Entanglement generation through local field and quantum dissipation

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Abstract
Entanglement in a Gaussian two-mode system can be generated by local driving if additional non-local features are introduced to the dynamics. We demonstrate that weak to moderate ohmic friction arising from a dissipative environment can enable entanglement generation in a driven system. This synergy of driving and dissipation is highly sensitive to the pulse shape; several simple pulse shapes fail to produce this effect at all or deposit large amounts of energy in the system as a side effect. Complex pulse shapes, determined by optimal control techniques, however, are effective without detrimental side effects.

Keywords: Gaussian states, optimal control, open quantum system

(Some figures may appear in colour only in the online journal)

1. Introduction

Entanglement is a key resource in virtually all proposed applications of quantum information and quantum computation. It is frequently needed in the initial state of quantum algorithms. It must be provided on a large scale in the ancilla degrees of freedom in most proposed schemes for quantum error correction, and it is indispensable to ensure privacy in quantum encryption.

There are many different technologies aiming for the realization of the first universal quantum computer; among them several solid-state approaches have been appreciated for some time due to their use of highly scalable technologies. More recently, significant progress has been made in significantly extending the decoherence times of solid-state devices [1, 2].

The dissipation mechanisms found in solid-state devices, in particular circuit damping, correspond closely to the generic Ohmic damping model [3]. Here we study the effects of this dissipation model in a minimal mechanism of entanglement generation between two harmonic degrees of freedom. We find that the dissipative reservoir, by itself a source of decoherence and entanglement decay, can be turned into a source of entanglement when combined with local external driving of the constituents of the entangled pair which is being formed. Optimal control theory is used to find pulse shapes which yield significant entanglement in a setting where the only non-local interaction is quantum friction mediated by the environment.

We compare our optimal control results to several simple guesses for driving fields which might similarly induce entanglement. Of the three alternative driving scenarios considered, two fail to produce entanglement. One further scenario investigated results in entanglement; however, this comes at the price of significantly heating the system.

2. Quantum information in continuous-variable systems

2.1. Gaussian quantum information

Most practical proposals and algorithms in quantum information refer to the concept of a qubit, i.e., the notion of two orthogonal quantum states and their possible superpositions. The fundamental concepts of quantum information, however, do not require this constructive approach, and the important concepts of entanglement and non-classical correlations
equally apply to quantum systems with arbitrary level structure, and can be defined without referring to a specific basis in Hilbert space. In particular, Gaussian states of harmonic or near-harmonic systems have recently attracted significant attention in this context [4, 5]. In spite of their apparent simplicity and near-classical appearance, these states can be viable sources of entanglement. In this paper, we demonstrate a mechanism of entanglement generation based on simple ingredients; control of local fields and a common, Ohmic (featureless) thermal reservoir [3].

In the context of quantum optics, Gaussian states with non-negative Wigner distribution are often labeled classical; however, when more than one degree of freedom is considered, such states that look ‘classical’ in their single-particle properties may display highly non-classical features such as two-mode squeezing and entanglement [6]. Several models have recently been proposed for the generation or stabilization of these quantum resources through suitably chosen couplings and/or external fields [6–10]. Here we explore a scenario so simple that it could almost be called a circumvention of the well-known theorem that entanglement cannot be generated by a LOCC mechanism (local operations, classical communication) [11]. In essence, we propose to entangle two harmonic degrees of freedom, each weakly coupled to the same dissipative reservoir, without any direct coupling to each other, by applying suitable classical fields locally. In this paper, we investigate the sensitivity of this mode of entanglement generation to particular choices for the time dependence of external fields and their potential side effects.

