On invariants for the Poincaré equations and applications

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Abstract

We extend the Noether theory of invariants to the Poincaré equations. We apply this extension to the Maxwell-Lorentz equations coupled to the Abraham rotating extended electron with the configuration space \textit{SO}(3).

Keywords: Poincaré equations; conservation laws; Noether theory of invariants; Abraham’s rotating extended electron; Maxwell-Lorentz equations; Hamilton’s least action principle.
1 Introduction

For the Maxwell-Lorentz equations with a rotating charged particle (see Eqs. (2.3)-(2.6) below) Hamilton’s least action principle is justified in [3]. The main contribution of [3] is variational derivation of the Lorentz torque equation (2.6). While the equations (2.3)-(2.5) follow by standard Euler-Lagrange arguments, the Lorentz torque equation (2.6) follows by variational Poincaré equations [2, 6].

Our main result is a suitable generalization of the Noether theory of invariants to the Poincaré equations on the Lie groups. Moreover, we apply this generalization to a formal derivation of conservation laws for the Maxwell-Lorentz equations with a rotating charged particle. We show that the corresponding ”Poincaré invariants” coincide with classical known expressions considered in [4] where their conservation was shown by direct calculation.

We consider solutions for which all our formal differentiations and integration by parts hold true.

2 Maxwell-Lorentz equations

The Maxwell fields $E(x, t)$ and $B(x, t)$ are generated by motion of a rotating charge. External fields $E^{ext}$ and $B^{ext}$ are generated by the corresponding external charges and currents. Let the rotating charge be centered at the position $q$ with the velocity $\dot{q}$. For simplicity we assume that the mass distribution, $m \rho(x)$, and the charge distribution, $e \rho(x)$, are proportional to each other. Here $m$ is the total mass, $e$ is the total charge, and we use a system of units such that $m = 1$ and $e = 1$. The coupling function $\rho(x)$ is a sufficiently smooth radially symmetric function of fast decay as $|x| \to \infty$,

$$\rho(x) = \rho_0(|x|). \tag{C}$$

2.1 Angular velocity

Let us denote by $\omega(t) \in \mathbb{R}^3$ the angular velocity “in space” (in the terminology of [2]) of the charge. Namely, let us fix a “center” point $O$ of the rigid body. Then the trajectory of each fixed point of the body is described by

$$x(t) = q(t) + R(t)(x(0) - q(0)),$$

where $q(t)$ is the position of $O$ at the time $t$, and $R(t) \in SO(3)$. Respectively, the velocity reads

$$\dot{x}(t) = \dot{q}(t) + \dot{R}(t)(x(0) - q(0)) = \dot{q}(t) + \dot{R}(t)R^{-1}(t)(x(t) - q(t)) = \dot{q}(t) + \omega(t) \wedge (x(t) - q(t)), \tag{2.1}$$

where $\omega(t) \in \mathbb{R}^3$ corresponds to the skew-symmetric matrix $\dot{R}(t)R^{-1}(t)$ by the rule

$$\dot{R}(t)R^{-1}(t) = J\omega(t) := \begin{pmatrix}
0 & -\omega_3(t) & \omega_2(t) \\
\omega_3(t) & 0 & -\omega_1(t) \\
-\omega_2(t) & \omega_1(t) & 0
\end{pmatrix}. \tag{2.2}$$

We assume that $x$ and $q$ refer to a certain Euclidean coordinate system in $\mathbb{R}^3$, and the vector product $\wedge$ is defined in this system by standard formulas. The identification (2.2) of a skew-symmetric matrix and the corresponding angular velocity vector is true in any Euclidean coordinate system of the same orientation as the initial one.
2.2 Dynamical equations

Then the system of Maxwell-Lorentz equations with spin reads, see \cite{1,7}
\[
\begin{align*}
\dot{E} &= \nabla \wedge B - (\dot{q} + \omega \wedge (x - q)) \rho(x - q) \quad (a), \\
\dot{B} &= -\nabla \wedge E \quad (b), \\
\nabla \cdot E(x, t) &= \rho(x - q(t)) \quad (a), \\
\nabla \cdot B(x, t) &= 0 \quad (b), \\
\dot{q} &= \int [E + E^{\text{ext}} + (\dot{q} + \omega \wedge (x - q)) \wedge (B + B^{\text{ext}})] \rho(x - q) \, dx, \\
I \dot{\omega} &= \int (x - q) \wedge [E + E^{\text{ext}} + (\dot{q} + \omega \wedge (x - q)) \wedge (B + B^{\text{ext}})] \rho(x - q) \, dx,
\end{align*}
\]

where $I$ is the moment of inertia defined by
\[
I = \frac{2}{3} \int x^2 \rho(x) \, dx. 
\]

Here the equations \cite{2,3} are Maxwell equations with the corresponding charge density and current, equations \cite{2.4} are constraints. The back reaction of the field onto the particle is given through the Lorentz force equation \cite{2.5}, and the Lorentz torque equation \cite{2.6} deals with rotational degrees of freedom.

2.3 The variational Hamilton principle

Let us introduce electromagnetic potentials $A = (A_0, A)$, $A^{\text{ext}} = (A_0^{\text{ext}}, A^{\text{ext}})$:
\[
B = \nabla \wedge A, \quad E = -\nabla A_0 - \dot{A}.
\]

\[
B^{\text{ext}} = \nabla \wedge A^{\text{ext}}, \quad E^{\text{ext}} = -\nabla A_0^{\text{ext}} - \dot{A}^{\text{ext}}.
\]

Next we define the Lagrangian
\[
L(A, q, R, \dot{A}, \dot{q}, \dot{R}) = \frac{1}{2} \int \left( E^2(x) - B^2(x) \right) \, dx + \frac{1}{2} \dot{q}^2 + \frac{1}{2} I \omega^2 \\
- \int [A_0(x) + A_0^{\text{ext}}(x)] \rho(x - q) \, dx + \int (\dot{q} + \omega \wedge (x - q)) \cdot [A(x) + A^{\text{ext}}(x)] \rho(x - q) \, dx,
\]

where $E(x)$ and $B(x)$ are expressed in terms of $A(x)$ and $\dot{A}(x)$ according to \cite{2.8}, and $\omega = J^{-1} \dot{R} \dot{R}^{-1}$ by \cite{2.2}.

