \in \mathcal{E}_N^x$, thus the collapsed function $\overline{V}_{A,B} := \overline{V}_{A,B}$, satisfying $\overline{V}_{A,B}(o) = h_{A,B}(x)$, can be defined.
Lemma 6.19. It holds that

\[ \| \Psi_{V_{A,B}} \|_C^2 = (1 + o_N(1) + o_\epsilon(1)) N^{-(1+\alpha)} \text{cap}_Y(A, B). \]

Proof. By Lemma 6.13 and Lemma 6.11, we obtain

\[ \| \Psi_{V_{A,B}} \|_C^2 = \| \Psi_{V_{A,B}} \|_C^2 = \| \Psi_{V_{A,B}} \|_C^2, \]

where the second equality follows since by the elementary calculation we are able to check that the flow \( \Psi_{V_{A,B}} \) satisfies the equality conditions (6.27) and (6.28) of Lemma 6.11. It now suffices to invoke Corollary 6.9 to finish the proof.

Let \( \Phi_{A,B} := \Phi_{A,B} \) be the collapsed flow of \( \Phi_{A,B} \).

Lemma 6.20. For two disjoint and non-empty subsets \( A \) and \( B \) of \( S_* \setminus \{x\} \) that satisfy \( A \cup B = S_* \setminus \{x\} \), it holds that

\[ \sum_{\eta \in \mathbb{P}_N \setminus \mathcal{E}_N(A \cup B)} H_{\mathcal{E}_N(A), \mathcal{E}_N(B)}(\eta)(\text{div } \Phi_{A,B})(\eta) = o_N(1) N^{-(1+\alpha)}. \]

Proof. As \( \mathbb{P}_N \setminus \mathcal{E}_N(A \cup B) = \Delta_N \cup \{o\} \), Lemma 6.12 implies that the absolute value of the left-hand side is bounded above by

\[ \sum_{\eta \in \Delta_N} |(\text{div } \Phi_{A,B})(\eta)| + |(\text{div } \Phi_{A,B})(o)| = \sum_{\eta \in \Delta_N} |(\text{div } \Phi_{A,B})(\eta)| + |(\text{div } \Phi_{A,B})(\mathcal{E}_N^x)|. \]

The last expression is \( o_N(1) N^{-(1+\alpha)} \) by (2) and (3) of Proposition 6.7.

Proof of Proposition 6.17. The proof is similar to that of Theorem 6.3. We start by recalling the functions \( f, g \) and the flows \( \phi, \psi \) from (6.15) and (6.22). Then, by the definition of the collapsing procedure, it can be verified that

\[ \bar{f} \in \mathcal{C}_{1,0}(\mathcal{E}_N(A), \mathcal{E}_N(B)) \text{ and } \bar{\phi} \in \overline{\mathcal{E}_{\alpha_N}}(\mathcal{E}_N(A), \mathcal{E}_N(B)), \]

where \( \alpha_N = o_N(1) N^{-(1+\alpha)} \), and that

\[ \bar{\Phi}_{f} - \bar{\phi} = \Psi_{V_{A,B}} - \bar{\Theta}_N - \bar{\Theta}_N^*, \]

where \( \bar{\Theta}_N \) and \( \bar{\Theta}_N^* \) are the collapsed flows of \( \Theta_N \) and \( \Theta_N^* \) defined in (6.17), respectively. By Lemma 6.11, we have

\[ \left\| \Theta_N \right\|_C^2 = (o_N(1) + o_\epsilon(1)) N^{-(1+\alpha)} \text{ and } \left\| \Theta_N^* \right\|_C^2 = (o_N(1) + o_\epsilon(1)) N^{-(1+\alpha)}. \]

Thus, by Theorem 5.3, Lemma 6.19, and Lemma 6.20, we have that

\[ \text{cap}_N(\mathcal{E}_N(A), \mathcal{E}_N(B)) \leq (1 + o_N(1) + o_\epsilon(1)) N^{-(1+\alpha)} \text{cap}_Y(A, B). \]
For the reversed inequality, it suffices to take
\[ \mathbf{g} \in \mathcal{C}_{0,0}(\mathcal{E}_N(A), \mathcal{E}_N(B)) \text{ and } \overline{\psi} \in \mathcal{C}_{1+o_N(1)}(\mathcal{E}_N(A), \mathcal{E}_N(B)), \]
and then repeat the same arguments so that we obtain
\[ \text{cap}_N(\mathcal{E}_N(A), \mathcal{E}_N(B)) \geq (1 + o_N(1) + o_\epsilon(1)) N^{-(1+\alpha)} \text{cap}_Y(A, B). \tag{6.33} \]
By (6.32) and (6.33), the proof is completed. \[ \square \]

6.6. Estimate of mean jump rate and proof of Theorem 3.4. In view of (6.12), the probability \( \mathbb{P}^N_\sigma[\tau_{\mathcal{E}_N(A)} < \tau_{\mathcal{E}_N(B)}] \) should be estimated to obtain the sharp asymptotics of the mean jump rate \( r_N(x, y) \). This estimate follows from the following proposition.

Proposition 6.21. For two disjoint and non-empty subsets \( A, B \) of \( S_\star \setminus \{x\} \), we have that
\[ \lim_{N \to \infty} \mathbb{P}^N_\sigma[\tau_{\mathcal{E}_N(A)} < \tau_{\mathcal{E}_N(B)}] = h_{A,B}(x). \]

Proof. The proof relies on Proposition 6.17 and Lemma 6.19. Recall the equilibrium potential \( \mathbb{H}_{\mathcal{E}_N(A), \mathcal{E}_N(B)} \) between \( \mathcal{E}_N(A) \) and \( \mathcal{E}_N(B) \), with respect to the collapsed chain \( \eta_N(\cdot) \). Then, by Proposition 6.17,
\[ \left\| \overline{\psi}_{\mathbb{H}_{\mathcal{E}_N(A), \mathcal{E}_N(B)}} \right\|_{\mathcal{C}}^2 = \text{cap}_N(\mathcal{E}_N(A), \mathcal{E}_N(B)) = (1 + o_N(1) + o_\epsilon(1)) N^{-(1+\alpha)} \text{cap}_Y(A, B). \tag{6.34} \]
By Lemma 6.19,
\[ \left\| \overline{\psi}_{\mathbb{V}_{A,B}} \right\|_{\mathcal{C}}^2 = (1 + o_N(1) + o_\epsilon(1)) N^{-(1+\alpha)} \text{cap}_Y(A, B). \tag{6.35} \]
Finally, by (6.30) and (6.31),
\[ \left\langle \overline{\psi}_{\mathbb{V}_{A,B}}, \overline{\psi}_{\mathbb{H}_{\mathcal{E}_N(A), \mathcal{E}_N(B)}} \right\rangle_{\mathcal{C}} = \left\langle \overline{\psi}_{\mathbb{V}_{\mathcal{E}_N(A), \mathcal{E}_N(B)}}, \overline{\psi}_{\mathbb{H}_{\mathcal{E}_N(A), \mathcal{E}_N(B)}} \right\rangle_{\mathcal{C}} = (1 + o_N(1) + o_\epsilon(1)) N^{-(1+\alpha)}, \tag{6.36} \]
where \( \overline{\psi} \) and \( \overline{\phi} \) are the objects defined in the proof of Proposition 6.17. By the same computation as in (5.11),
\[ \left\langle \overline{\psi}_{\mathbb{V}_{\mathcal{E}_N(A), \mathcal{E}_N(B)}}, \overline{\psi}_{\mathbb{H}_{\mathcal{E}_N(A), \mathcal{E}_N(B)}} \right\rangle_{\mathcal{C}} = \text{cap}_N(\mathcal{E}_N(A), \mathcal{E}_N(B)) - \sum_{\eta \in \mathcal{H}_N \setminus \mathcal{E}_N(A) \cup B} \mathbb{H}_{\mathcal{E}_N(A), \mathcal{E}_N(B)}(\eta)(\text{div} \overline{\phi})(\eta) \tag{6.37} \]
Thus, by (6.36), (6.37), and Proposition 6.17,
\[ \left\langle \overline{\psi}_{\mathbb{V}_{A,B}}, \overline{\psi}_{\mathbb{H}_{\mathcal{E}_N(A), \mathcal{E}_N(B)}} \right\rangle_{\mathcal{C}} = (1 + o_N(1) + o_\epsilon(1)) N^{-(1+\alpha)} \text{cap}_Y(A, B). \tag{6.38} \]
Define \( \mathbf{u} = \mathbf{h}_{\mathcal{E}_{N}(A), \mathcal{E}_{N}(B)} - \nabla_{A, B} \). Then, by \((6.34), (6.35)\) and \((6.38)\),
\[
\begin{aligned}
\| \nabla \mathbf{u} \|^2_{C} &= \| \nabla \mathbf{h}_{\mathcal{E}_{N}(A), \mathcal{E}_{N}(B)} \|^2_{C} + \| \nabla \nabla_{A, B} \|^2_{C} - 2 \langle \nabla \nabla_{A, B}, \nabla \mathbf{h}_{\mathcal{E}_{N}(A), \mathcal{E}_{N}(B)} \rangle_{C} \\
&= (\alpha_{N}(1) + \alpha_{r}(1)) N^{-1(1+\alpha)}.
\end{aligned}
\]

As \( \mathbf{u}(0) = \mathbf{h}_{\mathcal{E}_{N}(A), \mathcal{E}_{N}(B)}(0) - \mathbf{h}_{A, B}(x) \) and \( \mathbf{u}(\eta) = 0 \) for all \( \eta \in \mathcal{E}_{N}(A \cup B) \), we can write
\[
\mathbf{u} = \left( \mathbf{h}_{\mathcal{E}_{N}(A), \mathcal{E}_{N}(B)}(0) - \mathbf{h}_{A, B}(x) \right) \mathbf{u}_{0}
\]
for some \( \mathbf{u}_{0} \in \overline{C}^{1}(\{0\}, \mathcal{E}_{N}(A \cup B)) \). Thus,
\[
\| \nabla \mathbf{u} \|^2_{C} = \mathcal{D}_{N}(\mathbf{u}) = \left( \mathbf{h}_{\mathcal{E}_{N}(A), \mathcal{E}_{N}(B)}(0) - \mathbf{h}_{A, B}(x) \right)^{2} \mathcal{D}_{N}(\mathbf{u}_{0}).
\]

By the Dirichlet principle for reversible dynamics, Lemma \((6.15)\), Lemma \((6.16)\) and Theorem \((6.3)\) we have
\[
\mathcal{D}_{N}(\mathbf{u}_{0}) \geq \text{cap}^{s}_{N}(\mathbf{e}, \mathcal{E}_{N}(A \cup B)) \geq C_{0}^{-1} \text{cap}_{N}(\mathbf{e}, \mathcal{E}_{N}(A \cup B))
\]
\[
= C_{0}^{-1} \text{cap}_{N}(\mathcal{E}_{N}^{x}, \mathcal{E}_{N}(A \cup B))
\]
\[
= C_{0}^{-1} (1 + \alpha_{N}(1) + \alpha_{r}(1)) N^{-1(1+\alpha)} \text{cap}(x, A \cup B)
\]
\[
(6.41)
\]

Therefore, by \((6.39), (6.40)\) and \((6.41)\),
\[
\left[ \mathbf{h}_{\mathcal{E}_{N}(A), \mathcal{E}_{N}(B)}(0) - \mathbf{h}_{A, B}(x) \right]^{2} \leq \alpha_{N}(1) + \alpha_{r}(1).
\]

Thus, the proof is completed by taking \( \limsup_{N \to \infty} \) on both sides and then letting \( \epsilon \to 0 \).

The following proposition completes the proof of Theorem \((3.4)\).

**Proposition 6.22.** For all \( x, y \in S_{*} \), we have that
\[
\lim_{N \to \infty} N^{1+\alpha} r_{N}(x, y) = a(x, y).
\]

**Proof.** By Theorem \((3.1), (6.11)\), and Corollary \((6.4)\) we have
\[
\lambda_{N}(x) = \frac{\text{cap}_{N}(\mathcal{E}_{x}^{N}, \mathcal{E}_{x}^{N})}{\mu(\mathcal{E}_{x}^{N})} = (1 + \alpha_{N}(1)) N^{-1(1+\alpha)} \frac{1}{M_{*} \Gamma(\alpha) I_{0}} \sum_{y \in S_{*} \setminus \{x\}} \text{cap}(x, y).
\]

Recall from \((6.14)\) the definition of \( h_{y, S_{*} \setminus \{x\}} \) and from \((6.13)\) the definition of chain \( \hat{Y}(\cdot) \).

Write \( \tau = \inf \{ t : \hat{Y}(t) \neq \hat{Y}(0) \} \). Then, one can observe that
\[
\mathbf{h}_{y, S_{*} \setminus \{x,y\}}(x) = \mathbf{Q}_{x}(\hat{Y}(\tau) = y) = \frac{\text{cap}(x, y)}{\sum_{y \in S_{*} \setminus \{x\}} \text{cap}(x, y)}.
\]

Thus, by Proposition \((6.21)\)
\[
\frac{r_{N}(x, y)}{\lambda_{N}(x)} = (1 + \alpha_{N}(1)) h_{y, S_{*} \setminus \{x,y\}}(x) = (1 + \alpha_{N}(1)) \frac{\text{cap}(x, y)}{\sum_{y \in S_{*} \setminus \{x\}} \text{cap}(x, y)}.
\]

\[(6.43)\]
The proof is completed by multiplying (6.42) and (6.43).

7. Approximation of equilibrium potentials

The construction of the approximation of the equilibrium potential between valleys, in reversible set-up was carried out in [4]. The corresponding construction is presented in this section. The following comments are valid throughout the remaining of the paper.

