A Liouville theorem for $p$-harmonic functions on exterior domains

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Abstract We prove Liouville type theorems for $p$-harmonic functions on exterior domains of $\mathbb{R}^d$, where $1 < p < \infty$ and $d \geq 2$. We show that every positive $p$-harmonic function satisfying zero Dirichlet, Neumann or Robin boundary conditions and having zero limit as $|x|$ tends to infinity is identically zero. In the case of zero Neumann boundary conditions, we establish that any semi-bounded $p$-harmonic function is constant if $1 < p < d$. If $p \geq d$, then it is either constant or it behaves asymptotically like the fundamental solution of the homogeneous $p$-Laplace equation.

Keywords Elliptic boundary-value problems · Liouville-type theorems · $p$-Laplace operator · $p$-Harmonic functions · Exterior domain

Mathematics Subject Classification 35B53 · 35J92 · 35B40

1 Introduction and main results

Assume that $\Omega$ is a general exterior domain of $\mathbb{R}^d$, that is, a connected open set such that $\Omega^c = \mathbb{R}^d \setminus \Omega$ is compact and nonempty. We assume that the boundary $\partial \Omega$ is the disjoint union of the sets $\Gamma_1, \Gamma_2$, where $\Gamma_1$ is closed. We denote by $\nu$ the outward pointing unit normal vector on $\partial \Omega$ and $\mathcal{H}$ the $(d - 1)$-dimensional Hausdorff measure on $\partial \Omega$. For $1 < p < \infty$ define the $p$-Laplace operator $\Delta_p$ by $\Delta_p v := \text{div}(|\nabla v|^{p-2} \nabla v)$. 
The aim of this paper is to establish a Liouville theorem for weak solutions of the elliptic boundary-value problem

\[-\Delta_p v = 0 \quad \text{in} \quad \Omega,\]
\[B v = 0 \quad \text{on} \quad \partial \Omega,\]  

where

\[B v := \begin{cases} v|_{\Gamma_1} & \text{on} \quad \Gamma_1 (\text{Dirichlet b.c.)}, \\ |\nabla v|^{p-2}\frac{\partial v}{\partial \nu} + h(x, v) & \text{on} \quad \Gamma_2 (\text{Robin/Neumann b.c.).} \end{cases}\]

Here we assume that \(h : \Gamma_2 \times \mathbb{R} \to \mathbb{R}\) is a Carathéodory function (see [23]) satisfying

\[h(\cdot, v) \in L^{p/(p-1)}(\Gamma_2) \quad \text{and} \quad h(x, v)v \geq 0 \quad \text{for} \quad \mathcal{H}\text{-a.e.} \quad x \in \Gamma_2,\]  

for every \(v \in L^p(\Gamma_2)\). Note that the first condition in (1.2) implicitly implies a growth condition on the function \(v \mapsto h(\cdot, v)\); see [17]. As usual, a function \(v \in W^{1,p}_{\text{loc}}(\Omega) \cap C(\Omega)\) is said to be \(p\)-harmonic (or simply harmonic if \(p = 2\)) on \(\Omega\) if \(\Delta_p v = 0\) in \(\Omega\) in the weak sense, that is,

\[\int_\Omega |\nabla v|^{p-2}\nabla v \nabla \varphi \, dx = 0\]

for every \(\varphi \in C^\infty_c(\Omega)\); see [20]. Throughout, we call a \(p\)-harmonic function \(v\) positive if \(v \geq 0\).

The classical Liouville theorem asserts that every harmonic function on the whole space \(\mathbb{R}^d\) is constant if it is bounded from below or from above; see for instance [3, Theorem 3.1] or [19, p. 111]. The classical Liouville theorem was generalised to \(p\)-harmonic functions on the whole space \(\mathbb{R}^d\) for \(1 < p < \infty\); see [22, Theorem II] or [15, Corollary 6.11]. The result extends to \(d\)-harmonic function on \(\mathbb{R}^d \setminus \{0\}\) for \(d \geq 2\); see [3, Corollary 3.3] for \(p = d = 2\) or [16, Corollary 2.2] for \(p = d \geq 2\). In our investigation, the fundamental solution

