WEAKLY LINDELOF DETERMINED BANACH SPACES NOT CONTAINING $\ell^1(\mathbb{N})$

by

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ABSTRACT: The class of countably intersected families of sets is defined. For any such family we define a Banach space not containing $\ell^1(\mathbb{N})$. Thus we obtain counterexamples to certain questions related to the heredity problem for W.C.G. Banach spaces. Among them we give a subspace of a W.C.G. Banach space not containing $\ell^1(\mathbb{N})$ and not being itself a W.C.G. space.

INTRODUCTION In the present paper we deal with Banach spaces not containing isomorphically the space $\ell^1(\mathbb{N})$. The motivation for this study was a problem, posed to us by S. Merkourakis, related to the heredity problem for weakly compactly generated (W.C.G.) Banach spaces. The problem in question is the following: Is every W.C.G. Banach space $X$ not containing $\ell^1(\mathbb{N})$ hereditarily W.C.G.? It is well known by a classical example due to Rosenthal [R] that the heredity problem for W.C.G. has negative answer. On the other hand, M. Fabian has shown in [F] that if the conjugate $X^*$ satisfies the Radon-Nikodym property (R.N.P.) then the space $X$ is hereditarily W.C.G. Actually M. Fabian proved the stronger result that if $X$ is weakly countably determined (W.C.D.) and $X^*$ satisfies RNP then $X$ is W.C.G. space. Later M. Valdivia in [V] extended this result to the class of weakly
Lindelöf determined (W.L.D.) Banach spaces. (For the definitions we use here and related results we refer the reader to [A-M-N], [A-M], [M-N] and [D-G-Z]). For alternative proofs of these results we refer the reader to [O-V-S] or [S] where the topological analogue of the above results is also proved; that is, that every Corson compact set which is also Radon-Nikodym set is Eberlein compact.

A class extending the W.L.D. Banach spaces with conjugate space satisfying the RNP is W.L.D. spaces not containing \( \ell^1(\mathbb{N}) \) and it is natural to ask for the solutions of the same problems for this wider class. Our aim is to give counterexamples to most of these questions. Thus in the first section we introduce the countably intersected families of sets. This class of families includes the family of segments of the dyadic tree and more generally the segments of any tree whose each branch is a countable set. The most interesting example of such a family is Recničenko’s space, which is presented in detail in the first section of the paper.

In the second section we introduce the James norm for a family of sets, which is the natural extension of James’ norm for James’ tree space [J]. We prove that if the family is countably intersected, then the resulting Banach space \( X \) does not contain \( \ell^1(\mathbb{N}) \). From this we conclude that there exists a weakly \( \mathcal{K} \)-analytic space \( X \) not containing \( \ell^1(\mathbb{N}) \) such that \( X \) is not a subspace of a W.C.G. Banach space as well as a W.L.D. Banach space \( X \) not containing \( \ell^1(\mathbb{N}) \) and not being W.C.D. space. Thus we cannot extend the results of Fabian and Valdivia to the class of Banach spaces that do not contain \( \ell^1(\mathbb{N}) \).

The third section of the paper contains certain results related to subspaces of W.C.G. spaces. By Rosenthal’s example there exists a probability measure \( \mu \) such that the space \( L^1(\mu) \) has a subspace which is not W.C.G.. Here we introduce the quasi-Eberlein sets and with each one of them we associate a Banach space \( X \) which is a subspace of a W.C.G. space but it is not itself a W.C.G. space. We thus produce counterexamples to the heredity problem for W.C.G. Banach spaces which are different from Rosenthal’s space. It should be noted that Merkourakis has recently shown that under MA every weakly \( \mathcal{K} \)-analytic Banach space \( X \) with \( \dim X < 2^\omega \) is a W.C.G. space. Thus it is not possible to find a Z.F.C. counterexample to the heredity problem for W.C.G. spaces of dimension \( \omega_1 \). Finally we give a Banach space \( X \) not containing \( \ell^1(\mathbb{N}) \) which is a subspace of a W.C.G. Banach space but it is not a W.C.G. space.
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1 Countably intersected families

This section is devoted to the definition and certain examples of countably
intersected families.

1.1. Definition: A family \( S \) of subsets of a set \( \Gamma \) is said to be countably
intersected (C.I.) if the following conditions are satisfied.

(a) \( S \) is a pointwise closed family of countable subsets of \( \Gamma \) containing all
singletons.

(b) For every \( s, t \) in \( S \), \( s \setminus t \) is the finite union of pairwise disjoint elements
of \( S \).

(c) For every \( t \) in \( S \) there exists \( s_t \) in \( S \) containing \( t \) and such that for
every \( s, t_1, \ldots, t_n \) in \( S \) there exists \( t \) in \( S \), \( t \subset s \setminus \bigcup_{i=1}^n s_{t_i} \) and

\[
s \setminus \left( \bigcup_{i=1}^n s_{t_i} \cup t \right)
\]

is a finite set.

(d) For every \( s \) in \( S \) the set \( L_s = \{ s \cap t : t \in S \} \) is countable.

1.2. Remarks: (i) Condition (a) in Definition 1.1 refers to the pointwise
closeness of the family \( \{ \chi_s : s \in S \} \) as a subset of \( \{0, 1\}^\Gamma \). Since every \( s \) in
\( S \) is a countable set, \( S \) is a Corson compact set ([A-M-N]).

(ii) Easy examples of C.I. families are pointwise closed families containing
only finite sets. These are called strong Eberlein compact. We are interested
in C.I. families lying beyond the class of Eberlein compact sets. Next we will
present certain such examples.
1.3. **Lemma:** If \( \{s_i\}_{i=1}^n \) is a subfamily of a C.I. family \( S \), there exists a pairwise disjoint subfamily \( \{t_j\}_{j=1}^l \) of \( S \) such that each \( t_j \) is contained in some \( s_i \) and \( \bigcup_{i=1}^n s_i = \bigcup_{j=1}^l t_j \).

**Proof.** We prove it by induction on the length \( n \) of the family \( \{s_i\}_{i=1}^n \).

For \( n = 1 \) it is clear. Suppose that \( \bigcup_{i=1}^n s_i = \bigcup_{j=1}^l t_j \) and \( \{t_j\}_{j=1}^l \) satisfies the properties listed in the statement of the lemma. Then for a given \( s_{n+1} \),

\[
\bigcup_{i=1}^{n+1} s_i = \left( \bigcup_{j=1}^l t_j \right) \cup s_{n+1} = \bigcup_{j=1}^l t_j \cup \left( \cdots \left( (s_{n+1} \setminus t_1) \setminus t_2 \right) \cdots \setminus t_l \right).
\]

Since \( S \) is a C.I. family, condition (b) in Definition 1.1 allows us to replace \( s_{n+1} \setminus t_1 \) by a disjoint union of elements of \( S \), say \( \bigcup_{j=1}^l t_j \). Then \( (s_{n+1} \setminus t_1) \setminus t_2 = \bigcup_{j=1}^l (t_j \setminus t_2) \) and again we can replace \( t_j \setminus t_2 \) by a disjoint union of elements of \( S \). Continuing in this manner, we write \( s_{n+1} \setminus \bigcup_{j=1}^l t_j \) as a disjoint union of elements of \( S \).

Q.E.D.

