A QUASI-POISSON STRUCTURE ON THE MULTIPLICATIVE GROTHENDIECK–SPRINGER RESOLUTION

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Abstract. In this note we show that the multiplicative Grothendieck–Springer space has a natural quasi-Poisson structure. The associated group-valued moment map is the resolution morphism, and the quasi-Hamiltonian leaves are the connected components of the preimages of Steinberg fibers. This is a multiplicative analogue of the standard Poisson structure on the additive Grothendieck–Springer resolution, and an explicit illustration of a more general procedure of reduction along Dirac realizations which was developed in recent work of the author and Mayrand.

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Introduction

The multiplicative Grothendieck–Springer resolution of a complex reductive group $G$ is the incidence variety

$$\tilde{G} = \{(g, B) \in G \times B \mid g \in B\},$$

where $B$ is the flag variety of $G$, viewed as the space of all Borel subgroups or, equivalently, of all Borel subalgebras. The first projection

$$\mu : \tilde{G} \rightarrow G$$

is a generically finite map which restricts to a resolution of singularities along each Steinberg fiber, and its restriction to the unipotent cone was originally constructed by Springer [Sp].

The group $G$ is an example of a quasi-Poisson manifold. Such manifolds are generalizations of Poisson structures in which the Jacobi identity is twisted by a canonical trivector field, and they were introduced by Alekseev, Kosmann-Schwarzbach, and Meinrenken [AKSM]. They are equipped with group-valued moment maps and are foliated by nondegenerate leaves, each of which carries a quasi-Hamiltonian 2-form. In particular, the quasi-Hamiltonian leaves of $G$ are the conjugacy classes, which vary in dimension. Therefore, the quasi-Poisson structure on $G$ is singular in the sense of foliation theory.
In this note we show that the Grothendieck–Springer resolution $\tilde{G}$ also has a natural quasi-Poisson structure, whose group-valued moment map is $\mu$. The leaves of this structure are the connected components of the preimages of Steinberg fibers under $\mu$, and they form a regular foliation of $\tilde{G}$. The map $\mu$ is therefore a desingularization in two ways—it resolves each singular Steinberg fiber to a smooth variety, and it resolves the singular quasi-Hamiltonian foliation of $G$ to a regular quasi-Hamiltonian foliation of $\tilde{G}$.

The additive Grothendieck–Springer resolution. These observations are a multiplicative analogue of the Grothendieck–Springer resolution of the Lie algebra $\mathfrak{g}$ of $G$, which is the variety of pairs

$$\bar{\mathfrak{g}} = \{(x, b) \in \mathfrak{g} \times B \mid x \in b\}$$

consisting of an element in the Lie algebra and a Borel subalgebra that contains it. We briefly recall its geometry, and we refer to [ChGi] for more details. Fixing a Borel subgroup $B$ of $G$ and writing $\mathfrak{b}$ for its Lie algebra, there is a natural isomorphism

$$G \times_B \mathfrak{b} \xrightarrow{\sim} \bar{\mathfrak{g}}$$

$$[g : x] \mapsto (\mathrm{Ad}_g(x), \mathrm{Ad}_g(b))$$

which makes $\bar{\mathfrak{g}}$ into a vector bundle over $G/B$.

Let $t$ be a fixed maximal Cartan in $\mathfrak{b}$ with corresponding subgroup $T \subset B$, and let $W$ be its Weyl group. Decompose $\mathfrak{b} = t \oplus u$, where $u = [\mathfrak{b}, \mathfrak{b}]$ is the nilpotent radical of $\mathfrak{b}$. The resolution $\bar{\mathfrak{g}}$ fits into the diagram

$$\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\mu} & \bar{\mathfrak{g}} \\
& \searrow & \swarrow \lambda \\
t/W, & \xleftarrow{\kappa} & \mathfrak{t}
\end{array}$$

where $\mu$ is the first projection, $\lambda$ is the projection of the bundle $G \times_B \mathfrak{b}$ onto the $t$-component of the fiber $\mathfrak{b}$, and $\kappa$ is the composition of the adjoint quotient map with the Chevalley isomorphism

$$\mathfrak{g}/G \cong t/W.$$

For any $s \in \mathfrak{t}$, write $\bar{s}$ for its image in $t/W$. When $s$ is not regular, $\kappa^{-1}(\bar{s})$ is a singular algebraic variety and the map $\mu$ restricts to a resolution of singularities

$$G \times_B (s + u) \cong \lambda^{-1}(s) \longrightarrow \kappa^{-1}(\bar{s}).$$

