I. INTRODUCTION

There are three types of consistent statistics in greater than two space dimensions: para-Bose, para-Fermi and infinite statistics, with the former two statistics as a direct generalization of Bose and Fermi statistics. The algebras of Bose statistics, Fermi statistics and infinite statistics can be viewed as the special cases of the $q$-deformed commutation relation $a_q a_q^\dagger - q a_q^\dagger a_q = \delta_{ij}$ with $q = 1, -1$ and $0$. While Bose and Fermi statistics are familiar in the standard model of particle physics, infinite statistics has become increasingly attractive in recent years. Infinite statistics with $a_q a_q^\dagger = \delta_{kl}$ has a great many interesting properties. Though there is an obvious absence of locality in the theory of particles obeying infinite statistics, other important properties like cluster decomposition and the CPT theorem still hold which makes it capable to be a sensible field theory. The nonlocality of the theory of infinite statistics might be a virtue in the context of quantum gravity for that it provides a new way in searching new physics beyond local quantum field theory which is based on bosons and fermions. In recent years, infinite statistics has been applied on the discussions of black hole statistics and dark energy quanta. Attempts to attach the nonlocality of infinite statistics to that of quantum gravity are also made there.

Holographic principle suggests that the information contained within a region should not exceed the area of its boundary in Planck units. There have been many verifications that the degrees of freedom of bosonic and fermionic systems under the gravitational stability condition are limited by the entropy bound $A^{3/4}$ for local quantum field theory and the holographic entropy bound $A$. This entropy gap even has its cosmological counterpart, which brings about a huge numerical differences from $10^9$ to $10^{20}$ at the present era of the universe.

For infinite statistics, all representations of the particle permutation group can occur. There are no special restrictions on the symmetric or anti-symmetric properties of the particle states. Therefore, the dimension of the Hilbert space of the quantum fields obeying infinite statistics is much larger than these of bosonic and fermionic fields. What is the total number of degrees of freedom of the quantum fields obeying infinite statistics? Should it be limited by the holographic entropy bound $A$? We shall show the answer is affirmative.

The paper is organized as follows. In Sec. II we shall introduce the elementary ingredients of infinite statistics and propose a $\mathcal{N}$-coincident brane scenario for constructing the state space of the quantum fields obeying infinite statistics. In Sec. III all those physically permitted field configurations in the Hilbert space are counted out. We show that the system obeying infinite statistics should be bounded by the holographic entropy $A$. In Sec. IV we analyze the thermodynamical properties of the systems obeying infinite statistics and verify the results in Sec. III. We point out for a general system obeying infinite statistics with energy $E$, the entropy bound is the Bekenstein bound $E l$. In Sec. V we conclude with a discussion of the applications of infinite statistics on the issues of quantum gravity.

II. INFINITE STATISTICS

We begin with a compact review of the elementary ingredients of infinite statistics. Then we propose a $\mathcal{N}$-coincident brane scenario for constructing the state space of the system which manifests the nonlocality of the quantum states.

The basic commutation relation of infinite statistics is

$$a_k a_l^\dagger = \delta_{kl}. \quad (1)$$
Assume the existence of the vacuum state $|0\rangle >$ annihilated by all the annihilators: $a_k|0\rangle = 0$. The entire space can be constructed by creation operators acting on the vacuum state in sequence. A general $N$ particle state can be written as 

$$a_{j_N}^\dagger \cdots a_{j_1}^\dagger |0\rangle > .$$  

(2)

The number operator $n_i$ is taken as the form 

$$n_i = a_i^\dagger a_i + \sum_k a_k^\dagger a_i a_k + \cdots.$$  

(3)

From the basic commutation relation (11), one can check 

$$[n_i, a_j] = -\delta_{ij} a_j.$$  

(4)

The subtlety here is the existence of a recursion pattern: 

$$n_i = a_i^\dagger a_i + \sum_k a_k^\dagger n_i a_k.$$  

The operator $n_i$ acting on a state gives the correct occupation number at the energy level $\varepsilon_i$. The total number operator and energy operator are given by 

$$N = \sum_i n_i, \quad E = \sum_i \varepsilon_i n_i.$$  

(5)

Note that two states obtained by acting on the vacuum state with creation operators in different order are orthogonal to each other. Taking two $N$ particle states for example, there is 

$$<0|a_{i_1}^\dagger \cdots a_{i_N}^\dagger a_{j_N}^\dagger \cdots a_{j_1}^\dagger |0\rangle > = \delta_{i_1 j_1} \cdots \delta_{i_N j_N}.$$  

(6)

Since changing the order of the particles gives another state orthogonal to the original one, the particles obeying infinite statistics are virtually distinguishable.

