NEW ESTIMATES FOR THE HARDY CONSTANT OF MULTIPOLAR SCHRÖDINGER OPERATORS WITH BOUNDARY SINGULARITIES

CRISTIAN CAZACU

Abstract. In this paper we study the optimization problem

$$\mu^*(\Omega) := \inf_{u \in H^1_0(\Omega)} \frac{\int_{\Omega} |\nabla u|^2 \, dx}{\int_{\Omega} V u^2 \, dx},$$

where the multi-singular Schrödinger potential $V$ is defined by

$$V := \sum_{1 \leq i < j \leq n} \frac{|a_i - a_j|^2}{|x - a_i|^2 |x - a_j|^2},$$

and the singular poles $a_1, \ldots, a_n$ arise at the boundary $\Gamma$ of a smooth domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$. In the case of interior singularities it was shown in [6] that $\mu^*(\Omega) = (N - 2)/n^2$ independent on the location of the poles $a_1, \ldots, a_n$.

Here we prove that, when moving all the poles on the boundary, we obtain a new critical barrier for the Hardy constant $\mu^*(\Omega)$ which is $N^2/n^2$. Our results depend on the entire geometry of the domain but not necessary on the location or the distances between the poles. In addition, we also discuss the attainability of $\mu^*(\Omega)$.

Keywords: Hardy inequality, multipolar potentials, boundary singularities, optimal constants.

2010 Mathematics Subject Classification: 46E35, 26D10, 35J75, 35B25.

1. INTRODUCTION AND MAIN RESULTS

In this paper we consider the multipolar Schrödinger operator

$$L_\mu := -\Delta - \mu \sum_{1 \leq i < j \leq n} \frac{|a_i - a_j|^2}{|x - a_i|^2 |x - a_j|^2}, \quad \mu \in \mathbb{R},$$

in smooth domains $\Omega \subset \mathbb{R}^N$, $N \geq 2$, such that the quadratic singularities $a_1, \ldots, a_n$, $n \geq 2$, are located on the boundary $\Gamma$ of $\Omega$.

We analyze the range of the parameters $\mu$ that ensures the existence of a lower bound for the spectrum of the operator $L_\mu$ with Dirichlet boundary conditions (denoted by $L_\mu^D$). Namely, in the spirit of Davies [7], we are looking for those $\mu$’s for which there exists a finite constant $c \in \mathbb{R}$ (which might also be negative) such that the operator inequality

$$L_\mu^D \geq c$$

holds.

Research group of the projects PN-II-ID-PCE-2011-3-0075 and PN-II-ID-PCE-2012-4-0021, “Simion Stoilow” Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania.

E-mail: cristi_cazacu2002@yahoo.com.
is verified in $L^2(\Omega)$ quadratic forms. This is equivalent to the validity of the following Hardy-Poincaré type inequality that holds for $L^D_\mu$:

\begin{align}
(1.2) \quad \int_\Omega |\nabla u|^2 \, dx - \mu \sum_{1 \leq i < j \leq n} \int_\Omega \frac{|a_i - a_j|^2}{|x - a_i|^2|x - a_j|^2} u^2 \, dx \geq c \int_\Omega u^2 \, dx, \quad \forall u \in H^1_0(\Omega).
\end{align}

This problem is motivated by the previous related results in Bosi-Dolbeault-Esteban [3] and Zuazua and myself in [6]. Firstly, it was shown in [3] that, in the whole space $\mathbb{R}^N$, $N \geq 3$, it is true that

\begin{align}
L^D_\mu > 0, \quad \forall \mu \leq \frac{(N - 2)^2}{4n^2},
\end{align}

and the location of the singular poles does not matter. More exactly, the authors in [3] proved that

\begin{align}
(1.3) \quad \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \geq \frac{(N - 2)^2}{4n^2} \sum_{1 \leq i < j \leq n} \int_{\mathbb{R}^N} \frac{|a_i - a_j|^2}{|x - a_i|^2|x - a_j|^2} u^2 \, dx \\
+ \frac{(N - 2)^2}{4n} \sum_{i=1}^n \int_{\mathbb{R}^N} \frac{u^2}{|x - a_i|^2} \, dx, \quad \forall u \in H^1(\mathbb{R}^N),
\end{align}

for any fixed configuration $a_1, \ldots, a_n \in \mathbb{R}^N$, with $a_i \neq a_j$ for $i \neq j$. It occurs that the constant $(N - 2)^2/(4n^2)$ in (1.3) is not optimal. More precisely, the new Hardy inequality (cf. [6])

\begin{align}
(1.4) \quad \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \geq \frac{(N - 2)^2}{n^2} \sum_{1 \leq i < j \leq n} \int_{\mathbb{R}^N} \frac{|a_i - a_j|^2}{|x - a_i|^2|x - a_j|^2} u^2 \, dx, \quad \forall u \in H^1(\mathbb{R}^N),
\end{align}

holds true and the new constant $(N - 2)^2/n^2$ is optimal. Moreover, it is not attained in $H^1_{\text{loc}}(\mathbb{R}^N)$ for $n = 2$. Besides, remark that inequalities (1.3)-(1.4) are also true in bounded domains $\Omega \subset \mathbb{R}^N$, $N \geq 2$, for functions in the Sobolev space $H^1_0(\Omega)$ and $a_1, \ldots, a_n \in \overline{\Omega}$. If all the singularities $a_i$ are located in the interior of $\Omega$, inequality (1.4) restricted to $\Omega$ is still optimal in $H^1_0(\Omega)$.

The Hardy-type inequalities have been intensively studied in the literature. In particular, they play a crucial influence to the spectral properties of singular Hamiltonian operators which arise in mathematical physics or quantum mechanics (cf. Davies [8], Laptev et al. [2], [14], Krejčířík-Zuazua [13], Brezis-Vazquez [4], Pinchover-Tintarev [15], Fefferman [9], etc.). For Hamiltonians with multipolar singularities we refer to the papers by Bosi et al. [3], Terracini et al. [12], [10], [11] and the references therein. In particular, important remarks, comparisons and applications of such Hamiltonians were emphasized in more details in the Introduction of [6].

As we mentioned before, in this paper we are concerned with the singular potential

\begin{align}
V = V(x; a_1, \ldots, a_n) := \sum_{1 \leq i < j \leq n} V_{ij}(x), \quad n \geq 2,
\end{align}

where

\begin{align}
(1.5) \quad V_{ij}(x) = \frac{|a_i - a_j|^2}{|x - a_i|^2|x - a_j|^2}, \quad i, j \in \{1, \ldots, n\}, \quad i \neq j,
\end{align}

and the singular poles $a_1, \ldots, a_n$ (where $a_i \neq a_j$ for any $i \neq j$) arise at the boundary $\Gamma$ of a bounded smooth domain $\Omega \subset \mathbb{R}^N$ (at least $C^2$-regularity), $N \geq 2$. 

The corresponding optimal Hardy constant is defined by
\[ \mu^*(\Omega; a_1, \ldots, a_n) := \inf_{u \in H^1_0(\Omega)} \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega V u^2 \, dx}. \]

When there is no risk of confusion, we will write \( \mu^*(\Omega) \) and \( V \) instead of \( \mu^*(\Omega; a_1, \ldots, a_n) \) respectively.

Before stating the main results let us recall some useful notations and notions. As usual, we denote by \( B_r(x_0) \) the ball of radius \( r > 0 \) centered at a point \( x_0 \in \mathbb{R}^N \), and by \( B^c_r(x_0) \) the exterior of such ball \( B_r(x_0) \). The upper half-space in \( \mathbb{R}^N \) is defined by the set \( \mathbb{R}^N_+ := \{ x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_N > 0 \} \) which is supported by the hyperplane \( x_N = 0 \). In a more general sense, a half-space in \( \mathbb{R}^N \) can be defined as a support of an affine hyperplane of the form \( \alpha_1 x_1 + \ldots + \alpha_N x_N = b \), with fixed constants \( \alpha_i, b \in \mathbb{R}, i \in \{1, \ldots, N\} \). For simplicity, in this paper we will only refer to \( \mathbb{R}^N_+ \). However, our next results obtained for \( \mathbb{R}^N_+ \) could also be extended to any general half-space.

**Main results**

The main results of the paper improve and extend the previous findings for interior singularities proven in [6] to the case of boundary singularities. To be more precise, when moving the singularities from the interior to the boundary, a better critical constant appears in the Hardy inequality as follows.