**Examples of countably intersected families**

Since all the examples we will give below are related to trees, we briefly recall some basic definitions and notation for trees.

A **tree** is a partially ordered set \((T, \leq)\) such that for \( \alpha \in T \) the set \( \{\beta \in T : \beta < \alpha\} \) is well ordered. The **segments** of \( T \) are defined as follows; for \( \beta \leq \alpha \) the segment \([\beta, \alpha]\) consists of all \( \gamma \in T \) such that \( \beta \leq \gamma \leq \alpha \). The segment \([\beta, \alpha)\) is defined similarly. If \( T \) has a unique minimal element \( \alpha_0 \), an **initial segment** is a segment of the form \([\alpha_0, \alpha)\) with \( \alpha \in T \). A **branch** is any maximal segment. Finally, **tail** is any segment of the form \( b \setminus s \), where \( b \) is a branch and \( s \) an initial segment contained in \( b \).

**A. The C.I. family for the dyadic tree**

The dyadic tree is the set

\[
\Delta = \bigcup_{n=0}^{\infty} \{(n, k) : k = 0, 1, \ldots, 2^n - 1\}
\]

ordered by the relation \((n, k) < (n+1, l)\) if and only if \( l = 2k \) or \( l = 2k + 1 \).

It is easy to see that the family \( S \) of all branches and all tails of \( \Delta \) defines a countably intersected family. To check condition (c) of Definition 1.1 one should use as \( s_t \) the set \( t \).
B. The C.I. family for Todorcević tree

Consider a stationary subset $A$ of $\omega_1$ ($\omega_1$ is the first uncountable ordinal number) such that $\omega_1 \setminus A$ is also stationary. We set $T$ the set of closed subsets of $A$. Recall that a subset $K$ of $A$ is closed if and only if it is a closed subset of $\omega_1$ with the topology induced by the order. We define a partial order on $T$ by the rule: $K_1 \leq K_2$ if and only if $K_1$ is an initial segment of $K_2$. Since, by definition, every stationary subset of $\omega_1$ meets every closed uncountable subset of $\omega_1$ and $A$, $\omega_1 \setminus A$ are both stationary, we have that every branch of $T$ is countable. The countably intersected family for Todorcević tree is the family $S$ of all segments of $T$. Again, to check that this family is a C.I. family is straightforward, the only point we want to note is that for $t$ in $S$, the $s_t$ appearing in condition (c) of Definition 1.1 is any initial segment containing $t$.

Remark. As we defined the C.I. family for Todorcević tree we could define a C.I. family for any tree every branch of which is countable.

We turn now to Recničenko's space which is more complicated than the previous examples. We proceed to a detailed presentation of it since, to the best of our knowledge, there is no English reference for this example. A brief presentation is given in [Ny].

B. The C.I. family for Recničenko's space

We set $\Gamma = \omega_1 \times 2^\omega$, where $\omega_1$ and $2^\omega$ are considered as ordinal numbers. There exists a sequence $(A_n)_{n \in \mathbb{N}}$, where each $A_n$ is a subset of $\Gamma$, satisfying the following properties.

(a) $A_n$ is a tree of height $\omega$ and, if we denote by $A^k_n$ the (incomparable) elements of $A_n$ that belong to level $k$ of the tree, the following properties are fulfilled.

(i) $A^0_n = \{(0, n)\}$.

(ii) If $(\xi, t) \in A^k_n$ then the set $(\xi, t)^+$ of the immediate successors of $(\xi, t)$ has cardinality $2^\omega$ and every $(\zeta, t)$ in $(\xi, t)^+$ satisfies the condition $\xi < \zeta$.

(iii) For every $(\xi, t)$ in $A_n$, $\xi > 0$, there exists exactly one $(\zeta, s)$ in $A_n$ with $(\xi, t) \in (\zeta, s)^+$.
(b) The trees $A_n$, $n < \omega$, are connected by the following properties.

(i) If $A_n$ is an initial segment of $A_m$, $A_n$ is an initial segment of $A_m$ and $n \neq m$ then $#A_n \cap A_m \leq 1$.

(ii) For every $I$ infinite subset of $\mathbb{N}$, $A_n$ initial segments of $A_m$, $n \in I$, such that $A_n \cap A_m = \emptyset$ there exist uncountably many $(\xi, t)$ in $\Gamma$ such that each $(\xi, t)$ extends $A_n$ for all $n$ in $I$.

Construction of the trees $(A_n)_{n \in \mathbb{N}}$

Each $A_n$ will be realized as $\bigcup_{\xi < \omega_1} A^\xi_n$ with $A^\xi_n$ satisfying the following inductive properties.

(i) $A^\xi_n$ is a tree and $A^\xi_n \subset [0, \xi] \times 2^\omega$.

(ii) For $\zeta < \xi < \omega_1$, $A^\zeta_n$ is a subtree of $A^\xi_n$.

(iii) If $\xi = \zeta + 1$, $I$ is an infinite subset of $\mathbb{N}$ and, for $n \in I$, $A_n$ is an initial segment of $A^\zeta_n$ such that $A_n \cap A_m = \emptyset$ then there exists $t < 2^\omega$ such that $(\xi, t)$ extends $A_n$ for all $n$ in $I$.

(iv) If $A_i$ is an initial segment of $A^\xi_i$, $i = n, m$, $n \neq m$, then $#A_n \cap A_m \leq 1$.

It is clear that the inductive properties listed above imply that the family $\{A_n\}_{n < \omega}$ satisfies the desired properties.

The inductive construction

We set $A^0_n = \{(0, n)\}$.

If $\xi$ is a limit ordinal and $A^\zeta_n$ has been constructed for all $\zeta < \xi$, we set $A^\xi_n = \bigcup_{\zeta < \xi} A^\zeta_n$.

If $\xi = \zeta + 1$ and $A^\zeta_n$ has been constructed, we consider all sequences $K = (A_n)_{n \in I}$ such that $I$ is an infinite subset of $\mathbb{N}$, $A_n$ is an initial segment of $A^\zeta_n$ and, for $n \neq m$, $A_n \cap A_m = \emptyset$. The cardinality of all such $K$ is $2^\omega$ and we order them as $\{K^\xi_t\}_{t < 2^\omega}$. We define $A^\xi_n$ as follows.

$$A^\xi_n = A^\zeta_n \cup \{(\eta, t)^+ : (\eta, t) \in A^\zeta_n\},$$

where $(\eta, t)^+ = \{(\xi, t') : \text{there exists } A \in K^\xi_t, A \subset A_n \text{ and } \max A = (\eta, t)\}$.

We order $A^\xi_n$ in the obvious way.
Thus the inductive construction is complete.

**Reecničenko’s space** is the following family.

\[ R = \{ s : s \text{ is a segment for some } A_n \}. \]

It is easy to check that \( R \) is a family of countable subsets of \( \Gamma \) which is countably intersected.

1.4. **Proposition:** \( R \) is a Talagrand compact set which is not Eberlein compact.

The proof will follow from the next two lemmata.

Before we state the first lemma we need some notation. We denote by \( FS(\mathbb{N}) \) the (countable) set of all finite sequences of natural numbers. Also, for \( d \in \mathbb{N}^\mathbb{N} \), we denote by \( dn \) the first \( n \) terms of the sequence \( d \).

