Analytical coexistence of s, p, s + p phases of a holographic superconductor

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Abstract: We study analytically the critical phase of a mixed system of the U(2) gauge fields with a global symmetry on the boundary using gauge/gravity using the variational calculus minimization problem. In the agreement with the numerical results, we show that there exists a minimum of the chemical potentials in which both scalar (s-wave) and vector (p-wave) condensates exist in a mixture as well as in the distinct phases s,p. This is a result on symmetry breaking to $U(1)$ symmetry and also rotational symmetry. We obtain the condensates and charge densities for both cases of the balanced and unbalanced holographic superconductors analytically. This is the first analytical study of the coexistence of two modes of the superconductivity with different order parameters. The realistic model consists of two different phases of the superfluidity in Helium.

Keywords: Holography and condensed matter physics (AdS/CMT); Sturm-Liouville variational method.
I. INTRODUCTION

Anti de-Sitter/ conformal field theory (AdS/CFT) proposed by Maldacena conjecture that any weakly gravitational model in the bulk has a dual boundary quantum field theoretic description through a set of the dual quantum relevant operators $\mathcal{O}$. Especially a high temperature superconductor as a strongly coupled system has a good well defined gravitational dual using this conjecture. The conjecture has been named as the gauge/gravity duality which has been studied in several papers. Depending upon the kind of the condensates used, we have scalar (s-wave), vector broken rotational symmetry (p-wave) or d-wave and more. From the point of the view of the symmetry spontaneously breaking, different kinds of the holographic superconductors correspond to different schemes of the breaking of different symmetries. In brief, in the scalar condensates, the symmetry is broken to $U(1)$. Further, p-wave is the result of the rotational symmetry. But a mixed phase $s + p$ is due to the breaking of the two $U(1)$ symmetries to two distinct phases of s,p. It comes from the symmetry breaking cascade in the form of $U(2) \rightarrow U(1) \times U(1)$. Bulk action can be included by higher order curvature corrections like Gauss-Bonnet, Weyl and etc. The role of these higher order corrections and their dual descriptions on the boundary using AdS/CFT has been widely studied in the literature.

Very recently the problem of the existence or coexistence of two modes s,p-waves simultaneously has been investigated in the literature. It has been showed that there is a possibility to have two s-wave phases with two different scalar condensates. This study also extended to include the case of the spontaneously symmetry breaking of the rotational symmetry by a vector as a possibility for the coexistence of two p-wave order parameters. But there is an intermediate phase between s,p. This phase is so called as the $s + p$. In which we have simultaneously two kinds of the relevant operators on the boundary and the CFT description of it has a direct interpretation as a candidate for $s + p$ as a mixed phase. This model is based on the $U(2)$ gauge fields and it was numerically studied the s+p wave phase in a typical holographic high temperature superconductor.

Firstly the existence or coexistence problem was related to usage of two vector order parameters was explained in terms of the AdS/CFT in Ref. As it was showed that, in this mixed system, the phase transition can be controlled by the form of the interaction.

The model which we used in this work is based on a toy model of the multiband holo-
graphic superconductors proposed originally in \cite{7} and extended in details in \cite{8}. This multiband model was used to describe the AdS/CFT description of two-component superfluidity and it has been showed that the system under a perturbation becomes unstable. Actually, the system under such kind of the instability moves towards the $s+p$ wave phase as an intermediate phase in which the free energy of the system minimized and this intermediate phase is preferred to the other pure phases.

So far it is believed that under some restrictions, upper than a critical chemical potential $\mu_c$, it exists a possibility to have both $s,p$ phases in a mixture, it is so called as the $s+p$ phase. It is a transitive mode and in the very highly condensed regime of the dual system in the boundary it undergoes to the purely $p$-wave phase. In this paper, we study some basic analytic properties of such mixed $s+p$ phase using a highly motivated semi analytical method of functional theory. We show that the system of highly non-linear coupled differential equation can be transformed in a self-adjoint system of the pair of auxiliary functions $F$ and $G$. By using the variational methods we obtain the minimum of the chemical potentials of the different modes of the superfluidity. In the case of $s+p$ we used analytic expressions for condensates which remain valid only near the critical point. Also we study the non-linear relation between the charge densities and the chemical potentials near the critical point. We show that the linear approximations so far is valid as an approximation. A qualitative comparison between values of the free energy shows when the mixed phase $s+p$ is preferred.

II. ON GAUGE/GRAVITY PICTURE OF THE HOLOGRAPHIC SUPERCONDUCTORS

Consider a weakly gravitational bulk action in four dimension as the large colour number limit of a super string theory. In this limit the gravity sector decoupled and behaves like a classical matter field in the spacetime manifold. We assume that the metric has the form of a static and spherically symmetric tensor. Asymptotically the metric must be Anti de-Sitter (AdS). If we calculate the action of the bulk we will be able to compute the point functions (two,..) as local functions of the fields. Here by fields we mean scalar fields mainly. With action we are able to compute the partition function $Z$ of the bulk. The key note here is that the asymptotic behavior of the two point functions of the scalar fields can be read off in terms of the source terms of some relevant operators of a typical conformal field theory.
(a field theory of operators with traceless energy momentum tensor and with a definite centeral charge) on the boundary. Mathematically it is equivalent to say that the system in the bulk (for example a black hole with non zero temperature on horizon or a soliton with zero temperature) remains in the thermal equilibrium with the boundary under the following neccesary and suffiecent conditions:

\[ Z_{\text{Bulk}} = Z_{\text{CFT}} \Rightarrow T_{\text{BH}} = T_{\text{CFT}}. \]  

(1)

The above brief introductionary section of the relation between the correlation functions of some scalar matter fields on a bulk and the expectation values (the source terms) of some quantum operators (strictly speaking some relevant quantum operators) is called as the gauge/gravity duality in it’s very simple interpretaion. A thchically discussion must be done to cover all lessly clear points like which kind of the duality exists if the gravity deals with some higher order corrections like Gauss Bonnet terms or Weyl ones. This duality is between a bulk and a boundary with planar toplogy of the horizon. We need to review some mathematical steps towards this duality in this section.

