Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background

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Abstract

We construct the covariant $\kappa$-symmetric superstring action for type $IIB$ superstring on plane wave space supported by Ramond-Ramond background. The action is defined as a $2d$ sigma-model on the coset superspace. We fix the fermionic and bosonic light-cone gauges in the covariant Green-Schwarz superstring action and find the light-cone string Lagrangian and the Hamiltonian. The resulting light-cone gauge action is quadratic in both the bosonic and fermionic superstring $2d$ fields, and therefore, this model can be explicitly quantized. We also obtain a realization of the generators of the basic superalgebra in terms of the superstring $2d$ fields in the light-cone gauge.
1 Introduction

Until recently precisely two maximally supersymmetric solutions of the IIB supergravity were known. The first solution is the flat ten dimensional Minkowski space (and its toroidal compactification) and the second solution is the famous $AdS_5 \times S^5$ space supported by Ramond-Ramond charges [1]-[3], which plays a distinguished role in the dual description of gauge theories. Surprisingly, very recently a new maximally supersymmetric solution of IIB supergravity was found [4] that turns out to be a ten dimensional plane wave space supported by the RR 5-form flux. This new solution is a ten dimensional counterpart of the solution found for eleven dimensional supergravity in [5] (see also [6]).

Applying arguments similar to those in [7], one can expect that as in the case of the $AdS_5 \times S^5$ space, the plane wave RR background is an exact solution not only of the equations of motion of IIB supergravity but also of the equations of motion for massless modes of the type IIB superstring. The action of the $AdS_5 \times S^5$ superstring constructed in [8] (see also [9]-[12]) turns out to be very complicated for explicit quantization. In this paper, motivated by desire to find explicitly quantizable superstring model in curved target space time supported by the Ramond-Ramond flux, we investigate classical mechanics of the superstring propagating in the plane wave Ramond-Ramond background. We find that the light-cone action of this superstring model is quadratic in bosonic as well as fermionic superstring 2$d$ fields and can therefore be quantized in a rather straightforward way. The interest in plane wave RR background also comes from the fact that this relatively simple model may serve as a training ground for the study of a more interesting case of superstring in the $AdS_5 \times S^5$/RR-charge background.

One possible approach to the construction of the action is to start with the general type IIB superstring action in [13] and plug in the type IIB superfields [14] representing the plane wave RR background. This approach, however, is indirect and somewhat complicated because it does not explicitly use the basic symmetries of the problem. Our strategy is instead to use the basic superalgebra underlying the symmetry of the plane wave RR background. As in the previous construction of the type IIB superstring action in $AdS_5 \times S^5$ [8], we obtain the space-time supersymmetric and $\kappa$-symmetric action in terms of the invariant Cartan one-forms defined on the appropriate coset superspace. The supercoset construction turned out to be very effective and fruitful and was used to construct $AdS(2)$ and $AdS(3)$ superstring actions [15]-[18], brane actions in super $AdS/RR$ backgrounds [10],[19], and various $AdS$ supermembrane actions [20]-[22].

This paper is organized as follows.

In section 2, we describe the structure of the basic symmetry superalgebra of the plane wave RR background and the invariant Cartan 1-forms on the coset superspace $(x, \theta)$.

In section 3, we present the covariant superstring action on the plane wave RR background in the coordinate free form, i.e. in terms of the Cartan 1-forms. As in the cases of flat and $AdS_5 \times S^5$ spaces, it is given by the sum of the ‘kinetic’ or ‘Nambu’ term (2$d$ integral of the quadratic term in Cartan 10-beins) and Wess-Zumino term (3$d$ integral of a closed 3-form $H$ on the superspace) with the coefficient of the WZ term fixed by the requirement of $\kappa$-symmetry (Siegel symmetry).

In section 4, we find the explicit 2$d$ form of the action by choosing a specific WZ parametrization of the coset superspace. As in the $AdS_5 \times S^5$ space, the resulting action is given by the covariantization of the flat-space GS action plus terms containing higher
powers of the fermionic coordinates $\theta$.

In section 5, we evaluate Cartan 1-forms in fermionic light-cone gauge, which we use to derive the superstring action in the fermionic $\kappa$-symmetry light-cone gauge. After this, we fix the $2d$ diffeomorphism symmetry by choosing bosonic light-cone gauge. The resulting light-cone action turns out to be quadratic in bosonic and fermionic $2d$ string fields. This allows us to solve the equations of motion explicitly. We discuss the superstring action in various parametrization of the plane wave background and in the WZ and the Killing parametrizations of superspace.

Section 6 is devoted to the light-cone phase space approach to superstring theory. We fix the analog of the GGRT [23] bosonic light-cone gauge and derive the phase space analog of the superstring Lagrangian of Section 5 and the corresponding light-cone gauge Hamiltonian.

In Section 7, we obtain a realization of the generators of the basic symmetry superalgebra as Noether charges expressed in terms of the $2d$ fields that are coordinates of the plane wave superstring in the light-cone gauge.

Section 8 summarizes our conclusions. Some technical details are collected in two Appendices. In Appendix A, we summarize our notation and definitions. In Appendix B, we explain the construction of dynamical supercharges realized in terms of $2d$ superstring fields.

## 2 Symmetry Superalgebra of plane wave RR background

We start with reviewing the plane wave RR background of the type IIB supergravity and its symmetry superalgebra [4]. The bosonic sector of IIB supergravity includes the graviton, a complex scalar, a complex two-index antisymmetric tensor and a real four-index antisymmetric tensor whose five-index field strength is (anti)self-dual. The line element of the plane wave background given by

$$ ds^2 = 2dx^+ dx^- - m^2 x^2_+ dx^+ x^- + dx_I^2, \quad I = 1, \ldots, 8, $$

describes ten dimensional Lorentzian symmetric space. Here $m$ is a dimensional parameter and $x^+$ is taken to be light-cone evolution parameter. This metric is supported by non-vanishing four-index antisymmetric RR field whose five-index field strength takes the values

$$ F^{-i_1 \ldots i_4} = 2m \epsilon^{i_1 \ldots i_4}, \quad F^{-i'_1 \ldots i'_4} = 2m \epsilon^{i'_1 \ldots i'_4}, $$

where $i, j = 1, \ldots, 4$, $i', j' = 5, \ldots, 8$. All the remaining bosonic fields (as well as fermionic fields) are set to be equal zero. Surprisingly, this background has 32 Killing spinors and therefore preserves 32 supersymmetries. This can easily be checked by considering the equations for Killing spinors $D\epsilon = 0$ with the covariant differential $^1$

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^1 We adopt the normalization in which the Einstein equations take the form

$$ R^{\mu \nu} = (1/24) F^{\mu \rho_1 \ldots \rho_4} F^{\nu \rho_1 \ldots \rho_4} $$

and take the five-index field strength to be anti-self dual: $F^{\mu_1 \ldots \mu_5} = -(1/120) \epsilon^{\mu_1 \ldots \mu_5 \rho_1 \ldots \rho_8} F^{\rho_1 \ldots \rho_8}$. The expression for $D$ in (2.3) is restricted to 16-component $\epsilon$ that is a half of a 32-component positive chirality spinor. The indices $\mu, \nu, \rho = 0, 1, \ldots, 9$ are $so(9,1)$ vector indices, i.e. tangent space indices. In light cone frame these indices take values $+, -, I$ where $I = i, i'$. 

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\[ D = d + \frac{1}{4} \omega^{\mu\nu} \gamma_{\mu\nu} - \frac{i}{960} \gamma^{\mu_1 \ldots \mu_5} \gamma^\mu F_{\mu_1 \ldots \mu_5}, \]  

(2.3)

where \( e^\mu \) are 10-beins of plane wave space (2.1) and \( \omega^{\mu\nu} \) is the corresponding Lorentz connection. Inserting the above expressions for five-index field strength (2.2) in (2.3), one verifies that the integrability condition of the Killing spinors equations, i.e. \( D^2 = 0 \), is satisfied automatically. Because all the other conditions are also satisfied automatically, there are 32 Killing spinors. In what follows, unless otherwise specified, we set

\[ m = 1 \]  

(2.4)
to simplify our expressions.

We now describe the basic symmetry superalgebra of the plane wave RR background (2.1), (2.2). The even (bosonic) part of the superalgebra includes ten translation generators \( P^\mu \), the \( SO(4) \) rotation generators \( J^{ij} \), \( i, j = 1, \ldots, 4 \), the \( SO'(4) \) rotation generators \( J^{i'j'} \), \( i', j' = 5, \ldots, 8 \) and eight rotation generators in the \((x^-, x^+)\) plane \( J^{+I} \), \( I = 1, \ldots, 8 \). The odd (fermionic) part of the superalgebra consists of the complex 16-component spinor \( Q_\alpha \), \( \alpha = 1, \ldots, 16 \), which is half of 32-component negative chirality spinor.

Commutation relations between the even generators are given by

\[ [P^-, P^I] = -J^{+I}, \]  

(2.5)

\[ [P^I, J^{+J}] = -\delta^{IJ} P^+, \quad [P^-, J^{+I}] = P^I, \]  

(2.6)

\[ [P^i, J^{jk}] = \delta^{ij} P^k - \delta^{ik} P^j, \quad [P^{i'}, J^{j'k'}] = \delta^{i'j'} P^{k'} - \delta^{i'k'} P^{j'}, \]  

(2.7)

\[ [J^{+i}, J^{jk}] = \delta^{ij} J^{+k} - \delta^{ik} J^{+j}, \quad [J^{+i'}, J^{j'k'}] = \delta^{i'j'} J^{+k'} - \delta^{i'k'} J^{+j'}, \]  

(2.8)

\[ [J^{ij}, J^{kl}] = \delta^{ij} J^{kl} + 3 \text{ terms}, \quad [J^{i'j'}, J^{k'l'}] = \delta^{i'j'} J^{k'l'} + 3 \text{ terms}. \]  

(2.9)

Commutation relations between the even and odd parts are

\[ [J^{ij}, Q_\alpha] = \frac{1}{2} Q_\beta (\gamma^{ij})^\beta_\alpha, \quad [J^{i'j'}, Q_\alpha] = \frac{1}{2} Q_\beta (\gamma^{i'j'})^\beta_\alpha, \]  

(2.10)

\[ [J^{+I}, Q_\alpha] = \frac{1}{2} Q_\beta (\gamma^{+I})^\beta_\alpha, \]  

(2.11)

\[ [P^I, Q_\alpha] = \frac{i}{2} Q_\beta (\Pi \gamma^+ \bar{\gamma}^I)^\beta_\alpha. \]  

(2.12)

together with the commutators that follow from these by complex conjugation. Here, \( \gamma^\mu \) are 16 \times 16 gamma matrices, \( \gamma^{+I} = \gamma^{[+I]} \), and \( \Pi \) is the product of four gamma matrices (see Appendix A for details).

