THE CUNTZ SEMIGROUP OF THE TENSOR PRODUCT OF C*-ALGEBRAS

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Abstract. We calculate the Cuntz semigroup of the tensor product of A with A. We restrict our attention to C*-algebras which are unital, simple, nuclear, stably finite, satisfy the UCT and absorbs the Jiang-Su algebra tensorially.

On calcule le semigroupe de Cuntz de l’algèbre du produit tensoriel $A \otimes A$. On considère seulement les C*-algèbres simples, nucléaires, à élément unité, stablement finies, $Z$-stables et satisfaisant au UCT.

Key words and phrases. C*-algebra, Cuntz semigroup, $K_0$-group, tensor product.
1. Introduction

The Cuntz semigroup has been studied since the late seventies but only recently has it proved to be an important invariant for C*-algebras. First, in the early 2000s, M. Rordam and A. Toms constructed, in separate works, examples of C*-algebras that appeared to be counter-examples to the Elliott conjecture. Shortly afterwards, Toms realized that the Cuntz semigroup distinguishes some of the newly constructed algebras; hence, the Cuntz semigroup should be added to the Elliott invariant. Toms’s discovery obviously prompted major questions, such as: What is the relation between the Cuntz semigroup and the Elliott invariant? or What is the range of the Cuntz semigroup? or What are the properties of the Cuntz semigroup? In this paper we propose to study one property of the Cuntz semigroup, namely how the Cuntz semigroup of the tensor product, $A \otimes A$, of two identical algebras relates to the Cuntz semigroup of $A$. It is well-known that not every positive element of a tensor product is equivalent to a tensor product of positive elements. Thus, the bilinear tensor product map from $A^+ \times A^+$ to $(A \otimes A)^+$ is in general not surjective. It seems interesting that, as we shall see, in some cases this map becomes surjective if we pass to Cuntz equivalence classes. In the language of mathematical physics — see, e.g., the introduction of [10] — one could say that in these cases, entanglements disappear when passing to the Cuntz semigroup. In a recent preprint [1], the question of determining surjectivity, at the level of Cuntz semigroups, of the natural tensor product map is posed; and left as an open problem. In that preprint, they observe that surjectivity does hold in the cases of AF algebras and $O_\infty$-stable algebras. We will consider the case of simple, unital, exact and stably finite C*-algebras, with strict comparison of positive elements.

Brown, Perera and Toms, [4], showed an important representation result for the original version of the Cuntz semigroup. This result was extended to the stabilized version of the Cuntz semigroup by Elliott, Robert and Santiago, [7], and with more abstract hypothesis by Tikuisis and Toms, [12]. Their results say that for certain simple exact C*-algebras, a part of the Cuntz semigroup is order isomorphic to an ordered semigroup of lower semicontinuous functions defined on a compact Hausdorff space.
2. The Cuntz semigroup

Let $A$ be a separable C*-algebra. For positive elements $a, b \in A \otimes K$, we say that $a$ is Cuntz subequivalent to $b$ and write $a \preceq b$, if $v_n b v_n^* \to a$, in the norm topology, for some sequence $(v_n)$ in $A \otimes K$. We say that $a$ is Cuntz equivalent to $b$ and write $a \sim b$, if $a \preceq b$ and $b \preceq a$. Denote by $Cu(A)$ the set of Cuntz equivalence classes of $A \otimes K$, i.e. $Cu(A) = (A \otimes K)_+/\sim$. The partial order $a \preceq b$ defined for the positive elements of $A \otimes K$ induces a partial order on $Cu(A)$: $[a] \leq [b]$ if $a \preceq b$, where $[a]$ denotes the Cuntz equivalence class of the positive element $a$. Note that (see [11, pg.151]) this partial order does not need to be the algebraic order with respect to the addition operation defined by setting $[a] + [b] := [a' + b']$, where $a'$ and $b'$ are orthogonal positive elements. It turns out that we can always find such orthogonal representatives, i.e. in $A \otimes K$ we have $a \sim a'$, $b \sim b'$ with $a'b' = 0$. Moreover, the choice of the orthogonal representatives does not affect the Cuntz class of their sum. So the ordered set $Cu(A)$ becomes an abelian semigroup. There also exists a transitive relation, $a \bowtie b$, termed compact containment, and elements $a$ that satisfy $a \bowtie a$ are termed compact elements. Compact containment will be further discussed later.