\[\mu_p(x) := \begin{cases} |x|^{(p-d)/(p-1)} & \text{if} \quad p \neq d, \\ \log|x| & \text{if} \quad p = d. \end{cases}\]  

on \(\mathbb{R}^d \setminus \{0\}\) plays an important role. If \(1 < p < d\), then \(\mu_p\) provides an example of a non-constant \(p\)-harmonic function bounded from below. Another example valid for \(1 < p < d\) is given by \(v(x) := 1 - \mu_p(x)\) for every \(x \in \mathbb{R}^d \setminus B_1\), where \(B_1\) denotes the open unit ball. In this case \(v\) is a positive \(p\)-harmonic function on the exterior domain \(\Omega := \mathbb{R}^d \setminus \overline{B}_1\) satisfying zero Dirichlet boundary conditions at \(\partial \Omega\). Hence, in order to have a chance of proving a Liouville type theorem for exterior domains we need to make use of the boundary conditions and the behavior of a \(p\)-harmonic function near infinity.
First, we consider the case $1 < p < d$. Then by [21, Corollary, p.84] or [2, Theorem 2 and Theorem 3] and by rescaling if necessary, we know that for every positive $p$-harmonic function $v$ on an exterior domain $\Omega \subseteq \mathbb{R}^d$, the limit
\[
\lim_{|x| \to \infty} v(x) = b := \lim_{|x| \to \infty} v(x) \quad \text{exists and} \quad |v(x) - b| \leq c_1 \mu_p(x) \quad \text{whenever} \quad |x| \geq 2
\]
where $c_1 > 0$. With this in mind, our first result is a kind of maximum principle for weak solution of (1.1) on an unbounded domain. A precise definition of weak solutions of (1.1) is given in Definition 3.5 below.

**Theorem 1.1** Let $\Omega$ be an exterior domain with Lipschitz boundary and let $1 < p < \infty$. Suppose that (1.2) is satisfied and that $v$ is a positive weak solution of (1.1) such that $\lim_{|x| \to \infty} v(x) = 0$. Then $v \equiv 0$.

If $p \geq d$, the conclusion of the theorem is valid without any restrictions on the boundary conditions or regularity of $\Omega$ due to a result in [12]; see Sect. 4. If $1 < p < d$, then under some additional assumptions on $v$ we can remove the assumption that $\partial \Omega$ is Lipschitz. The condition is that $v$ has a trace in some weak sense which is in $L^p(\Gamma_2)$. Such a condition is satisfied in the setting discussed in [1, 6, 10, 11].

The proof of Theorem 1.1 relies on the asymptotic decay estimates for positive $p$-harmonic functions on exterior domains as stated in (1.4). We give a simple alternative proof of such estimates in case $p = 2$ and $d \geq 3$ in Sect. 2.

If $p \geq d$, then there are two alternatives for a positive $p$-harmonic function $v$: Either $v$ is bounded in a neighbourhood of infinity and has a limit as $|x| \to \infty$, or $v \sim \mu_p$ near infinity, that is,
\[
\lim_{|x| \to \infty} \frac{v(x)}{\mu_p(x)} = c
\]
for some constant $c > 0$; see [12, Theorem 2.3]. In the first case, if $v > 0$, then the limit is strictly positive; see [12, Lemma A.2]. See also the related work in [13]. As an example let $\Omega := \mathbb{R}^d \setminus \overline{B}_1$ and set $v_p := \mu_p + 1$ if $p = d$ and $v_p := \mu_p$ if $p > d$. Then $v_p$ is a positive $p$-harmonic function on $\Omega$ satisfying zero Robin boundary conditions
\[
|\nabla v|^{p-2} \frac{\partial v}{\partial v} + |v|^{p-2}v = 0 \quad \text{on} \ \partial \Omega.
\]

Similarly, $w_p := \mu_p$ if $p = d$ and $w_p := \mu_p - 1$ if $p > d$ satisfies zero Dirichlet boundary conditions on $\partial \Omega$ and is a positive unbounded $p$-harmonic function on $\Omega$.

Our second main result is a Liouville theorem for $p$-harmonic functions on exterior domains with zero Neumann boundary conditions, that is, the case $\Gamma_2 = \partial \Omega$ and $h \equiv 0$.

