VIRTUAL ENRICHING OPERATORS

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Abstract. We construct bounded linear operators that map $H^1$ conforming Lagrange finite element spaces to $H^2$ conforming virtual element spaces in two and three dimensions. These operators are useful for the analysis of nonstandard finite element methods.

1. Introduction

Let $\Omega \in \mathbb{R}^d$ ($d = 2, 3$) be a bounded polygonal/polyhedral domain, $\mathcal{T}_h$ be a simplicial triangulation of $\Omega$ and $V_h \subset H^1(\Omega)$ be the Lagrange $P_k$ finite element space with $k \geq 3$. The mesh dependent semi-norm $\| \cdot \|_h$ is defined by

$$
\|v\|_h^2 = \|D^2_h v\|_{L^2(\Omega)}^2 + J(v,v) \quad \forall v \in V_h,
$$

where $D^2_h v$ is the piecewise Hessian of $v$ with respect to $\mathcal{T}_h$, and

$$
J(w,v) = \sum_{e \in \mathcal{E}^i_h} h_e^{-1} \int_e [\partial w/\partial n] [\partial v/\partial n] ds \quad \text{for } d = 2,
$$

$$
J(w,v) = \sum_{F \in \mathcal{F}^i_h} h_F^{-1} \int_F [\partial w/\partial n] [\partial v/\partial n] dS \quad \text{for } d = 3.
$$

Here $\mathcal{E}^i_h$ (resp., $\mathcal{F}^i_h$) is the set of interior edges (resp., faces) of $\mathcal{T}_h$, $h_e$ (resp., $h_F$) is the diameter of the edge $e$ (resp., face $F$), $[\partial w/\partial n]$ is the jump of the normal derivative across an edge $e$ (resp., face $F$), and $ds$ (resp., $dS$) is the infinitesimal length (resp., area).

Our goal is to construct a linear operator $E_h : V_h \rightarrow H^2(\Omega)$ such that

$$
\|v - E_h v\|_h \leq C^\flat \sqrt{J(v,v)} \quad \forall v \in V_h,
$$

$$
\sum_{\ell=0}^2 h^\ell \| \zeta - E_h \Pi_h \zeta \|_{H^\ell(\Omega)} \leq C^\flat h^{k+1} \| \zeta \|_{H^{k+1} (\Omega)} \quad \forall \zeta \in H^{k+1}(\Omega),
$$

where $\Pi_h : C(\bar{\Omega}) \rightarrow V_h$ is the Lagrange nodal interpolation operator, and the positive constants $C^\sharp$ and $C^\flat$ only depend on the shape regularity of $\mathcal{T}_h$ and $k$. Moreover, the operator $E_h$ maps $V_h \cap H_0^1(\Omega)$ into $H^2(\Omega) \cap H_0^1(\Omega)$.

Enriching operators that satisfy (1.4) and (1.5) are useful for a priori and a posteriori error analyses for fourth order elliptic problems [8, 17, 6, 7, 9], and they also play an important role in fast solvers for fourth order problems [4, 5, 10]. A recent application to Hamilton-Jacobi-Bellman equations can be found in [21].

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In the two dimensional case, one can construct $E_h$ through the $C^1$ macro finite elements in [14, 13, 23, 22, 15]. This was carried out in [8] for the quadratic element and in [17] for higher order elements. However macro elements of order higher than 3 are not available in three dimensions and therefore this approach can only be carried out for quadratic and cubic Lagrange elements (cf. [21]) using the three dimensional cubic Clough-Tocher macro element from [25].

We take a different approach in this paper by connecting the $k$-th order Lagrange finite element space to a $k$-th order $H^2$ conforming virtual element space. In two dimensions such spaces are already in the literature [11, 12], and we will develop a version of three dimensional $H^2$ conforming virtual element spaces that are sufficient for the construction of $E_h$.

**Remark 1.1.** The assumption that the order of the Lagrange finite element space is at least 3 allows a uniform construction of $E_h$. For the quadratic Lagrange finite element space we can simply take $E_h$ to be the restriction of the cubic enriching operator.

The rest of the paper is organized as follows. The construction of $E_h$ in two dimensions is carried out in Section 2, followed by the construction in three dimensions in Section 3 and some concluding remarks in Section 4. Appendix A contains some technical results concerning inverse trace theorems that are needed for the construction of $H^2$ conforming virtual elements.

A list of notations and conventions that will be used throughout the paper is provided here for convenience.

- A polygon/polyhedron is an open subset in $\mathbb{R}^2/\mathbb{R}^3$, an edge of a polygon/polyhedron does not include the endpoints and a face of a polyhedron does not include the vertices and edges. These conventions apply in particular to triangles and tetrahedra.
- Let $G$ be an open line segment, a triangle or a tetrahedron, and $k$ be an integer. $P_k(G)$ is the space of polynomials of total degree $\leq k$ restricted to $G$ if $k \geq 0$ and $P_k(G) = \{0\}$ if $k < 0$. We say that two functions $u$ and $v$ defined on $G$ have identical moments up to order $\ell$ if the integral of $(u - v)q$ on $G$ vanishes for all $q \in P_\ell(G)$.
- The orthogonal projection from $L_2(G)$ onto $P_k(G)$ is denoted by $Q_{G,k}$.
- $V_h$ is the set of all the vertices of the triangles/tetrahedra in $T_h$, $V^i_h$ is the set of vertices in $\Omega$ and $V^b_h$ is the set of vertices on $\partial \Omega$.
- $E_h$ is the set of all the edges of the triangles/tetrahedra in $T_h$, $E^i_h$ is the set of edges in $\Omega$ and $E^b_h$ is the set of edges that are subsets of $\partial \Omega$.
- $F_h$ is the set of all the faces of the tetrahedra in $T_h$, $F^i_h$ is the set of faces in $\Omega$ and $F^b_h$ is the set of faces that are subsets of $\partial \Omega$.
- $T_p$ is the set of all the triangles/tetrahedra in $T_h$ that share $p$ as a common vertex.
- $T_e$ is the set of all the triangles/tetrahedra in $T_h$ that share $e$ as a common edge.
- $T_F$ is the set of all the tetrahedra in $T_h$ that share $F$ as a common face.
- $F_e$ is the set of the faces of the tetrahedra in $T_h$ that share $e$ as a common edge.
- If $v$ is a function defined on a triangle/tetrahedron, then $v_e$ (resp., $v_F$) is the restriction of $v$ to an edge $e$ (reps., a face $F$).
- If $v$ is a function defined on a triangle/tetrahedron, then $\partial v/\partial n$ denotes the outward normal derivative of $v$ along $\partial T$. In the case of a triangle (resp. tetrahedron), $\partial v/\partial n$ is double-valued at the vertices (resp., edges) of $T$. 
If \( e \) is an edge of the triangle \( T \), then \( n_{e,T} \) is the unit vector orthogonal to \( e \) and pointing towards the outside of \( T \). If \( e \) is an edge of a face \( F \) of a tetrahedron \( T \), then \( n_{e,F} \) is the unit vector tangential to \( F \), orthogonal to \( e \) and pointing towards the outside of \( F \).

If \( F \) is a face of the tetrahedron \( T \), then \( n_{F,T} \) is the unit vector orthogonal to \( F \) and pointing towards the outside of \( T \).

2. The Two Dimensional Case

The construction of \( E_h \) is based on the characterizations of trace spaces associated with a triangle and the construction of polynomial data on the skeleton of \( T_h \) that satisfy these characterizations on all the triangles of \( T_h \).

2.1. Trace Spaces for a Triangle. Let \( T \) be a triangle with vertices \( p_1, p_2 \) and \( p_3 \), \( e_i \) be the edge of \( T \) opposite \( p_i \), \( n_i \) be the unit outer normal along \( e_i \), and \( t_i \) be the counterclockwise unit tangent of \( e_i \). Let \( \ell \) be a nonnegative number. A function \( u \) belongs to the piecewise Sobolev space \( H^\ell(\partial T) \) if and only if \( u_i \), the restriction of \( u \) to \( e_i \), belongs to \( H^\ell(e_i) \) for \( 1 \leq i \leq 3 \). It follows from the Sobolev Embedding Theorem [1, Theorem 4.12] that we can define a linear operator \( \text{Tr} : H^2(T) \rightarrow H^{\frac{3}{2}}(\partial T) \times H^{\frac{1}{2}}(\partial T) \) by

\[
\text{Tr} \zeta = \langle \zeta, \partial \zeta / \partial n \rangle|_{\partial T},
\]

where the restrictions of \( \zeta \) and \( \partial \zeta / \partial n \) are in the sense of trace and defined piecewise with respect to the edges. For the subspace \( H^{\frac{3}{2}}(T) \) of \( H^2(T) \), we have \( \text{Tr} H^{\frac{3}{2}}(T) \subset H^2(\partial T) \times H^1(\partial T) \). Our first task is to identify the image of \( H^{\frac{3}{2}}(T) \).

**Definition 2.1.** A pair \((f, g) \in H^2(\partial T) \times H^1(\partial T)\) belongs to the space \((H^2 \times H^1)(\partial T)\) if and only if the following conditions are satisfied:

\[
f_j(p_i) = f_k(p_i) \quad \text{for } 1 \leq i \leq 3 \text{ and } j, k \in \{1, 2, 3\} \setminus \{i\},
\]

and there exist \( w_1, w_2, w_3 \in \mathbb{R}^2 \) (which depend on \((f, g)\)) such that

\[
(\partial f_j / \partial t_j)(p_i) = w_i \cdot t_j \quad \text{for } 1 \leq i \leq 3 \text{ and } j \in \{1, 2, 3\} \setminus \{i\},
\]

\[
g_j(p_i) = w_i \cdot n_j \quad \text{for } 1 \leq i \leq 3 \text{ and } j \in \{1, 2, 3\} \setminus \{i\}.
\]

Note that the compatibility conditions (2.3)–(2.4) are equivalent to

\[
(\partial f_j / \partial t_j) t_j + g_j n_j = (\partial f_k / \partial t_k) t_k + g_k n_k \quad \text{at } p_i
\]

for \( 1 \leq i \leq 3 \) and \( j, k \in \{1, 2, 3\} \setminus \{i\} \).

