Nonlinear Network Autoregression

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Abstract

We study general nonlinear models for time series networks of integer and continuous valued data. The vector of high dimensional responses, measured on the nodes of a known network, is regressed non-linearly on its lagged value and on lagged values of the neighboring nodes by employing a smooth link function. We study stability conditions for such multivariate process and develop quasi maximum likelihood inference when the network dimension is increasing. In addition, we study linearity score tests by treating separately the cases of identifiable and non-identifiable parameters. In the case of identifiability, the test statistic converges to a chi-square distribution. When the parameters are not-identifiable, we develop a supremum-type test whose \( p \)-values are approximated adequately by employing a feasible bound and bootstrap methodology. Simulations and data examples support further our findings.

Keywords: contraction, hypothesis testing, increasing dimension, multivariate count time series.

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1 Introduction

The availability of network data recorded over a timespan in several applications (social networks, GPS data, epidemics, etc.) requires assessing the effect of a network structure to a multivariate time series process. This problem has attracted considerable attention. Recently [72] proposed a Network Autoregressive model (NAR), under Independent Identically Distributed (IID) innovation sequence, where a continuous response variable is observed for each node of a network. The high dimensional vector of such responses is modelled linearly on the past values of the response, measured on the node itself and the average lagged response of the neighbours connected to the node. Motivated by the fact that real data networks are usually of large dimension, the authors study least squares inference under two large-sample regimes (a) with increasing time sample size, i.e. $T \rightarrow \infty$, and fixed network dimension, say $N$, and (b) with $N \rightarrow \infty$ and $T_N \rightarrow \infty$, where the temporal size depends on $N$. In this contribution, we extend this work in various directions by proposing appropriate inference and testing methodology which is applicable to continuous and integer valued data.

1.1 Related works

The NAR model has been the focus of recent research e.g. logistic network models [69], network quantile model [73], grouped least squares estimation [71], network GARCH models [70], and applications [8]. In addition, [36] consider the Generalized NAR model, for the continuous case, which incorporates the effect of several layers of connections between nodes of the network. This study is accompanied by appropriate R software. As pointed out by [72], discrete response variables are frequently encountered in applications and are strongly related to network data. For example, in the social network analysis data are counts, e.g. number of characters contained in posts of single users, number of posts shared, etc. Models for count time series have been studied by [7] who introduced linear and log-linear Poisson network autoregression models (PNAR). Such extensions show that the NAR model is a member of the broad class of Generalized Linear Models (GLM) [49]; for the count case, the observations are in fact marginally Poisson distributed, conditionally to their past history. The joint dependence is imposed by employing a copula construction, as introduced by [32] and is outlined in the Supplement S-1. In addition, [7] have studied thoroughly the two related types of asymptotic inference (a)-(b) discussed above, in the context of quasi maximum likelihood inference (QMLE) [13].

1.2 Nonlinear models & testing linearity

Theory related to NAR model relies on the assumption of linearity. However, there are many real world examples where a non-linear model might be more appropriate. For instance, in economics, the theory supports occasionally nonlinear behaviour, see [59, Ch. 2] for several examples. In modelling economic/financial time series, it seems natural to allow for the existence of different states, or regimes (e.g. expansion/crisis), such that dynamics depend upon the specific regime; [74, Ch. 18]. Government agencies, research institutes and central banks may typically employ nonlinear models [59, p. 16]. As far as social network analysis concerned, nonlinear phenomena are frequently observed e.g. ”superstars” with huge number of followers having an exponentially higher impact on other users’ behaviour when compared to the ”standard” user [72]. So from both theoretical and
applied point of view, there exists a necessity to develop non-linear autoregression model theory for the case of network time series. Literature in univariate non-linear time series is vast and well developed, in particular for continuous random variables. The interested reader is referred to [62], [12] and [29], among many others. For integer-valued data the literature is more recent, although still flourishing; see [17], [31], [33], [19], [13], [64] and [40]. General results are given by [67], [22], [1], [18], [23] and [9].

Estimation for non-linear models has been traditionally accompanied by tests who examine the assumption of linearity. Such tests are routinely used because they provide a sound framework where evidence of linearity can be examined thoroughly. In addition, they offer guidance about the specific non-linear model to be fitted; see [61, Sec. 3] who suggests that "The first step of a specification strategy for any type of nonlinear model should therefore consist of testing linearity". Furthermore, proper inference is developed especially when the linear model is nested within a non-linear model and as such the resulting estimators obtained after fitting a non-linear model may be inconsistent; see [59, Sec. 5.1,5.5]. Finally, it is always important to have additional tools for both practical usefulness and for explanatory data analysis; detecting latent variables, change point testing etc. These points motivated the growth of a large literature on linearity tests, especially for continuous-valued random variables. A survey of general type results for test statistics whose application depend on identifiable/non-identifiable parameters are given by [34] and [44]. Finally, [6] and [42] established a general framework for testing linearity when some parameters are non-identifiable under the hypothesis of linearity. Non-linear models for count time series and the associated testing linearity problem has been studied by [14] who suggest a quasi-score test for (mixed) Poisson random variables. All the above works are concerned with univariate time series. Related literature on multivariate observation-driven models for discrete-valued data considers only linear cases, see [52; 53; 54] and [32], among others.

1.3 Main challenges

Existing theory does not cover the case of NAR models which are multivariate and their properties depend on both $N$ (size) and $T$ (time). Therefore, asymptotically, both indices, $N$ and $T$, tend to infinity and it is a great challenge to address the properties of such multivariate processes. Moreover, QMLE inference breaks down when estimating network models because $N$ is large. Consequently, non standard proofs are required for establishing stationarity of infinite-dimensional processes and to obtain sound inference within the double asymptotic regime (b). Note that even a simple weak law does not hold under regime (b). In particular, quantities related to the inference are of the order $O(N)$. Consider, for example, the sample information matrix which depends on the network structure and diverges with $N \to \infty$. Then the covariance of estimators explodes and proving existence of limiting Hessian and Fisher matrices is a challenging problem. These issues become more persistent when testing linearity, especially in the case of non-identifiable parameters. Then, a double indexed asymptotic theory with infinite dimensional vectors over a uniform metric space for the score and related matrices is relevant and asks for development.

1.4 Our contribution

The main results of our contribution are the following: i) Specification of a novel general nonlinear network autoregressive model for both continuous and discrete valued multivariate network obser-
vations (Section 2). ii) Under mild conditions, stationarity results (Section 2.2) and asymptotic theory of QMLE are established, when both time and network dimensions increases (Section 3). These are new results because nonlinear NAR models have not been treated in the literature. iii) Development of testing procedures for examining linearity of the model by employing a quasi-score based test under both asymptotic regimes (a)-(b); see Sec. 4. We focus on score tests, as they require fitting NAR models under the null hypothesis. This is computationally simpler task. Their asymptotic distribution is (non-central) chi-square even when the parameters to be tested lie on the boundary of the parameter space. iv) Finally, we consider the situation where non identifiable parameters, say $\gamma$, are present under the null. In such case the results of Section 2-4 are extended. This is done by proving stochastic equicontinuity of the score with diverging number of nodes and double-indexed convergence of Hessian/information matrices uniformly over $\gamma$. Then, as $N \to \infty$ and $T_N \to \infty$, we show that the quasi-score linearity test asymptotically approximates a (non-central) chi-square process (Section 5). We discuss two ways to approximate the $p$-values of the tests: by the upper bound developed in [16], and by bootstrap approximation relying on stochastic permutations [42]. The double asymptotic convergence of bootstrap $p$-values to their theoretical counterpart is also established. We are not aware of other contributions, to the best of our knowledge, attacking the problem of general asymptotic inference with increasing dimension network time series models for both count and continuous data. The last sections discuss results of a simulation study (Section 6) and real data examples (Section 7). All the methodology is implemented in the new released R package PNAR [63; 10]. The paper concludes with Appendix A containing the proofs for Sections 2 and 5. A Supplementary Material S (henceforth Supp. Mat.) contains the proofs for Sections 3 and 4 and additional results for Threshold Network Autoregressive model, under asymptotic regime (a).

1.5 Notation

We denote $|x|_r = \left(\sum_{j=1}^p |x_j|^r\right)^{1/r}$ the $l^r$-norm of a $p \times 1$ vector $x$. If $r = \infty$, $|x|_\infty = \max_{1 \leq j \leq p} |x_j|$. Let $\|X\|_r = \left(\sum_{j=1}^p \text{E}(|X_j|^r)\right)^{1/r}$ the $L^r$-norm for a random vector $X$. For a $q \times p$ matrix $M = (m_{ij})$, $i = 1, \ldots, q$, $j = 1, \ldots, p$, denote the generalized matrix norm $\|M\|_r = \max_{1 \leq j \leq p} |Mx|_r$. If $r = 1$, then $\|M\|_1 = \max_{1 \leq j \leq p} \sum_{i=1}^q |m_{ij}|$. If $r = 2$, $\|M\|_2 = \rho^{1/2}(M'M)$, where $\rho(\cdot)$ is the spectral radius. If $r = \infty$, $\|M\|_\infty = \max_{1 \leq i \leq q} \sum_{j=1}^p |m_{ij}|$. If $q = p$, these norms are matrix norms. Define the entry-wise norms $|M|_r = \left(\sum_{i=1}^q \sum_{j=1}^p |m_{ij}|^r\right)^{1/r}$. If $q = p$ and $1 \leq r \leq 2$, these are matrix norms. Define $|x|_{vec}^r = (|x_1|^r, \ldots, |x_p|^r)'$, $\|X\|_{r,vec} = \left((E^{1/r} |X_1|^{r}, \ldots, E^{1/r} |X_p|^{r})', |M|_{vec} = (|m_{ij}|)_{i,j}\right)$ and $\preceq$ a partial order relation on $x, y \in \mathbb{R}^p$ such that $x \preceq y$ means $x_i \leq y_i$ for $i = 1, \ldots, p$. The same notation holds for random vectors $X, Y$ such that $X \preceq Y$ means $X_i \leq Y_i$ almost surely (a.s.) for $i = 1, \ldots, p$. Set the compact notation $\max_{1 < i \leq \infty} x_i = \max_{i \geq 1} x_i$. The notations $C_r$ denote a constant which depend on $r$, where $r \in \mathbb{N}$, and $C$ is a generic constant. The symbol $I$ denotes an identity matrix, $1 (0)$ a vector of ones (zeros), whose dimension depends on context. Let $\Rightarrow$ denote weak convergence with respect to the uniform metric. Finally, the notation $\{N, T_N\} \to \infty$ will be used as a shorthand for $N \to \infty$ and $T_N \to \infty$. 
2 Nonlinear NAR model specification

Consider a network with \(N\) nodes (network size) indexed by \(i = 1, \ldots, N\). The neighborhood structure of the network is explicitly described by its adjacency matrix \(A = (a_{ij}) \in \mathbb{R}^{N \times N}\) where \(a_{ij} = 1\), if there is a directed edge from \(i\) to \(j\) (e.g. user \(i\) follows \(j\) on Twitter, a flight take off from airport \(i\) landing to airport \(j\)), and \(a_{ij} = 0\) otherwise. Undirected graphs are allowed (\(A = A'\)) but self-relationships are excluded i.e. \(a_{ii} = 0\) for any \(i = 1, \ldots, N\). This is a typical and realistic assumption, e.g. social networks, see [66] and [47], among others. The network structure, equivalently the matrix \(A\), is assumed to be non-random. A row-normalized adjacency matrix is defined by \(W = \text{diag}\{n_1, \ldots, n_N\}^{-1} A\) where \(n_i = \sum_{j=1}^{N} a_{ij}\) is the so called out-degree, the total number of edges starting from the node \(i\). Then, \(W\) satisfies \(\|W\|_{\infty} = 1\) and \(W1 = 1\). Moreover, define \(e_i\) the \(N\)-dimensional unit vector with 1 in the \(i^{th}\) position and 0 everywhere else, such that \(w_i = e_i' W = (w_{i1}, \ldots, w_{iN})\) the \(i^{th}\) row of the matrix \(W\), with \(w_{ij} = a_{ij}/n_i\).

Define a \(N\)-dimensional vector of time series \(\{Y_t = (Y_{1,t}, \ldots, Y_{N,t})', t = 1, 2, \ldots, T\}\) which is observed on a given network; i.e. a univariate time series is measured for each node, with rate \(\lambda_{i,t}\). Denote by \(\{\lambda_i \equiv E(Y_t|F_{t-1}) = (\lambda_{1,t}, \ldots, \lambda_{i,t}, \ldots, \lambda_{N,t})', t = 1, 2, \ldots, T\}\), the corresponding conditional expectation vector, and denote the history of the process by \(F_t = \sigma(Y_s: s \leq t)\). Assume that \(\{Y_t: t \in \mathbb{Z}\}\) is integer-valued and consider the following first order nonlinear Poisson Network Autoregression (PNAR)

\[
Y_t = N_t(\lambda_t), \quad \lambda_t = f(Y_{t-1}, W, \theta),
\]

(1)

where \(\{N_t\}\) is a sequence of \(IID\) \(N\)-variate copula-Poisson process with intensity 1, counting the number of events in \([0, \lambda_{1,t}] \times \cdots \times [0, \lambda_{N,t}]\) and \(f(\cdot)\) is a deterministic function depending on the past lag values of the count process, the known network structure \(W\) and an \(m\)-dimensional parameter vector \(\theta\). Examples will be given below. More precisely, the conditional marginal probability distribution of the count variables is \(Y_{i,t}|F_{t-1} \sim \text{Poisson}(\lambda_{i,t})\), for \(i = 1, \ldots, N\), and the joint distribution is generated by a copula, which depends on a parameter \(\rho\), say \(C(\cdot, \rho)\) and it is imposed on waiting times of a Poisson process specified as in [7, Sec. 2.1]; See Supp. Mat. S-1.1. Several alternative models resembling multivariate Poisson distributions have been proposed in the literature; see [30, Sec. 2] for a discussion about the issues of available multivariate count distributions. A copula-based approach for the data generating process (henceforth DGP) is used throughout this paper. Results for higher order models are derived straightforwardly; see Remark 5.

Similar to the case of integer-valued time series, we define the nonlinear Network Autoregression (NAR) for continuous-valued time series by

\[
Y_t = \lambda_t + \xi_t, \quad \lambda_t = f(Y_{t-1}, W, \theta)
\]

(2)

where \(\xi_{i,t} \sim IID(0, \sigma^2)\), for \(1 \leq i \leq N\) and \(1 \leq t \leq T\) and \(\lambda_t = E(Y_t|F_{t-1})\).

Denote by \(X_{i,t} = n_i^{-1} \sum_{j=1}^{N} a_{ij} Y_{j,t}\) the network effect, i.e. the average impact of node \(i\)’s connections. Consider the partition of the parameter vector \(\theta = (\theta^{(1)}, \theta^{(2)})'\), where the vectors \(\theta^{(1)}\) and \(\theta^{(2)}\) are of dimension \(m_1\) and \(m_2\), respectively, such that \(m_1 + m_2 = m\). For \(t = 1, \ldots, T\), both (1)-(2) have element-wise components

\[
\lambda_{i,t} = f_i(X_{i,t-1}, Y_{i,t-1}; \theta^{(1)}, \theta^{(2)}), \quad i = 1, \ldots, N,
\]

(3)

where \(f_i(\cdot)\) is the \(i^{th}\) component of the function \(f(\cdot)\) depending on the specific model of interest, which can contain linear and nonlinear effects. In general, \(\theta^{(1)}\) will denote an \(m_1 \times 1\) vector associated...
with linear model parameters, whereas $\theta^{(2)}$ will denote the $m_2 \times 1$ vector of nonlinear parameters. Some examples are given below.

### 2.1 Examples

**Example 1.** Consider (2) and the first order linear NAR(1),

$$\lambda_{i,t} = \beta_0 + \beta_1 X_{i,t-1} + \beta_2 Y_{i,t-1},$$

(4)

which is a special case of (3), with $\theta^{(1)} = (\beta_0, \beta_1, \beta_2)'$. Model (4) was originally introduced by [72] for the case of continuous random variables $Y_t$, such that $Y_{i,t} = \lambda_{i,t} + \xi_{i,t}$. For each single node $i$, model (4) allows the conditional mean of the process to depend on the past of the variable itself, for the same node $i$, and the average of the other nodes $j \neq i$ by which the focal node $i$ is connected. Implicitly, only the nodes directly connected with the focal node $i$ can impact on the conditional mean process $\lambda_{i,t}$. This is reasonable assumption in many applications; for example, in the social network analysis, if the focal node $i$ does not follows a node $l$, so $a_{il} = 0$, the effect of the activity related to the latter do not affect the former. The parameter $\beta_1$ is called network effect, as it measures the average impact of the $i$'th node connections. The coefficient $\beta_2$ is called autoregressive effect because it determines the impact of the lagged variable $Y_{i,t-1}$. Model (4) has been extended to the case of count time series by [7]; it is called the linear PNAR(1) with $Y_{i,t} | \mathcal{F}_{t-1} \sim \text{Poisson}(\lambda_{i,t})$ for $i = 1, \ldots, N$ and the copula-based DGP, as described earlier.

**Example 2.** A nonlinear deviation of (4), when $Y_t$ takes integer values is given by

$$\lambda_{i,t} = \frac{\beta_0}{(1 + X_{i,t-1})^\gamma} + \beta_1 X_{i,t-1} + \beta_2 Y_{i,t-1},$$

(5)

where $\gamma \geq 0$. Clearly, (5) approaches a linear model for small values of $\gamma$, and $\gamma = 0$ reduces to the linear model (4). Instead, when $\gamma$ is larger than zero, (5) introduces a perturbation, deviating from the linear model (4). Hence, (5) is a special case of (3), with $\theta^{(1)} = (\beta_0, \beta_1, \beta_2)'$ and $\theta^{(2)} = \gamma$. Model (5) introduces a nonlinear drift in the intercept so that the baseline effect varies over time as a function of the network. If $Y_{i,t}$ counts activities of users in a social network (likes, reactions, etc.) and the community becomes more active, then the average magnitude of $X_{i,t-1}$ grows and thus the baseline for each node $i$ varies. When $Y_t \in \mathbb{R}^N$, the following model

$$\lambda_{i,t} = \frac{\beta_0}{(1 + |X_{i,t-1}|)^\gamma} + \beta_1 X_{i,t-1} + \beta_2 Y_{i,t-1},$$

(6)

is analogous to (5) but for continuous valued time series. To the best of our knowledge, we are not aware of any stability or inferential results for models (5)-(6) when $\{N, T_N\} \to \infty$. When $N = 1$ and $T \to \infty$, such non-linear models have been studied by [37, 33], among others.

**Example 3.** Another example of (3) is given by the Smooth Transition version of the NAR model, say STNAR(1),

$$\lambda_{i,t} = \beta_0 + (\beta_1 + \alpha \exp(-\gamma X_{i,t-1}^2))X_{i,t-1} + \beta_2 Y_{i,t-1},$$

(7)

where $\gamma \geq 0$; see [60] for an introduction to STAR models. This models introduces a smooth regime switching behavior of the network effect making it possible to vary smoothly from $\beta_1$ to $\beta_1 + \alpha$, as $\gamma$ varies from large to small values. When $\alpha = 0$ in (7), the linear NAR model (4) is obtained. Moreover, (7) is a special case of (3), with $\theta^{(1)} = (\beta_0, \beta_1, \beta_2)'$ and $\theta^{(2)} = (\alpha, \gamma)'$. In the case of univariate count time series see [31, 33], for more.
Example 4. Define the Threshold NAR model \((48)\), say TNAR(1), by

\[
\lambda_{i,t} = \beta_0 + \beta_1 X_{i,t-1} + \beta_2 Y_{i,t-1} + (\alpha_0 + \alpha_1 X_{i,t-1} + \alpha_2 Y_{i,t-1}) I(X_{i,t-1} \leq \gamma),
\]

where \(I(\cdot)\) is the indicator function and \(\gamma\) is the threshold parameter. When \(\alpha_0 = \alpha_1 = \alpha_2 = 0\), model \((8)\) reduces to the linear model \((4)\). In this case, \(\theta^{(1)} = (\beta_0, \beta_1, \beta_2)\) and \(\theta^{(2)} = (\alpha_0, \alpha_1, \alpha_2, \gamma)\) show that \((8)\) is a special case of \((3)\). In the case of univariate count time series see \([67, 22, 64, 14]\) for more.

Nonlinear functions, such as \((7)-(8)\), provide examples of switching models accounting for regime specific dynamics of the observed process. The switching mechanism depends on the network effect. For example, consider \(A\) as the network matrix connecting regional districts which share at least a border. Let \(Y_{i,t}\) denote the numbers of reported cases for some disease in each of these districts. Then, for each district \(i\), the historical average of neighbours \((X_{i,t-1})\) determines a switching effect, say from exponentially expanding pandemic to dying out pandemic (and vice versa). Note that for \((7)\), the network effect is regime-dependent but this can be modified suitably as in \((8)\). In conclusion, the dichotomy between STNAR and TNAR models is that the former accounts for smooth transitions while the latter models sudden changes; see \([59]\) for more on nonlinear modeling of time series.

### 2.2 Stability conditions for fixed network size

Set \(f(\cdot, W, \theta) = f(\cdot)\).

**Theorem 2.1.** Consider model \((1)\), with fixed \(N\). Define \(G = \mu_1 W + \mu_2 I\), where \(\mu_1, \mu_2\) are non-negative constants such that \(\rho(G) < 1\) and assume that for \(y, y^* \in \mathbb{N}^N\)

\[
|f(y) - f(y^*)|_{vec} \leq G |y - y^*|_{vec}.
\]

Then, the process \(\{Y_t, t \in \mathbb{Z}\}\) is stationary, ergodic and \(E|Y_{i,a}|^\alpha < \infty\) for any \(\alpha \geq 1\).

The parallel result for continuous variables is also established.

**Theorem 2.2.** Consider model \((2)\), with fixed \(N\). Define \(G = |\mu_1||W| + |\mu_2||I\), where \(\mu_1, \mu_2\) are real constants such that \(\rho(G) < 1\) and the contraction condition \((9)\) holds. Then, the process \(\{Y_t, t \in \mathbb{Z}\}\) is stationary ergodic with \(E|Y_t|_1 < \infty\). Moreover, if \(E|\xi_{i,a}^\alpha| < \infty\) for some \(\alpha \geq 8\), then \(E|Y_{i,a}|^\alpha < \infty\).

