Kinematical Lepton Transmutation in $e^+e^-$ Collision and Vector Boson Decay

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Abstract

The change in orientation in generation space (rotation) of the fermion mass matrix with changing scales can lead to flavour-violations through just the kinematics of a non-diagonal mass matrix. Such effects for the reactions: $e^+e^- \rightarrow e^\pm\mu^\mp, e^\pm\tau^\mp, \mu^\pm\tau^\mp$, and for the decays of vector bosons into the same channels, are calculated following a method suggested earlier which gives the differential cross section for each reaction and the branching ratio for each decay mode in terms of an overall normalization depending only on the speed at which the mass matrix rotates. A rotation speed estimated earlier, under certain assumptions from the fermion mixing angles and mass ratios, is found to give the above effects at a level readily detectable in modern high sensitivity experiments such as Bepc, Cleo, BaBar and Belle, at least in principle. The observation of these effects would not only confirm the concept of a rotating mass matrix with a significance on par with the running coupling constant, but also offer some valuable insight into the origin of fermion generations. However, a negative result cannot unfortunately rule out the rotating mass matrix since the effects deduced here from this mechanism alone could in principle be cancelled by other rotation effects.
1 Introduction

With the running coupling constant now a familiar concept already amply verified in experiment, one would not be surprised that the fermion mass matrix also varies with changing energy scales. That its eigenvalues, namely the fermion masses, actually do run has already been experimentally verified in certain circumstances [1]. What we consider in this paper is the scenario when the fermion mass matrix changes also its orientation (rotates) in generation space as the scale changes.

Theoretically, of course, there are good reasons to expect that the fermion mass matrix will rotate with changing scales. Even in the Standard Model mass matrix rotation occurs so long as there is nontrivial mixing between the up and down fermion states. The renormalization group equation there satisfied by the mass matrix \( U \) of the up fermions [2]:

\[
16\pi^2 \frac{dU}{dt} = -\frac{3}{2} DD^\dagger U + ...
\]  

contains already at leading order a term on the right which is nondiagonal in the eigenstates of \( U \) when the mass matrix \( D \) of the down fermions is related to \( U \) by a nontrivial mixing matrix, so that a \( U \) matrix diagonalized at one scale can no longer remain diagonal at another scale, or in other words, it will rotate as claimed. The same argument holds for the mass matrix \( D \) for the down fermions. This fact was pointed out for quarks already long ago in, for example, [3], and now that nontrivial mixing for leptons has been strongly indicated [4, 5, 6, 7, 8, 9], if not already confirmed by experiment, the same conclusion can be drawn for leptons also. Next, looking beyond the present Standard Model, one encounters further possible mechanisms for driving the mass matrix rotation. Indeed, the mere fact that different generations of fermions can rotate into one another, as the mass matrix rotation implies, already means that they are not distinct entities as once conceived but just different manifestations of the same object, like the different colours of a quark, related presumably by some “horizontal” symmetry [10]. This suggests new forces which can change the generation index, and hence drive directly mass matrix rotations, in addition and in contrast to the “indirect” mechanism in (1) via mixing.

However, even apart from theoretical prejudices as above indicated, there are, in our opinion, already some empirical indications which argue strongly, though perhaps as yet only circumstantially, in favour of fermion mass matrix rotation. These come about as follows. As is well-known, quarks and leptons
exhibit remarkable mass and mixing patterns. First, both their mass spectra are hierarchical, with the masses falling by one to two orders in magnitude from generation to generation. Secondly, the mixing matrices (i.e. CKM for quarks and MNS for leptons) which parametrize the relative orientations between up and down states seem roughly similar in shape for quarks and leptons, at least as far as is already known about the latter from the data in neutrino oscillation, only with the off-diagonal elements generally much larger for leptons than for quarks. Neither of these features have any explanation in the present Standard Model, in which they are just taken for granted, and between them they account for some two-thirds of the Standard Model’s 20-odd empirical (“fundamental”) parameters. However, as was pointed out in a recent note, if one assumes that the fermion mass matrix rotates at a certain speed, then these features can all be very simply understood.

That this is the case can be summarized as follows. Once the mass matrix is allowed to rotate with changing scales, then the usual definition of fermion flavour states as its eigenstates will have to be refined since, the eigenstates being now scale-dependent, one has to specify at what scale(s) the fermion flavour states are taken as the eigenstates. If the fermion masses are defined as the eigenvalues of the mass matrix at the scales equal in value to the masses themselves, as is usually done, then it would be natural to define the corresponding fermion flavour states as the eigenstates at the same scales also. It follows then that the state vectors of, say, the $t$ and $b$ quarks will be defined at different scales, so that even if the $U$- and $D$-quark mass matrices share always the same orientation at the same scale, the rotation of the mass matrix from the scale of the $t$ mass to that of the $b$ mass will already imply a disorientation between the $t$ and $b$ state vectors, or in other words a CKM matrix element $V_{tb}$ represented by the direction cosine between the $t$ and $b$ vectors which is different from unity. More generally, it can be seen along the same lines that a rotating mass matrix will generate not only a nontrivial mixing (CKM or MNS) matrix but also nonzero masses for lower generation fermions even when one starts with neither. Indeed, what was shown in was that all existing empirical information on the fermion mass hierarchy and mixing pattern, (excepting for the moment only $CP$-violation) can now be understood as consequences a rotating mass matrix in the above manner, at least qualitatively but in some cases even quantitatively. We regard this as a rather strong though indirect empirical indication in favour of fermion mass matrix rotation. Indeed, if this interpretation of fermion mixing and mass hierarchy is accepted, then it implies a rotation speed considerably...
greater than that driven by the mechanism (1) in the current Standard Model framework and suggests a driving mechanism from beyond that.

Given these indications, both empirical and theoretical, it would be natural to enquire what other physical implications a rotating fermion mass matrix may have which can be tested directly by experiment. One obvious candidate is flavour-violation, for a rotating mass matrix will not remain diagonal in the flavour states at scales other than the scale(s) at which these flavour states are defined. And since reaction amplitudes depend in general on the fermion mass matrices, these too can become flavour-nondiagonal leading thus to flavour-violating reactions. This possibility has already been considered in general terms in, for example, [16] in some detail. We need here only to give an outline of its physical significance.

The importance of flavour-conservation as a possible fundamental concept, of course, has long been recognized and its consequences subjected to rigorous experimental tests. In particular, the very stringent bounds set on $\mu \rightarrow e\gamma$ and $\mu \rightarrow eee\bar{e}$ decays, which are at present respectively $1.2 \times 10^{-11}$ and $1.0 \times 10^{-12}$ for the branching ratio over the predominant, but already weak, decay for $\mu$, tend to give the impression that any violation of lepton flavour would have to be extremely small. However, such a conclusion may be premature for not having taken account of the possibility that the lepton mass matrix rotates with the energy scale. If the mass matrix does rotate, then the fact that flavour-conservation has been stringently tested in $\mu \rightarrow e\gamma$ and $\mu \rightarrow eee\bar{e}$ at the $\mu$ mass scale, where the $\mu$ state is by definition diagonal, is by itself no guarantee that flavour-violation will be equally small in another reaction at another scale where the mass matrix may have rotated to another orientation so that the $\mu$ state is no longer diagonal there. Such flavour-violating effects can in principle occur by virtue of the mass matrix rotation even when there are no explicit flavour-changing neutral current (FCNC) couplings in the action, and can be sizeable even when FCNC effects are small. We have therefore suggested for them the term “transmutation” in [16] for distinction, which we shall adopt also in this paper. The size of transmutation effects, if any are observed, and their variation with energy will give indications on how the rotation of the mass matrix is driven, the knowledge of which may in turn shed light on the origin of generations itself, a basic question in particle physics that has already been with us for many years. For this reason, we suggest that flavour-violation be routinely tested in experiment whenever conditions are favourable.

