A physics perspective on geometric Langlands duality

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Abstract

We review the approach to the geometric Langlands program for algebraic curves via S-duality of an $\mathcal{N}=4$ supersymmetric four dimensional gauge theory, initiated by Kapustin and Witten in 2006. We sketch some of the central further developments. Placing this four dimensional gauge theory into a six dimensional framework, as advocated by Witten, holds the promise to lead to a formulation which makes geometric Langlands duality a manifest symmetry (like covariance in differential geometry). Furthermore, it leads to an approach toward geometric Langlands duality for algebraic surfaces, reproducing and extending the recent results of Braverman and Finkelberg.

1 Introduction

In April 2006 Kapustin and Witten published their pathbreaking work [KW] which led to a completely new perspective on geometric Langlands duality for algebraic curves. It starts from a four dimensional $N = 4$ supersymmetric gauge theory. Assuming that $S$-duality, a certain symmetry which generalizes the electric-magnetic duality of the Maxwell equations to the case of a nonabelian gauge theory, holds for this theory, it is possible to derive geometric Langlands duality for algebraic curves from this. $S$-duality is conjectural
but very well supported. In this sense, we get a reformulation of geometric Langlands duality for algebraic curves.

In the first part of this contribution, we will review the approach of Kapustin and Witten. We will continue by very briefly sketching some of the new developments which this approach has initiated. Finally, we will place the four dimensional gauge theory in a six dimensional string theory framework. This perspective (see [Wit 2007b], [Wit 2009b]) holds the promise to lead to a formulation, making geometric Langlands duality a manifest symmetry (like covariance is manifest in differential geometry and has no longer to be verified by calculations on specific coordinate transformations). The six dimensional view also leads to an approach toward geometric Langlands duality for algebraic surfaces, reproducing and extending the recent results of Braverman and Finkelberg (see [BF], see also [Nak]).

This article is intended as an introduction to the gauge and string theory approach to the geometric Langlands program for mathematicians. As such, it focuses on a short, non-technical, overview of the central ideas and concepts and does not contain any original research results. Neither do we pretend to give a complete overview of this rapidly developing and highly promising field. To keep the article in a sufficiently focused form, some exciting developments (e.g. the appearance of Arthur’s $SL(2)$ in this framework, see [BN] and [Wit 2009a]) will completely be left out. For another recently published review on the topic, see [Fre 2009].

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\section{N=4 supersymmetric gauge theory}

Let $G$ be a compact Lie group. For simplicity (and to keep all the formulae valid in precisely the form used here, without any extra factors), we will assume that $G$ is from the $ADE$-series. Later, for the six dimensional framework, this assumption will be essential and no longer a technical assumption for simplicity.

Let $X_4$ be the four dimensional space-time (again, we make a technical assumption for simplicity, assuming that the signature of $X_4$ is Euclidean.
rather than Minkowskian). We use Greek indices, e.g
\[ \mu = 0, 1, 2, 3 \]
to label space-time indices and Latin ones
\[ i = 1, \ldots, 6 \]
for an internal set of indices. Let \( A \) be a connection of a \( G \)-bundle over \( X_4 \) and \( F \) the corresponding curvature form. With
\[ D_\mu \]
we denote the covariant derivatives and with
\[ \phi_i \]
a set of adjoint-valued scalar fields (i.e. functions valued in the adjoint representation of the Lie algebra of \( G \)). The action of the \( N = 4 \) supersymmetric gauge theory which we want to consider is then given by
\[
S_4 = \frac{1}{e^2} \int d^4x \mathrm{Tr} \left( \frac{1}{2} \sum_{\mu, \nu}^3 F_{\mu \nu} F^{\mu \nu} + \sum_{\mu=0}^3 \sum_{i=1}^6 D_\mu \phi_i D^\mu \phi_i + \frac{1}{2} \sum_{i,j=1}^6 [\phi_i, \phi_j]^2 \right) + \ldots
\]
where the dots indicate the fermionic part of the action and \( e \) the coupling constant of the theory (just as Newton’s constant in gravity). The fermionic part is necessary for supersymmetry but we will not consider it, here. Though we can explain the essential ideas without considering it explicitly, one should nevertheless keep in mind that the whole construction does only hold with \( N = 4 \) supersymmetry implemented. Here, \( N = 4 \) denotes the degree of supersymmetry. The supersymmetry algebra is determined by a choice of representation of the – in this case four dimensional – spin group, i.e. the double cover of the Lorentz group. The simplest degree of supersymmetry is denoted by \( N = 1 \) while \( N = 4 \) means (very roughly speaking) that we have four copies of the simplest representation involved in the definition of the supersymmetry algebra.