**Theorem 1.1.** Assume \( \Omega \) is either a ball, an exterior of a ball, or a half-space in \( \mathbb{R}^N, N \geq 2 \), such that \( a_1, \ldots, a_n \in \Gamma, n \geq 2 \). Then
\[ \mu^*(\Omega) = \frac{N^2}{n^2}. \]
Moreover, the constant in (1.6) is not achieved when \( n = 2 \) but when \( n \geq 3 \).

**Theorem 1.2.** Assume \( \Omega \subset \mathbb{R}^N, N \geq 2 \), is a bounded domain with \( C^2 \)-regularity with \( a_1, \ldots, a_n \in \Gamma, n \geq 2 \). Then, for any parameter \( \mu < N^2/n^2 \), there exists a finite constant \( c_\mu \in \mathbb{R} \) which also depends on \( \Omega \) and \( a_1, \ldots, a_n \), so that the inequality
\[ c_\mu \int_\Omega u^2 \, dx + \int_\Omega |\nabla u|^2 \, dx \geq \mu \sum_{1 \leq i < j \leq n} \int_\Omega \frac{|a_i - a_j|^2}{|x - a_i||x - a_j|^2} u^2 \, dx, \]
is verified for any \( u \in H^1_0(\Omega) \).

As a consequence of Theorem 1.2 we obtain

**Corollary 1.1.** Assume \( \Omega \subset \mathbb{R}^N, N \geq 2 \), is a bounded domain with \( C^2 \)-regularity such that \( a_1, \ldots, a_n \in \Gamma \) with \( n \geq 2 \). Then
\[ \mu^*(\Omega) > 0. \]
In particular, the embedding
\[ H^1_0(\Omega) \hookrightarrow L^2(V; \, dx) \]
is continuous.
In view of inequality (1.4) and the mentioned remarks, $\mu^*(\Omega) \geq (N - 2)^2/n^2$ and therefore, Corollary 1.1 is non-trivial if $N = 2$. Particularly, Corollary 1.1 whose proof is given at the end of Section 5 constitutes a first step to prove the third item of the following Theorem.

**Theorem 1.3.** Assume $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with $C^2$-regularity such that $a_1, \ldots, a_n \in \Gamma$, $n \geq 2$. We claim

(i). If $n = 2$ then

$$
\frac{(N - 2)^2}{4} < \mu^*(\Omega; a_1, a_2) \leq \frac{N^2}{4}.
$$

(ii). If $n \geq 3$ then

$$
\frac{(N - 2)^2}{n^2} \leq \mu^*(\Omega; a_1, \ldots, a_{n-1}, a_n) \leq \mu^*(\Omega; a_1, \ldots, a_{n-1}) \leq \ldots \leq \frac{N^2}{4}.
$$

(iii). In addition, if $\mu^*(\Omega) < N^2/n^2$, $n \geq 2$, then $\mu^*(\Omega)$ is attained.

Remark that Theorem 1.2 is a weaker version of Theorem 1.1 by paying the price of adding an $L^2$-reminder term on the gradient side. Moreover, the potential involved in Theorem 1.2 is subcritical compared with the one in Theorem 1.1.

The proof of Theorem 1.1 is inspired from a general strategy to prove Hardy-inequalities. The idea is based on building suitable weights in a general identity which match to both the multi-singular potential and the entire geometry of the domain. This method is presented in Section 2 and applies in Section 3 in the proof of Theorem 1.1. The most consistent part of Theorem 1.3 is the upper bound in (1.10) whose proof requires to build adequate approximating sequences as shown in Section 5. Theorem 1.2 applies the same strategy as Theorem 1.1 slightly modified in the neighborhood of the boundary which, in view of the Hardy inequality involving the distance to the boundary, allows to absorb the unbounded potentials. Then we can trivially glue the terms far from the singularities where the potentials are bounded. Section 4 is dedicated to the proof of Theorem 1.2. Finally, in Section 6 we discuss some important remarks, comments and open problems.

## 2. Hardy inequalities with multi-singular potentials

**General strategy.** Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be an open smooth domain with $a_1, \ldots, a_n \in \mathbb{R}$, $n \geq 2$.

As pointed out in [6], for any $u \in C_0^1(\Omega \setminus \{a_1, a_2, \ldots, a_n\})$ and any $\phi \in C^2(\Omega \setminus \{a_1, \ldots, a_n\})$ with $\phi > 0$ in $\Omega$ ($\phi$ is allowed to be singular at any of the points $a_1, \ldots, a_n$), it holds that

$$
\int_{\Omega} \left( |\nabla u|^2 + \frac{\Delta \phi}{\phi} u^2 \right) \, dx = \int_{\Omega} \left| \nabla u - \frac{\nabla \phi}{\phi} u \right|^2 \, dx = \int_{\Omega} \phi^2 |\nabla (u \phi^{-1})|^2 \, dx.
$$

The proof of (2.1) can be rigorously justified using integration by parts. In consequence, we obtain

$$
\int_{\Omega} |\nabla u|^2 \, dx \geq \int_{\Omega} \left( -\frac{\Delta \phi}{\phi} \right) u^2 \, dx, \quad \forall u \in H_0^1(\Omega),
$$

which holds a priori for any $u \in C_0^1(\Omega \setminus \{a_1, \ldots, a_n\})$. The extension to the appropriate Sobolev spaces in inequality (2.2) is possible since $C_0^1(\Omega \setminus \{a_1, \ldots, a_n\})$ is dense in $H_0^1(\Omega)$. 

Next we will apply (2.2) to study the positivity of the Schrödinger operator $-\Delta - \mu V$, $\mu > 0$, where $V$ is the potential defined in (1.5) with $a_1, \ldots, a_n \in \partial \Omega$.

For our purpose, we have to make a suitable election of $\phi$ and (up to now, as we proceeded in [6]) we consider $\phi$ in (2.1) of the form

\[ \phi = (\phi_1 \phi_2 \ldots \phi_n)^{\frac{1}{n}}, \]

which verifies

\[ \frac{-\Delta \phi}{\phi} = \frac{1}{n^2} \sum_{1 \leq i < j \leq n} \left| \frac{\nabla \phi_i}{\phi_i} - \frac{\nabla \phi_j}{\phi_j} \right|^2 - \frac{1}{n} \sum_{i=1}^{n} \frac{\Delta \phi_i}{\phi_i}, \quad \forall x \in \Omega \setminus \{1, \ldots, a_n\}, \]

where the weights $\phi_i$, $i \in \{1, \ldots, n\}$, satisfy some admissible conditions. In particular, if

\[ \begin{cases} 
\phi_i \in C^2(\Omega \setminus \{a_1, a_2, \ldots, a_n\}), \\
\phi_i(x) > 0 \text{ in } \Omega, \\
-\Delta \phi_i(x) \geq 0, \quad \forall x \in \Omega \setminus \{a_1, \ldots, a_n\}.
\end{cases} \]

for any $i \in \{1, \ldots, n\}$, due to (2.4) we get the pointwise inequality

\[ \frac{-\Delta \phi}{\phi} \geq \frac{1}{n^2} \sum_{1 \leq i < j \leq n} \left| \frac{\nabla \phi_i}{\phi_i} - \frac{\nabla \phi_j}{\phi_j} \right|^2, \quad \forall x \in \Omega \setminus \{a_1, \ldots, a_n\}. \]

Hence, for $\phi$ satisfying (2.5) and (2.8), from (2.2) we conclude that

\[ \int_{\Omega} |\nabla u|^2 \, dx \geq \frac{1}{n^2} \sum_{1 \leq i < j \leq n} \int_{\Omega} \left| \frac{\nabla \phi_i}{\phi_i} - \frac{\nabla \phi_j}{\phi_j} \right|^2 u^2 \, dx, \quad \forall u \in H_0^1(\Omega). \]

**Interior singularities.** In [6], in the context of interior singularities $a_1, \ldots, a_n \in \Omega$, we chose

\[ \phi_i = |x - a_i|^{-(N-2)}, \]

(this weight corresponds to the fundamental solution of the Laplacian at the point $a_i$) which satisfies conditions (2.5) for all $i \in \{1, \ldots, n\}$ and the following identity holds true:

\[ \sum_{1 \leq i < j \leq n} \left| \frac{\nabla \phi_i}{\phi_i} - \frac{\nabla \phi_j}{\phi_j} \right|^2 = (N-2)^2 \sum_{1 \leq i < j \leq n} \frac{|a_i - a_j|^2}{|x - a_i|^2|x - a_j|^2}. \]

According to (2.8) and (2.7) we have that

\[ \int_{\Omega} |\nabla u|^2 \, dx \geq \frac{(N-2)^2}{n^2} \sum_{1 \leq i < j \leq n} \int_{\Omega} \frac{|a_i - a_j|^2}{|x - a_i|^2|x - a_j|^2} u^2 \, dx, \quad \forall u \in H_0^1(\Omega), \]

as shown in [6].

