Stable Submanifolds in the Product of Projective Spaces

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Abstract
We provide a classification theorem for compact stable minimal immersions (CSMI) of codimension 1 or dimension 1 (codimension 1 and 2 or dimension 1 and 2) in the product of a complex (quaternionic) projective space with any other Riemannian manifold. We characterize the complex minimal immersions of codimension 2 or dimension 2 as the only CSMI in the product of two complex projective spaces. As an application, we characterize the CSMI of codimension 1 or dimension 1 (codimension 1 and 2 or dimension 1 and 2) in the product of a complex (quaternionic) projective space with any compact rank one symmetric space.

Keywords
Product projective spaces · Minimal submanifolds · Stable submanifolds · Complex projective space · Quaternionic projective space · Cayley plane · Sphere

Mathematics Subject Classification 53C42 · 53C40

1 Introduction
Let \( M \) be a Riemannian manifold of dimension \( n + d \). It is a really interesting problem to know what are the submanifolds \( \Sigma \) of dimension \( n \) of \( M \) that minimizes area under perturbations. For example in the Euclidean space \( \mathbb{R}^3 \), we can see intuitively that a plane has less area than any of its perturbations (see [1, 2]).

Since it is a minimization problem, one condition \( \Sigma \) must satisfy is that it be a critical point of the area functional (i.e. minimal). Returning to the case of the plane in

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In $\mathbb{R}^3$, we can see then that the plane is minimal. But contrary to the previous example, there are many examples where the submanifold is minimal and it does not minimize area. For example, if we perturb an equator in the Euclidean sphere along normal direction with constant height, we get a circle with smaller radius, and thus, smaller length.

Therefore, we have to study the second variation of the area functional. More specifically, we need the second variation to be non negative for all possible perturbations of $\Sigma$. If $\Sigma$ is as described above, we say that $\Sigma$ is stable. The second variation of the area functional defines an operator acting on normal sections of $\Sigma$ (see Preliminaries), called the Jacobi or stability operator. The number of negative eigenvalues counting multiplicity is what we call the Morse index. Then, $\Sigma$ is stable if its Morse index is zero.

For a long time, it has been object of study if a Riemannian manifold has stable submanifolds and if the case to characterize them. In particular when the Riemannian manifold is well known, as the Euclidean space, the Euclidean sphere, and projective spaces. For instance, in the 3-dimensional Euclidean space, Fischer-Colbrie and Schoen [3], Do Carmo and Peng [2], and Pogorelov [4] proved independently that planes are the only stable complete minimal two-sided surfaces in $\mathbb{R}^3$. Moreover, Ros [5] proved that there are no complete one-sided stable minimal surfaces in the 3-dimensional Euclidean space. Recently, in the same direction, Chodosh and Li [1] proved that a complete, two-sided, stable minimal hypersurface in $\mathbb{R}^4$ must be flat. For the case of compact stable minimal immersions (CSMI), Simons in [6] proved that there are no CSMI in the Euclidean sphere.

**Theorem 1** [6] Let $\Sigma$ be a compact, minimal $n$-dimensional submanifold immersed in $S^{n+d}$. Then, the Morse index of $\Sigma$ is greater than or equal to $d$, and equality holds only when $\Sigma$ is $S^n$.

He used the fact that $S^{n+d}$ is a hypersurface of $\mathbb{R}^{n+d+1}$. More precisely, the stability operator was evaluated on the projections $w$ of the constant parallel vector fields $v \in \mathbb{R}^{n+d+1}$ in $N^{S^{n+d}}_{\Sigma}$, where in general $N^M_{\Sigma}$ denote the normal space of $\Sigma$ in $M$.

Afterwards, Lawson and Simons in [7] characterized the complex submanifolds (in the sense that each tangent space of the submanifold is invariant under the complex structure) as the only CSMI in the complex projective spaces. This was followed by Ohnita [8], who completed the classification of CSMI in all compact rank one symmetric spaces.

**Theorem 2** [8] Let $\Sigma$ be a compact minimal $n$-dimensional submanifold immersed in the real projective space $\mathbb{R}P^{n+d}$ with the standard metric. Then, $\Sigma$ is stable if and only if $\Sigma$ is a real projective subspace $\mathbb{R}P^n$ of $\mathbb{R}P^{n+d}$.

Specifically, putting together the results of those authors in a single statement, we get the following result.

**Theorem 3** [7, 8] Let $\Sigma$ be a compact stable minimal $n$-dimensional submanifold immersed in $\mathbb{F}P^n$, where $\mathbb{F} = \mathbb{C}, \mathbb{H}, \mathbb{O}$ (complex, quaternion, octonion numbers, respectively). Then,
• If $F = \mathbb{C}$, then $n = 2l$ for some integer $l$ and $\Sigma$ is a complex submanifold, in the sense that each tangent space is invariant under the complex structure of the complex projective space (Lawson and Simons).

• If $F = \mathbb{H}$, then $n = 4l$ for some integer $l$ and $\Sigma$ is a quaternionic projective subspace $\mathbb{HP}^l$ of $\mathbb{HP}^m$ (Ohnita).

• If $F = \emptyset$ and $m = 2$, then $n = 8$ and $\Sigma$ is a Cayley projective line (Ohnita).

The projective space $\mathbb{F}P^m$ can be isometrically immersed in some Euclidean space $\mathbb{R}^l$ by the generalized Veronese imbedding (Sakamoto [9]). Therefore, in order to prove the previous theorem, the stability operator was evaluated on the projections of the constant parallel vector fields $v \in \mathbb{R}^l$ in $N^F\mathbb{P}^m$, as Simons did in the proof of Theorem 1. This was done because after running $v$ in an orthonormal basis of $\mathbb{R}^l$, a term $T$ given by the sum of the second variations of the associated normal sections is non-positive. Since $T$ is also non-negative (because the submanifold is stable), $T$ must be zero. Thus, using that $T = 0$, we can obtain geometric information about the submanifold.

After classifying the CSMI of these well known Riemannian manifolds, a natural direction is to study CSMI in their products. Along these lines, Torralbo and Urbano proved a classification theorem of CSMI in the product of a sphere and any Riemannian manifold whenever the dimension of the sphere is at least three or the immersion has codimension 1.

**Theorem 4** [10] Let $M$ be any Riemannian manifold and $\Phi = (\phi, \psi) : \Sigma \rightarrow S^m \times M$ be a compact minimal immersion of dimension $n$ in $S^m \times M$, $n \geq 2$, satisfying either $m \geq 3$ or $m = 2$ and $\Phi$ is of codimension 1. Then, $\Phi$ is stable if and only if one of the following holds

• $\Sigma = S^m$ and $\Phi(\Sigma)$ is a slice $S^m \times \{q\}$ with $q$ a point in $M$.

• $\Sigma$ is a covering of $M$ and $\Phi(\Sigma)$ is a slice $\{p\} \times M$ with $p$ a point of $S^m$.

• $\psi : \Sigma \rightarrow M$ is a stable minimal submanifold and $\Phi(\Sigma) = \{p\} \times \psi(\Sigma)$ with $p$ a point of $S^m$.

• $\Sigma = S^m \times \overline{\Sigma}$, $\Phi = (Id, \psi)$ and $\psi : \overline{\Sigma} \rightarrow M$ is a stable minimal submanifold.

Moreover, they complete the classification of CSMI in the product of two spheres.

**Theorem 5** [10] Let $\Phi = (\phi, \psi) : \Sigma \rightarrow S^{n_1}(r_1) \times S^{n_2}(r_2)$ be a compact minimal immersion of dimension $n$ in $S^{n_1}(r_1) \times S^{n_2}(r_2)$, $n \geq 2$. Then, $\Phi$ is stable if and only if one of the following possibilities occurs

• $\Sigma = S^{n_1}(r_1)$ and $\Phi(\Sigma)$ is a slice $S^{n_1}(r_1) \times \{q\}$ with $q$ a point of $S^{n_2}(r_2)$.

• $\Sigma = S^{n_2}(r_2)$ and $\Phi(\Sigma)$ is a slice $\{p\} \times S^{n_2}(r_2)$ with $p$ a point of $S^{n_1}(r_1)$.

• $n_1 = n_2 = n = 2$, $\Sigma$ is orientable and $\Phi$ is a complex immersion of the Riemann surface $\Sigma$ in $S^2(r_1) \times S^2(r_2)$ with respect to one of the two complex structures that $S^2(r_1) \times S^2(r_2)$ has.

To prove the last two theorems, the stability operator was evaluated on the projections of the vector fields $(v, 0)$ to the normal part of the immersion in the Riemannian product, where $v$ are the constant parallel vector fields used in the proof of Theorem 1.
In [11], Chen and Wang generalize Theorem 4 to the product of any hypersurface \( M \) of the Euclidean space \( \mathbb{R}^{m+1} \) with certain conditions, and a Riemannian manifold. Similar to Torralbo and Urbano, the stability operator was evaluated on projections of the vector fields \((v, 0)\) to the normal part of the immersion in the Riemannian product, where \( v \in \mathbb{R}^{m+1} \) are the constant parallel vector fields in the Euclidean space.

**Theorem 6** [11] Let \( \Phi = (\phi, \psi) : \Sigma \to \tilde{M} := M_1 \times M_2 \) be a compact minimal immersion in \( \tilde{M} \), where \( M_1 \) is a compact connected hypersurface in \( \mathbb{R}^{m+1} \) and \( M_2 \) is any Riemannian manifold. Assume that the sectional curvature \( K_{M_1} \) of \( M_1 \) satisfies

\[
\frac{1}{\sqrt{m_1-1}} \leq K_{M_1} \leq 1.
\]

Then, \( \Phi \) is stable if and only if one of the following holds

- \( \Sigma = M_1 \) and \( \Phi(\Sigma) \) is a slice \( M_1 \times \{ q \} \) with \( q \) a point in \( M_2 \).
- \( \Sigma \) is a covering of \( M_2 \) and \( \Phi(\Sigma) \) is a slice \( \{ p \} \times M_2 \) with \( p \) a point of \( M_1 \).
- \( \psi : \Sigma \to M_2 \) is a stable minimal submanifold and \( \Phi(\Sigma) \) is \( \{ p \} \times \psi(\Sigma) \) with \( p \) a point of \( M_1 \).
- \( \Sigma = M_1 \times \tilde{\Sigma} \), \( \Phi = (Id, \psi) \) and \( \psi : \tilde{\Sigma} \to M_2 \) is a stable minimal submanifold.

This motivates us to evaluate the stability operator on the projections of the vector fields \((v, 0)\) to the normal part of an immersion in the Riemannian product of a complex or quaternionic projective space with any other Riemannian manifold, where \( v \) are the constant parallel vector fields used in the Theorem 3. Notice that we can not use Theorem 6 because the complex and quaternionic projective spaces are not hypersurfaces of Euclidean space (see [12, 13]). Our main results for \( \mathbb{C} P^{m_1} \times M \), are summarized in the following statements (see Theorems 16 and 26 for details).

**Theorem A** Let \( \Phi = (\psi, \phi) : \Sigma \to \mathbb{C} P^{m_1} \times M \) be a compact stable minimal immersion of codimension \( d \) and dimension \( n \), where \( M \) is any Riemannian manifold of dimension \( m_2 \). Then,

- If \( d = 1 \), \( \Sigma = \mathbb{C} P^{m_1} \times \tilde{\Sigma} \), \( \Phi = Id \times \hat{\phi} \) where \( \hat{\phi} : \tilde{\Sigma} \to M \) is a stable minimal immersion of codimension 1, and therefore \( \Phi(\Sigma) = \mathbb{C} P^{m_1} \times \hat{\phi}(\tilde{\Sigma}) \). In particular, for \( m_2 = 1 \), \( \Sigma = \mathbb{C} P^{m_1} \), \( \hat{\phi} \) is a constant function, and \( \Phi(\Sigma) = \mathbb{C} P^{m_1} \times \{ q \} \), for \( q \in M \).
- If \( n=1 \), \( \phi : \Sigma \to M \) is a stable geodesic, \( \psi \) is a constant function, and therefore \( \Phi(\Sigma) = \{ r \} \times \phi(\Sigma) \) for \( r \in \mathbb{C} P^{m_1} \).

For the particular case \( m_2 = 2 \) in Theorem 4, Torralbo and Urbano classified the CSMI of codimension \( d = 1 \) in \( S^2 \times M \). Applying Theorem A for \( m_1 = 2 \) and using the fact that \( S^2 \) is isometric to \( \mathbb{C} P^1 \), we obtain the same classification result. It is worth pointing out that Theorem A gives a complete classification of CSMI in \( \mathbb{C} P^1 \times M \), where \( M \) is a Riemannian manifold of dimension 1.

The last theorem tells us that the CSMI of either codimension 1 or dimension 1 in the Riemannian product \( \mathbb{C} P^{m_1} \times M \) are products of trivial CSMI of \( \mathbb{C} P^{m_1} \) with CSMI of \( M \). As an application, since we know the CSMI of \( M = S^3, \mathbb{R} P^5, \mathbb{C} P^5, \mathbb{H} P^5, \mathbb{O} P^2 \), we derived the following statements (see Corollaries 18, 19, 27, and 28 for details).
Corollary A There are no compact stable minimal immersions of

- codimension \( d = 1 \) in the product manifold \( \mathbb{C}P^{m_1} \times S^s \), \( \mathbb{C}P^{m_1} \times \mathbb{C}P^2 \), or \( \mathbb{C}P^{m_1} \times \mathbb{K}P^s \) other than \( \mathbb{C}P^{m_1} \times \{q\} \) in \( \mathbb{C}P^{m_1} \times S^1 \), for \( q \in S^1 \),
- or dimension \( n = 1 \) in the product manifold \( \mathbb{C}P^{m_1} \times S^s \), \( \mathbb{C}P^{m_1} \times \mathbb{C}P^2 \), or \( \mathbb{C}P^{m_1} \times \mathbb{K}P^s \) other than \( \{r\} \times S^1 \) in \( \mathbb{C}P^{m_1} \times S^1 \), for \( r \in \mathbb{C}P^{m_1} \),

where \( \mathbb{K} \in \{\mathbb{C}, \mathbb{H}\} \).

The particular case in the previous corollary of CSMI of codimension \( d = 1 \) in \( \mathbb{C}P^{m_1} \times S^s \) for \( s \geq 2 \), can be also obtained as a consequence of Theorem 4 from Torralbo and Urbano by setting \( M = \mathbb{C}P^{m_1} \) and applying Theorem 3.

Corollary B The only compact stable minimal immersion of

- codimension \( d = 1 \) in the product space \( \mathbb{C}P^{m_1} \times \mathbb{R}P^s \) is \( \mathbb{C}P^{m_1} \times \mathbb{R}P^{s-1} \),
- or dimension \( n = 1 \) in the product space \( \mathbb{C}P^{m_1} \times \mathbb{R}P^s \) is \( \{r\} \times \mathbb{R}P^1 \), \( r \in \mathbb{C}P^{m_1} \).

The first item in Theorem 3 tells us that the CSMI in the complex projective space behave well under the complex structure of the complex projective space. Therefore, it is expected (see [7], and [10] and references therein) that if the manifold has a complex structure, named \( J \), then the CSMI in this manifold also behave well under \( J \). In the specific case of \( \mathbb{C}P^{m_1} \times M \), where \( M \) is an arbitrary manifold, we do not know if this product manifold has a complex structure. Thus, there is no natural complex structure for the minimal submanifold to be well behaved with. However, we can indeed expect the CSMI to have a complex behaviour in the projections in the first component (the component associated to the complex projective space) of some vectors associated to the immersion. In fact, we have the following Lemma,

Lemma 7 Under the same conditions than Theorem A we have:

- If \( d = 2 \), \( \eta_1^j = \pm J(\eta_1^j) \), where \( \{\eta_1^j, \eta_2^j\} \) is an orthonormal basis of \( N_p \Sigma, p \in \Sigma \).
- If \( n = 2 \), \( e_2^j = \pm J(e_1^j) \), where \( \{e_1^j, e_2^j\} \) is an orthonormal basis of \( T_p \Sigma, p \in \Sigma \).

Here, \( J \) is the complex structure of \( \mathbb{C}P^{m_1} \) and \( w^1 \) is the projection of \( w \) in \( T_{\psi(p)}\mathbb{C}P^{m_1} \).

If \( M = \mathbb{C}P^{m_2} \), the Riemannian product \( \mathbb{C}P^{m_1} \times \mathbb{C}P^{m_2} \) has two complex structures, \( J_1 \) and \( J_2 \) induced by the complex structure \( J \), of the complex projective space (see Definition 5). Therefore, according to what was mentioned before, it is expected that the CSMI in \( \mathbb{C}P^{m_1} \times \mathbb{C}P^{m_2} \) behave well under a complex structure of \( \mathbb{C}P^{m_1} \times \mathbb{C}P^{m_2} \).

