Convergence rate to a lower tail dependence coefficient
of a skew-t distribution

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Abstract

We examine the rate of decay to the limit of the tail dependence coefficient of a bivariate skew t distribution which always displays asymptotic tail dependence. It contains as a special case the usual bivariate symmetric t distribution, and hence is an appropriate (skew) extension. The rate is asymptotically power-law. The second-order structure of the univariate quantile function for such a skew-t distribution is a central issue.

Keywords: Bivariate skew-t distribution, lower tail dependence coefficient, quantile function, convergence rate.

1 Background and Motivation

The coefficient of lower tail dependence of a random vector $\mathbf{X} = (X_1, X_2)^T$ with marginal inverse distribution function $F_1^{-1}$ and $F_2^{-1}$ is defined as

$$\lambda_L = \lim_{u \to 0^+} \lambda_L(u), \quad \text{where} \quad \lambda_L(u) = P(X_1 \leq F_1^{-1}(u) | X_2 \leq F_2^{-1}(u)).$$

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\( \mathbf{X} \) is said to have asymptotic lower tail dependence if \( \lambda_L \) exists positive. If \( \lambda_L = 0 \), then \( \mathbf{X} \) is said to be asymptotically independent in the lower tail. This quantity provides insight on the tendency for the distribution to generate joint extreme event since it measures the strength of dependence (or association) in the lower tails of a bivariate distribution. If the marginal distributions of these random variables are continuous, then from (1), it follows that \( \lambda_L(u) \) can be expressed in terms of the copula of \( \mathbf{X} \), \( C(u_1, u_2) \), as

\[
\lambda_L(u) = \frac{P(X_1 \leq F_1^{-1}(u), X_2 \leq F_2^{-1}(u))}{P(X_2 \leq F_2^{-1}(u))} = \frac{C(u, u)}{u}.
\]  

(2)

In this paper we investigate the rate of convergence to 0 of \( \lambda_L(u) - \lambda_L \) as \( u \to 0^+ \), in an important case when \( \lambda_L > 0 \). Heffernan (2000) provides a summary of coefficients for many commonly employed bivariate distributions, but the specific situation which we study is not considered.

In the sequel we refer to the bivariate skew-\( t \) as that distribution resulting from variance-mixing of the bivariate skew normal, \( \mathbf{Z} \sim SN_2(\theta, R) \) (see Azzalini and Dalla Valle (1996)), inversely with a gamma random variable \( V \sim \Gamma(\eta_2, \eta_2) \), with \( \eta > 0 \):

\[
\mathbf{X} = V^{-\frac{1}{2}} \mathbf{Z},
\]

where \( \mathbf{Z} \) is independently distributed of \( V \).

This skew distribution was originally introduced in multivariate form in Branco and Dey (2001) and studied extensively in Azzalini and Capitanio (2003). Some recent reviews on this area of study can be found in Azzalini and Genton (2008), Azzalini and Capitanio (2010) and in the book edited by Genton (2004).

The bivariate skew-\( t \) always satisfies \( \lambda_L > 0 \) (See Fung and Seneta (2010)). This was also considered in Bortot (2010) and Padoan (2011) with an approach initiated by Cheng and Genton (2007) which is quite different from that of Fung and Seneta (2010). The case \( \theta_1 = \theta_2 = 0 \) reduces to the symmetric bivariate \( t \) distribution. In this sense, the bivariate skew-\( t \) distribution defined by (3) is a more appropriate generalisation of the symmetric case.

The motivation for our investigation of the rate of convergence in the present specific case of bivariate skew-\( t \) arose from the following. Ramos and Ledford (2009), continuing
the work of Ledford and Tawn (1997), studied intensively a family of bivariate distributions (which they characterised) which satisfied in particular the condition

\[ \lambda_L(u) = u^{\frac{1}{\alpha}-1}L(u). \]  

Here \( L(u) \) is a slowly varying function as \( u \to 0^+ \), and \( \alpha \in (0, 1] \), so that, in fact, the value of \( \alpha \) could be used for comparison of the degree of tail dependence structure between members of the family. The standard bivariate extreme value models correspond to \( \alpha = 1 \).

Expression (4) may also be regarded as the rate of convergence to \( \lambda_L(u) \) when \( \lambda_L = 0 \), but when \( \lambda_L(u) \to \lambda_L > 0 \), which is also covered by (4) with \( \alpha = 1 \) and \( L(u) \to \lambda_L \), the rate of convergence is more appropriately studied by considering the rate of convergence to 0 as \( u \to 0^+ \) of

\[ |\lambda_L(u) - \lambda_L|. \]

Our study of an important special case is an early step in this direction.