It follows from the Sobolev Embedding Theorem that \( \text{Tr} \zeta \in (H^2 \times H^1)(\partial T) \) for \( \zeta \in H^{\frac{3}{2}}(T) \), where \( w_i = \nabla \zeta(p_i) \), and we can recover \( \nabla \zeta \) on \( \partial T \) from \((f, g) = \text{Tr} \zeta \) by

\[
\nabla \zeta = (\partial f_i / \partial t_i) t_i + g_i n_i \quad \text{on } e_i \text{ for } 1 \leq i \leq 3.
\]

We want to show that in fact \( \text{Tr} H^{\frac{3}{2}}(T) = (H^2 \times H^1)(\partial T) \). For this purpose it is useful to construct a linear isomorphism \( \Phi : (H^2 \times H^1)(\partial T) \rightarrow (H^2 \times H^1)(\partial T) \) such that

\[
\text{Tr} (\zeta \circ \Phi) = \Phi^* (\text{Tr} \zeta) \quad \forall \zeta \in H^{\frac{3}{2}}(T),
\]
where Φ is an orientation preserving affine transformation that maps the triangle ˜T onto T. We assume that Φ maps the vertex ˜p_i of ˜T to the vertex p_i of T and hence it also maps the edge ˜e_i of ˜T to the edge e_i of T.

First we note that, by the chain rule,
\[ \nabla(\zeta \circ \Phi) = J_\Phi(\nabla \zeta \circ \Phi) \quad \forall \zeta \in H^{1/2}(T), \]
where \( J_\Phi \) (a constant 2 \times 2 matrix with a positive determinant) is the Jacobian of \( \Phi \).

Let \((f, g) \in H^{1/2}(T)\). Motivated by (2.6)–(2.8), we define \( \Phi^*(f, g) = (\tilde{f}, \tilde{g}) \), where
\[ \tilde{f} = f \circ \Phi, \]
and
\[ \tilde{g} = J_\Phi^*(g \circ \Phi) \cdot \tilde{n}_i \quad \text{on} \ e_i \quad \text{for} \ 1 \leq i \leq 3, \]
where \( \tilde{n}_i \) is the outward pointing unit normal along the edge ˜e_i and the vector field \( g \) on \( \partial T \) is given by
\[ g = (\partial f_i/\partial t_i) \mathbf{t}_i + g_i \mathbf{n}_i \quad \text{on} \ e_i \quad \text{for} \ 1 \leq i \leq 3. \]

It is straightforward to check that \((\tilde{f}, \tilde{g}) \in (H^2 \times H^1)(\partial \tilde{T})\), \( \Phi^* : (H^2 \times H^1)(\partial T) \rightarrow (H^2 \times H^1)(\partial \tilde{T}) \) is a bijection, and that (2.7) follows from (2.6) and (2.8)–(2.11).

We are now ready to characterize \( \text{Tr} \ H^{1/2}(T) \).

**Lemma 2.2.** The image of \( H^{1/2}(T) \) under \( \text{Tr} \) is the space \( (H^2 \times H^1)(\partial T) \).

**Proof.** We already know that \( \text{Tr} \ H^{1/2}(T) \subset (H^2 \times H^1)(\partial T) \). In the other direction, we want to construct \( \zeta \in H^{1/2}(T) \) that satisfies (2.1) for a given \((f, g) \in (H^2 \times H^1)(\partial T)\).

If \( f \) and \( g \) vanish near the vertices, we can use the operator \( L_1 \) in Lemma A.1 and cut-off functions to obtain \( \zeta \). Therefore, by using a partition of unity, we can reduce the construction to a neighborhood of a vertex and, by an affine transformation (cf. (2.7)), we can further assume that the angle at the vertex is a right angle. The existence of \( \zeta \) near such a vertex then follows from Lemma A.3. \( \square \)

2.2. **Affine Invariant \( H^2 \) Virtual Element Spaces.** The construction of the virtual element spaces involves polynomial subspaces of \( (H^2 \times H^1)(\partial T) \).

**Definition 2.3.** Let \( T \) be a triangle. We will denote the intersection of \( (H^2 \times H^1)(\partial T) \) and \( P_k(\partial T) \times P_{k-1}(\partial T) \) by \( (H^2 \times H^1)_{k,k-1}(\partial T) \).

**Remark 2.4.** It follows from the compatibility conditions (2.2)–(2.4) that \((f, g) \in (H^2 \times H^1)_{k,k-1}(\partial T)\) is determined by (i) the values of \( f \) at the vertices, (ii) the tangential derivatives of \( f \) at the vertices, (iii) moments of \( f \) on \( e_i \) up to order \( k - 4 \) that together with (i) and (ii) determine \( f_i \in P_k(e_i) \), and (iv) moments of \( g \) up to order \( k - 3 \) that together with (ii) (through (2.4)) determine \( g_i \in P_{k-1}(e_i) \). These degrees of freedom (dofs) are depicted in Figure 2.1 for \( k = 3 \) and 4, where (i) the values of \( f \) at the vertices and the moments of \( f \) on the edges are represented by solid dots, and (ii) the tangential derivatives of \( f \) at the vertices and the moments of \( g \) on the edges are represented by arrows. Altogether we have \( \dim(H^2 \times H^1)_{k,k-1}(\partial T) = 6(k - 1) \).
Remark 2.5. Since polynomial spaces are preserved by an affine transformation, the map \( \Phi^*: (H^2 \times H^1)(\partial T) \rightarrow (H^2 \times H^1)(\partial \hat{T}) \) defined by (2.9)–(2.10) maps \((H^2 \times H^1)_{k,k-1}(\partial T)\) one-to-one and onto \((H^2 \times H^1)_{k,k-1}(\partial \hat{T})\).

2.2.1. Virtual Element Spaces on the Reference Triangle. We begin with a simple well-posedness result for the biharmonic problem.

Lemma 2.6. Given any \((f,g) \in (H^2 \times H^1)(\partial T)\) and \(\rho \in L^2(T)\), there exists a unique \(\xi \in H^2(T)\) such that
\[
(\Delta \xi, \Delta z)_{L^2(T)} = (\rho, z)_{L^2(T)} \quad \forall z \in H^2_0(T) \quad \text{and} \quad \text{Tr} \xi = (f,g).
\]

Proof. Let \(\zeta \in H^2_0(T)\) satisfy (2.1) and \(\eta \in H^2_0(T)\) be defined by
\[
(\Delta \eta, \Delta z)_{L^2(T)} = (\rho - \Delta \zeta, z)_{L^2(T)} \quad \forall z \in H^2_0(T).
\]
Then \(\xi = \eta + \zeta\) is the unique solution of (2.6). \(\square\)

Let \(\hat{T}\) be the reference triangle with vertices \((0,0), (1,0)\) and \((0,1)\). In view of Lemma 2.6 and the fact that \((H^2 \times H^1)_{k,k-1}(\partial \hat{T})\) is a subspace of \((H^2 \times H^1)(\partial \hat{T})\), we can now define the reference virtual element spaces \(\mathcal{V}^k(\hat{T})\), which are identical to the virtual element spaces in [11] for the special case of the reference triangle.

Definition 2.7. A function \(\hat{\xi} \in H^2(\hat{T})\) belongs to the virtual element space \(\mathcal{V}^k(\hat{T})\) if and only if \(\text{Tr} \hat{\xi} \in (H^2 \times H^1)_{k,k-1}(\partial \hat{T})\) and the distributional derivative \(\Delta^2 \hat{\xi}\) belongs to \(P_{k-4}(\hat{T})\), i.e., there exists \(\hat{\rho} \in P_{k-4}(\hat{T})\) such that
\[
(\Delta \hat{\xi}, \Delta \hat{z})_{L^2(\hat{T})} = (\hat{\rho}, \hat{z})_{L^2(\hat{T})} \quad \forall \hat{z} \in H^2_0(\hat{T}).
\]

Remark 2.8. According to Remark 2.4 and Lemma 2.6, we have
\[
\dim \mathcal{V}^k(\hat{T}) = \dim (H^2 \times H^1)_{k,k-1}(\partial \hat{T}) + \dim P_{k-4}(\hat{T}) = \frac{k^2 + 7k - 6}{2}.
\]

The following result is well-known (cf. [11]). We provide a proof here for self-containedness.