The anticommutator takes the form

\[ \{Q_\alpha, \bar{Q}_\beta\} = -2i\gamma^\mu_{\alpha\beta} P^\mu - 2(\bar{\gamma}^i \Pi)_{\alpha\beta} J^{+i} - 2(\bar{\gamma}^{i'} \Pi')_{\alpha\beta} J^{+i'} \]  

\[ + (\bar{\gamma}^+ \gamma^{ij} \Pi)_{\alpha\beta} J^{ij} + (\bar{\gamma}^+ \gamma^{i'j'} \Pi')_{\alpha\beta} J^{i'j'}. \]  

(2.13)
All the other commutators and anticommutators vanish. The bosonic generators are assumed to be antihermitean while the fermionic generators are conjugated to each other, \( \bar{Q}_\alpha = (Q_\alpha)^\dagger \). The generators of the full transformation group \( G \) of the plane wave RR superspace are \( P^\mu, Q_\alpha, \bar{Q}_\alpha, J^{ij}, J^{i'j'}, J^{+I} \). The generators of the stability subgroup \( H \) are \( J^{ij}, J^{i'j'}, J^{+I} \). The plane wave RR superspace is defined then as coset superspace \( G/H \).

A few remarks are in order.

(i) The (anti)commutation relations of the superalgebra are invariant under \( U(1) \) transformation of supercharges: \( Q \to e^{i\phi} Q, \bar{Q} \to e^{-i\phi} \bar{Q} \). This \( U(1) \) symmetry reflects the fact that the plane wave RR background respects the original \( U(1) \) symmetry of \( IIB \) supergravity.

(ii) In contrast to flat and \( AdS_5 \times S^5 \) cases, the above superalgebra does not include rotation generator in the \((x^+, x^-)\) plane, usually denoted by \( J^{+I} \), and the rotation generators in the \((x^+, x^I)\) planes which are \( J^{-I} \). We note that for the bosonic string in a plane wave NS-NS background, it is the absence of \( J^{-I} \) generators that explains why the critical dimension cannot be obtained from the usual operator formalism argument [24]. The interesting fact [4] is that the dimension of the bosonic subalgebra of the superalgebra under consideration, which is equal to 30, coincides with the dimension of the isometry algebra of the \( AdS_5 \times S^5 \) space.

(iii) The dimensionful parameter of the plane wave geometry can be introduced by rescaling the generators as \( P^\mu \to P^\mu/m^2, Q_\alpha \to Q_\alpha/m \). The limit as \( m \to 0 \) then gives the subalgebra of \( d = 10, IIB \) Poincaré superalgebra.

### 2.1 Cartan 1-forms

To find the super-invariant and \( \kappa \)-invariant string action we use the formalism of Cartan forms defined on the coset superspace.\(^2\) The left-invariant Cartan 1-forms

\[
L^A = dX^M L^A_M, \quad X^M = (x^\mu, \theta^\alpha, \bar{\theta}^\alpha)
\]

are given by

\[
G^{-1} dG = L^\mu P^\mu + L^\alpha \bar{Q}_\alpha + \bar{L}^\alpha Q_\alpha + \frac{1}{2} L^\mu \nu J^{\mu \nu},
\]

where we impose the conditions

\[
L^{+\mu} = 0, \quad L^{i'j'} = 0,
\]

which simply reflect the fact that the generators \( J^{-\mu} \) and \( J^{ij'} \) are not included in superalgebra. \( L^\mu \) are the 10-beins, \( L^\alpha, \bar{L}^\alpha \) are the two spinor 16-beins, and \( L^{\mu \nu} \) are the Cartan \( H \) connections.\(^3\) They satisfy the Maurer-Cartan equations implied by the structure of the superalgebra,

\[
dL^\mu = -L^{\mu \nu} L^\nu - 2i \bar{L} \bar{\gamma}^\mu L, \quad \text{(2.17)}
\]

\[
dL^\alpha = -\frac{1}{4} L^{\mu \nu} (\gamma^{\mu \nu})^\alpha_\beta L^\beta + \frac{i}{2} L^\mu (\Pi \bar{\gamma}^\mu)^\alpha_\beta L^\beta, \quad \text{(2.18)}
\]

\(^2\)For some applications of the formalism of the Cartan forms on coset superspaces see [25].

\(^3\)\( L^\alpha \) is a half of 32-component positive chirality spinor.
\[ dL^\alpha = -\frac{1}{4} L^\mu (\gamma^{\mu})^\alpha_\beta L^\beta - \frac{i}{2} L^\mu (\Pi^{+}_{\gamma^\mu})^\alpha_\beta L^\beta. \]  

We note that we suppress the exterior products symbols for 1-forms and use the following sign conventions under permutations of Cartan 1-forms:

\[ L^\mu L^\nu = -L^\nu L^\mu, \quad L^\mu L^\alpha = -L^\alpha L^\mu, \quad L^\alpha L^\beta = L^\beta L^\alpha. \]  

The dependence on the dimensionful parameter \( m \) can be restored by simultaneously rescaling the Cartan 1-forms and coordinates as \( L^\mu \to m^2 L^\mu, \quad L^\mu\nu \to L^\mu\nu, \quad L^\alpha \to mL^\alpha, \quad x^\mu \to m^2 x^\mu, \quad \theta \to m \theta. \)

For comparison, we note that in the flat superspace case,

\[ G(x, \theta) = \exp(x^\mu P^\mu + L^\alpha \bar{Q}_\alpha + \bar{L}^\alpha Q_\alpha), \]  

\[ [P^\mu, P^\nu] = 0, \quad \{Q_\alpha, \bar{Q}_\beta\} = -2i\gamma^{\mu}_{\alpha\beta} P^\mu \]  

and thus the coset space (super)vielbeins in \( G^{-1} dG = L^A T_A \) are therefore given by

\[ L^\mu_0 = dx^\mu - i\theta \bar{\gamma}^\mu d\theta - i\bar{\theta} \gamma^\mu d\bar{\theta}, \quad L_0 = d\theta. \]  

### 3 Superstring action as sigma model on \( G/H \) coset superspace

In this section, closely following the general method suggested in [8], we construct the superstring action that satisfies the following conditions (some of which are not completely independent):

a) its bosonic part is the standard \( \sigma \)-model with the plane wave geometry as a target space;

b) it has global super-invariance with respect to supersymmetry algebra above described;

c) it is invariant under the local \( \kappa \)-symmetry;

d) it reduces to the standard Green-Schwarz type IIB superstring action in the flat-space \( (m \to 0) \) limit.

As in [8] we find that such an action exists and is \emph{unique}. Its leading \( \theta^2 \) fermionic term contains the required coupling to the RR 5-form field background.

It is useful to recall that the flat-space GS superstring Lagrangian [26] can be written in the manifestly supersymmetric form in terms of (super)vielbeins (2.23) as a sum of the ‘kinetic’ term and the WZ term [27],

\[ \mathcal{L}_0 = \mathcal{L}_{0\, \text{kin}} + \mathcal{L}_{0\, \text{WZ}}, \]  

where the kinetic and WZ terms are given by

\[ \mathcal{L}_{0\, \text{kin}} = -\frac{1}{2} \sqrt{g} g^{ab} L^\mu_{0b} L^\mu_{0a}, \quad \mathcal{L}_{0\, \text{WZ}} = d^{-1} \mathcal{H}_0, \]  

\footnote{We use Minkowski signature 2d world-sheet metric \( g_{ab} \) with \( g \equiv -\det g_{ab} \).}
and $L^A_{\alpha a} = \partial_a X^M L^A_{\alpha M}$. The 3-form $H_0$ being closed and exact allows manifestly supersymmetric representation\(^5\)

\[ H_0 = iL^\mu_0 L_0 \bar{\gamma}^\mu L_0 + \text{h.c.} \tag{3.3} \]

The coefficient of the WZ term is fixed by the condition of the local $\kappa$-invariance [26]. Using the explicit representation for (super)vielbeins (2.23) one observes that the 3-form in the WZ term is indeed exact and thus finds the explicit 2d form of the WZ part of the superstring Lagrangian [26]

\[ L_{WZ} = -i\epsilon^{ab}(\partial_a x^\mu - \frac{i}{2}\bar{\theta} \gamma^\mu \partial_a \theta - \frac{i}{2}\theta \gamma^\mu \partial_a \bar{\theta}) \theta \gamma^\mu \partial_b \theta + \text{h.c.}, \tag{3.4} \]

which is invariant under global supersymmetry only up to a total derivative. The action that we find below is the generalization of (3.1) to the case where the free bosonic term is replaced by the sigma model on the plane wave RR background.

We now turn to superstring in the plane wave RR background. As in the flat space, the Lagrangian is given by the sum of $\sigma$-model term $L_{\text{kin}}$ and the WZ term $L_{WZ} = d^{-1}H$. To satisfy invariance with respect to the symmetry superalgebra, both $L_{\text{kin}}$ and $H$ should be constructed in terms of the Cartan 1-forms $L^\mu$ and $L^\alpha$. The basic observation is that under the action of an arbitrary element of the group $G$ these forms transform as tangent vectors (spinors) of the stability group $H$. Therefore any invariant of stability group constructed in terms of $L^\mu$ and $L^\alpha$ is automatically invariant under full transformations of $G$.

The structure of $L_{\text{kin}}$ is fixed by conditions (a) and (b) and can be obtained from $L_{\text{0kin}}$ in (3.2) by replacing $L^\mu_0$ with the Cartan 1-forms of the plane wave RR background $L^\mu$. As to the WZ part, it turns out that the only relevant closed 3-form built out of $L^\mu, L^\alpha$ that is invariant under transformations of $SO(4) \otimes SO'(4)$ and the those generated by the light-cone boost generators $J^+ I$ is given by

\[ H = H^q + \bar{H}^q, \quad H^q = (\bar{H}^q)^\dagger, \tag{3.5} \]

where

\[ H^q = iL^\mu L\bar{\gamma}^\mu L. \tag{3.6} \]

The fact that this form is indeed closed can be demonstrated as follows. Using Maurer-Cartan equations (2.17)-(2.19), we find terms in $dH^q$ that are proportional to the (super)vielbeins $L^\mu, L^\alpha$, and the Cartan $H$ connections $L^\mu$. Taking the relation

\[ \gamma^\mu \gamma^\nu = \gamma^\mu + \eta^\mu \eta^\nu - \eta^\mu \eta^\nu \tag{3.7} \]

into account and using that $(\gamma^\mu)^{\alpha \beta}$ are antisymmetric in $\alpha, \beta$ we find that the terms proportional to $L^\mu$ cancel out. The remaining part of $dH^q$ is then given by

\[ dH^q = L^\mu L^\nu L^\alpha (\gamma^\mu \Pi^+ \gamma^\nu)^{\alpha \beta} L^\beta. \tag{3.8} \]