1.5. **Lemma:** There exists a partition \( \{ \Gamma_\phi : \phi \in FS(\mathbb{N}) \} \) of \( \Gamma \) such that for every \( d \in \mathbb{N}^\mathbb{N} \) and every \( (\xi_n, t_n) \in \Gamma_\phi \) the set \( \{(\xi_n, t_n)\}_{n \in \mathbb{N}} \cap s \) is finite for all \( s \) in \( R \).

It is well known that every pointwise closed family satisfying the conclusion of the lemma defines a Talagrand compact set (see [A-N]).

**Proof.** For \( n \in \mathbb{N}, k = 0, 1, 2, \ldots \), we define \( \Gamma^{n,k} = \{(\xi, t) : (\xi, t) \in \mathcal{A}_n^k\} \) (recall \( \mathcal{A}_n^k \) is the \( k \)th level of \( \mathcal{A}_n \)) and for \( \phi \in FS(\mathbb{N}), \phi = (n_1, \ldots, n_s), \) we set

\[ \Gamma_\phi = \bigcup_{i=1}^s \Gamma^{i,n_i}. \]

It is easy to check that \( \{ \Gamma_\phi : \phi \in FS(\mathbb{N}) \} \) satisfies the desired properties. \( \square \)

The following lemma will show that \( R \) is not an Eberlein compact set. Indeed, if a totally disconnected set is an Eberlein compact set, as S. Merkourakis noticed, the algebra of clopen subsets is a \( \sigma \)-weakly compact set. This follows from Rosenthal’s characterization of the Eberlein compacta [R]. Therefore, if \( R \) were an Eberlein compactum and we considered \( R \) as a subset of \( \{0, 1\}^\Gamma \), we could find a partition of \( \Gamma \) into a countable family \( \Gamma_n \) such
that for every \( \{(\xi_k, t_k)\}_{k \in \mathbb{N}} \subset \Gamma_n \) there exists a subsequence \( \{\Gamma(\xi_k, t_k)\}_{k \in M} \) converging pointwise to zero. But this contradicts the conclusion of the next lemma. For alternative proofs see [So], [Fa].

1.6. Lemma: For every partition \((D_i)_{i \in \mathbb{N}}\) of the set \( \Gamma \) there exists \( M \in \mathbb{R} \) and \( i_0 \in \mathbb{N} \) such that \( M \cap D_{i_0} \) is an infinite set.

**Proof.** We define the following set

\[
L = \{ i \in \mathbb{N} : \text{there exists an infinite countable family} \{M_k\}_{k \in \mathbb{N}} \text{ of pairwise disjoint elements of} \ A \text{ and} \ (\xi, t) \in D_i \text{ such that} \ M_k \cup \{(\xi, t)\} \in A \text{ for all} \ k \in \mathbb{N} \}.
\]

It is clear that \( L \) is non empty set. For each \( i \in L \) and some \( (M_k)_{k=1}^{\infty} \) witnessing the belongingness of \( i \) to \( L \) we set

\[
E_i = \{n_k : M_k \in A_{n_k}\}.
\]

Each \( E_i \) is an infinite set.

Assuming the failure of the conclusion, we examine the following two mutually exclusive cases.

*Case 1.* The set \( L \) is an infinite set.

Then we could define \( \phi : L \rightarrow \mathbb{N} \) one-to-one function such that for all \( i \in L \), \( \phi(i) \in E_i \). Notice that our assumptions imply that for every \( i \in L \) there exists \( M_i \in A_{\phi(i)} \) such that \( M_i \cap D_i \neq \emptyset \), \( \max M_i \in D_i \) and \( M_i \) does not have extensions \((\xi, t)\) with \((\xi, t) \in D_i \). But the family \( \{M_i\}_{i \in L} \) consists of pairwise disjoint elements of \( A \), hence it has at least one extension; that is, there exists \((\xi, t) \in \Gamma \) with \( M_i \cup \{(\xi, t)\} \in A \) for all \( i \in L \). Since \( \{D_i\}_{i \in \mathbb{N}} \) is a partition of \( \Gamma \), there exists \( i_0 \) in \( \mathbb{N} \) with \((\xi, t) \in D_{i_0} \). Further \( i_0 \in L \), hence \((\xi, t)\) extends \( M_{i_0} \) which contradicts the maximality of \( M_{i_0} \) and the proof in this case is complete.

*Case 2.* The set \( L \) is a finite set.

In this case we produce inductively a set \( M \) such that \( M \in \mathbb{R} \) and \( M \cap D_i \) is an infinite set for some \( i \in L \).

Indeed, we start with the family \( \{(0, n)\}_{n \in \mathbb{N}} \). Then there exists \( i_1 \in L \) and an infinite set \( \{(\xi_n^1, t_n^1)\}_{n \in \mathbb{N}} \subset D_{i_1} \) such that \((0, n), (\xi_n^1, t_n^1) \in A \) and \((\xi_n^1, t_n^1) \neq (\xi_m^1, t_m^1) \) for \( n \neq m \). In the next step we consider the family
\{(0, n), (\xi_n^1, t_n^1)\}\}_{n \in \mathbb{N}} and we select \(i_2 \in L\), \{(\xi_n^2, t_n^2)\}_{n \in \mathbb{N}} \subset D_{i_2} such that
\[(0, n), (\xi_n^1, t_n^1), (\xi_n^2, t_n^2) \in \mathcal{A} \text{ and } (\xi_n^2, t_n^2) \neq (\xi_m^2, t_m^2) \text{ for } n \neq m.\]

We proceed inductively in the same manner. Since \(L\) is a finite set, we find \(i_0 \in L\) and \((n_k)_{k \in \mathbb{N}}\) an infinite set such that for all \(n \in \mathbb{N}\), \(((\xi_n^{k_0}, t_n^{k_0}) \in D_{i_0}\).

Then clearly for all \(n \in \mathbb{N}\), \(((0, n), (\xi_n^1, t_n^1), \ldots, (\xi_n^k, t_n^k), \ldots) \cap D_{i_0}\) is an infinite set, a contradiction and the proof of the lemma is complete. Q.E.D.

Actually the space \(w\) satisfies the following stronger condition that we will use in the third section.

**1.7. Lemma:** For every \((\Gamma_d)_{d \in D}\) partition of \(\Gamma\) into pairwise disjoint countable sets and every \((D_i)_{i \in \mathbb{N}}\) countable partition of \(\Gamma\) there exists \(M \in w\) and \(i_0 \in \mathbb{N}\) such that \(M \cap D_{i_0}\) is an infinite set and \(# M \cap \Gamma_d \leq 1\) for all \(d \in D\).

**Proof.** The proof is essentially the same as in the previous lemma. We only have to take into account the additional property that \(# M \cap \Gamma_d \leq 1\).

Thus we define the set \(L\) as:

\[L = \{i \in \mathbb{N} : \text{there exists an infinite countable family } \{M_k\}_{k \in \mathbb{N}} \text{ of pairwise disjoint elements of } \mathcal{A} \text{ and } (\xi, t) \in D_i \text{ such that } M_k \cup \{(\xi, t)\} \in \mathcal{A} \text{ and } # M_k \cup \{(\xi, t)\} \cap \Gamma_d \leq 1 \text{ for all } k \in \mathbb{N} \text{ and } d \in D\}.\]

Using conditions (b)-(ii) of the definition of the families \((\mathcal{A}_n)_{n \in \mathbb{N}}\), we proceed with the proof in the same manner as in Lemma 1.6. Q.E.D.

