ON THE DEEPLY NEGATIVE RANGE $p < -n - 1$ FOR $L_p$-MINKOWSKI PROBLEM

SHI-ZHONG DU

ABSTRACT. In this paper, we study the $L_p$-Minkowski problem
\begin{equation}
\det(\nabla^2 u + u I) = f u^{p-1}, \quad \text{on } \mathbb{S}^n
\end{equation}
in the deeply negative range $p \in (-\infty, -n - 1)$. Two long standing problems concerning solvability and uniqueness were considered. For the critical case $p = -n - 1$, insolvability of (0.1) for some positive smooth function $f$ has been observed by Jian-Lu-Wang in \cite{25} (see also \cite{29} by Lu). The first main purpose of this paper is to show a same result holds for some Hölder function $f$ which is positive outside two polar of $\mathbb{S}^n$ in the deeply negative case $p < -n - 1$. Turning to the problem of uniqueness, we obtained the existence of non-constant positive smooth solution of (0.1) for $f \equiv 1$, in the deeply negative case
\[ p < \mathcal{P}_n \equiv -2n - 5 \]
for all dimensions $n \geq 1$. Our result for higher dimension cases $n \geq 2$ generalizes a famous non-uniqueness result by Andrews (\cite{1}, Page 444, Theorem 1.5) for the planar case $n = 1$ and $p \in (-\infty, -7)$.

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1. Introduction

The Minkowski problem is to determine a convex body with prescribed curvature or other similar geometric data. It plays a central role in the theory of convex bodies. Various Minkowski problems \([1, 8, 12, 18, 19, 22, 36]\) have been studied especially after Lutwak \([30, 31]\), who proposes two variants of the Brunn-Minkowski theory including the dual Brunn-Minkowski theory and the \(L_p\) Brunn-Minkowski theory. Besides, there are singular cases such as the logarithmic Minkowski problem and the centro-affine Minkowski problem \([4, 10]\).

The support function \(h(x)\) of a convex body \(\{0\} \in \Omega \subseteq \mathbb{R}^{n+1}\) is given by

\[ h(x) \equiv r(x) \cdot x, \quad \forall x \in \mathbb{S}^n \]

for each \(x \in \mathbb{S}^n\), where \(r(x)\) is a point on \(\partial \Omega\) whose outer unit normal is \(x\). Its assigned matrix \([\nabla^2_{ij} h + h\delta_{ij}]\) is positive definite if \(\Omega\) is uniformly convex, where \([\nabla^2_{ij} h]\) stands for the Hessian tensor of \(h\) acting on an orthonormal frame of \(\mathbb{S}^n\). Conversely, any \(C^2\) function \(h\) satisfying \([\nabla^2_{ij} h + h\delta_{ij}] > 0\) determines a uniformly convex \(C^2\) body \(\Omega_h\). Direct calculation shows the standard surface measure of \(\partial \Omega\) is given by

\[ dS \equiv \frac{1}{n+1} \det(\nabla^2_{ij} h + h\delta_{ij}) dH^n|_{\mathbb{S}^n} \]

for \(n\)-dimensional Hausdorff measure \(dH^n\). It’s well known that classical Minkowski problem looks for a convex body such that its standard surface measure matches a given Radon measure on \(\mathbb{S}^n\). In \([31]\), Lutwak introduces the \(L_p\)-surface measure \(dS_p \equiv h^{1-p} dS\) on \(\partial \Omega\), and the corresponding \(L_p\)-Minkowski problem which looks for a convex body whose \(L_p\)-surface measure equaling to a prescribed function.

Parallel to the classical Minkowski problem, the \(L_p\)-Minkowski problem boils down to solve the fully nonlinear equation

\[ \det(\nabla^2_{ij} h + h\delta_{ij}) = fh^{p-1}, \quad \forall x \in \mathbb{S}^n \]

in the smooth category. In their cornerstone paper \([10]\), Chou and Wang study the \(L_p\)-Minkowski problem and obtain various existence results for different exponents \(p\). The existence theorems of Chou-Wang analyze only the range \(p \geq -n - 1\). The deeply negative case \(p < -n - 1\) has been left open yet. The first purpose of this paper is to discuss the solvability of \((1.1)\) in the case of \(p \leq -n - 1\). In fact, we will prove the following result.

