CODIMENSION ESTIMATES IN MEAN CURVATURE FLOW

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Abstract

We show that the blow-ups of compact solutions to the mean curvature flow in $\mathbb{R}^N$ initially satisfying the pinching condition $|H| > 0$ and $|A|^2 < c|H|^2$ for some suitable constant $c = c(n)$ must be codimension one.

1. Introduction

In this note, we are interested in studying the codimension of blow-ups for the mean curvature flow in codimension greater than one. In general, this is a difficult problem. In higher codimension, the second fundamental form is much more complicated and useful preserved curvature conditions for the mean curvature flow are, so far, relatively rare. Colding and Minicozzi [CM19] have shown that if the asymptotic shrinker of an ancient solution is a multiplicity one cylinder, then the solution must be mean curvature flow are, so far, relatively rare. Colding and Minicozzi [CM19] have shown that if the asymptotic shrinker of an ancient solution is a multiplicity one cylinder, then the solution must be mean curvature flow.

Andrews, Baker, and Nguyen in [AB10], [Ngu18] and we show that blow-ups are codimension one directly. Here we take an alternative approach. We work with the preserved curvature pinching investigated by Andrews, Baker, and Nguyen in [AB10], [Ngu18] and we show that blow-ups are codimension one directly.

Suppose $M_0 \subset \mathbb{R}^N$ is an $n$-dimensional closed submanifold. In [AB10], it was shown that curvature pinching of the form $|H| > 0$ and $|A|^2 < c|H|^2$ is preserved by the mean curvature flow if $c \leq \frac{4}{3n}$. This bound is a technical constraint used in their proof of preservation. A more natural condition is to take $c = \frac{4}{3n}$ or $c = \frac{4}{3n}$. Note that $\frac{4}{3n} \leq \frac{4}{3n}$ if $n \geq 4$ and $\frac{4}{3n} \leq \frac{4}{3n}$ if $n \geq 8$. In codimension one, taking $c = \frac{4}{3n}$ implies convexity of the hypersurface and taking $c = \frac{4}{3n}$ implies two-convexity of the hypersurface. The study of the mean curvature flow of convex and two-convex hypersurfaces are the foundational works of Huisken [Hui84] and Huisken-Sinestrari [HS99],[HS09]. The results of [AB10] and [Ngu18] are extensions of these results to higher codimension assuming these stronger pinching conditions.

In the seminal paper [Hui84] (which draws upon Hamilton’s foundational work on the Ricci flow [Ham82]), Huisken proved the mean curvature flow evolves compact convex hypersurfaces into spherical singularities. Using the techniques developed there (in particular the delicate Stampacchia iteration), Andrews and Baker in [AB10] proved that the mean curvature flow in $\mathbb{R}^N$ will deform compact $n$-dimensional initial data satisfying

$$|H| > 0, \quad |A|^2 < c_n|H|^2 \quad c_n = \begin{cases} \frac{4}{3n} & n = 2, 3 \\ \frac{1}{n-1} & n \geq 4 \end{cases}$$

to a point in finite time. In particular, the flow is asymptotic to a family of shrinking spheres contained in some $(n+1)$-dimensional affine space in $\mathbb{R}^N$.

Because $\frac{4}{3n} \leq \frac{1}{n-1}$ for $n \leq 4$ and the pinching is only preserved for $c \leq \frac{4}{3n}$, the previous result is the best currently possible for preserved pinching if $n \leq 4$. Suppose now we have a compact initial manifold satisfying the weaker pinching condition of Nguyen

$$|H| > 0, \quad |A|^2 < c_n|H|^2 \quad c_n = \begin{cases} \frac{4}{3n} & n = 5, 6, 7 \\ \frac{1}{n-1} & n \geq 8 \end{cases}.$$ 

By the work of Huisken and Sinestrari in [HS99], [HS09], if we evolve a two-convex hypersurface by the mean curvature flow, then the blow-ups must be weakly convex (by the almost convexity estimate) and the only singularities that can form along the flow are neck-pinch singularities (by the cylindrical estimate). Also very important in their work is the pointwise gradient estimate. In higher codimension, we no longer have a notion of convexity, but the gradient and cylindrical estimates still make sense. By first proving a pointwise gradient estimate using the pinching condition, Nguyen in [Ngu18] managed to prove a cylindrical estimate in higher codimension. Specifically, his quantitative cylindrical estimates show the following alternative: either a blow-up at the first singular time is compact or there are regions...
of the manifold \( M_t \) which are becoming arbitrary close to the codimension one cylinder \( S^{n-1} \times \mathbb{R} \) up to the first singular time.

The result of [Ngu18] leaves open the possibility that the “cap” of a forming cylindrical singularity may not lie in an \((n+1)\)-dimensional subspace. Presently, we rule out this possibility and show the pinching implies all blow-ups are codimension one. Results of this type have been obtained before: Altschuler proved singularities of the curve shortening flow in \( \mathbb{R}^2 \) must be planar (see [Alt91]).

Here is our setting. We suppose \( n \geq 5 \) and our initial data is a compact \( n \)-dimensional submanifold \( M_0 \subset \mathbb{R}^N \) satisfying

\[
|H| > 0, \quad |A|^2 < c_n|H|^2, \quad c_n = \begin{cases} \frac{3(n+1)}{2(n+2)} & n = 5, 6, 7 \\ \frac{4}{3n} & n \geq 8 \end{cases}
\]

For \( n = 5 \) and \( n = 6 \), \( c_n \) is strictly between \( \frac{1}{3} \) and \( \frac{4}{15} \). This value of \( c_n \) in these dimensions is the largest we can allow in our new estimates in the proof of our main theorem below. For \( n \geq 7 \), \( \frac{3(n+1)}{2(n+2)} \geq \frac{4}{15} \), with equality for \( n = 7 \). The value of \( c_n \) in higher dimensions is the largest allowed by estimates in the preservation of pinching in [AB10]. We use these estimates as well. Under these assumptions, we consider a maximal solution \( M_t, t \in [0, T) \) to the mean curvature flow where \( T \) is the first singular time.

For the purpose of studying codimension, we define a tensor \( \hat{A} \) by

\[
\hat{A}(X, Y) = A(X, Y) - \frac{\langle A(X, Y), H \rangle}{|H|^2} H.
\]

for vector fields \( X \) and \( Y \) tangent to \( M_t \). As \( |H| > 0 \) initially (and is preserved) this tensor is well-defined. The importance of \( \hat{A} \) is that under our pinching assumption \( \hat{A} \) vanishes identically if and only if our submanifold is a hypersurface inside an \((n+1)\)-dimensional affine subspace of \( \mathbb{R}^N \). See Proposition 2.4 in Section 2.

Here is our main theorem.

**Theorem 1.1.** Suppose \( n \geq 5 \). Let \( c_n = \frac{1}{n-2} \) if \( n \geq 8 \) and \( c_n = \frac{3(n+1)}{2n(n+2)} \) if \( n = 5, 6, \) or \( 7 \). Suppose \( M_t, t \in [0, T) \) is a smooth, compact, \( n \)-dimensional solution to mean curvature flow in \( \mathbb{R}^N \) initially satisfying \( |H| > 0 \), and \( |A|^2 < c_n|H|^2 \). Then there are constants \( \sigma = \sigma(n, M_0) > 0 \) and \( C = C(n, M_0) < \infty \), depending upon \( n \) and the initial submanifold \( M_0 \), such that

\[
|\hat{A}|^2 \leq C|H|^{2-\sigma}
\]

for all \( t \in [0, T) \).

Together with Proposition 2.4 below, this result shows that at the first singular time, blow-ups must be codimension one. Since \( \frac{1}{n-2} \leq \frac{1}{3n} \) for \( n \geq 8 \), the pinching condition considered in [Ngu18] is included in the theorem above. Our result also applies for weaker pinching constants of the form \( c = \frac{1}{n-k} \), for \( n \geq 4k \) sufficiently large so that \( \frac{1}{n-k} \leq \frac{4}{15} \). These weaker pinching constants will allow a wider range of singularities models and our result shows these must also be codimension one. We note that for \( c = \frac{1}{n-2} \), if we knew that singularities were also noncollapsed, then the classification in codimension one by Brendle and Choi in [BC18] would give a complete classification of singularity models under this pinching as well.

The structure of this note is as follows. In Section 2, we record various notation and standard identities for the higher codimension mean curvature flow that we will use. We also show prove if \( \hat{A} \) vanishes, the submanifold is codimension one. In Section 3, we derive the evolution equation for \( |A|^2 \). In Section 4, we prove Theorem 1.1 via the maximum principle.

There is a connection (observed, for example, in [Bre19]) between the mean curvature flow of convex and two-convex hypersurfaces and the Ricci flow of initial data with positive isotropic curvature. Positive isotropic curvature was introduced by Micallef and Moore [MM88] for the study of minimal two-spheres and has been studied in the Ricci flow since Hamilton’s fundamental paper [Ham97]. If the pinching constant \( c = \frac{1}{n-2} \) (or one has a two-convex hypersurface), then the induced metric on \( M_0 \) has positive isotropic curvature (denoted PIC). Consequently, if \( c = \frac{1}{n-2} \) (or one has a convex hypersurface) the induced metric on \( M \times \mathbb{R} \) has positive isotropic curvature (this property is called PIC1). Brendle in [Bre08] showed that the Ricci flow of PIC1 initial data flows into round spheres. As for PIC initial data, one of Hamilton’s breakthroughs in [Ham97] was that in dimension four the Ricci flow of PIC manifolds only develops neck-like singularities. The study of PIC initial data for the Ricci flow in higher dimensions \((n \geq 12)\) has recently been solved by Brendle in [Bre19]. Both of these results in Ricci flow are of course analogous to the results of Huisken and Sinestrari and consequently to Andrews, Baker, and Nguyen as well.

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2. Notation and Preliminaries

In this section we record notation and identities we will use in the proofs of our results. We will let \( \nabla \) denote both the ambient connection on \( \mathbb{R}^N \) and its restriction to the tangent bundle, \( TM \). We will let \( \nabla^\perp \) denote the connection on the normal bundle, \( N.M. \) We will do our computations in a local orthonormal frame. For a fixed time \( t \in [0,T) \), we let \( e_1, \ldots, e_n \) denote a local orthonormal frame in a neighborhood of a point \( p \in M_t \). We may assume that \( \nabla_t e_j = \nabla_{e_j} e_j = 0 \) at a point \( p \).

Repeated indices will indicate summation. Sometimes we will include the summation symbol to emphasize its presence. Since we work with an orthonormal basis we can raise or lower indices freely (except for the metric tensor). For example, \( T_{ik} S_{jk} = T_{ik} S_{jk}^k = g^{kl} T_{ik} S_{jl} = \sum_{k=1}^n T_{ik} S_{jk}. \)

For taking the time derivative of traced tensors, the third form above above is best. We recall that the evolution equations for the metric and its inverse in higher codimension are

\[
\frac{\partial}{\partial t} g_{ij} = -2\langle A_{ij}, H \rangle, \quad \frac{\partial}{\partial t} g^{ij} = 2\langle A^{ij}, H \rangle = 2\langle A_{ij}, H \rangle.
\]

We will not use indices for the components of tensors valued in the normal bundle. Instead we will use \( \langle A, \omega \rangle \) where \( \omega \) is a local frame for the normal bundle and \( A_{ij} = \langle A(e_i, e_j), \omega \rangle \), then our convention is

\[
|\langle A_{ij}, A_{kl} \rangle|^2 = \langle A_{ij}, A_{kl} \rangle \langle A_{ij}, A_{kl} \rangle = \sum_{i,j,k,l=1}^n A_{ijk\alpha} A_{ij\beta} A_{kl\alpha} A_{k\beta}.
\]

For the norm of traced tensors, summation will always take place inside the norm. For example,

\[
|\langle A_{ik}, A_{jk} \rangle|^2 = \left| \sum_{k=1}^n \langle A_{ik}, A_{jk} \rangle \right|^2 = \langle A_{ik}, A_{jk} \rangle \langle A_{il}, A_{jl} \rangle = \sum_{i,j,k,l=1}^n \left( \sum_{k=1}^n \langle A_{ik}, A_{jk} \rangle \right) \left( \sum_{l=1}^n \langle A_{il}, A_{jl} \rangle \right).
\]

The curvature and normal curvature are denoted by \( R \) and \( R^+ \) respectively, and our sign convention is that

\[
R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{\langle Y,X \rangle} Z, \quad R^+(X,Y)\nu = \nabla_\nu \nabla_X \nu - \nabla_X \nabla_\nu \nu - \nabla_{\langle Y,X \rangle} \nu.
\]

In higher codimension, the Gauss, Codazzi, and Ricci equations in a local frame take the form

\[
R_{ijkl} = \langle A_{ik}, A_{jl} \rangle - \langle A_{il}, A_{jk} \rangle, \quad \nabla^\perp_X A_{jk} = \nabla^\perp_j A_{ik}, \quad R^+_{ijkl}(\nu) = \langle A_{ik}, \nu \rangle A_{jk} - \langle A_{jk}, \nu \rangle A_{ik}.
\]

