ON THE ITERATED HAMILTONIAN FLOER HOMOLOGY

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Abstract. The focus of the paper is the behavior under iterations of the filtered and local Floer homology of a Hamiltonian on a symplectically aspherical manifold. The Floer homology of an iterated Hamiltonian comes with a natural cyclic group action. In the filtered case, we show that the supertrace of a generator of this action is equal to the Euler characteristic of the homology of the un-iterated Hamiltonian. For the local homology the supertrace is the Lefschetz index of the fixed point. We also prove an analog of the classical Smith inequality for the iterated local homology.

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1. INTRODUCTION

1.1. Introduction. The main theme of this paper is the behavior of the filtered and local Hamiltonian Floer homology under iterations. In particular, we establish some lower bounds on the rank of the Floer homology of the iterated Hamiltonian via the homology of the original Hamiltonian.

Virtually nothing was known, prior to now, about the behavior of the filtered Floer homology under iterations of a Hamiltonian \( H \). With local Floer homology the situation has been more encouraging. Namely, consider an isolated one-periodic orbit \( x \) of the Hamiltonian diffeomorphism \( \varphi_H \) generated by a Hamiltonian \( H \). As was shown in [GG10], the local Floer homology group of the iterated \( k \)-periodic orbit \( x^k \) is independent of the order of iteration \( k \), up to a shift of degree, as long

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as \( k \) is admissible, i.e., the algebraic multiplicity of the eigenvalue 1 is the same for \( D\varphi_H \) and \( D\varphi_H^k \). Without the latter assumption, the question is much more subtle and again essentially nothing was known about the relation between the local Floer homology groups for \( x \) and \( x^k \) prior to now.

The key ingredient in our arguments is the \( \mathbb{Z}_k \)-action on the filtered or local Floer homology of \( \varphi_H^k \). This action has a canonical generator \( g \) given, roughly speaking, by the time-shift. The existence of such an action has been known for quite some time. For instance, in the framework of persistence modules this action is used in [PS, PSS, Zh] to study the placement of the iterated Hamiltonians in the group of all Hamiltonian diffeomorphisms in terms of the Hofer distance. In [To] a slightly different version of the action (less suitable for our purposes) is considered and a supertrace relation, similar to the one proved in this paper, is established for the Floer of homology of commuting symplectomorphisms.

Our first result equates the supertrace of \( g \) to the Euler characteristic of the corresponding Floer homology group for \( \varphi_H \). (In the filtered case the manifold is assumed to be symplectically aspherical.) For the local homology, this Euler characteristic is simply the Lefschetz index of the fixed point.

The second result concerns a Floer theoretical version of the Smith inequality. The classical version of this inequality relates the rank of the total homology of a compact \( \mathbb{Z}_p \)-space \( X \) with coefficients in the field \( \mathbb{F} = \mathbb{F}_p \) and the rank of the homology of the fixed point set \( X^{\mathbb{Z}_p} \), where \( p \) is prime and \( \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \); see, e.g., [Bo, Br] and references therein. Namely,

\[
\dim H_*(X^{\mathbb{Z}_p}; \mathbb{F}) \leq \dim H_*(X; \mathbb{F}),
\]

where on the right and left hand sides we have the sum of dimensions for all degrees.

From various perspectives, it is reasonable to expect a similar inequality to hold for the filtered Floer homology of a Hamiltonian diffeomorphism \( \varphi_H \) of a symplectically aspherical closed symplectic manifold when the role of the right hand side is taken by the filtered Floer homology of \( \varphi_H^p \) and the left hand side is the filtered Floer homology of the original Hamiltonian diffeomorphism \( \varphi_H \). (For instance, \( \mathbb{Z}_p \) acts on a basis of the Floer complex of \( \varphi_H^p \) and the elements of this basis, fixed up to a sign by the action, form a basis for the Floer complex of \( \varphi_H \).) For \( p = 2 \), such an inequality is proved in [Se] by introducing and utilizing a \( \mathbb{Z}_2 \)-equivariant version of the pair-of-pants product. The proof of an analogous result for all primes \( p \geq 2 \) is a work in progress, [ShZh].

Our goal here is much less ambitious: we prove a version of the Smith inequality for the local Floer homology of an isolated periodic orbit. (This, of course, will also follow from [ShZh].) The proof is rather simple. We identify the local Floer homology with the generating function homology (see [Vi]) and then use the fact that the latter is the homology of an actual \( \mathbb{Z}_p \)-space, where \( p \) is the iteration order. It remains to apply (1.1) to this space. The \( \mathbb{Z}_p \)-action on the Floer homology does not explicitly enter the statement, but it is essential as a motivation and, somewhat disguised, it also plays a central role in the argument.

