Relation between exponential behavior and energy denominators-Weak Coupling Limit

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Abstract

We show some interesting properties of tridiagonal and pentadiagonal matrices in the weak coupling limits. In the former case of this limit the ground state wave function amplitudes are identical to the Taylor expansion coefficients of the exponential function $e^{(-v/E)}$. With regards to transition rates a dip in the pentadiagonal case which is not present in the tridiagonal case is explained. An intimate connection between energy denominators and exponential behavior is demonstrated.

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1 Introduction

This work is a continuation of work done before on matrix models of strength distributions[1,2]. Matrix mechanics was of course introduced into quantum mechanics by Heisenberg[2] and Born and Jordan[3]. In the previous work we had a matrix in which the diagonal elements were $E_n = nE$ with $E=1$. We introduced a constant coupling $v$ which for a level $E_n$ occurs only with the nearest neighbors $E_{(n-1)}$ and $E_{(n+1)}$. The matrix is shown in Table 1. Note that the only relevant parameter is $v/E$. Here we extend the work to pentadiagonal matrices. In both cases we discuss for the first time the weak coupling limits for the ground state wave functions.
The eigenfunctions resulting from the diagonalizations of the above matrices are of the form \( \{a_0, a_1, a_2, \ldots, a_{10}\} \). We are interested in the values of these \( a_n \) in the limit where \( v/E \) is very small. (NB In previous publications we used the notation \( a_1 \) to \( a_{11} \)).

We ran Mathematica programs for small \( v \) to get an idea of the coefficients of the ground state eigenfunctions. For the tridiagonal case we used \( v/E=0.01 \). The results were as follows:

\[
\{0.99995, -0.00999, 0.0000499933, -1.6664*10^{-7}, 4.16592*10^{-10}, -8.33171*10^{-13}, 1.3886*10^{-15}, -1.98269*10^{-18}, 2.47958*10^{-21}, -2.75506*10^{-24}, 2.75504*10^{-27}\}
\]

These numbers can be put in a more suggestive way with a bit of rounding up.

\[
1, -v/E, (v/E)^2/2, -(v/E)^3/6, \ldots, (-1)^n (v/E)^n/n!
\]

We recognize these \( a_i \)'s as coefficients in the Taylor series of \( e^{-v/E} \).

To derive this result we should realize that to get \( a_n \) we have to go the \( n \)th order in perturbation theory. Let \( H=H_0+V \) with \( H_0 \) the diagonal part of the matrix and \( V \) the off diagonal.

\[
\Psi = \Phi + 1/(E_0-H_0)QV\Psi = (1+1/(E_0-H_0)Q V +1/(E_0-H_0) QV 1( E_0-H_0)Q \]

\section{The calculation}

The eigenfunctions resulting from the diagonalizations of the above matrices are of the form \( \{a_0, a_1, a_2, \ldots, a_{10}\} \). We are interested in the values of these \( a_n \) in the limit where \( v/E \) is very small. (NB In previous publications we used the notation \( a_1 \) to \( a_{11} \)). We ran Mathematica programs for small \( v \) to get an idea of the coefficients of the ground state eigenfunctions. For the tridiagonal case we used \( v/E=0.01 \). The results were as follows:

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where Q prevents the unperturbed ground state from being an intermediate state.

To get $a_n$ to the lowest power in $v/E$ we have to go in $n$ steps 0 to 1, 1 to 2, ... $n$-1 to $n$. In the numerator all the matrix elements $< (n+1) Q V_n >$ are the same, namely $v$. In the denominator we get $(E_0-E_1)....(E_0-E_n)$. Since we have $E_n = nE$ the denominator is $n!$. This proves the result and shows an intimate connection between exponential behavior and equally spaced levels.

We next consider the pentadiagonal case. This time we chose $v/E$ to be $10^{-10}$. The results are as follows:

$$\{1,-1.00006*10^{-10},-5.0*10^{-11}, 5.00091*10^{-21}, 1.25*10^{-21},-1.25071*10^{-31},-2.08333*10^{-32}, 2.08951*10^{-42}, 2.60417*10^{-43},-2.63376*10^{-53},-2.60417*10^{-54}\},$$

These numbers can be put in a more suggestive way with a bit of rounding up.

$$\{1,-v/E, -v/2E, (v/E)^2/2, (v/E)^2/8, -(v/E)^3/8 \ldots \}$$

We now have 2 types of non-vanishing matrix elements - one from $m$ to $(m+1)$ and another from $m$ to $(m+2)$. Both have a value $v$ so it is still the energy denominators which come into play. We now consider selected $a_n$:

$a_1$: We get $v/(E_0-E_1)=- v/E$.

$a_2$: In order to get a result linear in $v$ we have only the direct connection from 0 to 2 which yields a value $-1/2 v$

$a_3$: there are 2 paths:

A: 0 to 2 and 2 to 3

B: 0 to 1 and 1 to 3.