- The dependency on \( N \) and \( \epsilon \) will be ignored for the subsets of \( H_N \), functions on \( H_N \), and flows on \( H^3_N \) when there is no risk of confusion. For instance, the notation \( E^x \) will be used instead of \( E_N^x \) or \( D^x \) instead of \( D_N^x,\epsilon \) (cf. (7.3)). Of course, the sets such as \( H_N \) or \( H_{N,S_0} \) defined in Section 7.1 the subscript is retained to stress the dependency, as \( N \) is occasionally replaced with some other number such as \( N\epsilon \).
- We shall assume that \( N \) is sufficiently large so that \( N\epsilon > \pi_N > \ell_N \). Recall that \( \pi_N = \lfloor N^{\frac{1}{d} + \frac{1}{2}} \rfloor \ll N \). For notational simplicity, it will be assumed that \( N\epsilon \) is an integer. Of course, all the arguments are valid without this assumption.

This section is organized as follows. In Section 7.1 several basic properties of invariant measure that are frequently used are investigated. In Section 7.2 a global geometry of \( H_N \) is presented that is well-suited for describing the metastability of non-reversible zero-range processes. In Section 7.3 certain auxiliary functions are introduced that play a fundamental role in the construction of test flows. In Section 7.4 the approximation of equilibrium potential on tubes is constructed, and is finally extended into a global object in Section 7.5 thus completing the construction. The proof of Proposition 6.6 is also given in that subsection as well.

7.1. Estimates related to invariant measure. For a non-empty set \( S_0 \subseteq S \) and \( k \in \mathbb{N} \), let \( H_{k,S_0} \) be the set of particle configuration on \( S_0 \) with \( k \) particles, i.e.,

\[
H_{k,S_0} = \left\{ \zeta = (\zeta_x)_{x \in S_0} \in \mathbb{N}^{S_0} : \sum_{x \in S_0} \xi_x = k \right\}.
\]

Using the notations introduced in (2.4), let

\[
Z_{k,S_0} = k^\alpha \sum_{\zeta \in H_{k,S_0}} \prod_{x \in S_0} \frac{m_*(x)^{\zeta_x}}{a(\zeta_x)} = k^\alpha \sum_{\zeta \in H_{k,S_0}} m_\zeta \frac{\zeta}{a(\zeta)}.
\]

Hence, \( H_{N,S} = H_N \) and \( Z_{N,S} = Z_N \). By the same principle as in Proposition 2.1, the following result is obtained

**Lemma 7.1.** For all non-empty set \( S_0 \subseteq S \), we have

\[
\lim_{k \to \infty} Z_{k,S_0} = |S_0 \cap S_\star| \Gamma(\alpha) |S_0 \cap S_\star|^{-1} \prod_{x \in S_0 \setminus S_\star} \Gamma_x.
\]
In this subsection, let us fix a sequence \((d_N)_{N \in \mathbb{N}}\) of positive integer satisfying \(1 \ll d_N \ll N\), namely \(\lim_{N \to \infty} d_N = +\infty\) and \(\lim_{N \to \infty} d_N/N = 0\). In the applications, \(d_N\) is either \(\pi_N\) or \(\ell_N\).

**Lemma 7.2.** For all non-empty sets \(S_0 \subseteq S\), we have

\[
\lim_{N \to \infty} \sum_{k=0}^{d_N} \sum_{\zeta \in H_{k,S_0}} \frac{m^\zeta}{a(\zeta)} = \prod_{x \in S_0 \setminus S^*} \Gamma_x .
\]

**Proof.** By Lemma 7.1,

\[
\sum_{k=d_N+1}^{\infty} \sum_{\zeta \in H_{k,S_0}} \frac{m^\zeta}{a(\zeta)} \leq C \sum_{k=d_N+1}^{\infty} \frac{1}{k^{\alpha}} = o_N(1) .
\]

Therefore, it suffices to verify that

\[
\sum_{k=0}^{\infty} \sum_{\zeta \in H_{k,S_0}} \frac{m^\zeta}{a(\zeta)} = \prod_{x \in S_0 \setminus S^*} \Gamma_x = \prod_{x \in S_0} \Gamma_x .
\]

This is obvious because if we express \(\Gamma(\alpha)\) and \(\Gamma_x\) as infinite series, and expand the right-hand side, then the left-hand side is obtained. \(\Box\)

For \(d < k\), let

\[
H_{k,S_0}(d) = \{\zeta \in H_{k,S_0} : \zeta_x < k - d \text{ for all } x \in S_0 \cap S^*\} .
\]

**Lemma 7.3.** For a non-empty set \(S_0 \subseteq S\) and sufficiently large \(N\), we have

\[
\sum_{\zeta \in H_{N,S_0}(d_N)} \frac{m^\zeta}{a(\zeta)} < \frac{C}{N^\alpha d_N^{\alpha-1}} .
\]

**Proof.** It should be noted that

\[
H_{N,S_0}(d_N) \subseteq \mathcal{H}_{N,S_0}(d_N) \cup \left( \bigcup_{y \in S_0 \setminus S^*} H^y_{N,S_0}(d_N) \right) ,
\]

where

\[
\mathcal{H}_{N,S_0}(d_N) = \{\zeta \in H_{N,S_0} : \zeta_x < N - d_N \text{ for all } x \in S_0\} ,
\]

\[
H^y_{N,S_0}(d_N) = \{\zeta \in H_{N,S_0} : \zeta_y \geq N - d_N \} ; y \in S_0 \setminus S^* .
\]

The set \(\mathcal{H}_{N,S_0}(d_N)\) differs from \(H_{N,S_0}(d_N)\) as it is additionally imposed that \(\zeta_x < N - d_N\) for \(x \in S_0 \setminus S^*\) on this set. By [4] Lemma 3.2,

\[
\sum_{\zeta \in \mathcal{H}_{N,S_0}(d_N)} \frac{m^\zeta}{a(\zeta)} < \frac{C}{N^\alpha d_N^{\alpha-1}} . \tag{7.1}
\]
Let \( \hat{m}_* = \max \{ m_*(y) : y \in S \setminus S_* \} < 1 \). As \( d_N \ll N \), there exists sufficiently large \( N_0 \) such that for all \( N > N_0 \),

\[
\frac{\hat{m}^{N-d_N}}{(N-d_N)\alpha} < \frac{1}{N^{\alpha}d_N^{\alpha-1}}
\]

Hence, for \( y \in S_0 \setminus S_* \) and \( N > N_0 \),

\[
\sum_{\zeta \in \mathcal{H}_N} \frac{m^\zeta}{a(\zeta)} \leq \frac{m_*(y)^{N-d_N}}{a(N-d_N)} < \frac{1}{N^{\alpha}d_N^{\alpha-1}}.
\]

The proof is completed by (7.1) and (7.2).

\[\square\]

7.2. Enlarged valleys and saddle tubes. Herein several subsets of \( \mathcal{H}_N \) are defined that are suitably designed to capture typical metastable transitions among valley and in turn play a central role in the construction of approximations of equilibrium potentials and optimal flows. Figure 7.1 shows a visualization of the sets that are defined below.

The enlarged valley is defined by

\[
\mathcal{D}^x = \{ \eta \in \mathcal{H}_N : \eta_x \geq N(1-2\epsilon) \} ; \ x \in S_* .
\]

Thus, \( \mathcal{E}^x \subset \mathcal{D}^x \) for all \( x \) as it is assumed that \( N \) is sufficiently large so that \( N\epsilon > \ell_N \). The set \( \mathcal{D}^x \) can be regarded as a metastable well corresponding to \( \mathcal{E}^x \) in the sense of [2]. For
$x, y \in S_*$, the tube between valleys $E^x$ and $E^y$ is defined by
\[
\mathcal{T}^{x,y} = \{ \eta \in \mathcal{H}_N : \eta_x + \eta_y \geq N - \pi_N \},
\]
and the saddle tube is defined by
\[
\mathcal{J}^{x,y} = \mathcal{T}^{x,y} \setminus (\mathcal{D}^{x} \cup \mathcal{D}^{y}).
\]
\[
= \{ \eta \in \mathcal{H}_N : \eta_x + \eta_y \geq N - \pi_N \text{ and } \eta_x, \eta_y < N(1 - 2\epsilon) \}.
\]
As $\pi_N < N\epsilon$, one can observe that
\[
\eta_x, \eta_y \in [N\epsilon, N(1 - 2\epsilon)] \text{ for all } \eta \in \mathcal{J}^{x,y}.
\]
We claim that $\mathcal{J}^{x,y} \cap \mathcal{J}^{x',y'} = \emptyset$ unless $\{x, y\} = \{x', y'\}$. To prove this claim, it suffices to verify that for three points $x, y, z \in S_*$, we have
\[
\mathcal{T}^{x,y} \cap \mathcal{T}^{x,z} \subset \mathcal{D}^{x}.
\]
This is obvious because if $\eta \in \mathcal{T}^{x,y} \cap \mathcal{T}^{x,z}$, then
\[
2(N - \pi_N) \leq (\eta_x + \eta_y) + (\eta_x + \eta_z) \leq \eta_x + N,
\]
and thus $\eta_x \geq N - 2\pi_N > N(1 - 2\epsilon)$.

Let
\[
\mathcal{G} = \left( \bigcup_{x \in S_*} \mathcal{D}^{x} \right) \bigcup \{ \bigcup_{\{x, y\} \subset S_*} \mathcal{J}^{x,y} \}.
\]
It should be noticed that the unions in the previous definition are disjoint. The construction of approximating functions and optimal flows is focused on $\mathcal{G}$, particularly on each of its components. The definition of approximating objects outside $\mathcal{G}$ hardly affects the computation except for the discontinuity along the boundary of $\mathcal{G}$. Hence, the boundary of $\mathcal{G}$ is carefully analyzed. It can be decomposed into several parts. The inner and outer boundaries of the saddle tube $\mathcal{J}^{x,y}$, $x, y \in S_*$, are defined by
\[
\partial^{\text{in}} \mathcal{J}^{x,y} = \{ \eta \in \mathcal{J}^{x,y} : \eta_x + \eta_y = N - \pi_N \},
\]
\[
\partial^{\text{out}} \mathcal{J}^{x,y} = \{ \eta \in \mathcal{G}^c : \eta_x + \eta_y = N - \pi_N - 1 \},
\]
respectively. The corresponding boundaries of the enlarged valleys $\mathcal{D}^{x}$, $x \in S_*$, are defined by
\[
\partial^{\text{in}} \mathcal{D}^{x} = \{ \eta \in \mathcal{D}^{x} : \eta_x = N(1 - 2\epsilon) \} \setminus \left( \bigcup_{y \in S^* \setminus \{x\}} \mathcal{T}^{x,y} \right),
\]
\[
\partial^{\text{out}} \mathcal{D}^{x} = \{ \eta \in \mathcal{G}^c : \eta_x = N(1 - 2\epsilon) - 1 \} \setminus \left( \bigcup_{y \in S^* \setminus \{x\}} \mathcal{T}^{x,y} \right).
Finally, the inner and outer boundaries of $\mathcal{G}$ are defined by
\[
\partial^{\text{in}} \mathcal{G} = \left( \bigcup_{x \in S_*} \partial^{\text{in}} \mathcal{D}^x \right) \cup \left( \bigcup_{\{x, y\} \subset S} \partial^{\text{in}} \mathcal{J}^{x, y} \right),
\]
\[
\partial^{\text{out}} \mathcal{G} = \left( \bigcup_{x \in S_*} \partial^{\text{out}} \mathcal{D}^x \right) \cup \left( \bigcup_{\{x, y\} \subset S} \partial^{\text{out}} \mathcal{J}^{x, y} \right),
\]
respectively. In addition, the interior of $\mathcal{J}^{x, y}$, $x, y \in S_*$, and of $\mathcal{D}^x$, $x \in S_*$, are defined by
\[
\mathcal{J}^{x, y}_{\text{int}} = \mathcal{J}^{x, y} \setminus \partial^{\text{in}} \mathcal{J}^{x, y} \quad \text{and} \quad \mathcal{D}^x_{\text{int}} = \mathcal{D}^x \setminus \partial^{\text{in}} \mathcal{D}^x,
\]
respectively. Thus, the set $\mathcal{G}$ can be further decomposed as
\[
\mathcal{G} = \left( \bigcup_{x \in S_*} \mathcal{D}^x_{\text{int}} \right) \cup \left( \bigcup_{\{x, y\} \subset S} \mathcal{J}^{x, y}_{\text{int}} \right) \cup \partial^{\text{in}} \mathcal{G}.
\]
(7.6)
The interior of the set $\mathcal{G}^c = \mathcal{H}_N \setminus \mathcal{G}$ is defined by
\[
(\mathcal{G}^c)_{\text{int}} = \mathcal{G}^c \setminus \partial^{\text{out}} \mathcal{G}.
\]
(7.7)
To control the discontinuity of approximating objects along the boundary, the following lemma is required.

