Abstract

We show that it is decidable, given an automatic sequence $s$ and a constant $c$, whether all prefixes of $s$ have a string attractor of size $\leq c$. Using a decision procedure based on this result, we show that all prefixes of the period-doubling sequence of length $\geq 2$ have a string attractor of size 2. We also prove analogous results for other sequences, including the Thue-Morse sequence and the Tribonacci sequence.

We also provide general upper and lower bounds on string attractor size for different kinds of sequences. For example, if $s$ has a finite appearance constant, then there is a string attractor for $s[0..n−1]$ of size $O(\log n)$. If further $s$ is linearly recurrent, then there is a string attractor for $s[0..n−1]$ of size $O(1)$. For automatic sequences, the size of the smallest string attractor for $s[0..n−1]$ is either $\Theta(1)$ or $\Theta(\log n)$, and it is decidable which case occurs. Finally, we close with some remarks about greedy string attractors.

1 Introduction

Recently Kempa and Prezza [14] introduced the notion of string attractor. Let $w = w[0..n−1]$ be a finite word, indexed beginning at position 0. A string attractor of $w$ is a subset $S \subseteq \{0,1,\ldots,n−1\}$ such that every nonempty factor $f$ of $w$ has an occurrence in $w$ that touches one of the indices of $S$. For example, $\{2,3,4\}$ is a string attractor for the English word alfalfa, and no smaller string attractor exists for that word. Also see Kociumaka et al. [15].

It is of interest to estimate the minimum possible size $\gamma(w)$ of a string attractor for classically-interesting words $w$. This has been done, for example, for the finite Thue-Morse words $\mu^n(0)$, where $\mu$ is the map sending $0 \rightarrow 01$ and $1 \rightarrow 10$, and $\mu^n$ denotes the $n$-fold
composition of $\mu$ with itself. Indeed, Mantaci et al. [17] proved that $\mu^n(0)$ has a string attractor of size $n$ for $n \geq 3$, and this was later improved to the constant 4, for all $n \geq 4$, by Kutsukake et al. [16].

In this note we first show that it is decidable, given an automatic sequence $s$ and a constant $c$, whether all prefixes of $s$ have a string attractor of size at most $c$. Furthermore, if this is the case, we can construct an automaton that, for each $n$, provides the elements of a minimal string attractor for the length-$n$ prefix of $s$. We illustrate our ideas by proving that the minimal string attractor for length-$n$ prefixes of the period-doubling sequence is of cardinality 2 for $n \geq 2$, and we obtain analogous results for the Thue-Morse sequence, the Tribonacci sequence, and two others.

We use the notation $[i..j]$ for $\{i, i+1, \ldots, i+j\}$, and $w[i..j]$ for the factor $w[i]w[i+1] \cdots w[j]$.

2 Automatic sequences

A numeration system represents each natural number $n$ uniquely as a word over some finite alphabet $\Sigma$. If further the set of canonical representations is regular, and the relation $z = x+y$ is recognizable by a finite automaton, we say that the numeration system is regular. Examples of regular numeration systems include base $b$, for integers $b \geq 2$ [1]; Fibonacci numeration [12]; Tribonacci numeration [20]; and Ostrowski numeration systems [2].

Finally, a sequence $s = (s_n)_{n \geq 0}$ is automatic if there exists a regular numeration system and an automaton that, on input the representation of $n$, computes $s_n$. We have the following result [8]:

**Theorem 1.** Let $s$ be an automatic sequence.

(a) There is an algorithm that, given a well-formed formula $\varphi$ in the FO($\mathbb{N}, +, 0, 1, n \rightarrow s[n]$) having no free variables, decides if $\varphi$ is true or false.

(b) Furthermore, if $\varphi$ has free variables, then the algorithm constructs an automaton recognizing the representation of the values of those variables for which $\varphi$ evaluates to true.

We are now ready to prove

**Theorem 2.**

(a) It is decidable, given an automatic sequence $s$ and a constant $c$, whether all prefixes of $s$ have a string attractor of size at most $c$.

(b) Furthermore, if this is the case, we can construct an automaton that, for each $n$, provides the elements of a minimal string attractor for the length-$n$ prefix of $s$.

**Proof.**
(a) From Theorem 1, it suffices to create a first-order formula \( \varphi \) asserting that the length-\( n \) prefix of \( s \) has a string attractor of size at most \( c \). We construct \( \varphi \) in several stages.

First, we need a formula asserting that the factors \( s[i..j] \) and \( s[k + i - j..k] \) coincide. We can do this as follows:

\[
\text{faceq}(i, j, k) := \forall u, v \,(u \geq i \land u \leq j \land v + j = u + k) \implies s[u] = s[v].
\]

Next, let us create a formula asserting that \( \{i_1, i_2, i_3, \ldots, i_c\} \) is a string attractor for the length-\( n \) prefix of \( s \). We can do this as follows:

\[
\text{sa}(i_1, i_2, \ldots, i_c, n) := (i_1 < n) \land (i_2 < n) \land \cdots \land (i_c < n) \land (\forall k, l \,(k \leq l \land l < n) \implies (\exists r, s \, r \leq s \land s < n \land (s + k = r + l) \land \text{faceq}(k, l, s) \land ((r \leq i_1 \land i_1 \leq s) \lor (r \leq i_2 \land i_2 \leq s) \lor \cdots \lor (r \leq i_c \land i_c \leq s))).
\]

Notice here that we do not demand that the \( i_j \) be distinct, which explains why this formula checks that the string attractor size is \( \leq c \) and not equal to \( c \).

Finally, we can create a formula with no free variables asserting that every prefix of \( s \) has a string attractor of cardinality \( \leq c \) as follows:

\[
\forall n \exists i_1, i_2, \ldots, i_c \text{ sa}(i_1, i_2, \ldots, i_c, n).
\]

(b) The algorithm in [8] (which is essentially described in [7]) constructs an automaton for \( \text{sa} \), which recognizes \( \text{sa}(i_1, i_2, \ldots, i_c, n) \). It is now easy to find the lexicographically first set of indices \( i_1, \ldots, i_c \) corresponding to any given \( n \).

One advantage to this approach is that it lets us (at least in principle) compute the size of the smallest string attractor for all prefixes of an automatic word, not just the ones of (for example) length \( 2^n \) (as in the case of the Thue-Morse words). Notice that knowing \( \gamma \) on prefixes of length \( 2^n \) of the Thue-Morse word \( t \) doesn’t immediately give \( \gamma \) on all prefixes, since \( \gamma \) need not be monotone increasing on prefixes.