For pursuing this program, a method was developed in [17] for calculating the flavour-violation effects due just to the kinematics of a rotating mass ma-
trix. In this, one treats the rotating mass matrix as a proposition in isolation without enquiring from what mechanism this rotation originates and without taking account of other possible rotation effects which might in principle accompany the mass matrix rotation. Such a preliminary attitude, we think, is reasonable given that the only empirical evidence one has so far is for the rotating mass matrix alone \cite{15} with no direct hints yet of the mechanism driving it. The result of such a calculation will serve to gauge what size flavour-violation effects might in principle be expected from a mass matrix rotating at a given speed with the aim of providing an indicator for experimenters planning an analysis along these line. However, it is not to be taken as a necessary prediction of a rotating mass matrix under general circumstances for the following reason. Reaction amplitudes depend on quantities other than the fermion mass matrix, such as, say, interaction vertices, which may in principle also rotate, and any theoretical mechanism for driving the mass matrix rotation may also imply vertex rotations. And these other rotational effects may modify or even cancel the effects computed by the above method from the kinematics of the rotating mass matrix alone. Indeed, in a detailed calculation performed specifically in the Dualized Standard Model (DSM) framework that we ourselves suggest which is reported in a separate paper \cite{19}, such a cancellation is found in fact to occur. Hence, calculations done with the method of \cite{16, 17} as those in the the present paper have to be taken with this reservation in mind, which reservation was unfortunately not made clear in the earlier references because it was not clear then even to ourselves.

In the present paper, we choose to investigate transmutation effects in the following reactions by the method suggested in \cite{17} which, though originally developed for photo-transmutation, can be adapted to the present case with but minor modifications:

\[
e^+e^- \rightarrow e^+\tau^-, \tau^+e^-; \quad (2)
\]

\[
e^+\mu^- \rightarrow e^+\mu^-, \mu^+e^-; \quad (3)
\]

\[
e^+e^- \rightarrow \mu^+\tau^-, \tau^+\mu^-.
\]  \quad (4)

The obvious practical reason for investigating these reactions is that there are several high intensity machines in operation, such as BEPC, CESR (Cleo), PEP II (BaBar), and KEK II (Belle), which appear capable of observing these flavour-violating effects to high accuracy, besides LEP, which though now turned off, has left still masses of data which could be useful for the same purpose.
Lepton transmutation in reactions (2)—(4) can proceed by, for example, the processes represented by the Feynman diagrams in Figure 1. At the energy at which an experiment is performed, the amplitudes for these processes, being dependent on the lepton masses, are diagonal in the eigenstates $j = 1, 2, 3$ of the mass matrix at that scale but not in general, by the reasoning above, diagonal in the flavour states $e, \mu, \tau$. And this fact alone could be enough to give lepton transmutation as a result. The transmutation reactions (2)—(4) can of course occur also via other processes, such as higher order photon-exchange diagrams or $Z_0$-exchange, but the effects from these for the energy range of present interest are small and will be neglected. What can give sizeable contributions, however, is the formation of vector bosons in the intermediate state followed by their subsequent (transmutational) decays, as represented by Figure 2 (b).

![Feynman diagrams](image)

Figure 1: Feynman diagrams for the transmutation amplitude.

Figures 1 and 2, then, are the only processes we shall consider in this paper, where we shall show that, by following the procedure suggested in [17], one can calculate explicitly the differential cross sections for the 3 transmutational reactions (2), (3), and (4), given any rotating mass matrix. The rotating mass matrix itself will figure only in the normalization of the cross sections, not in their angular or spin dependence both of which are given essentially just by kinematics.

As a numerical example for the sort of cross sections one might expect for these transmutation reactions, let us consider in particular the reaction (4). The normalization of the cross section of this reaction is given by the rotation angle between the $\mu$ and $\tau$ states at the energy scale of the experiment, a
good estimate for which can be read already from the existing data on fermion mass and mixing patterns interpreted as rotation effects as in [15], say from Figure 3 of that paper. Specifically, for $\sqrt{s} = 10.58 \text{ GeV}$ at the mass of the $\Upsilon(4S)$ at which BaBar is run, one obtains in this way an estimate for the integrated cross section of reaction (4) of around 80 fb which translates to as many as a thousand events in the data they have already collected from their first year’s run of 20 fb$^{-1}$, assuming 100 percent detection efficiency, and if so should be readily detectable.

However, as already stressed above, these estimates for transmutation cross sections are not to be interpreted as definitive predictions for the quoted rotating mass matrix since they can be modified by other rotation effects. Nevertheless, they are of interest in giving an idea of the size of flavour-violation effects that a rotating mass matrix can in principle generate. Given the smallness of flavour-violation at the $\mu$ mass scale in $\mu \rightarrow e\gamma$ and $\mu \rightarrow eee$ decay, any detection of flavour-violation in the reactions (2)–(4) at BaBar or similar experiment at a different scale would be a positive indication for mass matrix rotation, although a negative result, by virtue of the preceding observation, would not be able at present to rule it out.

2 The Reaction Amplitude (a)

Consider first the one-photon exchange diagram of Figure 1(a), which will be seen to give the dominant contribution to the two reactions (2) and (3). It does not contribute to the reaction (4), which can proceed by one-photon exchange in $e^+e^-$ collision only when both $e^+$ and $e^-$ transmute, but this will be so far down in magnitude as to be negligible for present consideration.
Then according to the procedure suggested in [17], at any given energy $\sqrt{s}$, the transmutation amplitude for the reaction:

$$e^+ \ell^- \rightarrow e^+ \ell^-,$$

is given by a rotation in generation space, thus:

$$M^{(a)} = \sum_j S^\dagger_{\beta j} M^{(a)}_j S_{\alpha j},$$

from the diagonal amplitudes $M^{(a)}_j$ for the reaction:

$$e^+ \ell^-_j \rightarrow e^+ \ell^-_j$$

for the mass eigenstate $j$ at the scale $\sqrt{s}$ with eigenvalue $m_j$, where $S_{\alpha j} = \langle j | \alpha \rangle$ is the rotation matrix in generation space which relates the triad of lepton flavour states $\alpha = e, \mu, \tau$ to the eigentriad $j$ at the scale $\sqrt{s}$. Explicitly, for the one-photon exchange diagram of Figure 3(a), we have:

$$(M_j^{(a)})^{\gamma r}_{s' s} = -ie^2 [\bar{u}_{s'}(p'_j) \gamma^\mu u_s(p_j)] \frac{1}{(p'_j - p_j)^2} [\bar{v}_r(q) \gamma_\mu v_{r'}(q')],$$

where $s$ and $s'$ denote the spins of the incoming and outgoing lepton $\ell^-$ and $r$ and $r'$ those of the antileptons $\ell^+$.