In fact, Lemma 7 gives us information about how the complex structure of the complex projective space behaves on projections onto the first component of normal or tangent vectors. Using the same technique as in the proof of Lemma 7 when \( M = \mathbb{C}P^{m_2} \), we can get information about how the complex structure of the complex projective space behaves on projections onto the second component of normal or tangent vectors. Using the behaviours of those projections under \( J \), we can determine that for all points \( p \) in \( \Sigma \), \( T_p \Sigma \) has the structure \( J_1 \) or the structure \( J_2 \). We then prove that in fact every point has the same complex structure. More precisely, we proceed by contradiction and we
assume there are two points with different structures. This allows us to construct a real function \( g : (t - \epsilon, t + \epsilon) \to \mathbb{R} \) such that it changes sign and vanishes to infinite order at \( t \). We prove that, in our setting, the function \( g \) is real analytic, therefore \( g \) must vanish in the interval, which contradicts the change of sign. Theorems 21 and 25 combined give the following result.

**Theorem B** The only compact stable minimal immersions of codimension \( d = 2 \) or dimension \( n = 2 \) in the product manifold \( \bar{M} := \mathbb{C}P^{m_1} \times \mathbb{C}P^{m_2} \) are the complex ones, in the sense that each tangent space is invariant under the complex structure \( J_1 \) or \( J_2 \) of \( \bar{M} \) (see Definition 5).

For the particular case \( n_1 = n_2 = 2 \) in Theorem 5, Torralbo and Urbano characterized the CSMI of dimension \( 2 \leq n \leq 3 \) in \( S^2 \times S^2 \). Applying Corollary A, Theorem B and using again the fact that \( \mathbb{C}P^1 \) is isometric to \( S^2 \), we obtain the same characterization. Also, notice that Corollary A and Theorem B give a complete characterization of CSMI in \( \mathbb{C}P^1 \times \mathbb{C}P^1 \).

The results for the case \( H \mathbb{P}^{m_1} \times M \) are summarized as follow (see Theorems 30, 33, 36, and 39 for more details)

**Theorem C** Let \( \Phi = (\psi, \phi) : \Sigma \to H \mathbb{P}^{m_1} \times M \) be a compact stable minimal immersion of codimension \( d \) and dimension \( n \), where \( M \) is any Riemannian manifold of dimension \( m_2 \). Then,

- If \( d = 1 \), \( \Sigma = H \mathbb{P}^{m_1} \times \hat{\Sigma} \), \( \Phi = Id \times \hat{\phi} \) where \( \hat{\phi} : \hat{\Sigma} \to M \) is a compact stable minimal immersion of codimension 1, and therefore \( \Phi(\Sigma) = H \mathbb{P}^{m_1} \times \hat{\phi}(\hat{\Sigma}) \).
  In particular, for \( m_2 = 1 \), \( \Sigma = H \mathbb{P}^{m_1} \), \( \hat{\phi} \) is a constant function, and \( \Phi(\Sigma) = H \mathbb{P}^{m_1} \times \{q\} \), for \( q \in M \).

- If \( d = 2 \), \( \Sigma = H \mathbb{P}^{m_1} \times \hat{\Sigma} \), \( \Phi = Id \times \hat{\phi} \) where \( \hat{\phi} : \hat{\Sigma} \to M \) is a compact stable minimal immersion of codimension 2, and therefore \( \Phi(\Sigma) = H \mathbb{P}^{m_1} \times \hat{\phi}(\hat{\Sigma}) \).
  In particular, for \( m_2 = 1 \), there are no compact stable minimal immersions of codimension 2 in \( H \mathbb{P}^{m_1} \times M \). And for \( m_2 = 2 \), \( \Sigma = H \mathbb{P}^{m_1} \), \( \hat{\phi} \) is a constant function, and \( \Phi(\Sigma) = H \mathbb{P}^{m_1} \times \{q\} \), for \( q \in M \).

- If \( n = 1 \), \( \phi : \Sigma \to M \) is a stable geodesic, \( \psi \) is a constant function, and therefore \( \Phi(\Sigma) = \{r\} \times \phi(\Sigma) \) with \( r \) a point of \( H \mathbb{P}^{m_1} \).

- If \( n = 2 \), \( \phi : \Sigma \to M \) is a stable minimal immersion of dimension 2, \( \psi \) is a constant function, and therefore \( \Phi(\Sigma) = \{r\} \times \phi(\Sigma) \) with \( r \) a point of \( H \mathbb{P}^{m_1} \).

As a consequence of the previous theorem, we have a complete classification of CSMI in \( H \mathbb{P}^1 \times M^1 \).

As in the complex case, the last theorem tells us that the CSMI of codimension 1 and 2 or dimension 1 and 2 in \( H \mathbb{P}^{m_1} \times M \) are the product of trivial CSMI of \( H \mathbb{P}^{m_1} \) with CSMI of \( M \). Therefore, as an application, using Theorems 2 and 3, we get the following result (see Corollaries 31, 34, 37, and 40 for details):
Corollary C  There are no compact stable minimal immersions of

- codimension \( d = 1 \) in the product manifold \( \mathbb{H}P^{\frac{m_1}{2}} \times S^s \), \( \mathbb{H}P^{\frac{m_1}{2}} \times \mathbb{O}P^2 \), or \( \mathbb{H}P^{\frac{m_1}{2}} \times \mathbb{K}P^s \), other than \( \mathbb{H}P^{\frac{m_1}{2}} \times \{q\} \) in \( \mathbb{H}P^{\frac{m_1}{2}} \times S^1 \), for \( q \in S^1 \).
- codimension \( d = 2 \) in the product manifold \( \mathbb{H}P^{\frac{m_1}{2}} \times S^s \), \( \mathbb{H}P^{\frac{m_1}{2}} \times \mathbb{O}P^2 \), or \( \mathbb{H}P^{\frac{m_1}{2}} \times \mathbb{K}P^s \), other than \( \mathbb{H}P^{\frac{m_1}{2}} \times \{q\} \) in \( \mathbb{H}P^{\frac{m_1}{2}} \times S^2 \), for \( q \in S^2 \).
- dimension \( n = 1 \) in the product manifold \( \mathbb{H}P^{\frac{m_1}{2}} \times S^s \), \( \mathbb{H}P^{\frac{m_1}{2}} \times \mathbb{O}P^2 \), or \( \mathbb{H}P^{\frac{m_1}{2}} \times \mathbb{K}P^s \), other than \( \{r\} \times S^1 \) in \( \mathbb{H}P^{\frac{m_1}{2}} \times S^1 \), for \( r \in \mathbb{H}P^{\frac{m_1}{2}} \).
- dimension \( n = 2 \) in the product manifold \( \mathbb{H}P^{\frac{m_1}{2}} \times S^s \), \( \mathbb{H}P^{\frac{m_1}{2}} \times \mathbb{O}P^2 \), or \( \mathbb{H}P^{\frac{m_1}{2}} \times \mathbb{K}P^s \), other than \( \{r\} \times S^2 \) in \( \mathbb{H}P^{\frac{m_1}{2}} \times S^2 \), for \( r \in \mathbb{H}P^{\frac{m_1}{2}} \).

Here, \( \mathbb{K} \in \{\mathbb{C}, \mathbb{H}\} \).

The particular cases in Corollary C of CSMI of codimension \( d = 1 \) in \( \mathbb{H}P^{\frac{m_1}{2}} \times S^s \) for \( s \geq 2 \), or the case of codimension 2 or dimension 2 in \( \mathbb{H}P^{\frac{m_1}{2}} \times S^s \), for \( s \geq 3 \), can be also obtained as a consequence of Theorem 4 from Torralbo and Urbano by setting \( M = \mathbb{H}P^{\frac{m_1}{2}} \) and applying Theorem 3.

Finally, we complete the picture described above with a result that brings together the Corollaries 32, 35, 38, and 41.

Corollary D  The only compact stable minimal immersion of

- codimension \( d = 1 \) in the product space \( \mathbb{H}P^{\frac{m_1}{2}} \times \mathbb{R}P^s \) is \( \mathbb{H}P^{\frac{m_1}{2}} \times \mathbb{R}P^{s-1} \).
- codimension \( d = 2 \) in the product space \( \mathbb{H}P^{\frac{m_1}{2}} \times \mathbb{R}P^s \) is \( \mathbb{H}P^{\frac{m_1}{2}} \times \mathbb{R}P^{s-2} \), and in \( \mathbb{H}P^{\frac{m_1}{2}} \times \mathbb{C}P^s \) is \( \mathbb{H}P^{\frac{m_1}{2}} \times M \), where \( M \) is a complex submanifold of dimension \( 2s - 2 \) immersed in \( \mathbb{C}P^s \).
- dimension \( n = 1 \) in the product space \( \mathbb{H}P^{\frac{m_1}{2}} \times \mathbb{R}P^s \) is \( \{r\} \times \mathbb{R}P^1 \), \( r \in \mathbb{H}P^{\frac{m_1}{2}} \).
- dimension \( n = 2 \) in the product space \( \mathbb{H}P^{\frac{m_1}{2}} \times \mathbb{R}P^s \) is \( \{r\} \times \mathbb{R}P^2 \), and in \( \mathbb{H}P^{\frac{m_1}{2}} \times \mathbb{C}P^s \) is \( \{r\} \times M \), where \( M \) is a complex submanifold of dimension 2 immersed in \( \mathbb{C}P^s \) and \( r \in \mathbb{H}P^{\frac{m_1}{2}} \).

This paper is structured as follows:

In the second section, we state important notations, formulae, and theorems needed for the developments in the third and fourth section. The second section is divided in four subsections.

- In Sect. 2.1, we define the Jacobi operator and the Morse index.
- In Sect. 2.2, we set up notation.
- In Sect. 2.3, we state some formulae involving the geometry of the complex and quaternionic projective spaces which are needed for computations presented in Sects. 3.1 and 4.1.
- In Sect. 2.4, we recall some important definitions and propositions related to Riemannian submersions. We also establish Lemma 12 which states a sufficient condition for a Riemannian submersion to be trivial. This lemma will be used in the proof of classification theorems of CSMI presented in Sects. 3.2 and 4.2.
The third section studies the CSMI in the product of a complex projective space with any other Riemannian manifold. The third section is divided in four subsections.

- In Sect. 3.1, we derive a general formula that will be used throughout the current section.
- In Sect. 3.2, we prove a classification theorem where the codimension of the immersion is 1. Moreover, we obtain some corollaries when the second manifold is a compact rank one space.
- In Sect. 3.3, we use the general formula obtained in Sect. 3.1 when the codimension or dimension of the immersion is 2. This allows us to prove a characterization of CSMI in the product of two complex projective spaces.
- In Sect. 3.4, we give a classification theorem where the dimension of the immersion is 1. Additionally, we obtain some corollaries when the second manifold is a compact rank one space.

The fourth section is dedicated to the study of CSMI in the product of a quaternionic projective space with any other Riemannian manifold. The fourth section is divided in three subsections.

- In Sect. 4.1, we derive a general formula that will be used throughout this section.
- In Sect. 4.2, we prove a classification theorem where the codimension of the immersion is 1 and 2. Moreover, we obtain some corollaries for when the second manifold is a compact rank one space.
- Analogously, in Sect. 4.3, we give a classification theorem where the dimension of the immersion is 1 and 2. Additionally, we obtain some corollaries for when the second manifold is a compact rank one space.

2 Preliminaries

2.1 Jacobi Operator

Let $\Phi: \Sigma \to M$ be a compact Riemannian immersion, where $\Sigma$ and $M$ are Riemannian manifolds of dimensions $n$ and $n+d$, respectively. Let $F: \Sigma \times (-\epsilon, \epsilon) \to M$ be a smooth map such that $F(\cdot, 0) = \Phi(\cdot)$. We denote $F_t(x) := F(x, t)$ and $\Sigma_t := F_t(\Sigma)$. Then, we get the first variational formula

**Theorem 8** First variational formula:

$$\left. \frac{d}{dt} |\Sigma_t| \right|_{t=0} = -\int_{\Sigma} \langle X, H \rangle d\Sigma,$$

where $|\Sigma_t|$ denotes the area of $\Sigma_t$, $H$ is the mean curvature vector of $\Sigma$, $W(p) = \frac{\partial F}{\partial t}(p, 0)$ and $X := W^N$.

**Definition 1** We say that $\Sigma$ is minimal if $H = 0$.

**Remark 1** Notice that $X \in \Gamma(N\Sigma)$, i.e., it is a section in the normal bundle of $\Sigma$.  

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When studying minimal immersions, it is natural to ask about the extent to which they locally minimize area. This leads us to consider the second variation of the area functional.

**Theorem 9** Second Variation formula.  
If \( \Phi : \Sigma \rightarrow M \) is a compact minimal immersion, then:

\[
\frac{d^2}{dt^2} | \Sigma_t | \bigg|_{t=0} = - \int_{\Sigma} \langle J_\Sigma X, X \rangle d \Sigma,
\]

where \( J_\Sigma \) is the elliptic Jacobi operator defined by

\[
J_\Sigma(X) := \Delta^\perp X + \left( \sum_{i=1}^{n} R^M(X, e_i) e_i \right)^\perp + \sum_{i,j=1}^{n} (B(e_i, e_j), X) B(e_i, e_j),
\]

and the normal Laplacian is given by

\[
\Delta^\perp X = \sum_{i=1}^{n} (\nabla_{e_i} \nabla^\perp_{e_i} X - \nabla^\perp_{(\nabla_{e_i} e_i)_T} X).
\]

Here, \( \{e_1, \ldots, e_n\} \) is an orthonormal basis of \( T \Sigma \), \( \nabla \) is the connection of \( M \), \( \nabla^\perp \) is the normal connection of \( \Sigma \) in \( M \), \( B \) is the second fundamental form of \( \Sigma \) in \( M \), and \( R^M \) is the curvature tensor of \( M \).

**Definition 2** The Morse index of \( \Sigma \) is the number of negative eigenvalues of \( J_\Sigma \) counting multiplicities. We say that \( \Sigma \) is stable if it has Morse index 0, i.e.

\[
- \int_{\Sigma} \langle J_\Sigma X, X \rangle d \Sigma \geq 0, \text{ for all } X \in \Gamma(N(M)).
\]

**Remark 2** The Morse index gives us information about the number of directions in which our submanifold fails to minimize area to second order.

### 2.2 Notation

Let \( \tilde{M} := M_1 \times M_2 \) and \( p = (p_1, p_2) \in \tilde{M} \) where \( p_i \) is a point in \( M_i \) for \( i = 1, 2 \), and \( M_1 \) and \( M_2 \) are \( m_1 \) and \( m_2 \)-dimensional Riemannian manifolds, respectively. Then, we have the splitting

\[
T_p(\tilde{M}) = T_{p_1}(M_1) \oplus T_{p_2}(M_2)
\]

i.e., if \( x \in T_p(\tilde{M}) \), we have

\[
x = (x^1, x^2)
\]

where \( x^1 = P^1(x) \in T_{p_1}(M_1) \), \( x^2 = P^2(x) \in T_{p_2}(M_2) \) and \( P^1 \) and \( P^2 \) are the projections on \( T_{p_1}(M_1) \) and \( T_{p_2}(M_2) \) respectively.

Let us denote by \( \tilde{\nabla}, \nabla^1 \) and \( \nabla^2 \) the Riemannian connections of \( \tilde{M}, M_1 \) and \( M_2 \) respectively. Then, for \( X, Y \in \mathfrak{X}(\tilde{M}) \),

\[
\tilde{\nabla}_XY(p) = (\nabla^1_{x^1}Y^1(p), \nabla^2_{x^2}Y^2(p)).
\]
2.3 Projective Spaces

Let $\mathbb{CP}^{m_1}$ be the complex projective space of real dimension $m_1$ with the Fubini-Study metric, and $\mathbb{HP}^{m_1}$ be the quaternionic projective space of real dimension $m_1$ with the standard metric. Let us consider the composition $\Phi_1 := i \circ \phi_1$, where $i$ is the inclusion map of $S^d$ in $R^{d+1}$ and $\phi_1$ is the generalized Veronese embedding $\phi_1 : \mathbb{KP}^{m_1} \to S^d$. Here, $l_d = m_1 + 1$, $K \in \{ \mathbb{C}, \mathbb{H} \}$, and $d = \dim_{\mathbb{R}}(K)$ (see [9, Sect. 2]). We will denote $m = l_d + 1$.

The constant holomorphic sectional curvature of $\mathbb{CP}^{m_1}$ is given by $\lambda_2 := \frac{m_1}{m_1 + 1}$. Moreover, if $R$ is the curvature tensor of $\mathbb{CP}^{m_1}$, then

$$\langle R(X, Y)Z, W \rangle = \frac{\lambda_2^2}{4} \left\{ \langle Y, Z \rangle \langle X, W \rangle - \langle X, Z \rangle \langle Y, W \rangle + \langle JY, Z \rangle \langle JX, W \rangle - \langle JX, Z \rangle \langle JY, W \rangle + 2 \langle X, JY \rangle \langle JZ, W \rangle \right\},$$

where $J$ is the complex structure of $\mathbb{CP}^{m_1}$ (see [8, Eq. (1.1)]). Notice that from Eq. (2), $\lambda_2^2$ is also the maximum of the sectional curvatures on $\mathbb{CP}^{m_1}$.