The proof of Theorems 2.1, 2.2 is given in Appendix A.1. Theorem 2.1 extends [7] Prop. 1, which was established for the linear PNAR model \((4)\). Theorem 2.2 similarly, extends [72] Thm. 1. In particular, existence of some moments for \(\{\xi_t: t \in \mathbb{Z}\}\) guarantees the conclusions of Theorem 2.2. Such assumption is not necessary in the linear case considered by [72]—see their eq. (2.1)—because of the assumed normality.

For each \(i = 1, \ldots, N\), the contraction condition \((9)\) follows by assuming that for \(x_i, x^*_i \in \mathbb{R}^+\) and \(y_i, y^*_i \in \mathbb{N}\)

\[
|f_i(x_i, y_i) - f_i(x^*_i, y^*_i)| \leq 1 \sum_{j=1}^N w_{ij}(y_j - y^*_j) + \mu_2 |y_i - y^*_i|
\]

because the left hand side of \((10)\) is bounded by \(1 \sum_{j=1}^N w_{ij}(y_j - y^*_j) + \mu_2 |y_i - y^*_i| \leq (\mu_1 w^i_1 + \mu_2 \epsilon^i_1)|y - y^*|_{vec}\), where \(\mu_1 w^i_1 + \mu_2 \epsilon^i_1 = \epsilon^i G\) is the \(i^{\text{th}}\) row of the matrix \(G\). Condition \((10)\) is verified element-wise. When the nonlinear functions \(f_i(\cdot)\) cannot be expressed in a vector form, e.g. \(f =
(f_1, \ldots, f_N)'$, verification of (10) is helpful; see (5)-(7). Moreover, the condition $\rho(G) < 1$ of Theorem 2.1 is implied by (10) when $\mu_1 + \mu_2 < 1$, because $\rho(G) \leq ||G||_\infty \leq \mu_1 ||W||_\infty + \mu_2 \leq \mu_1 + \mu_2$, since $||W||_\infty = 1$, by construction.

Theorem 2.2 follows again by (10) but with $|\mu_s|$, for $s = 1, 2$ and assuming that $|\mu_1| + |\mu_2| < 1$. Some illustrative examples are given below.

Example 1 (continued). For model (4), $\lambda_t = \beta_0 1 + G Y_{t-1}$, with $G = \beta_1 W + \beta_2 I$. In this case, the sharp condition $\rho(G) < 1$ is easily verifiable and, under (9), it implies the results of Theorem 2.1. However, the assumptions of the theorem are also satisfied by the set of sufficient conditions (10) with $\mu_1 = \beta_1$, $\mu_2 = \beta_2$ and $\beta_1 + \beta_2 < 1$, for integer-valued processes. For the continuous-valued case, a similar argument shows that $|\beta_1| + |\beta_2| < 1$.

Example 2 (continued). Consider model (5). By the mean value theorem (MVT)

$$|f(x_i, y_i) - f(x_i^*, y_i^*)| \leq \max_{x_i \in \mathbb{R}^+} \left| \frac{\partial f(x_i, y_i)}{\partial x_i} \right| |x_i - x_i^*| + \max_{y_i \in \mathbb{N}} \left| \frac{\partial f(x_i, y_i)}{\partial y_i} \right| |y_i - y_i^*|$$

where $\beta_i^* = \max \{|\beta_1, \beta_0 \gamma - \beta_1\}$. Theorem 2.1 holds with $G = \beta_1^* W + \beta_2 I$ and $\beta_1^* + \beta_2 < 1$. Similarly to model (5), by considering all the possible combinations of signs of $x, \beta_0$ and $\beta_1$ in model (6), we have $|\partial f(x_i, y_i)/\partial x_i| = |\beta_1 - \beta_0 \gamma/(1 + x_i)^{\gamma+1} x_i/|x_i|| \leq \beta_1 \equiv \max \{|\beta_1|, |\beta_0 \gamma - \beta_1|, |\beta_1 - \beta_0 \gamma\}|$. Theorem 2.2 holds with $G = \beta_1 W + |\beta_2| I$ and $\beta_1 + |\beta_2| < 1$.

Example 3 (continued). In the integer-valued case, Theorem 2.1 applies to model (7) with $G = (\beta_1 + \alpha) W + \beta_2 I$ and $\beta_1 + \alpha + \beta_2 < 1$, which coincides with the stationarity condition developed for the standard STAR model (60). By considering all the possible combinations of signs for $\beta_1$ and $\alpha$, it is not difficult to show that Theorem 2.2 is verified, for model (7), under the similar sufficient condition $\beta_1^* + |\beta_2| < 1$, where $\beta_1^* = \max \{|\beta_1|, |\beta_1 + \alpha|\}$.

Example 4 (continued). The threshold model (8) does not satisfy the contraction conditions (9)-(10). For the case of count data and $N$ fixed, we develop a different proof to show that $\{Y_t\}$ is stationary and ergodic provided that it has a positive conditional probability mass function and $\|G\|_1 < 1$, where $G = (\beta_1 + \alpha_1) W + (\beta_2 + \alpha_2) I$. Analogous result holds also for continuous data; see Supp. Mat. S-4.

2.3 Stability conditions for increasing network size

In this section, following the works by [72] and [7], we investigate the stability conditions of the process $\{Y_t \in E^N\}$, with $E = \mathbb{R}$ or $E = \mathbb{N}$, respectively, when the network size diverges ($N \to \infty$). We use a working definition of stationarity for increasing dimensional processes following [72 Def. 1]; see Supp. Mat. S-1.2

**Theorem 2.3.** Consider model (1) and $N \to \infty$. Define $G = \mu_1 W + \mu_2 I$, where $\mu_1, \mu_2 \geq 0$ are constants such that $\mu_1 + \mu_2 < 1$ and the contraction condition (9) holds, with $\max_{i \geq 1} f_i(0, 0) < \infty$. Then, there exists a unique strictly stationary solution $\{Y_t \in E^N, t \in \mathbb{Z}\}$ to the nonlinear PNAR model, with $\max_{i \geq 1} E |Y_{i,t}|^a \leq C_a < \infty$, for any $a \geq 1$. 

Theorem 2.4. Consider model (2) and $N \to \infty$. Define $G = |\mu_1| W + |\mu_2| I$, where $\mu_1$, $\mu_2$ are real constants such that $|\mu_1| + |\mu_2| < 1$ and the contraction condition (9) holds, with $\max_{i \geq 1} |f_i(0,0)| < \infty$. Then, there exists a unique strictly stationary solution $\{Y_t \in \mathbb{R}^N, t \in \mathbb{Z}\}$ to the nonlinear NAR model. In addition, if $\max_{i \geq 1} \mathbb{E}|\xi_{i,t}|^a \leq C_{\xi,a} < \infty$ for some $a \geq 8$, then $\max_{i \geq 1} \mathbb{E}|Y_{i,t}|^a \leq C_{a} < \infty$.

Theorems 2.3, 2.4 (whose proof is given in Appendices A.2 and A.3) extend the increasing network-type results of [7, Thm. 1] and [72, Thm. 2] to nonlinear versions of the PNAR and NAR models, respectively. For models (4)-(8), $\max_{i \geq 1} |f_i(0,0)| = \beta_0$. Moreover, the contraction condition (10), with $\mu_1 + \mu_2 < 1$ ($|\mu_1| + |\mu_2| < 1$), fulfills the conditions of Theorem 2.3 (Theorem 2.4), i.e. we obtain identical sufficient conditions which guarantee stationarity with fixed and diverging $N$. We emphasize again that (8) does not satisfy (9)-(10). This fact makes difficult to show stationarity, when $N$ increases. More importantly, all stability results do not depend on the network structure, as specified by the matrix $W$, and on the data generating process describing the joint dependence.

3 Quasi maximum likelihood inference

Consider model (3). Estimation for the unknown parameter vector $\theta$ is developed by means of QMLE. Define the quasi-log-likelihood function for $\theta$ by

$$l_{NT}(\theta) = \sum_{t=1}^{T} \sum_{i=1}^{N} l_{i,t}(\theta),$$

where $l_{i,t}(\theta)$ is the log-likelihood contribution of a single network node whose form depends on the type of data (discrete or continuous). Observe that (11) is not necessarily the true log-likelihood. The QMLE is denoted by $\hat{\theta}$ and maximizes (11). It is obtained by solving the system of equations $S_{NT}(\theta) = 0$, where

$$S_{NT}(\theta) = \frac{\partial l_{NT}(\theta)}{\partial \theta} = \sum_{t=1}^{T} s_{Nt}(\theta)$$

is the quasi-score function. Moreover define the following matrices

$$H_{NT}(\theta) = -\frac{\partial^2 l_{NT}(\theta)}{\partial \theta \partial \theta'},$$

$$B_{NT}(\theta) = \sum_{t=1}^{T} \mathbb{E}(s_{Nt}(\theta)s_{Nt}^t(\theta) | \mathcal{F}_{t-1}),$$

as the sample Hessian matrix and the conditional information matrix, respectively. Henceforth we drop the dependence on $\theta$ when a quantity is evaluated at the true value $\theta_0$.

3.1 Inference for PNAR models

Consider model (1). In this case the QMLE estimator, $\hat{\theta}$, maximizes

$$l_{NT}(\theta) = \sum_{t=1}^{T} \sum_{i=1}^{N} \left(Y_{i,t} \log \lambda_{i,t}(\theta) - \lambda_{i,t}(\theta) \right),$$

which is the log-likelihood obtained if all time series were contemporaneously independent. This simplifies computations allowing to establish consistency and asymptotic normality of the resulting
estimator. It is worth noting that the joint copula structure, say \( C(\cdot, \rho) \), with set of parameters \( \rho \), do not enter into the maximization problem of the working log-likelihood (14). However, this does not imply that inference does not take into account dependence among observations. The corresponding score function is given by

\[
S_{NT}(\theta) = \sum_{t=1}^{T} \sum_{i=1}^{N} \left( \frac{Y_{i,t}}{\lambda_{i,t}(\theta)} - 1 \right) \frac{\partial \lambda_{i,t}(\theta)}{\partial \theta} = \sum_{t=1}^{T} s_{NT}(\theta). \tag{15}
\]

Define \( \partial \lambda_i(\theta)/\partial \theta' \) the \( N \times m \) matrix of derivatives, \( D_1(\theta) \) the \( N \times N \) diagonal matrix with elements equal to \( \lambda_{i,t}(\theta) \), for \( i = 1, \ldots, N \) and \( \xi_t(\theta) = Y_t - \lambda_t(\theta) \) is a Martingale Difference Sequence (MDS) at \( \theta = \theta_0 \). Then, the empirical Hessian and conditional information matrices are given, respectively, by

\[
H_{NT}(\theta) = \sum_{t=1}^{T} \sum_{i=1}^{N} \left( \frac{Y_{i,t}}{\lambda_{i,t}(\theta)} \frac{\partial \lambda_{i,t}(\theta)}{\partial \theta} \right) - \sum_{t=1}^{T} \sum_{i=1}^{N} \left( \frac{Y_{i,t}}{\lambda_{i,t}(\theta)} - 1 \right) \frac{\partial^2 \lambda_{i,t}(\theta)}{\partial \theta \partial \theta'}, \tag{16}
\]

\[
B_{NT}(\theta) = \sum_{t=1}^{T} \frac{\partial \lambda_i(\theta)}{\partial \theta} D_1^{-1}(\theta) \Sigma_t(\theta) D_1^{-1}(\theta) \frac{\partial \lambda_i(\theta)}{\partial \theta'}, \tag{17}
\]

where \( \Sigma_t(\theta) = E(\xi_t(\theta) \xi_t(\theta) | F_{t-1}) \) is the conditional covariance matrix evaluated at \( \theta \). We impose the following standard assumptions:

A The parameter space \( \Theta \) is compact and the true value \( \theta_0 \) belongs to its interior.

B For \( i = 1, \ldots, N \) the function \( f_i(\cdot) \) is three times differentiable with respect to \( \theta \) and satisfies, for \( x_i, y_i^* \in \mathbb{R}_+ \) and \( y_i, y_i^* \in \mathbb{N} \)

\[
\left| \frac{\partial f_i(x_i, y_i, \theta)}{\partial \theta_g} - \frac{\partial f_i(x_i^*, y_i^*, \theta)}{\partial \theta_g} \right| \leq c_{1g} |x_i - x_i^*| + c_{2g} |y_i - y_i^*|, \quad g = 1, \ldots, m,
\]

\[
\left| \frac{\partial^2 f_i(x_i, y_i, \theta)}{\partial \theta_g \partial \theta_l} - \frac{\partial^2 f_i(x_i^*, y_i^*, \theta)}{\partial \theta_g \partial \theta_l} \right| \leq c_{1gl} |x_i - x_i^*| + c_{2gl} |y_i - y_i^*|, \quad g, l = 1, \ldots, m,
\]

\[
\left| \frac{\partial^3 f_i(x_i, y_i, \theta)}{\partial \theta_g \partial \theta_l \partial \theta_s} - \frac{\partial^3 f_i(x_i^*, y_i^*, \theta)}{\partial \theta_g \partial \theta_l \partial \theta_s} \right| \leq c_{1gls} |x_i - x_i^*| + c_{2gls} |y_i - y_i^*|, \quad g, l, s = 1, \ldots, m.
\]

Furthermore, \( \forall g, l, s, \max_{i \geq 1} |\partial f_i(0, 0, \theta)/\partial \theta_g| < \infty, \max_{i \geq 1} |\partial^2 f_i(0, 0, \theta)/\partial \theta_g \partial \theta_l| < \infty, \max_{i \geq 1} |\partial^3 f_i(0, 0, \theta)/\partial \theta_g \partial \theta_l \partial \theta_s| < \infty, \sum_g (c_{1g} + c_{2g}) < \infty, \sum_g (c_{1gl} + c_{2gl}) < \infty, \sum_{g,l,s} (c_{1gls} + c_{2gls}) < \infty \). In addition, the components of \( \partial f_i/\partial \theta \) are linearly independent.

C For \( i = 1, \ldots, N \), \( f_i(x_i, y_i, \theta) \geq C > 0 \), where \( C \) is a generic constant.

Such regularity conditions have been employed in the literature to guarantee consistency and asymptotic normality of the QMLE in the context of nonlinear time series models; see [58, Ch. 3], among others. We now give additional assumptions employed for developing inference when \( \{N, T_N\} \to \infty \) and the necessary network properties. Define

\[
H_N(\theta) = E \left[ \frac{\partial \lambda_i(\theta)}{\partial \theta} D_1^{-1}(\theta) \frac{\partial \lambda_i(\theta)}{\partial \theta'} \right], \tag{18}
\]
as, respectively, (minus) the expected Hessian matrix and the information matrix. Consider the following assumptions.

**H1** The process \{\xi_t, F_t : N \in \mathbb{N}, t \in \mathbb{Z}\} is \(\alpha\)-mixing with mixing coefficients \{\alpha(J)\}.

**H2** Define the standardized random process \(\hat{Y}_t = D_t^{-1/2}(Y_t - \lambda_t)\). There exists a non negative, non increasing sequence \{\(\varphi_h\)\}_{h=1,\ldots,\infty} such that \(\sum_{h=1}^{\infty} h \varphi_h = \Phi < \infty\) and, for \(i < j < k < l\), a.s.,

\[
\left| \text{Cov}(\hat{Y}_{i,t}, \hat{Y}_{j,t}, \hat{Y}_{k,t}, \hat{Y}_{l,t} \mid F_{t-1}) \right| \leq \varphi_{j-i}, \quad \left| \text{Cov}(\hat{Y}_{i,t}, \hat{Y}_{j,t}, \hat{Y}_{k,t} \mid F_{t-1}) \right| \leq \varphi_{k-j}, \quad \left| \text{Cov}(\hat{Y}_{i,t}, \hat{Y}_{j,t} \mid F_{t-1}) \right| \leq \varphi_{j-i}.
\]

**H3** For model (1) with network \(W\), the following limits exist, at \(\theta = \theta_0\):

1. \(\lim_{N \to \infty} N^{-1}H_N = H\), with \(H\) a \(m \times m\) positive definite matrix.
2. \(\lim_{N \to \infty} N^{-1}B_N = B\).
3. The third derivative of the quasi-log-likelihood is bounded by functions \(m_{i,t}\) which satisfy \(\lim_{N \to \infty} N^{-1} \sum_{i=1}^{N} E(m_{i,t}) = M\), where \(M\) is a finite constant.

Assumption **H1** is useful for studying processes with dependent errors. When \(N\) is fixed, a combination of Theorem 1-2 in [26] and Remark 2.1 in [24] shows that the process \{\xi_t : t \in \mathbb{Z}\}, is \(\alpha\)-mixing, with exponentially decaying coefficients, provided that \(||G||_1 < 1\). Analogous conclusion follow by [32, Prop. 3.1-3.4]. Condition **H2** represents a contemporaneous weak dependence assumption. Indeed, even in the simple case of the independence model, i.e. \(\lambda_{i,t} = \beta_0\), for all \(i = 1, \ldots, N\), the reader can easily verify that, without any further constraints, \(N^{-1}B_N = O(N)\), so the limiting variance of the QMLE diverges. Note that **H2** does not guarantee finiteness of the Hessian and information matrices, as \(N \to \infty\). Such requirement is imposed by Assumption **H3**. Obviously, such properties depend on the structure of \(W\) and on the functional form of \(f(\cdot)\) in (1); without the knowledge of these components it cannot be simplified any further. We present a detailed example involving the nonlinear PNAR model in Section 3.2 to offer further insight about **H3**. Proofs for all the following results are given in Supp. Mat. S-2.

**Lemma 3.1.** Consider model (1) with \(S_{NT}, H_{NT}\) and \(B_{NT}\) defined as in (15), (16) and (17), respectively. Let \(\theta \in \Theta \subset \mathbb{R}^m_+\). Suppose the conditions of Theorem 2.3, Assumption **B**, **C** and **H1-H3** hold. Then, as \(\{N, T_N\} \to \infty\)

1. \((NT_N)^{-1}H_{NT_N} \overset{p}{\to} H\),
2. \((NT_N)^{-1}B_{NT_N} \overset{p}{\to} B\),
3. \((NT_N)^{-\frac{1}{2}}S_{NT_N} \overset{d}{\to} N(0, B)\),
4. \(\max_{g,l,s} \sup_{\theta \in \Theta(0)} \left| \frac{1}{NT_N} \sum_{t=1}^{T_N} \sum_{i=1}^{N} \frac{\partial^3 l_{i,t}(\theta)}{\partial \theta_g \partial \theta_l \partial \theta_s} \right| \leq M_{NT_N} \overset{p}{\to} M\),
where $M_{NTN} := (NT)^{-1} \sum_{t=1}^{T_N} \sum_{i=1}^{N} m_{i,t}$ and $O(\theta_0) = \{ \theta : |\theta - \theta_0|_2 < \delta \}$ is a neighbourhood of $\theta_0$.

**Theorem 3.2.** For model (1), suppose that Assumption $[A]$ and conditions of Lemma 3.1 hold. Then, there exists a fixed open neighbourhood $O(\theta_0) = \{ \theta : |\theta - \theta_0|_2 < \delta \}$ of $\theta_0$ such that with probability tending to 1, as $\{N, T_N\} \to \infty$, the equation $S_{NTN}(\theta) = 0$ has a unique solution, denoted by $\hat{\theta}$, such that $\hat{\theta} \overset{p}{\to} \theta_0$ and $\sqrt{NTN} (\hat{\theta} - \theta_0) \overset{d}{\to} N(0, H^{-1}BH^{-1})$.

Thm 3.2 follows by Lemma 3.1 as proved by [7, Sec. S-3.3]. In addition, it extends the results of [7, Thm. 3] to nonlinear Poisson NAR models. The novelty of Theorem 3.2 is that both $N$ and $T$ tend to infinity as opposed to standard case (when $N$ is fixed). Additional conditions guarantee strong consistency of the estimators, i.e.

**Theorem 3.3.** If $T_N = \lambda N$, for some $\lambda > 0$ and Assumption $[H1]$ is such that the mixing coefficients satisfy $\alpha(J)^{1-1/r} = O(J^{-3-\epsilon})$, for some $r > 2$ and some $\epsilon > 0$, then, as $\{N, T_N\} \to \infty$, all the convergences “$\overset{p}{\to}$” in Lemma 3.1 are replaced by “$\overset{a.s.}{\to}$” and Theorem 3.2 holds with $\hat{\theta} \overset{a.s.}{\to} \theta_0$.

For instance, exponential decay of the mixing coefficients $\alpha(J)$ satisfies the assumption for any $r$ and $\epsilon$. Theorem 3.3 is a new result, to the best of our knowledge, as strong laws of large numbers for generally dependent double-indexed processes are scarce in the literature (for an exception, see [20]); see the discussion in [2, Com. 6] and [4, p. 256]. It is pointed out again that the proof of Theorem 3.2 does not depend on the specification of the data generating process for the joint dependence of $\{Y_t\}$.