If one has higher than \( N = 1 \) supersymmetry, it is generally possible to derive the theory from an \( N = 1 \) supersymmetric theory, living in a higher
dimensional space, by dimensional reduction. In this case, the higher dimensional theory is \( N = 1 \) supersymmetric gauge theory in ten dimensions. Dimensional reduction (which will appear over and over again in this article) means that one assumes some of the dimensions (in this case six) to be scrolled up to a compact space of very small volume (in this case, six dimensions are scrolled up to small circles). Sending the volume to zero (i.e. sending the radii of the circles to zero) results in an induced lower dimensional theory. The fact that we can get the action \( S_4 \) from ten dimensions by reducing on six circles is the reason for the appearance of the adjoint-valued scalar fields and the internal indices \( i = 1, ..., 6 \). Indeed, the first term in \( S_4 \) is the well known gauge theory term while the other two arise from the dimensional reduction of the corresponding gauge theory term in ten dimensions.

To get the most general \( N = 4 \) supersymmetric gauge theory in four dimensions, it is possible to add the so called topological term \( S_\theta \) to \( S_4 \). This is given by

\[
S_\theta = -\frac{\theta}{8\pi^2} \int d^4x \; Tr (F \wedge F)
\]

It is referred to as topological since the integral just gives the second Chern class of the \( G \)-bundle. The coupling constant \( e \) of \( S_4 \) and the parameter \( \theta \) of the topological term are combined into the complex coupling constant

\[
\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}
\]

Observe that the imaginary part of \( \tau \) is always positive, i.e. \( \tau \) is from the upper half plane \( \mathbb{H} \).

3  S-duality

In the quantized theory there is a natural symmetry given by the generator

\[
T: \tau \mapsto \tau + 1
\]

which results from the fact that – roughly speaking – the complex coupling constant appears only in the form

\[
e^{2\pi i \tau}
\]
in the path integral. On the other hand, there is a natural action of $SL(2, \mathbb{R})$ on $\mathbb{H}$.

The $S$-duality conjecture states the following:

There exists a second symmetry (i.e. generator) $S$, generating together with $T$ a discrete subgroup of $SL(2, \mathbb{R})$ such that $S_\theta + S_\vartheta$ is invariant under the following combination of operations:

- $\tau \mapsto S(\tau)$
- exchange of electric and magnetic charges
- $G \mapsto L_G$

The latter two operations are not unrelated: In 1977 Goddard, Nuyts, and Olive investigated how electric and magnetic charges are classified in a nonabelian gauge theory (see [GNO]). The result is that one set of charges is given by the weight lattice of the Lie algebra of the gauge group $G$ while the other one is given by the root lattice. Of course, exchanging weight and root lattice is precisely what defines the Langlands dual $L_G$ of the group $G$.

In the same year Montonen and Olive presented $S$-duality as a conjectural symmetry for nonabelian gauge theory (see [MO]). While $S$-duality does not hold in the non-supersymmetric case, there is strong evidence that it holds with $N = 4$ supersymmetry.

