Abstract

The analysis of progressively censored data has received considerable attention in the last few years. In this paper, we consider the joint progressive censoring scheme for two populations. It is assumed that the lifetime distribution of the items from the two populations follows Weibull distribution with the same shape but different scale parameters. Based on the joint progressive censoring scheme, first, we consider the maximum likelihood estimators of the unknown parameters whenever they exist. We provide the Bayesian inferences of the unknown parameters under a fairly general priors on the shape and scale parameters. The Bayes estimators and the associated credible intervals cannot be obtained in closed form, and we propose to use the importance sampling technique to compute the same. Further, we consider the problem when it is known a priori that the expected lifetime of one population is smaller than the other. We provide the order-restricted classical and Bayesian inferences of the unknown parameters. Monte Carlo simulations are performed to observe the performances of the different estimators and the associated confidence and credible intervals. One real data set has been analyzed for illustrative purpose.

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1 Introduction

In any life testing experiment, it is a common practice to terminate the experiment before all specimens under observation fail. In a type-I censoring scheme, the test is terminated at a prefixed time point whereas in a type-II censoring scheme, the experiment continues until a certain number of failures occurs. In a practical scenario, it might be necessary to remove some of the experimental units during the experiment. Different progressive censoring schemes allow removal of experimental units during the experiment.
A progressive type-II censoring scheme can be briefly described as follows. It is assumed that $n$ items are put on a test. Suppose $k, R_1, \ldots, R_k$ are non-negative integers such that $n - k = R_1 + \ldots + R_k$. At the time of the first failure say $t_1$, $R_1$ units are chosen at random from the remaining $(n - 1)$ items and they are removed from the experiment. Then, at the time of the second failure, say $t_2$, $R_2$ units are chosen at random from the remaining $n - 2 - R_1$ units and they are removed. The process continues, finally at the time of the $k$-th failure, all the remaining $R_k$ items are removed from the experiment, and the experiment stops.

During the Second World War, due to the demand of the highly reliable military equipment, engineers started laboratory investigation instead of long-awaited field experiments. Due to tremendous pressure on cost and time, various schemes were introduced to reduce the cost of testing and multistage censoring was one of them. As it has been mentioned in Herd (1956) that in many laboratory evaluations, due to limited allocation of funds, there are attempts to study the factors contributing to either the reliability or the unreliability of the components or the whole systems under study, as well as to estimate the reliability of the items. In some investigations, in order to measure these auxiliary variables, some systems need to be disassembled or subject to measurement which are destructive in nature or might change the expected lifetime. In certain evaluations, systems are removed from the main experimental setup in order to measure certain specific characteristics. In most of the cases, systems are subjected to interaction of human operator, and to assess the effect of the human operator on the efficiency of the active system, sometimes, it requires to withdraw few systems from the main experimental setup. During the reliability study of production prototype systems, the addition of engineering modifications also requires analyses on multistage censoring.

In Montanari and Cacciari (1988), experimenters conducted testing to evaluate endurance of XLPE-insulated cable to electrical and thermal stress along with the aging mechanism. Some specimens were removed from the test at selected times or at a time of breakdowns for the measurements of electrical, mechanical, and chemical properties along with micro structural analyses in order to evaluate aging process. These measurements are destructive in nature. In this experiment, the data obtained consist of failure time and censored time of specimens. Ng and Wang (2009) presented a clinical study where multistage censored data arise quite naturally. In a study with plasma cell myeloma at National Cancer Institute on 112 patients, few patients were dropping out at the end of certain intervals whose survival was ensured at that time but no further follow-up was available. In his thesis,
Herd (1956) discussed estimation under multistage censoring scheme and referred the scheme as “multi-censored samples.” Later on, different multistage censoring schemes were referred as progressive censoring schemes. Cohen (1963) studied the importance of progressive censoring scheme in reliability experiment and Cohen (1966) discussed several cases where censored data occurred quite naturally.

To be more precise, in favor of progressive censoring scheme, the following points can be mentioned. In practical scenario, experimental units might get damaged due to some unrelated reasons than the normal failure mechanism. To incorporate these information into inference study, we can rely on progressive censoring schemes. Sometimes, the multistage censoring is also intentionally done to use censored units from one experiment to another related experiment due to budgetary constraints.

The progressive censoring scheme has received considerable attention in the literature. Mann (1971) and Lemon (1975) studied on estimation for Weibull parameters under progressive censoring scheme. Viveros and Balakrishnan (1994) provided interval estimation on progressively censored data. Ng et al. (2004) studied on optimal progressive censoring plan when the underlined distribution is Weibull distribution whereas Kundu (2008) provided Bayesian inference of the Weibull population under progressive censoring scheme. Wang et al. (2010) studied the inference of certain lifetime distributions under progressive type-II right censored scheme. Recently published book by Balakrishnan and Cramer (2014) provided an excellent overview of the different developments on different aspects of the progressive censoring scheme, which have taken place on this topic over the last 20 years. Although type-II progressive censoring scheme is the most popular one, several other progressive censoring schemes have also been introduced in the literature. In this article, we have restricted our attention to the type-II progressive censoring scheme, although most of our results can be extended for other progressive censoring schemes also.

Although extensive work has been done on the progressive censoring scheme for one group, not much work has been done when two or more groups are present. Rasouli and Balakrishnan (2010) introduced a joint progressive censoring (JPC) scheme, which can be used to compare the lifetime distributions of two products manufactured in different units under the same environmental conditions. The JPC scheme proposed by Rasouli and Balakrishnan (2010) can be briefly described as follows. Suppose $m$ units of product A (Group 1) and $n$ units of product B (Group 2) are put on a test simultaneously at time zero. It is assumed that $R_1, \ldots, R_k$ are $k$ non-negative integers such that $R_1 + \ldots + R_k = n + m - k$. At the time of the first failure,
which may be either from Group 1 or Group 2, \( R_1 \) items from the remaining \( n + m - 1 \) remaining items have been selected at random, and they have been removed from the experiment. These \( R_1 \)-censored units consist of \( S_1 \) units from Group 1 and \( R_1 - S_1 \) units from Group 2. Here, \( S_1 \) is random. The time and group of the first failed item are recorded. Similarly, at the time of the second failure, \( R_2 \) items from the remaining \( n + m - R_1 - 2 \) items have been chosen at random, and they have been removed from the experiment. Among censored \( R_2 \) units, random number of \( S_2 \) units come from Group 1. The time and group of the second failed items are recorded. The process continues till the \( k \)-th failure takes place, when all the remaining items are removed and the experiment ends.

Based on the assumptions that the lifetime distributions of the experimental units of the two populations follow exponential distribution with different scale parameters, Rasouli and Balakrishnan (2010) provided the exact distributions of the maximum likelihood estimators (MLEs) of the unknown parameters and suggested several confidence intervals. Some of the related works on JPC scheme are by Shafay et al. (2014), Parsi and Bairamov (2009), and Doostparast et al. (2013). In most of these cases, it has been assumed that the lifetime distributions of the items in the two groups follow exponential distribution.

The exponential distribution has the constant hazard rate, which may not be very reasonable in a practical scenario. Again, when two similar kind of products are tested, it is quite expected to have some common parameters from the underlined distributions. To justify these scenario, in this paper, it is assumed that the lifetime distributions of the individual items of the two different groups follow Weibull distribution with the same shape parameter, but different scale parameters.

Our aim is to compare the lifetime distributions of the two populations. The problem is a typical two-sample problem. It can appear in a accelerated life testing problem, or in estimating the stress-strength parameter of a system.

First, we consider the MLEs of the unknown parameters based on the data obtained from a JPC scheme as proposed by Rasouli and Balakrishnan (2010). It has been shown that the MLEs exist under a very general condition and they are unique. The MLE of the common shape parameter can be obtained by solving one non-linear equation, and given the MLE of the common shape parameter, the MLEs of the scale parameters can be obtained in explicit forms. Although the performances of the MLEs are quite satisfactory, the associated confidence intervals are not very easy to obtain. Hence, it seems the Bayesian inference is a natural choice in this case. It
may be treated as an extension of the work of Kundu (2008), where the Bayesian inference of the unknown Weibull parameters for one sample problem was considered, and in this case, the results have been generalized to two-sample problem. Clearly, the generalization is a non-trivial generalization. Although the whole development in this paper is for two groups, the results can be easily generalized to more than two groups also.

For the Bayesian inference, we need to assume some priors on the unknown parameters. If the common shape parameter is known, the most convenient but a fairly general conjugate prior on the scale parameters can be the beta-gamma prior, as it was suggested by Pena and Gupta (1990). In this case, the explicit form of the Bayes estimates of the scale parameters can be obtained. When the shape parameter is unknown, the conjugate prior does not exist. In this case, following the approach of Berger and Sun (1993) or Kundu (2008), it is assumed that the prior on the shape parameter has a support on $(0, \infty)$, and it has a log-concave density function. It may be mentioned that many well-known distribution functions for example log-normal, Weibull, and gamma may have log-concave probability density function. Based on the prior distributions, the joint posterior density function can be obtained. As expected, the explicit expressions of the Bayes estimates cannot be obtained in explicit forms. We propose to use importance sampling technique to compute the Bayes estimate of any function of the unknown parameters, and also to construct the associated highest posterior density (HPD) credible interval. Monte Carlo simulations are performed to see the performances of the proposed method, and one data analysis has been performed for illustrative purposes.

