Solitary waves on flows with an exponentially sheared current and stagnation points

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Abstract
While several articles have been written on water waves on flows with constant vorticity, little is known about the extent to which a nonconstant vorticity affects the flow structure, such as the appearance of stagnation points. In order to shed light on this topic, we investigate in detail the flow beneath solitary waves propagating on an exponentially decaying sheared current. Our focus is to analyse numerically the emergence of stagnation points. For this purpose, we approximate the velocity field within the fluid bulk through the classical Korteweg-de Vries asymptotic expansion and use the Matlab language to evaluate the resulting streamfunction. Our findings suggest that the flow beneath the waves can have zero, one or two stagnation points in the fluid body. We also study the bifurcation between these flows. Our simulations indicate that the stagnation points emerge from a streamline with a sharp corner.

Keywords: Variable vorticity, Solitary waves, Water waves, Stagnation points.

1 Introduction
In the last three decades a great attention has been given to the study of waves propagating on linearly sheared currents (constant vorticity). Many advances have been made in a better understand of the effects of this kind of current on the free surface and in the bulk of the fluid.

It is well-known that the flow beneath a rotational water wave with constant vorticity is characterized mainly by: i) the appearance of stagnation points – fluid particles with zero velocity in the wave’s moving frame; ii) critical layers – a horizontal layer with closed streamlines separating the fluid into two disjoint regions; iii) the arising of pressure anomalies such as the occurrence of maxima and minima of pressure in locations other than below the crest and trough respectively. These results have been showed numerically \cite{5, 27, 29, 30}, asymptotically \cite{8, 20} and also proved rigorously \cite{14, 33, 24}. The constant vorticity also affects the shape of the free surface, for instance its profile can become rounder and possibly form an exotic overhanging wave \cite{31, 32, 9, 10}.

Although many realistic situations can be represented by constant vorticity (e.g. when waves are long compared with the water depth or when waves are short compared with lengthscale of the vorticity distribution \cite{30}) there are cases that cannot. For instance, when the current is generated by wind stress at the surface a highly sheared current near the water surface with non-uniform vorticity results \cite{28}.

The study of waves with arbitrary vorticity distribution started in 1934 with Dubreil-Jacotin \cite{8}. She introduced a change of variables that transforms the domain of the fluid into a rectangle, then by using power series she computed small-amplitude waves with general vorticity. Since then, a number of authors have studied this problem by considering different current profiles (vorticity distribution) and approaches.

Starting from the results on infinitesimal rotational waves, we mention the works of Peregrine \cite{25} and Dalrymple \cite{7}. In these works the authors present a detailed discussion on simplified models for currents used in several studies (currents that are modelled as a bilinear, a sine, a cosine or a one-seventh power law as function of the depth).

In an attempt of moving on the direction of nonlinear waves, Freeman and Johnson \cite{15} deduced a Korteweg-de Vries equation for waves on the surface of a general sheared current. A rigorous theoretical result on the existence of large-amplitude waves with general vorticity distributions was given only 70 years after Dubreil-Jacotin \cite{8} findings in the seminal work of Constantin and Strauss \cite{6}. Their results relies on the hypothesis of
periodic waves over a finite depth channel and vorticity-stream-function in $C^{1,\alpha}$. This work opened the way to several theoretical and numerical fruitful researches in the coming years. Among them, we mention the findings of Hur [18] and Groves and Wahlén [16] who proved the existence of solitary waves of small amplitude for an arbitrary vorticity. Besides, the contributions of Hur [19] who proved the existence of large-amplitude periodic waves on deep water regime.