2.2. Dynamics, decoherence and relaxation

The type of entanglement generation we envisage is a dynamical process, where the only non-local feature is the common dissipative environment. Strong coupling to a dissipative reservoir has previously been demonstrated to promote or statically support entanglement in particular cases [9, 12]. Most of such approaches rely on the simple fact that a strong dissipative mechanism will force a quantum system into a corresponding pointer state [13]. In a somewhat more subtle approach, there is a decoherence-free subspace instead of effective projections on individual pointer states [9]. For these entangled states to be useful, they need to be decoupled or transferred away from the reservoir after preparation, with high fidelity required for this secondary process. Here we do not rely on such a controlled switching of dissipation, nor on any reliable transfer mechanism, but consider weak to moderate coupling to the reservoir from the outset.

This means that the corresponding equilibrated two-mode state will typically be separable. A thermal reservoir can also display effects far more subtle than merely ‘pulling towards’ pointer states. This holds in particular for driven systems, which allow the ‘mutual friction’ of the two modes induced by the reservoir to be leveraged for entanglement generation.

Quantum dynamics under the combined influence of strong driving and a dissipative reservoir is a challenging theoretical subject even in the case of weak dissipative coupling. In the absence of strong driving, the reduced dynamics of the system is easily described by a Lindblad-type master equation in most cases. However, since Lindblad operators represent transitions between time-independent, unperturbed energy eigenstates of the system, this approach fails to reproduce the true dynamics even qualitatively if strong driving is taken into account by merely changing Hamiltonian part of the Liouvillian, while keeping the Lindblad dissipator unchanged [14].

An alternative approach to open system dynamics, which keeps external driving and dissipation conceptually separate, relies on Feynman–Vernon influence functionals [3, 15], which have been used extensively in studying mesoscopic quantum phenomena [16]. Feynman–Vernon influence functionals are Gaussian functionals of the integrand in a path integral. Due to this property, even influence functionals representing a quantum reservoir can be constructed from Gaussian noise which is classical in the sense that it is represented by c-numbers, but has correlation functions which match the quantum case. Equivalently, they can be seen as representing averages of sample states $\rho_0$ propagated under a fictitious stochastic force [17–19].

In the case of a simple Ohmic reservoir, the dynamics of sample states for a single particle, given by the Feynman–Vernon path integral, translates into a stochastic Liouville equation with dissipation [20],

$$
\frac{d}{dt}\rho \approx \frac{1}{i\hbar} \left( \left[ H_S, \rho \right] - \xi(t) \left[ q, \rho \right] \right) + \frac{\gamma}{2\hbar} \left[ q, \{ p, \rho \} \right].
$$

(1)

Here $\xi(t)$ is a real-valued stochastic force with a spectrum determined by the quantum fluctuation–dissipation theorem. In general, this spectrum is non-white, and the corresponding time correlation function decays only on the finite thermal time scale $\hbar \theta$. One might interpret the last term in (1) as velocity-dependent friction; however, the partial term $\{ p, \rho \}$ does not represent objective or certain information about the momentum. In the classical limit, on the other hand, this interpretation is quite obvious, and (1) is closely related to the Klein–Kramers equation. A phase-space representation of $\rho_0$ and an average over $\xi(t)$ (which turns into white noise for $\hbar \to 0$) are all that is needed to make this connection.

The key advantage of our approach in the context of driven systems lies in the fact that neither $\xi(t)$ nor the friction term change when the system Hamiltonian $H_S$ is modified by external driving. For two-mode Gaussian states and a quadratic Hamiltonian

$$
H_S \equiv \sum_{k=A,B} \frac{p_k^2}{2m} + \frac{m\omega^2}{2} q_k^2 + \frac{u(t)}{2} q_k^2,
$$

(2)

with parametric driving $u(t)$, (1) can be mapped to a set of ordinary differential equations for the parameters of the Gaussian state. Here we choose the first and second