We let \( H_1 = |H| \) denote the norm of the mean curvature. Since we assume \( H_1 > 0 \), we define \( \nu_1 = H_1^{-1} H \) to denote the principal normal direction. Note that \( |\nu_1| = 1 \). Then the tensor \( h_{ij} = \langle A_{ij}, \nu_1 \rangle \) is the component of the second fundamental form in the principal normal direction (and the only nonzero component if our submanifold is codimension one). With this notation we have

\[
A_{ij} = \hat{A}_{ij} + h_{ij} \nu_1
\]

\[
\hat{A}_{ij} = \hat{A}_{ij} + \hat{h}_{ij} \nu_1 + \frac{1}{n} H_1 g_{ij} \nu_1, \quad H_1 \nu_1, \quad \langle A \rangle^2 = |\hat{A}|^2 + |\hat{h}|^2 + \frac{1}{n} H_1^2, \quad |H|^2 = H_1^2.
\]

We will use often that \( \hat{A}_{ij} \) is traceless and the orthogonality relations

\[
\langle \hat{A}_{ij}, \nu_1 \rangle = \langle \nabla^\perp_1 \nu_1, \nu_1 \rangle = 0.
\]
We define a new connection for the orthogonal decomposition of $\mathcal{N} = E_1 \oplus \hat{E}$ where $\hat{E}$ consists of normal vectors $\hat{\nu}$ everywhere orthogonal to $\nu_1$, $\langle \hat{\nu}, \nu_1 \rangle = 0$, and $E_1 = C^\infty(M)\nu_1$. Define $\hat{\nabla}^\perp$ on $\hat{E}$ by

$$\hat{\nabla}^\perp_i \hat{\nu} = \nabla^\perp_i \hat{\nu} - \langle \nabla^\perp_i \hat{\nu}, \nu_1 \rangle \nu_1.$$ 

Since by definition $\hat{A}$ maps $TM \otimes TM$ into $\hat{E}$, we can define the connection $\hat{\nabla}^\perp$ on $\hat{A}$ by

$$\hat{\nabla}^\perp_i \hat{A}_{jk} = \nabla^\perp_i \hat{A}_{jk} - \langle \nabla^\perp_i \hat{A}_{jk}, \nu_1 \rangle \nu_1.$$ 

The are various relations between our connections $\nabla$, $\nabla^\perp$, $\hat{\nabla}^\perp$. For example, by viewing $A$ and $H$ as sections of $\mathbb{R}^N$, we can decompose the tensors $\nabla A$ and $\nabla H$ into the tangential and normal components using $\mathbb{R}^N = TM \oplus \mathcal{N}$ to get

$$\nabla_i A_{jk} = (\nabla^\perp_i A_{jk})^\perp + (\nabla_i A_{jk})^\top = \nabla^\perp_i A_{jk} - \sum_{l=1}^n \langle A_{jk}, A_{il} \rangle e_l,$$

$$\nabla_i H = (\nabla^\perp_i H)^\perp + (\nabla_i H)^\top = \nabla^\perp_i H - \sum_{j=1}^n \langle H, A_{ij} \rangle e_j.$$

Similarly, and more relevant for the coming computations, we can decompose the tensors $\nabla^\perp A$, $\nabla^\perp H$, and $\nabla^\perp A$ via the decomposition $\mathcal{N} = E_1 \oplus \hat{E}$ to get

$$\nabla^\perp_i A_{jk} = \hat{\nabla}^\perp_i \hat{A}_{jk} + h_{jk} \nabla_i \nu_1 + (\langle \hat{\nabla}^\perp_i \hat{A}_{jk}, \nu_1 \rangle + \nabla_i h_{jk}) \nu_1,$$

$$\nabla^\perp_i H = H_i \nabla^\perp_i \nu_1 + \nabla_i H \nu_1,$$

$$\nabla^\perp_i \hat{A}_{jk} = \hat{\nabla}^\perp_i \hat{A}_{jk} + \langle \hat{\nabla}^\perp_i \hat{A}_{jk}, \nu_1 \rangle \nu_1.$$

Consequently, we have

**Proposition 2.1** (Decomposition of gradients).

$$|\nabla^\perp A|^2 = |\nabla^\perp_i A_{jk} + h_{jk} \nabla_i \nu_1|^2 + |\langle \nabla^\perp_i \hat{A}_{jk}, \nu_1 \rangle + \nabla_i h_{jk}|^2.$$ 

$$|\nabla^\perp H|^2 = |H_i \nabla^\perp_i \nu_1|^2 + |\nabla_i H|^2.$$ 

$$|\nabla^\perp A|^2 = |\nabla^\perp_i A|^2 + |\langle \nabla^\perp_i \hat{A}, \nu_1 \rangle|^2.$$ 

We will use these identities in Sections 3 and 4.

It is useful to consider the Codazzi equation under the decomposition of $\nabla^\perp_i A_{jk}$ above. Projecting the Codazzi equation onto $E_1$ and $\hat{E}$ implies the tensors

$$\nabla_i h_{jk} + \langle \nabla^\perp_i \hat{A}_{jk}, \nu_1 \rangle,$$

$$\hat{\nabla}^\perp_i \hat{A}_{jk} + h_{jk} \nabla_i \nu_1$$

are symmetric in all of their indices. In particular, tracing different pairs of indices, we arrive at

$$\sum_{k=1}^n \nabla_k h_{ik} + \langle \nabla^\perp_k \hat{A}_{ik}, \nu_1 \rangle = \nabla_i H_1,$$

$$\sum_{k=1}^n \nabla_k \hat{A}_{ik} + h_{ik} \nabla^\perp_1 \nu_1 = H_i \nabla^\perp_1 \nu_1.$$ 

Next, we review the evolution equations of $A$ and $H$ in higher codimension. For derivations of these equations see [Smo12] or [AB10]. If we let $\frac{\partial}{\partial t}^\perp$ denote the projection of the time derivative onto the normal bundle and $\Delta^\perp$ the Laplacian with respect to the connection $\nabla^\perp$, then:

**Proposition 2.2** (Evolution of $A$ and $H$). With the summation convention, the evolution equations of $A_{ij}$ and $H$ are

$$\left( \frac{\partial}{\partial t}^\perp - \Delta^\perp \right) A_{ij} = -\langle H, A_{ik} \rangle A_{jk} - \langle H, A_{jk} \rangle A_{ik} + \langle A_{ij}, A_{kl} \rangle A_{kl} - 2 \langle A_{ik}, A_{jl} \rangle A_{kl} + \langle A_{ik}, A_{kl} \rangle A_{jl} + \langle A_{ij}, A_{kl} \rangle A_{ik},$$

$$\left( \frac{\partial}{\partial t}^\perp - \Delta^\perp \right) H = \langle A_{kl}, H \rangle A_{kl}.$$
The evolution equations of $|A|^2$ and $|H|^2$ are
\[
\frac{\partial}{\partial t} |A|^2 = \Delta |A|^2 - 2|\nabla^\perp A|^2 + 2|\langle A_{ij}, A_{kl} \rangle|^2 + 2|R_{ij}^+|^2, \\
\frac{\partial}{\partial t} |H|^2 = \Delta |H|^2 - 2|\nabla^\perp H|^2 + 2|\langle A_{ij}, H \rangle|^2.
\]

Let us describe these reaction terms in greater detail. By the Ricci equation, we can view the normal curvature as a section of $NM \otimes NM$ and in this case,
\[
R_{ij}^+ = A_{ik} \otimes A_{jk} - A_{jk} \otimes A_{ik} = \sum_{k=1}^n A_{ik} \otimes A_{jk} - A_{jk} \otimes A_{ik}.
\]

If we define the inner product on $NM \otimes NM$ in the usual way, $\langle \nu \otimes \mu, \tilde{\nu} \otimes \tilde{\mu} \rangle = \langle \nu, \tilde{\nu} \rangle \langle \mu, \tilde{\mu} \rangle$, then we have the following formulas for each of our reaction terms above
\[
\begin{align*}
|\langle A_{ij}, H \rangle|^2 &= \langle A_{ij}, H \rangle \langle A_{ij}, H \rangle, \\
|\langle A_{ij}, A_{kl} \rangle|^2 &= \langle A_{ij}, A_{kl} \rangle \langle A_{ij}, A_{kl} \rangle, \\
|R_{ij}^+|^2 &= \langle A_{ik} \otimes A_{jk} - A_{jk} \otimes A_{ik} , A_{il} \otimes A_{jl} - A_{jl} \otimes A_{il} \rangle \\
&= 2\langle A_{ik}, A_{il} \rangle \langle A_{jk}, A_{jl} \rangle - 2\langle A_{ik}, A_{jl} \rangle \langle A_{il}, A_{jk} \rangle.
\end{align*}
\]

Note that in each of these formulas, all of the indices are being summed over.

For the coming computations, it will be useful to expand the right hand sides of the formulas above in terms of $\hat{A}$, $\mathring{h}$, and $H_1$ using $A = \hat{A} + \mathring{h} \nu + \frac{1}{n} H_1 g \nu$. Doing so, we arrive at
\[
\begin{align*}
\langle A_{ij}, H \rangle &= H_1 \mathring{h}_{ij} \\
&= \frac{1}{n} H_1^2 g_{ij} + H_1 \mathring{h}_{ij}, \\
\langle A_{ij}, A_{kl} \rangle &= \mathring{h}_{ij} \mathring{h}_{kl} + (\hat{A}_{ij}, \hat{A}_{kl}) \\
&= \frac{1}{n^2} H_1^2 (\mathring{h}_{ij})^2 + \frac{1}{n} H_1 (\mathring{h}_{ij} \mathring{h}_{kl} + \hat{A}_{ij} \mathring{h}_{kl}) \mathring{h}_{ij} + \hat{A}_{ij} \mathring{h}_{kl} + (\hat{A}_{ij}, \hat{A}_{kl}), \\
R_{ij}^+(\nu) &= \sum_{k=1}^n \mathring{h}_{ik} \mathring{A}_{jk} - \hat{h}_{jk} \mathring{A}_{ik}, \\
R_{ij}^+(\nu) &= \sum_{k=1}^n (\hat{A}_{ik} \otimes \mathring{A}_{jk} - \mathring{A}_{jk} \otimes \hat{A}_{ik}) + R_{ij}^+(\nu) \otimes \nu + \nu \otimes R_{ij}^+(\nu).
\end{align*}
\]

As a consequence, we have the following proposition which we record for use in later sections.

**Proposition 2.3** (Decomposition of reaction terms).
\[
\begin{align*}
|\langle A_{ij}, H \rangle|^2 &= H_1^2 |\mathring{h}|^2 \\
&= \frac{1}{n} H_1^2 + \mathring{H}_1^2 |\mathring{h}|^2 \\
|\langle A_{ij}, A_{kl} \rangle|^2 &= |\mathring{h}|^4 + 2 \sum_{i,j=1}^n h_{ij} \hat{A}_{ij}^2 + |\langle \hat{A}_{ij}, \hat{A}_{kl} \rangle|^2 \\
&= \frac{1}{n^2} H_1^4 + \frac{2}{n} H_1^2 |\hat{h}|^2 + |\mathring{h}|^4 + 2 \sum_{i,j=1}^n \mathring{h}_{ij} \hat{A}_{ij}^2 + |\langle \hat{A}_{ij}, \hat{A}_{kl} \rangle|^2 \\
|R_{ij}^+(\nu)|^2 &= \sum_{k=1}^n h_{ik} \mathring{A}_{jk} - \hat{h}_{jk} \mathring{A}_{ik}^2 \\
&= \sum_{k=1}^n (\hat{A}_{ik} \otimes \mathring{A}_{jk} - \mathring{A}_{jk} \otimes \hat{A}_{ik})^2 + 2|R_{ij}^+(\nu)|^2 \\
|R_{ij}^+|^2 &= \sum_{k=1}^n |\langle \hat{A}_{ik} \otimes \mathring{A}_{jk} - \mathring{A}_{jk} \otimes \hat{A}_{ik} \rangle|^2 + 2|R_{ij}^+(\nu)|^2.
\end{align*}
\]

From here on, except in lemma statements, we will drop the summation symbols in preference of
more concise presentation. In particular, we will use
\[ |\hat{A}_{ik} \otimes A_{jk} - \hat{A}_{ijk} \otimes A_{ik}|^2 = | \sum_{k=1}^{n} (\hat{A}_{ik} \otimes A_{jk} - \hat{A}_{ijk} \otimes A_{ik}) |^2, \]
\[ |\hat{h}_{ij} \hat{A}_{ij}|^2 = | \sum_{i,j=1}^{n} \hat{h}_{ij} \hat{A}_{ij} |^2. \]

We end by giving a proof that the vanishing of \( \hat{A} \) implies codimension one. In application, the work of Nguyen shows that if the initial manifold is pinched with constant \( c = \frac{1}{n-1} \), then blow-ups will be pinched with \( c = \frac{1}{n-1} \). The argument below works as long as the tensor \( H \) is positive definite (which is equivalent to \( h_{ij} \) be \( (n-1) \)-convex).