This Lagrangian functional depends on $R$ only through $\omega$ due to the spherical symmetry of the charge and mass distributions (C). Respectively, the dynamical equations \cite{2.3}–\cite{2.6} involve $R$ only through $\omega$ as well. On the other hand, in the case of non-radial densities the Lagrangian and the equations involve $R$ explicitly, and the moment of inertia $I$ becomes a matrix with $x \otimes x$ instead of $x^2$ in \cite{2.7}.

The corresponding action functional has the form
\[
S = S(A, q, R) := \int_{t_1}^{t_2} L(A(t), q(t), R(t), \dot{A}(t), \dot{q}(t), \dot{R}(t)) \, dt 
\]

Then the Hamilton’s least action principle reads
\[
\delta S(A, q, R) = 0, \quad (2.12)
\]
where the variation is taken over $A(t), q(t), R(t)$ with the boundary conditions

$$(\delta A, \delta q, \delta R)|_{t=t_1} = (\delta A, \delta q, \delta R)|_{t=t_2} = 0. \quad (2.13)$$

**Regular solutions and external potential.** Everywhere below we consider regular solutions to the system (2.3)–(2.6). This means that $q \in C^2(\mathbb{R}, \mathbb{R}^3)$, $\omega \in C^1(\mathbb{R}, \mathbb{R}^3)$, and all the involved functions and fields/potentials are sufficiently smooth and have (with all the necessary derivatives) a sufficient decay as $|x| \to \infty$ so that the partial integrations below are allowed.

In [3, Theorem 2.1] we have shown that for regular solutions, the Maxwell-Lorentz system (2.3)–(2.6) is equivalent to the least action principle (2.12)–(2.13) In detail, consider the variational equations

$$\frac{\delta S}{\delta A} = 0 \quad (a), \quad \frac{\delta S}{\delta q} = 0 \quad (b), \quad \frac{\delta S}{\delta R} = 0 \quad (c). \quad (2.14)$$

Then (2.14), (a), (b) are equivalent respectively to the standard Euler-Lagrange equations

$$\frac{d}{dt} \frac{\delta L}{\delta \dot{A}} = L_A \quad (a) \quad \frac{d}{dt} L_q = L_q \quad (b) \quad (2.15)$$

for the Lagrangian (2.10). Further, the equation (2.15), (a) is equivalent to the Maxwell equations (2.3) with the constraints (2.4), and the equation (2.15), (b) is equivalent to the Lorentz force equation (2.5).

Note that the equations (2.14), (a), (b) are equivalent to standard Euler-Lagrange equations (2.15) because the variables $A, \dot{A}, q, \dot{q}$ vary in the corresponding linear spaces. So, we will call these variables the “Lagrange variables”.

On the other hand, $R \in SO(3)$, and respectively, the variational equation (2.14) (c) cannot be transformed to a Euler-Lagrange equation since $SO(3)$ is not a linear space. We have shown in [3, Theorem 2.1] that (2.14) (c) is equivalent to the Lorentz torque equation (2.6) using the variational Poincaré equations with the Lagrangian $L$ expressed in suitable coordinates on $T SO(3)$.

In detail, consider an orthonormal basis $\{e_k\}$ with the right orientation in $\mathbb{R}^3$. Then

$$e_1 \land e_2 = e_3, \quad e_2 \land e_3 = e_1, \quad e_3 \land e_1 = e_2. \quad (2.16)$$

Let us express the angular velocity in $\{e_k\}$: $\omega(t) = \sum \omega_k(t)e_k$. The algebra $so(3)$ of skew-symmetric $3 \times 3$ matrices with the matrix commutator is isomorphic to the algebra $\mathbb{R}^3$ with the vector product, through the isomorphism $\mathcal{J}$ of (2.2):

$$\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} = \mathcal{J}(\omega_1, \omega_2, \omega_3). \quad (2.17)$$

Namely, let $A, B \in so(3), a, b \in \mathbb{R}^3$, and $A = \mathcal{J}a, B = \mathcal{J}b$. Then

$$AB - BA = \mathcal{J}(a \land b). \quad (2.18)$$

Further, $\dot{R} R^{-1} \in T_E SO(3)$ is the tangent vector $\dot{R}$ of $SO(3)$ at the point $R$ translated to the unit $E$ of $SO(3)$ by the right translation $R^{-1}$. By the linear isomorphism (2.17),

$$\dot{R} R^{-1} = \sum \omega_k \bar{e}_k, \quad \bar{e}_k := \mathcal{J} e_k. \quad (2.19)$$

Then

$$\dot{R} = \dot{R} R^{-1} R = \sum \omega_k v_k(R), \quad v_k(R) := \bar{e}_k R. \quad (2.20)$$
As the result,  \( \hat{R} \) has the same coordinates w.r.t. the vector fields \( v_k \) at the point \( R \) as \( \omega \) in the basis \( \{ e_k \} \). The fields \( v_k(R) \) are right translations of \( \hat{e}_k \) and hence are right-invariant.

In [3 Lemma 6.1] it is shown that for the vector fields \( v_k \) on \( SO(3) \) the following commutation relations hold:
\[
[v_1, v_2] = -v_3, \quad [v_2, v_3] = -v_1, \quad [v_3, v_1] = -v_2.
\] (2.21)

We will identify vector fields with the corresponding operators of differentiation. According to the Poincaré theory [2, 6], the equation (2.14) (c) is equivalent to the Poincaré equations
\[
\frac{d}{dt} \hat{L}_{\omega_k}(Y(t)) = \sum_{i,j} c_{ik}^j \omega_i(t) \hat{L}_{\omega_j}(Y(t)) + v_k \hat{L}(Y(t)), \quad k = 1, 2, 3,
\] (2.22)

where \( Y(t) := (A(t), q(t), \dot{A}(t), \dot{q}(t), \omega(t)) \) and \( \hat{L}(A, q, \dot{A}, \dot{q}, \omega) \) is defined as the right hand side of (2.10), and the constants \( c_{ik}^j \) arise from commutation relations
\[
[v_i, v_k](R) = \sum c_{ik}^j v_j(R).
\]

In Appendix A, we recall the calculation of the Poincaré equations (2.22). These calculations will be used throughout the paper.

