Abstract

We develop a general theory for Markov chains whose transition probabilities are the coefficients of descent operators on combinatorial Hopf algebras. These model the breaking-then-recombining of combinational objects. Examples include the various card-shuffles of Diaconis, Fill and Pitman, Fulman’s restriction-then-induction chains on the representations of the symmetric group, and a plethora of new chains on trees, partitions and permutations. The eigenvalues of these chains can be calculated in a uniform manner using Hopf algebra structure theory, and there is a simple expression for their stationary distributions. For an important subclass of chains analogous to the top-to-random shuffle, we derive a full right eigenbasis, from which follow exact expressions for expectations of certain statistics of interest. This greatly generalises the coproduct-then-product chains previously studied in joint work with Persi Diaconis and Arun Ram.

1 Introduction

There has long been interest in using algebra and combinatorics to study Markov chains [DS05, SD09, DS18]. One highly successful technique is the theory of random walks on groups [Dia88, SC04] and on monoids [BHR99, Bro00, ASST15]. The transition matrix of one of these chains is the matrix for a multiplication map in the group algebra or the monoid algebra. The eigenvalues, and sometimes eigenfunctions, of the transition matrix can be calculated in terms of the representation theory of the group or monoid. Such algebraic data has implications for the long term behaviour of the chain, such as its stationary distribution and convergence rate.

The purpose of this paper is to execute similar eigen-data analysis when the transition matrix comes from a descent operator on a combinatorial Hopf algebra, instead of from multiplication in a group or monoid. The chains describe the breaking, or breaking-then-recombining, of some combinatorial object, such as a graph or a tree. To illustrate the general theory, this paper will concentrate on the following two examples.

Example 1.1 (A chain on organisational structures). A company starts with \( n_0 \) employees in a tree structure, where each employee except the boss has exactly one direct superior. The boss is the direct superior of the department heads. For example, Figure 1.1 is a company with \( n_0 = 8 \). It has two departments: A heads the accounting department, consisting of himself and B, and C heads the consulting department, of C, D, E, F and G. And C is the direct superior to D, E and F.

Let \( q \) be a parameter between 0 and 1. The monthly performance of each employee is, independently, uniformly distributed between 0 and 1, and each month all employees with performance below \( 1 - q \) are fired. Each firing causes a cascade of promotions; for the specifics, see the third
paragraph of Section 5, and Figure 5.1. The chain keeps track of the tree structure of the company, but does not see which employee is taking which position.

It is clear that the stationary distribution of the chain is concentrated at the company consisting of only the boss. The eigenvalues of this chain are 1 and \( q^n \), for \( 2 \leq n' \leq n_0 \), and their multiplicities are the number of “connected” subsets of size \( n' \) of the starting tree containing the boss. (So, for the chain starting at the tree in Figure 1.1, the eigenvalue \( q^3 \) has multiplicity 5, corresponding to the subsets of the boss with A and B, with A and C, with C and D, with C and E, and with C and F.) Theorem 5.4.iii gives a formula for a full right eigenbasis of this chain. One application of these expressions is as follows: suppose the company in Figure 1.1 has a project that requires \( s_1 \) accountants and \( s_2 \) consultants. The expected number of such teams falls roughly by a factor of \( q^{1+s_1+s_2} \) monthly (see Corollary 5.5 for the precise statement). Such results are obtained by relating this chain to a decorated version of the Connes-Kreimer Hopf algebra of trees.

Example 1.2 (Relative time on a to-do list). You keep an electronic to-do list of \( n \) tasks. Each day, you complete the task at the top of the list, and are handed a new task, which you add to the list in a position depending on its urgency (more urgent tasks are placed closer to the top). Assume the incoming tasks are equally distributed in urgency relative to the \( n-1 \) tasks presently on the list, so they are each inserted into the list in a uniform position. To produce from this process a Markov chain on permutations, relabel the tasks by \( \{1, \ldots, n\} \) at the end of each day so that 1 indicates the newest addition to the list, 2 indicates the next newest task, i.e. the newest amongst the \( n-1 \) tasks not labelled 1, and so on, so that \( n \) is the task spending the longest time on the list.

Figure 1.2 shows a possible trajectory of this chain over three days. On the first day, the top task, labelled 3, is completed, and tasks 1, 5, 4, 2 remain, in that order. In this example, the random urgency of the new incoming task places it in position 3. This incoming task is now the task spending the shortest time on the list, so it is relabelled 1. The task spending the next shortest time on the list is that previously labelled 1, and its label is now changed to 2. Similarly, the task spending the third-shortest time on the list is that previously labelled 2, and its label is now changed to 3, and so on. The next two days are similar.

The stationary distribution of this chain is the uniform distribution. Its eigenvalues are \( 0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-2}{n}, 1 \), and the multiplicity of \( \frac{j}{n} \) is \( (n-j)! - (n-j-1)! \), the number of permutations of \( n \) objects fixing pointwise 1, 2, \ldots, \( j \) but not \( j+1 \). Theorem 6.6 gives a full right eigenbasis indexed by such permutations. One consequence of such eigenvectors (Corollary 6.8) is as follows: assume the tasks started in ascending order (i.e. newest at the top and oldest at the bottom). After \( t \) days, consider the \( n-j \) tasks at the bottommost positions of the list. The position of the newest task among
Figure 1.2: A possible three-day trajectory of the Markov chain of “relative time on a to-do list”, for \( n = 5 \). The horizontal arrows indicate the (random) positions of incoming tasks.

these is distributed as follows:

\[
\text{position } j + 1 \text{ with probability } \frac{1}{n-j} \left( 1 + \frac{j}{n}(n-j-1) \right);
\]

\[
\text{positions } j + 2, j + 3, \ldots, n \text{ each with probability } \frac{1}{n-j} \left( 1 + \frac{j}{n} \right).
\]

Remarkably, the above formula also gives the distribution of the card of smallest value after \( t \) top-to-random shuffles of \([AD86]\) - this is because the probability distribution of a deck under top-to-random shuffling, after \( t \) steps starting at the identity permutation, agrees with those of the to-do list chain \([Pan18, Th. 3.14]\). (Note however that the trajectories for the two chains do not correspond.) So the new to-do list chain opens up a whole new viewpoint to study the top-to-random shuffle. \([Pan18, Th. 3.11]\) also relates the to-do list chain to the restriction-then-induction chains of Fulman \([Ful04]\) (see below), namely that observing the RSK shape of the to-do list chain gives the restriction-then-induction chain.

The descent operators \( m \Delta_P \) are variants of the coproduct \( \Delta \) followed by the product \( m \), indexed by a probability distribution \( P \) on compositions. In terms of the associated Markov chains, \( P \) is the distribution of the piece sizes in the breaking step. For example, when \( P \) is concentrated at the composition \((1, n-1)\), the chain models removing then reattaching a piece of size 1, analogous to the “top-to-random” card-shuffle. The associated descent operator is \( T2R_n := \frac{1}{n} m(\text{Proj}_1 \otimes \text{id}) \Delta \), where \( \text{Proj}_1 \) denotes projection to the degree 1 subspace. The case where \( P \) is a multinomial distribution was covered in \([DPR14]\); the present paper is a vast extension of that framework. As in \([DPR14]\), all our results rely heavily on the structure theory of cocommutative Hopf algebras; however, unlike \([DPR14]\), some basic Markov chain theory plays an essential part in the proofs of entirely algebraic statements (Theorems 3.5, 4.4 and 4.6).

This paper is organised as follows: Section 2 skims the necessary probabilistic and algebraic background, including much non-standard, probability-inspired notation regarding descent operators. Section 3 finds the eigenvalues, multiplicities, and stationary distributions of all descent operator Markov chains. Section 4 proves an eigenbasis formula for the chains driven by \( T2R_n \) and related maps, and establishes a “recursive lumping” property for such chains. Sections 5 and 6 applies these general results to, respectively, the decorated Connes-Kreimer Hopf algebra of trees and the Malvenuto-Reutenauer algebra of permutations to obtain the results of Examples 1.1 and 1.2 above.
The main results of this paper were announced in the extended abstract [Pan15a].

Below are some vignettes of T2R_n chains on various Hopf algebras, to demonstrate the diverse range of chains that this framework covers. Note that the long term behaviour varies greatly depending on the properties of the associated Hopf algebra.

Card-shuffling The descent operators $m \Delta_P$ on the shuffle algebra (see Example 2.5) induce precisely the “cut and interleave” shuffles of [DFP92]: cut the deck according to the distribution $P$, then drop the cards one by one from the bottom of piles chosen with probability proportional to pile size. In particular, T2R_n corresponds to the much-studied “top-to-random” shuffle [AD86, Sta02]: take the top card off the deck, then re-insert it at a uniformly chosen position. (The time-reversal of this chain is the very important Tsetlin library model [Tse63] of storage allocation, see [Pan19] for an overview of the many results on this chain and pointers to diverse applications.) A multi-graded version of Theorem 3.5 recovers (the unweighted case of) [Pha91] on the spectrum of the top-to-random shuffle: if all $n$ cards in the deck are distinct, then the multiplicity of the eigenvalue $\frac{j}{n}$ is the number of permutations of $n$ objects with $j$ fixed points. From the eigenbasis formula of Theorem 4.4 one obtains explicit expectation formulae for a variety of “pattern functions” on the bottom $n-j$ cards of the deck. For example, the case $n-j=2$ gives the following analogue of [DPR14, Ex. 5.8]: after $t$ top-to-random shuffles of a deck of $n$ distinct cards, starting in ascending order, the probability that the bottommost card has lower value than the card immediately above is $\left(1 - \left(\frac{n-2}{n}\right)^t\right)^{\frac{1}{2}}$.

Restriction-then-induction, or box moves The descent operators on the Schur basis of the algebra of symmetric functions [Sta99, Sec. 7.10] induce the chains of [Ful04], on the irreducible representations of the symmetric group $S_n$. In the case of T2R_n, this chain is: restrict to $S_{n-1}$, induce back to $S_n$, and pick an irreducible constituent with probability proportional to the dimension of its isotypic component. Because the Littlewood-Richardson coefficients involved in this case are particularly simple, this chain has a neat interpretation in terms of partition diagrams: remove a random box using the hook walk of [GNW79], then add a random box according to the complementary hook walk of [GNWS04]. See [Pan18, Sec. 3.2.1] for a detailed derivation.

The unique stationary distribution of this chain is the ubiquitous Plancherel measure $\pi(x) = \frac{(\dim x)^2}{n!}$. The eigenfunctions show that, after $t$ moves, the expected character ratio $\frac{\chi(\sigma)}{\chi(id)}$ on an $n'$-cycle $\sigma$ is $\left(\frac{n-n'}{n}\right)^t$ times its initial value. [Ful05] uses similar results to study central limit theorems for character ratios.

Rock-breaking and coupon collection Work in the elementary basis of the algebra of symmetric functions. The states of the descent operator chains are partitions, viewed as a multiset of integers recording the sizes of a collection of rocks. [DPR14, Sec. 4] analysed the chain where each rock breaks independently according to the binomial distribution. The T2R_n chain is as follows: pick a rock in the collection with probability proportional to its size, and chip off a piece of size 1 from this rock. As noted in [Cha18a], starting the T2R_n chain from one single rock gives a rephrasing of the classical coupon-collection problem: the number of chipped-off pieces correspond to the different coupons collected, and the size of the large rock corresponds to the number of uncollected coupons.

It is clear that both binomial-breaking and chipping have the same stationary distribution - concentrated at rocks all of size 1. Indeed, because the algebra of symmetric functions is
both commutative and cocommutative, both chains have a common right eigenbasis. One consequence for the $T_{2R_n}$ chain (analogous to [DPR14, Cor. 4.10]) reads: after $t$ steps starting from $\lambda$, the probability that there remains a rock of size at least $n' > 1$, is at most 

$$\left( \frac{|\lambda| - n'}{|\lambda|} \right)^t \sum_i \left( \frac{\lambda_i}{n'} \right).$$

Or, in the coupon-collection formulation, the probability of still needing $n'$ or more of the $n$ different coupons is at most $$\left( \frac{n - n'}{n} \right)^t \binom{n}{n'}.$$

**Mining**
The operator $T_{2R_n}$ on the Hopf algebra of subword complexes [BC17] gives rise to an intriguing variant of rock-chipping, as sketched in [BC17, App. B]. Here is a simplified version which sprouted from discussion with those authors. The states of this chain are collections of “gems”, where each gem is a distinct direction in $\mathbb{R}^n$ (or an element of the projective space $\mathbb{P}^n$). The linear dependences between these directions indicate which gems are “entangled” in rock and must be “mined”. At each time step, pick a gem $v$ in the collection to mine, and pick a complementary subspace $W$ spanned by some other gems in the collection. Then remove from the collection all gems which are not in $v \cup W$. (The probability of selecting the “2-flat decomposition” $(v, W)$ is proportional to the number of bases of $\mathbb{R}^n$ using the gems in $v \cup W$ - we aim to find a more natural interpretation in future work.)

This chain is absorbing at any state with $n$ linearly independent gems, and there are usually multiple such states. For example, starting with three non-collinear gems in $\mathbb{R}^2$, any pair of them is a possible absorbing state. This is the first instance of a descent operator Markov chain with multiple absorbing states, which merits further investigation.

**Phylogenetic trees**
Following [Hol99], a phylogenetic tree is a rooted tree on labelled vertices, recording the ancestry connections of several biological species. Since much ancestry data is conjectural, it is natural to consider random walks on the set of such trees. In the simplest models, each species has at most two descendants, and this can be represented by complete binary trees (each internal vertex has exactly two children, and only internal vertices are labelled). If, in addition, left and right children are distinguished (i.e. the tree is planar), then these trees form a basis for (a labelled version of) $YSym$, the graded dual to the Loday-Ronco Hopf algebra [LR98, AS06].

The $T_{2R_n}$ chain on this algebra is on such trees with $n$ internal vertices: remove the leftmost leaf and combine its parent with its sibling, keeping the label of the sibling. Next, uniformly choose a leaf, give it the label of the parent, and add to it two children. This is a variant of the chain of [Ald00, Sch02], where we have restricted the edge contraction and insertion operations to specific edges. Figure 1.3 demonstrates the possible moves from a particular state in the case $n = 3$.

The unique stationary distribution is the number of ways to label the vertices with $\{1, \ldots, n\}$, each label occurring once, such that a parent has a smaller label than both its children. Since this algebra is not commutative, Theorem 4.4 provides many right eigenfunctions but not a complete basis. [Cha18b] proved using one eigenfunction that, after $t$ steps, the probability of the root having no right children is bounded above by $$\left( 1 + \left( \frac{n^2}{n} \right)^t \right) \frac{1}{2}.$$ 

Since $YSym$ is a quotient of the Malvenuto-Reutenauer Hopf algebra of permutations via taking the decreasing tree, by [Pan18, Th. 3.6], the unlabelled version of this chain on trees precisely records the decreasing trees for permutations under the to-do list chain of Example 1.2 / Section 6.

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Since this is the author’s main piece of work during her postdoctoral stay in Montreal, this seems a good place to thank all the members and visitors of LaCIM, and her colleagues and students at McGill, for an unimaginably wonderful two years. A special mention goes to Mathieu Guay-Paquet and Franco Saliola for their invaluable advice and support, both mathematical and personal.

2 Background and Notation

Since this paper is intended for multiple audiences, this section provides both probabilistic (Sections 2.2 and 2.3) and algebraic (Sections 2.4 and 2.5) preliminaries. Two points are of particular interest: Section 2.3 outlines an obscure use of the Doob transform to create transition matrices out of non-negative matrices, which underlies the construction of the descent operator chains. Section 2.5 is a non-standard treatment of the descent operators, in order to facilitate our new probabilistic connections. Readers familiar with these operators are encouraged to skim this section nonetheless.

2.1 Linear algebra notation

Given a matrix $K$, let $K(x, y)$ denote its entry in row $x$, column $y$, and write $K^T$ for the transpose of $K$, so $K^T(x, y) = K(y, x)$.

For a vector space $V$ with basis $B$, and a linear map $T : V \to V$, write $[T]_B$ for the matrix of $T$ with respect to $B$, satisfying

$$T(x) = \sum_{y \in B} [T]_B(y, x)y.$$

$V^*$ is the dual vector space to $V$, the set of linear functions from $V$ to $\mathbb{R}$. (Because of the probability applications, take $\mathbb{R}$ to be the ground field of all vector spaces.) Its natural basis is $B^* := \{x^* | x \in B\}$, where $x^*$ satisfies $x^*(x) = 1$, $x^*(y) = 0$ for all $y \in B$, $y \neq x$. The dual map to $T : V \to V$ is the
linear map \( T^* : V^* \to V^* \) satisfying \( (T^*f)(v) = f(Tv) \) for all \( v \in V, f \in V^* \). Dualising a linear map is equivalent to transposing its matrix: \( [T^*]_{B^*} = [T]^T_{B^*} \).

### 2.2 Markov chains

All Markov chains in this work are in discrete time, are time-homogeneous and have a finite state space \( \Omega \). Hence they are each described by a \(|\Omega|\)-by-\(|\Omega|\) transition matrix \( K \). We follow the probability community’s convention that the rows index the source state and the columns index the destination state, so the probability of moving from \( x \) to \( y \) is \( K(x, y) \). (Combinatorialists sometimes take the opposite convention, for example [ASST15].) Note that a matrix \( K \) specifies a Markov chain in this manner if and only if \( K(x, y) \geq 0 \) for all \( x, y \in \Omega \), and \( \sum_{y \in \Omega} K(x, y) = 1 \) for each \( x \in \Omega \). We refer the reader to [LPW09] for the basics of Markov chain theory.

This paper is primarily interested in the stationary distributions of a Markov chain, and its left and right eigenfunctions. These are functions \( \pi, f, g : \Omega \to \mathbb{R} \) satisfying respectively

\[
\sum_{x \in \Omega} \pi(x)K(x, y) = \pi(y), \quad \sum_{x \in \Omega} g(x)K(x, y) = \beta g(y), \quad \sum_{y \in \Omega} K(x, y)f(y) = \beta f(x).
\]

(For brevity, we will occasionally write \( \beta \)-eigenfunction to mean that the eigenvalue is \( \beta \), and similarly for eigenvectors of linear maps.) So the stationary distributions are precisely the distributions that are left 1-eigenfunctions. [DPR14, Sec. 2.1] lists many applications of both left and right eigenfunctions; this work will concentrate on their Use A, which is immediate from the definitions of expectation and eigenfunction:

**Proposition 2.1** (Expectations from right eigenfunctions). The expected value of a right eigenfunction \( f \) with eigenvalue \( \beta \) is \( \text{Expect}(f(X_t)|X_0 = x_0) := \sum_{y \in \Omega} K^t(x_0, y)f(y) = \beta^t f(x_0) \).

\(
\)

In Sections 4.2 and 4.3, the computation of eigenfunctions boils down to finding stationary distributions of certain related chains, and our main tool for doing so will be detailed balance:

**Lemma 2.2.** [LPW09, Prop. 1.19] Let \( \{X_t\} \) be a Markov chain on the state space \( \Omega \) with transition matrix \( K \). If a distribution \( \pi \) on \( \Omega \) is a solution to the detailed balance equation \( \pi(x)K(x, y) = \pi(y)K(y, x) \) for each \( x, y \in \Omega \), then \( \pi \) is a stationary distribution.

**Proof.**

\[
\sum_{x \in \Omega} \pi(x)K(x, y) = \sum_{x \in \Omega} \pi(y)K(y, x) = \pi(y) \sum_{x \in \Omega} K(y, x) = \pi(y).
\]

Note that the detailed balance condition is far from necessary for a distribution to be stationary; there are plenty of Markov chains which are not reversible, meaning they admit no solutions to their detailed balance equations. In fact, the descent operator chains of this paper are in general not reversible; it is only their related chains, for computing eigenfunctions, which are reversible.
2.3 The Doob $h$-transform

This section briefly explains a very general method of constructing a transition matrix out of a linear map satisfying certain positivity conditions. Section 3.1 will expost this construction in full detail in the case where these linear maps are descent operators, so readers interested solely in descent operator chains should feel free to skip this section.

The Doob $h$-transform is a very general tool in probability, used to condition a process on some event in the future [Doo57]. The simple case of relevance here is conditioning a (finite, discrete-time) Markov chain on non-absorption. The Doob transform constructs the transition matrix of the conditioned chain out of the transition probabilities of the original chain between non-absorbing states, or, equivalently, out of the original transition matrix with the rows and columns for absorbing states removed. As observed in the multiple references below, the same recipe essentially works for an arbitrary non-negative matrix. In the case where this matrix comes from a linear operator, the transform has an elegant interpretation in terms of a rescaling of the basis.

**Theorem 2.3** (Doob $h$-transform for linear maps). [Pan14, Th. 3.1.1] [KSK66, Def. 8.11, 8.12] [LPW09, Sec.17.6.1] Let $V$ be a finite-dimensional vector space with basis $B$, and $T : V \to V$ be a linear map for which $K := [T]_B$ has all entries non-negative. Suppose there is an eigenvector $\eta \in V^*$ of the dual map $T^* : V^* \to V^*$, with eigenvalue 1, taking only positive values on $B$. Then

$$\tilde{K}(x, y) := \frac{\eta(y)}{\eta(x)} K(x, y)$$

defines a transition matrix. Equivalently, $\tilde{K} := [T]_{\tilde{B}}^T$, where $\tilde{B} := \{ \frac{x}{\eta(x)} | x \in B \}$.

The Markov chain with transition matrix $\tilde{K}$ above is called the Markov chain on $B$ driven by $T$.

**Proof.** First note that $K = [T^*]_{B^*}$ by definition, so $T^* \eta = \eta$ translates to $\sum_y K(x, y) \eta(y) = \eta(x)$. (Functions satisfying this latter condition are called harmonic, hence the name $h$-transform.) Since $\eta(x) > 0$ for all $x$, it is clear that $\tilde{K}(x, y) \geq 0$. It remains to show that the rows of $\tilde{K}$ sum to 1:

$$\sum_y \tilde{K}(x, y) = \frac{\sum_y K(x, y) \eta(y)}{\eta(x)} = \frac{\eta(x)}{\eta(x)} = 1.$$

The function $\eta : V \to \mathbb{R}$ above is the rescaling function. Different choices of $\eta$ for the same linear operator $T$ can lead to different Markov chains, but the notation suppresses the dependence on $\eta$ because, for $T$ a descent operator, there is a canonical choice of $\eta$ (Lemma 3.3 below). The assumption that $\eta$ has eigenvalue 1 can be easily relaxed by scaling the transformation $T$, see [Pan18, Th. 2.3]. We choose to impose this assumption here as it unclutters the eigenfunction formulae in Proposition 2.4 below.

The main advantage of fashioning a transition matrix using the Doob transform, as opposed to some other manipulation on $K$ (such as scaling each row separately) is that the diagonalisation of the Markov chain is equivalent to identifying the eigenvectors of $T$ and its dual $T^*$:

**Proposition 2.4** (Eigenfunctions for Doob transform chains). [Pan14, Prop. 3.2.1] [Zha08, Lemma 4.4.1.4] [Swa12, Lem. 2.11] Let $\{X_t\}$ be the Markov chain on $B_n$ driven by $T : V \to V$ with rescaling function $\eta$. Then:
The left $\beta$-eigenfunctions $g : \mathcal{B}_n \to \mathbb{R}$ for $\{X_i\}$ are in bijection with the $\beta$-eigenvectors $g \in V$ of $T$, through the vector space isomorphism

$$g(x) := \text{coefficient of } x \text{ in } \eta(x)g.$$ 

The right $\beta$-eigenfunctions $f : \mathcal{B}_n \to \mathbb{R}$ for $\{X_i\}$ are in bijection with the $\beta$-eigenvectors $f \in V^*$ of the dual map $T^*$, through the vector space isomorphism

$$f(x) := \frac{1}{\eta(x)}f(x).$$

Remark. In the Markov chain literature, the term “left eigenvector” is often used interchangeably with “left eigenfunction”, but this work will be careful to make a distinction between the eigenfunction $g : \mathcal{B} \to \mathbb{R}$ and the corresponding eigenvector $g \in V$ (and similarly for right eigenfunctions).

2.4 Combinatorial Hopf algebras

The application of Hopf algebras to combinatorics originated from [JR79]; much general theory have since been developed [Sch93, Hiv07, ABS06, AM10, BS17], and a plethora of examples constructed and analysed in detail [PR95, AS05a, NT07, CP17]. Loosely speaking, a combinatorial Hopf algebra is a graded vector space $\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$ with a basis $\mathcal{B} = \bigsqcup \mathcal{B}_n$ indexed by combinatorial objects, such as graphs, trees, or permutations. The grading reflects the “size” of the objects.

