Limits of atomic entanglement by cavity feedback: From weak to strong coupling

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Abstract – We theoretically investigate the entangled states of an atomic ensemble that can be obtained via cavity feedback, varying the atom-light coupling from weak to strong, and including a systematic treatment of decoherence. In the strong-coupling regime for small atomic ensembles, the system is driven by cavity losses into a long-lived, highly entangled many-body state that we characterize analytically. In the weak-coupling regime for large ensembles, we find analytically the maximum spin-squeezing that can be achieved by optimizing both the coupling and the atom number. This squeezing is fundamentally limited by spontaneous emission to a constant value, which is independent of the atom number.

Harnessing entanglement in many-body systems is of fundamental interest [1] and is the key requirement for quantum enhanced technologies, in particular quantum metrology [2]. With this goal, many efforts have been devoted to prepare entangled states in atomic ensembles because of their high degree of coherence and their potential for precision measurement. Spin-squeezed states as well as number states have been produced following methods based either on coherent evolution in the presence of a nonlinearity in the atomic field [3–5], or on quantum non-demolition measurement [6–9]. Among methods of the first kind, cavity feedback [5,10] is one of the most promising: it has already allowed for the creation of highly squeezed states [5] and the effective nonlinearity introduced by the atom-cavity coupling can be easily switched off, making it very attractive for metrology applications.

In this letter, we analyze the entangled states that can be produced by cavity feedback in different coupling regimes from weak to strong, and derive the ultimate limits of the metrology gain, extending the optimization of squeezing to unexplored domains of parameters values. After optimization of both the coupling strength and the atom number, we find a maximum squeezing in the limit $N \to \infty$ that depends only on the atomic structure.

Cavity feedback relies on the dispersive interaction between one mode of an optical cavity and an ensemble of three level atoms, e.g. alkali atoms with two hyperfine ground states $|0\rangle$ and $|1\rangle$ (see fig. 1). The atom-cavity system is characterized by the atom-cavity coupling $g$, the cavity linewidth (HWHM) $\kappa$, the atom-cavity detuning that is opposite for the two transitions $\omega_{0e} - \omega_c = \Delta$, $\omega_{1e} - \omega_c = -\Delta$ and equal in absolute value to half of the hyperfine energy splitting in the ground state, and the spontaneous emission rate $\Gamma$ with $\Delta \gg \Gamma$. The dynamics of entanglement is governed by the two dimensionless quantities $C = g^2/(\kappa \Gamma)$ and $\phi_0 = 2g^2/(\kappa \Delta)$. The cooperativity $C$ gives the ratio between the number of photons emitted in the cavity mode to the spontaneously emitted photons as can be seen by a Fermi golden rule argument [11], and a large $C$ is favorable to entanglement because it minimizes the role of spontaneous emission. The parameter $\phi_0$ represents the cavity frequency shift, normalized to the cavity linewidth, when a single atom changes its hyperfine state. In the regime $\phi_0 \gg 1$, photons leaking from the cavity precisely measure the atom number difference between the two hyperfine states and therefore destroy the coherence between them. One could expect that this may prevent the apparition of atomic entanglement. However, this is not the case and we identify the condition to produce entanglement in this regime and characterize the produced states. They appear to have a potential for metrology as signaled by their quantum Fisher information. In the opposite regime $\phi_0 \ll 1$, spin coherence can be maintained and our calculations confirm
that this regime is optimal for producing spin-squeezed states. One important result is that the maximum squeezing is limited by the ratio of the excited state linewidth to the hyperfine splitting that should be as small as possible. As \( \phi_0/C = 2\Gamma/\Delta \), the condition \( \Gamma/\Delta \ll 1 \) allows to maintain \( \Phi_0 \) small while maximizing \( C \).