Note that the Lagrangian \( \hat{L} \) does not depend explicitly on \( R \), and hence \( v_k(\hat{L}) = 0, k = 1, 2, 3 \). Then the corresponding Poincaré equations read
\[
\frac{d}{dt} \frac{\partial \hat{L}(\omega(t))}{\partial \omega_k} = \sum_{i,j} c_{ik}^j \omega_i(t) \frac{\partial \hat{L}(\omega(t))}{\partial \omega_j}, \quad k = 1, 2, 3.
\] (2.23)

In our case (2.21) and (A.6) imply
\[
c_{21}^3 = c_{32}^1 = c_{13}^2 = 1, \quad c_{31}^2 = c_{12}^3 = c_{23}^1 = -1, \quad \text{all the rest } c_{ik}^j = 0.
\]

Thus, we can rewrite (2.23) as
\[
\frac{d}{dt} \frac{\partial \hat{L}(\omega(t))}{\partial \omega} = \omega \wedge \frac{\partial \hat{L}(\omega(t))}{\partial \omega},
\] (2.24)

where \( \frac{\partial \hat{L}}{\partial \omega} \) is the column vector with the components \( \frac{\partial \hat{L}}{\partial \omega_k} \), \( k = 1, 2, 3 \).

We summarize the situation as follows, see [3]. The Lagrangian \( \hat{L} \) depends on two groups of variables: on the “Lagrangian variables” \( A, \dot{A}, q, \dot{q} \) and on the variables \( \omega_k \) which we will call the “Poincaré variables”. The variational equations (2.14) (a), (b) imply the Maxwell-Lorentz equations (2.3)–(2.5), while (2.14) (c) give the Lorentz torque equations (2.6).

## 3 Invariants for the Poincaré equations

When the external fields possess a symmetry with respect to the Lagrangian variables, the corresponding conservation laws are given by the Noether theorem on invariants [1]. In this section we extend the Noether theory to the Poincaré equations.

Let \( v_1(g), \ldots, v_n(g) \) be vector fields on an \( n \)-dimensional manifold \( M \) which are linearly independent at each point \( g \in M \). In particular such vector fields exist for any open region \( M \subset \mathbb{R}^n \). Then \( TM \) is isomorphic to \( M \times \mathbb{R}^n \), and any function \( L(g, \dot{g}) \) on \( TM \) can be expressed in the Poincaré variables \( g, \omega \):
\[
\hat{L}(g, \omega) := L(g, \dot{g}), \quad \dot{g} = \sum \omega_k v_k(g).
\] (3.1)
In [6], Poincaré discovered that the corresponding Hamilton least action principle is equivalent to the equations

\[
\frac{d}{dt} \hat{L}_{\omega_k}(g(t), \omega(t)) = \sum_{ij} c^j_{ik}(g) \dot{\omega}_i \hat{L}_{\omega_j}(g, \omega) + v_k(g) \hat{L}(g, \omega), \quad k = 1, \ldots, n. \tag{3.2}
\]

where the ”structure constants” \( c^j_{ik}(g) \) arise from commutation relations

\[
[v_i, v_j](g) = \sum c^k_{ij}(g) v_k(g), \quad g \in M.
\]

see details in Appendix A. Obviously, (2.22) is the particular case of (3.2).

Here we develop the corresponding theory of invariant for the Poincaré equations. Let us start with the energy conservation.

**Theorem 3.1** The ”energy”

\[
E := \hat{L}_{\omega} \cdot \omega - \hat{L} = \sum_k \hat{L}_{\omega_k} \omega_k - \hat{L} \tag{3.3}
\]

is conserved along the paths of the Poincaré equations (3.2).

**Proof** Let a smooth path \((g(t), \omega(t))\) satisfy Poincaré equations (3.2). Let us compute

\[
\frac{d}{dt}(\hat{L}_{\omega} \cdot \omega - \hat{L}) = \frac{d}{dt} \hat{L}_{\omega} \cdot \omega + \hat{L}_g \cdot \dot{\omega} - \hat{L}_\omega \cdot \dot{\omega} = \sum_k \frac{d}{dt} \hat{L}_{\omega_k} \omega_k - \hat{L}_g \cdot \dot{\omega} = \sum_k \sum_{ij} c^j_{ik}(g) \hat{L}_{\omega_i} + v_k(g) \hat{L}(g, \omega) - \hat{L}_g \cdot \dot{\omega}
\]

by (3.2). Note that \( \hat{L}_g \cdot \dot{\omega} = \hat{L}_g \cdot \sum_k \omega_k v_k = \sum_k \omega_k \hat{L}_g \cdot v_k = \sum_k v_k(\hat{L}) \omega_k \). Thus, we obtain

\[
\frac{d}{dt}(\hat{L}_{\omega} \cdot \omega - \hat{L}) = \sum_k \sum_{ij} c^j_{ik}(g) \hat{L}_{\omega_i} \omega_k = \sum_j \hat{L}_{\omega_j} \sum_k c^j_{ik}(g) \omega_k = 0, \tag{3.5}
\]

since \( \sum_k c^j_{ik}(g) \omega_k = 0 \) by skew-symmetry property \((A.1)\) of the coefficients \( c^j_{ik}(g) \). \( \square \)

**Remark 3.2** In the Lagrangian case (i.e., when \( M \) is a linear space and \( \omega = \dot{\omega} \)), the invariant (3.3) coincides with the standard energy functional.

Now let us consider general case of a one-parametric group of diffeomorphisms \( h^s : M \to M \) (in particular, \( h^0 = Id_M \)).