**Theorem 1.2** Let $\Omega$ be an exterior domain with no regularity assumption on $\partial \Omega$. Suppose that $v$ is a weak solution of (1.1) on $\Omega$ that is bounded from below or from above. Moreover, assume that $v$ satisfies homogeneous Neumann boundary conditions, that is, $h(x, v) \equiv 0$ and $\Gamma_2 = \partial \Omega$. If $1 < p < d$, then $v$ is constant. If $p \geq d$, then $v$ is either constant or $v \sim \pm \mu_p$ near infinity.
The proofs of the theorems are based on a general criterion for Liouville type theorems established in Sect. 3. We fully prove the two Theorems in Sect. 4.

There is an intimate relationship between Liouville-type theorems and pointwise a priori estimates of solutions of boundary value problems. On the one hand, Liouville’s theorem for some semi-linear equations on \( \mathbb{R}^d \) can be seen as a corollary of pointwise a priori estimates; see [8, Lemma 1]. On the other hand, Liouville’s theorem can be used to derive universal upper bounds for positive solutions on bounded domains. These connections were outlined in [22, p. 82] and recently revisited in [18]. More precisely, it is shown in [18, p. 556] that Liouville’s theorem and universal boundedness theorems are equivalent for semi-linear equations and systems of Lane-Emden type; see also [16]. This relationship becomes again apparent in this paper. This article was motivated by application to domain perturbation problems for semi-linear elliptic boundary value problems on domains with shrinking holes; see [9].

2 Estimates near infinity in the linear case

In this section we establish pointwise decay estimates for semi-bounded harmonic functions on an exterior domain \( \Omega \subseteq \mathbb{R}^d \) when \( d \geq 3 \). The result is a special case of estimates proved in [2], but it seems appropriate to provide a much shorter proof in the linear case.

**Proposition 2.1** Let \( v \) be harmonic on the exterior domain \( \Omega \subseteq \mathbb{R}^d \) \( d \geq 3 \). Further assume that \( v \) is bounded from below or from above. Then \( b := \lim_{|x| \to \infty} v(x) \) exists. Moreover, there exist positive constants \( r_0 \) and \( C_1 \) such that

\[
|v(x) - b| \leq C_1 |x|^{2-d}
\]  

for all \( |x| \geq r_0 \).

To prove the proposition let \( v \) be a harmonic function on the exterior domain \( \Omega \), and suppose that \( v \) is bounded from below or from above. Since by assumption \( \overline{\Omega}^c \) is bounded, by translating and rescaling if necessary, we can assume without loss of generality that \( \overline{\Omega}^c \) is contained in the unit ball \( B_1 \), and that \( 0 \in \overline{\Omega}^c \). Furthermore, without loss of generality we can consider non-negative harmonic functions on \( \Omega \). Indeed, if \( v \) is bounded from below we consider \( v + \inf v \geq 0 \), and if \( v \) is bounded from above we consider \( -v + \sup v \geq 0 \).

If \( v \) is positive and harmonic on \( \overline{B}_1 \), then in particular \( v \) is positive and harmonic on \( \overline{B}_1 \). Hence, the Kelvin transform \( K[v] \) of \( v \) given by \( K[v](x) := |x|^{2-d} v(x/|x|^2) \) for \( x \in \overline{B}_1 \setminus \{0\} \) is positive and harmonic on \( B_1 \setminus \{0\} \); see [3, Theorem 4.7]. By Bôcher’s theorem there exist a harmonic function \( w \) on \( B_1 \) and a constant \( b \geq 0 \) such that

\[
K[v](x) = w(x) + b |x|^{2-d} \quad \text{or} \quad K[v - b](x) = w(x)
\]
for every $x \in B_1 \setminus \{0\}$ see [3, Theorem 3.9]. Applying the Kelvin transform again yields

$$v(x) - b = K[w](x) = |x|^{2-d} \cdot w(x/|x|^2)$$  \hspace{1cm} (2.2)

for every $x \in \overline{B_1}$. Note that $w(x/|x|^2) \to w(0)$ as $|x| \to \infty$. Hence (2.2) implies the existence of constants $C_1, r_0 > 0$ such that (2.1) holds whenever $|x| > r_0$.

