Higher order curvature generalisations of Bartnick-McKinnon and coloured black hole solutions in $d = 5$

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Abstract

We construct globally regular as well as non-abelian black hole solutions of a higher order curvature Einstein-Yang-Mills (EYM) model in $d = 5$ dimensions. This model consists of the superposition of the first two members of the gravitational hierarchy (Einstein plus first Gauss-Bonnet(GB)) interacting with the superposition of the first two members of the $SO(d)$ Yang–Mills hierarchy.

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I. INTRODUCTION

In a previous paper [1] we discussed regular solutions to the $d$ dimensional Einstein–Yang-Mills (EYM) system consisting of the first two members of each hierarchy in increasing orders of the curvatures, the YM fields with gauge group $SO(d)$. Concretely, we constructed these solutions for $d = 6, 7$ and 8. In [1] we restricted to those dimensions in which the solutions constructed had nontrivial flat-space limits. Due to Derrick’s scaling requirement, $d = 5$ was excluded. Here we consider this system in dimension $d = 5$, and in addition to regular ones, we construct also black hole solutions.

It is well known that Derrick’s theorem forbids static, globally regular, finite energy solutions in pure YM theory, and that pure Einstein gravity has also no globally regular solutions. Finite energy solutions in the combined EYM system however, are not obstructed by this scaling requirement and indeed spherically symmetric particle-like solutions in this system in $d = 4$ dimensions [2] were discovered. In fact, Bartnik and McKinnon discovered an infinite series of solutions indexed by the number $k$ of nodes of the gauge field function. The energy of these solutions is however not bounded from below by any topological charge, and as expected are in this sense sphaleron- like [3]. Shortly after this discovery, the corresponding black hole solutions, so-called “coloured black holes” where constructed [4–6]. These provided one of the first examples of the fact that the famous “No-hair” conjecture cannot be generalised to theories which involve non-linear matter sources. In the years that followed, solutions of $d = 4$ EYM theory have been studied in great detail (see [7] for a review).

Field theories in more than 4 dimensions gained increasing interest in past years since it now seems accepted that many aspects of field theory depend on the higher (than four) dimensional aspects of some of these theories. Superstring theory [8], which is considered to be a good candidate for a “Theory of Everything” (TOE), is only consistent in $d = 10$. In the low energy effective action limit of (Super)string theories, new terms appear. In the gravity sector, these are the Gauss-Bonnet terms, while e.g. for Yang-Mills fields a hierarchy appears, which results from the replacement of the standard Yang-Mills action by the corresponding Born-Infeld action.

Recently moreover, the Randall-Sundrum (RS) models [9,10] have pushed forward the study of solutions in dimensions $d > 4$ even further. These models involve either two (RSI) [9] or one (RSII) [10] 3-brane(s) in a 5-dimensional space-time. While gravity “lives” in the full 5 dimensions, the matter fields are confined to the brane.

Consequently, Einstein-Yang-Mills solutions in $d = 5$ have been studied [11,12]. In [11] it was found that no static, globally regular, finite energy solutions exist in an asymptotically flat space-time. Only if $\partial/\partial x_4$ with $x_4$ being the coordinate of the 5th dimension is chosen to be a Killing vector of the model, finite energy solutions are possible. In [12] globally regular as well as non-abelian black hole solutions of the $d = 5$ EYM model in asymptotically flat, in de Sitter (dS) and in Anti-de Sitter (AdS) space-time have been constructed. It was found that –like in asymptotically flat space- non globally regular, finite energy solutions are possible in a space-time with negative cosmological constant (AdS space).

We give the model, ansatz, classical field equations and boundary conditions for generic $d$ in Section II. We discuss Reissner-Nordström-like solutions in Section III, the generalised Bartnik-McKinnon solutions in Section IV and the generalised coloured black hole solutions.
in Section V. We give our conclusions in Section VI.

