GLOBAL WELL-POSEDNESS FOR THE DEFOCUSING HARTREE EQUATION WITH RADIAL DATA IN $\mathbb{R}^4$

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Abstract. By $I$-method, the interaction Morawetz estimate, long time Strichartz estimate and local smoothing effect of Schrödinger operator, we show global well-posedness and scattering for the defocusing Hartree equation

$$\begin{cases}
  iu_t + \Delta u = F(u), \ (t, x) \in \mathbb{R} \times \mathbb{R}^4 \\
  u(0) = u_0(x) \in H^s(\mathbb{R}^4),
\end{cases}$$

where $F(u) = (V * |u|^2)u$, and $V(x) = |x|^{-\gamma}$, $3 < \gamma < 4$, with radial data in $H^s(\mathbb{R}^4)$ for $s > s_c := \gamma/2 - 1$. It is a sharp global result except of the critical case $s = s_c$, which is a very difficult open problem.

1. Introduction

In this paper, we consider the Hartree equation

$$\begin{cases}
  iu_t + \Delta u = F(u), \ (t, x) \in \mathbb{R} \times \mathbb{R}^n \\
  u(0) = u_0(x) \in H^s(\mathbb{R}^n),
\end{cases}$$

where $F(u) = \iota(V * |u|^2)u$, with $V(x) = |x|^{-\gamma}$, $2 < \gamma < n$, $\iota = \pm 1$ and $H^s$ denotes the usual inhomogeneous Sobolev space of order $s$. The Hartree equation arises in the study of Boson stars and other physical phenomena, and in chemistry, it appears as a continuous-limit model for mesoscopic molecular structures, see for example [44] and references therein.

Define scaling transformation

$$u^\lambda(t, x) = \lambda^{(n+2-\gamma)/2}u(\lambda^2 t, \lambda x).$$

Clearly, it leaves the equation (1.1) invariant, and the $H^s-$norm of initial data behaves as

$$\|u_0^\lambda\|_{H^s(\mathbb{R}^n)} = \lambda^{s-\gamma/2+1}\|u_0\|_{H^s(\mathbb{R}^n)}.$$  

Hence, $\dot{H}^{\gamma/2-1}$ is invariant under the scaling transformation and it is called as the critical Sobolev space.

Local well-posedness for (1.1) in $H^s$ for any $s > \gamma/2 - 1$ was established in [39] where the maximal time interval of existence depends on the $H^s$ norm of initial data. A local solution also exists for $\dot{H}^{\gamma/2-1}$ initial data, however the time of existence depends not only on the $\dot{H}^{\gamma/2-1}$ norm of $u_0$, but also on the profile of $u_0$. Ill-posedness in some specific sense for (1.1) in $H^s$ for any $s < \max(0, \gamma/2 - 1)$ was also established. For more details on local well-posedness, see [39].

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It is well known that $H^1$ solutions of (1.1) conserve the mass and energy

$$\left\|u(t, \cdot)\right\|_{L^2(\mathbb{R}^n)} = \left\|u_0(\cdot)\right\|_{L^2(\mathbb{R}^n)},$$

$$E(u)(t) := \frac{1}{2}\left\|\nabla u(t)\right\|^2_{L^2(\mathbb{R}^n)} + \frac{\kappa}{4} \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(t,x)|^2|u(t,y)|^2}{|x-y|^\gamma} \; dxdy = E(u)(0).$$

We refer to (1.1) as the defocusing case when $\iota = 1$, and as the focusing case when $\iota = -1$. The local well-posedness along with the above two conservation laws immediately yields global well-posedness for (1.1) in $H^1$ with $\iota = +1$, where $0 < \gamma < 4$ if $n = 4$ and $0 < \gamma \leq 4$ if $n \geq 5$.

As for the long time dynamics of (1.1), there are many results so far. The global well-posedness and scattering of the defocusing $H^1$-subcritical Hartree equation ($\iota = 1, 2 < \gamma < \min(4, n)$) in the energy space were firstly solved in [19, 52] based on the classical Morawetz estimate. With the development of the induction on energy strategy in [1, 2, 8, 58] and the concentration-compactness argument in [27, 28, 30, 31], the defocusing $H^1$-critical Hartree equation ($\iota = 1, 4 = \gamma < n$) has been completely settled in [40, 44] and the long time dynamics for the focusing $H^1$-critical Hartree equation ($\iota = -1, 4 = \gamma < n$) under the condition that the energy is less than that of the ground state have also been characterized in [35, 41, 45]. Of course, the global well-posedness and scattering for the $L^2$-critical Hartree equation ($\gamma = 2 < n$) with radial data was similarly solved in [42], however, the non-radial case is still open.

For other works on the global well-posedness and scattering for the Hartree equation, see [9, 16, 20, 21, 22, 24, 43, 53].

The long-time Strichartz estimate is a powerful tool in dealing with nonlinear Schrödinger equations with semilinear inhomogeneous local term, see [10, 12, 13, 14, 32, 38, 48, 49, 50, 62]. In this paper, we adopt the long-time Strichartz estimate to study the scattering theory of nonlinear Hartree equation, where the nonlinearity is nonlocal due to the convolution with $V$. We combine this implement with $I$-method [6, 59], the interaction Morawetz estimate [7, 8, 44, 61] and local smoothing effect of Schrödinger operator [26, 55], to obtain our main result which reads

**Theorem 1.** Let $n = 4$, $\iota = 1$, $3 < \gamma < 4$, $s > \gamma/2 - 1$ and $u_0 \in H^s(\mathbb{R}^4)$ be spherically symmetric. Then the Cauchy problem (1.1) is globally well-posed. Moreover the solution satisfies

$$\sup_{t \in \mathbb{R}} \left\|u(t)\right\|_{H^s(\mathbb{R}^4)} \leq C\left(\left\|u_0\right\|_{H^s}\right),$$

and the solution scatters to a free wave, that is, there exist $u_0^\pm \in H^s(\mathbb{R}^4)$ such that

$$\lim_{t \to \pm \infty} \left\|u(t) - e^{i\Delta}u_0^\pm\right\|_{H^s(\mathbb{R}^4)} = 0.$$

**Remark 1.1.** This is an unconditional global result from the perspective of regularity assumption on initial data, where we do not assume any a priori uniform boundedness on the solution in Sobolev norms with respect to time. In particular, the initial data is allowed to have infinite energy. If the energy of the initial data is finite, then it is well known that one easily obtains the scattering result by the interaction Morawetz estimate.

**Remark 1.2.** According to local well-posedness result in [39], our result is sharp except of the critical case $s = s_c$ for $3 < \gamma < 4$. As for the restriction about $\gamma > 3$ (that is, $s_c > \frac{3}{2}$), it...
is corresponding to the $H^{1/2}$-regularity of the solution in the interaction Morawetz estimate (see Proposition 2.12).

**Remark 1.3.** Analogous unconditional global existence and scattering for the critical case $s = s_c$ for $2 < \gamma < 4$ is much more difficult. On one hand, there is no conserved quantity to be used. On the other hand, the $I$-method would also break down as can be seen in our proof. In this case, it is well-known in the literature that for the semilinear Schrödinger equations with radial data, the uniform boundedness of the critical norm $\dot{H}^{s_c}$ implies scattering [29]. An interesting problem is to relax this assumption by considering a discrete time sequence tending to the maximal time of existence, along which the solution is bounded in certain critical Sobolev norms, and showing that this weaker assumption also implies scattering, as investigated by Duyckaerts and the third author in [15] for wave equations.

**Remark 1.4.** The result is restricted to the radial setting because we need the radial Sobolev inequality in the frequency localized version, see Proposition 2.7. The argument here can also be extended to all higher dimensions $n \geq 5$, where by using the double Duhamel formula, Miao Xu and Zhao [44] established the scattering theory for energy critical case $\gamma = 4$. Thus, it is interesting to remove the radial assumption in this low regularity problems for $\gamma$ smaller than but close to 4.

Before giving some further remarks on our theorem, we briefly review the $I$-method on which the proof of Theorem 1 is based.

The study of a low regularity problem stimulates the development of the scattering in $L^2(\mathbb{R}^d)$ for the mass-critical problem. Dodson developed so-called longtime Strichartz estimates to prove the global well-posedness and scattering in $L^2$-space by making use of a concentration-compactness approach and the idea of $I$-method. The $I-$method consists in smoothing out the $H^s$-initial data with $0 < s < 1$ in order to access a good local and global theory available at the $H^1$-regularity. To do this, one defines the Fourier multiplier $I$ by

$$\hat{Iu}(\xi) := m(\xi)\hat{u}(\xi),$$

where $m(\xi)$ is a smooth radial decreasing cut-off function such that

$$m(\xi) = \begin{cases} 1, & |\xi| \leq N, \\ \left(\frac{|\xi|}{N}\right)^{s-1}, & |\xi| \geq 2N. \end{cases}$$

Thus, $I$ is the identity operator when acting on functions whose frequencies are localized to $|\xi| \leq N$ and behaves like a multiplier of order $s-1$ with respect to higher frequencies. It is easy to show that the $I$ operator maps $H^s$ to $H^1$. Moreover, we have

$$\|u\|_{H^s} \lesssim \|Iu\|_{H^1} \lesssim N^{1-s}\|u\|_{H^s}.$$ 

Thus, the energy is well-defined for $Iu(t)$ and to prove the problem (1.1) is globally well-posed in $H^s$, it suffices to show that $E(Iu(t)) < +\infty$ for all $t \in \mathbb{R}$. Since $Iu$ is not a solution to (1.1), the modified energy $E(Iu)(t)$ is not conserved. Thus the key idea is to show that $E(Iu)$ is “almost conserved” in the sense that its derivative $\frac{d}{dt}E(Iu(t))$ will decay with respect to a large parameter $N$. This will allow us to control $E(Iu)$ on time interval where the local solution exists, which
allows us to iterate this procedure to obtain a global-in-time control of the solution by means of the bootstrap argument.

Turning to the proof of our main theorem, we are inspired by a recent work of Dodson [14], where he first implements the long-time Strichartz estimate into the theory of I-method. Notice that the long-time Strichartz estimate appeared already in Dodson’s previous works on the scattering theory of mass critical NLS [10, 11, 12, 13]. It is natural to compare these techniques with that in [14]. In fact, the proof of the scattering of mass critical NLS was all based on a contradiction argument that assuming the global existence and scattering fails, one must have a minimal mass blow-up solution which is a critical element with various additional properties. The long-time Strichartz estimate in [10, 13, 12, 11] was established for this kind of solutions and hence is not an a priori estimate which should hold for an arbitrary solution. On the contrary, the longtime Strichartz estimate introduced in [14] under the same name with Dodson’s previous works, was proved, in the framework of I-method, for every solutions, which satisfying certain assumption of the boundedness of the I-energy. Notice also that Dodson adopted long time Strichartz estimate with I-method, the interaction Morawetz estimate and local smoothing effect to show the lower regularity of the defocusing nonlinear Schrödinger equations in [14], where a remarkable point is the use of $U^k_\Delta$, $V^k_\Delta$ spaces incorporated with the local smoothing effect of Schrödinger operators.

The crux in the proof of Theorem 1 is the deduction of the long time Strichartz estimate for Hartree equations and the difficulty arises naturally from handling the convolution operator in the nonlinear term of (1.1). To overcome these problems, we shall employ some fractional order inequalities in weighted norms established in [47] as well as a modified Coifman-Meyer theorem. See Section 2 for details.

By the end of this section, we outline the organization of this paper as following: In Section 2, we introduce some notation and a couple of propositions which will play important roles in the later context. We will also review the local well-posedness theory for the Cauchy problem (1.1). In Section 3, we review and outline the I-method at our disposal. We will also obtain in this section a uniform local estimate. In Section 4, we prove long time Strichartz estimate and obtain the boundedness of high frequency part of $\nabla Iu$ in the endpoint Strichartz space $L^4_t L^4_x(J \times \mathbb{R}^4)$. Finally in Section 5, we use long time Strichartz estimate to control the increment of the modified energy $E(Iu)(t)$, which will conclude the proof of Theorem 1 by the bootstrap argument.

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2. Notation and preliminaries

Throughout this paper, we will use the following notation for the sake of brevity in exposition. The tempered distribution is denoted by $\mathcal{S}'(\mathbb{R}^n)$. We use $A \lesssim B$ to denote an estimate of the form $A \leq CB$ for some constant $C$. If $A \lesssim B$ and $B \lesssim A$, we say that $A \approx B$. We write $a \pm$ to mean $a \pm \epsilon$ where $\epsilon$ may be taken arbitrarily small. We use $2^k$ to denote the set of dyadic integers of the
form $2^j$ with $j \in \mathbb{Z}$. We use $\langle f, g \rangle$ to denote the inner product $\int_{\mathbb{R}^4} f(x) \overline{g(x)} dx$. Given $\alpha \in \mathbb{R}$, we denote by $\lfloor \alpha \rfloor$ the largest integer bounded by $\alpha$.

