FEDERATED LEARNING FROM BIG DATA OVER NETWORKS

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ABSTRACT

This paper formulates and studies a novel algorithm for federated learning from large collections of local datasets. This algorithm capitalizes on an intrinsic network structure that relates the local datasets via an undirected “empirical” graph. We model such big data over networks using a networked linear regression model. Each local dataset has individual regression weights. The weights of close-knit sub-collections of local datasets are enforced to deviate only little. This lends naturally to a network Lasso problem which we solve using a primal-dual method. We obtain a distributed federated learning algorithm via a message passing implementation of this primal-dual method. We provide a detailed analysis of the statistical and computational properties of the resulting federated learning algorithm.

Index Terms— machine learning, federated learning, convex optimization, estimation, complex networks

1. INTRODUCTION

Federated learning is a recent paradigm for training machine learning models in a collaborative fashion using distributed local datasets [13, 11, 12]. These methods are appealing for sensitive applications, such as healthcare, as they do not require local raw data to be revealed to others [2]. In contrast to existing federated learning techniques, we study a novel federated learning method that leverages a well-defined network structure relating local datasets.

Network structures between local datasets can arise from spatio-temporal proximity or statistical dependencies. The network structure allows to learn tailored models for coherent subsets, or clusters, of local datasets instead of a single global model which is delivered by most existing federated learning methods [13]. As a case in point, consider the high-precision model which is delivered by most existing federated learning techniques, we study a novel federated learning method that leverages a well-defined network structure relating local datasets.

This paper represents networked data conveniently using an undirected “empirical” or “similarity” graph. Each node of this graph represents a local dataset which is constituted by feature vectors and labels. We model the relation between features and labels of a local dataset using a local Lasso problem (sparse linear regression).

The local regression models are coupled by requiring their parameter or weight vector to have a small total variation. This essentially requires weight vectors to be approximately constant for all local datasets within the same well-connected subset or cluster. We frame federated learning as a convex network Lasso optimization problem which we solve using an efficient and robust primal-dual method [16, 7, 10].

This paper significantly extends our previous work on localized linear regression and classification [10, 20, 3]. In particular, we allow local datasets to contain many individual data points. Moreover, our method can be used with non-linear local models such as Lasso and its generalizations [9].

Notation. The identity matrix of size $n \times n$ is denoted $I_n$ with the subscript omitted if the size $n$ is clear from context. The Euclidean norm of a vector $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$ is $\|w\|_2 := \sqrt{\sum_{r=1}^{n} w_r^2}$ and the $\ell_1$ norm $\|w\|_1 := \sum_{r=1}^{n} |w_r|$. It will be convenient to use the notation $(1/2\tau)$ instead of $(1/(2\tau))$. We will need the scalar clipping function $T^\lambda(w) := \lambda w/|w|$ for $|w| \geq \lambda$ and $T^\lambda(w) := w$ otherwise.

2. PROBLEM FORMULATION

We model local datasets by an undirected “empirical” graph $G = (\mathcal{V}, \mathcal{E}, A)$ (see Figure 1).

![Empirical graph of local datasets](image)

Fig. 1: Empirical graph of local datasets $\mathcal{X}^{(i)}$.

Each node $i \in \mathcal{V}$ represents a local dataset

$$\mathcal{X}^{(i)} := (x^{(i,1)}, y_1^{(i)}), \ldots, (x^{(i,m_i)}, y_{m_i}^{(i)}).$$

We assume that features $x^{(i,j)}$ of any data point can be determined easily. In contrast, the acquisition of labels $y_j^{(i)}$ is
costly (requiring human expert labour). Thus, we assume to have access only to the labels of a small training set
\[ M = \{i_1, \ldots, i_M\} \subseteq V. \] (1)

The undirected edges \( \{i, j\} \in E \) of the empirical graph connect similar datasets \( X^{(i)}, X^{(j)} \) with the strength of the similarity quantified by the entry \( A_{ij} > 0 \). The neighbourhood of the node \( i \in V \) is denoted \( N_i := \{ j \in V : \{i, j\} \in E \} \).

