A Sampling Control Framework and Its Applications to Robust and Adaptive Control

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Abstract—In this article, we propose a novel sampling control framework based on the emulation technique where the sampling error is regarded as an auxiliary input to the emulated system. Utilizing the supremum norm of sampling error, the design of periodic sampling and event-triggered control law renders the error dynamics bounded-input-bounded-state, and when it is coupled with system dynamics, achieves global or semiglobal stabilization. The proposed framework is then extended to tackle the stabilization problem for a system where the state of dynamic uncertainty is not available for feedback and the dynamics is subject to parameter uncertainties. The proposed framework is further extended to solve two classes of event-triggered adaptive control problems where the emulated closed-loop system does not admit an input-to-state stability Lyapunov function. For the first class of systems with linear parameterized uncertainties, even-triggered global adaptive stabilization is achieved without the global Lipschitz condition on nonlinearities as often required in the literature. For the second class of systems with uncertainties whose bound is unknown, the event-triggered adaptive (dynamic) gain controller is designed for the first time. Finally, theoretical results are verified by three numerical examples.

Index Terms—Adaptive control, event-triggered control, nonlinear systems, robust control, sampled-data control.

I. INTRODUCTION

The majority of modern control relies on its digital implementation in microprocessors and/or is deployed in a networked environment. And the sampled-data control scheme [2], [3], thus, arises from the demand of more efficient controller execution to reduce computation cost and save communication bandwidth in applications, such as multirobot systems, electrical power systems, and chemical processes. Sampled-data control schedules the update of a digital controller output at periodic or aperiodic sampling instances. Then, input-delay approach [10], [11], [12] is usually adopted to design the sampled-data controller for linear systems, while two other methods are mainly used for nonlinear systems, namely controller emulation, involving digital implementation of a continuous-time stabilizing control law, and plant discretization consisting of the discretization of the plant model and discrete-time control law design [23]. However, the sampled-data control approach may oversample in some cases and motivates event-triggered control. It suggests scheduling update only when necessary and specified by the occurrence of the designated triggering event and may achieve a more efficient sampling pattern. Sampled-data and event-triggered control have been developed for stabilization and tracking of individual systems, e.g., [19], [21], [27], [28] and cooperative control of networked systems, e.g., [5], [9], [25], [32]. In this article, these two control methods are uniformly called sampling control.

The two-step digital emulation, constituted by continuous-time control law design and its digital implementation, is a common technique for analysis and design of sampling control systems especially for nonlinear systems. In [15] and [24], when the emulation of periodic sampled-data control is formulated in the hybrid system setting, the maximum allowable sampling period (MASP) that guarantees asymptotic stability of sampled-data systems can be explicitly computed. In [4], a new small gain theorem is proposed to reveal the quantitative tradeoff between robustness and sampling bandwidth of the sampled-data control. Emulation is also commonly adopted for the design of event-triggered laws where the continuous-time controller is assumed to ensure dissipativity [23] or input-to-state stability (ISS) of the closed-loop system with the sampling error as input, see, e.g., [1], [7], [18], [19], [22], [27]. The ISS condition can be specified in a max-form, e.g., [19], [34] or in an ISS-Lyapunov form, e.g., [1], [18], [22], [27], [30] for the closed-loop sampling system and the small gain conditions are then proposed to ensure stability of the event-triggered system. In [17], the event-triggered technique in [27] is interpreted as a stabilization problem of interconnected hybrid systems for which each subsystem admits an ISS-Lyapunov function...
and a hybrid small gain condition was proposed. As another variant of small gain theorem, the cyclic small gain theorem has been proved effective for event-triggered control of large-scale systems [7], [18], [19].

Achieving Zeno free sampling and robustness to disturbance and uncertainties simultaneously is not trivial. In [8], a dynamic output-based event-triggered law is proposed to achieve a finite $L_p$-gain and a strictly positive lower bound on the interevent times when the system is subjected to nonvanishing external disturbances. In [19], the event-triggered control scheme is designed for systems subject to disturbances. In order to handle uncertainties, Liu and Jiang [18] proposed a dynamic event-triggered controller to achieve stabilization when the system has dynamic uncertainties whose state is not available for feedback. The problem is solved in [34] by a static event-triggered controller using a small-gain theorem method.

Recently, a few works on the event-triggered adaptive control have been proposed when the systems possess parameter uncertainties. Since the ISS condition is not guaranteed by the continuous-time stabilization controller and how to design an event-triggered adaptive control scheme is still a challenging problem. In [31], an event-triggered adaptive control scheme was proposed for a class of nonlinear systems based on the $\sigma$-modification scheme. The $\sigma$-modification scheme results in an ISS Lyapunov function, but only practical stabilization can be achieved, i.e., the trajectories converge to a bounded set in the neighborhood of the origin. The event-triggered law in [31] was improved in [13] in the sense that the adaptation dynamics was also sampled on the assumption of global Lipschitz conditions. In [29], a novel event-triggered adaptive control scheme is proposed for a class of nonlinear systems with an unknown control direction and unknown sensor faults, while only practical stabilization is achieved.

In this article, we propose a novel sampling control framework that can lead to event-triggered and periodic sampling (PS) robust and adaptive stabilization of nonlinear systems. The main contribution of this article is three-fold. First, this article proposes a sampling control framework based on the Lyapunov function method for the closed-loop system composed of the plant and the sampling error dynamics. The work in [8] and [30] also considers the error dynamics explicitly but requires the derivatives of Lyapunov candidate functions satisfy some specific forms and becomes restrictive for nonlinear systems. By exploiting the changing supply function technique, this article relaxes the restrictive conditions and it only requires the existence of an ISS Lyapunov function for state dynamics and a differentiable positive function for error dynamics whose derivative is bounded by $K_{\infty}$ functions. By integrating the bound of the derivative, we can design an event-triggered law that depends on functions of the sampling error and acts as the small-gain condition such that the error dynamics is bounded-input-bounded-state (BIBS). When it is coupled with system dynamics, global or semiglobal stabilization is achieved. The proposed method also facilitates avoidance of Zeno behavior and PS control design.

Second, the proposed framework is then extended to tackle the event-triggered and PS control for the systems where only partial state is available for feedback and the system dynamics is subject to parameter uncertainties. The dynamics containing immeasurable state is called dynamic uncertainty that is difficult to handle in the setting of sampling control. The event-triggered stabilization problem of systems with dynamic uncertainty was solved in [18] and [20] by designing an auxiliary dynamic system to estimate the decay rate of immeasurable states. However, the initial value of the dynamic uncertainty is also required for the controller design. The proposed approach in this article brings a static event-triggered controller, which is easier to design and implement in practice and does not rely on any measurement of the dynamic uncertainty. Because the proposed event-triggered law only relies on the sampling error, it can explicitly estimate the MASP and construct a PS controller. PS control for systems with partial measurement has yet to be studied in the existing literature, such as [18], [19], [20], [34].

Third, the proposed framework is extended to solve two classes of event-triggered adaptive control problems where the emulated closed-loop system does not admit an ISS Lyapunov function. The first class of systems contains linear parameterized uncertainties. Only practical stabilization can be achieved for the problem in [13], [29], and [33], even in the absence of external disturbances. However, this article achieves global asymptotic stabilization without the global Lipschitz condition on nonlinearities as required in [13]. We can also prove that the estimated value of the uncertainty converges to its real value under a persistent exciting (PE) condition. The second class of systems has the uncertainties whose bound is unknown and requires the adaptive/dynamic gain technique. To the best of authors’ knowledge, the event-triggered dynamic gain control is solved in this article for the first time.

The rest of this article is organized as follows. Section II formulates the sampling control problem for nonlinear systems, which is solved by the sampling control framework proposed in Section III when an ISS condition is assumed. The proposed framework is extended to tackle the event-triggered and PS control for the systems with dynamic uncertainty in Section IV. It is further extended to solve event-triggered adaptive stabilization of two types of uncertain systems in Section V. The numerical simulation is conducted in Section VI. Finally, Section VII concludes this article.

Notations: $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{> 0}$ denote the set of nonnegative and positive real numbers, respectively. $\mathbb{R}^n$ denotes the real space of dimension $n$, $\mathbb{R}^{n \times m}$ denotes the set of real matrices with dimension $n \times m$, and $\mathbb{N}$ denotes the set of nonnegative integers. Let $\|x_{[a,b]}\| := \sup_{t \in [a,b]} \|x(t)\|$ be the supremum norm of a given signal $x : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}^n$, over the interval $[a, b]$, and $x'_{[a,b]}$ be the signal slice of $x$ over the interval $[a, b]$. Denote a function by $\alpha \in K_{\infty}$ if it is a $K_{\infty}$ function, by $\alpha \in K L$ if it is a $K L$ function, by $m \in SP$ if it is a smooth positive function, and by $m \in S V^r$ if it is a smooth nonnegative function. A continuous function $\alpha : [0, \infty) \times (0, \infty) \mapsto [0, \infty)$ is said to be a parameterized $K_{\infty}$ function if, for each fixed $s > 0$, the function $\alpha(s, \cdot)$ is a $K_{\infty}$ function, and for each fixed $r > 0$, the function $\alpha(r, \cdot)$ is nondecreasing. For $K_{\infty}$ functions $\alpha'(s)$ and $\alpha(s)$, $\alpha'(s) = \mathcal{O}(\alpha(s))$ as $s \rightarrow 0^+$ means that $\lim \sup_{s \rightarrow 0^+} \frac{\alpha'(s)}{\alpha(s)} < \infty$. Let $V : \mathbb{R}^n \mapsto \mathbb{R}_{> 0}$ be a continuously differentiable function. It is called an ISS-Lyapunov function for the system $\dot{x} = f(x, u, d)$ if the derivative of
$V(x)$ along the $x$-dynamics satisfies, for all $x \in \mathbb{R}^n$, $\varpi \in \mathbb{R}^m$ and $d \in \mathbb{D}$

$$\alpha(||x||) \leq V(x) \leq \bar{\alpha}(||x||)$$

$$V(x) \leq -\alpha(||x||) + \sigma(||\varpi||)$$ (1)

where $\bar{\alpha}, \alpha, \sigma \in \mathbb{K}_\infty$. The inequalities in (1) are simplified as $V(x) \sim \{\bar{\alpha}, \alpha, \sigma \mid x = f(x, u, d)\}$. A bounded piecewise continuous function $f : [0, \infty) \mapsto \mathbb{R}^n$ is said to be PE if there exist positive constants $\epsilon$, $t_0$, $T_0$ such that

$$\frac{1}{T_0} \int_{t}^{t+T_0} f(s) \, ds \geq \epsilon^2 I, \forall t \geq t_0.$$ (2)