1.2. Main results. Let \( H: S^1 \times M \to \mathbb{R} \) be a one-periodic in time Hamiltonian on a closed symplectically aspherical manifold \((M, \omega)\). The filtered Floer homology of the iterated Hamiltonian \( H^{2k} \) (or the local Floer homology of an iterated orbit \( x^k \)) carries a \( \mathbb{Z}_k \)-action, which comes with a preferred generator

\[
g_*: \text{HF}_*(x^{ka, kb})(H^{2k}; \mathbb{F}) \to \text{HF}_*(x^{ka, kb})(H^{2k}; \mathbb{F}).
\]
Here the coefficient ring \( \mathbb{F} \) is a field and the endpoints \( ka \) and \( kb \) are not in the action spectrum \( S(H^{2k}) \) of \( H^{2k} \). Roughly speaking, \( g \) arises from the time-shift map \( t \mapsto t + 1 \) applied to \( k \)-periodic orbits of \( H^{2k} \). In Section 3.1 we recall in detail the definition of the action, paying particular attention to the role of orientations and sign changes which is central to our results. The discussion of the action in that section is essentially self-contained albeit brief.

Then, in Section 3.2, we show that the supertrace
\[
\text{str}(g) := \sum (-1)^i \text{tr} \left(g_i: \text{HF}_i^{(ka, kb)}(H^{2k}; \mathbb{F}) \to \text{HF}_i^{(ka, kb)}(H^{2k}; \mathbb{F})\right)
\]
of \( g \) is equal to the Euler characteristic
\[
\chi(\text{HF}_*^{(a, b)}(H; \mathbb{F})) = \sum (-1)^i \text{dim} \text{HF}_i^{(a, b)}(H; \mathbb{F})
\]
of \( \text{HF}_*^{(a, b)}(H; \mathbb{F}) \), viewed as an element of \( \mathbb{F} \). In other words, \( \chi(\text{HF}_*^{(a, b)}(H; \mathbb{F})) \) is the image in \( \mathbb{F} \) of the Euler characteristic under the natural map \( \mathbb{Z} \to \mathbb{F} \) or, equivalently, the supertrace of the identity map on the homology of \( H \). For instance, \( \chi(\text{HF}_*^{(a, b)}(H; \mathbb{F})) \) is precisely equal to the Euler characteristic when \( \mathbb{F} = \mathbb{Q} \) and is the Euler characteristic modulo \( p \) when \( \mathbb{F} = \mathbb{F}_p \) and \( p \) is prime.

The equality of the supertrace and the Euler characteristic is the first result of this paper:

**Theorem 1.1.** Let \( H \) be a one-periodic Hamiltonian on a closed symplectically aspherical manifold \((M, \omega)\). Assume that \( \mathbb{F} \) is a field. Then the supertrace of \( g \) is equal to \( \chi(\text{HF}_*^{(a, b)}(H; \mathbb{F})) \).

Since both the supertrace and the Euler characteristic are additive with respect to the action filtration, the proof of Theorem 1.1 reduces to checking the orientation change at non-degenerate iterated orbits \( x^k \) under the time-shift map \( t \mapsto t + 1 \). For the local Floer homology of an iterated orbit \( x^k \), Theorem 1.1 takes the following form.

**Corollary 1.2.** Let \( M^{2n} \) be a symplectic manifold and \( x: S^1 \to M \) be an isolated one-periodic orbit of a Hamiltonian \( H \) on \( M \). Assume that \( \mathbb{F} \) is a field. For isolated iterations \( x^k \) of \( x \), the supertrace of the generator is, up to the sign \( (-1)^n \), equal to the image in \( \mathbb{F} \) of the Lefschetz index of the fixed point \( x(0) \).

The corollary readily follows from Theorem 1.1 and its proof.

**Remark 1.3.** Note that in Theorem 1.1, the Floer homology groups of \( \varphi_H \) and \( \varphi_H^k \) “count” only contractible periodic orbits and thus the theorem gives information only about such orbits. On the other hand, in Corollary 1.2 we do not need to require the orbit \( x \) to be contractible. Indeed, recall that the composition of the flow of \( H \) near \( x \) with a local loop of Hamiltonian diffeomorphisms does not change the local Floer homology, up to an even shift of grading; see [Gi, GG10]. (Here one should view the flow and the loop as defined on a neighborhood of the image of \( x \) in the extended phase-space \( M \times S^1 \).) Then, composing the flow with a suitably chosen loop, we reduce the general case to the case where \( x \) is a constant orbit.

Corollary 1.2 implies that if the Lefschetz index of an orbit \( x \) (as a fixed point) is non-zero, then the local Floer homology of its isolated iterations with field coefficients cannot be zero, cf. Remark 1.6. For prime iterations \( x^p \) with \( \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \), coefficients a stronger result holds.
Theorem 1.4 (Smith Inequality in Local Floer Homology). Let $M$ be a symplectic manifold and $x$ be an isolated one-periodic orbit of a Hamiltonian on $M$. Then for all isolated prime iterations $x^p$ of $x$, we have the following inequality of total dimensions:

$$\dim HF_* (x; \mathbb{F}_p) \leq \dim HF_* (x^p; \mathbb{F}_p).$$

(1.2)

This theorem is proved in Section 4.