### 2.1. Definition of spaces and Strichartz estimates

We use $L^r_x(\mathbb{R}^n)$ to denote the Lebesgue space of functions $f : \mathbb{R}^n \to \mathbb{C}$ whose norm

$$\|f\|_{L^r_x} := \left( \int_{\mathbb{R}^n} |f(x)|^r dx \right)^{\frac{1}{r}}$$

is finite, with the standard modification when $r = \infty$. We also define the space-time Lebesgue spaces $L^q_t L^r_x([a,b] \times \mathbb{R}^n)$ which are endowed with the norm

$$\|u\|_{L^q_t L^r_x([a,b] \times \mathbb{R}^n)} := \left( \int_a^b \|u(t, \cdot)\|_{L^q_x}^r dt \right)^{\frac{1}{r}}$$

for any space-time slab $[a, b] \times \mathbb{R}^n$, with the standard modification when either $q$ or $r$ is infinite. If $q = r$, we abbreviate $L^q_t L^r_x$ by $L^q_{t,x}$.

We define the Fourier transform of $f(x) \in L^1_x$ by

$$\hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx.$$ 

The fractional differential operator $|\nabla|^\alpha$ of order $\alpha$ is defined via Fourier transform

$$|\nabla|^\alpha f(\xi) := |\xi|^\alpha \hat{f}(\xi).$$

Define the Schrödinger semi-group $e^{it\Delta}$ as

$$e^{it\Delta} u(x) = \int_{\mathbb{R}^n} e^{ix \cdot \xi} \hat{u}(\xi) e^{it |\xi|^2} d\xi.$$ 

We define the Sobolev space $W^{s,r}(\mathbb{R}^d)$ as the class of distributions $f$ satisfying

$$\|f\|_{W^{s,r}} = \|(1 + |\nabla|^s) f\|_{L^r(\mathbb{R}^d)} < +\infty.$$ 

In particular, we denote $W^{s,2}$ by $H^s$ the Hilbert Sobolev space.

We will use the following Littewood-Paley decompositions. Let $\varphi(\xi) \geq 0$ be a smooth function supported in the ball $\{ \xi \in \mathbb{R}^n : |\xi| \leq 2 \}$ which equals to 1 when $|\xi| \leq 1$. For each dyadic integer $M$, the Littewood-Paley projector $P_{\leq M}, P_{> M}$ and $P_M$ are defined via Fourier transform respectively as followings

$$\widehat{P_{\leq M}} f(\xi) = \varphi \left( \frac{\xi}{M} \right) \hat{f}(\xi), \quad P_{> M} f = f - P_{\leq M} f, \quad P_M f = P_{\leq 2M} f - P_{< M} f.$$ 

For brevity, we will also write $f_{\leq M}$ instead of $P_{\leq M} f$ and write $f_M$ rather than $P_M f$.

Consequently, we have the inhomogeneous Littlewood-Paley decomposition

$$f(x) = P_{\leq 1} f(x) + \sum_{N \geq 1} P_N f(x),$$

where $N$ is dyadic integers. The homogeneous Littlewood-Paley decomposition reads

$$f(x) = \sum_{N \in 2^\mathbb{Z}} P_N f(x).$$
Now, we state the following Strichartz estimate for $n = 4$. Let $2 \leq q, r \leq \infty$. We say $(q, r)$ is $\alpha$-admissible and write $(q, r) \in \Lambda_\alpha$ if
\[
4\left(\frac{1}{2} - \frac{1}{r}\right) - \frac{2}{q} = \alpha.
\]
In particular, we say $(q, r)$ is an admissible pair when $\alpha = 0$.

**Proposition 2.1.** [17] [18] [3] [25] Let $(q, r)$ and $(\tilde{q}, \tilde{r})$ be two arbitrary admissible pairs and $J \subset \mathbb{R}$. Suppose $u$ is a solution to
\[
\begin{aligned}
iu_t + \Delta u &= G(t, x), (t, x) \in J \times \mathbb{R}^4, \\
u(0) &= u_0(x) \in L^2(\mathbb{R}^4).
\end{aligned}
\]
Then there exists $C_0 > 0$ such that we have
\[
\|u\|_{L_t^4 L_x^\infty(J \times \mathbb{R}^4)} \leq C_0 \left(\|u_0\|_{L^2(\mathbb{R}^4)} + \|G\|_{L_t^\infty L_x^\infty(J \times \mathbb{R}^4)}\right),
\]
where the primed exponents denote Hölder dual exponents.

We recall the local well-posedness theory for (1.1).

**Proposition 2.2.** Let $2 < \gamma < 4$ and $\frac{\gamma}{2} - 1 < s < 1$. Then for every $u_0 \in H^s(\mathbb{R}^4)$, there exists $T = T(\|u_0\|_{H^s}) > 0$ and a unique solution $u(t, x)$ of (1.1) such that
\[
u \in C_t H^s_x([0, T] \times \mathbb{R}^4) \cap L_t^4 W_x^{s, r}([0, T] \times \mathbb{R}^4),
\]
for all $(q, r) \in \Lambda_0$.

Moreover, if we denote by $T_* = \sup\{T : T = T(\|u_0\|_{H^s})\}$ given as above, then we have the conservation of mass
\[
u(t, \cdot)\|_{L^2(\mathbb{R}^4)} = \|u_0\|_{L^2(\mathbb{R}^4)}, \quad \text{for all } 0 < t < T_*,
\]
and the blow-up criterion that $T_* < +\infty$ implies
\[
\lim_{t \to T_*} \|u(t, \cdot)\|_{H^s(\mathbb{R}^4)} = +\infty.
\]

**Proof.** Let
\[
(\rho, \sigma) = \left(\frac{8}{\gamma - 2s}, \frac{16}{8 + 2s - \gamma}\right).
\]
It suffices to show the map
\[
\Phi_{u_0} : u \mapsto e^{it\Delta}u_0(x) - i \int_0^t e^{i(t-\tau)\Delta}F(u(\tau, x))d\tau
\]
defined on the space
\[
\mathcal{X}_T = \left\{u \in L_t^{\rho} W_x^{s, \sigma}([0, T] \times \mathbb{R}^4) : \|u\|_{L_t^{\rho} W_x^{s, \sigma}([0, T] \times \mathbb{R}^4)} \leq 2MC_0\right\}
\]
is a contraction for $T$ small enough, where $C_0$ is the constant in (2.1) and $M = \|u_0\|_{H^s}$.

To see this, it suffices to show the following estimate
\[
\|\Phi_{u_0}(u)\|_{L_t^{\rho} W_x^{s, 2}([0, T] \times \mathbb{R}^4)} \leq C_0\|u_0\|_{H^s(\mathbb{R}^4)} + CT^{s - \frac{\gamma}{2} + 1}\|u\|^3_{L_t^{\rho} W_x^{s, \sigma}([0, T] \times \mathbb{R}^4)}.
\]
This is reduced by Strichartz’s estimate further to
\[
\|(V * |u|^2)u\|_{L_t^2 W_x^{s,s'}([0,T] \times \mathbb{R}^4)} \lesssim T^{s-\frac{7}{2}+1} \|u\|_{L_t^6 W_x^{2,s}([0,T] \times \mathbb{R}^4)}^3,
\]
where \(1 \leq \rho' = \rho/(\rho - 1)\) and \(s' = \sigma/(\sigma - 1)\). By the chain rule of the fractional order derivatives and Hardy-Littlewood-Sobolev inequality, we get
\[
\|(V * |u|^2)u\|_{W_x^{\frac{p}{2},s'}([0,T] \times \mathbb{R}^4)} \lesssim \|u\|_{W_x^{\frac{p}{2},s}} \|u\|_{L_t^3}^{1+1/p}.
\]
By using Sobolev embedding \(W_x^{s,\sigma}(\mathbb{R}^4) \subset L^{16/(8-2s-\gamma)}(\mathbb{R}^4)\) and Hölder’s inequality, we get
\[
\|(V * |u|^2)u\|_{L_t^{16} W_x^{\frac{16}{7},\frac{16}{7}}([0,T] \times \mathbb{R}^4)} \lesssim \left( \int_0^T \|u(t)\|_{L_t^{36} W_x^{\frac{36}{17},\frac{36}{17}}} dt \right)^{1/\rho'} \lesssim T^{s-\frac{7}{2}+1} \|u\|_{L_t^6 W_x^{2,s}([0,T] \times \mathbb{R}^4)}.
\]
\[
\square
\]
We shall use the \(U_{\Delta}^p\) and \(V_{\Delta}^p\) spaces adapted to Schrödinger equations. Denote by \(\mathcal{Z}\) the set of finite partitions \(-\infty < t_0 < t_1 < \cdots < t_K \leq \infty\) of the real line. If \(t_K = \infty\), we use the convention that \(v(t_K) := 0\) for all functions \(v : \mathbb{R} \to L^2(\mathbb{R}^n)\). The idea and techniques of the \(U_{\Delta}^p, V_{\Delta}^p\) spaces was first used in [33].

**Definition 2.3.** Let \(1 \leq p < \infty\). For \(\{t_k\}_{k=0}^K \in \mathcal{Z}\) and \(\{\phi_k\}_{k=0}^{K-1} \subset L^2(\mathbb{R}^n)\) with \(\sum_{k=0}^{K-1} \|\phi_k\|_{L^2(\mathbb{R}^n)} = 1\), we define a \(U_{\Delta}^p\)-atom \(a(t, x)\) as a piecewise solution to the linear Schrödinger equation
\[
a(t, x) = \sum_{k=1}^K \mathbf{1}_{[t_k-1, t_k]}(t) e^{it\Delta} \phi_{k-1}(x).
\]
The atomic space \(U_{\Delta}^p(\mathbb{R}; L^2(\mathbb{R}^n))\) consists of all \(u : \mathbb{R} \to L^2(\mathbb{R}^n)\) such that there exists a series of \(U_{\Delta}^p\)-atoms \(\{a_j\}_j\) along with \(\{\lambda_j\}_j\)
\[
u = \sum_{j=1}^\infty \lambda_j a_j, \quad \sum_{j=1}^\infty |\lambda_j| < \infty.
\]
For any \(1 \leq p < \infty\), we define \(U_{\Delta}^p\)-norm as
\[
\|u\|_{U_{\Delta}^p} = \inf \left\{ \sum_{j=1}^\infty |\lambda_j| : u = \sum_{j=1}^\infty \lambda_j a_j, \text{ \(U_{\Delta}^p\)-atoms} \right\}.
\]
The normed spaces \(U_{\Delta}^p(\mathbb{R}; L^2(\mathbb{R}^n))\) are complete and \(U_{\Delta}^p \subset L^\infty(\mathbb{R}; L^2(\mathbb{R}^n))\). Moreover, each \(u \in U_{\Delta}^p\) is right continuous and continuous except at countably many points.

**Lemma 2.4.** [13] Suppose \(I = I_1 \cup I_2, I_1 = [a, b], I_2 = [b, c]\) with \(a \leq b \leq c\). Then
\[
\|u\|_{U_{\Delta}^p(I \times \mathbb{R}^n)} \leq \|u\|_{U_{\Delta}^p(I_1 \times \mathbb{R}^n)} + \|u\|_{U_{\Delta}^p(I_2 \times \mathbb{R}^n)}.
\]

**Definition 2.5.** Let \(1 \leq p < \infty\). We define \(V_{\Delta}^p(\mathbb{R}; L^2(\mathbb{R}^n))\) as the space of all right continuous functions \(v \in L_t^\infty L_x^2\) such that the following norm is finite
\[
\|v\|_{V_{\Delta}^p} := \|v\|_{L_t^\infty L_x^2} + \sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}} \sum_k \|e^{-it_k\Delta}v(t_k) - e^{-it_{k+1}\Delta}v(t_{k+1})\|_{L_x^2}^p.
\]
These function spaces enjoy several well-known embedding relations as summarized below.
Proposition 2.6. [14, 23, 34] For $1 \leq p < q < \infty$, we have
\[ U^p_\Delta \subset V^p_\Delta \subset U^q_\Delta \subset L^2_t L^2_x. \]

Let $DU^p_\Delta$ be the space induced by $U^p_\Delta$, namely
\[ DU^p_\Delta = \{(i\partial_t + \Delta)u : u \in U^p_\Delta\}. \]
Then the dual space of $DU^p_\Delta$ is $V^q_\Delta$, i.e.
\[ (DU^p_\Delta)^* = V^q_\Delta, \]
and we have
\[ \|u\|_{U^p_\Delta} \lesssim \|u(0)\|_{L^q} + \|(i\partial_t + \Delta)u\|_{DU^p_\Delta}. \]
Moreover, we have $U^p_\Delta \subset L^q_t L^r_x(J \times \mathbb{R}^4)$ for any $(q, r) \in \Lambda_0$ and $L^q_t L^r_x(J \times \mathbb{R}^4) \subset DU^p_\Delta(J \times \mathbb{R}^4)$ when $(q, r)$ is an admissible pair and $q > 2$. These spaces are stable under truncation in time by multiplying a characteristic function of a time interval $J$
\[ \chi_J : U^p_\Delta \rightarrow U^p_\Delta, \ \chi_J : V^p_\Delta \rightarrow V^p_\Delta. \]

2.2. Some known estimates. In this part, we collect several well known results which will be used later.

The Littlewood-Paley projectors commute with derivative operators, the free propagator $e^{i\Lambda}$ and the conjugation operation. Moreover, they are self-adjoint and bounded on every $L^p_x$ space for $1 \leq r \leq \infty$ and $s \geq 0$. In addition, they obey the following Sobolev and Bernstein estimates
\[
\|P_{\geq M} f\|_{L^p} \lesssim M^{-s} \|\nabla^s P_{\geq M} f\|_{L^p}, \\
\|\nabla^s P_{\leq M} f\|_{L^q} \lesssim M^{s+n(\frac{1}{q} - \frac{1}{2})} \|P_{\leq M} f\|_{L^p}, \\
\|\nabla^s P_M f\|_{L^q} \lesssim M^{s+n(\frac{1}{p} - \frac{1}{2})} \|P_M f\|_{L^p},
\]
whenever $s \geq 0$ and $1 \leq p \leq q \leq \infty$.

