A Novel Parametrization of $\tau$-lepton Dominance and Simplified One-loop Renormalization-group Equations of Neutrino Mixing Angles and CP-violating Phases

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Abstract

We point out that the $\tau$-lepton dominance in the one-loop renormalization-group (RG) equations of neutrino mixing quantities allows us to set a criterion to choose the most suitable parametrization of the lepton flavor mixing matrix $U$: its elements $U_{3i}$ (for $i = 1, 2, 3$) should be as simple as possible. Such a novel parametrization is quite different from the “standard” one used in the literature and can lead to greatly simplified RG equations for three mixing angles and the physical CP-violating phase(s) in the standard model or its minimal supersymmetric extension, no matter whether neutrinos are Dirac or Majorana particles. Some important features of our analytical results and their phenomenological consequences are also discussed.

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I. INTRODUCTION

The fact that neutrinos have non-vanishing masses is a clean signal of new physics beyond the standard model (SM). To understand the small neutrino mass-squared differences and the large lepton flavor mixing angles observed in solar and atmospheric neutrino oscillation experiments [1–4], many models based on either new flavor symmetries or some unspecified interactions have been proposed at some superhigh energy scales [5]. Their phenomenological consequences at low energy scales can be confronted with current experimental data, after the renormalization-group (RG) effects on those neutrino mixing parameters are properly taken into account. Such radiative corrections can be very significant in some cases, for instance, when the masses of three light neutrinos are nearly degenerate or the value of \( \tan \beta \) is very large in the minimal supersymmetric standard model (MSSM).

An elegant idea to explain the smallness of left-handed neutrino masses is to introduce very heavy right-handed neutrinos and lepton number violation into the SM or MSSM and to make use of the famous seesaw mechanism [6]. Below the seesaw scale, where heavy Majorana neutrinos become decoupled, the effective neutrino coupling matrix \( \kappa \) obeys the following one-loop RG equation [7]:

\[
16\pi^2 \frac{d\kappa}{dt} = \alpha_M \kappa + C \left[ (Y_i Y_i^\dagger) \kappa + \kappa (Y_i Y_i^\dagger)^T \right],
\]

where \( t \equiv \ln(\mu/\Lambda_{SS}) \) with \( \mu \) being an arbitrary renormalization scale between the electroweak scale \( \Lambda_{EW} \sim 10^2 \text{ GeV} \) and the typical seesaw scale \( \Lambda_{SS} \sim 10^{10–14} \text{ GeV} \), and \( Y_i \) is the charged-lepton Yukawa coupling matrix. In the SM, \( C = -1.5 \) and \( \alpha_M \approx -3g_2^2 + 6\lambda^2 + \lambda \); and in the MSSM, \( C = 1 \) and \( \alpha_M \approx -1.2g_1^2 - 6g_2^2 + 6g_3^2 \), where \( g_1 \) and \( g_2 \) denote the gauge couplings, \( y_t \) stands for the top-quark Yukawa coupling, and \( \lambda \) is the Higgs self-coupling in the SM.

There are also some good reasons to speculate that massive neutrinos might be the Dirac particles [8]. In this case, the Dirac neutrino Yukawa coupling matrix \( Y_\nu \) must be extremely suppressed in magnitude, so as to reproduce the light neutrino masses of \( \mathcal{O}(1) \text{ eV} \) or smaller at the electroweak scale. \( Y_\nu \) can run from a superhigh energy scale down to \( \Lambda_{EW} \) via the one-loop RG equation

\[
16\pi^2 \frac{d\omega}{dt} = 2\alpha_D \omega + C \left[ (Y_i Y_i^\dagger) \omega + \omega (Y_i Y_i^\dagger)^T \right],
\]

where \( \omega \equiv Y_\nu Y_\nu^\dagger \), \( \alpha_D \approx -0.45g_1^2 - 2.25g_2^2 + 3g_3^2 \) in the SM or \( \alpha_D \approx -0.6g_1^2 - 3g_2^2 + 3g_3^2 \) in the MSSM [8]. In obtaining Eq. (2), we have safely neglected those tiny terms of \( \mathcal{O}(\omega^2) \).

Eq. (1) or (2) allows us to derive the explicit RG equations for all neutrino mass and mixing parameters in the flavor basis where \( Y_i \) is diagonal and real (positive). In this basis, we have \( \kappa = \mathcal{V}_M \overline{\kappa} \mathcal{V}_M^T \) with \( \overline{\kappa} = \text{Diag}\{\kappa_1, \kappa_2, \kappa_3\} \) for Majorana neutrinos; or \( \omega = \mathcal{V}_D \overline{\omega} \mathcal{V}_D^T \) with \( \overline{\omega} = \text{Diag}\{\omega_1, \omega_2, \omega_3\} \) for Dirac neutrinos. \( \mathcal{V}_M \) or \( \mathcal{V}_D \) is just the lepton flavor mixing matrix. At \( \Lambda_{EW} \), Majorana neutrino masses are given by \( m_i = v^2 \kappa_i \) (SM) or \( m_i = v^2 \kappa_i \sin^2 \beta \) (MSSM), while Dirac neutrino masses are given by \( m_i = \nu y_i \) (SM) or \( m_i = \nu y_i \sin \beta \) (MSSM) with \( v \approx 174 \text{ GeV} \).