In many practical situations, it is known a priori that one population is better than the other in terms of the expected lifetime of the experimental units. In accelerated life testing, if units of one sample are put on higher stress than the units of other sample, it is quite expected to get shorter lifetime of the specimens under higher stress. We incorporate this information by considering the order-restricted classical and Bayesian inference of the unknown parameters based on joint progressively censored samples. We obtain the MLEs of the unknown parameters based on the order restriction. The construction of the confidence intervals of the unknown parameters can be obtained using bootstrap method. For Bayesian inference, we propose a new order-restricted beta-gamma prior. In this case, the explicit expressions of the Bayes estimates cannot be obtained, and we propose to use importance sampling technique to compute the Bayes estimates and the associated HPD credible intervals. We re-analyze the data set based on the order restriction.
Rest of the paper is organized as follows. In Section 2, we present the notations, preliminaries, and the priors. Maximum likelihood estimators are presented in Section 3. Posterior analysis is provided in Section 4. In Section 5, we present the order-restricted inferences of the unknown parameters. In Section 6, we present the simulation results and data analysis. Finally, we conclude the paper in Section 7.

2 Notations, Model Assumptions, and Priors

2.1. Notations and Model Assumptions.

PDF : Probability density function
HPD : Highest posterior density
CDF : Cumulative distribution function
MLE : Maximum likelihood estimator
i.i.d. : Independent and identically distributed

- $F_1$: CDF of the lifetime distribution of the items of Group 1
- $F_2$: CDF of the lifetime distribution of the items of Group 2
- $T_1$: Random variable with CDF $F_1$
- $T_2$: Random variable with CDF $F_2$
- $k_1$: Number of failures from Group 1
- $k_2$: Number of failures from Group 2
- $k$: Total number of failures: $k = k_1 + k_2$

- $GA(\alpha, \lambda)$: Gamma random variable with PDF:
  
  \[ \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}; \quad x > 0. \]

- $WE(\alpha, \lambda)$: Weibull random variable with PDF:
  
  \[ \alpha \lambda x^{\alpha-1} e^{-\lambda x^{\alpha}}; \quad x > 0. \]

- $Bin(n, p)$: Binomial random variable with probability mass function:
  
  \[ \binom{n}{i} p^i (1 - p)^{n-i} \]

- $Beta(a, b)$: Beta random variable with PDF:
  
  \[ \frac{(\Gamma(a + b)/\Gamma(a)\Gamma(b))} a^{a-1} (1 - p)^{b-1}; \quad 0 < p < 1. \]

Suppose $m$- and $n$-independent units are placed on a test with the corresponding lifetimes being identically distributed with PDF $f_1(\cdot)$ and $f_2(\cdot)$, and CDF $F_1(\cdot)$ and $F_2(\cdot)$, respectively. It is assumed that the lifetime distribution of the items from Group 1 and Group 2 follows $WE(\alpha, \lambda_1)$ and $WE(\alpha, \lambda_2)$, respectively. For a given $(R_1, \ldots, R_k)$, as described before, in a JPC scheme, the observations are as follows:

\[ \{ (t_1, \delta_1, s_1), \ldots, (t_k, \delta_k, s_k) \}. \] (2.1)

Here, for $j = 1, \ldots, k$, $\delta_j = 1$, if the failure at $t_j$ occurs from Group 1, otherwise $\delta_j = 0$, and $s_j$ denotes the number of items from Group 1, which
have been removed at the time \( t_j \). Therefore, the likelihood function can be written as follows:

\[
L(\text{data}|\alpha, \lambda_1, \lambda_2) \propto \alpha^k \lambda_1^{k_1} \lambda_2^{k_2} \prod_{j=1}^{k} t_j^{\alpha-1} e^{-\lambda_1 U(\alpha)} e^{-\lambda_2 V(\alpha)},
\]

where \( k_1 = \sum_{j=1}^{k} \delta_j \) and \( k_2 = k - k_1 \),

\[
C_1 = \{j; \delta_j = 1\}, \quad C_2 = \{j; \delta_j = 0\},
\]

\[
U(\alpha) = \sum_{j=1}^{k} s_j t_j^\alpha + \sum_{C_1} t_j^\alpha, \quad V(\alpha) = \sum_{j=1}^{k} w_j t_j^\alpha + \sum_{C_2} t_j^\alpha,
\]

where \( w_j = R_j - s_j \).

2.2. Prior Assumptions: Without Order Restriction. The following prior assumptions are made on the common shape parameter \( \alpha \) and on the scale parameters \( \lambda_1 \) and \( \lambda_2 \), when there is no order restriction on \( \lambda_1 \) and \( \lambda_2 \). If we denote \( \lambda = \lambda_1 + \lambda_2 \), then similarly as in Pena and Gupta (1990), it is assumed that \( \lambda \sim \text{GA}(a_0, b_0) \), with \( a_0 > 0, b_0 > 0 \) and \( p = \lambda_1/(\lambda_1 + \lambda_2) \sim \text{beta}(a_1, a_2) \), with \( a_1 > 0, a_2 > 0 \), and they are independently distributed. The joint PDF of \( \lambda_1 \) and \( \lambda_2 \) can be obtained as follows:

\[
\pi_1(\lambda_1, \lambda_2|a_0, b_0, a_1, a_2) = \frac{\Gamma(a_1 + a_2)}{\Gamma(a_0)} (b_0 \lambda)^{a_0-a_1-a_2} \times \frac{b_0^{a_1}}{\Gamma(a_1)} \lambda_1^{a_1-1} e^{-b_0 \lambda_1} \times \frac{b_0^{a_2}}{\Gamma(a_2)} \lambda_2^{a_2-1} e^{-b_0 \lambda_2}; \quad 0 < \lambda_1, \lambda_2 < \infty.
\]

It is known as the beta-gamma PDF, and it will be denoted by BG\((a_0, b_0, a_1, a_2)\).

The above beta-gamma prior is a very flexible prior on the scale parameters. Depending on the values of the hyper-parameters, the joint prior on \( \lambda_1 \) and \( \lambda_2 \) can take variety of shapes. Moreover, for a given \( \alpha \), it is a conjugate prior on \((\lambda_1, \lambda_2)\). The correlation between \( \lambda_1 \) and \( \lambda_2 \) can be both positive and negative depending on the values of \( a_0, a_1, \) and \( a_2 \). If \( a_0 = a_1 + a_2 \), then \( \lambda_1 \) and \( \lambda_2 \) are independent. If \( a_0 > a_1 + a_2 \), then they are positively correlated, and for \( a_0 < a_1 + a_2 \), they are negatively correlated. The following results will be useful for further development, and they can be established very easily.
Result 1. If \((\lambda_1, \lambda_2) \sim BG(a_0, b_0, a_1, a_2)\), then for \(i = 1, 2\),

\[
E(\lambda_i) = \frac{a_0a_i}{b_0(a_1 + a_2)} \quad \text{and} \quad V(\lambda_i) = \frac{a_0a_i}{b_0(a_1 + a_2)} \times \left\{ \frac{(a_i + 1)(a_0 + 1)}{a_1 + a_2 + 1} - \frac{a_0a_i}{a_1 + a_2} \right\}.
\]

(2.4)

Moreover, the generation from a beta-gamma distribution is quite simple using the property that \((\lambda_1, \lambda_2) \sim BG(a_0, b_0, a_1, a_2)\), if and only, \(\lambda_1 + \lambda_2\) has a gamma distribution and \(\frac{\lambda_1}{\lambda_1 + \lambda_2}\) has a beta distribution and they are independently distributed (see for example Kundu and Pradhan (2011)).

No specific form of prior has been assumed here on the common shape parameter \(\alpha\). Following the idea of Berger and Sun (1993), it is assumed that \(\pi(\alpha)\), the prior on \(\alpha\) has a support on the positive real line, and it has a log-concave PDF. Moreover, \(\pi(\alpha)\) and \(\pi(\lambda_1, \lambda_2|b_0, a_0, a_1, a_2)\) are independently distributed. It may be mentioned that many well-known distribution has log-concave PDFs. \(\pi(\alpha)\) also has its hyper-parameters. We do not make it explicit here; whenever it is needed, we will make it explicit.