Using that

\[ 5\text{For fermionic coordinates we assume the convention } (\theta_1, \theta_2)^\dagger = \bar{\theta}_2 \bar{\theta}_1, \theta_1 \theta_2 = -\theta_2 \theta_1, \text{ while for fermionic Cartan 1-forms we adopt } (L_1 \wedge L_2)^\dagger = -L_2 \wedge L_1, L_1 \wedge L_2 = L_2 \wedge L_1. \]
\((\gamma^i \Pi)_{\alpha\beta}, \ (\Pi \gamma^+ \gamma^j)_{\alpha\beta}, \ (\Pi \gamma^+ \gamma^j \gamma')_{\alpha\beta}\) are antisymmetric in \((\alpha, \beta)\) (3.9)

together with the symmetry properties of the products of Cartan 1-forms given in (2.20),
we find that \(dH^q = 0\). Because all Cartan 1-forms of plane wave RR background reduce
to flat Cartan 1-forms in the flat-space limit the 3-form \(H\) (3.5),(3.6) also reduces to the
3-form in the GS action (3.3). As in flat space, the value of the overall coefficient in
front \(H\) is fixed to be 1 by the requirement of \(\kappa\)-symmetry of the whole action (which is
proved below). The final expression for the Lagrangian written in the manifestly invariant
form in terms of the (super)vielbeins \(L^\mu\) and \(L^\alpha\) thus has the same structure as the GS
Lagrangian (3.2),

\[
\mathcal{L} = \mathcal{L}_{kin} + \mathcal{L}_{WZ},
\]

\[
\mathcal{L}_{kin} = -\frac{1}{2} \sqrt{g} g^{ab} L_b^\mu L_a^\mu, \quad \mathcal{L}_{WZ} = d^{-1}\mathcal{H}
\]

and, indeed, reduces to (3.2) in the flat-space limit. Because the 3-form \(H\) is closed it can
be represented as \(H = dB\) in a local coordinate system; the action then takes the usual
2d sigma-model form, which is considered in what follows.

### 3.1 The \(\kappa\)-symmetry invariance

The action (3.10) is invariant with respect to the local \(\kappa\)-transformations [26],[28]. These
can be conveniently written in terms of the Cartan 1-form and the variations defined by

\[
\tilde{\delta} x^\mu \equiv \delta X^M L_M^\mu, \quad \tilde{\delta} x^{\mu\nu} \equiv \delta X^M L_M^{\mu\nu}, \quad \tilde{\delta} \theta^\alpha \equiv \delta X^M L_M^\alpha.
\]

To formulate the \(\kappa\)-transformations we introduce a complex 16-component spinor \((\kappa^a)^\alpha\)
(the corresponding 32-component spinor has negative chirality) that is a 2d vector on the
world-sheet. The \(\kappa\)-transformation then takes the form

\[
\tilde{\delta} x^\mu = 0, \quad \tilde{\delta} \theta = 2L_a^\mu \gamma^\mu \kappa_a,
\]

\[
\delta \left(\sqrt{g} g^{ab}\right) = -8i \sqrt{g} (L^a \kappa^b + L^b \kappa^a - \frac{1}{2} g^{ab} L_c \kappa^c) + h.c.,
\]

where \(\kappa = \kappa^\dagger\). The \(\kappa\)-transformation parameter satisfies the (anti) self duality constraints
which in complex notation that we use take the form\(^6\)

\[
\frac{\epsilon^{ab}}{\sqrt{g}} \kappa_b = -\bar{\kappa}^a, \quad \frac{\epsilon^{ab}}{\sqrt{g}} \bar{\kappa}_b = -\kappa^a.
\]

To demonstrate the \(\kappa\)-invariance we use the following expressions for the variations of
the Cartan 1-forms:

\(^6\)The selfduality constraints (3.15) are counterpart of standard constraints formulated in terms of two
Majorana Weyl spinors \(\kappa^1, \kappa^2\): \(\epsilon^{ab} \kappa_b = -\sqrt{g} \kappa^1, \epsilon^{ab} \kappa_b^2 = \sqrt{g} \kappa^2\).
\[ \delta L^\mu = d\delta x^\mu + L^\nu \delta x^\nu + L^{\mu\nu} \delta x^\nu + 2i\bar{L}\gamma^\mu \delta \theta - 2i\bar{\theta}\gamma^\mu L, \]  
(3.16)

\[ \delta L = d\delta \theta - \frac{1}{2}L^\mu \Pi\gamma^+\gamma^\mu \delta \theta + \frac{1}{4}L^{\mu\nu}\gamma^{\mu\nu} \delta \theta \]
\[ + \frac{i}{2}\delta x^\mu \Pi\gamma^+\gamma^\mu L - \frac{1}{4}\delta x^{\mu\nu}\gamma^{\mu\nu} L. \]  
(3.17)

The crucial relation that allows us to check the \( \kappa \)-invariance of the superstring action directly in terms of the Cartan 1-forms is

\[ \delta \mathcal{H}^q = d\Lambda^q, \quad \Lambda^q = i\delta x^\mu L\bar{\gamma}^\mu L + 2iL^\mu L\bar{\gamma}^\mu \bar{\delta} \theta. \]  
(3.18)

4 \hspace{1cm} \textbf{Explicit 2-dimensional form of the action}

To find the explicit form of the WZ part of the superstring Lagrangian in terms of the coordinate 2d field \( \theta \) which generalizes (3.4) we choose a particular parametrization of the coset representative \( G \) in (2.15):

\[ G(x, \theta) = g(x)g(\theta), \quad g(\theta) = \exp(\theta^{\alpha} \bar{Q}_{\alpha} + \bar{\theta}^{\dot{\alpha}} Q_{\dot{\alpha}}). \]  
(4.1)

Here, \( g(x) \) is a coset representative of \( G_{\text{bos}}/H \), i.e., \( x = x^\mu \) provides a certain parametrization of the plane wave geometry (which we do not need to specify in this section). We note that choosing the coset representative in form (4.1) corresponds to the Wess-Zumino type gauge in the plane wave/RR superspace, while another, \( G(x, \theta) = g(\theta)g(x) \), corresponds to the Killing gauge. These “gauges” (better to be called “parametrizations”) do not reduce the number of the fermionic degrees of freedom but only specialize a choice of fermionic coordinates. The covariant action given in this Section corresponds to the WZ parametrization.

To represent the WZ term in (3.11) as an density over the 2-dimensional space we use the standard trick of rescaling \( \theta \rightarrow \theta_t = t\theta \),

\[ \mathcal{L}_{WZ} = \mathcal{L}_{WZ}(t = 1), \quad \mathcal{L}_{WZ}(t) = d^{-1}(\mathcal{H}_t^q + \text{h.c.}), \quad \mathcal{H}_t^q = \mathcal{H}^q(\theta_t). \]  
(4.2)

We then have the obvious relation

\[ \mathcal{H}_{t=1}^q = \mathcal{H}_{t=0}^q + \int_0^1 dt \partial_t \mathcal{H}_t^q. \]  
(4.3)

Inserting \( G_t \equiv G(x, t\theta) \) given by (4.1) in definition of Cartan 1-forms and setting \( L_t^A = L^A(x, t\theta) \) we obtain the equations for the ‘shifted’ Cartan one-forms \( L_t^A \)

\[ \partial_t L_t^\mu = d\theta + \frac{1}{4}L_t^{\mu\nu}\gamma^{\mu\nu} \theta - \frac{i}{2}L_t^\mu \Pi\gamma^+\gamma^\mu \theta, \]  
(4.4)

\[ \partial_t L_t^\mu = -2\theta\gamma^\mu \bar{L}_t - 2i\bar{\theta}\gamma^\mu L_t, \]  
(4.5)

\[ \partial_t L_t^{-i} = 2\theta \bar{\gamma}^i \Pi L_t - 2\bar{\theta}\gamma^i \Pi L_t, \]  
(4.6)
\[
\partial_t L_i^\tau = 2 \theta \bar{\gamma}^\mu \Pi L_t - 2 \bar{\theta} \bar{\gamma}^\nu \Pi L_t, \\
\partial_t L_i^\tau = -2 \theta \bar{\gamma}^\mu \bar{\gamma}^{ij} \Pi L_t + 2 \bar{\theta} \bar{\gamma}^\nu \gamma^{ij} \Pi L_t, \\
\partial_t L_i^{\nu} = -2 \theta \bar{\gamma}^\mu \gamma^{\nu j} \Pi L_t + 2 \bar{\theta} \bar{\gamma}^\nu \gamma^{\mu j} \Pi L_t.
\]

These equations should be supplemented by the initial conditions

\[
L_{\nu t=0} = e^\mu, \quad L_{\mu \nu t=0} = \omega^{\mu \nu}, \quad L_{t=0} = 0,
\]

\(\omega^{\tau \mu} = 0, \quad \omega^{ij} = 0\). Here \(e^\mu\) are the 10-beins of the plane wave geometry and \(\omega^{\mu \nu}\) is the corresponding Lorentz connection. Using these differential equations we prove that

\[
\partial_t H_{q t} = -2i d(L^\mu_0 \bar{\theta} \bar{\gamma}^\mu L_t).
\]

Taking into account that \(H_{q t} = 0\), we obtain the desired representation for \(L_{WZ}\),

\[
L_{WZ} = -2i \int_0^1 dt L^\mu_0 \bar{\theta} \bar{\gamma}^\mu L_t + h.c.
\]

To summarize the complete superstring Lagrangian is given by the sum of the kinetic part (3.11) and the 2d form of the WZ part (4.12),

\[
\mathcal{L} = -\frac{1}{2} \sqrt{g} g^{ab} L_a^\mu L_b^\mu - 2i \int_0^1 dt \epsilon^{ab} L^\mu_0 (\theta \bar{\gamma}^\mu L_b + \bar{\theta} \bar{\gamma}^\mu \bar{L}_b).
\]

In the next section we use this representation to derive the light-cone gauge superstring action.

As a side remark we note that the equations for the Cartan 1-forms can be solved in a rather straightforward way. For this, we collect the fermionic Cartan 1-forms and \(\theta^s\) in two vectors\(^7\)

\[
\mathbf{L} = \begin{pmatrix} L \\ \bar{L} \end{pmatrix}, \quad \Theta = \begin{pmatrix} \theta \\ \bar{\theta} \end{pmatrix}.
\]

The solution to equations (4.4)-(4.9) and initial conditions (4.10) then leads to the Cartan 1-forms \(L = L_{t=1}\) given by

\[
L^\mu = e^\mu - 2i \bar{\theta} \bar{\gamma}^\mu \frac{\cosh M}{M} - \frac{1}{M^2} D\Theta,
\]

where the covariant derivative \(D\) is\(^8\)

\[
D\Theta = (d + \frac{1}{4} \omega^{\mu \nu} \gamma^{\mu \nu} - \frac{i}{2} e^\mu_0 \Pi \gamma^+ \bar{\gamma}^\mu \sigma_3)\Theta.
\]

The square of the covariant differential \(D\) is equal to zero, \(D^2 = 0\), and the Killing spinor equation \(D\epsilon = 0\) is therefore integrable. The block 32 \times 32 matrix \(M^2\) is given by\(^9\)

\(^7\)These \(\mathbf{L}\) and \(\Theta\) should not be confused with 32-component spinors.

