A non-distributive logic for semiconcepts of a context and its modal extension with semantics based on Kripke contexts

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Abstract

A non-distributive two-sorted hypersequent calculus PDBL and its modal extension MPDBL are proposed for the classes of pure double Boolean algebras and pure double Boolean algebras with operators respectively. A relational semantics for PDBL is next proposed, where any formula is interpreted as a semiconcept of a context. For MPDBL, the relational semantics is based on Kripke contexts, and a formula is interpreted as a semiconcept of the underlying context. The systems are shown to be sound and complete with respect to the relational semantics. Adding appropriate sequents to MPDBL results in logics with semantics based on reflexive, symmetric or transitive Kripke contexts. One of these systems is a logic for topological pure double Boolean algebras. It is demonstrated that, using PDBL, the basic notions and relations of conceptual knowledge can be expressed and inferences involving negations can be obtained. Further, drawing a connection with rough set theory, lower and upper approximations of semiconcepts of a context are defined. It is then shown that, using the formulae and sequents involving modal operators in MPDBL, these approximation operators and their properties can be captured.

Keywords: Formal concept analysis, Rough set theory, Double Boolean algebra, Non-distributive Modal logic, Conceptual knowledge.

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1. Introduction

Rough set theory \cite{26} and formal concept analysis (FCA) \cite{31} are both well-established domains of study with applications in a wide range of fields, including knowledge representation and logic. In FCA, a context (also called a polarity) \cite{31,13} is a triple $\mathbb{K} := (G, M, R)$, where $G$, $M$ are the sets of objects and properties respectively, and $R \subseteq G \times M$. For $g \in G$ and $m \in M$, $g Im$ is read as “the object $g$ has the property $m$”. A context $\mathbb{K}$ induces a Galois connection \cite{7,12}: $(\cdot) \preceq \mathcal{P}(G) \rightleftharpoons \mathcal{P}(M) : (\cdot)'$, where for any $A \subseteq G, B \subseteq M$, $A' := \{m \in M : \forall g \in G (g \in A \implies gRm)\}$, and $B' := \{g \in G : \forall m \in M (m \in B \implies gRm)\}$. A pair $(A, B)$ is called a concept of $\mathbb{K}$ provided $A' = B$ and $B' = A$; $A$ is then called its extent, denoted as $ext((A, B))$, and $B$ its intent, denoted as $int((A, B))$. In this work, we are interested in formulating logics with which one can reason about objects and properties in a context, concepts of a context as well as ‘negations of concepts’.

As seen in \cite{24}, if the ‘negation of a concept’ is formalized using set-complement, there is a problem of closure. The notion of a concept was then generalized to that of a semiconcept by Wille \cite{24,32}.

Definition 1. \cite{33} A pair $(A, B)$ is called a semiconcept of $\mathbb{K}$ if and only if $A' = B$ or $B' = A$.

For a context $\mathbb{K}$, $B(\mathbb{K})$ and $\mathcal{F}(\mathbb{K})$ denote the sets of all concepts and semiconcepts of $\mathbb{K}$, respectively. It can be shown that $B(\mathbb{K}) \subseteq \mathcal{F}(\mathbb{K})$. Our focus is on defining a logic for semiconcepts of a context.

$B(\mathbb{K})$ forms a complete lattice, called the concept lattice of $\mathbb{K}$, where the order relation $\preceq$ giving the lattice structure on $B(\mathbb{K})$ is obtained as follows: for $(A_1, B_1), (A_2, B_2) \in B(\mathbb{K}), (A_1, B_1) \preceq (A_2, B_2)$ if and only if $A_1 \subseteq $
A_2 \) (equivalent to \( B_2 \subseteq B_1 \)). Furthermore, every complete lattice is isomorphic to the concept lattice of some context. On the other hand, the following operations are defined on \( \mathcal{H}(K) \). For \((A_1, B_1)\) and \((A_2, B_2)\) in \( \mathcal{H}(K) \),

\[
\begin{align*}
(A_1, B_1) \cap (A_2, B_2) & := (A_1 \cap A_2, (A_1 \cap A_2)' ) \\
(A_1, B_1) \cup (A_2, B_2) & := ((B_1 \cap B_2)', B_1 \cap B_2) \\
\neg (A, B) & := (G \setminus A, (G \setminus A)' ) \\
\prec (A, B) & := ((M \setminus B)', M \setminus B) \\
\top & := (G, \emptyset) \\
\bot & := (\emptyset, M).
\end{align*}
\]

\(-\) and \(\cup\) are negation operators in \( \mathcal{H}(K) \). A semiconcept \( x \in \mathcal{H}(K) \) is called a left semiconcept if it is idempotent with respect to \( \cap \), i.e. \( x \cap x = x \). A right semiconcept is defined dually. In general, the set \( \mathcal{H}(K) \) does not form a lattice with respect to the operations \( \cup \) and \( \cap \). However, \( \mathcal{H}(K) := (\mathcal{H}(K), \cap, \cup, \neg, \prec, \top, \bot) \) is an instance of a rich algebraic structure called pure double Boolean algebra; it is called the algebra of semiconcepts of the context \( K \). Further, every pure double Boolean algebra is embeddable in the algebra of semiconcepts of some context, as proved in [3]. Considering these facts, our goal in this work is to formulate a logic that is sound and complete with respect to the class of all pure double Boolean algebras.

The language of the proposed logic is taken to consist of two disjoint countably infinite sets of propositional variables - the set \( OV \) of object variables and the set \( PV \) of property variables. More precisely, we propose a hypersequent calculus \( PDBL \) whose language \( \mathcal{L} \) consists of the following: propositional constants \( \top, \bot \), logical connectives \( \cup, \cap, \neg, \prec \), and the two sets \( OV, PV \) as mentioned above. Formulae are built over \( OV \cup PV \). \( PDBL \) is a non-distributive logic, that is, \( \cap \) does not necessarily distribute over \( \cap \) and vice versa. It is shown that \( PDBL \) is sound and complete with respect to the class of all pure double Boolean algebras.

Relational semantics of non-distributive logics where models are two-sorted and defined using contexts, have been extensively investigated. Concepts of a context are potential interpretants for logical formulae over models in such studies. Gehrke [14] introduced a two-sorted approach to the relational semantics for the implication-fusion fragment of various sub-structural logics. The interpretation is encoded by two relations: satisfaction (\( \models \)) and “a part of” relation (\( \succ \)). Satisfaction is defined at a world (an object) and “a part of” relation at a co-world (a property). Conradie et al. [3, 11, 13] investigated a two-sorted approach to the relational semantics for non-distributive modal logic. The definition of interpretation in [9] is the same as in [14], and the “a part of” relation (\( \succ \)) is renamed as co-satisfaction. Later, Hartonas [13, 16] also studied two-sorted approaches to relational semantics for non-distributive modal systems.

Along similar lines, a two-sorted approach to relational semantics is also defined for \( PDBL \) in this work – only a formula is interpreted here as a semiconcept of a context. In particular, an object variable is interpreted as a left semiconcept, while a property variable is interpreted as a right semiconcept. The interpretation of any formula is then defined inductively with the help of relations of satisfaction (\( \models \)) and co-satisfaction (\( \succ \)).

Using the concept lattice of a context, a mathematical representation is given to conceptual knowledge; such a representation is termed as a conceptual knowledge system [24, 52]. Wille adopted the idea from traditional philosophy that there are three basic notions of conceptual knowledge, namely, objects, attributes, and concepts. Moreover, these three are linked by four basic relations: “an object has an attribute”, “an object belongs to a concept”, “an attribute abstracts from a concept”, and “a concept is a subconcept of a concept”. As concepts of a context are semiconcepts, it is expected that a logic for semiconcepts of a context should be one in which the three basic notions and four relations of conceptual knowledge can be expressed. Indeed, one is able to demonstrate that using \( PDBL \) and its relational semantics, the basic notions and relations of conceptual knowledge are expressible. In particular, (\( \models \)) represents the relation “an object belongs to a concept” and (\( \succ \)) represents the relation “an attribute abstracts from a concept”. Wille [24] has shown through an example that the negations \( \neg, \prec \) aid conceptual knowledge representation. Employing sequents of \( PDBL \) comprising negations, we are also able to capture conceptual knowledge involving negation. An example is given in this regard as well.

In [19], we have expanded the notion of a context to that of a Kripke context which links two Kripke frames by a relation:

**Definition 2.** [19] A Kripke context based on a context \( K := (G, M, I) \) is a triple \( KC := ((G, R), (M, S), I) \), where \( R, S \) are relations on \( G \) and \( M \) respectively.
The motivation of a Kripke context lies in the intersection of the frameworks of rough set theory and FCA. Two basic notions of rough set theory are those of approximation spaces and approximation operators. A pair \((W, E)\) is a Pawlakian approximation space, where \(W\) is a set and \(E\) is an equivalence relation on \(W\). \((W, E)\) is called a generalised approximation space [34], when \(E\) is any binary relation on \(W\). For \(x \in W\), \(E(x) := \{y \in W : xRy\}\). The lower and upper approximations of any \(A(\subseteq W)\) are defined respectively as \(A_L := \{x \in W : E(x) \subseteq A\}\), and \(\overline{A} := \{x \in W : E(x) \cap A \neq \emptyset\}\). Kent [21], Saquer et al. [20] and Hu et al. [29] incorporated the idea of approximation space in FCA, and discussed approximations of concepts, albeit from different perspectives. The basic motivation is that in some cases, the objects and properties that define a context are indistinguishable in terms of certain attributes. For example, two diseases may have symptoms that are indistinguishable.

In this work, we propose lower and upper approximations of semiconcepts. The base approximation space that we use is the same as the one in [29].

Many researchers [5, 34, 23, 11, 6] have worked on various modal logics and their rough set semantics, in which modalities are interpreted as approximation operators. This extensive literature prompted us to look for appropriate modal systems with semantics based on the Kripke context which could, moreover, express the approximations of semiconcepts. For a Kripke context \(KC\), the notion of complex algebra \(\Sigma^+(KC)\) of semiconcepts is defined in [19].

In order to understand the equational theory of the complex algebras, pure double Boolean algebras with operators and topological pure double Boolean algebras are proposed in [19]. Here, the language \(\mathcal{L}\) of PDBL is extended to \(\mathcal{L}_1\) by adding unary modal connectives \(\square\) and \(\lozenge\), and we get the modal system MPDBL. \(\Box\) and \(\Diamond\) are introduced as duals of \(\square\) and \(\lozenge\) respectively. MPDBL corresponds to the class of pure double Boolean algebras with operators. Further, the logic MPDBL\(\Sigma\) is defined, where \(\Sigma\) is any set of sequents in MPDBL. Taking an appropriate \(\Sigma\), we get the logic for topological pure double Boolean algebras. As a natural next course, the relational semantics of PDBL is extended to that for the modal systems. Formulæ in the language \(\mathcal{L}_1\) are interpreted as semiconcepts of the underlying context \(K\) of a Kripke context \(KC\). It is shown that formulæ of the form \(\Box\alpha, \Diamond\alpha\) translate into the lower approximations of a left semiconcept and right semiconcept respectively, while \(\lnot\alpha\), \(\Diamond\alpha\) translate into the upper approximations of a left semiconcept and right semiconcept respectively. Moreover, valid sequents involving modal operators express properties of lower and upper approximations of semiconcepts. The deductive systems are shown to be sound and complete for the respective semantics.

Section 2 gives the preliminaries required for this work. Results related to pure double Boolean algebras are given in Section 2.1. Section 2.2 outlines work on approximation operators in rough set theory and approximations of concepts. Pure double Boolean algebras with operators, topological pure double Boolean algebras and Kripke contexts and related results are given in Section 2.3. In Section 3 the logics corresponding to the algebras are studied. PDBL for the class of all pure double Boolean algebras is discussed in Section 3.1. In Section 3.2 the modal systems are given. Section 4 presents the relational semantics for the logics, with the semantics for PDBL discussed in Section 4.1 and that for the modal systems given in Section 4.2. A study of PDBL and conceptual knowledge is done in Section 5. Section 6 describes the approximations of semiconcepts and demonstrates how fundamental properties of the approximations are captured by using one of the proposed modal systems, MPDBL5. Section 7 concludes the paper.

In our presentation, the symbols \(\iff, \implies\), and, or and not will be used with the usual meanings in the metalanguage. Throughout, \(P(X)\) denotes the power set of any set \(X\), and the complement of \(A \subseteq X\) in a set \(X\) is denoted \(A^c\).

2. Preliminaries

Our key references for this section are [13, 33, 21, 8].

2.1. Pure double Boolean algebra

Let us first give the definition of a double Boolean algebra.

**Definition 3.** [33] An algebra \(D := (D, \cup, \cap, \neg, \top, \bot)\), satisfying the following properties for any \(x, y, z \in D\), is called a double Boolean algebra (dBa).
Lemma 1. \[ \text{For all} \]

\[ \text{For any} \]

\[ \text{In every pdBa} \]

Proposition 1. \[ \text{Let} \]

\[ \text{Where} \]

Proposition 2. \[ \text{Every pdBa} \]

\[ \text{Let us give some notations that shall be used:} \]

\[ \text{D is pure if for all} \]

D is \textit{contextual} if the quasi-order is a partial order.

The abbreviations pdBa and cdBa stand for pure dBa and contextual dBa, respectively.

Proposition 1. \[ \text{In every pdBa} D, \subseteq \text{is a partial order.} \]

Corollary 1. Every pdBa D is a cdBa.

In the following, let \[ D := (D_{\cap}, \cup, \cap, \rightarrow, \top, \bot) \] be a dBa. Let us give some notations that shall be used: \[ D_{\cap} := \{ x \in D : x \cap x = x \}, \ D_{\cup} := \{ x \in D : x \cup x = x \}, D_{\cap \cup} := D_{\cap} \cup D_{\cup}. \] For \[ x \in D, \ x_{\cap} := x \cap x \] and \[ x_{\cup} := x \cup x. \]

Proposition 2. \[ \text{Let} \]

\[ \text{Then the following hold.} \]

Proposition 3. \[ \text{Let} \]

\[ \text{Then the following hold.} \]

Proposition 4. \[ \text{For any} \]

\[ \text{For any} \]

Lemma 1. \[ \text{For all} \]

1. \[ x \cap \bot = \bot \text{ and } x \cup \bot = x \cap x \text{ that is } \bot \subseteq x. \]
2. \[ x \cup \top = \top \text{ and } x \cap \top = x \cap x \text{ that is } x \subseteq \top. \]
3. \[ x = y \text{ implies that } x \subseteq y \text{ and } y \subseteq x. \]
4. \[ x \subseteq y \text{ and } y \subseteq x \text{ if and only if } x \cap x = y \cap y \text{ and } x \cup x = y \cup y. \]
5. \[ x \subseteq y \text{ implies } x \cap a \subseteq y \cap a \text{ and } x \cup a \subseteq y \cup a. \]

1. \[ \neg x \cap \neg x = \neg x \text{ and } \neg x \cup \neg x = \neg x \text{ that is } \neg x = (\neg x)_{\cap} \in D_{\cap} \text{ and } \neg x = (\neg x)_{\cup} \in D_{\cup}. \]
2. \[ x \subseteq y \text{ if and only if } \neg y \subseteq \neg x \text{ and } y \subseteq x. \]
3. \[ \neg x = x \cap x \text{ and } \neg x = x \cup x. \]
4. \[ x \vee y \in D_{\cap}, x \wedge y \in D_{\cup}. \]
5. \[ \neg \bot = \bot \text{ and } \neg \top = \top. \]

1. \[ a \cap b = \bot \text{ then } a \cap a = \neg b. \]
2. \[ a \cap a \subseteq \neg b \text{ then } a \cap b = \bot. \]
3. If $a \cup b = \top$ then $\mathcal{J}b \subseteq a \cup a$.
4. If $\mathcal{J}b \subseteq a \cup a$ then $\top \subseteq a \cup b$.

In particular, if $\mathcal{D}$ is a cdBa then $a \cap b = \bot$ if and only if $a \cap a \subseteq \neg b$, and $a \cup b = \top$ if and only if $\mathcal{J}b \subseteq a \cup a$.

**Definition 4.** A subset $F$ of $D$ is a filter in $\mathcal{D}$ if and only if $x \cap y \in F$ for all $x, y \in F$, and for all $z \in D$ and $x \in F$, $x \subseteq z$ implies that $z \in F$. An ideal in a cdBa is defined dually.

A filter $F$ (ideal $I$) is proper if and only if $F \neq D$ ($I \neq D$). A proper filter $F$ (ideal $I$) is called primary if and only if $x \in F$ or $\neg x \in F$ ($x \in I$ or $\neg x \in I$), for all $x \in D$.

The set of primary filters is denoted by $\mathcal{F}_{pr}(\mathcal{D})$; the set of all primary ideals is denoted by $\mathcal{I}_{pr}(\mathcal{D})$.

A base $F_0$ for a filter $F$ is a subset of $D$ such that $F = \{ x \in D : z \subseteq x \text{ for some } z \in F_0 \}$. A base for an ideal is defined similarly.

For a subset $X$ of $D$, $F(X)$ and $I(X)$ denote the filter and ideal generated by $X$ respectively.

To prove representation theorems for dBas, the following are introduced in $\mathcal{33}$.

\[ \mathcal{F}_{p}(\mathcal{D}) := \{ F \subseteq D : F \text{ is a filter of } \mathcal{D} \} \]

\[ \mathcal{I}_{p}(\mathcal{D}) := \{ I \subseteq D : I \text{ is an ideal of } \mathcal{D} \} \]

In fact, we have

**Proposition 5.** $\mathcal{33}$ $\mathcal{F}_{p}(\mathcal{D}) = \mathcal{F}_{pr}(\mathcal{D})$ and $\mathcal{I}_{p}(\mathcal{D}) = \mathcal{I}_{pr}(\mathcal{D})$.

**Lemma 2.** $\mathcal{33}$

1. For any filter $F$ of $\mathcal{D}$, $F \cap D_{\cap}$ and $F \cap D_{\cup}$ are filters of the Boolean algebras $D_{\cap}$, $D_{\cup}$ respectively.

2. Each filter $F_0$ of the Boolean algebra $D_{\cap}$ is the base of some filter $F$ of $\mathcal{D}$ such that $F_0 = F \cap D_{\cap}$. Moreover if $F_0$ is prime, $F \in \mathcal{F}_{p}(\mathcal{D})$.

Similar results can be proved for ideals of dBas.

Recall the operations on the the set $\mathcal{H}(\mathbb{K})$ of all semiconcepts given in the Introduction.

**Theorem 1.** $\mathcal{33}$ $\mathcal{H}(\mathbb{K}) := (\mathcal{H}(\mathbb{K}), \cap, \cup, \neg, \wedge, \vee, \top, \bot)$ is a pdBa.

For $\mathcal{H}(\mathbb{K})$, consider the sets $\mathcal{H}(\mathbb{K})_{\cap}$ and $\mathcal{H}(\mathbb{K})_{\cup}$. Note that the elements of the set $\mathcal{H}(\mathbb{K})_{\cap}$ are of the form $(A, A')$ and the elements of the set $\mathcal{H}(\mathbb{K})_{\cup}$ are of the form $(B', B)$.

**Theorem 2.** $\mathcal{33}$

1. $\mathcal{H}(\mathbb{K}) := \mathcal{H}(\mathbb{K})_{\cap} \cup \mathcal{H}(\mathbb{K})_{\cup}$.