3 A general criterion for Liouville type theorems

The proofs of the main theorems are based on a general criterion showing that a function satisfying suitable integral conditions is constant. We generalise an idea from [5, Lemma 2.1]. Similar ideas were used for instance in [4,22] or in [7, Theorem 19.8] for $p = 2$.

**Proposition 3.1** Let $\varphi \in C_c^\infty(R^d)$ such that $0 \leq \varphi \leq 1$ on $R^d$, $\varphi \equiv 1$ on $B(0, 1)$ and with support contained in $B(0, 2)$. For $x \in R^d$ and $r > 0$ let $\varphi_r(x) := \varphi(x/r)$. Let $v \in W^{1, p}_{loc}(\Omega)$ and suppose that there exist constants $b \in R$ and $C_0, C_1, r_0 > 0$ such that

$$\int_{\Omega \cap B_{2r}} |\nabla v|^p \varphi_r^p \, dx \leq C_0 \left( \frac{r}{r} \right)^{(p-1)/p} \left( \frac{r}{r} \right)^{1/p} \left( \int_{(\Omega \cap B_{2r}) \setminus B_r} |v-b|^p \, dx \right)^{1/p}$$  \hspace{1cm} (3.1)

and

$$\frac{1}{r^p} \int_{(\Omega \cap B_{2r}) \setminus B_r} |v-b|^p \, dx \leq C_1$$  \hspace{1cm} (3.2)

for all $r > r_0$. Then $v$ is constant.

The above proposition is a direct consequence of the following stronger result. To prove Proposition 3.1, we set $C := C_0 C_1^{1/p}$ and $\delta = (p - 1)/p$ in the lemma below. Then inequality (3.3) follows from (3.1) and (3.2).

**Lemma 3.2** Let $\varphi$ and $\varphi_r$ be as in Proposition 3.1. Let $v \in W^{1, p}_{loc}(\Omega)$ and suppose that there exist constants $C$, $r_0 > 0$ and $\delta \in (0, 1)$ such that

$$\int_{\Omega \cap B_{2r}} |\nabla v|^p \varphi_r^p \, dx \leq C \left( \frac{r}{r} \right)^{\delta} \left( \frac{r}{r} \right)^{1/p} \left( \int_{(\Omega \cap B_{2r}) \setminus B_r} |\nabla v|^p \varphi_r^p \, dx \right)^{\delta}$$  \hspace{1cm} (3.3)

for all $r > r_0$. Then $v$ is constant.

**Proof** In a first step we show that $\nabla v \in L^p(\Omega)^d$. In a second step we then prove that $\nabla v \in L^p(\Omega)^d$ and (3.3) imply that $\nabla v = 0$. As $\Omega$ is assumed to be connected we can apply [14, Lemma 7.7] to conclude that $v$ is constant on $\Omega$. 


(i) We first show that $\nabla v \in L^p(\Omega^d)$. If $\nabla v = 0$, then there is nothing to show, so assume that $\nabla v \neq 0$. By possibly increasing $r_0$ we can assume that

$$
\int_{\Omega^d \cap B_{2r}} |\nabla v|^p \varphi_r^p \, dr > 0
$$

for all $r > r_0$. Rearranging inequality (3.3) and using that $\delta < 1$ yields

$$
\int_{\Omega^d} |\nabla v|^p \varphi_r^p \, dx = \int_{\Omega^d \cap B_{2r}} |\nabla v|^p \varphi_r^p \, dx \leq C^{1/(1-\delta)}
$$

for all $r > r_0$. Note that $\varphi_r^p \to 1_{\mathbb{R}^d}$ pointwise and monotonically increasing as $r \to \infty$. Hence, the monotone convergence theorem implies that

$$
\int_{\Omega^d} |\nabla v|^p \, dx = \lim_{r \to \infty} \int_{\Omega^d \cap B_{2r}} |\nabla v|^p \varphi_r^p \, dx \leq C^{1/(1-\delta)} < \infty. \tag{3.4}
$$

In particular $\nabla v \in L^p(\Omega^d)$ as claimed.

