Generalised Virasoro Constructions from
Affine Inönü-Wigner Contractions

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ABSTRACT

We present a new method to find solutions of the Virasoro master equations for any affine Lie algebra \( \hat{g} \). The basic idea is to consider first the simplified case of an Inönü-Wigner contraction \( \hat{g}_c \) of \( \hat{g} \) and to extend the Virasoro constructions of \( \hat{g}_c \) to \( \hat{g} \) by a perturbative expansion in the contraction parameter. The method is then applied to the orthogonal algebras, leading to fixed-level multi-parameter Virasoro constructions, which are the generalisations of the one-parameter Virasoro construction of \( \hat{su}(2) \) at level four.

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1. Introduction

An important ingredient in the study of two-dimensional conformal field theories is the realisation of their chiral symmetry algebras in terms of simpler structures such as free fields or affine currents. The interest in realisations of the two-dimensional conformal symmetry algebra, the Virasoro algebra, on affine Lie algebras has increased considerably since it has been noted that the Sugawara and coset constructions are but specific examples of far more general “Virasoro constructions.” The most general realisations of the Virasoro algebra as a quadratic combination

\[ T(z) = L_{ab}(J^a J^b)(z), \tag{1} \]

with \( L_{ab} = L_{ba} \), of affine currents \( J^a(z) \), generating the affine Lie algebra \( \hat{g} \)

\[ J^a(z) J^b(w) = g^{ab}(z-w)^2 + \frac{i f_{abc} J^c(w)}{z-w} + \text{regular}, \tag{2} \]

leads to a set of coupled algebraic equations [1,2], the so-called Virasoro master equations for \( \hat{g} \),

\[ L_{ab} = 2 \ell L_{ac} g^{cd} L_{db} - f^{kl}_{a} f^{cd}_{b} L_{kc} L_{ld} - f^{ld}_{c} f^{kc}_{(a} L_{b) d} L_{kl}, \tag{3} \]

where the brackets denote symmetrisation, \( X_{(ab)} = X_{ab} + X_{ba} \). The central charge of the resulting Virasoro algebra is then given by

\[ c = 2 \ell g^{ab} L_{ab}. \tag{4} \]

An important property of the Virasoro master equations for (products of) simple algebras is that all its solutions come in “conjugate pairs”: if \( L_{ab} \) is a solution (with central charge \( c \)), then so is \( L_{Sug}^{ab} - L_{ab} \) (with central charge \( c_{Sug} - c \)), where \( L_{Sug}^{ab} \) denotes the Sugawara solution (and \( c_{Sug} \) the Sugawara central charge).

Even though only quadratic the Virasoro master equations (3) have so far resisted any attempt at a general solution. In fact the only simple algebra for which the complete solution is known is \( \hat{s}u(2) \) [2]. For all other algebras one has to recur to specific ansatze in order to reduce the number of nonlinear equations in such a way that the resulting simplified set can be solved. Such approaches have led to numerous new Virasoro constructions (see e.g. [1–7]). So far, the only method leading to a systematic treatment of these “simplified master equations” is a perturbative expansion, developed by Halpern and Obers, in the inverse level [4]. Solutions of the zeroth order equations, which reduce to constructions on (products of) \( \hat{u}(1) \) algebras, can be straightforwardly extended to higher order, often leading to new constructions. This “high-level expansion” has, in particular, been very succesful in the case of orthogonal algebras, leading to a surprising connection with graph theory [5]. The major drawback of this perturbative method, however, is that, by construction, solutions which exist only for fixed values of the level of the affine algebra cannot be obtained. The
canonical example of such a fixed-level Virasoro construction is the only $\hat{su}(2)$ solution that is not a Sugawara or coset construction. It occurs at level four, contains a free parameter and has central charge $c = 1$.

In this letter we present a new perturbative approach which gives solutions for generic as well as for specific values of the level. The starting point is to solve the master equations for an In"on"u-Wigner contraction of the original algebra and then extend these solutions to the original algebra by expanding in the contraction parameter.