II. THE MODEL

We consider here the Lagrangian employed in [1], containing only the first two terms of both the gravitational and Yang-Mills hierarchies, namely

\[ \mathcal{L} = \mathcal{L}_{\text{grav}} + \mathcal{L}_{\text{YM}} \]  (1)

with

\[ \mathcal{L}_{\text{grav}} = \sqrt{-g} \left( \frac{k_1}{2} R(1) + \frac{k_2}{4} R(2) \right) \]  (2)

and

\[ \mathcal{L}_{\text{YM}} = \sqrt{-g} \left( \frac{\tau_1}{4} \text{Tr}F^2(2) + \frac{\tau_2}{48} \text{Tr}F^2(4) \right). \]  (3)

\( g \) denotes the determinant of the \( d \)-dimensional metric tensor, \( R(1) \) is the standard Ricci scalar appearing in the Einstein-Hilbert action, while \( R(2) \) is the 2-Ricci scalar associated with the first Gauss-Bonnet (GB) term. \( F(2) \) denotes the standard Yang-Mills 2-form, while \( F(4) \) is the Yang-Mills 4-form resulting from the total antisymmetrisation of \( F(2) \) (see [1] for details).

Though we are primarily interested in \( d = 5 \) in this paper, we give the ansatz, the equations and the boundary conditions in this section for generic \( d \). This is because, before proceeding to construct solutions numerically in Sections IV and V in \( d = 5 \), we will consider the question of Reissner-Nordström-like solutions in section III, for generic \( d \).

A. The Ansatz

The choice of gauge group is somewhat flexible. In [1] the gauge group chosen was \( SO(d) \), in \( d \) dimensions. But the gauge field of the static solutions in question took their values in \( SO(d-1) \). Thus in effect, it is possible to choose \( SO(d) \) in the first place. Now for even \( d \), it is convenient to choose \( SO(d) \) since we can then avail of the chiral representations of the latter, although this is by no means obligatory. Adopting this criterion, namely to employ chiral representations, also for odd \( d \), it is convenient to choose the gauge group to be \( SO(d-1) \). We shall therefore denote our representation matrices by \( SO_\pm(\tilde{d}) \), where \( \tilde{d} = d \) and \( d = d-1 \) for even and odd \( d \) respectively.

In this unified notation (for odd and even \( d \)), the spherically symmetric Ansatz for the \( SO_\pm(\tilde{d}) \)-valued gauge fields then reads:

\[ A_0 = 0, \quad A_i = \left( \frac{w(r) - 1}{r} \right) \Sigma_{ij}^{(\pm)} \hat{x}_j, \quad \Sigma_{ij}^{(\pm)} = -\frac{1}{4} \left( \frac{1 \pm \Gamma_{\tilde{d}+1}}{2} \right) [\Gamma_i, \Gamma_j]. \]  (4)

The \( \Gamma \)'s denote the \( \tilde{d} \)-dimensional gamma matrices and \( 1, j = 1, 2, ..., d-1 \) for both cases.

For the metric, we choose the Schwarzschild-type spherically symmetric Ansatz:
\[ ds^2 = -\sigma^2(r)N(r)dt^2 + N^{-1}(r)dr^2 + r^2d\Omega_{d-2}^2 \]  

where \( \sigma \) and \( N \) are only functions of the \( d-2 \) dimensional radial coordinate \( r \). The consistency of this Ansatz for this system has been checked [1]. The function \( N(r) \) is related to the mass function \( m(r) \) in the following way:

\[
m(r) = n_d^{-1}[\kappa_1 r^{d-3}(1 - N) + \frac{1}{4}\kappa_2 r^{d-5}(1 - N)^2],
\]

where \( n_d \) is the dimension of the \( \Gamma \) matrices entering in (4). \( n_d = 2^{(d-2)/2} \) for \( d \) even and \( n_d = 2^{(d-1)/2} \) for \( d \) odd. For \( \kappa_2 = 0 \) and \( d = 4 \) (6) reduces to the standard definition of the mass function \( m(r) \).

### B. The classical field equations

Inserting the Ansatz into the action and varying with respect to the matter field \( w(r) \) and the metric fields \( N(r), \sigma(r) \), we obtain the classical field equations:

\[
2\tau_1 \left( r^{d-4}\sigma N w' \right)' - (d - 3)r^{d-6}\sigma(w^2 - 1)w + 3\tau_2(d - 3)(d - 4)(w^2 - 1) \left( r^{d-8}\sigma N (w^2 - 1)w' \right)' - (d - 5)r^{d-10}\sigma(w^2 - 1)^2w = 0 \tag{7}
\]

\[
m' = \frac{1}{8} r^{d-4} \left( \tau_1 \left[ N w'^2 + \frac{1}{2}(d - 3) \left( \frac{w^2 - 1}{r} \right)^2 \right] + \frac{3}{2}\frac{\tau_2}{r^2}(d - 3)(d - 4) \left( \frac{w^2 - 1}{r} \right)^2 \left[ N w'^2 + \frac{1}{4}(d - 5) \left( \frac{w^2 - 1}{r} \right)^2 \right] \right) \tag{8}
\]

\[
\left[ \kappa_1 + \frac{\kappa_2}{2r^2}(d - 3)(d - 4)(1 - N) \right] \left( \frac{\sigma'}{\sigma} \right) = \frac{n_d}{8r} \left[ \tau_1 + \frac{3}{2}\frac{\tau_2}{r^2}(d - 3)(d - 4) \left( \frac{w^2 - 1}{r} \right)^2 \right] w'^2. \tag{9}
\]

