A Completely Monotonic Function
Used in an Inequality of Alzer

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Abstract. The function
\[ G(x) = \left( 1 - \frac{\ln(x)}{\ln(1 + x)} \right) x \ln(x) \]
has been considered by Alzer and by Qi and Guo. We prove that \( G' \) is completely monotonic by finding an integral representation of the holomorphic extension of \( G \) to the cut plane. A main difficulty is caused by the fact that \( G' \) is not a Stieltjes transform.

Keywords. Completely monotonic function, Stieltjes transform.

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1. Introduction and results

In a recent paper [1], Alzer proved a number of inequalities involving the volume of the unit ball in \( \mathbb{R}^n \),

\[ \Omega_n = \frac{\pi^{n/2}}{\Gamma(1 + \frac{n}{2})}, \quad n = 1, 2, \ldots. \]  

That paper contains many references to earlier results about \( \Omega_n \). We mention in particular that Anderson and Qiu [2] proved that the sequence \( f(n) = \Omega_n^{1/(n \ln n)} \), \( n \geq 2 \), is strictly decreasing and converges to \( e^{-1/2} \). It is therefore of interest to study the function

\[ f(x) = \left( \frac{\pi^{x/2}}{\Gamma(1 + \frac{x}{2})} \right)^{1/(x \ln x)}, \]
and in [10] the authors have given an integral representation of $\ln f(x+1)$, $x > 0$ by considering its holomorphic extension (denoted by $\log f(z+1)$) to the cut plane $A = \mathbb{C} \setminus (-\infty, 0]$. From this representation it has been possible to deduce that $f(n+2)$ is a Hausdorff moment sequence, in particular decreasing and convex.

The papers [2, 3] have also been an inspiration for several papers about the functions

$$F_a(x) = \frac{\ln \Gamma(x+1)}{x \ln(ax)}, \quad x > 0, \ a > 0, \quad (3)$$

see [1, 4, 8, 9, 10, 11, 12, 13]. In particular, [10] contains an integral representation of the meromorphic extension of $F_a$ to $A$. From this representation it is possible to deduce that $F_a$ is a Pick function if and only if $a \geq 1$. The relation between $F_a$ and $f$ is given by

$$\log f(z+1) = \frac{\ln \sqrt{\pi}}{\text{Log}(z+1)} - \frac{1}{2} \, F_2 \left( \frac{z+1}{2} \right),$$

where $\text{Log} z = \ln |z| + i \text{Arg} z$ is the principal logarithm in $A$ and $-\pi < \text{Arg} z < \pi$ for $z \in A$.

Alzer found the best constants $a^*, b^*$ such that for all $n \geq 2$

$$\exp \left( \frac{a^*}{n (\log n)^2} \right) \leq \frac{f(n)}{f(n+1)} < \exp \left( \frac{b^*}{n (\log n)^2} \right). \quad (4)$$

In the proof of this result Alzer considered the function

$$G(x) = \left( 1 - \frac{\ln(x)}{\ln(1+x)} \right) x \ln(x), \quad (5)$$

and in [1, Lem. 2.3] it was proved that $2/3 < G(x) < 1$ for $x \geq 3$. Qi and Guo observed in [13] that $G$ is strictly increasing on $(0, \infty)$ with $G((0, \infty)) = (-\infty, 1)$ and that $G(3) > 2/3$, which gave another proof of the inequality $2/3 < G(x) < 1$. Furthermore, in [13, Rem. 4] it was conjectured that

$$(-1)^{k-1} G^{(k)}(x) > 0 \quad \text{for} \ x > 0, \ k = 1, 2, \ldots, \quad (6)$$

or equivalently that $G'$ is a completely monotonic function.