**Lemma 7.4.** We have that
\[
\mu_N(\partial^{\text{in}} \mathcal{G}) = o_N(1) N^{-(1+\alpha)} \quad \text{and} \quad \mu_N(\partial^{\text{out}} \mathcal{G}) = o_N(1) N^{-(1+\alpha)}.
\]
**Proof.** Only the first estimate will be proved; the proof of the second is identical. By (7.5),
\[
\mu_N(\partial^{\text{in}} \mathcal{G}) = \sum_{x \in S_*} \mu_N(\partial^{\text{in}} \mathcal{D}^x) + \sum_{\{x, y\} \subset S} \mu_N(\partial^{\text{in}} \mathcal{J}^{x, y}).
\]
For the first summation, let us fix $x \in S_*$ and let us temporarily denote $\zeta = \zeta(\eta) \in N^{S_\setminus\{x\}}$ the particle configuration of $\eta$ conditioned on $S \setminus \{x\}$, i.e., $\zeta_y = \eta_y$ for all $y \in S \setminus \{x\}$. Then, for $\eta \in \partial^{\text{in}} \mathcal{D}^x$, it follows from the definition of $\partial^{\text{in}} \mathcal{D}^x$ that
\[
|\zeta| = \sum_{y \in S \setminus \{x\}} \zeta_y = N - \eta_x = 2N \epsilon.
\]
That is, $\zeta \in \mathcal{H}_{2N_*, S \setminus \{x\}}$. Moreover, as $\eta_x + \eta_y < N - \pi_N$ for all $\eta \in \partial^{\text{in}} \mathcal{D}^x$ and $y \in S_*$, we have
\[
\zeta_y = \eta_y < N - \eta_x - \pi_N < 2N \epsilon - \pi_N ; \quad y \in S_* \setminus \{x\}.
\]
Therefore, $\zeta \in \mathcal{H}_{2N_*, S \setminus \{x\}}(\pi_N)$. Hence, by Proposition 2.1 and Lemma 7.3,
\[
\mu_N(\partial^{\text{in}} \mathcal{D}^x) = \frac{N^\alpha}{Z_N} \frac{1}{a(N - 2N \epsilon)} \sum_{\zeta \in \mathcal{H}_{2N_*, S \setminus \{x\}}(\pi_N)} \frac{m_\zeta}{a(\zeta)} \leq \frac{C}{(2N \epsilon)^\alpha(\pi_N)^{\alpha-1}}.
\]
This proves that $\mu_N(\partial^{\text{in}} \mathcal{D}^x) = o_N(1) N^{-(1+\alpha)}$, as $\pi_N^{\alpha-1} \gg N$. 
For the second estimate of the lemma, it suffices to demonstrate that \( \mu_N(\partial^{in} J^{x,y}) = o_N(1) N^{-(1+\alpha)} \) for all \( x, y \in S_\star \). Let us fix \( x, y \in S_\star \), and let us temporarily write \( \xi = \xi(\eta) \in \mathbb{H}^{S_\star \setminus \{x,y\}} \) the particle configuration of \( \eta \) on \( S \setminus \{x,y\} \). Then, by the definition of \( \partial^{in} J^{x,y} \), we have that 

\[
|\xi| = \sum_{z \in S \setminus \{x,y\}} \xi_z = N - \eta_x - \eta_y = \pi_N.
\]

That is, \( \xi \in \mathcal{H}_{\pi_N, S_\star \setminus \{x,y\}} \). Hence, by Proposition 2.1 and (7.4), we have

\[
\mu_N(\partial^{in} J^{x,y}) \leq CN^{\alpha} \sum_{\xi \in \mathcal{H}_{\pi_N, S_\star \setminus \{x,y\}}} \frac{m_\xi^{N(1-2\epsilon)}}{a(\xi)} \sum_{i=1}^{N\epsilon} \frac{1}{a(i) a(N - \pi_N - i)}.
\]

(7.8)

By Lemma 8.10, the first summation is bounded above by \( C \pi_N^{2\alpha} \), whereas the second summation is bounded by

\[
\sum_{i=1}^{N\epsilon} \frac{1}{a(i) a(N - \pi_N - i)} \leq \frac{N(1-3\epsilon)}{a(N\epsilon) a(N\epsilon)} = \frac{C}{N^{2\alpha-1}}.
\]

(7.9)

Consequently, \( \mu_N(\partial^{in} J^{x,y}) \leq C \pi_N^{\alpha} N^{-(\alpha-1)} \). Thus the proof is completed as \( \pi_N \gg N^2 \). \( \Box \)

The mass of \( J^{x,y} \) satisfies the following estimate.

**Lemma 7.5.** For all \( x, y \in S_\star \), there exists a constant \( C > 0 \) such that

\[
\mu_N(J^{x,y}) \leq CN^{-(\alpha-1)}.
\]

**Proof.** By Proposition 2.1 (7.4), and computations as in (7.8) and (7.9), we obtain

\[
\mu_N(J^{x,y}) \leq CN^{\alpha} \sum_{k=0}^{\pi_N} \frac{N(1-2\epsilon)}{a(N-k-i)} \sum_{\xi \in \mathcal{H}_{k, S_\star \setminus \{x,y\}}} \frac{m_\xi^{N(1-2\epsilon)}}{a(\xi)}
\]

\[
\leq CN^{-(\alpha-1)} \sum_{k=0}^{\pi_N} \sum_{\xi \in \mathcal{H}_{k, S_\star \setminus \{x,y\}}} \frac{m_\xi}{a(\xi)}.
\]

Therefore, the proof is completed by Lemma 7.2. \( \Box \)

**7.3. Auxiliary functions.** To introduce approximations of equilibrium potentials, an important function is

\[
V(t) = \frac{1}{t_\alpha} \int_0^t s^\alpha (1-s)^\alpha
\]

which essentially captures the one-dimensional projection of the equilibrium potential along the tube. However, this function cannot be used in its own form, and thus an approximated version is required. We introduce this object below.
For $\epsilon \in (0, 1/8)$, a continuous piece-wise linear function $\hat{\gamma}_\epsilon : \mathbb{R} \to [0, 1]$ is defined by

$$\hat{\gamma}_\epsilon(t) = \begin{cases} 0 & \text{if } t \in (-\infty, 4\epsilon] \\ (t - 3\epsilon)/(1 - 6\epsilon) & \text{if } t \in [4\epsilon, 1 - 4\epsilon] \\ 1 & \text{if } t \in [1 - 4\epsilon, \infty) . \end{cases}$$

where $I_\alpha$ has been introduced in (3.2). Let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth, symmetric, non-negative function supported on $[-1, 1]$ such that $\int_{-1}^{1} \phi(t) dt = 1$, and let $\phi_\delta(t) = (1/\delta) \phi(t/\delta)$ for $\delta > 0$. Namely, $(\phi_\delta)_{\delta > 0}$ is a sequence of standard smooth mollifiers. Furthermore, let $\gamma_\epsilon : \mathbb{R} \to [0, 1]$ be defined by $\gamma_\epsilon = \hat{\gamma}_\epsilon \ast \phi_\epsilon$.

**Lemma 7.6.** There exists $\epsilon_0 > 0$ such that the following properties of $\gamma$ hold for all $\epsilon \in (0, \epsilon_0)$:

1. $\gamma_\epsilon$ is a smooth increasing function and satisfies
   
   $$\gamma_\epsilon(t) = \begin{cases} 0 & \text{if } t \in (-\infty, 3\epsilon] \\ (t - 3\epsilon)/(1 - 6\epsilon) & \text{if } t \in [3\epsilon, 1 - 3\epsilon] \\ 1 & \text{if } t \in [1 - 3\epsilon, \infty) . \end{cases}$$

2. $\gamma_\epsilon(1 - t) = 1 - \gamma_\epsilon(t)$.
3. $\gamma_\epsilon'(t) \leq 1 + \epsilon^{1/2}$ for all $t \in [0, 1]$.
4. $\gamma_\epsilon(t)/t \leq 1 + \epsilon^{1/2}$ for all $t \in (0, 1]$.
5. $\gamma_\epsilon(t)/t \geq 1 - 4\epsilon^{1/2} > 0$ for all $t \in [\epsilon^{1/2}, 1]$.

The proof is elementary and left to the reader. Henceforth, we shall assume that $\epsilon \in (0, \epsilon_0)$, so that the above properties hold, and the notation $\gamma$ will be used instead of $\gamma_\epsilon$. Let a non-decreasing smooth function $H = H_\epsilon : [0, 1] \to [0, 1]$ be defined by

$$H(t) = \frac{1}{I_\alpha} \int_0^{\gamma(t)} s^\alpha (1 - s)^\alpha ds = V(\gamma(t)) . \quad (7.10)$$

This function is an approximation of $V(\cdot)$. By (1) of Lemma 7.6, $H(\cdot)$ satisfies

$$H(t) = \begin{cases} 0 & \text{if } t \in [0, 3\epsilon] \\ 1 & \text{if } t \in [1 - 3\epsilon, 1] . \end{cases} \quad (7.11)$$

Let

$$U(t) = (1/I_\alpha) t^\alpha (1 - t)^\alpha = V'(t) .$$

The following basic results will be useful later.

**Lemma 7.7.** For all $t \in [0, 1]$, it holds that

$$U(\gamma(t)) \leq (1 + o_\epsilon(1)) U(t) \quad \text{and} \quad H'(t) \leq (1 + o_\epsilon(1)) U(t) .$$
Lemma 7.9. □

For the upper bound of \( \gamma \)

By (5) of Lemma 7.6, for all \( \epsilon \), we have \( U(\gamma(t)) = \gamma(t)^{\alpha} \gamma(1-t)^{\alpha} \leq \left\{ (1 + \epsilon^{1/2})t \right\}^{\alpha} \left\{ (1 + \epsilon^{1/2})(1 - t) \right\}^{\alpha} = (1 + o_{\epsilon}(1)) U(t) \).

The second inequality is now obvious by this and (3) of Lemma 7.6, since

\[
H'(t) = \gamma'(t) U(\gamma(t)).
\] (7.12)

Lemma 7.8. For all \( t \in [\epsilon^{1/2}, 1 - \epsilon^{1/2}] \), it holds that

\[
H'(t) = (1 + o_{\epsilon}(1)) U(t).
\]

Proof. By (5) of Lemma 7.6 for all \( t \in [\epsilon^{1/2}, 1 - \epsilon^{1/2}] \), we have

\[
U(\gamma(t)) = \gamma(t)^{\alpha} \gamma(1-t)^{\alpha} \geq \left( 1 - 4 \epsilon^{1/2} \right)^{\alpha} \left( 1 - 4 \epsilon^{1/2} \right)^{\alpha} = (1 + o_{\epsilon}(1)) U(t).
\]

As \( \gamma'(t) \geq 1 \), by the above computation and (7.12),

\[
H'(t) - U(t) \geq U(\gamma(t)) - U(t) \geq o_{\epsilon}(1) U(t).
\]

For the upper bound of \( H'(t) - U(t) \), it suffices to use the second inequality of Lemma 7.7

Lemma 7.9. For all \( x, y \in S \), and for all \( \eta \in T^{x,y} \), it holds that

\[
0 \leq U \left( \frac{\eta_x}{N} \right) - \frac{a(\eta_x) a(\eta_y)}{N^{2\alpha} I_{\alpha}} \leq C \frac{\pi_N}{N}.
\]

Proof. The left inequality is trivial. For the right inequality, by the mean-value theorem,

\[
U \left( \frac{\eta_x}{N} \right) - \frac{a(\eta_x) a(\eta_y)}{N^{2\alpha} I_{\alpha}} = \frac{\eta_x^{\alpha}}{N^{\alpha} I_{\alpha}} \left[ \left( \frac{N - \eta_x}{N} \right)^{\alpha} - \left( \frac{\eta_y}{N} \right)^{\alpha} \right] \leq C \frac{(N - \eta_x) - \eta_y}{N} = C \frac{\pi_N}{N}.
\]

Lemma 7.10. We have that

\[
\sum_{\eta \in T^{x,y}} \mu_N(\eta) U \left( \frac{\eta_x}{N} \right)^{2} \leq (1 + o_N(1)) N^{-(\alpha - 1)} \frac{1}{\kappa_{\alpha} I_{\alpha} \Gamma(\alpha)}.
\]

Proof. By Lemmas 7.5 and 7.9 we obtain

\[
\left| \sum_{\eta \in T^{x,y}} \mu_N(\eta) \left[ U \left( \frac{\eta_x}{N} \right)^{2} - \frac{a(\eta_x)^{2} a(\eta_y)^{2}}{N^{4\alpha} I_{\alpha}^{2}} \right] \right| \leq C \frac{\pi_N}{N} \sum_{\eta \in T^{x,y}} \mu_N(\eta) = o_N(1) N^{-(\alpha - 1)}.
\]

By Proposition 2.1, we have

\[
\sum_{\eta \in T^{x,y}} \mu_N(\eta) \frac{a(\eta_x)^{2} a(\eta_y)^{2}}{N^{4\alpha} I_{\alpha}^{2}} \leq \frac{1 + o_N(1)}{2} \sum_{k=0}^{N-k} \sum_{\zeta \in H_{k,S}(x,y)} \frac{m_{\zeta}}{a(\zeta)^{\alpha}}.
\] (7.13)
For \( k \leq \pi N \ll N \), we have that
\[
\sum_{i=0}^{N-k} t^\alpha (N-k-i)^\alpha = (1 + o_N(1)) N^{2\alpha+1} \int_0^1 t^\alpha (1-t)^\alpha \, dt = (1 + o_N(1)) N^{2\alpha+1} I_\alpha.
\]
Therefore, the right-hand side of (7.13) is bounded above by
\[
(1 + o_N(1)) \frac{1}{Z I_\alpha N^{\alpha-1}} \sum_{k=0}^{\pi N} \sum_{\zeta \in \mathcal{H}_{k,S} \{x,y\}} m_\zeta \frac{\zeta}{a(\zeta)}.
\]
The proof is completed by Lemma 7.2 and the definition (2.5) of the constant \( Z \).

7.4. Construction on tubes. Throughout this subsection, we fix two points \( x, y \in S_* \). Then, we shall define a function \( W_{x,y}(\cdot) \) corresponding to the approximation of the equilibrium potential \( h_{x,y}(\cdot) \) on the tube \( T^{x,y} \). Indeed, this task has been carried out in \([4]\) for the reversible case, and the definitions as well as concomitant estimates for the non-reversible case are similar to those for the reversible case.