Note that \( \mathcal{V}_M \) (or \( \mathcal{V}_D \)) can be parametrized in terms of three mixing angles and a few CP-violating phases. Their RG equations consist of the flavor-dependent contributions from
Because of $y_e^2 \ll y_\mu^2 \ll y_\tau^2$, where $y_e$, $y_\mu$, and $y_\tau$ correspond to the electron, muon and tau Yukawa couplings, we only need to take account of the dominant $\tau$-lepton contribution to those one-loop RG equations of neutrino mixing angles and CP-violating phases in an excellent approximation. A careful analysis shows that the $\tau$-dominance is closely associated with the matrix elements $(V_M)_{3i}$ or $(V_D)_{3i}$ (for $i = 1, 2, 3$). This important observation implies that very concise RG equations can be obtained for those flavor mixing and CP-violating parameters, if $V_M$ (or $V_D$) is parametrized in such a way that its elements $(V_M)_{3i}$ (or $(V_D)_{3i}$) are as simple as possible. One may then make use of this criterion to choose the most suitable parametrization of $V_M$ or $V_D$ in deriving the one-loop RG equations.

We find that the so-called “standard” parametrization (advocated by the Particle Data Group [9]), which has extensively been used in describing lepton flavor mixing, does not satisfy the above criterion. Instead, the parametrization recommended in Ref. [10] fulfills our present requirement 1:

\[
U = \begin{pmatrix}
    c_l & s_l & 0 \\
    -s_l & c_l & 0 \\
    0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
    e^{-i\phi} & 0 & 0 \\
    0 & c & s \\
    0 & -s & c
\end{pmatrix} \begin{pmatrix}
    c_\nu & -s_\nu & 0 \\
    s_\nu & c_\nu & 0
\end{pmatrix},
\]

where $c_l \equiv \cos \theta_l$, $s_l \equiv \sin \theta_l$, $c_\nu \equiv \cos \theta_\nu$, $s_\nu \equiv \sin \theta_\nu$, $c \equiv \cos \theta$ and $s \equiv \sin \theta$. In general, we have $V_M = Q_M U P_M$ for Majorana neutrinos or $V_D = Q_D U P_D$ for Dirac neutrinos, where $P_M$ (or $P_D$) and $Q_M$ (or $Q_D$) are two diagonal phase matrices. It is clear that $U_{3i}$ (for $i = 1, 2, 3$) shown in Eq. (3) are simple enough to describe the $\tau$-dominant terms in those one-loop RG equations of $\theta_l$, $\theta_\nu$, $\theta$ and $\phi$ (as well as two Majorana phases of $V_M$ coming from $P_M$). In the approximation that solar and atmospheric neutrino oscillations are nearly decoupled [13], three mixing angles of $U$ can simply be related to those of solar, atmospheric and CHOOZ neutrino oscillations [1–3]: $\theta_{\text{sun}} \approx \theta_\nu$, $\theta_{\text{atm}} \approx \theta$ and $\theta_{\text{CHOOZ}} \approx \theta_l \sin \theta$. Hence our parametrization is also a convenient option to describe current neutrino oscillation data.

The main purpose of this paper is to show that Eq. (3) is actually a novel parametrization of $\tau$-dominance in the one-loop RG equations of neutrino mixing angles and CP-violating phases. Compared with the “standard” parametrization used in the literature, Eq. (3) leads to greatly simplified results for relevant RG equations. The latter can therefore allow us to understand the RG running behaviors of lepton flavor mixing parameters in a much simpler and more transparent way, which is of course useful for model building at a superhigh energy scale to explore possible flavor symmetries or flavor dynamics responsible for the origin of neutrino masses and CP violation.

In section II, we use Eqs. (1) and (3) to derive the one-loop RG equations of three mixing angles and three CP-violating phases for Majorana neutrinos. Section III is devoted

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1This parametrization may naturally arise from the parallel (and probably hierarchical) textures of charged-lepton and neutrino mass matrices [10]. It is phenomenologically possible to obtain $\theta_l \approx \arctan \left( \sqrt{m_\tau/m_\mu} \right) \approx 4^\circ$ [11] and a suggestive relationship between $\theta_\nu$ and $m_1/m_2$ [12].
to the one-loop RG equations of three mixing angles and one CP-violating phase for Dirac neutrinos, and to a brief comparison between the Jarlskog invariants of CP violation in Dirac and Majorana cases. Some concluding remarks are given in section IV.

