GENERALIZATION OF THE HARTOGS-BOCHNER THEOREM FOR FORMS TO UNBOUNDED DOMAINS

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ABSTRACT. We give some generalizations of the $\mathcal{C}^\infty$ Hartogs-Bochner extension theorem for differential forms defined in unbounded domains of a complex analytic manifold in relation with the vanishing of some $\partial\bar{\partial}$ cohomologies groups.

Mots clés: CR differential forms, holomorphic extension, paracompactifying family of closed, $\partial\bar{\partial}$-cohomology.

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1. Introduction

Let $X$ be a complex manifold of complex dimension $n$, and $\Omega \subset X$ a domain with boundary $b\Omega$ of class $\mathcal{C}^\infty$. We assume that $\Omega$ is unbounded. If $f$ is a CR differential form of class $\mathcal{C}^\infty$ on $b\Omega$ or with coefficients in $L^p_{\text{loc}}(bD)$, does it exist a differential form $F$, $\partial\bar{\partial}$-closed on $\Omega$, of class $\mathcal{C}^\infty(\Omega)$ or with coefficients in $L^p_{\text{loc}}(X)$ such that $F|_{b\Omega} = f$ on $b\Omega$?

In the case where $\Omega$ is bounded, the answer was given by Christine in [2].

Our objective is to give not only an isomorphism result between $H_{\Phi}^{l,r}(X)$ and $H_{\Phi,\text{cur}}^{l,r}(X)$ but to give a geometric characterization on $\Omega$ to obtain the Hartogs-Bochner extension theorem for CR differential forms of class $\mathcal{C}^\infty$ or with coefficients in $L^p_{\text{loc}}(bD)$. More precisely, we give a Hartogs-Bochner extension result for CR differential forms of class $\mathcal{C}^\infty$ or with coefficients in $L^p_{\text{loc}}(bD)$ in an unbounded domain which is related to the vanishing of the some Dolbeault cohomologies groups of differential forms of class $\mathcal{C}^\infty$ with support in $\Phi$ denoted by $H_{\Phi}^{l,r}(X)$ or with coefficients in $L^p_{\text{loc}}(bD)$ denoted by $H_{\Phi,\text{cur}}^{l,r}(X)$ where $\Phi$ is a closed paracompactifying family of closed set of $X$.

2. Preliminaries

Definition 2.1. Let $X$ be a complex manifold of complex dimension $n$ and $S$ a real hypersurface of $M$. For $0 \leq p \leq n$ and $0 \leq q \leq n-1$, a $(p,q)$-form $f$ of class $\mathcal{C}^\infty$ on a real hypersurface $S$ of $X$ is called CR
of class $C^\infty$ if $q = n - 1$ or if $0 \leq q \leq n - 2$ and

$$\int_S f \wedge \bar{\partial} \varphi = 0$$

for all $\varphi \in \mathcal{E}^{np,nq-2}(X)$ such that $S \cap \text{Supp} \varphi$ is compact.

**Definition 2.2.** Let $X$ be a complex manifold of complex dimension $n$ and $D \subset X$ be a domain. The pair $(X, D)$ is said to have the $C^\infty$ Hartogs-Bochner extension property for $(l, r)$-form if: for any $(l, r)$-form $f$ of class $C^\infty$ on $bD$ and $\partial$, there exists a $(l, r)$-form $F$ of class $C^\infty$ on $\overline{D}$ and $\partial$-closed on $D$ such that $F|_{\partial D} = f$.

**Definition 2.3.** (see [7]) A family $\Phi$ of closed subset of $X$ is called a paracompactifying family of $X$, if it satisfies the following conditions:

1. Any closed subset of an element of $\Phi$ is an element of $\Phi$.
2. $\Phi$ is closed under finite unions.
3. Every element of $\Phi$ has neighborhoods which is in $\Phi$.