\(^8\)This is essentially the same derivative which appeared in the Killing spinor equations of IIB supergravity(2.3)

\(^9\)Action of the asterisk on the product of fermions in (4.18) is defined to be \((F_1 F_2)^* = \bar{F}_1 \bar{F}_2\).
\[(\mathcal{M}^2)^{\alpha}_\beta = \begin{pmatrix} A^\alpha_\beta & B^\alpha_\beta \\ -(B^\alpha_\beta)^* & -(A^\alpha_\beta)^* \end{pmatrix}, \] (4.18)

where the $16 \times 16$ matrices $A$ and $B$ are given by

\[A^\alpha_\beta = -(\Pi^{\gamma+\bar{\gamma}^i\theta})^\alpha(\bar{\theta}^{\bar{\gamma}^i\mu})_\beta - (\gamma^{+i}\theta)^\alpha(\bar{\theta}^{\bar{\gamma}^i\Pi}_\beta),(4.19)\]

\[B^\alpha_\beta = -(\Pi^{\gamma+\bar{\gamma}^i\theta})^\alpha(\theta^{\bar{\gamma}^i\mu})_\beta + (\gamma^{+i}\theta)^\alpha(\bar{\theta}^{\bar{\gamma}^i\Pi}_\beta) - \frac{1}{2}(\gamma^{+ij}\theta)^\alpha(\theta^{\bar{\gamma}^i\bar{\gamma}^j\Pi}_\beta) - \frac{1}{2}(\gamma^{+ij}\theta)^\alpha(\bar{\theta}^{\bar{\gamma}^i\bar{\gamma}^j\Pi}_\beta).\] (4.20)

5 **Light-cone superstring action in plane wave R-R background**

In this section we find the form of the type IIB superstring action in the plane wave background with R-R 5-form flux in the light-cone gauge. Our discussion of the light-cone gauge fixing closely repeats the same steps as in Refs. [11, 12], where the $AdS_5 \times S^5$ case was treated.

In flat space the superstring light-cone gauge fixing procedure consists of the two stages:

(I) fermionic light-cone gauge choice, i.e., fixing the $\kappa$-symmetry by $\bar{\gamma}^+\theta = 0$

(II) bosonic light-cone gauge choice, i.e., using the conformal gauge $\sqrt{gg}^{ab} = \eta^{ab}$ and fixing the residual conformal diffeomorphism symmetry by $x^+(\tau, \sigma) = p^+\tau$.

Our fermionic $\kappa$-symmetry light-cone gauge is the same as in flat superstring $\bar{\gamma}^+\theta = 0$. One usually imposes the $\kappa$-symmetry light-cone gauge by starting with the explicit representation for the superstring Lagrangian (4.13) in terms of $\theta'$s. However it is convenient to first impose the light-cone gauge at the level of the Cartan forms $L^\mu, L^\alpha$ and then to use them in (4.13). In what follows we adopt this strategy.\(^{10}\)

5.1 **Cartan 1-forms in Wess-Zumino parametrization in fermionic light-cone gauge**

We first consider fixing the fermionic $\kappa$-symmetry in the Cartan 1-forms written in the WZ parametrization. The fermionic $\kappa$-symmetry light-cone gauge is defined as

\[\bar{\gamma}^+\theta = 0.\] (5.1)

To simplify our expressions, we choose the parametrization of the plane wave space such that the bosonic bodies of the Cartan $H$ connections $L^i$, $L^{ij}$ (defined in (4.10)) are equal to zero.

\(^{10}\)This strategy was used in [29],[30] while deriving $\kappa$-symmetry fixed action for long superstring in $AdS_5 \times S^5$ and for $\kappa$-symmetry fixed light-cone $AdS$ superstring in [11].
The $\kappa$-symmetry gauge fixed bosonic Cartan 1-forms $L^\mu$ are then found to be

\[
L^+ = e^+ \quad \text{and} \quad L^I = e^I , \quad \text{while} \quad L^- = e^- - i(\bar{\theta}\gamma^- d\theta + \theta\gamma^- d\bar{\theta}) - 2e^+\bar{\theta}\gamma^- \Pi \theta ,
\]

while the fermionic 16-beins $L^\alpha$ take the form

\[
L = d\theta - ie^+ \Pi \theta .
\]

These expressions for the Cartan 1-forms are valid for an arbitrary parametrization of the bosonic 10-beins $e^\mu$ satisfying relations (5.2). In what follows, we use the 10-beins\(^\text{11}\)

\[
ee^+ = dx^+ , \quad e^- = dx^- - \frac{1}{2}x^2 dx^+ , \quad e^I = dx^I .
\]

Expressions (5.3)-(5.5) can obviously be obtained from the general solution for the Cartan 1-forms given in (4.15),(4.16). However the most convenient procedure is to derive them from differential equations (4.4),(4.9) using the fermionic $\kappa$-symmetry light-cone gauge (5.1). We now outline this procedure.

First of all, using (5.1) in (4.5) for $\mu = +$, we find the equation $\partial_t L^+_t = 0$, which together with (4.10) gives

\[
L^+_t = e^+ .
\]

Second, multiplying equations (4.4) by $\bar{\gamma}^+$ and taking (5.1) and (2.16) into account, we find the equation $\partial_t (\bar{\gamma}^+ L_t) = 0$, which together with (4.10) gives

\[
\bar{\gamma}^+ L_t = 0 .
\]

Now inserting the decomposition of unity $1 = (\gamma^+ \bar{\gamma}^- + \gamma^- \bar{\gamma}^+)/2$ between $L_t$ and $\theta$ in the r.h.s. of (4.5) and taking (5.1) and (5.8) into account we obtain the equation $\partial_t L^I_t = 0$, which together with (4.10) leads to

\[
L^I_t = e^I .
\]

Third, using (5.1) in (4.8),(4.9) gives the equations $\partial_t L^i_t = \partial_t L^{i'}_t = 0$. Because of (5.2) these equations imply

\[
L^i_t = 0 , \quad L^{i'}_t = 0 .
\]

Using of these relations and gauge (5.1) in equations (4.4) gives

\[
\partial_t L_t = d\theta - ie^+ \Pi \theta .
\]

We thus find

\(^{11}\)Note that such 10-beins can be obtained by using the following coset representative of bosonic body in (4.1): $g(x) = \exp(x^+ P^-) \exp(x^- P^+ + x^I P^I)$.  

12
\[ L_t = t(d\theta - i e^+ \Pi \theta). \]  \hspace{1cm} (5.12)

Inserting this \( L_t \) in the equation for \( L_t^- \), we obtain

\[ \partial_t L_t^- = -2it(\bar{\theta} \gamma^- d\theta + \theta \gamma^- d\bar{\theta}) - 4te^+ \bar{\theta} \gamma^- \Pi \theta. \] \hspace{1cm} (5.13)

The solution of this equations for \( t = 1 \) gives the expression for \( L^- \) in (5.4).

### 5.2 \( \kappa \)-symmetry gauge fixed superstring action in WZ parametrization

Because we use coset parametrization (4.1) the \( \kappa \)-symmetry gauge fixed action given in this Section corresponds to the WZ parametrization of superspace. The light-cone gauge action in the Killing parametrization is discussed in Section 5.5.

Inserting the above expressions for Cartan 1-forms (5.3)-(5.5) into Lagrangian (4.13) and using (5.6) we obtain the \( \kappa \)-symmetry gauge fixed superstring Lagrangian in terms of the light-cone supercoset coordinates

\[ L = L_{\text{kin}} + L_{\text{WZ}}, \] \hspace{1cm} (5.14)

where the kinetic and WZ parts are given by

\[ L_{\text{kin}} = -\frac{1}{2} \sqrt{g} g^{ab} \left( 2 \partial_a x^+ \partial_b x^- - x_1^2 \partial_a x^+ \partial_b x^+ + \partial_a x^I \partial_b x^I \right) \]
\[ - i \sqrt{g} g^{ab} \partial_b x^+ \left( \bar{\theta} \gamma^- \partial_a \theta + \theta \gamma^- \partial_a \bar{\theta} + 2i \partial_a x^+ \bar{\theta} \gamma^- \Pi \theta \right), \] \hspace{1cm} (5.15)

\[ L_{\text{WZ}} = i \epsilon^{ab} \partial_a x^+ \theta^I \partial_b \theta + \text{h.c.} \] \hspace{1cm} (5.16)

The kinetic terms in (5.15) can be obtained in a straightforward way. To find the WZ part we make rescaling \( \theta \rightarrow t \theta \) in (5.3)-(5.5) and then integrate over \( t \) in (4.13). Note that in deriving representations for \( L_{\text{kin}}, L_{\text{WZ}} \) in (5.15), (5.16) we made the redefinition \( x^\mu \rightarrow -x^\mu \). This form of the Lagrangian is most convenient for deriving the Noether charges (see Section 7).

The superstring Lagrangian given by equations (5.14)-(5.16) can obviously be rewritten as

\[ \mathcal{L} = \mathcal{L}_B + \mathcal{L}_F. \] \hspace{1cm} (5.17)

Here, \( \mathcal{L}_B = -\frac{1}{2} \sqrt{g} g^{ab} C_{\mu \nu} \partial_a x^\mu \partial_b x^\nu \) given in the first line in (5.15) is the standard bosonic sigma model with the plane wave geometry as the target space and \( \mathcal{L}_F \) given by the second line in (5.15) and by (5.16) is the fermionic part of the Lagrangian

\[ \mathcal{L}_F = -i \sqrt{g} g^{ab} \partial_b x^+ \left( \bar{\theta} \gamma^- \partial_a \theta + \theta \gamma^- \partial_a \bar{\theta} + 2i \partial_a x^+ \bar{\theta} \gamma^- \Pi \theta \right) \]
\[ + i \epsilon^{ab} \partial_a x^+ \left( \theta^I \gamma^- \partial_b \theta + \bar{\theta}^I \gamma^- \partial_b \bar{\theta} \right). \] \hspace{1cm} (5.18)
5.3 Bosonic light-cone gauge and “2d spinor” form of the action

As in the flat space case [26], the resulting action can then be put into the “2-d spinor” form. Indeed, the 8+8 fermionic degrees of freedom can be organized into 8 Majorana 2d spinors, defined in flat 2d geometry. This form of the action may be useful in establishing the relation to the NSR formulation.

