The Feynman identity (FI) of a planar graph relates the Euler polynomial of the graph to an infinite product over the equivalence classes of closed non-periodic signed cycles in the graph. The main objectives of this paper are to compute the number of equivalence classes of nonperiodic cycles of given length and sign in a planar graph and to interpret the data encoded by the FI in the context of free Lie superalgebras. This solves in the case of planar graphs a problem first raised by S. Sherman and sets the FI as the denominator identity of a free Lie superalgebra generated from a graph. Other results are obtained. For instance, in connection with zeta functions of graphs.

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1 Introduction

Denote by $\theta_\pm(N)$ the number of equivalence classes of nonperiodic cycles of length $N$ with sign $\pm 1$ in a finite connected and non-oriented planar graph $G$. Definitions are given in section 2. The univariate Feynman identity (FI) for a planar graph is the formal relation in the indeterminate $z$ that can be expressed as

$$E_G^2(z) = \prod_{N=1}^{+\infty} (1 + z^N)^{\theta_+(N)}(1 - z^N)^{\theta_-(N)}$$

(1.1)

where

$$E_G(z) := 1 + \sum_{N=1}^{|E|} a(N) z^N,$$

(1.2)

is the Euler polynomial of $G$. The coefficient $a(N)$ is the number of subgraphs of $G$ with $N$ edges that have all the vertices with even degree, called eulerian subgraphs.
The FI (1.1) was first conjectured by R. Feynman and proved by S. Sherman [22]. It is an important element of the combinatorial formalism of the Ising model in two dimensions, much studied by physicists [4]. The FI in the general non planar case is more complicated and demands advanced ideas from algebraic topology to be understood. It was proved in full generality by M. Loebl [18] and D. Cimasoni [2]. See also the paper by T. Helmuth [9]. In the present paper, we consider the FI for planar graphs, only.

In [23] Sherman considered a special case of the multivariate version of (1.1) where the graph has a single vertex and several loops hooked to it. He raised the problem of interpreting the FI in this special case in algebraic terms after pointing out similarities with the Witt identity of Lie algebra theory. The Witt identity, the $(+, -, +)$ case of relation (1.3) below, encodes informations about the dimensions of the vector spaces of a free Lie algebra, which are given by Witt formula (1.4), the enveloping algebra and the vector space that generates the algebra. See [21]:

**Proposition 1.1** If $V$ is an $R$-dimensional vector space and $L$ is the free Lie algebra generated by $V$, then $L = \bigoplus_{N=1}^{\infty} L_N$, $L_N$ has dimension given by $\mathcal{M}(N; R)$, and

$$\prod_{N=1}^{\infty} (1 - z^N)^{\pm \mathcal{M}(N; R)} = 1 \mp \sum_{N=1}^{\infty} f_\pm(N) z^N, \tag{1.3}$$

$$\mathcal{M}(N; R) = \frac{1}{N} \sum_{g|N} \mu(g) R^\frac{N}{g}, \tag{1.4}$$

where $f_+(1) = R$, $f_+(N) = 0$, if $N > 1$, $f_-(N) = R^N$. The summation is over all divisors of $N$ and $\mu$ is the Möbius function: $\mu(+) = 1$, $\mu(N) = 0$, if $g = p_1^{a_1}...p_q^{a_q}$ with $a_i > 1$, and $\mu(p_1...p_q) = (-1)^q$, $p_1$, ..., $p_q$ primes. In the $(-, +, -)$ case, (1.3) is the generating function for the dimensions of the homogeneous subspaces of the enveloping algebra of $L$.

In a series of papers S.-J. Kang and collaborators generalized Proposition 1.1 to more general free Lie algebras and superalgebras [11-13]. In this more general setting Sherman’s problem for the special case of (1.1) was solved in [5,6]. Now, we turn our attention to (1.1) for general planar graphs. This is now possible thanks to the formulation of the FI using the edge adjacency matrix [24] and the Kac-Ward transition matrix [2] of a planar graph. Using these matrices and ideas from [5] and [16] we derive counting formulas for the numbers $\theta_\pm(N)$ in (1.1). Then, using these counting formulas as generalized Witt formulas and results from [7] and [11-13] we solve Sherman’s problem.

The paper is organized as follows. In section 2, the edge adjacency matrix and the Kac-Ward transition matrix of a planar graph are defined and used to obtain counting formulas for the numbers of signed cycles (Theorem 2.1) and of
equivalence classes of nonperiodic cycles (the $\theta_{\pm}$ in (1.1)) (Theorem 2.2). In section 3, the results are encapsulated in three theorems and two remarks. In Theorem 3.1, which is important for section 4, other forms of the FI are given. The other theorems and remarks are complimentary or of general nature. Section 4 has two subsections. In the first one, we define a zeta function associated with the Kac-Ward transition matrix and show how it relates to the Ihara zeta function of a graph. In the second subsection, the results in previous sections and subsection are put together with results from [11-13]. These are collected in two propositions. They are the base of our interpretation of the FI (1.1) as the denominator (or generalized Witt) identity of a Lie superalgebra generated by a general finite connected planar graph and the signed cycles in them. In a previous paper [7], inspired by Sherman’s problem and the results in [5,6], we have considered the same problem in connection with identities which involve non-signed cycles in general graphs. They are associated with the Ihara and Bowen-Lanford zeta functions. New interpretations of these functions arose from relating them with Lie algebras. Some results in that paper will be needed in section 4.

Let’s add three final comments to this introduction. The combinatorial formalism of the Ising model was invented to avoid the complications of the algebraic formalism based on Lie algebras used by L. Onsager [20] in the calculation of the exact partition function for the two-dimensional Ising model. The present paper shows that there is a connection, which deserves further investigation, of the combinatorial formalism of the Ising model with Lie algebraic ideas. The results on the Sherman’s problem in [5,6] and in the present paper establishes a connection going from graph theoretical ideas to some of the foundational results in [11-14]. However, the results can be understood as a graph theoretical representation of some results in [11-14]. In the case of non-planar graphs, the right hand side of (1.1) becomes a finite sum of infinite products. It is possible to rewrite it in terms of a single infinite product and then use the same ideas of [11-13] to interpret the result algebraically, as in section 4.

2 Basic definitions and cycle counting formulas

In this section, we define cycles and the graph matrices needed to derive the counting formulas given in Theorems 2.1 and 2.2.