2.3. Prior Assumptions: Order-Restricted. If we have a order restriction on the scale parameters as \(\lambda_1 < \lambda_2\), then we make the following prior assumption on \(\lambda_1\) and \(\lambda_2\).

\[
\pi_2(\lambda_1, \lambda_2|a_0, b_0, a_1, a_2) = \frac{\Gamma(a_1 + a_2)}{\Gamma(a_0)\Gamma(a_1)\Gamma(a_2)} b_0^{a_0} \lambda_0^{a_0-a_1-a_2} e^{-b_0(\lambda_1+\lambda_2)}
\]

\[
\times \left( \lambda_1^{a_1-1}\lambda_2^{a_2-1} + \lambda_1^{a_1-1}\lambda_2^{a_2-1} \right) \quad ; \quad 0 < \lambda_1 < \lambda_2 < \infty.
\]

(2.5)

We will call it as the ordered beta-gamma PDF, and it will be denoted by \(\text{OBG}(a_0, b_0, a_1, a_2)\). Note that Eq. 2.5 is the PDF of the ordered random variable \((\lambda_1, \lambda_2)\), where \((\lambda_1, \lambda_2) = (\lambda_1, \lambda_2)\) if \(\lambda_1 < \lambda_2\), \((\lambda_1, \lambda_2) = (\lambda_2, \lambda_1)\) if \(\lambda_2 < \lambda_1\) and \((\lambda_1, \lambda_2)\) follow \(\sim\) \(BG(a_0, b_0, a_1, a_2)\). It may be noted that the generation from a ordered beta-gamma distribution is also quite straightforward. First, we generate sample from a beta-gamma distribution, and then by ordering them, we obtain a random sample from a ordered beta-gamma distribution. We assume the same prior \(\pi(\alpha)\) on \(\alpha\) as before and \(\alpha\) and \((\lambda_1, \lambda_2)\) are assumed to be independently distributed.
3 Maximum Likelihood Estimators

Based on the observations described in Eq. 2.1, the log-likelihood function without the additive constant becomes the following:

\[ l(\text{data}|\alpha, \lambda_1, \lambda_2) = k \ln \alpha + k_1 \ln \lambda_1 + k_2 \ln \lambda_2 + (\alpha - 1) \sum_{j=1}^{k} \ln t_j - \lambda_1 U(\alpha) - \lambda_2 V(\alpha), \]

where, \( k_1, k_2, U(\alpha), \) and \( V(\alpha) \) are same as defined before. The following result provides the uniqueness of the MLEs of \( \lambda_1 \) and \( \lambda_2 \) for a given \( \alpha \).

**Theorem 1.** If \( k_1 > 0 \) and \( k_2 > 0 \), then for a fixed \( \alpha > 0 \), \( g_1(\lambda_1, \lambda_2) = l(\text{data}|\alpha, \lambda_1, \lambda_2) \) is a unimodal function of \((\lambda_1, \lambda_2)\).

**Proof.** Note that \( g_1(\lambda_1, \lambda_2) \) is a concave function as the Hessian matrix of \( g_1(\lambda_1, \lambda_2) \) is a negative definite matrix. Now, the result follows because for fixed \( \lambda_1(\lambda_2) \), \( g_1(\lambda_1, \lambda_2) \) tends to \(-\infty\), as \( \lambda_2(\lambda_1) \) tends to 0, or \( \infty \).

For known \( \alpha \), the MLEs of \( \lambda_1 \) and \( \lambda_2 \), say \( \hat{\lambda}_1(\alpha) \) and \( \hat{\lambda}_2(\alpha) \), respectively, can be obtained as follows:

\[ \hat{\lambda}_1(\alpha) = \frac{k_1}{U(\alpha)} \quad \text{and} \quad \hat{\lambda}_2(\alpha) = \frac{k_2}{V(\alpha)}. \]

When \( \alpha \) is unknown, first the MLE of \( \alpha \) can be obtained by maximizing the profile log-likelihood function of \( \alpha \) without the additive constant, and that is

\[ p_1(\alpha) = l(\text{data}|\alpha, \hat{\lambda}_1(\alpha), \hat{\lambda}_2(\alpha)) = k \ln \alpha - k_1 \ln U(\alpha) - k_2 \ln V(\alpha) + (\alpha - 1) \sum_{j=1}^{k} \ln t_j. \]

The following result will provide the existence and uniqueness of the MLE of \( \alpha \).

**Theorem 2.** If \( k_1 > 0 \) and \( k_2 > 0 \), \( p_1(\alpha) \) is a unimodal function of \( \alpha \).

**Proof.** See in the Appendix A.

Therefore, combining Theorem 1 and Theorem 2, it is immediately obtained that for \( k_1 > 0 \) and \( k_2 > 0 \), the MLEs of \( \alpha, \lambda_1, \) and \( \lambda_2 \) are unique. It is quite simple to compute the MLE of \( \alpha \) as \( p_1(\alpha) \) is a unimodal function. Use the bisection or Newton-Raphson method to compute the MLEs of \( \alpha \), and once the MLE of \( \alpha \) is obtained, the MLEs of \( \lambda_1 \) and \( \lambda_2 \) can be obtained from Eq. 3.2. Although the MLEs can be obtained quite efficiently
in this case, the exact distribution of the MLEs is not possible to obtain. Hence, the construction of confidence intervals of the unknown parameters may not be very simple. The Fisher information matrix may be used to construct the asymptotic confidence intervals of the unknown parameters, and it is provided in Appendix B. Alternatively, we propose to use the bootstrap method to construct the confidence intervals of the unknown parameters in this case. Since the exact confidence intervals cannot be obtained, the Bayesian inference seems to be a natural choice in this case.

4 Bayes Estimates and Credible Intervals

In this section, we provide the Bayes estimates of the unknown parameters, and the corresponding credible sets based on JPC scheme as described before. We mainly assume the squared error loss function, although any other loss function can be easily incorporated. Now, to compute the Bayes estimates of the unknown parameters, we need to assume some specific form of the prior distribution of \( \alpha \), and it is assumed that \( \pi(\alpha) \) has a \( GA(a, b) \) distribution. Hence, the joint posterior density function of \( \lambda_1, \lambda_2 \) and \( \alpha \) can be written as follows:

\[
\pi(\lambda_1, \lambda_2, \alpha | \text{data}) \propto (\lambda_1 + \lambda_2)^{a_0-a_1-a_2}\lambda_1^{a_1+k_1-1}\lambda_2^{a_2+k_2-1}e^{-\lambda_1(b_0+U(\alpha))}e^{-\lambda_2(b_0+W(\alpha))}e^{-\lambda_1+\lambda_2(b_0+W(\alpha))}e^{-\lambda_1(b_0+U(\alpha))}e^{-\lambda_2(b_0+W(\alpha))}.
\]

(4.1)

We re-write (4.1) in the following manner:

\[
\pi(\lambda_1, \lambda_2, \alpha | \text{data}) \propto (\lambda_1 + \lambda_2)^{a_0-a_1-a_2}\lambda_1^{a_1+k_1-1}\lambda_2^{a_2+k_2-1}e^{-\lambda_1+\lambda_2(b_0+W(\alpha))}e^{-\lambda_1(b_0+W(\alpha))}e^{-\lambda_2(b_0+W(\alpha))}e^{-\lambda_1(U(\alpha) - W(\alpha))}e^{-\lambda_2(V(\alpha) - W(\alpha))}.
\]

(4.2)

Here, \( W(\alpha) = \min\{U(\alpha), V(\alpha)\} \). The posterior density function of \( \alpha, \lambda_1 \), and \( \lambda_2 \) can be written as follows:

\[
\pi(\alpha, \lambda_1, \lambda_2 | \text{data}) \propto \pi_1^*(\lambda_1, \lambda_2 | \text{data}, \alpha) \times \pi_2^*(\alpha | \text{data}) \times g(\alpha, \lambda_1, \lambda_2 | \text{data}).
\]

(4.3)

Here, \( \pi_1^*(\lambda_1, \lambda_2 | \alpha, \text{data}) \) is the PDF of a \( BG(a_0 + k_1 + k_2, b_0 + W(\alpha), a_1 + k_1, a_2 + k_2) \), and

\[
\pi_2^*(\alpha) \propto \frac{\alpha^{k+a-1}e^{-\alpha(b-\sum_{i=1}^{k}\ln t_i)}}{(b_0 + W(\alpha))^{a_0+k}},
\]

\[
g(\alpha, \lambda_1, \lambda_2 | \text{data}) = e^{-\lambda_1(U(\alpha) - W(\alpha))}e^{-\lambda_2(V(\alpha) - W(\alpha))}.
\]

(4.4)
Therefore, the Bayes estimate of $h(\alpha, \lambda_1, \lambda_2)$, any function of $\alpha, \lambda_1, \text{ and } \lambda_2$ with respect to squared error loss function, is follows:

$$E_{\pi(\alpha,\lambda_1,\lambda_2|\text{data})}(h(\alpha, \lambda_1, \lambda_2)) = \int_0^\infty \int_0^\infty \int_0^\infty h(\alpha, \lambda_1, \lambda_2) \pi(\alpha, \lambda_1, \lambda_2|\text{data}) \times d\alpha d\lambda_1 d\lambda_2 = \frac{K_1}{K_2},$$

provided it exists. Here,

$$K_1 = \int_0^\infty \int_0^\infty \int_0^\infty h(\alpha, \lambda_1, \lambda_2) \times \pi^*_1(\lambda_1, \lambda_2|\text{data}, \alpha) \times \pi^*_2(\alpha|\text{data}) \times g(\alpha, \lambda_1, \lambda_2|\text{data}) d\alpha d\lambda_1 d\lambda_2.$$  

and

$$K_2 = \int_0^\infty \int_0^\infty \int_0^\infty \pi^*_1(\lambda_1, \lambda_2|\text{data}, \alpha) \times \pi^*_2(\alpha|\text{data}) \times g(\alpha, \lambda_1, \lambda_2|\text{data}) d\alpha d\lambda_1 d\lambda_2.$$  

It is clear that Eq. 4.5 cannot be obtained in closed form. We may use Lindley’s approximation to compute (4.5), but we may not be able to compute the associated credible interval using that. We propose to use importance sampling technique to compute simulation consistent Bayes estimate and the associated credible interval. The details will be explained later. We use the following result for that purpose.