2. $\mathcal{B}(\mathbb{K}) := \mathcal{H}(\mathbb{K})_{\cap} \cap \mathcal{H}(\mathbb{K})_{\cup}$.

Theorem 2(1) thus implies that a semiconcept is either a left semiconcept or a right semiconcept. Next we move to representation theorem for pdBAs. The notations and results listed below are required. Let $\mathcal{D}$ be a dBa. For any $x \in D$, $F_x := \{ F \in \mathcal{F}_p(\mathcal{D}) : x \in F \}$ and $I_x := \{ I \in \mathcal{I}_p(\mathcal{D}) : x \in I \}$.

**Lemma 3.** $\mathcal{33}$ $\mathcal{17}$ Let $x \in D$. Then the following hold.

1. $(F_x)^c = F_{\neg x}$ and $(I_x)^c = I_{\neg x}$.

2. $F_{x \cap y} = F_x \cap F_y$ and $I_{x \cap y} = I_x \cap I_y$.

To prove the representation theorem, Wille uses the standard context corresponding to the dBa $\mathcal{D}$, defined as $\mathcal{K}(\mathcal{D}) := (\mathcal{F}_p(\mathcal{D}), \mathcal{I}_p(\mathcal{D}), \Delta)$, where for all $F \in \mathcal{F}_p(\mathcal{D})$ and $I \in \mathcal{I}_p(\mathcal{D})$, $F \Delta I$ if and only if $F \cap I = \emptyset$. Then we have

**Lemma 4.** $\mathcal{33}$ For all $x \in D$, $F'_{x} = I_{x \cup} = I_{x \cup}$.

**Theorem 3.** $\mathcal{33}$ Let $\mathcal{D}$ be a pdBa. Then the map $h : \mathcal{D} \rightarrow \mathcal{H}(\mathcal{D})$ defined by $h(x) = (F_{x}, I_{x})$ for all $x \in \mathcal{D}$ is an injective homomorphism.
2.2. Rough set theory and approximations

In rough set theory, a Pawlakian approximation space $K := (W, E)$ can be understood to represent a basic classification skill of an “intelligent” agent [27]. $K$ models “knowledge”, which is this ability to classify objects by agents. In the induced quotient set $W/E$ of equivalence classes due to the equivalence relation $E$, each equivalence class is called a basic category. It is not always the case that a subset $A$ of $W$ is a category. Then the task in rough set theory is to approximate the set $A$ with respect to $K$. This is done by using the lower and upper approximation operators defined in the Introduction. In the knowledge $K$, the lower approximation $A_E$ of the set $A$ is interpreted as the set of all elements of $W$ that can be classified as elements of $A$ with certainty, while $A$’s upper approximation $A_{E}^{\prime\prime}$ is the set of elements of $W$ that can be possibly classified as elements of $A$. We will omit the subscript and superscript, and denote $A_E$ by $A$, $A_{E}^{\prime\prime}$ by $A$, if the relation is evident from the context. Recall that $(W, E)$ is termed a generalised approximation space, when $E$ is any binary relation on $W$.

Proposition 6. [34, 27] Let $(W, E)$ be a generalised approximation space, and $A, B \subseteq W$.

I. The following hold.

(i) $A = ((A^c)^c)$, $A = ((A^c)^c)$.

(ii) $W = W$.

(iii) $A \cap B = A \cap B$, $A \cup B = A \cup B$.

(iv) $A \subseteq B$ implies that $A \subseteq B$, $A \subseteq B$.

II. If $E$ is a reflexive and transitive relation, the following hold.

(v) $A \subseteq A$ and $A \subseteq A$.

(vi) $(A) = A$ and $(A) = A$.

III. If $(W, E)$ is a Pawlakian approximation space, the following hold.

(vii) $A = A$ implies $A$ is a category.

(viii) $A = A$ implies $A$ is a category.

(ix) $A \cap B$ is a category, if $A, B$ are categories.

Motivated by rough set theory, several authors have introduced approximation operators in FCA. Let $\mathbb{K} := (G, M, I)$ be a context. Kent [21] considered an approximation space $(G, E)$ and defined lower and upper approximations of concepts of the relation $I$. Using these contexts, he defined lower and upper approximations of concepts of the context $\mathbb{K}$.

On the other hand, Hu et al. [28] defined relations $J_1, J_2$ on $G$ and $M$ respectively, as follows.

(a) For $g_1, g_2 \in G$, $g_1 J_1 g_2$ if and only if $I(g_1) \subseteq I(g_2)$.

(b) For $m_1, m_2 \in M$, $m_1 J_2 m_2$ if and only if $I^{-1}(m_1) \subseteq I^{-1}(m_2)$.

The relations $J_1, J_2$ are partial orders. It is shown that $UI := \{(I(g), I(g')) : g \in G\}$ is the set of join irreducible elements of $B(\mathbb{K})$ and $MI := \{(I^{-1}(m), I^{-1}(m)) : m \in M\}$ is the set of meet irreducible elements of $B(\mathbb{K})$. For $A \subseteq G (B \subseteq M)$, the lower and upper approximations of $A$ ($B$) are defined in terms of members of $UI$ ($MI$).

In [29], relations $E_1, E_2$ on $G$ and $M$ respectively were defined by Saquer et al. as follows.

(a) For $g_1, g_2 \in G$, $g_1 E_1 g_2$ if and only if $I(g_1) = I(g_2)$.

(b) For $m_1, m_2 \in M$, $m_1 E_2 m_2$ if and only if $I^{-1}(m_1) = I^{-1}(m_2)$.

$E_1, E_2$ are equivalence relations. $A \subseteq G$ and $B \subseteq M$ are called feasible, if $A'' = A$ and $B'' = B$. The concept approximations of $A, B$ are then defined:

- If $A$ is feasible, the concept approximation of $A$ is $(A, A')$.

- If $A$ is not feasible, it is treated as a rough set of the approximation space $(G, E_1)$, and its concept approximations are constructed using its lower approximation $A_{E_1}$ and upper approximation $A_{E_1}^{\prime\prime}$, respectively. The pair $((A_{E_1})'', (A_{E_1})')$ is the lower concept approximation of $A$, whereas $((A_{E_1})'', (A_{E_1})')$ is the upper concept approximation of $A$. 

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- if $B$ is feasible, the concept approximation of $B$ is $(B^I, B^E)$.

- if $B$ is not feasible, the lower and upper concept approximations of $B$ are defined as $((\overline{B}^{E_2})', (\overline{B}^{E_2})'')$ and $((B_2^{E_2})', (B_2^{E_2})'')$, respectively.

If a pair $(A, B)$ is not a concept of the context $K$, it is called a non-definable concept. If the extent of a concept approximates $A$ and the intent approximates $B$, it is said to approximate such a pair $(A, B)$. The following are the four possible scenarios for $(A, B)$: (i) both $A$ and $B$ are feasible, (ii) $A$ is feasible and $B$ is not, (iii) $B$ is feasible and $A$ is not, and (iv) both $A$ and $B$ are not feasible. If both $A$ and $B$ are feasible and $A' = B$, the pair $(A, B)$ is a concept itself, and no approximations are required. In the other cases, the lower (upper) approximation of $(A, B)$ is obtained in terms of the meet (join) of the lower (upper) concept approximations of its individual components. For example, in case (iii) where $B$ is feasible and $A$ is not, the lower approximation $(A, B) := ((A^{E_1})' \cap B', (A^{E_1})' \cap B')$, while the upper approximation $(A, B) := ((A^{E_1})' \cap B')'$.

2.3. Pure double Boolean algebras with operators and Kripke contexts

Let us recall the definitions of $pdBa$ with operators and topological $pdBa$ \cite{19} and its representation results \cite{19}, which are used in Sections 2.2 and 4.2.

**Definition 5.** \cite{19} A structure $\mathfrak{O} := (D, \sqcup, \cap, \neg, \exists, \top, \bot, I, C)$ is a $pdBa$ with operators (pdBao) provided

1. $(D, \sqcup, \cap, \neg, \exists, \top, \bot)$ is a $pdBa$ and
2. $I, C$ are monotonic operators on $D$ satisfying the following for any $x, y \in D$.

\begin{align*}
1a & \quad I(x \cap y) = I(x) \cap I(y) \\
1b & \quad I(x \sqcup y) = I(x) \cup I(y) \\
2a & \quad I(\neg \bot) = \neg \bot \\
2b & \quad I(\neg \top) = \top \\
3a & \quad I(x \cap x) = I(x) \\
3b & \quad I(x \sqcup x) = I(x)
\end{align*}

Moreover, a pdBao is called a topological pdBa (tpdBa) if the following hold.

\begin{align*}
4a & \quad I(x) \subseteq x \\
4b & \quad I(x) \subseteq C(x) \\
5a & \quad I(x) = I(x) \\
5b & \quad C(x) = C(x)
\end{align*}

The duals of $I$ and $C$ with respect to $\neg$, $\circ$ are defined as $I^\circ(a) := \neg I(\neg a)$ and $C^\circ(a) := \circ C(\circ a)$ for all $a \in D$.

Some essential features of the operators $I^\circ, C^\circ$ for a pdBao are:

**Lemma 5.** \cite{19} Let $\mathfrak{O}$ be a pdBao. Then for all $a \in D$,

1. $I^\circ I(a) = I^\circ(a)$ and $C^\circ C(a) = C^\circ(a)$.
2. $a \subseteq I^\circ(a)$ and $C^\circ(a) \subseteq a \cup a$.

Recall Definition 2 of a Kripke context from Section 1. In \cite{19}, we show that for a Kripke context $KK := ((G, R), (M, S), I)$, we can define two unary operators $f_R$ and $f_S$ on $\mathfrak{S}(KK)$ as follows.

For any $(A, B) \in \mathfrak{S}(KK)$,

- $f_R((A, B)) := (A^R, (A^R)^')$,
- $f_S((A, B)) := ((B^S)', B^S)$.

$f_R, f_S$ are well-defined, as $(A^R, (A^R)^')$ and $((B^S)', B^S)$ are both semiconcepts of $KK$. This implies that the set $\mathfrak{S}(KK)$ of semiconcepts is closed under the operators $f_R, f_S$. We have

**Definition 6.** \cite{19} Let $KK := ((G, R), (M, S), I)$ be a Kripke context. The complex algebra of $KK$, $\mathfrak{S}^+(KK) := (\mathfrak{S}(KK), \sqcup, \cap, \neg, \exists, \top, \bot, f_R, f_S)$, is the expansion of the algebra $\mathfrak{S}(KK)$ of semiconcepts with the operators $f_R$ and $f_S$.

Let $f_R^\circ, f_S^\circ$ denote the operators on $\mathfrak{P}(KK)$ that are dual to $f_R, f_S$ respectively. In other words, for each $x := (A, B) \in \mathfrak{S}(KK)$, $f_R^\circ(x) := \neg f_R(\neg x) = \neg f_R((A', A'')) = \neg (A^R, (A^R)^') = ((A'^R)^c, (A'^R)^'c) = (A'^R, (A'^R)^')$, by Proposition 3.1.

Similarly $f_S^\circ(x) := f_S(\neg x) = ((B^S)^c, B^S)$. Again, note that $f_R^\circ(x) = (A'^R, (A'^R)^')$ and $f_S^\circ(x) = ((B^S)^c, B^S)$ are semiconcepts of $KK$. 

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Theorem 4. \[19\] Let \( \mathbb{K}C := ((G, R), (M, S), I) \) be a Kripke context based on the context \( \mathbb{K} := (G, M, I), \mathfrak{H}^+ (\mathbb{K}C) := (\mathfrak{H}(\mathbb{K}), \sqcup, \sqcap, \neg, \cdot, \top, \bot, f_R, f_S) \) is a pdBao.

Different kinds of Kripke contexts have been defined, depending on the properties of the relations giving the Kripke contexts.

Definition 7. \[19\] Let \( \mathbb{K}C := ((G, R), (M, S), I) \) be a Kripke context.

1. \( \mathbb{K}C \) is reflexive from the left, if \( R \) is reflexive.
2. \( \mathbb{K}C \) is reflexive from the right, if \( S \) is reflexive.
3. \( \mathbb{K}C \) is reflexive, if it is reflexive from both left and right.

The cases for symmetry and transitivity of \( \mathbb{K}C \) are similarly defined.

Theorem 5. \[19\] Let \( \mathbb{K}C := ((G, R), (M, S), I) \) be a reflexive and transitive Kripke context. Then \( \mathfrak{H}^+ (\mathbb{K}C) \) is a tpdBa.

The following are introduced in order to prove representation theorems for pdBaos in \[19\]. For every pdBao \( \mathcal{O} := (D, \sqcup, \sqcap, \neg, \cdot, \top, \bot, I, C) \), we construct a Kripke context based on the standard context \( \mathbb{K}C(D) := (\mathcal{F}(\mathcal{D}), \mathcal{I}(\mathcal{D}), \Delta) \) corresponding to the underlying pdBa \( \mathcal{D} \). For that, relations \( R \) and \( S \) are defined on \( \mathcal{F}(\mathcal{D}) \) and \( \mathcal{I}(\mathcal{D}) \) respectively as follows.

- For all \( u, u_1 \in \mathcal{F}(\mathcal{D}), uRa_1 \) if and only if \( 1(a) \in u \) for all \( a \in u_1 \).
- For all \( v, v_1 \in \mathcal{I}(\mathcal{D}), vSa_1 \) if and only if \( C(a) \in v \) for all \( a \in v_1 \).

The following results are required to get (Representation) Theorem [6].

Lemma 6. \[19\] Let \( \mathcal{O} := (D, \sqcup, \sqcap, \neg, \cdot, \top, \bot, I, C) \) be a pdBao. The following hold.

1. For all \( u, u_1 \in \mathcal{F}(\mathcal{D}), uRa_1 \) if and only if for all \( a \in D, Ia \in u \) implies that \( a \in u_1 \).
2. For all \( v, v_1 \in \mathcal{I}(\mathcal{D}), vSa_1 \) if and only if for all \( a \in D, Ca \in v \) implies that \( a \in v_1 \).

Lemma 7. \[19\] Let \( \mathcal{O} \) be a pdBao and \( \mathbb{K}C(\mathcal{O}) := ((\mathcal{F}(\mathcal{D}), R), (\mathcal{I}(\mathcal{D}), S), \Delta) \). For all \( a \in D 

1. \( \overline{F}_a^R = F_{1(a)}^R \) and \( \overline{F}_a^S = F_{1(a)}^S \).
2. \( \overline{I}_a^S = I_{C(a)}^S \) and \( \overline{I}_a^S = I_{C(a)}^S \).

The Kripke context \( \mathbb{K}C(\mathcal{O}) \) of Lemma [7] is used to obtain the representation theorem.

Theorem 6 (Representation theorem). \[19\] A pdBao \( \mathcal{O} := (D, \sqcup, \sqcap, \neg, \cdot, \top, \bot, I, C) \) is embeddable into the complex algebra \( \mathfrak{H}^+ (\mathbb{K}C(\mathcal{O})) := (\mathfrak{H}(\mathbb{K}(\mathcal{O}))), \sqcup, \sqcap, \neg, \cdot, \top, \bot, f_R, f_S \) of the Kripke context \( \mathbb{K}C(D) \). The map \( h : D \rightarrow \mathfrak{H}(\mathbb{K}(\mathcal{O})) \) defined by \( h(x) := (F_x, I_x) \) for all \( x \in D \), is the required embedding.

For a tpdBa \( \mathcal{O} \), we now have

Theorem 7. \[19\] \( \mathbb{K}C(\mathcal{O}) := ((\mathcal{F}(\mathcal{D}), R), (\mathcal{I}(\mathcal{D}), S), \Delta) \) is a reflexive and transitive Kripke context.

We get the representation results for tpdBaos in terms of reflexive and transitive Kripke contexts by combining Theorems [5] and [7].

Theorem 8. \[19\] A tpdBa \( \mathcal{O} \) is embeddable into the complex algebra \( \mathfrak{H}^+ (\mathbb{K}C(\mathcal{O})) \) of the reflexive and transitive Kripke context \( \mathbb{K}C(D) \).

3. The logic PDBL and its modal extensions

We now formulate the logic PDBL for pdBaos, followed by the modal extensions MPDBL for the class of pdBaos and MPDBL4 for tpdBaos.
3.1. PDBL

Recall that a pdBa is obtained from a dBa D, by adding the property: for all \( a \in D \), \( a = a \cap a \) or \( a = a \cup a \) in D (Definition 3.3). By Proposition 3.5, for all \( a \in D \), \( a \cap a \subseteq a \) and \( a \subseteq a \cup a \) are anyway valid in the class of pdBAs. So \( ((a = a \cap a) \) or \( (a = a \cup a) \)) is valid in the class of pdBAs if and only if \( ((a \subseteq a \cap a) \) or \( (a \cup a \subseteq a) \)) is valid in the class. Thus PDBL will need to be a system in which the last disjunction is provable. To facilitate this, one makes use of hypersequent calculus.

Hypersequents were introduced by Pottinger [28] and independently studied by Avron [1]. For a language L, a hypersequent \( \{2,25\} \) is a finite sequence of sequents of the form:

\[ \Gamma_1 \vdash \Delta_1 \mid \Gamma_2 \vdash \Delta_2 \mid \ldots \mid \Gamma_n \vdash \Delta_n, \]

where \( \Gamma_i \) and \( \Delta_i \) are finite sequences of formulae in L, \( \Gamma_i \vdash \Delta_i \) are called the components of the hypersequent. A hypersequent is called empty when \( n = 0 \). It is called single-conclusioned or an s-hypersequent, if \( \Delta_i \) consists of a single formula for all \( i \). For PDBL, a special case of s-hypersequents is considered \( \Gamma_i \) also consists of a single formula for all \( i \). The intended interpretation of an s-hypersequent, that is, the definition of its satisfaction by a valuation (given formally in Definition 10 below), is in terms of the satisfaction of any of its components by the valuation. This fact enables us to deal with the disjunction of statements giving the pdBa axiom.

The language \( \Sigma \) of PDBL consists of a countably infinite set \( \text{OV} := \{p,q,r,\ldots\} \) of object variables, a countably infinite set \( \text{PV} := \{P,Q,R,\ldots\} \) of property variables, propositional constants \( \bot,\top \), and logical connectives \( \land,\lor,\neg,\iff \). The set \( \mathfrak{F} \) of formulae is given by the following scheme:

\[ T \mid \bot \mid p \mid P \mid \alpha \lor \beta \mid \alpha \land \beta \mid \neg \alpha \mid \iff \beta \]

where \( p \in \text{OV} \) and \( P \in \text{PV} \). \lor and \land are definable connectives: \( \alpha \lor \beta := \neg (\neg \alpha \land \neg \beta) \) and \( \alpha \land \beta := \iff (\alpha \lor \beta) \) for all \( \alpha, \beta \in \mathfrak{F} \). A sequent in PDBL is a pair of formulae denoted by \( \alpha \vDash \beta \) for \( \alpha, \beta \in \mathfrak{F} \). If \( \alpha \vDash \beta \) and \( \beta \vDash \alpha \), we use the abbreviation \( \iff \alpha \beta \).

**Definition 8.** An s-hypersequent in PDBL is a finite sequence of sequents of the form:

\[ \alpha_1 \vDash \beta_1 \mid \alpha_2 \vDash \beta_2 \mid \ldots \mid \alpha_n \vDash \beta_n, \]

where for all \( i \in \{1,2,\ldots,n\} \), \( \alpha_i, \beta_i \in \mathfrak{F} \).