**Boundary singularities.** Let us switch next to the case of boundary singularities, i.e. $a_1, \ldots, a_n \in \Gamma$. Of course, the choice of the weights $\phi_i = |x - a_i|^{-(N-2)}$ above is still admissible in this new case. Despite of that, these weights do not provide an optimal Hardy constant $\mu^*(\Omega)$, only ensuring that $\mu^*(\Omega) \geq \frac{(N - 2)^2}{n^2}$.

In order to get optimal results we design new weights $\phi_i$ with the profile adapted to both the whole geometry of $\Omega$ and the corresponding singularity $a_i$, $i \in \{1, \ldots, n\}$.

In view of that, for removing the singularities we firstly introduce the weights

\[ \phi_i(x) = f(x)|x - a_i|^{-N}, \quad x \in \Omega, \quad i \in \{1, \ldots, n\}, \]
where $f$ is a function independent of $a_i$ such that $f > 0$ in $\Omega$ and $f \in C^2(\Omega)$. Then we obtain

$$
\frac{\nabla \phi_i}{\phi_i} = \frac{\nabla f}{f} - \frac{N(x - a_i)}{|x - a_i|^2}, \quad \left| \frac{\nabla \phi_i}{\phi_i} - \frac{\nabla \phi_j}{\phi_j} \right|^2 = N^2 \frac{|a_i - a_j|^2}{|x - a_i|^2 |x - a_j|^2}
$$

respectively

$$
\frac{\Delta \phi_i}{\phi_i} = \left( \frac{\Delta f |x - a_i|^2 - 2N \nabla f \cdot (x - a_i) + 2N f}{f} \right) \frac{1}{|x - a_i|^2}.
$$

From (2.3), (2.10)-(2.12) and (2.4) we get

$$
\phi = f \prod_{i=1}^n |x - a_i|^{-\frac{n}{2}}
$$

and

$$
-\frac{\Delta \phi}{\phi} = \frac{N^2}{n^2} \sum_{1 \leq i < j \leq n} \frac{|a_i - a_j|^2}{|x - a_i|^2 |x - a_j|^2}
$$

$$
+ \frac{1}{n} \sum_{i=1}^n \int_{\Omega} \frac{(-\Delta f |x - a_i|^2 + 2N \nabla f \cdot (x - a_i) - 2N f)}{f} \frac{u^2}{|x - a_i|^2} \, dx
$$

Therefore, combining (2.13) and (2.3), for any positive function $f \in C^2(\Omega)$ the identity

$$
\int_{\Omega} |\nabla u|^2 \, dx = \frac{N^2}{n^2} \sum_{1 \leq i < j \leq n} \int_{\Omega} \frac{|a_i - a_j|^2}{|x - a_i|^2 |x - a_j|^2} u^2 \, dx
$$

$$
+ \frac{1}{n} \sum_{i=1}^n \int_{\Omega} \left( \frac{(-\Delta f |x - a_i|^2 + 2N \nabla f \cdot (x - a_i) - 2N f)}{f} \right) \frac{u^2}{|x - a_i|^2} \, dx
$$

$$
+ \int_{\Omega} \nabla \left[ u \left( f \prod_{i=1}^n |x - a_i|^{-\frac{n}{2}} \right)^{-1} \right]^2 \frac{n}{2} \prod_{i=1}^n |x - a_i|^{-\frac{2n}{\pi}} \, dx
$$

is verified for any $u \in H^1_0(\Omega)$.

In particular, if we are able to design a function $f$ satisfying the conditions

$$
\begin{cases}
  f \in C^2(\Omega), \\
f(x) > 0, \\
S(f, a_i) := -\Delta f |x - a_i|^2 + 2N \nabla f \cdot (x - a_i) - 2N f \geq 0, & \forall x \in \Omega,
\end{cases}
$$

for any $i \in \{1, \ldots, n\}$, from (2.16) and (2.15) we get the improved Hardy inequality

$$
\int_{\Omega} |\nabla u|^2 \, dx \geq \frac{N^2}{n^2} \sum_{1 \leq i < j \leq n} \int_{\Omega} \frac{|a_i - a_j|^2}{|x - a_i|^2 |x - a_j|^2} u^2 \, dx, \quad \forall u \in H^1_0(\Omega),
$$

which is verified in several particular cases as shown in Theorem 1.1.
3. Proof of Theorem 1.1

3.1. The case of a ball. Let $\Omega = B_r(x_0)$ be the ball of a fixed radius $r > 0$ centered at an arbitrary point $x_0 \in \mathbb{R}^N$, with the singular poles $a_1, \ldots, a_n$ arising at the boundary $\Gamma$.

Step 1. Firstly, we show that inequality (2.17) is verified and therefore $\mu^*(B_r(x_0)) \geq N^2/n^2$.

For that to be true, due to (2.15) it is sufficient to build a weight function $f$ satisfying conditions (2.16) in $B_r(x_0)$. In view of that, let us introduce

$$ f := r^2 - |x - x_0|^2 $$

which satisfies (2.16). Indeed, the first two conditions are trivial. The third condition is also true: expanding the square we get for any $x \in B_r(x_0)$ that

$$ S(f; a_i) = 2N|x-a_i|^2 - 4N(x - x_0) \cdot (x - a_i) - 2N(r^2 - |x - x_0|^2) $$

$$ = -2Nr^2 + 2N|x-a_i| - (x - x_0)^2 $$

$$ = -2Nr^2 + 2N|a_i - x_0|^2 $$

$$ = 2N(|a_i - x_0|^2 - r^2) $$

$$ = 0, $$

In the last identity we applied the fact that the singularities $a_i$ are localized on the boundary.

Step 2. We prove the reverse inequality $\mu^*(B_r(x_0)) \leq N^2/n^2$ by building a minimizing sequence, i.e. there exists $\{u_\varepsilon\}_{\varepsilon > 0} \subset H^1_0(\Omega)$ such that

$$ \lim_{\varepsilon \searrow 0} \frac{\int_\Omega |\nabla u_\varepsilon|^2 \, dx}{\int_\Omega V u_\varepsilon^2 \, dx} = \frac{N^2}{n^2}. $$

Since the function $f$ introduced in (3.1) verifies $S(f, a_i) = 0$ in (2.16) for any $i \in \{1, \ldots, n\}$, identity (2.15) becomes

$$ \int_\Omega |\nabla|^2 \, dx - \frac{N^2}{n^2} \int_\Omega V^2 \, dx $$

$$ = \int_\Omega \left| \nabla \left[ u \left( f \prod_{i=1}^n |x - a_i|^{-\frac{N}{2}} \right)^{-1} \right] \right|^2 \, dx $$

$$ = f^2 \prod_{i=1}^n |x - a_i|^{-\frac{N}{2}} \, dx, \quad \forall u \in H^1_0(\Omega). $$

The case $n \geq 3$. Attainability.

In view of (3.4) the constant sequence (with respect to $\varepsilon$) defined by

$$ u_\varepsilon = u := (r^2 - |x - x_0|^2) \prod_{i=1}^n |x - a_i|^{-\frac{N}{2}} \in H^1_0(\Omega), $$

satisfies

$$ \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega V u^2 \, dx} = \frac{N^2}{n^2}. $$

Therefore $\mu^*(\Omega) = N^2/n^2$ and it is achieved by $u$ in (3.5).

The case $n = 2$. Non-attainability.
Observe that the function $u$ in (3.5) does not belong to $H^1_0(\Omega)$ when $n = 2$ (it can be noticed through the estimates later in (5.18)), so the Hardy constant $N^2/4$ is not attained. However, we prove its optimality by building a minimizing sequence as in (3.3). Such a sequence is a truncation in the neighborhood of the singular poles of $u$ in (3.5) as follows.