2 The Bivariate Skew-\( t \) Distribution

From Branco and Dey (2001), the random vector \( \mathbf{X} \), defined by (3), has probability density:

\[
 f_{\mathbf{X}}(\mathbf{x}) = \frac{2\Gamma\left(\frac{\eta+2}{2}\right)}{\pi\eta\Gamma\left(\frac{\eta}{2}\right)\sqrt{1-\rho^2}} F_{t_{\eta+2}}\left(\theta^T \mathbf{x} \sqrt{\frac{\eta+2}{\eta+\mathbf{x}^T \mathbf{R}^{-1} \mathbf{x}}} \right),
\]

where \( F_{t_{\eta+2}}(\cdot) \) is the distribution function of the (symmetric) \( t \) distribution with \( \eta + 2 \) degrees of freedom, \( \mathbf{R} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \), and \( \theta = (\theta_1, \theta_2)^T \) is a vector that controls the asymmetry of the distribution.

The marginal density of \( X_1 \) can then be found as

\[
 f_{X_1}(x) = 2f_{\eta}(x)F_{t_{\eta+1}}(\lambda_1 x \sqrt{\frac{\eta+1}{\eta+x^2}}),
\]
where

\[ f_{\eta}(x) = \frac{\Gamma\left(\frac{\eta+1}{2}\right)}{(\pi\eta)^{\frac{\eta}{2}}\Gamma\left(\frac{\eta}{2}\right)} \left(1 + \frac{x^2}{\eta}\right)^{-\frac{\eta+1}{2}} \]  

(5)

is the density of the (symmetric) t distribution with \( \eta \) degrees of freedom and \( \lambda_1 = (\theta_1 + \rho \theta_2)/\sqrt{1 + \theta_2^2(1 - \rho^2)} \). \( X_2 \) has a similar marginal density, except its marginal skewness parameter, \( \lambda_2 \), takes the form of \( \lambda_2 = (\theta_2 + \rho \theta_1)/\sqrt{1 + \theta_1^2(1 - \rho^2)} \).

From (2) and using some basic properties of copulas, it can be shown that

\[ \lambda_L = \lim_{u \to 0^+} \lambda_L(u) = \lim_{y \to -\infty} \left[ P(X_2 \leq F_2^{-1}(F_1(y))|X_1 = y) + P(X_1 \leq F_1^{-1}(F_2(y))|X_2 = y) \right]. \]  

(6)

Fung and Seneta (2010) showed that if \( X = (X_1, X_2)^T \) is a random vector defined by (3), then

\[ \lim_{y \to -\infty} P(X_2 \leq F_2^{-1}(F_1(y))|X_1 = y) = \int_{-\infty}^{-a_{2.1}} f_{t_{\eta+2}}(z) \frac{F_{t_{\eta+2}}\left(\left(\theta_2 \sqrt{\frac{(1 - \rho^2)}{\eta+1}} z - (\theta_1 + \rho \theta_2)\right) \sqrt{\frac{\eta+2}{1 + \frac{z^2}{\eta+1}}} \right)}{F_{t_{\eta+1}}(-\lambda_1 \sqrt{\eta + 1})} dz; \]  

(7)

and

\[ \lim_{y \to -\infty} P(X_1 \leq F_1^{-1}(F_2(y))|X_2 = y) = \int_{-\infty}^{-a_{1.2}} f_{t_{\eta+2}}(z) \frac{F_{t_{\eta+2}}\left(\left(\theta_1 \sqrt{\frac{(1 - \rho^2)}{\eta+1}} z - (\theta_2 + \rho \theta_1)\right) \sqrt{\frac{\eta+2}{1 + \frac{z^2}{\eta+1}}} \right)}{F_{t_{\eta+1}}(-\lambda_2 \sqrt{\eta + 1})} dz; \]  

(8)

where \( a_{2.1} = \left(\frac{F_{t_{\eta+1}}(-\lambda_2 \sqrt{\eta + 1})}{F_{t_{\eta+1}}(-\lambda_1 \sqrt{\eta + 1})}\right)^\frac{1}{\eta} - \rho \sqrt{\frac{\eta+1}{1 - \rho^2}} \), and \( a_{1.2} = \left(\frac{F_{t_{\eta+1}}(-\lambda_1 \sqrt{\eta + 1})}{F_{t_{\eta+1}}(-\lambda_2 \sqrt{\eta + 1})}\right)^\frac{1}{\eta} - \rho \sqrt{\frac{\eta+1}{1 - \rho^2}} \).

We shall show that

\[ |\lambda_L(u) - \lambda_L| = \left| \frac{C(u, u)}{u} - \lambda_L \right| \sim \text{Const.} u^{\frac{2}{\eta}}, \]  

(9)

as \( u \to 0^+ \).

The rest of this paper is set out as follows. In Section 3, we derive an accurate lower quantile result for the skew t distribution defined in (3). In Section 4, we derive the rate
of convergence in the form of (9) for the skew-t distribution.

3 Lower Quantile results

In our subsequent theoretical development, both the asymptotic behaviour of \( F_i(y) \) and its inverse \( F_i^{-1}(y) \) as \( y \to -\infty \) with higher order terms are needed. We begin by discussing the behaviour of \( F_i(y) \) as \( y \to -\infty \). Without loss of generality, set \( i = 1 \). The result is summarised into the following theorem, only the first term of which is given in eqn.(28) of Fung and Seneta (2010).