### 3.2 A detailed example

We give a detailed discussion for proving Theorem 3.2 for the nonlinear PNAR model (5) case. Let $\Sigma_\xi = E[\xi_t \xi_t^\prime_{vec}]$ and $\lambda_{max}(X)$ the largest absolute eigenvalue of an arbitrary symmetric matrix $X$. Consider the vector form of model (5): $\lambda_t = \beta_0 \lambda_{t-1} + \xi_t = \lambda_{t-1} + \gamma_t$, where $\beta = \beta_1 W + \beta_2 I$ and $\lambda_{t-1} = (1 + X_{t-1})^{-1}$. Under the conditions of Theorem 2.3, and by using a infinite backward substitution argument on $Y_{t-1}$ we can rewrite the model as $Y_t = \mu_{t-1} + \lambda_t$, where $\mu_{t-1} = \beta_0 \sum_{j=0}^\infty G^j \lambda_{t-1-j}$ and $\lambda_t = \sum_{j=0}^\infty G^j \lambda_{t-j}$. The proof of such representation is given in Supp. Mat. S-2.3 Define the following quantities: $L_{t-1} = \log(1 + X_{t-1}), E_{t-1} = \sigma \sigma_{t-1} \odot \sigma_{t-1}, F_{t-1} = \log^2(1 + X_{t-1}) \odot \sigma_{t-1}, J_{t-1} = \sigma_2^2(1 + X_{t-1}) \odot \sigma_{t-1}$, where $\odot$ is the Hadamard product [57, Sec. 11.7]; $I_{1,t-1} = \sigma_{t-1}, I_{2,t-1} = \sigma_1 \sigma_1, I_{3,t-1} = \sigma_2 \sigma_2, \sigma_{t-1} = \sum_{i=1}^2 D_i^{-1}, \Gamma(0) = E[\sigma_1 (Y_{t-1} - \mu_{t-1}) (Y_{t-1} - \mu_{t-1})^\prime \sigma_1], \Delta(0) = E\left[\sigma_1 W (Y_{t-1} - \mu_{t-1}^\prime) (Y_{t-1} - \mu_{t-1}^\prime)^\prime W^\prime \sigma_1^\prime\right], \Gamma_{1,t-1} = I_{1,t-1}, \Gamma_{2,t-1} = I_{2,t-1}, \Gamma_{3,t-1} = Y_{t-1}, \Gamma_{4,t-1} = E_{t-1}$; moreover let $(\hat{\gamma}, \hat{\xi}, \hat{\kappa}) = \arg\max_{\gamma, \xi, \kappa} \left| \sum_{i=1}^N \sigma_i \partial_i \xi_i / \partial \theta_i \right|$, $\Pi_{j,k} = N^{-1} \sum_{i=1}^N E(\Gamma_{j,i,t-1} \Gamma_{i,k,t-1} / \lambda_{i,t})$, $\Pi_{F,k} = N^{-1} E(F_{j,t}^\prime D^{-1} F_{k,t-1})$, $\Pi_{J} = N^{-1} E(J_{j,t}^\prime D^{-1} i_{vec})$, for $j, l, k = 1, \ldots, 4$. Consider the following assumptions.

**Q1** Let $W$ be a sequence of matrices with non-stochastic entries indexed by $N$.

**Q1.1** Consider $W$ as a transition probability matrix of a Markov chain, whose state space is defined as the set of all the nodes in the network (i.e., $\{1, \ldots, N\}$). The Markov chain is assumed to be irreducible and aperiodic. Further, define $\pi = (\pi_1, \ldots, \pi_N)^\prime \in \mathbb{R}^N$ as the stationary distribution of the Markov chain, where $\pi \geq 0$, $\sum_{i=1}^N \pi_i = 1$ and $\pi = W^\prime \pi$. Furthermore, assume that $\lambda_{max}(\Sigma_\xi) \sum_{i=1}^N \pi_i^2 \to 0$ as $N \to \infty$. 

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Q1.2 Define $W^* = W + W'$ and assume $\lambda_{\text{max}}(W^*) = O(\log N)$ and $\lambda_{\text{max}}(\Sigma_\xi) = O((\log N)^{\delta})$, for some $\delta \geq 1$.

Q2 Assume that the following limits exist: $l_{kj}^B = \lim_{N \to \infty} N^{-1}E(I'_{k,t-1}N_{j,t-1})$, for $k,j = 1,\ldots,4$, $u_1^B = \lim_{N \to \infty} N^{-1}E(I'_{k,t-1}N_{j,t-1})$, $u_2^B = \lim_{N \to \infty} N^{-1}E[I'_{k,t-1}I_{j,t-1}]$, $u_3^B = \lim_{N \to \infty} N^{-1}E[I'_{k,t-1}I_{j,t-1}]$.

Theorem 3.4. Consider [5] and suppose the conditions of Theorem 2.8 Assumption A, C, H1, H2 and Q1, Q2 hold. Then, the conclusions of Theorem 3.2 hold true for model (5), with corresponding limiting matrices:

$$H = \left(\begin{array}{cccc}
    l_{11}^H & l_{12}^H & l_{13}^H & l_{14}^H \\
l_{22}^H + u_1^H & l_{23}^H + u_3^H & l_{24}^H & -\beta_0 v_{14} \\
l_{33}^H + u_3^H & l_{34}^H & -\beta_0 v_{24} \\
l_{44}^H & -\beta_0 v_{34} & \beta_0^2 v_{44} \\
\end{array}\right), \quad B = \left(\begin{array}{cccc}
l_{11}^B & l_{12}^B & l_{13}^B & l_{14}^B \\
l_{22}^B + u_1^B & l_{23}^B + u_3^B & l_{24}^B & -\beta_0 v_{14} \\
l_{33}^B + u_3^B & l_{34}^B & -\beta_0 v_{24} \\
l_{44}^B & -\beta_0 v_{34} & \beta_0^2 v_{44} \\
\end{array}\right),$$

where the elements of the Hessian matrix are obtained by the elements of the information matrix with $\Sigma_t = D_t$.

Remark 1. Clearly, the network structure influences the results of Theorem 3.4. Indeed, Assumption Q1 requires a well-behaved underlying network: i) there should exists a non-zero probability to connect each pair of nodes; this allows the network to converge to its stationary distribution, i.e. $\lim_{N \to \infty} W^N = 1p'$. ii) The growth of the network should be such that certain regularity properties hold. For instance, the covariances of the errors do not diverge fast, as $N \to \infty$. The proof in Supp. Mat. S-2.3 shows that the leading terms of Hessian and information matrices depend on the error component $\xi_{t-j}$ and the pseudo covariance matrix $\Sigma_\xi$ and are asymptotically negligible (compare also with [7, Lem. S-1]). In this way, the remaining terms appearing in Assumption Q2 show existence of the limiting Hessian matrix $H$ and (together with Assumption H2) of the limiting information $B$. Without any assumptions for the network, the structure of matrices $H$ and $B$ is unknown and conditions of finiteness of the limiting matrices could not be specified explicitly.

3.3 Inference for NAR models

In this case, define $\hat{\theta}$, as the maximizer of the least squares criterion

$$l_{NT}(\theta) = -\sum_{t=1}^{T} (Y_t - \lambda_t(\theta))^T (Y_t - \lambda_t(\theta)).$$

It follows that

$$S_{NT}(\theta) = \sum_{t=1}^{T} \frac{\partial \lambda_t'(\theta)}{\partial \theta} (Y_t - \lambda_t(\theta)) = \sum_{t=1}^{T} s_{NT}(\theta).$$
The empirical Hessian and information matrices are respectively

\[
H_{NT}(\theta) = \sum_{t=1}^{T} \sum_{i=1}^{N} \frac{\partial \lambda_{i,t}(\theta)}{\partial \theta} \frac{\partial \lambda_{i,t}(\theta)}{\partial \theta'} - \sum_{t=1}^{T} \sum_{i=1}^{N} (Y_{i,t} - \lambda_{i,t}(\theta)) \frac{\partial^{2} \lambda_{i,t}(\theta)}{\partial \theta \partial \theta'},
\]

(23)

\[
B_{NT}(\theta) = \sum_{t=1}^{T} \frac{\partial \lambda'_{i}(\theta)}{\partial \theta} \Sigma_{t}(\theta) \frac{\partial \lambda_{i}(\theta)}{\partial \theta'},
\]

(24)

where notation is as in Section 3.1. In addition

\[
H_{N}(\theta) = \text{E} \left( \frac{\partial \lambda'_{i}(\theta)}{\partial \theta} \frac{\partial \lambda_{i}(\theta)}{\partial \theta'} \right),
\]

(25)

\[
B_{N} = \text{E} \left( \frac{\partial \lambda'_{i}(\theta)}{\partial \theta} \Sigma_{t}(\theta) \frac{\partial \lambda_{i}(\theta)}{\partial \theta'} \right),
\]

(26)

and the latter equals \( \sigma^{2} H_{N} \), when \( \theta = \theta_{0} \), because \( \xi_{t} \) an is IID \((0, \sigma^{2})\) process. For the same reasons Assumption \( \text{H1}, \text{H2} \) hold trivially. Assumption \( \text{H3} \) is modified as

**H3’** For model (2) with network \( W \), the following limits exist, at \( \theta = \theta_{0} \):

**H3’1** \( \lim_{N \to \infty} N^{-1} H_{N} = H \), with \( H \) a \( m \times m \) positive definite matrix.

**H3’2** The third derivative of the quasi-log-likelihood (21) is bounded by functions \( m_{i,t} \) which satisfy \( \lim_{N \to \infty} N^{-1} \sum_{t=1}^{N} \text{E}(m_{i,t}) = M \), where \( M \) is a finite constant.

**Theorem 3.5.** Consider model (2) with \( S_{NT} \), \( H_{NT} \) and \( B_{NT} \) defined as in (22), (23) and (24), respectively. Let \( \theta \in \Theta \subset \mathbb{R}^{m} \). Suppose that the conditions of Theorem 2.4, Assumption A, B and \( \text{H3} \) hold. Then, there exists a fixed open neighbourhood \( O(\theta_{0}) = \{ \theta : |\theta - \theta_{0}|_{2} < \delta \} \) of \( \theta_{0} \) such that with probability tending to 1 as \( \{N, T_{N}\} \to \infty \), the equation \( S_{NT,N}(\theta_{0}) = 0 \) has a unique solution, denoted by \( \hat{\theta} \), such that \( \hat{\theta} \xrightarrow{p} \theta_{0} \) and \( \sqrt{NT_{N}}(\hat{\theta} - \theta_{0}) \xrightarrow{d} N(0, B^{-1}) \), where \( B = \sigma^{2} H \) and \( H \) is defined as in (20), with \( \sum_{t=1}^{N} \Sigma_{t} = D_{t} = I \).

The proof is omitted since it is analogous to the proof of Theorem 3.2. Theorem 3.5 generalises the results of [72] Thm. 3] to nonlinear NAR models, and it can be proved to entail results analogous to Proposition 3.4 by considering [72] Assumption C2] instead of [Q1] and [Q2] holding, with \( D_{t} = \Lambda_{t} = I \) and \( \Pi_{jkl} = 0 \), for \( j, k, l = 1, \ldots, 4 \). See [72] Thm. 3] for a detailed proof concerning the case of model (4). A result similar to Theorem 3.3 for model (2) is also established by setting \( T_{N} = \lambda N \).

**Remark 2.** Reiterating the discussion following Theorems 2.3, 2.4 and noting that Assumption B does not hold for (8), the double asymptotic based inference derived in this section and the associated testing theory (see Section 5) do not hold for threshold models when \( N \) is increasing. However, Supp. Mat. S-4 provides all these results for the threshold model if \( N \) is fixed.

**Remark 3.** The asymptotic theory of this section applies for parameter values satisfying the conditions of Theorems 2.3, 2.4. In practical applications, the QMLE is obtained using constrained optimization where the constraints satisfy such conditions. In the integer-valued case, additional constraints should be introduced so that the mean process is positive.
4 Hypothesis testing on network autoregressive models

With the same notation as in Sections 2 and 3, recall (3) and consider the following testing problems

\[ H_0 : \theta^{(2)} = \theta_0^{(2)} \quad \text{vs.} \quad H_1 : \theta^{(2)} \neq \theta_0^{(2)}, \quad \text{componentwise,} \quad (27) \]

against the Pitman’s local alternatives

\[ H_0 : \theta^{(2)} = \theta_0^{(2)} \quad \text{vs.} \quad H_1 : \theta^{(2)} = \theta_0^{(2)} + \frac{\delta_2}{\sqrt{NT}}, \quad \delta_2 \in \mathbb{R}^{m_2}. \quad (28) \]

To develop a test statistic for testing (27)-(28), we employ a quasi-score test based on (11). An appealing property of the score test is that it is computed under the null, which is computationally simpler. Moreover, the asymptotic distribution of the test is not affected when \( \theta^{(2)} \) belongs to the boundary of the parameter space. Define \( \tilde{\theta}^{(1)}, \tilde{\theta}^{(2)} \) the constrained quasi-likelihood estimator of \( \theta^{(1)}, \theta^{(2)} \), under the null hypothesis, and \( S_{NT}(\theta) = (S_{NT}^{(1)}(\theta), S_{NT}^{(2)}(\theta))' \) denote the corresponding partition of the quasi-score function. Because we study a quasi-score test, we correct the test statistic to obtain thoroughly its limiting distribution; see [35], among others. Accordingly the test statistic is given by (Supp. Mat. S-3)

\[ LM_{NT} = S_{NT}^{(2)}(\tilde{\theta}) \Sigma^{-1}_{NT}(\tilde{\theta}) S_{NT}^{(2)}(\tilde{\theta}). \quad (29) \]

Here \( (NT)^{-1} \Sigma_{NT}(\bar{\theta}) \) is a suitable estimator for the covariance matrix defined as \( \Sigma = \text{Var}[(NT)^{-1/2}S_{NT}^{(2)}(\bar{\theta})] \), where

\[ \Sigma = B_{22} - H_{21}H_{11}^{-1}B_{12} - B_{21}H_{11}^{-1}H_{12} + H_{21}H_{11}^{-1}B_{11}H_{11}^{-1}H_{12}, \quad (30) \]

with \( B_{gl}, H_{gl}, g, l = 1, 2 \) with dimension \( m_g \times m_l \), are blocks of the matrices \( H, B \) such that

\[ H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}. \]

If (11) is the true likelihood, then \( LM_{NT} \) reduces to the standard score test with \( B \equiv H \) and \( \Sigma = B_{22} - B_{21}B_{11}^{-1}B_{12} =: \Sigma_B \).

**Remark 4.** Following [7], the estimator \( \Sigma_{NT}(\bar{\theta}) \) of (29) is computed as the sample counterpart of (30), obtained by replacing the partitioned matrices \( H \) and \( B \) respectively by \( H_{NT}(\bar{\theta}) \) and \( B_{NT}(\bar{\theta}) \), where \( H_{NT}(\theta) \) is defined in (13) and \( B_{NT}(\theta) \) is the sample information matrix.

**A’** The parameter space \( \Theta \) is compact. Define the partition of the parameter space \( \Theta^{(1)} \) such that \( \theta^{(1)} \in \Theta^{(1)} \) and the true value \( \theta_0^{(1)} \) belongs to its interior.

**Theorem 4.1.** Suppose that model (3) admits a stationary solution, for \( N \to \infty \). Consider \( l_{NT}, S_{NT}, H_{NT} \) and \( B_{NT} \) defined by (11), (12) and (13), respectively. Assume that, under \( H_0 \), [A] is satisfied such that, as \( \{N, T_N\} \to \infty \), Lemma 3.1 and Theorem 3.2 holds. Recall the testing problem (27). Then, as \( \{N, T_N\} \to \infty \), the quasi-score test statistic (29) converges to a chi-square random variable,

\[ LM_{NTN} \xrightarrow{d} \chi^2_{m_2}. \]
under $H_0$. Moreover, under the alternative \textsuperscript{28}, \textsuperscript{29} converges to a non-central chi-square random variable,
\[ LM_{NTN} \xrightarrow{d} \chi^2_{m_2}(\delta^2_2\bar{\Delta}), \]
where $\bar{\Delta} = \bar{\Sigma}_H \bar{\Sigma}_H^{-1} \bar{\Sigma}_H$ and $\Sigma_H := H_{22} - H_{21} H_{11}^{-1} H_{12}$; $\bar{\Sigma}$ and $\bar{\Sigma}_H$ are sample counterparts of $\Sigma$ and $\Sigma_H$, respectively, evaluated at $\hat{\theta}$.

Theorem 4.1 extends the results of \textsuperscript{13} for the case of multivariate discrete and continuous network autoregressive models with infinite dimensional data. In addition, it implies that even though $\theta^{(2)}$ belongs to the boundary of the parameter space, the asymptotic $\chi^2$ distribution remains unaffected. Instead, the asymptotic distribution of the Wald and likelihood ratio tests depends on the null hypothesis and do not converge to $\chi^2$ distributed when $N$ is fixed; see \textsuperscript{35} Sec. 8.3.2 and \textsuperscript{1}. We illustrate some applications of Theorem 4.1 to the network models (1)-(2) but we emphasize that its conclusion applies to more general settings.

**Proposition 4.2.** Assume $Y_t$ follows \textsuperscript{1} and the process $\lambda_t$ is defined as in (3). Consider the test $H_0 : \theta^{(2)} = \theta_0^{(2)}$ vs. $H_1 : \theta^{(2)} \neq \theta_0^{(2)}$. Then, under $H_0$, $A^1$ and the conditions of Lemma 3.1, Theorem 4.1 is true.

Proposition 4.2 follows by Lemma 3.1, Theorems 2.3 and 4.1.

### 4.1 A detailed example (continued)

For model \textsuperscript{5}, the linearity test \textsuperscript{27} is equivalent to testing $H_0 : \gamma = 0$ versus $H_1 : \gamma > 0$. Convergence for all necessary asymptotic quantities is required only under the null. Recall the notation of Section 3.2. Then, under $H_0$, $C_{t-1} = 1$, so the decomposition of the count process simplifies to $Y_t = \mu + \sum_{j=0}^{\infty} G^j \xi_t - j$, since $\mu_{t-1} = \mu = \beta_0 / (1 - \beta_1 - \beta_2)^{-1}$. This entails that $\Gamma_t = 1, \Gamma_{4t-1} = L_{t-1}, F_t = \log^2(1 + X_{t-1})$ and $J_t = \log^2(1 + X_{t-1})$. Moreover, set $\Lambda = E(\Lambda'_t \Lambda_t), \Gamma(0) = E[\Lambda_t (Y_{t-1} - \mu)(Y_{t-1} - \mu)\Lambda'_t]$ and $\Delta(0) = E[\Lambda_t W(Y_{t-1} - \mu)(Y_{t-1} - \mu)\Lambda'_t W\Lambda'_t]$. So condition Q2 simplifies as follows:

Q2' Assume that the following limits exist: $f_1 = \lim_{N \to \infty} N^{-1}(1' \Lambda_1), f_2 = \lim_{N \to \infty} N^{-1} \text{tr} [\Gamma(0)], f_3 = \lim_{N \to \infty} N^{-1} \text{tr} [W \Gamma(0)], f_4 = \lim_{N \to \infty} N^{-1} \text{tr} [\Delta(0)], \hat{v}^B_{k4} = \lim_{N \to \infty} N^{-1} E(\Gamma'_{k4t-1} \Lambda'_t \Lambda_t L_{t-1}), d^* = \lim_{N \to \infty} \Pi_{j^*, l^*, k^*}$. If at least two indices among $(j^*, l^*, k^*)$ equal 4, $d^*_F = \lim_{N \to \infty} \Pi_{F, s^*}$, where $s^* = j^*, l^*, k^*$. Moreover, if all three indices $(j^*, l^*, k^*)$ equal 4, $d^*_J = \lim_{N \to \infty} \Pi_{J}$.

In this case, the limiting Hessian and information matrices in (20) are equal to the respective matrices obtained by the linear model fitting \textsuperscript{7}, Eq. 22 plus the addition of the fourth row and column whose elements are given by $(-\beta_0)^{\nu} \hat{v}^B_{k4}$, for $k = 1, \ldots, 4$ and $\nu = 2$, when $k = 4$ and $\nu = 1$, otherwise.

**Proposition 4.3.** Assume $Y_t$ follows \textsuperscript{1} and the process $\lambda_t$ is defined as in (3). Suppose the conditions of Theorem 2.3, Assumptions A1, B1, H1 and Q1, Q2' hold. Consider the test $H_0 : \gamma = 0$ vs. $H_1 : \gamma > 0$. Then, Theorem 4.1 holds true.
Denoting the constrained QMLE by \( \tilde{\theta} = (\tilde{\theta}^{(1)}, 0)' \), where \( \tilde{\theta}^{(1)} \) is the QMLE of the linear model (1), the partial quasi-score (29) is given by \( S_{NT}^{(2)}(\tilde{\theta}) = \sum_{t=1}^{T} \sum_{i=1}^{N} (Y_{i,t}/\lambda_{i,t}(\tilde{\theta}) - 1) \partial \lambda_{i,t}(\tilde{\theta})/\partial \gamma \), with \( \partial \lambda_{i,t}(\tilde{\theta})/\partial \gamma = -\hat{\beta}_0 \log(1 + X_{i,t-1}) \), where \( \hat{\beta}_0 \) is the QMLE of the intercept \( \beta_0 \) in the linear model (1). Furthermore, the covariance estimator \( \Sigma_{NT}(\tilde{\theta}) \) for the test statistic (29) is defined as in Remark 4. Note that in Proposition 4.3, the nonlinear perturbation is due to the network structure. Moreover, since the asymptotic distribution of the score test (29) depends on the convergence of sample Hessian and information matrices to (20), the approximation to the chi-square distribution depends by the convergence of the network according to the regularity properties given by Q1 and information matrices to (20). The approximation to the chi-square distribution depends on the sample Hessian and information matrices convergence to (20). Therefore, the approximation to the chi-square distribution depends on the convergence of the network according to the regularity properties given by Q1 and information matrices convergence to (20).

Analogous result and conclusions are obtained for (2), by using Thm. 2.4, 3.5 and 4.1, and therefore it is omitted. Consider the following condition:

**Q2'** For \( Y_t \) defined as in (2) and \( \lambda_t \) following (6), Assumption Q2 holds, with \( D_t = \Lambda_t = I \) and \( \Pi_{jlk} = 0 \), for \( j, k, l = 1, \ldots, 4 \).

**Proposition 4.4.** Assume \( Y_t \) follows (2) and the process \( \lambda_t \) is defined as in (6). Suppose the conditions of Theorem 2.4. Assumptions A'\(B\)H1H2 [72, Assumption C2] and Q2' hold. Consider the test \( H_0 : \gamma = 0 \) vs \( H_1 : \gamma > 0 \). Then, Theorem 4.1 holds true.