If $A$ is unital, denote by $T(A)$ the set of bounded traces on $A$. In the case $A$ is not a unital algebra, $T(A)$ will denote the set of lower semicontinuous densely defined traces on $A$. The set $T(A)$ turns out to be a compact Hausdorff space. If $A$ is a separable C*-algebra then $T(A)$ is also a metrizable space. By $V(A)$ we denote the projection semigroup defined by the Murray von Neumann equivalence classes of projections in $A \otimes K$. $Lsc(T(A), (0, \infty))$ denotes the set of lower semicontinuous, affine, strictly positive functions on the tracial state space of $A$.

2.1. Representations of the Cuntz semigroup. Brown, Perera and Toms’s representation result [4] for the Cuntz semigroup states:

**Theorem 2.1.** Let $A$ be a simple, unital, exact, stably finite $\mathcal{Z}$-stable C*-algebra. Then

$$W(A) \cong V(A) \bigcup Lsc(T(A), (0, \infty)).$$
Here \( W(A) \) is the original definition of the Cuntz semigroup, i.e., \( W(A) = M_\infty(A)/\sim \).

Elliott, Robert and Santiago’s representation result \cite{7} is very similar, and uses the stabilized Cuntz semigroup. In this result, the functions that appear may take infinite values.

**Theorem 2.2.** Let \( A \) be a simple, exact, stably finite \( \mathbb{Z} \)-stable \( C^* \)-algebra. Then
\[
Cu(A) \cong V(A) \bigcup Lsc(T(A), (0, \infty)).
\]

These theorems show that the Cuntz semigroup is the disjoint union of the semigroup of positive elements coming from projections in \( (A \otimes K)_+ \), denoted \( V(A) \), and the set of lower semicontinuous, affine, strictly positive, possibly infinite, functions on the tracial state space of \( A \), denoted by \( Lsc(T(A), (0, \infty)) \). In \cite{4}, the elements of the Cuntz semigroup that correspond to lower semicontinuous, affine, strictly positive, possibly infinite, functions on the tracial state space are termed *purely positive* elements.

This representation is very useful. For instance, it can be used to better explain the partial order on the Cuntz semigroup. To prove the representation result, a map \( i : Cu(A) \to Lsc(T(A), (0, \infty)) \) is being used, \( i([a])(\tau) = d_\tau(a) \) with \( d_\tau \) to be explained below. Then on \( V(A) \bigcup Lsc(T(A), (0, \infty)) \) we have a binary operation \(+\) given by BDF addition of projections when both elements are in \( V(A) \), point-wise addition when both elements are in \( Lsc(T(A), (0, \infty)) \), and in the mixed case
\[
p + f = i(p) + f.
\]

The partial order given by Cuntz subequivalence reduces to the usual partial order in each component, and for the mixed components we have:

(i) \( p \leq f \) if and only if \( i(p) < f \)

(ii) \( p \geq f \) if and only if \( i(p) \geq f \).

Motivated by Theorem 2.2, we introduce the following definition:

**Definition 2.3.** A *function semigroup* is a semigroup of the form \( Lsc(X, [0, \infty]) \) or \( Lsc(X, (0, \infty)) \), where \( X \) is a compact Hausdorff space.
Theorem 2.2 then shows that the subsemigroup of purely positive elements of certain Cuntz semigroups is isomorphic to a function semigroup. Urysohn’s Theorem from topology implies that the space $X$ is metrizable if and only if it is second countable. In general, the space $X$ is given by the set of all dimension functions; but as has been seen, in some cases, the space $X$ can be taken to be the set of lower semicontinuous densely defined traces on a C*-algebra $A$. In such cases, and perhaps in general, the separability of $A$ implies the metrizability of $X$.

Function semigroups have on them the topology of pointwise convergence. Cuntz semigroups have the weak topology induced by the dimension functions. Since the isomorphism between function semigroups and the subsemigroup of purely positive elements is such that evaluation at a point corresponds to pairing with a dimension function, it follows that this isomorphism, when it exists, preserves the natural topologies.

An important aspect of the Cuntz semigroup is the transitive relation given by compact containment. We say that $x$ is compactly contained in $y$, and write $x \vDash y$, if, given an increasing sequence $(z_n)$ with $y \leq \sup z_n$, there is an integer $m$ such that $x \leq z_m$. We note that this definition applies equally to Cuntz semigroups and to function semigroups.

In the context of function semigroups, we present the following characterization of compact containment:

**Lemma 2.4.** Assume that $f$ and $g$ are two elements in the function semigroup of a metrizable space. Then $f$ is compactly contained in $g$ if, and only if,

$$f < (g - \epsilon)_+ = \max\{g - \epsilon, 0\}$$

for some $\epsilon > 0$, where $1$ is the constant function equal to one.