For $G$ from the $ADE$-series (as we do assume), the generator $S$ has to take the form

$$S : \tau \mapsto -\frac{1}{\tau}$$

i.e. the discrete subgroup of $SL(2, \mathbb{R})$ generated by $T$ and $S$ is the modular group $SL(2, \mathbb{Z})$.

In order to derive the geometric Langlands duality for algebraic curves from the $S$-duality conjecture, one has to perform two essential steps on the four dimensional gauge theory: First, one has to perform a topological twist and than a dimensional reduction to a two dimensional theory. We will discuss both steps very briefly, in a non-technical manner, in the next two sections. After that we will introduce the operators of the four dimensional gauge theory which – after performing the two steps on them – will lead to geometric Langlands duality. For technical details, we refer the reader to [KW].
4 Topological twisting

Topological twisting means that one retains only part of the state space of the original theory. For this one introduces the cohomology with respect to a certain differential $Q$ (what physicists call a BRST-operator) with

$$Q^2 = 0$$

To find a $Q$ suitable for the topological twist, supersymmetry is essential. Passing to the cohomology with respect to $Q$ – i.e. forgetting all $Q$-exact terms – one retains only part of the information of the original theory. The resulting theory is called topological. For a pure mathematician this nomenclature might be slightly disturbing since the theory is not independent of all non-topological information, e.g. we will see that after dimensional reduction it is still dependent on certain holomorphic and symplectic structures. Topological theory in this context means that on the $Q$-cohomology we have independence from the choice of metric.

Concretely, in this case $Q$ is determined by a choice of homomorphism

$$\chi : \text{Spin}(4) \to \text{Spin}(6)$$

which is related to the fact that the gauge theory arises from a ten dimensional theory by dimensional reduction and a decomposition of $\text{Spin}(10)$ into $\text{Spin}(4)$ and $\text{Spin}(6)$ components. The approach is very similar to the introduction of Donaldson invariants for four dimensional manifolds by using a topological twist for $N = 2$ supersymmetric four dimensional gauge theory.

It turns out that the topological twist is not determined uniquely but there arises a whole family of suitable topological twists, parametrized by the topological twisting parameter

$$t \in \mathbb{C}P^1$$

The complex coupling constant $\tau$ and the topological twisting parameter $t$ are then combined into the canonical parameter

$$\psi = \frac{\tau + \overline{\tau}}{2} + \frac{\tau - \overline{\tau}}{2} \left( \frac{t - t^{-1}}{t + t^{-1}} \right)$$

The reason for introducing the canonical parameter is that the correlation functions of the observables of the topological theory (i.e. of the $Q$-cohomology classes) do only depend on $\psi$ and not on the parameters $e$, $\theta$, and $t$ separately.
It is a small lemma to show that on $\psi$ the two generators $T$ and $S$ operate, again, as

$$T : \psi \mapsto \psi + 1$$

and

$$S : \psi \mapsto -\frac{1}{\psi}$$

### 5 Dimensional reduction

Let now

$$X_4 \cong \Sigma \times C$$

with $\Sigma$ a (compact or non-compact) Riemann-surface (which will become the two dimensional space-time after dimensional reduction) and $C$ a compact Riemann surface. As a technical assumption, we will require that

$$\text{genus} (C) \geq 2$$

We now perform the dimensional reduction by assuming that

$$\text{vol} (C) \ll \text{vol} (\Sigma)$$

In this limit, the four dimensional $N = 4$ supersymmetric gauge theory induces a two dimensional field theory on $\Sigma$. In two dimensions there do not exist non-trivial gauge theories and the resulting field theory turns out to be a nonlinear sigma model, i.e. a field theory where the (bosonic) fields are given by maps from $\Sigma$ to the so called target space. Roughly speaking, the action is given by a minimal area requirement for the image of $\Sigma$ under these maps into the target space. So, the essential information to determine the nonlinear sigma model is to specify the target space. One shows that in this case the resulting target space is the Hitchin moduli space $\text{Hit} (G, C)$ for the gauge group $G$ and the complex curve $C$ (see [Hit]).

