Functional central limit theorems for spatial averages of the parabolic Anderson model with delta initial condition in dimension $d \geq 1$

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Abstract

Let $\{u(t, x)\}_{t \geq 0, x \in \mathbb{R}^d}$ denote the solution to a $d$-dimensional parabolic Anderson model with delta initial condition and driven by a multiplicative noise that is white in time and has a spatially homogeneous covariance given by a nonnegative-definite measure $f$. Let $S_{N, t} := N^{-d} \int_{[0, N]^d} [U(t, x) - 1] dx$ denote the spatial average on $\mathbb{R}^d$. We obtain various functional central limit theorems (CLTs) for spatial averages based on the quantitative analysis of $f$ and spatial dimension $d$. In particular, when $f$ is given by Riesz kernel, that is, $f(x) = ||x||^{-\beta} dx$, $\beta \in (0, 2 \wedge d)$, the functional CLT is also based on the index $\beta$.

Keywords: parabolic Anderson model, functional central limit theorem, Malliavin calculus, Stein’s method, delta initial condition.

1 Introduction

Consider the following parabolic Anderson model:

$$\begin{cases}
\partial_t u(t, x) = \frac{1}{2} \Delta u(t, x) + u(t, x) \eta(t, x) & \text{for } (t, x) \in (0, +\infty) \times \mathbb{R}^d, \\
\text{subject to } & u(0) = \delta_0, 
\end{cases}$$

(1)
where $\eta$ denotes a centered, generalized Gaussian random field such that for all $s, t \geq 0$ and $x, y \in \mathbb{R}^d$,

$$\text{Cov}[\eta(t, x), \eta(s, y)] = \delta_0(t-s)f(x-y) \text{ for all } s, t \geq 0 \text{ and } x, y \in \mathbb{R}^d,$$

(2)

for a non-zero, nonnegative-definite, tempered Borel measure $f$ on $\mathbb{R}^d$. In order to avoid degeneracies, we assume that $f(\mathbb{R}^d) > 0$.

Following from [1], the equation (1) has a mild solution that satisfies the integral equation:

$$u(t, x) = p_t(x) + \int_{(0,t) \times \mathbb{R}^d} p_{t-s}(x-y)u(s, y)\eta(ds, dy) \text{ for } t > 0, \text{ and } x \in \mathbb{R}^d,$$

(3)

where $p_t(x)$ is the heat kernel defined by

$$p_t(x) = (2\pi t)^{-d/2}e^{-\|x\|^2/2t} \text{ for } t > 0, \text{ and } x \in \mathbb{R}^d.$$

Because

$$\frac{p_{t-s}(a)p_s(b)}{p_t(a+b)} = p_{s(t-s)/t}\left(b - \frac{s}{t}(a+b)\right) \text{ for all } 0 < s < t, \text{ and } a, b \in \mathbb{R}^d,$$

(3) can be recast as the following evolution equation:

$$U(t, x) = 1 + \int_{(0,t) \times \mathbb{R}^d} p_{s(t-s)/t}\left(y - \frac{s}{t}x\right)U(s, y)\eta(ds, dy),$$

(4)

where $U(t, x) := u(t, x)/p_t(x)$.

In order to ensure the existence and uniqueness of $u$, hence also $U$, we assume that the Fourier transform $\hat{f}$ satisfies the integrability condition:

$$\Upsilon(\beta) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\hat{f}(dy)}{\beta + \|y\|^2} < \infty \text{ for all } \beta > 0.$$

The purpose of this paper is to establish specific functional CLTs for spatial averages of $U$. The reason we do not consider the spatial average of $u$ is that $\{u(t, x)\}_{x \in \mathbb{R}}$ lacks spatial stationarity, whereas the renormalized process $\{U(t, x)\}_{x \in \mathbb{R}}$ is spatial stationary [2, 3]. Let us start by introducing the spatial average:

$$S_{N,t} = \frac{1}{N^d} \int_{[0,N]^d} [U(t, x) - 1]dx.$$

(5)

In order to present the main theorems, we introduce a quantity associated with $f$. For every real number $m > 0$, define

$$I_m(x) := m^{-d}1_{[0,m]^d}(x), \quad \tilde{I}_m(x) := I_m(-x) \text{ for } x \in \mathbb{R}^d,$$

(6)
and

\[ R(f) : = \frac{1}{\pi^d} \int_{0}^{\infty} ds \int_{\mathbb{R}^d} \hat{f}(dz) \prod_{j=1}^{d} \frac{1 - \cos(sz_j)}{(sz_j)^2} \]

\[ = \frac{1}{(2\pi)^d} \int_{0}^{\infty} dr \int_{\mathbb{R}^d} \hat{f}(dz) \left( \left( I_1 \ast \hat{I}_1 \right)(\bullet/r) \right)(z). \]  

(7)

**Theorem 1.** Assume \( R(f) < \infty \), then for any fixed real number \( T > 0 \), we have

\[ \sqrt{N} \mathcal{S}_{N,t} \xrightarrow{C[0,T]} \{ G_t \}_{t \in [0,T]} \quad \text{as} \quad N \to \infty, \]  

where \( C[0,T] \) denotes weak convergence on the space of all continuous functions \( C[0,T] \), \( \{ G_t \}_{t \geq 0} \) is a centered Gaussian process with covariance \( E[G_t G_{t_2}] = g_{t_1, t_2} \), which is defined explicitly in Proposition 7.

**Theorem 2.** Assume \( d = 1 \) and \( f \) is a Rajchman measure, that is, \( f \) satisfies \( f(\mathbb{R}) < \infty \) and \( \lim_{x \to \infty} \hat{f}(x) = 0 \), then for any fixed real number \( T > 0 \), we have

\[ \sqrt{N} \log N \mathcal{S}_{N,t} \xrightarrow{C[0,T]} \{ \sqrt{f(\mathbb{R})} B_t \}_{t \in [0,T]}, \]

(9)

where \( B \) denotes a standard Brownian motion.

**Remark 1.** The collection of Rajchman measures [4] is very rich. For example, \( f \) is given by a Gaussian kernel \( f(dx) = p_1(x)dx \) and \( \hat{f}(dx) = e^{-|x|^2/2}dx \) or a Cauchy kernel \( f(dx) = (1 + |x|^2)^{-1}dx \) and \( \hat{f}(dx) = e^{-|x|}dx \).

**Remark 2.** In fact, the assumption \( \lim_{x \to \infty} \hat{f}(x) = 0 \) is not necessary. We can obtain the functional CLT provided the limit of the covariance exists. It is worth noting that the limit can only take values between \( (t_1 \wedge t_2) f(\mathbb{R}) \) and \( 2(t_1 \wedge t_2) f(\mathbb{R}) \); see Proposition 8.

**Remark 3.** One can see from Lemma 5.9 in [3] that \( R(f) = \infty \) if \( d = 1 \). Therefore, Theorem 1 holds only for \( d \geq 2 \). And there does not seem to be a canonical functional CLT when \( R(f) = \infty \) (\( d \geq 2 \)), or \( f(\mathbb{R}) = \infty \). In this situation, we present a special case that \( f \) is given by Riesz kernel as an example.

**Theorem 3.** (Riesz kernel). Assume \( f(dx) = \|x\|^{-\beta} dx \), for some \( \beta \in (0, d \wedge 2) \), then for any fixed real number \( T > 0 \), we have

(A) If \( \beta \in (0, 1) \), then

\[ \sqrt{N} \mathcal{S}_{N,t} \xrightarrow{C[0,T]} \{ C_t^{(1)} \}_{t \in [0,T]} \quad \text{as} \quad N \to \infty. \]

(10)

(B) If \( \beta = 1 \), then
both the asymptotic behavior of the variance asymptotic covariance of the spatial averages in Section 3, which is an extension on
dition and the measure
its related variations for other types of equations can be found in
The results depend on not only the dimension already well-developed
tic process. Although the methods for proving functional central limit theorems is
upper bound and tightness of the spatial averages. In order to p rove our main theo-
function CLT. As an example of the condition
implies a standard form of CLT
quantitative CLT and the related functional CLT were introduced first in [2] by using
in the case that
δ
(0), this also affects the form of the functional CLT. Later, [3] generalized the
CLT to the multidimensional case, which is much more involved than the case d = 1.
The results depend on not only the dimension d but also the behavior of f. CLT and
its related variations for other types of equations can be found in [9–12]. For other
limit theories, we refer to [13–18].

In this paper, we establish the functional CLTs in detail under the condition d ≥ 1
mainly based on Malliavin–Stein’s method, Fourier analysis and Poincaré-type inequal-
ity. The Functional CLT is an extension of the CLT to function spaces, providing a
more comprehensive understanding of the limiting behavior of {SN,t}t>0 as a stochas-
tic process. Although the methods for proving functional central limit theorems is
already well-developed [2, 3, 5, 7, 8, 11, 12], it is undeniable that the δ0 initial con-
dition and the measure f introduce significant challenges to our proof. We derive the
asymptotic covariance of the spatial averages in Section 3, which is an extension on
both the asymptotic behavior of the variance [3, Section 5] and the asymptotic behav-
ior of the covariance for f = δ0 [2, Section 4]. In Section 4, we prove the uniform
upper bound and tightness of the spatial averages. In order to prove our main theo-
remas, it remains to establish the convergence of finite dimensional distributions, which
is shown in Section 5. And the last section is Appendix, we introduce a few technical
lemmas that are used throughout the paper.

Throughout this paper, we write "g1(x) ≤ g2(x) for all x ∈ X" when there exists
a real number L independent of x such that g1(x) ≤ Lg2(x) for all x ∈ X. And for
every Z ∈ L^k(Ω), we write ∥Z∥_k instead of ∥Z∥_{L^k(Ω)}.
2 Preliminaries

Let us begin by introducing the fundamentals of Malliavin calculus and stochastic integral. Let $\mathcal{H}_0$ denote the reproducing kernel Hilbert space, spanned by all real-valued functions on $\mathbb{R}^d$, that corresponds to the scalar product $\langle \phi, \psi \rangle := \langle \phi, \psi \ast f \rangle_{L^2(\mathbb{R}^d)}$, and let $\mathcal{H} = L^2(\mathbb{R}_+ \times \mathcal{H}_0)$. The Gaussian random field $\{W(h)\}_{h \in \mathcal{H}}$ formed by such Wiener integrals

$$W(h) := \int_{\mathbb{R}_+ \times \mathbb{R}^d} h(s, y) \eta(ds, dy)$$

defines an isonormal Gaussian process on the Hilbert space $\mathcal{H}$. Therefore, we can develop the Malliavin calculus (see, for instance, [19]). We denote by $D$ the Malliavin derivative operator and by $\delta$ the corresponding divergence operator, then we have the following important property:

Let $F$ denote a predictable and square-integrable random field valued in the Gaussian Sobolev space $D^{1,2}$, and also, $F$ belongs to the domain of $\delta$ denoted by $\text{Dom}[\delta]$, then $\delta(F)$ coincides with the Walsh integral:

$$\delta(F) = \int_{\mathbb{R}_+ \times \mathbb{R}^d} F(s, y) \eta(ds, dy).$$

Moreover, for all $p \geq 2$, there exists $c_p > 0$, such that for all $t > 0$, the following Burkholder-Davis-Gundy (BDG) inequality holds:

$$\mathbb{E} \left[ \left( \int_{(0,t) \times \mathbb{R}^d} F(s, y) \eta(ds, dy) \right)^p \right] \leq c_p \mathbb{E} \left[ \left( \int_{(0,t) \times \mathbb{R}^d} F(s,y)F(s,y+y')f(dy')dyds \right)^{\frac{p}{2}} \right].$$

Let us recall the following inequalities from Theorem 1.1 and Proposition 4.1 in [3].

Lemma 4. Suppose $U = \{U(t,x)\}_{t>0, x \in \mathbb{R}^d}$ is a predictable solution to the integral equation (4), then, $U$ is the only predictable solution to (4) that satisfies the following for all $\varepsilon \in (0,1)$, $t > 0$, and $k \geq 2$:

$$\sup_{x \in \mathbb{R}^d} \|U(t,x)\|_k \leq \left( \frac{2}{\varepsilon} \right) \exp \left\{ \frac{4}{t} \Gamma^{-1} \left( \frac{1-\varepsilon}{4z^2_k} \right) \right\} := c_{t,k},$$

where $z_k$ denotes the optimal constant in the BDG inequality. And $\{U(t,x)\}_{x \in \mathbb{R}^d}$ is a stationary random field for every $t > 0$. Moreover, for almost every $(s, y) \in (0, t) \times \mathbb{R}^d$, there exists a real number $C_{t,k}$ such that

$$\|D_{s,y}U(t,x)\|_k \leq C_{t,k} \mathcal{P}_{h(t-s)/t} \left( y - \frac{s}{t} x \right).$$

The following lemma, which is a generalization of a result of Theorem 6.1.2 in [20], plays an important role in the proofs of our theorems.

Lemma 5. Let $F = (F^{(1)}, \ldots, F^{(m)})$ denote a random vector such that for every $i = 1, \ldots, m$, $F^{(i)} = \delta(v^{(i)})$ for some $v^{(i)} \in \text{Dom}[\delta]$. Assume additionally that $F^{(i)} \in D^{1,2}$
for \( i = 1, \ldots, m \). Let \( G \) be a centered \( m \)-dimensional Gaussian random vector with covariance matrix \((C_{i,j})_{1 \leq i,j \leq m}\). Then, for any \( h \in C^2(\mathbb{R}^m) \) that has bounded partial derivatives, we have

\[
|\mathbb{E}(h(F)) - \mathbb{E}(h(G))| \leq \frac{1}{2} \|h''\|_\infty \sqrt{\sum_{i,j=1}^{m} \mathbb{E}\left( |C_{i,j} - \langle DF^{(i)}, v^{(j)} \rangle_H|^2 \right)},
\]

where

\[
\|h''\|_\infty := \max_{1 \leq i,j \leq m} \sup_{x \in \mathbb{R}^m} \left| \frac{\partial^2 h(x)}{\partial x_i \partial x_j} \right|.
\]

Finally, we recall the following Poincaré-type inequality [3, Section 2.2]:

**Lemma 6.** Suppose that \( F, G \in D^{1,2} \) and \( DF \) and \( DG \) are real-valued random variables, then we have

\[
|\text{Cov}(F,G)| \leq \int_{0}^{\infty} ds \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} f(dy') \|D_{s,y}F\|_2 \|D_{s,y+y'}G\|_2,
\]

(16)

### 3 Asymptotic behavior of the covariance

Recall that the spatial average \( S_{N,t} \) and the quantity \( \mathcal{R}(f) \) are defined respectively in (5) and (7). In this section, we will estimate the asymptotic behavior of the covariance functions of \( S_{N,t} \).