**Theorem 1.1.** For any dimension \(n\), if \(p = -n - 1\), there exists a positive function \(f \in C^\alpha(\mathbb{S}^n), \alpha \in (0, 1)\) such that \((1.1)\) admits no solution. If \(p < -n - 1\), a same result holds for some Hölder function \(f\) which is positive outside two polar of \(\mathbb{S}^n\).
The insolvability of (1.1) for some positive smooth function \( f \) in the critical case \( p = -n - 1 \) has been observed by Jian-Lu-Wang in [25] (see also [29] by Lu). We give a simple argument here for sake of completeness and convenience of the readers. For the case of \( p < -n - 1 \), Theorem 1.1 in the planar has already been shown in [13] using the technics of ordinary differential equations. The higher dimensional case is much more complicated to be explored. A personal conversation with Professor Chou draws the attention of the author to an individual necessary condition proven in [10] for \( p = -n - 1 \). For technical reason, we generalize the condition to a full version for all \( p \leq -n - 1 \), which plays a central role in proving of our main theorem 1.1.

Theorem 1.2. (Pohozaev-Chou-Wang type identity) For \( p \leq -n - 1 \) and each solution \( h \) of (1.1), there holds

\[
\int_{S^n} K_f h^p d\sigma_{S^n} = 0, \quad K_f \equiv \nabla_\xi f + \beta f,
\]

where

\[
\xi = T^* \left[ (C^T x) x + (A x - D x) - B \right]
\]

is the projective vector field for constant \( D \), vectors \( C, B \) and trace free matrix \( A \), and

\[
\beta \equiv T^* (\text{div}\xi + p\sigma) - (p + n + 1) T^* \left( \frac{x_k \xi^k}{1 + |x|^2} \right)
\]

\[
\equiv \frac{p + n + 1}{n + 1} T^* \left\{ \frac{(C_l x_l - D)(n - |x|^2) - (n + 1)(A_{kj} x_j x_k - B_k x_k)}{1 + |x|^2} \right\}
\]

is a Lipschitz function on \( S^n \).

After this paper was finished, Professor Lu [28] pointed out to me another more simpler proof for Theorem 1.2 using only the Chou-Wang condition [10] in the critical case \( p = -n - 1 \).

Next we discuss the uniqueness of positive solutions to (1.1). Using the maximum principle, Chou-Wang have shown uniqueness result for \( p > n + 1 \) in [10] for all dimension. The uniqueness result certainly can not be expected for \( p = n + 1 \) due to the homogeneity of equation. When \( p < n + 1 \), the uniqueness problem is much more subtle since lack of the maximum principle. There are only some partial results are known on the past. Chow has shown in [6] for all \( n \geq 1 \) and \( p = 1 - n \), the uniqueness holds true for constant function \( f \equiv 1 \). Using the invertible result for linearized equation, Lutwak showed in [31] for \( n \geq 1, p > 1 \), uniqueness holds for some special symmetric \( f \). When \( p \in (-n - 1, -n - 1 + \sigma), 0 < \sigma \ll 1 \), a counter example of uniqueness have also obtained by Chou-Wang in [10]. More recently,
Jian-Lu-Wang [24] have proven that for \( p \in (-n-1, 0) \), there exists at least a smooth positive function \( f \) such that (1.1) admits two different solutions. While a partial uniqueness result was established by Chen-Huang-Li-Liu [7] on origin symmetric convex bodies for \( p \in (p_0, 1) \), \( p_0 \in (0, 1) \), using the \( L_p \)-Brunn-Minkowski inequality. For the lower dimensional cases, Andrews showed uniqueness for \( n = 2 \), \( p = 0 \) and arbitrary positive function \( f \). Subsequently, Dohmen-Giga [14] and Gage [17] have extended the result to \( n = 1 \), \( p = 0 \) for some symmetric function \( f \). Contrary to the results in [14, 17], Yagisita [39] showed a surprising non-uniqueness result for \( n = 1 \), \( p = 0 \) and non-symmetric function \( f \).

The deeply negative cases \( p \leq -n-1 \) are much more complicated. As it is known that for \( n \geq 1 \), \( p = -n-1 \), all ellipsoids with the volume of the unit ball are all solutions of (1.1) with \( f \equiv 1 \), the uniqueness fails in this case. When \( p < -n-1 \), Andrews classifies all planer curves in [2] and shows that uniqueness holds for \( p \in [-7, -2) \) and fails to hold for \( p < -7 \) in case \( f \equiv 1 \). Turning to higher dimensional case, He-Li-Wang showed in [20] that for \( p < -n \) and \( N \in \mathbb{N} \), there exists at least a smooth positive function \( f \) on \( S^n \) such that the equation (1.1) admits at least \( N \) different smooth solutions.

For its importance and difficulty, the issue of uniqueness of solutions has attracted much attentions. The problems have been conjectured for a number of special cases. This is the main purpose for us to discuss the problem. We will consider the deeply negative range \( p < -n-1 \) to

\[
(1.5) \quad \det(\nabla^2 h + hI) = h^{p-1}, \quad \forall x \in S^n
\]

and prove the following result.

