How to parameterize a light and broad resonance (the $\sigma$ meson)$^*$

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We point out that a commonly used parameterization form to describe the $\sigma$ resonance is problematic by introducing a spurious singularity below two particle threshold, on the second sheet. The spurious singularity violates chiral symmetry and leads to sizable contribution when the resonance is light and broad – as what happens to the $\sigma$ resonance, hence must be removed.

The evidence for the existence of a light and broad resonance in $I=J=0$ $\pi\pi$ scattering process – named as the $\sigma$ resonance – has been accumulated, and more and more physicists working on the subject accepted the renaissance of the $\sigma$ resonance. Among those works, model independent analysis$^1,2$ are vital in clarifying the issue whether the $\sigma$ resonance exists or not, since one has to distinguish and separate the contribution of a broad resonance to the phase motion from background contributions, and the latter is often very difficult, if not impossible, to estimate. However, there are different opinions in the literature against the existence of the light $\sigma$ resonance$^3$, based upon the following observation: the $\pi\pi$ phase shift does not pass $90^\circ$ at low energies, which would have occurred, according to Eq. (1.7) given below, at $s=M^2$ with $M$ the Breit–Wigner mass which should be light. This question is actually already answered, in one aspect, by Ishida and collaborators.$^4$ who pointed out that chiral symmetry requires the background contribution to the phase shift in $I=J=0$ $\pi\pi$ scatterings to be negative,$^*$\textsuperscript{**} Ishida and collaborators argue that the background phase cancels the large phase motion generated by the Breit–Wigner resonance and therefore there is no contradiction between the observed $\pi\pi$ phase shift and the existence of a light and broad resonance. In this talk I will further point out that a conventional Breit-Wigner description to the $\sigma$ resonance actually violates chiral symmetry and should be abandoned, and it is not even necessary for a light and broad resonance to develop a phase shift passing $90^\circ$.

§1. The Breit–Wigner formula as a narrow width approximation

In principal the Breit–Wigner description of resonance only works for infinitely small width. This is clearly understood from the following heuristic derivation of the Breit–Wigner formula.$^5$ First of all, for the propagator of a stable particle we have

$$\Delta(p^2) = \frac{1}{p^2 - m^2 + i\epsilon}.$$  \hspace{1cm} (1.1)
Now let us focus on the following chain decay process,
\[ A \to B + C, \quad B \to D + E, \] (1.2)
described by the effective Hamiltonian \( H_{\text{eff}} = fABC + gBDE \) where \( D \) and \( E \) are massless particles and \( B \) is a narrow resonance. The decay of particle \( A \) can be viewed either as a two body decay process, \( A \to B + C \), which has a decay width\(^*\)
\[ \Gamma_A = \frac{f^2}{16\pi} \frac{m_A^2 - m_B^2}{m_A^3}, \] (1.3)
or as a three body decay, \( A \to B + C + E \). To calculate the 3 body decay we need a knowledge on the form of the propagator of the unstable particle \( B \). By comparing with Eq. (1.1) we may assume the propagator to be the following form:
\[ \Delta_B(p^2) = \frac{1}{p^2 - \alpha + i\beta}, \] (1.4)
where \( \alpha \) and \( \beta \) are constants. Now the partial decay width of the process \( A \to C + D + E \) can be calculated since we know the effective Hamiltonian and Eq. (1.4) (in the calculation \( m_A >> \Gamma_A \) is assumed!),
\[ \Gamma_A = \frac{f^2}{16\pi} \frac{g^2}{16\pi} \frac{m_A^2 - \alpha}{m_A^3\beta}. \] (1.5)
The two different calculation of \( \Gamma_A \) have to yield the same result, which leads to
\[ \Delta_{BW}(p^2) = \frac{1}{p^2 - M^2 + i\rho(s)\Gamma_B}. \] (1.6)
This equation is the well known Breit–Wigner formula. From the way we derive it we realize that the Breit–Wigner description of resonances can only be exact in the case of infinitely small width, and one expects that it works reasonably well in narrow width approximation. However, there is a problem associated with Eq. (1.6), since it missed the threshold effect and the parameterization form does not respect a nice property of Feynman amplitudes called real analyticity.\(^**\) Therefore, one often adopts another parameterization form instead of Eq. (1.6),
\[ \Delta_{\text{BW}}'(p^2) = \frac{1}{p^2 - M^2 + \rho(s)G}, \] (1.7)
where \( \rho(s) \) is the kinematic factor. For equal mass case like \( \pi\pi \) scattering it is \( \rho(s) = \sqrt{1 - 4m_\pi^2/s} \). Both Eq. (1.6) and Eq. (1.7) are frequently used in the literature and are recommended by the Particle Data Group. It is worth emphasizing that when the resonance’s width is narrow and the pole locates far above the threshold \( (M^2 >> 4m_\pi^2) \), in the vicinity of \( s = M^2 \) the Eq. (1.6) and Eq. (1.7) gives almost identical
\(^*\) Assuming \( B \) is stable in obtaining \( \Gamma_A \). The decay width of \( B \) is also obtainable: \( \Gamma_B = \frac{g^2}{16\pi} \frac{1}{m_B^3} \)
\(^**\) The latter is not crucial but the former is really a shortcoming of the standard Breit–Wigner parameterization.
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results. It is noticed that in more complicated applications people sometimes also treat the constant $G$ in Eq. (1.7) as a function of $s$.

