FRAME ANALYSIS AND APPROXIMATION IN REPRODUCING KERNEL HILBERT SPACES

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Abstract. We consider frames $F$ in a given Hilbert space, and we show that every $F$ may be obtained in a constructive way from a reproducing kernel and an orthonormal basis in an ambient Hilbert space. The construction is operator-theoretic, building on a geometric formula for the analysis operator defined from $F$. Our focus is on the infinite-dimensional case where a priori estimates play a central role, and we extend a number of results which are known so far only in finite dimensions. We further show how this approach may be used both in constructing useful frames in analysis and in applications, and in understanding their geometry and their symmetries.

1. Introduction

Frames are redundant bases which turn out in certain applications to be more flexible than the better known orthonormal bases (ONBs) in Hilbert space. The frames allow for more symmetries than ONBs do, especially in the context of signal analysis, and of wavelet constructions; see, e.g., [CoDa93, BDP05, Dut06]. Since frame bases (although containing redundancies) still allow for efficient algorithms, they have found many applications, even in finite dimensions; see, for example, [BeFi03, CaCh03, Chr03, Eld02, FJKO05, VaWa05].

As is well known, when a vector $f$ in a Hilbert space $\mathcal{H}$ is expanded in an orthonormal basis $B$, there is then automatically an associated Parseval identity. In physical terms, this identity typically reflects a stability feature of a decomposition based on the chosen ONB $B$. Specifically, Parseval’s identity reflects a conserved quantity for a problem at hand, for example, energy conservation in quantum mechanics.

The theory of frames (see Definitions 6.1) begins with the observation that there are useful vector systems which are in fact not ONBs but for which a Parseval formula still holds. In fact, in applications it is important to go beyond ONBs. While this viewpoint originated in signal processing (in connection with frequency bands, aliasing, and filters), the subject of frames appears now to be of independent interest in mathematics.

On occasion, we may have a system of vectors $S$ in $\mathcal{H}$ for which Parseval’s identity is still satisfied, but such that a generalized Parseval’s identity might only hold up to a fixed constant $c$ of scale. (For example, in sampling theory, a scale might be introduced as a result of “oversampling”.) In this case, we say that the constant

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c scales the expansion. Suppose a system of vectors $S$ in a given Hilbert space $H$ allows for an expansion, or decomposition of every $f$ in $H$, but the analogue of Parseval’s identity holds only up to a fixed constant $c$ of scale. In that case, we say that $S$ is a *tight frame* with frame constant $c$. So the special case $c = 1$ is the case of a Parseval frame. For precise definitions of these terms, we refer to Section 2 below, or to the book literature, e.g., [Chr03].

Aside from applications, at least three of the other motivations for frame theory come from: (1) wavelets, e.g., [CoDa93], [BJMP05], and [BJMP06]; (2) from non-harmonic Fourier expansions [DuSc52]; and (3) from computations with polynomials in several variables, and their generalized orthogonality relations [DuXu01].

While frames already have impressive uses in signal processing (see, e.g., [ALTW04, Chr99]), they have recently [CCLV05, CKL04] been shown to be central in our understanding of a fundamental question in operator algebras, the Kadison–Singer conjecture. We refer the reader to [CFTW06] for up-to-date research, and to [Chr99, KaRi97, Nel57, Nel59] for background.

In all these cases, the authors work with recursive algorithms, and the issue of *stability* plays a crucial role. Stability, however, may obtain in situations that are much more general than the context of traditional ONBs, or even tight frames. In fact, stability may apply even when we have only *a priori* estimates, as opposed to identities: for example, when the scaled version of Parseval’s identity is replaced with a pair of estimates, a fixed lower bound and an upper bound; see (6.1) below. If such bounds exist, they are called lower and upper frame bounds.

If a system $S$ of vectors in a Hilbert space $H$ satisfies such a pair of *a priori* estimates, we say that $S$ is simply a *frame*. And if such an estimate holds only with an *a priori* upper bound, we say that $S$ is a *Bessel sequence*. It is known (see, e.g., [AkGl93]) that for a fixed Hilbert space $H$, the various classes of frames $S$ in $H$ may be obtained from some ambient Hilbert space $K$ and an orthonormal basis $B$ in $K$, i.e., when the pair $(S, H)$ is given, there are choices of $K$ such that the frame $S$ may be obtained from applying a certain bounded operator $T$ to a suitable ONB $B$ in $K$.

Passing from the given structure in $H$ to the ambient Hilbert space is called *dilation* in operator theory. The properties of the operator $T$ which does the job depend on the particular frame in question. For example, if $S$ is a Parseval frame, then $T$ will be a projection of the ambient Hilbert space $K$ onto $H$. But this operator-theoretic approach to frame theory has been hampered by the fact that the ambient Hilbert space is often an elusive abstraction. Starting with a frame $S$ in a fixed Hilbert space $H$, then by dilation, or extension, we pass to an ambient Hilbert space $K$.

In this paper we make concrete the selection of the “magic” operator $T$: $K \to H$ which maps an ONB in $K$ onto $S$. While existence is already known, the building of a dilation system $(K, T, ONB)$ is often rather non-constructive, and the various methods for getting $K$ are fraught with choices that are not unique.

Nonetheless, it was shown recently [Dut04b, Dut06] that when the dilation approach is applied to Parseval frames of wavelets in $H = L^2(\mathbb{R})$, i.e., to wavelet bases which are not ONBs, then the ambient Hilbert space $K$ can be made completely explicit, and the constructions are algorithmic. Moreover, the “inflated” ONB in $K$ then takes the form of a traditional ONB-wavelet basis, a so-called “super-wavelet”. For details, see [BMP05, Dut04b], and also the papers [Kol04, BMP05, BMP06].

It is the purpose of the present paper to show that the techniques which work well in this restricted context, “super-wavelets” and redundant wavelet frames, apply to
a more general and geometric context, one which is motivated in turn by extension
principles in probability theory; see, e.g., PaSc72, Dut04a, and Jor06.

A key idea in our present approach is the use of reproducing Hilbert spaces, and
their reproducing kernels in the sense of Aro50. See also Nel57, Nel59 for an
attractive formulation. We show that for every Hilbert space \( H \), and every frame
\( S \) in \( H \) (even if \( S \) is merely a Bessel sequence), there is a way of constructing
the ambient Hilbert space \( K \) in such a way that the operator \( T \) has a concrete
reproducing kernel.

Finally, we mention that a recent paper VaWa05 serves as a second motivation
for our work; in fact VaWa05 contains finite-dimensional cases of two of our present
theorems. These results in VaWa05 are Theorems 2.9 and 6.4 in that paper: The
results in VaWa05 are concerned with symmetries of tight frames, and with
associated families of unitary representations of the symmetry groups. It turns out
that this approach to symmetry is natural in the context of operator theory; see,
e.g., PaSc72, Dut04a, Dut06.