**Proof.** Assume $g$ is continuous. If $f$ is compactly contained in $g$, defining $h_n := (g - \frac{1}{n})_+ = \max\{g - \frac{1}{n}, 0\}$, we have that $g \leq \sup h_n$. But then compact containment implies that there is an $n$ such that $f \leq (g - \frac{1}{n})_+$.

Conversely, assume $f \leq (g - 1\epsilon)_+$ for some $\epsilon > 0$. Assume a sequence of continuous functions $(h_n)$ is given such that $\sup h_n > g$. We can arrange that the sequence is nondecreasing by taking, if necessary, the pointwise maximum of increasingly large finite subsets of the sequence. Still denoting the sequence by $(h_n)$,
the pointwise minimum $\min(h_n, g)$ defines a nondecreasing sequence of continuous functions converging pointwise to $g$. By Dini’s theorem the convergence is uniform, so for some specific $n$ it is true that $\min(h_n(x), g(x))$ is within $\epsilon$ of $g(x)$. But then $h_n > g - 1\epsilon$ and hence $h_n > f$. This shows that $f$ is compactly contained in $g$.

The case that the functions $h_n$ are not continuous can be reduced to the continuous case, and it is here that we will use the hypothesis that the underlying space is metrizable. The functions $h_n$ are lower semicontinuous and hence each $h_n$ is the supremum of some infinite set of continuous functions. The algebra of continuous functions on a metrizable space is separable, and thus we may suppose that the infinite set of continuous functions considered is countable. We can thus replace the countable set $\{h_n\}$ by the (countable) union of the countable sets of continuous functions that we have obtained. Denoting this countable collection of continuous functions by $\{h'_n\}$, and repeating the argument above with $h'_n$ instead of $h_n$, we will obtain an $n$ and an $h'_n$ such that $h'_n > f$. Since $h'_n \leq h_m$ for some $m$, the conclusion follows.

\begin{remark}
Recall that an element $f$ is called compact if $f$ is compactly contained in $f$. It is a consequence of the above Lemma that under the hypotheses of Theorem 2.1 or Theorem 2.2, there are no compact elements except algebra projections: cf. [5, Corollary 5].
\end{remark}

As an application of the above criterion, we have:

\begin{lemma}
Assume $f'$ and $f$ are functions in a function semigroup over a metrizable space $X$. Let $g$ be an element of a function semigroup over a metrizable space $Y$. If $f'$ is compactly contained in $f$ then $f' \otimes g$ is compactly contained in $f \otimes g$.
\end{lemma}

\begin{proof}
Assume $g$ is not zero under any dimension function. This is possible as $g$ is lower semicontinuous, hence bounded below. Then add a small multiple of 1 to $g$ if necessary.

We have $f' \leq (f - \epsilon)_+$. Define $f'' := \min\{f', (f - \epsilon)\}$. $f''$ equals $f'$ wherever $f'$ is nonzero and may be negative only where $f'$ is zero. Furthermore

$$f'' < f - \epsilon.$$
It follows that

\[ f'' \otimes g \leq f \otimes g - \epsilon \mathbf{1} \otimes g. \]

Since \( g \) is lower semicontinuous, and the base space is compact, it follows that \( g \) attains its minimum. Since \( g \) is not zero at any point, the minimum is therefore nonzero, and there exists an \( \epsilon' > 0 \) such that \( g \geq \epsilon' \).

But then it follows that \( f'' \otimes g \leq f \otimes g - \epsilon \epsilon' \).

Consider the function \( h : \mathbb{R} \to \mathbb{R} \) that is equal to zero on the left half-line and is equal to \( f(x) = x \) on the right half line. The usual functional calculus simplifies to composition of functions when dealing with a function algebra. We can apply the nondecreasing function \( h \) to both sides of the above inequality of functions, obtaining again an inequality.

Applying \( h \) makes the left hand side zero wherever it was negative, and the left hand side is negative exactly when \( f'' \) is negative. We obtain \( f' \otimes g \leq h(f \otimes g - \epsilon \epsilon') \).