Many families of combinatorial objects (graphs, trees) have a single member of size 1, so $\mathcal{H}_1$ is often one-dimensional. In such cases, $\bullet$ will denote this sole object of size 1, so $\mathcal{B}_1 = \{\bullet\}$.

One simple, instructive, example of a combinatorial Hopf algebra is the shuffle algebra of [Ree58], whose associated Markov chains describe the cut-and-interlace card-shuffles of [DFP92].

Example 2.5. The shuffle algebra $\mathcal{S}$, as a vector space, has basis the set of all words in the letters $\{1, 2, \ldots, N\}$ (for some $N$, whose exact value is often unimportant). View the word $[w_1 \ldots w_n]$ as the deck of cards with card $w_1$ on top, card $w_2$ second from the top, and so on, so card $w_n$ is at the bottom (the bracket notation is non-standard). The degree of a word is its number of letters, i.e. the number of cards in the deck. The product of two words, also denoted by $\sqcup$, is the sum of all their interleavings (with multiplicity), and the coproduct is deconcatenation, or cutting the deck. For example:

$$m([15] \otimes [52]) = [15] \sqcup [52] = 2[1552] + [1525] + [5152] + [5125] + [5215];$$

$$\Delta([316]) = [3] \otimes [316] + [3] \otimes [16] + [31] \otimes [6] + [316] \otimes [\cdot].$$

Given a graded connected Hopf algebra $\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$, the symmetry of the Hopf axioms allows the definition of a Hopf structure on the (graded) dual vector space $\mathcal{H}^* := \bigoplus_{n \geq 0} \mathcal{H}^*_n$: for $f, g \in \mathcal{H}^*$, set

$$m(f \otimes g)(x) := (f \otimes g)(\Delta x), \quad \Delta(f)(w \otimes z) = f(wz),$$

with $x, z, w \in \mathcal{H}$. (Here, $(f \otimes g)(a \otimes b) = f(a)g(b)$.)
Example 2.6. The dual of the shuffle algebra $S$ is the free associative algebra $S^*$, whose basis is also indexed by words in the letters $\{1, 2, \ldots, N\}$. The product in $S^*$ is concatenation, and the coproduct is “deshuffling”; for example:

\[
\begin{align*}
m([15] \otimes [52]) &= [152]; \\
\Delta([316]) &= [\ldots [36] + [1] \otimes [36] + [6] \otimes [31] + [31] \otimes [6] + [36] \otimes [1] + [16] \otimes [3] + [316] \otimes [\ldots].
\end{align*}
\]

The associated Markov chains are the (unweighted) pop-shuffles of [BHR99, Sec. 2], the time-reversal of the cut-and-interleave shuffles. First, take some cards out of the deck to form a separate pile, keeping their relative order. Repeat this a few times, then place the piles one on top of another. This viewpoint is useful for the proof of the spectrum of descent operators (see Remark 2 after Lemma 3.9).

Because many combinatorial objects have “symmetric” assembling or breaking rules, many combinatorial Hopf algebras are commutative ($wz = zw$ for all $w, z \in H$) or cocommutative (if $\Delta(x) = \sum_i w_i \otimes z_i$, then $\sum_i w_i \otimes z_i = \sum_i z_i \otimes w_i$). For example, shuffle algebra is commutative but not cocommutative, and dualising means that the free associative algebra is noncommutative and cocommutative. The descent operators are better behaved on such algebras, so under a (co)commutativity hypothesis, stronger results hold - for example, the Markov chains are diagonalisable (Theorem 3.5).

The eigenvectors of our Markov chains will be constructed from primitive elements: $x \in H$ satisfying $\Delta(x) = 1 \otimes x + x \otimes 1$. It is easy to show that the primitive elements of $H$ form a subspace and a Lie algebra. Write $P$ for a basis of this subspace. Such a basis has been computed for many combinatorial Hopf algebras [AS06, Sec. 5] [Fis10, Sec. 3.1.3].

2.5 Descent operators

Here is a non-standard definition of descent operators, which will be useful for our probabilistic applications.

Definition 2.7. Let $H$ be a graded Hopf algebra.

i) Given a weak-composition $D = (d_1, \ldots, d_{l(D)})$ of $n$ (i.e. non-negative integers $d_i$ summing to $n$), define the refined coproduct $\Delta_D : H_n \to H_{d_1} \otimes \cdots \otimes H_{d_{l(D)}}$ to be the composition of the iterated coproduct $\Delta_{[i(D)]:} = (\Delta \otimes \text{id} \otimes \ldots) \circ \ldots \circ (\Delta \otimes \text{id}) \circ \Delta$ followed by the projection $H_{\otimes l(D)} \to H_{d_1} \otimes \cdots \otimes H_{d_{l(D)}}$.

ii) The descent operators are the linear combinations of the composite maps $m_{\Delta_D} : H_n \to H_n$ (abusing notation here and writing $m : H^l \to H$ for the multiplication of arbitrarily many elements).

iii) Given a probability distribution $P$ on the set of weak-compositions of $n$, define $m_{\Delta_P} : H_n \to H_n$ as

\[
m_{\Delta_P} := \sum \frac{P(D)}{n(D)} m_{\Delta_D},
\]

where $\binom{n}{D}$ is the multinomial coefficient $\binom{n}{d_1 \ldots d_{l(D)}}$.  

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On a combinatorial Hopf algebra, $\Delta_D$ captures the notion of breaking an object into pieces of sizes $d_1, d_2, \ldots, d_{\ell(D)}$. So each step of the Markov chain driven by $m\Delta_D$ first picks a weak-composition $D$ according to the distribution $P$, then breaks the current state into pieces of sizes $d_1, d_2, \ldots, d_{\ell(D)}$, then reassembles these pieces (see Theorem 3.3 for a precise statement). For example, when $P$ is the binomial distribution on weak-compositions with two parts, and zero on all other weak-compositions, the map $m\Delta_D$ is simply $m\Delta$. On the shuffle algebra, this describes the Gilbert-Shannon-Reeds model of riffle-shuffling, as analysed in [BD92]: cut the deck into two piles binomially, then combine them by repeatedly dropping the bottommost card from either pile, chosen with probability proportional to the current pile size.

**Example 2.8.** In the shuffle algebra $S$,

$\Delta_{1,1}[1552] = [1] \otimes [5] \otimes [52].$

$\Delta_{2,0}[1552] = [15] \otimes [] \otimes [52].$

$m\Delta_{1,3}[1552] = m([1] \otimes [552]) = [1552] + [5152] + [5512] + [5521].$

And in the free associative algebra $S^*$,

$m\Delta_{1,2}[316] = m([3] \otimes [16] + [1] \otimes [36] + [6] \otimes [31])$

$\quad = [316] + [136] + [631].$

The notation $m\Delta_D$ is from AM10 and is recent; the same operator is written $B_D$ in Pat94, and $\xi_D$ in Gri16. AM13 Prop. 88 gives a version for Hopf monoids. These and other sources mostly consider $m\Delta_D$ when $D$ is a (strict) composition (i.e. no $d_i$ is zero). Indeed, on a graded connected Hopf algebra, this is sufficient, since removing parts of size 0 from a weak-composition $D$ does not change the map $m\Delta_D$. However, the probability distributions are more natural if parts are allowed to have size 0.

**Remarks.**

1. Every positive descent operator (that is, a non-negative linear combination of $m\Delta_D$) is a multiple of $m\Delta_P$ for some probability distribution $P$. Hence the results in Sections 3 and 4 concerning $m\Delta_P$ have analogues for arbitrary positive descent operators.

2. The dual of a descent operator $m\Delta_P$ is simply the same operator on the dual Hopf algebra. This observation will be useful for deriving right eigenfunctions of the associated Markov chains.

The descent operators are so named because, on a commutative or cocommutative Hopf algebra, their composition is equivalent to the product on Solomon’s descent algebra [Sol76]. For this work, it will be more useful to express the latter as the internal product $\cdot$ in the algebra of noncommutative symmetric functions $\text{Sym}$ [GKL+95]. Let $\theta : \text{Sym} \to \text{End}(\mathcal{H})$ denote the linear map sending the complete noncommutative symmetric functions $S^D$ to the descent operator $m\Delta_D$. (Here, End$(\mathcal{H})$ is the algebra of linear maps $\mathcal{H} \to \mathcal{H}$; these maps need not respect the product or coproduct.)

**Proposition 2.9** (Compositions of descent operators). [Pat94, Th. II.7] [GKL+95 Prop. 5.1] Let $\mathcal{H}$ be a graded connected Hopf algebra, and $\theta : \text{Sym} \to \text{End}(\mathcal{H})$ be the linear map with $\theta(S^D) = m\Delta_D$.

i) If $\mathcal{H}$ is commutative, then, for any $F, G \in \text{Sym}$, the composite of their images is $\theta(F) \cdot \theta(G) = \theta(G \cdot F)$. 

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ii) If $\mathcal{H}$ is cocommutative, then, for any $F, G \in \text{Sym}$, the composite of their images is $\theta(F) \circ \theta(G) = \theta(F \cdot G)$.

In particular, if $\mathcal{H}$ is commutative (resp. cocommutative), then the set of descent operators on $\mathcal{H}$ is closed under composition. Indeed,

$$m \Delta_D \circ m \Delta_{D'} = \sum_M m \Delta_{D''(M)},$$

where the sum runs over all $l(D)$-by-$l(D')$ matrices $M$ of non-negative integers, with row $i$ summing to $d_i$ and column $j$ summing to $d'_j$. And $D''(M)$ is the weak-composition formed from all the entries of $M$, by reading down each column from the leftmost column to the rightmost column (resp. by reading left to right across each row from the top row to the bottom row):

$$D''(M) = (M(1,1), M(2,1), \ldots, M(l(D), 1), M(1,1)\ldots, M(1, l(D')), \ldots, M(l(D), l(D')))$$

(resp. $D''(M) = (M(1,1), M(1,2), \ldots, M(1, l(D')), M(2,1)\ldots, M(l(D), 1), \ldots, M(l(D), l(D'))$).

(The case $\mathcal{H} = S$, concerning compositions of cut-and-interleave shuffles, was proved in [DFP92, Th. 5.1].) Consequently, the problem of finding eigenvalues and eigenvectors of descent operators is closely connected to the determination of orthogonal idempotents of subalgebras (under the internal product) of $\text{Sym}$ (see the remark after Lemma 3.11).

Remark. The use of $\cdot$ to denote the internal product is non-standard, chosen to evoke the standard symbol $\circ$ for composition on $\text{End}(\mathcal{H})$. The usual notation of $*$ is confusing here, as it usually indicates convolution product ($T * T' := m(T \otimes T') \Delta$), which corresponds under $\theta$ to the external product on $\text{Sym}$, not the internal product.

3 Markov Chains from Descent Operators

Section 3.1 applies the Markov chain construction of the previous section to the descent operators $m \Delta_P$ of Section 2.5. Sections 3.2 and 3.3 give respectively the spectrum and stationary distributions of these chains, and Section 3.4 relates the quasisymmetric invariants of [ABS06] to the absorption probabilities of certain chains.

3.1 Construction

Recall that, for a probability distribution $P$ on weak-compositions of a fixed integer $n$, the descent operator $m \Delta_P : \mathcal{H}_n \rightarrow \mathcal{H}_n$ is

$$m \Delta_P = \sum_n \frac{P(D)}{(d_1 \ldots d_{l(D)})} m \Delta_D.$$

To apply the Doob $h$-transform (Theorem 2.3) to the linear map $m \Delta_P$, it is first necessary to find a basis $\mathcal{B}_n$ of $\mathcal{H}_n$ with respect to which $m \Delta_P$ has a nonnegative matrix. One stronger condition that is more natural, at least for combinatorial Hopf algebras, is to mandate that $\mathcal{B}$ has non-negative product and coproduct structure constants in the relevant degrees - this is the essence of conditions i and ii in the definition below. As for condition iii: informally, this insists that every combinatorial object indexing the basis may be broken into pieces of size 1. Such a restriction is necessary since Lemma 3.3 will show that $\eta$ is a suitable rescaling function (in the sense of Theorem 2.3).
Definition 3.1. Let \( \mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n \) be a graded connected Hopf algebra over \( \mathbb{R} \) with each \( \mathcal{H}_n \) finite-dimensional. Let \( D = (d_1, \ldots, d_{l(D)}) \) be a weak-composition of some fixed integer \( n \). A basis \( \mathcal{B} = \Pi_{n \geq 0} \mathcal{B}_n \) of \( \mathcal{H} \) is a state space basis for \( D \) (or for \( m\Delta_D \)) if:

i) for all \( z_1 \in \mathcal{B}_{d_1}, z_2 \in \mathcal{B}_{d_1}, \ldots, z_l(D) \in \mathcal{B}_{d_{l(D)}} \), their product is \( z_1 z_2 \cdots z_l(D) = \sum_{y \in \mathcal{B}_n} \xi_{z_1,\ldots,z_l(D)} y \) with \( \xi_{z_1,\ldots,z_l(D)} \geq 0 \) (non-negative product structure constants);

ii) for all \( x \in \mathcal{B}_n \), its coproduct is \( \Delta_D(x) = \sum_{z_i \in \mathcal{B}_{d_i}} \eta_{x,z_1,\ldots,z_l(D)} z_1 \otimes \cdots \otimes z_l(D) \) with \( \eta_{x,z_1,\ldots,z_l(D)} \geq 0 \) (non-negative coproduct structure constants);

iii) for all \( x \in \mathcal{B} \), the function

\[
\eta(x) := \text{sum of coefficients (in the } B_1 \otimes \cdots \otimes B_1 \text{ basis) of } \Delta_{1,\ldots,1}(x)
\]

evaluates to a positive number.

If \( P \) is a probability distribution on weak-compositions of \( n \), then a basis \( \mathcal{B} = \Pi_{n \geq 0} \mathcal{B}_n \) of \( \mathcal{H} \) is a state space basis for \( P \) (or for \( m\Delta_P \)) if it is a state space basis for all \( D \) with non-zero probability under \( P \).

Note that, if all structure constants of \( \mathcal{B} \) are non-negative regardless of degree, then \( \mathcal{B} \) is a state space basis for all distributions \( P \). (It is in fact sufficient to check that all \( \xi_{z_1,z_2}^y \) and all \( \eta_{x,z_1,z_2}^{c,z} \) are non-negative, because of associativity and coassociativity, see [Pan14 Lem. 4.2.1].) In this case, [Pan14 Th. 4.3.7.i] shows that condition iii is equivalent to \( \mathcal{B} \) not containing any primitive elements of degree greater than 1. (\( \mathcal{H} \) may contain primitive elements of any degree, so long as those of degree greater than one are not in the basis \( \mathcal{B} \).) In general, the absence of primitives in the basis \( \mathcal{B} \) is necessary but not sufficient.

Example 3.2. Let \( P \) be the distribution that is concentrated at \((1, n-1)\) - that is, \( P((1, n-1)) = 1 \), and \( P(D) = 0 \) for all other weak-compositions \( D \). (Recall from the introduction that, on the shuffle algebra, this distribution induces the top-to-random card-shuffle.) Then conditions i and ii in Definition 3.1 simply require \( m : \mathcal{H}_1 \otimes \mathcal{H}_{n-1} \to \mathcal{H}_n \) and \( \Delta_{1,n-1} \) to have non-negative structure constants. (In other words, the requirement \( \xi_{z_1,z_2}^y \geq 0 \), \( \eta_{x,z_1,z_2}^{c,z} \geq 0 \) is only for \( z \in \mathcal{B}_{n-1}, c \in \mathcal{B}_1, x,y \in \mathcal{B}_n \).

All bases of Hopf algebras in this paper have all structure constants non-negative. For examples which satisfy the conditions in Example 3.2 and yet have some negative structure constants, see the plethora of “schurlike” bases in noncommutative symmetric functions [BLvW11 BBS+14].

Remark. If \( \mathcal{H}_1 = \emptyset \) (so there are no combinatorial objects of size 1), then, according to Definition 3.1, \( \mathcal{H} \) has no state space bases. However, it is still possible, at least theoretically, to define descent operator Markov chains on \( \mathcal{H} \). There are currently no known examples of such chains, so we do not go into the technical details here - see the last paragraph of [Pan14 Sec. 4.3].

Having scrutinised the non-negativity condition on structure constants, focus now on the function \( \eta \), which rigourises the concept of “number of ways to break into singletons”. It is usually a well-investigated number: for example, for the irreducible representations of the symmetric group (inducing Fulman’s restriction-then-induction chain), it is the dimension of the representation [Pan14 Ex. 4.4.3]. Proposition 3.3 below verifies that \( \eta \) is indeed a possible rescaling function for the Doob transform - in fact, [Pan14 Th. 4.3.7] shows that, in some sense, this \( \eta \) is the optimal rescaling function.
Lemma 3.3. Let \( \mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n \) be a graded connected Hopf algebra over \( \mathbb{R} \) with each \( \mathcal{H}_n \) finite-dimensional, and \( B_1 \) a basis of \( \mathcal{H}_1 \). The linear function \( \eta : \mathcal{H} \to \mathbb{R} \) with

\[
\eta(x) := \text{sum of coefficients (in the } B_1 \otimes \cdots \otimes B_1 \text{ basis) of } \Delta_{1,\ldots,1}(x)
\]

is a 1-eigenvector of the dual map to the descent operator \( m\Delta_P : \mathcal{H}_n \to \mathcal{H}_n \), for any probability distribution \( P \).

Proof. Let \( \bullet^* \in \mathcal{H}_1^* \) denote the linear map on \( \mathcal{H}_1 \) taking value 1 on each element of \( B_1 \). (So, if \( B_1 = \{\bullet\} \), then this map is the dual basis element \( \bullet^* \), hence the notation.) Since multiplication in \( \mathcal{H}^* \) is dual to the coproduct on \( \mathcal{H} \), it is true that \( \eta \) restricted to \( \mathcal{H}_n \) is \( (\bullet^*)^n \).

As noted in Remark 2 of Section 2.5 the dual map to a descent operator \( m\Delta_P : \mathcal{H}_n \to \mathcal{H}_n \) is the same descent operator on the dual Hopf algebra \( \mathcal{H}_n^* \). So it suffices to show that \( (\bullet^*)^n \) is a 1-eigenvector of \( m\Delta_P : \mathcal{H}_n^* \to \mathcal{H}_n^* \). By linearity, this will follow from \( (\bullet^*)^n \) being a \( (\binom{n}{D}) \)-eigenvector of \( m\Delta_D : \mathcal{H}_n^* \to \mathcal{H}_n^* \) for each weak-composition \( D \).

Write \( l \) for the number of parts in \( D \). As \( \deg(\bullet^*) = 1 \), the iterated coproduct sends \( \bullet^* \) to \( \Delta[l](\bullet^*) = \bullet^* \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes \bullet^* \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes \bullet^* \), i.e. the sum of \( l \) terms, each with \( l \) tensorands, one of which is \( \bullet^* \) and all others are 1. Because of the compatibility of product and coproduct,

\[
\Delta[l](\bullet^*)^n = (\bullet^* \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes \bullet^* \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes \bullet^*)^n
\]

\[
= \sum_{i_1,\ldots,i_l} \left( \frac{n}{i_1 \cdots i_l} \right) (\bullet^*)^{i_1} \otimes \cdots \otimes (\bullet^*)^{i_l}.
\]

Hence \( \Delta_D((\bullet^*)^n) = \binom{n}{D} (\bullet^*)^{d_1} \otimes \cdots \otimes (\bullet^*)^{d_l} \), so \( m\Delta_D((\bullet^*)^n) = \binom{n}{P}(\bullet^*)^n \).

So it is indeed possible to apply the Doob transform to \( m\Delta_P \) in a state space basis, with this choice of \( \eta \).

To obtain a more intuitive interpretation of the Markov chain driven by \( m\Delta_P \), appeal to this description of the cut-and-interleave shuffles of [DFP92] (recall from Example 2.5 that this is the case with the shuffle algebra):

1. Choose a weak-composition \( (d_1,\ldots,d_{l(D)}) \) of \( n \) according to the distribution \( P \).
2. Cut the deck so the first pile contains \( d_1 \) cards, the second pile contains \( d_2 \) cards, and so on.
3. Drop the cards on-by-one from the bottom of one of the \( l(D) \) piles, chosen with probability proportional to the current pile size.

Theorem 3.4 gives an analogous description of the chain driven by \( m\Delta_P \) on any Hopf algebra, separating it into a breaking part (steps 1 and 2) followed by a recombination (step 3). The probabilities involved in both stages are expressed in terms of the rescaling function \( \eta \) and the structure constants of \( \mathcal{H} \): these are the numbers \( \xi_{z_1,\ldots,z_l}^y \) defined by

\[
\sum_{y \in B} \xi_{z_1,\ldots,z_l}^y, \quad \Delta[l](x) = \sum_{z_1,\ldots,z_l \in B} \eta_{z_1,\ldots,z_l}^z, \quad z_1 \cdots z_l
\]

for \( x, y, z_1, \ldots, z_l \) in the distinguished basis \( B \).
**Theorem 3.4** (Definition of descent operator chains). [Pan15a, Def. 3.1] Let $P$ be a probability distribution on the weak-compositions of a fixed integer $n$. Let $\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$ be a graded connected Hopf algebra over $\mathbb{R}$ with each $\mathcal{H}_n$ finite-dimensional, and $\mathcal{B} = \prod_{n \geq 0} \mathcal{B}_n$ a state space basis of $\mathcal{H}$ for $P$. As above, define functions $m\Delta_P : \mathcal{H}_n \to \mathcal{H}_n$ and $\eta : \mathcal{H} \to \mathbb{R}$ by

$$m\Delta_P := \sum \frac{P(D)}{\binom{m}{d_1, \ldots, d_{\ell(D)}}} m\Delta_D;$$

$$\eta(x) := \text{sum of coefficients (in the } \mathcal{B}_1 \otimes \cdots \otimes \mathcal{B}_1 \text{ basis) of } \Delta_1, \ldots, 1(x).$$

Then

$$\check{K}(x, y) := \frac{\eta(y)}{\eta(x)} \text{ coefficient of } y \text{ in } m\Delta_P(x)$$

gives a transition matrix. Each step of this Markov chain, starting at $x \in \mathcal{B}_n$, is equivalent to the following three-step process:

1. Choose a weak-composition $(d_1, \ldots, d_{\ell(D)})$ of $n$ according to the distribution $P$.

2. Choose $z_1 \in \mathcal{B}_{d_1}, z_2 \in \mathcal{B}_{d_2}, \ldots, z_{\ell(D)} \in \mathcal{B}_{d_{\ell(D)}}$ with probability $\frac{1}{\eta(x)} \eta^{z_1, \ldots, z_{\ell(D)}}(z_1) \ldots \eta(z_{\ell(D)}).

3. Choose $y \in \mathcal{B}_n$ with probability $\left(\binom{n}{d} \eta(z_1) \ldots \eta(z_{\ell(D)})\right)^{-1} \xi_{z_1, \ldots, z_{\ell(D)}}(y).

Note that the probabilities of the choices in steps 2 and 3 depend only on the Hopf algebra, not on the probability distribution $P$.

**Proof of Theorem 3.4** That $\check{K}(x, y)$ is a transition matrix follows from Theorem 2.3, the Doob transform for linear maps. What follows will check that the probabilities under the three-step process agree with $\check{K}(x, y)$. This is easiest using the alternative characterisation of $\eta$ from the proof of Proposition 3.3: $\eta = (\bullet)^n$ where $\bullet \in \mathcal{H}_1^*$ is the linear map sending all elements of $\mathcal{B}_1$ to 1. Recall also that $\Delta^{[l]}$ is the iterated coproduct $(\Delta \otimes \text{id}^{\otimes (D-1)}) \circ \cdots \circ (\Delta \otimes \text{id} \otimes \text{id}) \circ (\Delta \otimes \text{id}) \circ \Delta$.