As in the flat case, we should eliminate the $\partial x^+$-factors from the kinetic terms of fermionic fields (5.18). In flat space this was possible by choosing the bosonic light-cone gauge. In the BDHP formulation [31, 32] that we are using, this can be done by fixing the conformal gauge as

$$\sqrt{g} g^{ab} = \eta^{ab}, \quad -\eta^{00} = \eta^{11} = 1 \quad (5.19)$$

and then noting that because the resulting equation $(\partial_0^2 - \partial_1^2) x^+ = 0$ has the general solution $x^+(\tau, \sigma) = f(\tau - \sigma) + h(\tau + \sigma)$ we can fix the residual conformal diffeomorphism symmetry on the plane by choosing

$$x^+(\tau, \sigma) = \tau, \quad (5.20)$$

where we put $p^+ = 1$. Using (5.19) and (5.20) in $L_B$ and $L_F$ (5.17),(5.18) we find the bosonic and fermionic Lagrangians

$$L_B = -\frac{1}{2} \partial_a x^I \partial^a x^I - \frac{1}{2} x_I^2, \quad (5.21)$$

$$L_F = i(\bar{\psi} \gamma^- \partial_0 \theta + \theta \gamma^- \partial_0 \bar{\theta} + \theta \gamma^- \partial_1 \theta + \bar{\theta} \gamma^- \partial_1 \bar{\theta}) - 2 \bar{\theta} \gamma^- \Pi \theta. \quad (5.22)$$

Similarly to the flat space case the fermionic Lagrangian $L_F$ can be rewritten as

$$L_F = -i \bar{\psi} \gamma^- \sigma^a \partial_a \psi + i \bar{\psi} \gamma^- \Pi \psi. \quad (5.23)$$

Here $\sigma^a$ are 2 × 2 Dirac matrices

$$\sigma^0 \equiv -i \sigma_2, \quad \sigma^1 \equiv \sigma_1, \quad (5.24)$$

$\bar{\psi} \equiv \psi_T \theta^0$, $\psi^T$ denotes transposition of 2d spinor and $\psi'$s are related to the (2d scalar) fermionic fields $\theta$'s by

$$\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}, \quad \psi^1 \equiv \theta^1, \quad \psi^2 \equiv \theta^2, \quad (5.25)$$

where $\theta^1$ and $\theta^2$ are real spinors related to the original complex spinors $\theta$ and $\bar{\theta}$ by (A.17). From this, it is clear that $\psi'$s are 8 Majorana 2d spinors.

If we restore a dependence on the dimensionful parameter $m$ (2.1) then we obtain

$$L = -\frac{1}{2} \partial_a x^I \partial^a x^I - \frac{m^2}{2} x_I^2 - i \bar{\psi} \gamma^- \sigma^a \partial_a \psi + im \bar{\psi} \gamma^- \Pi \psi. \quad (5.26)$$

Thus the total superstring Lagrangian describes 8 free massive 2d scalars and 8 free massive Majorana 2d fermionic fields propagating in a flat two dimensional world-sheet. This superstring model can obviously be quantized in a rather straightforward way.

\[\text{In our notation } \bar{\psi} \sigma^a \partial_a \psi = -\psi^1 (\partial_0 + \partial_1) \psi^1 - \psi^2 (\partial_0 - \partial_1) \psi^2.\]
5.4 Solution to superstring equations of motion

In contrast to the $AdS_5 \times S^5$ case the superstring in the plane wave Ramond-Ramond background has free equations of motion that can be easily solved. To analyze the equations of motion we use Lagrangian formulated in terms of $(x^I, \theta, \bar{\theta})$ (see (5.17),(5.21),(5.22)):

\[
\mathcal{L} = -\frac{1}{2} \partial_a x^I \partial^a x^I - \frac{1}{2} x_I^2 + i(\bar{\theta} \gamma^0 \partial_0 \theta + \theta \gamma^0 \partial_0 \bar{\theta} + \theta \gamma^i \partial_i \theta + \bar{\theta} \gamma^i \partial_i \bar{\theta}) - 2\bar{\theta} \gamma^i \Pi \theta .
\]  
(5.27)

The equations of motion and periodicity conditions for bosonic fields are given by

\[
(-\partial_0^2 + \partial_1^2)x^I - x^I = 0 ,
\]  
(5.28)

\[
x^I(\sigma + 1, \tau) = x^I(\sigma, \tau), \quad 0 \leq \sigma \leq 1 .
\]  
(5.29)

The solution to these equations can be written as

\[
x^I(\sigma, \tau) = \cos \tau x_0^I + \sin \tau p_0^I + i \sum_{n \neq 0} \frac{1}{\omega_n} \left( \varphi^1_n(\sigma, \tau) \alpha^1_n + \varphi^2_n(\sigma, \tau) \alpha^2_n \right),
\]  
(5.30)

where the $x_0^I$ and $p_0^I$ are zero modes, while $\alpha^1_n$ and $\alpha^2_n$ are string oscillators modes. The base functions $\varphi^{1,2}_n(\sigma, \tau)$ for the right and left movers are given by

\[
\varphi^1_n(\sigma, \tau) = \exp(-i(\omega_n \tau - k_n \sigma)) , \quad \varphi^2_n(\sigma, \tau) = \exp(-i(\omega_n \tau + k_n \sigma)) ,
\]  
(5.31)

where the frequencies $\omega_n$ are defined by

\[
\omega_n = \sqrt{k_n^2 + 1}, \quad n > 0 ; \quad \omega_n = -\sqrt{k_n^2 + 1}, \quad n < 0 ;
\]  
(5.32)

\[
k_n = 2\pi n , \quad n = \pm 1, \pm 2 \ldots
\]  
(5.33)

Using the rules implied by standard canonical Hamiltonian approach (which is systematically developed in Section 6.1) we obtain the Poisson brackets

\[
[p_0^I, x_0^J] = \delta^{IJ}, \quad [\alpha^1_m, \alpha^1_n] = \frac{i}{2} \omega_m \delta_{m+n,0} \delta^{1J}, \quad [\alpha^2_m, \alpha^2_n] = \frac{i}{2} \omega_m \delta_{m+n,0} \delta^{2J} .
\]  
(5.34)

We now turn to fermionic fields. Lagrangian (5.27) gives the equations of motion for $\theta$'s

\[
\partial_0 \theta + \partial_1 \bar{\theta} + i \Pi \theta = 0 , \quad \partial_0 \bar{\theta} + \partial_1 \theta - i \Pi \theta = 0 ,
\]  
(5.35)

that should be supplemented by the periodicity conditions $\theta(\sigma + 1, \tau) = \theta(\sigma, \tau)$. We prefer to re-formulate these equations in terms of two real fermionic fields $\theta^1$ and $\theta^2$ (see (A.17))

\[
(\partial_0 + \partial_1)\theta^1 - \Pi \theta^2 = 0 ,
\]  
(5.36)
\[
(\partial_0 - \partial_1)\theta^2 + \Pi \theta^1 = 0. \tag{5.37}
\]

Solutions to these equations are found to be

\[
\theta^1(\sigma, \tau) = \cos \tau \theta^1_0 + \sin \tau \Pi \theta^2_0 + \sum_{n \neq 0} c_n \left( \varphi^1_n(\sigma, \tau) \theta^1_n + i(\omega_n - k_n) \varphi^2_n(\sigma, \tau) \Pi \theta^2_n \right), \tag{5.38}
\]

\[
\theta^2(\sigma, \tau) = \cos \tau \theta^2_0 - \sin \tau \Pi \theta^1_0 + \sum_{n \neq 0} c_n \left( \varphi^2_n(\sigma, \tau) \theta^2_n - i(\omega_n - k_n) \varphi^1_n(\sigma, \tau) \Pi \theta^1_n \right), \tag{5.39}
\]

where \(\theta^1_0\) and \(\theta^2_0\) are fermionic zero modes, while \(\theta^1_n\) and \(\theta^2_n\) are fermionic string oscillators. The coefficients \(c_n\) are fixed to be

\[
c_n = \frac{1}{\sqrt{1 + (\omega_n - k_n)^2}}, \quad n = \pm 1, \pm 2, \ldots \tag{5.40}
\]

With this normalization for the coefficients \(c_n\) the canonical Hamiltonian approach (see Section 6.1) gives the Poisson-Dirac anticommutation relations

\[
\{ \theta^1_m, \theta^1_n \} = \frac{i}{4} (\gamma^+)^{\alpha\beta} \delta_{m+n,0}, \quad \{ \theta^2_n, \theta^2_n \} = \frac{i}{4} (\gamma^+)^{\alpha\beta} \delta_{m+n,0} \tag{5.41}
\]

\[
m, n = 0, \pm 1, \pm 2, \ldots
\]

We can now quantize our 2d string fields \(x^I\) and \(\theta^I\)s by promoting the various Fourier components appearing in (5.30),(5.38),(5.39) to operators. Supplementing this by promoting classical (anti)brackets (5.34)(5.41) to equal time (anti)commutators according to the rules \(\{ \ldots \}_{\text{clas}} \rightarrow i \{ \ldots \}_{\text{quant}}, \quad [ \ldots ]_{\text{clas}} \rightarrow i [ \ldots ]_{\text{quant}}\) we obtain the final form of the quantum equal-time (anti)commutators. We note that if we restore dependence on the dimensionful parameter \(m\) (see (2.1)), then in flat space limit \((m \to 0)\), the solutions to the equations of motion reduces to the well known solutions of the flat superstring equations of motion.

### 5.5 Superstring action in various coordinates and Killing parametrization of superspace

In this section we discuss an alternative form of the superstring Lagrangian that may be useful in various applications. This form of the Lagrangian is based on the parametrization of plane wave space given by

\[
ds^2 = 2dx^+ dx^- + \cos^2 x^+ dx_I^2. \tag{5.42}
\]

As is well known this line interval can be obtained from the one in (2.1) by replacing the variables as

\[
x^- \rightarrow x^- + \frac{1}{4} x_I^2 \sin 2x^+, \quad x^I \rightarrow \cos x^+ x^I. \tag{5.43}
\]
With simultaneous replacing of the fermionic field\(^{13}\)

\[ \theta \rightarrow e^{-ix^\Pi \theta} \]  

(5.44)
equations (5.15),(5.16) lead to the following superstring Lagrangian in fermionic light-cone gauge

\[ L_{\text{kin}} = -\frac{1}{2}\sqrt{g} g^{ab}\left(2\partial_a x^+ \partial_b x^- + \cos^2 x^+ \partial_a x^I \partial_b x^I \right) - i\sqrt{g} g^{ab}\partial_b x^+ \left(\bar{\theta} \gamma^- \partial_a \theta + \theta \gamma^- \partial_a \bar{\theta} \right) \]  

(5.45)

\[ L_{WZ} = i\epsilon^{ab} \partial_a x^+ \theta \gamma^- e^{-2ix^\Pi \partial_b \theta} + \text{h.c.} \]  

(5.46)
The corresponding Lagrangian in the bosonic light-cone gauge can be obtained using gauge (5.19) and fixing the residual conformal diffeomorphism symmetry as in (5.20). The resulting Lagrangian is

\[ L = \frac{1}{2} \cos^2 x^+ \partial_a x^I \partial^a x^I \]

\[ + \ i(\bar{\theta} \gamma^- \partial_a \theta + \theta \gamma^- \partial_a \bar{\theta} + \theta \gamma^- e^{-2ix^\Pi \partial_1 \theta} + \bar{\theta} \gamma^- e^{2ix^\Pi \partial_1 \bar{\theta}}) . \]  

(5.47)

Compared to (5.27) this superstring Lagrangian does not involve mass like terms for superstring 2d fields. One can expect that this parametrization is most convenient for establishing the connection with the NSR formulation. Solution to equations of motion obtained from (5.47) can easily be obtained from (5.30),(5.38)(5.39) using transformations (5.43),(5.44).

