Quantum Diagrams and Quantum Networks

Louis H. Kauffman\textsuperscript{a} and Samuel J. Lomonaco Jr.\textsuperscript{b}

\textsuperscript{a} Department of Mathematics, Statistics and Computer Science (m/c 249), 851 South Morgan Street, University of Illinois at Chicago, Chicago, Illinois 60607-7045, USA
\textsuperscript{b} Department of Computer Science and Electrical Engineering, University of Maryland Baltimore County, 1000 Hilltop Circle, Baltimore, MD 21250, USA

ABSTRACT

This paper explores how diagrams of quantum processes can be used for modeling and for quantum epistemology.

Keywords: quantum process, measurement, diagram, graph, knot, matrix, network.

1. INTRODUCTION

This paper is an introduction to diagrammatic methods for representing quantum processes and quantum computing. We review basic notions for quantum information and quantum computing in Sections 2 and 3. In Section 3, we discuss topological diagrams and some issues about using category theory in representing quantum computing and teleportation. In Section 4 we analyze very carefully the diagrammatic meaning of the usual representation of the Mach-Zehnder interferometer, and we show how it can be generalized to associate to each composition of unitary transformations a “laboratory thought experiment diagram” such that particles moving through the many alternate paths in this diagram will mimic the quantum process represented by the composition of unitary transformations. This is a finite dimensional way to think about the Feynman Path Integral. We call our representation result the Path Theorem. In Section 5 we go back to the basics of networks and matrices and show how elements of quantum measurement can be represented with network diagrams.

2. QUANTUM MECHANICS AND QUANTUM COMPUTATION

We shall indicate the basic principles of quantum mechanics. The quantum information context encapsulates a concise model of quantum theory:

The initial state of a quantum process is a vector $|v\rangle$ in a complex vector space $H$. Measurement returns basis elements $\beta$ of $H$ with probability

$$\langle \beta | v \rangle^2 / \langle v | v \rangle$$

where $\langle v | w \rangle = v^\dagger w$ with $v^\dagger$ the conjugate transpose of $v$. A physical process occurs in steps $|v\rangle \rightarrow U |v\rangle = |Uv\rangle$ where $U$ is a unitary linear transformation. Measurement and quantum process are kept separate in the model.

Note that since $\langle Uv | Uw \rangle = \langle v | U^\dagger U | w \rangle = \langle v | w \rangle$ when $U$ is unitary, it follows that probability is preserved in the course of a quantum process.

One of the details required for any specific quantum problem is the nature of the unitary evolution. This is specified by knowing appropriate information about the classical physics that supports the phenomena. This information is used to choose an appropriate Hamiltonian through which the unitary operator is constructed via a correspondence principle that replaces classical variables with appropriate quantum operators. (In the path integral approach one needs a Langrangian to construct the action on which the path integral is based.) One needs to know certain aspects of classical physics to solve any specific quantum problem.

Further author information: L.H.K. E-mail: kauffman@uic.edu, S.J.L. Jr.: E-mail: lomonaco@umbc.edu
A key concept in the quantum information viewpoint is the notion of the superposition of states. If a quantum system has two distinct states \(|v\rangle\) and \(|w\rangle\), then it has infinitely many states of the form \(a|v\rangle + b|w\rangle\) where \(a\) and \(b\) are complex numbers taken up to a common multiple. States are “really” in the projective space associated with \(H\). There is only one superposition of a single state \(|v\rangle\) with itself. On the other hand, it is most convenient to regard the states \(|v\rangle\) and \(|w\rangle\) as vectors in a vector space. We then take it as part of the procedure of dealing with states to normalize them to unit length. Once again, the superposition of a state with itself is again itself.

Dirac introduced the “bra -(c)-ket” notation \(\langle A \mid B \rangle = A^\dagger B\) for the inner product of complex vectors \(A, B \in H\). He also separated the parts of the bracket into the bra \(\langle A \mid\) and the ket \(|B\rangle\). Thus

\[
\langle A \mid B \rangle = (A \mid B)
\]