II. Problem Formulation of Sampling Control

Consider the nonlinear system

$$\dot{x}(t) = f(x(t), u(t), d(t))$$

$$y(t) = h(x(t))$$ (2)

where $x \in \mathbb{R}^n$ is the system state, $y \in \mathbb{R}^q$ the output, $u \in \mathbb{R}^m$ the control input, and $d \in \mathbb{R}^{n_d}$ system uncertainty. Suppose the function $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n_d} \mapsto \mathbb{R}^n$ is a continuously differentiable function satisfying $f(0, 0, d(t)) = 0$ and $h(0) = 0$, and the uncertainty $d$ belongs to a compact set $d \in \mathbb{D}$. Suppose the equilibrium point of the system can be stabilized by a continuous-time feedback controller

$$u(t) = \kappa(y(t))$$ (3)

with a continuously differentiable function $\kappa$. It becomes a continuous-time state feedback system when $y = x$. In this article, we use the emulation technique to study the sampling version of (3) as follows:

$$u(t) = \kappa(y(t)), \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}$$ (4)

where $\{t_k\} \subseteq \mathbb{N}$ is a sequence of sampling time instances. The sampling instances can be determined by an event-triggered law specified as follows:

$$t_{k+1} = \inf_{t \in [t_k, t_{k+1})} \left\{ \Xi(t, t_k, x_{[t_k, t)}) \geq 0 \right\}$$ (5)

for some function $\Xi \in \mathcal{R}$ representing event occurrence to be designed. A PS law with sampling period $T > 0$ can be considered as a special case of (5) with $\Xi(t, t_k, x_{[t_k, t)}) = t - t_k - T$, i.e., $t_{k+1} = t_k + T$. Note that the error caused by the sampling mechanism is

$$\bar{x}(t) := \kappa \circ h(x(t_k)) - \kappa \circ h(x(t)), \quad t \in [t_k, t_{k+1}).$$ (6)

For the emulation, the closed-loop system composed of dynamics (2) and a sampling version of feedback controller (4) is put into the impulsive form

$$\dot{x} = f_c(x, \varpi, d) := f(x, \kappa \circ h(x) + \varpi, d)$$

$$\varpi = \psi(x, \varpi, d) := -\frac{\partial \kappa \circ h(x)}{\partial x} f_c(x, \varpi, d) \quad \forall t \in [t_k, t_{k+1})$$

$$\varpi(t_{k+1}) = 0.$$ (7)

The objective of the sampling control is to design the sampling law (5), either PS or event-triggered law, such that the closed-loop system composed of (2) and (4) achieves the following two objectives:

1) stabilization: the original system is asymptotically stable at the origin, in particular, $\lim_{t \to \infty} x(t) = 0$, semiglobally or globally;

2) Zeno-free behavior: a finite number of events are triggered in a finite amount of time.

In general, the continuous-time feedback controller in (3) is assumed to ensure that the $x$-subsystem in (7) has some properties such as ISS when regarding sampling error $\varpi$ as the input. The motivation of using the sampling error of the actuation $\varpi$ rather than state or output measurement is explained in [16]. It is exploited in our previous papers [16], [34] to design an event-triggered control law where $x$-dynamics is assumed to have the ISS condition specified in a max-form. As is known, the classic analysis tools developed for the adaptive control rely on the Lyapunov function method, typically along the gradient of a Lyapunov function. In order to solve event-triggered adaptive control in Section V, this article first considers the ISS condition specified in terms of ISS-Lyapunov function and assumes the existence of a positive function on $\varpi$-subsystem, which plays a similar role of a Lyapunov function. As a result, the sampling control framework proposed in following section can be naturally extended to event-triggered adaptive control in Section V where the continuous-time adaptive control does not admit an ISS condition.

Assumption II.1: The $x$-subsystem in (7) has an ISS-Lyapunov function $V(x) \sim \{\bar{\alpha}, \alpha, \sigma \mid \bar{x} = f(x, \varpi, d)\}$. There exists a continuously differentiable $U(\varpi) : \mathbb{R}^m \mapsto \mathbb{R} \geq 0$ such that the derivative of $U(\varpi)$ along the $\varpi$-subsystem, for all $x \in \mathbb{R}^n$, $\varpi, d \in \mathbb{R}^{n_d}$, satisfies

$$\alpha_{\varpi}(||\varpi||) \leq U(\varpi) \leq \bar{\alpha}_{\varpi}(||\varpi||)$$

$$\dot{U}(\varpi) = \left( \frac{\partial U(\varpi)}{\partial \varpi} \right)^T \psi(x, \varpi, d) \leq \alpha_{\varpi}(||\varpi||) + \sigma_{\varpi}(||x||)$$ (8)

for some functions $\alpha_{\varpi}, \bar{\alpha}_{\varpi}, \sigma_{\varpi} \in \mathbb{K}_\infty$.

Remark II.1: Note that the existence of the function $U(\varpi)$ condition is mild. Due to $\bar{\kappa}(0) = 0$ and, thus, $f_c(0, 0, d) = 0$, the function $\psi$ in (7) is continuously differentiable and satisfies $\psi(0, 0, d) = 0$. Applying [6, Lemma 11.1], one has $||\psi(x, \varpi, d)|| \leq m_1(x, d)||x|| + m_2(\varpi, d)||\varpi||$ for some functions $m_1, m_2 \in \mathcal{L}^1$. As a result, $U(\varpi) \leq m_1(x, d)||\partial U(\varpi)/\partial \varpi|| ||x|| + m_2(\varpi, d)||\partial U(\varpi)/\partial \varpi||$. Then, it is always possible to find functions $\alpha_{\varpi}, \sigma_{\varpi} \in \mathbb{K}_\infty$ such that the second inequality of (8) holds.

Remark II.2: In [34], a pair of ISS and input-to-output stability (IOS) conditions are given in max-norm form for the closed-loop system (7) where $\varpi$ is regarded as the input and $\psi$ as the output. ISS and IOS properties, for any $x(t_0)$, are stated as follows:

$$||x(t)|| \leq \max \left\{ \beta(||x(t_0)||, t - t_0), \tilde{\beta}(||\varpi([t_0, t])||) \right\}$$ (9)

$$||\psi(t)|| \leq \max \left\{ \beta(||x(t_0)||, t - t_0), \tilde{\beta}(||\varpi([t_0, t])||) \right\}$$ (10)

for $t \geq t_0$, where $\beta, \tilde{\beta} \in \mathbb{K}_\infty$ and $\beta, \tilde{\beta}$ are $\mathcal{K}$ functions. Although the ISS-Lyapunov function $V(x)$ in Assumption II.1 implies (9), (10), and (8) do not necessarily imply each other. Therefore, the conditions in Assumption II.1 are different from (9) and (10).
III. SAMPLING CONTROL DESIGN

In this section, PS and event-triggered control are proposed for the emulation system (7). We first present Theorem III.1 for the design of event-triggered law, which provides a common guideline for PS control design.

A. Event-Triggered Control

In order to solve the sampling stabilization problem, we apply the changing supply function technique (see [26] and [6, Lemma 2.5]) to the ISS-Lyapunov function $V(x)$ in Assumption II.1. Let $K^\omega$ function $\alpha_\omega(s)$ be selected such that $\alpha_\omega(s) > 2\sigma_\omega(s)$. If $\alpha_{\omega}(s) = O[\alpha(s)]$ as $s \to 0^+$, the changing supply function technique shows that there exists another ISS Lyapunov function $V_q(x) \sim \{\alpha_q, \alpha_q, \alpha_q, \sigma_q | \dot{x} = f_c(x, \omega, d)\}$ where the functions $\alpha_q, \alpha_q, \sigma_q \in K^\omega$ can be calculated accordingly. Then, the event-triggered control law is presented as follows.

Theorem III.1: Suppose the system composed of (2) and (4) satisfies Assumption II.1. Let $K^\omega$ functions $\alpha_\omega$ and $\gamma$ be $\alpha_\omega(s) > \sigma_\omega(s)/2$ and $\gamma(s) > \alpha_\omega(s) + \alpha_\omega(s)$. Suppose $\omega(t_0) = 0$, $\sigma_\omega(s) = O[\alpha(s)]$ and $\gamma(s) = O[\alpha_\omega(s)]$ as $s \to 0^+$. If the event-triggered law in (5) is designed as

$$
t_{k+1} = \inf_{t > t_k} \{2(t - t_k)\gamma(\|\omega(t_k)\|) + \alpha_\omega(\|\omega(t_k)\|)\}
$$

(11)

the equilibrium point $x = 0$ of the system is globally asymptotically stable and Zeno behavior is avoided. Moreover, $V_q(x(t)) \leq V_q(x(t_k))$, $\forall t \in [t_k, t_{k+1}]$ and $k \in \mathbb{N}$.

Proof: The closed-loop system consisting of (2) and (4) is put into the form (7). The proof will be divided into three parts. First, the boundedness of all signals is proved. The fact that $\sigma_\omega(s) = O[\alpha(s)]$ as $s \to 0^+$ implies that it is always possible to find a new supply function $V_q(x) \sim \{\alpha_q, \alpha_q, \alpha_q, \alpha_q | \dot{x} = f_c(x, \omega, d)\}$ and $\alpha_\omega(s) > 2\sigma_\omega(s)$. Integrating both sides of (8) in Assumption II.1 gives

$$
U(\omega(t)) = \int_{t_k}^{t} \dot{U}(\omega(\tau)) d\tau \leq \int_{t_k}^{t} \alpha_\omega(\|\omega(\tau)\|) d\tau + \int_{t_k}^{t} \sigma_\omega(\|\omega(\tau)\|) d\tau,
$$

$t \in [t_k, t_{k+1}]$

which leads to

$$
\max_{r \in [t_k, t_{k+1}]} \{U(\omega(\tau))\} \leq \int_{t_k}^{t} \alpha_\omega(\|\omega(\tau)\|) d\tau + \int_{t_k}^{t} \sigma_\omega(\|\omega(\tau)\|) d\tau,
$$

t \in [t_k, t_{k+1}]

(12)

due to $\alpha_\omega(\|\omega(\tau)\|) \geq 0$ and $\sigma_\omega(\|\omega(\tau)\|) \geq 0, \forall \tau \in [t_k, t_{k+1}]$. It further implies that

$$
\max_{\tau \in [t_k, t_{k+1}]} \{U(\omega(\tau))\} \leq -\int_{t_k}^{t} \alpha_\omega(\|\omega(\tau)\|) d\tau + \int_{t_k}^{t} \sigma_\omega(\|\omega(\tau)\|) d\tau + \int_{t_k}^{t} \alpha_\omega(\|\omega(\tau)\|) d\tau + \int_{t_k}^{t} \sigma_\omega(\|\omega(\tau)\|) d\tau \leq 2\max \left\{ -\int_{t_k}^{t} \alpha_\omega(\|\omega(\tau)\|) d\tau + \int_{t_k}^{t} \sigma_\omega(\|\omega(\tau)\|) d\tau \right\}
$$

(13)

where $\alpha_\omega(s) > \sigma_\omega(s)/2$. The sampling law (11) implies

$$
U(\omega(t)) \leq \max_{\tau \in [t_k, t_{k+1}]} \{U(\omega(\tau))\} \leq -2\int_{t_k}^{t} \alpha_\omega(\|\omega(\tau)\|) d\tau + 2\int_{t_k}^{t} \sigma_\omega(\|\omega(\tau)\|) d\tau
$$

(14)

Let

$$
\tilde{V}(x, \omega) = V_q(x) + U(\omega)
$$

(15)

be the Lyapunov function candidate for the closed-loop system (7). Since $x(t)$ and, hence, $V_q(x(t))$ are continuous in $t$ and $U(\omega(t))$ is piecewise continuous in $t$ and has jump at $t_k$, one has

$$
\tilde{V}(x(t_k^+), \omega(t_k^+)) = V_q(x(t_k^+)) \leq \tilde{V}(x(t_k), \omega(t_k))
$$

$$
= U(\omega(t)) + V_q(x(t_k)) \forall k \in \mathbb{N}
$$

by noting $\omega(t_k^+) = 0$. For $t \in [t_k, t_{k+1}]

$$
\tilde{V}(x(t), \omega(t)) - \tilde{V}(x(t_k), \omega(t_k)) = U(\omega(t_k)) + \int_{t_k}^{t} \dot{V}_q(x(t), \omega(t)) d\tau
$$

$$
\leq -2\int_{t_k}^{t} \alpha_\omega(\|\omega(\tau)\|) d\tau + 2\int_{t_k}^{t} \sigma_\omega(\|\omega(\tau)\|) d\tau
$$

$$
- \int_{t_k}^{t} \alpha_\omega(\|\omega(\tau)\|) d\tau + \int_{t_k}^{t} \sigma_\omega(\|\omega(\tau)\|) d\tau
$$

$$
\leq -\int_{t_k}^{t} \alpha_\omega(\|\omega(\tau)\|) d\tau - \int_{t_k}^{t} \gamma_\alpha(\|\omega(\tau)\|) d\tau
$$