II. RG EQUATIONS FOR MAJORANA NEUTRINOS

The general strategy and tactics about how to derive the one-loop RG equations for Majorana neutrino mixing parameters have been outlined in Refs. [14–17]. To be specific, we take \( P_M = \text{Diag}\{e^{i\rho}, e^{i\sigma}, 1\} \) and \( Q_M = \text{Diag}\{e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3}\} \). The phase parameters \( \rho \) and \( \sigma \) are physical and referred to as the Majorana phases. The phase parameters \( \phi_i \) (for \( i = 1, 2, 3 \)) are unphysical, but they have their own RG evolution. Following the procedure described in Ref. [14] and taking the \( \tau \)-dominance approximation, we obtain the RG equations of \( \kappa_i \) (for \( i = 1, 2, 3 \)) from Eq. (1):

\[
\dot{\kappa}_i = \frac{\kappa_i}{16\pi^2} (\alpha_M + 2C \gamma^2 \|U_{3i}\|^2) ,
\]

where \( \dot{\kappa}_i \equiv \frac{d\kappa_i}{dt} \). In addition, the quantities \( \rho \), \( \sigma \), \( \phi_i \) and \( U_{ij} \) (for \( i, j = 1, 2, 3 \)) satisfy the following equations:

\[
\sum_{j=1}^{3} \left[U_{j1}^* \left(i\dot{U}_{j1} - U_{j1}\dot{\phi}_j\right)\right] = \dot{\rho} ,
\]

\[
\sum_{j=1}^{3} \left[U_{j2}^* \left(i\dot{U}_{j2} - U_{j2}\dot{\phi}_j\right)\right] = \dot{\sigma} ,
\]

\[
\sum_{j=1}^{3} \left[U_{j3}^* \left(i\dot{U}_{j3} - U_{j3}\dot{\phi}_j\right)\right] = 0 ;
\]

and

\[
\sum_{j=1}^{3} \left[U_{j1}^* \left(\dot{U}_{j2} + iU_{j2}\dot{\phi}_j\right)\right] = -\frac{C \gamma^2}{16\pi^2} e^{\rho-\sigma} \left[\zeta_{12} \Re \left(U_{31}^* U_{32} e^{i(\rho-\sigma)}\right) + i\zeta_{12} \Im \left(U_{31}^* U_{32} e^{i(\rho-\sigma)}\right)\right] ,
\]

\[
\sum_{j=1}^{3} \left[U_{j1}^* \left(\dot{U}_{j3} + iU_{j3}\dot{\phi}_j\right)\right] = -\frac{C \gamma^2}{16\pi^2} e^{\rho} \left[\zeta_{13} \Re \left(U_{31}^* U_{33} e^{-i\rho}\right) + i\zeta_{13} \Im \left(U_{31}^* U_{33} e^{-i\rho}\right)\right] ,
\]

\[
\sum_{j=1}^{3} \left[U_{j2}^* \left(\dot{U}_{j3} + iU_{j3}\dot{\phi}_j\right)\right] = -\frac{C \gamma^2}{16\pi^2} e^{\sigma} \left[\zeta_{23}^{-1} \Re \left(U_{32}^* U_{33} e^{-i\sigma}\right) + i\zeta_{23} \Im \left(U_{32}^* U_{33} e^{-i\sigma}\right)\right] ,
\]