### 5 Testing under non identifiable parameters

We develop testing theory when the parameters are not identifiable under the linearity hypothesis. A case in point is model (7), with \( \theta^{(1)} = (\beta_0, \beta_1, \beta_2)' \) and \( \theta^{(2)} = (\alpha, \gamma)' \). Then testing \( H_0 : \alpha = 0 \), makes \( \gamma \) non-identifiable but the score partition (15) and consequently the test statistic still depends on the value of \( \gamma \). Hence the theory of Sec. 4 does not apply any more. Similar remarks hold for the threshold parameter \( \gamma \) of model (8), when testing \( H_0 : \alpha_0 = \alpha_1 = \alpha_2 = 0 \). Assigning a fixed arbitrary value for \( \gamma \) resolves such issues but this approach might lack power as the test is sensitive to the choice of \( \gamma \), especially when \( \gamma \) is far from its true value. It is well known (see [59, Sec. 5.1,5.5]) that testing linearity is an important issue because non-identifiable parameters have tremendous impact on properties of estimators. Usually a sup-type test, say \( g_{NT} = \sup_{\gamma \in \Gamma} LM_{NT}(\gamma) \), is employed in applications, where \( \Gamma = [\gamma_L, \gamma_U] \) is a compact domain for \( \gamma \); e.g. [16; 34] and [14, Par. 3.2].

Define \( Z \) a random variable, and suppose that the function \( g(\cdot) : \Gamma \rightarrow \mathbb{R} \) is continuous with respect to the uniform metric, monotonic for each \( \gamma \), and such that, as \( Z \rightarrow \infty \), then \( g(Z) \rightarrow \infty \) in a subset of \( \Gamma \) with a non-zero probability \( \mathbb{P} \). For the standard asymptotics, i.e. \( T \rightarrow \infty \), such functions have been employed in applications. Examples include \( g_T = g(LM_T) \) [42] and [6] who considered \( g(LM_T) = \int_{\Gamma} LM_T(\gamma)d\mathbb{P}(\gamma) \) and \( g(LM_T) = \log(\int_{\Gamma} \exp(1/2LM_T(\gamma))d\mathbb{P}(\gamma)) \). We extend this theory to the case of both \( T, N \rightarrow \infty \).

#### 5.1 Specification

In this section, we use a more convenient notation. Accordingly, consider the nonlinear PNAR model defined in [1] as

\[
Y_t = N_t(\lambda_t(\gamma)), \quad \lambda_t(\gamma) = Z_{1t}(W)\beta + h(Y_{t-1}, W, \gamma)\alpha, \tag{31}
\]

where \( \beta \) is a \( k_1 \times 1 \) vector of identifiable parameters associated with the linear component of the model, \( \alpha \) is a \( k_2 \times 1 \) vector of identifiable non-linear parameters and \( \gamma \) denote nuisance parameters.
We set \( \theta = (\phi', \gamma')', \phi = (\beta', \alpha')' \). With this notation, the dimension of \( \theta \) is \( m = k + m^* \), where \( m^* \) is the dimension of \( \gamma \) and \( k = k_1 + k_2 \). In addition, \( Z_{it}(W) = (1, WY_{i,t-1}, Y_{i,t-1}) \) is a \( N \times k_1 \) matrix associated to the linear part of the network autoregressive model (for the order 1 model \( k_1 = 3 \)), and \( h(Y_{i,t-1}, W, \gamma) \) is a \( N \times k_2 \) matrix describing the nonlinear part of the model. Set \( Z_{it} = Z_{it}(W), h_t(\gamma) = h(Y_{i,t-1}, W, \gamma) \) and \( h_t(\gamma) = (h^1_t(\gamma) \ldots h^b_t(\gamma) \ldots h^k_t(\gamma)) \), where each column indicates a nonlinear regressor \( h^b_t(\gamma) \), for \( b = 1, \ldots, k_2 \), being a \( N \times 1 \) vector whose elements are \( h^b_{i,t}(\gamma) \), where \( i = 1, \ldots, N \). Then, the conditional expectation of (31) is \( \lambda_t(\gamma) = Z_{it}(\gamma)\phi \) where \( Z_t(\gamma) = (Z_{it}, h_t(\gamma)) \) is the \( N \times k \) matrix of regressors. Analogously, for continuous-valued time series and \( \xi_t \sim \text{IID}(0, \sigma^2) \), equation (2) becomes

\[
Y_t = \lambda_t(\gamma) + \xi_t, \quad \lambda_t(\gamma) = Z_{it}(W)\beta + h(Y_{i,t-1}, W, \gamma)\alpha.
\]

Many nonlinear models are included in this general frameworks provided by (31)-(32); for example, the STNAR model (7), where \( k_2 = 1 \) and \( h_{i,t}(\gamma) = \exp(-\gamma X^2_{i,t-1})X_{i,t-1} \), for \( i = 1, \ldots, N \), and the TNAR model (8), where \( k_2 = 3 \) and \( h_{i,t}^1(\gamma) = I(X_{i,t-1} \leq \gamma), h_{i,t}^2(\gamma) = X_{i,t-1} I(X_{i,t-1} \leq \gamma) \) and \( h_{i,t}^3(\gamma) = Y_{i,t-1} I(X_{i,t-1} \leq \gamma) \); see [42, p. 414].

### 5.2 Testing linearity

For models (31)-(32), consider testing linearity in the presence of a non identifiable parameters \( \gamma \)

\[
H_0 : \alpha = 0, \quad \text{vs.} \quad H_1 : \alpha \neq 0, \quad \text{elementwise.}
\]

Consider first the case of count time series, i.e. eq. (31). In this case, the score (15), Hessian (16), and the sample information matrix (17), for the quasi-log-likelihood (14) are \( S_{NT}(\gamma) = \sum_{t=1}^{N} s_{Nt}(\gamma), \ H_{NT}(\gamma_1, \gamma_2) = \sum_{t=1}^{T} \sum_{i=1}^{N} Y_{i,t} \bar{Z}_{i,t}(\gamma_1) \bar{Z}_{i,t}'(\gamma_2), B_{NT}(\gamma_1, \gamma_2) = \sum_{t=1}^{T} E[s_{it}(\gamma_1)s_{it}'(\gamma_2)|F_{t-1}] \) where \( s_{Nt}(\gamma) = \bar{Z}_{i,t}'(\gamma_1)D^{-1}_1(\gamma)(Y_t - Z_t(\gamma)\phi) \) and \( \bar{Z}_{i,t}(\gamma) = Z_{i,t}(\gamma)/\lambda_{i,t}(\gamma) \). The theoretical counterpart of such quantities are then denoted by \( H_N(\gamma_1, \gamma_2) = \sum_{i=1}^{N} E[Y_{i,t} \bar{Z}_{i,t}(\gamma_1) \bar{Z}_{i,t}'(\gamma_2)], H(\gamma_1, \gamma_2) = \lim_{N \to \infty} N^{-1} H_N(\gamma_1, \gamma_2), \) and \( B_N(\gamma_1, \gamma_2) = E[s_{i,t}(\gamma_1)s_{i,t}'(\gamma_2)], B(\gamma_1, \gamma_2) = \lim_{N \to \infty} N^{-1} B_N(\gamma_1, \gamma_2) \).

Following the discussion of Section 4 the quasi-score function is partitioned again in two components: the part concerning linear parameters and the component associated with the nonlinear part of the model. We denote this by \( S_{NT}(\gamma) = (S^{(1)}_{NT}, S^{(2)}_{NT}(\gamma))' \). Moreover, consider \( S(\gamma) = (S^{(1)}(\gamma), S^{(2)}(\gamma))' \) a mean zero Gaussian process with covariance kernel \( B(\gamma_1, \gamma_2) \). Define the matrix \( \Sigma(\gamma_1, \gamma_2) \) as in (30), with partitioned matrices \( B_{gl}, H_{gl} \), for \( g, l = 1, 2 \), of dimension \( k_g \times k_l \), being blocks of the matrices \( B(\gamma_1, \gamma_2), H(\gamma_1, \gamma_2) \), with obvious rearrangement of the notation. Then, \( S^{(2)}(\gamma) \) is a Gaussian process with covariance kernel \( \Sigma(\gamma_1, \gamma_2) \). Define \( \tilde{\phi} = (\tilde{\beta}', \tilde{\alpha}') \) the constrained estimator under the null hypothesis and use the tilde notation for all quantities which correspond to constrained QMLE. Then, for testing (33) we consider the test statistic

\[
LM_{NT}(\gamma) = \tilde{S}^{(2)}_{NT}(\gamma) \bar{\Sigma}^{-1}_{NT}(\gamma, \gamma) \tilde{S}^{(2)}_{NT}(\gamma),
\]

where, according to Remark 4, \( \bar{\Sigma}_{NT}(\gamma, \gamma) \) is the estimator of \( \Sigma(\gamma, \gamma) \), obtained by substituting \( H(\gamma, \gamma), B(\gamma, \gamma) \) with \( \bar{H}_{NT}(\gamma, \gamma), \bar{B}_{NT}(\gamma, \gamma) \), respectively.

Define \( Z_{i,t} = (1, X_{i,t-1}, Y_{i,t-1})' \) and \( \eta_{NT} = N^{-1/2} \sum_{i=1}^{N} Y_{i,t}(Z_{i,t}'Z_{i,t} - 1) + X_{1,i,t} + Y_{1,i,t} \). An extra condition is required:
B' Assumption $\mathbf{B}$ holds with all constants not depending on $\gamma \in \Gamma$, where $\Gamma$ is compact, and $\|\eta_{NT}\|_q < \infty$, for some $q > \max \{1 + \delta, m^*\}$, with $0 < \delta < 1$.

Assumption $\mathbf{B'}$ is similar to assumption $\mathbf{B}$ for the particular case we consider. An extra moment assumption is required to guarantee stochastic equicontinuity of the score. It can be easily shown that a sufficient condition for obtaining $\|\eta_{NT}\|_q < \infty$ would be, for example, the weak dependence condition $|E(Y_{it}^r Y_{jt}^s | F_{t-1})| \leq \phi_{j-i}$, such that $\sum_{i=1}^{\infty} \phi_i^{1/r} < \infty$, where $r = q/2$, if $q$ is even, and $r = (q + 1)/2$, if $q$ is odd. For instance, in the STNAR model $\eta$, $m^* = 1, q = 2$ and $r = 1$, so the condition simplifies to a special case of Assumption $\mathbf{H2}$. From $\mathbf{B'}$ Assumption $\mathbf{C}$ holds trivially for $\mathbf{(31)}$, because a.s. $\lambda_{it}(\gamma) \geq \beta_0 + h'_t(0, \gamma)\alpha = C > 0$, for $i = 1, \ldots, N$. Define $\delta_2 \in \mathbb{R}^k_+$ and $J_2 = (O_{k_2 \times k_1}, I_{k_2})$, where $I_s$ is a $s \times s$ identity matrix and $O_{a \times b}$ is a $a \times b$ matrix of zeros.

**Theorem 5.1.** Assume $Y_t$ is integer-valued, following $\mathbf{(31)}$ and suppose the conditions of Theorem $\mathbf{2.3}$ Assumption $\mathbf{A'}, \mathbf{B'}$ and $\mathbf{H1}, \mathbf{H3}$ hold. Consider the test $H_0 : \alpha = 0$ vs. $H_1 : \alpha > 0$, componentwise. Then, under $H_0$, as $\{N, T_N\} \to \infty$, $S_{NT_N}(\gamma) \Rightarrow S(\gamma)$, $LM_{NT_N}(\gamma) \Rightarrow LM(\gamma)$ and $g_{NT_N} \Rightarrow g = g(LM(\gamma))$ where

$$LM(\gamma) = S^{(2)}(\gamma)\Sigma^{-1}(\gamma, \gamma)S^{(2)}(\gamma).$$

Moreover, the same result holds under local alternatives $H_1 : \alpha = (NT_N)^{-1/2}\delta_2$, with $S^{(2)}(\gamma)$ having mean $J_2H^{-1}(\gamma, \gamma)J_2\delta_2$.

Theorem 5.1 the proof is given in Appendix A.4 extends [42] Thm. 1 in three directions: i) develops testing for NAR models; ii) proves convergence to asymptotic process, where both time and network dimension diverge together; iii) the results holds for both continuous-valued data (see below) and integer-valued multivariate random variables. In line with Section 4.1, for each single model encompassed in $\mathbf{(31)}$ one can substitute $\mathbf{H3}$ with network conditions $\mathbf{Q1}$ and suitable limits existence as in $\mathbf{Q2'}$. An analogous result holds for continuous valued time series, as in $\mathbf{(32)}$. Its proof is omitted. Consider $s_t(\gamma) = Z'_t(\gamma)\xi_t$ and $H_T(\gamma_1, \gamma_2) = \sum_{t=1}^T \sum_{i=1}^N Z_{i,t}(\gamma_1)Z'_{i,t}(\gamma_2)$. In this case no additional weak dependence assumption is required since the error sequence is independent.

$\mathbf{B''}$ Assumption $\mathbf{B}$ holds with all constants not depending on $\gamma \in \Gamma$, where $\Gamma$ is compact.

**Theorem 5.2.** Assume $Y_t$ is continuous-valued, following $\mathbf{(32)}$ and suppose the conditions of Theorem $\mathbf{2.4}$ Assumptions $\mathbf{A'}, \mathbf{B'}$ and $\mathbf{H3}$ hold. Consider the test $H_0 : \alpha = 0$ vs. $H_1 : \alpha \neq 0$, componentwise. Then, the results of Theorem 5.1 hold true.

**Remark 5.** The results of this paper extend straightforwardly to the case of model order $p > 1$, i.e. $\lambda_t = f(Y_{t-1}, \ldots, Y_{t-p}, W, \theta)$. Indeed, the proof of stability conditions of Theorem 2.1 and 2.2 are based on the fact that the process $\{Y_t : t \in \mathbb{Z}\}$ is a first order Markov chain. All proofs adapt directly to a Markov chain of generic order $p$, by suitable adjustment of the contraction property $\mathbf{(9)}$. Similar remark holds for asymptotic properties of the QMLE and Theorems 3.2, 5.2 by a suitable extension.

### 5.3 Computations of $p$-values

The null distribution of the process $g(\cdot)$ cannot be tabulated in general, apart from special cases; $\mathbf{[5]}$. To overcome this obstacle we consider two different approaches. Consider the sup-type test,
\[ g = \sup_{\gamma \in \Gamma}(LM(\gamma)) \].

By Theorems \[5.1, 5.2\] under \( H_0 \), \( LM(\gamma) \) is a chi-square process with \( k_2 \) degrees of freedom. If the nuisance parameter \( \gamma \) is scalar, \[16\] proves that the \( p \)-value of the sup-test is approximately bounded by

\[
P\left( \sup_{\gamma \in \Gamma} (LM(\gamma)) \geq M \right) \leq P\left( \chi^2_{k_2} \geq M \right) + VM^{\frac{1}{2}(k_2-1)} \exp\left(-\frac{M}{2}\right) 2^{-k_2} \frac{\Gamma\left(\frac{k_2}{2}\right)}{\Gamma\left(\frac{k_2}{2}\right)},
\]

where \( M \) is the maximum of the test statistic \( LM_{NT}(\gamma) \), with \( \gamma \in \Gamma_F \) and \( \Gamma_F = (\gamma_L, \gamma_1, \ldots, \gamma_I, \gamma_U) \) is a grid of values for \( \Gamma \). The quantity \( V \) is the approximated total variation, defined by

\[ V = \left| LM_{NT}^{1}(\gamma_1) - LM_{NT}^{1}(\gamma_L) \right| + \cdots + \left| LM_{NT}^{1}(\gamma_U) - LM_{NT}^{1}(\gamma_I) \right|.
\]

Such method is attractive because of its simplicity and its computational speed. This last point is of great importance in network models, especially when the dimension \( N \) is large. However, the method suffers from three main drawbacks. First, \[35\] leads to a conservative test, because usually the \( p \)-values are smaller than their bound. Second, the results of \[16\] hold only for scalar nuisance parameters. Though this observation applies to several models discussed so far, like the STNAR model \[7\], more complex models may require inclusion of more than one nuisance parameter. Finally, \[35\] cannot be applied to the TNAR model \[8\], because \( LM(\gamma) \), under the null hypothesis, has to be differentiable \[16\, p. 36], \[42\, Sec. 4\]. Following \[42\], we develop a bootstrap method based on stochastic permutations.

Define \( F(\cdot) \) the distribution function of the process \( g \) with \( p_{NT} = 1 - F(g_{NT}) \). From Theorem 5.1 and the Continuous Mapping Theorem (CMT), \( p_{NT} \Rightarrow p \), where \( p = 1 - F(g) \) and \( p \sim U(0,1) \), under the null. Hence, the test rejects \( H_0 \) if \( p_{NT} \leq a_{H_0} \), where \( a_{H_0} \) is the asymptotic size of the test. Define \( \{\nu_t : t = 1, \ldots, T\} \sim IIDN(0,1) \), such that \( \hat{S}_{NT}^{\nu}(\gamma) = \sum_{t=1}^T \hat{s}_{NT}^{\nu}(\gamma) \), with \( \hat{s}_{NT}^{\nu}(\gamma) = \hat{s}_{NT}(\gamma)\nu_t \), is the version of the estimated score perturbed by a Gaussian noise. Similarly, the perturbed score test is defined by \( LM_{NT}^{\nu} = \hat{S}_{NT}^{\nu}(\gamma) \hat{S}_{NT}^{\nu}(\gamma) \hat{S}_{NT}^{\nu}(\gamma) \) and \( g_{NT} = g(LM_{NT}^{\nu}) \). Finally, \( \hat{p}_{NT} = 1 - \hat{F}_{NT}(g_{NT}) \) is the approximation of \( p \)-values obtained by stochastic permutations, where \( \hat{F}_{NT}(\cdot) \) denotes the distribution function of \( g_{NT} \), conditional to the available sample. The following result shows that such a bootstrap approximation provides adequate approximation to the null distribution:

**Theorem 5.3.** Assume the conditions of Theorems \[3.3, 5.1\] hold. Then, \( \hat{p}_{NT} - p_{NT} = o_p(1) \) and \( \hat{p}_{NT} \Rightarrow p \). Moreover, under \( H_0 \), \( \hat{p}_{NT} \overset{d}{\rightarrow} U(0,1) \).

The proof of this theorem is given in Appendix A.5. An analogous result is obtained for continuous-valued network models and it is omitted. Although \( \hat{p}_{NT} \) is close to \( p_{NT} \) asymptotically, the conditional distribution \( \hat{F}_{NT}(\cdot) \) is not observed. We can approximate this by Monte Carlo simulations following (i)-(iv) of \[42\, p. 419\]. A Gaussian sequence \( \{\nu_{t,j} : t = 1, \ldots, T\} \sim IIDN(0,1) \) is generated and at each iteration, compute the quantities \( \hat{S}_{NT}^{\nu}(\gamma), LM_{NT}^{\nu}(\gamma) \) and \( \hat{g}_{NT}^j = g(LM_{NT}^{\nu}(\gamma)) \), for \( j = 1, \ldots, J \). Hence, an approximation of the \( p \)-values is obtained by \( \hat{p}_{NT}^J = J^{-1} \sum_{j=1}^J I(\hat{g}_{NT}^j \geq g_{NT}) \). The Glivenko-Cantelli Theorem implies that \( \hat{p}_{NT}^J \xrightarrow{P} \hat{p}_{NT} \), as \( J \to \infty \), and choosing \( J \) large enough allows to make \( \hat{p}_{NT}^J \) arbitrary close to \( \hat{p}_{NT} \).

The proposed bootstrap methodology provides a direct approximation of the \( p \)-values instead of an approximate bound, given by \[35\]. Furthermore, it is suitable even when testing linearity in the presence of more than one nuisance parameter. As a final remark, the stochastic permutation bootstrap method has been preferred instead of parametric bootstrap as it requires only the generation
of standard univariate normal sequences at each step. This reduces considerably the computational burden of generating a $N \times 1$ vector of observations at each step of the procedure. This is especially relevant in the case of count data, since the simulation of copula can be time consuming.

**Remark 6.** Following up Remark 2, note that the previous results do not apply to TNAR model $(8)$, if $N \to \infty$. The stochastic equicontinuity and uniform convergence assumptions require $h_t(\gamma)$ to be continuous with respect to $\gamma$ which is not satisfied for $(8)$. For instance, when trying to establish stochastic equicontinuity of the score, it can be proved that for $(8)$, the Lipschitz property $(36)$ can be obtained in expectation but with magnitude $\lambda = 1/(2q)$. However, to establish the result of Theorem 5.1 we need $\lambda > m^*/q$ [43, p. 357]. This can happen only when $m^* < 1/2$ but for the TNAR model $m^* = 1$, so the condition is not satisfied. Supp. Mat. S-4 provides properties, estimation and testing for TNAR models when $N$ is fixed.

### 6 Simulations

We provide two different cases for the network generating mechanism to verify empirically the above results. Additional results are reported in Supp. Mat. S-5.

**Example N-1.** (Stochastic Block Model (SBM)). First consider the stochastic block model, see [55] and [51], among others. A block label ($k = 1, \ldots, K$) is assigned for each node with equal probability and $K$ is the total number of blocks. Then, set $P(a_{ij} = 1) = N^{-0.3}$ if $i$ and $j$ belong to the same block, and $P(a_{ij} = 1) = N^{-1}$ otherwise. Practically, the model assumes that nodes within the same block are more likely to be connected with respect to nodes from different blocks. We assume $K \in \{2, 5\}$.

**Example N-2.** (Erdős-Rényi (ER) Model). Introduced by [28] and [38], this graph model is simple. The network is constructed by connecting $N$ nodes randomly. Each edge is included in the graph with probability $p = P(a_{ij} = 1) = N^{-0.3}$.