2. Preliminary notions and definitions

Let \( S \) be a countable set, finite or infinite, and let \( \mathcal{H} \) be a complex or real
Hilbert space. We shall be interested in a class of spanning families of vectors
\( (v(s)) \) in \( \mathcal{H} \) indexed by points \( s \in S \). Their properties will be defined precisely
below, and the families are termed frames. The simplest instance of this is when
\( \mathcal{H} = \ell^2(S) \) = the Hilbert space of all square-summable sequences, i.e., all \( f : S \rightarrow \mathbb{C} \)
such that \( \sum_{s \in S} |f(s)|^2 < \infty \). In that case, set
\[
\langle f_1 | f_2 \rangle := \sum_{s \in S} \overline{f_1(s)} f_2(s)
\]
for all \( f_1, f_2 \in \ell^2(S) \).

It is then immediate that the delta functions \( \{ \delta_s | s \in S \} \) given by
\[
\delta_s(t) = \begin{cases} 1, & t = s, \\ 0, & t \in S \setminus \{s\}, \end{cases}
\]
form an orthonormal basis (ONB) for \( \mathcal{H} \), i.e., that
\[
\langle \delta_{s_1} | \delta_{s_2} \rangle = \begin{cases} 1, & \text{if } s_1 = s_2 \text{ in } S, \\ 0, & \text{if } s_1 \neq s_2, \end{cases}
\]
and that this is a maximal orthonormal family in \( \mathcal{H} \). Moreover,
\[
f = \sum_{s \in S} f(s) \delta_s \quad \text{for all } f \in \ell^2(S).
\]

It also is immediate from (2.1) that Parseval’s formula
\[
\|f\|^2 = \sum_{s \in S} |\langle \delta_s | f \rangle|^2
\]
holds for all \( f \in \ell^2(S) \).

We shall consider pairs \((S, \mathcal{H})\) and indexed families
\[
\{ v(s) | s \in S \} \subset \mathcal{H}
\]
such that for some $c \in \mathbb{R}_+$, the identity

$$\|f\|^2 = c \sum_{s \in S} |\langle v(s) \mid f \rangle|^2$$

holds for all $f \in \mathcal{H}$.

When a Hilbert space $\mathcal{H}$ is given, our main result states that solutions to (2.7) exist if and only if $\mathcal{H}$ is isometrically embedded in $\ell^2(S)$. But we further characterize these embeddings, and we use this in understanding the geometry of tight frames (details below).

**Definition 2.1.** Let $(S, \mathcal{H}, c, (v(s))_{s \in S})$ be as above. We shall say that this system constitutes a **tight frame** with frame constant $c$ if (2.7) holds.

(Note that if $(v(s))_{s \in S}$ satisfies (2.7), then the scaled system $(\sqrt{c}v(s))_{s \in S}$ has the property with frame constant one.)

**Example 2.2.** (S finite.) Let $\mathcal{H}$ be the two-dimensional real Hilbert space, and let $n \geq 3$. Set $S := \{1, 2, \ldots, n\} =: S_n$, and

$$\langle v(s) := \begin{pmatrix} \cos \left( \frac{2\pi s}{n} \right) \\ \sin \left( \frac{2\pi s}{n} \right) \end{pmatrix}, \quad s \in S. \right.$$ 

Then it is easy to see that this constitutes a tight frame with frame constant $c = \frac{2}{n}$. Examples are presented in Figures 1 and 2.
Definition 2.3. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces over $\mathbb{C}$ or $\mathbb{R}$, and let $V: \mathcal{H} \rightarrow \mathcal{K}$ be a linear mapping. We say that $V$ is an isometry, and that $\mathcal{H}$ is isometrically embedded in $\mathcal{K}$ (via $V$) if

$$\|Vf\|_\mathcal{K} = \|f\|_\mathcal{H}, \quad f \in \mathcal{H}.$$  

Given a linear operator $V: \mathcal{H} \rightarrow \mathcal{K}$, we then denote the adjoint operator $V^*: \mathcal{K} \rightarrow \mathcal{H}$. It is easy to see that $V$ is isometric iff $V^*V = I_\mathcal{H}$, where $I_\mathcal{H}$ denotes the identity operator in $\mathcal{H}$. Moreover, if $V$ is isometric then $P = PV = VV^*: \mathcal{K} \rightarrow \mathcal{K}$ is a projection, i.e.,

$$P = P^* = P^2$$

holds, and the subspace

$$PK \subset \mathcal{K}$$

may be identified with $\mathcal{H}$ via the isometric embedding.

We state our next result only in the case of frame constant $c = 1$, but as noted it easily generalizes.

Theorem 2.4. Let $S$ be a countable set, and let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$ (or $\mathbb{R}$).

Then the following two conditions are equivalent:

(i) There is a tight frame $\{v(s) | s \in S\} \subset \mathcal{H}$ with frame constant $c = 1$;

(ii) $\mathcal{H}$ is isometrically embedded (as a closed subspace) in $\ell^2(S)$.

Proof. The details of proof will follow in the next section. □

Remark 2.5. Tight frames with frame constant equal to one are called Parseval frames.

3. Proof of Theorem

Let the data be as specified in the theorem: A pair $(S, \mathcal{H})$ is given where $S$ is a set and $\mathcal{H}$ is a Hilbert space. The conclusion is that (i) the $S$-tight-frame property for $\mathcal{H}$ is equivalent to (ii) the (isometric) embedding of $\mathcal{H}$ into $\ell^2(S)$.

Assume (i), and let $\{v(s) | s \in S\} \subset \mathcal{H}$ be the frame system which is asserted. Now define

$$V: \mathcal{H} \ni f \mapsto (\langle v(s) | f \rangle_{\mathcal{H}})_{s \in S}.$$  

This operator is called the analysis operator for the frame.

Equivalently, the analysis operator $V$ may be written in terms of the canonical ONB $(\delta_s)_{s \in S}$ for $\ell^2(S)$ as follows:

$$Vf = \sum_{s \in S} \langle v(s) | f \rangle \delta_s \quad (\in \ell^2(S)).$$

Our first two observations are that the range of $V$ is contained in $\ell^2(S)$, and that $V$ is isometric. But note that both these conclusions are immediate consequences of identity (2.7) for the special case $c = 1$.

We formalize this in the following lemma. (The proof of converse implication in Theorem 2.4 will be resumed after the lemma.)