## 2 Banach spaces not containing \(l^1(\mathbb{N})\)

Let \(\Gamma\) be a non empty set and \(\mathcal{S}\) a family of subsets of \(\Gamma\). We denote by \(\Phi\) the linear space of all finitely supported real valued functions defined on \(\Gamma\). For \(a \in \Gamma\) we denote by \(e_a\) the characteristic function of the one point set \(\{a\}\). Clearly the family \(\{e_a\}_{a \in \Gamma}\) is a Hamel basis for the space \(\Phi\).

**2.1. Definition:** Let \(\Gamma\) be a non empty set and \(\mathcal{S}\) a family of subsets of \(\Gamma\). For \(\phi\) a finitely supported real valued function on \(\Gamma\), we define the James-S...
or \( J - S \) norm as:

\[
\phi = \sup \left( \sum_{i=1}^{n} \left( \sum_{a \in s_i} \phi(a) \right)^2 \right)^{1/2},
\]

where the supremum is taken over all families \( \{s_1, s_2, \ldots, s_n\} \) of pairwise disjoint elements of \( S \). We denote by \( JS \) the completion of \( \Phi \) in the above defined norm.

Our goal is to prove the next result.

2.2. Theorem: For every C.I. family \( S \) the space \( JS \) does not contain isomorphically \( \ell^1(\mathbb{N}) \).

Remark. (i) This result is known for certain C.I. families. For example, the classical James Tree space \([J]\) is the space defined by \( J - S \) norm for the C.I. family of the dyadic tree. (More precisely, the family used in the original James’ definition is that \( S \) that consists of all segments of the dyadic tree.

(ii) The proof of the theorem is divided into two parts. In the first we will give a representation of the extreme points of the unit ball of the dual \( JS^* \). In the second part we show that every bounded sequence in \( JS \) has a weak Cauchy subsequence. For this we use the fact that the family \( S \) is countably intersected together with the representation of the extreme points of the first step and Rainwater’s theorem.

We begin with some notation and definitions.

With each \( s \) in \( S \) we associate a functional \( s^* \) in \( JS^* \) defined by \( s^*(\phi) = \sum_{a \in s} \phi(a) \) for every \( \phi \) finitely supported real valued function. It follows from the definition of \( JS \) norm that \( s^*(\phi) \leq \phi \), hence \( s^* \leq 1 \) and if \( s \neq \emptyset \) then \( s^* = 1 \). We also define the set

\[
D = \left\{ \sum_{i=1}^{n} \lambda_i s_i^* : \lambda_i \in \mathbb{R}, s_i \in S, \{s_i\}_{i=1}^{n} \text{ are pairwise disjoint and } \sum_{i=1}^{n} \lambda_i^2 \leq 1 \right\}.
\]

A simple argument shows that for \( \{s_i\}_{i=1}^{n} \) pairwise disjoint

\[
\left( \sum_{i=1}^{n} \lambda_i s_i^* \right) \leq \left( \sum_{i=1}^{n} \lambda_i^2 \right)^{1/2},
\]
hence $D$ is a subset of the unit ball of $J^*$. 

2.3. Lemma: The $w^*$ closure of $D$ contains the extreme points of the unit ball of $J^*$. 

Proof. Suppose that the conclusion is false. Then there exists an extreme point $x^*$ of $B_{J^*}$ with $x^* \not\in \overline{D}_{w^*}$. Hence we could find a $w^*$-neighborhood of $x^*$ in $B_{J^*}$ disjoint from $\overline{D}_{w^*}$. But since the $w^*$ slices of $x^*$ define a neighborhoods basis for the $w^*$ topology, there exists a slice $S(x^*, x, t)$ disjoint from $D$. Further we may assume that $x$ is a finitely supported function on $\Gamma$ with $x = 1$. From all the above we get that 

(i) For some $\epsilon > 0$, $x^*(x) > \sup_{y^* \in D} y^*(x) + \epsilon$. 

(ii) There exists $\{s_i\}_{i=1}^n$ pairwise disjoint and such that 

$$1 = x = \left( \sum_{i=1}^n \left( \sum_{a \in s_i} x(a) \right)^2 \right)^{1/2}.$$ 

We set $\lambda_i = \sum_{a \in s_i} \phi(a)$ and $y^* = \sum_{i=1}^n \lambda_is_i^*$. Then $y^* \in D$ and 

$$1 = y^*(x) < x^*(x) - \epsilon \leq 1 - \epsilon,$$

a contradiction and the proof is complete. Q.E.D. 

In the sequel we identify each $s \in S$ with its characteristic function on $\{0, 1\}^\Gamma$, and $S$ with the corresponding (closed) subset of $\{0, 1\}^\Gamma$. Recall that for $a \in \Gamma$, $e_a$ denotes $\chi_{\{a\}}$ and note that $e_a = 1$ and the family $\{e_a\}_{a \in \Gamma}$ generates the space $J^*$. Finally, we denote by $B_{\ell^2}$ the unit ball of $\ell^2(\mathbb{N})$ endowed with the weak topology. We set 

$$V = \{(\lambda_n)_{n \in \mathbb{N}} \subset B_{\ell^2} : (\lambda_n)_{n \in \mathbb{N}} \text{ is a decreasing sequence }\},$$

which is a closed subset of $B_{\ell^2}$.

2.4. Lemma: Let $E \subset S^\Gamma$ be defined by $(s_n)_{n \in \mathbb{N}} \in E$ if and only if $(s_n)_{n \in \mathbb{N}}$ is a pairwise disjoint family. Then 

(i) $E$ is closed subset of $S^\Gamma$. 

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(ii) The function

\[ T : V \times E \longrightarrow (D^w, w^*) \] defined by \( T((\lambda_n)_{n \in \mathbb{N}}, (s_n)_{n \in \mathbb{N}}) = \sum_{n=1}^{\infty} \lambda_n s^*_n \) is continuous and onto.

**Proof.** (i) It is straightforward and it is left to the reader.

(ii) Let \( x_i = ((\lambda_{n,i})_{n \in \mathbb{N}}, (s_{n,i})_{n \in \mathbb{N}}) \) and \( x = ((\lambda_n)_{n \in \mathbb{N}}, (s_n)_{n \in \mathbb{N}}) \) and suppose that the net \((x_i)_{i \in I}\) converges to \( x \).

We will show that \( T x_i \) converges in the \( w^* \) topology to \( T x \). For this it is enough to show that \( T x_i(e_a) \longrightarrow T x(e_a) \) for all \( a \in \Gamma \).

Indeed, there are two cases.

**Case 1.** \( a \in \bigcup_{n \in \mathbb{N}} s_n \). Then there exists a unique \( n_0 \in \mathbb{N} \) such that \( a \in s_{n_0} \).