The conforml field theory is a metric conformally invariance theory on the boundary. Maldacena gauge/gravity interpretation is related to such quantum theory on boundary and a higher dimensional gravity model. To have a CFT, we need that the total Hamiltonian of a typical quantum theory remains invariant under a ceratin non singural metric conformal transformations as the following:

\[ g_{\mu\nu} \rightarrow \Omega(x^{\alpha})g_{\mu\nu}. \]  

(2)

We consider a typical general coordinate transformation like \( x^{\mu} \rightarrow x^{\mu} + a^{\mu} \). Here \( a^{\mu} \) has three different parts as boost, rotation (the Lorentz group) and also an additional symmetry is so called as the scaling symmetry. Under this local transformation, we assume that it is possible to write down a conserved current by contraction of the \( a^{\mu} \) with the energy momentum (EM) tensor of this field theory \( T^{\mu\nu} \) like:

\[ J^{\beta} = T^{\alpha\beta}a_{\alpha}. \]  

(3)

By EM tensor we mean the effective EM tensor of the CFT. We apply the continuity equation on the current and since all quantities are local quantities so the continuity equation is local and it can be written as:

\[ \partial_{\beta}J^{\beta} = 0. \]  

(4)
By substituting the current we find that EM tensor must be traceless and in the following form $T_{\beta\mu}g^{\beta\mu} = 0$. The strategy of using the CFT is useful also to compute the degrees of the freedom of the fields or central charge as well. Back to the conformal transformation, we have three kinds of the generators for three kinds of the possible symmetries. The last one corresponds to the scaling symmetry is called as the dilatation operators. The operation of this quantum operator is understood by applying it to a unitary operator of CFT as $O$ satisfying the following operator equation:

$$DO = \Delta O.$$  (5)

This equation is in the form of a eigen value-eigen function operator equation. The eigen value $\Delta$ is called as the conformal dimension and as it has been shown that it’s value is defined by the mass of the scalar fields on the boundary. In the four dimensional bulk theory where the boundary is three dimensional, from this preliminary treatment of the $AdS_4/CFT_3$, we have:

$$\Delta(\Delta - 3) = m^2L^2.$$  (6)

Here $m^2$ is the mass of the scalar fields in bulk and $L^2 = -\frac{6}{\Lambda}$ is the AdS radius of the spacetime(commonly we set $L=1$). For vector fields we need to replace $\Delta(\Delta - 3) \rightarrow \Delta(\Delta - 1)$.

To study the condensed matter physics we need scaling invariant quantum theories. From the renormalization group point of view, such theories can be addressed as the ending points of the flows of the renormalization group. To have a scaling invariant theory it is required to keep the operators under the scaling invarince as the following:

$$O = \lambda^D O(\lambda x).$$  (7)

Three kinds of the quantum operators live on the boundary treating the CFT. relevant, irrelevant and marginal. To writing the CFT Hamiltonian $H_{CFT}$ and to avoiding from a non renormalizable quantum theory we only use the relevant operators. So, the total action including the free partion and the interaction action reads:

$$S = S_{free} + \int d^4x \Sigma_i g_i O_i.$$  (8)

The first term denoted the free fields contribution and the last part is the interaction between fields and the coupling contants $g_i$ are the interaction coupling constants. If the model be
renormalizable it is needed that the interaction couplings satisfy the following first order (always) integrable system:

$$\frac{dg_i}{d \log \epsilon} = \beta_i(g_j). \quad (9)$$

The integral curves are the solutions (first integrals) of the renormalization group flow equations. This dynamical system has a set of the fixed points located at $\beta_i(g_j^*) = 0$. Near these fixed points the quantum theory is scale invariant and we are able to explain the evolutionary dynamics of the system using a quantum field theory with invariant structure under scaling transformations. Any small perturbation of this kind of CFT induces an extra flow. With this statement we can say that gauge/gravity is a duality and it represents a one to one map between any gauge theory described by CFT on the boundary to a bulk matter action in thermal equilibrium with the boundary.

To have a better view on AdS/CFT we start by a simple model of a single scalar field $\phi$ with the following action:

$$S \sim \int d^4x \sqrt{-g} \left( \mathcal{L}_{CFT} + \mathcal{L}_m \right) \quad (10)$$

Here we set the AdS radius $L = 1$. The first term corresponds to the CFT and the second term induces the quantum operators on the boundary. Any small deformation from this CFT action, produces a flow. After this emergence of the flow, the full Hamiltonian of the system by including the quantum operators (induced on the boundary by the matter action) is written as the below:

$$\hat{H} = H_{CFT} + \sum_i \lambda_i \mathcal{O}_i. \quad (11)$$

Here the summation is done over all relevant or irrelevant operators. By the definition, a holographic superconductor is a high temperature ($T \sim 138K$) superconductor which it has a dual gravitational model. The core of the gauge/duality picture of the holographic superconductor is that we can show that the quantum vacuum expectation value of a relevant operator $\mathcal{O}_r$ has the following form versus temperature and the critical temperature $T_c$:

$$\langle \mathcal{O}_r \rangle \sim \sqrt{1 - \frac{T}{T_c}}. \quad (12)$$

The main purpose of this dualization is to find the basic properties like critical exponent, critical temperature and critical chemical potentials. Since the system of the bulk field
equations is highly non linear, so the common method to solve these equations is numerical methods based on the shooting tool. It is desired to find the temperature dependence of the vacuum expectation values of the boundary operators $<O_\pm>$ or the chemical potentials $\mu_c$ at the critical points which a phase transition occurs.