Recall that the function \( h_{x,y}(\cdot) \) represents the equilibrium potential between two points \( x \) and \( y \) with respect to the random walk \( X(\cdot) \). Enumerate points of \( S \) by \( x = z_1, z_2, \ldots, z_\kappa = y \) in such a manner that
\[
1 = h_{x,y}(z_1) \geq h_{x,y}(z_2) \geq \cdots \geq h_{x,y}(z_\kappa) = 0.
\]
For \( \eta \in \mathcal{H}_N \) and \( 1 \leq i \leq \kappa \), define
\[
\eta^{(i)} = \eta_{z_1} + \eta_{z_2} + \cdots + \eta_{z_i}.
\]
The function \( W_{x,y} = W_{x,y}^{N,\epsilon} : T^{x,y} \to \mathbb{R} \) is defined by
\[
W_{x,y}(\eta) = \sum_{i=1}^{\kappa-1} \left[ h_{x,y}(z_i) - h_{x,y}(z_{i+1}) \right] H\left( \frac{\eta^{(i)}}{N} \right) ; \quad \eta \in T^{x,y},
\]
where \( H = H_\epsilon \) is the function introduced in (7.10). By (7.11), it is obvious that
\[
W_{x,y}(\eta) = \begin{cases} 
1 & \text{if } \eta \in \mathcal{D}^{x} \cap T^{x,y}, \\
0 & \text{if } \eta \in \mathcal{D}^{y} \cap T^{x,y}.
\end{cases} \tag{7.14}
\]
The following lemma is useful.

Lemma 7.11. For \( \eta \in \mathcal{J}^{x,y} \), we have
\[
\left| H\left( \frac{\eta^{(u)}}{N} \right) - H\left( \frac{\eta^{(u)} + 1}{N} \right) \right| \leq (1 + o_N(1)) \frac{a(\eta_x) a(\eta_y)}{N^{2\alpha} I_\alpha} + C \frac{\pi N}{N^2}.
\]
Lemma 7.12. For $\epsilon$ and Lemma 7.7, we obtain that

$$H \left( \frac{\eta^{(u)}}{N} \right) - H \left( \frac{\eta^{(u)} - 1}{N} \right) = \frac{1}{N} H' \left( \frac{\eta^{(u)} - \delta}{N} \right).$$

By the mean-value theorem again, the fact that $|\eta^{(u)} - \delta| - \eta| \leq \max\{\delta, \eta_z + \cdots + \eta_{z+1}\} < \pi_N$,

and Lemma 7.7, we obtain that

$$\left| H \left( \frac{\eta^{(u)}}{N} \right) - H \left( \frac{\eta^{(u)} - 1}{N} \right) \right| \leq \frac{1}{N} H' \left( \frac{\eta}{N} \right) + C \frac{\pi N}{N^2} \leq \frac{1 + o(1)}{N} U \left( \frac{\eta}{N} \right) + C \frac{\pi N}{N^2}, \quad (7.15)$$

where the constant $C$ appeared in this expression is the $L^\infty$ norm of $H''$, which depends on $\epsilon$. Finally, by Lemma 7.9, the term $U(\eta_z/N)$ can be replaced with $a(\eta_z) a(\eta_y)/(N^{2\alpha} I_\alpha)$, without changing the order of the error term. This completes the proof. □

The neighborhood of a set $A \subseteq H_N$ is defined by

$$\mathcal{A} = \{ \eta : \eta = \sigma^z,w \zeta \text{ for some } \zeta \in A \text{ and } z, w \in S \}.$$

For $f : \mathcal{A} \to \mathbb{R}$, the Dirichlet form of $f$ on $A$ is defined by

$$\mathcal{D}_N(f; A) = \frac{1}{2} \sum_{\eta \in \mathcal{A}} \sum_{z,w \in S} \mu_N(\eta) g(\eta_z) r(z, w) [f(\sigma^z,w \eta) - f(\eta)]^2.$$

The right-hand side can be evaluated since $f$ is defined on $\mathcal{A}$.

Lemma 7.12. For all $x, y \in S$, we have

$$\mathcal{D}_N(W_{x,y}; J_{int}^x) \leq (1 + o_N(1) + o(1)) N^{-(1+\alpha)} \frac{\text{cap}_X(x, y)}{M_* \kappa_* I_\alpha \Gamma(\alpha)}.$$

Proof. By the definition of $W_{x,y}$, we can write

$$W_{x,y}(\eta) - W_{x,y}(\sigma^z,j \eta)$$

$$= \begin{cases} \sum_{i=j}^{j-1} [h_{x,y}(z_i) - h_{x,y}(z_{i+1})] \left[ H(\eta^{(u)}/N) - H\left( \frac{(\eta^{(u)} - 1)/N}{N} \right) \right] & \text{if } i < j, \quad (7.16) \\
\sum_{j=i}^{i-1} [h_{x,y}(z_i) - h_{x,y}(z_{i+1})] \left[ H(\eta^{(u)}/N) - H\left( \frac{(\eta^{(u)} - 1)/N}{N} \right) \right] & \text{if } i > j, \end{cases}$$

provided that $\eta_z \geq 1$. Of course, this quantity is equal to 0 if $\eta_z = 0$.

The case $i < j$ is first considered. By Lemma 7.11 and (7.16),

$$|W_{x,y}(\eta) - W_{x,y}(\sigma^z,j \eta)| \leq (1 + o(1)) \frac{a(\eta_x) a(\eta_y)}{N^{2\alpha+1} I_\alpha} (h_{x,y}(z_i) - h_{x,y}(z_{i+1})) + C \frac{\pi N}{N^2}. \quad (7.17)$$
Therefore, by Lemma 7.5
\[
\frac{1}{2} \sum_{\eta \in \mathcal{J}_{\text{int}}^x} \mu_N(\eta) g(\eta_i) r(z_i, z_j) [W_{x,y}(\eta) - W_{x,y}(\sigma_{z_i}^{z_j} \eta)]^2
\leq \frac{1 + \epsilon(1)}{2} \sum_{\eta \in \mathcal{J}_{\text{int}}^x, \eta_i \geq 1} \mu_N(\eta) g(\eta_i) r(z_i, z_j) \left[ \frac{a(\eta_x) a(\eta_y)}{N^{2\alpha+1}} f_\alpha(h_{x,y}(z_i) - h_{x,y}(z_j)) \right]^2
\]
\[+ C \sum_{\eta \in \mathcal{J}_{\text{int}}^x, \eta_i \geq 1} \mu_N(\eta) \frac{\pi_N}{N^3}
\leq \frac{1 + \epsilon(1)}{2Z_N I_\alpha^2 N^{3\alpha+2}} \sum_{\eta \in \mathcal{J}_{\text{int}}^x, \eta_i \geq 1} \frac{m^\eta_{\eta - \omega^{z_i}}}{a(\eta - \omega^{z_i})} a(\eta_x)^2 a(\eta_y)^2 \leq (1 + o_N(1)) N^{2\alpha+1} I_\alpha \Gamma(\alpha) \kappa_{-2} \prod_{x \in S \setminus S_*} \Gamma_x. \tag{7.19}
\]
By inserting (7.19) into the line of (7.18), and applying Proposition 2.1 we obtain
\[
\frac{1}{2} \sum_{\eta \in \mathcal{J}_{\text{int}}^x} \mu_N(\eta) g(\eta_i) r(z_i, z_j) [W_{x,y}(\eta) - W_{x,y}(\sigma_{z_i}^{z_j} \eta)]^2
\leq (1 + o_N(1) + \epsilon(1)) \frac{1}{2M_* \kappa_* I_\alpha \Gamma(\alpha) N^{\alpha+1}} m(z_i) r(z_i, z_j) [h_{x,y}(z_i) - h_{x,y}(z_j)]^2.
\]
The case \(i > j\) can be similarly treated and the same form of estimate is obtained. Hence, by summing this result over all \(1 \leq i, j \leq \kappa\), and by using
\[
\frac{1}{2} \sum_{i,j=1}^{\kappa} m(z_i) r(z_i, z_j) (h_{x,y}(z_i) - h_{x,y}(z_j))^2 = \text{cap}_X(x, y),
\]
we complete the proof. \(\square\)

**Remark 7.13** (Construction of \(W_{y,x}\)). Suppose that the enumeration \(x = z_1, z_2, \ldots, z_\kappa = y\) is used in the construction of \(W_{x,y}\). If all \(h_{x,y}(z_i), 1 \leq i \leq \kappa\), are different, then the construction of \(W_{y,x}\) is unambiguous as \(h_{y,x} = 1 - h_{x,y}\), and we obtain \(W_{y,x} = 1 - W_{x,y}\). By contrast, if \(h_{x,y}(z_i) = h_{x,y}(z_{i+1})\) for some \(i\), then there are several possibilities in the selection of the enumeration for the construction of \(W_{y,x}\). In this case, the rule is to select \(y = w_1, w_2, \ldots, w_\kappa = x\) for the enumeration, where \(w_i = z_{\kappa+1-i}, 1 \leq i \leq \kappa\). It can be thereby verified that \(W_{y,x} = 1 - W_{x,y}\); therefore,
\[
\mathcal{D}_N(W_{x,y}; \mathcal{J}_{\text{int}}^x) = \mathcal{D}_N(W_{y,x}; \mathcal{J}_{\text{int}}^y).
\]
Construction for adjoint dynamics. The function $W^*_{x,y}$ on $T^{x,y}$ is similarly defined. Recall $h^*_{x,y}$ the equilibrium potential between $x$ and $y$ with respect to the adjoint random walk $X^*(\cdot)$, and enumerate points of $S$ by $x = z^*_1, \ldots, z^*_\kappa = y$ in such a manner that

$$1 = h^*_{x,y}(z^*_1) \geq h^*_{x,y}(z^*_2) \geq \cdots \geq h^*_{x,y}(z^*_\kappa) = 0 .$$

Then, let

$$W^*_{x,y}(\eta) = \sum_{i=1}^{\kappa-1} \left[ h^*_{x,y}(z^*_i) - h^*_{x,y}(z^*_{i+1}) \right] H \left( \frac{\eta(i)}{N} \right) .$$

This function $W^*_{x,y}$ also satisfies the property (7.14), and the following variant of Lemma 7.12, whose proof is identical to that of Lemma 7.12.

**Lemma 7.14.** For all $x, y \in S_*$, we have

$$\mathcal{D}_N(W^*_{x,y} \mid J^{x,y}) \leq (1 + o_N(1) + o_\epsilon(1)) N^{-(1+\alpha)} \frac{\text{cap}_X(x,y)}{M_* \kappa_* I_\alpha \Gamma(\alpha)} .$$

The rule for selecting the enumeration corresponding to $W^*_{y,x}$ is the same as that in Remark 7.13; hence, $W^*_{y,x} = 1 - W^*_{x,y}$.

7.5. **Global construction of $V_{A,B}, V^*_{A,B}$ and proof of Proposition 6.6.** For two disjoint non-empty subsets $A, B$ of $S_*$, let the function $V_{A,B} : \mathcal{H}_N \to \mathbb{R}$ be defined as follows:

$$V_{A,B}(\eta) = \begin{cases} h_{A,B}(x) & \text{if } \eta \in \mathcal{D}^x, x \in S_* , \\ h_{A,B}(y) + [h_{A,B}(x) - h_{A,B}(y)] W_{x,y}(\eta) & \text{if } \eta \in \mathcal{J}^{x,y} , x, y \in S_* , \\ 0 & \text{if } \eta \in \mathcal{G}^c . \end{cases}$$

In this expression, the definition on $\mathcal{J}^{x,y}$ does not depend on the order of $x$ and $y$, owing to Remark 7.13 in the sense that

$$h_{A,B}(y) + [h_{A,B}(x) - h_{A,B}(y)] W_{x,y}(\eta) = h_{A,B}(x) + [h_{A,B}(y) - h_{A,B}(x)] W_{y,x}(\eta) .$$

The function $V^*_{A,B}(\cdot)$ is defined by replacing $W_{x,y}$ in the definition of $V_{A,B}$ with $W^*_{x,y}$. Then, it is immediate from the definition that $V_{A,B}$ and $V^*_{A,B}$ satisfy part (1) of Proposition 6.6. Hence, to complete the proof of Proposition 6.6 it suffices to prove part (2). This will be verified only for the function $V_{A,B}$, as the proof for $V^*_{A,B}$ is essentially the same.