In this interpretation, the ket \(|B\rangle\) is identified with the vector \(B \in H\), while the bra \(\langle A \mid\) is regarded as the element dual to \(A\) in the dual space \(H^*\). The dual element to \(A\) corresponds to the conjugate transpose \(A^\dagger\) of the vector \(A\), and the inner product is expressed in conventional language by the matrix product \(A^\dagger B\) (which is a scalar since \(B\) is a column vector). Having separated the bra and the ket, Dirac can write the “ket-bra” \(|A\rangle \langle B| = AB^\dagger\). In conventional notation, the ket-bra is a matrix, not a scalar, and we have the following formula for the square of \(P = |A\rangle \langle B|\):

\[
P^2 = |A\rangle \langle B| |A\rangle \langle B| = A(B^\dagger A)B^\dagger = (B^\dagger A)AB^\dagger = \langle B \mid A \rangle P.
\]

The standard example is a ket-bra \(P = |A\rangle \langle A|\) where \(\langle A \mid A \rangle = 1\) so that \(P^2 = P\). Then \(P\) is a projection matrix, projecting to the subspace of \(H\) that is spanned by the vector \(|A\rangle\). In fact, for any vector \(|B\rangle\) we have

\[
P|B\rangle = |A\rangle \langle A | B \rangle = |A\rangle \langle A | B \rangle = \langle A \mid B \rangle |A\rangle.
\]

If \(\{|C_1\rangle, |C_2\rangle, \cdots |C_n\rangle\}\) is an orthonormal basis for \(H\), and

\[
P_i = |C_i\rangle \langle C_i|,
\]

then for any vector \(|A\rangle\) we have

\[
|A\rangle = \langle C_1 \mid A \rangle |C_1\rangle + \cdots + \langle C_n \mid A \rangle |C_n\rangle.
\]

Hence

\[
\langle B \mid A \rangle = \langle B \mid C_1\rangle \langle C_1 \mid A \rangle + \cdots + \langle B \mid C_n\rangle \langle C_n \mid A \rangle
\]

One wants the amplitude for starting in state \(|A\rangle\) and ending in state \(|B\rangle\). The probability for this event is equal to \(|\langle B \mid A \rangle|^2\). This can be refined if we have more knowledge. If the intermediate states \(|C_i\rangle\) are a complete set of orthonormal alternatives then we can assume that \(\langle C_i \mid C_i \rangle = 1\) for each \(i\) and that \(\Sigma_i |C_i\rangle \langle C_i| = 1\). This identity now corresponds to the fact that 1 is the sum of the amplitudes of an arbitrary state projected into one of these intermediate states.

If there are intermediate states between the intermediate states this formulation can be continued until one is summing over all possible paths from \(A\) to \(B\). This becomes the path integral expression for the amplitude \(\langle B \mid A \rangle\).
2.1. What is a Quantum Computer?

A quantum computer is a composition $U$ of unitary transformations, together with an initial state and a choice of measurement basis. One runs the computer by repeatedly initializing it, and then measuring the result of applying the unitary transformation $U$ to the initial state. The results of these measurements are then analyzed for the desired information that the computer was set to determine.

Let $H$ be a given finite dimensional vector space over the complex numbers $C$. Let

$$\{W_0, W_1, ..., W_n\}$$

be an orthonormal basis for $H$ so that with $|i\rangle := |W_i\rangle$ denoting $W_i$ and $(i)$ denoting the conjugate transpose of $|i\rangle$, we have

$$(i|j) = \delta_{ij}$$

where $\delta_{ij}$ denotes the Kronecker delta (equal to one when its indices are equal to one another, and equal to zero otherwise). Given a vector $v$ in $H$ let $|v|^2 := \langle v|v \rangle$. Note that $(i|v$ is the $i$-th coordinate of $v$.

An measurement of $v$ returns one of the coordinates $|i\rangle$ of $v$ with probability $|\langle i|v \rangle|^2$. This model of measurement is a simple instance of the situation with a quantum mechanical system that is in a mixed state until it is observed. The result of observation is to put the system into one of the basis states.

When the dimension of the space $H$ is two ($n = 1$), a vector in the space is called a qubit. A qubit represents one quantum of binary information. On measurement, one obtains either the ket $|0\rangle$ or the ket $|1\rangle$. This constitutes the binary distinction that is inherent in a qubit. Note however that the information obtained is probabilistic. If the qubit is

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,$$

then the ket $|0\rangle$ is observed with probability $|\alpha|^2$, and the ket $|1\rangle$ is observed with probability $|\beta|^2$. In speaking of an idealized quantum computer, we do not specify the nature of measurement process beyond these probability postulates.