III. MODEL

As we mentioned before our aim here is to study the analytical properties of a mixed phase $s + p$ of a holographic superconductor. We start by writing the action of a 3 + 1 dimensional bulk with AdS boundary, by including a $U(2)$ gauge field and a scalar doublet field, instead of the usual scalar field:

$$S = \int d^4x \sqrt{-g}\left(\frac{R - 2\Lambda}{2\kappa^2}\right) + \int d^4x \sqrt{-g}\left(-\frac{1}{4} F_{\mu\nu}^c F_{\mu\nu}^c - m^2 |\Psi|^2 - |D^\mu \Psi|^2\right).$$  

(13)

The mass of the scalar-doublet remains upper than the Breitenlohner and Freedman (BF) bound [10]. It is believed that this action enables to explain a mixture of two phases $(s,p)$ both, meanwhile by studying and read off the vacuum field expectation values on the boundary conformal fields using AdS/CFT. Here the notation is the same as the usual p-wave holographic superconductors, like $D_\mu = \partial_\mu - iqA_\mu$, where for simplicity and in the probe limit we neglect the back reaction of all matter fields, we set $q = 1$. Also the gauge field is expanded in the basis of the units (generators) of the SU(2) group, Pauli’s matrices $\sigma_i$. The only difference is that here we have a normalized doublet scalar $\Psi$:

$$\Psi = \left(\begin{array}{c} \frac{w(r)}{\sqrt{2}} \\ \frac{\psi(r)}{\sqrt{2}} \end{array}\right)$$

where to preserve the symmetry of the AdS metric, we assume that all the functions are well defined as functions (single valued) of the radial coordinate $r$. It enables us to perform a radial quantization for a AdS/CFT construction of the dual model. It is a vital and fundamentally important feature of our gauge/gravity approach. We assume that the bulk is an AdS spacetime in the following form:

$$g_{\mu\nu} = diag[-f(r), f(r)^{-1}, r^2 \Sigma_2], \quad \Sigma_2 = diag(1, \sin^2 \theta).$$

(14)

The metric represents a black hole bounced in an AdS and with planar topology of the horizon. By using the same assumptions on the fields and following the [9], we can write
the next set of the field equations for our system in the probe limit, by ignoring from the back reactions (in units $2\kappa^2 = 1$):

$$
\psi'' + \left(\frac{f'}{f} + \frac{2}{r}\right)\psi' + \left(\frac{(\Phi - \Theta)^2}{4f^2} - \frac{m^2}{f} - \frac{w^2}{4r^2f}\right)\psi = 0 \quad (15)
$$

$$
\Phi'' + \frac{2\Phi'}{r} - \psi^2\Phi = 0 \quad (16)
$$

$$
\Theta'' + \frac{2}{r}\Theta' + \psi^2(\Phi - \Theta) - \frac{w^2}{r^2f}\Theta = 0 \quad (17)
$$

$$
w'' + \frac{f'}{f}w' + \frac{\Theta^2}{f^2}w - \psi^2w = 0. \quad (18)
$$

Here prime denotes the derivative with respect to the radial coordinate $r$. Previously full numerical solutions for condensates (scalar and vector) presented, it has been shown that the mixed phase $s + p$ exists as a sub dominant phase under $s$ wave phase for both cases of the balanced and unbalanced systems. Our aim here is to study the phase transitions in system (15-18) using the analytical method, proposed in [11]. In this method, the key note is we write down the equations near the criticality as a variationa self-adjoint problem with appropriate boundary conditions. The eigenvalue of the variational problem denotes the chemical potential in the dual CFT description. To minimize the chemical potential, we minimize the functional integral using an arbitrary trial function. The minimum of the chemical potential is obtained by finding the optimum values for the variational parameters. If we increase the number of the variational parameters, the values of our semi analytical estimation becomes close to the numerical results, which they have been obtained by solving the system of the coupled non linear differential equations by shooting method [9].

The normal phase, when the temperature of the system $T > T_c$, is defined by the following solutions of the full system of the field equations (15-18):

$$
f = r^2 - \frac{r^3}{r}, \psi = w = 0, \Theta = \mu'(1 - \frac{1}{r}), \Phi = \mu(1 - \frac{1}{r}). \quad (19)
$$

Here $\mu$ is the dual chemical potential for $U(1)$ but $\mu'$ comes from the $SU(2)$. They have the same description using group theory of the U(2) group. If we set $\mu' = 0$, the model is a balanced holographic superconductor. In the unbalanced case, $\mu' \neq 0$. We keep always this last one. The only change that is vital is to perform a recomputation of free energy density using the on-shell action, in which in the case of the balanced it has not the constant term $-\frac{1}{2}\mu' \rho'$. We are able to modify the equations given by (15-18) with a non unit $r_+$, but to compare our results with the numerical [9], we also set $r_+ = 1$. 
We have two kinds of the condensates, one is the leading term of the scalar field $\psi$ on the AdS boundary:

$$\psi \simeq \frac{<O_+>}{r^{\Delta_+}} + \frac{<O_->}{r^{\Delta_-}}. \quad (20)$$

Where as the usual, $\Delta_\pm$ denotes the conformal dimension, as for renormalizability, we take $\Delta_\pm \geq 1$. It corresponds to the choice of the scalar doublet mass $m^2 = -2$, that is upper than the BF bound $m^2_{BF} = -\frac{9}{2}$. As we know $O_\pm$ denotes the scalar order parameter. Here we have also a current operator $J_x$, which it is related (as the source term) to the AdS asymptotically solution of field $w$ as the following:

$$w \simeq \frac{<J^1_x>}{r} + <J^0_x> . \quad (21)$$

We need to fix our quantization scheme. We choice to switch-off $<O_-> = 0, <J^0_x> = 0, <J^1_x> = <J_x>$, so the set of the asymptotic solutions in the AdS boundary read:

$$\Phi = \mu - \rho r, \quad \Theta = \mu' - \rho' r, \quad w = \frac{<J_x>}{r}, \quad \psi = \frac{<O_+>}{r^2}. \quad (22)$$

Our main goal is to find the explicit expressions with respect to the chemical potentials of the $\{<J_x>, <O_+>\}$ (we choice x as the preferred direction). We have two regime in which in the balanced case of the superconductors we put $\mu' = 0$ and in unbalanced case $\mu' \neq 0$. We keep $\mu'$ as a non zero parameter. The case of the unbalanced superconductors will be studied as a limiting case $\mu' \to 0$.