The main goal of this paper is to prove this conjecture. We do this by considering $G$ as a holomorphic function in the cut plane $A$. We put

$$G(z) = \left( 1 - \frac{\text{Log}(z)}{\text{Log}(1+z)} \right) z \text{Log}(z), \quad (7)$$

for $z \in A$. Using the same Cauchy integral formula technique as in [10], we shall establish the following theorem.
Theorem 1. The function $G$ from (7) has the representation

$$(8) \quad G(z) = 1 - \int_{0}^{\infty} \frac{\rho(t)}{z + t} \, dt, \quad z \in \mathcal{A},$$

where

$$(9) \quad \rho(t) = \begin{cases} 
-\frac{t \ln(\frac{1-t}{t^2})}{\ln(1-t)}, & \text{if } 0 < t < 1, \\
-\frac{t(\ln(\frac{1}{t})^2}{(\ln(t-1))^2 + \pi^2}, & \text{if } 1 < t < \infty.
\end{cases}$$

Notice that $\rho(1^{-}) = \rho(1^{+}) = -1$ so that $\rho$ is continuous on the positive half-line. It is decreasing from $\infty$ to $-1$ on the interval $(0,1)$ with $\rho'(1^{-}) = -1$, and increasing from $-1$ to $0$ on the interval $(1,\infty)$ with $\rho'(1^{+}) = \infty$. We have $\rho((\sqrt{5} - 1)/2) = 0$. Notice also that $\rho$ is integrable over $(0,\infty)$ because of the asymptotics

$$\rho(t) \sim -2 \ln t \quad \text{for } t \to 0^+, \quad \rho(t) \sim -\frac{1}{t(\ln t)^2} \quad \text{for } t \to \infty.$$ 

The graph of $\rho$ is shown in Figure 1.

Since $\rho$ assumes positive and negative values, $G$ as well as $1 - G$ are not Stieltjes transforms. Nevertheless $1 - G$ turns out to be completely monotonic, because it is the Laplace transform of a positive function, as described in the following theorem. In particular, $G'$ is completely monotonic so (6) holds. For properties about completely monotonic functions and Stieltjes transforms we refer to [7, 16].
Theorem 2. For Re $z > 0$ the function $1 - G$ has the representation

$$1 - G(z) = \int_0^\infty e^{-zs} \varphi(s) \, ds,$$

where

$$\varphi(s) = \int_0^\infty e^{-st} \rho(t) \, dt > 0 \quad \text{for } s \geq 0.$$  

Remark 1. The relations (10) and (11) yield that $1 - G(z)$ is an iterated Laplace transform, i.e.

$$1 - G(z) = \mathcal{L}(\mathcal{L}(\rho))(z).$$

This is a special case of a general result in [15]. The key assertion in Theorem 2 is the positivity of $\varphi$.

The graph of $\varphi$ is given in Figure 2.

![Figure 2. The graph of $\varphi$](image)

The function $\varphi$ given in (11) is continuous and bounded on $[0, \infty)$, but it is not integrable because $1 - G(x) \to \infty$ for $x \to 0^+$.

Setting $z = a + it$ in (10) with $a > 0$ we get the following result.

Corollary 3.

(i) For each $a > 0$

$$1 - G(a + it) = \int_0^\infty e^{-its} e^{-as} \varphi(s) \, ds, \quad t \in \mathbb{R},$$
is an analytic positive definite function of $t$, and it is the Fourier transform of
\begin{equation}
\phi(s)1_{[0,\infty)}(s).
\end{equation}
(ii) $t \mapsto G(a+it) - G(a)$ is a continuous negative definite function of $t$ for each $a > 0$. In particular
\begin{equation}
\Re G(a + it) \geq G(a), \quad a > 0, \ t \in \mathbb{R}.
\end{equation}
(iii) $t \mapsto G(a+it)$ is a continuous negative definite function of $t$ for $a \geq 1$.

Concerning continuous positive and negative definite functions we refer to e.g. [7].