In the case of general dimension $n + 1$ of the space $H$, we will call the vectors in $H$ quunits. It is quite common to use spaces $H$ that are tensor products of two-dimensional spaces (so that all computations are expressed in terms of qubits) but this is not necessary in principle. One can start with a given space, and later work out factorizations into qubit transformations.

A quantum computation consists in the application of a unitary transformation $U$ to an initial qubit

$$|\psi\rangle = a_0|0\rangle + ... + a_n|n\rangle$$

with $|\psi|^2 = 1$, plus an measurement of $U|\psi\rangle$. A measurement of $U|\psi\rangle$ returns the ket $|i\rangle$ with probability $|\langle i|U|\psi\rangle|^2$. In particular, if we start the computer in the state $|i\rangle$, then the probability that it will return the state $|j\rangle$ is $|\langle j|U|i\rangle|^2$.

It is the necessity for writing a given computation in terms of unitary transformations, and the probabilistic nature of the result that characterizes quantum computation. Such computation could be carried out by an idealized quantum mechanical system.

We end this section with two diagrams that show some of the ideas behind quantum computation and also illustrate the diagrammatic issues of this paper. In Figure 1 we show the diagram for a quantum computer based on knot theoretic structure. In this diagram the crossings in the knot represent unitary transformations that have topological properties. The cups and the caps denote preparations and measurements respectively. The point we make about this sort of diagram is that it promotes thinking back and forth between topology and
quantum computing. One can use the language of category theory\(^2\) to clarify these interdisciplinary issues. The language of category theory is particularly useful when there is a geometric/topological object, such as a knot, that is subject to being divided into parts that represent sub-processes of a larger process. Our attitude toward the use of category theory in quantum computing is based on this notion of decomposition. When a process is specified by a space or a graph, there may be many ways to factor it into subprocesses. An appropriate choice of category and generators of that category (here the crossings, the cups and the caps) can act to organize and clarify the structure of the situation and the relationship between the disciplines that are under discussion.

Figure 2 illustrates a diagram for teleportation.\(^1,8,10,12\) Here the cap \(\langle M |\rangle\) represents a measurement, while the cup \(|\delta\rangle\) represents an an entangled (EPR) state. The direction for process and measurement proceeds from the bottom of the diagram to the top, through which the initial state \(|\phi\rangle\) is transformed to \(M|\phi\rangle\). Here \(M\) denotes a unitary transformation that is naturally associated with the measurement. On the other hand, from a graphical point of view, the composition of transformations is mathematically equivalent to what one finds on traversing the one-dimensional arc of the curve of the cap followed by the cup. This corresponds to the composition of the matrices \(M\) and \(\delta\) (an identity matrix in this case) and shows directly how the state \(|\phi\rangle\) was transformed to \(M|\phi\rangle\). Underneath any particular point of view about category, direction (or time in the physical interpretation) is the structure embodied in compositions of transformations. These are directly represented by the diagrams. Multiple interpretations of the diagrams furnish a multiplicity of viewpoints for the problem at hand.

3. HALF-SILVERED MIRRORS, A MACH-ZEHNDER INTERFEROMETER AND THE PATH THEOREM

In this section we give examples of diagramming quantum processes and we prove that any finite dimensional quantum process can be configured as a thought experiment that generalizes the diagrammatics for the Mach-Zehnder Interferometer.

View Figure 3. In this figure we indicate a matrix \(H\) that acts on a qubit space via the formulas

\[
H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \\
H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).
\]

This is a unitary transformation. In the same figure we indicate a diagram that can be interpreted as a half-silvered mirror that admits or transmits a particle that is labeled either as \(|0\rangle\) or as \(|1\rangle\). We shall refer to these
as the 0 and 1 states of the particle. The rules by which the mirror operates are that when it transmits a particle, it does not change the state of the particle, but it may change the phase. In this case when the mirror transmits a 1 it changes the phase corresponding to the negative sign in the second formula. But this mirror is supposed to be a “quantum mirror,” and so if a state of 0 or 1 enters the mirror, what “leaves” the mirror is a superposition of 0 and 1 along the two possible paths of transmission and reflection. As the reader can see, the diagram we have used for the half-silvered mirror is a combination of what one might diagram from a laboratory situation, combined with the principles of the quantum model. The diagram includes information that mimics the laboratory, with the understanding that the mirror action means that detection in a certain direction indicates reflection, while detection in another direction indicates transmission.