In the next section, we shall illustrate the basic ideas on the $\hat{su}(2)$ example, recovering the one-parameter solution at level four. In the third section we discuss the general case and derive a necessary condition for the existence of fixed-level solutions. In the fourth section we show that this necessary condition is also sufficient for orthogonal algebras and find fixed-level multi-parameter Virasoro constructions. More specifically, we find for any $\hat{so}(n)$ algebra $n - 2$ solutions with integer central charges $c = k$, $k = 1, 2, \ldots, n - 2$, containing $(k - 1)(n - k)$ arbitrary (complex) parameters. Finally, in the last section we present some conclusions.

2. An illustrative example

Consider as an example the case $\hat{g} = \hat{su}(2) \cong \hat{so}(3)$ generated by $J^a(z), a = 1, 2, 3$. Take $g^{ab} = \frac{1}{2} \delta^{ab}$ and $f^{abc} = \epsilon_{abc}$ so that $\ell$ is the level of the affine algebra. In this case, the most general Virasoro constructions are given by the diagonal ones: $L_{ab} = \lambda_a \delta_{ab}$ [1,2]. Even though it is a simple exercise to solve the corresponding master equations completely, we shall only do so indirectly.

Perform the transformation

$$J^{1,2}(z) \rightarrow \sqrt{\epsilon} J^{1,2}(z),$$
$$J^3(z) \rightarrow J^3(z).$$

The limit $\epsilon \rightarrow 0$ is an In"on"u-Wigner contraction [8] and leads to the affine two-dimensional Euclidean algebra $\hat{so}(2) \otimes \hat{T}_2 \equiv \hat{e}(2)$. The master equations are after this rescaling

$$\lambda_1 = \epsilon \ell \lambda_1^2 + 2 \epsilon \lambda_1 \lambda_2 + 2 \lambda_1 \lambda_3 - 2 \lambda_2 \lambda_3,$$
$$\lambda_2 = \epsilon \ell \lambda_2^2 + 2 \epsilon \lambda_1 \lambda_2 + 2 \lambda_2 \lambda_3 - 2 \lambda_1 \lambda_3,$$
$$\lambda_3 = \ell \lambda_3^2 + 2 \epsilon \lambda_1 \lambda_3 + 2 \epsilon \lambda_2 \lambda_3 - 2 \epsilon^2 \lambda_1 \lambda_2.$$  

(6)

Expanding

$$\lambda_a = \sum_{s=0}^{\infty} \epsilon^s \lambda_a^{(s)},$$

(7)
leads, at order zero, to the master equations for $\tilde{e}(2)$:

$$
\lambda_1^{(0)} = 2\lambda_3^{(0)}(\lambda_1^{(0)} - \lambda_2^{(0)}),
$$

$$
\lambda_2^{(0)} = 2\lambda_3^{(0)}(\lambda_2^{(0)} - \lambda_1^{(0)}),
$$

$$
\lambda_3^{(0)} = \ell \left( \lambda_3^{(0)} \right)^2.
$$

These equations are clearly easier to solve than the corresponding ones for $\tilde{su}(2)$. The solution $\lambda_3^{(0)} = 0$ from (10) leads to the trivial solution $\lambda_1^{(0)} = \lambda_2^{(0)} = 0$. The other solution of (10), $\lambda_3^{(0)} = 1/\ell$, implies in general $\lambda_1^{(0)} = \lambda_2^{(0)} = 0$, except when $\ell = 4$, when $\lambda_1^{(0)} = -\lambda_2^{(0)} = \alpha_0$ is a solution for an arbitrary parameter $\alpha_0$.

The higher-order equations in $\epsilon$ are now all linear and hence straightforward to solve. The trivial solution at zeroth order remains trivial at all orders, while the $\lambda_1^{(0)} = \lambda_2^{(0)} = 0$, $\lambda_3^{(0)} = 1/\ell$ solution for generic $\ell$ gives $\lambda_i^{(n \geq 1)} = 0$ and thus the construction corresponding to the energy-momentum tensor of the $\tilde{u}(1)$ subalgebra generated by $J^3(z)$. Finally, the solution at level four has an expansion

$$
\lambda_1 = \alpha_0 + \alpha_1 \epsilon + \alpha_2 \epsilon^2 + \alpha_3 \epsilon^3 + \alpha_4 \epsilon^4 + o(\epsilon^5),
$$

