ALTERNATING CONVOLUTIONS OF CATALAN NUMBERS

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Abstract. A new class of alternating convolutions concerning binomial coefficients and Catalan numbers are evaluated in closed forms.

1. Introduction and Motivation

The Catalan numbers

\[ C_n = \frac{1}{n+1} \binom{2n}{n} \text{ for } n \in \mathbb{N}_0 \]

are probably the most frequently encountered sequence in mathematics. There exist numerous interpretations in enumerative combinatorics and remarkable identities about them that can be found in the monographs by Koshy [6], Roman [10] and Stanley [12] as well as in [4,5,13]. For instance, these numbers satisfy the nonlinear recurrence relation

\[ C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k} \]

and the Touchard identity

\[ C_{n+1} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} 2^{-2k} \binom{n}{2k} C_k. \]

Here and forth, \( \lfloor x \rfloor \) denotes the integer part for the real number \( x \). For \( m \in \mathbb{N}_0 \) and \( i, j \in \mathbb{Z} \), we shall utilize the notation \( "i \equiv m j" \) for "\( i \) is congruent to \( j \) modulo \( m". \) The logical function \( \chi \) is defined for brevity by \( \chi(\text{true}) = 1 \) and \( \chi(\text{false}) = 0. \)

Recently, Mikić [7] 2019 found, by combinatorial bijections, the following unusual convolution identities:

\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} C_k C_{n-k} = \frac{2\chi(n \equiv 2 0)}{n+2} \left( \frac{n}{\lfloor \frac{n}{2} \rfloor} \right)^2, \tag{1} \]

\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left( \frac{2n - 2k}{n - k} \right) C_k = \left( \frac{n}{\lfloor \frac{n}{2} \rfloor} \right)^2. \tag{2} \]

Prodinger [8] provided different proofs by making use of Zeilberger’s algorithm, Dixon’s formula and its variants. Let \((x)_n\) be the Pochhammer symbol given by

\( (x)_0 = 1 \) and \( (x)_n = x(x+1) \cdots (x+n-1) \) for \( n \in \mathbb{N}. \)

The objective of this paper is to show the following generalizations.

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Theorem 1 \((n, \lambda \in \mathbb{N}_0)\).
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} C_{k+\lambda} C_{n-k+\lambda} = \frac{\lambda! \chi(n \equiv 2 \mod 0)}{(2+n)\lambda} \binom{2\lambda}{\lambda} \left(\frac{n}{\lfloor \frac{n}{2} \rfloor}\right) C_{\lambda + \lfloor \frac{n}{2} \rfloor}.
\]

Theorem 2 \((n, \lambda \in \mathbb{N}_0)\).
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(2n - 2k + 2\lambda)}{n - k + \lambda} C_{k+\lambda} = \frac{\lambda! (2\lambda)}{(2+n)\lambda} \binom{2\lambda}{\lambda} \left(\frac{n}{\lfloor \frac{n}{2} \rfloor}\right) \left(\frac{n + 2\lambda}{\lambda + \lfloor \frac{n}{2} \rfloor}\right).
\]

Three further binomial identities of alternating convolutions will also be established.

Theorem 3 \((n, \lambda \in \mathbb{N}_0)\).
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k + 2\lambda}{k + \lambda} \binom{2n - 2k + 2\lambda}{n - k + \lambda} = \frac{\lambda! \chi(n \equiv 2 \mod 0)}{(1+n)\lambda} \binom{2\lambda}{\lambda} \left(\frac{n}{\lfloor \frac{n}{2} \rfloor}\right) \left(\frac{n + 2\lambda}{\lambda + \lfloor \frac{n}{2} \rfloor}\right).
\]

Theorem 4 \((n, \lambda \in \mathbb{N}_0)\).
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{2k + 2\lambda}{k + \lambda} \frac{2n - 2k + 2\lambda}{n - k + \lambda} (n - k + \lambda) = \frac{\lambda! (n \equiv 2 \mod 0)}{(n)\lambda} \binom{2\lambda}{\lambda} \left(\frac{n}{\lfloor \frac{n}{2} \rfloor}\right) \left(\frac{n + 2\lambda}{\lambda + \lfloor \frac{n}{2} \rfloor}\right) \chi(n \equiv 2 \mod 0) \chi(n \equiv 2 \mod 1).
\]