We use \( B,C,D,\ldots \) as meta variables for s-hypersequents. Note that a sequent \( \alpha \vDash \beta \) is a special case of an s-hypersequent \( B \mid \alpha \vDash \beta \mid D \), where \( B \) and \( D \) are empty s-hypersequents.

The axioms of PDBL are given by the following schema. \( \alpha, \beta, \gamma \in \mathfrak{F} \), \( p \in \text{OV} \), \( P \in \text{PV} \).

1. \( \alpha \vDash \alpha \).

**Axioms for \( \land \) and \( \lor \):**

\[ \begin{align*}
2a & \alpha \land \beta \vDash \alpha \\
2b & \alpha \lor \beta \\
3a & \alpha \land \beta \vDash \beta \\
3b & \beta \lor \alpha \\
4a & \alpha \land \beta \vdash \left( (\alpha \land \beta) \land \left( (\alpha \land \beta) \right) \right) \\
4b & \left( \alpha \lor \beta \right) \lor \left( (\alpha \lor \beta) \right) \lor \left( \alpha \lor \beta \right)
\end{align*} \]

**Axioms for \( \neg \) and \( \iff \):**

\[ \begin{align*}
5a & \neg (\alpha \land \alpha) \vdash \neg \alpha \\
5b & \alpha \vdash \alpha \iff \alpha \\
6a & \alpha \land \neg \alpha \vdash \bot \\
6b & \top \vdash \alpha \iff \alpha \\
7a & \neg (\alpha \land \beta) \iff (\alpha \land \beta) \\
7b & \alpha \iff (\alpha \lor \beta) \iff (\alpha \lor \beta)
\end{align*} \]

**Generalization of the law of absorption:**

\[ \begin{align*}
8a & \alpha \cap \alpha \vdash \bot \\
8b & \alpha \lor \left( (\alpha \land \beta) \right) \vdash \alpha \lor \alpha \\
9a & \alpha \land \alpha \vdash \bot \\
9b & \alpha \lor \left( (\alpha \land \beta) \right) \vdash \alpha \lor \alpha
\end{align*} \]

**Laws of distribution:**

\[ \begin{align*}
10a & \alpha \lor (\beta \lor \gamma) \vdash \left( (\alpha \lor \beta) \lor (\alpha \lor \gamma) \right) \\
10b & \alpha \lor \left( (\beta \lor \gamma) \right) \lor \left( (\alpha \lor \beta) \lor (\alpha \lor \gamma) \right)
\end{align*} \]

**Axioms for \( \top, \bot \):**

\[ \begin{align*}
11a & \bot \vdash \alpha \\
11b & \alpha \vdash \top \\
12a & \neg \top \vdash \bot \\
12b & \top \vdash \bot \\
13a & \neg \bot \vdash \top \\
13b & \bot \vdash \bot \bot
\end{align*} \]
The compatibility axiom:

\[ (\alpha \cup \alpha) \cap (\alpha \cup \alpha) \vdash (\alpha \cap \alpha) \cup (\alpha \cap \alpha) \]

Special axioms for variables: \[ 15 \]
For \[ \text{CDBL} \]

Axioms 1-14 are lookalikes of the axioms defining \text{CDBL} \[ 19 \], the logic for cdBas – this is expected because a pdBa is also a cdBa. However, it may be pointed out that \text{PDBL} and \text{CDBL} are different syntactically, the fact also begin highlighted by the \text{PDBL} axioms 15a and 15b.

Rules of inference of \text{PDBL} are as follows. \( B, C, D, E, F, G, H, X \) are possibly empty s-hypersequents.

For \( \cap \) and \( \cup \):

\[
\begin{align*}
\frac{B \mid \alpha \vdash \beta \mid C}{B \mid \alpha \cap \gamma \vdash \beta \cap \gamma \mid C} \quad (R1) \\
\frac{B \mid \alpha \vdash \beta \mid C}{B \mid \alpha \cap \gamma \vdash \beta \cup \gamma \mid C} \quad (R2) \\
\frac{B \mid \alpha \cup \gamma \vdash \beta \cup \gamma \mid C}{B \mid \alpha \cap \gamma \vdash \beta \cup \gamma \mid C} \quad (R2)'
\end{align*}
\]

For \( \neg \), \( \exists \):

\[
\begin{align*}
\frac{B \mid \alpha \vdash \beta \mid C}{B \mid \neg \beta \vdash \neg \alpha \mid C} \quad (R3) \\
\frac{B \mid \alpha \vdash \beta \mid C}{B \mid \forall \beta \vdash \exists \alpha \mid C} \quad (R3)'
\end{align*}
\]

Transitivity:

\[
\begin{align*}
\frac{B \mid \alpha \vdash \beta \mid C \quad D \mid \beta \vdash \gamma \mid E}{B \mid D \vdash \gamma \mid C \mid E} \quad (R4) \\
\frac{\alpha \vdash \alpha \cap \alpha \mid \alpha \cup \alpha \vdash \alpha}{(Sp)}
\end{align*}
\]

\[
\begin{align*}
\frac{B \mid \alpha \cap \beta \vdash \alpha \cap \alpha \mid C \quad D \mid \alpha \cap \alpha \vdash \alpha \cap \beta \mid E \quad F \mid \alpha \cup \beta \vdash \beta \cup \beta \mid G \quad H \mid \beta \cup \beta \vdash \alpha \cup \beta \mid X}{B \mid D \vdash F \mid H \vdash \beta \mid C \mid E \mid G \mid X} \quad (R5)
\end{align*}
\]

We shall see in the sequel that (Sp) corresponds to the defining axiom for pdBas, while (R5) captures the order relation of the pdBas.

External rules of inference:

\[
\begin{align*}
\frac{B \mid D \mid D \mid C}{B \mid D \mid C} \quad \text{(External contraction-EC)} \\
\frac{B \mid D \mid E \mid C}{B \mid E \mid D \mid C} \quad \text{(External exchange-EE)} \\
\frac{B}{B \mid C} \quad \text{(External weakening-EW)}
\end{align*}
\]

Derivability is defined in the standard manner: an s-hypersequent \( S \) is \textit{derivable} (or \textit{provable}) in \text{PDBL}, if there exists a finite sequence of s-hypersequents \( S_1, ..., S_m \) such that \( S_m \) is the s-hypersequent \( S \) and for all \( k \in \{1, ..., m\} \) either \( S_k \) is an axiom or \( S_k \) is obtained by applying rules of \text{PDBL} to elements from \( \{S_1, ..., S_{k-1}\} \). Let us give a few examples of derived rules and sequents.

Proposition 7. The following rules are derivable in \text{PDBL}.

\[
\begin{align*}
\frac{B \mid \alpha \vdash \beta \mid C \quad D \mid \alpha \vdash \gamma \mid E}{B \mid D \vdash \alpha \cap \beta \cap \gamma \mid C \mid E} \quad (R6) \\
\frac{B \mid \beta \vdash \alpha \mid C \quad D \mid \gamma \vdash \alpha \mid E}{B \mid D \vdash \beta \cup \gamma \vdash \alpha \cup \alpha \mid C \mid E} \quad (R7)
\end{align*}
\]

\[ \text{Proof.} \quad (R6) \text{ is derived using } (R1), (R1)', \text{ and } (R4), \text{ while for } (R7) \text{ one uses } (R2), (R2)', \text{ and } (R4). \]

Theorem 9. For \( \alpha, \beta, \gamma \in \mathfrak{F} \), the following are provable in \text{PDBL}.
1a \((\alpha \cap \beta) \dashv \vdash (\beta \cap \alpha)\).
1b \(\alpha \cup \beta \vdash \beta \cup \alpha\).
2a \(\alpha \cap (\beta \cap \gamma) \vdash (\alpha \cap \beta) \cap \gamma\).
2b \(\alpha \cup (\beta \cup \gamma) \vdash (\alpha \cup \beta) \cup \gamma\).
3a \((\alpha \cap \alpha) \cap \beta \vdash (\alpha \cap \beta)\).
3b \((\alpha \cup \alpha) \cup \beta \vdash \alpha \cup \beta\).
4a \(\neg \alpha \vdash \neg (\alpha \cap \alpha)\).
4b \(\alpha \cup \alpha \vdash \neg \alpha\).
5a \(\alpha \cap (\alpha \cup \beta) \vdash (\alpha \cap \alpha) \cup \beta\).
5b \(\alpha \cup \alpha \cup \beta \vdash (\alpha \cap \alpha) \cup \beta\).
6a \(\alpha \cap (\alpha \cup \beta) \vdash \alpha \cap \alpha \cup \beta\).
6b \(\alpha \cup \alpha \vdash \alpha \cap (\alpha \cup \beta)\).
7a \(\perp \vdash \alpha \cap \neg \alpha\).
7b \(\alpha \cup \alpha \vdash \perp\).
8a \(\perp \vdash \neg \perp\).
8b \(\perp \vdash \perp\).
9a \(\neg (\alpha \cap \alpha) \vdash \alpha \cap \alpha\).
9b \(\alpha \cup \alpha \vdash \neg (\alpha \cap \alpha)\).
10a \(\neg \alpha \vdash \alpha \cup \alpha\).
10b \(\alpha \cup \alpha \vdash \neg \alpha\).
11a \(\beta \vdash \alpha \lor \beta \mid \beta \lor \beta \vdash \beta\).
11b \(\beta \vdash \beta \land \beta \mid \alpha \land \beta \vdash \beta\).

Proof. The proofs of 1-9 are obtained in a similar way as the proofs given in Theorem 25 in [19] in case of CDBL. Nevertheless, we include the proofs in the appendix to make the paper self-contained.

10a is obtained using axiom 5a, Theorem 9, 4a, 9a) and rules (R3), (R4). We get 10b by using the dual axiom and rules. The proof of 11a is given below. 11b is again obtained by using duals of axioms and rules that prove 11a.

Proof of 11a:

\[
\begin{align*}
\alpha & \vdash \alpha \ (\text{axiom 11a)} \\
\neg \alpha & \vdash \neg \perp \ (\text{R3)} \\
(R1) & \neg \alpha \cap \neg \beta \vdash \neg \perp \cap \neg \beta \\
\neg \alpha & \vdash \neg \alpha \cap \neg \beta \ (\text{axiom 3a)} \\
(R4) & \neg \alpha \cap \neg \beta \vdash \neg \beta \\
(10a, \text{this theorem}) & \beta \cap \beta \vdash \neg \beta \\
\neg \alpha & \vdash \neg \alpha \cap \neg \beta \ (\text{R3 on previous step)} \\
\beta & \vdash \beta \land \beta \mid \beta \lor \beta \vdash \beta \ (\text{using definition of } \lor \text{ in previous step)} \\
\beta & \vdash \alpha \lor \beta \mid \beta \lor \beta \vdash \beta \ (\text{R4}
\end{align*}
\]

\(\square\)

Definition 9. Let \(D := (D, \cap, \cup, \neg, \top, \bot, \perp, \top_D, \bot_D)\) be a pdBa. A valuation \(v : OV \cup PV \cup \{\top, \bot\} \to D\) on \(D\) is a map such that \(v(p) \in D_{\gamma}\), for all \(p \in OV, v(P) \in D_{\cup}\), for all \(P \in PV, v(\top) := \top_D\) and \(v(\bot) := \bot_D\). \(v\) is extended to the set \(\mathfrak{S}\) of formulæ by the following.

1. \(v(\alpha \cup \beta) := v(\alpha) \cup v(\beta)\).
2. \(v(\alpha \cap \beta) := v(\alpha) \cap v(\beta)\).
3. \(v(\neg \alpha) := \neg v(\alpha)\).
4. \(v(\alpha) := v(\alpha)\).

Definition 10. A sequent \(\alpha \vdash \beta\) is said to be satisfied by a valuation \(v\) on a pdBa \(D\) if and only if \(v(\alpha) \subseteq v(\beta)\).

An s-hypersequent \(B\) is said to be satisfied by a valuation \(v\) on a pdBa \(D\) if and only if \(v\) satisfies one of the components of the s-hypersequent \(B\). An s-hypersequent \(B\) is true in \(D\) if and only if for all valuations \(v\) on \(D\), \(v\) satisfies the s-hypersequent \(B\). An s-hypersequent \(B\) is valid in the class of all pdBas if and only if it is true in every pdBa.

Theorem 10 (Soundness). If an s-hypersequent \(G\) is provable in PDBL then it is valid in the class of all pdBas.

Proof. The proof that all the axioms of PDBL are valid in the class of all pdBas is straightforward and can be obtained using Proposition 4 and Definition 3. As examples, we give proofs for 15a and 15b. Let \(p \in OV, P \in PV\), \(D\) a pdBa and \(v\) a valuation on \(D\). As \(v(p) \in D_{\gamma}\), \(v(P) \in D_{\cup}\), for all \(P \in PV, v(\top) := \top_D\) and \(v(\bot) := \bot_D\). \(v\) is extended to the set \(\mathfrak{S}\) of formulæ by the following.

1. \(v(\alpha \cup \beta) := v(\alpha) \cup v(\beta)\).
2. \(v(\alpha \cap \beta) := v(\alpha) \cap v(\beta)\).
3. \(v(\neg \alpha) := \neg v(\alpha)\).
4. \(v(\alpha) := v(\alpha)\).

The completeness theorem is proved using the Lindenbaum-Tarski algebra of PDBL, which is constructed in the usual fashion as follows. A relation \(\equiv\) is defined on \(\mathfrak{S}\) by: \(\alpha \equiv \beta\) if and only if \(\alpha \vdash \beta, \beta \vdash \alpha, \alpha, \beta \in \mathfrak{S}\). \(\equiv\) is a
Lemma 8. \[I\]

1. Let \(I\) be a congruence relation on \(\mathfrak{F}\) with respect to \(\top, \land, \lor, \neg, \bot\). The quotient set \(\mathfrak{F}/I\) with operations induced by the logical connectives, give the Lindenbaum-Tarski algebra \(L(\mathfrak{F}) := (\mathfrak{F}/\equiv, \lor, \land, \neg, \top, \bot, \exists, \forall)\). The axioms in PDBL and Theorem ensure that \(L(\mathfrak{F})\) is a dBa such that \([p]_\top = [p] \land [p] = [p]\) and \([P\top] = [P] \lor [P] = [P]\) for all \(p \in \text{OV}\) and \(P \in \text{PV}\), respectively. One then obtains

**Proposition 8.** For any formulae \(\alpha\) and \(\beta\), the following are equivalent.

1. \([\alpha] \subseteq [\beta]\) in \(L(\mathfrak{F})\).
2. \(\alpha \vdash \beta\) is provable in PDBL.

The proof of Proposition is obtained in a similar way as that of Proposition 27 in [19].

For \(L(\mathfrak{F})\), the corresponding standard context is \(\mathbb{K}(L(\mathfrak{F})) := (\mathcal{F}_p(L(\mathfrak{F})), \mathcal{I}_p(L(\mathfrak{F})), \Delta)\). Let us note

**Lemma 8.**

1. For \(p \in \text{OV}\), \(F'_p = I[p]\) and \(P \in \text{PV}\), \(I'_p = F[p]\).
2. For \(\phi, \sigma \in \mathfrak{F}\), \(F'_\phi \cap \sigma = I(\phi) \cap I[\sigma] = I(\phi \cap \sigma)\).

**Proof.** Let \(p \in \text{OV}\). By Lemma 4, \(F'_p = I[p] \cup p\), as \([p] = [p]\) in \(L(\mathfrak{F})\). By Lemma 3, \(I[p] \cup [p] = I[p]\), implying \(F'_p = I[p]\). Similarly, we can show that for \(P \in \text{PV}\), \(I'_p = F[p]\).

2. In the following equations, we use Lemmas 4 and 3.

\[
F'_\phi \cap \sigma = I(\phi) \cap I[\sigma] = I(\phi \cap [\sigma])
\]

\[
I'_\phi \cup \sigma = I(\phi) \cup I[\sigma] = I(\phi \cup [\sigma])
\]

\[
F'_\phi \cap \sigma = I(\phi) \cap I[\sigma] = I(\phi \cap [\sigma]).
\]

Using Lemma 8, we obtain a well-defined map:

**Definition 11.** The map \(v_0 : \text{OV} \cup \text{PV} \cup \{\top, \bot\} \rightarrow \mathfrak{I}(\mathbb{K}(L(\mathfrak{F})))\) is defined by \(v_0(x) := (F[x], I[x], \Delta)\), for all \(x \in \text{OV} \cup \text{PV} \cup \{\top, \bot\}\).

**Lemma 9.** \(v_0\) is a valuation on \(\mathfrak{I}(\mathbb{K}(L(\mathfrak{F})))\). Moreover, for each formula \(\alpha \in \mathfrak{F}\), \(v_0(\alpha) = (F[\alpha], I[\alpha], \Delta)\).

**Proof.** Let \(p \in \text{OV}\), \(v_0(p) = (F[p], I[p]) \cap I[p] = (F[p] \cap I[p])\). Similarly, for \(P \in \text{PV}\), \(v_0(P) = (F[P], I[P], \Delta)\). By Definition 11, \(v_0(\top) = (F[\top], I[\top], \Delta)\). So \(v_0(\top) = (F[\top], I[\top])\). Similarly, we can show that \(v_0(\bot) = \bot(\mathbb{K}(L(\mathfrak{F})))\). So \(v_0\) is a valuation on \(\mathfrak{I}(\mathbb{K}(L(\mathfrak{F})))\).

The second part of the lemma is proved by induction on the number of connectives in \(\alpha\), and using Definition 1 and Lemma 8(2), one gets the result in each case.

**Proposition 9.** The following are equivalent.

1. An s-hypersequent \(G : \alpha_1 \vdash \beta_1 | \alpha_2 \vdash \beta_2 | \alpha_3 \vdash \beta_3 | \ldots | \alpha_n \vdash \beta_n\) is provable in PDBL.
2. \(\alpha_i \vdash \beta_i\) is provable in PDBL for some \(i \in \{1, 2, \ldots, n\}\).

**Proof.** \(1 \implies 2\) : Let \(G = \alpha_1 \vdash \beta_1 | \alpha_2 \vdash \beta_2 | \alpha_3 \vdash \beta_3 | \ldots | \alpha_n \vdash \beta_n\) be provable in PDBL. If possible, let us assume that for all \(i \in \{1, 2, \ldots, n\}\), \(\alpha_i \not\subseteq \beta_i\) is provable in PDBL. By Proposition 8 for all \(i \in \{1, 2, \ldots, n\}\), \(\alpha_i \not\subseteq \beta_i\) in \(L(\mathfrak{F})\). By Proposition 2, either \(\alpha_i \cap \beta_i\) is not provable in PDBL. By Proposition 2, \(\alpha_i \not\subseteq \beta_i\) is provable in PDBL. If \(\alpha_i \not\subseteq \beta_i\) is provable in PDBL, then there exists a prime filter \(F_0i\) in \(L(\mathfrak{F})\) such that \(\alpha_i \cap \beta_i \notin F_0i\). Therefore \(F_0i\) is a prime filter in \(L(\mathfrak{F})\) such that \(\alpha_i \cap \beta_i \notin F_0i\). Therefore \(F_0i \cap [\beta_i] \notin \beta_i\). If \(\alpha_i \cup \beta_i \not\subseteq \beta_i\), then dually, we can show that there exists \(F_i \in \mathcal{I}(L(\mathfrak{F}))\) such that \(\alpha_i \not\subseteq F_i\) and \(\beta_i \subseteq F_i\). Therefore \(F_i \cap [\beta_i] \notin \beta_i\).
So in either case, $v_0(\alpha_1)\overset{\text{L}}{=}v_0(\beta_i)$ for all $i \in \{1, 2, 3, \ldots, n\}$, which is not possible, as $G$ is valid in the class all pdBas by Theorem [10]. So there exists $i \in \{1, 2, 3, \ldots, n\}$ such that $\alpha_i \vdash \beta_i$ is provable in PDBL.

