Modulational instability and bright solitary wave solution for Bose–Einstein condensates with time-dependent scattering length and harmonic potential

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Abstract. We consider the one-dimensional Gross–Pitaevskii (GP) equation, which governs the dynamics of Bose–Einstein condensate (BEC) matter waves with time-dependent scattering length and a harmonic trapping potential. We present the integrable condition for the one-dimensional GP equation and obtain the exact analytical solution which describes the modulational instability and the propagation of a bright solitary wave on a continuous wave (cw) background. Moreover, by employing the adiabatic perturbation theory for a bright soliton, we obtain approximative bright solitary wave solutions under near-integrable conditions. Both the exact and approximative solutions show that the amplitude of a bright solitary wave with zero boundary condition depends on the scattering length while its motion depends on the external trapping potential.
1. Introduction

Since the realization of Bose–Einstein condensates (BEC) in dilute alkali-metal atomic vapours, we have seen a rapid development in this field [1]. In particular, the experimental observations of solitons and vortices [2]–[7] in BEC have stimulated intensive studies of the nonlinear excitations of BEC matter waves, including such aspects as the soliton propagation [8], vortex dynamics [9], interference patterns [10], domain walls in binary BEC [11] and the modulational instability [12]–[14]. One of the most important aspects of the BEC solitary wave is its dynamics, since it is believed that the generation, dynamics and management of a BEC solitary wave is important for a number of BEC applications, like atomic interferometry [15], and different kinds of quantum phase transitions [16], as well as in the context of the nonlinear physics, including nonlinear optics and hydrodynamics.

It is well known that at zero temperature the dynamics of weakly interacting bosonic gases trapped in a potential $V$ is well described by the Gross–Pitaevskii (GP) equation for the order parameter $\Psi(r, t)$ [1, 17]

$$i\hbar \frac{\partial \Psi(r, t)}{\partial t} = \left[-\frac{\hbar^2}{2m} \nabla^2 + V(r, t) + g_0 |\Psi(r, t)|^2\right] \Psi(r, t). \tag{1}$$

Here $g_0 = \frac{4\pi \hbar^2 a_s}{m}$ characterizes the strength of interatomic interaction and is defined in terms of the $s$-wave scattering length $a_s$, the atomic mass $m$, and the Planck constant $\hbar$. The trapping potential we consider here is the cigar-shaped harmonic trapping potential with the elongated axis in the $x$-direction as used in experiment [2], which takes the form $V = \frac{m}{2} \left[\omega^2 x^2 + \omega_\perp^2 (y^2 + z^2)\right]$ with $\omega \ll \omega_\perp$. In this case, since $\frac{\omega}{\omega_\perp}$ is small, one can expect that the variation of the profile of the order parameter is slow in the elongated direction. On the other hand, due to the strong confinement in the transverse ($y$ and $z$) directions, the motion of the order parameter in the transverse directions is frozen, and can be described by the ground-state wavefunction of a two-dimensional quantum harmonic oscillator. Thus, by introducing the following transformation [18]–[21]

$$\Psi(r, t) = \frac{1}{\sqrt{2\pi \hbar a_\perp}} \exp \left(-i\omega_\perp t - \frac{y^2 + z^2}{2a_\perp^2}\right) u \left(\frac{x}{a_\perp}, \omega_\perp t\right),$$
one can obtain the model equation for the cigar-shaped BEC [22]–[24], which is the well-known one-dimensional nonlinear Schrödinger equation with an external harmonic trapping potential

$$i\frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + g|u|^2u - kx^2u = 0,$$