Due to technical complications, the numerical investigations for large-amplitude waves with nonconstant vorticity have been conducted under the premise that there is no stagnation point in the bulk of the fluid. Among the works on this topic, we highlight the findings of Ko and Strauss [22, 23]. These authors computed periodic water waves for: a shear vorticity layer of some uniform thickness, a continuous vorticity and for the case in which the vorticity is represented by a step function. Their studies focused on certain characteristic of the flow such as the probable location of the first stagnation point and the shape of the nearly stagnant waves. Later, Amann and Kalimeris [2] improved the results of Ko and Strauss [22, 23] in the sense that they computed waves closer to stagnation. They also constructed waves for flows with vorticity that varies linearly and quadratically with respect the stream function. More recently, Chen et al. [4] presented an complete description of the flow beneath periodic water waves propagating on rotational flows with piece-wise constant vorticity. They took into account the effects of the vorticity on the free surface, on the streamlines, on the pressure distribution in the fluid and also showed particle trajectories.

A full description of the flow pattern beneath a large-amplitude wave with nonconstant vorticity in the presence of internal stagnation points is still an open problem. The state of the art of this topic is restricted to linear waves and for the single case in which the vorticity depends linearly upon the stream function. Regarding this subject, we refer to the study of Ehrnsrtröm et al. [12] in which the authors give a qualitative description of the cat’s eye structure. They show that arbitrarily many critical layers with cat’s vortices are possible beneath linear waves with an affine vorticity. This is also proved by Ehrnsrtröm et al. [13] whose work is later revisited by Aasen and Varholm [1] through a slightly different approach.

In this work we study numerically the flow structure beneath solitary waves propagating on an exponentially sheared current. Our goal is to capture with details the appearance of stagnation points. For this purpose, we consider the classical Korteweg-de Vries model to obtain an approximation of the velocity field in the fluid body. We find parameter regimes in which the flow has zero, one or two stagnation points. Differently from the constant vorticity case, in which the stagnation points gives rise to the Kelvin’s cat’s eye flow pattern, now they form a flow characterized by a “loop-pattern”, namely, there is a closed flow region with its corresponding separatrix is in the shape of a loop. By varying the parameters of the flow, we show that the two stagnation points (a centre and a saddle) that define the loop-pattern get closer, until they collide, then vanish, resulting in a saddle-centre bifurcation. Besides, we notice that a streamline with a sharp corner results from the collision of the two stagnation points.

This paper is organized as follows. In section 2 we revisit the classical KdV equation for waves on a general sheared current and present the approximation of the velocity field. At the beginning of the section 3 we discuss the ODE system for the particle trajectories. Subsequently, the numerical approach and the results are shown. Concluding remarks are given in section 4.

2 Governing equations

Consider a two-dimensional incompressible flow of an inviscid fluid of constant density $\rho$ in a finite channel with depth $h_0$. Denote by $(u(x, y, t), v(x, y, t))$ the velocity field in the bulk of fluid and $\zeta(x, t)$ the free surface profile. Choosing the set of dimensionless variables, where $\lambda$ is a typical wavelength, $a$ is a typical wave amplitude, $h_0$ is a typical depth and $a/\sqrt{gh_0}$ is a typical time-scale we have the dimensionless governing Euler equations

\begin{equation}
\begin{align*}
&u_x + uu_x + vv_y = -p_x, \quad \text{for } 0 < y < 1 + \epsilon \zeta(x, t),
&\mu^2\left(v_t + \epsilon(u_x + vv_y)\right) = -p_y, \quad \text{for } 0 < y < 1 + \epsilon \zeta(x, t),
&u_x + v_y = 0, \quad \text{for } h(x) < y < 1 + \epsilon \zeta(x, t),
&p = \epsilon \zeta, \quad \text{at } y = 1 + \epsilon \zeta(x, t),
&v = \epsilon (\zeta^t + u \zeta_x), \quad \text{at } y = 1 + \epsilon \zeta(x, t),
&v = 0, \quad \text{at } y = 0,
\end{align*}
\end{equation}

where $\mu = h_0/\lambda$ is the dispersive parameter and $\epsilon = a/h_0$ is the nonlinearity parameter.

