SYNCHRONIZATION OF FIRST-ORDER AUTONOMOUS OSCILLATORS ON RIEMANNIAN MANIFOLDS

Simone Fiori
Dipartimento di Ingegneria dell’Informazione, Università Politecnica delle Marche
Via Brecce Bianche, I-60131 Ancona (Italy)

Abstract. The present research work recalls a control-theoretic approach to the synchronization of a first-order master/slave oscillators pair on $\mathbb{R}^3$ and extends such technique to the case of curved Riemannian manifolds. As theoretical results, this paper proves the asymptotic convergence of the feedback controller and studies the entity of the ‘control effort’. As a case study, the complete equations for the controller of a slave oscillator on the unit hypersphere $S^{n-1}$ are laid out and are illustrated by numerical examples for $n = 3$ and $n = 10$, even in the hypothesis of noisy master-system state measurement.

1. Introduction. The theory of coupled non-linear oscillators is an active research area in sciences, medicine and engineering [10]. Non-linear oscillators arise in the modeling of complex scientific phenomena, as in the management of spatio-temporally distributed sensor networks [13] and in smart grids applications [6].

An interesting scientific problem that has attracted a wide attention in the field of applied and computational mathematics is the synchronization of dynamical systems whose state-spaces are either $\mathbb{R}^n$ or $\mathbb{C}^n$ [4, 18, 13, 20, 6]. Synchronization is a rich phenomenon and a multi-disciplinary subject with a broad range of applications [2, 19]. To acquire information on this research topic from a broad context, readers might consult, e.g., [3, 15].

In the framework studied in [14], it is assumed that two non-linear oscillators, a master oscillator and a slave oscillator, both of the same kind, evolve at the same time but start with different initial conditions. Synchronization in a master-slave system aims at inducing the slave oscillator to change its dynamics in such a way that its state syncs with the state of the master oscillator. In the case of non-linear oscillators evolving on the state-space $\mathbb{R}^n$, a number of methodological solutions are available in the literature, while the problem of syncing non-linear oscillators whose state evolves on a curved manifold is still an open question.

The present paper aims at contributing to the theory of synchronization of a master-slave system whose state space is a curved Riemannian manifold. The present paper, in fact, generalizes the well-grounded synchronization theory, discussed in [5, 14], available for first-order master/slave oscillators pairs on $\mathbb{R}^3$, and

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extends it to curved Riemannian manifolds by means of key concepts in Riemannian geometry. In particular, the present paper contributes by setting up a control-theoretic approach to synchronize a slave dynamical system identical to a given first-order dynamical system evolving on a fairly-generic curved Riemannian manifold. A further contribution of this paper is a study about the speed of synchronization that results to be exponential according to a Lyapunov-function stability theory tailored to curved state-spaces. Such a general approach to synchronization may be applied to a number of scientific problems as, for instance:

- **synchronization of the attitude of a fleet of flying bodies**: the attitude of a rigid body (such as a drone) may be described by a matrix in the manifold SO(3) of special orthogonal matrices, therefore the synchronized trajectory of flying rigid objects in a fleet may be described in terms of dynamical systems and synchronization control laws formulated over the manifold SO(3);

- **secure transmission of information over digital networks**: transmission of information over digital networks is harmed by the relative easiness of interception; modern transmission techniques embody encryption algorithms that make transmitted signals hard to decipher; a possible encryption scheme relies on signal masking, which consists in ‘corrupting’ the transmitted signal by noise generated through a non-linear oscillator at the transmission stage: in order to decipher the signal, an identical oscillator at the receiver needs to be remotely synchronized to the oscillator at the transmitter; multichannel oscillators may be realized/synchronized through non-linear systems and controllers whose state space is a curved manifold.

The present author is currently pursuing these applications which will be discussed in a forthcoming scientific paper. In addition, the present paper contributes with an analysis of the behavior of the control effort necessary to synchronize the slave/master coupled subsystems.

The problem of synchronizing two or more dynamical systems (on manifolds) is tightly related to consensus optimization. As a reference discussing this problem in details, the paper [17] considers distributed consensus algorithms that involve a number of ‘agents’ whose state evolves on a connected compact homogeneous manifold. These agents communicate their relative state to one another according to a communication graph. The consensus problem is formulated in terms of a specifically designed cost function and is solved by means of gradient-based optimization algorithms to synchronize (by maximizing the consensus) or balance (by minimizing the consensus) the agents. The cost function is defined on the basis of a specific centroid definition on manifolds.

The present research paper is organized as follows: In Section 2, we review a control-theoretic approach to the synchronization of two non-linear oscillators on $\mathbb{R}^3$ described in [5, 14] and extend such control-theoretic design to the case of first-order autonomous systems whose state evolves on Riemannian manifolds. As a case study, in Section 3 we apply the devised theory to first-order dynamical systems whose state evolves on the unit hypersphere $\mathbb{S}^{n-1}$. Section 4 concludes the paper.

### 2. Control-theoretic approach to synchronization

The papers [5, 14] describe a synchronization approach of a slave oscillator with respect to a master oscillator based on the design of an ad hoc nonlinear control law. In particular, the master/slave oscillators are taken as various flows whose state evolves over the spaces $\mathbb{R}^3$ and $\mathbb{R}^4$. The controller is designed on the basis of the Lyapunov stability
theory and, remarkably, the speed of synchronization results to be exponential. We recall such control-theoretic synchronization technique in Subsection 2.1 and extend such approach to the case of Riemannian manifolds in Subsection 2.2.