![Feynman Diagram](image-url)

**Figure 3:** Feynman diagram for the diagonal amplitudes.

To actually evaluate (8), we have still to specify the values of the momenta $p_j$ and $p'_j$ entering there. The point is that the amplitude for the two-body
reaction (7) is of course a function of only two variables, which we may take to be the standard Mandelstam variables $s$ and $t$, so that all components of the various momenta appearing in (8) must be expressible in terms of them. The reasoning required to arrive at these expressions is not entirely trivial, but following the considerations given in [17] which apply as well to the present case with but minor modifications, we obtain the following relationships between the different momenta to be used later for deriving the required expressions:

$$
\begin{align*}
\mathbf{p}_j &= a_j \mathbf{q} + b_j \mathbf{q}' + c_j \mathbf{p}_i \\
\mathbf{p}_j' &= a_j \mathbf{q}' + b_j \mathbf{q} + c_j \mathbf{p}_i',
\end{align*}
$$

(9)

where

$$
\begin{align*}
c_j &= \sqrt{\frac{(s - m^2_j + m^2)^2 + st_0}{(s - m^2)^2 + st}}, \\
b_j &= c_j(s + m^2) - (s - m^2_j + m^2) \frac{t_0}{t}, \\
a_j &= c_j(s + m^2 + t_0) - (s - m^2_j + m^2 + t_0) \frac{t_0}{t},
\end{align*}
$$

(10)

with $t_0 = t - 4m^2$, $m$ the positron mass, and $m_i$ put equal to zero for reasons to be made apparent.

In this paper, we shall be interested mainly in unpolarized cross sections which means that we shall need to evaluate sums of the absolute values squared of the amplitudes (6) over all the spins $s, r, s', r'$. These spin-sums, as was explained in [17], are not so readily performed as usual by the standard method of taking traces of $\gamma$-matrices because of the crossed terms between channels labelled by different $j$’s obtained in squaring (8). We shall therefore follow the tactics adopted in [17] of explicitly performing the spin-sums in a specific Lorentz frame with a specific representation of the $\gamma$-matrices. As in [17], we choose to work in the cm frame of the channel $i = 3$ which in our convention denotes the mass eigenstate with the lowest mass $m_3$, which, being in all cases considered at most of the order of the electron mass and therefore negligible, is put equal to zero. This gives:

$$
\begin{align*}
\mathbf{q}\mu &= (E, 0, 0, \omega), \\
\mathbf{q}'\mu &= (E, 0, \omega \sin \theta'_3, \omega \cos \theta'_3), \\
\mathbf{p}'_3 &= (\omega, 0, 0, -\omega),
\end{align*}
$$
\[
p_3' = (\omega, 0, -\omega \sin \theta_3', -\omega \cos \theta_3'), \\
p_j' = (E_j, 0, -\omega_j \sin \theta_j, -\omega_j \cos \theta_j), \\
p_j'' = (E_j, 0, -\omega_j \sin \theta_j', -\omega_j \cos \theta_j'), \tag{11}
\]

with
\[
E = \frac{s + m^2}{2\sqrt{s}}, \\
\omega = \frac{s - m^2}{2\sqrt{s}}, \\
\cos \theta_3' = 1 + \frac{2st}{(s - m^2)^2}. \tag{12}
\]

Further, from (9)–(11), one obtains
\[
E_j = \frac{1}{2\sqrt{s}t_0} \left\{ 2\sqrt{[(s - m^2) + st][s - m_j^2 + m^2 + st_0]} - (s + m^2)[2(s - m^2 - m_j^2) + t] \right\}, \tag{13}
\]

with
\[
\omega_j = \sqrt{E_j^2 - m_j^2}, \tag{14}
\]

and
\[
\sin \theta_j = \frac{-\sqrt{-st}}{2st_0\omega_j} \left\{ (s + m^2)\sqrt{(s - m_j^2 + m^2)^2 + st_0} - (s - m_j^2 + m^2)\sqrt{(s - m^2)^2 + st} \right\}, \\
\sin \theta_j' = \frac{-\sqrt{-st}}{2st_0\omega_j} \left\{ (s + m^2)\sqrt{(s - m_j^2 + m^2)^2 + st_0} - (s - m_j^2 + m^2 + t_0)\sqrt{(s - m^2)^2 + st} \right\}, \tag{15}
\]

with
\[
\theta_3' = \theta_j + \theta_j'. \tag{16}
\]

These formulae (12)–(15), together with the formulae in the last paragraph, all reduce to the corresponding formulae derived in [17] for photo-transmutation if we put the mass \( m \) of the e\(^+\) in (7) equal to zero, which will indeed be a very good approximation in most applications. We have kept the dependence
on \( m \) explicit only for the sake of generality in case the formulae are to be applied in future to other circumstances, such as lepton transmutations in \( \mu^+ \mu^- \) collisions.

We choose again for \( \gamma \)-matrices the Pauli–Dirac representation:

\[
\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}.
\] (17)

The spins of the incoming and outgoing leptons \( j \) we quantize along the direction \( p_3 \) and \( p'_3 \) respectively, while the spins of the \( e^+ \), whether incoming or outgoing, we quantize along its direction of motion. With these specifications, the wave functions of the leptons \( j \) are given by:

\[
u_+(q) = \frac{1}{\sqrt{2(E + m)}} \begin{pmatrix} 0 \\ -\omega \\ 0 \end{pmatrix},
\]

\[
u_-(q) = \frac{1}{\sqrt{2(E + m)}} \begin{pmatrix} 0 \\ -\omega \\ E + m \end{pmatrix},
\]

while for the \( e^+ \), we have the wave functions:

\[
u_+(q) = \frac{1}{\sqrt{2(E + m)}} \begin{pmatrix} (E + m)(1 - e^{-i\theta_j}) \\ (E + m)(1 + e^{-i\theta_j}) \\ \omega_j(e^{i\theta_j} - e^{-i\theta_j}) \\ \omega_j(e^{i\theta_j} + e^{-i\theta_j}) \end{pmatrix},
\]

\[
u_-(q) = \frac{1}{\sqrt{2(E + m)}} \begin{pmatrix} (E + m)(1 + e^{-i\theta_j}) \\ (E + m)(1 - e^{-i\theta_j}) \\ -\omega_j(e^{i\theta_j} + e^{-i\theta_j}) \\ -\omega_j(e^{i\theta_j} - e^{-i\theta_j}) \end{pmatrix};
\] (19)
\[ v_-(q) = \frac{1}{\sqrt{2(E + m)}} \begin{pmatrix} \omega & 0 \\ 0 & E + m \end{pmatrix}, \]  \hspace{1cm} (20)

and:

\[ v_+(q') = \frac{1}{2\sqrt{2(E + m)}} \begin{pmatrix} -\omega(1 - e^{-i\theta_3'}) & -\omega(1 + e^{-i\theta_3'}) \\ (E + m)(1 - e^{-i\theta_3'}) & (E + m)(1 + e^{-i\theta_3'}) \end{pmatrix}, \]
\[ v_-(q') = \frac{1}{2\sqrt{2(E + m)}} \begin{pmatrix} \omega(1 + e^{-i\theta_3'}) & \omega(1 - e^{-i\theta_3'}) \\ (E + m)(1 + e^{-i\theta_3'}) & (E + m)(1 - e^{-i\theta_3'}) \end{pmatrix}. \]  \hspace{1cm} (21)