In order to simplify the exposition, for any \( t_1, t_2 > 0 \) we define

\[
\tau := \frac{2t_1 t_2}{t_1 + t_2}, \quad \tau_1 := \frac{2t_2}{t_1 + t_2}, \quad \text{and} \quad \tau_2 := \frac{2t_1}{t_1 + t_2}.
\]

(17)

**Proposition 7.** \((d \geq 2)\). Assume \( \mathcal{R}(f) < \infty \), then for any \( t_1, t_2 > 0 \), we have

\[
\lim_{N \to \infty} \text{Cov}(\sqrt{N}S_{N,t_1}, \sqrt{N}S_{N,t_2}) = g_{t_1,t_2},
\]

(18)

where

\[
g_{t_1,t_2} = \frac{\tau}{(2\pi)^d} \int_{0}^{\infty} ds \int_{\mathbb{R}^d} f(sz) \left[ \left( I_{(\tau_1 \wedge \tau_2)} * \tilde{I}_{(\tau_1 \vee \tau_2)} \right)(\zeta) \right](z) f(dz).
\]

**Proposition 8.** \((d = 1)\). Assume \( f(\mathbb{R}) < \infty \), then for any \( t_1, t_2 > 0 \), we have

\[
\liminf_{N \to \infty} \text{Cov} \left( \sqrt{\frac{N}{\log N}} S_{N,t_1}, \sqrt{\frac{N}{\log N}} S_{N,t_2} \right) \geq (t_1 \wedge t_2)f(\mathbb{R}),
\]

\[
\limsup_{N \to \infty} \text{Cov} \left( \sqrt{\frac{N}{\log N}} S_{N,t_1}, \sqrt{\frac{N}{\log N}} S_{N,t_2} \right) \leq 2(t_1 \wedge t_2)f(\mathbb{R}).
\]

(19)
In particular, if \( \lim_{x \to \infty} f(x) = 0 \), then the limit of the covariance exists, we have that

\[
\lim_{N \to \infty} \text{Cov} \left( \sqrt{\frac{N}{\log N}} S_{N, t_1}, \sqrt{\frac{N}{\log N}} S_{N, t_2} \right) = (t_1 \land t_2) f(R). \tag{20}
\]

**Proposition 9.** (Riesz kernel). Assume \( f(dx) = \|x\|^{-\beta} dx \) and \( f(dx) = \kappa_{\beta, d}\|x\|^{\beta - d} dx \) for \( 0 < \beta < 2 \land d \), where \( \kappa_{\beta, d} \) is a real number depending on \( \beta \) and \( d \), we have

(A) If \( \beta \in (0, 1) \), then

\[
\lim_{N \to \infty} \text{Cov} \left( \sqrt{N^\beta} S_{N, t_1}, \sqrt{N^\beta} S_{N, t_2} \right) = c_{t_1, t_2}^{(1)}, \tag{21}
\]

where

\[
c_{t_1, t_2}^{(1)} = \frac{(t_1 \land t_2)^{1-\beta} \tau^\beta}{1-\beta} \int_{\mathbb{R}^d} \left( I_{(t_1 \land t_2)} * \tilde{I}_{(t_1 \lor t_2)} \right)(z) \|z\|^{-\beta} dz.
\]

(B) If \( \beta = 1 \), then

\[
\lim_{N \to \infty} \text{Cov} \left( \sqrt{\frac{N}{\log N}} S_{N, t_1}, \sqrt{\frac{N}{\log N}} S_{N, t_2} \right) = c_{t_1, t_2}^{(2)}, \tag{22}
\]

where

\[
c_{t_1, t_2}^{(2)} = \frac{2\tau \kappa_{1, d}}{(2\pi)^d} \int_{\mathbb{R}^d} \left( I_{(t_1 \land t_2)} * \tilde{I}_{(t_1 \lor t_2)} \right)(z) \|z\|^{1-d} dz.
\]

(C) If \( \beta \in (1, 2 \land d) \), then

\[
\lim_{N \to \infty} \text{Cov} \left( \sqrt{N^{2-\beta}} S_{N, t_1}, \sqrt{N^{2-\beta}} S_{N, t_2} \right) = c_{t_1, t_2}^{(3)}, \tag{23}
\]

where

\[
c_{t_1, t_2}^{(3)} = \frac{\tau^{2-\beta} \kappa_{d, d}}{(2\pi)^d} \int_{\mathbb{R}^d} \left( I_{(t_1 \land t_2)} * \tilde{I}_{(t_1 \lor t_2)} \right)(z) \|z\|^{2-\beta-d} dz \int_0^\infty r^{\beta-2} e^{-r} dr.
\]

**Remark 4.** Unlike the case that \( u(0) \equiv 1 \), we can’t obtain the asymptotic behavior of the covariance from the asymptotic variance directly due to the complexity of the condition probability density \( p_{s(t-x)/t}(s/x) \). Therefore, we need to prove the above propositions as follows.

For every \( y \in \mathbb{R}^d \), and \( s > 0 \), we first denote

\[
\chi_s(y) := \text{Cov}[U(s, 0), U(s, y)] = E[U(s, 0)U(s, y)] - 1. \tag{24}
\]

By (4), the stationary property of \( \{U(t, x)\}_{x \in \mathbb{R}^d} \) and the semigroup property of the heat kernel,

\[
\text{Cov}(S_{N, t_1}, S_{N, t_2}) = \frac{1}{N^{2d}} \int_{[0, N]^{2d}} dx_1 dx_2 \text{Cov}[U(t_1, x_1) - 1, U(t_2, x_2) - 1] = \frac{1}{N^{2d}} \int_{[0, N]^{2d}} dx_1 dx_2 \int_{(0, t_1 \land t_2) \times \mathbb{R}^{2d}} f(dy') dy ds E[U(s, y)U(s, y + y')]
\]
\[
\times p_s[(t_1 \wedge t_2) - s]/(t_1 \wedge t_2) \left( y - \frac{s}{t_1 \wedge t_2} x_1 \right) p_s[(t_1 \vee t_2) - s]/(t_1 \vee t_2) \left( y + y' - \frac{s}{t_1 \vee t_2} x_2 \right)
\]
\[
= \frac{1}{N^{2d}} \int_{[0, N]^{2d}} dx_1 dx_2 \int_{(0, t_1 \wedge t_2) \times \mathbb{R}^d} f(dy') ds \left( 1 + \chi_s(y') \right)
\times p_s[(t_1 - s)/t_1 + s(t_2 - s)/t_2] \left( y' - \left( \frac{s}{t_1 \wedge t_2} x_2 - \frac{s}{t_1 \vee t_2} x_1 \right) \right).
\]

Suppose now that \( t_1 < t_2 \), then we can write
\[
\text{Cov}(S_{N, t_1}, S_{N, t_2}) = V_{N, t_1, t_2}^{(1)} + V_{N, t_1, t_2}^{(2)},
\]
where
\[
V_{N, t_1, t_2}^{(1)} = \frac{1}{N^{2d}} \int_{[0, N]^{2d}} dx_1 dx_2 \int_0^{t_1} ds \int_{\mathbb{R}^d} f(dy') \ p_{2s(\tau - s)/\tau} \left( y' - \frac{s}{\tau} (\tau_2 x_2 - \tau_1 x_1) \right),
\]
\[
V_{N, t_1, t_2}^{(2)} = \frac{1}{N^{2d}} \int_{[0, N]^{2d}} dx_1 dx_2 \int_0^{t_1} ds \int_{\mathbb{R}^d} f(dy') \times p_{2s(\tau - s)/\tau} \left( y' - \frac{s}{\tau} (\tau_2 x_2 - \tau_1 x_1) \right) \chi_s(y').
\]

Furthermore, by making the change of variables \([x_i \mapsto N x_i/\tau_i, \text{ for } i = 1, 2] \), since \( \chi_t(x) \leq \chi_t(0) \) for all \( t > 0 \) and \( x \in \mathbb{R}^d \) [3, Lemma 5.6], we have
\[
V_{N, t_1, t_2}^{(1)} = \int_{\mathbb{R}^d} dz \left( \tilde{I}_{\tau_2} * \tilde{I}_{\tau_1} \right) (z) \int_0^{t_1} ds \left( p_{2s(\tau - s)/\tau} \ast f \right) \left( \frac{sN}{\tau} z \right),
\]
\[
V_{N, t_1, t_2}^{(2)} \leq \int_{\mathbb{R}^d} dz \left( \tilde{I}_{\tau_2} * \tilde{I}_{\tau_1} \right) (z) \int_0^{t_1} ds \chi_s(0) \left( p_{2s(\tau - s)/\tau} \ast f \right) \left( \frac{sN}{\tau} z \right),
\]
where \( \tau, \tau_1, \tau_2 \) are defined in (17). Next, we define
\[
\psi_{\tau_1, \tau_2}(z) := \left( \tilde{I}_{\tau_2} * \tilde{I}_{\tau_1} \right) (z) = \widehat{\left( \tilde{I}_{\tau_2} \ast \tilde{I}_{\tau_1} \right)} (z) \text{ for all } z \in \mathbb{R}^d.
\]
From Lemma A.1 (1), we have
\[
|\psi_{\tau_1, \tau_2}(z)| \leq 1 \wedge \|z\|^{-2},
\]
which we use several times in computing integrals. Following the argument for \( V_{N, t_1, t_2}^{(2)} \) in Section 5 of [3], we can see that the main asymptotic behavior of the covariance is \( V_{N, t_1, t_2}^{(1)} \) by replacing \( t_1 \) with \( \tau \) in (28) and using the inequality (30). Therefore, it remains to consider the behavior of \( V_{N, t_1, t_2}^{(1)} \) only.
3.1 Analysis in $d \geq 2$

Proof of Proposition 7. By Fubini’s theorem and a change of variables, we can see that

$$V^{(1)}_{N,t_1,t_2} = \frac{\tau}{N} \int_0^{t_1/N} ds \int_{\mathbb{R}^d} \left( I_{t_2} \ast \widetilde{I}_{\tau_1} \right) (z) \left( p_{\frac{2\pi}{N}(1 - s)} \ast f \right) (sz)$$

$$= \frac{\tau}{N} \int_0^{t_1/N} \frac{ds}{s^d} \int_{\mathbb{R}^d} \left( I_{t_2} \ast \widetilde{I}_{\tau_1} \right) \left( \frac{z}{s} \right) \left( p_{\frac{2\pi}{N}(1 - s)} \ast f \right) (z)$$

$$= \frac{\tau}{(2\pi)^d N} \int_0^{t_1/N} \frac{ds}{s^d} \int_{\mathbb{R}^d} f(dz) \left[ \left( I_{t_2} \ast \widetilde{I}_{\tau_1} \right) \left( \frac{z}{s} \right) \right] (z) \exp \left[ -\frac{s\tau}{N} \left( 1 - \frac{s}{N} \right) \right].$$

Then we apply Lemma A.1 (2) and the dominated convergence theorem to find that

$$\lim_{N \to \infty} NV^{(1)}_{N,t_1,t_2} = \frac{\tau}{2\pi} \int_0^{\infty} ds \int_{\mathbb{R}^d} \hat{f}(dz) \left[ \left( I_{t_2} \ast \widetilde{I}_{\tau_1} \right) \left( \frac{z}{s} \right) \right] (z).$$

This implies (18). □

3.2 Analysis in $d = 1$

Lemma 10. For all $t_1, t_2 > 0$,

$$(t_1 \land t_2)f(\mathbb{R}) \leq \liminf_{N \to \infty} \frac{N}{\log N} V^{(1)}_{N,t_1,t_2} \leq \limsup_{N \to \infty} \frac{N}{\log N} V^{(1)}_{N,t_1,t_2} \leq 2(t_1 \land t_2)f(\mathbb{R}). \ (31)$$

Proof. Recalling (27), we can write

$$V^{(1)}_{N,t_1,t_2} = \frac{\tau}{2\pi N} \int_0^{t_1} \frac{ds}{s} \int_{-\infty}^{\infty} dz \psi_{\tau_1,\tau_2}(z) \exp \left( \frac{-\tau(z - s)^2}{N^2 s} \right) \hat{f} \left( \frac{\tau z}{N^2} \right)$$

$$\leq \frac{\tau}{2\pi N} \int_0^{t_1} \frac{ds}{s} \int_{-\infty}^{\infty} dz \psi_{\tau_1,\tau_2}(z) \exp \left( \frac{-\tau(z - s)^2}{N^2 s} \right) \hat{f} \left( \frac{\tau z}{N^2} \right).$$

Then, by using Lemma A.3, the dominated convergence theorem and

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \psi_{\tau_1,\tau_2}(z) dz = 1/\tau_1,$$

we obtain the third inequality in (31).

Next, we prove the first inequality. With a change of variable $s = t_1 r N^{-2}$, the term $V^{(1)}_{N,t_1,t_2}$ can be recast as

$$V^{(1)}_{N,t_1,t_2} = \frac{\tau}{2\pi N} \int_0^{N^2} \frac{dr}{r} \int_{-\infty}^{\infty} dz \psi_{\tau_1,\tau_2}(z) \exp \left( -\tau z^2 \left( \frac{1 - \frac{r}{\tau_1 N^2}}{r} \right) \right) \hat{f} \left( \frac{N \tau_1 z}{r} \right)$$

$$= \frac{\tau}{2\pi N} (T_1 + T_2).$$
where

\[ T_1 = \int_0^1 \frac{dr}{r} \int_{-\infty}^{\infty} dz \, \psi_{\tau_1, \tau_2}(z) \exp \left[ -\tau z^2 \left( \frac{1 - \frac{r}{\tau_1 \tau_2}}{r/\tau_1} \right) \right] \hat{f} \left( N \tau_1 \frac{z}{r} \right), \tag{32} \]

\[ T_2 = \int_1^{N^2} \frac{dr}{r} \int_{-\infty}^{\infty} dz \, \psi_{\tau_1, \tau_2}(z) \exp \left[ -\tau z^2 \left( \frac{1 - \frac{r}{\tau_1 \tau_2}}{r/\tau_1} \right) \right] \hat{f} \left( N \tau_1 \frac{z}{r} \right). \tag{33} \]

We can see that, for all \( N \geq 1 \)

\[ \int_0^1 \frac{dr}{r} \exp \left[ -\tau \tau_1 z^2 \left( \frac{1 - \frac{r}{\tau_1 \tau_2}}{r} \right) \right] \leq \log_{+} \left( \frac{e}{\tau \tau_1 z^2} \right), \]

where \( \log_{+}(x) = \log(e + x) \). Hence,

\[ T_1 \leq f(\mathbb{R}) \int_{-\infty}^{\infty} |\psi_{\tau_1, \tau_2}(z)| \log_{+} \left( \frac{e}{\tau \tau_1 z^2} \right) dz < \infty, \]

which implies that

\[ \limsup_{N \to \infty} N \frac{\tau}{\log N 2\pi N} T_1 = 0. \tag{34} \]

Then, it remains to prove

\[ \liminf_{N \to \infty} N \frac{\tau}{\log N 2\pi N} T_2 \geq t_1 f(\mathbb{R}). \]

We make the change of variable \( s = \tau N^{-2} \) to see that

\[ T_2 \geq \int_{-\infty}^{\infty} dz \int_1^{1/N} \frac{ds}{s} \psi_{\tau_1, \tau_2}(z) \exp \left[ -\tau z^2 \left( \frac{1 - \frac{s}{\tau_1 \tau_2}}{s/\tau_1} \right) \right] \hat{f} \left( \frac{\tau_1 z}{\tau N} \right), \tag{35} \]

\[ = T_{2,1} + T_{2,2}, \tag{36} \]

where

\[ T_{2,1} = \int_{-\log N}^{\log N} dz \int_1^{1/N} \frac{ds}{s} \psi_{\tau_1, \tau_2}(z) \exp \left[ -\tau z^2 \left( \frac{1 - \frac{s}{\tau_1 \tau_2}}{s/\tau_1} \right) \right] \hat{f} \left( \frac{\tau_1 z}{\tau N} \right), \]

\[ T_{2,2} = \int_{|z| > \log N} dz \int_1^{1/N} \frac{ds}{s} \psi_{\tau_1, \tau_2}(z) \exp \left[ -\tau z^2 \left( \frac{1 - \frac{s}{\tau_1 \tau_2}}{s/\tau_1} \right) \right] \hat{f} \left( \frac{\tau_1 z}{\tau N} \right). \]