Let $\varepsilon > 0$ aimed to be small ($\varepsilon < \min\{1, d := |a_1 - a_2|/2\}$) and consider the sequence $\{u_\varepsilon\}_{\varepsilon > 0}$ defined by

$$u_\varepsilon := \theta_\varepsilon f \prod_{i=1}^2 |x - a_i|^{-\frac{N}{2}} \in H^1_0(\Omega),$$

where $f = r^2 - |x - x_0|^2$ and the cut-off function $\theta_\varepsilon \in C(\mathbb{R}^N)$, supported far from the singular poles, is given by

$$\theta_\varepsilon(x) = \begin{cases} 0, & \log|x-a_i|/\varepsilon^2, \\ \frac{\log|x-a_i|/\varepsilon^2}{\log 1/\varepsilon}, & \varepsilon^2 \leq |x-a_i| \leq \varepsilon, \\ 1, & \text{otherwise.} \end{cases}$$

Then, from (3.4) and (3.6) we obtain

$$\int \nabla u_\varepsilon^2 \, dx - \frac{N^2}{4} \int \frac{|a_1 - a_2|^2}{|x - a_1|^2 |x - a_2|^2} u_\varepsilon^2 \, dx = \int \nabla \theta_\varepsilon^2 f^2 \prod_{i=1}^2 |x - a_i|^{-N} \, dx,$$

which is equivalent to

$$\frac{\int \nabla u_\varepsilon^2 \, dx}{\int \nabla \theta_\varepsilon^2 \, dx} \cdot \frac{N^2}{4} = \frac{\int \nabla \theta_\varepsilon^2 f^2 \prod_{i=1}^2 |x - a_i|^{-N} \, dx}{\int \nabla u_\varepsilon^2 \, dx}.$$

We conclude the proof of Step 2 by showing that the right hand side in (3.9) converges to zero as $\varepsilon$ tends to zero.

Indeed, we observe that there exists a constant $C > 0$ (independent of $\varepsilon$) such that

$$\int \nabla \theta_\varepsilon^2 \, dx > C > 0, \quad \forall \varepsilon > 0.$$

On the other hand we remark that

$$f(x) = \rho(x)(2r - \rho(x)) \leq 2r|x-a_i|, \quad \forall x \in B_r(x_0), \forall i \in \{1, 2\},$$

where $\rho(x)$ denotes the distance function to the boundary $\rho(x) = \text{dist}(x, \Gamma)$ (here, $\rho(x) = r - |r - x_0|$). Then, taking into account the support of $\nabla \theta_\varepsilon$ and (3.11) we successively obtain

$$I_1 := \int \nabla \theta_\varepsilon^2 f^2 \prod_{i=1}^2 |x - a_i|^{-N} \, dx$$

$$\leq 4r^2 \sum_{i=1}^2 \int_{B_\varepsilon(a_i) \setminus B_{2\varepsilon}(a_i)} \nabla \theta_\varepsilon^2 |x - a_i|^2 \prod_{j=1}^2 |x - a_j|^{-N} \, dx$$

$$= 4r^2 \sum_{i=1}^2 \int_{B_\varepsilon(a_i) \setminus B_{2\varepsilon}(a_i)} \frac{1}{\log^2(1/\varepsilon)} \prod_{j=1}^2 |x - a_j|^{-N} \, dx.$$
Since
\[(3.13)\quad |x - a_j| \geq \frac{d}{2}, \quad \forall x \in B_\varepsilon(a_i), \quad \forall j \neq i, \quad \forall i, j \in \{1, 2\},\]
from \((3.12)\) we deduce that
\[
I_1 \leq \frac{4r^2 \left(\frac{d}{2}\right)^N}{\log^2(1/\varepsilon)} \sum_{i=1}^2 \int_{B_\varepsilon(a_i) \setminus B_{\varepsilon^2}(a_i)} |x - a_i|^{-N} \, dx
\]
\[
= \frac{8r^2 \left(\frac{d}{2}\right)^N}{\log^2(1/\varepsilon)} \int_{\varepsilon^2}^{\varepsilon} s^{N-1} \int_{S^{N-1}} s^{-N} d\sigma \, ds
\]
\[
= \frac{8r^2 \omega_N \left(\frac{d}{2}\right)^N}{\log^2(1/\varepsilon)} \int_{\varepsilon^2}^{\varepsilon} s^{-1} \, ds,
\]
where \(\omega_N\) is the \((N - 1)\)-Hausdorff measure of the unit sphere \(S^{N-1}\) in \(\mathbb{R}^N\). From \((3.14)\) we obtain that
\[(3.15)\quad I_1 = O \left(\frac{1}{\log(1/\varepsilon)}\right), \quad \text{as} \quad \varepsilon \to 0.
\]
According to \((3.15)\), \((3.10)\) and \((3.9)\) we finally obtain the validity of \((3.3)\).

\[\square\]

3.2. The case of the exterior of a ball. Let us consider \(\Omega = B^c_\varepsilon(x_0) = \mathbb{R}^N \setminus B_r(x_0)\) for some \(r > 0\) and fixed \(x_0 \in \mathbb{R}^N\). The proof is very similar to the one in Subsection 3.1 as follows.

**Step 1.** Let us consider
\[(3.16)\quad f := |x - x_0|^2 - r^2.
\]
Remark that \(f\) in \((3.16)\) is exactly the function in \((3.1)\) with the opposite sign. Due to \((3.2)\), \(f\) in \((3.16)\) satisfies \((2.16)\) and therefore we deduce that \(\mu^*(B^c_\varepsilon(x_0)) \geq N^2/n^2\).

**Step 2.** The optimality of the constant \(N^2/n^2\) goes smoothly as in the previous situation corresponding to a ball. If \(n \geq 3\), the constant is attained by the function
\[(3.17)\quad u = (|x - x_0|^2 - r^2)^n \prod_{i=1}^n |x - a_i|^{-\frac{n}{n-2}} \in H^1_0(\Omega).
\]
Otherwise, if \(n = 2\), the function in \((3.17)\) does not belong to \(H^1_0(\Omega)\) and therefore, we have to proceed with an approximation argument as in \((3.6)\).

However, the cut-off function \(\theta_\varepsilon\) defined in \((3.7)\) requires a different definition at infinity in this present case. So, let us introduce
\[(3.18)\quad \theta_\varepsilon(x) = \begin{cases} 0, & \text{if } |x - a_i| \leq \varepsilon^2, \quad \forall i \in \{1, 2\}, \\ \frac{\log \frac{|x - a_i|}{\varepsilon^2}}{\log 1/\varepsilon}, & \varepsilon^2 \leq |x - a_i| \leq \varepsilon, \quad \forall i \in \{1, 2\}, \\ 1, & \text{if } x \in B_1/\varepsilon(x_0) \setminus \bigcup_{i=1}^n B_\varepsilon(a_i), \\ \frac{\log 1/\varepsilon^2}{\log 1/\varepsilon}, & x \in B_{1/\varepsilon^2}(x_0) \setminus B_1/\varepsilon(x_0), \\ 0, & |x - x_0| \geq 2/\varepsilon^2. \end{cases}
\]
Then we define the sequence \( u_\varepsilon := \theta_\varepsilon f \prod_{i=1}^{2} |x - a_i|^{-N/2} \) for \( f \) in (3.16) and \( \theta_\varepsilon \) in (3.18).

Applying (2.15) for these new weights we get

\[
\int_{\Omega} |\nabla u_\varepsilon|^2 \, dx - \frac{N^2}{n^2} \int_{\Omega} \frac{|a_1 - a_2|^2}{|x - a_1|^2|x - a_2|^2} u_\varepsilon^2 \, dx = \int_{\Omega} |\nabla \theta_\varepsilon|^2 f^2 \prod_{i=1}^{2} |x - a_i|^{-N} \, dx.
\]

Remark that \( \nabla \theta_\varepsilon \) is supported in \( \cup_{i=1}^{2} (B_\varepsilon(a_i) \setminus B_{2\varepsilon}(a_i)) \) respectively in \( B_{1/\varepsilon}(x_0) \setminus B_{1/\varepsilon}(x_0) \). Next let us denote by \( I_1 \) and \( I_2 \) the integrals of the right hand side in (3.19) restricted to \( \cup_{i=1}^{2} (B_\varepsilon(a_i) \setminus B_{2\varepsilon}(a_i)) \) respectively to \( B_{1/\varepsilon}(x_0) \setminus B_{1/\varepsilon}(x_0) \). Observe that \( I_1 \) was already computed in the case of the ball and stands for a quantity which converges to zero as \( \varepsilon \) tends to zero.