With the wave functions in (18)–(21), it is straightforward to evaluate the diagonal amplitudes in (8). We obtain the following:

\[
(M_{j}^{(a)})_{++} = (M_{j}^{(a)})_{--} = \frac{-ie^2}{t} \left\{ \frac{E}{4}(1 + \cos \theta_3')[(E_j + m_j) + (E_j - m_j)e^{-2i\theta_j}] + \omega \omega_j (1 - \cos \theta_3') e^{-i\theta_j} + \frac{\omega \omega_j}{2}(1 + \cos \theta_3') e^{-i\theta_j} \right\},
\]

\[
(M_{j}^{(a)})_{+-} = (M_{j}^{(a)})_{-+} = \frac{-ie^2}{t}(1 + \cos \theta_3') \left\{ \frac{E}{4}[(E_j + m_j) + (E_j - m_j)e^{-2i\theta_j}] + \frac{\omega \omega_j}{2} e^{-i\theta_j} \right\},
\]

\[
(M_{j}^{(a)})_{++} = (M_{j}^{(a)})_{-+} = (M_{j}^{(a)})_{+-} = (M_{j}^{(a)})_{-+} = \frac{-ie^2 m}{t} (1 - \cos \theta_3') \left\{ (E_j + m_j) - (E_j - m_j)e^{-2i\theta_j} \right\},
\]

\[
(M_{j}^{(a)})_{++} = (M_{j}^{(a)})_{--} = (M_{j}^{(a)})_{-+} = (M_{j}^{(a)})_{-+} = \frac{e^2 m}{t} \sin \theta_3 \left\{ (E_j + m_j) + (E_j - m_j)e^{-2i\theta_j} \right\},
\]

\[
(M_{j}^{(a)})_{++} = (M_{j}^{(a)})_{+-} = (M_{j}^{(a)})_{-+} = (M_{j}^{(a)})_{-+} = -\frac{e^2 E}{t} \sin \theta_3' \left\{ (E_j + m_j) - (E_j - m_j)e^{-2i\theta_j} \right\}, \hspace{1cm} (22)
\]
where subscripts denote the spins of the leptons \( j \) and superscripts the spins of the \( e^+ \), with the right index pertaining to the incoming and the left index to the outgoing particle. Using the formulae derived earlier in (12)–(15), these amplitudes can then be expressed in terms of the Mandelstam invariants \( s \) and \( t \) as desired.

### 3 The Reaction Amplitude (b)

Turning next to the diagram of Figure 3(b) which contributes to all three reactions (2)–(4), we proceed in a similar manner. The transmutational amplitude for the reaction:

\[
e^+e^- \longrightarrow \ell^+_\alpha \ell^-_{\beta},
\]

is given by a rotation in generation space, thus:

\[
\mathcal{M}^{(b)} = \sum_j S_{\alpha j} S_{\beta j}^\dagger \mathcal{M}_j^{(b)},
\]

from the diagonal amplitudes \( \mathcal{M}_j^{(b)} \) depicted in Figure 3(b) for the reaction:

\[
e^+e^- \longrightarrow \ell^+_j \ell^-_j
\]

where

\[
(M_j^{(b)})_{s's}^{r'r} = ie^2[\bar{u}_{s'}(p'_j)\gamma\mu v_{r'}(q'_j)]\frac{1}{(p+q)^2}[\bar{v}_r(q)\gamma\mu u_s(p)].
\]

We work now in the cm of the incoming \( e^+ \) and \( e^- \) system with

\[
q^\mu = (E, 0, 0, \omega),
\]

\[
p^\mu = (E, 0, 0, -\omega),
\]

\[
p_{j}^{\mu} = (E_j, 0, -\omega_j \sin \theta'_j, -\omega_j \cos \theta'_j),
\]

\[
q_{j}^{\mu} = (E_j, 0, \omega_j \sin \theta'_j, \omega_j \cos \theta'_j),
\]

and

\[
E = E_j = \sqrt{s}/2,
\]

\[
\omega = \sqrt{E^2 - m^2}; \quad \omega_j = \sqrt{E_j^2 - m_j^2},
\]

\[
\cos \theta'_3 = 1 + \frac{t}{2E^2},
\]

\[
\cos \theta'_j = \frac{t - m^2 - m_j^2 + 2E^2}{2\omega\omega_j}.
\]
Further, with the spins of the outgoing particles $\ell^+_j, \ell^-_j$ quantized along $q'_3$ and $p'_3$ respectively, and those of the incoming $e^+, e^-$ along their directions of motion $q$ and $p$ respectively, the wave functions are given by:

\[
\begin{align*}
\mathbf{u}^+ (p) &= \frac{1}{\sqrt{2(E + m)}} \begin{pmatrix}
0 \\
E + m \\
0 \\
\omega
\end{pmatrix}, \\
\mathbf{u}^- (p) &= \frac{1}{\sqrt{2(E + m)}} \begin{pmatrix}
E + m \\
0 \\
-\omega \\
0
\end{pmatrix}, \\
\mathbf{v}^+ (q) &= \frac{1}{\sqrt{2(E + m)}} \begin{pmatrix}
0 \\
-\omega \\
0 \\
E + m
\end{pmatrix}, \\
\mathbf{v}^- (q) &= \frac{1}{\sqrt{2(E + m)}} \begin{pmatrix}
\omega \\
0 \\
E + m \\
0
\end{pmatrix},
\end{align*}
\]

and

\[
\begin{align*}
\mathbf{u}^+ (p'_j) &= \frac{1}{2\sqrt{2(E_j + m_j)}} \begin{pmatrix}
(E_j + m_j)(1 - e^{-i\theta'_j}) \\
(E_j + m_j)(1 + e^{-i\theta'_j}) \\
\omega_j(1 - e^{-i\theta'_j}) \\
\omega_j(1 + e^{-i\theta'_j})
\end{pmatrix}, \\
\mathbf{u}^- (p'_j) &= \frac{1}{2\sqrt{2(E_j + m_j)}} \begin{pmatrix}
(E_j + m_j)(1 + e^{-i\theta'_j}) \\
(E_j + m_j)(1 - e^{-i\theta'_j}) \\
-\omega_j(1 + e^{-i\theta'_j}) \\
-\omega_j(1 - e^{-i\theta'_j})
\end{pmatrix}, \\
\mathbf{v}^+ (q'_j) &= \frac{1}{2\sqrt{2(E_j + m_j)}} \begin{pmatrix}
-\omega_j(1 - e^{-i\theta'_j}) \\
-\omega_j(1 + e^{-i\theta'_j}) \\
(E_j + m_j)(1 - e^{-i\theta'_j}) \\
(E_j + m_j)(1 + e^{-i\theta'_j})
\end{pmatrix},
\end{align*}
\]

(29)
\[ v_-(q'_j) = \frac{1}{2 \sqrt{2(E_j + m_j)}} \begin{pmatrix}
\omega_j(1 + e^{-i\theta'_j}) \\
\omega_j(1 - e^{-i\theta'_j}) \\
(E_j + m_j)(1 + e^{-i\theta'_j}) \\
(E_j + m_j)(1 - e^{-i\theta'_j})
\end{pmatrix}. \]

Hence, one obtains the amplitudes:

\[
\begin{align*}
(M_j^{(b)})_{++} & = (M_j^{(b)})_{--} = -\frac{ie^2}{s} mm_j \cos \theta'_j, \\
(M_j^{(b)})_{+-} & = (M_j^{(b)})_{-+} = \frac{ie^2}{s} EE_j(1 + \cos \theta'_j), \\
(M_j^{(b)})_{+-} & = (M_j^{(b)})_{-+} = \frac{e^2}{s} mm_j \cos \theta'_j, \\
(M_j^{(b)})_{-+} & = (M_j^{(b)})_{+-} = \frac{e^2}{s} Em_j \sin \theta'_j, \\
(M_j^{(b)})_{++} & = (M_j^{(b)})_{--} = -\frac{e^2}{s} Em_j \sin \theta'_j, \\
(M_j^{(b)})_{+-} & = (M_j^{(b)})_{-+} = \frac{e^2}{s} mE_j \sin \theta'_j, \\
(M_j^{(b)})_{-+} & = (M_j^{(b)})_{+-} = -\frac{e^2}{s} mE_j \sin \theta'_j,
\end{align*}
\]

which again can all be expressed in terms of the Mandelstam invariants \( s \) and \( t \) by means of the formulae in (28).

