Counting carefree couples

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Abstract

A pair of natural numbers \((a, b)\) such that \(a\) is both squarefree and coprime to \(b\) is called a carefree couple. A result conjectured by Manfred Schroeder (in his book ‘Number theory in science and communication’) on carefree couples and a variant of it are established using standard arguments from elementary analytic number theory. Also a related conjecture of Schroeder on triples of integers that are pairwise coprime is proved.

1 Introduction

It is well known that the probability that an integer is squarefree is \(6/\pi^2\). Also the probability that two given integers are coprime is \(6/\pi^2\). (More generally the probability that \(n\) positive integers chosen arbitrarily and independently are coprime is well-known \([17, 22, 27]\) to be \(1/\zeta(n)\), where \(\zeta\) is Riemann’s zeta function. For some generalizations see e.g. \([3, 4, 12, 23, 25]\).) One can wonder how ‘statistically independent’ squarefreeness and coprimality are. To this end one could for example consider the probability that of two random natural numbers \(a\) and \(b\), \(a\) is both squarefree and coprime to \(b\). Let us call such a couple \((a, b)\) carefree. If \(b\) is also squarefree, we say that \((a, b)\) is a strongly carefree couple.

Let us denote by \(C_1(x)\) the number of carefree couples \((a, b)\) with both \(a \leq x\) and \(b \leq x\) and, similarly, let \(C_2(x)\) denote the number of strongly carefree couples \((a, b)\) with both \(a \leq x\) and \(b \leq x\).

The purpose of this note is to establish the following result, part of which was conjectured, on the basis of heuristic arguments, by Manfred Schroeder \([26]\, p. 54]\). (In it and in the rest of the paper the mathematical symbol \(p\) is exclusively used to denote primes.)

Theorem 1 We have

\[
C_1(x) = \frac{x^2}{\zeta(2)} \prod_p \left(1 - \frac{1}{p(p+1)}\right) + O(x \log x),
\]

and

\[
C_2(x) = \frac{x^2}{\zeta(2)^2} \prod_p \left(1 - \frac{1}{(p+1)^2}\right) + O(x^{3/2}).
\]
The interpretation of Theorem 1 is that the probability for a couple to be carefree is
\[ K_1 := \frac{1}{\zeta(2)} \prod_p \left( 1 - \frac{1}{p(p+1)} \right) \approx 0.4282495056770944022 \]  
and to be strongly carefree is
\[ K_2 := \frac{1}{\zeta(2)^2} \prod_p \left( 1 - \frac{1}{(p+1)^2} \right) \approx 0.28674742843447873411 \]  

Using the identity \( \zeta(n) = \prod_p (1 - p^{-n})^{-1} \) valid for \( n > 1 \) we can alternatively write
\[ K_2 = \frac{1}{\zeta(2)^2} \prod_p \left( 1 - \frac{2}{p(p+1)} \right) = \prod_p \left( 1 - \frac{1}{p} \right)^2 \left( 1 + \frac{2}{p} \right). \]  

For \( m \geq 3 \) and \( 0 \leq k \leq m \) we put
\[ Z_k(m) = \prod_p \left( 1 + \frac{k - 1}{p^m} - \frac{k}{p^m - 1} \right). \]  

Note that \( Z_2(3) = K_1 \) and \( Z_3(3) = K_2 \).

The constants \( K_1 \) and \( K_2 \) we could call the carefree, respectively strongly carefree constant, cf. [10, Section 2.5].

Assuming independence of squarefreeness and coprimality we would expect that \( K_1 = \zeta(2)^{-2} \) and \( K_2 = \zeta(2)^{-3} \). Now note that
\[ K_1 = \frac{1}{\zeta(2)^2} \prod_p \left( 1 + \frac{1}{(p+1)(p^2 - 1)} \right), \quad K_2 = \frac{1}{\zeta(2)^3} \prod_p \left( 1 + \frac{2p + 1}{(p+1)^2(p^2 - 1)} \right). \]  

We have \( \zeta(2)^2 K_1 \approx 1.15876 \) and \( \zeta(2)^3 K_2 \approx 1.27627 \). Thus, there is a positive correlation between squarefreeness and coprimality.

Let \( I_3(x) \) denote the number of triples \((a, b, c)\) with \( a \leq x, \ b \leq x, \ c \leq x \) such that \((a, b) = (a, c) = (b, c) = 1\). Schroeder [26, Section 4.4] claims that \( I_3(x) \sim K_2 x^3 \). Indeed, in Section 2.2 we will prove the following result.