But this right hand side is equal to \( \max(f \otimes g - \epsilon \epsilon', 0) \). This proves that tensoring with elements that are not zero under any dimension function preserves compact inclusion.

\[ \square \]

2.2. Dimension functions and a conjecture of Blackadar and Handelman.

We just saw that the map \( i \) is useful in describing the order on the Cuntz semigroup. The map \( i \) is defined by means of the function \( d_\tau \), \( i(a) = d_\tau(a) \). We define \( d_\tau(a) \) to be an extended version of the rank of \( a \): \( d_\tau(a) = \lim_{n \to \infty} \tau(a^{1/n}) \), for \( \tau \in T(A) \). This map, \( d_\tau \), also called a dimension function, is lower semicontinuous, possibly taking infinite values, and a state on the Cuntz semigroup. Being a lower semicontinuous dimension function is equivalent (in the abstract setting) to saying that the function preserves suprema of increasing sequences, or that the corresponding trace is lower semicontinuous.

The set of all states on the Cuntz semigroup is the set of all additive and order preserving maps \( d : W(A) \to \mathbb{R}^+ \) such that \( d([1]) = 1 \). (In the case of the stabilized Cuntz semigroup, infinite values may occur.) We will denote the set of states by \( D(A) \). In 1982, Blackadar and Handelman conjectured, see [3], that the set of lower semicontinuous dimension functions is weakly dense in the set of dimension
functions (or states on the Cuntz semigroup). The conjecture is known to be true for a large class of C*-algebras that includes the algebras that we propose to study in this paper, namely: simple unital exact and stably finite C*-algebras with strict comparison of positive elements. From now on, we thus assume that all dimension functions come from a trace. In other words, if \( d: Cu(A) \to [0, \infty] \), is a dimension function, then there is a trace, \( \tau \), on \( A \) such that \( d(a) = d_\tau(a) \) for any positive element \( a \in A \).

Consider the map \( t: A^+ \times A^+ \to Cu(A \otimes A) \) defined by
\[
t(a, b) = [a \otimes b].
\]

We now check that the above map \( t \) respects Cuntz equivalence.

**Lemma 2.7.** Let \( A \) be a \( \sigma \)-unital C*-algebra. Given positive elements \( a, a', b \) in \( A \) such that \( a' \preceq a \) we have \( a \otimes b \preceq a' \otimes b \).

**Proof.** Let \( e_n \) be a countable approximate unit. Since \( a' \preceq a \), let \( c_n \) be such that \( c_nac_n^* \to a' \). We have
\[
(c_n \otimes e_n)(a \otimes b)(c_n \otimes e_n)^* - a' \otimes b = c_nac_n^* \otimes e_nbe_n^* - a' \otimes b.
\]

Then it follows from the properties of approximate units that
\[
||(c_n \otimes e_n)(a \otimes b)(c_n \otimes e_n)^* - a' \otimes b||
\]
goes to zero as \( n \) goes to infinity. \( \square \)

If \( a \sim a' \) then \( a \otimes b \sim a' \otimes b \) by applying the Lemma twice, and thus we obtain the Corollary:

**Corollary 2.8.** Consider the map \( t: A^+ \times A^+ \to Cu(A \otimes A) \) defined by \( t(a, b) = [a \otimes b] \). If \( a \) and \( a' \) are positive elements of \( A \) that are Cuntz equivalent, then \( t(a, b) = t(a', b) \).

### 3. Main result

**Lemma 3.1.** Let \( F \) be a continuous and positive function defined on the product of compact Hausdorff spaces \( X \) and \( Y \). There exists an approximation in the supremum norm, of \( F \), composed of continuous positive function of the form \( \sum_i f_i(x)g_i(y) \) with \( f_i \) and \( g_i \) continuous positive functions defined on \( X \) and \( Y \),
respectively. In addition, the approximation is bounded above by \( F \) with respect to the usual ordering of functions.

**Proof.** Let \( d_x \) and \( d_y \) be the metrics on \( X \) and \( Y \), respectively. Let \( d \) be the product metric on \( X \times Y \) given by

\[
d((x_1, y_1), (x_2, y_2)) = \max(d(x_1, x_2), d(y_1, y_2)).
\]

Let \( \epsilon > 0 \). Because \( F \) is a continuous function on a compact space, it has the uniform continuity property and hence there exists \( \delta > 0 \) such that \( |F(x_1, y_1) - F(x_2, y_2)| < \epsilon \) for any \( (x_1, y_1), (x_2, y_2) \) with \( d((x_1, y_1), (x_2, y_2)) < \delta \).

