Generalized Richardson-Gaudin Nuclear Models

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The exact solvability of several nuclear models with non-degenerate single-particle energies is outlined and leads to a generalization of integrable Richardson-Gaudin models, like the $su(2)$-based fermion pairing, to any simple Lie algebra. As an example, the $so(5) \sim sp(4)$ model of $T = 1$ pairing is discussed and illustrated for the case of $^{68}$Ge with non-degenerate single-particle energies.

Exactly solvable models (ESM) built upon a dynamical symmetry have a long history of providing important insights into the structure of nuclei. The two main advantages of ESM are: (1) They can describe in an analytical or exact numerical way a wide variety of elementary phenomena. (2) They can be and have been used as a testing ground for various many-body approaches. The simplest example of ESM is the rank-one $su(2)$ model of fermions in one orbit or in several degenerate orbits with a constant pairing interaction, often used to introduce nuclear superconductivity (see e.g., Ref. \cite{1}).

In general, a quantum system has a dynamical symmetry if the Hamiltonian can be expressed in terms of the Casimir operators of a chain of nested algebras. A typical example of a model with rank-two Lie algebra dynamical symmetry is Elliott’s $su(3)$ model, introduced to describe the phenomenon of nuclear deformation \cite{2}. Elliott’s Hamiltonian is a linear combination of the quadratic Casimir operator of the $su(3)$ Lie algebra (involving a quadrupole-quadrupole interaction) and of the Casimir of its $so(3)$ subalgebra, associated with angular momentum. Lie algebras with higher rank lead to more complex ESM like the $so(5)$ model of $T = 1$, isovector pairing \cite{3}, the $so(8)$ model of $T = 0,1$ isoscalar and isovector pairing \cite{4,5}. Ginocchio’s model with $so(8)$ and $sp(6)$ structure \cite{6}, also known as the Fermion Dynamical Symmetry Model (FDSM) \cite{7}. The three dynamical symmetries of the Interacting Boson Model (IBM) \cite{8} provide another example of this approach.

The concept of quantum integrability goes beyond the limits of the dynamical symmetry approach. A quantum system is integrable if there exist as many commuting Hermitian operators (integrals of motion) as quantum degrees of freedom. \cite{9}. The set of Casimir operators of a chain of nested algebras fulfills this condition. There are, however, also well-known examples of integrable models without dynamical symmetry \cite{10}.

Usually dynamical symmetry models are defined for degenerate single-particle levels. Changes in the single-particle energies break the dynamical symmetry but may still preserve integrability. The pairing model with non-degenerate single-particle levels, with an exact solution found by Richardson in the sixties, represents a unique example of an ESM with such characteristics \cite{11}. Recently, Richardson’s model has been shown to be integrable by finding the complete set of integrals of motion constructed in terms of the generators of the $su(2)$ algebra \cite{12}. Subsequently, more general exactly solvable pairing models, both for fermions and for bosons, called the Richardson-Gaudin (RG) models, have been proposed \cite{13}. Since then, a great deal of work has been devoted to understand the properties of these ESM and to apply them to a wide variety of problems in nuclear, condensed-matter, atomic, and molecular physics \cite{14}.

The aim of this Letter is to present the generalization of the RG models to those symmetry algebras that have given rise to the well-known ESM in nuclear physics. In most cases this generalization will allow non-degenerate single-particle energies, as well as other symmetry breaking one-body operators. As an example, we shall discuss the exact solution of the $so(5)$ proton-neutron (pn) isovector pairing model with non-degenerate orbits, for which an exact solution has been proposed by Richardson \cite{15}. However, it has been recently shown that the Richardson’s solution was incorrect \cite{16}.

We also mention that $so(5)$ has been proposed as the symmetry underlying high $T_c$-superconductivity \cite{17}. The exactly solvable RG models discussed in this Letter could be used to generalize $so(5)$ condensed-matter models \cite{18} by the explicit addition of non-degenerate single-particle symmetry-breaking terms.