We consider \( N \) atoms, with two (hyperfine) ground states \( |0\rangle \) and \( |1\rangle \) equally coupled with a constant \( g \) and opposite detunings \( \pm \Delta \) to an excited manifold \(|e\rangle\) by a single cavity mode (see Fig. 1(a)). We introduce the collective spin operators \( \hat{S}_z = i\hat{S}_y = \sum_{i=1}^{N}|1\rangle\langle 1|_i - |0\rangle\langle 0|_i \) obtained by summing the effective spin-(1/2) operators for each atom. The initial atomic state is a coherent spin state, each atom being in an even superposition of \(|0\rangle\) and \(|1\rangle\). A single off-resonant cavity photon shifts the energy of these levels in opposite directions by an amount \( \Delta \). Through these opposite light shifts, the energy difference between levels \(|0\rangle\) and \(|1\rangle\) depends on the cavity photon number \( \hat{c}^{\dag}\hat{c} \), that depends on its turn on the population difference \( \hat{S}_z \) as the atoms change the index of refraction in the cavity. Spin-squeezing in this scheme occurs in the following way: the atomic quantum noise in \( \hat{S}_z \), induces fluctuations of the cavity field intensity (see Fig. 1), which during the evolution are imprinted into the phases of each atom, thus correlating \( \hat{S}_z \) with \( \hat{S}_z \) that is the population imbalance. Assuming low saturation of the optical transition \( g^2(\hat{c}^{\dag}\hat{c})/\Delta^2 \ll 1 \), we eliminate the excited manifold \(|e\rangle\) and describe each atom within the \(|0\rangle\)-\(|1\rangle\) subspace. Unitary evolution is governed by

\[
\hat{H}_0 = \hbar (\delta + \kappa \phi_0 \hat{S}_z) \hat{c}^{\dag}\hat{c} + i \eta (\hat{c}^{\dag} - \hat{c}),
\]

where \( c \) annihilates a photon of the cavity mode, \( \eta \) is the cavity pumping rate, \( \delta = (\omega_c - \omega_p) \) is the empty cavity detuning, and we already introduced \( \phi_0 = 2g^2/(\kappa\Delta) \) that is the properly normalized atomic light shift due to a single photon. Cavity losses and spontaneous emission including the possibility of scattering outside the \(|0\rangle\)-\(|1\rangle\) subspace are described by jump operators:

\[
\hat{d}_c = \sqrt{2\kappa c} \quad \text{and} \quad \hat{d}_{i,\text{el}} = \sqrt{\frac{\Gamma_{\text{Ray}}}{2}} \langle |1\rangle\langle 1|_i - |0\rangle\langle 0|_i \rangle \hat{c}; \quad \Gamma_{\text{Ray}} = \frac{\Gamma \kappa \phi_0}{2\Delta} \quad \text{(2)}
\]

\[
\hat{d}_{i,\sigma'\sigma} = \sqrt{\frac{\Gamma_{\text{Ram}}}{2}} |\sigma'\rangle\langle \sigma|_i \hat{c}; \quad \Gamma_{\text{Ram}} = \frac{\Gamma \kappa \phi_0}{2\Delta} \quad \text{a}_{\text{Ram}} \quad \text{X} \quad \text{numbers that depend on the atomic structure and field polarization. The operators} \quad \hat{d}_{i,\text{el}}, \hat{d}_{i,\sigma'\sigma}, \hat{d}_{i,\text{X} \sigma} \quad \text{refer to Rayleigh, Raman and scattering outside the} \quad |0\rangle\)-\(|1\rangle\) \quad \text{subspace processes for the atom} \quad i, \quad \text{respectively. We assume that once out of the} \quad |0\rangle\)-\(|1\rangle\) \quad \text{subspace the atoms do not interact with the field anymore and they are simply lost for our system.}

In the following we assume that the cavity is weakly pumped, \( \eta^2/\kappa_{\text{eff}} \ll 1 \), where \( \kappa_{\text{eff}} \) is the cavity linewidth in the presence of the atoms.

