An APOS analysis of student’s understanding of quadratic function graph

E Listiawati\(^{1,2}\) and D Juniati\(^3\)
\(^1\)Postgraduate Study of Mathematics Education, Universitas Negeri Surabaya, Indonesia
\(^2\)Department of Mathematics Education, STKIP PGRI Bangkalan, Indonesia
\(^3\)Department of Mathematics, Universitas Negeri Surabaya, Indonesia

Enny.19001@mhs.unesa.ac.id, dwijuniati@unesa.ac.id

Abstract. The purpose of this study is to describe students' understanding of the graph of quadratic functions. In this study, APOS theory (Action, Process, Object and Schema) used as a theoretical framework. The subjects of this study were two students at a college of education, one male and one female who had equal mathematics abilities. Data collection techniques were written tests and semi-structured interviews. The data generated were analyzed through the stages of data reduction, data presentation and conclusion. The results showed that at action of male subjects drawing graphs directly in the Cartesian plane and uses the general form of quadratic functions while female subject makes a table of points to make graphs and used the general form. At process, male subject enters several numbers to the equation then connects the points but female determines the intersection points with the coordinate axis, looks for other points, looks at the direction of the graph then draws the graph. At object, male subjects make a graph by shifting the top of the other graph, while female determine the function formula by adding constants from the original graph. In this study the two subjects were not yet at the stage of the scheme.

1. Introduction

Function is a central topic of calculus that relates to other courses with higher level on their hierarchy, such as limits, continuity, derivative, integral, and the others dealing with function [1]. In learning about functions, of course understanding function graphs cannot be dismissed. Function graphs play an significant part in understanding the nature and characteristics of the relationship between two variables in a function. By observing the function graphs, students can determine a relationship between two variables and make predictions [2]. Understanding the graph of the quadratic function is very important for students because it is a fundamental subject before students work with functions with higher degrees. On the other hand, it is important to study quadratic functions because the concept of quadratic functions is used later, when student learn the more complex polynomial functions [3]. However, in reality, students have difficulty solving problems related to the graph of the quadratic function. This is because students have difficulty understanding the quadratic function [4]. Students have difficulty in making a relationship between the two existing variables [5]. Besides, they also do not understand the benefits and applications of the subject in everyday life [6]. Quadratic function problems have been widely used in investigating problems [7]. Likewise, this material is also used in secondary and early years of college education in both science and mathematics education. Therefore, the material needs to be discussed further on the reasons why many students fail to solve problems related to this topic [8].

Related to the quadratic function graph, there are four activities carried out by students in translating from function to symbolic graph [9]. These activities are unloading resources, initial
coordination, establishing targets, and determining equity. Further research on the translation of mathematical representations other than graphs to symbols is also needed in completing a more detailed examination in the translation process [9]. Create key concepts that are constantly emphasized in mathematics planning and learning. If students understand quadratic functions and their properties and applications, it will be easier for them to build and develop a good understanding of the various types of functions that are more complex and different [10]. The belief that students' understanding of mathematics is built on experience is also a direction for teachers to help students move forward to graphing the quadratic function. A review of the student learning literature on quadratic functions and student management of quadratic equations revealed that existing research is based on the procedural aspects of solving quadratic equations, with a small amount of research on how to understand variables and graph functions [11]. Very little is known about students' understanding of how graphs and quadratic functions are related.

Thus, further research is needed to see the mental structure of students in solving problems related to graphs of quadratic functions. Dubinsky [12] proposed a theory of mathematical concept construction through 4 (four) stages, namely action, process, object, and schema. These four stages are then called the APOS framework. Investigating the understanding of a concept using the APOS stages will make it easier to find out what students understand step by step and detect which parts are not yet understood [13]. By using this theory, it will be known how a person understands a concept from starting to take external action, processing knowledge internally, strengthening knowledge as an object and finally, until a schema is formed about the concept being constructed.

The purpose of this study was to analyze students' understanding of the concept of quadratic functions using the APOS (Action, Process, Object, Schema) theory as a lens.