where \( \zeta_{ij} \equiv (\kappa_i - \kappa_j) / (\kappa_i + \kappa_j) \). Obviously, those \( y^2 \)-associated terms only consist of the matrix elements \( U_{3i} \) (for \( i = 1, 2, 3 \)). If a parametrization of \( U \) assures \( U_{3i} \) to be as simple as possible, then the resultant RG equations of relevant neutrino mixing angles and CP-violating phases will be as concise as possible. One can see that the parametrization of \( U \) given in Eq. (3) just accords with such a criterion, while the “standard” parametrization advocated in Ref. [9] and used in many papers (e.g., Refs. [14–18]) does not satisfy this requirement.
Combining Eq. (3) with Eqs. (4), (5) and (6), we arrive at
\[
\begin{align*}
\dot{\kappa}_1 &= \frac{\kappa_1}{16\pi^2} \left( \alpha_M + 2 C y_r^2 s_l^2 s^2 \right), \\
\dot{\kappa}_2 &= \frac{\kappa_2}{16\pi^2} \left( \alpha_M + 2 C y_r^2 c_l^2 s^2 \right), \\
\dot{\kappa}_3 &= \frac{\kappa_3}{16\pi^2} \left( \alpha_M + 2 C y_r^2 c^2 \right); \\
\end{align*}
\]
and
\[
\begin{align*}
\dot{\theta}_l &= \frac{C y_r^2}{16\pi^2} c_l s_l c \left[ \left( \zeta_{13} c_l c_{(\rho-\phi)} + \zeta_{23} s_l s_{(\rho-\phi)} \right) - \left( \zeta_{23} s_l c_{(\sigma-\phi)} + \zeta_{23} s_{(\sigma-\phi)} \right) \right], \\
\dot{\theta}_\nu &= \frac{C y_r^2}{16\pi^2} c_l \left[ s_l^2 \left( \zeta_{12} c_{(\sigma-\rho)} + \zeta_{12} s_{(\sigma-\rho)} \right) + s_l^2 \left( \zeta_{13} c_{(\rho-\phi)} + \zeta_{13} s_{(\rho-\phi)} \right) - c_l^2 \left( \zeta_{23} c_{(\sigma-\phi)} + \zeta_{23} s_{(\sigma-\phi)} \right) \right], \\
\dot{\theta} &= \frac{C y_r^2}{16\pi^2} c_s \left[ s_l^2 \left( \zeta_{13} c_l c_{(\rho-\phi)} + \zeta_{23} s_l s_{(\rho-\phi)} \right) + c_l^2 \left( \zeta_{23} c_{(\sigma-\phi)} + \zeta_{23} s_{(\sigma-\phi)} \right) \right]; \\
\end{align*}
\]
as well as
\[
\begin{align*}
\dot{\phi}_l &= \frac{C y_r^2}{16\pi^2} \left[ s_l^2 \zeta_{13} c_l c_{(\rho-\phi)} + \zeta_{13} s_l s_{(\rho-\phi)} + \zeta_{23} s_l s_{(\rho-\phi)} \right], \\
\dot{\phi}_\nu &= \frac{C y_r^2}{16\pi^2} c_l s_l c \left[ s_l^2 \zeta_{12} c_{(\sigma-\rho)} + \zeta_{12} s_{(\sigma-\rho)} \right], \\
\dot{\phi} &= \frac{C y_r^2}{16\pi^2} c_s \left[ s_l^2 \zeta_{13} c_{(\rho-\phi)} + \zeta_{23} s_l s_{(\rho-\phi)} \right] + \frac{C y_r^2}{16\pi^2} c_s \left[ s_l^2 \zeta_{23} s_{(\sigma-\phi)} \right], \\
\end{align*}
\]
where \( \zeta_{ij} \equiv \zeta_{ij} - 4 \kappa_i \kappa_j / (\kappa_2^2 - \kappa_1^2) \), \( c_a \equiv \cos a \) and \( s_a \equiv \sin a \) (for \( a = \rho, \sigma, \sigma - \rho, \rho - \phi \) or \( \sigma - \phi \)). Comparing the RG equations of three mixing angles and three CP-violating phases obtained in Eqs. (8) and (9) with their counterparts given in Refs. [14–18], which were derived by using the “standard” parametrization, we find that great simplification and conciseness have been achieved for our present analytical results.

As a by-product, the RG equations of three unphysical phases \( \phi_i \) are listed below:
\[
\begin{align*}
\dot{\phi}_1 &= + \frac{C y_r^2}{16\pi^2} \left[ c_l s_{l_1} c_l s_l c \left( \zeta_{13} c_l c_{(\rho-\phi)} - \zeta_{23} s_l s_{(\rho-\phi)} \right) - c_l^2 \left( \zeta_{13} c_l c_{(\sigma-\phi)} + \zeta_{23} s_l s_{(\sigma-\phi)} \right) \right], \\
\dot{\phi}_2 &= - \frac{C y_r^2}{16\pi^2} \left[ c_l s_{l_1} c_l s_l c \left( \zeta_{13} c_l c_{(\rho-\phi)} - \zeta_{23} s_l s_{(\rho-\phi)} \right) - c_l^2 \left( \zeta_{13} c_l c_{(\sigma-\phi)} + \zeta_{23} s_l s_{(\sigma-\phi)} \right) \right], \\
\dot{\phi}_3 &= + \frac{C y_r^2}{16\pi^2} \left[ s_l^2 \left( \zeta_{13} c_l c_{(\rho-\phi)} + \zeta_{23} s_l s_{(\rho-\phi)} \right) \right]. \\
\end{align*}
\]
It is easy to check that the relationship \( \dot{\phi} = \dot{\rho} + \dot{\sigma} + \dot{\phi}_1 + \dot{\phi}_2 + \dot{\phi}_3 \) holds. That is why \( \phi_i \) should not be ignored in deriving the RG equations of other physical parameters, although these three phases can finally be rotated away via rephasing the charged-lepton fields.
Some qualitative comments on the basic features of Eqs. (7)–(10) are in order.

(a) The RG running behaviors of three neutrino masses $m_i$ (or equivalently $\kappa_i$) are essentially identical and determined by $\alpha_M$ [15], unless $\tan \beta$ is large enough in the MSSM to make the $y_{\tau}^2$-associated term is competitive with the $\alpha_M$ term. In our phase convention, $\kappa_i$ or $\bar{m}_i$ (for $i = 1, 2, 3$) are independent of the CP-violating phase $\phi$.

