Homotopical interpretation of link invariants from finite quandles

Takefumi Nosaka

Abstract

This paper demonstrates a topological meaning of quandle cocycle invariants of links with respect to finite connected quandles $X$, from a perspective of homotopy theory: Specifically, for any prime $\ell$ which does not divide the type of $X$, the $\ell$-torsion of this invariants is equal to a sum of the colouring polynomial and a $\mathbb{Z}$-equivariant part of the Dijkgraaf-Witten invariant of a cyclic branched covering space. Moreover, our homotopical approach involves application of computing some third homology groups and second homotopy groups of the classifying spaces of quandles, from results of group cohomology. Furthermore we develop an approach for automorphism groups associated to quandles.

Keywords  Quandle, group homology, homotopy group, link, branched covering, bordism group, orthogonal and symplectic group, mapping class group

1 Introduction

A quandle, $X$, is a set with a binary operation $\triangleleft : X^2 \to X$ the axioms of which were motivated by knot theory. Actually, given an oriented link $L$ in the 3-sphere $S^3$, we can define the link quandle $Q_L$ [Joy], which is, roughly speaking, the conjugacy classes of $\pi_1(S^3\setminus L)$ including the meridians; A quandle homomorphism $C : Q_L \to X$ is practical and is called an $X$-coloring of $L$. Fenn, Rourke and Sanderson [FRS1, FRS2] introduced a space $BX$ as an analogy of the classifying spaces of groups; In addition they defined a fundamental class in $\pi_2(BQ_L)$ diagrammatically, and considered its push-forwards, $\Xi_X(L;C) \in \pi_2(BX)$ by $X$-colorings $C : Q_L \to X$. Furthermore, when $X$ is of finite order, the quandle homotopy invariant $\Xi_X(L) \in \mathbb{Z}[\pi_2(BX)]$ of a link $L$ is defined as the formal sum of $\Xi_X(L;C)$ running over all $X$-colorings $C$ of $L$, i.e., $\Xi_X(L) := \sum_C \Xi_X(L;C)$. Then the author [N1, N2] studied quantitatively $\pi_2(BX)$ and the invariant $\Xi_X(L)$ for some quandles $X$.

From a cohomological viewpoint, Carter et.al [CJKLS, CEGS, CKS] used 2-cocycles $\phi$ of $X$ with a local coefficient group $A$ to introduce quandle cocycle invariants $\Phi_\phi(L) \in \mathbb{Z}[A]$ of links. The cocycle invariants were much studied since they can be computed relatively, compared with the homotopy group $\pi_2(BX)$. However, as is known [RS], any such cocycle invariants can be derived from the quandle homotopy invariant via the Hurewicz map $\pi_2(BX) \to H_2(BX;A)$ with local system (see [N1] §2 for the detailed formula).

However these invariants of links were constructed from link diagrams combinatorially. So two papers [Kab, HN] tried to provide a topological meaning of these invariant; however, the second homology $H_2(BX;\mathbb{Z})$ is an obstacle to study the space $BX$, and accordingly successful works are only for the simplest quandle of the form $X = \mathbb{Z}/(2m-1)$ with $x \triangleleft y := 2y - x$. In fact, so far, there are only a few topological studies of $BX$ with respect to general quandles $X$ (However, we refer to [FRS2, Cla1, N2] as some topological approaches to $BX$).

In this paper we demonstrate a topological meaning of these invariants from, more generally, “connected” quandles, together with computing the homotopy groups $\pi_2(BX)$. For this, we
shall focus on the quandle homotopy invariant $\Xi_X(L)$, since $\Xi_X(L)$ is universal among all the cocycle invariants as mentioned above. To state our main theorem, we fix two simple $x$ such that $(\cdots (x < a_1) < \cdots) < a_n = y$: The type $t_X$ of $X$ is the minimal $N$ such that $x = (\cdots (x < y) < \cdots) < y$ $[N$-times on the right with $y]$ for any $x, y \in X$.

**Theorem 1.1** (Theorem 3.5). Let $X$ be a connected quandle of type $t_X$ and of finite order. Let $H_X : \pi_2(BX) \to H_2(BX; \mathbb{Z})$ be the Hurewicz map. Then there is a homomorphism $\Theta_X$ from $\pi_2(BX)$ to the third group homology $H_3^g(\pi_1(BX); \mathbb{Z})$ for which the sum

$$\Theta_X \oplus H_X : \pi_2(BX) \to H_3^g(\pi_1(BX); \mathbb{Z}) \oplus H_2(BX; \mathbb{Z})$$

is an isomorphism after localization at $\ell$, where $\ell$ is relatively prime to $t_X$.

Moreover, in this paper, the sum $\Theta_X \oplus H_X$ which we call the TH-map plays an important role as follows: We will show (Corollary 3.6) that, for any link $L \subset S^3$, the homotopy invariant $\Xi_X(L) \in \mathbb{Z}[\pi_2(BX)]$ is sent to a sum of two invariants via the TH-map $H_X \oplus \Theta_X$: precisely, the original quandle cycle invariant $\Xi_X(L)$ is $\Xi_X(L)$ in $\mathbb{C}[H_2(BX)]$ in [CJKLS] and $a$ $Z$-equivariant of Dijkgraaf-Witten invariant of $\tilde{C}_L^{tx}$ (see [10] for the definition), where $\tilde{C}_L^{tx}$ is the $t_X$-fold cyclic covering space of $S^3$ branched over the link $L$. Denoting these two invariants by $DW_{As(X)}(\tilde{C}_L^{tx})$ and $\Phi_X(L)$ respectively, the corollary 3.6 is formally summarized to an equality

$$(\Theta_X \oplus H_X)(\Xi_X(L)) = DW_{As(X)}(\tilde{C}_L^{tx}) + \Phi_X(L) \mod t_X$$

We here remark that the former part $DW_{As(X)}(\tilde{C}_L^{tx})$ is roughly defined to be a sum of push-forwards of the fundamental class of the space $\tilde{C}_L^{tx}$ via some $Z$-equivariant homomorphisms $\pi_1(\tilde{C}_L^{tx}) \to \pi_1(BX)$ by definition [10]; furthermore, as is shown [12], the latter invariant $\Phi_X(L)$ is characterized by longitudes of $X$-colored links (see [15] for details). In conclusion, via the TH-map, the homotopy invariant $\Xi_X(L)$ without $t_X$-torsion is reduced to the two topological invariants as desired. As a result of our theorem, we here emphasize that a minimal obstacle in studying $\pi_2(BX)$ and topological meanings is the type $t_X$ of $X$, rather than the second homology $H_2(BX)$.

Moreover, our theorem leads to concrete computations of most parts of $\pi_2(BX)$ with respect to concrete quandles. For this, we have to study the image of the TH-map mentioned in Theorem 1.1 in particular the fundamental group $\pi_1(BX)$. We then develop a simple and applicable method for determining $\pi_1(BX)$ in terms of ‘universal central extensions of groups modulo $t_X$-torsion’ (see [14]). Actually, in some cases, we determine $\pi_1(BX)$ and compute the homology $H_3^g(\pi_1(BX); \mathbb{Z})$ concretely. Furthermore we can determine the second homology $H_2(BX)$ in terms of $\pi_1(BX)$, thanks to a method of Eisermann [12] (see Theorem 5.7.2 and Appendix 14). In summary, we conclude a computation of the $\pi_2(BX)$ up to $t_X$-torsion from computations of $H_3^g(\pi_1(BX))$ and $H_2(BX)$.
This paper further investigates and computes some \( t_X \)-torsion of \( \pi_2(BX) \) of some quandles \( X \). To be precise we will show that the TH-map is an isomorphism for several quandles: “regular Alexander quandles”, most “symplectic quandles over \( \mathbb{F}_q \)”, and most connected quandles of order \( \leq 8 \) (see Theorems 3.11, 3.13). Hence, we have computed the homotopy group \( \pi_2(BX) \), from which follows computing \( H_2(BX) \) and \( H_3^\text{gr}(\pi_1(BX)) \). For example, regarding a symplectic quandle over \( \mathbb{F}_q \) “in a certain stable range”, the \( \pi_2(BX) \) is isomorphic to \( \mathbb{Z} \oplus K_3(\mathbb{F}_q) \), where \( K_3(\mathbb{F}_q) \cong \mathbb{Z}/(q^2-1) \) is the Quillen-K-group of \( \mathbb{F}_q \). So, in general, we conjecture that the TH-map is an isomorphism for many connected quandles \( X \). Incidentally, as an application, regarding a “Dehn quandle \( \mathcal{D}_g \)”, which is a certain conjugacy class of the mapping class group and is useful for Lefschetz fibrations, we show that \( \pi_2(BD_g) \) is either \( \mathbb{Z} \oplus \mathbb{Z}/24 \) or \( \mathbb{Z} \oplus \mathbb{Z}/48 \) for \( g \geq 7 \) (Theorem 8.1).

Moreover, as an application, our approach to \( \pi_2(BX) \) involves a new method for computing the third homology \( H_3(BX) \), and establish a relation between third quandle homology and group homology. As a general result, with respect to a finite connected quandle \( X \), we solve some torsion subgroups of the \( H_3(BX) \) in terms of group homology (Theorem 3.14). Furthermore, as examples, we compute most torsions of the third homology groups \( H_3(BX) \) of the symplectic quandles and spherical quandles over \( \mathbb{F}_q \) in a stable range (see Theorem 3.17). In addition, letting \( X \) be an Alexander quandle, we will show an isomorphism \( H_3(BX) \cong H_3^\text{gr}(\pi_1(BX)) \oplus (H_2(BX) \wedge H_2(BX)) \) up to \( 2t_X \)-torsion. We here note that most of known methods for computing \( H_3(BX) \) by hand was a result of Mochizuki [Moc] with respect to Alexander quandles of the forms \( X = \mathbb{F}_q[T]/(T - \omega) \), although his presentation of \( H^3(BX; \mathbb{F}_q) \) was a little complicated (see [Moc]). However our result implies that the complexity of \( H^3(BX; \mathbb{F}_q) \) stems from that of \( H_3^\text{gr}(\pi_1(BX)) \).

Furthermore, we study a close relation between the homology \( H_3(B\tilde{X}) \) and \( \pi_2(B\tilde{X}) \) with respect to “extended quandles \( \tilde{X} \)”. Here such a quandle \( \tilde{X} \) is constructed from a connected quandle \( X \) of type \( t_X \); see §3.3 for the definition. As we see in Theorem 3.18 if \( X \) is of finite order, our approach above provides isomorphisms

\[
\pi_2(B\tilde{X}) \cong H_3(B\tilde{X}) \cong H_3^\text{gr}(\text{As}(X)) \oplus \mathbb{Z} \quad \text{up to } t_X\text{-torsion.}
\]

This result and viewpoint from extended quandles \( \tilde{X} \) is of vital importance in the proof of Theorem 1.1 (see §6.1) and in a subsequent paper [N4].

Finally, we emphasize two benefits from our study on \( \pi_2(BX) \) in Theorem 1.1. First, our theorem suggests a simple computation of the Dijkgraaf-Witten invariant with respect to a finite group \( G \). Although the definition seems very simple (see [29]), it is not so easy to compute this invariants exactly. Actually, most known computations of the invariants are in the cases where \( G \) are abelian (see, e.g., [DW, Kab, HN]). However, as mentioned above, the TH-isomorphism \( \Theta_X \oplus \mathcal{H}_X \) implies that we can deal with some \( \mathbb{Z} \)-equivariant parts of this invariants via the quandle cocycle invariants of links. In fact, in the paper [N4] we will compute the invariants of some knots using Alexander quandles \( X \) over \( \mathbb{F}_q \), whose the \( \pi_1(BX) \) are nilpotent groups (see [37] for the lower central series). As a result, triple Massey products of some Brieskorn manifolds \( \Sigma(n, m, l) \) will be calculated; see [N4, §5].

On the other hand, for algebraic topology, our proof of Theorem 1.1 determines the second
Postnikov invariant, \( k^3(BX) \in H^3_{gr}(\pi_1(BX); \pi_2(BX)) \), modulo \( t_X \)-torsion. Indeed, Theorem 1.1 is shown by using the Postnikov tower of \( BX \) written in an exact sequence

\[
H_3(BX) \xrightarrow{c_*} H^3_{gr}(\pi_1(BX)) \xrightarrow{\tau} \pi_2(BX) \xrightarrow{H_X} H_2(BX) \xrightarrow{c_*} H^2_{gr}(\pi_1(BX)) \to 0,
\]

where \( \tau \) is the transgression map (see §6.1 for details). To be more precise, we will show (Theorem 6.1) that the both maps \( c_* \) are annihilated by \( t_X \), and (Proposition 6.13) that the map \( \Theta_X \) in Theorem 1.1 gives a splitting from \( \pi_2(BX) \) for general finite quandles. Here we compare the sequence (1) with results in [N2] where the \( \pi_2(BX) \) were analyzed with local systems with respect to only “regular” Alexander quandles \( X \).

This paper is organized as follows. Section 2 reviews the quandle homotopy invariant. Section 3 states our results. Section 4 constructs the homomorphism \( \Theta_X \). Section 5 reviews a method to compute the second quandle homology according to [E2]. Section 6 proves the Theorem 1.1. Section 7 contains the proofs of Theorems 3.11, 3.13 precisely, we will compute some \( \pi_2(BX) \) and \( H_3(BX) \) concretely. Section 8 determines \( \pi_2(BX) \) of the Dehn quandle in a stable range. Section 9 is devoted to computing the third homology \( H_3(BX) \). In addition, Appendix A proposes a calculating for automorphism groups of quandles. Appendix B computes some second quandle homology groups.

Conventional notation Throughout this paper, most homology groups are with (trivial) integral coefficients; so we often omit writing coefficients, e.g., \( H_n(X) \). We denote the group homology of a group \( G \) by \( H_{gr}^n(G) \). Furthermore, we denote a \( \mathbb{Z} \)-module \( M \) localized at a prime \( \ell \) by \( M_{(\ell)} \). Moreover, a homomorphism \( f : A \to B \) between abelian groups is said to be an isomorphism modulo \( N \)-torsion, denoted by \( f : A \cong B \pmod{N} \), if the localization of \( f \) at \( \ell \) is an isomorphism for any prime \( \ell \) that does not divide \( N \). In addition, we assume that every manifolds are in \( C^\infty \)-class and oriented, and that any fields is not of characteristic 2.

2 Review of quandles and the quandle homotopy invariants

To establish our results in §3, we will review quandles in §2.1 and quandle homotopy invariants of links in §2.2.

2.1 Review of quandles

A quandle is a set, \( X \), with a binary operation \( \triangleleft : X \times X \to X \) such that

(i) The identity \( a \triangleleft a = a \) holds for any \( a \in X \).

(ii) The map \( (\bullet \triangleleft a) : X \to X \) defined by \( x \mapsto x \triangleleft a \) is bijective for any \( a \in X \).

(iii) The identity \( (a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c) \) holds for any \( a, b, c \in X \).
A quandle $X$ is said to be of type $t_X$, if $t_X > 0$ is the minimal $N$ such that $x = x \triangleleft^N y$ for any $x, y \in X$, where we denote by $\bullet \triangleleft^N y$ the $N$-times on the right operation with $y$. Note that, if $X$ is of finite order, it is of type $t_X$ for some $t_X \in \mathbb{Z}$. A map $f: X \to Y$ between quandles is a (quandle) homomorphism, if $f(a \triangleleft b) = f(a) \triangleleft f(b)$ for any $a, b \in X$. We now refer to some examples of quandles:

**Example 2.1** (Alexander quandle). Any $\mathbb{Z}[T, T^{-1}]$-module $M$ is a quandle with the operation $x \triangleleft y = y + T(x - y)$ for $x, y \in M$, called an Alexander quandle. This operation $\bullet \triangleleft y$ is roughly a $T$-multiple at $y$. The type is the minimal $N$ such that $T^N = \text{id}_{M}$ since $x \triangleleft^N y = y + T^n(x - y)$. As a typical example, with a choice of an element $\omega \in \mathbb{F}_q \setminus \{0, 1\}$, the quandle of the form $X = \mathbb{F}_q[T]/(T - \omega)$ is called an Alexander quandle on a finite field $\mathbb{F}_q$.

**Example 2.2** (Symplectic quandle). Let $K$ be a field. Let $\Sigma_g$ be the closed surface of genus $g$, and let $X$ be the first homology with $K$-coefficients outside 0, that is, $X = H^1(\Sigma_g; K) \setminus \{0\} = K^{2g} \setminus \{0\}$. Denote the standard symplectic form by $\langle \cdot, \cdot \rangle : H^1(\Sigma_g; K) \times H^1(\Sigma_g; K) \to K$. Then this set $X$ is made into a quandle by the operation $x \triangleleft y := \langle x, y \rangle y + x \in X$ for any $x, y \in X$, and is called a symplectic quandle (over $K$). The operation $\bullet \triangleleft y : X \to X$ is commonly called the transvection of $y$. Note that the quandle $X$ is of type $p = \text{Char}(K)$ since $x \triangleleft^N y = N\langle x, y \rangle y + x$. When $K$ is a finite field $\mathbb{F}_q$, we denote the quandle by $Sp_q^g$.

**Example 2.3** (Spherical quandle). Let $K$ be a field of characteristic $\neq 2$. Let $\langle \cdot, \cdot \rangle : K^{n+1} \otimes K^{n+1} \to K$ be the standard symmetric bilinear form. Consider a set of the form

$$S_K^n := \{ x \in K^{n+1} \mid \langle x, x \rangle = 1 \}.$$  

We define the operation $x \triangleleft y$ to be $2\langle x, y \rangle y - x \in S_K^n$. This pair $(S_K^n, \triangleleft)$ is a quandle, and is referred to as a spherical quandle (over $K$). This operation $\bullet \triangleleft y$ can be interpreted as a rotation of 180-degrees with the center $y$. Hence the quandle is of type 2. In the finite case $K = \mathbb{F}_q$, we denote the quandle by $S_q^n$.

As observed above, quandle consists of, figuratively speaking, ‘operations itself centered at $y \in X$’, which can be described as homogenous spaces (see [Joy, §7] for detail).

Next we review the associated group denoted by $\text{As}(X)$ [FRS1]. This group is the abstract group defined by generators $e_x$ labeled by $x \in X$ modulo the relations $e_x \cdot e_y = e_y \cdot e_{x \triangleleft y}$ with $x, y \in X$. To be precise, the group $\text{As}(X)$ is presented by

$$\text{As}(X) = \langle e_x \ (x \in X) \mid e_{x \triangleleft y}^{-1} \cdot e_y^{-1} \cdot e_x \cdot e_y \ (x, y \in X) \rangle.$$  

We fix an action $\text{As}(X)$ on $X$ defined by $x \cdot e_y := x \triangleleft y$ for $x, y \in X$. Note the equality

$$e_{x \triangleleft y} = g^{-1} e_x g \in \text{As}(X) \quad (x \in X, \ g \in \text{As}(X)),$$  

by definitions. The orbits of the above action of $\text{As}(X)$ on $X$ are called connected components of $X$, denoted by $O(X)$. If the action of $\text{As}(X)$ on $X$ is transitive, $X$ is said to be connected. For example, it is known [LN, Proposition 1] that an Alexander quandle $X$ in Example 2.1 is
connected if and only if \((1-T)X = X\). Furthermore it can be easily seen that all the quandles over \(K\) in Examples 2.2 2.3 are connected.

Note that the group \(\text{As}(X)\) is of infinite order. Actually, there is a split epimorphism
\[
\varepsilon_X : \text{As}(X) \longrightarrow \mathbb{Z}
\]
which sends each generators \(e_x\) to \(1 \in \mathbb{Z}\). Furthermore, if \(X\) is connected, this \(\varepsilon_X\) gives the abelianization \(\text{As}(X)_{ab} \cong \mathbb{Z}\) because of (2). The reader should be keep in mind this epimorphism \(\varepsilon_X\).

2.2 Review; Quandle homotopy invariant of links.

We begin reviewing \(X\)-colorings. Let \(X\) be a quandle, and \(D\) an oriented link diagram of a link \(L \subset S^3\). An \(X\)-coloring of \(D\) is a map \(C : \{\text{arcs of } D\} \rightarrow X\) such that \(C(\gamma_k) = C(\gamma_i) \cdot C(\gamma_j)\) at any crossing of \(D\) such as Figure 1. Let \(\text{Col}_X(D)\) denote the set of all \(X\)-colorings of \(D\). As is well known, if two diagrams \(D_1, D_2\) are related by Reidemeister moves, we easily obtain a canonical bijection \(\text{Col}_X(D_1) \simeq \text{Col}_X(D_2)\); see, e.g., [Joy, CJKLS].

![Figure 1: Positive and negative crossings.](image)

We now study a topological meaning of an \(X\)-coloring \(C\) of \(D\). Let us correspondence each arc \(\gamma\) to the generator \(\Gamma_C(\gamma_i) := e_{C(\gamma_i)} \in \text{As}(X)\), which defines a group homomorphism
\[
\Gamma_C : \pi_1(S^3 \setminus L) \longrightarrow \text{As}(X)
\]
by Writinger presentation. We then have a map \(\text{Col}_X(D) \rightarrow \text{Hom}(\pi_1(S^3 \setminus L), \text{As}(X))\) which carries \(C\) to \(\Gamma_C\), leading to a topological meaning of the set \(\text{Col}_X(D)\) as follows:

**Proposition 2.4** (cf. [E2, Lemma 3.14] in the knot case). Let \(X\) be a quandle. Let \(D\) be a diagram of an oriented link \(L\). We fix a meridian-longitude pair \((m_i, l_i) \in \pi_1(S^3 \setminus L)\) of each link-component which is compatible with the orientation. Then the previous map which sends \(C\) to \(\Gamma_C\) gives rise to a bijection between the \(\text{Col}_X(D)\) and a set
\[
\{(x_1, \ldots, x_{\#L}, f) \in X^{\#L} \times \text{Hom}_{Gr}(\pi_1(S^3 \setminus L), \text{As}(X)) \mid f(m_i) = e_{x_i}, \ x_i \cdot f(l_i) = x_i\}. \tag{5}
\]

This proof will appear in Appendix D although it is not so difficult.

Next, we briefly recall the quandle homotopy invariant of links (our formula is a modification the formula in [FRS1]). Let us consider the set, \(\Pi_2(X)\), of all \(X\)-colorings of all diagrams subject to Reidemeister moves and the concordance relations illustrated in Figure 2. Then disjoint unions of \(X\)-colorings make \(\Pi_2(X)\) into an abelian group, which is closely related to a homotopy group \(\pi_2(BX)\); see (19). For any link diagram \(D\), we have a map \(\Xi_{X,D} : \text{Col}_X(D) \rightarrow\)
$\Pi_2(X)$ taking $C$ to the class $[C]$ in $\Pi_2(X)$. If $X$ is of finite order and $D$ is a diagram of a link $L$, then the quandle homotopy invariant of $L$ is defined as the expression

$$\Xi_X(L) := \sum_{C \in \text{Col}_X(D)} \Xi_{X,D}(C) \in \mathbb{Z}[\Pi_2(X)].$$

(6)

Moreover, as is known $[RS]$ (see also $[N1, \S2]$), the homotopy invariant is universal among all “the quandle cocycle invariants with local coefficients” (see $[CJKLS, CKS, CEGS, Kab]$ for these definitions). Hence, to answer what the cocycle invariants are, instead, hereafter we may focus on the study of the homotopy invariant and the abelian group $\Pi_2(X)$ in details.