### 3. An Isomorphism Result

Let $\Phi$ be a paracompactifying family of closed sets of a complex analytic manifold $X$. Let $H_{\Phi, \text{cur}}^{l,r}(X)$ denote the $(l, r)$ Dolbeault cohomology group of currents with support in $\Phi$. We know in the classical case that there is a natural isomorphism between $H^{l,r}(X)$ and $H_{\Phi, \text{cur}}^{l,r}(X)$ called the Dolbeault isomorphism. What can be said about the natural map between $H_{\Phi, \text{loc}}^{l,r}(X)$ and $H_{\Phi, \text{cur}}^{l,r}(X)$? According to [6], the cohomological groups $H_{\Phi, \text{loc}}^{l,r}(X)$, $H_{\Phi, L^\infty}^{l,r}(X)$, $H_{\Phi, \text{cur}}^{l,r}(X)$ and $H_{\Phi, C^k}$ are isomorphic. In this part, the objective is to prove this result using Chirka’s Theorem (cf [1]). We have the following result:

**Theorem 3.1.** Let $X$ be a complex manifold of complex dimension $n$. Then the natural map between

$$H_{\Phi, \text{loc}}^{l,r}(X) \to H_{\Phi, \text{cur}}^{l,r}(X)$$

is an isomorphism.

**Proof:** According to [1], we have

$$R_\varepsilon : \mathcal{D}^{l,r}(X) \to \mathcal{E}^{l,r}(X)$$

and

$$A_\varepsilon : \mathcal{D}^{l,r}(X) \to \mathcal{D}^{l,r-1}(X)$$

such that for all $T \in \mathcal{D}^{l,r}(X)$, we have

$$T - R_\varepsilon T = \bar{\partial} A_\varepsilon T + A_\varepsilon \bar{\partial} T.$$  

(3.1)

(1) Injectivity: Let $[h] \in H_{\Phi, \text{loc}}^{l,r}(X)$ such that its rang belongs to the null class in $H_{\Phi, \text{cur}}^{l,r}(X)$. Then there exists a $(l, r - 1)$-current $u$ with support in $\Phi$ such that $\bar{\partial} u = h$. According to (3.1) and by continuity of the operator $A_\varepsilon$ on $(L^p_{\text{loc}})^*(X)$, we
have \( h = \bar{\partial}(R_T^e + A_e^e h) \) and \((R_T^e + A_e^e h) \in (L^p_{loc})^*(X)\) with support in a \( \varepsilon \)-neighborhood of the support of \( h \), for \( \varepsilon > 0 \) small enough, the support of \((R_T^e + A_e^e h)\) is an element of \( \Phi \). So \([h] = [0]\) in \( H_{\Phi, l,r}^{l,r}(X) \). Which proves the injectivity of the natural map.

(2) Surjectivity: Let \([h] \in H_{\Phi, cour}^{l,r}(X)\), we have \( \bar{\partial}h = 0 \). According (3.1), we have \( h - R_e^e h = \bar{\partial}A_e^e h \), where \( R_e^e h \in \mathcal{E}^{l,r}(X) \subset (L^p_{loc})^{(l,r)}(X) \), \( A_e^e h \in \mathcal{D}^{l,r-1}(X) \) and their supports are in a \( \varepsilon \)-neighborhood of the support of \( h \). For \( \varepsilon > 0 \) small enough, the supports of \( R_e^e h \) and of \( A_e^e h \) are in \( \Phi \). Then \([h] = [R_e^e h]\) in \( H_{\Phi, l,r}^{l,r}(X) \). Which proves the surjectivity of the natural map.

\[ \square \]

Remark 3.1. Analogously, we show that \( H_{\Phi, l,r}^{l,r}(X) \) is isomorphic to \( H_{\Phi,C^k}^{l,r}(X) \) using the \( C^k \) estimates for the \( \bar{\partial} \).

We can have the following consequence of Proposition 2.2 of [6].