Call $G = (V, E)$ a finite connected and non oriented planar graph, $V$ is the set of vertices with $|V|$ elements and $E$ is the set of non oriented edges with $|E|$ elements labelled $e_1, ... , e_{|E|}$. The graph may have multiple edges and loops but no 1-degree vertices. Consider the graph $G'$ built from $G$ by fixing an orientation for the edges of $G$ and adding in the opposing oriented edges $e_{|E|+1} = (e_1)^{-1}, ... , e_{2|E|} = (e_{|E|})^{-1}$, $(e_i)^{-1}$ being the oriented edge opposite to $e_i$ and with origin (end) the end (origin) of $e_i$. In the case that $e_i$ is an oriented loop, $e_{i+|E|} = (e_i)^{-1}$ is just an additional
oriented loop hooked to the same vertex. Thus, $G'$ has $2|E|$ oriented edges. An edge with vertices $v_i$ and $v_j$ with the orientation from $v_i$ to $v_j$ is said to have origin at $v_i$ and end at $v_j$. A path in $G$ is given by an ordered sequence of edges $(e_{i_1}, ..., e_{i_N})$, $i_k \in \{1, ..., 2|E|\}$, in $G'$ such that the end of $e_{i_k}$ is the origin of $e_{i_k+1}$.

In this paper we call a cycle a closed path, that is, the end of $e_{i_N}$ coincides with the origin of $e_{i_1}$, subjected to the non-backtracking condition that $e_{i_k+1} \neq e_{i_k} + |E|$. In another words, a cycle never goes immediately backwards over a previous edge. The length of a cycle is the number of edges in its sequence. A cycle $p$ is called periodic if $p = q^r$ for some $r > 1$ and subsequence $q$. If there is no such $r > 1$, the cycle is called non periodic. Number $r$ is called the period of $p$. The cycle $(e_{i_N}, e_{i_1}, ..., e_{i_{N-1}})$ is called a circular permutation of $(e_{i_1}, ..., e_{i_N})$ and $(e_{i_{N-1}}^{-1}, ..., e_{i_1}^{-1})$ is an inversion of the latter. The circular permutations of a sequence represent the same cycle $p$, hence, they constitute an equivalence class denoted by $[p]$. Equivalent cycles have the same length. We will consider a cycle and its inversion as distinct. This is the reason for the square on the left hand side of (1.1). The sign of a cycle $p$ is given by the formula

$$s(p) = (-1)^{1+n(p)}$$

where $n(p)$ is the number of integral revolutions of a vector tangent to $p$. Equivalent cycles have the same sign. A cycle and the inverse cycle have the same sign. (The sign of a cycle $p$ can be obtained drawing a normal curve compatible with the cycle. Then, Whitney’s theorem says that the sign is given by $(-1)^{V_0}$ where $V_0$ is the number of self-intersections of the curve. See [3]).

In order to count cycles of a given length and sign in a non oriented graph $G$ we need the edge adjacency matrix [24] and the Kac-Ward transition matrix of $G$ [2]. The edge adjacency matrix is the $2|E| \times 2|E|$ matrix $T(G)$ with entries indexed by the edges of $G'$ defined by

$$T(G)_{e,e'} = \begin{cases} 1 & \text{if } f(e) = s(e') \text{ but } e' \neq e^{-1}; \\ 0 & \text{otherwise,} \end{cases}$$

where $f(e)$ is the end vertex of edge $e$ and $s(e')$ is the vertex at the origin of $e'$. The transition matrix of $G$ is the $2|E| \times 2|E|$ matrix $S(G)$ with entries also indexed by the edges of $G'$ and given by

$$S(G)_{e,e'} = \begin{cases} \exp\left(\frac{i}{2} \alpha(e, e')\right) & \text{if } f(e) = s(e') \text{ but } e' \neq e^{-1}; \\ 0 & \text{otherwise.} \end{cases}$$

Fix an orienting frame on $\mathbb{R}^2$ and a unit circle with center at the origin $(0,0)$. Define the turning angle $\alpha(e, e')$ as the net number of rotations around the unit circle made by a vector equivalent to the unit tangent vector to the edge $e$ as it starts, for instance, from the middle of this edge and ends at the middle point of $e'$. The angle is positive for anticlockwise rotations and negative for clockwise ones. The structure of matrix $S$, except for the complexity of the entries, resembles very much that of $T$.
given in [10] and [24] and can be obtained similarly. The matrix can be expressed as
\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\]
where \(A, B, C\) and \(D\) are \(|E| \times |E|\) matrices. Denote by \(\overline{M}\) the transpose conjugate of \(M\). Given the graph \(G'\) with the edges labeled as in the introduction, a) \(S_{e,e'} = 0 = S_{e',e}\) if \(e' = e^{-1}\), by definition of \(S\), b) \(B = \overline{B}\), c) \(C = \overline{C'}\). The diagonal entries of \(B\) and \(C\) are zero. d) \(D = \overline{A}\). The diagonals of \(A\) and \(D\) are zero if the graph has no loops.

Theorems 2.1 and 2.2 below give counting formulas for the number of cycles with given sign and length in a planar graph in terms of powers of the matrices \(T\) and \(S\). The counting formulas for the exponents \(\theta_{\pm}\) in (1.1) are given in Theorem 2.2.

**Theorem 2.1** Given a graph \(G\), denote by \(K_{\pm}(N)\) the number of cycles with sign \(\pm 1\). Then,
\[
\text{Tr} \, T^N = K_{+}(N) + K_{-}(N),
\]
\[
\text{Tr} \, S^N = K_{-}(N) - K_{+}(N),
\]
\[
K_{\pm}(N) = \frac{1}{2} \text{Tr} \left[ T^N \mp S^N \right].
\]