In terms of the new coordinate $z = \frac{1}{r}$, the field equations (15-18) for $m^2 = -2$ have been written as the following forms:

$$\psi'' - \frac{(2 + z^3)}{z(1 - z^3)} \psi' + \left[ \frac{(\Phi - \Theta)^2}{4(1 - z^3)^2} + \frac{2}{z^2(1 - z^3)} - \frac{w^2}{4(1 - z^3)^2} \right] \psi = 0 \quad (23)$$

$$\Phi'' - \frac{\psi^2}{z^2(1 - z^3)} (\Phi - \Theta) = 0 \quad (24)$$

$$\Theta'' + \frac{\psi^2(\Phi - \Theta)}{z^2(1 - z^3)} - \frac{w^2 \Theta}{1 - z^3} = 0 \quad (25)$$

$$w'' - \frac{3z^2}{1 - z^3} w' + \left[ \frac{\Theta^2}{(1 - z^3)^2} - \frac{\psi^2}{z^2(1 - z^3)} \right] w = 0. \quad (26)$$

Our aim is to solve $\{23,26\}$ semi analytically.

IV. VARIATIONAL METHOD

Based on a study of $[11]$, we can find the minimum of the $\mu, \mu'$ using a variational method. The starting point is we make the following substitutions for the fields which are valid only
near the critical point:

\[ \Phi \approx \mu(1-z), \quad \Theta \approx \mu'(1-z), \quad (27) \]
\[ w \approx \langle J_x \rangle F(z), \quad \psi \approx D_+ z^2 G(z), \quad D_+ \equiv \mathcal{O}_+. \quad (28) \]

We apply these functions in the equations (23,26). Here, \( F, G \) are two trial functions to minimize the variational integral:

\[ F'' + \left( -3z^3 + \frac{\mu'^2}{1 + z + z^2} - \frac{D_+^2 z^2 G^2}{1 - z^3} \right) F = 0 \quad (29) \]
\[ G'' + \left( -3z^3 + \frac{(\mu - \mu')^2}{4(1 + z + z^2)} - \frac{\langle J_x \rangle^2 z^2 F^2}{4(1 - z)^2} \right) G = 0. \quad (30) \]

We call these equations as "'F'" and "'G'" respectively. Now we write the system (29,30) in the Sturm-Liouville (SL) form as the following:

\[ (z^2(1 - z^3) F')' + \left( -3z^3 + \frac{\mu'^2 z^2(1 - z)}{1 + z + z^2} - D_+^2 z^4 G^2 \right) F = 0 \quad (31) \]
\[ (z^2(1 - z^3) G')' + \left( -4z^3 + \frac{z^2(1 - z)(\mu - \mu')^2}{4(1 + z + z^2)} - \frac{\langle J_x \rangle^2 z^2 F^2}{4(1 - z)^3} \right) G = 0. \quad (32) \]

Any (SL) system in the form of \((Py')' + (\lambda W + Q)y = 0\), can be obtained from the Lagrangian density in the form of:

\[ L = Py'^2 - (\lambda W + Q)y^2. \quad (33) \]

From (33) it comes that the minimum of the eigen value \( \lambda \) reads as the following:

\[ \lambda_{min} = \text{Min}\left\{ \frac{\int_0^1 dz(Py'^2 - Qy^2)}{\int_0^1 dzWy^2} \right\}. \quad (34) \]

For system (31,32) we have:

\[ P_F = z^2(1 - z^3), \lambda_1 = \mu'^2, W_F = \frac{z^2(1 - z)}{(1 + z + z^2)}, Q_F = -D_+^2 z^4 G^2 - 3z^3. \quad (35) \]

and for the G equation:

\[ P_G = z^2(1 - z^3), \lambda_2 = (\mu - \mu')^2, W_G = \frac{z^2(1 - z)}{4(1 + z + z^2)}, Q_G = -\frac{\langle J_x \rangle^2 z^2 F^2}{4(1 - z^3)} - 4z^3. \quad (36) \]

Now, we will evaluate the integral (34) for \( \lambda_1, \lambda_2 \), with trial functions

\[ F(z) \equiv y_F = 1 - bz^2, G(z) \equiv y_G, \quad (37) \]
The first trial function \( y_F \) is choisen since in the F equation we required to avoide from the singularity. It is needed to fix \( y_F \) with auxiliary boundary conditions as the following:

\[
y_F(0) = 1, \quad y_F'(0) = 0. \tag{38}
\]

But about \( y_G \) the situation is little different. As we observe from the G equation, it has an essential pole located at \( z = 1 \). The pole is second order one. So, it is required that the following expression remains finite:

\[
\frac{G(z)z^2F(z)^2}{(1-z^3)^2} < \infty. \tag{39}
\]

In the variational integral also we need to have a \( y_G \) which it be able to remove the singularity safely. So, the one of the best simple candidates may be is written as the following trial function:

\[
y_G = a + bz + cz^2, \quad y_G(1) = 0, \quad y_G'(0) = 0. \tag{40}
\]

By imposoing the above boundary conditions we find \( c = -a, b = 0 \), so the trial function for \( G \) is written as the following

\[
y_G = a(1 - z^2). \tag{41}
\]

There are so many others of the acceptable trial functions, but to keep the simplicity of the results, we choice these two forms.

We are able to mention here something about the integrals involving the variational technique from the purely mathematical point of view. Look at the forms of the \( Q_F, Q_G \). Since near the critical point both \( D_x < J_x > \) treat like the small parameters, it is possible to define the smallness parameters \( \epsilon_1 = D_x^2, \epsilon_2 = < J_x >^2 \).

\[
\epsilon_1 = D_x^2, \epsilon_2 = < J_x >^2. \tag{42}
\]

In these forms the dominators of the integral reads as the following:

\[
I^i_Q = \int_0^1 dz Q_i y^2 = \int_0^1 dz Q^0_i y^2 + \epsilon_i \delta I^i_Q, \quad i = \{F, G\}. \tag{43}
\]

Here

\[
Q^0_F = -3z^2, \quad \delta I^F_Q = \int_0^1 dz z^2 y_G^2 y_F^2. \tag{44}
\]

\[
Q^0_G = -4z^3, \quad \delta I^G_Q = \int_0^1 dz \frac{z^2 y_F^2 y_G^2}{4(1-z^3)}. \tag{45}
\]
The integral $\delta I_G^Q$ diverges at the boundary point $z = 1$, except that

$$y_G|_{z \to 1} \approx a(1 - z^2) + O((1 - z^2)^2).$$  \hspace{1cm} (46)

We observe that the form of $y_G$ supports this avoidance of the divergency.