**Lemma 7.15.** For two disjoint non-empty subsets $A, B$ of $S_*$, we have

$$\mathcal{D}_N(V_{A,B}) \leq (1 + o_N(1) + o_\epsilon(1)) N^{-(1+\alpha)} \text{cap}_Y(A,B) .$$
Proof. By (7.6) and (7.7) the Dirichlet form \( D_N(V_{A,B}) \) can be decomposed as
\[
\sum_{x \in S_*} D_N(V_{A,B}; D^x_{\text{int}}) + \sum_{\{x,y\} \subset S_*} D_N(V_{A,B}; J^x_{\text{int}}) + D_N(V_{A,B}; \partial \text{in} \mathcal{G}) + D_N(V_{A,B}; \partial \text{out} \mathcal{G}) .
\]

(7.20)

It is first observed that for all \( z, w \in S \), we have \( V_{A,B}(\sigma^{z,w} \eta) - V_{A,B}(\eta) = 0 \) for all \( \eta \in D^x_{\text{int}} \), \( x \in S_* \), and for all \( \eta \in (G^c)_{\text{int}} \). Therefore,
\[
\sum_{x \in S_*} D_N(V_{A,B}; D^x_{\text{int}}) = 0 \quad \text{and} \quad D_N(V_{A,B}; (G^c)_{\text{int}}) = 0 .
\]

(7.21)

Moreover, as \( V_{A,B}(\eta) \in [0, 1] \), and \( g(\cdot), r(\cdot, \cdot) \) are bounded, it holds that
\[
D_N(V_{A,B}; \partial \text{in} \mathcal{G}) \leq C \mu_N(\partial \text{in} \mathcal{G}) \quad \text{and} \quad D_N(V_{A,B}; \partial \text{out} \mathcal{G}) \leq C \mu_N(\partial \text{out} \mathcal{G})
\]

for some constant \( C \). Hence, by Lemma 7.4
\[
D_N(V_{A,B}; \partial \text{in} \mathcal{G}) + D_N(V_{A,B}; \partial \text{out} \mathcal{G}) = o_N(1) N^{-(1+\alpha)} .
\]

(7.22)

Finally, by Lemma 7.12 for \( x, y \in S_* \),
\[
D_N(V_{A,B}; J^x_{\text{int}}) = [h_{A,B}(y) - h_{A,B}(x)]^2 D_N(W_{x,y}; J^x_{\text{int}}) \leq (1 + o_N(1) + o_\alpha(1)) N^{-(1+\alpha)} [h_{A,B}(y) - h_{A,B}(x)]^2 \frac{\text{cap}_X(x, y)}{M_* \kappa_* I_\alpha(\Gamma(\alpha))} .
\]

(7.23)

The proof is completed by combining (7.20), (7.21), (7.22), (7.23), and the fact that
\[
\sum_{\{x,y\} \subset S_*} \frac{\text{cap}_X(x, y)}{M_* \kappa_* I_\alpha(\Gamma(\alpha))} [h_{A,B}(y) - h_{A,B}(x)]^2 = \mathcal{D}_Y(h_{A,B}) = \text{cap}_Y(A, B) .
\]

\[ \square \]

8. Correction procedure for test flows

In this section, for two disjoint non-empty subsets \( A \) and \( B \) of \( S_* \), we shall construct suitable approximations of the optimal flows \( \Phi^*_{h_{\mathcal{E}(A), \mathcal{E}(B)}} \) and \( \Phi^*_{h_{\mathcal{E}(A), \mathcal{E}(B)}} \), denoted by \( \Phi_{A,B} \) and \( \Phi^*_{A,B} \), respectively. The focus is only on the former because the construction of the latter is entirely parallel. In particular, it is demonstrated that \( \Phi_{A,B} \) satisfies four conditions presented in Proposition 6.7. Henceforth, we fix two disjoint non-empty subsets \( A \) and \( B \) of \( S_* \).
8.1. **Analysis of the flow** $\Phi_{V_{A,B}}^\ast$. Recall the definition of $\Phi_{V_{A,B}}^\ast$ from (5.4). By (1) of Proposition 5.1, we have

$$\left(\text{div} \, \Phi_{V_{A,B}}^\ast \right)(\eta) = \sum_{z, w \in S} \mu_N(\eta) g(\eta_z) r(z, w) \left[ V_{A,B}(\eta) - V_{A,B}(\sigma^{z,w}\eta) \right].$$

(8.1)

This expression can be used to derive several basic facts about the flow $\Phi_{V_{A,B}}^\ast$.

**Proposition 8.1.** The flow $\Phi_{V_{A,B}}^\ast$ has the following properties.

1. The flow is divergence-free on $D_{int}^x, x \in S_\ast$, and on $(G^c)_{int}$, i.e.,

$$\left(\text{div} \, \Phi_{V_{A,B}}^\ast \right)(\eta) = 0 \quad \text{for all} \quad \eta \in \bigcup_{x \in S_\ast} D_{int}^x \quad \text{and for all} \quad \eta \in (G^c)_{int}.$$  

2. The divergence on boundaries is negligible in the sense that

$$\sum_{\eta \in \partial_{\text{aux}} G} \left| \left(\text{div} \, \Phi_{V_{A,B}}^\ast \right)(\eta) \right| + \sum_{\eta \in \partial G} \left| \left(\text{div} \, \Phi_{V_{A,B}}^\ast \right)(\eta) \right| = o_N(1) N^{-(1+\alpha)}.$$  

*Proof.* Part (1) is obvious, as for all $z, w \in S$, we have $V_{A,B}(\eta) = V_{A,B}(\sigma^{z,w}\eta)$ for all $\eta$ belonging to $D_{int}^x, x \in S_\ast$, or $(G^c)_{int}$. For part (2), it suffices to observe from (8.1) that $\left|\left(\text{div} \, \Phi_{V_{A,B}}^\ast \right)(\eta)\right| \leq C \mu_N(\eta)$ and use Lemma 7.4. □

The previous proposition reveals a crucial drawback regarding the flow $\Phi_{V_{A,B}}^\ast$, namely,

$$\left(\text{div} \, \Phi_{V_{A,B}}^\ast \right)(\mathcal{E}(A)) = \left(\text{div} \, \Phi_{V_{A,B}}^\ast \right)(\mathcal{E}(B)) = 0,$$

as one of the main requirements of the test flow in the application of Theorem 5.3 is that the total divergence on $\mathcal{E}(A)$ of the test flow is approximately $N^{-(1+\alpha)} \text{cap}_Y(A, B)$. In addition, one can easily verify that the divergence of $\Phi_{V_{A,B}}^\ast$ on $\mathcal{J}_{x,y}^z, x, y \in S_\ast$, is not negligible, i.e., not of order $o_N(1) N^{-(1+\alpha)}$. This is the second serious problem, as one would hope that the divergence on $\Delta_N$ is of order $o_N(1) N^{-(1+\alpha)}$, but $\mathcal{J}_{x,y}^z \subset \Delta_N$.

These two defects are closely related. The non-negligible divergences on the saddle tube $\mathcal{J}_{x,y}^z$ should be carefully sent to the valleys $\mathcal{E}^x$ and $\mathcal{E}^y$, so that after this correction procedure, the divergence on saddle tubes is negligible whereas the divergence on valleys $\mathcal{E}(A)$ is close to the desired values.

Under the assumption (8.2), this correction procedure is first carried out for each tube $\mathcal{T}_{x,y}^z$ in Section 8.2 and then globally in Section 8.3. The proof of Proposition 6.7 is also presented in Section 8.3. Several technical computations are summarized in Section 8.4. The special assumption (8.2) is imposed to simplify cumbersome notations and does not affect the validity of the main arguments. In Section 8.5 the general case without this special assumption is considered.
8.2. Correction on tube: special case. In Sections 8.2, 8.3, and 8.4, it will be assumed that
\[ r(u, v) > 0 \quad \text{for all} \quad u, v \in S. \quad (8.2) \]
The general result without this redundant assumption will be explained in Section 8.5, as mentioned earlier. We fix two points \( x, y \in S_* \) throughout this subsection in order to focus on the construction on the tube \( T^{x,y} \).

Localization of the correction procedure. Recall the enumeration \( x = z_1, z_2, \ldots, z_\kappa = y \) and the function \( W_{x,y} \) from Section 7.4. As the function \( W_{x,y} \) is defined only on the tube \( T^{x,y} \), the flow \( \Phi^{*}_{W_{x,y}} \) cannot be defined in the usual manner. Hence, let us first extend \( W_{x,y} \) to a function on \( H_N \) by
\[
\tilde{W}_{x,y}(\eta) = W_{x,y}(\eta) 1\{\eta \in T^{x,y}\}.
\]
Then, the flow \( \Phi^{*}_{\tilde{W}_{x,y}} \) can be defined. With a slight abuse of notation, this flow can be written as \( \Phi^{*}_{W_{x,y}} \). Then, by the definition of \( V_{A,B} \), the flow \( \Phi^{*}_{V_{A,B}} \) satisfies
\[
\Phi^{*}_{V_{A,B}}(\eta, \zeta) = [h_{A,B}(x) - h_{A,B}(y)] \Phi^{*}_{W_{x,y}}(\eta, \zeta) \quad \text{for all} \quad \eta, \zeta \in T^{x,y}. \quad (8.3)
\]
It should be noted that this relation is valid not only for \( \eta, \zeta \in J^{x,y} \) but also for \( \eta, \zeta \in T^{x,y} \), as both sides are equal to 0 if either \( \eta \in T^{x,y} \) or \( \zeta \in T^{x,y} \) does not belong to \( J^{x,y} \). The interior of the tube is defined by
\[
T_{\text{int}}^{x,y} = \{ \eta : \eta_x + \eta_y > N(1 - \epsilon) \}.
\]
Then, by (8.3),
\[
(\text{div} \: \Phi^{*}_{V_{A,B}})(\eta) = [h_{A,B}(x) - h_{A,B}(y)] (\text{div} \: \Phi^{*}_{W_{x,y}})(\eta) \quad \text{for all} \quad \eta \in T_{\text{int}}^{x,y}. \quad (8.4)
\]
Therefore, the correction procedure for the divergence of the flow \( \Phi^{*}_{V_{A,B}} \) on the tube \( T_{\text{int}}^{x,y} \) is reduced to that of \( \Phi^{*}_{W_{x,y}} \) on \( T_{\text{int}}^{x,y} \).

The divergence of the flow \( \Phi^{*}_{W_{x,y}} \) on \( T_{\text{int}}^{x,y} \) is now investigated. By (4.1), the divergence \( \text{div} \: \Phi^{*}_{W_{x,y}} \) at \( \eta \in T_{\text{int}}^{x,y} \) can be written as
\[
(\text{div} \: \Phi^{*}_{W_{x,y}})(\eta) = \sum_{z, w \in S} \mu_N(\eta) g(\eta_z) r(z, w) \left[ W_{x,y}(\eta) - W_{x,y}(\sigma^{z, w}; \eta) \right]
\begin{align}
&= a_N \sum_{z, w \in S} \mu_{N-1}(\eta - \omega^z) m(z) r(z, w) \left[ W_{x,y}(\eta) - W_{x,y}(\sigma^{z, w}; \eta) \right]. \quad (8.5)
\end{align}
\]
By (7.16), the last expression can be written as
\[
(\text{div} \: \Phi^{*}_{W_{x,y}})(\eta) = a_N \sum_{i=1}^{\kappa} \mu_{N-1}(\eta - \omega^{z_i}) m(z_i) B(\eta; z_i) 1\{\eta_{z_i} \geq 1\}, \quad (8.6)
\]
where
\[ B(\eta; z_i) = \sum_{j : i < j} r(z_i, z_j) \sum_{u = i}^{j-1} [h_{x,y}(z_u) - h_{x,y}(z_{u+1})] \left[ H\left(\frac{\eta(u)}{N}\right) - H\left(\frac{\eta(u) - 1}{N}\right)\right] + \sum_{j : i > j} r(z_i, z_j) \sum_{u = i}^{j-1} [h_{x,y}(z_u) - h_{x,y}(z_{u+1})] \left[ H\left(\frac{\eta(u)}{N}\right) - H\left(\frac{\eta(u) + 1}{N}\right)\right]. \] (8.7)

Estimates on \( B(\cdot, \cdot) \) will now be provided.

**Lemma 8.2.** For \( \eta \in \mathcal{T}_{x,y} \), there exists a constant \( C \geq 0 \) such that
\[
|B(\eta; z_i)| \leq C \frac{\pi_N}{N^2} ; \quad 2 \leq i \leq \kappa - 1,
\]
\[
|B(\eta; z_1) - \frac{1}{N} H'\left(\frac{\eta_x}{N}\right) \frac{1}{M_x} \text{cap}_X(x, y)| \leq C \frac{\pi_N}{N^2}, \quad \text{and}
\]
\[
|B(\eta; z_n) + \frac{1}{N} H'\left(\frac{\eta_x}{N}\right) \frac{1}{M_x} \text{cap}_X(x, y)| \leq C \frac{\pi_N}{N^2}.
\]
In particular, the constant \( C \) can be chosen to be 0 if \( \eta \notin \mathcal{J}^{x,y} \).

**Proof.** The argument presented in the proof of Lemma 7.11 based on the mean-value theorem yields
\[
\left| \left\{ H\left(\frac{\eta(u)}{N}\right) - H\left(\frac{\eta(u) + 1}{N}\right)\right\} \pm \frac{1}{N} H'\left(\frac{\eta_x}{N}\right) \right| \leq C \frac{\pi_N}{N^2}.
\]
Applying this bound to each term in \( B(\eta; z_i) \) and using (2.6) and (2.7) provide the desired estimates. For \( \eta \in \mathcal{J}^{x,y} \), both \( B(\eta; z_i) \), \( 1 \leq i \leq \kappa \), and \( H'(\eta_x/N) \) are equal to 0; therefore we can select \( C = 0 \). \( \square \)

At first glance, these estimates imply that the right-hand side of (8.6) is small; hence, the divergence of \( \Phi_{\mathcal{W}_{x,y}} \) on \( \mathcal{J}^{x,y} \) is small. As the flow \( \Phi_{\mathcal{W}_{x,y}} \) is divergence-free on \( \mathcal{J}^{x,y} \), this heuristic observation supports the claim that \( \mathcal{W}_{x,y} \) approximates the equilibrium potential \( h_{x,y} \) on the saddle tube. However, the word small used here is not quite correct in some sense. To be more precise, these estimates along with the expression (8.6) imply that (\( \text{div} \Phi_{\mathcal{W}_{x,y}}(\eta) \)) is of order \( \mu_N(\eta) (\pi_N/N^2) \) for \( \eta \in \mathcal{J}^{x,y} \). Therefore, in view of Lemma 7.5, the divergence on \( \mathcal{J}^{x,y} \) is not negligible, i.e., not of order \( o_N(1) N^{-(1+\alpha)} \).