Letting $a \to 0^+$ in (12), we formally get that $1 - G(it)$ is the Fourier transform of $\phi(s)1_{[0,\infty)}(s)$. This is true in the $L^2$-sense because of Plancherel’s Theorem. In fact, we have

**Proposition 4.** The function $\phi$ in (11) is square integrable and
\begin{equation}
\lim_{a \to 0^+} \int_{-\infty}^{\infty} |1 - G(a+it)|^2 \frac{dt}{2\pi} = \int_{-\infty}^{\infty} |1 - G(it)|^2 \frac{dt}{2\pi} = \int_{0}^{\infty} \phi^2(s) ds.
\end{equation}

The function $G$ is one-to-one when considered on the positive real line. It is shown below that $G$ is conformal when defined in a sector containing the positive real line. We put
\[ S(a, b) = \{ z \neq 0 : a < \Arg z < b \}. \]

**Proposition 5.** The function $G$: $S(-\pi/3, \pi/3) \to \mathbb{C}$ is a conformal mapping.

Based on computer experiments it seems that $G$ is conformal in the right half plane, but we have not been able to verify this. On the other hand, $G$: $\mathcal{A} \to \mathbb{C}$ is not conformal.

2. Proof of the properties of $G$

In the first lemma the behaviour of $G$ close to zero and infinity is investigated.

**Lemma 6.** We have

(i) There exist constants $A, B > 0$ such that
\[ |G(z)| \leq A |\Log z| + B |\Log z|^2 \quad \text{for } z \in \mathcal{A}, |z| \leq \frac{1}{2}, \]

(ii) $zG(z) \to 0$ for $z = \varepsilon e^{i\theta}$, $\varepsilon \to 0$, uniformly for $-\pi < \theta < \pi$.

(iii) There exists a constant $C > 0$ such that
\[ |1 - G(z)| \leq \frac{C}{|z|} \quad \text{for } z \in \mathcal{A}, |z| \geq \varepsilon, \]

(iv) $G(z) \to 1$ for $z = Re^{i\theta}$, $R \to \infty$, uniformly for $-\pi < \theta < \pi$. 

Proof. We have for $z \in \mathcal{A}$

$$G(z) = \frac{z}{\Log(1 + z)} \left( \Log(1 + z) \Log z - (\Log z)^2 \right),$$

hence for $|z| \leq 1/2$

$$|G(z)| \leq \max_{|z| \leq 1/2} \left| \frac{z}{\Log(1 + z)} \right| \left( |\Log z| \max_{|z| \leq 1/2} |\Log(1 + z)| + |\Log z|^2 \right),$$

which shows (i).

(ii) follows from (i) since $z(\Log z)^n \rightarrow 0$ for $n \geq 1$ and $|z| = \varepsilon \rightarrow 0$.

To see (iii), we note that the power series (in $1/z$)

$$\Log\left(1 + \frac{1}{z}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{nz^n}, \quad |z| > 1$$

yields

$$\Log\left(1 + \frac{1}{z}\right) \leq \sum_{n=1}^{\infty} \frac{1}{|z|^n} = \frac{1}{|z| - 1} \leq \frac{1}{e - 1}, \quad |z| \geq e.$$

The power series also yields

$$z \Log\left(1 + \frac{1}{z}\right) = 1 + \frac{\alpha(z)}{z}, \quad |\alpha(z)| \leq \frac{e}{2(e - 1)}, \quad |z| \geq e.$$

Note also that $|\Log z| \geq 1$ for $z \in \mathcal{A}, |z| \geq e$.

Writing

$$\frac{\Log z}{\Log(1 + z)} = 1 + \beta(z) \Log\left(1 + \frac{1}{z}\right),$$

with

$$\beta(z) = \frac{-1}{\Log(1 + z)}$$

we find for $z \in \mathcal{A}, |z| \geq e$,

$$|\beta(z)| = \frac{1}{|\Log(1 + z)|} \leq \frac{1}{\ln|1 + z|} \leq \frac{1}{\ln(|z| - 1)} \leq \frac{1}{\ln(e - 1)}.$$