\[
\kappa_{\text{eff}} = \kappa + \frac{N}{4} (\Gamma_{\text{Ray}} + \Gamma_{\text{Ram}} + \Gamma_X) \quad \eta \ll \frac{\kappa_0}{\kappa} \quad \kappa \quad \text{(5)}
\]

so that the mean photon number in the cavity is always small. Under this condition, the probability of occurrence of a quantum jump in the transient time \( 1/\kappa_{\text{eff}} \) is negligible, and for an initial state with the field in vacuum and the atoms in an eigenstate of the atomic operator \( \hat{S}_z \) with eigenvalue \( m \in [-N/2, N/2] \), the cavity field reaches in a time \( 1/\kappa_{\text{eff}} \) a coherent state \((\alpha(m))\) of amplitude \( \alpha(m) \)

\[
\alpha(m) = \frac{\eta}{\kappa_{\text{eff}}} + i(\delta + \kappa \phi_0 m) \quad \text{(6)}
\]

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1Spin-squeezing is indeed a quantum correlation between the two transverse components \( \hat{S}_y \) and \( \hat{S}_z \) of the collective spin.

2We follow the procedure explained in [12,13]. The starting point are optical Bloch equations, in the rotating wave approximation, for a single atom of density matrix \( \sigma \) and a coherent field of frequency \( \omega_c \) coupling with opposite detunings \( g_0, g_1 \) to an excited manifold \( e \) of lifetime \( \Gamma^{-1} \). To obtain effective equations in the ground state \( i \) we use the fact that there are “slow” and “fast” variables (e.g. the ground-state populations and coherences \( \sigma_{00}, \sigma_{11}, \sigma_{1} \) (exp\([-2i\Delta t])) are slow variables) and ii) we limit to first order in the saturation parameter. A subtle point concerns the ground-excited states coherences \( \sigma_{01} = \sigma_{10}(t) e^{i\epsilon L} \) \( (g = g_0, g_1) \) for which we use an Ansatz of the form \( \sigma_{01}(t) = \sigma_0(t) + \sigma_{1}(t) \exp(2\Delta t) + \sigma_{1}(t) \exp(-2\Delta t) \) with \( \sigma_0(t), \sigma_1(t) \) and \( \sigma_{1}(t) \) slow variables after a transient time \( \Gamma^{-1} \). The obtained equations in the ground state have an Hamiltonian part, representing light shifts, and a non-Hamiltonian part that we can interpret as coming from a master equation in the Limblad form. In the case where only two ground-state levels \(|0\rangle\) and \(|1\rangle\) are coupled to \(|e\rangle\) with symmetric couplings, we arrive at eqs. (1)–(4). The Rayleigh and Raman operators that we obtain (2), (3) are in agreement with [14].
we calculate the change in the system purity after tracing out one atom [15,16] that we note PC (purity change):

$$PC \equiv \text{Tr}_{1,2,\ldots,N}[|\rho^2|] - \text{Tr}_{2,\ldots,N}[|\text{Tr}_1 |\rho)|\rho|^\dagger|.$$  (9)

If all the atoms are correlated, tracing one of them can strongly influence the purity. Indeed, one can show that PC > 0 implies that the state is not separable. PC’s maximum value is 1/2 obtained for a Schrödinger cat state. In a purely Hamiltonian model $\hat{H} = \chi S^z_1$ one finds $PC = \frac{1}{2}[1 - (\cos \chi)^2(N^{-1})]$. The main advantage of the quantity (9), is that it only requires the calculation of a trace and it can be computed even for relatively large atom numbers despite the large size of the Hilbert space. Using eq. (7) with $\langle \hat{c}_1 | \hat{\rho} | \hat{c}_2 \rangle (0) = 1/2N$, we obtain fig. 2 where we show PC as a function of time and of $\phi_0/\sqrt{N}$ that is the cavity detuning induced by the quantum fluctuations of $S_1$ (as $\Delta S_2 = \sqrt{N}/2$). On the same plot we show the isolines for the spin-squeezing parameter [17]

$$\xi^2 = \frac{N\Delta^2 \hat{S}_\perp}{|\langle \hat{S} \rangle|^2},$$  (10)

where $\Delta^2 \hat{S}_\perp$ is the minimal variance of the collective spin orthogonally to the mean spin direction and $|\langle \hat{S} \rangle|$ is the mean spin length. Spin-squeezed states $\xi^2 < 1$ appear for coupling values such that $\phi_0/\sqrt{N} < 1$, this conclusion illustrated for 50 atoms in fig. 1, holds for much larger atom numbers (see fig. 4).