Corollary 3.2. Let \( X \) be a complex manifold of complex dimension \( n \) and \( D \subset X \) a domain such that \( \overline{D} \in \Phi \), \( H_{\Phi}^{l,r}(X) = 0 \) and \( H_{\Phi}^{l,r-1}(X \setminus D) = 0 \). Then

\[
H_{D}^{l,r}(X) = 0, \quad H_{D,L^p_{loc}}^{l,r}(X) = 0 \quad \text{and} \quad H_{D,C^k}^{l,r}(X) = 0.
\]

Proof: Let \( f \) be a \((l, r)\)-form of class \( C^\infty \), \( \bar{\partial} \)-closed and defined on \( X \) with support in \( D \). Since \( H_{\Phi}^{l,r}(X) = 0 \), there exists a \((l, r-1)\)-form \( g \) of class \( C^\infty \) and defined on \( X \) with support in \( \Phi \) such that \( \bar{\partial}g = f \). Then \( \text{supp}(f) \subset \text{supp}(g) \) and \( \bar{\partial}g|_{X \setminus \text{supp}f} = 0 \). If \( \text{supp}g = \overline{D} \), then \( H_{\Phi}^{l,r}(X) = 0 \). Otherwise, we have \( \bar{\partial}g|_{X \setminus \overline{D}} = 0 \) leads to \( \bar{\partial}g|_{X \setminus D} = 0 \). For \( r = 1 \), from Corollary 2.1 of [8], we have \( H_{\Phi}^{l,1}(X) = 0 \). For \( r > 1 \), since \( H_{\Phi}^{l,r-1}(X \setminus D) = 0 \), there exists a \((l, r-2)\)-form \( u \) of class \( C^\infty \) and defined on \( X \setminus D \) such that \( \bar{\partial}u = g \). Let \( \hat{u} \) be an extension of class \( C^\infty \) from \( u \) to \( D \). Thereby

\[
\hat{g} = \begin{cases} 
\bar{\partial}\hat{u} \text{ sur } D \\
g \text{ sur } X \setminus D
\end{cases}
\]

is \( \bar{\partial} \)-closed on \( D \) and \( \hat{g} = g - \hat{g} \) is supported in \( \overline{D} \). We have

\[
\bar{\partial}\hat{g} = f
\]

on \( X \). Thus

\[
H_{\Phi}^{l,r}(X) = 0.
\]

By analogy we have

\[
H_{D,L^p_{loc}}^{l,r}(X) = 0 \quad \text{and} \quad H_{D,C^k}^{l,r}(X) = 0.
\]

\[ \square \]
4. Extension of CR differential forms of class $C^\infty$

In this part, the object is to show that the Hartogs-Bochner extension phenomenon for differential forms of class $C^\infty$ or differential forms with coefficients in $L^p_{loc}$ is equivalent to $H^{l,r}_\Phi(X) = 0$. For this, we start with the case of differential forms of class $C^\infty$ giving the following result:

**Theorem 4.1.** Let $X$ be a non compact complex manifold of complex dimension $n$ such that $H^{l,r}_\Phi(X) = 0$ with $0 \leq l \leq n$ and $0 \leq r \leq n$. Let $\Phi$ be a paracompactifying family of closed sets of $X$ not containing $X$. Let $D \subset X$ be a domain with connected boundary, smooth of class $C^\infty$ which is not relatively compact and that $\overline{D} \in \Phi$. We assume that $H^{l,r-1}(X \setminus D) = 0$. Then the Hartogs-Bochner extension phenomenon for differential forms is equivalent to $H^{l,r}_\Phi(X) = 0$ for all $0 \leq l \leq n$ and $0 \leq r \leq n$.