For example, here is the size \( s_n \) of the smallest string attractor for length-\( n \) prefixes of the Thue-Morse infinite word \( t \), \( 1 \leq n \leq 32 \):

| \( n \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|-------|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|
| \( s_n \) | 1 | 2 | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 4 | 4 |

| \( n \) | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 |
|-------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| \( s_n \) | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |

Note that \( \gamma(t[0..15]) = 4 \), while \( \gamma(t[0..16]) = 3 \). We will obtain a closed form for \( \gamma(t[0..n-1]) \) in Section 4.
3 The period-doubling sequence

Now let’s apply these ideas to a famous 2-automatic infinite word, the period-doubling sequence \( \mathbf{pd} = 10110101011 \cdots \) [13, 4, 11]. It is the fixed point of the morphism \( 1 \to 10, 0 \to 11 \).

We will use the theorem-proving software \texttt{Walnut} [19] created by Hamoon Mousavi; it implements the decision procedure alluded to in Theorem 1. \texttt{Walnut} allows the user to enter the logical formulas we created above and determine the results.

Here is the \texttt{Walnut} translation of our logical formulas. In \texttt{Walnut} the period-doubling sequence is represented by the automaton named PD. The universal quantifier \( \forall \) is represented by \( A \), and the existential quantifier \( \exists \) is represented by \( E \). The macro \texttt{pdfaceq} tests whether \( \mathbf{pd}[i..j] = \mathbf{pd}[k-j+i..k] \). The rest of the translation into \texttt{Walnut} is probably self-explanatory.

\[
\text{def pdfaceq} \ "k+i\geq j \land \ A \ u,v \ (u\geq i \land u\leq j \land v+j=u+k) \Rightarrow PD[u]=PD[v]"
\]

\[
\text{def pdsa2} \ "(i1\lt n) \land (i2\lt n) \land A k,l \ (k\leq l \land l\lt n) \Rightarrow (E r,s \ r\leq s \land s\lt n \land (s+k=r+l) \land \text{pdfaceq}(k,l,s) \land ((r\leq i1 \land i1\leq s) \lor (r\leq i2 \land i2\leq s)))"
\]

\[
\text{def pdfa1} \ "A n \ (n\geq 2) \Rightarrow E i1,i2 \ \text{pdsa2}(i1,i2,n)"
\]

\[
\text{def pdfa2} \ "$\text{pdsa2}(i1,i2,n) \land i1 < i2"
\]

\[
\text{def pdfa3} \ "$\text{pdfa2}(i1,i2,n) \land (A i3, i4 \ \text{pdfa2}(i3,i4,n) \Rightarrow i4 \geq i2)"
\]

\[
\text{def pdfa4} \ "$\text{pdfa3}(i1,i2,n) \land (A i3 \ \text{pdfa3}(i3,i2,n) \Rightarrow i3=i1)"
\]

Here the \texttt{Walnut} command \texttt{pdfa1} returns \texttt{true}, which confirms that every length-\( n \) prefix of the period-doubling sequence, for \( n \geq 2 \), has a string attractor of size at most 2. Since every word containing two distinct letters requires a string attractor of size at least 2, we have now proved

\textbf{Theorem 3.} The minimum string attractor for \( \mathbf{pd}[0..n-1] \) is 2 for \( n \geq 2 \).

Achieving this was a nontrivial computation in \texttt{Walnut}. Approximately 45 gigabytes of storage and 60 minutes of CPU time were required. The number of states in each automaton, and the time in milliseconds required to compute it, are given as follows:
Here is the automaton produced for pdfa4. It takes as input the binary representation of triples of natural numbers, in parallel. If it accepts \((i_1, i_2, n)\), then \(\{i_1, i_2\}\) is a string attractor of size 2 for the length-\(n\) prefix of pd.

For example, consider the prefix pd[0..25] of length 26: 10111010101110111011101010. In the automaton we see that the base-2 representation of (7, 15, 26) is accepted, in parallel, so one string attractor is \(\{7, 15\}\):

10111010101110111011101010.

By inspection of this automaton we easily get

**Theorem 4.** Let \(n \geq 6\). Then a string attractor for pd[0..n – 1] is given by

\[
\begin{cases}
\{3 \cdot 2^{i-3} – 1, 3 \cdot 2^{i-2} – 1\}, & \text{if } 2^i \leq n < 3 \cdot 2^i; \\
\{2^i – 1, 2^{i+1} – 1\}, & \text{if } 3 \cdot 2^i \leq n < 2^{i+1}.
\end{cases}
\]

4 Back to Thue-Morse

We would like to perform the same kind of analysis for the Thue-Morse sequence, finding the size of the minimal string attractor for each of its finite prefixes. Although clearly achievable in theory, unfortunately, a direct translation of what we did for the period-doubling sequence fails to run to completion on Walnut within reasonable time and space bounds. So we use a different approach.
Theorem 5. Let \( a_n \) denote the size of the smallest string attractor for the length-\( n \) prefix of the Thue-Morse word \( t \). Then

\[
a_n = \begin{cases} 
1, & \text{if } n = 1; \\
2, & \text{if } 2 \leq n \leq 6; \\
3, & \text{if } 7 \leq n \leq 14 \text{ or } 17 \leq n \leq 24; \\
4, & \text{if } n = 15, 16 \text{ or } n \geq 25.
\end{cases}
\]

Proof. For \( n \leq 59 \) we can verify the values of \( a_n = \gamma(t[0..n-1]) \) with a machine computation.

Define \( \rho_n(w) \) to be the number of distinct length-\( n \) factors of the word \( w \), and set \( \delta(w) = \max_{1 \leq i \leq n} \rho_i(w)/i \). Christiansen et al. [10, Lemma 5.6] proved that \( \delta(w) \leq \gamma(w) \) for all words \( w \).

Next, we observe that \( t[0..59] \), the prefix of \( t \) of length 60, contains 40 factors of length 13. Thus \( \gamma(w) \geq \delta(w) \geq 40/13 > 3 \) for all prefixes \( w \) of \( t \) of length \( \geq 60 \).