For \( \kappa_2 = \tau_2 = 0 \) and \( d = 4 \), (7)-(9) coincide with the ordinary differential equations of [2,7].

### III. REISSNER-NORDSTÖRM-LIKE SOLUTIONS

For \( d = 4 \) the Reissner-Nordström (RN) solution is a solution of the Einstein-Yang-Mills equations [15] with \( w \equiv 0 \) and \( \sigma \equiv 1 \). Interestingly, it was found that the RN solution is the limiting solution of the sequence of Bartnik-McKinnon (BM) solutions for the number of gauge field nodes going to infinity. Thus, a charged solution (the RN solution) represents the limiting solution of a sequence of non-charged solutions (the BM solutions). However, it should be noted that through suitable rescalings in the \( d = 4 \) EYM system, the only parameter that can be changed for spherically symmetric solutions is the number of nodes.
of the gauge field. In this paper, we have additional parameters, which result from the inclusion of higher order curvature and Yang-Mills hierarchy terms. Here we study whether the RN solution is still a possible solution of the generalised EYM equations.

In this Section, we give an answer to this question, for generic $d$. For our considerations, it is immaterial how many GB terms the gravitational sector contains. What is important is the YM content of increasing orders. For this purpose, it is useful to consider the most general version of equation (8), but in practice we restrict to the case with the first three members of the YM hierarchy, from which it is easy to conclude the maximal superposition of YM terms.

The generalisation of (8) for the case with YM content up to the $p = 3$ member is

$$m' = \frac{1}{8} r^{d-2} \left( \tau_1 \left[ N \left( \frac{w'}{r} \right)^2 + \frac{1}{2} (d-3) \left( \frac{w^2-1}{r^2} \right)^2 \right] + \frac{3}{2} \tau_2 (d-3)(d-4) \left( \frac{w^2-1}{r^2} \right)^2 \left[ N \left( \frac{w'}{r} \right)^2 + \frac{1}{4} (d-5) \left( \frac{w^2-1}{r^2} \right)^2 \right] + \frac{147}{160} \tau_3 (d-3)(d-4)(d-5)(d-6) \left( \frac{w^2-1}{r^2} \right)^4 \left[ N \left( \frac{w'}{r} \right)^2 + \frac{1}{6} (d-7) \left( \frac{w^2-1}{r^2} \right)^2 \right] \right)$$

in which the mass function $m(r)$ can incorporate all possible GB terms allowed in the given dimensions [13].

Setting $w(r) \equiv 0$ in (10), and then integrating it, we find for $d = 5$:

$$m(r) = \frac{\tau_1}{8} \ln r + C_5$$

while for $d = 9$, we have:

$$m(r) = \frac{3}{32} \tau_1 r^4 + \frac{45}{8} \tau_2 \ln r - \frac{441}{128} \tau_3 r^{-4} + C_9$$

and for $d = 13$:

$$m(r) = \frac{5}{16} \tau_1 r^8 + \frac{135}{4} r^4 + \frac{9261}{16} \ln r + C_{13}$$

with $C_5$, $C_9$ and $C_{13}$ being the respective integration constants. Clearly, for $d = 5$, 9, 13, these solutions have infinite energy and are not regular at the origin. A $d = 5$ solution with logarithmically divergent mass was also found in [12] and was named “quasi-asymptotically” flat.