Remark 1.5. In Theorem 1.4 the order of iteration is set to be equal exactly $p$ only for the sake of notational convenience. As in Smith theory for group actions on topological spaces, one can replace the iteration order $p$ by its power $p^\ell$, $\ell \in \mathbb{N}$, while still keeping the coefficient field $\mathbb{F}_p$. This, however, is a formal consequence of Theorem 1.4:

$$\dim HF_* (x; \mathbb{F}_p) \leq \dim HF_* (x^p; \mathbb{F}_p) \leq \dim HF_* (x^{p^2}; \mathbb{F}_p) \leq ...$$

as long as $x^{p^\ell}$ is isolated, where in the second inequality we applied (1.2) to $x^p$, etc.

We prove Theorem 1.4 in Section 4. The proof relies on the classical Smith theory. We use generating functions to interpret $HF_* (x^p; \mathbb{F}_p)$ as the singular homology of a pair of sufficiently nice topological spaces $(U, U_-)$ that carry a $\mathbb{Z}_p$-action; and $HF_* (x; \mathbb{F}_p)$ as the singular homology of the fixed point set of this action. Then Theorem 1.4 follows from the relative version of the Smith inequality, (1.1), applied to this pair:

$$\dim H_* (U^{\mathbb{Z}_p}, U_+^{\mathbb{Z}_p}; \mathbb{F}_p) \leq \dim H_* (U, U_-; \mathbb{F}_p).$$

Remark 1.6. Somewhat surprisingly, Corollary 1.2 seems to have no obvious topological counterpart. Namely, there exists a diffeomorphism $\varphi$ of $\mathbb{R}^n$, $n \geq 3$, with an isolated, for all iterations, fixed point $x$ such that the index of $\varphi$ at $x$ is non-zero, but the index of $\varphi^k$ at $x^k$ vanishes for some $k$. (The construction of $\varphi$, described to us by Hernández-Corbato, is non-obvious.) We do not know if this can also happen in the Hamiltonian setting, but Corollary 1.2 does not rule out the existence of such Hamiltonian diffeomorphisms. Overall, the question if the local Floer homology of a homologically non-trivial orbit can vanish for some (isolated) iteration is still open; cf. [GG10].

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2. Preliminaries

2.1. Conventions and basic definitions. Let, as above, $(M, \omega)$ be a closed symplectic manifold and $H: S^1 \times M \to \mathbb{R}$ be a one-periodic in time Hamiltonian on $M$. Here we identify $S^1$ with $\mathbb{R}/\mathbb{Z}$. The Hamiltonian vector field $X_H$ of $H$ is defined by $i_{X_H} \omega = -dH$. The time-one map of the time-dependent flow of $X_H$ is denoted by $\varphi_H$ and referred to as a Hamiltonian diffeomorphism. In this paper we work with iterated Hamiltonians. By $k$-th iteration $H^{tk}$ of $H$, we simply mean $H$ treated as $k$-periodic. (We acknowledge the fact that strictly speaking it is the map $\varphi_H$ rather than the Hamiltonian $H$ that is iterated.) Time-dependent flows of $H$ and $H^{tk}$ agree and the time-$k$ map of the latter is equal to $\varphi_H^k$. 
A capping of a contractible loop \( x: S^1 \to M \) is a map \( A: D^2 \to M \) such that \( A|_{S^1} = x \). The action of a Hamiltonian \( H \) on a capped closed curve \( \tilde{x} = (x, A) \) is

\[
A_H(\tilde{x}) = -\int_A \omega + \int_{S^1} H(t, x(t)) \, dt.
\]

When \( \omega|_{\pi_2(M)} = 0 \), the action \( A_H(\tilde{x}) \) is independent of the capping. The critical points of \( A_H \) on the space of capped closed curves are exactly the capped one-periodic orbits of \( X_H \). The set of critical values of \( A_H \) is called the action spectrum \( S(H) \) of \( H \). These definitions extend to Hamiltonians of any period in an obvious way. Note that the action functional is homogeneous with respect to iteration, i.e., for iterated capped orbits \( \tilde{x}^k \) we have

\[
A_{H^k}(\tilde{x}^k) = k A_H(\tilde{x}).
\]

A periodic orbit \( x \) of \( H \) is called non-degenerate if the linearized return map \( D\varphi_H: T_x(0)M \to T_x(0)M \) has no eigenvalues equal to one. The Conley-Zehnder index \( \mu_{\text{CZ}}(\tilde{x}) \in \mathbb{Z} \) of a non-degenerate capped orbit \( \tilde{x} \) is defined as in [Sa, SaZe]. The index satisfies the determinant identity

\[
\text{sign} \left( \det(D\varphi_H(x(0)) - I) \right) = (-1)^{n - \mu_{\text{CZ}}(\tilde{x})},
\]

where \( \dim M = 2n \) (see Section 2.4 in [Sa]), and is independent of capping when \( c_1(TM) |_{\pi_2(M)} = 0 \).

A symplectic manifold \((M, \omega)\) is called symplectically aspherical if both \( \omega \) and \( c_1(TM) \) vanish on \( \pi_2(M) \). In this case we drop capping from the notation of the action and the index.