Next, we need to see that \( \gamma(w) \leq 4 \) for all prefixes \( w \) of \( t \). For prefixes of length \( < 12 \) this can be verified by a computation. Otherwise, we claim that

(a) \( \{2^i - 1, 3 \cdot 2^{i-1} - 1, 2^{i+1} - 1, 3 \cdot 2^i - 1\} \) is a string attractor for \( t[0..n-1] \) for \( 13 \cdot 2^{i-2} - 1 \leq n \leq 5 \cdot 2^i \) and \( i \geq 2 \);

(b) \( \{3 \cdot 2^{i-1} - 1, 2^{i+1} - 1, 3 \cdot 2^i - 1, 2^{i+2} - 1\} \) is a string attractor for \( t[0..n-1] \) for \( 9 \cdot 2^{i-1} - 1 \leq n \leq 3 \cdot 2^i \) and \( i \geq 1 \);

(c) \( \{3 \cdot 2^{i-1} - 1, 2^{i+1} - 1, 2^{i+2} - 1, 5 \cdot 2^i - 1\} \) is a string attractor for \( t[0..n-1] \) for \( 3 \cdot 2^{i+1} - 1 \leq n \leq 13 \cdot 2^{i-1} \) and \( i \geq 1 \).

As can easily be checked, these intervals cover all \( n \geq 12 \).

To verify these three claims, we use Walnut again. Here are the ideas, discussed in detail for the first of the three claims. Since we cannot express \( 2^i \) directly in Walnut, instead we define a variable \( x \) and demand that the base-2 representation of \( x \) be of the form \( 10^* \). We can do this with the command

\texttt{reg power2 msd_2 "0*10*";}

which defines a macro \texttt{power2}.

Next, we define the analogue of \texttt{pdfaceq} above; it tests whether \( t[k..l] = t[q - l + k..q] \).

\texttt{def tmfaceq \"(q+k>=l) & At Au (t>=k & t<=l & l+u=q+t) => T[t]=T[u]\";}

Next we define four macros, \texttt{testa1}, \texttt{testa2}, \texttt{testa3}, \texttt{testa4} with arguments \( k, l, n, x \). These test, respectively, if the factor \( t[k..l] \) has an occurrence in \( t[0..n-1] \) as \( t[p..q] \) such that

1. \( x - 1 \in [p..q] \);
2. \(3x/2 - 1 \in [p..q];\)

3. \(2x - 1 \in [p..q];\)

4. \(3x - 1 \in [p..q].\)

Here \(x = 2^i.\)

def testa1 "Ep,q (k+q=l+p) & (p<=q) & (q<n) & \$tmfaceq(k,l,q) & (p+1=x) & (q+1=x)";
def testa2 "Ep,q (k+q=l+p) & (p<=q) & (q<n) & \$tmfaceq(k,l,q) & (2*p+2<=3*x) & (2*q+2>=3*x)";
def testa3 "Ep,q (k+q=l+p) & (p<=q) & (q<n) & \$tmfaceq(k,l,q) & (p+1=x+2*x) & (q+1=x+2*x)";
def testa4 "Ep,q (k+q=l+p) & (p<=q) & (q<n) & \$tmfaceq(k,l,q) & (p+1=x+3*x) & (q+1=x+3*x)";

Finally, we evaluate the formula \texttt{checka}, which asserts that for all \(n, x, k, l\) with \(n \geq 12, x = 2^i, i \geq 2, 13 \cdot 2^{i-2} - 1 \leq n \leq 5 \cdot 2^i\), and \(k \leq l < n\), the factor \(t[k..l]\) has an occurrence in \(t[0..n-1]\) matching one of the four possible values of the string attractor.

eval checka "An Ax Ak Al ((n>=12) & $power2(x) & (x>=4) & (13*x<=4*n+4) & (n <= 5*x) & (k=1) & (l<n)) => ($testa1(k,l,n,x) | $testa2(k,l,n,x) | $testa3(k,l,n,x) | $testa4(k,l,n,x))":

Then \texttt{Walnut} evaluates this as true, so claim (a) above holds.

Claims (b) and (c) can be verified similarly. We provide the \texttt{Walnut} code below:

def testb1 "Ep,q (k+q=l+p) & (p<=q) & (q<n) & \$tmfaceq(k,l,q) & (2*p+2<=3*x) & (2*q+2>=3*x)";
def testb2 "Ep,q (k+q=l+p) & (p<=q) & (q<n) & \$tmfaceq(k,l,q) & (p+1=x+2*x) & (q+1=x+2*x)";
def testb3 "Ep,q (k+q=l+p) & (p<=q) & (q<n) & \$tmfaceq(k,l,q) & (p+1=x+3*x) & (q+1=x+3*x)";
def testb4 "Ep,q (k+q=l+p) & (p<=q) & (q<n) & \$tmfaceq(k,l,q) & (p+1=x+4*x) & (q+1=x+4*x)";

eval checkb "An Ax Ak Al ((n>=12) & power2(x) & (x>=2) & (9*x<=2*n+2) & (n <= 6*x) & (k=1) & (l<n)) =>
($testb1(k,l,n,x) | $testb2(k,l,n,x) | $testb3(k,l,n,x) | $testb4(k,l,n,x))":

def testc1 "Ep,q (k+q=l+p) & (p<=q) & (q<n) & \$tmfaceq(k,l,q) & (2*p+2<=3*x) & (2*q+2>=3*x)";
def testc2 "Ep,q (k+q=l+p) & (p<=q) & (q<n) & \$tmfaceq(k,l,q) & (p+1=x+2*x) & (q+1=x+2*x)";
def testc3 "Ep,q (k+q=l+p) & (p<=q) & (q<n) & \$tmfaceq(k,l,q) & (p+1=x+4*x) & (q+1=x+4*x)";
def testc4 "Ep,q (k+q=l+p) & (p<=q) & (q<n) & \$tmfaceq(k,l,q) & (p+1=x+5*x) & (q+1=x+5*x)";

eval checkc "An Ax Ak Al ((n>=12) & $power2(x) & (x>=2) & (6*x<=n+1) & (2*n<=13*x) & (k=1) & (l<n)) =>
($testc1(k,l,n,x) | $testc2(k,l,n,x) | $testc3(k,l,n,x) | $testc4(k,l,n,x))":

The reason why this approach succeeded (and ran much faster than for the period-doubling sequence) is that we could find a good guess as to what the string attractors look like, and then \texttt{Walnut} can be used to verify our guess.
5 Going even further

In this last section, we sketch the proof of two additional results, which can be achieved in the same way we proved the result for the Thue-Morse word.