Finally, we briefly review the original quandle cocycle invariant $[CJKLS]$. Given a finite quandle $X$, we set its quandle homology $\mathcal{H}_Q^2(X)$ with trivial coefficients, which is a quotient of the free module $\mathbb{Z}\langle X \times X \rangle$ (see §5 for the definition). For an $X$-coloring $C \in \text{Col}_X(D)$, we consider a sum $\sum \epsilon_\tau(C(\gamma_i), C(\gamma_j)) \in \mathbb{Z}\langle X \times X \rangle$, where $\tau$ runs over all crossing of $D$ as shown in Figure 1 and the symbol $\epsilon_\tau \in \{\pm1\}$ denotes the sign of the crossing $\tau$. As is known (see $[RS, N1]$), this sum is a 2-cycle, and homology classes of these sums in $\mathcal{H}_Q^2(X)$ are independent of Reidemeister moves and the concordance relations; Hence we obtain a homomorphism

$$\mathcal{H}_X : \Pi_2(X) \rightarrow \mathcal{H}_Q^2(X).$$

(7)

Using the formula (6), the quandle cycle invariant of a link $L$, denoted by $\Phi_X(L)$, is then defined to be the image $\mathcal{H}_X(\Xi_X(L))$ valued in the group ring $\mathbb{Z}[\mathcal{H}_Q^2(X)]$. Namely

$$\Phi_X(L) := \mathcal{H}_X(\Xi_X(L)) \in \mathbb{Z}[\mathcal{H}_Q^2(X)].$$

(8)

As is known $[RS, N1, CKS]$, given a quandle 2-cocycle $\phi : X^2 \rightarrow A$, the paring between this $\phi$ and the cycle invariant $\Phi_X(L)$ coincides with the original cocycle invariant in $[CJKLS, \text{Theorem 4.4.}]$.

Although this $\Phi_X(L)$ is constructed from link diagrams, in §5 we later explain its topological meaning, together with a computation of $\mathcal{H}_Q^2(X)$ following from Eisermann $[E1, E2]$.

### 3 Results on the quandle homotopy invariants

The purpose in this section is to state our results on the $\Pi_2(X)$. In §3.1 we will set up a homomorphism $\Theta_X$. In §3.2 we state our results on the group $\Pi_2(X)$. As an application, we will see the computations of third quandle homology groups in §3.3.
3.1 A key homomorphism $\Theta_X$

Before stating our main results, we will set up a homomorphism $\Theta_X$ in Theorem 3.1 which plays a key role in this paper. Furthermore we discuss Corollary 3.4 which proposes a necessary condition to obtain topological interpretations of the quandle homotopy invariants and, hence, of any quandle cocycle invariants.

To state the theorem, for any link $L \subset S^3$, we denote by $p : \tilde{S}^3 \setminus L \rightarrow S^3 \setminus L$ the $t_X$-fold cyclic covering associated to the homomorphism $\pi_1(S^3 \setminus L) \rightarrow \mathbb{Z}/t_X$ sending each meridian of $L$ to 1. Furthermore, given an $X$-coloring $C$ and using the map $\Gamma_C$ in (1), we consider the composite $p_* \circ \Gamma_C : \pi_1(S^3 \setminus L) \rightarrow \text{As}(X)$. Then

**Theorem 3.1.** Let $X$ be a connected quandle of type $t_X$. Then, for any diagram $D$ of any link $L \subset S^3$, the composite $p_* \circ \Gamma_C : \pi_1(S^3 \setminus L) \rightarrow \text{As}(X)$ induces a group homomorphism

$$\theta_{X,D}(C) : \pi_1(\tilde{C}_L^{t_X}) \rightarrow \text{As}(X),$$

where we denote by $\tilde{C}_L^{t_X}$ the $t_X$-fold cyclic covering space of $S^3$ branched over the link $L$.

Moreover, consider the pushforward of the fundamental class $[\tilde{C}_L^{t_X}]$ by the $\theta_{X,D}(C)$, that is, $(\theta_{X,D}(C))_*([\tilde{C}_L^{t_X}]) \in H^3_{\text{gr}}(\text{As}(X))$. Then, these pushforwards with running over all $X$-colorings of all diagrams $D$ yield an additive homomorphism $\Theta_X : \Pi_2(X) \rightarrow H^3_{\text{gr}}(\text{As}(X))$.

As is seen in the proof in [4], we later reconstruct these maps $\theta_{X,D}$ and $\Theta_X$ concretely. By construction, they provide a commutative diagram, functorial in $X$, described as

$$\begin{array}{ccc}
\text{Col}_X(D) & \xrightarrow{\theta_{X,D}} & \text{Hom}_{\text{gr}}(\pi_1(\tilde{C}_L^{t_X}), \text{As}(X)) \\
\cong_{X,D} & & \cong_{\Theta_X}
\Pi_2(X) & \xrightarrow{\Theta_X} & H^3_{\text{gr}}(\text{As}(X)).
\end{array}$$

**Remark 3.2.** As is seen in [4] for any $X$-coloring $C \in \text{Col}_X(D)$, the homomorphism $\theta_{X,D}(C) : \pi_1(\tilde{C}_L^{t_X}) \rightarrow \text{As}(X)$ factors through the kernel $\text{Ker}(\varepsilon_X) \subset \text{As}(X)$ in (3) and is $\mathbb{Z}$-equivariant with respect to the covering transformation $\mathbb{Z} \acts \tilde{C}_L^{t_X}$ and the action $\mathbb{Z} \acts \text{Ker}(\varepsilon_X)$ from the splitting (3).

The map plays an important role in this paper. Thus we fix terminologies:

**Definition 3.3.** Let $X$ be a connected quandle of type $t_X$. We call the map $\Theta_X$ $T$-map, and the sum $\Theta_X \oplus \mathcal{H}_X$ $TH$-map. Here $\mathcal{H}_X$ is the map $\mathcal{H}_X : \Pi_2(X) \rightarrow H^2_{\text{gr}}(X)$ defined in (7).

Next, to state Corollary 3.4, we briefly review the Dijkgraaf-Witten invariant [DW]. Given a finite group $G$ and a group 3-cocycle $\kappa \in H^3_{\text{gr}}(G; A)$, the Dijkgraaf-Witten invariant of $M$ is defined as a formal sum of some pairings expressed as

$$\text{DW}_\kappa(M) := \sum_{f \in \text{Hom}(\pi_1(M), G)} \langle \kappa, f_*([M]) \rangle \in \mathbb{Z}[A].$$

Here $\mathbb{Z}[A]$ is a group ring of $A$. Furthermore, when $X$ is of finite order, as the image of the $\theta_{X,D}$ in Theorem 3.1 we define a certain ($\mathbb{Z}$-equivariant) part of the Dijkgraaf-Witten invariant of...
branched covering spaces $\hat{C}_L^{t_X}$ as the formula

$$DW_{\text{As}(X)}(\hat{C}_L^{t_X}) := \sum_{C \in \text{Col}_X(D)} [\Theta_X(\Xi_{X,D}(C))] = \sum_{C \in \text{Col}_X(D)} \theta_{X,D}(C)_*(\hat{C}_L^{t_X}) \in \mathbb{Z}[H_3^{gr}(\text{As}(X))]. \quad (10)$$

Using this, we discuss the quandle homotopy invariant under an assumption as follows:

**Corollary 3.4.** Let $X, \hat{C}_L^{t_X}, \Theta_X$ be as above and let $\ell \in \mathbb{Z}$ be a prime. Let $|X| < \infty$. Take the Hurewicz homomorphism $H_X : \Pi_2(X) \to H_2^Q(X)$ in (7). If the TH-map $\Theta_X \oplus H_X : \Pi_2(X) \to H_3^{gr}(\text{As}(X)) \oplus H_2^Q(X)$ is an isomorphism after $\ell$-localization, then the $\ell$-torsion of the quandle homotopy invariant of any link $L$ is decomposed as

$$(\Theta_X \oplus H_X)(\ell)(\Xi_X(L)) = DW_{\text{As}(X)}(\hat{C}_L^{t_X})(\ell) + \Phi_X(L)(\ell) \in \mathbb{Z}[\Pi_2(X)(\ell)]. \quad (11)$$

To conclude, under the assumption on the TH-map, we succeed in providing a topological interpretation of the quandle homotopy invariant $\Xi_X(L)$ as mentioned in the introduction. Actually, the two invariants in the right hand side of (11) are defined topologically.

### 3.2 Results on the TH-maps $\Theta_X \oplus H_X$

Following Corollary 3.4, it is thus significant to find quandles such that the localized TH-map $(\Theta_X \oplus H_X)(\ell)$ are isomorphisms. This section lists such quandles.

To begin with, we state the main theorem as a general statement. To be specific,

**Theorem 3.5.** Let $X$ be a connected quandle of type $t_X < \infty$. If the homology $H_3^{gr}(\text{As}(X))$ is finitely generated (e.g., if $X$ is of finite order), then the TH-map is an isomorphism modulo $t_X$-torsion.

As a result, combining this theorem with Corollary 3.4 we have obtained the interpretation of some torsion of the quandle homotopy (cocycle) invariant. Precisely,

**Corollary 3.6.** Let $X$ be a finite connected quandle of type $t_X$. Then the equality (11) in Corollary 3.4 holds for any prime $\ell$ which does not divide $t_X$.

Note that there are many quandles whose types $t_X$ are powers of some prime, e.g., the quandles in Examples 2.2, 2.3 and connected quandles of order $\leq 8$. In conclusion, for such quandles $X$, we determine most subgroups of $\pi_2(BX)$ by Theorem 3.5.

**Remark 3.7.** Corollary 3.6 is a strong generalization of some results in [Kab, HN]. Indeed, the results dealt with only the Alexander quandle of the from $X = \mathbb{Z}[T^{\pm 1}]/(2n - 1, T + 1)$, and were based on peculiar properties of the quandle.

We moreover address some $t_X$-torsion subgroups of $\Pi_2(X)$. First we discuss an easy condition of vanishing of these $t_X$-torsions:

**Proposition 3.8.** Let $X$ be a connected quandle of type $t_X$. If the $t_X$-torsion part of the image $H_3^{gr}(\text{As}(X)) \oplus H_2^Q(X)$ is assumed to be zero, then that of the TH-map is zero.

While the easy proof will appear in §6.1, we now observe examples satisfying the assumption.
**Example 3.9.** We now discuss regular Alexander quandles $X$ of finite order. Here, $X$ is said to be *regular*, if $X$ is connected and its type is relatively prime to the order $|X|$, e.g., the Alexander quandles on $\mathbb{F}_q$ since $\omega^{q-1} = 1$. Then the previous assumption holds. Actually, the homology $H^3_2(X)$ is a certain quotient of $X \otimes \mathbb{Z} X$ (see Proposition [B.1]), and it can be easily seen that $t_X$-torsion of the group homology $H^*_X(\text{As}(X))$ is zero by the lower central series of $\text{As}(X)$ [see (37)]. In particular, by Theorem [3.5], we immediately have

**Corollary 3.10.** Let $X$ be a regular Alexander quandle of finite order. Then, the TH-map is an isomorphism.

We however remark that the torsion $\Pi_2(X) \otimes \mathbb{Z}/p$ of the Alexander quandles on $\mathbb{F}_q$ with $p \neq 2$ were computed from another direction (see [N2, Appendix]).

Next, we deal with several quandles $X$ such that the $t_X$-torsions of $\Pi_2(X)$ are non-zero. Let us discuss the symplectic and spherical quandles over $\mathbb{F}_q$ in Examples [2.2] 2.3. To describe this, we call $q = p^d \in \mathbb{N}$ exceptional, if the $q$ is one of $\{3, 3^2, 3^3, 5, 7\}$, that is, $d(p-1) \leq 6$ (cf. the condition in Theorem [7.3] later).

**Theorem 3.11.** Let $q = p^d$ be odd, and be not exceptional.

(I) Let $X$ be the symplectic quandle $S^a_q$ over $\mathbb{F}_q$. Then, the TH-map is an isomorphism. Furthermore, $\Pi_2(X) \cong \mathbb{Z}/(q^2 - 1)$ for $n > 1$ and $\Pi_2(X) \cong \mathbb{Z}/(q^2 - 1) \oplus (\mathbb{Z}/p)^d$ for $n = 1$.

(II) Let $X$ be the spherical quandle $S^a_q$ over $\mathbb{F}_q$. The TH-map is an isomorphism modulo 2-torsion. Moreover, $\Pi_2(X) \cong \mathbb{Z}/(q^2 - 1) \mod 2$ for $n > 2$, and $\Pi_2(X) \cong \mathbb{Z}/(q^2 - 1) \oplus \mathbb{Z}/(q - \delta_q) \mod 2$ for $n = 2$. Here $\delta_q = \pm 1$ is according to $q \equiv \pm 1 \pmod{4}$.

**Remark 3.12.** We comment the exceptional cases of $q$. As seen in the proofs in §7.1, the TH-map is an isomorphism modulo 2p-torsion; Furthermore, if $n$ is enough large, then the $p$-torsion part of the TH-map is an isomorphism (see §7.1 and Remark [7.3]).

Furthermore, we focus on connected quandles $X$ of order $\leq 8$ as follows:

**Theorem 3.13.** For any connected quandle $X$ of order $\leq 8$, the TH-map gives an isomorphism $\Pi_2(X) \cong H^*_3(\text{As}(X)) \oplus \text{Im}(\mathcal{H}_X)$.

See §7.2 for the proof and for the computations of $\Pi_2(X)$. This theorem generalizes some results in the previous paper [N1, §4]. Indeed, the author only estimated the group $\Pi_2(X)$ for quandles $X$ with $|X| \leq 6$.

In conclusion, for such quandles discussed above, we obtain a topological meaning of the quandle homotopy (cocycle) invariants from the viewpoint of Corollary [3.4].

### 3.3 Application; some computations of third quandle homology

As an application of computing the (homotopy) group $\Pi_2(X)$, we develop a new method for computing the third quandle homology $H^*_3(X)$; see §A for the definition. Moreover, we will give explicit computations of $H^*_3(X)$ of some quandles. Statements in this subsection will be proven in §9.
To describe our results, we briefly review the inner automorphism group, \(\text{Inn}(X)\), of a quandle \(X\). Recalling the action of \(\text{As}(X)\) on \(X\), we thus have a group homomorphism \(\psi_X\) from \(\text{As}(X)\) to the symmetric group \(\mathfrak{S}_X\). The group \(\text{Inn}(X)\) is defined by the image \((\subset \mathfrak{S}_X)\).

Hence we have a group extension

\[
0 \longrightarrow \ker(\psi_X) \longrightarrow \text{As}(X) \xrightarrow{\psi_X} \text{Inn}(X) \longrightarrow 0 \quad \text{(exact).}
\]

By the equality (12), this kernel \(\ker(\psi_X)\) is contained in the center. Furthermore, as we later show (Corollary 6.4) that, if \(X\) is of type \(t_X\) and connected, then \(\ker(\psi_X) \cong \mathbb{Z} \oplus H^2_2(\text{Inn}(X))\) modulo \(t_X\)-torsion.

The following theorem is an estimate on the third homology \(H^Q_3(X)\).

**Theorem 3.14.** Let \(X\) be a connected quandle of finite order. Then, there is the following isomorphism up to \(2|\text{Inn}(X)|/|X|\)-torsion:

\[
H^Q_3(X) \cong H^gr_3(\text{As}(X)) \oplus (\ker(\psi_X) \wedge \ker(\psi_X)).
\]

Here note that the type \(t_X\) divides the order \(|\text{Inn}(X)|/|X|\); see Lemma A.9. In summary, many torsion subgroups of the third quandle homology are determined after computing the group homologies of \(\text{As}(X)\) and \(\text{Inn}(X)\).

**Remark 3.15.** The isomorphism does not hold in the 2-torsion subgroup. See Remark [7,6] for a counterexample. Furthermore, as mentioned in the introduction, the most known results on the third \(H^Q_3(X)\) are with respect to the Alexander quandles over \(\mathbb{F}_q\) and are due to Mochizuki [Moc].

To be more concrete, we consider the third quandle homologies of some quandles. Notice that Theorem 3.14 is of use with respect to quandles \(X\) such that the order \(|\text{Inn}(X)|/|X|\) is small. For example, in Alexander case, the order \(|\text{Inn}(X)|/|X|\) equals \(\text{Type}(X)\) exactly (see [N2, Lemma 5.6]). We then describe third homologies of regular Alexander quandles.

**Theorem 3.16.** Let \(X\) be a regular Alexander quandle of finite order. Then, there is the isomorphism \(H^Q_3(X) \cong H^gr_3(\text{As}(X)) \oplus (\wedge^2 \ker(\psi_X))\) modulo 2-torsion.

Moreover, if the order of \(X\) is odd, then the 2-torsion subgroups of the both sides are zero.

Here remark that, the associated group \(\text{As}(X)\) and the kernel \(\ker(\psi_X)\) have been calculated by Clauwens [Cla2] (see also Appendix B). In particular, the group \(\text{As}(X)\) is a nilpotent group of degree 2 (see (37) for the lower central series); hence the group (co)homology is not simple, e.g., it contains some Massey products (see [N4, §4]).

Next, we further determine explicitly the third quandle homologies of symplectic quandles and of spherical quandles over \(\mathbb{F}_q\), although the orders \(|\text{Inn}(X)|/|X|\) are not simple.

**Theorem 3.17.** Let \(q = p^d\) be odd and not exceptional.

(I) Let \(X = \text{Sp}^n_q\) be the symplectic quandle of order \(q^{2n} - 1\) in Example [2,2] If \(n = 1\), then

\[
H^Q_2(X) \cong (\mathbb{Z}/p)^d, \quad H^Q_3(X) \cong \mathbb{Z}/(q^2 - 1) \oplus (\mathbb{Z}/p)^{d(d+1)/2}.
\]

On the other hand, if \(n > 1\), then \(H^Q_2(X) \cong H^Q_3(X) \cong 0\).
Let $X = S^n_q$ be the spherical quandle over $\mathbb{F}_q$ in Example 2.3. Let $\delta_q \in \{\pm 1\}$ be according to $q \equiv \pm 1 \pmod{4}$. If $n = 2$, then
\[
H^Q_2(X) \cong \mathbb{Z}/(q - \delta_q), \quad H^Q_3(X) \cong \mathbb{Z}/(q^2 - 1) \oplus \mathbb{Z}/(q - \delta_q)
\]
modulo 2-torsion. If $n > 2$, then $H^Q_3(X)$ and $H^Q_2(X)$ are elementary abelian 2-groups.

This theorem mostly settles a problem posed by Kabaya [ILDT] for computing the homology $H^Q_3(\text{Sp}_n^q)$ with $n = 1$.

Finally, we will mention some extended quandles considered in [Joy, §7], which plays a role to prove Theorem 3.5, and describe these quandle homologies. Recall the homomorphism $\varepsilon_X : \text{As}(X) \to \mathbb{Z}$ in (3). Given a connected quandle $X$ with $a \in X$, we equip the kernel $\text{Ker}(\varepsilon_X)$ with a quandle operation by setting
\[
g \triangleleft h := e^{-1}_a gh e^{-1}_a \quad \text{for } g, h \in \text{Ker}(\varepsilon_X).
\]
We denote the quandle $(\text{Ker}(\varepsilon_X), \triangleleft)$ by $\tilde{X}$, which is called the extended quandle (of $X$). We easily see the independence of the choice of $a \in X$ up to quandle isomorphisms. Using the restricted action $X \acts \text{Ker}(\varepsilon_X) \subset \text{As}(X)$, the canonical map $p : \tilde{X} \to X$ sending $g$ to $a \cdot g$ is known to be a quandle homomorphism (see [Joy, Theorem 7.1]), and is called the universal (quandle) covering of $X$, according to Eisermann [E2, §5]. We see that, when $X$ is finite and of type $t_X$, so is the extended one $\tilde{X}$. We later show the connectivity of $\tilde{X}$ (Lemma 6.9).

We now discuss the groups $H^Q_3(\tilde{X})$ and $\Pi_2(\tilde{X})$ of extended quandles $\tilde{X}$ as follows:

**Theorem 3.18.** Let $X$ be a connected quandle of type $t_X$. Let $p : \tilde{X} \to X$ be the universal covering mentioned above. If $H^\text{gr}_3(\text{As}(X))$ is finitely generated, then there are isomorphisms
\[
H^Q_3(\tilde{X}) \cong \Pi_2(\tilde{X}) \cong H^\text{gr}_3(\text{As}(X)) \text{ modulo } t_X\text{-torsion}.
\]
Here the second isomorphism $\Pi_2(\tilde{X}) \cong H^\text{gr}_3(\text{As}(X))$ is obtained from the composite $\Theta_X \circ p_*$.

**Remark 3.19.** Furthermore we will determine the second homology $H^Q_2(\tilde{X})$ (Theorem 6.12).

In a subsequent paper [N4], this theorem on the quandle $\tilde{X}$ will be used to understand the T-map $\Theta_X$ from a viewpoint of complexes of groups and quandles.

### 4 The homomorphism $\Theta_X$

From now on, we will prove the results mentioned in the previous section.

Our purpose in this section is to prove Theorem 3.1 and, is to construct a homomorphism from $\Pi_2(X)$ to a bordism group (Lemma 4.2), which plays a key role in this paper. The construction is a modification of a certain map in [HN, §4,5], where we dealt with only a class of “4-fold symmetric quandles”.

For the purpose, we first describe a presentation of the fundamental group $\pi_1(\tilde{C}_L^t)$, where $\tilde{C}_L^t$ denotes the $t$-fold cyclic covering of $S^3$ branched along a link $L$. Put a link diagram $D$ of $L$. Let $\gamma_0, \ldots, \gamma_n$ be the arcs of this $D$. Let $\tilde{S^3} \setminus \tilde{L}$ be the $t$-fold cyclic covering space of...
$S^3 \setminus L$ associated to the homomorphism $\pi_1(S^3 \setminus L) \to \mathbb{Z}/t$ sending each $\gamma_i$ to 1. For any index $s \in \mathbb{Z}/t$, we take a copy $\gamma_{i,s}$ of the arc $\gamma_i$. Then, by Reidemeister-Schreier method (see, e.g., [Rol, Appendix A] and [Kab, §3]), the fundamental group $\pi_1(S^3 \setminus L)$ can be presented by

- **generators:** $\gamma_{i,s}$ $(0 \leq i \leq n, s \in \mathbb{Z}),$
- **relations:** $\gamma_{k,s} = \gamma_{j,s}^{-1} \gamma_{i,s-1} \gamma_{j,s}$ for each crossings such as Figure 1 and, $\gamma_{0,0} = \gamma_{0,1} = \cdots = \gamma_{0,t-2} = 1$.