First check that, for each weak-composition $D$, the probabilities in step 2 do sum to 1:

$$\sum_{z_1 \in \mathcal{B}_{d_1}, \ldots, z_{\ell(D)} \in \mathcal{B}_{d_{\ell(D)}}} \eta^{z_1, \ldots, z_{\ell(D)}}(z_1) \ldots \eta(z_{\ell(D)})$$

$$= \left( (\bullet)^{d_1} \otimes \cdots \otimes (\bullet)^{d_{\ell(D)}} \right) \left( \sum_{z_1 \in \mathcal{B}_{d_1}, \ldots, z_{\ell(D)} \in \mathcal{B}_{d_{\ell(D)}}} \eta^{z_1, \ldots, z_{\ell(D)}} z_1 \otimes \cdots \otimes z_{\ell(D)} \right)$$

$$= \left( (\bullet)^{d_1} \otimes \cdots \otimes (\bullet)^{d_{\ell(D)}} \right) (\Delta_D(x))$$

$$= \left( (\bullet)^{d_1} \otimes \cdots \otimes (\bullet)^{d_{\ell(D)}} \right) (\Delta^{[\ell(D)]}(x))$$

$$= (\bullet)^n (x)$$

$$= \eta(x),$$

where the third equality is because $(\bullet)^d (z_i) = 0$ if $\deg(z_i) \neq d$, and the fourth equality is by
Theorem 3.5 (Eigenvalues of descent operators). Let $\mathcal{H} = \bigoplus \mathcal{H}_n$ be a graded connected Hopf algebra over $\mathbb{R}$, and $P$ a probability distribution on weak-compositions of a fixed integer $n$. As usual, write $m\Delta_P$ for the associated descent operator

$$m\Delta_P := \sum_D \frac{P(D)}{\binom{n}{D}} m\Delta_D.$$ 

where the last equality again relies on the fact that $(\bullet)^d(z_i) = 0$ if $\deg(z_i) \neq d$. Finally, the probability of moving from $x$ to $y$ under the three-step process is

$$\sum_D P(D) \sum_{z_i \in B_i} \eta_{x z_i \cdots z_{l(D)}} \eta(z_1) \cdots \eta(z_{l(D)}) \eta(y) \frac{\xi_y^{z_1 \cdots z_{l(D)}}}{\binom{n}{D} \eta(z_1) \cdots \eta(z_{l(D)})}$$

$$= \hat{K}(x, y).$$

3.2 Eigenvalues and multiplicities

Recall from Proposition 2.4 that the eigenvalues for a Markov chain from the Doob transform are simply the eigenvalues of the associated linear map. Hence, to obtain the spectrum of the breaking-and-recombination chains of the previous section, it suffices to calculate the spectrum of the descent operators $m\Delta_P$. The completely general spectrum formula, valid for all $m\Delta_P$ and all $\mathcal{H}$, is rather unsightly, but it simplifies neatly for many examples of interest, such as Examples 3.6, 3.7.

The eigenvalues of $m\Delta_P$ are indexed by partitions - these are usually written as tuples $\lambda = (\lambda_1, \ldots, \lambda_l(\lambda))$ of integers with $\lambda_1 \geq \cdots \geq \lambda_l(\lambda) > 0$, but it will be more convenient here to forget the decreasing ordering and view them simply as multisets of positive integers. The values of these eigenvalues themselves are related to set compositions (also known as ordered set partitions): a set composition $B_1 \cdots | B_l$ of a set $S$ is simply an $l$-tuple of disjoint subsets of $S$ with $B_1 \cdots B_l = S$. The blocks $B_i$ are allowed to be empty (so perhaps the correct terminology is “weak set composition”). The type of a set composition is the weak-composition of cardinalities ($|B_1|, \ldots, |B_l|$). If $S = \{1, \ldots, n\}$, then the symmetric group $\mathcal{S}_n$ acts on the set compositions of $S$ of any given type. For example, $B = \{2, 5\} | \{1, 4\} | \{3\}$ is a set composition of $\{1, 2, 3, 4, 5\}$ of type $(2, 0, 2, 1)$. The permutation $\sigma = 42351$ sends $B$ to $\sigma(B) = \{\sigma(2), \sigma(5)\} | \{\sigma(1), \sigma(4)\} | \{\sigma(3)\} = \{1, 2\} | \{4, 5\} | \{3\}$, and the transpositions $(25)$ and $(14)$ both fix $B$.

Theorem 3.5 (Eigenvalues of descent operators). Let $\mathcal{H} = \bigoplus \mathcal{H}_n$ be a graded connected Hopf algebra over $\mathbb{R}$, and $P$ a probability distribution on weak-compositions of a fixed integer $n$. As usual, write $m\Delta_P$ for the associated descent operator

$$m\Delta_P := \sum_D \frac{P(D)}{\binom{n}{D}} m\Delta_D.$$
The eigenvalues of $m\Delta_P : \mathcal{H}_n \to \mathcal{H}_n$ are

$$\beta^P_\lambda := \sum_D \frac{P(D)}{\binom{n}{D}} \beta^D_\lambda,$$

where $\beta^D_\lambda$ is the number of set compositions $B_1 \cdots |B_{|D|}$ of $\{1, 2, \ldots, |\lambda]\}$ such that, for each $i$, we have $\sum_{j \in B_i} \lambda_j = d_i$. The multiplicity of the eigenvalue $\beta^P_\lambda$ is the coefficient of $x_\lambda := x_{\lambda_1} \cdots x_{\lambda_{|\lambda|}}$ in the generating function $\prod_i (1 - x_i)^{-b_i}$, where the numbers $b_i$ satisfy

$$\sum_n \dim \mathcal{H}_n x^n = \prod_i (1 - x^i)^{-b_i}.$$

Furthermore, $m\Delta_P$ is diagonalisable if $\mathcal{H}$ is commutative or cocommutative.

Observe that, under the mild condition $b_i > 0$ for all $i$ (i.e. $(\text{gr} \mathcal{H})^*$ contains primitives in every degree, by five paragraphs below), the eigenvalues of a descent operator depend only on the associated probability distribution $P$, not on the Hopf algebra it acts on. By contrast, in the generic case where all $\beta^P_\lambda$ are distinct, their multiplicities depend only on the Hopf algebra (in fact, only on the dimensions of its graded subspaces) and not on the distribution $P$.

The following two interpretations of the eigenvalues $\beta^P_\lambda$ are sometimes useful:

1. $\beta^P_\lambda$ is the number of set compositions of $\{1, 2, \ldots, n\}$ of type $D$ which are fixed under the action of any particular permutation of cycle type $\lambda$ (since this forces each cycle to lie in the same block). Hence $\beta^P_\lambda$ is the probability that a particular permutation of cycle type $\lambda$ fixes a random set composition chosen in the following way: choose a weak-composition $D$ according to $P$, then choose uniformly amongst the set compositions of type $D$. For many interesting probability distributions $P$, this choice procedure is not as contrived as it may sound - see Example 3.7.

2. By [Sta99] Prop. 7.7.1, Eq. 7.30, $\beta^P_\lambda = \langle S^P, p_\lambda \rangle$, the inner product of the power sum symmetric function $p_\lambda$ with the commutative image of the noncommutative symmetric function $S^P := \sum_D \frac{P(D)}{\binom{n}{D}} S^D$ (i.e. with the linear combination of complete symmetric functions $\sum_D \frac{P(D)}{\binom{n}{D}} h_D$).

Note that the numbers $\beta^P_\lambda$ depend only on the sizes of the parts of $D$, not on their order. Also, the eigenvalues $\beta^P_\lambda$ need not be distinct for different partitions $\lambda$; see the example below.

**Example 3.6.** Take $P$ to be concentrated at $(1, n-1)$, so $m\Delta_P$ induces the top-to-random card-shuffle. Then $\beta_{\lambda^{(1,n-1)}}$ is the number of parts of size 1 in $\lambda$, which can be $0, 1, \ldots, n-2, n$. So the eigenvalues of a top-to-random chain on any Hopf algebra are $\beta^P_\lambda = \frac{1}{n^{|\lambda|}} \beta_{\lambda^{(1,n-1)}} = 0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-2}{n}, 1$. Alternatively, by Interpretation 1 above, $\beta^P_\lambda$ is the proportion of set compositions of type $(1, n-1)$ fixed by any particular permutation of cycle type $\lambda$ - this is simply the proportion of fixed points of the permutation, since set compositions of type $(1, n-1)$ are entirely determined by the single element in their first block.

The top-to-random chain is one of the rare examples of a descent operator chain that admits an explicit diagonalisation on cocommutative Hopf algebras, see Theorem 4.4.

**Example 3.7.** We apply Interpretation 1 above to two examples.
First, take $P$ to be the binomial distribution on weak-compositions with two parts, so $m\Delta_P = m\Delta$, inducing the riffle-shuffle (see the paragraph after Definition 2.7). Then the process in Interpretation 1 uniformly chooses one of the $2^n$ set compositions with two parts. Since each such set composition is entirely determined by its first block, Interpretation 1 says that the eigenvalues $\beta_\lambda^P$ are the proportions of subsets of $\{1, \ldots, n\}$ fixed by a permutation of cycle type $\lambda$. Being fixed under the permutation means that these are subsets of its cycles - hence $\beta_\lambda^P = \frac{2^{(\lambda)}}{2^n}$, as shown in [DPR14] Th. 3.15, 3.16.

Now consider a variant where $P$ is supported only on distributions of the form $(1^r, n-r)$ for $0 \leq r \leq n$, and let $r$ be binomially distributed. (Here, $1^r$ denotes $r$ consecutive parts of size 1.) For this “binomial-top-to-random” operator (Definition 4.11 with $q = \frac{1}{2}$), $\beta_\lambda^P$ is the proportion of subsets fixed pointwise by a permutation of cycle type $\lambda$. These fixed subsets are precisely the subets of the fixed points of the permutation. So, if there are $j$ fixed points (i.e. $\lambda$ has $j$ parts of size 1), then the eigenvalue $\beta_\lambda^P$ is $\frac{2^j}{2^n}$.

So both these descent operators have non-positive powers of 2 as their eigenvalues, but with different multiplicities. Each fixed partition $\lambda$ has more parts in total than parts of size 1, so its corresponding eigenvalue is larger for $m\Delta$ than for the binomial-top-to-random operator. In the case of card-shuffling, this agrees with intuition: having cut the deck according to a symmetric binomial distribution, reinserting the top half of the deck without preserving the relative order of the cards will randomise the deck faster. [BD92] Proof of Cor. 3 made the same comparison; instead of eigenvalues, they looked at the mixing time, which is $\frac{3}{2} \log_2 n$ for the riffle-shuffle, and $\log_2 n$ for binomial-top-to-random.

Below are two very different proofs of the spectrum of a descent operator. The first is probabilistically-inspired, and its key ideas aid in the construction of eigenvectors in Sections 4.2 and 4.3. The second comes from assembling known theorems on noncommutative symmetric functions; this proof was outlined by the reviewer of [Pan15a]. Both are included in the hope that they lead to generalisations for different classes of operators.

Both proofs begin by reducing to the case where $H$ is cocommutative; by duality, this will also imply the case for commutative $H$. This reduction follows the argument of [AL15] Th. 3. As explained in their Section 1.3, the coradical filtration of a graded connected Hopf algebra $H$ is defined as $H^{(k)} = H_0 \oplus \bigoplus_D \ker \Delta_D$, where the sum ranges over all (strict) compositions $D$ with $k$ parts. The associated graded algebra $\text{gr}(H)$ of $H$ with respect to this filtration, written $\text{gr}(H)$, is a Hopf algebra. Every linear map $T : H \to H$ preserving the coradical filtration induces a map $\text{gr}(T) : \text{gr}(H) \to \text{gr}(H)$ with the same eigenvalues and multiplicities. Now $m\Delta_P$ is a (linear combination of) convolution product of projections $\text{Proj}_d$ to the graded subspace $H_d$. Since $\text{gr}(\text{Proj}_d) = \text{Proj}_d$, and $T \to \text{gr}T$ preserves convolution products, it must be that $\text{gr}(m\Delta_P) = m\Delta_P$. So it suffices to show that $m\Delta_P : \text{gr}(H) \to \text{gr}(H)$ has the claimed eigenvalues and multiplicities. By [Swe69] Th. 11.2.5.a, $\text{gr}(H)$ is commutative. (So this argument shows that the eigenvalues and multiplicities of Theorem 3.5 also apply to any $T : H \to H$ with $\text{gr}(T) = m\Delta_P$, even if $T$ itself is not a descent operator.)

**First proof of Theorem 3.5: Poincare-Birkhoff-Witt straightening algorithm and Perron-Frobenius theorem**

By the Cartier-Milnor-Moore theorem [Car07] Th. 3.8.1, a graded connected cocommutative Hopf algebra $H$ is the universal enveloping algebra of its subspace of primitives. Consequently, $H$ has a Poincare-Birkhoff-Witt (PBW) basis: if $(\mathcal{P}, \preceq)$ is an ordered basis of the primitive subspace of $H$, then $\{p_1 \ldots p_k | k \in \mathbb{N}, p_1 \preceq \cdots \preceq p_k \in \mathcal{P}\}$ is a basis of $H$. The basis element $p_1 \ldots p_k$ has length $k$. 

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We will need the following fact, which follows easily from the “straightening algorithm”:

**Lemma 3.8.** ([Kna02, Lem. III.3.9] Let \( \mathcal{P}, \preceq \) be an ordered basis of the primitive subspace of \( \mathcal{H} \). If \( p_1, \ldots, p_k \in \mathcal{P} \) with \( p_1 \preceq \cdots \preceq p_k \), then, for any \( \sigma \in \mathfrak{S}_k \),

\[
p_{\sigma(1)} \cdots p_{\sigma(k)} = p_1 \cdots p_k + \text{terms of length less than } k.
\]

In particular, the coefficient of the leading term is 1.

The key to this proof is the following variant of [DPR14, Th. 3.10]:

**Lemma 3.9 (Symmetrisation Lemma).** Let \( p_1, \ldots, p_k \) be primitive elements of \( \mathcal{H} \) and let \( \deg(p) \) denote the partition \((\deg p_1, \ldots, \deg p_k)\). Then \( \text{span}\{p_{\sigma(1)} \cdots p_{\sigma(k)} | \sigma \in \mathfrak{S}_k\} \) is an invariant subspace under \( m \Delta_P \), and contains a \( \beta_{\deg(p)}^P \)-eigenvector of the form \( \sum_{\sigma \in \mathfrak{S}_k} \kappa_{\sigma} p_{\sigma(1)} \cdots p_{\sigma(k)} \) with all \( \kappa_{\sigma} \geq 0 \).

**Proof.** Work first in the free associative algebra generated by primitive elements \( p'_1, \ldots, p'_k \), with \( \deg p'_i = \deg p_i \). (Equivalently, treat \( p_1, \ldots, p_k \) as formal variables, ignoring any algebraic relationships between them.) Since the \( p'_i \) are primitive,

\[
(3.10) \quad m \Delta_D (p'_1 \cdots p'_k) = \sum_{B_1, \ldots, B_i(\ell)} \left( \prod_{i \in B_1} p'_i \right) \left( \prod_{i \in B_2} p'_i \right) \cdots \left( \prod_{i \in B_i(\ell)} p'_i \right),
\]

summing over all set compositions \( B_1 | \ldots | B_i(\ell) \) of \( \{1, 2, \ldots, k\} \) such that \( \sum_{j \in B_i} \deg(p'_j) = d_i \). So each summand \( m \Delta_D \) of \( m \Delta_P \) fixes the subspace \( W := \text{span}\{p'_1(1) \cdots p'_k(1) \sigma \in \mathfrak{S}_k\} \), and hence so does \( m \Delta_P \) itself. Consider the matrix of the restricted map \( m \Delta_P |_W \) with respect to the basis \( \{p'_1(1) \cdots p'_k(1) \sigma \in \mathfrak{S}_k\} \). (This is indeed a basis because the \( p'_i \) generate a free associative algebra.) From taking the appropriate linear combination of Equation \( 3.10 \) we see that the sum of the entries in the column corresponding to \( p'_1 \cdots p'_k \) is \( \beta_{\deg(p)}^P \). Note that the partition \( \deg(p) \), and hence \( \beta_{\deg(p)}^P \), is independent of the ordering of the \( p'_i \), so all columns of the matrix of \( m \Delta_P |_W \) sum to \( \beta_{\deg(p)}^P \). So the left (row) vector \((1, 1, \ldots, 1)\) is an eigenvector of this matrix with eigenvalue \( \beta_{\deg(p)}^P \). Since this vector has all components positive, and all entries of this matrix of \( m \Delta_P |_W \) are non-negative, the Perron-Frobenius theorem ([Gan59, Ch. XIII Th. 3]) states that \( \beta_{\deg(p)}^P \) is the largest eigenvalue of \( m \Delta_P |_W \), and \( m \Delta_P |_W \) has a (right, column) eigenvector of this eigenvalue with non-negative entries (note that it is in general not unique). Let \( \sum_{\sigma \in \mathfrak{S}_k} \kappa_{\sigma} p'_1(1) \cdots p'_k(1) \sigma \in \mathfrak{S}_k \) denote this eigenvector, so

\[
m \Delta_P \left( \sum_{\sigma \in \mathfrak{S}_k} \kappa_{\sigma} p'_1(1) \cdots p'_k(1) \right) = \beta_{\deg(p)}^P \left( \sum_{\sigma \in \mathfrak{S}_k} \kappa_{\sigma} p'_1(1) \cdots p'_k(1) \right).
\]

Apply to both sides the Hopf morphism sending \( p'_i \) to \( p_i \); this shows that \( \sum_{\sigma \in \mathfrak{S}_k} \kappa_{\sigma} p_{\sigma(1)} \cdots p_{\sigma(k)} \) is a \( \beta_{\deg(p)}^P \)-eigenvector of \( m \Delta_P \) on our starting Hopf algebra.

**Remarks.**

1. The transpose of the matrix in the above proof, of \( m \Delta_P |_W \) with respect to the basis \( \{p'_1(1) \cdots p'_k(1) | \sigma \in \mathfrak{S}_k\} \), is the transition matrix of a hyperplane walk [BHR99], scaled by \( \beta_{\deg(p)}^P \). Informally, this walk is the pop shuffle associated to \( m \Delta_P \) (see Example 2.6 above) where the primitive \( p'_i \) behave like \( \deg(p'_i) \) cards glued together. So the distribution \( \pi(\sigma) = \kappa_{\sigma} \) is a stationary...
distribution of this chain. The idea of expressing each member of an eigenbasis in terms of the stationary distribution of a different chain is also integral to the recent left eigenfunction formulae for hyperplane walks [Sal12, Den12].

In extremely simple cases, this view of the coefficients $\kappa_\sigma$ as the stationary distribution is surprisingly powerful: take $p_1 = \cdots = p_j \in \mathcal{H}_1$, and let $p_{j+1}, \ldots, p_k$ be primitives of degree greater than 1. The hyperplane walk that $m \Delta_{1,n-1}$ induces on $\text{span}\{p_{\sigma(1)} \cdots p_{\sigma(k)}\}$ is the “random-to-top shuffle”: uniformly choose a card to remove from the deck and place it on top. Since $p_{j+1}, \ldots, p_k$ represent multiple cards glued together, these cards are never moved, so in the stationary distribution, they must be at the bottom of the deck, in the same relative order as they started. And the order of the single cards $p_1, \ldots, p_j$ at the top of the deck is immaterial since these cards are identical. So $p_1 \cdots p_k$ is an eigenvector for $m \Delta_{1,n-1}$. In the common scenario where $\dim \mathcal{H}_1 = 1$, all multisets of $\mathcal{P}$ have this form, so the simple argument above produces a full eigenbasis.

Theorem 1.4 is a more complex argument along the same lines, making use of symmetry to simplify the required hyperplane walk (see point 3 below). However, for general descent operators $m \Delta_P$, the formula for this stationary distribution [BD98, Th. 2b] is notoriously difficult to compute with.

2. If $\mathcal{H}$ is commutative as well as cocommutative, then the Symmetrisation Lemma shows that any product of primitive elements is an eigenvector for $m \Delta_P$ for all distributions $P$. Hence $\{p_1 \cdots p_k | p_1 \leq \cdots \leq p_k \in \mathcal{P}\}$ is an eigenbasis for all $m \Delta_P$, and all descent operators commute (which is also clear from Proposition 2.9).

3. The entries of the matrix of $m \Delta_P|_W$ depend only on the degrees of the primitives $p_i$. Hence permuting the labels of the $p_i$ with the same degree does not change this matrix. So the eigenvector $\sum_{\sigma \in S_k} \kappa_\sigma p_{\sigma(1)} \cdots p_{\sigma(k)}$ can be arranged to be symmetric in the primitives of same degree - in other words, $\kappa_\sigma$ depends only on the tuple $(\deg p_{\sigma(1)}, \ldots, \deg p_{\sigma(k)})$, not on $(p_{\sigma(1)}, \ldots, p_{\sigma(k)})$.

The final step of the proof, to deduce the existence of an eigenbasis with the claimed eigenvalues and multiplicities, goes as follows. Apply the Symmetrisation Lemma to each multiset $\{p_1 \leq \cdots \leq p_k\} \subseteq \mathcal{P}$ to get an eigenvector, whose highest length term, by Lemma 3.3 is $\sum_{\sigma \in S_k} \kappa_\sigma p_1 \cdots p_k$. Since $\sum_{\sigma \in S_k} \kappa_\sigma > 0$, this set of eigenvectors is triangular with respect to the PBW basis of $\mathcal{H}$, hence is itself a basis of $\mathcal{H}$. The number of such eigenvectors with eigenvalue $\beta^P_\lambda$ is the number of multisets $\{p_1 \leq \cdots \leq p_k\} \in \mathcal{P}$ with $\lambda = (\deg p_1, \ldots, \deg p_k)$. Since $b_i$ counts the elements of $\mathcal{P}$ of degree $i$, the generating function for such multisets is indeed $\prod_i (1 - x_i)^{-b_i}$.

Second proof of Theorem 3.5: descent algebras and noncommutative symmetric functions

Recall from Proposition 2.9 that, on a cocommutative Hopf algebra $\mathcal{H}$, the composition of descent operators is equivalent to the internal product of noncommutative symmetric functions: $\theta(f) \circ \theta(g) = \theta(f \cdot g)$. (Recall that $\theta$ is defined as the linear map sending the complete noncommutative symmetric function $S^D$ to the descent operator $m \Delta_P$.) Write $S^P$ for $\sum_D P(D) S^D$ (so $\theta(S^P) = m \Delta_P$), and focus on the linear map $L_{SP} : \text{Sym} \to \text{Sym}$ given by internal product on the left by $S^P$. Since the internal product of $\text{Sym}$ is equivalent to the product in the descent algebra, [BP08, Prop. 3.10] [BD98, Th. 1] asserts that $L_{SP}$ is diagonalisable, and [KLT97, Prop. 3.12] shows that its eigenvalues are the $\beta^P_\lambda$ in the theorem statement. (In what follows, assume $P$ is “generic” so
that all $\beta^P_\lambda$ are distinct. This suffices since the characteristic polynomial of a matrix is continuous in its entries.)

To see that $\beta^P_\lambda$ are also eigenvalues of $\theta(S^P) = m\Delta_P : \mathcal{H}_n \rightarrow \mathcal{H}_n$, consider the orthogonal projections to each eigenspace of $L_{SP}$. These are polynomials in $L_{SP}$, and are therefore of the form $L_{E^P_\lambda}$ for some $E^P_\lambda \in \text{Sym}$. Now the image of $\theta(E^P_\lambda) : \mathcal{H}_n \rightarrow \mathcal{H}_n$ consists of eigenvectors of $m\Delta_P$, since

$$\theta(S^P)(\theta(E^P_\lambda)x) = \theta(S^P \cdot E^P_\lambda)(x) = \beta^P_\lambda \theta(E^P_\lambda)x.$$ 

Hence, $\beta^P_\lambda$ are indeed the eigenvalues of $m\Delta_P$.

It remains to determine the multiplicities of the eigenvalues (and deduce by dimension counting that no other eigenvalues can exist). An extra piece of notation is useful here: consider the linear symmetric function $h_\lambda$ (the left-internal-product map on symmetric functions) is the orthogonal projection to the algebra of symmetric functions \[\text{Sta}99\] Chap. 7], sending $S^P$ to the complete symmetric function $h_D$. The image of $F \in \text{Sym}$ under this map is its commutative image $\bar{F}$.

**Lemma 3.11.** \[\text{KLT97} \] Th. 3.21] Suppose $E^P_\lambda, E^P_\lambda'$ are two noncommutative symmetric functions, idempotent under the internal product, whose commutative images $\bar{E^P_\lambda}, \bar{E^P_\lambda}'$ are both the normalised power sum $p^\lambda_\lambda$. Let $\mathcal{H}$ be a graded connected cocommutative Hopf algebra. Then the linear map $\theta(S^H - E^\lambda - E^\lambda') : \mathcal{H}_n \rightarrow \mathcal{H}_n$ is invertible, and sends the image of $\theta(E^\lambda)$ to the image of $\theta(E^\lambda')$. In particular, these two images have the same dimension.