(16)

where the functions $\gamma_\alpha$ and $\gamma_\alpha$ are defined as

$$
\gamma_\alpha(s) = 2\alpha_\omega(s) - \gamma(s) > 0
$$

$$
\gamma_\alpha(s) = \alpha_\omega(s) - 2\sigma_\omega(s) > 0
$$

As a result, the bound of $\tilde{V}(x(t), \omega(t))$ is monotonically decreasing except when $\cos(\omega) = 0$, and therefore, the signals $x$ and $\omega$ are bounded for $t \geq t_0$ whose bound depends on the initial value $x(t_0)$. Denote the bound $\omega$ by $R_\omega$. The derivative of $\omega$ denoted as $q := \dot{x}$ is also bounded and its bound is denoted by $|q(t)| \leq R(x(t_0)), \forall t \geq t_0$. The notation $R$ will be used instead of $R(x(t_k))$ for the notation simplicity. Note that $V_q(x(t_k)) = V_q(x(t_k^+), \omega(t_k^+)) \geq \tilde{V}(x(t), \omega(t)) \geq V_q(x(t))$ for any $t \in [t_k, t_{k+1}]$ and $k \in \mathbb{N}$.

Then, it will be shown that the event-triggered law is free of Zeno behavior by showing that $t_{k+1} - t_k > c$ for some constant $c > 0$ possibly depending on the initial condition. Note that $\gamma(s) = O[\alpha_\omega(s)]$ as $s \to 0^+$ implies that $\lim_{s \to 0^+} \gamma(s)/\alpha_\omega(s) < \infty$. Denote $C := \lim_{s \to 0^+} \gamma(s)/\alpha_\omega(s)$. Due to $\omega(t) = -\int_{t_k}^{t} q(\tau) d\tau, \forall t \in [t_k, t_{k+1}]$, one has $\|\omega(t_k^\omega)\| \leq \|x(t_k^\omega)\| \leq t(t_k) - t_{k+1}$ for $t \in [t_k, t_{k+1}].$
where we used (17). Therefore, the sampling interval can be selected to be lower bounded irrespective of \( k \), i.e., \( t_{k+1} - t_k \geq T_c = t_d \) for \( k \in \mathbb{N} \) and the bound \( t_d \) depends on \( R(x(t_0)) \) and, thus, the initial condition \( x(t_0) \).

Finally, the convergence of the signal \( x \) to zero, i.e., \( \lim_{t \to \infty} x(t) = 0 \), is proved. If it is not true, there is a positive constant \( c_2 \) such that for every \( t > 0 \), we can find \( t > t \) with \( \|x(t)\| \geq c_1 \). It is noted from (16) that \( \dot{V}(x(t_{k+1} + \tau), x(t_{k+1})) = V(x(t_{k+1} + \tau), x(t_{k+1})) - f(t_{k+1}) \alpha_\gamma\|x(\tau)\|d\tau \) and \( \lim_{\tau \to \infty} \int_{t}^{t+1} \alpha_\gamma\|x(\tau)\|d\tau \) exists and is finite. Since \( x(t) \) is a uniformly continuous function, applying Barbalat’s lemma, we can show that \( \lim_{t \to \infty} x(t) = 0 \).

Remark III.1: Let us examine the role of the event-triggered law (11) in stabilization. In the first step of the proof, integrating both sides of (8) results in (13) for the error dynamics. The event-triggered law (11) then acts as a small gain condition as in [14, Th. 10.6.1], and simplifies (13) into (14), which can be treated as the “integral” version of the ISS condition for the error dynamics. It also leaves \( \alpha_\gamma(s) \) to be determined for the stabilization. By the integration, the event-triggered law can be designed only depending on the sampling error \( \overline{x} \). When the error dynamics is coupled with state dynamics also admitting an ISS condition, a well-designed \( \alpha_\gamma(s) \) can render the small gain condition for the interconnected systems and the stabilization is then achieved. This approach that explicitly considers the error dynamics will also be adopted in Sections III-B, IV, and V for robust and adaptive sampling control.

Remark III.2: The bound of \( \overline{x} \) is explicitly derived below. Inequality (12) and the event-triggered law (11) together with the fact \( \gamma(s) > \gamma_\alpha(s) + \alpha(s) \) lead to

\[
2(t - t_k)\alpha_\gamma\|x(t_{k+1})\| \leq 2(t - t_k)\gamma\|x(t_{k+1})\| \\
\leq \max_{\tau \in [t_k, t_{k+1}]} \{U(\overline{x}(\tau))\} \\
\leq \int_{t_k}^{t} \alpha_\gamma\|x(\tau)\|d\tau + \int_{t_k}^{t} \alpha_\gamma\|x(\tau)\|d\tau \\
\leq (t - t_k)\alpha_\gamma\|x(t_{k+1})\| + (t - t_k)\gamma\|x(t_{k+1})\|, \quad t \in [t_k, t_{k+1}]
\]

which further implies that

\[
\alpha_\gamma\|x(t_{k+1})\| \leq \sigma_\gamma(\|x(t_{k+1})\|) \quad \forall t \in [t_k, t_{k+1}].
\]

Note that \( \sigma_\gamma(\|x(t)\|) \leq V_\gamma(x(t)) \leq \dot{V}(x(t), x(t)) \leq \ddot{V}(x(t_0), x(t_0)) = V_\gamma(x(t_0)), \) therefore, one has \( \|x(t)\| \leq \sigma_\gamma^{-1}(V_\gamma(x(t))) \). Hence, it follows that \( \|x(t)\| \leq \sigma_\gamma^{-1}(V_\gamma(x(t))) \). Therefore, the event-triggered law in (11) renders the \( \overline{x} \)-dynamics BIBS when regarding \( x \) as the input.

Remark III.4: The condition \( \gamma(s) = O(\alpha_\gamma(s)) \) as \( s \to 0^+ \) in Theorem III.1 can be implied by two conditions \( \sigma(s) = O(\alpha(s)) \) and \( \gamma(s) = O(\alpha(s)) \) as \( s \to 0^+ \). Note that \( \sigma(s) = O(\alpha(s)) \) as \( s \to 0^+ \) implies \( \sigma(s) = O(\alpha(s)) \) as \( s \to 0^+ \). It together with \( \alpha(s) = O(\alpha(s)) \) as \( s \to 0^+ \) implies that one can find a \( \gamma(s) \) such that \( \lim_{s \to 0^+} \gamma(s)/\alpha(s) < \infty \) or equivalently \( \gamma(s) = O(\alpha(s)) \) as \( s \to 0^+ \).

To see we can make \( \alpha(s) = O(\alpha(s)) \) as \( s \to 0^+ \), let \( U(\overline{x}) = 1/2\|\overline{x}\|^2 \) and its derivative is calculated as

\[
U'(\overline{x}) \leq m_1(x, d)\|x\|\|\overline{x}\| + m_2(x, d)\|\overline{x}\|^2 \leq \frac{1}{4} + m_2(x, d)\|\overline{x}\|^2
\]

where we used the bound of \( \|\psi(x, \overline{x}, d)\| \) in Remark II.1. Then, inequality (8) in Assumption II.1 is satisfied with \( \alpha_\gamma(\|\overline{x}\|) \geq \frac{1}{4} + m_2(x, d)\|\overline{x}\| \). Note that in this case \( \alpha_\gamma(\|\overline{x}\|) = 1/2\|\overline{x}\|^2 \) and then \( \lim_{s \to 0^+} \alpha(s)/\alpha(s) > 0 \) is satisfied due to \( m_2(x, d) > 0 \).

The conditions \( \sigma(s) = O(\alpha(s)) \) and \( \sigma_\gamma(s) = O(\alpha(s)) \) as \( s \to 0^+ \) can be made satisfied during the continuous-time feedback controller design phase. From (20), we can choose the function \( \sigma_\gamma \) in (8) to be \( \sigma_\gamma(\|x\|) \geq \bar{m}(x, d)\|x\|^2 \) for some function \( \bar{m}(x, d) > 0 \). Then, if one designs the continuous-time feedback controller that renders \( \lim_{s \to 0^+} \alpha(s)/s^2 < \infty \) and \( \lim_{s \to 0^+} \alpha_\gamma(s)/s^2 < \infty \), it leads to \( \sigma(s) = O(\alpha(s)) \) and \( \sigma_\gamma(s) = O(\alpha(s)) \) as \( s \to 0^+ \). Such kind of feedback controller can always be found for a large class of nonlinear systems such as those in strict feedback form and lower triangular form, to be presented in Sections IV and V.

Remark III.5: It is observed from the proof of Theorem III.1 that if the next sampling time is selected as \( t_{k+1} = t_k + T_k \), for any \( T_k \leq T_{k+1} - t_k \), where \( t_{k+1} \) is the next sampling time calculated from Theorem III.1, i.e.,

\[
t_{k+1} := \inf_{t \geq t_k} \{2(t - t_k)\gamma(\|x(t_{k+1})\|)\} \\
\geq \max_{\tau \in [t_k, t_{k+1}]} \{U(\overline{x}(\tau))\}, \quad \|\overline{x}(t_{k+1})\| \neq 0
\]

results of Theorem III.1 still hold.

The following proposition reveals that the sampling interval of event-triggered control approaches a constant as \( t \to \infty \). In other words, it tends to behave like a PS control as the system approaches origin.

Proposition III.1: Suppose the system is composed of (2) and (4) and the event-triggered control law is designed according to (11) in Theorem III.1. Let \( \mu := \lim_{s \to 0^+} \alpha_\gamma(s)/\gamma(s) \). Then, \( \lim_{s \to \infty} (t_{k+1} - t_k) > \mu/2 \). Moreover, if the function \( U(\overline{x}) \) is
specified as a $K_\infty$ function, i.e., $U(\varpi) = U(\|\varpi\|)$ and $\mu := \lim_{s \to 0^+} U(s)/\gamma(s)$, then, $\lim_{s \to \infty} (t_{k+1} - t_k) = \mu/2$.

Proof: The proof is similar to that of Proposition 2.1 in our paper [34] and omitted here.

IV. ROBUST SAMPLING CONTROL

In this section, we will apply the sampling control method proposed in Section III to solve the sampling robust stabilization problem of a class of nonlinear systems, called strict feedback systems of a relative degree one, as follows:

$$\dot{z} = q(z, x, d)$$
$$\dot{x} = f(z, x, d) + u$$

where $z \in \mathbb{R}^p$ and $x \in \mathbb{R}$ are state variables, $u \in \mathbb{R}$ is the input, and $d \in \mathbb{D}$ is the uncertainties belonging to a compact set $\mathbb{D} \subset \mathbb{R}^d$. The functions $q$ and $f$ are sufficiently smooth with $q(0, 0, d) = 0$ and $f(0, 0, d) = 0$ for all $d \in \mathbb{D}$. Note that the state $z$ is assumed to be not available for feedback control and thus the $z$-dynamics is called dynamic uncertainty. A common assumption on $z$-dynamics is given to make the problem tractable.