The following Sobolev type inequality for radial functions will be used in Section 3.

Proposition 2.7. Assume $n \geq 2$ and $M > 0$ is dyadic. Then there is a constant $C > 0$ such that
\[
\sup_{x \in \mathbb{R}^n} |x|^{\frac{n-1}{2}} |P_M u(x)| \leq C M^{\frac{n}{2}} \|P_M u\|_{L^2(\mathbb{R}^n)}, \quad (2.4)
\]
for every radial function $u \in L^2(\mathbb{R}^n)$.

Proof. Since $u(x)$ is spherically symmetric, we may write by Fourier transform
\[
|x|^{\frac{n-1}{2}} |P_M u(x)| = 2\pi |x|^{\frac{1}{2}} \int_0^\infty 1_{[M/2, 4M]}(r) \widehat{P_M u}(r) J_{\frac{n-1}{2}}(2\pi |x|r) r^{\frac{n}{2}} dr,
\]
where $J_m$ denotes the Bessel function of order $m$. If $|x|r < 1$, we have $|x|^{\frac{1}{2}} < M^{-\frac{1}{2}}$ and hence
\[
|x|^{\frac{n-1}{2}} |P_M u(x)| \leq C \int_M^{4M} |\widehat{P_M u}(r)| r^{\frac{n-1}{2}} dr. \quad (2.5)
\]
If $|x|r \geq 1$, (2.5) remains valid from the asymptotic behavior of the Bessel function. Now the proposition follows from (2.3), Cauchy-Schwarz’s inequality and the Plancherel Theorem. \qed
In [47], Muckenhoupt and Wheeden extended the classical Hardy-Littewood-Sobolev's inequality to the weighted Lebesgue spaces.

**Proposition 2.8.** For $0 < \gamma < n$, we denote by

$$I_\gamma f(x) = \int_{\mathbb{R}^n} f(x - y) |y|^{-\gamma} dy.$$

Assume $V(x) \geq 0$ and

$$1 + \frac{1}{q} = \frac{1}{p} + \frac{\gamma}{n}, \quad q < \infty, \quad 1 < p < \frac{n}{n - \gamma}.$$

Then there exist a constant $C > 0$ independent of the function $f$ such that

$$\|I_\gamma f(x) V(x)\|_{L^q(\mathbb{R}^n)} \leq C \|f(x) V(x)\|_{L^p(\mathbb{R}^n)}$$

if and only if there is a $K > 0$ such that

$$\left(\frac{1}{|Q|} \int_Q [V(x)]^q dx\right)^\frac{1}{q} \left(\frac{1}{|Q|} \int_Q [V(x)]^{-p'} dx\right)^\frac{1}{p'} \leq K,$$

for all cubes $Q \subset \mathbb{R}^n$, where $|Q|$ is the volume of $Q$.

As a consequence of this proposition, we have the following estimate concerning the fractional order integration in weighted norms.

**Corollary 2.9.** Let $\psi(x)$ be the characteristic function of the unit annulus $\mathcal{C} = \{x \in \mathbb{R}^n \mid 1 \leq |x| \leq 2\}$ and $\psi_R(x) = \psi(R^{-1}x)$ for $R \geq 1$. Then, for $\beta > 0$ and $p, q$ satisfying the condition in Proposition 2.8, we have

$$\|\psi_R(x)|x|^\beta I_\gamma f(x)\|_{L^q(\mathbb{R}^n)} \leq C \|\psi_R(x)|x|^\beta f(x)\|_{L^p(\mathbb{R}^n)}.$$

**Proof.** Assume $0 < \beta$ and $p, q$ satisfy the condition in Proposition 2.8. It is easy to see that (2.6) is fulfilled by $V(x) = \psi_R(x)|x|^\beta$. Indeed, denote by $\mathcal{C}_R = \{x \in \mathbb{R}^n \mid R \leq |x| \leq 2R\}$. Then $V(x) = |x|^\beta$ on $\mathcal{C}_R$ and $V(x)$ vanishes outside $\mathcal{C}_R$. The left side of (2.6) is clearly bounded by

$$\left(\frac{|\mathcal{C}_R \cap Q|}{|Q|}\right)^\frac{1}{q} \leq 1.$$

□

Next, we recall the local smoothing estimate of Schrödinger propagator.

**Proposition 2.10.** [26, 55] Suppose $f \in L^2(\mathbb{R}^n)$ and $n \geq 2$. Then there is a constant $C > 0$ independent of $f$ such that

$$\sup_{R > 0} \sup_Q \frac{1}{R} \int_{\mathbb{R}} \int_Q |\nabla|^\frac{1}{2} e^{it\Delta} f(x)^2 dx dt \leq C \|f\|_{L^2(\mathbb{R}^n)}^2,$$

where $Q$ is taken over all cubes in $\mathbb{R}^n$ of side length $R$.

Interpolating (2.7) and the trivial estimate $\|e^{it\Delta} f\|_{L^\infty(\mathbb{R}; L^2(\mathbb{R}^n))} \lesssim \|f\|_{L^2}$, we get

$$\|\nabla|^\frac{1}{p} e^{it\Delta} f\|_{L^p(\mathbb{R}; L^2(Q))} \lesssim R^\frac{1}{p} \|f\|_2,$$

where $p \geq 2$ and $Q$ is of size $R$. 
The local smoothing effect can be expressed in the $U^p_\Delta$ and $V^p_\Delta$ space as follows.

**Corollary 2.11.** Let $2 \leq p \leq \infty$ and $Q$ is a cube of size $R$. Then we have

$$
\| |\nabla|^{1/p} u \|^p_{L^p_tL^2_x(\mathbb{R} \times Q)} \lesssim R^\frac{1}{p} \|\;\|U^p_\Delta\|. \tag{2.8}
$$

**Proof.** Assume $u$ is a $U^p_\Delta$-atom and $p \geq 2$,

$$
u(t, x) = \sum_k 1_{[t_{k-1}, t_k)}(t) e^{it\Delta} \phi_{1-1}(x), \quad \sum_k \|\phi_k\|^p_{L^2} = 1.
$$

From (2.3), we have

$$\| |\nabla|^{1/p} u \|^p_{L^p_tL^2_x((t_0, t_k); L^2(Q))} \lesssim \sum_{k=1}^K \| |\nabla|^{1/p} e^{it\Delta} \phi_{k-1} \|^p_{L^p_t((t_{k-1}, t_k); L^2(Q))} \lesssim R.
$$

The general case follows from this special case and the expression $u = \sum \lambda_j a_j$ with $a_j$ being $U^p_\Delta$-atoms.

From this corollary and the embedding relation $V^q_\Delta \subset U^p_\Delta$ for $2 \leq q < p$, we have for cubes $Q$ of size $R$

$$\| |\nabla|^{1/p} u \|^p_{L^p_tL^2_x(\mathbb{R} \times Q)} \lesssim R^\frac{1}{p} \|\;\|V^q_\Delta\|. \tag{2.9}
$$

An additional key estimate is the following interaction Morawetz estimate for Hartree equations.

**Proposition 2.12.** Let $\mu = 1$ and $u$ be a Schwartz solution to (1.1). Then we have

$$\| |\nabla|^{-1/4} u \|^2_{L^4_{t,x}(\mathbb{R} \times \mathbb{R}^4)} \leq C \|u_0\|^{\frac{1}{2}}_{L^2(\mathbb{R}^4)} \sup_{t \in J} \|u(t)\|^{\frac{1}{2}}_{H^{\frac{3}{2}}(\mathbb{R}^4)}.
$$

**Proof.** It is proved in [43] that

$$\| |\nabla|^{-1/2} |u|^{2} \|^2_{L^2_{t,x}(\mathbb{R} \times \mathbb{R}^4)} \leq C \|u\|_{L^\infty(J; L^2(\mathbb{R}^4))} \|u\|_{L^\infty(J; H^{\frac{1}{2}}(\mathbb{R}^4))}.
$$

We conclude the result of this proposition by the following fact

$$\| |\nabla|^{-1/4} u \|^2_{L^4_{t,x}(\mathbb{R} \times \mathbb{R}^4)} \leq C \| |\nabla|^{-1/2} |u|^{2} \|^2_{L^2_{t,x}(\mathbb{R} \times \mathbb{R}^4)};
$$

and the conservation of mass.

The Morawetz estimate is an essential tool in the proof of scattering for the nonlinear dispersive equations. A classical version of this inequality was first derived by Morawetz [46] for the nonlinear Klein-Gordon equation and then extended by Lin and Strauss [37] to the nonlinear Schrödinger equation with $d \geq 3$. Nakanishi [51] extended the above Morawetz inequality to the dimension $d = 1, 2$ by considering certain variants of the Morawetz estimate with space-time weights and consequently he proved the scattering in low dimension. Morawetz estimates play an important role in the proof of scattering for NLS in the energy-subcritical case, but it does not work in the energy-critical case. An essential breakthrough came from Bourgian [1] who exploited the “induction on energy” technique and the the spatial-localized Morawetz inequality. Colliander, Keel, Staffilani, Takaoka and Tao [8] removed the radial symmetry assumption based on Bourgian’s “induction on energy” technique and the frequency localized type of the interaction Morawetz estimates. This interaction Morawetz estimate was first derived by Colliander, Keel, Staffilani, Takaoka and Tao.
Then we have by the Hardy-Littlewood-Sobolev inequality
\[ m \quad \text{and} \quad R \]
Moreover, in that proof, Young’s inequality was used to obtain the following estimate
\[ \tau \in L^{q_1,\infty}(\mathbb{R}^n), \text{ namely } V \text{ has finite weak } L^{q_1} \text{ norm} \]
Then there exists a constant \( C > 0 \) independent of the function \( f, g, h \) and \( V \) such that
\[ \| T(f, g, h) \|_p \leq C \| V \|_{q_1,\infty} \| g \|_{s_1} \| h \|_{s_2} \| f \|_r. \]  
(2.10)
Proof. This lemma is proved in the case \( n = 3 \) and \( V \in L^{q_1} \) in [9]. It is not hard to see that their argument is dimensional independent and can be extended to higher dimensions directly. Moreover, in that proof, Young’s inequality was used to obtain the following estimate
\[ \left\| \left( \sum_{j=1}^{\infty} |V \ast (g_j h_j)|^2 \right)^{\frac{1}{2}} \right\|_{q_2} \leq C \| V \|_{q_1} \left\| \sum_j |g_j h_j| \right\|_{q_2} \]
However, we may apply the weak Young inequality, (see [36], instead of Young’s inequality in this estimate to conclude (2.10).