Further we can define the inclusion $p : \pi_1(S^3 \setminus L) \hookrightarrow \pi_1(S^3 \setminus L)$ by $\iota(\gamma_{i,s}) = \gamma_{i,s} \gamma_{i,0}^{-s}$, with a choice of an appropriate base point. Moreover, the fundamental group $\pi_1(C_L)$ is obtained from this presentation by adding the relation $\gamma_{0,t-1} = 1$.

Next, given a quandle $X$ of type $t_X$, we now construct a map (13) below. For this end, we note the following lemma, which is often used later.

**Lemma 4.1.** Let $X$ be a connected quandle of type $t$. Then, for any $x, y \in X$, we have the identity $(e_x)^t = (e_y)^t$ in the center of $\text{As}(X)$.

**Proof.** For any $b \in X$, note the equalities $(e_x)^{-t} e_b e_x^t = e_{(\cdot-b \cdot <x \cdot)<x} = e_b$ in $\text{As}(X)$. Namely $(e_x)^t$ lies in the center. Furthermore the connectivity admits $g \in \text{As}(X)$ such that $x \cdot g = y$. Hence it follows from (2) that $(e_x)^t = g^{-1}(e_x)^t g = (e_x g)^t = (e_y)^t$ as desired. 

Thus, by the above presentation of $\pi_1(C_L)$, the map $\Gamma_C$ induces a homomorphism $\hat{\Gamma}_C : \pi_1(C_L) \to \text{Ker}(\varepsilon_X)$, where the homomorphism $\varepsilon_X : \text{As}(X) \to \mathbb{Z}$ sending the generators $e_x$ to 1 in $\mathbb{Z}$ [see (3)]. Precisely, this $\hat{\Gamma}_C$ is defined by the formula $\hat{\Gamma}_C(\gamma_{i,s}) = e_{c(\gamma_{0,s})}^{s-1} e_{c(\gamma)} e_{c(\gamma_0)}^{-s}$. In summary, we obtain the map stated in Theorem 3.1

$$\theta_{X,D} : \text{Col}_X(D) \to \text{Hom}_{\text{Gr}}(\pi_1(C_L), \text{Ker}(\varepsilon_X)), \quad (\mathcal{C} \mapsto \hat{\Gamma}_C). \quad (13)$$

We here remark that this map depends on the choice of the arc $\gamma_0$; however it does not up to conjugacy of $\text{Ker}(\varepsilon_X)$ by construction, if $X$ is connected.

Finally, in order to state Lemma 4.2 below, we briefly recall the oriented bordism group, $\Omega_n(G)$, of a group $G$. We consider a pair consisting of a closed connected oriented $n$-manifold $M$ without boundary and a homomorphism $\pi_1(M) \to G$. Then the set, $\Omega_n(G)$, is defined to be the quotient set of such pairs $(M, \pi_1(M) \to G)$ subject to $G$-bordant equivalence. Here, such two pairs $(M_i, f_i : \pi_1(M_i) \to G)$ are $G$-bordant, if there exist an oriented $(n+1)$-manifold $W$ which bounds the connected sum $M_1 \# (-M_2)$ and a homomorphism $\tilde{f} : \pi_1(W) \to G$ so that $f_1 \ast f_2 = \tilde{f} \circ (i_W)_*$, where $i_W : M_1 \# (-M_2) \to W$ is the natural inclusion. An abelian group structure is imposed on $\Omega_n(G)$ by connected sum. Note that this $\Omega_n(G)$ agrees with the usual oriented $(SO)$-bordism group of the Eilenberg-MacLane space $K(G, 1)$.

**Lemma 4.2.** Let $X$ be a connected quandle of type $t$. Then, by considering all link diagrams $D$, the maps $\theta_{X,D}$ in (13) give rise to an additive homomorphism $\Theta_{\Pi_2} : \Pi_2(X) \to \Omega_3(\text{Ker}(\varepsilon_X))$. 

$$\Theta_{\Pi_2} : \Pi_2(X) \to \Omega_3(\text{Ker}(\varepsilon_X)). \quad (14)$$
A sketch of the proof. Since the proof is analogous to [HN] Lemma 5.3 and Proposition 4.3 essentially, we will sketch it. To obtain the homomorphism \( \Theta_{\text{H0}} \), it suffices to show that the maps take the concordance relations to the bordance ones.

First, to deal with the local move in the right of Figure 2, we recall that the \( t \)-fold cyclic covering of \( S^3 \) branched over the 2-component trivial link \( T_2 \) is \( S^2 \times S^1 \to S^3 \) (see [Rol, §10.2]). It suffices to show that any \( f : \pi_1(S^2 \times S^1) \to \text{Ker}(\varepsilon_X) \) is \( G \)-bordant. Indeed, \( f : \pi_1(B^3 \times S^1) \to \text{Ker}(\varepsilon_X) \) provides its bordance, where \( B^3 \) is a ball.

Next, for two \( X \)-colorings \( C_1 \) and \( C_2 \) related by the left in Figure 3, we will show that the connected sum \( \theta_{X,D}(C_1 \#(-C_2)^*) : \pi_1(\hat{C}_{L_1}^t \# \hat{C}_{L_2}^t) \to \text{Ker}(\varepsilon_X) \) is null-bordance. Let \( N_{C_1} \subset S^3 \) be a neighborhood around the local move. Then we put a canonical saddle \( \mathcal{F} \) in \( N_{C_1} \times [0, 1] \) which bounds the four arcs illustrated in Figure 3. Define an embedded surface \( W \subset S^3 \times [0, 1] \) to be \((L_1 \setminus N_{C_1}) \times [0, 1] \cup \mathcal{F}\). Then the \( t \)-fold cyclic covering \( \mathcal{W} \to S^3 \times [0, 1] \) branched over \( W \) bounds \( \hat{C}_{L_1}^t \sqcup \hat{C}_{L_2}^t \). Moreover, we can verify that the sum \( \theta_{X,D}(C_1 \#(-C_2)^*) \) extends to a group homomorphism \( \pi_1(\mathcal{W}) \to \text{Ker}(\varepsilon_X) \), which gives the desired null-bordance.

Finally, in order to prove Theorem 3.1, we recall Thom homomorphism \( T_G : \Omega_n(G) \to H_n(K(G, 1)) = H_n^{gr}(G) \) obtained by assigning to every pair \((M, f : \pi_1(M) \to G)\) the image of the orientation class under \( f_* : H_n(M) \to H_n(K(G, 1)) \). It is widely known that, if \( n = 3 \), the map \( T_G \) is an isomorphism \( \Omega_3(G) \cong H_3^{gr}(G) \).

Proof of Theorem 3.1. Let \( G \) be \( \text{As}(X) \). Put the inclusion \( \iota : \text{Ker}(\varepsilon_X) \hookrightarrow \text{As}(X) \). Consequently, defining the T-map \( \Theta_X \) to be the composite \( T_{\text{As}(X)} \circ \iota_* \circ \Theta_{\text{H0}} : \Pi_2(X) \to H_3^{gr}(\text{As}(X)) \), we can see that this \( \Theta_X \) satisfies the desired properties.

5 Preliminaries; quandle homology and cocycle invariant.

As a preparation, we now review some properties of the (co)homology of the rack space, and a topological interpretation of the cocycle invariants. There is nothing new in this section.

We start reviewing the (action) rack space introduced by Fenn-Rourke-Sanderson [FRS1 Example 3.1.1]. Let \( X \) be a quandle. We further fix a set \( Y \) acted on by \( \text{As}(X) \), which is called \( X \)-set. For example, the quandle \( X \) is itself an \( X \)-set, referred as to the primitive \( X \)-set, from the canonical action \( X \curvearrowright \text{As}(X) \) mentioned in [2]. We further equip a quandle \( X \) and the \( X \)-set \( Y \) with their discrete topology. We put a union \( \bigcup_{n \geq 0} (Y \times ([0, 1] \times X)^n) \), and consider the relations given by

\[
(y, t_1, x_1, \ldots, x_j-1, 1, x_j, t_{j+1}, \ldots, t_n, x_n) \sim (y \cdot \varepsilon_{x_j}, t_1, x_1 \triangleleft x_j, \ldots, t_{j-1}, x_j-1 \triangleleft x_j, t_{j+1}, x_j+1, \ldots, t_n, x_n),
\]

\[
(y, t_1, x_1, \ldots, x_j-1, 0, x_j, t_{j+1}, \ldots, t_n, x_n) \sim (y, t_1, x_1, \ldots, t_{j-1}, x_j-1, t_{j+1}, x_{j+1}, \ldots, t_n, x_n).
\]

Figure 3: \( \mathcal{F} \) is a saddle in the neighborhood \( N_{C_1} \times [0, 1] \).
Then the rack space \( B(X,Y) \) is defined to be the quotient space. When \( Y \) is a single point, we denote the space by \( BX \) for short. By construction, we have a cell decomposition of \( B(X,Y) \) by regarding the projection \( \bigcup_{n \geq 0} (Y \times ([0,1] \times X)^n) \rightarrow B(X,Y) \) as characteristic maps (see [N1 §2] or [N2 §2.2] for a detailed picture of the 3-skeleton of \( BX \)).

Furthermore, we briefly review the rack and quandle (co)homologies (our formula relies on [AG [CEGS]]). Let \( X \) be a quandle, and \( Y \) an \( X \)-set. Let \( C_{-n}^R(X,Y) \) be the free right \( \mathbb{Z} \)-module generated by \( Y \times X^n \). Define a boundary \( \partial_n^R : C_{-n}^R(X,Y) \rightarrow C_{-n-1}^R(X,Y) \) by

\[
\partial_n^R(y, x_1, \ldots, x_n) = \sum_{1 \leq i \leq n} (-1)^i ((y < x_i, x_1 < x_i, \ldots, x_{i-1} < x_i, x_{i+1}, \ldots, x_n) - (y, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)).
\]

The composite \( \partial_{n-1}^R \circ \partial_n^R \) is known to be zero. The homology is denoted by \( H_n^R(X,Y) \) and is called rack homology. As is known, the cellular complex of the rack space \( B(X,Y) \) above is isomorphic to the rack complex \( (C_{-n}^R(X,Y), \partial_n^R) \).

**Remark 5.1.** If \( Y \) is the primitive \( X \)-set \( Y = X \), we have the chain isomorphism \( C_{-n}^R(X,X) \rightarrow C_{-n+1}^R(X,pt) \) induced from the identification \( X \times X^n \simeq X^{n+1} \); see, e.g., [Clai Proposition 2.1]. In particular, we obtain an isomorphism \( H_n^R(X,X) \cong H_{n+1}^R(X,pt) \).

Furthermore, let \( C_{-n}^D(X,Y) \) be a submodule of \( C_{-n}^R(X,Y) \) generated by \((n+1)\)-tuples \((y, x_1, \ldots, x_n)\) with \( x_i = x_{i+1} \) for some \( i \in \{1, \ldots, n-1\} \). It can be easily seen that the submodule \( C_{-n}^D(X,Y) \) is a subcomplex of \( C_{-n}^R(X,Y) \). Then the quandle homology, \( H_n^Q(X,Y) \), is defined to be the homology of the quotient complex \( C_{-n}^R(X,Y)/C_{-n}^D(X,Y) \).

We will review some properties of these homologies in the case where \( Y \) is a single point (in such a case, we omit the symbol \( Y \)). Let us decompose \( X \) as \( X = \sqcup_{i \in O(X)} X_i \) by the connected components. The following direct sum decompositions were shown [LN]:

\[
H_1^R(X) \cong \mathbb{Z}^{O(X)}, \quad H_2^R(X) \cong H_2^Q(X) \oplus \mathbb{Z}^{O(X)}.
\] (15)

Furthermore, Eisermann [E2] gave a computation of the second quandle homologies \( H_2^Q(X) \) with trivial \( \mathbb{Z} \)-coefficients. To see this, for \( i \in O(X) \), define a homomorphism

\[
\varepsilon_i : \text{As}(X) \rightarrow \mathbb{Z} \quad \text{by} \quad \begin{cases} 
\varepsilon_i(e_x) = 1 \in \mathbb{Z}, & \text{if } x \in X_i, \\
\varepsilon_i(e_x) = 0 \in \mathbb{Z}, & \text{if } x \in X \setminus X_i.
\end{cases}
\] (16)

Note that the sum \( \oplus_{i \in O(X)} \varepsilon_i \) yields the abelianization \( \text{As}(X)_{\text{ab}} \cong \mathbb{Z}^{O(X)} \) by [2]. Furthermore

**Theorem 5.2 ([E2 Theorem 9.9]).** Let \( X \) be a quandle. Decompose \( X = \sqcup_{i \in O(X)} X_i \) as the orbits by the action of \( \text{As}(X) \). Fix an element \( x_i \in X_i \) for each \( i \in O(X) \). Let \( \text{Stab}(x_i) \subset \text{As}(X) \) be the stabilizer of \( x_i \). Then the quandle homology \( H_2^Q(X) \) is isomorphic to the direct sum of the abelianizations of \( \text{Stab}(x_i) \cap \ker(\varepsilon_i) \): Precisely, \( \bigoplus_{i \in O(X)} (\text{Stab}(x_i) \cap \ker(\varepsilon_i))_{\text{ab}} \).

Eisermann showed topologically this result by using a certain CW-complex. However, in [B.1] we later give another proof as a slight application of Proposition 9.2.

We furthermore change to the study of the cycle invariant \( \Phi_X(L) \) explained in [8]. We now briefly explain a topological interpretation of this invariant shown by Eisermann [E1, E2]. Decompose \( X = \sqcup_{i \in O(X)} X_i \) as above. Given an \( X \)-coloring \( \mathcal{C} \in \text{Col}_X(D) \), with respect to a link component of \( L \), we fix an arc \( \gamma_j \) on \( D \) for \( 1 \leq j \leq \#L \). Let \( x_j := \mathcal{C}(\gamma_j) \in X_j \), and fix a longitude \( l_j \) of the component. Recall from [1] the associated group homomorphism \( \Gamma_{\mathcal{C}} : \pi_1(S^3 \setminus L) \rightarrow \text{As}(X) \). Remark that each longitude \( l_j \) commutes with the meridian in the same link component. Accordingly, \( \Gamma_{\mathcal{C}}(l_j) \) commutes with \( e_{x_j} \) in \( \text{As}(X) \) in other wards,
\[ \Gamma_C(t_j) \in \text{Stab}(x_j). \] Furthermore, since the class of the longitude \( l_j \) in \( H_1(S^3 \setminus L) \) is zero, the \( \Gamma_C(t_j) \) is contained in the kernel \( \text{Ker}(\varepsilon_j) \) [see (16)]. Therefore the \( \Gamma_C(t_j) \) lies in \( \text{Stab}(x_j) \cap \text{Ker}(\varepsilon_j) \). Further, consider the class \( \Gamma_C(t_j) \) in the abelianization of this \( \text{Stab}(x_j) \cap \text{Ker}(\varepsilon_j) \). In summary, we obtain

\[ ([\Gamma_C(t_1)], \ldots, [\Gamma_C(t_{\#L})]) \in \bigoplus_{1 \leq j \leq \#L} (\text{Stab}(x_j) \cap \text{Ker}(\varepsilon_j))_{\text{ab}}. \] (17)

Here note that each direct summand in the right side is contained in \( H_2^Q(X) \) by Theorem 5.2.

We then put the product \([\Gamma_C(t_1) \cdots \Gamma_C(t_{\#L})] \in H_2^Q(X)\). By the discussion in [E1, Theorems 3.24 and 3.25], it can be seen that the product coincides with the value \( \mathcal{H}_X(C) \) in (7) exactly.

Hence, when \( |X| < \infty \), the cycle invariant \( \Phi_X(L) \) written in (8) is reformulated as

\[ \Phi_X(L) = \sum_{C \in \text{Col}_X(D)} [\Gamma_C(t_1) \cdots \Gamma_C(t_{\#L})] \in \mathbb{Z}[H_2^Q(X)]. \] (18)

This was called “colouring polynomials” in [E1, §1]. As a result, this formula suggests an easy computation and a topological meaning of the cycle invariant as desired.

Finally, we observe a relation between this formula (15) and the Hurewicz homomorphism of \( BX \). Recall the map \( \mathcal{H}_X : \Pi_2(X) \rightarrow H_2^Q(X) \) in (7). Using the isomorphisms (15) and (19), we consider a composite

\[ \pi_2(BX) \cong \Pi_2(X) \oplus \mathbb{Z}^{O(X)} \xrightarrow{\text{proj}} \Pi_2(X) \xrightarrow{\mathcal{H}_X} H_2^Q(X) \rightarrow H_2^Q(X) \oplus \mathbb{Z}^{O(X)} \cong H_2(BX). \]

From the definitions of the maps \( \mathcal{H}_X \) and the 2-skeleton of the rack space \( BX \), we can easily verify that this composite coincides with the Hurewicz map of \( BX \) modulo the direct summand \( \mathbb{Z}^{O(X)} \) (see [RS] and [N1, Proposition 3.12] for details). In conclusion, the formula (18) enables us to compute the Hurewicz map of \( BX \).

6 Proof of Theorem 3.5

This section proves Theorem 3.5. Since the proof is ad hoc, the hasty reader may read only the outline in §6.1 and skip the details in other subsections. In §6.2 we state a \( t_X \)-vanishing theorem following the outline. In §6.3 we observe some properties of quandle coverings since they play a key role in the proof. In §6.4 we will investigate the homomorphism \( \Theta_X \) in terms of relative bordism groups, and complete the proof.

6.1 Outline of proofs of Theorem 3.5

We roughly outline the proof of Theorem 3.5 to compute \( \Pi_2(X) \).

As an approach to the homotopy group \( \pi_2(BX) \), the reader should keep in mind the following isomorphism shown by [FRS2] (see also [N1, Theorem 6.2] for the detailed description):

\[ \pi_2(BX) \cong \Pi_2(X) \oplus \mathbb{Z}^{O(X)}, \] (19)

where the symbol \( O(X) \) is the set of the connected components of \( X \). According to the isomorphism (19), to compute \( \Pi_2(X) \), we will change a focus on computing the homotopy group \( \pi_2(BX) \) from a standard discussion of “Postnikov tower on \( BX \)”. To illustrate, let
$c : BX \hookrightarrow K(\pi_1(BX), 1)$ be an inclusion obtained by killing the higher homotopy groups of $BX$. Notice that the homotopy fiber of $c$ is the universal covering of $BX$. Thanks to the fact that the action of $\pi_1(BX)$ on $\pi_n(BX)$ is trivial (see also [Clau1, Proposition 2.16]), the Leray-Serre spectral sequence of the map $c$ provides an exact sequence

$$H_3(BX) \xrightarrow{c_*} H_3^{gr}(\pi_1(BX)) \xrightarrow{\tau} \pi_2(BX) \xrightarrow{\mathcal{H}} H_2(BX) \xrightarrow{c_*} H_2^{gr}(\pi_1(BX)) \rightarrow 0 \quad \text{(exact)}, \quad (20)$$

where $\mathcal{H}$ is the Hurewicz map of $BX$ and the $\tau$ is the transgression (see, e.g., [McC] §8.3bis], [Bro] §II.5 for details).

We now reduce this (20) to (21) below. Recalling the isomorphism $H_2(BX) \cong \mathbb{Z}O(X) \oplus H_2^Q(X)$ (see (15)), the restriction of the Hurewicz map $\mathcal{H}$ on the summand $\mathbb{Z}O(X) \subset \pi_2(BX)$ is shown to be an isomorphism [N1, Proposition 3.12]. Therefore, recalling the isomorphism $\text{As}(X) \cong \pi_1(BX)$, the sequence (20) is reformulated as

$$H_3(BX) \xrightarrow{c_*} H_3^{gr}(\text{As}(X)) \xrightarrow{\tau} \Pi_2(X) \xrightarrow{\mathcal{H}} H_2^Q(X) \xrightarrow{c_*} H_2^{gr}(\text{As}(X)) \rightarrow 0 \quad \text{(exact)}. \quad (21)$$

Since this paper often uses this sequence, we call it the $P$-sequence (of $X$).

Using the $P$-sequence, we outline the proof of Theorem 3.5. Let $X$ be connected and of type $t_X < \infty$. We later show Theorem 6.1 which says that the maps $c_* : H_n(BX) \rightarrow H_n^{gr}(\pi_1(BX))$ in (20) are annihilated by $t_X$ for $n \leq 3$. Thus, the $P$-sequence (21) becomes a short exact sequence modulo $t_X$ (Corollary 6.3). Hence, in order to show that the TH-map $\Pi_2(X) \rightarrow H_3^{gr}(\text{As}(X)) \oplus H_2^Q(X)$ without $t_X$-torsion is an isomorphism as stated in Theorem 3.5, we shall show that the T-map $\Theta_X : \Pi_2(X) \rightarrow H_3^{gr}(\text{As}(X))$ constructed in Theorem 3.1 turns out to be a splitting of the exact sequence (21).

To this end, we first show the splitting with respect to the extended quandles $\tilde{X}$ (Proposition 6.13). So we will study properties of the $\tilde{X}$ in 6.3. The point is that, using these properties, the transgression $\tau$ in (21) can be regarded as an inverse mapping of the T-map $\Theta_{\tilde{X}}$ in a (relative) bordism theory. After that, for general connected quandles $X$, the functoriality of the projection $\tilde{X} \rightarrow X$ completes the proof of Theorem 3.5.

Before going to the next subsection, we immediately prove Proposition 3.8.

**Proof of Proposition 3.8.** Since the $t_X$-torsion of $H_3^{gr}(\text{As}(X)) \oplus H_2^Q(X)$ is zero by assumption, that of $\Pi_2(X)$ vanishes by (21). Hence, that of the TH-map $\Theta_X \oplus \mathcal{H}_X$ is zero as desired. \qed

### 6.2 The vanishing of the $t_X$-multiple of the map $c_*$

As the first step in the preceding outline, we state Theorem 6.1.

**Theorem 6.1.** Let $X$ be a connected quandle of type $t_X$, and let $t_X < \infty$. For $n = 2$ and $3$, the induced map $c_* : H_n(BX) \rightarrow H_n^{gr}(\text{As}(X))$ in (20) is annihilated by $t_X$.

**Remark 6.2.** This theorem is still more powerful than a result of Clauwens [Clau1, Proposition 4.4], which stated that, if a finite quandle $X$ satisfies a certain strong condition, then the composite $(\psi_X)_* c_* : H_n(BX) \rightarrow H_n^{gr}(\text{As}(X)) \rightarrow H_n^{gr}(\text{Inn}(X))$ is annihilated by $|\text{Inn}(X)|/|X|$ for any $n \in \mathbb{Z}_{\geq 0}$. Here note that $t_X$ is a divisor of the order $|\text{Inn}(X)|/|X|$ (Lemma A.9).
The proof will appear in Appendix C. Instead, we now give some corollaries.

**Corollary 6.3.** Let $X$ be as above, and $\ell$ be a prime which is relatively prime to the $t_X$. Then the $P$-sequence localized at $\ell$ is reduced to a short exact sequence

$$0 \rightarrow H^\text{gr}_2(\mathbb{A}(X))_{(\ell)} \rightarrow \pi_2(BX)_{(\ell)} \xrightarrow{\partial} H_2(BX)_{(\ell)} \rightarrow 0.$$  

**Corollary 6.4.** For any connected quandle $X$ of type $t_X$, the second group homology $H^\text{gr}_2(\mathbb{A}(X))$ is annihilated by $t_X$. Furthermore, the abelian kernel $\text{Ker}(\psi_X)$ in $[12]$ is isomorphic to $\mathbb{Z} \oplus H^\text{gr}_2(\text{Inn}(X))$ modulo $t_X$-torsion.