**Theorem 1.** The asymptotic behaviour of the marginal distribution function of \( X_1 \) is

\[
F_1(y) = P(X_1 \leq y) = c_1 |y|^{\eta/2} (1 + d_1 y^{-2} + O(y^{-4})), \quad \text{as} \quad y \to -\infty,
\]

where

\[
c_1 = \frac{2 \Gamma(\frac{\eta+1}{2}) \Gamma(\frac{\eta}{2})}{(\pi \eta)^{1/2} \Gamma(\frac{\eta+1}{2})},
\]

\[
d_1 = \frac{-\eta^2 (\eta + 1)}{2(\eta + 2)} + \frac{\eta^2 f_{t+1}(-\lambda_1 \sqrt{\eta + 1} \lambda_1 \sqrt{\eta + 1})}{2(\eta + 2) F_{t+1}(-\lambda_1 \sqrt{\eta + 1})}.
\]

**Proof.** For \( x < 0 \) and by using a second-order Mean Value Theorem, we have

\[
F_{t+1}(\lambda_1 x \sqrt{\eta + 1}) = F_{t+1}(-\lambda_1 \sqrt{\eta + 1} (1 + \frac{\eta}{x^2})^{-\frac{1}{2}})
\]

\[
= F_{t+1}(-\lambda_1 \sqrt{\eta + 1}) + f_{t+1}(-\lambda_1 \sqrt{\eta + 1}) \lambda_1 \sqrt{\eta + 1} [1 - (1 + \frac{\eta}{x^2})^{-\frac{3}{2}}]
\]

\[
+ (\lambda_1 \sqrt{\eta + 1})^2 [1 - (1 + \frac{\eta}{x^2})^{-\frac{3}{2}}] F''_{t+1}(\delta_1(x))/2,
\]

for some \( \delta_1(x) \) contained in the interval \( \min(-\lambda_1 \sqrt{\eta + 1} (1 + \frac{\eta}{x^2})^{-\frac{1}{2}}, -\lambda_1 \sqrt{\eta + 1}), \max(-\lambda_1 \sqrt{\eta + 1} (1 + \frac{\eta}{x^2})^{-\frac{1}{2}}, -\lambda_1 \sqrt{\eta + 1}) \);

\[
= F_{t+1}(-\lambda_1 \sqrt{\eta + 1}) + f_{t+1}(-\lambda_1 \sqrt{\eta + 1}) \lambda_1 \sqrt{\eta + 1} \frac{\eta}{2x^2} + O(\frac{1}{x^4})
\]

\[
+ (\lambda_1 \sqrt{\eta + 1})^2 [1 - (1 + \frac{\eta}{x^2})^{-\frac{3}{2}}] F''_{t+1}(\delta_1(x))/2.
\]

Since \( |F''_{t+1}(\delta_1(x))| = \left| \frac{\Gamma(\frac{\eta+1}{2})}{(\pi \eta)^{1/2} \Gamma(\frac{\eta}{2})} \left( \eta + 1 \eta \right) \delta_1(x) (1 + \frac{\delta_1^2(x)}{\eta})^{-\frac{\eta}{2}} \right| \leq k_1 \)
for some constant \( k_1 \) as the function is bounded for large \( |x| \) and

\[
(1 - (1 + \frac{\eta}{x^2})^{-\frac{1}{2}})^2 = O(\frac{1}{x^4}).
\]

Therefore the dominating term of

\[
(\lambda_1 \sqrt{\eta + 1})^2 [1 - (1 + \frac{\eta}{x^2})^{-\frac{1}{2}}]^2 F''_{\eta+1}(\delta_1(x))/2
\]
is in the order of \( x^{-4} \) and hence,

\[
F_{\eta+1}(\lambda_1 x \sqrt{\frac{\eta + 1}{\eta + x^2}})
\]

\[
= F_{\eta+1}(-\lambda_1 \sqrt{\eta + 1}) + f_{\eta+1}(-\lambda_1 \sqrt{\eta + 1}) \lambda_1 \sqrt{\eta + 1} \frac{\eta}{2x^2} \left(1 + O(\frac{1}{x^2})\right).
\]

Then for any \( y < 0 \),

\[
F_1(y) = P(X_1 \leq y) = \int_{-\infty}^{y} 2f_{\eta}(x) F_{\eta+1}(\lambda_1 x \sqrt{\frac{\eta + 1}{\eta + x^2}}) \, dx
\]

\[
= \int_{-\infty}^{y} 2f_{\eta}(x) F_{\eta+1}(-\lambda_1 \sqrt{\eta + 1}) \, dx
\]

\[
+ \int_{-\infty}^{y} f_{\eta}(x) f_{\eta+1}(-\lambda_1 \sqrt{\eta + 1}) \lambda_1 \sqrt{\eta + 1} x^{-2} \left(1 + O(\frac{1}{x^2})\right) \, dx
\]

(11)