Lemma 2.9. A function \(\hat{\xi} \in \mathcal{V}^k(\hat{T})\) is uniquely determined by \(\text{Tr} \hat{\xi} \in (H^2 \times H^1)_{k,k-1}(\partial \hat{T})\) and \(Q_{\hat{T},k-4}\hat{\xi} \in P_{k-4}(\hat{T})\).
Proof. In view of Remark 2.8, it suffices to show that \( \hat{\xi} = 0 \) is the only function in \( \mathcal{V}^k(\hat{T}) \) with the properties that \( \text{Tr} \hat{\xi} = (0,0) \) and \( Q_{T,k-4}\hat{\xi} = 0 \). Indeed, using integration by parts and the fact that the distributional derivative \( \Delta^2 \hat{\xi} \in P_{k-4}(\hat{T}) \), we have
\[
(\Delta \hat{\xi}, \Delta \hat{\xi})_{L_2(\hat{T})} = (\Delta^2 \hat{\xi}, \hat{\xi})_{L_2(\hat{T})} = 0.
\]
Therefore \( \hat{\xi} \in H^2(\hat{T}) \) is a harmonic function that vanishes on \( \partial \hat{T} \) and hence \( \hat{\xi} = 0 \). \( \square \)

Remark 2.10. The definition of the virtual element space \( \mathcal{V}^k(\hat{T}) \) relies on the fact that \((H^2 \times H^1)_{k,k-1}(\partial \hat{T})\) is a subspace of \( \text{Tr} H^2(\hat{T}) \subset \text{Tr} H^2(\hat{T}) \). One can show by using macro elements of order \( k \) that a pair \((f, g) \in P_k(\partial T) \times P_{k-1}(\partial \hat{T})\) satisfying the compatibility conditions (2.2)–(2.4) automatically belongs to \( \text{Tr} H^2(\hat{T}) \). Hence Lemma 2.2 is not necessary for the definition of the virtual element space \( \mathcal{V}^k(\hat{T}) \) in two dimensions. However, the definition of the virtual element spaces in three dimensions requires the characterization of the trace of \( H^2(\hat{T}) \) for the reference tetrahedron \( \hat{T} \), since macro elements of arbitrary order are not available. The approach here provides a preview of the three dimensional case.

2.2.2. Virtual Element Spaces for a General Triangle. We now define \( \mathcal{V}^k(T) \) for an arbitrary triangle \( T \) in terms of \( \mathcal{V}^k(\hat{T}) \).

Definition 2.11. Let \( T \) be an arbitrary triangle and \( \Phi \) be an orientation preserving affine transformation that maps \( \hat{T} \) onto \( T \). Then \( \xi \in \mathcal{V}^k(T) \) if and only if \( \xi \circ \Phi \in \mathcal{V}^k(\hat{T}) \).

Remark 2.12. The definition of \( \mathcal{V}^k(T) \) is independent of the choice of \( \Phi \). The polynomial space \( P_k(T) \) is a subspace of \( \mathcal{V}^k(T) \) since \( P_k(\hat{T}) \) is obviously a subspace of \( \mathcal{V}^k(\hat{T}) \). The dimension of \( \mathcal{V}^k(T) \) is also given by the formula in (2.14).

We have an analog of Lemma 2.9.

Lemma 2.13. A function \( \xi \) in \( \mathcal{V}^k(T) \) is uniquely determined by \( \text{Tr} \xi \in (H^2 \times H^1)_{k,k-1}(\partial T) \) and \( Q_{T,k-4}\xi \in P_{k-4}(T) \).

Proof. This is a direct consequence of (2.7), Remark 2.5, Lemma 2.9 and the relation \( Q_{T,k-4}(\xi \circ \Phi) = (Q_{T,k-4}\xi) \circ \Phi \). \( \square \)

Remark 2.14. Our definition of \( \mathcal{V}^k(T) \), which is invariant under affine transformations, differs from the one in [11] for a general triangle. The affine invariance simplifies the proofs of (1.4) and (1.5) in Section 2.4. We note that it is also possible to use the virtual finite element spaces from [11] in the construction of \( E_h \). But then the proofs of (1.4) and (1.5) will become more involved.

Remark 2.15. The definition of \( H^2 \) virtual element spaces on polygons and their applications to the plate bending problem can be found in [11, 12].

2.3. Construction on the Skeleton \( \Gamma = \bigcup_{T \in T_h} \partial T \). Given \( v \in V_h \), our goal is to define \( f_{v,T} \) representing (the desired) \( E_h v \big|_{\partial T} \) and \( g_{v,T} \) representing (the desired) \( (\partial E_h v / \partial n) \big|_{\partial T} \) for all \( T \in T_h \), such that
\[
(f_{v,T}, g_{v,T}) \in (H^2 \times H^1)_{k,k-1}(\partial T) \text{ for all } T \in T_h,
\]
and the following conditions are satisfied:

(2.16) if $T_1, T_2 \in \mathcal{T}_p$ (resp., $T_1, T_2 \in \mathcal{T}_e$), then $f_{v,T_1}(p) = f_{v,T_2}(p)$ (resp., $f_{v,T_1} = f_{v,T_2}$ on $e$),

(2.17) if two distinct $T_1$ and $T_2$ belong to $\mathcal{T}_e$, then $g_{v,T_1} + g_{v,T_2} = 0$ on $e$,

(2.18) if $e \in \mathcal{E}_h^b$ is an edge of $T$ and $v \in H^1_0(\Omega)$, then $f_{v,T} = 0$ on $e$.

Note that (2.16) and (2.17) imply any piecewise $H^2$ function $\xi$ satisfying $(\xi_T, \partial \xi_T / \partial n)|_{\partial T} = (f_{v,T}, g_{v,T})$ for all $T \in \mathcal{T}_h$ will belong to $H^2(\Omega)$, and (2.18) implies that $\xi \in H^2(\Omega) \cap H^1_0(\Omega)$ if $v \in H^1_0(\Omega)$.

2.3.1. Construction at the Vertices. In view of the compatibility conditions (2.3) and (2.4), we need to define vectors $w_p \in \mathbb{R}^2$ associated with the vertices $p$ of $\mathcal{T}_h$. There are three cases: (i) $p$ is an interior vertex, (ii) $p$ is boundary vertex that is not a corner of $\Omega$ and (iii) $p$ is a corner of $\Omega$.

Case (i) For an interior vertex $p$, we define $w_p$ to be $\nabla v_T$, where $T$ is any triangle in $\mathcal{T}_p$.

Case (ii) For a boundary vertex $p$ that is not a corner of $\Omega$, we define $w_p$ to be $\nabla v_T$, where $T$ is one of the triangles in $\mathcal{T}_p$ that has an edge on $\partial \Omega$. This choice ensures that $w_p \cdot t = 0$ if $v \in H^1_0(\Omega)$, where $t$ is any vector tangential to $\partial \Omega$ at $p$.

Case (iii) At a corner $p$ of $\Omega$, we define $w_p$ by

\begin{equation}
(2.19) \quad w_p \cdot t_i = (\partial v / \partial t_i)(p) \quad \text{for } i = 1, 2,
\end{equation}

where $e_1, e_2 \in \mathcal{E}_h^b$ are the two edges emanating from $p$ and $\partial / \partial t_i$ is the derivative in the direction of the unit tangent $t_i$ of $e_i$. Note that $w_p = 0$ at a corner $p$ of $\Omega$ if $v \in H^1_0(\Omega)$.

The choices of the triangles and tangent vectors in Case (i)–Case (iii) are illustrated in Figure 2.2.

**Figure 2.2.** Triangles and tangent vectors in the definition of $w_p$

**Remark 2.16.** If the condition

\begin{equation}
(2.20) \quad \nabla v_{T_1}(p) = \nabla v_{T_2}(p) \quad \forall T_1, T_2 \in \mathcal{T}_p
\end{equation}

is satisfied at a vertex $p$, then obviously $w_p = \nabla v_T(p)$ for all $T \in \mathcal{T}_p$. 
2.3.2. **Construction on the Edges.** On any edge $e \in \mathcal{E}_h$, we define a polynomial $g_e \in P_{k-1}(e)$ as follows: First we choose $T \in \mathcal{T}_e$ and then we specify that

\begin{align}
(2.21) & \quad g_e(p) = w_p \cdot n_{e,T} \text{ at an endpoint } p \text{ of } e. \\
(2.22) & \quad g_e \text{ and } \partial v_T / \partial n \text{ have the same moments up to order } k - 3 \text{ on } e.
\end{align}

2.3.3. **Construction on the Triangles.** We are now ready to define $(f_{v,T}, g_{v,T}) \in H^2(\partial T) \times H^1(\partial T)$ for any $T$ as follows. Given any edge $e$ of $T$, the function $f_{v,T}$ on $e$ is the unique polynomial in $P_k(e)$ with the following properties:

\begin{align}
(2.23) & \quad f_{v,T} \text{ agrees with } v \text{ at the two endpoints of } e \text{ and shares the same moments up to order } k - 4 \text{ with } v, \\
(2.24) & \quad \text{the directional derivative of } f_{v,T} \text{ at an endpoint } p \text{ of } e \text{ in the direction of the tangent } t_e \text{ of } e \text{ is given by } w_p \cdot t_e.
\end{align}

**Remark 2.17.** If the condition (2.20) is satisfied at both endpoints of $e$, then $w_p = \nabla v_T(p)$ at the two endpoints $p$ of $e$ by Remark 2.16 and then conditions (2.23) and (2.24) imply $f_{v,T} = v$ on $e$.

Given any edge $e$ of $T$, we define

\begin{align}
(2.25) & \quad g_{v,T} = g_e \text{ if } T \text{ is the triangle chosen in the definition of } g_e \text{ (cf. Section 2.3.2),} \\
& \quad \text{otherwise } g_{v,T} = -g_e.
\end{align}

**Remark 2.18.** If the condition (2.20) is satisfied at both endpoints of $e$ and $v$ is $C^1$ across $e$, then Remark 2.16 and (2.21)–(2.22) imply that $g_{v,T} = \partial v_T / \partial n$ on $e$.

By construction, the condition (2.15) is satisfied because the compatibility conditions (2.2)–(2.4) follow from (2.21) and (2.23)–(2.24). The condition (2.16) follows from (2.23)–(2.24) and the condition (2.17) follows from (2.25). The choices we make in the definition of $w_p$ for $p \in \partial \Omega$ (cf. Case (ii) and Case (iii) in Section 2.3.1 and (2.23)–(2.24)) also implies (2.18).