We begin by introducing a set of commuting operators, the RG operators (integrals of motion) $R_i$ \cite{19}:

\begin{equation}
R_i = \sum_{j(\neq i)} X_i \cdot X_j + \xi, \tag{1}
\end{equation}

where the index $i$ ($j$) refers to the $i$-th ($j$-th) copy of a Lie algebra $\mathcal{L}$ with generators $X_i^\alpha$, $z_i$ are fixed parameters which later will be related to the single-particle energies, and $\xi$ is a generic element of the Cartan subalgebra of $\mathcal{L}$ (see below). The scalar product is defined through the $\mathcal{L}$-invariant metric tensor $g_{\alpha\beta} = c_{\alpha\rho} c_{\beta\sigma}^\rho$ where $c_{\alpha\beta}^\gamma$ are

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the structure constants of $\mathcal{L}$, e.g., $X_i \cdot X_j \equiv X_i^\rho g_{\alpha \beta} X_j^\beta$. Since the $R_i$ commute, any function of these operators can be used as a model Hamiltonian for an integrable system. In particular, any linear combination of the $R_i$ is integrable and is at most quadratic in the generators.

For any simple algebra $\mathcal{L}$, the eigenvalues of the $R_i$ have been given in Ref. \[10\] or can be derived from the Gaudin-algebra approach \[20\]:

$$r_i = \Lambda_i \cdot \xi + \sum_{j(\neq i)} \frac{\Lambda_j \cdot \Lambda_i}{z_j - z_i} + \sum_{a=1}^{M^a} \frac{\Lambda_i \cdot \pi^a}{z_i - e_{a,\alpha}},$$

(2)

where the parameters $e_{a,\alpha}$ are solutions of the generalized Richardson equations \[10\], \[20\]:

$$\sum_{k=1}^{r} \frac{\pi^b \cdot \pi^a}{e_{b,\beta} - e_{a,\alpha}} - \sum_{i=1}^{L} \frac{\Lambda_i \cdot \pi^a}{z_i - e_{a,\alpha}} = \xi \cdot \pi^a \equiv \rho^a.$$

The primed sum means that the singular term $(a, \alpha) = (b, \beta)$ is omitted and $L$ is the number of copies of the algebra $\mathcal{L}$ of rank $r$. The Cartan subalgebra $\mathcal{L}^C \subset \mathcal{L}$ has the elements $h_1, \ldots, h^r$ and the $\rho^a$ are the components of $\xi$ in this generally non-orthonormal basis. The $\pi^a$ are the components of the $r$ simple roots $\pi_1, \ldots, \pi_r$ of $\mathcal{L}$ in the Cartan-Weyl basis, $[h_a, E^a] = \pi^a E^a$. Each $\Lambda_i$ is a vector of highest weight (usually of the fundamental representation) for the $i$-th copy of $\mathcal{L}$ and the expression $\Lambda_i \cdot \pi^a$ corresponds to the eigenvalue of $H^a = \sum_i \pi_i^a h_i^a$ in this highest-weight state of the $i$-th copy of $\mathcal{L}$. The $\rho^a$ are, in a sense, the strength components of the symmetry-breaking one-body operator $\xi$. Finally, the $M^a$ are positive numbers related to the eigenvalues $m^a$ of $H^a$ in the eigenstate of the integrals of motion \[10\], \[20\]:

$$M^a = \sum_i (\Lambda_i \cdot \pi^a - m_i^a).$$

Equations (2) and (3) were derived by Asorey et al. \[19\] under the assumption that $\mathcal{L}$ is the Lie algebra of a simply-connected, compact Lie group. We arrived at the same equations within Ushveridze’s framework \[20\] which assumes that $\mathcal{L}$ is a singular semi-simple Lie algebra (i.e., a classical algebra or one of the exceptional algebras $E_6$ or $E_7$). Since Eqs. (2) and (3) involve a dot product, well defined for any Lie algebra, one may expect that the equations themselves are valid for any semi-simple Lie algebra and possibly for an even larger class. In fact, it is desirable to find a formulation of these equations that involves arbitrarily chosen Cartan algebra generators instead of the simple roots of the algebra.