(The reference treats only the case where $\mathcal{H}$ is the free associative algebra, but the proof - that the eigenvalues $\langle S^H - E^\lambda - E^\lambda', p_\mu \rangle$ are non-zero - holds for any cocommutative $\mathcal{H}$.)

Set $E^P_\lambda$ to be the eigenspace projector $E^P_\lambda$. Its commutative image $\bar{E^P_\lambda}$ is indeed $p^\lambda_\lambda$, because $L_{E^P_\lambda}$ (the left-internal-product map on symmetric functions) is the orthogonal projection to the $\beta^P_\lambda$-eigenspace for $L_{SP}$. Take $E^\lambda$ to be the Garsia-Reutenauer idempotents \[\text{KLT97} \] Sec. 3.3], so the image of $\theta(E^\lambda) : \mathcal{H}_n \rightarrow \mathcal{H}_n$ has basis $\{\sum_{\sigma \in \mathcal{S}(\lambda)} p_{\sigma(1)} \cdots p_{\sigma(|\lambda|)} | p_1 \leq \cdots \leq p_{|\lambda|} \in \mathcal{P}, \deg p_i = \lambda_i\}$, where $\mathcal{P}$ is an ordered basis of primitives of $\mathcal{H}$. The cardinality of these sets are precisely as given by the generating functions in Theorem 3.5.

**Remark.** Note that, if $x \in \text{im} \theta(E^\lambda)$, then $\theta(S^H - E^\lambda - E^\lambda')x = \theta(-E^\lambda')x$, since $\theta(S^H)$ and $\theta(E^\lambda)$ both act as the identity map on $\text{im} \theta(E^\lambda)$. Hence the proof above supplies the following eigenbasis for $m\Delta_P$:

$$\left\{ \theta(E^P_{\ deg(p)} ) \left( \sum_{\sigma \in \mathcal{S}(k)} p_{\sigma(1)} \cdots p_{\sigma(k)} \right) | k \in \mathbb{N}, p_1 \leq \cdots \leq p_k \in \mathcal{P}, \deg p_i = \lambda_i \right\},$$

where $L_{E^P_{\ deg(p)}}$ are the orthogonal projections to the eigenspaces of $L_{SP}$. However, this formula may not lead to easy computation, since expressions for the $E^P_\lambda$ are usually fairly complicated, see \[\text{DFP92} \] Eq. 4.5).

### 3.3 Stationary distribution

All descent operator Markov chains on the same state space basis share the same stationary distributions. These have a simple expression in terms of the product structure constants and the rescaling function $\eta$ of Lemma 3.3. Informally, $\pi_{c_1, \ldots, c_n}(x)$ enumerates the ways to build $x$ out of $c_1, \ldots, c_n$ (in any order) using the multiplication of the combinatorial Hopf algebra, and to then
break it into singletons. (The theorem below restricts to probability distributions $P$ taking a non-zero value on some weak-composition with at least two non-zero parts, so the chain driven by $m\Delta P$ is not trivial - else every distribution is a stationary distribution.)

**Theorem 3.12** (Stationary distributions of descent operator chains). Let $\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$ be a graded connected Hopf algebra over $\mathbb{R}$ with each $\mathcal{H}_n$ finite-dimensional, and $B = \prod_{n \geq 0} B_n$ a basis of $\mathcal{H}$. Fix an integer $n$, and let $P$ be any probability distribution on the weak-compositions of $n$ such that $B$ is a state space basis for $P$, and $P$ is non-zero on some weak-composition with at least two non-zero parts. For each multiset $\{c_1, \ldots, c_n\}$ in $B_n$, define the function $\pi_{c_1, \ldots, c_n}(x) : B_1 \to \mathbb{R}$ by

$$\pi_{c_1, \ldots, c_n}(x) := \frac{\eta(x)}{n!^2} \sum_{\sigma \in S_n} \xi_{x_{\sigma(1)}, \ldots, x_{\sigma(n)}}^{c_1, \ldots, c_n} = \frac{\eta(x)}{n!^2} \sum_{\sigma \in S_n} \text{coefficient of } x \text{ in the product } c_{\sigma(1)} \cdots c_{\sigma(n)}.$$

If $\pi_{c_1, \ldots, c_n}(x) \geq 0$ for all $x \in B_n$, then $\pi_{c_1, \ldots, c_n}$ is a stationary distribution for the Markov chain on $B_n$ driven by $m\Delta P$, and any stationary distribution of this chain can be uniquely written as a linear combination of these $\pi_{c_1, \ldots, c_n}$.

Many Hopf algebras satisfy $\dim \mathcal{H}_1 = 1$, in which case the stationary distribution is unique and given by

$$\pi(x) := \frac{\eta(x)}{n!^2} \xi_{\bullet^{n}} = \frac{\eta(x)}{n!} \text{ coefficient of } x \text{ in the product } \bullet^n,$$

where $\bullet$ denotes the sole element of $B_1$. This simplified formula applies to both extended examples in Sections 5 and 6.

**Proof.** By Proposition 2.4.L, the theorem follows from the following three assertions:

i) Each function $\pi_{c_1, \ldots, c_n}$ has images summing to 1, so if it takes non-negative values, it is indeed a probability distribution.

ii) For any probability distribution $P$ on weak-compositions which is non-zero on some weak-composition with at least two non-zero parts, the partition $(1, 1, \ldots, 1)$ is the only $\lambda$ for which $\beta_\lambda^P = 1$.

iii) The set of symmetrised products $\{\sum_{\sigma \in S_n} c_{\sigma(1)} \cdots c_{\sigma(n)}\}$, over all choices of multisets $\{c_1, \ldots, c_n\} \subseteq B_1$, gives a basis of the $\beta_\lambda^P$-eigenspace of $m\Delta P$. In other words, each symmetrised product is a $\beta_\lambda^P$-eigenvector of $m\Delta P$, and this set is linearly independent and has cardinality equal to the multiplicity of $\beta_\lambda^P$ specified in Theorem 3.5.

For i, to see that $\sum_{x \in B_n} \pi_{c_1, \ldots, c_n}(x) = 1$, appeal to the second displayed equation of the proof of Theorem 3.4. Taking $z_i = c_i$, it shows that, for each $\sigma \in S_n$,

$$\sum_{x \in B_n} \xi_{x_{\sigma(1)}, \ldots, x_{\sigma(n)}}^{c_1, \ldots, c_n} \eta(x) = \left(\deg c_{\sigma(1)} \cdots \deg c_{\sigma(n)}\right) \eta(c_1) \cdots \eta(c_n) = n! \cdot 1 \cdots 1.$$

Now turn to ii. Recall that $\beta_\lambda^P := \sum_D P(D) \beta_\lambda^D$, so it suffices to show, for each weak-composition $D$, that $\beta_\lambda^D(1, \ldots, 1) = \binom{n}{D}$, and that $\beta_\lambda^D \leq \binom{n}{D}$ for all other partitions $\lambda$, with a strict inequality if $D$ has more than one non-zero part. The first assertion follows from the definition of $\beta_\lambda^D$ as the number of set compositions of $\{1, 2, \ldots, n\}$ into $l(D)$ blocks such that block $i$ contains $d_i$ elements. Observe that $\beta_\lambda^D$ counts the subsets of these set compositions such that $1, 2, \ldots, \lambda_1$ are in the same
block, $\lambda_1 + 1, \lambda_1 + 2, \ldots, \lambda_1 + \lambda_2$ are in the same block, and so on. If $\lambda \neq (1, \ldots, 1)$ and $D$ has more than one non-zero part, this imposes a non-trivial restriction, so the count is strictly smaller.

As for iii, recall from the proof of the Symmetrisation Lemma (Lemma 3.9) that, for each weak-composition $D$,

$$m \Delta_D(c_1 \ldots c_n) = \sum_{B_1,\ldots,B_{l(D)}} \left( \prod_{i \in B_1} c_i \right) \left( \prod_{i \in B_2} c_i \right) \cdots \left( \prod_{i \in B_{l(D)}} c_i \right),$$

where the sum runs over all set compositions $B_1 \ldots B_{l(D)}$ of $\{1,2,\ldots,n\}$ with $d_i$ elements in $B_i$. So the symmetrised product $\sum_{\sigma \in S_n} c_{\sigma(1)} \cdots c_{\sigma(n)}$ is a $\beta^D_{(1,\ldots,1)}$-eigenvector of $m \Delta_D$, for all weak-compositions $D$, and hence is a $\beta^P_{(1,\ldots,1)}$-eigenvector of $m \Delta_P$. Applying the Poincare-Birkhoff-Witt straightening algorithm to these symmetrised products give different highest length terms, so they are linearly independent (see Lemma 3.8).

It remains to check that the number of such symmetrised products is equal to the multiplicity of the eigenvalue $\beta^P_{(1,\ldots,1)}$ as specified by Theorem 3.5. Clearly the number of such symmetrised products is $(\binom{|B_1|+n-1}{n})$, the number of ways to choose $n$ unordered elements, allowing repetition, from $B_1$. On the other hand, the eigenvalue multiplicity is $(\binom{b_1+n-1}{n})$, since choosing $n$ elements whose degrees sum to $n$ constrains each element to have degree 1. By equating the coefficient of $x$ in the equality $\prod_i (1-x)^{-b_i} = \sum_n \dim \mathcal{H}_n x^n$, it is clear that $b_1 = \dim \mathcal{H}_1 = |B_1|$. So the number of such symmetrised products is indeed the multiplicity of the eigenvalue $\beta^P_{(1,\ldots,1)}$. [\qed]

### 3.4 Absorption probabilities and quasisymmetric functions

In this section, assume that $\mathcal{H}$ is commutative, so no symmetrisation is necessary in the expression for the stationary distributions in Theorem 3.12

$$\pi_{c_1,\ldots,c_n}(x) = \frac{\eta(x)}{n!} \text{ coefficient of } x \text{ in the product } c_1 \ldots c_n.$$  

Clearly, if $c_1 \ldots c_n \in \mathcal{B}$ (as opposed to being a linear combination of more than one basis element), then this state is absorbing. In general, there may be many absorbing states, and also stationary distributions supported on multiple, non-absorbing states. One sufficient condition for the absence of the latter is when $\mathcal{H}$ is freely generated as an algebra (i.e. $\mathcal{H} = \mathbb{R}[x_1, x_2, \ldots]$ where $x_i$ may have any degree) and $\mathcal{B} = \{x_{i_1} \ldots x_{i_k}\}$ is the set of products in the generators. This is the case for the chain on organisational tree structures of Example 1.1 (eventually all employees leave) and also for the rock-chipping model in the introduction (eventually all rocks have size 1). Under these conditions, the probability of absorption can be rephrased in terms of the fundamental Hopf morphism of [ABS06] Th. 4.1. This connection is a generalisation of [DPR14] Prop. 3.25 and [Pan14] Prop. 5.1.18; as remarked there, this result does not seem to give an efficient way to compute these absorption probabilities.

Define a function $\zeta$ on the generators of $\mathcal{H}$

$$\zeta(x_i) = \begin{cases} 1, & \text{if } \deg x_i = 1; \\ 0, & \text{if } \deg x_i > 1, \end{cases}$$

and extend it linearly and multiplicatively to a character $\zeta : \mathcal{H} \to \mathbb{R}$. So, on $\mathcal{B}_n$, we have $\zeta(x) = 1$ if $x$ is an absorbing state, and 0 otherwise. [ABS06] Th. 4.1 asserts that there is a unique Hopf morphism $\chi$ from $\mathcal{H}$ to $QSym$, the algebra of quasisymmetric functions [Ges84], such that $\zeta$ agrees with the composition of $\chi$ with evaluation at $z_1 = 1, z_2 = z_3 = \cdots = 0$. (Here, $z_i$ are the variables of $QSym$.)

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Theorem 3.13 (Absorption probabilities of certain descent operator chains). Let $\mathcal{H}$ be a Hopf algebra isomorphic to $\mathbb{R}[x_1, x_2, \ldots]$ as an algebra, and $\mathcal{B} = \{x_i, \ldots x_n\}$. Let $\{X_t\}$ be the descent operator Markov chain on $\mathcal{B}_n$ driven by $m\Delta_P$, started at $x_0$. Then the probability that $\{X_t\}$ is absorbed in $t$ steps is

$$\frac{n!}{\eta(x_0)} \langle S^P, S^P, \ldots, S^P, \chi(x_0) \rangle,$$

More explicitly, this absorption probability is: start with the noncommutative symmetric function $S^P := \sum D \Delta_P \frac{P(D)}{\langle D \rangle}$, take its $t$-fold internal product with itself, then take its inner product with $\chi(x_0)$ (since $QSym \ni \chi(x_0)$ and $\text{Sym} \ni S^P$ are dual Hopf algebras), multiply by $n!$ and divide by the rescaling function evaluated at the initial state.

Note that this theorem only gives the probability of reaching the set of absorbing states; the above formulation does not calculate the different probabilities of being absorbed at different states. A variant which partially addresses this is [Pan14, Prop. 5.1.19].

Proof. The proof is simply a matter of unpacking definitions.

First, reduce to the case of $t = 1$ using Proposition 2.9 since $\mathcal{H}$ is commutative, $t$ steps of the chain from $S^P$ is equivalent to a single step of the chain from $S^P \cdot S^P \cdot \ldots \cdot S^P$ (with $t$ factors). So it suffices to show that

$$\sum_y \tilde{K}(x_0, y) = \frac{n!}{\eta(x_0)} \langle S^P, \chi(x_0) \rangle,$$

where the sum runs over all absorbing states $y$, and $\tilde{K}$ is the transition matrix of the chain driven by $m\Delta_P$. Unravelling the definition of the Doob transform,

$$\tilde{K}(x_0, y) = \frac{\eta(y)}{\eta(x_0)} \text{coefficient of } y \text{ in } m\Delta_P(x_0)$$

$$= \frac{\eta(y)}{\eta(x_0)} \text{coefficient of } y \text{ in } (\theta(S^P))(x_0),$$

where $\theta$ is the (external product) algebra homomorphism from noncommutative symmetric functions to descent operators. So Equation 3.14 is linear in $S^P$, and thus it suffices to work with the complete noncommutative symmetric function $S^D$, i.e. to show

$$\sum_y \text{coefficient of } y \text{ in } m\Delta_D(x_0) = \frac{n!}{\eta(x_0)} \langle S^D, \chi(x_0) \rangle.$$

Now the inner product of a quasisymmetric function with $S^D$ is simply its coefficient of $M_D$, the monomial quasisymmetric function. And [ABS06, Th. 4.1] defines the $M_D$ coefficient of $\chi(x_0)$ to be $\zeta^{\otimes l(D)} \circ \Delta_D(x_0)$. Since $\zeta$ is multiplicative and linear, this is

$$\zeta(m\Delta_D(x_0)) = \sum_{y \in \mathcal{B}_n} \zeta(y) \times \text{coefficient of } y \text{ in } m\Delta_D(x_0)$$

$$= \sum_{y \text{ absorbing}} \text{coefficient of } y \text{ in } m\Delta_D(x_0).$$

Finally, note that, by the compatibility of product and coproduct, the rescaling function $\eta$ evaluates to $n!$ on each absorbing state $c_1 \ldots c_n$. \qed

24
4 The Top-or-Bottom-to-Random Chains

This section examines Markov chains driven by various generalisations of the top-to-random operator $m\Delta_{1,n-1}$. These are of particular interest because, on many combinatorial Hopf algebras, the refined coproduct $\Delta_{1,n-1}$ and refined product $m : \mathcal{H}_1 \otimes \mathcal{H}_{n-1} \rightarrow \mathcal{H}_n$ are much easier to understand than the full coproduct and product. As a result, the chains arising from $m\Delta_{1,n-1}$ (remove and reattach a piece of size 1) are more natural than those from other descent operators. Furthermore, these chains have very tractable eigendata (Theorem 4.4): many of their eigenvalues collide, resulting in at most $n$ distinct eigenvalues, and there is an extremely simple formula for many of their eigenvectors.

4.1 The top-or-bottom-to-random operators

This section defines the operators of interest and details the relationships between them. In a weak-composition, write $1^r$ to denote $r$ consecutive parts of size 1. (Take the convention that $1^0$ denotes a single part of size 0.)

Definition 4.1.

- The top-to-random distribution is concentrated at $(1, n-1)$.
- The top-$r$-to-random distribution is concentrated at $(1^r, n-r)$ (see [DFP92, Sec. 2]).
- The binomial-top-to-random distribution, with parameter $q_2$, assigns probability $\binom{n}{r}(1-q_2)^r q_2^{n-r}$ to $(1^r, n-r)$, for $0 \leq r \leq n$ (see [DFP92, Sec. 3 Exs. 2,3]).

The related descent operators are:

$$T_{2R_n} = \frac{1}{n} m\Delta_{1,n-1};$$
$$T_{r\ 2R_n} = \frac{1}{n(n-1)\ldots(n-r+1)} m\Delta_{1^r,n-r};$$
$$\text{BinT}_{2R_n}(q_2) = \sum_{r=0}^{n} \frac{1}{r!} (1-q_2)^r q_2^{n-r} m\Delta_{1^r,n-r}.$$ 

An alternative definition of $\text{BinT}_{2R_n}$, in terms of the formal series for the exponential function (under the convolution product)

$$\exp_*(x) = 1 + x + \frac{1}{2!} x \ast x + \frac{1}{3!} x \ast x \ast x + \ldots,$$

is

$$\sum_n \text{BinT}_{2R_n}(q_2) := \left( \exp_* \left( \frac{1-q_2}{q_2} \text{Proj}_{1} \right) \right) \ast \left( \sum_n q_2^n \text{Proj}_{n} \right).$$

This formulation will not be necessary for what follows.

We will informally refer to all three operators above as top-to-random maps. Here is the first way in which they are simpler than the arbitrary descent operator. Recall that a descent operator induces a Markov chain on $\mathcal{B}_n$ only if $\mathcal{B}_n$ is a state space basis (Definition 3.1), where the relevant product and coproduct structure constants are non-negative. When using a top-to-random map, there are fewer structure constants to check. Thanks to associativity and coassociativity, the following conditions suffice:
i) for all \( c \in B_1, z \in B \), their product is \( cz = \sum_{y \in B} \xi^y_{c,z} y \) with \( \xi^y_{c,z} \geq 0 \);

ii) for all \( x \in B \), its refined coproduct is \( \Delta_{1,\deg(x)-1}(x) = \sum_{c,z \in B} \eta^c_{x} z \) with \( \eta^c_{x} \geq 0 \);

iii) for each \( x \in B \), at least one of the \( \eta^c_{x} \) above is non-zero; in other words, \( \Delta_{1,\deg(x)-1}(x) \neq 0 \).

Observe that performing a \( T \circ r \) 2R card-shuffle followed by a 2R shuffle results in either a \( T(r+1) \) 2R shuffle (if the top card after \( T \circ r \) 2R is one which has not yet been touched) or a \( T \circ r \) 2R shuffle (if this top card was already touched). Thus, as operators on the shuffle algebra,

\[
T_{2R_n} \circ T \circ r \circ 2R_n = \frac{n-r}{n} T(r+1) 2R_n + \frac{r}{n} T \circ r \circ 2R_n.
\]

Iterating this shows that the \( T \circ r \) 2R shuffle is a degree \( r \) polynomial (under composition) in the 2R shuffle (see Proposition 4.3 below.) And the BinT2R shuffle is clearly a linear combination of \( T \circ r \) 2R shuffles. Hence all three types of shuffles can be understood just by examining the top-to-random shuffle.

In fact, the argument above goes through for all commutative Hopf algebras, by the composition rule (Proposition 2.9). It holds more generally for the following maps:

**Definition 4.2.**

- The *top-or-bottom-to-random* distribution, with parameter \( q \), assigns probability \( q \) to \((1, n-1)\) and probability \( 1-q \) to \((n-1, 1)\). ([DFP92, Sec. 6 Ex. 4] mentions the symmetric case, of \( q = \frac{1}{2} \)).

- The *binomial-top-or-bottom-r-to-random* distribution, with parameter \( q \), assigns probability \( \binom{r}{r_1} q^{r_1} (1-q)^{r_3} \) to \((1^{r_1}, n-r, 1^{r_3})\), for \( r_1, r_3 \geq 0 \) with \( r_1 + r_3 = r \).

- The *trinomial-top-or-bottom-r-to-random* distribution, with parameters \( q_1, q_2, q_3 \) summing to 1, assigns probability \( \binom{n}{r_1 r_2 r_3} q_1^{r_1} q_2^{r_2} q_3^{r_3} \) to \((1^{r_1}, r_2, 1^{r_3})\), for \( r_1, r_2, r_3 \geq 0 \) with \( r_1 + r_2 + r_3 = n \). (The corresponding shuffle was termed “trinomial top and bottom to random” in [DFP92, Sec. 6 Ex. 6], but we change the conjunction from “and” to “or” to highlight its relationship to the top-or-bottom-to-random distribution, and to distinguish it from the top-and-bottom-to-random distributions of Section 4.3)

The related descent operators are:

\[
T/B2R_n(q) := \frac{q}{n} m \Delta_{1,n-1} + \frac{1-q}{n} m \Delta_{n-1,1};
\]

\[
\text{BinT}/B \circ r \circ 2R_n(q) := \frac{1}{n(n-1) \ldots (n-r+1)} \sum_{r_1 + r_3 = r} \binom{r}{r_1} q^{r_1} (1-q)^{r_3} m \Delta_{1^{r_1}, n-r, 1^{r_3}};
\]

\[
\text{TrinT}/B2R_n(q_1, q_2, q_3) := \sum_{r_1 + r_2 + r_3 = n} \frac{1}{r_1! r_3!} q_1^{r_1} q_2^{r_2} q_3^{r_3} m \Delta_{1^{r_1}, r_2, 1^{r_3}}.
\]

Observe that

\[
T_{2R_n} = T/B2R_n(1);
\]

\[
T \circ r \circ 2R_n = \text{BinT}/B \circ r \circ 2R_n(1);
\]

\[
\text{BinT}/B2R_n(q_2) = \text{TrinT}/B2R_n(1 - q_2, q_2, 0).
\]
So it is no surprise that $\text{TrinT}/B2R_n$ also admits an equivalent definition in terms of $\exp_*$ (which again will not be necessary for this work):

$$
\sum_n \text{TrinT}/B2R_n(q_1, q_2, q_3) := \left(\exp_* \left(\frac{q_1}{q_2} \text{Proj}_1\right)\right) \ast \left(\sum_n q_2^n \text{Proj}_n\right) \ast \left(\exp_* \left(\frac{q_3}{q_2} \text{Proj}_1\right)\right).
$$

This more general triple of operators still enjoy simplified state space basis axioms, namely the two-sided analogue of the conditions for top-to-random: $\xi^y_{c,z}, \xi^y_{z,c}, \eta^c_{x,c}, \eta^{z,c} \geq 0$ for all $x, y, z \in B$ and all $c \in \mathcal{B}_1$, and $\Delta_{1, \deg(x) - 1}(x) \neq 0$ for all $x \in \mathcal{B}$.

The precise relationship between the top-or-bottom-to-random operators, coming from the composition rule [Proposition 2.9], is

**Proposition 4.3** (Relationship between top-or-bottom-to-random operators). On all Hopf algebras,

$$
\text{TrinT}/B2R_n(q_1, q_2, q_3) = \sum_{r=0}^{n} \binom{n}{r} q_2^{n-r} (1 - q_2)^r \text{BinT}/B r 2R_n \left(\frac{q_1}{q_1 + q_3}\right).
$$

And, on all commutative or cocommutative Hopf algebras, $\text{BinT}/B r 2R_n(q)$ is a polynomial of degree $r$ in $\text{T}/B2R_n(q)$, namely the falling factorial

$$
\text{BinT}/B r 2R_n = \left(\frac{n \text{T}/B2R_n}{n}\right) \circ \left(\frac{n \text{T}/B2R_n - \text{id}}{n - 1}\right) \circ \cdots \circ \left(\frac{n \text{T}/B2R_n - (r - 1) \text{id}}{n - (r - 1)}\right).
$$

Hence, on all commutative or cocommutative Hopf algebras, $\text{TrinT}/B2R_n(q_1, q_2, q_3)$ is the polynomial

$$
\sum_{r=0}^{n} \binom{n}{r} q_2^{n-r} (1 - q_2)^r
$$

evaluated at $x = \text{T}/B2R_n \left(\frac{q_1}{q_1 + q_3}\right)$.

Note that, if $x$ is an integer between 0 and $n$, then the polynomial above simplifies to $q_2^{n-x}$. (This is false if $x$ is outside this range.) These powers of $q_2$ will turn out to be the eigenvalues of $\text{TrinT}/B2R_n(q_1, q_2, q_3)$.