2.1. Control-theoretic approach to synchronization on $\mathbb{R}^n$. In the $\mathbb{R}^n$ case, the master oscillator is an autonomous, first-order dynamical system described by

$$\dot{z}^m(t) = f(z^m(t)), \text{ with } z^m(0) = z^m_0 \in \mathbb{R}^n, \ t \geq 0,$$

where $z^m \in \mathbb{R}^n$ is the oscillator’s state variable and $f : \mathbb{R}^n \to \mathbb{R}^n$ denotes a state-transition operator. The slave oscillator is described by

$$\dot{z}^s(t) = f(z^s(t)) + u(t), \text{ with } z^s(0) = z^s_0 \in \mathbb{R}^n, \ t \geq 0,$$

where $z^s \in \mathbb{R}^n$ is the oscillator’s state variable, $f : \mathbb{R}^n \to \mathbb{R}^n$ denotes the same state-transition operator of the master oscillator, the initial state $z^s_0$ is supposed to differ from the initial state $z^m_0$, and the signal $u \in \mathbb{R}^n$ denotes the input from the controller that makes the slave oscillator sync asymptotically with the master oscillator. The master/slave/controller configuration is illustrated in the Figure 1.

![Figure 1. Master/slave/controller configuration: The feedback-type control chain is designed to make the slave oscillator sync asymptotically with the master oscillator. The symbol $\mathbb{M}$ denotes the state space of the master/slave pair which, in the classical setting, is $\mathbb{M} = \mathbb{R}^n$.](image)

In the contributions [5, 14], the controller is designed in order to reduce the magnitude of the error $e(t) := z^m(t) - z^s(t)$. The state of the master oscillator is taken as a reference signal and the controller is designed in order to make the controller-slave chain subsystem’s state converge asymptotically to the reference signal. The resulting control law and the corresponding synchronization speed are summarized in the following

**Theorem 2.1.** The control law $u := f(z^m) - f(z^s) + z^m - z^s$ synchronizes the slave oscillator (2) to the master oscillator (1) exponentially fast.

**Proof.** Let us define the function

$$V(t) := \frac{1}{2}e^T(t)e(t).$$ (3)
Its derivative with respect to the time $t$ reads

$$\frac{dV}{dt} = e^T de = (z^m - z^s)^T (\dot{z}^m - \dot{z}^s).$$  \hspace{1cm} (4)$$

By plugging-in the right-hand sides of the equations (1) and (2), one gets

$$\dot{V} = (z^m - z^s)^T[f(z^m) - f(z^s) - u].$$ \hspace{1cm} (5)

The key idea to design a synchronizing controller is to make sure that $V$ is a Lyapunov function for the controller-slave chain. This may be achieved by setting $u$ in such a way that the sum of terms $f(z^m) - f(z^s) - u$ in (5) equals $-(z^m - z^s)$, namely

$$u := f(z^m) - f(z^s) + z^m - z^s. \hspace{1cm} (6)$$

The function $V$ associated to such control law obeys

$$\dot{V} = (z^m - z^s)^T (z^s - z^m) = -e^T e = -2V,$$ \hspace{1cm} (7)

which implies that $V$ is a Lyapunov function for the controller-slave subsystem, which converges exponentially by the law $V(t) \propto \exp(-2t)$. Therefore, the error $\|z^m(t) - z^s(t)\|$ tends to zero as fast as $\exp(-2t)$.

### 2.2. Extension to a curved Riemannian manifold

The above-recalled design of a control strategy for a slave oscillator to sync to a master oscillator may be extended to the case of oscillators whose state evolves over curved manifolds by properly placing the systems equations and the above-recalled key idea within the frame of differential geometry. Up-to-date information about these kinds of dynamical systems may be acquired by readers, e.g., by consulting papers [8, 9].

We assume that the first-order oscillators evolve on a finite-dimensional Riemannian manifold $\mathbb{M}$ whose tangent bundle is denoted by $T\mathbb{M}$. The manifold plays the role of state-space of the oscillators, while the tangent bundle plays the role of phase-space. Since the purpose is to synchronize the trajectories of oscillators that start from different states over the manifold $\mathbb{M}$, in order not to restrict how far the initial states may be, the state-space is assumed to be a geodesically complete and path-connected Riemannian manifold (that is, the manifold has no boundary nor any singular point that can be reached in a finite time). The class of geodesically complete Riemannian manifolds appears as of prime importance in applications as some important examples of geodesically complete Riemannian manifolds are compact Riemannian manifolds, such as the unit hypersphere and the group of special orthogonal matrices (generalized rotations).

The fundamental mathematical device to derive differential-geometric operators on Riemannian manifolds is the Levi-Civita connection associated to an inner product on its tangent bundle [12, Chapter 5]. Then, the notions of geodesic arc, parallel transport, exponential map and its inverse logarithmic map, and Riemannian gradient follow naturally. The manifold exponential/logarithmic maps exchange nonlinear information in the manifold and linear information in its associated tangent bundle, while parallel translation connects different tangent spaces. In particular, the logarithmic map plays a crucial role in the design of the control law in this paper (a comprehensive study of the exponential/logarithmic maps may be found, e.g., in [11, 12]).

Given a point $z \in \mathbb{M}$, the tangent space to $\mathbb{M}$ at $z$ will be denoted as $T_z \mathbb{M}$. Moreover, the symbol $d : \mathbb{M}^2 \to \mathbb{R}_+$ denotes a Riemannian distance over the manifold $\mathbb{M}$ associated to an inner product $\langle \cdot, \cdot \rangle : (T\mathbb{M})^2 \to \mathbb{R}$. For a point $z \in \mathbb{M}$, the inner product of two tangent vectors $v, w \in T_z \mathbb{M}$ shall be written as $\langle v, w \rangle_z$. 

The inner product defines a local norm \( \|v\|_z := \sqrt{\langle v, v \rangle_z} \). Throughout this paper, the operator \( \log : M^2 \to TM \) denotes a manifold logarithmic map, namely, the inverse of the manifold exponential map associated to the metric (that is, for every \( z \in M \), the logarithmic map \( \log_z : M \to T_zM \) is the inverse of the exponential map \( \exp_z : T_zM \to M \)). The parallel transport of a tangent vector at a point \( x \in M \) to the tangent space at a point \( y \in M \) along the geodesic arc connecting \( x \) to \( y \) shall be denoted by \( P^x \to y : T_xM \to T_yM \).