In particular, in the special case when $s = 0$ this map is the Springer resolution

$$G \times_B \mathfrak{u} \longrightarrow \mathcal{N}$$

of the nilpotent cone $\mathcal{N}$ of $\mathfrak{g}$. 
One can equip $\mathfrak{g}$ with a Poisson structure through Hamiltonian reduction as follows. The action of $G$ on itself by right multiplication induces Hamiltonian actions of $G$ and $B$ on the cotangent bundle $T^*_G$. The corresponding moment maps fit into the commutative diagram

$$
\begin{array}{ccc}
T^*_G & \cong & G \times \mathfrak{g} \\
\pi & \downarrow & \nu \\
\mathfrak{g}/u. & \overset{}{\leftarrow} & \mathfrak{g}
\end{array}
$$

Here we use the left trivialization of the cotangent bundle and we identify $\mathfrak{g}^*$ with $\mathfrak{g}$ and $\mathfrak{b}^*$ with $\mathfrak{g}/u$ through the Killing form, so that the moment map $\nu$ is just the second projection. Since each point in the subset $\mathfrak{b}/u \subset \mathfrak{g}/u$ is fixed under the action of $\mathfrak{b}$ on $\mathfrak{g}/u$, the reduced space

$$
\nu^{-1}(\mathfrak{b}/u)/B = \nu^{-1}(\mathfrak{b})/B = G \times_B \mathfrak{b}
$$

inherits a natural Poisson structure from the canonical symplectic structure on $T^*_G$. Its symplectic leaves are the individual symplectic reductions

$$
\nu^{-1}(s + u)/B = G \times_B (s + u)
$$

for $s \in \mathfrak{t}$, each of which is an affine bundle over $G/B$. Therefore the Grothendieck–Springer resolution $\mathfrak{g}$ is a regular Poisson variety, and each resolution of singularities

$$
G \times_B (s + u) \longrightarrow \kappa^{-1}(s)
$$

is a symplectic resolution.

The group-valued analogue. In the quasi-Poisson setting, the role of the cotangent bundle is played by the internal fusion double $\mathbb{D}_G = G \times G$. A priori there is no analogue of Hamiltonian reduction with respect to the action of the Borel subgroup, because the $G$-valued moment map of $\mathbb{D}_G$ does not descend to a moment map with values in $B$—in other words, a quasi-Poisson $G$-space is not generally quasi-Poisson for the action of a subgroup of $G$.

However, by work of Bursztyn and Crainic [BuCr1, BuCr2], quasi-Poisson manifolds are an example of a more general class of structures known as Dirac manifolds. In [BaMa], the author and Mayrand give a general procedure for Dirac reduction along a certain class of maps known as Dirac realizations. In this note, we show explicitly how this theory specializes to the Grothendieck–Springer space $\tilde{G}$, using only the basics of Dirac structures. We prove the following result, as Theorems 2.7 and 2.9.

**Theorem.** There is a natural quasi-Poisson structure on the multiplicative Grothendieck–Springer resolution

$$
\tilde{G} \cong G \times_B B,
$$

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which is inherited from the quasi-Hamiltonian structure on $\mathbb{D}_G$ and for which the map $\mu$ is a group-valued moment map. Its quasi-Hamiltonian leaves are the twisted unipotent bundles

$$G \times_B tU \quad \text{for } t \in T.$$ 

The fiber bundle $G \times_B tU$ resolves the singularities of the Steinberg fiber $F_t$, which is a possibly singular quasi-Hamiltonian variety. This resolution is quasi-Hamiltonian, in the sense that the pullback of the corresponding 2-form along the desingularization map

$$G \times_B tU \to F_t$$

agrees with the quasi-Hamiltonian structure on the leaf $G \times_B tU$. This quasi-Hamiltonian structure first appeared in work of Boalch [Bo, Section 4] in the setting of moduli spaces, and the quasi-Poisson structure on $\tilde{G}$ which induces it can also be constructed using results of Li-Bland and Severa [LBS, Theorem 5].

We remark that the resolution $\tilde{G}$ also carries a Poisson structure, which was introduced by Evens and Lu [EvLu] and which is obtained from a Poisson structure on the double $\mathbb{D}_G$ through coisotropic reduction. In this work, the authors equip $G$ with a compatible Poisson structure that can be viewed as the semi-classical limit of the quantum group $U_q(\mathfrak{g})$, and with respect to which the resolution map $\mu$ is Poisson. The $T$-orbits of the symplectic leaves of $G$ are intersections of conjugacy classes and Bruhat cells, and the singularities of their closures are resolved by the $T$-orbits of symplectic leaves in $\tilde{G}$, which are regular Poisson manifolds.