Now we proceed with the estimates for \( I_2 \). For \( \varepsilon > 0 \) small enough such that \( \varepsilon \leq 1/2r \), it holds

\[
|x - a_i| \geq \frac{1}{2} |x - x_0|, \quad \forall x \in B_{1/\varepsilon}(x_0) \setminus B_{1/\varepsilon}(x_0), \forall i \in \{1, 2\}.
\]

Due to (3.20) we obtain

\[
I_2 = \left( \log \frac{1}{\varepsilon} \right)^{-2} \int_{B_{1/\varepsilon}(x_0) \setminus B_{1/\varepsilon}(x_0)} |x - x_0|^{-2} \left( \frac{|x - x_0|^2 - r^2}{2} \right)^{2} \prod_{i=1}^{2} |x - a_i|^{-N} \, dx
\]

\[
\leq \left( \log \frac{1}{\varepsilon} \right)^{-2} \int_{B_{2/\varepsilon}(x_0) \setminus B_{1/\varepsilon}(x_0)} |x - x_0|^2 \prod_{i=1}^{2} \left( \frac{1}{2} |x - x_0| \right)^{-N} \, dx
\]

\[
= \left( \log \frac{1}{\varepsilon} \right)^{-2} 2^{2N} \int_{B_{1/\varepsilon}(x_0) \setminus B_{1/\varepsilon}(x_0)} |x - x_0|^{2-2N} \, dx
\]

\[
(3.21) \quad = \left( \log \frac{1}{\varepsilon} \right)^{-2} 2^{2N} \omega_N \int_{1/\varepsilon}^{1/2} s^{1-N} \, ds.
\]

From (3.21) we get that

\[
(3.22) \quad I_2 = \begin{cases} 
O\left( \left( \log \frac{1}{\varepsilon} \right)^{-1} \right), & N = 2, \\
O\left( \varepsilon^{N-2} \left( \log \frac{1}{\varepsilon} \right)^{-2} \right), & N \geq 3,
\end{cases} \quad \text{as } \varepsilon \to 0.
\]

Combining (3.22) with the similar estimates for \( I_1 \) discussed before, we end up with

\[
\int_{\Omega} |\nabla \theta_\varepsilon|^2 f^2 \prod_{i=1}^{2} |x - a_i|^{-N} \, dx \to 0, \quad \text{as } \varepsilon \to 0.
\]

In addition, the uniform estimate in (3.10) remains valid in this case, which together with (3.23) yield to the optimality of \( N^2/n^2 \) for any \( n \geq 2 \).

\[\square\]

3.3. The case of a half-space. For simplicity, let us focus on the upper-half space \( \Omega = \mathbb{R}^N_+ \). Once we have dealt with Subsections 3.1 and 3.2, the proof of this case is easy and natural.

\textit{Step 1.} The inequality \( \mu^*(\mathbb{R}^N_+) \geq N^2/n^2 \) derives from (2.15) for

\[
f = x_N,
\]

\[
(3.24) \quad f = x_N.
\]
which verifies (2.16).

Step 2. If \( n \geq 3 \) then \( \mu^*(\mathbb{R}^N) = N^2/n^2 \), since the function

\[
(3.25) \quad u = x_N \prod_{i=1}^{n} |x - a_i|^{-\frac{N}{n}} \in H^1_0(\Omega)
\]

satisfies

\[
\int_{\Omega} |\nabla u|^2 \, dx = \frac{N^2}{n^2} \int_{\Omega} V u^2 \, dx.
\]

If \( n = 2 \), the function \( u \) in (3.25) does not belong to \( H^1_0(\Omega) \). The fact that \( \mu^*(\mathbb{R}^N) = N^2/4 \) (which is not attained) is a consequence of the approximation process for the sequence \( u_\varepsilon := \theta_\varepsilon f \prod_{i=1}^{2} |x - a_i|^{-N/2} \), with \( f \) as in (3.24) and \( \theta_\varepsilon \) verifying

\[
(3.26) \quad \theta_\varepsilon(x) = \begin{cases} 
0, & |x - a_i| \leq \varepsilon^2, \quad \forall i \in \{1, 2\}, \\
\frac{\log|x-a_i|/\varepsilon^2}{\log 1/\varepsilon}, & \varepsilon^2 \leq |x - a_i| \leq \varepsilon, \quad \forall i \in \{1, 2\}, \\
1, & x \in B_{1/j}(0) \setminus \bigcup_{i=1}^{2} B_{\varepsilon}(a_i), \\
\frac{\log 1/(\varepsilon^2|x|)}{\log 1/\varepsilon}, & x \in B_{1/j^2}(0) \setminus B_{1/\varepsilon}(0), \\
0, & |x| \geq 1/\varepsilon^2.
\end{cases}
\]

Then the computations are similar to those in Subsection 3.2. The details are let to the reader. Therefore, the proof of Subsection 3.3 is complete and Theorem 1.1 is totally proven.

\[\square\]

4. Proof of Theorem 1.2

We split the proof in two main steps.

In the first step we deal with the distance function to the boundary defined by the mapping \( \Omega \ni x \mapsto \rho(x) := \min\{|x-y| \mid y \in \Gamma\} \). For any \( \beta > 0 \) let be \( \Omega_\beta := \{x \in \Omega \mid \rho(x) \leq \beta\} \). It is well known that \( \rho \in C^2(\Omega_{\beta_0}) \) for some \( \beta_0 \) small enough since \( \Omega \) is smooth and \( \text{card}\{N(x) := \{y \in \Gamma \mid \rho(x) = |x-y|\}\} = 1 \) for \( x \) close enough to the boundary \( \Gamma \).

Step 1. We claim there exists a constant \( C_\Omega \), depending on \( \Omega \), such that

\[
(4.1) \quad C_\Omega \int_{\Omega} \frac{u^2}{\rho} \, dx + \int_{\Omega} |\nabla u|^2 \, dx \geq \frac{N^2}{n^2} \int_{\Omega} V u^2 \, dx, \quad \forall u \in H^1_0(\Omega).
\]

The proof of (4.1) is a consequence of (2.13) by building proper weights \( f \) satisfying

\[
(4.2) \quad \left( -\frac{\Delta f}{f} |x - a_i|^2 + \frac{2N \nabla f \cdot (x - a_i)}{f} - 2N \right) \frac{1}{|x - a_i|^2} \geq -\frac{C_\Omega}{\rho(x)}
\]

for any \( x \in \Omega, i \in \{1, \ldots, n\} \) and some uniform constant \( C_\Omega \). Next we build such \( f \) satisfying (4.2). In order to justify (4.2), let us consider \( f > 0, f \in C^2(\Omega) \) such that

\[
(4.3) \quad \begin{cases} 
f(x) = \rho(x), & x \in \Omega_{\beta_0}, \\
f(x) \geq \beta_0, & x \in \Omega \setminus \Omega_{\beta_0}.
\end{cases}
\]

In particular, we observe that \( f \) vanishes on the boundary \( \Gamma \).
Let us now focus on $\Omega_{\beta_0}$ where $f$ coincides with the distance function $\rho$. Therefore, according to Lemma 3.1. in [5], we can show that there exists a constant, say $D_{\Omega} > 0$, such that

$$\left| \nabla \rho(x) \cdot (x - a_i) - \rho(x) \right| \leq D_{\Omega} |x - a_i|^2, \quad \forall x \in \Omega_{\beta_0}, \; \forall i \in \{1, \ldots, n\}. \quad (4.4)$$

In convex domains, the proof of (4.4) is straightforwardly obtained by just taking the Taylor expansion of function $\rho$ in the neighborhood of each pole $a_i$. Otherwise, in non-convex domains the proof is more technical.

According to (4.3) and (4.4) we get

$$\left| -\Delta f \frac{|x - a_i|^2}{f} + 2N \nabla f \cdot (x - a_i) - 2N \frac{1}{|x - a_i|^2} \right| \leq \frac{C_{\Omega}}{\rho(x)} \quad x \in \Omega_{\beta_0},$$

$$\leq C_{\Omega}, \quad x \in \Omega \setminus \Omega_{\beta_0}, \quad \frac{(|x - a_i|^2)}{f}$$

for a large enough constant $C_{\Omega} > D_{\Omega}$. The estimates in $\Omega \setminus \Omega_{\beta_0}$ are trivial since $f$ is smooth enough and strictly positive far from the boundary and the singularities $a_i$ are avoided. Therefore, reconsidering the constants, (4.5) implies (4.2) and the proof of (4.1) is finished.