The first concerns the Tribonacci word $\text{TR} = 0102010\ldots$, the fixed point of the morphism sending $0 \rightarrow 01$, $1 \rightarrow 02$, $2 \rightarrow 0$, which was studied, for example, in [9, 3, 21]. Define the Tribonacci numbers, as usual, to be $T_0 = 0$, $T_1 = 1$, $T_2 = 1$, and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 3$. Furthermore, define $W_n = T_n + T_{n-3} + T_{n-6} + \cdots + T_{2 + ((n-2) \mod 3)}$ for $n \geq 4$.

**Theorem 6.** The size of the smallest string attractor for the length-$n$ prefix of the Tribonacci word is 3 for $n \geq 4$. Furthermore, for $i \geq 4$ and $W_i \leq n < W_{i+1}$ a smallest string attractor of size 3 for $\text{TR}[0..n-1]$ is $\{T_{i-2} - 1, T_{i-1} - 1, T_i - 1\}$.

**Proof.** The proof proceeds almost exactly like the proof of Theorem 5. Every prefix of length $n \geq 4$ contains the three distinct symbols 0, 1, 2, so clearly $\gamma(\text{TR}[0..n-1]) \geq 3$ for $n \geq 4$.

It remains to show the claim about the string attractor. To do so we create a Walnut formula, given below. We use the so-called Tribonacci representation of numbers, described in [6]. It is easy to see that the Tribonacci representation of $T_i$ is of the form $10^{i-2}$, and the Tribonacci representation of $W_i$ is the length-$(i-1)$ prefix of the word $100100100\ldots$. We create regular expressions $\text{istolib}$ and $\text{t100}$ to match these numbers. The formula $\text{threetrib}$ accepts $x, y, z$ if $x = T_i$, $y = T_{i+1}$, $z = T_{i+2}$ for some $i \geq 2$. The formula $\text{adj}$ accepts $w_1, w_2$ if $w_1 = W_i$, $w_2 = W_{i+1}$ for some $i \geq 4$.

```latex
\text{reg istolib msd_trib "0*10*":}
\text{reg t100 msd_trib "0*(100)*100|0*(100)*1001|0*(100)*10010":}
\text{def threetrib "?msd_trib $\text{istolib}(x) \& $\text{istolib}(y) \& $\text{istolib}(z) \& (x<y) \& (y<z) \& (A(x<u \& u<y) => "$\text{istolib}(u)") \& (A(v<y \& v<z) => "$\text{istolib}(v)")":}
\text{def adj "?msd_trib $\text{t100}(x) \& $\text{t100}(y) \& A(t>x \& t<y) => "$\text{t100}(t)":}
\text{def testtr "?msd_trib Ep,q (k=q+1+p) \& (p=q) \& (q<n) \& \text{tribfaceq}(k,1,q) \& (p=q) \& (q>n) \& \text{tribfaceq}(k,1,q) \& (p=n) \& (q=n):}
\text{eval checktrib "?msd_trib An,x,y,z,w1,w2,k,1 ((n=4) \& $\text{threetrib}(x,y,z) \& (x=1) \& (n=w1) \& (n+1=w2) \& $\text{adj}(w1,w2) \& (z=w1) \& (z+1=w2) \& (k=1) \& (l=n)) => (\text{testtr}(k,1,n,x-1) \| \text{testtr}(k,1,n,y-1) \| \text{testtr}(k,1,n,z-1))":}
```

Since $\text{checktrib}$ returns true, the result is proven. This was a large computation in Walnut, requiring 20 minutes of CPU time and 125 Gigs of storage.

We now turn to another word, a variant of the Thue-Morse word sometimes called $\text{vtm}$ and studied by Berstel [5]. It is the fixed point of the map $2 \rightarrow 210$, $1 \rightarrow 20$, and $0 \rightarrow 1$. 

8
Theorem 7. The size of the smallest string attractor for the length-$n$ prefix of the ternary infinite word $vtm$ is

\[
\begin{cases}
1, & \text{if } n = 1; \\
2, & \text{if } n = 2; \\
3, & \text{if } 3 \leq n \leq 6; \\
4, & \text{if } n \geq 7.
\end{cases}
\]

Proof. The result can easily be checked by a short computation for $n \leq 12$.

For a lower bound, observe that $vtm[0..13]$ contains 10 distinct factors of length 3, so $\gamma(w) \geq \delta(w) \geq 10/3 > 3$ for all prefixes $w$ of $vtm$ of length $\geq 14$.

We now claim that

(a) $\{2^i - 1, 3 \cdot 2^i - 1, 1, 2^i - 1\}$ is a string attractor for $vtm[0..n-1]$ for $13 \cdot 2^i \leq n < 5 \cdot 2^i$ and $i \geq 2$;

(b) $\{2^i - 1, 2^{i-1} - 1, 3 \cdot 2^i, 19 \cdot 2^i - 1\}$ is a string attractor for $vtm[0..n-1]$ for $5 \cdot 2^i \leq n < 6 \cdot 2^i$ and $i \geq 2$;

(c) $\{3 \cdot 2^{i-1} - 1, 2^{i-1} - 1, 1, 2^i - 1\}$ is a string attractor for $vtm[0..n-1]$ for $6 \cdot 2^i \leq n < 13 \cdot 2^{i-1}$ and $i \geq 2$.