[DFP92] Sec. 6 Ex. 6] observed that, if $q_1 = q_3$, then $\text{TrinT}/B2R_n(q_1, q_2, q_3)$ (on a commutative or cocommutative Hopf algebra) spans a commutative subalgebra of operators. The polynomial expression above shows that this is true whenever the ratio between $q_1$ and $q_3$ is fixed. Hence the eigendata theorems below apply to $\text{TrinT}/B2R_n(q_1, q_2, q_3)$ chains where $q_2$ varies over time, or chains where different steps are driven by different top-or-bottom-to-random operators, for instance alternating between $\text{T}/B2R_n \left(\frac{q_1}{q_1 + q_3}\right)$ and $\text{TrinT}/B2R_n(q_1, q_2, q_3)$.

**Remark.** Interestingly, the expression of $\text{T} r 2R$ as a polynomial in $\text{T}2R$ (i.e. the case $q = 1$) still holds on a noncommutative and noncocommutative Hopf algebra, so long as $\dim \mathcal{H}_1 = 1$. This is a feature of the theory of dual graded graphs [Fom94, BLL12], see [Pan15b, Lem. 6.3] for a proof in the language of combinatorial Hopf algebras.

### 4.2 A construction for eigenvectors

The goal of this section is to prove Theorem [14] a particularly simple way to construct many eigenvectors for the top-or-bottom-to-random maps. On a cocommutative Hopf algebra, all eigenvectors are of this special form. By Proposition [2.4] this leads to a full basis of left eigenfunctions...
for chains on cocommutative Hopf algebras, and a full basis of right eigenfunctions for chains on commutative Hopf algebras.

Two special cases of this theorem exist in the literature. For shuffling a distinct deck of cards, [DFP92, Sec. 6, Exs. 1 and 4] identified the spectrum for \( T_r 2R_n \) and \( T/B_2R_n(\frac{1}{2}) \) respectively. Unrelated, the eigenvectors for the top-to-random maps \((q = 1)\) follow from applying Schocker’s derangement idempotents [Sch03] to symmetrised products of primitives, as described in the final remark of the present Section 3.2.

**Theorem 4.4** (Eigenvectors for the top-or-bottom-to-random family). Let \( H = \bigoplus_{n \geq 0} H_n \) be a graded connected Hopf algebra over \( \mathbb{R} \) with each \( H_n \) finite-dimensional.

1. The distinct eigenvalues for the top-or-bottom-to-random operators are \( \beta_j \) where:
   - for \( T_2R_n \) and \( T/B_2R_n(q) \), \( \beta_j = \frac{j}{n} \) with \( j \in [0, n-2] \cup \{n\} \);
   - for \( T_r 2R_n \) and \( \text{Bin}T/B_r 2R_n(q) \), \( \beta_j = \frac{j(j-1)\ldots(j-r+1)}{n(n-1)\ldots(n-r+1)} \) with \( j \in \{0\} \cup [r, n-2] \cup \{n\} \);
   - for \( \text{Bin}T2R_n(q_2) \) and \( \text{Trin}T/B2R_n(q_1, q_2, q_3) \), \( \beta_j = q_2^{n-j} \) with \( j \in [0, n-2] \cup \{n\} \).

   The multiplicity of the eigenvalue \( \beta_j \) is the coefficient of \( x^{n-j}y^j \) in \( \left(\frac{1-x}{1-y}\right)^{\dim H_1} \sum_n \dim H_n x^n \).

   (Exception: for \( T_r 2R_n \) and \( \text{Bin}T/B_r 2R_n(q) \), the multiplicity of \( \beta_0 \) is the sum of the coefficients of \( x^n, x^{n-1}y, \ldots, x^{n-r+1}y^{r-1} \) in \( \left(\frac{1-x}{1-y}\right)^{\dim H_1} \sum_n \dim H_n x^n \).) In particular, if \( B_1 = \{\bullet\} \), then these multiplicities are \( \dim H_{n-j} - \dim H_{n-j-1} \).

2. Fix \( j \in [0, n-2] \cup \{n\} \). For any \( p \in H_{n-j} \) satisfying \( \Delta_{1,n-j-1}(p) = 0 \), and any \( c_1, \ldots, c_j \in H_1 \) (not necessarily distinct),
   \[
   \sum_{\sigma \in S_j} c_{\sigma(1)} \ldots c_{\sigma(j)} p
   \]
   is an eigenvector for:
   - \( T_2R_n \) and \( \text{Bin}T2R_n(q_2) \), with eigenvalue \( \beta_j \);
   - \( T_r 2R_n \), with eigenvalue \( \beta_j \) if \( j \geq r \), and 0 otherwise.

For any \( p \in H_{n-j} \) satisfying \( \Delta_{1,n-j-1}(p) = \Delta_{n-j-1,1}(p) = 0 \), and any \( c_1, \ldots, c_j \in H_1 \) (not necessarily distinct),
   \[
   \sum_{i=0}^{j} \sum_{\sigma \in S_j} \left(\begin{array}{c} j \\ i \end{array}\right) q^i (1-q)^{j-i} c_{\sigma(1)} \ldots c_{\sigma(i)} p c_{\sigma(i+1)} \ldots c_{\sigma(j)}
   \]
   is an eigenvector for:
   - \( T/B_2R_n(q) \), with eigenvalue \( \beta_j \);
   - \( \text{Bin}T/B_r 2R_n(q) \), with eigenvalue \( \beta_j \) if \( j \geq r \), and 0 otherwise.

and
   \[
   \sum_{i=0}^{j} \sum_{\sigma \in S_j} \left(\begin{array}{c} j \\ i \end{array}\right) q_1^{i} q_3^{j-i} c_{\sigma(1)} \ldots c_{\sigma(i)} p c_{\sigma(i+1)} \ldots c_{\sigma(j)}
   \]
   is a \( \beta_j \)-eigenvector for \( \text{Trin}T/B2R_n(q_1, q_2, q_3) \).
iii) Let \( \mathcal{P} \) be a (graded) basis of the primitive subspace of \( \mathcal{H} \). Write \( \mathcal{P} \) as the disjoint union \( \mathcal{P}_1 \sqcup \mathcal{P}_{>1} \), where \( \mathcal{P}_1 \) has degree 1. Set

\[
\mathcal{E}_j(1) := \left\{ \sum_{\sigma \in \mathcal{E}_j} c_{\sigma(1)} \cdots c_{\sigma(i)} \sum_{\tau \in \mathcal{E}_{k-j}} p_{\tau(1)} \cdots p_{\tau(k-j)} \right\},
\]

\[
\mathcal{E}_j(q) := \left\{ \sum_{i=0}^{j} \sum_{\sigma \in \mathcal{E}_j} \binom{j}{i} q^i (1-q)^{j-i} c_{\sigma(1)} \cdots c_{\sigma(i)} \left( \sum_{\tau \in \mathcal{E}_{k-j}} p_{\tau(1)} \cdots p_{\tau(k-j)} \right) c_{\sigma(i+1)} \cdots c_{\sigma(j)} \right\},
\]

\[
\mathcal{E}_j(q_1, q_2, q_3) := \left\{ \sum_{i=0}^{j} \sum_{\sigma \in \mathcal{E}_j} \binom{j}{i} q_1^i q_2^{j-i} c_{\sigma(1)} \cdots c_{\sigma(i)} \left( \sum_{\tau \in \mathcal{E}_{k-j}} p_{\tau(1)} \cdots p_{\tau(k-j)} \right) c_{\sigma(i+1)} \cdots c_{\sigma(j)} \right\},
\]

where each set ranges over all multisets \( \{c_1, \ldots, c_j\} \) of \( \mathcal{P}_1 \), and all multisets \( \{p_1, \ldots, p_{k-j}\} \) of \( \mathcal{P}_{>1} \) with \( \deg p_1 + \cdots + \deg p_{k-j} = n - j \). Then

- \( \mathcal{E}_j(1) \) (resp. \( \mathcal{E}_j(q) \), \( \mathcal{E}_j(q_1, q_2, q_3) \)) consists of linearly independent \( \beta_j \)-eigenvectors for \( T^{2R_n} \) and \( \text{BinT} T^{2R_n}(q_2) \) (resp. \( \text{T/B} 2R_n(q) \), \( \text{TrinT/B} 2R_n(q_1, q_2, q_3) \));

- for \( j \geq r \), the set \( \mathcal{E}_j(1) \) (resp. \( \mathcal{E}_j(q) \)) consists of linearly independent \( \beta_j \)-eigenvectors for \( T r 2R_n \) (resp. \( \text{BinT/B} r 2R_n(q) \), \( \text{II}^{-1}_j \mathcal{E}_j(1) \) (resp. \( \text{II}^{-1}_j \mathcal{E}_j(q) \)) consists of linearly independent \( 0 \)-eigenvectors for \( T r 2R_n \) (resp. \( \text{BinT/B} r 2R_n(q) \)).

iv) In addition, if \( \mathcal{H} \) is cocommutative, then \( \Pi^{-2}_j \mathcal{E}_j \sqrt{n} \mathcal{E}_n \) is an eigenbasis for the above maps.

The symmetrisation of the \( p_i \) in iii above is unnecessary: its only advantage is to put all \( p_i \) in the chosen multiset on equal footing. In other words, if the basis of primitives \( \mathcal{P} \) admits a natural order \( \preceq \), then setting \( \mathcal{E}_j(1) = \left\{ \sum_{\sigma \in \mathcal{E}_j} c_{\sigma(1)} \cdots c_{\sigma(i)} p_1 \cdots p_{k-j} \right\} \) over all multisets \( \{c_1, \ldots, c_j\} \) of \( \mathcal{P}_1 \), and all multisets \( \{p_1 \preceq \cdots \preceq p_{k-j}\} \subseteq \mathcal{P}_{>1} \) with \( \deg p_1 + \cdots + \deg p_{k-j} = n - j \), would also give linearly independent eigenvectors (and similarly for \( \mathcal{E}_j(q) \) and \( \mathcal{E}_j(q_1, q_2, q_3) \)). (Indeed, by Lemma 3.8 on the leading term under the PBW straightening algorithm, it is possible to use any linear combination of the products \( p_{\tau(1)} \cdots p_{\tau(k-j)} \) so long as the coefficient sum is non-zero.) The symmetrisation of the \( c_i \), however, is necessary.

Proof. We start by proving the series of implications \( \text{ii} \Rightarrow \text{iii} \Rightarrow \text{iv} \Rightarrow \text{i} \), and tackle the proof of \( \text{ii} \) at the end.

\text{ii} \Rightarrow \text{iii}: Taking \( p = \sum_{\tau \in \mathcal{E}_{k-j}} p_{\tau(1)} \cdots p_{\tau(k-j)} \) shows that each \( \mathcal{E}_j(q) \) consists of eigenvectors. Their linear independence follows from a PBW-straightening and triangularity argument in the subalgebra \( \mathcal{U}(\mathcal{P}) \), the universal enveloping algebra of the primitive subspace (see the last paragraph of the first proof of Theorem 3.5).

\text{iii} \Rightarrow \text{iv}: If \( \mathcal{H} \) is cocommutative, then \( \mathcal{H} = \mathcal{U}(\mathcal{P}) \) [Car07] Th. 3.8.1, so the \( \mathcal{E}_j \) also span.

\text{iv} \Rightarrow \text{i}: Recall from the proof of Theorem 3.5 that the multiplicity of each eigenvalue on \( \mathcal{H} \) is equal to its multiplicity on the cocommutative Hopf algebra \( \text{gr}(\mathcal{H})^* \). Hence these multiplicities are the product of two numbers: the number of multiset of \( \mathcal{P}_1 \) (for \( \text{gr}(\mathcal{H})^* \)) with \( j \) elements, and the number of multiset of \( \mathcal{P}_{>1} \) (for \( \text{gr}(\mathcal{H})^* \)) with total degree \( n - j \). As in Theorem 3.5 write \( b_i \) for the number of elements of degree \( i \) in \( \mathcal{P} \) (for \( \text{gr}(\mathcal{H})^* \)), so \( \sum_n \dim \mathcal{H}_n x^n = \prod_i (1 - x_i)^{-b_i} \). Then the required multiplicities are the coefficients of \( x^{n-j} y^j \) in \( (1-y)^{-b_1} \prod_{i>1} (1-x_i)^{-b_i} = \left(\frac{1-z}{1-y}\right)^{b_1} \sum_n \dim \mathcal{H}_n x^n \), and \( b_1 = \dim \mathcal{H}_1 \), as noted in the proof of Theorem 3.12.
Proof of ii: This can be checked by direct calculation, but that doesn’t explain where the formula comes from. So here’s a more circutous proof to demonstrate how one might discover such a formula. (It is a more complicated version of the \( q = 1 \) argument in Remark 1 after Lemma \[3.9\]) The proof first concentrates on \( T/B2R_n \). The result for \( \text{BinT}/Br2R_n \) and \( \text{TrinT}/B2R_n \) are not then immediate - Proposition \[4.3\] doesn’t apply as there is no commutativity or cocommutativity hypothesis. A more careful argument (the final sentence of the proof) is necessary to make this extension.

The Symmetrisation Lemma (Lemma \[3.9\]) asserts that there is an eigenvector for each multiset. Experiment with a simplest case where this multiset is necessary to make this extension.

\( m \Delta_{1,n-1}(c_1 c_2 c_3 pc_{4+5}) = c_1 c_2 c_3 pc_{4+5} + c_2 c_1 c_3 pc_{4+5} + c_3 c_1 c_2 pc_{4+5} + c_4 c_1 c_2 c_3 pc_{4+5} + c_5 c_1 c_2 c_3 pc_{4+5} \) (see Equation \[3.10\] for the general formula of a descent operator acting on a product of primitives), it is not hard to see that

\[
m \Delta_{1,n-1} \left( \sum_{\sigma \in \mathcal{E}_j} c_{\sigma(1)} \cdots c_{\sigma(i)} pc_{\sigma(i+1)} \cdots c_{\sigma(j)} \right) = i \left( \sum_{\sigma \in \mathcal{E}_j} c_{\sigma(1)} \cdots c_{\sigma(i)} pc_{\sigma(i+1)} \cdots c_{\sigma(j)} \right) + (j - i) \left( \sum_{\sigma \in \mathcal{E}_j} c_{\sigma(1)} \cdots c_{\sigma(i+1)} pc_{\sigma(i+2)} \cdots c_{\sigma(j)} \right),
\]

and

\[
m \Delta_{n-1,1} \left( \sum_{\sigma \in \mathcal{E}_j} c_{\sigma(1)} \cdots c_{\sigma(i)} pc_{\sigma(i+1)} \cdots c_{\sigma(j)} \right) = i \left( \sum_{\sigma \in \mathcal{E}_j} c_{\sigma(1)} \cdots c_{\sigma(i-1)} pc_{\sigma(i)} \cdots c_{\sigma(j)} \right) + (j - i) \left( \sum_{\sigma \in \mathcal{E}_j} c_{\sigma(1)} \cdots c_{\sigma(i)} pc_{\sigma(i+1)} \cdots c_{\sigma(j)} \right).
\]

So, treating \( \omega_i := \sum_{\sigma \in \mathcal{E}_j} c_{\sigma(1)} \cdots c_{\sigma(i)} pc_{\sigma(i+1)} \cdots c_{\sigma(j)} \) as a state, the linear map \( T/B2R_n(q) = \frac{q}{n} m \Delta_{1,n-1} + \frac{1-q}{n} m \Delta_{n-1,1} \), acting on \( \text{span}(\omega_i | 0 \leq i \leq j) \), induces (a multiple by \( \frac{j}{n} \) of) a birth-and-death process \[LPW09\] Sec. 2.5], with transition “probabilities”

\[
\text{Prob}(\omega_i \rightarrow \omega_{i+1}) = \frac{q}{n}(j-i),
\]

\[
\text{Prob}(\omega_i \rightarrow \omega_i) = \frac{q}{n}(1 - q)(j-i),
\]

\[
\text{Prob}(\omega_i \rightarrow \omega_{i-1}) = \frac{1-q}{n}.i.
\]

The standard formula \[LPW09\] Prop. 2.8] for the “stationary distribution” of such a chain then simplifies to \( \pi(\omega_i) = \frac{1}{j} q^i (1 - q)^{j-i} \). So \( \text{sum}(\pi(\omega_i) \omega_i) \) is an eigenvector for \( T/B2R_n(q) \), of eigenvalue \( \frac{j}{n} \). Even without using the theory of birth-and-death processes, it is not hard to guess a solution to the detailed balance equations

\[
\pi(\omega_i) \frac{q}{n}(j-i) = \pi(\omega_{i+1}) \frac{1-q}{n}(i+1), \quad 0 \leq i < j.
\]
Note that this argument did not require \( x \) to be primitive - thanks to the compatibility between product and coproduct, \( c_{\sigma(1)} \cdots c_{\sigma(i)}x \sigma(i+1) \cdots c_{\sigma(j)} \) will behave like a product of primitives under \( T/B2R_n \), \( \text{BinT}/B \rightarrow 2R_n \) and \( \text{TrnT}/B2R_n \) (i.e. satisfying Equation \[3.10\] so long as \( \frac{1}{n} \Delta_{1,n-1}(x) = \frac{1-q}{n} \Delta_{n-1,1}(x) = 0 \).

\[\square\]

Note that, unless \( c_1 = \cdots = c_j \), the chain on the \( \omega_i \), with \( j + 1 \) states, is not the chain from the Symmetrisation Lemma, which is on all orderings of the multiset \( \{ \bullet_1, \ldots, \bullet_j, p \} \), so in general has \( (j + 1)! \) states. The latter chain is not reversible and does not have solutions to the detailed balance equation.

**Remark.** In the case where \( \dim H_1 = 1 \) (so \( B_1 = \{ \bullet \} \)), part iii of the Theorem above also follows from the eigenspace algorithm for dual graded graphs \[Fom94\] since, on cocommutative Hopf algebras, the maps \( U_n : H_n \rightarrow H_{n+1} \) and \( D_n : H_n \rightarrow H_{n-1} \) defined by

\[U_n(x) := q \bullet x + (1-q)x \bullet;
\]

\[\Delta_{1,n-1}(x) = \bullet \otimes D_n(x)
\]

(so \( \Delta_{n-1,1}(x) = D_n(x) \otimes \bullet \) by cocommutativity) satisfy the relation \( D_{n+1}U_n - U_{n-1}D_n = \text{id} \). This is a \( q \)-deformation of the canonical dual graded graph structure on a combinatorial Hopf algebra, as detailed in \[BLL12\]. Then \( T/B2R_n(q) = U_{n-1}D_n \). In the case \( q = 1 \) (or \( q = 0 \)), cocommutativity is not necessary - hence, whenever \( \dim H_1 = 1 \), part iii gives an eigenbasis for \( T2R_n, Tr2R_n \) and \( \text{BinT2R}_n(q_2) \).

### 4.3 Eigenvectors for top-and-bottom-to-random chains

Much of the analysis in the last two sections generalises to a wider class of chains, where the distribution \( P \) is non-zero only on weak-compositions with at most one part of size larger than 1. (That is, \( P \) is non-zero only on \( (1^{r_1}, r_2, 1^{r_3}) \) and “paddings” of these compositions by parts of size zero.) These chains model removing pieces of size 1 from either “end” of a combinatorial object then reattaching them.

The conditions for \( B \) to be a state space basis for these distributions \( P \) are the same simplified conditions for the top-or-bottom-to-random maps. The chains again have at most \( n \) eigenvalues, with the same multiplicities as in Theorem \[4.4\].i. The eigenvector algorithm goes through to a lesser extent: \( m \Delta_P \) would still induce (a multiple of) a Markov chain on \( W := \text{span}\{\omega_i := \sum_{\sigma \in \mathcal{E}_j} c_{\sigma(1)} \cdots c_{\sigma(i)} p c_{\sigma(i+1)} \cdots c_{\sigma(j)} | 0 \leq i \leq j\} \), whose “stationary distribution” \( \pi \) gives the eigenvector

\[\sum_{i=0}^{j} \sum_{\sigma \in \mathcal{E}_j} \pi(\sigma) c_{\sigma(1)} \cdots c_{\sigma(i)} p c_{\sigma(i+1)} \cdots c_{\sigma(j)}.
\]

(As above, \( p \in \ker \Delta_{1,n-j-1} \cap \ker \Delta_{n-j-1,1} \), and \( c_1, \ldots, c_j \in H_1 \).) In theory, this would again give a full eigenbasis on cocommutative Hopf algebras. However, the chains on \( W \) are in general not birth-and-death processes, and it appears to be rather rare for such “stationary distributions” to have as simple an expression as in Theorem \[4.4\]. We conclude this section with one exception.

**Definition 4.5.**

- The **top-and-bottom-to-random** distribution is concentrated at \( (1, n - 2, 1) \).
- The **top-and-bottom-r-to-random** distribution is concentrated at \( (1^r, n - 2r, 1^r) \).
The related descent operators are:

\[ T + B2R_n := \frac{1}{n(n-1)^m} \Delta_{1,n-2,1}; \]
\[ T + Br 2R_n := \frac{1}{n(n-1) \ldots (n-2r+1)^m} \Delta_{1^r,n-2r,1^r}. \]

Unlike the top-or-bottom-to-random maps, \( T + Br 2R_n \) are not polynomials and specialisations of a single descent operator. (Indeed, the eigenvectors in Theorem 4.6 below depend on \( r \).) Polynomials in \( T + B2R_n \) involve compositions \((1^r, r, 1^r)\) where \( r_1 \neq r_2 \), and the distributions associated to these polynomials don’t seem natural.

The detailed balance equations for the “Markov chain” that \( T + Br 2R_n \) induces on \( \text{span}\{\omega_i := \sum_{\sigma \in \mathfrak{S}_j} c_{\sigma(1)} \ldots c_{\sigma(i)} p_{\sigma(i+1)} \ldots c_{\sigma(j)}|0 \leq i \leq j\} \) read

\[ \pi(\omega_i) \begin{pmatrix} j - i \\ r + i' - i \end{pmatrix} \begin{pmatrix} i \\ r - i' + i \end{pmatrix} = \pi(\omega_{i'}) \begin{pmatrix} j - i' \\ r + i - i' \end{pmatrix} \begin{pmatrix} i' \\ r - i + i' \end{pmatrix}, \]

and admit the solution

\[ \pi(\omega_i) = \begin{pmatrix} j - i \\ r \end{pmatrix} \begin{pmatrix} j - r \\ i - r \end{pmatrix}. \]

(This \( \pi \) is not normalised to be a distribution - i.e. \( \sum_{i=0}^{j} \pi(\omega_i) \neq 1 \) - but that is immaterial for the eigenvector construction.) This generates the eigenvectors in the theorem below. In the case of card-shuffling, the eigenvalues of \( T + B2R_n \) previously appeared in [DFP92 Sec. 6 Ex. 5].

**Theorem 4.6 (Eigenvectors for the top-and-bottom-to-random family).** Let \( \mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n \) be a graded connected Hopf algebra over \( \mathbb{R} \) with each \( \mathcal{H}_n \) finite-dimensional.

i) The distinct eigenvalues for \( T + Br 2R_n \) are \( \beta_j = \frac{j(j-1) \ldots (j-2r+1)}{n(n-1) \ldots (n-2r+1)} \) with \( j \in \{0\} \cup [2r, n-2] \cup \{n\} \). For \( j \neq 0 \), the multiplicity of \( \beta_j \) is the coefficient of \( x^{n-j} y^j \) in \( \left( \frac{1-x}{1-y} \right)^{\dim \mathcal{H}_1} \sum_n \dim \mathcal{H}_n x^n. \)

The multiplicity of \( \beta_0 \) is the sum of the coefficients of \( x^n, x^{n-1} y, \ldots, x^{n-r+1} y^{r-1} \) in \( \left( \frac{1-x}{1-y} \right)^{\dim \mathcal{H}_1} \sum_n \dim \mathcal{H}_n x^n. \)

ii) For any \( p \in \mathcal{H}_{n-j} \) satisfying \( \Delta_{1,n-j-1}(p) = \Delta_{n-j-1,1}(p) = 0 \), and any \( c_1, \ldots, c_j \in \mathcal{H}_1 \) (not necessarily distinct):

- if \( j < 2r \), then
  \[ \sum_{\sigma \in \mathfrak{S}_j} c_{\sigma(1)} \ldots c_{\sigma(j)} p \]
  is a 0-eigenvector for \( T + Br 2R_n \).