We assume that the flow is in the presence of a depth-dependent imposed current $(U(y), 0)$, which dominates the the velocity field. We are interested in investigating particle trajectories in the bulk of the fluid beneath a solitary wave. To this end, we consider the asymptotical approximation for the velocity field as well as for the free surface reported by Johnson [21]. In order to ease the read of the article, we summarise bellow the main steps necessary to obtain the reduced system of governing equations.
We consider the scaling

\[ u \to U(y) + \epsilon u, \quad v \to \epsilon v, \quad p \to \epsilon p, \]  

(2)

and the standard long-wave approximation \( \mu^2 = \epsilon \). The condition (2) yields

\[
\begin{align*}
& u_t + U(y)u_x + U'(y)v + \epsilon(\mu u_x + \nu u_y) = -p_x, \quad \text{for } 0 < y < 1 + \epsilon \zeta(x,t), \\
& \epsilon \left\{ (v_t + U(y)v_x + \mu v_x + \nu v_y) \right\} = -p_y, \quad \text{for } 0 < y < 1 + \epsilon \zeta(x,t), \\
& u_x + v_y = 0, \quad \text{for } 0 < y < 1 + \epsilon \zeta(x,t), \\
& p = \zeta, \quad \text{at } y = 1 + \epsilon \zeta, \\
& v = \zeta(t) + U(y)\xi_x + \epsilon \mu \zeta, \quad \text{at } y = 1 + \epsilon \zeta, \\
& v = 0, \quad \text{at } y = 0.
\end{align*}
\]

Introducing the variables \( \xi = x - ct \) and \( \tau = ct \), where \( c \) is to be determined \textit{a posteriori}, we obtain

\[
\begin{align*}
& (U(y) - c)u_{\xi} + U'(y)v + \epsilon(u_\tau + \mu u_{\xi} + \nu u_y) = -p_{\xi}, \quad \text{for } 0 < y < 1 + \epsilon \zeta, \\
& \epsilon \left\{ (U(y) - c)v_{\xi} + \epsilon(v_\tau + \mu v_{\xi} + \nu v_y) \right\} = -p_y, \quad \text{for } 0 < y < 1 + \epsilon \zeta, \\
& u_{\xi} + v_y = 0, \quad \text{for } 0 < y < 1 + \epsilon \zeta, \\
& p = \zeta, \quad \text{at } y = 1 + \epsilon \zeta, \\
& v = (U(y) - c)\xi + \epsilon(\xi_\tau + \mu \xi), \quad \text{at } y = 1 + \epsilon \zeta, \\
& v = 0 \quad \text{at } y = 0. \\
\end{align*}
\]

(3)

Seeking for asymptotic solutions of (3), we define the power series expansion

\[ q(\xi, y, \tau; \epsilon) = \sum_{n=0}^{\infty} \epsilon^n q_n(\xi, y, \tau), \quad \zeta(\xi, \tau) = \sum_{n=0}^{\infty} \epsilon^n \zeta_n(\xi, \tau), \]

(4)

where \( q = u, v \) or \( p \). Substituting (4) in (3) and proceeding as in (21) we obtain as first approximation for the velocity field in bulk of the fluid

\[
\begin{align*}
& v_0 = \left\{ \frac{(U(y) - c)}{U(y) - c} \int_0^y \frac{dz}{(U(y) - c)^2} \right\} \zeta_0, \\
& u_0 = -\left\{ \frac{1}{U(y) - c} + \frac{U'(y)}{U(y) - c} \int_0^y \frac{dz}{(U(y) - c)^2} \right\} \zeta_0,
\end{align*}
\]

(5)

(6)

where \( c \) satisfies the compatibility condition known as Burns condition:

\[
\int_0^1 \frac{1}{(U(y) - c)^2} dy = 1,
\]

(7)

and the free surface \( \zeta_0 \) is solution of the sheared Korteweg-de Vries equation

\[
-2I_{31}\zeta_0 + 3I_{41}\zeta_0\zeta_0 + J_1\zeta_0\zeta_0 = 0,
\]

