Combinatorial aspects of the conserved quantities of the tropical periodic Toda lattice

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Abstract
The tropical periodic Toda lattice (trop p-Toda) is a dynamical system attracting attention in the area of the interplay of integrable systems and tropical geometry. We show that the Young diagrams associated with trop p-Toda given by two very different definitions are identical. The first definition is given via a Lax representation of the discrete periodic Toda lattice, and the second one is associated with a generalization of the Kerov–Kirillov–Reshetikhin bijection in the combinatorics of the Bethe ansatz. By means of this identification, it is shown for the first time that the Young diagrams given by the latter definition are preserved under time evolution. This result is regarded as an important first step in clarifying the iso-level set structure of this dynamical system in general cases, i.e. not restricted to generic cases.

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1. Introduction
The Toda lattice is one of the most famous integrable systems in classical mechanics [1]. Recently, one of its variations has been attracting attention in the context of the connections between tropical geometry and integrable systems [2]. We call this system the tropical periodic Toda lattice (trop p-Toda) [3–5]. Its evolution equation is known as the ultra-discretization of the discrete periodic Toda equation [6]. In [7, 8], Inoue and Takenawa studied this system and clarified its iso-level set structure under a certain condition which they call generic. From the viewpoint of tropical geometry, this condition is related to the smoothness of the tropical spectral curve determined by the conserved quantities of the system.
In this paper we study conserved quantities of trop p-Toda without the generic condition. In particular, we show that the Young diagrams associated with trop p-Toda given by two very different definitions are identical. From one of the definitions, one immediately sees that the common Young diagram is preserved under time evolution. In the context of the integrable cellular automaton explained below, this Young diagram represents the content of solitons in the system, and the generic condition requires that no two solitons have a common amplitude. We believe that this identification of the Young diagrams is a first step in clarifying the iso-level set structure of this dynamical system in general cases, i.e. not restricted to generic cases.

The two definitions of the Young diagrams are as follows. The first is related to the Lax representation of the discrete periodic Toda lattice (dp-Toda). Here the $k$th conserved quantity of dp-Toda is defined as a sum of products of $k$ dependent variables whose indices obey a nearest-neighbor-exclusion condition. Then the corresponding conserved quantity of trop p-Toda is defined as its tropical limit or tropicalization [9–11]. We show that the above condition leads to a condition of weak convexity relating the conserved quantities, which is also a new result of this paper that enables us to represent them as a Young diagram. The second is related to a generalization of the Kerov–Kirillov–Reshetikhin (KKR) bijection in the combinatorics of the Bethe ansatz [12, 13], especially one of its variations in the $\mathfrak{sl}_2$ case [14]. It is also considered as a continuous analogue of the ‘10-elimination’ algorithm for conserved quantities of an integrable cellular automaton known as the periodic box–ball system (pBBS) [15].

We note that for a special case there has already been attention paid to this remarkable equivalence of Young diagrams. The pBBS is regarded as a case of trop p-Toda where the values of its dependent variables are restricted to positive integers. For this case, the equivalence of the Young diagrams has been pointed out by Iwao and Tokihiro [16]. On the basis of their idea of drawing diagrams associated with the second definition of the Young diagram, we give a proof of our main theorem for the conserved quantities of general trop p-Toda.

Here we explain why we expect the identification of the Young diagrams from the Lax representation and those from the generalized KKR bijection to be a first step in clarifying the iso-level set structure of trop p-Toda without the generic condition. In the pBBS case, the KKR bijection gives the action–angle variables of this dynamical system [17]. While the action variables are the conserved quantities, the angle variables yield certain time evolutions which turn out to be flows on the iso-level set. By means of this fact the author has succeeded in clarifying the iso-level set structure of pBBS without the generic condition [18]. Since the trop p-Toda is a generalization of the pBBS, it is reasonable to consider the corresponding generalization of the KKR bijection. We note that in the pBBS case the conservation of the Young diagrams defined by the KKR bijection is directly proved by using crystal theory, combinatorial $R$ maps, and Yang–Baxter relations (see Theorem 2.2 and Proposition 3.4 of [17]). Since these methods are not developed for the trop p-Toda case, our main result in this paper is so far the only proof of the conservation of the Young diagrams defined by the generalized KKR.

Readers may wonder why trop p-Toda is worth studying independently of the already known many results for pBBS. The most remarkable difference between pBBS and trop p-Toda is that the iso-level set of the latter is not a finite set but is an algebraic variety. This implies that while any state comes back to the same state in pBBS, that is not true in trop p-Toda. Actually, when the lengths of the solitons are linearly independent over the field of rational numbers, the phase flow can be dense in the iso-level set as in the case of classical mechanics [19]. However, this is just one aspect of the fact that their iso-level sets are totally...
different mathematical objects. A really important problem here is that the structure of the iso-
level set of trop p-Toda has not yet been fully clarified in general. As we have mentioned, it
has been clarified only for the case where it reduces to a real torus under the generic
condition. Without this condition, we have no suitable description of their connected com-
ponents or invariant tori, which have different sizes according to their internal symmetries.
We expect that they should not be regarded as mere subsets as in the pBBS case [18], but
should be regarded as lower dimensional tori embedded in the whole iso-level set. The
present work is a starting point for developing such a description, which will contribute to
making progress in the studies on tropical geometry and integrable systems.

This paper is organized as follows. In section 2.1 we derive the nearest-neighbor-
exclusion condition on the indices of variables from a determinant formula of a matrix. In
section 2.2 definitions of dp-Toda and trop p-Toda are given, and we show that the matrix
given above is related to the Lax matrix of dp-Toda. Here we obtain conserved quantities of
dp-Toda and trop p-Toda from the formulas described by the nearest-neighbor-exclusion
condition. In section 2.3 we show that the conserved quantities of trop p-Toda obey the weak
convexity condition (theorem 9), which enables us to describe the conserved quantities as a
Young diagram. In section 2.4 another algorithm for constructing the Young diagram is
introduced, and the main result of this paper (theorem 13) is presented. We devote our efforts
to proving this theorem in section 3. In section 3.1 we introduce an algorithm for drawing
diagrams of trees that visualizes the algorithm in section 2.4. In section 3.2 some elementary
lemmas on the properties of the diagrams are presented. Using these lemmas, we give a proof
of the main theorem in section 3.3, leaving the proofs of two more lemmas which we call the
Close Packing Lemma and the Forest Realization Lemma. We devote our efforts to proving
these lemmas in section 3.4 and section 3.5. A continuous analogue of the KKR bijection is
discussed in section 4. Some concluding remarks are given in section 5.

2. Discrete and tropical periodic Toda lattices

2.1. Determinant formulas and the nearest-neighbor-exclusion condition

Throughout this paper we use the symbol $\triangleleft$ with the following meaning:

\[ i \triangleleft j \iff i + 1 < j. \]  

(1)

Given a sequence of real numbers $a_0=0, a_1, a_2, \ldots$, let $c_k^{(N)}$ be the numbers defined by the

recursion relation

\[ c_k^{(N)} = c_k^{(N-1)} + (a_{2N-1} + a_{2N})c_k^{(N-1)} - a_{2N-2}a_{2N-1}c_{k-2}^{(N-2)}, \]  

(2)

and the boundary conditions

\[ c_k^{(N)} = 0 \quad \text{for } k < 0 \quad \text{or} \quad k > N, \quad c_0^{(N)} = 1 \quad \text{for } N \geq 0. \]  

(3)

Then it is easy to see that the unique solution of (2) under (3) is given by

\[ c_k^{(N)} = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq 2N} a_{i_1}a_{i_2} \cdots a_{i_k}. \]  

(4)

Let $e_k^{(N)}$ be the numbers defined by the relations

\[ e_1^{(N)} = c_1^{(N)}, \quad e_2^{(N)} = c_2^{(N)} - a_1a_{2N}, \]  

(5)
\[ e_k^{(N)} = c_k^{(N)} = a_1 a_{2N} \sum_{3 \leq i_1 < i_2 < \cdots < i_{k-2} \leq 2N-2} a_{i_1} a_{i_2} \cdots a_{i_{k-2}}, \]  

for \( 3 \leq k \leq N \). Then it is easy to see that

\[ e_k^{(N)} = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq 2N} a_{i_1} a_{i_2} \cdots a_{i_k}. \]  

Let \( F_1(x; a_1, a_2) = x + a_1 + a_2 \) and

\[ F_N(x; a_1, \ldots, a_{2N}) = \det \begin{pmatrix} x + a_1 + a_2 & 1 \\ a_2 a_3 & x + a_3 + a_4 & 1 \\ & a_4 a_5 & \ddots & \ddots \\ & & \ddots & 1 \\ & & & a_{2N-2} a_{2N-1} & x + a_{2N-1} + a_{2N} \end{pmatrix}. \]  

for \( N \geq 2 \).