We aim at learning a local (node-wise) predictor \( h(x; w^{(i)}) \) for each node \( i \in V \). The local predictors are parametrized by the weight vectors \( w^{(i)} \). For numeric labels \( y \in \mathbb{R} \), we use the output \( h(x; w^{(i)}) \) directly as the predicted label \( \hat{y} = h(x; w^{(i)}) \). For binary labels \( y \in \{0, 1\} \) we classify \( \hat{y} = 1 \) for \( h(x; w^{(i)}) \geq 0 \) and \( \hat{y} = 0 \) otherwise.

Our approach to learning the weights \( w^{(i)} \) exploits the intrinsic network structure relating the local datasets. We use a clustering assumption of having similar statistical properties of local datasets that form tight-knit subsets or clusters. We show theoretically and empirically that enforcing the learnt weight vectors \( w^{(i)} \) to be similar for well-connected datasets allows to share statistical strength between local datasets.

We interpret the weight vectors \( w^{(i)} \) as the values of a graph signal \( w : V \to \mathbb{R}^n \) which assigns node \( i \in V \) the vector \( w^{(i)} \in \mathbb{R}^n \). With a slight abuse of notation, we denote the set of all vector-valued node signals as \( \mathbb{R}^{n|V|} := \{w : V \to \mathbb{R}^n : i \mapsto w^{(i)}\} \). Each graph signal \( w \in \mathbb{R}^{n|V|} \) represents a network predictor, parametrized by \( w^{(i)} \in \mathbb{R}^n \) for \( i \in V \).

A good predictor should have small empirical loss
\[ \hat{E}(w) := \sum_{i \in M} \mathcal{L}(X^{(i)}, w^{(i)}). \] (2)

The loss function \( \mathcal{L}(X^{(i)}, w^{(i)}) \) measures the quality of applying the predictor \( h^{(i)}(x) := x^T w^{(i)} \) to the local dataset \( X^{(i)} \) and \( y^{(i)} \). Section 4 discusses three choices for the loss function.

### 3. NETWORK LASSO

The criterion (2) by itself is not enough for guiding the learning of a predictor \( w \) since (2) completely ignores the weights \( w^{(i)} \) at unlabeled nodes \( i \in V \backslash M \). We need to impose some additional structure on the predictor \( w \). To this end, we require the predictor \( w \) to conform with the cluster structure of the empirical graph \( G \).

To obtain similar predictors \( w^{(i)} \approx w^{(j)} \) for nodes \( i, j \in V \) belonging to the same cluster, we enforce a small total variation (TV)
\[ \|w\|_{TV} := \sum_{i,j \in E} A_{ij} \|w^{(j)} - w^{(i)}\|_1. \] (3)

Minimizing TV forces weights \( w^{(i)} \) to only change over few edges with relatively small weights \( A_{i,j} \). The predictor optimally balancing between empirical error (2) with TV is
\[ \hat{w} \in \arg \min_{w \in \mathbb{R}^{n|V|}} \hat{E}(w) + \lambda \|w\|_{TV}. \] (4)

Note that (4) does not enforce the predictions \( \hat{y}^{(i)} \) themselves to be clustered, but the weight vectors \( \hat{w}^{(i)} \) of predictors.

The convex optimization problem (4) is a special case of nLasso [7]. The parameter \( \lambda > 0 \) in (2) allows to trade small TV \( \|\hat{w}\|_{TV} \) against small error \( \hat{E}(\hat{w}) \) (2). The choice of \( \lambda \) can be guided by cross validation [8].

Let us define the block incidence matrix \( D \in \mathbb{R}^{n|E| \times n|V|} \) as follows:
\[ D_{e,i} = 1 \] for \( e = \{i,j\} \in E \) with some \( j > i \),
\[ D_{e,i} = -1 \] for \( e = \{i,j\} \) with some \( i > j \) and \( D_{e,i} = 0 \) otherwise. Then we can reformulate (4) as
\[ \hat{w} \in \arg \min_{w \in \mathbb{R}^{n|V|}} f(w) + g(Dw) \] (5)

with
\[ f(w) := \sum_{i \in M} \mathcal{L}(X^{(i)}, w^{(i)}), \quad \text{and} \quad g(u) := \lambda \sum_{e \in E} A_e \|u^{(e)}\|_1 \]
using \( u = (u^{(1)}, \ldots, u^{(E)})^T \in \mathbb{R}^{n|E|}. \) (6)