**Proposition 2.4.** Let \( n \geq 5 \) and \( c_n \leq \frac{4}{3n} \). Suppose \( M \subset \mathbb{R}^N \) is a connected complete \( n \)-dimensional submanifold satisfying \( |H| > 0 \), \( |A|^2 \leq c_n |H|^2 \), and \( \hat{A} \equiv 0 \). Then \( M \) is a hypersurface in some \( (n+1) \)-dimensional affine space in \( \mathbb{R}^N \).

**Proof.** Let \( N = n + \bar{n} \). Because \( |H| > 0 \), the principal normal \( \nu_1 \) is well-defined. The vanishing of \( \hat{A} \) in addition to our pinching assumption implies \( \nu_1 \) is parallel with respect to \( \nabla^\perp \). Specifically, by the Codazzi identity, we have
\[ H_1 \nabla^\perp_1 \nu_1 = \sum_{k=1}^{n} \nabla^\perp_1 \hat{A}_{ik} + h_{ik} \nabla^\perp_1 \nu_1 = h_{ik} \nabla^\perp_1 \nu_1. \]

Since \( |h| \leq \frac{1}{n} H^2 \), the tensor \( H_1 \hat{g}_{ij} - h_{ik} \) is positive definite, which shows \( \nabla^\perp \nu_1 = 0 \).

Now pick a point \( p \in M \) and define \( \nu_2, \ldots, \nu_{n} \) to be the completion of \( \nu_1(p) \) to an orthonormal basis of \( N_p M \). Let \( \beta = 2, \ldots, n \). Consider an arbitrary point \( q \in M \) and let \( \gamma : [0, 1] \rightarrow M \) be a path connecting \( p \) to \( q \). Define \( \nu_\beta(t) \) along \( \gamma \) to be the parallel transport of \( \nu_\beta \) with respect to \( \nabla^\perp \). Because \( \nu_1 \) is parallel with respect to \( \nabla^\perp \) and \( \langle \nu_1(0), \nu_1(p) \rangle = \langle \nu_1, \nu_1(p) \rangle = 0 \), we have \( \langle \nu_\beta(t), \nu_1(\gamma(t)) \rangle = 0 \) for all \( t \). If we let \( e_1, \ldots, e_n \) denote a parallel basis of \( T_{\gamma(t)} M \) along \( \gamma \), then
\[ (\nabla^\perp_\gamma(t) \nu_\beta(t))^\top = \sum_{i=1}^{n} (\nabla^\perp_\gamma(t) \nu_\beta(t), e_i) e_i = -\sum_{i=1}^{n} (\nu_\beta(t), A(\gamma', e_i)) e_i = 0 \]

since \( \nu_\beta(t) \) is orthogonal to \( A = \nu \nu_1 \). It follows that
\[ \nabla^\perp_\gamma(t) \nu_\beta(t) = \nabla^\perp_\gamma(t) \nu_\beta(t) + (\nabla^\perp_\gamma(t) \nu_\beta(t))^\top = 0, \]

which shows \( \nu_\beta(t) \) is parallel along \( \gamma \) with respect to the ambient connection \( \nabla \) as well. On the other hand, the constant unit vector field \( \omega_\beta \) in \( \mathbb{R}^N \) defined by the condition \( \omega_\beta(p) = \nu_\beta \), is also parallel along \( \gamma(t) \) with respect to \( \nabla \). By uniqueness of parallel transport, this implies \( \nu_\beta(1) \) agrees with the restriction of \( \omega_\beta \). Since \( q \) was arbitrary, we see that the restriction of the vector fields \( \omega_2, \ldots, \omega_n \) form a parallel orthonormal basis of the complement of \( \nu_1 \) in \( NM \) at every point on \( M \). It follows that \( M \) must lie in a translation of the \( (n+1) \)-dimensional subspace of \( \mathbb{R}^N \) orthogonal to \( \omega_2, \ldots, \omega_n \). \( \square \)

3. Evolution of \( |\hat{A}|^2 \)

In this section, we compute the evolution equation of \( |\hat{A}|^2 \). We do this by using the formulas stated in Section 2. To begin, we note the useful standard identity
\[ \left( \frac{\partial}{\partial t} - \Delta \right) \left( \frac{u}{v} \right) = \frac{1}{v} \left( \frac{\partial}{\partial t} - \Delta \right) u - \frac{u}{v} \left( \frac{\partial}{\partial t} - \Delta \right) v + \frac{2}{v} \nabla_k v \nabla_k \left( \frac{u}{v} \right). \]

Now from the definition, we have
\[ |\hat{A}|^2 = |A|^2 - |(A_{ij}, H)|^2|H|^{-2}. \]

So we will compute the evolution equations of \( |A|^2 \), and \( |(A_{ij}, H)|^2|H|^{-2} \). We have already recorded the evolution equations of \( |A|^2 \) and \( |H|^2 \) in the second part of Proposition 2.2. Namely,
\[ \left( \frac{\partial}{\partial t} - \Delta \right) |A|^2 = -2|\nabla^\perp A|^2 + 2|\langle A_{ij}, A_{kl} \rangle|^2 \]
\[ \left( \frac{\partial}{\partial t} - \Delta \right) |H|^2 = -2|\nabla^\perp H|^2 + 2|\langle A_{ij}, H \rangle|^2. \]
As for the remaining gradient term, we have
\[ \frac{2}{H} \left( \frac{\partial}{\partial t} - \Delta \right) \frac{|\langle A_{ij}, H \rangle|^2}{|H|^2} = |H|^{-2} \left( \frac{\partial}{\partial t} - \Delta \right) |\langle A_{ij}, H \rangle|^2 - |H|^{-4} |\langle A_{ij}, H \rangle|^2 \left( -2 |\nabla^\perp H|^2 + 2 |\langle A_{kl}, H \rangle|^2 \right) + 2 |H|^{-2} \left( \nabla_k |H|^2, \nabla_k \frac{|\langle A_{ij}, H \rangle|^2}{|H|^2} \right) \]

Before computing the evolution of $|\langle A_{ij}, H \rangle|^2$, we simplify the other terms using Propositions 2.1 and 2.3. In particular, using $|\langle A_{ij}, H \rangle|^2 = H_1^2 |h|^2$ and the formula for $|\nabla^\perp H|^2$, we write
\[
2 |H|^{-4} |\langle A_{ij}, H \rangle|^2 |\nabla^\perp H|^2 = 2 |h|^2 |\nabla^\perp \nu_1|^2 + 2 H_1^{-2} |h|^2 |\nabla H_1|^2, \\
-2 |H|^{-4} |\langle A_{ij}, H \rangle|^4 = -2 |h|^4. \]

As for the remaining gradient term, we have $\nabla_k |H|^2 = 2 (\nabla_k H, H)$ and $\nabla_k (|H|^2 |\langle A_{ij}, H \rangle|^2) = \nabla_k |h|^2 = 2 h_{ij} \nabla_k h_{ij}$. Therefore, since $H = H_1 \nu_1$ and $\langle \nabla_k \nu_1, \nu_1 \rangle = 0$, we have
\[
2 |H|^{-2} \left( \nabla_k |H|^2, \nabla_k \frac{|\langle A_{ij}, H \rangle|^2}{|H|^2} \right) = 8 H_1^{-2} \nabla_k^\perp H, H \right) h_{ij} \nabla_k h_{ij} \\
= 8 H_1^{-1} h_{ij} \nabla_k H_1 \nabla_k h_{ij}. \]

To summarize, we have shown so far that
\[
\left( \frac{\partial}{\partial t} - \Delta \right) \frac{|\langle A_{ij}, H \rangle|^2}{|H|^2} = H_1^{-2} \left( \frac{\partial}{\partial t} - \Delta \right) |\langle A_{ij}, H \rangle|^2 - 2 |h|^4 \\
+ 2 |h|^2 |\nabla^\perp \nu_1|^2 + 2 H_1^{-2} |h|^2 |\nabla H_1|^2 + 8 H_1^{-1} h_{ij} \nabla_k H_1 \nabla_k h_{ij}. \]

For the evolution of $\langle A_{ij}, H \rangle$, we have the following lemma.

**Lemma 3.1.**
\[
H_1^{-2} \left( \frac{\partial}{\partial t} - \Delta \right) |\langle A_{ij}, H \rangle|^2 = 4 h_{ij} \hat{A}_{ij} + 2 |H_1|^2 \nu_1^2 + 4 |h|^4 \\
- 4 H_1^{-1} \hat{h}_{ij} \nabla_k H_1 (\nabla^\perp \hat{A}_{ij}, \nabla^\perp \nu_1) - 4 \hat{h}_{ij} (\nabla^\perp \hat{A}_{ij}, \nabla^\perp \nu_1) \\
- 4 |h|^2 |\nabla^\perp \nu_1|^2 - 2 H_1^{-2} |h|^2 |\nabla H_1|^2 - 8 H_1^{-1} h_{ij} \nabla_k H_1 \nabla_k h_{ij} - 2 |\nabla h|^2. \]

**Proof.** Note that any time $h$ is traced with $\hat{A}$ or its derivative, we may replace $h$ with $\hat{h}$ because $\hat{A}$ is traceless. To begin, we substitute the formulas in the first part of Proposition 2.2 which gives
\[
\left( \frac{\partial^{\perp}}{\partial t} - \Delta^{\perp} \right) A_{ij, H} = - \langle H, A_{ik} \rangle \langle A_{jk, H} \rangle - \langle H, A_{jk} \rangle \langle A_{ik, H} \rangle + \langle A_{ij}, H \rangle \langle A_{kl}, H \rangle \\
- 2 \langle A_{ik}, H \rangle \langle A_{jk, H} \rangle + \langle A_{ik}, H \rangle \langle A_{jk}, H \rangle + \langle A_{jk}, H \rangle \langle A_{ik}, H \rangle, \\
\langle A_{ij}, H \rangle \left( \frac{\partial^{\perp}}{\partial t} - \Delta^{\perp} \right) H \rangle = \langle A_{kl}, H \rangle \langle A_{ij}, H \rangle. \]

Tracing each of the equations with a copy of $\langle A_{ij}, H \rangle$, we get
\[
\left( \frac{\partial^{\perp}}{\partial t} - \Delta^{\perp} \right) A_{ij, H} \langle A_{ij}, H \rangle = - 2 \langle A_{ik}, H \rangle \langle A_{jk, H} \rangle \langle A_{ij}, H \rangle + \langle A_{ij}, H \rangle \langle A_{kl}, H \rangle \langle A_{ij}, H \rangle \\
- 2 \langle A_{ik}, H \rangle \langle A_{jk, H} \rangle \langle A_{ij, H} \rangle + 2 \langle A_{ik}, H \rangle \langle A_{jk}, H \rangle \langle A_{ij}, H \rangle, \\
\langle A_{ij}, H \rangle \left( \frac{\partial^{\perp}}{\partial t} - \Delta^{\perp} \right) H \rangle \langle A_{ij}, H \rangle = \langle A_{ij}, H \rangle \langle A_{ij}, H \rangle \langle A_{kl}, H \rangle. \]

Putting these equations together,
\[
\left( \frac{\partial}{\partial t} - \Delta \right) \langle A_{ij}, H \rangle \langle A_{ij}, H \rangle = - 2 \langle A_{ik}, H \rangle \langle A_{jk, H} \rangle \langle A_{ij}, H \rangle + 2 \langle A_{ij}, H \rangle \langle A_{kl}, H \rangle \langle A_{ij}, H \rangle \\
- 2 \langle A_{ik}, H \rangle \langle A_{jk, H} \rangle \langle A_{ij, H} \rangle + 2 \langle A_{ik}, H \rangle \langle A_{jk}, H \rangle \langle A_{ij}, H \rangle \\
- 2 \langle A_{ik}, H \rangle \langle A_{jk}, H \rangle \langle A_{ij}, H \rangle. \]
Therefore, including the time derivative of the inverse of the metric we have
\[
\left( \frac{\partial}{\partial t} - \Delta \right) \left| (A_{ij}, H) \right|^2 = 2 \left( \frac{\partial}{\partial t} \langle k i i \rangle \right) g^{ij} \langle A_{ik}, H \rangle \langle A_{ij}, H \rangle + 2 \left( \left( \frac{\partial}{\partial t} - \Delta \right) \langle A_{ij}, H \rangle \right) \langle A_{ij}, H \rangle - 2 |\nabla (A_{ij}, H)|^2
\]
\[
= 4 \langle A_{ij}, A_{ki} \rangle \langle A_{ij}, H \rangle \langle A_{ij}, H \rangle
- 4 \langle A_{ik}, A_{ij} \rangle \langle A_{ij}, H \rangle \langle A_{ij}, H \rangle
+ 4 \langle A_{ik}, A_{ki} \rangle \langle A_{ij}, H \rangle \langle A_{ij}, H \rangle
- 4 \langle \nabla^i_i A_{ij}, \nabla^k_k H \rangle \langle A_{ij}, H \rangle - 2 |\nabla (A_{ij}, H)|^2.
\]

To finish the proof, we multiply by \( H_t^{-2} \) and then rewrite each of the remaining terms using \( A = \bar{A} + h H_t \).