(b) Among three mixing angles, only the derivative of $\theta_{\nu}$ contains a term proportional to $\zeta_{12}^{-1}$. Note that $\zeta_{12}^{-1} = -(m_1 + m_2)^2/\Delta m^2_{12}$ with $\Delta m_{12}^2 \equiv m_2^2 - m_1^2$ holds, and current solar and atmospheric neutrino oscillation data yield $\Delta m_{21}^2 \approx 8 \times 10^{-5}$ eV$^2$ and $|\Delta m_{31}^2| \approx |\Delta m_{32}^2| \approx 2.5 \times 10^{-3}$ eV$^2$ [13]. Thus $\theta_{\nu}$ is in general more sensitive to radiative corrections than $\theta_l$ and $\theta$. The RG running of $\theta_{\nu}$ can be suppressed through the fine-tuning of $(\sigma - \rho)$. The smallest mixing angle $\theta_l$ may get radiative corrections even if its initial value is zero, thus it can be radiatively generated from other mixing angles and CP-violating phases.

(c) The RG running behavior of $\phi$ is quite different from those of $\rho$ and $\sigma$, because it includes a peculiar term proportional to $s_\nu^{-1}$. This term, which dominates $\dot{\phi}$ when $\theta_l$ is sufficiently small, becomes divergent in the limit $\theta_l \to 0$. Indeed, $\phi$ is not well-defined if $\theta_l$ is exactly vanishing. But both $\theta_l$ and $\phi$ can be radiatively generated. We may require that $\dot{\phi}$ should remain finite when $\theta_l$ approaches zero, implying that the following necessary condition can be extracted from the expression of $\dot{\phi}$ in Eq. (9):

$$
\zeta_{13}^{-1} c_\rho s_{(\rho - \phi)} - \zeta_{13} s_\rho c_{(\rho - \phi)} - \zeta_{23}^{-1} c_\sigma s_{(\sigma - \phi)} + \zeta_{23} s_\sigma c_{(\sigma - \phi)} = 0.
$$

It turns out that

$$
\tan \phi = \frac{\hat{\zeta}_{13} \sin 2\rho - \hat{\zeta}_{23} \sin 2\sigma}{(\zeta^{-1}_{13} + \zeta_{13} \cos 2\rho) - (\zeta^{-1}_{23} + \zeta_{23} \cos 2\sigma)}
$$

holds, a result similar to the one obtained in Eq. (25) of Ref. [15]. Note that the initial value of $\theta_l$, if it is exactly zero or extremely small, may immediately drive $\phi$ to its quasi-fixed point (see Ref. [19] for a relevant study of the quasi-fixed point in the “standard” parametrization of lepton flavor mixing). In this interesting case, Eq. (12) can be used to understand the relationship between $\phi$ and two Majorana phases $\rho$ and $\sigma$ at the quasi-fixed point.

(d) On the other hand, the RG running behaviors of $\rho$ and $\sigma$ are relatively mild in comparison with that of $\phi$. A remarkable feature of $\dot{\rho}$ and $\dot{\sigma}$ is that they will vanish, if both $\rho$ and $\sigma$ are initially vanishing. This observation indicates that $\rho$ and $\sigma$ cannot simultaneously be generated from $\phi$ via the one-loop RG evolution. In contrast, a different conclusion was drawn in Ref. [18], where the “standard” parametrization with a slightly changed phase convention was utilized.

(e) As for three unphysical phases, $\phi_2$ and $\phi_3$ only have relatively mild RG running effects, while the running behavior of $\phi_1$ may be violent for sufficiently small $\theta_l$. A quasi-fixed point of $\phi_1$ is also expected in the limit $\theta_l \to 0$ and under the circumstance given by Eq. (11) or (12).

III. RG EQUATIONS FOR DIRAC NEUTRINOS

Now let us derive the one-loop RG equations for Dirac neutrino mixing parameters. To be specific, we take $P_D = \text{Diag} \{ e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3} \}$ and $Q_D = \text{Diag} \{ e^{i\alpha}, e^{i\beta}, 1 \}$. The phase
Following the procedure described in Refs. [8,20] and taking the $\tau$-dominance approximation, we get the RG equations of $y_i$ (for $i = 1, 2, 3$) from Eq. (2):

$$\dot{y}_i = \frac{y_i}{16\pi^2} \left( \alpha_D + Cy_i^2 |U_{3i}|^2 \right),$$  \hspace{1cm} (13)

where $\dot{y}_i \equiv dy_i/dt$. On the other hand, the quantities $\alpha$, $\beta$ and $U_{ij}$ (for $i,j = 1,2,3$) satisfy the following equations:

$$\sum_{j=1}^{3} \left( U_{j1}^* \dot{U}_{j2} + i \left( \dot{\alpha} U_{j1}^* U_{12} + \dot{\beta} U_{j1}^* U_{22} \right) \right) = -\frac{Cy_1^2}{16\pi^2} \xi_{12} U_{31}^* U_{32},$$

$$\sum_{j=1}^{3} \left( U_{j2}^* \dot{U}_{j3} + i \left( \dot{\alpha} U_{j2}^* U_{13} + \dot{\beta} U_{j2}^* U_{23} \right) \right) = -\frac{Cy_2^2}{16\pi^2} \xi_{13} U_{31}^* U_{33},$$