The essence of the correction procedure hereafter presented is to send these small, but non-negligible divergences on \( \mathcal{J}^{x,y} \) to \( \mathcal{E}^x \) and \( \mathcal{E}^y \), without excessively perturbing the flow \( \Phi_{\mathcal{W}_{x,y}} \), in the sense of the flow norm. This procedure is carried out through the correction flow \( \chi_{x,y} \) defined below.

**Correction flow \( \chi_{x,y} \) and corrected flow \( \Phi_{x,y} \).** Two subsets of \( \mathcal{H}_N \) are now defined by
\[
\mathcal{V}^x = \{ \eta \in \mathcal{T}^{x,y} : \eta_y = 0 \} \quad \text{and} \quad \mathcal{V}^y = \{ \eta \in \mathcal{T}^{x,y} : \eta_x = 0 \},
\]
so that $\mathcal{V}^x \subset \mathcal{D}^x$ and $\mathcal{V}^y \subset \mathcal{D}^y$. For $\eta \in \mathcal{H}_N$, let $\tilde{\eta} \in \mathbb{N}^{S \setminus \{x, y\}}$ be the configuration on $S \setminus \{x, y\}$ obtained from $\eta$ by neglecting two sites $x$ and $y$. Then, let

$$C(\eta) := \frac{\text{cap}_X(x, y)}{N^{\alpha+1} Z_N M_* I_\alpha a(\eta)} = \frac{m^y}{N^{\alpha+1} Z_N M_* I_\alpha a(\eta)}.$$ \hfill (8.8)

The correction flow $\chi_{x,y}$ will now be defined. Recall $B(\eta; z_i)$ from (8.7).

We first define a flow $\chi_{x,y}^{(1)}$. If $\eta \in \mathcal{T}^{x,y}$ satisfies $\eta_i \geq 1$ for some $2 \leq i \leq \kappa - 1$ and $\zeta = \sigma^x \eta$ or $\sigma^y \eta$, then

$$\chi_{x,y}^{(1)}(\eta, \zeta) = -\chi_{x,y}^{(1)}(\zeta, \eta) = -\frac{1}{2} a_N \mu_N(\eta - \omega z_i) m(z_i) B(\eta; z_i).$$

Otherwise, $\chi_{x,y}^{(1)}(\eta, \zeta) = 0$.

Now we define a flow $\chi_{x,y}^{(2)}$. If $\eta \in \mathcal{T}^{x,y}$ satisfies $\eta_y \geq 1$ and $\zeta = \sigma^y \eta$, then

$$\chi_{x,y}^{(2)}(\eta, \zeta) = -\chi_{x,y}^{(2)}(\zeta, \eta) = \frac{1}{2} a_N \mu_N(\eta - \omega z_i) m(x) B(\sigma^z \eta; z_1) - m(y) B(\eta; z) - C(\eta).$$

Otherwise, $\chi_{x,y}^{(2)}(\eta, \zeta) = 0$. Finally,

$$\chi_{x,y} = \chi_{x,y}^{(1)} + \chi_{x,y}^{(2)}.$$

It should be noted here that $\chi_{x,y}(\eta, \zeta) = 0$ unless $\eta, \zeta \in \mathcal{T}^{x,y}$.

**Proposition 8.3.** The flow $\chi_{x,y}$ is negligible in the sense that

$$\|\chi_{x,y}\|^2 = (o_N(1) + o_\epsilon(1)) N^{-(1+\alpha)}.$$

**Proof.** The estimate for $\chi_{x,y}^{(1)}$ is not complicated. By Lemma 8.2, it is seen that

$$|\chi_{x,y}^{(1)}(\eta, \zeta)| \leq C \mu_N(\eta) \frac{\pi N}{N^2}$$

for all $\eta \in \mathcal{J}^{x,y}$, and that $\chi_{x,y}^{(1)}(\eta, \zeta) = 0$ for all $\eta \notin \mathcal{J}^{x,y}$. As $c^y_N(\eta, \zeta) \geq C \mu_N(\eta)$ for all $(\eta, \zeta) \in \mathcal{H}^{y}_N$, we have

$$\left\|\chi_{x,y}^{(1)}\right\|^2 = \frac{1}{2} \sum_{\eta \in \mathcal{H}_N} \sum_{\zeta, \zeta \sim \eta} \left(\chi_{x,y}^{(1)}(\eta, \zeta)\right)^2 \leq C \sum_{\eta \in \mathcal{J}^{x,y}} \mu_N(\eta) \frac{\pi^2 N}{N^4}.$$ \hfill (8.9)

Hence, by Lemma 7.5

$$\|\chi_{x,y}^{(1)}\|^2 = o_N(1) N^{-(1+\alpha)}. \hfill (8.10)$$

The estimate for $\chi_{x,y}^{(2)}$ is rather complicated. For $\eta \in \mathcal{T}^{x,y}$, by Lemma 8.2,

$$\chi_{x,y}^{(2)}(\eta, \sigma^y \eta) = \mu_N(\eta) \frac{1}{N} \text{cap}_X(x, y) \frac{m^y}{M_* I_\alpha} \left[H' \left(\frac{\eta_x}{N}\right) g(\eta_x) - \frac{a(\eta_x) a(\eta_y)}{N^{2\alpha} I_\alpha}\right] + \mu_N(\eta) \frac{o_N(1)}{N} \mathbf{1}\{\eta \in \mathcal{J}^{x,y}\}.$$ \hfill (8.11)
Since \( H'(\eta_x/N) = 0 \) for \( \eta \notin \mathcal{J}^{x,y} \), and since \( g(\eta_x) = 1 + o_N(1) \) for \( \eta \in \mathcal{J}^{x,y} \), we have

\[
H'(\frac{\eta_x}{N}) g(\eta_x) = H'(\frac{\eta_x}{N}) + o_N(1) \chi^{(2)} \eta \in \mathcal{J}^{x,y}.
\]

Therefore, the term \( g(\eta_x) \) in (8.11) can be replaced with 1, without changing the type of the error term. Thus, for \( \eta \in \mathcal{T}^{x,y} \), we have

\[
\chi^{(2)}_{x,y}(\eta, \sigma^{y,x}\eta) = \mu_N(\eta) \frac{1}{N} \frac{\text{cap}_X(x,y)}{M_*} \left[ H'(\frac{\eta_x}{N}) - \frac{a(\eta_x) a(\eta_y)}{N^{2\alpha} I_{\alpha}} \right] + \mu_N(\eta) \frac{\beta_N}{N} \chi^{(2)} \eta \in \mathcal{J}^{x,y},
\]

where \( \beta_N = o_N(1) \).

The flow norm of \( \chi^{(2)}_{x,y} \) is now considered by decomposing it into three flows. The first flow is defined by

\[
\chi^{(2a)}_{x,y}(\eta, \sigma^{y,x}\eta) = \chi^{(2)}_{x,y}(\eta, \sigma^{y,x}\eta) \chi^{(2)} \eta \in \mathcal{J}^{x,y}, \eta_x \in [N^{\epsilon/2}, N(1 - \epsilon^{1/2})].
\]

If \( \eta \in \mathcal{T}^{x,y} \) and \( \eta_x \in [N^{\epsilon/2}, N(1 - \epsilon^{1/2})] \), then by Lemmas 7.9 and 7.8

\[
\|\chi^{(2a)}_{x,y}\| \leq \mu_N(\eta) \frac{1}{N} \frac{\text{cap}_X(x,y)}{M_*} \left[ H'(\frac{\eta_x}{N}) - U(\frac{\eta_x}{N}) \right] + \mu_N(\eta) \frac{o_N(1)}{N} U(\frac{\eta_x}{N}) = \frac{1}{N} \left[ o_N(1) + o_\epsilon(1) U(\frac{\eta_x}{N}) \right] \mu_N(\eta).
\]

Thus, by the same argument as in (8.9),

\[
\|\chi^{(2a)}_{x,y}\|^2 \leq \frac{1}{N^2} \sum_{\eta \in \mathcal{J}^{x,y}} \left[ o_N(1) + o_\epsilon(1) U(\frac{\eta_x}{N}) \right] \mu_N(\eta).
\]

Hence, by Lemmas 7.5 and 7.10

\[
\|\chi^{(2a)}_{x,y}\|^2 \leq (o_N(1) + o_\epsilon(1)) N^{-(\alpha + 1)}.
\]

For \( \eta \in \mathcal{T}^{x,y} \) with \( \eta_x \notin [N^{\epsilon/2}, N(1 - \epsilon^{1/2})] \), the second flow is defined by,

\[
\chi^{(2b)}_{x,y}(\eta, \sigma^{y,x}\eta) = \mu_N(\eta) \frac{1}{N} \frac{\text{cap}_X(x,y)}{M_*} \left[ H'(\frac{\eta_x}{N}) - \frac{a(\eta_x) a(\eta_y)}{N^{2\alpha} I_{\alpha}} \right],
\]

and \( \chi^{(2b)}_{x,y} \equiv 0 \) otherwise. By the trivial bound

\[
\left| H'(\frac{\eta_x}{N}) - \frac{a(\eta_x) a(\eta_y)}{N^{2\alpha} I_{\alpha}} \right| \leq H'(\frac{\eta_x}{N}) + \frac{a(\eta_x) a(\eta_y)}{N^{2\alpha} I_{\alpha}} \leq H'(\frac{\eta_x}{N}) + U(\frac{\eta_x}{N})
\]

and by Lemma 7.7 we obtain

\[
\left| \chi^{(2b)}_{x,y}(\eta, \zeta) \right| \leq C \frac{\mu_N(\eta)}{N} U(\frac{\eta_x}{N}).
\]
Therefore, by the same argument as in (8.9) and in the proof of Lemma 7.10,
\[ \|\chi_{x,y}^{(2b)}\|^2 \leq \frac{C}{N^2} \sum_{\eta \in T^{x,y}, \eta_x \notin [N\epsilon^{1/2}, N(1-\epsilon^{1/2})]} \mu_N(\eta) U^2 \left( \frac{\eta_x}{N} \right) \]
\[ \leq \frac{C}{N^{\alpha+1}} \left[ \int_0^{\epsilon^{1/2}} t^\alpha (1-t)^\alpha \, dt + \int_{1-\epsilon^{1/2}}^1 t^\alpha (1-t)^\alpha \, dt \right]. \]

It should be noticed that, in the last bound, the constant \( C \) can be chosen to be the one independent of \( \epsilon \). Consequently,
\[ \|\chi_{x,y}^{(2b)}\|^2 \leq o(1) N^{-(\alpha+1)}. \] (8.14)

Finally, the third flow is defined by
\[ \chi_{x,y}^{(2c)}(\eta, \sigma^{y,x} \eta) = \mu_N(\eta) \frac{\beta_N}{N} 1\{\eta \in J^{x,y}, \eta_x \notin [N\epsilon^{1/2}, N(1-\epsilon^{1/2})]\}. \]

Then, by the same computations as before,
\[ \|\chi_{x,y}^{(2c)}\|^2 \leq \frac{o_N(1)}{N^2} \mu_N(J^{x,y}) = o_N(1) N^{-(\alpha+1)}. \] (8.15)

Since
\[ \chi_{x,y}^{(2)} = \chi_{x,y}^{(2a)} + \chi_{x,y}^{(2b)} + \chi_{x,y}^{(2c)}, \]
the estimate of the flow norm of \( \chi_{x,y}^{(2)} \) can be completed by combining (8.13), (8.14), and (8.15).

The divergence of \( \chi_{x,y} \) is now considered.

**Proposition 8.4.** The correction flow \( \chi_{x,y} \) has the followings properties:

1. The flow \( \chi_{x,y} \) is divergence-free on \((V^x \cup V^y \cup J^{x,y})^c\).
2. The divergence of \( \chi_{x,y} \) on \( \partial^{\text{in}} J^{x,y}, V^x \backslash E^x \), and \( V^y \backslash E^y \) is negligible in the sense that

\[ \left( \sum_{\eta \in \partial^{\text{in}} J^{x,y}}, \sum_{\eta \in V^x \backslash E^x}, \sum_{\eta \in V^y \backslash E^y} \right) |(\text{div} \chi_{x,y})(\eta)| = o_N(1) N^{-(\alpha+1)}. \]

3. The divergence of \( \chi_{x,y} \) on \( E^x \) and \( E^y \) satisfies

\[ (\text{div} \chi_{x,y})(E^x) = (1 + o_N(1)) N^{-(\alpha+1)} \frac{\text{cap}_X(x,y)}{M, \Gamma(\alpha) I_\alpha} \text{ and } \]
\[ (\text{div} \chi_{x,y})(E^y) = - (1 + o_N(1)) N^{-(\alpha+1)} \frac{\text{cap}_X(x,y)}{M, \Gamma(\alpha) I_\alpha}. \]

In addition, \( (\text{div} \chi_{x,y})(\eta) > 0 \) for all \( \eta \in E^x \) and \( (\text{div} \chi_{x,y})(\eta) < 0 \) for all \( \eta \in E^y \).