Now, we minimize the integral with respect to the $\{a, b\}$. This evaluation for both integrals has been done in the below:

$$\mu'^2 = \text{Min}\left\{ \frac{D^2}{2} \left( 0.37296a^2b^2 - 1.38528a^2b + 1.52381a^2 + 40.5b^2 - 60b + 45 \right) \right\}. \hspace{1cm} (47)$$

And for another case:

$$(\mu - \mu')^2 = \text{Min}\left\{ 0.114305 \left( (6.18929b^2 - 19.8343b + 18.3936) < J_x >^2 + 784 \right) \right\}. \hspace{1cm} (48)$$

From these last equations, we are able to complete the analysis of the different phases. We mention here that the transition phases from s-wave to p-wave in the unbalanced cases are in the first order phase transition. We mention here near the critical point, $D_+ = D_+(\mu, \mu'|\mu_c, \mu'_c), < J_x > >= < J_x(\mu, \mu'|\mu_c, \mu'_c) >$. So the correct minimization must be following a simultaneously minimization scheme applied on these two functions. In next section when we study the critical behavior and the critical exponent of these expectation values analytically we will deduce some expressions for these. Generally in this case of the superconductors and due to the mixing of homogenous and inhomogenous superconductors we have the following expressions, valid only near the critical point $(\mu_c, \mu'_c)$:

$$D_+ \sim (\mu - \mu_c)^{1/2}, \hspace{1cm} < J_x > \sim \Sigma i,j \gamma_{i,j}(\mu - \mu_c)^i(\mu' - \mu'_c)^j. \hspace{1cm} (49)$$

Here

$$\gamma_{i,j} = \frac{\partial^2 < J_x >}{\partial (\mu - \mu_c) \partial (\mu' - \mu'_c)} |_{(0,0)}. \hspace{1cm} (50)$$

Where we ignore the proportionality constants in these definitions.

V. BALANCED SUPERCONDUCTORS

In this case, we set $\mu' = 0$. Our aim here is to show that there exists any lower bound $\mu^*$ such that for at least two values of $\mu \geq \mu^*$ we can have two separate regimes of holographic
superconductors as like purely s-wave (with non zero $\psi$ but $w = 0$) or s+p (with non zero $\psi, w$) or not.

**Normal phase:** The normal phase of the system when $T > T_c$ exists as an exact solution to the field equations $\psi = w = \Theta = 0$ and $\Phi = \mu(1 - z)$. The system under a small perturbation undergoes to an excited superconducting phase.

**s-wave phase:** In this case with $w = 0$, from minimization of using "$G$" equation with $< J > = 0$, we find that for

$$\mu_{\text{min}}^{s\text{-wave}} = 9.46653.$$  \hspace{1cm} (51)

The numerical result as it has been reported in [9] is $\mu_{\text{min}} \approx 8.127$. As we observe that our analytical estimation is in the same order and in a very good agreement with the numeric. It realizes a two component s-wave superfluidity.

**VI. UNBALANCED SUPERCONDUCTORS**

A simple holographic model of a s-wave unbalanced superconductors proposed [14]. Before introducing the analytical approach to the holographic model we review the basics of an unbalanced superconductor in condensed matter physics. In unbalanced superconductors the phase transition happens from a normal superconducting phase to another superconducting phase but this last phase has a non zero intrinsic mechanical angular momentum. This phase transition is first order but in the balanced one is second order. The final superconducting phase with angular momentum is so called as the LOFF (Larkin-Orchinnikov-Fulde-Ferrel) phase. This phase transition is happen because the populations of two groups of the fermions are inequal. So also the second phase has a non zero (inequal) chemical potential. From condensed matter point of the view, this phase transition occures in some strongly coupled systems and specially in the quark systems in QCD. So, exprimentaly it is possible to be test by labratory. In babalanced superconductors the system intinally when $T > T_c$ is located at the normal phase as the ground state phase. After that in $T < T_c$ the system fall down to the superconducting phaseas an excited phase due to the energy considerations thermodynamically. But in unbalanced superconductors the first order phase transition is between two superconducting phases: from usual superconducting with zero angular momentum to the non zero one (LOFF) configuration. In holographic picture we need an additional dual pair of the $(\mu', \rho')$ corresponding to the new gauge scalar field $\Theta$ in
the bulk. Although this pair comes from another dual field but the condensation affected by it. From another view, the phase structure on an unbalanced superconductor has three different phases: One is normal phase $T > T_c$. The superconductor phase has two distinct phases of the homogenous superconductor in which the BCS theory basically is valid. But the tail of the superconductivity looks like a small island in which the superfluidity is LOFF kind. To cross the border of the BCS superconducting country to this LOFF tribe, the phase transition is first order but the situation is different from normal phase to the BCS superconductivity in which the phase transition is a second order. But at the border of the LOFF to the normal phase, system has another second order phase transition as well as the usual superconductors. This picture remains the same in any weakly coupled superconductor.

Following the model proposed in last section to have a LOFF phase, we relax $\mu' = 0$, so we have two variational integrals to find the minimum of $\{\mu, \mu'\}$. In this case, we have also three different phases, excluding the normal phase as the following:

**s-wave:** In this case with $w = 0, \psi \neq 0$, from minimization of G equation and by remembering that in this case $< J_x > = 0$ (switch-off) we find the same result as the balanced case, as the following:

$$(\mu - \mu')_{\min} = 9.46653. \quad (52)$$

We used the same quantities as the balanced s-wave phase. The numerical result as it has been reported in [9] is $(\mu - \mu') \approx 8.127$. So, the result is very close to the numerical estimates.