The maximum of the sectional curvatures on $\mathbb{HP}^{m_1}$ is given by $\lambda_2 := \frac{2m_1}{m_1 + 4}$. Moreover, if $R$ is the curvature tensor of $\mathbb{HP}^{m_1}$, then

$$\langle R(X, Y)Z, W \rangle = \frac{\lambda_2^2}{4} \left\{ \langle Y, Z \rangle \langle X, W \rangle - \langle X, Z \rangle \langle Y, W \rangle + \sum_{k=1}^{3} \langle X, J_k(Y) \rangle \langle J_k(Z), W \rangle + \sum_{k=1}^{3} \langle J_k(Y), Z \rangle \langle J_k(X), W \rangle - \langle J_k(X), Z \rangle \langle J_k(Y), W \rangle \right\},$$

where $J_k$, $k = 1, 2, 3$, is a canonical local basis of the quaternionic Kähler structure of $\mathbb{HP}^{m_1}$ (see [8, Eq. (1.2)]).

Let $B$ be the second fundamental form of $\Phi_1$ (for both cases $K = \mathbb{C}, \mathbb{H}$). Then,

$$3 \langle B(X, Y), B(Z, W) \rangle = \langle R(X, Z)W, Y \rangle + \langle R(X, W)Z, Y \rangle + \lambda_2^2 \langle X, Y \rangle \langle Z, W \rangle + \lambda_2^2 \langle X, W \rangle \langle Y, Z \rangle + \lambda_2^2 \langle X, Z \rangle \langle W, Y \rangle,$$

where $X, Y, Z, W \in T_q \mathbb{KP}^{m_1}$, $q \in \mathbb{KP}^{m_1}$ (see [8, Eq. (3.10)]).

2.4 Riemannian Submersions

In this subsection we prove an important lemma (Lemma 12) which is used in the proof of classification theorems of compact stable minimal immersions of codimension 1 in
Fig. 1 Isometries induced by a Riemannian submersion

$\mathbb{C}P^{m_1} \times M$ (see Theorem 16) and codimension 1 and 2 in $\mathbb{H}P^{m_1} \times M$ (see Theorems 30 and 33). To prove Lemma 12, we need some definitions and some important theorems found in O’Neill [14] and Hermann [15].

We summarized Theorem 1, Proposition 3.1, Proposition 3.2, and Proposition 3.3 from Hermann in [15] in the following statement.

**Proposition 10** [15] Let $\Pi : M \to B$ be an onto Riemannian submersion. If $M$ is complete, so is $B$. In particular, if $\sigma : [a, b] \to B$ is a geodesic segment in $B$, then for each point $m \in M$ with $\Pi(m) \in \sigma(a)$, there exists a unique horizontal lift $\sigma^m : [a, b] \to M$ of $\sigma$ such that,

$$\sigma^m(a) = m$$

and $\sigma^m$ is also geodesic.

Now let $\phi_\sigma : \Pi^{-1}(\sigma(a)) \to \Pi^{-1}(\sigma(b))$ be the function given by $\phi_\sigma(m) = \sigma^m(b)$ (see Fig. I). This function $\phi_\sigma$ is a diffeomorphism. If the fibers of $\Pi$ are totally geodesic submanifolds, then $\phi_\sigma$ is an isometry and $\Pi$ is a fibre bundle.

**Remark 3** Notice that since $\sigma^m$ is a lift of $\sigma$, then

$$\sigma(c) = \Pi(\sigma^m(c)),$$

for all $c \in [a, b]$.

Now, we give the definition of the group of a submersion found in Section 6 before Theorem 5 in [14].

**Definition 3** [14] Let $\Pi : M \to B$ be an onto Riemannian submersion and $M$ complete. Fixed a point $o \in B$. The group of the submersion $\Pi$, $G_\Pi$, is given by
$G_{\Pi} := \{ \phi_{\sigma} : \Pi^{-1}(o) \to \Pi^{-1}(o); \sigma \text{ is a geodesic loop at } o \}$,

with the composition of functions as the group operation.

**Definition 4** Let $\Pi : M \to B$ be a Riemannian submersion. We said that $\Pi$ is a trivial submersion if $M = F \times B$, where $F$ is a Riemannian manifold, and $\Pi(f, b) = b$ is the projection of the Riemannian product $F \times B$ onto the factor $B$, where $f \in F$ and $b \in B$.

We recall Theorem 5 from [14] in the following proposition.

**Proposition 11** [14] Let $\Pi : M \to B$ be an onto Riemannian submersion of a complete Riemannian manifold $M$. Then, $\Pi$ is trivial if and only if the fibers of $\Pi$ are totally geodesic and the group of the submersion vanishes.

Given the technicality of the proof of Lemmas 12 and 13, the reader may wish to skip them on a first reading and continue to Sect. 3.

**Lemma 12** Let $M$ be a complete Riemannian manifold, $B$ be a simply connected Riemannian manifold and $\Pi : M \to B$ an onto Riemannian submersion such that the fibers of $\Pi$ are totally geodesic and the horizontal distribution is integrable. Then, $\Pi$ is the trivial submersion.

**Proof** Using Proposition 11, it is enough to prove that $G_{\Pi}$ is trivial. For a fixed element $o \in B$, let $\sigma : [a, b] \to B$ be a geodesic loop at $o$, i.e., $\sigma(a) = \sigma(b) = o$. Then, we will prove that

$$\phi_{\sigma} : \Pi^{-1}(o) \to \Pi^{-1}(o)$$

is the identity function. For $m \in \Pi^{-1}(\sigma(a))$, there exists a unique horizontal lift $\sigma^m : [a, b] \to M$ of $\sigma$ such that $\sigma^m(a) = m$. Moreover, $\sigma^m$ is also geodesic.

Since the horizontal distribution is integrable, through $m \in M$, there passes a unique maximal connected integral manifold of the horizontal distribution, denoted by $(N, \varphi)$, where $\varphi : N \to M$ is the inclusion map. Now we use the following lemma that will be proved at the end:

**Lemma 13** Under the conditions of Lemma 12, the restriction

$$\Pi|_{\varphi(N)} : (\varphi(N), \tau_N) \to B$$

is a homeomorphism, where $\tau_N$ is the topology induced by $(N, \varphi)$, i.e., such that $\varphi : N \to (\varphi(N), \tau_N)$ is a continuous map (notice that $\varphi : N \to (\varphi(N), \tau_N)$ is an open map because $\varphi$ is injective).

We have $\sigma^m([a, b]) \subset \varphi(N)$ because $\sigma^m$ is horizontal and $\sigma^m([a, b])$ is connected.

Since $\sigma(a) = \sigma(b)$, we have that $\Pi(\sigma^m(a)) = \Pi(\sigma^m(b))$, by Remark 3. Consequently, $\Pi|_{\varphi(N)}(\sigma^m(a)) = \Pi|_{\varphi(N)}(\sigma^m(b))$. By Lemma 13, $\sigma^m(a) = \sigma^m(b)$ and thus,

$$\phi_{\sigma}(m) = \sigma^m(b) = \sigma^m(a) = m.$$  

Since $m$ was arbitrary, $\phi_{\sigma}$ is the identity. This proves the Lemma 12. \qed

Now, let us prove Lemma 13:
Proof Let us denote $\pi := \Pi|_{\varphi(N)}$. First, we will prove that $\pi$ is a covering map of $B$.

- **$\pi$ is continuous.** Let $W$ be an open set in $B$. Since $\Pi \circ \varphi$ is continuous,

$$\left(\Pi \circ \varphi\right)^{-1}(W) = \varphi^{-1}(\Pi^{-1}(W)) = \varphi^{-1}(\varphi(N) \cap \Pi^{-1}(W))$$

is an open set in $N$. Therefore, $\varphi(N) \cap \Pi^{-1}(W) = \pi^{-1}(W)$ is open in $(\varphi(N), \tau_N)$.

- **$\pi$ is onto.** Let $b \in B, n \in \varphi(N)$ and $q := \Pi(n)$. Since $B$ is arcwise connected, there is a geodesic segment $\gamma : [0, 1] \rightarrow B$ with $\gamma(0) = q$ and $\gamma(1) = b$. Consequently, there exists a unique horizontal lift $\gamma^n : [0, 1] \rightarrow M$ of $\gamma$ such that

$$\gamma^n(0) = n, \quad \gamma^n(1) \in \Pi^{-1}(b).$$

We have that $\gamma^n([0, 1]) \subset \varphi(N)$, because $\gamma^n$ is horizontal, $\gamma^n(0) \in \varphi(N)$, and $\gamma'^n([0, 1])$ is connected. Therefore, $b = \Pi(\gamma^n(1)) = \pi(\gamma^n(1))$, for $\gamma^n(1) \in \varphi(N)$. Since $b$ was arbitrary, $\pi$ is onto.

- **Disjoint union.** Let $x \in \bar{B}$. By [15, Proposition 10], $\Pi$ is a fiber bundle. Since the horizontal distribution is integrable, there exists a connected open set $U$ of $B$ containing $x$, such that

$$\Pi^{-1}(U) = U \times F \quad \text{(6)}$$

$$\Pi(u, f) = u, \quad \text{(7)}$$

for $(u, f) \in U \times F$, where $F$ is a typical fiber of $\Pi (F = \Pi^{-1}(b)$, for some $b \in B$). Thus,

$$\pi^{-1}(U) = \Pi^{-1}(U) \cap \varphi(N) = \bigcup_{f \in F} (U \times \{f\}) \cap \varphi(N).$$

It is evident that the last union is disjoint. Let us consider the following set,

$$F' := \{f \in F : (U \times \{f\}) \cap \varphi(N) \neq \emptyset\}.$$

Notice that $F' \neq \emptyset$, because $\pi$ is onto, and then $\pi^{-1}(U) \neq \emptyset$. For $f \in F'$, we have that $U \times \{f\} \subset \varphi(N)$, because $U \times \{f\}$ is a connected integral manifold of the horizontal distribution, and $(U \times \{f\}) \cap \varphi(N) \neq \emptyset$. Therefore,

$$\pi^{-1}(U) = \bigcup_{f \in F'} U \times \{f\}.$$

- **The set $U \times \{f\}$, for $f \in F'$ is an open set of $(\varphi(N), \tau_N)$.** Notice that $(U \times \{f\}, \tau_u)$ is a topological manifold with smooth structure such that $i : U \times \{f\} \rightarrow M$ is a smooth embedding, where $\tau_u$ is the subspace topology induced by $M$ (equivalently induced by $U \times F$). By [16, Theorem 1.62], there exists a unique $C^{\infty}$ map $i : (U \times \{f\}, \tau_u) \rightarrow N$ such that $\varphi \circ i = i$.

The map $i : (U \times \{f\}, \tau_u) \rightarrow N$ is nonsingular, because $i : U \times \{f\} \rightarrow M$ is nonsingular. Applying the inverse function theorem (see [17, Proposition 5.16]) to the $C^{\infty}$ function $\tilde{i} : (U \times \{f\}, \tau_u) \rightarrow N$, we have that $\tilde{i}$ is a local diffeomorphism.

Therefore, $\tilde{i}$ is an open map, and then $\tilde{i}(U \times \{f\})$ is an open set of $N$. Since $\varphi : N \rightarrow (\varphi(N), \tau_N)$ is an open map, it follows that $\varphi(\tilde{i}(U \times \{f\})) = i(U \times \{f\}) = U \times \{f\}$ is open in $(\varphi(N), \tau_N)$. 

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The mapping,

\[ \pi|_{U \times \{f\}} : (U \times \{f\}, \tau_{\varphi(N)}) \to B \]  

(8)

is a homeomorphism, where \( \tau_{\varphi(N)} \) is the subspace topology in \( U \times \{f\} \) induced by \( (\varphi(N), \tau_N) \). Notice that from Eq. (7) the mapping, 

\[ \pi|_{U \times \{f\}} : (U \times \{f\}, \tau_u) \to U \]

is a homeomorphism. Therefore, it is enough to prove that \( \tau_u = \tau_{\varphi(N)} \).

Let \( W \in \tau_u \). The set \( \varphi(\bar{i}(W)) = i(W) = W \) is open in \( (\varphi(N), \tau_N) \), because \( \bar{i} \) and \( \varphi \) are open maps. Since \( W = W \cap (U \times \{f\}) \), we have that \( W \in \tau_{\varphi(N)} \).

Now, let \( W \in \tau_{\varphi(N)} \). There exists an open set \( \omega \) in \( (\varphi(N), \tau_N) \) such that \( W = (U \times \{f\}) \cap \omega \). Since \( U \times \{f\} \) is open in \( (\varphi(N), \tau_N) \), it follows that \( W \) is an open set of \( (\varphi(N), \tau_N) \), and therefore \( \varphi^{-1}(W) \) is open in \( N \). Thus, \( \bar{i}^{-1}(\varphi^{-1}(W)) \in \tau_u \), because \( \bar{i} \) is continuous. From the following equality,

\[ \bar{i}^{-1}(\varphi^{-1}(W)) = (\varphi \circ \bar{i})^{-1}(W) = i^{-1}(W) = W, \]

we conclude that \( W \in \tau_u \).

Until now, we have proved that \( \Pi|_{\varphi(N)} : (\varphi(N), \tau_N) \to B \) is a covering map of \( B \). We recall the following technical result.

**Lemma 14** [18, Exercise 6.1, Chapter 5] If \( X \) is a simply connected space and \( (\tilde{X}, p) \) is a covering space of \( X \), then \( p \) is a homeomorphism of \( \tilde{X} \) onto \( X \).

Since \( B \) is simply connected and applying the previous lemma, we have the map \( \Pi|_{\varphi(N)} : (\varphi(N), \tau_N) \to B \) is a homeomorphism.

### 3 Minimal Stable Submanifolds in \( \mathbb{C}P^{m_2} \times M \)

#### 3.1 General Formula

In this subsection we prove Lemma 15, which is fundamental for the development of main theorems in Sects. 3.2, 3.3, 3.4.

Let \( \Phi = (\psi, \phi) : \Sigma \to \tilde{M} := \mathbb{C}P^{m_2} \times M \) be a compact minimal immersion of codimension \( d \) and dimension \( n \), where \( M \) is any Riemannian manifold of dimension \( m_2 \) and \( \Phi_1 : \mathbb{C}P^{m_2} \to \mathbb{R}^m \) is the immersion described in Sect. 2.3. For each \( v \in \mathbb{R}^m \) let us consider the following:

\[ v := (v, 0) \in T(\mathbb{R}^m \times M) \]

\[ N_v := [v]^N, \]

where \([.]^N\) is projection in the orthogonal complement, \( N_p \Sigma \), of \( T_p \Sigma \) in \( T_{\Phi(p)}\tilde{M} \), \( p \in \Sigma \).
Lemma 15  Let \( p \in \Sigma, \{e_1, \ldots, e_n\} \) be an orthonormal basis of \( T_p \Sigma, \{\eta_1, \ldots, \eta_d\} \) be an orthonormal basis of \( N_p \Sigma, \) and \( \{E_1, \ldots, E_m\} \) be the usual canonical basis of \( \mathbb{R}^m. \) Then,

\[
\sum_{A=1}^{m} -\langle N_{E_A}, J_{\Sigma}(N_{E_A}) \rangle = \lambda^2 \left( \sum_{k=1}^{d} \sum_{l=1}^{d} \langle J(\eta^1_k), \eta^1_l \rangle^2 - \langle \eta^1_k, \eta^1_l \rangle^2 \right) \quad (9)
\]

\[
= \lambda^2 \left( \sum_{j=1}^{n} \sum_{i=1}^{n} \langle J(e^1_j), e^1_i \rangle^2 - \langle e^1_j, e^1_i \rangle^2 \right), \quad (10)
\]

where \( e^1 \) denotes the projection of the tangent vector \( x \) onto the first factor.