Although one can use any function of the $R_i$ as a Hamiltonian, the following particular linear combination yields a simple expression for the eigenvalues:

$$H = \sum_i z_i R_i = \sum_i z_i \xi_i - \frac{1}{2} \left( C_2 - \sum_i C_2^{(i)} \right).$$

(4)

With the same linear combination of the eigenvalues $r_i$ \[2\] and using the Richardson equations \[3\], one can show that the eigenvalues of $H$ are linear in the spectral parameters $e_{a,\alpha}$ with coefficients $\delta^a$ that depend on the symmetry-breaking strengths $\rho^a$.

The Hamiltonian $H$ shows how the global $\mathcal{L}$-symmetry represented by the total second-order Casimir operator $C_2$ is broken by the first term containing the elements $\xi_i$ of the Cartan subalgebras $\mathcal{L}^C_i$. In the general case of $z_i \neq z_j$ the first term does not commute with $C_2$. Thus, the eigenstates of $H$ are spread over different irreducible representations of $\mathcal{L}$ and they are not eigenstates of the total Casimir operator $C_2$.

Using Cartan’s classification of semi-simple Lie algebras, one can now generalize many nuclear physics models within this framework (see Table I). Note that models based on a fermion realization of the type $X_{a,\beta} \sim a_{\alpha}^\dagger a_{\beta}^\dagger$, such as $sp(2n)$ or $so(n)$, are well adapted for the generalization towards non-degenerate single-particle energies because in that case the Cartan generators are sums of fermion number operators. In contrast, models based on realizations of the form $a_{\alpha}^\dagger a_{\beta}$, such as $su(n)$, contain differences of number operators.

To illustrate the present formalism, we shall discuss the example of $T = 1$ pairing in systems with protons and neutrons and non-degenerate single-particle levels. The dynamical symmetry limit of pn-pairing in a single degenerate shell was first considered by Flowers \[1\] as an important extension of the original pairing problem between identical particles. It yields a seniority classification of protons and neutrons in $jj$ coupling in terms of $pp$, $nn$, and pn pairs. In doing so, the concepts of seniority $v$ (the number of nucleons not in pairs coupled to angular momentum zero) and of reduced isospin $t$ (the isospin of these nucleons) are established. The application of this $so(5)$ formalism has given rise to many analytic reduction formulae for shell-model matrix elements \[22\].

For simplicity, in the RG extension of the $so(5)$ dynamical symmetry model we will restrict ourselves to seniority $v = 0$ (and consequently reduced isospin $t = 0$), though states with broken nucleon pairs can be easily incorporated in the formalism. We begin by specifying the Cartan generators of $so(5)$:

$$h_{1,i} = \frac{1}{2} (p_i^\dagger p_i + p_i p_i^\dagger) - \frac{1}{2} (n_i^\dagger n_i + n_i n_i^\dagger),$$

$$h_{2,i} = \frac{1}{2} \sum_{\rho = p,n} (p^\dagger_\rho p_\rho + p_\rho^\dagger p_\rho) - 1,$$
where $\rho$ labels protons ($p$) or neutrons ($n$), $i$ is the $i$-th copy of $so(5)$ and $\bar{i}$ is the time reversal of the state $i$. We identify $h_1$ with the third component of the isospin and $h_2$ with the nucleon number.

The positive-root vectors are the $T=1$ creation operators $\{b_{-1,i}^+, n_1 n_i^+ b_{1,i}^+ \} \equiv \left\{ (n_1^+ p_i^+ + p_i^+ n_1^+) / \sqrt{2} , b_{1,i}^+ \right\}$ plus the isospin-raising operator $T_{+,i} = (p_i^+ n_i^+ + n_i^+ p_i^+) / \sqrt{2}$.

The simple-root vectors are $\{b_{-1,i}^+, T_{+,i} \}$ and $b_{1,i}^+$ is the singular-root vector $20$. These, together with the conjugate operators, close the $so(5)$ algebra. The isospin subalgebra $su_T(2) \subset so(5)$ is generated by the $so(5)$ elements that are not related to $b_{1,i}^+$. That is, $su_T(2)$ is generated by $T_{+,-i} = (T_{+,i})^\dagger$, and $[T_{+,i}, T_{-,i}] = T_{0,i} = h_1, i$.