Lemma 3.1. Let $(v(s))_{s \in S}$ be a system in the Hilbert space $\mathcal{H}$, and let $V$ be defined by (3.1). Then (2.7) holds if and only if $\sqrt{c}V$ is isometric.
An easy calculation yields the formula for the adjoint operator:

\[(3.3) \quad V^* (c_s)_{s \in S} = \sum_{s \in S} c_s v(s), \quad (c_s) \in \ell^2(S).\]

Hence

\[(3.4) \quad f = V^* V f = \sum_{s \in S} \langle v(s) | f \rangle v(s)\]

holds for all \( f \in \mathcal{H}. \)

Moreover, the projection operator \( P := V V^* \) is given by the formula

\[(3.5) \quad (P(c_s))_t = \sum_{s \in S} \langle v(t) | v(s) \rangle_H c_s;\]

in other words, \( P \) has a concrete matrix representation in the Hilbert space \( \ell^2(S). \)

Specifically, \( P \) is represented as multiplication on column vectors \((c_s)_{s \in S}\), and the matrix of \( P \) is

\[(3.6) \quad P(t,s) = \langle v(t) | v(s) \rangle_H .\]

To prove the converse, assume (ii). Hence, there is an isometry \( V : \mathcal{H} \to \ell^2(S) \).

As we noted, this means that \( P := V V^* \) is a projection in \( \ell^2(S) \), and that

\[(3.7) \quad P\ell^2(S) = \{ Vf | f \in \mathcal{H} \} .\]

Now observe that \( v(s) := P(\delta_s) \), where \( \delta_s \) for each \( s \in S \) is the delta function of \(2.2\). We claim that \((2.7)\) holds for \( c = 1 \). To see this, let \( f \in \mathcal{H} \) be given. Then

\[
\|f\|^2 = \|Vf\|^2 = \sum_{s \in S} |\langle \delta_s | Vf \rangle|^2 \\
= \sum_{s \in S} |\langle \delta_s | PVf \rangle|^2 \\
= \sum_{s \in S} |\langle P\delta_s | Vf \rangle|^2 \\
= \sum_{s \in S} |\langle v(s) | f \rangle|^2 ,
\]

which is the desired conclusion. \( \square \)

**Remark 3.2.** Even when no restricting assumptions are placed on a given system of vectors \( (v(s))_{s \in S} \), the function \( P \) from \((3.6)\) is always positive semidefinite; see Definition \( \text{5.1} \). Moreover, the following converse implication holds (Theorem \( \text{5.2} \) below), known as the **reconstruction principle**: Every positive semidefinite function has the form \((3.6)\) for some Hilbert space and an associated system of vectors. As we show in Section \( \text{4} \) when this is specialized, we find that there is a graduated system of frame properties which may or may not hold for a given system of vectors \( (v(s))_{s \in S} \). Moreover we show that each of these properties reflects a corresponding axiom for the associated function \((3.6)\), Gramian, or correlation matrix. Our results in Sections \( \text{5} \) and \( \text{6} \) below serve three purposes: (1) one is to use reproducing kernels \( \text{Aro50, Nel57, PaSc72} \) and their spectral theory to study classes of frames and their symmetries; (2) another (e.g., Theorems \( \text{6.5 and 6.2} \) is to use an idea from operator theory to establish results about frame deformations and their stability; and finally (3) these results serve to tie the operator theory to more current applications.
Corollary 3.3. For \( i = 1, 2 \), consider two systems \((S_i, \mathcal{H}_i)\) of sets and Hilbert spaces, and assume that \((v_i(s_i))_{s_i \in S_i}\) is a pair of tight frames with respective frame constants \(c_i\). Then

\[
w(s_1, s_2) := v_1(s_1) \otimes v_2(s_2)
\]
defines a tight frame for the tensor-product Hilbert space \(\mathcal{H}_1 \otimes \mathcal{H}_2\) with frame constant \(c = c_1c_2\).

Proof. The result follows immediately from the theorem and Lemma 3.1. To see this, note that if \(V_i, i = 1, 2\), are the operators defined in (3.1) above, then both

\[
\sqrt{c_i} V_i : \mathcal{H}_i \rightarrow \ell^2(S_i)
\]
are isometric embeddings. It then follows that

\[
\sqrt{c_1} V_1 \otimes \sqrt{c_2} V_2 = \sqrt{c_1c_2} V_1 \otimes V_2
\]
is an isometric embedding of \(\mathcal{H}_1 \otimes \mathcal{H}_2\) into \(\ell^2(S_1 \times S_2)\). From the fact that each \(V_i\) is defined from the system \((v_i(s_i))_{s_i \in S_i}\), it follows readily that the tensor operator \(V_1 \otimes V_2\) is defined from the system in (3.8).

Specifically,

\[
w(s_1, s_2) = (V_1 \otimes V_2)^* (\delta_{s_1} \otimes \delta_{s_2}) = V_1^* \delta_{s_1} \otimes V_2^* \delta_{s_2} = v_1(s_1) \otimes v_2(s_2),
\]

which is the desired result. \(\square\)

Definition 3.4. Let \((S, \mathcal{H})\) be a pair as above: \(S\) is a set and \(\mathcal{H}\) is a Hilbert space. Then the matrix

\[
k(t, s) := \langle v(t) | v(s) \rangle_{\mathcal{H}}, \quad s, t \in S,
\]
is called the Gramian, or the Gram matrix.

Corollary 3.5. Let the pair \((S, \mathcal{H})\) be as in the statement of Theorem 2.4. Let \(\{v(s) | s \in S\}\) be a system of vectors in \(\mathcal{H}\) and let \(k(t, s) = \langle v(t) | v(s) \rangle\) be the corresponding Gram matrix. Then \((v(s))_{s \in S}\) is a tight frame with frame constant \(c\) if and only if the following two conditions hold:

\[
\begin{align*}
(a) \quad & k(t, s) = k(s, t), \text{ and} \\
(b) \quad & \sum_{t \in S} k(s_1, t) k(t, s_2) = c^{-1} k(s_1, s_2) \text{ for all } s_1, s_2 \in S.
\end{align*}
\]

Proof. It is immediate that the stated conditions are equivalent to the matrix \(P\) in (3.6) defining a projection when \(P = cK\) and \(K(t, s) := \langle v(t) | v(s) \rangle\). As a consequence, we see that the two conditions \(\mathbf{0}\) and \(\mathbf{1}\) are a restatement of (2.10), i.e., the definition of a projection. (We shall consider only orthogonal projections \(P\), i.e., operators \(P\) where both of the conditions in (2.10) are assumed.) \(\square\)

Remark 3.6. As an application, we are now able to verify the assertion in Example 2.2. First note that the Gram matrix of (2.8) is

\[
k(s_1, s_2) = \cos \left( \frac{2\pi (s_1 - s_2)}{n} \right), \quad s_1, s_2 \in S_n.
\]
Hence property (a) in the corollary is immediate. To verify (b), note that
\[
\sum_{t \in S_n} k(s_1, t) k(t, s_2) = \sum_{t \in S_n} \cos\left(\frac{2\pi (s_1 - t)}{n}\right) \cos\left(\frac{2\pi (t - s_2)}{n}\right)
\]
\[
= \frac{1}{2} \sum_{t \in S_n} \left( \cos\left(\frac{2\pi (s_1 - s_2)}{n}\right) + \cos\left(\frac{2\pi (s_1 + s_2 - 2t)}{n}\right) \right)
\]
\[
= \frac{n}{2} \sum_{t \in S_n} \cos\left(\frac{2\pi (s_1 - s_2)}{n}\right) = c^{-1} k(s_1, s_2)
\]
with \( c = \frac{2}{n} \).