Finally, since

$$G(z) = \left( z \Log\left(1 + \frac{1}{z}\right) \right) \frac{\Log z}{\Log(1 + z)} = \left( 1 + \frac{\alpha(z)}{z} \right) \left( 1 + \beta(z) \Log\left(1 + \frac{1}{z}\right) \right),$$

we see that

$$z(G(z) - 1) = \alpha(z) + \beta(z)z \Log\left(1 + \frac{1}{z}\right) + \alpha(z)\beta(z) \Log\left(1 + \frac{1}{z}\right),$$

which by (17) and (18) is bounded by some constant $C > 0$ for $|z| \geq e$, showing (iii). Property (iv) follows immediately from (iii).
**Proof of Theorem 1.** For fixed \( z \in \mathcal{A} \) choose \( \varepsilon, R \) with \( 0 < \varepsilon < |z| < R \) and consider the positively oriented contour \( \gamma(\varepsilon, R) \) in \( \mathcal{A} \) consisting of the half-circle \( z = \varepsilon e^{i\theta}, \theta \in [-\pi/2, \pi/2] \) and the half-lines \( z = x \pm i\varepsilon, x \leq 0 \) until they cut the circle \( |z| = R \), which closes the contour at the points \( -R(\varepsilon) \pm i\varepsilon \), where \( 0 < R(\varepsilon) \to R \) for \( \varepsilon \to 0 \). (See Figure 3.)

![Figure 3. The contour \( \gamma(\varepsilon, R) \).](image)

By Cauchy’s Integral Theorem we have

\[
G(z) = \frac{1}{2\pi i} \int_{\gamma(\varepsilon, R)} \frac{G(w)}{w - z} \, dw.
\]

Letting \( \varepsilon \) tend to zero, the contribution corresponding to the half-circle with radius \( \varepsilon \) tends to 0 by (ii) of Lemma 6.

Concerning the boundary behaviour of \( G \) on the negative real line we obtain

\[
G(t + i0) := \lim_{\varepsilon \to 0^+} G(t + i\varepsilon) = \begin{cases} 
1 - \frac{\ln(-t) + i\pi}{\ln |1 + t| + i\pi} t(\ln(-t) + i\pi), & \text{if } t < -1 \\
1 - \frac{\ln(-t) + i\pi}{\ln(1 + t)} t(\ln(-t) + i\pi), & \text{if } -1 < t < 0.
\end{cases}
\]
Note that $G(t + i0)$ is continuous at $t = -1$ with value $-i\pi$. Using that $G(\pi) = \overline{G(\pi)}$, (19) yields
\begin{equation}
G(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{G(Re^{i\theta})}{Re^{i\theta} - z} Re^{i\theta} d\theta + \frac{1}{\pi} \int_{-R}^{R} \frac{\text{Im} G(t + i0)}{t - z} dt.
\end{equation}
In the last integral we replace $t$ by $-t$ and use that $(-1/\pi) \text{Im} G(-t + i0) = -\rho(t)$. Letting $R \to \infty$ and using (iv) of Lemma 6, we finally get (8).

\textbf{Remark 2.} Feng Qi has kindly informed us about the following elementary proof of the observation that $1 - G$ is not a Stieltjes function. In fact, if it were, then also $h(x) = 1/(x(1 - G(x))$ would be a Stieltjes transform by the Stieltjes-Reuter-Itô Theorem, cf. [14, 5] or [6, p. 25]. In particular, $h$ would be decreasing, which is contradicted by the simple fact that $1 = h(1) < h(2) = 1.02 \ldots$.

\textbf{Proof of Theorem 2.} The formulas (10) and (11) follow immediately from Theorem 1 and it remains to prove that $\varphi$ is positive. Let $t_0 = (\sqrt{5} - 1)/2$. Then $\rho(t) > 0$ for $0 < t < t_0$ and $\rho(t) < 0$ for $t_0 < t < \infty$ and hence
\begin{equation}
A = \int_0^{t_0} \rho(t) \, dt > 0, \quad B = \int_{t_0}^{\infty} \rho(t) \, dt < 0.
\end{equation}
Using this notation we get
\[
\varphi(s) = \int_0^{t_0} e^{-st} \rho(t) \, dt + \int_{t_0}^{\infty} e^{-st} \rho(t) \, dt 
\geq \int_0^{t_0} e^{-st_0} \rho(t) \, dt + \int_{t_0}^{\infty} e^{-st_0} \rho(t) \, dt = (A + B) e^{-st_0}.
\]
In the following lemma it is established that $A + B > 0$, and hence $\varphi(s) > 0$ for all $s \geq 0$.