Since $so(5)$ is a rank-two algebra, there are two types of spectral parameters: $e_{1,\alpha}$ and $e_{2,\beta}$ which shall be denoted as $w_\alpha$ and $e_\beta$. The upper bounds $M_1$ and $M_2$ for the indices $\alpha$ and $\beta$, respectively, are related to the isospin $T$ and the total number of pairs $N$ via the expressions $M_1 = N - T$ and $M_2 = N$. The scalar products of the simple roots are $\pi_2 \cdot \pi_2 = 2$, $\pi_1 \cdot \pi_1 = 1$, and $\pi_2 \cdot \pi_1 = -1$. In the spherical shell model, protons and neutrons occupy single-particle states with quantum numbers $(j,m)$. The index $i$ then corresponds to $jm$ and $\bar{i}$ to $\bar{jm}$. Alternatively, due to the rotational symmetry, the angular momentum $j$ can be used as a label instead of $i$ but then the corresponding degeneracy $\Omega_j = (2j + 1)/2$ should be taken into account.

Finally, the weights $\Lambda_\mu$ are the same for all $i$ and are those of the fundamental representation of $so(5)$, that is, $\Lambda_1 = 0$ and $\Lambda_2 = 1$. Inserting these in Eq. (4), together with the choice $\rho_1 = 0$, $\rho_2 = -1/g$, and $\varepsilon_i = 2\varepsilon_\alpha$, we obtain the generalized Richardson equation for the $T=1$ pn-pairing:

$$g^{-1} = \sum_{j=1}^{1} \frac{\Omega_j}{2\varepsilon_j - e_\alpha} + \sum_{\beta \neq \alpha}^{N} \frac{2}{e_\alpha - e_\beta} + \sum_{\gamma=1}^{N-T} \frac{1}{w_\gamma - e_\alpha},$$

$$0 = \sum_{\alpha=1}^{N} \frac{1}{e_\alpha - w_\gamma} - \sum_{\delta \neq \gamma}^{N-T} \frac{1}{w_\gamma - w_\delta}.$$ (5)

The particular case of $\Omega_j = 1$ was derived by Links et al. 23 using the algebraic Bethe ansatz. Each solution of the equations (4) gives an eigenstate of the pn-pairing Hamiltonian

$$H_{pn} = \sum_{jm} 2\varepsilon_j (h_2, jm + 1) - g \sum_{\mu, jm} b_{\mu, jm}^+ b_{\mu, jm}^+, \quad (6)$$

with eigenvalues $E = \sum_{\alpha=1}^{N} 1/e_\alpha$. The spectral parameters $e_\beta$ are interpreted as pair energies as in the case of $su(2)$ pairing. However, due to the larger rank of the $so(5)$, a new set of spectral parameters $w_\delta$ appears in the equations (5). These new parameters $w$ are associated with the $su_T(2)$ isospin subalgebra. For each possible isospin $T$ there are $N-T$ new $w$-parameters. The meaning of this new set of parameters $w$, which do not appear in the expression for the eigenvalues, becomes evident when analyzing the eigenstates of (5). The Bethe ansatz for the $so(5)$ eigenstates of the RG model is a factorized product wave function. It consists, for $N < \sum_j \Omega_j$, of a neutron-pair product state related to the lowest-weight state of $su_T(2)$ and an isospin-raising product operator state that guarantees the required total isospin of the ansatz:

$$|w,e;\varepsilon\rangle = \prod_{\gamma=1}^{N-T} T_{+\gamma} (w_\gamma) \prod_{\alpha=1}^{N} b_{-1}^+ (e_\alpha) |\emptyset\rangle,$$

where the spectral dependence of the generators is given by the expressions

$$T_{+\gamma} (w_\gamma) = \sum_i \frac{T_{+\gamma}}{2\varepsilon_i - w_\gamma}, \quad b_{-1}^+ (e_\alpha) = \sum_i \frac{b_{-1}^+ (e_\alpha)}{2\varepsilon_i - e_\alpha}$$

