Abstract. At present, the vast majority of known image models are varieties of random fields defined on rectangular flat grids or grids of higher dimension. In some practical situations, images have a circular or radial-circular structure. For example, an image of a facies (a thin film of dried biological fluid), an eye, a biological cell, a flower, a slice of a tree trunk, etc. This circumstance requires the development of appropriate models of random fields describing images of this type. This paper proposes autoregressive models of homogeneous and inhomogeneous random fields defined on a circular spiral. The problem of correlation analysis of such models is considered. Examples of imitation of circular images are given. This paper proposes linear autoregressive models of homogeneous and inhomogeneous random fields defined on a circle. The samples of these fields are defined on a circular or spiral grid. The next field value is a linear combination of several previous values and a standard random variable. Expressions of the correlation function of these fields are given depending on the values of the autoregression parameters. A graphic representation of the correlation function on a circle and examples of simulated images are presented in the figures.

1. Introduction
Various models of random fields are used as mathematical models of images. However, in most well-known models, the fields are defined on rectangular grids or grids of higher dimension. To represent images of a particular class, one has to select an appropriate model of a random field, for example, fields with a given type of covariance function (CF) or normalized CF. For this task of correlation synthesis, autoregressive, wave, Gibbs, polynomial, morphological, and other models can be applied [1-6]. In [7-11], models of images defined on curved surfaces are given. These models served as the basis for creating image processing algorithms, for example, [1, 2, 4, 11-13]. Much attention is paid to medical images processing, for example, [14-19].

In some practical situations, the processed images have a radial, circular or radial-circular structure. For example, images of the biological fluid, eye, biological cell, flower, the slice of a tree trunk and a map of Moscow shown in figure 1. In addition, radar, ultrasound and other images are physically obtained in polar or spherical coordinates. These circumstances require their consideration in random field models. This paper proposes autoregressive models of random fields defined on a circle.

2. Models of random fields on a cylinder
We first consider the autoregressive model of a random field on a cylinder. It will be the basis for circular fields. In [9], the following model was used to represent images on a cylindrical grid similar to Habibi autoregression model [11] of a flat image:
where $k$ is a spiral turn number; $l$ is a node number $l = 0, ..., T - 1$; $x_{k,l} = x_{k+1,l,T}$ when $l \geq T$; $T$ is the period, i.e. the number of points in one turn; $\xi_{k,l}$ are independent standard random variables.

Figure 1. Examples of images of radial-ring structure.

For the convenience of analyzing this model, we will assume that the pixels are numbered and located on a cylindrical spiral (figure 2 (a)). Then model (1) can be represented in an equivalent form as a model of a random process, which is a scan of the image along the spiral:

$$x_n = a x_{n-1} + b x_{n-T} - a b x_{n-T-1} + c \xi_n,$$

where $n = kT + l$. The characteristic equation of model (2) is

$$z^T - a z^{T-1} - a z + a = 0 \quad \text{or} \quad (z^T - b)(z - a) = 0.$$

Applying the z-transform, we have

$$(z^T - b)(z - a) x_n = c \xi_n \quad \text{or} \quad x_n = \frac{c}{(z^T - b)(z - a)} \xi_n.$$

Therefore, CF $V(n) = M[x_{n,n}]$ of random process $\{x_n, n = 0, 1, ...\}$ is expressed as follows

$$V(n) = c^2 \int \frac{z^{n-1}dz}{2\pi i \sum_{j=0}^{T-1} (z^T - b)(z - a)(z - b)(z - a)} ,$$

where integration is carried out along a unit complex circle. Using residues, we obtain

$$V(n) = c^2 \left[ \frac{1}{(1-b^T)(1-a^T)(1-a^T)(1-b^T)} \right] \left[ \sum_{k=0}^{T-1} \frac{z_k}{(1-a^T)(1-a^T)} z_k^{n^*} + \frac{s}{(1-s^T)(1-b^T)(1-b^T)(1-b^T)} \rho^* \right],$$

where $z_k = e^{i2\pi k/T}$ and $s = a^T$. In particular, when $n = kT$ we obtain

$$V(kT) = \frac{c^2}{(1-a^T)(1-b^T)(1-b^T)(1-b^T)} \left[ (1-s^T)b^{k+1} - (1-b^T)s^{k+1} \right]$$

and the variance, when $k=0$:

$$\sigma^2 = \frac{c^2(1+bs)}{(1-a^T)(1-b^T)(1-b^T)}.$$

To reduce the calculations, it is possible to calculate only $V(0), V(1), ..., V(T)$ by formula (3), and for the rest of values use recurrent formula

$$V(n) = a V(n-1) + b V(n-T) - a b V(n-T-1).$$
This CF decreases with increasing distance \( n \), but at distances divisible by period \( T \), it is high (figure 2 (b)). Figure 2 (c) shows an example of a cylindrical image cut lengthwise and unfolded, simulated using model (1).