3. I-Method and modified local wellposedness

In this section, we will utilize the strategy of I-method to prove Theorem [11]. In addition, by time reversibility, it suffices to show the result on \( \mathbb{R}^+ = [0, +\infty) \). First we introduce the operator \( I \). Let \( \tau = 1, \gamma/2 - 1 < s < 1 \) in the remainder parts and \( m(\xi) \) be a smooth monotone multiplier such that
\[ m(\xi) = \begin{cases} 1, & |\xi| \leq 1, \\ |\xi|^{s-1}, & |\xi| \geq 2, \end{cases} \]
and \( m_N(\xi) = m(N^{-1}\xi) \). Define \( I_N \) by
\[ \hat{I}_Nf(\xi) = m_N(\xi)\hat{f}(\xi). \]
Then we have by the Hardy-Littlewood-Sobolev inequality
\[ \| I_Nu_0 \|_{\dot{H}^1(\mathbb{R}^4)} \lesssim N^{1-s} \| u_0 \|_{\dot{H}^s(\mathbb{R}^4)}. \]
Standard scaling manipulations yield
\[ E(I_N u_0) \lesssim \|\nabla I_N u_0\|_{L^2}^2 + \|I_N u_0\|_{L^4}^2 \|I_N u_0\|_{L^8}^2. \]

From \( \dot{H}^1 \subset L^1(\mathbb{R}^4) \) and \( \dot{H}^{\gamma/2 - 1} \subset L^{2\gamma} \), we get
\[ E(I_N u_0) \leq C N^{2(1-s)} (1 + \|u_0\|_{L^{\gamma/2-1}}^2) \|u_0\|_{H^s}. \]

From the local wellposedness theory and the scaling invariance, we have on \([0, T_*/\lambda^2)\)
\[ iu_t^\lambda + \Delta u^\lambda = (V * |u^\lambda|^2) u^\lambda, \quad u^\lambda(0, x) = u_0^\lambda(x). \]

Hence, we have for \( t \in [0, T_*/\lambda^2) \)
\[ i\partial_t (I_N u^\lambda) + \Delta I_N u^\lambda = I_N ((V * |u^\lambda|^2) u^\lambda), \quad I_N u^\lambda(0, x) = I_N u_0^\lambda(x). \]

Standard scaling manipulations yield
\[ \|\nabla I_N u^\lambda(0)\|_2 = \lambda^{4-\gamma} \|\nabla I_N u_0\|_2, \]
\[ E(I_N u^\lambda(0)) = \lambda^{4-\gamma} E(I_N u_0) \leq C(\|u_0\|_{H^s}) (N/\lambda)^{2(1-s)} \lambda^{4-\gamma}. \]

Choosing
\[ \lambda \approx N^{-\frac{\gamma}{2(1-s)}}, \tag{3.1} \]
with the implicit constant depending only on \( \|u_0\|_{H^s} \), we have
\[ E(I_N u^\lambda(0)) \leq 1/2. \tag{3.2} \]

Based on the observation of \( (3.2) \), the idea of \( I^- \) method arises as follows. If we define
\[ J = \left\{ t \in [0, T_*/\lambda^2) : E(I_N u^\lambda)(t) \leq 1 \right\}, \tag{3.3} \]
then by the above argument, we see clearly that \( J \) is non-empty and closed subset of \([0, T_*/\lambda^2)\)
by the continuity of the solution \( u(t, x) \). The strategy of \( I^- \) method is to derive the openness of \( J \)
relative to \([0, T_*/\lambda^2)\) and as a result, we must have \( J = [0, T_*/\lambda^2) \) and in particular
\[ \|\nabla I_N u^\lambda(T_*/\lambda^2, \cdot)\|_{L^2} \lesssim 1. \]

By rescaling, we obtain the following \textit{a priori} bound on the original smooth solution \( u(t, x) \)
\[ \sup_{t \in [0, T_*)} \|u(t)\|_{H^s(\mathbb{R}^4)} \leq C(N, \|u_0\|_{H^s}). \]

Indeed, we have by using Bernstein’s inequality and rescaling and conservation of mass for \( (1.1) \)
\[
\|u(T_*/\cdot)\|_{H^s(\mathbb{R}^4)} = \lambda^{-s+\frac{2}{4}} \|u^\lambda(T_*/\lambda^2, \cdot)\|_{H^s} \\
\lesssim N^{1-s} \left( N^s \|P_{\leq N} u^\lambda(T_*/\lambda^2, \cdot)\|_{L^2} \\
+ N^{s-1} \|\nabla I_N P_{\geq N} u^\lambda(T_*/\lambda^2, \cdot)\|_{L^2} \right) \\
\lesssim N^{-\frac{s+1}{2s+1}} \|u_0\|_{H^s} + 1 \\
< + \infty.
\]

Hence we have \( T_* = +\infty \). Standard argument yields scattering.
The openness of \( J \) is achieved by estimating the energy increment of \( I_N u^\lambda \) on \( J \)
\[
\int_J \frac{d}{dt} E(I_N u^\lambda)(t) \, dt.
\] (3.4)

In order to estimate the energy increment \((3.4)\), we need establish some \textit{a priori} estimates. Firstly, applying the interaction Morawetz estimate to \( u^\lambda(t, x) \) in place of \( u(t, x) \), we can deduce that

**Lemma 3.1.** Let \( J \) be as in \((3.3)\) and \( u^\lambda(t, x) \) solve the equation \((1.1)\). Then we have
\[
\left\| \nabla |^{-1/4} u^\lambda \right\|_{L^2_{t,x}(J \times \mathbb{R}^4)} \leq C \left( \|u_0\|_{H^s} \right) N^{(1-s)(\gamma/2-1)+\frac{1}{4}}.
\] (3.5)

**Remark 3.2.** If \( J = [0, +\infty) \), we can obtain the scattering result by the above proposition and the well-known argument in [7]. Hence we only show that \( J = [0, \infty) \) in the remainder parts.

**Proof of Lemma 3.1.** Writing \( u^\lambda(t, x) = P_{\leq 2N} u^\lambda(t, x) + P_{\geq 2N} u^\lambda(t, x) \), we have
\[
\left\| u^\lambda(t, x) \right\|_{H^s_{t,x}(\mathbb{R}^4)} \leq \left\| P_{\leq 2N} u^\lambda(t, \cdot) \right\|_{H^s_{t}(\mathbb{R}^4)} + \left\| P_{\geq 2N} u^\lambda(t, \cdot) \right\|_{H^s_{t}(\mathbb{R}^4)}
\]
\[
\lesssim N^{\frac{1}{2}} \left\| u^\lambda(t, \cdot) \right\|_{L^2(\mathbb{R}^4)} + N^{-\frac{3}{2}} \left\| \nabla I_N u^\lambda(t, \cdot) \right\|_{L^2(\mathbb{R}^4)}.
\]

Since \( t \in J \), we have \( \left\| \nabla I_N u^\lambda(t, \cdot) \right\|_{L^2(\mathbb{R}^4)} \leq 1 \). On the other hand, we have by scaling \((1.2)\) and the conservation of mass
\[
\left\| u^\lambda(t, \cdot) \right\|_{L^2(\mathbb{R}^4)} = \lambda^{-(\gamma/2-1)} \| u_0 \|_{L^2(\mathbb{R}^4)}.
\] (3.6)

Now \((3.5)\) can follow from \((3.1)\) and Proposition 2.12. \(\square\)

As the consequence of the above result, we may partition the interval \( J \) in \((3.5)\) into approximately
\[
L \approx \rho_0 N^{(2\gamma-4)(1-s)+1\gamma s}
\]
many consecutive intervals \( J_\ell \) with \( \ell = 1, 2, \ldots, L \) such that
\[
\left\| u^\lambda \right\|_{L^4(J_\ell; \dot{H}^{-\frac{1}{4}4\gamma s}(\mathbb{R}^4))} \leq \rho_0,
\]
on each interval \( J_\ell \) with \( \rho_0 \) a universal small number in \((0, 1)\), which will be determined in the next proposition. From this \textit{a priori} estimate, we can also obtain many useful estimates of \( I_N u^\lambda \) on every \( J_\ell \). Precisely, we have the following proposition.

**Proposition 3.3.** Let \( s > \gamma/2 - 1 \) and \( 2 < \gamma < 4 \). Consider for sufficiently large \( N \)
\[
i \partial_t I_N u^\lambda + \Delta I_N u^\lambda = I_N (V * |u^\lambda|^2) u^\lambda,
\] (3.7)

Then for any \( u_0^\lambda = \lambda u^\lambda(t_0) \in H^s \), there exists a time interval \( J_0 = [t_0, t_0 + \delta] \) with \( \delta = \delta(\|I_N u_0^\lambda\|_{H^s}) \) and there is a unique solution \( u^\lambda \) of \((3.7)\) such that \( I_N u^\lambda \in U^2_{\lambda}(J_0 \times \mathbb{R}^4) \) and the flow map is Lipschitz continuous.

Moreover, there exists a small universal constant \( \rho_0 \) such that if
\[
\left\| |\nabla|^{-\frac{1}{4}} u^\lambda \right\|_{L^4(J_0; L^4(\mathbb{R}^4))} \leq \rho_0,
\]
then we have
\[ \| \nabla I_N u^\lambda \|_{L^2(I_0 \times \mathbb{R}^4)} \leq C \| \nabla I_N u^\lambda(t_0) \|_{L^2(\mathbb{R}^4)}. \]

Proof. We suppress \( N \) and \( \lambda \) for brevity unless necessary. We shall also denote \( [I_0] = (I_0 \times \mathbb{R}^4) \) for short. The first part of this proposition is a consequence of Proposition 2.2 and the following estimates (see similar argument in [43]). We omit the details and proceed on to the second part of this proposition. Writing (3.7) into the Duhamel form
\[ Iu(t) = e^{it-t_0}\Delta Iu(t_0) - i \int_{t_0}^{t} e^{i(t-\tau)\Delta} I((\cdot | - \gamma * |u|^2)u)(\tau)d\tau, \]
we have from Strichartz’s estimate
\[ \| \nabla Iu \|_{L^\infty_t[I_0]} \lesssim \| \nabla Iu_0 \|_2 + \| \nabla I((\cdot | - \gamma * |u|^2)u) \|_{DU^\lambda_{\Delta}(I_0)}. \] (3.8)
Using \( 4 \frac{1}{4-\gamma} < \infty, \frac{1}{3} \in \Lambda_0 \) and
\[ L^\frac{4}{3}(I_0; L^{\frac{8}{3-\gamma}}(\mathbb{R}^4)) \subset DU^\lambda_{\Delta}(I_0 \times \mathbb{R}^4), \]
we have from the Leibnitz rule enjoyed by \( \nabla I \)
\[ \| \nabla I((\cdot | - \gamma * |u|^2)u) \|_{DU^\lambda_{\Delta}(I_0 \times \mathbb{R}^4)} \lesssim \| \nabla I u \|_{L^\frac{4}{3}(I_0; L^{\frac{8}{3-\gamma}}(\mathbb{R}^4))} \]
\[ + \| (\cdot | - \gamma * (\nabla I u \cdot \vec{u}) \cdot u \|_{L^\frac{4}{3}(I_0; L^{\frac{8}{3-\gamma}}(\mathbb{R}^4))}. \] (3.9)
From the Hölder and Hardy-Littlewood-Sobolev inequalities along with \( U^\lambda_{\Delta}[J_0] \subset L^p_tL^r_x[J_0] \) for \( (q,r) \in \Lambda_0, \) we have
\[ \| \nabla I u \|_{L^\frac{4}{3}(I_0; L^{\frac{8}{3-\gamma}}(\mathbb{R}^4))} \lesssim \| \nabla Iu \|_{U^\lambda_{\Delta}[J_0]} \| u \|_{L^\frac{4}{3}(I_0; L^{\frac{8}{3-\gamma}}(\mathbb{R}^4))}, \]
(3.10)
and
\[ \| (\cdot | - \gamma * (\nabla I u \cdot \vec{u}) \cdot u \|_{L^\frac{4}{3}(I_0; L^{\frac{8}{3-\gamma}}(\mathbb{R}^4))} \lesssim \| \nabla I u \cdot \vec{u} \|_{L^\frac{4}{3}(I_0; L^{\frac{8}{3-\gamma}}(\mathbb{R}^4))} \| u \|_{L^\frac{4}{3}(I_0; L^{\frac{8}{3-\gamma}}(\mathbb{R}^4))}, \]
where we have used
\[ 1 + \frac{4-\gamma}{4} = \frac{\gamma}{4} + 2 \times \frac{4-\gamma}{4}, \quad 1 + \frac{\gamma}{8} = \frac{\gamma}{4} + \frac{8-\gamma}{8}, \]
\[ 1 + \frac{\gamma}{8} = \frac{\gamma}{4} + 2 \times \frac{8-\gamma}{8}, \quad \frac{8-\gamma}{8} = \frac{\gamma}{8} + \frac{4-\gamma}{4} \]
As a consequence, the Duhamel part of (3.8) can be controlled with
\[ \| \nabla I u \|_{U^\lambda_{\Delta}[J_0]} \| u \|_{L^\infty_t[I_0; L^{\frac{8}{3-\gamma}}(\mathbb{R}^4)]}, \] (3.11)
where \( \left( \frac{4-\gamma}{4}, \frac{1}{3} \right) \in \Lambda_{\gamma/2-1}. \)
To estimate $\|u\|_{L_t^{\frac{3}{2}} L_x^{\frac{3}{2}}[J_0]}$, we write for dyadic $N_j \geq N/2$ with $j = 1, 2, \ldots$

$$u = P_{\leq N} u + \sum_{j=1}^{\infty} P_{N_j} u,$$

and handle the following two cases in different ways.