**Theorem 2** We have \( I_3(x) = K_2 x^3 + O(x^2 \log^2 x) \).

The work described in this note was carried out in 2000 and with some improvement in the error terms was posted on the arXiv in September of 2005 [21], with the remark that it was not intended for publication in a research journal as the methods used involve only rather elementary and standard analytic number theory. Over the years various authors referred to [21], and this induced me to try to publish it in a mathematical newsletter. (For publications in this area after 2005 see, e.g., [1, 6, 7, 8, 9, 14, 15, 16, 30, 31].) In [21] there was a mistake in the proof of (2) leading to an error term of \( O(x \log^3 x) \), rather than \( O(x^{3/2}) \). Except for this, the present version has essentially the same mathematical content as the earlier one, but is written in a less carefree way and with the mathematical details more spelled out.
2 Proofs

As usual we let $\mu$ denote the M"obius function and $\varphi$ Euler’s totient function. Note that $n$ is squarefree if and only if $\mu(n)^2 = 1$. We will repeatedly make use of the basic identities

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1; \\ 0 & \text{otherwise,} \end{cases} \quad (7)$$

and

$$\frac{\varphi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d} = \prod_{p|n} \left(1 - \frac{1}{p}\right). \quad (8)$$

We will also use several times that if $s$ is a complex number and $f$ a multiplicative function such that $\sum_p \sum_{\nu \geq 1} |f(p^\nu) p^{-\nu s}| < \infty$, then

$$\sum_{n=1}^\infty \frac{f(n)}{n^s} = \sum_p \sum_{\nu \geq 1} \frac{f(p^\nu)}{p^{\nu s}}. \quad (9)$$

(For a proof see, e.g., Tenenbaum [28, p. 107].)

In the proof of Theorem 1 we will make use of the following lemma.

**Lemma 1** Let $d \geq 1$ be arbitrary. Put

$$S_d(x) = \sum_{n \leq x, (d,n)=1} \mu(n)^2.$$

We have

$$S_d(x) = \frac{x}{\zeta(2)} \prod_{p|d} \left(1 + \frac{1}{p}\right) + O(2^{\omega(d)} \sqrt{x}), \quad (10)$$

where $\omega(d)$ denotes the number of distinct prime divisors of $d$.

**Proof.** Let $T_d(x)$ denote the number of natural numbers $n \leq x$ that are coprime to $d$. Using (7) and (8) and $[x] = x + O(1)$ we deduce that

$$T_d(x) = \sum_{\substack{n \leq x \atop (n,d)=1}} 1 = \sum_{\substack{n \leq x \atop \alpha|d}} \mu(\alpha) = \sum_{\alpha|d} \mu(\alpha) \left[\frac{x}{\alpha}\right] = \frac{\varphi(d)}{d} x + O(2^{\omega(d)}). \quad (11)$$

By the principle of inclusion and exclusion we find that

$$S_d(x) = \sum_{m \leq \sqrt{x}, (d,m)=1} \mu(m) T_d\left(\frac{x}{m^2}\right).$$

Hence, on invoking (11), we find

$$S_d(x) = x \frac{\varphi(d)}{d} \sum_{m \leq \sqrt{x}, (d,m)=1} \frac{\mu(m)}{m^2} + O(2^{\omega(d)} \sqrt{x}).$$
and hence, on completing the sum,

\[ S_d(x) = x \frac{\varphi(d)}{d} \sum_{m=1}^{\infty} \frac{\mu(m)}{m^2} + O(2^{\omega(d)} \sqrt{x}) \]

Note that

\[ \sum_{m=1}^{\infty} \frac{\mu(m)}{m^2} = \prod_{p \nmid d} \left( 1 - \frac{1}{p^2} \right) = \frac{1}{\zeta(2) \prod_{p \mid d} (1 - 1/p^2)}. \]

Using this and (8) the proof is completed. \(\square\)

Let \(d(n)\) denote the number of divisors of \(n\). We have \(2^{\omega(n)} \leq d(n)\) with equality iff \(n\) is squarefree. The estimates below also hold with \(2^{\omega(n)}\) replaced by \(d(n)\).