Consider testing $H_0 : \gamma = 0$ versus $H_1 : \gamma > 0$, for models (5) and (6). Under $H_0$, the model reduces to (4). For the continuous-valued case, we test linearity of the NAR against the nonlinear version in (6). The random errors $\xi_{i,t}$ are simulated from $N(0, 1)$. For the data generating process of the vector $Y_t$, the initial value $Y_0$ is randomly simulated according to its stationary distribution [72, Prop. 1], which is Gaussian with mean $\mu = \beta_0(1 - \beta_1 - \beta_2)^{-1}$ and covariance matrix $\text{vec}[\text{Var}(Y_t)] = (I_{N^2} - G \otimes G)^{-1}\text{vec}(I)$, where $\text{vec}(.)$ denotes the vec operator and $\otimes$ denotes the Kronecker product. We set $\theta^{(1)} = (\beta_0, \beta_1, \beta_2)' = (1.5, 0.4, 0.5)'$. This procedure is replicated $S = 1000$ times. Then, $\hat{\theta}^{(1)}$ is computed for each replication. By Proposition 4.4 the quasi-score statistic (29) is evaluated and compared with the critical values of a $\chi^2_1$ distribution. Results of this simulation study are reported in Table 1. The empirical size of the test does not exceed the nominal significance level in all cases considered. When $N$ is small and $T$ is large enough, the power of the test statistics tends to 1. In the case of small temporal size $T$ and large network dimension $N$, the approximation suffers. This is expected and is explained by i) the double asymptotic results of Section 4 hold when $T_N \to \infty$ as $N \to \infty$—see the proof of Lemma 3.1 ii) the temporal dependence induced by the error term requires a sufficiently large $T$ for successful model identification; iii) the quasi-likelihood might not approximate the true likelihood, see also [7, Sec. 4.1]. When both $N, T$ are large enough, the test approximates adequately its asymptotic distribution. As expected,
when $\gamma = 1$, the test statistic’s power improves, because $\gamma$ is far from 0. Improved performance of the test statistic is observed when either $K = 5$ or when the Erdős-Rényi model is employed. Histograms and Q-Q plots of the simulated score test against the $\chi^2_1$ distribution are plotted in Supp. Mat. Fig. S-2. For all the network models, the histogram is positively skewed and approximates satisfactorily the $\chi^2_1$ distribution. The Q-Q plots lie into the confidence bands quite satisfactorily and the empirical mean and variance of the simulated score tests are close to 1 and 2, respectively. Further simulations results for the integer-valued case can be found in Supp. Mat. S-5 together with simulation results related to the non-identifiable case.

Table 1: Empirical size and power of the test statistics (29) for testing $H_0 : \gamma = 0$ versus $H_1 : \gamma > 0$, in model (6), with $S = 1000$ simulations, for various values of $N$ and $T$. Data are continuous-valued and generated from the linear model (4).

| Model | $K$ | $N$ | $T$ | 10% | 5%  | 1%  | 10% | 5%  | 1%  | 10% | 5%  | 1%  |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| SBM 2 | 4   | 500 | 0.093 | 0.043 | 0.009 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 | 1.000 |
|       | 500 | 10  | 0.019 | 0.004 | 0.000 | 0.158 | 0.063 | 0.002 | 0.164 | 0.067 | 0.001 |
|       | 200 | 300 | 0.110 | 0.044 | 0.006 | 0.495 | 0.337 | 0.125 | 0.994 | 0.990 | 0.933 |
|       | 500 | 300 | 0.101 | 0.048 | 0.009 | 0.716 | 0.583 | 0.288 | 1.000 | 1.000 | 0.995 |
|       | 500 | 400 | 0.105 | 0.050 | 0.006 | 0.751 | 0.619 | 0.311 | 1.000 | 1.000 | 0.999 |
| SBM 5 | 10  | 500 | 0.119 | 0.062 | 0.015 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
|       | 200 | 300 | 0.091 | 0.051 | 0.006 | 0.667 | 0.542 | 0.268 | 1.000 | 1.000 | 1.000 |
|       | 500 | 300 | 0.098 | 0.047 | 0.006 | 0.847 | 0.748 | 0.448 | 1.000 | 1.000 | 1.000 |
|       | 500 | 400 | 0.086 | 0.039 | 0.006 | 0.885 | 0.807 | 0.541 | 1.000 | 1.000 | 1.000 |
| ER   - | 30  | 500 | 0.066 | 0.029 | 0.004 | 0.272 | 0.156 | 0.048 | 0.888 | 0.802 | 0.565 |
|       | 500 | 30  | 0.026 | 0.005 | 0.000 | 0.392 | 0.235 | 0.044 | 0.935 | 0.847 | 0.523 |
|       | 200 | 300 | 0.085 | 0.031 | 0.004 | 0.411 | 0.272 | 0.080 | 0.974 | 0.949 | 0.798 |
|       | 500 | 300 | 0.082 | 0.042 | 0.004 | 0.649 | 0.476 | 0.192 | 0.999 | 0.998 | 0.974 |
|       | 500 | 400 | 0.089 | 0.051 | 0.008 | 0.666 | 0.519 | 0.206 | 1.000 | 1.000 | 0.992 |

7 Empirical example

We discuss an example of the testing methods for integer data. For an example concerning continuous data see Supp. Mat. S-5.3. The dataset consists of monthly number of burglaries on the south side of Chicago from 2010-2015, i.e. $T = 72$ and $N = 552$ census block groups of Chicago; see [15], https://github.com/nick3703/Chicago-Data. To predict future number of burglaries, the ordinary Vector Autoregressive (VAR) model can be applied but we should take into account that data are counts and dimensionality issues because the number of VAR parameters is large compared to the sample size. A simple method, like fitting AR(1) models separately to each individual census blocks, is applicable but still requires $2N$ parameters to be fitted. More crucially, the relationship across different time series is not taken into account. To overcome such issues, we appeal to geographic network information between blocks to fit a PNAR model which takes into account dependence among count valued data. An undirected network structure is defined by
geographical proximity: two blocks are connected if they share at least a border. The density of this network is 1.74%. The median number of connections is 5. The QMLE is employed for fitting linear PNAR model (4). The results are summarized in Table 2. The magnitude of the network effect $\beta_1$ shows that an increasing number of burglaries in a block can lead to a growth in the same type of crime committed in a neighborhood area. The effect of the lagged variable has an upwards impact on the number of burglaries, as well. We evaluate the out-of-sample forecasting performance of the linear PNAR(1) model versus a baseline AR(1) model fitted separately to each individual census block. We evaluate the one step ahead forecast by computing its Root Mean Square Error (RMSE). The RMSE for the PNAR model is 0.038. This is considerably smaller than the RMSE obtained by the AR(1) models (which is 0.167). In conclusion, the PNAR model gives significant accuracy improvement of the one-step prediction and at the same time achieves parsimony. We apply now the proposed linearity tests. A quasi-score linearity test is computed according to (29), by using the asymptotic chi-square test, for the nonlinear model (5), testing $H_0 : \gamma = 0$ vs. $H_1 : \gamma > 0$. We also test linearity against the presence of smooth transition effects, as in (7), with $H_0 : \alpha = 0$ vs. $H_1 : \alpha > 0$. A grid of 100 equidistant values in an interval of values $\Gamma_F = [\gamma_L, \gamma_U]$ is selected for the nuisance parameter $\gamma$, where the extremes are defined as in [63, p. 9]. According to the results of Theorem 5.1, the p-values are computed with the Davies bound approximation (35) for the test $\sup_{\gamma \in \Gamma_T} LM_T(\gamma)$ as well as through bootstrap approximation procedure. The number of bootstrap replications is $J = 299$. Finally, a linearity test against threshold effects, as in (8), is also performed, which leads to the test $H_0 : \alpha_0 = \alpha_1 = \alpha_2 = 0$ vs. $H_1 : \alpha_l > 0$, for some $l = 0, 1, 2$. A feasible range values for the non identifiable threshold parameter has been considered as in [63, p. 11]. From Table 2, the linearity test against (5) is rejected at standard levels. This gives an intuition for possible nonlinear drifts in the intercept. Davies bound gives evidence in favour of STNAR effects at 5% level. Conversely, bootstrap sup tests reject nonlinearity coming from both smooth (7) and abrupt transitions (8) models. We conclude that there is no clear evidence of regime switching effect.

Table 2: QMLE estimates of the linear model (4) for Chicago burglaries counts. Standard errors in brackets. Linearity is tested against the nonlinear model (5), with $\chi^2$ asymptotic test (29); against the STNAR model (7), with p-values computed by (DV) Davies bound (35), bootstrap p-values of sup-type test; and versus TNAR model (8).

| Models | $\beta_0$ | $\beta_1$ | $\beta_2$ |
|--------|----------|----------|----------|
| (4)    | 0.455    | 0.322    | 0.284    |
| SE     | (0.022)  | (0.013)  | (0.008)  |

| Models | Chi-sq. | DV | Bootstrap |
|--------|---------|----|-----------|
| (5)    | 8.999   | -  | -         |
| (7)    | -       | 0.038 | 0.515 |
| (8)    | -       | -  | 0.498     |
A. Appendix

A.1 Proof of Theorem 2.1

Consider the $N \times 1$ Markov chain $Y_t = F(Y_{t-1}, N_t)$ where $\{N_t, t \in \mathbb{Z}\}$ defined in (1) is a sequence of IID $N$-dimensional count processes such that $N_{i,t}$, for $i = 1, \ldots, N$, are Poisson processes with intensity 1. $F(\cdot)$ is a measurable function such that $F(y, N_t) = (N_t[y])$ and $f(\cdot)$ is defined in (1) for $y \in \mathbb{N}^N$. By [9], $f(y) \leq C + Gy$, where $C = f(0)$, we have

$$E[F(y, N_1)]_1 = 1'f(y) \leq 1'[C + Gy] < \infty$$

since the expectation of the Poisson process is $E N_1(\lambda) = \lambda$. Moreover, for $y, y^* \in \mathbb{N}^N$

$$E[F(y, N_1) - F(y^*, N_1)]_{vec} \leq G\|y - y^*\|_{vec}$$

as $E[N_1(\lambda_1) - N_1(\lambda_2)]_{vec} = |\lambda_1 - \lambda_2|_{vec}$. Note that $\rho(G) < 1$, Therefore, by [21] Thm. 1, $\{Y_t, t \in \mathbb{Z}\}$ is a stationary and ergodic process with $E|Y_t|_1 < \infty$. Now set $\delta > 0$ such that $\rho(G_\delta) < 1$, if $G_\delta = (1 + \delta)G$. From [21] Lemma 2 we have that

$$\|N_t[f(y)]\|_{a, vec} \leq (1 + \delta)\|f(y)\|_{vec} + b1 \leq C_{db} + G_\delta|y|_{vec}$$

by recalling that $\|f(y)\|_{vec} \leq C + G|y|_{vec}$ and $\rho(G_\delta) < 1$, where $b > 0$ and $C_{db} = (1 + \delta)C + b$. Then, by [21] Thm. 1 we get $E|Y_t|^a < \infty, \forall a \geq 1$. Theorem 2.2 follows analogously.

A.2 Proof of Theorem 2.3

For any arbitrary $N$, $E(Y_t) = E(\lambda_t) = E[f(Y_{t-1})] \leq c1 + GE(Y_{t-1})$, by [9], where $\max_{i \geq 1} f_i(0, 0) = c > 0$. Define $\mu = c(1 - \mu_1 - \mu_2)^{-1}$. Note that $\rho(G) \leq |||G|||_{\infty} \leq \mu_1|||W|||_{\infty} + \mu_2 \leq \mu_1 + \mu_2$. This is so because $|||W|||_{\infty} = 1$, by construction. Since $\mu_1 + \mu_2 < 1$ we have $|||G|||_{\infty} < 1$ and by [57] 19.16(a) $(I - G)^{-1}$ exists. Moreover, $(I - G)^{-1}1 = (1 - \mu_1 - \mu_2)^{-1}1$, implying that $E(Y_t) \leq \mu_1$ and $\max_{i \geq 1} E(Y_{i,t}) \leq \mu$. It holds that $\xi_t = Y_t - \lambda_t$, $E|\xi_t| \leq 2E(Y_{i,t}) \leq 2\mu < \infty$. Furthermore, by using backward substitution and (9), we have $Y_t \leq \mu_1 + \sum_{j=0}^{\infty} G_j \xi_{t-j} = \sum_{j=0}^{\infty} G_j (c1 + \xi_{t-j})$.

From the definition in Supp. Mat. S-1.2 we have that $\mathcal{W} = \{\omega \in \mathbb{R}^\omega : \omega_\infty = \sum |\omega_i| < \infty\}$, where $\omega = (\omega_i) = \sum_{1 \leq i \leq \infty} \omega_i \in \mathbb{R}^\infty$. For each $\omega \in \mathcal{W}$, let $\omega_N = (\omega_1, \ldots, \omega_N) \in \mathbb{R}^N$ be its truncated $N$-dimensional version. For any $\omega \in \mathcal{W}$, $E[c1 + \xi_t]_{vec} \leq (c + 2\mu)1 = C1 < \infty$, $G_{j1} = (\mu_1 + \mu_2)^j$ and $E|\omega_{N,y}| \leq E(|\omega_{N,y}|_{vec} \sum_{j=0}^{\infty} G_j (c1 + \xi_{t-j})) \leq C\omega_\infty \sum_{j=0}^{\infty} (\mu_1 + \mu_2)^j = C_\infty$, since $\mu_1 + \mu_2 < 1$, which implies that $Y_t^{\omega} = \lim_{N \to \infty} \omega_N Y_t < \infty$ with probability 1. Moreover, $Y_t^{\omega}$ is strictly stationary and therefore $\{Y_t\}$ is strictly stationary, following Supp. Mat. S-1.2. To verify the uniqueness of the solution take another stationary solution $Y_t^*$ to the PNAR model. Then $E|\omega_{N,y} Y_t - \omega_{N,y} Y_t^*| \leq \omega_{N,y} \leq E|N_t(\lambda_t - N_t(\lambda_t^*))|_{vec} \leq |\omega_{N,y} |||GE|||_{vec} Y_{t-1} - Y_{t-1}^{*}|_{vec} = 0$, by infinite backward substitution, for any $N$ and weight $\omega$. So $Y_t^{\omega} = Y_t^{\omega,\omega}$ with probability one. In addition, $\mu_1 w_i^t + \mu_2 e_i^t = e_i^t G$ and condition (9) is equivalent to require, for $i = 1, \ldots, N$, a.s.

$$|\lambda_{i,t} - \lambda_{i,t}^*| = e_i^t |f(Y_{t-1}) - f(Y_{t-1}^*)|_{vec} \leq (\mu_1 w_i^t + \mu_2 e_i^t) |Y_{t-1} - Y_{t-1}^{*}|_{vec}$$

$$= \mu_1 \sum_{j=1}^{N} w_{ij} |Y_{j,t-1} - Y_{j,t-1}^{*}| + \mu_2 |Y_{i,t-1} - Y_{i,t-1}^{*}|$$

25
which leads a.s. to \( \lambda_{i,t} = f_i(X_{i,t-1}, Y_{i,t-1}) \leq c + \mu_1 X_{i,t-1} + \mu_2 Y_{i,t-1} \). Then, when \( N \) is increasing, [7 Prop. 2] applies directly by a recursion argument [7 Sec. S-1.1] and all moments of the process \( \{Y_t\} \) are uniformly bounded. \qed

### A.3 Proof of Theorem 2.4

Similar to [A.1], by assuming that \( \max_{i \geq 1} E |\xi_{i,t}|^a \leq C \xi,a < \infty \), the first \( a \)th moments of \( Y_t \) are uniformly bounded. By (9), \( |Y_t|_{vec} \leq \sum_{j=0}^{\infty} G(\|c_1 + |\xi_{t-j}|_{vec}\|) \), where \( c = \max_{i \geq 1} |f_i(0,0)| \). Analogously to [A.2], since \( G^1 = (|\mu_1| + |\mu_2|)^q 1 \) and \( |\mu_1| + |\mu_2| < 1 \), \( \{Y_t\} \) defined as in (2), is strictly stationary, following [72 Def. 1]. The uniqueness of the solution follows by \( |Y_t - Y^*_t|_{vec} = |\lambda_t - \lambda^*_t|_{vec} \) and the infinite backward substitution argument. \qed

### A.4 Proof of Theorem 5.1

First we show the weak convergence of \( (NT)^{-1/2} S_{NT}(\gamma) \) to a Gaussian process with kernel \( B(\gamma_1, \gamma_2) \). For all non-null \( \eta \in \mathbb{R}^k \), consider the triangular array \( s_{NT}^*(\gamma) = \eta' (N^{-1/2} \sum_{i=1}^N s_{i,t}(\gamma)) \). By Assumption \( \Gamma \) is compact, and by the continuity of the score \( s_{NT}^*(\gamma) \) is compact. Note that \( s_{NT}^*(\gamma) \) is a martingale difference array. So by the results of Lemma 3.1 the multivariate pointwise central limit theorem and \( (NT)^{-1} B_{NT}(\gamma_1, \gamma_2) \) establish the finite dimensional convergence. It remains to show the stochastic equicontinuity, i.e. ([13 Thm. 2])

\[
|s_{NT}^*(\gamma) - s_{NT}^*(\gamma^*)| \leq \delta_{NT} |\gamma - \gamma^*|^{\lambda}, \tag{36}
\]

a.s. with \( \|\delta_{NT}\|_q < \infty \) and \( \|s_{NT}^*(\gamma)\|_q < \infty \), where \( q \geq 2 \) and \( \lambda \) such that \( q > m^*/\lambda \). By [7 Sec. S-6] and Assumption \( [H1][H3] \|s_{NT}^*(\gamma)\|_4 < \infty \). For \( q > 4 \) a similar result can be obtained following the arguments of [68 Rem. 2.3] by requiring higher order covariances in Assumption \( [H2] \). To prove (36) we recall the following uniform bounds. By Assumption \( [B'] \) for \( i = 1, \ldots, N \), \( |\partial f_i(x_i, y_i, \theta)/\partial \alpha_b| = h^b_i(\gamma) \leq c_b + c_{10} x_i + c_{2b} y_i \) for \( b = 1, \ldots, k \) and \( l = 1, \ldots, m^* \), where \( c_b = h^b_i(0,0) \) \( \forall \gamma \in \Gamma \). Let \( C, C_0, C_1, C_2 > 0 \) be generic constants varying from place to place, which do not depend on \( \gamma \). Then, a.s. \( |h_{i,t}(\gamma)|_1 \leq C_0 + C_1 X_{i,t-1} + C_2 Y_{i,t-1} \). Similar bounds hold for \( \lambda_{i,t}(\gamma) \) and \( |Z_{i,t}(\gamma)|_1 \). By Theorem 2.3 all the moments of the Poisson process \( Y_t \) exist as well as those associated to the error \( \xi_t(\gamma) = Y_t - \lambda_t(\gamma) \). This fact and the multinomial theorem imply that every moment of all the previously defined random variables is uniformly bounded. Define \( h_{i,t}(0,0) = c, \forall \gamma \in \Gamma \), and \( h^*_{i,t}(\gamma) = h_{i,t}(\gamma) - h_{i,t}(0,0) \). For \( i = 1, \ldots, N \), and MVT \( |h_{i,t}(\gamma) - h_{i,t}(\gamma^*)|_1 = |h^*_{i,t}(\gamma) - h^*_{i,t}(\gamma^*)|_1 \leq |\partial h^*_{i,t}(\gamma)/\partial \gamma|_1 |\gamma - \gamma^*|_1 \leq A_{i,t-1} |\gamma - \gamma^*|_1 \) a.s. where \( A_{i,t-1} = C_1 X_{i,t-1} + C_2 Y_{i,t-1} \), \( \tilde{\gamma} \) are intermediate points between \( \gamma_l \) and \( \gamma^*_l \), for \( l = 1, \ldots, m^* \), and the last inequality holds by Assumption \( [B'] \) since \( |\partial h^*_{i,t}(\gamma)/\partial \gamma| = |\partial^2 f_i(x_i, y_i, \theta)/\partial \alpha \partial \gamma| - |\partial^2 f_i(0,0, \theta)/\partial \alpha \partial \gamma| \). For all \( \gamma, \gamma^* \in \Gamma \), standard algebra and previous bounds show that a.s. \( |s_{NT}^*(\gamma) - s_{NT}^*(\gamma^*)| \leq \delta_{NT} |\gamma - \gamma^*|_1 \) where \( \delta_{NT} = C/\sqrt{N} \sum_{i=1}^N A_{i,t-1} (1 + C_0 X_{i,t} + C_1 Y_{i,t} + C_2 Y_{i,t-1}) \) proving (36) with \( \lambda = 1 \). By Assumption \( [B'] \) \( \|\delta_{NT}\|_q < \infty \) since \( \delta_{NT} \leq C \eta_{NT} \), then \( (NT)^{-1/2} S_{NT}(\gamma) = T_N^{-1/2} \sum_{t=1}^{T_N} N^{-1/2} s_{NT}(\gamma) \) is stochastically equicontinuous and, as \( \{N,T_N\} \to \infty \), \( (NT)^{-1/2} S_{NT}(\gamma) \to S(\gamma) \).

