Extended loop quantum gravity*

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Received 16 March 2010, in final form 1 July 2010
Published 3 August 2010
Online at stacks.iop.org/CQG/27/185016

Abstract
We discuss a constraint structure of extended theories of gravitation (also known as \( f(\mathcal{R}) \) theories) in the vacuum self-dual formulation introduced in Fatibene et al (2010 Class. Quantum Grav. 27 165021).

PACS numbers: 04.50.Kd, 04.20.Fy

1. Introduction
We have recently investigated a formulation of \( f(\mathcal{R}) \) theories (in a metric-affine framework) based on nonlinear actions similar to the Holst Lagrangian; see [1]. These actions are in fact written in terms of the scalar curvature \( \beta \mathcal{R} \) of the Barbero–Immirzi connection with the parameter \( \beta \) (see [2, 3]) and are dynamically equivalent to the corresponding ‘classical’ \( f(\mathcal{R}) \) theory. For the linear case \( f(\beta \mathcal{R}) = \beta \mathcal{R} \), one obtains the standard Holst action. Hence these new actions are to be understood as Barbero–Immirzi formulations of the corresponding classical \( f(\mathcal{R}) \) theory.

This could be interesting for at least two reasons. From the point of view of loop quantum gravity (LQG) this new formulation provides a family of models which are classically well understood and investigated in detail (see [4, 5]). There are many classical effects known in \( f(\mathcal{R}) \) theories that should be traced in their quantum genesis. The minisuperspace of these models is quite well understood and should be studied in loop quantum cosmology (LQC) formulation (see [6]), to contribute to a better understanding of the classical limit of LQG models. Moreover, as in all metric-affine models, matter has a non-trivial feedback on the gravitational field which would also be interesting to trace in its quantum origin. It is often said that matter in LQG simply adds new labels to spin networks, while in these models one could

* This paper is published despite the effects of the Italian law 133/08 (http://groups.google.it/group/scienceaction). This law drastically reduces public funds to public Italian universities, which is particularly dangerous for free scientific research, and it will prevent young researchers from getting a position, either temporary or tenured, in Italy. The authors are protesting against this law to obtain its cancellation.
expect a more complicated mechanism that would be certainly interesting to be discussed in detail. Finally, there are a number of equivalences, e.g. with scalar–tensor models (see [7]), that again would be interesting to discuss in detail at a quantum level. Let us stress that these equivalences are known to hold at the classical level and, as usual, one should investigate whether they still hold at the full quantum level or just emerge classically.

From the classical viewpoint we provide here a route to define a quantization in the manner of loop of $f(R)$ theories. Of course classical effects of these extended theories of gravitation have been extensively investigated. It is therefore interesting to investigate also their quantum effects. For example it would be interesting to see whether the removal of singularities that has been shown to hold in standard loop quantization of GR is preserved generically in these extended gravitational models.

For the sake of simplicity we restrict here our attention to the Euclidean signature and to the self-dual formulation (which in the Euclidean sector is in fact a special case of the Barbero–Immirzi formulation) and show that one can apply LQG methods (see [8]) also to the quantization of these theories. In vacuum we obtain something similar to Einstein gravity with a cosmological constant. This is very well expected on the basis of a classical equivalence (see [9]); however, let us stress that our result seems to establish a stronger equivalence at the quantum level and not only at the classical level.

Moreover, let us stress that the classical equivalence holds only in vacuum, while the equivalence is broken when generic matter is considered and the extended models are equivalent to scalar–tensor theories; see [7]. Tracing the mechanism which leads to this shift of equivalence at the quantum level would be therefore rather interesting and will be investigated in forthcoming papers. We follow the notation introduced in [1] and [8].

The aim of this paper is to go toward a quantum description of $f(R)$ theories; we however have to mention the reverse problem of giving a (semi)-classical account of the quantum effect of ordinary standard LQG models; see [10]. The two approaches are somehow complementary and are based on similar techniques.