In the last step, we used the trigonometric formula
\[
\sum_{l=1}^{n} \cos\left(\frac{4\pi l}{n} - \theta\right) = 0.
\]
(We refer to Figures 1 to 2 for simple illustrations.)

**Corollary 3.7.** Let the pair \((S, H)\) be as in Corollary 3.5 and Theorem 2.4. Suppose \(\{v(s) \mid s \in S\}\) is a tight frame in \(H\) with frame constant \(c\). Then every \(f\) in \(H\) has the representation
\[
f = c \sum_{s \in S} \langle v(s) \mid f \rangle v(s),
\]
where the sum converges in the norm of the Hilbert space \(H\).

**Proof.** In view of the argument in the proof of Corollary 3.5, we may reduce to the case where the frame constant \(c\) is one, i.e., \(c = 1\). (In general, the Gram matrix \(K(t, s) = \langle v(t) \mid v(s) \rangle_H\) satisfies \(K = c^{-1} P\) where \(P\) is a projection in the Hilbert space \(l^2(S)\).) The reduction to the case \(c = 1\) means that
\[
V : H \ni f \mapsto (\langle v(s) \mid f \rangle)_{s \in S} \in l^2(S)
\]
is isometric. The projection \(P := VV^*\) is multiplication by the matrix
\[
\left(\langle v(t) \mid v(s) \rangle\right)_{s, t \in S}.
\]
Since \(P\) is the projection onto the range of \(V\) in (3.15), we conclude that
\[
\langle v(t) \mid f \rangle = \sum_{s \in S} \langle v(t) \mid v(s) \rangle \langle v(s) \mid f \rangle,
\]
where the convergence is in \(l^2(S)\). But the vectors \(\{v(t) \mid t \in S\}\) span a dense subspace in \(H\), so we get the desired formula
\[
f = \sum_{s \in S} \langle v(s) \mid f \rangle v(s),
\]
now referring to the norm and the inner product in \(H\). Recall that on the range of \(V\), the respective inner products of \(H\) and of \(l^2(S)\) coincide. \(\square\)

**Corollary 3.8.** Let the pair \((S, H)\) be as in Corollary 3.7 and suppose that
\[
\{v(s) \mid s \in S\}
\]
is a tight frame for $\mathcal{H}$ with frame constant $c$. Further suppose that some $\xi: S \to \mathbb{C}$ satisfies
\begin{equation}
 f = c \sum_{s \in S} \xi_s v(s),
\end{equation}
where the sum is convergent in $\mathcal{H}$. Then
\begin{equation}
 \sum_{s \in S} |\xi_s|^2 \geq \sum_{s \in S} |\langle v(s) \mid f \rangle|^2.
\end{equation}

Proof. With the assumptions in the corollary, apply the mapping $V$ from (3.15) to both sides in (3.18). We get the formula
\begin{equation}
 (P\xi)_t = \langle v(t) \mid f \rangle \in \ell^2(S).
\end{equation}
Hence
\begin{equation}
 \xi = P\xi + (I - P)\xi
\end{equation}
is an orthogonal decomposition, i.e.,
\begin{equation}
 \|\xi\|_{\ell^2}^2 = \|P\xi\|_{\ell^2}^2 + \|(I - P)\xi\|_{\ell^2}^2,
\end{equation}
and the conclusion (3.19) is immediate. \hfill \Box

4. Shannon’s example

In the Hilbert space $L^2(\mathbb{R})$ we will consider the usual Fourier transform
\begin{equation}
 \hat{f}(t) := \int_{\mathbb{R}} e^{-i2\pi tx} f(x) \, dx,
\end{equation}
where convergence is understood in the sense of $L^2$. The familiar interpolation formula of Shannon \cite{Ash90} applies to band-limited functions, i.e., to functions $f$ on $\mathbb{R}$ such that the Fourier transform $\hat{f}$ is of compact support. For the present purpose, pick the following normalization
\begin{equation}
 \supp(\hat{f}) \subset \left[ -\frac{1}{2}, \frac{1}{2} \right],
\end{equation}
and let $\mathcal{H}$ denote the subspace in $L^2(\mathbb{R})$ defined by this support condition. In particular, $\mathcal{H}$ is the range of the projection operator $P$ in $L^2(\mathbb{R})$ defined by
\begin{equation}
 (Pf)(x) := \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{i2\pi xt} \hat{f}(t) \, dt.
\end{equation}
Shannon’s interpolation formula applies to $f \in \mathcal{H}$, and it reads:
\begin{equation}
 f(x) = \sum_{n \in \mathbb{Z}} f(n) \frac{\sin \pi (x - n)}{\pi (x - n)}.
\end{equation}

Definitions 4.1. Let $S \subset \mathbb{R}$, and set
\begin{equation}
 v(s)(x) := v(s, x) = \frac{\sin \pi (x - s)}{\pi (x - s)}, \quad s \in S.
\end{equation}
Hence if we take as index set $S := \mathbb{Z}$, then we may observe that the functions on the right-hand side in Shannon’s formula \cite{lea} are $v(n)$ frame vectors, $n \in \mathbb{Z}$. We shall be interested in other index sets $S$, so-called sets of sampling points. We shall
view the functions $v(s)$ as vectors in $\mathcal{H}$. The inner product in $\mathcal{H}$ will be that which is induced from $L^2(\mathbb{R})$, i.e.,

$$\langle f_1 \mid f_2 \rangle := \int_{\mathbb{R}} f_1(x) f_2(x) \, dx.$$  

The following is well known but is included as an application of Corollary 3.5.

**Proposition 4.2.** Let $S \subset \mathbb{R}$ be a fixed discrete subgroup, and assume that $Z \subset S$. Then $\{v(s) \mid s \in S\}$ is a tight frame in $\mathcal{H}$ if and only if the group index $(S:Z)$ is finite, and in that case the frame constant $c$ is $c = (S:Z)^{-1}$. For the Gram matrix, we have:

$$K(s_1, s_2) = \begin{cases} \sin \pi (s_1 - s_2) \\ \pi (s_1 - s_2) \end{cases} \quad \text{for } s_1, s_2 \in S, \ s_1 \neq s_2,$$

$$1 \quad \text{if } s_1 = s_2.$$  

Proof. Formula (4.7) for the Gram matrix follows from Fourier transform and the following computation of the inner products:

$$\langle v(s_1) \mid v(s_2) \rangle_{\mathcal{H}} = \int_{\mathbb{R}} \frac{\sin \pi (x - s_1)}{\pi (x - s_1)} \frac{\sin \pi (x - s_2)}{\pi (x - s_2)} \, dx.$$  