Theorem 5 \((n, \lambda, \mu \in \mathbb{N}_0)\).
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{2k + 2\lambda}{k + \lambda} \frac{2n - 2k + 2\mu}{n - k + \mu} \binom{2\lambda + 2\mu}{\lambda + \mu} = \frac{\lambda! (n + \lambda + \mu)}{(n)\lambda} \left(\frac{n}{\lfloor \frac{n}{2} \rfloor}\right) \left(\frac{n + \lambda + \mu}{\lambda + \lfloor \frac{n}{2} \rfloor}\right) \chi(n \equiv 2 \mod 0) \chi(n \equiv 2 \mod 1) \chi(n \equiv 2 \mod 2) \chi(n \equiv 2 \mod 3) \chi(n \equiv 2 \mod 4).
\]

The rest of the paper will be organized as follows. As a preliminary, we shall prove, in the next section, a product formula for confluent hypergeometric \(1F_1\) and \(2F_2\)-series, which may serve as a counterpart of the product formulae due to Bailey [1]. By extracting the coefficients of \(x^n\) across these hypergeometric equations, we derive, in Section 3, three binomial formulae of alternating sums, which contain the five summation theorems just displayed as particular cases. Finally, the paper will end with Section 4, where two equivalent integral formulae are proposed as problems.

2. PRODUCTS OF HYPERGEOMETRIC SERIES

Recall that the \(\Gamma\)-function (see [9, §8] for example) is given by the beta integral
\[
\Gamma(x) = \int_{0}^{\infty} u^{x-1}e^{-u}du \quad \text{for} \quad \Re(x) > 0.
\]

It satisfies Euler’s reflection property
\[
\Gamma(x) \times \Gamma(1-x) = \frac{\pi}{\sin \pi x}
\]
and Legendre’s duplicate formula
\[\Gamma(2x) = \Gamma(x)\Gamma(x + \frac{1}{2})\frac{2^{2x-1}}{\sqrt{\pi}}.\]

For the sake of brevity, we shall utilize the following multiparameter expression
\[\Gamma \left[ \alpha, \beta, \ldots, \gamma \middle| A, B, \ldots, C \right]_{n} = \frac{\Gamma(\alpha)_{n}\Gamma(\beta)_{n}\cdots\Gamma(\gamma)_{n}}{\Gamma(A)_{n}\Gamma(B)_{n}\cdots\Gamma(C)_{n}}.\]

Analogously, the quotient of the Pochhammer symbol will be abbreviated to
\[\left[ \alpha, \beta, \ldots, \gamma \middle| A, B, \ldots, C \right]_{n} = \frac{(\alpha)_{n}(\beta)_{n}\cdots(\gamma)_{n}}{(A)_{n}(B)_{n}\cdots(C)_{n}}.\]

According to Bailey [2, §2.1], the classical hypergeometric series reads as
\[pF_{q} \left[ \begin{array}{c} a_{1}, a_{2}, \ldots, a_{p} \\ b_{1}, b_{2}, \ldots, b_{q} \end{array} \middle| x \right] = \sum_{k=0}^{\infty} \frac{a_{1}, a_{2}, \ldots, a_{p}}{b_{1}, b_{2}, \ldots, b_{q}} \frac{x^{k}}{k!}.\]