$2 \Longrightarrow 1$: Let $\alpha_i \vdash \beta_i$ be provable for some $i \in \{1, 2, \ldots, n\}$. The s-hypersequent $G$ is then proved by repeatedly applying the external rules EW and EE.

Lemma 10. $\mathcal{L}(\mathfrak{N})$ is a pdBa.

Proof. By Propositions 8 and (Sp), it follows that $[\alpha] \sqsubseteq [\alpha \cap \alpha]$ or $[\alpha \cup \alpha] \subseteq [\alpha]$ in $\mathcal{L}(\mathfrak{N})$. By Proposition 3, $[\alpha] \cap [\alpha] \subseteq [\alpha]$ and $[\alpha] \subseteq [\alpha] \cup [\alpha]$. So $[\alpha] = [\alpha] \cap [\alpha] = [\alpha \cup \alpha]$ or $[\alpha] = [\alpha] \cup [\alpha] = [\alpha \cup \alpha]$, which implies that $\mathcal{L}(\mathfrak{N})$ is a pdBa.

Theorem 11 (Completeness). If a hyper-sequent $G : \alpha_1 \vdash \beta_1 | \alpha_2 \vdash \beta_2 | \ldots | \alpha_n \vdash \beta_n$ is valid in the class all pdBas then $G$ is provable in PDBL.

Proof. Suppose if possible, $G$ is not provable. Then $\alpha_i \vdash \beta_i$ are not provable for all $i \in \{1, 2, \ldots, n\}$ by Proposition 9. Rest of the proof is similar to the proof of $1 \Longrightarrow 2$ part of Proposition 9.

3.2. MPDBL

The language $\mathcal{L}_1$ of MPDBL adds two unary modal connectives $\Box$ and $\mathbf{L}$ to the language $\mathcal{L}$ of PDBL. The formulae are given by the following scheme.

$$\top \mid \bot \mid p \mid P \mid \alpha \uplus \beta \mid \alpha \cap \beta \mid \neg \alpha \mid \lozenge \alpha \mid \blacklozenge \alpha,$$

where $p \in OV$ and $P \in PV$. The set of formulae is denoted by $\mathfrak{N}_1$. The axiom schema for MPDBL consists of all the axioms of PDBL and the following.

16a $\Box \alpha \cap \Box \beta \vdash \Box(\alpha \cap \beta)$
16b $\Box \alpha \uplus \Box \beta \vdash \Box(\alpha \uplus \beta)$
17a $\Box(\neg \bot) \vdash \neg \bot$
17b $\Box(\bot \top) \vdash \bot \top$
18a $\Box(\alpha \cap \alpha) \vdash \Box(\alpha)$
18b $\Box(\alpha \uplus \alpha) \vdash \Box(\alpha)$

Rules of inference: All the rules of PDBL and the following.

$$\frac{B \mid \alpha \vdash \beta \mid C}{B \mid C \vdash \alpha \uplus \beta \mid C}\quad \text{(R8)}$$
$$\frac{B \mid \alpha \vdash \Box \beta \mid C}{B \mid C \vdash \Box \alpha \vdash \Box \beta \mid C}\quad \text{(R9)}$$

Definable modal operators are $\Diamond$, $\blacklozenge$, given by $\Diamond \alpha := \neg \Box \neg \alpha$ and $\blacklozenge \alpha := \Box \Box \alpha$. It is immediate that

Theorem 12. If an s-hypersequent $G := \alpha_1 \vdash \beta_1 | \alpha_2 \vdash \beta_2 | \ldots | \alpha_n \vdash \beta_n$ is provable in PDBL then $G$ is also provable in MPDBL.

Proof. As all the axioms of PDBL are also axioms of MPDBL and the rules of inference of PDBL are also rules of inference of MPDBL, a proof for the s-hypersequent $G$ in PDBL is also a proof for $G$ in MPDBL.

As we observed in the case of PDBL, the modal axioms defining MPDBL are also lookalikes of those defining MCDBL [19].

Definition 12. A valuation $v$ on a pdBao $\mathfrak{D} := (D, \sqcup, \sqcap, \neg, \top, \bot, D, \bot, D, I, C)$, is a map from $OV \cup PV \cup \{\top, \bot\}$ to $D$ that satisfies the conditions in Definition [9] and the following for the modal operators:

$v(\Box \alpha) := I(v(\alpha))$ and $v(\blacklozenge \alpha) := C(v(\alpha))$.

The satisfaction, truth, and validity of s-hypersequents are defined in the same way as before.

3.2.1. MPDBLΣ

We give a scheme of logics that can be obtained using sequents of MPDBL.

Definition 13. Let $\Sigma$ be any set of sequents in MPDBL. MPDBLΣ is the logic obtained from MPDBL by adding all the sequents in $\Sigma$ as axioms.
If $\Sigma = \emptyset$, $\text{MPDBL}_\Sigma$ is the same as $\text{MPDBL}$. At the end of this section, we give the set $\Sigma$ defining $\text{MPDBL}_4$. Let us go over some of the properties of $\text{MPDBL}_\Sigma$ for any $\Sigma$, which would be applicable to both $\text{MPDBL}$ and $\text{MPDBL}_4$.

The class of pdBaos in which the sequents of $\Sigma$ are valid is denoted by $V_\Sigma$.

**Theorem 13 (Soundness).** If an s-hypersequent $G : \alpha_1 \vdash \beta_1 \mid \alpha_2 \vdash \beta_2 \mid \ldots \mid \alpha_n \vdash \beta_n$ is provable in $\text{MPDBL}_\Sigma$ then it is valid in the class $V_\Sigma$.

**Proof.** To complete the proof it is sufficient to show that all axioms of $\text{MPDBL}_\Sigma$ are valid in $V_\Sigma$ and rules of inference preserve validity. The propositional cases follow from Theorem 10. Here, we check the validity of axioms 16a, 17a, 18a, 16b, 17b and 18b in $V_\Sigma$, and show that rules of inference (R9) and (R10) preserve validity. Let $\mathfrak{D}$ be a pdBa belonging to the class $V_\Sigma$ and $v$ be a valuation on $\mathfrak{D}$. Let $\alpha, \beta \in \mathfrak{D}$. Then $v(\square(\alpha \land \beta)) = I(v(\alpha) \land v(\beta)) = I(v(\alpha) \land v(\beta)) = v(\square\alpha) \land v(\square\beta) = v(\square\alpha \land \square\beta)$. Now $v(\square(\alpha \land \alpha)) = I(v(\alpha \land \alpha)) = I(v(\alpha) \land v(\alpha)) = I(v(\alpha)) = v(\square\alpha)$. Therefore the axioms 16a and 18a are true in $\mathfrak{D}$, which implies that 16a and 18a are valid in the class $V_\Sigma$. Dually, we can show that the axioms 16b and 18b are valid in the class $V_\Sigma$. Now $v(\square(\lnot\perp)) = I(v(\lnot\perp)) = I(\lnot(\perp)) = \lnot\perp_D$ and $v(\blacksquare(j\top)) = C(v(j\top)) = C(v(\top)) = C(j\top_D) = j\top_D$. Therefore axioms 17a and 17b are true in $\mathfrak{D}$, which implies that 17a and 17b are valid in the class $V_\Sigma$.

To show that (R9) preserves validity, let $B \mid \alpha \vdash \beta \mid C$ be valid in $V_\Sigma$. Let $\mathfrak{D} \in V_\Sigma$ and $v$ be a valuation on $\mathfrak{D}$. Now $B \mid \alpha \vdash \beta \mid C$ is true in $\mathfrak{D}$, as $B \mid \alpha \vdash \beta \mid C$ is valid in $V_\Sigma$. So $v$ satisfies the s-hypersequent $B \mid \alpha \vdash \beta \mid C$. Now there are two possibilities. (1) $v$ satisfies a component from $B$ or $C$. (2) $v$ satisfies the component $\alpha \vdash \beta$. Now if (1) holds then $v$ also satisfies the s-hypersequent $B \mid \square\alpha \vdash \square\beta \mid C$, and if (2) holds then $v(\alpha) \subseteq v(\beta)$ in $\mathfrak{D}$, which implies that $I(v(\alpha)) \subseteq I(v(\beta))$, by the monotonicity of $I$. So $v(\square\alpha) \subseteq v(\square\beta)$, which implies that $v$ satisfies the s-hypersequent $B \mid \square\alpha \vdash \square\beta \mid C$. So (R9) preserves validity in $V_\Sigma$.

Showing (R10) preserves validity is similar to the above.

As before, the Lindenbaum-Tarski algebra $\mathcal{L}_\Sigma(\mathfrak{F}_1)$ is obtained for $\text{MPDBL}_\Sigma$; the modal operators in $\mathcal{L}_1$ have induced new unary operators. More precisely, $\mathcal{L}_\Sigma(\mathfrak{F}_1) := (\mathfrak{F}_1/ \equiv, \cup, \cap, \neg, \top, \bot, \blacksquare, \boxtimes, f^\circ, f^\bullet)$, where $f^\circ, f^\bullet$ are defined as: $f^\circ([a]) := [\square a]$, $f^\bullet([a]) := [\blacksquare a]$.

**Note 1.** $f^\circ([a]) = \neg f^\circ([\neg a]) = [\neg\neg a] = [\neg a]$ and, we denote $f^\circ([a]) := f^\circ([a])$.

Similar to the above $f^\bullet([a]) = [\blacksquare a]$ and, we denote $f^\bullet([a]) := f^\bullet([a])$.

Propositions 8 and 9 extend to this case also:

**Proposition 10.** For any formulæ $\alpha$ and $\beta$ the following are equivalent.

1. $[\alpha] \subseteq [\beta]$ in $\mathcal{L}_\Sigma(\mathfrak{F}_1)$.
2. $\alpha \vdash \beta$ is provable in $\text{MPDBL}_\Sigma$.

**Proposition 11.** The following are equivalent.

1. An s-hypersequent $G : \alpha_1 \vdash \beta_1 \mid \alpha_2 \vdash \beta_2 \mid \ldots \mid \alpha_n \vdash \beta_n$ is provable in $\text{MPDBL}_\Sigma$.
2. $\alpha_i \vdash \beta_i$ is provable in $\text{MPDBL}_\Sigma$ for some $i \in \{1, 2, \ldots, n\}$.

The operators $f^\circ, f^\bullet$ are monotonic, according to Proposition 10 and rules (R9), (R10):

**Lemma 11.** For $\alpha, \beta \in \mathfrak{F}_1$, $[\alpha] \subseteq [\beta]$ in $\mathcal{L}_\Sigma(\mathfrak{F}_1)$ implies that $f^\circ([\alpha]) \subseteq f^\circ([\beta])$ and $f^\bullet([\alpha]) \subseteq f^\bullet([\beta])$.

**Proof.** Let $[\alpha] \subseteq [\beta]$. $\alpha \vdash \beta$ by Proposition 10. So $\square\alpha \vdash \square\beta$ and $\blacksquare\alpha \vdash \blacksquare\beta$ by (R9) and (R10). Again using Proposition 10, $[\square\alpha] \subseteq [\square\beta]$ and $[\blacksquare\alpha] \subseteq [\blacksquare\beta]$. Hence $f^\circ([\alpha]) \subseteq f^\circ([\beta])$ and $f^\bullet([\alpha]) \subseteq f^\bullet([\beta])$.

$(\mathfrak{F}_1/ \equiv, \cup, \cap, \neg, \top, \bot, \blacksquare, \boxtimes)$ is a pdBa; Lemma 11 along with axioms 16a, 16b, 17a, 17b and Proposition 11 gives

**Theorem 14.** $\mathcal{L}_\Sigma(\mathfrak{F}_1) \in V_\Sigma$.
Proof. By Propositions \[10\] and \[11\] \((\#_1/\equiv, \sqcup, \sqcap, \neg, [\top],[\bot])\) is a pdBa. Lemma \[11\] implies that the operators \(\sqcap, \sqcup\) are monotonic. Now let \(\alpha, \beta \in \#_1/\equiv\). Then \(\sqcap((\alpha \sqcap \beta)) = \sqcap((\alpha \sqcap \beta)) = \sqcap((\alpha \sqcap \beta)) = \sqcap((\alpha \sqcap \beta))\) by axiom 16a, which implies that \(\sqcap((\alpha \sqcap \beta)) = \sqcap((\alpha \sqcap \beta)) \sqcap((\alpha \sqcap \beta)) = \sqcap((\alpha \sqcap \beta))\). Dually, we can show that \(\sqcup((\alpha \sqcup \beta)) = \sqcup((\alpha \sqcup \beta))\). The others axioms can be proved directly using axioms 17a, 17b, 18a, and 18b, which implies that \(\Sigma_{\Sigma}(\#_1)\) is a pdBa. □

Now, we define \(v_+: \text{OV} \cup \text{PV} \cup \{\top, \bot\} \rightarrow \#_1/\equiv\), by \(v_+(x) := [x]\) for all \(x \in \text{OV} \cup \text{PV} \cup \{\top, \bot\}\).

Lemma 12. \(v_+\) is a valuation on \(\Sigma_{\Sigma}(\#_1)\). Moreover, for all \(\alpha \in \#_1\), \(v_+((\alpha)) = [\alpha]\).

Proof. For any \(p \in \text{OV}\), \(v_+(p) := [p] \in \Sigma_{\Sigma}(\#_1)_{\alpha}\), using axiom 15a. Similarly, using axiom 15b, for any \(P \in \text{PV}\), \(v_+(P) := [P] \in \Sigma_{\Sigma}(\#_1)_{\alpha}\). By definition of \(v_+\), \(v_+(\top) = [\top] = \top_{\Sigma_{\Sigma}(\#_1)}\). Similarly, \(v_+(\bot) = [\bot] = \bot_{\Sigma_{\Sigma}(\#_1)}\). So \(v_+\) is a valuation on \(\Sigma_{\Sigma}(\#_1)\).

The second part of the lemma is proved by mathematical induction on the number of connectives in \(\alpha\); using the definition of \(v_+\) and the induction hypothesis, the cases are obtained. □

Theorem 15 (Completeness). If an s-hypersequent \(G : \alpha_1 \vdash \beta_1 \mid \alpha_2 \vdash \beta_2 \mid \ldots \mid \alpha_n \vdash \beta_n\) is provable in \(\text{MPDL}_{\Sigma}\) then it is provable in \(\text{MPDBL}_{\Sigma}\).

Proof. Let \(G : \alpha_1 \vdash \beta_1 \mid \alpha_2 \vdash \beta_2 \mid \ldots \mid \alpha_n \vdash \beta_n\) be not provable in \(\text{MPDBL}_{\Sigma}\). By Proposition \[11\] for all \(i \in \{1, 2, \ldots, n\}\), \(\alpha_i \vdash \beta_i\) is not provable in \(\text{MPDL}_{\Sigma}\). By Proposition \[11\] for all \(i \in \{1, 2, \ldots, n\}, [\alpha_i]_{\Sigma_{\Sigma}(\#_1)}\), which implies that for all \(i \in \{1, 2, \ldots, n\}, v_+(\alpha_i) \notin v_+(\beta_i)\). So \(G\) is not true in \(\Sigma_{\Sigma}(\#_1)\), which is not possible, as \(G\) is valid in \(\Sigma_{\Sigma}\). So \(G\) is provable in \(\text{MPDBL}_{\Sigma}\). □

\(\text{MPDBL}_{4}\) is defined as the logic \(\text{MPDL}_{\Sigma}\) where \(\Sigma\) contains the following:

\[19a \square \alpha \vdash \alpha \quad 19b \alpha \vdash \blacksquare \alpha \]
\[20a \square \blacksquare \alpha \vdash \blacksquare \alpha \quad 20b \blacksquare \blacksquare \alpha \vdash \blacksquare \alpha \]

We have thus obtained

Theorem 16 (Soundness and Completeness).

1. An s-hypersequent \(G : \alpha_1 \vdash \beta_1 \mid \alpha_2 \vdash \beta_2 \mid \ldots \mid \alpha_n \vdash \beta_n\) is provable in \(\text{MPDBL}\) if and only if \(G : \alpha_1 \vdash \beta_1 \mid \alpha_2 \vdash \beta_2 \mid \ldots \mid \alpha_n \vdash \beta_n\) is valid in the class of all pdBao.

2. An s-hypersequent \(G : \alpha_1 \vdash \beta_1 \mid \alpha_2 \vdash \beta_2 \mid \ldots \mid \alpha_n \vdash \beta_n\) is provable in \(\text{MPDL}_{4}\) if and only if \(G : \alpha_1 \vdash \beta_1 \mid \alpha_2 \vdash \beta_2 \mid \ldots \mid \alpha_n \vdash \beta_n\) is valid in the class of all tpdBao.

4. Relational semantics for the logics

In this section, a two-sorted approach to give a relational semantics for the logics is proposed. Instead of taking the Kripke frame as a relational structure, we work with a context \(K := (G, M, R)\) or a Kripke context \(K C\) based on \(K\). We consider the pdBa \(\Sigma_{\Sigma}(K)\) and as defined earlier, a valuation on \(\Sigma_{\Sigma}(K)\) is a map \(v : \text{OV} \cup \text{PV} \cup \{\top, \bot\} \rightarrow \Sigma_{\Sigma}(K)\) such that \(v(p) \in \Sigma_{\Sigma}(K)_{\alpha}\), \(v(P) \in \Sigma_{\Sigma}(K)_{\beta}\), \(v(\top) = (G, \emptyset)\) and \(v(\bot) = (\emptyset, M)\).

4.1. Relational semantics for \(\text{PDBL}\)

First, we define a model for \(\text{PDBL}\). Interpretation is defined in the same way as in \[14\]. Instead of saying "a part of" relation \(\triangleright\), we refer to it as "co-satisfaction" as in \[9\].

Definition 14. A model for \(\text{PDBL}\) is a triple \(\langle K, v \rangle\), where \(K = (G, M, R)\) is a context and \(v\) is a valuation on \(\Sigma_{\Sigma}(K)\).