(ii) Assuming that $\nabla v \in L^p(\Omega^d)$ we now show that $\nabla v = 0$. We can rewrite (3.3) in the form

$$
\int_{\Omega^d \cap B_{2r}} |\nabla v|^p \varphi_r^p \, dx \leq C \left( \int_{\Omega^d} |\nabla v|^p \, dx - \int_{\Omega^d \cap B_r} |\nabla v|^p \varphi_r^p \, dx \right)^\delta.
$$

Letting $r \to \infty$, making use of (3.4) and the fact that $\delta > 0$, we deduce that $\|\nabla v\|_p \leq 0$, that is, $\|\nabla v\|_p = 0$. \qed

Remark 3.3 Suppose that $v \in W^{1,p}_{\text{loc}}(\Omega)$ satisfies inequality (3.1), and that there exists $r_0 > 0$ such that $v \in L^\infty(\Omega \cap B_{r_0}^c)$. Then, for every $b \in \mathbb{R}$

$$
\frac{1}{r^p} \int_{B_{2r} \setminus B_r} |v - b|^p \, dx \leq \frac{\|v - b\|_\infty^p}{r^p} \int_{B_{2r} \setminus B_r} 1 \, dx \leq \frac{\omega_d}{d} (2^d - 1) \|v - b\|_\infty^p r^{d-p}
$$

for all $r \geq r_0$, where $\|v - b\|_\infty := \|v - b\|_{L^\infty(B_{r_0}^c)}$ and $\omega_d$ is the surface area of the unit sphere in $\mathbb{R}^d$. If $p \geq d$, then Proposition 3.1 implies that $v$ is constant.

We next show that weak solutions of (1.1) satisfy (3.1). Before we can do that we want to state our precise assumptions and give a definition of weak solutions of boundary-value problem (1.1).

**Assumption 3.4** By assumption, an exterior domain $\Omega$ as defined in the introduction is an open connected set such that $\Omega^c$ is compact. In particular $\partial \Omega$ is compact. Thus there exists $r_0 > 0$ such that $\partial \Omega \subseteq B_r := B(0, r)$ for all $r \geq r_0$. We consider solutions of (1.1) that lie in

$$
V^{1,p}(\Omega) := \left\{ v \in W^{1,p}_{\text{loc}}(\Omega) : v \in W^{1,p}(\Omega \cap B_r) \text{ for all } r > r_0 \right\}.
$$
For simplicity we now assume that $\partial \Omega$ is Lipschitz. We assume that $\Gamma_1, \Gamma_2$ are disjoint subsets of $\partial \Omega$ such that $\Gamma_1$ is closed and $\Gamma_1 \cup \Gamma_2 = \partial \Omega$. We let $V_{\Gamma_1}^{1,p}(\Omega)$ be the closure of the vector space

$$\{ v \in V^{1,p}(\Omega) : v = 0 \text{ in a neighbourhood of } \Gamma_1 \}$$

in $V^{1,p}(\Omega)$. If $h(x, v) \equiv 0$ no regularity assumption on $\partial \Omega$ is needed.

We use the space $V^{1,p}(\Omega)$ because we do not want to assume that the solutions of (1.1) are in $L^p(\Omega)$.

**Definition 3.5** We say that a function $v$ is a weak solution of the boundary value problem (1.1) on $\Omega$ if $v \in V_{\Gamma_1}^{1,p}(\Omega)$ and

$$\int_{\Omega} |\nabla v|^p - 2 \nabla v \nabla \varphi \, dx + \int_{\Gamma_2} h(x, v) \varphi \, d\mathcal{H} = 0 \quad (3.5)$$

for every $\varphi \in V_{\Gamma_1}^{1,p}(\Omega)$ with $\text{supp}(\varphi) \subseteq B_r$.

The above definitions have to be modified in an obvious manner for non-smooth domains. In particular, when using the setting from [1,10,11] we require that $v$ is in the Maz’ya space $W^{1,p}_{p,\text{loc}}(\Omega \cap B_r, \partial \Omega)$ for all $r$ large enough.