Let $A$ be a $G$-connection on $C$ and $F$ the corresponding curvature form. Let $\phi$ be an adjoint-valued 1-form on $C$. Consider the set of equations

$$F - \phi \wedge \phi = 0$$
and

\[ D\phi = D^*\phi = 0 \]

The solutions to this set of equations, modulo \( G \)-gauge transformations, define the Hitchin moduli space \( \text{Hit}(G, C) \). For those readers with a knowledge of Higgs-bundles one can simply define it as the moduli space of \( G \)-Higgs-bundles on \( C \). That we have used the letter \( \phi \) for the adjoint-valued 1-form is not by accident. The topological twist shifts the degree of some fields and we get the adjoint-valued 1-form from the adjoint-valued scalar fields of the original gauge theory. With

\[ g = \text{genus}(C) \]

one has

\[ \dim_C \text{Hit}(G, C) = (2g - 2) \dim G \]

and \( \text{Hit}(G, C) \) is a Hyperkähler manifold. So, we have a representation of the quaternion algebra on the tangent bundle and complex structures \( I, J, K \) with corresponding symplectic structures \( \omega_I, \omega_J, \omega_K \). When we refer to complex or symplectic structures in the sequel, we will have to keep in mind that we have to make precise to which of these structures we refer.

## 6 Wilson operators

We are now ready to introduce the needed operators in the four dimensional gauge theory. Usual operators in a quantum field theory, as you remember them from any introductory course on the subject, are attached to points (i.e. they are zero dimensional objects): They are operators \( M(x), M(z) \) attached to points \( x, z \) and satisfying the well known commutation relations (e.g. \( M(x) \) and \( M(z) \) commute if \( x \) and \( z \) are space-like separated). Physicists have learned in recent decades that there are other operators, attached to lines (one dimensional objects), containing essential information in a quantum field theory (in solid state physics or in the study of phase transitions these are prominent operators).

Recall that \( A \) is a connection on a \( G \)-bundle over \( X_4 \). Let \( S \) be an oriented loop in \( X_4 \), \( R \) an irreducible representation of \( G \). With \( Tr_R \) we denote taking the trace in the representation \( R \). We define the Wilson operator \( W_0(R, S) \)
as the holonomy of $A$ around $S$:

$$W_0(R, S) = Tr_R \exp \left( - \oint_S A \right)$$

Since we want to perform the two steps, topological twisting and dimensional reduction, on the operators, the next question is if these operators induce well defined operators on $Q$-cohomology. Unfortunately, the answer is no and this problem can not be resolved for general values of the topological twisting parameter $t$. But for the special values $t = i$ and $t = -i$ there exists a solution: For these values there exists a linear combination of $A$ with the adjoint-valued 1-form $\phi$, such that the holonomy of the linear combination induces a well defined operator on cohomology, i.e. we have topological Wilson operators. Concretely, the topological Wilson operators are defined by

$$W(R, S) = Tr_R \exp \left( - \oint_S A + i\phi \right)$$

for $t = i$ and

$$W(R, S) = Tr_R \exp \left( - \oint_S A - i\phi \right)$$

for $t = -i$.

Next, replace the loop $S$ with a line $L$ from $p$ to $q$. Replace the trace $Tr_R$ with the matrix of parallel transport from the fiber $E_p$ of the $G$-bundle on $X_4$ to the fiber $E_q$, with both fibers considered in the representation $R$ of $G$. The parallel transport is taken with respect to the connection

$$\mathcal{A} = A + i\phi$$

and

$$\overline{\mathcal{A}} = A - i\phi$$

for $t = \pm i$, respectively. This corresponds to the canonical parameter $\psi = \infty$. In conclusion, for $\psi = \infty$ we have topological Wilson operators, defined by representations $R$ of $G$.