Given a model \(\langle K, v \rangle\), the relations \(\models\) of satisfaction and \(\triangleright\) of co-satisfaction of formulae with respect to \(\models\) are defined recursively as follows. For any \(g \in G\) and formula \(\alpha\), \(\models g \models \alpha\) denotes \(\alpha\) is satisfied at \(g\) in \(M\), while for any \(m \in M\), \(m \triangleright \alpha\) denotes \(\alpha\) is co-satisfied at \(m\) in \(M\).
Definition 15 (Satisfaction and co-satisfaction). For each \( g \in G \) and for each \( m \in M \),

1. \( \mathcal{M}, g \models p \) if and only if \( g \in \text{ext}(v(p)) \), where \( p \in \mathcal{OV} \).
2. \( \mathcal{M}, g \models P \) if and only if \( g \in \text{ext}(v(P)) \), where \( P \in \mathcal{PV} \).
3. \( \mathcal{M}, g \models \top \) for all \( g \in G \) and \( \mathcal{M}, m \not\models \top \) for all \( m \in M \).
4. \( \mathcal{M}, g \not\models \bot \) for all \( g \in G \) and \( \mathcal{M}, m \models \bot \) for all \( m \in M \).
5. \( \mathcal{M}, g \models \alpha \cap \beta \) if and only if \( \mathcal{M}, g \models \alpha \) and \( \mathcal{M}, g \models \beta \).
6. \( \mathcal{M}, g \models \neg \alpha \) if and only if \( \mathcal{M}, g \not\models \alpha \).
7. \( \mathcal{M}, m \models p \) if and only if \( m \in \text{int}(v(p)) \), where \( p \in \mathcal{OV} \).
8. \( \mathcal{M}, m \models P \) if and only if \( m \in \text{int}(v(P)) \), where \( P \in \mathcal{PV} \).
9. \( \mathcal{M}, m \models \alpha \cup \beta \) if and only if \( \mathcal{M}, m \models \alpha \) and \( \mathcal{M}, m \models \beta \).
10. \( \mathcal{M}, m \models \neg \alpha \) if and only if \( \mathcal{M}, m \not\models \alpha \).
11. \( \mathcal{M}, g \models \alpha \cup \beta \) if and only if for all \( m \in M \), \( \mathcal{M}, m \not\models \alpha \implies gRm \).
12. \( \mathcal{M}, g \models \neg \alpha \) if and only if for all \( m \in M \), \( \mathcal{M}, m \models \alpha \implies gRm \).
13. \( \mathcal{M}, m \models \neg \alpha \) if and only if for all \( g \in G \), \( \mathcal{M}, g \not\models \alpha \implies gRm \).
14. \( \mathcal{M}, m \models \alpha \cap \beta \) if and only if for all \( g \in G \), \( \mathcal{M}, g \models \alpha \land \beta \implies gRm \).

Definition 16. For any model \( \mathcal{M} := (\mathbb{K}, v) \) and \( \alpha \in \mathfrak{F} \), the maps \( v_1 : \mathfrak{F} \to \mathcal{P}(G) \) and \( v_2 : \mathfrak{F} \to \mathcal{P}(M) \) are defined as \( v_1(\alpha) := \{ g \in G \mid \mathcal{M}, g \models \alpha \} \), and \( v_2(\alpha) := \{ m \in M \mid \mathcal{M}, m \models \alpha \} \).

Proposition 12. Let \( \mathcal{M} := (\mathbb{K}, v) \) be a model and \( \alpha, \beta \in \mathfrak{F} \).

1. \( v_1(\alpha \cap \beta) = v_1(\alpha) \cap v_1(\beta) \) and \( v_2(\alpha \cap \beta) = (v_1(\alpha) \cap v_1(\beta))' \).
2. \( v_1(\alpha \cup \beta) = (v_2(\alpha) \cup v_2(\beta))' \) and \( v_2(\alpha \cup \beta) = v_2(\alpha) \cap v_2(\beta) \).
3. \( v_1(\neg \alpha) = v_2(\alpha)^c \) and \( v_2(\neg \alpha) = (v_1(\alpha))' \).
4. \( v_1(\omega) = (v_2(\alpha))' \) and \( v_2(\omega) = (v_2(\alpha))^c \).
5. \( v_1(\bot) = G \) and \( v_2(\bot) = \emptyset \).
6. \( v_1(\top) = 0 \) and \( v_2(\top) = M \).

Proof. We given the proofs of 1 and 3. 2 and 4 follow dually. 5 and 6 follow directly from Definition 15.
Corollary 2. For any formula $\alpha \in \mathcal{F}$, $(v_1(\alpha), v_2(\alpha)) \in \mathcal{S}(K)$.

Proof. The result is obtained by using mathematical induction on the number of connectives that occur in $\alpha$. Let $M := (K, v)$ be a model, where $v : \mathcal{O} \cup \mathcal{P} \cup \{\top, \bot\} \rightarrow \mathcal{S}(K)$ is the valuation.

For the base case let the number of connectives in $\alpha$, $n = 0$. Then either $\alpha$ belongs to $\mathcal{O} \cup \mathcal{P}$ or $\alpha$ is a propositional constant.

Let $\alpha = p$. Then $v_1(p) = \{g \in G \mid M, g \models p\} = \text{ext}(v(p))$ and $v_2(p) = \{m \in M \mid M, m \succ p\} = \text{int}(v(p))$ by the definition of satisfaction and co-satisfaction. So $(v_1(p), v_2(p)) = (\text{ext}(v(p)), \text{int}(v(p))) = (v(p)) \in \mathcal{S}(K)$.

If $\alpha = \top$ then similar to the above, $(v_1(\alpha), v_2(\alpha)) = (\text{ext}(v(P)), \text{int}(v(P))) = (v(P)) \in \mathcal{S}(K)$.

For $\alpha = \lor$, $v_1(\top) = G$ and $v_2(\top) = \emptyset$. So $(v_1(\top), v_2(\top)) = (G, \emptyset) \in \mathcal{S}(K)$, and if $\alpha = \bot$ then $v_1(\bot) = \emptyset$ and $v_2(\bot) = M$. So $(v_1(\bot), v_2(\bot)) = (\emptyset, M) \in \mathcal{S}(K)$.

Now, we assume that the result is true for all formulae with number of connectives less or equal to $n$. Let $\alpha$ be a formula with $n + 1$ connectives. Then $\alpha$ is of one of the forms $\beta \land \delta$, $\beta \lor \delta$, $\neg \beta$, or $\beta$, where $\beta, \delta$ are formulae with number of connectives less or equal to $n$. By induction hypothesis $(v_1(\beta), v_2(\beta))$ and $(v_1(\delta), v_2(\delta))$ belong to $\mathcal{S}(K)$. Therefore $(v_1(\beta), v_2(\beta)) \cap (v_1(\delta), v_2(\delta))$, $(v_1(\beta), v_2(\beta)) \cup (v_1(\delta), v_2(\delta))$, $\neg(v_1(\beta), v_2(\beta))$ and $\beta(\beta, v_2(\beta))$ all belong to $\mathcal{S}(K)$, as $\mathcal{S}(K)$ is closed under $\cap, \cup, \neg, \beta$.

By Proposition 12 it follows that

$$(v_1(\beta \land \delta), v_2(\beta \land \delta)) = (v_1(\beta) \cap v_1(\delta), v_2(\beta) \cap v_2(\delta)) = (v_1(\beta), v_2(\beta)) \cap (v_1(\delta), v_2(\delta)),$$

$$(v_1(\beta \lor \delta), v_2(\beta \lor \delta)) = ((v_2(\beta) \lor v_2(\delta)), v_2(\beta) \lor v_2(\delta)) = (v_1(\beta), v_2(\beta)) \cup (v_1(\delta), v_2(\delta)),$$

$$(v_1(\neg \beta), v_2(\neg \beta)) = (v_1(\beta), v_2(\beta)) = (v_2(\beta), v_2(\beta)) = \beta(\beta, v_2(\beta)).$$

So $(v_1(\alpha), v_2(\alpha)) \in \mathcal{S}(K)$. \qed

Corollary 3. For any formulae $\alpha, \beta \in \mathcal{F}$,

1. $(v_1(\alpha \land \beta), v_2(\alpha \land \beta)) = (v_1(\alpha), v_2(\alpha)) \cap (v_1(\beta), v_2(\beta)).$
2. $(v_1(\alpha \lor \beta), v_2(\alpha \lor \beta)) = (v_1(\beta), v_2(\beta)) \cup (v_1(\alpha), v_2(\alpha)).$
3. $(v_1(\neg \alpha), v_2(\neg \alpha)) = (v_1(\neg \alpha), v_2(\neg \alpha))$ and $\beta(\beta, v_2(\beta)) = (v_1(\neg \alpha), v_2(\neg \alpha)).$

Proof. The proof follows from Proposition 12 Corollary 2 and the definitions of the operations $\cap, \cup, \neg, \beta$ on $\mathcal{S}(K)$ as given in Section I. \qed

Corollary 4. For any model $M := (K, v)$ and $\alpha \in \mathcal{F}$, $v(\alpha) = (v_1(\alpha), v_2(\alpha))$.

Proof. Follows by induction on the number of connectives that occur in a formula, using definition of extension of a valuation and Corollary 3. Note that $v(p) = (\text{ext}(v(p)), \text{int}(v(p))) = (v_1(p), v_2(p))$ and similarly, $v(P) = (v_1(P), v_2(P))$. $v(\top) = (G, \emptyset) = (v_1(\top), v_2(\top))$ by Proposition 12. Similarly, $v(\bot) = (v_1(\bot), v_2(\bot))$. \qed

Definition 17. Let $M$ be a model. For $\alpha, \beta \in \mathcal{F}$ a sequent $\alpha \vdash \beta$ is said to be satisfied in the model $M$ if and only if the following hold.

1. For all $g \in G \ M, g \models \alpha$ implies that $M, g \models \beta$.
2. For all $m \in M \ M, m \sim \beta$ implies that $M, m \sim \alpha$.

Proposition 13. Let $M := (K, v)$ be a model. For $\alpha, \beta \in \mathcal{F}$ a sequent $\alpha \vdash \beta$ is satisfied in the model $M$ if and only if $v(\alpha) \subseteq v(\beta)$ in $\mathcal{S}(K)$, that is, $\alpha \vdash \beta$ is satisfied by $v$ on the pdBa $\mathcal{S}(K)$.

Proof. Let $\alpha \vdash \beta$ be satisfied in the model $M$. Let $g \in v_1(\alpha)$. Then $M, g \models \alpha$, which implies that $M, g \models \beta$. So $g \in v_1(\beta)$, whence $v_1(\beta) \subseteq v_1(\beta)$. Let $m \in v_2(\beta)$. Then $M, m \models \beta$, which implies that $M, m \models \alpha$. So $m \in v_2(\alpha)$, which implies that $v_2(\beta) \subseteq v_2(\alpha)$. Therefore $v(\alpha) \subseteq v(\beta)$.

Conversely, let $v(\alpha) \subseteq v(\beta)$. Then $v_1(\alpha) \subseteq v_1(\beta)$ and $v_2(\beta) \subseteq v_2(\alpha)$. Let $M, g \models \alpha$. Then $g \in v_1(\alpha)$, which implies that $g \in v_1(\beta)$. So $M, g \models \beta$. Similar to the above, we can show that $M, m \models \beta$ implies that $M, m \models \alpha$. Therefore $\alpha \vdash \beta$ is satisfied in the model $M$. \qed

Next we define satisfaction of an $s$-hypersequent in a model and validity of an $s$-hypersequent in the class $K$ of all contexts. Then we show that PDBL is sound and complete with respect to the class $K$. 17
Definition 18. A hyper-sequent \( G : \alpha_1 \vdash \beta_1|\alpha_2 \vdash \beta_2|\ldots|\alpha_n \vdash \beta_n \) is said to be satisfied in a model \( M \) if and only if at least one of the components of \( G \) is satisfied in \( M \).

\( G \) is said to be valid in \( K \) if and only if \( G \) is satisfied in every model \( M \) based on \( K \).

Proposition 14. Let \( M := (K, v) \) be a model. A hyper-sequent \( G : \alpha_1 \vdash \beta_1|\alpha_2 \vdash \beta_2|\alpha_3 \vdash \beta_3|\ldots|\alpha_n \vdash \beta_n \) is satisfied in \( M \) if and only if \( G \) is satisfied by the valuation \( v \) on the pdBa \( \Omega(K) \).

Proof. Proof follows from Definition \[10\] and Proposition \[13\].

Theorem 17 (Soundness). If a hyper-sequent \( G : \alpha_1 \vdash \beta_1|\alpha_2 \vdash \beta_2|\alpha_3 \vdash \beta_3|\ldots|\alpha_n \vdash \beta_n \) is provable in PDBL then it is valid in \( K \).

Proof. Let \( M := (K, v) \) be a model based on \( K \). Theorem \[10\] and Proposition \[14\] ensure that \( G \) is satisfied in \( M \) as \( v \) is also a valuation on the pdBa \( \Sigma(K) \).

Now, we consider the pair \((K(\mathcal{L}(\mathcal{F})), v_0)\), where \( v_0 \) is as defined in Definition \[11\]. By Lemma \[9\] \((K(\mathcal{L}(\mathcal{F})), v_0)\) is a model for PDBL. The model is denoted by \( M(\mathcal{F}) := (K(\mathcal{L}(\mathcal{F})), v_0) \).

Theorem 18 (Completeness). If an s-hyper-sequent \( G : \alpha_1 \vdash \beta_1|\alpha_2 \vdash \beta_2|\ldots|\alpha_n \vdash \beta_n \) is valid in \( K \) then \( G \) is provable in PDBL.

Proof. Let \( G := \alpha_1 \vdash \beta_1|\alpha_2 \vdash \beta_2|\ldots|\alpha_n \vdash \beta_n \) be valid in \( K \). If possible, let us assume that \( G \) is not provable in PDBL. By Proposition \[8\] for all \( i \in \{1, 2, \ldots, n\}, \alpha_i \vdash \beta_i \) is not provable in PDBL. By Proposition \[8\] \([\alpha_i] \nsubseteq [\beta_i] \) in \( \mathcal{L}(\mathcal{F}) \), for all \( i \in \{1, 2, \ldots, n\} \). Now, we consider the model \( M(\mathcal{F}) := (K(\mathcal{L}(\mathcal{F})), v_0) \). Using the same argument as the proof of part 1 in Proposition \[9\], we can show that \( v_0(\alpha_i) \nsubseteq v_0(\beta_i) \) for all \( i \in \{1, 2, 3 \ldots n\} \), which is not possible, as \( G \) is valid in the class \( K \). So \( G \) is provable in PDBL.

We end the section with an example of characterisation of a class of contexts.

Theorem 19. Let \( K \) be a class of contexts. Then the sequent \( \top \sqcap \top \leftarrow \bot \sqcup \bot \) is valid in the class \( K \) if and only if \( K \) contains only contexts of the kind \( K := (G, M, R) \), where \( R = G \times M \).

Proof. For any model \( M := (K, v) \), \( \top \sqcap \top \leftarrow \bot \sqcup \bot \) is satisfied in \( M \) if and only if \( v(\top \sqcap \top) = v(\bot \sqcup \bot) \), i.e. if and only if \( (G, G') = (M', M) \), i.e. if and only if \( G' = M \) and \( G = M' \), which is equivalent to \( R = G \times M \).

4.2. Relational semantics for MPDBL

Definition 19. A model for MPDBL is a pair \( M := (K, v, K) \), where \( K := ((G, R), (M, S), I) \) is a Kripke context and \( v \) is a valuation on \( \Omega(K) \).

Satisfaction and co-satisfaction for propositional formulae are given in a similar manner as in Definition \[15\] adding the following for modal formulæ.

Definition 20. Let \( M := (K, v) \) be a model based on the Kripke context \( K := ((G, R), (M, S), I) \). In the following \( g \in G \) and \( m \in M \).

1. \( M, g \models \Box \alpha \) if and only if for all \( g_1 \in G \) \((gRg_1 \Rightarrow M, g_1 \models \alpha)\).
2. \( M, m \models \Diamond \alpha \) if and only if for all \( g, g_1 \in G ((gRg_1 \Rightarrow M, g_1 \models \alpha) \Rightarrow gM)\).
3. \( M, m \models \Box \alpha \) if and only if for all \( m_1 \in M ((mSm_1 \Rightarrow M, m_1 \models \alpha) \Rightarrow gM)\).
4. \( M, g \models \Diamond \alpha \) if and only if for all \( m \in M ((mSm_1 \models M, m_1 \models \alpha) \Rightarrow gM)\).

Lemma 13. For all \( g \in G \) and for all \( m \in M \), the following hold.

1. \( M, g \models \Diamond \alpha \) if and only if there exists \( g_1 \in G \) such that \( gRg_1 \) and \( M, g_1 \models \alpha \).
2. \( M, m \models \Diamond \alpha \) if and only if for all \( g \in G ((\text{there exists } g_1 \in G \text{ with } M, g_1 \models \alpha \text{ and } gRg_1 \Rightarrow gM) \).
3. \( M, m \models \Box \alpha \) if and only if there exists \( m_1 \in M \) such that \( mSm_1 \) and \( M, m_1 \models \alpha \).
4. \( M, g \models \Box \alpha \) if and only if for all \( m \in M ((\text{there exists } m_1 \in M \text{ with } mSm_1 \text{ and } M, m_1 \models \alpha) \Rightarrow gM) \).
Proof. 1. Let $g \in G$. Then $M, g \models \Box \alpha$ if and only if $M, g \models \neg \Box \neg \alpha$, which is equivalent to $M, g \not\models \Box \neg \alpha$. Now $M, g \not\models \Box \neg \alpha$ if and only if there exists $g_1 \in G$ such that $gRg_1$ and $M, g_1 \not\models \neg \alpha$, which is equivalent to say that there exists $g_1 \in G$ such that $gRg_1$ and $M, g_1 \models \alpha$.

2. Let $m \in M$. Then $M, m \models \Box \alpha$ if and only if $M, m \models \neg \Box \neg \alpha$, which is equivalent to say that for all $g \in G(M, g \not\models \Box \neg \alpha$ implies $gIm)$. Now $M, g \not\models \Box \neg \alpha$ if and only if there exists $g_1 \in G$ such that $gRg_1$ and $M, g_1 \not\models \neg \alpha$. So $M, m \models \Box \alpha$ if and only if for all $g \in G$ ((there exists $g_1 \in G$ such that $gRg_1$ and $M, g_1 \models \alpha$) $\implies$ $gIm$).

3. Let $m \in M$. Then $M, m \models \neg \Box \alpha$ if and only if for all $g \in M$ ((there exists $m_1 \in M$ such that $mSm_1$ and $M, m_1 \models \alpha$). So $M, g \models \neg \Box \alpha$ if and only if for all $g \in M$ such that $M \models \Box \neg \alpha$ and the cases are $\Box \neg \alpha$.

4. Let $g \in G$. Then $M, g \models \neg \Box \alpha$ if and only if $M, g \models \neg \Box \alpha$, which is equivalent to say that for all $m \in M$ ($M, m \not\models \Box \alpha$ implies $gIm$). Now $M, m \models \Box \alpha$ if and only if there exists $m_1 \in M$ such that $mSm_1$ and $M, m_1 \models \alpha$.

Given a formula $\alpha$, recall the definitions of $v_1(\alpha)$ and $v_2(\alpha)$. Let $M := (K, v)$ be a model. Then we have the following.

**Proposition 15.**

1. If $\alpha = \Box \phi$ then $v_1(\alpha) = v_1(\phi)_R$ and $v_2(\alpha) = (v_1(\phi)_I)'$.
2. If $\alpha = \neg \Box \phi$ then $v_1(\alpha) = (v_2(\phi)_R)'$ and $v_2(\alpha) = v_2(\phi)_S$.
3. If $\alpha = \Box \phi$ then $v_1(\alpha) = v_1(\phi)_R$ and $v_2(\alpha) = (v_1(\phi)_R)'$.
4. If $\alpha = \neg \Box \phi$ then $v_1(\alpha) = (v_1(\phi)_R)'$ and $v_2(\alpha) = v_2(\phi)_S$.