(8)

whose coefficients \( I_{n1} = I_n(1), n = 1, 2, 3, 4 \), are given by

\[
I_n(y) = \int_0^y \frac{dy}{(U(y) - c)^n},
\]

(9)

and

\[
J_1 = \int_0^1 \int_y^1 \int_y^{y_1} \frac{(U(y_1) - c)^2}{(U(y) - c)^2(U(y_2) - c)^2} dy_2 dy_1 dy.
\]

(10)

Solitary wave solutions of (8) are described by the formula

\[
\zeta_0(\xi, \tau) = A \text{sech}^2(k(\xi - C\tau)), \quad \text{where } k = \sqrt{\frac{A I_{41}}{4 J_1}} \text{ and } C = -\frac{2J_1 k^2}{I_{31}}.
\]

(11)

In the Euler coordinates, the solitary wave solution is written as

\[
\zeta_0(x, t) = \epsilon A \text{sech}^2 \left( k(x - (c + \epsilon C)t) \right), \quad \text{where } k = \sqrt{\frac{A I_{41}}{4 J_1}} \text{ and } C = -\frac{2J_1 k^2}{I_{31}}.
\]

(12)
3 Computing particle trajectories

Particle trajectories beneath the solitary wave (12) can be approximated by the solution of the system of ordinary differential equations (ODEs)

\[
\begin{align*}
\frac{dx}{dt} &= U(y) + \epsilon u_0(x, y, t), \\
\frac{dy}{dt} &= \epsilon v_0(x, y, t).
\end{align*}
\]

(13)

It is convenient to solve (13) in the wave frame \( X = x - (c + \epsilon C) t \) and \( Y = y \). In this framework, particle trajectories are solutions of the autonomous ODE system

\[
\begin{align*}
\frac{dX}{dt} &= U(Y) - (c + \epsilon C) + \epsilon u_0(X, Y), \\
\frac{dY}{dt} &= \epsilon v_0(X, Y).
\end{align*}
\]

(14)

It is easy to see that streamlines are solutions of (14). Thus, particle trajectories are the level curves of the stream function \( \Psi(X, Y) \), which is given by

\[
\Psi(X, Y) = \int_Y^0 (U(s) + \epsilon u_0(X, s)) \, ds - (c + \epsilon C)Y.
\]

(15)

In order to compute the streamlines (15) we need to solve the integrals (7), (9) and (10). These integrals are obtained through the function integral that is implemented in MATLAB. Following we present the results for a linear and an exponentially vertically sheared current.

3.1 Linear current

Guan [17] studied particle trajectories beneath solitary waves in the presence of a linear sheared current through the classical asymptotic long-wave limit. He showed that the orbits obtained from the asymptotic approximation agree well with the ones produced by full Euler equations when the wave amplitude is small. Based on his results in all simulations presented in this article we fix \( \epsilon = 0.1 \).

![Figure 1: Coefficients of KdV in terms of a for the linear sheared current (\( \tilde{a} \approx 2.1092 \)).](image1)

![Figure 2: Typical phase portrait of (14) for a linear sheared current.](image2)
In order to verify our mathematical formulation for the particle trajectories, we start considering the linear sheared current $U(y) = ay$ and analyse how the parameters of the KdV (8) varies according to the value of $a$. This is shown in figure 1. We see that for positive values of $a$ the coefficients $J_1$ (that controls the dispersion) and $I_{41}$ (that controls the nonlinearity effects) reproduce different behaviours. While $J_1$ is decreasing in the interval $(0, \tilde{a})$ and approaches to zero as $a \to \tilde{a}$, $I_{41}$ is increasing in the same interval. Furthermore, we notice that $J_1$ and $I_{41}$ have a discontinuity at $a = \tilde{a}$. This shows us that the KdV model is not appropriate to study flows with $a \gg 0$. With this in mind, we reproduce a typical phase portrait for $a < 0$. Figure 2 shows the phase portrait of the ODE (14) for $U(y) = -30y$. This case is featured by the existence of a critical layer close to the bottom with three stagnation points, two saddles and a centre, with the centre beneath the crest of the wave – and this ties in with the results in the literature [17, 30, 27].