Assume, now, that $S$-duality holds for the $N = 4$ supersymmetric gauge theory. This means that there has to exist a second set of topological line operators which exchange with the Wilson operators on lines under $S$-duality, i.e. for $\psi = 0$ there should exist topological line operators, defined by representations of $L^*G$. Indeed, these operators can be constructed in the form of
the so called ’t Hooft operators which we are not going to discuss explicitly, here.

Finally, we perform the dimensional reduction on the topological line operators. Consider a two dimensional theory and a line operator $\hat{L}$ on a line $L$, close to a boundary with specified boundary condition (i.e. what physicists call a $D$-brane for a two dimensional nonlinear sigma model). Imagine $L$ approaching the boundary more and more closely. In the limit, $L$ will be absorbed by the boundary and the operator $\hat{L}$ disappears, resulting in a change of boundary conditions. Of course, this is a heuristic picture but it can be validated in a calculation. The boundary condition is given by a submanifold ($D$-brane) of the target space to which the one dimensional boundary of $\Sigma$ has to be mapped under the fields, together with a vector bundle $W$ on this submanifold. One can show the operator $\hat{L}$ to change boundary conditions by changing this vector bundle $W$. So, we can view the line operators in the two dimensional theory as abstract operators, operating on boundary conditions.

We call a boundary condition, given by $W$, an eigenbrane of $\hat{L}$ if there exists a fixed vector space $V$ such that $\hat{L}$ acts as

$$\hat{L} : W \mapsto V \otimes W$$

This is similar to eigenfunctions for operators in quantum mechanics, with the function replaced by a vector bundle and the eigenvalue replaced by the fixed vector space $V$. As in quantum mechanics, we can pose the question if line operators $\hat{L}_1$ and $\hat{L}_2$ on lines $L_1$ and $L_2$ can have simultaneous eigenbranes. The answer is that they have simultaneous eigenbranes iff

$$[\hat{L}_1, \hat{L}_2] = 0$$

For the dimensional reduction of the topologically twisted $N = 4$ supersymmetric gauge theory, one can show that there exist simultaneous eigenbranes of all topological Wilson operators. These eigenbranes are called electric eigenbranes. Similarly, there exist simultaneous eigenbranes of all topological ’t Hooft operators and these are called magnetic eigenbranes.
7 Mirror symmetry

Without defining the three complex structures $I$, $J$, $K$ (and corresponding symplectic structures) explicitly for $Hit(G, C)$ (see [KW]), we recall that we have to keep them apart when referring to a complex or a symplectic structure. For a nonlinear sigma model on a Ricci flat Kähler manifold there exist two types of topological twists, called the $A$- and the $B$-model (see [Wit 1991]). The $A$-model couples only to the symplectic structure of the target space and the $B$-model only to the holomorphic structure. One proves that

*Electric eigenbranes are elements of the bounded derived category of coherent sheaves in complex structure $J$ on $Hit(G, C)$.*

The elements of the bounded derived category of coherent sheaves are the $D$-branes for the $B$-model, referred to as $B$-branes in the physics literature. Including reference to the complex structure $J$, they are called $J_B$-branes.

Similarly, one shows that

*Magnetic eigenbranes are elements of the Fukaya category in symplectic structure $\omega_K$. *

In physics terminology, this means magnetic eigenbranes are $K_A$-branes. Mirror symmetry exchanges the $A$- and the $B$-model. One mathematically rigorous formulation of mirror symmetry, called *homological mirror symmetry* (see [Kon 1994]) states that Calabi-Yau manifolds $X$ and $Y$ form a mirror pair if there is a suitable equivalence between the Fukaya category of $X$ and the bounded derived category of coherent sheaves of $Y$ and vice versa (to make this technically precise, one does not really work with simple categories but with a triangulated version of $A_{\infty}$-categories). $S$-duality of the four dimensional gauge theory induces homological mirror symmetry for the $Hit(G, C)$ sigma model on $\Sigma$ or in more physics oriented language, $S$-duality induces mirror symmetry between the $B$-model on $Hit(G, C)_J$ (corresponding to $\psi = \infty$) and the $A$-model on $Hit(LG, C)_K$ (corresponding to $\psi = 0$). Here, the subscripts refer to the complex, respectively symplectic structure, used on Hitchin moduli space.