We now prove uniform convergence of \( \hat{\Sigma}_{NT}(\gamma_1, \gamma_2) \) by showing stochastic equicontinuity for Hessian and information matrices. For all \( \eta \in \mathbb{R}^k, \eta \neq 0 \), consider the triangular array \( b_{NT}(\gamma_1, \gamma_2) = \eta' (N^{-1} B_{NT}(\gamma_1, \gamma_2)) \eta \) where \( B_{NT}(\gamma_1, \gamma_2) \) is the single summand of \( B_{NT}(\gamma_1, \gamma_2) \). Define \( \rho_{ij}(\gamma_1, \gamma_2) = \eta' (\gamma_1, \gamma_2) \eta \).
Rewriting in matrix form we have

\[ |b_{Nt}(\gamma_1, \gamma_2) - b_{Nt}(\gamma_1^*, \gamma_2^*)| \]

\[ \leq \eta' \left( \frac{1}{N} \sum_{i,j=1}^{N} Z_{i,t}(\gamma_1) \rho_{ijt}(\gamma_1, \gamma_2) Z_{j,t}^*(\gamma_2) \right) \eta \]

\[ \leq C \sum_{r=1}^{5} D_r, \]

and the inequality follows since for a matrix \( M, \eta' M \eta \leq |\eta|_1 |M|_1 \), and by \( \lambda_{i,t}(\gamma) \geq C \forall \gamma \in \Gamma \). The elements \( D_r \) are obtained by consecutive addition and subtraction. We focus on one element (say \( D_1 \)), the other terms are treated analogously. Some tedious algebra shows that, a.s.

\[ |\rho_{ijt}(\gamma_1, \gamma_2) - \rho_{ijt}(\gamma_1^*, \gamma_2^*)| \]

\[ \leq \left| \lambda_{i,t}^{\frac{1}{2}}(\gamma_1) - \lambda_{i,t}^{\frac{1}{2}}(\gamma_1^*) \right| |\rho_{ijt}(\gamma_1^*, \gamma_2^*)| \lambda_{j,t}^{\frac{1}{2}}(\gamma_2) + \left| \lambda_{j,t}^{\frac{1}{2}}(\gamma_2) - \lambda_{j,t}^{\frac{1}{2}}(\gamma_2^*) \right| |\rho_{ijt}(\gamma_1, \gamma_2)| \lambda_{i,t}^{\frac{1}{2}}(\gamma_1) \]

\[ \leq C_1 |\lambda_{i,t}(\gamma_1) - \lambda_{i,t}(\gamma_1^*)| \varphi_{-i} A_{j,t}^{*} - C_2 |\lambda_{j,t}(\gamma_2) - \lambda_{j,t}(\gamma_2^*)| \varphi_{-j} A_{i,t}^{*} \]

\[ \leq C_1 \varphi_{-i} \tilde{A}_{i,j,t} - 1 |\gamma_1 - \gamma_1^*|_1 + C_2 \varphi_{-j} \tilde{A}_{j,i,t} - 1 |\gamma_2 - \gamma_2^*|_1 \]

where \( A_{i,t}^{*} = A_{i,t} + C_0 \) and \( \tilde{A}_{i,j,t} = A_{i,t} - A_{i,t}^{*} \). The first inequality follows by addition and subtraction. The second inequality is a consequence of Assumption \( \text{H2} \) and \( \sqrt{|x - y|} = (|x| + |y|) / (|x| + |y|) \); the third is due to Lipschitz continuity of \( h_{i,t}(\gamma) \). Let us define \( \pi_{ijt}(\gamma, \gamma^*) = \left| \sqrt{\lambda_{i,t}(\gamma_1^*) \lambda_{j,t}(\gamma_2^*)} Z_{i,t}(\gamma_1) Z_{j,t}^*(\gamma_2) \right| \leq \pi_{ijt} \) a.s. with the inequality coming from previous uniform bounds where \( \pi_{ijt} \) is a linear combination of \( X_{i,t} \) and \( Y_{i,t} \) not depending on \( \gamma \). Then,

\[ D_1 = \frac{1}{N} \sum_{i,j=1}^{N} |\rho_{ijt}(\gamma_1, \gamma_2) - \rho_{ijt}(\gamma_1^*, \gamma_2^*)| \pi_{ijt}(\gamma, \gamma^*) \]

\[ \leq C_1^{*} \sum_{i,j=1}^{N} \varphi_{-i} \tilde{A}_{i,j,t} - 1 \pi_{ijt} |\gamma_1 - \gamma_1^*|_1 + C_2^{*} \sum_{i,j=1}^{N} \varphi_{-j} \tilde{A}_{j,i,t} - 1 \pi_{ijt} |\gamma_2 - \gamma_2^*|_1 . \]

This shows that \( |b_{Nt}(\gamma_1, \gamma_2) - b_{Nt}(\gamma_1^*, \gamma_2^*)| \leq b_{1,N} |\gamma_1 - \gamma_1^*|_1 + b_{2,N} |\gamma_2 - \gamma_2^*|_1 \) a.s. with \( b_{s,N} \) defined by obvious notation, not depending on \( \gamma \) and such that \( E(b_{s,N}^*) < \infty \), for \( s = \{1, 2\} \). Rewriting in matrix form we have \( b_{s,N}^* = \eta'(N^{-1} B_{s,Nt}(\gamma_1, \gamma_2)) \eta \). According to \[4\] Lem. 1 this is a sufficient condition for the information matrix to be stochastic equicontinuous and by \[4\] Thm. 1 \( (NT_N)^{-1} B_{NT,Nt}(\gamma_1, \gamma_2) \overset{P}{\to} B(\gamma_1, \gamma_2) \) uniformly over \( \gamma_1, \gamma_2 \in \Gamma \), as \( \{N, T_N\} \to \infty \). An analogous result for the Hessian follows by MVT with respect to \( \gamma_1, \gamma_2 \) and the uniform boundedness of the third derivative (Lemma 3.1). By standard Taylor expansion arguments, and CMT, \( (NT_N)^{-1} \Sigma_{NT,N}(\gamma_1, \gamma_2) \overset{P}{\to} \Sigma(\gamma_1, \gamma_2) \) uniformly over \( \gamma_1, \gamma_2 \in \Gamma \). Following analogous steps of Supp. Mat. \[3.1\] for the identifiable parameters \( \phi \), equation \[S-2\] leads to

\[ \frac{\tilde{S}_{NT,N}^{(2)}(\gamma)}{\sqrt{NT_N}} \doteq P(\gamma, \gamma) \frac{S_{NT,N}(\gamma)}{\sqrt{NT_N}} \Rightarrow P(\gamma, \gamma) S(\gamma) \doteq S^{(2)}(\gamma) \equiv N(0, \Sigma(\gamma, \gamma)), \quad (37) \]
with \( P(\gamma, \gamma) = \left[ -J_2 H(\gamma, \gamma)J_1'(J_1 H(\gamma, \gamma)J_1')^{-1}, I_{k_0} \right] \). Finally, the CMT shows that \( LM_{NT_N}(\gamma) \Rightarrow LM(\gamma) \) and \( g_{NT_N} \Rightarrow g \). A similar conclusion is obtained for the local alternatives \( \alpha = (NT)^{-1/2}\delta_2 \), where \( \delta_2 \in \mathbb{R}^{k_2} \), by (S-4), with \( S(2)(\gamma) \equiv N(J_2 H^{-1}(\gamma, \gamma)J_2\delta_2, \Sigma(\gamma, \gamma)) \) in \([37]\). This ends the proof.

A.5 Proof of Theorem 5.3

Following the results of Section A.4 the information matrix \( B_{NT}(\gamma_1, \gamma_2) \) is Lipschitz for \( \gamma_1, \gamma_2 \) with constants \( B_{1,NT}, B_{2,NT} \) for \( \gamma_2 \) having finite absolute moments. Moreover, by Supp. Mat. S-2.2, \( B_{NT}(\gamma_1, \gamma_2) \overset{a.s.}{\rightarrow} B(\gamma_1, \gamma_2) \forall \gamma_1, \gamma_2 \in \Gamma \). Define \( B_{s,NT} = T^{-1}\sum_{t=1}^{T} B_{s,NT}^{*} \), for \( s = \{1, 2\} \). Following the same arguments of Supp. Mat. S-2.1-S-2.2 it can be proved that \( B_{s,NT}^{*} \rightarrow 0 \) where \( B_{s,N}^{*} = E(B_{s,NT}^{*}) \). Assumptions H1-H3 imply that \( B_{s,N}^{*} \rightarrow B_{s,N}^{*} \). This is a sufficient condition for \( B_{NT} \) to be strongly stochastically equicontinuous [4] Lem. 1 and, together with pointwise almost sure convergence, [4] Thm. 2 shows that \( B_{NT}(\gamma_1, \gamma_2) \overset{a.s.}{\rightarrow} B(\gamma_1, \gamma_2) \) uniformly over \( \gamma_1, \gamma_2 \).

Consider \( \omega \in \Omega \), where \( \Omega \) denotes a set of samples. We operate conditionally on the sample \( \omega \), so randomness is through the IID standard normal process \( \nu_t \). Set \( S_{NT}(\gamma) = \sum_{t=1}^{T} s_{NT}^{\nu}(\gamma) \), with \( s^{\nu}_{NT}(\gamma) = s_{NT}(\gamma)\nu_t \). Then, \( \bar{S}_{NT}(\gamma) = \overline{S}_{NT}(\gamma) + \bar{S}_{NT}(\gamma), \bar{S}_{NT}(\gamma) = \sum_{t=1}^{T}(s^{\nu}_{NT}(\gamma) - s^{\nu}_{NT}(\gamma)) = \sum_{t=1}^{T}\sum_{i=1}^{N}(i,N_{t})(\gamma) - s^{\nu}_{i,N_t}(\gamma)) \), a.s.

\[
\bar{S}_{NT}(\gamma) = \sum_{t=1}^{T}\sum_{i=1}^{N} \left( \frac{Z_{i,t}(\gamma)\xi_{i,t} \nu_t}{\lambda_{i,t}(\gamma)} - \frac{Z_{i,t}(\gamma)\xi_{i,t} \nu_t}{\lambda_{i,t}(\gamma)} \right) \nu_t \\
= \sum_{t=1}^{T}\sum_{i=1}^{N} \left( Z_{i,t}(\gamma) \left( \frac{\lambda_{i,t}(\gamma)\xi_{i,t} - \lambda_{i,t}(\gamma)\xi_{i,t}}{\lambda_{i,t}(\gamma)} \right) \right) \nu_t \\
\leq \beta_{0}^{-2}\sum_{t=1}^{T}\sum_{i=1}^{N} Z_{i,t}(\gamma)Z'_{i,t}(\gamma) \left( \phi - \phi \right) Y_{i,t} \nu_t.
\]

Set \( G_{NT}(\gamma) := \beta_{0}^{-2}\sum_{t=1}^{T}\sum_{i=1}^{N} Y_{i,t}Z_{i,t}(\gamma)Z'_{i,t}(\gamma) \nu_t \), so

\[
\sup_{\gamma \in \Gamma} \left| \frac{\bar{S}_{NT}(\gamma)}{\sqrt{NT}} \right| \leq \sup_{\gamma \in \Gamma} \left| \frac{G_{NT}(\gamma)}{\sqrt{NT}} \right| \leq \left| \sqrt{NT} (\tilde{\phi} - \phi) \right|.
\]

By Section A.4 \( s_{t}(\gamma) \) is \( L^2 \) integrable. Then, from the assumptions of Theorems 3.3, 5.1 and Pollard’s central limit theorem for triangular empirical processes [35] Thm. 10.6], the arguments in [42] pp. 426-427 prove that \( (NT)^{-1/2}S_{NT}(\gamma) \Rightarrow P(\gamma, \gamma) \). Furthermore, we have \( (NT)^{-1/2}G_{NT}(\gamma) \overset{a.s.}{\rightarrow} O_{k \times k} \). Then, \( (NT)^{-1/2}\bar{S}_{NT}(\gamma) \Rightarrow P(0), (NT)^{-1/2}\bar{S}_{NT}(\gamma) \Rightarrow P(\gamma), LM_{NT}(\gamma) \Rightarrow P LM(\gamma), g_{NT} \Rightarrow P g, \bar{F}_{NT}(x) \overset{p}{\Rightarrow} F(x) \), uniformly over \( x \), and \( \bar{p}_{NT} = 1 - \bar{F}_{NT}(g_{NT}) = 1 - F(g_{NT}+o_{P}(1)) = P_{NT} + o_{P}(1) \).
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S Supplementary Material

The supplement contains proofs for Sections 3 and 4 and detailed study of the TNAR model (8) with fixed network dimension $N$. It also includes key concepts used in the main paper, simulations and further data analysis, as described in Sections 3-7.

S-1 Key concepts

S-1.1 Joint copula-Poisson process

We describe the joint copula-Poisson DGP for the nonlinear PNAR (1). Consider a network $W$, a set of values for the parameters $\theta$ of (1) and a starting vector at time $t = 0$, say $\lambda_0 = (\lambda_{1,0}, \ldots, \lambda_{N,0})'$,

1. Let $U_l = (U_{1,l}, \ldots, U_{N,l})$, for $l = 1, \ldots, K$ a sample from a $N$-dimensional copula $C(u_1, \ldots, u_N)$, where $U_{i,l}$ follows a $Uniform(0,1)$ distribution, for $i = 1, \ldots, N$.

2. The transformation $E_{i,l} = -\log U_{i,l}/\lambda_{i,0}$ follows the exponential distribution with parameter $\lambda_{i,0}$, for $i = 1, \ldots, N$.

3. If $E_{i,1} > 1$, then $Y_{i,0} = 0$, otherwise $Y_{i,0} = \max \left\{ k \in [1, K] : \sum_{l=1}^{k} E_{i,l} \leq 1 \right\}$, by taking $K$ large enough. Then, $Y_{i,0}|\lambda_0 \sim Poisson(\lambda_{i,0})$, for $i = 1, \ldots, N$. So, $Y_0 = (Y_{1,0}, \ldots, Y_{N,0})'$ is a set of (conditionally) marginal Poisson processes with mean $\lambda_0$.

4. By updating model (1), $\lambda_1$ is obtained.

5. Return back to step 1 to obtain $Y_1$, and so on.

In applications, the sample size $K$ should be taken large, e.g. $K = 100$. Its value clearly depends, in general, on the magnitude of observed data. Moreover, the copula function $C(\ldots)$ depends on one or more unknown parameters, say $\rho$, which express the contemporaneous correlation among the variables.

The proposed DGP ensures that all marginal distributions of $Y_{i,t}$ are univariate Poisson, conditionally to the past, as described in (1), while it introduces arbitrary dependence among them, in
a flexible and general way, using the copula construction. We point out that the assumed marginal
Poisson distribution can be replaced by other assumptions, like Negative Binomial and more
generally mixed Poisson distribution, by modifying suitable the above algorithm. See [7] and [32] for
further details on copula-Poisson joint distributions.

S-1.2 Stationarity for increasing dimensional processes

We follow [72]. Define \{X_t \in \mathbb{R}^N\} be an \(N\)-dimensional time series with \(N \to \infty\) and \(W = \{\omega \in \mathbb{R}^\infty : \omega_\infty = \sum |\omega_i| < \infty\}\), where \(\omega = (\omega_i \in \mathbb{R} : 1 \leq i \leq \infty)' \in \mathbb{R}^\infty\). For each \(\omega \in W\), let \(\omega_N = (\omega_1, \ldots, \omega_N)' \in \mathbb{R}^N\) be its truncated \(N\)-dimensional version. Then \{\(X_t\)\} is said to be strictly stationary if the following conditions hold:

- \(X_t^\omega = \lim_{N \to \infty} \omega_N^T X_t < \infty\), a.s. (almost surely)
- \(\{X_t^\omega, t \in \mathbb{Z}\}\) is strictly stationary.

S-2 Proofs for Section 3

S-2.1 Proof of Lemma 3.1

We provide a sketch of the proof of Lemma 3.1 because the proof is analogous the corresponding
proof of the linear PNAR model, as described in [7, Lemma 1-2]. Set \(\mu_1 + \mu_2 = d\). For \(J > 0\), define \(\tilde{Y}_t = f(\hat{Y}_{t-1}, \theta)\), if \(t > 0\), and \(\tilde{Y}_t = Y_0\), if \(t = 0\). Moreover, \(\tilde{Y}_{t-j} = f(\hat{Y}_{t-1-j}, \theta) + \xi_s\), if \(\max\{t - J, 0\} < s \leq t\), and \(\tilde{Y}_{t-j} = Y_s\), if \(s \leq \max\{t - J, 0\}\) where \(f(\hat{Y}_{t-1-j}, \theta) = \hat{\lambda}_{t-j}\). Let \(\tilde{Y}_t = \hat{Y}_t + (1 - \delta)\hat{Y}_t\) with \(0 \leq \delta \leq 1\). Then, a.s

\[
\left| Y_t - \tilde{Y}_{t-j} \right|_{\infty} \leq d^j \sum_{j=0}^{t-j-1} d^j |\xi_{t-j-j}|_{\infty}, \tag{S-1}
\]

where \(|\xi_{t}|_{\infty} = \max_{1 \leq i \leq N} |\xi_{t,i}|\) For [9, Lemma 3-2] applies to \(\hat{Y}\) and \(\tilde{Y}\) holds true.

Define \(\hat{Y}_{i,t}, \hat{\lambda}_{i,t}\) the \(i\)th elements of \(\hat{Y}_{t-j}\) and \(\hat{\lambda}_{t-j}\). Set \(l_{i,t} = Y_{i,t} \hat{\lambda}_{i,t} d_{i,g} d_{i,gl}\), where \(d_{i,g} = \partial f_i(X_{i,t-1}, Y_{i,t-1}, \theta)/\partial \theta_g\) and \(d_{i,gl} = \partial^2 f_i(X_{i,t-1}, Y_{i,t-1}, \theta)/\partial \theta_g \partial \theta_l\). Set \(a \geq 4\), a generic integer which can take different values. Recall that the notations \(C_a\), \(l_a\) denote constants which depend on \(a\) and \(c\), a generic constant. By Theorem 2.3 and the a.s. inequality of Appendix [A.2] \(\hat{\lambda}_{i,t} = f_i(X_{i,t-1}, Y_{i,t-1}) \leq c + \mu_1 X_{i,t-1} + \mu_2 Y_{i,t-1}\), for every \(i = 1, \ldots, N\), we have that \(\max_{i \geq 1} \|\hat{\lambda}_{i,t}\|_a \leq \max_{i \geq 1} \|Y_{i,t}\|_a \leq C_a\), by the conditional Jensen’s inequality. Similarly, \(\max_{i \geq 1} \|\hat{\lambda}_{i,t}\|_a \leq \max_{i \geq 1} \|Y_{i,t}\|_a\). By (S-1),

\[
\max_{i \geq 1} \left| Y_{i,t} - \hat{Y}_{i,t} \right|_a \leq d^j 2C_a^{1/a}/(1 - d). \tag{S-2}
\]

An analogous recursion shows that \(\max_{i \geq 1} \|\hat{Y}_{i,t}\|_a \leq (2c + 2C_a)/ (1 - d) = \Delta < \infty\). By Assumption [B] \(|\partial f_i(x_i, y_i, \theta)/\partial \theta_g| \leq \max_{i \geq 1} |\partial f_i(0, 0, \theta)/\partial \theta_g| + c_1 x_i + c_2 y_i\); so \(\max_{i \geq 1} \|\partial f_i(x_i, y_i, \theta)/\partial \theta_g\|_a \leq \Delta_g < \infty\), \(\max_{i \geq 1} \|\partial f_i(x_i, y_i, \theta)/\partial \theta_g\|_a \leq \Delta_g < \infty\) for \(i = 1, \ldots, N\) and \(r = 1, \ldots, m\), where \(\Delta_g, \Delta_g\) are constants depending on index \(g\). Similar arguments apply to second and third derivatives. Then, by Hölder’s inequality, \(|l_{i,t}|_a \leq l_a < \infty\). Set \(W_t = (Y_t, Y_{t-1})', \hat{W}_{t-j} = (\hat{Y}_{t-j}, \hat{Y}_{t-j-1})'. \) Consider the following triangular array \(\{g_{Nt}(W_t) : 1 \leq t \leq T_N, N \geq 1\}\), where \(T_N \to \infty\) as \(N \to \infty\). For any \(\eta \in \mathbb{R}^m\), \(g_{Nt}(W_t) = \)
\[ \sum_{g=1}^{m} \sum_{i=1}^{m} \eta_i \eta_h l_{gl}, \quad N^{-1} h_{Nt} = (l_{gl})_{1 \leq g, l \leq m} \quad \text{and} \quad h_{Nt} = \sum_{i=1}^{N} -\partial^2 l_{i,t}(\theta)/\partial \theta \partial \theta', \] where the second derivative is defined as in [16]. We take a single element, \( h_{gl} \), the result is similarly proved for the other elements. By recalling (9) and Assumptions B-C following [7 Sec. A.2], it can be proved that \( \| h_{gl} - h_{gl-t,j} \| \leq 2l_1 c_1^l a/(1 - d) d^{-1} := c_{gl} \nu_j \), with \( \nu_j = d^{-1} \). This holds true for any \( g, l = 1, \ldots, m \). Therefore, the triangular array process \( \{ \tilde{W}_{Nt} = g_{Nt}(W_t) - \mathbb{E} [g_{Nt}(W_t)] \} \) is \( L^p \)-near epoch dependence \( (L^p \text{-NED}) \), with \( p \in [1,2] \). Moreover, using [3 p. 464], we have that \( \{ \tilde{W}_{Nt} \} \) is a uniformly integrable \( L^1 \)-mixingale and, by Assumption H3 the law of large numbers of [3 Thm. 2] shows that \( (NT^{-1/2})^{-1}\eta' H_{NT} \eta \overset{p}{\to} \eta' H \eta \) as \( N, T \to \infty \). By the last inequality of [2] an analogous argument, along the lines of [7 Sec. A.2-A.3], applies to the convergence of the information matrix \( (NT^{-1/2})^{-1}\eta' B_{NT} \eta \overset{p}{\to} \eta' B \eta \) and to the third log-likelihood derivative defined by Lemma 3.1 Assumption H2 the arguments in [7 Sec. A.3, Supp. Mat. S-6] and the central limit theorem for martingale arrays in [11 Cor. 3.1] yields \( (NT^{-1/2}) S_{NT} \overset{d}{\to} N(0, B) \), as \( N \to \infty \), leading to the desired result.

\[ \square \]