A second computation shows that $K(s_1, s_2) = \langle v(s_1) \mid v(s_2) \rangle$ satisfies the two conditions (a) and (b) of Corollary 3.5 i.e.,

$$K(s_1, s_2) = K(s_2, s_1),$$

and

$$\sum_{t \in S} K(s_1, t) K(t, s_2) = (S:Z) K(s_1, s_2) \quad \text{for all } s_1, s_2 \in S.$$

The argument behind this formula uses the known fact that the functions $\{v(n) \mid n \in \mathbb{Z}\}$ in (4.3) form an ONB in $\mathcal{H}$; in particular, that

$$\langle v(n_1) \mid v(n_2) \rangle_{\mathcal{H}} = \delta_{n_1, n_2} \quad \text{for } n_1, n_2 \in \mathbb{Z}.$$  

Since $\mathbb{Z} \subset S$, we get the following summation formula:

$$\sum_{t \in S} K(s_1, t) K(t, s_2) = \sum_{t \in S} \langle v(s_1) \mid v(t) \rangle_{\mathcal{H}} \langle v(t) \mid v(s_2) \rangle_{\mathcal{H}}$$

$$= \sum_{k \in S/\mathbb{Z}} \sum_{n \in \mathbb{Z}} \langle v(s_1) \mid v(k + n) \rangle_{\mathcal{H}} \langle v(k + n) \mid v(s_2) \rangle_{\mathcal{H}}$$

$$= \sum_{k \in S/\mathbb{Z}} \sum_{n \in \mathbb{Z}} \frac{\sin \pi (s_1 - k - n)}{\pi (s_1 - k - n)} \frac{\sin \pi (n - (s_2 - k))}{\pi (n - (s_2 - k))}$$

$$= \sum_{k \in S/\mathbb{Z}} \frac{\sin \pi (s_1 - s_2)}{\pi (s_1 - s_2)}$$

$$= (S:Z) \langle v(s_1) \mid v(s_2) \rangle_{\mathcal{H}} = (S:Z) K(s_1, s_2),$$

which is the desired formula (4.10). The remaining conclusions in the proposition now follow immediately from Corollary 3.5. □
Remark 4.3. The significance of using a larger subgroup $S$, i.e., $\mathbb{Z} \subset S$, in a modified version of Shannon’s interpolation formula (4.4) is that a larger (discrete) group represents “oversampling”. However, note that the oversampling changes the frame constant.

As a contrast showing stability, we now recast a result on oversampling from [BJMP05] in the present context. It is for tight frames of wavelet bases in $L^2(\mathbb{R})$, and it represents an instance of stability: a case when oversampling leaves invariant the frame constant.

Proposition 4.4. Let $\psi \in L^2(\mathbb{R})$, and suppose that the family
\begin{equation}
\psi_{j,k}(x) := 2^{j/2} \psi \left(2^j x - k\right), \quad j, k \in \mathbb{Z},
\end{equation}
is a Parseval frame in $L^2(\mathbb{R})$. Let $p \in \mathbb{N}$ be odd, $p > 1$, and set
\begin{equation}
\tilde{\psi}_p(x) := \frac{1}{p} \psi \left(\frac{x}{p}\right),
\end{equation}
and
\begin{equation}
\tilde{\psi}_{p,j,k}(x) := 2^{j/2} \tilde{\psi}_p \left(2^j x - k\right), \quad j, k \in \mathbb{Z}.
\end{equation}
Then the “oversampled” family is again a Parseval frame in the Hilbert space $L^2(\mathbb{R})$.

Proof. We refer the reader to the argument in Section 2 of [BJMP05], and to Example 6.9 below. □

5. Symmetries

A basic fact of Hilbert space is that permutations induce unitary operators. By this we mean that a permutation of the vectors in an orthonormal basis for a Hilbert space $\mathcal{H}$ induces a unitary transformation $U$ in $\mathcal{H}$. In this section we explore generalizations of this to frames.

Definition 5.1. Let $S$ be a set and let $k: S \times S \to \mathbb{C}$ be a function. We say that $k$ is positive definite, or more precisely positive semidefinite, if the following holds for all finite sums:
\begin{equation}
\sum_{s \in S} \sum_{t \in S} \xi_s \xi_t^* k(s,t) \geq 0.
\end{equation}

While the following result is known, it is not readily available in the literature, at least not precisely in the form in which we need it. Thus we include here its statement to save readers from having to track it down.

Theorem 5.2. (Kolmogorov, Parthasarathy–Schmidt [PaSc72])

(a) Let $k: S \times S \to \mathbb{C}$ be positive definite. Then there are a Hilbert space $\mathcal{H}$ and vectors $\{v(s) \mid s \in S\} \subset \mathcal{H}$ such that
\begin{equation}
k(s,t) = \langle v(s) \mid v(t) \rangle_{\mathcal{H}}, \quad s, t \in S,
\end{equation}
and $\mathcal{H}$ is the closed linear span of $\{v(s) \mid s \in S\}$.

(b) Let $\pi: S \to S$ be a bijection. Then
\begin{equation}
U_\pi: v(s) \mapsto v(\pi(s))
\end{equation}
extends to a unitary operator in $\mathcal{H}$ ($=: \mathcal{H}(k)$) if and only if
\begin{equation}
k(\pi(s), \pi(t)) = k(s, t) \quad \text{for all } s, t \in S.
\end{equation}
(c) If $\lambda: S \rightarrow T = \{z \in \mathbb{C} | |z| = 1\}$ and if $\pi$ is as in (b), then

\begin{equation}
U_{\pi, \lambda}: v(s) \mapsto \lambda(s) v(\pi(s))
\end{equation}

extends to a unitary operator in $\mathcal{H}$ if and only if

\begin{equation}
\lambda(s) \lambda(t) k(\pi(s), \pi(t)) = k(s, t) \quad \text{for all } s, t \in S.
\end{equation}

Proof. The reader is referred to [PaSc72] for details. □

In fact, there are important applications of Theorem 5.2 to frames with continuous index set. The best known (also included in [PaSc72]) is the example when $v(t)$ is a Wiener process, i.e., a mathematical realization of Einstein’s Brownian motion.

Remark 5.3. As an application of Theorem 5.2(b), note that in each of the Figures 1 and 2, the frames are unitarily equivalent as the angle $\theta$ varies. But they are inequivalent as $n$ varies from 3 to 4. More generally, turning to Example 2.2, if $n$ is fixed and a new system $(v_\theta(s))$ is defined by translating the argument in (2.8) by $\theta$, then the assignment $v(s) \rightarrow v_\theta(s)$ extends to a unitary operator in the two-dimensional Hilbert space $\mathcal{H}$. But as $n$ varies, we get families that are not unitarily equivalent. In fact, it follows from Remark 5.6 that the examples from Figure 1 are not equivalent to those in Figure 2 even when equivalence is defined in the less restrictive sense of (5.5) in Theorem 5.2(c), i.e., allowing a phase factor in the transformation of the respective systems of frame vectors.