Proof  Recall that \( R \) is the curvature tensor of \( \mathbb{C}P^{\frac{m}{2}} \) and \( B \) is the second fundamental form of \( \mathbb{C}P^{\frac{m}{2}} \) in \( \mathbb{R}^m \) (see Sect. 1.2). By [11, Eq. (2.8)], we have the following

\[
\sum_{A=1}^{m} -\langle N_{E_A}, J_{\Sigma}(N_{E_A}) \rangle = \sum_{j=1}^{n} \sum_{k=1}^{d} 2|B(e^1_j, \eta^1_k)|^2 - \langle B(\eta^1_k, \eta^1_k), B(e^1_j, e^1_j) \rangle. \quad (11)
\]

Using Eq. (4), we have:

\[
3|B(e^1_j, \eta^1_k)|^2 = \langle R(e^1_j, \eta^1_k)e^1_j, \eta^1_k \rangle + 2\lambda^2 \langle e^1_j, \eta^1_k \rangle^2 + \lambda^2 |e^1_j|^2 |\eta^1_k|^2
\]

and

\[
3\langle B(\eta^1_k, \eta^1_k), B(e^1_j, e^1_j) \rangle = -2\langle R(e^1_j, \eta^1_k)e^1_j, \eta^1_k \rangle + \lambda^2 |e^1_j|^2 |\eta^1_k|^2 + 2\lambda^2 \langle e^1_j, \eta^1_k \rangle^2.
\]

Now using the last two equalities in Eq. (11),

\[
\sum_{A=1}^{m} -\langle N_{E_A}, J_{\Sigma}(N_{E_A}) \rangle
\]

\[
= \sum_{j=1}^{n} \sum_{k=1}^{d} \frac{2}{3} \left( \langle R(e^1_j, \eta^1_k)e^1_j, \eta^1_k \rangle + 2\lambda^2 \langle e^1_j, \eta^1_k \rangle^2 + \lambda^2 |e^1_j|^2 |\eta^1_k|^2 \right)
\]

\[
- \frac{1}{3} \left( -2\langle R(e^1_j, \eta^1_k)e^1_j, \eta^1_k \rangle + \lambda^2 |e^1_j|^2 |\eta^1_k|^2 + 2\lambda^2 \langle e^1_j, \eta^1_k \rangle^2 \right)
\]

\[
= \sum_{j=1}^{n} \sum_{k=1}^{d} -\frac{4}{3} \langle R(e^1_j, \eta^1_k)e^1_j, \eta^1_k \rangle + \frac{2\lambda^2}{3} |e^1_j|^2 |\eta^1_k|^2 + \frac{\lambda^2}{3} |e^1_j|^2 |\eta^1_k|^2. \quad (12)
\]

Using Eq. (2) in Eq. (12),

\[
\sum_{A=1}^{m} -\langle N_{E_A}, J_{\Sigma}(N_{E_A}) \rangle = \lambda^2 \left( \sum_{j=1}^{n} \sum_{k=1}^{d} \langle e^1_j, \eta^1_k \rangle^2 - \langle e^1_j, J(\eta^1_k) \rangle^2 \right). \quad (13)
\]
For \( k \in \{1, \ldots, d\} \),

\[
|\eta_k^1|^2 = |J(\eta_k^1)|^2 = |(J(\eta_k^1), 0)|^2 = \sum_{j=1}^{n} (\langle J(\eta_k^1), e_j \rangle)^2 + \sum_{l=1}^{d} (\langle J(\eta_k^1), \eta_l \rangle)^2 = \sum_{j=1}^{n} (J(\eta_k^1), e_j^1)^2 + \sum_{l=1}^{d} (J(\eta_k^1), \eta_l^1)^2.
\]

Then,

\[
-\sum_{j=1}^{n} \langle J(\eta_k^1), e_j^1 \rangle^2 = -|\eta_k^1|^2 + \sum_{l=1}^{d} (J(\eta_k^1), \eta_l^1)^2,
\]

and summing in \( k \),

\[
-\sum_{j=1}^{n} \sum_{k=1}^{d} \langle J(\eta_k^1), e_j^1 \rangle^2 = -\sum_{k=1}^{d} |\eta_k^1|^2 + \sum_{k=1}^{d} \sum_{l=1}^{d} (J(\eta_k^1), \eta_l^1)^2. \tag{14}
\]

On the other hand, again for \( k \in \{1, \ldots, d\} \)

\[
|\eta_k^1|^2 = |(\eta_k^1, 0)|^2 = \sum_{j=1}^{n} (\langle \eta_k^1, e_j \rangle)^2 + \sum_{l=1}^{d} (\langle \eta_k^1, \eta_l \rangle)^2 = \sum_{j=1}^{n} (\eta_k^1, e_j^1)^2 + \sum_{l=1}^{d} (\eta_k^1, \eta_l^1)^2.
\]

Therefore,

\[
\sum_{j=1}^{n} (\eta_k^1, e_j^1)^2 = |\eta_k^1|^2 - \sum_{l=1}^{d} (\eta_k^1, \eta_l^1)^2,
\]

and summing in \( k \),

\[
\sum_{j=1}^{n} \sum_{k=1}^{d} (\eta_k^1, e_j^1)^2 = \sum_{k=1}^{d} |\eta_k^1|^2 - \sum_{k=1}^{d} \sum_{l=1}^{d} (\eta_k^1, \eta_l^1)^2. \tag{15}
\]

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Then, in order to prove Eq. (9), we replace Eqs. (14) and (15) in (13),

\[
\sum_{A=1}^{m} -\langle N_{E_A}, J \Sigma (N_{E_A}) \rangle
\]

\[
= \lambda^2 \left( \sum_{k=1}^{d} |\eta_k^1|^2 - \sum_{k=1}^{d} \sum_{l=1}^{d} \langle \eta_k^1, \eta_l^1 \rangle^2 \\
- \sum_{k=1}^{d} |\eta_k^1|^2 + \sum_{k=1}^{d} \sum_{l=1}^{d} \langle \eta_k^1, \eta_l^1 \rangle^2 \right)
\]

\[
= \lambda^2 \left( \sum_{k=1}^{d} \sum_{l=1}^{d} \langle \eta_k^1, \eta_l^1 \rangle^2 - \langle \eta_k^1, \eta_l^1 \rangle^2 \right).
\]

Now, let us prove Eq. (10). For \( j \in \{1, \ldots, n\} \),

\[
|e_j^1|^2 = |J(e_j^1)|^2 = |(J(e_j^1), 0)|^2
\]

\[
= \sum_{i=1}^{n} \langle (J(e_j^1), 0), e_i \rangle^2 + \sum_{k=1}^{d} \langle (J(e_j^1), 0), \eta_k \rangle^2
\]

\[
= \sum_{i=1}^{n} \langle J(e_j^1), e_i^1 \rangle^2 + \sum_{k=1}^{d} \langle J(e_j^1), \eta_k^1 \rangle^2.
\]

Then,

\[
- \sum_{k=1}^{d} \langle J(e_j^1), \eta_k^1 \rangle^2 = -|e_j^1|^2 + \sum_{i=1}^{n} \langle J(e_j^1), e_i^1 \rangle^2,
\]

and summing in \( j \),

\[
- \sum_{j=1}^{n} \sum_{k=1}^{d} \langle J(e_j^1), \eta_k^1 \rangle^2 = - \sum_{j=1}^{n} |e_j^1|^2 + \sum_{j=1}^{n} \sum_{i=1}^{n} \langle J(e_j^1), e_i^1 \rangle^2. \tag{16}
\]

On the other hand, again for \( j \in \{1, \ldots, n\} \)

\[
|e_j^1|^2 = |(e_j^1, 0)|^2
\]

\[
= \sum_{i=1}^{n} \langle (e_j^1, 0), e_i \rangle^2 + \sum_{k=1}^{d} \langle (e_j^1, 0), \eta_k \rangle^2
\]

\[
= \sum_{i=1}^{n} \langle e_j^1, e_i^1 \rangle^2 + \sum_{k=1}^{d} \langle e_j^1, \eta_k^1 \rangle^2.
\]

Therefore,
\[
\sum_{k=1}^{d} \langle e_j^k, \eta_k \rangle^2 = |e_j^1|^2 - \sum_{i=1}^{n} \langle e_j^1, e_i^1 \rangle^2,
\]

summing in \(j\),

\[
\sum_{j=1}^{n} \sum_{k=1}^{d} \langle e_j^k, \eta_k \rangle^2 = \sum_{j=1}^{n} |e_j^1|^2 - \sum_{j=1}^{n} \sum_{i=1}^{n} \langle e_j^1, e_i^1 \rangle^2.
\]  

(17)

Replacing Eqs. (16) and (17) in (13)

\[
\sum_{A=1}^{m} -\langle N_{E_A}, J_{\Sigma}(N_{E_A}) \rangle = \lambda^2 \left( \sum_{j=1}^{n} |e_j^1|^2 - \sum_{j=1}^{n} \sum_{i=1}^{n} \langle e_j^1, e_i^1 \rangle^2 - \sum_{j=1}^{n} |e_j^1|^2 + \sum_{j=1}^{n} \sum_{j=1}^{n} \langle J(e_j^1), e_i^1 \rangle^2 \right)
\]

\[
= \lambda^2 \left( \sum_{j=1}^{n} \sum_{j=1}^{n} \langle J(e_j^1), e_i^1 \rangle^2 - \langle e_j^1, e_i^1 \rangle^2 \right).
\]

\[\square\]

3.2 Codimension 1

In this subsection, we will use the general formula proved in the last subsection to prove a classification theorem for compact stable minimal immersions of codimension 1 in the product of a complex projective space with an arbitrary Riemannian manifold. Moreover, as an application, we obtain some corollaries when the second manifold is a compact rank one space.

**Theorem 16** Let \(\Phi = (\psi, \phi) : \Sigma \rightarrow \tilde{M} := \mathbb{C}P^{m_1} \times M\) be a compact stable minimal immersion of codimension \(d = 1\), where \(M\) is any Riemannian manifold of dimension \(m_2\). Then, \(\Sigma = \mathbb{C}P^{m_1} \times \hat{\Sigma}, \Phi = 1d \times \hat{\phi}\) where \(\hat{\phi} : \hat{\Sigma} \rightarrow M\) is a compact stable minimal immersion of codimension 1, and therefore \(\Phi(\Sigma) = \mathbb{C}P^{m_1} \times \hat{\phi}(\hat{\Sigma})\). In particular, for \(m_2 = 1, \Sigma = \mathbb{C}P^{m_1}, \hat{\phi}\) is a constant function, and \(\Phi(\Sigma) = \mathbb{C}P^{m_1} \times \{q\}, \) for \(q \in M\).

**Proof** Since \(d = 1\), Eq. (9) becomes

\[
\sum_{A=1}^{m} -\langle N_{E_A}, J_{\Sigma}(N_{E_A}) \rangle = \lambda^2 (\langle J(\eta^1), \eta^1 \rangle^2 - \langle \eta^1, \eta^1 \rangle^2) = -\lambda^2 |\eta^1|^4,
\]

where \(\eta\) is a unit vector in \(N_p \Sigma\), for \(p \in \Sigma\). Therefore,
\[ 0 \leq \sum_{A=1}^{m} \int_{\Sigma} \langle N_{E_{A}}, J_{\Sigma}(N_{E_{A}}) \rangle d \Sigma = -\lambda^2 \int_{\Sigma} |\eta|^4 d \Sigma \leq 0, \]

where we have used the fact that \( \Sigma \) is stable in the first inequality. Hence, for \( p \in \Sigma, \eta^1 = 0 \), and therefore \( \eta = (0, \eta^2) \). Then,

\[ d \Phi_p(T_p \Sigma) = \tilde{D}_1(p) \bigoplus \tilde{D}_2(p), \]

where \( \tilde{D}_1 \) and \( \tilde{D}_2 \) are given by:

\[ \tilde{D}_1(p) = \{(x, 0) : x \in T_{\psi(p)}C P^{m1} \}, \]

and

\[ \tilde{D}_2(p) = \{(0, w) : w \in [\eta^2]^{1-m} \}, \]

where \([z]^{1-m} \) is the orthogonal complement of \( z \) in \( T_{\psi(p)}M \). Since \( d \Phi_p(T_p \Sigma) \) is isometric to \( T_{\psi(p)} \Sigma \), \( \tilde{D}_1 \) and \( \tilde{D}_2 \) induce orthogonal complementary smooth distributions \( D_1 \) and \( D_2 \) on \( \Sigma \) given by

\( D_1(p) = \{h \in T_p \Sigma : d \Phi_p(h) \in \tilde{D}_1(p)\} = \{h \in T_p \Sigma : d \phi_p(h) = 0\} = \ker(d \phi_p) \)

\( D_2(p) = \{v \in T_p \Sigma : d \Phi_p(v) \in \tilde{D}_2(p)\} = \{v \in T_p \Sigma : d \psi_p(v) = 0\} = \ker(d \psi_p). \)

**Lemma 17** The function \( \psi : \Sigma \to C P^{m1} \) is an onto Riemannian submersion, with horizontal and vertical distributions given by \( D_1 \) and \( D_2 \), respectively. Moreover, \( D_1 \) and \( D_2 \) are totally geodesic distributions.

**Proof**

- **The mapping** \( d \psi_p : T_p \Sigma \to T_{\psi(p)}C P^{m1} \) **is onto.** Let \( x \in T_{\psi(p)}C P^{m1} \). Since \( (x, 0) \in \tilde{D}_1(p) \subset d \Phi_p(T_p \Sigma) \), there exists \( h \in T_p \Sigma \) such that

\[ d \Phi_p(h) = (d \psi_p(h), d \phi_p(h)) = (x, 0). \]

Therefore, \( d \psi_p(h) = x \).

- By definition the vertical vectors \( v \) of \( \psi \) at \( p \in \Sigma \) are such that \( d \psi_p(v) = 0 \). Therefore, \( D_2(p) \) consists of the vertical vectors, and thus \( D_1(p) \) consists of the horizontal vectors.

- **\( d \psi_p \) preserves the length of horizontal vectors.** Let \( h_1, h_2 \) be horizontal vectors. Therefore, \( h_1, h_2 \in D_1(p) \) and then \( d \phi_p(h_1) = d \phi_p(h_2) = 0 \). Since \( \Phi \) is an isometric immersion,

\[ \langle d \phi(h_1), d \phi(h_2) \rangle = \langle h_1, h_2 \rangle. \]

But,

\[ \langle d \phi(h_1), d \phi(h_2) \rangle = \langle (d \psi(h_1), d \phi(h_1)), (d \psi(h_2), d \phi(h_2)) \rangle \]

\[ = \langle d \psi(h_1), d \psi(h_2) \rangle. \]

Then,
\[ \langle d\psi_p(h_1), d\psi_p(h_2) \rangle = \langle h_1, h_2 \rangle. \]

- **\( \psi \) is onto.** Until now, we have that \( \psi \) is a Riemannian submersion. By properties of submersions [17, Proposition 5.18], \( \psi(\Sigma) \) is an open set in \( \mathbb{C}P^{m_1} \). Now \( \psi : \Sigma \to \psi(\Sigma) \) is an onto Riemannian submersion. Since \( \Sigma \) is complete, \( \psi(\Sigma) \) is complete by Hermann [15] and then closed. Therefore, \( \psi(\Sigma) = \mathbb{C}P^{m_1} \) because \( \psi(\Sigma) \) is a closed and open set of the connected set \( \mathbb{C}P^{m_1} \).

- **\( D_1 \) and \( D_2 \) are totally geodesic distributions.** Let \( \nabla \) and \( \tilde{\nabla} \) be the connections of Levi-Civita on \( \Sigma \) and \( \tilde{M} \), respectively. Let \( q = (q_1, q_2) \in \tilde{M} \) and \( P : T_q\tilde{M} \to T_q\tilde{M} \) be a mapping given by

\[
P(v_1, v_2) = (v_1, -v_2), \text{ where } v_1 \in T_{q_1}\mathbb{C}P^{m_1} \text{ and } v_2 \in T_{q_2}M.
\]

The map \( P \) is a linear isometry that is parallel, \( \tilde{\nabla}P = 0 \), i.e. \( (\tilde{\nabla}_AP)C = 0 \), for all \( A, C \in T_q\tilde{M} \).

Let us define at every point \( p \in \Sigma \) the mapping \( P^\Sigma : T_p\Sigma \to T_p\Sigma \) given by

\[
P^\Sigma(w_1 + w_2) = w_1 - w_2, \text{ where } w_1 \in D_1(p) \text{ and } w_2 \in D_2(p).
\]

Notice that \( P^\Sigma \) is a Riemannian almost product structure on \( \Sigma \), where the eigenspaces of the eigenvalues 1 and \(-1\) of the operator \( P^\Sigma \) are precisely given by \( D_1(p) \) and \( D_2(p) \), respectively. We have the following properties:

1. \( d\Phi(P^\Sigma) = P(d\Phi) \). Let \( x = x_1 + x_2 \in T_p\Sigma \), where \( x_i \in D_i(p), i = 1, 2 \).

Then,

\[
d\Phi_p(P^\Sigma(x)) = d\Phi_p(x_1 - x_2) = d\Phi_p(x_1) - d\Phi_p(x_2)
= (d\psi_p(x_1), 0) - (0, d\phi_p(x_2)) = (d\psi_p(x_1), -d\phi_p(x_2))
= P(d\psi_p(x_1), d\phi_p(x_2)) = P(d\psi_p(x_1 + x_2), d\phi_p(x_1 + x_2))
= P(d\Phi_p(x)).
\]

2. \( P([x]^\Sigma) = [P(x)]^\Sigma \), where \([\cdot]^\Sigma \) is the projection onto \( d\Phi(T\Sigma) \). Let \( x \in T_{\Phi(p)}\tilde{M} \) and \( \{e_i\}_{i=1}^n \) be an orthonormal basis of \( T_p \Sigma \). Then,

\[
P([x]^\Sigma) = P(\sum_{i=1}^n \langle x, d\Phi_p(e_i) \rangle d\Phi_p(e_i))
= \sum_{i=1}^n \langle x, d\Phi_p(e_i) \rangle P(d\Phi_p(e_i))
= \sum_{i=1}^n \langle P(x), P(d\Phi_p(e_i)) \rangle P(d\Phi_p(e_i))
= \sum_{i=1}^n \langle P(x), d\Phi_p(P^\Sigma(e_i)) \rangle d\Phi_p(P^\Sigma(e_i))
= [P(x)]^\Sigma,
\]
where we have used the previous fact that $P$ is an isometry and $\{ P^\Sigma (e_i) \}_{i=1}^n$ is still an orthonormal basis of $T_p \Sigma$.

Let $X, Y$ be vector fields in $D_1$. Then

$$d \Phi((\nabla_X P) Y) = d \Phi(\nabla_X P Y) - P^{\Sigma}(\nabla_X Y)$$

$$= d \Phi(\nabla_X P^\Sigma (Y)) - d \Phi(\nabla_X Y)$$

$$= [\tilde{\nabla}_d \Phi(X) d \Phi(\nabla_X Y)]^\Sigma - P(\nabla_X Y)$$

$$= [\tilde{\nabla}_d \Phi(X) P(\nabla_X Y)]^\Sigma - P(\tilde{\nabla}_d \Phi(X) d \Phi(\nabla_X Y))$$

$$= [\tilde{\nabla}_d \Phi(X) P(\nabla_X Y) - P(\tilde{\nabla}_d \Phi(X) d \Phi(\nabla_X Y))]^\Sigma$$

$$= [\tilde{\nabla}_d \Phi(X) P d \Phi(Y)]^\Sigma = 0.$$

Since $d \Phi$ is one to one, $(\nabla_X P^\Sigma) Y = 0$. Therefore, by [19, Proposition 2], $D_1$ is a totally geodesic distribution.