**Proof:** Let $f$ be a differential CR-form of bidegree $(l, r-1)$, of class $C^\infty$ and defined on $bD$. Let $\tilde{f}$ be an extension of class $C^\infty$ of $f$ to $D$ and set $g = \chi(\overline{D}) \hat{\partial} \tilde{f}$ is a $(l, r)$-differential form of class $C^\infty$ with support in $\overline{D}$ and $\hat{\partial}$-closed. Suppose $H^{l,r}_\Phi(X) = 0$, since $\overline{D} \in \Phi$, there is a $(l, r-1)$-form $u$ of class $C^\infty$ and defined on $X$ such that $\hat{\partial}u = g$ and $K = \text{supp}(u) \in \Phi$. We have $\overline{D} \subset K$ which leads to $X \setminus K \subset X \setminus \overline{D}$. We have $\hat{\partial}u_{|X \setminus \overline{D}} = 0$ leads to $\hat{\partial}u_{|X \setminus D} = 0$. For $r = 1$, $f$ is a CR function of class $C^\infty$, defined on $bD$ and we suppose that its support is not compact, then according to Proposition 2.4 of [8], there exists a function $F \in \mathcal{O}(D) \cap C^\infty(\overline{D})$ such that $F_{|bD} = f$. For $r > 1$, since $H^{l,r}(X \setminus D) = 0$, there exists a $(l, r-2)$-form $h$ of class $C^\infty$ and defined on $X \setminus D$ such that $\hat{\partial}h = u$. Let $\tilde{h}$ be an extension of class $C^\infty$ from $h$ to $D$. Let $\tilde{u} = u - \hat{\partial}h$, we have $\tilde{u}_{|bD} = 0$. We take $F = \tilde{f} - \tilde{u}$. It is a $(l, r-1)$-form, defined on $D$ and of class $C^\infty$ on $\overline{D}$ such that $F_{|bD} = f$. The extension therefore takes place for any domain $D$ not bounded with boundary of class $C^\infty$ such that $\overline{D} \in \Phi$.

Suppose that the extension takes place for any unbounded domain $D$ with boundary of class $C^\infty$ such that $\overline{D} \in \Phi$. Let $[h] \in H^{l,r}_\Phi(X)$, there exists $g \in \mathcal{E}^{l,r-1}(X)$ such that $\hat{\partial}g = h$. Since $\text{supp}(h) \in \Phi$, there exists an open neighborhood $D$ of $\text{supp}(h)$ with smooth boundary $bD$ of class $C^\infty$ such that $\overline{D} \in \Phi$, (see definition 2.3). Since $\text{supp}(h) \subset \overline{D}$, we have $\hat{\partial}g_{|X \setminus \overline{D}} = 0$ this results in $\hat{\partial}g_{|X \setminus D} = 0$. For $r = 1$, then according to Theorem 2.8 of [8], we have $H^{l,1}_\Phi(X) = 0$ for all $0 \leq l \leq n$ and $0 \leq r \leq n$. For $r > 1$, since $H^{l,r}(X \setminus D) = 0$, there exists a $(l, r-2)$-form $u$ of class $C^\infty$ and set to $X \setminus D$ such that $\hat{\partial}u = g$. So

\[
\tilde{g} = \begin{cases} 
G & \text{on } D \\
\hat{\partial}u & \text{on } X \setminus D 
\end{cases}
\]
is $\bar{\partial}$-closed on $D$ and $\hat{g} = g - \check{g}$ is supported in $\bar{D}$. We have
\[ \bar{\partial} \hat{g} = \partial \check{g} = h \]
on $X$. Which means that $\lfloor h \rfloor = 0$ in $H_{l,r}^{l,r}(X)$. We thus have
\[ H_{l,r}^{l,r}(X) = 0. \]

As a consequence of Theorem 4.1, we have the following result:

**Corollaire 4.2.** Under the assumptions of Theorem 4.1, the Hartogs-Bochner extension phenomenon for forms is equivalent to
\[ H_{l,r}^{l,r}(X) = H_{l,r}^{l,r}(X) = H_{l,r}^{l,r}(X) = 0. \]

**Proof:** By Theorem 4.1, the Hartogs-Bochner extension phenomenon for differential forms is equivalent to
\[ H_{l,r}^{l,r}(X) = 0. \]
From [6], $H_{l,r}^{l,r}(X)$, $H_{l,r}^{l,r}(X)$, $H_{l,r}^{l,r}(X)$ and $H_{l,r}^{l,r}(X)$ are isomorphic so the Hartogs-Bochner expansion phenomenon for differential forms is equivalent to
\[ H_{l,r}^{l,r}(X) = H_{l,r}^{l,r}(X) = H_{l,r}^{l,r}(X) = 0. \]

**Remark 4.1.** The Theorem 4.1 remains valid if $X$ is a stein manifold.

We also have the $L_{loc}^{p}(X)$ version of Theorem 4.1 given by the following result:

**Theorem 4.3.** Let $X$ be a non compact complex manifold of complex dimension $n$ such that $H^{l,r}(X) = 0$ with $0 \leq l \leq n$ and $0 \leq r \leq n$. Let $\Phi$ be a paracompactifying family of closed sets of $X$ not containing $X$. Let $D \subset X$ be a domain with boundary connected, smooth of class $C^{\infty}$ which is not relatively compact and that $\bar{D} \in \Phi$. We assume that $H_{l,r}^{l,r-1}(X \setminus \bar{D}) = 0$. So then the Hartogs-Bochner extension phenomenon for the forms with coefficients in $L_{loc}^{p}(bD)$ is equivalent to $H_{l,r}^{l,r}(X) = 0$ for all $0 \leq l \leq n$ and $0 \leq r \leq n$.

**Proof:** Let $f$ be a CR-form $f$ of bidegree $(l, r - 1)$, of class $C^{\infty}$ on $bD$ with coefficients in $L_{loc}^{p}(bD)$. Let $\hat{f}$ be an extension of class $C^{\infty}$ from $f$ to $\bar{D}$, and set $g = \chi(\bar{D}) \partial \hat{f}$, it is a $(l, r)$-form with coefficients in $L_{loc}^{p}(X)$, with support in $\partial X$ and $\bar{\partial}$-closed. Since $H_{l,r}^{l,r}(X) = 0$ which implies $H_{l,r}^{l,r}(X) = 0$, then $g = \bar{\partial} \lambda$ where $\lambda$ is a differential form of bidegree $(l, r - 1)$ with coefficients in $L_{loc}^{p}(X)$ with $K = \text{supp}(\lambda) \in \Phi$. We have $\partial D \subset K$ which leads to $X \setminus K \subset X \setminus D$. We have $\bar{\partial} \lambda|_{X \setminus D} = 0$. For $r = 1$, $f$ is a CR function with coefficients in $L_{loc}^{p}(bD)$ and we assume that its support is not compact, then according to Theorem 1.3
from [9], there is a function $F \in \mathcal{O}(D) \cap L^p_{loc}(\overline{D})$ such that $F_{|bD} = f$.

For $r > 1$, since $H^{l, r}_{\Phi, L^p_{loc}}(X \setminus \overline{D}) = 0$, there is a $(l, r - 2)$-form $u$ with coefficients in $L^p_{loc}(X \setminus \overline{D})$ such that $\bar{\partial}u = \lambda$. Let

$$\hat{u} = \begin{cases} u & \text{on } X \setminus \overline{D} \\ 0 & \text{on } D \end{cases}$$

and

$$\hat{\lambda} = \lambda - \bar{\partial}\hat{u},$$

we have $\hat{\lambda}_{|bD} = 0$. We take $F = \hat{f} - \hat{\lambda}$. It is a $(l, r - 1)$-form with coefficients in $L^p_{loc}(\overline{D})$, $\bar{\partial}$-closed on $D$ and $F_{|bD} = f$.

Suppose the extension takes place for any unbounded domain $D$ with boundary of class $C^\infty$ such that $\overline{D} \in \Phi$. Let $[h] \in H^{l, r}_{\Phi, L^p_{loc}}(X)$, which implies $\bar{\partial}h = 0$. According to Proposition 1.1 of [7], we have $H^{l, r}_{\Phi, L^p_{loc}}(X) = H^{l, r}(X) = 0$, then there exists a $(l, r - 1)$-form $g$ with coefficients in $L^p_{loc}(\overline{D})$ such that $\bar{\partial}g = h$. Since $supp(h) \subset \Phi$, there exists an open neighborhood $U$ of $X \setminus D$ with smooth boundary $bD$ of class $C^\infty$ such that $\overline{D} \in \Phi$, (see definition 2.3). Since $supp(h) \subset \overline{D}$, we have $\bar{\partial}g_{|bD} = 0$ and therefore $\bar{\partial}g = 0$ on $bD$. By hypothesis, there exists a $(l, r - 1)$-form $G$ with coefficients in $L^p_{loc}(\overline{D})$, $\bar{\partial}$-closed in $D$ such that $G_{|bD} = g$. For $r = 1$, then according to Theorem 2.8 of [8], we have $H^{1, 1}_{\Phi, L^p_{loc}}(X) = 0$ for all $0 \leq l \leq n$ and by isomorphism $H^{l, 1}_{\Phi, L^p_{loc}}(X) = 0$. For $r > 1$, let