**Theorem 1.3.** When \( p < \mathcal{P}_n \) for

\[
\mathcal{P}_n \equiv -2n - 5, \quad \forall n \geq 1,
\]

there exists at least one positive smooth solution of (1.5) which is not constant.

Due to the uniqueness result of Andrew [1] for \( n = 1 \) and \( p \in [-7, -2) \), we believe that the range of \( p \) in Theorem 1.3 is optimal. Our contents of this paper are organized as follows. After projecting \( S^n \) into \( \{x_{n+1} = \pm 1\} \), we will show a Pohozaev-Chou-Wang type identity Theorem 1.2 in Section 2 and Section 3. A resolution of a first order differential equation on \( S^n \) gives the proof of Theorem 1.1 in Section 4. Turning to the uniqueness problem of (1.5), we introduce a new variational scheme in the deeply negative range under the \((n + 2)\text{-symmetricity} \) at Section 5 and prove the existence of smooth minimizer in Section 6 for \( p \leq -n \). With the helps of instability of constant function for \( p < \mathcal{P}_n \) and a calculation of best constant of Poincaré inequality on \( S_{n+2}(q) \) for given \( n+2 \) points \( q_i \in S^n \), \( i = 1, 2, \cdots, n+2 \) which
spread evenly on $S^n$, we complete the proof of Theorem 1.3 in Section 7 and Section 8.

2. Projection of a $L_p$-Minkowski equation onto $\{x_{n+1} = \pm 1\}$

Considering the solution $h$ of $L_p$-Minkowski problem (1.1), if one projects south hemisphere

$$S^n_+ \equiv \{(x, x_{n+1}) \in S^{n+1} \mid x_{n+1} < 0\},$$

to the hyperplane $x_{n+1} = -1$ by mapping

$$T_*(X) = x, \quad X \equiv \left( \frac{x}{\sqrt{1 + |x|^2}}, \frac{-1}{\sqrt{1 + |x|^2}} \right), \quad x \in \mathbb{R}^n$$

and sets $u(x) = \sqrt{1 + |x|^2}h(X)$, there holds

$$\det(D^2u) = (1 + |x|^2)^{-\frac{n+1}{2}} \det(\nabla^2 h + hI).$$

So, the equation satisfied by $u$ is given by

$$\det(D^2u) = gu^{p-1}, \quad \forall x \in \mathbb{R}^n$$

for

$$g(x) = (1 + |x|^2)^{-\frac{p+1}{2}} f \left( \frac{x}{\sqrt{1 + |x|^2}}, \frac{-1}{\sqrt{1 + |x|^2}} \right).$$

Moreover, the relation between volume elements of $S^n$ and $\mathbb{R}^n$ is given by

$$d\sigma_{S^n} = (1 + |x|^2)^{-\frac{n+1}{2}} dx_{\mathbb{R}^n}.$$  

To proceed further, one may also need the following inverse projection mapping $T^* : \mathbb{R}^n \rightarrow S^n$ defined by

$$T^*(x) = X \equiv \left( \frac{x}{\sqrt{1 + |x|^2}}, \frac{-1}{\sqrt{1 + |x|^2}} \right) \in S^n.$$

Its induced pull back mapping on vector fields is given by

$$T^*(\xi) = \left( \frac{\xi}{\sqrt{1 + |x|^2}}, \frac{(x \cdot \xi)x}{(1 + |x|^2)^{\frac{3}{2}}}, \frac{x \cdot \xi}{(1 + |x|^2)^{\frac{3}{2}}} \right)$$

for each vector field $\xi$ of $\mathbb{R}^n$. In this paper, if no special indication, we may not distinguish $\xi$ with $T^*(\xi)$. 
3. POHOZAEV-CHOU-WANG TYPE IDENTITY ON $\mathbb{S}^n$

In this section, we will perform calculations in spirit of Pohozaev-Chou-Wang to give a simplified proof to Theorem 1.2. Let us first take a vector field $\xi^k$ and a function $\sigma$ on $\mathbb{R}^n$. Multiplying (2.2) by $\xi^k u_k$, integrating over $B_R$ and then performing integration by parts, one concludes that

$$\int_{B_R} \xi^k u_k \det(D^2 u) = \frac{1}{n} \int_{B_R} \xi^k u_k u^{ij} u_{ij}$$

$$= -\frac{1}{n} \int_{B_R} D_i \xi^k u_k u^{ij} u_{ij} - \frac{1}{n} \int_{\partial B_R} \xi^k u_k u^{ij} u_{ij} + \frac{1}{n} \int_{\partial B_R} \xi^k u_k \nu^i u^{ij} u_{ij}$$