Step 2. We will combine the Hardy inequality involving the distance to the boundary with (4.1). Indeed, cf. [1], there exists a positive constant $E_{\Omega}$ such that

$$\int_{\Omega} |\nabla u|^2 \, dx \geq E_{\Omega} \int_{\Omega} \frac{u^2}{\rho^2(x)} \, dx, \quad \forall u \in H^1_0(\Omega). \quad (4.6)$$

Let $\mu < N^2/n^2$ and $\varepsilon_\mu > 0$ small enough such that $\varepsilon_\mu \leq 1 - \mu n^2/N^2$. Splitting the gradient we successfully get

$$\int_{\Omega} |\nabla u|^2 - \mu \int_{\Omega} V u^2 \, dx = \varepsilon_\mu \int_{\Omega} |\nabla u|^2 \, dx + (1 - \varepsilon_\mu) \int_{\Omega} |\nabla u|^2 \, dx - \mu \int_{\Omega} V u^2 \, dx$$

$$\geq \varepsilon_\mu E_{\Omega} \int_{\Omega} \frac{u^2}{\rho^2} \, dx + \left( (1 - \varepsilon_\mu) \frac{N^2}{n^2} - \mu \right) \int_{\Omega} V u^2 \, dx$$

$$- C_{\Omega} (1 - \varepsilon_\mu) \int_{\Omega} \frac{u^2}{\rho} \, dx$$

$$\geq \varepsilon_\mu E_{\Omega} \int_{\Omega} \frac{u^2}{\rho^2} \, dx - C_{\Omega} (1 - \varepsilon_\mu) \int_{\Omega} \frac{u^2}{\rho} \, dx.$$

For the first gradient term above we have applied the Hardy inequality (4.6) whereas for the second gradient term we have used inequality (4.2) in Step 1.

On the other hand, for $\beta \leq \min\{\beta_0, \varepsilon_\mu E_{\Omega}/|C_{\Omega} (1 - \varepsilon_\mu)|\}$ we get the pointwise inequality

$$\frac{\varepsilon_\mu E_{\Omega}}{\rho^2} - \frac{C_{\Omega} (1 - \varepsilon_\mu)}{\rho} \geq 0, \quad \forall x \in \Omega_{\beta}.$$

In consequence, for such small $\beta$ from (4.7) we end up with

$$\int_{\Omega} |\nabla u|^2 - \mu \int_{\Omega} V |u|^2 \, dx \geq - \frac{|C_{\Omega} (1 - \varepsilon_\mu)|}{\beta} \int_{\Omega \setminus \Omega_{\beta}} u^2 \, dx$$

$$\geq - \frac{|C_{\Omega} (1 - \varepsilon_\mu)|}{\beta} \int_{\Omega} u^2 \, dx,$$

which finishes the proof of Theorem 1.2. \qed
I. Verification of item (ii).

The left inequality of (1.10) in non strict form is trivial due to (1.4). Roughly speaking, we can trivially extend a function \( u \in H^1_0(\Omega) \) to \( u \in H^1(\mathbb{R}^N) \) to maintain the inequality in bounded domains.

Let us now assume by contradiction that \( \mu^*(\Omega; a_1, a_2) = (N - 2)^2/4 \). Then, it follows from item (iii) that \( \mu^*(\Omega) \) is attained. Extending trivially with zero, we get that inequality (1.4) is attained for \( n = 2 \) by a function in \( H^1(\mathbb{R}^N) \). Contradiction! (cf. Section 2) if the constant \( (N - 2)^2/n^2 \) is attained by, say \( u \), then \( u = \prod_{i=1}^n |x - a_i|^{-(N-2)/n} \) which escapes from \( H^1_{loc}(\mathbb{R}^N) \) precisely when \( n = 2 \).

The most tricky part of item (i) concerns the justification of the upper bound

(5.1) \[
\mu^*(\Omega; a_1, a_2) \leq \frac{N^2}{4}.
\]

For that, it suffices to build a sequence \( \{u_\varepsilon\}_{\varepsilon > 0} \subset H^1_0(\Omega) \) such that

(5.2) \[
\frac{\int_\Omega |\nabla u_\varepsilon|^2 \, dx}{\int_\Omega V |u_\varepsilon|^2 \, dx} \to \frac{N^2}{4}, \quad \text{as } \varepsilon \to 0.
\]

Next we verify (5.2), for suitable choices of \( f_\varepsilon \) and \( u_\varepsilon \) in (2.15).

Firstly, let \( \varepsilon > 0 \) be fixed (small enough) and consider \( f_\varepsilon > 0, \ f_\varepsilon \in C^2(\Omega) \) such that

(5.3) \[
\begin{cases}
  f_\varepsilon = \rho^{1+\varepsilon}, & \rho(x) \leq \beta_0 \\
  \beta_0^{1+\varepsilon} \leq f_\varepsilon \leq 2\beta_0, & \beta_0 \leq \rho(x) \leq 2\beta_0, \\
  f_\varepsilon = 2\beta_0, & \rho(x) \geq 2\beta_0,
\end{cases}
\]

where \( \rho \) denotes the distance function to the boundary and \( \beta_0 \) is small enough such that \( \rho \in C^2(\Omega_{\beta_0}) \), as in the beginning of Section 4. In addition, such a profile of \( f_\varepsilon \) can be easily built to check the uniform bound

(5.4) \[
\|f_\varepsilon\|_{W^{2,\infty}(\Omega; \Omega_{\beta_0})} \leq C, \quad \text{as } \varepsilon \to 0,
\]

where \( C \) is a constant depending only on \( \Omega \) and \( \beta_0 \).

Then we consider

(5.5) \[ u_\varepsilon := f_\varepsilon \prod_{i=1}^n |x - a_i|^{-N} \in H^1_0(\Omega), \]

with \( f \) as in (5.3)-(5.4). In this case, according to (2.15) \( u_\varepsilon \) satisfies

(5.6) \[
\int_\Omega |\nabla u_\varepsilon|^2 \, dx - \frac{N^2}{4} \int_\Omega V |u_\varepsilon|^2 \, dx = \frac{1}{2} \sum_{i=1}^2 \int_\Omega \left( -\Delta f_\varepsilon |x - a_i|^2 + 2N \nabla f_\varepsilon \cdot (x - a_i) - 2N f_\varepsilon \right) \frac{u_\varepsilon^2}{|x - a_i|^2} \, dx.
\]

Due to (5.3) and (5.4), we get that

(5.7) \[
\sum_{i=1}^2 \left| -\Delta f_\varepsilon |x - a_i|^2 + 2N \nabla f_\varepsilon \cdot (x - a_i) - 2N f_\varepsilon \right| \frac{1}{|x - a_i|^2} \leq C_1, \quad \forall x \in \Omega \setminus \Omega_{\beta_0},
\]
for some uniform constant $C_1 > 0$ as $\varepsilon \to 0$.

On the other hand, for $x \in \Omega_{\beta_0}$, using the Eikonal equation $|\nabla \rho(x)| = 1$ we obtain

$$
\nabla f_\varepsilon = (1 + \varepsilon)\rho^\varepsilon \nabla \rho, \quad \Delta f_\varepsilon = \varepsilon(1 + \varepsilon)\rho^{\varepsilon-1} + (1 + \varepsilon)\rho^\varepsilon \Delta \rho.
$$

Then, according to (4.4) we have

$$
-\Delta f_\varepsilon |x - a_i|^2 + 2N\nabla f_\varepsilon \cdot (x - a_i) - 2Nf_\varepsilon \leq C_2 \left( \frac{\varepsilon}{\rho^2} + \frac{1}{\rho} \right),
$$

for some constant $C_2$ depending on $N, \beta_0$ and $R_{\Omega} (= \sup_{x, y \in \Omega} |x - y|)$.

Then, combining (5.7), (5.6) and (5.8) we get (reconsidering the uniform constant in $\varepsilon$)

$$
\int_{\Omega} |\nabla u_\varepsilon|^2 \, dx - \frac{N^2}{n^2} \int_{\Omega} V u_\varepsilon^2 \, dx \leq C \int_{\Omega} \left( \frac{\varepsilon}{\rho^2} + \frac{1}{\rho} \right) u_\varepsilon^2 \, dx.
$$