5 In the classical limit, this stochastic force can be identified as thermal noise.
cumulants of position $q$ and momentum $p$ as parameters. In the simple case of two independent reservoirs, the damping term $(\gamma/2i\hbar)[q, [p, \rho_k]]$ is duplicated like the Hamiltonian, and we obtain the set of equations ($k = A, B$)

$$\frac{d}{dt} \langle p_k \rangle = \langle p_k \rangle / m, \quad (3)$$

$$\frac{d}{dt} \langle q_k \rangle = -m\omega^2 + u(t) \langle q_k \rangle - \gamma \langle p_k \rangle + \dot{\xi}(t), \quad (4)$$

$$\frac{d}{dt} \langle q_k^2 \rangle = 2\langle p_k q_k \rangle / m, \quad (5)$$

$$\frac{d}{dt} \langle p_k^2 \rangle = -2\gamma \langle p_k^2 \rangle - 2m\omega^2 \langle p_k q_k \rangle, \quad (6)$$

$$\frac{d}{dt} \langle p_k q_k \rangle = -m\omega^2 \langle q_k^2 \rangle + \langle p_k^2 \rangle / m - \gamma \langle p_k q_k \rangle. \quad (7)$$

Symmetric (Weyl) operator ordering is assumed for all mixed products.

In the more interesting case of a shared reservoir coupling equally to both modes, $\xi(t)$ loses its index $k$, and the new damping term $(\gamma/2i\hbar)[q_A + q_B, \{p_A + p_B, \rho_k\}]$ in (1) is not itself a sum, but depends on sums of position and momentum coordinates. Equations (3)–(7) must then be modified by A–B cross terms proportional to $\gamma$ and augmented by additional equations of motion for cumulants denoting correlations between modes A and B. We obtain an expanded system of equations [21] of the form

$$\ddot{x} = M(u(t)) \cdot \ddot{x} + \dot{x}(t) \dot{\epsilon}, \quad (8)$$

with

$$\ddot{x} = \left( \langle q_A \rangle, \langle p_A \rangle, \langle q_B \rangle, \langle p_B \rangle, \langle q_A^2 \rangle, \langle p_A^2 \rangle, \langle p_A q_A \rangle, \langle p_A q_B \rangle, \langle q_B q_A \rangle, \langle q_B q_B \rangle \right) \cdot.$$ (9)

$$M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix},$$

$$M_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\gamma & 0 & 0 & 0 & -\gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\gamma & 0 & 0 & 0 \\ 0 & 0 & -\gamma & 0 & 0 & 0 & 0 & -\gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\gamma & 0 \\
0 & 0 & 0 & 0 & -\gamma & 0 & 0 & 0 & 0 & -\gamma \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\gamma & 2a & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ a & 1 & -\gamma & 0 & 0 & 0 & 0 & 0 & 0 & -\gamma \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\gamma & 2a & 0 & -2\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 1 & -\gamma & 0 & 0 & -\gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & -\gamma & 0 & 0 & -\gamma & 0 & 0 & 0 & 0 & -2\gamma \\ 0 & 0 & 0 & 0 & -\gamma & a & 1 & -\gamma & 0 & 0 \\ 0 & 0 & -\gamma & 0 & 0 & a & 1 & -\gamma & 0 & -\gamma \end{pmatrix}.$$ (12)

$a = -1 - u$ and

$$\dot{\epsilon} = (0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) \cdot$$ (13)

For the sake of brevity, these expressions for $M$ have been given here using natural units of position and momentum (i.e., $m = 1$ and $\omega = 1$).

The diagonalization of $M$ is time-dependent, with the parametric driving $u(t)$ yet unknown. Equation (8) must therefore be solved numerically for any given $u(t)$.

3. External control and optimization of entanglement generation

3.1. Optimal control theory

Freedom to choose the time dependence of the field $u(t)$, combined with the intent to maximize the entanglement of the final two-mode state, defines an elaborate optimization problem [22]. Maximization of the final-state entanglement is subject to a dynamical constraint given by (8). This constraint is valid at any time, and, in our case, for any realization of the stochastic process $\xi(t)$. Lagrange multipliers $\lambda(t)$ referring to these constraints are therefore (i) functions of time and (ii) random variables in the same probability space as $\xi(t)$. Variations $\delta \xi(t)$ are independent for different realizations of the noise $\xi(t)$. The control field $u(t)$, on the other hand, is not a random variable, since the optimization objective is based on the final quantum state, resulting from an average over realizations.