**Definition 3.3** The Poincaré invariant \( I \) and the corresponding ”current” \( w = (w_1, \ldots, w_n) \) are defined as

\[
I(g, \omega) := \sum \hat{L}_{\omega_k} w_k(g), \quad \frac{dh^s g}{ds} \bigg|_{s=0} = \sum w_k(g) v_k(g). \tag{3.6}
\]

These definitions generalize the corresponding Noether formulas \([1]\) to the case of Poincaré equations.

Let the Lagrangian \( L \) be invariant with respect to the diffeomorphisms \( h^s \), i.e.

\[
L(h^s g, dh^s \dot{g}) = L(g, \dot{g}), \quad (g, \dot{g}) \in TM, \quad s \in \mathbb{R}. \tag{3.7}
\]
Theorem 3.4 Let condition (3.7) hold. Then the function $I(g, \omega)$ is conserved along the paths of the Poincaré equations (3.2).

Proof Let a smooth path $(g(t), \omega(t))$ satisfy Poincaré equations (3.2). Let us denote $g(s, t) := h^s g(t)$ and write

$$\dot{g}(s, t) = dh^s \dot{g}(t) = \sum \omega_k(s, t)v_k(g(s, t)).$$

In particular, $g(0, t) = g(t)$ and $\dot{g}(t) := \sum \omega_k(t)v_k(g(t))$. By (3.7), the quantity

$$\dot{L}(g(s, t), \omega(s, t)) := L(g(s, t), \sum \omega_k(s, t)v_k(g(s, t))) = L(g(s, t), dh^s \dot{g}(t))$$

does not depend on $s$; here $\omega(s, t) = (\omega_1(s, t), ..., \omega_n(s, t))$. Denote by prime the derivative in $s$, and by dot the derivative in $t$. Then we obtain

$$0 = \frac{d}{ds} \dot{L}(g(s, t), \omega(s, t)) = \dot{L}_g \cdot g' + \sum \dot{L}_{\omega_k} \omega_k' = \dot{L}_g \cdot g' + \sum \dot{L}_{\omega_k} \left( \sum c_{ij}^k \omega_i \dot{L}_{\omega_k} w_j + \dot{w}_k \right) =: S \quad (3.8)$$

by the formula (A.2) of Appendix A. First we change the order of summation on the right-hand side:

$$S = \dot{L}_g \cdot g' + \sum \dot{L}_{\omega_k} \dot{w}_k + \sum \left( \sum c_{ij}^k \omega_i \dot{L}_{\omega_k} w_j \right). \quad (3.9)$$

Next we wish to evaluate the term $\sum c_{ij}^k \omega_i \dot{L}_{\omega_k}$ for $s = 0$. Namely, $g(0, t) = g(t)$ together with $\omega(0, t) = \omega(t)$ satisfy the Poincaré equations (3.2). Hence, for $s = 0$

$$S = \dot{L}_g \cdot g' + \sum \dot{L}_{\omega_k} \dot{w}_k + \sum \left( \frac{d}{dt} \dot{L}_{\omega_j} - v_j(\dot{L}) \right) w_j$$

$$= \sum \dot{L}_{\omega_k} \dot{w}_k + \sum \frac{d}{dt} \dot{L}_{\omega_j} \cdot w_j + \dot{L}_g \cdot g' - \sum v_j(\dot{L}) w_j. \quad (3.10)$$

However, the definition of the current $w$ in (3.6) implies that

$$\dot{L}_g \cdot g' - \sum v_j(\dot{L}) w_j = \sum \dot{L}_g \cdot v_j w_j - \sum v_j(\dot{L}) w_j = 0$$

Therefore, (3.10) gives

$$S = \sum \dot{L}_{\omega_k} \dot{w}_k + \sum \frac{d}{dt} \dot{L}_{\omega_k} \cdot w_k = \frac{d}{dt} \left( \sum \dot{L}_{\omega_k} w_k \right) = \dot{I}(t). \quad (3.11)$$

The proof is complete, since $S = 0$ by (3.8). □

Remark 3.5 Let $M = \mathbb{R}^n$ and the vector fields $v_k = \nabla g_k$ be the commuting fields of differentiations w.r.t. coordinates $g_k$. Then Poincaré equations (3.2) read as the Euler-Lagrange equations, and the Poincaré invariant (3.6) coincides with the Noether invariant $L_\dot{g} \cdot \left. \frac{dh^s \dot{g}}{ds} \right|_{s=0}$. 
4 Invariants for the Lagrange-Poincaré equations

Here we generalize the theory of the previous section to systems with the configuration space $Y \times M$, where $Y$ is a Hilbert space either of finite or infinite dimension, while $M$ is a finite-dimensional manifold endowed with the vector fields $v_k(g)$ as above. Then $TY \simeq Y \times Y$ and $TM \simeq M \times \mathbb{R}^n$.

Let $L(X, V, g, \dot{g})$ be a differentiable Lagrangian which is defined on $TY \times TM$. Let us define

$$\hat{L}(X, V, g, \omega) := L(X, V, g, \dot{g}), \quad \dot{g} = \sum \omega_k v_k(g).$$

(4.1)

Let a smooth path $(X(t), V(t)), g(t), \omega(t)$ satisfy standard Euler-Lagrange equations w.r.t. the variables $(X, V)$ and Poincaré equations w.r.t. the variables $(g, \omega)$:

$$\frac{d}{dt} \hat{L}_V = L_X,$$

$$\frac{d}{dt} \hat{L}_{\omega_k} = \sum c^j_{ik}(g) \omega_i \hat{L}_{\omega_j} + v_k(g) \hat{L}, \quad k = 1, ..., n.$$

(4.2)

**Theorem 4.1** Let (4.2) hold. Then the energy

$$E := \hat{L}_V \cdot V + \hat{L}_\omega \cdot \omega - \hat{L}$$

is conserved along the path.