Next we will estimate the right hand side in (5.9). For that, let $\delta > 0$ such that $\delta < \min\{\beta_0, |a_1 - a_2|/4\}$ and $\bigcup_{i=1}^2 B_\delta(a_i) \subset \Omega_{\beta_0}$. Then, we split the first term to write

$$
\varepsilon \int_{\Omega} \frac{u_\varepsilon^2}{\rho^2} \, dx = \varepsilon \int_{\Omega} \rho^{2\varepsilon} \prod_{i=1}^2 |x - a_i|^{-N} \, dx
$$

$$
= \varepsilon \int_{\bigcup_{i=1}^2 B_\delta(a_i)} \ldots + \varepsilon \int_{\Omega \setminus \bigcup_{i=1}^2 B_\delta(a_i)}
$$

$$
= I_1 + I_2.
$$

We can easily observe that

$$
I_2 \leq \varepsilon R_{\Omega} \delta^{-N} |\Omega \setminus \bigcup_{i=1}^2 B_\delta(a_i)|
$$

$$
= O(\varepsilon), \text{ as } \varepsilon \to 0.
$$

On the other hand, we successively have

$$
I_1 = \varepsilon \sum_{i=1}^2 \int_{B_\delta(a_i)} \rho^{2\varepsilon} \prod_{j=1}^2 |x - a_j|^{-N} \, dx
$$

$$
\leq \varepsilon \sum_{i=1}^2 \delta^{-N} \int_{B_\delta(a_i)} \rho^{2\varepsilon} |x - a_i|^{-N} \, dx
$$

$$
\leq \varepsilon \delta^{-N} \sum_{i=1}^2 \int_{B_\delta(a_i)} |x - a_i|^{-N+2\varepsilon} \, dx
$$

$$
\leq 2\varepsilon \omega_N \delta^{-N} \int_0^\delta r^{-1+2\varepsilon} \, dr
$$

$$
= \omega_N \delta^{-N+2\varepsilon}
$$

$$
= O(1), \text{ as } \varepsilon \to 0.
$$

From (5.10)-(5.12) we obtain

$$
\varepsilon \int_{\Omega} \frac{u_\varepsilon^2}{\rho^2} \, dx = O(1), \text{ as } \varepsilon \to 0.
$$
Similar estimates apply for the term $\int_{\Omega} \frac{u_\varepsilon^2}{\rho} \, dx$ to conclude from (5.9) that

\begin{equation}
\int_{\Omega} \left| \nabla u_\varepsilon \right|^2 \, dx - \frac{N^2}{4} \int_{\Omega} V u_\varepsilon^2 \, dx \leq C,
\end{equation}

for some constant $C > 0$ independent of $\varepsilon$.

Next we will show that

\begin{equation}
\lim_{\varepsilon \to 0} \int_{\Omega} V u_\varepsilon^2 \, dx = \infty,
\end{equation}

which combined with (5.13) yield to (5.2) and the proof of item [ii] is complete.

In order to certify (5.14) we need to introduce some notation and notions as follows.

For any $\gamma > 0$ aimed to be small and $i \in \{1, 2\}$, the set $\Sigma_{\alpha_i}^\gamma \subset S^{N-1}$ defines the surface

\begin{equation}
\Sigma_{\alpha_i}^\gamma := \{ \sigma \in S^{N-1} : d_g(\sigma, \vec{n}(a_i)) \leq \gamma \},
\end{equation}

where $d_g$ denotes the geodesic distance on the sphere $S^{N-1}$ and $\vec{n}(a_i)$ is the inward normal unit vector on $\Gamma$ at the point $a_i$.

Then for $R > 0$ we introduce the cone-like domain

\begin{equation}
C_{\Sigma_{\alpha_i}^\gamma}^R := \{ x = a_i + r \sigma \mid r \in (0, R), \, \sigma \in \Sigma_{\alpha_i}^\gamma \}.
\end{equation}

Therefore, for $\gamma$ and $R$ small enough one can check that

\begin{equation}
\int_{\Omega} V u_\varepsilon^2 \, dx \geq |a_1 - a_2| \int_{\Omega} \rho^{2+2\varepsilon} \prod_{i=1}^{2} |x - a_i|^{-N-2} \, dx
\end{equation}

\begin{equation}
\geq |a_1 - a_2| R^{-N-2} \sum_{i=1}^{2} \int_{C_{\Sigma_{\alpha_i}^\gamma}^R} \rho^{2+2\varepsilon} |x - a_i|^{-N-2} \, dx.
\end{equation}

Applying (5.16)-(5.17) and the co-aria formula we obtain

\begin{equation}
\int_{\Omega} V u_\varepsilon^2 \, dx \geq |a_1 - a_2| R^{-N-2} \sum_{i=1}^{2} c_i^{2+2\varepsilon} \int_{C_{\Sigma_{\alpha_i}^\gamma}^R} |x - a_i|^{2\varepsilon - N} \, dx
\end{equation}

\begin{equation}
= |a_1 - a_2| R^{-N-2} \sum_{i=1}^{2} \rho_i^{2+2\varepsilon} \int_{C_{\Sigma_{\alpha_i}^\gamma}^R} \frac{R}{\rho} r^{N-1} \int_{\Sigma_{\alpha_i}^\gamma} \rho^{2\varepsilon - N} \, d\sigma \, dr
\end{equation}

\begin{equation}
= |a_1 - a_2| R^{-N-2} \sum_{i=1}^{2} c_i^{2+2\varepsilon} \int_{\Sigma_{\alpha_i}^\gamma} \int_{0}^{R} \rho^{-1+2\varepsilon} \, dr \, d\sigma
\end{equation}

\begin{equation}
= O(\varepsilon^{-1}), \text{ as } \varepsilon \to 0.
\end{equation}
This ensures the validity of the limit in (5.14) and implicitly item (i) is justified.

II. Verification of item (ii).

Again, the left inequality is trivial due to (1.4). The anti-monotonicity of the constant \( \mu^*(\Omega) \) with respect to \( n \) can be easily seen by looking at the corresponding potentials. Indeed,

\[
V(x; a_1, \ldots, a_n) = V(x; a_1, \ldots, a_{n-1}) + \sum_{i=1}^{n-1} \frac{|a_n - a_i|^2}{|x - a_n|^2|x - a_i|^2} > V(x; a_1, \ldots, a_{n-1}).
\]

Then, we simply get

\[
\mu^*(\Omega; a_1, \ldots, a_{n-1}, a_n) \leq \mu^*(\Omega; a_1, \ldots, a_{n-1}).
\]

Hence, from item (i) we obtain the upper bound in (1.10).

III. Verification of item (iii).

Let \( \{u_n\}_{n \in \mathbb{N}} \subset H^1_0(\Omega) \) be a normalized minimizing sequence such that

\[
(5.19) \quad \int_{\Omega} |\nabla u_n|^2 \, dx = \mu^*(\Omega),
\]

\[
(5.20) \quad \int_{\Omega} V u_n^2 \, dx = 1 + o(1), \quad \text{as } n \to \infty.
\]

Then there exists \( u \in H^1_0(\Omega) \) s.t.

\[
(5.21) \quad u_n \rightharpoonup u \text{ in } H^1_0(\Omega).
\]

Since the inclusion \( H^1_0(\Omega) \hookrightarrow L^2(V; \, dx) \) is continuous (due to Corollary 1.1) and \( H^1_0(\Omega) \) is compact embedded in \( L^2(\Omega) \) we get

\[
(5.22) \quad u_n \to u \text{ in } L^2(\Omega), \quad \text{and } u_n \to u \text{ in } L^2(V; \, dx).
\]

Put \( e_n := u_n - u \). We apply (5.19) and (5.20) to obtain

\[
(5.23) \quad \int_{\Omega} |\nabla u|^2 \, dx + \int_{\Omega} |\nabla e_n|^2 \, dx = \mu^*(\Omega) + o(1),
\]

\[
(5.24) \quad \int_{\Omega} V u^2 \, dx + \int_{\Omega} V e_n^2 \, dx = 1 + o(1), \quad \text{as } n \to \infty.
\]

Now, let us fix \( \delta > 0 \) such that \( \mu^*(\Omega) + \delta < N^2/n^2 \) and choose \( \mu \in (\mu^*(\Omega) + \delta, N^2/n^2) \).

Then, cf. Theorem 1.2 it follows

\[
(5.25) \quad \int_{\Omega} |\nabla e_n|^2 \, dx \geq \mu \int_{\Omega} V e_n^2 \, dx + o(1),
\]

since \( e_n \to 0 \) in \( L^2(\Omega) \) due to (5.22).
Therefore, applying (5.24), the election of $\mu$ and (5.24) we successively have
\[
\mu^*(\Omega) \int_\Omega V u^2 \, dx \leq \int_\Omega |\nabla u|^2 \, dx \leq \mu^*(\Omega) - (\mu^*(\Omega) + \delta) \int_\Omega V e_n^2 \, dx + o(1)
\leq \mu^*(\Omega) \left( \int_\Omega V u^2 \, dx + \int_\Omega V e_n^2 \, dx \right) - (\mu^*(\Omega) + \delta) \int_\Omega V e_n^2 \, dx + o(1)
\leq \mu^*(\Omega) \int_\Omega V u^2 \, dx - \delta \int_\Omega V e_n^2 \, dx + o(1).
\]
(5.26)

Since $\delta > 0$, we must have $\int_\Omega V e_n^2 \, dx \to 0$, as $n \to \infty$. Coming back to (5.23) and (5.24) we obtain $\int_\Omega V u^2 \, dx = 1$ (so $u \neq 0$) and $\int_\Omega |\nabla u|^2 \, dx \leq \mu^*(\Omega)$. Taking into account the definition of $\mu^*(\Omega)$ we get that $u$ satisfies
\[
\mu^*(\Omega) \int_\Omega V u^2 \, dx = \int_\Omega |\nabla u|^2 \, dx,
\]
so the constant is attained by a non-trivial function $u \in H_0^1(\Omega)$. The proof is finished. □