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1. Dirac manifolds and quasi-Poisson structures

1.1. Dirac structures. We begin by giving some background on Dirac manifolds, and we refer the reader to [Bu] for more details. Let $M$ be a (real or complex) manifold and let $\eta \in \Omega^3(M)$ be a closed 3-form. A $\eta$-twisted Dirac structure on $M$ is a subbundle $L \subset T_M \oplus T_M^*$ such that

- $L$ is Lagrangian with respect to the nondegenerate symmetric pairing
  $$\langle (X, \alpha), (Y, \beta) \rangle = \alpha(Y) + \beta(X), \text{ and}$$
- the space of sections $\Gamma(L)$ is closed under the $\eta$-twisted Dorfman bracket
  $$[[X, \alpha], (Y, \beta)] = ([X, Y], L_X \beta + i_Y \alpha + i_{X \wedge Y} \eta).$$

While the Dorfman bracket is not a Lie bracket on $T_M \oplus T_M^*$, it restricts to a Lie bracket on the sections of the subbundle $L$. In this way $L$ becomes a Lie algebroid over $M$, with anchor map the first projection

$$p_T : L \to T_M$$

$$(X, \alpha) \mapsto X.$$
Example 1.1. Let $\omega \in \Omega^2(M)$ be any 2-form, and let $\eta = -d\omega$. Then the graph

$$L_\omega = \{(X, \omega^\flat(X)) \mid X \in T_M\}$$

is a $\eta$-twisted Dirac structure on $M$. Such Dirac structures are called nondegenerate or presymplectic, and are characterized by the property

$$L_\omega \cap T_M^* = 0.$$

Note that here and throughout we view $T_M$ and $T_M^*$ as subbundles of $T_M \oplus T_M^*$ via the coordinate embeddings. In particular, a Dirac structure $L$ on $M$ is induced by a symplectic form if and only if it is non-twisted and

$$L \cap T_M^* = L \cap T_M = 0.$$

More generally, if $(M, L)$ is a $\eta$-twisted Dirac manifold, the image of the anchor map $p_T$ is an involutive generalized distribution which integrates to a foliation of $M$ by nondegenerate or presymplectic leaves. Each leaf $S$ carries a natural presymplectic form $\omega_S$, which has the property that

$$d\omega_S = -\eta|_S.$$

Example 1.2. Let $\pi \in \mathcal{X}^2(M)$ be a bivector on $M$ and suppose that there is a 3-form $\eta \in \Omega^3(M)$ such that

$$[\pi, \pi] = 2\pi^\#(\eta),$$

where the left-hand side denotes the Schouten–Nijenhuis bracket. Then $\pi$ is a twisted Poisson structure, and its graph

$$L_\pi = \{(\pi^\#(\alpha), \alpha) \mid \alpha \in T_M^*\}$$

is a $\eta$-twisted Dirac structure on $M$. Conversely, a Dirac structure $L$ on $M$ is induced by a Poisson bivector if and only if it is non-twisted and

$$L \cap T_M = 0.$$

In this case, the foliation of $M$ by nondegenerate leaves is exactly the symplectic foliation.

Given a map $f : M \to N$ and a $\eta_N$-twisted Dirac structure $L_N$ on $N$, the pullback of $L_N$ under $f$ is the generalized distribution

$$f^*L_N = \{(X, f^*\alpha) \in T_M \oplus T_M^* \mid (f_*X, \alpha) \in L_N\}.$$

If $f^*L_N$ is a smooth bundle, it defines a $f^*\eta_N$-twisted Dirac structure on $M$ [Bu, Proposition 1.10]. When $M$ is equipped with a twisted Dirac structure $L_M$, the map $f$ is called backward-Dirac (or b-Dirac) if

$$L_M = f^*L_N.$$

Such maps generalize the pullbacks of differential forms. In particular, when $(M, \omega_M)$ and $(N, \omega_N)$ are nondegenerate Dirac manifolds, $f$ is b-Dirac if and only if

$$f^*\omega_N = \omega_M.$$
Similarly, if $L_M$ is a $\eta_M$-twisted Dirac structure on $M$, the pushforward of $L_M$ under $f$ is the generalized distribution

$$f_* L_M = \{(f_* X, \alpha) \in T_N \oplus T_N^* \mid (X, f^* \alpha) \in L_M\},$$

as long as it is well-defined. When $N$ is equipped with a twisted Dirac structure $L_N$, the map $f$ is called forward-Dirac (or $f$-Dirac) if at every point it satisfies