**Proof** Differentiating formally, we get

$$\frac{d}{dt} \left( \hat{L}_V \cdot V + \hat{L}_\omega \cdot \omega - \hat{L} \right) =$$

$$= \left( \frac{d}{dt} \hat{L}_V (X, V) \cdot V - \hat{L}_X \cdot \dot{X} \right) + \left( \frac{d}{dt} \hat{L}_\omega (g, \omega) \cdot \omega - \hat{L}_g \cdot \dot{g} \right) = 0.$$

Indeed, here the first bracket of the last line vanishes by the first equation of (4.2). The second bracket vanishes by the second equation of (4.2) that follows by the calculations (3.4)–(3.5). $\square$

Further, consider a one-parametric group of diffeomorphisms

$$h^s : (X, g) \mapsto (h^s_1(X), h^s_2(g)).$$

(4.4)

Let us suppose that the Lagrangian functional is $h^s$-invariant, i.e.

$$L(h^s_1 X, dh^s_1 V, h^s_2 g, dh^s_2 \dot{g}) \equiv L(X, V, g, \dot{g}).$$

(4.5)

**Theorem 4.2** Let (4.2), (4.5) hold. Then the sum

$$\hat{L}_V \cdot \frac{dh^s_1 X}{ds} \bigg|_{s=0} + \sum \hat{L}_{\omega_k} w_k(g)$$

is conserved along the path.

**Proof** Let $X(s, t) := h^s_1 X$, $g(s, t) := h^s_2 g$, and let $\omega(s, t)$ be defined as above. Then $\dot{g}(s, t) = \sum k v_k(g(s, t)) \omega_k(s, t)$, and formally,

$$0 = \frac{d}{ds} \hat{L}(X(s, t), \dot{X}(s, t), g(s, t), \omega(s, t)) = \hat{L}_X \cdot X' + \hat{L}_\dot{X} \cdot \dot{X}(s, t)' + \hat{L}_g \cdot g' + \sum \hat{L}_{\omega_k} \omega_k'.$$

At $s = 0$, the sum of the first two terms reduces to $\frac{d}{dt}(\hat{L}_X \cdot \frac{dh^s_1 X}{ds} \bigg|_{s=0})$ like in the proof of the standard theorem on Noether invariants [1]. The sum of the last two terms transforms to $\frac{d}{dt}(\sum \hat{L}_{\omega_k} w_k(g))$ like in the proof of Theorem 3.3 (calculations (3.8)–(3.11)). $\square$
5 Conservation laws for Maxwell-Lorentz equations

We now apply the theory of Noether invariants and Poincare invariants for our system of Maxwell-Lorentz equations with rotating charge. As above, we denote

\[ \hat{L}(A, q, \dot{A}, \dot{q}, \omega) = L(A, q, R, \dot{A}, \dot{q}, \dot{R}) \] (5.1)

where \( \omega = (\omega_1, \omega_2, \omega_3) \) is defined by (2.20), i.e., \( \omega_k \) are coordinates of \( \dot{R} \) in the basis \( v_1(R), v_2(R), v_3(R) \); recall that \( \hat{L} \) does not depend explicitly on \( R \).

5.1 Energy

Let us note that \( L \) does not depend on \( \dot{A}_0 \). By Theorem 4.1, we come formally to the following statement:

**Corollary 5.1** Suppose \( A^\text{ext}_0 \) and \( A^\text{ext} \) do not depend on time. Then the functional

\[ E(A, q, \dot{A}, \dot{q}, R, \omega) := \hat{L}_\dot{A} \cdot \dot{A} + \hat{L}_\dot{q} \cdot \dot{q} + \hat{L}_\omega \cdot \omega - \hat{L} \] (5.2)

is conserved along the regular solutions of the Maxwell-Lorentz system (2.3)–(2.5).

5.2 Momentum

Let the external field

\[ A^\text{ext}(x) = (A^\text{ext}_0(x), A^\text{ext}(x)) \] do not depend on \( x_k \) for some \( k \). (5.3)

Then the Lagrangian (5.1) is invariant w.r.t to the one-parametric group of spatial translations

\[ h^k_s(A(x), q) = (A(x - s e_k), q + s e_k), \] (5.4)

where \( e_k \in \mathbb{R}^3 \) is the corresponding basis vector. Since the group acts only on the Lagrange coordinates \( X := (A, q), V := (\dot{A}, \dot{q}) \), we may formally apply the Noether theory \[1, 5\] and obtain

**Corollary 5.2** Under the condition (5.3) the functional

\[ P_k = P_k(X, V, R, \omega) := \hat{L}_V \cdot \frac{dh^k_s X}{ds} \bigg|_{s=0} \] (5.5)

is conserved for regular solutions to the Maxwell-Lorentz system (2.3)–(2.5).

**Definition 5.3** \( P_k \) is called \( k \)-th component of momentum of the state \( (X, V, R, \omega) \).

5.3 Angular momentum

Let the external potential \( A^\text{ext} \) be axially symmetric,

\[ A^\text{ext}_0(U_k x) = A^\text{ext}_0(x), \quad A^\text{ext}(U_k x) = U_k A^\text{ext}(x), \] (5.6)

where \( U_k \) is any rotation around the axis \( O x_k \).
Lemma 5.4 Let (5.6) hold. Then the Lagrangian (2.10) is invariant w.r.t. the axial rotations

\[ A_0(x) \mapsto A_0(U_k^{-1}x), \quad A(x) \mapsto U_kA(U_k^{-1}x), \quad \hat{A}(x) \mapsto U_k\hat{A}(U_k^{-1}x), \quad (5.7) \]

\[ R \mapsto U_kR, \quad \hat{R} \mapsto U_k\hat{R}, \quad (5.8) \]

\[ q \mapsto U_kq, \quad \hat{q} \mapsto U_k\hat{q}. \quad (5.9) \]

**Proof** By (2.8) the transforms (5.7) of the potentials induce the following transforms of the fields:

\[ E(x) \mapsto U_kE(U_k^{-1}x), \quad B(x) \mapsto U_kB(U_k^{-1}x). \quad (5.10) \]

Further, we have, in operator notations, \( J\omega = \omega \wedge \), where \( \omega \wedge \) is the operator of the vector product by \( \omega \) in \( \mathbb{R}^3 \). Then it is easy to check that \( J(U_k\omega) = U_kJ(\omega)U_k^{-1} \). Thus, for \( \omega = J^{-1}\hat{R}R^{-1} \) we obtain \( J\omega = \hat{R}R^{-1} \) and hence \( J(U_k\omega) = U_k(J\hat{R}R^{-1})U_k^{-1} = (U_k\hat{R})(U_kR)^{-1} \). Finally,

\[ U_k\omega = J^{-1}(U_k\hat{R})(U_kR)^{-1}. \]