For simplicity, return here to the case when the set $S$ is assumed countable and discrete.

Definition 5.4. (Following [Ne57, Ne59].) A closed subspace $\mathcal{H}$ in $\ell^2(S)$ is said to be a reproducing kernel subspace if for every $s \in S$, the mapping

\begin{equation}
\mathcal{H} \ni f \mapsto f(s) \in \mathbb{C}
\end{equation}

is continuous. Note that by Riesz’s lemma this means that there is for each $s$ a unique element $v(s) \in \mathcal{H}$ such that

\begin{equation}
f(s) = \langle v(s) | f \rangle.
\end{equation}

And by Schwarz’s inequality we get

\begin{equation}
|f(s)| \leq \|v(s)\|_\mathcal{H} \|f\|_\mathcal{H}.
\end{equation}

Theorem 5.5. Let the pair $(S, \mathcal{H})$ be as in the statement of Theorem 2.4. Specifically, we assume that there is a tight frame $(v(s))_{s \in S}$ for $\mathcal{H}$ with frame constant $c$.

Then it follows that $\mathcal{H}$ is a reproducing kernel Hilbert space, and that

\begin{equation}
|f(s)| \leq k(s, s)^{1/2} \|f\|_\mathcal{H} \quad \text{for all } s \in S \text{ and } f \in \mathcal{H},
\end{equation}

where $k(s, t) = \langle v(s) | v(t) \rangle$ is the Gram matrix of $\mathcal{H}$.

Proof. Let $(S, \mathcal{H})$ be as stated. Then by Corollary 3.7, we have the representation

\begin{equation}
f = c \sum_{s \in S} \langle v(s) | f \rangle v(s),
\end{equation}

referring to the isometric embedding $\mathcal{H} \hookrightarrow \ell^2(S)$. The Gram matrix of $(v(s))_{s \in S}$ induces an operator $K$, and $P = cK$ is the projection of $\ell^2(S)$ onto the subspace
Moreover, \( \mathbf{v}(s) = P(\delta_s) \) for all \( s \in S \). Now apply both sides of (5.11) to some point \( t \in S \). Via the isometric embedding, we know that
\[
\mathcal{H} = \{ f \in \ell^2(S) \mid Pf = f \}. 
\]
Hence, if \( f \in \mathcal{H} \), we get \( f(t) = \langle \delta_t \mid f \rangle = \langle P \delta_t \mid f \rangle = \langle \mathbf{v}(t) \mid f \rangle \); and by (5.11),
\[
f(t) = c \sum_{s \in S} \langle \mathbf{v}(s) \mid f \rangle \underbrace{\langle \mathbf{v}(t) \mid \mathbf{v}(s) \rangle}_P.
\]

An application of Corollary 3.5 and of Schwarz for \( \ell^2(S) \), now yields
\[
|f(t)| \leq c \left( \sum_{s \in S} |\langle \mathbf{v}(s) \mid f \rangle|^2 \right)^{1/2} \left( \sum_{s \in S} |k(t,s)|^2 \right)^{1/2} 
= c \cdot c^{-1/2} \|f\|_\mathcal{H} c^{-1/2} k(t,t)^{1/2} 
= \|f\|_\mathcal{H} k(t,t)^{1/2}.
\]

**Remark 5.6.** It is clear that the converse to the theorem also holds: If \( \mathcal{H} \xrightarrow{\sim} \ell^2(S) \) is a reproducing kernel Hilbert space, then for each \( s \in S \), and \( f \in \mathcal{H} \), \( f(s) = \langle \mathbf{v}(s) \mid f \rangle_\mathcal{H} \) holds for a unique family \( \{\mathbf{v}(s)\}_{s \in S} \) in \( \mathcal{H} \). This family will be a tight frame with frame constant one. If \( c \in (0,1) \), then \( \mathbf{w}_c(s) := c^{-1/2} \mathbf{v}(s) \) will define a tight frame with frame constant \( c \).

6. More General Frames

Since the condition (6.7) which defines tight frames is rather rigid, it is of interest to consider how it can be relaxed in a way which still makes it useful.

As before, we will consider a pair \( (S, \mathcal{H}) \), where \( S \) is a fixed countable set, and where \( \mathcal{H} \) is a Hilbert space

**Definitions 6.1.** A system of vectors \( \{\mathbf{v}(s)\}_{s \in S} \) in \( \mathcal{H} \) is called a frame for \( \mathcal{H} \) if there are constants \( 0 < A_1 \leq A_2 < \infty \) such that
\[
A_1 \|f\|^2 \leq \sum_{s \in S} |\langle \mathbf{v}(s) \mid f \rangle|^2 \leq A_2 \|f\|^2 \quad \text{for all } f \in \mathcal{H}.
\]

It is called a Bessel sequence if only the estimate on the right-hand side in (6.1) is assumed, i.e., if for some finite constant \( A \),
\[
\sum_{s \in S} |\langle \mathbf{v}(s) \mid f \rangle|^2 \leq A \|f\|^2 \quad \text{for all } f \in \mathcal{H}.
\]

Recall that when (6.2) is assumed, then the analysis operator \( V = V(\mathbf{v}(s)) \) given by
\[
\mathcal{H} \ni f \mapsto ((\mathbf{v}(s) \mid f) )_{s \in S} \in \ell^2(S)
\]
is well defined and bounded. Hence, the adjoint operator \( V^* : \ell^2(S) \to \mathcal{H} \) is bounded as well, and
\[
V^*(\xi_s) = \sum_{s \in S} \xi_s \mathbf{v}(s) \quad \text{for all } (\xi_s) \in \ell^2(S),
\]
where the sum on the right-hand side in (6.4) is convergent in \( \mathcal{H} \) for all \( (\xi_s) \in \ell^2(S) \).

The tight frames for which \( A_1 = A_2 = 1 \) are called Parseval frames.
Theorem 6.2. Let \((S, \mathcal{H})\) be as above, and let \((v(s))_{s \in S}\) be a Bessel sequence with Bessel constant \(A\).

(a) Then the closed span \(\mathcal{H}_{in}\) of \((v(s))_{s \in S}\) contains a derived Parseval frame.
(b) The derived Parseval frame is a Parseval frame for \(\mathcal{H}\) if and only if \((v(s))_{s \in S}\) is a frame for \(\mathcal{H}\), i.e., iff \(\mathcal{H}_{in} = \mathcal{H}\).
(c) In the general case when \((v(s))_{s \in S}\) is a Bessel sequence, the operator \(W := V (V^* V)^{-1/2}\) is well defined and isometric on \(\mathcal{H}_{in}\), and

\[w(s) := (V^* V)^{-1/2} v(s), \quad s \in S,\]

is a Parseval frame in \(\mathcal{H}_{in}\).