$$L_N = f_* L_M.$$

Such maps generalize pushforwards of vector fields. In particular, when $(M, \pi_M)$ and $(N, \pi_N)$ are Poisson manifolds, $f$ is $f$-Dirac if and only if

$$f_* \pi_M = \pi_N.$$

When $f : M \rightarrow N$ is a diffeomorphism, it is $f$-Dirac if and only if it is $b$-Dirac, and in this case it is called a Dirac diffeomorphism. Suppose now that a group $G$ acts on $M$ by Dirac diffeomorphisms and that $M/G$ has the structure of a manifold such that the quotient map

$$q : M \rightarrow M/G$$

is a smooth submersion. Write $\mathfrak{g}$ for the Lie algebra of $G$ and

$$\rho_M : \mathfrak{g} \rightarrow \mathcal{X}(M)$$

$$\xi \mapsto \xi_M$$

for the infinitesimal action map. In this case the pushforward $f_* L_M$ is a Dirac structure on $M/G$ if the action of $G$ on $M$ is regular—that is, if the generalized distribution

$$\rho_M(\mathfrak{g}) \cap L_M \subset T_M$$

has constant dimension [Bu, Proposition 1.13].

1.2. Quasi-Poisson manifolds. Let $G$ be a (real or complex) Lie group whose Lie algebra $\mathfrak{g}$ carries an invariant, nondegenerate, symmetric bilinear form $(\cdot, \cdot)$. The Cartan 3-form of $G$ is the invariant 3-form induced by the element $\eta \in \wedge^3 \mathfrak{g}^*$ defined by

$$\eta(x, y, z) = \frac{1}{12} (x, [y, z]) \quad \text{for all } x, y, z \in \mathfrak{g},$$

and we denote by $\chi \in \wedge^3 \mathfrak{g}$ the 3-tensor which corresponds to it under the induced identification $\mathfrak{g} \cong \mathfrak{g}^*$.

Through the infinitesimal action map, $\chi$ generates a canonical bi-invariant trivector field

$$\chi_M \in \mathcal{X}^3(M).$$

A quasi-Poisson structure on $M$ is a bivector field $\pi \in \mathcal{X}^2(M)$ whose Schouten bracket satisfies

$$[\pi, \pi] = \chi_M.$$
This notion was first introduced in a series of papers by Alekseev, Malkin, and Meinrenken [AMM], by Alekseev and Kosmann-Schwarzbach [AKS], and by Alekseev, Kosmann-Schwarzbach, and Meinrenken [AKSM]. Viewed as a skew-symmetric bracket on the space of functions on $M$, a quasi-Poisson structure is a biderivation which satisfies a $\chi_M$-twisted version of the Jacobi identity. In particular, when the action of $G$ on $M$ is trivial, the Cartan trivector field $\chi_M$ vanishes and we recover the usual definition of a Poisson manifold.

Consider the map
\[
\sigma : g \to T_G^*, \\
\xi \mapsto \frac{1}{2}(\xi^R + \xi^L)^\vee,
\]
where $\xi^R$ and $\xi^L$ are the right- and left-invariant vector fields induced by the Lie algebra element $\xi$, and $v^\vee \in T_G^*$ is the 1-form corresponding to the vector field $v \in T_G$ under the isomorphism given by left-invariant bilinear form induced by $(\cdot, \cdot)$ on $T_G$. We let
\[
\sigma^\vee : T_G^* \to g
\]
be its adjoint. The quasi-Poisson manifold $(M, \pi)$ is Hamiltonian if it is equipped with a $G$-equivariant map
\[
\Phi : M \to G,
\]
called a group-valued moment map, which satisfies the condition
\[
\pi^# \circ \Phi^* = \rho_M \circ \sigma^\vee. \tag{1.3}
\]

The Hamiltonian quasi-Poisson manifold $(M, \pi, \Phi)$ is nondegenerate if the bundle map
\[
T_M^* \oplus g \to T_M \tag{1.4}
\]
\[
(\alpha, \xi) \mapsto \pi^#(\alpha) + \xi_M
\]
is surjective. In this case, $M$ carries a quasi-Hamiltonian 2-form $\omega \in \Omega^2(M)$ which satisfies the compatibility condition
\[
\pi^# \circ \omega^\flat = C, \tag{1.5}
\]
where
\[
C := 1 - \frac{1}{4} \rho_M \circ \rho^\vee \circ \Phi^*
\]
and
\[
\rho^\vee : T_G \to g
\]
is the adjoint of the infinitesimal action of $G$ on itself by conjugation [AKSM, Lemma 10.2]. This 2-form has the property that
\[
d\omega = -\Phi^* \eta,
\]
and the moment map condition (1.3) can be rewritten as
\[
\omega^\flat \circ \rho_M = \Phi^* \circ \sigma. \tag{1.6}
\]
Moreover, if (1.4) fails to be surjective, its image is an integrable generalized distribution and the quasi-Poisson manifold $M$ is foliated by quasi-Hamiltonian leaves.