### 2.2. Floer homology

In this section we recall the basics of filtered (and local) Floer homology and orientations in Hamiltonian Floer homology. We refer the reader to, e.g., [FO, GG09, HS, MS, Sa, SaZe] for a detailed treatment of Floer homology and to [Ab, FH, Za] for a thorough discussion of orientations.

#### 2.2.1. Filtered and local Floer homology

Let, as above, \((M, \omega)\) be a closed symplectically aspherical manifold and \( H: S^1 \times M \to \mathbb{R} \) be a non-degenerate one-periodic Hamiltonian on \((M, \omega)\). For a generic time-dependent almost complex structure \( J \), the pair \((H, J)\) satisfies the transversality conditions; see [FHS]. Pick two points \( a \) and \( b \in \mathbb{R} \) not in the action spectrum \( S(H) \) of \( H \). The filtered Floer homology \( HF^a, b(H) \) of \( H \) is defined as the homology of the Floer chain complex \( CF_\ast(H, J) \) restricted to the generators \( x \) with \( A_H(x) \in (a, b) \). (We omit the ground ring from the notation of the homology when it is not essential.) If \( H \) is degenerate, we take a \( C^2 \)-small non-degenerate perturbation \( \tilde{H} \) of \( H \) such that \( a \) and \( b \notin S(\tilde{H}) \) and define \( HF^a, b_\ast(\tilde{H}) \) as the filtered Floer homology \( HF^a, b_\ast(\tilde{H}) \) of \( \tilde{H} \).

The local Floer homology \( HF_\ast(H, x) \) (or just \( HF_\ast(x) \)) of an isolated one-periodic orbit \( x: S^1 \to M \) is defined as in [Gi, GG09, GG10]. Note that here we do not require \( x \) to be contractible; see Remark 1.3. The absolute grading of \( HF_\ast(H, x) \) depends on the choice of a trivialization of \( TM|_x \). In Section 3.1, introducing the \( \mathbb{Z}_k \)-action on filtered Floer homology, we imposed the condition that \((M, \omega)\) is symplectically aspherical. In the case of local Floer homology this assumption on \((M, \omega)\) is not needed. The reason is that after composing \( \varphi_H \) with a local loop of Hamiltonian diffeomorphisms we can always assume that \( x \) is a constant orbit of a local Hamiltonian diffeomorphism of \((\mathbb{R}^{2n}, \omega_0)\); cf. Remark 1.3.
2.2.2. Orientations. We describe the setup following [Ab] and [Za]. Let \((M, \omega)\) be a closed symplectically aspherical manifold and \((H, J)\) be a regular pair on \((M, \omega)\). For every contractible one-periodic orbit \(x\) of \(H\), we choose a unitary trivialization \(\Psi\) of \(J\) along \(x\) that comes from a capping of \(x\). Using \(\Psi\), we can linearize the Hamiltonian vector field along \(x\) and write it as \(J_0 S_x\), where \(J_0\) is the multiplication by \(i\) on \(\mathbb{C}^n = \mathbb{R}^{2n}\) and \(S_x\) is a map from \(S^1\) to the space of (symmetric) \(2n \times 2n\) real matrices. Next consider the asymptotic operator
\[
D_x = J_0 \partial_t + S_x,
\]
where \(t \in S^1\); see [Sa, Sect. 2.2]. We extend \(D_x\) to an operator
\[
\tilde{D}_x: W^{1,p}(\mathbb{C}, \mathbb{R}^{2n}) \to L^p(\mathbb{C}, \mathbb{R}^{2n}),
\]
\[
\tilde{D}_x(X) = \partial_t X + J_0 \partial_t X + B_x \cdot X,
\]
where \(p > 2\) and \(B_x \in C^\infty(\mathbb{C}, \mathbb{R}^{2n \times 2n})\) is a function on \(\mathbb{C}\) taking values in the vector space \(\mathbb{R}^{2n \times 2n}\) of \(2n \times 2n\) matrices such that
\[
B_x(e^{-s-2\pi it}) = S_x(t)
\]
for \(s \ll 0\). Since \(x\) is non-degenerate, \(\tilde{D}_x\) is a Fredholm operator and the index of \(\tilde{D}_x\) is equal to \(n - \mu_{CZ}(x)\); see [Ab]. Let
\[
\text{det}(\tilde{D}_x) = \text{det} \left( \text{coker}^* (\tilde{D}_x) \right) \otimes \text{det}(\text{ker}(\tilde{D}_x))
\]
be the determinant line for the extended operator \(\tilde{D}_x\). An orientation choice for an orbit \(x\) is an orientation choice for the determinant line \(\text{det}(\tilde{D}_x)\). A Floer trajectory that connects orbits \(x\) and \(y\) gives a canonical (up to multiplication by a positive number) map between the determinant lines \(\text{det}(\tilde{D}_x)\) and \(\text{det}(\tilde{D}_y)\). This map is then used to determine the signs in the Floer differential by comparing the chosen orientations.