4 Spin-summed Differential Cross Sections

Substituting the diagonal amplitudes in (22) and (31) into respectively the formulae (6) and (24) and adding the two contributions, one obtains the spin-amplitudes for the actual transmutation reaction (2), (3) and (4). Hence, taking the absolute values squared of these amplitudes and summing over all spins, one obtains the spin-summed differential cross sections as desired:

\[
\frac{d\sigma}{d\Omega} = \frac{1}{4\pi^2} \frac{1}{s} \omega' \frac{1}{4} \sum_{r,r',s,s'} |(M_j^{(b)})_{r'r}^{r's}|^2 \times 0.3894 \text{ mb/sr},
\]

which is the desired result.

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for energies measured in GeV, where $\omega$ and $\omega'$ are respectively the cm momenta of the actual incoming and outgoing particles in the transmutation reaction. Although this is in principle straightforward, a few practical observations are in order.

First, in performing the sum in (6) and (24) over the diagonal state $s_j$, it is convenient to make use of the unitary property of the rotation matrix $S_{\alpha j}$ to write, for $\alpha \neq \beta$:

$$
\sum_j S_{\alpha j} S^\dagger_{\beta j} M_j = S_{\alpha 1} S^\dagger_{\beta 1} [M_1 - M_3],
$$

(33)

which, as explained in [17], is a good approximation whenever the mass eigenvalues $m_j$ are hierarchical and avoids the need to know the rotation matrix to unreasonably high accuracy. Besides, as we shall see, it gives us a clearer picture of how transmutation cross sections behave as functions of the Mandelstam invariants $s$ and $t$.

Second, as can be seen in (33), the transmutation amplitude being proportional to $M_1 - M_3$ with the two amplitudes differing just by the mass values, i.e., whether $m_1$ or $m_3$, the cross section for transmutation is at most of order $m_2^2/s$ and decreases rapidly with increasing energy. For high $s$, therefore, $M_1$ and $M_3$ will largely cancel leading potentially to inaccuracy in a direct calculation with the formula (33). Indeed, this was exactly what we found in our actual calculations, especially in the amplitude (a) where other large cancellations occur in the exact formulae. This computational difficulty can be avoided just by expanding the amplitudes to order $m_2^2/s$ giving:

$$
(M^{(a)})^{++} = (M^{(a)})^{--} = \frac{ie^2}{t} \left\{ \frac{m_1^2}{2s}(s - t) - im_1 \sqrt{-t} \right\},
$$

$$
(M^{(a)})^{+-} = (M^{(a)})^{-+} = \frac{ie^2}{t} \left\{ \frac{m_1}{2s} - i \frac{m_1}{s} \sqrt{-t} \right\},
$$

$$
(M^{(a)})^{++} = (M^{(a)})^{+-} = (M^{(a)})^{+-} = (M^{(a)})^{--} = \frac{e^2}{2} \sqrt{\frac{s + t}{-t}} \left\{ \frac{m_1}{\sqrt{s}} - i \frac{m_1}{s} \sqrt{-t} \right\},
$$

(34)

and all other components zero, where we have also neglected terms of the order of the electron mass $m$. This approximation is already very good by $\sqrt{s}$
of order 10 GeV, at least near the forward direction where the amplitudes are large, and becomes eventually necessary above this energy for computations without double precision. In Figure 4 is shown the spin-summed differential cross sections for the reaction $e^+e^- \rightarrow e^+\tau^-$ at $\sqrt{s} = 10$ GeV, 100 GeV calculated with the rotation matrix element $S_{\alpha_1}$ of the DSM scheme taken from ref. [16, 20]. The curve at 10 GeV is calculated with the exact formulae (22) which is seen to be almost indistinguishable from the crosses calculated with the approximate formulae (34). The curve at 100 GeV is calculated with (34) where the exact formulae is found to have problems with accuracy in application.

Third, we note that the two sets of diagonal amplitudes (22) and (31) were each calculated in a particular Lorentz frame, namely for the diagram (a) in the cm frame of the $e^+\ell_3$ system and for the diagram (b) in the cm frame of the incoming $e^+e^-$ system. Although the amplitudes were all converted in the end into functions of the invariants $s$ and $t$, the directions of spin quantization are still frame-dependent. Hence, strictly speaking, the two frames for (a) and (b) being different, the two respective spin-amplitudes could not be added in the manner that we have done above. However, the electron mass $m$ is so small compared to the energies we are interested in that this difference in frame is entirely negligible for practical purposes. Were the present formalism to be adapted in future to say $\mu^+\mu^-$ collisions, then this would be a point to be borne in mind.

With these points clarified, we have not encountered any more practical difficulties in computing the cross sections of the three transmutation reactions (2)–(4). Rather than presenting our results in a wide range of $s$ and $t$, which could be confusing, we shall instead first give here a description of the general features, and then in the next section a detailed report on the result at $\sqrt{s} = 10.58$ GeV, namely at the $\Upsilon(4S)$ where BaBar [21], Belle [22], and Cleo [23] have already collected a massive amount of data, in principle ready to be confronted with the above predictions.

Consider first the reactions (2) and (3) which receive contributions from both diagrams (a) and (b), and are very similar except for the difference in normalization due to the different values of the rotation matrix elements $S_{\alpha_1}$. As in ordinary Bhabha scattering, the amplitude (a) is dominated by the pole at $t = 0$ which gives the cross section a sharp forward peak, as can be seen in the examples of Figure 4. Except at large scattering angles where $t$ is of order $s$, this peak overshadows the contribution from the (b) diagram. However, the forward peak for the transmutation reactions (2) and (3) is nowhere near as sharp as for ordinary Bhabha scattering, as can be seen in
Figure 4: Spin-summed differential cross sections for the reaction $e^+e^-\rightarrow e^+\tau^-$ at $\sqrt{s} = 10$ GeV, 100 GeV. For details see text.
Figure 5: The ratio at 10 GeV of the cross section of reaction $e^+e^- \rightarrow e^+\tau^-$ over that of ordinary Bhabha scattering $e^+e^- \rightarrow e^+e^-$ as a function of $t$.

Figure 5. The reason for this difference is seen in (34), where one notices that the normally dominant spin non-flip amplitudes with $1/t$ behaviour are of order $m_1^2/s$, while the spin flip with a weaker $1/\sqrt{-t}$ behaviour are of order $m_1/\sqrt{s}$. The same formulae (34) explains also the sharp decline of the cross section with increasing energy as well as its change in $t$-dependence as the spin flip terms become ever more dominant, both of which effects can be seen in Figure 4 by comparing the curves at $\sqrt{s} = 10$ and 100 GeV.

The other reaction (4) receives contributions only from the (b) diagram which has no peak in the forward direction. It is distinguished from the same diagram in ordinary Bhabha scattering by the fact that, like the (a) transmutation amplitude, it is also dominated by the spin flip terms at high energy. Without the sharp singular peak in the forward direction, it gives, in contrast to reactions (2) and (3), a finite total cross section, the rough energy dependence of which is shown in Figure 4, where one sees that, as in photo-transmutation [17], the cross section rises shortly after threshold to a
Figure 6: Cross section for the reaction $e^+e^- \rightarrow \mu^+\tau^-$ integrated over all scattering angles as a function of $\sqrt{s}$.

peak and then declines as $\sqrt{s}$ increases.

We note that since the rotation matrix elements $S_{\alpha j}$ enter only in the normalization of the cross section and are themselves only weakly dependent on energy, their actual values do not affect the discussion above on the the $t$-dependence, and qualitatively also on the $s$-dependence, of the cross section.