1.3. **Quasi-Poisson bivectors as Dirac structures.** The quasi-Poisson structure on the $G$-manifold $(M, \pi, \Phi)$ is encoded [BuCr1, Theorem 3.16] by the $\Phi^*\eta$-twisted Dirac bundle

$$L = \{ (\pi^\flat(\alpha) + \xi_M, C^\flat(\alpha) + \Phi^*\sigma(\xi)) \mid \alpha \in T_M^*, \xi \in \mathfrak{g} \}.$$

The image of its projection onto the tangent component is the generalized distribution given by the map (1.4), and the nondegenerate leaves associated to this Dirac structure are precisely the quasi-Hamiltonian leaves. In particular, if $M$ is quasi-Hamiltonian with $2$-form $\omega \in \Omega^2(M)$, by (1.5) and (1.6) this bundle can be written

$$L = \{ (X, \omega^b(X)) \mid X \in T_M \}.$$

**Example 1.7.** [AKSM, Proposition 3.1] The group $G$ has a natural quasi-Poisson structure relative to the conjugation action, called the *Cartan–Dirac structure*, whose moment map is the identity. The associated Dirac bundle [BuCr1, Example 3.4] is

$$L_G = \{ (\rho(\xi), \sigma(\xi)) \mid \xi \in \mathfrak{g} \} = \{ (\xi^L - \xi^R, \sigma(\xi)) \mid \xi \in \mathfrak{g} \}.$$  

Projecting onto the tangent component, we see that the nondegenerate leaves of this structure are the conjugacy classes. Therefore the Cartan–Dirac structure on $G$ can be seen as a multiplicative analogue of the classical Kirillov–Kostant–Souriau Poisson structure on $\mathfrak{g}^* \cong \mathfrak{g}$.

**Example 1.9.** [AKSM, Example 5.3] The *internal fusion double* $D_G := G \times G$ has a natural nondegenerate quasi-Poisson structure relative to the $G \times G$-action

$$(g_1, g_2)(a, b) = (g_1 a g_2^{-1}, g_2 b g_2^{-1}).$$

This structure is a multiplicative counterpart of the canonical symplectic structure on the cotangent bundle of $G$, viewed under the identification

$$T_G^* \cong G \times \mathfrak{g}$$

induced by left-trivialization and by the invariant inner product. Its moment map is given by

$$\Phi : D_G \mapsto G \times G$$

$$(a, b) \mapsto (aba^{-1}, b^{-1}).$$

Viewing the quasi-Poisson $G$-manifold $(M, \pi, \Phi)$ as a Dirac manifold with $\Phi^*\eta$-twisted Dirac structure $L$, the moment map $\Phi$ is a $f$-Dirac map which satisfies the additional nondegeneracy condition

$$\ker \Phi_* \cap L = 0.$$  

In fact, this property completely characterizes Hamiltonian quasi-Poisson manifolds [BuCr1, Proposition 3.20]—suppose that $(M, L)$ is a Dirac manifold and that

$$\Phi : (M, L) \mapsto (G, L_G)$$
is a f-Dirac map which satisfies (1.11). Then, for any \((\rho(\xi), \sigma(\xi)) \in L_G\), there is a unique pair \((X_\xi, \alpha_\xi) \in L\) such that

\[
\Phi_\ast X_\xi = \rho(\xi) \quad \text{and} \quad \Phi_\ast \sigma(\xi) = \alpha_\xi.
\]

This gives an infinitesimal action of the Lie algebra \(g\) on \(M\) via

\[
\rho_M : g \longrightarrow \mathcal{X}(M) \quad (1.12)
\]

\[
\xi \longmapsto X_\xi.
\]

Moreover, for any \(\alpha \in T^* M\), there is a unique \(X_\alpha \in T_M\) such that

\[
\Phi_\ast X_\alpha = \sigma^{\vee} \rho_M^\ast (\alpha) \quad \text{and} \quad (X_\alpha, C^\ast (\alpha)) \in L.
\]

The map

\[
\pi^\# : T_M^* \longrightarrow T_M
\]

\[
\alpha \longmapsto X_\alpha
\]

defines a quasi-Poisson bivector \(\pi \in \mathcal{X}^2(M)\) relative to the action (1.12), whose associated moment map is \(\Phi\).