3. The \(\mathbb{Z}_k\)-action on Floer homology

3.1. Definition of the action. In this section we recall the definition of the \(\mathbb{Z}_k\)-action on the \(k\)-iterated Floer homology, treating in detail the role of orientations and sign changes – a point of particular importance to us here. As before, let \((M, \omega)\) be a closed symplectically aspherical manifold and \(H: S^1 \times M \to \mathbb{R}\) be a one-periodic in time Hamiltonian on \((M, \omega)\). Denote by \(H^{2k}\) the \(k\)th iteration of \(H\). Our goal is to define, for \(a\) and \(b\) not in \(S(H^{2k})\), a generator
\[
g: \text{HF}^*_a(b)(H^{2k}) \to \text{HF}^*_a(b)(H^{2k})
\]
of the action and show that \(g^k = id\). The definition works for any coefficient ring, and it extends word-for-word to the local Floer homology \(\text{HF}_*^a(x^k)\) of an isolated iterated orbit \(x^k\).

Fix such \(a\) and \(b\). If the iterated Hamiltonian \(H^{2k}\) is degenerate, we perturb \(H\) so that the \(k\)th iteration of the perturbed Hamiltonian \(\tilde{H}^{2k}\) is non-degenerate and \(C^2\)-close to \(H^{2k}\). Let \(J\) be a \(k\)-periodic almost complex structure, which is one-periodic in a tubular neighborhood of every \(k\)-periodic orbit of \(\tilde{H}^{2k}\). For a generic almost complex structure \(J\) in this class, the pair \((\tilde{H}^{2k}, J)\) satisfies the
Next we show that the induced map \( g \) the chosen orientations. The sign is positive if they agree and negative otherwise.

In the target CF\(_t\), \( x(t) \mapsto \pm x(t + 1) \)

with a continuation map

\[
C: \text{CF}^{(a, b)}_k(\tilde{H}^{2k}_t, J_t) \to \text{CF}^{(a, b)}_k(\tilde{H}^{2k}_1, J_1)
\]

which is constant in the Hamiltonian: \( \tilde{H}^{2k}_{t+1} = \tilde{H}^{2k}_t \). (Here and throughout we use the same notation \( g \) for the chain map and the induced map in homology.) Note that the domain and the target complexes are identical as graded vector spaces, but in general not as complexes. Furthermore, these Floer complexes come with a preferred basis and \( R \) sends generators to generators, up to a sign. When \( F = \mathbb{Z}_2 \), the time-shift map \( R \) is clearly a chain map since all the data, including Floer trajectories, is just shifted in time. If \( J \) is one-periodic in time, the domain and the target complexes agree and we have \( C = \text{id} \). Then \( g = R \) is obviously \( k \)-periodic already on the level of complexes. This need not be the case in general.

Below we will discuss how the signs in the time-shift map are determined and then show in Proposition 3.1 that \( CR \) induces a \( k \)-periodic map, i.e., a \( \mathbb{Z}_k \)-action, on the level of homology; cf. [To, Lemma 2.6]. (A proof of the proposition can also be found in, e.g., [PS, Zh], but there the signs are implicit.)

Let us first explain how to fix the orientations for periodic orbits generating the domain \( \text{CF}^{(a, b)}_k(\tilde{H}^{2k}_t, J_t) \) of \( g \). For the orbits that are time-shifts of each other, e.g. \( x(t) \) and \( x(t + 1) \), we use the same capping to trivialize \( J \). Since all the data is one-periodic in tubular neighborhoods of the orbits, the asymptotic operator for \( x(t + 1) \) becomes the time-shift of the asymptotic operator \( D_x \) for \( x(t) \). Hence we may use the time-shift of the extended operator (see Section 2.2.2) for \( x(t) \) to determine the orientation line for \( x(t + 1) \). We choose extended operators for uniterated \( k \)-periodic orbits according to the rule above. If a periodic orbit is iterated, we require its extended operator to have the same period as the orbit. In other words, we extend asymptotic operators by preserving their minimal period. Once the extended operators are fixed, we choose any orientation of their determinant lines.

In the target \( \text{CF}^{(a, b)}_k(\tilde{H}^{2k}_{t+1}, J_{t+1}) \) we use the same extended operators and we make the same orientation choices as above. Now we have a natural time-shift map between the determinant lines for \( x(t) \) and \( x(t + 1) \). Using this map, we compare the chosen orientations. The sign is positive if they agree and negative otherwise. Next we show that the induced map \( g \) in homology generates a \( \mathbb{Z}_k \)-action.

**Proposition 3.1.** The map \( g \) generates a \( \mathbb{Z}_k \)-action in the Floer homology. In other words, \( g^k = \text{id} \).