**Proof.** By Corollary 6.3, the map $H_2(BX) \rightarrow H^\text{gr}_2(\mathbb{A}(X))$ is surjective; hence it follows from Theorem 6.1 that $H^\text{gr}_2(\mathbb{A}(X))$ is annihilated by $t_X$. Furthermore, the desired isomorphism $\text{Ker}(\psi_X) \cong \mathbb{Z} \oplus H^\text{gr}_2(\text{Inn}(X))$ (mod $t_X$) is immediately obtained from the inflation-restriction exact sequence with respect to the $\psi_X$ and $H^\text{gr}_1(\mathbb{A}(X)) \cong \mathbb{Z}$. \(\square\)

**Corollary 6.5.** Let $X$ and $\ell \in \mathbb{Z}$ be as above. Let $X$ be of finite order. Then the quandle cycle invariant $\Phi_X$ in $[8]$ is non-trivial in the $\ell$-torsion part. That is, for any class $[O] \in H_2(BX)_{(\ell)}$, there exists some $X$-coloring $\mathcal{C}$ of some link such that $\mathcal{H}_X([\mathcal{C}]) = [O]$.

**Proof.** By Corollary 6.3, the map $\mathcal{H}_X$ localized at $\ell$ is surjective. Since the $\Pi_2(X)$ is generated by $X$-colorings of links by definition, we have $\mathcal{H}_X([\mathcal{C}]) = [O]$ for some $X$-coloring $\mathcal{C}$. \(\square\)

**Remark 6.6.** By this discussion, we see that, more generally, for a quandle $X$ with $H^\text{gr}_2(\mathbb{A}(X)) = 0$, any class $[O] \in H_2(BX)$ ensures some $X$-coloring $\mathcal{C}$ such that $\mathcal{H}_X([\mathcal{C}]) = [O]$.

### 6.3 Some properties of quandle coverings

As preliminaries, we will explore some properties of quandle coverings. Here, recall that a quandle epimorphism $p : Y \rightarrow Z$ is a (quandle) covering in the sense of [22], if the equality $p(\bar{x}) = p(\bar{y}) \in Z$ implies $\bar{a} \triangleleft \bar{x} = \bar{a} \triangleleft \bar{y} \in Y$ for any $\bar{a}, \bar{x}, \bar{y} \in Y$. For example,

**Example 6.7.** The universal quandle covering $\tilde{X} \rightarrow X$ explained in [22] is a covering.

**Proposition 6.8.** For any quandle covering $p : Y \rightarrow Z$, the induced group surjection $p_* : \mathbb{A}(Y) \rightarrow \mathbb{A}(Z)$ is a central extension. Furthermore, if $Y$ and $Z$ are connected and $Z$ is of type $t_X$, then the abelian kernel $\text{Ker}(p_*)$ is annihilated by $t_X$.

**Proof.** For any $y \in Z$, put arbitrary $y_i, y_j \in p^{-1}(y)$. Since $p$ is a covering, we notice

$$e_{y_i}^{-1}e_{y_i} = e_{y_j}^{-1}e_{y_j} = e_{y_i}^{-1}e_{y_j} \in \mathbb{A}(Y)$$

for any $b \in Y$. Namely, for any indices $i, j$, there are central elements $z_{ij} \in \mathbb{A}(Y)$ such that $e_{y_i} = z_{ij}e_{y_j}$. Hence $\mathbb{A}(Y)$ is generated by $e_{y_i}$ for $y \in Z$ and the central elements $z_{ij}$ with $i \neq j$; Consequently the surjection $p_*$ is a central extension.

We will show the latter part. Take the inflation-restriction exact sequence, i.e.,

$$H^\text{gr}_2(\mathbb{A}(Z)) \rightarrow \text{Ker}(p_*) \rightarrow H^\text{gr}_1(\mathbb{A}(Y)) \rightarrow H^\text{gr}_1(\mathbb{A}(Z)) \rightarrow 0 \quad (\text{exact}).$$

By connectivities the third map from $H^\text{gr}_1(\mathbb{A}(Y)) = \mathbb{Z}$ is an isomorphism. Further, since Corollary 6.4 says that $H^\text{gr}_2(\mathbb{A}(Z))$ is annihilated by $t_X$, so is the kernel $\text{Ker}(p_*)$ as desired. \(\square\)
induces an isomorphism \( p \). Furthermore, since the particular, the quandle \( \tilde{X} \).

**Lemma 6.9.** For any connected quandle \( X \), the action \( \mu \) of \( \text{As}(X) \) on \( \tilde{X} \) is transitive. In particular, the quandle \( \tilde{X} \) is also connected.

**Proof.** It is enough to show that the identity \(1_{\tilde{X}} \in \tilde{X} = \text{Ker}(\epsilon_X)\) is transitive to any element \( h \) in the quandle \( \tilde{X} \). Expand \( h \in \tilde{X} \subset \text{As}(X) \) as \( h = e_{x_1}^{\sigma_1} \cdots e_{x_n}^{\sigma_n} \) for some \( x_i \in X \) and \( \sigma_i \in \{\pm 1\} \) with \( i \leq n \). Since \( h \in \text{Ker}(\epsilon_X) \), note \( \sum \sigma_i = 0 \). The connectivity ensures some \( g_i \in \text{As}(X) \) so that \( a \cdot g_i^{\sigma_i} = x_i \). Therefore \( g_i^{-\sigma_i} e_a g_i^{\sigma_i} = e_a g_i^{\sigma_i} = e_{x_i}^{\sigma_i} \) (see (2)). In the sequel, we have

\[
1_{\tilde{X}} \cdot (g_i^{\sigma_1} \cdots g_i^{\sigma_n}) = e_a^{\sum \sigma_i} 1_{\tilde{X}} (g_i^{-\sigma_i} e_a g_i^{\sigma_i}) \cdots (g_n^{-\sigma_n} e_a g_n^{\sigma_n}) = e_{x_1}^{\sigma_1} \cdots e_{x_n}^{\sigma_n} = h \in \tilde{X}.
\]

This equalities imply the transitivity. \( \square \)

Hence we have

**Proposition 6.10.** For any connected quandle \( X \) of type \( t_X \), the universal covering \( p : \tilde{X} \to X \) induces an isomorphism \( p_* : H^3_{gr}(\text{As}(\tilde{X})) \cong H^3_{gr}(\text{As}(X)) \mod t_X \).

**Proof.** By connectivity of \( \tilde{X} \) in Lemma 6.9, \( H^2_{gr}(\text{As}(\tilde{X})) \) and \( H^2_{gr}(\text{As}(X)) \) are annihilated by \( t_X \). Furthermore, since the \( p_* : \text{As}(\tilde{X}) \to \text{As}(X) \) is a central extension whose kernel is annihilated by \( t_X \) (Proposition 6.8), we easily obtain the isomorphism \( p_* : H^3_{gr}(\text{As}(\tilde{X})) \cong H^3_{gr}(\text{As}(X)) \mod t_X \) from the Lyndon-Hochschild sequence of \( p_* \).

Finally we will determine the second quandle homology of \( \tilde{X} \) (Theorem 6.12). For this,

**Proposition 6.11.** Let \( X \) be a connected quandle, and \( p_* : \text{As}(\tilde{X}) \to \text{As}(X) \) be the epimorphism induced from the covering \( p : \tilde{X} \to X \). Then, under the canonical action of \( \text{As}(\tilde{X}) \) on \( \tilde{X} \), the stabilizer \( \text{Stab}(1_{\tilde{X}}) \) of \( 1_{\tilde{X}} \) is equal to \( \mathbb{Z} \times \text{Ker}(p_*) \) in \( \text{As}(\tilde{X}) \). Furthermore, the summand \( \mathbb{Z} \) is generated by \( 1_{\tilde{X}} \).

**Proof.** We easily see that the stabilizer of \( 1_{\tilde{X}} \) via the action \( \text{Ker}(\epsilon_X) = \tilde{X} \subset \text{As}(X) \) above is \( \text{Stab}(1_{\tilde{X}}) = \{e_a \}_{a \in \mathbb{Z}} \subset \text{As}(X) \) exactly. Notice that any central extension of \( \mathbb{Z} \) is trivial; Since the \( p_* \) is a central extension (Proposition 6.8), the restriction \( p_* : \text{Stab}(1_{\tilde{X}}) \to \text{Stab}(1_{\tilde{X}}) = \mathbb{Z} \) implies the required identity \( \text{Stab}(1_{\tilde{X}}) = \mathbb{Z} \times \text{Ker}(p_*) \).

**Theorem 6.12.** Let \( X \) be a connected quandle. Then the second quandle homology of the extended quandle \( \tilde{X} \) is isomorphic to the kernel of the \( p_* : \text{As}(\tilde{X}) \to \text{As}(X) \). Namely \( H^2_{gr}(\tilde{X}) \cong \text{Ker}(p_*) \). In particular, \( H^2_{gr}(\tilde{X}) \) is annihilated by its type \( t_X \), according to Proposition 6.8.

**Proof.** Note that \( \tilde{X} \) is connected (Lemma 6.9) and the kernel \( \text{Ker}(p_*) \) is abelian (Proposition 6.8). Accordingly, the desired isomorphism \( H^2_{gr}(\tilde{X}) \cong (\text{Ker}(\epsilon_{\tilde{X}}) \cap \text{Stab}(1_{\tilde{X}})) \) follows immediately from Proposition 6.11 and Theorem 5.2.
6.4 The T-map $\Theta_X$ as a splitting

To prove the main theorem 3.5, we now fix some notation in this section:

**Notation.** Let $X$ be a connected quandle of type $t_X < \infty$ (possibly, $X$ is of infinite order), and let $p : \tilde{X} \to X$ be the universal covering of the $X$. Furthermore, a symbol $\ell$ means a prime which is relatively prime to the $t_X$.

We first prove Theorem 3.5 by using the following proposition.

**Proposition 6.13.** Let $p : \tilde{X} \to X$ be as above. If the homology $H_3^g(\text{As}(X))$ is finitely generated, then the T-map $\Theta_{\tilde{X}} : \Pi_2(\tilde{X}) \to \Omega_3(\text{As}(\tilde{X}))$ constructed in Theorem 3.4 is an isomorphism modulo $t_X$-torsion and is a splitting in the short exact sequence in Corollary 6.3.

**Proof of Theorem 3.5.** We put the $P$-sequences associated to the $p : \tilde{X} \to X$:

$$
\begin{array}{cccc}
H_3^g(\text{As}(\tilde{X}))(\ell) & \xrightarrow{\tau} & \Pi_2(\tilde{X})(\ell) & \xrightarrow{p} & H_2^g(\tilde{X})(\ell) \\
p_* & & p_* & & p_* \\
H_3^g(\text{As}(X))(\ell) & \xrightarrow{\tau} & \Pi_2(X)(\ell) & \xrightarrow{p} & H_2^g(X)(\ell) & \xrightarrow{p} & H_2^g(\text{As}(X))(\ell) = 0
\end{array}
$$

Since the left $p_*$ between group homologies is an isomorphism modulo $t_X$ by Proposition 6.10, the TH-map $\Theta_X \oplus H_X : \Pi_2(X)(\ell) \to H_3^g(\text{As}(X))(\ell) \oplus H_2^g(X)(\ell)$ is an isomorphism by the functoriality of $\Theta_X$ and Proposition 6.13. 

Here we shall aim to prove Proposition 6.13 with respect to the extended quandles $\tilde{X}$.

For the proof, we will review a classical bordism theory (see [CF, §1.4]). Given a space-pair $(Y, A)$ with $A \subset Y$, consider a continuous map

$$f : (M, \partial M) \to (Y, A),$$

where $M$ is an oriented compact $n$-manifold. Such two maps $f_1, f_2$ are $G$-bordant, if there exist an oriented compact manifold $W$ of dimension $n + 1$ and a map $F : W \to Y$ for which

(I) There is an $n$-dimensional submanifold $M' \subset \partial W$ satisfying $F(\partial W \setminus M') \subset A.$

(II) There is a diffeomorphism $g : (-M_1) \cup M_2 \to M'$ preserving orientation such that $(-f_1) \cup f_2 = (F|M') \circ g.$

Then the bordism group of $(Y, A)$, denoted by $\Omega_n(Y, A)$, is defined to be the set of all such map $f$ modulo the $G$-bordant relations. We make this $\Omega_n(Y, A)$ into an abelian group by disjoint union. Furthermore, $\{\Omega_n(Y, A)\}_{n \geq 0}$ gives a homology theory (see [CF, §1.6]), and the isomorphism $\Omega_n(Y, A) \cong H_n(Y, A)$, for $n \leq 3$, is obtained by the Atiyah-Hirzebruch spectral sequence. Furthermore, if $Y$ is the Eilenberg-MacLane space $K(G, 1)$ and $A$ is the empty set, then $\Omega_n(Y, A)$ is isomorphic to the group $\Omega_n(G)$ introduced in §4.

Using the bordism groups, we will construct a homomorphism (22) below. Given an $\tilde{X}$-coloring $C$ with $#L$ link components, we take $t_X$-copies of $C$, and denote them by $C_j$ for $1 \leq j \leq m$. Let us fix an arc of each link component of $C$, and consider a sum of these
\(C_1, \ldots, C_m\) connected by the arcs (see Figure 4). Denote the resulting link by \(\overline{L}\) and the associated \(\tilde{X}\)-coloring of \(\overline{L}\) by \(\overline{C}\). We then set a homomorphism \(\Gamma_{\tau} : \pi_1(S^3 \setminus \overline{L}) \to \text{As}(\tilde{X})\) discussed in [3]. Note that each meridian of \(\overline{L}\) is sent to be \(e_g\) for some \(g \in \tilde{X}\) by definition. Furthermore each longitudes \(l_j \in \pi_1(S^3 \setminus \overline{L})\) are sent to be zero. Actually the formula \(17\) says that the \(\Gamma_{\tau}(l_j)\) lies in \(\text{Ker}(\varepsilon_{\tilde{X}}) \cap \text{Stab}(1_{\tilde{X}})\), which is equal to the abelian kernel \(\text{Ker}(p_*)\) and is annihilate by \(t_X\) following from Propositions 6.8 and 6.11. Consequently, the map \(\Gamma_{\tau}\) sends every boundaries of \(S^3 \setminus \overline{L}\) to a 1-cell of \(B\tilde{X}\). Here note that the 1-skeleton \(B\tilde{X}_1\) is, by definition, a bouquet of circles labeled by elements of \(\tilde{X}\). In the sequel, considering all \(\tilde{X}\)-coloring \(\overline{C}\) and such homomorphisms \(\Gamma_{\tau}\) modulo the bordance relations, the map \(C \mapsto \Gamma_{\tau}\) defines the desired homomorphism

\[\Upsilon_{\tilde{X}} : \Pi_2(\tilde{X}) \longrightarrow \Omega_3(K(\text{As}(\tilde{X}), 1), B\tilde{X}_1).\]  

(22)

Hereafter, we denote by \(\Omega_3^{\text{rel}}(\tilde{X})\) this relative bordism \(\Omega_3(K(\text{As}(\tilde{X}), 1), B\tilde{X}_1)\), for simplicity.

![Figure 4: Construction of \(\overline{C}\) from \(C\), when the link components of \(C\) are two.](image)

We now prove Proposition 6.13 by using the following lemma:

**Lemma 6.14.** For any connected quandle \(X\) of type \(t_X\), the homomorphism \(\Upsilon_{\tilde{X}} : \Pi_2(\tilde{X}) \rightarrow \Omega_3^{\text{rel}}(\tilde{X})\) is surjective up to \(t_X\)-torsion.

**Proof of Proposition 6.13.** We first explain the following diagram:

\[
\begin{array}{ccc}
\Omega_3^{\text{rel}}(\tilde{X})_{(\ell)} & \xrightarrow{\delta_*} & \Omega_3^{\text{rel}}(\tilde{X})_{(\ell)} \\
\Upsilon_{\tilde{X}} & | & | \\
0 \rightarrow H_3(\text{As}(\tilde{X}))_{(\ell)} & \xrightarrow{\tau_*} & \Pi_2(\tilde{X})_{(\ell)} \\
& | & | \\
& & H_2^Q(\tilde{X})_{(\ell)} = 0
\end{array}
\]

Here the top sequence is derived by the homology theory \(\Omega_n\) with considering the pair \(B\tilde{X}_1 \hookrightarrow K(\text{As}(\tilde{X}), 1)\), and the bottom one is obtained from the \(P\)-sequence of \(\tilde{X}\) with Theorems 6.1 and 6.12. Since \(\Pi_2(\tilde{X})\) is isomorphic to the finitely generated module \(\Omega_3(\text{As}(\tilde{X}))\) modulo \(t_X\) by assumption, the localized map of \(\Upsilon_{\tilde{X}}\) is an isomorphism by Lemma 6.14.

Therefore, to accomplish the proof, it is sufficient to show the equality

\[t_X \cdot (\delta_* \circ \Theta_{\tilde{X}}([C])) = t_X \cdot \Upsilon_{\tilde{X}}([C]) \in \Omega_3^{\text{rel}}(\tilde{X}),\]  

(23)

for any \(\tilde{X}\)-coloring \(C\). For this, put the resulting link \(\overline{L}\) and coloring \(\overline{C}\) explained above. Furthermore, take the \(t_X\)-fold cyclic covering \(p : \overline{C}^{t_X} \rightarrow S^3 \setminus \overline{L}\) and the natural inclusion \(i_C : \overline{C}^{t_X} \subset \overline{C}^{t_X}\) by gluing the 2-handles along the boundary tori. Here notice that the
composite \( \theta_{\bar{X},D}(\overline{C}) \circ (i_c)_* : \pi_1(C^L_{\overline{T}}) \to \text{As}(X) \) coincides with \( \Gamma_\overline{C} \circ p_* \) by the definition (13).

Furthermore notice that the inclusion \( i_c \) gives a bordance relation between the \( \theta_{\bar{X},D}(\overline{C}) \) and this composite \( \theta_{\bar{X},D}(\overline{C}) \circ (i_c)_* \). Since the above map \( \delta_* \) comes from the correspondences with \( (M,f) \) to \( (M,f) \) itself by definition, we thus have

\[
t_X \cdot \delta_* \circ \Theta_X([C]) = \delta_* \circ \Theta_X(\overline{C}) = [\theta_{\bar{X},D}(\overline{C})] = [\Gamma_{\overline{C}} \circ p_*] \in \Omega_3^\text{rel}(\bar{X}),
\]

where the first equality is derived from \( t_X[C] = \overline{C} \in \Pi_2(\bar{X}) \) from the definition of \( \overline{L} \). We notice \( \Gamma_{\overline{C}} \circ p_* \) is the projection \( p \) takes the (relative) fundamental class of \( C^L_{\overline{T}} \) to the \( t_X \)-multiple of that of \( S^3 \setminus \overline{T} \). Hence, since \( \Upsilon_X([C]) = \Gamma_{\overline{C}} \) by definition, we have the desired equality (23).

To conclude this section, we will work out the proof of Lemma 6.14.

Proof of Lemma 6.14. To begin with, we claim that the \( \mathbb{Z}_{(\ell)} \)-module \( \Omega_3^\text{rel}(\bar{X})_{(\ell)} \) is generated by bordism classes represented by (normal) 3-submanifolds in \( S^3 \) with torus boundary components.

For this, we first explain the isomorphism (24) below. Here refer to the fact [Cla1, §2.5] that the universal covering of \( B\bar{X} \) is a topological monoid; hence, it is a based loop space of some space \( \mathcal{L}_X \). We therefore have two homotopy fibrations

\[
\Omega \mathcal{L}_X \longrightarrow \bar{B}\bar{X} \xrightarrow{c} K(\text{As}(\bar{X}),1), \quad \bar{B}\bar{X} \xrightarrow{c} K(\text{As}(\bar{X}),1) \xrightarrow{p_\ell} \mathcal{L}_X.
\]

From the right map \( p_\ell \), we have an isomorphism after localization at \( \ell \):

\[
\left( p_\ell \right)_*: \Omega_3(B\bar{X}, K(\text{As}(\bar{X}),1))_{(\ell)} \cong \Omega_3(\mathcal{L}_X)_{(\ell)}. \tag{24}
\]

However, since the \( \mathcal{L}_X \) is 2-connected by definition, the Hurewicz theorem \( \pi_3(\mathcal{L}_X) \cong \Omega_3(\mathcal{L}_X) \) implies that the \( \Omega_3(\mathcal{L}_X) \) is generated by maps \( S^3 \to \mathcal{L}_X \). Noticing that the map \( (p_\ell)_* \), can be regarded as a map coming from collapse of each boundaries of manifolds, this isomorphism (24) implies that generators of the \( \Omega_3(K(\text{As}(\bar{X}),1),B\bar{X})_{(\ell)} \) are derived from 3-submanifolds in \( S^3 \).

Next, so as to verify the claim above, consider the inclusions

\[
S^1 \subset B\bar{X}_1 \subset B\bar{X} \xrightarrow{c} K(\text{As}(\bar{X}),1),
\]

where the first is obtained by taking the circle labeled by \( a \in \bar{X} \). Notice from Corollary 6.4 that these inclusions \( S^1 \subset B\bar{X}_1 \subset K(\text{As}(\bar{X}),1) \) induce isomorphisms

\[
H_2(S^1)_{(\ell)} \cong H_1(B\bar{X}_1)_{(\ell)} \cong H^0_2(\text{As}(\bar{X}))_{(\ell)} \cong 0.
\]

Therefore, they yield isomorphisms

\[
\Omega_3(K(\text{As}(\bar{X}),1),S^1)_{(\ell)} \cong \Omega_3^\text{rel}(\bar{X})_{(\ell)} \cong \Omega_3(K(\text{As}(\bar{X}),1),B\bar{X})_{(\ell)}.
\]

Here note that, since the last term is generated by classes from 3-manifolds in \( S^3 \) as observed above and \( \pi_1(S^1) \cong \mathbb{Z} \) is abelian, the first term is generated by classes from 3-manifolds in \( S^3 \) with torus boundaries\(^3\). Hence, so is the \( \Omega_3^\text{rel}(\bar{X})_{(\ell)} \) as claimed.