**Lemma 2** We have

\[ \sum_{d \leq x} 2^{\omega(d)} \frac{d}{d^{3/2}} = O(1), \quad \sum_{d \leq x} 4^{\omega(d)} \frac{d}{\sqrt{d}} = O(x \log x), \quad \sum_{d \leq x} 4^{\omega(d)} \frac{d}{d} = O(\log^3 x). \]

**Proof.** Using the convergence of \(\sum_p p^{-3/2}\) we find by (9) that \(\sum_{d=1}^{\infty} 2^{\omega(d)} d^{-3/2} = O(1)\). The remaining estimates follow on invoking Theorem 1 at p. 201 of Tenenbaum’s book [28] together with partial integration. \(\square\)

### 2.1 Proof of Theorem 1

Note that

\[ C_1(x) = \sum_{a \leq x} \sum_{b \leq x} \mu(a)^2 \sum_{d \mid (a, db)} \mu(d) = \sum_{d \leq x} \mu(d) \sum_{a \leq x \atop d \mid a} \mu(a)^2 \sum_{b_1 \leq x/d} 1, \]

after swapping the summation order. Using \([x/d] = x/d + O(1)\), we then obtain

\[ C_1(x) = x \sum_{d \leq x} \frac{\mu(d)}{d} \sum_{a \leq x \atop d \mid a} \mu(a)^2 + O(x \log x). \]

On noting that

\[ \sum_{a \leq x \atop d \mid a} \mu(a)^2 = \mu(d)^2 \sum_{n \leq x/d \atop (d, n) = 1} \mu(n)^2 = \mu(d)^2 S_d\left(\frac{x}{d}\right) \quad (12) \]

and \(\mu(d) = \mu(d)^3\), we find

\[ C_1(x) = x \sum_{d \leq x} \frac{\mu(d)}{d} S_d\left(\frac{x}{d}\right) + O(x \log x). \]

On using Lemma 1 we obtain the estimate

\[ C_1(x) = \frac{x^2}{\zeta(2)} \sum_{d \leq x} \frac{\mu(d)}{d^2} \prod_{p \mid d} (1 + 1/p) + O(\sqrt{x} \sum_{d \leq x} \frac{2^{\omega(d)}}{\sqrt{d}}) + O(x \log x). \]
On completing the latter sum and noting that
\[ \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2 \prod_{p|d} (1 + 1/p)} = \prod_p \left(1 - \frac{1}{p(p+1)}\right), \]
we obtain
\[ C_1(x) = \frac{x^2}{\zeta(2)} \prod_p \left(1 - \frac{1}{p(p+1)}\right) + O(\sqrt{x} \sum_{d \leq x} \frac{2^{\omega(d)}}{\sqrt{d}}) + O(x \log x). \]

Estimate (1) now follows on invoking Lemma 2.

The proof of (2) is very similar to the proof of (1). We start by noting that
\[ C_2(x) = \sum_{a \leq x} \sum_{b \leq x} \mu(a)^2 \mu(b)^2 \sum_{d|a} \sum_{d|b} \mu(d). \]

On swapping the summation order, we obtain
\[ C_2(x) = \sum_{d \leq x} \mu(d) \sum_{a \leq x} \frac{\mu(a)^2}{d_a} \sum_{b \leq x} \frac{\mu(b)^2}{d_b}. \]  

(13)

On noting that \( \mu(d) = \mu(d)^5 \) and invoking (12) we obtain
\[ C_2(x) = \sum_{d \leq x} \mu(d) S_d \left(\frac{x}{d}\right)^2. \]

(14)

On using Lemma 1 we obtain the estimate
\[ C_2(x) = \frac{x^2}{\zeta(2)^2} \sum_{d \leq x} \frac{\mu(d)}{d^2 \prod_{p|d} (1 + 1/p)^2} + O(x^{3/2} \sum_{d \leq x} \frac{2^{\omega(d)}}{d^{3/2}}) + O(x \sum_{d \leq x} \frac{4^{\omega(d)}}{d}). \]

On completing the first sum and noting that
\[ \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2 \prod_{p|d} (1 + 1/p)^2} = \prod_p \left(1 - \frac{1}{p(p+1)^2}\right), \]
we find
\[ C_2(x) = \frac{x^2}{\zeta(2)^2} \prod_p \left(1 - \frac{1}{(p+1)^2}\right) + O(x^{3/2} \sum_{d \leq x} \frac{2^{\omega(d)}}{d^{3/2}}) + O(x \sum_{d \leq x} \frac{4^{\omega(d)}}{d}). \]

On invoking Lemma 2 estimate (2) is then established. \( \square \)

2.2 Proof of Theorem 2

We write \([n, m]\) for the least common multiple of \(n\) and \(m\), and \((n, m)\) for the greatest common divisor. Recall that \((n, m)[n, m] = nm\).