**Proof.**

1. Let $\alpha = \Box \phi$. Then $v_1(\alpha) = \{ g \in G : M, g \models \Box \phi \} = \{ g \in G : \text{for all } g_1 \in G, (Rgg_1 \implies M, g_1 \models \phi) \} = \{ g \in G : g \in v_1(\phi)_R \} = v_1(\phi)_R$. $v_2(\alpha) = \{ m \in M : M, m \models \Box \phi \} = \{ m \in M : \text{for all } g, g_1 \in G, ((Rgg_1 \implies M, g_1 \models \phi) \implies gIm) \} = (v_1(\phi)_R)'$.

2. Let $\alpha = \neg \Box \phi$. Then $v_2(\alpha) = \{ m \in M : M, m \models \neg \Box \phi \} = \{ m \in M : \text{for all } g, m_1 \in M((Smm_1 \implies M, m_1 \models \phi) \implies gIm) \} = (v_2(\phi)_S)'$.

3. Let $\alpha = \Box \phi$. Then by Lemma 13.1, 13.2, $v_1(\alpha) = (v_2(\neg \phi)_R)$ and $v_2(\alpha) = (v_1(\neg \phi)_R)'$, which implies that $v_1(\alpha) = v_1(\neg \phi)_R = (v_1(\neg \phi)_R)$ and $v_2(\alpha) = (v_1(\neg \phi)_R)' = (v_1(\phi)_R)'$.

4. The proof is similar to that of 3.

**Corollary 5.** For any formula $\alpha$, $(v_1(\alpha), v_2(\alpha)) \in S_1(K)$.

**Proof.** The corollary is proved using mathematical induction on the number of connectives in $\alpha$, and the cases are obtained using Corollary 2 Proposition 13 and the induction hypothesis.

The result analogous to Corollary 3 is also true here. Moreover, the following holds.

**Corollary 6.**

1. $(v_1(\Box \alpha), v_2(\Box \alpha)) = f_R(v_1(\alpha), v_2(\alpha))$.
2. $(v_1(\neg \Box \alpha), v_2(\neg \Box \alpha)) = f_S(v_1(\alpha), v_2(\alpha))$.

**Proof.** The proof follows from the definition of extension of a valuation, Proposition 13 and Corollary 5.

**Corollary 7.** For any model $M := (K, v)$ and formula $\alpha$, $v(\alpha) = (v_1(\alpha), v_2(\alpha))$.

**Proof.** The proof follows from the definition of extension of a valuation, Corollaries 5 and 6.

Satisfaction in a model $M := (K, v)$ of a sequent $\alpha \vdash \beta$ is given in a similar manner as in Definition 17. Let $K$ denote the class of all Kripke contexts. For an s-hypersequent, the definition of satisfaction in a model and validity in the class $K$ is given as: 

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Definition 21. $\alpha_1 \vdash \alpha_2 \vdash \beta_2 \ldots | \alpha_n \vdash \beta_n$ is said to be satisfied in a model $M$ if and only if at least one of the components of $G$ is satisfied in $M$.

$G$ is said to be true in $K\mathcal{C}$ if and only if $G$ is satisfied in every model $M$ based on $K\mathcal{C}$.

$G$ is said to be valid in $K\mathcal{C}$ if and only if $G$ is true in every $K\mathcal{C}$ for all $K\mathcal{C} \in K\mathcal{C}$.

Results analogous to Propositions 13 and 14 are also true here:

Proposition 16. Let $M := (K\mathcal{C}, v)$ be a model. For $\alpha, \beta \in \mathfrak{F}_1$, a sequent $\alpha \vdash \beta$ is satisfied in $M$ if and only if $v(\alpha) \sqsubseteq v(\beta)$ in $\mathfrak{S}(K\mathcal{C})$.

Proposition 17. An $s$-hypersequent $G := \alpha_1 \vdash \alpha_2 \vdash \beta_2 \ldots | \alpha_n \vdash \beta_n$ is satisfied in a model $M := (K\mathcal{C}, v)$ if and only if $G$ is satisfied by the valuation $v$ on the pdBao $\mathfrak{S}^+(K\mathcal{C})$.

Theorem 20 (Soundness). If an $s$-hypersequent $G := \alpha_1 \vdash \beta_1 | \ldots | \alpha_n \vdash \beta_n$ is provable in MPDBL then it is valid in the class $K\mathcal{C}$.

Proof. Let $M := (K\mathcal{C}, v)$ be a model based on $K\mathcal{C}$. By Theorem 16(1), $G$ is satisfied by the valuation $v$ on the pdBao $\mathfrak{S}^+(K\mathcal{C})$. So by Proposition 17, $G$ is satisfied in $M$.

To prove completeness, we recall the Lindenbaum-Tarski algebra $L(\mathfrak{F}_1)$ and the Kripke context $K\mathcal{C}(L(\mathfrak{F}_1)) := ((F_p(L(\mathfrak{F}_1)), R), (F_p(L(\mathfrak{F}_1)), S), \Delta)$ based on the context $L(\mathfrak{F}_1)$ defined in Section 5 and Section 2 respectively. Recall the definitions of $R$ and $S$. Using Notation 1 we have the following.

- For all $F, F_1 \in F_p(L(\mathfrak{F}_1))$, $FRF_1$ if and only if $\varphi(\alpha) \in F$ for all $\alpha \in F_1$.
- For all $I, I_1 \in F_p(L(\mathfrak{F}_1))$, $ISON_1$ if and only if $\varphi(\alpha) \in F$ for all $\alpha \in I_1$.

Recall the map $v_0 : OV \cup PV \cup \{\top, \bot\} \rightarrow \mathfrak{S}(L(\mathfrak{F}_1))$ given in Definition 11 $(K\mathcal{C}(L(\mathfrak{F}_1)), v_0)$ is a model for MPDBL and denoted as $M(\mathfrak{F}_1)$.

Lemma 14. For any formula $\alpha$, $v_0(\alpha) = (F_{[\alpha]}, I_{[\alpha]})$.

Proof. We will use mathematical induction on the number of connectives in $\alpha$ to prove this lemma. The base case follows from the definition of $v_0$. Let the claim be true for all formulae $\alpha$ with number of connectives less or equal to $n$.

Let $\alpha$ be a formula with $n + 1$ connectives. Then $\alpha \in \{\beta \cup \gamma, \beta \cap \gamma, \neg \beta, \neg \gamma\}$ or $\alpha \in \{\square \beta, \Box \beta\}$, where $\beta$ and $\gamma$ are formulae with number of connectives less or equal to $n$. Proof for the propositional cases is similar to the proof of Lemma 9. To complete the proof it is sufficient to show that $v_0(\alpha) = (F_{[\alpha]}, I_{[\alpha]})$, for $\alpha \in \{\square \beta, \Box \beta\}$.

Case I: Let $\alpha = \square \beta$. By Proposition 15, $v_1(\alpha) = v_1(\beta)$. By induction hypotheses $v_1(\beta) = F_{[\beta]}$. Therefore $v_1(\alpha) = F_{[\alpha]}$. By Lemma 11, $v_1(\alpha) = F_{[\alpha]} = F_{[\beta]} = F_{[\beta]} = F_{[\alpha]}$. By Proposition 15, $v_2(\alpha) = (v_1(\beta))^\prime = (F_{[\alpha]})^\prime$. By Lemmas 3 and axiom 17a, $v_2(\alpha) = (F_{[\alpha]})^\prime = I_{[\alpha]}^\prime = I_{[\alpha]}$. Therefore $v_0(\alpha) = (F_{[\alpha]}, I_{[\alpha]})$.

Case II: Let $\alpha = \Box \phi$. Then similar to the above, we can show that $v_2(\alpha) = I_{[\alpha]}$ and $v_1(\alpha) = (v_2(\alpha))^\prime = F_{[\alpha]}$. In this case we use Lemmas 4 and axiom 17b. Therefore $v_0(\alpha) = (F_{[\alpha]}, I_{[\alpha]})$.

Theorem 21 (Completeness). If an $s$-hypersequent $G := \alpha_1 \vdash \beta_1 | \ldots | \alpha_n \vdash \beta_n$ is valid in $K\mathcal{C}$ then $G$ is provable in MPDBL.

Proof. Let the $s$-hypersequent $G := \alpha_1 \vdash \beta_1 | \ldots | \alpha_n \vdash \beta_n$ be valid in $K\mathcal{C}$. If possible, assume that $G$ is not provable in MPDBL. By Propositions 11 and 11 $[\alpha_i] \not\subseteq [\beta_i]$ for all $\alpha_i \in \{1, 2, 3, \ldots, n\}$. The rest of the proof is similar to the proof of Theorem 18.

Let $K\mathcal{C}_{RT}$ be the class of all reflexive and transitive Kripke contexts. In the following theorems, we show that MPDBL4 is sound and complete with respect to the class $K\mathcal{C}_{RT}$.

Theorem 22 (Soundness). If an $s$-hypersequent $G := \alpha_1 \vdash \beta_1 \ldots | \alpha_n \vdash \beta_n$ is provable in MPDBL then $G$ is valid in $K\mathcal{C}_{RT}$.
Proof. The proof is a consequence of Theorem 16(2) and Proposition 17 as for a model $M := (KC, v)$, $v$ is also a valuation on the tpdBa $\mathfrak{S}^+(KC)$. 

\[\text{Theorem 23} \text{ (Completeness).} \text{ If an s-hypersequent } G := \alpha_1 \vdash \beta_1 \ldots \alpha_n \vdash \beta_n \text{ is valid in } KC_{RT} \text{ then } G \text{ is provable in } MPDBL_4.\]

Proof. In case of $MPDBL_4$, $L_C(\mathfrak{A}_1)$ forms a tpdBa. By Theorem 7 it follows that $KC(L(\mathfrak{A}_1))$ is reflexive and transitive. Rest of the proof is similar to the proof of Theorem 21.

Let $KC_{RST}$ be the class of all reflexive, symmetric and transitive Kripke contexts. Next, we propose a logic $MPDBL_5$ for the class $KC_{RST}$. The logic $MPDBL_5$ is obtained from $MPDBL_4$ by adding the following sequents as axioms.

\[21a \quad \triangledown \alpha \vdash \Box \Box \alpha \quad 21b \quad \Box \triangledown \alpha \vdash \triangledown \alpha.\]

\[\text{Theorem 24} \text{ (Soundness).} \text{ If an s-hypersequent } G := \alpha_1 \vdash \beta_1 \ldots \alpha_n \vdash \beta_n \text{ is provable in } MPDBL_5 \text{ then } G \text{ is valid in } KC_{RST}.\]

Proof. To complete the proof it is sufficient to show that 21a and 21b are valid in $KC_{RST}$. To show 21a is valid in $KC_{RST}$, let $KC \in KC_{RST}$ and $M := (KC, v)$ be a model based on $KC$.

By Corollary 6 and 7 $v(\Box \alpha) = f_{R^2}^1((v_1(\alpha), v_2(\alpha))) = f_{R}^1(v_1(\alpha)^R, (v_1(\alpha)^R)^R) = (v_1(\alpha)^R, (v_1(\alpha)^R)^R)^R$. Let $x \in v_1(\alpha)^R$. Then $R(x) \cap v_1(\alpha) \neq \emptyset$. Let $z_0 \in R(x) \cap v_1(\alpha)$ for some $z_0 \in G$. Then $xRz_0$. Let $y \in R(x)$. Then $xRy$, which implies that $yRx$, as $R$ is symmetric. So $yRz_0$, as $R$ is transitive, which implies that $R(y) \cap v_1(\alpha) \neq \emptyset$. So $y \in v_1(\alpha)^R$, whence $R(x) \subseteq v_1(\alpha)^R$. So $x \in (v_1(\alpha)^R)_{R}$, which implies that $v_1(\alpha)^R \subseteq (v_1(\alpha)^R)_{R}$. So $(v_1(\alpha)^R)_{R}' \subseteq (v_1(\alpha)^R)'$.

By Proposition 19 $\Box \alpha \vdash \Box \Box \alpha$ is satisfied in $M$, which implies that $\Box \alpha \vdash \Box \Box \alpha$ is true in $KC$. So $\triangledown \alpha \vdash \Box \Box \alpha$ is valid in $KC_{RST}$.

Similar to the above proof, we can show that $\Box \triangledown \alpha \vdash \triangledown \alpha$ is valid in $KC_{RST}$. 

\[\text{Theorem 25} \text{ (Completeness).} \text{ If an s-hypersequent } G := \alpha_1 \vdash \beta_1 \ldots \alpha_n \vdash \beta_n \text{ is valid in } KC_{RST} \text{ then } G \text{ is provable in } MPDBL_5.\]

Proof. We show that $KC(L_S(\mathfrak{A}_1))$ is a symmetric Kripke context and rest of the proof is similar to the proof of Theorems 21 and 23.

To show $R$ is symmetric, let $F, F_1 \in F_p(L_S(\mathfrak{A}_1))$ and $FRF_1$. Then for all $[\alpha] \in F_1$, $f_0([\alpha]) = [\alpha] \in F$. By axiom 21a and Proposition 18 $f_0([\alpha]) = [\alpha] \subseteq [\Box \Box \alpha] = [\Box \alpha] = f_\Box f_0([\alpha])$. By Lemma 31, $[\alpha] \cap [\alpha] \subseteq f_0([\alpha])$, as $L_S(\mathfrak{A}_1)$ is a pdBao. So $[\alpha] \cap [\alpha] \subseteq f_\Box f_0([\alpha])$. Let $[\alpha] \in F$. Then $[\alpha] \cap [\alpha] \subseteq F$, as $F$ is a filter. So $f_\Box f_0([\alpha]) \in F$, as $F$ is a filter. By Lemma 31, $f_0([\alpha]) \in F_1$, which implies that $RF_1F$. So $R$ is symmetric. Similar to the above, we can show that $S$ is also symmetric. 

5. PDBL and conceptual knowledge

In this part, we look at PDBL through the lens of conceptual knowledge [24]. As discussed in [24], the main assumption for conceptual knowledge is that it must be expressible by the three basic notions of objects, attributes and concepts, and these three are linked by the four basic relations “an object has an attribute”, “an object belongs to a concept”, “an attribute abstracts from a concept” and “a concept is a subconcept of another concept”. These three basic notions and relations are represented mathematically using the concept lattice of a context. Let us recall this briefly. For each object $g \in G$, there is $\{g\}^\prime, \{g\}^\prime \in B(\mathbb{K})$, the smallest concept containing $g$ in its extent and for each attribute $m \in M$, the largest concept $\{m\}^\prime, \{m\}^\prime \in B(\mathbb{K})$ containing $m$ in its intent. Additionally, $g_1m \Leftrightarrow \{(g_1)^\prime, \{g\}^\prime \} \subseteq \{m\}^\prime, \{m\}^\prime \} \subseteq B(\mathbb{K})$. Now the concept lattice $B(\mathbb{K})$ depicts the objects and attributes of the context $\mathbb{K}$, if each object $g$ is identified with $\{(g)^\prime, \{g\}^\prime \}$ and each attribute $m$ with $\{(m)^\prime, \{m\}^\prime \}$. On the other hand, the four relations are describable in $B(\mathbb{K})$ as follows. The object $g$ has the attribute $m$ if and only if $\{(g)^\prime, \{g\}^\prime \} \subseteq \{m\}^\prime, \{m\}^\prime \}$, the object $g$ belongs to the concept $(A, B)$ if and only if $\{(g)^\prime, \{g\}^\prime \} \subseteq (A, B)$, the
attribute \( m \) abstracts from the concept \((A, B)\) if and only if \((A, B) \subseteq (\{m\}', \{m\}'')\), and the concept \((A_1, B_1)\) is a subconcept of the concept \((A_2, B_2)\) if and only if \((A_1, B_1) \subseteq (A_2, B_2)\).

Lukeh et al. [24] assume that all conceptual knowledge for a given field of interest is derived from a comprehensive formal context \(U := (G_U, M_U, I_U)\), which is referred to as a conceptual universe for the field of interest.

Theorem 2 implies that a concept can be expressed using semiconcepts. As PDBL is a logic for semiconcepts of a context, it is natural to expect that all the three basic notions and the four basic relations of conceptual knowledge can also be represented in the system. In the rest of the section, we establish this. Recall that a model for PDBL is \(M := (\mathbb{K}, v)\), where \(v : \mathbb{OV} \cup \mathbb{PV} \cup \{\top, \bot\} \to \mathcal{S}(\mathbb{K})\) is a valuation. From Corollary 2 it follows that each formula \(\alpha\) represents a semiconcept \(v(\alpha) := (v_1(\alpha), v_2(\alpha))\), where \(v_1(\alpha) := \{g \in G : M, g \models \alpha\}\) and \(v_2(\alpha) := \{m \in M : M, m \succ \alpha\}\). Let \(\alpha\) be a formula and \(g \in G\). Then \(M, g \models \alpha\) represents that \(\alpha\) is satisfied at \(g\) in a model \(M\). In other words, the satisfaction relation \(\models\) may be considered as a relation between \(G\) and \(\mathbb{G}\), i.e., \(\models \subseteq G \times \mathbb{G}\). \(M, g \models \alpha\) if and only if \(g \in v_1(\alpha)\), which is equivalent to “the object \(g\) belongs to the semiconcept \(v(\alpha)\)”. Similar to the above, the co-satisfaction relation \(\succ \subseteq M \times \mathbb{G}\) represents the relation “the property \(m\) abstracts from the semiconcept \(v(\alpha)\)”.

The next proposition tells us when a formula represents a concept.

**Proposition 18.** Let \(\alpha \in \mathbb{G}\) and \(M := (\mathbb{K}, v)\) be a model. Then the sequents \(\alpha \land \alpha \not\models \alpha \lor \alpha, \alpha \land \alpha \not\models \alpha\) and \(\alpha \not\models \alpha \lor \alpha\) are satisfied in \(M\) if and only if \(v(\alpha)\) is a concept of \(\mathbb{K}\).

**Proof.** Let \(\mathbb{K} := (G, M, I)\) and \(A \subseteq G, B \subseteq M\). Now observe that the pair \((A, B)\) is a concept of \(\mathbb{K}\) if and only if \((A, B) \cup (A, B) = (A, B) \cap (B, A) = (A, B)\). As \(v\) is a valuation, \(v\) preserve \(\cap, \cup\) and in \(\mathcal{S}(\mathbb{K})\) partial order. The proof is a consequence of the observations.

**Definition 22.** For a model \(M := (\mathbb{K}, v)\), \(\mathfrak{F}_M := \{\alpha \in \mathbb{G} : \alpha \land \alpha \not\models \alpha \lor \alpha, \alpha \land \alpha \not\models \alpha, \alpha \not\models \alpha \lor \alpha\}\) are satisfied in \(M\).

In other words, \(\mathfrak{F}_M = \{\alpha \in \mathbb{G} : v(\alpha) \in \mathcal{B}(\mathbb{K})\}\). In the next proposition, we characterize the set \(\mathfrak{F}_M\) for a class of models that is based on the class \(\mathcal{K}_*\) of contexts defined as \(\mathcal{K}_* := \{\mathbb{K} := (G, M, I) : I = G \times M\}\).

**Proposition 19.** Let \(M := (\mathbb{K}, v)\) be a model based on the context \(\mathbb{K} \in \mathcal{K}_*\). The following hold.