(2)

where $t$ and $x$ are the temporal and spatial coordinates measured in units $\omega_\perp^{-1}$ and $a_\perp = \sqrt{\hbar / m \omega_\perp}$, respectively. The nonlinearity parameter $g = -\frac{2a_s}{a_B}$, which is negative (positive) for repulsive (attractive) interatomic interaction, the potential parameter $k = \frac{\omega^2}{2a^2}$, and $a_B = \frac{a_\perp}{2} \int_{-\infty}^{\infty} |\Psi_1|^2 dx$. It should be pointed out that in the relevant experiments, a bright soliton has been created by utilizing a Feshbach resonance to manipulate the sign of the s-wave scattering length from positive to negative [3, 4]. In this situation, the s-wave scattering length is allowed to be a function of time $t$ [25]–[28]. On the other hand, in order to control the dynamics of BEC in the trap [29]–[31], the trapping frequency in the elongated axis $\omega$ can also be a function of time $t$ (while in deriving equation (2) from (1), $\omega_\perp$ is kept constant). Therefore, the nonlinearity parameter $g$ and potential parameter $k$ can be time-dependent, and equation (2) can be used to describe the control and management of BEC by properly choosing the two time-dependent parameters.

Indeed, the temporal periodic modulation and the exponential controlling of s-wave scattering length have been described [26, 27] and the evolution of periodic matter waves in one-dimensional BEC with time-dependent scattering length has been analysed [28]. Moreover, the evolution of the condensate in a time-dependent trap has been addressed in many different ways [29, 30], and the modulational instability of a one-dimensional BEC system in a time-dependent harmonic potential has been analysed [31]. Recently, based on equation (2), dynamically stabilized bright solitons in two-dimensional BEC [23], breathing bright solitons [24] and the controllable compression of bright soliton matter waves [22] in onedimensional BEC have been demonstrated. However, although the GP equation (2) is widely accepted as a valid model for the dynamics of the BEC with low matter wave density at low temperature, knowledge of the dynamics of the condensate with time-dependent s-wave scattering length and time-dependent harmonic trapping potential is scarce, and no explicit solutions are known except for $g(t)$ and $k(t)$ having special form such as $g(t) = \exp(\lambda t)$, $k(t) = -\lambda^2/2$ [18], and $g(t) = 1 + \tanh(\Omega t)$, $k(t) = \tanh(\Omega t) - 1$ [22]. In this paper, we aim to present the explicit, analytical solution to describe the dynamics of a bright solitary wave on a continuous wave (cw) background for equation (2) with time-dependent s-wave scattering length and time-dependent harmonic trapping potential. Based on the exact analytical solution, we investigate the modulational instability and discuss the dynamics of bright solitary waves in the one-dimensional BEC system.

This paper is organized as follows. In section 2, we present the integrable condition for equation (2), and reduce equation (2) to the standard nonlinear Schrödinger equation by employing a modified lens-type transformation. Then we obtain the general expression of the bright solitary wave solution on a cw background for equation (2) with time-dependent s-wave scattering length and time-dependent harmonic trapping potential, and discuss the modulational instability and the dynamics of a bright one-solitary wave in section 3. In section 4, we consider the near-integrable conditions and present the approximative analytical bright solitary wave solutions. Finally, we conclude with the discussion in section 5.
2. Integrable condition

In order to get the exact analytical expression to describe the modulational instability and the propagation of a bright solitary wave on a cw background, we need to find the integrable condition for equation (2) first. We begin with the following modified lens-type transformation

\[ u(x, t) = \frac{1}{\sqrt{g(t)\ell(t)}} \Phi(X, T) \exp \left[ i f(t)x^2 \right], \]

where \( f(t) \) is a real function of \( t \), \( X = x/\ell(t) \) and \( T = T(t) \). In particular, when \( g \) is time independent, the transformation (3) is just the so-called lens-type transformation reported in [31, 32]. Also, it should be pointed out that the transformation (3) implicitly assumed that \( g(t) > 0 \), which corresponds to the attractive interatomic interaction.