Note that
\[ I_3(x) = \sum_{a, b, c \leq x} \sum_{d_1|a} \mu(d_1) \sum_{d_2|b} \mu(d_2) \sum_{d_3|c} \mu(d_3), \]
which can be rewritten as 

$$I_3(x) = \sum_{\frac{x}{d_1}, d_2 \leq x} \mu(d_1)\mu(d_2)\mu(d_3)\left[\frac{x}{d_1}, d_2 \leq x\right] \left[\frac{x}{d_1}, d_3 \leq x\right] \left[\frac{x}{d_2}, d_3 \leq x\right].$$

Now put 

$$J_1(x) = \sum_{\frac{x}{d_1}, d_2 \leq x} \frac{\mu(d_1)\mu(d_2)\mu(d_3)}{[d_1, d_2][d_1, d_3][d_2, d_3]}, \quad J_2(x) = \sum_{\frac{x}{d_1}, d_2 \leq x} \frac{1}{[d_1, d_2][d_1, d_3]},$$

$$J_3(x) = \sum_{\frac{x}{d_1}, d_2 \leq x} \frac{1}{[d_1, d_2]} \quad \text{and} \quad J_4(x) = \sum_{\frac{x}{d_1}, d_2 \leq x} 1.$$

Using that $[x] = x + O(1)$ we find that 

$$I_3(x) = x^3J_1(x) + O(x^2J_2(x)) + O(xJ_3(x)) + O(J_4(x)). \quad (15)$$

We will show first that 

$$J_1(x) = \sum_{d_1=1}^{\infty} \sum_{d_2=1}^{\infty} \sum_{d_3=1}^{\infty} \frac{\mu(d_1)\mu(d_2)\mu(d_3)}{[d_1, d_2][d_1, d_3][d_2, d_3]} + O\left(\frac{\log x}{x}\right).$$

To this end it is enough, by symmetry of the argument of the sum, to show that 

$$\sum_{\frac{x}{d_1}, d_2 > x} \sum_{d_3 \geq 1} \frac{1}{[d_1, d_2][d_1, d_3][d_2, d_3]} = O\left(\frac{\log x}{x}\right). \quad (16)$$

Put $(d_1, d_2) = \alpha, (d_1, d_3) = \beta$ and $(d_2, d_3) = \gamma$. Since $\alpha|d_1$ and $\beta|d_1$, we can write 

$$d_1 = [\alpha, \beta]\delta_1 \quad \text{for some integer} \quad \delta_1 \geq 1, \quad \text{and similarly} \quad d_2 = [\alpha, \gamma]\delta_2, \quad d_3 = [\beta, \gamma]\delta_3.$$

Note that any triple $(d_1, d_2, d_3)$ corresponds to a unique 6-tuple $(\alpha, \beta, \gamma, \delta_1, \delta_2, \delta_3)$. Since $\alpha(\delta_1, \delta_2)$ divides $([\alpha, \beta]\delta_1, [\alpha, \gamma]\delta_2)$ on the one hand and $([\alpha, \beta]\delta_1, [\alpha, \gamma]\delta_2) = (d_1, d_2) = \alpha$ on the other, it follows that $(\delta_1, \delta_2) = 1$ and likewise $(\delta_1, \delta_3) = (\delta_2, \delta_3) = 1$. Write $u = \alpha\beta\gamma/(\alpha, \beta, \gamma)^2$. On noting that $((d_1, d_2), (d_2, d_3)) = (d_1, d_2, d_3) = ((d_1, d_2), (d_1, d_3), (d_2, d_3))$ we infer that $(\alpha, \beta) = (\alpha, \gamma) = (\beta, \gamma) = (\alpha, \beta, \gamma)$ and hence we find that $[d_1, d_2] = u\delta_1\delta_2, [d_1, d_3] = u\delta_1\delta_3$ and $[d_2, d_3] = u\delta_2\delta_3$. Now 

$$\sum_{\frac{x}{d_1}, d_2 > x} \sum_{d_3 \geq 1} \frac{1}{[d_1, d_2][d_1, d_3][d_2, d_3]} \leq \sum_{\alpha, \beta, \gamma} \frac{1}{u^3} \sum_{\delta_1, \delta_2 > x/u} \sum_{\delta_3 \geq 1} \frac{1}{(\delta_1\delta_2\delta_3)^2},$$