5 Transmutation Cross Sections at $\sqrt{s} = 10.58$ GeV

The reason for selecting this particular energy corresponding to the mass of the $\Upsilon(4S)$ for detailed analysis is that two experiments of ultra-high sensitivity, namely BaBar and Belle, are running and have collected already up to 20 fb$^{-1}$ of luminosity, with much more expected in the near future $^{21,22}$. Although these experiments were designed originally to look for other rare
effects like CP-violation from $B$ decay, their data could conveniently be used also to search for the transmutation reactions of interest to us here.

Consider first the reaction (4) with contributions only from Figure 1(b). The normalization of the cross section is, according to (33), given by the rotation matrix elements $S_{\mu_1}S_{\tau_1}$. If one accepts the contention of [15] that it is the rotation which is giving rise to fermion mixing and mass hierarchy, then these rotation elements should be related to the data on the mixing parameters and mass ratios. Indeed, as explained in that paper, the quantity $S_{\mu_1}S_{\tau_1}$ would then be given to a good approximation by $\sin \theta \cos \theta$ where $\theta$ is the rotation angle from the $\tau$-mass scale where the mass matrix is diagonal in the lepton flavour states to the scale 10.58 GeV of say the BaBar experiment. Its value can thus be read off from Figure 3 of [16], either directly by interpolating the actual data points or equivalently, since the rotation curve from the DSM scheme is seen there to be a very good fit to the data, by taking the values from the DSM calculation, in either case leading to an estimate $S_{\mu_1}S_{\tau_1} \sim 0.043$. With this then for the normalization, one obtains Figure 7 for the spin-summed differential cross section of reaction (4). Integrating over the whole angular range, one obtains a cross section of around 80 fb. This can also be read in Figure 6 which was in fact calculated already with this normalization. In practical terms, this could mean as many as 1600 events in the data sample of 20 fb$^{-1}$ already collected by BaBar, assuming 100 percent efficiency.

Consider next the reaction (2) which receives contribution from both diagrams in Figure 1. Its normalization depends now instead on the rotation matrix elements $S_{e_1}S_{\tau_1}$ which is considerably smaller and harder to estimate directly from data. Let us then just insert the values obtained from the DSM scheme [20] which was shown [15] to fit the data very well and can be taken as a convenient interpolation device. The actual value can be read in Figure 4 of [17] at 10.58 GeV to be around 0.0092. This then gives Figure 8 for the spin-summed differential cross section for this reaction. As in ordinary Bhabha scattering, the cross section for (2) is divergent at $t = 0$. However, this region cannot be explored experimentally, the detectors being insensitive to the forward region with $|t| \lesssim 5$ GeV$^2$ [21]. A rough estimate from Figure 8 then yields for the integrated cross section for reaction (2) over the range $|t| < 5$ GeV$^2$ a value of about 20 fb, which means about 400 events in the sample of 20 fb$^{-1}$ already collected in last year’s run assuming again 100 percent detection efficiency. Compared to reaction (4) above, we note the very different angular dependence which is here dominated by the diagram (a) of Figure 1 absent in reaction (4). The reason why the cross section for (2)
Figure 7: Spin-summed differential cross section for the reaction $e^+ e^- \rightarrow \mu^+ \tau^-$ at $\sqrt{s} = 10.58$ GeV. For the normalization, see text.
is smaller than that for (4) despite the fact that (4) receives contribution only from the sub-dominant diagram (b) is the very different sizes of the rotation matrix elements involved, \((S_{\mu 1}/S_{e1})^2\) being \(\sim 20\), as read from Figure 4 in [17].

However, we should perhaps stress once more that the Figures 7 and 8 calculated by the method of [17] represent only the kinematic effects of the rotating mass matrix. In particular, though calculated with DSM rotation matrices, the transmutation cross sections given are not those predicted by the DSM scheme in which, as seen in [19], the above kinematic effects are largely cancelled by other rotation effects implied concurrently by the renormalization mechanism driving the mass matrix rotation.

Apart from minor kinematic differences, the spin-summed differential cross section for the reaction (3) has the same \(t\)-dependence as (2) but a different normalization, namely with \(S_{\tau 1}\) in (2) replaced by \(S_{\mu 1}\) in (3). For instance, taking again the rotation matrix elements from Figure 4 of [17], one obtains a cross section for (3) a factor \(\sim 2 \times 10^{-3}\) smaller than for (2), making it probably difficult in any case to observe in the near future.

For all 3 reactions, there is in principle also a contribution from the transmutational decay of the \(\Upsilon(4S)\) resonance with this mass, but this will be seen in the following section to be negligible in comparison with the above contributions.

From the above results, it would appear that if one accepts the interpretation as given in [15] of fermion mixing and mass hierarchy as rotation effects, and that there are no other rotation effects than that of the mass matrix, then there will be lepton flavour-violating transmutation effects in \(e^+e^-\) collisions at BaBar, Belle, and Cleo energy, which are of a magnitude to be observable by these experiments. On the other hand, if the interpretation of [15] is not accepted, one knows at present of no other empirical means for estimating the rotation angles and thus no estimate for the absolute rates of transmutation can yet be made. However, since the rotation matrix elements enter only in the normalization of the cross section, the calculation above is easily adaptable to any rotation matrix obtained from whatever source, whether empirical or theoretical.

Suppose that the reactions (4) and (2) are indeed observed, can one be sure that they are due to transmutation and not some other lepton-violating effect? The answer to this question would seem to be quite affirmative since one has here the differential cross sections as functions of 2 variables, each with distinctive characteristics. For example, it is predicted that the cross section for (2) should be peaked sharply forwards as seen in Figure 5, while for
Figure 8: Spin-summed differential cross section for the reaction $e^+e^- \rightarrow e^+\tau^-$ at $\sqrt{s} = 10.58$ GeV. For the normalization see text.
it should have a roughly $\sin^2 \theta$ behaviour, as seen in Figure [7]. And both are predicted to be spin-flip dominated, which assertion may be verifiable with the decaying $\tau$ serving as its own spin-analyser and the spin-dependent cross section calculable from the amplitudes (22) and (24) when the occasion demands. Thus, if the reactions are observed at all with any reasonable statistics, the signatures for transmutation would seem to be quite unmistakable.

We have restricted the discussion in this section to only the operation energy of BaBar, Belle and Cleo, but very similar remarks apply also to the BEPC energy [24]. The expected transmutation cross sections at BEPC, as seen in Figure 3 for reaction (4), are even larger, but this advantage is unfortunately more than offset by the lower luminosity so far achieved by the machine. For this reason, for BEPC, we think a search for transmutation in $\psi$ decay will be more immediately profitable, as we shall elucidate in the following section. As for LEP, according to Figure 4 for example, transmutation cross sections would have fallen much below the fb level by that energy even for the optimistic scenario of [13] and are thus unlikely to show up in the data collected.

6 Vector Boson Decay

Vector boson formation and decay occur naturally in $e^+e^-$ collision so that they will have in any case to be taken into account in studying transmutation from this process. Besides, being essentially a single particle effect, vector boson decays are easier to analyse theoretically and can give more succinct conclusions than the reactions previously considered. We propose therefore to examine in this section the following transmutation decays of vector bosons into leptonic final states:

$$V \rightarrow \ell_\alpha \bar{\ell}_\beta,$$

(35)

for $\ell_\alpha (\bar{\ell}_\beta)$ being $e, \mu$ or $\tau$. Order of magnitude estimates for some of these decays have already been made in [16]. Now, with the procedure developed in [17] an explicit calculation can be carried out.