Because

\[ |T_{2,2}| \leq f(\mathbb{R}) \log N \int_{|z| > \log N} |\psi_{\tau_1, \tau_2}(z)| dz = o(\log N), \tag{37} \]
we focus on the behavior of $T_{2.1}$. First, we can write

$$\psi_{\tau_1,\tau_2} = \text{Re}\{\psi_{\tau_1,\tau_2}\} + i \text{Im}\{\psi_{\tau_1,\tau_2}\},$$

where $\text{Re}\{\psi_{\tau_1,\tau_2}\}$ and $\text{Im}\{\psi_{\tau_1,\tau_2}\}$ denote the real and imaginary parts of the function $\psi_{\tau_1,\tau_2}$ respectively. Moreover, we decompose $\text{Re}\{\psi_{\tau_1,\tau_2}\}$ into two parts such that

$$\text{Re}\{\psi_{\tau_1,\tau_2}(z)\} = \psi_{\tau_1,\tau_2}^+(z) - \psi_{\tau_1,\tau_2}^-(z),$$

(38)

where $\psi_{\tau_1,\tau_2}^+$ and $\psi_{\tau_1,\tau_2}^-$ denote respectively the positive and negative parts of the function $\psi_{\tau_1,\tau_2}$. A few lines of computation show that $\text{Im}\{\psi_{\tau_1,\tau_2}\}$ is an odd function, then by making a change of variable $r = sN$, we have

$$T_{2.1} = \int_{-\log N}^{\log N} dz \int_1^N \frac{dr}{r} \psi_{\tau_1,\tau_2}(z) \exp \left[ -\frac{\tau z^2}{N} \left( \frac{\tau_1}{r} - \frac{1}{N} \right) \right] \hat{f} \left( \frac{\tau_1 z}{r} \right),$$

$$= T_{2.1,1} - T_{2.1,2},$$

where

$$T_{2.1,1} = \int_{-\log N}^{\log N} dz \int_1^N \frac{dr}{r} \psi_{\tau_1,\tau_2}^+(z) \exp \left[ -\frac{\tau z^2}{N} \left( \frac{\tau_1}{r} - \frac{1}{N} \right) \right] \hat{f} \left( \frac{\tau_1 z}{r} \right),$$

$$T_{2.1,2} = \int_{-\log N}^{\log N} dz \int_1^N \frac{dr}{r} \psi_{\tau_1,\tau_2}^-(z) \exp \left[ -\frac{\tau z^2}{N} \left( \frac{\tau_1}{r} - \frac{1}{N} \right) \right] \hat{f} \left( \frac{\tau_1 z}{r} \right).$$

Notice that

$$T_{2.1,1} \geq \exp \left[ -\frac{\tau_1}{N} (\log N)^2 \right] \int_{-\log N}^{\log N} dz \int_1^N \frac{dr}{r} \psi_{\tau_1,\tau_2}^+(z) \hat{f} \left( \frac{\tau_1 z}{r} \right).$$

Hence,

$$T_{2.1,1} \geq (1 + o(1)) \int_{-\log N}^{\log N} dz \int_1^N \frac{dr}{r} \psi_{\tau_1,\tau_2}^+(z) \hat{f} \left( \frac{\tau_1 z}{r} \right) = (f(\mathbb{R}) + o(1)) \int_{-\log N}^{\log N} dz \int_1^N \frac{dr}{r} \psi_{\tau_1,\tau_2}^+(z)$$

$$= (f(\mathbb{R}) + o(1)) \log N \int_{-\infty}^{\infty} dz \psi_{\tau_1,\tau_2}^+(z).$$

Recall the definition of $T_{2.1,2}$. We deduce that

$$T_{2.1,2} \leq f(\mathbb{R}) \int_{-\infty}^{\infty} \psi_{\tau_1,\tau_2}^-(z) dz \int_1^N \frac{dr}{r} = f(\mathbb{R}) \log N \int_{-\infty}^{\infty} \psi_{\tau_1,\tau_2}^-(z) dz.$$
Therefore,
\[ T_{2,1} \geq f(R) \log N \int_{-\infty}^{\infty} \psi_{r_1,r_2}(z) dz + o(\log N) = \frac{2\pi f(R)}{r_2} \log N + o(\log N), \quad (39) \]

where we apply the Parseval's identity in the equality. Combining (34)-(39), we obtain the first inequality in (31).

**Proof of Proposition 8.** According to Lemma 10, it remains to prove the case \( \lim_{x \to \infty} f(x) = 0 \) by showing that
\[ \limsup_{N \to \infty} \frac{N}{\log N} \frac{\tau}{2\pi N} T_2 \leq t_1 f(R). \]

Recall the definition of \( T_2 \) from (33) that
\[
T_2 = \int_{-\infty}^{\infty} dz \psi_{r_1,r_2}(z) \int_{1/N^2}^{1} ds \ \exp \left[-\tau z^2 \left(1 - \frac{s}{sN^2/r_1}\right)\right] f \left(\frac{\tau_1 z}{sN}\right)
\]
\[
= T_{2,1} + T_{2,2},
\]

where
\[
T_{2,1} = \int_{-\log N}^{\log N} dz \psi_{r_1,r_2}(z) \int_{1/N^2}^{1} ds \ \exp \left[-\tau z^2 \left(1 - \frac{s}{sN^2/r_1}\right)\right] f \left(\frac{\tau_1 z}{sN}\right)
\]
\[
T_{2,2} = \int_{|z| > \log N} dz \psi_{r_1,r_2}(z) \int_{1/N^2}^{1} ds \ \exp \left[-\tau z^2 \left(1 - \frac{s}{sN^2/r_1}\right)\right] f \left(\frac{\tau_1 z}{sN}\right).
\]

Using the same computation as the proof of (37), it is possible to prove that \( T_{2,2} = o(\log N) \). Then we move on to study the behavior of \( T_{2,1} \). By a change of variable \( r = sN \),
\[
T_{2,1}^{r} = \int_{-\log N}^{\log N} dz \psi_{r_1,r_2}(z) \int_{1/N^2}^{(\log N)^2} dr \ \exp \left[-\tau z^2 \frac{1}{N^2} \left(1 - \frac{1}{r}\right)\right] f \left(\frac{\tau_1 z}{r}\right)
\]
\[
+ \int_{-\log N}^{\log N} dz \psi_{r_1,r_2}(z) \int_{(\log N)^2}^{N} dr \ \exp \left[-\tau z^2 \frac{1}{N^2} \left(1 - \frac{1}{r}\right)\right] f \left(\frac{\tau_1 z}{r}\right)
\]
\[
:= T_{2,1,1}^{r} + T_{2,1,2}^{r}.
\]

According to the proof of Theorem 5.2, item 2 in [3], since \( \lim_{x \to \infty} \hat{f}(x) = 0 \) and \( \int_{-\infty}^{\infty} |\psi_{r_1,r_2}(z)|dz < \infty \), it is easy to see that \( T_{2,1,1}^{r} = o(\log N) \). Recall the function \( \psi_{r_1,r_2}^{+} \) and \( \psi_{r_1,r_2}^{-} \) defined in (38). In order to estimate \( T_{2,1,2}^{r} \), we rewrite one more time to find that
\[
T_{2,1,2}^{r} = \int_{-\log N}^{\log N} dz \psi_{r_1,r_2}^{+}(z) \int_{(\log N)^2}^{N} dr \ \exp \left[-\tau z^2 \frac{1}{N^2} \left(1 - \frac{1}{r}\right)\right] f \left(\frac{\tau_1 z}{r}\right)
\]
This proves Proposition 8.

**Remark 5.** Comparing with the proof of asymptotic variance [3], one of the difficulties of our estimation is that \( \psi_{t_1, t_2} \) may not be a real function and the real part of the function \( \psi_{t_1, t_2} \) might be negative unless \( t_1 = t_2 \). So we use highly technical computations to obtain the results. For example, the values of the expressions can be reduced or enlarged by adjusting the upper and lower limits of integration with respect to time variable, rather than the space variable; see (37). And we need to decompose the function \( \psi_{t_1, t_2} \) in two parts; see (38).

### 3.3 Analysis of Riesz kernel case

**Proof of Proposition 9.** **Case 1:** \( \beta \in (0, 1) \). Recall (25). Since \( f(dx) = \|x\|^{-\beta} \) we can write

\[
V_{N,t_1,t_2}^{(1)} = \int_{\mathbb{R}^d} dz \left( I_{t_2} * I_{t_1} \right) (z) \int_0^{t_1} ds E \left( \left\| \frac{2s(\tau - s)}{\tau} Z + \frac{Ns}{\tau} \right\|^{-\beta} \right)
\]

Then applying a similar argument in proving (39), which we skip, we conclude that

\[
\limsup_{N \to \infty} \frac{N \tau}{\log N} T_{2,1,2} = t_1 f(\mathbb{R}).
\]

This proves Proposition 8.

\[
\text{Remark 5. Comparing with the proof of asymptotic variance [3], one of the difficulties of our estimation is that } \psi_{t_1, t_2} \text{ may not be a real function and the real part of the function } \psi_{t_1, t_2} \text{ might be negative unless } t_1 = t_2. \text{ So we use highly technical computations to obtain the results. For example, the values of the expressions can be reduced or enlarged by adjusting the upper and lower limits of integration with respect to time variable, rather than the space variable; see (37). And we need to decompose the function } \psi_{t_1, t_2} \text{ in two parts; see (38).}
\]

**Proof of Proposition 9.** **Case 1:** \( \beta \in (0, 1) \). Recall (25). Since \( f(dx) = \|x\|^{-\beta} \) we can write

\[
V_{N,t_1,t_2}^{(1)} = \int_{\mathbb{R}^d} dz \left( I_{t_2} * I_{t_1} \right) (z) \int_0^{t_1} ds E \left( \left\| \frac{2s(\tau - s)}{\tau} Z + \frac{Ns}{\tau} \right\|^{-\beta} \right)
\]

where \( Z \) is a \( d \)-dimensional standard normal random variable. Since

\[
\int_{\mathbb{R}^d} \|z\|^{-\beta} \left( I_{t_2} * I_{t_1} \right) (z) \, dz < \infty,
\]

by using Lemma 3.1 in [8] and the dominated convergence theorem, we have

\[
\lim_{N \to \infty} N^\beta V_{N,t_1,t_2}^{(1)} = \int_{\mathbb{R}^d} dz \left( I_{t_2} * I_{t_1} \right) (z) \int_0^{t_1} ds \, \tau^\beta s^{-\beta} \|z\|^{-\beta}.
\]

Next, recall (27). Before deducing the last two parts, we rewrite the expression \( V_{N,t_1,t_2}^{(1)} \) to see that

\[
V_{N,t_1,t_2}^{(1)} = \frac{\kappa_d \beta d}{(2\pi)^d} \int_0^{t_1} ds \int_{\mathbb{R}^d} dz \psi_{t_1,t_2}(z) \left( \frac{\tau}{sN} \right)^d \exp \left( -\frac{\tau(\tau - s)}{Ns^2} \|z\|^2 \right) \frac{\tau}{sN}^\beta - d
\]

\[
= \frac{\kappa_d \beta d}{(2\pi)^d N^\beta} \int_0^{t_1} \left( \frac{\tau}{s} \right)^d \, ds \int_{\mathbb{R}^d} dz \psi_{t_1,t_2}(z) \exp \left( -\frac{\tau(\tau - s)}{Ns^2} \|z\|^2 \right) \|z\|^{-\beta - d}
\]

\[
= \frac{\tau \kappa_d \beta d}{(2\pi)^d N^\beta} \int_{\mathbb{R}^d} dz \|z\|^{-\beta - d} \psi_{t_1,t_2}(z) \int_{t_1}^{\infty} dr \, (1 + r)^{\beta - 2} \exp \left( -\frac{\tau r}{N^2} \|z\|^2 \right)
\]

\[
\text{Case 1: } \beta \in (0, 1)
\]

\[
\limsup_{N \to \infty} \frac{N \tau}{\log N} T_{2,1,2} = t_1 f(\mathbb{R}).
\]
and the dominated convergence theorem, the following:

\[
\frac{N}{\log N} V^{(1)}_{N,t_1,t_2} = \frac{\kappa_{1,d}}{(2\pi)^d \log N} \int_{\mathbb{R}^d} ds \int_{\mathbb{R}^d} dz \psi_{r_1,r_2}(z) \exp \left( -\frac{\tau(r-s)}{N^2 \kappa_{s}} \right) \left| z \right|^{1-d} \\
= \frac{\tau \kappa_{1,d}}{(2\pi)^d \log N} \int_{t_1}^{t_2} \frac{ds}{s} \int_{\mathbb{R}^d} dz \psi_{r_1,r_2}(z) \exp \left( -\frac{\tau(r-s)}{N^2 \kappa_{s}} \right) \left| z \right|^{1-d} \\
- \frac{\tau \kappa_{1,d}}{(2\pi)^d \log N} \int_{t_1}^{t_2} \frac{ds}{s} \int_{\mathbb{R}^d} dz \psi_{r_1,r_2}(z) \exp \left( -\frac{\tau(r-s)}{N^2 \kappa_{s}} \right) \left| z \right|^{1-d} \\
:= \frac{1}{\log N} (A_1 - A_2).
\]

According to Lemma A.3 and the dominated convergence theorem, the following:

\[
\lim_{N \to \infty} \frac{A_1}{\log N} = \frac{2\tau \kappa_{1,d}}{(2\pi)^d} \int_{\mathbb{R}^d} \psi_{r_1,r_2}(z) \left| z \right|^{1-d} dz
\]

holds due to \(\int_{\mathbb{R}^d} \left| z \right|^{1-d} |\psi_{r_1,r_2}(z)| \log_+ (1/|z|) dz < \infty\). Moreover, we observe that

\[
0 \leq A_2 \leq \frac{\tau \kappa_{1,d}}{(2\pi)^d} \log \left( \frac{\tau}{t_1} \right) \int_{\mathbb{R}^d} |\psi_{r_1,r_2}(z)| \left| z \right|^{1-d} dz = o(\log N),
\]

which together with (43) proves (22).

Case 2: \(\beta = 1\). We apply (41) with \(\beta = 1\) to see that

\[
\frac{N}{\log N} V^{(1)}_{N,t_1,t_2} = \frac{\kappa_{1,d}}{(2\pi)^d \log N} \int_{\mathbb{R}^d} ds \int_{\mathbb{R}^d} dz \psi_{r_1,r_2}(z) \exp \left( -\frac{\tau(r-s)}{N^2 \kappa_{s}} \right) \left| z \right|^{1-d} \\
= \frac{\tau \kappa_{1,d}}{(2\pi)^d \log N} \int_{t_1}^{t_2} \frac{ds}{s} \int_{\mathbb{R}^d} dz \psi_{r_1,r_2}(z) \exp \left( -\frac{\tau(r-s)}{N^2 \kappa_{s}} \right) \left| z \right|^{1-d} \\
- \frac{\tau \kappa_{1,d}}{(2\pi)^d \log N} \int_{t_1}^{t_2} \frac{ds}{s} \int_{\mathbb{R}^d} dz \psi_{r_1,r_2}(z) \exp \left( -\frac{\tau(r-s)}{N^2 \kappa_{s}} \right) \left| z \right|^{1-d} \\
:= \frac{1}{\log N} (A_1 - A_2).
\]

\[
\lim_{N \to \infty} \frac{A_1}{\log N} = \frac{2\tau \kappa_{1,d}}{(2\pi)^d} \int_{\mathbb{R}^d} \psi_{r_1,r_2}(z) \left| z \right|^{1-d} dz
\]

holds due to \(\int_{\mathbb{R}^d} \left| z \right|^{1-d} |\psi_{r_1,r_2}(z)| \log_+ (1/|z|) dz < \infty\). Moreover, we observe that

\[
0 \leq A_2 \leq \frac{\tau \kappa_{1,d}}{(2\pi)^d} \log \left( \frac{\tau}{t_1} \right) \int_{\mathbb{R}^d} |\psi_{r_1,r_2}(z)| \left| z \right|^{1-d} dz = o(\log N),
\]

which together with (43) proves (22).