This means that the transforms (5.8) induce the following transform of \( \omega \):

\[ \omega \mapsto U_k\omega. \quad (5.11) \]

Now it is easy to check, in view of axial symmetry of \( \mathcal{A}^{\text{ext}} \), the invariance of \( \hat{L} \) w.r.t. the transforms (5.10), (5.9), (5.11), since \( \rho \) is spherically symmetric.  \( \square \)

Recall that \( \tilde{e}_k \) is the image of the basis vector \( e_k \) w.r.t. the isomorphism (2.17). By Lemma 5.4 the Lagrangian \( \hat{L} \) (5.1) is invariant w.r.t. the spatial rotations (5.9), (5.10), (5.11). In particular, \( \hat{L} \) is invariant under the transform group \( h_k^\# = e^{\tilde{e}_k} \in SO(3) \).

In detail, we have the situation of previous section when \( \hat{L} \) depends on Lagrangian variables \( (X; V) = (\mathcal{A}, q; \dot{\mathcal{A}}, \dot{q}) \) and on Poincaré variables \( (R, \omega) \). The action of this group on the state \( (X, R) \) reads

\[ h_k^\#(X, R) = (\alpha_k^\#X, \beta_k^\#R) : \quad \alpha_k^\#X = (A_0(e^{-\tilde{e}_k}x), e^{\tilde{e}_k}A(e^{-\tilde{e}_k}x), e^{\tilde{e}_k}q); \quad \beta_k^\#R = e^{\tilde{e}_k}R. \]

The currents \( w_1^k(R), w_2^k(R), w_3^k(R) \) are defined from

\[ \left. \frac{d\beta_k^\#R}{ds} \right|_{s=0} = \sum_{j=1}^3 w_j^k(R)v_j(R), \quad R \in SO(3). \quad (5.12) \]

Hence, by Theorem 4.2 we come to the following statement:

**Corollary 5.5** Under the condition (5.6) the quantity

\[ M_k = M_k(X, V, R, \omega) := \hat{L}_V \cdot \left. \frac{d\alpha_k^\#X}{ds} \right|_{s=0} + \sum_{j=1}^3 \hat{L}_{\omega_j}w_j^k(R) \quad (5.13) \]

is conserved for regular solutions to the Maxwell-Lorentz system (2.3)–(2.9).

**Definition 5.6** \( M_k \) is called \( k \)-th component of angular momentum of the state \( (X, V, R, \omega) \).
6 Expressions for energy and momenta

Let us show that the Poincaré invariants from previous section coincide with classical known expressions considered in [4] (where their conservation was shown by direct calculation).

Proposition 6.1 The invariants for the Maxwell-Lorentz system (2.3)–(2.5) read as follows:

(i) The energy reads

\[ E = \frac{1}{2} \int (|E(x)|^2 + |B(x)|^2) \, dx + \frac{1}{2} q^2 + \frac{1}{2} I \omega^2 + \int A_0^\text{ext}(x) \rho(x-q) \, dx. \]  

(ii) The momentum reads

\[ P = \dot{q} + \int E(x) \land B(x) \, dx + \int A^\text{ext}(x) \rho(x-q) \, dx. \]

(iii) The angular momentum reads

\[ M = q \land \dot{q} + I \omega + \int x \land E(x) \land B(x) \, dx + \int x \land A^\text{ext}(x) \rho(x-q) \, dx. \]

Proof

(i) By (5.1) and (2.10), one has

\[ \hat{\mathcal{L}}_A \cdot \dot{A} = \int E \cdot \dot{A} \, dx, \quad \hat{\mathcal{L}}_q \cdot \dot{q} = \frac{1}{2} q^2 + \frac{1}{2} I \omega^2 + \int (|B|^2 - |E|^2) \, dx \]

\[ + \int (-E \cdot \dot{A} + A_0 \rho(x-q)) \, dx + \int A_0^\text{ext} \rho(x-q) \, dx. \]  

Then

\[ E = \hat{\mathcal{L}}_A \cdot \dot{A} + \hat{\mathcal{L}}_q \cdot \dot{q} + \hat{\mathcal{L}}_\omega \cdot \dot{\omega} = \frac{1}{2} q^2 + \frac{1}{2} I \omega^2 + \int (|B|^2 - |E|^2) \, dx \]

\[ + \int (-E \cdot \dot{A} + A_0 \rho(x-q)) \, dx + \int A_0^\text{ext} \rho(x-q) \, dx. \]  

Since

\[ \int (-E \cdot \dot{A} + A_0 \rho(x-q)) \, dx = \int (-E \cdot \dot{A} + A_0 \cdot \nabla E) \, dx \]

\[ = - \int E(\dot{A} - \nabla A_0) \, dx = \int E^2 \, dx, \]

formula (6.4) reads (6.1).

(ii) Let us compute \( P_j \). Formula (5.4) implies

\[ \frac{dh_j^S(X)}{ds} \bigg|_{s=0} = -(e_j \cdot \nabla A(x), \ e_j). \]

Then

\[ P_j = L_V \cdot \frac{dh_j^S(X)}{ds} \bigg|_{s=0} = -L_A \cdot (e_j \cdot \nabla)A + L_\dot{q} \cdot e_j \]

\[ = - \int (\nabla A_0 + \dot{A}) \cdot (e_j \cdot \nabla)A \, dx + \dot{q} \cdot e_j + \int e_j \cdot A \rho(x-q) \, dx + \int A_j^\text{ext} \rho(x-q) \, dx \]

\[ = \dot{q}_j + \int A_j \rho(x-q) \, dx - \int (\nabla A_0 + \dot{A}) \cdot \partial_j A \, dx + \int A_j^\text{ext} \rho(x-q) \, dx. \]  