When \(p \leq q\), this series is said confluent. In 1928, Bailey [1] found, among others, two product formulae for confluent hypergeometric series
\[1_{F} \left[ \begin{array}{c} a \\ c \end{array} \middle| x \right] \times 1_{F} \left[ \begin{array}{c} a \\ c - x \end{array} \middle| x \right] = 2_{F} \left[ \begin{array}{c} a, c - a \times x^{2} \\ c, \frac{1+a}{2} \end{array} \middle| x \right], \quad (3)\]
\[1_{F} \left[ \begin{array}{c} a \\ 2a \end{array} \middle| x \right] \times 1_{F} \left[ \begin{array}{c} c \\ 2c \end{array} \middle| x \right] = 2_{F} \left[ \begin{array}{c} a + 1 \times \frac{a+1}{2}, a + 1 \times \frac{a+1}{2} \times x^{2} \\ a + c, a + 1 \times \frac{a+1}{2}, a + 1 \times \frac{a+1}{2} \end{array} \middle| x \right]. \quad (4)\]

They resemble the following beautiful product formula (cf. Bailey [2, §10.1]) discovered by Clausen one century earlier
\[2_{F} \left[ \begin{array}{c} a, c \\ a + c + \frac{1}{2} \end{array} \middle| x \right] = 3_{F} \left[ \begin{array}{c} a + c, 2a, 2c \\ a + c + \frac{1}{2}, 2a + 2c \end{array} \middle| x \right]. \]

By inserting an extra linear factor \(\lambda + k\) in (3), we find the extended formula.

**Lemma 6** \((a, c, \lambda \in \mathbb{R})\).
\[1_{F} \left[ \begin{array}{c} a \\ c \end{array} \middle| x \right] \times 2_{F} \left[ \begin{array}{c} 1 + \lambda, a \\ \lambda, c \end{array} \middle| x \right] = 3_{F} \left[ \begin{array}{c} 1 + \lambda, a, c - a \times x^{2} \\ \lambda, c, \frac{1+a}{2} \end{array} \middle| x \right] - \frac{ax}{c\lambda} 2_{F} \left[ \begin{array}{c} 1 + a, c - a \times x^{2} \\ c, \frac{1+a}{2}, \frac{2+a}{2} \end{array} \middle| x \right]. \]

In particular for \(\lambda = c - 1\), we get a variant formula of (3):
\[1_{F} \left[ \begin{array}{c} a \\ c - 1 \end{array} \middle| x \right] \times 1_{F} \left[ \begin{array}{c} a \\ c \end{array} \middle| x \right] = 2_{F} \left[ \begin{array}{c} a, c - a \times x^{2} \\ c - 1, c, \frac{1+a}{2} \end{array} \middle| x \right] - \frac{ax}{c(c-1)} 2_{F} \left[ \begin{array}{c} 1 + a, c - a \times x^{2} \\ c, \frac{1+a}{2}, \frac{2+a}{2} \end{array} \middle| x \right]. \quad (5)\]