This can be verified in exactly the same way that we verified the claims for the Thue-Morse sequence $t$. The Walnut code is given below:

```walnut
reg power2 msd_2 "0*10*":

def vtmfaceq "(q+k>=l) & At Au (t>=k & t<=l & l+u=q+t) => VTM[t]=VTM[u]":

def testva1 "Ep,q (k+q=l+p) & (p<=q) & (q<n) & $vtmfaceq(k,l,q) & (p+1<=x) & (q+1>=x)":

def testva2 "Ep,q (k+q=l+p) & (p<=q) & (q<n) & $vtmfaceq(k,l,q) & (2*p+2<=3*x) & (2*q+2>=3*x)":

def testva3 "Ep,q (k+q=l+p) & (p<=q) & (q<n) & $vtmfaceq(k,l,q) & (p+1<=2*x) & (q+1>=2*x)":

def testva4 "Ep,q (k+q=l+p) & (p<=q) & (q<n) & $vtmfaceq(k,l,q) & (p+1<=3*x) & (q+1>=3*x)":

eval checkva "An Ax Ak Al ((n>=13) & $power2(x) & (x>=4) & (13*x<=4*n) & (n<5*x) & (k<=1) & (l<n)) => ($testva1(k,l,n,x) | $testva2(k,l,n,x) | $testva3(k,l,n,x) | $testva4(k,l,n,x))":

def testvb1 "Ep,q (k+q=l+p) & (p<=q) & (q<n) & $vtmfaceq(k,l,q) & (p+1<=y) & (q+1>=x)":

def testvb2 "Ep,q (k+q=l+p) & (p<=q) & (q<n) & $vtmfaceq(k,l,q) & (p+1<=2*x) & (q+1>=2*x)":

def testvb3 "Ep,q (k+q=l+p) & (p<=q) & (q<n) & $vtmfaceq(k,l,q) & (p+1<=3*x) & (q+1>=3*x)":

def testvb4 "Ep,q (k+q=l+p) & (p<=q) & (q<n) & $vtmfaceq(k,l,q) & (2*p+2<=9*x) & (2*q+2>=9*x)":

eval checkvb "An Ax Ak Al ((n>=20) & $power2(x) & (x>=4) & (5*x<=n) & (n<6*x) & (k<=1) & (l<n)) => ($testvb1(k,l,n,x) | $testvb2(k,l,n,x) | $testvb3(k,l,n,x) | $testvb4(k,l,n,x))":

def testvc1 "Ep,q (k+q=l+p) & (p<=q) & (q<n) & $vtmfaceq(k,l,q) & (2*p+2<=3*x) & (2*q+2>=3*x)":
```

9
The formulas checkva, checkvb, and checkvc all return true, so the result is proved. □

6 Non-constant string attractor size

So far all the infinite words we have discussed have prefix string attractor size bounded by a constant. However, if we can guess the form of a string attractor, then we can verify it with a first-order formula, even for increasing size.

Let us consider the characteristic sequence of the powers of 2: $p = 11010001 \cdots$. More precisely, $p[i+1] = 1$ if $i$ is a power of 2, and 0 otherwise. This is an automatic sequence, generated by a 3-state automaton in base 2.

**Theorem 8.** Suppose $n \geq 3$. Then

(a) $\{2 \cdot 4^i - 1 : 0 \leq j \leq i\} \cup \{3 \cdot 4^i - 1\}$ is a string attractor of cardinality $i + 2$ for $p[0..n-1]$ and $3 \cdot 4^i \leq n < 6 \cdot 4^i$ and $i \geq 0$;

(b) $\{4^i - 1 : 0 \leq j \leq i + 1\} \cup \{6 \cdot 4^i - 1\}$ is a string attractor of cardinality $i + 3$ for $p[0..n-1]$ and $6 \cdot 4^i \leq n < 12 \cdot 4^i$ and $i \geq 0$.

**Proof.** Again, we use Walnut. Here P2 is the abbreviation for $p$.

def p2faceq "(q+k>=l) & At Au (t>=k & t<=l & l+u=q+t) => P2[t]=P2[u]":

def test1 "An Ax Ak Al
($power4(x) & (3*x<=n) & (n<6*x) & (k<=l) & (l<n))
=> (Ep Eq Ey (k+q=l+p) & (p<=q) & (q<n) & $p2faceq(k,1,q) & (($power4(y) & y<=x & p+1<=2*y & 2*y<=q+1) | (y=3*x & p+1<=y & y<=q+1)))":

def test2 "An Ax Ak Al
($power4(x) & (6*x<=n) & (n<12*x) & (k<=l) & (l<n))
=> (Ep Eq Ey (k+q=1+p) & (p<=q) & (q<n) & $p2faceq(k,1,q) & (($power4(y) & y<=4*x & p+1<=y & y<=q+1) | (y=6*x & p+1<=y & y<=q+1)))":

Both test1 and test2 return true, so the theorem is proved. □

Of course, this theorem only gives an upper bound on the string attractor size for prefixes of $p$. For a matching lower bound, other techniques must be used.
7 Span

After seeing a draft of this paper, Marinella Sciortino asked one of us about a certain feature of string attractors: span. The span of a string attractor $S = \{i_1, i_2, \ldots, i_k\}$ with $i_1 < i_2 < \cdots < i_k$ is defined to be $\text{span}(S) = i_k - i_1$. It measures the distance between the largest and smallest elements of a string attractor.

One can then ask, among all string attractors of minimum cardinality for a finite string $x$, what are the maximum and minimum span? Call these quantities $\text{maxspan}(x)$ and $\text{minspan}(x)$. We then have the following theorem:

**Theorem 9.** If $s$ is an automatic sequence with $O(1)$ string attractor size, then $f(n) := \text{maxspan}(s[0..n-1])$ and $g(n) := \text{minspan}(s[0..n-1])$ are synchronized functions of $n$. That is, there is an automaton that recognizes the representations of $(n, f(n))$ and $(n, g(n))$.

**Proof.** It suffices to give a first-order formula for $\text{minspan}$ and $\text{maxspan}$. Let $c$ be an upper bound on string attractor size for all prefixes of $s$. Then we can create formulas as follows:

$$\text{saspan}(n, r) := \exists i_1, i_2, \ldots, i_c \text{ sa}(i_1, i_2, \ldots, i_c, n) \land (i_1 \leq i_2) \land (i_2 \leq i_3) \land \cdots \land (i_{c-1} \leq i_c) \land i_c = i_1 + r$$

$$\text{minspan}(n, r) := \text{saspan}(n, r) \land \forall s (s < r) \implies \neg \text{saspan}(n, s)$$

$$\text{maxspan}(n, r) := \text{saspan}(n, r) \land \forall s (s > r) \implies \neg \text{saspan}(n, s).$$

As an example, we can carry this out for the period-doubling word with the following Walnut commands:

```walnut
def pdsaspan "E i1, i2 $pdsa2(i1,i2,n) & i2=i1+r":
def pdsamind "$pdsaspan(n,r) & As (s<r) => ~$pdsaspan(n,s)":
def pdsamaxd "$pdsaspan(n,r) & As (s>r) => ~$pdsaspan(n,s)":
```