The sets of open balls \( B(x, \delta/2), x \in X \) and \( B(y, \delta/2), y \in Y \) are open covers of \( X \) and \( Y \), respectively. Because \( X \) and \( Y \) are compact, we can find finite open sub-covers \( U = B(x_i, \delta/2) \) and \( V = B(y_j, \delta/2) \) for \( X \) and \( Y \), respectively. Let \( f_i \) and \( g_j \) be partitions of unity subordinated to the open covers \( U \) and \( V \), respectively.

The product set \( B(x_i, \delta/2) \times B(y_j, \delta/2) \) has the property that, for any two points \( (x_1, y_1), (x_2, y_2) \) in the set:

\[
d((x_1, y_1), (x_2, y_2)) \leq d((x_1, y_1), (x_i, y_j)) + d((x_i, y_j), (x_2, y_2))
\]

\[
\leq \max(d(x_1, x_i), d(y_j, y_2)) + \max(d(x_1, x_2), d(y_j, y_2)) < \delta.
\]

Moreover, because of uniform continuity of \( F \), \( |F(x_1, y_1) - F(x_2, y_2)| < \epsilon \). If we define \( a_{ij} \) to be the infimum value of \( F \) on the product set \( B(x_i, \delta/2) \times B(y_j, \delta/2) \), then \( 0 \leq F(x_1, y_1) - a_{ij} < \epsilon \) for any \( (x_1, y_1) \) in the product set.

We consider the positive function \( \sum a_{ij} f_i(x) g_j(y) \). Using the partition of unity property, we have for any \( (x, y) \in X \times Y \)

\[
F(x, y) - \sum a_{ij} f_i(x) g_j(y) = \sum (F(x, y) - a_{ij}) f_i(x) g_j(y) < \sum \epsilon f_i(x) g_j(y) = \epsilon.
\]

This shows that for any \( \epsilon > 0 \) we can find a positive function of the form \( \sum a_{ij} f_i(x) g_j(y) \) that is bounded above by \( F \) and within supremum norm distance \( \epsilon \) from \( F \). This completes the proof of the lemma. \( \square \)

The next Theorem is of interest in the setting of stably projectionless C*-algebras.
Theorem 3.2. Assume \( A \) and \( B \) are \( C^* \)-algebras such that \( \text{Cu}(A) \cong \text{Lsc}(T(A), (0, \infty)) \) and \( \text{Cu}(B) \cong \text{Lsc}(T(B), (0, \infty)) \). Then the tensor product map \( t \) from \( \text{Cu}(A) \times \text{Cu}(B) \) to \( \text{Cu}(A \otimes B) \) is surjective.

Proof. Define \( \Phi : \text{Lsc}(T(A), (0, \infty)) \times \text{Lsc}(T(B), (0, \infty)) \to \text{Lsc}(T(A \times T(B), (0, \infty)) \), a semigroup morphism that acts on the elements \( (f, g) \) of \( \text{Lsc}(T(A), (0, \infty)) \times \text{Lsc}(T(B), (0, \infty)) \) as

\[
\Phi(f, g)(x, y) = f(x)g(y).
\]

Note that the product of non-negative lower semicontinuous functions is a lower semicontinuous function. The given representations of the Cuntz semigroups intertwine \( \Phi \) with the tensor product map \( t \).

We now consider the surjectivity of the map \( \Phi \). Let \( g \) be an element of \( \text{Lsc}(T(A \otimes B), (0, \infty)) \cong \text{Lsc}(T(A), (0, \infty)) \otimes \text{Lsc}(T(B), (0, \infty)) \). Lemma 3.1 provides an element \( F_n \) of the form \( \sum_i^n f_i(x)g_i(y) \) that is bounded above by \( g \) and approximates \( g \) within \( \frac{1}{n} \). Since the functions \( f_i \) and \( g_i \) are positive, the element \( F_n \) is in the image of the map \( \Phi \). The supremum of the sequence \( (F_n) \) in the function representation of \( \text{Cu}(A \otimes B) \) is \( g \). \( \square \)

3.1. Beyond the purely positive case. The next step is to assume our algebra \( A \) has non-trivial projections, hence \( V(A) \), the projection monoid in the representation of the Cuntz semigroup of \( A \), is non-trivial.

We make use of the assumption that the Cuntz semigroup is the disjoint union of projection elements from the Cuntz semigroup, denoted \( V(A) \), and purely positive elements, denoted \( \text{Lsc}(T(A), (0, \infty)) \). Ara, Perera and Toms \([2]\), prove that that for algebras of stable rank one, the Cuntz class of a positive element is given by a projection if and only if \( \{0\} \) is not in the spectrum or if it is an isolated point of the spectrum, see \([2]\).