4. The divergence of \( \chi_{x,y} \) on \( J^{x,y} \) satisfies

\[ (\text{div} \chi_{x,y})(\eta) = -(\text{div} \Phi_{W_{x,y}}^*)(\eta) \text{ for all } \eta \in J_{\text{int}}^{x,y}. \]
The proof is postponed to Section 8.4. The corrected flow is defined by

$$\Phi_{x,y} = \Phi^*_{W_{x,y}} + \chi_{x,y}.$$  

An interpretation of the previous proposition is now given in terms of the correction procedure. By (3) of Proposition 8.4 and the fact that $\Phi^*_{W_{x,y}}$ is divergence-free on $\mathcal{E}^x$ and $\mathcal{E}^y$, it follows that

$$\left(\text{div} \, \Phi_{x,y}\right)(\mathcal{E}^x) = (1 + o_N(1)) N^{-\alpha} \frac{\text{cap}_x(x,y)}{\kappa_\alpha M_\alpha \Gamma(\alpha)} I_\alpha \quad \text{and} \quad (8.16)$$

Moreover, by (4) of Proposition 8.4, the flow $\Phi_{x,y}$ is divergence-free on $J_{x,y}$. Hence, the divergence of $\Phi^*_{W_{x,y}}$ on $J_{x,y}$ was cleaned out by sending it to $\mathcal{E}^x$ and $\mathcal{E}^y$. By (1) and (2) of Proposition 8.4, this procedure has negligible effect on the divergence of the remaining part, and by Proposition 8.3 it does not essentially change the flow norm.

8.3. Global correction of $\Phi^*_{V_{A,B}}$ and proof of Proposition 6.7: special case. Herein, the global correction for the flow $\Phi^*_{V_{A,B}}$ is carried out. This procedure relies on the correction flows $\{\chi_{x,y} : x, y \in S_*\}$ defined in the previous subsection.

By following the rule stated in Remark 7.13 it can be verified that $\chi_{x,y} = -\chi_{y,x}$; hence, the following summation is well-defined:

$$\chi_{A,B} = \sum_{\{x,y\} \in S_*} \left[ h_{A,B}(x) - h_{A,B}(y) \right] \chi_{x,y}.$$  

(8.17)

The test flow is finally defined by

$$\Phi_{A,B} = \Phi^*_{V_{A,B}} + \chi_{A,B}.$$  

(8.18)

The following lemma is believed to hold in typical metastability situations. Recall from Section 6.1 the notation $\xi^*_{N} \in \mathcal{E}_x, x \in S_*$, which indicates the configuration for which all the particles are located at site $x$.

Lemma 8.5. Suppose that $A$ and $B$ are disjoint non-empty subsets of $S_*$ and that $x \in S_* \setminus (A \cup B)$. Then, it holds that

$$\lim_{N \to \infty} \sup_{\eta \in \mathcal{E}_x} \left| h_{\xi(A),\xi(B)}(\eta) - h_{\xi(A),\xi(B)}(\xi^*_{N}) \right| = 0.$$  

Proof. Fix $A$, $B$, and $x \in S_* \setminus (A \cup B)$. For $\eta \in \mathcal{E}_x \setminus \{\xi^*_{N}\}$, define

$$q_N(\eta) = \mathbb{P}^N_\eta \left[ \tau_{\xi(A \cup B)} \left| \tau_{\xi^*_{N}} \right] \text{ and } p_N(\eta) = \mathbb{P}^N_\eta \left[ \tau_{\xi(A)} \left| \tau_{\xi(B)} \right| \tau_{\xi(A \cup B)} \left| \tau_{\xi^*_{N}} \right] \right.$$
As $h_{E(A),E(B)}(\eta) = \mathbb{P}_\eta^N[\tau_{E(A)} < \tau_{E(B)}]$, by the Markov property,

$$h_{E(A),E(B)}(\eta) = \mathbb{P}_\eta^N[\tau_{E(A)} < \tau_{E(B)}|\tau_{E(A\cup B)}] (1 - q_N(\eta)) + p_N(\eta) q_N(\eta) = h_{E(A),E(B)}(\xi_N^x) (1 - q_N(\eta)) + p_N(\eta) q_N(\eta).$$

Therefore,

$$\left| h_{E(A),E(B)}(\eta) - h_{E(A),E(B)}(\xi_N^x) \right| = q_N(\eta) \left| p_N(\eta) - h_{E(A),E(B)}(\xi_N^x) \right| \leq q_N(\eta). \quad (8.19)$$

It is well known (for instance, [23, display (3.2)]) that

$$q_N(\eta) \leq \text{cap}_N(\eta, E(A \cup B)) \text{cap}_N(\eta, \xi_N^x). \quad (8.20)$$

By the monotonicity of the capacity, we have $\text{cap}_N(\eta, E(A \cup B)) \leq \text{cap}_N(E^x, \tilde{E}^x)$. Hence, by (6.2) and (8.20),

$$\lim_{N \to \infty} \sup_{\eta \in E^x} q_N(\eta) = 0.$$ 

In view of (8.19), the last estimate completes the proof. \hfill \Box

Now we are ready to prove Proposition 6.7.

**Proof of Proposition 6.7.** We claim that $\Phi_{A,B}$ defined in (8.18) fulfills all the requirements presented in the statement of the proposition. By (8.17) and the Cauchy-Schwarz inequality,

$$||\chi_{A,B}||^2 \leq \frac{\kappa_*(\kappa_* - 1)}{2} \sum_{\{x,y\} \subset S^*} [h_{A,B}(x) - h_{A,B}(y)]^2 \||\chi_{x,y}\||^2.$$

Hence, by (8.18) and Proposition 8.1,

$$||\Phi_{A,B} - \Phi^*_{A,B}||^2 = ||\chi_{A,B}||^2 = (o_N(1) + o_\epsilon(1)) N^{-(1 + \alpha)}. \quad (8.19)$$

This proves part (1).

Now we consider part (2). The set $\Delta_N$ can be decomposed as

$$(G^c)_{\text{int}} \cup \partial^{\text{out}} G \cup \partial^{\text{in}} G \cup \left( \bigcup_{x,y \in S^*} J_{\text{int}}^{x,y} \right) \cup \left( \bigcup_{x \in S^*} (D_{\text{int}}^x \setminus E^x) \right).$$

On $(G^c)_{\text{int}}$, both $\Phi^*_{A,B}$ and $\chi_{A,B}$ are divergence-free, by (1) of Proposition 8.1 and (1) of Proposition 8.4, respectively. Thus,

$$\text{(div } \Phi_{A,B})(\eta) = 0 \quad \text{for all } \eta \in (G^c)_{\text{int}}. \quad (8.21)$$

On $\partial^{\text{out}} G$, the flow $\chi_{A,B}$ is divergence-free by (1) of Proposition 8.4, hence,

$$\text{(div } \Phi_{A,B})(\eta) = \text{(div } \Phi^*_{A,B})(\eta) \quad \text{for all } \eta \in \partial^{\text{out}} G.$$
Thus, by (2) of Proposition 8.4
\[ \sum_{\eta \in \partial^{\text{out}} G} |(\text{div} \Phi_{A,B})(\eta)| = o_N(1) N^{-(1+\alpha)}. \] (8.22)

For \( \partial^{\text{in}} G \), by the triangle inequality, by (2) of Proposition 8.1, and by (1), (2) of Proposition 8.4, we have
\[ \sum_{\eta \in \partial^{\text{in}} G} |(\text{div} \Phi_{A,B})(\eta)| \leq \sum_{\eta \in \partial^{\text{in}} G} |(\text{div} \Phi^*_{V_{A,B}})(\eta)| + \sum_{\eta \in \partial^{\text{in}} G} |(\text{div} \chi_{x,y})(\eta)| = o_N(1) N^{-(1+\alpha)}. \] (8.23)

On \( J_{x,y} \), by (8.3), the flow \( \Phi_{A,B} \) can be written as
\[ \Phi_{A,B}(\eta, \zeta) = [h_{A,B}(y) - h_{A,B}(x)] (\Phi^*_{W_{x,y}} + \chi_{x,y})(\eta, \zeta) ; \eta \in J_{x,y}^{x,y}. \]

Hence, by (4) of Proposition 8.4,
\[ (\text{div} \Phi_{A,B})(\eta) = 0 \text{ for all } \eta \in J_{x,y}^{x,y}, x, y \in S_* \]. (8.24)

Finally, on \( D_{\text{int}}^x \setminus \mathcal{E}^x \), the flow \( \Phi_{V_{A,B}}^* \) is divergence-free by (1) of Proposition 8.1. Hence, by (1) and (2) of Proposition 8.4 for all \( x \in S_* \), we have
\[ \sum_{\eta \in D_{\text{int}}^x \setminus \mathcal{E}^x} |(\text{div} \Phi_{A,B})(\eta)| = \sum_{\eta \in V^x \setminus \mathcal{E}^x} |(\text{div} \chi_{A,B})(\eta)| = o_N(1) N^{-(1+\alpha)}. \] (8.25)

Combining (8.21)–(8.25) yields the proof of part (2).

Part (3) is now considered. As \( \Phi_{V_{A,B}}^* \) is divergence-free on \( \mathcal{E}(S_*) \), it follows from (1) and (3) of Proposition 8.4 that for \( x \in S_* \setminus (A \cup B) \),
\[ (\text{div} \Phi_{A,B})(\mathcal{E}^x) = \sum_{y \in S_* \setminus \{x\}} (h_{A,B}(x) - h_{A,B}(y)) (\text{div} \chi_{x,y})(\mathcal{E}^x) \]
\[ = N^{-(1+\alpha)} \left[ o_N(1) + \sum_{y \in S_* \setminus \{x\}} (h_{A,B}(x) - h_{A,B}(y)) \frac{\text{cap}_X(x, y)}{\kappa_* M_* \Gamma(\alpha) I_\alpha} \right] \] (8.26)
\[ = N^{-(1+\alpha)} [o_N(1) - \mu(x) (\mathcal{L}_Y h_{A,B})(x)] = o_N(1) N^{-(1+\alpha)}, \]
where the last equality follows from the fact that \( \mathcal{L}_Y h_{A,B} \equiv 0 \) on \( S_* \setminus (A \cup B) \). This proves the first identity. To prove the second identity, it is first claimed that
\[ \sum_{\eta \in \mathcal{E}^x} |(\text{div} \Phi_{A,B})(\eta)| \leq C N^{-(1+\alpha)}. \] (8.27)

To prove this, based on the trivial bound \( |h_{A,B}(y) - h_{A,B}(x)| \leq 1 \), we have
\[ \sum_{\eta \in \mathcal{E}^x} |(\text{div} \Phi_{A,B})(\eta)| \leq \sum_{y \in S_* \setminus \{x\}} \sum_{\eta \in \mathcal{E}^x} |(\text{div} \chi_{x,y})(\eta)|. \]
It should be noted that $\nabla \chi_{x,y}(\eta) = 0$ for all $\eta \in E^x$ by (3) of Proposition 8.4. Thus, the bound \ref{8.27} is a direct consequence of the estimate in (3) of Proposition 8.4. Now we prove part (3). By the triangle inequality and Lemma 8.5,

$$\left| \sum_{\eta \in E} h_{E(A), E(B)}(\eta) \left( \nabla \Phi_{A,B}(\eta) \right) \right| \leq o_N(1) \sum_{\eta \in E} \left( \nabla \Phi_{A,B}(\eta) \right) + h_{E(A), E(B)}(\xi^x) \sum_{\eta \in E} \left( \nabla \Phi_{A,B}(\eta) \right).$$

The first term of the right-hand side is $o_N(1) N^{-1+\alpha}$ by \ref{8.27}, whereas the second term is $o_N(1) N^{-1+\alpha}$ by \ref{8.26}. This verifies the second identity of part (3).

For part (4), by a computation as in \ref{8.26},

$$\left( \nabla \Phi_{A,B}(E(A)) \right) = \sum_{x \in A} \sum_{y \in S \setminus \{x\}} \left[ h_{A,B}(x) - h_{A,B}(y) \right] \left( \nabla \chi_{x,y}(E^x) \right) = (1 + o_N(1)) N^{-1+\alpha} \sum_{x \in A} \mu(x) \left( -\mathcal{L}_y h_{A,B} \right)(x) = (1 + o_N(1)) N^{-1+\alpha} \text{cap}_y(A, B),$$

where the last line follows from elementary properties of the capacity. The proof of the estimate for $\left( \nabla \Phi_{A,B}(E(B)) \right)$ is identical. \hfill \( \square \)

It should be remarked that, the approximation $\Phi_{A,B}^*$ of $\Phi_{h_{E(A), E(B)}}$ can be constructed by an identical procedure and it can be verified that this flow enjoys all the corresponding properties in Proposition 6.7.

8.4. Proof of Proposition 8.4. The proof of Proposition 8.4 is divided into a series of lemmas. The correspondence between these lemmas and Proposition 8.4 will be explained at the end of the this subsection.