2. Self-dual formulation for extended theories

In [1] we introduced

$$\beta R := R^{ab}_{\mu\nu} e^a_\mu e^b_\nu + \beta R^{ab}_{\mu\nu} e^a_\mu e^b_\nu e_{cdab},$$

(2.1)

where $e^a_\mu$ is a spin frame (see [11]), $R^{ab}_{\mu\nu}$ is the curvature of a spin connection $\omega^a_{\mu\nu}$ on a four-dimensional (spin) manifold $M$ and $\beta \neq 0$ is a real parameter. Indices $a, c, \ldots = 0 \ldots 3$ and $\mu, \nu, \ldots = 0 \ldots 3$, while $i, j, \ldots = 1 \ldots 3$.

In the Euclidean sector one obtains for $\beta = \frac{1}{2}$ the standard self-dual curvature

$$^+R := R^{ab}_{\mu\nu} e^a_\mu e^b_\nu + \frac{1}{2} R^{ab}_{\mu\nu} e^a_\mu e^b_\nu e_{cdab},$$

(2.2)

which can be written in terms of the curvature $F^i_{\mu\nu} := p^{ij}_{ab} R^{ab}_{\mu\nu}$ of the usual self-dual connection $A^i_{\mu} := p^{ij}_{ab} \omega^a_{\mu} = \omega^0_{\mu} + \frac{1}{2} \epsilon^i_{jk} \omega^j_{\mu}$ as follows:

$$\frac{1}{2}^+R = \frac{1}{2} R^{cd}_{\mu\nu} \left( \delta^a_{[c} e^b_{d]} + \frac{1}{2} \epsilon^a_{cd} e^b_{d} \right) e^a_{\mu} e^b_{\nu} = R^{cd}_{\mu\nu} p^{ij}_{ab} e^a_{\mu} e^b_{\nu} = p^{ij}_{ab} F^i_{\mu\nu} e^0_{\mu} e^0_{\nu} := F.$$  

(2.3)

Here, $p^{ij}_{ab}$ denotes the algebra projector $p : \text{spin}(4) \to \text{su}(2)$ on self-dual forms. It is given by

$$p^{ij}_{ab} = \frac{1}{2} \delta^{ij}_{ab}, \quad p^{ij}_{a0} = -p^{ij}_{0a}, \quad p^{jk}_{ij} = \frac{1}{2} \epsilon^{ijk}.$$  

(2.4)

and the inverse projector $p^{ij}_{ab}$ is defined by

$$p^{0i}_{0j} = \frac{1}{2} \delta^{ij}_{00}, \quad p^{0i}_{00} = -p^{0i}_{00}, \quad p^{0j}_{0k} = \frac{1}{2} \epsilon^{ijk}.$$  

(2.5)
One can easily prove that
\[ p_{ab}^{(j)} = δ^{j}_{a}, \quad p_{ab}^{(d)} = \frac{1}{2} (δ^{ja}_{b} + \frac{1}{2} ε_{abcd}). \] (2.6)

One is then led to consider the following family of Lagrangians:
\[ L^* = \frac{1}{2κ} e f (F) + L_m, \] (2.7)
where κ = 8πG, e is the determinant of the frame matrix, f is a generic analytic function and \( L_m \) encodes the matter contribution. Usually matter is assumed to couple only with \( g \) (and possibly to its derivatives up to some finite order, usually, in view of minimal coupling principle, at most 1) and not to the connection \( ω_{ab} \). Hereafter we shall just consider the vacuum sector, i.e. we set \( L_m = 0 \).