**p-wave:** In this case with $\psi = 0, w \neq 0$ we set $D_+ = 0$, from minimization we find that:

$$|\mu'|_{\min} = 3.71044. \quad (53)$$

Where we used the minimization technique firstly on the G equation to find the minimum of the $\mu'$. The numerical value is $|\mu'| \geq 3.65 \ [9]$. We have a great agreement between the results. This phase transition corresponds to the breaking of the $U(1) \times U(1)$ to $U(1)$.

**VII. EXISTENCE OF $s + p$ PHASE**

We study the mixed phase $s + p$ separately in this section. Due to the mixing terms of $\mu', D_+$ in F equation and $\mu - \mu', < J_x >$ in the G equation in the case of the unbalanced it is a hard job to find a numeric for $\mu'$ since we are not aware about the functionality of the
$D_+, < J_x >$ versus the chemical potentials. Before to studying this numerical estimation we rewrite the system of the F,G equations in the following integral form:

$$\int_0^1 dz F G \delta Q^0 + \int_0^1 dz F G (\lambda W) + \epsilon_1 \int_0^1 dz Q_F^1 F G^3 - \epsilon_2 \int_0^1 dz Q_G^1 F^3 G = 0. \quad (54)$$

Here $\delta f = f_F - f_G$, and the up script 0 refers to that part of the quantity without the perturbation parameters $\epsilon_i$. Since in the numerical study of the s+p phase we distinguish three regimes depending on the value of the $\delta = \mu' - \mu_c$.

**Balanced case:** Here $w, \psi \neq 0$, so consequently, $\{ < J_x >, D_+ \} \neq 0$, we will have a minimum but in a two variable function $\mu(a, b)$ for a set of the trial functions as $\{y_G, y_F\}$. Actually when we look at the $(47)$, we observe that in the balanced case with $\mu' = 0$ always it exists a minimum $D_+$. The key note here is that in the balanced case always $< J_x > \sim (\mu - \mu_c)$, as we can observe from the $(49)$. Indeed in this case we always have

$$< J_x > \sim (\mu - \mu_c) = \gamma_{1,-1/2} (\mu - \mu_c), \quad \gamma_{1,-1/2} = (0.2157 a^2 |\mu'_c - \mu_c|)^{-1/2}. \quad (55)$$

The proportionality coefficient is a constant. Since we know only about the approximated expression of $< J_x >$ near the criticality and since if we apply the minimization we will have a non linear expression which can not be solved analytically we will obtain the order of the critical chemical potential. We obtain the following expressions for $\mu_m = \text{Min} \mu$:

$$\mu_m^\pm = \frac{\mu^*}{2} \pm \sqrt{\Delta}, \quad \Delta = \mu_0^2 - \mu^* \mu_c + \frac{(\mu^*)^2}{4}. \quad (56)$$

We have two values of $\mu_m$ as the minimum of the s+p phase chemical potential. It shows that always exist an intermediate phase but due to the leakage of the complete form of the $< J_x >$ a more precise estimation is not possible. So, qualitatively

$$\mu^{(s+p)-wave} = \frac{\mu^*}{2} \pm \sqrt{\Delta}. \quad (57)$$

The numeric is more precise than our estimation. In fact the numerical estimation still now is valid for another value of the $< J_x >$. The reason of the phase transition is due to the breaking of the rotational symmetry [9].

**Unbalanced configurations:** Here both $w, \psi \neq 0$, so consequently $\{ < J >, D_+ \} \neq 0$. In this intermediate case we will use both of the F,G equations simultaneously. In a similar method as the previous section we use the approximation formulas of the condensates near the criticality and we obtain:

$$\left[ \frac{\mu'_m}{\mu_m - \mu'_c} \right]^2 = \frac{\mu_3 \mu_4 (\mu'_m - \mu'_c)(\mu_m - \mu_c + \mu_1)}{\mu_2 \mu_4 (\mu_m - \mu_c)^2 + \mu_2 \mu_3 (\mu'_m - \mu'_c)^2 + \beta \mu_2 \mu_3 \mu_4 (\mu'_m - \mu'_c)}. \quad (58)$$
Here $\mu_i, \beta$ are constants. The general solution for the cubic equation (58) can be written as quadratures. But physically we also can solve it for the following three distinct cases:

Case $\mu >> \mu'$ but $O(\mu') = 0$: In this case we have:

$$\mu_m \sim \mu_c - \mu_1.$$  (59)

It gives us the minimum of the $\mu$ independent from the $\mu'_m$.

Case $\mu >> \mu'$ but $O(\mu'^2) = 0$: In this case we have:

$$\mu_m \sim \sqrt[3]{-\mu'_c(-\mu'_c^2 + 2\mu_4\mu_c\mu_m - \mu_4\mu'_m + \mu_3\mu'_c + \beta\mu_3\mu_4\mu'_c)}.$$  (60)

It gives us the minimum of the $\mu$ in terms of the $\mu'_m$.

Case $\mu << \mu'$. In this case we have:

$$\mu'_c \sim \frac{\mu_2\mu_4(\mu_c - \mu_m)}{\mu_3\mu_2 - \mu_1\mu_4 + \beta\mu_2\mu_4 + \mu_4\mu_c - \mu_4\mu_m}.$$  (61)

It gives us the minimum of the $\mu'_c$ in terms of the $\mu_i$.

So, the existence of $s + p$ phase is verified qualitatively by these solutions.