In the former case the value is $1/(2*3)$ and in the latter $1/3$. So we get $1/6+1/3=1/2$ - answer $-1/2 v$.

$a_4$: We look for $(v/E)^2$ terms. There is actually only one: 0 to 2 and 2 to 4. This gives a value $1(2*4)=1/8$-answer $1/8 \sqrt{(v/E)^2}$

$a_5$: There are 3 paths:

C: 0 to 1, 1 to 3, 3 to five: $-1/(1*3*5)$

D: 0 to 2, 2 to 3, 3 to 5: $-1/(2*3*5)$

E 0 to 2, 2 to 4, 4 to 5: $-(1/(2*4*5)$

sum = $-1/8$- answer $-1/8 \sqrt{(v/E)^3}$

And so it goes.

3 Results for $<nT_1(n+1)>$and $<nT_2(n+1)>$

In previous works we introduced 2 possible transition operators:

$<nT_1(n+1)>$ is a constant. We choose this constant to be one.

$<(n+1)T_2 n>$ is equal to $\sqrt{(n+1)}$

In both cases all other transition elements are taken to be zero.

The strength matrix element between a state $\{a\}$ and a state $\{b\}$ for $T_1$ is simply

$(a_0 b_1+...+a_9 b_{10}) + (b_0 a_1+...+b_9 a_{10}) <n T_1(n+1)>$ with the last factor taken to be unity. (1). For $T_2$ we take the previous expression and multiply the $a_n b_{(n+1)}$ by $\sqrt{(n+1)}$-likewise $b_n a_{(n+1)}$.
In Fig.1 and 2 we show results for $T_{1}$, $v=0.1$ for the tridiagonal and pentadiagonal cases. In Fig 3 and 4 we show corresponding results for $T_{2}$. We see a near exponential decrease of the strength in the the tridiagonal case with $T_{1}$. This shows up on our log plot as a near straight line with a negative slope, however the last point (the 10th excited state) is much lower than the exponential projection. This has been discussed in [1,2] where it was shown that for $T_{1}$ the value is actually minus infinity.

It should be pointed out that, although, as seen in Fig. 1 for $v=0.1$ the tri curve for $T_{1}$ looks exponential and this persists until about $v=2$, this is not the case for extremely small $v$ or extremely large $v$. In Table 3 we show what happens for very small $v$. In the weak coupling limit the $T_{1}$ matrix element $O$ can be shown to be of the form $O(1 \rightarrow n) = (v/E)^{m} A(1 \rightarrow n)$ where $A$ is independent of $v/E$. We define $g_{k}=1/k!$ and give the values of $A$ in the 3rd column of the table. In the 4th column of Table 2 we give the value of $\ln(O^{2})$ using the expressions in the second and third columns i.e. keeping only the lowest powers of $v/E$. We do this for $v/E=0.0001$. The behavior is somewhat complicated. The results are not monotonic in $m$. For example although $m$ increases from $n=1$ to 5, there is a decease in going from $n=5$ to $n=6$. Thus we get a non-exponential behavior with the value of $\ln(O^{2})$ being larger for $n=6$ than for $n=5$. In the 5th column we show the exact results for $v=0.0001$. They are reasonably close to the approximate expressions in column 4.

As a counterpoint we show in the last column the exact results for $v=0.1$. The are reasonably linear in $n$ indicating a good exponential behavior.

### Table 3 Weak Coupling expressions for the $T_{1}$ amplitude and $\ln(O^{2})$

| n  | $A(1 \rightarrow n)$ | $\ln(O^{2})|v=0.0001|$, lowest power of $v$ | $v=0.0001$ exact | $v=0.1$ exact |
|----|----------------------|--------------------------------------------|-----------------|--------------|
| 1  | 0                    | 0                                          | $-2 \times 10^{-8}$ | $-0.020$    |
| 2  | $g_{2} x$           | $-56.07$                                   | $-57.46$        | $-16.06$    |
| 3  | $g_{3} x^2 + g_{4} x^3$ | $-118.75$                               | $-120.46$       | $-35.95$    |
| 4  | $g_{4} x^4$         | $-161.46$                                  | $-163.85$       | $-42.26$    |
| 5  | $g_{5} x^5$         | $-195.67$                                  | $-198.68$       | $-44.14$    |
| 6  | $g_{6} x^6$         | $-219.48$                                  | $-220.46$       | $-45.27$    |
| 7  | $g_{7} x^7 + g_{8} x^8$ | $-140.46$                               | $-141.52$       | $-60.57$    |
| 8  | $g_{8} x^8$         | $-161.46$                                  | $-161.46$       | $-70.05$    |
| 9  | $g_{9} x^9$         | $-195.67$                                  | $-195.67$       | $-101.67$   |