We can look for horizons of these solutions, at which $N(r = r_h) = 0$. For $d = 5$ and $\kappa_2 = 0$ (which is the case, we are interested in mainly in this paper), we then have from (6)

$$N(r_h = 0) \Rightarrow \frac{n_d}{\kappa_1 r^2} \left( \frac{\tau_1}{8} \ln(r) + C_5 \right) = 1$$

By maximal superposition we mean the superposition of $F(2p)^2$ terms for up to $p = \frac{d}{2}$ for even $d$ and $p = \frac{d-1}{2}$ for odd $d$, above which these terms become total divergences.
which has solutions if $C_5 > 0$. For an extremal solution, for which in addition we have $rac{\partial N}{\partial r}|_{r=r_{h}^{ex}} = 0$, we find that

$$r_{h}^{ex} = \sqrt{\frac{\Pi d \tau_1}{16 \kappa_1}}.$$  \hspace{1cm} (15)

Thus, indeed, magnetically charged black hole solutions exist in $d = 5$. However, these have logarithmically divergent energy.

For $d \neq 5, 9, 13$ we find from (10):

$$r^{-d+1} m(r) = \frac{d-3}{16} \left( C r^{-d+1} + \frac{\tau_1}{d-5} r^{-4} + \frac{3}{4} \frac{\tau_2}{d-9} (d-4)(d-5)r^{-8} + \frac{49}{160} \frac{\tau_3}{d-13} (d-4)(d-5)(d-6)(d-7)r^{-12} \right)$$  \hspace{1cm} (16)

where $C$ is an integration constant.

Now gravitational equations, for example (10), support RN-type solutions [14,13] if the right hand side of (16) is proportional to:

$$a r^{-d+1} - b r^{4-2d}, \hspace{0.5cm} a, b \hspace{0.5cm} \text{constants}.$$  \hspace{1cm} (17)

The $r^{-d+1}$ expression is the Schwarzschild-like term and $a$ is related to the mass of the solution, while the $r^{4-2d}$ part denotes a RN type term with $b$ being related to the global charge. Comparing the powers of $r$ on the right hand side of (16) with (17) shows that in the example at hand there are only three cases where this can be fulfilled for $a \neq 0, b \neq 0$

- $\tau_2 = \tau_3 = 0$ and $d = 4$: This is the case of the usual EYM equations in 4 dimensions [2], which was shown to have an embedded RN solution [15].

- $\tau_1 = \tau_3 = 0$ and $d = 6$: In this case, an embedded RN solution is possible, however due to the absence of the standard YM term it seems rather unphysical.

- $\tau_1 = \tau_2 = 0$ and $d = 8$: In this case too, an embedded RN solution is possible, which again is rather unphysical in the absence of the standard YM term.

It follows by induction from the above argument that RN type solutions can be embedded by the above prescription, only in those generalised EYM systems consisting of a single YM member, namely that with $p = \frac{d-2}{2}$. One conclusion that may follow from this is, that for EYM systems with multi-member YM sector, there may be no multi-node solutions at all, since in $d = 4$ the existence of multinode solutions is intimately tied up with the existence of limiting RN field configurations.

For all other cases, no RN type solutions are possible. However, there exists an embedded solution for all $d$, namely the Schwarzschild solution. This can be easily seen by setting $w = \pm 1$ in (8) and using the relation between $m$ and $N$. 
IV. GENERALISED BARTNIK-MCKINNON SOLUTIONS

A. The boundary conditions

We require asymptotic flatness and finite energy. Thus the boundary conditions at infinity read:

\[ \sigma(r = \infty) = 1, \quad w(r = \infty) = (-1)^k \]

with \( k = 1, 2, 3, \ldots \) being the node number of the gauge field function \( w(r) \). For odd \( k \), the gauge field function at infinity behaves like:

\[ w(r \to \infty) = -1 + \frac{C}{r^{d-3}}, \quad (19) \]

while for even \( k \), we have:

\[ w(r \to \infty) = 1 - \frac{C}{r^{d-3}}. \quad (20) \]

The requirement of regularity at the origin leads to the conditions [1]:

\[ m(0) = 0, \quad w(0) = 1. \quad (21) \]

From here on, we specialise to the spacetime dimension \( d = 5 \).

B. Numerical results

In this section we present the results we obtained for the two cases corresponding to \( p = 1 \) and \( p = 2 \) gravity in \( d = 5 \) space-time dimensions.

When the two mixing parameters \( \tau_1, \tau_2 \) are non-zero we can set them equal to particular values without losing generality. This can be done by an appropriate rescaling of the overall Lagrangian density and of the radial variable. So we take advantage of this freedom and choose \( \tau_1 = 8, \tau_2 = 8/3 \) (this simplifies the numerical coefficients in the equations).