$$\sum_{j=1}^{3} \left( U_{j3}^* \dot{U}_{j3} + i \left( \dot{\alpha} U_{j3}^* U_{13} + \dot{\beta} U_{j3}^* U_{23} \right) \right) = -\frac{Cy_3^2}{16\pi^2} \xi_{23} U_{32}^* U_{33},$$  \hspace{1cm} (14)

where $\xi_{ij} \equiv (y_i^2 + y_j^2) / (y_i^2 - y_j^2)$. Again, the $y_i^2$-associated terms in Eqs. (13) and (14) only contain $U_{3i}$ (for $i = 1, 2, 3$). These RG equations can therefore be specified in a relatively concise way, if the parametrization of $U$ shown in Eq. (3) is taken into account.

Explicitly, the Yukawa coupling eigenvalues of three Dirac neutrinos obey the one-loop RG equations

$$\dot{y}_1 = \frac{y_1}{16\pi^2} \left( \alpha_D + Cy_1^2 s_\nu^2 s_\phi^2 \right),$$

$$\dot{y}_2 = \frac{y_2}{16\pi^2} \left( \alpha_D + Cy_2^2 c_\nu^2 s_\phi^2 \right),$$

$$\dot{y}_3 = \frac{y_3}{16\pi^2} \left( \alpha_D + 2Cy_3^2 c_\phi^2 \right).$$  \hspace{1cm} (15)

The RG equations of three neutrino mixing angles and one (physical) CP-violating phase are given by

$$\dot{\theta}_l = +\frac{Cy_1^2}{16\pi^2} c_\nu s_\nu c_\phi (\xi_{13} - \xi_{23}),$$

$$\dot{\theta}_\nu = +\frac{Cy_2^2}{16\pi^2} c_\nu s_\nu \left[ s_\nu^2 \xi_{12} + c_\nu^2 (\xi_{13} - \xi_{23}) \right],$$

$$\dot{\theta} = +\frac{Cy_3^2}{16\pi^2} c_s \left[ s_\nu^2 \xi_{13} + c_\nu^2 \xi_{23} \right],$$

$$\dot{\phi} = -\frac{Cy_1^2}{16\pi^2} \left( c_l^2 - s_l^2 \right) c_l^{-1} s_l^{-1} c_\nu s_\nu c_\phi (\xi_{13} - \xi_{23}),$$  \hspace{1cm} (16)

where $c_\phi \equiv \cos \phi$ and $s_\phi \equiv \sin \phi$. The RG equations of two unphysical phases $\alpha$ and $\beta$ read

$$\dot{\alpha} = -\frac{Cy_1^2}{16\pi^2} c_l s_l^{-1} c_\nu s_\nu c_\phi (\xi_{13} - \xi_{23}),$$

$$\dot{\beta} = +\frac{Cy_3^2}{16\pi^2} c_l^{-1} s_l c_\nu s_\nu c_\phi (\xi_{13} - \xi_{23}).$$  \hspace{1cm} (17)
The relationship $\dot{\phi} = \dot{\alpha} + \dot{\beta}$ holds obviously, implying that $\alpha$ and $\beta$ are not negligible in deriving the RG equations of other physical parameters. One can see that our analytical results are really concise, thanks to the novel parametrization of $U$ that we have taken.

Some qualitative remarks on the main features of Eqs. (15), (16) and (17) are in order.

(1) Like the Majorana case, the RG running behaviors of three Dirac neutrino masses $m_i$ (or equivalently $y_i$) are nearly identical and determined by $\alpha_D$ [8], unless $\tan \beta$ is sufficiently large in the MSSM. It is also worth mentioning that $\dot{y}_i$ or $\dot{m}_i$ (for $i = 1, 2, 3$) are independent of both the CP-violating phase $\phi$ and the smallest mixing angle $\theta_l$ in our parametrization.

(2) The derivative of $\theta_\nu$ consists of a term proportional to $\xi_{12} = -(m_1^2 + m_2^2)/\Delta m^2_{21}$. Hence $\theta_\nu$ is in general more sensitive to radiative corrections than $\theta_l$ and $\theta$, whose derivatives are only dependent on $\xi_{13} = -(m_2^2 + m_3^2)/\Delta m^2_{31}$ and $\xi_{23} = -(m_2^2 + m_3^2)/\Delta m^2_{32}$. Given $\theta_\nu$ and $\theta$ at a specific energy scale, the smallest mixing angle $\theta_l$ can be radiatively generated at another energy scale. In this case, however, it is impossible to simultaneously generate the CP-violating phase $\phi$ (see Ref. [8] for a similar conclusion in the “standard” parametrization of $U$). The reason is simply that $\phi$ can always be rotated away when $\theta_l$ is exactly vanishing, and the proportionality relationship between $\phi$ and $\sin \phi$ forbids $\phi$ to be generated even when $\theta_l$ becomes non-vanishing.