$$
\lambda_2 = -\alpha_0 + (4\alpha_0^2 - \alpha_1) \epsilon + (-16\alpha_0^3 + 8\alpha_0\alpha_1 - \alpha_2) \epsilon^2 
+ (128\alpha_0^4 - 48\alpha_0^2\alpha_1 + 4\alpha_2^2 + 8\alpha_0\alpha_2 - \alpha_3) \epsilon^3 
+ (-1024\alpha_0^5 + 512\alpha_0^3\alpha_1 - 48\alpha_0\alpha_1^2 - 48\alpha_0^2\alpha_2 + 8\alpha_1\alpha_2 + 8\alpha_0\alpha_3 - \alpha_4) \epsilon^4 + o(\epsilon^5),
$$

$$
\lambda_3 = \frac{1}{4} - 4\alpha_0^2 \epsilon^2 + (16\alpha_0^3 - 8\alpha_0\alpha_1) \epsilon^3 
+ (-128\alpha_0^4 + 48\alpha_0^2\alpha_1 - 4\alpha_2^2 - 8\alpha_0\alpha_2) \epsilon^4 + o(\epsilon^5).
$$

By redefining

$$
\alpha_0 = \tilde{\alpha}_0 - \alpha_1 \epsilon - \alpha_2 \epsilon^2 - \alpha_3 \epsilon^3 - \alpha_4 \epsilon^4 + o(\epsilon^5)
$$

this can be rewritten as

$$
\lambda_1 = \tilde{\alpha}_0,
$$

$$
\lambda_2 = -\tilde{\alpha}_0 + 4\tilde{\alpha}_0^2 \epsilon - 16\tilde{\alpha}_0^3 \epsilon^2 + 128\tilde{\alpha}_0^4 \epsilon^3 - 1024\tilde{\alpha}_0^5 \epsilon^4 + o(\epsilon^5),
$$

$$
\lambda_3 = \frac{1}{4} - 4\tilde{\alpha}_0^2 \epsilon^2 + 16\tilde{\alpha}_0^3 \epsilon^3 - 128\tilde{\alpha}_0^4 \epsilon^4 + o(\epsilon^5),
$$

which explicitly shows that there is only one relevant parameter.

This solution satisfies

$$
\epsilon(\lambda_1 + \lambda_2) + \lambda_3 = \frac{1}{4} + o(\epsilon^5),
$$

$$
\epsilon^2(\lambda_1^2 + \lambda_2^2) + \lambda_3^2 = \frac{1}{16} + o(\epsilon^5)
$$

(14)
and has central charge $c = 1 + o(\epsilon^5)$. It is precisely the one-parameter $su(2)$ Virasoro construction at level four.

3. The general case

Consider an affine Lie algebra $\hat{g}$ with generators $J^\alpha(z)$. Split the generators of $\hat{g}$ into two sets as follows

$$\{J^\alpha(z)\} = \{J^\alpha(z)\} \cup \{J^A(z)\}$$

and perform the scale transformation

$$J^\alpha(z) \rightarrow J^\alpha(z),$$
$$J^A(z) \rightarrow \epsilon J^A(z).$$

The structure constants then rescale as

$$f^{AB}_C \rightarrow \epsilon f^{AB}_C,$$
$$f^{AB}_\gamma \rightarrow \epsilon^2 f^{AB}_\gamma,$$
$$f^{A\beta}_C \rightarrow f^{A\beta}_C,$$
$$f^{A\beta}_\gamma \rightarrow \epsilon f^{A\beta}_\gamma,$$
$$f^{\alpha\beta}_C \rightarrow \epsilon^{-1} f^{\alpha\beta}_C,$$
$$f^{\alpha\beta}_\gamma \rightarrow f^{\alpha\beta}_\gamma,$$

and the Killing form as

$$g^{AB} \rightarrow \epsilon^2 g^{AB},$$
$$g^{A\alpha} \rightarrow \epsilon g^{A\alpha},$$
$$g^{\alpha\beta} \rightarrow g^{\alpha\beta}.$$

Existence of the limit $\epsilon \rightarrow 0$ requires $f^{\alpha\beta}_C = 0$, or thus the currents $J^\alpha(z)$ to form a subalgebra $\hat{h}$. This limit then corresponds to an Inönü-Wigner contraction \cite{8} $\hat{g}_c$ of $\hat{g}$.