By demanding that

\[ \frac{df}{dt} + 2f^2 + k(t) = 0, \]

\[ \frac{d\ell}{dt} - 2f\ell = 0, \]

\[ \frac{dT}{dt} - \ell^{-2} = 0, \]

equation (2) is converted to the perturbed nonlinear Schrödinger equation

\[ i \frac{\partial \Phi}{\partial T} + \frac{1}{2} \frac{\partial^2 \Phi}{\partial X^2} + |\Phi|^2 \Phi = i \epsilon(t) \Phi, \]

where \( \epsilon(t) = [f(t) + \frac{1}{2g} \frac{dg}{dt}] \ell^2. \)

From the expression of \( \epsilon(t) \), we can clearly see that if \( f(t) = -\frac{1}{2g} \frac{dg}{dt} \), equation (2) can be reduced to the standard nonlinear Schrödinger equation

\[ i \frac{\partial \Phi}{\partial T} + \frac{1}{2} \frac{\partial^2 \Phi}{\partial X^2} + |\Phi|^2 \Phi = 0. \]

That is to say, equation (2) is integrable when the nonlinearity parameter \( g(t) \) and the harmonic trapping potential parameter \( k(t) \) satisfy the following integrable condition

\[ \frac{dg}{dt} \frac{d^2 g}{dt^2} - 2 \left( \frac{dg}{dt} \right)^2 - 2kg^2 = 0. \]

Thus, the virtue of the modified lens-type transformation (3) is that, without much complicated calculation, we not only find the integrable condition (9) for equation (2), but also retrieve the well-known standard nonlinear Schrödinger equation (8), which has been well studied and various solutions such as Jacobian elliptic function solutions, bright \( N \)-solitary wave...
solutions and trigonometric function solutions have been known [32, 33]. In this situation, by solving equations (5) and (6), we obtain

\[ \ell = g(t)^{-1}, \quad T = \int_0^t g^2(\tau) \, d\tau. \] (10)

Here, we have assumed \( \ell(0) = g(0) = 1, T(0) = 0 \) for simplicity. Thus under the integrable condition (9), the solution for equation (2) is of the form

\[ u(x, t) = \sqrt{g(t)/\Phi_1} \left[ \int_0^t g^2(\tau) \, d\tau \right] \exp \left[ -\frac{i}{2g} \frac{dg}{dt} x^2 \right], \] (11)

where \( \Phi_1(X, T) \) is the solution of equation (8) and \( X = g(t)x \). Note that the special form of \( g(t) \) and \( k(t) \) such as \( g(t) = \exp(\lambda t) \), \( k(t) = -\lambda^2/2 \) [18] and \( g(t) = 1 + \tanh(\Omega_1 t) \), \( k(t) = \tanh(\Omega_1 t) - 1 \) [22] are all satisfying the integrable condition (9), and the solutions therein can be recovered by combining the solutions of equation (8) and transformation (11).

3. Modulational instability and exact bright solitary wave solution under the integrable condition

As mentioned in section 2, well-known bright solitary wave solutions exist for equation (8), which is solved in the vanishing boundary conditions [34]. But in order to examine the modulational instability, we need to find a solution on the non-vanishing cw background \( \Phi_c(X, T) = \Lambda_c \exp(i\phi_c) \), where \( \phi_c = k_c X + (A^2_c - k^2_c)T/2 \). After performing the Darboux transformation [35] for equation (8), we obtain the following solution

\[ \Phi(X, T) = \left( A_c + A_s \frac{d \cosh \theta + \cos \phi}{\cosh \theta + d \cos \phi} + iA_s \frac{b \sinh \theta + c \sin \phi}{cosh \theta + d \cos \phi} \right) \exp(i\phi_c), \] (12)

where

\[ \theta = M_R X - [(k_c + k_s)M_R - A_s M_1]T/2 - \theta_0, \]

\[ \phi = M_1 X - [(k_c + k_s)M_1 + A_s M_R]T/2 - \phi_0, \]

and \( b = A_c(k_c - k_s + M_1)/\Delta, \quad c = 1 - 2A^2_c/\Delta, \quad d = A_c(M_R - A_s)/\Delta \) with \( M = \sqrt{(-A_s + ik_s - ik_c)^2 - 4A^2_c} \equiv M_R + iM_1 \) and \( \Delta = A^2_c + (M_R - A_s)^2/4 + (k_s - k_c + M_1)^2/4 \), and \( \theta_0 \) and \( \phi_0 \) are arbitrary real constants. The solution (12) has some novel properties. When \( A_s \) vanishes, it turns into the cw background solution \( \Phi_c(X, T) \). On the other hand, when \( A_c \) vanishes, which means there is no cw background, it turns into the bright soliton solution

\[ \Phi(X, T) = A_s \text{sech}[A_s(X - X_0 - k_s T)] \exp\{i[k_s X + (A^2_s - k^2_s)T/2 + \phi_0]\}. \] (13)

Therefore in general, the exact solution (12) describes a solitary wave embedded on a cw background. Thus, by employing the expression (11), we can obtain the bright solitary solution which propagates on a cw background for equation (2) under the integrable condition (9). Here,
we are interested in the following three cases.