6 Light-cone Hamiltonian approach to superstring in plane wave geometry

In this section, following GGRT approach [23], we develop phase space formulation of superstring in the plane wave RR background. This formulation is most convenient for deriving of Noether charges of basic symmetry superalgebra.

The superstring Lagrangian (5.14),(5.15)(5.16) can be represented as

\[ L = L_1 + L_2 , \]

(6.1)
where the two parts are

\[ L_1 = -h^{ab} \partial_a x^+ \partial_b x^- + \frac{1}{2} h^{ab} \partial_a x^+ \partial_b x^+ B + \partial_a x^+ A^a + C , \]  

(6.2)

\[ L_2 = -\frac{1}{2} h^{ab} \partial_a x^I \partial_b x^I \]

(6.3)
and \( h^{ab} \) is defined by

\(^{13}\)One can make sure that passing to the new \( \theta' \)’s (5.44) corresponds to usage of Killing parametrization of superspace: \( G(x, \theta) = g(\theta)g(x) \).
\[ h^{ab} \equiv \sqrt{g} g^{ab}, \quad h^{00} h^{11} - (h^{01})^2 = -1. \] (6.4)

The functions \( A^a, B, \) and \( C \) take the form

\[
A^a = -ih^{ab}(\bar{\theta} \gamma^- \partial_b \theta + \theta \gamma^- \partial_b \bar{\theta}) + i\epsilon^{a1}(\theta \gamma^- \dot{\theta} + \bar{\theta} \gamma^- \dot{\bar{\theta}}),
\] (6.5)
\[
B = x_I^2 + 4\bar{\theta} \gamma^\Pi \theta,
\] (6.6)
\[
C = -i\dot{x}^+ (\theta \gamma^- \dot{\theta} + \bar{\theta} \gamma^- \dot{\bar{\theta}}),
\] (6.7)

where dot and prime are derivatives over \( \tau \) and \( \sigma \) (see Appendix A for definitions). Decomposition (6.1) is made so that the functions \( A^a, B, C, \) depend on (i) the anticommuting coordinates and their derivatives with respect to both world-sheet coordinates \( \tau \) and \( \sigma \) and (ii) the bosonic coordinates and their derivatives with respect to the world-sheet spatial coordinate \( \sigma \) only. The reason for this decomposition is that we use the phase space description with respect to the bosonic coordinates only, i.e. we not make the Legendre transformation with respect to the fermionic coordinates.

### 6.1 Phase space Lagrangian

Computing the canonical momenta for the bosonic coordinates

\[
P_\mu = \frac{\partial L}{\partial \dot{x}_\mu},
\] (6.8)

we obtain

\[
P^+ = -h^{00} \dot{x}^+ - h^{01} \dot{x}^+,
\] (6.9)
\[
P^I = -h^{00} \dot{x}^I - h^{01} \dot{x}^I,
\] (6.10)
\[
P^- = -h^{00} \dot{x}^- - h^{01} \dot{x}^- + A^0 - B P^+.
\] (6.11)

where we use the conventions \( P^\pm \equiv P_\pm \) and \( P^I \equiv P_I \). Applying the standard procedure, we then find the phase space Lagrangian \( L = L_1 + L_2, \)

\[
L_1 = P^+ \dot{x}^+ + P^- \dot{x}^- + \frac{1}{2h^{00}}(2P^+ P^- + 2\dot{x}^+ \dot{x}^- + (P^{+2} - \dot{x}^{+2})B)
+ \frac{h^{01}}{h^{00}}(P^+ \dot{x}^- + P^- \dot{x}^+) - \frac{1}{h^{00}}(P^+ + h^{01} \dot{x}^+) A^0 + \dot{x}^+ A^1 + C,
\] (6.12)
\[
L_2 = P^I \dot{x}^I + \frac{1}{2h^{00}}(P^2 + \dot{x}^2_I) + \frac{h^{01}}{h^{00}} P^I \dot{x}^I.
\] (6.13)

We next impose the light-cone gauge

\[
x^+ = \tau, \quad P^+ = p^+.
\] (6.14)

Using these gauge conditions in the action and integrating over \( P^- \) we arrive at
\[ h_{00} = -p^+ . \] (6.15)

Inserting this in (6.12) and (6.13) we obtain the general form of the phase space light-cone gauge Lagrangian

\[ \mathcal{L}_1 = -\frac{1}{2p^+}(p^+ B) - \frac{h_{01}}{p^+}(p^+ \dot{x}^-) + A^0, \] (6.16)

\[ \mathcal{L}_2 = p^I \dot{x}^I - \frac{1}{2p^+}(P^2_I + \dot{x}^2_I) - \frac{h_{01}}{p^+} p^I \dot{x}^I. \] (6.17)

In deriving these expressions we used that the function \( C \) given in (6.7) is equal to zero in the light-cone gauge (6.14). Now, in order to obtain the light-cone gauge phase Lagrangian we need only to insert the appropriate functions \( A_0 \) and \( B \) (6.5),(6.6). The result is

\[ \mathcal{L} = p^I \dot{x}^I + ip^+(\bar{\theta}\gamma^- \dot{\dot{\theta}} + \theta\bar{\gamma}^- \dot{\theta}) - \frac{1}{2p^+} (P^2_I + \dot{x}^2_I + p^+ (x^2_I + 4\bar{\theta}\bar{\gamma}^- \Pi \theta)) + i(\theta\bar{\gamma}^- \dot{\dot{\theta}} + \bar{\theta}\gamma^- \dot{\theta}). \] (6.18)

This Lagrangian gives the Hamiltonian

\[ P^- = \int d\sigma \ P^-, \] (6.19)

where the Hamiltonian density \( \mathcal{P}^- \) is

\[ \mathcal{P}^- = -\frac{1}{2p^+} (P^2_I + \dot{x}^2_I + p^+ (x^2_I + 4\bar{\theta}\bar{\gamma}^- \Pi \theta)) + i(\theta\bar{\gamma}^- \dot{\dot{\theta}} + \bar{\theta}\gamma^- \dot{\theta}). \] (6.20)

It should be supplemented by the constraint

\[ p^+ \dot{\theta} + p^I \dot{x}^I + ip^+(\bar{\theta}\gamma^- \dot{\dot{\theta}} + \theta\bar{\gamma}^- \dot{\theta}) = 0. \] (6.21)

As usual, this constraint allows one to express the non-zero modes of the bosonic coordinate \( x^- \) in terms of the transverse physical modes.

The equations of motion corresponding to the phase space superstring Lagrangian (6.18) takes the form

\[ \dot{x}^I = \frac{1}{p^+} p^I, \] (6.22)

\[ \dot{p}^I = \frac{1}{p^+} \ddot{x}^I - p^+ x^I, \] (6.23)

\[ \dot{\theta} = -\frac{1}{p^+} \dot{\theta} - i\Pi \theta, \] (6.24)

\[ \dot{\bar{\theta}} = -\frac{1}{p^+} \dot{\bar{\theta}} + i\Pi \bar{\theta}. \] (6.25)
These equations can be written in the Hamiltonian form. Introducing the notation $X$ for the phase space variables $(P^I, x^I, \theta, \bar{\theta})$ we have the Hamiltonian equations

$$\dot{X} = \{X, P^-\},$$

(6.26)

where the phase space variables satisfy the (classical) Poisson-Dirac brackets

$$\{P^I(\sigma), x^J(\sigma')\} = \delta^{IJ} \delta(\sigma, \sigma'),$$

(6.27)

$$\{\bar{\theta}^\alpha(\sigma), \theta^\beta(\sigma')\} = \frac{i}{4p^+} (\gamma^+)_{\alpha\beta} \delta(\sigma, \sigma'),$$

(6.28)

$$[x^-_0, \theta^\alpha] = \frac{1}{2p^+} \theta^\alpha, \quad [x^-_0, \bar{\theta}^\alpha] = \frac{1}{2p^+} \bar{\theta}^\alpha.$$  

(6.29)

Here $x^-_0$ is the zero mode of $x^-$ and therefore $[p^+, x^-_0] = 1$. All the remaining brackets are equal to zero. To derive (6.28), (6.29) we must recall that the Lagrangian (6.18) has the fermionic second class constraints

$$p^\alpha + ip^+ \gamma^-_{\alpha\beta} \bar{\theta}^\beta = 0, \quad \bar{p}^\alpha + ip^+ \gamma^-_{\alpha\beta} \theta^\beta = 0,$$

(6.30)

where $p^\alpha$ and $\bar{p}^\alpha$ are the canonical momenta of the respective fermionic coordinates $\theta^\alpha$ and $\bar{\theta}^\alpha$. Starting with the Poisson brackets

$$\{p_{\beta}(\sigma), \theta^\alpha(\sigma')\}_{P.B.} = \frac{1}{2} (\gamma^+ \gamma^-)_{\alpha\beta} \delta(\sigma, \sigma'), \quad \{\bar{p}_{\beta}(\sigma), \bar{\theta}^\alpha(\sigma')\}_{P.B.} = \frac{1}{2} (\gamma^+ \gamma^-)_{\alpha\beta} \delta(\sigma, \sigma'),$$

(6.31)

and $[p^+, x^-_0]_{P.B.} = 1$ we then obtain the Poisson-Dirac brackets given in (6.28), (6.29). The projector $(\gamma^+ \gamma^-)/2$ in (6.31) is to respect light-cone gauge imposed on $\theta^\alpha$ and $\bar{\theta}^\alpha$ (5.1).

### 7 Noether charges as generators of the basic supersymmetry algebra

The Noether charges play an important role in analysis of the symmetries of dynamical systems. The choice of the light-cone gauge spoils manifest global symmetries, and in order to demonstrate that these global invariances are still present, one needs to find the Noether charges that generate them.\(^{14}\) These charges play a crucial role in formulating superstring field theory in the light-cone gauge (see [33, 34]).

In the light-cone formalism, the generators (charges) of the basic superalgebra can be split into two groups:

$$P^+, \quad P^I, \quad J^{+I}, \quad J^{ij}, \quad J^{i'j'}, \quad Q^+, \quad Q^+, \quad \bar{Q}^+, \quad \bar{Q}^-,$$

(7.1)

which we refer to as kinematical generators, and

$$P^-, \quad Q^-, \quad \bar{Q}^-.$$  

(7.2)

\(^{14}\text{In what follows “currents” and “charges” will mean both bosonic and fermionic ones, i.e. will include supercurrents and supercharges.}\)
which we refer to as dynamical generators. The \( Q^+ \) and \( Q^- \) are expressible in terms of the original supercharges \( Q \) as

\[
Q^+ \equiv \frac{1}{2} \bar{\gamma} \gamma^+ Q, \quad Q^- \equiv \frac{1}{2} \bar{\gamma} \gamma^- Q.
\] (7.3)

For \( x^+ = 0 \), the kinematical generators in the superfield realization are quadratic in the physical string fields, while the dynamical generators receive higher-order interaction-dependent corrections. The first step in the construction of superstring field theory is to find a free (quadratic) superfield representation of the generators of the basic superalgebra. The charges that we obtain below can be used to obtain (after quantization) these free superstring field charges.