In the same Figure 3 we have indicated a more abstract diagram for the Hadamard matrix that just indicates an input or preparation line and shows the linear superposition on the output line. This sort of diagram can be used to indicate quantum processes quite independently of extra laboratory descriptions.

In Figure 4 we show a diagram for a quantum computer, called the Hadamard Test that can be used to find matrix entries of the form $\langle \psi | U | \psi \rangle$. The probability of observing $|0\rangle$ at the output of the test can be calculated to be $\frac{1}{2} + \frac{1}{2} \text{Re} \langle \psi | U | \psi \rangle$. This means that repeated use of the Test shown here will yield approximate values for $\langle \psi | U | \psi \rangle$. An insertion of an appropriate phase in the network, allows one to obtain approximations to the imaginary part. The language of this sort of diagrammatic is quite independent of any laboratory specifics. The input lines are in parallel and indicate an algebraic tensor product of the input to the unitary transformation $U$ and an external line above it that carries a single qubit. To the right of the inputs is a trivalent node and connection of the single qubit line with the unitary $U$, indicating that $U$ has been extended to a controlled $U$, where a $|0\rangle$ on the single qubit line replaces $U$ by an identity transformation and a $|1\rangle$ on the line leaves $U$ operative. The controlled $U$ is itself a unitary operation and the diagram as a whole represents the composition of unitary operations. This is a typical example of a “quantum circuit diagram”. We shall return to the general form of this kind of diagram in the next section.

In Figure 5 we have indicated a quantum apparatus made from half-silvered and ordinary mirrors. This diagram is a spatial composition of the half-silvered mirror diagram of Figure 3 and another mirror diagram.
\[
H = \begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix} / \text{Sqrt}(2)
\]

\[H|0> = (|0> + |1>) / \text{Sqrt}(2)\]

\[H|1> = (|0> - |1>) / \text{Sqrt}(2)\]

Figure 3. Hadamard as Half-Silvered Mirror

Figure 4. Hadamard Test
(indicated with dark shading) and matrix $M$ where

$$M|0\rangle = |1\rangle,$$

$$M|1\rangle = |1\rangle.$$  

Thus, as a mirror, $M$ only reflects and changes type 0 to type 1 and vice-versa. We can continue to use the idea of a particle “reflected” by this mirror, but it is understood that it just as well reflects a linear superposition as in the formula $M(a|0\rangle + b|1\rangle) = a|1\rangle + b|0\rangle$. Each mirror represents a unitary transformation in a sequence of unitary transformations that move from the preparation to the detection diagonally from left in the diagram to the right. The diagram has been configured so that the reader can imagine particles moving through it in an imaginary laboratory. All four paths that the initial $|0\rangle$ particle can take through the apparatus have been indicated on the diagram. The reader will see that a “destructive interference” occurs in the reflection output from the second half-silvered mirror, while a “reinforcement” occurs in the transmission output from the second mirror. Thus we conclude, thinking of summing over all the paths, that a $|0\rangle$ qubit input will result in a $|0\rangle$ qubit output. This “sum over paths” account of the quantum process for the Mach-Zehnder interferometer is exactly coincident with the abstract process corresponding to the composition $HMH$. What is the source of this coincidence?

The coincidence is no coincidence. There is a $1-1$ correspondence between the indicated paths for a particle in the diagram for the interferometer and the product terms in the the matrix $HMH$ as it is applied to $|0\rangle$ (or to any other qubit as preparation). The key to this observation is the definition of matrix multiplication:

$$(AB)_{ij} = \Sigma_k A_{ik} B_{kj}.$$ 

With respect to a chosen basis the matrix element $A_{rs}$ represents the amplitude for a transition from state $r$ to states and so the matrix entry $(AB)_{ij}$ is given by the sum over all products of amplitudes across all possibilities
for state transition. In the case of the interferometer, the laboratory diagram is designed to encompass all these possibilities. But there is an important observation to make about this diagram. The diagram contains two copies of the middle mirror $M$. From the point of view of the quantum model we are using these two copies of $M$ are identical in every way, in fact they are indistinguishable. If an observer made a measurement to distinguish these two mirrors in the course of our laboratory thought experiment, paths in the process would be disturbed and we would not get the same result as $HMH|0\rangle$.

The laboratory diagram for the Mach-Zehnder interferometer contains features that are specific to interpreting $H$ and $M$ as mirrors. Thus we have set up the rules so that qubits will traverse certain lines when transmitted and other lines when reflected. In Figure 6 we have shown how the laboratory circuit we have described is unfolded from the straightline quantum circuit for the same process. In this diagram, we give the output lines roles according to reflection and transmission.