Analogously, redefining $P^\Sigma : T_p \Sigma \to T_p \Sigma$ by $P^\Sigma (v_1 + v_2) = -v_1 + v_2$, we conclude that $D_2$ is also a totally geodesic distribution. □

By Lemma 12, we get that $\Sigma = \mathbb{CP}^{m_1}_r \times \psi^{-1}(s)$ up to an isometry, where $s \in \mathbb{CP}^{m_1}_r$ is a fixed element and that $\psi$ is the trivial submersion, i.e. $\psi (p) = r$, for $p = (r, q)$ where $r \in \mathbb{C}P^{m_1}$ and $q \in \psi^{-1}(s)$ (we pick an arbitrary element $s$ because the fibers of $\psi$ are isometric, see [15]). Notice that $\psi^{-1}(s)$ is a compact Riemannian manifold of dimension $m_2 - 1$ and that

$$d \psi_p (x, w) = x \text{ for } x \in T_r \mathbb{CP}^{m_1}_r \text{ and } w \in T_q \psi^{-1}(s).$$

Now, we will show that the function $\phi$ does not depend on $r \in \mathbb{CP}^{m_1}_r$. It suffices to prove that $d \phi_p = 0$ for $x \in T_r \mathbb{CP}^{m_1}_r$. Let $x \in T_r \mathbb{CP}^{m_1}_r$. Then,

$$d \Phi_p (x, 0) = (d \psi_p (x, 0), d \phi_p (x, 0)) = (x, d \phi_p (x, 0)),$

which implies that

$$|d \Phi_p (x, 0)|^2 = |x|^2 + |d \phi_p (x, 0)|^2.$$

On the other hand, since $\Phi$ is an isometric immersion,

$$|d \Phi_p (x, 0)|^2 = |(x, 0)|^2 = |x|^2.$$

Thus, $|d \phi_p (x, 0)|^2 = 0$. Since $\phi$ does not depend of $r \in \mathbb{CP}^{m_1}_r$, we can fix $r = s$ and denote

$$\hat{\phi} : \psi^{-1}(s) \to M$$

$$q \to \hat{\phi}(q) := \phi(s, q),$$

with $d \hat{\phi}_q (w) = d \phi_p (0, w)$ for $p = (s, q), q \in \psi^{-1}(s)$. Now, it only remains to prove that $\hat{\phi}$ is a stable minimal immersion in $M$.

- If $d \hat{\phi}_q (w) = 0$, for $w \in T_q \psi^{-1}(s)$, then
\[d \Phi_p(0, w) = (d \psi_p(0, w), d \phi_p(0, w)) = (0, 0).\]

Since \(\Phi\) is an immersion, \((0, w) = (0, 0)\), and thus \(w = 0\).

- Let \(w, v \in T_q \psi^{-1}(s)\),

\[
\langle d\hat{\phi}_q(w), d\hat{\phi}_q(v) \rangle = \langle d\phi_p(0, w), d\phi_p(0, v) \rangle
\]

\[
= \langle (0, d\phi_p(0, w)), (0, d\phi_p(0, v)) \rangle
\]

\[
= \langle (d\psi_p(0, w), d\phi_p(0, w)), (d\psi_p(0, v), d\phi_p(0, v)) \rangle
\]

\[
= \langle d\Phi_p(0, w), d\Phi_p(0, v) \rangle
\]

\[
= \langle (0, w), (0, v) \rangle
\]

\[
= \langle w, v \rangle
\]

- Let \(H\) be the mean curvature vector of \(\Phi\), \(p = (r, q) \in \Sigma, N = (0, N^2)\) be a local unit normal vector field around \(p\), \(X_1, \ldots, X_{m_1}\) be local orthonormal vector fields around \(r\) in \(\mathbb{C}P^{m_1}\), and \(W_1, \ldots, W_{m_2-1}\) be local orthonormal vector fields around \(q\) in \(\psi^{-1}(s)\). We will omit the evaluation at \(p\) in the following computation,

\[
0 = H = \sum_{i=1}^{m_1} [\tilde{\nabla}_{d\phi(X_i, 0)} d\Phi(X_i, 0)]^N + \sum_{j=1}^{m_2-1} [\tilde{\nabla}_{d\phi(0, W_j)} d\Phi(0, W_j)]^N
\]

\[
= \sum_{i=1}^{m_1} (\tilde{\nabla}_{X_i}(X_i, 0)(0, N^2))(0, N^2) + \sum_{j=1}^{m_2-1} (\tilde{\nabla}_{d\phi(0, W_j)}(0, d\phi(W_j)))(0, N^2)(0, N^2)
\]

\[
= \sum_{i=1}^{m_1} (\tilde{\nabla}_{X_i}(X_i, 0)(0, N^2))(0, N^2) + \sum_{j=1}^{m_2-1} (\tilde{\nabla}_{d\phi(0, W_j)}(0, d\phi(W_j)))(0, N^2)(0, N^2)
\]

\[
= \sum_{j=1}^{m_2-1} (0, d\phi(0, W_j))(0, N^2)(0, N^2)
\]

\[
= \left(0, \sum_{j=1}^{m_2-1} (\tilde{\nabla}_{d\phi(W_j)} d\phi(W_j), N^2)N^2\right)
\]

where \(\nabla^1\) and \(\nabla^2\) are the connections on \(\mathbb{C}P^{m_1}\) and \(M\), respectively. Then,

\[
A := \sum_{j=1}^{m_2-1} (\tilde{\nabla}_{d\phi(W_j)} d\phi(W_j), N^2)N^2 = 0.
\]

Notice that \(A\) is the mean curvature vector of the immersion \(\hat{\phi}\), and thus \(\hat{\phi}\) is minimal. Finally, \(\hat{\phi}\) is stable because \(\Sigma\) is stable (see Preliminaries Section in [10]).

\(\square\)
Corollary 18 There are no compact stable minimal hypersurfaces in the product manifold $\mathbb{CP}^{\frac{m_1}{2}} \times S^1$, $\mathbb{CP}^{\frac{m_1}{2}} \times \mathbb{O}P^2$, or $\mathbb{CP}^{\frac{m_1}{2}} \times \mathbb{K}P^s$, where $\mathbb{K} \in \{ \mathbb{C}, \mathbb{H} \}$ other than $\mathbb{CP}^{\frac{m_1}{2}} \times \{ q \}$ in $\mathbb{CP}^{\frac{m_1}{2}} \times S^1$, where $q \in S^1$.

Proof We apply Theorem 16, and notice that there are no stable minimal hypersurfaces in $S^s$, $\mathbb{CP}^s$, $\mathbb{HP}^s$, or $\mathbb{OP}^2$ (see Theorems 1 and 3) other than a point $q$ in $S^1$.

In this sense, we also obtain the following result.

Corollary 19 The only compact stable minimal hypersurface in the product space $\mathbb{CP}^{\frac{m_1}{2}} \times \mathbb{R}P^s$ is $\mathbb{CP}^{\frac{m_1}{2}} \times \mathbb{R}P^{s-1}$.

Proof The proof is completed by using Theorem 16, and noticing that the only stable minimal hypersurface in $\mathbb{R}P^s$ is $\mathbb{R}P^{s-1}$ (see Theorem 2).

3.3 Codimension 2 or Dimension 2

In this subsection, we will use the general formula obtained in Sect. 3.1 to prove a characterization of compact stable minimal immersions of codimension or dimension two in the product of two complex projective spaces.

Lemma 20 Let $\Phi = (\psi, \phi) : \Sigma \to \mathbb{CP}^{\frac{m_1}{2}} \times M$ be a compact stable minimal immersion of codimension $d = 2$, where $M$ is any Riemannian manifold of dimension $m_2$. Then, $\eta_2 = \pm J(\eta_1)$, where $\{ \eta_1, \eta_2 \}$ is an orthonormal basis of $N_p \Sigma$, $p \in \Sigma$, $J$ is the complex structure of $\mathbb{CP}^{\frac{m_1}{2}}$ and $\eta_1$ is the projection of $\eta_i$ in $T_{\psi(p)} \mathbb{CP}^{\frac{m_1}{2}}$.

Proof Let $p \in \Sigma$. Since $d = 2$, Eq. (9) becomes

$$\sum_{A=1}^{m} -\langle N_{E_A}, J_\Sigma (N_{E_A}) \rangle = \lambda^2 2 \sum_{k=1}^{2} \sum_{l=1}^{2} (J(\eta_k^1), \eta_l^1)^2 - (\eta_k^1, \eta_l^1)^2$$

$$= \lambda^2 \left( -|\eta_1^1|^4 - |\eta_2^1|^4 - 2(\eta_1^1, \eta_2^1)^2 + 2(J(\eta_1^1), \eta_2^1)^2 \right).$$

(19)

If $\eta_1^1 = 0$,

$$\sum_{A=1}^{m} -\langle N_{E_A}, J_\Sigma (N_{E_A}) \rangle = -\lambda^2 |\eta_2^1|^4 \leq 0.$$

(20)

Or either, $\eta_1^1 \neq 0$, we can write $\eta_2$ in terms of $\eta_1^1$, $J(\eta_1^1)$ and $X$, for some vector $X \in T_{\psi(p)} \mathbb{CP}^{\frac{m_1}{2}}$ which is unit and orthogonal to $\eta_1^1$ and $J(\eta_1^1)$. In fact,

$$\eta_2^1 = \langle \eta_2^1, \eta_1^1 \rangle \frac{\eta_1^1}{|\eta_1^1|^2} + \langle \eta_2^1, J(\eta_1^1) \rangle \frac{J(\eta_1^1)}{|\eta_1^1|^2} + \langle \eta_2^1, X \rangle X.$$

(21)

Hence,
\[ |\eta_2|^2 = \frac{\langle \eta_2, \eta_1 \rangle^2}{|\eta_1|^2} + \frac{\langle \eta_2, J(\eta_1) \rangle^2}{|\eta_1|^2} + \langle \eta_1, X \rangle^2, \]

and multiplying both sides by \(|\eta_1|^2\),

\[ |\eta_1|^2 |\eta_2|^2 = \langle \eta_2, \eta_1 \rangle^2 + \langle \eta_2, J(\eta_1) \rangle^2 + |\eta_1|^2 \langle \eta_2, X \rangle^2. \]

Therefore,

\[ \langle \eta_2, J(\eta_1) \rangle^2 = |\eta_1|^2 |\eta_2|^2 - \langle \eta_2, \eta_1 \rangle^2 - |\eta_1|^2 \langle \eta_2, X \rangle^2. \]

Replacing this last equation in Eq. (19), we have

\[
\sum_{A=1}^{m} - \langle N_{E_A}, J_\Sigma(N_{E_A}) \rangle = \\
\lambda^2 \left(-|\eta_1|^4 - |\eta_2|^4 - 2\langle \eta_1, \eta_2 \rangle^2 + 2|\eta_1|^2 |\eta_2|^2 - 2\langle \eta_1, \eta_1 \rangle^2 - 2|\eta_1|^2 \langle \eta_2, X \rangle^2 \right) \\
= \lambda^2 \left(-|\eta_1|^4 - |\eta_2|^4 - 4\langle \eta_1, \eta_2 \rangle^2 + 2|\eta_1|^2 |\eta_2|^2 - 2|\eta_1|^2 \langle \eta_2, X \rangle^2 \right) \leq 0. \tag{22}
\]

From Eqs. (20) and (22), we have

\[
\sum_{A=1}^{m} - \langle N_{E_A}, J_\Sigma(N_{E_A}) \rangle \leq 0. \tag{23}
\]

Integrating Eq. (23) and using the stability of \( \Sigma \) gives us that

\[
\int_\Sigma \sum_{A=1}^{m} - \langle N_{E_A}, J_\Sigma(N_{E_A}) \rangle d\Sigma = 0.
\]

From Eq. (23), we know the integrand of the last equality has a sign, and hence

\[
\sum_{A=1}^{m} - \langle N_{E_A}, J_\Sigma(N_{E_A}) \rangle = 0. \tag{24}
\]

We have two options: if a point is such that \( \eta_1^1 = 0 \), using Eq. (24) in Eq. (20), gives that \( \eta_2^1 = 0 \). On the other hand, if a point is such that \( \eta_1^1 \neq 0 \), using Eq. (24) in Eq. (22), we thus get:

- \( |\eta_2^1| = |\eta_2^1| \)
- \( \langle \eta_2^1, \eta_1^1 \rangle = 0 \)
- \( \langle \eta_2^1, X \rangle = 0. \)

Using the last two items in Eq. (21), yields

\[ \eta_2^1 = \langle \eta_2^1, J(\eta_1^1) \rangle \frac{J(\eta_1^1)}{|\eta_1^1|^2}. \]
Therefore, \( \eta^1_2 \) and \( J(\eta^1_1) \) are parallel, and since \( |J(\eta^1_1)| = |\eta^1_1| = |\eta^1_2| \),

\[ \eta^1_2 = \pm J(\eta^1_1). \]

Notice we can include the case \( \eta^1_1 = \eta^1_2 = 0 \) in the last equality.

**Definition 5** Let \( J_1 \) and \( J_2 \) be two almost complex structures on \( \mathbb{C}P^{m_1} \times \mathbb{C}P^{m_2} \) given by:

\[ J_1(X, Y) := (J(X), J(Y)) \text{ and } J_2(X, Y) := (J(X), -J(Y)). \]

**Definition 6** Let \( \Phi : \Sigma \to \mathbb{C}P^{m_1} \times \mathbb{C}P^{m_2} \) be an immersion and \( p \in \Sigma \). For \( i \in \{1, 2\} \) fixed, we said that \( T_p \Sigma = d\Phi_p(T_p \Sigma) \) has structure \( J_i \) if \( J_i(T_p \Sigma) = T_p \Sigma \) or equivalently \( J_i(N_p \Sigma) = N_p \Sigma \). If for all \( p \in \Sigma \), we have that \( T_p \Sigma \) has structure \( J_i \), we say that \( \Phi \) is a complex immersion under the structure \( J_i \).

**Theorem 21** Let \( \Phi : \Sigma \to \tilde{\mathcal{M}} := \mathbb{C}P^{m_1} \times \mathbb{C}P^{m_2} \) be a compact stable minimal immersion of codimension \( d = 2 \) and dimension \( n \). Then, \( \Phi \) is a complex immersion under the structure \( J_1 \) or \( J_2 \).

**Proof** From Lemma 20, we have that for \( q \in \Sigma \) and \( \{\eta_1, \eta_2\} \) an orthonormal basis of \( N_q \Sigma \),

\[ \eta^1_2 = \pm J(\eta^1_1) \text{ and } \eta^2_2 = \pm J(\eta^2_1), \]

and then

\[ |\eta^1_1| = |\eta^1_2| \text{ and } |\eta^2_1| = |\eta^2_2|. \]

**Remark 4** Notice that if \( q \in \Sigma \) is such that \( |\eta^1_1| = |\eta^2_1| = 0 \) or \( |\eta^1_2| = |\eta^2_2| = 0 \), \( T_q \Sigma \) has both structures \( J_1 \) and \( J_2 \).

If for all \( q \in \Sigma \), we have that \( |\eta^1_1| = |\eta^2_1| = 0 \) or \( |\eta^1_2| = |\eta^2_2| = 0 \), then all points of \( \Sigma \) have both structures and then we are done. Otherwise there exists a point \( p \in \Sigma \) such that

\[ |\eta^1_1| = |\eta^1_2| \neq 0 \text{ and } |\eta^2_1| = |\eta^2_2| \neq 0, \tag{25} \]

for \( \{\eta_1, \eta_2\} \) an orthonormal basis for \( N_p \Sigma \). Denote by \( \text{cut}(p) \) the cut locus of the point \( p \) in \( \Sigma \).

In Part I we will show that \( \Sigma \setminus \text{cut}(p) \) has a single complex structure and in Part II we will show that we can extend this complex structure to the set \( \text{cut}(p) \).

**Part I**

By parallel transport of the orthonormal basis \( \{\eta_1, \eta_2\} \) in \( N_p \Sigma \) along geodesics of \( \Sigma \) under the normal connection of \( \Sigma \) in \( \tilde{\mathcal{M}} \), we define normal vector fields \( N_1 \) and \( N_2 \) in \( \Sigma \setminus \text{cut}(p) \). Using Lemma 20, we have that for an arbitrary point \( \tau \in \Sigma \setminus \text{cut}(p) \),

\[ J(N^1_1)(\tau) = \pm N^1_2(\tau) \text{ and } J(N^2_1)(\tau) = \pm N^2_2(\tau). \]

We collect the four possible options in the next table, with their respective implication in the last column.