![Figure 2](image)

**Figure 2.** Cylindrical image: (a) cylindrical mesh, (b) image of the correlation function, (c) example of simulation.

Note that the described model is only the simplest case. By introducing additional terms into equation (1), one can obtain random fields with more complex CF.

### 3. Models of random fields on a circle

A polar coordinate system \((r, \phi)\) is convenient for circular images representation. To do this, we will consider the turns of the cylindrical spiral of model (1) as turns of a circle grid (figure 3 (a)). In other words, index \( k \) is converted into a polar radius, and index \( l \) into a polar angle. Thus, the value \( x_{k,l} \) in the pixel \((k,l)\) of the cylindrical image is converted to the same value in the pixel \((k\Delta r, l\Delta \phi)\) of the circular image. When using model (1), it is also convenient to use a spiral grid (figure 3(b)), similar to the cylindrical spiral in figure 2 (a). Note that this representation in the form of a spiral is made conditionally to facilitate analysis. In the image, a sequence of turns of a conditional spiral is a sequence of expanding circles.

The parameters \( a \) and \( b \) of model (3) set the degree of correlation in the radial and circular direction. When \( ab < 1 \), the image will have a higher correlation in the radial directions. In figure 4 (a), the simulated image is shown at \( a = 0.95 \) and \( b = 0.99 \). When \( ab > 1 \), the image will have a higher correlation in the circular direction. Figure 4 (b) shows the simulated image with \( a = 0.99 \) and \( b = 0.95 \). In the case \( ab \approx 1 \), the image is approximately equally correlated in both directions. Figure 4 (c) shows such a simulated image with \( ab = 0.95 \).

![Figure 3](image)

**Figure 3.** Grids on a circle: (a) circular, (b) spiral.

The CF of the circular images is (3), since they are determined by model (2). But this CF is for linear pixel numbering. Therefore, the resulting image is homogeneous only for this numbering. If we consider the Euclidean distance, the image correlation weakens with distance from the center. And this is quite consistent with the specifics of the radial-ring images.
Figure 4. Simulated images on a circle for various values of parameters models (2): (a) $a = 0.95$, $b = 0.99$, (b) $a = 0.99$, $b = 0.95$, (c) $a = b = 0.95$.

Let us consider the CF view of the circular image. The circular image in this model is actually a geometric transformation of a cylindrical one, so its CF can be obtained from the CF of a cylindrical image (figure 5(a) shows the values of CF relative to the central pixel of this figure. The lighter areas correspond to larger values. The isocovariation lines $V(m,n) = \text{const}$ are shown in black. It is natural that these lines are circles. Figure 5(b) shows the values of CF and its isolines relative to the pixel in the middle of a radius (maximum brightness). Isolines at short distances are close to rhombuses, which is the typical property of Habibi model.

Note that by applying more complex types of autoregression, it is possible to obtain circular images with very different properties (that is, with covariance functions other than (3) types).

Figure 5. Normalized correlation function of a circular image: (a) relative to the center, (b) relative to the middle of a radius.

4. Doubly stochastic models of inhomogeneous random fields on a circle

In [10], doubly stochastic models were used to represent inhomogeneous images with random inhomogeneities. In these models, some random field sets the parameters of the resulting random field. The same method can be used to represent inhomogeneous circular images.

Let $Y = \{y_n\}$ be the realization of a circular image obtained using a model of the form (2). We take $Y$ for the control image, which forms the variable parameters of the controlled image $X = \{x_n\}$ in accordance with the model (2), in which instead of the constant $a$ and $b$ are used autoregressions

$$a_{n+1} = r_1 a_n + \gamma_1 y_n, \quad b_{n+1} = r_2 b_n + \gamma_2 y_n.$$  \hspace{1cm} (4)

Figure 6 shows an example of such a simulation with $r_1 = r_2 = 0.999$, $\gamma_1 = \gamma_2 = 0.005$ and the control image of a predominantly radial structure.