2. Theoretical Framework

This study uses the APOS Theory framework which will study the level of cognitive development of students who complete the calculus course on the quadratic function graph material. The use of the APOS theory is due to studying students from a variety of different mathematical concepts, many studies have used this theory and it has been shown to provide important insights for students studying mathematics [4]. It has been tested in the classroom and has proven effective in teaching students various concepts and developing classroom activities. APOS theory postulates that mathematical concepts develop when a person tries to change an existing physical or mental object [14]. Thus even though students work in the same mathematical situation, the mental structures built by each student can cause differences in the way students solve existing problems. It involves general descriptions of mental and mental structures [15]. There are five types of mental types namely interiorization, coordination, inversion, encapsulation, and thematization which allow mental construction; actions, processes, objects, and schemas [16]. The description of Action, Process, Object, and Schema are given below is based on that given in Arnon et al [15].

**Action:** a transformation is first implemented as an action when it is a reaction to a stimulus perceived as an external individual. It consists of following specific external rules, and the need to make each step of the transformation explicit. **Process:** when an individual examines and reflects on an action or series of actions, that action can be internalized into a mental process. The process is a mental structure that performs the same operations as an action or group of actions but in the mind of the individual. In particular, individuals can bring about transformation without having to take each step explicitly. The process can be coordinated into a new and reversed process. **Object:** If one realizes a process as a totality, realizes that we can act as acting on that totality, and can actually construct such a transformation (explicitly or in a person), then one will summarize the process into a cognitive object. **Schema:** construction of mathematical concepts often involves several actions, processes, objects, and the preceding. Mathematical schemes are coherent collections of actions, processes, objects, and other schemes built beforehand, which are synthesized to form mathematical structures used in problem situations [17]. A structured schema is structured and presented into a coherent framework, called a schema. It is coherent because it provides the individual with a means of deciding when presented with a particular mathematical situation, whether the schema is valid or not. Schemas are dynamic structures,
they are constructed and reconstructed through individual learning processes, and are used to work in mathematical situations.

APOS theory emphasizes the important role of one's mathematical schema in constructing new knowledge. Previously mastered concepts in mathematics, if not summarized to a stage where students can apply them to new situations, can work against the acquisition of new concepts because old concepts and new concepts are seen as separate, difficult to work together [18]. The phrase "before the object" refers to a cycle of individuals developing mathematical ideas, which "begins when a new idea is themed and ends when it becomes the basis for higher-level concepts." This cycle adds another layer in the system of mathematical concepts.

One of the important tools used in APOS-based research is genetic decomposition which is a hypothetical model of mental constructs that students may need to create to learn mathematical concepts [15], [19]. In this study, the authors' learning and teaching experiences, along with the study by Arnon et al. [15], was used to compile the initial genetic decomposition to describe the possible mental constructs that students need in solving problems related to graphing quadratic functions.

3. Method
The type of the research was qualitative research using descriptive data. The subjects in this study were two students at STKIP PGRI Bangkalan who had taken a calculus course which consisted of one male subject and one female subject. The subjects were selected from 15 students who had taken calculus lectures. Subjects selection was based on the equivalent mathematical ability of the subject. The instruments in this study were the task about quadratic function graph and the interviews. The data methods used were tests, observations, and interviews [20].

Data collected by quadratic function graphs test and interview guidelines used to obtain data about student understanding of quadratic function graphs based on APOS theory, interview-based assignment guidelines as semi-structured interviews as researchers' guidelines for obtaining data. The quadratic function graphing assignment was designed in an essay test and was developed by researchers and consulted by experts. The data analysis stage includes reduction, data presentation, and concluding. Data reduction is done by sharpening, grouping, and organizing the data obtained from tests and interviews related to student understanding of the quadratic function graph based on the APOS theory based on gender. This means that at this stage, the process of selecting and simplifying data during the research process is carried out by removing unnecessary data and focusing on important information. Data presentation is the organization of data about the information that has been collected to conclude. At this stage, the researcher presents the data from data reduction. Data are identified and presented descriptively. The data that has been presented makes it easier for researchers to understand the data to conclude. The conclusion in this study is to describe students’ understanding of the quadratic function graph based on the APOS theory based on gender.