$$= -\frac{1}{n} \int_{B_R} D_i \xi^k u_k u^{ij} u_{ij} - \frac{1}{n} \int_{B_R} \xi^k \det(D^2 u) \delta_{jk} u_j + \frac{1}{n} \int_{\partial B_R} \xi^k u_k \nu^i u^{ij} u_{ij}$$

$$= -\frac{1}{n} \int_{B_R} D_i \xi^k u_k u^{ij} u_{ij} - \frac{1}{n} \int_{B_R} \xi^k u_k \det(D^2 u) + \frac{1}{n} \int_{\partial B_R} \xi^k u_k \nu^i u^{ij} u_{ij}$$

by the divergence free property $D_i u^{ij} = 0$. Therefore, one gets

$$(n + 1) \int_{B_R} \xi^k u_k \det(D^2 u) = - \int_{B_R} D_i \xi^k u_k u^{ij} u_{ij} + \int_{\partial B_R} \xi^k u_k \nu^i u^{ij} u_{ij}$$

$$= \int_{B_R} U^{ij} D_i \xi^k u_k u^{ij} + \int_{B_R} D_i \xi^k u_k u^{ij} u_{ij} + \int_{\partial B_R} \xi^k u_k \nu^i u^{ij} u_{ij} - \int_{\partial B_R} D_i \xi^k u_k \nu^i u^{ij} u_{ij}$$

Subtracting $(n + 1) \int_{B_R} \sigma u \det(D^2 u)$ from both sides, it yields that

$$(n + 1) \int_{B_R} Q \det(D^2 u) = \int_{B_R} (\text{div}(\xi) - (n + 1)\sigma) u \det(D^2 u)$$

$$+ \int_{B_R} U^{ij} D_i \xi^k u_k u^{ij} + \int_{\partial B_R} \xi^k u_k \nu^i u^{ij} u_{ij} - \int_{\partial B_R} D_i \xi^k u_k \nu^i u^{ij} u_{ij},$$

where $Q \equiv \xi^k u_k - \sigma u$. So, if one chooses

$$\xi^k = C_l x_l x_k + (A_{lj} - D \delta_{kj}) x_j - B_k$$

and

$$(n + 1)\sigma = \text{div}(\xi) = D_k \xi^k$$

(3.3) $\iff (n + 1)\sigma = nC_l x_l + A_{kk} - nD$

$$\iff \sigma = \frac{n}{n + 1} (C_l x_l - D),$$
where \( tr(A) = A_{kk} \) can be assumed to be zero by adjusting the constant \( D \), there holds

\[
(n + 1) \int_{B_R} Q \det(D^2 u) = \int_{B_R} U^{ij} D_{ij} \xi^k u u_k
\]

\[
+ \int_{\partial B_R} \xi^k u_k v_i U^{ij} u_j - \int_{B_R} D_i \xi^k u_k v_j U^{ij} u
\]

\[
= \frac{1}{2} \int_{\partial B_R} u^2 U^{ij} (v_i C_j + v_j C_i) + \int_{\partial B_R} \xi^k u_k v_i U^{ij} u_j - \int_{B_R} D_i \xi^k u_k v_j U^{ij} u.
\]

Combining (3.4) with (3.5), one concludes that

\[
\int_{B_R} Q u^p = \frac{1}{p} \int_{\partial B_R} \xi^k v_k u^p + \frac{1}{p} \int_{\partial B_R} \xi^k v_k u^p.
\]

On another hand,

\[
\int_{B_R} Q u^p = \frac{1}{p} \int_{\partial B_R} \xi^k v_k u^p + \frac{1}{p} \int_{\partial B_R} \xi^k v_k u^p.
\]

Combining (3.4) with (3.5), one concludes that

\[
(n + 1) \int_{B_R} \left[ \frac{D_i (\xi^k g)}{p} + \sigma g \right] u^p = \frac{n+1}{p} \int_{\partial B_R} \xi^k v_k u^p
\]

\[
- \frac{1}{2} \int_{\partial B_R} u^2 U^{ij} (v_i C_j + v_j C_i) - \int_{\partial B_R} \xi^k u_k v_i U^{ij} u_j + \int_{\partial B_R} D_i \xi^k u_k v_j U^{ij} u.
\]