**Theorem 3.** The density function $\pi^*_2(\alpha|\text{data})$ is log-concave.

**Proof.** It can be obtained along the same line as in Theorem 1, the details are avoided.

Therefore, it is quite simple to generate samples from $l(\alpha|\text{data})$ using the method of Devroye (1984) or Kundu (2008), and for a given $\alpha$, random samples from $l(\lambda_1, \lambda_2|\text{data}, \alpha)$ can be easily generated using the method of Kundu and Pradhan (2011). The following algorithm can be used to compute the Bayes estimate and also the associated HPD credible interval of $h(\alpha, \lambda_1, \lambda_2)$.

**Algorithm**

- Step 1: Generate $\alpha$ from $\pi^*_2(\alpha|\text{data})$.
- Step 2: For a given $\alpha$, generate $\lambda_1$ and $\lambda_2$ from $\pi^*_1(\lambda_1, \lambda_2|\text{data}, \alpha)$.
• Step 3: Repeat the procedure $N$ times to generate $(\alpha_1, \lambda_{11}, \lambda_{21}), \ldots, (\alpha_N, \lambda_{1N}, \lambda_{2N})$.

• Step 4: To obtain Bayes estimate of $h(\alpha, \lambda_1, \lambda_2)$, compute $(h_1, \ldots, h_N)$, where $h_i = h(\alpha_i, \lambda_{1i}, \lambda_{2i})$ as well as compute $(g_1, \ldots, g_N)$, where $g_i = g(\alpha_i, \lambda_{1i}, \lambda_{2i})$.

• Step 5: Bayes estimate of $h(\beta, \lambda_1, \lambda_2)$ can be approximated as $\frac{\sum_{i=1}^N g_i h_i}{\sum_{j=1}^N g_j}$.

• Step 6: To compute 100$(1 - \beta)$% CRI of $h(\alpha, \lambda_1, \lambda_2)$, arrange $h_i$ in ascending order to obtain $(h_{(1)}, \ldots, h_{(N)})$ and record the corresponding $v_i$ as $(v_{(1)}, \ldots, v_{(N)})$. A 100$(1 - \gamma)$% CRI can be obtained as $(h_{(j_1)}, h_{(j_2)})$ where $j_1, j_2$ such that

$$j_1 < j_2, \quad j_1, j_2 \in \{1, \ldots, N\} \quad \text{and} \quad \sum_{i=j_1}^{j_2} v_i \leq 1 - \beta < \sum_{i=j_1}^{j_2+1} v_i. \quad (4.8)$$

The 100$(1 - \beta)$% highest posterior density (HPD) CRI can be obtained as $(h_{(j_1^*)}, h_{(j_2^*)})$, such that $h_{(j_2^*)} - h_{(j_1^*)} \leq h_{(j_2)} - h_{(j_1)}$ and $j_1^*, j_2^*$ satisfying (4.8) for all $j_1, j_2$ satisfying (4.8).

5 Order-Restricted Inference

In this section, we consider the inference on the unknown parameters under the restriction $\lambda_1 < \lambda_2$. In many practical situations, experimenter may have the information that one population has a smaller expected lifetime than the other. In our case, this leads to the above restriction. Therefore, our problem can be stated as follows. Based on the same set of assumptions as in Section 2.2 and with $\lambda_1 < \lambda_2$, the problem is to estimate the unknown parameters $\alpha$, $\lambda_1$, and $\lambda_2$ using the data (2.1).

5.1. Maximum Likelihood Estimators. In this case, also, we proceed along the same line as before. For a given $\alpha$, the log-likelihood function (3.1) is concave as a function of $\lambda_1$ and $\lambda_2$ and it has a unique maximum. The maximum value of the function (3.1) is obtained at the point $(\hat{\lambda}_1(\alpha), \hat{\lambda}_2(\alpha))$. Clearly, if $\tilde{\lambda}_1(\alpha) < \hat{\lambda}_1(\alpha)$, then the order-restricted MLEs of $\lambda_1$ and $\lambda_2$, say $\tilde{\lambda}_1(\alpha) = \lambda_1(\alpha)$ and $\tilde{\lambda}_2(\alpha) = \lambda_2(\alpha)$, respectively. On the other hand, if $\tilde{\lambda}_1(\alpha) \geq \hat{\lambda}_1(\alpha)$, then the maximum value of $g_1(\lambda_1, \lambda_2)$ will be on the line $\lambda_1 = \lambda_2$ under the order restriction $\lambda_1 < \lambda_2$. Therefore, in this case

$$\tilde{\lambda}_1(\alpha) = \tilde{\lambda}_2(\alpha) = \arg \max g_1(\lambda, \lambda).$$
Hence, for a given $\alpha$, the order-restricted MLEs of $\lambda_1$ and $\lambda_2$ become

$$
(\tilde{\lambda}_1(\alpha), \tilde{\lambda}_2(\alpha)) = \begin{cases} 
(\hat{\lambda}_1(\alpha), \hat{\lambda}_2(\alpha)) & \text{if } \hat{\lambda}_1(\alpha) < \hat{\lambda}_2(\alpha) \\
\left( \frac{\lambda_{j+1}^k}{\sum_{j=1}^k (R_j+1)} \right), \left( \frac{\lambda_{j+1}^k}{\sum_{j=1}^k (R_j+1)} \right) & \text{if } \hat{\lambda}_1(\alpha) \geq \hat{\lambda}_2(\alpha).
\end{cases}
$$

(5.1)

The MLE of $\alpha$, say $\tilde{\alpha}$, can be obtained by maximizing $p_2(\alpha) = l(data|\alpha, \tilde{\lambda}_1(\alpha), \tilde{\lambda}_2(\alpha))$ with respect to $\alpha$. The following result provides the existence and uniqueness of the MLE of $\alpha$.

**Theorem 4.** If $k_1 > 0$ and $k_2 > 0$, $p_2(\alpha)$ is a unimodal function of $\alpha$.

**Proof.** The result follows along the same line as in Theorem 2 by observing the fact that $p_2(\alpha)$ is log-concave in both the region, and $p_2(\alpha)$ is a continuous function of $\alpha$.

Once the MLE of $\alpha$ is obtained, the MLEs of $\lambda_1$ and $\lambda_2$ can be obtained from Eq. 5.1 explicitly. We propose to use bootstrap method to construct the confidence intervals of the unknown parameters.

5.2. Bayes Estimates and Credible Intervals. In this section, we will provide the order-restricted Bayesian inference of the unknown parameters based on the prior assumptions as provided in Section 2.4. Similarly as before, for specific implementation, we assume that $\pi(\alpha)$ has a $GA(a, b)$ distribution. The joint posterior density function of $\alpha$, $\lambda_1$, and $\lambda_2$ for $\alpha > 0, 0 < \lambda_1 < \lambda_2$, can be written as follows:

$$
\pi(\lambda_1, \lambda_2, \alpha|data) \propto (\lambda_1 + \lambda_2)^{a_0-a_1-a_2} \left( \lambda_1^{a_1-1} \lambda_2^{a_2-1} + \lambda_1^{a_2-1} \lambda_2^{a_1-1} \right)^k \lambda_1^{k_1} \lambda_2^{k_2} e^{-\lambda_1(b_0+U(\alpha))} e^{-\lambda_2(b_0+V(\alpha))} \alpha^{k+a-1} e^{-ba} \prod_{i=1}^k t_i^\alpha.
$$

(5.2)

We re-write (5.2) as follows:

$$
\pi(\lambda_1, \lambda_2, \alpha|data) \propto (\lambda_1 + \lambda_2)^{a_0-a_1-a_2} \left( \lambda_1^{a_1+J-1} \lambda_2^{a_2+J-1} + \lambda_1^{a_2+J-1} \lambda_2^{a_1+J-1} \right) e^{-\lambda_1(b_0+W(\alpha))} \lambda_1^{k_1-J} \lambda_2^{k_2-J} e^{-\lambda_1(U(\alpha)-W(\alpha))} e^{-\lambda_2(V(\alpha)-W(\alpha))} \alpha^{k+a-1} e^{-\alpha(b-\sum_{i=1}^k \ln t_i)},
$$

(5.3)

here, $J = \min\{k_1, k_2\}$ and $W(\alpha)$ is same as defined before.