Since \( E \subset (\{0, 1\}^{\Gamma})^\mathbb{N} \), denoting by \((a, n_0)\) the \( a^{th} \) coordinate of \( \Gamma \) appearing in the \( n_0 \) copy of it, we have \( x_i(a, n_0) \longrightarrow x(a, n_0) \), hence there exists \( i_0 \) such that for all \( i \geq i_0 \), \( x_i(a, n_0) = x(a, n_0) = 1 \) and \( x_i(a, m) = 0 \) for all \( m \in \mathbb{N}, m \neq n_0 \). Consequently, for \( i \geq i_0 \), \( T x_i(e_a) = \lambda_{i(n_0, i)} \) which converges to \( \lambda_{n_0} = T x(e_a) \).

**Case 2.** \( a \not\in \bigcup_{n \in \mathbb{N}} s_n \).

In this case we will show that \( T x_i(e_a) \) converges in the \( w^* \) topology to 0.

Indeed, let \( \varepsilon > 0 \). Since for each \( i \in I \), \( \sum_{n=1}^{\infty} \lambda^2_{(n, i)} \leq 1 \) and \((\lambda_{(n, i)})_{n \in \mathbb{N}} \) is decreasing, there exists \( n_0 \in \mathbb{N} \) such that for all \( i \in I \) and \( m > n_0 \), \( \lambda_{(m, i)} < \varepsilon \).

Also, since \( a \not\in \bigcup_{n \in \mathbb{N}} s_n \), there exists \( i_0 \) such that \( a \not\in s_{(k,i)} \) for all \( i \geq i_0 \) and \( k = 1, 2, \ldots, n_0 \).

Then for \( i \geq i_0 \), \( T x_i(e_a) \leq \sup\{\lambda_{(m, i)} : m > n_0\} \leq \varepsilon \) and this proves the result in the second case.

Q.E.D.

We turn to some consequences of the previous results.

**2.5. Corollary:** Every extreme point \( x^* \) of the unit ball of \( JS^* \) is represented as

\[ x^* = \sum_{n=1}^{\infty} \lambda_n s^*_n, \]
where $\sum_{n=1}^{\infty} \lambda_n^2 = 1$, $(s_n)_{n \in \mathbb{N}}$ is a pairwise disjoint sequence of elements of $S$ and the series converges in the norm topology of $JS^*$.

2.6. Corollary: If the family $S$ defines a Talagrand compact set, then the space $JS$ is weakly $K$-analytic.

Proof. It is well known that the class of Talagrand compact sets is closed with respect to closed subsets, countable products and continuous images [T]. Hence $D^{w^*}$ is a Talagrand compact set containing the extreme points of $B_{JS^*}$. Therefore $JS$ is isometric to a closed subspace of the space $C(D^{w^*})$ and the proof is complete. Q.E.D.

2.7. Corollary: Let $(x_n)_{n \in \mathbb{N}}$ be a bounded sequence in $JS$. If for each $s$ in $S$ the sequence $(s^*(x_n))_{n \in \mathbb{N}}$ is Cauchy, then $(x_n)_{n \in \mathbb{N}}$ is weakly Cauchy.

Proof. It follows from Corollary 2.5 that for every $x^*$ extreme point of $B_{JS^*}$, $(x^*(x_n))_{n \in \mathbb{N}}$ is Cauchy. The result now follows from Rainwater’s theorem [Rai]. Q.E.D.

The second part of the proof of the main theorem is contained in the next lemma.

2.8. Lemma: Every norm bounded sequence $(\phi_n)_{n \in \mathbb{N}}$ in $JS$ consisting of finitely supported functions contains a subsequence $(\phi_n)_{n \in B}$ such that for all $s \in S$, $(s^*(\phi_n))_{n \in B}$ is Cauchy.

Proof. Assume that for each $n \in \mathbb{N}$, $\phi_n$ $\leq 1$. We claim the following.

Claim 1. For every $\epsilon > 0$ and every $(\phi_n)_{n \in B}$ subsequence of the given sequence there exists a finite set $\{s_1, \ldots, s_k\}$ subset of $S$ and an infinite $B'$ subset of $B$ such that for every $s \in S$ with $s \cap s_i = \emptyset$, $i = 1, \ldots, k$, we have

$$\limsup_{n \in B'} s^*(\phi_n) \leq \epsilon.$$ 

To see this, we begin with an infinite $B$ subset of $\mathbb{N}$ and we choose $s_1$ in $S$, if such exists, so that

$$\limsup_{n \in B} s^1_*(\phi) > \epsilon.$$
Then we choose $B_1$ infinite such that

$$s_1^*(\phi_n) > \epsilon \quad \text{for all } n \in B_1.$$ 

If the pair $\{s_1\}, B_1$ does not satisfy the conclusion of the claim, we repeat the procedure to find an $s_2$ in $S$ disjoint from $s_1$ such that

$$\limsup_{n \in B_1} s_2^*(\phi_n) > \epsilon.$$ 

Choose $B_2$ subset of $B_1$ such that $s_2^*(\phi_n) > \epsilon$ for all $n \in B_2$.

Notice that for each $n \in B_2$, $\phi \epsilon \sqrt{2}$. Therefore, if $k = \min\{n : \epsilon \sqrt{n} > 1\}$ and we repeat the procedure at most $k$ times, we will get $s_1, s_2, \ldots, s_l$, for some $l \leq k$ and an infinite $B'$ subset of $B$ such that the conclusion of the claim is satisfied.

Next we apply the claim infinitely many times to choose a decreasing sequence $(B_m)_{m \in \mathbb{N}}$ of subsets of $\mathbb{N}$, and families $\{s_{1,m}, \ldots, s_{k_m,m}\}$ subsets of $S$ such that the pair $(\phi_n)_{n \in B_m}, (s_{1,m}, \ldots, s_{k_m,m})$ satisfies the conclusion of the claim for $\epsilon = 1/m$. Choose a sequence $(\phi_n)_{n \in B'}$ which is almost contained in every $B_m$. The family $S$ is countably intersected. Therefore, for every $s_{i,m}$ there exists $\tilde{s}_{i,m}$ containing $s_{i,m}$ and satisfying condition (c) of Definition 1.1.

Notice also that the set

$$L = \bigcup_{m=1}^{\infty} \bigcup_{i=1}^{k_m} L_{\tilde{s}_{i,m}}$$

is countable (condition (d) of Definition 1.1). Further, for $s, t$ in $S$, condition (b) of Definition 1.1 ensures that $(s \setminus t)^*$ belongs to $JS^*$, hence $(s \cap t)^* = s^* \setminus (s \setminus t)^*$ is also in $JS^*$. Therefore, using a diagonal argument, we may choose a subsequence $(\phi_n)_{n \in B'}$ of $(\phi_n)_{n \in B'}$ such that $(t^*(\phi_n))$ converges for all $t^* \in L$ and all $t^*$ finite subsets of $\bigcup_{n \in B'} \text{supp} \phi_n$.

Notice that the last property implies that $(f^*(\phi_n))_{n \in B'}$ converges for all $f$ in the class of finite sets.

*Claim 2.* The sequence $(s^*(\phi_n))_{n \in B'}$ is Cauchy for all $s \in S$.