**Strong-coupling regime.**—The purity change PC in fig. 2 detects a larger and larger region of entangled states as $\phi_0/\sqrt{N}$ increases, even after very short evolution times. To understand their nature, let us first consider the case without spontaneous emission. In this case decoherence is only due to cavity losses that affect the atoms collectively and preserve the symmetry of the initial coherent state with respect to the exchange of two atoms. A consequence is that one can use the Fock basis $|m\rangle$ to express the atomic density matrix (7) whose off-diagonal elements decay as $|\langle m | \alpha(m')\rangle|^{1+2\kappa t}$. If $\phi_0 >>> 1$, any state with $m \neq 0$ shifts the cavity out of resonance so that the cavity is practically dark, while the only distinguished state is the twin-Fock state $m = 0$ which does not detune the cavity. Coherences between this state and all the others vanish in a time $t_0 \approx T_0^{-1} = 2\kappa/e^2$ after which the initial coherent spin state4 $|\psi\rangle = \sqrt{p_0}|m = 0\rangle + \sqrt{1-p_0}|m = 1\rangle$ is mapped onto the mixture:

$$\hat{\rho} = p_0|m = 0\rangle\langle m = 0| + (1 - p_0)|\psi^\perp\rangle\langle \psi^\perp|.$$  (11)

The subspace $|\psi^\perp\rangle\langle \psi^\perp|$ of states $m \neq 0$ is preserved, both from decoherence and unitary evolution, for a time $t_1 \approx T_1^{-1} = (\kappa/e)^2\phi_0^2/4$ by which the cavity starts to distinguish the next Fock state. At longer times more and more coherences between different Fock states are

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3We tested this approximation numerically by solving exactly the initial master equation including the transient and found it to be accurate for $(\eta/\kappa_{\text{eff}})^2 \ll 1$.

4From the initial state we have $p_m = \sqrt{\frac{1}{2^m}(\frac{N}{m+\frac{1}{2}})}$. 

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Fig. 2: (Color online) Map of entangled states obtained from eq. (7), at different times and values of the phase shift $\phi_0$. The purity change (9) is shown in color and the isolines of the squeezing parameter are solid black lines. Parameters: $\delta = \kappa$, $N = 50$, $(\eta/\kappa)^2 = 10^{-2}$, $a_{\text{Ray}} = 2$, $a_{\text{Ram}} = a_\chi = 0$, $\Delta/\Gamma = 500$.
for the

Rayleigh scattering and cavity losses. The horizontal lines give the analytical predictions for $I_F$ and PC for the state (11). Other parameters are as in fig. 2.

killed until a complete mixture is reached. Note that $t_1/t_0 = \phi_0^2/8 \gg 1$ for strong coupling. Interestingly, the long-lived state (11) with partially removed coherences is highly entangled, its Fisher information scaling as $I/N^2$ and PC reach those of the state (11) around $t\sim 1$, indicating that our picture still holds in the presence of spontaneous emission for small values of $\kappa t$. Spontaneous emission and cavity losses are included. Fisher information and PC reach those of the state (11) (around $t\sim 1$) 250 $\approx 1.25\kappa t_0$ and stay close to these values until $\kappa t = 2 \times 10^3 \approx 2\kappa t_1$ indicating that our picture still holds in the presence of spontaneous emission for small atom ensembles. For the purity change, a similar conclusion holds for $N = 50$.