**Proof.** Let \(a\) and \(b\) be two edges of \(G\). The \((a, b)^{th}\) entry of matrix \(T^N\) is
\[
(T^N)_{(a,b)} = \sum_{e_{i_1} \ldots e_{i_{N-1}}} T_{(a,e_{i_1})} T_{(e_{i_1},e_{i_2})} \ldots T_{(e_{i_{N-1}},b)}.
\]
The definition of \(T\) gives that \((T^N)_{(a,b)}\) counts the number of paths of length \(N\) with no backtracks from edge \(a\) to edge \(b\). For \(b = a\), only closed paths are counted. Taking the trace gives the number of cycles with every edge taken into account as starting edge, hence, the trace overcounts cycles because every edge in the cycle is taken into account as starting edge. The cycles counted by the trace are tail-less, that is, \(e_{i_1} \neq e_{i_1}^{-1}\); otherwise, \(\text{Tr} T^N = \sum_a (T^N)_{(a,a)}\) would have a term with entry \((a, a^{-1})\) which is not possible. From the definition of \(K_{\pm}(N)\), relation (2.4) follows. On the other hand, the \((a, b)^{th}\) entry of matrix \(S^N\) is
\[
(S^N)_{(a,b)} = \sum_{e_{i_1} \ldots e_{i_{N-1}}} S_{(a,e_{i_1})} S_{(e_{i_1},e_{i_2})} \ldots S_{(e_{i_{N-1}},b)}.
\]
From (2.3) it follows that for \(b = a\) each non zero product in the summand equals
\[
e^{i\pi} = (-1)^n = (-1)^{\text{sign}(p)},
\]
for some integer \( n \) and cycle \( p \) of length \( N \). Hence, taking the trace gives the total number of positive signs which is equal to the number of positive cycles, \( \mathcal{K}_+(N) \), times \(-1\), plus the total number of negative signs which is equal to \(-\mathcal{K}_-(N)\), times \(-1\), and we get (2.5). From this and (2.4), (2.6) will follow. \( \square \)

In [16] M. Lin investigates combinatorial aspects of a multivariate special case of (1.1) when the graph has one vertex and several loops hooked to it. We apply ideas from [16] to prove relations (2.7) and (2.8) below.

**Theorem 2.2** Denote by \( \theta(N) \) the number of equivalence classes of non periodic cycles of length \( N \) in a graph \( G \) and by \( \theta_{\pm}(N) \) the number of equivalence classes of non periodic cycles of length \( N \) with sign \( \pm 1 \), \( \theta(N) = \theta_+(N) + \theta_-(N) \). Then,

\[
\theta_+(N) = \frac{1}{N} \sum_{g \text{ odd}|N} \mu(g)\mathcal{K}_+ \left( \frac{N}{g} \right),
\]

(2.7)

\[
\theta_-(N) = \frac{1}{N} \sum_{g \text{ even}|N} \mu(g)\mathcal{K}_+ \left( \frac{N}{g} \right) + \frac{1}{N} \sum_{g|N} \mu(g)\mathcal{K}_- \left( \frac{N}{g} \right),
\]

(2.8)

\[
\theta(N) = \frac{1}{N} \sum_{g|N} \mu(g)\text{Tr}_{T} \left( \frac{N}{g} \right).
\]

(2.9)

where \( \mathcal{K}_{\pm} \) is given by (2.6).

**Proof.** See [16], section 2.2. The sign of a cycle of period \( g \), \( p = (h)^g \), can be expressed as

\[
\text{sign}(p) = (-1)^{g+1}(\text{sign}(h))^g.
\]

For \( g \) odd, \( \text{sign}(p) = +1 \) if and only if \( \text{sign}(h) = +1 \), hence,

\[
\mathcal{K}_+(N) = \sum_{g \text{ odd}|N} \frac{N}{g} \theta_+ \left( \frac{N}{g} \right).
\]

(2.10)

Inverting this relation gives (2.7). The proof is similar to the one of Möbius inversion formula [15]. Now, \( \text{sign}(p) = -1 \) whenever \( g \) is even or, for any \( g \), we have that \( \text{sign}(h) = -1 \). Therefore,

\[
\mathcal{K}_-(N) = \sum_{g \text{ even}|N} \frac{N}{g} \theta_+ \left( \frac{N}{g} \right) + \sum_{g|N} \frac{N}{g} \theta_- \left( \frac{N}{g} \right),
\]

(2.11)

and

\[
\mathcal{K}(N) := \mathcal{K}_+(N) + \mathcal{K}_-(N) = \sum_{g|N} \frac{N}{g} \theta \left( \frac{N}{g} \right).
\]

(2.12)
By Möbius inversion,
\[
\theta(N) = \theta_+(N) + \theta_-(N) = \frac{1}{N} \sum_{g|N} \mu(g) \mathcal{K} \left( \frac{N}{g} \right).
\]

Using (2.6), we get (2.9). Besides,
\[
\theta_-(N) = \frac{1}{N} \sum_{g|N} \mu(g) \mathcal{K}_+ \left( \frac{N}{g} \right) - \theta_+(N)
\]
\[
= \frac{1}{N} \sum_{g|N} \mu(g) \left( \mathcal{K}_+ \left( \frac{N}{g} \right) + \mathcal{K}_- \left( \frac{N}{g} \right) \right) - \frac{1}{N} \sum_{g \text{ odd}|N} \mu(g) \mathcal{K}_+ \left( \frac{N}{g} \right)
\]
\[
= \frac{1}{N} \sum_{g \text{ even}|N} \mu(g) \mathcal{K}_+ \left( \frac{N}{g} \right) + \frac{1}{N} \sum_{g|N} \mu(g) \mathcal{K}_- \left( \frac{N}{g} \right).
\]

\[
\Box
\]

3 Other forms of the FI

In this section we derive some identities which connect the FI (1.1) to an exponential, a determinant, formal Taylor expansions and an explicit formula for the coefficients. These other forms of the FI are given in Theorem 3.1 below. Together with Theorem 2.2, Theorem 3.1 is important for the algebraic interpretation of the FI in section 4. The determinantal identity (3.3) below is called the Kac-Ward formula and it is well known in the literature about the Ising model. See [2], [17], and references therein, for other derivations of this formula. The meaning of $\Omega(N)$ in (3.3) will be clear from Theorem 3.2. In Theorem 3.3 we collect several recursions relating the coefficients $a(N)$ in (1.2), the Taylor expansions (3.2) and the exponents $\Omega(N)$. Remarks 3.1-2 give complimentary results.