Assumption IV.1: The $z$-subsystem in (25) has an ISS-Lyapunov function $V(z) \sim \{\bar{R}, \bar{\alpha}, \alpha, \sigma \} \dot{z} = q(z, x, d)$ and functions $\alpha(s)$ and $\sigma(s)$ satisfy $\limsup_{s \to 0^+} s^2/\alpha(s) < \infty$, $\limsup_{s \to 0^+} \sigma(s)/s^2 < \infty$.

The sampling robust stabilization problem is to design a sampling controller $u$ such that $\lim_{t \to \infty} \text{col}(z(t), x(t)) = 0$. Since $f(z, x, d)$ is sufficiently smooth, one has

$$|f(z, x, d)| \leq m_1(z)\|z\| + m_2(x)\|x\| \quad \forall d \in \mathbb{D}$$

for some sufficiently smooth functions $m_1, m_2 \in \mathcal{S}'$ depending on the size of $\mathbb{D}$. For continuous-time stabilization of the system, a high-gain controller can be adopted to dominate the uncertainties when the size of $\mathbb{D}$ is known. The case that the size of $\mathbb{D}$ is unknown will be handled using the dynamic gain technique in Section V. The continuous-time controller usually takes the form of $u = \kappa(x) = -\rho(x)x$ with the high-gain term $\rho(x)$ to be specified. We adopt the method developed in Section III and propose the sampling controller as follows:

$$u(t) = \kappa(x(t_k)) = -\rho(x(t_k))x(t_k), \quad t \in [t_k, t_{k+1})$$

for $k \in \mathbb{N}$. Define the sampling error $\varpi(t)$ as $\varpi(t) = \kappa(x(t_k)) - \kappa(x(t)), \quad t \in [t_k, t_{k+1})$. Then, the sampled-data closed-loop system is rewritten as

$$\dot{x} = q(z, x, d)$$
$$\dot{z} = f(z, x, d) - \rho(x)x + \varpi$$
$$\dot{\varpi} = -\frac{\rho(x)}{dx}(f(z, x, d) - \rho(x)x + \varpi)$$

$$\varpi(t_k^n) = 0.$$
the equilibrium point col(z, x) = 0 of the system is globally asymptotically stable and Zeno-behavior is avoided.

Proof: Denote ξ = col(z, x). Let us first consider the ξ-dynamics. Let Δ(z) = m^2(z) + 1. By the changing supply function technique, there exists another ISS Lyapunov function for z-dynamics, \( V_z(z) \sim \{ \alpha_z, \alpha_z, \Delta(z) \| z \|^2, \varpi(x), z^2 \} \) for some functions \( \alpha_z, \alpha_z \in \mathcal{K}_{\infty} \) and \( \varpi \in \mathcal{S}_V \), that can be calculated accordingly. Let
\[
\rho(x) \geq [\kappa(x) + m_2(x) + 3/2]
\]
and the Lyapunov function candidate be
\[
V_z(\xi) = V_z(z) + x^2/2. \tag{30}
\]
The calculation of the derivative of \( V_z(\xi) \), along the trajectory of \( \xi \)-dynamics, obtains
\[
\dot{V}_z(\xi) \leq - \Delta(z) \| z \|^2 + \kappa(x) x^2 + x (m_1(z) \| z \| + m_2(x) \| x \| - \rho(x)x + \varpi(x)) \leq - \| \xi \|^2 + \varpi^2.
\]
Now, let us examine the \( \varpi \)-dynamics. Note that \( \varpi = \frac{dx(x(t), z(t))}{dt} = - \pi(x) \dot{x} \) where \( \tau(x) = \frac{d\rho(x)}{dt} + \rho(x) \). Let \( \dot{\tau}(x) = \tau(x) \) and decompose \( \dot{\tau} \) to be \( \dot{\tau}(x) = \tau_a(x) + \tau_c \) where \( \tau_a(0) = 0 \) and \( \tau_c \geq 0 \). As a result, \( \tau_a(x) \leq m_3(x) \| x \| \) for some function \( m_3 \in \mathcal{S}_V \). Let
\[
U(\varpi) = \frac{1}{2} \varpi^2. \tag{31}
\]
We claim that
\[
\dot{U}(\varpi) \leq \alpha_{\varpi}(\| \varpi \|) + \sigma_{\varpi}(\| \xi \|) \tag{32}
\]
with some functions \( \alpha_{\varpi}, \sigma_{\varpi} \in \mathcal{K}_{\infty} \). In fact, one possible calculation of the derivative of \( U(\varpi) \) is given below
\[
\dot{U}(\varpi) \leq \| \tau \| (m_1(\xi) \| z \| + m_2(x) \| x \| + \rho(x) \| x \| + \| \varpi \|) \leq \| \tau \| \left[ \alpha_{\varpi}(\| \varpi \|) + \sigma_{\varpi}(\| \xi \|) \right] \]
\[
\leq \alpha_{\varpi}(\| \varpi \|) + \sigma_{\varpi}(\| \xi \|). \tag{33}
\]

Let
\[
\nu(x) = \frac{1}{2} \varpi^2 + \sigma(\| \xi \|) \| \xi \|^2
\]
where \( \sigma(\| \xi \|) \) and \( \alpha(\| \varpi \|) \) are selected as
\[
\sigma(\| \xi \|) \geq \sigma_{\varpi}(\| \varpi \|) \| \varpi \|^2 \text{ and } \sigma_{\varpi}(\| \xi \|) \geq \alpha_{\varpi}(\| \varpi \|) \| \varpi \|^2. \tag{34}
\]
Then, (32) is satisfied.

Let function \( \alpha_{\varpi} \) be \( \alpha_{\varpi}(\| \xi \|) \geq 2\sigma_{\varpi}(\| \varpi \|) + \| \varpi \|^2 \). Since there exists an ISS Lyapunov function \( V_q(\xi) \) for (28), by the changing supply function method, there exists another ISS Lyapunov function \( V_q(\xi) \sim \{ \alpha_q, \alpha_q, \alpha_q, \alpha_q, \alpha_q \} \) for some \( \mathcal{K}_{\infty} \) functions \( \alpha_q, \alpha_q, \alpha_q \), and \( \alpha_q \) that are calculated accordingly. Moreover, let \( \sup_{x \to 0} \sigma_q(s) / s^2 > 0 \). Select the \( \mathcal{K}_{\infty} \) function \( \gamma \) as
\[
\gamma(s) \geq \sigma_q(s) / 2 + \alpha_q(s) + \frac{1}{2} s^2. \tag{35}
\]
Applying Theorem III.1 completes the proof.

As has been done in Section III, the PS law can be found by following the idea of Theorem IV.1. The proof is straightforward and omitted here.

Proposition IV.1: Consider the system composed of (25) and (27) under Assumption IV.1. Let functions \( V_q, \gamma, \alpha_m, \sigma_m \) be defined in the proof of Theorem IV.1. Let \( \mathcal{X} := \{ \xi \in \mathbb{R}^{n+1} \| \| \xi \| \leq \tau_c, \| x \| \leq q \} \) and \( R_0 = \max_{\xi, \varpi} \{ \alpha_m^{-1} \circ \sigma_m \circ \alpha_q^{-1}(V_q(\xi)) \} \) where \( \mathcal{X} \) is the closure of \( \mathcal{X} \). Find \( T \) such that
\[
2T \gamma(s) < \alpha_{\varpi}(s) \quad \forall 0 < s \leq R_0 \tag{36}
\]
the equilibrium point \( \xi = 0 \) of the system is asymptotically stable for any initial condition \( \xi(t_0) \in \mathcal{X} \).

The discussion in Remark III.6 shows that the initial condition of the signal \( \xi \) must be known for the design of PS. Although the signal \( z \) is not available for the feedback, Proposition IV.1 requires the initial condition of the signal \( z \) or at least its bound be known, which is used to estimate \( T \) in (36).

Remark IV.1: In [18] and [19], the decay rate of immeasurable states \( z \) is estimated by an auxiliary dynamic system and then used for constructing the event-triggered law. In comparison, a static sampling controller is proposed in Theorem IV.1, that is easier to design and implement in practice. Moreover, our method can also be used to derive the PS control law when the initial bound of the signal \( z \) is known.

V. EVENT-TRIGGERED ADAPTIVE CONTROL

In this section, we will solve two types of classical adaptive control problem in the event-triggered setting exploiting the sampling control scheme proposed in Section III.

A. Adaptive Control With Uncertain Parameters

We consider the event-triggered adaptive stabilization problem of a class of nonlinear systems with unknown parameters as follows:
\[
\dot{x} = f(x)\theta + u \tag{37}
\]
where \( x \in \mathbb{R} \) is the state and \( \theta \in \mathbb{R}^l \) is an unknown constant parameter vector. Note that function \( f(x) : \mathbb{R} \to \mathbb{R}^l \) does not necessarily vanish at \( x = 0 \). When \( f(0) = 0 \) and the bound of the uncertainty \( \theta \) is known, the high-gain approach in Section IV can be applied to solve the stabilization problem. In this section, we further consider the case that \( f(0) \neq 0 \) and \( \theta \) is bounded,
for which the high-gain approach is not applicable and adaptive control is required.

Assumption VI.1: 1) The unknown parameter $\theta$ is bounded with a known bound $\theta_0$, i.e., $\|\theta\| \leq \theta_0$; 2) $f(0) \neq 0$.

In this case, the argument in Remark II.1 does not apply and inequality (8) in Assumption II.1 might not hold. Therefore, the development of the controller design in Section III must be modified to suit the problem. Similar to the continuous-time adaptive control for system (37), we propose the event-triggered controller as follows:

$$
\dot{u} = \kappa \left( x(t_k), \hat{\theta}(t_k) \right), \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}
$$

$$
\dot{\hat{\theta}} = \Lambda x \rho(x) + \Lambda \varsigma \left( \hat{\theta}, \varpi \right)
$$

where $\kappa(x, \hat{\theta}) = -f^T(x) \hat{\theta} - cx + \varpi$ and $c \in \mathbb{R}^+, \Lambda > 0 \in \mathbb{R}^{l \times 1}$ is a diagonal matrix and the functions $\rho(x)$, $\varsigma(\hat{\theta}, \varpi)$ are to be designed. Note that the term $\varsigma(\hat{\theta}, \varpi)$ does not appear in traditional continuous-time adaptive control and is introduced particularly for the event-triggered control. In this article, we do not consider the challenging case where the adaptation dynamics $\dot{\theta}$ is sampled and it remains our future research direction.

Define the sampling error $\varpi(t)$ as $\varpi(t) = \kappa(x(t_k), \hat{\theta}(t_k)) - \kappa(x(t), \hat{\theta}(t))$, $t \in [t_k, t_{k+1})$. Then, the closed-loop system is rewritten as

$$
\begin{aligned}
\dot{x} &= -f^T(x) \hat{\theta} - cx + \varpi \\
\dot{\hat{\theta}} &= \Lambda x \rho(x) + \Lambda \varsigma \left( \hat{\theta}, \varpi \right) \\
\varpi(t_k) &= 0
\end{aligned}
$$

where $\hat{\theta} = \hat{\theta} - \theta$ is estimation error. Based on the event-triggered controller design in Section III, the following theorem is obtained.