We shall consider these two terms separately. Focusing on the first term, i.e. (11), we have

\[
\int_{-\infty}^{y} 2f_{\eta}(x) F_{\eta+1}(-\lambda_1 \sqrt{\eta + 1}) \, dx
\]

by setting \( c = \frac{2\Gamma(\frac{\eta+1}{2})\Gamma(\frac{\eta+2}{2})}{(\pi\eta)^\frac{\eta+1}{2}} F_{\eta+1}(-\lambda_1 \sqrt{\eta + 1}); \)

\[
= c \frac{|y|^{-\eta}}{\eta} \left(1 - (\frac{\eta + 1}{2}) \frac{\eta}{y^2} + O(\frac{1}{y^4})\right) + c(\eta + 1) \left\{ \frac{|y|^{-(\eta+2)}}{\eta + 2} \left(1 + O(\frac{1}{y^2})\right) \right\}
\]

by applying integration by parts as suggested in Soms (1976). Thus,

\[
\int_{-\infty}^{y} 2f_{\eta}(x) F_{\eta+1}(-\lambda_1 \sqrt{\eta + 1}) \, dx
\]
\[ \Gamma\left(\frac{n+1}{2}\right)\eta^{\frac{n+1}{2}} F_{\eta+1}\left(-\sqrt{\eta+1}\lambda_1 \sqrt{\eta+1} x^{-2} \left(1 + O\left(\frac{1}{x^2}\right)\right) \right) \]

The second term, i.e. (12), can be treated similarly to get

\[ \int_{-\infty}^{y} f_{n}(x)f_{n+1}\left(-\sqrt{\eta+1}\lambda_1 \sqrt{\eta+1} x^{-2} \left(1 + O\left(\frac{1}{x^2}\right)\right) \right) dx 
= \Gamma\left(\frac{n+1}{2}\right)\eta^{\frac{n+3}{2}} f_{\eta+1}\left(-\sqrt{\eta+1}\lambda_1 \sqrt{\eta+1} \frac{y}{\eta+1} \left(1 + O\left(\frac{1}{y^2}\right)\right) \right), \quad (14) \]

Hence, by combining (13) and (14) the result follows. \(\square\)

**Theorem 2.** The inverse of \(P(X_1 \leq y)\), \(F^{-1}_1(u)\), satisfies:

\[ F^{-1}_1(u) = -c_1^{-1} u^{\frac{1}{\eta}}(1 + d_1 \frac{u^\frac{2}{\eta}}{c_1^\eta} + O(u^\frac{4}{\eta})), \quad \text{as } u \rightarrow 0^+ \]  

**Proof.** On account of (10), to find the inverse of \(F_1(\cdot)\) i.e. \(F^{-1}_1(\cdot)\), it is sufficient to consider the function \(H(y) = F_1(-y)\), \(y > 0\), so that, by Theorem 1

\[ H(y) = c_1 y^{-\eta}(1 + d_1 y^{-2} + O(y^{-4})). \]

where \(\eta > 0\) and \(c_1, d_1 \neq 0\), as \(y \rightarrow \infty\). Now define \(G(y) = 1/H(y)\) so that

\[ G(y) = c_1^{-1} y^{\eta}(1 - d_1 y^{-2} + O(y^{-4})) = c_1^{-1} y^{\eta} S(y) \]  

which defines \(S(y)\), and we note \(S(y) \rightarrow 1\), as \(y \rightarrow \infty\). Noting that \(G(y)\) is strictly increasing and continuous, denote its inverse by \(G^{-1}(y)\). (We shall use this notation for inverses, to avoid confusion, only in this proof.) Then:

\[ y = G\left(G^{-1}(y)\right) = c_1^{-1}\left(G^{-1}(y)\right)^{\eta} \left(1 - d_1\left(G^{-1}(y)\right)^{-2} + O\left((G^{-1}(y))^{-4}\right)\right) \]

so that

\[ G^{-1}(y) = \left\{ \frac{c_1 y}{\left(1 - d_1\left(G^{-1}(y)\right)^{-2} + O\left((G^{-1}(y))^{-4}\right)\right)} \right\}^{\frac{1}{\eta}} \]

\[ = \frac{1}{\eta} \left(1 + \frac{d_1}{\eta} (G^{-1}(y))^{-2} + O((G^{-1}(y))^{-4})\right) \]  

(17)
\[ y = G^-(G(y)) = ((G(y)c_1)^{\frac{1}{\eta}} S^*(G(y)) = (c_1^{-1} y^\eta S(y)c_1)^{\frac{1}{\eta}} S^*(G(y)) \]

so that \( S^*(G(y)) = S^{-\frac{1}{\eta}}(y) \), whence, since \( S(y) \to 1 \) as \( y \to \infty \)

\[ \lim_{y \to \infty} S^*(y) = 1. \]  

Hence, substituting expression (17) for \( G^-(y) \) into the right hand side of (17) (recursively), and using (19), as \( y \to \infty \),

\[ G^-(y) = c_1^{\frac{1}{\eta}} y^{\frac{1}{\eta}} \left( 1 + \frac{d_1}{c_1^{\frac{1}{\eta}}} y^{-\frac{2}{\eta}} + O(y^{-\frac{4}{\eta}}) \right). \]

The final result follows as \( H(y) = 1/G(y) \) implies that \( H^-(y) = G^-(1/y) \). 