- if \( j \geq 2r \), then
  \[ \sum_{i=r}^{j-r} \sum_{\sigma \in \mathfrak{S}_j} \begin{pmatrix} j - r \\ i \end{pmatrix} \begin{pmatrix} j - r \\ i - r \end{pmatrix} c_{\sigma(1)} \ldots c_{\sigma(i)} p c_{\sigma(i+1)} \ldots c_{\sigma(j)} \]
  is a \( \beta_j \)-eigenvector for \( T + Br 2R_n \).
iii) Let \( \mathcal{P} \) be a (graded) basis of the primitive subspace of \( \mathcal{H} \). Write \( \mathcal{P} \) as the disjoint union \( \mathcal{P}_1 \sqcup \mathcal{P}_{>1} \), where \( \mathcal{P}_1 \) has degree 1. Set

\[
\mathcal{E}_j := \left\{ \left( \sum_{\sigma \in \mathcal{E}_j} c_{\sigma(1)} \ldots c_{\sigma(j)} \right) \left( \sum_{\tau \in \mathcal{E}_{k-j}} p_{\tau(1)} \ldots p_{\tau(k-j)} \right) \right\}
\]

if \( j < 2r \), and

\[
\mathcal{E}_j := \left\{ \sum_{i=r}^{j-r} \left( \sum_{\sigma \in \mathcal{E}_j} \binom{j-r}{i} \binom{j-r}{i-r} c_{\sigma(1)} \ldots c_{\sigma(i)} \left( \sum_{\tau \in \mathcal{E}_{k-j}} p_{\tau(1)} \ldots p_{\tau(k-j)} \right) c_{\sigma(i+1)} \ldots c_{\sigma(j)} \right) \right\}
\]

if \( j \geq 2r \), ranging (in both cases) over all multisets \( \{c_1, \ldots, c_j\} \) of \( \mathcal{P}_1 \), and all multisets \( \{p_1, \ldots, p_k\} \) of \( \mathcal{P}_{>1} \) with \( \deg p_1 + \cdots + \deg p_k = n - j \). Then \( \prod_{j=0}^{r-1} \mathcal{E}_j \) consists of linearly independent \( \beta \)-eigenvectors for \( T + B r 2\mathbb{R}_n \), and, for \( j > 2r \), \( \mathcal{E}_j \) consists of linearly independent \( \beta \)-eigenvectors for \( T + B r 2\mathbb{R}_n \).

iv) In addition, if \( \mathcal{H} \) is cocommutative, then \( \prod_{j=0}^{r-2} \mathcal{E}_j \prod_{\mathcal{E}_n} \) is an eigenbasis for \( T + B r 2\mathbb{R}_n \). \( \square \)

As in the case of the top-or-bottom-to-random shuffles (Theorem 4.4), the symmetrisation of the high degree primitives \( p_t \) in part iii is unnecessary. The symmetrisation of the \( c_i \) in the \( j < 2r \) case, in both parts ii and iii, are also unnecessary - indeed, any linear combination of products in \( c_i \) and \( p_t \) is a 0-eigenvector when \( j < 2r \).

### 4.4 Recursive lumping property

Return to the top-to-random chains of Definition 4.1. Assume that the underlying Hopf algebra \( \mathcal{H} \) satisfies \( \mathcal{H}_1 = \text{span}\{\bullet\} \), together with a new condition: for each \( x \in \mathcal{B}_n \), we have \( \Delta_{1,n-1}(x) = \bullet \otimes x' \) for \( x' \in \mathcal{B}_{n-1} \). (Note this forces \( \eta \equiv 1 \).) Then, there is a well-defined map \( D : \mathcal{B}_n \to \mathcal{B}_{n-1} \) satisfying \( \Delta_{1,n-1}(x) = \bullet \otimes D(x) \). (This \( D \) is the down operator of the dual graded graph associated to \( \mathcal{H} \) in [BLL12]'s construction.) Iterates \( D^{n-k} \) of \( D \) satisfy \( \Delta_{1,n-k,k}(x) = \bullet \otimes \cdots \otimes \bullet \otimes D^{n-k}(x) \). View \( D^{n-k} \) as a “forgetful function” on \( \mathcal{B}_n \), observing only a size \( k \) part of these size \( n \) objects.

Theorem 4.10 below proves that the image under \( D^{n-k} \) of the top-to-random chain on \( \mathcal{B}_n \) is a lazy version of the analogous chain on \( \mathcal{B}_k \). See Theorem 6.3 for an intuitive card-shuffling example. Informally, observing a subobject of the chain gives a smaller copy of the same chain (with laziness), hence “recursive”. In order to state this result precisely, the following definitions are necessary:

**Definition 4.7.** Given a Markov chain \( \{X_t\} \) with transition matrix \( K \) and a number \( \alpha \in [0,1] \), the \( \alpha \)-lazy version of \( \{X_t\} \) is the Markov chain with transition matrix \( \alpha \text{id} + (1 - \alpha)K \).

Equivalently, at each time step, the \( \alpha \)-lazy version of \( \{X_t\} \) stays at the same state with probability \( \alpha \), and with probability \( 1 - \alpha \) it evolves according to \( \{X_t\} \).

From the “basis scaling” interpretation of the Doob transform \( (K := [T]^T_{\mathcal{R}}) \), it is immediate that the Doob transform “commutes with lazying” in the following sense:

**Lemma 4.8.** Let \( \{X_t\} \) be the Markov chain on \( \mathcal{B} \) driven by \( T \) with rescaling function \( \eta \). The \( \alpha \)-lazy version of \( \{X_t\} \) is driven by \( \alpha \text{id} + (1 - \alpha)T \), with the same rescaling function \( \eta \). \( \square \)

Since \( \text{id} : \mathcal{H}_n \to \mathcal{H}_n \) may be expressed as the descent operator \( m \Delta_{\{n\}} \), a lazy version of a descent operator chain is also a descent operator chain.
**Definition 4.9.** Let \( \{X_t, \hat{X}_t\} \) be Markov chains on state spaces \( \Omega, \bar{\Omega} \) respectively, and let \( \theta : \Omega \to \bar{\Omega} \) be a surjection. Then \( \{X_t\} \) is said to **lump via** \( \theta \) to \( \{\hat{X}_t\} \) if the process \( \{\theta(X_t)\} \) is a Markov chain with the same transition matrix as \( \{\hat{X}_t\} \).

**Theorem 4.10** (Recursive lumping of top-to-random chains). Let \( \mathcal{H} \) be a graded connected Hopf algebra with basis \( B \) such that \( B_1 = \{\bullet\} \), and \( \Delta_{1,n-1}(x) = \bullet \otimes D(x) \) for some \( D : B_n \to B_{n-1} \). Then:

i) the Markov chain on \( B_n \) driven by \( T2R_n \) lumps via \( \frac{n-k}{n} \)-lazy version of the chain on \( B_k \) driven by \( T2R_k \);

ii) the Markov chain on \( B_n \) driven by \( \text{BinT}2R_n(q_2) \) lumps via \( D^{n-k} \) to the chain on \( B_k \) driven by \( \text{BinT}2R_k(q_2) \).

**Proof.** For i: by Lemma 4.8 we wish to prove that the chain driven by \( T2R_n = \frac{1}{n}m\Delta_{1,n-1} : B_n \to B_n \) lumps via \( D^{n-k} \) to the chain driven by \( \frac{n-k}{n}\text{id} + \frac{k}{n}T2R_k = \frac{n-k}{n}\text{id} + \frac{1}{n}m\Delta_{1,k-1} : B_k \to B_k \). By [Pan18, Th. 2.7], it suffices to show that

\[
D^{n-k} \circ \left( \frac{1}{n}m\Delta_{1,n-1} \right) = \left( \frac{n-k}{n}\text{id} + \frac{1}{n}m\Delta_{1,k-1} \right) \circ D^{n-k};
\]

the other hypotheses are clearly satisfied. As noted at the beginning of this section, \( D \) is the down operator of a dual graded graph following the general construction in [BLL12]. The corresponding up operator \( U \) satisfies \( m_{1,n-1}(\bullet \otimes x) = U(x) \). Hence Equation 4.11 may be rephrased in terms of dual graded graphs as follows:

\[
\frac{1}{n}D^{n-k} \circ U \circ D = \frac{1}{n}( (n-k)\text{id} + U \circ D ) \circ D^{n-k}.
\]

And this is proved by repeated application of the dual graded graph condition \( DU = UD + \text{id} \):

\[
D^{n-k} \circ U \circ D = D^{n-k-1} \circ (UD + \text{id}) \circ D
\]

\[
= D^{n-k-2} \circ (DU + \text{id}) \circ D^2
\]

\[
= D^{n-k-2} \circ ((UD + \text{id}) + \text{id}) \circ D^2
\]

\[
= D^{n-k-3} \circ (DU + 2\text{id}) \circ D^3 = \ldots
\]

\[
= ( (n-k)\text{id} + U \circ D ) \circ D^{n-k}.
\]

For ii: by [Pan18] Th. 2.7, we wish to show

\[
D^{n-k} \circ \text{BinT}2R_n = \text{BinT}2R_k \circ D^{n-k}
\]

which, by Proposition 4.3, is equivalent to

\[
D^{n-k} \circ \left( \sum_{r=0}^{n} \begin{pmatrix} n \\ r \end{pmatrix} q_2^{n-r}(1-q_2)^r \left( \frac{nT2R_n}{n} \right) \circ \left( \frac{nT2R_n-\text{id}}{n-1} \right) \circ \ldots \circ \left( \frac{nT2R_n-(r-1)\text{id}}{n-(r-1)} \right) \right)
\]

\[
= \left( \sum_{r=0}^{k} \begin{pmatrix} k \\ r \end{pmatrix} q_2^{k-r}(1-q_2)^r \left( \frac{kT2R_k}{k} \right) \circ \left( \frac{kT2R_k-\text{id}}{k-1} \right) \circ \ldots \circ \left( \frac{kT2R_k-(r-1)\text{id}}{k-(r-1)} \right) \right) \circ D^{n-k}.
\]

From Equation 4.11

\[
D^{n-k} \circ (nT2R_n - i\text{id}) = ((n-k)\text{id} + kT2R_k) \circ D^{n-k},
\]

34
and so the left hand side of Equation 4.12 is
\[
\left( \sum_{r=0}^{n} \frac{1}{r!} q_2^{n-r} (1 - q_2)^r ((n - k) \text{id} + k T2R_k) \circ ((n - k - 1) \text{id} + k T2R_k) \circ \cdots \circ ((n - k - (r - 1)) \text{id} + k T2R_k) \right) \circ D^{n-k} \text{ evaluated at } x = k T2R_k.
\]

For every integer \( x > k \), we have \( (n-k+x) = \sum_{r'=0}^{r'} (\binom{n-k}{r'}) \binom{r}{r'} \), so this must be a polynomial identity in \( x \). Hence the above is
\[
\left( \sum_{r=0}^{n} \sum_{r'=0}^{r} \binom{n-k}{r-r'} \binom{r}{r'} q_2^{n-r}(1-q_2)^r \circ D^{n-k} \text{ evaluated at } x = k T2R_k \right)
\]
\[
= \left( \sum_{r'=0}^{n} \binom{x}{r'} q_2^{k-r'}(1-q_2)^{r'} \sum_{r'=0}^{n} \binom{n-k}{r-r'} q_2^{n-k-(r-r')}(1-q_2)^{r-r'} \circ D^{n-k} \text{ evaluated at } x = k T2R_k \right)
\]
\[
= \left( \sum_{r'=0}^{k} \binom{x}{r'} q_2^{k-r'}(1-q_2)^{r'} \sum_{r-r'=0}^{n} \binom{n-k}{r-r'} q_2^{n-k-(r-r')}(1-q_2)^{r-r'} \circ D^{n-k} \text{ evaluated at } x = k T2R_k, \right)
\]
where we have neglected the terms with \( r' > k \) because then evaluating \( \binom{r}{r'} \) at \( x = k T2R_k \) gives a factor \( k T2R_k - i \text{id} \) for all the eigenvalues \( i = 0, 1, \ldots k \) of \( k T2R_k \), hence the evaluation is 0. And when \( r' \leq k \), the second sum runs up to \( r - r' = n - r' \geq n - k \), but \( \binom{n-k}{r-r'} = 0 \) for \( r - r' > n - k \), so we can truncate the sum at \( n - k \). Then this second sum yields 1, and the remainder of the expression is exactly the right hand side of Equation 4.12. 

\[
\square
\]

5 A Chain on Organisational Structures

The goal of this section is to employ the present Hopf-algebraic framework to analyse a “leaf-removal” or “employee firing” chain, as outlined in Example 1.1. We note that, as a result of SY89 Th. 1 (repeated as Theorem 5.2 below), our leaf-removal step also occurs in the chain of Ful09, where it is followed by a leaf-attachment step that is absent here.

The situation is as follows: A company has a tree structure, so each employee except the boss has exactly one direct superior. Each month, some employees are fired (details in three paragraphs below), and each firing independently causes a cascade of promotions: first, someone further down the chain of superiority from the fired employee is uniformly selected to replace him. Then, if the promoted employee was superior to anyone, then one of those is uniformly selected and promoted to his position. This process continues until someone who is not superior to anyone (a leaf) is promoted. Figure 5.1 shows the probabilities of all the possible scenarios after C is fired.

The chain keeps track of the tree structure of the company, but does not know which employee is taking which position. For example the rightmost two possibilities in Figure 5.1 represent the same state. More specifically, if \( T_0 \) denotes the starting state, then the state space of the chain is the set of rooted subtrees of \( T_0 \) - that is, the connected subsets of vertices of \( T_0 \) which include the root (the boss). Let \( T_t \) denote the structure of the company after \( t \) months, and \( n_t \) the number of employees after \( t \) months (the number of vertices in \( T_t \)).

We consider two ways in which employees are fired:
Figure 5.1: All possible promotion scenarios after an employee is fired, and their respective probabilities.

- The **single model**: with probability \( \frac{n_t}{n_0} \), one uniformly chosen employee is fired at the start of month \( t + 1 \). (It does not matter whether the fired employee is chosen from all \( n_t \) employees or from only the \( n_t - 1 \) non-boss employees; the promotion cascades ensure that both options result in the same chain.) With the complementary probability \( \frac{n_0 - n_t}{n_0} \), there is no change to the company structure.

- The **binomial model**: the monthly performance of each employee (including the boss) is, independently, uniformly distributed between 0 and 1. For a fixed parameter \( q_2 \), all employees with performance below \( 1 - q_2 \) are fired one-by-one in a random order. Hence the number of fired employees follows a binomial distribution with parameter \( 1 - q_2 \).

In both models, no firing occurs when only the boss remains - this is the unique absorbing state. Below are the transition matrices for these two models, starting from the four-person company on the far right. (All empty entries are zeroes.)
### 5.1 A connection to a decorated Connes-Kreimer Hopf algebra of trees

The Connes-Kreimer Hopf algebra of trees arose independently from the study of renormalisation in quantum field theory [Kre98] and of Runge-Kutta numerical methods of solving ordinary differential equations [But72]. Relevant here is the decorated variant from [FU13, end of Sec. 2.1], where each vertex is labelled. In the present application, the labels are the job positions, not the employees currently holding each position. These labels are necessary to distinguish abstractly isomorphic trees without reference to the starting state $T_0$.

Take a quotient of this decorated tree algebra so the root becomes unlabelled. This ensures that there is a unique rooted forest on one vertex - call it • - which will simplify the notation slightly. Below are the aspects of this Hopf algebra relevant to the present Markov chain application; its full Hopf structure follows easily from [CK98, Sec. 2]. (It is probably possible to run the subsequent analysis using a Hopf monoid [AM10, Sec. 13.3.1] instead.)

- A basis is the set of all decorated rooted forests - that is, each connected component has a distinguished root vertex, and each non-root vertex is assigned one of a finite set of labels.

$$
\begin{array}{cccc}
1 & \cdot & \cdot & \cdot \\
(1 - q)(1 + q) & q^2 & & \\
(1 - q)(1 + q) & & q^2 & \\
(1 - q)^2(1 + 3q) & 3q^2(1 - q) & q^3 & \\
(1 - q)^2(1 + 3q) & \frac{3}{2}q^2(1 - q) & \frac{3}{2}q^2(1 - q) & q^3 \\
(1 - q)^3(1 + 4q) & 2q^2(1 - q)^2 & 4q^2(1 - q)^2 & \frac{4}{3}q^3(1 - q) \quad \frac{8}{3}q^3(1 - q) \quad q^4
\end{array}
$$

Section 5.1 recasts these chains as the $T2R_{n_0}$ and $BinT2R_{n_0}$ chains respectively on a decorated variant of the Connes-Kreimer Hopf algebra of rooted trees. Section 5.2 applies Theorem 4.4 to produce a full basis of right eigenfunctions (Theorem 5.4), using which we bound the expected numbers of “inter-departmental teams” (Corollary 5.5). Section 5.3 extends the binomial model to a $TrinT/B2R$ chain, where employees with outstanding work are promoted to a special status, independent of the firings.

Another possible extension, not discussed here, is to give the company a more general poset structure, where each employee may have multiple direct supervisors. This uses a Hopf algebra of (unranked) posets [ABS06, Ex. 2.3] which contains the Connes-Kreimer algebra of trees. If the initial company structure $T_0$ and all its order ideals are $d$-complete [Pro09], then the promotion cascade algorithm above agrees with the $T2R$ and $BinT2R$ chains on the poset algebra. However, for more general posets, there is no known method to generate a uniform linear extension, and it is unclear how to define a firing/promotion process to keep the Hopf-algebraic connection.

We note also that much of the following analysis can be easily adapted for Hopf algebras whose state space basis is a free-commutative monoid (what [Pan14, Chap. 5] termed a “free-commutative” state space basis.) The class of associated chains includes the rock-chipping process of the introduction (a $T2R$ analogue of [DPR14, Sec. 4]) and the mining variant. They do not have a combining step. A few results below require the additional hypothesis that $\Delta_{1,n-1}$ “preserves atomicity” (i.e. all terms in $\Delta_{1,n-1}$ of a tree has a tree in the second factor, as opposed to forests with more than one connected component.) This is true for rock-chipping, but not for mining.
The degree of a forest is its number of vertices.

The product of two forests is their disjoint union (preserving all labels). Hence this Hopf algebra is commutative.

The partial coproduct of a forest $x$ on $n$ vertices is $\Delta_{1,n-1}(x) = \sum \bigotimes x \setminus v$ (preserving all labels in the second factor), where the sum runs over all leaves $v$ of $x$. For example,

$$\Delta_{1,7}(\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}) = \bigotimes \left( \begin{array}{c} 1 \\ 4 \\ 5 \\ 8 \\ 2 \\ 3 \\ 6 \end{array} + \begin{array}{c} 1 \\ 3 \\ 4 \\ 6 \\ 2 \\ 7 \\ 8 \end{array} + \begin{array}{c} 1 \\ 2 \\ 3 \\ 5 \\ 7 \\ 8 \end{array} \right).$$

Because of coassociativity, $\Delta_{1,\ldots,1,n-r}(x) = \sum \bigotimes x \setminus \{v_1, \ldots, v_r\}$, summing over all choices of $v_1, \ldots, v_r$ such that $v_i$ is a leaf of $x\setminus\{v_1, \ldots, v_{i-1}\}$. In particular, the coefficient of $\bigotimes v$ in $\Delta_{1,\ldots,1}(x)$ enumerates the ways to successively remove leaves from $x$, or, equivalently, the increasing labellings of the vertices of $x$ - that is, each of the labels $1, 2, \ldots, n$ occur once, and the label of a parent is less than the label of the child. An example is in Figure 5.2. By Theorem 3.4, this coefficient is the reweighting function $\eta$. (In the more general case of posets, it is the number of linear extensions.)

[SY89, Eq. 1.1] gives a hook length formula for the number of increasing labellings of a forest $x$:

$$\eta(x) = \frac{\deg x!}{\prod_{v \in x} h(v)},$$

where $h(v)$ is the number of vertices in the hook $H(v)$ of vertex $v$: the hook consists of $v$, its children, its grandchildren, ... ; see the top of Figure 5.3. (In the interpretation of a tree as a company structure, the hook of employee $v$ consists of everyone further down in superiority from $v$, including $v$ himself.) One proof of this formula goes via an algorithm for uniformly generating an increasing labelling, which corresponds to the “promotion cascade” process described above.

**Theorem 5.2 (Sagan-Yeh hook walk).** [SY89, Th. 1] For a tree $T$ on $n$ vertices, the Sagan-Yeh hook walk is the following recursive process:

1. Choose a vertex $v$ of $T$ uniformly.

2. If $v$ is a leaf, assign the label $n$ to $v$.

3. Else, uniformly choose a vertex $w$ in the hook of $v$, and return to step 2 with $w$ in place of $v$.

Hence the walk terminates with some leaf receiving the label $n$. Remove this leaf and re-apply the walk to the remaining tree on $n-1$ vertices, and repeat until the root is assigned the label 1. This generates an increasing labelling of $T$ uniformly.
The remainder of this section will prove the following.

**Theorem 5.3** (Interpretation of top-to-random chains on trees). The chains driven by \( T_{2R} \) and \( \text{BinT2R}_{n_0}(q_2) \) on the decorated Connes-Kreimer Hopf algebra of trees, starting from a tree \( T_0 \) on \( n_0 \) vertices, remain in the subset of states given by

\[
\{ \bullet^{n_0-n} \Pi T | T \text{ is a rooted subtree of } T_0 \text{ on } n \text{ vertices} \}.
\]

Under the equivalence \( \bullet^{n_0-n} \Pi T \to T \), these chains have the “employee firing” description given at the beginning of Section 5.

It is crucial that all Hopf-algebraic calculations use the form \( \bullet^{n_0-n} \Pi T \) as opposed to \( T \). And note that the equivalence sends \( \bullet^{n_0} \) to \( \bullet \), not to the empty tree.

**Proof.** Begin by substituting the definition of product and coproduct into the 3-step description of Theorem 3.4. For \( T_{2R} \), the first and third steps are trivial, and the second translates to: for each leaf \( v \) of a forest \( x \) on \( n_0 \) vertices, move from \( x \) to \( v \Pi x \setminus v \) with probability \( \frac{n(x)\eta(x \setminus v)}{\eta(x)} = \frac{n(x \setminus v)}{\eta(x)} \). (It is impossible to move to forests not of the form \( x \setminus v \).) Notice that the increasing labellings of \( x \setminus v \) are precisely the increasing labellings of \( x \) where \( v \) has label \( n_0 \). Hence each step of the chain is: uniformly pick an increasing labelling of \( x \setminus v \), and isolate the vertex receiving label \( n_0 \). Similarly, the \( \text{BinT2R}_{n_0}(q_2) \) chain uniformly picks an increasing labelling of \( x \), then isolates the vertices with labels \( n_0, n_0 - 1, \ldots, n_0 - r + 1 \), where \( r \in [0, n_0] \) has a binomial distribution with parameter \( 1 - q_2 \).

Now specialise to the case where the rooted forest \( x \) is of the form \( \bullet^{n_0-n} \Pi T \), for \( T \) a tree on \( n \) vertices. The key observation is this: a uniform choice of an increasing labelling of \( \bullet^{n_0-n} \Pi T \) is equivalent to a uniform (ordered) choice of \( n_0 - n \) distinct labels from \( \{1, \ldots, n_0\} \) for the singletons, plus an independent uniform choice of increasing labelling for \( T \) (the “standardisation” of the original labelling for \( \bullet^{n_0-n} \Pi T \)).

In the case of \( T_{2R} \), there is a probability of \( \frac{n_0-n}{n_0} \) that the label \( n_0 \) is amongst those chosen for the singletons: in this case, the chain remains at \( \bullet^{n_0-n} \Pi IT \). With the complementary probability \( \frac{n_0}{n_0} \), the chain isolates the vertex with the largest label in a random increasing labelling of \( T \), which, by Theorem 5.2, is precisely the firing of a uniformly chosen employee and the subsequent promotion cascade.

As for the \( \text{BinT2R}_{n_0} \) chain: if \( r' \) of the \( r \) labels \( n_0, n_0 - 1, \ldots, n_0 - r + 1 \) were assigned to singletons, then the chain would remove \( r - r' \) vertices from \( T \) according to the hook walk /
promotion cascade. Hence it suffices to show that, if \( r \in [0, n_0] \) follows a binomial distribution with parameter \( 1 - q_2 \), then \( r - r' \in [0, n] \) is also binomial with parameter \( 1 - q_2 \). Because the choice of labels for the singletons is uniform (independent of the value of the label), the situation has this alternative description: pick \( n - n_0 \) distinct labels from \( n \) labels, and \( r' \) is the number of \( r \) “labels of interest” that were picked. \( r \) having a binomial distribution is equivalent to each label in independently having probability \( 1 - q_2 \) of being a “label of interest”. Thus \( r' \) is binomial with parameter \( 1 - q_2 \), and so is the number of unpicked labels of interest \( r - r' \).