For very large $v$ there is an even-odd effect resulting in two exponential behaviors-one for even $n$ and one or odd $n$. This is not shown here but is displayed in great detail in ref [2]. For the $T_{2}$ we get a somewhat simpler behavior in the weak coupling limit. The amplitude $O(1 \rightarrow n) = (v/E)^{(n-1)} B(1 \rightarrow n)$. We find $\ln(O^{2}) = 2(n-1) \ln(v/E)+\ln(B^2)$. The values of $\ln(B^2)$ from 1 to 10 are respectively

$\{0, -1.763, -6.065, -9.721, -13.473, -9.536, -9.137, -10.423, -12.606, -14.929\}$

Note the linear behavior of the first term which supports an exponential behavior of $O^{2}$. As $v$ goes to zero the first term becomes very large and negative and so the $B$ terms can be neglected. In that case we get a perfect linear dependance on a log plot which implies perfect exponential behavior.
In Fig. 2 we show corresponding results for the pentadiagonal case. There is one significant difference between tri and penta for $T_1$. As seen in Fig 2 in the transition from the ground state to the second excited state there is a big dip for the pentadiagonal case. This is not the case for the tridiagonal case. After the dip in the pentadiagonal case one gets a near exponential behavior.

To explain the dip we examine the second excited state for the pentadiagonal case in the weak coupling limit. Recall that for the ground state the values of \( \{a_0, a_1, a_2\} \) are respectively \( \{1,-v/E,-v/(2E)\} \). For the second excited state the values of \( \{b_0,b_1,b_2\} \) can be shown to be \( \{v/(2E), v, 1\} \). For the $T_1$ case note that $a_0 b_1 = v/E$ while $a_1 b_2 = -v/E$ so we have complete cancellation of the leading terms.

To get \( \{b_0,b_1,b_2\} \) we note that clearly $b_2 = 1$ in the weak coupling limit. For $b_0$ we go from state 2 to state 0 so that we get a contribution (from the energy denominator) $v/2E$ i.e. $(E_2-E_0)=2$. For $b_1$ we go from state 2 to state 1 and so we get $v/E$ i.e. $(E_2-E_1)=1$.

For the $T_2$ case, figs 3 and 4, the dip is not as pronounced. Because of the factor $\sqrt{(n+1)}$ there is only a partial cancellation.

We had previously discussed the strong coupling limit for $T_1 [1,2]$. In that
case all transitions vanished. This was explained by the fact that in the strong coupling limit the transition matrix becomes identical to the Hamiltonian.

4 All v’s.

We briefly consider the case where on the diagonal we still have nE but all other matrix elements are v (there are no zeros). One can easily show that the ground state column vector \( \{a_0, a_1, \ldots, a_{10}\} \) in the weak coupling limit is \( \{1, -v/E, -v/(2E), -v/(3E), \ldots, -v/(10E)\} \).

There are other matrix models which address problems related to, but different from, what we have here considered. As previously mentioned, [1] Bohr and Mottelson [5] used matrix models to derive the Breit-Wigner formula for a resonance. Brown and Bolsterli described the giant dipole resonances in nuclei in a schematic model using a delta interaction[6]. In that work they made the approximation that certain radial integrals were constant. Abbas and Zamick [7] removed this restriction. Generally speaking matrix models are very useful for casting insights into the physics of given problems where the more accurate but involved calculations fail.

Although the models here are not geared to fit specific experiments we do keep our eyes on the empirical data. In refs [8-11] we cite works in which exponential behavior is seen and explained. Here we show that with unperturbed equally spaced energy levels we generate factorials which are needed to get exponential behavior. These come from the energy denominators. Also, our perturbations are simple enough that we can easily perform nth order perturbation theory for any n and get analytic results in the weak coupling limit.

5 Appendix

In table 4 we give all the wave functions for the tridiagonal case in the weak coupling limit. Let \( u = v/E \).

Table 4 Wave function components for the tridiagonal case in terms of \( u = v/E \) weak coupling limit.
\begin{tabular}{cccccccccccc}
\hline
$a_0$ & $a_1$ & $a_2$ & $a_3$ & $a_4$ & $a_5$ & $a_6$ & $a_7$ & $a_8$ & $a_9$ & $a_{10}$ \\
\hline
1 & -u & $u^2/2!$ & -$u^3/3!$ & $u^4/4!$ & -$u^5/5!$ & $u^6/6!$ & -$u^7/7!$ & $u^8/8!$ & -$u^9/9!$ & $u^{10}/10!$
\hline
\end{tabular}

\begin{tabular}{cccccccccccc}
\hline
$u^2/2!$ & u & 1 & -u & $u^2/2!$ & -$u^3/3!$ & $u^4/4!$ & -$u^5/5!$ & $u^6/6!$ & -$u^7/7!$ & $u^8/8!$ & -$u^9/9!$ & $u^{10}/10!$
\hline
\end{tabular}

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