**Complete the proof of Theorem 1.2.** At first, we need to handle the boundary terms vanishes as \( R \) tends to infinity. Actually, we have the asymptotic behaviors

\[
\left| \frac{u(x)}{\sqrt{1 + |x|^2}} \right| + |D u(x)| \leq C, \quad \forall x \in \mathbb{R}^n
\]

and

\[
C^{-1} I \leq \sqrt{1 + |x|^2} D^2 u(x) \leq CI, \quad \forall x \in \mathbb{R}^n,
\]

together with

\[
C^{-1} d\sigma_{S^\theta} \geq (1 + R^2)^{-\frac{n-1}{2}} d\sigma_{\partial B_R} \geq C d\sigma_{S^\theta}, \quad R = -\cot \theta
\]

for

\[
S_{\theta} \equiv \{(x, x_{n+1}) \in \mathbb{S}^n_{\theta} \mid x_{n+1} = \sin \theta\}.
\]

More precisely,

\[
\lim_{R \to \infty} \int_{\partial B_R} \xi^k v_k u^p = \int_{\partial_0^\mathbb{S}^n} (C_i p_i) p_k v_k f h^p
\]

\[
\lim_{R \to \infty} \int_{\partial B_R} u^2 U^{ij} (v_i C_j + v_j C_i) = \int_{\partial_0^\mathbb{S}^n} h^2 U^{ij} (v_i C_j + v_j C_i)
\]

\[
\lim_{R \to \infty} \int_{\partial B_R} \xi^k u_k v_i U^{ij} u_j = \int_{\partial_0^\mathbb{S}^n} (C_i p_i) p_k (h_k + h p_k) v_i U^{ij} (h_j + h p_j)
\]

\[
\lim_{R \to \infty} \int_{\partial B_R} D_i \xi^k u_k v_j U^{ij} u = \int_{\partial_0^\mathbb{S}^n} [(C_i p_i) (h_i + h p_i) + C_i p_k (h_k + h p_k)] v_j U^{ij} h,
\]

where \( L_p \)-Minkowski problem
where \( \nu \) is the unit normal of \( \partial S^n_- \) pointing to north polar, and
\[
\partial S^n_- \equiv \{(q, x_{n+1}) \in \mathbb{S}^n \mid x_{n+1} = 0\}, \quad q \equiv \frac{x}{\sqrt{1 + |x|^2}}.
\]
Next, one can construct a mirror vector field \( \xi \) on north hemisphere and argue as above to deduce a similar formula like (3.6), except the normal \( \nu \) of \( \partial S^n_+ \) is now pointing to the south polar. Adding two analogue formulas, and then writing back to sphere, one finally arrives at
\[
(n + 1) \int_{\mathbb{S}^n} T^\ast \left\{ (1 + |x|^2)^{\frac{n+1}{2}} \left[ \frac{D_k(\xi^k g)}{p} + \sigma g \right] \right\} u^p d\sigma_{\mathbb{S}^n} = 0
\]
as the boundary terms are canceled each other. This is exactly the desired identity (1.2) by rearrangement. The proof of Theorem 1.2 was done. \( \Box \)

Let us reformulate the critical part \( p = -n - 1 \) for Theorem 1.1 in the following corollary and then give its proof.

**Corollary 3.1.** Considering (1.1) for \( p = -n - 1 \), there exists some positive function \( f \in C^0(\mathbb{S}^n), \alpha \in (0, 1) \), such that the equation admits no classical solution.

**Proof.** For \( p = -n - 1 \), if one chooses \( \xi^k = -Dx_k \), the unique positive solution \( f \) of
\[
Df = \frac{\xi}{(1 + |x|^2)^{\frac{n+1}{2}}}, \quad \forall x \in \mathbb{R}^n
\]
is given by
\[
f(x) = (1 + |x|^2)^{-\frac{n}{2}} + C
\]
for some positive constant \( C \). When \( D \) is chosen large, the pull back function \( T^\ast f \) of \( f \) is Lipschitz continuous on closure of south hemisphere. Arguing similarly in north hemisphere or by mirror symmetrization, one finally gets a Lipschitz function \( f \) on the whole sphere \( \mathbb{S}^n \) such that
\[
\nabla_\xi f = T^\ast \left( \frac{||\xi||^2}{(1 + |x|^2)^{\frac{n+1}{2}}} \right), \quad \forall x \in \mathbb{S}^n.
\]
One thus concludes from the necessary condition (3.7) that
\[
\int_{\mathbb{S}^n} T^\ast \left[ \frac{||\xi||^2}{(1 + |x|^2)^{\frac{n-1}{2}}} \right] u^{-n-1} d\sigma_{\mathbb{S}^n} = 0,
\]
which gives a contradiction. The conclusion was drawn. \( \Box \)
4. Special solutions of (1.2) for \( p < -n - 1 \)

In this section, we will give the proof of the following result.