In the special case \( f(F) = F \), one obtains an equivalent formulation of the usual self-dual action
\[ \mathbb{L}^* = \frac{1}{8κ} R^{ab} e^{e} ∧ e^{d} e_{abcd} = \frac{1}{16κ} \left( R^{ab}_{\ μν} + \frac{1}{2} ε_{ab} ε_{ef} R^{ef}_{\ μν} \right) e^{e}_{\ μν} e^{d}_{\ μν} e_{abcd} d\!s \]
\[ = \frac{1}{14κ} R^{ij}_{\ μν} \left( δ^{ab}_{\ [ij]} + \frac{1}{2} ε_{ab} ε_{ef} \right) ε^{i}_{\ μν} ε^{j}_{\ μν} e_{abcd} d\!s = \frac{1}{8κ} R^{ij}_{\ μν} p^{ab}_{ij} p^{ef}_{ij} e^{i}_{\ μν} e^{f}_{\ μν} e_{abcd} d\!s \]
\[ = \frac{1}{8κ} p^{ab}_{ij} F^{ij}_{\ μν} e^{i}_{\ μν} e^{f}_{\ μν} e_{abcd} d\!s = \frac{e}{2κ} p^{ab}_{ij} F^{ij}_{\ μν} e^{i}_{\ μν} d\!s = \frac{e}{2κ} F d\!s, \] (2.8)
where \( d\!s \) is the standard local basis of 4-forms on \( M \) induced by coordinates.

The field equations of the Lagrangian \( \mathbb{L}^* \) are
\[ \begin{align*}
p^{ab}_{ij} F^{ij}_{\ μν} e^{i}_{\ μν} e^{f}_{\ μν} e_{abcd} & = 0 \\
p^{ab}_{ij} \left( e^{i}_{\ μν} e^{f}_{\ μν} \right) e_{abcd} & = 0.
\end{align*} \] (2.9)

Let us now consider a Cauchy (boundary) surface \( i : S \rightarrow M : k^A \mapsto x^\mu(k), A, B, \ldots = 1 \ldots 3; \) in the coordinates \( x^\mu = (t, k^A) \) adapted to the submanifold \( S \), one has \( i : k^A \mapsto k^A \) and \( \partial_\alpha x^\mu = δ^\mu_\alpha \). The unit covector normal to \( S \) is given by \( n = dx^0 \). One can use anti-self-dual transformations to define a canonical-adapted frame \( e_a = x^A_\partial_\mu \) and coframe \( e^\mu = \bar{\tau}^\mu d\!x^\mu \) (see [12]) given by
\[ \begin{align*}
e^0 & = N^{-1} \quad e^i = 0 \\
e^0 & = N^{-1} N^i \\
e^i & = \bar{\alpha}^i, \quad \bar{\tau}^0 = N \\
\bar{\tau}^i & = - N^j \bar{\alpha}^j. \end{align*} \] (2.10)

Tetrads (or better spin frames; [11]) adapted to \( S \) define the triads \( \epsilon_i = e_i = x^A_\partial_\mu \) on \( S \). Also the self-dual connection can be projected onto \( S \) to define a connection \( A^A_\mu = A^A_\mu \partial_\lambda x^\mu \) on \( S \). Let us denote by \( F^j_{\ μν} = F^j_{\ μν} \partial_\lambda x^\mu \partial_\sigma x^\nu \) the projected curvature (which is the same as the curvature of the projected connection); for later convenience, let us also define the tangent–normal projection of the curvature \( F^j_{\ μν} = F^j_{\ μν} \partial_\lambda x^\mu n^\nu \) (of course the normal–normal projection vanishes due to the skew-symmetry of \( F \)).

Let us also set \( E^A_i = ε_{i}^A \) for the momentum conjugated to the connection \( A^A_\mu \) written in terms of the triad \( ε^A_\nu \) tangent to \( S \), with \( ε \) being the determinant of the (co)triad \( ε^A_{\mu} \).