We shall expand $L_{ab}$ and $c$ in powers of $\epsilon$

$$L_{ab} = \sum_{s=0}^{\infty} \epsilon^s L^{(s)}_{ab},$$
$$c = \sum_{s=0}^{\infty} \epsilon^s c^{(s)}.$$

The zeroth order equations then reduce to the master equations for the contracted algebra $\hat{g}_c$. Even though still quadratic, they are easier to solve than the corresponding equations for $\hat{g}$, since at that order one can put $f^{AB}_C = f^{AB}_\gamma = f^{A\beta}_\gamma = 0$ and $g^{AB} = g^{A\beta} = 0$. The higher order equations now all become linear and are hence straightforward to solve order by order.

Instead of writing the general form of these equations down, we shall only consider the “natural” ansatz

$$L^{(0)}_{\alpha\beta} = 0,$$
which is sufficient for our purposes. The zeroth order master equations then become

\[
L_{\mu
u}^{(0)} = 2\ell L_{\mu\nu}^{(0)}g^{\gamma\delta}L_{\delta\nu}^{(0)} - f^{\alpha\beta}\mu_{\nu}L_{\alpha\gamma}^{(0)}L_{\beta\delta}^{(0)} + f^{\beta\delta}\gamma_{\nu}L_{\mu\gamma}^{(0)}L_{\alpha\beta}^{(0)},
\]

\[
L_{MN}^{(0)} = L_{\alpha\gamma}^{(0)}f^{B\alpha}_{\ (M}L_{\gamma\delta}^{(0)}f^{D\alpha}_{\ N)}L_{\beta\delta}^{(0)}L_{\alpha\beta}^{(0)}.
\]

The equations (21) are precisely the master equations for the subalgebra \( \hat{h} \). Once a solution for \( \hat{h} \) has been chosen, (22) become homogeneous linear equations in the remaining variables \( L_{MN}^{(0)} \). A nontrivial solution exists only if the determinant of the coefficient matrix is zero.

Let us work this out in more detail for a specific example. Suppose that \( g \) is simple and choose a specific solution \( L_{\alpha\beta}^{(0)} \) for (21). Assume the diagonal ansatz

\[
L_{AB} = g_{AB}\lambda_B \quad \text{(no sum)}.
\]

After multiplication by \( g_{NM} \) and summation over \( N \), (22) can be rewritten as \( \mathcal{M}_M^P\lambda_P = 0 \) where we have defined the matrix \( \mathcal{M} \) as

\[
\mathcal{M}_M^P = \sum_N g_{MN}g_{NM}\delta_M^P + 2\sum_{N,B,\alpha,\gamma}L_{\alpha\gamma}^{(0)}g_{NM}f^{AB}_{\ (M}f^{\gamma\delta}_{\ B}g_{NP} + f^{\gamma\delta}_{\ N}g_{PB}).
\]

(For convenience, we have written out all the summation indices explicitly.) A necessary condition for a solution is hence

\[
\det(\mathcal{M}) = 0.
\]

Note that for symmetric spaces \( \sum_N g_{AN}g_{NB} = \delta^B_A \). In the next section we shall consider specific ansatze for orthogonal algebras where this condition will turn out to be sufficient as well.

4. Orthogonal algebras

Consider the orthogonal affine Lie algebra \( \hat{g} = \hat{so}(n) \). Choose as a basis the currents \( J^{a\bar{a}}(z) \) with vector indices \( 1 \leq a < \bar{a} \leq n \) and operator product expansions

\[
J^{a\bar{a}}(z)J^{b\bar{b}}(w) = \frac{\ell g^{a\bar{a},b\bar{b}}}{(z-w)^2} + \frac{if^{a\bar{a},b\bar{b}}_{\ c\bar{c}}J^{c\bar{c}}}{z-w} + \text{regular},
\]