If $\Omega$ admits the divergence theorem and the solution $v$ is smooth enough, then an integration by parts shows that $v$ is a weak solution of (1.1) if and only if $v$ satisfies (1.1) in a classical sense. We next show that positive solutions of (1.1) satisfy (3.1).

**Proposition 3.6** Let Assumption 3.4 be satisfied and let $\varphi_r$ be as in Proposition 3.1, and $r_0 > 0$ such that $\Omega^c \subseteq B_{r_0}$. Suppose that (1.2) is satisfied and that $v$ is a weak solution of (1.1). Then, inequality (3.1) holds with $b = 0$. In the case of homogeneous Neumann boundary conditions, that is, if $h(x, v) \equiv 0$ and $\Gamma_2 = \partial \Omega$, then every weak solution of (1.1) satisfies (3.1) for every $b \in \mathbb{R}$.

**Proof** Let $r \geq r_0$ and let $\varphi_r$ be the same test-function as in Proposition 3.1. Then $v\varphi_r \in W^{1,p}_{\text{loc}}(\Omega)$ with support in $B_{2r}$. Moreover, by definition of $\varphi_r$ we have $v\varphi_r = v$ on $\Omega \cap B_r$. Hence $v\varphi_r$ is a suitable test function to be used in (3.5). Using that $v$ is a weak solution of (1.1) gives

$$0 = \int_{\Omega \cap B_{2r}} |\nabla v|^p - 2 \nabla v \nabla (v\varphi_r) \, dx + \int_{\Gamma_2} h(x, v) v \varphi_r \, d\mathcal{H}$$

$$= \int_{\Omega \cap B_{2r}} |\nabla v|^p \varphi_r - \frac{p}{r} \int_{(\Omega \cap B_{2r}) \setminus B_r} v\varphi_r^{p-1} |\nabla v|^p \nabla \varphi(\cdot/r) \, dx$$

$$+ \int_{\Gamma_2} h(x, v) v \varphi_r \, d\mathcal{H}.$$
Rearranging this equation we arrive at
\[
\int_{\Omega \cap B_{2r}} |\nabla v|^p \varphi_r^p \, dx + \int_{\Gamma_2} h(x, v) v \varphi_r^p \, d\mathcal{H} = -\frac{p}{r} \int_{(\Omega \cap B_{2r}) \setminus B_r} v \varphi_r^{p-1} |\nabla v|^{p-2} \nabla v \nabla \varphi(x/r) \, dx.
\]

By assumption (1.2) we have \( h(x, v)v \geq 0 \). Setting \( C_0 := p \|\nabla \varphi\|_{L^\infty(B_2)} \) and applying Hölder’s inequality we obtain
\[
\int_{\Omega \cap B_{2r}} |\nabla v|^p \varphi_r^p \, dx \leq \frac{C_0}{r} \left( \int_{(\Omega \cap B_{2r}) \setminus B_r} |\nabla v|^p \varphi_r^p \, dx \right)^{(p-1)/p} \left( \int_{(\Omega \cap B_{2r}) \setminus B_r} |v|^p \, dx \right)^{1/p},
\]
which is (3.1) with \( b = 0 \). In the case of homogeneous Neumann boundary conditions, for every \( b \in \mathbb{R} \), the function \( v - b \) is another weak solution of (1.1). Hence we can replace \( v \) by \( v - b \) in the above calculations to obtain (3.1).

\[\Box\]

**Remark 3.7** Note that the above proof only uses that
\[
0 \leq \int_{\Gamma_2} h(x, v)v \varphi_r^p \, d\mathcal{H} = \int_{\Gamma_2} h(x, v) v \, d\mathcal{H} < \infty.
\]

### 4 Proofs of the main theorems

This section is dedicated to the proofs of Theorems 1.1 and 1.2. By rescaling, we can assume without loss of generality that \( \Omega^c \subseteq B_1 \) and that \( v \) is \( p \)-harmonic on \( \overline{B_1^c} \).