(1) In the case of \( A_c = 0 \), the solution for equation (2) becomes

\[
    u(x, t) = A_s \sqrt{g(t)} \text{sech} \left[ A_s \left( g(t)x - k_s \int_0^t \frac{g^2(\tau)}{2g(\tau)} d\tau + X_0 \right) \right] e^{i\psi},
\]

where \( \psi = k_s g(t)x + \frac{(A_s^2 - k_s^2)}{2} \int_0^t g^2(\tau) d\tau - \frac{s^2}{2g(t)} \frac{dg}{dt} + \phi_0 \), and \( A_s, X_0, k_s \) and \( \phi_0 \) are arbitrary real constants. This is just the bright one-solitary wave solution for equation (2). From the solution (14) we can conclude that: (i) the amplitude of bright solitary wave is proportional to \( \sqrt{g(t)} \), while the width is inversely proportional to \( g(t) \); (ii) the centre of bright solitary wave is

\[
    \xi = \left[ k_s \int_0^t g^2(\tau) d\tau + X_0 \right]/g(t),
\]

which satisfies the following equation

\[
    \frac{d^2 \xi}{dt^2} + 2k(t)\xi = 0.
\]

Equation (15) means that the centre of mass of the macroscopic wave packet behaves like a classical particle, and allows one to manipulate the motion of bright solitary waves in BEC systems by controlling the external harmonic trapping potential. In the following we take some examples to demonstrate the dynamics of bright solitary waves in one-dimensional BEC systems with different kinds of scattering length and harmonic trapping potential.

Firstly, we consider the time-independent harmonic potential which was used in the creation of bright BEC solitons [3]. In that experiment, \( \omega = 2\pi i \times 70 \text{ Hz} \), \( \omega_\perp = 2\pi \times 710 \text{ Hz} \), so \( k = -2\kappa \) (\( \kappa \approx 0.05 \)). According to equations (4) and (9), we have \( f(t) = \pm \kappa \) and \( g(t) = e^{2\pi t} \), thus we get the bright one-solitary wave solution

\[
    u(x, t) = A_s e^{2\pi x/\kappa} \text{sech} \left[ A_s (X - k(t)T - X_0) \right] e^{i\psi},
\]

where \( X = e^{2\pi x \kappa}, and\) \( T = \pm \frac{1 - e^{2\pi x \kappa}}{4\kappa} \). In particular, when \( f(t) = -\kappa \) and \( g(t) = e^{2\pi t} \), the solution (16) is in agreement with that in [18]. The dynamics of a bright one-solitary wave in the harmonic trapping potential is shown in figure 1. From figure 1 it can be seen that, with the increasing (decreasing) of the absolute value of the s-wave scattering length, the bright solitary wave has an increase (decrease) in the peak value and a compression (broadening) in its width. Besides, according to equation (15), we obtain the velocity for the bright solitary wave

\[
    V = k_s \cosh(2\kappa t) \pm 2\kappa X_0 \exp (\pm 2\kappa t).
\]

In particular, when \( X_0 = 0 \), the velocity increases as the bright solitary wave propagates along the longitudinal direction due to the repulsive trapping potential.