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Table 1 Complex structure at $\tau \in \Sigma \setminus \text{cut}(p)$

| Value of $J(N^1_1)(\tau)$ | Value of $J(N^2_1)(\tau)$ | Value of $J_1$ or $J_2$ at $\tau$ |
|----------------------------|----------------------------|---------------------------------|
| $N^1_2(\tau)$              | $N^2_2(\tau)$              | $J_1(N_1)(\tau) = N_2(\tau)$  |
| $-N^1_2(\tau)$             | $N^2_2(\tau)$              | $J_2(N_1)(\tau) = -N_2(\tau)$ |
| $N^1_2(\tau)$              | $-N^2_2(\tau)$             | $J_2(N_1)(\tau) = N_2(\tau)$  |
| $-N^1_2(\tau)$             | $-N^2_2(\tau)$             | $J_1(N_1)(\tau) = -N_2(\tau)$ |

Therefore, for $\tau \in \Sigma \setminus \text{cut}(p)$, $T_\tau \Sigma$ has the structure $J_1$ or $J_2$. Without loss of generality, assume that $p$ has structure $J_1$. We will prove that all the points in $\Sigma \setminus \text{cut}(p)$ have the same structure $J_1$.

If all the points in $\Sigma \setminus \text{cut}(p)$ are such that their tangent space has the structure $J_1$, then we are done. Otherwise there exists a point $r$ in $\Sigma \setminus \text{cut}(p)$ such that $T_r \Sigma$ has complex structure $J_2$ and not $J_1$. Recall that from Eq. (25) and the construction of $N_1$ and $N_2$ we have

$$|N^1_1|(p) = |N^2_1|(p) \neq 0 \text{ and } |N^2_1|(p) = |N^2_2|(p) \neq 0.$$  \hfill (30)

And since $T_r \Sigma$ does not have the structure $J_1$, by Remark 4,

$$|N^1_1|(r) = |N^2_1|(r) \neq 0 \text{ and } |N^2_1|(r) = |N^2_2|(r) \neq 0.$$  \hfill (31)

Without loss of generality, suppose that $p$ satisfies Eq. (29), i.e. $J_1(N_1)(p) = -N_2(p)$ and $r$ satisfies Eq. (28), i.e. $J_2(N_1)(r) = N_2(r)$ (see Table 2). According to the Table 1,

$$J(N^1_1)(p) = -N^1_2(p), \quad J(N^2_1)(p) = -N^2_2(p),$$  \hfill (32)

and

$$J(N^1_1)(r) = N^1_2(r), \quad J(N^2_1)(r) = -N^2_2(r).$$  \hfill (33)

Let $\gamma := \gamma(s, p, v) : [0, 1] \to \Sigma$ be the unique geodesic contained in $\Sigma \setminus \text{cut}(p)$ such that $\gamma(0) = p$ and $\gamma(1) = r$. Let us define $f : [0, 1] \to \mathbb{R}$ in the following way,

$$f(s) := \langle J(N^1_1), N^1_2 \rangle_{\gamma(s)}.$$  

Using Eqs. (32) and (30) and Eqs. (33) and (31), $f$ is such that

$$f(0) = \langle J(N^1_1), N^1_2 \rangle_p = -|N^1_2|^2(p) < 0,$$

and

$$f(1) = \langle J(N^1_1), N^1_2 \rangle_r = |N^1_2|^2(r) > 0,$$

respectively. Since $f$ is smooth, there exists a point $t \in (0, 1)$ such that,

$$f(t) = 0, \quad f(s) > 0 \text{ for } s \in (t, t + \epsilon) \text{ and } f(s) \leq 0 \text{ for } s \in (t - \epsilon, t).$$
Let \( g : (t - \epsilon, t + \epsilon) \to \mathbb{R} \) be the function given by \( g(s) := |N_2^1|_{2y(s)} \). Since \( J(N_1^1) = \pm N_2^1 \),

\[
f(s) = \begin{cases} 
eg g(s) & \text{if } s \in (t - \epsilon, t) \\ g(s) & \text{if } s \in (t, t + \epsilon). \end{cases}
\]

Since \( f \) is smooth, for all \( k \geq 0 \), \( k \in \mathbb{N} \)

\[
f^{(k)}(s) = \begin{cases} 
eg g^{(k)}(s) & \text{if } s \in (t - \epsilon, t) \\ g^{(k)}(s) & \text{if } t \in (t, t + \epsilon). \end{cases}
\]

is continuous, and thus \( g^{(k)}(t) = 0 \).

The function \( g \) is real analytic because it is a composition of real analytic functions (see the technical Lemma 22). Since \( g \) is an analytic function such that \( g^{(k)}(t) = 0 \) for all \( k \geq 0 \), \( f \equiv 0 \). This is a contradiction because \( g(t + \frac{\epsilon}{2}) = f(t + \frac{\epsilon}{2}) > 0 \). Therefore, all the points in \( \Sigma \setminus cut(p) \) are such that their tangent space has the structure \( J_1 \).

The proof of Lemma 22 can be omitted on a first reading; the reader may wish to continue to Part II.

**Lemma 22** The function \( g : (t - \epsilon, t + \epsilon) \to \mathbb{R} \) given by \( g(s) := |N_2^1|_{2y(s)} \) is a real analytic function.

**Proof** Let

\[
Y : U \subset \mathbb{R}^{n+d} \to \tilde{U} \subset \tilde{M} \\
(y^1, \ldots, y^{n+d}) \to Y(y^1, \ldots, y^{n+d})
\]

be a coordinate chart of \( \tilde{M} \) around \( \Phi(y(t)) \) compatible with the real analytic structure of \( \tilde{M} \) and

\[
X : W \subset \mathbb{R}^n \to \tilde{W} \subset \Sigma \\
(x^1, \ldots, x^n) \to X(x^1, \ldots, x^n)
\]

be the coordinate chart of \( \Sigma \) around \( y(t) \). Notice that we can assume that \( y(t - \epsilon, t + \epsilon) \subset \tilde{W} \), otherwise we can just modify \( \epsilon \). If \( y \) is the local representation of \( \Phi \) in the coordinate charts \( X \) and \( Y \),

\[
y : W \to \mathbb{R}^{n+d} \\
x = (x^1, \ldots, x^n) \to y(x) = (y^1(x), \ldots, y^{n+d}(x)).
\]

We also can assume that \( y(W) \subset U \), otherwise we can make the open set \( W \) smaller.

Since \( \Phi \) is a minimal immersion, \( y \) satisfies the following non-linear elliptic system of partial differential equations (see [20, Section 52]):

\[
g^{ij} \frac{\partial^2 y^\beta}{\partial x^i \partial x^j} + \frac{g^{ij}a^\beta_\sigma}{2} \left( \frac{\partial a_\mu_\sigma}{\partial y^\nu} + \frac{\partial a_\nu_\sigma}{\partial y^\mu} - \frac{\partial a_\mu_\nu}{\partial x^i} \frac{\partial y^\mu}{\partial x^j} \frac{\partial y^\nu}{\partial x^j} \right) = 0, \quad \beta = 1, \ldots, n + d,
\]

(35)
where \( i, j \) are summing in \( \{1, \ldots, n\} \), \( \sigma, \mu, v \) are summing in \( \{1, \ldots, n + d\} \), \( a_{\alpha \beta} \) is the Riemannian metric in \( M \), and \( g_{ij} \) is the metric in \( \Sigma \) which is given by:

\[
g_{ij} = a_{\alpha \theta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\theta}{\partial x^j}.
\]  

(36)

The system (35) is real analytic because \( \tilde{M} \) is a real analytic Riemannian manifold. Then, using the main result in [21], the local representation of \( \Phi, y(x) \), is also real analytic. We can also conclude, from (36) that the metric \( g_{ij} \) is real analytic as a function of \( x \). In the same manner we can see that the local representation \( \tilde{\gamma} := X^{-1} \circ \gamma \) of the geodesic \( \gamma \) in \( \Sigma \) around \( \gamma(t) \) is real analytic.

Using the local coordinates described above we can see that \( g(s) \) is the composition of the following functions:

\[
s \rightarrow \tilde{\gamma}(s) \xrightarrow{N_2} N_2(\tilde{\gamma}(s)) \xrightarrow{P^1} N^1_2(\tilde{\gamma}(s)) \xrightarrow{|\cdot|^2} |N^1_2(\tilde{\gamma}(s))|^2.
\]

We already proved that \( \tilde{\gamma} \) is a real analytic function. So we now we have to prove that the other functions described above are real analytic too.

For convenience, we will take the coordinate chart (34) given by slice coordinates for \( \Sigma \) in \( \tilde{M} \) around \( \Phi(\gamma(t)) \). By using the process of Gram–Schmidt we can construct in \( U \) vector fields \( E_1, E_2 \) that are real analytic as functions of \( (y^1, \ldots, y^{n+d}) \in U \) and such that for every point \( x \in W \), \( \{E_1(y(x)), E_2(y(x))\} \) is an orthonormal basis of \( N_x \Sigma \). Therefore, \( N_2 \) can be seen as,

\[
N(s) := N_2(\tilde{\gamma}(s)) = c^z(s)E_z(y(\tilde{\gamma}(s))); \quad z = 1, 2.
\]

(37)

Recall \( N_2 \) is the parallel transport of a fixed unit vector \( \eta_2 \in N_p \Sigma \) along the geodesic \( \gamma(s) = \gamma(s, p, v) \) under the normal connection. Then,

\[
\nabla^\perp_{\frac{d}{ds}} N(s) = 0, \quad N(s_0) = \eta,
\]

(38)

for some \( s_0 \in (t - \epsilon, t + \epsilon) \) and \( \eta \in N_{\tilde{\gamma}(s_0)} \Sigma \). If \( \eta = a^z E_z(y(\tilde{\gamma}(s_0))) \), for \( a^z \in \mathbb{R} \), from (38) we have the following system of ordinary differential equations,

\[
\begin{cases}
\frac{dc^z(s)}{ds} = -c^l(s)\langle \nabla_{\frac{d}{ds}} E_l(y(\tilde{\gamma}(s))), E_z(y(\tilde{\gamma}(s))) \rangle := G_Z(c, s) \\
\end{cases}
\]

where \( c = (c^1, c^2) \). Since \( G := (G_1, G_2) \) is real analytic as a function of \( (c, s) \), the solution \( c(s) \) is real analytic, and therefore \( N(s) = N_2(\tilde{\gamma}(s)) \) is real analytic.

It is straightforward to see that the projection \( P^1 \) of a real analytic vector field is also real analytic. Therefore,

\[
N^1_2(\tilde{\gamma}(s)) = \sum_{\beta=1}^{n+d} b^\beta(y(\tilde{\gamma}(s))) \frac{\partial}{\partial y^\beta} 
\]

\( y(\tilde{\gamma}(s)) \).
Table 2  Definition of \( f \) and \( g \) for other possible cases

| Equation satisfied by \( p \) | Equation satisfied by \( r \) | \( f(s) \) | \( g(s) \) |
|-----------------------------|-----------------------------|-------|-------|
| (29)                        | (27)                        | \( (J(N_1^2), N_2^2)_{\gamma(s)} \) | \( |N_2^2|_{\gamma(s)}^2 \) |
| (26)                        | (28)                        | \( -(J(N_1^2), N_2^2)_{\gamma(s)} \) | \( |N_2^2|_{\gamma(s)}^2 \) |
| (26)                        | (27)                        | \( -(J(N_1^1), N_2^1)_{\gamma(s)} \) | \( |N_2^1|_{\gamma(s)}^2 \) |

where \( b^\beta \) are real analytic functions. Now,

\[
|N_2^1(\bar{\gamma}(s))|^2 = \sum_{\alpha, \beta=1}^{n+d} b^\alpha(y(\bar{\gamma}(s)))b^\beta(y(\bar{\gamma}(s)))a_{\alpha\beta}(y(\bar{\gamma}(s))),
\]

which involves only products, sums, and compositions of real analytic functions. Therefore, \( g(s) \) is real analytic. \( \Box \)

Part II

Now we will show that we can extend that structure to the points in \( \text{cut}(p) \). Notice that in part I we have indeed shown the following lemma,

Lemma 23  If a point \( q \in \Sigma \) is such that \( T_q \Sigma \) has a structure \( (J_1 \text{ or } J_2) \), and if for \( \{\eta_1, \eta_2\} \) a basis of \( N_q \Sigma \) we have that

\[
|\eta_1|^2 = |\eta_2|^2 \neq 0 \quad \text{and} \quad |\eta_1|^2 = |\eta_2|^2 \neq 0,
\]

then \( \Sigma \setminus \text{cut}(q) \) has the same structure as \( q \).

Let \( b \) be a point in \( \text{cut}(p) \). If \( b \) is such that one of the projections of its normal vectors is zero, by Remark 4, \( b \) has both structures and we are done. Otherwise, \( b \) is such that (39) is satisfied for \( \{\eta_1, \eta_2\} \) a basis of \( N_b \Sigma \). Let \( V \) be a normal neighborhood around \( b \) in \( \Sigma \), where we can define orthonormal normal vector fields \( N_1 \) and \( N_2 \) such that

\[
|N_1^1| = |N_2^1| \neq 0 \quad \text{and} \quad |N_1^2| = |N_2^2| \neq 0,
\]

with \( N_1(b) = \eta_1 \) and \( N_2(b) = \eta_2 \). Let \( \alpha : [0, l] \to \Sigma \) be a geodesic of \( \Sigma \), such that \( \alpha(0) = p, \alpha(l) = b \) and \( \alpha([0, l]) \cap \text{cut}(p) = b \). There exists \( a < l \) such that \( \alpha(a) \in V \). Since \( \alpha(a) \notin \text{cut}(p) \), \( \alpha(a) \) has structure \( J_1 \). Moreover, since \( \alpha(a) \in V \), \( b \notin \text{cut}(\alpha(a)) \), and therefore by Remark 23 and (40), \( b \) has the same structure as \( \alpha(a) \), i.e., \( J_1 \). \( \Box \)

Employing the same arguments used in the proofs of Lemma 20 and Theorem 21 and using Eq. (10), we have the following:

Lemma 24  Let \( \Phi = (\psi, \phi) : \Sigma \to \mathbb{C}P^{m_1} \times M \) be a compact stable minimal immersion of dimension \( n = 2 \), where \( M \) is any Riemannian manifold of dimension \( m_2 \). 

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Then, $e_1^2 = \pm J(e_1^1)$, where $\{e_1, e_2\}$ is an orthonormal basis of $T_p \Sigma$, $p \in \Sigma$, $J$ is the complex structure of $\mathbb{C}P^{m_2}$, and $e_1^1$ is the projection of $e_1$ in $T_{\psi(p)}\mathbb{C}P^{m_2}$.

**Theorem 25** Let $\Phi : \Sigma \to \mathbb{C}P^{m_1} \times \mathbb{C}P^{m_2}$ be a compact stable minimal immersion of dimension $n = 2$. Then, $\Phi$ is a complex immersion under the structure $J_1$ or $J_2$.

### 3.4 Dimension 1

In this subsection, we will use the general formula proved in Sect. 3.1 to prove a classification theorem for compact stable minimal immersions of dimension 1 (geodesics) in the product of a complex projective space with any other Riemannian manifold. Moreover, as an application, we obtain some corollaries when the second manifold is a compact rank one space.

**Theorem 26** Let $\Phi = (\psi, \phi) : \Sigma \to \mathbb{C}P^{m_1} \times M$ be a compact stable minimal immersion of dimension $n = 1$, where $M$ is any Riemannian manifold of dimension $m_2$. Then, $\phi : \Sigma \to M$ is a stable geodesic, $\psi$ is a constant function, and therefore $\Phi(\Sigma) = \{r\} \times \phi(\Sigma)$ with $r$ a point of $\mathbb{C}P^{m_1}$.