where the triple sum is over all 6-tuples $(\alpha, \beta, \gamma, \delta_1, \delta_2, \delta_3)$ and is of order 

$$O\left(\sum_{\alpha, \beta, \gamma} \frac{1}{u^3} \sum_{\delta_1, \delta_2 > x/u} \frac{1}{(\delta_1\delta_2)^2}\right) = O\left(\sum_{\alpha, \beta, \gamma} \frac{1}{u^3} \sum_{n > x/u} \frac{d(n)}{n^2}\right) = O\left(\frac{\log x}{x} \sum_{\alpha, \beta, \gamma} \frac{1}{u^2}\right),$$

where we used the well-known estimate $\sum_{n > x} d(n)n^{-2} = O(\log x/x)$. Now 

$$\sum_{\alpha, \beta, \gamma} \frac{1}{u^2} = \sum_{\alpha, \beta, \gamma} \frac{1}{u^2} \frac{(\alpha, \beta, \gamma)^4}{(\alpha\beta\gamma)^2} = O\left(\sum_{d=1}^{\infty} \frac{1}{d^2} \sum_{\alpha', \beta', \gamma'} \frac{1}{(\alpha'\beta'\gamma')^2}\right) = O(1), \quad (17)$$

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where we have written \((\alpha, \beta, \gamma) = d, \alpha = d\alpha', \beta = d\beta'\) and \(\gamma = d\gamma'\). Thus we have established equation (16).

In the same vein \(J_2(x)\) can be estimated to be

\[
J_2(x) = O\left(\sum_{\alpha,\beta,\gamma} \frac{1}{[d_1, d_2][d_3]}\right) = O\left(\sum_{\alpha,\beta,\gamma} \frac{1}{u^2} \sum_{\delta_1, \delta_2, \delta_3 \leq (x/u)^{3/2}} \frac{1}{\delta_1^2 \delta_2 \delta_3}\right).
\]

Using the classical estimate \(\sum_{n \leq x} d(n)/n = O(\log^2 x)\) and (17), one obtains \(J_2(x) = O(\log^2 x)\).

Note that \(0 \leq J_4(x) \leq x \cdot J_3(x) \leq x^2 J_2(x)\). Using (15) we see that it remains to evaluate the triple infinite sum, which we rewrite as

\[
\sum_{d_1=1}^{\infty} \sum_{d_2=1}^{\infty} \sum_{d_3=1}^{\infty} \mu(d_1)\mu(d_2)\mu(d_3)(d_1, d_2)(d_1, d_3)(d_2, d_3),
\]

which can be rewritten as

\[
\sum_{d_1=1}^{\infty} \frac{\mu(d_1)}{d_1^2} \sum_{d_2=1}^{\infty} \frac{\mu(d_2)(d_1, d_2)}{d_2^3} \sum_{d_3=1}^{\infty} \frac{\mu(d_3)(d_1, d_3)(d_2, d_3)}{d_3^2}.
\]

Note that the argument of the inner sum is multiplicative in \(d_3\). By Euler’s product identity (9) it is zero if \((d_1, d_2) > 1\) and \(\zeta(2)^{-1} \prod_{p|d_1 d_2} (1 + 1/p)^{-1}\) otherwise. Thus the latter triple sum is seen to yield

\[
\frac{1}{\zeta(2)} \sum_{d_1=1}^{\infty} \frac{\mu(d_1)}{d_1^2} \sum_{d_2=1}^{\infty} \frac{\mu(d_2)(d_1, d_2)}{d_2^3} \sum_{d_3=1}^{\infty} \frac{\mu(d_3)(d_1, d_3)(d_2, d_3)}{d_3^2} \prod_{p|d_1 d_2} (1 + 1/p),
\]

the argument of the inner sum is multiplicative in \(d_2\) and proceeding as before we obtain that it equals

\[
\frac{1}{\zeta(2)} \prod_p \left(1 - \frac{1}{p(p + 1)}\right) \sum_{d_1=1}^{\infty} \frac{\mu(d_1)}{d_1^2} \prod_{p|d_1} \left(1 + \frac{1}{p}\right) \prod_{p|d_1} \left(1 - \frac{1}{p(p + 1)}\right),
\]

which is seen to equal

\[
\frac{1}{\zeta(2)} \prod_p \left(1 - \frac{2}{p(p + 1)}\right),
\]

which by equation (3) equals \(K_2\).