**Theorem 3.1** Define
\[
g(z) := \sum_{N=1}^{+\infty} \frac{\text{Tr} S^N}{N} z^N,
\]
\[
(3.1)
\]
Then,
\[
\mathcal{E}_G^{\pm 2}(z) = e^{\mp g(z)} = 1 + \sum_{i=1}^{+\infty} c_{\pm}(i) z^i
\]
\[
= \prod_{N=1}^{+\infty} \left( 1 - z^N \right)^{\pm \Omega(N)} = [\det (I - zS)]^{\pm 1},
\]
\[
(3.3)
\]
where
\[ \Omega(N) := \frac{1}{N} \sum_{g \mid N} \mu(g) \text{Tr} S_N^g \]  
(3.4)

and
\[ c_\pm(i) = \sum_{m=1}^i \lambda_\pm(m) \sum_{a_1 + 2a_2 + \ldots + ia_i = i} \prod_{k=1}^i (\text{Tr} S_k^a_k)^{a_k} a_k^{k a_k} \]  
(3.5)

where \( \lambda_+(m) = (-1)^{m+1} \), \( \lambda_-(m) = +1 \). Furthermore,
\[ \text{Tr} S_N = N \sum_{s = (s_i)_{i \geq 1}} \frac{(\pm 1)^{|s| + 1} (|s| - 1)!}{s!} \prod_{i} c_\pm(i)^{s_i} \, |s| = \sum s_i, \quad s! = \prod s_i! \]  
(3.6)

Proof. See [5], [16] and [11-13]. Take the formal logarithm of both sides of (1.1) to get
\[ \ln \mathcal{E}_G^{\pm 2}(z) = \pm \sum_{N'=1}^{+\infty} \left[ \theta_+(N') \ln(1 + z^{N'}) + \theta_-(N') \ln(1 - z^{N'}) \right] \]
\[ = \pm \sum_{N'=1}^{+\infty} \sum_{l=1}^{+\infty} \left[ (-1)^{l-1} z^{N' l} \right] \frac{\theta_+(N')}{l} - \theta_-(N') \frac{z^{N' l}}{l} \]
\[ = \pm \sum_{N'=1}^{+\infty} \sum_{l=1}^{+\infty} \left[ (-1)^{l-1} \theta_+(N') + \theta_-(N') \right] \frac{z^{N' l}}{l} \]
\[ = \pm \sum_{N=1}^{+\infty} \mathcal{L}(N) z^{N}, \]

where
\[ \mathcal{L}(N) := \sum_{g \mid N} \frac{1}{g} \left[ (-1)^g \theta_+ \left( \frac{N}{g} \right) + \theta_- \left( \frac{N}{g} \right) \right] . \]

Decompose \( \mathcal{L}(N) \) as a sum over the even divisors of \( N \) plus a sum over the odd divisors of \( N \). Using \( \theta = \theta_+ + \theta_- \) it results that
\[ -\mathcal{L}(N) = - \sum_{g \mid N \text{ even}} \frac{1}{g} \left( \frac{N}{g} \right) + \sum_{g \mid N \text{ odd}} \frac{1}{g} \left[ \theta_+ \left( \frac{N}{g} \right) - \theta_- \left( \frac{N}{g} \right) \right] \]
\[ = - \frac{1}{N} \sum_{g \mid N} \frac{N}{g} \theta \left( \frac{N}{g} \right) + \frac{1}{N} \sum_{g \mid N \text{ odd}} N \theta_+ \left( \frac{N}{g} \right) . \]
Using (2.12) and (2.6),
\[-N \mathcal{L}(N) = -\mathcal{K}(N) + 2\mathcal{K}_+(N) = \mathcal{K}_+(N) - \mathcal{K}_-(N) = \text{Tr}(-S^N),\]
so that
\[\mathcal{L}(N) = \frac{\text{Tr}S^N}{N}.\]
By Jacobi trace formula,
\[\ln \mathcal{E}_G^{\pm 2}(z) = \pm \sum_{N=1}^{\infty} \frac{\text{Tr}(-S^N)}{N} z^N = \pm \text{Tr} \ln (1 - zS) = \pm \ln \det (1 - zS),\]
\[\mathcal{E}_G^{\pm 2}(z) = [\det (I - zS)]^\pm.\]
Set
\[\Omega(N) = \sum_{g | N} \frac{\mu(g)}{g} \mathcal{L} \left( \frac{N}{g} \right)\]
so
\[\mathcal{L}(N) = \sum_{g | N} \frac{1}{g} \Omega \left( \frac{N}{g} \right).\]
Then,
\[\ln \mathcal{E}_G^{\pm 2}(z) = \pm \sum_{N=1}^{\infty} \sum_{g | N} \frac{1}{g} \Omega \left( \frac{N}{g} \right) z^N = \pm \sum_N \Omega(N) \ln (1 - z^N),\]
and we get (3.3).

The coefficients \(c_\pm(i)\) are given by
\[c_\pm(i) = \frac{1}{i!} \frac{d^i}{dz^i} [(1 - e^{\mp g})]_{z=0}.\]
Using Faa di Bruno’s formula as in [5], the derivatives can be computed explicitly to give (3.5).

To prove (3.6) write
\[\sum_{k=1}^{\infty} \frac{\text{Tr} S^k}{k} z^k = \pm \ln \left( 1 + \sum_{i=1}^{\infty} c_\pm(i) z^i \right) = \pm \sum_{l=1}^{\infty} \frac{(-1)^l}{l} \left( \pm \sum_i c_\pm(i) z^i \right)^l.\]
Expand the right hand side in powers of \(z\) to get:
\[\sum_{k=1}^{\infty} z^k \sum_{s = (s_i)_{i \geq 1}, s_i \in \mathbb{Z}_{\geq 0}}^{(s_i)_{i \geq 1}} (\pm 1)^{|s|+1} \frac{(|s|-1)!}{s!} \prod_{i} c_\pm(i)^{s_i}.\]
Comparing the coefficients give the result. □

The meaning of $\Omega(N)$ becomes clear by the following result:

**Theorem 3.2**

$$\Omega(N) = \begin{cases} 
\theta_-(N) - \theta_+(N) & \text{if } N \text{ is odd} \\
\theta_-(N) - \theta_+(N) + \theta_+\left(\frac{N}{2}\right) & \text{if } N \text{ is even}
\end{cases}$$ (3.7)

**Proof.** From (2.7) and (2.8),

$$\theta_+(N) - \theta_-(N) = \frac{1}{N} \sum_{g \text{ odd}\mid N} \mu(g) K_+\left(\frac{N}{g}\right) - \frac{1}{N} \sum_{g \text{ even}\mid N} \mu(g) K_+\left(\frac{N}{g}\right)$$

$$- \frac{1}{N} \sum_{g\mid N} \mu(g) K_-\left(\frac{N}{g}\right)$$

$$= \frac{1}{N} \sum_{g \text{ odd}\mid N} \mu(g) \left[ K_+\left(\frac{N}{g}\right) - K_-\left(\frac{N}{g}\right) \right]$$

$$- \frac{1}{N} \sum_{g \text{ even}\mid N} \mu(g) \left[ K_+\left(\frac{N}{g}\right) + K_-\left(\frac{N}{g}\right) \right].$$