The condition for applying the Breit–Wigner approximation is actually well known among physicists working in the field, what I will focus on in this talk is only the $\sigma$ resonance, which is light and broad\(^1\),\(^2\),\(^6\). Therefore one hesitates to use the parameterization form like Eq. (1.7) in the experimental fit to extract the $\sigma$ pole position. Unfortunately this is not the case in many of the research works. Before demonstrating Eq. (1.7) should not be used to parameterize the $\sigma$ resonance, it is necessary to study some general properties of the scattering $S$ matrix.

§2. The factorized $S$ matrix

2.1. The simplest partial wave $S$ matrices

The starting point is the dispersion relations set up by the present author and collaborators on $\pi\pi$ scattering partial wave $S$ matrix elements\(^1\),\(^7\),

\[
\sin(2\delta_\pi) \equiv \rho F ,
\]

\[
F(s) = \alpha - \sum_j \frac{1/2i\rho(z_j^{II})}{S'(z_j^{II})(s - z_j^{II})} + \frac{1}{\pi} \int_L \frac{\text{Im}F(s')}{s' - s} ds' + \frac{1}{\pi} \int_R \frac{\text{Im}F(s')}{s' - s} ds' ,
\]

in which we did not include the bound state contribution for simplicity, since it is irrelevant here. In the above expression $L = (-\infty, 0], R = [4m_K^2, \infty)$, $\alpha$ is a subtraction constant and $z_j$ denotes the pole position on the second sheet. When $z_j$ is real it represents a virtual state pole, when $z_j$ is complex it must appear in one pair together with $z_j^{*}$, representing a resonance. The experimental curve of the function $F$ is convex, yet chiral perturbation theory predicts a negative and concave left hand integral contribution.\(^1\) This fact undoubtfully establish the existence of the $\sigma$ resonance. Similarly we have\(^8\),

\[
\cos(2\delta_\pi) \equiv \tilde{F} = \tilde{\alpha} + \sum_j \frac{1}{2S'(z_j^{II})(s - z_j^{II})} + \frac{1}{\pi} \int_L \frac{\text{Im}\tilde{F}(s')}{s' - s} ds' + \frac{1}{\pi} \int_R \frac{\text{Im}\tilde{F}(s')}{s' - s} ds' .
\]

Eqs. (2.1) and (2.2) defines the analytic continuation of the scattering $S$ matrix, $S = \cos(2\delta_\pi) + i\sin(2\delta_\pi)$ and further, the generalized unitarity equation,

\[
\sin^2 2\delta_\pi + \cos^2 2\delta_\pi \equiv 1 ,
\]
Before studying more complicated case of $S$ matrix in reality, let us first focus
on those simplest solutions of the $S$ matrix. Here ‘simplest’ means there is no
‘dynamical’ cut and the number of poles is minimal. In the absence of bound state
there are two such simplest solutions (taking the mass of the scattering particle to
be unity):