\(^2\)To be precise, since the first homology of any closed surfaces is generated by homology classes from some tori, given a submanifold \( M \subset S^3 \) with \( f : \pi_1(M) \to \text{As}(X) \) such that \( f(\pi_1(\partial M)) \subset \mathbb{Z} \), we can obtain another \( M' \subset S^3 \) with torus boundaries by attaching some 2-handles to \( M \), and the \( f \) extend to \( f : \pi_1(M') \to \text{As}(X) \) such that \( f(\pi_1(\partial M')) \subset \mathbb{Z} \).
Finally, to show the required surjectivity of $\Upsilon_{\widetilde{X}}$, we will prove that any generator $O$ of $\Omega_{3}^{\text{rel}}(\widetilde{X})_{(l)}$ comes from some $\widetilde{X}$-coloring via the bijection (3). By the previous claim, $\iota_{\widetilde{X}}^{-2} \cdot O$ is represented by a homomorphism $f : \pi_{1}(S^{3} \setminus L) \to \text{As}(\widetilde{X})$ for some link $L \subset S^{3}$. Furthermore put the resulting link $\overline{L}$ in constructing $\Upsilon_{\widetilde{X}}$ in (22). By repeating the process, we have another $\overline{L}$. Then the $f$ extends to two maps $\overline{f} : \pi_{1}(S^{3} \setminus \overline{L}) \to \text{As}(\widetilde{X})$ and $\overline{f} : \pi_{1}(S^{3} \setminus \overline{L}) \to \text{As}(\widetilde{X})$ canonically, where note that the class of the latter in $\Omega_{3}^{\text{rel}}(\widetilde{X})_{(l)}$ equals the $O$. Notice that, for each link component $1 \leq i \leq \#L$, with a choice of meridian element $m_{i}$, the $\overline{f}(m_{i}) \in \text{As}(\widetilde{X})$ is conjugate to $e_{\overline{y}}^{n_{i}}$ for some $n_{i} \in \mathbb{N}$, from the definition of $\Omega_{3}^{\text{rel}}(\widetilde{X})_{(l)}$: Namely, in Wirtinger presentation of $\pi_{1}(S^{3} \setminus L)$, each arc $\alpha$ is labeled by $e_{\overline{y}}^{n_{i}}$ for some $y_{\alpha} \in \widetilde{X}$; see (2). Accordingly, replacing the $i$-th component of the link $\overline{L}$ by $n_{i}$-parallel copies of the component, we have another link $\overline{L}_{p}$ and, then, can construct a canonical homomorphism $\overline{f}_{p} : \pi_{1}(S^{3} \setminus \overline{L}_{p}) \to \text{As}(\widetilde{X})$ by which each meridian of $\overline{L}_{p}$ is sent to $e_{y}$ for some $y \in \widetilde{X}$ (see Figure 5). We remark that, this $\overline{f}_{p}$ sends the associated longitude $l_{i}$ of $\overline{L}_{p}$ to $e_{y}^{l_{i} \cdot n_{y}}$ for some $n_{y} \in \mathbb{Z}$, by the reason similar to the construction of $\Upsilon_{\widetilde{X}}$ in (22). In particular, we have $y : \overline{f}_{p}(l_{i}) = y \cdot e_{y}^{l_{i} \cdot n_{y}} = y \in \widetilde{X}$. Hence, the bijection (3) admits an $\widetilde{X}$-coloring $C_{x}$ such that $\Gamma_{C_{x}} = \overline{f}_{p} : \pi_{1}(S^{3} \setminus \overline{L}_{p}) \to \text{As}(\widetilde{X})$. Consequently, we have the equality $\Upsilon_{\widetilde{X}}([C_{x}]) = O \in \Omega_{3}^{\text{rel}}(\widetilde{X})_{(l)}$ by construction, which implies the surjectivity.

![Figure 5: Construction from the $\overline{f} : \pi_{1}(S^{3} \setminus L) \to \text{As}(\widetilde{X})$ to $\overline{f}_{p} : \pi_{1}(S^{3} \setminus \overline{L}_{p}) \to \text{As}(\widetilde{X})$.](image)

### 7 Proof of Theorems 3.11, 3.13.

We will compute the $\Pi_{2}(X)$ with respect to symplectic and orthogonal quandles in §7.1 and do $\Pi_{2}(X)$ of connected quandles of order $\leq 8$ in §7.2 as stated in Theorems 3.11, 3.13 respectively. The fundamental line of the proofs is to study the exact sequence in (21) that we called the $P$-sequence. Actually, the proofs result from computations of the terms $H_{3}^{gr}(\text{As}(X))$ and $H_{2}^{gr}(X)$ concretely including these $t_{X}$-torsions.

### 7.1 On symplectic and orthogonal quandles over $\mathbb{F}_{q}$

We will prove Theorem 3.11 about computing the $\Pi_{2}(X)$ for symplectic and orthogonal quandles $X$ over finite fields. Our computation essentially relies on some works of Quillen and Friedlander, who had calculated group homologies of some groups of Lie type over $\mathbb{F}_{q}$. We list their works after the proof.

**Proof of Theorem 3.11** Let $q = p^{d}$ be odd and not exceptional, i.e., $d(p - 1) < 6$. 


For (I), we will mention some homologies associated with the symplectic quandle $X = Sp^n$. As is shown, As($X) \cong \mathbb{Z} \times Sp(2n; \mathbb{F}_q)$ (Proposition A.6); hence Theorems 7.2, 7.3 below tell us the group homology $H^gr_3(As(X)) \cong \mathbb{Z}/(q^2 - 1)$. In addition, when $n \geq 2$ the second quandle homology $H^Q_2(X)$ vanishes; see Proposition B.3. Therefore the $P$-sequence (20) is reduced to be an epimorphism $\mathbb{Z}/(q^2 - 1) \rightarrow \Pi_2(X) \rightarrow 0$. Since $X$ is of type $p$, Theorem 3.5 with $n \geq 2$ thus concludes the isomorphism $\mathbb{Z}/(q^2 - 1) \rightarrow \Pi_2(X)$ as required.

Next, we work out the case $n = 1$. Note that the second quandle homology $H^Q_2(X)$ is $(\mathbb{Z}/p)^d$; see Proposition B.3 again. Thereby the $P$-sequence (20) is rewritten as

$$\mathbb{Z}/(q^2 - 1) \rightarrow \Pi_2(X) \rightarrow (\mathbb{Z}/p)^d \rightarrow 0.$$ (25)

Using Theorem 3.5 as well, we have $\Pi_2(X) \cong \mathbb{Z}/(q^2 - 1) \oplus (\mathbb{Z}/p)^d$ as required.

For (II), we similarly deal with the spherical quandle $X = S^n$ with $n \geq 3$. Note $H^gr_3(As(X)) \cong \mathbb{Z}/(q^2 - 1)$ without 2-torsion (Example A.8). Furthermore, if $n \geq 3$, $H^Q_2(X)$ is an elementary abelian 2-group (Proposition B.2). Hence, the desired isomorphism $\Pi_2(X) \cong \mathbb{Z}/(q^2 - 1)$ results from the $P$-sequence (20) and Theorem 3.5.

Finally, we deal with the case $n = 2$. By Proposition B.2, the quandle homology $H^Q_2(X)$ is $\mathbb{Z}/(q - \delta_q)$, where $\delta_q = \pm 1$ is according to $q \equiv \pm 1 \pmod{4}$. Thereby the $P$-sequence (20) is

$$\mathbb{Z}/(q^2 - 1) \rightarrow \Pi_2(X) \rightarrow \mathbb{Z}/(q - \delta_q) \rightarrow 0 \pmod{2}.$$ (26)

Using Theorem 3.5 again, we reach the goal $\Pi_2(X) \cong \mathbb{Z}/(q^2 - 1) \oplus \mathbb{Z}/(q - \delta_q) \pmod{2}$. □

As mentioned above, we review the group homologies of the symplectic groups $Sp(2g, \mathbb{F}_q)$ and the orthogonal groups $O(n; \mathbb{F}_q)$. There is nothing new until the end of this subsection. We start recalling the homologies of $Sp(2, \mathbb{F}_q)$ and $O(3; \mathbb{F}_q)$.

**Proposition 7.1.** If $p \neq 2$ and $q \neq 3, 9$, then the first and second homologies of $Sp(2g; \mathbb{F}_q)$ vanish, i.e., $H^gr_1(Sp(2g, \mathbb{F}_q)) \cong H^gr_2(Sp(2g, \mathbb{F}_q)) \cong 0$.

Furthermore, the $\ell$-torsions of the third homology of $Sp(2, \mathbb{F}_q)$ are expressed by

$$H^gr_3(Sp(2, \mathbb{F}_q))_{(\ell)} \cong (\mathbb{Z}/(q^2 - 1))_{(\ell)}, \quad \text{for } \ell \neq p.$$ (As($X)$

On the other hand, the homologies $H^gr_1(O(3, \mathbb{F}_q))$ and $H^gr_2(O(3, \mathbb{F}_q))$ are annihilated by 2. Furthermore, the $\ell$-torsions of the third homology of $O(3, \mathbb{F}_q)$ are expressed by

$$H^gr_3(O(3, \mathbb{F}_q))_{(\ell)} \cong (\mathbb{Z}/(q^2 - 1))_{(\ell)}, \quad \text{for } \ell \neq p, 2.$$ (As($X)$

**Proof.** See [FP] VIII. §4 or [Fri], noting the order $|O(3, \mathbb{F}_q)| = 2q(q^2 - 1)$. □

We moreover review the group homologies of $Sp(2g; \mathbb{F}_q)$ and $O(n, \mathbb{F}_q)$ as follows:

**Theorem 7.2** ([FP], [Fri]). Let $q = p^d$ be odd. The inclusion $Sp(2, \mathbb{F}_q) \hookrightarrow Sp(2n, \mathbb{F}_q)$ induces isomorphisms $H^gr_3(Sp(2, \mathbb{F}_q)) \cong_{(\ell)} H^gr_3(Sp(2n, \mathbb{F}_q)) \cong_{(\ell)} \mathbb{Z}/(p^2 - 1)$ localized at $\ell \neq p$. Furthermore, for $n \geq 3$, the inclusion $O(3, \mathbb{F}_q) \hookrightarrow O(n, \mathbb{F}_q)$ induces isomorphisms $H^gr_3(O(3, \mathbb{F}_q)) \cong_{(\ell)} H^gr_3(O(n, \mathbb{F}_q)) \cong_{(\ell)} \mathbb{Z}/(p^2 - 1)$ localized at $\ell \neq p, 2$. □
Proof. According to [FP], the inclusions induces isomorphisms their cohomology with \( \mathbb{Z}/\ell \)-coefficients. Taking limits as \( n \to \infty \), their homologies are known to be \( H^2_3(Sp(\infty; \mathbb{F}_q)) \cong (\mathbb{Z}/\ell) \mathbb{Z}/q^2 - 1 \) and \( H^2_3(O(\infty; \mathbb{F}_q)) \cong (\mathbb{Z}/\ell) \mathbb{Z}/q^2 - 1 \) [Fri] Theorem 1.7. Hence, by Propositions 7.1 the induced maps on homologies localized at \( \ell \) are isomorphisms.

Finally, we focus on these \( p \)-torsion parts, and state the vanishing theorem.

**Theorem 7.3** (Quillen, see [Fri §4]). Let \( q = p^d \) be odd. If \( d(p - 1) > 6 \), then the \( p \)-torsion parts \( H^2_3(Sp(2n; \mathbb{F}_q))_{(p)} \) and \( H^2_3(O(n + 2; \mathbb{F}_q))_{(p)} \) vanish for any \( n \geq 1 \). Furthermore, if \( n \) is enough large, the \( p \)-vanishing holds even for \( d(p - 1) \leq 6 \).

**Remark 7.4.** As a result in [Fri Corollary 1.8], the inclusions \( Sp(2n; \mathbb{F}_q) \hookrightarrow GL(2n; \mathbb{F}_q) \) induce the isomorphism between the third homology \( H^2_3(Sp(2n; \mathbb{F}_q)) \) and the Quillen \( K \)-group \( K_3(\mathbb{F}_q) \cong H^2_3(GL(\infty; \mathbb{F}_q)) \cong \mathbb{Z}/(q^2 - 1) \), if \( d(p - 1) > 6 \) or \( n \) is enough large. For instance, following [Fri §4], we can see that, when \( q = p = 5 \) and \( n \geq 7 \), the third homology \( H^2_3(Sp(2n; \mathbb{F}_q)) \) is \( \mathbb{Z}/(5^2 - 1) = \mathbb{Z}/24 \). We later use this result in [N].

### 7.2 Connected quandles of order \( \leq 8 \) (Proof of Theorem 3.13)

We will prove Theorem 3.13 which stated that, for connected quandles \( X \) of order \( \leq 8 \), the TH-maps \( \Theta_X \oplus \mathcal{H}_X : \Pi_2(X) \to H^2_3(\mathrm{As}(X)) \oplus \mathrm{Im}(\mathcal{H}_X) \) are isomorphisms. The proof is obtained by computing concretely \( \Pi_2(X) \) from the list of connected quandles of order \( \leq 8 \). In the proofs, we will often use the T-map \( \Theta_{\Pi_0} : \Pi_2(X) \to \Omega_3(\ker(\varepsilon_X)) \) in Lemma 4.2.

#### 7.2.1 Connected Alexander quandles of order 4, 8

We first determine the \( \Pi_2(X) \) of the connected quandle of order 4 as follows:

**Proposition 7.5.** If \( X \) is the Alexander quandle of the form \( \mathbb{Z}[T]/(2, T^2 + T + 1) \), then \( \Pi_2(X) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/8 \).

**Proof.** We first recall the fact [N1] Proposition 4.5 which says that \( \Pi_2(X) \) is either \( \mathbb{Z}/2 \oplus \mathbb{Z}/4 \) or \( \mathbb{Z}/2 \oplus \mathbb{Z}/8 \) and \( H^2_3(X) \cong \mathbb{Z}/2 \). By Theorem 3.5, hence it suffices to construct an isomorphism \( H^2_3(\mathrm{As}(X)) \cong \mathbb{Z}/8 \mod 3 \). For this, note \( \mathrm{As}(X) \cong Q_8 \times \mathbb{Z} \), where \( Q_8 \) is the quaternion group of order 8 (see [N1] Lemma 4.8 for details). Noting \( \mathrm{Type}(X) = 3 \), consider the quotient \( Q_8 \times \mathbb{Z}/3 \), which is isomorphic to \( Sp(2; \mathbb{F}_3) \). By Proposition 7.1 and the transfer, we thus have \( H^2_3(\mathrm{As}(X)) \cong H^2_3(Sp(2; \mathbb{F}_3)) \cong \mathbb{Z}/8 \mod 3 \)-torsion as desired.

**Remark 7.6.** We here note a relation between \( \Pi_2(X) \) and quandle homology groups. As is known, \( H^2_3(X) \cong \mathbb{Z}/2 \) and \( H^2_3(X) \cong \mathbb{Z}/4 \) [CJKLS Remark 6.10]. Namely the summand \( \mathbb{Z}/8 \) of \( \Pi_2(X) \) is evaluated not by the quandle cohomology, but by the group cohomology \( H^3_3(Q_8; \mathbb{Z}/8) \). It is therefore sensible to deal with 2-torsion of the groups \( \Pi_2(X) \) in general.

We next consider two Alexander quandles of order 8 of the forms \( X = \mathbb{Z}[T]/(2, T^3 + T^2 + 1) \) and \( X = \mathbb{Z}[T]/(2, T^3 + T + 1) \). Then the both \( \Pi_2(X) \) were shown to be \( \mathbb{Z}/2 \) [N2 Table 1]. We remark that, as is known (see, e.g., [Cla3]), a connected Alexander quandle \( X \) of order 4 or 8 is one of the three quandles above; hence the \( \Pi_2(X) \) is determined.
### 7.2.2 Two conjugate quandles of order 6

In this subsection, we calculate $\Pi_2(X)$ of two quandles $S_6, S'_6$ of order 6. Here the quandle $S_6$ (resp. $S'_6$) is defined to be the set of elements of a conjugacy class in the symmetric group $\mathfrak{S}_4$ including $(12) \in \mathfrak{S}_4$ (resp. $(1234) \in \mathfrak{S}_4$) with the binary operation $x \triangleleft y = y^{-1} xy$.

![Figure 6: An $S_6$-coloring $C_3$ of the trefoil knot $3_1$, and an $S'_6$-coloring $C_4$ of $T_{3,4}$.

**Proposition 7.7.** For the quandle $S_6$, $\Pi_2(S_6) \cong \mathbb{Z}/24 \oplus \mathbb{Z}/4$. The first summand $\mathbb{Z}/24$ is generated by $\Xi_{S_6,3_1}(C_3)$, where $C_3$ is a coloring of the trefoil knot shown as Figure 6.

On the other hand, for another quandle $S'_6$, we have $\Pi_2(S'_6) \cong \mathbb{Z}/12$. The generator is represented by $\Xi_{S'_6,T_{3,4}}(C_4)$, where $C_4$ is a coloring of the torus knot $T_{3,4}$ shown as Figure 6.

**Proof.** We show the sequence (27). It follows from the proof of [N1, Proposition 4.9] that $H^Q_2(S_6) \cong \mathbb{Z}/4$ and $H^P_3(\text{As}(S_6)) \cong 0$, and that $H^P_3(\text{As}(S_6))$ is a quotient of $\mathbb{Z}/24$. Hence the $P$-sequence (21) becomes

$$\mathbb{Z}/24 \rightarrow \Pi_2(S_6) \xrightarrow{\mathcal{H}} H^Q_2(X) \cong \mathbb{Z}/4 \rightarrow 0.$$ (27)

Next we will show that $\Pi_2(S_6)$ surjects onto $\mathbb{Z}/24$. It is shown [N1, Lemma 4.10] that the kernel $\text{Ker}(\varepsilon_X)$ is the binary tetrahedral group $D_{24} = \text{Sp}(2;\mathbb{F}_3)$ whose third homology is $\mathbb{Z}/24$ (see Proposition 7.1). Let $D_{24}$ act canonically the sphere $S^3$. Since the 4-fold covering branched over the trefoil is $S^3/D_{24}$ (see [Rol, §10.D]), it can be easily seen that the map $\theta_{X,D}$ in (13) sends the $X$-coloring $C_3$ to an isomorphism $\pi_1(S^3/D_{24}) \rightarrow \text{Sp}(2;\mathbb{F}_3)$. Since $\Omega_3(D_{24}) \cong \mathbb{Z}/24$ is known to be generated by the pair $(S^3/D_{24}, \text{id}_{D_{24}})$ (see [Bro] VI. Examples 9.2)), the T-map $\Theta_H : \Pi_2(S_6) \rightarrow \Omega_3(D_{24})$ in Lemma 4.2 is surjective.

Finally, for proving the decomposition $\Pi_2(S_6) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/24$, it is enough to show that the class $[C_3] \in \Pi_2(S_6)$ is sent to $2 \in H^Q_2(X) \cong \mathbb{Z}/4$ by the map $\mathcal{H}_X$. By the formula (18),

$$\mathcal{H}_X([C_3]) = \Gamma_{C_3}(1) = e_{(1432)}^{-2} e_{(1432)} e_{(123)} \in \text{Ker}(\varepsilon_X) \cap \text{Stab}(x_0) \cong \mathbb{Z}/4.$$ 

An elementary calculation can show the square $\mathcal{H}([C_3])^2 = 1$ and $\mathcal{H}([C_3]) \neq 1$ in $\text{As}(X)$, although we will not go into the details. In the sequel, $\mathcal{H}([C_3]) = 2 \neq 0$ as desired.

Changing the subject, we will compute another $\Pi_2(S'_6)$. For this, we now explain the sequence (28) below. It is shown [N1, Lemma 4.12, Appendix A] that $H^Q_2(S'_6) \cong \mathbb{Z}/2$, and further, the order $|H^P_3(\text{As}(S'_6))| \leq 12$, and $H^P_2(\text{As}(S'_6)) \cong \mathbb{Z}/2$. Hence, as a routine work, the $P$-sequence (21) becomes

$$H^P_3(\text{As}(S'_6)) \rightarrow \Pi_2(S'_6) \xrightarrow{\mathcal{H}} \mathbb{Z}/2 \rightarrow \mathbb{Z}/2 \rightarrow 0.$$ (28)
To complete the proof of the $\Pi_2(S'_6) \cong \mathbb{Z}/12$, it is sufficient to show the surjectivity of the $T$-map $\Theta_{\Omega^2} : \Pi_2(S'_6) \to \mathbb{Z}/12$. As is shown [10] Lemma 4.12, the kernel $\ker(\varepsilon_X)$ is the alternating group $A_4$ of order 12 whose third homology is $\mathbb{Z}/12$. Since the quandle $S'_6$ is of type 2 and the double cover branched over the knot $T_{3,4}$ is $S^3/D_{24}$ (see [Rol, §10. D and E]), we can show that the map $\theta_{X,D}$ in (13) sends the $X$-coloring $C_4$ to the epimorphism $\pi_1(S^3/D_{24}) = D_{24} \to A_4$, which is a central extension. Hence, the class $[\theta_{X,D}(C_4)]$ is a generator of $\Omega_3(A_4) \cong \mathbb{Z}/12$. This means the surjectivity of $\Theta_{\Omega^2} : \Pi_2(S'_6) \to \mathbb{Z}/12$; recalling $|H_3^{gr}(\text{As}(S'_6))| \leq 12$ and the sequence (28) above, the $T$-map $\Theta_{\Omega^2}$ is an isomorphism $\Pi_2(S'_6) \cong \mathbb{Z}/12$. Furthermore, by this process, the generator is represented by the coloring $C_4$. 

7.2.3 The remaining quandle and the proof of Theorem 3.13

Finally, the rest of connected quandle of order 8 is the (extended) quandle $\tilde{X}$ explained in §3.3 where $X$ is the Alexander quandle of the form $X = \mathbb{Z}[T]/(2, T^2 + T + 1)$. Since $\text{As}(X) \cong Q_8 \rtimes \mathbb{Z}$ (see the proof of Proposition 7.5), we have $|\tilde{X}| = 8$ by definition.

Proposition 7.8. Let $\tilde{X}$ be the above quandle of order 8. Then $\Pi_2(\tilde{X}) \cong \mathbb{Z}/8$.

Proof. We first show that the induced map $p_* : \text{As}(\tilde{X}) \to \text{As}(X)$ is an isomorphism. As is seen in the proof of Proposition 7.5 we note $H_2^{gr}(\text{As}(X)) \cong 0$ and $H_1^{gr}(\text{As}(X)) \cong \mathbb{Z}$. Since the $p_*$ is a central extension (see Proposition 6.8), $p_*$ is an isomorphism. Next we will show $\Pi_2(\tilde{X}) \cong \mathbb{Z}/8$. Note $H_3^{gr}(\tilde{X}) \cong 0$ from Theorem 7.5 and $H_3^{gr}(\text{As}(\tilde{X})) \cong H_3^{gr}(Q_8 \rtimes \mathbb{Z}) \cong \mathbb{Z}/8$ from the proof of Proposition 7.7 Therefore the $P$-sequence is reduced to be an epimorphism $\mathbb{Z}/8 \to \Pi_2(\tilde{X})$. By Theorem 3.18 we conclude $\Pi_2(\tilde{X}) \cong \mathbb{Z}/8$. 

Proof of Theorem 3.13. One first deals with quandles $X$ with $|X| = 3, 5, 7$. Then $X$ is shown to be an Alexander quandle over the finite field $\mathbb{F}_{|X|}$ [EGS]. By Theorem 3.10 the isomorphism $\Theta_X \oplus \mathcal{H}_X$ holds for such $X$. Next, a connected quandle of even order with $|X| \leq 8$ is known to be one of the quandles in the previous subsections (see [Cla3]). Hence the above proposition completes the proof. 