(More generally, for forests, \( T_{2R_{n_0}} \) selects a tree with probability proportional to its number of vertices and removes a vertex as per the hook walk. \( \text{Bin}T_{2R_{n_0}} \) removes a binomial number of vertices independently from each tree in the forest.)

5.2 The spectrum and right eigenbasis

This section seeks to prove the following:

**Theorem 5.4 (Eigenfunctions of employee-firing chains).** Consider the chain on organisational structures, started from \( T_0 \) with \( n_0 \) employees/vertices.

i) The eigenvalues of the single and binomial models are \( \frac{1}{n_0} \) and \( q_2^{n_0-j} \) respectively, for \( 0 \leq j \leq n_0 - 2 \) and \( j = n_0 \).

ii) The eigenvalue \( 1 \) (\( j = n_0 \)) has multiplicity 1. For \( j < n_0 \), the multiplicity of the eigenvalue indexed by \( j \) is the number of rooted subtrees of \( T_0 \) on \( n_0 - j \) vertices.

iii) For \( T' \neq \bullet \), the right eigenfunction \( f_{T'} \) corresponding to the rooted subtree \( T' \) on \( n' \) vertices is

\[
f_{T'} := \binom{n}{n'} \Prob\left( \text{After firing } n-n' \text{ uniformly chosen employees (and the subsequent promotion cascades), the remaining company structure is } T' \right).
\]

These, together with the constant function 1, form an eigenbasis.

For the chain explicited at the start of Section 5, the basis of right eigenfunctions constructed in part iii above are the columns of the following table (all empty entries are zeroes):

|   | 1 | f | f | f | f |
|---|---|---|---|---|---|
| 1 |   | 1 |   |   |   |
| 1 |   |   | 3 | 1 |   |
| 1 | 3 | 3 | 3 | 1 |   |
| 1 | 2 | 4 | 4 | 3 | 1 |

The corresponding eigenvalues for the single model are \( 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0 \), and for the binomial model are \( 1, q_2^2, q_2^2, q_3^2, q_3^2, q_4^2 \).

Here is a quick way to see parts i and ii of the theorem: the transition matrices for both chains are triangular if the states are ordered by the number of employees. Then the eigenvalues are the diagonal entries, i.e. the probability of remaining at a state \( T' \) (on \( n' \) vertices), which is \( \frac{n_0-n'}{n_0} \) for the single model and \( q_2^{n'} \) for the binomial model. The proof of iii is at the end of this section.

Note that part ii does not follow from the general result on the spectrum of descent operator chains (Theorem 3.5): that gives the eigenvalue multiplicities on the full state space basis of all
rooted forests on $n_0$ vertices, of which the states of the employee-firing chains are a proper subset. Similarly, part iii is not an immediate consequence of Theorem 4.4.iv. Instead, the proof below will use the general theory to construct the eigenfunctions, then show they stay linearly independent when restricted to this smaller state space.

Here’s an application before the proof. Recall from Example 1.1 that a department head is an employee whose direct superior is the boss, and a department is everyone further down the chain of superiority from a department head. (So a department is a connected component of the tree with the root removed). Write $n^{(i)}$ for the number of employees in department $i$. For example, the company in Figure 1.1 has two departments, with $n^{(1)} = 2$ and $n^{(2)} = 5$. Because the boss is not in any department, it always holds that $1 + \sum n^{(i)} = n$.

Suppose the company receives a project that requires a team of $s_i$ employees from department $i$. The number of ways to choose such a team is $\prod_i \binom{n^{(i)}_{s_i}}{s_i}$. For example, the company in Figure 1.1 has two departments, with $n^{(1)} = 2$ and $n^{(2)} = 5$.

**Corollary 5.5.** Let $s_i$ be any sequence of integers, and use the $n^{(i)}$ notation for department sizes as above.

1) $f(T) := n \prod_i \binom{n^{(i)}_{s_i}}{s_i}$ is a right eigenfunction of eigenvalue $\frac{n_0 - 1 - \sum s_i}{n_0}$ (resp. $q_2^{1+\sum s_i}$) for the single (resp. binomial) model.

2) After $t$ months under the single (resp. binomial) model, starting from a company of $n_0$ employees,

$$\text{Expect} \left( n_t \prod_i \binom{n^{(i)}_{s_i}}{s_i} \right) = \beta^t n_0 \prod_i \binom{n^{(i)}_{s_i}}{s_i},$$

where $\beta^t = \frac{n_0 - 1 - \sum s_i}{n_0}$ for the single model, and $\beta^t = q_2^{1+\sum s_i}$ for the binomial model.

3) In particular, the expected number of teams consisting of $s_i$ employees in department $i$ satisfies

$$\beta^t \prod_i \binom{n^{(i)}_{s_i}}{s_i} \leq \text{Expect} \left( \prod_i \binom{n^{(i)}_{s_i}}{s_i} \right) \leq \beta^t \frac{n_0}{1 + \sum s_i} \prod_i \binom{n^{(i)}_{s_i}}{s_i}$$

for both the single and binomial models. Also, in the single model,

$$\text{Expect} \left( \prod_i \binom{n^{(i)}_{s_i}}{s_i} \right) \leq \beta^t \frac{n_0}{n_0 - t} \prod_i \binom{n^{(i)}_{s_i}}{s_i}.$$
Part ii then follows by applying Proposition\ref{prop:2.1}. To see part iii, note that \( n_t \leq n_0 \), and, if \( f(T_t) \neq 0 \), it must be that \( 1 + \sum s_i \leq n_t \). The last statement uses the alternative upper bound \( n_t \geq n_0 - t \) for the single model.

\textit{Proof of Theorem\ref{thm:5.4}}. The proof follows the argument of \cite[Proof of Th. 3.19]{DPR14} to show that the functions \( f_{T'} \), as defined in part iii of the theorem, are right eigenfunctions for the Markov chain, of the claimed eigenvalues, and are linearly independent. Parts i and ii will then follow immediately.

Recall that right eigenfunctions of a descent operator chain come from eigenvectors in the dual Hopf algebra. By the “duality of primitives and generators” \cite[Sec. 3.8.24]{HGK10}, the elements \( T^* \) that are dual to trees are primitive in this dual algebra. Hence, for any tree \( T' \neq \bullet \) on \( n' \) vertices, Theorem\ref{thm:4.4} iii asserts that \( \bullet^{n_0-n'} T'^* \) is a \( \frac{n_0-n'}{n_0} \)-eigenvector of \( T2\mathbb{R}_{n_0} \), and a \( q'_2 \)-eigenvector of \( \text{Bin}T2\mathbb{R}_{n_0}(q_2) \). The corresponding right eigenfunction, by Proposition\ref{prop:2.4} R, is

\[
 f_{T'}(\bullet^{n_0-n} \Pi T) = \frac{1}{\eta(\bullet^{n_0-n} \Pi T)} \bullet^{n_0-n'} T'^* \text{ evaluated on } \bullet^{n_0-n} \Pi T \\
 = \frac{1}{\eta(\bullet^{n_0-n} \Pi T)} (\bullet^* \otimes \cdots \otimes \bullet^* \otimes T'^*) \Delta_1,\ldots,1,n_r(\bullet^{n_0-n} \Pi T).
\]

This is the probability that a uniformly chosen increasing labelling of \( \bullet^{n_0-n} \Pi T \) has the vertices labelled \( 1, 2, \ldots, n' \) forming a copy of \( T' \). As before, view each increasing labelling of \( \bullet^{n_0-n} \Pi T \) as a (ordered) choice of \( n_0 - n \) labels for the singletons, together with an increasing labelling of \( T \). Since \( T' \neq \bullet \), the desired condition is equivalent to the singletons all having labels greater than \( n' \), and the smallest \( n' \) labels in \( T' \) being assigned to the vertices of \( T' \). These two subconditions are independent; the first happens with probability

\[
 \frac{(n_0 - n')(n_0 - n' - 1) \ldots (n - n' + 1)}{n_0(n_0 - 1) \ldots (n + 1)} = \binom{n}{n'} / \binom{n_0}{n'},
\]

and the second is the probability that, after \( n - n' \) firings, the remaining company structure is \( T' \). So the product of these two numbers give an eigenfunction. Since any scalar multiple of an eigenfunction is again an eigenfunction, we can multiply this by \( \binom{n_0}{n'} \) (which is independent of \( T \)) and obtain \( f_{T'} \).

The linear independence of \( f_{T'} \) comes from a triangularity argument. Clearly both factors in \( f_{T'}(T) \) are zero if \( T \) has fewer than \( n' \) vertices (i.e. fewer vertices than \( T' \)). And if \( T \) has exactly \( n' \) vertices, then \( f_{T'}(T) \) is the probability that \( T = T' \), so it evaluates to 1 at \( T \) and 0 otherwise.

\textbf{5.3 An employee-firing chain with VPs}

This section studies the TrinT/B2R\(_n\)(\( q_1, q_2, q_3 \)) chains on the Connes-Kreimer algebra (where \( q_1 + q_2 + q_3 = 1 \)). This chain involves both \( \Delta_{1,n-1} \), the firing operator from above, as well as a new operator \( \Delta_{n-1,1} \), which removes from a forest the root of one of its connected components. So now, in addition to firing employees whose performance is in the interval \([0, q_1]\), there is a reward for employees performing in the interval \([q_1 + q_2, 1] = [1 - q_3, 1]\) - a special VP status is given to the person highest in the chain of superiority above this good employee, who isn’t yet a VP. The chain keeps track of the forest structure of the non-VP employees of the company; since VP status is for life, the chain ignores any positions once it becomes VP status.

To state a result using the eigenfunctions of Theorem\ref{thm:4.4} iii, more tree terminology is necessary. Recall that, if \( v \) is a vertex of the forest \( x \), then \( H(v) \) is the hook of \( v \), consisting of \( v \), its children, its grandchildren, \( \ldots \) , and \( h(v) \) is the number of vertices in \( H(v) \). For the company interpretation,
$H(v)$ is the set of employees under indirect supervision of $v$. A complementary idea is the ancestors $A(v)$ of $v$, consisting of $v$, its parent, its grandparent, ...; let $a(v)$ denote the number of ancestors of $v$. This is the length of the superiority chain from $v$ upwards, not including any VPs. $A(v) \setminus \{v\}$ is the strict ancestors of $v$.

Theorem 5.6 below gives a family of functions $f_j$ whose average value falls by approximately $q_2^{n_0-j}$ with each step of the chain. $f_j$ is the expected number of teams, of size $n_0 - j$, that some employee $u$ can assemble from his/her “hook”. The project leader $u$ is chosen with probability proportional to $\left(\frac{q_3}{q_1 + q_3}\right)^{a(u)-1} \left(\frac{q_1}{q_1 + q_3}\right)^{h(u)}$. One may justify these probabilities as follows: we disfavour a project leader with large $h(u)$ as he/she has many employees to manage, that will take time away from this project. Large $a(u)$ is also undesirable as he/she may have many other projects from these supervisors.

**Theorem 5.6** (Approximate eigenfunction for the employee-firing chain with VPs). Let $\{X_i\}$ denote the employee-firing chain with VPs as detailed above, starting with a company of $n_0$ employees (and no VPs). For each integer $j \in [0, n-2]$, define the following functions on the forest structure of the non-VP employees:

$$f_j(x) := \sum_{u \in x \setminus \{n_0 - j\}} \left(\frac{h(u)}{n_0 - j}\right) \left(\frac{q_3}{q_1 + q_3}\right)^{a(u)-1} \left(\frac{q_1}{q_1 + q_3}\right)^{h(u)}.$$

(The binomial coefficient is 0 if $h(u) < n_0 - j$.) Then

$$\text{Expect} \left\{ f_j(X_i) \right\} \leq q_2^{(n_0-j)^2} f_j(X_0) \max_{u \in X_0; h(u) \geq n_0 - j} \left\{ \left(\frac{n_0}{a(u)-1}\right) \right\}.$$

The proof requires yet more definitions: a trunk of a forest consists of rooted subtrees of its constituent trees. For example, in Figure 5.4 $S$ is a trunk of $x$, and $S'$ and $S' \cup \{w\}$ are both trunks of $x'$. $T'$ is not a trunk of $x$, but rather a trunk of $x \setminus S$. Since we will consider the hook of $v$ both within the full forest $x$ and of a trunk $S$ (or of other subtrees), write $H_x(v)$ and $H_S(v)$ for these respectively, and similarly $h_x(v)$ and $h_S(v)$. For example, in Figure 5.4 $h_x(v_2) = 10$ whilst $h_S(v_2) = 3$.

For the proof of this theorem, work in the non-decorated Connes-Kreimer algebra, where the vertices are unlabelled. The following coassociativity result will be useful:

**Lemma 5.7.** Suppose $x$ is a forest of degree $n$, and $T'$ is a tree of degree $n-j$. Fix an integer $i \in [0, j]$. Then a ratio of coproduct coefficients may be expressed as follows:

$$\frac{\eta_x}{\eta(x)} = \frac{1}{(n-j)!} \frac{\prod_{v \in S,T'} h_x(v)}{\prod_{v \in S,T'} h_S(v)} \sum_{S,T'} \left(\prod_{v \in S} h_x(v)\right) \left(\prod_{v \in T'} h_x(v)\right),$$

where the sum runs over all trunks $S$ of $x$ with degree $i$, and all copies of $T'$ within $x$ that are trunks of $x \setminus S$. In particular, taking $T' = \emptyset$ (so $n = j$) shows that

$$\binom{n}{i} = \sum_{S \in S} \prod_{v \in S} h_S(v),$$

where the sum runs over all trunks $S$ of $x$ with degree $i$. 

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Proof. Recall that the numerator on the left hand side is the coefficient of \( \bullet^\otimes j-i \otimes T' \otimes \bullet^\otimes i \) in \( \Delta_{1,\ldots,1,n-j,1,\ldots,1}(x) \). By coassociativity, and then the definition of \( \eta \):

\[
\Delta_{1,\ldots,1,n-j,1,\ldots,1}(x) = (\Delta_{1,\ldots,1} \otimes \text{id} \otimes \Delta_{1,\ldots,1}) \circ \Delta_{j-i, n-j, i}(x)
= \left( \sum_{S,T} \eta(x \setminus (S \cup T)) \eta(S) \right) \bullet^\otimes j-i \otimes T \otimes \bullet^\otimes i,
\]

summing over all trunks \( S \) of \( x \) with degree \( i \), and all trunks \( T \) of \( x \setminus S \) with degree \( n-j \). So it suffices to show that, for a specific trunk \( S \) of \( x \) with degree \( i \) and a copy of \( T' \) within \( x \) that is a trunk of \( x \setminus S \),

\[
\eta(x \setminus (S \cup T')) \eta(S) \frac{1}{\eta(x)} = \frac{1}{(n-j)!} \left( \prod_{v \in S} \frac{h_x(v)}{h_S(v)} \right) \prod_{v \in T'} h_{x}(v).
\]

The key is Equation \( 5.1 \):

\[
\eta(x) = \frac{\deg x!}{\prod_{v \in x} h_{x}(v)},
\]

so the left hand side of Equation \( 5.8 \) is

\[
\frac{(j-i)!}{\prod_{v \in x \setminus (S \cup T')} h_x(v)} \frac{i!}{\prod_{v \in S} h_S(v)} \prod_{v \in T'} \frac{n!}{h_x(v)}.
\]

Observe that each \( v \in x \) is exactly one set out of \( S, T' \) and \( x \setminus (S \cup T') \). Further, if \( v \in x \setminus (S \cup T') \), then all descendants of \( v \) within \( x \) are within \( x \setminus (S \cup T') \), so \( h_x(v) = h_{x \setminus (S \cup T')}(v) \). This proves Equation \( 5.8 \).

To see the “in particular” claim in the lemma, note that \( \eta_x \bullet^{n-i} \cdot \cdot \cdot \cdot \cdot = \eta_x \bullet^{n-i} \cdot \cdot \cdot = \eta(x) \), by coassociativity. \( \square \)

Proof of Theorem \( 5.6 \). The first step is to calculate a right eigenfunction by applying Theorem \( 4.4 \)ii to the dual algebra. Let the kernel element \( p \) be \( T'^* \) where \( T' \) is a tree, of degree \( j \). Then the associated eigenfunction is

\[
f_{T'}(x) = \frac{1}{\eta(x)} \sum_{i=0}^{j} \left( \begin{array}{c} j \\ i \end{array} \right) q_1^i q_3^{j-i} \bullet^i T'^* \bullet^{j-i} \text{ evaluated on } x
\]

\[
= \frac{1}{\eta(x)} \sum_{i=0}^{j} \left( \begin{array}{c} j \\ i \end{array} \right) q_1^i q_3^{j-i} \bullet^i T', \bullet^{j-i} \text{ (renaming } j-i \text{ as } i) \]

\[
= \sum_{i=0}^{j} \left( \begin{array}{c} j \\ i \end{array} \right) q_1^i q_3^{j-i} \frac{1}{(n-j)!} \left( \sum_{S,T'} \prod_{v \in S} \frac{h_x(v)}{h_S(v)} \prod_{v \in T'} h_x(v) \right),
\]

summing over all trunks \( S \) of \( x \) with degree \( i \), and all copies of \( T' \) within \( x \) that are trunks of \( x \setminus S \).

We exchange the order of summation in \( S \) and \( T' \) - rather than first choosing the trunk \( S \), and then letting \( T' \) be a trunk of \( x \setminus S \), we instead start by specifying the copy of \( T' \) in \( T \), then let \( u \) denote its root, and let \( S \) be a trunk of \( x \setminus T' \). The condition that \( T' \) is a trunk of \( x \setminus S \) translates to \( S \) containing all strict ancestors of \( u \). Note also that \( S \) cannot contain any vertices in \( H(u) \). Thus the degree \( i \) of \( S \) must range between \( a(u) - 1 \) and \( n - h_x(u) \).
It would be ideal to simplify Equation 5.9 using the second statement of Lemma 5.7. This requires removing the condition $S \supseteq A(u) \setminus \{u\}$. To do so, define $\bar{x} := x \setminus (A(u) \cup H(u))$, and $\bar{S} = S \setminus \{A(u) \setminus \{u\}\}$; see Figure 5.4. Then the trunks $S$ of $x$ that contain all strict ancestors of $u$ are in bijection with the trunks $\bar{S}$ of $\bar{x}$. Moreover, $h_\bar{S}(v) = h_S(v)$ for all $v \in \bar{S}$, and $h_\bar{S}(v) = h_x(v)$ if $v \in \bar{x}$. And for $v \in \bar{S} \setminus \bar{\bar{S}} = \{A(u) \setminus \{u\}\}$, it is true that $h_\bar{S}(v) > h_S(v)$. So

$$\prod_{v \in S} \frac{h_x(v)}{h_S(v)} = \left( \prod_{v \in S} \frac{h_x(v)}{h_S(v)} \right) \left( \prod_{v \in A(u) \setminus \{u\}} \frac{h_x(v)}{h_S(v)} \right) = \left( \prod_{v \in S} \frac{h_x(v)}{h_S(v)} \right).$$

Now sum over all trunks $S$ of $x$ with degree $i$ and containing all strict ancestors of $u$:

$$\sum_{S} \prod_{v \in S} \frac{h_x(v)}{h_S(v)} \geq \sum_{S} \prod_{v \in S} \frac{h_x(v)}{h_S(v)} = \left( \frac{\deg \bar{x}}{\deg \bar{S}} \right) = \left( \frac{n_0 - a(u) - h_x(u) + 1}{i - a(u) + 1} \right),$$

using the second statement of Lemma 5.7. Substitute into Equation 5.9 keeping in mind that $i = \deg S$ ranges between $a(u) - 1$ and $n_0 - h_x(u)$ (and viewing $u$ as a function of $T'$):

$$f_{T'}(x) \geq \sum_{T'} \prod_{v \in T'} h_x(v)^{n_0 - h_x(u)} \sum_{i = a(u) - 1}^{n_0 - h_x(u)} q_1^{j - i} q_3^i \frac{1}{(n_0 - j)! (n_0 - j - i)! (n_0 - j) !} \left( \frac{n_0 - a(u) - h_x(u) + 1}{i - a(u) + 1} \right)$$

$$= \sum_{T'} \prod_{v \in T'} h_x(v)^{n_0 - h_x(u)} \frac{1}{(n_0 - j)!} q_1^{n_0 - h_x(u) - a(u) + j} q_3^{i - a(u) + 1} \left( \frac{n_0 - a(u) - h_x(u) + 1}{i - a(u) + 1} \right)$$

$$= \sum_{T'} \prod_{v \in T'} h_x(v)^{n_0 - j} q_1^{n_0 - j} q_3^{i - a(u) - h_x(u) + j} q_1^{n_0 - a(u) - h_x(u) + 1}.$$

$f_{T'}$ has eigenvalue $q_2^{n_0 - j}$ whenever $\deg T' = n_0 - j$. Take the following linear combination of such $T'$, over all $T'$ with degree $n_0 - j$ (as opposed to only over copies within $x$ of a fixed $T'$):

$$\sum_{T'} (n_0 - j)! \frac{q_1^{n_0 - j} q_3^{i - a(u) - h_x(u) + j}}{q_1 + q_3} \prod_{v \in T'} h_{T'}(v) f_{T'}(x) \geq \sum_{T'} \prod_{v \in T'} h_x(v)^{n_0 - j} q_3^{i - a(u) - h_x(u) + j} q_1^{n_0 - a(u) - h_x(u) + 1}$$

$$= \sum_{u \in x} \left( \frac{h_x(u)}{n_0 - j} \right)^{n_0 - j} q_1^{n_0 - j} q_3^{i - a(u) - h_x(u) + j} q_1^{n_0 - a(u) - h_x(u) + 1}$$

$$= f_j(x)$$
where obtaining the second line uses Equation 5.9 for a sum over trunks \( T' \) of \( H_x(u) \) with degree \( n_0 - j \).

So, by Proposition 2.1

\[
\text{Expect}(\vec{f}_j(X_t)|X_0 = x_0) \leq \text{Expect}(\vec{f}_j(X_t)|X_0 = x_0) = q_2^{(n_0-j)t} \sum_{T'} (n_0 - j)! \left( \begin{array}{c} n_0 \\ j \end{array} \right) \frac{q_1^{n_0-j}}{q_1 + q_3} \prod_{v \in T'} h_{T'(v)} f_{T'}(x_0)
\]

and it remains to upper-bound the right-hand side. We do so by finding an upper bound \( C \) for \( \prod_{v \in A(u) \setminus \{u\}} \frac{h_x(v)}{h_S(v)} \); then we have

\[
\prod_{v \in S} \frac{h_x(v)}{h_S(v)} = \left( \prod_{v \in S} \frac{h_x(v)}{h_S(v)} \right) \left( \prod_{v \in A(u) \setminus \{u\}} \frac{h_x(v)}{h_S(v)} \right) \geq C \left( \prod_{v \in S} \frac{h_x(v)}{h_S(v)} \right)
\]

and using this in place of Equation 5.10 in the first four paragraphs of this proof will show

\[
q_2^{(n_0-j)t} \sum_{T'} (n_0 - j)! \left( \begin{array}{c} n_0 \\ j \end{array} \right) \frac{q_1^{n_0-j}}{q_1 + q_3} \prod_{v \in T'} h_{T'(v)} f_{T'}(x_0) \leq q_2^{(n_0-j)t} C f_j(X_0).
\]

To obtain the upper bound \( C \), let \( v_1, v_2, \ldots, v_{a(u)-1} \) be the vertices of \( A(u) \setminus \{u\} \), such that \( v_1 \) is a root, \( v_2 \) is a child of \( v_1 \), \( v_3 \) is a child of \( v_2 \), ... . So \( \text{Anc}(v_k) = \{v_1, \ldots, v_k\} \), and, except \( v_k \) itself, these are not in the hook of \( v_1 \); hence \( h_x(v_k) \leq n_0 - (k-1) \). For the denominator, note that \( H_S(v_k) \supseteq v_k, v_{k+1}, \ldots, v_{a(u)-1} \), so \( h_S(v_i) \geq a(u) - k \). Thus

\[
\prod_{v \in A(u) \setminus \{u\}} \frac{h_x(v)}{h_S(v)} \leq \frac{n_0(n_0-1) \ldots (n_0 - a(u) + 2)}{(a(u) - 1)(a(u) - 2) \ldots 1} \leq \max_{u \in X_0 : h(u) \geq n_0 - j} \left\{ \left( \begin{array}{c} n_0 \\ a(u) - 1 \end{array} \right) \right\},
\]

which proves the theorem. \[\square\]

6 Relative time on a to-do list

This section applies the Hopf-algebraic framework of Sections 3.1 to the Malvenuto-Reutenauer Hopf algebra to analyse variations of the “to-do list” chain of Example 1.2, also known as top-to-random-with-standardisation. As shown in [Pan18], the significance of this family of chains is two-fold: first, the distribution after \( t \) steps is equal to that of a card shuffle if started from the identity permutation. So any result below phrased solely in terms of the distribution at time \( t \) (and starting at the identity) also applies to the corresponding shuffle; see Theorem 6.4 and Corollary 6.8. Second, information about these chains can be used to analyse descent operator Markov chains on the numerous subquotients of the Malvenuto-Reutenauer Hopf algebra, which are all lumpings of the present chains. These lumped chains include the phylogenetic tree example from the introduction (Loday-Ronco algebra of binary trees), an unbump-and-reinsert chain on tableaux (Poirier-Reutenauer algebra) and a remove-and-readd-a-box chain on partitions (symmetric functions with Schur basis) [Pan18] Sec. 3.2-3.3].
6.1 Markov chains from the Malvenuto-Reutenauer Hopf algebra

Following [DHT02], let \( \text{FQSym} \) (free quasisymmetric functions) denote the Malvenuto-Reutenauer Hopf algebra of permutations [MR95, Sec. 3] (the other common notation is \( \mathcal{S}\text{Sym} \) [AS05b]). Its basis in degree \( n \) consists of the permutations in \( \mathcal{S}_n \), written in one-line notation: \( \sigma = (\sigma_1 \ldots \sigma_n) \). The \( \sigma_i \) are called letters, in line with the terminology of words.