**Lemma 1.** ([20, Proposition 7.1])

\[ F_N(x; a_1, \ldots, a_{2N}) = \sum_{k=0}^{N} c_k^{(N)} x^{N-k}. \]  

**Proof.** Let \( F_N(x; a_1, \ldots, a_{2N}) = \sum_{k=0}^{N} b_k^{(N)} x^{N-k} \). Then we have \( b_0^{(N)} = 1 \).

By expanding (8) with respect to its \( N \)th row one obtains

\[ F_N(x; a_1, \ldots, a_{2N}) = (x + a_{2N-1} + a_{2N}) F_{N-1}(x; a_1, \ldots, a_{2N-2}) - a_{2N-2} a_{2N-1} F_{N-2}(x; a_1, \ldots, a_{2N-4}). \]  

Defining \( b_k^{(N)} = 0 \) for \( k < 0 \) or \( k > N \), one can deduce from (10) that the \( b_k^{(N)} \)'s satisfy the same recursion relation, (2), as the \( c_k^{(N)} \)'s for \( 1 \leq k \leq N \). Hence \( b_k^{(N)} = c_k^{(N)} \). \( \square \)

Let

\[ G_N(x; a_1, \ldots, a_{2N}) \]

\[ = \det \begin{pmatrix} x + a_1 + a_2 & 1 \\ a_2 a_3 & x + a_3 + a_4 & 1 \\ & a_4 a_5 & \ddots & \ddots \\ & & \ddots & 1 \\ & & & a_{2N-2} a_{2N-1} & x + a_{2N-1} + a_{2N} \end{pmatrix}, \]  

for \( N \geq 3 \).
Lemma 2. \[ G_N(x; a_1, ..., a_{2N}) = y + \left( \prod_{i=1}^{2N} a_i \right) y + \sum_{k=0}^{N} c_k^{(N)} x^{N-k}. \]  

Proof. By expanding (11) with respect to its \(N\)th column one obtains  
\[ G_N(x; a_1, ..., a_{2N}) = (x + a_{2N-1} + a_{2N}) F_{N-1}(x; a_1, ..., a_{2N-2}) + y - a_{2N-2} a_{2N-1} F_{N-2}(x; a_1, ..., a_{2N-4}) + \frac{(a_1 ... a_{2N})}{y} - a_{2N} F_{N-2}(x; a_1, ..., a_{2N-2}) = F_N(x; a_1, ..., a_{2N}) - a_1 a_{2N} F_{N-2}(x; a_3, ..., a_{2N-2}) + y + (a_1 ... a_{2N})/y, \]  where we have used (10). Let \( G_N(x; a_1, ..., a_{2N}) = y + (\prod_{i=1}^{2N} a_i)/y + \sum_{k=0}^{N} s_k^{(N)} x^{N-k}. \) From lemma 1 and (13), one finds that the \(s_k^{(N)}\)'s satisfy the same relations, (5) and (6), as the \(e_k^{(N)}\)'s for \(1 \leq k \leq N\). Hence \( s_k^{(N)} = e_k^{(N)}. \)

2.2. The evolution equation and Lax representation

On the basis of [3], we briefly review the derivation of discrete and tropical periodic Toda lattice equations. Let \( \{x_n(t)\} \in \mathbb{Z}_N \) be a set of smooth functions of time \( t \in \mathbb{R} \). Set \( a_{2n} = a_{2n}(t) = 1 + \delta x_n(t), a_{2n+1} = a_{2n+1}(t) = \delta^2 e^{\delta x_{n+1}(t)} \) with \( \delta > 0 \) and \( a_j = a_j(t + \delta) \) for \( j \in \mathbb{Z}_N \). Then we have \( \lim_{\delta \to 0^+} \frac{1}{\delta} (a_{2n+1} a_{2n+1} - a_{2n+1} a_{2n+2}) = 0 \). Suppose that the \( x_n = x_n(t) \) satisfy the Toda lattice equation  
\[ x_{n+1} = e^{x_{n+1} - x_n} - e^{x_n - x_{n-1}}. \]  Then we have \( \lim_{\delta \to 0^+} \frac{1}{\delta} (a_{2n-1} + a_{2n} - a_{2n} a_{2n+1}) = 0 \). In this consideration, we define the evolution equations for the discrete periodic Toda lattice as  
\[ \tilde{a}_{2n-1} + \tilde{a}_{2n} = a_{2n} + a_{2n+1}, \quad \tilde{a}_{2n} \tilde{a}_{2n+1} = a_{2n+1} a_{2n+2}, \]  where the \( a_n = a_n^+, \tilde{a}_n = a_n^{+1} \) are dependent variables which depend on the discrete spatial coordinate \( n \in \mathbb{Z}_N \) and discrete time \( t \in \mathbb{Z} \). Obviously, \( \prod_{i=1}^{2N} a_i \) is a conserved quantity. By lemma 4 we will find that \( b_N = \prod_{i=1}^{N} a_{2i-1} + \prod_{i=1}^{N} a_{2i} \) is also a conserved quantity. This implies that \( (\prod_{i=1}^{N+1} a_{2i-1}, \prod_{i=1}^{N} a_{2i}) = (\prod_{i=1}^{N+1} a_{2i-1}, \prod_{i=1}^{N} a_{2i}) \) or \( (\prod_{i=1}^{N+1} a_{2i-1}, \prod_{i=1}^{N} a_{2i}) \). While the former leads to the trivial solution \( \tilde{a}_n = a_n^+ \), the latter leads to a non-trivial solution  
\[ a_{2n} = a_{2n+1} + a_{2n-1} \prod_{i=1}^{N} (a_{2i-1} a_{2i}) / \sum_{k=0}^{N} \prod_{i=1}^{k} (a_{2n+k} a_{2n-k}) \]  or \( a_{2n} = a_{2n+1} a_{2n+2} / a_{2n-2} \). For a derivation of this solution, see Proposition 6.13 of [3]. Now we consider its tropicalization, which is a procedure for replacing \( x \) by \( + \), and \( + \) by \( \min \). Note that the numerator in (16) is a conserved quantity. By regarding it as a positive constant and setting it to be zero under the tropicalization with trivial valuation [11], we obtain a dynamical system.
given by the piecewise linear evolution equations

\[
\begin{align*}
\bar{A}_{2n} &= \min\left\{ A_{2n+1}, A_{2n} - \min_{0 \leq k < N-1} \left( \sum_{j=1}^{k} (A_{2(n-k)+1} - A_{2(n-k)}) \right) \right\}, \\
\bar{A}_{2n+1} &= A_{2n+1} + A_{2n+2} - \bar{A}_{2n},
\end{align*}
\]

on the phase space \( \mathcal{T} = \left\{ (A_n)_{n \in \mathbb{Z}} \mid \sum_{i=1}^{N} |A_{2i} - \sum_{i=1}^{N} A_{2i-1}| \right\} \subset \mathbb{R}^{2N} \). We call this system the tropical periodic Toda lattice \([3]\).

**Remark 3.** We have changed the notation to \( a_{n} = q_n, A_{2n-2} = Q_n, A_{2n-1} = W_n \) from that adopted in \([3]\), since this enables us to describe the conserved quantities neatly.

Without loss of generality, we can assume that all the \( A \)-variables in (17) take their values in \( \mathbb{R}_{>0} \). This enables us to represent the time evolution of trop p-Toda by a sequence of two-colored (white and black) strips, where the lengths of the white (resp. black) segments are denoted by \( A_{2n-1} \)'s (resp. \( A_{2n} \)'s). See figure 1 for an example.

Now, on the basis of \([7]\) we briefly review the Lax representation for the discrete periodic Toda lattice equation. Let

\[
R(\lambda) = \begin{pmatrix}
1 & a_1 \\
\vdots & \ddots & \ddots \\
\vdots & \ddots & a_{2N-1} & 1
\end{pmatrix}
\quad
M(\lambda) = \begin{pmatrix}
a_2 & 1 \\
\ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
1 & a_{2N}
\end{pmatrix},
\]

We denote by \( \bar{R}(\lambda), \bar{M}(\lambda) \) the matrices obtained from \( R(\lambda), M(\lambda) \) by replacing \( a_i \) by \( \bar{a}_i \) for all \( i \). Then the evolution equations of the discrete periodic Toda lattice (15) are equivalent to \( R(\lambda) \bar{M}(\lambda) = M(\lambda) R(\lambda) \). Let \( L(\lambda) = R(\lambda) M(\lambda), \bar{L}(\lambda) = \bar{R}(\lambda) \bar{M}(\lambda) \). Then we have the Lax representation for the discrete periodic Toda lattice as
This implies that the polynomial $\det(xI + L(\lambda))$ is invariant under time evolution. Hence its coefficients are conserved quantities. Since $G_N(x; a_1, \ldots, a_{2N})$ in (11) is expressed as $G_N(x; a_1, \ldots, a_{2N}) = \det(xI + L((-1)^{N-1}y))$, we have the following:

Lemma 4. The discrete periodic Toda lattice (15) has $N + 1$ conserved quantities

$$h_k = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq 2N, (i_1, i_2) \neq (1, 2N)} a_{i_1} a_{i_2} \ldots a_{i_k},$$

for $1 \leq k \leq N$ and $h_{N+1} = \prod_{i=1}^{2N} a_i$.

By means of their tropicalization we obtain:

Lemma 5. The tropical periodic Toda lattice (17) has $N + 1$ conserved quantities

$$H_k = \min_{1 \leq i_1 < i_2 < \ldots < i_k \leq 2N} \left(A_{i_1} + A_{i_2} + \cdots + A_{i_k}\right),$$

for $1 \leq k \leq N$ and $H_{N+1} = \sum_{i=1}^{2N} A_i$.

Proof. As was explained in section 4.1 of [3], the tropicalization can be realized as a concrete limiting procedure. In the present setting we let $a_i = e^{-A_i}/\varepsilon$ with $\varepsilon > 0$, apply the map $x \mapsto -\varepsilon \log x$ and take the limit $\varepsilon \to 0$. This procedure transforms (16) to (17), which implies the claim of this lemma on the basis of the previous one.

Remark 6. We derived lemmas 4 and 5 directly through the Lax representation. An equivalent result was obtained by using a different method in [16], Proposition 3.9.

2.3. The weak convexity condition relating the conserved quantities

The iso-level set structure of trop p-Toda has been clarified by means of tropical geometry [7, 8] for the case where the strong convexity condition $H_k + H_{k+2} > 2H_{k+1}$ (which they call generic) is satisfied by the conserved quantities. In this section we prove that in general cases only the weak convexity condition $H_k + H_{k+2} \geq 2H_{k+1}$ holds. For this purpose we first consider a lemma in elementary combinatorics.

Put two kinds of symbols, $\circ$’s and $\bullet$’s, on a circle. Say two $\circ$’s are adjacent if there are no other $\bullet$’s between them.