The representations (16) and (18) are those for a regularly varying function with index \( \eta \), and its inverse \( G^-(\cdot) \), regularly varying with index \( 1/\eta \). (See Proposition 0.8 on p. 22 of Resnick (1987)). However, the specialized form of the slowly varying function \( S(y) \) needs to be invoked in our self-contained proof.

Similarly, the inverse of \( P(X_2 \leq y) \), i.e. \( F_2^{-1}(u) \), is thus

\[ F_2^{-1}(u) = -c_2^{\frac{1}{\eta}} u^{\frac{1}{\eta}} \left( 1 + \frac{d_2}{c_2^{\frac{1}{\eta}}} u^{\frac{2}{\eta}} + O(u^{\frac{4}{\eta}}) \right), \quad \text{as } u \to 0^+, \]  

where

\[ c_2 = \frac{2\Gamma(\eta + 1)\eta^{\eta + 1}}{(\pi \eta)^{\frac{1}{2}} \Gamma(\frac{\eta}{2})} \frac{F_{t_{\eta + 1}}(-\lambda_2 \sqrt{\eta + 1})}{F_{t_{\eta + 1}}(-\lambda_1 \sqrt{\eta + 1})}, \]

and

\[ d_2 = -\eta^2(\eta + 1) + \eta^2 f_{t_{\eta + 1}}(-\lambda_2 \sqrt{\eta + 1})\lambda_2 \sqrt{\eta + 1} \frac{\lambda_2 \sqrt{\eta + 1}}{2(\eta + 2)F_{t_{\eta + 1}}(-\lambda_2 \sqrt{\eta + 1})}. \]

A result which we shall need repeatedly in the sequel is that

\[ c(y) \overset{\text{def}}{=} F_2^{-1}(F_1(y)) = \left( \frac{F_{t_{\eta + 1}}(-\lambda_2 \sqrt{\eta + 1})}{F_{t_{\eta + 1}}(-\lambda_1 \sqrt{\eta + 1})} \right)^{\frac{1}{\eta}} y \left( 1 - \frac{d_1 - d_2(c_2^{\frac{1}{\eta}})^{\frac{2}{\eta}}}{\eta y^2} + O(y^{-4}) \right) \]  

as \( y \to -\infty \), which follows after some algebra by combining (10) and (20).

Notice that when \( \lambda_1 = \lambda_2 = \lambda \), then the first order term in (21) vanishes as

\[
d_1 - d_2 \left( \frac{c_1}{c_2^2} \right)^{\frac{1}{\eta}} = -\frac{\eta^2(\eta + 1)}{2(\eta + 2)} + \frac{\eta^2 f_{\eta+1}(-\lambda_1 \sqrt{\eta + 1}) \lambda_1 \sqrt{\eta + 1}}{2(\eta + 2) F_{\eta+1}(-\lambda_1 \sqrt{\eta + 1})} - \left( \frac{F_{\eta+1}(-\lambda_1 \sqrt{\eta + 1})}{F_{\eta+1}(-\lambda_2 \sqrt{\eta + 1})} \right)^{\frac{1}{\eta}}
\]

\[
\times \left( -\frac{\eta^2(\eta + 1)}{2(\eta + 2)} + \frac{\eta^2 f_{\eta+1}(-\lambda_2 \sqrt{\eta + 1}) \lambda_2 \sqrt{\eta + 1}}{2(\eta + 2) F_{\eta+1}(-\lambda_2 \sqrt{\eta + 1})} \right)
\]

\[
= -\frac{\eta^2(\eta + 1)}{2(\eta + 2)} + \frac{\eta^2 f_{\eta+1}(-\lambda \sqrt{\eta + 1}) \lambda \sqrt{\eta + 1}}{2(\eta + 2) F_{\eta+1}(-\lambda \sqrt{\eta + 1})}
\]

\[
- \left( -\frac{\eta^2(\eta + 1)}{2(\eta + 2)} + \frac{\eta^2 f_{\eta+1}(-\lambda \sqrt{\eta + 1}) \lambda \sqrt{\eta + 1}}{2(\eta + 2) F_{\eta+1}(-\lambda \sqrt{\eta + 1})} \right) = 0,
\]

so that \( \lambda_1 = \lambda_2 \Rightarrow c(y) = F_2^{-1}(F_1(y)) = y(1 + O(y^{-3})) \). Finally, one can show that: \( \lambda_1 = \lambda_2 \Leftrightarrow \theta_1 = \theta_2 \). This case of “equiskewness” in particular covers the symmetric case \( \theta_1 = \theta_2 = 0 \).