For the term on the first line we have
\[
4H_t^{-2} \langle A_{ij}, A_{ki} \rangle \langle A_{ij}, H \rangle \langle A_{ij}, H \rangle = 4H_t^{-2} H_t^2 h_{ij} h_{kl} \langle A_{ij}, A_{kl} \rangle
= 4|\hbar|^4 + 4 h_{ij} h_{kl} \langle \bar{A}_{ij}, \bar{A}_{kl} \rangle
= 4|\hbar|^4 + 4 \bar{h}_{ij} \bar{h}_{kl} \langle \bar{A}_{ij}, \bar{A}_{kl} \rangle
= 4|\hbar|^4 + 4 \bar{h}_{ij} \bar{A}_{ij}.\]

For the difference of terms on the second line, we notice the resemblance to \( |R_{ij}^{(2)}(\nu_1)|^2 \) (see Section 2).

Working backwards we compute
\[
|R_{ij}^{(2)}(\nu_1)|^2 = |\bar{h}_{ik} \bar{A}_{jk} - \bar{h}_{jk} \bar{A}_{ik}|^2
= |\bar{h}_{ik} A_{jk} - \bar{h}_{jk} A_{ik}|^2
= |h_{ik} A_{jk} - h_{jk} A_{ik}|^2
= |h_{ik} A_{jk} - h_{jk} A_{ik} - h_{ij} A_{ik} + h_{ij} A_{ik}|^2
= 2 h_{ik} h_{kl} (A_{jk}, A_{ji}) - 2 h_{ik} h_{kl} (A_{jk}, A_{ik})
= 2H_t^{-2} (\langle A_{jk}, A_{ji} \rangle / A_{ij}, H \rangle \langle A_{kl}, H \rangle - \langle A_{ik}, A_{id} \rangle / A_{ij}, H \rangle \langle A_{kl}, H \rangle)\]

After reindexing, we have
\[
4H_t^{-2} (\langle A_{ik}, A_{ki} \rangle / A_{ij}, H \rangle \langle A_{ij}, H \rangle - 4 \langle A_{ik}, A_{ji} \rangle / A_{ij}, H \rangle \langle A_{ij}, H \rangle) = 2 |R_{ij}^{(2)}(\nu_1)|^2.
\]

Thus we have shown the reaction terms on the first line of our lemma statement are correct. For the gradient terms, it follows from the identities in Section 2 that
\[
\langle \nabla^k_k A_{ij}, \nu_1 \rangle = \langle \nabla^k_k \bar{A}_{ij}, \nu_1 \rangle + \nabla_k h_{ij},
\langle \nabla^k_k A_{ij}, \nabla^k_k \nu_1 \rangle = \langle \nabla^k_k \bar{A}_{ij}, \nabla^k_k \nu_1 \rangle + h_{ij} |\nabla^k_k \nu_1 |^2,
\nabla^k_k H = \nabla_k H_1 \nu_1 + H_1 \nabla^k_k \nu_1.
\]

Therefore, we have
\[
-4H_t^{-2} \langle \nabla^k_k A_{ij}, \nabla^k_k H \rangle \langle A_{ij}, H \rangle = -4H_t^{-2} h_{ij} \nabla_k H_1 (\nabla^k_k A_{ij}, \nu_1) - 4 h_{ij} (\nabla^k_k A_{ij}, \nabla^k_k \nu_1)
= -4H_t^{-2} \hat{h}_{ij} \nabla_k H_1 (\nabla^k_k \bar{A}_{ij}, \nu_1) - 4H_t^{-2} h_{ij} \nabla_k H_1 \nabla_k h_{ij}
\quad -4 \hat{h}_{ij} (\nabla^k_k \bar{A}_{ij}, \nabla^k_k \nu_1) - 4 |h_1|^2 |\nabla^k_k \nu_1 |^2
\quad -2H_t^{-2} |\nabla \langle A_{ij}, H \rangle |^2
\quad = -2H_t^{-2} |\nabla H_1 |^2 - 2 |\nabla h_1 |^2 - 4H_t^{-1} h_{ij} \nabla_k H_1 \nabla_k h_{ij},
\]

which together give the correct six gradient terms in the lemma statement. \(\blacksquare\)

Substituting the result of the above lemma into our equation for the evolution of \( |H|^{-2} (A_{ij}, H) |^2 \) and combining like terms yields
\[
\left( \frac{\partial}{\partial t} - \Delta \right) \frac{|(A_{ij}, H)|^2}{|H|^2} = 4 \bar{h}_{ij} \bar{A}_{ij} |^2 + 2 |R_{ij}^{(2)}(\nu_1)|^2 + 2 |\hbar|^4
\quad - 4H_t^{-1} \bar{h}_{ij} \nabla_k H_1 (\nabla^k_k \bar{A}_{ij}, \nu_1) - 4 \hat{h}_{ij} (\nabla^k_k \bar{A}_{ij}, \nabla^k_k \nu_1)
\quad -2 |\nabla h_1 |^2 - 2 |\nabla^k_k \nu_1 |^2.
We negate the expression above and add in the evolution of $|A|^2$ to get
\[
\left( \frac{\partial}{\partial t} - \Delta \right)|\dot{A}|^2 = -2|\nabla^\perp A|^2 + 2|\langle A_{ij}, \dot{A}_{kl} \rangle|^2 + 2|R_{ij}^k|^2 \\
- 4\dot{h}_{ij}\dot{A}_{ij}|^2 - 2|R_{ij}(\nu_t)|^2 - 2|h|^4 \\
+ 4H_1^{-1}\dot{h}_{ij}\nabla_k H_1(\nabla^\perp_k \dot{A}_{ij}, \nu_t) + 4\dot{h}_{ij}(\nabla^\perp_k \dot{A}_{ij}, \nabla^\perp_k \nu_t) \\
+ 2|\nabla h|^2 + 2|h|^2|\nabla^\perp \nu_t|^2.
\]
By the identities in Proposition 2.3, the reaction terms satisfy
\[
2|\langle A_{ij}, \dot{A}_{kl} \rangle|^2 - 4\dot{h}_{ij}\dot{A}_{ij}|^2 - 2|h|^2 = 2|\langle \dot{A}_{ij}, \dot{A}_{kl} \rangle|^2, \\
2|R_{ij}^k|^2 - 2|R_{ij}(\nu_t)|^2 = 2|\dot{A}_{jk} \otimes \dot{A}_{ij} - \dot{A}_{ij} \otimes \dot{A}_{jk}|^2 + 2|R_{ij}(\nu_t)|^2.
\]
As for the gradient terms, taking the norm of $\nabla^\perp A_{ij} = \nabla^\perp \dot{A}_{jk} + \nabla_i h_{jk} \nu_t + h_{jk} \nabla^\perp \nu_t$, we see
\[
|\nabla^\perp A|^2 = |\nabla^\perp \dot{A}|^2 + |\nabla h|^2 + |h|^2|\nabla^\perp \nu_t|^2 + 2\dot{h}_{ij}(\nabla^\perp_k \dot{A}_{ij}, \nabla^\perp_k \nu_t) + 2\nabla_k \dot{h}_{ij}(\nabla^\perp_k \dot{A}_{ij}, \nu_t).
\]
Thus,
\[
-2|\nabla^\perp A|^2 + 2|\nabla h|^2 + 2|h|^2|\nabla^\perp \nu_t|^2 + 2\dot{h}_{ij}(\nabla^\perp_k \dot{A}_{ij}, \nabla^\perp_k \nu_t) = -2|\nabla^\perp \dot{A}|^2 - 4\nabla_k \dot{h}_{ij}(\nabla^\perp_k \dot{A}_{ij}, \nu_t).
\]
Putting this all together gives
\[
\left( \frac{\partial}{\partial t} - \Delta \right)|\dot{A}|^2 = 2|\langle \dot{A}_{ij}, \dot{A}_{kl} \rangle|^2 + 2|\dot{A}_{jk} \otimes \dot{A}_{ij} - \dot{A}_{ij} \otimes \dot{A}_{jk}|^2 + 2|R_{ij}(\nu_t)|^2 \\
- 2|\nabla^\perp \dot{A}|^2 + 4H_1^{-1}\dot{h}_{ij}\nabla_k H_1(\nabla^\perp_k \dot{A}_{ij}, \nu_t) - 4\nabla_k \dot{h}_{ij}(\nabla^\perp_k \dot{A}_{ij}, \nu_t).
\]
Note that $\nabla_k \dot{h}_{ij} = -\nabla_k \dot{A}_{ij} - (\nabla_k \dot{A}_{ij}, \nu_t) - (\nabla_k \dot{A}_{ij}, \nu_t)$ and
\[
\nabla_k \dot{h}_{ij} = (\nabla^\perp_k \dot{A}_{ij}, \nu_t) - (\nabla^\perp_k \dot{A}_{ij}, \nu_t).
\]
To simplify our final expression, let us define the tensor
\[
Q_{ijk} = \langle \nabla^\perp_k \dot{A}_{ij}, \nu_t \rangle - \langle \nabla^\perp_k \dot{A}_{ij}, \nu_t \rangle - H_1^{-1}\dot{h}_{ij}\nabla_k H_1.
\]
Then in conclusion we have
**Proposition 3.2** (Evolution of $|\dot{A}|^2$).
\[
\left( \frac{\partial}{\partial t} - \Delta \right)|\dot{A}|^2 = 2|\langle \dot{A}_{ij}, \dot{A}_{kl} \rangle|^2 + 2\sum_{k=1}^{n}|A_{jk} \otimes \dot{A}_{ij} - \dot{A}_{ij} \otimes \dot{A}_{jk}|^2 + 2|R_{ij}(\nu_t)|^2 \\
- 2|\nabla^\perp \dot{A}|^2 + 4\sum_{i,j,k=1}^{n}Q_{ijk}(\dot{A}_{ij}, \nabla^\perp_k \nu_t)
\]
where
\[
Q_{ijk} = \langle \nabla^\perp_k \dot{A}_{ij}, \nu_t \rangle - \langle \nabla^\perp_k \dot{A}_{ij}, \nu_t \rangle - H_1^{-1}\dot{h}_{ij}\nabla_k H_1.
\]

4. Proof of Theorem 1.1.

As in [Ngu18], we consider the function $f = c_n|H|^2 - |A|^2$. The assumption of the theorem is that $f > 0$ (and consequently $|H| > 0$) everywhere on $M_0$. As $M_0$ is compact, there exist constants $\varepsilon_0, \varepsilon_1 > 0$ depending on $M_0$ such that $f \geq \varepsilon_1|H|^2 + \varepsilon_0$ on $M_0$. By Theorem 2 in [AB10], $f \geq \varepsilon_1|H|^2 + \varepsilon_0$ on $M_t$ for all $t \in [0, T)$ and consequently $|H| > 0$ is preserved as well. Recall that
\[
c_n = \frac{4}{3n} \text{ if } n \geq 8 \text{ and } c_n = \frac{3(n+1)}{2n(n+2)} \text{ if } n = 5, 6 \text{ or } 7.
\]
We will need a bit more breathing room for our estimates when $n = 5, 6, 7$. Since we have that $|A|^2 + \varepsilon_0 \leq (c_n - \varepsilon_1)|H|^2$ for all $t \in [0, T)$, without loss of generality, we may replace $c_n$ by $c_n - \varepsilon_1$ and we can assume throughout the proof that
\[
c_n \leq \frac{4}{3n} \text{ if } n \geq 8 \text{ and } c_n < \frac{3(n+1)}{2n(n+2)} \text{ if } n = 5, 6 \text{ or } 7.
\]
The strictness of the latter inequality depends on initial data through \( \varepsilon_1 \). We still have \( f \geq \varepsilon_0 > 0 \) and \(|H| > 0\) for all \( t \).

Let \( \delta > 0 \) be a small constant to be determined later in the proof. We computed the evolution equation of \(|A|^2\) in the previous section. By work in Section 2, the evolution equation for \( f \) is

\[
\frac{\partial}{\partial t} - \Delta \left( \frac{\partial |A|^2}{\partial t} - \Delta \right) f = 2(|\nabla^+ A|^2 - c_n(|\nabla^+ H|^2) + 2(c_n|\langle A_{ij}, H\rangle| - |\langle A_{ij}, A_{kl}\rangle|^2 - |R_{ij}^+|^2).
\]

The pinching condition implies that both terms on the right hand side of the equation for \( f \) are nonnegative at each point in space-time (see Lemma 2.3 in [Ngu18] and also the ensuing arguments). The first step of the proof, and the main effort, is to analyze the evolution equation of \(|A|^2\). We will show this ratio satisfies a favorable evolution equation with a right hand side that has a nonpositive term. Specifically, we will show that

\[
\frac{\partial}{\partial t} - \Delta \left( \frac{\partial |A|^2}{\partial t} - \Delta \right) \left( \frac{|A|^2}{f} \right) \leq 2\left\langle \nabla \frac{|A|^2}{f}, \nabla \log f \right\rangle - \delta \frac{|A|^2}{f} \left( \frac{\partial}{\partial t} - \Delta \right) f.
\]

Then we will analyze the evolution of \(|A|^2\). We will show for \( \sigma \) sufficiently small, the nonpositive term above can be used to control the nonnegative terms introduced by the additional factor of \( f^{\sigma} \). The result will then follow from the maximum principle.