If \( a \in M_\infty(A)_+ \), \( b \in M_\infty(B)_+ \) are positive elements then it follows that \( a \otimes b \) is a positive element in \( M_\infty(A \otimes B) \). This induces a bilinear morphism from \( \text{Cu}(A) \times \text{Cu}(B) \) to \( \text{Cu}(A \otimes B) \) which in turn induces a natural Cuntz semigroup map

\[
\psi : \text{Cu}(A) \otimes \text{Cu}(B) \to \text{Cu}(A \otimes B)
\]

\[
\psi([a] \otimes [b]) = [a \otimes b].
\]
In the case that $A = B$ and $A$ is a simple, unital, nuclear, $\mathcal{Z}$-stable C*-algebra, with vanishing $K_1$ group, we now show that the map $\psi$ is surjective:

**Lemma 3.3.** *If a C*-algebra $A$ is unital, simple, stably finite, satisfies the UCT, and has $K_1(A) = \{0\}$, then the natural map*

$$\psi : Cu(A) \times Cu(A) \to Cu(A \otimes A),$$

$$\psi([a] \otimes [b]) = [a \otimes b]$$

*is an isomorphism when restricted to $V(A \otimes A) \cong V(A) \otimes V(A)$.*

**Proof.** $K_0(A)$ is determined by the subsemigroup $V(A)$ of the Cuntz semigroup, in the sense that $K_0(A)$ is the Grothendieck group generated by $V(A)$:

$$K_0(A) = G(V(A)).$$

Our algebra $A$ is assumed to satisfy the UCT, so then $A$ will satisfy the K"unneth formula for tensor products in $K$-theory, which means that there is a short exact sequence

$$0 \to K_0(A) \otimes K_0(A) \oplus K_1(A) \otimes K_1(A) \to K_0(A \otimes A) \to$$

$$Tor(K_0(A), K_1(A)) \oplus Tor(K_1(A), K_0(A)) \to 0$$

Since we assume $K_1(A) = 0$ it follows that

$$K_0(A) \otimes K_0(A) \to K_0(A \otimes A)$$

is an isomorphism. Moreover, since the algebra $A$ has stable rank one, it follows that the above isomorphism restricts to an isomorphism between the projection monoids

$$V(A) \otimes V(A) \to V(A \otimes A).$$

$\square$

**Remark 3.4.** The argument of the above Lemma can be adapted to provide a class of counter-examples to the possible surjectivity of the tensor product map
$t: Cu(A) \times Cu(B) \to Cu(A \otimes B)$. If an algebra is in the UCT class, and the $K$-theory groups are such that the last term in the Künneth sequence does not vanish, then we see that the first map in the short exact sequence

$$0 \to K_0(A) \otimes K_0(A) \oplus K_1(A) \otimes K_1(A) \to K_0(A \otimes A) \to Tor(K_0(A), K_1(A)) \oplus Tor(K_1(A), K_0(A)) \to 0$$

will not be surjective. But then, in particular, the tensor product map from $K_0(A) \otimes K_0(A)$ to $K_0(A \otimes A)$ will not be surjective. Hence the tensor product map at the level of projection semigroups will not be surjective either, and this is an obstacle to the surjectivity of the tensor product map at the level of Cuntz semigroups (since the tensor product of elements that are equivalent to a projection is an element that is equivalent to a projection).

**Theorem 3.5.** If a C*-algebra $A$ is unital, simple, stably finite C*-algebra, $\mathcal{Z}$-stable and satisfies the UCT, and has $K_1(A) = \{0\}$, then the natural Cuntz semigroup map

$$t : Cu(A) \times Cu(A) \to Cu(A \otimes A)$$

given by $t([a], [b]) = [a \otimes b]$ is a surjective map.

**Proof.** Under these hypotheses, any element $x$ in $Cu(A \otimes A)$ is either in $V(A \otimes A)$ or in $Lsc(T(A \otimes A), (0, \infty))$.

In the case that $x$ is in $V(A \otimes A)$, because $V(A \otimes A) \cong V(A) \otimes V(A)$, Lemma 3.3 gives an element $y \times z$ with $y \in M_\infty(A)^+$, $z \in M_\infty(B)^+$ so that

$$t(y, z) = x.$$ 

In the case that $x$ is in $Lsc(T(A \otimes A), (0, \infty))$, by Theorem 3.2, there are $y, z \in Lsc(T(A), (0, \infty))$ such that

$$t(y, z) = x$$

which completes the proof. \qed
4. The kernel of a semigroup map

Since we have only a semigroup structure, a definition for the kernel of a map is required. One might be tempted to use the inverse image of a neutral element, motivated perhaps by the special case of group homomorphisms. An example may clarify this situation.