### 4. FEDERATED LEARNING ALGORITHM

We solve (5) jointly with the dual problem
\[ \max_{u \in \mathbb{R}^{n|E|}} -g^*(u) - f^*(-D^T u). \] (7)

The problem (7) optimizes a vector-valued signal \( u \in \mathbb{R}^{n|E|} \) which maps each edge \( e \in E \) to some vector \( u^{(e)} \in \mathbb{R}^n \). The objective function (7) is composed of the convex conjugates
\[ g^*(u) := \sup_{z \in \mathbb{R}^E} u^T z - g(z) \]
\[ = \sup_{z \in \mathbb{R}^E} u^T z - \lambda \sum_{e \in E} A_e \|z^{(e)}\|_1 \]
\[ = \begin{cases} \infty & \text{if } |u^{(e)}| > \lambda A_e \text{ for some } e \in E \\ 0 & \text{otherwise}, \end{cases} \] (8)

and
\[ f^*(w) := \sup_{z \in \mathbb{R}^{n|E|}} w^T z - f(z). \] (9)

The duality between (5) and (7) is made precise in [18, Ch. 31] (see also [6, Sec. 3.5]). The optimal values of both problems are the same [18, Cor. 31.2.1].

\[ \min_{w \in \mathbb{R}^{n|V|}} f(w) + g(Dw) = \max_{u \in \mathbb{R}^{n|E|}} -g^*(u) - f^*(-D^T u). \] (10)

A necessary and sufficient condition for \( \hat{w} \) to solve (5) and \( \hat{u} \) to solve (7) is
\[ -D^T \hat{u} \in \partial f(\hat{w}) \text{, and } D \hat{w} \in \partial g^*(\hat{u}). \] (11)
The coupled conditions (11) are, in turn, equivalent to
\[
\dot{w} - TD^T u \in (I + T \partial f)(\dot{w}), \\
\dot{u} + \Sigma D \dot{w} \in (I + \Sigma \partial g^*)(\dot{u}).
\] (12)

The positive semi-definite block-diagonal matrices
\[
(\Sigma)_{e,e} = \sigma(e) I_n, \text{ for } e \in E, \quad (\Sigma)_{k,i} = \tau(i) I \text{ for } i \in V,
\]
with \(\sigma(e) = 1/2\) for \(e \in E\) and \(\tau(i) = 1/|N_i|\) for \(i \in V\), serve as a pre-conditioning to speed up the convergence of the resulting iterative algorithm (see [17]).

The coupled conditions (12) represent a fixed-point characterization of the solutions to nLasso (5) and its dual (7). We solve the optimality condition (12) iteratively using the following fixed-point iterations (see [6])
\[
\begin{align*}
\tilde{w}_{k+1} & := (I + T \partial f)^{-1}(w_k - TD^T u_k) \\
\tilde{u}_{k+1} & := (I + \Sigma \partial g^*)^{-1}(u_k + \Sigma D (2w_{k+1} - w_k)).
\end{align*}
\] (14) (15)

The updates (14) can be evaluated by using the identity [17]
\[
(I + T \partial f)^{-1}(w) = \arg\min_{v \in \mathbb{R}^{|V|}} f(v) + (1/2)\|v - w\|^2_{T^{-1}}
\] (16)
and a similar identity for \((I + \Sigma \partial g^*)^{-1}\).

Inserting (6) into (16) reveals that the primal update decomposes into separate updates at each node \(i \in V\),
\[
\tilde{w}^{(i)}_{k+1} = \mathcal{P} \mathcal{U}_i \{ w^{(i)}_k - \tau(i) \sum_{e \in E} D_{e,i} u^{(e)}_k \}
\] (17)
with the primal update operator
\[
\mathcal{P} \mathcal{U}_i \{ v \} := \arg\min_{z \in \mathbb{R}^n} \mathcal{L}(\mathcal{X}^{(i)}, z) + (1/2\tau_i)\|z - v\|^2.
\] (18)

The operator (18) and associated node-wise primal update and (17) depend (via (6)) on the choice for the loss function \(\mathcal{L}\).