Lemma 7. Put $k$ $\circ$’s and $k + 2$ $\bullet$’s on a circle such that their positions do not coincide and there are at most two $\bullet$’s between adjacent $\circ$’s. Then on the circle we always have a configuration such as $-\bullet - (\circ - \bullet)^n - \bullet$ for some $n \geq 1$.

Proof. If there is more than one $\circ$ between any adjacent $\bullet$’s on the circle, remove the $\circ$’s until there remains only one. Suppose that the number of $\circ$’s that we have removed is $\alpha$. Since the number of $\circ$’s is now $k - \alpha$, the number of the configuration $-\bullet - \bullet$ on the circle is $(k + 2) - (k - \alpha) = \alpha + 2$. By construction, we have such configurations as $-\bullet - \bullet - (\circ - \bullet)^n - \bullet$ for some $n \geq 1$ between all $(\alpha + 2)$ adjacent $-\bullet - \bullet - \bullet$, and at least two of them will be left unchanged when we put all $(\alpha)$ removed $\circ$’s back into the original positions. \qed
Example 8. See figure 2.

Now we present one of our new results in this paper.

**Theorem 9.** For $x = (x_1, \ldots, x_{2N}) \in (\mathbb{R}_{>0})^{2N}$ let $H_0(x) = 0$ and

$$H_k(x) = \min_{1 \leq i_1 < i_2 < \cdots < i_k \leq 2N} \left( x_{i_1} + x_{i_2} + \cdots + x_{i_k} \right),$$

for $1 \leq k \leq N$. Then the relations $H_k(x) + H_{k+2}(x) \geq 2H_{k+1}(x)$ are satisfied for $0 \leq k \leq N - 2$.

**Proof.** Suppose that we have $H_k(x) = x_{i_1} + \cdots + x_{i_k}$ and $H_{k+2}(x) = x_{i_1} + \cdots + x_{i_{k+2}}$. Let $S_1 = \{i_1, \ldots, i_k\}$ and $S_2 = \{i_1, \ldots, i_{k+2}\}$ be the sets of their indices. To begin with we assume that $S_1 \cap S_2 = \emptyset$. For $a, b \in \{1, \ldots, 2N\}$, say that $b$ is next to $a$ if $|a - b| = 1$ or $2N - 1$. If there exists $b \in S_2$ such that $b$ is not next to any $a \in S_1$, then we have $H_k(x) + x_b \geq H_{k+1}(x)$ and $H_{k+2}(x) - x_b \geq H_{k+1}(x)$. Hence the claim follows.

Suppose otherwise, i.e. we assume that every $b \in S_2$ is next to some $a \in S_1$. Draw a circle of circumference $2N$ with a spatial coordinate $1, \ldots, 2N$ assigned counterclockwise on it. Put $\circ$’s on the circle at the positions given by $S_1$, and put $\bullet$’s at those given by $S_2$. Since every $b \in S_2$ is next to some $a \in S_1$, there exist at most two $\bullet$’s between adjacent $\circ$’s on the circle.

Hence by lemma 7 we have such a configuration as $\cdots \circ \bullet \cdots (\circ \bullet) \cdots \cdots \cdots \cdots$ for some $n \geq 1$ on the circle. Replace this configuration by $\cdots \circ \circ \cdots (\circ \bullet) \cdots \cdots \cdots \cdots$. Now let $I_1$ (resp. $I_2$) be the set of the positions of $\circ$’s (resp. $\bullet$’s) on the circle. Then since $|I_1| = |I_2| = k+1$, we have $H_k(x) + H_{k+2}(x) = \sum_{i \in I_1} x_i + \sum_{i \in I_2} x_i \geq 2H_{k+1}(x)$.

When $S_1 \cap S_2 \neq \emptyset$ but we do not have $S_1 \subset S_2$, we can prove the statement by replacing $S_i$ by $S_i \setminus (S_1 \cap S_2)$ for $i = 1, 2$ and repeating the above arguments. This is because no elements of $S_1 \cap S_2$ are next to any $a \in S_i \setminus (S_1 \cap S_2)$ for $i = 1, 2$. The case $S_1 \subset S_2$ is simpler, because no elements of $S_2 \setminus S_1$ are next to any $a \in S_1$. The proof is completed. \qed
Recall the conserved quantity of the tropical periodic Toda lattice (20). Let $\tilde{I}_i = H_{N+1-i} - H_{N-1}$ for $1 \leq i \leq N$. Then by theorem 9 we have $\tilde{I}_1 \geq \ldots \geq \tilde{I}_N > 0$. Let \{l_i\} be the set of real numbers satisfying $l_1 > \ldots > l_s$ such that for any $1 \leq j \leq N$ there exists $1 \leq i \leq s$ such that $\tilde{I}_j = l_i$. And let $m_i = \{j \mid 1 \leq j \leq N, \tilde{I}_j = l_i\}$. Now we can express the conserved quantities using a Young diagram (figure 3) in which the lengths of the horizontal edges are not necessarily integers.

In the context of the integrable cellular automaton, this Young diagram represents the content of solitons in the system. From this point of view, we have $m_i$ solitons of amplitude $l_i$ for $1 \leq i \leq s$. We note that the lengths of the black segments in figure 1 are not always identical to the amplitudes of solitons represented by $\lambda$, since there are intermediate states of collisions of several solitons.

**2.4. Another algorithm for the Young diagram and the main theorem**

We introduce another algorithm for constructing Young diagrams from states of trop p-Toda. This algorithm is related to the KKR bijection in the $\mathfrak{sl}_2$ case which was applied to the inverse scattering transform of pBBS [17]. It is also regarded as a continuous analogue of the ‘10-elimination’ procedure. We shall prove that the Young diagram obtained here coincides with the one that was defined in the previous subsection.

Say that any sequence of non-negative real numbers $x_1, \ldots, x_{2n}$ obeys the highest weight condition if the inequalities

$$\sum_{j=1}^{k}(x_{2j-1} - x_{2j}) \geq 0, \quad (22)$$

are satisfied for $1 \leq k \leq n$. Fix time $t$ and denote the dependent variables of trop p-Toda $A_i^t$ by $A_i$. By shifting their indices cyclically, one can make the sequence $A_1, \ldots, A_{2N}$ obey the highest weight condition. This is due to the phase space condition under (17); suppose that we have done this.

Let $x^{(1)} = (x_1^{(1)}, \ldots, x_{2N}^{(1)})$ with $x_1^{(1)} = A_1$ and $N^{(1)} = N$. Given an array of positive real numbers $x^{(0)} = (x_1^{(0)}, \ldots, x_{2N}^{(0)})$ satisfying the highest weight condition, define
\[ y^{(i)} = (y_1^{(i)}, \ldots, y_{2N_1}^{(i)}) \] by \[ y_j^{(i)} = x_j^{(i)} - \mu^{(i)} \] where \[ \mu^{(i)} = \min_{1 \leq j \leq 2N_1} \{ x_j^{(i)} \} \]. In the array of non-negative real numbers \( y_1^{(i)}, \ldots, y_{2N_1}^{(i)} \), suppose that there are \( k^{(i)} \) sequences of zeros. Here we regard a lone zero also as a sequence. We denote by \( n_j^{(i)} (1 \leq j \leq k^{(i)}) \) the length of the \( j \)th sequence of zeros. Let \( N^{(i+1)} = N^{(i)} - \sum_{j=1}^{k^{(i)}} \left\lceil \frac{n_j^{(i)}}{2} \right\rceil \) where \( \lceil c \rceil \) is the smallest integer satisfying \( \lceil c \rceil \geq c \). If \( N^{(i+1)} = 0 \) then we stop. Otherwise we define an array \( x^{(i+1)} = (x_1^{(i+1)}, \ldots, x_{2N_1^{(i+1)}}) \) of positive real numbers satisfying the highest weight condition by the following procedure.

(i) If the first sequence of \( n_1^{(i)} \) zeros in the array \( y_1^{(i)}, \ldots, y_{2N_1}^{(i)} \) is at the left end, then remove these zeros. We see that \( n_1^{(i)} \) must be even because the array satisfies the highest weight condition.

(ii) For any \( j \) such that the \( j \)th sequence of \( n_j^{(i)} \) zeros is between positive neighbors \( a, b, \ldots, a, 0, \ldots, 0, b, \ldots \), replace it by \( a, b, \ldots \) if \( n_j^{(i)} \) is even, or by \( a + b, \ldots \) if \( n_j^{(i)} \) is odd. More precisely, we simply remove \( n_j^{(i)} \) zeros in the former case, and in the latter case we further replace the neighbors \( a, b \) by a single number \( a + b \).

(iii) Suppose that the last sequence of \( n_k^{(i)} \) zeros is at the right end after a positive neighbor \( a, \ldots, a, 0, \ldots, 0 \). Remove these zeros. If \( n_k^{(i)} \) is odd then also remove \( a \), and add it to the first element of the array.

Let \( x_j^{(i+1)} \) be the \( j \)th element of the resulting array of \( 2N^{(i+1)} \) positive integers. By the following lemma we see that the sequence \( x_1^{(i+1)}, \ldots, x_{2N_1^{(i+1)}} \) satisfies the highest weight condition.

**Lemma 10.** The highest weight condition is preserved under the following procedures.

(i) Insert or remove two consecutive zeros.

(ii) Split any positive term into two positive numbers and insert a zero between them, or remove a zero and join its neighbors into one term.
Suppose that $N^{(i)} > 0$ for $1 \leq i \leq n$ and $N^{(i+1)} = 0$. Let $\nu^{(i)} = N^{(i)} - N^{(i+1)}$. Obviously the set of numbers $\left\{ (\mu^{(i)}, \nu^{(i)}) \right\}_{1 \leq i \leq n}$ determines a Young diagram $\tilde{\lambda}$ in which the lengths of the horizontal edges are positive real numbers (figure 4).