**Case 1.** $2 < \gamma < 3$. From Hölder’s inequality, Sobolev embedding and the Gagliardo-Nirenberg inequality, we have

$$\|P_{\leq N} u\|_{L_t^{\frac{3}{2}} L_x^{\frac{3}{2}}[J_0]} \lesssim \|u\|_{L_t^3 L_x^3[J_0]} \|I u\|_{L_t^4 L_x^4[J_0]}^{\gamma-2}$$

$$\lesssim \|u\|_{L_t^\gamma L_x^\gamma[J_0]} \||\nabla|^{\frac{1}{2}} IP_{\leq N} u\|_{L_t^2 L_x^2[J_0]} \||\nabla|^\frac{1}{2} I u\|_{L_t^4 L_x^4[J_0]}^{(\gamma-2)/3}$$

$$\lesssim \rho_0 (\gamma-2)/6 \lambda^{3/2} \|u\|_{L_t^2 L_x^2[J_0]} \||\nabla I u\|_{U^2_{\lambda}[J_0]}^{(\gamma-2)/3}$$

$$\lesssim \rho_0 (\gamma-2)/6 N^{(1-s)(\gamma-2)/(3-\gamma)} \|\nabla I u\|_{U^2_{\lambda}[J_0]}^{(\gamma-2)/3}.\quad (3.12)$$

For $j \geq 1$, we have by Hölder’s inequality and Bernstein’s inequalities

$$\|P_{N_j} u\|_{L_t^{\frac{3}{2}} L_x^{\frac{3}{2}}[J_0]} \lesssim \|P_{N_j} u\|_{L_t^3 L_x^3[J_0]} \|P_{N_j} u\|_{L_t^\infty L_x^2[J_0]}^{\gamma-2}$$

$$\lesssim \left( \frac{N_j}{N} \right)^{1-s} \|IP_{N_j} u\|_{L_t^2 L_x^2[J_0]} \|IP_{N_j} u\|_{L_t^\infty L_x^2[J_0]}^{\gamma-2}$$

$$\lesssim \left( \frac{N_j}{N} \right)^{1-s} N_j^{\frac{1}{2}((\gamma-2)-(3-\gamma))} \|\nabla|^\frac{1}{2} IP_{N_j} u\|_{L_t^4 L_x^4[J_0]}^{(\gamma-2)/2} \|\nabla IP_{N_j} u\|_{L_t^4 L_x^4[J_0]}^{3-\gamma}$$

$$\lesssim N^{s-1} N_j^{-(s-(\gamma/2)-1)} \|\nabla I u\|_{U^2_{\lambda}[J_0]}^{(\gamma-2)/3}.\quad (3.13)$$

Summing up (3.12) and (3.13), we obtain

$$\|u\|_{L_t^{\frac{3}{2}} L_x^{\frac{3}{2}}[J_0]} \lesssim \rho_0 (\gamma-2)/6 N^{(1-s)(\gamma-2)/(3-\gamma)} \|\nabla I u\|_{U^2_{\lambda}[J_0]}^{(\gamma-2)/3}$$

$$+ N^{-2+s} \|\nabla I u\|_{U^2_{\lambda}[J_0]}^{(\gamma-2)/3}.\quad (3.14)$$

**Case 2.** $3 \leq \gamma < 4$. From Hölder’s inequality, Sobolev embedding and the Gagliardo-Nirenberg inequality, we have

$$\|P_{\leq N} u\|_{L_t^{\frac{3}{2}} L_x^{\frac{3}{2}}[J_0]} \lesssim \|IP_{\leq N} u\|_{L_t^{\frac{4}{2-\gamma}} L_x^{\frac{4}{2-\gamma}}[J_0]}$$

$$\lesssim \|\nabla I u\|_{L_t^2 L_x^2[J_0]} \|\nabla|^\frac{1}{2} I u\|_{L_t^4 L_x^4[J_0]}^{2(4-\gamma)/3} \|\nabla|^\frac{1}{2} I u\|_{L_t^4 L_x^4[J_0]}^{(4-\gamma)/3}$$

$$\lesssim \|\nabla|^\frac{1}{2} I u\|_{L_t^4 L_x^4[J_0]}^{2(4-\gamma)/3} \|\nabla I u\|_{U^2_{\lambda}[J_0]}^{(2\gamma-5)/3}.\quad (3.15)$$
For \( j \geq 1 \), we have by Hölder’s inequality, Bernstein’s inequalities and Sobolev embedding

\[
\|P_N u\|_{L_t^{q_{\ell}} L_x^{r_{\ell}}[J_0]} \lesssim \|P_N u\|_{L_t^{\frac{4-\gamma}{\ell}} L_x^{s_{\ell}}[J_0]}^{\frac{4-\gamma}{\ell}} \|P_N u\|_{L_t^{\frac{4}{\ell}} L_x^{\frac{s}{\ell}-\gamma}[J_0]}^{\frac{4}{\ell}} \|P_N u\|_{L_t^{\frac{4}{\ell}} L_x^{\frac{s}{\ell}-\gamma}[J_0]}^{\frac{4}{\ell}} \lesssim \left( \frac{N_j}{N} \right)^{1-s} \|IP_{N_j} u\|_{L_t^{\frac{4-\gamma}{\ell}} L_x^{s_{\ell}}[J_0]}^{\frac{4-\gamma}{\ell}} \|IP_{N_j} u\|_{L_t^{\frac{4}{\ell}} L_x^{s_{\ell}-\gamma}[J_0]}^{\frac{4}{\ell}} \|\nabla IP_{N_j} u\|_{L_t^{\frac{4}{\ell}} L_x^{s_{\ell}-\gamma}[J_0]}^{\frac{4}{\ell}} \lesssim N^{s-1} N^{-s(\gamma/2-1)} \|\nabla I u\|_{U^3_4[J_0]},
\]

Summing up (3.15) and (3.16), we obtain

\[
\|u\|_{L_t^{\frac{4-\gamma}{\ell}} L_x^{s_{\ell}}[J_0]} \lesssim \rho_0^{\frac{4-\gamma}{\ell}} \|\nabla I u\|_{U^3_4[J_0]}^{\frac{2\gamma-5}{2}} + N^{-2+\frac{2}{\ell}} \|\nabla I u\|_{U^3_4[J_0]}.
\]

Substituting (3.14) for \( 2 < \gamma < 3 \) and (3.17) for \( 3 \leq \gamma < 4 \) to (3.11) and then back to (3.8), we close the bootstrap by choosing \( N \) large enough and \( \rho_0 \) small, say

\[
0 < \rho_0 \ll \min \left\{ 1, N^{-\frac{6(1-\gamma)(3-\gamma)}{2+\gamma}} \right\}.
\]

The proof is complete. \( \square \)

As the consequence of this proposition, we obtain that

\[
\|\nabla I_N u^\lambda\|_{U^3_4(J_t \times \mathbb{R}^4)} \lesssim 1.
\]

on each interval \( J_\ell \). Using (2.3) to sum up (3.18) with respect to \( \ell \), we have

\[
\|\nabla I_N u^\lambda\|_{U^3_4(J \times \mathbb{R}^4)} \leq C(\|u_0\|_{H^s}, \rho_0) N^{\frac{(\gamma-2)(1-\gamma)}{\gamma(\gamma-1)} + \frac{2}{\gamma}}.
\]

It grows up with \( N \) polynomially and will be the base point of the induction on frequency argument for long time Strichartz estimate proved in the next section.

4. The long-time Strichartz estimate

In this section, we deduce long time Strichartz estimate for \( I_N u^\lambda \). The main result in this section is the following proposition.

**Proposition 4.1.** Assume \( 1 \leq M \leq N \) and \( E(I_N u^\lambda(t)) \leq 1 \) for \( t \in J \). If \( 3 < \gamma < 4 \), then there is a constant for any \( \epsilon > 0 \), we have

\[
\|\nabla I_N P_{>M} u^\lambda\|_{U^3_4(J \times \mathbb{R}^4)} \lesssim \gamma \epsilon + 1 + M^{6(1-\gamma)} N^\epsilon \|\nabla I_N P_{>M} u^\lambda\|_{U^3_4(J \times \mathbb{R}^4)}.
\]

**Remark 4.2.** We have to afford an \( \epsilon \)-loss to avoid the failure of the Hardy-Littlewood-Sobolev inequality in the end-point case.

Comparing with (3.19), we can obtain uniform boundedness of the high frequency part of \( \nabla I_N u^\lambda \) in the endpoint Strichartz space.
Corollary 4.3. Let \(3 < \gamma < 4\) and \(E(I_N u^\lambda(t)) \leq 1\) for \(t \in J\), and \(N\) large enough. Then
\begin{equation}
\| \nabla I_N u^\lambda \|_{L^2_t L^4_x(J \times \mathbb{R}^4)} \lesssim \| \nabla I_N u^\lambda \|_{U^2_{\Delta}(J \times \mathbb{R}^4)} \lesssim \| u_0 \|_{H^s_1}. \tag{4.2}
\end{equation}

Proof. The technique is standard. The first inequality follows by \(U^2_{\Delta}(J \times \mathbb{R}^4) \hookrightarrow L^2_t L^4_x(J \times \mathbb{R}^4)\). Now we show the second inequality by iteration. Given \(\gamma < 4\), we take \(\epsilon\) small enough and set
\[M = 0.01 N, \quad c_\gamma = \frac{4 - \gamma}{2}\]
to get
\begin{equation}
\| \nabla I_N P >_{N_{100}} u^\lambda \|_{U^2_{\Delta}(J \times \mathbb{R}^4)} \leq C_0 + C_0 N^{-\epsilon_0} \| \nabla I_N P >_{N_{100}} u^\lambda \|_{U^2_{\Delta}(J \times \mathbb{R}^4)}, \tag{4.3}
\end{equation}
where \(C_0 \geq 1\) depends only on \(\gamma\) and \(\epsilon\). Let
\[N \geq 10^4 C_0^{10},\]
and
\[L = \left\lfloor \frac{\log N}{10 \log 8} \right\rfloor.\]
Iterating (4.3) \(L\) times and using (3.19), we get
\begin{align*}
\| \nabla I_N P >_{N_{100}} u^\lambda \|_{U^2_{\Delta}(J \times \mathbb{R}^4)} & \leq \sum_{\ell = 1}^{L-1} C_0 \left( \frac{N}{100} \right)^{-\ell \epsilon_0} \| \nabla I_N P >_{N_{100}} u^\lambda \|_{U^2_{\Delta}(J \times \mathbb{R}^4)} \\
& \quad + C_0^L \left( \frac{N}{100} \right)^{-L \epsilon_0} \| \nabla I_N P >_{N_{100}} u^\lambda \|_{U^2_{\Delta}(J \times \mathbb{R}^4)} \\
& \quad \leq 2C_0 + C(\| u_0 \|_{H^s_1}, \rho_0)(N^{c_\gamma_2} C_0)^{-L} N^{\frac{(\gamma - 2)(1 - \gamma)}{s(\gamma/2 - 1)} + \frac{1}{2}}. \tag{4.4}
\end{align*}
Now, it is easy to see that the proof is concluded by taking \(N\) sufficiently large and.

Proof of Proposition 4.7. For the sake of brevity, we suppress \(N\) in \(I_N\) and \(\lambda\) in \(u^\lambda\). Applying \(P > M\) to both sides of the modified system, we get
\[I u > M(t) = e^{i \Delta} I u > M(0) - i \int_0^t e^{i(t-\tau) \Delta} IP > M((| \cdot |^{-\gamma} * |u|^2)u)(\tau) d\tau,\]
and hence
\begin{equation}
\| \nabla I u > M \|_{U^2_{\Delta}(J \times \mathbb{R}^4)} \lesssim \| I u > M(0) \|_{H^1} + \| \nabla IP > M((| \cdot |^{-\gamma} * |u|^2)u) \|_{DU^2_{\Delta}(J \times \mathbb{R}^4)}. \tag{4.5}
\end{equation}
Noting that
\[P > M((| \cdot |^{-\gamma} * u_{\leq M}(\cdot)u_{\leq M})) = 0,\]
we only have to estimate contributions to (4.5) from the following two terms
\begin{align*}
\| \nabla IP > M((| \cdot |^{-\gamma} * u_{\leq M}(\cdot)u) \|_{DU^2_{\Delta}(J \times \mathbb{R}^4)}, \tag{4.6}
\| \nabla IP > M((| \cdot |^{-\gamma} * u_{\leq M}(\cdot)u) \|_{DU^2_{\Delta}(J \times \mathbb{R}^4)}. \tag{4.7}
\end{align*}
It is no need for us to distinguish \(u\) and \(\bar{u}\) below, so we adopt the notion that \(u\) means either \(u\) or \(\bar{u}\). To perform nonlinear estimates, we will introduce a sequence of small parameters \(\epsilon_1, \cdots, \epsilon_7\). It will be clear from the context how they depend on each other. All these parameters will be
taken sufficiently small in the end.