The quantity \( \exp_z(v) \) associates to each tangent vector \( v \in T_zM \) the point of a Riemannian manifold \( M \) reached by the geodesic departing from \( x \) with tangent \( v \) after a unit time. The exponential map is a local diffeomorphism from \( 0 \in T_xM \) to \( M \) (meaning that it is a smooth map with local smooth inverse); its inverse, \( v := \log_z(y) \), for \( y \in M \), may be conceived as the shortest tangent vector \( v \in T_zM \) that ‘shoots’ a geodesic arc from the point \( x \) to the point \( y \). In fact, any such geodesic arc may be written as \( \exp_z(tv) \) and may be extended from \( t = 0 \) to \( t = t_v \), but not past \( t_v \). Whenever the limit value \( t_v \) is finite, the vector \( t_vv \in T_xM \) is termed tangential cut-point and \( \exp_z(t_vv) \) is termed cut point. A domain of injectivity \( \mathbb{D}_x \subseteq T_xM \) of the exponential map (i.e., a subset of tangent directions at a point \( x \) where the exponential map is one-to-one) may be extended up to the tangential cut locus \( \partial \mathbb{D}_x \). Such domain covers the whole manifold \( M \) except for the cut locus \( \exp_z(\partial \mathbb{D}_x) \) which, noticeably, has null Riemannian measure. A cut locus associated to a point \( x \) is (the closure of) the set of points where several minimizing geodesics starting from \( x \) will meet. On the sphere \( S^2 \), for instance, the cut locus of a point \( x \) distinct from itself is its antipodal point \(-x\). The same proviso holds for the parallel transport operator, in fact, given \((x, v) \in T^2M \) and a point \( y \in M \), to compute \( P^x \to y(v) \) it is necessary to find a geodesic arc connecting \( x \) to \( y \), which is uniquely determined only if \( y \notin \exp_x(\partial \mathbb{D}_x) \).

For a geodesically complete manifold, it holds that \( \mathbb{D}_x = T_xM \), that is, the exponential \( \exp_x(v) \) is defined for every \( x \in M \) and for every \( v \in T_xM \). In other terms, a Riemannian manifold is geodesically complete if and only if every geodesic can be extended indefinitely. The Hopf-Rinow theorem states that there always exists at least one minimizing geodesic between any two points of a geodesically complete path-connected manifold and that, therefore, the exponential map is surjective. However, an exponential map \( \exp_x \) is injective only in \( M - \exp_x(\partial \mathbb{D}_x) \), hence its inverse \( \log_x \) is defined only in the manifold deprived of the cut locus. A noticeable result is that, in \( M - \exp_x(\partial \mathbb{D}_x) \), the Riemannian gradient of the function \( d^2(x, y) \) with respect to \( y \) is \(-2 \log_x y \) [16].

In order to extend the control problem about synchronization to a Riemannian manifold, the master oscillator will be described by the equation

\[
\dot{z}^m(t) = f(z^m(t)), \quad \text{with } z^m(0) = z^m_0 \in M, \quad t \geq 0, \tag{8}
\]

where \( z^m \in M \) is the oscillator’s state variable and \( f : M \to TM \) denotes a state-transition operator. Likewise, the controlled slave oscillator will be described by

\[
\dot{z}^s(t) = f(z^s(t)) + u(t), \quad \text{with } z^s(0) = z^s_0 \in M - \exp_{z^s_0}(\partial \mathbb{D}_{z^s_0}), \quad t \geq 0, \tag{9}
\]

where \( z^s \in M \) is the oscillator’s state variable, \( f : M \to TM \) denotes the same state-transition operator of the master oscillator, the initial state \( z^s_0 \) is supposed to differ from the initial state \( z^m_0 \) and is assumed not to belong to the cut locus of
z_0^m$ (which will ensure the control law to be well-defined$^1$), and the signal $u \in TM$ denotes the driving signal from the controller. Notice that, when $M = \mathbb{R}^n$, it holds that $TM \cong \mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{R}^{2n}$, hence the present general setting trivializes to the case recalled in Subsection 2.1.

In words, the states of the master and of the slave oscillators vary in the curved manifold $M$, while the additive control signal lives in the tangent bundle (in fact, unlike in the flat-space case, on a manifold it is impossible to control the state of a system directly in the state space via an additive control signal): At a state $z^s$, the control signal $u \in T_z M$, henceforth, the control signal appears as a vector field $(z^s(t), u(t))$ over the tangent bundle parametrized by the time $t$.

In the present context, we may define the notion of synchronization as follows:

**Definition 2.2.** We say that the state $z^s$ of the slave oscillator synchronizes (asymptotically) to the state $z^m$ of the master oscillator if

$$\lim_{t \to \infty} d((z^s(t), z^m(t))) = 0.$$  \hspace{1cm} (10)

The main result of the present contribution is laid out in the following

**Theorem 2.3.** The control law $u := P^{z^m \to z^s}(f(z^m)) - f(z^s) + \log_{z^s}(z^m)$ synchronizes the slave oscillator (9) to the master oscillator (8) exponentially fast.

**Proof.** In order to design a tangent-bundle-controller, let us define the function

$$V(t) := \frac{1}{2}d^2(z^m(t), z^s(t)).$$  \hspace{1cm} (11)

On a geodesically complete Riemannian manifold, it holds that

$$d(x, y) = \|\log_y x\|_y.$$  \hspace{1cm} (12)

In order to compute the time-derivative of the function $V$, we may make use of the chain rule of derivation $\frac{d}{dt} F(x(t)) = (\nabla_x F, x)_x$, for $x \in M$ and $F : M \to \mathbb{R}$, and the expression of the Riemannian gradient $\nabla_x d^2(x, y) = -2\log_x y$ for $x, y \in M$ such that $y \notin \exp_x(\partial D_x)$. The derivative of the function $V$ with respect to the time $t$ now reads

$$\frac{dV}{dt} = \left\langle (-\log_{z^m}(z^s), z^m)_z, \dot{z}^s \right\rangle \cdot \frac{1}{2}d^2(z^m(t), z^s(t)),$$

$$= -\left\langle \log_{z^m}(z^s), f(z^m) \right\rangle_{z^m} - \left\langle \log_{z^s}(z^m), f(z^s) + u \right\rangle_{z^s},$$  \hspace{1cm} (13)

after plugging in the right-hand sides of the equations (8) and (9). Since the two scalar products in (13) are evaluated in different tangent spaces, it is convenient to move the calculations to the tangent space attached to the trajectory of the slave oscillator by means of the **parallel transport operator**. Since parallel transport is an isometry, the following important property holds:

$$\langle \log_{z^m}(z^s), f(z^m) \rangle_{z^m} = \langle P^{z^m \to z^s}(\log_{z^m}(z^s)), P^{z^m \to z^s}(f(z^m)) \rangle_{z^s}.$$  \hspace{1cm} (14)