**Proof.** We will show that

\[
(CR)^k : \text{CF}^{(a, b)}_k(\tilde{H}^{2k}_t, J_t) \to \text{CF}^{(a, b)}_k(\tilde{H}^{2k}_1, J_1)
\]

can be written as a composition of continuation maps, and hence is the identity on the level of homology. Denote by \( R_i, i \in \mathbb{Z} \), the time-shift map

\[
R_i : \text{CF}^{(a, b)}_k(\tilde{H}^{2k}_{t+i}, J_{t+i}) \to \text{CF}^{(a, b)}_k(\tilde{H}^{2k}_{t+(i+1)}, J_{t+(i+1)})
\]

transversality conditions (see [FHS, Rmk. 5.2]). On the chain level, the generator \( g \) is the composition \( CR \) of the time-shift map

\[
R : \text{CF}^{(a, b)}_k(\tilde{H}^{2k}_t, J_t) \to \text{CF}^{(a, b)}_k(\tilde{H}^{2k}_{t+1}, J_{t+1})
\]

\[
x(t) \mapsto \pm x(t + 1)
\]
and by \( C_{i+1} \) the continuation map
\[
C_{i+1}: \text{CF}^{(a, b)}_\star(\tilde{H}^{2k}_{i+1}, J_{i+1}) \to \text{CF}^{(a, b)}_\star(\tilde{H}^{2k}_{i+1}, J_{i+1})
\]
given by shifting in time the continuation data in \( C \) (i.e., a homotopy in \( H \) and \( J \)). With this new notation \( C = C_1 \) and \( R = R_0 \). Note that all \( R_i \)'s are essentially the same map and \( C_{i+1} \) is the time-shift of \( C_i \), up to signs determined by the relation
\[
\langle x, y \rangle_{C_i} = \langle R_i(x), R_{i-1}(y) \rangle_{C_{i+1}}
\]
where \( \langle x, y \rangle_{C_i} \) denotes the coefficient of \( y \) in the image \( C_i(x) \) of \( x \). In other words the identity
\[
R_{i-1}C_i = C_{i+1}R_i
\]
holds. By applying (3.1) to \((C_1R_0)^k\) we conclude that
\[
(C_1R_0)^k = (C_1C_2 \cdots C_k)(R_{k-1} \cdots R_1R_0) = C_1C_2 \cdots C_k,
\]
which induces the identity map on the level of homology. \( \square \)

### 3.2. Calculation of the supertrace – Proof of Theorem 1.1.

In this section we show that the supertrace of the map
\[
g: \text{HF}^{(ka, kb)}_\star(H^{2k}; \mathbb{F}) \to \text{HF}^{(ka, kb)}_\star(H^{2k}; \mathbb{F})
\]
is equal to the Euler characteristic of \( \text{HF}^{(a, b)}_\star(H; \mathbb{F}) \) and thus prove Theorem 1.1. Here the coefficient ring \( \mathbb{F} \) is required to be a field. This allows us to split the chain complex as a direct sum of kernels and images and conclude that the supertrace at the chain level is equal to the supertrace on the level of homology. We reduce the problem to the case where \( H^{2k} \) is non-degenerate and the interval \((ka, kb)\) contains a single action value. These are non-restrictive assumptions since one can compute the supertrace \( \text{str}(g) \) at the chain level and both \( \text{str}(g) \) and the Euler characteristic are additive with respect to the action filtration.

We further simplify the problem by choosing a continuation
\[
C: \text{CF}^{(ka, kb)}_\star(H^{2k}_{i+1}, J_{i+1}) \to \text{CF}^{(ka, kb)}_\star(H^{2k}_{i+1}, J_{i+1})
\]
with a homotopy constant in the almost complex structure on a tubular neighborhood of every \( k \)-periodic orbit. For a generic homotopy in this class the transversality conditions are satisfied (see [FHS, Rmk. 5.2]). As a result, since \((ka, kb)\) contains a single action value, the continuation part of \( g \) becomes the identity. We have
\[
\text{str}(g) = \text{str}(CR) = \text{str}(R) = \sum_{x \in P(H)} \pm (-1)^{\mu_{C\mathbb{Z}}(x^k)},
\]
where \( \pm \)'s are the signs in the time-shift map \( R \) (see Section 3.1). Note that only the iterated one-periodic orbits appear in the trace formula. Next we show that the sign of a \( k \)-iterated orbit \( x^k \) is equal to \((-1)^{\mu_{C\mathbb{Z}}(x^k)} \mu_{C\mathbb{Z}}(x^k) \) (cf. [BM, Thm. 3]), and hence conclude that
\[
\text{str}(g) = \sum_{x \in P(H)} (-1)^{\mu_{C\mathbb{Z}}(x)} = \chi\left( \text{HF}^{(a, b)}_\star(H; \mathbb{F}) \right).
\]

**Proposition 3.2.** The sign for a \( k \)-iterated orbit \( x^k \) is equal to \((-1)^{\mu_{C\mathbb{Z}}(x)} \mu_{C\mathbb{Z}}(x^k) \).
Proof. Recall that the extended operator $\hat{D}_{x^k}$ for a $k$-iterated orbit $x^k$ is one-periodic; see Section 3.1. The time-shift map generates a $\mathbb{Z}_k$-action on the kernel $\ker\hat{D}_{x^k}$ and the cokernel of $\hat{D}_{x^k}$. Irreducible representations of this action are rotations, and the multiplications by one and negative one. We are interested in the parity of the dimension of $(−1)$-eigenspace.