The sum over the odd divisors of $N$ equals

$$\frac{1}{N} \sum_{g\mid N} \mu(g) \text{Tr}\left(-S^N\right) - \frac{1}{N} \sum_{g\mid N} \mu(g) \left[ K_+\left(\frac{N}{g}\right) - K_-\left(\frac{N}{g}\right) \right].$$

We get

$$\theta_+(N) - \theta_-(N) = -\Omega(N) - \frac{2}{N} \sum_{g \text{ even}\mid N} \mu(g) K_+\left(\frac{N}{g}\right).$$

Thus, $\Omega(N) = \theta_+(N) - \theta_-(N)$, if $N$ is odd. If $N$ is even, the even divisors of $N = 2^j n$ are $2^k$, $k = 1, \ldots, j$, and $2^i p$, $i = 1, 2, \ldots, j$, and $p$ are the odd divisors of $n$. However, $\mu = 0$ for the cases $k, i \geq 2$, hence, using that $\mu(2p) = \mu(2)\mu(p) = -\mu(p)$, we get that

$$\frac{2}{N} \sum_{g \text{ even}\mid N} \mu(g) K_+\left(\frac{N}{g}\right) = \frac{2}{N} \sum_{p \text{ odd}\mid N} \mu(2p) K_+\left(\frac{N}{2p}\right)$$

$$= -\frac{2}{N} \sum_{p \text{ odd}\mid N/2} \mu(p) K_+\left(\frac{N}{2p}\right)$$

$$= -\theta_+\left(\frac{N}{2}\right).$$
Remark 3.1. There are graphs with the property that $\Omega(N) = 0$ for all $N \geq N_0$, for some $N_0$. In another words, $\theta_+(N) = \theta_-(N)$, for all odd $N \geq N_0$, and $\theta_-(N) = \theta_+(N) - \theta_+((N/2))$, for all even $N \geq N_0$. This is the case of the graph with one vertex and $R$ loops hooked to it. This is proved in reference [5]. A simpler proof is as follows. From Theorem 3.1, relation (3.2) and (1.1),

$$-2z \frac{d}{dz} \ln \mathcal{E} = \sum_{N \geq 1} \text{Tr} S^N z^N.$$  

Using that $\mathcal{E}(z) = (1 + z)^R$,

$$-2z \frac{d}{dz} \ln \mathcal{E} = \sum_{N \geq 1} (-1)^N 2R z^N,$$

we get $\text{Tr} S^N = (-1)^N 2R$. Then, by (3.4), $\Omega(1) = -2R$, $\Omega(2) = 2R$, $\Omega(N) = 0$, if $N \geq 3$, and

$$\prod_{N=1}^{+\infty} (1 - z^N) \Omega(N) = (1 - z)^{-2R}(1 - z^2)^{2R} = (1 + z)^{2R}.$$  

Another example is provided by the graph obtained from $R$ copies of the graph with vertices and two edges linking them, glued at the vertices. It has one vertex of degree 2 at the far left and another one at the far right and $R - 2$ vertices of degree 4 in between them. The Euler polynomial is $\mathcal{E}(z) = (1 + z^2)^R$ so

$$-2z \frac{d}{dz} \ln \mathcal{E} = \sum_{N \geq 1} (-1)^N 4R z^N.$$  

We get $\text{Tr} S^N = (-1)^{N/2} 4R$, if $N$ is even, and $\text{Tr} S^N = 0$, if $N$ is odd, so that $\Omega(N) = 0$, if $N$ is odd, $\Omega(2) = -2R$, $\Omega(4) = +2R$, $\Omega(N) = 0$, if $N$ is even and $N \geq 6$, so

$$\prod_{N=1}^{+\infty} (1 - z^N) \Omega(N) = (1 - z^2)^{-2R}(1 - z^4)^{2R} = (1 + z^2)^{2R}.$$  

Remark 3.2. By the Schur decomposition method there is a matrix $P$ and an upper triangular matrix $J$ with the eigenvalues $\lambda_i$ of $S$ along the diagonal such that $S = PJP^{-1}$, hence,

$$\text{Tr} S^N = \text{Tr}(PJP^{-1})^N = \text{Tr} J^N = \sum_{i=1}^{2|E|} \lambda_i^N,$$

and

$$\Omega(N) = \frac{1}{N} \sum_{g|N} \mu(g) \text{Tr} S^\frac{N}{g} = \sum_{i=1}^{2|E|} \frac{1}{N} \sum_{g|N} \mu(g) \lambda_i^\frac{N}{g} = \sum_{i=1}^{2|E|} \mathcal{M}(N; \lambda_i),$$

\(\Box\)
where $\mathcal{M}$ is given in (1.4). Using Witt identity (see (1.3), case $(+, - , +)$),

$$\det(1 - zS) = \prod_{N=1}^{+\infty} (1 - z^N)^{\Omega(N)} = \prod_{i=1}^{2|E|} (1 - \lambda_i z).$$

The zeros of $\mathcal{E}_G^2(z)$, as a complex function in $z$, are the reciprocals of the eigenvalues of $S$. Expanding the product,

$$\det(1 - zS) = \prod_{i=1}^{2|E|} (1 - \lambda_i z) = 1 - \text{Tr} Sz + \cdots + \det Sz^{2|E|}.$$

This polynomial is the square of the Euler polynomial $1 + \sum_{i=1}^{E} a(i) z^i$ so that $-\text{Tr} S = 2a(1)$ and $\det S = a^2(|E|)$, where $a(i)$ is the number of eulerian subgraphs with $i$ edges, hence, $-\text{Tr} S \geq 0$ and $-\text{Tr} S$ is the number of loops in the graph so that $\sum_i \lambda_i = 0$ if the graph $G$ has no loops and $\sum_i \lambda_i < 0$, otherwise. Also, $\det S \geq 0$ and the square root of $\det S$ is the number of eulerian graphs with $|E|$ edges, hence, $\det S = 0$ if and only if the graph is not itself eulerian and $\det S = 1$, otherwise. We may conclude that $S$ has zero as an eigenvalue if and only if the graph itself is not eulerian.