1. A virtual state pole at $s = s_0$ ($0 < s_0 < 4$): scattering length
$a = \sqrt{\frac{s_0}{4-s_0}}$.

$$
\text{Re}_R T(s) = \frac{1}{4} \sqrt{s_0(4-s_0)} \frac{s}{s-s_0},
$$

$$
\text{Im}_R T(s) = \rho(s) \frac{ss_0}{4(s-s_0)}.
$$

2. A (pair of) resonance: poles locate at $z_0$ and $z_0^*$ on the second sheet. The
solution is:

$$
\text{Re}_R T(s) = \Delta \text{Re}[\sqrt{z_0(z_0-4)}] \frac{s(r_0-s)}{(s-z_0)(s-z_0^*)},
$$

$$
\text{Im}_R T(s) = \Delta \text{Im}[z_0] \rho(s) \frac{s^2}{(s-z_0)(s-z_0^*)},
$$

where

$$
\Delta = \frac{\text{Im}[z_0]}{(\text{Re}[\sqrt{z_0(z_0-4)}])^2 + (\text{Im}[z_0])^2},
$$

$$
r_0[z_0] = \text{Re}[z_0] + \text{Im}[z_0] \frac{\text{Im}[\sqrt{z_0(z_0-4)}]}{\text{Re}[\sqrt{z_0(z_0-4)}]}.
$$

The $S$ matrix can be rewritten as,

$$
S^R(s) = \frac{r_0[z_0] - s + i\rho(s)s}{r_0[z_0] - s - i\rho(s)s} \frac{\text{Im}[z_0]}{\text{Re}[\sqrt{z_0(z_0-4)}]},
$$

where

$$
r_0[z_0] = \text{Re}[z_0] + \text{Im}[z_0] \frac{\text{Im}[\sqrt{z_0(z_0-4)}]}{\text{Re}[\sqrt{z_0(z_0-4)}]}.
$$

This solution is unique for the $S$ matrix with only one resonance and without
the so called ‘dynamical cuts’.

It is very interesting to see $r_0$ as a function of $z_0$, as shown in Fig. 1. When
$\text{Re}[z_0] >> \text{Im}[z_0]$, $r_0$ is very close to $\text{Re}[z_0]$ and the result of Eq. (2.7) is similar
to the Breit–Wigner formula Eq. (1.7), as can be seen from Fig. 2. However when
decreasing $\text{Re}[z_0]$ while keeping $\text{Im}[z_0]$ fixed $r_0[z_0]$ does not decrease monotonously.
On the contrary, it increases when $\text{Re}[z_0]$ gets small enough. When the resonance
becomes light and broad, the typical phase motion it exhibits is depicted in Fig. 3.
That is the phase increases slowly and reach $90^\circ$ at very distant place. I believe,
this simple picture as shown by Fig. 3, nicely reveals qualitatively what happens in
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Fig. 1. $r_0[z_0]$ as a function of $\text{Re}[z_0]$, fixing $\text{Im}[z_0] = 1$. The vertical line corresponds to $\text{Re}[z_0] = 2$ and the horizontal line corresponds to $r_0 = 4$.

Fig. 2. The qualitative behavior of a narrow resonance with $\text{Re}[z_0] = 10$, $\text{Im}[z_0] = .5$. Line 1 represents the real part of the $T$ matrix, line 2 represents the imaginary part of the $T$ matrix whereas line 3 represents the phase shift.

I=$J=0$ $\pi\pi$ scattering at low energies$^\text{*)}$. Since $s = r_0$ is the point where the phase pass $90^\circ$, Eq. (2.7) characterizes the drastic difference between the phase motion generated by a light and broad resonance and a narrow resonance located far away from the threshold.