**Example 11.** For $N = 6$, let $A_1 = \sqrt{2}$, $A_2 = A_{12} = 2$, $A_3 = A_8 = A_{11} = \sqrt{2}$, $A_4 = A_5 = A_6 = A_7 = A_{10} = 1$. Then we have

$$
\begin{align*}
&x^{(1)} = (2\sqrt{2}, 2, \sqrt{2}, 1, 1, 1, \sqrt{2}, 1, 1, \sqrt{2}, 2), \\
&y^{(1)} = (2\sqrt{2} - 1, 1, \sqrt{2} - 1, 0, 0, 0, \sqrt{2} - 1, 0, 0, \sqrt{2} - 1, 1), \\
&x^{(2)} = (2\sqrt{2} - 1, 1, \sqrt{2} - 1, \sqrt{2} - 1, \sqrt{2} - 1, 1), \\
&y^{(2)} = (\sqrt{2}, 2 - \sqrt{2}, 0, 0, 0, 2 - \sqrt{2}), \\
&x^{(3)} = (\sqrt{2}, 4 - 2\sqrt{2}), \\
&y^{(3)} = (3\sqrt{2} - 4, 0),
\end{align*}
$$

and $\mu^{(1)} = 1$, $\mu^{(2)} = 2$, $\mu^{(3)} = 4 - 2\sqrt{2}$, $\nu^{(1)} = 3$, $\nu^{(2)} = 2$, $\nu^{(3)} = 1$.

**Example 12.** For $N = 4$, let $A_1 = 4$, $A_2 = A_3 = 3$, $A_4 = A_5 = 2$, $A_6 = A_7 = A_8 = 1$. Then we have

$$
\begin{align*}
&x^{(1)} = (4, 3, 3, 2, 2, 1, 1, 1), \\
&y^{(1)} = (3, 2, 2, 1, 1, 0, 0, 0), \\
&x^{(2)} = (4, 2, 2, 1), \\
&y^{(2)} = (3, 1, 1, 0), \\
&x^{(3)} = (4, 1), \\
&y^{(3)} = (3, 0),
\end{align*}
$$

and $\mu^{(1)} = \mu^{(2)} = \mu^{(3)} = 1$, $\nu^{(1)} = 2$, $\nu^{(2)} = \nu^{(3)} = 1$.

Now we present the main result of this paper.

**Theorem 13.** For any integer $1 \leq k \leq N$, the area of the part of the Young diagram $\tilde{\lambda}$ between its bottom line and the horizontal line above it at height $k$ is given by $H_k$, the $k$th conserved quantity of the trop $p$-Toda defined as in (20).

In other words, the Young diagram $\tilde{\lambda}$ in figure 4 coincides with the Young diagram $\lambda$ in figure 3. We shall give a proof of this theorem in the next section.

### 3. Proof of the main theorem

#### 3.1. Drawing a diagram of trees

By generalizing the method of Appendix A. 1 of [16], we introduce a way to draw a graph $\Phi_A$ associated with a state of the trop $p$-Toda $A = (A_1, \ldots, A_{2N})$. It has several connected components called trees. See figure 5 for an example, that is for the $A$-variables of example 11. At the end we rewrite the assertion of theorem 13 in terms of the trees.

Recall the algorithm from section 2.4 where $N = N^{(1)}$. Place the symbols $A_1, \ldots, A_{2N}$ or $x^{(1)}_1, \ldots, x^{(1)}_{2N}$ at a horizontal level, called level 0. Draw a vertical line from each symbol
upward to a certain horizontal level, called level 1. We associate the non-negative real numbers $y_1^{(i)}, \ldots, y_{2N}^{(i)}$ with the lines. If $y_1^{(i)} = 0$, we put a symbol $\times$ at the top of the line. Then, change the symbol $\times$ into another symbol $\otimes$ if it is an isolated one, or is at an ‘odd-th’ position of a sequence of consecutive $\times$’s. In what follows we will pay attention to $\otimes$’s only and ignore the other $\times$’s. Each $\otimes$ is called the top of a tree of level 1. For each isolated $\otimes$ or sequence of $\otimes$’s placed at every other position, we let the lines adjacent to $\otimes$’s join together to straddle the $\otimes$’s. Here we respect the periodic boundary condition, so the leftmost and rightmost ends are regarded as adjacent. We call each joining point a branching point of level 1. After this procedure, the number of lines is reduced to $N_2^{(i)}$.

Now we describe a general procedure for drawing the diagram from level $i - 1$ to level $i$. We associate the positive real numbers $x_1^{(i)}, \ldots, x_{2N}^{(i)}$ with the tops of the $N_2^{(i)}$ lines at level $i - 1$. Extend the lines upward to level $i$. We associate the non-negative real numbers $y_1^{(i)}, \ldots, y_{2N}^{(i)}$ with the lines. If $y_1^{(i)} = 0$, we put a symbol $\times$ at the top of the line. By repeating the procedure of the previous paragraph, we obtain the tops of the trees, as well as the branching points, of level $i$.

At the end we obtain a graph $\Phi_A$ for the state $A = (A_1, \ldots, A_{2N})$. Denote by $\text{Tree}(\Phi_A)$ the set of all trees in the graph $\Phi_A$. Define the level of a tree by the level of its top point marked by $\otimes$. By construction, the number of level-$i$ trees is $\nu^{(i)}$. Hence there are $\sum_{i=1}^N \nu^{(i)} = N$ trees in total. Let $t \in \text{Tree}(\Phi_A)$ be a tree of level $i$. We define its height by $\text{Ht}(t) = \sum_{j=1}^{i} \mu^{(j)}$.

Definition 14. We label all the trees in $\text{Tree}(\Phi_A)$ as $t_1, \ldots, t_N$ such that their heights are in weakly increasing order, i.e. $i < j \Rightarrow \text{Ht}(t_i) \leq \text{Ht}(t_j)$. Let $\text{Tree}^{(k)} = \{t_i, \ldots, t_k\}$.

Now we see that the assertion of theorem 13 is equivalent to the relation

$$
\sum_{j=1}^{k} \text{Ht}(t_j) = \min_{1 \leq i_1 < i_2 < \cdots < i_k \leq 2N} \left( A_{i_1} + A_{i_2} + \cdots + A_{i_k} \right),
$$

for $1 \leq k \leq N$.

3.2. Elementary lemmas

We present several elementary lemmas on the graph $\Phi_A$ that are necessary for proving theorem 13. Given $t \in \text{Tree}(\Phi_A)$, we define a set of indices of the $A$-variables as
For $T \subset \text{Tree}(\Phi_A)$ we write $\text{Root}(T) = \bigcup_{i \in T} \text{Root}(t)$. If $P$ is a branching point of $t$ at level $i$, we define its height by $\text{Ht}(P) = \sum_{j=1}^{m} \mu^{(j)}$. (Note that we regard a tree and a branching point of the same level as having common height, though they are not so depicted in the figures, for technical reasons.) Say $P$ has multiplicity $m_P$ if there are $m_P + 2$ lines going out from $P$. If the tree $t$ has branching points $P_1, \ldots, P_q$ with multiplicities $m_{P_1}, \ldots, m_{P_q}$, define the weight of $t$ by

$$\text{wt}(t) = \text{Ht}(t) + \sum_{j=1}^{q} \text{Ht}(P_j)m_{P_j}. \quad (25)$$

Then we have:

**Lemma 15.** ([16], Lemma A.1) For any $t \in \text{Tree}(\Phi_A)$ it holds that

$$\text{wt}(t) = \sum_{i \in \text{Root}(t)} A_i. \quad (26)$$

The set $\text{Tree}(\Phi_A)$ becomes a partially ordered set on introducing the following partial order. Denote by $t > s$ that $s$ is straddled by $t$. We denote by $t \geq s$ a case where either $t > s$ or $t = s$ is satisfied. For $t \in \text{Tree}(\Phi_A)$ we define

$$\text{Sub}(t) = \{ s \in \text{Tree}(\Phi_A) \mid t > s \}, \quad (27)$$

$$\text{Sub}_{\max}(t) = \{ s \in \text{Tree}(\Phi_A) \mid t \geq s \} = \text{Sub}(t) \cup \{ t \}. \quad (28)$$

Each element of $\text{Sub}(t)$ takes either an ‘odd-th’ position or an ‘even-th’ position in its nesting structure, regarding $t$ itself as taking the first position and the order of the nesting as increasing inwardly. We denote by $\text{Sub}_{\max}(t) \subset \text{Sub}(t)$ the set of all trees at ‘odd-th’ positions, and let $\text{Sub}_e(t) = \text{Sub}(t) \setminus \text{Sub}_{\max}(t)$. Accordingly, we define

$$\text{Root}_o(t) = \text{Root}(\text{Sub}_o(t)), \quad (29)$$

$$\text{Root}_e(t) = \text{Root}(\text{Sub}_e(t)) \quad (30)$$

and $\text{Root}(t) = \text{Root}_o(t) \cup \text{Root}_e(t)$. Then we have:

**Lemma 16.** ([16], Lemma A.2) For any $t \in \text{Tree}(\Phi_A)$ it holds that

$$\sum_{s \in \text{Sub}(t)} \text{Ht}(s) = \sum_{i \in \text{Root}(t)} A_i. \quad (31)$$

We also use the following lemmas in the next subsection.