**Proof** Since $n = 1$, Eq. (10) becomes

$$\sum_{A=1}^{m} -\langle N_{E_A}, J_{\Sigma}(N_{E_A}) \rangle = -\lambda^2 |e^1|^4,$$

where $e = (e_1^1, e_2^1)$ is a unit vector in $d\Phi_p(T_p \Sigma)$, for $p \in \Sigma$, and $m$ is the dimension of the Euclidean space where $\mathbb{C}P^{m_2}$ is embedded. Therefore,

$$0 \leq \sum_{A=1}^{m} \int_{\Sigma} \langle N_{E_A}, J_{\Sigma}(N_{E_A}) \rangle d\Sigma = -\lambda^2 \int_{\Sigma} |e^1|^4 d\Sigma \leq 0,$$

where we have used the fact that $\Sigma$ is stable in the first inequality. Hence, for $p \in \Sigma$, $e^1 = 0$. Therefore,

$$d\Phi_p(T_p \Sigma) = \{\alpha(0, e^2) : \alpha \in \mathbb{R}\}.$$  

(42)

Let $x \in T_p \Sigma$ arbitrary. Then, for some $\alpha \in \mathbb{R}$,

$$d\Phi_p(x) = (d\psi_p(x), d\phi_p(x)) = (0, \alpha e^2).$$

Therefore, $d\psi_p(x) = 0$, which implies that $\psi$ is constant. Now we will prove that $\phi : \Sigma \to M$ is a stable minimal immersion of dimension 1. Since $\Phi$ is an isometric immersion and $\psi$ is constant, $\phi$ is an isometric immersion. From Eq. (42), we have that at $p \in \Sigma$,

$$N_p \Sigma = \{(v, 0) : v \in T_{\psi(p)}\mathbb{C}P^{m_2}\} \oplus \{(0, w) : w \in [e^2]^{1-m}\}.$$
Let $H$ be the mean curvature vector of $\Phi$, $E = (0, E^2)$ a local unit vector field tangent to $\Sigma$ around $p$ with $E(p) = (0, e^2)$, and take the orthonormal basis of $N_p\Sigma$ given by

\begin{align*}
\{(v_i, 0) : i = 1, \ldots, m_1\} \cup \{(0, w_j) : j = 1, \ldots, m_2 - 1\},
\end{align*}

where $v_i \in T_{\psi(p)}CP^{m_1}$ and $w_j \in [e^2]^{\perp M}$. We will omit the evaluation at $p$ in the following computation,

\begin{equation}
0 = H = (\bar{\nabla} E)_{N\Sigma} = (0, \nabla^2_{E^2}E^2)_{N\Sigma}
\end{equation}

\begin{align*}
&= \sum_{i=1}^{m_1} \langle (0, \nabla^2_{E^2}E^2), (v_i, 0) \rangle (v_i, 0) + \sum_{j=1}^{m_2-1} \langle (0, \nabla^2_{E^2}E^2), (0, w_j) \rangle (0, w_j) \\
&= \sum_{j=1}^{m_2-1} \langle (0, \nabla^2_{E^2}E^2), (0, w_j) \rangle (0, w_j) = 0, \sum_{j=1}^{m_2-1} \langle \nabla^2_{E^2}E^2, w_j \rangle w_j.
\end{align*}

Therefore,

\begin{equation}
A := \sum_{j=1}^{m_2-1} \langle \nabla^2_{E^2}E^2, w_j \rangle w_j = 0.
\end{equation}

But notice that $A$ is the mean curvature vector of $\phi$ as an immersion in $M$, and thus $\phi$ is minimal. Moreover, since $\Phi$ is stable then $\phi$ is stable (see Preliminaries Section in [10]).

\textbf{Corollary 27} There are no compact stable geodesics in the product space $CP^{m_1} \times S^s$, $CP^{m_1} \times O P^2$, or $CP^{m_1} \times \mathbb{K} P^s$, where $\mathbb{K} \in \{\mathbb{C}, \mathbb{H}\}$ other than $\{r\} \times S^1$ in $CP^{m_1} \times S^1$, where $r \in CP^{m_1}$.

\textbf{Proof} We use Theorem 26 and notice that there are no stable geodesics in $S^s$, $CP^s$, $\mathbb{H} P^s$, or $OP^2$ (see Theorems 1 and 3) other than $S^1$ in $S^1$. \hfill \Box

\textbf{Corollary 28} The only compact stable geodesic in the product space $CP^{m_1} \times \mathbb{R} P^s$ is $\{r\} \times \mathbb{R} P^1$, $r \in CP^{m_1}$.

\textbf{Proof} It follows from Theorem 26 and the fact that the only stable geodesic in $\mathbb{R} P^s$ is $\mathbb{R} P^1$ (see Theorem 2). \hfill \Box

\section{4 Minimal Stable Submanifolds in $HP^{m_1} \times M$}

The strategies that will be used in this section are similar to those found in Sect. 3. Some details are presented again for completeness purposes.

\subsection{4.1 General Formula}

Let $\Phi = (\psi, \phi) : \Sigma \to \tilde{M} := HP^{m_1} \times M$ be a compact minimal immersion of codimension $d$ and dimension $n$, where $M$ is any Riemannian manifold of dimension...
\[ m_2 \text{ and } \Phi_1 : \mathbb{H} P^{m_1} \to \mathbb{R}^m \text{ is the immersion described in Sect. 2.3. For each } v \in \mathbb{R}^m \text{ let us consider the following:} \]

\[
v := (v, 0) \in T(\mathbb{R}^m \times M) \quad N_v := [v]^N,\]

where \([.]^N \) is projection in the orthogonal complement, \( N_p \Sigma, \) of \( T_p \Sigma \) in \( T_{\Phi(p)} \tilde{M}, \) \( p \in \Sigma. \)

**Lemma 29** Let \( p \in \Sigma, \{e_1, \ldots, e_n\} \) be an orthonormal basis of \( T_p \Sigma, \{\eta_1, \ldots, \eta_d\} \) be an orthonormal basis of \( N_p \Sigma, \) and \( \{E_1, \ldots, E_m\} \) be the usual canonical basis of \( \mathbb{R}^m. \) Then, for \( s \in \{1, 2, 3\} \)

\[
\sum_{A=1}^{m} -\langle N_{E_A}, J_{\Sigma}(N_{E_A}) \rangle = \lambda^2 \left( - \sum_{k=1}^{3} \sum_{j=1}^{n} \sum_{\beta=1}^{d} \frac{1}{3} \langle R(e_j^1, \eta_{\beta}^1)\eta_{\beta}^1, e_j^1 \rangle + \frac{2\lambda^2}{3} \langle e_j^1, \eta_{\beta}^1 \rangle^2 \right. \\
\left. + \frac{\lambda^2}{3} \langle e_j^1, \eta_{\beta}^1 \rangle^2 \right). \tag{43}
\]

where \( x^1 \) denotes the projection of the tangent vector \( x \) onto the first factor.

**Proof** Recall that \( R \) is the curvature tensor of \( \mathbb{H} P^{m_1} \) and \( B \) is the second fundamental form of \( \mathbb{H} P^{m_1} \) in \( \mathbb{R}^m \) (see Sect. 1.2). Using [11, Eq. (2.8)] and proceeding as in the beginning of the proof of Lemma 15 (changing the index \( k \) for \( \beta \)) we have the following equation (same expression than Eq. (12))

\[
\sum_{A=1}^{m} -\langle N_{E_A}, J_{\Sigma}(N_{E_A}) \rangle = \sum_{j=1}^{n} \sum_{\beta=1}^{d} \frac{4}{3} \langle R(e_j^1, \eta_{\beta}^1)\eta_{\beta}^1, e_j^1 \rangle + \frac{2\lambda^2}{3} \langle e_j^1, \eta_{\beta}^1 \rangle^2 \\
+ \frac{\lambda^2}{3} \langle e_j^1, \eta_{\beta}^1 \rangle^2. \tag{45}
\]

Using Eq. (3) in Eq. (45),

\[
\sum_{A=1}^{m} -\langle N_{E_A}, J_{\Sigma}(N_{E_A}) \rangle = \lambda^2 \sum_{j=1}^{n} \sum_{\beta=1}^{d} \left( \langle e_j^1, \eta_{\beta}^1 \rangle^2 - \sum_{k=1}^{3} \langle e_j^1, J_k(\eta_{\beta}^1) \rangle^2 \right)
\]

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\[
\lambda^2 \left( \sum_{j=1}^{n} \sum_{\beta=1}^{d} \langle e_j^1, \eta_{\beta}^1 \rangle^2 - \sum_{j=1}^{n} \sum_{\beta=1}^{d} \sum_{k=1}^{3} \langle e_j^1, J_k(\eta_{\beta}^1) \rangle^2 \right) = \lambda^2 \left( \sum_{j=1}^{n} \sum_{\beta=1}^{d} \langle e_j^1, \eta_{\beta}^1 \rangle^2 - \sum_{j=1}^{n} \sum_{\beta=1}^{d} \langle e_j^1, J_s(\eta_{\beta}^1) \rangle^2 - \sum_{j=1}^{n} \sum_{\beta=1}^{d} \sum_{k=1}^{3} \langle e_j^1, J_k(\eta_{\beta}^1) \rangle^2 \right).
\]

Ignoring the last term in the last equality and replacing \( \beta \) with \( k \) and \( J_s \) with \( J \), we now proceed in the same way as in the proof of Lemma 15 (from Eq. (13) on) to study the first two terms in the last equality above. For \( \beta \in \{1, \ldots, d\} \)

\[
|\eta_{\beta}^1|^2 = |J_s(\eta_{\beta}^1)|^2 = |(J_s(\eta_{\beta}^1), 0)|^2
\]

\[
= \sum_{j=1}^{n} \langle (J_s(\eta_{\beta}^1), 0), e_j \rangle^2 + \sum_{l=1}^{d} \langle (J_s(\eta_{\beta}^1), 0), \eta_l \rangle^2
\]

\[
= \sum_{j=1}^{n} \langle J_s(\eta_{\beta}^1), e_j^1 \rangle^2 + \sum_{l=1}^{d} \langle J_s(\eta_{\beta}^1), \eta_l^1 \rangle^2.
\]

Then,

\[
- \sum_{j=1}^{n} \langle J_s(\eta_{\beta}^1), e_j^1 \rangle^2 = -|\eta_{\beta}^1|^2 + \sum_{l=1}^{d} \langle J_s(\eta_{\beta}^1), \eta_l^1 \rangle^2,
\]

and summing in \( \beta \),

\[
- \sum_{\beta=1}^{d} \sum_{j=1}^{n} \langle J_s(\eta_{\beta}^1), e_j^1 \rangle^2 = - \sum_{\beta=1}^{d} |\eta_{\beta}^1|^2 + \sum_{\beta=1}^{d} \sum_{l=1}^{d} \langle J_s(\eta_{\beta}^1), \eta_l^1 \rangle^2.
\]

On the other hand, again for \( \beta \in \{1, \ldots, d\} \)

\[
|\eta_{\beta}^1|^2 = |(\eta_{\beta}^1, 0)|^2 = \sum_{j=1}^{n} \langle (\eta_{\beta}^1, 0), e_j \rangle^2 + \sum_{l=1}^{d} \langle (\eta_{\beta}^1, 0), \eta_l \rangle^2
\]

\[
= \sum_{j=1}^{n} \langle \eta_{\beta}^1, e_j^1 \rangle^2 + \sum_{l=1}^{d} \langle \eta_{\beta}^1, \eta_l^1 \rangle^2.
\]

Therefore,

\[
\sum_{j=1}^{n} \langle \eta_{\beta}^1, e_j^1 \rangle^2 = |\eta_{\beta}^1|^2 - \sum_{l=1}^{d} \langle \eta_{\beta}^1, \eta_l^1 \rangle^2.
\]
and summing in $\beta$,

\[
\sum_{\beta=1}^{d} \sum_{j=1}^{n} \langle \eta_\beta^1, e_j^1 \rangle^2 = \sum_{\beta=1}^{d} |\eta_\beta^1|^2 - \sum_{\beta=1}^{d} \sum_{l=1}^{d} \langle \eta_\beta^1, \eta_l^1 \rangle^2. \tag{48}
\]

Then, in order to prove Eq. (43), we replace Eqs. (47) and (48) in (46), given us that

\[
\sum_{A=1}^{m} -\langle N_{E_A}, J_S(N_{E_A}) \rangle = \lambda^2 \left( \sum_{\beta=1}^{d} |\eta_\beta^1|^2 - \sum_{\beta,l=1}^{d} \langle \eta_\beta^1, \eta_l^1 \rangle^2 - \sum_{\beta=1}^{d} \sum_{j=1}^{n} \langle J_s(e_j^1), \eta_\beta^1 \rangle^2 \right)
\]

\[
= \lambda^2 \left( - \sum_{\beta=1}^{d} \sum_{j=1}^{n} \langle \eta_\beta^1, \eta_j^1 \rangle^2 + \sum_{\beta=1}^{d} \sum_{l=1}^{d} \langle J_s(e_j^1), \eta_l^1 \rangle^2 - \sum_{j=1}^{n} \sum_{l=1}^{d} \sum_{k=1}^{d} \langle e_j^1, J_k(e_l^1) \rangle^2 \right). \tag{49}
\]

Now, let us prove Eq. (44). For $j \in \{1, \ldots, n\}$,

\[
|e_j^1|^2 = |J_s(e_j^1)|^2 = |(J_s(e_j^1), 0)|^2
\]

\[
= \sum_{i=1}^{n} \langle (J_s(e_j^1), 0), e_i \rangle^2 + \sum_{\beta=1}^{d} \langle (J_s(e_j^1), 0), \eta_\beta \rangle^2
\]

\[
= \sum_{i=1}^{n} \langle J_s(e_j^1), e_i^1 \rangle^2 + \sum_{\beta=1}^{d} \langle J_s(e_j^1), \eta_\beta^1 \rangle^2.
\]

Then,

\[
- \sum_{\beta=1}^{d} \langle J_s(e_j^1), \eta_\beta^1 \rangle^2 = -|e_j^1|^2 + \sum_{i=1}^{n} \langle J_s(e_j^1), e_i^1 \rangle^2,
\]

and summing in $j$,

\[
- \sum_{j=1}^{n} \sum_{\beta=1}^{d} \langle J_s(e_j^1), \eta_\beta^1 \rangle^2 = - \sum_{j=1}^{n} |e_j^1|^2 + \sum_{j=1}^{n} \sum_{i=1}^{n} \langle J_s(e_j^1), e_i^1 \rangle^2. \tag{49}
\]
On the other hand, again for \( j \in \{1, \ldots, n\} \)
\[
|e_j^1|^2 = |(e_j^1, 0)|^2
\]
\[
= \sum_{i=1}^{n} (e_j^1, e_i)^2 + \sum_{\beta=1}^{d} (e_j^1, \eta_{\beta})^2
\]
\[
= \sum_{i=1}^{n} (e_j^1, e_i^1)^2 + \sum_{\beta=1}^{d} (e_j^1, \eta_{\beta}^1)^2.
\]

Therefore,
\[
\sum_{\beta=1}^{d} (e_j^1, \eta_{\beta}^1)^2 = |e_j^1|^2 - \sum_{i=1}^{n} (e_j^1, e_i^1)^2,
\]
summing in \( j \),
\[
\sum_{j=1}^{n} \sum_{\beta=1}^{d} (e_j^1, \eta_{\beta}^1)^2 = \sum_{j=1}^{n} |e_j^1|^2 - \sum_{i=1}^{n} \sum_{j=1}^{n} (e_j^1, e_j^1)^2.
\] (50)

Replacing Eqs. (49) and (50) in (46),
\[
\sum_{A=1}^{m} -\langle N_{E_A}, J_\Sigma(N_{E_A}) \rangle
\]
\[
= \lambda^2 \left( \sum_{j=1}^{n} |e_j^1|^2 - \sum_{j,i=1}^{n} (e_j^1, e_j^1)^2 - \sum_{j=1}^{n} |e_j^1|^2 - \sum_{j=1}^{n} \sum_{\beta=1}^{d} (e_j^1, \eta_{\beta}^1)^2 \right)
\]
\[
= \lambda^2 \left( - \sum_{i=1}^{n} \sum_{j=1}^{n} (e_j^1, e_j^1)^2 + \sum_{j=1}^{n} \sum_{i=1}^{n} (J_\Sigma J_j(e_j^1), e_j^1)^2 - \sum_{j=1}^{n} \sum_{\beta=1}^{d} (e_j^1, \eta_{\beta}^1)^2 \right).
\]

\[ \Box \]

4.2 Codimension 1 and 2

The arguments that will be used in the proof of Theorem 30 are analogous to those used in the proof of Theorem 16. The ideas that will be presented in the first part of the proof of Theorem 33 are similar to the ideas presented in the proof of Lemma 20. Then, after obtaining a characterization of the tangent space of the immersion, the second part of the proof follows similarly the proof of Theorem 16.

Theorem 30 Let \( \Phi = (\psi, \phi) : \Sigma \to \tilde{M} := \mathbb{H} P^m \times M \) be a compact stable minimal immersion of codimension \( d = 1 \) and dimension \( n \), where \( M \) is any Riemannian manifold of dimension \( m_2 \). Then, \( \Sigma = \mathbb{H} P^m \times \hat{\Sigma}, \Phi = I_d \times \hat{\phi} \) where \( \hat{\phi} : \hat{\Sigma} \to M \)
is a compact stable minimal immersion of codimension 1, and therefore $\Phi(\Sigma) = \mathbb{H}P^{m_1} \times \hat{\phi}(\hat{\Sigma})$. In particular, for $m_2 = 1$, $\Sigma = \mathbb{H}P^{m_1}$, $\hat{\phi}$ is a constant function, and $\Phi(\Sigma) = \mathbb{H}P^{m_1} \times \{q\}$, for $q \in M$.

**Proof** Since $d = 1$, Eq. (43) becomes

$$\sum_{A=1}^{m} -\langle NE_A, J_\Sigma(NE_A) \rangle = \lambda^2 \left( -\sum_{k=1}^{3} \sum_{j=1}^{n} (J_k(\eta^1), e_j^1)^2 + \langle J_s(\eta^1), \eta^1 \rangle^2 - \langle \eta^1, \eta^1 \rangle^2 \right)$$

$$= \lambda^2 \left( -\sum_{k=1}^{3} \sum_{j=1}^{n} (J_k(\eta^1), e_j^1)^2 - |\eta^1|^4 \right),$$

where $\eta$ is an unit vector at $N_p \Sigma$, for $p \in \Sigma$. The last expression differs from Eq. (18) in the first term. This term does not add any new difficulties because it is non-positive. In fact, integrating both sides of the last equality, we have

$$0 \leq \sum_{A=1}^{m} -\int_\Sigma \langle NE_A, J_\Sigma(NE_A) \rangle d\Sigma = -\lambda^2 \int_\Sigma \sum_{k=1}^{3} \sum_{j=1}^{n} (J_k(\eta^1), e_j^1)^2 + |\eta^1|^4 d\Sigma \leq 0,$$

where we have used the fact that $\Sigma$ is stable in the first inequality. Hence, for $p \in \Sigma$, $\eta^1 = 0$, and therefore $\eta = (0, \eta^2)$. Thus,

$$d\Phi_p(T_p \Sigma) = \tilde{D}_1(p) \bigoplus \tilde{D}_2(p),$$

where $\tilde{D}_1$ and $\tilde{D}_2$ are given by

$$\tilde{D}_1(p) = \{(x, 0) : x \in T_{\psi(p)}\mathbb{H}P^{m_1}\},$$

and

$$\tilde{D}_2(p) = \{(0, w) : w \in [\eta^2]^{\perp_M}\},$$

where $[z]^{\perp_M}$ is the orthogonal complement of $z$ in $T_{\psi(p)}M$. Since $\mathbb{H}P^{m_1}$ is simply connected, the rest of the proof follows by applying the same proof of Theorem 16.