Proof. It is easy to see that both \(V\) and \(V^*\) are bounded; \(V\) is everywhere defined on \(\mathcal{H}\), and \(V^*\) everywhere defined on \(\ell^2(S)\). For the operator norms, we have \(\|V\| = \|V^*\| = \|V^* V\|^{1/2} \leq \sqrt{A}\). The initial space of \(V\), \(\mathcal{H}(V)\), is defined as

\[\mathcal{H} \cap \{ f \in \mathcal{H} \mid Vf = 0 \} = \overline{R(V^*)},\]

where the over-bar stands for norm-closure in \(\mathcal{H}\), and where \(R(V^*)\) denotes the range of \(V^*\).

We let \(\mathcal{H}_{in}\) denote the closed span of the given Bessel sequence \((v(s))_{s \in S}\). Our claim is that

\[(6.7) \quad \mathcal{H}_{in} = \mathcal{H}(V).\]

To see this, note that \((6.4)\) implies the inclusion \((\supseteq)\) in \((6.7)\). But all finite linear combinations \(\sum_s \xi_s v(s)\) are contained in \(R(V^*)\), and by closure, we get \(\mathcal{H}_{in} \subseteq \overline{R(V^*)}\), which is the second inclusion in \((6.7)\). Hence \((6.7)\) holds.

We now apply the polar decomposition (from operator theory), see \cite{KaRi97}, to the operator \(V\). The conclusion is that there is a partial isometric \(W\) such that

\[(6.8) \quad V = W (V^* V)^{1/2} = (VV^*)^{1/2} W\]

and such that the initial space of \(W\) is \(\mathcal{H}_{in}\) and the final space of \(W\) is \(\overline{R(V)}\).

Implied in this are the following assertions:

(i) Both of the operators \(W^* W\) and \(W W^*\) are projections;
(ii) The range of \(W^* W\) is \(\mathcal{H}_{in}\);
(iii) The range of \(W W^*\) is \(\overline{R(V)}\);
(iv) The operator \(V^* V\) is selfadjoint, and \((V^* V)^{1/2}\) is defined by the spectral theorem applied to \(V^* V\).

Now apply Theorem 5.5 to the restriction operator

\[(6.9) \quad W : \mathcal{H}_{in} \rightarrow \ell^2(S).\]

In view of \((6.1)\) above, this is an isometry. Recall \(W^* W \mathcal{H} = \mathcal{H}_{in}\) by \((6.7)\) and the polar decomposition \((6.8)\). As a result, we get that the vectors \(w(s) = W^* \delta_s\), for \(s \in S\), form a Parseval frame for \(\mathcal{H}_{in}\). But we also have the formula \(V^* \delta_s = v(s)\) as an application of \((6.4)\). Using now the two facts \((6.9)\) and \((6.8)\), we get \(w(s) = W^* \delta_s = (V^* V)^{-1/2} V^* \delta_s = (V^* V)^{-1/2} v(s)\), which is the desired formula \((6.5)\) from the conclusion in the theorem. The remaining conclusions stated in the theorem are already implied by the reasoning above. \(\square\)
Remark 6.3. The operators $V^*V$ and $VV^*$. Except for the point $\lambda = 0$, the two operators $V^*V$ and $VV^*$ have the same spectrum. But in the general case, this spectrum could be discrete or continuous; or it could even be singular.

The fact that the spectrum minus $\{0\}$ is the same follows from (6.8). In the frame case, it follows from (6.1) that the spectrum is contained in the interval $[A_1, A_2]$. But if $(v(s))_{s \in S}$ is only a Bessel sequence, (6.2), the best that can be said is that the spectrum is contained in $[0, A]$. In fact, the lower bound in spectrum $(V^*V)$ is also the best lower frame bound, referring to (6.1).

Computationally, however, the two operators $V^*V$ and $VV^*$ are quite different. The first one operates in $H$, while the second one maps $\ell^2(S)$ into itself.

The formula (6.10) $V^*Vf = \sum_{s \in S} \langle v(s) \mid f \rangle v(s)$ for $f \in H$ shows that $V^*V$ serves to decompose vectors in $H$ relative to the given frame $(v(s))_{s \in S}$. In contrast, $VV^*$ has an explicit matrix representation:

**Proposition 6.4.** Relative to the ONB $(\delta_s)_{s \in S}$, $VV^*$ is simply multiplication with the Gram matrix

(6.11) $(\langle v(s) \mid v(t) \rangle_{H})_{s,t \in S}$.

**Proof.** To compute the $(s, t)$-matrix entry for the matrix which represents the operator $VV^*$, we use the inner product in $\ell^2(S)$ as follows: The $(s, t)$-matrix entry is

(6.12) $\langle \delta_s \mid VV^*\delta_t \rangle_{\ell^2} = \langle V^*\delta_s \mid V^*\delta_t \rangle_{H} = \langle v(s) \mid v(t) \rangle_{H}$,

which is the Gram matrix. □

**Example 6.5.** Example 2.2 revisited for $n = 3$.

In the case of Example 2.2 (see also Fig. 1) for $n = 3$, $VV^*$ is the $3 \times 3$ matrix

$$
\begin{pmatrix}
1 & -1/2 & -1/2 \\
-1/2 & 1 & -1/2 \\
-1/2 & -1/2 & 1
\end{pmatrix},
$$

while $V^*V$ may be represented as

$$
\begin{pmatrix}
3/2 & 0 \\
0 & 3/2
\end{pmatrix},
$$

i.e., a $2 \times 2$ matrix. Moreover, $\text{spec}(VV^*) = \{0, \frac{3}{2}\}$, and $\text{spec}(V^*V) = \{\frac{3}{2}\}$.

**Corollary 6.6.** Consider a system $(\mathcal{H}, S, (v(s))_{s \in S})$ as above, and let

$$V = V_{(v(s))} : \mathcal{H} \rightarrow \ell^2(S)$$

be the associated analysis operator. Then the respective lower and upper frame estimates (6.1) are equivalent to

(6.13) $\text{spec} \left( (\langle v(s) \mid v(t) \rangle) \setminus \{0\} \right) \subset [A_1, A_2],$

where $(\langle v(s) \mid v(t) \rangle)_{s,t \in S}$ is the Gram matrix in (6.11).
Proof. Since we noted in Remark 6.3 that
\[
\text{spec } (V^* V) \setminus \{0\} = \text{spec } (V V^*) \setminus \{0\},
\]
the assertion in (6.13) follows immediately from Proposition 6.4. Specifically, the two estimates in (6.1) are equivalent to the following system of operator inequalities:
\[
A_1 I_H \leq V^* V \leq A_2 I_H,
\]
where the ordering “\(\leq\)" in (6.15) refers to the familiar ordering on the set of self-adjoint operators [KaRi97]. Now an application of the spectral theorem [KaRi97] to \(V^* V\) shows that (6.15) is also equivalent to the set-containment \(\text{spec } (V^* V) \subseteq [A_1, A_2]\). In fact, the lower estimate in (6.1) holds iff \(\text{spec } (V^* V) \subseteq [A_1, \infty)\), while the upper estimate is equivalent to the containment \(\text{spec } (V^* V) \subseteq [0, A_2]\). □

While Corollary 6.6 is rather abstract, it has a variety of specific uses which serve to make it clear how certain initial frame systems are transformed into more detailed frames having additional structure. A case in point is the transformation of frames used in the analysis of signals into frequency bands, for example mutually non-interfering low-pass and high-pass bands. A convenient tool for accomplishing this is the discrete wavelet transform. As noted in, for example, [Chr03] and [Jor06], the discrete wavelet transform serves to create nested families of resolution subspaces (details below) in the Hilbert spaces of sequences which are used in our representation of time-series, or of speech signals. When more sub-bands are introduced, this same idea works for the analysis of images.