\[\square\]

3 Numerical aspects

Direct evaluation of the constants \(K_1\) and \(K_2\) through (3), respectively (4) yields only about five decimal digits of precision. By expressing \(K_1\) and \(K_2\) as infinite products involving \(\zeta(k)\) for \(k \geq 2\), they can be computed with high precision. To
this end Theorem 1 of [20] can be used. The error analysis can be dealt with using Theorem 2 of [20]. Using [20, Theorem 1] it is inferred that

\[ K_1 = \prod_{k \geq 2} \zeta(k)^{-e_k}, \text{ where } e_k = \frac{\sum_{d|k} b_d \mu(d/k)}{k} \in \mathbb{Z}, \]

with the sequence \( \{b_k\}_{k=0}^{\infty} \) defined by \( b_0 = 2 \) and \( b_1 = -1 \) and \( b_{k+2} = -b_{k+1} + b_k \). Using the same theorem, it is seen that

\[ K_2 = \frac{1}{2} \prod_{k \geq 2} \{\zeta(k)(1 - 2^{-k})\}^{-f_k}, \text{ where } f_k = \frac{\sum_{d|k} (-2)^d \mu(d/k)}{k} \in \mathbb{Z}. \]

Typically in analytic number theory constants of the form \( \prod_p f(1/p) \) with \( f \) rational arise as densities. Their numerical evaluation was considered by the author in [20]. By similar methods any constant of the form \( \prod_p f(1/p) \) with \( f \) an analytic function on the unit disc satisfying \( f(0) = 1 \) and \( f'(0) = 0 \) can be evaluated [19].

4 Related problems

Let us call a couple \((a, b)\) with \( a, b \leq x \), \( a \) and \( b \) coprime and either \( a \) or \( b \) squarefree, weakly carefree. A little thought reveals that \( C_3(x) = 2C_1(x) - C_2(x) \). By Theorem 1 it then follows that the probability \( K_3 \) that a couple is weakly carefree equals

\[ K_3 = 2K_1 - K_2 \approx 0.5697515829. \]

The problem of estimating \( I_3(x) \) has the following natural generalisation. Let \( k \geq 2 \) be an integer and let \( I_k(x) \) be the number of \( k \)-tuples \((a_1, \ldots, a_k)\) with \( 1 \leq a_i \leq x \) for \( 1 \leq i \leq k \) such that \( (a_i, a_j) = 1 \) for every \( 1 \leq i \neq j \leq k \). The number of \( k \)-tuples such that none of the gcd’s is divisible by some fixed prime \( p \) is easily seen to be

\[ \sim x^k \left( \left(1 - \frac{1}{p}\right)^k + \frac{k}{p} \left(1 - \frac{1}{p}\right)^{k-1} \right) = x^k \left( 1 - \frac{1}{p} \right)^{k-1} \left( 1 + \frac{k - 1}{p} \right). \]

Thus, it seems plausible that

\[ I_k(x) \sim x^k \prod_p \left(1 - \frac{1}{p}\right)^{k-1} \left( 1 + \frac{k - 1}{p} \right), \quad (x \to \infty). \tag{18} \]

For \( k = 2 \) and \( k = 3 \) (by Theorem 2 and equation (13) this is true. In 2000 I did not see how to prove this for arbitrary \( k \), however the conjecture (18) was established soon afterwards (in 2002) by L. Tóth [29], who proved that for \( k \geq 2 \) we have

\[ I_k(x) = x^k \prod_p \left(1 - \frac{1}{p}\right)^{k-1} \left( 1 + \frac{k - 1}{p} \right) + O(x^{k-1} \log^{k-1} x). \tag{19} \]

Let \( I_k^{(u)}(x) \) denote the number of \( k \)-tuples \((a_1, \ldots, a_k)\) with \( 1 \leq a_i \leq x \) that are pairwise coprime and moreover satisfy \((a_i, u) = 1\) for \( 1 \leq i \leq k \). It is easy to
see that

\[ I_{k+1}^{(u)}(n) = \sum_{j=1}^{n} I_{k}^{(ju)}(n). \]

Note that \( I_{1}^{(u)}(n) = T_{u}(n) \) can be estimated by (11). Then by recursion with respect to \( k \) an estimate for \( I_{k}^{(u)}(n) \) can be established that implies (19).