\textbf{Lemma 7.}
\[
\int_0^{\infty} \rho(t) \, dt > 0.
\]

\textbf{Proof.} We first establish
\begin{equation}
\int_0^{1} \rho(t) \, dt > \frac{\pi^2}{6} - \frac{1}{2}.
\end{equation}
Since
\[
\sum_{n=0}^{\infty} t^n \int_0^{1} \binom{x}{n} dx = \int_0^{1} (1 + t)^x dx = \left[ \frac{(1 + t)^x}{\ln(1 + t)} \right]_0^1 = \frac{t}{\ln(1 + t)}
\]
we obtain the power series expansion
\begin{equation}
\frac{t}{\ln(1 + t)} = 1 + \sum_{n=1}^{\infty} b_n t^n, \quad |t| < 1, \quad b_n = \int_0^{1} \binom{x}{n} dx.
\end{equation}
The numbers \( b_n \) are sometimes called Cauchy numbers. Note that for \( n \geq 1 \)
\begin{equation}
0 < (-1)^{n-1}b_n = \int_0^1 \frac{x(1-x)\cdots(n-1-x)}{n!} \, dx \leq \frac{1}{n} \int_0^1 x \, dx = \frac{1}{2n}.
\end{equation}

By (23) we get
\begin{align*}
\int_0^1 \rho(t) \, dt &= -\frac{1}{2} - 2 \int_0^1 \ln t \, dt + 2 \sum_{n=1}^\infty (-1)^{n-1}b_n \int_0^1 t^n \ln t \, dt \\
&= \frac{3}{2} - 2 \sum_{n=1}^\infty (-1)^{n-1} \frac{b_n}{(n+1)^2}
\end{align*}
and hence using (24)
\begin{align*}
\int_0^1 \rho(t) \, dt &> \frac{3}{2} - \sum_{n=1}^\infty \frac{1}{n(n+1)^2} = \frac{3}{2} - \sum_{n=1}^\infty \frac{1}{n+1} \left( \frac{1}{n} - \frac{1}{n+1} \right) \\
&= \frac{1}{2} + \sum_{n=1}^\infty \frac{1}{(n+1)^2} = \frac{\pi^2}{6} - \frac{1}{2}.
\end{align*}

We next show that
\begin{equation}
(25) \quad \int_1^2 \rho(t) \, dt > -\frac{1}{12} - \frac{2(1 + \ln 2) \ln 2}{\pi^2},
\end{equation}
by using the rough estimate
\begin{align*}
\int_1^2 \rho(t) \, dt &> -\frac{1}{\pi^2} \int_1^2 t \left( \ln \left( 1 - \frac{1}{t} \right) \right)^2 \, dt \\
\int_1^2 t \left( \ln \left( 1 - \frac{1}{t} \right) \right)^2 \, dt &= \int_1^2 \left( t(\ln(t-1))^2 + t(\ln t)^2 - 2t \ln(t-1) \ln t \right) \, dt.
\end{align*}

The integral of the first two terms can be calculated because
\begin{equation*}
\int t(\ln t)^2 \, dt = \frac{t^2}{2} \left( (\ln t)^2 - \ln t + \frac{1}{2} \right),
\end{equation*}
and for the integral of the third term we have
\begin{align*}
2 \int t \ln(t-1) \ln t \, dt &= \left( t^2 \ln t - \frac{1}{2}t^2 + \frac{1}{2} \right) \ln(t-1) - \frac{1}{2}t^2 \ln t + \frac{1}{2}t^2 \\
&\quad - t \ln t + \frac{3}{2} t + \text{dilog}(t),
\end{align*}
where
\begin{equation*}
\text{dilog}(t) = \int_1^t \frac{\ln x}{1-x} \, dx.
\end{equation*}
Since
\[
dilog(2) = -\int_0^1 \frac{\ln(1+u)}{u} \, du = -\sum_{n=0}^{\infty} \int_0^1 (-1)^n \frac{u^n}{n+1} \, du = -\frac{\pi^2}{12},
\]
this leads to the expression in (25).