**Theorem VI.1:** Suppose the system composed of (37) and (38) satisfies Assumption VI.1. Then, there exist smooth positive functions $\rho(x) : \mathbb{R} \to \mathbb{R}^+, \varsigma(\hat{\theta}, \varpi) : \mathbb{R}^T \times \mathbb{R} \to \mathbb{R}$ for the controller (38) and $\tilde{\gamma}(x, \theta, \varpi) : \mathbb{R} \times \mathbb{R}^T \times \mathbb{R} \to \mathbb{R}$ such that the event-triggered law is designed as

$$
t_{k+1} = \inf_{t > t_k} \left\{ 2t - t_k \right\} \max_{\tau \in [t_k, t]} \left\{ \tilde{\gamma}(x(\tau), \hat{\theta}(\tau), \varpi(\tau)) \varpi^2(\tau) \right\}
$$

$$
\geq \frac{1}{2} \varpi^2(t), \quad \|\varpi(t_k, t)\| \neq 0
$$

then Zeno behavior is avoided and the equilibrium point $x = 0$ is globally asymptotically stable. Moreover, $\lim_{t \to \infty} \hat{\theta}(t) = \theta$.

The algorithm of event-triggered adaptive controller design is summarized in Algorithm 1.

**Proof:** First, let us consider the $\varpi$-dynamics. Denote $\tau(x, \hat{\theta}) = \frac{\partial \tilde{\gamma}(x, \hat{\theta})}{\partial \hat{\theta}} + c$ and

$$
U(\varpi) = \frac{1}{2} \varpi^2.
$$

Note that

$$
\varpi = -\frac{d\kappa(x(t), \theta(t))}{dt} = \tau(x, \hat{\theta}) \dot{x} + f^T(x) \dot{\hat{\theta}}
$$

$$
= -\tau(x, \hat{\theta}) f^T(x) \dot{\hat{\theta}} - c \tau(x, \hat{\theta}) \dot{x} + x + \tau(x, \hat{\theta}) \varpi + f^T(x) \lambda x \rho(x) + f^T(x) \Lambda \varsigma \left( \hat{\theta}, \varpi \right).
$$

Decompose $f(x)$ as $f(x) = \tilde{g}_x(x) + \bar{g}_x$, where $\tilde{g}_x(x)$ depends on $x$ satisfying $\tilde{g}_x(0) = 0$ and $\bar{g}_x$ is a constant vector. Decompose $\tau(x, \hat{\theta})$ as $\tau(x, \hat{\theta}) = \tilde{\tau}_x(x) + \bar{\tau}_x(\hat{\theta})$ where $\tilde{\tau}_x(x)$ depends on $x$ satisfying $\tilde{\tau}_x(0) = 0$ and $\bar{\tau}_x(\hat{\theta})$ is a scalar possibly depending on $\hat{\theta}$. Denote $g_x(x) = \|\tilde{g}_x(x)\|$, $g_c = \|\bar{g}_c\|$, $\tau_c(x) = \|\tilde{\tau}_x(x)\|$, $\tau_c(\hat{\theta}) = \|\bar{\tau}_x(\hat{\theta})\|$ and $\chi(\hat{\theta}) = \|f^T(x)\lambda x\|$. Note that $\|\theta\|^2 \leq 2\|\theta\|^2 + 2\|\theta\|^2 \leq 2\theta^2_0 + 2\|\theta\|^2$. As a result

$$
\begin{aligned}
\dot{U}(\varpi) &= -\tau(x, \hat{\theta}) f^T(x) \dot{\hat{\theta}} - c \tau(x, \hat{\theta}) \dot{x} + x + \tau(x, \hat{\theta}) \varpi + f^T(x) \lambda x \rho(x) + f^T(x) \Lambda \varsigma \left( \hat{\theta}, \varpi \right) \\
&\leq [\tilde{\tau}_x(x)] \|f^T(x)\| \|\theta\| + g_x(x) \tau_c(\hat{\theta}) \|\theta\| \|\varpi\| - \bar{g}_c \tau_c(\hat{\theta}) \dot{\theta} \varpi \\
&\quad + \left( \bar{c}_x(\hat{\theta}) \|\varpi\| + \tau_c(\hat{\theta}) \right) \|x\| \varpi + \tau(x, \hat{\theta}) \varpi \varpi \\
&\quad + \chi(\hat{\theta}) \|\varpi\| \varpi + f^T(x) \Lambda \varsigma \left( \hat{\theta}, \varpi \right) \varpi \\
&\leq \tilde{\tau}_x^2(x) \|f^T(x)\|^2 / 4 + g^2(x) / 4 + c^2 / 4 \tau_c^2(\hat{\theta}) \varpi^2 \\
&\quad + \chi^2(\hat{\theta}) x^2 \varpi^2 + \varpi^2 \varpi^2 + \|\tau(x, \hat{\theta})\| \varpi \varpi + \|\rho(x)\| \varpi \varpi \\
&\quad - \bar{g}_c \tau_c(\hat{\theta}) \dot{\theta} \varpi + f^T(x) \Lambda \varsigma \left( \hat{\theta}, \varpi \right) \varpi.
\end{aligned}
$$

(42)

Since $\tau_x(0) = 0$ and $g_x(0) = 0$, $\tau_c(x) \leq m_1(x) |x|$ and $\bar{g}_x \leq m_2(x) |x|$ for some functions $m_1, m_2 \in \mathcal{S} \mathcal{N}$. Select two smooth positive functions $\sigma_\varpi$ and $\alpha_\varpi$ as

$$
\sigma_\varpi(x) \geq \frac{1}{4} m_1^2(x) \|f(x)\|^2 + \frac{1}{4} m_2^2(x) + c^2 / 4 \\hat{\tau}_c^2(\hat{\theta}) \varpi^2 + \frac{\tilde{c}^2}{4} + \chi^2(\hat{\theta}) / 4
$$

$$
\alpha_\varpi(x, \hat{\theta}) \geq \left( \|\theta\|^2 + \hat{\tau}_c^2(\hat{\theta}) \right) \left( 2 \theta^2_0 + 2 \|\theta\|^2 \right)
$$

Moreover, $\varpi_\varpi(x, \hat{\theta})$ is bounded when signals $x$ and $\hat{\theta}$ are bounded. As a result, it follows from (42) that

$$
\dot{U}(\varpi) \leq \sigma_\varpi(x) \varpi \varpi - \bar{g}_c \tau_c(\hat{\theta}) \dot{\theta} \varpi + f^T(x) \Lambda \varsigma \left( \hat{\theta}, \varpi \right) \varpi.
$$

It should be noted that the selection of $\sigma_\varpi(x)$ only depends on the function $f(x)$ and not on function $\rho(x)$ in $\theta$-dynamics. In other words, $\rho(x)$ can be selected to be dependent on $\sigma_\varpi(x)$.

Then, let us consider the $x$-dynamics. Let

$$
V_x(x) = \frac{1}{2} x^2
$$

The derivative of $V_x(x)$ along the $x$-dynamics is

$$
\dot{V}_x(x) = x \left( -f^T(x) \dot{\theta} - cx + \varpi \right) \leq -x^2 + \varpi^2 - x f^T(x) \dot{\theta}
$$

(45)
when \( c \geq 5/4 \). Following changing supply function lemma (in [6, Lemma 2.5]), for \( \Delta(x) = 2\bar{\sigma}_\varpi(x) + 1 \), there exists a non-decreasing function \( \rho_q \) such that

\[
\frac{1}{2} \rho_q \left( \frac{1}{2} x^2 \right) - \frac{1}{2} \bar{\sigma}_\varpi(x) + 1 \geq \Delta(x) \tag{46}
\]

and a new supply function

\[
V'_q(x) = \int_0^{V_q(x)} \rho_q(s) ds \tag{47}
\]

such that its derivative along the \( x \)-dynamics satisfies

\[
\dot{V'_q}(x) \leq -\Delta(x)x^2 + \sigma(\varpi)x^2 - \rho_q \left( \frac{1}{2} x^2 \right) x f^T(x) \theta \tag{48}
\]

for some \( \sigma \in \mathcal{P} \) satisfying \( \sigma(\varpi) \geq \rho_q(\varpi^2) \). The second inequality of (48) is due to (46) and facts in the following two cases:

1. \( x^2 \geq \bar{\varpi} \omega \): \( \dot{V'_q}(x) \leq -\frac{1}{2} \rho_q \left( \frac{1}{2} x^2 \right) - \rho_q \left( \frac{1}{2} x^2 \right) x f^T(x) \theta \); 
2. \( x^2 < \bar{\varpi} \omega \): \( \dot{V'_q}(x) \leq -\frac{1}{2} \rho_q \left( \frac{1}{2} x^2 \right) + \rho_q(\varpi^2) - \rho_q \left( \frac{1}{2} x^2 \right) x f^T(x) \theta \).

Since \( \bar{\sigma}_\varpi(x) \) is not dependent on \( \rho_q(x) \), so is function \( \rho_q(x) \). Then, one can choose \( \rho(x) \) and \( \zeta(\bar{\theta}, \varpi) \) in (38) to be

\[
\rho(x) = \rho_q \left( \frac{1}{2} x^2 \right) f(x) \tag{49}
\]

\[
\zeta(\bar{\theta}, \varpi) = \bar{\theta} \bar{\varpi} \bar{\tau}(\bar{\theta}) \varpi. \tag{50}
\]

Now, let us consider the \( \bar{\theta} \)-dynamics. Let

\[
\dot{\bar{\theta}} = \bar{\theta}^T \Lambda^{-1} \bar{\theta} / 2
\]

and its derivative along \( \bar{\theta} \)-dynamics becomes

\[
\dot{\bar{\theta}} = \rho_q \left( \frac{1}{2} x^2 \right) x f^T(x) \bar{\theta} + \bar{\theta}^T \bar{\varpi} \bar{\tau}(\bar{\theta}) \varpi. \tag{51}
\]

Denote \( \xi = \text{col}(x, \bar{\theta}) \). Letting

\[
V(\xi) = V'_q(x) + V_{\bar{\theta}}(\bar{\theta}) \tag{52}
\]

leads to

\[
\dot{V}(\xi) = \bar{\theta}^T \Lambda^{-1} \bar{\theta} / 2 + \bar{\theta}^T \bar{\varpi} \bar{\tau}(\bar{\theta}) \varpi. \tag{53}
\]

By the selection of \( \rho(x) \) and \( \zeta(\bar{\theta}, \varpi) \) in (49), inequality (44) further becomes

\[
U(\varpi) \leq \bar{\sigma}_\varpi(x) x^2 + \bar{\sigma}_\varpi(x, \bar{\theta}) \varpi^2 + \bar{\varpi} \bar{\tau}(\bar{\theta}) \bar{\theta} \varpi \tag{54}
\]

where

\[
\alpha_\varpi(x, \bar{\theta}) \geq \bar{\alpha}_\varpi(x, \bar{\theta}) + \| f^T(x) \Lambda \bar{\varpi} \bar{\tau}(\bar{\theta}) \| \tag{55}
\]

Finally, the event-triggered law design is presented. Let

\[
\bar{\gamma}(x, \bar{\theta}, \varpi) = \sigma(\varpi)/2 + \alpha_\varpi(x, \bar{\theta}) + c_1 \tag{56}
\]

where \( \sigma(\varpi) \) is given in (48) and \( c_1 = 1/2 \). One has

\[
U(\varpi(t)) = \int_{t_k}^{t} \dot{U}(\varpi(\tau)) d\tau \tag{57}
\]

\[
\leq \int_{t_k}^{t} \left( \bar{\sigma}_\varpi(x(\tau)) x^2(\tau) - \bar{\varpi} \bar{\tau}(\bar{\theta}(\tau)) \bar{\theta}(\tau) \varpi(\tau) \right) d\tau \tag{58}
\]

The above inequality is due to (57) and the second inequality is due to (58). Therefore, (40) implies that \( t_{k+1} \geq t_k + \min \{ t_{k,c}, t_{c} \} > t_k \), and thus, it is Zeno free. The convergence of the signal \( \text{col}(x(t), \varpi(t)) \) to zero follows similar argument in the proof of Theorem III.1. When \( x \equiv 0 \) and \( \varpi \equiv 0 \), it shows that \( \lim_{t \to \infty} f^T(x(t)) \bar{\theta}(t) = 0 \). Because \( f(0) \) is PE, it is...
Algorithm 1: Event-Triggered Adaptive Control.