**Example 4.1.** Let \( S = \mathbb{N} \cup \{\infty\} \) and \( T = \{0, \infty\} \) be two semigroups and \( \phi : S \to T \) be given by \( \phi(0) = 0 \) and \( \phi(a) = \infty \) if \( a \neq 0 \). If the kernel is taken to be the inverse image of 0, then \( \{ s \in S : \phi(s) = 0 \} = 0 \), even though \( \phi \) is far from being one to one.

Thus we revert to the basic set-theoretical definition of injectivity: a map \( f \) is injective if and only if the property \( f(x) = f(y) \) implies that \( x = y \). This motivates the following definition of the kernel of a map.

**Definition 4.2.** Assume \( S \) and \( T \) are two ordered abelian semigroups and \( \phi : S \to T \) a semigroup map that preserves the order. We define the kernel of \( \phi \), written \( \text{Ker} \phi \), to be \( \{(s_1, s_2) \in S \times S : \phi(s_1) = \phi(s_2)\} \).

The kernel set we have just defined will always contain the diagonal elements of \( S \times S \), and the map \( \phi \) is injective if and only if the diagonal elements are the only elements in the kernel. Moreover, the kernel set is a sub-semigroup of the semigroup \( S \times S \). Note that in the special case of homomorphisms of finite groups, the kernel set that we just defined will generally have larger cardinality than the usual kernel of a group map. The above definition is very similar to — but not identical to — a definition used in [6].

5. Tensor products of Cuntz semigroups

In this paper, we study Cuntz semigroups through their representations as a disjoint union of projection elements from \( V(A) \) and purely positive elements in \( Lsc(T(A), (0, \infty)) \). Thus, the most immediate way to define a tensor product of Cuntz semigroups for algebras in our class is simply to define it through the evident tensor product(s) on the representations. This approach to the tensor product has the merit of being obviously compatible with the usual tensor product of functions.
in \( Lsc(T(A), (0, \infty)) \), and the usual tensor product on \( K \)-theory. Even though this approach is sufficient and well suited to our needs, it seems reasonable to make a few further comments. An alternative approach to the tensor product \( Cu(A) \otimes Cu(B) \), based on Grothendieck’s inductive (or injective) tensor product\[^9\] \( \text{Définition 3} \), is as follows. First form the algebraic tensor product of abelian semigroups, see \[^8\], and then view an element of \( Cu(A) \otimes Cu(B) \) as a function on \( D(A) \times D(B) \), where \( D(A) \) and \( D(B) \) denotes the dimension functions on the Cuntz semigroup(s).

The inductive topology is the topology induced by this embedding. Thus, in the inductive tensor product \( Cu(A) \otimes Cu(B) \), a sequence \( t_n \) converges to \( x \) if and only if \( (d_1 \otimes d_2)(t_n) \) converges to \( (d_1 \otimes d_2)(x) \) for all \( d_1 \in D(A) \) and \( d_2 \in D(B) \). In general, of course, since we regard \( Cu(A) \) as being topologically a disjoint union of two sets, namely the set of elements having the same class as a projection, and the set of purely positive elements, the tensor product of Cuntz semigroups will be a disjoint union of four sets.

The surjective bilinear map \( t : Cu(A) \times Cu(A) \rightarrow Cu(A \otimes A) \) induces a map, also denoted by \( t \), from \( Cu(A) \otimes Cu(A) \rightarrow Cu(A \otimes A) \). In general, when considering the tensor product map \( t \), the following three cases will appear:

Case 1: projection elements tensored with projection elements,

Case 2: purely positive tensored with purely positive elements, and

Case 3: projection elements tensored with purely positive elements.