$$\tilde{g} = \begin{cases} \bar{\partial}u & \text{on } X \setminus \overline{D} \\ G & \text{on } \overline{D} \end{cases}$$

is $\bar{\partial}$-closed on $X$ and $\tilde{g} = g - \tilde{g}$ is supported in $\overline{D}$. We have

$$\bar{\partial}\tilde{g} = \bar{\partial}g = h$$

on $X$. Which means that $[h] = 0$ in $H^{l, r}_{\Phi, L^p_{loc}}(X)$. We thus have

$$H^{l, r}_{\Phi, L^p_{loc}}(X) = 0.$$  

\[ \square \]

### 5. Vanishing conditions of $H^{l, r}_{\Phi}(X)$

**Definition 5.1.** Let $X$ be a complex manifold of complex dimension $n$. We say that $X$ is a generalized $q$-concave extension of a domain $D \subset X$ $(0 < q < n - 1)$ if:

1. $D$ meets all the connected components of $X$.
2. There is a fonction $\rho : \mathbb{R} \times U \to \mathbb{R}$, where $U$ is a neighborhood of $X \setminus D$ such that:
   a) For all $t \in \mathbb{R}$, $\rho(t, .)$ is $(q + 1)$-concave.
   b) For all $z \in U$, $\rho(., z)$ is a decreasing function.
   c) The map $t \mapsto \rho(t, .)$ is continuous from $\mathbb{R}$ in $C^\infty(U, \mathbb{R})$. 

d) \( D \cap U = \{ z \in U | \rho(0, z) < 0 \} \) and for all \( t > 0 \), the set \( \{ z \in U | \rho(t, z) < 0 \} \cap \overline{C(D)} \) is relatively compact in \( X \).

**Theorem 5.2.** Let \( X \) be a Stein manifold of complex dimension \( n \geq 2 \), suppose that the paracompactifying family \( \Phi \) of sets of \( X \) not containing \( X \) and verifying: for all \( K \in \Phi \), \( \exists \tilde{K} \in \Phi \) with \( K \subset \tilde{K} \) such that \( X \) is a Generalized \( q \)-concave extension of \( X \setminus \tilde{K} \) and \( H_{\tilde{\Phi}}^{l,r}(X \setminus \tilde{K}) = 0 \). Then \( H_{\Phi}^{l,r}(X) = 0 \) for all \( 0 \leq r \leq q - 1 \).

**Proof:** Let \([f] \in H_{\Phi}^{l,r}(X)\), then \( \partial f = 0 \) so there is \( u \in \mathcal{E}^{l,r-1}(X) \) such that \( \partial u = f \). Let \( K = \text{supp}(f) \), then there exists \( \tilde{K} \in \Phi \) with \( K \subset \tilde{K} \) such that \( X \) is a Generalized \( q \)-concave extension of \( X \setminus \tilde{K} \). According to [3], the restriction map of \( H^{l,r}(X) \rightarrow H_{\tilde{\Phi}}^{l,r}(X \setminus \tilde{K}) \) is an isomorphism for \( 0 \leq r \leq q - 1 \). Since \( \partial u_{|X \setminus K} = 0 \), we have \( \partial u_{|X \setminus \tilde{K}} = 0 \).

For \( r = 1 \), then according to the Theorem 3.2 of [8],

\[
H_{\Phi}^{1,1}(X) = 0.
\]

For \( r > 1 \), since \( H_{\Phi}^{l,r-1}(X \setminus \tilde{K}) = 0 \), there is \((l, r-2)\)-form \( h \) of class \( C^\infty \) and defined on \( X \setminus \tilde{K} \) such that \( \partial h = u \). Let \( \hat{h} \) be an extension of class \( C^\infty \) from \( h \) to \( \tilde{K} \). Let \( \hat{u} = u - \partial \hat{h} \), it is a \((l, r-1)\)-form of class \( C^\infty \) on \( X \) and \( \text{supp}(\hat{u}) \subset \tilde{K} \), so \( \text{supp}(\hat{u}) \in \Phi \). Also, \( \partial \hat{u} = f \). Thus

\[
H_{\Phi}^{l,r}(X) = 0
\]

for all \( 1 \leq r \leq q - 1 \).