**Lemma 17.**

$$\#\text{Sub}(t) = \#\text{Root}_o(t). \quad (32)$$

**Proof.** Denote all the branching points in $\text{Sub}(t)$ by $P_1, \ldots, P_q$ and their multiplicities by $m_{P_1}, \ldots, m_{P_q}$. By looking down the graph, we see that the number of trees increases by $m$ at a
branching point with multiplicity $m$. Hence $\#\text{Sub}(t) = 1 + \sum_{i=1}^{m} m_i$. In the same way, we see that the number of vertical lines increases by $2m$ at the branching point. Hence $\#\text{Root}(t) = 1 + \sum_{i=1}^{m} 2m_i$, which implies that $\#\text{Root}_c(t) = 1 + \sum_{i=1}^{m} m_i$. 

Recall the definition of the set $\text{Tree}^{(k)}$ in definition 14. One can prove the following lemma by induction on $k$.

**Lemma 18.** For any $t \in \text{Tree}^{(k)}$ the relation $\text{Sub}(t) \subset \text{Tree}^{(k)}$ holds.

Say $\xi$ is a maximal element of a partially ordered set $X$ if $\xi \supseteq \xi'$ is satisfied for any $\xi' \in X$ that is comparable to $\xi$ with respect to the partial order.

**Definition 19.** Let $\text{MTree}^{(k)} \subset \text{Tree}^{(k)}$ be the set of all maximal elements of $\text{Tree}^{(k)}$.

Then we have:

**Lemma 20.**

$$\bigcup_{t \in \text{MTree}^{(k)}} \text{Sub}(t) = \text{Tree}^{(k)}.$$  

(33)

**Proof.** We show the inclusion $\subset$ since the opposite inclusion is almost trivial. Suppose that $u$ is an element of LHS. Then there exists $t \in \text{MTree}^{(k)} \subset \text{Tree}^{(k)}$ such that $u \in \text{Sub}(t)$. This implies that $u \in \text{Tree}^{(k)}$ by lemma 18. 

**Definition 21.** Let $\text{MSub}(t) \subset \text{Sub}(t)$ be the set of all maximal elements of $\text{Sub}(t)$.

Then it is easy to see that the following relations are satisfied:

$$\text{Sub}(t) = \bigcup_{s \in \text{MSub}(t)} \text{Sub}(s),$$  

(34)

$$\text{Root}_c(t) = \bigcup_{s \in \text{MSub}(t)} \text{Root}_c(s).$$  

(35)

From (28), (34), (35) and lemma 16 we have:

**Lemma 22.**

$$\sum_{i \in \text{Root}_c(t)} A_i = \text{Ht}(t) + \sum_{i \in \text{Root}(t)} A_i.$$  

(36)

Let $t \in \text{Tree}(\Phi_A)$ be a tree and $P \in t$ be one of its branching points. Consider a subtree $t_1 \subset t$ that extends downward from $P$. See figure 6. Define $\text{Root}(t_1)$, $\text{Sub}(t_1)$, and so on by extending the definitions (24), (27), and so on in an obvious way. Note that $t_1$ itself is not an element of $\text{Tree}(\Phi_A)$. Then we have:
Lemma 23.

$$\sum_{i \in \text{Root}(t_1)} A_i \geq \text{Ht}(P) + \sum_{i \in \text{Root}(t_1)} A_i.$$  

(37)

**Proof.** If the subtree $t_1$ has branching points $Q_1, \ldots, Q_s$ with multiplicities $m_{Q_1}, \ldots, m_{Q_s}$, define its weight by

$$\text{wt}(t_1) = \text{Ht}(P) + \sum_{j=1}^{s} \text{Ht}(Q_j)m_{Q_j}.$$  

(38)

Then by the algorithm for drawing the graph $\Phi_A$, we can deduce $\text{wt}(t_1) \leq \sum_{i \in \text{Root}(t_1)} A_i$. We define $\bar{t}_1$, called the completion of $t_1$, as a tree obtained by extending the top of $t_1$ by the length $\sum_{i \in \text{Root}(t_1)} A_i - \text{wt}(t_1)$. In other words $\bar{t}_1$ is a tree that shares all the branching/bottom points with $t_1$ but satisfies lemmas 15 and 16. Then the claim follows from applying lemma 22 on the tree $\bar{t}_1$ and using $\text{wt}(t_1) \geq \text{wt}(P)$.

\[ \square \]

### 3.3. Proof of Theorem 13

Now we give a proof of the main theorem, leaving the proofs of two more lemmas to appear afterwards in the following subsections.

Given $N$, we define the sets of nearest-neighbor-excluding indices as

$$B(k, N) = \left\{ \{i_1, \ldots, i_k\} \subset \mathbb{Z}^k \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq 2N, (i_1, i_k) \neq (1, 2N) \right\},$$  

(39)

$$B(N) = \bigcup_{1 \leq k \leq N} B(k, N).$$  

(40)

Then the RHS of (23) can be written as $\min_{B(k, N)} \{ \sum_{i \in B} A_i \}$. The Forest Realization Lemma from section 3.5 claims that for any $1 \leq k \leq N$ there exists $T \subset \text{Tree}(\Phi_A)$ such that the relation
is satisfied. Then, from the *Close Packing Lemma* from section 3.4 we can deduce that such a \( T \) must be written as a disjoint union, as

\[
T = \bigcup_{\mathcal{U}} \sum_{i} \mathsf{Sub}_{\mathcal{U}}(i), \quad \mathcal{U} \subset \mathsf{Tree}(\Phi_A).
\]  

(42)

This implies that

\[
\sum_{i \in \mathsf{Root}(\mathcal{T})} A_i = \sum_{i \in \mathcal{U} \cap \mathsf{Root}(\mathcal{T})} \sum_{j \in \mathsf{Sub}(\mathcal{T})} A_j = \sum_{i \in \mathcal{U} \cap \mathsf{Root}(\mathcal{T})} \sum_{j \in \mathsf{Sub}(\mathcal{T})} Ht(s) \geq \sum_{j=1}^{k} Ht(t_j),
\]

(43)

where we have used lemma 16 and the relation \( \sum_{i \in \mathcal{U}} \#(\mathsf{Sub}(i)) = k \) which was verified by lemma 17. Hence it suffices to show that there exists \( \mathcal{U} \subset \mathsf{Tree}(\Phi_A) \) such that the relation

\[
\bigcup_{\mathcal{U}} \mathsf{Sub}(\mathcal{U}) = \{ t_1, \ldots, t_k \},
\]

(44)

is satisfied. By lemma 20 one finds that this relation is satisfied when \( \mathcal{U} = \mathsf{MTree}^{(k)} \). This completes the proof of theorem 13.

### 3.4. The Close Packing Lemma

Recall that \( B(N) = \bigcup_{1 \leq k \leq N} B(k, N) \) is the set of nearest-neighbor-excluding indices. Given \( t \in \mathsf{Tree}(\Phi_A) \), take any \( \mathcal{V} \subset \mathsf{Sub}(t) \) satisfying \( \mathcal{V} \cap \mathsf{Root}(\mathcal{T}) = \emptyset \) and \( \sum_{i \in \mathcal{V}} A_i < \sum_{i \in \mathsf{Root}(\mathcal{T})} A_i \).

**Lemma 24.** If \( \mathcal{V} \) is not closely packed with respect to \( t \), there exists \( \mathcal{U} \subset B(N) \) such that \( \mathcal{U} \subset \mathsf{Root}(\mathcal{T}) \), \( |\mathcal{U}| = |\mathsf{Root}(\mathcal{T})| \) and \( \sum_{i \in \mathcal{U}} A_i < \sum_{i \in \mathsf{Root}(\mathcal{T})} A_i \).

**Proof.** Let \( n \) be the order of the nesting of the trees in \( \mathsf{Sub}(\mathcal{T}) \). We prove the lemma by induction on \( n \). If \( n=1,2 \) one necessarily has \( \mathcal{V} = \mathsf{Sub}(\mathcal{T}) \) so there is nothing to prove. Suppose that \( n \geq 3 \). Let \( \mathcal{V}' = \mathsf{Sub}(\mathcal{T}) \setminus \mathcal{V} \) and \( a = \min \mathsf{Root}(\mathsf{Sub}(\mathcal{T}) \setminus \mathcal{V}) \). If such a \( a \) does not exist, then \( \mathcal{V} \) is closely packed. Suppose otherwise. We denote by \( s \in \mathsf{Sub}(\mathcal{T}) \cap \mathcal{V} \) the tree satisfying \( a \in \mathsf{Root}(s) \).

(i) Suppose that there exists \( u \in \mathsf{MSub}(s) \) such that both \( u \in \mathcal{V} \) and \( \mathsf{Sub}(u) \setminus \mathcal{V} \neq \mathsf{Sub}(u) \) are satisfied. Then \( \mathcal{V} = \mathsf{Sub}(\mathcal{T}) \) is not closely packed with respect to \( u \). From the induction hypothesis, there exists \( \mathcal{S}_u \subset B(N) \) such that \( \mathcal{S}_u \subset \mathsf{Root}(u) \), \( |\mathcal{S}_u| = |\mathsf{Root}(\mathcal{T})| \) and \( \sum_{i \in \mathcal{S}_u} A_i < \sum_{i \in \mathsf{Root}(\mathcal{T})} A_i \). Now the assertion of the lemma follows on taking \( \mathcal{S} = (\mathsf{Root}(\mathcal{T}) \setminus \mathsf{Root}(\mathcal{T})) \cup \mathcal{S}_u \).

(ii) Suppose otherwise, i.e. for any \( u \in \mathsf{MSub}(s) \) either \( u \notin \mathcal{V} \) or \( \mathsf{Sub}(\mathcal{T}) \subset \mathcal{V} \) is satisfied.