Moreover, it holds that $P^{z^m \to z^s}(\log_{z^m}(z^s)) = -\log_{z^s}(z^m)$ because the $P$-operator and the log-operator are referred to the same geodesic arc connecting the endpoints and to the same metric (this argument is elaborated in the Appendix A). Therefore, the expression (11) may be recast as

$$\dot{V} = \langle \log_{z^s}(z^m), P^{z^m \to z^s}(f(z^m)) \rangle_{z^s} - \langle \log_{z^s}(z^m), f(z^s) + u \rangle_{z^s},$$

$$= \langle \log_{z^s}(z^m), P^{z^m \to z^s}(f(z^m)) - f(z^s) - u \rangle_{z^s}.$$  \hspace{1cm} (15)

$^1$The exclusion of the cut locus from the set of possible initial states makes the synchronization law of global flavor, with the exception of a null-measure subset of the state space $M$. 

We may now apply the same technique recalled in the Subsection 2.1: Design a synchronizing controller that ensures $V$ to be a Lyapunov function for the controller-slave chain. This may be achieved by setting $u$ in such a way that the sum $Pz^{m} \rightarrow z^{*}(f(z^{m})) - f(z^{*}) - u$ of tangent terms in (15) equals $-\log_{z^{*}}(z^{m})$, namely

$$u := Pz^{m} \rightarrow z^{*}(f(z^{m})) - f(z^{*}) + \log_{z^{*}}(z^{m}).$$

(16)

Since the inner product, in a Riemannian manifold, is positive-definite, the function $V$ associated to such control law obeys

$$V = -\langle \log_{z^{*}}(z^{m}), \log_{z^{*}}(z^{m})\rangle_{z^{*}} = -\| \log_{z^{*}}(z^{m})\|_{z^{*}}^{2} \leq 0,$$

(17)

which implies that $V$ is a Lyapunov function for the controller-slave subsystem. Notice that equality may hold only when $\log_{z^{*}}(z^{m}) = 0$, which happens only when the master and the slave oscillators are perfectly synchronized. Since $d(z^{m}, z^{*}) = \| \log_{z^{*}}(z^{m})\|_{z^{*}}$, the right-hand side of (17) coincides with $-2V$, hence the controller is again expected to converge exponentially by the law

$$V(t) = V_{0} \exp(-2t) \text{ with } V_{0} := \frac{1}{2}d^{2}(z^{m}_{0}, z^{*}_{0}).$$

(18)

Notice that if $\mathbb{M}$ is taken as $\mathbb{R}^{n}$ endowed with the standard Euclidean geometry, the parallel transport operator $P$ trivializes to the identity map, while the logarithmic map $\log_{x} y$ trivializes to $y - x$, for any $x, y \in \mathbb{R}^{n}$, therefore, for $\mathbb{M} = \mathbb{R}^{n}$, the control law (16) would trivialize to the control law (6).

The master/controller/slave chain equations read:

$$\begin{cases}
\dot{z}^{m} = f(z^{m}), \\
u = Pz^{m} \rightarrow z^{*}(f(z^{m})) - f(z^{*}) + \log_{z^{*}}(z^{m}), \\
\dot{z}^{*} = f(z^{*}) + u.
\end{cases}$$

(19)

The equation governing the controlled slave oscillator has a quite straightforward interpretation: The dynamics is governed by the state of the master oscillator, transported over the trajectory of the slave oscillator, represented by the terms $Pz^{m} \rightarrow z^{*}(f(z^{m}))$, and by the ‘difference’ between the state of the master and the state of the slave, embodied by the term $\log_{z^{*}}(z^{m})$.

Further remarks on the speed of synchronization are reported in the Appendix B which, furthermore, elaborates briefly the non-uniqueness of the control laws derivable by the expression (17).

2.3. Behavior of the control field upon synchronization. The magnitude of the control field (16) may be evaluated and it may be proven that, under mild conditions, it tends to zero upon synchronization. This result is made precise in the following

**Theorem 2.4.** If the function $\varphi(x, y) := \| P^{x} \rightarrow y(f(x)) - f(y)\|_{y}$, with $x \in \mathbb{M}$ and $y \in \mathbb{M} - \exp_{x}(\partial \mathbb{D}_{x})$, satisfies the condition that for any $x \in \mathbb{M}$ and for any $\varepsilon > 0$, there exists a value $\delta > 0$, independent of $x$, such that for any $y \in \mathbb{M} - \exp_{x}(\partial \mathbb{D}_{x})$ with $d(y, x) < \delta$ it holds that $\varphi(x, y) < \varepsilon$, then the norm $\| u \|_{z^{*}}$ of the control field tends to zero upon synchronization.