In the geometric Langlands program for algebraic curves $C$ one considers two different moduli spaces: The moduli space $\mathcal{M}$ of flat $LG_C$-bundles on $C$ and the moduli space $\tilde{\mathcal{M}}$ of holomorphic $G$-bundles on $C$. On $\mathcal{M}$ one considers sheaves with support at a point of $\mathcal{M}$ (skyscraper sheaves) and
on \( \tilde{M} \) one considers the so called Hecke eigensheaves. It is a central part of \( \text{[KW]} \) to show that the skyscrapers are in one-to-one correspondence to the electric eigenbranes and the Hecke eigensheaves to the magnetic eigenbranes. In consequence, if \( S \)-duality holds for the four dimensional \( N = 4 \) supersymmetric gauge theory, one can derive geometric Langlands duality for algebraic curves.

At this point, the attentive reader might ask why one needs \( S \)-duality of the four dimensional gauge theory for this and why one does not start directly from the homological mirror symmetry conjecture for the \( Hit (G, C) \) sigma model. The answer is that mathematicians no very well that \( \tilde{M} \) is not a true moduli space (and can not be for geometric Langlands duality to hold true) but a stack. The nonlinear sigma model treats the target space in a first approach as a proper space. If one takes the stacky nature into account, one rediscovers that one actually derived the model from the four dimensional gauge theory, i.e. the four dimensional viewpoint is essential for the geometric Langlands program (see \( \text{[KW]} \)).

There are further examples for the deep interplay between the structures, naturally emerging from physics, and those needed for the mathematics of the geometric Langlands program, in this approach. E.g. the Fukaya category as it is originally defined (see \( \text{[Fuk]} \), see \( \text{[FOOO]} \) for an approach to a rigorous treatment), involves the Lagrangian submanifolds of \( Hit (G, C) \). But there exist additional \( A \)-branes on \( Hit (G, C) \) which are only coisotropic submanifolds. A special such \( A \)-brane (called the canonical coisotropic brane or c.c. brane, for short), corresponding to a coisotropic submanifold of full dimensionality (i.e. isomorphic to \( Hit (G, C) \) itself) and of rank one (i.e. the vector bundle \( W \) on the brane is a line bundle) is used in \( \text{[KW]} \) to show that the magnetic eigenbranes satisfy the \( D \)-module property which is so important for the Hecke eigensheaves in the geometric Langlands program.

Finally, there exists another physics motivated approach to the geometric Langlands program for algebraic curves, using two dimensional conformal field theory on \( C \) to construct Hecke eigensheaves (see \( \text{[Fre 2005]} \) for a beautiful review and the original literature). One might wonder how the two approaches are related, one leading to a two dimensional nonlinear sigma model on \( \Sigma \), the other to a conformal field theory on \( C \). It would be possible to derive the conformal field theory approach on \( C \) also from the four dimensional gauge theory if one could prove the following property to hold: The dual brane (under \( S \)-duality, respectively mirror symmetry) of the c.c.
brane should be a brane which has support on the space of opers of [BD] ([Wit]). The dual of the c.c. brane is a coisotropic brane of rank > 1 and is calculated in the gauge theory setting in [GaW 2008b].

After this review of some of the central parts of [KW], we are now ready to take a brief look on some of the further developments in 2006-2009.

8 Higher dimensional operators

As we have seen, beyond the usual zero dimensional operators (attached to points) there are one dimensional line operators in a quantum field theory, containing fundamental information. One might ask if there are further even higher dimensional operators. In a four dimensional theory, these could be two dimensional (attached to surfaces) or three dimensional (attached to volumes). Four dimensional operators would be trivial.