By a discrete wavelet transform with perfect reconstruction we shall mean (following [Jor06]) a Hilbert space \(H\) and a system of operators \(F_0, F_1\) in \(H\) such that
\[
F_i F_j^* = \delta_{i,j} I, \quad \sum_{i=0}^{1} F_i^* F_i = I, \quad \text{and } F_0^n f \to 0 \quad \text{as } n \to \infty \quad \text{for all } f \in H.
\]

As further noted in [Jor06], each system of quadrature mirror filters from standard signal processing defines an operator system \((F_i)\) as in (6.16).

Remarks 6.7. (a) The subscript convention for the two operators \(F_0\) and \(F_1\) in (6.16) comes from engineering. The index value \(i = 0\) corresponds to a chosen low-pass filter followed by downsampling, while \(F_1\) is the operation of a high-pass filtering followed by downsampling. Hence “low” is indexed by zero. This index convention is not related to that of the two frame bounds \(A_i\) in (6.1). These two numbers are usually called \(A_1\) and \(A_2\).

(b) [Referring to the sum-formula in (6.16).] In stating our quadrature conditions (6.16) we have for reasons of clarity restricted attention to the simplest case: dyadic and orthogonal filters. (The dyadic case refers to the dyadic wavelets from Proposition 4.4 above.) But our present discussion (both in Sections 4 and 6) easily generalizes to systems with more than two bands, and even to less restrictive quadrature conditions. In case of more, say \(N\) bands, the indexing of the operators \(F_i\) is \(i = 0, 1, \ldots, N - 1\). And instead of using the operator \(F_i^*\) in the sum in (6.16) at the \(i\)'th place, we may for some applications rather use a second operator \(G_i\) (not equal to \(F_i^*\)); see for example [JoKr03] for details on these more general systems.

Corollary 6.8. Let \((v(s))_{s \in S}\) be a frame with frame bounds \(A_i, i = 1, 2, \) in a Hilbert space \(\mathcal{H}\), and let \((F_i)\) be a discrete wavelet transform system (as in (6.16)).
Set
\[ v(k,s) := F_0^* k F_1^* v(s), \quad s \in S, \quad k = 0, 1, 2, \ldots. \]

Then the subdivided vector system \((v(k,s))\) is a frame in \(H\) with the same frame bounds \(A_1\) as \((v(s))\). Moreover the Gramian of \((v(k,s))\) has the form \(I \otimes G\) where \(I\) is the (infinite) identity matrix, and \(G\) is the Gramian for \((v(s))\).

Moreover, if two frames \((v(s))\) and \((v'(s))\) are unitarily equivalent in the sense of Theorem 5.2, then two refined systems \((v(k,s))\) and \((v'(k,s))\) resulting from \((6.17)\) are also unitarily equivalent.

**Proof.** Since in general, by Corollary 6.6, the two frame bounds (upper and lower) coincide with the spectral bounds for the corresponding Gramian, we only need to study the Gramian of the new system \((v(k,s))\) in (6.17). Recall that the Gramian for the new system is the matrix of inner products, with the only modification to the formula in Theorem 5.5 being that the row and column indices are now double indices.

The significance of the third condition from (6.16) is related to the kind of subspace structure in \(H\) which models resolutions of signals. Since \(F_0 F_0^* = I\), we get a nested system of projections
\[ P_k := F_0^* k F_1^* \quad k = 0, 1, \ldots, \]

and
\[ \cdots P_{k+1} \leq P_k \leq \cdots \leq P_1 \leq P_0 = I. \]

Recall that the ordering of projections coincides with the associated ordering of the range subspaces \(H_k := P_k H\) in \(H\), i.e.,
\[ \cdots H_{k+1} \subseteq H_k \subseteq \cdots \subseteq H_1 \subseteq H. \]

The third condition in (6.16) is equivalent to the assertion that
\[ \bigcap_{k \in \{0, 1, \ldots\}} H_k = \{0\}. \]

Specifically,
\[ \langle v(j,s) | v(k,t) \rangle = \left\langle F_0^* j F_1^* v(s) | F_0^* k F_1^* v(t) \right\rangle = \delta_{j,k} \langle v(s) | v(t) \rangle. \]

Since the last expression is the tensor product of \(I\) with \(G\), the result follows.

The last part of the conclusion in the corollary about preservation of unitary equivalence in passing to the refinement \((v(s)) \rightarrow (v(k,s))\) follows from this and Theorem 5.2. \(\square\)

**Example 6.9.** (Following the terminology from (4.12) in Proposition 4.4) Suppose some dyadic wavelet \((\psi_{j,k})\) comes from a scaling function \(\phi\). Set \(S := \mathbb{Z}\), and
\[ v(k) := \phi(x-k), \quad k \in \mathbb{Z}. \]

Then there are known conditions on such a scaling function \(\phi\) for the frame estimates (6.1) to hold; see, e.g., CoDa93. To apply the operator system (6.16) from above, choose the Hilbert space \(H\) be the closed subspace in \(L^2(\mathbb{R})\) spanned by the integral translates \(\{\phi(x-k) | k \in \mathbb{Z}\}\). Following Jor06 and above, we may then construct operators \(F_0, F_1\) and introduce the associated resolution frame \((v(j,k))\), double-indexed as in Corollary 6.8 as follows:
Set
\[ v(j, k) := F^*_0 F^*_1 v(k) = \psi_{-j-1,k} \quad \text{for } j = 0, 1, 2, \ldots, \text{ and } k \in \mathbb{Z}. \]

Using the corollary, we then conclude that for each \( j \), the functions \( \{ v(j, k) \mid k \in \mathbb{Z} \} \) generate the relative complement subspace \( H_j \ominus H_{j+1} \) from the nested resolution system we introduced there. By this we mean that for each \( j \), \( H_j \ominus H_{j+1} \) is the closed linear span of \( \{ v(j, k) \mid k \in \mathbb{Z} \} \).

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