Indeed, consider any $s$ in $S$ and $m \in \mathbb{N}$. We set

$$s_m = s \setminus \bigcup_{i=1}^{k_m} \tilde{s}_{i,m}.$$
Also, we choose \( \{ t_{j,m} \}_{j=1}^{l_m} \) pairwise disjoint subfamily of \( S \) such that \( \bigcup_{i=1}^{k_m} \tilde{s}_{i,m} = \bigcup_{j=1}^{l_m} t_{j,m} \) (Lemma 1.3) and each \( t_{j,m} \) is contained in some \( \tilde{s}_{i,m} \). Finally, we set \( d_{j,m} = t_{j,m} \cap s \) and we choose \( t_m \subset s_m \) with \( s_m \setminus t_m \) being a finite set. Then

\[
\limsup_{n \in B''} s^*(\phi_n) - \liminf_{n \in B''} s^*(\phi_n) = \limsup_{n \in B''} t_m^*(\phi_n) + \lim_{n \in B''} \sum_{j=1}^{l_m} d_{j,m}^*(\phi_n) + \lim_{n \in B''} (s_m \setminus t_m)^*(\phi_n) \\
- \left( \liminf_{n \in B''} t_m^*(\phi_n) + \lim_{n \in B''} d_{j,m}^*(\phi_n) + \lim_{n \in B''} (s_m \setminus t_m)^*(\phi_n) \right) \\
= \limsup_{n \in B''} t_m^*(\phi_n) - \liminf_{n \in B''} t_m^*(\phi_n) \leq \frac{1}{m}.
\]

Since the last inequality holds for every \( m \in \mathbb{N} \), we get that \( (s^*(\phi_n))_{n \in B''} \) is Cauchy.

**Proof of Theorem 2.2.** It is well known that a Banach space \( X \) does not contain isomorphically \( \ell^1(\mathbb{N}) \) if every bounded sequence \( (x_n)_{n \in \mathbb{N}} \) has a weakly Cauchy subsequence. Given a bounded sequence \( (x_n)_{n \in \mathbb{N}} \) in \( JS \), we approximate each \( x_n \) by some \( \phi_n \), a function of finite support such that \( x_n - \phi_n \leq 1_n \). Applying Lemma 2.8 and Corollary 2.7, we find a subsequence \( (\phi_n)_{n \in B} \) which is weakly Cauchy. Since \( x_n - \phi_n \) converges in norm to zero, we get that \( (x_n)_{n \in B} \) is also weakly Cauchy.

**Proof.**

This is the \( JS \) space for \( S \) the C.I. family for Recničenko’s space. Indeed, in Corollary 2.6 we proved that \( JS \) is weakly \( K \)-analytic and the unit ball of \( JS^* \) contains a subset which is homeomorphic in the \( w^* \) topology to \( S \). Hence it is not Eberlein compact set and \( X \) is not a subspace of a W.C.G.

**2.9. Corollary:** There exists a Banach space \( X \) not containing \( \ell^1(\mathbb{N}) \) such that \( X \) is weakly \( K \)-analytic but not a subspace of a W.C.G.

**Proof.**

This is the \( JS \) space for \( S \) the C.I. family for Recničenko’s space.

**2.10. Corollary:** There exists a Banach space \( X \) not containing \( \ell^1(\mathbb{N}) \) such that \( (B_X^*, w^*) \) is Corson compact set, or equivalently \( X \) is a W.L.D. space, and \( X \) is not weakly \( K \)-analytic.
Proof. This is the $JS$ space for $S$ the C.I. family for Todorcević tree. Indeed, $S$ is not a Talagrand compact set and it is homeomorphic to a closed subset of the unit ball of $J^*$. This shows that $JS$ is not weakly $K$-analytic. On the other hand $S$ has property (M) [A-M (Proposition 3.10)] and hence $B_{JS^*}$ in the $w^*$- topology is a Corson compact set. Q.E.D.

3 Subspaces of W.C.G. spaces

In this section we give examples of subspaces of W.C.G. spaces which are not W.C.G. The first part contains a general method of producing such examples. In the second part we construct an example of this kind that satisfies the additional property that $\ell^1(N)$ is not isomorphically embedded into the space.

We begin with the definition of a class of compact spaces.

3.1. Definition: A pointwise closed family of countable subsets of a set $\Gamma$ is said to be quasi-Eberlein if the following conditions are satisfied

(i) The singletons of $\Gamma$ are contained in $S$, and each $s \in S$ is an increasing union of finite sets $t$ from $S$.

(ii) $S = \bigcup_{n=1}^{\infty} S_n$, where each $S_n$ is an Eberlein compact set and $\Gamma$ is a subset of $S_1$.

(iii) For all partitions $(\Gamma_d)_{d \in D}$, $(\Gamma_n)_{n=1}^{\infty}$ of $\Gamma$ such that $\Gamma_d$ are countable and pairwise disjoint there exists an $s \in S$ and $n_0 \in \mathbb{N}$ such that $s \cap \Gamma_{n_0}$ is an infinite set while $\#s \cap \Gamma_d \leq 1$ for all $d \in D$.

3.2. Remark: (a) The first part of condition (i) and the second part of condition (ii) are not so important. If $S$ does not satisfy these two conditions, we may consider the family $S' = S \cup \Gamma$ and $S'_1 = S_1 \cup \Gamma$ and the family $S'$ is a quasi-Eberlein set.

(b) The fact in condition (iii) that for every partition $(\Gamma_n)_{n \in \mathbb{N}}$ of $\Gamma$ there exists $s \in S$ and $n_0 \in \mathbb{N}$ such that $s \cap \Gamma_{n_0}$ is an infinite set shows, as we explained in the case of Recničenko's space, that $S$ is not an Eberlein
compact set. The second part of condition (iii) will be used to prove that
certain Banach spaces defined by means of quasi-Eberlein sets are not W.C.G.

3.3. Lemma: If \( S \) is a quasi-Eberlein set and \((S_n)_{n \in \mathbb{N}}\) its partition into
Eberlein compact sets, then the set

\[ SE = \left\{ \frac{1}{n}\chi_s : s \in S_n \right\} \]

is Eberlein compact.

Proof. It is easy to see that \( SE \) is a closed subset of \([0,1]^\Gamma\) and using
Rosenthal’s criterion [R] we get the desired result.

3.4. Definition: We call the set \( SE \) the **Eberleinization** of the quasi-
Eberlein set \( S \).

3.5. Examples: We give two examples of quasi-Eberlein sets. Notice that
by a result due to Sokolov [S], every quasi-Eberlein set is a Talagrand
compact set.

**A. Talagrand’s space:** Recall the definition of Talagrand’s example
that is given in [T]. We set \( \Gamma = \mathbb{N}^\mathbb{N} \); a subset \( s \) of \( \Gamma \) is said to be admissible
if and only if there exists \( n \in \mathbb{N} \) such that for \( \gamma_1, \gamma_2 \in s, \gamma_1 \neq \gamma_2 \), we have
\( \gamma_1(1) = \gamma_2(1), \gamma_1(2) = \gamma_2(2), \ldots, \gamma_1(n-1) = \gamma_2(n-1) \) and \( \gamma_1(n) \neq \gamma_2(n) \).