4. Results and Discussion
This section will discuss the results and discuss the results based on the data obtained related to student understanding of the quadratic function graph based on the APOS theory according to gender. The research data was obtained by giving the quadratic function assignment to the subject and then task-based interviews. The following questions are used in this study as a task for the subject regarding the quadratic function material:

Question 1:
   a. Graph \( f(x) = x^2 \)
   b. Graph \( f(x) = x^2 - 1 \)
   c. Graph \( f(x) = 2x^2 \) and \( f(x) = 2x^2 - 2 \)

Question 1 is used to see the mental structure of students when drawing a graph of the function if the function formula is given.
Question 2:
Determine the equation for the function from the following graph:

![Graph Image]

4.1 Understanding of Male Subjects on the Quadratic Function Graph
When graphing a quadratic function when given the quadratic function formula, the subject has no difficulty. The male subject directly draws a graph on the Cartesian plane as shown below:

![Graph Image]

Figure 1. Answers to Male Subjects to Questions 1b

From the subject's answer, it can be seen that in this case, a mental process of the action occurs on the subject. Furthermore, the researcher asked the subject again how to draw the graphic. Then the subject answered:

Dena: I enter the equation for the function equation $f(x) = x^2 - 1$, after the calculation is complete, then I mark it in the cartesian coordinates, then I draw a line through the points.

Based on the subject's answer, the subject explains in detail how to draw a graph, namely by looking for points that fulfill the given function equation, then the subject is connected to these points to form a graph. Thus, the subject is in the cognitive structure of the process according to the APOS theory.

In addition to drawing a graph by finding points that satisfy the equation of the function, the subject also draws a graph of $f(x) = 2x^2 - 2$ by shifting the vertex of the graph of the function $f(x) = 2x^2$ to point $(0, -2)$ so that the graphic image $f(x) = 2x^2 - 2$ is obtained as shown in figure 2 below:
Figure 2. Answers to Male Subjects to Question 1c

Based on Figure 2, it can be seen that the subject is wrong in drawing the graph of the function of the given function because the subject does not look for points that fulfill the function equation but by looking for the relationship between the given function and the graphic image of the previous problem which has a similar formula, thus the subject has a mental construction at the object stage. The following are the results of the interview with the subject:

Researcher: How did you draw the graph?
Dena: You can do this by looking at the previous graph, you can also analyze it first, for example there is a graph of $f(x) = 2x^2$ the vertex is at (0,0), then for the graph $f(x) = 2x^2 - 2$ the vertex is at (0, –2), so you can service that the key is in the back number if there are more examples like $f(x) = 2x^2 - 10$ so you can be sure the peak is at (0, -10).

Based on the interview, it can be seen that the subject found a relationship between the constants in the function equation and the vertex of a graph. The subject of the statement is that if the constants in the function equation change, the vertices of the graph also change. The subject can also provide other examples that have similar formulas to the given problem. At this stage, the construction of the mental subject is at the object stage.

4.2 Understanding of Female Subjects on the Quadratic Function Graph

When graphing the quadratic function when given the quadratic function formula, the subject did not experience difficulties. The subject uses the table to determine the points that satisfy the quadratic function equation in the problem. The subject is also the coefficient of $x^2$, in the example, the coefficient is 1 which is more than zero, so that the subject can identify that the graph of $f(x) = x^2$ is open upward. Then with the help of several known points, the subject immediately draws a graph in the Cartesian plane as shown below:

Figure 3. Answers to Female Subjects to Question 1a
From the subject's answer, it can be seen that in this case, a mental process of the action occurs on the subject. Furthermore, the researcher asked the subject again how to draw the graphic. Then the subject answered:

Desi: The first is to determine the points by assuming $x = 0$, then $f(x) = 0$ is also obtained after entering into the equation $f(x) = x^2$. Because $f(x) = x^2$ is a quadratic function then the graph will be in the form of a parabola and since $a > 0$ the graph will open upwards. Then plot the points obtained so that the parabola is produced as shown.

Based on the subject's answer, the subject explains in detail how to draw a graph, namely by looking for points that meet the given function equation, then the subject is connected to these points to form a graph. Thus, the subject is in the cognitive structure of the process according to the APOS theory. When the subject is given another problem to graph the quadratic function from the quadratic function formula, namely $f(x) = 2x^2$, the subject also works in the same way as the previous problem. Like figure 4 below:

![Figure 4. Answers to Female Subjects to Question 1c](image)

Based on Figure 4, it can be seen that the subject uses the same procedure for all the questions to draw a quadratic function graph, the subject does not find a relationship pattern between the given function formula and the graph. Thus the subject is not yet in the structure of object cognition and schema according to APOS theory. Whereas in question 2, which is to determine the quadratic function formula if the graph is given, the subject uses the general form of the quadratic function $y = a(x - x_1)(x - x_2)$ as shown below:

![Figure 5. Answers to Female Subjects to Question 2b](image)
Based on Figure 5, it can be seen that the subject in this case is a mental process of action on the subject. Then the researcher asked the subject again how to get the quadratic function formula. Then the subject answered:

Desi: For number 2 I first determine the equation which corresponds to what is known on the graph and the graph is tangent to the x-axis with the vertex (2,0) and through another point at (0,4), then the equation fits the graph this will be in the form \( y = a(x - x_p)^2 + y_p \). Then first look for the value of \( a \) and get \( a = 1 \) then substitute point (0,4), and vertex (2,0) to the equation \( y = a(x - x_p)^2 + y_p \) and the equation \( y = x^2 - 4x + 4 \).