\section{On relation of Condensates and Chemical Potentials}

In this section we want to find an approximation for the relation of $\{<J_x>, D_+\}$ and $\{\mu - \mu', \mu'\}$ qualitatively. Following the series expansions obtained in \cite{13}, and using the asymptotically AdS solutions for $\{w, \psi\}$ we write down the equations (24,25) as perturbations around the following zero order solutions:

$$\Phi \approx \mu_c + D_+\chi(z), \Theta \approx \mu'_c + <J_x> \zeta(z)$$  (62)

as the following forms:

$$\chi'' - \frac{D_+ z^2 G^2}{1 - z^3} (\mu - \mu'_c) = 0$$  (63)

$$\zeta'' + \frac{z^2 G^2}{1 - z^3} (\mu - \mu'_c) \frac{D^2_+}{<J_x>} - \frac{z^2 F^2}{1 - z^3} \mu'_c <J_x> = 0.$$  (64)

With trial functions $y_G, y_F$ the solutions evaluated at $z = 0$ for the (63,64). Since

$$\Phi \approx \mu - \rho z \approx \mu_c + D_+ \left(\chi(0) + \chi'(0)z + \frac{1}{2}\chi''(0)z^2 + ..\right).$$  (65)
So, we have
\[
\mu \approx \mu_c + D_+ \chi(0)
\] (66)
and since the general solution for (63) reads as the following:
\[
\chi(z) = c_0 + c_1 z + D_+ \left( -0.05 z^5 a^2 \mu + 0.33 z^3 a^2 \mu - 0.5 z^2 a^2 \mu
- 0.5 a^2 \mu z \ln (z^2 + z + 1) + z \mu a^2 - 0.25 a^2 \mu \ln (z^2 + z + 1)
+ 1.7321 a^2 \mu \arctan (1.1547 z + 0.57736) z - 0.75 a^2 \mu \ln ((1.1547 z + 0.57736)^2 + 1)
+ 0.05 z^5 a^2 \mu' - 0.33 z^3 a^2 \mu' + 0.5 z^2 a^2 \mu' + 0.5 a^2 \mu' z \ln (z^2 + z + 1) - z \mu' a^2
+ 0.25 a^2 \mu' \ln (z^2 + z + 1) - 1.7321 a^2 \mu' \arctan (1.1547 z + 0.57736) z
+ 0.75 a^2 \mu' \ln ((1.1547 z + 0.57736)^2 + 1) \right).\] (67)

From this solution we obtain:
\[
\chi(0) \approx c_0 + 0.2157 a^2 (\mu - \mu') c D_+
\] (68)
Without any loss of the generality we set \(c_0 = 0\). Consequently we obtain:
\[
D_+ \approx \sqrt{\frac{\mu - \mu_c}{0.2157 a^2 (\mu'_c - \mu_c)}} = C \sqrt{\mu - \mu_c}.
\] (69)
It shows that the critical exponent \(\frac{1}{2}\) is recovered also in this case.
For the current $< J_x >$ we repeat the calculation by integration of $\zeta$ equation:

\[
\zeta(z) = c_0 + c_1 z - 1/3 C + 1/2 B z^2 a^2 - 1/2 B a^2 \arctan\left(\frac{2/3 \sqrt{3} z + 1/3 \sqrt{3}}{\sqrt{3}}\right) \sqrt{3} \tag{70}
\]
+ 1/6 C b^2 \arctan\left(\frac{2/3 \sqrt{3} z + 1/3 \sqrt{3}}{\sqrt{3}}\right) \sqrt{3} + 2/3 C b \ln (z - 1) z - 1/3 B z^2 a^2 - 1/3 C b z \ln (z^2 + z + 1)
+ 1/6 C b^2 z \ln (z^2 + z + 1) + 1/3 C b \arctan\left(\frac{2/3 \sqrt{3} z + 1/3 \sqrt{3}}{\sqrt{3}}\right) \sqrt{3} + 1/2 B a^2 z \ln (z^2 + z + 1)
+ 2/3 C b \arctan\left(\frac{2/3 \sqrt{3} z + 1/3 \sqrt{3}}{\sqrt{3}}\right) \sqrt{3} z - 1/2 C b^2 z^2 + 1/20 B z^5 a^2 - 1/20 z^5 C b^2
+ 1/2 z^3 C b - 1/3 C \sqrt{3} \arctan\left(\frac{1/3 (2 z + 1) \sqrt{3}}{\sqrt{3}}\right) + 3/4 B a^2 \ln \left(\frac{2/3 \sqrt{3} z + 1/3 \sqrt{3}}{\sqrt{3}}\right)^2 + 1
- 1/4 C b^2 \ln \left(\frac{2/3 \sqrt{3} z + 1/3 \sqrt{3}}{\sqrt{3}}\right)^2 + 1 - 1/3 C b \ln (z - 1) z - 1/3 C z \ln (z^2 + z + 1)
- 1/2 C b \ln \left(\frac{2/3 \sqrt{3} z + 1/3 \sqrt{3}}{\sqrt{3}}\right)^2 + 1 - B a^2 \arctan\left(\frac{2/3 \sqrt{3} z + 1/3 \sqrt{3}}{\sqrt{3}}\right) \sqrt{3} z
+ 1/3 C b^2 \arctan\left(\frac{2/3 \sqrt{3} z + 1/3 \sqrt{3}}{\sqrt{3}}\right) \sqrt{3} z - 1/3 C b^2 \ln (z - 1) z
+ 2/3 C b - 1/3 C b^2 - 1/3 \sqrt{3} \arctan\left(\frac{1/3 (2 z + 1) \sqrt{3}}{\sqrt{3}}\right) C b + 1/3 C b \arctan\left(\frac{1/3 (2 z + 1) \sqrt{3}}{\sqrt{3}}\right) C b - 1/6 \sqrt{3} \arctan\left(\frac{1/3 (2 z + 1) \sqrt{3}}{\sqrt{3}}\right) C b^2 - z B a^2 + 1/4 \ln (z^2 + z + 1) B a^2 - 1/6 \ln (z^2 + z + 1) C b^2 / 2 - 3 C \ln (z - 1) b + 1/3 C \ln (z - 1) b^2
- 1/6 \ln (z^2 + z + 1) C + 1/3 C \ln (z - 1) + z C
\]