Consider then in general a decay of the type (35) as depicted in Figure 2(a). The decay amplitude is to be evaluated at the scale $\sqrt{s} = M$, with $M$ being the mass of the decaying boson. At this scale, the fermion mass matrix $m$ is in general not diagonal in the flavour states $e, \mu, \tau$ but in some other states, say, $i = 1, 2, 3$ with masses (eigenvalues of $m$) $m_i$, the two triads of state vectors being related by a rotation matrix $S_{ai} = \langle \alpha | i \rangle$. To calculate
the decay amplitude, we first evaluate the amplitudes for the decay into the diagonal states $i$, namely $V \rightarrow \ell_i \bar{\ell}_i$ as depicted in Figure 2(b). For a vector boson with polarization vector $\epsilon_{\alpha}^\mu$, this amplitude is:

$$M_i = g \epsilon_{\alpha}^\mu(k) \bar{u}_r(p_i) \gamma^\mu v_s(q_i),$$

(36)

where $r$ and $s$ denote the spin respectively of $\ell_i$ and $\bar{\ell}_i$. We do not need to specify the coupling strength $g$ of the vector boson to the lepton pair for in our branching ratios calculation it will be cancelled out. According to the procedure suggested in [17], the amplitude for the decay of interest (35) is then given just by a rotation of these diagonal amplitudes $M_i$ to the appropriate lepton flavour states, thus:

$$M_{\alpha\beta} = \sum_i S_{\alpha i}^\dagger M_i S_{\beta i}.$$  

(37)

To obtain the decay rate, with which alone we shall be concerned at present, we average over the initial polarization $a$ of the decaying boson and sum over the final spins $r, s$ of the product leptons, obtaining:

$$\frac{1}{3} \sum_{a, r, s} |M_{\alpha\beta}^{a\mu}|^2 = \frac{1}{3} g^{\mu\nu} \sum_{r, s} M_{\alpha \beta}^{\mu rs}(M_{\alpha \beta}^{\nu rs})^\dagger + \frac{1}{3} M^2 \sum_{r, s} |M_{\alpha \beta}^{(k) rs}|^2,$$

(38)

where

$$M_{\alpha \beta}^{\mu rs} = g \sum_i S_{\alpha i}^\dagger \bar{u}_r(p_i) \gamma^\mu v_s(q_i) S_{\beta i},$$

(39)

$$M_{\alpha \beta}^{(k) rs} = g \sum_i S_{\alpha i}^\dagger \bar{u}_r(p_i) k^\mu v_s(q_i) S_{\beta i}.$$  

(40)

The remaining sum over lepton spins again cannot readily be done as an invariant trace because of the crossed terms between different internal channels $i$. We proceed as for the reactions (2) — (4) above.

We choose to work in the rest frame of the decaying vector boson:

$$k = p_i + q_i = 0,$$

(41)

so that:

$$p_i^\mu = (E, 0, 0, -\omega_i),$$

$$q_i^\mu = (E, 0, 0, \omega_i),$$

$$k^\mu = (M, 0, 0, 0),$$

(42)
with
\[ E = M/2, \quad \omega_i = \sqrt{E^2 - m_i^2}. \tag{43} \]

As a result, we have
\[
\begin{align*}
u_+(p_i) &= \frac{1}{\sqrt{2(E + m_i)}} \begin{pmatrix} 0 \\ E + m_i \\ 0 \end{pmatrix}; \\
u_-(p_i) &= \frac{1}{\sqrt{2(E + m_i)}} \begin{pmatrix} 0 \\ -\omega_i \\ 0 \end{pmatrix}; \\
u_+(q_i) &= \frac{1}{\sqrt{2(E + m_i)}} \begin{pmatrix} 0 \\ -\omega_i \\ 0 \end{pmatrix}; \\
u_-(q_i) &= \frac{1}{\sqrt{2(E + m_i)}} \begin{pmatrix} E + m_i \\ 0 \\ 0 \end{pmatrix}. \tag{44}
\end{align*}
\]

With these explicit expressions for the lepton wave functions, it is easy to evaluate the spin amplitudes and perform the sum over the lepton spins giving:
\[
\frac{1}{3} \sum_{a,r,s} |M_{ars}^{\alpha\beta}|^2 = \frac{4g^2}{3} \delta_{\alpha\beta} E^2 + \frac{2g^2}{3} \sum_{i,j} S_{\alpha i}^\dagger S_{\beta j} m_i m_j S_{\beta j}^\dagger S_{\alpha i}. \tag{45}
\]

The states \( i \) being by definition the eigenstates of the mass matrix \( m \) at the scale of the decaying boson mass \( M \), and \( S_{\alpha i} \) the rotation matrix relating these states \( i \) to the lepton flavour state \( \alpha = e, \mu, \) or \( \tau \), the sum \( \sum_{i,j} S_{\alpha i}^\dagger S_{\beta j} m_i \) is just the element \( \langle \alpha|m|\beta \rangle \) of the matrix \( m \) at the boson mass scale. Hence,
\[
\frac{1}{3} \sum_{a,r,s} |M_{ars}^{\alpha\beta}|^2 = \frac{4g^2}{3} \delta_{\alpha\beta} E^2 + \frac{2g^2}{3} |\langle \alpha|m|\beta \rangle|^2. \tag{46}
\]

This gives the total width for the decay \( V \rightarrow \ell_\alpha \bar{\ell}_\beta \) as:
\[
\Gamma = g^2 \frac{1}{\pi} \frac{\omega_{\alpha\beta}^2}{4M E_\alpha E_\beta} \frac{1}{3} \left[ M^2 \delta_{\alpha\beta} + 2|\langle \alpha|m|\beta \rangle|^2 \right], \tag{47}
\]
where we note that the phase space factor \( \omega_{\alpha\beta}^2/E_\alpha E_\beta \) refers as per [17] to the freely propagating “external” leptons, with momentum and energy respectively:
\[
\begin{align*}
\omega_{\alpha\beta} &= \frac{1}{2M} \sqrt{M^4 + m_\alpha^4 + m_\beta^4 - 2M^2 m_\alpha^2 m_\beta^2 - 2M^2 m_\alpha^2 - 2M^2 m_\beta^2}, \\
E_\alpha &= \frac{1}{2M} (M^2 + m_\alpha^2 - m_\beta^2), \\
E_\beta &= \frac{1}{2M} (M^2 - m_\alpha^2 + m_\beta^2). \tag{48}
\end{align*}
\]
More conveniently, since the coupling $g$ has not been specified, one writes for \( \alpha \neq \beta \), i.e. transmutational decays:

\[
\frac{\Gamma(V \rightarrow \ell_\alpha \bar{\ell}_\beta)}{\Gamma(V \rightarrow e^+e^-)} = \frac{\omega_{\alpha\beta}^2}{E_\alpha E_\beta} \frac{2}{M^2} |\langle \alpha | m | \beta \rangle|^2, \tag{49}
\]

where we have neglected terms of the order of the electron mass $m_e$ compared to $M$. Multiplying then this ratio by the experimental branching ratio, if known, of the boson $V$ decaying into $e^+e^-$ gives the branching ratio of the transmutational $\ell_\alpha \bar{\ell}_\beta$ mode.

One notices that apart from a numerical factor of 2 and the phase space factor $\omega_{\alpha\beta}^2/E_\alpha E_\beta$, the formula for the branching ratio (49) for transmutational decays is the same as the order-of-magnitude estimate given in [16]. This formula is supposed to hold in general and can be applied to calculate the branching ratio of transmutational decays of the type (35) given the matrix element $\langle \alpha | m | \beta \rangle$ at scale $M$.