Case 3: \(\beta \in (1,2 \wedge d)\). Recall (42). Since \(\int_{\mathbb{R}^d} \left| z \right|^{2-\beta} |\psi_{r_1,r_2}(z)| dz < \infty\) and \(\int_0^\infty r^{\beta-2} e^{-r} dr < \infty\), the dominated convergence theorem implies that

\[
\lim_{N \to \infty} N^{2-\beta} V^{(1)}_{N,t_1,t_2} = \frac{\tau^{2-\beta} \kappa_{1,d}}{2\pi^d} \int_{\mathbb{R}^d} \psi_{r_1,r_2}(z) \left| z \right|^{2-\beta} dz \int_0^\infty r^{\beta-2} e^{-r} dr.
\]

The proof is completed.

\[\square\]

4 Tightness

We will show the upper bound and tightness of the spatial averages in this section. To simplify the expressions, let \(L\) denote a real number which may take different values and depend on different variables, this notation will also be used in Chapter 5.
4.1 Uniform upper bound

In this subsection, we prove quantitative upper bound of the spatial averages in $L^k(\Omega)$. Apparently, the rates of convergence are coincident with the rates that were ensured by Propositions 7-9.

**Lemma 11.** $(d \geq 2)$. Suppose $R(f) < \infty$, then for any $k \geq 2$ and $T \geq 0$,

$$\sup_{N \geq e} \left\| \sqrt{N} S_{N,t} \right\|_k \leq L_{T,k} t^{1/2} \text{ uniformly for all } t \in (0,T).$$

**Lemma 12.** $(d = 1)$. Suppose $f(\mathbb{R}) < \infty$, then for any $k \geq 2$ and $T \geq 0$,

$$\sup_{N \geq e} \left\| \sqrt{\log N} S_{N,t} \right\|_k \leq L_{T,k} t^{1/2} \text{ uniformly for all } t \in (0,T).$$

**Lemma 13.** (Riesz kernel). Suppose $f(dx) = \|x\|^{-\beta}$, where $0 < \beta < 2 \wedge d$, then for any $k \geq 2$ and $T \geq 0$, we have

(A) If $\beta \in (0,1)$, then

$$\sup_{N \geq e} \left\| \sqrt{N^\beta} S_{N,t} \right\|_k \leq L_{T,k,\beta} t^{1/2} \text{ uniformly for all } t \in (0,T),$$

(B) If $\beta = 1$, then

$$\sup_{N \geq e} \left\| \sqrt{\log N} S_{N,t} \right\|_k \leq L_{T,k} t^{1/2} \text{ uniformly for all } t \in (0,T),$$

(C) If $\beta \in (1, 2 \wedge d)$, then

$$\sup_{N \geq e} \left\| \sqrt{N^{2-\beta}} S_{N,t} \right\|_k \leq L_{T,k,\beta} t^{(2-\beta)/2} \text{ uniformly for all } t \in (0,T).$$

**Proof of Lemmas 11-13.** Denote $V_{N,t}^{(1)} := V_{N,t,t}^{(1)}$, and define

$$g_{N,t}(s,y) := \frac{1}{N^d} \int_{[0,N]^d} p_{s(t-s)/t} \left( y - \frac{s}{t}x \right) \, dx.$$  

Recall (4) and (5). Thanks to stochastic Fubini’s theorem and the BDG inequality,

$$\|S_{N,t}\|_k^2 = \left\| \int_{(0,t) \times \mathbb{R}^d} \left( \frac{1}{N^d} \int_{[0,N]^d} p_{s(t-s)/t} \left( y - \frac{s}{t}x \right) \, dx \right) U(s,y) \eta(ds,dy) \right\|_k^2 \leq L_k \int_{(0,t) \times \mathbb{R}^d} dy'ds \ g_{N,t}(s,y) g_{N,t}(s,y + y') \|U(s,y)U(s,y + y')\|_{k/2}.$$ 

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Then, we use the moment inequality (14) and the semigroup property of the heat kernel to find that
\[
\|S_{N,t}\|_2^2 \leq L_{T,k} \|V_{N,t}^{(1)}\|.
\]
First, we analyze the case that \( d \geq 2 \). According to the proof of proposition 7, we observe that
\[
\|S_{N,t}\|_2^2 \leq L_{T,k} t N \int_0^\infty ds \int_{\mathbb{R}^d} f(dz) \left| \left( I_k * I_k \right) \left( \frac{z}{s} \right) \right|(z) = L_{T,k} t N.
\]
This proves Lemma 11, and the rest Lemma 12 and Lemma 13 can be proved by using similar arguments, we skip the details. \( \square \)

4.2 Moment estimate

In this subsection, we establish tightness by the following moment estimates.

**Proposition 14.** \((d \geq 2)\). Assume \( R(f) < \infty \), then for any \( T > 0 \), \( k \geq 2 \), and \( \gamma \in (0, 1/8) \), there exists a real number \( L = L_{T,k,\gamma} > 0 \) such that for all \( \varepsilon \in (0, 1) \),
\[
\sup_{t \in (0, T)} E \left( |S_{N,t+\varepsilon} - S_{N,t}|^k \right) \leq L \varepsilon^{\gamma k} N^{-k/2} \text{ uniformly for all } N \geq e. \quad (50)
\]

**Proposition 15.** \((d = 1)\). Assume \( f(\mathbb{R}) < \infty \), then for any \( T > 0 \), \( k \geq 2 \), and \( \gamma \in (0, 1/4) \), there exists a real number \( L = L_{T,k,\gamma} > 0 \) such that for all \( \varepsilon \in (0, 1) \),
\[
\sup_{t \in (0, T)} E \left( |S_{N,t+\varepsilon} - S_{N,t}|^k \right) \leq L \varepsilon^{\gamma k} \left( \frac{N}{\log N} \right)^{-k/2} \text{ uniformly for all } N \geq e. \quad (51)
\]

**Proposition 16.** (Riesz kernel). Assume \( f(dx) = \|x\|^{-\beta} dx \), \( \beta \in (0, 2 \wedge d) \), for any \( T > 0 \), \( k \geq 2 \) and \( \varepsilon \in (0, 1) \) we have

(A) If \( \beta \in (0, 1) \), then for any \( \gamma \in (0, 1/4] \), there exists \( L = L_{T,k,\gamma,\beta} > 0 \) such that
\[
\sup_{t \in (0, T)} E \left( |S_{N,t+\varepsilon} - S_{N,t}|^k \right) \leq L \varepsilon^{\gamma k} (N)^{-\beta k/2} \text{ uniformly for all } N \geq e. \quad (52)
\]

(B) If \( \beta = 1 \), then for any \( \gamma \in (0, 1/4) \), there exists \( L = L_{T,k,\gamma} > 0 \) such that
\[
\sup_{t \in (0, T)} E \left( |S_{N,t+\varepsilon} - S_{N,t}|^k \right) \leq L \varepsilon^{\gamma k} \left( \frac{N}{\log N} \right)^{-k/2} \text{ uniformly for all } N \geq e. \quad (53)
\]
(C) If $\beta \in (1, 2 \wedge d)$, then for any $\gamma \in (0, (2 - \beta)/(6 - 2\beta)]$, there exists $L = L_{T,k,\gamma,\beta} > 0$ such that

$$
\sup_{t \in (0,T)} \mathbb{E} \left( |S_{N,t+\varepsilon} - S_{N,t}|^k \right) \leq L \varepsilon^k (N)^{-2(2-\beta)k/2} \quad \text{uniformly for all } N \geq e. \quad (54)
$$

The proofs of Propositions 14-16 hinge on the following lemmas, which will be used to analyze the behavior when $t$ stays away from zero.

**Lemma 17.** $(d \geq 2)$. Suppose $R(f) < \infty$, then for any $T > 0$, $k \geq 2$, and $\alpha \in (0, 1/4)$, there exists a real number $L = L_{T,k,\alpha} > 0$ such that for all $\varepsilon \in (0, 1)$, $t \in (0, T)$ and $N \geq e$,

$$
\mathbb{E} \left( |S_{N,t+\varepsilon} - S_{N,t}|^k \right) \leq \frac{L \varepsilon^k}{(t \wedge 1)^{k/2}} N^{-k/2}. \quad (55)
$$

**Lemma 18.** Suppose $f(\mathbb{R}) < \infty$, then for any $T > 0$, $k \geq 2$, and $\delta > 0$, there exists a real number $L = L_{T,k,\delta} > 0$ such that for all $\varepsilon \in (0, 1)$, $t \in (0, T)$ and $N \geq e$,

$$
\mathbb{E} \left( |S_{N,t+\varepsilon} - S_{N,t}|^k \right) \leq \frac{L \varepsilon^{k/2}}{(t \wedge 1)^{(1+\delta)/2}} \left( \frac{N}{\log N} \right)^{-k/2}. \quad (56)
$$

**Lemma 19.** (Riesz kernel). Suppose $f(dx) = ||x||^{-\beta}dx$, $\beta \in (0, 2 \wedge d)$, for any $T > 0$, $k \geq 2$ and $\varepsilon \in (0, 1)$ we have

(A) If $\beta \in (0, 1)$, then there exists a real number $L = L_{T,k,\beta} > 0$ such that for all $t \in (0, T)$ and $N \geq e$,

$$
\mathbb{E} \left( |S_{N,t+\varepsilon} - S_{N,t}|^k \right) \leq \frac{L \varepsilon^{k/2}}{(t \wedge 1)^{k/2}} N^{-k\beta k/2}. \quad (57)
$$

(B) If $\beta = 1$, then for any $\delta > 0$, there exists a real number $L = L_{T,k,\delta} > 0$ such that for all $t \in (0, T)$ and $N \geq e$,

$$
\mathbb{E} \left( |S_{N,t+\varepsilon} - S_{N,t}|^k \right) \leq \frac{L \varepsilon^{k/2}}{(t \wedge 1)^{(1+\delta)/2}} \left( \frac{N}{\log N} \right)^{-k/2}. \quad (58)
$$

(C) If $\beta \in (1, 2 \wedge d)$, then there exists a real number $L = L_{T,k,\beta} > 0$ such that for all $t \in (0, T)$ and $N \geq e$,

$$
\mathbb{E} \left( |S_{N,t+\varepsilon} - S_{N,t}|^k \right) \leq \frac{L \varepsilon^{k/2}}{(t \wedge 1)^{k/2}} N^{-(2-\beta)k/2}. \quad (59)
$$

We now aim to prove Lemmas 17-19. Recall from (4) and (5) that

$$
S_{N,t+\varepsilon} - S_{N,t} = \frac{1}{Nd} \int_{[0,N]^d} [U(t+\varepsilon,x) - U(t,x)]dx
$$

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\[
= \int_{(0,t) \times \mathbb{R}^d} U(s, y) A(s, y) \eta(ds, dy) + \int_{(t, t+\varepsilon) \times \mathbb{R}^d} U(s, y) B(s, y) \eta(ds, dy),
\]
where
\[
A(s, y) = \frac{1}{N^d} \int_{[0,N]^d} \left[ p_{s(t+\varepsilon-s)/t+\varepsilon} \left( y - \frac{sx}{t+\varepsilon} \right) - p_{s(t-s)/t} \left( y - \frac{sx}{t} \right) \right] dx,
\]
\[
B(s, y) = \frac{1}{N^d} \int_{[0,N]^d} p_{s(t+\varepsilon-s)/(t+\varepsilon)} \left( y - \frac{sx}{t+\varepsilon} \right) dx.
\]
Thus,
\[
\| S_{N,t+\varepsilon} - S_{N,t} \|_k \leq T_A + T_B,
\]
where
\[
T_A = \left\| \int_{(0,t) \times \mathbb{R}^d} U(s, y) A(s, y) \eta(ds, dy) \right\|_k
\]
and
\[
T_B = \left\| \int_{(t, t+\varepsilon) \times \mathbb{R}^d} U(s, y) B(s, y) \eta(ds, dy) \right\|_k.
\]
Owing to the BDG inequality, (14) and Parseval’s identity,
\[
T_A^2 \leq \frac{L_{T,k}}{(2\pi)^d} \int_0^t ds \int_{\mathbb{R}^d} \hat{f}(d\xi) \left| \widehat{A(s)}(\xi) \right|^2,
\]
\[
T_B^2 \leq L_{T,k} \int_{(t, t+\varepsilon) \times \mathbb{R}^{2d}} dy f(dy')ds \ B(s, y) B(s, y + y').
\]

**Proof of Lemma 17.** We will estimate \( T_A \) and \( T_B \) separately. First, by using a change of variable \([s \mapsto (t+\varepsilon)s/N]\) we can see that
\[
T_A^2 \leq \frac{L_{T,k}(t+\varepsilon)}{N} \int_0^{tN/(t+\varepsilon)} ds \int_{\mathbb{R}^d} \hat{f}(d\xi) \left| A \left( \frac{(t+\varepsilon)s}{N} \right) \right|^2.
\]
Notice that
\[
A \left( \frac{(t+\varepsilon)s}{N} \right) = \frac{1}{N^d} \int_{[0,N]^d} \left[ \exp \left( \frac{isx \cdot \xi}{N} - \frac{s(t+\varepsilon)(N-s)\|\xi\|^2}{2N^2} \right) - \exp \left( \frac{i(t+\varepsilon)sx \cdot \xi}{Nt} - \frac{s(t+\varepsilon)(Nt-(t+\varepsilon)s)\|\xi\|^2}{2N^2t} \right) \right] dx
= J_1 + J_2.
\]
where
\[ J_1 = \int_{[0,1]^d} e^{i\alpha x} \xi \, dx \left[ \exp \left( -\frac{\varepsilon (t + \varepsilon)x^2 \|\xi\|^2}{2N^2t} \right) - 1 \right] \exp \left[ -\frac{s(t + \varepsilon)\|\xi\|^2}{2Nt} \left( 1 - \frac{s(t + \varepsilon)}{Nt} \right) \right], \]
\[ J_2 = \int_{[0,1]^d} \left[ \exp(i\alpha x \cdot \xi) - \exp \left( -\frac{i\varepsilon(t + \varepsilon)x \cdot \xi}{t} \right) \right] \, dx \exp \left[ -\frac{s(t + \varepsilon)\|\xi\|^2}{2Nt} \left( 1 - \frac{s(t + \varepsilon)}{Nt} \right) \right]. \]

Hence,
\[
T_A^2 \leq \frac{L_{T,k}}{N} \int_0^\infty \! ds \int_{\mathbb{R}^d} f(d\xi) |J_1 + J_2|^2
\]
\[
\leq \frac{L_{T,k}}{N} \int_0^\infty \! ds \int_{\mathbb{R}^d} f(d\xi) |J_1|^2 + \frac{L_{T,k}}{N} \int_0^\infty \! ds \int_{\mathbb{R}^d} f(d\xi) |J_2|^2
\]
\[
:= T_{A,1}^2 + T_{A,2}^2. \quad (64)\]