2.4. **The Operator** $E_h$. Let $v \in V_h$ and $T \in \mathcal{T}_h$ be arbitrary, and $(f_{v,T}, g_{v,T}) \in (H^2 \times H^1)_{k,k-1}(\partial T)$ be the function pair constructed in Section 2.3. We define $E_h v \in \mathcal{T}^k(T)$ by the following conditions (cf. Lemma 2.13):

\begin{align}
(2.26) & \quad (E_h v, \partial E_h v / \partial n) = (f_{v,T}, g_{v,T}) \text{ on } \partial T \quad \text{and} \quad Q_{T,k-4}(E_h v) = Q_{T,k-4}(v).
\end{align}

It follows from (2.16)–(2.17) that the piecewise $H^2$ function $E_h v$ belongs to $H^2(\Omega)$, and (2.18) implies $E_h v \in H^1_0(\Omega)$ if $v \in H^1_0(\Omega)$. It only remains to establish the estimates (1.4) and (1.5).

Note that Remark 2.17 and Remark 2.18 imply

\begin{align}
(2.27) & \quad (f_{v,T}, g_{v,T}) = (v_T, \partial v_T / \partial n) \text{ on } \partial T \text{ if } v \text{ is } C^1 \text{ on } \partial T,
\end{align}

and hence $v = E_h v$ if $v$ is $C^1$ on $\partial T$, which is the rationale behind (1.4) and (1.5).

**Theorem 2.19.** *The estimate* (1.4) *holds with a positive constant* $C_s$ *that only depends on* $k$ *and the shape regularity of* $\mathcal{T}_h$. 

Proof. All the constants (explicit or hidden) that appear below will only depend on the minimum angle of $T_h$.

Let $T \in T_h$ be arbitrary. In view of Remark 2.4, Lemma 2.13 and the equivalence of norms on finite dimensional vector spaces, we have, by scaling,

$$\|\xi\|^2_{L^2(T)} \approx \|Q_{T,k-4}\xi\|^2_{L^2(T)} + \sum_{e \in E_T} h_T^2|Q_{e,k-4}\xi|^2_{L^2(e)} + h_T^3|Q_{e,k-3}(\partial\xi/\partial n)|^2_{L^2(e)},$$

(2.28)

$$+ \sum_{p \in V_T} \left(h_T^2|\nabla(v - E_h v)(p)|^2 + h_T^4|\nabla\xi(p)|^2\right) \quad \forall \xi \in \mathcal{V}^k(T),$$

where $h_T$ is the diameter of $T$ and $V_T$ (resp., $E_T$) is the set of the three vertices (resp., edges) of $T$. Moreover the affine invariance of $\mathcal{V}^k(T)$ (cf. Definition 2.11) together with (2.10) and (2.11) implies that the hidden constants in (2.28) only depend on the shape regularity of $T$.

It follows from (2.23), (2.26) and (2.28) that

$$\|v - E_h v\|^2_{L^2(T)} \approx \sum_{p \in V_T} h_T^4|\nabla(v - E_h v)(p)|^2 + \sum_{e \in E_T} h_T^3|Q_{e,k-3}(\partial v - E_h v)/\partial n|^2_{L^2(e)};$$

(2.29)

and we also have, by the construction of $w_p$ in Section 2.3.1, (2.22), (2.24), and (2.25),

$$|\nabla(v - E_h v)(p)|^2 \leq C_1 \sum_{e \in E_p} h_T|\nabla v/\partial n|^2_{L^2(e)},$$

$$|Q_{e,k-3}(\partial v - E_h v)/\partial n|^2_{L^2(e)} \leq \|\nabla v/\partial n\|^2_{L^2(e)},$$

where $E_p$ is the set of all the edges in $E_T$ that share $p$ as a common vertex, and hence

$$\|v - E_h v\|^2_{L^2(T)} \leq C_2 h_T^3 \sum_{p \in V_T} \sum_{e \in E_p} \|\nabla v/\partial n\|^2_{L^2(e)};$$

(2.30)

We then deduce from (2.30) and scaling that

$$\|D^2(v - E_h v)\|^2_{L^2(T)} \leq C_3 h_T^{-1} \sum_{p \in V_T} \sum_{e \in E_p} \|\nabla v/\partial n\|^2_{L^2(e)}.$$

(2.31)

Note that, because of the affine invariance of $\mathcal{V}^k(T)$, the scaling constants behind (2.31) only depend on the shape regularity of $T$.

The estimate (1.4) follows immediately from (1.1) and (2.31). □

Theorem 2.20. The estimate (1.5) holds with a positive constant $C_5$ that only depends on $k$ and the shape regularity of $T_h$. □

Proof. Let $T \in T_h$ be arbitrary and $S_T$ (the star of $T$) be the interior of the union of the closures of all the triangles in $T_h$ that share a common vertex with $T$. If $\zeta \in H^{k+1}(\Omega)$ belongs to $P_k(S_T)$, then $\Pi_h \zeta = \zeta$ in $S_T$ and hence $\zeta - E_h \Pi_h \zeta = \Pi_h \zeta - E_h \Pi_h \zeta = 0$ on $T$ by (2.30). The estimate (1.5) can then be established through the Bramble-Hilbert lemma [3, 16]. □

3. The Three Dimension Case

The construction of $E_h$ in three dimensions follows the same strategy as in Section 2, and our treatment will be brief regarding the results and arguments that are (almost) identical with the two dimensional case.
3.1. Trace Spaces for a Tetrahedron. Let \( T \) be a tetrahedron with vertices \( p_1, p_2, p_3, p_4 \), and \( F_i \) be the face of \( T \) opposite \( p_i \). Let \( \ell \) be a nonnegative number. A function \( u \) belongs to the piecewise Sobolev space \( H^\ell(\partial T) \) if and only if \( u_i \), the restriction of \( u \) to \( F_i \), belongs to \( H^\ell(F_i) \) for \( 1 \leq i \leq 4 \).

For a function \( \phi \) defined on a face \( F \) of the tetrahedron \( T \), the planar gradient \( \nabla_F \phi \) is defined by

\[
\nabla_F \phi = \nabla \tilde{\phi} - (\nabla \tilde{\phi} \cdot n_{F,T}) n_{F,T},
\]

where \( \tilde{\phi} \) is any extension of \( \phi \) to a neighborhood of \( F \) in \( \mathbb{R}^3 \).

The operator \( \text{Tr} : H^2(T) \to H^2(\partial T) \times H^1(\partial T) \) is again defined by (2.1) in a piecewise sense. We want to characterize the image of \( H^{\frac{5}{2}}(T) \) in \( (H^2 \times H^1)(\partial T) \) under the operator \( \text{Tr} \), for which we will need more notations and definitions.

The common edge of \( F_i \) and \( F_j \) is denoted by \( e_{ij} (= e_{ji}) \) and \( e^\perp_{ij} \) denotes the two dimensional subspace of \( \mathbb{R}^3 \) perpendicular to \( e_{ij} \). The outward unit normal on \( F_j \) is denoted by \( n_j \), and we denote by \( t_{j,i} \) the unit vector tangential to \( F_j \), perpendicular to \( e_{ij} \) and pointing outside \( F_j \) (cf. Figure 3.1).

**Figure 3.1.** Faces, normals, edge and orthogonal subspace

**Definition 3.1.** The space \( H^{\frac{1}{2}}(e_{ij}, e^\perp_{ij}) \) consists of all vector functions \( w \) defined on \( e_{ij} \) with image in \( e^\perp_{ij} \) such that \( w \cdot z \in H^{\frac{1}{2}}(e_{ij}) \) for all \( z \in e^\perp_{ij} \).

**Definition 3.2.** A pair \((f, g) \in H^2(\partial T) \times H^1(\partial T)\) belongs to the space \((H^2 \times H^1)(\partial T)\) if and only if the following conditions are satisfied:

\[
\begin{align*}
(3.1) & \quad f_i = f_j \quad \text{on} \quad e_{ij} \quad \text{for} \quad 1 \leq i \neq j \leq 4, \\
(3.2) & \quad \nabla F_j f_j \cdot t_{j,i} = w_{ij} \cdot t_{j,i} \quad \text{on} \quad e_{ij} \quad \text{for} \quad 1 \leq i \neq j \leq 4, \\
(3.3) & \quad g_j = w_{ij} \cdot n_j \quad \text{on} \quad e_{ij} \quad \text{for} \quad 1 \leq i \neq j \leq 4.
\end{align*}
\]

Note that we can replace the compatibility conditions (3.2)–(3.3) by the condition

\[
\begin{align*}
(3.4) & \quad \nabla F_i f_i + g_i n_i = \nabla F_j f_j + g_j n_j \quad \text{on} \quad e_{ij} \quad \text{for} \quad 1 \leq i \neq j \leq 4.
\end{align*}
\]
It follows from the Sobolev Embedding Theorem that \( \text{Tr} \, \zeta \in (H^2 \times H^1)(\partial T) \) for \( \zeta \in H^3 \tilde{T}(T) \), where \( w_{ij} \) is the orthogonal projection of \( \nabla \zeta \) along \( e_{ij} \) onto the subspace \( e_{ij}^\perp \), and we can recover \( \nabla \zeta \) on \( F_i \) from \((f,g) = \text{Tr} \, \zeta \) through the relation
\[
\nabla \zeta = \nabla_{F_i} f_i + g_i n_i \quad \text{on} \ F_i \text{ for } 1 \leq i \leq 4.
\]

We want to show that \( \text{Tr} \, H^3 \tilde{T}(T) = (H^2 \times H^1)(\partial T) \).