1. Select smooth positive functions $\bar{\sigma}$ and $\bar{\sigma}$ to satisfy (43);
2. Find a positive function $\rho_2$ to satisfy (46) where $\Delta(x) = 2\bar{\sigma}(x) + 1$ and $\bar{\sigma}$ to satisfy (48);
3. Choose $\rho(x)$ and $\varsigma(\theta, \bar{\sigma})$ to be in (49);
4. Find a positive function $\bar{\sigma}(x, \hat{\theta})$ to satisfy (54) and $\gamma(x, \hat{\theta}, \bar{\sigma})$ to be in (55);
5. Construct the event-triggered law in (40).

proved that $\lim_{t \to \infty} \hat{\theta}(t) = 0$ or $\lim_{t \to \infty} \hat{\theta}(t) = \theta$ by [6, Lemma 2.4].

Remark V.1: In [13], the event-triggered adaptive control problem is solved when the global Lipschitz condition is assumed on the function $f(x)$, while it is not required in our method. This is because we explicitly consider the error dynamics of $\bar{\sigma}$ in the event-triggered controller design, which allows us to directly take into account of the impact of nonlinearity $f(x)$ on the error $\bar{\sigma}$. It is noted from the proof that the additional term $\varsigma(\theta, \bar{\sigma})$ in adaptation dynamics (38) plays an important role of canceling the term appearing in the bound of $\dot{V}(x)$ in (53) and leads to global asymptotical stabilization. It would be very interesting to consider that the adaptation dynamics is also sampled as shown in [13] in our future work. Compared with the development in Section III, the derivative of $\dot{V}(x)$ is upper bounded as in (44) which is more complicated than (8) in Assumption II.1. Therefore, the event-triggered law in (40) is also more involved and not merely determined by $\bar{\sigma}$ but also $x$ and $\theta$.

B. Event-Triggered Stabilization With Dynamic Gain

In this section, we consider the dynamic system

$$\begin{align*}
\dot{z} &= q(z, x, d) \\
\dot{x} &= f(z, x, d) + bu, \ b > 0
\end{align*}$$

(59)

where $z \in \mathbb{R}^n$ and $x \in \mathbb{R}$ are state variables, $u \in \mathbb{R}$ is the input, and $d \in \mathbb{D}$ is the uncertainties belonging to a compact set $\mathbb{D} \subset \mathbb{R}$. The functions $q$ and $f$ are sufficiently smooth with $q(0, 0, d) = 0$ and $f(0, 0, d) = 0$ for all $d \in \mathbb{D}$. In contrast to Section IV, the size of $\mathbb{D}$ and controller gain $b$ are not known and a dynamic gain technique is required. The state $x$ is also assumed to be not available for feedback control, and thus, $z$-dynamics is the dynamic uncertainty. A few assumptions are listed as follows.

Assumption V.2: $0 < b \leq \bar{b}$ for some known constant $\bar{b}$.

Assumptions V.3: The $z$-subsystem in (25) has an ISS-Lyapunov function $V(z) \sim \{a, \bar{a}, \alpha, \sigma \mid z = q(z, x, d)\}$ and functions $\alpha(s)$ and $\sigma(s)$ satisfy $\limsup_{s \to 0} s^2/\alpha(s) < \infty$, $\limsup_{s \to 0} \sigma(s)/s^2 < \infty$. Moreover, the functions $a$, $\bar{a}$, $\alpha$, and a known function $\sigma$ such that $\sigma = \sigma\alpha$.

Note that system (59) differs from that studied in Section IV in some aspects. First, in Section IV, since the controller gain is unity, we can apply sufficient high gain specified by $\rho(x)$ to dominate uncertainties and stabilize the system. Here, the controller gain $b$ is not known. Although the upper bound of $b$ is known, it is still not possible to calculate the controller gain that is sufficiently high to stabilize the system. Second, although the $z$-subsystem in (59) admits an ISS-Lyapunov function, the input gain $\sigma$ is only known to a constant factor. Third, since the size of $\mathbb{D}$ is unknown, by [6, Corollary 11.1], there exists a positive number $c$, which depends on the size of $\mathbb{D}$ and is also unknown, and two positive and sufficiently smooth known functions $m_1$ and $m_2$, such that

$$|f(x, z, d)| \leq cm_1(z)z + cm_2(x)x| \quad \forall d \in \mathbb{D}. \tag{60}$$

These three differences call for the dynamic gain stabilization technique.

For the continuous-time stabilization of system (59), dynamic gain controller using universal adaptive control technique is proposed in [6]. It takes the form of

$$\begin{align*}
\dot{\theta} &= \dot{\theta} = \lambda \rho(x)x^2, \ \lambda > 0
\end{align*}$$

(61)

where $\rho(x)$ is to be specified. The second equation of (61) is the adaptation dynamics for dynamic gain. For the event-triggered control, we adopt the method developed in Section III and propose the controller as follows:

$$\begin{align*}
u(t) &= \kappa(x(t_k), \theta(t_k)), \ t \in [t_k, t_{k+1}), \ k \in \mathbb{N}
\end{align*}$$

(62)

Define the sampling error $\bar{x}(t)$ as $\bar{x}(t) = \kappa(x(t_k), \theta(t_k)) - \kappa(x(t), \theta(t)), \ t \in [t_k, t_{k+1})$. Then, the event-triggered closed-loop system is rewritten as

$$\begin{align*}
\dot{\bar{x}} &= -\partial x(\theta) \hat{\theta} - \partial \kappa(x, \hat{\theta}) \hat{\theta} \\
\bar{x}(t_k^+) &= 0
\end{align*}$$

(63)

Theorem V.2: Suppose the system composed of (59) and (62) satisfies Assumption V.2 and V.3. Then, there exist smooth positive functions $\rho(x) : \mathbb{R} \to \mathbb{R}_{>0}$, and $\gamma(x, \theta, \bar{\sigma}) : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_{>0}$ such that event-triggered law is designed as

$$\begin{align*}
t_{k+1} &= \inf_{t > t_k} \left\{ 2(t - t_k) \max_{r \in [t_k, t_k^+]} \left\{ \gamma(x(r), \theta(r), \bar{x}(r)) \bar{x}^2(r) \right\} \right\} \\
&\geq \frac{1}{2} \|\bar{x}(t_k^+)\|^2, \text{ and } \|\bar{x}(t_k^+)\| \neq 0
\end{align*}$$

(64)

then the equilibrium point $\text{col}(z, x) = 0$ of the system is globally asymptotically stable and Zeno behavior is avoided.

Proof: Denote $\xi = \text{col}(z, x)$ and $\Xi = \text{col}(z, x, \theta)$. First, let us examine $\Xi$-dynamics. By changing supply function method, for $\Delta(z) = m_2^2(z) + 1$, there exists another ISS Lyapunov function, $V_2(z) \sim \{a_2, \bar{a}_2, \Delta(z)\|z\|^2, \sigma_2(x)x^2 \mid \xi = q(z, x, d)\}$ for some $\sigma_2$ functions $\bar{a}_2, a_2$, and a smooth function $\sigma_2$ are calculated accordingly. Let $\hat{\theta} = \theta - \theta$ where $\theta$ is a positive
number to be specified later. Define the Lyapunov function candidate as
\[ V(z, x, \hat{\theta}) = V_z(z) + x^2/2 + b\hat{\theta}^2/(2\lambda). \]  
(65)

The calculation of the derivative of \( V(z, x, \hat{\theta}) \), along the trajectory of (63), shows that
\[ \dot{V}(z, x, \hat{\theta}) \leq -\Delta(z)\|z\|^2 + \kappa(x)x^2 + x(cm_1(z))|z| \]
\[ + cm_2(x)|x| - b\hat{\theta} \rho(x)x + b\kappa \]
\[ \leq -\|z\|^2 + (\kappa(x) + c^2/4 + cm_2(x) + b^2 q/4) \]
\[ - b\hat{\theta} \rho(x)x^2 + \omega^2/q \]  
(66)

where \( q \) is to be specified. Let \( \rho(x) \) be
\[ \rho(x) = \max\{1, \kappa(x), m_2(x)\} \]
and \( \theta \) be
\[ \theta \geq (1 + c^2/4 + c)/b + bq/4. \]

As a result, one has
\[ \dot{V}(z, x, \hat{\theta}) \leq -\|\xi\|^2 + \omega^2/q. \]  
(67)

Next, the \( \omega \)-dynamics is examined as follows:
\[ \dot{\omega} = -\frac{dn(x(t), z(t))}{dt} = \hat{\theta} \rho(x)x + \hat{\theta} \frac{d\rho(x)}{dx} x \dot{x} + \hat{\theta} \rho(x) \dot{x} \]
\[ = \lambda \rho^2(x)\dot{x}^3 + \hat{\theta} \pi(x) \left( f(z, x, d) - b\hat{\theta} \rho(x)x + b\omega \right) \]
where \( \pi(x) = \frac{d\rho(x)}{dx}x + \rho(x) \). Let \( \tilde{\tau}(x) = |\tau(x)| \) and decompose \( \tilde{\tau}(x) \) as \( \tilde{\tau}(x) = \tau_2(x) + \tau_\epsilon(x) \) where \( \tau_\epsilon(x) = 0 \) and \( \tau_2(x) \geq 0 \). As a result, \( \tau_2(x) \leq m_3(x)|x| \) for some function \( m_3 \in \mathcal{S}_N \). Let
\[ U(\omega) = \frac{1}{2} \omega^2 \]  
(68)

and one has
\[ U(\omega) \leq |\omega||\dot{\omega}| \leq |\omega| \left( |\lambda \rho^2(x)| |x|^3 + b\hat{\theta}^2 \pi(x) \rho(x) |x| \right) \]
\[ + b\tilde{\tau}(x)|\theta| |\omega| + \hat{\theta} \pi(x) m_2(x) |x| \]
\[ + c^2(\hat{\theta} \tau_2(x))^2/4 + c\hat{\theta} \tau_\epsilon(x) m_3(x) |x| + m_1(x) |x| \]
\[ \leq \lambda \rho^2(x)|x|^6 + b\hat{\theta}^2 \pi(x) \rho(x) |x|^2 \]
\[ + c^2(\hat{\theta} \tau_2(x))^2/4 + c\hat{\theta} \tau_\epsilon(x) m_3(x) |x| + m_1(x) |x| \]
\[ \leq \lambda \rho^2(x)|x|^6 + b\hat{\theta}^2 \pi(x) \rho(x) |x|^2 \]
\[ + c^2(\hat{\theta} \tau_2(x))^2/4 + c\hat{\theta} \tau_\epsilon(x) m_3(x) |x| + m_1(x) |x| \]
\[ + (\hat{\theta} \tau_2(x))|\theta| + 2\hat{\theta}^2 + 2 \omega^2 \]  
(69)