Secondly, we consider the temporal periodic modulation of the s-wave scattering length [23, 24], and the nonlinearity parameter takes the form \( g(t) = 1 + m \sin(\omega t) \), with \( 0 < m < 1 \). According to the integrable condition (9), we have \( k(t) = -\omega m [m + \sin(\omega t) + m \cos^2(\omega t)]/[2 + 2m \sin(\omega t)] \), and the BEC system has a bright solitary wave solution

\[
    u(x, t) = A_s \sqrt{1 + m \sin(\omega t)} \text{sech} \left[ A_s (X - k_c T - X_0) \right] e^{i\psi},
\]

where \( T = [4m + (2 + m^2) \omega t - 4m \cos(\omega t) - m^2 \cos(\omega t) \sin(\omega t)]/2\omega \) and \( X = [1 + m \sin(\omega t)] x \). Figure 2 shows the solitary wave trajectories oscillate due to the temporal periodic

\[
    \sin m \omega t\]

\[
    T = [4m + (2 + m^2) \omega t - 4m \cos(\omega t) - m^2 \cos(\omega t) \sin(\omega t)]/2\omega \]
Figure 1. The dynamics of a bright solitary wave in a time-independent harmonic trapping potential given by equation (16) with the parameters $\kappa = 0.05$, $A_s = k_s = 1$, $X_0 = 0$. (a) $g(t) = e^{2\kappa t}$; (b) $g(t) = e^{-2\kappa t}$.

Figure 2. The evolution plot of a bright solitary wave in a time-dependent harmonic trapping potential given by equation (17) with the parameters $m = 0.1$, $\omega = 1$, $A_s = k_s = 1$, and $X_0 = 0$.

modulation of the s-wave scattering length and trapping potential. For a better understanding, we plotted the velocity diagram, as shown in figure 3.

(2) In the case of $k_s = k_c$, and $A_s^2 < 4A_c^2$, the solution for equation (2) becomes

$$u(x, t) = \sqrt{g(t)} \left( A_c + A_s \frac{-A_s \cosh \theta + 2A_c \cos \phi + i \eta \sinh \theta}{2A_c \cosh \theta - A_s \cos \phi} \right) \exp \left[ i \left( \psi_c - \frac{x^2}{2g(t)} \frac{dg}{dt} \right) \right], \quad (18)$$
where
\[ \eta = \sqrt{4A_s^2 - A_c^2}, \quad \theta = A_s \eta \int_0^t g^2(\tau) \, d\tau - 2\theta_0, \quad \phi = \eta g(t) x - k_c \eta \int_0^t g^2(\tau) \, d\tau - \phi_0. \]

From the expressions of \( \theta \) and \( \phi \), we can see that the solution (18) is periodic with the period \( 2\pi/|\eta g(t)| \) in the spatial coordinate and aperiodic in the temporal coordinate, as shown in figure 4. From figure 4 we can see that the cw background becomes unstable, therefore, the solution (18) can be used to describe the modulational instability process of a nonlinear matter wave governed by equation (2) [33]. In order to understand this modulational instability process, we linearize the initial value of solution (18) as follows [35]

\[ |u(x, 0)| \simeq |\rho + \epsilon_0 \chi \cos(\eta X - \phi_0)|, \quad (19) \]

where \(|\rho|^2 = A_c^2\), \(|\chi|^2 = A_s^2 \eta^2 / A_c^2\) and \(\epsilon_0 = e^{-\theta_0}\) is small for \(\theta_0 > 0\). Equation (19) means that a small periodic perturbation in the cw background may lead to modulational instability. It should be noted that the period of small perturbation is \(\sqrt{4A_c^2 - A_s^2} < 2A_c\), which is just the modulational instability region obtained by linear instability analysis for the nonlinear Schrödinger equation [31].

(3) In the case of \(k_s = k_c\), and \(A_s^2 < 4A_c^2\), the solution for equation (2) becomes

\[ u(x, t) = \sqrt{g(t)} \left( -A_c + \eta \cos \phi - iA_s \sin \phi \right) \exp \left[ i \left( \psi_c - \frac{g'(t)}{2g(t)} \chi^2 \right) \right], \quad (20) \]

where
\[ \eta = \sqrt{A_s^2 - 4A_c^2}, \quad \theta = \eta g(t) x - \eta k_c \int_0^t g^2(\tau) \, d\tau - \theta_0, \quad \phi = -\eta A_s \int_0^t g^2(\tau) \, d\tau / 2 - \phi_0. \]