**Proof.** By the definition (16) of the control field $u(t)$, for any $t \geq 0$, we have that

$$\| u(t) \|_{z^{*}(t)} \leq \| Pz^{m}(t) \rightarrow z^{*}(t)(f(z^{m}(t)) - f(z^{*}(t)))\|_{z^{*}(t)} + \| \log_{z^{*}(t)}(z^{m}(t))\|_{z^{*}(t)}.$$  

(20)

The first term on the right-hand side may be rewritten as $\varphi(z^{m}(t), z^{*}(t))$, while the second term on the right-hand side, by virtue of the property (12), corresponds to
the master-slave states distance $d(z^m(t), z^s(t))$. Therefore, the inequality (20) may be rewritten as
\[
\|u(t)\|_{z^s(t)} \leq \varphi(z^m(t), z^s(t)) + d(z^m(t), z^s(t)).
\] (21)

Chosen an arbitrary value $\varepsilon > 0$, by virtue of the hypothesis on the function $\varphi$, there exists a value $\delta > 0$, possibly dependent on the value $\varepsilon$ but independent of the trajectory $z^m(t)$, such that, as long as $d(z^m(t), z^s(t)) < \delta$, it holds that $\varphi(z^m(t), z^s(t)) < \varepsilon$. According to Theorem 2.3, $d(z^m(t), z^s(t)) = d_0 e^{-t}$, where $d_0$ denotes the initial distance $d(z^m_0, z^s_0)$. Therefore, the condition $d(z^m(t), z^s(t)) < \delta$ is equivalent to the condition $t \geq -\ln(\delta(\varepsilon)/d_0)$, where ‘ln’ denotes natural logarithm. The inequality (21) is therefore equivalent to
\[
\|u(t)\|_{z^s(t)} < \varepsilon + d_0 e^{-t}, \text{ for } t \geq \ln(d_0/\delta(\varepsilon)).
\] (22)

As a result, we have that
\[
\lim_{t \to +\infty} \|u(t)\|_{z^s(t)} < \varepsilon,
\] (23)
for every $\varepsilon > 0$. Namely, the control effort $\|u(t)\|_{z^s(t)}$ becomes asymptotically smaller than any positive number, hence it tends to zero. \hfill \Box

It is worth remarking that the inequality (22) means that, waiting for a sufficiently long time, the control efforts become arbitrarily small, in a way that is completely independent of the trajectory in $\mathbb{M}$. This follows from the assumption that the function $\varphi(x, y)$ is continuous with respect to $y$ about $x$ uniformly. We underline that the properties of the function $\varphi$ depend on the geometry of the manifold $\mathbb{M}$ jointly with the structure of the master/slave systems encoded in the function $f$.

3. A case-study: Synchronization of first-order oscillators on the unit hypersphere. We now consider the special case that the manifold $\mathbb{M} = S^{n-1}$, namely, the unit-radius hyper-sphere in $n$ dimensions (for instance, the ordinary 3D sphere is $S^2$, where the superscript reminds us that, geometrically speaking, the ordinary sphere is a manifold of dimension 2).

The present numerical experiment is a case-study that was developed in the context of secure transmission of information over digital networks, as described in the Introduction. In particular, it refers to the encryption scheme based on multichannel signal masking: in this context, the signal $z^m(t) \in S^{n-1}$ denotes the masking noise generated – at the transmitter – by a non-linear oscillator with state-space $S^{n-1}$, while the signal $z^s(t) \in S^{n-1}$ denotes the unmasking noise generated – at the receiver – by an identical non-linear oscillator; the two oscillators need to achieve synchronization (remotely) in order for the masking/unmasking process to take place correctly.

Let us recall that
\[
S^{n-1} := \{x \in \mathbb{R}^n \mid x^T x = 1\}, \quad (24)
\]
\[
T_xS^{n-1} = \{v \in \mathbb{R}^n \mid x^T v = 0\}. \quad (25)
\]

An example of first-order system on $S^{n-1}$ embedded in $\mathbb{R}^n$ is
\[
\dot{z} = A(z \circ z \circ z) - zz^T A(z \circ z \circ z), \text{ with } z(0) = z_0 \in S^{n-1}, \ t \geq 0,
\] (26)
where $z$ denotes the $n \times 1$ state-vector, $A$ denotes a constant $n \times n$ matrix and $\circ$ denotes Hadamard (i.e., component wise) product. The state-transition operator is given by
\[
f(z) := (I_n - zz^T) A(z \circ z \circ z),
\] (27)
where $I_n$ denotes a $n \times n$ identity matrix. It is easy to prove that, for any $z \in S^{n-1}$, it holds that $z^T f(z) = 0$, therefore the dynamics of the system (26) takes place over the hypersphere. The Figure 2 illustrates an example of dynamics over the sphere $S^2$. The Figure illustrates a trajectory of the system, the values of the kinetic energy $K(t) := \frac{1}{2} f^T(z(t)) f(z(t))$ of the system and the temporal evolution of the three components of the state-vector $z$. The kinetic energy function takes a scalar value no matter how large is the state-manifold. When the dimension $n$ of the embedding space of the state-manifold $S^{n-1}$ exceeds 3, the state-trajectory generated by a non-linear system cannot get rendered graphically anymore, while the kinetic energy may always be represented graphically. Therefore, the kinetic energy may be taken as a scalar indicator which summarizes the behavior of a non-linear system that evolves on a high-dimensional manifold.

In order to apply the synchronization approach devised in Section 2, it is necessary to recall the expressions of the parallel transport operator and of the logarithmic map that appear in the control law (16). We assume that the manifold $S^{n-1}$ is endowed with the metric inherited from the Euclidean metric, therefore the necessary formulas read

$$\begin{align*}
\log_x(y) &= (I_n - xx^T)y(sinc d(x, y))^{-1}, \\
P^{x \rightarrow y}(v) &= I_n - \left[ \left( I_n - xx^T \right) yy^T - xx^T \right] v,
\end{align*}$$

where $P^{x \rightarrow y} : T_x M \rightarrow T_y M$ denotes again the parallel transport of a tangent vector at a point $x \in M$ to the tangent space at a point $y \in M$ along the geodesic arc.
connecting \( x \) to \( y \). In the above formulas, it is assumed that \( x^T y \neq -1 \) (namely, that \( y \) does not belong to the cut locus associated to \( x \)) and the symbol \( \text{sinc} : \mathbb{R} \to \mathbb{R} \) denotes the cardinal sine function, defined as:

\[
\text{sinc}(x) = \begin{cases} 
\sin \frac{x}{x} & \text{for } x \neq 0, \\
1 & \text{for } x = 0.
\end{cases}
\]

In order to measure the distance between the state vectors of the master oscillator and of the slave oscillator (or, equivalently, in the case that one needs to check the numerical behavior of the Lyapunov function), we can use the distance function associated to the standard inner product

\[
d(x, y) = |\arccos(x^T y)|,
\]

where ‘\( \arccos \)’ denotes the inverse cosine function.