where the structure constants are

\[
f^{a\bar{a},b\bar{b}}_{\ c\bar{c}} = \delta^{ab}\delta^{\bar{a}\bar{b}}_{[c\bar{c}]} + \delta^{ab}\delta^{\bar{a}\bar{b}}_{[c\bar{c}]} - \delta^{a\bar{a}}\delta^{\bar{b}b}_{[c\bar{c}]} - \delta^{a\bar{a}}\delta^{\bar{b}b}_{[c\bar{c}]} \quad \text{(27)}
\]
and the Killing form is
\[ g^{\bar{a},b} = \kappa_n \delta^{[b}[\delta]\bar{a}], \quad (28) \]
the square brackets denoting antisymmetrisation, \( X_{[ab]} = X_{ab} - X_{ba} \). Taking the normalisation \( \kappa_3 = 1/2 \) and \( \kappa_{n \geq 4} = 1 \), fixes \( \ell \) as the level of the \( \hat{\mathfrak{so}}(n) \) algebra. The quadratic combination
\[ T(z) = L_{a\bar{a},bb}(J^a\bar{a}J^{b\bar{b}})(z), \quad L_{a\bar{a},bb} = L_{bb,a\bar{a}}, \quad (29) \]
with the diagonal ansatz
\[ L_{a\bar{a},bb} = L_{a\bar{a}}\delta_{[a}[\delta]b]\bar{a}, \quad (30) \]
generates a Virasoro algebra provided [5]
\[ L_{ab} = 2\kappa_n \ell L^2_{ab} + 2 \sum_{c \neq a,b} [L_{ab}(L_{ac} + L_{bc}) - L_{ac}L_{bc}]. \quad (31) \]
The central charge is then given by
\[ c = 2\ell \kappa_n \sum_{a < b} L_{ab}, \quad (32) \]
Consider now a subalgebra \( \hat{h} = \hat{\mathfrak{so}}(k) \) of \( \hat{\mathfrak{so}}(n) \) generated by the currents \( J^{\alpha\bar{a}}(z) \) with \( 1 \leq \alpha < \bar{\alpha} \leq k < n \). Rescale the currents as
\[ J^{\alpha\bar{a}}(z) \rightarrow J^{\alpha\bar{a}}(z), \quad J^{\alpha B}(z) \rightarrow \sqrt{\epsilon} J^{\alpha B}(z), \quad J^{C\bar{C}}(z) \rightarrow \sqrt{\epsilon} J^{C\bar{C}}(z), \quad (33) \]
with \( k+1 \leq B \leq n \) and \( k+1 \leq C < \bar{C} \leq n \). Since the currents \( J^{\alpha\bar{a}}(z) \) generate a subalgebra, the limit \( \epsilon \rightarrow 0 \) exists. When \( k = n - 1 \) this yields \( \hat{\mathfrak{e}}(n - 1) \), the affine \( (n - 1) \)-dimensional Euclidean algebra. As usual we expand
\[ L_{ab} = \sum_{s=0}^{\infty} \epsilon^s L^{(s)}_{ab}. \quad (34) \]
As explained in the previous section, we can choose for \( L^{(0)}_{\alpha\beta} \) an arbitrary solution of the Virasoro master equations for \( \hat{\mathfrak{so}}(k) \). In this letter we shall consider the Sugawara construction:
\[ L^{(0)}_{\alpha\beta} = \frac{1}{2\kappa_n(\ell + k - 2)}, \quad \text{for all } 1 \leq \alpha < \beta \leq k. \quad (35) \]
Plugging (27), (28), (30) and (35) into (24) gives the following simple result for the matrix \( \mathcal{M} \):
\[ \mathcal{M}_{\alpha A}^{\beta B} = \left\{ \left[ 1 - \frac{k}{\kappa_n(\ell + k - 2)} \right] \delta_{\alpha}^{\beta} + \frac{1}{\kappa_n(\ell + k - 2)} \right\} \delta_{A}^{B}. \quad (36) \]
The determinant of this matrix can be easily computed:

\[
\det(M) = \left[1 - \frac{k}{\kappa_n(\ell + k - 2)}\right]^{(k-1)(n-k)}.
\]  

(37)

Hence we deduce that possible continuous Virasoro constructions might arise at fixed level

\[
\ell = 2 + \frac{1 - \kappa_n}{\kappa_n} k,
\]

(38)
or thus for \(\tilde{so}(3)\) at level 4 and for \(\tilde{so}(n \geq 4)\) at level 2. More precisely, (37) suggests that, given \(n\), there might be \(n - 2\) different constructions corresponding to a specific choice of a subalgebra \(\tilde{so}(k)\), \(k = 2, 3, \ldots, n - 1\), each one having \((k - 1)(n - k)\) arbitrary parameters.