The posterior density function of $\alpha$, $\lambda_1$, and $\lambda_2$ in this case can be written as follows:

$$
\pi(\alpha, \lambda_1, \lambda_2|data) \propto \pi_1^*(\lambda_1, \lambda_2|data, \alpha) \times \pi_2^*(\alpha|data) \times g(\alpha, \lambda_1, \lambda_2|data).
$$

(5.4)
Here, $\pi^*_1(\lambda_1, \lambda_2|\alpha, \text{data})$ is the PDF of a OBG($a_0 + 2J, b_0 + W(\alpha), a_1 + J, a_2 + J$), and

$$\pi^*_2(\alpha) \propto \alpha^{k+a-1}e^{-\alpha\left(b-\sum_{i=1}^{k} \ln t_i\right)} \left(b_0 + W(\alpha)\right)^{a_0+2J}$$

(5.5)

and

$$g(\alpha, \lambda_1, \lambda_2|\text{data}) = \lambda_1^{k_1-J}\lambda_2^{k_2-J}e^{-\lambda_1(U(\alpha)-W(\alpha))}e^{-\lambda_2(V(\alpha)-W(\alpha))}.$$  

(5.6)

Since $\pi^*_2(\alpha|\text{data})$ is a log-concave function, and the generation from a $\pi^*_1(\lambda_1, \lambda_2|\alpha)$ can be performed quite conveniently, we can use the same importance sampling technique as in Section 4, to compute the Bayes estimate and the associated HPD credible interval of any function of $\alpha$, $\lambda_1$, and $\lambda_2$.

6 Numerical Experiments and Data Analysis

6.1. Numerical Experiments. In this section, we have performed some simulation experiments to see the effectiveness of the proposed methods and also to observe how the order-restricted inference behaves in practice. We have considered $m = 20$, $n = 25$, $\lambda_1 = 1.0$, and $\lambda_2 = 0.5$. We have taken different effective sample sizes, $k = 20$, 25, different censoring schemes, and different $\alpha$ values, $\alpha = 1$ and 2.

In each case, we obtain the MLEs of the unknown parameters. We compute the average estimates (AE) and the mean squared errors (MSE) based on 10,000 replications. The results are reported in Tables 1 and 2. In each case, we also compute 90% symmetric percentile bootstrap confidence interval based on 500 bootstrap samples. We repeat the process 1000 times and obtain the average lengths (AL) of the confidence intervals and their coverage percentages (CP). The results are reported in Tables 3 and 4.

We further compute the Bayes estimates and the associated 90% credible intervals based on both informative priors (IP) and non-informative priors (NIP). In case of non-informative priors, it is assumed that $b_0 = a_0 = a_1 = a_2 = a = b = 0$. For informative priors, when $\alpha = 1$, then $b_0 = 1$, $a_0 = \frac{3}{2}b_0$, $a_1 = 2$, $a_2 = 4$, $a = 2$, $b = 2$, and when $\alpha = 2$, $b_0 = 1$, $a_0 = \frac{3}{2}b_0$, $a_1 = 2$, $a_2 = 4$, $a = 4$, $b = 2$. We compute the Bayes estimates and the associated credible intervals based on 1000 samples. We report the average Bayes estimates and the corresponding MSEs in Tables 1 and 2. The average lengths of the credible intervals and the associated coverage percentages are reported in Tables 3 and 4. We use the following notation for a particular progressive censoring scheme. For example, $k = 6$ and $R = (4,0_{(5)})$ means $R_1 = 4$, $R_2 = R_3 = R_4 = R_5 = R_6 = 0$. 
From Tables 1 and 2, it is clear that as the effective sample size increases in all the cases the average biases and the MSEs decrease. The performances of the Bayes estimators with respect to the NIPs are slightly better than those of the MLEs both in terms of biases and MSEs in most of the cases investigated here. Also, the Bayes estimators with respect to IPS are performing better than the Bayes estimators with respect to NIPs in terms of the biases and MSEs, as expected. In Tables 3 and 4, we report the performances of the percentile bootstrap confidence intervals and the Bayes credible intervals. In most of the cases, the coverage percentages are very close to the nominal values. The average credible lengths based on the non-informative priors are larger than the informative priors, but they are slightly lower than the average bootstrap confidence intervals.

Therefore, it is clear in this case that the Bayesian inference is preferable than the classical inference. If we have some prior information on the unknown parameters, then we should use the Bayes estimates and the associated credible intervals with respect to the informative priors; otherwise, we should use the non-informative priors.

| Censoring scheme | Parameter | MLE | Bayes IP | Bayes NIP |
|------------------|-----------|-----|----------|-----------|
|                  |           | AE  | MSE      | AE        | MSE       |
|                  |           |     |          |           |           |
| $k = 20,$       | $\alpha$ | 1.097 | 0.063 | 1.075 | 0.050 | 1.095 | 0.064 |
| $R = (7,0_{(18)},15)$ | $\lambda_1$ | 0.554 | 0.057 | 0.536 | 0.035 | 0.543 | 0.047 |
|                  | $\lambda_2$ | 1.102 | 0.147 | 1.066 | 0.086 | 1.086 | 0.123 |
| $k = 20,$       | $\alpha$ | 1.101 | 0.067 | 1.062 | 0.048 | 1.066 | 0.055 |
| $R = (0_{(9)},7,0_{(9)},15)$ | $\lambda_1$ | 0.562 | 0.062 | 0.535 | 0.031 | 0.557 | 0.057 |
|                  | $\lambda_2$ | 1.123 | 0.185 | 1.064 | 0.077 | 1.081 | 0.127 |
| $k = 20,$       | $\alpha$ | 1.105 | 0.072 | 1.068 | 0.052 | 1.064 | 0.060 |
| $R = (0_{(18)},7,15)$ | $\lambda_1$ | 0.569 | 0.074 | 0.548 | 0.035 | 0.551 | 0.052 |
|                  | $\lambda_2$ | 1.126 | 0.191 | 1.077 | 0.092 | 1.085 | 0.117 |
| $k = 25,$       | $\alpha$ | 1.077 | 0.044 | 1.063 | 0.043 | 1.064 | 0.040 |
| $R = (7,0_{(23)},10)$ | $\lambda_1$ | 0.532 | 0.036 | 0.523 | 0.025 | 0.532 | 0.036 |
|                  | $\lambda_2$ | 1.062 | 0.095 | 1.042 | 0.068 | 1.044 | 0.085 |
| $k = 25,$       | $\alpha$ | 1.078 | 0.044 | 1.062 | 0.036 | 1.075 | 0.044 |
| $R = (0_{(11)},7,0_{(12)},10)$ | $\lambda_1$ | 0.537 | 0.038 | 0.524 | 0.025 | 0.530 | 0.034 |
|                  | $\lambda_2$ | 1.071 | 0.099 | 1.053 | 0.069 | 1.064 | 0.101 |
| $k = 25,$       | $\alpha$ | 1.080 | 0.049 | 1.073 | 0.043 | 1.079 | 0.047 |
| $R = (0_{(23)},7,10)$ | $\lambda_1$ | 0.540 | 0.040 | 0.529 | 0.026 | 0.535 | 0.035 |
|                  | $\lambda_2$ | 1.071 | 0.099 | 1.059 | 0.075 | 1.069 | 0.098 |
Further, we perform the order-restricted inference of the unknown parameters. In this case, we have taken the same set of parameter values, the sample sizes, and the censoring schemes mainly for comparison purposes. For the Bayesian inference, we have considered the same set of hyper-parameters also. The average estimates and the associated MSEs are reported in Tables 2 and 3. The confidence and credible intervals are reported in Tables 4 and 5.