**Theorem 4.1.** For each \( p < -n - 1 \),

\[
\xi^k = -x_k, \quad \sigma = -\frac{n}{n + 1},
\]

the first order differential equation (1.2) on \( \mathbb{S}^n \) admits a Hölder solution \( f \) which is positive outside two polar, such that \( K_f \) is positive almost everywhere.

When \( \xi, \sigma \) is given by (4.1) and \( -T_\sigma(K_f) \) is given by a positive function \( \varphi = \varphi(|x|) \in C(\mathbb{R}^n) \) satisfying

\[
\lim_{r \to \infty} \varphi(r) = \varphi_\infty \in (0, \infty), \quad \varphi(r) \text{ decays to zero rapidly as } r \to 0,
\]

the first order differential equation (1.2) has been projected to an equation

\[
\xi^k D_k f + (p + n + 1) \tilde{\beta} f = -\varphi, \quad \forall x \in \mathbb{R}^n,
\]

where

\[
\tilde{\beta} \equiv -\frac{n + |x|^2}{(n + 1)(1 + |x|^2)}.
\]

Along the characterization lines

\[ x = (r, \omega), \quad r \in (0, \infty), \quad \omega \in \mathbb{S}^{n-1}, \]

(4.3) changes to

\[
-r \frac{df}{dr} + \frac{(p + n + 1)(-n + r^2)}{(n + 1)(1 + r^2)} f = -\varphi(r).
\]

A resolution of this ordinary differential equation (4.5) yields that

\[
f(r) = \left[ \frac{(1 + r^2)^{\frac{p + 1}{2}}}{r^n} \right]^\gamma \left[ \int_1^r \varphi(r) \left[ \frac{(1 + r^2)^{\frac{p + 1}{2}}}{r^n} \right]^{-\gamma} dr + \beta \right], \quad \forall r \in (0, \infty),
\]

where \( \beta > 0 \) will be chosen later and

\[
\gamma \equiv \frac{p + n + 1}{n + 1} < 0.
\]

**Lemma 4.1.** Under the assumptions of Theorem 4.1, the \( C^1 \)-function \( f \) given by (4.6) satisfies that

\[
\lim_{r \to \infty} \frac{\varphi_\infty}{|y|} = \varphi, \quad f(0) = 0, \quad f(r) > 0, \quad \forall r \in (0, \infty),
\]

as long as

\[
\beta > \int_0^1 \frac{\varphi(r)}{r \left[ \frac{(1 + r^2)^{\frac{p + 1}{2}}}{r^n} \right]^{-\gamma}} dr.
\]
Until now, we have constructed a $C^{\gamma}$-function $f$ which is positive outside south polar. By an inversion transformation, one can extend $f$ to be a $C^{\gamma}$-function on the whole sphere $\mathbb{S}^n$, which is positive outside two polar. The proof of Theorem 4.1 was completed. □

**Corollary 4.1.** For any dimension $n$, if $p < -n - 1$, there exists a Hölder function $f$ which is positive outside two polar of $\mathbb{S}^n$, such that (1.1) admits no solution.

**Proof.** By our construction of $f$, there holds

$$K_f = \nabla_\xi f + \beta f = -T^*(\varphi), \quad \forall x \in \mathbb{S}^n$$

for some nonnegative function $\varphi$ which is positive almost everywhere. Combining with Theorem 1.2, we conclude the insolvability of (1.1) for this $f$. The proof of Theorem 1.1 was done. □

5. \((n + 2)\)--symmetricity and new variational scheme in the deeply negative range

Let us now discuss the uniqueness problem of (1.5) in the deeply negative range. In the first stage, we shall introduce a new variational scheme for $p < -n - 1$. Since lacking of Blaschke-Santalo’s inequality, we need restricting the functions into a \((n + 2)\)--symmetricity class defined as follows.

**Definition 5.1.** Fixing $n + 2$ points $q_i \in \mathbb{S}^n, i = 1, 2, \cdots, n + 2$ which are spreading evenly on $\mathbb{S}^n$, we denote all orthonormal mappings $\phi \in SO(n+1)$ which map $\{q_j\}_{j=1}^{n+2}$ to $\{q_j\}_{j=1}^{n+2}$ by

$$\phi \in S_{n+2}(q), \quad q \equiv \{q_1, q_2, \cdots, q_{n+2}\}.$$  

A function $\xi \in C^\infty(\mathbb{S}^n)$ (or a function $h \in C^\infty(\mathbb{R}^{n+1})$ is called to be \((n + 2)\)--symmetric and denoted by $\xi \in \tilde{S}_{n+2}(q)$ (or $h \in \tilde{S}_{n+2}(q)$) if and only if it is invariant under all mappings $\phi \in \tilde{S}_{n+2}(q)$.