**Proof of Lemma 6** By means of the linear relation
\[\frac{\lambda + k}{k} = \frac{\lambda - a}{\lambda} + \frac{a + k}{\lambda}\]
it is not hard to get the contiguous relation
\[ 4F_3 \left[ \begin{array}{c} a, c, e, 1 + \lambda \\ 1 + a - c, 1 + a - e, \lambda \end{array} \right] = \frac{\lambda - a}{\lambda} \times 3F_2 \left[ \begin{array}{c} a, c, e \\ 1 + a - c, 1 + a - e \end{array} \right] \]
where the condition \( \Re(\frac{\lambda}{2} - c - e) > 0 \) is provided for convergence. Evaluating the former \( 3F_2 \)-series by the Dixon formula (cf. Bailey [2, §3.1])
\[ 3F_2 \left[ \begin{array}{c} a, c, e \\ 1 + a - c, 1 + a - e \end{array} \right] = \frac{\pi}{\Gamma} \left[ \begin{array}{c} 1 + a - c, 1 + a - e \end{array} \right] \frac{\omega}{\lambda} \frac{\omega - c}{\lambda} \frac{\omega - e}{\lambda} \frac{1}{1 + a - c, 1 + a - e} \]
and the latter \( 3F_2 \)-series by “\( D_{-1,-1} \)” due to the author [3, Example 18]
\[ 3F_2 \left[ \begin{array}{c} 1 + a, c, e \\ 1 + a - c, 1 + a - e \end{array} \right] = \frac{\omega^2 - 2\omega - 1}{\pi} \Gamma \left[ \begin{array}{c} 1 + a - c, 1 + a - e \end{array} \right] \frac{\omega}{\lambda} \frac{\omega - c}{\lambda} \frac{\omega - e}{\lambda} \frac{1}{1 + a - c, 1 + a - e} \]
and then simplifying the result, we get the expression
\[ 4F_3 \left[ \begin{array}{c} a, c, e, 1 + \lambda \\ 1 + a - c, 1 + a - e, \lambda \end{array} \right] = \frac{\lambda - a}{\lambda} \Gamma \left[ \begin{array}{c} 1 + a - c, 1 + a - e \end{array} \right] \frac{\omega}{\lambda} \frac{\omega - c}{\lambda} \frac{\omega - e}{\lambda} \frac{1}{1 + a - c, 1 + a - e} \]
In particular, when the series is terminated by \( a = -n \) with \( n \in \mathbb{N}_0 \), we have
\[ 4F_3 \left[ \begin{array}{c} -n, c, e, 1 + \lambda \\ 1 - e - n, 1 - e - n, \lambda \end{array} \right] = \frac{\lambda + n}{\lambda} \times \left\{ \begin{array}{ll} -n, 1 - c - e - n \\ 1 - c - n, 1 - e - n \end{array} \right\} \]
Now we turn to examine, by letting \( i + k = n \), the product
\[ 1F_1 \left[ \begin{array}{c} a \\ c \end{array} \right] \times 2F_2 \left[ \begin{array}{c} 1 + \lambda, a \\ \lambda, c \end{array} \right] = \frac{\omega}{\lambda} \times \left\{ \begin{array}{ll} -n, a, 1 - c - n, \lambda + 1 \\ c, 1 - a - n, \lambda \end{array} \right\} \]
Writing the last sum in terms of \( 4F_3 \)-series and then evaluating it by (7)
\[ 4F_3 \left[ \begin{array}{c} -n, a, 1 - c - n, \lambda + 1 \\ c, 1 - a - n, \lambda \end{array} \right] = \frac{\lambda + n}{\lambda} \times \left\{ \begin{array}{ll} -n, c - a \\ c, 1 - a - n \end{array} \right\} \]
we confirm, in view of the parity of \( n \), the product formula in Lemma [4]

3. Binomial Convolution Formulae

By extracting the coefficient of \( x^n \) across the equations [3, 4] and [5] on hypergeometric products, we find the following three identities.
Proposition 7 \((n \in \mathbb{N}_0 \text{ and } a, \ c \in \mathbb{R})\). 

\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(a)_{k}(c)_{n-k}}{(c)_{k}(c)_{n-k}} = \frac{n!}{(c)_n} \binom{a, c-a}{1, c} \chi(n \equiv 2 \ 0). \]

Proposition 8 \((n \in \mathbb{N}_0 \text{ and } a, \ c \in \mathbb{R})\). 

\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(a)_{k}(c)_{n-k}}{(2a)_{k}(2c)_{n-k}} = \binom{1, a+c}{2a, 2c} \binom{a, c}{1, a+c} \chi(n \equiv 2 \ 0). \]

Proposition 9 \((n \in \mathbb{N}_0 \text{ and } a, \ c \in \mathbb{R})\). 

\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(a)_{k}(c)_{n-k}}{(c)_{k}(c-1)_{n-k}} = \frac{n!}{(c-1)_{n+1}} \binom{a, c-a}{1, c} \times \begin{cases} c + \frac{n-2}{2}, & n \equiv 2 \ 0; \\ a + \frac{n-1}{2}, & n \equiv 2 \ 1. \end{cases} \]