1. \( \kappa_1 \neq 0, \kappa_2 = 0 \)

This case corresponds to the case of pure Einstein-Hilbert gravity. We define \( \alpha^2 \equiv n_d/(2\kappa_1) \). The limit \( \alpha^2 \to 0 \) then corresponds to the flat case. As was already noticed in [1], Derrick’s theorem does not allow regular solutions in the flat limit, and our numerical analysis indeed confirms this expectation. However for \( \alpha^2 > 0 \), we were able to construct regular solutions. This is remarkable, since it was observed in [11,12], that no finite energy solutions exist for \( \tau_2 = 0 \). As expected the inclusion of the second term of the Yang-Mills hierarchy leads to the possibility of having finite energy solutions. These solutions exist up to a maximal value, \( \alpha_{\text{max}} \), of the parameter \( \alpha \). We find numerically \( \alpha_{\text{max}}^2 \approx 0.28242 \).

Although no regular solution exist in the flat limit, some comments are in order here. In our equations only the coupling to gravity fixes the scale and prevents the solution from
shrinking to zero. Rescaling the radial variable according to $y = \alpha r$, we see that the right hand side of (9) vanishes in the $\alpha \to 0$ limit while the term proportionnal to $\tau_2$ in (7) gets multiplied by $\alpha^4$. Therefore, taking the limit $\alpha = 0$ leads to the following form for (7):

$$\frac{d}{dy} \left( y \frac{dw}{dy} \right) - 2w(w^2 - 1) = 0$$

(22)

which is nothing else but the equation of the YM instanton [17] in 4-dimensional flat space. Its solution is $w(y) = (1 - y^2)/(1 + y^2)$ and it has mass $M = 8/3$, in complete agreement with our numerical solution.

When $\alpha^2$ increases, we observe that the mass decreases, and that both, the value $\sigma(0)$ and the minimum $N_m$ of the function $N(r)$ decrease, as indicated in Fig. 1. The stopping of the main branch of solutions at $\alpha^2 = \alpha^2_{\text{max}}$ strongly suggest the existence of a second branch of solutions. We indeed confirmed this numerically. We found another branch of solutions on the interval $\alpha^2 \in [\alpha^2_{\text{cr}(1)}, \alpha^2_{\text{max}}]$, with $\alpha^2_{\text{cr}(1)} \approx 0.1749$. On this second branch of solutions, both $\sigma(0)$ and $N_m$ continue to decrease but stay finite. However, a third branch of solutions exists for $\alpha^2 \in [\alpha^2_{\text{cr}(1)}, \alpha^2_{\text{cr}(2)}]$, $\alpha^2_{\text{cr}(2)} \approx 0.1778$ on which the two quantities decrease further. Progressing on this succession of branches the main observation is that the value $\sigma(0)$ decreases much quicker that $N_m$ as illustrated in Fig. 1. In Fig. 2, we compare the profiles of the functions $N$, $\sigma$ and $w$ for the same value of $\alpha^2$ on the first and third branch. The pattern strongly suggests that after a finite (or infinite) number of branches the solution terminates into a singular solution with $\sigma(0) = 0$. The existence of a series of branches in $d = 5$, albeit in a different model, has already been observed in [11].

Since our equations generalise the Bartnik-McKinnon (BM) equations, which admit a sequence of regular, finite energy solutions indexed by the number of nodes of the gauge function $w(r)$, we have tried to construct the counterparts of these multi-node solutions for $d = 5$. However, we have not succeed in constructing them. Our numerical technique was to start from the two-node BM solution for $d = 4$ and to solve the equations while increasing the dimension $d$ gradually. At $d = 4$ the $p = 2$ YM term trivialises, but when $d$ increases to values higher than 4, the $p = 2$ YM terms comes into play. We found that the BM solution is indeed deformed for $5 > d > 4$ and that the position of the two nodes is a function of $d$. In the limit $d \to 5$ our numerical procedure indicates that the position of the second node reaches out to infinity. This provides an argument that - if multi-node solution would exist at all in the $d = 5$ case - it will be very difficult to construct them.