By what we have shown thus far, the evolution equation of \(|A|^2\) is

\[
\frac{\partial}{\partial t} - \Delta \left( \frac{\partial |A|^2}{\partial t} - \Delta \right) \left( \frac{|A|^2}{f} \right) = \frac{1}{f} \left( \frac{\partial}{\partial t} - \Delta \right) \left( \frac{|A|^2}{f} \right) \left( \frac{\partial}{\partial t} - \Delta \right) f + 2\left\langle \nabla \frac{|A|^2}{f}, \nabla \log f \right\rangle
\]

\[
= \frac{1}{f} \left( 2|\langle \hat{A}_{ij}, \hat{A}_{kl}\rangle|^2 + 2|\hat{A}_{ik} \otimes \hat{A}_{jk} - \hat{A}_{jk} \otimes \hat{A}_{ik}|^2 + 2|R_{ij}^+|^2(\nu_1)^2 \right)
\]

\[
- |\hat{A}|^2 \frac{1}{f} \left( 2(|\nabla^+ A|^2 - c_n|\nabla^+ H|^2) \right)
\]

\[
+ 2\left\langle \nabla \frac{|A|^2}{f}, \nabla \log f \right\rangle.
\]

Rearranging these terms, we have

\[
\frac{\partial}{\partial t} - \Delta \left( \frac{\partial |A|^2}{\partial t} - \Delta \right) \left( \frac{|A|^2}{f} \right) = \frac{1}{f} \left( 2|\langle \hat{A}_{ij}, \hat{A}_{kl}\rangle|^2 + 2|\hat{A}_{ik} \otimes \hat{A}_{jk} - \hat{A}_{jk} \otimes \hat{A}_{ik}|^2 + 2|R_{ij}^+|^2(\nu_1)^2 \right)
\]

\[
- \frac{1}{f} \left( 2|\langle A_{ij}, H\rangle|^2 - |\langle A_{ij}, A_{kl}\rangle|^2 - |R_{ij}^+|^2 \right)
\]

\[
+ \frac{1}{f} \left( 4Q_{ijk}(\hat{A}_{ij}, \nabla^+ \nu_1) - 2|\nabla^+ \hat{A}|^2 - 2|\langle A_{ij}, H\rangle|^2 - c_n|\nabla^+ H|^2 \right)
\]

\[
+ 2\left\langle \nabla \frac{|A|^2}{f}, \nabla \log f \right\rangle.
\]

We analyze the right hand side in two steps. We must estimate the reaction terms on the first line by the reaction terms on the second line and the gradient term \( 4Q_{ijk}(\hat{A}_{ij}, \nabla^+ \nu_1) \) by the good Bochner terms coming from the evolution of \(|A|^2\) and \( f \).

We begin by estimating the reaction terms. We will make use of the following estimates (see [AB10] Section 3 and [LL92]).

**Lemma 4.1.**

\[
|\sum_{i,j=1}^n \hat{h}_{ij} \hat{A}_{ij}|^2 + |R_{ij}^+(\nu_1)|^2 \leq 2|\hat{A}|^2|\hat{A}|^2,
\]

\[
|\langle \hat{A}_{ij}, \hat{A}_{kl} \rangle|^2 + \left| \sum_{k=1}^n (\hat{A}_{ik} \otimes \hat{A}_{jk} - \hat{A}_{jk} \otimes \hat{A}_{ik}) \right|^2 \leq \frac{3}{2}|\hat{A}|^4.
\]
Consequently, we have the following estimate for the reaction terms coming from the evolution of $|\hat{A}|^2$.

**Lemma 4.2** (Upper bound for the reaction terms of $(\partial_t - \Delta)|\hat{A}|^2$),

$$\langle |\hat{A}_{ij}, \hat{A}_{kl}|^2 + \sum_{k=1}^{n} (\hat{A}_{ik} \otimes \hat{A}_{jk} - \hat{A}_{ik} \otimes \hat{A}_{ik})^2 + |R_{ij}^1(v_1)|^2 \leq \frac{3}{2} |\hat{A}|^4 + 2|\hat{h}|^2 |\hat{A}|^2.$$  

Next we express the reaction term in the evolution of $f$ in terms of $\hat{A}, \hat{h},$ and $H_1$. Recall by Proposition 2.3 in Section 2,

$$\langle |A_{ij}, H|^2 \rangle = \frac{1}{n} H_1^2 + |\hat{h}|^2 R_{ij}^1,$$

$$\langle |A_{ij}, A_{kl}|^2 \rangle = |\hat{h}|^4 + \frac{2}{n} |\hat{h}|^2 H_1^2 + \frac{1}{n^2} H_1^4 + 2|\hat{h}|^2 |A_{ij}|^2 + |\langle A_{ij}, A_{kl} \rangle|^2,$$

$$|R_{ij}^1|^2 = |\hat{A}_{ik} \otimes \hat{A}_{jk} - \hat{A}_{ik} \otimes \hat{A}_{ik}|^2 + 2|R_{ij}^1(v_1)|^2.$$  

Also observe that

$$f = c_n H_1^2 - |A|^2$$

$$= (c_n - \frac{1}{n}) H_1^2 - |\hat{A}|^2 - |\hat{h}|^2.$$  

Then we have following lower bound for the reaction terms in the evolution of $f$.

**Lemma 4.3** (Lower bound for the reaction terms of $(\partial_t - \Delta)f$). If $c_n \leq \frac{1}{3n}$, then

$$\frac{|\hat{A}|^2}{f} (c_n \langle A_{ij}, H \rangle - \langle A_{ij}, A_{kl} \rangle - |R_{ij}^1|^2) \geq \frac{2}{n c_n - 1} |\hat{A}|^4 + \frac{n c_n}{n c_n - 1} |\hat{h}|^2 |\hat{A}|^2.$$  

**Proof.** We begin with the observation that

$$\left( c_n - \frac{1}{n} \right) H_1^2 = f + |\hat{A}|^2 + |\hat{h}|^2.$$  

Now we do a computation that is similar to the computation in [AB10] without throwing away the pinching term. By the identities above, we have

$$c_n \langle A_{ij}, H \rangle^2 - \langle A_{ij}, A_{kl} \rangle^2 - |R_{ij}^1|^2$$

$$= \frac{1}{n} c_n H_1^2 + c_n |\hat{h}|^2 R_{ij}^1$$

$$- |\hat{h}|^4 - \frac{2}{n} |\hat{h}|^2 H_1^2 - \frac{1}{n^2} H_1^4 - 2|\hat{h}|^2 |\hat{A}_{ij}|^2 - |\langle A_{ij}, A_{kl} \rangle|^2$$

$$- |\hat{A}_{ik} \otimes \hat{A}_{jk} - \hat{A}_{ik} \otimes \hat{A}_{ik}|^2 - 2|R_{ij}^1(v_1)|^2$$

$$= \frac{1}{n} (c_n - \frac{1}{n}) H_1^2 + (c_n - \frac{1}{n}) |\hat{h}|^2 H_1^2 - \frac{1}{n^2} |\hat{h}|^2 H_1^2 - |\hat{h}|^4$$

$$- 2|\hat{h}_{ij} A_{ij}|^2 - 2\left( |R_{ij}^1(v_1)|^2 - |\langle A_{ij}, A_{kl} \rangle|^2 - |\hat{A}_{ik} \otimes \hat{A}_{jk} - \hat{A}_{ik} \otimes \hat{A}_{ik}|^2 \right).$$

Replace $(c_n - \frac{1}{n}) H_1^2$ with $f + |\hat{A}|^2 + |\hat{h}|^2$, and cancel terms to get

$$c_n \langle A_{ij}, H \rangle^2 - \langle A_{ij}, A_{kl} \rangle^2 - |R_{ij}^1|^2$$

$$= \frac{1}{n} (f + |\hat{A}|^2 + |\hat{h}|^2) H_1^2 + (f + |\hat{A}|^2 + |\hat{h}|^2) |\hat{h}|^2 - \frac{1}{n} |\hat{h}|^2 H_1^2 - |\hat{h}|^4$$

$$- 2|\hat{h}_{ij} A_{ij}|^2 - 2\left( |R_{ij}^1(v_1)|^2 - |\langle A_{ij}, A_{kl} \rangle|^2 - |\hat{A}_{ik} \otimes \hat{A}_{jk} - \hat{A}_{ik} \otimes \hat{A}_{ik}|^2 \right)$$

$$= \frac{1}{n} (f + |\hat{A}|^2) H_1^2 + (f + |\hat{A}|^2) |\hat{h}|^2$$

$$- 2|\hat{h}_{ij} A_{ij}|^2 - 2\left( |R_{ij}^1(v_1)|^2 - |\langle A_{ij}, A_{kl} \rangle|^2 - |\hat{A}_{ik} \otimes \hat{A}_{jk} - \hat{A}_{ik} \otimes \hat{A}_{ik}|^2 \right).$$
Replacing $H^2_4$ by $(c_n - \frac{1}{n})^{-1}(f + |\hat{A}|^2 + |\hat{h}|^2)$ once more gives
\[
    c_n |\langle A_{ij}, H \rangle|^2 - |\langle A_{ij}, A_{kl} \rangle|^2 - |R_{ij}^+|^2 \\
    = \frac{1}{n} (f + |\hat{A}|^2) \left(c_n - \frac{1}{n}\right)^{-1} (f + |\hat{A}|^2 + |\hat{h}|^2) + (f + |\hat{A}|^2)|\hat{h}|^2 \\
    - 2|\hat{h}_{ij} \hat{A}_{ij}|^2 - 2|R_{ij}^+|^2 |\langle \hat{h}, \hat{A}_{ij} \rangle|^2 - |\langle \hat{A}_{ij}, \hat{A}_{kl} \rangle|^2 - |\hat{A}_{ik} \otimes \hat{A}_{jk} - \hat{A}_{jk} \otimes \hat{A}_{ik}|^2 \\
    = \frac{1}{nc_n - 1} (f + 2|\hat{A}|^2 + |\hat{h}|^2) + f|\hat{h}|^2 + \frac{1}{nc_n - 1} |\hat{A}|^4 + \frac{nc_n}{nc_n - 1} |\hat{A}|^2 |\hat{h}|^2 \\
    - 2|\hat{h}_{ij} \hat{A}_{ij}|^2 - 2|R_{ij}^+|^2 |\langle \hat{h}, \hat{A}_{ij} \rangle|^2 - |\langle \hat{A}_{ij}, \hat{A}_{kl} \rangle|^2 - |\hat{A}_{ik} \otimes \hat{A}_{jk} - \hat{A}_{jk} \otimes \hat{A}_{ik}|^2.
\]

Now by Lemma 4.1,
\[
    2|\hat{h}_{ij} \hat{A}_{ij}|^2 + 2|R_{ij}^+|^2 |\langle \hat{h}, \hat{A}_{ij} \rangle|^2 + |\langle \hat{A}_{ij}, \hat{A}_{kl} \rangle|^2 + |\hat{A}_{ik} \otimes \hat{A}_{jk} - \hat{A}_{jk} \otimes \hat{A}_{ik}|^2 \leq 4|\hat{h}|^2 |\hat{A}|^2 + \frac{3}{2} |\hat{A}|^4.
\]

Therefore
\[
    \frac{1}{nc_n - 1} |\hat{A}|^4 + \frac{nc_n}{nc_n - 1} |\hat{A}|^2 |\hat{h}|^2 - 2|\hat{h}_{ij} \hat{A}_{ij}|^2 - 2|R_{ij}^+|^2 |\langle \hat{h}, \hat{A}_{ij} \rangle|^2 - |\langle \hat{A}_{ij}, \hat{A}_{kl} \rangle|^2 - |\hat{A}_{ik} \otimes \hat{A}_{jk} - \hat{A}_{jk} \otimes \hat{A}_{ik}|^2 \\
    \geq \left( \frac{1}{nc_n - 1} - \frac{3}{2} \right) |\hat{A}|^4 + \left( \frac{nc_n}{nc_n - 1} - 4 \right) |\hat{h}|^2 |\hat{A}|^2 \geq 0
\]
since $c_n \leq \frac{1}{3n}$ and so
\[
    \frac{1}{nc_n - 1} \geq \frac{3}{2}, \quad \frac{nc_n}{nc_n - 1} - 4 \geq 0.
\]

After throwing away the nonnegative term involving $f^2$, we have shown
\[
    c_n |\langle A_{ij}, H \rangle|^2 - |\langle A_{ij}, A_{kl} \rangle|^2 - |R_{ij}^+|^2 \geq \frac{2}{nc_n - 1} |\hat{A}|^2 + \frac{nc_n}{nc_n - 1} f|\hat{h}|^2.
\]