Spin-squeezed states. – We now concentrate on spin-squeezing and large atom numbers. In fig. 4 we show the squeezing parameter optimized over time, $\xi_{\min}^2$ as a function of $\phi_0\sqrt{N}$ for $N = 10^6$, in a realistic configuration for $^{87}$Rb where we choose the clock transition $|F = 1, m_F = 0 \rangle - |F = 2, m_F = 0 \rangle$, $\pi$-polarized light on the D2 line and a detuning close to half of the hyperfine energy splitting so that there are symmetric couplings and no Raman processes. For this configuration we find\footnote{For $\sigma, \sigma' = 0, 1$ we find $a_{R\sigma} = 2a_{\sigma\sigma}$, $a_{R\sigma} = |a_{\sigma\sigma}|^2/a_{\sigma\sigma}$ and $\chi = \sum_{\sigma} |a_{\sigma}\rangle |a_{\sigma}\rangle^2/a_{\sigma\sigma}$, where $a_{\sigma\sigma}$ are probability amplitudes connecting state $\sigma'$ to $\sigma$ via the excited manifold that we calculate for the D2 Rb line and $\pi$ polarization as in [18].} $a_{R\sigma} = 1.404$, $a_{R\sigma} = 0$ and $a_\chi = 0.352$. The red solid curve, calculated from (7) includes cavity losses and Rayleigh spontaneous emission while the red dashed curve includes cavity losses only. We see that spontaneous emission is here important especially for small values of $\phi_0\sqrt{N}$.

In order to obtain explicit analytical results and have a physical insight in the different regimes, we derive effective $\chi S_2^z$ models in which the photonic field is eliminated from the beginning from the description. This is done by replacing, in the Hamiltonian (1) and in the jump operators $d_\sigma$ and $d^\dagger_\sigma$ defined in (2)–(4), the operators $\hat{c}$ and $\hat{c}^\dagger$ by their average stationary values (6), that still depend on the atomic state via the operator $S_2$. This procedure gives results in agreement with the solution (7) of the master equation as long as $\phi_0 \ll 1$ and $\kappa_{eff} \simeq \kappa$ with $a_{R\sigma} = 1.404$, $a_{R\sigma} = 0$ and $a_\chi = 0.352$. Horizontal gray dotted line: $\xi_{\min}^2$ in the large $N$ limit (12).

Fig. 4: (Color online) Spin-squeezing optimized over time as a function of $\phi_0\sqrt{N}$ in a realistic configuration for $^{87}$Rb (see text) with $N = 10^6$, $\Delta/\Gamma = 563.39$ and $(\eta/\kappa)^2 = 10^{-2}$ and $\delta = \kappa$. Red solid line: full model (7) with cavity losses and Rayleigh jumps ($\kappa_{R\sigma} = 1.404$). Red dashed line: full model without spontaneous emission. Blue dash-dotted line: effective model in the regime $\phi_0\sqrt{N} \ll 1$ and $\kappa_{eff} \simeq \kappa$ with $a_{R\sigma} = 1.404$, $a_{R\sigma} = 0$, and $a_\chi = 0.352$. Horizontal gray dotted line: $\xi_{\min}^2$ in the large $N$ limit (12).
Fig. 5: (Color online) Spin-squeezing optimized over time as a function of $\phi_0\sqrt{N}$, for $\Delta/\Gamma = 10$ and $(\eta/\kappa)^2 = 10^{-2}$. Solid lines: full model (7) with cavity losses and Rayleigh scattering with $a_{\text{Ray}} = 1.404$. Dot-dashed lines: analytical results in the regime $\kappa_{\text{eff}} \approx \kappa$ and $\phi_0\sqrt{N} \ll 1$. Dotted lines: analytical results in the regime $\kappa_{\text{eff}} \gg \kappa$ and $\phi_0\sqrt{N} \gg 1$. Horizontal gray dotted line: $\xi_{\text{min}}^2$ in the large $N$ limit (14). From top to bottom: $N = 10^4$, $10^5$, $10^6$, $10^7$.

the condition $\kappa_{\text{eff}} \phi_0\sqrt{N} = \kappa_{\text{eff}}$ can be written as
\[
\phi_0\sqrt{N}(1 - \sqrt{\frac{N}{N_c}}) = 1; \quad N_c \equiv \left(\frac{4\Delta/\Gamma}{a_{\text{Ray}} + a_{\text{Ram}} + a_X}\right)^2.
\]
(13)