The estimation of (4.6). We are aimed to show for any $\epsilon > 0$, there holds

\[
(4.6) \lesssim N^\epsilon M^{-(4-\gamma)} \| \nabla I u_{> \frac{M}{8}} \|_{U^2_\Delta (J \times \mathbb{R}^4)}.
\]

As observed in [7], $\nabla I$ obeys the Leibnitz rule, and we need to handle

\[
\| (| \cdot |^{-\gamma} * |u_{> \frac{M}{8}}|^2) \nabla I u \|_{DU^2_\Delta (J \times \mathbb{R}^4)},
\]

and

\[
\| (| \cdot |^{-\gamma} * (\nabla I u_{> \frac{M}{8}} * u_{> \frac{M}{8}})) u_{\leq N} \|_{DU^2_\Delta (J \times \mathbb{R}^4)},
\]

and

\[
\| (| \cdot |^{-\gamma} * (\nabla I u_{> \frac{M}{8}} * u_{> \frac{M}{8}})) u_{> N} \|_{DU^2_\Delta (J \times \mathbb{R}^4)}.
\]

Let us start with (4.9). Using $L^q(J; L^{r'}(\mathbb{R}^4)) \subset DU^2_\Delta (J \times \mathbb{R}^4)$, for $q > 2, (q, r) \in \Lambda_0$, and the Hölder and Hardy-Littlewood-Sobolev inequalities, we have

\[
(4.9) \lesssim \| (| \cdot |^{-\gamma} * |u_{> \frac{M}{8}}|^2) \nabla I u \|_{L^q(J; L^{r'}(\mathbb{R}^4))} \lesssim \| \nabla I u \|_{L^{\infty-} - L^{2+}_x (J \times \mathbb{R}^4)} \| u_{> \frac{M}{8}} \|_{L^q_1 L^{r_1}_x (J \times \mathbb{R}^4)} \| u_{> \frac{M}{8}} \|_{U^2_\Delta (J \times \mathbb{R}^4)},
\]

where

\[
(q, r) = \left( 2 + \epsilon_1, \frac{4 + 2\epsilon_1}{1 + \epsilon_1} \right) \in \Lambda_0,
\]

\[
(\infty-, 2+) = \left( \frac{2(2 + \epsilon_1)}{\epsilon_1}, \frac{8 + 4\epsilon_1}{4 + \epsilon_1} \right) \in \Lambda_0,
\]

\[
(q_1, r_1) = \left( 2, \frac{4}{5 - \gamma} \right),
\]

\[
1 - \frac{1}{r'} = 1 - \frac{2 + \gamma - 4}{4} + \frac{1}{r_1} - \frac{1}{q_1}
\]

By Hölder’s inequality and (3.19), we have by taking $\epsilon_1$ small enough

\[
\| \nabla I u \|_{L^{\infty-} - L^{2+}_x (J \times \mathbb{R}^4)} \leq \| \nabla I u \|_{L^{q_1}_x L^{r_1}_x (J \times \mathbb{R}^4)} \| u_{> \frac{M}{8}} \|_{U^2_\Delta (J \times \mathbb{R}^4)} \lesssim N^\epsilon.
\]

On the other hand, Sobolev embedding and interpolation yield

\[
\| u_{> \frac{M}{8}} \|_{L^2(J; L^{\frac{4}{4-\gamma}}(\mathbb{R}^4)))} \lesssim \| \nabla |^{\gamma/2-1} u_{> \frac{M}{8}} \|_{L^2(J; L^4(\mathbb{R}^4)))} \lesssim \| \nabla |^{\gamma/2-1} u_{> \frac{M}{8}} \|_{L^2(J; L^4(\mathbb{R}^4)))} \| \nabla |^{\gamma/2-1} u_{> \frac{M}{8}} \|_{L^2(J; L^4(\mathbb{R}^4)))},
\]

where, we may estimate by the definition of $\nabla I$

\[
\| \nabla |^{\gamma/2-1} u_{> \frac{M}{8}} \|_{L^2(J; L^4(\mathbb{R}^4)))} \lesssim M^{-(2-\frac{2}{\gamma})} \| \nabla I u_{> \frac{M}{8}} \|_{U^2_\Delta (J \times \mathbb{R}^4)},
\]

and

\[
\| \nabla |^{\gamma/2-1} u_{> \frac{M}{8}} \|_{L^\infty(J; L^4(\mathbb{R}^4)))} \lesssim M^{-(2-\frac{2}{\gamma})}.
\]
Therefore, we may substitute these estimates to (4.12) and get
\[ M^{-(4-\gamma)} N^\epsilon \| \nabla I u_{> M/\varepsilon} \| U^1_2(J \times \mathbb{R}^4). \] (4.13)

Next, we deal with (4.10). Similar to the argument for (4.9), we have
\[ \| \nabla I u_{> M/\varepsilon} \| L^q_t L^r_x(J \times \mathbb{R}^4) \leq \| u \| L^q_t L^r_x(J \times \mathbb{R}^4)^N \| \nabla I u \| L^q_t L^r_x(J \times \mathbb{R}^4) \lesssim \| \nabla I u \| L^q_t L^r_x(J \times \mathbb{R}^4). \] (4.14)

Now, we tackle (4.11). Choose \((q, r) = \left(2 + \varepsilon_2, \frac{2 + \varepsilon_2}{1 + \varepsilon_2} \right) \in \Lambda_0\), \((\infty-, 4+) = \left(\frac{2 + \varepsilon_2}{\varepsilon_2}, 4 + 2\varepsilon_2 \right) \in \Lambda_1\).

By Sobolev embedding and the assumption \(\gamma > 3\), we have
\[ \| u \| L^q_t L^r_x(J \times \mathbb{R}^4) \lesssim \| \nabla I u \| L^q_t L^r_x(J \times \mathbb{R}^4). \]

By Sobolev embedding and H"older’s inequality, we have
\[ \| u \| L^q_t L^r_x(J \times \mathbb{R}^4) \lesssim \| \nabla I u \| L^q_t L^r_x(J \times \mathbb{R}^4). \]

Next, we estimate by interpolation and Sobolev embedding inequalities
\[ \| u \| L^q_t L^r_x(J \times \mathbb{R}^4) \lesssim \| \nabla I u \| L^q_t L^r_x(J \times \mathbb{R}^4) \lesssim N^{-(\gamma-2)} \| u \| L^q_t L^r_x(J \times \mathbb{R}^4). \]

By Sobolev embedding and the definition of the \(\nabla I\) operator, we get
\[ \| u \| L^q_t L^r_x(J \times \mathbb{R}^4) \lesssim \| \nabla I u \| L^q_t L^r_x(J \times \mathbb{R}^4). \]

In view of (3.19) and the definition of \(\nabla I\), we have
\[ \| u \| L^q_t L^r_x(J \times \mathbb{R}^4) \lesssim N^{-(\gamma-2)} \| u \| L^q_t L^r_x(J \times \mathbb{R}^4) \lesssim N^{-2+\frac{\gamma}{2}} N^{\left(\gamma-2\right)\left(\frac{1-\varepsilon}{\varepsilon}\right)+\frac{1}{2}}. \]
Hence, we get
\[ \|u_N\|_{L_t^\infty L_x^{\frac{8}{3}}} \lesssim M^{-2+\frac{2}{7}}N^\epsilon, \]
by choosing \( \epsilon_3 \) sufficiently small. Thus, we have
\[ (4.11) \lesssim M^{-(4-\gamma)}N^\epsilon \|\nabla Iu_M\|_{U_\Delta^1}. \] 
Collecting (4.13) (4.14) and (4.15), we arrive at (4.8).

**The estimation of (4.7).** In this part, we will use local smoothing estimates for the Schrödinger operators and the radial Sobolev embedding along with the duality relation \( V_\Delta^2 = (DU_\Delta^2)^* \). Without loss of generality, taking \( v \in V_\Delta^2 (J \times R^4) \) with \( v = P_{>M}v \) and \( \|v\|_{V_\Delta^2} = 1 \), we see (4.7) is bounded by
\[
\int_J \left\langle v, \nabla I P_M \left( (|\cdot|^{-\gamma} \ast |u_{\leq M}|^2) \cdot u_{>M} \right) \right\rangle dt.
\]
The Leibnitz rule obeyed by \( \nabla I \) reduces (4.16) to estimating
\[
\int_J \left\langle v, |\cdot|^{-\gamma} \ast (\nabla I u_{\leq M} \cdot u_{>M}) \right\rangle dt,
\]
\[
\int_J \left\langle v, (|\cdot|^{-\gamma} \ast |u_{\leq M}|^2) \cdot \nabla I u_{>M} \right\rangle dt.
\]
Let us deal with (4.17) first. By Hölder and Hardy-Littwood-Sobolev’s inequalities, we have
\[
(4.17) \lesssim \left\| u_{>M} \right\|_{L_t^\infty L_x^{\frac{4}{3}}} \left( J \times R^4 \right) \left\| u_{\leq M} \right\|_{L_t^{\infty} L_x^{2+\frac{4}{7}}} \left( J \times R^4 \right) \left\| \nabla Iu_{\leq M} \right\|_{L_t^{\infty} L_x^{2+\frac{4}{7}}} \left( J \times R^4 \right) \left\| \nabla Iu_{\leq M} \right\|_{L_t^{\infty} L_x^{2+\frac{4}{7}}} \left( J \times R^4 \right),
\]
where
\[
\left( \infty - \frac{4}{4-\gamma} \right) = \left( \frac{4+2\epsilon_4}{\epsilon_4}, \frac{4+2\epsilon_4}{4-\gamma + \frac{3-\gamma+\epsilon_4}{2}} \right).
\]
We have by definition of \( \nabla I \)
\[
\left\| u_{>M} \right\|_{L_t^\infty L_x^{\frac{4}{3}}} \left( J \times R^4 \right) \lesssim \frac{1}{M} \left\| \nabla I u_{>M} \right\|_{U_\Delta^1 \left( J \times R^4 \right)}.
\]
From Bernstein’s inequality and interpolation, we have by letting \( \epsilon_4 \) small enough
\[
\left\| u_{\leq M} \right\|_{L_t^{\infty} L_x^{2+\frac{4}{7}}} \left( J \times R^4 \right) \lesssim \left\| \nabla |^{-\gamma} I u_{\leq M} \right\|_{L_t^{\infty} L_x^{2+\frac{4}{7}}} \left( J \times R^4 \right) \lesssim M^{\gamma-3}N^\epsilon.
\]
Thus, we obtain
\[
(4.17) \lesssim M^{-(4-\gamma)}N^\epsilon \|\nabla Iu_M\|_{U_\Delta^1}. \]

To estimate (4.18), we shall use local smoothing and radial Sobolev embedding. Let \( \chi \) be the characteristic function of the ball \( \{ x \in R^4 : |x| \leq 1/M \} \) and for \( j \geq 0 \) write \( \psi_j(x) = \chi(2^{-(j+1)}x) - \chi(2^{-j}x) \) such that
\[
1 = \chi(x) + \sum_{j=0}^{+\infty} \psi_j(x).
\]
We need to estimate
\[
\int_j \left\langle \chi v, \chi \nabla I u_{\frac{M}{4}} \cdot (| \cdot |^{-\gamma} \ast | u_{\frac{M}{4}} |^2) \right\rangle \ dt,
\]  
(4.19)
and
\[
\int_j \left\langle \psi_j(x) v, \psi_j(x) \nabla I u_{\frac{M}{4}} \cdot (| \cdot |^{-\gamma} \ast | u_{\frac{M}{4}} |^2) \right\rangle \ dt.
\]  
(4.20)

For (4.19), we may use Hölder’s inequality to bound it by
\[
\| \chi v \|_{L_t^{2+\epsilon_5} L_x^2(J \times \mathbb{R}^4)} \| \chi \nabla I u_{\frac{M}{4}} \|_{L_t^{2+\epsilon_5} L_x^2(J \times \mathbb{R}^4)} \| (| \cdot |^{-\gamma} \ast | u_{\frac{M}{4}} |^2) \|_{L_t^{-\infty} L_x^\infty(J \times \mathbb{R}^4)},
\]  
(4.21)
where
\[
\infty- = \frac{2(2 + \epsilon_5)}{\epsilon_5}.
\]

From the local smoothing estimate in Corollary 2.11 we have
\[
\| \chi v \|_{L_t^{2+\epsilon_5} L_x^2(J \times \mathbb{R}^4)} \lesssim M^{-1} M^{\frac{\gamma}{\gamma + 5}},
\]  
(4.22)
\[
\| \chi \nabla I u_{\frac{M}{4}} \|_{L_t^{2+\epsilon_5} L_x^2(J \times \mathbb{R}^4)} \lesssim M^{-1} \| \nabla I u_{\frac{M}{4}} \|_{L_t^{2+\epsilon_5} L_x^2(J \times \mathbb{R}^4)}.
\]  
(4.23)

For the third factor in (4.21), since the Fourier transform of \(| \cdot |^{-\gamma} \ast u_{\frac{M}{4}}(t, \cdot)(x)\) with respect to \(x\) is supported in a ball of radius approximately \(M\) and centered at the origin, we use Hölder and Bernstein’s inequalities and then Hardy-Littlewood-Sobolev’s inequality to get
\[
\| (| \cdot |^{-\gamma} \ast | u_{\frac{M}{4}} |^2) \|_{L_t^{-\infty} L_x^\infty(J \times \mathbb{R}^4)}
\lesssim \| (| \cdot |^{-\gamma} \ast | u_{\frac{M}{4}} |^2) \|_{L_t^{2+\epsilon_5} L_x^2(J \times \mathbb{R}^4)} \| (| \cdot |^{-\gamma} \ast | u_{\frac{M}{4}} |^2) \|_{L_t^{-\infty} L_x^\infty(J \times \mathbb{R}^4)}
\lesssim M^{\gamma - 3} M^{\frac{\epsilon_5}{2 + \epsilon_5}} \| u_{\frac{M}{4}} \|_{L_t^{2+\epsilon_5} L_x^2(J \times \mathbb{R}^4)} \| u_{\frac{M}{4}} \|_{L_t^{2+\epsilon_5} L_x^2(J \times \mathbb{R}^4)}
\lesssim M^{\gamma - 2} M^{\frac{-\epsilon_5}{2 + \epsilon_5}} \| \nabla I u_{\frac{M}{4}} \|_{L_t^{2+\epsilon_5} L_x^2(J \times \mathbb{R}^4)} \| \nabla I u_{\frac{M}{4}} \|_{L_t^{2+\epsilon_5} L_x^2(J \times \mathbb{R}^4)}.
\]
Therefore, we have for sufficiently small \(\epsilon_5\)
\[
\text{(4.21)} \lesssim M^{\gamma - 4} N^\epsilon \| \nabla I u_{\frac{M}{4}} \|_{L_t^{2+\epsilon_5} L_x^2(J \times \mathbb{R}^4)}.
\]