The following subsections show numerical outcomes of the devised synchronization method by some cases study. The following numerical examples were chosen so as to illustrate the behavior of the devised theory in a clear and amenable way.

3.1. **Exact synchronization of first-order oscillators.** Hereafter, we take the dimension \( n - 1 \) of the hypersphere to be 2. For the sole purpose of numerical simulation, the complete master/controller/slave chain equations, obtained by plugging the control law (16) into the system equation (9), may be simplified to:

\[
\begin{cases} 
\dot{z}_m = f(z_m), \\
\dot{z}_s = P z_m \to z_s(f(z_m)) + \log z_s(z_m).
\end{cases}
\]

Notice that the term \( f(z_s) \) cancels out from the second equation while, in reality, the controller and the slave system are separate physical objects and their equations cannot be compound as shown.

As usual, the initial state of the master oscillator is denoted as \( z_{m0} \), and the initial state of the slave oscillator is denoted as \( z_{s0} \). The initial value \( z_{m0} \) belongs to the whole manifold \( S^2 \), while the initial value \( z_{m0}^\prime \) may belong to the sphere \( S^2 \) except for the cut locus \( \partial D z_{m0} \). As already mentioned, and as it may be directly verified from the formulas (28), such cut locus is \( \partial D z_{m0} = \{ -z_{m0}^\prime \} \), which corresponds to a geodesic distance (29) \( d(z_{m0}^\prime, -z_{m0}^\prime) = \pi \).

The nonlinear oscillator is (26) with

\[
A := \begin{bmatrix} 
0.2037 & 0.1907 & -1.0671 \\
-0.7120 & -2.2201 & 0.0799 \\
-0.2505 & -1.9182 & 0.2028
\end{bmatrix}.
\]

The Figure 3 illustrates an example of synchronization of a master oscillator and a slave oscillator whose state-space is the ordinary sphere \( S^2 \) (the timescale is in arbitrary units). It is readily appreciated how the state vectors sync in time exponentially fast. The Figure 4 illustrates the synchronizing trajectories over the ordinary sphere \( S^2 \). It is readily appreciated how, upon synchronization and in the absence of any perturbation to the system, the two subsystems stay synchronized.

The Figure 5 illustrates an example of synchronization of a master oscillator and a slave oscillator whose state-space is the unit hyper-sphere \( S^9 \). In this experiment, the \( 10 \times 10 \) matrix \( A \) was chosen randomly. It is again readily appreciated how state vectors sync in time exponentially fast. In this experiment, it was assumed that at \( t = 25 \) the state of the master oscillator changes abruptly (this simulates a sudden but temporary connection loss in the communication between the master
Figure 3. Example of synchronization of oscillators over the sphere $S^2$. The top panel on the left-hand side illustrates the components of the vector $z^m(t)$, while the bottom panel on the left-hand side illustrates the components of the state-vector $z^s(t)$. The top panel on the right-hand side illustrates the kinetic energies $K_s(t)$ and $K_m(t)$ of the oscillators. The bottom panel on the right-hand side illustrates the theoretical Lyapunov function versus the actual one.

Figure 4. Example of synchronization of oscillators over the sphere $S^2$. Synchronizing trajectories over the state-space. Black solid line: Master oscillator. Red solid line: Slave oscillator.

and the slave oscillator): In the figure, it may be readily appreciated how the slave oscillator quickly re-syncs to the master.
Figure 5. Example of synchronization of oscillators over the sphere $S^9$ subjected to temporary connection loss at $t = 25$. The top panel on the left-hand side illustrates the components of the vector $z^m(t)$, while the bottom panel on the left-hand side illustrates the components of the state-vector $z^s(t)$. The top panel on the right-hand side illustrates the kinetic energies $K_s(t)$ and $K_m(t)$ of the oscillators, from which it is quite apparent the behavior of the control chain during the simulated connection loss. The bottom panel on the right-hand side illustrates the theoretical Lyapunov function versus the actual one.

3.2. Synchronization of first-order oscillators under measurement uncertainties. Whenever the state of the master oscillator may not be ‘measured’ reliably, the synchronization of the slave oscillator is more difficult. It is worth testing the devised synchronization approach in the case that the master’s state is corrupted by measurement errors or by ‘disturbances’ introduced in the system on purpose. An example of the latter case is the signal masking utilized in the secure transmission of information [5].

In order to test the robustness of the synchronization approach against measurement errors in the master oscillator’s state variable, it would be desirable to mimic the familiar ‘additive disturbance’ [7] which, however, is not directly applicable to systems on manifolds. Such a scheme may be replaced by an exponentiated additive disturbance formulation. Namely, for a system evolving over a manifold $\mathbb{M}$, given a state signal $z(t) \in \mathbb{M}$, its observed version, $\zeta(t) \in \mathbb{M}$, may be defined as

$$\zeta(t) := \exp_{z(t)}(v(t)),$$

where $\exp : T\mathbb{M} \to \mathbb{M}$ denotes the exponential map and $v(t) \in T_{z(t)}\mathbb{M}$ is a tangent-bundle-valued disturbance.
In the case of the unit hyper-sphere $S^{n-1}$, whose tangent bundle is endowed with the Euclidean metric, the exponential map reads

$$\exp_x(v) = \begin{cases} x \cos(|v|) + v \text{sinc}(|v|) & \text{if } v \neq 0, \\ x & \text{if } v = 0 \end{cases}$$

with $x \in S^{n-1}$ and $v \in T_xS^{n-1}$. The equations of the master/controller/slave chain then become

$$\begin{cases} \dot{z}^m = f(z^m), \\ \dot{\zeta}^m = \exp_{z^m}(v), \\ u = Pcz^m \rightarrow z^s(f(\zeta^m)) - f(z^s) + \log(z^s(\zeta^m)), \\ \dot{z}^s = f(z^s) + u, \end{cases}$$

where $z^m(t)$ denotes again the (unobservable) state of the master oscillator, $\zeta^m(t)$ denotes a noisy observation of the state of the master oscillator corrupted by a noise component represented by the tangent field $v(t)$, $u(t)$ and $z^s(t)$ denote again the tangent control field and the state of the slave oscillator, respectively. In the numerical experiments, the formula used to generate a random disturbance in $T_xS^{n-1}$ was

$$v = r - (x^T r)x,$$

with $r \in \mathbb{R}^n$ being any random vector (e.g., any of its components is drawn from a zero-mean Gaussian distribution with variance $\alpha^2$).