S-2.2 Proof of Theorem 3.3

Recall that \( c \) is a constant and \( \mu_1 + \mu_2 = d \). By the assumptions of Lemma 3.1, \( S_{Nt}/N, H_{Nt}/N, B_{Nt}/N \) and \( M_{Nt}/N \) have finite absolute fourth moments. Moreover, following the results of Section S-2.1 \( H_{Nt}/N, B_{Nt}/N \) and \( M_{Nt} \) are \( L^p \text{-NED} \). Arguing as in [20 Def. 1] and [3 p. 464], they are \( L^2 \)-mixingales with coefficients \( c_{Nt} = c \) and \( \psi(J) = d^{[J/2]} - 1 + 6\alpha([J/2])^{-1/3} \), where \([ \cdot ]\) denotes the integer part operator. Note that \( S_{Nt}/N \) is a martingale difference array so it is trivially \( L^2 \)-mixingale with \( \psi(J) = 0 \) [3 Com. (2), Example 1]. Define \( \zeta_N = [N^{1/3+\eta}] \) and \( \mu_N = [N^{1/6-2\eta}] \), for some \( \eta > 0 \) such that \( m_N \geq 1 \). Then, \( \sum_{N=1}^{\infty} T_{N}^{-1} \sum_{t=1}^{T_{N}} \mathbb{E}[H_{Nt}/N]/N^{1/3+\eta} \sum_{N=1}^{\infty} \zeta_N^3 \leq C_4 \sum_{N=1}^{\infty} \zeta_N^3 \leq C_4 \sum_{N=1}^{\infty} (N^{1/3+\eta})^{-3} < \infty \). Moreover, for all \( \delta > 0 \)

\[ \sum_{N=1}^{\infty} m_N \exp \left( -\delta^2 T_{N}^2 m_N^{-2} \sum_{t=1}^{T_{N}} \zeta_N^2 \right) = \sum_{N=1}^{\infty} m_N \exp \left( -\delta^2 T_{N} (m_N \zeta_N)^{-2} \right) \]

\[ \leq \sum_{N=1}^{\infty} m_N \exp \left( -\delta^2 T_{N} (N^{1/6-2\eta} N^{1/3+\eta})^{-2} \right) \leq \sum_{N=1}^{\infty} m_N \exp \left( -\delta^2 T_{N} \right) < \infty . \]

where the last inequality follows by \( T_{N} = \lambda N \). Finally, note that \( \sum_{N=1}^{\infty} (T_{N}^{-1} \sum_{t=1}^{T_{N}} c_{Nt} \psi(m_N)^2) \leq \sum_{N=1}^{\infty} \psi(m_N)^2 \) is convergent if \( \psi(J) = O(J^{-3-\epsilon}) \), by appropriate choice of \( \eta > 0 \); since \( d^{[J/2]} - 1 = O(J^{-3-\epsilon}) \), the assumption of Theorem 3.3 provides the result. Then, conditions 1.(c), 2-3 of [20 Thm. 4] are satisfied and, by Assumption H3 as \( \{ N, T_N \} \to \infty, \) \( T_N^{-1} \sum_{t=1}^{T_N} N^{-1} H_{Nt} = (NT^{-1})^{-1} H_{NT} \overset{a.s.}{\to} H \). Analogous results hold for the other mixingales. An application of [58 Thm. 3.2.23] ends the proof.

\[ \square \]

S-2.3 Proof of Theorem 3.4

The process \( Y_t = \lambda_t + \xi_t \), where \( \lambda_t \) is defined as in [5], can be rewritten as \( Y_t = \beta_0 C_{t-1} + G Y_{t-1} + \xi_t \). By backward substitution, \( Y_t = \beta_0 \sum_{j=0}^{t-1} G^j C_{t-1-j} + G^t Y_0 + \sum_{j=0}^{t-1} G^j \xi_{t-j} \). Under the conditions of Theorem 2.3 as \( t \to \infty \), the second term is negligible and the last term is well-defined as
\[ \tilde{Y}_t := \sum_{j=0}^{\infty} G^j \xi_{t-j}. \] It remains to show the finiteness of the first summand \( \mu_{t-1} := \beta_0 \sum_{j=0}^{t-1} G^j C_{t-1-j} \). Note that \( \mu_{t-1} \) is increasing sequence with respect to \( t \), moreover \( C_t \leq 1 \), so \( \mu_{t-1} \leq \beta_0 \sum_{j=0}^{t-1} G^j 1 = \beta_0 \sum_{j=0}^{t-1} (\beta_1 + \beta_2)^j \leq \mu 1 \), where \( \mu := \beta_0 (1 - \beta_1 - \beta_2)^{-1} \), then \( \mu_{t-1} := \beta_0 \sum_{j=0}^{t-1} G^j C_{t-1-j} \) is \( \mu \) a.s.

and \( Y_t = \mu_{t-1} + \tilde{Y}_t \). The proof of Theorem 3.4 follows by Lemma 3.1 we only need to show that assumptions Q1, Q2 imply H3 for model (I). We consider one element of the Hessian matrix, since we can work analogously for all other elements; see also [7, Sec. A.2]. Consider the element in position (1, 2) of matrix (18) and the decomposition \( Y_t = \mu_{t-1} + \tilde{Y}_t \), so \( H_{12} = H_{12a} + H_{12b} \), where \( N^{-1} H_{12a} = N^{-1} E(\Gamma_{1,t-1} D_t - \Gamma_{2,t-1}) \) converges to \( h_{12} \), by Q2 indeed

\[
\left| \frac{H_{12}}{N} \right| \leq \frac{1}{N} E \left( \sum_{i=1}^{N} \frac{w'_i Y_{t-1}|_{\text{vec}}}{\lambda_{t-1}(1 + X_{i,t-1})^\gamma} \right) \leq \frac{1}{CN} E \left( W | Y_{t-1}|_{\text{vec}} \right) \xrightarrow{N \to \infty} 0,
\]

where the last inequality follows by Assumption C, the convergence holds by Q1, [7, Eq. A-2] and an application of [7, Lemma S-1]. For the information matrix (19) similar results hold true, by considering Assumption H2. The third derivative of the likelihood (14) has the form derived in [32, Supp. Mat. S-4]. Moreover, \( \partial \lambda_{i,t}/\partial \theta_k = \Gamma_{ik,t-1} \), for \( k = 1, 2, 3 \) and \( \lambda_{i,t}/\partial \theta_k = -\beta_0 \Gamma_{ik,t-1} \); the second derivatives are all zeros apart from \( \partial^2 \lambda_{i,t}/\partial \theta_0 \partial \gamma = -\Gamma_{i,t-1} \) and \( \partial^2 \lambda_{i,t}/\partial \gamma^2 = 0 \); the third derivatives are all zeros apart from \( \partial^2 \lambda_{i,t}/\partial \theta_0 \partial \gamma^2 = \Gamma_{i,t-1} \) and \( \partial^2 \lambda_{i,t}/\partial \gamma^3 = 0 \). Condition of H3 is satisfied by recalling the \( \delta^* \) conditions in Assumption Q2 We omit the details.

\[ \square \]

**S-3 Proofs for Section 4**

**S-3.1 Proof of Theorem 4.1**

Under the condition of Lemma 3.1 Theorem 3.2 and A the constrained estimator \( \tilde{\theta} \) is consistent for \( \theta_0 \), when \( H_0 \) holds. See, for example, [38, Thm. 3.2.23]. So \( \tilde{\theta}^{(1)} \) is consistent for \( \theta_0 \) and, for \( \{N, T_N\} \) large enough, we have \( S_{NT}^{(1)}(\tilde{\theta}) = 0 \) with \( \tilde{\theta}^{(1)} \neq 0 \). Let \( J_1 = (J_{m_1}, O_{m_1 \times m_2}), \) \( J_2 = (O_{m_2 \times m_1}, J_{m_2}) \), where \( I_s \) is a \( s \times s \) identity matrix and \( O_{a \times b} \) is a \( a \times b \) matrix of zeros. Therefore, \( 0 = S_{NT}^{(1)}(\tilde{\theta}) = J_1 S_{NT}^{(1)}(\tilde{\theta}) \) and \( S_{NT}^{(1)}(\tilde{\theta}) = J_2 S_{NT}^{(1)}(\tilde{\theta}) \). Lemma 3.1 and Taylor’s theorem provides \( 0 = (NT)^{-1/2} S_{NT}^{(1)}(\tilde{\theta}) \hat{=} \hat{(NT)^{-1/2}} S_{NT}^{(1)}(\theta_0) - J_1 H(NT)^{1/2}(\tilde{\theta} - \theta_0) \), and \( (NT)^{-1/2} S_{NT}^{(1)}(\tilde{\theta}) \hat{=} \hat{(NT)^{-1/2}} S_{NT}^{(1)}(\theta_0) - J_2 H(NT)^{1/2}(\tilde{\theta} - \theta_0) \), where \( \hat{=} \) means equality up to an \( o_p(1) \) term. Since \( \tilde{\theta} - \theta_0 = J_1^t (\tilde{\theta}^{(1)} - \tilde{\theta}^{(1)}) \) we have \( 0 \hat{=} \hat{(NT)^{-1/2}} S_{NT}^{(1)}(\theta_0) - J_1 H J_1^t (NT)^{1/2}(\tilde{\theta}^{(1)} - \tilde{\theta}^{(1)}) \) and \( (NT)^{1/2}(\tilde{\theta}^{(1)} - \tilde{\theta}^{(1)}) \hat{=} \hat{H_{11}^t (NT)^{-1/2}} S_{NT}^{(1)}(\theta_0) \), where \( H_{11} = J_1 H J_1^t \). Combining the above results we obtain \( (NT)^{-1/2} S_{NT}^{(1)}(\tilde{\theta}) \hat{=} \hat{V_2 - H_{21} H_{11}^t V_1 = PV} \), where \( J_2 H J_1^t = H_{21}, V = (NT)^{-1/2} S_{NT}(\theta_0) \) and \( P = \left( -H_{21} H_{11}^t I_{m_2} \right) \). Then, \( V \xrightarrow{d} N(0, B) \), as \( N, T_N \to \infty \) leading to

\[
\frac{S_{NT}^{(2)}(\tilde{\theta})}{\sqrt{NT}} \hat{=} \hat{PV \frac{d}{N, T_N \to \infty} N(0, \Sigma)} \tag{S-2}
\]

where \( \Sigma = P B P^t \) has the expression (30). Following [27, 36, and 35], among others, the general formulation of the score test statistic for the nonlinear parameters of the model is

\[
LM_{NT} = \frac{S_{NT}^{(2)}(\tilde{\theta})}{\sqrt{NT}} \tag{S-3}
\]
Then, by replacing $\Sigma$ in (S-3) with the result follows by Lemma 3.1, $A'$ and (S-2), under $H_0$,

$$LM_{NT} = S_{NT}^{(2)}(\bar{\theta})\Sigma_{NT}^{-1}(\bar{\theta})S_{NT}^{(2)}(\bar{\theta}) \xrightarrow{d_{N,TN \to \infty}} \chi^2_{m_2}.$$  

To determine the asymptotic distribution of the test statistic $LM_{NT}$ under $H_1$ in (28) note that $\theta = \theta_0 + (NT)^{-1/2}\delta$, where $\delta = (\delta_1, \delta_2)'$. A Taylor expansion around $\theta$ shows that $(NT)^{-1/2}S_{NT}(\theta_0) = (NT)^{-1/2}S_{NT}(\theta) = (NT)^{-1/2}S_{NT}(\theta) + (NT)^{-1/2}H_{NT}(\theta_0 - \theta) + (NT)^{-1}H_{NT}(\theta_0)\delta$. Since, as $\{N, T_N\} \to \infty$, $\theta \xrightarrow{p} \theta_0$, and, by mean value theorem and Lemma 3.1 $(NT)^{-1/2}S_{NT}(\theta) \xrightarrow{d} N(0, B)$ and $(NT)^{-1}H_{NT}(\theta) \xrightarrow{d} H$, under $H_1$, $V \xrightarrow{d} N(H\delta, B)$. Hence,

$$\frac{S_{NT}^{(2)}(\bar{\theta})}{\sqrt{NT}} \xrightarrow{d_{N,TN \to \infty}} N(\Sigma_H\delta_2, \Sigma),$$  

under $H_1$, where $\Sigma_H \equiv (H^{22})^{-1}$, with $H^{22} = J_2H^{-1}J_2$, and $\Sigma_H\delta_2 = PH\delta$. By (S-3),

$$LM_{NT} = S_{NT}^{(2)}(\bar{\theta})\Sigma_{NT}^{-1}(\bar{\theta})S_{NT}^{(2)}(\bar{\theta}) \xrightarrow{d_{N,TN \to \infty}} \chi^2_{m_2}(\delta_2', \tilde{\Delta}_{\delta 2}).$$  

under $H_1$, where $\chi^2_{m_2}(\cdot)$ is the noncentral chi-square distribution and $\tilde{\Delta}$ is the sample counterpart of $\Delta = \Sigma_H^{-1}\Sigma_H$ computed according to Remark 4.  

\[ \square \]

### S-4 Threshold models

In this section we give a detailed study for the TNAR model (8). In this case the function $f(\cdot)$ is not continuous with respect to the threshold variable $X_t$. Therefore, the contraction condition (9) does not hold. In addition, the Lipschitz conditions, stated in Assumption B, are not satisfied due to the discontinuity of the process. However, we establish stability conditions for fixed network dimension (stationarity and geometric ergodicity) for model (8), by establishing absolute regularity of the process.

**Proposition S-4.1.** Consider model (31), with process $\{\lambda_t\}$ defined by (8) and $||G||_1 < 1$, where $G = (\beta_1 + \alpha_1)W + (\beta_2 + \alpha_2)I$. Suppose the probability mass function of $Y_t | F_{t-1}$ is positive. Then, the TNAR model is stationary, ergodic and absolutely regular with geometrically decaying coefficients.

**Proof of Proposition S-4.1** Consider model (8), for $i = 1, \ldots, N$, a.s.

$$|f_i(X_{i,t-1}, Y_{i,t-1})| \leq (\beta_0 + \alpha_0) + (\beta_1 + \alpha_1)X_{i,t-1} + (\beta_2 + \alpha_2)Y_{i,t-1} = (\beta_0 + \alpha_0) + [(\beta_1 + \alpha_1)e_i'W + (\beta_2 + \alpha_2)e_i]'|Y_{t-1}|_{vec},$$

implying that $|f(Y_{t-1})|_{vec} \leq c1 + G|Y_{t-1}|_{vec}$, with $G = (\beta_1 + \alpha_1)W + (\beta_2 + \alpha_2)I$ and $c = \beta_0 + \alpha_0$. It is immediate to see that the process $\{Y_t : t \in \mathbb{Z}\}$, defined by (31) with $\lambda_t$ specified as in (8) is a Markov chain on countable state space $\mathbb{N}^N$. Since $Y_t$ given $Y_{t-1}$ has a positive probability mass function, the first order transition probabilities of the Markov chain are $p_{m,l} = P(Y_t = m | Y_{t-1} = l) > 0$, for all $m, l$. This implies irreducibility and aperiodicity of the chain. See, for example, [50 Ch. 4-5]. A geometric drift condition is established, by the property of the Poisson process, $\forall y \in \mathbb{N}^N$,

$$E(|Y_t| + 1 | Y_{t-1} = y) = 1 + E|N_t[f(y)]|_1 = 1 + |f(y)|_1 \leq c^* + d_1(|y|_1 + 1),$$

33
where \( c_0 = Nc, \ c^* = c_0 + 1 < \infty \) and \( d_1 = \| G \|_1 < 1 \). Consider the non-empty compact sets \( S^1_M = \{ y : |y|_1 + 1 \leq M \} \), with real constants \( M \). Since the state space is countable and we have that \( \epsilon = \sum_{m \in M} \inf_{t \in S^1_M} p_{m,t} > 0 \), by [50, eq. 8-9], every non-null compact set \( S^1_M \) is small. By considering the small set \( S^1 = \{ y : |y|_1 + 1 \leq c^*/\delta_1 \} \), with \( \delta_1 = 1/2(1 - d_1) \), [50] Thm 15.0.1 implies that the stochastic process \( \{ Y_t : t \in \mathbb{Z} \} \) is stationary and absolutely regular with geometrically decaying coefficients.

An analogous result holds for the continuous case. Define \( \hat{\beta}_1 = \max \{|\beta_1|, |\beta_1 + \alpha_1|\} \) and \( \hat{\beta}_2 = \max \{|\beta_2|, |\beta_2 + \alpha_2|\} \).

**Proposition S-4.2.** Consider model \((32)\), with process \( \{\lambda_t\} \) defined as in \((8)\) and \( \hat{\beta}_1 + \hat{\beta}_2 < 1 \). The marginal cumulative distribution of \( \xi_t \) is absolutely continuous, with positive probability density function, and \( E|\xi_t|_1 < \infty \). Then, the TNAR model is stationary, ergodic and absolutely regular with geometrically decaying coefficients.

**Proof of Proposition S-4.2** Consider model \((8)\). Define \( \hat{\beta}_0 = \max \{|\beta_0|, |\beta_0 + \alpha_0|\} \) and \( \hat{d} = \hat{\beta}_1 + \hat{\beta}_2 \). For \( i = 1, \ldots, N, \) a.s.

\[
|f_t(X_{i,t-1}, Y_{i,t-1})| \leq \hat{\beta}_0 + \hat{\beta}_1 X_{i,t-1} + \hat{\beta}_2 Y_{i,t-1} = \hat{\beta}_0 + \left( \hat{\beta}_1 c_W + \hat{\beta}_2 c_Y \right) Y_{i-1} \| Y_{i-1} \|_{\text{vec}},
\]

implying that \( |f(Y_{i-1})|_{\text{vec}} \leq \hat{\beta}_0 1 + G |Y_{i-1}|_{\text{vec}} \) with \( G = \hat{\beta}_1 W + \hat{\beta}_2 I \). By [11] the Markov chain \( \{ Y_t : t \in \mathbb{Z} \} \) defined by \((32)\), with \( \lambda_t \) defined in \((8)\), is irreducible and aperiodic; moreover, since the p.d.f of \( \xi_t \) is lower semi-continuous and the function \( f(\cdot) \) is compact, all the non-empty compact sets \( S_M = \{ y : |y|_1 + 1 \leq M \} \) with real constant \( M \), are small. \( \forall y \in \mathbb{R}^N, \)

\[
E(|Y_{i-1} + 1| Y_{i-1} = y) \leq c + \hat{d} |y|_1 + 1 \leq c^* + \hat{d}(|y|_1 + 1), \quad (S-5)
\]

where \( E|\xi_t|_1 < \infty \), \( c = \hat{\beta}_0 + c_\xi, \ c^* = c + 1 < \infty \) and \( 0 < \hat{d} < 1 \). Then, the process \( \{ Y_t : t \in \mathbb{Z} \} \) is stationary and absolutely regular with geometrically decaying coefficients.

**S-4.1 Inference and testing for TNAR models**

We state an analogous result to Theorem [3.2] for \( N \) fixed and \( T \to \infty \). In this case, we drop the dependence of \( N \) to point out that the inference is for \( T \to \infty \). The following result implies, in particular, good large sample properties of the QMLE for model \((8)\) when the threshold parameter \( \gamma \) is known.

**Corollary S-4.3.** Consider model \((1)\). Let \( \theta \in \Theta \subset \mathbb{R}^m \). Suppose the conditions of Theorem [2.4], Assumption [A] [B] (with \( x^*_t = y^*_t = 0 \)) and [C] hold. Then, there exists a fixed open neighbourhood \( \mathcal{O}(\theta_0) = \{ \theta : |\theta - \theta_0|_2 < \delta \} \) of \( \theta_0 \) such that with probability tending to 1 as \( T \to \infty \), for the score function \((15)\), the equation \( S_T(\theta) = 0 \) has a unique solution, called \( \hat{\theta} \), which is (strongly) consistent and asymptotically normal:

\[
\sqrt{T}(\hat{\theta} - \theta_0) \overset{d}{\to} N(0, H^{-1}B H^{-1}),
\]

where the matrices \( H \equiv H_N \) and \( B \equiv B_N \) are defined in \((18)-(19)\), respectively.
This follows by an application of the ergodic theorem and analogous arguments to [7, Sec. 3.3]. Assumption \( A \) is imposed with \( x_i^* = y_i^* = 0 \) to provide bounds of all necessary moments. Corollary S-4.3 shows weak consistency of QMLE but strong consistency is proved under the same assumptions. Recall from Section 5 that \( \theta = (\phi', \gamma)' \), where \( \gamma \) is the threshold parameter and \( \phi \) is the vector of, linear and nonlinear, identifiable parameters.

Note that \( B \) with \( x_i^* = y_i^* = 0 \) is satisfied for \( \theta = \phi \) under the threshold model (8). Also condition \( C \) holds for TNAR models. Then, we have the following result.

**Corollary S-4.4.** Consider model (31), with process \( \{\lambda_i\} \) defined as in (8) where \( \gamma \) is known. Suppose the conditions of Proposition S-4.1 and Assumption \( A \) hold. Then, Corollary S-4.3 holds true.

Estimation of the \( \gamma \) can be tackled by profiling likelihood. This is a common practice; see for example [64] among others. A similar result is established for continuous-valued data.

We now provide the related result for testing the linearity of model (4) versus (8), i.e. 

\[
\begin{align*}
H_0 & : \alpha_0 = \alpha_1 = \alpha_2 = 0 \text{ vs. } H_1 : \alpha_l > 0, \text{ for some } l = 0, 1, 2.
\end{align*}
\]

**Proposition S-4.5.** Consider model (31), with process \( \{\lambda_i\} \) defined as in (8). Suppose conditions of Proposition S-4.1 hold true. Moreover, assume that \( X_t \) has marginal probability density function \( f_{X_t} \), such that \( \sup_{x_i \in \mathbb{R}} f_{X_t}(x_i) = \bar{f}_X < \infty \). Then, under \( A' \), Theorem 5.1 holds true for \( T \rightarrow \infty \) and fixed \( N \).

**Proof of Proposition S-4.5**. Note that \( I(X_{i,t-1} \leq \gamma) \), as a function of \( \gamma \), is upper semi-continuous, then \( \sup_{\gamma \in \Gamma} I(X_{i,t-1} \leq \gamma) \) exists and equals 1. Finally, the rows of \( -Z_t(\gamma) \) are linearly independent for some \( t \), uniformly over \( \gamma \in \Gamma \equiv [\gamma_L, \gamma_U] \), by considering that \( \inf_{\gamma \in \Gamma} -h_{i,t}(\gamma) = -\sup_{\gamma \in \Gamma} I(X_{i,t-1} \leq \gamma)Z_{it} \neq 0 \) componentwise, for some \( t \), by the ergodicity of \( Y_t \). So we have that \( \inf_{\gamma \in \Gamma} \det(H(\gamma, \gamma)) > 0 \). This fact, the finiteness of all the moments of \( Y_t \) and \( \xi_t(\gamma) \), and the results of Proposition S-4.1 satisfy [42, Asm. 1]. Set \( \gamma > \gamma^* \), for \( a > 1 \),

\[
\|h_{i,t}(\gamma) - h_{i,t}(\gamma^*)\|_a \leq \|(1 + X_{i,t-1} + Y_{i,t-1}) I(\gamma^* < X_{i,t-1} \leq \gamma)\|_a \tag{S-6}
\]

\[
\leq (1 + \|X_{i,t-1}\|_2 + \|Y_{i,t-1}\|_2) \left( \int_{\gamma}^{\gamma*} f_{X_t}(x_i)dx_i \right)^{\delta} 
\]

\[
\leq B_1|\gamma - \gamma^*|^{\delta}
\]

where \( \delta = 1/(2a) \) and \( B_1 = (1 + 2C) \bar{f}_X^{\delta} \). The same holds for \( \gamma < \gamma^* \). Then, [42, Asm. 2] holds true.