Our approach allow us to go beyond constant vorticity flows and study the existence of stagnation points in the presence of more complex types of currents such as exponentially decaying ones.

### 3.2 Exponentially decaying current

An interesting physical problem in water waves over a vertical variable current is surface shear waves [26]. Surface shear wave flows are idealized assuming a thin fast-moving sheet of fluid at the surface and still water beneath it. In order to investigate particle trajectories under these conditions, we choose as the current profile the function

$$U(y) = a \exp \left( -\sigma (y - 1)^8 \right),$$

where $a \in \mathbb{R}$ and $\sigma > 0$ is chosen so that the horizontal velocity decays rapidly with the depth. In figure 3 we depict the profile of the current (16) for some values of $a$. In the simulations presented here, we consider $\sigma = 10^6$. This choice of parameters represent a current acting roughly in 20% of the water column.

![Figure 3: Current profile $U(y) = a \exp \left( -\sigma (y - 1)^8 \right)$ for different values of $a$ ($\sigma = 10^6$).](image)

Figure 4 displays the graph of parameters of the KdV (8) as a function of $a$ for the exponentially decaying current (16). As in the linear case, we notice that the KdV model is not appropriate for values of $a \gg 0$.

![Figure 4: Coefficients of KdV in terms of $a$ for the exponentially decay current ($\tilde{a} \approx 2.7207$).](image)

For flows with constant vorticity, it is known that for $a \ll 0$ there are three stagnation points: a centre below the crest (inside the flow) and two saddles on the bottom, resulting in a cat’s eye structure below the crest and attached to the bottom (see figure 2 or the figures in [17]). As $a$ increases the three stagnation points coalesce at $x = 0$ and disappear in a saddle-centre bifurcation [27].

Differently from the constant vorticity case, for values of $a \ll 0$ the phase portrait of the flow associated with the exponentially decaying current (16) has only two stagnation points: one centre and one saddle – both of them bellow the crest of the wave forming a recirculation zone, see figure 5 (top). As $a$ increases, the saddle moves up and the centre moves down shrinking the recirculation zone, then these points coalesce forming a recirculation zone.
single stagnation point at \( a^* \approx -1.897 \). The phase portrait for \( a = a^* \) indicates the existence of a streamline with a sharp corner at \( x = 0 \) where its located a stagnation point. This happens when the two stagnation points collide. The saddle-centre bifurcation occurs from this streamline. Besides, it calls our attention the similarity between the phaseportrait for \( a = -3 \) and the one illustrated by Escher et al. [11](Figure 3) in the study of stratified water waves with density varying linearly along the streamlines. These pictures indicate that somehow there might exist a certain correspondence between stratification and vorticity as a mechanism of formation of critical layers. Figure 5 depict the saddle-centre bifurcation in a different perspective. For \( a > a^* \) there are no stagnation points.

Figure 5: Phase portraits for flows associated to the exponentially decaying current (16) for different values of \( a \) (\( a^* \approx -1.897 \)).

### 4 Conclusions

In this work, the focus has been on describing the main features of the flow beneath a solitary wave on an exponentially sheared current. The velocity field within the fluid bulk was approximated by the classical weakly nonlinear and weakly dispersive regime. This allowed to compute the location of stagnation points and the streamlines. Of particular interest we reported the following results. Flows may have a single stagnation point in the fluid body located at the top of a streamline with a sharp corner. By varying the intensity of the current we obtained a flow with two stagnation points (a centre and a saddle) and a recirculation zone closed by a separatrix loop shaped. The visualization of such features can hopefully provide valuable insights for both theoretical and numerical works on water waves on flows with variable vorticity. Besides, although the results
presented here are just for an exponential sheared current, the numerical procedure can be easily adjusted for other profiles of sheared currents.

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