7.1 Currents for \( \kappa \)-symmetry light-cone gauge fixed superstring action

As usual, symmetry generating charges can be obtained from conserved currents. Because the currents themselves may be helpful in some applications, we first derive them starting with the \( \kappa \)-symmetry gauge fixed Lagrangian in the form given in (5.15),(5.16). To obtain the currents, we use the standard Noether method (see, e.g., [35]) based on the localization of the parameters of the associated global transformations. We let \( \epsilon \) be a parameter of some global transformation that leaves the action invariant. Replacing it by a function of world-sheet coordinates \( \tau, \sigma \), the variation of the action takes the form

\[
\delta S = \int d^2 \sigma \ G^a \partial_a \epsilon, \tag{7.4}
\]

where \( G^a \) is the corresponding current. Using this formula in what follows, we find those currents that are related to symmetries that do not involve compensating \( \kappa \)-symmetry transformations. The remaining currents are found in the next subsection starting from the action (6.18) where both the \( \kappa \)-symmetry and the bosonic light-cone gauges are fixed.

We start with the translation invariance. Transformations of the coordinate under \( P^\pm \) translations take the form

\[
\delta x^\pm = \epsilon^\pm, \tag{7.5}
\]

while under \( P^I \) translations, we have the transformations

\[
\delta x^- = \sin x^+ \epsilon^I x^I, \quad \delta x^I = \cos x^+ \epsilon^I. \tag{7.6}
\]

Applying (7.4) to the Lagrangian (5.15),(5.16) gives the translation currents

\[
\begin{align*}
P^+ & = -\sqrt{g} g^{ab} \partial_b x^+, \tag{7.7} \\
\mathcal{P}^I & = -\sqrt{g} g^{ab} \cos x^+ \partial_b x^I + \sin x^+ x^I \mathcal{P}^+ a, \tag{7.8} \\
\mathcal{P}^a & = -\sqrt{g} g^{ab}(\partial_b x^- + i \bar{\theta} \gamma^- \partial_b \theta + i \theta \gamma^- \partial_b \bar{\theta}) - x^2 \mathcal{P}^+ - 4 \mathcal{P}^+ \bar{\theta} \gamma^- \Pi \theta \\
& \quad + (i \epsilon^{ab} \theta \gamma^- \partial_b \theta + h.c.). \tag{7.9}
\end{align*}
\]

Invariance of the action (5.15),(5.16) with respect to rotations in the \( (x^-, x^I) \) plane
\[ \delta x^- = \cos x^+ \epsilon^{-I} x^I, \quad \delta x^I = -\sin x^+ \epsilon^{-I} \] (7.10)

gives the conserved currents
\[ \mathcal{J}^{+Ia} = -\sqrt{g} g^{ab} \sin x^+ \partial_b x^I - \cos x^+ x^I P^+ a. \] (7.11)

Invariance with respect to the \(SO(4)\) and \(SO'(4)\) rotations
\[ \delta x^i = \epsilon^{ij} x^j, \quad \delta \theta = \frac{1}{4} \epsilon^{ij} \gamma^{ij} \theta, \quad \delta x'^i = \epsilon'^{ij} x^j, \quad \delta \bar{\theta} = \frac{1}{4} \epsilon'^{ij} \gamma^{ij} \bar{\theta}, \] (7.12)

\[ \epsilon^{ij} = -\epsilon^{ji}, \quad \epsilon'^{ij} = -\epsilon'^{ji}, \] gives the conserved currents
\[ \mathcal{J}^{ij} = -\sqrt{g} g^{ab} (x^i \partial_b x^j - x^j \partial_b x^i) - i \theta \bar{\gamma}^i \gamma^j \theta P^+ a - \left( \frac{i \epsilon^{ab}}{2 \sqrt{g}} P^+_0 \theta \bar{\gamma}^i \gamma^j \theta + \text{h.c.} \right), \] (7.13)
\[ \mathcal{J}^{ij'} = -\sqrt{g} g^{ab} (x^i' \partial_b x^j' - x^j' \partial_b x^i') - i \bar{\theta} \gamma^i \gamma^j' \theta P^+ a - \left( \frac{i \epsilon^{ab}}{2 \sqrt{g}} P^+_0 \theta \gamma^i \gamma^j' \theta + \text{h.c.} \right), \] (7.14)

where \( P^+_0 \equiv g_{ab} P^+ b \).

Invariance with respect to the super transformations
\[ \delta \theta = e^{-ix^+ \Pi} \epsilon, \quad \delta \bar{\theta} = e^{ix^+ \Pi} \bar{\epsilon}, \quad \delta x^- = -ie \bar{\gamma} e^{-ix^+ \Pi} \bar{\theta} - i \bar{\epsilon} \gamma e^{ix^+ \Pi} \theta, \] (7.15)

leads to the conserved supercurrents
\[ Q^+ a = 2 \bar{\gamma} e^{ix^+ \Pi} (P^+ a \theta + \frac{\epsilon^{ab}}{\sqrt{g}} P^+_0 \bar{\theta}), \] (7.16)
\[ \bar{Q}^+ a = 2 \bar{\gamma} e^{-ix^+ \Pi} (P^+ a \bar{\theta} + \frac{\epsilon^{ab}}{\sqrt{g}} P^+_0 \theta). \] (7.17)

### 7.2 Charges for bosonic and \(\kappa\)-symmetry light-cone gauge fixed superstring action

In the previous section, we have found (super)currents starting with the \(\kappa\)-symmetry light-cone gauge fixed action given in (5.15), (5.16). These currents can be used to find currents for the action where both the fermionic \(\kappa\)-symmetry and the bosonic reparametrization symmetry are fixed by light-cone gauges (6.18). To find the components of the currents in the world-sheet time direction \(G^0\) we should use relations for the canonical momenta (6.9)–(6.11) and then to insert the light-cone gauge conditions (6.14) and (6.15) in the expressions for the currents given in the previous subsection. The charges are then given by
\[ G = \int d\sigma \, G^0. \] (7.18)

We start with the kinematical generators (charges) (7.1). The results for the currents imply the following representations for some of them
\( P^+ = p^+ \), \( P^I = \int \cos x^+ p^I + \sin x^+ x^I p^+ \), \( J^{+I} = \int \sin x^+ p^I - \cos x^+ x^I p^+ \), \( Q^+ = \int 2p^+ \bar{\gamma} e^{ix^+ \Pi \theta} \), \( \bar{Q}^+ = \int 2\bar{p}^+ \bar{\gamma} e^{-ix^+ \Pi \bar{\theta}} \).

We note that these charges depend only on the zero modes of string coordinates. In (7.19)–(7.21) the integrands are \( G^0 \) parts of the corresponding currents in world-sheet time direction: \( p^{I0} \), \( J^{+I0} \), \( Q^{+0} \), \( \bar{Q}^{+0} \), and \( P^{+0} = p^+ \).

The remaining kinematical charges depend on non-zero string modes and are given by

\( J^{ij} = \int x^i p^j - x^j p^i - ip^+ \bar{\theta} \gamma^{ij} \theta \), \( J^{i'j'} = \int x^i p^{j'} - x^{j'} p^i - ip^+ \bar{\theta} \gamma^{i'j'} \theta \).

The dynamical charge \( P^− \) is given by (6.19), while the supercharges \( Q^- \) and \( \bar{Q}^- \) are given by

\( Q^- = \int 2p^I \bar{\gamma}^I \theta - 2 \bar{x}^I \bar{\gamma}^I \bar{\theta} + 2ip^+ x^I \bar{\gamma}^I \Pi \theta \), \( \bar{Q}^- = \int 2\bar{p}^I \bar{\gamma}^I \bar{\theta} - 2 \bar{x}^I \bar{\gamma}^I \theta - 2ip^+ x^I \bar{\gamma}^I \Pi \theta \).

The derivation of these supercharges can be found in Appendix B.

8 Conclusions

We have developed the \( \kappa \)-symmetric and light-cone gauge formulations of type IIB superstring in the plane wave Ramond-Ramond background. We restricted our consideration to the study of classical superstring dynamics. Because in light-cone gauge the superstring action is quadratic in \( 2d \) fields, this superstring model can be explicitly quantized in a rather straightforward way. We have presented various forms of the light-cone gauge superstring Lagrangian. In coordinates (2.1) the light-cone gauge Lagrangian given by (5.26) (or (5.27)) describes 8 massive bosonic \( 2d \) scalars and 8 massive Majorana \( 2d \) fermions propagating in flat \( 2d \) world-sheet. In coordinates (5.42) the light-cone gauge Lagrangian given by (5.47) describes 8 massless bosonic and 8+8 fermionic \( 2d \) superstring fields.

The fact that the light-cone gauge action is quadratic in physical fields implies that the corresponding plane wave background with RR 5-flux should be an exact string solution.

As noted above, the relatively simple model of superstring in the plane wave RR background may serve as a training ground for the study of the more interesting case of superstring in \( AdS_5 \times S^5 \). Because the plane wave superstring action is invariant under the global transformations of the basic symmetry superalgebra of the plane wave RR background, the string spectrum should be classified by unitary representations of this superalgebra. These unitary representations appear as Fourier modes of solutions to free equations of motion for quantized fields propagating in the plane wave RR background. The fields in the plane wave background have the following two features in common with
the fields in the $AdS$ space time: (i) The spectrum of the energy operator is discrete; (ii) The spectrum of the energy operator even for massless representations is bounded from below by a nonzero value [36]. From this line of reasoning we believe that the study of the plane wave RR superstring will be useful for better understanding strings in $AdS/RR$-charge backgrounds. We note that in this paper we have developed GS formulation of plane wave/RR superstring. NSR formulation of this superstring is still to be understood.