Now we estimate (4.20) by Hölder’s inequality,
\[
\text{(4.20)} \lesssim \| \psi_j(x) |x|^{-\frac{4}{\epsilon_6}} \|_{L_t^{2+\epsilon_6} L_x^2(J \times \mathbb{R}^4)} \| \psi_j(x) |x|^{-\frac{4}{\epsilon_6}} \|_{L_t^{2+\epsilon_6} L_x^2(J \times \mathbb{R}^4)} \times \| \psi_j(x) |x|^{-\gamma} \ast | u_{\frac{M}{4}} |^2 \|_{L_t^{-\infty} L_x^\infty(J \times \mathbb{R}^4)},
\]
where \(\psi_j(x) = \psi_j(x) |x|^{\frac{2}{\epsilon_6}}\) and
\[
(\infty-, \infty-) = \left(\frac{2}{\epsilon_6}, \frac{2}{\epsilon_6} + 4 \epsilon_7\right).
\]

From local smoothing estimates (2.9) and (2.8), we have respectively
\[
\| \psi_j(x) |x|^{-\frac{4}{\epsilon_6}} \|_{L_t^{2+\epsilon_6} L_x^2(J \times \mathbb{R}^4)} \lesssim M^{-\frac{4}{\epsilon_6}} M^{\frac{\epsilon_6}{2 + \epsilon_6}} 2^{\frac{\epsilon_6}{2 + \epsilon_6}} \|
\]  
(4.24)
\[ \| \psi_j(x) |x|^{\frac{3}{2}} \nabla Iu_{\geq \frac{M}{8}} \|_{L^p_t L^2_x (\mathbb{R}^4)} \leq \| \psi_j(x) |x|^{-\frac{3}{4}} \nabla Iu_{\geq \frac{M}{8}} \| L^2_t L^{2+\gamma} (\mathbb{R}^4) \| \psi_j(x) |x|^{\frac{3}{2}} \nabla Iu_{\geq \frac{M}{8}} \|_{L^2_t L^2_x (\mathbb{R}^4)} \]

Next, by Corollary 2.3 and Hölder's inequality, we have

\[ \| V_j(x) (| \cdot |^{-\gamma} \ast u_{\leq \frac{M}{8}})^2 \|_{L^p_t L^{2\gamma} (\mathbb{R}^4)} \leq \| V_j(x) |u_{\leq \frac{M}{8}}^2 \|_{L^p_t L^{2\gamma} (\mathbb{R}^4)} \]

To tackle the first factor in (1.26), we use the Sobolev type inequality for spherically symmetric functions (2.4), Hölder and Bernstein’s inequality to obtain

\[ \| \psi_j(x) |x|^{\frac{3}{2}} u_{\frac{M}{8}} (t, x)^2 \|_{L^p_t L^2_x (\mathbb{R}^4)} \leq \sum_{N_2 \leq N_1 \leq M} \| |x|^{\frac{3}{2}} u_{N_1} (t, x) \|_{L^p_t L^2_x} \| \psi_j(x) u_{N_2} (t, x) \|_{L^p_t L^2_x} \]

where \( N_1, N_2 \) are dyadic integers and

\[ \frac{3}{2} - \frac{4}{p} = \gamma - 5 - \frac{2\epsilon_7}{2 + \epsilon_7}. \]

If \( \gamma > 3 \), then we have

\[ \sum_{N_2 \leq N_1 \leq M} \left( \frac{N_2}{N_1} \right)^{1/2} N_2^{-3-\frac{2\epsilon_7}{2+\epsilon_7}} \leq M^{\gamma - 3 - \frac{2\epsilon_7}{2+\epsilon_7}}. \]

Schur’s Lemma yields

\[ \| V_j(x) |u_{\leq \frac{M}{8}} (t, x)|^2 \|_{L^p_t L^{2\gamma} (\mathbb{R}^4)} \leq \left( \frac{2^j}{M} \right)^{\gamma - 3 - \frac{2\epsilon_7}{2+\epsilon_7}}. \]
\begin{equation}
\lesssim \left( \frac{2^j}{M} \right)^{\frac{3}{2}} M^{\gamma - 3} \frac{2^{2\gamma}}{2^{\gamma + 2}} \| \nabla Iu \|_{L^2_t L^2_x(J \times \mathbb{R}^4)} \lesssim 2^{\frac{3}{2}j} \| \nabla Iu \|^{2}_{U^{3}_{\Delta}(J \times \mathbb{R}^4)}. \tag{4.28}
\end{equation}

Collecting (4.24) (4.25) (4.26) (4.27) and (4.28), we have
\begin{equation}
(4.20) \lesssim M^{\gamma - 4} M^{\frac{6}{\gamma} + \frac{2\gamma}{\gamma + 2}} 2^{\frac{\epsilon}{2} + \frac{\epsilon}{\gamma + 2}} \| \nabla Iu \|_{U^{3}_{\Delta}(J \times \mathbb{R}^4)}.
\end{equation}

For any $\gamma \in (3, 4)$ and $\epsilon > 0$, we first take $\epsilon_7$ small enough such that
\[ \gamma - 3 > \frac{2\epsilon_7}{2 + \epsilon_7}, \]
and then take $\epsilon_6$ much smaller such that
\[ \frac{\epsilon_7}{2 + \epsilon_7} - \frac{\epsilon_6}{2 + \epsilon_6} > \frac{\epsilon_7}{10}. \]

Thus for $\epsilon_7 > 0$ small enough, we have
\begin{equation}
(4.20) \lesssim M^{-(4-\gamma)} N^\epsilon 2^{-j\epsilon} \| \nabla Iu \|_{U^{3}_{\Delta}(J \times \mathbb{R}^4)}.
\end{equation}

Summing over $j \geq 0$ in (4.20) along with the estimate of (4.19), we obtain
\begin{equation}
(4.18) \lesssim M^{-(4-\gamma)} N^\epsilon \| \nabla Iu \|_{U^{3}_{\Delta}(J \times \mathbb{R}^4)}.
\end{equation}

The proof is complete. \hfill \Box

5. Energy increment of $Iu$

In this section, we estimate the energy increment
\[ \int_J \frac{d}{dt} E(I_N u^\lambda)(t) dt, \]
by the long time Strichartz estimate in Section 4. Since this part is pretty standard in the literature, we will only sketch the proof, where we again suppress $N, \lambda$ and the time interval $J$ for brevity.

**Proposition 5.1.** Let $3 < \gamma < 4$. Then there is a constant $C$ depending only on $\|u_0\|_{H^s}$ such that
\[ \int_J \left| \frac{d}{dt} E(I_N u^\lambda)(t) \right| dt \leq CN^{-\frac{\gamma - 4}{2}}. \]

**Remark 5.2.** From this proposition, we may deduce that $J$ given by (3.3) is relatively open provided $N$ is large enough. Hence $J = [0, \infty)$, and this concludes the proof of Theorem 1.

**Proof of Proposition 5.1.** By definition of $E(Iu(t))$ and direct calculation, we get
\begin{align}
\frac{d}{dt} E(Iu(t)) &= \text{Im} \int_{\mathbb{R}^4} \Delta u \left[ (| \cdot |^{-\gamma} * |Iu|^2) Iu - I((| \cdot |^{-\gamma} * |u|^2) u) \right] (t, x) dx \tag{5.1} \\
&\quad - \text{Im} \int_{\mathbb{R}^4} \frac{I((| \cdot |^{-\gamma} * |u|^2) u)}{|(\cdot |^{-\gamma} * |Iu|^2) Iu - I((| \cdot |^{-\gamma} * |u|^2) u)} (t, x) dx. \tag{5.2}
\end{align}
Observing that the definition of $I$ implies
\begin{equation}
Iu \leq \frac{N}{8} (| \cdot |^{-\gamma} * |Iu \leq \frac{N}{8}|^2) - I \left( u \leq \frac{N}{8} (| \cdot |^{-\gamma} * |u \leq \frac{N}{8}|^2) \right) = 0,
\end{equation}
we are reduced to dealing with the contributions from the following five terms to (5.1) and (5.2).

\begin{align}
Iu \leq \frac{N}{8} (| \cdot |^{-\gamma} * |Iu \leq \frac{N}{8}|^2) - I \left( u \leq \frac{N}{8} (| \cdot |^{-\gamma} * |u \leq \frac{N}{8}|^2) \right), \\
Iu \leq \frac{N}{8} | \cdot |^{-\gamma} * (Re Iu \leq \frac{N}{8} Iu \leq \frac{N}{8}) - I \left( u \leq \frac{N}{8} | \cdot |^{-\gamma} * (Re u \leq \frac{N}{8} u \leq \frac{N}{8}) \right), \\
Iu \leq \frac{N}{8} | \cdot |^{-\gamma} * (Re Iu \leq \frac{N}{8} Iu \leq \frac{N}{8}) - I \left( u \leq \frac{N}{8} | \cdot |^{-\gamma} * (Re u \leq \frac{N}{8} u \leq \frac{N}{8}) \right), \\
Iu \leq \frac{N}{8} (| \cdot |^{-\gamma} * |Iu \leq \frac{N}{8}|^2) - I \left( u \leq \frac{N}{8} (| \cdot |^{-\gamma} * |u \leq \frac{N}{8}|^2) \right), \\
Iu \leq \frac{N}{8} (| \cdot |^{-\gamma} * |Iu \leq \frac{N}{8}|^2) - I \left( u \leq \frac{N}{8} (| \cdot |^{-\gamma} * |u \leq \frac{N}{8}|^2) \right).
\end{align}

Note that (5.4) and (5.5) can be handled by the same argument while (5.6) and (5.7) can be treated in the similar way. All these terms will be estimated by using (4.2).

**Estimation of (5.1).** Using Fourier transforms, we may write (5.1) as a multilinear integration on hypersurfaces
\begin{equation}
\int_\Sigma \hat{I}u(t, \xi_1) q(\xi_2, \xi_3, \xi_4) |\xi_3 + \xi_4|^{-4+\gamma} \hat{I}u(t, \xi_2) \hat{I}u(t, \xi_3) \hat{I}u(t, \xi_4) d\sigma(\xi),
\end{equation}
where $\Sigma = \{ (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 \times \cdots \times \mathbb{R}^4 : \xi_1 + \xi_2 + \xi_3 + \xi_4 = 0 \}$ with
\begin{equation}
q(\xi_2, \xi_3, \xi_4) = \left( 1 - \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2) m(\xi_3) m(\xi_4)} \right).
\end{equation}
We will neglect the conjugation operation since this is irrelevant in following estimations.

The contribution of (5.4) to (5.1). We will not exploit the cancellation property. Instead, we use Minkowski’s inequality to reduce the problem to estimating each factors in the difference (5.4). By interpolation, (3.19), and the definition of $I$ along with $(2, \frac{8}{4-\gamma}) \in \Lambda_{\gamma/2-1}$, we get for any $\epsilon > 0$
\begin{equation}
\int_J \left| \nabla Iu, \nabla Iu \leq \frac{N}{8} \cdot (| \cdot |^{-\gamma} * |Iu \leq \frac{N}{8}|^2) \right| dt \lesssim \| \nabla Iu \|_{L^8_t L^2_x} \| \nabla Iu \leq \frac{N}{8} \|_{L^{1+\frac{8}{4-\gamma}}_t L^{2+\epsilon}_x} \| Iu \leq \frac{N}{8} \|_{L^{2+\epsilon}_t L^{\frac{8}{4-\gamma}}_x},
\end{equation}
where
\begin{equation}
\frac{8}{4-\gamma} = \frac{8}{4-\gamma + \frac{2\epsilon}{2+\epsilon}}.
\end{equation}
By interpolation, Bernstein inequality and (4.2), we have
\begin{equation}
\int_J \left| \nabla Iu, \nabla Iu \leq \frac{N}{8} \cdot (| \cdot |^{-\gamma} * |Iu \leq \frac{N}{8}|^2) \right| dt \lesssim N^{2\epsilon} \| \nabla |^{-\gamma/2-1} Iu \leq \frac{N}{8} \|_{L^2_t L^4_x}^2 \
\lesssim N^{-4+\gamma} N^\epsilon.
\end{equation}
Similarly, we have
\begin{equation}
\int_J \left| \nabla Iu, Iu \leq \frac{N}{8} \cdot |^{-\gamma} * (\nabla Iu \leq \frac{N}{8} \cdot Iu \leq \frac{N}{8}) \right| dt \lesssim \| \nabla Iu \|_{L^\infty_t L^2_x} \| Iu \leq \frac{N}{8} \|_{L^\infty_t L^2_x} \| \nabla Iu \leq \frac{N}{8} \|_{L^2_t L^4_x} \| Iu \leq \frac{N}{8} \|_{L^2_t L^4_x} \frac{8}{4-\gamma}.
\end{equation}
\[ \lesssim \| \nabla |^{-3} Iu_{\frac{N}{8}} \|_{L_t^2 L_x^4} \lesssim N^{-4+\gamma}. \]

Taking \( \epsilon \) small enough, we see the contribution from (5.4) to (5.1) is at most \( N^{-4+\gamma} \).