We finally show
\[
(26) \quad \int_2^\infty \rho(t) \, dt > -\frac{1}{2}.
\]

Squaring the power series for \(\ln(1-u)\) yields
\[
(27) \quad (\ln(1-u))^2 = u^2 \sum_{n=0}^{\infty} c_n u^n, \quad |u| < 1; \quad c_n = \sum_{k=0}^{n} \frac{1}{(k+1)(n+1-k)}.
\]

The relation \(0 < c_n \leq 1\) for all \(n\) is proved in Lemma 8 below. Therefore, and using (27) with \(u = 1/t\) it follows that
\[
\int_2^\infty \rho(t) \, dt = -\int_2^\infty \sum_{n=0}^{\infty} \frac{c_n}{t^{n+1}} \left(\frac{\ln(t-1)^2 + \pi^2}{\ln(t-1)^2 + \pi^2}\right) \, dt
\]
\[
= -\int_2^\infty \left(\frac{1}{t^{n+1}}\right) \left(\frac{\ln(t-1)^2 + \pi^2}{\ln(t-1)^2 + \pi^2}\right) \, dt
\]
\[
= -\int_2^\infty \left(\frac{(t-1)(\ln(t-1)^2 + \pi^2)}{1 + x^2}\right) \, dt
\]
\[
= -\frac{1}{\pi} \int_0^\infty \frac{dx}{1 + x^2} = -\frac{1}{2}.
\]

Combining (22), (25) and (26) we get
\[
\int_0^\infty \rho(t) \, dt > \frac{\pi^2}{6} - \frac{1}{2} - \frac{1}{12} - \frac{2(1 + \ln 2) \ln 2}{\pi^2} - \frac{1}{2} \approx 0.3238 > 0
\]

and the lemma is proved.

\[\Box\]

**Remark 3.** A numerical computation yields
\[\varphi(0) = \int_0^\infty \rho(t) \, dt \approx 0.5192.\]

**Lemma 8.** The numbers
\[c_n = \sum_{k=0}^{n} \frac{1}{(k+1)(n+1-k)}, \quad n \geq 0,
\]
can be written in the form
\[c_n = \frac{2H_{n+1}}{n+2},\]
where \( H_n = \sum_{k=1}^{n} \frac{1}{k} \) is the \( n \)th harmonic number. Moreover,
\[
c_{n-1} - c_n = \frac{2(H_n - 1)}{(n + 1)(n + 2)} \geq 0,
\]
whence \( 1 = c_0 > c_1 > c_2 > c_3 \ldots \).

**Proof.** The expression for \( c_n \) follows from the relation
\[
\frac{1}{(k+1)(n+1-k)} = \frac{1}{n+2} \left( \frac{1}{k+1} + \frac{1}{n+1-k} \right).
\]
Using that expression we find
\[
c_{n-1} - c_n = \frac{2}{(n+1)(n+2)} ((n+2)H_n - (n+1)H_{n+1})
\]
\[
= \frac{2(H_n - 1)}{(n+1)(n+2)}
\]
which proves the lemma.

**Proof of Corollary 3.** It is well-known that if \( F(t) \) is a continuous positive definite function on \( \mathbb{R} \), then \( F(0) - F(t) \) is continuous and negative definite, and a continuous negative definite function \( H(t) \) satisfies \( \text{Re } H(t) \geq H(0) \geq 0 \), see [7]. Therefore (ii) follows from (i), and (iii) follows from (ii) because \( G(a) \geq 0 \) for \( a \geq 1 \).