Looking at the results, we easily get

**Theorem 10.** For the period-doubling sequence the minspan of the length-$n$ prefix equals

$$\begin{cases} 
0, & \text{for } n = 1; \\
1, & \text{for } 2 < n < 5; \\
2^i, & \text{for } 3 \cdot 2^i \leq n < 3 \cdot 2^{i+1} \text{ and } i \geq 1. 
\end{cases}$$

The maxspan of the length-$n$ prefix equals

$$\begin{cases} 
0, & \text{for } n = 1; \\
1, & \text{for } 2 \leq n \leq 3; \\
2^i, & \text{if } 5 \cdot 2^{i-1} - 1 \leq n < 6 \cdot 2^{i-1} - 2 \text{ and } i \geq 1; \\
3 \cdot 2^i, & \text{if } 6 \cdot 2^i - 1 \leq n \leq 5 \cdot 2^{i+1} - 2 \text{ and } i \geq 0. 
\end{cases}$$
8 Asymptotic bounds on string attractor size

We have seen several $k$-automatic sequences that have $O(1)$-size string attractors for all prefixes. Others, like the characteristic sequence of powers of 2, have $\Theta(\log n)$-size string attractors for prefixes of length $n$. In this section, we show that these are the only two possibilities for $k$-automatic sequences: string attractors for length-$n$ prefixes have size $\Theta(1)$ or $\Theta(\log n)$ asymptotically.

We prove two upper bounds on the size of the string attractor in Theorem 15 and Theorem 17. These do not require the word to be $k$-automatic, but instead have conditions on the appearance function and recurrence function (see, for example, [1, §10.9, 10.10]), which happen to be satisfied for $k$-automatic sequences, as discussed in Corollary 12.

Recall that an infinite word $w$ is said to be recurrent if every finite factor of $w$ has infinitely many occurrences in $w$. If, in addition, there is a constant $C$ such that two consecutive occurrences (or from the beginning of the word to the first occurrence) of every length-$n$ factor are separated by at most $Cn$ positions, then $w$ is said to be linearly recurrent. Let $R_w$, the recurrence constant, be the least such $C$. The related notion of the appearance function records, for each $n$, the least smallest prefix that contains all length-$n$ factors. This, also, can be linearly bounded, i.e., there is a constant $C$ such that every length-$n$ appears in some prefix of at most $Cn$ symbols. When it exists, we call the least such $C$ the appearance constant, and denote it $A_w$.

The appearance and recurrence constants are computable for $k$-automatic sequences, so let us define the relevant first-order formulas:

- faceq($i, j, n$): the factor $w[i..i+n−1]$ is the same as $w[j..j+n−1]$
- appear($m, n$): all length-$n$ factors of $w$ have an occurrence in $[0..m−1]$
- leastappear($m, n$): $m$ is the least integer such that appear($m, n$) holds
- recurfac($i, n$): the factor $w[i..i+n−1]$ occurs infinitely often in $w$
- recur($m, n$): all length-$n$ factors of $w$ have an occurrence contained in every length-$m$ factor
- leastrecur($m, n$): $m$ is the least integer such that recur($m, n$) holds

Here are the definitions in first-order logic:

$$\text{faceq}(i, j, n) := \forall t \ (t < n) \implies w[i + t] = w[j + t]$$
$$\text{appear}(m, n) := \forall i \ \exists j \ \text{faceq}(i, j, n) \land j < m$$
$$\text{leastappear}(m, n) := \text{appear}(m, n) \land \neg \text{appear}(m - 1, n)$$
$$\text{recur}(m, n) := \forall i \ \forall k \ \exists j \ \text{faceq}(i, j, n) \land k \leq j \land j + n \leq k + m$$
$$\text{leastrecur}(m, n) := \text{recur}(m, n) \land \neg \text{recur}(m - 1, n)$$
Lemma 11. Let \( S \subseteq \mathbb{N}_{>0} \times \mathbb{N}_{>0} \) be a \( k \)-automatic set such that for every \( y \in \mathbb{N}_{>0} \), there are only finitely many \( x \in \mathbb{N}_{>0} \) such that \((x, y) \in S\). Then \( \sup_{(x, y) \in S} \frac{x}{y} \) is finite and computable.

Proof. Briefly, \( \frac{x}{y} \leq k^{s+1} \) for all \((x, y) \in S\) where \( s \) is the number of states in the automaton accepting \( S \). Otherwise, the base-\( k \) representation of \( x \) is longer than the representation of \( y \) by more than a pumping length \((s)\) and thus \((x, y) \in S\) can be pumped to infinitely many pairs \((x', y) \in S\), contradicting an assumption. This proves the supremum is finite; for computability, see our paper on critical exponents [22]. \( \square \)

Corollary 12. Let \( w \in \Sigma^\omega \) be \( k \)-automatic. Then \( A_w \) exists and is computable. If \( w \) is linearly recurrent, then \( R_w \) exists and is computable.

Proof. As we have seen, there are first-order predicates for appearance and recurrence. That is, predicates that accept \((m, n)\) if \( m \) is the minimum length such that the window(s) (i.e., the prefix of length \( m \), or every window of length \( m \)) intersects the desired factors (all or just recurrent factors) of length \( n \). By the previous lemma, the appearance constant and recurrence constant are computable. \( \square \)

Let us specialize the definition of a string attractor to a limited set of lengths.

Definition 13. Given a word \( w \) and a set \( L \subseteq \mathbb{N} \), a string attractor of \( w \) for lengths \( L \) is a set of integers \( S \) such that for every nonzero \( \ell \in L \), every length-\( \ell \) factor of \( w \) has some occurrence crossing an index in \( S \). When \( L \) is not specified, we take it to be \( \mathbb{N} \), i.e., the string attractor property holds for all lengths.

Lemma 14. Let \( w \in \Sigma^\omega \) be an infinite word with appearance constant \( A_w < \infty \). For all integers \( n, s \geq 1 \), there is a string attractor of \( w[0..n-1] \) for lengths \([s..2s]\) having size at most \( 2A_w \).

Proof. All factors of length \( \leq 2s \) in \( w \) intersect in a prefix \( x \) of length at most \( 2A_w s \) by the definition of the appearance constant. Take any arithmetic progression of stride \( s \) through this interval, and observe that it has at most \( 2A_w \) terms.