8 Dehn quandle of genus $\geq 7$

This section deals with Dehn quandles. To discuss this, we fix some notation:

Notation Denote by $\Sigma_{g,k}$ the closed surface of genus $g$ with $k$ boundaries as usual. Let $\mathcal{M}_{g,k}$ denote the mapping class group of $\Sigma_{g,k}$ which is the identity on the $k$-boundaries. In the case $k = 0$, we often suppress the symbol $k$, e.g., $\Sigma_{g,0} = \Sigma_g$.

We now review Dehn quandles [Y]. Consider the set, $\mathcal{D}_g$, defined by

$$\mathcal{D}_g := \{ \text{ isotopy classes of (unoriented) non-separating simple closed curves } \gamma \text{ in } \Sigma_g \}.$$ 

For $\alpha, \beta \in \mathcal{D}_g$, we define $\alpha \triangleleft \beta \in \mathcal{D}_g$ by $\tau_\beta(\alpha)$, where $\tau_\beta \in \mathcal{M}_g$ is the positive Dehn twist along $\beta$. The pair $(\mathcal{D}_g, \triangleleft)$ is a quandle, and called (non-separating) Dehn quandle. As is well-known, any two non-separating simple closed curves are related by some Dehn twists. Hence,
the quandle $D_g$ is connected, and is not of any type $t_X$. In addition, since the Dehn twists are transvections in the view of the cohomology $H^1(\Sigma_g; \mathbb{F}_p)$, for any prime $p$, we have a quandle epimorphism $\mathcal{P}_p$ from $D_g$ to the symplectic quandle $\mathfrak{sp}_p^g$. Precisely, $\mathcal{P}_p : D_g \to \mathfrak{sp}_p^g$. The Dehn quandle $D_g$ is applicable to study 4-dimensional Lefschetz fibrations (see, e.g., [Y, Zab, N3]).

We now aim to compute the second homotopy groups $\pi_2(BD_g)$ in a stable range as follows:

**Theorem 8.1.** Let $g \geq 7$. The homotopy group $\pi_2(BD_g)$ is isomorphic to either $\mathbb{Z} \oplus \mathbb{Z}/24$ or $\mathbb{Z} \oplus \mathbb{Z}/48$. Furthermore, the torsion subgroup is generated by a $D_g$-coloring in Figure 4.

**Proof.** We first observe homologies of the associated group $As(D_g)$. Note the well-known facts $H^1_{tr}(\mathcal{M}_g) \cong 0$ and $H^2_{tr}(\mathcal{M}_g) \cong \mathbb{Z}$ (see [AM]). Gervais [Ger] showed the isomorphism $As(D_g) \cong \mathbb{Z} \times T_g$ (cf. Proposition A.4), where $T_g$ is the universal central extension of $\mathcal{M}_g$. Then, Lemma 8.2 below and Kunneth theorem immediately imply isomorphisms $H^2_{tr}(As(D_g)) \cong 0$ and $H^3_{tr}(As(D_g)) \cong \mathbb{Z}/24$.

We next study the $P$-sequences in respect to the above epimorphism $\mathcal{P}_5 : D_g \to \mathfrak{sp}_5^g$ with $p = 5$. Noting the isomorphism $H^2_Q(D_g) \cong \mathbb{Z}/2$ is shown by Proposition B.5. Furthermore, recalling the vanishing $H^2_Q(\mathfrak{sp}_5^g) \cong 0$ from Proposition B.3 these $P$-sequences are written in

$$
\begin{array}{cccccc}
\mathbb{Z}/24 & \rightarrow & P_2(D_g) & \rightarrow & H^2_Q(D_g)(\cong \mathbb{Z}/2) & \rightarrow & 0 \\
\downarrow (\mathcal{P}_5)_* & & \downarrow \sim & & \downarrow (\mathcal{P}_5)_* & \\
H^3_{tr}(As(\mathfrak{sp}_5^g)) & \rightarrow & P_2(\mathfrak{sp}_5^g)(\cong \mathbb{Z}/24) & \rightarrow & H^2_Q(\mathfrak{sp}_5^g)(\cong 0) & \rightarrow & 0
\end{array}
$$

Here the proof of Theorem 3.11 says that the bottom $\delta_*$ is an isomorphism $H^3_{tr}(As(\mathfrak{sp}_5^g)) \rightarrow \Pi_2(\mathfrak{sp}_5^g) \cong \mathbb{Z}/24$.

We next claim that the middle map $(\mathcal{P}_5)_* : \Pi_2(D_g) \rightarrow \Pi_2(\mathfrak{sp}_5^g)$ is surjective. By combing the T-map $\Theta_{10}$ in [14] with the epimorphism $\mathcal{P}_5 : D_g \rightarrow \mathfrak{sp}_5^g$ above, we have a composite

$$
\Pi_2(D_g) \xrightarrow{(\mathcal{P}_5)_*} \Pi_2(\mathfrak{sp}_5^g) \xrightarrow{\Theta_{10}} \Omega_3(\text{Sp}(2g; \mathbb{F}_5)).
$$

To show the claim, it is enough to prove the surjectivity of this composite $\Theta_{10} \circ (\mathcal{P}_5)_*$. Recall the isomorphisms $\Omega_3(\text{Sp}(2g; \mathbb{F}_5)) \cong H^3_{tr}(\text{Sp}(2g; \mathbb{F}_5)) \cong \mathbb{Z}/24$ from Remark 7.4. Consider the $D_g$-coloring $C$ illustrated below. Notice that the quandle $\mathfrak{sp}_5^g$ is of type 5, and that the 5-fold cover of $S^3$ branched along the trefoil is the Poincaré sphere $\Sigma(2, 3, 5)$ (see [Rol, §10.D]), whose $\pi_1$ is $\text{Sp}(2; \mathbb{F}_5)$ exactly. Using the map $\theta_{X, D}$ in [13], we can see that the associated homomorphism $\theta_{X, D}(C) : \pi_1(\Sigma(2, 3, 5)) \rightarrow \text{Sp}(2; \mathbb{F}_5)$ is an isomorphism. Hence, the class $[\theta_{X, D}(C)]$ is a generator of $\Omega_3(\text{Sp}(2; \mathbb{F}_5))$. It follows from Theorem 7.2 that the inclusion $\text{Sp}(2; \mathbb{F}_5) \hookrightarrow \text{Sp}(2g; \mathbb{F}_5)$ induces an isomorphism between these homologies without 5-torsion, which means the claimed surjectivity of $(\mathcal{P}_5)_*$.

However, the left vertical map $(\mathcal{P}_5)_* : \mathbb{Z}/24 \rightarrow H^3_{tr}(As(\mathfrak{sp}_5^g))$ is not surjective (see Lemma 8.3 below). Hence, the purpose $\Pi_2(D_g)$ turns out to be either $\mathbb{Z}/48$ or $\mathbb{Z}/24$ by observing the above commutative diagram carefully.

We now show two lemmas which are used in the proof above.
Lemma 8.2. Let $T_g$ be the universal central extension on the group $M_g$. If $g \geq 3$, then $H^3_{gr}(T_g)$ vanishes. Furthermore, if $g \geq 7$, then $H^3_{gr}(T_g) \cong \mathbb{Z}/24$.

We now prove Lemma 8.2 by using Quillen plus constructions and Madsen-Tillmann [MT].

Proof. We first immediately have $H^3_{gr}(T_g) \cong 0$, since $T_g$ is the universal central extension of $M_g$ and the group $M_g$ is perfect (see, e.g., [Ros, Corollary 4.1.18]).

We next focus on $H^3_{gr}(T_g)$ with $g \geq 3$. Let $BM^+_{g,k}$ denote Quillen plus construction of Eilenberg-MacLane space of $M_{g,k}$ (see, e.g., [Ros, Chapter 5.2] for details). Since $M_{g,k}$ is perfect, the space $BM^+_{g,k}$ is simply connected. As a basic property of plus constructions (see [Ros, Theorem 5.2.7]), the homotopy group $\pi_3(BM^+_{g})$ is isomorphic to $H^3_{gr}(T_g)$.

It is therefore to calculate $\pi_3(BM^+_{g})$ for $g \geq 7$. For this, we set up some preliminaries. Consider the inclusion $M_{g,1} \to M_{g+1,1}$ obtained by gluing the surface $\Sigma_{1,2}$ along one of its boundary components. Let $M_{\infty,1} := \lim_{g \to \infty} M_{g,1}$. Furthermore put an epimorphism $\delta_g : M_{g,1} \to M_g$ induced by gluing a disc to the boundary component of $\Sigma_{1,2}$. According to the Harer-Ivanov stability theorem improved by [RW], the inclusion $\iota_\infty : M_{g,1} \to M_{\infty,1}$ induces an isomorphism $H^3_{gr}(M_{g,1}) \cong H^3_{gr}(M_{\infty,1})$, and the map $\delta_g$ does $H^3_{gr}(M_g) \cong H^3_{gr}(M_{g,1})$, for $j \leq 3$.

Finally, we consider the maps $\delta^+_g : BM^+_{g,1} \to BM^+_g$ and $\iota^+_\infty : BM^+_{g,1} \to BM^+_{\infty,1}$ induced by $\delta_g$ and $\iota_\infty$, respectively. By Whitehead theorem, these maps induce isomorphisms

$$(\delta^+_g)_* : \pi_3(BM^+_{g,1}) \cong \pi_3(BM^+_g), \quad (\iota^+_\infty)_* : \pi_3(BM^+_{g,1}) \cong \pi_3(BM^+_{\infty,1}).$$

However the $\pi_3(BM^+_{g,1}) \cong \mathbb{Z}/24$ was shown by Madsen and Tillmann [MT] (see also [Eb]). In summary, we have $H^3_{gr}(T_g) \cong \pi_3(BM^+_{\infty,1}) \cong \mathbb{Z}/24$ as required.

Lemma 8.3. The induced map $(P_5)_* : H^3_{gr}(As(D_g)) \to H^3_{gr}(As(Sp^g_1))$ with $g \geq 7$ is not surjective.

Proof. Recall the reduction of $P_5 : As(D_g) \to As(Sp^g_1)$ to $\mathbb{Z} \times T_g \to \mathbb{Z} \times Sp(2g; \mathbb{F}_3)$. We easily see that it factors through $\mathbb{Z} \times M_g$. However $H^3_{gr}(M_g) \cong \mathbb{Z}/12$ is known [MT] (see also [Eb, §1]). Since $H^3_{gr}(T_g) \cong H^3_{gr}(Sp(2g; \mathbb{F}_3)) \cong \mathbb{Z}/24$ as above, the map $(P_5)_*$ is not a surjection.

9 An application; third quandle homologies

As an application of the study of the homotopy group $\pi_2(BX)$, we compute some torsion subgroups of third quandle homologies $H^3_{gr}(X)$ of finite connected quandles $X$. First, we
prove Theorem 9.14 owing to the facts explained in §9.1. Next, in §9.2 we later determine \( H^Q_3(X) \) of some quandles.

We briefly explain a basic line to study \( H^Q_3(X) \) in this section. Let \( B(X, X) \) be the rack space associated to the primitive \( X \)-set. Note the following isomorphisms:

\[
H_2(B(X, X)) \cong H^R_2(X) \cong H^Q_2(X) \oplus H^Q_2(X) \oplus \mathbb{Z},
\]

(30)

where the first isomorphism is derived from Remark 5.1, and the second was shown \([\text{LN, Theorem } 2.2]\). Composing this (30) with the result on \( \pi_2(BX) = \pi_2(B(X, X)) \) from Theorem 9.5 can compute some torsion of the quandle homology \( H^Q_3(X) \) [see Lemma (9.4)].

Following this line, we first prove Theorem 9.14 as a general statement rewritten as

**Theorem 9.1 (Theorem 3.14).** Let \( X \) be a connected quandle with \( |X| < \infty \). Let \( \text{Ker}(\psi_X) \) be the abelian kernel in (12). Then an isomorphism \( H^Q_3(X) \cong (\text{H}_2^S(\text{As}(X)) \oplus (\text{Ker}(\psi_X) \wedge \text{Ker}(\psi_X))) \) holds after localization at any prime \( \ell \) which does not divide \( 2|\text{Inn}(X)|/|X| \).

**Proof.** By Lemma 9.4 and the isomorphism (33) below, we have an isomorphism

\[
\pi_2(BX)_{(\ell)} \oplus H^S_2(\text{Ker}(\psi_X))_{(\ell)} \cong H_2(B(X, X))_{(\ell)}.
\]

(31)

Recall from Theorem 3.5 the isomorphism \( \pi_2(BX)_{(\ell)} \cong H^S_2(\text{As}(X))_{(\ell)} \oplus H^Q_2(X)_{(\ell)} \oplus \mathbb{Z}_{(\ell)}. \)

Hence, together with (30) above, the isomorphism (31) is rewritten in

\[
H^S_2(\text{As}(X))_{(\ell)} \oplus H^Q_2(X)_{(\ell)} \oplus \mathbb{Z}_{(\ell)} \oplus H^S_2(\text{Ker}(\psi_X))_{(\ell)} \cong H^Q_2(X)_{(\ell)} \oplus H^Q_2(X)_{(\ell)} \oplus H^Q_2(X)_{(\ell)}.
\]

Since the second group homology \( H^S_2(\text{Ker}(\psi_X)) \) is the exterior product \( \wedge^2(\text{Ker}(\psi_X)) \) [see \([\text{Bro, \S V.6}])], by a reduction of the both hand sides, we reach at the conclusion. \( \square \)

### 9.1 Preliminaries

We now recall basic properties of the rack space \( B(X, Y) \) introduced in §6.1 which are used in the preceding proof. To begin, we review

**Proposition 9.2 (FRS1 Theorem 3.7 and Proposition 5.1).** Let \( X \) be a quandle, and \( Y \) an \( X \)-set. Decompose \( Y \) into the orbits as \( Y = \sqcup_{i \in I} Y_i \). For \( i \in I \) and an element \( y_i \in Y_i \), denote by \( \text{Stab}(y_i) \subset \text{As}(X) \) the stabilizer of \( y_i \). Then, the subspace \( B(X, Y_i) \subset B(X, Y) \) is path-connected, and the natural projection \( B(X, Y_i) \to BX \) is a covering. Furthermore, the \( \pi_1(B(X, Y_i)) \) is the stabilizer \( \text{Stab}(y_i) \), by observing the covering transformation group.

We next observe the spaces \( B(X, Y) \) in some cases of \( Y \), where \( X \) is assumed to be connected. First, since \( \pi_1(BX) \cong \text{As}(X) \) from the 2-skeleton of \( BX \), the projection \( B(X, Y) \to BX \) with \( Y = \text{As}(X) \) is the universal covering. Next, we let \( Y \) be the inner automorphism group \( \text{Inn}(X) \), and be acted on by \( \text{As}(X) \) via (12). Considering the surjections \( \text{Inn}(X) \to X \to \{pt\} \) as \( X \)-sets, they then yield a sequence of the coverings

\[
B(X, \text{Inn}(X)) \to B(X, X) \to BX.
\]

(32)

**Remark 9.3.** According to Proposition 9.2, \( \pi_1(B(X, X)) \) is the stabilizer \( \text{Stab}(x_0) \subset \text{As}(X) \), and \( \pi_1(B(X, \text{Inn}(X))) \) is the abelian kernel \( \text{Ker}(\psi_X) \) of \( \psi_X : \text{As}(X) \to \text{Inn}(X) \) in (12).
We further observe homologies of the space \( B(X, X) \). Let \( \ell \) be a prime which does not divide the order \(|\text{Inn}(X)|/|X|\). As is known, the action of \( \pi_1(BX) \) on the homology group \( H_* (BX) \) is trivial (see [Cla1]). Then the first covering in (32) induces an isomorphism between their homologies localized at \( \ell \). To be precise
\[
H_*(B(X, \text{Inn}(X)))_\ell \cong H_*(B(X, X))_\ell.
\] (33)
Actually the transfer map of the covering is an inverse mapping (see [Cla1, Proposition 4.2]).

Finally we observe a relation between the homotopy and homology groups of the rack space \( B(X, \text{Inn}(X)) \). Refer to the fact that Clauwens [Cla1, §2.5] gave a topological monoid structure on \( B(X, \text{Inn}(X)) \). Hence

**Lemma 9.4.** Let \( X \) be a connected quandle. Let \( BX_G \) denote the rack space \( B(X, \text{Inn}(X)) \) for short. Then the Hurewicz homomorphism \( \pi_2(BX) = \pi_2(BX_G) \rightarrow H_2(BX_G) \) gives a splitting up to modulo 2-torsion. In particular
\[
H_2(BX_G) \cong \pi_2(BX) \oplus H^g_2(\ker(\psi_X)) \pmod{2}.
\]

**Proof.** As is known, the second \( k \)-invariants of path-connected topological monoids with CW-structure are annihilated by 2 [Sou, AP]. Namely, the Hurewicz map \( H \) splits modulo 2-torsion. Noting \( \pi_1(BX_G) \cong \ker(\psi_X) \) by Remark 9.3, we have \( \text{Coker}(H) \cong H^g_2(\ker(\psi_X)) \), which implies the required decomposition. \( \square \)

### 9.2 Some calculations of third quandle homologies

We now prove Theorems 3.16, 3.17 based on the properties of rack spaces explained above: we will compute the third quandle homologies of some quandles in more details than Theorem 3.14 shown above. To begin, we will prove Theorem 3.16 which is in Alexander case.

**Proof of Theorem 3.16.** Let \( X \) be a regular Alexander quandle of finite order. Consider the rack space \( B(X, \text{Inn}(X)) \) whose \( \pi_1 \) is \( \ker(\psi_X) = \mathbb{Z} \times \mathcal{K} \) by Remark 9.3. It is shown that the torsion subgroups of the homology \( H_2(B(X, \text{Inn}(X))) \) and \( H_2(B(X, X)) \) are annihilated by \( |X|^3 \) (see [N2, Lemma 5.7] and [LN, Theorem 1.1]). Hence, \( H^Q_3(X) \) and \( H^g_3(\text{As}(X)) \) are annihilated by \( |X|^3 \), by repeating the proof of Theorem 9.1. Noticing \( t_X = |\text{Inn}(X)|/|X|\), the purpose \( H^Q_3(X) \cong H^g_3(\text{As}(X)) \oplus \mathcal{K} \oplus (\wedge^2 \mathcal{K}) \pmod{2} \) immediately follows from the regularity and Theorem 9.1.

Finally, we work out the case where \( |X| \) is odd. By repeating the above discussion, the homologies \( H_2(B(X, X)) \) and \( H_2(B(X, \text{Inn}(X))) \) have no 2-torsion; so does the \( H^Q_3(X) \) as well. \( \square \)

We next prove Theorem 3.17 which computes \( H^Q_3(X) \) for the symplectic and orthogonal quandles over \( \mathbb{F}_q \), where we exclude the exceptional cases of \( q \), i.e., \( q \neq 3, 3^2, 3^3, 5, 7 \).

**Proof of Theorem 3.17.** (I) Let \( X = \text{Sp}^n_q \) be the symplectic quandle over \( \mathbb{F}_q \). Recall \( \text{As}(X) \cong \mathbb{Z} \times Sp(2n; \mathbb{F}_q) \) by Proposition A.6. We particularly see the kernel \( \ker(\psi_X) \cong \mathbb{Z} \).
By the stability theorem 7.2, the left vertical map \( P_* \) is surjective onto \( H_2(B(X, X)) \). Therefore, the sequence is rewritten as

\[
H_3^{gr}(\mathcal{H}(B(X, X))) \cong \pi_2(B(X, X)) \rightarrow H_2(B(X, X)) \rightarrow H_2^{gr}(\mathcal{H}(B(X, X))) \rightarrow 0.
\]

Hence, compared with the isomorphism (30), we have \( H_3(Q) \cong (\mathbb{Z}/p)^{d(d+1)/2} \) as desired.

Next, when \( n \geq 2 \), we will show \( H_3^Q(X) = 0 \). Note that the second group homology of the stabilizer \( \mathcal{H}(B(X, X)) \) is zero (Lemma 3.4). Furthermore we note \( H_2(BX) \cong \mathbb{Z} \) by Proposition B.3. Therefore, the \( P \)-sequences of the projection \( P : B(X, X) \rightarrow BX \) are written as

\[
H_3^{gr}(\mathcal{H}(B(X, X))) \cong \pi_2(B(X, X)) \rightarrow H_2(B(X, X)) \rightarrow H_2^{gr}(\mathcal{H}(B(X, X))) \rightarrow 0.
\]

By the stability theorem 7.2, the left vertical map \( P_* \) surjects onto \( \mathbb{Z}/(q^2 - 1) \). Since \( \pi_2(BX) \cong \mathbb{Z} \oplus \mathbb{Z}/(q^2 - 1) \) by Theorem 3.11, the delta map \( \delta_* \) is surjective in torsion part. Therefore \( H_2(B(X, X)) = \mathbb{Z} \) by diagram chasing. Using (30) again, we have the goal \( H_3^Q(X) = 0 \).

II The calculations of the third homology \( H_3^Q(X) \) for the spherical quandle \( X = S_q^n \) over \( \mathbb{F}_q \) can be shown in a similar way to the symplectic case. The point is that the homology \( H_i^{gr}(\mathcal{H}(B(X, X))) \) is isomorphic to \( H_i^{gr}(O(n; \mathbb{F}_q)) \) modulo 2-torsion for \( i \leq 3 \) (see Proposition 7.1). Furthermore, we can complete the computation of \( H_2^Q(X) \) (see Proposition B.3 for details). To summarize, using results of Quillen and Friedlander explained in 7.1 we can complete the proof similarly, and omit the details.

Moreover we will prove Theorem 3.18 on extended quandles \( \tilde{X} \).

Proof of Theorem 3.18. We first construct the claimed isomorphism \( \Pi_2(\tilde{X}) \cong H_3^Q(\tilde{X}) \) modulo \( t_X \). Consider the rack space \( B(\tilde{X}, X) \) associated with the primitive \( X \)-set. Note that this \( \pi_1 \) is an abelian group \( \mathbb{Z} \times \text{Ker}(p_*) \) by Remark 9.3 and Proposition 6.11. Then the Postnikov tower is expressed by

\[
H_3^{gr}(\mathbb{Z} \times \text{Ker}(p_*)) \rightarrow \pi_2(B\tilde{X}) \rightarrow H_2(B(\tilde{X}, X)) \rightarrow H_2^{gr}(\mathbb{Z} \times \text{Ker}(p_*)) \rightarrow 0 \quad \text{(exact)}.
\]
By Proposition \[\text{6.11}\] again, \(H_i^{gr}(\mathbb{Z} \times \text{Ker}(p_*))\) is annihilated by \(t_X\) for \(i \geq 2\). Hence, the Hurewicz map \(H_\times\) takes the isomorphism \(\Pi_2(\tilde{X}) \cong H_3^{gr}(\tilde{X})\) modulo \(t_X\).