1. For \(\mathbb{K} \in \mathcal{K}_*, \mathcal{B}(\mathbb{K}) = \{(G, M)\}\),
2. \(\top \cap \top \in \mathfrak{F}_M\),
3. For any \(\alpha \in \mathbb{G}\), \(\alpha \in \mathfrak{F}_M\) if and only if for any \(\beta \in \mathfrak{F}_M\), \(\beta \not\models \alpha\) is satisfied in the model \(M\).

**Proof.** 1. Let \((A, B)\) be a concept of \(\mathbb{K}\), which implies that \(B = A' = M\) and \(A = B' = G\). So \((A, B) = (G, M)\), which implies that \(\mathcal{B}(\mathbb{K}) = \{(G, M)\}\).
2. \(v(\top \cap \top) = v(\top) \cap v(\top) = (G, \emptyset) \cap (G, \emptyset) = (G, G') = (G, M) \in \mathcal{B}(\mathbb{K})\). So \(\top \cap \top \in \mathfrak{F}_M\).
3. Let \(\alpha, \beta \in \mathfrak{F}_M\). Then \(v(\alpha)\) and \(v(\beta)\) are concepts of \(\mathbb{K}\), which implies that \(v(\alpha) = (G, M) = v(\beta)\). So \(\beta \not\models \alpha\) is satisfied in the model \(M\).

Conversely, let \(\alpha \in \mathbb{G}\), and take \(\beta \in \mathfrak{F}_M\) such that \(\beta \not\models \alpha\) is satisfied in the model \(M\). Then \(v(\alpha) = v(\beta) = (G, M) \in \mathcal{B}(\mathbb{K})\). So \(\alpha \in \mathfrak{F}_M\).

As a consequence of Proposition 19, for any \(\alpha \in \mathfrak{F}_M\), \(\top \cap \top \not\models \alpha\) is satisfied in each model \(M\) based on \(\mathbb{K} \in \mathcal{K}_*\).

From Proposition 19 it follows that the restriction \(\models_1 \) of the satisfaction relation \(\models \) to the set \(G \times \mathfrak{F}_M\) represents the relation “the object \(g \in G\) belongs to the concept \(v(\alpha), \alpha \in \mathfrak{F}_M\)”, while the restriction \(\succ_1 \) of the co-satisfaction relation \(\succ \) to the set \(M \times \mathfrak{F}_M\) represents the relation “the property \(m \in M\) abstracts from the concept \(v(\alpha), \alpha \in \mathfrak{F}_M\)”. Let \(M := (\mathbb{K}, v)\) be a model based on \(\mathbb{K}\) and \(\alpha \models \beta\) be a valid sequent in PDBL such that \(\alpha, \beta \in \mathfrak{F}_M\). This implies that \(v(\alpha) \subseteq v(\beta)\). Let \(\alpha, \beta \in \mathfrak{F}_M\). Then \(v(\alpha) \subseteq v(\beta)\) and \(v(\alpha), v(\beta)\) are concepts of \(\mathbb{K}\). So the restriction of the relation \(\models \subseteq \mathbb{G} \times \mathfrak{F}_M\) to the set \(\mathfrak{F}_M \times \mathfrak{F}_M\) represents the subconcept-superconcept relation.

Thus we have shown that using PDBL and its models, one can express the notion of concept, the relations “object belongs to a concept”, “property abstracts from a concept” and “a concept is a subconcept of another concept”. What about the other two basic notions of objects and attributes, and the relation “an object has an attribute”? These can also be represented in the system PDBL, by modifying the definition of a model.
For any context \( \mathbb{K} := (G, M, I) \) and for \( g \in G \), \( (\{g\}, \{g\}') \) is called the semiconcept of \( \mathbb{K} \) generated by the object \( g \) and for \( m \in M \), \( (\{m\}', \{m\}) \) is called the semiconcept of \( \mathbb{K} \) generated by the attribute \( m \). We note the following maps.

- \( \zeta : G \to \{((g), \{g\}') : g \in G\} \subseteq \mathcal{F}(\mathbb{K}) \), where \( g \mapsto ((g), \{g\}') \) for all \( g \in G \).
- \( \eta : M \to \{((m)', \{m\}) : m \in M\} \subseteq \mathcal{F}(\mathbb{K}) \), where \( m \mapsto ((m)', \{m\}) \) for all \( m \in M \).

\( \zeta \) and \( \eta \) are bijections. So the sets \( \mathcal{OS} := \{((g), \{g\}') : g \in G\} \) and \( \mathcal{PS} := \{((m)', \{m\}) : m \in M\} \) of semiconcepts can be used to describe respectively, objects and properties in \( \mathbb{K} \). Taking a cue from hybrid modal logic \( \mathbb{K} \) and the above observation, we have the following.

**Definition 23.** A PDBL model \( \mathbb{M} := (\mathbb{K}, v) \) is called a named model if for any \( p \in \mathcal{OV} \), \( v(p) = ((g), \{g\}') \) for some \( g \in G \), and for any \( P \in \mathcal{PV} \), \( v(P) = (\{m\}', \{m\}) \) for some \( m \in M \). Moreover, for each semiconcept \( ((g), \{g\}') \) generated by the object \( g \), there is \( p \in \mathcal{OV} \) such that \( v(p) = ((g), \{g\}') \), and for each semiconcept \( (\{m\}', \{m\}) \) generated by the attribute \( m \), there is \( P \in \mathcal{PV} \) such that \( v(P) = (\{m\}', \{m\}) \).

Under such an interpretation of PDBL, the object variables and property variables represent the objects and attributes of conceptual knowledge.

Next, we give an example of a named model based on the context given in Table 1.

### Table 1: \( \mathbb{K}_z \)

| Elfenbein Körpers | Leder Überzug | Permanent Überzug | Messungshallenk | aufgesetzter Mündungs- | eingedrehtes Mundstück | neue Form | gebeugt Form | d') (Simmangröße) | d' (Simmangröße) | g (Simmangröße) | b (Simmangröße) |
|-------------------|--------------|------------------|----------------|----------------------|-----------------------|----------|-------------|------------------|----------------|----------------|----------------|
| 1558              | *            | *                | *              | *                    | *                     | *        | *           | *                | *              | *              | *              |
| 1559              | *            | *                | *              | *                    | *                     | *        | *           | *                | *              | *              | *              |
| 1560              | *            | *                | *              | *                    | *                     | *        | *           | *                | *              | *              | *              |
| 1561              | *            | *                | *              | *                    | *                     | *        | *           | *                | *              | *              | *              |
| 1562              | *            | *                | *              | *                    | *                     | *        | *           | *                | *              | *              | *              |
| 1563              | *            | *                | *              | *                    | *                     | *        | *           | *                | *              | *              | *              |
| 1564              | *            | *                | *              | *                    | *                     | *        | *           | *                | *              | *              | *              |
| 1565              | *            | *                | *              | *                    | *                     | *        | *           | *                | *              | *              | *              |
| 4030              | *            | *                | *              | *                    | *                     | *        | *           | *                | *              | *              | *              |
| 1566              | *            | *                | *              | *                    | *                     | *        | *           | *                | *              | *              | *              |
| 1567              | *            | *                | *              | *                    | *                     | *        | *           | *                | *              | *              | *              |
| 1568              | *            | *                | *              | *                    | *                     | *        | *           | *                | *              | *              | *              |
| 1569              | *            | *                | *              | *                    | *                     | *        | *           | *                | *              | *              | *              |
| 1570              | *            | *                | *              | *                    | *                     | *        | *           | *                | *              | *              | *              |
| 1571              | *            | *                | *              | *                    | *                     | *        | *           | *                | *              | *              | *              |
| 4031              | *            | *                | *              | *                    | *                     | *        | *           | *                | *              | *              | *              |

**Example 1.** The context \( \mathbb{K}_z := (G_z, M_z, I_z) \) in Table 1 is part of the conceptual universe for the field of interest, a family of musical instruments described in [24]. The set \( G_z \) of objects contains some concrete zink and is represented by the numbers in the first column of Table 1. The set \( M_z \) of properties of the zink are written in the first row of Table 1. Each cell \((i, j)\) with * encodes the information that the zink \( g \) in the \( i \)th row has the property \( m \) in the \( j \)th column, that is, \( gI_zm \). Moreover, for this example, we assume that each empty cell \((i, j)\) encodes the information that the zink \( g \) in the \( i \)th row lacks the property \( m \) in the \( j \)th column, that is, \( gK \)\( \neg m \).
Let \( \{p_1, p_2, p_3, \ldots \} \) and \( \{P_1, P_2, P_3, \ldots \} \) be enumerations of the sets \( OV \) and \( PV \) respectively. The context contains 14 objects and so the set of semiconcepts generated by the objects is also finite. Let \( \{x_1, x_2, \ldots, x_{14} \} \) be an enumeration of the set of semiconcepts generated by the objects. Similarly, let \( \{y_1, y_2, \ldots, y_{13} \} \) be the enumeration of the set of semiconcepts generated by the properties. Now, consider the model \( M_z := (K_z, v_r) \), where \( v_r : OV \cup PV \cup \{\top, \bot\} \rightarrow S(K_z) \) is defined as follows:

\[
v_r(p_i) = x_i, \quad i \leq 14 \quad v_r(P_i) = y_i, \quad i \leq 13
\]
\[
\quad = x_1, \quad i \geq 15 \quad = y_1, \quad i \geq 14
\]
\[
v_r(\top) = \top_{S(K_z)}, \quad v_r(\bot) = \bot_{S(K_z)}.
\]

From the definition, it follows that \( M_z \) is a named model.

Gerader Zink and Stiller Zink are two typical instances of concepts from the zink family. A zink is a Gerader Zink if and only if it has the property gerade Form, while a zink is a Stiller Zink if and only if it has the properties gerade Form and eingedrehtes Mundstück. Let us assume that \( y_1 := \{(\text{gerade Form})', \{(\text{gerade Form}) = ((1558, 1559, 1560, 1561, 1562), \{(\text{gerade Form})\) and
\]
\[
y_2 := \{(\text{eingedrehtes Mundstück})', \{(\text{eingedrehtes Mundstück}) = ((1559, 1560, 1561, 1562), \{(\text{eingedrehtes Mundstück})\}.
\]

Then \( P_1 \) represents the concept Gerader Zink in the model \( M_z := (K_z, v_r) \), while \( P_2 \) represents the semiconcept generated by eingedrehtes Mundstück.

Observe that the concept \( \{(1559, 1560, 1561, 1562), \{(\text{gerade Form, eingedrehtes Mundstück})\} \) of \( K_z \) represents the concept Stiller Zink. Now
\[
\{(1559, 1560, 1561, 1562), \{(\text{gerade Form, eingedrehtes Mundstück})\} = \{(1558, 1559, 1560, 1561, 1562), \{(\text{gerade Form})\} \cap \{(1559, 1560, 1561, 1562), \{(\text{eingedrehtes Mundstück})\} = v_r(P_1) \cap v_r(P_2) = v_r(P_1 \cap P_2).
\]

So \( P_1 \cap P_2 \) represents the concept Stiller Zink in the model \( M_z \). In fact, we have the following.

**Proposition 20.** Let \( M := (K, v) \) be a named model based on a finite context \( K \) and \( (A, B) \) be a concept of \( K \). Then there is a formula \( \alpha \) in \( PDBL \) such that \( v) = (A, B) \).

Proof. Let \( K := (g_1, \ldots, g_n), \{m_1, \ldots, m_k\}, \alpha \) and \( (A, B) := (\{g_1, \ldots, g_n\}, \{m_1, \ldots, m_k\}) \) be a concept of \( K \). Then \( (A, B) = (\{g_1, \ldots, g_n\}, \{m_1, \ldots, m_k\}) \) and \( (A, B) = \{(g_1, \ldots, g_n), \{m_1, \ldots, m_k\} = (B', B'') = \cap_{i=1}^{k}\{m_i, m_i\} \). So there are some \( P_i, i = 1, 2, \ldots, k \), such that \( v(P_i) = (\{m_i, m_i\}) \) in the model \( M \). The required formula is \( \alpha := \cap_{i=1}^{k} P_i \).  

Let us return to the example. \( \eta(g(\text{Stimmgröße})) \cap \neg \eta(a(\text{Stimmgröße})) = ((1558, 1561, 1562), \{g(\text{Stimmgröße})\}) \) \( \eta(g(\text{Stimmgröße})) \cap \top \) encodes the fact that in \( K_z \), if an object has the property \( g(\text{Stimmgröße}) \) then it does not possess the property \( a(\text{Stimmgröße}) \). From this fact, we also derive that there is no object in \( K_z \) that has the properties \( a(\text{Stimmgröße}) \) and \( g(\text{Stimmgröße}) \) together, that is, \( \eta(a(\text{Stimmgröße})) \cap \eta(g(\text{Stimmgröße})) = \bot = (\emptyset, M) \). In fact, using \( PDBL \), we can show that \( \eta(a(\text{Stimmgröße})) \cap \eta(g(\text{Stimmgröße})) = \bot = (\emptyset, M) \) is deducible from \( \eta(g(\text{Stimmgröße})) \cap \neg \eta(a(\text{Stimmgröße})) = \eta(g(\text{Stimmgröße})) \). For that, we prove the following theorem.

**Theorem 26.** The following rules are derivable in \( PDBL \).

\[
\frac{\alpha \cap \neg \beta \vdash \alpha}{\vdash \alpha \cap \beta} \quad (R10) \quad \frac{\alpha \vdash \alpha \cap \neg \beta}{\alpha \cap \beta \vdash \bot} \quad (R11)
\]

Proof. The proof of (R11) is similar to that of (R10) and we only prove (R10).

Note that from Proposition 6 and Theorem 11 one obtains in \( PDBL \).

\[
\alpha \cap \bot \vdash \bot \quad \alpha \cap \bot \vdash \bot \quad (\ast)
\]

Then we have:

\[
\frac{\alpha \cap \neg \beta \vdash \alpha}{\vdash \alpha \cap \beta} \quad \alpha \cap \neg \beta \vdash \alpha \quad \frac{\alpha \cap \neg \beta \cap \beta \vdash \alpha \cap \neg \beta \cap \beta}{\alpha \cap \neg \beta \cap \beta \vdash \alpha \cap \beta \cap \beta \quad (R1) \quad \alpha \cap \neg \beta \cap \beta \vdash \alpha \cap \beta \quad \alpha \cap \beta \cap \beta \vdash \alpha \cap \beta \quad (R4)}\]

\[
\frac{\alpha \cap \bot \vdash \alpha \cap \bot \quad \alpha \cap \bot \vdash \alpha \cap \bot \quad (\ast)}{\bot \vdash \alpha \cap \beta \quad (\ast)}
\]

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Let \( y_3 := \{a(\text{Stimmgröße})\}' \) and \( y_4 := \{g(\text{Stimmgröße})\}' \) in Example 11. As \( \eta(g(\text{Stimmgröße})) \cap \neg \eta(a(\text{Stimmgröße})) = \eta(g(\text{Stimmgröße})) \), the sequents \( P_4 \land \neg P_3 \land P_2 \) and \( P_4 \land P_3 \land P_2 \) are satisfied in the model \( M_2 \). By soundness and Theorem 26, \( \vdash P_4 \land P_3 \land P_2 \) and \( P_4 \land P_3 \land P_2 \) are also satisfied in \( M_2 \), which implies that \( v_r(\bot) \subseteq v_r(P_4 \land P_3) \) and \( v_r(P_4 \land P_3) \subseteq v_r(\bot) \). So \( v_r(\bot) = v_r(P_4 \land P_3) = v_r(P_4) \land v_r(P_3) \), whence \( \eta(a(\text{Stimmgröße})) \cap \neg \eta(g(\text{Stimmgröße})) = \bot \).

We end this part by demonstrating that under the interpretation given in terms of named models, we can characterize the clarified context. Let us recall the definition of a clarified context.

**Definition 24.** A context \( \mathbb{K} := (G, M, I) \) is called a clarified context if \( I \) satisfies the following.

1. For all \( g_1, g_2 \in G \), \( \{g_1\}' = \{g_2\}' \Rightarrow g_1 = g_2 \).
2. For all \( m_1, m_2 \in M \), \( \{m_1\}' = \{m_2\}' \Rightarrow m_1 = m_2 \).

**Theorem 27.** The following inference rules are valid in the class \( K \) for any \( p, q \in OV \) and \( P, Q \in PV \) if and only if \( K \) is the class of all clarified contexts.

\[
\begin{align*}
\frac{p \land p \vdash p \land p}{P \land P \vdash Q \land Q} & \quad \ldots (1) & \frac{p \land p \vdash q \land q \land p \land p}{P \land P \vdash Q \land Q} & \quad \ldots (2) \\
\frac{q \vdash p}{P \vdash Q} & \quad \ldots (3) & \frac{q \vdash p}{P \land P \vdash Q} & \quad \ldots (4)
\end{align*}
\]

**Proof.** Let \( K \) be the class of all clarified contexts and \( p \land p \vdash q \land q \land p \land p \) be valid in the class \( K \). To show \( p \land p \vdash p \land p \) are valid in the class \( K \), let \( M := (\mathbb{K}, v) \) be a named model based on \( \mathbb{K} := (G, M, R) \in K \). Then \( v(p) = \{g\}' \) and \( v(q) = \{g\}' \) for some \( g, g_1 \in G \), and \( v(P) = \{m\}' \) for some \( m, m_1 \in M \). By Corollary 3, \( v(p \land p) = v(p) \lor v(p) = (\{g\}' \lor \{g\}' \lor (\{g\}' \lor v(Q)) = \{g\}' \lor \{g\}' \). Similarly \( v(q \land q) = \{g\}' \lor \{g\}' \). Then \( v(q \land q) = v(p \land p) \subseteq v(q \land q) = \{g\}' \lor \{g\}' \) and \( \{g\}' \lor \{g\}' \subseteq v(p \land p) = \{g\}' \lor \{g\}' \), which implies that \( \{g\}' \lor \{g\}' = \{g\}' \lor \{g\}' \). So \( \{g\}' = \{g\}' \), which implies that \( g = g_1 \), as \( \mathbb{K} \) is a clarified context. So \( v(p) = v(q) \), which implies that \( v(p) \subseteq v(q) \) and \( v(q) \subseteq v(p) \). So \( p \land p \vdash q \land q \land p \land p \) are satisfied in \( M \), which implies that \( p \land p \vdash q \) and \( q \land q \land p \land p \) are true in \( \mathbb{K} \). So \( p \land p \vdash q \) and \( q \land q \land p \land p \) are valid in the class \( \mathbb{K} \), which implies that \( (1) \) and \( (2) \) are valid in \( \mathbb{K} \).

Similar to the above proof we can show that \( (3) \) and \( (4) \) are valid in the class \( K \).

Conversely, let \( (1) \)-\( (4) \) be valid in the class \( K \). To show \( K \) is the class of all clarified contexts, let \( \mathbb{K} := (G, M, R) \in K \), and \( g, g_1 \in G \), \( m, m_1 \in M \) such that \( \{g\}' = \{g_1\}' \) and \( \{m\}' = \{m_1\}' \). Let \( M := (\mathbb{K}, v) \) be a named model such that \( v(p) = \{g\}' \lor \{g\}' \) and \( v(Q) = \{m\}' \lor \{m\}' \). Then \( p \land p \vdash q \land q \land p \land p \) and \( P \land P \vdash Q \land Q \) are satisfied in \( M \), as \( v(p \land p) = v(q \land q) \) and \( v(P \land P) = v(Q \land Q). \) So \( p \land p \) and \( P \land P \) are satisfied in \( M \), which implies that \( (1) \) and \( (2) \) are valid in \( \mathbb{K} \).