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Appendix A  Notation

In the main part of the paper, we use the following conventions for the indices:

\[ \begin{align*}
\mu, \nu, \rho &= 0, 1, \ldots, 9 \quad so(9,1) \text{ vector indices (tangent space indices)} \\
I, J, K, L &= 1, \ldots, 8 \quad so(8) \text{ vector indices (tangent space indices)} \\
i, j, k, l &= 1, \ldots, 4 \quad so(4) \text{ vector indices (tangent space indices)} \\
i', j', k', l' &= 5, \ldots, 8 \quad so'(4) \text{ vector indices (tangent space indices)} \\
\alpha, \beta, \gamma &= 1, \ldots, 16 \quad so(9,1) \text{ spinor indices in chiral representation} \\
a, b &= 0, 1 \quad 2d \text{ world-sheet coordinate indices}
\end{align*} \]

We suppress the flat space metric tensor $\eta_{\mu\nu} = (-,+,\ldots,+)$ in scalar products, i.e.

\[ X^\mu Y^\nu \equiv \eta_{\mu\nu} X^\mu Y^\nu \quad (A.1) \]

We decompose $x^\mu$ into the light-cone and transverse coordinates: $x^\mu = (x^+, x^-, x^I)$, $x^I = (x^i, x^{i'})$, where

\[ x^\pm \equiv \frac{1}{\sqrt{2}}(x^0 \pm x^9). \quad (A.2) \]

In this notation, scalar products of tangent space vectors are decomposed as

\[ X^\mu Y^\mu = X^+Y^- + X^-Y^+ + X^IY^I, \quad X^IY^I = X^iY^i + X^{i'}Y^{i'}. \quad (A.3) \]

Unless stated otherwise, we always assume the summation over repeated indices (irrespective of their position).

The derivatives with respect to the world-sheet coordinates $(\tau, \sigma)$ are

\[ \dot{x}^I \equiv \partial_\tau x^I, \quad \dot{x}^I \equiv \partial_\sigma x^I. \quad (A.4) \]

The world-sheet Levi-Civita symbol $\epsilon^{ab}$ is defined with $\epsilon^{01} = 1$.

We use the chiral representation for the $32 \times 32$ Dirac matrices $\Gamma^\mu$ in terms of the $16 \times 16$ gamma matrices $\gamma^\mu$.
\[ \Gamma^\mu = \begin{pmatrix} 0 & \gamma^\mu \\ \bar{\gamma}^\mu & 0 \end{pmatrix}, \] (A.5)

\[ \gamma^\mu \bar{\gamma}^\nu + \gamma^\nu \bar{\gamma}^\mu = 2 \eta^{\mu \nu}, \quad \gamma^\mu = (\gamma^\mu)^{\alpha \beta}, \quad \bar{\gamma}^\mu = \gamma^\mu_{\alpha \beta}, \] (A.6)

\[ \gamma^\mu = (1, \gamma^I, \gamma^9), \quad \bar{\gamma}^\mu = (-1, \gamma^I, \gamma^9), \quad \alpha, \beta = 1, \ldots, 16. \] (A.7)

We adopt the Majorana representation for \( \Gamma \)-matrices, \( C = \Gamma^0 \), which implies that all \( \gamma^\mu \) matrices are real and symmetric, \( \gamma^\mu_{\alpha \beta} = \gamma^\mu_{\beta \alpha}, (\gamma^\mu_{\alpha \beta})^* = \gamma^\mu_{\alpha \beta}. \)

We use the convention that \( \gamma^{\mu_1 \ldots \mu_k} \) are the antisymmetrized product of \( k \) gamma matrices normalized so that

\[ (\gamma^\mu \gamma^\nu)_{\alpha \beta}^\gamma \equiv \frac{1}{2}(\gamma^\mu \bar{\gamma}^\nu)_{\alpha \beta} - \left( \mu \leftrightarrow \nu \right), \]

\[ (\bar{\gamma}^\mu \gamma^\nu)_{\alpha \beta} \equiv \frac{1}{2}(\bar{\gamma}^\mu \gamma^\nu)_{\alpha \beta} - \left( \mu \leftrightarrow \nu \right). \] (A.8)

\[ (\gamma^\mu \gamma^\nu \gamma^\rho)_{\alpha \beta} \equiv \frac{1}{6}(\gamma^\mu \bar{\gamma}^\nu \gamma^\rho)_{\alpha \beta} \pm 5 \text{ terms}, \quad (\bar{\gamma}^\mu \gamma^\nu \bar{\gamma}^\rho)_{\alpha \beta} \equiv \frac{1}{6}(\bar{\gamma}^\mu \gamma^\nu \bar{\gamma}^\rho)_{\alpha \beta} \pm 5 \text{ terms}. \] (A.9)

We assume the normalization \( \gamma^0 \bar{\gamma}^1 \ldots \gamma^8 \bar{\gamma}^9 = 1 \), i.e.

\[ \Gamma_{11} \equiv \Gamma^0 \ldots \Gamma^9 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \] (A.10)

We use the following notation \( \Pi = \Pi^\alpha_{\beta}, \Pi' = (\Pi')^\alpha_{\beta} \) where

\[ \Pi^\alpha_{\beta} \equiv (\gamma^1 \gamma^2 \gamma^3 \bar{\gamma}^4)_{\alpha \beta}, \quad (\Pi')^\alpha_{\beta} \equiv (\gamma^5 \gamma^6 \gamma^7 \bar{\gamma}^8)_{\alpha \beta}. \] (A.11)

Because of the relation \( \gamma^0 \bar{\gamma}^9 = \gamma^+ - \gamma^- \) the normalization condition (A.10) takes the form

\[ \gamma^+ - \Pi \Pi' = 1. \] (A.12)

We have the relations

\[ (\gamma^+)^2 = \Pi^2 = (\Pi')^2 = 1, \] (A.13)

\[ \gamma^+ \gamma^\pm = \pm \gamma^\pm, \quad \bar{\gamma}^\pm \gamma^+ = \mp \gamma^\pm, \quad \gamma^+ \bar{\gamma}^+ = \gamma^- \bar{\gamma}^- = 0, \] (A.14)

\[ \bar{\gamma}^+(\Pi + \Pi') = (\Pi + \Pi') \gamma^- = 0, \quad \bar{\gamma}^- (\Pi - \Pi') = (\Pi - \Pi') \gamma^+ = 0. \] (A.15)

The 32-component positive chirality \( \theta \) and negative chirality \( Q \) spinors are decomposed in terms of the 16-component spinors as

\[ \theta = \begin{pmatrix} \theta^a \\ 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 \\ Q_\alpha \end{pmatrix}. \] (A.16)

Instead of one complex Weyl spinor \( \theta \), we sometimes prefer to use two real Majorana-Weyl spinors \( \theta^1 \) and \( \theta^2 \) defined by
\[ \theta = \frac{1}{\sqrt{2}}(\theta^1 + i\theta^2), \quad \bar{\theta} = \frac{1}{\sqrt{2}}(\theta^1 - i\theta^2). \] (A.17)

We use the shorthand notation like \( \bar{\theta}\gamma^\mu\theta \) and \( \gamma^\mu\theta \) which should read as \( \bar{\theta}\gamma_\alpha^\mu\theta^\beta \) and \( \gamma_\alpha^\mu\theta^\beta \) respectively.

### Appendix B  Derivation of supercharges

Here we demonstrate how the knowledge of the kinematical charges and commutation relations of the superalgebra allows one to obtain the dynamical supercharges \( Q^- \) systematically. Before proceeding we write down the (anti)commutation relation of the basic algebra (2.10)-(2.13) in terms of the \( Q^+ \) and \( Q^- \) supercharges defined in (7.3). Using (7.3) in (2.10)-(2.13) we have

\[
[J^{ij}, Q^\pm_\alpha] = \frac{1}{2} Q^\pm_\beta (\gamma^{ij})^\beta_\alpha, \quad [J'^{ij'}, Q^\pm_\alpha] = \frac{1}{2} Q^\pm_\beta (\gamma'^{ij'})^\beta_\alpha
\] (B.1)

\[
[J^+I, Q^-_\alpha] = \frac{1}{2} Q^+_\beta (\gamma^+I)^\beta_\alpha
\] (B.2)

\[
[P^I, Q^-_\alpha] = i Q^+_\beta (\Pi^+I)^\beta_\alpha, \quad [P^-, Q^+_\alpha] = i Q^-_\beta \Pi^\beta_\alpha
\] (B.3)

\[
\{Q^+_\alpha, Q^-_\beta\} = -2i \gamma^\alpha_\beta P^+, \quad \{Q^+_\alpha, \bar{Q}^-_\beta\} = -i(\bar{\gamma}^-\gamma^+\gamma^I)_{\alpha\beta}P^I - (\bar{\gamma}^-\gamma^+\gamma^I\Pi)_{\alpha\beta}J^+I - (\bar{\gamma}^-\gamma^+\gamma^I\Pi')_{\alpha\beta}J'^+I, \quad \{Q^-_\alpha, Q^-_\beta\} = -2i \gamma^\alpha_\beta P^- + (\gamma^+\gamma^ij\Pi)_{\alpha\beta}J^ij + (\gamma^+\gamma'^ij\Pi')_{\alpha\beta}J'^ij.
\] (B.4)

We now let \( Q^-a \) be the conserved supercurrent whose component in the world-sheet time direction gives the supercharge \( Q^- = \int Q^-0 \). We start with the anzats

\[
Q^-0 = \mathcal{P}^I A^+_I\theta + x^I A^-_I\theta + \dot{x} B^I\bar{\theta},
\] (B.7)

where the coefficients \( A^+_I, A^-_I, \) and \( B^I \) are assumed to be independent of the dynamical variables \( x^I, \mathcal{P}^I, \theta, \bar{\theta} \). Using the commutator (B.2), first commutator in (B.3), and Poisson-Dirac brackets (6.27),(6.28) we obtain the following equations

\[
p^+ A^+_I \cos x^+ + A^-_I \sin x^+ = 2p^+ \gamma^I e^{ix^+\Pi},
\] (B.8)

\[-p^+ A^+_I \sin x^+ + A^-_I \cos x^+ = 2ip^+ \gamma^I \Pi e^{ix^+\Pi}.
\] (B.9)

The solution to these equations is found to be

\[
A^+_I = 2\gamma^I, \quad A^-_I = 2ip^+ \gamma^I \Pi.
\] (B.10)

Thus we have

\[
Q^-0 = 2\mathcal{P}^I \gamma^I \theta + 2ip^+ x^I \gamma^I \Pi \theta + \dot{x} B^I \bar{\theta}.
\] (B.11)
The conservation law for the supercurrent $Q^{-a}$

$$\partial_0 Q^{-0} + \partial_1 Q^{-1} = 0 \quad (B.12)$$

gives $B^I = -2\gamma^I$ and fixes the $Q^{-0}$ and $Q^{-1}$ to be

$$Q^{-0} = 2p^I \gamma^I \theta - 2\dot{x}^I \gamma^I \bar{\theta} + 2ip^+_x \gamma^I \Pi \theta , \quad (B.13)$$

$$Q^{-1} = \frac{2}{p^+_x} p^I \gamma^I \bar{\theta} - \frac{2}{p^+_x} \dot{x}^I \gamma^I \theta + 2ix^I \gamma^I \Pi \bar{\theta} . \quad (B.14)$$

The expression for $Q^{-0}$ leads to the supercharges $Q^-$ given by (7.24). Finally, we check that the supercharges satisfy all the remaining (anti)commutation relation of the basic superalgebra. The corresponding (anti)commutation relations can be obtained from the (B.1)-(B.6) by changing signs in anticommutators $\{..\} \rightarrow -\{..\}$ there.
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