2. The multiplicative Grothendieck–Springer resolution

2.1. The Grothendieck–Springer space. From now on let \(G\) be a reductive complex group, so that the nondegenerate bilinear form on every simple factor of \(g\) is given by a scalar multiple of the Killing form, and once again write \(B\) for the flag variety of all Borel subgroups. The Grothendieck–Springer simultaneous resolution of \(G\) is the variety of pairs

\[
\tilde{G} := \{(g', B') \in G \times B \mid g' \in B'\}.
\]

There is a natural map

\[
\mu : \tilde{G} \longrightarrow G \quad (2.1)
\]

given by the first projection, whose general fiber is finite—when \(g' \in G\) is regular and semisimple, the points of \(\mu^{-1}(g')\) are permuted freely and transitively by \(W\), the Weyl group of \(G\).

Fixing a Borel subgroup \(B\) of \(G\), the isomorphism

\[
G \times_B B \longrightarrow \tilde{G}
\]

\[
[g : b] \longmapsto (gb^{-1}, gBg^{-1})
\]

realizes \(\tilde{G}\) as a fiber bundle over \(G/B \cong B\). Let \(T\) be a maximal torus of \(B\) and let \(U = [B, B]\) be its unipotent radical, so that we have a splitting \(B = TU\). Since \(B\) stabilizes each coset \(tU\), there is a well-defined map

\[
\lambda : G \times_B B \longrightarrow T
\]

\[
[g : tu] \longmapsto t
\]
whose fibers
\[ \lambda^{-1}(t) = G \times_B tU \]
are multiplicative analogues of twisted cotangent bundles over \( G/B \).

Let \( G//G = \text{Spec} \mathbb{C}[G]^G \) be the adjoint quotient of \( G \). By the Chevalley theorem, the restriction map gives an isomorphism of algebras
\[ \mathbb{C}[G]^G \cong \mathbb{C}[T]^W \]
and therefore an identification of varieties
\[ G//G \cong T/W. \]

The two quotient maps
\[ G \longrightarrow G//G \quad \text{and} \quad T \longrightarrow T/W \]
fit into a commutative diagram
\[
\begin{array}{ccc}
G & \xrightarrow{\lambda} & T \\
\downarrow{\kappa} & & \downarrow{\mu} \\
T/W & \xrightarrow{\lambda^{-1}} & G \\
\end{array}
\]
whose restriction to the regular locus of \( G \) is Cartesian.

For any \( t \in T \), let \( \bar{t} \) be its image in \( T/W \). The fibers of \( \kappa \), which are irreducible subvarieties of \( G \) of codimension equal to the rank, are called \textit{Steinberg fibers}, and each is a union of finitely many conjugacy classes. In particular, the Steinberg fiber
\[ F_t := \kappa^{-1}(\bar{t}) \]
contains a unique regular conjugacy class, which is open and dense, and the unique semisimple conjugacy class \( G \cdot t \), which is closed and of minimal dimension [St, Theorem 6.11 and Remark 6.15]. In particular, if \( t \in T \) is regular, then the fiber \( F_t \) is simply the orbit \( G \cdot t \) and therefore it is a smooth variety isomorphic to the quotient \( G/T \).

When \( t \in T \) is not regular, the Steinberg fiber \( F_t \) is generally singular. Diagram (2.2) gives a natural surjection \( \lambda^{-1}(t) \rightarrow \kappa^{-1}(\bar{t}) \), and we obtain a proper birational map
\[ G \times_B tU \rightarrow F_t \]
which is a resolution of singularities. In the special case of the identity element \( 1 \in T \), the Steinberg fiber \( \mathcal{U} := F_1 \) is the variety of all unipotent elements of \( G \), and this birational map is precisely the Springer resolution
\[ G \times_B U \rightarrow \mathcal{U}. \]
2.2. A quasi-Poisson structure on $\tilde{G}$. In this section we will show that $G \times B$ carries a natural quasi-Poisson structure for the left action of $G$ given by

$$h \cdot [g : b] = [hg : b].$$

(2.3)

Consider the diagram

$$
\begin{array}{ccc}
G \times B & \xrightarrow{j} & \mathbb{D}_G \\
\downarrow q & & \downarrow \Phi \\
G \times_B B & \xrightarrow{\mu} & G \\
\downarrow p & & \downarrow \Phi \\
G, & & \end{array}
$$

(2.4)

where $p$ and $q$ are the natural quotient maps, $\mu$ is the map defined in (2.1), and $\Phi$ is the quasi-Poisson moment map of the internal fusion double given by (1.10).