In classical statistics, it is known that Boltzmann statistics is based on distinguishable particles. The problem is that the entropy derived from it is a non-extensive one and contradicts with the properties of conventional thermodynamical systems. This is called “Gibbs paradox” and can be overcome by introducing the Gibbs factor $\frac{1}{N!}$ to offset the extra degrees of freedoms of particle interchanging. Along with the progress of quantum mechanics, quantum phase space has replaced the classical phase space, while the Bose statistics and Fermi statistic about identical particles have taken the place of classical statistics. However, the systems like black holes have been found to have non-extensive entropy. Thus the original reason for discarding Boltzmann statistics is not that adequate. The distinguishability of the particles obeying infinite statistics implies the rediscovery of Boltzmann statistics. Hence, infinite statistics is also called “quantum Boltzmannian statistics”, where “quantum” means the phase space is quantized according to quantum mechanics. The corresponding particle fields are called “quantum Boltzmannian fields”.

It has been suggested that infinite statistics can be viewed as the statistical property of the systems of identical particle with an infinite number of internal degrees of freedom, and the identical particles are distinguishable by their internal states [2]. In this regard, we suggest to write the particle states as 

$$\cdots a_{k_3}^\dagger |0\rangle >_3 \bigotimes a_{k_2}^\dagger |0\rangle >_2 \bigotimes a_{k_1}^\dagger |0\rangle >_1.$$  

(7)

This means there are $N \rightarrow \infty$ sets of vacuum states which correspond to the internal degrees of freedom. Such an intuitive notion makes one understand infinite statistics easily. One can even regain the infinite statistics with such a picture as the starting point.

For a concrete system of particles obeying infinite statistics, we can take the region where the particles are confined to as a 3-brane. We need $N$-coincident 3-branes as the base space to accommodate all the particles, with each brane attached with a distinguishable vacuum state $|0\rangle >_i$. Then the particles distributed on them are distinguishable. We at first introduce the $N$-coincident brane scenario for convenience of state counting. In fact it may have a deep physical origin rather than being just a notion. To see this, we move to the coordinate representation, where the number operator in Eq. (5) can be rewritten as 

$$N = \int d\xi \varphi^\dagger (\xi) \varphi (\xi) + \int d\xi' d\xi \varphi^\dagger (\xi') \varphi (\xi) \varphi (\xi') + \cdots.$$  

(8)

The wave fields are defined by 

$$\varphi (\xi) = \sum_k a_k \varphi_k (\xi),$$  

(9)

$$\varphi^\dagger (\xi) = \sum_k a_k^\dagger \varphi^\dagger_k (\xi),$$  

(10)

which satisfy 

$$\varphi (\xi) \varphi^\dagger (\xi') = \delta (\xi - \xi').$$  

(11)

Accordingly, the particle number can be counted as 

$$N = \sum_i \int d\eta \delta (\eta - \xi_i)$$  

$$+ \sum_{i_1,i_2} \int d\eta_1 \eta_2 \delta (\eta_1 - \xi_{i_1}) \delta (\eta_2 - \xi_{i_2}) + \cdots.$$  

(12)

From Eq. (8) or Eq. (12), one see that the particles are actually distributed as brane excitations with various dimensions onto these coincident base branes. The integrals in Eq. (8) first run over the coordinate of one base brane to collect 0-brane excitations (particles) at $\xi_i$, then run over the coordinates of two base branes to collect 1-brane excitations (strings) with its endpoints attached to $\xi_i$ and $\xi_{i_2}$, and so on.