Field equations (2.9) can be projected onto \( S \) to obtain a number of evolution equations and the following constraints on \( S \):
\[ \begin{align*}
\frac{A}{V_A} E^A_i & = 0 \\
F^j_{\ AB} E^B_k & = 0 \\
ε_{ljk} F^j_{\ AB} E^A_k & = 0.
\end{align*} \] (2.11)
These constitute the starting point of the LQG quantization scheme. The first equation is related to gauge covariance, the second to Diff(S)-covariance, while the third equation is called the Hamiltonian constraint; when quantized it becomes the so-called Wheeler–deWitt equation and it encodes the (quantum) dynamics. In order to solve the first and second equations one defines a Hilbert space spanned by spin knots (see [8]) so that the Wheeler–deWitt equation is implemented as an operator on that space and it defines physical states.

On this basis one expects to be able to perform the same steps with the extended models $f(F)$; since the extended models are still gauge and generally covariant, the first and second equations are expected to remain unchanged. This would mean that the definitions of area and volume operators are unchanged and ‘spacetime’ gets discretized in the extended models exactly as in standard LQG. Since the extended models are known to provide a modified dynamics with respect to standard GR, one also expects that the Wheeler–deWitt equation has to be modified.

We will hereafter compute the analogous of equations (2.11) for the action (2.7) in order to fully confirm our expectations.

3. Constraint structure

Let us then consider the Lagrangian

$$L^* = \frac{e}{2\kappa} f(F),$$

(3.1)

i.e. the purely gravitational part of (2.7).

The field equations are

$$\begin{cases}
  f' p^{ab}_{\mu} e_{\mu}^a e_{\nu}^b = \frac{1}{2} f e_{\nu}^b = 0 \\
  p_{\mu}^{ab} \nabla_{\mu} (ef^a e^b) = 0
\end{cases}$$

(3.2)

The master equation $f' F - 2 f = 0$ is obtained by tracing the first one by means of $e^b_{\nu}$; see [1] and [9]. This can be replaced back into the first equation to obtain

$$f' (p^{ab}_{\mu} e_{\mu}^a e_{\nu}^b - \frac{1}{2} F e_{\nu}^b) = 0 \Rightarrow \quad p_{\mu}^{ab} F^a_{\mu\nu} e_{\mu}^a - \frac{1}{2} F e_{\nu}^b = 0,$$

(3.3)

where we used the fact that generically $f' \neq 0$ on the zeros of the master equation. For simplicity let us assume that the master equation has only one (simple) zero $F = \rho$; when there are many (simple) zeros, each of them defines a sector of the quantum theory and one is supposed to sum over all sectors, which are in correspondence with the discrete zero structure of the analytic function $f$.

Let us also define a conformal tetrad $\tilde{e}_{\mu}^a = \sqrt{|f'|} e_{\mu}^a$, set $\sigma = \text{sgn}(f' (\rho))$ and use tilde to denote quantities depending on the conformal tetrad, e.g. $\tilde{E}_{iA} = \tilde{e}_{iA} = |f'| E_{iA}$ and

$$\tilde{F} = p^{ab}_{\mu} F_{\mu\nu} e_{\mu}^a e_{\nu}^b = \frac{\sigma}{f'} F.$$

(3.4)

The field equations are hence equivalent to

$$\begin{cases}
  p_{\mu}^{ab} F_{\mu\nu} e_{\mu}^a e_{\nu}^b = 0 \\
  f' F - 2 f = 0 \Rightarrow \quad F = \rho \\
  p_{\mu}^{ab} \nabla_{\mu} (\tilde{e}_{\mu}^a e_{\nu}^b) = 0.
\end{cases}$$

(3.5)