A detailed analysis of the higher-order equations indicates that all the above constructions do in fact exist! Even though we have no proof of this statement in the general case, we have performed extensive checks (up to sixth order in the expansion parameter and up to and including \(\tilde{so}(10)\)) and are confident that all these constructions are valid to all orders in \(\epsilon\). Since these perturbative expansions are quite complicated and are not very enlightening by themselves, we shall refrain from giving them explicitly and shall only comment on some of their main characteristics.

For \(\tilde{so}(3)\) we recover exactly the construction from section 2 after the identification \(J^{23}(z) \equiv J^1(z)\), \(J^{13}(z) \equiv J^2(z)\) and \(J^{12}(z) \equiv J^3(z)\). The two construction for \(\tilde{so}(4)\) have already been conjectured to exist [6]. They both contain two arbitrary parameters as expected. Using the form of the perturbative expansion, we have been able to find the explicit nonperturbative form of both these constructions. This result is presented in the appendix. For \(\tilde{so}(n \geq 5)\) we do not have a closed-form expression for the constructions, but, as already mentioned, sufficient evidence from the perturbative expansion. Again for all these constructions the number of arbitrary parameters is precisely the one indicated in (37). All these parameters are nontrivial.

Thus the vanishing of the determinant (37) is a sufficient condition for the existence of fixed-level multi-parameter Virasoro constructions. Let us, for definiteness, denote by \(V(n, k)\) \((k = 2, 3, \ldots, n - 1)\) the construction for \(\tilde{g} = \tilde{so}(n)\) which arises from the \(\tilde{h} = \tilde{so}(k)\) subalgebra. \(V(n, k)\) thus contains \((k - 1)(n - k)\) arbitrary (complex) parameters. Moreover (up to the order of the perturbative expansion that was checked and exactly for the \(n = 3\) and \(n = 4\) cases) the central charge of \(V(n, k)\) is \(c_{V(n, k)} = k - 1\). At the precise level at which these Virasoro constructions occur, the Sugawara construction for \(\tilde{so}(n)\) has integer central charge \(c_n = n - 1\). Since

\[
c_{V(n, k)} + c_{V(n,n+1-k)} = n - 1 = c_n
\]

(39)
and moreover $\mathcal{V}(n, k)$ and $\mathcal{V}(n, n+1-k)$ contain the same number of parameters, it is natural to conjecture that they form conjugate pairs. This implies in particular that $\mathcal{V}(2p+1, p+1)$ is closed under conjugation."

## 5. Conclusions

In this letter we have presented a new perturbative method, based on Inönü-Wigner contractions of affine Lie algebras, to find solutions of the Virasoro master equations. The major advantage of our method is that, contrary to the high-level analysis of Halpern and Obers [4], our method seems ideally suited to investigate the existence of fixed-level continuous Virasoro constructions. We have illustrated this on orthogonal algebras, resulting in a class of fixed-level multi-parameter deformations of coset and Sugawara constructions. These constructions are the natural generalisations of the one-parameter $\hat{su}(2)_4$ construction.

While completing this letter we encountered a preprint, [10], where these fixed-level multi-parameter orthogonal constructions are derived as deformations of coset constructions for $\hat{so}(n)/\hat{so}(k)$. The agreement on overlapping results is perfect.

Even though we have focussed in this letter on fixed-level constructions it is clear that our method applies as well to Virasoro constructions existing for generic values of the level. The major drawback of our method is, however, that it does not lead to an exhaustive list of these solutions (see section 2), so that it might be better suited for the search of new fixed-level constructions. This search, together with other applications, based on e.g. double Inönü-Wigner contractions, is under investigation.