### III. Entropy Bound for Infinite Statistics

By imposing only the periodic boundary condition for quantizing momentum modes and the gravitational sta-
bility requirement, we have accomplished the state counting for bosonic and fermionic fields which leads to the $A^{3/4}$ bound in [14]. In this section, we shall examine the entropy bound for the quantum Boltzmannian fields obeying infinite statistics in the same way.

Consider massless fields which are confined to a region of size $l$. Imposing periodic boundary conditions, the particle’s momentum will be quantized as $\vec{k} = \vec{p}(m_x, m_y, m_z)$. The elementary energy unit $l^{-1}$ is the infrared energy cutoff of the system. After introducing an additional effective ultraviolet cutoff $\Lambda$, the total number of these quantized modes and the corresponding creation operator $a_{k}^\dagger$ is

$$N_\Lambda = \sum_{\vec{k}} 1 \sim l^3 \int_0^\Lambda w^2 dw \sim l^3 \Lambda^3,$$  \hspace{1cm} (13)

Due to Eq. (13), the quantized wave vector $\vec{k}$ can be one-to-one labeled by a character $i$ with $i \in [1, N_\Lambda]$. The corresponding energy of the modes are $w_i = |\vec{k}_i| = \sqrt{m_x^2 + m_y^2 + m_z^2}$.

When the massless fields obey Bose statistics, we can construct the Fock states by assigning occupation number $n_i$ to these $N_\Lambda$ different modes, which is

$$|\Psi\rangle = |n_1, n_2, \cdots, n_{N_\Lambda}\rangle >, n_i \in \mathbb{N}, i \in [1, N_\Lambda].$$  \hspace{1cm} (14)

Each different set of occupancy $\{n_i\}$ determines an independent basis of the bosonic Hilbert space of the system. For that $n_i$ can be taken as arbitrary integer numbers, this Hilbert space is infinite dimensional which accounts for the infinite degrees of freedom of quantum field theory. However, a realistic system always has finite degrees of freedom and entropy. The reason is that these systems should obey certain energy or particle number constraints. In order to find the maximum realizable entropy for a physical system, we impose a non-gravitational collapse requirement, that is, the quantum states with energy more than the mass of a black hole of the same size is unstable and thus should be excluded from the Hilbert space. The requirement is written as

$$E = \sum_{i=1}^{N_\Lambda} n_i w_i \leq E_{bh},$$  \hspace{1cm} (15)

where $E_{bh}$ is the energy of the black hole with Schwarzschild radius $r_s = l/2$. The number of solutions $\{n_i\}$ satisfying the requirement (15) gives the dimension of the physical Hilbert space $W \equiv \text{dim} \mathbb{H}$, which is extremely large but finite now.

The entropy associated with the system is $S = -\sum_{i=1}^{W} \rho_i \ln \rho_i$, where $\rho_i$ is the probability distribution in the Hilbert space. Obviously the maximum value of the expression can be realized by a uniform distribution $\rho_i = \frac{1}{W}$. The corresponding entropy is

$$S_{\text{max}} = -\sum_{i=1}^{W} \frac{1}{W} \ln \frac{1}{W} = \ln W.$$  \hspace{1cm} (16)

The result for the bosonic counting is $S = \ln W \leq (E_{bh} l^3 \Lambda^2)^{1/2}$. (See [14] for details.) By taking the effective cutoff to be determined by $l^3 \Lambda^4 \leq E_{bh}$, the final entropy bound is $S \leq (E_{bh} l)^{3/4} \sim A^{3/4}$ with $A \sim l^{-3/2}$ at the maximum. The self-consistency of the choice of $\Lambda$ can be easily checked [14] and it is exactly the UV-IR relation for local quantum field theory first suggested by Cohen et al. [12].

Now we concentrate on the case of infinite statistics. The state basis of the Hilbert space for the quantum Boltzmannian fields can be generally written in the form

$$(a_{k_{3}}^\dagger)^{n_{m}} |0 >_{m} \otimes (a_{k_{2}}^\dagger)^{n_{2}} |0 >_{2} \otimes (a_{k_{1}}^\dagger)^{n_{1}} |0 >_{1},$$  \hspace{1cm} (17)

where $m$ is an arbitrary positive integer. Compared with the notion of [1], the neighboring branes with the same creation operators acting on them are collected as a brane cluster. So we call such states as clustered brane states. Different $k_i$ and $n_i$ correspond to different choices of momentum excitation and occupation number on the $i$-th clustered brane.