The bundle $j^* L_{\mathbb{D}_G}$ is the graph of the two-form $j^* \omega$, and is therefore a Dirac structure on $G \times B$. To check that it descends to the desired Dirac structure on the Grothendieck–Springer resolution, we will begin with the following simple lemma.

**Lemma 2.5.** Let $\xi \in b$. Then $T_B \subset \ker \sigma(\xi)$ if and only if $\xi \in u$.

**Proof.** Fix a point $tu \in B$, with semisimple part $t \in T$ and unipotent part $u \in U$, and write $\xi = s + n \in t \oplus u$. Any vector in $T_{tu} B$ is of the form $xR$ for some $x \in b$, and we have

$$x^R \in \ker (\xi^R + \xi^L)^\vee \quad \text{for all } x \in b \quad \iff \quad (\xi + \Ad_{tu}\xi, x) = 0 \quad \text{for all } x \in b$$

$$\iff \quad \xi + \Ad_{tu}\xi \in u$$

$$\iff \quad s = 0,$$

where the second equivalence follows from the fact that $b^\perp = u$ under the Killing form. \qed

**Proposition 2.6.** The action of $B$ on $G \times B$ defined by

$$h \cdot (g, b) = (gh^{-1}, hbb^{-1})$$

for $h \in B$ and $(g, b) \in G \times B$ is a regular Dirac action with respect to the Dirac structure $j^* L_{\mathbb{D}_G}$.

**Proof.** Since the second copy of $G$ acts on $\mathbb{D}_G$ by Dirac automorphisms, the same is true for the action of $B$ on the submanifold $G \times B$. It remains only to show that this action is regular.

We have

$$\rho_{\mathbb{D}_G}(0 \oplus b) \cap j^* L_{\mathbb{D}_G} = \{ \rho_{\mathbb{D}_G}(0, \xi) \mid \xi \in b \text{ and } T_{G \times B} \subset \ker \Phi^* \sigma(0, \xi) \}$$

$$= \{ \rho_{\mathbb{D}_G}(0, \xi) \mid \xi \in b \text{ and } T_{G \times B} \subset \ker \sigma(0, \xi) \}$$

$$= \{ \rho_{\mathbb{D}_G}(0, \xi) \mid \xi \in u \},$$
where the last equality follows from Lemma 2.5. Since the action of $B$ on $G \times B$ has trivial stabilizers, this implies that the intersection $\rho_{DG}(0 \oplus b) \cap j^*L_{DG}$ has constant rank and therefore that this action is regular. \hfill \Box

Because the Borel subalgebra satisfies
\[ \mathfrak{b}^\perp = [\mathfrak{b}, \mathfrak{b}], \]
the restriction of the Cartan 3-form $\eta$ to the subgroup $B$ vanishes. Chasing diagram (2.4), we see that
\[
q^* \mu^* \eta = \Phi^* p^* \eta \\
= \Phi^* r^*(\eta_1, \eta_2) \\
= j^* \Phi^*(\eta_1, \eta_2),
\]
so the twist of the Dirac structure $j^*L_{DG}$ is in the image of $q^*$. By [Bu, Proposition 1.13], Proposition 2.6 then implies that the pushforward
\[
L_G := q_* j^* L_{DG}
\]
is a smooth vector bundle and therefore a $\mu^* \eta$-twisted Dirac structure on $\tilde{G}$. We now show that it corresponds to a quasi-Poisson bivector.

**Theorem 2.7.** The Dirac structure $L_G$ is induced by a quasi-Poisson structure on $G \times B B$ whose moment map is $\mu$.

**Proof.** To show that $L_G$ defines a quasi-Poisson structure with moment map $\mu$, we must show (i) that $\mu$ is a $f$-Dirac map which satisfies the nondegeneracy condition (1.11), and (ii) that the induced $G$-action on $G \times B B$ coincides with the action given in (2.3).