Observe that $(+1)$-eigenspace corresponds to the kernel and the cokernel of the extended operator $\hat{D}_x$ for the un-iterated orbit $x$. So the parity of the dimension of the $(−1)$-eigenspace is equal to the parity of the difference between the Fredholm indices of $\hat{D}_{x^k}$ and $\hat{D}_x$, which in turn is equal to $\mu_{cz}(x) − \mu_{cz}(x^k)$.

We have the same supertrace formula for the generator $g$ of the $\mathbb{Z}_k$-action on the local Floer homology $HF_*(x^k;\mathbb{F})$ of an isolated iteration $x^k$ of a one-periodic orbit $x: S^1 → M$. In this case, using (2.1), one can also write the formula as

$$ \text{str}(g_∗: HF_*(x^k; \mathbb{F}) → HF_*(x^k; \mathbb{F})) = (−1)^n L(x(0)),$$

where $L(x(0))$ is the Lefschetz index of the fixed point $x(0)$ and $\dim M = 2n$.

Remark 3.3. It is easy to see that for a $\mathbb{Z}_k$-action on a (graded) vector space, the (super)trace of a generator depends in general on the generator. However, the above argument shows that this is not the case for the $\mathbb{Z}_k$-action on the filtered or local Floer homology: all generators of the action have the same supertrace.

4. Smith theory – Proof of Theorem 1.4

Let $(M, \omega)$ be a symplectic manifold and $x$ be a fixed point, a.k.a. a one-periodic orbit, of a Hamiltonian diffeomorphism $\varphi$ of $M$. In this section we show that for isolated prime iterations $x^p$ of $x$, the local Floer homology of $x$ and $x^p$ with $\mathbb{F}_p$-coefficients satisfy the Smith inequality

$$ \dim HF_*(x; \mathbb{F}_p) ≤ \dim HF_*(x^p; \mathbb{F}_p).$$

To prove this, the plan is to interpret $HF_*(x^p; \mathbb{F}_p)$ as the singular homology of a pair of topological spaces $(U, U_−)$ that carry a $\mathbb{Z}_p$-action; and $HF_*(x; \mathbb{F}_p)$ as the singular homology of the fixed point set of this action. Then the result will follow from the classical Smith inequality (1.1), which we restate for the reader’s convenience:

$$ \dim H_*(U_{2n}, U_{−}; \mathbb{F}_p) ≤ \dim H_*(U, U_−; \mathbb{F}_p);$$

see, e.g., [Br, Thm. 4.1] or [Bo].

As the first step we explain how we choose generating functions to pass from the Floer homology to the Morse homology. Since the assertion of the theorem is local, we may assume that $\varphi = \varphi_H$ is a Hamiltonian diffeomorphism defined in a neighborhood of the origin in $(\mathbb{R}^{2n}, \omega_0)$ and the Hamiltonian flow fixes the origin $x$; cf. Remark 1.3. Denote by $\text{gr}(\varphi) = (z, \varphi(z))$ the graph of $\varphi$ inside the twisted product $\mathbb{R}^{2n} × \mathbb{R}^{2n} = (\mathbb{R}^{2n} × \mathbb{R}^{2n}, −\omega_0 × \omega_0)$. We choose a particular identification $\tau$ of $\mathbb{R}^{2n} × \mathbb{R}^{2n}$ with $T^*\mathbb{R}^{2n}$:

$$ \tau(x_0, y_0, x_1, y_1) = (x_1, y_0, y_1 − y_0, x_0 − x_1).$$

Under $\tau$, the diagonal $Δ ⊂ \mathbb{R}^{2n} × \mathbb{R}^{2n}$ is mapped to the zero-section of $T^*\mathbb{R}^{2n}$, and the Lagrangian complement $N ⊂ \mathbb{R}^{2n} × \mathbb{R}^{2n}$ to $Δ$ that consists of vectors of the form $(x_0, 0, 0, y_1) ∈ \mathbb{R}^{2n} × \mathbb{R}^{2n}$ is mapped to the fibers. Below we utilize the composition formula (given for the identification $\tau$) to choose the generating functions for $\text{gr}(\varphi)$ and $\text{gr}(\varphi^p)$; see, e.g., [Ma] for details.
Remark 4.1. Note that even locally, near $x$, the graph $\text{gr}(\varphi)$ is not necessarily transverse to $N$. Hence the image of $\text{gr}(\varphi)$ under $\tau$ is not necessarily a section of $T^*\mathbb{R}^{2n}$. We could choose a different Lagrangian complement $N'$ to $\Delta$, which is transverse to $\text{gr}(\varphi)$ in a neighborhood of $x$, and then view $\text{gr}(\varphi)$ locally as the image of an exact 1-form $df$ in $T^*\mathbb{R}^{2n}$. Similarly, after choosing a different Lagrangian complement if necessary, we could choose a generating function $h : \mathbb{R}^{2n} \to \mathbb{R}$ for the iteration $\text{gr}(\varphi^p)$. However, then we would loose the $\mathbb{Z}_k$-action on the domain of $h$ and have no relation between the generating functions $f$ and $h$. Instead, to keep the action, we use the composition formula to choose generating functions.