4 Main result

Theorem 3. For the bivariate skew-t distribution:

\[
|\lambda_L(u) - \lambda_L| = u^{\frac{1}{\eta}} L(u),
\]

where \( L(u) \to k \), where \( k \) is a constant as \( u \to 0^+ \).

Proof. From (2) and using some basic properties of copulas, we have

\[
dC(u, u) \left/ \frac{du}{d u} \right. - \lambda_L
\]

\[
= \{ P(X_2 \leq F_2^{-1}(u)|X_1 = F_1^{-1}(u)) - \lim_{u \to 0^+} P(X_2 \leq F_2^{-1}(u)|X_1 = F_1^{-1}(u)) \} \tag{22}
\]

\[
+ \{ P(X_1 \leq F_1^{-1}(u)|X_2 = F_2^{-1}(u)) - \lim_{u \to 0^+} P(X_1 \leq F_1^{-1}(u)|X_2 = F_2^{-1}(u)) \} \tag{23}
\]

which allows for the distribution being skew. Without loss of generality, we focus on (22). Applying a change of variable of \( y = F_1^{-1}(u) \), so that \( y \to -\infty \) as \( u \to 0^+ \), (22) becomes

\[
P(X_2 \leq c(y)|X_1 = y) - \lim_{y \to -\infty} P(X_2 \leq c(y)|X_1 = y).
\]
Once again from Fung and Seneta (2010), these two terms can be expressed respectively
as
\[
P(X_2 \leq c(y) | X_1 = y) = \int_{-\infty}^{L_1(y)} f_{t_{\eta+1}}(z) \tau(z, y) \, dz,
\]
\[
= \int_{L_1}^{L_1(y)} f_{t_{\eta+1}}(z) \tau(z, y) \, dz + \int_{-\infty}^{L_1} f_{t_{\eta+1}}(z) \tau(z, y) \, dz,
\]
and
\[
\lim_{y \to -\infty} P(X_2 \leq c(y) | X_1 = y) = \int_{-\infty}^{L_1} f_{t_{\eta+1}}(z) \left\{ \frac{F_{t_{\eta+2}}(a(z) + b(z))}{\lambda_1 \sqrt{\eta + 1}} \right\} \, dz
\]
\[
= \int_{L_1}^{L_1} f_{t_{\eta+1}}(z) \tau(z, *) \, dz
\]
where
\[
L_1(y) = \frac{c(y) - \rho y}{\left( \frac{(y+y^2)(1-\rho^2)}{\eta+1} \right)^{\frac{1}{2}}}, \quad c(y) = F_{2}^{-1}(F_1(y)), \quad a(z) = \theta_2 \sqrt{\frac{1-\rho^2}{\eta+1}} \sqrt{\frac{\eta+2}{1+\frac{\xi^2}{\eta+1}}},
\]
\[
b(z) = -(\theta_1 + \rho \theta_2) \sqrt{\frac{\eta+2}{1+\frac{\xi^2}{\eta+1}}}, \quad \tau(z, y) = \frac{F_{t_{\eta+2}}(a(z) + b(z)(1 + \frac{y}{z})^{-\frac{1}{2}})}{F_{t_{\eta+1}}(-\lambda_1 \sqrt{\eta + 1}(1 + \frac{y}{z})^{-\frac{1}{2}})},
\]
\[
\tau(z, *) = \frac{F_{t_{\eta+2}}(a(z) + b(z))}{F_{t_{\eta+1}}(-\lambda_1 \sqrt{\eta + 1})} = \lim_{y \to -\infty} \tau(z, y), \text{ and } f_{t_{\eta+1}}(z) \text{ is defined by (5)}.
\]
Lastly,
\[
L_1 = \lim_{y \to -\infty} L_1(y) = - \left\{ \left( \frac{F_{t_{\eta+1}}(-\lambda_2 \sqrt{\eta + 1})}{F_{t_{\eta+1}}(-\lambda_1 \sqrt{\eta + 1})} \right)^{\frac{1}{2}} - \rho \right\} \frac{\eta+1}{1-\rho^2},
\]
by using (21). Notice that we made no assumption that $L_1(y) > L_1$ and the integral in (24) is still valid if $L_1(y) \leq L_1$ as $\int_{L_1}^{L_1(y)}$ is equivalent to $-\int_{L_1}^{L_1(y)}$. Thus,
\[
P(X_2 \leq F_2^{-1}(F_1(y)) | X_1 = y) = \lim_{y \to -\infty} P(X_2 \leq F_2^{-1}(F_1(y)) | X_1 = y)
\]
\[
= \int_{-\infty}^{L_1(y)} f_{t_{\eta+1}}(z) \tau(z, y) \, dz - \int_{-\infty}^{L_1} f_{t_{\eta+1}}(z) \tau(z, *) \, dz
\]
\[
= \int_{-\infty}^{L_1} f_{t_{\eta+1}}(z) \{ \tau(z, y) - \tau(z, *) \} \, dz + \int_{L_1}^{L_1(y)} f_{t_{\eta+1}}(z) \tau(z, y) \, dz.
\]
Treating these two summands separately, after some algebra,