As a consequence of Theorem 5.2, we have the following result:

**Corollaire 5.3.** Under the assumptions of Theorem 5.2, we have

\[
H_{\Phi, C^k}^{l,r}(X) = H_{\Phi, P_{loc}}^{l,r}(X) = H_{\Phi, C^{\text{loc}}}^{l,r}(X) = 0.
\]

**Proof:** By Theorem 5.2, we have \( H_{\Phi}^{l,r}(X) = 0 \).

From [6], \( H_{\Phi}^{l,r}(X) \), \( H_{\Phi, C^k}^{l,r}(X) \), \( H_{\Phi, P_{loc}}^{l,r}(X) \) and \( H_{\Phi, C^{\text{loc}}}^{l,r}(X) \) are isomorphic so we have

\[
H_{\Phi, C^k}^{l,r}(X) = H_{\Phi, P_{loc}}^{l,r}(X) = H_{\Phi, C^{\text{loc}}}^{l,r}(X) = 0.
\]

**Theorem 5.4.** Let \( X \) be a Stein manifold of complex dimension \( n \geq 2 \). Suppose that the paracompactifying family \( \Phi \) of sets of \( X \) does not contain \( X \) and for all \( K \in \Phi \), we have \( X \) is a generalized \( q \)-concave extension of \( X \setminus K \). Then we have the Hartogs-Bochner extension phenomenon for the \((l, r-1)\)-forms with coefficients in \( L_{loc}^{p} \) for all \( 1 \leq r \leq q - 1 \).
Proof: By Theorem 5.2, we have $H^{l,r}_{\Phi,L^p_{\text{loc}}}(X) = 0$, which implies $H^{l,r}_{\Phi,L^p_{\text{loc}}}(X) = 0$. Let $D \subset X$ be a domain with connected boundary such as $\overline{D} \in \Phi$. Let $f$ be a CR form of bidegree $(l, r-1)$ with coefficients in $L^p_{\text{loc}}(bD)$. Let $\tilde{f}$ be an extension of $f$ with coefficients in $L^p_{\text{loc}}(\overline{D})$ such that $\overline{\partial} \tilde{f}$ be a form of bidegere $(l, r)$ with coefficients in $L^p_{\text{loc}}(X)$. Let's pose $g = \chi(\overline{D}) \overline{\partial} \tilde{f}$, $g$ is a $(l, r)$-form with coefficients in $L^p_{\text{loc}}(X)$ and with support $\overline{D} \in \Phi$ and $\overline{\partial} g = 0$. Now $H^{l,r}_{\Phi,L^p_{\text{loc}}}(X) = 0$, then there exists a $(l, r - 1)$-form $\lambda$ with coefficients in $L^p_{\text{loc}}(X)$ such that $\overline{\partial} \lambda = g$ and $\text{supp} \lambda \in \Phi$. We have $\overline{\partial} \lambda|_{X \setminus \overline{D}} = 0$. For $r = 1$, $f$ is a CR function with coefficients in $L^p_{\text{loc}}(bD)$ and we assume that its support is not compact, then according the Theorem 1.2 of [9], there is a function $F \in \mathcal{O}(D) \cap L^p_{\text{loc}}(\overline{D})$ such that $F|_{bD} = f$. According to Proposition 1.1 of [4], $H^{l,r}(X)$ is isomorphic to $H^{l,r}_{\Phi,L^p_{\text{loc}}}(X)$ therefore $H^{l,r}(X \setminus \overline{D})$ is isomorphic to $H^{l,r}_{\Phi,L^p_{\text{loc}}}(X \setminus \overline{D})$. Since $X$ is a generalized $q$-concave extension of $X \setminus \overline{D}$, then the following restriction map $H^{l,r}_{\Phi,L^p_{\text{loc}}}(X) \rightarrow H^{l,r}_{\Phi,L^p_{\text{loc}}}(X \setminus \overline{D})$ is an isomorphism for $0 \leq r \leq q-1$, so $H^{l,r}_{\Phi,L^p_{\text{loc}}}(X \setminus \overline{D}) = 0$. Since $\overline{\partial} u|_{X \setminus \overline{D}} = 0$, there is a $(l, r - 2)$-form $\lambda$ with coefficients in $L^p_{\text{loc}}(X \setminus \overline{D})$ such that $\overline{\partial} \lambda = u$. Let
\[
\hat{\lambda} = \begin{cases} 
\lambda \text{ sur } X \setminus \overline{D} \\
0 \text{ sur } \overline{D}
\end{cases}
\]
and $\hat{u} = u - \overline{\partial} \hat{\lambda}$, we have $\hat{u}|_{bD} = 0$. Let $F = \tilde{f} - \hat{u}$. It is a $(l, r - 1)$-form with coefficients in $L^p_{\text{loc}}(\overline{D})$, $\overline{\partial}$-closed on $D$ and $F|_{bD} = f$. □