**Proof of Proposition 4.** By (i) and (iii) of Lemma 6 it is clear that
\[
\int_{-\infty}^{\infty} |1 - G(it)|^2 \frac{dt}{2\pi} < \infty
\]
and that dominated convergence can be applied to obtain the first equality in (15). By Plancherel’s Theorem \( 1-G(it) \) must be the Fourier transform of a square integrable function, which is the \( L^2 \)-limit of (13), hence equal to \( \varphi(s)1_{[0,\infty)}(s) \).

Before proving Proposition 5 we give Lemma 9.

**Lemma 9.** For \( z \in S(0, \pi/3) \) we have \( \text{Im } G'(z) < 0 \).

**Proof.** From the relation (8) it follows that
\[
\text{Im } G'(re^{i\theta}) = -2r \sin \theta \int_{0}^{\infty} \frac{r \cos \theta + t}{((r \cos \theta + t)^2 + (r \sin \theta)^2)^2} \rho(t) \, dt.
\]
We claim that for fixed \( r > 0 \) and \( \theta \in [0, \pi/3] \) the function
\[
k(t) = \frac{r \cos \theta + t}{((r \cos \theta + t)^2 + (r \sin \theta)^2)^2}
\]
is decreasing. Indeed,
\[
k'(t) = \frac{(r \sin \theta)^2 - 3(r \cos \theta + t)^2}{((r \cos \theta + t)^2 + (r \sin \theta)^2)^3}
\]
and the numerator is negative for all \( t > 0 \) because
\[
\sin^2 \theta \leq 3 \cos^2 \theta \quad \text{for } \theta \in \left[0, \frac{\pi}{3}\right].
\]
This implies
\[
\int_0^\infty \frac{r \cos \theta + t}{(r \cos \theta + t)^2 + (r \sin \theta)^2}^2 \rho(t) \, dt = \int_0^{t_0} k(t) \rho(t) \, dt + \int_{t_0}^\infty k(t) \rho(t) \, dt
\]
\[
\geq k(t_0) \left( \int_0^{t_0} \rho(t) \, dt + \int_{t_0}^\infty \rho(t) \, dt \right),
\]
where \( t_0 = (\sqrt{5} - 1)/2 \). From Lemma 7 it follows that the integral above is positive. From (28) we now obtain that
\[
\text{Im} G'(r e^{i\theta}) = -2r \sin \theta \int_0^\infty \frac{r \cos \theta + t}{(r \cos \theta + t)^2 + (r \sin \theta)^2}^2 \rho(t) \, dt < 0.
\]
This proves the lemma.

**Proof of Proposition 5.** From (8) it follows that
\[
\text{Im} G(x + iy) = y \int_0^\infty \frac{\rho(t)}{(x + t)^2 + y^2} \, dt.
\]
Here \( t \mapsto 1/((x + t)^2 + y^2) \) is a decreasing function of \( t \) and it follows as in Lemma 9 that \( \text{Im} G(x + iy) > 0 \) for \( x > 0 \) and \( y > 0 \) and also \( \text{Im} G(x + iy) < 0 \) for \( x > 0 \) and \( y < 0 \). Hence it is enough to show that \( G \) is one-to-one in the sector \( S(0, \pi/3) \).

For \( z_1 \) and \( z_2 \) belonging to the sector \( S(0, \pi/3) \) we have
\[
G(z_2) - G(z_1) = \int_{\gamma(z_1,z_2)} G'(w) \, dw,
\]
where \( \gamma(z_1,z_2) \) is the straight line segment from \( z_1 \) to \( z_2 \). Thus
\[
G(z_2) - G(z_1) = (z_2 - z_1) \int_0^1 G'(z_1 + t(z_2 - z_1)) \, dt \neq 0,
\]
when \( z_1 \neq z_2 \) since \( \text{Im} G'(w) < 0 \) for \( w \in S(0, \pi/3) \) by Lemma 9. This shows that \( G \) is one-to-one in \( S(0, \pi/3) \).

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