Based on the subject's answer, the subject explains in detail how to determine the functional formula from the given graph. The subject who uses the graph tangent to the x-axis, namely the vertex (2,0), then substitutes this value in the general form of the quadratic function. Thus, the subject is in the cognitive structure of the process according to the APOS theory.

In addition to using the general form formula for the function, the subject also realizes that the function formula can be obtained by looking at the graphical form of the questions in 2a and 2b which have the same graphical shape but different peak points, this can be seen in the following interview:

Desi: I see that in 2a the constant in the quadratic function formula is 3 and in 2b the constant in the function formula is 4. The vertices in problems 2a and 2b have the same x coordinate only differ in the y coordinate, so I see when \( y_p = -1 \), then the constant is 3 and when \( y_p = 0 \), then the constant is 4 so I think if with the same picture but the only \( y_p \) is different then if the vertex is shifted to (2,1) then the constant will increase by 1 which becomes 5, while the value of the \( x^2 \) and \( x \) coefficients does not change.

From the interview, it can be seen that the subject finds a relationship between the vertex on the two graphs with the constant in the obtained function formula, that is, if they coordinate at the vertex increases by 1, the constant in the function formula also increases by 1. Thus, the subject is in the mental construction of the object, based on the APOS theory.

Based on the research results, it was found that the two subjects had different methods both when drawing graphs of functions and when determining the quadratic function formula. This shows that there are differences in understanding between male and female subjects in solving math problems. Male subjects solve math problems directly without writing down the solution, while female subjects write in full the method used [21]. Another finding shows that male subjects have difficulty in drawing a graph of the quadratic function. This is because the subject experiences confusion in translating graphics into mathematical symbols in the form of function formulas. This is because students with different ability levels process translations differently and that students’ apparent difficulties with a translation may be directly related to the process and the obstacles they face during translation [9]. The difficulties the subject experiences both when graph drawing and when determining the formula for a function, may be related to the representation and algorithmic nature of the concepts involved. APOS theory asserts that a process occurs when a person interiorizes an action which allows one to reverse the process and coordinate the process with other processes. Thus algorithmic concepts can develop into conceptual ideas in the same way as action becomes a process through interiorization [22]. Algorty analysis is the first step to build a formal justification [23]. Therefore, the teacher must have a strategy to overcome these student difficulties so that students can work well when solving problems related to the quadratic function graph. Knowing the mental structure of students based on the APOS theory can make it easier for teachers to develop good learning strategies [24]. Research shows that it is very important to identify and correct students’ misconceptions on mathematical concepts or use them as a means of improving teaching [25]; [26]. When teachers focus on understanding students’ mistakes, teachers can increase their knowledge [27].
5. Conclusion
The study of the quadratic function graph received less attention from researchers. This study contributes to the knowledge of students’ understanding based on the APOS theory on the graph of quadratic functions. This underlines the importance of mental constructs when graphing a quadratic function when given a function formula and when determining a function formula when graphing a quadratic function is given as a means of giving meaning to the procedures involved when solving a given problem. Both subjects demonstrated that their mental constructs when working with the graphing of the quadratic function only got to the object stage. The results of the study also show that male subjects have difficulty formulating functions from a given graph. So that the description obtained from the understanding of the subject can be used by teachers to design didactic learning to improve student understanding. However, this research only discusses the mental construction of students, not the learning design. Thus, it is hoped that there will be further research on learning design based on the APOS theory on the quadratic function graph material.