For $N = N_c$ photon losses due spontaneous emission are equally important as those due to cavity mirrors. For $\sqrt{N}/N_c \ll 1$ which is the situation of fig. 4, the condition (13) tells us that for $\phi_0\sqrt{N} \geq 1$ the detuning induced by the quantum noise becomes larger than the cavity linewidth giving rise to nonlinear effects destroying squeezing. For $\sqrt{N}/N_c \gg 1$ the condition (13) is never satisfied and we always have $\phi_0\sqrt{N} \ll \kappa_{\text{eff}}$. In this regime dominated by absorption, the cavity linewidth increases linearly with the atom number $\kappa_{\text{eff}}/\kappa \approx \phi_0\sqrt{N}/N_c$, faster than the atoms-induced cavity detuning $\propto \phi_0\sqrt{N}$. To describe this situations we derive a second effective $\hat{H} = \chi\hat{S}^2_z$ model, this time for $\phi_0\sqrt{N} \gg 1$ and $\sqrt{N}/N_c \gg 1$, including cavity losses and Rayleigh jumps (details are given in the appendix). It predicts that i) the best squeezing becomes independent of $\phi_0$ for large $\phi_0\sqrt{N}$ and, most importantly, ii) the best squeezing has a non-zero limit for $N \to \infty$
\[
\xi^2_{\text{min}} \to \frac{\phi_0 N^2 a_{\text{Ray}} \Gamma}{\kappa} \left(1 - \frac{N}{N_c}\right)^2; \quad \eta_{\text{min}}^2 = \frac{\phi_0 N^2 a_{\text{Ray}} \Gamma}{128 \Delta} \left(1 - \frac{N}{N_c}\right),
\]
(14)

where we have assumed $\delta = \kappa$. We show the onset of this new regime as $N$ is increased in fig. 5 for $\Delta/\Gamma = 10$.

Conclusions. - We predict that highly entangled many-body states driven by cavity losses can be prepared by cavity feedback in the strong-coupling regime for small samples, and a very large amount of spin-squeezing is reachable for large atom numbers in the weak-coupling regime. We find that the ultimate squeezing limit of the cavity feedback scheme in the limit of an infinite atom number is given by the ratio between the spontaneous decay rate and the quantum detuning of the transition addressed by the cavity mode.

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Appendix: effective $\chi\hat{S}^2_z$ models. –

Quantum averages for an effective $\hat{S}^2_z$ model with decoherence. By eliminating the photon operators as described in the text in different regimes, we obtain effective models of the form
\[
\hat{H}_{\text{eff}}/\hbar = \chi\hat{S}^2_z - \frac{i}{2} \sum_{\alpha \in \text{jumps}} \hat{d}_\alpha^\dagger \hat{d}_\alpha,
\]
(1.1)
\[
\hat{d}_\alpha = \sqrt{\Lambda_{C,\alpha}}\hat{S}_{\alpha},
\]
(1.2)
\[
\hat{d}_{\text{cd}} = \sqrt{\frac{\Lambda_{\text{Ray}}}{2}}(|1\rangle\langle 1| - |0\rangle\langle 0|); \quad \sigma = 0, 1,
\]
(1.3)
\[
\hat{d}_{\text{c,d}} = \sqrt{\Lambda_{\text{Ram}}(\sigma')\langle \sigma|}; \quad \sigma, \sigma' = 0, 1; \quad \sigma, \sigma' \neq 0, 1,
\]
(1.4)
\[
\hat{d}_{\text{c},X\sigma} = \sqrt{\Lambda_X|X\rangle \langle \sigma|}; \quad \sigma = 0, 1; \quad X \neq 0, 1.
\]
(1.5)