Next, we estimate \( T_{A,2}^2 \) by analyzing the behavior of \(|J_1|^2\) and \(|J_2|^2\). Define
\[
\phi(x) := \frac{1 - \cos x}{x^2} \quad \text{for all } x \in \mathbb{R} \setminus \{0\}, \quad (65)
\]
and \( \phi(0) = 1/2 \) to preserve continuity. We have,
\[
|J_1|^2 \leq \left| \int_{[0,1]^d} e^{i\alpha x} \xi \, dx \right|^2 \left| 1 - \exp \left( -\frac{\varepsilon (t + \varepsilon)x^2 \|\xi\|^2}{2N^2t} \right) \right|^2
\]
\[
\leq L_\alpha \prod_{j=1}^d \phi(s\xi_j) \left[ \varepsilon (t + \varepsilon) \|\xi\|^2 (\varepsilon^2 t)^{2\alpha} \right] \leq L_\alpha \left[ \prod_{j=1}^d \left( 1 + \frac{1}{s\|\xi\|^2} \right) \right] \frac{e^{2\alpha(\|\xi\|^2)}}{(N^2t)^{2\alpha}}
\]
\[
\leq L_\alpha \left( 1 + \frac{1}{s\|\xi\|^2} \right) \frac{e^{2\alpha(\|\xi\|^2)}}{(N^2t)^{2\alpha}},
\]
where in the second inequality we use the fact that for any \( \alpha \in (0, 1/4) \) and \( x > 0 \), there exists a real number \( C_\alpha \) such that
\[
1 - e^{-x} \leq C_\alpha x^\alpha. \quad (66)
\]
And the last inequality holds by the following:
\[
\prod_{j=1}^d \left( 1 + |x_j|^{-1} \right) \leq 1 + |x|^{-1}. \quad (67)
\]
Thus, using a change of variable \([s \mapsto s/\|\xi\|]\), we have
\[
T_{A,1}^2 \leq L_{T,k} \varepsilon^{2\alpha} \frac{e^{2\alpha}}{L_\alpha N^{4\alpha+1}} \int_0^\infty \! ds \, s^{4\alpha} (1 + s^{-2}) \times \int_{\mathbb{R}^d} \frac{1}{\|\xi\|} f(d\xi).\]
Since $R(f) < \infty$, applying Lemma 5.9 in [3] we see that $\int_{\mathbb{R}^d} ||\xi||^{-1} \hat{f}(d\xi) < \infty$, and notice that for every $\alpha \in (0, 1/4)$, $\int_0^\infty s^{4\alpha} (1 \wedge s^{-2}) ds < \infty$, we finally get

$$T_{A,1}^2 \lesssim L_{T, k, \alpha} \frac{\varepsilon^{2\alpha}}{N^{4\alpha + 1}}.$$  \hfill (68)

Now, we turn to analyze $|J_2|^2$, a few lines of computation show that

$$|J_2|^2 \leq \left| \int_{[0, 1]^d} (e^{(1+\varepsilon/t)s\xi \cdot \xi} - e^{is\xi \cdot \xi}) \, dx \right|^2 \leq \prod_{j=1}^d \left| \frac{t}{i(t + \varepsilon)s\xi_j} e^{is\xi_j} - 1 + \frac{\varepsilon}{i(t + \varepsilon)s\xi_j} (1 - e^{is\xi_j}) \right|^2 \leq \prod_{j=1}^d \left\{ \frac{2t^2}{(t + \varepsilon)^2(s\xi_j)^2} [2 - 2\cos(\varepsilon s\xi_j/t)] + \frac{2\varepsilon^2}{(t + \varepsilon)^2(s\xi_j)^2} [2 - 2\cos(s\xi_j)] \right\}.$$  

Since $|1 - \cos x| \leq 2 \wedge (x^2/2)$, for every $x \in \mathbb{R}$, we rescale one more time to see that

$$|J_2|^2 \leq \prod_{j=1}^d \left\{ \frac{2t^2}{(t + \varepsilon)^2(s\xi_j)^2} \right\} \leq \prod_{j=1}^d \left[ \frac{8(t^2 + \varepsilon^2)}{(t + \varepsilon)^2(s\xi_j)^2} \wedge \frac{4\varepsilon^2}{(t + \varepsilon)^2} \right] \lesssim \left[ \frac{(t^2 + \varepsilon^2)}{(t + \varepsilon)^2(s||\xi||)^2} \wedge \frac{\varepsilon^2}{(t + \varepsilon)^2} \right] \leq \frac{1}{s||\xi||^2} \wedge \frac{\varepsilon}{t} \leq \frac{1}{s||\xi||} \left( \frac{1}{(s||\xi||)^2} \wedge \frac{\varepsilon}{t} \right),$$  

where we use (67) in the third inequality. Therefore, with a change of variable $[s \mapsto s/||\xi||]$ again, we obtain

$$T_{A,2}^2 \lesssim L_{T, k} \frac{(t \vee 1)}{Nt} \int_0^\infty ds \ (s^{-2} \wedge \varepsilon) \times \int_{\mathbb{R}^d} \frac{1}{||\xi||} \hat{f}(d\xi) \lesssim L_{T, k, \alpha} \frac{\varepsilon^{2\alpha}}{N^{2\alpha + 1}}, \quad \text{(69)}$$

where the last inequality follows from Lemma A.2 (1). Combining (64), (68) and (69), we conclude that

$$T_A^2 \lesssim L_{T, k, \alpha} \frac{\varepsilon^{2\alpha}}{N(t \wedge 1)}. \quad \text{(70)}$$
Recall (62). In order to estimate $T_B^2$, we use the semigroup property of the heat kernel and change of variables as follows:

$$T_B^2 \leq \frac{L_{t,k}}{N^2} \int_t^{t+\varepsilon} ds \int_{\mathbb{R}^d} f(dy') \int_{[0,N]^2d} dx_1 dx_2 \ p_{2s(t+\varepsilon-s)/(t+\varepsilon)} \left(y' - \frac{s}{t+\varepsilon}(x_2 - x_1)\right)$$

$$= L_{t,k} \int_t^{t+\varepsilon} ds \int_{\mathbb{R}^d} f(dy') \int_{[0,1]^2d} dx_1 dx_2 \ p_{2s(t+\varepsilon-s)/(t+\varepsilon)} \left(y' - \frac{sN}{t+\varepsilon}(x_2 - x_1)\right)$$

$$= L_{t,k} \frac{t+\varepsilon}{N} \int_{Nt/(t+\varepsilon)}^N ds \int_{\mathbb{R}^d} f(dy') \int_{[0,1]^2d} dx_1 dx_2 \ p_{s,N} \left(y' - s(x_2 - x_1)\right),$$

where $\gamma, N = \lfloor 2s(t+\varepsilon)(1 - s/N) \rfloor / N$. Furthermore, thanks to Parseval’s identity and (67), we may write the following:

$$T_B^2 \leq \frac{L_{t,k}}{(2\pi)^d N} \int_{Nt/(t+\varepsilon)}^N ds \int_{\mathbb{R}^d} \hat{f}(d\xi) \ e^{-\frac{2\pi^2}{N} ||\xi||^2} \int_{[0,1]^2d} dx_1 dx_2 \ \exp \left(is(x_2 - x_1) \cdot \xi\right)$$

$$\lesssim L_{t,k} \frac{1}{N} \int_{Nt/(t+\varepsilon)}^N ds \int_{\mathbb{R}^d} \hat{f}(d\xi) \ \prod_{j=1}^d \left(1 + |s\eta_j|^{-2}\right)$$

$$\lesssim L_{t,k} \frac{1}{N} \int_{Nt/(t+\varepsilon)}^N ds \int_{\mathbb{R}^d} \hat{f}(d\xi) \ \left(1 + (s||\xi||)^{-2}\right)$$

$$\lesssim L_{t,k} \frac{1}{N} \int_{Nt/(t+\varepsilon)}^N ds \int_{\mathbb{R}^d} \hat{f}(d\xi) \ (s||\xi||)^{-1} \lesssim L_{t,k} \frac{1}{N} \log \left(1 + \frac{\varepsilon}{t}\right)$$

This, together with (60) and (70) implies Lemma 17.

**Proof of Lemma 18.** We can use the same argument as the proof of Lemma 6.2 in [2] owing to $f(\mathbb{R}) < \infty$, which we skip here.

**Proof of Lemma 19.** In this case, recall (61) and (62). By using change of variables and semigroup property of the heat kernel we can write

$$T_A^2 \leq \frac{L_{t,k} \kappa_{\beta,d}}{N^3} \int_0^1 \frac{t^2}{s^d} ds \int_{\mathbb{R}^d} d\xi \ ||\xi||^{\beta-d} \left|\mathcal{A}(s)(t\xi/(Ns))\right|^2,$$  \hspace{1cm} (72)

$$T_A^2 \leq L_{t,k} \int_t^{t+\varepsilon} ds \int_{\mathbb{R}^d} dy' \left(I_N * \tilde{I}_N\right)(x) \ p_{2s(t+\varepsilon-s)/(t+\varepsilon)} \left(y' - \frac{s}{t+\varepsilon}x\right) \ ||y'||^{-\beta}.$$  \hspace{1cm} (73)

Similar to (63),

$$\mathcal{A}(s)(t\xi/(Ns)) = J_1 + J_2,$$
where
\[
J_1 = \int_{[0,1]^d} \exp \left( \frac{ix \cdot \xi}{t + \varepsilon} \right) dx \exp \left( - \frac{t(t-s)\|\xi\|^2}{2sN^2} \right) \left[ \exp \left( - \frac{\varepsilon t \|\xi\|^2}{2(t + \varepsilon)N^2} \right) - 1 \right],
\]
\[
J_2 = \int_{[0,1]^d} \left[ \exp \left( \frac{ix \cdot \xi}{t + \varepsilon} \right) - \exp(i x \cdot \xi) \right] dx \exp \left( - \frac{t(t-s)\|\xi\|^2}{2sN^2} \right).
\]

Therefore,
\[
T_{\mathcal{A}}^2 \lesssim \frac{L_{T,k,b}}{N^2} \int_{0}^{t} \frac{t^3}{s^2} ds \int_{\mathbb{R}^d} d\xi \|\xi\|^{\beta - d} |J_1|^2 + \frac{L_{T,k,b}}{N^2} \int_{0}^{t} \frac{t^3}{s^2} ds \int_{\mathbb{R}^d} d\xi \|\xi\|^{\beta - d} |J_2|^2
\]
\[
:= T_{\mathcal{A},1}^2 + T_{\mathcal{A},2}^2.
\] (74)

Apply (67) again to see that
\[
|J_1|^2 \lesssim \prod_{j=1}^{d} \frac{t\xi_j}{t + \varepsilon} \exp \left( - \frac{t(t-s)\|\xi\|^2}{sN^2} \right) \left[ \exp \left( - \frac{\varepsilon t \|\xi\|^2}{2(t + \varepsilon)N^2} \right) - 1 \right]^2
\]
\[
\lesssim \left( \frac{(t + \varepsilon)^2}{t^2} \right) \|\xi\|^2 \exp \left( - \frac{t(t-s)\|\xi\|^2}{sN^2} \right) \left[ 1 - \exp \left( - \frac{\varepsilon t \|\xi\|^2}{2(t + \varepsilon)N^2} \right) \right]^2
\] (75)

and
\[
|J_2|^2 \lesssim \left[ \left( \frac{\varepsilon^2}{t^2 + \varepsilon^2} \right) \wedge \left( \frac{t^2 + \varepsilon^2}{(t\|\xi\|)^2} \right) \right] \exp \left( - \frac{t(t-s)\|\xi\|^2}{sN^2} \right)
\]
\[
\lesssim \left( \frac{t^2 + \varepsilon^2}{t^2} \right) \left[ \left( \frac{\varepsilon^2 t^2}{(t^2 + \varepsilon^2)^2} \right) \wedge \left( \frac{1}{\|\xi\|^2} \right) \right].
\] (76)

First, we focus on \(T_{\mathcal{A},1}^2\). By making the following changes of variables: \(z' = \|\xi\|, z = z'/N, r = s/t\), we obtain that
\[
T_{\mathcal{A},1}^2 \lesssim L_{T,k,b} \frac{(t + \varepsilon)^2}{tN^2} \int_{1}^{\infty} \frac{1}{r^{2-\beta}} dr \int_{\infty}^{\infty} \frac{\varepsilon^{\beta-3}}{e^{t(r-1)z'}} \left| 1 - \exp \left( - \frac{\varepsilon t z^2}{2(t + \varepsilon)} \right) \right| dz
\]
\[
\lesssim L_{T,k,b} \frac{(t + \varepsilon)^2}{tN^2} \int_{1}^{\infty} \frac{1}{r^{2-\beta}} dr \int_{\infty}^{\infty} \frac{\varepsilon^{\beta-3}}{e^{t(r-1)z'}} \frac{\varepsilon t z^2}{2(t + \varepsilon)z'} dz
\]
\[
\lesssim L_{T,k,b} \frac{\varepsilon}{N^2} \int_{1}^{\infty} \frac{1}{r^{2-\beta}} dr \int_{\infty}^{\infty} \frac{\varepsilon^{\beta-3}}{e^{t(r-1)z'}} \frac{\varepsilon t z^2}{2(t + \varepsilon)z'} dz \lesssim L_{T,k,b} \frac{\varepsilon}{N^2 \beta/2},
\] (77)

where the last inequality holds by \(\int_{1}^{\infty} \frac{1}{r^{2-\beta}} \cdot \frac{1}{(r-1)^{\beta/2}} dr < \infty\) for all \(\beta \in (0, 2)\).

**Case 1:** \(\beta \in (0, 1)\). By Lemma A.2 (2), we have
\[
T_{\mathcal{A},2}^2 \lesssim L_{T,k,b} \frac{(t^2 + \varepsilon^2)}{t^2} \int_{0}^{t} \frac{t^3}{s^2} ds \int_{\mathbb{R}^d} d\xi \|\xi\|^{\beta - d} \left[ \left( \frac{\varepsilon^2 t^2}{(t^2 + \varepsilon^2)^2} \right) \wedge \left( \frac{1}{\|\xi\|^2} \right) \right]
\]

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\[
\lesssim L_{T,k,\beta} \frac{\varepsilon}{tN^\beta} \int_0^t \frac{t^\beta}{s^\beta} ds \lesssim \frac{L_{T,k,\beta}}{N^\beta} \varepsilon. \tag{78}
\]

Therefore, this together with (77) concludes that

\[
T_A^2 \lesssim L_{T,k} \left( \frac{\varepsilon}{t \wedge 1} \right) N^\beta. \tag{79}
\]

Now we turn to estimate \(T_B^2\). Recall (73). Define \(\varphi(x) := \prod_{j=1}^d (1 - |x_j|)\) for all \(x \in \mathbb{R}^d\). We observe that \((I_N \ast \tilde{I}_N)(x) = N^{-d} \varphi(x/N) 1_{[-N, N]^d}(x)\). Making a change of variables, it yields that

\[
T_B^2 \lesssim \frac{L_{T,k}}{N^\beta} \int_t^{t+\varepsilon} ds \int_{[-1,1]^d} \varphi(z) dz \left| \left| \frac{2s(t + \varepsilon - s)}{t + \varepsilon} Z + \frac{s}{t + \varepsilon} \right|^\beta \right| \left( \frac{1}{s} \right)^{-\beta} \log \left( \frac{1}{s} \right).
\]

(80)

Combine (60) with (79) and (80) to conclude (57).