For convex loss functions \(\mathcal{L}(\mathcal{X}^{(i)}, z)\) (see (2)), the sequences \(\tilde{w}_k\) and \(\tilde{u}_k\) obtained from iterating (14), (15) converge to solutions of (5) and (7). This convergence is guaranteed for any initialization \(w_0\) and \(u_0\) [17].

Another appealing property of the updates (14), (15) is that they are robust against errors. This property is important for applications where the resolvent operator (16) can be evaluated approximately only. An important example when this happens is obtained when using the logistic loss function (23) (see Section 4.3).

We summarize the primal-dual method for solving nLasso (5) and its dual (7) in Algorithm 1. This algorithm is to be understood as a template for specific learning algorithms that are obtained by evaluating (17) for given choice for the loss function in (2).

Algorithm 1 can be implemented as a message passing on the empirical graph. The application of the block-incidence matrix \(D\) and its transpose \(D^T\) involves, for each node and edge, only neighboring nodes and edges. Thus, Algorithm 1 is scalable to massive collections of local datasets.

Algorithm 1 combines the information contained in the local datasets with their network structure to iteratively improve the weight vectors \(\tilde{w}^{(i)}_k\) for each node \(i \in V\). Step 3 adapts the current weight vectors \(\hat{w}^{(i)}_k\) to better fit the labeled local datasets \(\mathcal{X}^{(i)}\) for \(i \in \mathcal{M}\). These updates are then propagated to the weight vectors at unlabeled nodes \(i \in V \setminus \mathcal{M}\) via steps 6 and 8.

### 4.1. Federated Networked Linear Regression

We now discuss how Algorithm 1 can be used to learn a node-wise linear predictor
\[
\tilde{y} = h(x; w^{(i)}) = x^T w^{(i)}. \tag{19}
\]
To measure how well a given network of node-wise linear predictor agrees with labeled datasets \(\mathcal{X}^{(i)}\), for \(i \in \mathcal{M}\), we use the squared error loss
\[
\mathcal{L}(\mathcal{X}^{(i)}, v) = (1/m_i) \sum_{r=1}^{m_i} (y^{(i)}_r - v^T x^{(i,r)})^2. \tag{20}
\]
Inserting (20) into (18), yields
\[
\mathcal{P} \mathcal{U}_i (v) = \left[ I + 2\tau(i) Q^{(i)} \right]^{-1} \left[ v + 2\tau(i) \tilde{y}^{(i)} \right]. \tag{21}
\]
Here, we used the squared feature matrix
\[
Q^{(i)} := (X^{(i)})^T X^{(i)} \text{ with } X^{(i)} := (x^{(i,1)}, \ldots, x^{(i,m_i)})^T
\]
and the normalized label vector
\[
\tilde{y}^{(i)} := (X^{(i)})^T y^{(i)} \text{ with } y^{(i)} := (y^{(i)}_1, \ldots, y^{(i)}_{m_i})^T.
\]
4.2. Federated Networked Lasso

Algorithm 1 for networked linear regression (see Section 4.2) can only be expected to work well if the number of data points in each local dataset is larger than the number of features \( n \).

Many application domains involve high-dimensional local datasets \( \mathcal{X}^{(i)} \), where the number of data points is much smaller than the length of feature vectors, \( m_i \ll n \) [5, 9]. This high-dimensional regime requires some form of regularization for learning a linear predictor (19). The Lasso is obtained from the regularized loss function [9]

\[
\mathcal{L}(\lambda^{(i)}, \mathbf{v}) := (1/m_i) \sum_{r=1}^{m_i} (\mathbf{v}^T \mathbf{x}^{(i,r)} - y_r^{(i)})^2 + \lambda \| \mathbf{v} \|_1.
\]

Plugging this into (18), yields the primal update operator

\[
\mathcal{P} \mathcal{U}_i (w^{(i)}) = \arg\min_{\mathbf{v} \in \mathbb{R}^n} (1/m_i) \sum_{r=1}^{m_i} (\mathbf{v}^T \mathbf{x}^{(i,r)} - y_r^{(i)})^2 + \lambda \| \mathbf{v} \|_1 + (1/2\tau_i) (\mathbf{v} - w^{(i)})^2.
\]

4.3. Federated Networked Logistic Regression

We discuss how Algorithm 1 can be used to learn a networked linear classifier for binary labels \( y \in \{0, 1\} \). As in Section 4.2, we aim at learning a networked linear predictor. In contrast to Section 4.2, we quantize its output to obtain the predicted label \( \hat{y} = 1 \) if \( h(x; \mathbf{w}^{(i)}) > 0 \) and \( \hat{y} = 0 \) otherwise.