**Theorem 3.3** Set $\omega(n) := \text{Tr} S^n$. Then,

$$c_{\pm}(1) = \omega(1) = \Omega(1), \quad (3.8)$$

$$nc_{\pm}(n) = \omega(n) \mp \sum_{k=1}^{n-1} \omega(n - k)c_{\pm}(k), \ n \geq 2, \quad (3.9)$$

$$c_{-}(n) = c_{+}(n) + \sum_{i=1}^{n-1} c_{+}(i)c_{-}(n - i), \ n \geq 2, \quad (3.10)$$

$$|c_{+}(n)| \leq c_{-}(n), \quad (3.11)$$

$$\Omega(n) = c_{+}(n) + \frac{1}{n} \sum_{k=1}^{n-1} \left( \sum_{g|k} g\Omega(g) \right) c_{+}(n - k) - \sum_{n\neq g|n} \frac{g}{n} \Omega(g), \quad (3.12)$$

$$2na(n) = -nc_{+}(n) + \sum_{k=1}^{n} (3k - n)a(k)c_{+}(n - k). \quad (3.13)$$

**Proof.** See reference [7].
4 The FI and Lie superalgebras

4.1 The zeta functions \( \zeta_I \) and \( \zeta_{KW} \)

In this subsection additional results are remarked. Some of them will be relevant in the next subsection. For instance, we define a zeta function \( \zeta_{KW} \) associated with the Kac-Ward transition matrix \( S \) and show how it relates to the Ihara zeta function of a graph and derive two product identities. We give examples of graphs with the same \( \zeta_{KW} \).

Remark 4.1. Calculations similar to those in Theorem 3.1 show that

\[
\prod_{N=1}^{+\infty} (1 - z^N)^{\theta(N)} = \det(I - zT). \tag{4.1}
\]

See [7,24]. The reciprocal of this relation is known in association with the Ihara zeta function \( \zeta_I(z) \) of a graph: \( \zeta_I(z) = \det(1 - zT)^{-1} \). Therefore, it seems natural to define \( \zeta_{KW}(z) = \det(1 - zS)^{-1} \). In subsection 4.2, this function will be associated to an algebra and the dimensions of its subspaces. Define

\[
g_{\pm}(z) = \sum_{N=1}^{+\infty} \frac{K_{\pm}(N)}{N} z^N. \tag{4.2}
\]

Then,

\[
\zeta_I(z) = e^{2g_+(z)} \zeta_{KW}(z), \quad \zeta_I(z) \zeta_{KW}(z) = e^{2g_-(z)}. \tag{4.3}
\]

From these two relations one can get the identities (3.3) and (4.1). Also,

\[
\prod_{N=1}^{+\infty} \left( \frac{1 + z^N}{1 - z^N} \right)^{\theta_+(N)} = \frac{\det(1 - zS)}{\det(1 - zT)}, \tag{4.4}
\]

\[
\prod_{N=1}^{+\infty} \left( \frac{1 + z^N}{1 - z^N} \right)^{\theta_-(N)} = \frac{\det(1 - z^2T)}{\det(1 - zT) \det(1 - zS)}. \tag{4.5}
\]

Other identities are possible to obtain.

Remark 4.2. Two non isomorphic graphs can have the same Ihara zeta function \( \zeta_I(z) \). Also, two non isomorphic graphs can have the same Euler polynomial (see [1,8,19], and references therein), hence, the same \( \zeta_{KW}(z) \). For instance, the graphs in Figure 1 of [8]. One of them is the graph in Remark (...), second example, with \( R = 3 \). The other one is the graph with 6 edges which has a circle subgraph with 4 vertices but a pair of the consecutive vertices has extra two edges linking them. They have the same Euler polynomial \( \mathcal{E}(z) = (1 + z^2)^3 \). From Remark 3.1, \( \text{Tr} \ S^N = 0 \), if \( N \) is odd, and \( \text{Tr} \ S^N = 12(-1)^{N/2} \), if \( N \) is even, so \( \Omega(N) = 0 \), if \( N \) is odd, \( \Omega(2) = -6 \), \( \Omega(4) = +6 \), \( \Omega(N) = 0 \), if \( N \) is even and \( N \geq 6 \). Both graphs have the same sequence
\( \{\Omega(N), N \geq 1\} \). It follows from the recursion relation (3.12) that two graphs with same \( \zeta_{KW}(z) \) will have in common the same sequence \( \{\Omega(N), N \geq 1\} \), and the same form (3.3) of the FI.

### 4.2 Feynman meets Lie

In this subsection the FI (1.1) is associated to a free Lie superalgebra, using results obtained by S.-J. Kang and collaborators in [11-14]. They generalized Proposition 1.1 (see section 1) to more general free Lie (super)algebras generated by infinite graded vector spaces. We apply their results to give a new interpretation of the FI (1.1) and of the associated function \( \zeta_{KW}(z) \). The results in this subsection can be understood as examples arising from graph theoretical ideas of some results in [11-14].

The results from [11-13] which are relevant for our objectives are summarized in the Propositions 4.1 and 4.2 below. Relations (4.6-7) and (4.11-13) are called the generalized Witt formulas; relations (4.8) and (4.14), the (+, −, +) cases, are called the denominator or generalized Witt identities of the algebras.

**Proposition 4.1** Let \( \mathcal{V} = \bigoplus_{N=1}^{\infty} \mathcal{V}_N \) be a \( \mathbb{Z}_{>0} \)-graded superspace with finite dimensions \( \dim \mathcal{V}_N = |t(N)| \) and superdimensions \( \dim \mathcal{V}_N = t(N) \in \mathbb{Z}, \forall i \geq 1 \). Let \( \mathcal{L} = \bigoplus_{N=1}^{\infty} \mathcal{L}_N \) be the free Lie superalgebra generated by \( \mathcal{V} \) with a \( \mathbb{Z}_{>0} \)-gradation induced by that of \( \mathcal{V} \). Then, the \( \mathcal{L}_N \) superdimension is

\[
\dim \mathcal{L}_N = \sum_{g|N} \frac{\mu(g)}{g} \, W \left( \frac{N}{g} \right),
\]

The summation ranges over all positive divisors \( g \) of \( N \) and \( W \) is given by

\[
W(N) = \sum_{s \in T(N)} \frac{(|s| - 1)!}{s!} \prod t(i)^{s_i},
\]

where \( T(N) = \{s = (s_i)_{i \geq 1} \mid s_i \in \mathbb{Z}_{\geq 0}, \sum is_i = N \} \) and \( |s| = \sum s_i, s! = \prod s_i! \).