The third equation implies the constraint

$$\nabla_{A} \tilde{E}^A_i = 0$$

(3.6)

as in the standard case, though for the conformal frame $\tilde{e}_{\mu}^a$. 
The second equation can now be expanded as
\[ \bar{F} = p_{i}^{ab} F_{\mu \nu}^{a b} e_{a}^{\mu} e_{b}^{\nu} = 2 p_{i}^{0} F_{\mu 0}^{a} e_{a}^{\mu} e_{i}^{\nu} + p_{i}^{jk} F_{\mu \nu}^{a j k} e_{a}^{\mu} e_{i}^{\nu} = -\bar{F}_{A}^{A} e_{i}^{A} + \frac{1}{2} \epsilon_{i}^{lk} F_{AB}^{i} e_{i}^{A} e_{k}^{B} = \frac{\sigma}{f^{j} \rho}, \] (3.7)
which allows us to express \( \bar{F}_{A}^{i} e_{i}^{A} \) as a function of constrained fields, i.e.
\[ \bar{F}_{A}^{i} e_{i}^{A} = \frac{1}{2} \epsilon_{i}^{lk} F_{AB}^{i} e_{i}^{A} e_{k}^{B} = \frac{\sigma}{f^{j} \rho}. \] (3.8)

Note that the first equation is really different from the standard case (i.e. LQG without the cosmological constant) due to the different coefficient \( \frac{1}{2} \) (which in the standard case is \( \frac{1}{2} \) and allows a complete cancellation of \( F_{A}^{i} e_{i}^{A} \)). The standard case in LQG can be recovered by setting \( f(F) = F \); in this case the master equation simply implies \( F = 0 \) and the standard case without the cosmological constant is obtained in particular. The first equation can be projected in the normal direction to the constraint to obtain
\[ \left( p_{i}^{ab} F_{\mu \nu}^{a b} - \frac{1}{4} \bar{F} e_{a}^{\mu} e_{b}^{\nu} n_{\alpha} \right) e_{i}^{\alpha} = 0 \] \Rightarrow \] (3.9)
\[ p_{i}^{0} F_{\mu 0}^{a} e_{a}^{\mu} e_{i}^{\nu} - \frac{1}{4} \bar{F} e_{a}^{\nu} \] \Rightarrow \] (3.10)
\[ F_{AB}^{i} e_{i}^{A} e_{i}^{B} + \frac{1}{2} \bar{F} e_{i}^{0} = 0. \] (3.11)

For \( d = k = 1 \ldots 3 \), one has
\[ F_{AB}^{i} e_{i}^{A} e_{i}^{B} = 0 \] \Rightarrow \] F_{AB}^{i} \bar{E}_{i}^{B} = 0. \] (3.12)

For \( d = 0 \), one instead has
\[ \bar{F}_{A}^{i} e_{i}^{A} + \frac{1}{2} \bar{F} = 0 \] \Rightarrow \] (3.13)

and, using (3.7) and (3.8), one obtains
\[ \bar{F}_{A}^{i} e_{i}^{A} = \frac{1}{2} \bar{F}_{A}^{i} e_{i}^{A} + \frac{1}{4} \epsilon_{i}^{lk} F_{AB}^{i} e_{i}^{A} e_{k}^{B} = \frac{1}{2} \bar{F}_{A}^{i} e_{i}^{A} + \frac{1}{4} \epsilon_{i}^{lk} F_{AB}^{i} e_{i}^{A} e_{k}^{B} \] \Rightarrow \] (3.14)
\[ \epsilon_{i}^{lk} F_{AB}^{i} e_{i}^{A} e_{k}^{B} = \frac{\sigma}{f^{j} \rho} \] \Rightarrow \] (3.15)
\[ \epsilon_{i}^{lk} F_{AB}^{i} \bar{E}_{i}^{A} \bar{E}_{i}^{B} = \frac{\sigma}{f^{j} \rho} \bar{E} = \frac{\sigma}{f^{j} \rho} \bar{E}. \] (3.16)

where \( \bar{E} := \text{det}(\bar{e}_{i}^{A}) = \bar{e}^{A} \bar{e}^{-1} = \bar{e}^{2} \) denotes the determinant of the conformal momentum \( \bar{E}_{i}^{A} \).