Multiplying both sides by $\frac{|\hat{A}|^2}{f}$ completes the proof of the lemma. □

Putting Lemmas 4.2 and 4.3 together, we have

**Lemma 4.4** (Reaction term estimate). For $\delta < \frac{1}{2}$ and $c_n \leq \frac{1}{3n}$, there holds
\[
    |\langle \hat{A}_{ij}, \hat{A}_{kl} \rangle|^2 + \left( \sum_{k=1}^{n} |\hat{A}_{ik} \otimes \hat{A}_{jk} - \hat{A}_{jk} \otimes \hat{A}_{ik}|^2 \right)^2 + |R_{ij}^+|^2 |\langle \hat{h}, \hat{A}_{ij} \rangle|^2 \\
    \leq (1 - \delta)^2 |\hat{A}|^4 \left( c_n |\langle A_{ij}, H \rangle|^2 - |\langle A_{ij}, A_{kl} \rangle|^2 - |R_{ij}^+|^2 \right).
\]

**Proof.**
\[
    |\langle \hat{A}_{ij}, \hat{A}_{kl} \rangle|^2 + \left( \sum_{k=1}^{n} |\hat{A}_{ik} \otimes \hat{A}_{jk} - \hat{A}_{jk} \otimes \hat{A}_{ik}|^2 \right)^2 + |R_{ij}^+|^2 |\langle \hat{h}, \hat{A}_{ij} \rangle|^2 \\
    \leq (1 - \delta)^2 \frac{|\hat{A}|^2}{f} \left( c_n |\langle A_{ij}, H \rangle|^2 - |\langle A_{ij}, A_{kl} \rangle|^2 - |R_{ij}^+|^2 \right) \\
    \leq \frac{3}{2} |\hat{A}|^4 + 2|\hat{h}|^2 |\hat{A}|^2 - \frac{2(1 - \delta)}{nc_n - 1} |\hat{A}|^4 - \frac{nc_n (1 - \delta)}{nc_n - 1} |\hat{h}|^2 |\hat{A}|^2 \\
    = \left( \frac{3}{2} - \frac{2(1 - \delta)}{nc_n - 1} \right) |\hat{A}|^4 + \left( 2 - \frac{nc_n (1 - \delta)}{nc_n - 1} \right) |\hat{h}|^2 |\hat{A}|^2.
\]

For $c_n \leq \frac{1}{3n}$, we have
\[
    \frac{1}{nc_n - 1} \geq 3, \quad \frac{nc_n}{nc_n - 1} \geq 4,
\]
and therefore
\[
    \frac{3}{2} - \frac{2(1 - \delta)}{nc_n - 1} \leq \frac{3}{2} - 6(1 - \delta) < 0, \quad 2 - \frac{nc_n (1 - \delta)}{nc_n - 1} \leq 2 - 4(1 - \delta) < 0.
\]

□

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Having analyzed the reaction terms, we turn our attention to the gradient terms. We begin by recalling the decomposition of gradients in Proposition 2.1.

\[ |\nabla^\perp A|^2 = |\nabla^\perp A_{jk} - (\nabla^\perp A_{jk}, \nu_1)|^2 + |(\nabla^\perp A_{jk}, \nu_1)|^2 \]

\[ = |\nabla^\perp A_{jk} + h_{jk} \nabla^\perp \nu_1|^2 + |\nabla_i h_{jk} + (\nabla^\perp A_{jk}, \nu_1)|^2. \]

\[ |\nabla^\perp H|^2 = H^2|^\nabla^\perp \nu_1|^2 + |\nabla^\perp H|^2, \]

\[ |\nabla^\perp \hat{A}|^2 = |\nabla^\perp \hat{A}_{jk}|^2 + |(\nabla^\perp \hat{A}_{jk}, \nu_1)|^2. \]

Next, using the Codazzi identity, we further decompose both of the terms in the identity for \(|\nabla^\perp A|^2\) into their traceless and traceless components

\[ |\nabla^\perp \hat{A}_{jk} + h_{jk} \nabla^\perp \nu_1|^2 = |\nabla^\perp \hat{A}_{jk} + h_{jk} \nabla^\perp \nu_1|^2 + \frac{1}{n} H^2|^\nabla^\perp \nu_1|^2, \]

\[ |\nabla_i h_{jk} + (\nabla^\perp \hat{A}_{jk}, \nu_1)|^2 = |\nabla_i h_{jk} + (\nabla^\perp \hat{A}_{jk}, \nu_1)|^2 + \frac{1}{n}|\nabla^\perp H|^2. \]

Because we will use it later, note that the gradient of the traceless second fundamental form in the \(\nu_1\) direction is

\[ |(\nabla^\perp \hat{A}_{jk}, \nu_1)|^2 = |\nabla_i h_{jk} + (\nabla^\perp \hat{A}_{jk}, \nu_1)|^2. \]

Now as observed in [Hui84] (and [Ham82]), the tensor

\[ E_{ijk} = \frac{1}{n+2} \left( g_{ij} h_{k} H + g_{jk} \nabla^\perp H + g_{ki} \nabla^\perp \nu_1 \right) \]

is an irreducible component of \(\nabla^\perp A_{jk}\) consisting of its various traces (by the Codazzi identity). This allows one to get an improved estimate over the trivial one:

\[ |E|^2 = \frac{3}{n+2} |\nabla^\perp H|^2 \leq |\nabla^\perp A|^2. \]

We apply this argument in both the \(\nu_1\) direction and its orthogonal complement. We observed in Section 2 that the Codazzi identity implies the tensors \(\nabla^\perp \hat{A}_{jk} + h_{jk} \nabla^\perp \nu_1\) and \(\nabla_i h_{jk} + (\nabla^\perp \hat{A}_{jk}, \nu_1)\) are symmetric in all indices. Therefore, an irreducible component of each tensor is given by

\[ \frac{1}{n+2} \left( g_{ij} H_{k} \nabla^\perp \nu_1 + g_{jk} H_{i} \nabla^\perp \nu_1 + g_{ki} H_{j} \nabla^\perp \nu_1 \right), \]

\[ \frac{1}{n+2} \left( g_{ij} \nabla_k H_{1} + g_{jk} \nabla_i H_{1} + g_{ki} \nabla_j H_{1} \right). \]

As above, this implies that

\[ \frac{3}{n+2} H^2|^\nabla^\perp \nu_1|^2 \leq |\nabla^\perp \hat{A}_{jk} + h_{jk} \nabla^\perp \nu_1|^2, \]

\[ \frac{3}{n+2} |\nabla^\perp H|^2 \leq |\nabla_i h_{jk} + (\nabla^\perp \hat{A}_{jk}, \nu_1)|^2. \]

Moreover, \(\frac{3}{n+2} - \frac{1}{n} = \frac{2(n-1)}{n(n+2)},\) so after subtracting the fully traceless component of the right hand side of each inequality above, we arrive at the estimates

\[ \frac{2(n-1)}{n(n+2)} H^2|^\nabla^\perp \nu_1|^2 \leq |\nabla^\perp \hat{A} + \hat{h} \nabla^\perp \nu_1|^2, \]

\[ \frac{2(n-1)}{n(n+2)} |\nabla^\perp H|^2 \leq |(\nabla^\perp \hat{A}, \nu_1)|^2. \]

The first of these two estimates implies the following useful lower bound.

**Lemma 4.5** (Lower bound for Bochner term of \((\delta_i - \Delta)|\hat{A}|^2\).

1. If \(n \geq 8\) and \(c_n \leq \frac{1}{n}\), then

\[ 2|\nabla^\perp \hat{A}|^2 \geq \frac{4n-10}{n+2} |\hat{h}|^2 |\nabla^\perp \nu_1|^2 + \frac{6(n-1)}{n+2} |\hat{A}|^2 |\nabla^\perp \nu_1|^2 + \frac{6(n-1)}{n+2} f |\nabla^\perp \nu_1|^2. \]

2. If \(n = 5, 6, \) or \(7\) and \(c_n \leq \frac{3(n+1)}{2n(n+2)},\) then

\[ 2|\nabla^\perp \hat{A}|^2 \geq 2|\hat{h}|^2 |\nabla^\perp \nu_1|^2 + 4|\hat{A}|^2 |\nabla^\perp \nu_1|^2 + 4f |\nabla^\perp \nu_1|^2. \]
Proof. We begin by applying Young's inequality
\[ | \nabla_i \hat{A}_{jk} + \hat{h}_{jk} \nabla_i \nu_1 |^2 = | \nabla_i \hat{A} |^2 + 2 \langle \nabla_i \hat{A}_{jk}, \hat{h}_{jk} \nabla_i \nu_1 \rangle + | \hat{h} |^2 | \nabla_i \nu_1 |^2 \]
\[ \leq 2 | \nabla_i \hat{A} |^2 + 2 | \hat{h} |^2 | \nabla_i \nu_1 |^2 \]
Next, we recall
\[ H_i^2 = \frac{n}{nc_n - 1} (f + | \hat{A} |^2 + | \hat{h} |^2). \]
Therefore,
\[ \frac{2(n - 1)}{n(n + 2)} H_i^2 = \frac{2(n - 1)}{(n + 2)(nc_n - 1)} (f + | \hat{A} |^2 + | \hat{h} |^2) \]
Because
\[ \frac{2(n - 1)}{n(n + 2)} H_i^2 | \nabla_i \nu_1 |^2 \leq | \nabla_i \hat{A}_{jk} + \hat{h}_{jk} \nabla_i \nu_1 |^2 \]
our observations give us that
\[ \frac{2(n - 1)}{(n + 2)(nc_n - 1)} (f + | \hat{A} |^2 + | \hat{h} |^2)| \nabla_i \nu_1 |^2 \leq 2 | \nabla_i \hat{A} |^2 + 2 | \hat{h} |^2 | \nabla_i \nu_1 |^2. \]
Subtracting the remaining $| \hat{h} |^2 | \nabla_i \nu_1 |^2$ term on the right gives
\[ \frac{2(n - 1)}{(n + 2)(nc_n - 1)} (f + | \hat{A} |^2)| \nabla_i \nu_1 |^2 + \left( \frac{2(n - 1)}{(n + 2)(nc_n - 1)} - 2 \right) | \hat{h} |^2 | \nabla_i \nu_1 |^2 \leq 2 | \nabla_i \hat{A} |^2. \]
For $c_n \leq \frac{4}{n^2}, (nc_n - 1) \leq \frac{1}{4}$ and substituting this into the inequality above gives the first estimate of the lemma. For $c_n \leq \frac{2n+1}{2n(n+2)}, nc_n - 1 \leq \frac{n-1}{2(n+2)}$, and hence
\[ \frac{2(n - 1)}{n + 2} \frac{1}{nc_n - 1} \geq 4. \]
This proves the second estimate in the lemma. \qed

Next we use our estimates for the gradient terms above to get lower bounds for the Bochner term in the evolution equation of $f$.

Lemma 4.6 (Lower bound for Bochner term of $(\partial_t - \Delta)f$).
1. If $n \geq 8$ and $c_n \leq \frac{4}{n^2}$, then
\[ 2 \frac{| \hat{A} |^2}{f} (| \nabla_i \hat{A} |^2 - c_n | \nabla_i H |^2) \geq \frac{5n - 8}{3(n - 1)} \frac{| \hat{A} |^2}{f} (| \nabla_i \hat{A} |^2 + 10n - 16) | \nabla_i \nu_1 |^2. \]
2. If $n = 5, 6, 7$ and $c_n \leq \frac{2n+1}{2n(n+2)}$, then
\[ 2 \frac{| \hat{A} |^2}{f} (| \nabla_i \hat{A} |^2 - c_n | \nabla_i H |^2) \geq \frac{3}{2} \frac{| \hat{A} |^2}{f} (| \nabla_i \hat{A} |^2 + 6) | \nabla_i \nu_1 |^2. \]

Proof. First we decompose the Bochner term into the $\nu_1$ direction and its orthogonal complement.
\[ | \nabla_i \hat{A} |^2 - c_n | \nabla_i H |^2 = | \langle \nabla_i \hat{A}_{jk}, \nu_1 \rangle |^2 - c_n | \nabla H_1 |^2 \]
\[ = | \nabla_i \hat{A}_{jk} + h_{jk} \nabla_i \nu_1 |^2 - c_n H_1^2 | \nabla_i \nu_1 |^2. \]
For the norm in the $\nu_1$ direction, we separate the fully trace component to get
\[ | \langle \nabla_i \hat{A}_{jk}, \nu_1 \rangle |^2 - c_n | \nabla H_1 |^2 = | \langle \nabla_i \hat{A}_{jk}, \nu_1 \rangle |^2 - \left( c_n - \frac{1}{n} \right) | \nabla H_1 |^2 \]
\[ = | \langle \nabla_i \hat{A}_{jk}, \nu_1 \rangle |^2 - \frac{nc_n - 1}{n} | \nabla H_1 |^2. \]
Now from the estimate
\[ \frac{2(n - 1)}{n(n + 2)} | \nabla H_1 |^2 \leq | \langle \nabla_i \hat{A}_{jk}, \nu_1 \rangle |^2, \]
it follows that
\[ |\langle \nabla_i A_{jk}, \nu_1 \rangle|^2 - c_n |\nabla H_1|^2 \geq \left( 1 - \frac{n(n+2)}{2(n-1)} \right) |\langle \nabla_i^j \hat{A}_{jk}, \nu_1 \rangle|^2 \]
\[ \quad = \left( 1 - \frac{(n+2)(nc_n - 1)}{2(n-1)} \right) |\langle \nabla_i^j \hat{A}_{jk}, \nu_1 \rangle|^2. \]

For \( c_n \leq \frac{4}{5n} \), \( nc_n - 1 \leq \frac{4}{5n} \) gives
\[ 1 - \frac{(n+2)(nc_n - 1)}{2(n-1)} \geq 1 - \frac{n+2}{6(n-1)} = \frac{5n-8}{6(n-1)}. \]

For \( c_n \leq \frac{3(n+1)}{2(n+2)} \), \( nc_n - 1 \leq \frac{n-1}{2(n+2)} \) and so
\[ 1 - \frac{(n+2)(nc_n - 1)}{2(n-1)} > 1 - \frac{1}{4} = \frac{3}{4}. \]

Multiplying by \( 2|\hat{A}|^2 \) gives us the first term in each of the inequalities of the lemma.