However, a simpler convention can be taken for a more universal translation of compositions of unitary matrices to laboratory diagrams. In Figure 7 we adopt the principle that the upper line on the diagram for $H$ indicates the $|0\rangle$ part of the output qubit, while the lower line indicates the $|1\rangle$ part of the output qubit. In diagramming $M$ we make no change, since $M$ just interchanges $|0\rangle$ and $|1\rangle$. In this Figure we indicate the paths, and the reader can see easily that for the preparation $|0\rangle$, there is destructive inference leading to only the output $|1\rangle$. In this case it is easy to see the $1-1$ correspondence between the path summation and the matrix multiplication. Of course this laboratory diagram is less physical since we do not imagine that we can make a device that sorts the 0 and 1 states without performing a measurement! The mechanics of matrix multiplication provide the system of paths and give us a story about the branching nature of the quantum process. In order to be quantum mechanically accurate about such diagrams we cannot assume that any of the paths are independently observed. Only a quantum measurement is made at the end of the process.

In Figure 8 we give the full generalization of a laboratory diagram node for a unitary transformation. Here we assume that we work with a finite dimensional complex vector space $W$ with orthonormal basis $B = \{|0\rangle, |1\rangle, \ldots |n\rangle\}$ and unitary transformation $U : W \rightarrow W$ with matrix formula

$$U|i\rangle = \sum_{k=0}^{n} U_{ki}|k\rangle.$$  

The multiplicity of lines emanating from the node represents the superposition of states in the sum above, and one can tell the story that an "$|i\rangle$ particle" enters the $U$-device on the left and is transformed into one of
MachZender = HMH

quantum circuit version

Standardized Laboratory Version

Figure 7. A Mach-Zehnder Circuit in Standardized Laboratory Form

Orthonormal Basis \{|0>,|1>,...,|n>\}

\[ U|i> = U_{1i}|1> + U_{2i}|2> + U_{3i}|3> + \ldots + U_{ni}|n> \]

Standardized Laboratory Diagram Element

Figure 8. The General Standardized Laboratory Form for a Unitary Element
the outputs in each of a set of parallel universes with the indicated complex number weighting. If we then plug a number of these devices into one another, we get a generalized Mach-Zehnder interferometer whose action can be described by the summation over all these weighted paths each in its own parallel universe. Of course upon measurement at the end of the process, we invoke the quantum postulate and find that the path summations have conspired to give us the correct answers. We have proved the following

Path Theorem. Every finite composition of unitary matrices can be represented by a “quantum laboratory diagram” such that the paths through the diagram are in 1 – 1 correspondence with the products that are summed in the formation of the matrix product. Thus, one can construct a thought experiment corresponding to the composition of unitary matrices where particles move along a garden of forking paths, and the final amplitude for a given state is obtained by summing over the (complex number) contributions of each path.

In Figure 9 we illustrate such a garden of forking paths for a composition of unitaries $A$, $B$ and $C$. Here the underlying space is a qubit space of dimension two, and so there are eight paths. The diagrams labeled $D$ and $D'$ represent the two possibilities for detection of the superposition at the end of the process. One will either detect $|0\rangle$ and take the summation for all paths ending in $D$, or one will detect $|1\rangle$ and take the summation for all paths ending in $D'$. These summations compute the amplitudes for the corresponding observations.

Note that in the composition of unitary transformations one must say that all paths occur at once. The paths do not happen in a multiplicity of worlds. The process happens in the Hilbert space for the quantum situation. There is no multiplicity of worlds. But in the path-space interpretation, one is led to think of many different paths and the possibility that each path is traversed in a different world. It is clear from our diagrammatic constructions that the many worlds are a convenient fiction to enable one to imagine parallel paths when there are, in actuality, no paths at all.

The Path Theorem is a finite dimensional analogue of the Feynman path integral. The formulation we have given raises questions about the relationship of the space of observations and spaces or graphs that may be standardly associated with a quantum process. We began this section with a version of the Mach-Zehnder interferometer whose geometry was, as a thought experiment, realistic in terms of the laboratory interpretation. In proving the Path Theorem we idealized each circuit element, removing the interpretation of where any given “particle” would be detected. This meant that each circuit element comes equipped with a set of output lines corresponding abstractly to the question - where does one find a given particle? With the graphical geometry of these lines, concatenations of elements produce path possibilities, just as in the Mach-Zehnder interferometer, and hence a thought experiment for each quantum process. In a sense we are letting the quantum process itself produce a space of paths for its own elucidation. We intend to explore further the possibility that spaces that arise in physical situations can be construed as spaces that are fitting to their underlying quantum processes.