(3) Different from the Majorana case, there is no non-trivial quasi-fixed point in the RG evolution of $\phi$ for Dirac neutrinos. If $\phi$ is required to keep finite when $\theta_l$ approaches zero, then $\phi$ itself must approach zero or $\pi$, as indicated by Eq. (16). On the other hand, $\dot{\theta}_l \propto \cos \phi$ implies that the RG running of $\theta_l$ has a turning point characterized by $\phi = \pi/2$ (i.e., $\dot{\theta}_l$ flips its sign at this point). Hence two interesting conclusions analogous to those drawn in Ref. [8] can be achieved: first, $\theta_l$ can never cross zero if $\theta_l \neq 0$ and $\sin \phi \neq 0$ hold at a certain energy scale; second, CP will always be a good symmetry if $\theta_l = 0$ or $\sin \phi = 0$ holds at a certain energy scale.

(4) The RG running behavior of $\alpha$ is quite similar to that of $\phi$, because $\dot{\phi} = \dot{\alpha} (1 - \tan^2 \theta_l)$ holds. In addition, $\dot{\beta} = -\dot{\alpha} \tan^2 \theta_l$ holds, implying that $\beta$ only gets some relatively mild RG corrections.

Let us remark that the Jarlskog invariant of CP violation [21] takes the same form for Dirac and Majorana neutrinos: $J = c_1 s_1 c_2 s_2 s_3 \phi$. If neutrinos are Dirac particles, the one-loop RG equation of $\mathcal{J}_D$ can be expressed as

$$\mathcal{J}_D = \frac{C y^2}{16 \pi^2} \mathcal{J}_D \left[ \left( c_\nu^2 - s_\nu^2 \right) s^2 \xi_{12} + \left( c^2 - s_\nu^2 s^2 \right) \xi_{13} + \left( c^2 - c_\nu^2 s^2 \right) \xi_{23} \right].$$

(18)

It becomes obvious that $\mathcal{J}_D = 0$ will be a stable result independent of the renormalization scales, provided $\theta_l$ or $\sin \phi$ initially vanishes at a given scale. In comparison, we have

$$\mathcal{J}_M = \frac{C y^2}{16 \pi^2} \left\{ \mathcal{J}_M \left[ \left( c_\nu^2 - s_\nu^2 \right) s^2 \left( \xi_{12}^{-1} c^2_{(\sigma - \rho)} + \xi_{12} s^2_{(\sigma - \rho)} \right) + \left( c^2 - s_\nu^2 s^2 \right) \left( \xi_{13}^{-1} c^2_\rho + \xi_{13} s^2_\rho \right) + \left( c^2 - c_\nu^2 s^2 \right) \left( \xi_{23}^{-1} c^2_\sigma + \xi_{23} s^2_\sigma \right) \right] + c_\nu s_\nu c s^2 \left( C_{12} \tilde{\mathcal{C}}_{12} + C_{13} \tilde{\mathcal{C}}_{13} + C_{23} \tilde{\mathcal{C}}_{23} \right) \right\}$$

(19)

for Majorana neutrinos, where

$$C_{12} = c_1 s_1 c_\phi c_{(\sigma - \rho)} s_{(\sigma - \rho)},$$

$$C_{13} = \left[ c_1 s_1 c_\phi \left( s_\rho^2 - c_\nu^2 c^2 \right) + \left( c^2 - s_\nu^2 \right) c_\nu s_\nu c \right] c_\rho s_\rho,$$

$$C_{23} = \left[ c_1 s_1 c_\phi \left( c_\nu^2 - s_\nu^2 c^2 \right) - \left( c^2 - s_\nu^2 \right) c_\nu s_\nu c \right] c_\sigma s_\sigma.$$
One can see that $J_M$ can be radiatively generated from two non-trivial Majorana phases $\rho$ and $\sigma$, even if it is initially vanishing at a specific scale. Taking $\rho = \sigma = 0$, we arrive at $C_{12} = C_{13} = C_{23} = 0$ as well as $\dot{\rho} = \dot{\sigma} = 0$. But it is impossible to obtain the equality $J_M(\rho = \sigma = 0) = J_D$, because $\zeta_{12}^{-1} = \xi_{12}$, $\zeta_{13}^{-1} = \xi_{13}$ and $\zeta_{23}^{-1} = \xi_{23}$ (or equivalently $m_1m_2 = m_1m_3 = m_2m_3 = 0$) cannot simultaneously hold. This observation demonstrates again that the RG running behavior of $J_M$ is essentially different from that of $J_D$.

IV. CONCLUDING REMARKS

We have pointed out that the $\tau$-lepton dominance in the one-loop RG equations of relevant neutrino mixing quantities allows us to set a criterion for the choice of the most appropriate parametrization of the lepton flavor mixing matrix $U$: its elements $U_{3i}$ (for $i = 1, 2, 3$) should be as simple as possible. Such a novel parametrization does exist, but it is quite different from the “standard” parametrization advocated by the Particle Data Group and used in the literature. We have shown that this parametrization can lead to greatly simplified RG equations for three mixing angles and the physical CP-violating phase(s), no matter whether neutrinos are Dirac or Majorana particles.