Two dimensional operators become important if the gauge connection \( A \) has singularities. So far, we have assumed \( A \) to be holomorphic but one can allow \( A \) to be meromorphic, only, and to have singularities along surfaces. The approach of [KW] can be extended to this case and surface operators take a central place, then. When Beilinson and Drinfeld developed the geometric Langlands program, it was intended as an analogue to the classical Langlands program, to get insights from a situation with additional smoothness property. The case of a meromorphic gauge connection \( A \) corresponds to what is called ramification in the classical Langlands program. If \( A \) has only simple poles, one has tame ramification, otherwise one has the case of wild ramification (see [GuW 2006], [GuW 2008], [Wit 2007a]; see [Fre 2005] for a discussion how structures in the classical and the geometric Langlands program are analogous). Especially, understanding wild ramification in the geometric case is believed to be important for comparison to the classical Langlands program.

Three dimensional operators live on volumes and therefore divide the four dimensional manifold \( X_4 \) into two halves. They correspond to what physicists call domain walls in a gauge theory. Domain walls allow to change the gauge group. On the one side, we have the gauge theory with gauge group \( G \) and on the other side the theory with gauge group \( \tilde{G} \). On the domain wall we have the three dimensional operator, corresponding to specifying a boundary
condition which ensures that the two gauge theories join consistently along the domain wall.

In the classical Langlands program, beyond Langlands duality, changing the group \( G \) is a central ingredient, giving rise to Langlands functoriality. It was an open question – again of tremendous importance for comparing the geometric to the classical case – what constitutes the counterpart of Langlands functoriality in the geometric Langlands program. Domain walls lead to geometric Langlands functoriality. This is a subject very much in its beginning. From the gauge theory side one has to get knowledge on the three dimensional boundary conditions which involve data given in the form of three dimensional quantum field theories (see [FW], [GaW 2008a], [GaW 2008b], [Wit 2009b]).

In conclusion, higher dimensional operators on surfaces and volumes have turned out to be very important for studying analogues of structures which are central for the classical Langlands program.

9 The six dimensional view

Remember that our four dimensional gauge theory lives on \( X_4 \). There is a conjecture, arising from string theory, which states that there exists a conformally invariant field theory on

\[ X_4 \times T^2 \]

such that in the limit of small \( T^2 \) (dimensional reduction), it induces precisely the \( N = 4 \) supersymmetric gauge theory on \( X_4 \).

On \( T^2 \) there is, of course, the natural action of \( SL(2,\mathbb{Z}) \). We can ask what compensates this action on \( X_4 \). It turns out that in this way the \( SL(2,\mathbb{Z}) \)-action on \( T^2 \) induces S-duality of the gauge theory on \( X_4 \). In consequence, if it would be possible to construct this six dimensional conformal field theory, one could prove S-duality for the \( N = 4 \) supersymmetric gauge theory on \( X_4 \) and, in consequence, geometric Langlands duality. One should stress that existence of the theory suffices: While for the four dimensional gauge theory one has to prove something (S-duality) to get geometric Langlands duality, for the six dimensional conformal field theory one only has to construct the theory since its very existence makes S-duality of the four dimensional theory (and, hence, geometric Langlands duality) manifest. In this sense, one can
view the search for this six dimensional theory as the search for the geometry behind the Langlands program, making Langlands duality a manifest symmetry. This would be very much like passing from a coordinate description to differential geometry where covariance becomes a manifest symmetry.

The problem is that it is known from string theory that this six dimensional theory can not exist consistently on its own. It actually has to be embedded into eleven dimensional $M$-theory as the world-volume theory of the $M5$-brane (a five dimensional extended object with a six dimensional world volume in $M$-theory, the central charges, leading to the $M5$-brane, arising as one of the components in the direct sum decomposition of the eleven dimensional supersymmetry algebra). So, its completion in the UV-regime is related to the so called six dimensional micro string theories (see [Dij 1998], [DVV] for an easily accessible introduction).