Obviously every factor of length \( \geq s \) in \( x \) intersects the arithmetic progression. On the other hand, all factors of length \( \leq 2s \) have an occurrence intersecting the prefix. It follows that \( S \) is a string attractor for \( w[0..n-1] \) for lengths \([s..2s]\). \( \square \)

Theorem 15. Let \( w \in \Sigma^\omega \) be an infinite word with appearance constant \( A_w < \infty \). There is a string attractor of \( w[0..n-1] \) of size \( O(A_w \log n) \).

Proof. Cover the range of factor lengths (1 up to \( n \)) with intervals

\[
[1..2] \cup [3..6] \cup [7..14] \cup \cdots \cup [2^k - 1..2^{k+1} - 2]
\]

where the last interval contains \( n \), so \( k = O(\log n) \). By Lemma 14, there is a string attractor of \( w \) for the lengths in each interval of size \( O(A_w) \). Define \( S \) to be the union of these partial string attractors, and note that it is a string attractor of \( w \) for lengths \([1..n]\), and has size \( O(A_w \log n) \). \( \square \)
It turns out that this is tight, as shown by an example of Mantaci et al. [17] using the characteristic sequence of powers of two. There is a family of $\Omega(\log n)$ factors (e.g., factors of the form $10^{2k-1}$) that have unique and pairwise disjoint occurrences in the prefix of length $n$, and thus every string attractor for this prefix must have at least $\Omega(\log n)$ indices.

The key point in the argument above is that some factors do not recur often (or at all), requiring dedicated indices in the string attractor just for those factors. If every factor recurs sufficiently often, then indices meant to cover long factors will also be near various short factors. We may be able to move the index slightly so that it covers all the necessary long factors, but also some of the short factors.

**Lemma 16.** Let $w \in \Sigma^\omega$ be an infinite word with $A_w, R_w < \infty$. Suppose $S$ is a string attractor of $w$ for lengths $L \subseteq [1..\lfloor \frac{s}{2R_w} \rfloor]$. For all $n, s \geq 1$, there is a string attractor of $w[0..n-1]$ for lengths $L \cup [3s..6s]$ having size at most $\max(|S|, 3A_w)$.

**Proof.** Similar to Lemma 14, we cover the window containing all factors of length $\leq 6s$, (i.e., the prefix of length $\leq \min(n, 6A_w s)$) with an array of points where there is no gap of length $3s$ or longer without points, and thus the array intersects any factor of length $\geq 3s$. The difference is that we pack the points closer, averaging $\leq 2s$ from the next point, so that we may perturb each point left or right (independently of the other points) by up to $\frac{s}{2}$ without introducing a gap of $\geq 3s$. Let $S'$ be this set of points, with the perturbations of the elements to be determined later. Since the spacing of the points is initially $2s$, we need at most $3A_w$ points to cover the appearance window for factors of length $\leq 6s$.

Recall that any window of length $s$ contains all factors of length $\leq \frac{s}{R_w}$ because the first occurrence $w$ is linearly recurrent with recurrent constant $R_w$. In particular, all factors of length $\leq \frac{s}{2R_w}$ that intersect some index $i \in S$ exist within an interval of length at most $\frac{s}{R_w}$, and that factor is contained somewhere in any window of length $s$. Thus, we can perturb any index in $S'$ within its window of length $s$ to intersect all factors of length $\leq \frac{s}{2R_w}$ that some index $i \in S$ is responsible for in $S$.

Clearly the union $S \cup S'$ is a string attractor for lengths $L \cup [3s..6s]$ as desired, but it is too large. By the argument above, we can remove elements of $S$ by perturbing elements of $S'$, until we run out of indices in one or the other. If we run out of indices in $S$ then only indices of $S'$ are left and the string attractor has size $|S'| \leq 3A_w$. Otherwise, we removed one element of $S$ for each element of $S'$ we added, so the string attractor has size $|S|$. In either case, the size is at most $\max(|S|, 3A_w)$, as required. \hfill \Box

**Theorem 17.** Let $w \in \Sigma^\omega$ be a linearly recurrent word with $A_w, R_w < \infty$. There is a string attractor of $w[0..n-1]$ having size $O(A_w \log R_w)$.

**Proof.** Cover the interval of possible lengths, $[1, n]$, with $O(\log n)$ intervals as follows

$$[1..2] \cup [3..6] \cup [6..12] \cup [12..24] \cup \cdots \cup [3 \cdot 2^k..6 \cdot 2^k].$$

By Lemma 14, there is a string attractor of $w$ for lengths $[1..2]$ having at most $2A_w$ elements. Thereafter, Lemma 16 says there is a string attractor for lengths $[3 \cdot 2^k..6 \cdot 2^k]$ of size $3A_w$, 14
which also obsoletes every string attractor for lengths \([1..[2^{i-1}/R_w]]\) having fewer than \(3A_w\) elements.

In other words, Lemma 16 says that the string attractor for lengths \([3 \cdot 2^i .. 6 \cdot 2^i]\) is no longer necessary once we have the string attractor for lengths \([3 \cdot 2^{i+k} .. 6 \cdot 2^{i+k}]\), where \(k = \log_2 R_w + O(1)\). It follows that the full string attractor \(S\) for \(w[0..n-1]\) is the union of only the last \(k\) string attractors (corresponding to the longest intervals), plus the string attractor for lengths \([1..2]\), and therefore

\[
|S| \leq 3kA_w + 2A_w = O(A_w \log R_w).
\]

**Theorem 18.** Let \(w\) be a \(k\)-automatic word. Let \(\gamma_w(n)\) be the size of the smallest string attractor of \(w[0..n-1]\). Then

- \(\gamma_w(n)\) is either \(\Theta(1)\) or \(\Theta(\log n)\),
- it is decidable whether \(\gamma_w(n)\) is \(\Theta(1)\) or \(\Theta(\log n)\), and
- in the case that \(\gamma_w(n) = \Theta(1)\), the sequence \(\gamma_w(1)\gamma_w(2)\cdots\) is \(k\)-automatic and the automaton can be computed.

**Proof.** By Corollary 12, we know \(A_w\) and \(R_w\) exist and are computable. From the appearance constant alone, Theorem 15 gives an \(O(\log n)\) upper bound for \(\gamma_w(n)\).