Denoting $C$ the family of support functions $u$ of convex bodies $\Omega_u$, we set

$$\mathcal{B} \equiv \left\{ u \in H^*_+(\mathbb{S}^n) \mid u \in \tilde{S}_{n+2}(q), \quad \int_{\mathbb{S}^n} u^p = \alpha_n \equiv |\mathbb{S}^n| \right\}$$

$$H^*_+(\mathbb{S}^n) \equiv \left\{ u \in C \mid u > 0, \quad |\Omega_u| < \infty \right\}$$
and introduce a variational scheme

\[ \inf_{u \in B} I(u), \quad I(u) \equiv -(n + 1)V(\Omega_u), \]

where \( V(\Omega) \) stands the volume of convex body \( \Omega \). We have the following a-priori upper bound for \( u \in B \).

**Lemma 5.1.** Letting \( n \geq 1 \) and \( p < 0 \), there exists a positive constant \( C_{n,p} \) depending only on \( n, p \), such that

\[ u(x) \leq C_{n,p}, \quad \forall x \in S^n \]

holds for each \( u \in B \).

**Proof.** Noting that if the maximum of a \((n + 2)\)-symmetric function \( u \) becomes large, one has the minimum of \( u \) must tend to infinity as well. So, it is inferred from

\[ \alpha_n = \int_{S^n} u^p = o(1) \]

a contradiction. The proof of the lemma was done. \( \square \)

Now, let us end this section by estimating the functions in \( B \) from below.

**Proposition 5.1.** Letting \( n \geq 1 \) and \( p \leq -n \), there exists a positive constant \( C_{n,p} \) depending only on \( n, p \), such that

\[ \|u\|_{L^p(S^n)} \leq C_{n,p} \]

and

\[ u(x) \geq C_{n,p}^{-1}, \quad \forall x \in S^n \]

hold for any function \( u \in B \).

The proof of this proposition needs the following interior gradient estimate for convex function.

**Lemma 5.2.** Let \( \Omega \) be a domain in \( \mathbb{R}^n \) and \( K \) be a compact subdomain of \( \Omega \). The estimates

\[ \|v\|_{L^p(K)} \leq \frac{o_{SC\Omega}(v)}{\text{dist}(K, \partial \Omega)} \]

holds for any convex function \( v \) on \( \Omega \).

**Proof of Proposition 5.1.** Projecting the south hemisphere

\[ S^{-}_n \equiv \{ x = (x', x_{n+1}) \in S^{n+1} \mid x_{n+1} < 0 \} \]

to the hyperplane \( x_{n+1} = -1 \) by mapping

\[ T_s(x) = y, \quad y = \left( \frac{x}{\sqrt{1 + |y|^2}}, \frac{-1}{\sqrt{1 + |y|^2}} \right), \quad y \in \mathbb{R}^n \]
and setting \( v(y) = \sqrt{1 + |y|^2} u(x) \) as in Section 2, one gets that
\[
\nu_{ij}(y) = \sqrt{1 + |y|^2} (\nabla_{ij}^2 u + u g_{ij}), \quad \forall y \in \mathbb{R}^n,
\]
where
\[
g_{ij} = \frac{\delta_{ij}}{1 + |y|^2} - \frac{y_i y_j}{(1 + |y|^2)^2}.
\]
Noting that \( \nabla_{ij}^2 u + u g_{ij} \) is semi-positive definite for convex body \( \Omega_u \), \( v \) should be a convex function on \( \mathbb{R}^n \). Application of Lemma 5.2 to \( v \) on \( B_1 \) and pulling back to \( u \) yield the estimate (5.3) after rotating the sphere. To show (5.4), one may assume that the minimum of \( u \) is attained at south polar of \( S^n \) and set \( m \equiv u(e) = \min_{x \in S^n} u(x) \) for \( e = (0, -1) \in S^n \). By Lemma 5.1
\[
|v(y) - m| \leq C|y|osc_{B_2}(v) \leq C_1|y|, \quad \forall y \in B_1,
\]
holds for some positive constant \( C_1 \) depending only on \( n, p \). Combining with
\[
\int_{B_1} v^p dy \leq C \int_{S^n} u^p \leq C_2
\]
for some positive constant \( C_2 \) depending only on \( n, p \), we conclude that
\[
C_2 \geq \int_{B_1} v^p dy \geq \int_{B_1} (m + C_1|y|)^p
\]
\[
\geq \begin{cases} C_3^{-1} \left( \frac{m}{C_1} \right)^{p+n} - C_4, & p < -n, \\ C_3^{-1} \log \left( \frac{C_1}{m} \right) - C_4, & p = -n \end{cases}
\]
for positive constants \( C_3, C_4 \) depending only on \( n \) and \( p \). So, the conclusion (5.4) follows and the proof was done. \( \square \)