We impose the non-gravitational collapse condition and are interested in the field configurations in (17) satisfying

$$\sum_{i=1}^{m} n_i w_i \leq E_{bh},$$  \hspace{1cm} (18)

with $j_1 \neq j_2, j_2 \neq j_3, \cdots$. All the field configurations satisfying this requirement are the physically accessible states and constitute the physical Hilbert space. So we have to count out the number of solutions $\{n_i\}$ of Eq. (18) to determine the dimension $W$ of the physical Hilbert space of quantum Boltzmannian fields. The difference of Eq. (15) with the bosonic counting (15) is that $w_i$ can be repeated in the summation if they are not neighboring. For example, $a_{k_{1}}^\dagger a_{k_{1}}^\dagger |0 >$ and $(a_{k_{1}}^\dagger)^2 a_{k_{1}} |0 >$ are the same state for the bosonic counting, but they are independent states for the quantum Boltzmannian counting. That is why there is no need to introduce the brane scenario for the bosonic fields. Mathematically speaking, Eq. (15) and Eq. (18) correspond respectively to the order independent and order dependent partition of $E_{bh}$ as various summations of $w_i$.

Just as that in [14], the question of counting solutions of Eq. (18) is equivalent to the counting of lattice points (points with integer coordinates) contained within the convex polytopes whose right-angle side lengths are respectively $l_{j_i} = \frac{E_{bh}}{w_{j_i}} \gg 1$. Thus the number of quantum states with $m$ clustered branes being occupied can be
estimated by the volume of the related polytopes $P^m$

\[ \text{Vol}(P^m) \sim \frac{1}{m!} \sum_{j_1 \neq j_2, j_3 \neq j_3, \ldots}^N l_{j_1} l_{j_2} \cdots l_{j_m} \leq \frac{1}{m!} z^m, \tag{19} \]

where $z$ is defined as

\[ z \equiv \sum_{i=1}^{N_A} l_i \sim l^3 \int_0^A \frac{E_{bh}}{w} \cdot w^2 dw \approx E_{bh} l^3 \Lambda^2. \tag{20} \]

Hence, the total number of physically permitted field configurations is

\[ W = \sum_{m=0}^{\infty} \text{Vol}(P^m) \leq \sum_{m=0}^{\infty} \frac{1}{m!} z^m = e^z. \tag{21} \]

The maximum entropy of the system obeying infinite statistics can be realized by a uniform probability distribution in the Hilbert space

\[ S_{\text{max}} = \ln W \leq z \sim E_{bh} l^3 \Lambda^2. \tag{22} \]

We should determine the self-consistent choice of $\Lambda$ for quantum Boltzmannian fields in the same spirit of that in the bosonic case [14]. First we notice in the summation of Eq. (21) the state density peaks at

\[ m_0 \sim z \sim E_{bh} l^3 \Lambda^2, \tag{23} \]

and when $m > m_0$ the state density drops dramatically to 0 due to $\frac{1}{m!} z^m \sim (\frac{E_{bh} l^3 \Lambda^2}{m})^m \sim 0$. Then consider the field configuration

\[ a_{k_3}^1 a_{k_2}^1 a_{k_1}^1 \cdots a_{k_3}^m a_{k_2}^m a_{k_1}^m |0>, \tag{24} \]

where $\vec{k}_1 = \frac{\pi}{l} (1, 0, 0)$, $\vec{k}_2 = \frac{\pi}{l} (1, 0, 0)$, $\vec{k}_3 = \frac{\pi}{l} (0, 0, 1)$. It is the state with the lowest energy when $m_0$ clustered branes have been excited. Now we require its energy $E$ is of the same order of $E_{bh}$ in order to physically insure no states with $m > m_0$ can satisfy $E < E_{bh}$ and thus no such states can contribute to the state counting. The requirement gives

\[ E = (E_{bh} l^3 \Lambda^2) l^{-1} \sim E_{bh}, \tag{25} \]

thus we obtain $\Lambda \sim l^{-1}$. Substitute it into Eq. (20) and Eq. (21), we get

\[ W \sim e^z = e^{E_{bh} l} = e^A \tag{26} \]