The Figure 6 illustrates an example of synchronization of a master oscillator and a slave oscillator whose state-space is the unit sphere $S^2$ in the case that the state of the master oscillator is observed under a disturbance of standard deviation $\alpha = 0.1$. The slave-oscillator syncs to the master oscillator exponentially fast in spite of the disturbance affecting the observations of the master state. The Figure 7 illustrates the synchronizing trajectories over the ordinary sphere $S^2$ in the presence of a random disturbance of standard deviation $\alpha = 0.1$. It is readily appreciated how the disturbance is rather strong, yet the control signal is able to drive the slave system to synchronize to the master system.

4. Conclusion. In the present contribution, we have tackled the problem of synchronizing two first-order autonomous oscillators on Riemannian manifolds by a control-theoretic approach. The control law was derived by defining a Lyapunov function for the master/controller/slave closed loop, which was designed to be compatible with the structure of the state-manifold. The obtained synchronizing control strategy may be applied to any first-order oscillator on any geodesically complete path-connected Riemannian manifold and was illustrated by numerical examples tailored to the unit hypersphere, even in the presence of exponentiated-additive state disturbances.

We wonder if it would be perhaps possible to prove some bounds on the synchronization precision under appropriate bounds for the disturbances and by appropriate technical assumptions on the system [1]. Another question that is currently open is as to whether it is possible to envisage alternative control laws besides (16) that ensure $V$ to be a Lyapunov function for the master-slave-controller chain and that affects positively the control effort figure (for example, by minimizing it). These topics will be a matter of investigation in future endeavors.

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Figure 6. Example of synchronization of oscillators over the sphere $S^2$ in the presence of an exponentiated-additive disturbance. The top panel on the left-hand side illustrates the components of the observable state-vector $\zeta^m(t)$, while the bottom panel on the left-hand side illustrates the components of the state-vector $z^s(t)$. The top panel on the right-hand side illustrates the kinetic energies $K_s(t)$ and $K_m(t)$ of the oscillators. The bottom panel on the right-hand side illustrates the theoretical Lyapunov function versus the actual one.

in Pairs’ cooperation project titled “Study of a new mathematical optimization technique on Riemannian manifolds based on the chaotic properties of second-order dynamical systems”.

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Appendix A. Parallel translation of a logarithmic map. For the sake of completeness, we provide the proof\(^2\) of a Lemma utilized in the proof of Theorem 2.3.

**Lemma A.1.** Given any two points $x$ and $y$ on a connected Riemannian manifold $\mathcal{M}$, a logarithmic map $\log : \mathcal{M}^2 \to T\mathcal{M}$ and a parallel transport operator $P : T\mathcal{M} \times \mathcal{M} \to T\mathcal{M}$ along geodesics, it holds that $P_{x \to y}(\log_y y) = -\log_x x$.

\(^2\)I am indebted to Erchuan Zhang (UWA) for this proof.
Figure 7. Example of synchronization of oscillators over the sphere \( S^2 \) in the presence of an exponentiated-additive disturbance. Synchronizing trajectories (noisy-observable master state and slave state) over the state-space obtained when the master oscillator state vector is observed under an exponentiated-additive random disturbance of standard deviation \( \alpha = 0.1 \).

**Proof.** Let \( \gamma : [0, 1] \rightarrow M \) denote the geodesic from \( x \) to \( y \) and \( \rho : [0, 1] \rightarrow M \) the geodesic from \( y \) to \( x \). The geodesics are the same up to a reparameterization, namely, \( \rho(t) = \gamma(1 - t) \), for every \( t \in [0, 1] \). It holds that \( \log_x y = \dot{\gamma}(0) \) and \( \log_y x = \dot{\rho}(0) \). Therefore, we have that \( P_{x ightarrow y}(\log_x y) = P_{x ightarrow y}(\dot{\gamma}(0)) = \dot{\gamma}(1) = -\dot{\rho}(0) = -\log_y x \).

As an example, one may work out the case, often encountered in applications, that \( M \) is a real matrix Lie group. Denote by \( \gamma(t) = x\text{Exp}(t\omega) \) the geodesic arc connecting two points \( x,y \in M \), where \( \omega = \text{Log}(x^{-1}y) \). In the above formulas, \text{Exp} denotes the matrix exponential, while \text{Log} denotes the principal matrix logarithm. Then, it holds that \( \log_x y = \dot{\gamma}(0) = x\omega = x\text{Log}(x^{-1}y) \). Therefore, \( P_{x ightarrow y}(\log_x y) = \dot{\gamma}(1) = x\text{Exp}(\omega)\omega = y\text{Log}(x^{-1}y) = -y\text{Log}(y^{-1}x) = -\log_y x \).

**Appendix B. Remarks on the speed of synchronization.** The expression (16) of the control law results in the simplest equation for the controlled slave oscillator, and it has been proven to lead to an exponentially fast synchronization dynamics governed by the Lyapunov function \( V(t) \propto \exp(-2t) \).

The control law may be generalized to achieve faster (or slower) convergence by setting

\[
u := P^{z \rightarrow z'}(f(z^m)) - f(z^s) + \frac{c}{2} \log_{z^s}(z^m),
\]

with \( c > 0 \). Calculations show that the corresponding Lyapunov function will converge to zero as fast as \( \exp(-ct) \). However, growing the synchronization exponential
c will also grow the control effort $\|u\|_z$, which might be limited by technological or cost bounds. Hence, in practice, the control system inherently limits the synchronization speed.