**Case 2:** \(\beta = 1\). According to Lemma A.3, we have

\[
T_{A,2}^2 \lesssim L_{T,k,\beta} \frac{(t^2 + \varepsilon^2)}{t^2 N^\beta} \int_0^t \frac{t^\beta}{s^\beta} ds \int_{\mathbb{R}^d} d\xi \cdot \|\xi\|^{1-d} \left[ \left( \frac{\varepsilon^2 t^2}{(t^2 + \varepsilon^2)^2} \right) \land \left( \frac{1}{\|\xi\|^2} \right) \right] \exp \left( -\frac{t(t - s)\|\xi\|^2}{s N^2} \right)
\]

\[
\lesssim L_{T,k,\beta} \frac{(t^2 + \varepsilon^2) \log N}{Nt} \log \left( \frac{1}{t} \right)
\]

\[
\times \int_{\mathbb{R}^d} d\xi \cdot \|\xi\|^{1-d} \left[ \left( \frac{\varepsilon^2 t^2}{(t^2 + \varepsilon^2)^2} \right) \land \left( \frac{1}{\|\xi\|^2} \right) \right] \log \left( \frac{1}{\|\xi\|} \right).
\]

Then, we made a change of variables \(z = \|\xi\|\) and use Lemma A.3 in [2] to obtain that

\[
T_{A,2}^2 \lesssim L_{T,k,\beta} \frac{\varepsilon \log N}{Nt} \log \left( \frac{1}{t} \right) \tag{81}
\]

Hence, for any \(\delta > 0\), there exists \(L_{T,k,\delta} > 0\) such that

\[
T_A^2 \lesssim L_{T,k,\delta} \frac{\varepsilon}{(t \wedge 1)^{1+\delta}}. \tag{82}
\]
Then it remains to consider the behavior of $T_B^2$. Recall (73). Since the Fourier transform of $I_N \ast \tilde{I}_N$ is $2^d \prod_{j=1}^d \phi(N \xi_j)$, by using a change of variables $z = N x$, we obtain

$$T_B^2 \lesssim \frac{L_{T,k,\beta}(t + \varepsilon)}{(2\pi)^d N} \int_t^{t + \varepsilon} ds \int_{\mathbb{R}^d} \prod_{j=1}^d \phi(\xi_j) \exp \left( -\frac{(t + \varepsilon)(t + \varepsilon - s)\|\xi\|^2}{sN^2} \right) \|\xi\|^{1-d} d\xi,$$

where $\phi(x)$ is defined in (65). Since $\int_{\mathbb{R}^d} \prod_{j=1}^d \phi(\xi_j) \|\xi\|^{1-d} dz < \infty$, we conclude that

$$T_B^2 \lesssim L_{T,k} \frac{\varepsilon}{Nt}. \quad (83)$$

This together with (82) proves (58).

**Case 3**: $\beta \in (1, 2 \wedge d)$. Recall (74) and (76).

$$T_{A,2}^2 \lesssim L_{T,k,\beta} \frac{(t^2 + \varepsilon)^2}{t^2 N^{2-\beta}}$$

$$\times \int_0^t \frac{t^\beta}{s^\beta} ds \int_{\mathbb{R}^d} d\xi \|\xi\|^\beta \left[ \left( \frac{\varepsilon^2 t^2}{(t^2 + \varepsilon)^2} \right) \wedge \left( \frac{1}{\|\xi\|^2} \right) \right] \exp \left( -\frac{t(t - s)\|\xi\|^2}{sN^2} \right).$$

Then, by using changes of variables $r' = (t - s)/s$ and $r = -t\|\xi\|^2 r'/N^2$ in order, we obtain that

$$T_{A,2}^2 \lesssim L_{T,k,\beta} \frac{(t^2 + \varepsilon)^2 \|\xi\|^{2-\beta}}{t^2 N^{2-\beta}} \int_{\mathbb{R}^d} d\xi \|\xi\|^{2-\beta} \left[ \left( \frac{\varepsilon^2 t^2}{(t^2 + \varepsilon)^2} \right) \wedge \left( \frac{1}{\|\xi\|^2} \right) \right] \int_0^\infty r^{3-\beta} e^{-r} dr$$

$$\lesssim L_{T,k,\beta} \frac{t^{2-\beta} \varepsilon}{tN^{2-\beta}}, \quad (84)$$

where the last inequality holds by Lemma A.2 (2). This, together with (77), implies that

$$T_A^2 \lesssim L_{T,k,\beta} \frac{\varepsilon}{(t \wedge 1)N^{2-\beta}}. \quad (85)$$

Next, we are going to estimate the term $T_B^2$. Recall (73). by making changes of variables we observe that

$$T_B^2 \lesssim \frac{L_{T,k,\beta}(t + \varepsilon)}{(2\pi)^d N^{3-\beta}} \int_t^{t + \varepsilon} ds \int_{\mathbb{R}^d} \prod_{j=1}^d \phi(\xi_j) \exp \left( -\frac{(t + \varepsilon)(t + \varepsilon - s)\|\xi\|^2}{sN^2} \right) \|\xi\|^{1-d} d\xi$$

$$\lesssim L_{T,k,\beta} \frac{t + \varepsilon}{N^{3-\beta}} \int_{\mathbb{R}^d} d\xi \|\xi\|^{2-d} \prod_{j=1}^d \phi(\xi_j) \int_0^{\varepsilon/t} dr (1 + r)^{\beta - 2} \exp \left( -\frac{r(t + \varepsilon)}{N^2} \|\xi\|^2 \right)$$

$$\lesssim L_{T,k,\beta} \frac{(t + \varepsilon) \varepsilon}{t N^{3-\beta}} \int_{\mathbb{R}^d} d\xi \|\xi\|^{2-d} \prod_{j=1}^d \phi(\xi_j) d\xi \lesssim L_{T,k,\beta} \frac{\varepsilon}{t N^{3-\beta}}. \quad (86)$$
Finally, (85) and (86) together imply (59).

Now, we can prove Propositions 14-16.

Proof of Proposition 14. We assume that $T \geq 1$ without losing generality. Choose and fix an arbitrary number $\lambda \in (0, 1)$. On the one hand, from Lemma 17, for any $\varepsilon \in (0, 1), N \geq c$ and $t \in (\varepsilon^\lambda, T)$ we have

$$\|S_{N,t+\varepsilon} - S_{N,t}\|_k \lesssim L_{T,k,\varepsilon} \varepsilon^{\alpha - \lambda/2} \sqrt{\frac{1}{N}}.$$  (4.43)

On the other hand, for any $t \in (0, \varepsilon^\lambda)$, Lemma 11 implies that

$$\|S_{N,t+\varepsilon} - S_{N,t}\|_k \leq \|S_{N,t+\varepsilon}\|_k + \|S_{N,t}\|_k \lesssim L_{T,k}\varepsilon^{\lambda/2} \sqrt{\frac{1}{N}}.$$  (4.44)

Choose $\lambda = \alpha$ to match the above exponents of $\varepsilon$ and define $\gamma := \alpha/2$ to finish the proof.

Proof of Proposition 15. Similar to the proof of Proposition 14, based on Lemma 12 and 18, for any $\delta > 0$,

$$\|S_{N,t+\varepsilon} - S_{N,t}\|_k \lesssim L_{T,k,\varepsilon}^{(1-\lambda(1+\delta))/2} \sqrt{\frac{\log N}{N}} \text{ for } t \in (\varepsilon^\lambda, T)$$  (87)

and for any $\alpha \in (0, 1),$

$$\|S_{N,t+\varepsilon} - S_{N,t}\|_k \lesssim L_{T,k,\varepsilon}^{\alpha\lambda/2} \sqrt{\frac{\log N}{N}} \text{ for } t \in (0, \varepsilon^\lambda).$$  (88)

Then, choose $\lambda = 1/(1 + \alpha + \delta)$, we prove (51).

Proof of Proposition 16. Similar to the proof of Proposition 14, based on Lemma 13 and 19, the following inequalities holds when $\beta \in (0, 1)$:

$$\|S_{N,t+\varepsilon} - S_{N,t}\|_k \lesssim L_{T,k,\varepsilon}^{(1-\lambda)/2} \sqrt{\frac{1}{N^\beta}} \text{ for } t \in (\varepsilon^\lambda, T)$$  (89)

and

$$\|S_{N,t+\varepsilon} - S_{N,t}\|_k \lesssim L_{T,k,\varepsilon}^{\lambda/2} \sqrt{\frac{1}{N^\beta}} \text{ for } t \in (0, \varepsilon^\lambda).$$  (90)

Also, if $\beta \in (1, 2 \land d)$, we have

$$\|S_{N,t+\varepsilon} - S_{N,t}\|_k \lesssim L_{T,k,\varepsilon}^{(1-\lambda)/2} \sqrt{\frac{1}{N^{2-\beta}} \text{ for } t \in (\varepsilon^\lambda, T)}$$  (91)
and
\[ \|S_{N,t+\varepsilon} - S_{N,t}\|_k \lesssim L_{T,k,\beta} \varepsilon^{\lambda(2-\beta)/2} \sqrt{\frac{1}{N^{2-\beta}}} \text{ for } t \in (0, \varepsilon^\lambda). \] (92)

Therefore, choose \( \lambda = 1/2 \) and \( \lambda = 1/(3 - \beta) \) respectively to prove (52) and (54). Finally, based on the proof of Proposition 15, we obtain the result in the case that \( \beta = 1 \). \( \square \)

5 Proofs of Theorems

In this section, we will establish the weak convergence of the finite-dimensional distributions to prove the functional CLTs. But first, we need to show a couple of technical lemmas.

Recall (5). Because of (4) and a stochastic Fubini argument, we can express \( S_{N,t} \) as an Itô-Walsh stochastic integral:
\[ S_{N,t} = \int_{\mathbb{R}^+ \times \mathbb{R}^d} v_{N,t}(s,y) \eta(ds,dy) = \delta(v_{N,t}), \] (93)
where
\[ v_{N,t}(s,y) = 1_{(0,t)}(s) U(s,y) \frac{1}{N^{d}} \int_{[0,N]^d} p_{s(t-s)/t}(y - \frac{s}{t}x) \, dx. \] (94)

Since \( S_{N,t} \) is Malliavin differentiable (19, Chapter 1.1.3), we can write
\[ D_{s,y}S_{N,t} = v_{N,t}(s,y) + \int_{(s,t) \times \mathbb{R}^d} D_{s,y}v_{N,t}(r,w) \eta(dr,dw). \] (95)

Lemma 20. \((d \geq 2)\). Suppose \( \mathcal{R}(f) < \infty \). For every \( T > 0 \), we have
\[ \sup_{t_1, t_2 \in (0,T)} \text{Var}(DS_{N,t_1}, v_{N,t_2})_H \leq L_N T^{-3} \text{ for all } N \geq e. \] (96)

Lemma 21. \((d = 1)\). Suppose \( f(\mathbb{R}) < \infty \). For every \( T > 0 \), we have
\[ \sup_{t_1, t_2 \in (0,T)} \text{Var}(DS_{N,t_1}, v_{N,t_2})_H \leq L_N \left( \frac{N}{\log N} \right)^{-3} \text{ for all } N \geq e. \] (97)

Lemma 22. \((\text{Riesz kernel})\). Suppose \( f(dx) = ||x||^{-\beta} dx \), \( \beta \in (0,2 \wedge d) \). for any \( T > 0 \) and \( t_1, t_2 \in (0, T) \), there exists a real number \( L = L_{T,t_1,t_2} > 0 \), such that
(A) If \( \beta \in (0,1) \), then
\[ \text{Var}(DS_{N,t_1}, v_{N,t_2})_H \leq LN^{-3\beta} \text{ for all } N \geq e. \] (98)
(B) If $\beta = 1$, then

$$\text{Var}(DS_{N,t_{1}},v_{N,t_{2}}) \leq L\left(\frac{N}{\log N}\right)^{-3} \text{ for all } N \geq e. \quad (99)$$

(C) If $\beta \in (1,2 \land d)$, then

$$\text{Var}(DS_{N,t_{1}},v_{N,t_{2}}) \leq LN^{-\alpha(2-\beta)} \text{ for all } N \geq e. \quad (100)$$

Before proving the above lemmas, we first decompose the quantity $(DS_{N,t_{1}},v_{N,t_{2}})$ by using (95) and stochastic Fubini argument,

$$(DS_{N,t_{1}},v_{N,t_{2}}) = \frac{1}{N^{2d}} \int_{0}^{t_{1} \land t_{2}} ds \int_{\mathbb{R}^{2d}} f(dz) dy \int_{[0,N]^{2d}} dx dx' U(s,y)U(s,y+z)$$

$$\times p_{s[(t_{1} \land t_{2})-s]/(t_{1} \land t_{2})}(y - \frac{s}{t_{1} \cap t_{2}}) p_{s[(t_{1} \lor t_{2})-s]/(t_{1} \lor t_{2})}(y + z - \frac{s}{t_{1} \lor t_{2}})$$

$$+ \frac{1}{N^{2d}} \int_{0}^{t_{1}} \eta(dr,du) \int_{0}^{r \land t_{2}} ds \int_{\mathbb{R}^{2d}} f(dz) dy \int_{[0,N]^{2d}} dx dx'$$

$$\times p_{r(t_{1}-r)/t_{1}} \left( w - \frac{r}{t_{1}} \right) p_{s(t_{2}-s)/t_{2}} \left( y + z - \frac{s}{t_{2}} \right) U(s,y+z)D_{s,y}U(r,w)$$

$$:= \mathcal{X}_{N,t_{1},t_{2}} + \mathcal{U}_{N,t_{1},t_{2}}.$$  

As a consequence,

$$\text{Var}(DS_{N,t_{1}},v_{N,t_{2}}) \leq 2 \left( \text{Var}\mathcal{X}_{N,t_{1},t_{2}} + \text{Var}\mathcal{U}_{N,t_{1},t_{2}} \right) = \frac{2}{N^{4d}} \left( \Phi_{N,t_{1},t_{2}}^{(1)} + \Phi_{N,t_{1},t_{2}}^{(2)} \right), \quad (101)$$

where

$$\Phi_{N,t_{1},t_{2}}^{(1)} = \int_{[0,t_{1} \land t_{2}]^{2}} ds_{1} ds_{2} \int_{\mathbb{R}^{2d}} f(dz_{1}) f(dz_{2}) dy_{1} dy_{2} \int_{[0,N]^{2d}} dx_{1} dx_{2} dx_{1}' dx_{2}'$$

$$\times p_{s_{1}[(t_{1} \land t_{2})-s_{1}]/(t_{1} \land t_{2})}(y_{1} - \frac{s_{1}x_{1}}{t_{1} \cap t_{2}}) p_{s_{2}[(t_{1} \land t_{2})-s_{2}]/(t_{1} \land t_{2})}(y_{2} - \frac{s_{2}x_{2}}{t_{1} \cap t_{2}})$$

$$\times p_{s_{1}[(t_{1} \lor t_{2})-s_{1}]/(t_{1} \lor t_{2})}(y_{1} + z_{1} - \frac{s_{1}x_{1}}{t_{1} \lor t_{2}}) p_{s_{2}[(t_{1} \lor t_{2})-s_{2}]/(t_{1} \lor t_{2})}(y_{2} + z_{2} - \frac{s_{2}x_{2}}{t_{1} \lor t_{2}})$$

$$\times \text{Cov}[U(s_{1},y_{1})U(s_{1},y_{1} + z_{1}), U(s_{2},y_{2})U(s_{2},y_{2} + z_{2})] \quad (102)$$

and

$$\Phi_{N,t_{1},t_{2}}^{(2)} = \int_{[0,t_{1}]^{2}} dw \int_{[0,r \land t_{2}]^{2}} ds_{1} ds_{2} \int_{\mathbb{R}^{2d}} f(dz_{1}) f(dz_{2}) dy_{1} dy_{2}$$

$$\times \int_{[0,N]^{2d}} dx_{1} dx_{2} dx_{1}' dx_{2}' p_{r(t_{1}-r)/t_{1}} \left( w - \frac{r}{t_{1}} \right) p_{r(t_{1}-r)/t_{1}} \left( w + b - \frac{r}{t_{1}} \right)$$