Proposition 5.5. Let $X$ be a Stein manifold and $\Phi$ a paracompactifying family of closed sets of $X$ not containing $X$. Let $D$ be a non-compact domain of $X$, with boundary $bD$ of class $C^\infty$, connected such that $\overline{D} \in \Phi$. We assume that $H^{l,r-1}(X \setminus \overline{D}) = 0$ and for any $(l, r - 1)$-form $f$ of class $C^\infty$, defined on $bD$, whose support is not compact and $\overline{\partial}_h$-closed, there is $F \in \mathcal{E}^{l,r-1}(\overline{D})$, $\overline{\partial}$-closed in $D$ such that $F|_{bD} = f$, then we have $H^{l,r}_{\Phi}(X) = 0$.

Proof: Let $[f] \in H^{l,r}_{\Phi}(X)$, then $[f] \in H^{l,r}(X) = 0$. There exists $g \in \mathcal{E}^{l,r-1}(X)$ such that $\overline{\partial} g = f$. Let $K = \text{supp}(f)$, we have $\overline{\partial} g|_{X \setminus K} = 0$. By the definition of $\Phi$, there exists $D_1$ an open neighborhood of $K$ such that $\overline{D}_1 \in \Phi$. Choose $D_2$ an unbounded domain with boundary of class $C^\infty$ such that $\overline{D}_2 \in \Phi$ and $\overline{D}_1 \subset D_2$. We have $X \setminus \overline{D}_2 \neq \emptyset$ because $X$ does not belong to $\Phi$. We have $\overline{\partial} g|_{X \setminus \overline{D}_2} = 0$ because $X \setminus \overline{D}_2 \subset X \setminus K$ and therefore $\overline{\partial} g = 0$ on $bD_2$. By hypothesis, there is $G \in \mathcal{E}^{l,r-1}(\overline{D}_2)$, $\overline{\partial}$-closed in $D_2$ such that $G|_{bD_2} = g$. For $r = 1$, $f$ is a CR function of class $C^\infty$ on $bD$ whose support is not compact, there is a function $F \in \mathcal{O}(D) \cap C^\infty(\overline{D})$ such that $F|_{bD} = f$. Then according to Proposition
For \( r > 1 \), since \( H_{l,r-1}^{l,r}(X \setminus \overline{D}_2) = 0 \), there exists \((l, r-2)\)-form \( u \) of class \( C^\infty \) and defined on \( X \setminus \overline{D}_2 \) such that \( \partial u = g \). Let’s pose

\[
\hat{g} = \begin{cases} 
\partial u & \text{ on } X \setminus \overline{D}_2 \\
G & \text{ on } \overline{D}_2
\end{cases}
\]

and

\[
\tilde{g} = g - \hat{g}.
\]

Thus \( \tilde{g} \) is a \((l, r-1)\)-form of class \( C^\infty \) with support in \( \overline{D}_2 \in \Phi \). Moreover \( \partial \tilde{g} = f \) on \( X \). Then we have

\[
H_{\Phi}^{l,r}(X) = 0.
\]

\[\square\]

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