2. $\kappa_1 = 0$, $\kappa_2 \neq 0$

When gravity just reduces to the Gauss-Bonnet term ($\kappa_1 = 0$) then from (6) we find the relation between $N$ and $m$ to be:

$$1 - 2\sqrt{m(r) \frac{4}{\kappa_2}} = N(r).$$

(23)

Clearly, this is not an asymptotically flat solution, since $m(\infty)$, which is proportional to the mass of the solution, should be non-zero. For large $r$, we can assume the mass function to be approximately constant $m(r \gg 1) \approx m_\infty = \text{const}$. Moreover, we can assume
\( \sigma(r \gg 1) \approx 1 \). Following a similar argument as in [16] and rescaling the \( t \) and \( r \) coordinate, we end up with a metric which has a deficit angle:

\[
ds^2 = -d\tilde{t}^2 + d\tilde{r}^2 + \left(1 - 2\sqrt{m_\infty \frac{4}{\kappa_2}}\right)\tilde{r}^2 d\Omega^2_3. \tag{24}\n\]

Thus a 3 dimensional hypersphere of radius \( r \) has surface area \( 2\pi^2 r^3 \left(1 - 2\sqrt{m_\infty \frac{4}{\kappa_2}}\right) \).

3. \( \kappa_1 \neq 0, \kappa_2 \neq 0 \)

Considering small values of \( \kappa_2 \), the solutions hardly differ from the pure Einstein-Hilbert gravity solutions. When the Gauss-Bonnet coupling constant becomes of the same order as \( \kappa_1 \) noticeable differences appear. For instance, for \( \kappa_2 = \kappa_1 \) we have constructed solutions on the first branch up to \( \alpha^2 \approx 0.5 \). The construction of further branches in this case appears to be numerically very difficult.

V. GENERALISED COLOURED BLACK HOLE SOLUTIONS

A. Boundary conditions

Since we are still considering asymptotically flat, finite energy solutions, the boundary conditions (18) are still valid.

Furthermore, black holes solutions are characterized by an horizon \( r \equiv x_h \) at which the metric function \( N \) vanishes : \( N(x_h) = 0 \). This leads (for \( \kappa_2 = 0 \)) to the condition :

\[
m(x_h) = \frac{\kappa_1}{n_\mathcal{D}} x_h^{d-3}. \tag{25}\n\]

The requirement of regularity of the gauge function \( w(r) \) at \( x_h \) leads to the following condition :

\[
\begin{align*}
\tau_1 (r^6 N' w' - (d - 3) r^4 w (w^2 - 1)) \\
+ \frac{\tau_2}{2} (d - 4)(d - 3)(w^2 - 1)(r^2 N' w'(w^2 - 1) - (d - 5) w (w^2 - 1)^2)|_{r=x_h} = 0.
\end{align*}
\tag{26}\n\]

B. Horizon properties

The surface gravity [18] of the \( d = 5 \) black hole, which is proportional to the temperature \( T \) of the black hole: \( T = \kappa_{sg}/(2\pi) \), is given by:

\[
\kappa_{sg}^2 = -\frac{1}{4} g^\mu \partial_\mu \left(\partial_r g_{tt}\right)^2 \tag{27}\n\]

which gives:
\[ \kappa_{\text{gh}} = \sigma_{xh} \left( \frac{1}{x_h} - \frac{\alpha^2}{x_h^2} (w_{xh}^2 - 1)^2 \right), \]  

where \( \sigma_{xh} \) and \( w_{xh} \), respectively, denote the values of the metric function \( \sigma \) and the gauge field function \( w \) at the horizon. As expected for spherically symmetric solutions, the expression (28) clearly shows that the surface gravity is constant at the horizon, thus the non-abelian black hole solutions fulfill the 0. Law of black hole mechanics [18].

Like the Gaussian curvature given by \( R \), the curvature of a black hole can also be described by the Kretschmann scalar. For the 5-dimensional Kretschmann scalar \( K = R_{ABCD} R_{ABCD}, A, B, C, D = 0, 1, 2, 3, 4 \), at the horizon \( x_h \) we find:

\[ K\big|_{x_h} = 9(N'|_{x_h})^2 \left( \frac{\sigma'}{\sigma}|_{x_h} \right)^2 + 6N'|_{x_h} N''|_{x_h} \frac{\sigma'}{\sigma}|_{x_h} + (N''|_{x_h})^2 + 6 \frac{(N'|_{x_h})^2}{x_h^2} + \frac{12}{x_h}. \]  

This can be further evaluated by inserting the following expressions into (29):

\[ \frac{\sigma'}{\sigma}|_{x_h} = 2\alpha^2 (w'|_{xh})^2 \left( 1 + \frac{(w_{xh}^2 - 1)^2}{x_h^4} \right), \]

\[ N'|_{x_h} = -\frac{2\alpha^2}{x_h^4} (w_{xh}^2 - 1)^2 + \frac{2}{x_h} \]

and

\[ N''|_{x_h} = \frac{10\alpha^2}{x_h^4} (w_{xh}^2 - 1)^2 - \frac{8\alpha^2 w_{xh} w'|_{xh}}{x_h^6} - \frac{2\alpha^2}{x_h^4} N'|_{x_h} (w'|_{xh})^2 \left( 1 + \frac{8}{x_h^4} (w_{xh}^2 - 1)^2 \right) \]

We can thus express the Kretschmann scalar at the horizon in terms of \( w_{xh}, w'|_{xh} \) and the horizon \( x_h \) itself.