Again we construct a linear bijection \( \Phi^* : (H^2 \times H^1)(\partial T) \rightarrow (H^2 \times H^1)(\partial \tilde{T}) \) so that (2.7) is valid, where \( \Phi \) is an orientation preserving affine transformation that maps the tetrahedron \( \tilde{T} \) onto \( T \). Let \((f,g) \in (H^2 \times H^1)(\partial T) \). Motivated by (2.7), (2.8) and (3.4), we define \( \Phi^*(f,g) = (\tilde{f}, \tilde{g}) \), where \( \tilde{f} \) is given by (2.9), \( \tilde{g} \) is given by (2.10) (where \( \tilde{n}_i \) is the outward pointing unit normal along the face \( \tilde{F}_i \)) and the vector field \( g \) on \( \partial T \) is given by
\[
g = \nabla_{F_i} f_i + g_i n_i \quad \text{on} \ F_i \text{ for } 1 \leq i \leq 4.
\]

It is straightforward to check that \((\tilde{f}, \tilde{g}) \in (H^2 \times H^1)(\partial \tilde{T}) \), \( \Phi^* : (H^2 \times H^1)(\partial T) \rightarrow (H^2 \times H^1)(\partial \tilde{T}) \) is a bijection, and that (2.7) follows from (2.8)–(2.10), (3.5) and (3.6).

We can now establish the following analog of Lemma 2.2.

**Lemma 3.3.** The image of \( H^3 \tilde{T}(T) \) under \( \text{Tr} \) is the space \((H^2 \times H^1)(\partial T)\).

**Proof.** Given \((f,g) \in H^2(\partial T) \times H^1(\partial T) \) that satisfies (3.1)–(3.3), we can reduce the construction of \( \zeta \) to the following three cases by a partition of unity. (i) \( f \) and \( g \) vanish near the vertices of \( T \) and the edges of \( T \), in which case we can use the operator \( L_2 \) in Lemma A.2 to obtain \( \zeta \). (ii) \( f \) and \( g \) are supported in a neighborhood of an edge and vanish near the vertices of \( T \), in which case we can assume through an affine transformation (cf. (2.7)) that the dihedral angle at the edge is a right angle and obtain \( \zeta \) through Lemma A.5. (iii) \( f \) and \( g \) are supported near a vertex of \( T \), in which case we can assume through an affine transformation that the angle at the vertex is a solid right angle and obtain \( \zeta \) through Lemma A.4. \( \Box \)

### 3.2. Affine Invariant \( H^2 \) Virtual Element Spaces.

We will use the same notation \((H^2 \times H^1)(\partial T)\) to denote \( \text{Tr} \, H^3 \tilde{T}(T) \) for a tetrahedron \( T \). But the definition of \((H^2 \times H^1)_{k,k-1}(\partial T)\) is different.

**Definition 3.4.** Let \( T \) be a tetrahedron. A pair \((f,g)\in(H^2 \times H^1)(\partial T)\) belongs to \((H^2 \times H^1)_{k,k-1}(\partial T)\) if and only if \((f_i,g_i)\in Y^k(F_i) \times P_{k-1}(F_i)\) for \(1 \leq i \leq 4\).

**Remark 3.5.** It follows from Lemma 2.9 and the constraints (3.1)–(3.3) that we need the following dofs for \((H^2 \times H^1)_{k,k-1}(\partial T)\): (i) The value of \( v \) at each vertex \( p \) together with the values of the three directional derivatives along the three edges emanating from \( p \), which requires \( 4 \times 4 \) dofs. (ii) The moments of \( v \) up to order \( k - 4 \) on each edge, which together with (i) ensure the constraint (3.1). This requires \( 6 \times (k - 3) \) dofs. (iii) The moments of order up to \( k - 3 \) on each edge in order to define, together with (i), a polynomial (vector) function of order \( \leq k - 1 \) on \( e \) with images in \( e^\perp \), which requires \( 6 \times 2(k - 2) \) dofs. We can then use this polynomial (vector) function to define \( \nabla v_F \cdot n_{e,F} \) on any edge \( e \) of \( F \) through (3.2) and \( \partial v/\partial n \) on \( \partial F \) through (3.3). (iv) On each face \( F \) we need to specify the moments.
of $v$ and $\partial v/\partial n$ up to order $k - 4$ in order to complete the definition of $v_F \in \mathcal{V}^k(F)$ and $\partial v/\partial n \in P_{k-1}(F)$, which requires $4 \times 2 \times \frac{(k-3)(k-2)}{2}$ dofs. Altogether we have

$$\dim (H^2 \times H^1)_{k,k-1}(\partial T) = 16 + 6(k - 3) + 12(k - 2) + 4(k - 3)(k - 2)$$

$$= 2(k - 1)(2k + 1).$$

The (visible) dofs of $(H^2 \times H^1)_{k,k-1}(\partial T)$ for $k = 3$ and 4 are depicted in Figure 3.2, where (i) the values of $f$ at the vertices and the moments of $f$ on the edges and faces are represented by solid dots, and (ii) the directional derivatives of $f$ at the vertices and the moments of $g$ on the edges and faces are represented by arrows.

**Figure 3.2.** Visible degrees of freedom for $(H^2 \times H^1)_{3,2}(\partial T)$ and $(H^2 \times H^1)_{4,3}(\partial T)$

The well-posedness result in Lemma 2.6 remains valid for a tetrahedron $T$ and the definition of the virtual element space $\mathcal{V}^k(\hat{T})$ on the reference tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ is identical to the one in Definition 2.7 for the reference triangle. The virtual element space $\mathcal{V}^k(T)$ for an arbitrary tetrahedron is then defined as in Definition 2.11 through an orientation preserving affine transformation $\Phi$ that maps $\hat{T}$ onto $T$, and Lemma 2.9 also holds for a tetrahedron.

The dimension of $\mathcal{V}^k(T)$ is now given by

$$\dim \mathcal{V}^k(T) = \dim (H^2 \times H^1)_{k,k-1}(\partial T) + \dim P_{k-4}(T)$$

$$= 2(k - 1)(2k + 1) + \frac{1}{6}(k - 3)(k - 2)(k - 1) = \frac{(k - 1)(k + 1)(k + 18)}{6}.$$

**Remark 3.6.** The definition of $\mathcal{V}^k(T)$ for a tetrahedron relies crucially on the fact that boundary data satisfying the compatibility condition (3.1)–(3.3) will belong to $\operatorname{Tr} H^2(T)$. Unlike the two dimensional case (cf. Remark 2.10), this cannot be taken for granted since macro elements of arbitrary order that share the same boundary data are yet to be developed.

**Remark 3.7.** Three dimensional $H^2$ virtual elements on arbitrary polyhedron have recently been proposed in [2].

### 3.3. Construction on the Skeleton $\Gamma = \bigcup_{T \in \mathcal{T}_h} \partial T$.

Given any $v \in V_h$, we want to define $f_{v,T}$ representing (the desired) $E_h v|_{\partial T}$ and $g_{v,T}$ representing (the desired) $(\partial E_h v/\partial n)|_{\partial T}$ for all $T \in \mathcal{T}_h$, such that

$$(f_{v,T}, g_{v,T}) \in (H^2 \times H^1)_{k,k-1}(\partial T) \quad \forall T \in \mathcal{T}_h$$
and the following conditions are satisfied:

\begin{equation}
3.10 \quad \text{if } T_1 \text{ and } T_2 \text{ belongs to } \mathcal{T}_p \text{ (resp., } \mathcal{T}_e \text{ or } \mathcal{T}_f), \text{ then } f_{v,T_1}(p) = f_{v,T_2}(p) \text{ (resp., } f_{v,T_1} = f_{v,T_2} \text{ on } e \text{ or } f_{v,T_1} = f_{v,T_2} \text{ on } F),
\end{equation}

\begin{equation}
3.11 \quad \text{if } T_1 \text{ and } T_2 \text{ are two distinct tetrahedra in } \mathcal{T}_F, \text{ then } g_{v,T_1} + g_{v,T_2} = 0 \text{ on } F,
\end{equation}

\begin{equation}
3.12 \quad \text{if } F \in \mathcal{F}_h^b \text{ is a face of } T \text{ and } v \in H_0^1(\Omega), \text{ then } f_{v,T} = 0 \text{ on } F.
\end{equation}

Note that (3.10) and (3.11) imply any piecewise $H^2$ function $\xi$ satisfying $(\xi_T, \partial \xi_T/\partial n)|_{\partial T} = (f_{v,T}, g_{v,T})$ for all $T \in \mathcal{T}_h$ will belong to $H^2(\Omega)$, and (3.12) implies that $\xi \in H^2(\Omega) \cap H_0^1(\Omega)$ if $v \in H_0^1(\Omega)$.

3.3.1. Construction at the Vertices. As in Section 2.3, we first define the vectors $w_p$ associated with the vertices $p$ of $\mathcal{T}_h$. There are three cases: (i) $p$ is an interior vertex, (ii) $p$ is a boundary vertex that belongs to a face of $\Omega$, (iii) $p$ is a boundary vertex that does not belong to any face of $\Omega$.

**Case (i)** For an interior vertex $p$, we choose a tetrahedron $T$ in $\mathcal{T}_p$ and define $w_p$ to be $\nabla v_T$.

**Case (ii)** For a boundary vertex $p$ that belongs to a face $F$ of $\Omega$, we define $w_p$ to be $\nabla v_T$, where $T$ is a tetrahedron in $\mathcal{T}_p$ that has a face on $F$. This choice ensures that $w_p \cdot t = 0$ if $v \in H_0^1(\Omega)$, where $t$ is any vector tangential to $\partial \Omega$ at $p$.

**Case (iii)** In this case $p$ is either a corner of $\Omega$ or $p$ belongs to an edge of $\Omega$. We define $w_p$ implicitly by

\begin{equation}
3.13 \quad w_p \cdot t_i = \frac{\partial v}{\partial t_i}(p) \quad \text{for } i = 1, 2, 3,
\end{equation}

where $\partial/\partial t_1$, $\partial/\partial t_2$ and $\partial/\partial t_3$ are the tangential derivatives along three edges $e_1, e_2, e_3 \in \mathcal{E}_h^b$ emanating from $p$ that are not coplanar. This choice of $e_1, e_2, e_3$ implies $w_p = 0$ if $v \in H_0^1(\Omega)$.