The set

\[
N := \{i \in \mathbb{N} : \text{for some } j > i, \ w[i..j] \text{ has finitely many occurrences in } w \text{ and } w[i..j] \text{ is the first such}\}.
\]

is first-order expressible. Suppose \(N\) is finite, with maximum element \(a = \max N\). Then the suffix \(w[a+1..\infty]\) is recurrent and has \(O(1)\) size string attractors by Theorem 17, to which we add \(\{1, \ldots, a\}\) to get \(O(1)\) string attractors for every prefix of \(w\). In fact, we get a computable upper bound for the size of the string attractors: \(\gamma_w(n) \leq B\) where \(B := a + O(A_w \log R_w)\). We can use the formulas from Section 2 to check the \(\leq B\) element string attractors and determine the minimum size for all \(n\). It follows that \(\gamma_w(1)\gamma_w(2)\cdots\) is a \(k\)-automatic sequence.

Otherwise, \(N\) is infinite. Hence \(N\) is \(k\)-automatic, and so is the set \(N' \subseteq \mathbb{N} \times \mathbb{N}\) where \((i, i') \in N'\) if \(i' \in N\) is the first element of \(N\) past the last occurrence of the shortest factor beginning at \(i \in N\) that has finitely many occurrences. In other words, we might as well consider the shortest non-recurrent factors starting at the positions in \(N\), and when we do, \((i, i') \in N\) if \(i'\) is next element of \(N\) such that all occurrences of the factor at \(i\) come before all occurrences of the factor at \(i'\), and thus no occurrences overlap.

The fact that \(i'\) is a \(k\)-synchronized function of \(i\) (i.e., \((i, i')\) is \(k\)-automatic, and for every \(i\) there is only one \(i')\) implies \(i' = O(i)\), so there exists a constant \(c\) such that \(i' \leq ci\) for all...
i ≥ 1. It follows that there is an infinite set of factors $w[i_1..i_1 + \ell_1 - 1], w[i_2..i_2 + \ell_2 - 1], \ldots$ where each $w[i_j..i_j + \ell_j - 1]$ occurs only in $w[i_j..i_{j+1} - 1]$ (and is therefore disjoint from occurrences of the others) and $i_j = O(c^j)$. The flip side is that $\Omega(\log n)$ of these factors have some occurrence in $w[0..n - 1]$, and must be intersected by $\Omega(\log n)$ distinct elements in the string attractor. Hence, $\gamma_w(n) = \Omega(\log n)$, matching the earlier $O(\log n)$ upper bound. □

9 Greedy string attractors

Finding the optimal string attractor for a word is known to be NP-complete [14]. Although this hardness result does not necessarily carry over to automatic sequences, it is still natural to consider approximations and heuristic algorithms. For example, we can define a string attractor greedily.

**Definition 19.** Let $w \in \Sigma^\omega$ be an infinite word. The greedy string attractor for $w$, $S$, is the limit of the sequence $\emptyset = S_0 \subseteq S_1 \subseteq \cdots \subseteq \mathbb{N}$, which is constructed iteratively as follows. For all $n \geq 0$, let $S_{n+1} = S_n \cup \{j\}$ where $j$ is the smallest integer such that $S_n$ is not a string attractor for $w[0..j]$ (or $S_{n+1} = S_n$ if no such integer exists).

We say the first occurrence of a factor in $w$ is novel, and prove that the elements of the greedy string attractor are related to minimal novel factors.

**Lemma 20.** Suppose $S \subseteq \mathbb{N}$ is a string attractor for a prefix $w[0..i]$ of an infinite word $w$, and suppose $i \in S$. Let $j$ be the smallest integer such that $S$ is not a string attractor for $w[0..j]$. Then $w[j - \ell + 1..j]$ is novel, no nonempty proper factor is novel. Also, $\ell$ is at most $j - i$.

**Proof.** Since $S$ is not a string attractor for $w[0..j]$, there is some factor $w[k..k+\ell-1]$ (let it be the shortest factor, i.e., take $\ell$ to be minimal) that has no occurrence in $w[0..j]$ intersecting $S$. Nearly everything else follows because of a contradiction argument.

- The factor ends at position $j = k + \ell - 1$ (i.e., is of the form $w[j - \ell + 1..j]$), otherwise $w[0..j - 1]$ is not a string attractor, contradicting the minimality of $j$.

- The factor is novel, since an earlier occurrence would either contradict that no occurrence intersects $S$ (if it intersected $S$), or the minimality of $j$ (if it did not intersect $S$).

- The prefix $w[j - \ell + 1..j - 1]$ is not novel because it would contradict the minimality of $j$.

- The suffix $w[j - \ell + 2..j]$ is not novel because it would contradict the minimality of $\ell$.

- Every other proper factor is contained in either $w[j - \ell + 1..j - 1]$ or $w[j - \ell + 2..j]$ and therefore not novel.
Finally, since $w[j - \ell + 1..j]$ does not intersect $i$, we have $i \leq j - \ell \implies \ell \leq j - i.$

**Theorem 21.** Let $S \subseteq \mathbb{N}$ be the greedy string attractor for a word $w \in \Sigma^\omega$. If $A_w < \infty$ then for all $n \geq 0$, $S \cap [0..n-1]$ is a string attractor for $w[0..n-1]$ of size $O(A_w \log n)$.

*Proof.* There is some iterate $S_k$ of the greedy construction, after which all indices added are $\geq n$. It is not hard to see that $S_k = S \cap \{0, \ldots, n-1\}$, and that $S_k$ is a string attractor for $w[0..n-1]$ because otherwise the next step of the greedy construction would add an index $< n$. All that remains is to argue that $S_k$ has $O(A_w \log n)$ elements.

By Lemma 20, two consecutive elements of $S$, $i$ and $j$, are separated by $\ell$, the length of a novel factor occurring after $i$. Since all factors of length $\leq i/A_w$ have some occurrence beginning in the first $i$ symbols of $w$, a novel factor must be longer, i.e., $\ell > i/A_w$. That is,

$$j \geq i + \ell \geq i + \frac{i}{A_w} \geq \left(1 + \frac{1}{A_w}\right)i.$$ 

This guarantees at least geometric growth in the elements of $S$, and therefore there are at most logarithmically many elements in $S \cap [0..n-1]$. Since the logarithm is base $1 + 1/A_w$, and

$$\frac{1}{\log(1 + 1/A_w)} \leq (A_w + \frac{1}{2}) ,$$

the size of the string attractor is $O(A_w \log n)$.

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