**Lemma 8.6.** The flow $\chi_{x,y}$ is divergence-free on $\left( V^x \cup \mathcal{V}^y \cup J^{x,y} \right)^c$.

**Proof.** In view of the definition of $\chi_{x,y}$, the only part that should be verified is $(T^{x,y}_{\text{int}} \cap D^x) \setminus V^x$ and $(T^{x,y}_{\text{int}} \cap D^y) \setminus \mathcal{V}^y$, as $B(\eta; z_i) = 0$ for $\eta \in D^x$ or $\eta \in D^y$ for all $1 \leq i \leq \kappa$. Hence, for $\eta \in (T^{x,y}_{\text{int}} \cap D^x) \setminus V^x$ or $(T^{x,y}_{\text{int}} \cap D^y) \setminus \mathcal{V}^y$, we have

$$(\nabla \chi_{x,y})(\eta) = \chi_{x,y}(\eta, \sigma^{x,y}\eta) + \chi_{x,y}(\eta, \sigma^{y,x}\eta) = C(\eta) - C(\sigma^{y,x}\eta).$$

The last expression equals to 0 since, by the definition \ref{8.8}, $C(\cdot)$ is a function of $\hat{\eta}$ only. \hfill \( \square \)

**Lemma 8.7.** It holds that

$$\sum_{\eta \in \partial^{\text{in}} J^{x,y}} \left| (\nabla \chi_{x,y})(\eta) \right| = o_N(1) N^{-(1+\alpha)}.$$
Proof. From the definition of $\chi_{x,y}$, it is immediate that $|(\text{div} \, \chi_{x,y})(\eta)| \leq C \mu_N(\eta)$ for some $C > 0$. Hence, the lemma is a direct consequence of Lemma 7.4. □

Lemma 8.8. It holds that

$$\sum_{\eta \in V^x \setminus \mathcal{E}^x} |(\text{div} \, \chi_{x,y})(\eta)| = o_N(1) N^{-(1+\alpha)}$$

and

$$\sum_{\eta \in V^y \setminus \mathcal{E}^y} |(\text{div} \, \chi_{x,y})(\eta)| = o_N(1) N^{-(1+\alpha)} .$$

Proof. By the same reasoning as in the proof of Lemma 8.6,

$$(\text{div} \, \chi_{x,y})(\eta) = \begin{cases} \chi_{x,y}(\eta, \sigma_{x,y} \eta) = C(\eta) & \text{if } \eta \in V_x , \\ \chi_{x,y}(\eta, \sigma_{y,x} \eta) = -C(\sigma_{y,x} \eta) &= -C(\eta) & \text{if } \eta \in V_y . \end{cases} \quad (8.28)$$

Thus, by the definition of $C(\cdot)$,

$$\sum_{\eta \in V^x \setminus \mathcal{E}^x} |(\text{div} \, \chi_{x,y})(\eta)| \leq C N^{-(\alpha+1)} \sum_{\eta \in V^x \setminus \mathcal{E}^x} \frac{m^x_\eta}{a(\eta)} . \quad (8.29)$$

For the configurations $\eta \in V^x$, we have that $m^x_\eta = 0$; hence, by Lemma 7.1,

$$\sum_{\eta \in V^x \setminus \mathcal{E}^x} \frac{m^x_\eta}{a(\eta)} = \sum_{k=\ell_{N}+1}^{\pi_N} \sum_{\zeta \in H_{k,S}(x,y)} \frac{m^\zeta_\eta}{a(\zeta)} \leq C \sum_{k=\ell_{N}+1}^{\pi_N} \frac{1}{k^{\alpha}} \leq C \ell_{N}^{-1} = o_N(1) \quad (8.30)$$

It should be noted that the fact that $m^x(x) = m^y(y) = 1$ was used at the first equality of (8.30). Thus, the first estimate of the lemma is obtained from (8.29) and (8.30). The proof for the second estimate is identical. □

Lemma 8.9. We have that

$$(\text{div} \, \chi_{x,y})(\mathcal{E}^x) = (1 + o_N(1)) N^{-(1+\alpha)} \frac{\text{cap}_X(x,y)}{\kappa_* M_* \Gamma(\alpha) I_\alpha} \text{ and }$$

$$(\text{div} \, \chi_{x,y})(\mathcal{E}^y) = - (1 + o_N(1)) N^{-(1+\alpha)} \frac{\text{cap}_X(x,y)}{\kappa_* M_* \Gamma(\alpha) I_\alpha} .$$

Proof. By (8.28) and an argument similar to that in the previous lemma,

$$(\text{div} \, \chi_{x,y})(\mathcal{E}^x) = \sum_{\eta \in \mathcal{E}^x} C(\eta) = \frac{\text{cap}_X(x,y)}{N^{\alpha+1}} \sum_{k=0}^{\ell_N} \sum_{\zeta \in H_{k,S}(x,y)} \frac{m^\zeta_\eta}{a(\zeta)} .$$

Hence, by Proposition 2.1 and Lemma 7.12, the first identity of the lemma is proved. The proof for the second identity is identical. □

An identity for the function $B(\cdot, \cdot)$ is now established.

Lemma 8.10. For all $\eta \in H_N$ such that $\eta_x \geq 1$,

$$\sum_{i=1}^{\kappa} m(z_i) B(\sigma^x_{z_i} \eta; z_i) = 0 .$$
Proof. For $1 \leq u < \kappa$, let

$$Q_u(\eta) = [h_{x,y}(z_u) - h_{x,y}(z_{u+1})] \left[ H \left( \frac{\eta(u)}{N} \right) - H \left( \frac{\eta(u) - 1}{N} \right) \right].$$

Then,

$$B(\sigma^x, z; z_i) = \sum_{j: j > i} r(z_i, z_j) \sum_{u=i}^{j-1} Q_u(\eta) - \sum_{j: j < i} r(z_i, z_j) \sum_{u=j}^{i-1} Q_u(\eta).$$

Therefore,

$$\sum_{i=1}^{\kappa} m(z_i) B(\sigma^x, z; z_i)$$

$$= \sum_{i=1}^{\kappa} \sum_{j: j > i} m(z_i) r(z_i, z_j) \sum_{u=i}^{j-1} Q_u(\eta) - \sum_{j: j < i} m(z_i) r(z_i, z_j) \sum_{u=j}^{i-1} Q_u(\eta)$$

$$= \sum_{u=1}^{\kappa-1} Q_u(\eta) \sum_{i=1}^{u} \sum_{j=u+1}^{\kappa} [m(z_i) r(z_i, z_j) - m(z_j) r(z_j, z_i)]$$

$$= \sum_{u=1}^{\kappa-1} Q_u(\eta) \sum_{i=1}^{u} \sum_{j=1}^{\kappa} [m(z_i) r(z_i, z_j) - m(z_j) r(z_j, z_i)] = 0.$$

It should be remarked that the last equality is a consequence of (2.1), whereas the third equality holds owing to the following which in turn holds by the symmetry of the summation:

$$\sum_{i=1}^{u} \sum_{j=1}^{u} [m(z_i) r(z_i, z_j) - m(z_j) r(z_j, z_i)] = 0 \ ; \ 1 \leq u \leq \kappa.$$

□

Lemma 8.11. The flow $\Phi_{x,y}$ is divergence-free on $J_{int}^{x,y}$.

Proof. We fix a configuration $\eta \in J_{int}^{x,y}$ so that $\eta_x, \eta_y \geq 1$. By the definition of $\chi_{x,y}$ we have

$$\text{(div } \chi_{x,y})(\eta) = \sum_{i=2}^{\kappa-1} \{ \chi_{x,y}(\eta, \sigma^{zi,x} \eta) + \chi_{x,y}(\eta, \sigma^{zi,y} \eta) \}$$

$$+ \sum_{i=2}^{\kappa-1} \chi_{x,y}(\eta, \sigma^{zzi} \eta) + \sum_{i=2}^{\kappa-1} \chi_{x,y}(\eta, \sigma^{zy} \eta) + \chi_{x,y}(\eta, \sigma^{xyz} \eta) + \chi_{x,y}(\eta, \sigma^{xyz} \eta).$$

(8.31)

The right-hand side can be computed term by term. The terms $\chi_{x,y}(\eta, \sigma^{zi,x} \eta), \chi_{x,y}(\eta, \sigma^{zi,y} \eta)$ and $\chi_{x,y}(\eta, \sigma^{zi,z1} \eta)$ are immediate from the definition, whereas the term $\chi_{x,y}(\eta, \sigma^{zi,z1} \eta), \chi_{x,y}(\eta, \sigma^{zi,z1} \eta)$,
2 \leq i \leq \kappa - 1, \text{ can be evaluated as }

\chi_{x,y}(\eta, \sigma^{x,z_i} \eta, \eta) = -\chi_{x,y}(\sigma^{x,z_i} \eta, \sigma^{x,z_i} \eta, \sigma^{x,z_i} \eta) = \frac{1}{2} \sum_{i=2}^{\kappa-1} a_N \mu_{N-1} (\sigma^{x,z_i} \eta - \omega^z) m(z_i) B(\sigma^{x,z_i} \eta; z_i)

= \frac{1}{2} \sum_{i=2}^{\kappa-1} a_N \mu_{N-1} (\eta - \omega^x) m(z_i) B(\sigma^{x,z_i} \eta; z_i).

It should be noted that the fact that \( \eta_x \geq 1 \) was implicitly used in this computation. Similarly, we can express \( \chi_{x,y}(\eta, \sigma^{y,z_i} \eta) \) and \( \chi_{x,y}(\eta, \sigma^{x,y} \eta) \). By inserting these results into (8.31), we obtain

\[
(\text{div } \chi_{x,y})(\eta) = -\sum_{i=2}^{\kappa-1} a_N \mu_{N-1} (\eta - \omega^z) m(z_i) B(\eta; z_i) \{ \eta_{z_i} \geq 1 \}
\]

Hence, by (8.6),

\[
(\text{div } \Phi_{x,y})(\eta) = (\text{div } \Phi_{x,y}^\ast W_{x,y})(\eta) + (\text{div } \chi_{x,y})(\eta)
\]

\[
= \frac{1}{2} a_N \mu_{N-1} (\eta - \omega^x) \sum_{i=1}^{\kappa} m(z_i) B(\sigma^{x,z_i} \eta; z_1)
\]

\[
+ \frac{1}{2} a_N \mu_{N-1} (\eta - \omega^y) \sum_{i=1}^{\kappa} m(z_i) B(\sigma^{y,z_i} \eta; z_1).
\]

The last expression is equal to 0 by Lemma 8.10. In particular, the second summation can be rewritten as

\[
\sum_{i=1}^{\kappa} m(z_i) B(\sigma^{x,z_i} (\sigma^{y,x} \eta); z_1).
\]

Thus, Lemma 8.10 can be applied by replacing \( \eta \) with \( \sigma^{y,x} \eta \) to verify that this summation is 0.

\[\square\]

The proof of Proposition 8.4 can be completed by combining the results obtained above.
Proof of Proposition 8.4. Part (1) is proven in Lemma 8.6, and part (2) is a direct consequence of Lemmas 8.7 and 8.8. Part (3) is immediate from Lemma 8.9 and (8.28). Part (4) has been verified in Lemma 8.11.

8.5. Proof in the general case. In the previous proof for the special case, the assumption (8.2) was used only in the construction of the correction flow \( \chi_{x,y} \). Namely, this assumption allowed the definition of \( \chi_{x,y}(\eta, \sigma^{u,v}\eta) \) for any \( u, v \in S \) without any restriction. In the general case, if \( r(u, v) = 0 \), we cannot define \( \chi_{x,y}(\eta, \sigma^{u,v}\eta) \); therefore, the cleaning of divergence on \( \eta \) is not immediate. This can be resolved by the canonical path introduced in Section 4.2. This argument is now explained in detail.

Let us fix two points \( x, y \in S_* \). Previously, for \( \eta \in T^x_{\text{int}} \) such that \( \eta_{z_i} \geq 1 \), we defined:

\[
\chi_{x,y}(\eta, \sigma^{z_i,x}\eta) = -\frac{1}{2}a_N \mu_N(\eta - \omega^{z_i}) m(z_i) B(\eta; z_i) .
\] (8.32)

As mentioned earlier, this object is meaningless if \( r(z_i, x) = 0 \). Hence, the canonical path \( z_i = w_1, w_2, \ldots, w_k = x \) between \( z_i \) and \( x \) should be invoked. By the property of the canonical path, we have:

\[
r(w_i, w_{i+1}) > 0 \quad \text{for all} \quad 1 \leq i \leq k-1 \quad \text{and} \quad k \leq \kappa .
\] (8.33)

For each \( \eta \) and \( 1 \leq i \leq k \), let \( \eta^i = \sigma^{w_{i+1},w_i}\eta \), so that \( \eta^1 = z_i \) and \( \eta^k = \sigma^{z_i,x}\eta \). With these notations, for \( \eta \in T^x_{\text{int}} \) with \( \eta_{z_i} \geq 1 \), the previous definition (8.32) of the correction flow can be replaced with:

\[
\hat{\chi}_{x,y}(\eta^1, \eta^2) = \cdots = \hat{\chi}_{x,y}(\eta^{k-1}, \eta^k) = -\frac{1}{2}a_N \mu_N(\eta - \omega^{z_i}) m(z_i) B(\eta; z_i) .
\] (8.34)

Of course, \( \hat{\chi}_{x,y}(\eta^{i+1}, \eta^i) \) is defined by \( -\hat{\chi}_{x,y}(\eta^i, \eta^{i+1}) \) for all \( i \). The crucial observation here is that:

\[
\eta^{i+1} = \sigma^{w_{i+1},w_i}\eta^i \quad \text{for all} \quad 1 \leq i \leq k-1 \quad \text{and} \quad \eta^i_{w_i} \geq 1 ,
\]

so that \( \hat{\chi}_{x,y}(\eta^i, \eta^{i+1}) \) is meaningful by (8.33). Furthermore, the construction (8.34) changes the divergence of \( \eta^1 = z_i \) and \( \eta^k = \sigma^{z_i,x}\eta \) only; the divergence at \( \eta^i, 2 \leq i \leq k-1 \), is not affected by this construction. Thus, as far as divergence is concerned, this new object plays the exact same role as (8.32). The flow \( \hat{\chi}_{x,y}(\eta, \sigma^{y,x}\eta) \) corresponding to \( \chi_{x,y}(\eta, \sigma^{y,x}\eta) \) can be constructed by a similar argument. The construction of the correction flow \( \hat{\chi}_{x,y} \) can be thereby completed in the general case.

The validity of Propositions 8.3 and 8.4 should now be verified for this general object. For Proposition 8.3, the same argument can be used; the only difference is that the same edge is used by several canonical paths. Here, the Cauchy-Schwarz inequality can be used for obtaining an upper bound, as the number of canonical paths using a certain edge is bounded by a uniform constant. In particular, the uniform bound on the length of canonical paths,
i.e., the condition $k \leq \kappa$ in (8.33), is crucially used here. Moreover, by the observation on the divergence in the previous paragraph, the validity of Proposition 8.4 is immediate.

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