**Remark 3.8.** Note that Remark 2.16 is also valid here, i.e., $w_p = \nabla v_T(p)$ for all $T \in \mathcal{T}_p$ if $v$ is $C^1$ at the vertex $p$.

3.3.2. Construction on the Edges. In view of (3.2) and (3.3), we also need to define polynomial vector functions $w_e : e \rightarrow e^\perp$ on the edges $e \in \mathcal{E}_h$. There are three cases: (i) $e$ is an interior edge of $\mathcal{T}_h$, (ii) $e$ is a subset of a face of $\Omega$ and (iii) $e$ is a subset of an edge of $\Omega$.

**Case (i)** Let $e$ belong to $\mathcal{E}_h^i$. We choose $T \in \mathcal{T}_e$, and then define $w_e$ by the following conditions:

\begin{equation}
3.14 \quad \text{at an endpoint } p \text{ of } e, w_e(p) \text{ is the projection of } w_p \text{ on } e^\perp.
\end{equation}

\begin{equation}
3.15 \quad w_e \text{ and the projection of } \nabla v_T \text{ on } e^\perp \text{ have the same moments along } e \text{ up to order } k - 3.
\end{equation}

**Case (ii)** Let $e$ be an edge of $\mathcal{T}_h$ that is a subset of a face $F$ of $\Omega$. We define $w_e$ again by (3.14)-(3.15), but with the stipulation that one of the faces of $T$ is a subset of $F$. This additional condition (together with the choices made in Cases (ii) and (iii) in Section 3.3.1) implies that $w_e \cdot t = 0$ on $e$ if $v \in H_0^1(\Omega)$, where $t$ is any vector tangential to $F$. 

VIRTUAL ENRICHING OPERATORS
Case (iii) Let $e$ be an edge of $T_h$ that is a subset of an edge of $\Omega$. Then there are two distinct faces $F_1, F_2 \in \mathcal{F}_h \cap \mathcal{F}_e$ and we define $w_e$ by (3.14) together with the condition that

$$
w_e \cdot n_{e,F_j} \quad \text{and} \quad \nabla_{F_j} v_{F_j} \cdot n_{e,F_j}$$

have identical moments up to order $k - 3$ for $j = 1, 2$.

Our choices of $F_1$ and $F_2$ (together with the choices made in Cases (ii) and (iii) in Section 3.3.1) ensures that $w_e = 0$ on $e$ if $v \in H^1_0(\Omega)$.

Remark 3.9. In the case where $v \in V_h$ is $C^1$ across an edge $e \in \mathcal{E}_h$ and at the endpoints of $e$, it follows from Remark 3.8 and (3.14)–(3.16) that the vector field $w_e$ is the projection of $\nabla v_T$ on $e^+$ for all $T \in \mathcal{T}_e$.

3.3.3. Construction on the Faces. We define $g_{v,F}$ on a face $F \in \mathcal{F}_h$ as follows. We choose $T \in \mathcal{T}_F$ and stipulate that

$$
g_{v,F} \quad \text{on an edge } e \text{ of } F, \quad g_{v,F} \in P_{k-1}(e) \text{ is given by } w_e \cdot n_{F,T},$$

(3.17) and $\partial v_T / \partial n$ have the same moments up to order $k - 4$ on $F$.

Remark 3.10. If $v$ is $C^1$ across $e \in \mathcal{E}_h$ and at the endpoints of $e$, then we have $g_{v,F} = \partial v_T / \partial n$ on $e$ for all $F \in \mathcal{F}_e$ and $T \in \mathcal{T}_F$ by Remark 3.9 and (3.17).

3.3.4. Construction on the Tetrahedra. We are now ready to define $(f_{v,T}, g_{v,T}) \in H^2(\partial T) \times H^1(\partial T)$ for any $T \in \mathcal{T}_h$ as follows. On any edge $e$ of a face $F$ of $T$, $f_{v,\partial F}$ is the unique polynomial in $P_k(e)$ with the following properties:

$$
f_{v,\partial F} \text{ agrees with } v \text{ at the two endpoints of } e \text{ and share the same moments up to order } k - 4,$$

(3.19) the directional derivative of $f_{v,\partial F}$ at an endpoint $p$ of $e$ in the direction of the tangent $t_e$ of $e$ is given by $w_p \cdot t_e$.

Remark 3.11. Remark 2.17 is also valid here, i.e., $f_{v,\partial F} = v$ on $e$ if $v$ is $C^1$ at the endpoints of $e$.

Let $F$ be a face of $T$ and $e$ be an edge of $F$, we define $q_{v,\partial F} \in P_{k-1}(e)$ by

$$q_{v,\partial F} = w_e \cdot n_{e,F}.$$ (3.21)

On each face $F$ of $T$, the pair $(f_{v,\partial F}, q_{v,\partial F})$ belongs to $(H^2 \times H^1)_{k,k-1}(\partial F)$ by (3.14) and (3.19)–(3.21). Hence we can define $f_{v,F} \in \mathcal{V}^k(F)$ to be the virtual element function (cf. Lemma 2.13) that satisfies the following conditions:

$$
\text{Tr } f_{v,F} = (f_{v,\partial F}, q_{v,\partial F}) \quad \text{on } \partial F \quad \text{and} \quad Q_{F,k-4} f_{v,F} = Q_{F,k-4} v.
$$

Remark 3.12. If $v$ is $C^1$ on $\partial F$, then $q_{v,\partial F} = \partial v_T / \partial n$ on $\partial F$ by Remark 3.9 and (3.21). It then follows from Remark 2.12, Remark 3.11 and (3.22) that $f_{v,F} = v$ on $F$.

Given any face $F$ of $T$, we define

$$g_{v,T} = g_{v,F} \text{ if } F \text{ is the tetrahedron chosen in the definition of } g_{v,F} \text{ (cf. Section 3.3.3), otherwise } g_{v,T} = -g_{v,F}.$$ (3.23)
**Remark 3.13.** If $v$ is $C^1$ on $\partial T$, then (3.18), Remark 3.10 and (3.23) imply $g_{v,T} = \partial v_T/\partial n$ on $\partial T$.

At the end of this process, we have constructed $(f_{v,T}, g_{v,T}) \in H^2(\partial T) \times H^1(\partial T)$ for every polyhedron $T \in \mathcal{T}_h$. The pair $(f_{v,T}, g_{v,T})$ belongs to $(H^2 \times H^1)_{k,k-1}(\partial T)$ because (i) the condition (3.1) is implied by (3.19)–(3.20), (ii) the condition (3.2) is implied by (3.21)–(3.22), and (iii) the condition (3.3) is implied by (3.17).

It follows from (3.19)–(3.22) that (3.10) is satisfied, and the condition (3.11) follows from (3.23). The choices we make in Section 3.3.1 and Section 3.3.2 ensure that $f_{v,\partial F}$ defined by (3.19)–(3.20) and $q_{v,\partial F}$ defined by (3.21) both vanish on $\partial F$ if the face $F$ of $T$ is a subset of $\partial \Omega$ and $v \in H^1_0(\Omega)$. The condition (3.12) then follows from (3.22).

In view of Remark 3.12 and Remark 3.13 the relation (2.27) remains valid, i.e., $E_h v = v$ if $v \in V_h$ is $C^1$ on $\partial T$, which is the basis for the estimates (1.4) and (1.5).

### 3.4. The Operator $E_h$.

We proceed as in Section 2.4. Let $v \in V_h$ and $T \in \mathcal{T}_h$ be arbitrary, and $(f_{v,T}, g_{v,T}) \in (H^2 \times H^1)_{k,k-1}(\partial T)$ be the function pair constructed in Section 3.3. We define $E_h v \in \mathcal{V}^k(T)$ again by the conditions in (2.26), i.e.,

$$
(E_h v, \partial E_h v/\partial n) = (f_{v,T}, g_{v,T}) \quad \text{on } \partial T \quad \text{and} \quad Q_{T,k-4}(E_h v) = Q_{T,k-4}(v).
$$

It follows from (3.10)–(3.11) that $E_h v \in H^2(\Omega)$, and (3.12) implies that $E_h v \in H^1_0(\Omega)$ if $v \in H^1_0(\Omega)$.

The estimates (1.4) and (1.5) are established by similar arguments as in Section 2.4, where the analog of (2.28) for a tetrahedron $T$ (cf. Lemma 2.13 and Remark 3.5) is given by

$$
\|\xi\|^2_{L_2(T)} \approx \|Q_{T,k-4}\xi\|^2_{L_2(T)} + \sum_{F \in \mathcal{F}_T} h_T \|Q_{F,k-4}\xi\|^2_{L_2(F)} + \sum_{F \in \mathcal{F}_T} h_T^3 \|Q_{F,k-4}(\partial \xi/\partial n)\|^2_{L_2(F)}
$$

$$
+ \sum_{e \in \mathcal{E}_T} h_T^2 \|Q_{e,k-4}\xi\|^2_{L_2(e)} + \sum_{e \in \mathcal{E}_T} h_T^4 \|Q_{e,k-3}(\nabla \xi)_{e_\perp}\|^2_{L_2(e)}
$$

$$
+ \sum_{p \in \mathcal{V}_T} [h_T^3 \xi^2(p) + h_T^5 |\nabla \xi(p)|^2]
$$

for all $\xi \in \mathcal{V}^k(T)$, where $\mathcal{F}_T$ (resp., $\mathcal{E}_T$ and $\mathcal{V}_T$) is the set of the four faces (resp., six edges and four vertices) of $T$ and $(\nabla \xi)_{e_\perp}$ is the orthogonal projection of $\nabla \xi$ onto the subspace of $\mathbb{R}^3$ perpendicular to $e$. The hidden constants in (3.25) only depend on the shape regularity of $\mathcal{T}_h$ because of the affine invariance of the virtual element spaces.