Note that application of $T_{+\gamma}$ on $b_{-1}^+$ results in $b_0^+$, and application of $T_{+\gamma}$ on $b_0^+$ results in $b_{+1}^+$ which provides us with Ushveridze’s linear combination of powers of $b_{-1}^+$.21

Further insight into the structure of the equations (5) can be gained by using the classical electrostatic analogy 24. They can be considered as the equilibrium condition for a 2D classical electrostatic problem involving three sets of charged particles: (a) the orbitons with charges $-\Omega_j/2$ and fixed positions $2\varepsilon_\alpha$; (b) the pairs with unit positive charges and free positions $e_\alpha$; and (c) the new $w$ particles with unit positive charges and free positions $w_\delta$. In addition, there is a uniform electric field in the vertical direction (the real axes) with strength $\sim 1/g$. Furthermore, the $w$ particles do not interact with the orbitons and do not feel the electric field, which can be seen from the second equation in (5).

As an example, we performed numerical calculations for $^{64}$Ge (4 protons and 4 neutrons) using the single-particle energies (in MeV) $\varepsilon_3/2 = 0.00$, $\varepsilon_5/2 = 0.77$, $\varepsilon_1/2 = 1.11$, and $\varepsilon_2/2 = 3.00$ and two pn-pairing strengths, $g = 0.1$ (weak) and 0.37 (strong). We first checked, in this non-trivial case, that the solutions (5) are indeed the exact solution of the Hamiltonian (4) by direct comparison with a standard nuclear shell-model calculation.

Figure 1 shows the solutions for the lowest $T=0$, 1, and 2 states. The $T=0$ solution corresponds to the ground state, while the $T=1$ and $T=2$ solutions are excited states in $^{64}$Ge. As in the $su(2)$ pairing case, the different configurations can be classified in the weak-coupling limit. As can be seen from Fig.1 at weak coupling four pairs occupy the $p_{3/2}$ (the four pairons are close to the $p_{3/2}$) in the $T=0$ state. This configuration is not allowed for $T=1$ and $T=2$ states due to the Pauli principle. Correspondingly, three pairons are close to the lowest $p_{3/2}$ orbiton and the fourth pairon approaches the orbiton $f_{5/2}$. In all cases the $w$ particles are intertwined with the pairons to minimize the Coulomb interaction. The number of $w$ particles $(N-T)$, together with the initial configuration at weak coupling, defines each eigenstate of the pn-pairing Hamiltonian. As $g$ increases, the free charges expand under the influence of the electric
field and their mutual interaction. The solutions are subject to numerical instabilities due to singularities arising when a real pair energy $e$ crosses a single-particle energy or when real $e$ and $w$ parameters cross. An example of the first class of crossings can be observed in Fig. 1 for $T = 2$ where the pairon associated with the $f_{5/2}$ orbiton at weak coupling goes down with increasing $g$ and crosses the $p_{3/2}$ orbiton. The $T = 1$ case shows an exchange of positions in the real axis of a pairon and a $w$ particle as an example of the second class of singularities. The first class of singularities were already present in the $su(2)$ pairing case and they precluded the practical use of the exact solution for a long time. Recently, a new method to overcome this numerical problem was proposed [25]. We believe that the same procedure can be used to treat the second class of singularities as well, making feasible the exact solution of the $so(5)$ model for very large systems.

In summary, we have presented the generalization of RG models to arbitrary semi-simple Lie algebras, which include most of the dynamical-symmetry models of nuclear physics. The generalized RG models allow the introduction of one-body symmetry breaking terms like non-degenerate single-particle energies. As an example of this approach, we gave the exact solution of the $so(5)$ pairing model. We emphasize that the exact solution for large systems with $so(5)$ symmetry could be of great importance in condensed-matter physics in addressing the phenomenon of high $T_c$-superconductivity [17, 18]. Finally, the treatment of higher-rank algebras like $sp(6)$ and $so(8)$ opens the possibility of exact nuclear structure calculations with more realistic quantum integrable models.

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