We call the number \( n \) the **characteristic** of the set \( s \). We also agree that
if \( \#s \leq 1 \) then every \( n \in \mathbb{N} \) is the characteristic of \( s \). The family \( S \) of all
admissible sets is pointwise closed and if \( s \) is admissible, \( t \) a subset of \( s \), then
\( t \) is also admissible. It is an application of Baire’s category theorem that if
\( \{\Gamma_n\}_{n \in \mathbb{N}} \) is a partition of \( \Gamma \), then there exists an infinite admissible \( s \) contained
in some \( \Gamma_n \). Therefore, if \( \{\Gamma_d\}_{d \in D} \) is a partition of \( \Gamma \) into countable pairwise
disjoint sets then there exists a partition \( \{\Delta_k\}_{k \in \mathbb{N}} \) of \( \Gamma \) such that \( \#\Delta_k \cap \Gamma_d \leq 1 \)
for all \( k \in \mathbb{N}, d \in \Delta \). Hence for given \( \{\Gamma_d\}_{d \in D}, \{\Gamma_n\}_{n \in \mathbb{N}} \) partitions of \( \Gamma \) as in
Definition 3.1 (iii), we consider as above the partition \( \{\Delta_k\}_{k \in \mathbb{N}} \) corresponding
to the family \( \{\Gamma_d\}_{d \in D} \) and the joint partition \( \{\Gamma_{(n,k)}\}_{(n,k) \in \mathbb{N} \times \mathbb{N}} \), where \( \Gamma_{(n,k)} = \Gamma_n \cap \Delta_k \). It is easy to see that every infinite admissible \( s \) contained in some
\( \Gamma_{(n,k)} \) satisfies condition (iii) of Definition 3.1. Finally we set

\[ S_n = \{ s \in S : \text{the characteristic of } s \text{ is } n \} \]
Then $S_n$ is an Eberlein compact set and $S = \bigcup_{n=1}^{\infty} S_n$. Therefore $S$ is a quasi-Eberlein compact set.

**B. Recničenko’s space**: We recall from the first section of the paper that Recničenko’s space is the set

$$R = \{ s : \text{there exists } n \in \mathbb{N} \text{ with } s \text{ a segment of the tree } A_n \}.$$ 

Also, the order type of each branch of $A_n$ is $\omega$. Therefore each $\gamma$ in $A_n$ belongs to some $A_n^k$, where $k \in \mathbb{N}$ denotes the $k^{th}$ level of $A_n$. Hence the correspondence $F : \Gamma \cap A_n \rightarrow c_0(\Gamma)$ defined as $A_n^k \ni \gamma \mapsto \frac{1}{k+1} e_\gamma$ is extended to a homeomorphism from the segments of $A_n$ to a subset of $(\overline{B}_{c_0(\Gamma)}, w)$. Thus the set

$$R_n = \{ s : s \text{ is a segment of } A_n \}$$

is an Eberlein compact set. We set

$$R' = R \cup \Gamma \text{ and } R'_1 = R_1 \cup \Gamma.$$ 

Then $R'$ is a quasi-Eberlein compact set (see Lemma 1.7).

**3.6. Theorem**: Let $S$ be a quasi-Eberlein set and $SE$ its Eberleinization that is naturally contained in the set $[0, 1]^{\Gamma}$. We denote by $e_\gamma$ the restriction of $\pi_\gamma$ projection of $[0, 1]^{\Gamma}$ on the space $SE$, and we consider the space $X$ generated by the vectors $\{ e_\gamma \}_{\gamma \in \Gamma}$ in the space $C(SE)$. Then $X$ is a subspace of a W.C.G. space but it is not itself a W.C.G. space.

Before we prove the theorem we give a combinatorial lemma which is necessary in proving the result.

**3.7. Lemma**: Let $\Gamma$ be a set and $\{ f_\delta \}_{\delta \in \Delta}$ a family of real-valued functions defined on $\Gamma$ and satisfying the following conditions:

(i) For every $\delta$ in $\Delta$, the function $f_\delta$ has countable support.

(ii) For every $\gamma \in \Gamma$ the set $\{ \delta : f_\delta(\gamma) \neq 0 \}$ is non empty and countable.

Then:

There exist partitions $(\Gamma_d)_{d \in D}, (\Delta_d)_{d \in D}$ of $\Gamma$ and $\Delta$, respectively, such that $\Gamma_d, \Delta_d$ are countable and if $d_1 \neq d_2, \Gamma_{d_1} \cap \Gamma_{d_2} = \emptyset$, $\Delta_{d_1} \cap \Delta_{d_2} = \emptyset$ and $f_\delta(\gamma) = 0$ for $\gamma \in \Gamma_d, \delta \in \Delta_d$, where $i = 1, j = 2$ or vice-versa.
Proof. We shall show that there exist $\Gamma_1, \Delta_1$ countable subsets of $\Gamma$ and $\Delta$, respectively, such that $\Gamma_1 = \bigcup\{\text{supp} f_\delta : \delta \in \Delta_1\}$ and $\Delta_1 = \{\delta \in \Delta : \text{there exists } \gamma \in \Gamma_1, f_\delta(\gamma) \neq 0\}$. If this has been done, then inductively we produce the desired families $(\Gamma_d)_{d \in D}, (\Delta_d)_{d \in D}$. To find the sets $\Gamma_1, \Delta_1$ we use the standard saturation argument. Indeed, we start with any $\delta_0 \in \Delta$. Set $\Delta^0 = \{\delta_0\}$ and $\Gamma^0 = \{\gamma \in \Gamma : f_{\delta_0}(\gamma) \neq 0\}$. Then $\Gamma^0$ is countable and the set $\Delta^1 = \{\delta \in \Delta : \text{there exists } \gamma \in \Gamma^0 \text{ with } f_\delta(\gamma) \neq 0\}$ is also countable. Next we define $\Gamma^1 = \{\gamma \in \Gamma : \text{there exists } \delta \in \Delta^1 \text{ with } f_{\delta}(\gamma) \neq 0\}$. We thus inductively produce $\Delta^0 \subset \Delta^1 \subset \cdots \subset \Delta^k \subset \cdots, \Gamma^0 \subset \Gamma^1 \subset \cdots \subset \Gamma^k \subset \cdots$ such that if $\delta \in \Delta^k$, the set $\{\gamma \in \Gamma : f_\delta(\gamma) \neq 0\} \subset \Gamma^k$ and if $\gamma \in \Gamma^k$, the set $\{\delta \in \Gamma : f_\delta(\gamma) \neq 0\} \subset \Delta^{k+1}$. It is clear that the sets $\Gamma_1 = \bigcup_{k=1}^{\infty} \Gamma^k$ and $\Delta_1 = \bigcup_{k=1}^{\infty} \Delta^k$ are the desired. Q.E.D.

Proof of the Theorem. Since $SE$ is an Eberlein compact set, $C(SE)$ is a W.C.G. space [A-L] and hence $X$ is, indeed, a subspace of a W.C.G. space.

To see that $X$ is not a W.C.G. space, notice first that for $s$ in $S$, $s \in S_n$ for some $n \in \mathbb{N}$ hence the functional $\frac{1}{n}s^*$ defined by
\[
\frac{1}{n}s^* \left( \sum_{i=1}^{n} \alpha_i e_{\gamma_i} \right) = \frac{1}{n} \sum \{\alpha_i : \gamma_i \in s\}
\]
is continuous and has $\frac{1}{n}s^*$. Therefore $s^*$.