There is a critical line corresponds to $\text{Re}[z_0] = 2$ on the $s$ plane. When the resonance locates on the right hand side of the line, i.e., $\text{Re}[z_0] > 2$, the phase shift can get larger than $\pi/2$; whereas when the resonance locates on the left hand side of the line, i.e., $\text{Re}[z_0] < 2$, the phase shift can never reach $\pi/2$. Of course, a resonance will always give a positive contribution to the phase shift increasing monotonously as (the physical value) $s$ increase. But a deeply bounded resonance corresponding to $\text{Re}[z_0] << 2$ behaves like a normal virtual state.

2.2. The violation of Levinson’s theorem

In the examples discussed above, the phase shift of a virtual state is

$$\tan(\delta_v) = \rho(s) \sqrt{\frac{s_0}{4 - s}}.$$  \hspace{1cm} (2.9)

For the phase shift of a resonance, it is

$$\tan(\delta_r) = \frac{\text{Im}[z_0]}{\text{Re}[\sqrt{z_0(z_0 - 4)}]} \frac{\rho(s)s}{r_0 - s}.$$ \hspace{1cm} (2.10)

If $r_0 > 4$ then the phase shift passes $90^\circ$ when $s$ passes $r_0$. At $s = \infty$,

$$\delta(\infty) = \pi - \tan^{-1}\left(\frac{\text{Im}[z_0]}{\text{Re}[\sqrt{z_0(z_0 - 4)}]}\right) < \pi.$$ \hspace{1cm} (2.11)

$^\text{*)}$ and even $\pi K$ scattering at low energies, though in $\pi K$ scattering the singularity structure of the $S$ matrix is more complicated due to unequal mass kinematics.
This disagrees with Levinson's theorem in scattering theory, where the theorem says that a resonance contributes a phase shift $180^\circ$ at $\infty$. The discrepancy comes from the fact that in here we have the left hand cut from the relativistic kinematics, which leads to physical effects violating Levinson's theorem. Actually, here we have

$$\delta(\infty) - \delta(-\infty) = \pi .$$ (2.12)

### 2.3. The factorized $S$ matrix

In above we have analyzed several simple $S$ matrices. For a given partial wave $S$ matrix, though very complicated, can be written as a product of the simple $S$ matrices:

$$S^{\text{Phys.}} = \prod_i S^{R_i} \cdot S^{\text{cut}} ,$$ (2.13)

since a unitary matrix divided by any unitary matrix is still unitary. In the above equation $S^{R_i}$ denotes simple $S$ matrices as described above and $S^{\text{cut}}$ contains only cut which can be parameterized in the following simple form,

$$S^{\text{cut}} = e^{2i\rho f(s)}$$

$$f(s) = f_0 + \frac{s - 4}{\pi} \int_L \frac{\text{Im}_L f(s')}{(s' - 4)(s' - s)}$$

$$+ \frac{s - 4}{\pi} \int_R \frac{\text{Im}_R f(s')}{(s' - 4)(s' - s)} ,$$ (2.14)

where $R$ denotes cuts at higher energies other than $2\pi$ cuts. The function $f$ must be non-vanishing in general and especially in $\pi\pi$ scatterings. It should be emphasized that there is no loss of generality in Eq. (2.13), since couple channel effects (or physics at sheet III, IV, etc.) are all hidden in the right hand cut integral in Eq. (2.14). The Eq. (2.13) is a re-derivation of the so-called Dalitz–Tuan parameterization, with the special treatment of $S$ matrix poles as discussed in sec. 2.1.
§3. The problem of a Breit–Wigner description of the $\sigma$ resonance

A frequently used parameterization form of a resonance in the literature is

$$S = \frac{M^2 - s + i\rho(s)M\Gamma}{M^2 - s - i\rho(s)M\Gamma}.$$  \hspace{1cm} (3.1)