Having introduced the dynamics through Lagrange multipliers, the optimization objective is now the sum of the entanglement measure we want to maximize and a Lagrange multiplier term in the form of a time integral of a suitably chosen expectation value in the probability space of the stochastic process $\xi(t)$,

$$\int_0^t dt \mathbb{E} \left[ -\langle \ddot{\lambda}, \dot{x} \rangle + \mathcal{H} \left( \ddot{\lambda}, \dot{x}, u, \xi \right) \right]$$ (14)

with

$$\mathcal{H} \left( \ddot{\lambda}, \dot{x}, u, \xi \right) = \langle \ddot{\lambda}, M(u) \cdot \dot{x} + \xi \dot{\epsilon} \rangle.$$ (15)

Round parentheses denote the scalar product. Variational calculus determines the time dependence of the Lagrange
multipliers to be governed by the equation of motion
\[ \dot{\lambda} = -M'(u)\lambda, \]  
with a boundary condition of the form \( \dot{x}_f = g(x_f) \) at the end time. The function \( g \) is determined by the variation of the final-state entanglement measure with respect to \( \delta x_f \). Since the Lagrange multipliers \( \lambda \) are dynamical variables, they are frequently referred to as co-states \([22]\). Even though (16) is a deterministic equation of motion, the co-states are true random variables since their boundary condition is random. Equations (8) and (16), together with
\[ \frac{\partial}{\partial u} \left[ H(\vec{\xi}, \lambda, u, \vec{x}) \right] = 0, \]  
again valid at arbitrary time \( t \), constitute necessary conditions for a local minimum of the constrained optimization problem. The approach of constructing co-states based on the ordinary differential equations (8) provides significant advantages over the direct application of control theory to the two-particle version of the Liouville equation (1) \([23]\).

Due to the mixed boundary conditions, the simultaneous solution of (8), (16) and (17) is usually computed by iteration, starting with an initial guess for \( u(t) \), propagating first in \( \vec{x} \), then \( \dot{\vec{x}} \), finally interpreting \( \partial E[H]/\partial u \) as the gradient associated with the constrained optimization problem. This provides a search direction for an updated test function \( u(t) \), which may be used, e.g., in a gradient search.

Here we do not use a simple gradient search, but the iterative algorithm of Krotov \([24-26]\), where the co-state dynamics has a slightly different role. The typical advantages of Krotov’s algorithm are decreased sensitivity to the initial guess, sizable improvements during the first few iterations, and monotonicity of the iteration without additional algorithmic elements such as line searches.

The entanglement measure we use is logarithmic negativity \([4, 27]\), which adequately quantifies entanglement for Gaussian states. In the present case, it can easily be computed from the covariance matrix \( \sigma \) associated with the observables \( q_{AB}, p_{AB}, q_{A} \) and \( p_{A} \). The logarithmic negativity can be obtained from the determinants of \( \sigma \) and its \( 2 \times 2 \) submatrices,
\[ \sigma = \begin{pmatrix} \alpha & \delta \\ \delta & \beta \end{pmatrix}, \]
\[ E_N = \max \left[ 0, -\ln \nu_- \right] \]
with
\[ \nu_- = \sqrt{2} \sqrt{\left| \alpha \right| + \left| \beta \right| - 2 \left| \delta \right| - \sqrt{\mu}} \]
and
\[ \mu = \left( \left| \alpha \right| + \left| \beta \right| - 2 \left| \delta \right| \right)^2 - 4 \left| \sigma \right|. \]

In the discussion of some of our results, we omit the maximum function and present plots of \(-\ln \nu_-\) instead of the logarithmic negativity. Negative values of \(-\ln \nu_-\) give some indication how far a separable state is from nearby entangled states.

\[ H_c(u) = \frac{\mu}{2} \left( q_{A}^2 + q_{B}^2 \right) = \frac{\mu}{2} \left( q_{A}^2 + q_{B}^2 \right) \]  
(22)

is clearly a local operator, therefore the external driving described by it does not generate entanglement. Without dissipation, the symmetric and antisymmetric modes (coordinates \( q_{+} \) and \( q_{-} \)) are squeezed equally. With a common dissipative reservoir, coupling only to \( q_{+} \), the symmetric and antisymmetric modes are no longer treated on an equal footing. Dissipation then curbs the build-up of squeezing in the symmetric mode (but not the antisymmetric mode). This effect, and the attendant generation of entanglement can be promoted by particular pulse shapes \( u(t) \).