Furthermore,

\[
\prod_{N=1}^{\infty} (1 - z^N)^{\pm \dim \mathcal{L}_N} = 1 \mp \sum_{N=1}^{\infty} f_{\pm}(N) z^N,
\]

with \( f_{+}(N) = t(N) \) and \( f_{-}(N) = \dim U(\mathcal{L}), \) where \( \dim U(\mathcal{L})_N \) is the dimension of the \( N\)-th homogeneous subspace of the universal enveloping algebra \( U(\mathcal{L}) \) and the generating function for the \( W \)'s,

\[
g(z) := \sum_{N=1}^{\infty} W(N) z^N,
\]
satisfies

\[ e^{-g(z)} = 1 - \sum_{N=1}^{\infty} t(N) z^N. \]  

(4.10)

□

See section 2.3 of [12]. Given a formal power series \( \sum_{N=1}^{\infty} t_N z^N \) with \( t_N \in \mathbb{Z} \), for all \( i \geq 1 \), the coefficients in the series can be interpreted as the superdimensions of a \( \mathbb{Z}_{>0} \)-graded superspace \( \mathcal{V} = \bigoplus_{N=1}^{\infty} \mathcal{V}_N \) with dimensions \( \dim \mathcal{V}_N = |t_N| \) and superdimensions \( \dim \mathcal{V}_N = t_N \in \mathbb{Z} \). Let \( \mathcal{L} \) be the free Lie superalgebra generated by \( \mathcal{V} \). Then, it has a gradation induced by \( \mathcal{V} \) and its homogeneous subspaces have dimensions given by (4.6) and (4.7). Let’s consider the \((+)\) case of (3.3). Apply the previous interpretation to \( \det(1 - zS) \) as a polynomial of degree \( 2|E| \) in the formal variable \( z \). This is a power series with coefficients \( c + (N) \), if \( N \leq 2|E| \), and \( t_N = 0 \), if \( N > 2|E| \). Comparison of the relations in Theorem 3.1 with those in Proposition 4.1 yields: given a graph \( G \), \( S \) its associated Kac-Ward transition matrix, let \( \mathcal{V} = \bigoplus_{N=1}^{\infty} \mathcal{V}_N \) be a \( \mathbb{Z}_{>0} \)-graded superspace with dimensions \( \dim \mathcal{V}_N = |c + (N)| \) and the superdimensions \( \dim \mathcal{V}_N = -c_+ (N) \) where \( -c_+ (N) \) is the coefficient of \( z^N \) in \( \det(1 - zS) \). Let \( \mathcal{L} = \bigoplus_{N=1}^{\infty} \mathcal{L}_N \) be the free Lie superalgebra generated by \( \mathcal{V} \). Then, the \( \mathcal{L}_N \) superdimension is \( \dim \mathcal{L}_N = \Omega(N) \) and \( \zeta_{KW}(z) \) is the generating function for the dimensions of the subspaces of the enveloping algebra of \( \mathcal{L} \). These can be computed recursively using the recursions in Theorem 3.3. If we raise both sides of the plus case of (3.3) to \( 1/2 \) its right hand side is the generating function of eulerian subgraphs, so in this case the vector space \( \mathcal{V}_N \) is generated by the eulerian subgraphs of size \( N \). In [7] we have already applied Proposition 4.1 to give an algebraic interpretation of (2.9) and (4.1).

Another interpretation follows from the next proposition:

**Proposition 4.2** Let \( \mathcal{V} = \bigoplus_{(n,a) \in \mathbb{Z}_{>0} \times \mathbb{Z}_2} \mathcal{V}_{(n,a)} \) be a \((\mathbb{Z}_{>0} \times \mathbb{Z}_2)\)-graded colored superspace with superdimensions \( \dim \mathcal{V}_{(n,a)} = t(n,a) \in \mathbb{Z}, \forall (n,a) \in \mathbb{Z}_{>0} \times \mathbb{Z}_2 \). Let \( \mathcal{L} = \bigoplus_{(n,a) \in \mathbb{Z}_{>0} \times \mathbb{Z}_2} \mathcal{L}_{(n,a)} \) be the free Lie superalgebra generated by \( \mathcal{V} \). Then, the dimensions of the homogeneous subspaces \( \mathcal{L}_{(n,a)} \) are given by

\[
\dim \mathcal{L}_{(n,0)} = \sum_{g \mid n} \frac{\mu(g)}{g} W \left( \frac{n}{g}, 0 \right) + \sum_{g \text{ even} \mid n} \frac{\mu(g)}{g} W \left( \frac{n}{g}, 1 \right),
\]

(4.11)

and

\[
\dim \mathcal{L}_{(n,1)} = \sum_{g \text{ odd} \mid n} \frac{\mu(g)}{g} W \left( \frac{n}{g}, 1 \right),
\]

(4.12)

where

\[
W(\tau, b) = \sum_{s \in T(\tau, b)} \frac{(|s| - 1)}{s!} \prod t(\tau_i, b_j)^{s_{ij}}
\]

(4.13)
and
\[ T(\tau, b) = \{ s = (s_{i,j})_{i,j \geq 1} \mid s_{i,j} \in \mathbb{Z}_{\geq 0}, \sum_{i,j} s_{i,j}(\tau_i, b_j) = (\tau, b) \}, \]
which is the set of partitions of \((\tau, b)\) into a sum of \((\tau_i, b_j)\)'s, \(|s| = \sum_{i,j} s_{i,j}, s! = \prod_{i,j} s_{i,j}!\). Furthermore,
\[ \prod_{(n, a) \in \mathbb{Z}_{>0} \times \mathbb{Z}_2} (1 - E^{(n,a)}_{\pm \dim \mathcal{L}_{(n,a)}}) = 1 \mp T^\pm_{\mathbb{Z}_{>0} \times \mathbb{Z}_2}, \quad (4.14) \]
where the \(E^{(n,a)}\) are basis elements of \(\mathbb{C}[\mathbb{Z}_{>0} \times \mathbb{Z}_2]\), \(E^{(n,a)}E^{(m,b)} = E^{(n+m,a+b)}\),
\[ f_+(n, a) = t(n, a), \text{ and } f_-(n, a) = \dim \mathcal{U}(\mathcal{L})_{(n,a)} \text{ is the superdimension of the homogeneous subspace } (n, a) \text{ of the enveloping algebra } \mathcal{U}(L). \]
The generating function for the \(W\)'s,
\[ g(z) := \sum_{(\tau, a) \in \mathbb{Z}_{>0} \times \mathbb{Z}_2} W(\tau, a)E^{(n,a)}, \quad (4.16) \]
satisfies
\[ e^{-g} = 1 - T^\pm_{\mathbb{Z}_{>0} \times \mathbb{Z}_2}. \quad (4.17) \]

\[ \square \]