For the norm over directions orthogonal to \( \nu_1 \), we use the estimate
\[ \frac{3}{n+2} H_1^2 |\nabla^+ \nu_1|^2 \leq |\nabla_i^j \hat{A}_{jk} + h_{jk} \nabla_i^j \nu_1|^2 \]
to conclude
\[ |\nabla_i^j \hat{A}_{jk} + h_{jk} \nabla_i^j \nu_1|^2 - c_n H_1^2 |\nabla^+ \nu_1|^2 \geq \left( \frac{3}{n+2} - c_n \right) H_1^2 |\nabla^+ \nu_1|^2. \]

Note \( c_n < \frac{3}{n+2} \). Now as in Lemma 4.5, we use
\[ H_1^2 = \frac{n}{nc_n - 1} (f + |\hat{A}|^2 + |\hat{h}|^2) \]
to get
\[ |\nabla_i^j \hat{A}_{jk} + h_{jk} \nabla_i^j \nu_1|^2 - c_n H_1^2 |\nabla^+ \nu_1|^2 \geq \frac{n}{nc_n - 1} \left( \frac{3}{n+2} - c_n \right) (f + |\hat{A}|^2 + |\hat{h}|^2) |\nabla^+ \nu_1|^2 \]
\[ \geq \frac{n}{nc_n - 1} \left( \frac{3}{n+2} - c_n \right) f |\nabla^+ \nu_1|^2. \]

As above, \( c_n \leq \frac{4}{5n} \) and \( c_n \leq \frac{3(n+1)}{2(n+2)} \) give
\[ \frac{n}{nc_n - 1} \left( \frac{3}{n+2} - c_n \right) \geq 3n \left( \frac{9n-4n-8}{(n+2)3n} \right) = \frac{5n-8}{n+2}, \]
\[ \frac{n}{nc_n - 1} \left( \frac{3}{n+2} - c_n \right) \geq \frac{2n(n+2)}{6n-3(n+1)} = 3. \]

Multiplying by \( 2|\hat{A}|^2 \) gives the second term in each of the inequalities of the lemma.

Finally, we must estimate the remaining gradient term in the evolution equation of \( |\hat{A}|^2 \). This gradient term comes with a factor of \( \langle \hat{A}_{ij}, \nabla_k^j \nu_1 \rangle \) and we estimate this
\[ |\langle \hat{A}_{ij}, \nabla_k^j \nu_1 \rangle|^2 = |\langle \hat{A}_{ij}, \nabla_k^j \nu_1 \rangle| |\langle \hat{A}_{ij}, \nabla_k^j \nu_1 \rangle| \]
\[ \leq |\hat{A}_{ij}| |\nabla_k^j \nu_1| |\hat{A}_{ij}| |\nabla_k^j \nu_1| \]
\[ \leq |\hat{A}|^2 |\nabla^+ \nu_1|^2. \]

**Lemma 4.7** (Upper bound for gradient term of \( (\partial_t - \Delta) |\hat{A}|^2 \)).

1. If \( n \geq 8 \) and \( c_n \leq \frac{4}{5n} \), then
\[ 4Q_{ij} \langle \hat{A}_{ij}, \nabla_k^j \nu_1 \rangle \leq 2|\langle \nabla_k^j \hat{A}, \nu_1 \rangle|^2 + \frac{5n-9}{3(n-1)} |\hat{A}|^2 |\langle \nabla_k^j \hat{A}, \nu_1 \rangle|^2 \]
\[ + 2|\hat{A}|^2 |\nabla^+ \nu_1|^2 + \frac{3(n-1)}{n-3} f |\nabla^+ \nu_1|^2 + \frac{2(n+2)}{n+3} |\hat{h}|^2 |\nabla^+ \nu_1|^2. \]
2. If $n = 5, 6, \text{ or } 7$ and $c_n \leq \frac{3(n + 1)}{2(n + 2)}$, then there exists $\varepsilon > 0$ sufficiently small depending only upon $M_0$ and $n$, such that

$$4Q_{ijk} \langle \hat{A}_{ij}, \nabla_k^+ \nu_1 \rangle \leq 2|\nabla^+ \hat{A}, \nu_1|^2 + (3 - \varepsilon) \left( \frac{1}{2} \frac{1}{f} |\hat{A}|^2 \right)|\nabla^+ \hat{A}, \nu_1|^2 + 2|\hat{A}|^2 |\nabla^+ \nu_1|^2 + 2f|\nabla^+ \nu_1|^2 + 2|\hat{h}|^2 |\nabla^+ \nu_1|^2.$$

**Proof.** Recall that

$$Q_{ijk} = \langle \nabla_k^+ \hat{A}_{ij}, \nu_1 \rangle - \langle \nabla_k^+ \hat{A}_{ij}, \nu_1 \rangle - H^{-1}_1 \hat{h}_{ij} \nabla_k H_1.$$

By the triangle inequality

$$|Q| \leq |\langle \nabla^+ \hat{A}, \nu_1 \rangle \rangle + |\langle \nabla^+ \hat{A}, \nu_1 \rangle \rangle + H^{-1}_1 \hat{h}||\nabla H_1||.$$

We will first treat the case $n \geq 8$ and $c_n \leq \frac{1}{6}$. Using the estimates

$$|\nabla H_1|^2 \leq \frac{n(n + 2)}{2(n - 1)} |\nabla^+ \hat{A}, \nu_1|^2$$

and

$$f \leq \left( c_n - \frac{1}{n} \right) H^2 \leq \frac{1}{3} H^2$$

we have

$$\frac{|\hat{A}|^2}{H^2_1} |\nabla H_1|^2 \leq \frac{n(n + 2)}{2(n - 1)} \frac{1}{3n} \frac{1}{f} |\nabla^+ \hat{A}, \nu_1|^2$$

$$= \frac{n + 2}{6(n - 1)} \frac{1}{f} |\nabla^+ \hat{A}, \nu_1|^2.$$

Using our estimate for the tensor $\langle \hat{A}_{ij}, \nabla_k^+ \nu_1 \rangle$ and Cauchy-Schwarz, we have

$$4Q_{ijk} \langle \hat{A}_{ij}, \nabla_k^+ \nu_1 \rangle \leq 4|Q||\langle \hat{A}, \nabla^+ \nu_1 \rangle|$$

$$\leq 4(\langle \nabla^+ \hat{A}, \nu_1 \rangle \rangle + |\langle \nabla^+ \hat{A}, \nu_1 \rangle \rangle + H^{-1}_1 \hat{h}||\nabla H_1|| |\nabla^+ \hat{A}, \nu_1||. $$

Now to each of these three summed terms we apply Young’s inequality with constants $a_1, a_2, a_3 > 0$ to be chosen momentarily. Specifically, we have

$$4|\nabla^+ \hat{A}, \nu_1||\hat{A}||\nabla^+ \nu_1| \leq 2a_1 |\nabla^+ \hat{A}, \nu_1|^2 + \frac{2}{a_1} |\hat{A}|^2 |\nabla^+ \nu_1|^2,$$

$$4|\nabla^+ \hat{A}, \nu_1||\hat{A}||\nabla^+ \nu_1| = 4|\nabla^+ \hat{A}, \nu_1| |\hat{A}|^2 f^2 |\nabla^+ \nu_1|$$

$$\leq 2a_2 \frac{|\hat{A}|^2}{f} |\nabla^+ \hat{A}, \nu_1|^2 + \frac{2}{a_2} f |\nabla^+ \nu_1|^2,$$

$$AH^{-1}_1 \hat{h}||\nabla H_1|| |\nabla^+ \nu_1| \leq 2a_3 \frac{|\hat{A}|^2}{H^2_1} |\nabla H_1|^2 + \frac{2}{a_3} |\hat{h}|^2 |\nabla^+ \nu_1|^2$$

$$\leq 2a_3 \frac{n + 2}{6(n - 1)} \frac{|\hat{A}|^2}{f} |\nabla^+ \hat{A}, \nu_1|^2 + \frac{2}{a_3} |\hat{h}|^2 |\nabla^+ \nu_1|^2.$$

Hence

$$4Q_{ijk} \langle \hat{A}_{ij}, \nabla_k^+ \nu_1 \rangle \leq 2a_1 |\nabla^+ \hat{A}, \nu_1|^2 + (2a_2 + 2a_3 \frac{n + 2}{6(n - 1)}) \frac{|\hat{A}|^2}{f} |\nabla^+ \hat{A}, \nu_1|^2$$

$$+ \frac{2}{a_1} |\hat{A}|^2 |\nabla^+ \nu_1|^2 + \frac{2}{a_2} f |\nabla^+ \nu_1|^2 + \frac{2}{a_3} |\hat{h}|^2 |\nabla^+ \nu_1|^2.$$

Now set

$$a_1 = 1,$$

$$a_2 = \frac{4n - 12}{6(n - 1)} = \frac{2n - 6}{3(n - 1)},$$

$$a_3 = \frac{n + 3}{n + 2}.$$
In this case,

\[
2a_2 + 2a_3 \frac{n + 2}{6(n - 1)} = \frac{4n - 12}{3(n - 1)} + \frac{n + 3}{n + 2} \frac{n + 2}{3(n - 1)} = \frac{5n - 9}{3(n - 1)}.
\]

Plugging these into our estimate above, we conclude

\[
4Q_{ijk}(\dot{A}_{ij}, \nabla^k_A \nu_1) \leq 2|\langle \nabla^k_A \nu_1 \rangle|^2 + \frac{5n - 9}{3(n - 1)} \frac{|\dot{A}|^2}{f} |\langle \nabla^k_A \nu_1 \rangle|^2
\]

\[
+ 2|\dot{A}|^2 |\nabla^k_A \nu_1|^2 + \frac{3(n - 1)}{n - 3} f|\nabla^k_A \nu_1|^2 + \frac{2(n + 2)}{n + 3} |\hat{h}|^2 |\nabla^k_A \nu_1|^2,
\]

as claimed.

For the case \( n = 5, 6, \) or \( 7 \) and \( c_n < \frac{3(n + 1)}{2n(n + 2)}, \) we have \( c_n - \frac{1}{n} < \frac{n - 1}{2(n + 2)}. \) We may assume \( \epsilon \) is sufficiently small depending upon initial data and \( n \) such that

\[
c_n - \frac{1}{n} \leq (1 - \epsilon) \frac{n - 1}{2n(n + 2)}.
\]

Then

\[
f \leq \left( c_n - \frac{1}{n} \right) H_1^2 \leq (1 - \epsilon) \frac{n - 1}{2n(n + 2)} H_1^2.
\]

It follows that

\[
\frac{|\dot{A}|^2}{H_1^2} |\nabla H_1|^2 \leq (1 - \epsilon) \frac{n(n + 2)}{2(n - 1)} \frac{n - 1}{2n(n + 2)} \frac{|\dot{A}|^2}{f} |\langle \nabla^k_A \nu_1 \rangle|^2
\]

\[
= \frac{1}{4} (1 - \epsilon) \frac{|\dot{A}|^2}{f} |\langle \nabla^k_A \nu_1 \rangle|^2.
\]

Now the proof is as before, but we set

\[
a_1 = 1, \quad a_2 = \frac{1}{2}, \quad a_3 = 1
\]

In this case,

\[
2a_2 + 2a_3 \frac{1}{4} (1 - \epsilon) = \frac{1}{2} (3 - \epsilon),
\]

\[
\frac{2}{a_2} = 4, \quad \frac{2}{a_3} = 2.
\]

Plugging these into the Young’s inequality estimates, we get

\[
4Q_{ijk}(\dot{A}_{ij}, \nabla^k_A \nu_1) \leq 2|\langle \nabla^k_A \nu_1 \rangle|^2 + \frac{1}{2} (3 - \epsilon) \frac{|\dot{A}|^2}{f} |\langle \nabla^k_A \nu_1 \rangle|^2
\]

\[
+ 2|\dot{A}|^2 |\nabla^k_A \nu_1|^2 + 4f|\nabla^k_A \nu_1|^2 + 2|\hat{h}|^2 |\nabla^k_A \nu_1|^2,
\]

as claimed.