Consider the six dimensional theory on another manifold $X_6$, now,

$$X_6 \cong \Sigma \times X_4$$

with $\Sigma$ a (compact or non-compact) Riemann surface, $X_4$ a compact Hyperkähler manifold, and

$$\text{vol}(X_4) \ll \text{vol}(\Sigma)$$

This is very similar to the situation we considered when reducing the four dimensional gauge theory to a two dimensional nonlinear sigma model. This time we get the reduction of the six dimensional theory to a two dimensional nonlinear sigma model and the target space turns out to be given by the instanton moduli space $\text{Inst}(X_4)$ on $X_4$, i.e. the space of all anti-self-dual $G$-connections on $X_4$ (remember that now, for the six dimensional view, $G$ definitely has to be from the $ADE$-series).

From the side of physics, there are some very nice relations behind this model. The Hitchin-Kobayashi correspondence relates $\text{Inst}(X_4)$ to $\text{Bun}_G(X_4)$, i.e. brings in a relation to Yang-Mills theory on $X_4$. On the other hand, for

$$G = U(k)$$

the space $X_4$ turns out to be related to the multi-center Taub-NUT solution $TN_k$ of the Einstein vacuum equations. This gives particular interest to the study of instantons on $TN_k$ (see [Che 2008], [Che 2009], [Wit 2009a]).

On the mathematical side, this model reproduces and extends – beyond the case $G = U(k)$ the results of Braverman and Finkelberg (see [BF],
on geometric Langlands duality for algebraic surfaces $X_4$ (see \cite{Tan}, \cite{WT2009b}). Let us review this in a little bit more detail (basically following \cite{WT2009b}).

One can show that the six dimensional theory can not have a Lagrangian description, it is a purely quantum field theoretic object. But dimensionally reducing the theory for small $S^1$ on

$$X_6 \cong X_5 \times S^1$$

one gets in the infrared limit a gauge theory description on $X_5$. One can now pass to the more complicated case with $X_6$ not being given as a Cartesian product, as above, but as a $U(1)$-bundle over $X_5$, i.e. we have a free action of $U(1)$ on $X_6$. This leads to an additional Chern-Simons like term in the dimensional reduction to $X_5$. Finally, one can pass to the case of a non-free action of $U(1)$ on $X_6$ and consider the singular quotient space $X_6/ U(1)$. Outside the non-free locus the dimensional reduction works as in the previous case. Consider the special case where the non-free locus has codimension four and consists only of fixed points of $U(1)$. In this case, the Chern-Simons term has an anomaly on the two dimensional non-free locus $W$, i.e. on $W$ a third term has to appear in the action of the dimensional reduction to $X_5$ which cancels this anomaly. This third term arises from a two dimensional quantum field theory on $W$, given by the holomorphic part of the $WZW$-model (at level one and for the group $G$). The affine Lie algebra of $G$, which mathematically is behind the $WZW$-model, naturally explains why the approach to the geometric Langlands program for algebraic surfaces (see \cite{BF}, \cite{Nak}) leads to Langlands duality for the affine case.

10 Conclusion

We have seen that the search for a six dimensional field theory (which has to be a purely quantum field theoretic structure, embedded into eleven dimensional $M$-theory) offers a fundamental perspective on the geometric Langlands program: It would lead to Langlands duality as a manifest symmetry, it would unite geometric Langlands duality for algebraic surfaces and algebraic curves into a single framework, and it would naturally include higher dimensional operators which are so important for studying the counterparts of ramification and Langlands functoriality on the geometric side. In physics
it has strong links to many areas (string- and $M$-theory, Yang-Mills theory, Taub-NUT solutions of the Einstein equations).

Last not least, though the full six dimensional theory has not been constructed so far, it is amenable to explicit calculations in dimensional reductions, leading to structures like $WZW$-models where a lot of results are available from the side of mathematical physics.

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