It follows from (3.19), (3.22), (3.24) and (3.25) that we have the following analog of (2.29):

$$
\|v - E_h v\|^2_{L_2(T)} \approx \sum_{p \in \mathcal{V}_T} h_T^5 |\nabla (v - E_h v)(p)|^2 + \sum_{e \in \mathcal{E}_T} h_T^4 \|Q_{e,k-3}(\nabla (v - E_h v))_{e_\perp}\|^2_{L_2(e)}
$$

$$
+ \sum_{F \in \mathcal{F}_T} h_T^3 \|Q_{F,k-4}[\partial v/\partial n]\|^2_{L_2(F)}.
$$

We can then establish the three-dimensional analogs of Theorem 2.19 and Theorem 2.20 as in Section 2.4.
4. Concluding Remarks

Following the approach of this paper (and with more patience and persistence), one can construct enriching operators $E_h$ that maps the totally discontinuous $P_k$ finite element space into $H^2(\Omega)$, where (1.4) and (1.5) are valid for $J(w, v)$ given by

$$J(w, v) = \sum_{e \in E_h} \left[ h_e^{-3} \int_e [w][v] ds + h_e^{-1} \int_e [\partial w/\partial n][\partial v/\partial n] ds \right]$$

for $d = 2$,

$$J(w, v) = \sum_{F \in F_h} \left[ h_F^{-3} \int_F [w][v] dS + h_F^{-1} \int_F [\partial w/\partial n][\partial v/\partial n] dS \right]$$

for $d = 3$.

One can also construct $E_h : V_h \cap H^0_0(\Omega) \rightarrow H^2_0(\Omega)$ such that (1.4) and (1.5) are valid, provided the sum in (1.2) (resp., (1.3)) is taken over $E_h$ (resp., $F_h$). This can also be carried out for the totally discontinuous $P_k$ finite element space.

Lemma 3.3 is also of independent interest, since inverse trace theorems for polyhedral domains in $\mathbb{R}^3$ do not appear to be readily available in the literature.

Appendix A. Inverse Trace Theorems for $\mathbb{R}^2_+$ and $\mathbb{R}^3_+$

We consider inverse trace theorems for $\mathbb{R}^2_+$ and $\mathbb{R}^3_+$ with data on the boundaries of these domains (cf. Figure A.1). We will rely on the results in Lemma A.1 and Lemma A.2 that follow from the construction of inverse trace operators through the Fourier transform [20, 24] and the Paley-Wiener theorem [19].

**Figure A.1.** Boundary data for $\mathbb{R}^2_+$ and $\mathbb{R}^3_+$

**Lemma A.1.** There exists a bounded linear operator $L_1 : H^2(\mathbb{R}) \times H^1(\mathbb{R}) \rightarrow H^2_0(\mathbb{R}^2)$ such that (i) $[L_1(\phi, \psi)](t, 0) = \phi(t)$, (ii) $[\partial L_1(\phi, \psi)/\partial x_2](t, 0) = \psi(t)$, and (iii) $L_1(\phi, \psi)(x_1, x_2)$ vanishes on the half plane $x_1 < 0$ if $\phi(t)$ and $\psi(t)$ vanish on the half line $t < 0$.

**Lemma A.2.** There exists a bounded linear operator $L_2 : H^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \rightarrow H^2_0(\mathbb{R}^3)$ with the following properties: (i) $[L_2(\phi, \psi)](x_1, x_2, 0) = \phi(x_1, x_2)$, (ii) $[\partial L_2(\phi, \psi)/\partial x_3](x_1, x_2, 0) = \psi(x_1, x_2)$, and (iii) $L_2(\phi, \psi)(x_1, x_2, x_3)$ vanishes on the half space $x_1 < 0$ (resp., $x_2 < 0$) if $\phi(x_1, x_2)$ and $\psi(x_1, x_2)$ vanish on the half plane $x_1 < 0$ (resp., $x_2 < 0$).
We begin with a two-dimensional inverse trace theorem. We note that similar results for $H^2(\mathbb{R}_+^2)$ can be found in [18, Section 1.5.2]. Our approach is simpler (since we are considering $H^\frac{5}{2}(\mathbb{R}_+^2)$) and therefore its extension to three dimensions is easier.

**Lemma A.3.** Let $(\phi_1, \psi_1)$ and $(\phi_2, \psi_2)$ belong to $H^2(\mathbb{R}_+) \times H^1(\mathbb{R}_+)$ such that

(A.1) \quad \phi_1(0) = \phi_2(0),
(A.2) \quad \psi_1(0) = \phi_2'(0),
(A.3) \quad \psi_2(0) = \phi_2'(0).

Then there exists $\zeta \in H^\frac{5}{2}(\mathbb{R}_+ \times \mathbb{R}_+)$ such that

(A.4) \quad (\zeta, \partial \zeta / \partial x_i) = (\phi_i, \psi_i) \quad \text{if } x_i = 0, \ 1 \leq i \leq 2.$

**Proof.** First we extend $\phi_1$ and $\psi_1$ to $\mathbb{R}$, so that the extensions (still denoted by $\phi_1$ and $\psi_1$) satisfy $\phi_1 \in H^2(\mathbb{R})$ and $\psi_1 \in H^1(\mathbb{R})$. This can be achieved by reflection (cf. [20, Theorem 2.3.9] and [1, Theorem 5.19]). Let $L_1$ be the lifting operator in Lemma A.1 and $\zeta = L_1(\phi_1, \psi_1) \in H^\frac{5}{2}(\mathbb{R}_+^2)$ so that

(A.5) \quad \zeta_1(0, x_2) = \phi_1(x_2) \quad \text{and} \quad (\partial \zeta_1 / \partial x_1)(0, x_2) = \psi_1(x_2).

Then we define $\tilde{\phi}_2(x_1) = \phi_2(x_1) - \zeta_1(x_1, 0)$, $\tilde{\psi}_2(x_1) = \psi_2(x_1) - (\partial \zeta_1 / \partial x_2)(x_1, 0)$ for $x_1 > 0$.

Note that $\tilde{\phi}_2 \in H^2(\mathbb{R}_+)$, and

$$\tilde{\phi}_2(0) = \phi_2(0) - \zeta_1(0, 0) = \phi_2(0) - \phi_1(0) = 0$$

by (A.1) and (A.5), and

$$\tilde{\phi}_2'(0) = \phi_2'(0) - (\partial \zeta_1 / \partial x_1)(0, 0) = \phi_2'(0) - \psi_1(0) = 0$$

by (A.2) and (A.5). Moreover we have $\tilde{\psi}_2 \in H^1(\mathbb{R}_+)$, and

$$\tilde{\psi}_2(0) = \psi_2(0) - (\partial \zeta_1 / \partial x_2)(0, 0) = \psi_2(0) - \phi_2'(0) = 0$$

by (A.3) and (A.5). Hence their trivial extensions (still denoted by $\tilde{\phi}_2$ and $\tilde{\psi}_2$) satisfy $\tilde{\phi}_2 \in H^2(\mathbb{R})$ and $\tilde{\psi}_2 \in H^1(\mathbb{R})$.

Let $\zeta_2 = L_1(\tilde{\phi}_2, \tilde{\psi}_2) \in H^\frac{5}{2}(\mathbb{R}_+^2)$ such that $\zeta_2(x_1, 0) = \tilde{\psi}_2(x_1, 0)$ and $(\partial \zeta_2 / \partial x_2)(x_1, 0) = \tilde{\psi}_1(x_1)$. Then $\zeta_2 = 0$ on the half plane $x_1 < 0$ by Lemma A.3, which implies

$$\zeta_2(0, x_2) = (\partial \zeta_2 / \partial x_1)(0, x_2) = 0 \quad \forall \ x_2 > 0.$$ We can now take $\zeta$ to be the restriction of $\zeta_1 + \zeta_2$ to $\mathbb{R}_+ \times \mathbb{R}_+$.

Next we consider the three dimensional analog of Lemma A.3.

**Lemma A.4.** Let $(\phi_1, \psi_1)$, $(\phi_2, \psi_2)$ and $(\phi_3, \psi_3)$ belong to $H^2(\mathbb{R}_+ \times \mathbb{R}_+) \times H^1(\mathbb{R}_+ \times \mathbb{R}_+)$ such that the following conditions are satisfied:

(A.6) \quad \phi_1(0, x_3) = \phi_2(0, x_3) \quad \forall \ x_3 > 0,
(A.7) \quad \phi_2(x_1, 0) = \phi_3(x_1, 0) \quad \forall \ x_1 > 0,
(A.8) \quad \phi_3(0, x_2) = \phi_1(x_2, 0) \quad \forall \ x_2 > 0,
Then there exists $\zeta \in H^2_2(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+)$ such that

\begin{align*}
(\zeta, \partial \zeta / \partial x_i) &= (\phi_i, \psi_i) \quad \text{if } x_i = 0, \ 1 \leq i \leq 3.
\end{align*}

**Proof.** First we extend $\phi_1$ and $\psi_1$ to $\mathbb{R}^2$ by reflection (twice) so that the extensions (still denoted by $\phi_1$ and $\psi_1$) satisfy $\phi_1 \in H^2(\mathbb{R}^2)$ and $\psi_1 \in H^1(\mathbb{R}^2)$. Let $L_2$ be the lifting operator in Lemma A.2 and $\zeta = L_2(\phi_1, \psi_1)$ so that

\begin{align*}
(\zeta, \partial \zeta / \partial x_1)(0, x_2, x_3) &= \psi_1(x_2, x_3), \\
(\partial \zeta / \partial x_1)(0, x_2, x_3) &= \psi_1(x_2, x_3).
\end{align*}