C. Numerical results

Here we limit our analysis to the case \( \kappa_2 = 0 \). We find that in analogy to the non-abelian black holes in \( d = 4 \) Einstein-Yang-Mills-Higgs (EYMH) theory [19], the \( d = 5 \) black hole solutions exist in a limited domain of the \( \alpha\times x_h \)-plane. For small values of the gravitational coupling \( \alpha^{(4)} \), the EYMH solutions exist up to a maximal value of the horizon \( x_h^{(4)} = x_h^{(4)(\text{max})} \) and on a second branch of solutions bifurcate with the branch of non-extremal Reissner-Nordström (RN) solutions at \( x_h^{(4)} = x_h^{(4)(\text{cr})} \). Both, \( x_h^{(4)(\text{max})} \) and \( x_h^{(4)(\text{cr})} \) are increasing functions of \( \alpha^{(4)} \). On the two branches, the value of the gauge field function at the horizon decreases monotonically from 1 to 0 at \( x_h^{(4)(\text{cr})} \) such that the limiting solution is identical to the RN solution on the full interval \( r^{(4)} \epsilon_{[x_h^{(4)(\text{cr})} : \infty]} \). For fixed \( x_h^{(4)} \) and large \( \alpha^{(4)} \), the situation changes and the solutions bifurcate with the branch of extremal RN solutions forming a degenerate horizon for \( \alpha^{(4)} = \alpha^{(4)(\text{cr})} = x_h^{(\text{RN})} \). Now, however, the limiting solution can only be described by the extremal RN solution for \( r^{(4)} \epsilon_{[x_h^{(\text{RN})} : \infty]} \). On the interval \( r^{(4)} \epsilon_{[x_h^{(4)} : x_h^{(4)(\text{RN})}]} \), the solution is non-trivial and non-singular. Since (as argued in a preceding section of this paper), there are no RN solutions in \( d = 5 \), we find a qualitatively different domain of existence for the \( d = 5 \) black holes. For fixed \( \alpha \), we find the following maximal values of \( x_h^{(\text{max})} \) up to which the non-abelian solutions exist:
This describes rather a semi-circle than the complicated pattern found in [19]. Extending backwards in \( x_h \), we find a second branch of solutions for \( x_h < x_{h(\max)} \). Progressing on this branch, the value \( \sigma(0) \) drastically decreases as shown in Fig. 3. We believe that the second branch stops at \( x_{h(\text{cr})}^{(1)} \approx 0.2329 \). This is indicated by the slight “curling” of \( w_{x_h} \). We believe that a third branch exists on which the value \( \sigma(0) \) continues to decrease further to zero. However, it is likely that the extension of this branch in \( x_h \) will be very small, which might be the reason why we were unable to find it so far. That we are close to the critical solution is also indicated by the surface gravity \( \kappa_{sg} \). It drops down very quickly on the second branch and likely tends to zero in the limit of the critical solution. We have also evaluated the Kretschmann scalar at the horizon. We find that a sharp increase in the curvature at the horizon is seen close to the critical solution. We find, for example, that on the second branch \( K\mid_{x_h} \approx 3970 \) for \( x_h = 0.235 \), while \( K\mid_{x_h} \approx 23992 \) for \( x_h = 0.23295 \approx x_{h(\text{cr})} \).

In Fig. 4, we show the functions \( w, \sigma \) and \( N \) for the same values of \( \alpha^2 \) and \( x_h \) on the two different branches. Clearly, these are distinct solutions.

Finally, in Fig. 5 we present the terminating solution that we were able to construct on the second branch for \( x_h = 0.23295 \). It is clearly seen in comparison to Fig. 4 that \( \sigma \) has a very large rise in value close to the horizon and then in a small interval of the coordinate \( r \) rises nearly linearly to its asymptotic value 1.