Finally, we combine the conclusions of Lemmas 4.5, 4.6, and 4.7 to get our desired estimate.

**Lemma 4.8** (Gradient term estimate). There exists \( \delta \) sufficiently small depending upon \( n \) and \( M_0 \) such that there holds

\[
4Q_{ijk}(\dot{A}_{ij}, \nabla^k_A \nu_1) \leq 2|\nabla^k_A |^2 + 2(1 - \delta) \frac{|\dot{A}|^2}{f} |\langle \nabla^k_A |^2 - c_n |\nabla^k A|^2 |.
\]

(If \( n \geq 8 \) and \( c_n \leq \frac{1}{3n}, \delta \leq \frac{1}{3n-8} \) is sufficient).
Proof. First suppose \( n \geq 8 \) and \( c_n \leq \frac{1}{5n} \). Including the component of \( |\nabla^\perp \hat{A}|^2 \) in the \( \nu_1 \)-direction in the first estimate of Lemma 4.5 gives us

\[
2|\nabla^\perp \hat{A}|^2 = 2|\nabla^\perp \hat{A}|^2 + 2|\langle \nabla^\perp \hat{A}, \nu_1 \rangle|^2 \\
\geq 2|\langle \nabla^\perp \hat{A}, \nu_1 \rangle|^2 + \frac{4n - 10}{n + 2} |\hat{h}|^2 |\nabla^\perp \nu_1|^2 + \frac{6(n - 1)}{n + 2} |\hat{A}|^2 |\nabla^\perp \nu_1|^2 + \frac{6(n - 1)}{n + 2} f |\nabla^\perp \nu_1|^2.
\]

Multiplying the first estimate of Lemma 4.6 by \( (1 - \delta) \) and using that \( 1 - \delta > \frac{1}{2} \) on the second term gives

\[
2(1 - \delta) \frac{|\hat{A}|^2}{f} ((|\nabla^\perp \hat{A}|^2 - c_n |\nabla^\perp H|^2) \geq (1 - \delta) \frac{5n - 8}{3(n - 1)} \frac{|\hat{A}|^2}{f} |\langle \nabla^\perp \hat{A}, \nu_1 \rangle|^2 + \frac{5n - 8}{n + 2} |\hat{A}|^2 |\nabla^\perp |\nu_1|^2.
\]

Putting these together, we get

\[
2|\nabla^\perp \hat{A}|^2 + 2(1 - \delta) \frac{|\hat{A}|^2}{f} ((|\nabla^\perp \hat{A}|^2 - c_n |\nabla^\perp H|^2) \geq \frac{11n - 14}{n + 2} |\nabla^\perp \hat{A}|^2 + 6(n - 1) f |\nabla^\perp |\nu_1|^2 + \frac{4n - 10}{n + 2} |\hat{h}|^2 |\nabla^\perp |\nu_1|^2.
\]

On the other hand, the first estimate of Lemma 4.7 gives us that

\[
4Q_{ijk} (\hat{A}_{ij}, \nabla^\perp_k \nu_1) \leq 2|\langle \nabla^\perp \hat{A}, \nu_1 \rangle|^2 + \frac{5n - 9}{3(n - 1)} \frac{|\hat{A}|^2}{f} |\langle \nabla^\perp \hat{A}, \nu_1 \rangle|^2 \\
+ 2|\hat{A}|^2 |\nabla^\perp \nu_1|^2 + \frac{3(n - 1)}{n - 3} f |\nabla^\perp \nu_1|^2 + \frac{2(n + 2)}{n + 3} |\hat{h}|^2 |\nabla^\perp \nu_1|^2.
\]

Therefore, it only remains to compare the coefficients of like terms in the two inequalities above. For the coefficients of \( \frac{|\hat{A}|^2}{f} |\langle \nabla^\perp \hat{A}, \nu_1 \rangle|^2 \), we need

\[
\frac{5n - 9}{3(n - 1)} \leq (1 - \delta) \frac{5n - 8}{3(n - 1)} \iff \delta \leq \frac{1}{5n - 8}.
\]

Comparing the coefficients of the remaining terms implies we need

\[
2n + 4 \leq 11n - 14, \\
n + 2 \leq 2(n - 3), \\
2(n + 2)^2 \leq (4n - 10)(n + 3).
\]

Each of these inequalities is true for \( n \geq 8 \) completing the proof for the first case.

Now suppose \( n = 5, 6, \) or \( 7 \) and \( c_n < \frac{3n + 1}{2(n + 2)} \). Arguing as above (again using \( \delta < \frac{1}{2} \)) to simplify the coefficient of \( |\hat{A}|^2 |\nabla^\perp \nu_1|^2 \) yields

\[
2|\nabla^\perp \hat{A}|^2 + 2(1 - \delta) \frac{|\hat{A}|^2}{f} ((|\nabla^\perp \hat{A}|^2 - c_n |\nabla^\perp H|^2) \geq 2|\langle \nabla^\perp \hat{A}, \nu_1 \rangle|^2 + (1 - \delta)^3 \frac{|\hat{A}|^2}{f} |\langle \nabla^\perp \hat{A}, \nu_1 \rangle|^2 \\
+ 7|\hat{A}|^2 |\nabla^\perp \nu_1|^2 + 4f |\nabla^\perp \nu_1|^2 + 2|\hat{h}|^2 |\nabla^\perp \nu_1|^2.
\]

Our estimate for the gradient term in this case is

\[
4Q_{ijk} (\hat{A}_{ij}, \nabla^\perp_k \nu_1) \leq 2|\langle \nabla^\perp \hat{A}, \nu_1 \rangle|^2 + (3 - \epsilon)^2 \frac{|\hat{A}|^2}{f} |\langle \nabla^\perp \hat{A}, \nu_1 \rangle|^2 \\
+ 2|\hat{A}|^2 |\nabla^\perp \nu_1|^2 + 4f |\nabla^\perp \nu_1|^2 + 2|\hat{h}|^2 |\nabla^\perp \nu_1|^2.
\]

Choosing \( \delta \leq \frac{1}{5n} \), completes the proof of the lemma. \( \square \)
We now complete the proof of Theorem 1.1. Let \( \delta \) be sufficiently small so that each of our above lemmas holds. We begin by splitting off the desired nonpositive term in the evolution equation

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \frac{|\dot{A}|^2}{f} = \frac{1}{f} \left( \frac{\partial}{\partial t} - \Delta \right) |\dot{A}|^2 - |\dot{A}|^2 \frac{1}{f^2} \left( \frac{\partial}{\partial t} - \Delta \right) f + 2 \left\langle \nabla \frac{|\dot{A}|^2}{f}, \nabla \log f \right\rangle 
\]

\[
= 2 \left\langle \nabla \frac{|\dot{A}|^2}{f}, \nabla \log f \right\rangle - \delta \frac{|\dot{A}|^2}{f^2} \left( \frac{\partial}{\partial t} - \Delta \right) f 
+ \frac{1}{f} \left( \frac{\partial}{\partial t} - \Delta \right) |\dot{A}|^2 - (1 - \delta) \frac{|\dot{A}|^2}{f^2} \left( \frac{\partial}{\partial t} - \Delta \right) f.
\]

By the inequalities of Lemmas 4.4 and 4.8, the sum of terms are the second line are nonpositive:

\[
\frac{1}{f} \left( \frac{\partial}{\partial t} - \Delta \right) |\dot{A}|^2 - (1 - \delta) \frac{|\dot{A}|^2}{f^2} \left( \frac{\partial}{\partial t} - \Delta \right) f 
= \frac{1}{f} \left( 2 |\langle A_{ij}, \dot{A}_{kl} \rangle|^2 + 2 |A_{ik} \otimes A_{jk} - \dot{A}_{ik} \otimes \dot{A}_{jk}|^2 + 2 |R_{ij}^k(\nu_1)|^2 \right) 
- \frac{1}{f} \left( (1 - \delta) \frac{|\dot{A}|^2}{f^2} (c_n |\langle A_{ij}, H \rangle|^2 - |\langle A_{ij}, A_{kl} \rangle|^2 - |R_{ij}^k|^2) \right) 
+ \frac{1}{f} \left( 4 Q_{ijkl}(\dot{A}_{ij}, \nabla \dot{c}_k \nu_1) - 2 |\nabla \dot{A}|^2 - 2(1 - \delta) \frac{|\dot{A}|^2}{f^2} (|\nabla \dot{A}|^2 - c_n |\nabla \dot{H}|^2) \right) 
\leq 0.
\]

Thus we have our initial claim

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \frac{|\dot{A}|^2}{f} \leq 2 \left\langle \nabla \frac{|\dot{A}|^2}{f}, \nabla \log f \right\rangle - \delta \frac{|\dot{A}|^2}{f^2} \left( \frac{\partial}{\partial t} - \Delta \right) f.
\]

Recall that \( \left( \frac{\partial}{\partial t} - \Delta \right) f \) is nonnegative at each point in space-time. Let \( \sigma = \delta \). We compute that

\[
\left( \frac{\partial}{\partial t} - \Delta \right) f^{1-\sigma} = (1 - \sigma) f^{-\sigma} \left( \frac{\partial}{\partial t} - \Delta \right) f + \sigma (1 - \sigma) f^{-\sigma - 1} |\nabla f|^2 
\geq (1 - \sigma) f^{-\sigma} \left( \frac{\partial}{\partial t} - \Delta \right) f.
\]

Then

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \frac{|\dot{A}|^2}{f^{1-\sigma}} = \frac{1}{f^{1-\sigma}} \left( \frac{\partial}{\partial t} - \Delta \right) |\dot{A}|^2 - |\dot{A}|^2 \frac{1}{f^{2-2\sigma}} \left( \frac{\partial}{\partial t} - \Delta \right) f^{1-\sigma} + 2 \left\langle \nabla \frac{|\dot{A}|^2}{f^{1-\sigma}}, \nabla \log f^{1-\sigma} \right\rangle 
\leq \frac{1}{f^{1-\sigma}} \left( \frac{\partial}{\partial t} - \Delta \right) |\dot{A}|^2 - |\dot{A}|^2 \frac{1}{f^{2-2\sigma}} (1 - \sigma) f^{-\sigma} \left( \frac{3}{\partial t} - \Delta \right) f 
+ 2 \left\langle \nabla \frac{|\dot{A}|^2}{f^{1-\sigma}}, \nabla \log f^{1-\sigma} \right\rangle 
= f^\sigma \left( \frac{1}{f} \left( \frac{\partial}{\partial t} - \Delta \right) |\dot{A}|^2 - \frac{|\dot{A}|^2}{f^2} \left( \frac{\partial}{\partial t} - \Delta \right) f \right) + c |\dot{A}|^2 f^\sigma \left( \frac{\partial}{\partial t} - \Delta \right) f 
+ 2 \left\langle \nabla \frac{|\dot{A}|^2}{f^{1-\sigma}}, \nabla \log f^{1-\sigma} \right\rangle.
\]

Now

\[
\frac{1}{f} \left( \frac{\partial}{\partial t} - \Delta \right) |\dot{A}|^2 - \frac{|\dot{A}|^2}{f^2} \left( \frac{\partial}{\partial t} - \Delta \right) f = \left( \frac{\partial}{\partial t} - \Delta \right) \frac{|\dot{A}|^2}{f} - 2 \left\langle \nabla \frac{|\dot{A}|^2}{f}, \nabla \log f \right\rangle 
\leq -\delta \frac{|\dot{A}|^2}{f^2} \left( \frac{\partial}{\partial t} - \Delta \right) f.
\]

Therefore

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \frac{|\dot{A}|^2}{f^{1-\sigma}} \leq -\delta \frac{|\dot{A}|^2}{f^2} f^\sigma \left( \frac{\partial}{\partial t} - \Delta \right) f + |\dot{A}|^2 f^\sigma \left( \frac{\partial}{\partial t} - \Delta \right) f 
+ 2 \left\langle \nabla \frac{|\dot{A}|^2}{f^{1-\sigma}}, \nabla \log f^{1-\sigma} \right\rangle 
= 2 \left\langle \nabla \frac{|\dot{A}|^2}{f^{1-\sigma}}, \nabla \log f^{1-\sigma} \right\rangle.
\]
Hence by the maximum principle, there exists a constant $C$ depending only on the initial manifold $M_0$ and $n$ such that $|\hat{A}|^2 \leq Cf^{1-\sigma}$ for all $t \in [0, T)$. Since $f \leq c_n|H|^2$, this implies $|\hat{A}|^2 \leq C|H|^{2-2\sigma}$, completing the proof of the theorem.

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