4. DIAGRAMMATIC METHODS FOR QUANTUM MEASUREMENT

The point of view of this paper is based on diagrammatic conventions for matrix multiplication and tensor composition. The purpose of this section is to describe these conventions and to show how they are used in our work. We take diagrams to represent matrices and products or concatenations of matrices. In this way a complex network diagram can represent a contraction of a collection of multi-indexed matrices, and so may represent a quantum state or a quantum amplitude. We regard each graph as both a possible holder for matrices, and hence as a vehicle for such a computation, and as a combinatorial structure. As a combinatorial structure the graph can be modified. An edge can be removed. A node can be inserted. Such modifications can be interpreted in terms of quantum preparation and measurement. One can then take the graph as a miniature “world” upon which such operations are performed. Since the graphs can also represent topological structures, this approach leads to a way to interface topology with the quantum mechanics. The diagrammatics in this section should be compared with work of Roger Penrose, the authors, Tom Etter and parts of our early paper on quantum knots.
First, consider the multiplication of matrices $M = (M_{ij})$ and $N = (N_{kl})$ where $M$ is $m \times n$ and $N$ is $n \times p$. Then $MN$ is $m \times p$ and

$$(MN)_{ij} = \sum_{k=1}^{n} M_{ik}N_{kj}.$$ 

We represent each matrix by a box, and each index for the matrix elements by a line segment that is attached to this box. The common index in the summation is represented by a line that emanates from one box, and terminates in the other box. This line segment has no free ends. By the definition of matrix multiplication, a line segment without free ends represents the summation over all possible index assignments that are available for that segment. Segments with free ends correspond to the possible index choices for the product matrix. See Figure 10.

The trace of an $m \times m$ matrix $M$ is given by the formula

$$tr(M) = \sum_{i=1}^{m} M_{ii}.$$
In diagrammatic terms the trace is represented by a box with the output segment identified with the input segment. See Figure 11 for this interpretation of the matrix trace.

\[
\text{tr}(M) = \text{tr}(M)
\]

**Figure 11 - Matrix Trace**

\[
\langle a | M | b \rangle = \text{tr}(\rho_{ab}M)
\]

\[
\rho_{ab} = |a \rangle \langle b |
\]

**Result of Measurement**

In Figure 12 we illustrate the diagrammatic interpretation of the formula

\[
\langle a | M | b \rangle = \text{tr}(\rho_{ab}M).
\]

This formula gives the amplitude for measuring the state \(|a\rangle\) from a preparation of \(|\psi\rangle = M|b\rangle\). Note that the state \(|\psi\rangle\) is obtained from the graphical structure of \(\text{tr}(M)\) by cutting the connection between the input line and output line of the box labeled \(M\), and inserting the ket \(|b\rangle\) on the output line. The resulting network is shown at the top of Figure 12. This network, with one free end, represents the quantum state \(|\psi\rangle\). This state is the superposition of all possible values (qubits) that can occur at the free end of the network. When we measure the state, one of the possible qubits occurs. The amplitude for the occurrence of \(|a\rangle\) is equal to \(\langle a | M | b \rangle\). When we insert \(|a\rangle\) at the free end of the network for \(|\psi\rangle\), we obtain the network whose value is this amplitude.
If $M$ is unitary, then we can interpret the formula as the amplitude for measuring state $|a\rangle$ from a preparation in $|b\rangle$, and an evolution of this preparation by the unitary transformation $M$. In the first interpretation the operator $M$ can be an observable, aiding in the preparation of the state. In this notation,

$$\rho_{ab} = |a\rangle \langle b|$$

is the ket-bra associated with the states $|a\rangle$ and $|b\rangle$. If $a = b$, then $\rho_{aa}$ is the density matrix associated with the pure state $|a\rangle$.