(6.6)
By partial integration
\[ \int A_j(x) \rho(x - q) \, dx = \int A_j(\nabla \cdot E) \, dx = \int A_j \nabla \cdot (-\nabla A_0 - \dot{A}) \, dx \]
\[ = \int A_j(-\Delta A_0 - \nabla \dot{A}) \, dx = \int (\nabla A_0 \cdot \nabla A_j + (\dot{A} \cdot \nabla) A_j) \, dx. \]

Hence,
\[ P_j = \dot{q}_j + \int (\nabla A_0 \cdot \nabla A_j + (\dot{A} \cdot \nabla) A_j) \, dx - \int (\nabla A_0 \cdot \partial_j A + \dot{A} \cdot \partial_j A) \, dx + \int A_j^{\text{ext}} \rho(x - q) \, dx. \quad (6.7) \]

On the other hand, the j-th component of the RHS of (6.2) equals
\[ \dot{q}_j + \int (E \wedge B)_j \, dx + \int A_j^{\text{ext}} \rho(x - q) \, dx. \]

Insert \( E = -\dot{A} - \nabla A_0, B = \nabla \wedge A \) and obtain
\[ \dot{q}_j + \int A_j^{\text{ext}} \rho(x - q) \, dx + \int \left((\dot{A} \cdot \nabla) A_j - \dot{A} \cdot \partial_j A + \nabla A_0 \cdot \nabla A_j - \nabla A_0 \cdot \partial_j A \right) \, dx \]
which coincides with (6.7).

iii) For concreteness let us compute \( M_1 \). Then
\[ \alpha_1^s(X) = (A_0(e^{-s\bar{e}_1} x), e^{s\bar{e}_1} A(e^{-s\bar{e}_1} x), e^{s\bar{e}_1} q). \]

One has
\[ \frac{d\alpha_1^s X}{ds} \bigg|_{s=0} = (-\bar{e}_1 e^{-s\bar{e}_1} x \cdot \nabla) A_0(e^{-s\bar{e}_1} x), \bar{e}_1 e^{s\bar{e}_1} A(e^{-s\bar{e}_1} x) + e^{s\bar{e}_1} (-\bar{e}_1 e^{-s\bar{e}_1} x \cdot \nabla) A(e^{-s\bar{e}_1} x), \bar{e}_1 e^{s\bar{e}_1} q) \bigg|_{s=0} \]
\[ = (\bar{e}_1 A_0(x), \bar{e}_1 A(x) - (\bar{e}_1 x \cdot \nabla) A(x), \bar{e}_1 q). \]

Further,
\[ \frac{d\beta^s R}{ds} \bigg|_{s=0} = \frac{d e^{s\bar{e}_1} R}{ds} \bigg|_{s=0} = \bar{e}_1 R = v_1(R) \]
by definition (2.20) of the fields \( v_k(R) \). Hence, for the currents \( w_j^1 \) of (5.12) we have \( w_1^1 = 1, w_2^1 = w_3^1 = 0 \). Then, since \( \hat{L} \) does not depend on \( \dot{A}_0 \),
\[ M_1 = \hat{L}_A \cdot (\bar{e}_1 A(x) - (\bar{e}_1 x \cdot \nabla) A(x)) + \hat{L}_{\dot{q}} \cdot (\bar{e}_1 q) + \hat{L}_{\omega_1} \]
\[ = \int \left( \hat{L}_A \cdot (\bar{e}_1 A(x) - (\bar{e}_1 x \cdot \nabla) A(x)) + \nabla A_0 \cdot (\bar{e}_1 A(x) - (\bar{e}_1 x \cdot \nabla) A(x)) \right) \, dx \]
\[ + \dot{q} \cdot (\bar{e}_1 q) + \int (\bar{e}_1 q) \cdot (A + A^{\text{ext}}) \rho(x - q) \, dx + I \omega \cdot e_1 + \int (e_1 \wedge (x - q)) \cdot [A + A^{\text{ext}}] \rho(x - q) \, dx \]
\[ = (q \wedge \dot{q})_1 + I \omega_1 + \int (x_2 A_3^{\text{ext}} - x_3 A_2^{\text{ext}}) \rho(x - q) \, dx \]
\[ + \int (x_2 A_3 - x_3 A_2) \rho(x - q) \, dx + \int (\dot{A} + \nabla A_0) \cdot ((0, -A_3, A_2) + (x_3 \partial_2 - x_2 \partial_3) A) \, dx. \quad (6.8) \]
We have to prove that this expression equals to the first component of the RHS of (6.3). It suffices to prove that the last line (6.8) equals to the first component of \( \int x \wedge (E \wedge B) \, dx \). Indeed, 
\[
\rho(x - q) = \nabla \cdot E = \nabla \cdot (-\nabla A_0 - \mathring{A}),
\]
hence
\[
\int (x_2 A_3 - x_3 A_2) \rho(x - q) \, dx = \int (x_2 A_3 - x_3 A_2) (-\nabla \mathring{A} - \nabla^2 A_0) \, dx = \int \nabla (x_2 A_3 - x_3 A_2) (\mathring{A} + \nabla A_0) \, dx. \tag{6.9}
\]
Then the line (6.8) transforms to
\[
\int \left( \partial_1 (x_2 A_3 - x_3 A_2) (\mathring{A}_1 + \partial_1 A_0) + x_2 \partial_2 A_3 (\mathring{A}_2 + \partial_2 A_0) - x_3 \partial_3 A_2 (\mathring{A}_3 + \partial_3 A_0) \right) \, dx
\]
\[
+ \int \left( (x_3 \partial_2 - x_2 \partial_3) A_1 (\mathring{A}_1 + \partial_1 A_0) - x_2 \partial_3 A_2 (\mathring{A}_2 + \partial_2 A_0) + x_3 \partial_2 A_3 (\mathring{A}_3 + \partial_3 A_0) \right) \, dx. \tag{6.10}
\]
On the other hand, substitute \( E = -\mathring{A} - \nabla A_0, B = \nabla \wedge A \) and obtain that the first component of \( \int x \wedge (E \wedge B) \, dx \) equals
\[
\int x_2 (\partial_1 A_3 - \partial_3 A_1) (\mathring{A}_1 + \partial_1 A_0) + (\partial_2 A_3 - \partial_3 A_2) (\mathring{A}_2 + \partial_2 A_0) \, dx
\]
\[
- \int x_3 (\partial_3 A_2 - \partial_2 A_3) (\mathring{A}_3 + \partial_3 A_0) + (\partial_1 A_2 - \partial_2 A_1) (\mathring{A}_1 + \partial_1 A_0) \, dx
\]
which coincides with (6.10). The proof is complete. \( \square \)

### A Poincaré equations

Poincaré suggested the form of the Hamilton least action principle for Lagrangian systems on manifolds [6]. We present the derivation of the Poincaré equations [2] since we use some of intermediate calculations.