In particular, when \(g(t) = e^{2\pi t}\), the solution (20) is in agreement with that in [18]. It should be noted that, when the background wave vanishes (i.e. \(A_c = 0\)), the solution (20) is reduced to the solution (14). Figures 5(a) and 5(b) show the evolutions of bright solitary waves on
Figure 4. The evolution of modulational instability given by equation (18) with the parameters $A_c = A_s = 1$, $k_c = 0.3$, $\theta_0 = 5$, $\phi_0 = 0$. (a) $g = e^{2\kappa t}$ with $\kappa = 0.05$; (b) $g(t) = 1 + m \sin(\omega t)$ with $m = 0.1$, $\omega = 1$.

Figure 5. The evolution of bright solitary waves on cw background given by equation (20) with the parameters $A_c = 1$, $A_s = 3$, $k_c = 1$, $\theta_0 = \phi_0 = 0$. (a) $g = e^{2\kappa t}$ with $\kappa = 0.05$; (b) $g(t) = 1 + m \sin(\omega t)$ with $m = 0.1$, $\omega = 1$.

cw backgrounds for $g(t) = e^{2\kappa t}$ and $g(t) = 1 + m \sin(\omega t)$, respectively. Analysis of equation (20) reveals that the velocity for such a solitary wave still satisfies equation (15), but the amplitude of such solitary waves are not proportional to $\sqrt{g(t)}$, as shown in figure 5. From figure 5 we can see the bright solitary wave behaves like a breather soliton propagating on a cw background, which corresponds to the atom exchanging between the bright solitary wave and the cw background, and this exchange keeps the bright solitary wave dynamically stable against the variation of the nonlinearity, as demonstrated in [18].

4. Approximative solitary wave solution under the near-integrable condition and numerical comparisons

It is worth noting that the existence of the exact analytical bright solitary wave solutions depends on the integrable condition (9), which presents a strict balance between the time-dependent nonlinear parameter $g(t)$ and the harmonic trapping potential parameter $k(t)$. In real physical experiments, however, it may be difficult to produce rigorously such a balance. Indeed, equation (2) is generally non-integrable. So we have to investigate the propagation of the solitary

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wave in non-integrable systems. Here we take the bright solitary wave with vanishing background as an example to demonstrate the dynamics of BEC solitary waves in the near-integrable condition (i.e. $|\epsilon(t)| \ll 1$).

Under the near-integrable condition $|\epsilon(t)| \ll 1$, the right-hand side of equation (7) can be treated as a small perturbation. Then, according to the adiabatic perturbation theory developed for bright solitons [36, 37], we obtain the approximative bright solitary wave solution for equation (7)

$$\Phi(X, T) = A_s(T)\text{sech}[A_s(T)(X - X_0(T))] \exp \{ik_s(T)X + i\phi_0(T)\}$$, \hspace{1cm} (21)

where

$$\frac{dX_0}{dT} = k_s(T), \quad \frac{d\phi_0}{dT} = \frac{i}{2} [A_s^2(T) - k_s^2(T)]$$,

$$\frac{dA_s}{dT} = 2\epsilon(t) A_s(T), \quad \frac{dk_s}{dT} = 0.$$  

Substituting equation (21) into (3), we finally obtain the approximative solution for equation (2)

$$|u(x, t)| = A_{s0}\sqrt{g(t)}\text{sech}\left\{A_{s0}g(t)(x - k_{s0}\ell(t))\int_0^t \epsilon^{-2}(\tau) \, d\tau - X_{00}\ell(t)\right\}, \hspace{1cm} (22)$$

where $\ell(t) = \exp \left(2\int_0^t f(\tau) \, d\tau\right)$ with $f(t)$ determined by equation (4), and $A_{s0} > 0$, $k_{s0}$, $X_{00}$ are arbitrary constants. From equation (22) we can conclude that: (i) the amplitude and width of a bright solitary wave in the near-integrable condition are the same as in the integrable condition (see equation (14)); (ii) the centre of the bright solitary wave is given by $\xi = [k_{s0}\int_0^t \epsilon^{-2}(\tau) \, d\tau + X_{00}]\ell(t)$ which also satisfies equation (15) resulting from the integrable case. That is to say, if we treat the macroscopic wave packet as a classical particle, the particle obeys Newton’s second law of motion (15) even in the near-integrable case, which suggests the motion is totally determined by the external trapping potential $k(t)x^2$, not the effective potential $k(t)x^2 - g(t)|u(x, t)|^2$.