The quantum averages needed to compute the squeezing can be obtained analytically
\[
\langle \hat{S}_x \rangle = \frac{N}{2} e^{-\Lambda C t/2} e^{-\Lambda X t} (H_{1}(t))^{N-1},
\]
(6)
\[
\langle \hat{S}_y \rangle = \frac{N}{4} e^{-\Lambda X t}; \quad \langle \hat{n}_0 \hat{n}_1 \rangle = \frac{N(N-1)}{4} e^{-2\Lambda X t},
\]
(7)
\[
\langle \hat{S}_z \rangle = \frac{N}{4} e^{-\Lambda X t} + \frac{(N-1)N}{8} e^{-2(\Lambda X + \Lambda_{\text{Ram}} + \Lambda_{\text{Ray}})t}
\]
\[
\times \left(1 - e^{-2\Lambda C t}(H_2(t))^{N-2}\right),
\]
(9)
\[
2\text{Re}\left(\hat{S}_z \hat{S}_y\right) = \frac{N(N-1)}{2} e^{-\Lambda C t/2} e^{-2(\Lambda X + \Lambda_{\text{Ram}} + 2\Lambda_{\text{Ray}})t}
\]
\[
\times G(t) (H_1(t))^{N-2},
\]
(10)

where for $\beta = 1, 2$ we have introduced the functions
\[
H_\beta(t) = \frac{\Lambda X (\Lambda X + 2\Lambda_{\text{Ram}})}{\beta^2 \chi^2 + \Lambda X (\Lambda X + 2\Lambda_{\text{Ram}})}
\]
\[
+ \frac{\beta^2 \chi^2 e^{-(\Lambda_{\text{Ram}} + \Lambda X)t}}{\Lambda_{\text{Ram}} + \Lambda X} \left(\cosh(u_\beta t) + \frac{\Lambda_{\text{Ram}} + \Lambda X}{u_{\beta}} \sinh(u_\beta t)\right),
\]
(11)
\[
G(t) = \frac{\chi \sinh(u_1 t)}{u_{1}}, \quad \text{where} \quad u_\beta = \sqrt{\Lambda_{\text{Ram}} - \beta^2 \chi^2}.
\]
(12)
Starting from these exact formulas for quantum averages, in the limit of large atom numbers and short evolution times and in the case of Rayleigh and Raman scatterings only (namely if $\Delta \chi = 0$), we recover the results given in eqs. (20)–(24) of ref. [19].

Case $N \ll N_c$ and $\phi_0 \sqrt{N} \ll 1$. In the limit $N \ll N_c$ and $\phi_0 \sqrt{N} \ll 1$, which implies $\kappa_{\text{eff}} \approx \kappa$, by expanding $\alpha(S_c)$ in powers of $\phi_0 S_c$ and taking $\delta = \kappa$, we obtain and effective $S_c^2$ model with the following parameters:

$$\chi = -\eta^2 \phi_0^3 \frac{\chi}{4\kappa}; \quad \Lambda_C = 2|\chi|, \quad (A.13)$$

$$\frac{\Lambda_{\text{Ram}}}{a_{\text{Ram}}} = \frac{\Lambda_{\text{Ray}}}{a_{\text{Ray}}} = \frac{\Lambda_X}{a_X} = \frac{|\chi|}{2\phi_0 \Delta} = \frac{|\chi|}{4\Delta}. \quad (A.14)$$

From (A.14) we see that for large cooperativity spontaneous emission rates vanish. Using the solution (A.6)–(A.12) one can show that if $N^{-1/10} \ll \phi_0 \sqrt{N} \ll 1$ then the squeezing is limited mostly by cavity losses, playing here a more important role than scattering into free space. The best squeezing and the time at which this squeezing is reached in this regime are given by eq. (12).

Case $N \gg N_c$ and $\phi_0 \sqrt{N} \gg 1$. In the limit $N \gg N_c$ and $\phi_0 \sqrt{N} \gg 1$, which implies $\kappa_{\text{eff}} \gg \kappa$, by expanding $\alpha(S_c)$ in powers of $\sqrt{N_c}/N$ and taking $\delta = \kappa_{\text{eff}}$, we obtain and effective $S_c^2$ model with the following parameters:

$$\chi = \eta^2 \left| \frac{\phi_0}{2\kappa \kappa_{\text{eff}}} \right|^3 \chi; \quad \Lambda_C = 2\chi; \quad \Lambda_{\text{Ray}} = \chi \left( \frac{\Gamma_{\text{Ray}}}{4\Delta} \right)^2 N. \quad (A.15)$$

The best squeezing limit and the time at which this squeezing is reached are then given by eq. (14).

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