(a) Suppose that \( \mathsf{Sub}(\mathcal{T}) \subset \mathcal{V} \) is satisfied for any \( u \in \mathsf{MSub}(s) \). This implies that \( \mathsf{Sub}(\mathcal{T}) \subset \mathcal{V} \) or equivalently \( \mathsf{Root}(\mathcal{T}) \subset \mathsf{Root}(\mathcal{T}) \). Let \( r \in \mathsf{Sub}(\mathcal{T}) \) be the tree that directly straddles \( s \), and \( q \in \mathsf{Sub}(\mathcal{T}) \) be the one that directly straddles \( r \). See figure 7. We denote by \( P \) the branch point of \( q \) at which it straddles \( r \). Note that \( Ht(P) = Ht(r) \). Let \( q_1 \) be the subtree of \( q \) that extends downward from \( P \) and is adjacent to \( r \) on its left side. By the definition of \( a \) we have \( \mathsf{Root}(q_1) \subset \mathsf{Root}(\mathcal{T}) \). Define \( \mathcal{S} \subset B(N) \) as
Then by lemmas 22 and 23 we have
\[
\sum_{s} - \sum_{t} \geq -H_t(P) - H_t(s) > 0.
\]
(b) Suppose otherwise, i.e. there exists \( s' \in \text{MSub}(s) \) such that \( s' \notin V \). Replace \( s \) by \( s' \) and repeat the above arguments. Since the order of the nesting is finite, this case (ii)b cannot repeat endlessly, and we will eventually arrive at case (i) or (ii)a. The proof is completed.

3.5. The Forest Realization Lemma

Given a state of the trop p-Toda, there is generally more than one \( s \in B_k \) satisfying the condition \( \sum_{i} A_i \). We want to show that it is always possible to find such a \( B \) that can be realized as the set of all the bottom points of a forest, a set of trees in \( \Phi_A \).

Example 25. Consider example 11 with \( k = 4 \). One can take \( B^* = \{4, 6, 8, 10\}, \{5, 7, 9, 11\}, \{3, 5, 7, 9\} \) or \( \{4, 6, 9, 11\} \). In figure 5 one finds that the former two are not realized by forests, but the latter two are.

Lemma 26. For any \( 1 \leq k \leq N \) there exists \( T \subset \text{Tree}(\Phi_A) \) such that both \( \text{Root}(T) \in B(k, N) \) and
\[
\min_{B \in B(k, N)} \left\{ \sum_{i \in B} A_i \right\} = \sum_{i \in \text{Root}(T)} A_i.
\]
are satisfied.

Proof. Recall that \( \Phi_A \) is a graph associated with \( A = (A_1, \ldots, A_{2N}) \). It is composed of several connected components called trees. We denote by \( N(\Phi_A) \) the set of all nodes of \( \Phi_A \). It is the set of top points, bottom points, and branching points of the trees in \( \Phi_A \). In the same way, we denote by \( L(\Phi_A) \) the set of all links of \( \Phi_A \). Note that a top point has only a downward link, a
A bottom point has only an upward link, and a branching point has an upward and several downward links going out from it.

Choose any $B^* \in B(k, N)$ that satisfies

$$\min_{B \in B(k, N)} \left\{ \sum_{i \in B} A_i \right\} = \sum_{i \in B^*} A_i.$$

(47)

We draw a subgraph of $\Phi_A$ associated with $B^*$, that is denoted by $\Phi_{B^*}(A)$ and is defined as follows. First we adopt the bottom points $\{A_i\}_{i \in B^*}$ as elements of $N(\Phi_A(B^*))$, and adopt the links connected to them as those of $L(\Phi_A(B^*))$. We also adopt the nodes at the other ends of these links as elements of $N(\Phi_A(B^*))$. If such an adopted node is a branching point of $\Phi_A$, then there are two cases to be distinguished.

(i) Filled branching point: all its downward links are adopted ones.

(ii) Unfilled branching point: some of its downward links are unadopted ones.

If there is a filled branching point, we also adopt its upward link and the node at the other end as elements of $\Phi_{B^*}(A)$ and $N(\Phi_A(B^*))$. Repeat this procedure as much as possible, and let $\Phi_{B^*}(A)$ be the graph obtained finally. If all the branching points in $\Phi_{B^*}(A)$ are filled ones, then there exists $T \subset \text{Tree}(\Phi_A)$ such that $B^* = \text{Root}(T)$. Hence we are done.

Suppose otherwise. It is enough to show that there exists a procedure for finding a $C^* \in B(k, N)$ such that the number of unfilled branching points in $\Phi_A(C^*)$ is smaller than the number of those in $\Phi_A(B^*)$ by 1, under the condition $\sum_{i \in C^*} A_i = \sum_{i \in B^*} A_i$. The claim of the lemma follows from using this procedure repeatedly. Now we start describing such a procedure. We denote by $P$ an arbitrary chosen unfilled branching point in $\Phi_A(B^*)$ with lowest height, and by $q \in \text{Tree}(\Phi_A)$ the tree in which $P$ lies. By definition, there are both adopted and unadopted subtrees of $q$ under $P$. It is enough to show that there exists a way to reduce the number of the adopted subtrees without changing the other conditions.

Among those subtrees, choose an adjacent pair of adopted/unadopted subtrees $q_1, q_2$. See figure 8 for an example. Then, there is an unadopted tree $r \in M\text{Sub}(q)$ under $P$ between $q_1$ and $q_2$. By lemma 24 and since there is no unfilled branching point under $P$, one sees that $\text{Sub}(q_1) \cap \Phi_A(B^*)$ must be closely packed with respect to $q_1$, because otherwise (47) is not satisfied.

![Figure 8](image-url)

Figure 8. An unfilled branching point $P$ lies in the tree $q$. Thick lines are adopted links, and thin lines are unadopted ones.
Let $B_r^* = (\text{Root}(q_1) \sqcup \text{Root}(r)) \cap B^\circ$. Then $B_r^* = \text{Root}_c(q_1) \sqcup \text{Root}(V)$ with some $V \subset \text{Sub}(r)$. We are to show that there exists a $C_r^* \in B(N)$ satisfying

$$|C_r^*| = |B_r^*|, \quad \sum_{i \in C_r^*} A_i = \sum_{i \in B_r^*} A_i,$$

and that can be written as $C_r^* = \text{Root}(V')$ with some $V' \subset \text{Sub}(q_1) \sqcup \text{Sub}(r) \subset \text{Tree}(\Phi_A)$.

(i) Suppose that there exists $s \in \text{MSub}(r)$ such that both $s \in V$ and $\text{Sub}_c(s) \cap V \neq \text{Sub}_c(s)$ are satisfied. See figure 9 (left) for an example. Then $V_i = V \cap \text{Sub}(s)$ is not closely packed with respect to $s$. By lemma 24, one finds that this case cannot happen because otherwise (47) is not satisfied.

(ii) Suppose otherwise, i.e. for any $s \in \text{MSub}(r)$ either $s \notin V$ or $\text{Sub}_c(s) \subset V$ is satisfied.

(a) Suppose that $\text{Sub}_c(s) \subset V$ is satisfied for any $s \in \text{MSub}(r)$. See figure 9 (middle).

This implies that $\text{Sub}_c(r) = V$; hence $B_r^* = \text{Root}_c(q_1) \sqcup \text{Root}_c(r)$. Let $C_r^* = \text{Root}_c(q_1) \sqcup \text{Root}_c(r)$. Then $|C_r^*| = |B_r^*|z$ and by lemmas 22, 23 we have

$$\sum_{i \in C_r^*} A_i - \sum_{i \in B_r^*} A_i \leq \text{Ht}(r) - \text{Ht}(P) = 0. \quad (49)$$

The inequality case is excluded because otherwise (47) is not satisfied.

(b) Suppose otherwise, i.e. there exists $r' \in \text{MSub}(r)$ such that $r' \notin V$. See figure 9 (right). Replace $r$ by $r'$ and repeat the above arguments. Since the order of the nesting is finite, this case (ii)b cannot repeat endlessly, and we will eventually arrive at case (i) or (ii)a. The case (i) has already been excluded. The case (ii)a can also be excluded because now we have $\text{Ht}(r') - \text{Ht}(P) < 0$ instead of the right equality of (49). Thus one finds that neither case can happen.

To summarize, the only possible case is (ii)a, under the condition that the equality in (49) holds. Then by replacing $B^*$ with $(B^* \setminus B_r^*) \sqcup C_r^*$, one can reduce the number of adopted subtrees under $P$ by 1 without changing the other conditions. The proof is completed.

4. A continuous analogue of Kerov–Kirillov–Reshetikhin bijection

4.1. A map from highest weight paths to rigged configurations

The Kerov–Kirillov–Reshetikhin (KKR) bijection is a bijection between the set of tensor products of crystals [21] and the set of combinatorial objects known as rigged configurations. In this section we consider a continuous analogue of the KKR map in the $\mathfrak{sl}_2$ case in order to explain the background of our algorithm given in section 2.4.
Given $L > 0$, let $P_\lambda = \coprod_{N=1}^{\infty} P_{\lambda,N}$ where

$$P_{\lambda,N} = \left\{ (x_1, \ldots, x_{2N}) \in (\mathbb{R}_{>0})^{2N} | \sum_{i=1}^{2N} x_i \leq L, \sum_{i=1}^{k} (x_{2i-1} - x_{2i}) \geq 0 \text{ for } 1 \leq k \leq N \right\}$$

and $M = \coprod_{s=1}^{\infty} M_s$, where

$$M_s = \left\{ \lambda = (l_i, m_i)_{1 \leq i \leq s} \in (\mathbb{R}_{>0} \times \mathbb{Z}_{\geq0})^s | l_1 > \cdots > l_s, \sum_{i=1}^{s} l_i m_i \leq L/2 \right\}.$$  

Each element of the set $M_s$ is depicted as a Young diagram with area $\leq L/2$.