Expressing the quotients of shifted factorials in terms of binomial coefficients

\[ \frac{\left(\frac{1}{2} + \lambda\right)_{k}}{(1 + \lambda)_{k}} = \frac{\binom{2k+2\lambda}{k+\lambda}}{4^k \binom{2\lambda}{\lambda}}, \quad \frac{\left(\frac{3}{2} + \lambda\right)_{k}}{(1 + \lambda)_{k}} = \frac{\binom{k+\lambda}{k+\lambda}}{4^k \binom{2\lambda}{\lambda}}; \]

\[ \frac{\left(\frac{1}{2} + \lambda\right)_{k}}{(2 + \lambda)_{k}} = \frac{C_{k+\lambda}}{4^k C_{\lambda}}, \quad \frac{\left(\frac{3}{2} + \lambda\right)_{k}}{(2 + \lambda)_{k}} = \frac{\binom{k+\lambda}{k+\lambda}}{4^k \binom{1+2\lambda}{\lambda}}; \]

we can confirm the five identities anticipated in the introduction as follows:

- Theorem 1 \(a = \frac{1}{2} + \lambda \text{ and } c = 2 + \lambda \) in Proposition 7.
- Theorem 2 \(a = \frac{1}{2} + \lambda \text{ and } c = 2 + \lambda \) in Proposition 8.
- Theorem 3 \(a = \frac{3}{2} + \lambda \text{ and } c = 1 + \lambda \) in Proposition 9.
- Theorem 4 \(a = \frac{3}{2} + \lambda \text{ and } c = 1 + \mu \) in Proposition 9.
- Theorem 5 \(a = \frac{1}{2} + \lambda \text{ and } c = 2 + \mu \) in Proposition 8.

Furthermore, we can derive four “reciprocal formulae” of those displayed in Theorems 1, 2, 3, and 4.

Corollary 10 \((a = 2 + \lambda \text{ and } c = \frac{1}{2} + \lambda \text{ in Theorem 7}) \ n, \ \lambda \in \mathbb{N}_0). \)

\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(n-1)(n-3)C_{\lambda}^2}{C_{k+\lambda}C_{n-k+\lambda}} = \frac{3C_{\lambda}(\frac{2\lambda}{\lambda}) \binom{n}{\frac{1}{2} \lambda}}{C_{\lambda+\frac{1}{2} \lambda}} \chi(n \equiv 2 \ 0). \]

Corollary 11 \((a = 2 + \lambda \text{ and } c = \frac{3}{2} + \lambda \text{ in Theorem 8}) \ n, \ \lambda \in \mathbb{N}_0). \)

\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{n^{(1+2\lambda)}_{k}}{C_{\lambda+1+2\lambda}C_{n-k+\lambda}} = \frac{(1 + n + 2\lambda)C_{\lambda}(\frac{1+2\lambda}{\lambda}) \binom{n}{\frac{1}{2} \lambda+\frac{1}{2} \lambda}}{\binom{(\lambda+n+1)_{\lambda+\frac{1}{2} \lambda}}{(\lambda+n+1)_{\lambda+\frac{1}{2} \lambda}} \times \begin{cases} \frac{n}{1-n}, & n \equiv 2 \ 0; \\ \frac{1+n-2n}{1+n-2n}, & n \equiv 2 \ 1. \end{cases} \]

Corollary 12 \((a = 1 + \lambda \text{ and } c = \frac{1}{2} + \lambda \text{ in Theorem 9}) \ n, \ \lambda \in \mathbb{N}_0). \)

\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(1-2a)(\frac{1}{2} \lambda)^2}{(2k+2\lambda)_{2n-2k+2\lambda}} = \frac{(\frac{2\lambda}{\lambda}) \binom{n}{\frac{1}{2} \lambda} \chi(n \equiv 2 \ 0)}{\binom{(\lambda+n)_{\lambda+\frac{1}{2} \lambda}}{(\lambda+n)_{\lambda+\frac{1}{2} \lambda}} \binom{2\lambda+n}{\lambda+\frac{1}{2} \lambda}}. \]
Corollary 13 \((a = 1 + \lambda \text{ and } c = \frac{1}{2} + \lambda \text{ in Theorem } \boxed{1} n, \ \lambda \in \mathbb{N}_0)\).
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{n(1 + 2\lambda)(1 + 2n + 2\lambda)(2\lambda)^2}{(1 + 2k + 2\lambda)(2k+2\lambda)(2n-2k+2\lambda)(n-k+\lambda)}
\]
\[
= \frac{(1 + 2\lambda)^2}{(\lambda+n)(2\lambda+2n)(2\lambda+\frac{1}{2})} \times \begin{cases} 
 n, & n \equiv 2 \mod 0; \\
 n + 1, & n \equiv 2 \mod 1.
\end{cases}
\]