It is worth mentioning that there exist ways to generalize the control law (16) that do not affect the synchronization speed and yet might possibly affect positively the control effort. One such way could be, for example, to add a transversal component to the control field, namely:

$$
\begin{align*}
\dot{z}_m &= f(z^m), \\
\tilde{u} &= P z^m \to z_s (f(z^m)) - f(z^s) + \log z_s (z^m) + \tau(z^s, z^m), \\
\dot{z}^s &= f(z^s) + \tilde{u},
\end{align*}
$$

(36)

with $\tau \in T_z \mathbb{M}$ such that $\tau(z^s, z^m) \perp \log z_s (z^m)$. One such choice would not affect the synchronization speed $\dot{V}$ at all, because $\langle \tau(z^s, z^m), \log z_s (z^m) \rangle_z = 0$, but would affect the control effort $\|\tilde{u}\|_z$. The additional transversal component would make the control law more complicated but it is possible that it could minimize the control effort. For example, on a sphere $\mathbb{S}^2$ embedded in $\mathbb{R}^3$, a way to generate such transversal component would be to set $\tau := \kappa z^s \times \log z_s (z^m)$, where $\kappa \in \mathbb{R}$ and $\times$ denotes vector cross product in $\mathbb{R}^3$.

The control problem may be further generalized even defining the function $V$ in a different way, namely, as

$$
V := \frac{1}{2p} d^2 (z^m, z^s), \quad p \in \mathbb{N}.
$$

(37)

Setting $p = 1$ takes us back to the framework studied in the Subsection 2.2. Setting $p \neq 1$ would lead, instead, to the equation $\dot{V} = -cpV$. This does not correspond, however, to a real acceleration of the synchronization dynamics, but rather to a different way to measure it.

REFERENCES

[1] S. Al-Azzawi, L. Jicheng and L. Xianming, Convergence rate of synchronization of systems with additive noise, *Discrete & Continuous Dynamical Systems - Series B*, 22 (2017), 227–245.

[2] A. Arenas, A. Díaz-Guilera, J. Kurths, Y. Moreno and C. Zhou, Synchronization in complex networks, *Physics Reports*, 469 (2008), 93–153.

[3] S. Boccaletti, J. Kurths, G. Osipov, D. L. Valladares and C. S. Zhou, The synchronization of chaotic systems, *Physics Reports*, 366 (2002), 1–101.

[4] I. Chueshov, P. E. Kloeden and Y. Meihua, Synchronization in couples sine-Gordon wave model, *Discrete & Continuous Dynamical Systems - Series B*, 21 (2016), 2969–2990.

[5] K. M. Cuomo, A. V. Oppenheim and S. H. Strogatz, Synchronization of Lorenz-based chaotic circuits with applications to communications, *IEEE Transactions on Circuits and Systems – Part II: Analog and Digital Signal Processing*, 40 (1993), 626–633.

[6] F. Dörfler, M. Chertkov and F. Bullo, Synchronization in complex oscillator networks and smart grids, in *Proceedings of the National Academy of Sciences*, 110 (2013), 2005–2010.

[7] Z. Cai, M. S. de Queiroz and D. M. Dawson, Robust adaptive asymptotic tracking of nonlinear systems with additive disturbance, *IEEE Transactions on Automatic Control*, 51 (2006), 524–529.

[8] S. Fiori, Nonlinear damped oscillators on Riemannian manifolds: Fundamentals, *Journal of Systems Science and Complexity*, 29 (2016), 22–40.

[9] S. Fiori, Nonlinear damped oscillators on Riemannian manifolds: Numerical simulation, *Communications in Nonlinear Science and Numerical Simulation*, 47 (2017), 207–222.

[10] J. M. González Miranda, *Synchronization and Control of Chaos*, Imperial College Press, London, 2004.

\[ ^3 \] This observation was suggested by one of the anonymous reviewers.
[11] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Volume I, Interscience Publishers, a division of John Wiley & Sons, New York-London, 1963.
[12] J. M. Lee, *Riemannian Manifolds*, Vol. 176 of Graduate Texts in Mathematics, Springer-Verlag, New York, 1997.
[13] T. E. Murphy, A. B. Cohen, B. Ravoori, K. R. B. Schmitt, A. V. Setty, F. Sorrentino, C. R. S. Williams, E. Ott and R. Roy, Complex dynamics and synchronization of delayed-feedback nonlinear oscillators, *Philosophical Transactions of the Royal Society A*, 368 (2010), 343–366.
[14] J. H. Park, Chaos synchronization of a chaotic system via nonlinear control, *Chaos, Solitons and Fractals*, 27 (2006), 1369–1375.
[15] L. M. Pecora and T. L. Carroll, Synchronization of chaotic systems, *Chaos: An Interdisciplinary Journal of Nonlinear Science*, (1996), 142–145.
[16] X. Pennec, Barycentric subspaces and affine spans in manifolds, *Geometric Science of Information, Lecture Notes in Comput. Sci.*, Springer, Cham, 9389 (2015), 12–21.
[17] A. Sarlette and R. Sepulchre, Consensus optimization on manifolds, *SIAM Journal on Control and Optimization*, 48 (2009), 56–76.
[18] J.-P. Yeh and K.-L. Wu, A simple method to synchronize chaotic systems and its application to secure communications, *Mathematical and Computer Modelling*, 47 (2008), 894–902.
[19] C. W. Wu, Synchronization in Complex Networks of Nonlinear Dynamical Systems, World Scientific Publishing Co. Pte. Ltd, 2007.
[20] X. Wu, C. Xu and J. Feng, Complex projective synchronization in drive-response stochastic coupled networks with complex-variable systems and coupling time delays, *Communications in Nonlinear Science and Numerical Simulation*, 20 (2015), 1004–1014.

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E-mail address: s.fiori@univpm.it