For $\lambda \in M_s$, define its $j$th vacancy number $p_j(\lambda)$ as

$$p_j(\lambda) = L - 2 \sum_{k=1}^{s} \min (l_j, l_k) m_k.$$  

Also we define the set of quantum numbers or riggings $\text{Rig}(\lambda)$ associated with $\lambda$ as

$$\text{Rig}(\lambda) = \left\{ (J_{ij})_{1 \leq i \leq m, 1 \leq j \leq s} \in \mathbb{R}^{m \times s} | 0 \leq J_{1 j} \leq \cdots \leq J_{m j} \leq p_j(\lambda) \right\}$$

for $1 \leq j \leq s$.

Let $\text{Rig} = \coprod_{\lambda \in M} \text{Rig}(\lambda)$.

Given $L > 0$ we define a pair of maps $\phi_1: P_\lambda \to M$ and $\phi_2: P_\lambda \to \text{Rig}$, such that $\phi = (\phi_1, \phi_2)$ gives a bijection $\phi: P_\lambda \to \coprod_{\lambda \in M} \{ \lambda \} \times \text{Rig}(\lambda)$.

4.2. The map $\phi_1$

In order to adjust the values of the quantum numbers to conventional ones, we slightly modify the algorithm given in section 2.4 by replacing item (iii) there by the following:

- Suppose that the last sequence of $n_{ij}$ zeros is at the right end after a positive neighbor $a$, as $\ldots, a, 0, \ldots, 0$. Remove these zeros. If $n_{ij}$ is odd, then also remove $a$.

We define the map $\phi_I: P_\lambda \to M$ using the algorithm given in section 2.4 with this modification. Given the $\lambda$-variables satisfying the highest weight condition, the Young diagram $\tilde{\lambda}$ constructed by the algorithm of section 2.4 does not change under this modification.

To be clearer, we call the algorithm of section 2.4 algorithm I, and the one in this section algorithm II. Then we have:

**Proposition 27.** The Young diagram constructed by algorithm I is equal to the one constructed by algorithm II.

**Proof.** To distinguish cases, we denote by $x^{(i)}(X)$, $y^{(i)}(X)$ the sequences $x^{(i)}$, $y^{(i)}$ constructed by algorithm $X$ (=I or II). By induction on $i$, it is easy to see that both algorithms preserve the highest weight condition, and that the lengths of the sequences are the same in the two algorithms. It is also easy to see that $x_{1}^{(I)}(I) \geq x_{1}^{(I)}(II)$, $y_{1}^{(I)}(I) \geq y_{1}^{(I)}(II)$, and $x_{j}^{(I)}(I) = x_{j}^{(I)}(II)$, $y_{j}^{(I)}(I) = y_{j}^{(I)}(II)$ for $j > 1$. This implies that the sets of numbers $\{(m^{(i)}, \nu^{(i)})\}_{1 \leq i \leq s}$ constructed by the two algorithms are the same. □
By this fact and theorem 13, we see that $\phi_1$ is a map that yields the conserved quantities of trop p-Toda.

**Example 28.** For the same state as in example 12, we have

$$\begin{align*}
  x^{(1)} &= (4, 3, 3, 2, 2, 1, 1, 1), \\
  y^{(1)} &= (3, 2, 2, 1, 1, 0, 0, 0), \\
  x^{(2)} &= (3, 2, 2, 1), \\
  y^{(2)} &= (2, 1, 1, 0), \\
  x^{(3)} &= (2, 1), \\
  y^{(3)} &= (1, 0),
\end{align*}$$

and $\mu^{(1)} = \mu^{(2)} = \mu^{(3)} = 1$, $\nu^{(1)} = 2$, $\nu^{(2)} = \nu^{(3)} = 1$.

The image of $P_\nu$ under the map $\phi_1$ is indeed in $\mathcal{M}$.

**Lemma 29.** $\phi_1(P_\nu) \subset \mathcal{M}$.

**Proof.** It suffices to show that the condition $\sum_{j=1}^{2N} x_j = L$ in (50) leads to the condition $\sum_{j=1}^{2N+1} x_j = L/2$ in (51). This is done by using the relations $y_j^{(i)} = x_j^{(i)} - \mu^{(i)}$ and $\sum_{j=1}^{2N+1} y_j^{(1)} = \sum_{j=1}^{2N} y_j^{(1)}$ as

$$L \geq 2 \sum_{j=1}^{2N} x_j^{(1)} = 2 \sum_{j=1}^{2N} x_j^{(1)} + \sum_{j=1}^{2N+1} y_j^{(1)} \geq 2 \sum_{j=1}^{2N} y_j^{(1)} + \sum_{j=1}^{2N+1} y_j^{(1)} \geq 2 \sum_{j=1}^{2N} \mu^{(j)} = 2 \sum_{j=1}^{2N+1} m_j.$$

$\square$

### 4.3. The map $\phi_2$

Consider the algorithm of section 2.4 with the modification given in section 4.2. With the $i$th block of the Young diagram $\lambda$, we associate $\nu^{(i)}$ non-negative real numbers called quantum numbers. Recall that in the array of non-negative real numbers $\ldots y_2^{(i)} y_1^{(i)}$, we have $k^{(i)}$ sequences of zeros, and the $j$th sequence has $n_j^{(i)}$ zeros. Denote by $l_j^{(i)}$ the position of the first element of the $j$th sequence. Let $r_j^{(i)} = \sum_{k=1}^{n_j^{(i)}} y_k^{(i)}$. Then the $\nu^{(i)}$ quantum numbers for the $i$th block are defined as

$$r_1^{(i)} \sigma_1^{(i)} \cdots r_k^{(i)} \sigma_k^{(i)} \cdots r_1^{(i)} \sigma_1^{(i)},$$

where $\sigma_j^{(i)} = \lfloor n_j^{(i)}/2 \rfloor$. Note that $r_j^{(i)} < r_k^{(i)}$ for $j < k$. Given any positive real number $L$ satisfying $\sum_{j=1}^{2N} x_j \leq L$, let $\lambda^{(i)} = \sum_{k=1}^s \nu^{(k)}$ and

$$q_j(\lambda) = L - 2 \sum_{k=1}^s \min \left( \lambda^{(k)}, \hat{\lambda}^{(k)} \right) \nu^{(k)},$$

for $1 \leq j \leq s$. By theorem 13, we have $\lambda^{(s+1-j)} = l_j$, $\nu^{(s+1-j)} = m_j$, and $q_{s+1-j}(\lambda) = p_j(\lambda)$.

By the following lemma, one sees that the quantum numbers for the $i$th block obey the condition $0 \leq r_j^{(i)} \leq q_j(\lambda)$. This enables us to define the map $\phi_2$: $P_\nu \to \text{Rig}$ by the procedure described above and by identifying the $m_{s+1-i} = \nu^{(i)}$ quantum numbers in (54) with $J_{s+1-i}^1, \ldots, J_{s+1-i}^{m_{s+1-i}}$ for $1 \leq i \leq s$ in (55).
Lemma 30. \( \phi_2(T_i) \subset \text{Rig} \).

Proof. By definition, the relation \( 0 \leq r^{(i)}_j \) holds trivially. Let \( L_0 = L - \sum_{j=1}^{2N} y_j \). It suffices to show that the relation \( \sum_{j=1}^{2N} y^{(i)}_j \leq q_i(\lambda) - L_0 \) is satisfied for \( 1 \leq i \leq s \) by induction on \( i \), since then we have \( r^{(i)}_j \leq \sum_{j=1}^{2N} y^{(i)}_j \leq q_i(\lambda) \). This is done by using the relations in the proof of lemma 29 as well as the relation \( q_j(\lambda) = q_{j-1}(\lambda) - 2\mu^{(i)} N^{(i)} \) where \( q_0(\lambda) = L \).

Example 31. For the \( A \)-variables of example 11, the quantum numbers for the first block of the Young diagram are \( 3\sqrt{2} - 1, 3\sqrt{2} - 1, 4\sqrt{2} - 2 \), those for the second are 2, 2, and that for the third is \( 3\sqrt{2} - 4 \).

4.4. The inverse map

Having obtained the pair \( \phi = (\phi_1, \phi_2) \) that gives a map \( \phi : P_n \rightarrow \bigcup_{\lambda \in \text{M}} \{ \lambda \} \times \text{Rig}(\lambda) \), now we consider its inverse map \( \phi^{-1} \). This is done by using the inverse of the algorithm of section 2.4 with the modification given in section 4.2. Let \( \lambda = (l_j, m_j)_{1 \leq j \leq s} \in \text{M} \) and \( J = (l^{(i)}_j)_{1 \leq j \leq s, 1 \leq i \leq s} \in \text{Rig}(\lambda) \). By the correspondence in the previous subsection, we regard \( \lambda \) as \( \lambda = ([\mu^{(i)}]_1 \in \text{M} \), and as \( J = ([\nu^{(i)}]_1 \in \text{M} \) is the multiplicity of \( r^{(i)}_j \). The quantum numbers obey the condition \( 0 \leq r^{(i)}_j < \cdots < r^{(i)}_k \leq q_i(\lambda) \) where \( q_i(\lambda) \) is the vacancy number defined by (55). Given \( (\lambda, J) \), its image \( x^{(i)} = (x^{(i)}_1, \ldots, x^{(i)}_{2N^{(i)}}) \) under the map \( \phi^{-1} \) is given by a step-by-step construction such as \( y^{(i)} \rightarrow x^{(i)} \rightarrow y^{(i-1)} \rightarrow x^{(i-1)} \rightarrow \cdots \rightarrow y^{(1)} \rightarrow x^{(1)} \). The first step goes as follows.

Let

\[
\sigma^{(i)}_1, \ldots, \sigma^{(i)}_k, \sigma^{(i)}_k, \ldots, \sigma^{(i)}_{2N^{(i)}} \in \text{M},
\]

be the \( \sigma^{(i)} \) quantum numbers for the \( s \)th (top) block. We define a sequence of non-negative real numbers \( y^{(i)}_1, \ldots, y^{(i)}_{2N^{(i)}} \) as

\[
r^{(i)}_j, 0, \ldots, 0, r^{(i)}_j - r^{(i)}_j, 0, \ldots, 0, r^{(i)}_k - r^{(i)}_k - 1, 0, \ldots, 0.
\]

Then we define a sequence of positive real numbers \( x^{(i)}_1, \ldots, x^{(i)}_{2N^{(i)}} \) by \( x^{(i)}_j = y^{(i)}_j + \mu^{(i)} \).