Then we define, for $(x_1, x_3) \in \mathbb{R}_+ \times \mathbb{R}_+$,

\begin{align*}
\tilde{\phi}_2(x_1, x_3) &= \phi_2(x_1, x_3) - \zeta_1(x_1, 0, x_3), \\
\tilde{\psi}_2(x_1, x_3) &= \psi_2(x_1, x_3) - (\partial \zeta / \partial x_2)(x_1, 0, x_3).
\end{align*}

Note that $\tilde{\phi}_2$ belongs to $H^2(\mathbb{R}_+ \times \mathbb{R}_+)$ and

$$
\tilde{\phi}_2(0, x_3) = \phi_2(0, x_3) - \zeta_1(0, 0, x_3) = \phi_2(0, x_3) - \phi_1(0, x_3) = 0 \quad \text{for} \quad x_3 > 0
$$

by (A.6), (A.16) and (A.18), and

$$
\frac{\partial \tilde{\phi}_2}{\partial x_1}(0, x_3) = \frac{\partial \phi_2}{\partial x_1}(0, x_3) - \frac{\partial \zeta_1}{\partial x_1}(0, 0, x_3) = \frac{\partial \phi_2}{\partial x_1}(0, x_3) - \psi_1(0, x_3) = 0
$$

by (A.9), (A.17) and (A.18). Furthermore $\tilde{\psi}_2$ belongs to $H^1(\mathbb{R}_+ \times \mathbb{R}_+)$ and

$$
\tilde{\psi}_2(0, x_3) = \psi_2(0, x_3) - \frac{\partial \zeta_1}{\partial x_2}(0, 0, x_3) = \psi_2(0, x_3) - \frac{\partial \phi_1}{\partial x_2}(0, x_3) = 0 \quad \text{for} \quad x_3 > 0
$$

by (A.12), (A.16) and (A.19).

Hence we can extend $\tilde{\phi}_2$ and $\tilde{\psi}_2$ to $\mathbb{R}_+ \times \mathbb{R}$ by reflection across $x_3 = 0$ (still denoted by $\tilde{\phi}_2$ and $\tilde{\psi}_2$) so that $\tilde{\phi}_2 \in H^2(\mathbb{R}_+ \times \mathbb{R})$, $\tilde{\psi}_2 \in H^2(\mathbb{R}_+ \times \mathbb{R})$, $\tilde{\phi}_2(0, x_3) = (\partial \tilde{\phi}_2 / \partial x_1)(0, x_3) = 0$ for $x_3 \in \mathbb{R}$ and $\tilde{\psi}_2(0, x_3) = 0$ for $x_3 \in \mathbb{R}$. Therefore the trivial extensions of $\tilde{\phi}_2$ and $\tilde{\psi}_2$ to $\mathbb{R}^2$ (still denoted by $\tilde{\phi}_2$ and $\tilde{\psi}_2$) belong to $H^2(\mathbb{R}^2)$ and $H^1(\mathbb{R}^2)$ respectively.
Let $\zeta_2 = L_2(\tilde{\phi}_2, \tilde{\psi}_2)$. Then we have, by Lemma A.2,
\begin{align*}
(A.20) \quad \zeta_2(x_1, 0, x_3) &= \tilde{\phi}_2(x_1, x_3), \\
(A.21) \quad (\partial \zeta_2/\partial x_2)(x_1, 0, x_3) &= \tilde{\psi}_2(x_1, x_3),
\end{align*}
and
\[\zeta_2(x_1, x_2, x_3) = 0 \quad \text{if } x_1 < 0,\]
which implies
\begin{align*}
(A.22) \quad \zeta_2 &= \partial \zeta_2/\partial x_1 = 0 \quad \text{if } x_1 = 0.
\end{align*}

We now define, for $(x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+$,
\begin{align*}
(A.23) \quad \tilde{\phi}_3(x_1, x_2) &= \phi_3(x_1, x_2) - \zeta_1(x_1, x_2, 0) - \zeta_2(x_1, x_2, 0), \\
(A.24) \quad \tilde{\psi}_3(x_1, x_2) &= \psi_3(x_1, x_2) - (\partial \zeta_1/\partial x_3)(x_1, x_2, 0) - (\partial \zeta_2/\partial x_3)(x_1, x_2, 0).
\end{align*}
Then $\tilde{\phi}_3$ (resp., $\tilde{\psi}_3$) belongs to $H^2(\mathbb{R}_+ \times \mathbb{R}_+)$ (resp., $H^1(\mathbb{R}_+ \times \mathbb{R}_+)$).

Moreover, it follows from (A.8), (A.16), (A.22) and (A.23) that
\[\tilde{\phi}_3(0, x_2) = \phi_3(0, x_2) - \zeta_1(0, x_2, 0) = \phi_3(0, x_2) - \phi_1(0, x_2, 0) = 0 \quad \text{for } x_2 > 0,\]
and (A.10), (A.17), (A.22) and (A.23) imply
\[\frac{\partial \tilde{\phi}_3}{\partial x_1}(0, x_2) = \frac{\partial \phi_3}{\partial x_1}(0, x_2) - \frac{\partial \zeta_1}{\partial x_1}(0, x_2, 0) = \frac{\partial \phi_3}{\partial x_1}(0, x_2) - \psi_1(x_2, 0) = 0 \quad \text{for } x_2 > 0.
\]
From (A.14), (A.16), (A.22) and (A.24) we also have
\[\tilde{\psi}_3(0, x_2) = \psi_3(0, x_2) - \frac{\partial \zeta_1}{\partial x_3}(0, x_2, 0) = \psi_3(0, x_2) - \frac{\partial \phi_1}{\partial x_3}(x_2, 0) \quad \text{for } x_2 > 0.
\]

Next we check the behavior of $\tilde{\phi}_3$ and $\tilde{\psi}_3$ at $x_2 = 0$. We have
\[\tilde{\phi}_3(x_1, 0) = \phi_3(x_1, 0) - \zeta_1(x_1, 0, 0) - \zeta_2(x_1, 0, 0) = \phi_3(x_1, 0) - \phi_2(x_1, 0) = 0 \quad \text{for } x_1 > 0\]
by (A.7), (A.18), (A.20) and (A.23);
\[\frac{\partial \tilde{\phi}_3}{\partial x_2}(x_1, 0) = \frac{\partial \phi_3}{\partial x_2}(x_1, 0) - \frac{\partial \zeta_1}{\partial x_2}(x_1, 0, 0) = \frac{\partial \phi_3}{\partial x_2}(x_1, 0) - \psi_2(x_1, 0) = 0 \quad \text{for } x_1 > 0\]
by (A.11), (A.18), (A.21) and (A.23);
\[\tilde{\psi}_3(x_1, 0) = \psi_3(x_1, 0) - \frac{\partial \zeta_1}{\partial x_3}(x_1, 0, 0) = \psi_3(x_1, 0) - \frac{\partial \phi_2}{\partial x_3}(x_1, 0) = 0 \quad \text{for } x_1 > 0\]
by (A.13), (A.18), (A.20) and (A.24).

The calculations above show that $\tilde{\phi}_3 = \partial \tilde{\phi}_3/\partial n = \tilde{\psi}_3 = 0$ on the boundary of $\mathbb{R}_+ \times \mathbb{R}_+$. Hence their trivial extensions to $\mathbb{R}^2$ (still denoted by $\tilde{\phi}_3$ and $\tilde{\psi}_3$) belongs to $H^2(\mathbb{R}^2)$ and $H^1(\mathbb{R}^2)$. 

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Let $\zeta_3 = L_2(\tilde{\phi}_1, \tilde{\psi}_1)$. Then we have, by Lemma A.2, $\zeta_3 \in H^3(\mathbb{R}^3)$,
\begin{align}
\zeta_3(x_1, x_2, 0) &= \tilde{\phi}_3(x_1, x_2), \\
(\partial \zeta_3/\partial x_3)(x_1, x_2, 0) &= \tilde{\psi}_3(x_1, x_2),
\end{align}
which implies
\begin{equation}
\zeta_3(x_1, x_2, x_3) = 0 \quad \text{if } x_1 < 0 \text{ or } x_2 < 0,
\end{equation}
\begin{align}
\zeta_3 &= \frac{\partial \zeta_3}{\partial x_1} = 0 \quad \text{if } x_1 = 0 \quad \text{and} \quad \zeta_3 &= \frac{\partial \zeta_3}{\partial x_2} = 0 \quad \text{if } x_2 = 0.
\end{align}

We can now take $\zeta$ to be the restriction of $\zeta_1 + \zeta_2 + \zeta_3$ to $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$, and (A.15) follows from (A.16)–(A.27), $\square$

Finally we have a three-dimensional result that is two-dimensional in nature and which can be derived by using the arguments in the proof of either Lemma A.3 or Lemma A.4.

**Lemma A.5.** Let $(\phi_1, \psi_1)$ and $(\phi_2, \psi_2)$ belong to $H^2(\mathbb{R}^+ \times \mathbb{R}) \times H^1(\mathbb{R}^+ \times \mathbb{R})$ such that
\begin{align}
\phi_1(0, x_3) &= \phi_2(0, x_3) \quad \forall x_3 \in \mathbb{R}, \\
\psi_1(0, x_3) &= \frac{\partial \phi_2}{\partial x_1}(0, x_3) \quad \forall x_3 \in \mathbb{R}, \\
\psi_2(0, x_3) &= \frac{\partial \phi_1}{\partial x_2}(0, x_3) \quad \forall x_3 \in \mathbb{R}.
\end{align}
Then there exists $\zeta \in H^{\frac{5}{2}}(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R})$ such that
\begin{align}
(\zeta, \partial \zeta/\partial x_i) &= (\phi_i, \psi_i) \quad \text{if } x_i = 0, 1 \leq i \leq 2.
\end{align}

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