To prove the isomorphism \(\Pi_2(\tilde{X}) \cong H_3^{gr}(\text{As}(X))\) modulo \(t_X\) and the latter part, we first note \(H_2^{gr}(\tilde{X}) \cong 0 \pmod{t_X}\) by Theorem \[\text{6.12}\]. Considering the isomorphisms \(p_* : H_3^{gr}(\text{As}(\tilde{X})) \to H_3^{gr}(\text{As}(X))\) and the T-map \(\Theta_X : \Pi_2(\tilde{X}) \to H_3^{gr}(\text{As}(\tilde{X}))\) modulo \(t_X\) in Proposition \[\text{6.10}\] and Theorem \[\text{3.5}\] respectively, we obtain the composite isomorphism \(p_* \circ \Theta_X : \Pi_2(\tilde{X}) \to H_3^{gr}(\text{As}(X))\) modulo \(t_X\) as desired.

Finally, we will show a lemma which will be used in a subsequent paper \[\text{[N4]}\].

**Lemma 9.5.** Let \(X\) be a connected Alexander quandle of type \(t_X\). Let \(p : \tilde{X} \to X\) be the universal covering. Then the induced map \(p_* : H_3^{gr}(\tilde{X}) \to H_3^{gr}(X)\) is injective up to \(2t_X\)-torsion.

**Proof.** By the proof of Theorem \[\text{3.16}\] in \[\text{[92]}\] the Hurewicz map \(\mathcal{H}_X : \pi_2(B(X,X)) \to H_2(B(X,X))\) is injective up to \(2t_X\)-torsion. Consider the Postnikov tower with respect to the \(p : \tilde{X} \to X\):

\[
\begin{array}{cccccc}
H_3^{gr}(\mathbb{Z} \times \text{Ker}(p_*))(\ell) & \overset{0}{\longrightarrow} & \pi_2(B\tilde{X})(\ell) & \overset{\mathcal{H}_X}{\longrightarrow} & H_2(B(\tilde{X}, \tilde{X}))(\ell) & \longrightarrow & H_2^{gr}(\mathbb{Z} \times \text{Ker}(p_*))(\ell) = 0 \\
\downarrow p_* & & \downarrow p_* & & \downarrow p_* & & \downarrow p_* \\
H_3^{gr}(\text{Stab}(x_0))(\ell) & \overset{0}{\longrightarrow} & \pi_2(BX)(\ell) & \overset{\mathcal{H}_X}{\longrightarrow} & H_2(B(X,X))(\ell) & \longrightarrow & H_2^{gr}(\text{Stab}(x_0))(\ell).
\end{array}
\]

Here a prime \(\ell\) is relatively prime to \(2t_X\). Since the top Hurewicz map \(\mathcal{H}_X\) is an isomorphism modulo \(2t_X\) (see the previous proof of Theorem \[\text{3.18}\]), the vertical map \(p_* : H_2(B(\tilde{X}, \tilde{X}))(\ell) \to H_2(B(X,X))(\ell)\) is injective. Hence, by \[\text{[30]}\] as usual, the \(p_* : H_3^{gr}(\tilde{X}) \to H_3^{gr}(X)\) turns out to be an injection up to \(2t_X\)-torsion.

**A Calculations of automorphism groups of quandles**

According to Theorem \[\text{3.5}\] it is significant to determine associated groups \(\text{As}(X)\). In this appendix, we develop a method for formulating \(\text{As}(X)\) concretely from the inner automorphism groups \(\text{Inn}(X)\), using the group extension \[\text{(12)}\] which is rewritten in

\[
0 \longrightarrow \text{Ker}(\psi_X) \longrightarrow \text{As}(X) \overset{\psi_X}{\longrightarrow} \text{Inn}(X) \longrightarrow 0 \quad \text{(central extension).} \quad (35)
\]

Here, we note that, when \(X\) is connected and of type \(t_X\), the kernel is isomorphic to \(\mathbb{Z} \oplus H_2^{gr}(\text{Inn}(X))\) up to \(t_X\)-torsion (Corollary \[\text{6.4}\]). In conclusion, to investigate \(\text{As}(X)\), we shall study the \(\text{Inn}(X)\) and \(H_2^{gr}(\text{Inn}(X))\) mod \(t_X\), metaphorically speaking, ‘universal central extensions’ of \(\text{Inn}(X)\) modulo \(t_X\)-torsion.

To study the \(\text{As}(X)\), we first propose a simple method of determining the group \(\text{Inn}(X)\).

**Lemma A.1.** Let a group \(G\) act on a quandle \(X\). Let a map \(\kappa : X \to G\) satisfy the followings:

(I) The identity \(x \triangleleft y = x \cdot \kappa(y) \in X\) holds for any \(x, y \in X\).

(II) The image \(\kappa(X) \subset G\) generates the group \(G\), and the action \(X \bigtriangleup G\) is effective.
Then there is an isomorphism $\text{Inn}(X) \cong G$, and the action $X \curvearrowright G$ agrees with the natural action of $\text{Inn}(X)$.

Proof. Identifying the action $X \curvearrowright G$ with a group homomorphism $F : G \to \mathfrak{G}_X$, this $F$ factors through $\text{Inn}(X)$ by (I). Notice, for any $x, y, z \in X$, the identities

$$z \cdot \kappa(x)\kappa(y) = (z \cdot x) \cdot y = (z \cdot y) \cdot (x \cdot y) = z \cdot \kappa(y)\kappa(x \cdot y) \in X,$$

which implies $\kappa(x)\kappa(y) = \kappa(y)\kappa(x \cdot y) \in G$ by the effectivity in (II). Hence the epimorphism $\psi_X$ in (II) is decomposed as $\text{As}(X) \to G \xrightarrow{\kappa} \text{Inn}(X)$. Moreover (II) concludes the bijectivity of $F : G \cong \text{Inn}(X)$ and, hence, the agreement of the two actions by construction. \qed

This lemma is applicable and practical; actually, for many quandles $X$, we write down the inner automorphism groups $\text{Inn}(X)$ in this way. For example, we now deal with the symplectic quandles $\text{Sp}_K^n$ and spherical quandles $S^n_K$ over $K$ (Examples 2.2, 2.3) as follows:

**Lemma A.2.** Let $K$ be a commutative field. Then $\text{Inn}(\text{Sp}_K^n)$ is isomorphic to the symplectic group $\text{Sp}(2n; K)$. Furthermore, if $n \geq 2$ and the characteristic of $K$ is not 2, then $\text{Inn}(S^n_K)$ is isomorphic to the orthogonal group $O(n + 1; K)$.

Proof. As is called the Cartan-Dieudonné theorem classically, the groups $\text{Sp}(2n; K)$ and $O(n + 1; K)$ are generated by transvections and symmetries $(\bullet \rhd y)$, respectively.

We will show the isomorphism $\text{Inn}(\text{Sp}_K^n) \cong \text{Sp}(2n; K)$. For any $y \in \text{Sp}_K^n$, the map $(\bullet \rhd y) : \text{Sp}_K^n \to \text{Sp}_K^n$ is a restriction of a linear map $K^{2n} \to K^{2n}$. It thus yields a map $\kappa : \text{Sp}_K^n \to \text{GL}(2n; K)$, which factors through the $\text{Sp}(2n; K)$ and satisfies the conditions in Lemma A.1. Indeed, e.g., the condition (II) follows from the previous classical theorem and the effectivity of the standard action $K^{2n} \curvearrowright \text{Sp}(2n; K)$. Therefore $\text{Inn}(\text{Sp}_K^n) \cong \text{Sp}(2n; K)$ as desired.

Turning to the orthogonal case, the isomorphism $\text{Inn}(S^n_K) \cong O(n + 1; K)$ can be shown by replacing $\text{Sp}_K^{2n}$ by $S^n_K$ and $\text{Sp}(2n; K)$ by $O(n + 1; K)$, respectively, in the previous proof. \qed

**Example A.3** (Symmetric space). As another application of Lemma A.1, let us consider a symmetric space $X$ as a typical example of quandles (see, e.g., [Joy, E2]). Note that, by definition, this smooth connected manifold $X$ is equipped with a Riemannian metric such that each point $x \in X$ admits an isometry $s_y : X \to X$ that reverses every geodesic line $\gamma : (\mathbb{R}, 0) \to (X, y)$, meaning that $s_y \circ \gamma(t) = \gamma(-t)$. Then we have a quandle structure on $X$ defined by $x \cdot y := s_y(x)$. Here recall the group $G \subset \text{Diff}(X)$ generated by the symmetries $s_y$ with compact-open topology. As is well known, this $G$ is a Lie group and has an effective $C^\infty$-action $X \curvearrowright G$. Considering the Cartan embedding $\kappa : X \to G$, we easily see that this action and the $\kappa$ satisfy the conditions in Lemma A.1. Consequently we conclude $G \cong \text{Inn}(X)$.

Next, using the extension (II), we calculate the associated groups of some quandles.

**Proposition A.4.** Let $X$ be a quandle, and $O(X)$ be the set of orbits of the action $X \curvearrowright \text{As}(X)$. Set the epimorphisms $\varepsilon_i : \text{As}(X) \to \mathbb{Z}$ associated with $i \in O(X)$ defined in (16). If the group $\text{Inn}(X)$ is perfect, i.e., $H_1^\text{gr}(\text{Inn}(X)) = 0$, then we have an isomorphism

$$\text{As}(X) \cong \text{Ker}(\bigoplus_{i \in O(X)} \varepsilon_i) \times \mathbb{Z}^{\oplus O(X)},$$

(36)
and this $\text{Ker}(\oplus_{i \in O(X)} \varepsilon_i)$ is a central extension of $\text{Inn}(X)$ and is perfect. In particular, if $X$ is connected and the $H_2^{gr}(\text{Inn}(X))$ vanishes, then $\text{As}(X) \cong \text{Inn}(X) \times \mathbb{Z}$.

Proof. We will show the (36). Since $H_1^{gr}(\text{Inn}(X)) = 0$, we obtain an epimorphism $\text{Ker}(\psi_X) \twoheadrightarrow \text{As}(X) \xrightarrow{\text{proj}} H_1^{gr}(\text{As}(X)) = \mathbb{Z}^{|O(X)|}$ from (41). Since $\mathbb{Z}^{|O(X)|}$ is free, we can choose a section $s : \mathbb{Z}^{|O(X)|} \to \text{As}(X)$ which factors through the $\text{Ker}(\psi_X)$. Hence, by the equality (2), the semi-product $\text{As}(X) \cong \text{Ker}(\oplus_{i \in O(X)} \varepsilon_i) \times \mathbb{Z}^{|O(X)|}$ is trivial, leading to the isomorphism (36) as desired. Furthermore the kernel $\text{Ker}(\oplus_{i \in O(X)} \varepsilon_i)$ is a central extension of $\text{Inn}(X)$ by construction, and is perfect by the Kunneth theorem and $\text{As}(X)_{ab} \cong \mathbb{Z}^{|O(X)|}$, which completes the proof. □

As a corollary, we consider the symplectic quandle over $F$ of positive characteristic.

**Proposition A.5.** Take a field $F$ of positive characteristic and with $|F| > 10$. Let $X$ be the symplectic quandle over $F$, and $Sp(2g; F)$ be the universal central extension of $Sp(2g; F)$. Then $\text{As}(X) \cong \mathbb{Z} \times Sp(2g; F)$.

Proof. Since $X$ is connected and $\text{Inn}(X) \cong Sp(2g; F)$ is perfect (Lemma A.2), Proposition A.4 implies $\text{As}(X) \cong \text{Ker}(\varepsilon) \times \mathbb{Z}$. Recall from Theorem 6.1 that $X$ is of type $p$ and $H_2^{gr}(\text{As}(X))$ is annihilated by $p$. Hence, following the fact [Sus] that $H_2^{gr}(Sp(2g; F))$ has no $p$-torsion, the kernel $\text{Ker}(\varepsilon)$ must be the universal central extension of $Sp(2g; F)$, which completes the proof. □

**Example A.6.** Let $q > 10$. Let $X$ be the symplectic quandles $Sp_q^n$ over $\mathbb{F}_q$. Then we see $\text{As}(X) \cong \mathbb{Z} \times Sp(2n; \mathbb{F}_q)$. In fact, noticing $\text{Inn}(X) \cong Sp(2n; \mathbb{F}_q)$ from Lemma A.2 the first and second homologies of $Sp(2n; \mathbb{F}_q)$ vanish (see Proposition 7.1 or [FP, Fri]).

In general, it is hard to calculate the associated groups $\text{As}(X)$ concretely; however, in some cases, we can calculate some torsion parts of their group homologies as follows:

**Lemma A.7.** Let $X$ be a connected quandle of type $t_X$. If $H_2^{gr}(\text{Inn}(X))$ is annihilated by $t_X$, then there is an isomorphism $H_3^{gr}(\text{As}(X)) \cong H_3^{gr}(\text{Inn}(X))$ modulo $t_X$-torsion.

Proof. By Corollary 6.4 we have $\text{Ker}(\psi_X) \cong \mathbb{Z} (\text{mod } t_X)$. Hence, the Lyndon-Hochschild spectral sequence of the $\psi_X$ leads to the required isomorphism modulo $t_X$-torsion. □

**Example A.8.** Let $n \geq 2$, and $q \neq 3, 9$. Let $X = S_q^n$ be the spherical quandle over $\mathbb{F}_q$ of type 2. Then $H_3^{gr}(\text{As}(X)) \cong H_3^{gr}(O(n+1; \mathbb{F}_q))$ without 2-torsion. Actually, the $H_1^{gr} \oplus H_2^{gr}(O(n+1; \mathbb{F}_q))$ is known to be annihilated by 2 (see Theorems 7.2, 7.3).

Incidentally, we compare the type with the order $|\text{Inn}(X)|/|X|$ as follows:

**Lemma A.9.** Let $X$ be a finite connected quandle. Then its type $t_X$ is a divisor of $|\text{Inn}(X)|/|X|$.

Proof. For $x, y \in X$, we define $m_{x,y}$ by the minimal $n$ satisfying $x \triangleleft^n y = x$. Note that ($\bullet \triangleleft^{m_{x,y}} y$) lies in the stabilizer $\text{Stab}(x)$. Since $|\text{Stab}(x)| = |\text{Inn}(X)|/|X|$ by connectivity, any $m_{x,y}$ divides $|\text{Inn}(X)|/|X|$; hence so does the type $t_X$. □
B Some calculations of second quandle homologies

This appendix calculates the second quandle homologies of some connected quandles \( X \), using the results on \( \text{As}(X) \) in the previous section. Our calculation relies on a result of Eisermann \cite{E2} (see Theorem \ref{thm:homomorphism}) which claims an isomorphism

\[
H^Q_2(X) \cong (\text{Ker}(\varepsilon_X) \cap \text{Stab}(x_0))_{ab}.
\]

We first consider the homology \( H^Q_2(X) \) of a connected Alexander quandle \( X \). Clauwens \cite{Cla2} determined the associated group \( \text{As}(X) \) as follows. Set up the homomorphism \( \mu_X : X \otimes X \to X \otimes X \) defined by \( \mu_X(x \otimes y) = x \otimes y - Ty \otimes x \). He defined a group operation on \( \mathbb{Z} \times X \times \text{Coker}(\mu_X) \) by setting

\[
(n, x, \alpha) \cdot (m, y, \beta) = (n + m, T^mx + y, \alpha + \beta + [T^mx \otimes y]),
\]

and showed that the homomorphism \( \text{As}(X) \to \mathbb{Z} \times X \times \text{Coker}(\mu_X) \) defined by sending \( e_x \) to \((1, x, 0)\) is a group isomorphism. The lower central series of \( \text{As}(X) \) is then described as

\[
\text{As}(X) \supset X \times \text{Coker}(\mu_X) \supset \text{Coker}(\mu_X) \supset 0.
\]

As a result, we see that the kernel of \( \psi_X : \text{As}(X) \to \text{Inn}(X) \) equals \( \mathbb{Z} \times \text{Coker}(\mu_X) \). Thanks to his presentation of \( \text{As}(X) \), we easily show a result of Clauwens:

**Proposition B.1** (Clauwens \cite{Cla2}). Let \( X \) be a connected Alexander quandle. The homology \( H^Q_2(X) \) is isomorphic to the quotient module \( \text{Coker}(\mu_X) = X \otimes_{\mathbb{Z}} X/(x \otimes y - Ty \otimes x)_{x,y \in X} \).

*Proof.* By definition we can see that the Ker(\( \varepsilon_X \)) \( \cap \text{Stab}(0) \) is the cokernel \( \text{Coker}(\mu_X) \).

Next, we focus on second homologies of spherical and symplectic quandles over \( \mathbb{F}_q \) as follows.

**Proposition B.2.** Let \( X = S^n_q \) be a spherical quandle over \( \mathbb{F}_q \). Let \( q \neq 3,9 \). For \( n \geq 3 \), the second homology \( H^Q_2(X) \) is annihilated by 2. If \( n = 2 \), then the homology \( H^Q_2(X) \) is the cyclic group \( \mathbb{Z}/(q - \delta_q) \) modulo 2-torsion, where \( \delta_q = \pm 1 \) is according to \( q \equiv \pm 1(\text{mod } 4) \).

*Proof.* Under the standard action \( X \curvearrowleft O(n + 1; \mathbb{F}_q) \), the stabilizer of \((1, 0, \ldots, 0) \in X \) is easily shown to be \( O(n; \mathbb{F}_q) \). By a similar discussion to the proof of Proposition \ref{prop:second_homology}, it follows from Theorem \ref{thm:homomorphism} that \( H^Q_2(X) \cong H^p_1(O(n; \mathbb{F}_q)) \) without 2-torsion. For \( n \geq 3 \), the abelianization of \( O(n; \mathbb{F}_q) \) is \((\mathbb{Z}/2)^2\); see \cite[II. §3]{FP}; hence the \( H^Q_2(X) \) is annihilated by 2 as required. Finally, when \( n = 2 \), the group \( O(2; \mathbb{F}_q) \) is cyclic and of order \( q - \delta_q \). Hence \( H^Q_2(X) \cong H^p_1(O(2; \mathbb{F}_q)) \cong \mathbb{Z}/(q - \delta_q) \) modulo 2.

**Proposition B.3.** Let \( X \) be the symplectic quandle \( Sp^n_q \) over \( \mathbb{F}_q \). Let \( q \neq 3,9 \). If \( n \geq 2 \), the second homology \( H^Q_2(X) \) vanishes. If \( n = 1 \), then \( H^Q_2(X) \cong (\mathbb{Z}/p)^d \), where \( q = p^d \).

*Proof.* Recall \( \text{As}(X) \cong \mathbb{Z} \times \text{Sp}(2n; \mathbb{F}_q) \) from Example \ref{ex:second_homology}. Considering the standard action \( X \curvearrowleft \text{Sp}(2n; \mathbb{F}_q) \), denote by \( G \) the stabilizer of \((1, 0, \ldots, 0) \in (\mathbb{F}_q)^{2n} \). Since Theorem \ref{thm:homomorphism} immediately means \( H^Q_2(X) \cong H^p_1(G) \), we will calculate \( H^p_1(G) \) as follows. First, for \( n = 1 \), it can be verified that the stabilizer \( G \) is exactly the product \((\mathbb{Z}/p)^d\) as an abelian group; hence \( H^Q_2(X) \cong (\mathbb{Z}/p)^d \) in the sequel. Next, for \( n \geq 2 \), the vanishing \( H^Q_2(X) = H^p_1(G) = 0 \) immediately follows from Lemma \ref{lem:vanishing} below.

---

36
Lemma B.4. Let $n \geq 2$ and $q \neq 3,9$. Let $G$ denote the stabilizer of the action $X = \text{Sp}_q^n \cap \text{Sp}(2n; \mathbb{F}_q)$ mentioned above. Then the homologies $H_1^{gr}(G)$ and $H_2^{gr}(G)$ vanish.

Proof. Recall from [FP, II. §6.3] the order of $\text{Sp}(2n; \mathbb{F}_q)$ as

$$|\text{Sp}(2n; \mathbb{F}_q)| = q^{2n} - 1)(q^{2n-2} - 1) \cdots (q^2 - 1).$$

Since $|X| = q^{2n} - 1$, the order of $G$ is then equal to $q^{2n-1} \cdot |\text{Sp}(2n-2; \mathbb{F}_q)|$. Thereby $H_1^{gr}(G)$ and $H_2^{gr}(G)$ are zero modulo $p$-torsion, it is because of the inclusion $\text{Sp}(2n-2; \mathbb{F}_q) \subset G$ by definitions and the vanishing $H_1^{gr} \oplus H_2^{gr}(\text{Sp}(2n-2; \mathbb{F}_q)) \cong 0$ modulo $p$.

Finally, we may focus on the $p$-torsion of $H_1^{gr} \oplus H_2^{gr}(G)$. Following the proof of [Fri. Proposition 4.4], there is a certain subgroup “$\Delta(\text{Sp}(2n; \mathbb{F}_q))$” of $G$ which contains a $p$-sylow group of $\text{Sp}(2n; \mathbb{F}_q)$, and this $\mathbb{Z}/p$-homology vanishes. Hence $H_1^{gr} \oplus H_2^{gr}(G) = 0$ as required.

Changing the subject to the Dehn quandle $\mathcal{D}_g$ (see [8] for the definition), we now show

Proposition B.5. For $g \geq 2$, the second quandle homology $H_2^Q(\mathcal{D}_g; \mathbb{Z})$ surjects onto $\mathbb{Z}/2$. Furthermore, if $g \geq 5$, then $H_2^Q(\mathcal{D}_g; \mathbb{Z}) \cong \mathbb{Z}/2$.

In the following proof, we often use the fact that an epimorphism $G \to H$ between groups induces an epimorphism $G_{ab} \to H_{ab}$, and the isomorphism $\text{As}((\mathcal{D}_g)) \cong \mathbb{Z} \times \mathcal{T}_g$ shown by [Ger].

Proof of Proposition B.5. Fixing $\alpha \in \mathcal{D}_g$, we first observe the stabilizer $\text{Stab}(\alpha) \subset \text{As}((\mathcal{D}_g))$. Note that the map $\mathcal{D}_g \to \mathcal{M}_g$ sending $\alpha$ to $\tau_\alpha$ yields a group epimorphism $\pi : \text{As}((\mathcal{D}_g)) \to \mathcal{M}_g$. Furthermore, by Proposition A.3.3, the restriction of $\pi$ to $\text{Ker}(\epsilon) \cong \mathcal{T}_g$ coincides with the projection $\mathcal{T}_g \to \mathcal{M}_g$. In particular, we thus have $\pi(\text{Stab}(\alpha)) = \pi(\text{Stab}(\alpha) \cap \text{Ker}(\epsilon)) \subset \mathcal{M}_g$.

To show the surjection $H_2^Q(\mathcal{D}_g; \mathbb{Z}) 	o \mathbb{Z}/2$, by virtue of Theorem 5.2, it is enough to construct a surjection from the previous $\pi(\text{Stab}(\alpha)) \subset \mathcal{M}_g$ to $\mathbb{Z}/2$ for $g \geq 2$. As is shown [PR Proposition 7.4], we have the following exact sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{M}_{g-1,2} \xrightarrow{\xi} \pi(\text{Stab}(\alpha)) \xrightarrow{\lambda} \mathbb{Z}/2. \quad (38)$$

Here $\xi$ is the homomorphism induced from the gluing $(\Sigma_{g-1,2}, \partial(\Sigma_{g-1,2})) \to (\Sigma_g, \alpha)$, and $\lambda$ is defined by the transposition of the connected components of boundaries of $\Sigma_g \setminus \alpha$. By considering a hyper-elliptic involution preserving the $\alpha$, the map $\lambda$ is surjective. Hence $\pi(\text{Stab}(\alpha) \cap \text{Ker}(\epsilon))$ surjects onto $\mathbb{Z}/2$ as desired.