\[ \int_{-\infty}^{L_1} f_{t_{y+1}}(z)\{\tau(z, y) - \tau(z, *)\} \, dz \]

\[ \sim -\frac{\eta}{2} y^{-2} \int_{-\infty}^{L_1} f_{t_{y+1}}(z) \left\{ \frac{f_{t_{y+1}}(-\lambda_1 \sqrt{\eta + 1}) F_{t_{y+2}}(a(z) + b(z))}{F_{t_{y+1}}(-\lambda_1 \sqrt{\eta + 1})} \lambda_1 \sqrt{\eta + 1} 
+ f_{t_{y+2}}(a(z) + b(z)) b(z) \right\} / F_{t_{y+1}}(-\lambda_1 \sqrt{\eta + 1}) \, dz, \quad \text{as } y \to -\infty. \quad (26) \]

Next, considering the second term of (25), by the mean value theorem,

\[ \int_{L_1}^{L_1(y)} f_{t_{y+1}}(z)\tau(z, y) \, dz = (L_1(y) - L_1) f_{t_{y+1}}(\xi_y) \tau(\xi_y, y) \]

\[ = \frac{y^{-2}}{(1-\eta^2)^{\frac{1}{2}}} \left\{ \left( \frac{F_{t_{y+2}}(-\lambda_2 \sqrt{\eta + 1})}{F_{t_{y+1}}(-\lambda_1 \sqrt{\eta + 1})} \right) \left( \left( \frac{d_1 - d_2((\alpha_1^2) \frac{1}{2})}{\eta} \right) + \frac{\eta}{2} \left( \frac{F_{t_{y+1}}(-\lambda_2 \sqrt{\eta + 1})}{F_{t_{y+1}}(-\lambda_1 \sqrt{\eta + 1})} \right)^{\frac{1}{2}} - \rho \right) \right\} 
\times f_{t_{y+1}}(\xi_y) \tau(\xi_y, y) \left( 1 + O\left( \frac{1}{\eta} \right) \right) \]

\[ \sim \frac{y^{-2}}{(1-\eta^2)^{\frac{1}{2}}} \left\{ \left( \frac{F_{t_{y+2}}(-\lambda_2 \sqrt{\eta + 1})}{F_{t_{y+1}}(-\lambda_1 \sqrt{\eta + 1})} \right) \left( \left( \frac{d_1 - d_2((\alpha_1^2) \frac{1}{2})}{\eta} \right) - \frac{\eta}{2} \left( \frac{F_{t_{y+1}}(-\lambda_2 \sqrt{\eta + 1})}{F_{t_{y+1}}(-\lambda_1 \sqrt{\eta + 1})} \right)^{\frac{1}{2}} - \rho \right) \right\} 
\times f_{t_{y+1}}(L_1) \frac{F_{t_{y+2}}(a(L_1) + b(L_1))}{F_{t_{y+1}}(-\lambda_1 \sqrt{\eta + 1})} (1 + O\left( \frac{1}{\eta} \right)), \quad (27) \]

as \( y \to -\infty \). Subsequently, if we combine (26) and (27), we have

\[ P(X_2 \leq c(y) | X_1 = y) - \lim_{y \to -\infty} P(X_2 \leq c(y) | X_1 = y) \]

\[ \sim y^{-2} \left\{ \int_{-\infty}^{L_1} f_{t_{y+1}}(z) \left\{ \frac{f_{t_{y+1}}(-\lambda_1 \sqrt{\eta + 1}) F_{t_{y+2}}(a(z) + b(z))}{F_{t_{y+1}}(-\lambda_1 \sqrt{\eta + 1})} \lambda_1 \sqrt{\eta + 1} \right\} 
+ f_{t_{y+2}}(a(z) + b(z)) b(z) \right\} / F_{t_{y+1}}(-\lambda_1 \sqrt{\eta + 1}) \, dz \]

\[ + \frac{1}{(1-\eta^2)^{\frac{1}{2}}} \left\{ \left( \frac{F_{t_{y+2}}(-\lambda_2 \sqrt{\eta + 1})}{F_{t_{y+1}}(-\lambda_1 \sqrt{\eta + 1})} \right) \left( \left( \frac{d_1 - d_2((\alpha_1^2) \frac{1}{2})}{\eta} \right) + \frac{\eta}{2} \left( \frac{F_{t_{y+1}}(-\lambda_2 \sqrt{\eta + 1})}{F_{t_{y+1}}(-\lambda_1 \sqrt{\eta + 1})} \right)^{\frac{1}{2}} - \rho \right) \right\} 
\times f_{t_{y+1}}(L_1) \frac{F_{t_{y+2}}(a(L_1) + b(L_1))}{F_{t_{y+1}}(-\lambda_1 \sqrt{\eta + 1})} \right\} \]

\[ = k_{2,1} y^{-2}. \]