**Theorem 5.1.** Suppose that \( A \) is a \( C^* \)-algebra that satisfies the Blackadar–Handelman conjecture, and has stable rank 1. Each of the maps

\[
\begin{align*}
t : Cu(A)|_{\text{pure}} \otimes Cu(A)|_{\text{pure}} &\rightarrow Cu(A \otimes A) \\
t : Cu(A)|_{\text{pure}} \otimes Cu(A)|_{\text{proj}} &\rightarrow Cu(A \otimes A) \\
t : Cu(A)|_{\text{proj}} \otimes Cu(A)|_{\text{pure}} &\rightarrow Cu(A \otimes A) \\
t : Cu(A)|_{\text{proj}} \otimes Cu(A)|_{\text{proj}} &\rightarrow Cu(A \otimes A)
\end{align*}
\]

is injective. The first three of these maps have range contained in the purely positive elements of \( Cu(A \otimes A) \), the last map has range contained in the projection-class elements of \( Cu(A \otimes A) \).
Proof. We first consider the range of these maps. A positive element is projection-class if and only if the spectrum of the element has a spectral gap at zero, and the spectrum of $a \otimes b$ is given by the set of all pairwise products \( \{ \lambda \mu \mid \lambda \in \text{Sp}(a), \mu \in \text{Sp}(b) \} \). It can thus be seen that $a \otimes b$ has the class of a projection if and only if both $a$ and $b$ have the class of a projection.

We now consider the injectivity of these maps. Suppose that $x, x' \in Cu(A) \otimes Cu(A)$ are such that $t(x) = t(x')$. Consider first the case where $t(x)$ and $t(x')$ belong to the purely positive part of $Cu(A) \otimes Cu(A)$. We have that $d \circ t(x) = d \circ t(x')$ for all dimension functions $d$ on $A \otimes A$. Choose a dimension function $d$ on $A \otimes A$ that comes from a tensor product $\tau_1 \otimes \tau_2$ of traces on $A$. Thus, if $t$ is the map taking $[a] \otimes [b]$ to $[a \otimes b]$, and $d(t(x)) = d(t(x'))$, we deduce that $(d_1 \otimes d_2)(x) = (d_1 \otimes d_2)(x')$ where $d_1$ and $d_2$ are the dimension states on $Cu(A)$ that come from $\tau_1$ and $\tau_2$ respectively. Since $A$ satisfies the Blackadar-Handelman conjecture, this means that $d_1$ and $d_2$ can be chosen to be (approximately) equal to any two dimension states of $A$. But this means that $x$ and $x'$ are equal in the tensor product $Cu(A) \otimes Cu(A)$. The two mixed cases are similar. The case of projections follows from the fact that the natural map from $V(A) \otimes V(A)$ to $V(A \otimes A)$ is an injection, just as in the proof of Lemma 3.3 in the presence of the stable rank one condition.

Combining the above with Theorem 3.5, we have the Corollary:

**Corollary 5.2.** If a C*-algebra $A$ is unital, simple, stably finite, Z-stable, satisfies the UCT, and has $K_1(A) = \{0\}$, then the natural Cuntz semigroup map

$$t : Cu(A) \otimes Cu(A) \to Cu(A \otimes A)$$

given by $t([a], [b]) = [a \otimes b]$ is surjective, and becomes an isomorphism when restricted to

$$Cu(A)_{\text{pure}} \otimes Cu(A)_{\text{pure}} \bigcup Cu(A)_{\text{proj}} \otimes Cu(A)_{\text{proj}}.$$

Thus the mixed elements of the tensor product are evidently an obstacle to injectivity of the unrestricted tensor product map. We note the following further Corollary:

**Corollary 5.3.** If a C*-algebra $A$ is unital, simple, stably projectionless, stably finite, Z-stable, satisfies the UCT, and has $K_1(A) = \{0\}$, then the natural Cuntz
semigroup map  

\[ t : Cu(A) \otimes Cu(A) \to Cu(A \otimes A) \]

given by \( t([a], [b]) = [a \otimes b] \) is an isomorphism.

Remark 5.4. The tensor product of two abelian semigroups is constructed by forming a free abelian semigroup and passing to the quotient in which \((a + a') \otimes b = a \otimes b + a' \otimes b\) and the symmetric identity holds: \((a' \otimes b')(a \otimes b) = aa' \otimes bb'\). One can instead define the tensor product as an abstract abelian semigroup on which additive maps are given by biadditive maps on the original semigroups. In a recent preprint [1], a definition of the tensor product of Cuntz semigroups is given which has a categorical flavor. However, Grillet in [8], Theorem 2.1, shows that, as for any universal property definition, the tensor product of abelian semigroups is unique up to semigroup isomorphism. In most applications of the Cuntz semigroup, the algebraic structure of the Cuntz semigroup is of primary importance. In such a case, in view of Grillet’s result, the topological considerations of this section are evidently irrelevant, and the inductive tensor product construction due to Grothendieck is effectively the same as the tensor product definition from [1].

In any case, the surjectivity result of Theorem 5.5 provides a version of our main result that is obviously independent of any choice of tensor product.

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