Let us stress that all this can be done also in the standard LQG framework, though in that case \( F_{i}^{i} \) does not enter other constraints and hence can be ignored.

Accordingly, the constraints can be written in terms of the conformal triad as follows:
\[ \begin{align*}
\bar{\nabla}_{A} \bar{E}_{i}^{A} &= 0 \\
F_{AB}^{i} \bar{E}_{i}^{B} &= 0 \\
\epsilon_{i}^{lk} F_{AB}^{i} \bar{E}_{i}^{A} \bar{E}_{i}^{B} &= \frac{\sigma}{f^{j} \rho} \bar{E}.
\end{align*} \] (3.17)
As expected, the first and second constraints are unchanged with respect to (2.11), while the Wheeler–deWitt equation is modified by the ‘source term’ \(\frac{2}{\ell} \rho \tilde{E}\), which explicitly depends on the nonlinearity of \(f(F)\). This is the quantum counterpart of what happens classically for \(f(R)\) theories and reflects also what happens in standard LQG with the cosmological constant \(\Lambda = -\frac{1}{2f(R)}\rho\); see appendix A. Let us also note that the third constraint is a density, which is fundamental in the approach to quantization proposed by Thiemann; see [13].

4. Conclusions and perspectives

We have shown that, in the generic extended models introduced in [1], constraints allow a loop theory approach to quantization formally similar to what one usually does in vacuum models with a cosmological constant. This shows that the equivalence between \(f(R)\) models and Einstein gravity with the cosmological constant (shown in [9] to hold in the classical theory) holds also at the quantum level.

Of course more attention should be paid when matter couplings are considered, when this equivalence is known to break and is replaced at least by a conformal equivalence.

Also the whole Hamiltonian structure of the theory should be verified in detail to exclude second-class constraints which might add further equations to set (3.17). These constraints (3.17) are in any case the necessary conditions on the boundary \(S\). Since from them discretization of ‘spacetime’ follows, one can claim in any event that extended spacetimes are discretized as in standard LQG.

Acknowledgments

We wish to thank C Rovelli for discussions about the Barbero–Immirzi formulation. We acknowledge the contribution of INFN (Iniziativa Specifica NA12) and the local research project Leggi di conservazione in teorie della gravitazione classiche e quantistiche (2010) of Dipartimento di Matematica of University of Torino, Italy.

Appendix A. LQG with a cosmological constant

Let us here briefly review the standard result for LQG quantization in vacuum with a cosmological constant in order to compare it with what we found for extended models.

Let us consider the Lagrangian

\[
L_\Lambda = \left( \mathcal{R}_{ab} + \frac{\Lambda}{6} e^a \wedge e^b \right) \wedge e^c \wedge e^d \epsilon_{abcd} = \left( \frac{1}{2} \mathcal{R}_{\mu \nu}^{ab} + \frac{\Lambda}{6} e^a_{(\mu} e^b_{\nu)} \right) e^c_{(a} e^d_{b)} \epsilon^{\mu \nu \rho \sigma} \epsilon_{abcd} d\mathbf{s}
\]

\[
= e \left( \frac{1}{2} \mathcal{R}_{\mu \nu}^{ab} e^a_{(\mu} e^b_{\nu)} \epsilon^{\rho \sigma} e_{abcd} + \frac{\Lambda}{6} \epsilon_{abcd} \right) d\mathbf{s} = 2 e \left( \mathcal{R}_{\mu \nu}^{ab} e^a_{\mu} e^b_{\nu} + 2\Lambda \right) d\mathbf{s},
\]  

(A.1)

which can also be written in terms of the self-dual curvature as

\[
L_\Lambda = \left( 2 p_i^{ab} F^i + \frac{\Lambda}{6} e^a \wedge e^b \right) \wedge e^c \wedge e^d \epsilon_{abcd}.
\]

(A.2)