\textbf{Proof of Corollary 1.1.} Let $u \in H_0^1(\Omega)$ and $0 < \mu < N^2/n^2$ be fixed. In view of Poincaré inequality we have
\[
\int_\Omega |\nabla u|^2 \, dx \geq \lambda_1(\Omega) \int_\Omega u^2 \, dx,
\]
where $\lambda_1(\Omega) > 0$ denotes the first eigenvalue of the Dirichlet Laplacian in $\Omega$. In view of Theorem 1.2 multiplying in (5.27) by $|C_\mu|/\lambda_1(\Omega)$ and summing up with (1.7) we get
\[
\left( 1 + \frac{|C_\mu|}{\lambda_1(\Omega)} \right) \int_\Omega |\nabla u|^2 \, dx \geq \frac{N^2}{n^2} \int_\Omega V u^2 \, dx.
\]
Therefore we obtain $\mu^*(\Omega) > \lambda_1(\Omega) N^2/n^2 (|C_\mu| + \lambda_1(\Omega)) > 0$ and the proof is complete. □

6. \textbf{OTHER REMARKS, COMMENTS AND OPEN QUESTIONS}

- We have shown that Theorem 1.1 provides optimal results for particular geometries like balls, exterior of balls or half-spaces. Furthermore, we may stress the question of determining more general classes of domains for which Theorem 1.1 applies. For instance, we may ask weather Theorem 1.1 is valid for convex domains. A positive answer to this latter question does not seem to be trivial at all. Indeed, the optimality of the Hardy inequality with one singular potential obtained in balls and half-spaces is enough to extend similar results to more general domains like convex domains. This is due to the comparison arguments which make use of the anti-monotonicity properties of the Hardy constant with respect to domains inclusions.

Generally, the anti-monotonicity property cannot be used in an efficient way for multipolar potentials. Therefore, one of the possibilities is to show directly that $\mu^*(\Omega) = N^2/n^2$ (or not) for convex domains, without using the results obtained for the particular geometries above. In view of that, it suffices to build weights $f$, super-solutions for (2.16). The existence of such weighted functions in any other domains would suffice to get $\mu^*(\Omega) = N^2/n^2$. 

In particular, there are non-convex domains for which \( \mu^*(\Omega) = \frac{N^2}{n^2} \). Indeed, in view of the proof of Theorem 1.1 this is true in any domain included in a ball, touching the boundary of the ball at the singular poles.

- Another interesting point is to analyze whether we can extend item (i) of Theorem 1.3 for any \( n \geq 3 \) to show that \( \mu^*(\Omega; a_1, \ldots, a_n) \leq \frac{N^2}{n^2} \) for any smooth domain with boundary singularities \( a_1, \ldots, a_n \). Our proof, which only works in the case \( n = 2 \), is very much related to the non-attainability of the optimal constant in Theorem 1.1.

- Moreover, we may wonder if the constant \( c_\mu \) in Theorem 1.2 is uniformly bounded as \( \mu \) tends to \( \frac{N^2}{n^2} \) so that we can obtain the validity of Theorem 1.2 even in the critical case \( \mu = \frac{N^2}{n^2} \).

- Finally, we have obtained the Hardy inequality with the optimal constant \( \frac{N^2}{n^2} \) in various domains with different geometries like the half-space, the ball, the exterior of a ball, etc.. In proving so, we had to make an adequate election for the weight \( f \) in (2.15). Previous examples suggest that \( f \) must be chosen in terms of the implicit equation of the surface \( \Gamma \) in the neighborhood of the singular poles \( a_i \).

It is important to remark that this is not the case of an ellipse. More precisely, assume \( \Omega \subset \mathbb{R}^2 \) is an ellipse given by

\[
\mathcal{E} : \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} \leq 1,
\]

with the singular poles \( a_i \) lying on its boundary.

Moreover, let us consider the weight function induced by the implicit equation of the ellipse, that is \( f = 1 - \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} \) and satisfies \( f > 0 \) in \( \mathcal{E} \) and \( f = 0 \) on the boundary of \( \mathcal{E} \). Then we get that

\[
-\Delta f|x - a_i|^2 + 2N \nabla f \cdot (x - a_i) - 2N f = \frac{2(b^2 - a^2)((x_2 - a_1^2)^2 - (x_1 - a_1^2)^2)}{a_1^2 b_1^2},
\]

where \( a_i = (a_1^i, a_2^i) \). Observe that quantity (6.1) does not have a constant sign in \( \mathcal{E} \). In consequence, we cannot make any statement about the Hardy constant in this case.

- To conclude, one of the main further challenges is to extend our analysis to multipolar potentials of the form

\[
V = \sum_{i=1}^{n} \frac{1}{|x - a_i|^2},
\]

for which, to our knowledge, the corresponding Hardy constant is not known for any particular domains. This problem has been intensively studied in the literature quoted below, but optimal results are still to be obtained. We believe that the optimal results of our paper could be a hint in order to handle other types of multi-singular potentials like (6.2).

Acknowledgements.

This work was partially supported by the both grants of the Ministry of National Education, CNCS-UEFISCEDI Romania, project PN-II-ID-PCE-2012-4-0021 and project PN-II-ID-PCE-2011-3-0075, and the Grant MTM2011-29306-C02-00 of the MICINN (Spain).
References

[1] A. Ancona, *On strong barriers and an inequality of Hardy for domains in $\mathbb{R}^n$*, J. London Math. Soc. (2) **34** (1986), no. 2, 274–290.

[2] A. Balinsky, A. Laptev, and A. V. Sobolev, *Generalized Hardy inequality for the magnetic Dirichlet forms*, J. Statist. Phys. **116** (2004), no. 1-4, 507–521.

[3] R. Bosi, J. Dolbeault, and M. J. Esteban, *Estimates for the optimal constants in multipolar Hardy inequalities for Schrödinger and Dirac operators*, Commun. Pure Appl. Anal. **7** (2008), no. 3, 533–562.

[4] H. Brezis and J. L. Vázquez, *Blow-up solutions of some nonlinear elliptic problems*, Rev. Mat. Univ. Complut. Madrid **10** (1997), no. 2, 443–469.

[5] C. Cazacu, *Null-controllability of the heat equation with an inverse square potential localized on the boundary*, http://arxiv.org/pdf/1201.3390.pdf.

[6] C. Cazacu and E. Zuazua, *Improved Multipolar Hardy Inequalities*, Studies in Phase Space Analysis of PDEs, Progress in Nonlinear Differential Equations and Their Applications, vol. **84**, Birkhäuser, New York, 2013, 37–52.

[7] E. B. Davies, *The Hardy constant*, Quart. J. Math. Oxford Ser. (2) **46** (1995), no. 184, 417–431.

[8] E. B. Davies, *A review of Hardy inequalities*, The Maz’ya anniversary collection, Vol. 2 (Rostock, 1998), Oper. Theory Adv. Appl., vol. **110**, Birkhäuser, Basel, 1999, 55–67.

[9] C. Fefferman, *The uncertainty principle*, Bull. Amer. Math. Soc. (N. S.) **9** (1983), no. 2, 129–206.

[10] V. Felli, A. Ferrero, and S. Terracini, *Asymptotic behavior of solutions to Schrödinger equations near an isolated singularity of the electromagnetic potential*, J. Eur. Math. Soc. **13** (2011), no. 1, 119–174.

[11] V. Felli, E. M. Marchini, and S. Terracini, *On Schrödinger operators with multisingular inverse-square anisotropic potentials*, Indiana Univ. Math. J. **58** (2009), no. 2, 617–676.

[12] V. Felli and S. Terracini, *Nonlinear Schrödinger equations with symmetric multi-polar potentials*, Calc. Var. Partial Differential Equations **27** (2006), no. 1, 25–58.

[13] D. Krejčířík and E. Zuazua, *The Hardy inequality and the heat equation in twisted tubes*, J. Math. Pures Appl. (9) **94** (2010), no. 3, 277–303.

[14] A. Laptev and T. Weidl, *Hardy inequalities for magnetic Dirichlet forms*, Oper. Theory: Advances and Applications **108** (1999), 299–305.

[15] Y. Pinchover and K. Tintarev, *Existence of minimizers for Schrödinger operators under domain perturbations with application to Hardy’s inequality*, Indiana Univ. Math. J. **54** (2005), no. 4, 1061–1074.