On the base of Proposition 4.2 we interpret the data defined on a graph in terms of the data in this proposition. First, let’s make the specialization \(E^{(n,0)} = z^n\) and \(E^{(n,1)} = z^nq\) with \(q^2 = 1\). It follows that
\[ \prod_{n=1}^{+\infty} (1 - z^n)^{\dim \mathcal{L}_{(n, 0)}} (1 - qz^n)^{\dim \mathcal{L}_{(n, 1)}} = 1 - \sum_{n=1}^{+\infty} (t(n, 0) + qt(n, 1))z^n. \]
In particular, for \(q = -1\), we get
\[ \prod_{n=1}^{+\infty} (1 - z^n)^{\dim \mathcal{L}_{(n, 0)}} (1 + z^n)^{\dim \mathcal{L}_{(n, 1)}} = 1 - \sum_{n=1}^{+\infty} (t(n, 0) - t(n, 1))z^n, \]
which has the same form of the FI, and, for \(q = 1\), we get
\[ \prod_{n=1}^{+\infty} (1 - z^n)^{\dim \mathcal{L}_{(n, 0)}} \mp \dim \mathcal{L}_{(n, 1)} = 1 - \sum_{n=1}^{+\infty} (t(n, 0) + t(n, 1))z^n. \]
Set
\[ t'(n) := t(n, 0) - t(n, 1), \quad t(n) := t(n, 0) + t(n, 1). \quad (4.18) \]
In the case \(q = 1\) define \(\mathcal{V}_n = \bigoplus_a \mathcal{V}_{(n,a)}\) and \(\mathcal{L}_n = \bigoplus_a \mathcal{L}_{(n,a)}\). Then, \(\mathcal{V} = \bigoplus_{n=1}^{+\infty} \mathcal{V}_n\) with dimensions given by \(t(n) = \sum_a t(n, a) = t(n, 0) + t(n, 1)\) becomes a graded
vector space and the free superalgebra $\mathcal{L}$ on $\mathcal{V}$ has a gradation $\mathcal{L} = \bigoplus_n \mathcal{L}_n$ induced by $\mathcal{V}$ with dimensions given by (4.6) and (4.7). Therefore, one gets the data in the Proposition 4.1.

In order to fix our algebraic interpretation of the counting formulas (2.7) and (2.8) we need to know the dimensions of the spaces $\mathcal{V}(n,a)$. This information comes from the data from a graph given by the matrices $T$ and $S$ as follows. Suppose one knows $t'(n)$ and $t(n)$ but not $t(n,0)$ and $t(n,1)$. In this case, $t(n,0)$ and $t(n,1)$ can be computed using

$$t(n,0) = \frac{1}{2}(t'(n) + t(n)), \quad t(n,1) = \frac{1}{2}(t(n) - t'(n)) \quad (4.19)$$

which follow from (4.18). The numbers $t(n)$ and $t'(n)$ are given by the coefficients of the non constant terms in $\det(1 - zT)$ and $\det(1 - zS)$, respectively. The numbers $t(n) \pm t'(n)$ are always even integers as proved next.

**Theorem 4.3** The coefficients in the polynomials in $z$ given by

$$\det(1 - zT) \pm \det(1 - zS) \quad (4.20)$$

are even integers.

**Proof.** Using (4.4),

$$\det(1 - zT) \pm \det(1 - zS) = \left[ 1 \pm \prod_{N \geq 1} (1 + 2 \sum_{k=1}^{+\infty} z^{Nk})^{\theta_+(N)} \right] \det(1 - zT).$$

Furthermore, putting $z^N = z'$,

$$(1 + 2 \sum_{k=1}^{+\infty} z'^k)^{\theta_+(N)} = \sum_{m \geq 0} \alpha_m z'^m,$$

where $\alpha_0 = 1$ and

$$\alpha_m = \frac{2}{m} \sum_{k=1}^{m} (k\theta_+(N) - m + k)\alpha_{m-k}.$$

□

With the specialization $q = -1$, the data in Proposition 4.2 together with relations (2.7) and (2.8) in Theorem 2.2 and those in Theorem 3.1 yields: given a graph $G$ with edge and transition matrices $T$ and $S$, respectively, let $\mathcal{V} = \bigoplus_{(n,i) \in \mathbb{Z}_{>0} \times \mathbb{Z}_2} \mathcal{V}_{(n,i)}$ be a $(\mathbb{Z}_{>0} \times \mathbb{Z}_2)$-graded colored superspace with superdimension

$$\text{Dim} \mathcal{V}_{(n,i)} = t(n,i) := \frac{1}{2}(t'(a) + (-1)^i t(a))$$
where the $t'$ are given by the coefficients of $\det(1 - zS)$ and $t$ by the coefficients of $\det(1 - zT)$. Let $\mathcal{L} = \bigoplus_{(n,i) \in \mathbb{Z}_{>0} \times \mathbb{Z}} \mathcal{L}_{(n,i)}$ be the free Lie superalgebra generated by $\mathcal{V}$. Then, the dimensions of the homogeneous subspaces $\mathcal{L}_{(n,i)}$ are given by (2.7) and (2.8), that is, $\text{Dim} \mathcal{L}_{(n,0)} = \theta_-(n)$ and $\text{Dim} \mathcal{L}_{(n,1)} = \theta_+(n)$, and these satisfy the FI which plays the role of the $(+, -, +)$ case of (4.14). The generating function for the dimensions of the subspaces of the enveloping algebra of $\mathcal{L}$ is given by $\zeta_{KW}(z)$, the $(-, +, -)$ case of (4.14).

**Remark 4.3.** Define the supermatrix

$$Q = \begin{pmatrix} S & 0 \\
0 & T \end{pmatrix}$$

(4.21)

and the supertrace $\text{Str} Q = \text{Tr} S - \text{Tr} T$ so that

$$\theta_+(N) = \frac{1}{2N} \sum_{g \text{ odd},|N|} \mu(g) \text{Str} Q_g^N$$

(4.22)

Then, the quotient of the two determinants in (4.4) can be expressed as the superdeterminant (the Berezinian) $\text{Ber}(1 - zQ)$. Using the superformalism one can make a connection with the algebras in [14] which suggests a concrete link of these algebras with graph theoretical ideas.

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