| Table 2: \( m = 20, n = 22, \alpha = 2, \lambda_1 = 0.5, \lambda_2 = 1 \) |
|---------------------------------------------------------------|
| **Parameter** | **MLE** | **Bayes IP** | **Bayes NIP** |
| | **AE** | **MSE** | **AE** | **MSE** | **AE** | **MSE** |
| \( k = 20, \) | \( \alpha \) | 2.191 | 0.252 | 2.127 | 0.157 | 2.185 | 0.243 |
| \( R = (7,0(18),15) \) | \( \lambda_1 \) | 0.555 | 0.057 | 0.545 | 0.035 | 0.550 | 0.049 |
| | \( \lambda_2 \) | 1.097 | 0.143 | 1.055 | 0.078 | 1.104 | 0.131 |
| \( k = 20, \) | \( \alpha \) | 2.209 | 0.264 | 2.108 | 0.145 | 2.147 | 0.228 |
| \( R = (0(9),7,0(9),15) \) | \( \lambda_1 \) | 0.563 | 0.063 | 0.536 | 0.034 | 0.556 | 0.049 |
| | \( \lambda_2 \) | 1.118 | 0.165 | 1.097 | 0.099 | 1.081 | 0.126 |
| \( k = 20, \) | \( \alpha \) | 2.207 | 0.273 | 2.120 | 0.157 | 2.160 | 0.251 |
| \( R = (0(18),7,15) \) | \( \lambda_1 \) | 0.561 | 0.067 | 0.548 | 0.035 | 0.541 | 0.046 |
| | \( \lambda_2 \) | 1.122 | 0.183 | 1.064 | 0.082 | 1.112 | 0.148 |
| \( k = 25, \) | \( \alpha \) | 2.149 | 0.169 | 2.133 | 0.151 | 2.139 | 0.171 |
| \( R = (7,0(23),10) \) | \( \lambda_1 \) | 0.534 | 0.037 | 0.523 | 0.025 | 0.524 | 0.034 |
| | \( \lambda_2 \) | 1.065 | 0.101 | 1.056 | 0.073 | 1.061 | 0.094 |
| \( k = 25, \) | \( \alpha \) | 2.150 | 0.176 | 2.127 | 0.135 | 2.146 | 0.192 |
| \( R = (0(11),7,0(12),10) \) | \( \lambda_1 \) | 0.5380 | 0.036 | 0.523 | 0.023 | 0.541 | 0.035 |
| | \( \lambda_2 \) | 1.066 | 0.102 | 1.045 | 0.068 | 1.047 | 0.088 |
| \( k = 25, \) | \( \alpha \) | 2.156 | 0.193 | 2.122 | 0.144 | 2.132 | 0.191 |
| \( R = (R = (0(23),7,10) \) | \( \lambda_1 \) | 0.537 | 0.038 | 0.527 | 0.025 | 0.545 | 0.039 |
| | \( \lambda_2 \) | 1.071 | 0.101 | 1.048 | 0.062 | 1.049 | 0.093 |

| Table 3: \( m = 20, n = 22, \alpha = 1, \lambda_1 = 0.5, \lambda_2 = 1 \) |
|---------------------------------------------------------------|
| **Parameter** | **90% HPD** | **90% HPD** | **90% Bootstrap** |
| | **CRI IP** | **CRI NIP** | **CI** |
| | **AL** | **CP (%)** | **AL** | **CP (%)** | **AL** | **CP (%)** |
| \( k = 20, R = (7,0(18),15) \) | \( \alpha \) | 0.621 | 86.6 | 0.663 | 82.8 | 0.804 | 82.2 |
| | \( \lambda_1 \) | 0.560 | 90.0 | 0.627 | 86.8 | 0.814 | 87.0 |
| | \( \lambda_2 \) | 0.894 | 89.6 | 1.018 | 87.6 | 1.358 | 83.8 |
| \( k = 20, R = (0(9),7,0(9),15) \) | \( \alpha \) | 0.603 | 84.6 | 0.624 | 85.0 | 0.818 | 83.0 |
| | \( \lambda_1 \) | 0.565 | 87.6 | 0.635 | 87.2 | 0.907 | 85.6 |
| | \( \lambda_2 \) | 0.900 | 91.6 | 0.977 | 89.2 | 1.454 | 82.8 |
| \( k = 20, R = (0(18),7,15) \) | \( \alpha \) | 0.615 | 87.0 | 0.624 | 82.0 | 0.854 | 83.6 |
| | \( \lambda_1 \) | 0.582 | 92.0 | 0.649 | 87.2 | 0.900 | 87.2 |
| | \( \lambda_2 \) | 0.903 | 91.4 | 0.981 | 89.4 | 1.463 | 86.0 |
Table 4: $m = 20, n = 22, \alpha = 2, \lambda_1 = 0.5, \lambda_2 = 1$

| Censoring scheme | Parameter | 90% HPD CRI | 90% HPD CRI | 90% Bootstrap NIP CI |
|------------------|-----------|-------------|-------------|---------------------|
|                  |           | AL CP (%)   | AL CP (%)   | AL CP (%)           |
| $k = 20, R = (7,0_{(18)},15)$ | $\alpha$ | 1.198 90.8 1.309 87.4 | 1.612 80.2 |
|                  | $\lambda_1$ | 0.562 88.2 0.639 85.0 | 0.827 87.2 |
|                  | $\lambda_2$ | 0.897 90.8 0.993 89.6 | 1.293 87.8 |
| $k = 20, R = (0_{(9)},7,0_{(9)},15)$ | $\alpha$ | 1.166 89.4 1.209 82.2 | 1.690 77.2 |
|                  | $\lambda_1$ | 0.569 91.4 0.614 85.0 | 0.903 88.4 |
|                  | $\lambda_2$ | 0.910 90.4 1.000 88.4 | 1.492 85.6 |
| $k = 20, R = (0_{(18)},7,15)$ | $\alpha$ | 1.171 89.0 1.258 81.8 | 1.758 81.0 |
|                  | $\lambda_1$ | 0.572 90.6 0.652 85.0 | 0.902 87.8 |
|                  | $\lambda_2$ | 0.907 91.8 0.988 89.0 | 1.468 84.4 |

From Tables 5 and 6, it is clear that the performances of the Bayes estimators based on the non-informative priors behave slightly better than the MLEs in terms of MSEs particularly for the scale parameters. In this case also the performances of the Bayes estimators with respect to the informative priors perform much better than the non-informative priors, as expected. From Tables 7 and 8, it is observed that coverage percentages of the bootstrap confidence intervals are very similar to the Bayes estimators based on the non-informative priors. The coverage percentages of the bootstrap confidence intervals are slightly closer to the nominal value than the

Table 5: $m = 20, n = 22, \alpha = 1, \lambda_1 = 0.5, \lambda_2 = 1$ (with order restriction)

| Censoring scheme | Parameter | MLE | Bayes inf-prior | Bayes noninf-prior |
|------------------|-----------|-----|-----------------|-------------------|
|                  |           | AE  | MLE            | AE               | MLE            |
| $k = 20,$ $R = (7,0_{(18)},15)$ | $\alpha$ | 1.093 0.063 | 1.080 0.051 | 1.078 0.056 |
|                  | $\lambda_1$ | 0.551 0.052 | 0.526 0.024 | 0.511 0.031 |
|                  | $\lambda_2$ | 1.113 0.155 | 1.041 0.067 | 1.103 0.110 |
| $k = 20,$ $R = (0_{(9)},7,0_{(9)},15)$ | $\alpha$ | 1.102 0.064 | 1.057 0.055 | 1.077 0.057 |
|                  | $\lambda_1$ | 0.553 0.058 | 0.527 0.026 | 0.506 0.034 |
|                  | $\lambda_2$ | 1.124 0.178 | 1.051 0.076 | 1.095 0.109 |
| $k = 20,$ $R = (0_{(18)},7,15)$ | $\alpha$ | 1.102 0.072 | 1.078 0.054 | 1.081 0.066 |
|                  | $\lambda_1$ | 0.560 0.062 | 0.535 0.028 | 0.504 0.028 |
|                  | $\lambda_2$ | 1.129 0.184 | 1.057 0.080 | 1.085 0.102 |
| $k = 25,$ $R = (7,0_{(23)},10)$ | $\alpha$ | 1.075 0.043 | 1.081 0.052 | 1.066 0.045 |
|                  | $\lambda_1$ | 0.529 0.033 | 0.523 0.020 | 0.508 0.022 |
|                  | $\lambda_2$ | 1.068 0.086 | 1.032 0.062 | 1.056 0.081 |
| $k = 25,$ $R = (0_{(11)},7,0_{(12)},10)$ | $\alpha$ | 1.079 0.045 | 1.053 0.035 | 1.079 0.048 |
|                  | $\lambda_1$ | 0.535 0.034 | 0.529 0.022 | 0.501 0.022 |
|                  | $\lambda_2$ | 1.072 0.099 | 1.034 0.058 | 1.073 0.080 |
| $k = 25,$ $R = (0_{(23)},7,10)$ | $\alpha$ | 1.082 0.050 | 1.061 0.042 | 1.070 0.046 |
|                  | $\lambda_1$ | 0.536 0.036 | 0.526 0.021 | 0.508 0.023 |
|                  | $\lambda_2$ | 1.075 0.101 | 1.029 0.058 | 1.065 0.075 |
Table 6: $m = 20, n = 22, \alpha = 2, \lambda_1 = 0.5, \lambda_2 = 1$ (with order restriction)