The contribution of (5.5) to (5.1). We use arguments similar to (5.4) to get

\[
\int_J \left| \langle \nabla Iu, \nabla Iu_{\frac{N}{8}} \rangle \cdot |^{-\gamma} \ast (\text{Re} \, Iu_{\leq \frac{N}{8}} \overline{Iu_{\frac{N}{8}}}) \right| \, dt \\
\lesssim \| \nabla Iu \|_{L_t^\infty L_x^{2^+}} \| \nabla Iu_{\leq \frac{N}{8}} \|_{L_t^\infty L_x^{2^+}} \| Iu_{\frac{N}{8}} \|_{L_t^2 L_x^{4+}} \\
\lesssim \| |^{-3} Iu_{\frac{N}{8}} \|_{L_t^2 L_x^4} \lesssim N^{-4+\gamma},
\]

and

\[
\int_J \left| \langle \nabla Iu, Iu_{\leq \frac{N}{8}} \rangle \cdot |^{-\gamma} \ast (\nabla Iu_{\leq \frac{N}{8}} \cdot Iu_{\frac{N}{8}}) \right| \, dt \\
\lesssim \| \nabla Iu \|_{L_t^\infty L_x^{2^+}} \| \nabla Iu_{\leq \frac{N}{8}} \|_{L_t^\infty L_x^{2^+}} \| Iu_{\frac{N}{8}} \|_{L_t^2 L_x^{4+}}^2 \lesssim N^{-4+\gamma}. 
\]

Thus the contribution of (5.5) to (5.1) is also at most \( N^{-4+\gamma} N^{0^+} \).

The contribution of (5.6) and (5.7) to (5.1) can be estimated in the same way and we only deal with (5.7), which is more difficult. In view of the relation on the frequencies on \( \Sigma \), the Fourier transform of (5.7) is supported outside the ball \( B(0, N/4) \). This allows us to put \( P_{\Delta Iu} \) to \( \Delta Iu(t, x) \) in the following estimate. To employ the modified Coifman-Meyer estimate in Lemma 2.13, we write by using Fourier transform and the inverse Fourier transform

\[
\int_J \int_{\Sigma} \Delta \overline{Iu}(t, \xi_1) q(\xi_2, \xi_3, \xi_4) |\xi_3 + \xi_4|^{-4+\gamma} \overline{Iu_{\leq \frac{N}{8}}}(t, \xi_2) Iu_{\leq \frac{N}{8}}(t, \xi_3) Iu_{\leq \frac{N}{8}}(t, \xi_4) d\sigma(\xi) dt \\
= \sum_{N \geq N_0 \geq N_4} \int_J \int_{\Sigma} T \left( \Delta Iu_{\leq \frac{N}{8}}, Iu_{N_4}, Iu_{N_4} \right) (t, x) Iu_{\leq \frac{N}{8}}(t, x) dx dt,
\]

where

\[
T(f, g, h)(x) = \int e^{i x \cdot (\xi_1 + \xi_3 + \xi_4)} \tilde{q}(\xi_1, \xi_3, \xi_4) \xi_3 + \xi_4 |^{-4+\gamma} \tilde{f}(\xi_1) \tilde{g}(\xi_3) \tilde{h}(\xi_4) \, d\xi_1 d\xi_3 d\xi_4,
\]

\[
\tilde{q}(\xi_1, \xi_3, \xi_4) = 1 - \frac{m(\xi_1)}{m(\xi_1 + \xi_3 + \xi_4)}.
\]

As in \( \mathbb{[7]} \), we use the fundamental theorem of calculus to see that on the dyadic supports

\[
\tilde{q}(\xi_1, \xi_3, \xi_4) \lesssim \frac{|\xi_3|}{|\xi_1|}.
\]

By Hölder’s inequality and Lemma 2.13 along with the same argument in \( \mathbb{[7]} \), we can deduce a bound on this term with

\[
\lesssim N^{-4+\gamma} N^{5e} \sum_{N \geq N_0 \geq N_4} N_1^2 \| \nabla Iu_{\leq \frac{N}{8}} \|_{L_t^2 L_x^4}^2 \| Iu_{\leq \frac{N}{8}} \|_{L_t^\infty L_x^{2^+}} \| Iu_{N_4} \|_{L_t^\infty L_x^4} \\
\lesssim N^{-4+\gamma} N^{5e} \sum_{N \geq N_0 \geq N_4} N_1^2 \| \nabla Iu_{\leq \frac{N}{8}} \|_{L_t^2 L_x^4}^2 \| \nabla Iu_{N_4} \|_{L_t^\infty L_x^4} \| \nabla Iu_{N_4} \|_{L_t^\infty L_x^4}.$
\[ \lesssim N^{-4+\gamma} N^{10\epsilon} \left\| \nabla I_{u_{>\frac{N}{8}}} \right\|_{L_t^2 L_x^4}^2 \left\| \nabla I_{u_{<\frac{N}{8}}} \right\|_{L_t^\infty L_x^2}^2. \]

Integrating over \( J \) in time, we are done.

The contribution from (5.8) is easier. By Hölder, Hardy-Littlewood-Sobolev's inequalities and Sobolev embedding, we obtain the following estimates

\[
\int_J \langle \nabla I_{u_{>\frac{N}{8}}}, \nabla I_{u_{>\frac{N}{8}}} (| \cdot |^{-\gamma} * |I_{u_{>\frac{N}{8}}}|^2) \rangle \, dt + \int_J \langle \nabla I_{u_{>\frac{N}{8}}}, | \cdot |^{-\gamma} * (\nabla I_{u_{>\frac{N}{8}}} \cdot I_{u_{>\frac{N}{8}}}) \rangle \, dt \\
\lesssim \left\| \nabla I_{u} \right\|_{L_t^\infty L_x^2} \left\| \nabla I_{u_{>\frac{N}{8}}} \right\|_{L_t^2 L_x^4} \left\| I_{u_{>\frac{N}{8}}} \right\|_{L_t^\infty L_x^2} \left\| I_{u_{>\frac{N}{8}}} \right\|_{L_t^2 L_x^{4+\gamma}} \\
\lesssim N^{-4+\gamma},
\]

and

\[
\int_J \langle \nabla I_{u_{>\frac{N}{8}}}, | \cdot |^{-\gamma} * (\nabla I_{u_{<\frac{N}{8}}} \cdot I_{u_{<\frac{N}{8}}}) \rangle \, dt \\
\lesssim \left\| \nabla I_{u} \right\|_{L_t^\infty L_x^2} \left\| \nabla I_{u_{>\frac{N}{8}}} \right\|_{L_t^2 L_x^4} \left\| I_{u_{<\frac{N}{8}}} \right\|_{L_t^\infty L_x^2} \left\| I_{u_{>\frac{N}{8}}} \right\|_{L_t^2 L_x^{4+\gamma}} \lesssim N^{-4+\gamma} N^{10\epsilon},
\]

and

\[
\int_J \langle \nabla I_{u_{<\frac{N}{8}}}, | \cdot |^{-\gamma} * (\nabla I_{u_{>\frac{N}{8}}} \cdot I_{u_{<\frac{N}{8}}}) \rangle \, dt \\
\lesssim \left\| \nabla I_{u} \right\|_{L_t^\infty L_x^2} \left\| \nabla I_{u_{>\frac{N}{8}}} \right\|_{L_t^2 L_x^4} \left\| I_{u_{<\frac{N}{8}}} \right\|_{L_t^\infty L_x^2} \left\| I_{u_{>\frac{N}{8}}} \right\|_{L_t^2 L_x^{4+\gamma}} \lesssim N^{-4+\gamma}.
\]

**Estimation of the sextilinear term (5.2).** Let us estimate the contribution of (5.2) to the energy increment and the proof of our main theorem will be completed. Recalling (5.3), we know there is at least one function \( u \) having Fourier support outside the ball \( B(0, N/8) \) in the difference term of (5.2).

As observed in the estimation of (5.1), it suffices to consider the contributions from (5.4) (5.6) and (5.8), where (5.5) and (5.7) can be estimated in a similar way. Moreover, it is easy to see that (5.8) is easier to handle since all the functions involved are frequency localized at \( |\xi| > N/8 \) and one can reduce the argument for this term to that of (5.4) and (5.6).

It remains to estimate the contributions from (5.4) and (5.6). Again, we note that the estimation for the first terms in (5.4) and (5.6) are the same as for the second ones. Writing

\[
|I(|\cdot|^{-\gamma} * |u_{\leq N/8}|^2)u_{\leq N/8}| \leq |I(|\cdot|^{-\gamma} * |u_{\leq N/8}|^2)u_{\leq N/8}| \tag{5.9}
\]

\[
+ |I(|\cdot|^{-\gamma} * |u_{< N/8}|^2)u_{> N/8}| \tag{5.10}
\]

\[
+ |I(|\cdot|^{-\gamma} * |u_{> N/8}|^2)u_{< N/8}| \tag{5.11}
\]

\[
+ |I(|\cdot|^{-\gamma} * |u_{< N/8}|^2)u_{> N/8}| \tag{5.12}
\]

and noting that (5.11) and (5.12) are easier to handle than (5.9) and (5.10), we are reduced to estimating the following four terms

\[
\int_J \left\langle I(|\cdot|^{-\gamma} * u_{\leq N/8}^2)u_{\leq N/8}, I(u_{\leq N/8}(|\cdot|^{-\gamma} * u_{\leq N/8}^2)) \right\rangle \, dt, \tag{5.13}
\]

\[
\int_J \left\langle I(|\cdot|^{-\gamma} * u_{\leq N/8}^2)u_{\leq N/8}, I(u_{\leq N/8}(|\cdot|^{-\gamma} * u_{> N/8}^2)) \right\rangle \, dt, \tag{5.14}
\]
\[
\int \langle I((\cdot | -\gamma \ast u_{\leq N/8}^2)u_{\leq N/8}), I(u_{\leq N/8} | -\gamma \ast (\text{Re} u_{\leq N/8} u_{\geq N/8})) \rangle \, dt, \quad (5.15)
\]

\[
\int \langle I((\cdot | -\gamma \ast u_{\leq N/8}^2)u_{\geq N/8}), I(u_{\leq N/8} | -\gamma \ast (\text{Re} u_{\leq N/8} u_{\geq N/8})) \rangle \, dt. \quad (5.16)
\]

By writing (5.15) into the multilinear integration over hypersurface \( S \) via Fourier transform and the Parseval identity

\[
S = \left\{ (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) \in \mathbb{R}^4 \times \cdots \times \mathbb{R}^4 : \xi_1 + \cdots + \xi_6 = 0 \right\},
\]

we know there is at least one term, among three \( u_{\leq N/8} \)'s in \( I((\cdot | -\gamma \ast u_{\leq N/8}^2)u_{\leq N/8}) \), with frequency localized at \( |\xi| > (\frac{N}{2} - \frac{N}{8} - \frac{N}{8}) / 3 = \frac{N}{12} \). Hence (5.15) may be reduced to (5.16).

The estimate of (5.14). We have by Hölder, Hardy-Littlewood-Sobolev’s inequality and Sobolev embedding

\[
\int \langle I((\cdot | -\gamma \ast u_{\leq N/8}^2)u_{\leq N/8}), I(u_{\leq N/8} | -\gamma \ast (\text{Re} u_{\leq N/8} u_{\geq N/8})) \rangle \, dt \\
\leq \int \| (\cdot | -\gamma \ast u_{\leq N/8}^2)u_{\leq N/8} \|_{L_{\xi}^{4/\gamma}} \| u_{\leq N/8} (\cdot | -\gamma \ast (\text{Re} u_{\leq N/8} u_{\geq N/8})) \|_{L_{\xi}^{4/\gamma}} \, dt \\
\lesssim \| u_{\leq N/8} \|^2_{L^2_{\xi}L^4_{\xi}} \| Iu_{\leq N/8} \|^4_{L^\infty_{\xi}L^2_{\xi}} \\
\lesssim N^{-4+\gamma}.
\]

The same argument applies to (5.14) equally well.

The estimate of (5.16). We have by Hölder, Hardy-Littlewood-Sobolev’s inequality and Sobolev embedding

\[
\int \langle I((\cdot | -\gamma \ast u_{\leq N/8}^2)u_{\geq N/8}), I(u_{\leq N/8} | -\gamma \ast (\text{Re} u_{\leq N/8} u_{\geq N/8})) \rangle \, dt \\
\leq \int \| (\cdot | -\gamma \ast u_{\leq N/8}^2)u_{\geq N/8} \|_{L_{\xi}^{4/\gamma}} \| Iu_{\leq N/8} \|^4_{L^\infty_{\xi}L^2_{\xi}} \, dt \\
\lesssim \| u_{\geq N/8} \|^2_{L^2_{\xi}L^4_{\xi}} \| Iu_{\leq N/8} \|^4_{L^\infty_{\xi}L^2_{\xi}} \| Iu_{\leq N/8} \|^4_{L^\infty_{\xi}L^2_{\xi}} \\
\lesssim N^{-4+\gamma}.
\]

The proof is complete. \( \square \)

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