The maximum entropy is surely the holographic entropy bound

\[ S_{\text{max}} \sim E_{bh} l \sim A. \tag{27} \]

We have introduced an effective ultraviolet cutoff $\Lambda$ and determine its value based on the consistency of mathematics and physics. Here we consider the field configurations with $k > \Lambda$ modes being occupied and explain that including them will not change our result of entropy bound. We approximately take $\Lambda_m \sim \frac{E_{bh}}{m}$ as the highest reachable momentum when we are counting the states with $m$ clustered branes simultaneously occupied. In this case, for the infinite statistics counting, replacing $\Lambda$ by $\Lambda_m$ and doing the summation, we find

\[ W \sim \sum_{m=0}^{\infty} \frac{1}{m! m^{2m}} (E_{bh} l)^m \sim e^{E_{bh} l}. \tag{28} \]

Thus there is still $S_{\text{max}} \sim E_{bh} l \sim A$. Now we have included those quantum states with $k > \Lambda \sim l^{-1}$ modes being occupied, but the final result of entropy bound has not been modified. The reason is that the dominant contribution to the above counting of states comes from $m_0 \sim E_{bh} l$ where $\Lambda_{m_0} \sim \frac{E_{bh}}{m_0} \sim l^{-1}$. The state density drops dramatically to 0 when $m > m_0$. After averaging these highest reachable momenta $\Lambda_m$ according to the state density, we get the effective ultraviolet cutoff $\Lambda \sim l^{-1}$ which can be taken as a macroscopic parameter. The thing is similar in the bosonic state counting where the dominant contribution comes from $\Lambda_{m_0} \sim \Lambda \sim l^{-1/2}$ [14]. Recalling that the standard canonical distribution for a system is of the form $e^{E/T}$, temperature $T$ also serves as a kind of effective cutoff. In Sec [LV] we shall see $\Lambda$ will not deviate much from the temperature of the corresponding system.

It is worth to note that the reason we can introduce a uniform effective ultraviolet cutoff $\Lambda$ to facilitate analysis is that we are considering a question of entropy bound. And the macroscopic state with the maximum entropy must be uniform in energy and momentum. Though one can have a non-uniform distribution and accumulate extremely high energy locally, the number of microscopic states consistent with this distribution must be far fewer than that of a uniform state. In other words, in principle one should count all the field configurations up to the modes with Planck energy $E_P$, however, the dominant contributions to the state counting which influence the behavior of entropy bound come from those states with the average occupying momentum $\Lambda \sim l^{-1}$. This is the meaning of effective cutoff.

We notice that the $e^A$ quantum states can be directly counted out. Since the effective ultraviolet cutoff in determining the maximum entropy is $\Lambda \sim l^{-1}$, we can directly consider a simplified physical system which contains only three lowest momentum modes $\vec{k}_1$, $\vec{k}_2$, $\vec{k}_3$. When the system has the critical energy $E = E_{bh}$, the particle number of the system is

\[ N = \frac{E_{bh}}{\pi l} \sim E_{bh} l \sim A. \tag{29} \]

Sine the three modes have the same energy $\pi l$, the most possible distribution of the particles on them is that each mode is occupied with $\frac{N}{3}$ particles. The number of possible microscopic states of the system is

\[ W = \frac{N!}{\left(\frac{N}{3}\right)! \left(\frac{N}{3}\right)! \left(\frac{N}{3}\right)!} \sim \frac{N^N}{(\frac{N}{3})^N} \sim 3^N \sim 3^A, \tag{30} \]
Thus the maximum entropy for a bosonic system with energy $E_{bh}$ and $\Lambda \sim T \sim l^{-1}$ can surely reach the holographic entropy. The $e^A$ dominant field configurations can be written as $a_1^1 a_2^1 a_1^1 \cdots |0>$, $a_3^1 a_4^1 a_3^1 \cdots |0>$ and so on.