Now we show uniform convergence of all quantities involved in inference. [42, Asm. 3]. Standard arguments show that the pointwise convergence of \( H_T(\gamma_1, \gamma_2) \) and \( B_T(\gamma_1, \gamma_2) \) hold, by the measurability of all the functions involved and the existence of moments of any order for the count process \( Y_t \). For all \( \gamma_1, \gamma_2 \in \Gamma \), define \( \tilde{\gamma} = (\gamma_1', \gamma_2') \), \( B(\tilde{\gamma}, \rho) = \{\tilde{\gamma}^* \in \Gamma : |\tilde{\gamma}^* - \tilde{\gamma}|_1 < \rho\} \) and consider the following random variables.

\[
H^*_t(\tilde{\gamma}, \rho) = \sup \{H_t(\gamma^*) : \gamma^* \in B(\tilde{\gamma}, \rho)\}, \quad H^*_t(\tilde{\gamma}, \rho) = \inf \{H_t(\gamma^*) : \gamma^* \in B(\tilde{\gamma}, \rho)\}.
\]

We need to prove, for all \( \tilde{\gamma} \in \Gamma \),

\[
\lim_{\rho \to 0} \sup_{t \geq 1} \frac{1}{T} \sum_{t=1}^{T} \left| EH^*_t(\tilde{\gamma}, \rho) - EH^*_t(\tilde{\gamma}) \right|_1 = 0, \tag{S-7}
\]
and the same limit should hold for \( H_i^*(\bar{\gamma}, \rho) \), as well. Set \( \bar{\gamma}^* \prec \bar{\gamma} \). For clarity in the notation, we prove (S-7) for the single element \( H_i^{(g, l)}(\bar{\gamma}) \), which is bounded by

\[
\lim_{\rho \to 0} \sum_{i=1}^{N} E \left( Y_{i,t} \left( \sup_{\gamma^* \in B(\bar{\gamma}, \rho)} \frac{h_{i,t}^g(\gamma_i^*) h_{i,t}^l(\gamma_j^*)}{\lambda_{i,t}(\gamma_i^*) \lambda_{i,t}(\gamma_j^*)} - \frac{h_{i,t}^g(\gamma_1 \rho h_{i,t}^l(\gamma_2)}{\lambda_{i,t}(\gamma_1) \lambda_{i,t}(\gamma_2)} \right) \right) 
\leq \lim_{\rho \to 0} \sum_{i=1}^{N} E \left( \frac{Y_{i,t}}{\beta^4} \sum_{s=1}^{4} \sup_{\gamma^* \in B(\bar{\gamma}, \rho)} | p_s | \right) .
\]

Note that \( \sup_{\gamma^* \in B(\bar{\gamma}, \rho)} h_{i,t}^g(\gamma_1^*) \leq h_{i,t}^g(\gamma_1 + \rho) \leq 1 \) and \( \inf_{\gamma^* \in B(\bar{\gamma}, \rho)} h_{i,t}^g(\gamma_1^*) \geq h_{i,t}^g(\gamma_1 - \rho) \geq 0 \) a.s.; the same holds for \( \gamma_j^* \). So \( \sup_{\gamma^* \in B(\bar{\gamma}, \rho)} | h_{i,t}^l(\gamma_i^*) - h_{i,t}^l(\gamma_i) | = \sup_{\gamma^* \in B(\bar{\gamma}, \rho)} (h_{i,t}^l(\gamma_1^*) - h_{i,t}^l(\gamma_2^*)) = h_{i,t}^l(\gamma_2) - h_{i,t}^l(\gamma_2 - \rho) \). Then,

\[
\lim_{\rho \to 0} \sum_{i=1}^{N} E \left( \frac{Y_{i,t}}{\beta^4} \sup_{\gamma^* \in B(\bar{\gamma}, \rho)} | p_1 | \right) 
\leq \lim_{\rho \to 0} \sum_{i=1}^{N} E \left( \frac{Y_{i,t}}{\beta^4} \lambda_{i,t}(\gamma_1) \lambda_{i,t}(\gamma_2) \sup_{\gamma^* \in B(\bar{\gamma}, \rho)} h_{i,t}^g(\gamma_i^*) \left| h_{i,t}^l(\gamma_1^*) - h_{i,t}^l(\gamma_2^*) \right| \right) 
\leq C \lim_{\rho \to 0} \| h_{i,t}^l(\gamma_2) - h_{i,t}^l(\gamma_2 - \rho) \|_a 
\leq CB_1 \lim_{\rho \to 0} \| \gamma_2 - \gamma_2 + \rho \|^{1/(2a)} 
= 0 ,
\]

where the second inequality holds by Hölder’s inequality and the third inequality holds by (S-6). Analogous results hold for \( p_s \), with \( s = 2, 3, 4 \); the same limit is obtained for \( \bar{\gamma} \prec \bar{\gamma}^* \) and the cross inequalities \( \gamma_i^* \prec \gamma_i, \gamma_j \prec \gamma_j^* \), with \( i, j = 1, 2 \) and \( i \neq j \). Similar arguments apply (inverted) to \( H_i^*(\bar{\gamma}, \rho) \) and to the information matrix. Then, the uniform law of large number of [42] holds, leading to \( T^{-1} H_T(\gamma_1, \gamma_2) \overset{a.s.}{\to} H(\gamma_1, \gamma_2), T^{-1} B_T(\gamma_1, \gamma_2) \overset{a.s.}{\to} B(\gamma_1, \gamma_2) \), uniformly in \( \gamma_1, \gamma_2 \in \Gamma \). Then, [42, Thm. 1] applies to model (5).

**Proposition S-4.6.** Consider model (32), with process \( \{ \lambda_i \} \) defined as in (8). Suppose the conditions of Proposition S-4.2 hold and \( E | \xi_i^* |^2 < \infty \). Moreover, assume that \( X_t \) has marginal probability density function \( f_{X_t} \), such that \( \sup_{X_t \in \mathbb{R}} f(x_i) = \bar{f}_X < \infty \). Then, under \( A' \) Theorem 5.2 holds true for \( T \to \infty \) and fixed \( N \).

The proof is omitted since is analogous to the integer-valued case. The convergence of bootstrap \( p \)-values as in Theorem 5.3 (with fixed \( N \)) is obtained under the conditions of Propositions S-4.5, S-4.6 by [42, Thm. 2].

**S-5 Additional simulations and empirical results**

**S-5.1 Simulations**

For the integer-valued case, we test linearity of the PNAR model against the nonlinear version in (5). The observed count time series \( \{ Y_t \} \) is generated as in (1), using the copula-based data
generating process of Section S-1.1. We consider a Gaussian copula \( C_{GaR}^{\ldots} \), with correlation matrix \( R = (R_{ij}) \), where \( R_{ij} = \rho^{|i-j|} \), the so called Toeplitz correlation matrix; \( \rho = 0.5 \) is the copula parameter. Set \( \lambda_0 = 1 \) and use a burnout sample of the 300 first observations to initiate the process. The true value of the linear parameters is set to \( \theta^{(1)} = (\beta_0, \beta_1, \beta_2)' = (1, 0.3, 0.2)' \). Then, the QMLE estimation \( \tilde{\theta}^{(1)} \) optimising (14) is computed for each replication. By considering the results of Proposition 4.3, the quasi-score statistic (29) is evaluated and compared with the critical values of a \( \chi^2_1 \) distribution. The same data generating process is employed for simulating observations \( Y_t \) from (5) to compute the empirical power of the test. All results of the simulation study are reported in Table S-1. Histograms and the Q-Q plots of the simulated score test for the integer-valued case are plotted in Fig. S-1. We discover similar findings as the ones discussed in the main text. The asymptotic approximation though is slower and therefore the asymptotic \( \chi^2 \) distribution of the test requires larger temporal and network sample dimensions. Hence, the empirical power of the test tends to 1 at a slower rate. Overall, the empirical evidence shows that, when \( N \) and \( T \) are large enough, the test performs quite satisfactory in the case of integer-valued data. The adequacy of the \( \chi^2 \) test is also confirmed by histograms and Q-Q plots of Table S-1.

Table S-1: Empirical size of the test statistics (29) for testing \( H_0 : \gamma = 0 \) versus \( H_1 : \gamma > 0 \), in model (4), with \( S = 1000 \) simulations, for various values of \( N \) and \( T \). Data are count time series generated from the linear model (4). The empirical power is also reported for data generated from model (5) with \( \gamma = \{0.5, 1\} \).

| Model | Size | Power (\( \gamma = 0.5 \)) | Power (\( \gamma = 1 \)) |
|-------|------|----------------|------------------|
|       |      | 10% | 5% | 1% | 10% | 5% | 1% | 10% | 5% | 1% |
|       |      |     |    |    | 10% | 5% | 1% | 10% | 5% | 1% |
| SBM   | 2    | 4   | 500 | 0.085 | 0.038 | 0.004 | 0.771 | 0.651 | 0.412 | 0.710 | 0.710 | 0.694 |
|       |      | 500 | 10  | 0.100 | 0.031 | 0.001 | 0.096 | 0.031 | 0.003 | 0.096 | 0.045 | 0.010 |
|       |      | 200 | 300 | 0.088 | 0.045 | 0.007 | 0.290 | 0.188 | 0.071 | 0.815 | 0.721 | 0.487 |
|       |      | 500 | 300 | 0.108 | 0.054 | 0.011 | 0.252 | 0.156 | 0.053 | 0.690 | 0.570 | 0.331 |
|       |      | 500 | 400 | 0.123 | 0.075 | 0.015 | 0.297 | 0.190 | 0.060 | 0.809 | 0.705 | 0.454 |
| SBM   | 5    | 10  | 500 | 0.077 | 0.038 | 0.008 | 0.976 | 0.948 | 0.857 | 0.935 | 0.935 | 0.935 |
|       |      | 500 | 10  | 0.089 | 0.027 | 0.002 | 0.114 | 0.049 | 0.004 | 0.196 | 0.099 | 0.009 |
|       |      | 200 | 300 | 0.108 | 0.042 | 0.008 | 0.901 | 0.838 | 0.638 | 1.000 | 1.000 | 1.000 |
|       |      | 500 | 300 | 0.094 | 0.039 | 0.008 | 0.795 | 0.693 | 0.427 | 1.000 | 0.999 | 0.999 |
|       |      | 500 | 400 | 0.108 | 0.059 | 0.011 | 0.863 | 0.771 | 0.549 | 1.000 | 1.000 | 1.000 |
| ER    | -    | 30  | 500 | 0.103 | 0.053 | 0.012 | 0.361 | 0.258 | 0.095 | 0.854 | 0.793 | 0.578 |
|       |      | 500 | 30 | 0.105 | 0.045 | 0.003 | 0.084 | 0.040 | 0.004 | 0.086 | 0.050 | 0.016 |
|       |      | 200 | 300 | 0.091 | 0.052 | 0.005 | 0.141 | 0.080 | 0.021 | 0.357 | 0.241 | 0.093 |
|       |      | 500 | 300 | 0.094 | 0.044 | 0.010 | 0.142 | 0.065 | 0.012 | 0.273 | 0.166 | 0.048 |
|       |      | 500 | 400 | 0.112 | 0.062 | 0.014 | 0.169 | 0.098 | 0.024 | 0.342 | 0.230 | 0.086 |
Figure S-1: Histogram and Q-Q plots for the score test (29), defined in Tab. S-1, against the $\chi^2_1$ distribution; $N = 500, T = 400$. Left: network N-1 with $K = 2$. Center: network N-1 with $K = 5$. Right: N-2 Red line: $\chi^2_1$ density. Dashed blue line: 5% confidence bands.

Figure S-2: Histogram and Q-Q plots for the score test (29), defined in Tab. 1, against the $\chi^2_1$ distribution; $N = 500, T = 400$. Left: network N-1 with $K = 2$. Center: network N-1 with $K = 5$. Right: N-2 Red line: $\chi^2_1$ density. Dashed blue line: 5% confidence bands.
S-5.2 Non-standard testing

Consider the hypothesis test \( H_0 : \alpha = 0 \) vs \( H_1 : \alpha > 0 \) \((\alpha \neq 0)\), for integer and continuous-valued random variables, respectively, on model (7). This is an example of a non-identifiable parameter; in this case it is \( \gamma \). Under the null, the model reduces to the linear NAR model (4). For the continuous case, we simulate data using the data generating process employed in Section 6. Set \((\beta_0, \beta_1, \beta_2)' = (1, 0.3, 0.2)'\). According to the results of Theorem 5.1-5.2, for each of the \( S \) replications, we can approximates the \( p \)-values of the sup-type test, \( \sup_{\gamma \in \Gamma_F} LM_T(\gamma) \), by the Davies bound (35), with \( k_2 = 1 \), where \( \Gamma_F \) is a grid of equidistant values computed as in Section 7. The fraction of cases in which the \( p \)-value approximation is smaller than the standard nominal levels defines the empirical size of the test. The empirical power of the test is obtained by computing the same fraction on data generated by the model (7) instead, with \( \alpha = \{0.1, 0.5\} \) and \( \gamma = \{0.05, 1\} \).

Results for the continuous case are summarized in Table S-2. The empirical size is close to the nominal level, although slightly smaller; this is expected as the approximation using the Davies bound (35) is conservative. For both SBM and Erdős-Rényi model, the empirical power is low, when the tested parameter is close to the tested value, \( \alpha = 0.1 \), and tends to increase for values away from the null (\( \alpha = 0.5 \)). The power of the test obviously improves for small values of the nuisance parameter (\( \gamma = 0.05 \)). When \( N \) and \( T \) are large, the test performs adequately.

The same analysis is replicated for the PNAR model. The data generating mechanism for the multivariate count process \( Y_t \) has been already discussed in Section S-5.1 but now the correlation matrix is given by \( R_{ij} = \rho \), for all \( i, j \), where \( \rho = 0.5 \) is the copula parameter. The results are presented in Table S-3. We obtain analogous results as those in the continuous case.

Finally, a linearity test concerning the threshold model (8), is empirically studied. In this case, \( H_0 : \alpha_0 = \alpha_1 = \alpha_2 = 0 \) vs. \( H_1 : \alpha_l > 0 \), for some \( l = 0, 1, 2 \). The \( p \)-values are computed from the bootstrap approximation procedure for the test \( \sup_{\gamma \in \Gamma_F} LM_T(\gamma) \). The number of bootstrap replications is set to \( J = 499 \). From Table S-4 we note that the empirical size is smaller than the expected nominal levels but the empirical power of the test shows that the test works satisfactorily. These outcomes are in line with previous empirical results in the literature [14, Tab. 5-6]. Results are also analogous to the Gaussian case presented in Table S-4 and therefore are omitted.

S-5.3 Additional empirical results

In this section we consider a continuous-valued dataset containing \( T = 721 \) wind speeds measured at each of \( N = 102 \) weather stations throughout England and Wales. The weather stations are the nodes of the potential network and two weather stations are connected if they share a border. In this way undirected network is drawn on geographic proximity. The dataset is taken by the GNAR R package [40] incorporating the time series data v windt s and the associated network v windnet. As the wind speed is continuous-valued, the NAR model (4) is estimated by OLS. The linearity test against model (6) is computed and compared with the \( \chi^2 \) distribution. Then, the same tests of Section 7 are evaluated, with \( H_0 : \alpha = 0 \) vs. \( H_1 : \alpha \neq 0 \), for the STNAR model (7) and \( H_0 : \alpha_0 = \alpha_1 = \alpha_2 = 0 \) vs. \( H_1 : \alpha_l \neq 0 \), for some \( l = 0, 1, 2 \), in the TNAR model (8). The results are summarized in Table S-5. All the estimated coefficients are significant at standard nominal levels. In this case, the magnitude of the lagged variable dominates the network effect but the latter still has considerable impact. An increasing positive effect is detected for the coefficients. This means that
Table S-2: Empirical size of the test statistics (34) for testing $H_0: \alpha = 0$ in $S = 1000$ simulations of model (7), for $N, T = \{200, 500\}$. Data are continuous-valued and generated from the linear model (4). The empirical power is also reported for data generated from model (7) with $\alpha = \{0.1, 0.5\}$ and $\gamma = \{0.05, 1\}$. The network is derived from Ex. N-1, for first two rows, with $K = 2$; second two: Ex. N-2.

| Model | Size | Power          |
|-------|------|----------------|
|       |      | $\alpha = 0.1, \gamma = 1$ | $\alpha = 0.1, \gamma = 0.05$ | $\alpha = 0.5, \gamma = 1$ | $\alpha = 0.5, \gamma = 0.05$ |
|       | $N$  | $T$ | 10% | 5% | 1% | 10% | 5% | 1% | 10% | 5% | 1% | 10% | 5% | 1% |
| SBM   | 200  | 200 | 0.078 | 0.045 | 0.012 | 0.092 | 0.044 | 0.007 | 0.099 | 0.055 | 0.014 | 0.322 | 0.194 | 0.061 | 0.921 | 0.746 | 0.214 |
|       | 500  | 500 | 0.084 | 0.036 | 0.006 | 0.088 | 0.0048 | 0.014 | 0.128 | 0.074 | 0.012 | 0.504 | 0.368 | 0.167 | 1.000 | 0.997 | 0.840 |
| ER    | 200  | 200 | 0.078 | 0.034 | 0.003 | 0.088 | 0.039 | 0.009 | 0.097 | 0.046 | 0.006 | 0.173 | 0.099 | 0.019 | 0.930 | 0.772 | 0.321 |
|       | 500  | 500 | 0.082 | 0.037 | 0.011 | 0.102 | 0.053 | 0.010 | 0.105 | 0.049 | 0.012 | 0.220 | 0.129 | 0.038 | 0.999 | 0.998 | 0.911 |

Table S-3: Empirical size of the test statistics (34) for testing $H_0: \alpha = 0$ in $S = 1000$ simulations of model (7), for $N, T = \{200, 500\}$. Data are integer-valued and generated from the linear model (4). The empirical power is also reported for data generated from model (7) with $\alpha = \{0.1, 0.5\}$ and $\gamma = \{0.05, 1\}$. The network is derived from Ex. N-1, for first two rows, with $K = 2$; second two: Ex. N-2.

| Model | Size | Power          |
|-------|------|----------------|
|       |      | $\alpha = 0.1, \gamma = 1$ | $\alpha = 0.1, \gamma = 0.05$ | $\alpha = 0.5, \gamma = 1$ | $\alpha = 0.5, \gamma = 0.05$ |
|       | $N$  | $T$ | 10% | 5% | 1% | 10% | 5% | 1% | 10% | 5% | 1% | 10% | 5% | 1% |
| SBM   | 200  | 200 | 0.052 | 0.027 | 0.006 | 0.059 | 0.024 | 0.002 | 0.088 | 0.045 | 0.007 | 0.129 | 0.072 | 0.013 | 0.601 | 0.460 | 0.179 |
|       | 500  | 500 | 0.058 | 0.025 | 0.005 | 0.054 | 0.028 | 0.005 | 0.146 | 0.072 | 0.015 | 0.218 | 0.130 | 0.034 | 0.902 | 0.804 | 0.580 |
| ER    | 200  | 200 | 0.068 | 0.025 | 0.001 | 0.055 | 0.019 | 0.002 | 0.095 | 0.043 | 0.005 | 0.136 | 0.051 | 0.008 | 0.538 | 0.393 | 0.143 |
|       | 500  | 500 | 0.068 | 0.026 | 0.002 | 0.071 | 0.028 | 0.004 | 0.123 | 0.063 | 0.015 | 0.200 | 0.108 | 0.023 | 0.880 | 0.794 | 0.557 |
Table S-4: Empirical size at of the test statistics \(34\) for testing \(H_0: \alpha_0 = \alpha_1 = \alpha_2 = 0\) in \(S = 200\) simulations of model \(8\), for \(N = 8\) and \(T = 1000\). Data are continuous-valued and generated from the linear model \(4\). The empirical power is also reported for data generated from model \(8\) with \(\alpha = (\alpha_0, \alpha_1, \alpha_2)' = (0.5, 0.2, 0.1)'\) and \(\gamma = 1\). The network is derived from Ex. N-1, with \(K = 2\).

| \(N\) | \(T\) | \(\% 10\) | \(5\) | \(1\) | \(\% 10\) | \(5\) | \(1\) |
|------|------|-------|-----|-----|-------|-----|-----|
| 8    | 1000 | 0.045 | 0.002 | 0.000 | 1.000 | 1.000 | 1.000 |

The expected wind speed for a weather station is positively correlated by its past speed and the past wind speeds detected on neighboring stations. The estimated variance of the errors is approximately 0.156; it is computed with the moment estimator \(\hat{\sigma}^2 = (NT)^{-1} \sum_{t=1}^{T} \sum_{i=1}^{N} (Y_{i,t} - \lambda_{i,t}(\hat{\theta}))^2\). The existence of nonlinear components is typically encountered in environmental time series studies. The null assumption of linearity is rejected against the alternative \(6\). This gives an indication of possible nonlinear shifts in the intercept. In addition, the linearity is also rejected when it is tested against the STNAR model \(7\) and TNAR model \(8\), at the usual significance levels, using the Davies bound and the bootstrap sup-type test. Such results suggest that a nonlinear regime switching effects could be present.

Table S-5: QMLE estimates of the linear model \(4\) for wind speed data. Standard errors in brackets. Linearity is tested against the nonlinear model \(6\), with \(\chi^2\) asymptotic test \(29\); against the STNAR model \(7\), with approximated \(p\)-values computed by (DV) Davies bound \(35\), bootstrap \(p\)-values of sup-type test; and versus TNAR model \(8\) (only bootstrap).

| Models | \(\beta_0\) | \(\beta_1\) | \(\beta_2\) | Std. | Chi-sq. | DV | Bootstrap |
|--------|-------------|-------------|-------------|------|--------|----|-----------|
| \(4\)  | 0.154       | 0.157       | 0.768       | (0.017) | 131.052 | -  | -         |
| \(7\)  | 0.154       | 0.157       | 0.768       | (0.011) | < 0.001 | 0.02 | < 0.001  |
| \(8\)  | 0.154       | 0.157       | 0.768       | (0.009) | -      | -  | -         |

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