It should be noted that the approximate solution (22) is valid only for the assumption $|\epsilon(t)| \ll 1$. Fortunately, for most cases, the assumption $|\epsilon(t)| \ll 1$ can always be satisfied, such as the slowly varying of $g(t)$, and the small value of $k(t)$. So equation (22) is fitting for more general cases because sometimes it may be difficult to match the scattering length and external trapping potential exactly in real experiments. In the following, we take two examples to demonstrate the dynamics of bright solitary waves in the near-integrable cases.

Firstly, we consider the real experiment [3], in which the $s$-wave scattering length and harmonic trapping potential remain unchanged (i.e. $g(t) = \text{constant}$, and $k(t) = -2\kappa^2$ with $\kappa \approx 0.05$). In this situation, taking $f = -\kappa$, which is independent of time $t$, we have $|\epsilon(t)| = |f\ell^2| = | -\kappa \exp(-2\kappa t)| \ll 1$, so the perturbation theory can be employed for equation (7). In figure 6(b) we plot the amplitude evolutions of the approximate solution (22) and direct numerical simulation of full PDE model (2), and find that they are in good agreement with the relative error $\leq 8\%$.

Secondly, we consider a BEC in the time-independent trapping potential ($\omega = 2\pi \times 71$ Hz, so $k(t) = 0.005$) with $g(t) = 1$. In this situation, taking $f(t) = -\frac{\sqrt{2}}{2} \tan(\sqrt{2}kt)$, we have $\ell(t) = \cos(\sqrt{2}kt)$ and $\epsilon(t) = -\frac{\sqrt{2}}{4} \sin(\sqrt{8}kt)$. Obviously, here $\epsilon(t)$ can be treated as a small perturbation. So according to equation (22), the amplitude and width of the bright solitary wave stay constant. Moreover, from equation (15) we can see that the solitary wave’s trajectory
oscillates harmonically with the period $20\pi$ in the dimensionless variables due to the attractive trapping potential, as shown in figure 7. We hope this can be checked in a real experiment with the following steps. (i) Create a bright solitary wave with the same parameters as in experiment [3] (the solitary wave has a speed due to the offset of external repulsive harmonic potential). (ii) Turn the repulsive harmonic potential to an attractive harmonic potential with $\omega = 2\pi \times 71$ Hz, then we hope to see the solitary wave’s trajectory oscillating harmonically with a period about 14 ms in real seconds.
5. Conclusions

In conclusion, we have considered the GP equation which describes the dynamics of the BEC matter waves with the time-dependent s-wave scattering length and time-dependent harmonic trapping potential. With the help of the modified lens-type transformation, we reduced the one-dimensional GP equation (2) to the standard nonlinear Schrödinger equation under the integrable condition (9), and derived the exact analytical expression for modulational instability and a bright one-solitary wave embedded in a cw background. The methodology presented here is powerful for systematically finding an infinite number of BEC bright solitary wave solutions by exactly matching the s-wave scattering length and external harmonic trapping potential. Furthermore, by employing the adiabatic perturbation theory, we have also presented the approximative bright solitary wave solutions of BEC systems for near-integrable cases. These solutions imply that control of the scattering length and external harmonic trapping potential allows us to manipulate the motion of bright solitary waves in BEC systems. Both our exact and approximative results show that the amplitude of a bright solitary wave with zero boundary condition depends on the absolute value of s-wave scattering while its motion depends on the external trapping potential. Besides, with the presented integrable condition and modified lens-type transformation, solitary wave–solitary wave interaction solutions can also be easily constructed.

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