In contrast, consider a bosonic system with the effective ultraviolet cutoff $\Lambda \sim l^{-1}$. There are still $N \sim A$ particles for the system with energy $E_{bh}$. But according to the indistinguishability of the particles, the number of independent distributions of these particles on the three modes $k_1$, $k_2$, $k_3$ is

$$W = C_2^{N+2} = \frac{(N+2)(N+1)}{2} \sim N^2,$$

Thus the maximum entropy for a bosonic system with cutoff $\Lambda \sim T \sim l^{-1}$ is

$$S = \ln W \sim \ln N \sim \ln A.$$  

It is far below the holographic bound of quantum Boltzmannian systems, and also far below the $A^{3/4}$ entropy bound for bosonic systems.

**IV. THERMODYNAMICAL ANALYSIS**

In this section, we determine the thermodynamical entropy bound for the system obeying infinite statistics by using the canonical ensemble method. The canonical ensemble method readily leads to a thermodynamical description of a system, thus the corresponding thermodynamical entropy can be easily obtained.

The entropy bound $A^{3/4}$ for local quantum field theory describing bosons and fermions has been recognized for many years [4, 12, 13, 14, 15]. It is first given by ’t Hooft by directly considering a thermal photon gas confined to a box of size $l$, which has $S \sim l^3 T^3$ and $E \sim l^3 T^4$. Together with the limitation $E \leq E_{bh} \sim l$ from general relativity, the entropy bound $S \sim (E l)^{3/4} \leq A^{3/4}$ can be easily obtained with $T \sim l^{-1/2}$ at the maximum.

As an supplement, to obtain the entropy bound $A^{3/4}$, we start from Boltzmann statistics with the Gibbs factor $1/N!$ for identical particles. This statistics can be viewed as the high temperature limit of Bose and Fermi statistics. For simplicity, we only consider massless particles and it is easy to show that the systems of particles with mass have less entropy than these composed of massless particles. The canonical partition function of a perfect gas of $N$ particles obeying this statistics is

$$Z_N = \frac{1}{N!} \left( \sum_i e^{-\beta \epsilon_i} \right)^N = \frac{1}{N!} \left( l^3 \int e^{-\beta w} w^2 dw \right)^N \sim \frac{1}{N!} \left( l^3 T^3 \right)^N,$$

where $T \equiv \beta^{-1}$. The free energy is thus $F = -T \ln Z_N \sim -NT \ln \left( \frac{l^3 T^3}{N} \right)$. The complete form of it is $-NT \left( \ln \left( \frac{l^3 T^3}{N} \right) + 1 \right)$, but we shall omit all those irrelevant coefficients to make the scaling behaviors clear. Thus we find the energy and entropy of the system

$$E = -\left( \frac{\partial \ln Z}{\partial \beta} \right)_{V,N} \sim NT,$$

$$S = -\left( \frac{\partial F}{\partial T} \right)_{V,N} \sim N \ln \left( \frac{l^3 T^3}{N} \right).$$

From Eq. (35) and Eq. (36), one can easily get

$$S \sim N \ln \left( \frac{l^3 T^3}{N} \right) \lesssim (E l)^{3/4},$$

with $N \sim (E l)^{3/4}$, $T \sim (E / l^3)^{1/4}$ at the maximum. When the system is on the verge of collapsing to form a black hole, we have $T \sim l^{-1/2}$ and

$$S_{\text{max}} \sim (E_{bh} l)^{3/4} \sim A^{3/4}$$

Whatever, one should notice Boltzmann statistics with the Gibbs factor $1/N!$ is not a realistic statistics, one still have to refer to Bose or Fermi statistics for a general description of identical particles especially at low temperature. Or else one may encounter with an embarrassing negative entropy. (The negative entropy is originated from terms like $e^{-\beta \epsilon_i}$. The negative entropy can be surely reach the holographic entropy bound.