By varying this Lagrangian one gets the following field equations:

\[
\begin{cases}
(p_i^{ab} F_i^{\mu \nu} + \frac{4\Lambda}{6} e^a_{\mu} e^b_{\nu}) e^c_{\rho} e^{\mu \nu \rho \sigma} \epsilon_{abcd} = 0 \\
(p_i^{ab} \nabla_\mu (e^c_{(a} e^d_{b)}) e^{\mu \nu \rho \sigma} \epsilon_{abcd} = 0.
\end{cases}
\]

(A.3)
By projecting on the boundary $S$, one gets the following constraints:

$$
\begin{align*}
A \nabla \mathcal{A}^A_i = 0 \\
F_{iAB}^i E^A_i = 0 \\
\epsilon_{ijk} F_{iAB}^j E^A_i E^B_j = -4\Lambda E,
\end{align*}
$$

which account for the value of the cosmological constant as claimed after (3.17) in which, however, the conformal frame was used.

Appendix B. Matter fields

We briefly comment here about some cases of models with matter. The general treatment of matter is quite difficult and it deserves further investigation. Matter fields in this context are complicated by two different reasons. First, in classical $f(R)$ theories generic matter modifies the master equations to become $f' R - 2 f = T$, where $T$ is the $g$-trace of the energy–momentum tensor of the matter $T_{\mu\nu}$ (i.e. Hilbert stress tensor). This generically still allows to determine $R$ (and thence $f'(R)$) as a function of matter fields. Accordingly, these quantities generically are not constant any longer, and they depend on the spacetime point $x$ through the matter fields. Of course, there are special cases in which the energy–momentum tensor remains traceless which still can be treated easily, essentially as in the vacuum case.

Second, in LQG, matter can be easily considered though the method is based on regarding it in terms of models so that matter contribution can be suitably encoded in terms of holonomies. This is trivial for Yang–Mills fields and easy in a number of relevant examples; see [8].

The connection $A^I_{\mu}$ in LQG is an $SU(2)$-connection, as described in detail in [14, 15]. Hence it is a principal connection on a suitable $SU(2)$-bundle $^+P$ over the spacetime $M$. If one couples with a Yang–Mills matter field $A^I_{\mu}$, a new gauge group $G$ is introduced and the gauge field is a principal connection over a bundle $P$ with the structure group $G$. Here, the Lie algebra $\mathfrak{g}$ of the (semisimple) Lie group $G$ is of dimension $n$ and $T_I$ denotes an orthonormal basis with respect to the Cartan–Killing metric on $G$.

The gauge field strength is assumed to be denoted as usual by $F^I_{\mu\nu}$ and the Yang–Mills Lagrangian is

$$
L_m = -\frac{1}{4} \sqrt{g} \delta_{IJ} \mathcal{F}^J_{\mu\nu} \mathcal{F}^I_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} ds
$$

and as usual we set $\mathcal{F}^I_{\mu\nu} := \delta_{IJ} \mathcal{F}^J_{\mu\nu} g^{\mu\rho} g^{\nu\sigma}$, the Greek word indices are lowered and raised by the associated metric $g$, while the Latin algebra indices are lowered and raised by the Cartan–Killing metric $\delta$ on $G$.

The energy–momentum tensor is

$$
T_{\mu\nu} = \mathcal{F}^I_{\mu\nu} \mathcal{F}^I_{\rho\sigma} \delta_{\rho\sigma} - \frac{1}{4} \mathcal{F}^I_{\mu\nu} \mathcal{F}^I_{\rho\sigma} \delta_{\rho\sigma}
$$

which is in fact traceless when the spacetime is in dimension $\text{dim}(M) = 4$.

The master equation is then exactly the same as in the vacuum case, and the conformal tetrad $e^a_\mu$ is defined exactly as in the vacuum case.