Let
\[ \alpha_q(|x|) = \rho^2(x)|x|^6 + \pi^2(x)\rho^2(x)|x|^2 \]
\[ + \pi_1^2(x) + \pi_2(x) m_2(x)|x|^2 \]
\[ \alpha_c = \max \left\{ \lambda^2/2, b^2/4, c^2/4, c^4/64 \right\} \]
\[ \beta(|\|z\||) = m_1^2(z) |z|^4 + m_1^2(z) |z|^2 \]
\[ \beta_c = \max \left\{ 1/4, c_1^2/4 \right\} \]
\[ \tilde{\alpha}(x, \hat{\theta}) = (\hat{\theta} \tau_2(x))|\theta| + 2\hat{\theta}^2 + 2 \omega^2 + 2 \]  
(69)

where \( \alpha_c \) and \( \beta_c \) are unknown. As a result, one has
\[ \dot{U}(\omega) \leq \alpha_c \alpha_q(|x|) + \beta_c (|\|z\||) + \tilde{\alpha}(x, \hat{\theta}) \omega^2 \]
which further leads to
\[ \dot{U}(\omega) \leq \tilde{\alpha} \tilde{\sigma}(\xi) |\xi|^2 + \alpha_c (x, \hat{\theta}) \omega^2 \]  
(70)

where \( \tilde{\alpha} = \max \left\{ \alpha_c, \beta_c \right\} \) is an unknown constant, and
\[ \tilde{\sigma}(\xi) = \rho^4(x)|x|^4 + \pi^2(x)\rho^2(x) + m_3^2(x)|x|^2 \]
\[ + \pi^2(x) m_2^2(x) + m_1^2(z) |z|^4 + m_1^2(z) |z|^2 \]

Let
\[ V(z, x, \hat{\theta}) = \begin{cases} 1 + 1/b \end{cases} V(z, x, \hat{\theta}) \]  
(71)

where \( V(z, x, \hat{\theta}) \) is given in (65) and its derivative also satisfies
\[ \dot{V}(z, x, \hat{\theta}) \leq -\|\xi\|^2 + \omega^2/q. \]

By Parameterized Changing Supply in Lemma A.1, for any smooth function \( \Delta(\Xi) \geq \tilde{\sigma}(\xi) + 1 \) and \( k = \max \{ \tilde{\alpha} \beta, 1 \} \), there exists a new supply function \( V_\Xi(\Xi) \) such that
\[ V_\Xi(\Xi) \leq -k \Delta(\Xi) |\xi|^2 + \bar{p} \tilde{\kappa}(\hat{\theta}, \omega) \omega^2/q \]
\[ \leq -k \Delta(\Xi) |\xi|^2 + \bar{p} \tilde{\kappa}(\hat{\theta}, \omega) \omega^2/q \]  
(72)

for some sufficiently smooth functions \( \tilde{\kappa}, \tilde{\kappa} \) and unknown constant \( \bar{p} \). Note that \( k \Delta(\Xi) \geq 2\alpha \tilde{\sigma}(\xi) + 1 \) and we specify \( q \) to be \( q \leq \tilde{p} \tilde{\kappa}(\hat{\theta}) \). It leads to
\[ V_\Xi(z, x, \hat{\theta}) \leq -k \Delta(\Xi) |\xi|^2 + \tilde{\kappa}(\hat{\theta}, \omega) \omega^2. \]  
(73)

Now, it is ready to design the event-triggered law. Let
\[ \gamma(x, \hat{\theta}, \omega) = \tilde{\kappa}(\hat{\theta}, \omega)/2 + \tilde{\alpha}(x, \hat{\theta}) + \frac{1}{2} \]
where \( \tilde{\alpha}(x, \hat{\theta}) \) is given in (69) and \( \tilde{\kappa}(\hat{\theta}, \omega) \) is given in (72). One has
\[ U(\omega(t)) = \int_{t_k}^t \dot{U}(\omega(t)) dt \leq \int_{t_k}^t \tilde{\alpha} \tilde{\kappa}(\xi(t)) |\xi(t)|^2 dt \]
\[ + \int_{t_k}^t \tilde{\kappa}(x(t), \hat{\theta}(t)) \omega^2 dt, \ t \in [t_k, t_{k+1}] \]

which together with event-triggered law (64) (similar to the proof of Theorem III.1) implies
\[ U(\omega(t)) \leq 2 \int_{t_k}^t \tilde{\alpha}(\xi(t)) |\xi(t)|^2 dt \]
\[ - \frac{1}{2} \tilde{\kappa}(\hat{\theta}(t), \omega(t)) \omega^2(t) - \frac{1}{2} \omega^2(t), \ t \in [t_k, t_{k+1}] \]

Let
\[ V(\Xi, \omega) = V_\Xi(z, x, \hat{\theta}) + U(\omega) \]  
(74)

be the Lyapunov function candidate for the closed-loop system (63). As a result
\[ V(\Xi(t), \omega(t)) - V(\Xi(t_{k+1}), \omega(t_{k+1})) \]
\[ \leq - \int_{t_k}^{t_{k+1}} |\xi(t)|^2 dt - \int_{t_k}^{t_{k+1}} \omega^2(t) dt \ \forall t \in [t_k, t_{k+1}] \]
Therefore, the bound of \( V(\Xi, \omega) \) is monotonically decreasing and all signal \( z, x, \hat{\theta}, \omega \) are bounded. The convergence and Zeno freeness can be proved similar to that of Theorems III.1 and IV.1 and are omitted here.  

Remark V.2: With \( x \) as the output, the systems in (25), (37), and (59) have relative degree of one. It is worth mentioning that the proposed sampling control method can be extended to handle the class of uncertain lower triangular systems of relative degree greater than one via the backstepping technique that is widely used in literature such as [6].
VI. NUMERICAL EXAMPLES

Example VI.1 (Robust Sampling Control of the Lorenz System): Consider the uncertain Lorenz system
\[
\dot{z} = \begin{bmatrix} -\alpha & 0 \\ x & -\beta \end{bmatrix} z + \begin{bmatrix} \alpha x \\ 0 \end{bmatrix}
\]
\[
\dot{x} = \begin{bmatrix} 1 & 0 \end{bmatrix} (\rho - \begin{bmatrix} 0 & 1 \end{bmatrix} z) - x + u
\]
where \( z = [z_1, z_2]^T \in \mathbb{R}^2 \), \( x \in \mathbb{R} \), and \( \rho, \alpha, \beta > 0 \) are unknown parameters. The sampling controller is designed as \( u(t) = -k x(t_k) \), \( t \in [t_k, t_{k+1}] \) for some \( k \) depending on the size of uncertainties. Note that this controller does not use any information from dynamic uncertainty, namely the \( z \)-dynamics. Define the sampling error \( \omega(t) = u(t_k) - u(t) \). According to Theorem IV.1, one can design the event-triggered mechanism (ETM) as in (29) with
\[
\gamma(s) = s^2 \max \left\{ 0, c_1 s^2 + c_2 + (k + 1)^2 + \rho_{\max}^2 - 1 \right\}
\]
for some parameters \( c_1 \) and \( c_2 \) depending on the size of uncertainties. In the simulation, we assume \( \alpha \in [0.5, 0.8], \beta \in [0.5, 1] \), and \( \rho \in [0.3, 0.8] \). Then, we design \( k = 0.25, c_1 = 3.2, \) and \( c_2 = 2.7 \).

For comparison, we use the method in [18] to design the event-triggered law for the Lorenz system (75) based on the sampling error of the state \( x \). Due to different controller design method in [18], the sampling controller differs from ours and becomes \( u(t) = -k_1 x(t_k) - k_2 x(t_k) \), \( t \in [t_k, t_{k+1}] \) for some \( k_1, k_2 > 0 \) depending on the size of uncertainties. Moreover, it is required to design an auxiliary dynamics as \( \hat{\eta} = -\Phi(\eta) \) with the initial value \( \eta(0) \) to be specified. Then, the event-triggered law is designed as \( t_{k+1} = \inf \{ t > t_k \mid \| \omega_x(t) \| \geq |\eta(t)| \} \) where \( \omega_x(t) = x(t_k) - x(t) \) is the sampling error of the state. Note that in order to specify \( \eta(0) \), the initial value of \( z \) must be known, while this is not required in our method for the event-triggered control.

Moreover, Proposition IV.1 can be used to design a PS controller, while [18] cannot. Let the initial condition be \( [z, x] = [0.1, 0.7, 0.5] \). According to Proposition IV.1, the period for the PS law in (22) can be calculated as \( T = 0.077s \). The samplings numbers for different sampling mechanisms in the simulation during \( t \in [0, 30] \) is summarized in Table I and the state trajectories are shown in Fig. 1.

It can be found that the ETM in [18] samples less but causes chattering state trajectory. This phenomenon can also be observed in the numerical example in [18]. Comparably, the state trajectories of event-triggered and PS control in this article are smoother, although a few more samplings are required for the ETM in (29). Compared with the PS mechanism in (22), the ETM in (29) can save the sampling numbers and thus is more efficient. It is observed that the minimal sampling interval is \( 0.345 \) s.

Example VI.2 (Event-triggered Adaptive Stabilization with Uncertain Parameters): Consider the uncertain system
\[
\dot{x} = \theta f(x) + u
\]
where \( x \in \mathbb{R}, f(x) = x^2 + 1, \) and \( \theta \) is an unknown parameter whose bounds is \( |\theta| < 0.5 \). Since \( f(x) \) is not globally Lipschitz, the method in [13] cannot apply. For comparison reason, we will use the method in [31] where global Lipschitz condition is not required, but it can merely achieve uniform bounded stability where the trajectories only converge to a small neighborhood of the origin. We propose the sampling controller as follows:
\[
u(t) = -\hat{\theta}(t_k)f(x(t_k)) - cx(t_k), t \in [t_k, t_{k+1}]
\]
\[
\dot{\hat{\theta}} = \lambda x \rho(x) + \lambda c \hat{\theta}(\omega, \varpi)
\]
with \( \rho(x) \) and \( \varsigma(\hat{\theta}, \varpi) \) to be designed and \( \lambda > 0 \) to be specified. Let the sampling error \( \varpi(t) \) be \( \varpi(t) = -f(x(t_k))\hat{\theta}(t_k) - cx(t_k) + f(x(t))\hat{\theta}(t) + cx(t), t \in [t_k, t_{k+1}] \). Let \( U(\varpi) = \frac{1}{2} \varpi^2 \) and inequality (53) is satisfied with \( U(\varpi) \leq \dot{\alpha}(\varpi) + \lambda (\omega \varsigma(\hat{\theta}, \varpi) - c\varpi) \) with \( \alpha = 1 + \lambda^2/4, b_s = 2(1 + \lambda^2/4) + \left( \frac{1}{4} + c^2 \right) \)
\[
c_s = (1 + \lambda^2/4) + \frac{c^2}{4}
\]
and \( \alpha(\varpi, \hat{\theta}) = (\| \varpi \|^2 + c^2)(0.5 + 2\| \hat{\theta} \|^2) + \| \hat{\theta} \|^2 - (2x\hat{\theta} + c) + |\rho(x)|^2 \). Let \( V(x) = \frac{1}{2}x^2 \) and \( \dot{V}_s \leq -x^2 + \varpi^2 - x\theta f(x) \). Let \( \Delta(x) = \tilde{a}_s s^4 + \tilde{b}_s s^2 + \tilde{c}_s \) with \( \tilde{a}_s = 2a_s, \tilde{b}_s = 2b_s, \tilde{c}_s = \)}
2c_\alpha + 1 \text{ and, thus, } \rho_\theta \text{ in } (46) \text{ should be } \rho_\theta = \bar{a}_s s^2 + 4\bar{b}_s \sigma + \bar{c}_s \text{ and } \sigma(\varpi) = \rho_\theta(\omega^2). \text{ Then, } \rho(x) \text{ and } \zeta(\bar{\theta}, \varpi) \text{ in } (38) \text{ is } \rho(x) = \rho_\theta\left(\frac{2}{s^2}\right)(s^2 + 1) \text{ and } \zeta(\bar{\theta}, \varpi) = c\varpi \omega. \text{ As a result, } \bar{\alpha}_\varpi(x, \bar{\theta}) \geq \bar{\alpha}_\varpi(x, \bar{\theta}) + c\lambda(x^2 + 1) \text{ is selected to make } (54) \text{ satisfied. Then, } \bar{\gamma}(x, \bar{\theta}) \text{ can be selected as in } (55). \text{ Finally, the event-triggered law is designed according to } (40) \text{ in Theorem V.1.}