4. INTEGRAL REPRESENTATIONS

According to the expressions of the beta integrals (cf. \[11\])
\[
\binom{2m}{m} = \frac{2^{2m}}{\pi} \beta\left(\frac{1}{2}, \frac{1}{2} + m\right) = \frac{2^{2m}}{\pi} \int_0^1 \frac{x^m - \frac{1}{2}}{\sqrt{1-x}} \, dx,
\]
\[
C_m = \frac{2^{1+2m}}{\pi} \beta\left(\frac{3}{4}, \frac{1}{2} + m\right) = \frac{2^{1+2m}}{\pi} \int_0^1 y^{m-\frac{1}{4}} \sqrt{1-y} \, dy;
\]
we can reformulate the sum in Theorem \[1\] as follows
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} C_{k+\lambda} C_{n-k+\lambda}
\]
\[
= \frac{4^{1+n+2\lambda}}{\pi^2} \int_0^1 \int_0^1 (xy)^{\lambda - \frac{1}{4}} \sqrt{(1-x)(1-y)} \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{n-k} y^k \, dx \, dy
\]
\[
= \frac{4^{1+n+2\lambda}}{\pi^2} \int_0^1 \int_0^1 (xy)^{\lambda - \frac{1}{4}} (x-y)^n \sqrt{(1-x)(1-y)} \, dx \, dy.
\]
Then we get the following integral formula equivalent to Theorem \[1\]
Corollary 14 \((n, \ \lambda \in \mathbb{N}_0)\).
\[
\int_0^1 \int_0^1 (xy)^{\lambda - \frac{1}{4}} (x-y)^n \sqrt{(1-x)(1-y)} \, dx \, dy = \frac{\pi^2 \lambda! \chi(n \equiv 2)}{4^{1+n+2\lambda}(2+n)\lambda} \binom{2\lambda}{\frac{n}{2}} (\frac{n}{2}) C_{\lambda+\frac{1}{2}}.
\]

The sum in Theorem \[2\] can analogously be manipulated:
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2n-2k+2\lambda}{n-k+\lambda} C_{k+\lambda}
\]
\[
= \frac{2^{1+2n+4\lambda}}{\pi^2} \int_0^1 \int_0^1 (xy)^{\lambda - \frac{1}{4}} \sqrt{\frac{1-y}{1-x}} \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{n-k} y^k \, dx \, dy
\]
\[
= \frac{2^{1+2n+4\lambda}}{\pi^2} \int_0^1 \int_0^1 (xy)^{\lambda - \frac{1}{4}} (x-y)^n \sqrt{\frac{1-y}{1-x}} \, dx \, dy,
\]
which leads us to another integral formula.
Corollary 15 \((n, \ \lambda \in \mathbb{N}_0)\).
\[
\int_0^1 \int_0^1 (xy)^{\lambda - \frac{1}{4}} (x-y)^n \sqrt{\frac{1-y}{1-x}} \, dx \, dy = \frac{\pi^2 \lambda! \chi(n \equiv 2)}{2^{1+2n+4\lambda}(2+n)\lambda} \binom{2\lambda}{\frac{n}{2}} (\frac{n}{2}) \left(\frac{n+2\lambda}{\lambda+\frac{1}{2}}\right).
\]

Two further integral formulae corresponding to Theorems \[3\] and \[4\] can be produced in a similar manner. Finally, an intriguing question is how to evaluate these integrals directly?
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