The field equations are then in the form

$$
\begin{align*}
p_{ab} \mathcal{F}^a_{\nu\mu} e^a_\nu - \frac{1}{4} \mathcal{F}^b_{\nu\mu} = \kappa T_{\nu\mu} e^b_\mu \\
f' F - 2 f = 0 \Rightarrow F = \rho \\
p_{ab} \nabla_\mu (e^a_\mu e^b_\nu) = 0 \Rightarrow \nabla_\mu e^b_\mu = 0
\end{align*}
$$

which is an ortho-normal basis with respect to the Cartan–Killing metric on $G$. The energy–momentum tensor is

$$
T_{\mu\nu} = \mathcal{F}^I_{\mu\nu} \mathcal{F}^I_{\rho\sigma} \delta_{\rho\sigma} - \frac{1}{4} \mathcal{F}^I_{\mu\nu} \mathcal{F}^I_{\rho\sigma} \delta_{\rho\sigma}
$$

and as usual we set $\mathcal{F}^I_{\mu\nu} := \delta_{IJ} \mathcal{F}^J_{\mu\nu} g^{\mu\rho} g^{\nu\sigma}$, the Greek word indices are lowered and raised by the associated metric $g$, while the Latin algebra indices are lowered and raised by the Cartan–Killing metric $\delta$ on $G$.

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$$

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p_{ab} \mathcal{F}^a_{\nu\mu} e^a_\nu - \frac{1}{4} \mathcal{F}^b_{\nu\mu} = \kappa T_{\nu\mu} e^b_\mu \\
f' F - 2 f = 0 \Rightarrow F = \rho \\
p_{ab} \nabla_\mu (e^a_\mu e^b_\nu) = 0 \Rightarrow \nabla_\mu e^b_\mu = 0
\end{align*}
$$

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f' F - 2 f = 0 \Rightarrow F = \rho \\
p_{ab} \nabla_\mu (e^a_\mu e^b_\nu) = 0 \Rightarrow \nabla_\mu e^b_\mu = 0
\end{align*}
$$
where, of course, $\nabla_\mu$ also takes care, when necessary, of the $G$-gauge transformations, besides spacetime diffeomorphisms.

For the sake of simplicity, let us consider hereafter the case of electromagnetism, i.e. taking $G = U(1)$ which being of dimension 1 leads to a systematic understanding of the algebra indices. These field equations can be shown to project on the Cauchy surface $\mathcal{S}$ to get the following equations:

\begin{align}
\frac{A}{\nabla_{\mu}} \bar{E}^{\mu}_{\lambda} & = 0 \\
F^{\lambda}_{AB} \bar{E}^{\mu}_{\lambda} & = \kappa \frac{2}{F} \mathbb{B} \times \mathbb{E} \\
\epsilon_{ijk} F^{\mu}_{AB} \bar{E}^{\lambda}_{i} \bar{E}^{\mu}_{j} & = -4 \Lambda \bar{E} + \kappa \frac{2}{F} (||\mathbb{E}||^2 + ||\mathbb{B}||^2) \\
\nabla_{A}(\mathbb{E}^{A}) & = 0,
\end{align}

where we defined $\mathbb{E}^{A} := F_{\mu\nu} \partial_{A} x^{\mu} n^{\nu}$ and $\mathbb{B}^{A} := \frac{1}{2} \epsilon_{ABC} F_{\mu\nu} \partial_{B} x^{\mu} \partial_{C} x^{\nu}$ for the electric and the magnetic field. These are the Hamiltonian constraints (together with $\nabla_{A}(\mathbb{B}^{A}) = 0$) and they are the starting point of LQG quantization with matter coupling. The system is described in terms of a connection of the group $SU(2) \times U(1)$ and thence in terms of its holonomies. This should lead to define spin networks with extra label of irreducible representation of $U(1)$. See [8] and references quoted therein.

Once again the conformal frame plays a preferred role. Except for that the model is equivalent to standard GR, with a cosmological constant, coupled with the electromagnetic field though with a modified coupling constant.

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