For a numerical example, we take again the interpretation in [15] of experimental data to estimate the rotation matrix elements required or, in case the data is insufficient, employ the DSM result of [20] as an interpolation formula, for which the required mass matrix elements $\langle \alpha | m | \beta \rangle$ can be read directly from Figure 3 of [16]. With these elements, the formula (49) can be applied immediately to calculate the branching ratios of transmutational modes in the decay of any vector boson by normalizing to the empirical branching ratios of the $e^+e^-$ mode (or to the $\mu^+\mu^-$ mode if more accurate) given in [14].

The result for the most experimentally interesting vector bosons listed in [14] is given in Table 1. (Again, the reader is reminded that these numbers represent only the transmutation effects due to kinematics of the rotating mass matrix alone but not, though calculated with DSM rotation matrices, predictions of the DSM scheme in which cancellations with other rotational effects occur [19].) We have calculated also the branching ratios for the other vector bosons listed in [14], such as the higher excitations of $\psi$, but for these, once the mass gets above the $D\bar{D}$ threshold, hadronic decays prevail, leading to large total widths and hence uninterestingly small branching ratios for the modes that concern us here, as can be seen in the example given of $\psi(3770)$. In any case, the predicted branching ratios for all higher excitations of $\psi$ and $\Upsilon$ are very similar to those given in Table 1 for $\psi(3770)$ and $\Upsilon(10860)$ and are therefore not given again there.

In Table 1, we note first that the predicted branching ratios for flavour-violating transmutational decays even for the fast rotation implied by [13] are not as large as one might fear at first sight. Indeed, all the estimates survive
| Boson     | Mode | Predicted Branching Ratio |
|-----------|------|---------------------------|
| $\phi(1020)$ | $e\mu$ | $2.5 \times 10^{-12}$ |
| $\psi(1S)$  | $\mu\tau$ | $6.3 \times 10^{-6}$ |
|            | $e\tau$    | $1.7 \times 10^{-7}$ |
|            | $e\mu$     | $1.1 \times 10^{-10}$ |
| $\psi(2S)$  | $\mu\tau$ | $1.2 \times 10^{-6}$ |
|            | $e\tau$    | $3.8 \times 10^{-8}$ |
|            | $e\mu$     | $3.1 \times 10^{-11}$ |
| $\psi(3770)$ | $\mu\tau$ | $1.7 \times 10^{-9}$ |
|            | $e\tau$    | $5.3 \times 10^{-11}$ |
|            | $e\mu$     | $4.3 \times 10^{-14}$ |
| $\Upsilon(1S)$ | $\mu\tau$ | $2.9 \times 10^{-6}$ |
|            | $e\tau$    | $1.4 \times 10^{-7}$ |
|            | $e\mu$     | $2.6 \times 10^{-10}$ |
| $\Upsilon(2S)$ | $\mu\tau$ | $1.3 \times 10^{-6}$ |
|            | $e\tau$    | $6.2 \times 10^{-8}$ |
|            | $e\mu$     | $1.2 \times 10^{-10}$ |
| $\Upsilon(3S)$ | $\mu\tau$ | $1.9 \times 10^{-7}$ |
|            | $e\tau$    | $9.3 \times 10^{-9}$ |
|            | $e\mu$     | $1.8 \times 10^{-11}$ |
| $\Upsilon(4S)$ | $\mu\tau$ | $2.8 \times 10^{-9}$ |
|            | $e\tau$    | $1.4 \times 10^{-10}$ |
|            | $e\mu$     | $2.8 \times 10^{-13}$ |
| $\Upsilon(10860)$ | $\mu\tau$ | $2.7 \times 10^{-10}$ |
|            | $e\tau$    | $1.4 \times 10^{-11}$ |
|            | $e\mu$     | $2.8 \times 10^{-14}$ |
| $Z_0$      | $\mu\tau$ | $1.0 \times 10^{-7}$ |
|            | $e\tau$    | $8.8 \times 10^{-9}$ |
|            | $e\mu$     | $3.5 \times 10^{-11}$ |

Table 1: Predicted branching ratios for transmutational decays. Modes which could be accessible to experiment in the near future are underlined.
existing experimental bounds comfortably, which are surprisingly weak for vector boson decays. Of the vector bosons listed, the data book [14] gives upper limits on lepton-flavour violations only for $Z_0$ decay, which for the modes $\mu\tau$, $e\tau$, and $e\mu$ are respectively $1.2 \times 10^{-5}$, $9.8 \times 10^{-6}$, and $1.7 \times 10^{-6}$ in branching ratios, which are seen to be all easily satisfied by the estimates in Table 1.

Secondly, we note several entries of branching ratios in Table 1 fall well within the sensitivity range of present experimental set-ups. In particular, the $\mu\tau$ decay modes of $\psi(1S), \psi(2S), \Upsilon(1S)$, and $\Upsilon(2S)$, are seen each to have a predicted branching ratio of several parts in a million and hence well within the present sensitivity range of BEPC [24], PEP II (BaBar) [21] and BELLE [22]. For instance, BEPC has already collected more than 20 million $\psi$'s to-date, and expects to collect twice as many more next year, which would mean for the predicted branching ratio of $6.3 \times 10^{-6}$ as many as 120 now, and 360 next year, of $\mu\tau$ decays assuming 100 percent detection efficiency. Similarly, PEP II has already accumulated for BaBar in a year over 20 fb$^{-1}$ in luminosity at $\Upsilon(4S)$, which means that if the machine is run at the $\Upsilon(1S)$ with a cross section of around 25 nb, it would collect in just a couple of months of running already enough $\Upsilon(1S)$ events to give, at a branching ratio of $1.2 \times 10^{-6}$, over 100 $\mu\tau$ decays, again assuming 100 percent detection efficiency. We note in passing, however, that at the energy 10.58 GeV at which the present BaBar experiment is run, the $\Upsilon(4S)$ resonance at that mass, being above the $B\bar{B}$ hadron threshold, has a branching ratio of only $2.8 \times 10^{-9}$ into the already most copious transmutational $\mu\tau$ mode, which makes the resonance effect negligible compared with direct transmutation via the process of Figure 1(b) as calculated in the preceding section.

7 Concluding Remarks

In summary, our conclusions are as follows. Although flavour conservation has been checked to very high accuracy in leptonic decays such as $\mu \rightarrow e\gamma$ and $\mu \rightarrow e e \bar{e}$, it may be premature to conclude that flavour-violation will always be small. The fact that the fermion mass matrix can rotate with changing scale can mean that flavour though accurately conserved at some energies are appreciably violated at other energies. We suggest therefore that lepton flavour violation should be routinely tested by experiment whenever conditions are favourable. Any sizeable signal (e.g. BR in vector boson of say order greater than $10^{-13}$ [25]) can mean mass matrix rotation, and any
information on rotation, and especially on its energy dependence, can give us new insight into the problem of fermion generations.

If we make the assumption that fermion mixing and mass hierarchy are due to mass matrix rotation, for which hypothesis there seems to be some empirical support [15], then the magnitude of the kinematic effects on flavour-violating from a rotating mass matrix can be estimated, and have been found to be appreciable and very likely detectable with the present generation of high sensitivity experiments such as BaBar, Belle, Bepc and Cleo. A detection of the effect at the estimated level would lend support to the hypothesis of [15] which would be a big step forward towards the solution of the generation puzzle. However, a negative result, unfortunately, cannot rule out the rotation hypothesis, for it can happen, as it is shown to be actually the case in a specific example in [19], that the effect of the rotating mass matrix calculated here is modified or even cancelled by other rotation effects accompanying the mass matrix rotation.

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