The subsequent steps go as follows. Given \( x^{(i+1)}_1, \ldots, x^{(i+1)}_{2N^{(i+1)}} \) and the quantum numbers in (54), we define a sequence of non-negative real numbers \( y^{(i+1)}_1, \ldots, y^{(i+1)}_{2N^{(i+1)}} \) in the following way. Let \( w^{(i+1)}_k = \sum_{j=1}^{k} x^{(i+1)}_j \) for \( 1 \leq k \leq 2N^{(i+1)} \) and \( w^{(i+1)}_0 = 0 \). For each \( r^{(i+1)}_j \), either \( w^{(i+1)}_k \leq r^{(i+1)}_j \) or \( w^{(i+1)}_k \geq w^{(i+1)}_k + 1 \) is satisfied. Roughly speaking, we split \( x^{(i+1)}_k \) and insert some zeros in the former case, while in the latter case we append \( r^{(i+1)}_j - w^{(i+1)}_k \) and some zeros at the end of the sequence. To be more precise, let us consider the case \( w^{(i+1)}_k \leq r^{(i+1)}_j \) and the case \( \mu^{(i)} N^{(i)} \) as examples, where we assumed that no other \( r^{(i+1)}_j \)’s exist in the (half-)intervals determined by \( w^{(i+1)}_k \)’s. In the former case we replace \( x^{(i+1)}_k = w^{(i+1)}_k \) by \( w^{(i+1)}_k \) by
Lemma 32. In the latter case we add the following sequence after $\lambda^{i+1}\\nolimits_{2N^{i+1}}$:

$$r^{(i+1)}_{i+1} = \lambda^{i+1}\\nolimits_{2N^{i+1}}, \quad 0, …, 0, \quad r^{(i+1)}_{i+1} = \lambda^{i+1}\\nolimits_{2N^{i+1}} - 1,$$

In the latter case we add the following sequence after $x^{(i+1)}\\nolimits_{2N^{i+1}}$:

$$r^{(i+1)}_{i+1} = \lambda^{i+1}\\nolimits_{2N^{i+1}}, \quad 0, …, 0, \quad r^{(i+1)}_{i+1} = \lambda^{i+1}\\nolimits_{2N^{i+1}} - 1.$$ 

It is easy to generalize these procedures for the cases where any number of different values of the quantum numbers exist in the (half-)intervals determined by $\lambda^{i+1}\\nolimits_{2N^{i+1}}$'s. Then we define a sequence of positive real numbers $x_1, …, x_{2N^{i+1}}$ by $x^{(i)} = \lambda^{(i)} + \mu^{(i)}$.

Given $\lambda = \{ (\mu^{(i)}, \nu^{(i)}) \}_{1 \leq i \leq s} \in \mathcal{M}$, and $J = \{ (r^{(i)}, \nu^{(i)}) \}_{1 \leq i \leq s, 1 \leq j \leq s} \in \text{Rig}(\lambda)$, we define the map $\phi^{-1}$ by $\phi^{-1}(\lambda, J) = (x_1^{(1)}, …, x_{2N^{(1)}}^{(1)})$. By construction, we see that it is indeed the inverse of the map $\phi$. Moreover we have the following:

Lemma 32. $\phi^{-1}(\bigsqcup_{\lambda \in \mathcal{M}} \{ \lambda \} \times \text{Rig}(\lambda)) \subset \mathcal{P}_t$.

Proof. From lemma 10, we see that the above algorithm for $\phi^{-1}$ preserves the highest weight condition. Hence it suffices to show that $w^{(1)}_{2N^{(1)}} = \sum_{j=1}^{2N^{(1)}} x^{(1)}_j \leq L$. To begin with, we prove $\sum_{j=1}^{2N^{(1)}} \nu^{(1)}_j \leq q_i(\lambda)$ for $1 \leq i \leq s$. For $i=s$, it is satisfied, as $\sum_{j=1}^{2N^{(1)}} \nu^{(1)}_j = r^{(1)}_k \leq q_s(\lambda)$.

Suppose that $\sum_{j=1}^{2N^{(1)}} \nu^{(1)}_j \leq q_i(\lambda)$ for some $i < s$. Then we have

$$w^{(1)}_{2N^{(1)i}} = \sum_{j=1}^{2N^{(1)}} \nu^{(1)}_j + 2N^{(1)i+1} \mu^{(1)} \leq q_{i+1}(\lambda) + 2N^{(1)i+1} \mu^{(1)} = q_i(\lambda),$$

and hence

$$\sum_{j=1}^{2N^{(1)}} \nu^{(1)}_j = \max \{ w^{(1)}_{2N^{(1)i}}, r^{(1)}_k \} \leq q_i(\lambda).$$

Thus by descending induction on $i$, this inequality holds for any $1 \leq i \leq s$. Now we obtain the desired result, as $w^{(1)}_{2N^{(1)}} = \sum_{j=1}^{2N^{(1)}} \nu^{(1)}_j + 2N^{(1)i+1} \mu^{(1)} \leq q_1(\lambda) + 2N^{(1)} \mu^{(1)} = q_0(\lambda) = L$. □

By lemmas 29, 30 and 32 we obtain the following result.

Theorem 33. The map $\phi$: $\mathcal{P}_t \rightarrow \bigsqcup_{\lambda \in \mathcal{M}} \{ \lambda \} \times \text{Rig}(\lambda)$ is a bijection.

So far we do not know whether one can regard this bijection as an isomorphism, i.e. we do not know what kinds of mathematical structures are preserved under this bijection.

5. Concluding remarks

In this paper, elucidated combinatorial aspects of the conserved quantities of the general tropical periodic Toda lattice beyond the generic condition. Let us summarize what we have done. The evolution equation of this dynamical system was given by (17), and the conserved quantities were written as in (20). We proved that the conserved quantities are related by a weak convexity condition (theorem 9), which enables us to write the set of conserved quantities as the Young diagram in figure 3. After introducing an algorithm related to the Kerov–Kirillov–Reshetikhin (KKR) bijection to construct another Young diagram shown in figure 4, we presented our main result (theorem 13), stating the identification of these Young diagrams. We gave a detailed proof of this theorem and a discussion on the KKR bijection in the subsequent sections.
The idea of our proof is based on [16], but it is not a straightforward extension. From our interpretation, we see that there are several ambiguous and/or incorrect descriptions in Appendix A.1 of [16]. To make our proof mathematically rigorous, we devise our own tools. The following are two of them. (i) In section 3.1 we devised our original rule for drawing lines in a graph when more than two consecutive $\times$’s appear in a given level. In the 10-elimination algorithm for pBBS, this corresponds to simultaneous disappearance of more than two consecutive blocks, for which the rule for drawing lines was ambiguous in [16].

(ii) We formulated our original lemmas in section 3.4 and section 3.5. Here we explain the latter. From our interpretation, equation (A.13) of [16] claims that any $B^*$ in (47) must be realized by a forest, on which the proof substantially depends. But as we have shown in example 25, this claim is not true. The correct statement is that at least one $B^*$ in (47) can be realized by a forest, as we proved in section 3.5.

In the original (discrete) KKR bijection in $\mathfrak{sl}_2$ case [17], the set $\mathcal{P}_L$ is a subset of the tensor product of crystals $\otimes B_L$ with $B = \{1, 2\}$ and its elements are expressed as

$$\ell_1 \cdots \ell_2 \cdots 2 \cdots 1 \cdots 2 \cdots 1 \cdots 1$$

for some $N$ with the condition in (50), where we omitted $\otimes$ symbols. In our algorithm, the integer $\ell_i$’s here have been replaced by continuous variables taking their values in the real numbers. Actually our algorithm is based on the algorithm in [14] which is a variation of the original algorithm. In the $\mathfrak{sl}_{n+1}$ case, the set $B$ is replaced by $B = \{1, \ldots, n+1\}$ and the highest weight condition is adequately modified. The algorithm of the KKR bijection for the $\mathfrak{sl}_{n+1}$ case was given in section 3.2 of [3]. In this case no analogues of the above mentioned variation in the $\mathfrak{sl}_2$ case have been developed yet. Therefore a promising way to construct a continuous analogue of the KKR bijection for $\mathfrak{sl}_{n+1}$ is to develop such a variation first. Then the remaining task will be rather straightforward.

Finally we would like to explain the difference in arguments between [17] and the present work. In [17] the conservation of the Young diagram under the time evolution of pBBS was shown in the following way. For any positive integer $l$ and any state $p$, we introduced a time evolution $T_l(p)$ and an energy $E_l(p)$ (Proposition 2.1) by using the crystal theory and its energy function. Then the conservation of the energy $E_l(T_l(p)) = E_l(p)$ and the commutativity of the time evolutions $T_l T_k(p) = T_k T_l(p)$ were shown (Theorem 2.2). Finally we proved that the set of data carried by the whole set of the values of the energy $E_l(p)$ ($l = 1, 2, \ldots$) was equivalent to the Young diagram constructed using the KKR map (Proposition 3.4). In the present work the author did not try to generalize this argument to the trop p-Toda case, because the crystal theory and its energy function have not been developed for this case. Therefore no counterpart of the above construction of $E_l(p)$ can be allocated in the present paper. However, the author thinks that one can generalize the above argument to the trop p-Toda case because the arguments in [4] used for constructing a commuting family of time evolutions may be used to develop an analogue of the energy function in this case. We hope to report any progress on this subject in the near future.

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