$\text{leq}n$. In particular, since $s_\gamma = \{\gamma\} \in S_1$, the functional $s^*_\gamma$ has $s^*_\gamma$ $\text{leq}1$ for all $\gamma \in \Gamma$. It is easy to see that the family $\{s^*_\gamma\}_{\gamma \in \Gamma}$ is $w^*$-total, $w^*$-discrete and $\{s^*_\gamma\}_{\gamma \in \Gamma} \cup \{0\}$ is $w^*$-compact.

Assume now that $X$ is a W.C.G. space. Then there exists a total $\{y_\delta\}_{\delta \in \Delta}$ subset of $X$ which is weakly discrete, and $\{y_\delta\}_{\delta \in \Delta} \cup \{0\}$ is weakly compact. Hence every countable subset of it defines a weakly null sequence in the space $X$. We apply Lemma 3.7 to the families $\{y_\delta\}_{\delta \in \Delta}$ and $\{s^*_\gamma\}_{\gamma \in \Gamma}$ to find partitions $(\Delta_d)_{d \in D}, (\Gamma_d)_{d \in D}$ of $\Delta$ and $\Gamma$, respectively, satisfying the conclusions of the lemma.

Each $\Gamma_d$ is countable hence we enumerate it as $\Gamma_d = \{\gamma^d_n\}_{n \in \mathbb{N}}$. We define a partition of $\Gamma$ into a countable family $\{\Gamma(n,k)\}_{(n,k) \in \mathbb{N} \times \mathbb{N}}$ so that $\gamma \in \Gamma(n,k)$ if and only if there exists $d \in D$ with $\gamma^d_n = \gamma$ and there exists $\delta \in \Delta_d$ with $e^*_\gamma(y_\delta) \geq \frac{1}{k}$. 

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Since $\mathcal{S}$ is a quasi-Eberlein set, there exists $s \in \mathcal{S}$ and $(n_0, k_0) \in \mathbb{N} \times \mathbb{N}$ such that $s \cap \Gamma_{(n_0, k_0)}$ is an infinite set and $\#s \cap \Gamma_d \leq 1$ for all $d \in D$ (Definition 3.1 (iii)). For every $\gamma \in s \cap G_{(n_0, k_0)}$ there exists unique $d \in D$ such that $\gamma = \gamma_d^{n_0}$ and there exists at least one $\delta_\gamma \in \Delta_d$ such that $e_\gamma^*(y_{\delta_\gamma}) \geq \frac{1}{k}$. Consider the countable set $\{y_{\delta_\gamma} : \gamma \in s \cap \Gamma_{(n_0, k_0)}\}$. As we have noticed, this set defines a weakly null sequence. On the other hand, $s^*(y_{\delta_\gamma}) = s^*_\gamma(y_{\delta_\gamma}) \geq \frac{1}{k}$ (by condition (i) of Definition 3.1) for all $\gamma$ in $s \cap \Gamma_{(n_0, k_0)}$. This is a contradiction and the proof is complete.

Q.E.D.

3.8. Remarks: (a) It is easy to see that if $\mathcal{S}$ is an adequate family, i.e. if $t$ is a subset of $s$ and $s$ is in $\mathcal{S}$ then $t$ is also in $\mathcal{S}$, then the family $\{e_\gamma\}_{\gamma \in \Gamma}$ that generates the space $X$ above is an unconditional basis. Hence in the case of Talagrand’s space, the space $X$ has an unconditional basis.

(b) It is well known that every subspace of a W.C.G. Banach space is a weakly $\mathcal{K}$-analytic space [T]. Rosenthal has proved that under Martin’s Axiom every subspace $X$ of $L^1(\mu)$, where $\mu$ is a probability measure, with $\dim X < 2^\omega$ is a W.C.G. space. Recently S. Merkourakis proved that this is a general phenomenon in the class of weakly $\mathcal{K}$-analytic Banach spaces. In particular, he showed that under MA every weakly $\mathcal{K}$-analytic Banach space $X$ with $\dim X < 2^\omega$ is a W.C.G. Banach space.

In the sequel we denote by $RE$ the Eberleinization of Recničenko’s space as it is defined in 3.5. We define the James-RE norm.

3.9. Definition: Let $\phi$ be a finitely supported real valued function on $\Gamma$. For $g$ in $RE$ we denote by $\langle \phi, g \rangle$ the real number $\sum_{a \in g} \phi(a)g(a)$. The $J-RE$ norm of the function $\phi$ is defined as

$$\phi = \sup \left( \sum_{i=1}^{n} \langle \phi, g_i \rangle^2 \right)^{1/2},$$

where $g_1, \ldots, g_n$ are in $RE$ with pairwise disjoint supports.

The space $J-RE$ is the completion in $J-RE$ norm of the linear space $\Phi$ of all finitely supported functions defined on $\Gamma$.

3.10. Theorem: The space $J-RE$ is a subspace of a W.C.G. space, it
does not contain isomorphically $\ell^1$ and it is not a W.C.G. space.

**Proof.** The proof of the theorem is analogous to the proof of certain previous results. We shall therefore only indicate how we get the result.

(a) **The space $J - RE$ does not contain $\ell^1$.**

To prove this we repeat step by step the proof of Theorem 2.2. The only change we have to make is to replace the functional $s^*$ by the functional $g^*$, where $g \in SE$.

(b) **The space $J - SE$ is a subspace of a W.C.G. space.**

As in the proof of Theorem 2.2, the space $J - SE$ is a subspace of the space $C(D^w)$, where in our case the set $D$ is defined as

$$D = \left\{ \sum_{i=1}^{n} \lambda_i g_i^* : \lambda_i \in \mathbb{R}, g_i \in SE, \{g_i\}_{i=1}^{n} \text{ have pairwise disjoint supports} \right\}.$$  

Then, as in Lemma 2.4, $(D^w, w^*)$ is the continuous image of $V \times E$, where $E$ is a closed subset of $SE^\mathbb{N}$ and $V$ is a closed subset of $(B_{\ell^2}, w)$. Hence $V \times E$ is an Eberlein compact set and the same holds for the set $(D^w, w^*)$ [B-R-W]. Hence $C(D^w)$ is W.C.G. [A-L] and the proof is complete.

(c) **The space $J - SE$ is not a W.C.G. space.**

For $s$ in $R$ there exists $n \in \mathbb{N}$ such that $s \in R_n$, hence $g^* = \frac{1}{n} s^* \leq 1$. Therefore $s^* \in J - RE^*$ and $s^*$

leqn Also, for $\gamma$ in $\Gamma$, $\gamma$ is in $R$, hence if $s^* = \{\gamma\}^*$, we have that $s^*$

leq1. It is easy to see that $\{s^*_\gamma\}_{\gamma \in \Gamma}$ is $w^*$-total in $J - RE^*$, $w^*$-discrete and $\{s^*_\gamma\}_{\gamma \in \Gamma} \cup \{0\}$ is $w^*$-compact. Assuming now that $J - RE$ is a W.C.G. space, there exists a subset $\{y_\delta\}_{\delta \in \Delta}$ of it weakly total, weakly discrete and $\{y_\delta\}_{\delta \in \Delta} \cup \{0\}$ weakly compact.

What remains is to repeat exactly in the same manner the part of Theorem 3.6 showing that $\{y_\delta\}_{\delta \in \Delta}$ has a countable subset which does not define a weakly null sequence, thus reaching a contradiction and completing the proof. Q.E.D.

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