$\square$

**Corollary 31** There are no compact stable minimal hypersurfaces in the product manifold $\mathbb{H}P^{m_1} \times S^s$, $\mathbb{H}P^{m_1} \times \mathbb{O}P^2$, or $\mathbb{H}P^{m_1} \times K^s$, where $K \in \{\mathbb{C}, \mathbb{H}\}$ other than $\mathbb{H}P^{m_1} \times \{q\}$ in $\mathbb{H}P^{m_1} \times S^1$, where $q \in S^1$.

**Proof** It follows from Theorem 30 and the fact that there are no stable minimal hypersurfaces in $S^s$, $\mathbb{C}P^s$, $\mathbb{H}P^s$, or $\mathbb{O}P^2$ (see Theorems 1 and 3) other than a point $q$ in $S^1$.

$\square$

Within the framework of these ideas, we also conclude that

\[\mathbb{C} Springer\]
Corollary 32 The only compact stable minimal hypersurface in the product space $\mathbb{H} P^{m_1} \times \mathbb{R} P^s$ is $\mathbb{H} P^{m_1} \times \mathbb{R} P^{s-1}$.

Proof The proof is completed by applying Theorem 30 and noticing that the only stable minimal hypersurface in $\mathbb{R} P^s$ is $\mathbb{R} P^s$ (see Theorem 2).

Theorem 33 Let $\Phi = (\psi, \phi) : \Sigma \to \mathbb{H} P^{m_1} \times M$ be a compact stable minimal immersion of codimension $d = 2$ and dimension $n$, where $M$ is any Riemannian manifold of dimension $m_2$. Then, $\Sigma = \mathbb{H} P^{m_1} \times \hat{\Sigma}$, $\Phi = Id \times \hat{\phi}$ where $\hat{\phi} : \hat{\Sigma} \to M$ is a compact stable minimal immersion of codimension 2, and therefore $\Phi(\Sigma) = \mathbb{H} P^{m_1} \times \hat{\phi}(\hat{\Sigma})$. In particular, for $m_2 = 1$, there are no compact stable minimal immersions of codimension 2 in $\mathbb{H} P^{m_1} \times M$. And for $m_2 = 2$, $\Sigma = \mathbb{H} P^{m_1}$, $\hat{\phi}$ is a constant function, and $\Phi(\Sigma) = \mathbb{H} P^{m_1} \times \{q\}$, for $q \in M$.

Proof Let $p \in \Sigma$. Since $d = 2$, for $s \in \{1, 2, 3\}$, Eq. (43) becomes

$$
\sum_{A=1}^{m} -\langle N_{E_A}, J_\Sigma(N_{E_A}) \rangle = \lambda^2 \left( - \sum_{k=1}^{3} \sum_{j=1}^{n} \sum_{l=1}^{2} \langle J_k(\eta^1_\beta), e_j^1 \rangle^2 + \sum_{l=1}^{2} \sum_{l=1}^{2} \langle J_s(\eta^1_\beta), \eta^1_l \rangle^2 - \langle \eta^1_\beta, \eta^1_l \rangle^2 \right) \\
= \lambda^2 \left( - \sum_{k=1}^{3} \sum_{j=1}^{n} \sum_{l=1}^{2} \langle J_k(\eta^1_\beta), e_j^1 \rangle^2 - |\eta^1_1|^4 - |\eta^1_2|^4 - 2\langle \eta^1_1, \eta^1_2 \rangle^2 + 2\langle J_s(\eta^1_2), \eta^1_1 \rangle^2 \right). 
$$

(51)

Notice that, disregarding the first term in the last equality, we have the same expression as in Eq. (19) (instead of $J$ we have the structure $J_\Sigma$). This term does not add any new difficulties because it is non-positive. From this point on, most of the arguments are analogous to those found in the proof of Lemma 20.

If $\eta^1_1 = 0$, then

$$
\sum_{A=1}^{m} -\langle N_{E_A}, J_\Sigma(N_{E_A}) \rangle = \lambda^2 \left( - \sum_{k=1}^{3} \sum_{j=1}^{n} \langle J_k(\eta^1_2), e_j^1 \rangle^2 - |\eta^1_2|^4 \right) \leq 0. 
$$

(52)

If $\eta^1_1 \neq 0$, we can write $\eta^1_2$ in terms of $\eta^1_1$, $J_s(\eta^1_1)$, and $X$ for some $X \in T_{\psi(p)} \mathbb{H} P^{m_1}$ which is unit and orthogonal to $\eta^1_2$ and $J_s(\eta^1_1)$. Then,

$$
\langle \eta^1_2, J_s(\eta^1_1) \rangle^2 = |\eta^1_1|^2 |\eta^1_2|^2 - \langle \eta^1_1, \eta^1_2 \rangle^2 - |\eta^1_1|^2 \langle \eta^1_2, X \rangle^2.
$$

Replacing the last equation in Eq. (51), we have

$$
\sum_{A=1}^{m} -\langle N_{E_A}, J_\Sigma(N_{E_A}) \rangle
$$
Corollary 35 The only compact stable minimal immersion of codimension 2 in the product space $\mathbb{H}P^{m_1} \times \mathbb{C}P^s$ is $\mathbb{H}P^{m_1} \times \mathbb{R}P^{s-2}$, and in the product space $\mathbb{H}P^{m_1} \times \mathbb{R}P^s$ is $\mathbb{H}P^{m_1} \times M^{2s-2}$, where $M$ is a complex submanifold of dimension $2s - 2$ immersed in $\mathbb{C}P^s$.

Proof The proof is completed by using Theorem 33 and observing that the only stable minimal immersions of codimension 2 in $\mathbb{R}P^s$ and $\mathbb{C}P^s$ are $\mathbb{R}P^{s-2}$ and $M^{2s-2}$, respectively, where $M$ is a complex submanifold of dimension $2s - 2$ immersed in $\mathbb{C}P^s$ (see Theorems 2 and 3).
4.3 Dimension 1 and 2

The reasoning that will be used in the proof of Theorem 36 is analogous to that used in the proof of Theorem 26. The arguments that will be presented in the first part of the proof of Theorem 39 are similar to those presented in the proof of Lemma 20. Then, after obtaining a characterization of the tangent space of the immersion, the second part of the proof follows similarly the proof of Theorem 26.

**Theorem 36** Let \( \Phi = (\psi, \phi) : \Sigma \to \mathbb{H}P^{m_1} \times M \) be a compact stable minimal immersion of dimension \( n = 1 \) and codimension \( d \), where \( M \) is any Riemannian manifold of dimension \( m_2 \). Then, \( \phi : \Sigma \to M \) is a stable geodesic, \( \psi \) is a constant function, and therefore \( \Phi(\Sigma) = \{r\} \times \phi(\Sigma) \) with \( r \) a point of \( \mathbb{H}P^{m_1} \).

**Proof** Since \( n = 1 \), Eq. (44) becomes

\[
\sum_{A=1}^{m} -\langle N_{E_A}, J_{\Sigma}(N_{E_A}) \rangle = \lambda^2 \left( -\sum_{k=1}^{3} \sum_{\beta=1}^{d} \langle J_{k}(\eta_{\beta}), e^1 \rangle^2 - |e^1|^4 \right),
\]

where \( e = (e^1, e^2) \) is a unit vector in \( d\Phi_p(T_p\Sigma) \), for \( p \in \Sigma \). The last expression just differs from Eq. (41) in the first term. As in the last subsection, this term does not add any new difficulties because it is non-positive. We have:

\[
0 \leq \sum_{A=1}^{m} -\int_{\Sigma} \langle N_{E_A}, J_{\Sigma}(N_{E_A}) \rangle d\Sigma = -\lambda^2 \int_{\Sigma} \sum_{k=1}^{3} \sum_{\beta=1}^{d} \langle J_{k}(\eta_{\beta}), e^1 \rangle^2 + |e^1|^4 d\Sigma \leq 0,
\]

where we have used the fact that \( \Sigma \) is stable in the first inequality. Hence, for \( p \in \Sigma \), \( e^1 = 0 \). Therefore,

\[
d\Phi_p(T_p\Sigma) = \{\alpha(0, e^2) : \alpha \in \mathbb{R}\},
\]

and thus,

\[
N_p\Sigma = \{(v, 0) : v \in T_{\psi(p)}\mathbb{H}P^{m_1} \} \bigoplus \{(0, w) : w \in [e^2]_{\perp M}\}.
\]

The rest of the proof follows in a similar way as in the proof of Theorem 26. \( \Box \)

**Corollary 37** There are no compact stable geodesics in the product space \( \mathbb{H}P^{m_1} \times S^s \), \( \mathbb{H}P^{m_1} \times \mathbb{C}P^{s} \), or \( \mathbb{H}P^{m_1} \times \mathbb{K}P^{s} \), where \( \mathbb{K} \in \{\mathbb{C}, \mathbb{H}\} \) other than \( \{r\} \times S^1 \) in \( \mathbb{H}P^{m_1} \times S^1 \), where \( r \in \mathbb{H}P^{m_1} \).

**Proof** We use Theorem 36 and observe that there are no stable geodesics in \( S^s \), \( \mathbb{C}P^{s} \), \( \mathbb{H}P^{s} \), or \( \mathbb{O}P^2 \) (see Theorems 1 and 3) other than \( S^1 \) in \( S^1 \). \( \Box \)

**Corollary 38** The only compact stable geodesic in the product space \( \mathbb{H}P^{m_1} \times \mathbb{R}P^{s} \) is \( \{r\} \times \mathbb{R}P^1 \), \( r \in \mathbb{H}P^{m_1} \).
Proof It follows from Theorem 36 and the fact that the only stable geodesic in $\mathbb{R}P^s$ is $\mathbb{R}P^1$ (see Theorem 2).

Theorem 39 Let $\Phi = (\psi, \phi) : \Sigma \to \mathbb{H}P_{m_1}^{m_2} \times M$ be a compact stable minimal immersion of dimension $n = 2$ and codimension $d$, where $M$ is any Riemannian manifold of dimension $m_2$. Then, $\phi : \Sigma \to M$ is a stable minimal immersion of dimension 2, $\psi$ is a constant function, and therefore $\Phi(\Sigma) = \{r\} \times \phi(\Sigma)$ with $r$ a point of $\mathbb{H}P_{m_1}^{m_2}$.

Proof Let $p \in \Sigma$. Since $n = 2$, for $s \in \{1, 2, 3\}$, Eq. (44) becomes

$$\sum_{A=1}^{m} -\langle NE_A, J\Sigma(NE_A) \rangle$$

$$= \lambda^2 \left( - \sum_{k=1}^{3} \sum_{j=1}^{2} \sum_{\beta=1}^{d} \langle J_k(\eta_\beta^1), e_j^1 \rangle^2 + \sum_{i=1}^{2} \sum_{j=1}^{2} \langle J_s(e_i^1), e_j^1 \rangle^2 - \langle e_i^1, e_j^1 \rangle^2 \right)$$

$$= \lambda^2 \left( - \sum_{k=1}^{3} \sum_{j=1}^{2} \sum_{\beta=1}^{d} \langle J_k(\eta_\beta^1), e_j^1 \rangle^2 - |e_1^1|^4 - |e_2^1|^4 - 2\langle e_1^1, e_2^1 \rangle^2 + 2\langle J_s(e_2^1), e_1^1 \rangle^2 \right).$$

If $e_1^1 = 0$,

$$\sum_{A=1}^{m} -\langle NE_A, J\Sigma(NE_A) \rangle = \lambda^2 \left( - \sum_{k=1}^{3} \sum_{\beta=1}^{d} \langle J_k(\eta_\beta^1), e_2^1 \rangle^2 - |e_2^1|^4 \right) \leq 0. \quad (56)$$

If $e_1^1 \neq 0$, we can write $e_2^1$ in terms of $e_1^1$, $J_s(e_1^1)$, and $X$ for some vector $X \in T_{\psi(p)}\mathbb{H}P_{m_1}^{m_2}$ which is unit and orthogonal to $e_1^1$ and $J_s(e_1^1)$. Then,

$$\langle e_2^1, J_s(e_1^1) \rangle^2 = |e_1^1|^2 |e_2^1|^2 - \langle e_2^1, e_1^1 \rangle^2 - |e_1^1|^2 \langle e_2^1, X \rangle^2.$$

Replacing the last equation in Eq. (56), we have

$$\sum_{A=1}^{m} -\langle NE_A, J\Sigma(NE_A) \rangle$$

$$= \lambda^2 \left( - \sum_{k=1}^{3} \sum_{\beta=1}^{d} \langle J_k(\eta_\beta^1), e_j^1 \rangle^2 - (|e_1^1|^2 - |e_2^1|^2)^2 - 4\langle e_1^1, e_2^1 \rangle^2 - 2|e_1^1|^2 \langle e_2^1, X \rangle^2 \right) \leq 0. \quad (57)$$

From Eqs. (56) and (57), we have
\[ \sum_{A=1}^{m} -(N_{E_A}, J_{\Sigma}(N_{E_A})) \leq 0. \]

We follow the proof of Lemma 20 (from Eq. (23) on) to obtain

\[ e_2^1 = \pm J_s(e_1^1) \quad (58) \]

for \( s \in \{1, 2, 3\} \).

From Eq. (58), we have

\[ \langle J_1(e_1^1), J_2(e_1^1) \rangle = \langle \pm e_2^1, \pm e_2^1 \rangle = \pm |e_2^1|^2. \]

On the other hand

\[ \langle J_1(e_1^1), J_2(e_1^1) \rangle = -\langle J_2(J_1(e_1^1)), e_1^1 \rangle = \langle J_3(e_1^1), e_1^1 \rangle = 0. \]

Hence \( e_2^1 = 0 \). Since \( J_s \) is an isometry from Eq. (58), \( e_1^1 = 0 \). Therefore, at \( p \in \Sigma \) we have that \( e_1 = (0, e_2^2) \) and \( e_2 = (0, e_2^2) \). Thus

\[ d\Phi_p(T_p\Sigma) = \{(0, x) : x \in \text{Gen}\{e_2^1, e_2^2\}\}, \]

and then,

\[ N_p\Sigma = \{(v, 0) : v \in T\psi(p)\mathbb{HP}^{m_1}] \bigoplus \{(0, w) : w \in [\text{Gen}\{e_2^1, e_2^2\}]^\perp M\}. \]

where \([.]^\perp M\) is the orthogonal complement in \( T\psi(p)M \) and \( \text{Gen}\{e_2^1, e_2^2\} \) is the subspace generated by the vectors \( e_2^1, e_2^2 \) in \( T\psi(p)M \). The rest of the proof follows by applying the same proof of Theorem 26.

\[ \square \]

**Corollary 40** There are no compact stable minimal immersions of dimension 2 in the product manifold \( \mathbb{HP}^{m_1} \times S^s, \mathbb{HP}^{m_1} \times \mathbb{OP}^2, \text{or } \mathbb{HP}^{m_1} \times \mathbb{CP}^s \) other than \( \{r\} \times S^2 \) in \( \mathbb{HP}^{m_1} \times S^2 \), where \( r \in \mathbb{HP}^{m_1} \).

**Proof** It follows immediately from Theorem 39 and the fact that there are no stable minimal immersions of dimension 2 in \( S^s, \mathbb{HP}^s, \text{or } \mathbb{OP}^2 \) (see Theorems 1 and 3) other than \( S^2 \) in \( S^2 \).

\[ \square \]

In the same spirit, we deduce that

**Corollary 41** The only compact stable minimal immersion of dimension 2 in the product space \( \mathbb{HP}^{m_1} \times \mathbb{RP}^s \) is \( \{r\} \times \mathbb{RP}^2 \), and in \( \mathbb{HP}^{m_1} \times \mathbb{CP}^s \) is \( \{r\} \times M \), where \( M \) is a complex submanifold of dimension 2 immersed in \( \mathbb{CP}^s \) and \( r \in \mathbb{HP}^{m_1} \).

**Proof** We use Theorem 39 and observe that the only stable minimal immersions of dimension 2 in \( \mathbb{RP}^s \) and \( \mathbb{CP}^s \) are \( \mathbb{RP}^2 \) and \( M \) respectively, where \( M \) is a complex submanifold of dimension 2 immersed in \( \mathbb{CP}^s \) (see Theorems 2 and 3).

\[ \square \]

**Declarations**

**Conflict of interest** The author has no conflict of interest.
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