Acknowledgments
This work was supported by Beasiswa Pendidikan Pascasarjana Dalam Negeri (BPPDN) Indonesian under grant no. B/67/D3/KD.02.00/2019

References
[1] Maf’ulah S and Juniati D 2019 Students’ Strategies to Solve Reversible Problems of Function: The Part of Reversible Thinking J. of Phys.: Conf. Series 1417 012051
[2] Astawa I W P, Budayasa I K, and Juniati D 2018 Processes of Student’s Cognition in Constructing Mathematical Conjecture J. on Math. Educ. 9 15
[3] Suzanne 2015 Students’ understanding of quadratic functions: Learning from students’ voice Diss. Abstr. Int. Sect. A Humanit. Soc. Sci. 76 4
[4] López J, Robles I, and Martínez-Planell R 2016 Students’ understanding of quadratic equations Int. J. of Math. Educ. in Sci. and Tech. 47 552
[5] Ma’rufi M et al 2018 Analysis of senior high school students’ emotional intelligence in cooperative based mathematics learning J. of Phys.: Conf. Series 1088, 012082
[6] Akguumil L 2011 Experiences of undergraduate students with literal symbols Scientific Research and Essays 6 1489
[7] Bannister V R P 2014 Flexible conceptions of perspectives and representations: An examination of pre-service mathematics teachers’ knowledge Int. J. of Educ. in Math. Sci. and Tech. 2
[8] Santia I 2019 Exploring Mathematical Representations in Solving Ill-Structured Problems: The Case of Quadratic Function J. on Math. Educ. 10 365
[9] Bossé M J, Adu-Gyamfi K and Chandler K 2014 Students’ Differentiated Translation Processes Int. J. for Math. Teaching & Learning
[10] Hoon T S, Singh P, and Halim U K A 2018 Understanding of Function and Quadratic Function among Secondary School Students in Selangor Asian Journal of University Education 14 77
[11] Nielsen L E J 2015 Understanding quadratic functions and solving quadratic equations: An analysis of student thinking and reasoning (Doctoral dissertation)
[12] Dubinsky E 2001 Using a Theory of Learning in College TaLUM 12 10
[13] Firdaus A M, Juniati D, and Wijayanti P 2020 Number pattern generalization process by provincial mathematics olympiad winner students J. for the Educ. of Gifted Young Sci. 8 991
[14] Bansilal S, Brijlall D, and Trigueros M 2017 An APOS study on pre-service teachers’ understanding of injections and surjections The J. of Math. Behav. 48 22
[15] Arnon I, Cottrill J, Dubinsky E, Oktac A, Roa Fuentes S, Trigueros M, and Weller K 2014 APOS theory. A framework for research and curriculum development in mathematics education,
[16] Dubinsky E, Arnon I, and Weller K 2013 Preservice Teachers’ Understanding of the Relation Between a Fraction or Integer and its Decimal Expansion: The Case of 0.9̅ and 1 Canadian J. of Sci. Math. and Tech. Educ. 13 232

[17] Baker B, Cooley L, and Trigueros M 2000 A calculus graphing schema J. for Res. in Math. Educ. 557-578

[18] Kazunga C and Bansilal S 2020 An APOS analysis of solving systems of equations using the inverse matrix method Educ. Stud. in Math. 1

[19] Meel D E 2003 Models and theories of mathematical understanding: Comparing Pirie and Kieren’s model of the growth of mathematical understanding and APOS theory CBMS Issues in Math. Educ. 12132

[20] Lestari N D S, Juniati D, and Suwarsono S T 2018 Gender differences in prospective teachers’ mathematical literacy: problem solving of occupational context on shipping company JPhCS, 1008 012074

[21] Listiawati E 2018 Analisis Pemahaman Siswa Sd Dalam Menyelesaikan Masalah Kalimat Matematika APOTEMA: Jurnal Program Studi Pendidikan Matematika 4 10

[22] Kazunga C and Bansilal S 2020 An APOS analysis of solving systems of equations using the inverse matrix method Educ. Stud. in Math. 1

[23] Bagley S, and Rabin J M 2016 Students’ use of computational thinking in linear algebra In. J.1 of Res. in Undergraduate Math. Educ. 2 83

[24] Listiawati E 2019 Implementasi Pembelajaran Berdasarkan Pendekatan M-APOS pada Matakuliah Aljabar Abstrak SNHRP 480

[25] Bray W S 2011 A collective case study of the influence of teachers’ beliefs and knowledge on error-handling practices during class discussion of mathematics J. for Res. in Math. Educ. 42 2

[26] Rach S, Ufer S, and Heinze A 2013 Learning from Errors: Effects of Teachers Training on Students’ Attitudes towards and Their Individual Use of Errors PNA 8 21

[27] Chauraya M and Brodie K 2018 Conversations in a professional learning community: An analysis of teacher learning opportunities in mathematics Pythagoras 39 1