For a system of $N$ noninteracting particles obeying infinite statistics, the procedure is almost the same. The Gibbs factor $\frac{1}{N!}$ in Eq. (34) must be absent for the distinguishable particles here. Now the partition function is written as

$$Z_N = (l^3 T^3)^N,$$

which leads to

$$S \sim N \ln \left( l^3 T^3 \right), E \sim NT.$$  

Thus we have

$$S \sim N \ln \left( \frac{l^3 E^3}{N^3} \right) \leq EL,$$  

with $N \sim EL$, $T \sim l^{-1}$ at the maximum. Such an entropy bound is exactly in the same form of the famous “Bekenstein entropy bound” [16, 17]. The Bekenstein bound is surely a general entropy bound in this perspective, since there are only three types of consistent statistics in greater than two space dimensions and all of them are limited by the Bekenstein entropy bound.

When taking $E \sim E_{bh}$ we have the holographic entropy bound $S_{\text{max}} \sim E_{bh} l \sim A$. This suggests the quanta
of black holes may really conform to infinite statistics. However, the system obeying infinite statistics is not necessary to be an extreme case like black hole. One can account for a system with arbitrary temperature \( T \gg t^{-1} \) and compute its entropy by \( S \sim \frac{4}{3} T^3 \) \( \propto A \). In this way, the theory of infinite statistics covers all the degrees of freedom from 0 to \( A \) as required by holographic principle.

As did in [3, 6, 7], for a system composed of particles with mass \( m \), by starting from the partition function \( Z_N = (\frac{1}{m^3} \frac{mT}{}^{3/2})^N \) for infinite statistics and setting \( m \sim l^{-1} \), one can still obtain an area entropy \( S \sim A \). The motivations of the works [5, 6, 7] are to find the counting of all the microscopic states on a constant energy sphere. By a concrete calculation, in Sec.III, we emphasized the non-gravitational collapse to that in Sec.III, with different physical interpretation. In Sec.III, we highlighted the non-gravitational collapse of such an unbounded entropy is that the above partition function is a non-relativistic one, which has omitted the contribution from the rest mass. The complete relativistic partition function for particles with mass is \( Z_N \sim (\frac{1}{m^3} \frac{mT}{}^{3/2} e^{-m/T})^N \), one can check that roughly there is \( S_{max} \sim \frac{m}{1+3mT} (El) < El \). We point out it is crucial to introduce the static mass term \( e^{-m/T} \) in the partition function to get an entropy bound.

Above all, the thermodynamical analysis based on canonical ensemble tells that the thermodynamical entropy bound for infinite statistics is the Bekenstein entropy bound \( S_{max} \sim El \). Considering the limitation from the black hole core, the final entropy bound is the holographic bound \( S_{max} \sim E_{bh} l \sim A \). We notice that the microscopic state counting method in Sec.III can also be applied to obtain the entropy bound \( El \) for the system obeying infinite statistics with energy \( E \). According to the microcanonical ensemble idea, we should count all the possible microscopic states that are consistent with the macroscopic parameters of the system. This involves the counting of all the microscopic states on a constant energy sphere with energy \( E \). It is very difficult, so general treatments are to count the microscopic states with energy \( E_{state} \), where \( \sum \eta_i w_i \leq E \). If the state density peaks near the given energy \( E \), we can use it as the state density on the constant energy sphere. By a concrete calculation, we obtain the number of quantum Boltzmannian field configurations with energy \( E \) is \( W \leq e^{El} \), then the statistical entropy is given by the Boltzmann entropy formula: \( S = lnW \leq El \). Thus the system is surely limited by the Bekenstein entropy bound. The calculation is similar to that in Sec.III, with different physical interpretation. In Sec.III, we emphasized the non-gravitational collapse condition in order to count all the physically permitted field configurations.

Comparing the microscopic state counting with the thermodynamical analysis of both the cases of Bose statistics and infinite statistics, one may have found the effective cutoff \( \Lambda \) is always at the same order of the temperature \( T \) of the corresponding systems. Besides, we notice that when taking \( \Lambda \sim T \), generally there is a coincidence between the thermodynamical and statistical entropy. For systems with fixed energy \( E \) and particle number \( N \), we can approximately count the microscopic states as \( W \sim \frac{1}{N!} z^N \) for bosonic systems and \( W \sim \frac{1}{N^N} z^N \) for infinite statistics systems, where \( z \) is defined as Eq.(20). Employing \( E \sim NT \sim N \Lambda \), we find the entropy formula for bosonic fields

\[
S = lnW \sim N \ln \left( \frac{E^3 A^2}{N^2} \right) \sim N \ln \left( \frac{1}{N^3} \right),
\]

and the entropy formula for the quantum Boltzmannian fields obeying infinite statistics

\[
S = lnW \sim N \ln \left( \frac{E^3 \Lambda^2}{N} \right) \sim N \ln \left( \frac{1}{N^3} \right).
\]

They coincide with the thermodynamical results Eq.(37) and Eq.(40) very well.