To measure the quality of a given linear classifier we use the logistic loss

\[
\mathcal{L}(\lambda^{(i)}, \mathbf{v}) = (-1/m_i) \sum_{r=1}^{m_i} y_r^{(i)} \log (\sigma(\mathbf{v}^T \mathbf{x}^{(i,r)}))
+ (1 - y_r^{(i)}) \log (1 - \sigma(\mathbf{v}^T \mathbf{x}^{(i,r)})).
\]

In general, there is no closed-form expression for the update (17) when using the logistic loss (23). However, the update (23) amounts to an unconstrained minimization of a smooth convex function (18). Such optimization problems can be solved efficiently with established iterative algorithms [8, Ch. 4].

5. NUMERICAL EXPERIMENTS

To empirically evaluate the accuracy of Algorithm 1, we apply it to a synthetic dataset. We generate the empirical graph \( \mathcal{G} \) using the stochastic block model with two clusters \( |\mathcal{C}_1| = |\mathcal{C}_2| = 150 \) [14, 1]. Two nodes within the same cluster are connected by an edge with probability \( p_{\text{in}} \). Two nodes from different clusters are connected by an edge with probability \( p_{\text{out}} \).

| Method Name                        | training MSE | test MSE |
|-----------------------------------|--------------|----------|
| our method                        | 1.7e-6       | 1.8e-6   |
| simple linear regression          | 4.04         | 4.51     |
| decision tree regression          | 4.21         | 4.87     |

Table 1: MSE achieved by Algorithm 1 which leverages the network structure encoded by the empirical graph \( \mathcal{G} \). We also report the MSE achieved by plain linear regression and decision tree regression applied to the concatenation of all local datasets, ignoring the network structure.

Each node \( i \in \mathcal{V} \) represents a local dataset consisting of 5 feature vectors \( \mathbf{x}^{(i,1)}, \ldots, \mathbf{x}^{(i,5)} \in \mathbb{R}^2 \). The feature vectors are i.i.d. realizations of a standard Gaussian random vector \( x \sim \mathcal{N}(0, \mathbf{I}) \). The labels \( y_1^{(i)}, \ldots, y_5^{(i)} \in \mathbb{R} \) of the nodes \( i \in \mathcal{V} \) are generated according to the linear model \( y_r^{(i)} = \mathbf{x}^{(i,r)^T} \mathbf{w}^{(i)} \) with weight vector \( \mathbf{w}^{(i)} = (2, 2)^T \) for \( i \in \mathcal{C}_1 \) and \( \mathbf{w}^{(i)} = (-2, 2)^T \) for \( i \in \mathcal{C}_2 \).

To learn the weight \( \mathbf{w}^{(i)} \), we apply Algorithm 1 to a training set \( \mathcal{M} \) obtained by randomly selecting 30 nodes. We run Algorithm 1 for different choices of \( p_{\text{out}} \) with a fixed \( p_{\text{in}} = 1/2 \).

By fixing \( p_{\text{out}} = 10^{-3} \), we run Algorithm 1 for different choices of \( \lambda \) and a fixed number of 500 iterations. We measure the quality of the learnt weight vectors \( \mathbf{w}^{(i)} \) by the mean-squared error (MSE) (fig 2)

\[
\epsilon(\tilde{\mathbf{w}}) := (1/|\mathcal{V}|) \sum_{i \in \mathcal{V} \setminus \mathcal{M}} \| \mathbf{w}^{(i)} - \tilde{\mathbf{w}}^{(i)} \|_2^2.
\]

The tuning parameter \( \lambda \) in (4) is manually chosen, guided by the resulting MSE, as \( \lambda = 10^{-3} \). We compare the MSE of Algorithm 1 with plain linear regression and decision tree regression in Table 1.

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