#### 4.1 Proof of Theorem 1.1

Assume that \( 1 < p < d \), and that \( v \) is a positive weak solution of (1.1) satisfying \( \lim_{|x| \to \infty} v(x) = 0 \). We need to show that \( v \equiv 0 \). Due to Propositions 3.1 and 3.6 we only need to show that there exists \( r_0 > 0 \) such that \( v \) satisfies (3.2) with \( b = 0 \) for all \( r \geq r_0 \). By (1.4) or Proposition 2.1 if \( p = 2 \), there are constants \( c_1, c_2 > 0 \) such that
\[
0 \leq v(x) \leq c_1 |x|^{(p-d)/(p-1)}
\]
for every \( x \in \overline{B_2^d} \). Hence,

\[
\frac{1}{r^p} \int_{B_2 \setminus B_r} |v|^p \, dx \leq \frac{c_1^p}{r^p} \int_{B_2 \setminus B_r} |x|^{p(p-d)/(p-1)} \, dx
\]
\[
= c_1^p \omega_d \int_r^{2r} s^{p(p-d)/(p-1)} s^{d-1} \, ds
\]
\[
= c_1^p \omega_d c_2 r^{(p-d)/(p-1)}
\]
(4.1)
for all \( r \geq r_0 := 2 \), where \( \omega_d \) is the surface area of the unit sphere in \( \mathbb{R}^d \) and \( c_2 = \ln 2 \) if \( d = p^2 \) and \( c_2 = \frac{p-1}{d-p^2}(2^{(p^2-d)/(p-1)} - 1) \) if \( d \neq p^2 \). As \( p < d \) we conclude that \( v \) satisfies (3.2) with \( b = 0 \) for every \( r \geq 2 \). As \( \lim_{|x| \to \infty} v(x) = 0 \) we conclude that \( v \equiv 0 \).

If \( p \geq d \), then every non-trivial positive bounded solution of (1.1) has a strictly positive limit as \( |x| \to \infty \); see [12, Lemma A.2]. Because we assume that the limit is zero, we must have \( v \equiv 0 \). Observe that these arguments do not make use of the boundary conditions. This completes the proof of Theorem 1.1.

4.2 Proof of Theorem 1.2

Let \( v \) be a semi-bounded weak solution of problem (1.1) with homogeneous Neumann boundary conditions, that is, \( \Gamma_2 = \partial \Omega \) and \( h(x, v) \equiv 0 \). Recall also that no regularity assumptions on \( \partial \Omega \) are needed.

Note that the \( p \)-Laplace operator \( \Delta_p \) is an odd operator, that is, \( \Delta_p(-v) = -\Delta_pv \). Hence, for every \( c \in \mathbb{R} \), the function \( c \pm v \) is another solution of problem (1.1). If \( v \) is bounded from below we can therefore replace \( v \) by \( v - \inf_{x \in \Omega} v(x) \), and if \( v \) is bounded from above we can replace \( v \) by \( \sup_{x \in \Omega} v(x) - v \). In either case we get a new solution \( v \geq 0 \) with \( \inf_{x \in \Omega} v(x) = 0 \). As before, we also assume that \( \Omega^c \subseteq B_1 \).

If \( 1 < p < d \), then by (1.4) the finite limit \( b := \lim_{|x| \to \infty} v(x) \) exists. By Proposition 3.6 inequality (3.1) is satisfied. To show that \( v \) satisfies (3.2) with \( b \) just defined we repeat the calculation (4.1) with \( v \) replaced by \( v - b \), using the decay estimate from (1.4). We can now apply Proposition 3.1 to conclude that \( v \) is constant.

It remains to deal with the case \( p \geq d \). Recall that by [12, Theorem 2.3], every positive \( p \)-harmonic function \( v \) on \( \Omega \) is either bounded in a neighbourhood of infinity and has a limit \( b := \lim_{|x| \to \infty} v(x) \) or \( v \sim \mu_p \) near infinity. In the second case the original solution considered is asymptotically equivalent to \( \pm \mu_p \) near infinity. Assume now that \( v \) has a limit as \( |x| \to \infty \). To show that \( v \) is constant we first note that by Proposition 3.6, \( v \) satisfies (3.1) with \( b = 0 \). As \( v \) is bounded in a neighbourhood of infinity and since \( p \geq d \), Remark 3.3 implies that \( v \) satisfies (3.2). Hence by Proposition 3.1, \( v \) is constant. This completes the proof of Theorem 1.2.

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