| Censoring scheme | Parameter | MLE | Bayes inf-prior | Bayes noninf-prior |
|------------------|-----------|-----|-----------------|--------------------|
|                  |           | AE  | MSE            | AE                 |
| $k = 20, R = (7,0_{(18)},15)$ | $\alpha$ | 2.185 | 0.247 | 2.123 | 0.160 |
|                  | $\lambda_1$ | 0.548 | 0.051 | 0.528 | 0.028 |
|                  | $\lambda_2$ | 1.107 | 0.147 | 1.056 | 0.075 |
| $k = 20, R = (0_{(9)},7,0_{(9)},15)$ | $\alpha$ | 2.197 | 0.266 | 2.108 | 0.153 |
|                  | $\lambda_1$ | 0.555 | 0.057 | 0.524 | 0.025 |
|                  | $\lambda_2$ | 1.119 | 0.167 | 1.046 | 0.074 |
| $k = 20, R = (0_{(18)},7,15)$ | $\alpha$ | 2.198 | 0.271 | 2.111 | 0.163 |
|                  | $\lambda_1$ | 0.558 | 0.058 | 0.525 | 0.025 |
|                  | $\lambda_2$ | 1.129 | 0.171 | 1.047 | 0.069 |
| $k = 25, R = (7,0_{(23)},10)$ | $\alpha$ | 2.149 | 0.175 | 2.127 | 0.171 |
|                  | $\lambda_1$ | 0.527 | 0.033 | 0.517 | 0.021 |
|                  | $\lambda_2$ | 1.061 | 0.094 | 1.034 | 0.063 |
| $k = 25, R = (0_{(11)},7,0_{(12)},10)$ | $\alpha$ | 2.149 | 0.176 | 2.115 | 0.126 |
|                  | $\lambda_1$ | 0.536 | 0.033 | 0.514 | 0.020 |
|                  | $\lambda_2$ | 1.073 | 0.101 | 1.024 | 0.063 |
| $k = 25, R = (0_{(23)},7,10)$ | $\alpha$ | 2.169 | 0.201 | 2.126 | 0.144 |
|                  | $\lambda_1$ | 0.540 | 0.039 | 0.530 | 0.022 |
|                  | $\lambda_2$ | 1.079 | 0.102 | 1.039 | 0.060 |

Bayes estimator with respect to the non-informative priors in most of the cases considered here.

From above discussion, we can conclude that when the order restriction is imposed on the scale parameters, and if we have some prior information about the unknown parameters, then the Bayesian inference with respect to the informative priors is preferable; otherwise, non-informative priors may be used.

Now, comparing the results between the ordered inference and the unordered inference, it is observed that the estimators obtained using ordered

Table 7: $m = 20, n = 22, \alpha = 1, \lambda_1 = 0.5, \lambda_2 = 1$ (with order restriction)

| Censoring scheme | Parameter | 90% HPD CRI | 90% Bootstrap CI |
|------------------|-----------|-------------|------------------|
|                  |           | inf-prior   | noninf-prior     |
|                  |           | AL | CP (%) | AL | CP (%) | AL | CP (%) |
| $k = 20, R = (7,0_{(18)},15)$ | $\alpha$ | 0.600 | 83.8 | 0.624 | 82.2 | 0.801 | 80.8 |
|                  | $\lambda_1$ | 0.502 | 88.6 | 0.515 | 86.2 | 0.740 | 89.2 |
|                  | $\lambda_2$ | 0.711 | 83.4 | 0.722 | 81.4 | 1.312 | 84.0 |
| $k = 20, R = (0_{(9)},7,0_{(9)},15)$ | $\alpha$ | 0.502 | 88.6 | 0.515 | 86.2 | 0.740 | 89.2 |
|                  | $\lambda_1$ | 0.515 | 88.5 | 0.537 | 87.2 | 0.763 | 89.8 |
|                  | $\lambda_2$ | 0.685 | 80.0 | 0.731 | 81.4 | 1.368 | 86.6 |
| $k = 20, R = (0_{(18)},7,15)$ | $\alpha$ | 0.502 | 88.6 | 0.515 | 86.2 | 0.740 | 89.2 |
|                  | $\lambda_1$ | 0.515 | 88.5 | 0.537 | 87.2 | 0.763 | 89.8 |
|                  | $\lambda_2$ | 0.685 | 80.0 | 0.731 | 81.4 | 1.368 | 86.6 |
Table 8: $m = 20, n = 22, \alpha = 2, \lambda_1 = 0.5, \lambda_2 = 1$ (with order restriction)

| Censoring scheme | Parameter | 90% HPD CRI | 90% HPD CRI | 90% Bootstrap CI |
|------------------|-----------|-------------|-------------|------------------|
|                  |           | inf-prior   | noninf-prior|                  |
|                  |           | AL         | CP (%)      | AL         | CP (%)         |
| $k = 20, \alpha$ | $\lambda_1$ | 0.499 | 89.2 | 0.517 | 86.8 | 0.743 | 88.8 |
| $R = (7,0_{(18)},15)$ | $\lambda_2$ | 0.724 | 83.4 | 0.746 | 79.8 | 1.314 | 85.4 |
| $k = 20, \alpha$ | $\lambda_1$ | 0.492 | 89.2 | 0.526 | 84.8 | 0.780 | 88.6 |
| $R = (0_{(9)},7,0_{(9)},15)$ | $\lambda_2$ | 0.703 | 82.8 | 0.740 | 80.2 | 1.378 | 88.0 |
| $k = 20, \alpha$ | $\lambda_1$ | 0.509 | 89.6 | 0.527 | 85.2 | 0.800 | 88.8 |
| $R = (0_{(18)},7,15)$ | $\lambda_2$ | 0.694 | 84.2 | 0.704 | 82.8 | 1.416 | 86.8 |

Information are slightly better than those of the unordered ones both in terms of biases and MSEs. Moreover, the confidence intervals and the credible intervals based on the ordered inference are slightly smaller than those on the unordered inference. Hence, if we have some knowledge about the ordering on the scale parameters, it is better to use that information.

6.2. Data Analysis. In this section, we present the analysis of real data sets to show how the different methods work in practice. The data represent the strength measured in GPA for single carbon fibers and can be obtained from Kundu and Gupta (2006). Single fibers were tested under tension at different gauge lengths. Data set 1 are measurements on single fiber of gauge length 20 mm, and Data set 2 are obtained from single fiber of gauge length 10 mm. They are presented below for an easy reference:

Data set 1:
1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.00, 2.021, 2.027, 2.055, 2.063, 2.098, 2.140, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.570, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.77, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.880, 2.954, 3.012, 3.067, 3.084, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585.

Data set 2:
Right arrow.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454,

Table 9: MLEs of parameters, K-S distance, and $p$ values for the fitted Weibull models to modified data sets 1 and 2

| Data     | MLE from complete sample | K-S distance | $p$ value |
|----------|--------------------------|--------------|-----------|
|          | Shape parameter | Scale parameter |             |             |
| Data set 1 | $\alpha = 3.843$ | $\lambda = 0.088$ | 0.046 | 0.998 |
| Data set 2 | $\alpha = 3.909$ | $\lambda = 0.025$ | 0.079 | 0.815 |
Table 10: Bayes estimates of parameters, expected posterior K-S, and $p$ values for the fitted Weibull models to modified data sets 1 and 2

| Data set | Bayes estimate from complete sample | expected posterior K-S distance |
|----------|-----------------------------------|-------------------------------|
|          | Shape parameter | Scale parameter | $p_b$(data) |
| Data set 1 | $\alpha = 3.930$ | $\lambda = 0.082$ | 0.061 | 0.877 |
| Data set 2 | $\alpha = 4.051$ | $\lambda = 0.022$ | 0.090 | 0.703 |

Kundu and Gupta (2006) subtract 0.75 from both the data sets and fit Weibull distributions for both the modified data sets separately. The MLEs of shape parameters and scale parameters and Kolmogorov-Smirnov (K-S) distances between empirical distribution functions and fitted distribution functions along with $p$ values are provided in Table 9 for both the data sets. The K-S distances and $p$ values clearly indicate both the data sets after subtracting 0.75 are well fitted by Weibull distributions.

Along with the classical K-S test, we also compute the Bayesian counterpart $p_b$(data), the posterior predictive $p$ value based on the K-S discrepancies (see for example Gelman et al. (1996)). We compute the $p_b$(data) based on the non-informative prior and the results are reported in Table 10.