In [13] Havas and Majewski considered the problem of counting the number of \( n \)-tuples of natural numbers that are pairwise not coprime. They suggested that the density \( \delta_{n} \) of these tuples should be

\[ \delta_{n} = \left(1 - \frac{1}{\zeta(2)}\right)^{\binom{n}{2}}. \]  

The probability that a pair of integers is not coprime is \( 1 - 1/\zeta(2) \). Since there are \( \binom{n}{2} \) pairs of integers in an \( n \)-tuple, one might naively expect the probability for this problem to be as given by (20).

T. Freiberg [11] studied this problem for \( n = 3 \) using my approach to estimate \( I_{3}(x) \) (it seems that the recursion method of Tóth cannot be applied here). Freiberg showed that the density of triples \((a, b, c)\) with \((a, b) > 1\), \((a, c) > 1\) and \((b, c) > 1\) equals

\[ F_{3} = 1 - \frac{3}{\zeta(2)} + 3K_{1} - K_{2} \approx 0.1742197830347247005, \]

whereas \( (1 - 1/\zeta(2))^{3} \approx 0.06 \). Thus the guess of Havas and Majewski for \( n = 3 \) is false. Indeed, it is easy to see (as Peter Pleasants pointed out to the author [24]) that for every \( n \geq 3 \) their guess is false. Since all \( n \)-tuples of even numbers are pairwise not coprime, \( \delta_{n} \), if it exists, satisfies \( \delta_{n} \geq 2^{-n} \). Since \( \binom{n}{2} \geq n \) and \( 1 - 1/\zeta(2) < 0.4 \) the predicted density by Havas and Majewski [13] satisfies \( \delta_{n} < 2^{-n} \) for \( n \geq 3 \) and so must be false.

In 2006 the author learned [18] that the result of Freiberg is implicit in the PhD thesis of R.N. Buttsworth [2] and indeed can be found there in more general form. Buttsworth showed that the density of relatively prime \( m \)-tuples for which \( k \) prescribed \((m - 1)\)-tuples have gcd 1 equals \( Z_{k}(m) \) given in (4). Consequently by inclusion and exclusion the set of relatively prime \( m \)-tuples such that every \((m - 1)\)-tuple fails to be relatively prime has density

\[ \sum_{k=0}^{m} (-1)^{k} \binom{m}{k} Z_{k}(m). \]

For \( m = 3 \) this yields \( 1/\zeta(3) - 3/\zeta(2) + 3K_{1} - K_{2} \). So the density of relatively prime 3-tuples such that at least one 2-tuple is relatively prime, is equal to \( 3/\zeta(2) - 3K_{1} + K_{2} \). However this is also equal to the density of 3-tuples such that at least one 2-tuple is relatively prime. Hence the density of 3-tuples such that all 2-tuples are not relatively prime is \( 1 - 3/\zeta(2) + 3K_{1} - K_{2} \), which is Freiberg’s formula.

To close this discussion, we like to remark that Freiberg established his result with error term \( O(x^{2}\log^{2}x) \) and that Buttsworth’s result gives only a density.

Some related open problems are as follows:
Problem 1
a) To compute the density of \( n \)-tuples such that at least \( k \) pairs are coprime.
b) To compute the density of \( n \)-tuples such that exactly \( k \) pairs are coprime.

Problem 2
To compute the density of \( n \)-tuples such that all pairs are not coprime.

Remark. Recently Jerry Hu [16] announced that he solved Problem 1.

5 Conclusion
In stark constrast to what experience from daily life suggests, (strongly) carefree couples are quite common...

Acknowledgement. The author likes to thank Steven Finch for bringing Schroeder’s conjecture to his attention and his instigation to write down these results. Also Finch and de Weger pointed out that one has \( \sum_{n \leq x} k(n) = \zeta(2)K_1x^{1/2} \big/ 2 + O(x^{3/2}) \), where \( k(n) = \prod_{p|n} p \), and that in [21] the \( K_1 \) was inadvertently dropped. For a proof of this formula see Eckford Cohen [5, Theorem 5.2].

The author likes to thank Tristan Freiberg and Jerry Hu for pointing out some references and helpful comments. Keith Matthews provided me kindly with very helpful information concerning the relevant results of his former PhD student Buttsworth. In particular he pointed out how Freiberg’s result follows from that of Buttsworth.

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