Let us write $\varphi$ as the composition of $C^1$-small Hamiltonian diffeomorphisms

$$\varphi = \varphi_k \circ \cdots \circ \varphi_1$$

so that the graphs $\text{gr}(\varphi_i)$ are transverse to $N$ and choose a generating function $f_i : \mathbb{R}^{2n} \to \mathbb{R}$ for each $\text{gr}(\varphi_i)$. By applying the composition formula we define the generating function $F : (\mathbb{R}^{2n})^k \to \mathbb{R}$ as

$$F(x_k, y_0; x_1, y_1, \ldots, x_{k-1}, y_{k-1}) = \sum_{i=1}^{k-1} (f_i(x_i, y_{i-1}) + x_i \cdot (y_{i-1} - y_i))$$

where the dot stands for the inner product. A direct computation shows that $\varphi_i(x_{i-1}, y_{i-1}) = (x_i, y_i)$ if and only if

$$\partial_{\xi} F(x_k, y_0; \xi) = 0,$$

$$\partial_{x_k} F(x_k, y_0; \xi) = y_k - y_0$$

and

$$\partial_{y_0} F(x_k, y_0; \xi) = x_0 - x_k;$$

which implies that

$$\tau(\text{gr}(\varphi)) = \{(x_k, y_0, \partial_{x_k} F, \partial_{y_0} F) \mid \partial_{\xi} F = 0\}.$$

For $\text{gr}(\varphi^p)$, we use the same decomposition

$$\varphi^p = (\varphi_k \circ \cdots \circ \varphi_1)^p$$

and the same choice of generating functions $f_i : \mathbb{R}^{2n} \to \mathbb{R}$ as above. The composition formula yields

$$G(x_0, y_0; x_1, y_1, \ldots, x_{kp-1}, y_{kp-1}) = \sum_{i \in \mathbb{Z}_{kp}} f_i(x_i, y_{i-1}) + x_i \cdot (y_{i-1} - y_i),$$

where $f_{i+k} = f_i$. Observe that $G$ is invariant under the $\mathbb{Z}_p$-action on $(\mathbb{R}^{2n})^{kp}$ generated by the shift

$$(x_i, y_i) \mapsto (x_{i+k}, y_{i+k})$$

of the coordinates. Moreover, the restriction of $G$ to the fixed point set of this action is equal to $pF$.

Using the generating functions $F$ and $G$ chosen above and the isomorphism between the Floer homology and the generating function homology (see [Vil]) we pass to the local Morse homology:

$$\text{HF}_*(x; \mathbb{F}_p) = \text{HM}_*(F, x; \mathbb{F}_p)$$

and

$$\text{HF}_*(x^p; \mathbb{F}_p) = \text{HM}_*(G, x; \mathbb{F}_p).$$
Note that these isomorphisms are up to a shift in degree. Now the problem reduces to showing that
\[ \dim \text{HM}_*(pF, x; F_p) \leq \dim \text{HM}_*(G, x; F_p). \]
Since multiplication by \( p \) does not change the local Morse homology, we replaced \( F \) by \( pF \) in this inequality.

As the last step we choose a \( \mathbb{Z}_p \)-invariant Gromoll–Meyer pair \((U, U^-)\) for \((G, x)\) (see [HHM, Prop. 2.4]) and use the fixed point set \((U^{\mathbb{Z}_p}, U^{\mathbb{Z}_p}_-)\) as a Gromoll–Meyer pair for \((pF, x)\). Theorem 1.4 follows now from the classical Smith inequality, (4.1), applied to \((U, U^-)\):
\[
\dim H_*(U^{\mathbb{Z}_p}, U^{\mathbb{Z}_p}_-; F_p) \leq \dim H_*(U, U^-; F_p).
\]

**Remark 4.2.** In the standard modern proof of the Smith inequality, (4.1), one obtains the inequality as an immediate consequence of the Borel localization theorem for \( \mathbb{Z}_p \)-actions (see, e.g., [Bo]) and one can also expect a version of this theorem to hold for the local or filtered, in the aspherical case, Floer homology. The argument above falls just a little bit short from establishing such a theorem in the local case. The missing part is an identification of the \( \mathbb{Z}_p \)-equivariant generating function homology and the \( \mathbb{Z}_p \)-equivariant Floer homology – an equivariant analog of a result from [Vi].

On other hand, some other refinements of (4.1) do not seem to have obvious Floer theoretic analogs. For instance, [Br, Thm. 4.1] replaces the rank of the total homology by the sum of dimensions for degrees above a fixed thresh. We do not see how to extend this result to the Floer theoretic setting. (The most naive attempts break down already for a strongly non-degenerate orbit.)

**Remark 4.3.** The method used in this section can be applied whenever the (iterated) Floer homology can be identified with the homology of a sufficiently nice topological \( \mathbb{Z}_p \)-space and this identification behaves well under iteration. In particular, one should be able to use it to establish the Smith inequality for the filtered Floer homology of Hamiltonians on the tori or cotangent bundles. However, some parts of the argument require modifications and we omit the details; for the main result of [ShZh] should hold in much greater generality.

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