Apply a change of variable \( u = F_1(y) \) to get

\[ P(X_2 \leq F_2^{-1}(u) | X_1 = F_1^{-1}(u)) \]

\[ - \lim_{u \to 0^+} P(X_2 \leq F_2^{-1}(u) | X_1 = F_1^{-1}(u)) \]
~k_{2,1}(F_1^{-1}(u))^{-2} \sim k_{2,1}(-c_1^u u^{-\frac{1}{\eta}})^{-2} = k_{2,1}^* u^{\frac{2}{\eta}}, \quad (28)

using (16), where

\begin{align*}
k_{2,1}^* &= \left( \frac{(\pi \eta)^{\frac{1}{2}} \Gamma(\frac{\eta}{2})}{2 \Gamma(\frac{\eta+1}{2}) \eta^{\frac{\eta}{2}} F_{t_{\eta+1}}(-\lambda_1 \sqrt{\eta} + 1)} \right)^{\frac{2}{\eta}} \left\{ - \int_{-\infty}^{L_1} f_{t_{\eta+1}}(z) \times \left\{ \frac{f_{t_{\eta+1}}(a(z) + b(z))}{F_{t_{\eta+1}}(-\lambda_1 \sqrt{\eta} + 1)} + \frac{f_{t_{\eta+1}}(-\lambda_1 \sqrt{\eta} + 1) F_{t_{\eta+1}}(a(z) + b(z))}{(F_{t_{\eta+1}}(-\lambda_1 \sqrt{\eta} + 1))^2} \times \lambda_1 \sqrt{\eta} + 1 \right\} \right\} + \frac{1}{(1-\rho^2)^2} \left\{ \frac{F_{t_{\eta+1}}(-\lambda_2 \sqrt{\eta} + 1)}{F_{t_{\eta+1}}(-\lambda_1 \sqrt{\eta} + 1)} \right\} \left( \frac{d_1 - d_2 \rho^2}{\eta} \right) + \frac{\eta}{2} \left( \frac{F_{t_{\eta+1}}(-\lambda_2 \sqrt{\eta} + 1)}{F_{t_{\eta+1}}(-\lambda_1 \sqrt{\eta} + 1)} \right)^{\frac{1}{\eta}} \left\{ \frac{f_{t_{\eta+1}}(L_1) F_{t_{\eta+1}}(a(L_1) + b(L_1))}{F_{t_{\eta+1}}(-\lambda_1 \sqrt{\eta} + 1)} \right\}.
\end{align*}

The rate of convergence for (23) can therefore be obtained similarly as

\[ P(X_1 \leq F_1^{-1}(u)|X_2 = F_2^{-1}(u)) \rightarrow_{u \rightarrow 0^+} P(X_1 \leq F_1^{-1}(u)|X_2 = F_2^{-1}(u)) \sim k_{1,2}^* u^{\frac{2}{\eta}}, \]

where \( k_{1,2}^* \) is defined analogously to \( k_{2,1}^* \).

Overall, by (22) and (23)

\[ \frac{C(u, u)}{u} - \lambda_L = C^*(u, u) = \frac{1}{u} \int_0^u dC^*(x, x) \frac{dx}{x} = \frac{u^{\frac{2}{\eta}} L(u)}{\eta/2 + 1}, \quad (29) \]

as \( u \rightarrow 0^+ \), using Karamata’s Theorem (See Resnick (1987), p. 17 or Seneta (1976), p.87) for regular variation at \( u = 0 \) for the final equality, where the slowly varying function \( L(u) \sim |k_{2,1}^* + k_{1,2}^*| \) as \( u \rightarrow 0^+ \) is asymptotically a constant. Thus (9) obtains.

Notice that if \( \theta_1 = \theta_2 = 0 \) (i.e. the symmetric \( t \) special case), then

\[ P(X_2 \leq F_2^{-1}(u)|X_1 = F_1^{-1}(u)) \rightarrow_{u \rightarrow 0^+} P(X_2 \leq F_2^{-1}(u)|X_1 = F_1^{-1}(u)) \sim f_{t_{\eta+1}}(-\sqrt{\frac{(\eta + 1)(1-\rho)}{1+\rho}}) \sqrt{\frac{(\eta + 1)(1-\rho)}{1+\rho}} \eta \left( \frac{\sqrt{\pi \Gamma(\frac{\eta}{2})}}{\Gamma(\frac{\eta+1}{2})} \right)^{\frac{\eta}{2}} u^{\frac{2}{\eta}}, \]

as \( L_1 = -\sqrt{\frac{(\eta + 1)(1-\rho)}{1+\rho}} \) and \( a(z) = b(z) = \lambda_1 = \lambda_2 = d_1 - d_2 \rho^2 \sqrt{\frac{\pi}{\Gamma(\frac{\eta}{2})}} \frac{2}{\eta} = 0 \). Comparing with (28), we can see that the slowly varying bits in both are asymptotically constant, and the
polynomial rate is the same. This consistency further supports the proposal that \( \mathcal{B} \) is a proper skew extension to the symmetric multivariate \( t \) distribution.

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