Let \( v_1, \ldots, v_n \) be vector fields on a \( n \)-dimensional manifold \( M \) which are linearly independent at every point \( g \in M \). Then the commutation relations hold,
\[
[v_i, v_j](g) = \sum c^k_{ij}(g) v_k(g), \quad g \in M
\]
where the commutator \([v_i, v_j]\) is defined by
\[
[v_i, v_j](f) := v_i(v_j(f)) - v_j(v_i(f)),
\]
and \( v(f) \) is the derivative of a smooth function \( f \) on \( M \) w.r.t. the vector field \( v \). Note that by the skew-symmetry property of the commutators one has
\[
c^k_{ij}(g) = -c^k_{ji}(g), \quad \forall \, k = 1, \ldots, n. \tag{A.1}
\]
If \( g(t) \) is a smooth path in \( M \) and \( f \) is a smooth function on \( M \), one has \( \dot{g}(t) = \sum \omega_i(t) v_i(g(t)) \) and
\[
\frac{d}{dt} f(g(t)) = f'(g(t)) \cdot \dot{g} = f'(g(t)) \cdot \sum \omega_i(t) v_i(g(t)) = \sum v_i(f) \omega_i(t).
\]
Now consider a variation \( g(s, t) \) of the path \( g(t) \). Then similarly,
\[
\partial_s f(g(s, t)) = \sum_j v_j(f) w_j(s, t),
\]
where \( w_j(s, t) \) are coordinates of \( \frac{\partial w}{\partial s}(s, t) \in T_{g(s, t)}M \). Hence

\[
\partial_s \partial_t f(g(s, t)) = \sum_i \sum_j v_j(v_i(f))w_j\omega_i + \sum_i v_i(f)\omega'_i,
\]

\[
\partial_t \partial_s f(g(s, t)) = \sum_j \sum_i v_i(v_j(f))w_j\omega_i + \sum_j v_j(f)\dot{w}_j,
\]

where the prime respectively the dot stand for the differentiation in \( s \) respectively in \( t \). However, the differentiations in \( t \) and \( s \) commute, hence we obtain by subtraction

\[
\sum_{k} v_k(f)\omega'_{k} = \sum_{k} \sum_{ij} c_{ij}^{k}\omega_{i}w_{j}(f) + \sum_{k} v_k(f)\dot{w}_k.
\]

Since \( f \) is an arbitrary smooth function, we come to the relations

\[
\omega'_{k}(s, t) = \sum_{ij} c_{ij}^{k}\omega_{i}w_{j} + \dot{w}_k.
\]

Further, let us consider a Lagrangian function \( L(g, \dot{g}) \) on \( TM \). Then \( L(g, \dot{g}) \) can be expressed in the variables \( \omega: L(g, \dot{g}) = \hat{L}(g, \omega) \). Let us compute the variation of the corresponding action functional taking (A.2) into account:

\[
\frac{d}{ds} \int_{t_1}^{t_2} \hat{L}(g(s, t), \omega(s, t)) dt = \int_{t_1}^{t_2} \left( \sum_{k} \frac{\partial \hat{L}}{\partial \omega_k} \omega'_{k} + \nabla_g \hat{L} \cdot g' \right) dt = \\
\int_{t_1}^{t_2} \left[ \sum_{k} \frac{\partial \hat{L}}{\partial \omega_k}(w_k + \sum_{ij} c_{ij}^{k}\omega_{i}w_{j}) + \nabla_g \hat{L} \cdot \sum_{k} w_kv_k \right] dt = \\
\sum_{k} \frac{\partial \hat{L}}{\partial \omega_k} w_k|^{t_2}_{t_1} + \int_{t_1}^{t_2} \sum_{k} \left[ -\frac{d}{dt} \frac{\partial \hat{L}}{\partial \omega_k} + \sum_{ij} c_{ij}^{k}\omega_{i} \frac{\partial \hat{L}}{\partial \omega_j} + v_k(\hat{L}) \right] w_k dt.
\]

The variation should be zero by the Hamilton least action principle, under the boundary value conditions

\[
g(s, t_1) = g_1, \quad g(s, t_2) = g_2. \tag{A.3}
\]

Since \( w_k(t_1) = w_k(t_2) = 0 \) by (A.3), we obtain the following Poincaré equations:

\[
\frac{d}{dt} \frac{\partial \hat{L}}{\partial \omega_k} = \sum_{ij} c_{ij}^{k}\omega_{i} \frac{\partial \hat{L}}{\partial \omega_j} + v_k(\hat{L}). \tag{A.4}
\]

**Remarks**

1. If \( g \) is expressed in a local map as \((g_1, ... , g_n) \in \mathbb{R}^n\), and \( v_k = \partial g_k \), then (A.4) reduce to the standard Euler-Lagrange equations.
2. If a Lagrangian \( L \) does not depend on \( g \), then \( \hat{L} = \hat{L}(\omega) \) and one has

\[
v_k(\hat{L}) = 0. \tag{A.5}
\]

Indeed, \( v_k(\hat{L}) = \nabla_g \hat{L} \cdot v_k(g) = 0 \).

3. Suppose \( M = G \) is a Lie group, and let \( v_k, k = 1, ..., n \) be independent either left-invariant or right-invariant vector fields on \( G \). Then \( c_{ij}^{k}(g) \) are constant:

\[
c_{ij}^{k}(g) \equiv c_{ij}^{k}, \quad g \in G. \tag{A.6}
\]

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