The key to this graphical model for preparation and measurement is the understanding that the diagram is both a combinatorial structure and a representative for the computation of either an amplitude or a state (via summation over the indices available for the internal lines and superposition over the possibilities for the free ends of the network). A diagram with free ends (no kets or bras tied into the ends) represents a state that is the superposition of all the possibilities for the values of the free ends. This superposition is a superposition of diagrams with different labels on the ends. In this way the principles of quantum measurement are seen to live in categories of diagrams. A given diagram can be regarded as a world that is subject to preparation and measurement. After such an operation is performed (Cut an edge. Insert a density matrix.), a new world is formed that is itself subject to preparation and measurement. This succession of worlds and states can be regarded as a description of the evolution of a quantum process.

**Remark.** One can generalize this notion of quantum process in networks by allowing the insertion of other operators into the network, and by allowing systematic operations on the graph. Techniques of this sort are used in spin foam models for quantum gravity, and in renormalization of statistical mechanics models. The reader may also find it fruitful to compare the approaches in the present paper with the work in where we work on the relationship of classical computation and network structures.

**REFERENCES**

1. B. Coecke, The logic of entanglement, quant-phy/0402014.
2. S. Abramsky, B. Coecke, Categorical quantum mechanics. Handbook of quantum logic and quantum structuresquantum logic, 261323, Elsevier/North-Holland, Amsterdam, 2009.
3. T. Etter, How to take apart a wire, part I, Anpa West Journal, Vol. 6, No. 2 (1996), 16–28.
4. Marius Buliga, Louis H. Kauffman, Chemlambda, universality and self-multiplication, arXiv:1403.8046.
5. Marius Buliga, Louis H. Kauffman, GLC actors, artificial chemical connectomes, topological issues and knots, arXiv:1312.4333.
6. Chen, G., L. Kauffman, and S. Lomonaco, (eds.), "Mathematics in Quantum Computation and Quantum Technology," Chapman & Hall/CRC , (2007).
7. L.H. Kauffman, Quantum computation and the Jones polynomial , in Quantum Computation and Information, S. Lomonaco, Jr. (ed.), AMS CONM/305, 2002, 101–137.
8. L. H. Kauffman and S. J. Lomonaco Jr., Quantum entanglement and topological entanglement, New Journal of Physics 4 (2002), 73.1–73.18 (http://www.njp.org/).
9. L. H. Kauffman and S. J. Lomonaco Jr., Quantum knots, in Quantum Information and Computation II, Proceedings of Spie, 12 -14 April 2004 (2004), ed. by Donkor Pirich and Brandt, pp. 268-284.
10. L. H. Kauffman and S. J. Lomonaco Jr., Entanglement Criteria - Quantum and Topological, in Quantum Information and Computation - Spie Proceedings, 21-22 April, 2003, Orlando, FL, Donkor, Pinch and Brandt (eds.), Volume 5105, 51–58.
11. L. H. Kauffman and S. J. Lomonaco Jr., Braiding operators are universal quantum gates, arXiv:quant-ph/0401090 v2 19 Jan 2004. New Journal of Physics 6 (2004) 134, pp. 1-39.
12. Louis H. Kauffman, Teleportation Topology, quant-ph/0407224, (in the Proceedings of the 2004 Byelorus Conference on Quantum Optics), Opt. Spectrosc. 9, 2005, 227-232.
13. Kauffman, Louis H.; Lomonaco, Samuel J., Jr. $q$-deformed spin networks, knot polynomials and anyonic topological quantum computation. J. Knot Theory Ramifications 16 (2007), no. 3, 267–332.
14. L.H. Kauffman, Knots and Physics, World Scientific Publishers (1991), Second Edition (1993), Third Edition (2002).
15. L. Kauffman, Spin networks and topology, in The Geometric Universe (Conference in Honor of Roger Penrose, Oxford, 1996), 277–289, Oxford Univ. Press, Oxford, 1998.
16. L.H. Kauffman, Quantum topology and quantum computing, in Quantum Computation, S. Lomonaco (ed.), AMS PSAPM/58, 2002, pp. 273–303.
17. L. H. Kaufman, Knot automata, Proceedings of the Twenty-Fourth International Conference on Multiple Valued Logic- Boston (1994), IEEE Computer Society Press, 328-333.
18. F. Markoupoulou, Coarse graining in spin foam models, arXiv:gr-qc/0203036 v1 12 Mar 2002.
19. R. Penrose, Angular momentum: An approach to combinatorial spacetime, in Quantum Theory and Beyond, T. Bastin (ed.), Cambridge Univ. Press, 1969.