According to the method in [31], we can design the following event-triggered controller:

\[ u(t) = \bar{m} \tanh\left(\frac{\bar{m}x(t_k)}{\epsilon}\right) - cx(t_k), \quad t \in [t_k, t_{k+1}) \]

\[
\dot{\bar{\theta}} = \lambda f(x) - \lambda \sigma \bar{\theta} \tag{76}
\]

with event-triggering law as

\[ t_{k+1} = \inf \{ |e(t)| \geq m \} \]

where \( e(t) = \bar{m} \tanh(\bar{m}x(t)/\epsilon) - cx(t) - \bar{m} \tanh(\bar{m}x(t_k)/\epsilon) + cx(t_k) \). The design parameters \( \bar{m}, m \) must satisfy \( \bar{m} > m > 0 \). The ultimate bound can be adjusted by parameters \( \bar{m}, m, \epsilon \).

The simulation result of our controller is illustrated in Fig. 2, which includes figures for state trajectories, input signal and sampling intervals. For \( t \in [0, 15] \), the sampling number is 1605 and the minimal sampling interval is 2 ms. Note that the estimated value \( \bar{\theta} \) converges to the real value of \( \bar{\theta} = 0.3 \) and the trajectories of system converge to the origin.

The simulation result of the event-triggered controller (76) is illustrated in Fig. 3. It is shown that trajectories of system only converge to a small neighborhood of the origin and the estimated value \( \bar{\theta} \) cannot converge to real value, while the controller input has a strong chattering. Although it has less sampling number

1169, the sampling number will significantly increase above ours when the control parameters are adjusted to decrease the ultimate bound.

**Example VI.3 (Event-triggered Stabilization with Dynamic Gain):** Consider the following nonlinear system:

\[
\dot{z} = -z + w_1x,
\]

\[
\dot{x} = w_1 \sin x + w_2x + bu
\]

where \( w_1, w_2 \), and \( w_3 \) are unknown parameters and \( 0 < b < 1 \). Note that when \( V(z) = z^2 \), one has \( \dot{V}(z) \leq -z^2 + px^2 \) for any unknown constant \( p \geq w_3^2 \), and thus, Assumption V.3 is satisfied. And \( |w_1 \sin x + w_2x| \leq c(|z| + |x|) \) for some \( c \geq \max\{|w_1|, |w_2|\} \), that is, inequality (60) is verified for \( m_1 = 1 \) and \( m_2 = 1 \). Then, we propose the sampling controller as in (61) and the closed-loop system is written as

\[
\dot{z} = -z + w_3x,
\]

\[
\dot{x} = w_1 \sin x + w_2x - b\tilde{\theta} \rho(x) x + b\varpi
\]

\[
\dot{\varpi}(t_k) = 0
\]

where \( \varpi(t) = -\bar{\theta}(t_k) \rho(x(t_k))x(t_k) + \tilde{\theta}(t) \rho(x(t))x(t) \) is the sampling error. Let \( V(z, x, \bar{\theta}) = \frac{1}{2}x^2 + \frac{1}{4}\lambda \tilde{\theta}^2 + \frac{1}{2}b^2q/4 + 2p - b\rho(x) \) and \( \dot{V}(z, x, \bar{\theta}) \leq -|z|^2 + (\varepsilon^2/4 + c + b^2q/4 + 2p - b\rho(x) \) \( x^2 + \varpi^2/q \). Select \( \rho(x) = 1 \) and \( \theta \geq (\varepsilon^2/4 + c + 2p)/b + bq/4 \). Then, \( \dot{V}(z, x, \bar{\theta}) \leq -\|z\|^2 + \varpi^2/q \). Note that \( \varpi = -\bar{\theta}x - 2\tilde{\theta} \dot{x} \) and let \( U(\varpi) = \frac{1}{2}\varpi^2 \). Inequality (70) is satisfied, i.e., \( \dot{\varpi}(\varpi) \leq \bar{\alpha}_\beta \tilde{\theta}(\xi)\|\xi\|^2 + \alpha(\bar{\theta})\varpi^2 \) where

\[
\alpha(\bar{\theta}) = 1 + \bar{\theta}^2 + \bar{\theta}^4 + 2b\bar{\theta}
\]

\[
\bar{\alpha}_\beta = \max \left\{ \lambda^2/4, \varepsilon^2 + b^2 \right\}
\]

\[
\tilde{\theta}(\xi) = 1 + \|\xi\|^4.
\]

Note that \( \frac{1}{2}x^2 + \frac{1}{4}\lambda \tilde{\theta}^2 + 2\varepsilon^2 \leq \dot{V}(z, x, \bar{\theta}) = (1 + 1/b) \) \( V(z, x, \bar{\theta}) \leq s(\frac{1}{2}x^2 + \frac{1}{2}\lambda \tilde{\theta}^2 + 2\varepsilon^2) \) where \( s \geq \max\{1 + 1/b, 1 + b\} \). Let \( \bar{k} = \max\{2\bar{\alpha}_\beta, 1\} \) and \( \Delta(\xi) = \bar{\sigma}(\xi) + 1 \). Let
design of PS and event-triggered control law utilizes the supremum norm of sampling error and renders the error dynamics BIBS, when coupled with system dynamics, achieves global or semiglobal stabilization. The proposed framework is then extended to tackle the event-triggered and PS stabilization for systems with system uncertainties. The proposed framework is further extended to solve two classes of event-triggered universal adaptive control problems. It would be interesting to consider the case where the adaptation dynamics is also sampled and periodic event-triggered control along this research line.

APPENDIX

Lemma A.1: (Parameterized Changing Supply Functions, Lemma 6.1 in [6]). Consider the system \( \xi = f(\xi, \varpi) \) with \( \xi = \text{col}(z, x) \in \mathbb{R}^n \) and \( \varpi \in \mathbb{R}^m \). Suppose there exists a supply function \( V(\xi) \) satisfying

\[
\alpha(||\xi||) \leq V(\xi) \leq \bar{\alpha}(||\xi||, s)
\]

\[ V(x) \leq -\alpha(||x||) + p\sigma(||\varpi||) \]

for \( K_\infty \) functions \( \alpha, \bar{\alpha}, \sigma \), a parameterized \( K_\infty \) function \( \bar{\alpha} \), and positive number \( p \) and \( s \). Then, for any smooth function \( \Delta : \mathbb{R}^n \rightarrow [0, \infty) \) and positive number \( k \), there exists a continuously differentiable function \( V'(\xi) \) satisfying

\[
\alpha'(||\xi||) \leq V'(\xi) \leq \bar{\alpha}'(||\xi||, s')
\]

\[ V'(x) \leq -k\Delta(\xi)\alpha(||x||) + p'\varpi(z, \varpi)\sigma(||\varpi||) \]

for a \( K_\infty \) functions \( \alpha', \bar{\alpha}', \sigma \), a parameterized \( K_\infty \) function \( \bar{\alpha}' \), a \( \mathcal{S}N \) function \( \varpi \), and positive numbers \( p' \) and \( s' \). Moreover, if the functions \( \alpha, \varpi, \bar{\alpha}, \sigma \) and \( \Delta \) are known, so are the functions \( \alpha', \bar{\alpha}', \varpi \). The positive numbers \( s, p, k, s' \), and \( p' \) are not necessarily known.

Proof of Theorem III.2: It suffices to show condition (24) imply (23) and meanwhile the boundedness of signal \( \varpi \) is guaranteed, i.e., \( ||\varpi(t)|| \leq R_0 \), \( \forall t \geq t_0 \). From (24), one has \( \omega_{\varpi,\varpi}(s) \leq 2\gamma(s) > 0, \forall 0 < s < R_0 \). Since \( \omega_{\varpi,\varpi} \) and \( \gamma \) are continuous functions, there exists a sufficiently small \( \bar{R} \) such that

\[ \omega_{\varpi,\varpi}(R_0 + s) - 2\gamma(R_0 + s) > 0 \quad \forall 0 < s \leq \bar{R}. \]

It, together with (24), further implies

\[ 2\gamma(s) < \omega_{\varpi,\varpi}(s) \quad \forall 0 < s \leq R_0 + \bar{R}. \]

We will prove that \( ||\varpi(t)|| \leq R_0 \), \( \forall t \geq t_0 \). If this is not true, there exists a finite time \( t_a > t_0 \) such that \( ||\varpi(t)|| \leq R_0 + \bar{R}, \forall t \in [t_a, t_b] \) but \( ||\varpi(t)|| > R_0 \). Due to (77) and the PS law in (22), one have

\[ 2(t-t_a)\gamma(||\varpi||_{t_a,t}) \leq 2T\gamma(||\varpi||_{t_a,t}) < \omega_{\varpi,\varpi}(||\varpi||_{t_a,t}) \]

\[
\leq \max_{t \in [t_a, t_b]} \{U(\varpi(t))\} \quad \forall t \in [t_a, t_b] + T
\]

\[ \forall t_i = \{t_0 + t_0 + \ldots + t_0 + (k + 1)T\} \]

where \( k = \max_{t_a} \{k | t_0 + (k + 1)T \leq t_a \} \). Therefore, (23) is valid for \( t < t_a \) and \( T \leq t_k - t_{k-1}, \forall k \in \mathbb{N} \) where \( t_k-1 \) is given in (21). By Remark III.5, results of Theorem III.1 hold, and one has \( V_q(x(t)) \leq V_q(x(t_0)) \leq \omega_{\varpi} \circ \omega_{\varpi}^{-1} \circ \omega_{\varpi}(R_0) \), that is \( \mathbb{S} \subset \mathbb{R}^n \) is a positively invariant set. By Remark III.2, one has \( ||\varpi(t)|| \leq R_0, \forall t_0 \leq t \leq t_a \) which causes a contradiction to \( ||\varpi(t_a)|| > R_0 \).
Therefore, $\|c(t)\| \leq R_0$, $\forall t \geq t_0$. As a result, conducting similar analysis as above for $t \geq t_0$ shows that (23) is valid for $t \geq t_0$ and using the discussion in Remark III.5 completes the proof.

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