Here $B = \frac{a^2 (\mu_e - \mu'_e) (\mu - \mu_e)}{< J_x >}, C = \mu'_e < J_x >$. By evaluating the solution at $z = 0$ we find:

\[
\zeta(0) = c_0 + (-2.0944 b + 1.0472 b^2 + 0.52284 b - 0.10294 b^2 - 0.63564 + 1.0472) C + 0.21574 B a^2 / 71 \tag{71}
\]

Since $B \gg 1, C \ll 1$ so,

\[
\mu' \approx \mu'_e + < J_x > \zeta(0) \tag{72}
\]

after a simple algebraic manipulation we find:

\[
< J_x > \approx \frac{24 C a^2 (\mu'_e - \mu_e) (\mu - \mu_e)}{\sqrt{\mu' - \mu'_e (\mu'_e)^{1/2} (-78578 b + 47213 b^2 + 20578)^{1/2}}} \tag{73}
\]

\[
+ \frac{223.61 \sqrt{\mu' - \mu'_e}}{\sqrt{(\mu'_e)^{1/2} (-78578 b + 47213 b^2 + 20578)^{1/2}}}. \]

It is a completely different expression and it shows that in the s+p phase why the vector current is a functions of the all chemical potentials $\mu, \mu'$.

Now we can compute the relation between the chemical potentials and the charge densities. Using the series expansion given in (65) we obtain:

\[
\rho = -D_+ \chi'(0). \tag{74}
\]
The same relation is valid, but with simple replacement of $< J_x > = D_+, \rho' \equiv \rho, \zeta \equiv \chi$.

Using (68) we find:

$$\chi'(0) = c_0 + 0.9069a^2D_+(\mu - \mu_0),$$

(75)

So,

$$\rho = -D_+ c_0 - 0.9069a^2D_+^2(\mu - \mu_0).$$

(76)

Using (69) we finally have:

$$\rho = -c_0 C(\mu - \mu_c)^{1/2} - 0.9069a^2C^2(\mu - \mu_0)(\mu - \mu_c) \equiv \beta_1(\mu - \mu_c)^{1/2} + \beta_1(\mu - \mu_c).$$

(77)

Here $\beta_i$ denotes the coefficient of $(\mu - \mu_c)^i$. When $c_0 = 0$ we have:

$$\rho \approx \beta_1(\mu - \mu_c).$$

(78)

The linear well know approximation $\rho = (\mu - \mu_c)$ still now is valid as a linear aproximation for $c_0 = 0$ or when $\mu \gg \mu_c$. But due to the mixing of the s,p phases the behavior of the $\rho - (\mu, \mu_0)$ is not linear due to the term of $-c_0 C(\mu - \mu_c)^{1/2}$. As we conclude that in the mixed phase when the system is far from the critical point the charge density of the mixed phase is larger than the pure phases of s or p. So, the free energy of the mixed phase $s + p$ is smaller than the pure single phases s or p. It means $F_{s+p} < F_{s,p}$. Since as it has been showed that in [9], the free energy is a linear function of the $-\rho, -\rho'$ and it has also a non local integral term as the following:

$$F = -\frac{1}{2}\Sigma_i \mu_i \rho_i + \int dr \{ \text{fields} + \text{fields}' + \ldots \}.$$  

(79)

Since the system thermodynamically goes to the minimum states with minimum of $F$, so in the competition between these phases, $s + p$ has a greater chance to be the dominant phase and it is the preffeed phase. In this duel between phases, the system prefer to remain in a mixed phases $(s+p)$. The same conclusion reported in [9] by numerically studying the behavior of the $F$. We mention here that since the analytical estimations of $\mu_c, \mu_c'$ are very close to the numerical ones so the evolutionary scheme of the system after these critical chemical potentials will be happen as a second order phase transition.

Now we compute the relation of $\rho', (\mu' - \mu_0')$ following the same strategy. We have:

$$\zeta'(0) = < J_x > (1.0472 - 0.74489b^2 + 2.6990b) \mu'_c - c_1 - 0.90695c^2a^2 \frac{(\mu_c - \mu'_c)(\mu - \mu_c)}{< J_x >}.$$  

(80)
So:

\[
\rho' = < J_x >^2 \left( 1.0472 + 0.74489 b^2 - 2.6990 b \right) \mu_c' + < J_x > c_1 + 0.90695 (\mu_c - \mu_c') c^2 (\mu - \mu_c) a^2. \tag{81}
\]

Since:

\[
< J_x > = \frac{\hat{A}(\mu - \mu_c)}{\sqrt{\mu' - \mu_c}} + \hat{B} \sqrt{\mu' - \mu_c}. \tag{82}
\]

Here \( \hat{A}, \hat{B} \) are two proportionality constants, so near the critical point when \( \mu \to \mu_c, \mu' \to \mu_c' \) we have:

\[
\rho' = \alpha_{2}^{-1}(\mu - \mu_c)^2(\mu' - \mu_c')^{-1} + \alpha_0^0(\mu - \mu_c) + \alpha_1^1(\mu' - \mu_c')(\mu - \mu_c) + \alpha_1^{-1/2} (\mu - \mu_c) + \alpha_0^{1/2} \sqrt{\mu' - \mu_c'}. \tag{83}
\]

Here \( \alpha_i^j \) denotes the coefficient of the term \( (\mu - \mu_c)^i (\mu' - \mu_c')^j \). So far also the linear approximation exists as the limiting case when the other non linear terms removed.

So, these generalized expressions show that why a mixed critical phase of \((s + p)\) exists in a system of U(2) fields under symmetry breaking to U(1).

IX. SUMMARY

In this paper, for the first time in the literature, we study the analytical mixed phase transitions from normal phase to an intermediate phase \(s+p\) and also p-wave in a newly proposed model of holographic superconductors. By variational method. We showed that in balanced and unbalanced configurations, a minimum of the critical chemical potential exists in which the system undergoes to the s-wave or p-wave and also in an intermediate s+p phase, due to the spontaneous symmetry breaking to the U(1) symmetry. Also, we find the condensation values as functions of the dual chemical potentials analytically. Our results have a reasonable good qualitatively agreement with the numerical results [9].
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