### Appendix

Here we present as an example the nonperturbative form of the two solutions for $\hat{g} = \hat{so}(4)_2$. The first one arises from $\hat{h} = \hat{so}(2)$ and contains 2 free (complex) parameters $\alpha$ and $\beta$. It is given by

\begin{align*}
L_{12} &= x_1, \\
L_{13} &= x_2, \\
L_{23} &= x_3, \\
L_{14} &= \frac{1}{4} - x_1 - x_2 - x_3 - \alpha - \beta, \\
L_{24} &= \beta, \\
L_{34} &= \alpha.
\end{align*}

(40)

Here

\begin{equation}
x_3 = \frac{-\alpha\beta[1 + 4(\alpha + \beta) - \Delta]}{8(\alpha^2 + \alpha\beta + \beta^2)},
\end{equation}

(41)

* Two-parameter constructions for $\hat{so}(2p+1)$, $p \geq 2$ with central charge $c = p$ have already been described in [5] (where they were called $SO(2p+1)_\frac{p}{2}[d,6]$); these constructions are closed under conjugation. It is conceivable that they form a special case of $\mathcal{V}(2p+1, p+1)$, which contains $p^2$ parameters.
with \( \Delta \) given by
\[
\Delta = \pm \sqrt{1 + 8(\alpha + \beta) - 48(\alpha^2 + \beta^2) - 32\alpha\beta}
\] (42)
and \( x_1 \) and \( x_2 \) by
\[
x_1 = \frac{A(\alpha, \beta) + B(\alpha, \beta)\Delta}{\mathcal{N}(\alpha, \beta)}, \quad x_2 = \frac{A(\beta, \alpha) + B(\beta, \alpha)\Delta}{\mathcal{N}(\beta, \alpha)},
\] (43)
with
\[
\begin{align*}
\mathcal{N}(\alpha, \beta) &= 8(\alpha^2 + \alpha\beta + \beta^2)[-2\beta(\alpha + \beta) + 4\beta(\alpha^2 + \beta^2) - \alpha^2 + \beta(\alpha + \beta)\Delta], \\
A(\alpha, \beta) &= \beta\{(4\alpha - 1)[\alpha^3 + 2\alpha(16\alpha + 3)\beta^2] + 4\alpha^2(-1 + \alpha + 8\alpha^2)\beta \\
&\quad + (-3 - 12\alpha + 128\alpha^2)\beta^3 + 4(1 + 32\alpha)\beta^4 + 32\beta^5\}, \\
B(\alpha, \beta) &= \beta(\alpha + \beta)[\alpha^2 + 3\beta(\alpha + \beta) - 8\beta(\alpha^2 + \beta^2)].
\end{align*}
\] (44)

This Virasoro construction has central charge \( c = 1 \). For some special values of the parameters these construction reduce to Sugawara or coset constructions: for \( (\alpha, \beta) = (0,1/4) \) we recover the Sugawara construction for the \( \tilde{\mathfrak{so}}(2) \) subalgebra generated by \( J^{24}(z) \), \( (\alpha, \beta) = (0, -1/12) \) gives a \( \tilde{\mathfrak{so}}(3)/\tilde{\mathfrak{so}}(2) \) coset construction, \( (\alpha, \beta) = (1/8, -1/8) \) an \( \tilde{\mathfrak{so}}(4)/(\tilde{\mathfrak{so}}(2) \times \tilde{\mathfrak{so}}(2)) \) coset construction, etc.