We work in the fundamental basis of \( \text{FQSym} \). The product \( \sigma \tau \) of two permutations is computed as follows: first, add to each letter in \( \tau \) the degree of \( \sigma \), then sum over all interleavings of the result with \( \sigma \). For example:

\[
(312)(21) = (312) \sqcup (54) = (31254) + (31524) + (35124) + (35142) + (35412) + (53124) + (53142) + (53412) + (54312).
\]

The coproduct is “deconcatenate and standardise”:

\[
\Delta(\sigma_1 \ldots \sigma_n) = \sum_{i=0}^{n} \text{std}(\sigma_1 \ldots \sigma_i) \otimes \text{std}(\sigma_{i+1} \ldots \sigma_n),
\]

where the standardisation map \( \text{std} \) converts an arbitrary string of distinct letters into a permutation by preserving the relative order of the letters. For example:

\[
\Delta(4132) = () \otimes (4132) + (1) \otimes (132) + (21) \otimes (21) + (312) \otimes (1) + (4132) \otimes ()
\]

Since, for every permutation \( \sigma \in \mathcal{S}_n \), its coproduct \( \Delta_{1,n-1}(\sigma) \) is of the form \( (1) \otimes \tau \) for some permutation \( \tau \in \mathcal{S}_n \), we see inductively that the rescaling function is \( \eta(\sigma) \equiv 1 \), i.e. no rescaling is required. Thus, by Theorem [3.3], each timestep of the top-\( r \)-to-random-with-standardisation chains, driven by \( T_r 2R_n = \frac{1}{n(n-1)...(n-r+1)} m \Delta_{1,r,n-r} \), has the following three-part description:

i) Remove the first \( r \) letters of \( \sigma \).

ii) Replace the letters of the result by \( r+1, \ldots, n \) such that they have the same relative order as before.

iii) Insert the letters \( 1, 2, \ldots, r \) in a uniformly chosen position.

Figure 6.1 shows a possible trajectory for \( r = 2, n = 5 \).

We give three interpretations of this chain. First, there is the to-do list formulation: every day you complete \( r \) tasks at the top of a to-do list comprising \( n \) tasks, then add \( r \) new tasks independently, each at a uniformly chosen position. You encode each daily to-do list as a permutation, writing 1 for the latest addition to the list, 2 indicates the next newest addition excluding task 1, and so on, so that \( n \) is the task spending the longest time on the list. See Example [1.2]. In other words, you record only the relative times that the tasks have spent on the list. In addition to fixed values of \( r \), we can let \( r \) be a binomial variable with parameter \( q_2 \). This corresponds to the descent operator \( \text{BinT}2R \), and models \( n \) managers independently handing you a task each with probability \( q_2 \). The variable work speed (that each day the number of completed tasks equals this variable number of incoming tasks) can be explained by procrastination when the workload is low, and a panic to overwork when workload is high, to avoid a growing to-do list.
The second formulation is in terms of card-shuffling - take $r$ cards from the top of an $n$-card deck, and reinsert them independently each at a uniformly chosen position. Such shuffles were studied in [DFP92], but here we make one crucial modification: instead of recording the values of the cards in each position as they are shuffled, we record the relative last times that each card was touched, 1 being the most recently moved card. So the number assigned to each card changes at each shuffle - more specifically, if the top, removed, card is labelled $i$, then we change its label to 1 during its reinsertion, and the cards previously labelled 1, 2, \ldots, $i-1$ now increase their labels each by 1, becoming 2, 3, \ldots, $i$ respectively.

The third formulation models a player’s hand during a card game. At each turn, the player plays the $r$ leftmost cards in his hand (thus removing them from the hand), then draws $r$ new cards to maintain a hand of $n$ cards. The newly-drawn cards are inserted into the hand in uniformly chosen positions, depending on when the player plans to play them (cards to be played sooner are placed towards the left). Again, the chain tracks the relative times that the cards have spent in the player’s hand.

This chain has a unique stationary distribution, namely the uniform distribution $\pi(\sigma) \equiv \frac{1}{n!}$. Indeed, since $\bullet = (1)$ is the unique permutation of 1, the comment after Theorem 3.12 applies: the stationary distribution is given by $\pi(\sigma) = \frac{1}{n!} \eta(\sigma) \times \text{coefficient of } \sigma \text{ in } \bullet^n$. And the required coefficient is 1 as there is a unique way of inserting the letters 1, 2, \ldots, $n$ in that order to obtain a given permutation.

### 6.2 Relationship to card-shuffles

For the readers’ convenience, we reproduce below a theorem and proof from [Pan18], which allows results for the to-do list chain to apply to cut-and interleave card-shuffles (as described after Lemma 3.3).

**Theorem 6.1** (Equidistribution under descent-operator chains on $\mathbb{FQSym}$ and $\mathcal{S}$). [Pan18, Th. 3.14] The distribution on permutations after $t$ iterates of the $m\Delta_p$-chain on $\mathbb{FQSym}$ (to-do list chain) is the same as that after $t$ iterates of the $m\Delta_p$-chain on the shuffle algebra $\mathcal{S}$ (card-shuffling), if both are started from the identity permutation.

**Proof.** First consider the case $t = 1$. In $\mathbb{FQSym}$, for any weak-composition $D$,

\[
m\Delta_D(1\ldots n) = m\left(\text{std}(12\ldots d_1) \otimes \text{std}(d_1 + 1) \ldots (d_1 + d_2) \otimes \cdots \otimes \text{std}(d_1 + \cdots + d_{l(D) - 1} + 1) \ldots n)\right)
\]

\[
= m\left(12\ldots d_1 \otimes 12\ldots d_2 \otimes \cdots \otimes 12\ldots d_{l(D)}\right)
\]

\[
= 1\ldots d_1 \sqcup (d_1 + 1)\ldots (d_1 + d_2) \sqcup \cdots \sqcup (d_1 + \cdots + d_{l(D) - 1} + 1)\ldots n,
\]

and $m\Delta_D$ calculated in $\mathcal{S}$ gives the same result (under the identification of $\sigma$ with $[\sigma]$). By linearity,
\[ m \Delta_P(1 \ldots n) = m \Delta_P([1 \ldots n]) \]

for all distributions \( P \). This proves the equidistribution in the case \( t = 1 \).

The key to showing equidistribution for larger \( t \) is to express \( t \) iterates of \( m \Delta_P \), in either \( \text{FQSym} \) or \( S \), as a single application of \( m \Delta_{P'} \) for the same distribution \( P' \). This uses the identification of the descent operator \( m \Delta_P \) with the homogeneous noncommutative symmetric function \( S^D \), see Section 2.5. Let \( S^P = \sum_D \frac{P(D)}{(D)} S^D \) be the noncommutative symmetric function associated to \( m \Delta_P \).

On a commutative Hopf algebra, such as \( S \), the composition \( (m \Delta_P) \circ (m \Delta_P) \) corresponds to the internal product of noncommutative symmetric functions \( S^P \cdot S^P \cdot \ldots \cdot S^P \), with \( t \) factors.

Now consider \( m \Delta_P \) on \( \text{FQSym} \). Note that the coproduct of the identity permutation is

\[
\Delta(1 \ldots n) = \sum_{r=0}^{n} (1 \ldots r) \otimes (1 \ldots n-r),
\]

and each tensor-factor is an identity permutation of shorter length. Thus the subalgebra of \( \text{FQSym} \) generated by identity permutations of varying length is closed under coproduct, and repeated applications of \( m \Delta_P \), starting at the identity, stays within this sub-Hopf-algebra. Equation 6.2 shows that this sub-Hopf-algebra is cocommutative, so by Proposition 2.9, the composition \( (m \Delta_P) \circ (m \Delta_P) \) corresponds to the internal product \( S^P \cdot S^P \cdot S^P \cdot \ldots \cdot S^P \), in the opposite order from for \( S \). However, we only concern the case \( P = P' \) where the order is immaterial: \( t \) iterates of the \( m \Delta_P \)-chain on \( \text{FQSym} \) are also driven by \( S^P \cdot S^P \cdot S^P \cdot \ldots \cdot S^P \), with \( t \) factors.

\[ \square \]

6.3 Recursive lumping

Since \( \Delta_{1,n-1} \) sends a permutation to \( \bullet \otimes \) another permutation, the recursive lumping of Theorem 4.10 applies. Note that the lumping map \( D^{n-k} : B_n \rightarrow B_k \) is simply the observation of the bottom \( k \) items on the to-do list, or the relative last-moved times of the bottom \( k \) cards of the deck. Hence we have

**Theorem 6.3** (Recursive lumping for to-do list chains). _Observing the last \( k \) items under the top-to-random-with-standardisation chain on \( n \) items gives a \( \frac{n-k}{n} \)-lazy version of the top-to-random-with-standardisation chain on \( k \) items._ Observing the last \( k \) items under the binomial-top-to-random-with-standardisation chain on \( n \) items gives the binomial-top-to-random-with-standardisation chain on \( k \) items, with the same parameter \( q_2 \). 

It is possible to see these two lumpings via an elementary argument. For each step of the top-to-random-with-standardisation chain on \( n \) cards, one of these two scenarios occur:

- With probability \( \frac{n-k}{n} \), the removed top card is reinserted in one of positions \( 1, 2, \ldots, n-k \) (counting from the top). When observing only the bottommost \( k \) cards, we see no change.

- With the complementary probability of \( \frac{k}{n} \), the removed top card is reinserted in one of positions \( n-k+1, n-k+2, \ldots, n \), chosen uniformly. The insertion pushes the \( n-k+1 \)th card into the \( n-k \)th position, so when observing only the bottommost \( k \) cards, the top card amongst these \( k \) appears removed. Although the inserted card is not the “top” card of this apparent removal, it is nevertheless the most recently touched card, whether we are observing the entire deck or just the bottommost \( k \) cards. And the chain tracks only the relative times that cards are last touched, so this difference in card is invisible.

As for binomial-top-to-random: view each step of the chain in terms of reinserted positions as follows:
1. Select each position out of \{1, 2, \ldots, n\} independently with probability \((1 - q_2)\). Let \(r\) denote the number of selected positions.

2. Select uniformly a permutation \((\tau_1 \ldots \tau_r) \in \mathfrak{S}_r\).

3. Remove the top \(r\) cards and reinsert into the positions chosen in 1., in the relative order chosen in 2. (i.e. the card previously in position \(\tau_1\) is reinserted to the topmost selected position, the card previously in position \(\tau_2\) is reinserted to the second-topmost selected position, \ldots).

When observing only the bottom \(k\) positions, we see that they are each independently selected as in 1.. The relative order of reinserted cards is that of \(\tau_{r'+1}\tau_{r'+2}\ldots\tau_r\) for some \(r'\), and this is uniform on \(\mathfrak{S}_{r-r'}\) when \(\tau\) is uniform. So the change in the bottom \(k\) positions follows exactly steps 1., 2., 3. above, with \(k\) cards that are pushed above and outside the observation area are not the cards that are reinserted into the observation area, but since they are the last-touched cards, the chain does not notice this difference.

In view of Theorem 6.4, this recursive-lumping theorem can be restated in terms of top-to-random shuffles (although the relationship is no longer a lumping):

**Theorem 6.4** (Distribution of bottommost cards under top-to-random shuffles). After \(t\) top-to-random shuffles (resp. binomial-top-to-random shuffles) of a deck of \(n\) cards, the probability distribution on \(\mathfrak{S}_n\) given by standardising the values of the bottommost \(k\) cards is equal to the distribution on a deck of \(k\) cards after \(t\) steps of an \(\frac{n-k}{n}\)-lazy version of the top-to-random shuffle (resp. after \(t\) steps of a binomial-top-to-random shuffle, with the same parameter \(q_2\)). \(\square\)

### 6.4 Eigenvalues and eigenvectors

An easy application of Theorem 4.4.i shows that

**Proposition 6.5.** The eigenvalues of the top-to-random-with-standardisation and binomial-top-to-random-with-standardisation chains are \(\beta_j = \frac{j}{n}\) and \(\beta_j = q_2^{n-j}\) respectively \((j \in [0, n-2] \cup \{n\})\), and the multiplicity of \(\beta_j\) is \(\dim \mathcal{H}_{n-j} - \dim \mathcal{H}_{n-j-1} = (n-j)! - (n-j-1)!\).

Observe that the multiplicity \((n-j)! - (n-j-1)!\) is precisely the number of permutations in \(\mathfrak{S}_n\) fixing pointwise 1, 2, \ldots, \(j\) but not \(j+1\). Indeed, we can associate an eigenvector of eigenvalue \(\beta_j\) to each such permutation \(\tau\):

**Theorem 6.6** (Eigenbasis of top-to-random-with-standardisation chains). Given a permutation \(\tau \in \mathfrak{S}_n\), let \(j+1\) be the smallest number in \(\{1, 2, \ldots n\}\) not fixed pointwise by \(\tau\). So the number \(i\) defined by \(\tau_i = j+1\) satisfies \(i > j+1\). Define the function \(f_\tau : \mathfrak{S}_n \to \mathbb{R}\) by

\[
    f_\tau(\sigma) = \begin{cases} 
        1 & \text{if } (\sigma_{j+1}\sigma_{j+2}\ldots\sigma_n) \text{ is in the same relative order as } (\tau_{j+1}\tau_{j+2}\ldots\tau_n); \\
        -1 & \text{if } (\sigma_{j+1}\sigma_{j+2}\ldots\sigma_n) \text{ is in the same relative order as } ((j+1)\tau_{j+1}\tau_{j+2}\ldots\tau_i-1\tau_i+1\tau_{i+2}\ldots\tau_n); \\
        0 & \text{otherwise.}
    \end{cases}
\]

Then \(f_\tau\) is a right eigenfunction of top-to-random-with-standardisation and binomial-top-to-random-with-standardisation with eigenvalue \(\beta_j\), and \(\{f_\tau | \tau \in \mathfrak{S}_n\}\) is a basis.

The proof is at the end of this section.
Example 6.7. Let $\tau = 12534$, so $j = 2$ and $i = 4$ (because $\tau_1 = 3$). Then $f_\tau(\sigma)$ is 1 if the last three letters of $\sigma$ are in the same relative order as 534, i.e. “high then low then middle”. And $f_\tau(\sigma)$ is $-1$ if the last three letters of $\sigma$ are in the same relative order as 354, i.e. “low then high then middle”. For example $f_\tau(35412) = 1$, $f_\tau(24513) = 1$, $f_\tau(25431) = 0$.

The eigenbasis of Theorem 6.6 yields the probability distribution of the position of the newest task amongst the bottom $n - j$ tasks, or the last card that was touched in those bottom $n - j$ cards.

Corollary 6.8. After $t$ iterates of top-to-random (resp. binomial-top-to-random), with or without standardisation, starting from the identity permutation, the probability that the smallest value among $\{\sigma_{j+1}, \sigma_{j+2}, \ldots, \sigma_n\}$ is $\sigma_{j+k}$ is

$$
\begin{cases}
\frac{1}{n-j} \left(1 + \beta_j(n-j-1)\right) & \text{if } k = 1; \\
\frac{1}{n-j} \left(1 - \beta_j^k\right) & \text{if } k \neq 1;
\end{cases}
$$

where $\beta_j = \frac{i}{n}$ (resp. $\beta_j = q_2^{n-j}$).

Proof. Fix $j$. Let $\bar{\sigma} = \text{std}(\sigma_{j+1} \ldots \sigma_n) \in \mathfrak{S}_{n-j}$, so the desired probabilities are of $\bar{\sigma}_k = 1$.

Consider first the case $k = 1$. Let $f_1 = \sum f_\tau$, summing over all $\tau$ that fix $1, 2, \ldots, j$ pointwise, but not $j + 1$. We show

$$
f_1(\sigma) = \begin{cases}
-(n-j-1) & \text{if } \bar{\sigma}_1 = 1; \\
1 & \text{otherwise.}
\end{cases}
$$

If $\bar{\sigma}_1 \neq 1$ (i.e. $\sigma_{j+1}$ is not the smallest amongst $\sigma_{j+2}, \ldots, \sigma_n$), then the only summand contributing to $f(\sigma)$ is $\tau = 12 \ldots j(\bar{\sigma}_1 + j)(\bar{\sigma}_2 + j) \ldots (\bar{\sigma}_{n-j} + j)$, and $f_\tau(\sigma) = 1$. If $\bar{\sigma}_1 = 1$, then the only non-zero summands in $f_1(\sigma)$ come from $\tau$ of the form $12 \ldots j(\bar{\sigma}_2 + j) \ldots (\bar{\sigma}_j + j)(1 + j)(\bar{\sigma}_{j+1} + j) \ldots (\bar{\sigma}_{n-j} + j)$, where $i \in (2, n-j)$. Hence there are $n - j - 1$ terms all contributing $-1$.

Writing $\mathbb{1}$ for an indicator function, we have

$$
\begin{align*}
\text{Prob}(\bar{\sigma}_1 = 1) &= \text{Expect}(\mathbb{1}\{\bar{\sigma}_1 = 1\}) \\
&= \text{Expect} \left( \frac{1}{n-j}(1 - f_1) \right) \\
&= \frac{1}{n-j} \left(1 - \beta_j^1 f_1(x_0)\right) \\
&= \frac{1}{n-j} \left(1 + \beta_j^1(n-j-1)\right).
\end{align*}
$$

(The third equality uses the linearity of expectations and that $f_1$ is an eigenvector with eigenvalue $\beta_j$.)

For $k > 1$, consider $f_k = \sum f_\tau$ summing over all $\tau$ that fix $1, 2, \ldots, j$ pointwise, and satisfy $\tau_{k+j} = j + 1$. Then

$$
f_k(\sigma) = \begin{cases}
1 & \text{if } \bar{\sigma}_k = 1; \\
-1 & \text{if } \bar{\sigma}_1 = 1; \\
0 & \text{otherwise},
\end{cases}
$$

with the contributions in the first two cases coming from $\tau = 12 \ldots j(\bar{\sigma}_1 + j)(\bar{\sigma}_2 + j) \ldots (\bar{\sigma}_{n-j} + j)$ and $\tau = 12 \ldots j(\bar{\sigma}_2 + j) \ldots (\bar{\sigma}_k + j)(1 + j)(\bar{\sigma}_{k+1} + j) \ldots (\bar{\sigma}_{n-j} + j)$ respectively. So

$$
\text{Prob}(\bar{\sigma}_k = 1) - \text{Prob}(\bar{\sigma}_1 = 1) = \text{Expect}(f_k) = \beta_j^1 f_k(x_0) = -\beta_j^k,
$$

and substitute for $\text{Prob}(\bar{\sigma}_1 = 1)$ from (6.9). \qed
Remark. Corollary 6.8 also follows from the following elementary argument. By the recursive lumpings of Theorem 6.3, it suffices to calculate the distribution of the card/task labelled 1 under a \( \frac{1}{n} \)-lazy version of the chain driven by \( T2R_{n-j} \), or under the chain driven by \( \text{Bin}T2R_{n-j}(q_2) \). Note that, because a reinserted card is relabelled 1 and placed in a uniformly chosen position, the card labelled 1 would be uniformly distributed in position at time \( t \) as long as at least one reinsertion has happened by time \( t \). Let \( P(t) \) be the probability that no reinsertion has happened by time \( t \); then card 1 is at the top at time \( t \) with probability \( (1 - P(t)) \frac{1}{n-j} \), and in any other position with probability \( (1 - P(t)) \frac{j}{n-j} \). To complete the proof,

\[
P(t) = P(1)^t = \begin{cases} \left( \frac{j}{n} \right)^t & \text{for the } \frac{1}{n} \text{-lazy version of the chain driven by } T2R_{n-j}; \\
\left( \frac{q_2}{n-j} \right)^t & \text{for the chain driven by } \text{Bin}T2R_{n-j}(q_2).
\end{cases}
\]

Proof of Theorem 6.6. To calculate right eigenfunctions, we must work in the dual \( \text{FQSym}^* \), where \( \Delta_{1,n-1} \) is the removal of the letter 1 (followed by standardisation) and \( m_{1,n-1}(\bullet^* \otimes \sigma^*) \) is the sum over all permutations whose last \( n-1 \) letters standardise to \( \sigma^* \).

Given \( \tau, j \) as in the theorem statement, let \( \bar{\tau} = \text{std}(\tau_{j+1} \ldots \tau_n) \in \mathfrak{S}_{n-j} \). Then the \( \bar{i} \) with \( \bar{\tau}_i = 1 \) satisfies \( \bar{i} > 1 \). Note that, in \( \text{FQSym}^* \), we have \( \Delta_{1,n-j-1}(\bar{\tau}^*) = (\bar{\tau}_1 \ldots \bar{\tau}_{i-1} \bar{\tau}_{i+1} \ldots \bar{\tau}_{n-j})^* = \Delta_{1,n-j-1}(1 \bar{\tau}_1 \ldots \bar{\tau}_{i-1} \bar{\tau}_{i+1} \ldots \bar{\tau}_{n-j})^* \). So \( p := \bar{\tau}^* - (1 \bar{\tau}_1 \ldots \bar{\tau}_{i-1} \bar{\tau}_{i+1} \ldots \bar{\tau}_{n-j})^* \in \ker \Delta_{1,n-j-1}^* \) (and \( p \) is nonzero because \( \bar{i} > 1 \)), and thus, by Proposition 2.4.R and Theorem 4.4, a right eigenfunction is given by

\[
f_{\bar{\tau}}(\sigma) = m_{1,,\ldots,1,n-j}(\bullet^* \otimes \ldots \otimes \bullet^* \otimes p) \text{ evaluated on } \sigma \\
= (\bullet^* \otimes \ldots \otimes \bullet^* \otimes p) \text{ evaluated on } \Delta_{1,,\ldots,1,n-j}^* \\
= p(\text{std}(\sigma_{j+1} \sigma_{j+2} \ldots \sigma_{n})) \\
= \begin{cases} 
1 & \text{if } \text{std}(\sigma_{j+1} \sigma_{j+2} \ldots \sigma_{n}) = \bar{\tau}; \\
-1 & \text{if } \text{std}(\sigma_{j+1} \sigma_{j+2} \ldots \sigma_{n}) = 1 \bar{\tau}_1 \ldots \bar{\tau}_{i-1} \bar{\tau}_{i+1} \ldots \bar{\tau}_n; \\
0 & \text{otherwise}.
\end{cases}
\]

To show that \( \{f_\tau | \tau \in \mathfrak{S}_n\} \) form a basis of eigenfunctions, it suffices to show that the \( f_\tau \) for a fixed \( j \) are linearly independent. Indeed, \( f_\tau(\tau) = 1 \) and \( f_{\tau'}(\tau) = 0 \) for any \( \tau' \neq \tau \) with the “same \( j \)” as \( \tau \) (i.e. the smallest number not fixed by \( \tau \)); hence, \( \tau ' \) is also the smallest number not fixed by \( \tau \). \( \square \)

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