V. DISCUSSIONS AND CONCLUSIONS

The theory of infinite statistics has intriguing properties, for example, possessing nonlocality and non-extensive entropy (without the Gibbs factor \( \frac{1}{N!} \)), which resemble these of gravitational systems. Now we have proved that infinite statistics really has a well-behaved holographic property, that is, the maximum entropy of the system obeying infinite statistics is equal to its boundary area in Planck units. This strongly favors infinite statistics as an important ingredient of quantum gravity.

Strominger [3] has suggested that the gas of external black holes should obey infinite statistics. Ng [6, 7] also has suggested that the compositions of dark energy/matter may obey infinite statistics. The underlying motivation of them is that all the three self-consistent statistics should show up in the nature. If the compositions of dark energy or dark matter are not the conventional matter obeying Bose and Fermi statistics, they may obey the only other self-consistent statistics. In addition, infinite statistics also has applications in the context of ADS/CFT correspondence [23]. It plays an important part in \( SU(\infty) \) gauge theories [4, 22, 23]. For example, the large \( N \) limit of \( SU(N) \) matrix theory can be effectively described by the master fields obeying infinite statistics [22, 24, 25, 26]. By intuition, the identical particles obeying Bose or Fermi statistics now gain infinite internal degrees of freedom through \( N \sim \infty \) coincident branes or say the \( SU(\infty) \) gauge group, which leads naturally to an infinite statistics description.
stated in [24], “large $N$ fermions and bosons are surprisingly similar, exhibiting some aspects of a Bose-Fermi equivalence”. The holographic property of infinite statistics implies that the large $N$ limit of matrix theory can be holographic in degrees of freedom, thus the theory is very different from the conventional local quantum field theory. Whatever, the exact role of infinite statistics in capturing the bulk physics in the context of ADS/CFT correspondence is seldom mentioned in the literature and is worthy of intense study.

When symmetrizing and anti-symmetrizing the state space of infinite statistics, one can obtain its bosonic and fermionic subspaces. Since the fundamental degrees of freedom of infinite statistics are brane excitations, there should be a condensation mechanism from brane states to bosons and fermions, which leads us to the conventional quantum field theory. We notice there has been an attempt in this direction [27], which suggested infinite statistics is related to the new physics in high energy scale and discussed the hierarchy problem from the electroweak scale to the Planck scale. As an inverse problem, one can also consider the process of a bulk of bosons or fermions accumulating to form a black hole, along with the evolvement of the entropy from $A^{3/4}$ to $A$. High entropic objects are seldom referred to in the literature except the extreme cases like black holes. Actually they are not realizable by the conventional local quantum field theory [21]. Now we suggest infinite statistics as the new physics to account for these highly entropic systems. But the question how could gravity be emergent in this framework is still far from clear, maybe gauge/gravity duality provides a better framework to address this question.

In conclusion, we have proved the the entropy bound for infinite statistics is $A$, while the entropy bound for local quantum filed theory describing bosons and fermions is $A^{3/4}$, by a careful examination on both the thermodynamical (canonical) entropy and statistical (microcanonical) entropy of the corresponding systems. Actually the two types of entropy well agree with each other. Our results indicate that infinite statistics can naturally fit into holographic principle and can shed light on the understanding of the gap between the $A^{3/4}$ entropy bound for local quantum field theory and the holographic entropy $A$, the corresponding degrees of freedom of which are very obscure before. Since one may expect this entropy gap is related to quantum gravitational degrees of freedom, it suggests a close relationship between infinite statistics and quantum gravity. However, at present there are only a limited number of conclusions on the relation between infinite statistics and quantum gravity. Thus far from systematized. Many open questions should be further clarified.

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