To test whether two data sets have common shape parameter, we perform the K-S test as well as conventional likelihood ratio (L-R) test. The MLE of the common shape parameter and two scale parameters along with $p$ values are recorded in Table 11. Similarly, we compute the posterior predictive $p$ values $p_b$(data) to test the equality of the shape parameters based on the K-S test. Results are provided in Table 12. We have obtained the expected

Table 11: MLEs of parameters, and $p$ value of L-R and K-S tests for common shape parameter on fitted Weibull models to modified data sets 1 and 2

| Shape parameter | Scale parameter | $p$ value(L-R test) | $p$ value(K-S test) |
|----------------|----------------|---------------------|---------------------|
| $\alpha = 3.876$ | $\lambda_1 = 0.0861$ | 0.895 | (Data set 1) 0.998 |
|                | $\lambda_2 = 0.026$ |          | (Data set 2) 0.852 |
Table 12: Bayes estimate of parameters, expected posterior K-S distance, and $p$ values for common shape parameter on fitted Weibull models to modified data sets 1 and 2

| Shape parameter | Scale parameter | Expected posterior K-S distance | $p_b$(data)         |
|-----------------|-----------------|---------------------------------|---------------------|
| $\alpha = 4.074$ | $\lambda_1 = 0.0881$ (Data set 1) | 0.102 (Data set 1) | 0.552               |
|                 | $\lambda_2 = 0.030$ (Data set 2) | 0.134 (Data set 2) | 0.351               |

Table 13: The MLEs and BEs of the unknown parameters for the real data set

| Parameter | MLE | BE  |
|-----------|-----|-----|
| $\alpha$  | 4.495 | 3.896 |
| $\lambda_1$ | 0.071 | 0.098 |
| $\lambda_2$ | 0.016 | 0.028 |

Table 14: The CIs and CRIs of the unknown parameters for the real data set

| Parameter | 90% HPD CRI | 90% bootstrap CI |
|-----------|-------------|-------------------|
|           | LL          | UL                | LL          | UL          |
| $\alpha$  | 3.472       | 4.338             | 3.461       | 6.693       |
| $\lambda_1$ | 0.070       | 0.167             | 0.030       | 0.117       |
| $\lambda_2$ | 0.007       | 0.049             | 0.0004      | 0.037       |

Table 15: The MLEs and BEs of the unknown parameters for the real data set with order restriction

| Parameter | MLE | BE  |
|-----------|-----|-----|
| $\alpha$  | 4.495 | 3.728 |
| $\lambda_1$ | 0.071 | 0.088 |
| $\lambda_2$ | 0.016 | 0.022 |
Table 16: The CIs and CRIs of the unknown parameters for the real data set with order restriction

| Parameter | 90% HPD CRI | 90% bootstrap CI |
|-----------|-------------|------------------|
|           | LL | UL | LL | UL |
| $\alpha$  | 3.442 | 4.360 | 3.461 | 6.774 |
| $\lambda_1$ | 0.078 | 0.112 | 0.028 | 0.117 |
| $\lambda_2$ | 0.007 | 0.033 | 0.004 | 0.038 |

$p$ value of the L-R test based on the posterior density and it is 0.41. Based on these values, it is reasonable to assume that both data sets are from Weibull distributions with equal shape parameter.

For illustrative purpose, we have generated a joint progressive type-II censored sample with $k = 20$, $R_i = 4$ for $i = 1, \ldots, 19$ and $R_{20} = 36$. The data set is as follows: (1.312,1,2), (1.314,1,2), (1.479,1,1), (1.552,1,1), (1.700,1,3), (1.861,1,1), (1.865,1,1), (1.901,0,4), (1.944,1,2), (1.966,1,3), (1.997,1,1), (2.006,1,2), (2.027,1,1), (2.055,1,3), (2.098,1,2), (2.132,0,3), (2.140,1,2), (2.179,1,2), (2.203,0,3), and (2.257,0,14).

First, we obtain the MLEs of the unknown parameters without any order restriction and the associate 90% confidence intervals. The Bayes estimators (BEs) and the associated 90% confidence and credible intervals without any order restriction are obtained based on non-informative priors setting $a_0 = 0$, $b_0 = 0$, $a_1 = 0$, $a_2 = 0$, and $a = 0$, $b = 4$. We set non-informative prior of $\alpha$ as GA(0,4) so that the posterior density $\pi_2^*(\alpha|\text{data})$ is integrable based on the given data. The results are presented in Tables 13 and 14. The corresponding results based on order restriction are presented in Tables 15 and 16.

Based on the joint type-II progressively censored data, among non-informative Bayes estimates and MLEs, non-informative Bayes estimates are more close to the estimates based on complete data sets (see Table 11). Also, the length of the HPD credible intervals are less than that of the bootstrap CIs.

7 Conclusions

In this paper, we consider the analysis of joint progressively censored data for two populations. It is assumed that the lifetime of the items of the two populations follows Weibull distributions with the same shape parameter but different scale parameters. We obtained the MLEs of the unknown parameters and showed that they exist under a very general set of conditions and they are unique. We also obtain the Bayes estimate and the associated credible interval of a function of the unknown parameters based on a
fairly general prior distribution both on the scale and shape parameters. We further consider the order restricted inference on the unknown parameters. Based on an extensive simulation experiment, it is observed that the Bayes estimators with respect to the informative priors perform significantly better than the corresponding Bayes estimators based on non-informative priors for point estimation in terms of average bias and mean-squared errors. Also, non-informative prior-based estimators perform slightly better than MLEs in terms of average bias and mean-squared errors. The credible intervals based on an informative prior perform better than the other two methods in terms of average lengths and coverage percentage. Also, it is observed that if the ordering information on the scale parameters is available, it is better to use it.

It may be mentioned that in this paper we have mainly considered two populations. It will be interesting to extend the results for more than two populations also. Su (2013) developed joint progressive censoring scheme for multiple populations. For multiple Weibull populations with common shape parameter, we can assume multivariate gamma-Dirichlet prior for scale parameters and a log-concave prior for common shape parameter as described in this article. More work is needed along that direction.

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Appendix A: Proof of Theorem 1

To prove Theorem 1, first, we show the following result. Suppose $a_j \geq 0$, for $j = 1, \ldots, k$, $g(\alpha) = \sum_{j=1}^{k} a_j t_j^\alpha$, then $-\ln(g(\alpha))$ is a concave function. Note that

$$g'(\alpha) = \sum_{j=1}^{k} a_j t_j^\alpha \ln t_j \quad \text{and} \quad g''(\alpha) = \sum_{j=1}^{k} a_j t_j^\alpha (\ln t_j)^2.$$ 

Moreover,

$$\left( \sum_{j=1}^{k} a_j t_j^\alpha (\ln t_j)^2 \right) \left( \sum_{j=1}^{m} a_j t_j^\alpha \right) - \left( \sum_{j=1}^{m} a_j t_j^\alpha \ln t_j \right)^2 = \sum_{1 \leq i < j \leq k} a_i a_j (\ln t_i - \ln t_j)^2 \geq 0.$$ 

Therefore,

$$- \frac{d^2 \ln g(\alpha)}{d\alpha^2} = - \frac{g''(\alpha)g(\alpha) - (g'(\alpha))^2}{(g(\alpha))^2} \leq 0.$$
Now, from the above observation, it immediately follows that $p(\alpha)$ is a concave function. The result follows by observing the fact that $p(\alpha) \to -\infty$, as $\alpha \to 0$ or $\alpha \to \infty$.

### Appendix B: Fisher Information Matrix

If we denote the matrix $A$ as

$$
A = \begin{bmatrix}
\frac{\partial^2 l}{\partial \alpha^2} & \frac{\partial^2 l}{\partial \alpha \partial \lambda_1} & \frac{\partial^2 l}{\partial \alpha \partial \lambda_2} \\
\frac{\partial^2 l}{\partial \lambda_1 \partial \alpha} & \frac{\partial^2 l}{\partial \lambda_1^2} & \frac{\partial^2 l}{\partial \lambda_1 \partial \lambda_2} \\
\frac{\partial^2 l}{\partial \lambda_2 \partial \alpha} & \frac{\partial^2 l}{\partial \lambda_2 \partial \lambda_1} & \frac{\partial^2 l}{\partial \lambda_2^2}
\end{bmatrix} = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix},
$$

then

$$
\begin{align*}
a_{11} &= \frac{k}{\alpha^2} + \lambda_1 \left[ \sum_{j=1}^{k} s_j t_j^\alpha (\ln t_j)^2 + \sum_{C_1} t_j^\alpha (\ln t_j)^2 \right] \\
&\quad + \lambda_2 \left[ \sum_{j=1}^{k} w_j t_j^\alpha (\ln t_j)^2 + \sum_{C_2} t_j^\alpha (\ln t_j)^2 \right] \\
a_{12} &= a_{21} = \left[ \sum_{j=1}^{k} s_j t_j^\alpha \ln t_j + \sum_{C_2} t_j^\alpha \ln t_j \right] \\
a_{13} &= a_{31} = \left[ \sum_{j=1}^{k} w_j t_j^\alpha \ln t_j + \sum_{C_1} t_j^\alpha \ln t_j \right] \\
a_{22} &= \frac{k_1}{\lambda_1^2}, \quad a_{33} = \frac{k_1}{\lambda_1^2}, \quad a_{23} = a_{32} = 0.
\end{align*}
$$

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Shuvashree Mondal
Debasis Kundu
Department of Mathematics
and Statistics, Indian Institute of Technology Kanpur Pin 208016, India
E-mail: kundu@iitk.ac.in

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