However, such an $S$ matrix contains three poles. For a sufficiently large $M^2$ and small $M\Gamma$ it contains a resonance and a virtual state. According to the discussion made in sec. 2.1, this $S$ matrix can be factorized in the product of two simpler $S$ matrices. The Eq. (3.1) contains only two parameters, therefore the pole location of the virtual state is determined by the pole location of the resonance. Qualitatively speaking, the virtual state pole get closer to the threshold when the resonance pole is light and broad, and vice versa. If denoting the resonance pole position as $z_0$, then the scattering length of the virtual state is,

$$a = \frac{\text{Im}[z_0]}{\text{Re}[\sqrt{z_0(z_0 - 4)}]}.$$  \hspace{1cm} (3.2)

According to the present parameterization, the scattering length of the virtual state pole and the resonance pole are additive and are both positive. Taking $M = 400\text{MeV}$, $\Gamma = 600\text{MeV}$ as an example ($z_0 \equiv (M + i\Gamma/2)^2$). The scattering length predicted by Eq. (2.7) is 0.23, and is 2.73 as predicted by Eq. (3.1)! It is easy to understand that such a virtual state pole is not allowed in $I=J=0\pi\pi$ scattering. Because the virtual state pole corresponds to an $S$ matrix zero on the physical sheet: $S = 1 + 2i\rho T = 0$, which implies $T(s_0) = -1/2i\rho(s_0)$. For $s_0$ not far from the threshold it predicts $T \sim O(1)$ in chiral power counting, yet chiral symmetry dictates that $T \sim O(m^2_\pi^2)$ near threshold.

To use Eq. (3.1) in studying the $\sigma$ resonance is therefore in error – we struggled to look for a pole rather far from the physical region but end up with a spurious pole much closer to the physical region and dominates the physics near the threshold! Unfortunately, this problem is overlooked by most studies which incorrectly make use of Eq. (1.7).

§4. Conclusions and prospects

Before jumping to the conclusion I would like to discuss a little bit more on the background phase in Eq. (2.14). One tries to determine the background term by a match between our parameterization and the $\chi$PT results in the region where $\chi$PT result is reasonable.

$$\prod_i S^{r_i} \cdot S^{\text{cut}} \simeq 1 + 2i\rho(s)T^{\chi PT}(s),$$  \hspace{1cm} (4.1)

where

$$T^{\chi PT} = T^{(2)} + T^{(4)} + T^{(6)} + \cdots.$$  \hspace{1cm} (4.2)

$^*$) In the $I=J=2$ channel, chiral symmetry however does predict a virtual state pole. But the pole locates very close to $s = 0$ and hence has only negligible effect.  

9)
On the other side we have \( f = f^{(2)} + f^{(4)} + \cdots \) and also \( S^{Ri} \) can be expanded. Taking the simpler \( I=2, J=0 \) channel for example (where there is no resonance pole), up to \( O(p^4) \), we have

\[
S^{\text{cut}} = e^{2i\rho f} \simeq 1 + 2i\rho(f^{(2)} + f^{(4)}) - 2\rho^2 f^{(2)2} \\
\simeq 1 + 2i\rho(T^{(2)} + T^{(4)}),
\]

which leads to \( f^{(2)} = T^{(2)}, f^{(4)} = T^{(4)} - i\rho T^{(2)2} = \text{Re}T^{(4)}, \) etc.. This procedure however has a problem by introducing an essential singularity to the approximate \( S \) matrix at \( s = 0 \). But the \( S \) matrix may be acceptable except in the vicinity of \( s = 0 \). This way of matching with \( \chi \)PT determines the background phase without introducing new parameters except those pole parameters, avoiding the disastrous physical sheet resonance poles predicted by Padé approximations. Therefore it is worthwhile to further study along this direction.

To conclude, it is suggested to use Eq. (2.7) to parameterize the propagator of the \( \sigma \) resonance. Notice that the parameterization form is \emph{not} in any sense unique. For example, in production processes, since there are difficulties in estimating the background contribution, one is free to absorb some of the background contribution into the propagator. But one thing is clear, that the spurious virtual state as introduced by Eq. (1.7) must be removed. There is strong evidence in the experimental fit to production processes that Eq. (2.7) considerably improves the total \( \chi^2 \) comparing with Eq. (1.7).

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\[ \text{References} \]

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* Related discussion may be found in Ref. 6.*