In the described model, two images are unequal: one controls the parameters of the other. In [10], a model of autoregressive images defined on a cylinder was proposed, jointly controlling the parameters of each other. Let’s apply this approach to circular images. Let image $Y$ determine the parameters for
the next pixel of image $X$, as in (4). At the same time, image $X$ sets the parameters for the next element of the image $Y$ in the same way. As a result, these two images together control the parameters of each other. Figure 7 shows an example of such images. Significant correlation of images is noticeable, which is a consequence of the mutual influence on the autoregressive parameters of their models.

Figure 6. Simulated images with control: (a) control image, (b) controlled image.

Figure 7. Simulated images with mutual influence.

5. Conclusions
In this paper, autoregressive models of random fields on a circle are proposed, which allow describing and simulating circular images of a radially circular structure. The covariance function expressions are given. For the representation of inhomogeneous images, doubly stochastic models with random inhomogeneities are applied. The choice of parameters of these models allows us to represent a wide class of such images. Examples of the application of the described models to simulate circular images are given.

6. References
[1] Soifer V A, Popov S B, Mysnikov V V and Sergeev V V 2009 Computer image processing. Part I: Basic concepts and theory (VDM Verlag Dr. Muller)
[2] Vizilter Y V, Pyt'ev Y P, Chulichkov A I and Mestetskiy L M 2015 Morphological image analysis for computer vision applications Computer Vision in Control Systems-1, ISRL 73
[3] Shalygin A S and Palagin Y I 1986 Applied methods of statistical modeling (Mechanical engineering, American society of mechanical engineers)
[4] Gonzalez R C and Woods R E 2007 Digital image processing (Prentice Hall)
[5] Gimel'farb G L 1999 Image Textures and Gibbs Random Fields (Kluwer Academic Publishers Dordrecht)
[6] Myasnikov V V 2018 Description of images using a configuration equivalence relation Computer Optics 42(6) 998-1007 DOI: 10.18287/2412-6179-2018-42-6-998-1007
[7] Krasheninnikov V R and Vasil'ev K K 2018 Multidimensional image models and processing Computer Vision in Control Systems-3, ISRL 135 11-64
[8] Krasheninnikov V R 2012 Models of random fields on surfaces Proceedings of the Samara
Scientific Center of the Russian Academy of Sciences 4 812-816

[9] Krasheninnikov V R, Kalinov D V and Pankratov Yu G 2001 Spiral autoregressive model of a quasi-periodic signal Pat Rec Im An 11 211-213

[10] Krasheninnikov V R and Subbotin A Yu 2018 Doubly stochastic model of a quasi-periodic process as an image on a cylinder Proceedings of the International Scientific and Technical Conference "Advanced Information Technologies" (PIT-2018) (Samara: Samara Scientific Center of the Russian Academy of Sciences) 1017-1021

[11] Habibi A 1972 Two-dimensional Bayesian estimate of images Proc IEEE 60 878-883

[12] Woods J W 1981 Two-dimensional Kalman filtering Topics in Applied Physic 42 11-64

[13] Maass P and Stark H-G 1994 Wavelets and Digital Image Processing (Surveys on Mathematics for Industry)

[14] Bourne R 2010 Fundamentals of Digital Imaging in Medicine (London: Springer)

[15] Lim J H, Ong S H and Xiong W 2015 Biomedical Image Understanding: Methods and Applications (Wiley)

[16] Ammari H 2012 Mathematical Modeling in Biomedical Imaging II: Optical, Ultrasound and Opto-Acoustic Tomographies (Springer)

[17] Shirokanov A S, Kirsh D V, Ilyasova N Yu and Kupriyanov A V 2018 Investigation of algorithms for coagulate arrangement in fundus images Computer Optics 42(4) 712-721 DOI: 10.18287/2412-6179-2018-42-4-712-721

[18] Khorin P A, Ilyasova N Yu and Paringer R A 2018 Informative feature selection based on the Zernike polynomial coefficients for various pathologies of the human eye cornea Computer Optics 42(1) 159-166 DOI: 10.18287/2412-6179-2018-42-1-159-166

[19] Krasheninnikov V R and Kopylova A S 2012 Algorithms for automated processing images of blood serum facies Pat Rec Im An 22 583-592

[20] Vasilyev K K and Dement'iev V E 2017 Presentation and processing of satellite multi-zone images (Ulyanovsk: Ulyanovsk State Technical University)

[21] Vasil'ev K K, Dement'iev V E and Andriyanov N A 2015 Doubly stochastic models of images Pat Rec Im An 25 105-110