VI. CONCLUSIONS AND SUMMARY

The idea that our space-time consists of more than four dimensions is not new and has been long discussed in string theory. It is by now an accepted fact that bosonic string theory is only consistent in \( d = 26 \) space-time dimensions, while fermionic string theory “lives” in \( d = 10 \) space-time dimensions. Extra dimensions have recently gained additional interest in the context of the Randall-Sundrum models [9,10]. These are basically 5-dimensional space-times in which gravity lives in the full 5 dimensions, while the remaining fields are confined to 3-branes.

The study of soliton solutions in classical field theories coupled to gravity leads in general to a rich pattern of phenomena. A proof that the study of non-abelian gauge fields interacting with gravity is indeed worthwhile was the discovery of particle-like solutions, the Bartnik-McKinnon solutions, in the coupled Einstein-Yang-Mills (EYM) system [2]. It is also remarkable that non-abelian black holes, i.e. black holes which violate the “No-hair” conjecture, can be constructed in non-abelian gauge theories coupled to gravity. One of the most famous examples are the coloured black hole solutions constructed in EYM theory.

All this gives motivation to study the counterparts of the Bartnik-McKinnon and coloured black hole solutions in non-abelian gauge field theories coupled gravity in more than four dimensions. Since in low energy effective actions higher order curvature terms (Gauss-Bonnet terms) as well as higher order terms from the Yang-Mills hierarchy (Born-Infeld terms) appear, these should be taken into account when studying solutions in \( d > 4 \). Here we

| \( \alpha^2 \) | 0.025 0.05 0.10 0.15 0.20 0.25 0.27 |
|----------------------|
| \( x_{h(\max)} \) | 0.470 0.468 0.431 0.385 0.322 0.220 0.141 |
have studied a 5 dimensional model which consists of the first two terms of the gravitational and Yang-Mills hierarchies, respectively. We have solved the system of coupled differential equations numerically.

While without higher order Yang-Mills curvature terms no globally regular, finite energy solutions are possible [11], we find that in our model they exist. A number of branches exist which terminate into a singular solution. These solutions, which are the analogues of the BK solution [2] and are hence expected like the latter to be sphaleron like, in other respects differ from these. Most importantly these solutions exhibit branches more akin to those in EYMH theory [19]. This is not too surprising since like the latter, our model features dimensionful constants.

In passing we remark that unlike the BK solutions which consist of an infinite sequence with multiple nodes, it seems most unlikely that there should be any multinode solutions in any EYM model featuring more than one YM term, as in our case.

We have also constructed non-abelian black holes. We find two branches of solutions. We were unable to reach the limiting solution in this case, however our numerical results indicate that the black hole solutions bifurcate with a singular solution which has zero surface gravity and a very huge, if not infinite curvature at the horizon.

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FIG. 1. The value of the minimum of the metric function $N$, $N_m$, as well as the value of the metric function $\sigma$ at the origin, $\sigma(0)$, are shown for the generalised Bartnik-McKinnon solutions as functions of $\alpha^2 = n_d/(8\kappa_1)$ for $p = 1$ gravity ($\kappa_2 = 0$).
FIG. 2. The gauge field function $w(r)$ as well as the metric functions $N(r)$, $\sigma(r)$ are shown for the generalised Bartnik-McKinnon solutions as functions of $r$ for $\alpha^2 = 0.176$ on the main and on the third branch of solutions, respectively.
FIG. 3. The value of the gauge field function at the horizon \( w(x_h) = w_{x_h} \), the value of the metric function \( \sigma \) at the horizon, \( \sigma(x_h) = \sigma_{x_h} \), as well as the mass divided by 3 and the surface gravity \( \kappa_{sg} \) divided by 12 are shown as functions of the horizon \( x_h \) for the generalised \( d = 5 \) dimensional coloured black hole solutions with \( \kappa_2 = 0 \) and \( \alpha^2 = 0.1 \).
FIG. 4. The gauge field function $w(r)$ as well as the metric functions $N(r)$, $\sigma(r)$ are shown for the generalised coloured black hole solutions as functions of $r$ for $\alpha^2 = 0.1$ and $x_h = 0.3$ on the main and on the second branch of solutions, respectively.
FIG. 5. The gauge field function $w(r)$ as well as the metric functions $N(r)$, $\sigma(r)$ are shown for the generalised coloured black hole solutions as functions of $r$ for $\alpha^2 = 0.1$ and $x_h = 0.2395$ close to the critical $x_h$. 