Hence \( \mathbb{K} \) is the class of all clarified contexts. \( \square \)

6. MPDBL and rough sets

In this section, we explain our approach to FCA from the perspective of rough set theory. For that, recall the approximation spaces \((\mathcal{G}, \mathcal{E}_1)\) and \((\mathcal{M}, \mathcal{E}_2)\) for a context \( \mathbb{K} := (G, M, I) \) given in Section 2.2. The relations \( \mathcal{E}_1, \mathcal{E}_2 \) are defined as: \( g_1, g_2 \in G \) if and only if \( I(g_1) = I(g_2) \) for \( g_1, g_2 \in G \), and \( m_1, m_2 \in M \) if and only if \( I^{-1}(m_1) = I^{-1}(m_2) \). We consider the Kripke context \( \mathbb{K} \in \mathcal{C}_{SD} := ((G, \mathcal{E}_1), (M, \mathcal{E}_2), I) \) based on \( \mathbb{K} \). \( \mathbb{K} \in \mathcal{C}_{SD} \) is an example of a reflexive, symmetric and transitive Kripke context, and may be understood to represent a basic classification skill of an agent about objects in \( G \) and properties in \( M \), with respect to information given in the context \( \mathbb{K} \). We now propose the notion of definability of a semiconcept \((A, B)\), \( A \subseteq G, B \subseteq M \).

**Definition 25.** A definable semiconcept of \( \mathbb{K} \in \mathcal{C}_{SD} := ((G, \mathcal{E}_1), (M, \mathcal{E}_2), I) \) is a semiconcept \((A, B)\) if \( A \) and \( B \) are categories in the Pawlakian approximation spaces \((G, \mathcal{E}_1)\) and \((M, \mathcal{E}_2)\) respectively.
As noted earlier after Theorem 2, a semiconcept is either a left semiconcept of the form \((A, A')\) or a right semiconcept of the form \((B', B)\). In Corollary 5 below, we shall see that the former kind is definable if and only if the first component is a category in \((G, E_1)\), and the latter one is definable if and only if the second component is a category in \((M, E_2)\).

**Proposition 21.** If a semiconcept \((A, B)\) is a concept of \(\mathbb{K}\) then it is a definable semiconcept of \(\mathbb{K}C_{SD}\).

**Proof.** Let \((A, B)\) be a concept of \(\mathbb{K}\). Then \(A' = A\) and \(B'' = B\). Let \(g \in A\) and \(g_1 \in E_1(g)\). By definition of \(E_1\), this means \(I(g) = I(g_1)\). As \(g \in A\), \(g_1m\) for any \(m \in A'\). So \(g_1m\) for all such \(m\), and thereby, \(g_1 \in A'' = A\). Thus \(E_1(g) \subseteq A\), which gives \(g \in A\). So \(A = A\), by Proposition 6(v), and by Proposition 6(vii), \(A\) is a category in \((G, E_1)\). Similarly, we can show that \(B\) is a category in \((M, E_2)\). Therefore, \((A, B)\) is a definable semiconcept. \(\square\)

It is not always the case that a semiconcept is a definable semiconcept. Let us use an example to demonstrate this fact.

**Example 2.** \((G, M, I)\) is a modified subcontext of a context provided by Wille [13], where \(G := \{\text{Leech, Bream, Frog, Dog, Cat}\}\) and \(M := \{a, b, c, g\}\), and \(a:=\) requires water to survive, \(b:=\) lives in water, \(c:=\) lives on land, and \(g:=\) can move. Table 2 provides \(I\), where \(*\) as an entry corresponding to object \(x\) and property \(y\) indicates that \(x\) if \(y\) holds.

|               | \(a\) | \(b\) | \(c\) | \(g\) |
|---------------|-------|-------|-------|-------|
| Leech         | *     | -     | -     | *     |
| Bream         | *     | *     | *     | -     |
| Frog          | *     | *     | -     | *     |
| Dog           | *     | -     | -     | *     |
| Cat           | *     | *     | *     | *     |

The induced Kripke context is \(\mathbb{K}_{SD}^G := ((G, \{\{\text{Leech, Bream}, \{\text{Frog}, \{\text{Dog, Cat}\}\}\}), (M, \{a, g\}, \{b\}, \{c\}\}), I)\). \(\mathbb{K}^G_{SD}\) represents that an agent cannot distinguish the properties \(a\) and \(g\), while Leech and Bream as well as Dog and Cat are indistinguishable for an agent based on the information given in \(\mathbb{K}\). \((\{\text{Leech, Bream, Dog}\}, \{a, g\}\) is a non-definable semiconcept of \(\mathbb{K}C_{SD}\).

The question then is: can we approximate a semiconcept of \(\mathbb{K}\) by definable semiconcepts of \(\mathbb{K}C_{SD}\)? For that purpose, the lower and upper approximations of a semiconcept are defined. We split the set \(\delta(\mathbb{K})\) as \(\delta(\mathbb{K}) := (\delta(\mathbb{K}) \cap \mathbb{B}(\mathbb{K})) \cup (\delta(\mathbb{K}) \cup \mathbb{B}(\mathbb{K})) \cup \mathbb{B}(\mathbb{K})\).

**Definition 26.**

(a) Let \((A, B) \in \mathbb{B}(\mathbb{K})\). The lower approximation and upper approximation of \((A, B)\) are \((A, B)\) itself.

(b) Let \((A, B) \in \delta(\mathbb{K}) \cap \mathbb{B}(\mathbb{K})\). The lower approximation and upper approximation of \((A, B)\) are defined as \((A, B) := (A_{E_1}, (A_{E_1})')\) and \((A, B) := (A^{E_1}, (A^{E_1})')\) respectively.

(c) Let \((A, B) \in \delta(\mathbb{K}) \cup \mathbb{B}(\mathbb{K})\). The lower approximation and upper approximation of \((A, B)\) are defined as \((A, B) := ((B^{E_2}), (B^{E_2})')\) and \((A, B) := ((B_{E_2}), B_{E_2})\) respectively.

**Observation 1.** Recall the operators \(f_R\) and \(f_S\) defined in Section 2.3

1. For \((A, B) \in \delta(\mathbb{K}) \cap \mathbb{B}(\mathbb{K})\), \((A, B) = f_{E_1}((A, B)), (A, B) = f_{E_1}^*(((A, B))\).

2. For \((A, B) \in \delta(\mathbb{K}) \cup \mathbb{B}(\mathbb{K})\), \((A, B) = f_{E_2}((A, B)), (A, B) = f_{E_2}^*((A, B))\).

**Example 3.** In Example 2 considered above, the lower and upper approximations of the semiconcept \((\{\text{Leech, Bream, Dog}\}, \{a, b, g\}\) are \((\{\text{Leech, Bream}\}, \{a, b, g\}\) and \((\{\text{Leech, Bream, Cat, Dog}\}, \{a, g\}\) respectively.

**Proposition 22.** Let \(A \subseteq G\) and \(B \subseteq M\).

1. \((A_{E_1})'_{E_2} = (A_{E_1})'\) and \((A^{E_1})'_{E_2} = (A^{E_1})'\).

2. \((B_{E_2})'_{E_1} = (B_{E_2})'\) and \((B^{E_2})'_{E_1} = (B^{E_2})'\).
Proposition 23. Let $\gamma \in \mathfrak{F}_1$. Then the following sequents are derivable in MPDBL5.

(1) $\gamma \vdash \gamma \cup \gamma$
(2) $\gamma \cap \gamma \vdash \emptyset$

Proof. We give the proof for (1). The proof of (2) is similar.

(1)
\[
\frac{\gamma \vdash \gamma}{\gamma \cup \gamma} \quad (\text{R3})' \quad \frac{\gamma \vdash \gamma \cup \gamma}{\gamma \vdash \gamma \cup \gamma} \quad (\text{R4})
\]

Proposition 24. For any $(A, B) \in \mathfrak{S}(\mathbb{K})$, $(A, B) \subseteq (A, B)$ and $(A, B) \subseteq (A, B)$.

Proof. Let $(A, B) \in \mathfrak{S}(\mathbb{K})$. If $(A, B) \in \mathcal{B}(\mathbb{K})$, the results are trivially true.

Case I: Let $(A, B) \in \mathfrak{S}(\mathbb{K}) \cap \mathcal{B}(\mathbb{K})$. We consider a model $\mathbb{M} := (\mathbb{K}, v)$ and $p \in \mathcal{P}$ such that $v(p) = (A, B)$. As $(p) \vdash p$ is a valid sequent of MPDBL5, $(p) \subseteq (p)$, by Proposition 16. Using Proposition 15 and Definition 20(b), $v(\neg p) = (A, B) = (A, B)$. So we get $v(p) \subseteq v(p)$, which implies that $(A, B) \subseteq (A, B)$.

Case II: Let $(A, B) \in \mathfrak{S}(\mathbb{K}) \cap \mathcal{B}(\mathbb{K})$. We consider a model $\mathbb{M} := (\mathbb{K}, v)$ and $p \in \mathcal{P}$ such that $v(p) = (A, B)$. By Proposition 23(1), $p \vdash p \cup p$. So we get $v(\neg p) \subseteq v(\neg p)$, by Proposition 16. Using Proposition 15 and Definition 20(b), $v(\neg p) = (A, B)$. So we get $v(p) \subseteq v(p)$, which implies that $(A, B) \subseteq (A, B)$.

Now, we prove sequents in MPDBL5 that will yield fundamental properties of the approximations of semiconcepts, as will be seen in Proposition 23 below.
\[ v(\mathbf{P}) = ((B^{E_2})', B^{E_2}) = v(P). \] Hence \( v(P) \subseteq v(P \cup P) = v(P) \cup v(P) = v(P) \), as \( v(P) \in \mathcal{H}(K) \cup \) which implies that \( (A, B) \subseteq (A, B) \).

Similar to the above derivations, using the valid sequents \( P \vdash \Box P \) and \( p \cap p \vdash \Diamond p \) of MPDBL5 (the last from Proposition 23.2), we get the property \( (A, B) \subseteq (A, B) \) in \( \mathcal{H}(K) \).

**Observation 2.** From the proof of Proposition 21 we have the following.

1. For a formula \( \alpha \), if \( v(\alpha) \) is a left semiconcept of \( K \) then \( v(\Box \alpha) \) is its lower approximation and \( v(\Diamond \alpha) \) is its upper approximation. Moreover, the valid sequents \( \Box \alpha \vdash \alpha \) and \( \alpha \cap \alpha \vdash \Diamond \alpha \) respectively represent the properties that the lower approximation of a left semiconcept lies “below” the left semiconcept, while its upper approximation lies “above” it.

2. On the other hand, if \( v(\alpha) \) is a right semiconcept of \( K \) then \( v(\Diamond \alpha) \) is its lower approximation and \( v(\Box \alpha) \) is its upper approximation. The valid sequents \( \Diamond \alpha \vdash \alpha \cap \alpha \) and \( \alpha \vdash \Box \alpha \) translate into properties of approximations of right semiconcepts: the lower approximation of a right semiconcept lies below it, while its upper approximation lies above it.

Let \( (A, B) = (X, Y) \). By Proposition 22 it follows that the extent \( X \) and int \( Y \) are categories in \( (G, E_1) \) and \( (M, E_2) \), respectively. By Proposition 21 we have \( X \subseteq A \) and \( B \subseteq Y \), which implies that \( V \subseteq A \subseteq A \) and \( I \subseteq Y \subseteq Y \). From \( X \subseteq A \subseteq A \), we can say that the objects in the extent \( X \) are certainly classified as objects in the extent \( A \) of the concept \( (A, B) \). Now \( I \subseteq Y \) implies that \( Y^c \subseteq (B^c) \). So we can say that properties in \( Y^c \) are certainly classified as properties in the extent \( B^c \) of \( (A, B) \). Similarly, we have \( A \subseteq \text{ext}(A, B) \) and \( \text{int}(A, B) \subseteq I \). So the properties in \( \text{int}(A, B) \) are the properties that possibly belong to the extent \( B \) of the concept \( (A, B) \). Now \( A \subseteq \text{ext}(A, B) \) implies that \( \text{ext}(A, B)^c \subseteq (A^c) \). So the objects in the set \( \text{ext}(A, B)^c \) are the objects that possibly belong to the extent \( A^c \) of \( \neg(A, B) \).

We end the section by making a comparison with the work in [22]. Consider a semiconcept \( (A, A') \) that is not a concept. According to [29], this is a non-definable concept such that \( A' \) is feasible and \( A \) is not. As mentioned in Section 22, the lower and upper approximations of \( (A, A') \) as given in [22] are \( (A, A') = ((A_{E_1})' \cap A_{E_1}), (A_{E_1})' \cap A_{E_1}) = ((A_{E_1})', (A_{E_1})') \) and \( (A, A') = ((A_{E_1})', (A_{E_1})') \).

Now observe that \( (A, A') \) and \( (A, A') \) are both concepts. In our case, the lower and upper approximations of a semiconcept that is not a concept, may not be a concept: recall Example 3. The lower approximation \( (\{\text{Leech, Bream}\}, \{a, b, g\}) \) of the semiconcept \( (\{\text{Leech, Bream, Dog}\}, \{a, g\}) \) is not a concept. Furthermore, in our case (as pointed out in Observation 2), the lower approximation of a semiconcept lies below the semiconcept, while its upper approximation lies above it — thereby justifying that these are lower and upper approximations of the semiconcept. In general, these properties do not hold for the lower and upper approximations of concepts defined in [24].

7. Conclusions

The hyper-sequent calculus PDBL is defined for the class of pdBas and extended to MPDBL for the class of pdBas. For any set \( \Sigma \) of sequents in MPDBL, MPDBL5 is defined. As particular cases of MPDBL5, the logics MPDBL4 (for tpdBAs) and MPDBL5 are obtained. This presents a technique for constructing various logics in general, that may represent attributes of pdBAs and associated classes of Kripke contexts.

For conceptual knowledge, it is established that using PDBL and its models, one can express the basic notion of concept, and the relations “object belongs to a concept”, “property abstracts from a concept” and “a concept is a subconcept of another concept”. Further, the basic notions of objects and attributes and the relation “an object has an attribute” are expressible by using the named PDBL models. When interpretations of PDBL are restricted to the collection of named models, the logic is sound with respect to the class of of all contexts. An open question then is, whether a logic may possibly be derived from PDBL that would be complete with respect to this restricted collection of models.

This work attempts to highlight the significance of semiconcepts of a context from different points of view. In particular, the observations of Section 5 indicate that a mathematical model for knowledge related to semiconcepts
may well be defined, akin to the conceptual knowledge system defined in \cite{24,32}. The significance of such a model may be worth exploring.

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Appendix 1. Proofs

Proof of Theorem 9. The proofs are straightforward and one makes use of axioms 2a, 3a, 4a, Propositions and the rule (R4) in most cases. 1a:

\[ 4a \ (\alpha \sqcap \beta) \vdash (\alpha \sqcap \beta) \sqcap (\alpha \sqcap \beta) \]
\[ 3a \ \alpha \sqcap \beta \vdash \beta \quad \alpha \sqcap \beta \vdash \alpha \]
\[ 2a \ (\alpha \sqcap \beta) \sqcap (\alpha \sqcap \beta) \]

Interchanging \( \alpha \) and \( \beta \) in the above, we get \( (\beta \sqcap \alpha) \vdash (\alpha \sqcap \beta) \).

2a.

\[ 2a \ (\alpha \sqcap \beta) \sqcap \gamma \vdash (\alpha \sqcap \beta) \quad \alpha \sqcap \beta \vdash \beta \quad \alpha \sqcap \beta \vdash \gamma \]
\[ 3a \]
\[ (\alpha \sqcap \beta) \sqcap (\alpha \sqcap \beta) \]

Now,

\[ 2a \ (\alpha \sqcap \beta) \sqcap \gamma \vdash \alpha \quad \alpha \sqcap \beta \vdash \alpha \quad \alpha \sqcap \beta \vdash \alpha \]
\[ 3a \]
\[ (\alpha \sqcap \beta) \sqcap (\alpha \sqcap \beta) \]

Similarly we can show that \( \alpha \sqcap (\beta \sqcap \gamma) \vdash (\alpha \sqcap \beta) \sqcap \gamma \).

3a.

\[ 2a \ (\alpha \sqcap \alpha) \sqcap \beta \vdash \alpha \quad \alpha \sqcap \alpha \vdash \alpha \quad \alpha \sqcap \alpha \vdash \alpha \]
\[ 3a \]
\[ (\alpha \sqcap \alpha) \sqcap \beta \]

\[ 4a \ (\alpha \sqcap \beta) \sqcap \gamma \vdash (\alpha \sqcap \beta) \quad ((\alpha \sqcap \beta) \sqcap \gamma) \quad ((\alpha \sqcap \beta) \sqcap \gamma) \]
\[ \alpha \sqcap \beta \vdash \gamma \quad (\alpha \sqcap \beta) \sqcap (\alpha \sqcap \beta) \]
\[ 3a \]

\[ (\alpha \sqcap \alpha) \sqcap \beta \]

(4a) follows from axiom 2a and (R3).

5a.

\[ 2a \ (\alpha \sqcap \gamma) \sqcup \alpha \quad \alpha \sqcap (\alpha \sqcup \beta) \vdash \alpha \]
\[ 2a \]
\[ (\alpha \sqcup (\alpha \sqcap \beta)) \]

6a. Proof is identical to that of 5a.

(7a), (8a) follow from axiom 11a.

Note that the proofs of (ib), \( i = 1, 2, 3, 4, 5, 6, 7, 8 \), are obtained using the axioms and rules dual to those used to derive (ia).

Proof of Proposition. For \( 1 \Rightarrow 2 \), we make use of (R1)', (R4), axiom 2a and Theorem 3a, 3a.)
\[
\alpha \vdash \beta \\
\alpha \cap \alpha \vdash \alpha \cap \beta \\
\alpha \cap \beta \vdash \alpha \\
\alpha \cap (\alpha \cap \beta) \vdash \alpha \cap \alpha \quad \alpha \cap \beta \vdash \alpha \cap (\alpha \cap \beta) \\
\alpha \cap \beta \vdash \alpha \cap \alpha
\]

So \(\alpha \cap \alpha \not\vdash \alpha \cap \beta\), which implies that \([\alpha] \cap [\alpha] = [\alpha \cap \alpha] = [\alpha \cap \beta] = [\alpha] \cap [\beta]\). Dually we can show that \([\alpha] \cup [\beta] = [\beta] \cup [\beta]\). Therefore \([\alpha] \subseteq [\beta]\).

For \(2 \Rightarrow 1\), suppose \([\alpha] \subseteq [\beta]\). Then \([\alpha] \cap [\beta] = [\alpha] \cap [\alpha]\). So \([\alpha \cap \beta] = [\alpha \cap \alpha]\). Similarly we can show that \([\alpha \cup \beta] = [\beta \cup \beta]\). Therefore \(\alpha \cap \beta \vdash \alpha \cap \alpha\) and \(\alpha \cup \beta \vdash \beta \cup \beta\). Now using (R5), \(\alpha \vdash \beta\).