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\[ \times \mathcal{P}_{s_1(t_2-s_1)/t_2} \left( y_1 + z_1 - \frac{s_1}{t_2} x_1 \right) P_{s_2(t_2-s_2)/t_2} \left( y_2 + z_2 - \frac{s_2}{t_2} x_2 \right) \]
\[ \times E[U(s_1, y_1 + z_1)U(s_2, y_2 + z_2)D_{s_1, y_1}U(r, w)D_{s_2, y_2}U(r, w + b)]. \quad (103) \]

Now we turn to estimate the terms \( \Phi_{N, t_1, t_2}^{(1)} \) and \( \Phi_{N, t_1, t_2}^{(2)} \). By the Poincaré-type inequality (16), Lemma 4, semigroup property and symmetry we can write

\[ \Phi_{N, t_1, t_2}^{(1)} \lesssim \int_{[0, \cdot]^{2d}} ds_1 ds_2 \int_0^{s_1 \wedge s_2} dr \int_{\mathbb{R}^{2d}} f(dz_1)f(dz_2)f(db) dy_1 dy_2 \]
\[ \times \int_{[0, N]^{4d}} dx_1 dx_2 dx_3 dx_4 \mathcal{P}_{s_1(t_2-t_1)/t_1} \left( y_1 - \frac{s_1}{t_1} x_1 \right) P_{s_2(t_2-s_2)/t_2} \left( y_2 - \frac{s_2}{t_2} x_2 \right) \]
\[ \times P_{r(s_1-r)/s_1 + r(s_2-r)/s_2} \left( b - \frac{r}{s_2} y_2 + \frac{r}{s_1} y_1 \right), \quad (104) \]

where we assume \( t_1 < t_2 \) to simplify the expressions. Similarly,

\[ \Phi_{N, t_1, t_2}^{(2)} \lesssim \int_0^{\cdot} dr \int_{[0, r \wedge t_2]^{2d}} ds_1 ds_2 \int_{\mathbb{R}^{2d}} f(db)f(dz_1)f(dz_2)dy_1 dy_2 dw \]
\[ \times \int_{[0, N]^{4d}} dx_1 dx_2 dx_3 dx_4 \mathcal{P}_{r(t_1-t)/t_1} \left( w + \frac{r}{t_1} x_1 \right) P_{r(s_2-s)/r} \left( y_2 - \frac{s_2}{r} (w + b) \right) \]
\[ \times P_{r(t_2-r)/t_2} \left( w + \frac{r}{t_2} x_2 \right) P_{s_1(t_2-s_1)/t_2 + s_1(r-s)/r} \left( z_1 - \frac{s_1}{t_2} x_1 + \frac{s_1}{r} w \right), \quad (105) \]

where in the first inequality we use the Hölder inequality and Lemma 4, and in the last equality we use the semigroup property of the heat kernel.

Now we begin to prove Lemmas 20-22. Actually, the proof is similar to that of Theorems 1.3-1.5 in [3], but there are still slight differences in detail.

**Proof of Lemma 5.1.** In order to prove (96), it suffices to show that

\[ \sup_{t_1, t_2 \in (0, T)} N^{-4d+3} (\Phi_{N, t_1, t_2}^{(1)} + \Phi_{N, t_1, t_2}^{(2)}) \leq L_T \quad \text{for all } N \geq c. \quad (106) \]
We first estimate the term $\Phi^{(1)}_{N,t_1,t_2}$, because the time variables $t_1, t_2$ may take different values, we can't use the elementary relation $p_i(x)p_i(y) = 2^d p_{2i}(x+y)p_{2i}(x-y)$ as the proof of Theorem 1.3 in [3]. Therefore, we proceed in the following order: using the changes of variables at first [(i)$m_1 = y_1 - s_1x_1/t_1$, $m_2 = y_2 - s_2x_2/t_1$. (ii)$x_i \mapsto N x_i$, $x'_i \mapsto N x'_i$ for $i = 1, 2$. (iii)$r_1 = s_1 N/\tau$, $r_2 = s_2 N/\tau$, $\sigma = r N/\tau$,], then applying Parseval's identity, to obtain that

$$N^{-4d+3} \Phi^{(1)}_{N,t_1,t_2} \lesssim L_T^{3} \int_{(0,\infty)^3} \frac{dr_1dr_2d\sigma}{2\pi^d} \int_{\mathbb{R}^3} f(d\xi_1) f(d\xi_2) f(d\xi_3) \Delta_1(\xi_1, \xi_2, \xi_3) \left| \right.$$ \n
$$\times \exp \left( -\frac{\gamma_1 N}{2} \left\| \frac{\sigma}{r_1} \xi_3 + \xi_1 \right\|^2 - \frac{\gamma_2 N}{2} \left\| \frac{\sigma}{r_2} \xi_3 - \xi_2 \right\|^2 - \frac{\gamma_3 N}{2} \left\| \xi_1 \right\|^2 \right)$$ \n
$$- \frac{\gamma_4 N}{2} \left\| \xi_2 \right\|^2 - \frac{\gamma_5 N}{2} \left\| \xi_3 \right\|^2 \right)$$ \n
$$\lesssim L_T^{3} \int_{(0,\infty)^3} dr_1dr_2d\sigma \int_{\mathbb{R}^3} f(d\xi_1) f(d\xi_2) f(d\xi_3) \Delta_1(\xi_1, \xi_2, \xi_3), \quad (107)$$

where

$$\Delta_1(\xi_1, \xi_2, \xi_3) := \int_{[0,1]^d} dx_1 dx'_1 dx_2 dx'_2$$ \n
$$\times \exp[-i(r_1 \xi_1 + \sigma \xi_3) \cdot (\tau_1 x_1) - i(r_2 \xi_2 - \sigma \xi_1) \cdot (\tau_2 x_2) + ir_1 \xi_1 \cdot (\tau_2 x'_2) + ir_2 \xi_2 \cdot (\tau_2 x'_2)]$$

and

$$\gamma_{1,N} = \frac{r_1 \tau}{N} \left( 1 - \frac{r_1 \tau}{N \tau_1} \right); \quad \gamma_{2,N} = \frac{r_2 \tau}{N} \left( 1 - \frac{r_2 \tau}{N \tau_2} \right); \quad \gamma_{3,N} = \frac{r_1 \tau}{N} \left( 1 - \frac{r_1 \tau}{N \tau_2} \right);$$ \n
$$\gamma_{4,N} = \frac{r_2 \tau}{N} \left( 1 - \frac{r_2 \tau}{N \tau_1} \right); \quad \gamma_{5,N} = \frac{\sigma \tau}{N} \left( 2 - \frac{\sigma}{\tau_1} - \frac{\sigma}{\tau_2} \right),$$

where $\tau_1, \tau_2$ and $\tau$ are defined in (17). Therefore, we can use the same computations as Theorem 1.3 in [3] to find that

$$N^{-4d+3} \Phi^{(1)}_{N,t_1,t_2} \lesssim L_T^{3} \int_{(0,\infty)^3} dr_1dr_2d\sigma \int_{\mathbb{R}^3} f(d\xi_3) f(d\xi_4) f(d\xi_5) \left( \left\| \xi_3 \right\| \left\| \xi_4 \right\| \left\| \xi_5 \right\| \right)^{-1}$$ \n
$$\times \left[ 1 \wedge (\tau_2 x)^{-1} \right] \left[ 1 \wedge (\tau_2 y)^{-1} \right] \left[ 1 \wedge (\tau_1 \left\| x e_3 + z e_5 \right\|)^{-1} \right] \left[ 1 \wedge (\tau_1 \left\| y e_4 - z e_5 \right\|)^{-1} \right]$$ \n
$$\lesssim L_T^{3} \frac{\tau_1 \tau_2}{\tau_1 \tau_2} \int_{(0,\infty)^3} dr_1dr_2d\sigma \int_{\mathbb{R}^3} f(d\xi_3) f(d\xi_4) f(d\xi_5) \left( \left\| \xi_3 \right\| \left\| \xi_4 \right\| \left\| \xi_5 \right\| \right)^{-1}$$ \n
$$\times \left[ 1 \wedge (x)^{-1} \right] \left[ 1 \wedge (y)^{-1} \right] \left[ 1 \wedge (\left\| x e_3 + z e_5 \right\|)^{-1} \right] \left[ 1 \wedge (\left\| y e_4 - z e_5 \right\|)^{-1} \right]$$ \n
$$\lesssim L_T^{3} \frac{\tau_1 \tau_2}{\tau_1 \tau_2}. \quad (108)$$

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As for $\Phi^{(2)}_{N,t_1,t_2}$, similar to the estimation of $\Phi^{(1)}_{N,t_1,t_2}$, we proceed in the following order: using the changes of variables at first [(i) $z = w - rx_i/t_1$. (ii) $x_i \mapsto N x_i / r$, $x_i' \mapsto N x_i'$ for $i = 1, 2$. (iii) $r_1 = s_1 N / \tau$, $r_2 = s_2 N / \tau$, $\sigma = r N / \tau$], then applying Parseval’s identity and some computations, to obtain that

$$N^{-4d+3} \Phi^{(2)}_{N,t_1,t_2} \lesssim L_T T^3 \int_{[0,\infty)^3} \mathcal{d} r_1 \mathcal{d} r_2 \mathcal{d} \sigma \int_{\mathbb{R}^3} \mathcal{f}(d\xi_1) \mathcal{f}(d\xi_2) \mathcal{f}(d\xi_3) |\Delta_2(\xi_1, \xi_2, \xi_3)|$$

where

$$\Delta_2(\xi_1, \xi_2, \xi_3) := \int_{[0,1]^{4d}} \mathcal{d} x_1 \mathcal{d} x'_1 \mathcal{d} x_2 \mathcal{d} x'_2 \times \exp[-i (r_1 \xi_1 + \sigma \xi_2) \cdot (\tau_1 x_1) + i (\sigma \xi_2 - r_2 \xi_3) \cdot (\tau_1 x_2) + i r_1 \xi_1 \cdot (r_2 x'_1) + i r_2 \xi_3 \cdot (r_2 x'_2)].$$

Combining (108) and (109), we obtain the desired result.

**Proof of Lemma 5.2.** Recall the decomposition (101) of $\text{Var}(DS_{N,t_1}, v_{N,t_2})$. In this case, we only need to show that

$$\sup_{t_1, t_2 \in (0, T)} (\Phi^{(1)}_{N,t_1,t_2} + \Phi^{(2)}_{N,t_1,t_2}) \leq L_T N (\log N)^3 \quad \text{for all } N \geq e. \quad (110)$$

Recall (104). One can follow the arguments in the proof of Theorem 1.4 in [3] to estimate $\Phi^{(1)}_{N,t_1,t_2}$ such that

$$\Phi^{(1)}_{N,t_1,t_2} \lesssim L_T t_1 t_2^2 N \left[ 2 \log N + \log \left( \frac{1}{t_1} + 1 \right) \right]^3. \quad (111)$$

Next, similarly, we discuss $\Phi^{(2)}_{N,t_1,t_2}$ as follows: integrating the variables $x_1$ and $x_2$ on $\mathbb{R}$, applying semigroup property and using the changes of variables and Plancherel’s identity, to obtain that

$$\Phi^{(2)}_{N,t_1,t_2} \lesssim L_T N f(\mathbb{R})^3 \int_{\mathbb{R}} \mathcal{f}(z) dz \int_0^{t_1} \mathcal{d} r \int_0^r \mathcal{d} s \frac{1}{s} e^{-\frac{[t_2 (t_2 - s)/s + \xi_2^2 (t_2 - r)/r]^2}{2 N^2}}$$

$$= L_T N f(\mathbb{R})^3 \int_{\mathbb{R}} \mathcal{f}(z) dz \int_0^{t_1 \wedge t_2} \mathcal{d} r \int_0^r \mathcal{d} s \frac{1}{s} e^{-\frac{[t_2 (t_2 - s)/s + \xi_2^2 (t_2 - r)/r]^2}{2 N^2}}$$

$$+ L_T N f(\mathbb{R})^3 \int_{\mathbb{R}} \mathcal{f}(z) dz \int_0^{t_1} \mathcal{d} r \int_0^{t_2} \mathcal{d} s \frac{1}{s} e^{-\frac{[t_2 (t_2 - s)/s + \xi_2^2 (t_2 - r)/r]^2}{2 N^2}}$$

$$:= \Phi^{(2,1)}_{N,t_1,t_2} + \Phi^{(2,2)}_{N,t_1,t_2} \quad (112)$$
With the changes of variables $\theta = (r - s)/s$ and $\xi = (t_2 - r)/r$, we have

$$\Phi_{N,t_1,t_2}^{(2,1)} \lesssim L_T N f(\mathbb{R})^3 t_1^2 t_2 \int_{\mathbb{R}} \phi(z) dz \left( \int_0^\infty \frac{1}{\theta + \frac{t_2}{N T}} e^{-\theta d\theta} \right)^3 \lesssim L_T N (\log N)^3. \quad (113)$$

As for $\Phi_{N,t_1,t_2}^{(2,2)}$, by using the changes of variables $\theta' = (t_2 - s)/s$ and $\theta = t_2 z^2 \theta'/2N^2$ in order, we can write

$$\Phi_{N,t_1,t_2}^{(2,2)} \lesssim L_T N f(\mathbb{R})^3 \int_{\mathbb{R}} \phi(z) dz \int_{t_1 \wedge t_2}^{t_1} \frac{t_1^2 t_2}{r} \left( \int_0^\infty \frac{1}{\theta + \frac{t_2}{2N^2 z^2}} e^{-\theta d\theta} \right)^2$$

$$= L_T N t_1 t_2 \log \left( \frac{t_1}{t_1 \wedge t_2} \right) \int_{\mathbb{R}} \phi(z) dz \left( \int_0^\infty \frac{1}{\theta + \frac{t_2}{2N^2 z^2}} e^{-\theta d\theta} \right)^2 \lesssim L_T N (\log N)^2. \quad (114)$$

The proof of Lemma 21 is now completed. \qed

**Remark 6.** Comparing with the case $t_1 = t_2 = t$, we need to deal with the term $\Phi_{N,t_1,t_2}^{(2,2)}$ additionally because of the possibility of the case that $t_2 < r < t_1$.