Finally, we will show $H_2^Q(\mathcal{D}_g; \mathbb{Z}) \cong \mathbb{Z}/2$ for $g \geq 5$. By Theorem 5.2 again, recall that $(\text{Stab}(\alpha) \cap \text{Ker}(\epsilon))_{ab} \cong H_2^Q(\mathcal{D}_g; \mathbb{Z})$. To compute this, put the inclusion $\iota : \pi(\text{Stab}(\alpha)) \to \mathcal{M}_g$. By the Harer-Ivanov stability theorem (see [Iva]), the composition $\iota \circ \xi : \mathcal{M}_{g-1,2} \to \mathcal{M}_g$ induces an epimorphism

$$(\iota \circ \xi)_* : H_2(\mathcal{M}_{g-1,2}; \mathbb{Z}) \to H_2(\mathcal{M}_g; \mathbb{Z}) \quad \text{for } g \geq 5. \quad (39)$$

Since $H_2(\mathcal{M}_{g-1,2}; \mathbb{Z}) \cong H_2(\mathcal{M}_g; \mathbb{Z}) \cong \mathbb{Z}$ is well-known (see, e.g., [FM]), the epimorphism is isomorphic. Let $(\iota \circ \xi)^*(\mathcal{T}_g)$ denote the central extension of $\mathcal{M}_{g-1,2}$ obtained by $\iota \circ \xi$. Since $\mathcal{M}_g$ and $\mathcal{M}_{g-1,2}$ are perfect, the group $(\iota \circ \xi)^*(\mathcal{T}_g)$ is also perfect by the isomorphism (39). Note that the group $\text{Stab}(\alpha) \cap \text{Ker}(\epsilon)$ is isomorphic to $\iota^*(\mathcal{T}_g)$. Hence the abelianization $(\text{Stab}(\alpha) \cap \text{Ker}(\epsilon))_{ab}$ never be bigger than $\mathbb{Z}/2$. Thus we arrive at the conclusion $H_2^Q(\mathcal{D}_g; \mathbb{Z}) \cong \mathbb{Z}/2$. \(\square\)
B.1 Proof of Theorem 5.2

We will show Theorem 5.2 as a result of Proposition B.6. This proposition provides an algorithm to compute the first rack homology as follows:

**Proposition B.6.** Let \( X \) be a quandle, and \( Y \) an \( X \)-set. Decompose \( Y \) into the orbits as \( Y = \sqcup_{i \in I} Y_i \). For \( i \in I \), choose an arbitrary element \( y_i \in Y_i \), and denote by \( \text{Stab}(y_i) \subset \text{As}(X) \) the stabilizer subgroup of \( y_i \). Then \( H_1^R(X,Y) \) is isomorphic to the direct sum of the abelianizations of \( \text{Stab}(y_i) \). Precisely, \( H_1^R(X,Y) \cong \bigoplus_{i \in I} (\text{Stab}(y_i))_{\text{ab}} \).

**Proof.** Recall from Proposition 9.2, that each connected component of the \( \text{Stab}(x) \) is isomorphic to the direct sum of the abelianizations \( \text{Stab}(y_i) \). Thereby \( H_1(B(X,Y)) \cong \pi_1(B(X,Y))_{\text{ab}} \cong \text{Stab}(y_i)_{\text{ab}} \). Hence we conclude

\[
H_1^R(X,Y) \cong H_1(B(X,Y)) \cong \bigoplus_{i \in I} H_1(B(X,Y)_i) \cong \bigoplus_{i \in I} \text{Stab}(y_i)_{\text{ab}}.
\]

**Proof of Theorem 5.2** We first show (40) below. Let \( Y = X \) be the primitive \( X \)-set. For each \( x_i \in X_i \), we have \( e_{x_i} \in \text{Stab}(x_i) \) since \( x_i \subset x_i = x_i \). Hence the restriction of \( \varepsilon_i : \text{As}(X) \to \mathbb{Z} \) on \( \text{Stab}(x_i) \) is also surjective, and permits a section \( s : \mathbb{Z} \to \text{Stab}(x_i) \) defined by \( s(n) = e_{x_i}^n \).

Here we remark that the action of \( \mathbb{Z} \) on \( \text{Stab}(x_i) \cap \ker(\varepsilon_i) \) induced by the section is trivial. Indeed, \( g^{-1}e_{x_i}g = e_{x_i} \in \text{As}(X) \) for any \( g \in \text{Stab}(x_i) \) by (2). We therefore have \( \text{Stab}(x_i)_{\text{ab}} \cong (\text{Stab}(x_i) \cap \ker(\varepsilon_i))_{\text{ab}} \). Hence it follows from Proposition B.6 that

\[
H_1^R(X,X) \cong \bigoplus_{i \in O(X)} \text{Stab}(x_i)_{\text{ab}} \cong \mathbb{Z}^{O(X)} \oplus \bigoplus_{i \in O(X)} (\text{Stab}(x_i) \cap \ker(\varepsilon_i))_{\text{ab}}.
\]

Finally it is sufficient to show that \( H_2^Q(X) \) is isomorphic to the last summand. Recall \( H_2^Q(X) \cong H_1^R(X,X) \) in Remark 9.3. It is known [LN, Theorem 2.1] that \( H_2^Q(X) \cong H_2^Q(X) \oplus \mathbb{Z}^{O(X)} \), and that a basis of the \( \mathbb{Z}^{O(X)} \) is represented by \( (x_i, x_i) \in C_2^R(X) \) for \( i \in O(X) \). By comparing the basis with the isomorphisms in (40), we complete the proof.

**C Proof of Theorem 6.1**

To prove Theorem 6.1 in an ad hoc way, we now observe concretely the map \( c_* : H_n(BX) \to H_n^{\text{gr}}(\text{As}(X)) \) for \( n \leq 3 \). Let us recall the rack complex \( C_n^R(X) \) in [5] and the (non-homogenous) standard complex \( C_n^{\text{gr}}(\text{As}(X)) \) of \( \text{As}(X) \); see e.g. [Bro, §1.5]. The map \( c_* \) was described in terms of their complexes by Kabaya [Kab, §8.4]. Actually he considered homomorphisms \( c_n : C_n^R(X) \to C_n^{\text{gr}}(\text{As}(X)) \) for \( n \leq 3 \) defined by setting

\[
c_1(x) = e_x,
\]

\[
c_2(x,y) = (e_x, e_y) - (e_y, e_{x \subset y}),
\]

\[
c_3(x,y,z) = (e_x, e_y, e_z) - (e_x, e_z, e_{y \subset z}) + (e_y, e_z, e_A) - (e_y, e_{x \subset y}, e_z) + (e_z, e_{x \subset z}, e_A) - (e_z, e_{y \subset z}, e_A),
\]

where we denote \( (x \subset y) \subset z \in X \) by \( A \) for short. As is known the induced map on homology coincides with the map above \( c_* \) up to homotopy (see [Kab, §8.4]).
We will construct a chain homotopy between $t \cdot c_n$ and zero, when $X$ is connected and of type $t$. Define a homomorphism $h_i : C^n_i(X) \to C^n_{i+1}(\text{As}(X))$ by

$$h_1(x) = \sum_{1 \leq j \leq t-1}(e_x, e_{e_j}^i),$$

$$h_2(x, y) = \sum_{1 \leq j \leq t-1}(e_x, e_y, e_{e_{j-1}}^i) - (e_x, e_{e_j}^i, e_y) - (e_y, e_{x<ty}, e_{e_{j-1}}^i) + (e_y, e_{e_j}^i, e_y),$$

$$h_3(x, y, z) = \sum_{1 \leq j \leq t-1}(e_x, e_y, e_z, e_{e_{j-1}}^i) - (e_x, e_y, e_{x<ty}, e_{e_{j-1}}^i) - (e_y, e_{x<ty}, e_z, e_{e_{j-1}}^i) + (e_y, e_{x<ty}, e_z, e_{e_{j-1}}^i)
++(e_x, e_y, e_{e_{j-1}}^i, e_{x<ty}) + (e_x, e_{x<ty}, e_{e_{j-1}}^i) + (e_x, e_{e_{j-1}}^i, e_z) - (e_x, e_{x<ty}, e_{e_{j-1}}^i, e_z)
++(e_y, e_{x<ty}, e_{e_{j-1}}^i, e_z) - (e_y, e_{x<ty}, e_{e_{j-1}}^i, e_z) + (e_y, e_{x<ty}, e_{e_{j-1}}^i, e_z).

Lemma C.1. Let $X$ be as above. Then we have the equality $h_1 \circ \partial^R_2 - \partial^R_3 \circ h_2 = t \cdot c_2$.

Proof. Compute the two terms $h_1 \circ \partial^R_2(x, y)$ and $\partial^R_3 \circ h_2$ in the left side:

$$h_1 \circ \partial^R_2(x, y) = \sum e_x - (e_y, e_{e_{j-1}}^i),$$

$$\partial^R_3 \circ h_2(x, y) = \partial^R_3 \left(\sum (e_x, e_y, e_{e_{j-1}}^i) - (e_x, e_{e_j}^i, e_y) - (e_y, e_{x<ty}, e_{e_{j-1}}^i) + (e_y, e_{e_j}^i, e_y)\right)
= \left(\sum (e_y, e_{e_{j-1}}^i) - (e_x e_y, e_{e_{j-1}}^i) + (e_x e_y, e_{e_{j-1}}^i) - (e_x, e_y) - (e_x, e_y) - (e_x, e_y) + (e_x, e_y) + (e_x, e_y) - (e_x, e_y)\right)
= t \left((e_y, e_{x<ty}) - (e_x, e_y)\right) + (e_x, e_y) - (e_y, e_{e_{j-1}}^i) - (e_y, e_{e_{j-1}}^i) + (e_y, e_{e_{j-1}}^i) + h_1 \circ \partial^R_2(x, y)
= -t \cdot c_2(x, y) + h_1 \circ \partial^R_2(x, y).$$

Here we use Lemma 4.1 for the last equality. Hence, the equalities complete the proof.

Lemma C.2. Let $X$ be a connected quandle of type $t$. The difference $h_2 \circ \partial^R_3 - \partial^R_4 \circ h_3$ is chain homotopic to $t \cdot c_3$.

Proof. This is similarly proved by a direct calculation. For this end, recalling the notation $A = (x<ty)z$, we remark two identities

$$e_x e_A = e_{x<ty}e_z, \quad e_y e_A = e_{x<ty}e_{y<zy} \in \text{As}(X).$$

Using them, a tedious calculation can show that the difference $(t \cdot c_3 - h_2 \circ \partial^R_3 - \partial^R_4 \circ h_3)(x, y, z)$ is equal to

$$(e_y, e_z, e_{e_A}^i) - (e_{e_A}^i, e_y, e_z) + (e_{e_A}^i, e_z, e_{e<zy}) - (e_y, e_{e_A}^i)
++(e_z, e_{e_A}^i, e_{x<zy}) - (e_z, e_{e<zy}, e_{e_A}^i) + \sum_{1 \leq j \leq t-1}(e_y, e_{e_A}^i, e_y) - (e_y, e_{e_A}^i, e_{e<zy}).$$

Note that this formula is independent of any $x \in X$ since the identity $(e_a)^t = (e_b)^t$ holds for any $a, b \in X$ by Lemma 4.1. However, the map $c_3(x, y, z)$ with $x = y$ is zero by definition. Hence, the $t \cdot c_3$ is null-homotopic as desired.
Proof of Theorem 6.7. The $t \cdot c_*$ are null-homotopic immediately by Lemmas C.1 and C.2.

The proof was a brute computation in an algebraic way; However this should be easily shown by a topological method:

Problem C.3. Does the $t$-vanishing of the map $c_* : H_n(BX) \to H_n^\text{gr}(\text{As}(X))$ hold for any $n \in \mathbb{N}$? Provide its topological proof.

D Proof of Proposition 2.4

Proof of Proposition 2.4. We first construct an $X$-coloring from any element $(x_1, \ldots, x_{\#L}, f)$ in $\mathcal{N}$. Let us denote by $\gamma_i$ the oriented arc associated to the meridian $m_i$, and color the $\gamma_i$ by the $x_i \in X$. For each $i$, we consider the path $\mathcal{P}_i$ along the longitude $l_i$ as illustrated in the figure below. Furthermore, let $\alpha_0(= \gamma_i), \alpha_1, \ldots, \alpha_{N_i-1}, \alpha_{N_i} = \alpha_0$ be the over-paths on this $\mathcal{P}_i$, and let $\beta_k \in \pi_1(S^3 \setminus L)$ be the meridian corresponding to the arc that divides the arcs $\alpha_{k-1}$ and $\alpha_k$. Then we define a map $\mathcal{C} : \{\text{over arcs of } D \} \to X$ by the formula

$$\mathcal{C}(\alpha_k) := x_i \cdot (f(\beta_1) \cdots f(\beta_{k-1})) \in X.$$  

Here $\epsilon_j \in \{\pm 1\}$ is the sigh of the crossing of $\alpha_j$ and $\beta_j$. Note $\mathcal{C}(\alpha_{N_i}) = \mathcal{C}(\alpha_0) = x_i$ since the longitude $l_i$ equals the product $\beta_1^j \beta_2^j \cdots \beta_{N_i}^{-j}$ by definition. Here we claim that this $\mathcal{C}$ is an $X$-coloring. For this purpose, using (2), we notice equalities

$$e_{\mathcal{C}(\alpha_k)} = e_{x_i \cdot f(\beta_1^j) \cdots f(\beta_{k-1}^j)}$$

$$= (f(\beta_1^j) \cdots f(\beta_{k-1}^j))^{-1} f(m_i) (f(\beta_1^j) \cdots f(\beta_{k-1}^j)) = f(\alpha_k) \in \text{As}(X). \quad (41)$$

Hence, with respect to the crossing between $\alpha_k$ and $\beta_k$ with $k \leq N_i$, we have the following equality in $X$:

$$\mathcal{C}(\alpha_k) = \mathcal{C}(\beta_k) = \mathcal{C}(\alpha_k) \cdot e_{\mathcal{C}(\beta_k)} = \mathcal{C}(\alpha_k) \cdot f(\beta_k^j) = x_i \cdot (f(\beta_1^j) \cdots f(\beta_{k-1}^j)) \cdot f(\beta_k^j) = \mathcal{C}(\alpha_{k+1}).$$

This equality means that this $\mathcal{C}$ turns out to be an $X$-coloring as desired. Here note the equality $f = \Gamma_\mathcal{C}$ for such an $X$-coloring $\mathcal{C}$ coming from the $f$ (see the previous (41)).

To summarize, we obtain a map from the set $\mathcal{N}$ to the Col$_X(D)$ which carries such $(x_1, \ldots, x_{\#L}, f)$ to the above $\mathcal{C}$. Moreover, by construction, it is the desired inverse mapping of the $\Gamma_\bullet$, which proves the desired bijectivity.

As a result, we further show a reduction for some quandles.
Corollary D.1. Let $X$ be a quandle such that the map $X \to \text{As}(X)$ sending $x$ to $e_x$ is injective. Let $D$ be a diagram of an oriented link $L$. We fix a meridian $m_i \in \pi_1(S^3 \setminus L)$ of each link-component. Then, the set of $X$-colorings of $D$ is bijective to the following set:

\[
\{ f \in \text{Hom}_{\text{gr}}(\pi_1(S^3 \setminus K), \text{As}(X)) \mid f(m_i) = e_{x_i} \text{ for some } x_i \in X \}.
\]

(42)

Proof. By Proposition 2.4, it is enough to show that such an $f$ in (42) satisfies the equality $x_i \cdot f(l_i) = x_i$. Actually, noting $e_{x_i} = f(m_i) = f(l_i)^{-1} f(m_i) f(l_i) = f(l_i)^{-1} e_{x_i} f(l_i) = e_{x_i \cdot f(l_i)}$ by (2), the injectivity $X \hookrightarrow \text{As}(X)$ concludes the desired $x_i \cdot f(l_i) = x_i$.

We remark that every quandle in Example 2.1, 2.2, 2.3 satisfies the injectivity of $X \hookrightarrow \text{As}(X)$.

Acknowledgment

The author expresses his gratitude to Tomotada Ohtsuki and Seiichi Kamada for comments and advice.

References

[AG] N. Andruskiewitsch, M. Graña. From racks to pointed Hopf algebras, Adv. Math., 178 (2003), 177–243.

[AP] D. Arlettaz, N. Pointet-Tischler, Postnikov invariants of H-spaces, Fund. Math. 161 (1-2) (1999), 17–35.

[Bro] K. S. Brown, Cohomology of Groups, Graduate Texts in Mathematics, 87, Springer-Verlag, New York, 1994.

[CEGS] J. S. Carter, J. S. Elhamdadi, M. Graña, M. Saito, Cocycle knot invariants from quandle modules and generalized quandle homology, Osaka J. Math. 42 (2005), 499–541.

[CF] P. E. Conner, E. N. Floyd, Differentiable Periodic Maps, Ergeb. Math. Grenzgeb. N. F., vol 33, Springer-Verlag, Berlin, Göttingen, Heidelberg 1964.

[CJKLS] J. S. Carter, D. Jelsovsky, S. Kamada, L. Langford, M. Saito, Quandle cohomology and state-sum invariants of knotted curves and surfaces, Trans. Amer. Math. Soc. 355 (2003), 3947–3989.

[CKS] J. S. Carter, S. Kamada, M. Saito, Geometric interpretations of quandle homology, J. Knot Theory Ramifications 10 (2001), 345–386.

[Cla1] F.J.-B.J. Clauwens, The algebra of rack and quandle cohomology, J. Knot Theory Ramifications 11 (2011), 1487–1535.

[Cla2] The adjoint group of an Alexander quandle, arXiv:math/1011.1587

[Cla3] Small simple quandles, arXiv:math/1011.2456

[DW] R. Dijkgraaf, E. Witten, Topological gauge theories and group cohomology, Comm. Math. Phys. 129 (1990), 393–429.

[E1] M. Eisermann, Knot colouring polynomials, Pacific Journal of Mathematics 231 (2007), 305–336.

[E2] Quandle coverings and their Galois correspondence, arXiv:math/0612459

[Eb] J. F. Ebert, The icosahedral group and the homotopy of the stable mapping class group, Münster Journal of Mathematics 3 (2010), 221–232.

[EGS] P. Etingof, R. Guralnick, A. Soloviev, Indecomposable set-theoretical solutions to the quantum Yang-Baxter equation on a set with a prime number of elements, J. Algebra 242 (2001) 709–719.
[FM] B. Farb, D. Margalit, *A primer on mapping class groups*, PMS 50, Princeton University Press, 2011.

[FRS1] R. Fenn, C. Rourke, B. Sanderson, *Trunks and classifying spaces*, Appl. Categ. Structures 3 (1995) 321–356.

[FRS2] ______, *The rack space*, Trans. Amer. Math. Soc. 359 (2007), no. 2, 701–740.

[FP] Z. Fiedorowicz, S. Priddy, *Homology of classical groups over finite fields and their associated infinite loop spaces*, Lecture Notes in Mathematics, 674, Springer, Berlin, 1978.

[Fri] E. M. Friedlander, *Computations of K-theories of finite fields*, Topology 15 (1976), no. 1, 87–109.

[Ger] S. Gervais, *Presentation and central extension of mapping class groups*, Trans. Amer. Math. Soc. 348 (1996) 3097–3132.

[HN] E. Hatakenaka, T. Nosaka, *Some topological aspects of 4-fold symmetric quandle invariants of 3-manifolds*, to appear Internat. J. Math.

[Iva] N. V. Ivanov, *Stabilization of the homology of Teichmüller modular groups*, Algebra i Analiz 1 (1989), 110–126 (Russian); translation in Leningrad Math. J. 1 (1990) 675–691.

[ILDT] T. Ohtsuki (ed), *Problems on Low-dimensional Topology*, in a conference “Intelligence of Low-dimensional Topology” 2010, Kokyuroku RIMS 1716, 119–128.

[Joy] D. Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra 23 (1982) 37–65.

[Kab] Y. Kabaya, *Cyclic branched coverings of knots and quandle homology*, Pacific Journal of Mathematics, 259 (2012), No. 2, 315–347.

[LN] R. Litherland, S. Nelson, *The Betti numbers of some finite racks*, J. Pure Appl. Algebra 178 (2003), no. 2, 187–202.

[McC] J. McCleary, *A user’s guide to spectral sequences*, Second edition. Cambridge Studies in Advanced Mathematics, 58. Cambridge University Press, Cambridge, 2001.

[Moc] T. Mochizuki, *The 3-cocycles of the Alexander quandles \( \mathbb{F}_q[T]/(T-\omega) \)*, Algebraic and Geometric Topology, 5 (2005), 183–205.

[MT] I. Madsen, U. Tillmann, *The stable mapping class group and \( Q(\mathbb{C}P_\infty^+) \)*, Invent. Math. 145 (2001), no. 3, 509–544

[N1] T. Nosaka, *On homotopy groups of quandle spaces and the quandle homotopy invariant of links*, Topology and its Applications 158 (2011), 996–1011.

[N2] ______, *Quandle homotopy invariants of knotted surfaces*, to appear Mathematische Zeitschrift, available at arXiv:math/1011.6035v2.

[N3] ______, *Quandle cocycle invariants of Lefschetz fibrations over the 2-sphere*, preprint

[N4] ______, *On third homologies of groups and of quandles via the Dijkgraaf-Witten invariant and Inoue-Kabaya map*, preprint

[PR] L. Paris, D. Rolfsen, *Geometric subgroups of mapping class groups*, J. Reine Angew. Math. 521 (2000), 47–83.

[Ros] J. Rosenberg, *Algebraic K-theory and its applications*. Graduate Texts in Mathematics, 147. Springer-Verlag, New York, (1994)

[Rol] D. Rolfsen, *Knots and links*, Math. Lecture Series, 7, Publish or Perish, Inc., Houston, Texas, 1990, Second printing, with corrections.

[RS] C. Rourke, B. Sanderson, *A new classification of links and some calculation using it*, arXiv:math.GT/0006062
[RW] O. Randal-Williams, *Resolutions of moduli spaces and homological stability*, arXiv:0909.4278

[Sou] C. Soulé, *Opérations en K-théorie algébrique*, Canad. J. Math. 37 (1985) 488–550.

[Sus] A. A. Suslin, *Torsion in $K_2$ of fields*, K-Theory, 1(1): 5–29, 1987.

[Y] D. N. Yetter, *Quandles and monodromy*, J. Knot Theory Ramifications 12 (2003), 523–541.

[Zab] J. Zablow, *On relations and homology of the Dehn quandle*, Algebr. Geom. Topol. 8 (2008), no. 1, 19–51.

Faculty of Mathematics, Kyushu University, 744, Motooka, Nishi-ku, Fukuoka, 819-0395, Japan

E-mail address: nosaka@math.kyushu-u.ac.jp