Optimized pulse shapes \( u(t) \) have been determined by repeated Krotov iterations. They typically display a complex structure (see figure 1, dotted line). It is not a priori obvious whether this structure is incidental, maybe due to a near degeneracy of the optimization problem, or if it is necessary and cannot be substituted for by simple pulse shapes. In order to address this question, we compare the optimized solution with the effect of simple periodic pulse shapes (i) \( u(t) = \sin(\omega t) \), (ii) \( u(t) = \sin^2(\omega t) \), and (iii) \( u(t) = \sin(2\omega t) \).

In cases (i) and (iii), \( u(t) \) briefly touches the value \( -1 \), transforming the oscillatory degree of freedom into a free

\[ \text{Figure 1. Control signals to be compared: iteratively optimized pulse [11], dotted/red, and periodic signals as indicated in the inset. The control field } u(t) \text{ is scaled by its natural unit } m\omega^2. \]

3.2. Numerical results

In a recent publication \([11]\), we have demonstrated the feasibility of entanglement generation by local control and a shared weak dissipative reservoir. Neither of these factors alone can induce entanglement; an entangled state results only if dissipation is combined with a properly shaped pulse. We observe that entanglement generation coincides with a build-up of two-mode squeezing. In a perfectly symmetric setting, the two-mode squeezing can be visualized as simple squeezing of the symmetric and antisymmetric normal modes \(^6\). However, two-mode squeezing is not sufficient for entanglement generation. The Hamiltonian control term

\[ H_c(u) = \frac{\mu}{2} \left( q_{A}^2 + q_{B}^2 \right) = \frac{\mu}{2} \left( q_{A}^2 + q_{B}^2 \right) \]  
(22)

\[^6\] These modes are degenerate in the absence of a static coupling.
particle for an instant. Cases (ii) and (iii) are examples of driving at frequency $2\omega$, which amounts to resonant driving for the case of parametric control. The resulting dynamics is illustrated by figure 2, indicating the amount of entanglement achieved, and figure 3, indicating how much kinetic and potential energy is accumulated by driving the system. The reservoir temperature has been chosen as $T_k = \gamma \hbar \omega / k_B$; damping is moderate, $\gamma / \omega = 0.1$.

Out of the three simple pulse shapes examined, only case (iii) creates a final state which is entangled. Figure 3 shows that this simplicity comes at a price: parametric driving at the first harmonic tends to add huge amounts of energy to the modes, a side effect that should be undesirable in most applications.

Our data indicates that the numerical optimization provides substantial benefits over several simple versions of an ansatz form the pulse shape: optimizing entanglement rather than guessing pulse shapes yields benefits even for figures of merit which were not explicitly included as objectives of the computation.

4. Conclusions and outlook

Numerical techniques of optimal control are a useful tool in quantum information. We have introduced a simple model with features that do not directly promote entanglement (weak dissipation, local driving), but do not contradict it, either (non-locality from the shared environment). Even though this system tends to equilibrate towards a separable state, its dissipative features can be leveraged by optimized parametric driving which provides two-mode squeezing, which is further modified by dissipation to yield entanglement. Pulse shapes found by numerical optimization are superior to the simple ansatz of periodic functions at the natural frequency of the system or its double. One might ask whether the shared reservoir could have the same effect if it has a significant spatial extension. If so, this would allow the entanglement of spatially separated modes in situ. Ohmic reservoirs in the parameter range considered here are typical of solid-state quantum devices, suggesting an experimental realization with existing resources.

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