**Proof of Lemma 5.3.** According to the proof of (6.14) and (6.21) in [3], we first write

$$\Phi_{N,t_1,t_2}^{(1)} \lesssim L_T N^{4d-3\beta} \int_{[0,t_1]^2} ds_1 ds_2 \int_0^{s_1 \wedge s_2} dr \left( \frac{r}{r_1} \right)^{\beta} \left( \frac{r}{r_2} \right)^{\beta} \int_{[0,1]^{4d}} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \times$$

$$\times \left[ \left| \frac{\tau}{N s_1} \sqrt{\frac{s_1(t_2 - s_1)}{t_2}} Z_3 - \frac{\tau}{N s_2} \sqrt{\frac{s_1(t_1 - s_1)}{t_1}} Z_1 - (\tau_1 x_1 - \tau_2 x_1') \right|^{\beta} \right.$$

$$\times \left. \left| \frac{\tau}{N s_2} \sqrt{\frac{s_2(t_2 - s_2)}{t_2}} Z_4 - \frac{\tau}{N s_2} \sqrt{\frac{s_2(t_1 - s_2)}{t_1}} Z_2 - (\tau_1 x_2 - \tau_2 x_2') \right|^{\beta} \right.$$

$$\times \left. \left| \frac{\tau}{N r_1} \sqrt{\frac{r_1(s_1 - r)}{s_1}} + \frac{r(s_2 - r)}{s_2} Z_5 + \frac{\tau}{N s_2} \sqrt{\frac{s_2(t_1 - s_2)}{t_1}} Z_2 
\right. \right.$$

$$\left. \left. \left. \left. - \frac{\tau}{N s_1} \sqrt{\frac{s_1(t_1 - s_1)}{t_1}} Z_1 + (\tau_1 x_2 - \tau_1 x_1) \right|^{\beta} \right] \right) \quad (115)$$

and

$$\Phi_{N,t_1,t_2}^{(2)} \lesssim L_T N^{4d-3\beta} \int_0^{t_1} dr \int_{[0,r \wedge t_2]^2} ds_1 ds_2 \left( \frac{r}{r_1} \right)^{\beta} \left( \frac{r}{r_2} \right)^{\beta} \int_{[0,1]^{4d}} d\xi_1 d\xi_2 d\xi_3 d\xi_4$$

$$\times \left[ \left| \frac{\tau}{N s_1} \sqrt{\frac{s_1(t_2 - s_1)}{t_2}} Z_3 - \frac{\tau}{N s_2} \sqrt{\frac{s_1(t_1 - s_1)}{t_1}} Z_1 - (\tau_1 x_1 - \tau_2 x_1') \right|^{\beta} \right.$$
\begin{align}
&\times \mathbf{E} \left[ \left\| \frac{\tau}{N r} \sqrt{\frac{r(t_1 - r)}{t_1}} Z_2 - \frac{\tau}{N r} \sqrt{\frac{r(t_1 - r)}{t_1}} Z_1 + (\tau x_2 - \tau x_1) \right\|^\beta \right] \\
&\times \left\| \frac{\tau}{N s_1} \sqrt{\frac{s_1(t_2 - s_1)}{t_2}} + \frac{s_1(r - s_1)}{r} Z_3 - \frac{\tau}{N r} \sqrt{\frac{r(t_1 - r)}{t_1}} Z_1 + (\tau x_2' - \tau x_1) \right\|^\beta \\
&\times \left\| \frac{\tau}{N s_2} \sqrt{\frac{s_2(t_2 - s_2)}{t_2}} + \frac{s_2(r - s_2)}{r} Z_4 - \frac{\tau}{N r} \sqrt{\frac{r(t_1 - r)}{t_1}} Z_2 + (\tau x_2' - \tau x_2) \right\|^\beta ,
\end{align}

where \( Z_1, Z_2, Z_3, Z_4, Z_5 \) denote i.i.d., \( d \)-dimension standard normal random variables. Then, in a similar way to the proof of Theorem 1.5 in [3], we conclude that

**Case 1:** \( \beta \in (0, 1) \).

\begin{align}
\Phi_{N,t_1,t_2}^{(1)} &\leq L_T \frac{N^{4d-3\beta}}{(\tau \tau_2)^2} \int_{[0,\tau]^2} ds_1 ds_2 \int_0^{s_1 \wedge s_2} dr \left( \frac{\tau}{r} \right)^\beta \left( \frac{\tau}{s_1} \right)^\beta \left( \frac{\tau}{s_2} \right)^\beta \lesssim L_{T,t_1,t_2} N^{4d-3\beta}, \\
\Phi_{N,t_1,t_2}^{(2)} &\leq L_T \frac{N^{4d-3\beta}}{(\tau \tau_2)^2} \int_{[0,\tau]^2} ds_1 ds_2 \left( \frac{\tau}{r} \right)^\beta \left( \frac{\tau}{s_1} \right)^\beta \left( \frac{\tau}{s_2} \right)^\beta \lesssim L_{T,t_1,t_2} N^{4d-3\beta},
\end{align}

which yield (98).

**Case 2:** \( \beta \in (1, 2 \wedge d) \).

\begin{align}
\Phi_{N,t_1,t_2}^{(1)} &\leq L_T \frac{N^{4d-3\beta}}{(\tau \tau_2)^2} \int_{[0,\tau]^2} ds_1 ds_2 \int_0^{s_1 \wedge s_2} dr \\
&\times \left[ N^\beta \left( \frac{s_2(t_2 - s_2)}{t_2} \right)^{-\beta/2} \right] \wedge \left( \frac{t_2}{s_2} \right)^\beta \left[ N^\beta \left( \frac{s_1(t_2 - s_1)}{t_2} \right)^{-\beta/2} \right] \wedge \left( \frac{t_2}{s_1} \right)^\beta \\
&\times \left[ N^\beta \left( \frac{r(s_1 - r)}{s_1} + \frac{r(s_2 - r)}{s_2} \right)^{-\beta/2} \right] \wedge \left( \frac{t_2}{r} \right)^\beta \\
&\lesssim L_{T,t_1,t_2} N^{4d-3\beta+6}, \\
\Phi_{N,t_1,t_2}^{(2)} &\leq L_T \frac{N^{4d-3\beta}}{(\tau \tau_2)^2} \int_{[0,\tau]^2} ds_1 ds_2 \\
&\times \left[ N^\beta \left( \frac{r(t_1 - r)}{t_1} \right)^{-\beta/2} \right] \wedge \left( \frac{\tau}{s_1} \right)^\beta \left[ N^\beta \left( \frac{s_1(r - s_1)}{r} \right)^{-\beta/2} \right] \wedge \left( \frac{\tau}{s_1} \right)^\beta \\
&\times \left[ N^\beta \left( \frac{s_2(r - s_2)}{r} \right)^{-\beta/2} \right] \wedge \left( \frac{\tau}{s_2} \right)^\beta,
\end{align}

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\[
\begin{align*}
    &\lesssim L_T \frac{N^{d-3\beta}}{(\tau_1^2)^{\beta}} \int_0^{t_1} dr \int_{[0,r]} ds_1 ds_2 \left(1 + \tau_1^2\right)^{3} \left[N^\beta \left(\frac{r(t_1 - r)}{t_1}\right)^{-\beta/2}\right] \wedge \left(t_1 \frac{\tau_1}{s_1}\right)^{\beta} \\
    &\times \left[N^\beta \left(\frac{s_1(r - s_1)}{s_1}\right)^{-\beta/2}\right] \wedge \left(t_1 \frac{\tau_1}{s_1}\right)^{\beta} \left[N^\beta \left(\frac{s_2(r - s_2)}{s_2}\right)^{-\beta/2}\right] \wedge \left(t_1 \frac{\tau_1}{s_2}\right)^{\beta}
\end{align*}
\]
which yield (100).

**Case 3:** $\beta = 1$. Notice that (119) and (120) still hold for $\beta = 1$, it is easy to see that

\[
\Phi_{N,t_1,t_2}^{(1)} \lesssim L_{T,t_1,t_2} N^{4d-3}(\log N)^3 \quad \text{and} \quad \Phi_{N,t_1,t_2}^{(2)} \lesssim L_{T,t_1,t_2} N^{4d-3}(\log N)^3.
\]

This proves (99).

Now, we proceed with the proofs of Theorems 1, 2 and 3.

**Proof of Theorems.** We first prove Theorem 1, and the proofs of Theorem 2 and 3 follow by the same arguments. Choose and fix some $T > 0$, by Lemma 11 and Proposition 14, a standard application of Kolmogorov’s continuity theorem and the Arzelà-Ascoli theorem ensures that $\{\sqrt{N}S_N \cdot \}_{N \geq e}$ is a tight net of processes on $C[0,T]$, so it remains to prove that the finite-dimensional distributions of the process $t \mapsto \sqrt{N}S_N t$ converge to those of $\{G_t\}_{t \in [0,T]}$.

Let us choose and fix some $T > 0$ and $m \geq 1$ points $t_1, \ldots, t_m \in (0,T)$. Consider the random vector $F_N = (F_N^{(1)}, \ldots, F_N^{(m)})$ defined by

\[
F_N^{(i)} := \sqrt{N}S_{N,t_i}, \quad \text{for} \ i = 1, \ldots, m,
\]
and let $G = (G_{t_1}, \ldots, G_{t_m})$ be a centered Gaussian random vector with covariance matrix $(g_{i,j})_{1 \leq i,j \leq m}$. Recall from (93) to see that $F_N^{(i)} = \delta(\sqrt{N}U_{N,t_i})$, for all $i = 1, \ldots, m$. Let $V_N^{(i)} = \sqrt{N}U_{N,t_i}$ and $V_N = (V_N^{(1)}, \ldots, V_N^{(m)})$. Lemma 5 ensures that

\[
|E(h(F_N)) - E(h(G))| \leq \frac{1}{2} \|h''\|_\infty \sqrt{\sum_{i,j=1}^{m} E\left|g_{i,t_i} - \langle DF_N^{(i)}, V_N^{(j)}\rangle_H\right|^2},
\]
for all $h \in C_b^2(\mathbb{R}^m)$. Therefore, it suffices to show that for any $i, j = 1, \ldots, m$,

\[
\lim_{N \to \infty} E\left|g_{i,t_j} - \langle DF_N^{(i)}, V_N^{(j)}\rangle_H\right|^2 = 0.
\]

On one hand, applying Lemma 5.2 in [2] and Proposition 7, we find that, as $N \to \infty$,

\[
E(DF_N^{(i)}, V_N^{(j)})_H = \text{Cov}(\sqrt{N}S_{N,t_i}, \sqrt{N}S_{N,t_j}) \to g_{i,t_j}.
\]
On the other hand, we can apply Lemma 20 to see that

$$\lim_{N \to \infty} \text{Var}(DF_N^{(i)}, V_N^{(j)})_{\mathcal{H}} = 0. \quad (125)$$

Combining (124) and (125), we finish the proof. \qed

**Appendix A**

**Lemma A.1.** (1). Let $\psi_{\tau_1, \tau_2}(z)$ be defined in (29), then we have

$$|\psi_{\tau_1, \tau_2}(z)| \lesssim 1 \wedge \|z\|^{-2},$$

(2). Assume $\mathcal{R}(f) < \infty$, then

$$\int_0^\infty \frac{ds}{s^d} \int_{\mathbb{R}^d} \hat{f}(dz) \left| \left[ \left( I_m + \hat{I}_n \right) \left( \frac{\cdot}{s} \right) \right](z) \right| < \infty.$$

**Proof.** (1). By a few computations, it is easy to find that there exists $L_{\tau_2, d} > 0$ such that

$$\left| \hat{I}_{\tau_2}(z) \right| = \prod_{j=1}^d \sqrt{\frac{2(1 - \cos(\tau_2 z_j))}{(\tau_2 z_j)^2}} \leq L_{\tau_2, d} \prod_{j=1}^d \left( 1 \wedge \frac{1}{(z_j)^2} \right) \leq L_{\tau_2, d} \sqrt{1 \wedge (\max_j z_j)^{-2}} \leq L_{\tau_2, d} \sqrt{1 \wedge \|z\|^{-2}}.$$

Similarly,

$$\left| \hat{I}_{\tau_1}(z) \right| \leq L_{\tau_1, d} \sqrt{1 \wedge \|z\|^{-2}}.$$

Hence,

$$|\psi_{\tau_1, \tau_2}(z)| \leq L_{\tau_1, \tau_2, d} \left| \hat{I}_{\tau_2}(z) \right| \left| \hat{I}_{\tau_1}(z) \right| \lesssim 1 \wedge \|z\|^{-2}.$$

(2). From Lemma 5.9 in [3], we know that $\mathcal{R}(f) < \infty$ is equivalent to $\int_{\mathbb{R}^d} \|z\|^{-1} \hat{f}(dz) < \infty$. Hence,

$$\int_0^\infty \frac{ds}{s^d} \int_{\mathbb{R}^d} \hat{f}(dz) \left| \left[ \left( I_m + \hat{I}_n \right) \left( \frac{\cdot}{s} \right) \right](z) \right|$$

$$\leq (mn)^d (C_{m,n})^d \int_0^\infty \frac{ds}{s^d} \int_{\mathbb{R}^d} \hat{f}(dz) \prod_{j=1}^d \left( 1 \wedge \frac{1}{(s z_j)^2} \right)$$

$$\lesssim (C_{m,n})^d \int_0^\infty ds \int_{\mathbb{R}^d} \hat{f}(dz) \left( 1 \wedge \frac{1}{(s \|z\|)^2} \right)$$
\[ \lesssim (C_{m,n})^d \int_0^\infty \left(1 \wedge \frac{1}{r^2}\right) \, dr \int_{\mathbb{R}^d} \|z\|^{-1} f(z) \, dz < \infty. \]

\[ \square \]

**Lemma A.2.** (1). For every \( \varepsilon \in (0,1) \),
\[ \int_0^\infty (s^{-2} \wedge \varepsilon) \, ds \leq 2\sqrt{\varepsilon}. \]

(2). For every \( \varepsilon \in (0,1) \) and \( \beta \in (0,1) \), there exists a real number \( C_\beta > 0 \), such that
\[ \int_{\mathbb{R}^d} \|\xi\|^{\beta-d} \left(\varepsilon \wedge \frac{1}{\|\xi\|^2}\right) \, d\xi \leq C_\beta \sqrt{\varepsilon}. \]

**Proof.** (1).
\[ \int_0^\infty (s^{-2} \wedge \varepsilon) \, ds \leq \int_0^1 \varepsilon \, ds + \varepsilon \int_1^\infty \left(\frac{1}{\varepsilon s^2} \wedge 1\right) \, ds \]
\[ \leq \varepsilon + \sqrt{\varepsilon} \left(\int_1^\infty \frac{1}{s^2} \, ds + \int_1^\infty \frac{1}{s} \, ds\right) = 2\sqrt{\varepsilon}, \]
where we use a change of variables \( s \mapsto s/\sqrt{\varepsilon} \) in the second inequality.

(2). Similarly,
\[ \int_{\mathbb{R}^d} \|\xi\|^{\beta-d} \left(\varepsilon \wedge \frac{1}{\|\xi\|^2}\right) \, d\xi = \int_0^\infty r^{\beta-1} \left(\varepsilon \wedge \frac{1}{\|\xi\|^2}\right) \, dr \]
\[ \leq \varepsilon \int_0^1 r^{\beta-1} \, dr + \varepsilon \int_1^\infty r^{\beta-1} \left(1 \wedge \frac{1}{(\sqrt{\varepsilon} r)^2}\right) \, dr \]
\[ = \frac{\varepsilon}{\beta} + (\sqrt{\varepsilon})^{2-\beta} \left(\int_0^1 r^{\beta-1} \, dr + \int_1^\infty r^{\beta-3} \, dr\right) \]
\[ = \left(\frac{1}{\beta} + \frac{1}{2-\beta}\right) \varepsilon^{1-\beta/2} \leq C_\beta \sqrt{\varepsilon}. \]

\[ \square \]

**Lemma A.3.** (2, Lemma A.1) Define
\[ G_{N,t}(x) := \frac{t}{\log N} \int_0^t \exp \left(-\frac{(t-s)t}{s} \cdot \frac{x^2}{N^2}\right) \, ds \quad \text{for all } N,t > 0 \text{ and } x \in \mathbb{R}\setminus\{0\}. \]

Then, for every \( t > 0 \) and \( x \in \mathbb{R}\setminus\{0\} \),
\[ \sup_{N \geq e} G_{N,t}(x) \leq 7t \log_+ (1/t) \log_+ (1/|x|), \]
where $\log_+(w) = \log(e + w)$ for all $w \geq 0$. Moreover,

$$\lim_{N \to \infty} G_{N,t}(x) = 2t, \quad \text{for every } t > 0 \text{ and } x \in \mathbb{R}.$$ 

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