The present work aims at the derivation of those one-loop RG equations and some generic discussions about their important features. A quantitative and detailed analysis of the RG running behaviors of relevant neutrino mixing angles and CP-violating phases is very desirable and will be done elsewhere. It is worth emphasizing that our analytical results are so concise that they can help understand radiative corrections to lepton flavor mixing parameters in a much simpler and more transparent way. In particular, they are expected to be very useful for model building at a superhigh energy scale (e.g., the seesaw scale) to explore possible flavor symmetries or flavor dynamics which are responsible for the origin of neutrinos masses and leptonic CP violation.

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REFERENCES

[1] SNO Collaboration, Q.R. Ahmad et al., Phys. Rev. Lett. 89, 011301 (2002).
[2] For a review, see: C.K. Jung et al., Ann. Rev. Nucl. Part. Sci. 51, 451 (2001).
[3] KamLAND Collaboration, K. Eguchi et al., Phys. Rev. Lett. 90, 021802 (2003); CHOOZ Collaboration, M. Apollonio et al., Phys. Lett. B 420, 397 (1998); Palo Verde Collaboration, F. Boehm et al., Phys. Rev. Lett. 84, 3764 (2000).
[4] K2K Collaboration, M.H. Ahn et al., Phys. Rev. Lett. 90, 041801 (2003).
[5] For recent reviews with extensive references, see: H. Fritzsch and Z.Z. Xing, Prog. Part. Nucl. Phys. 45, 1 (2000); S.F. King, Rept. Prog. Phys. 67, 107 (2004); G. Altarelli and F. Feruglio, New J. Phys. 6, 106 (2004); R.N. Mohapatra et al., hep-ph/0510213.
[6] P. Minkowski, Phys. Lett. B 67, 421 (1977); T. Yanagida, in Proceedings of the Workshop on Unified Theory and the Baryon Number of the Universe, edited by O. Sawada and A. Sugamoto (KEK, Tsukuba, 1979), p. 95; M. Gell-Mann, P. Ramond, and R. Slansky, in Supergravity, edited by F. van Nieuwenhuizen and D. Freedman (North Holland, Amsterdam, 1979), p. 315; S.L. Glashow, in Quarks and Leptons, edited by M. Lévy et al. (Plenum, New York, 1980), p. 707; R.N. Mohapatra and G. Senjanovic, Phys. Rev. Lett. 44, 912 (1980).
[7] P.H. Chankowski and Z. Pluciennik, Phys. Lett. B 316, 312 (1993); K.S. Babu, C.N. Leung, and J. Pantaleone, Phys. Lett. B 319, 191 (1993); S. Antusch, M. Drees, J. Kersten, M. Lindner, and M. Ratz, Phys. Lett. B 519, 238 (2001); Phys. Lett. B 525, 130 (2002).
[8] M. Lindner, M. Ratz, and M.A. Schmidt, hep-ph/0506280; and references therein.
[9] Particle Data Group, S. Eidelman et al., Phys. Lett. B 592, 1 (2004).
[10] H. Fritzsch and Z.Z. Xing, Phys. Rev. D 57, 594 (1998).
[11] See, e.g., H. Fritzsch and Z.Z. Xing, Phys. Lett. B 372, 265 (1996); Phys. Lett. B 440, 313 (1998); Phys. Rev. D 61, 073016 (2000); Phys. Lett. B 598, 237 (2004); Z.Z. Xing, Phys. Lett. B 530, 159 (2002); Phys. Lett. B 539, 85 (2002).
[12] H. Fritzsch and Z.Z. Xing, in preparation.
[13] See, e.g., M. Maltoni, T. Schwetz, M.A. Tortola, and J.W.F. Valle, New. J. Phys. 6, 122 (2004); A. Strumia and F. Vissani, hep-ph/0503246.
[14] J.A. Casas, J.R. Espinosa, A. Ibarra, and I. Navarro, Nucl. Phys. B 573, 652 (2000).
[15] S. Antusch, J. Kersten, M. Lindner, and M. Ratz, Nucl. Phys. B 674, 401 (2003).
[16] S. Antusch, J. Kersten, M. Lindner, M. Ratz, and M.A. Schmidt, JHEP 0503, 024 (2005).
[17] J.W. Mei, Phys. Rev. D 71, 073012 (2005).
[18] S. Luo, J. Mei, and Z.Z. Xing, Phys. Rev. D 72, 053014 (2005).
[19] S. Luo and Z.Z. Xing, hep-ph/0509065.
[20] H. Zhang, a research note in 2005 (unpublished).
[21] C. Jarlskog, Phys. Rev. Lett. 55, 1039 (1985); Z.Z. Xing, Phys. Rev. D 64, 033005 (2001); Int. J. Mod. Phys. A 19, 1 (2004).