(i) To see that $\mu$ is $f$-Dirac, we use diagram (2.4) to compute
\[
L_G = p_* \Phi_* L_{DG} \\
= p_* \Phi_* j^* L_{DG} \\
= \mu_* q_* j^* L_{DG} = \mu_* L_G,
\]
where the second equality follows from [Ba, Lemma 1.19]. Checking the nondegeneracy condition
\[
\ker \mu_* \cap L_G = 0,
\]
is equivalent to showing that
\[
\ker(p_* \Phi_*) \cap j^* L_{DG} \subset \rho_{DG}(0 \oplus \mathfrak{b}). \tag{2.8}
\]
Suppose that $(X, 0) \in j^*L_{DG}$ satisfies $p_* \Phi_*(X) = 0$. Then there is a 1-form $\alpha \in T_{G \times B}^0$ such that
\[
(X, \alpha) \in L_{DG}.
\]
Moreover, since $T_{G \times B}^0 = \Phi^* T_{G \times B}^0$ and since $\Phi$ is $f$-Dirac, there is a further $\beta \in T_{G \times B}^0$ such that $(\Phi \ast X, \beta)$ is an element of the Cartan–Dirac structure $L_{G \times G}$. Since $p_\ast \Phi \ast X = 0$ and since $X$ is tangent to $G \times B$, we obtain

$$\Phi \ast X = \rho(0, \xi)$$

for some $\xi \in \mathfrak{b}$. By (1.8), this means that $\beta = \sigma(0, \xi)$ and we get

$$(X, \Phi^\ast \beta) \in L_{\mathbb{D}_G} \quad \text{and} \quad (\rho_{\mathbb{D}_G}(0, \xi), \Phi^\ast \beta) \in L_{\mathbb{D}_G}.$$ 

Since $\rho_{\mathbb{D}_G}(\xi, 0)$ is tangent to $G \times B$, it follows that

$$(\rho_{\mathbb{D}_G}(\xi, 0), j^\ast \Phi^\ast \sigma(\xi, 0)) \in L_{G \times B}.$$ 

Moreover, by chasing diagram (2.4) we see that

$$j^\ast \Phi^\ast \sigma(\xi, 0) = \Phi^\ast i^\ast \sigma(\xi, 0) = \Phi^\ast p^\ast \sigma(\xi) = q^\ast \mu^\ast \sigma(\xi),$$

and we get

$$(q_\ast \rho_{\mathbb{D}_G}(\xi, 0), \mu^\ast \sigma(\xi)) \in L_{\mathbb{G}}.$$ 

Since $\mu \ast q_\ast \rho_{\mathbb{D}_G}(\xi, 0) = \rho(0, \xi)$, by (1.12) this implies that $\xi$ acts on $G \times B$ by the vector field $q_\ast \rho_{\mathbb{D}_G}(\xi, 0)$, which is precisely the vector field corresponding to $\xi$ under the action (2.3). \hfill \Box

**Theorem 2.9.** The quasi-Hamiltonian leaves of $L_{\mathbb{G}}$ are the twisted unipotent bundles $G \times_B tU$ for $t \in T$.

**Proof.** Let $X \in T_{G \times B}$. The pushforward $q_\ast X$ is contained in $p_T(L_{\mathbb{G}})$ if and only if

$$j^\ast \omega^\flat(X) \in \text{im} q^\ast \iff \omega(X, \rho_{\mathbb{D}_G}(0, \xi)) = 0 \quad \text{for all} \ \xi \in \mathfrak{b} \quad \text{(2.10)}$$

$$\iff X \in \ker \Phi^\ast \sigma(0, \xi) \quad \text{for all} \ \xi \in \mathfrak{b}$$

$$\iff \Phi_\ast X \in \ker \sigma(0, \xi) \quad \text{for all} \ \xi \in \mathfrak{b},$$

where the second equivalence follows from the moment map condition (1.6).

Keeping the notation of Lemma 2.5 and letting $x \in \mathfrak{b}$, we have

$$x^R \in \ker (\xi^R + \xi^L)^\vee \quad \text{for all} \ \xi \in \mathfrak{b} \iff (\xi + \text{Ad}_{\mathfrak{u}} \xi, x) = 0 \quad \text{for all} \ \xi \in \mathfrak{b} \quad \text{(2.11)}$$

$$\iff (\xi, x + \text{Ad}_{\mathfrak{u}}^{-1} x) = 0 \quad \text{for all} \ \xi \in \mathfrak{b}$$

$$\iff x \in \mathfrak{u},$$

where the last equivalence follows from Lemma 2.5.
Using (2.10) and (2.11),

\[ p_T(L_{G^*}) = \{ q_\ast X \mid j^\ast \omega^\flat(X) \in \text{im } q_\ast \} = \{ q_\ast X \mid X \in T_{G \times B} \}, \]

and so the leaves integrating the generalized distribution \( p_T(L_{G^*}) \) are precisely the submanifolds of \( G \times_B B \) of the form \( G \times_B tU \) for \( t \in T \). \qed

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