The second construction for \( \tilde{\mathfrak{so}}(4)_2 \), arising from \( \tilde{h} = \tilde{\mathfrak{so}}(3) \), also contains two free (complex) parameters and has central charge \( c = 2 \). Explicitly
\[
\begin{align*}
L_{12} &= \tilde{x}_1, \quad L_{13} = \tilde{x}_2, \quad L_{23} = \tilde{x}_3, \\
L_{14} &= \frac{1}{2} - \tilde{x}_1 - \tilde{x}_2 - \tilde{x}_3 - \tilde{\alpha} - \tilde{\beta}, \quad L_{24} = \tilde{\beta}, \quad L_{34} = \tilde{\alpha}.
\end{align*}
\] (45)

Here \( \tilde{\alpha} \) and \( \tilde{\beta} \) are the free (complex) parameters and
\[
\tilde{x}_3 = \frac{(8\tilde{\alpha} - 1)(8\tilde{\beta} - 1)\Delta - [4(\tilde{\alpha} + \tilde{\beta}) - 1][5 - 24(\tilde{\alpha} + \tilde{\beta}) + 64\tilde{\alpha}\tilde{\beta}]}{8[3 - 24(\tilde{\alpha} + \tilde{\beta}) + 64(\tilde{\alpha}^2 + \tilde{\alpha}\tilde{\beta} + \tilde{\beta}^2)]},
\] (46)
with \( \Delta \) given by (42), and \( \tilde{x}_1 \) and \( \tilde{x}_2 \) are
\[
\tilde{x}_1 = \frac{\tilde{A}(\tilde{\alpha}, \tilde{\beta}) + \tilde{B}(\tilde{\alpha}, \tilde{\beta})\Delta}{\tilde{\mathcal{N}}(\tilde{\alpha}, \tilde{\beta})}, \quad \tilde{x}_2 = \frac{\tilde{A}(\tilde{\beta}, \tilde{\alpha}) + \tilde{B}(\tilde{\beta}, \tilde{\alpha})\Delta}{\tilde{\mathcal{N}}(\tilde{\beta}, \tilde{\alpha})},
\] (47)
with
\[
\begin{align*}
\tilde{\mathcal{N}}(\tilde{\alpha}, \tilde{\beta}) &= 4[3 - 24(\tilde{\alpha} + \tilde{\beta}) + 64(\tilde{\alpha}^2 + \tilde{\alpha}\tilde{\beta} + \tilde{\beta}^2)]\{(8\tilde{\beta} - 1)[4(\tilde{\alpha} + \tilde{\beta}) - 1]\Delta \\
&\quad + 2[1 - 6\tilde{\alpha} - 8\tilde{\beta} + 8(1 + 8\tilde{\beta})(\tilde{\alpha}^2 + \tilde{\beta}^2) + 16\tilde{\alpha}\tilde{\beta}]}, \\
\tilde{A}(\tilde{\alpha}, \tilde{\beta}) &= (1 - 4\tilde{\alpha})[1 - 32\tilde{\alpha} + 224\tilde{a}^2 - 384\tilde{\alpha}^3 - 16(1 - 28\tilde{\alpha} + 160\tilde{a}^2 - 128\tilde{\alpha}^3)\tilde{\beta} \\
&\quad - 1024(3 - 16\tilde{\alpha} + 32\tilde{a}^2)\tilde{\beta}^3] + 16(17 - 28\tilde{\alpha} + 1792\tilde{a}^2 - 4096\tilde{\alpha}^3 + 2048\tilde{a}^4)\tilde{\beta}^2 \\
&\quad + 512(33 - 192\tilde{\alpha} + 256\tilde{a}^2)\tilde{\beta}^4 + 8192(16\tilde{\alpha} - 5)\tilde{\beta}^5 + 32768\tilde{\alpha}\tilde{\beta}^6, \\
\tilde{B}(\tilde{\alpha}, \tilde{\beta}) &= -(8\tilde{\beta} - 1)^2[4(\tilde{\alpha} + \tilde{\beta}) - 1][32(\tilde{\alpha}^2 + \tilde{\beta}^2) - 1].
\end{align*}
\] (48)
Special points of interest are \((\alpha, \beta) = (1/8, -1/8)\) giving an \(\mathfrak{so}(4)/\mathfrak{so}(2)\) coset construction, \((\alpha, \beta) = (1/8, 5/24)\) an \(\mathfrak{so}(4)/(\mathfrak{so}(3)/\mathfrak{so}(2))\) nested coset construction, \((\alpha, \beta) = (0, 1/4)\) an \(\mathfrak{so}(2) \times \mathfrak{so}(2)\) Sugawara construction, etc.

Conjugation with respect to \(\mathfrak{so}(4)\) transforms these solutions into one another.

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