6. Existence of smooth minimizer \( u_\infty \)

By Lemma 5.1, the functional \( I \) is bounded from below. Taking a minimizing sequence \( u_j \in \mathcal{B}, j \in \mathbb{N} \) of \( I \), it follows from Proposition 5.1 and Lemma 5.1 that for a subsequence, \( \{u_j\} \) tends to a limiting positive Lipschitz function \( u_\infty \) uniformly. Moreover, \( u_\infty \) belongs to \( \mathcal{B} \) and stands for a support function of a convex body \( \Omega_\infty \) with maximal volume. So, we achieve the following existence result.

**Proposition 6.1.** For \( n \geq 1 \) and \( p \leq -n \), the variational problem (5.1) admits a positive Lipschitz minimizer \( u_\infty \) of \( I \).
Noting that the limiting minimizer \( u_\infty \) may not be strict convex, it is not obviously to show that the minimizer satisfies the Euler-Lagrange equation (1.5) in weak sense. Fortunately, after modifying slightly an argument of Chou-Wang [10, section 5, Lemma 5.2-Lemma 5.6 and Corollary 5.3-Corollary 5.4], one still has the following result.

**Proposition 6.2.** For \( n \geq 1 \) and \( p \leq -n \), the maximizer \( v \) of the variational problem

\[
\sup_{v \in \mathcal{B}'} J(v), \quad J(v) \equiv \int_{\mathbb{S}^n} h^p
\]
on family

\[
\mathcal{B}' \equiv \left\{ v \in H^1_0(\mathbb{S}^n) \mid v \in \tilde{S}_{n+2}(q), \quad V(\Omega_\mathcal{V}) = 1 \right\}
\]
satisfies the Euler-Lagrange equation

\[
\det(\nabla^2 v + vI) = \lambda v^{p-1}, \quad \forall x \in \mathbb{S}^n
\]
in weak sense, where

\[
\lambda = (n + 1)\left( \int_{\mathbb{S}^n} v^p \right)^{-1}.
\]

Now, we can reformulate Proposition 6.2 into our cases.

**Proposition 6.3.** For \( n \geq 1 \) and \( p \leq -n \), the minimizer \( u_\infty \) derived in Proposition 6.1 satisfies the Euler-Lagrange equation

\[
\det(\nabla^2 u_\infty + u_\infty I) = \lambda_\infty u_\infty^{p-1}, \quad \forall x \in \mathbb{S}^n
\]
in the weak sense, where

\[
\lambda_\infty = (n + 1)|\mathbb{S}^n|^{-1}V(\Omega_{u_\infty}).
\]

Moreover, \( u_\infty \) is not a constant solution unless \( u_\infty \equiv 1 \).

**Proof of Proposition 6.3.** Using the duality of variational problems (5.1) and (6.1), if one sets

\[
v_\infty \equiv \frac{u_\infty}{V(\Omega_{u_\infty})},
\]

the resulting function \( v_\infty \) is a maximizer of (6.1). So, application of Proposition 6.2 to \( v_\infty \) yields the desired conclusion of (6.4). We claim that \( u_\infty \) is not a constant function unless it is identical to one. Otherwise, it is inferred from (6.4) that

\[
u_{\infty}^{n+1} = \lambda_\infty u_\infty^{p-1} = u_\infty^{n+p}
\]
and hence \( u_\infty \equiv 1 \). Contradiction holds and the proof of the proposition was done. □

Let us sum the above arguments into a theorem at the end of this section.
**Theorem 6.1.** For \( n \geq 1 \) and \( p \leq -n \), the variational problem \((5.1)\) admits a positive smooth minimizer \( u_\infty \) of \( I \), which satisfies the Euler-Lagrange equation \((1.5)\) up to a constant. Moreover, \( u_\infty \) can not be a constant function unless it is identical to one.

**Proof.** We need only to show that \( u_\infty \) is smooth. Actually, we have known that \( u_\infty \) is a positive Lipschitz solution of \((6.4)\) in weak sense. Projecting the south hemisphere into the hyperplane \( x_{n+1} = -1 \) by \( T^* (x) = y, \quad x \equiv \left( \frac{y}{\sqrt{1 + |y|^2}}, \frac{-1}{\sqrt{1 + |y|^2}} \right) \), \( y \in \mathbb{R}^n \) as in proof of Proposition 5.1 or Section 2, one has the function \( v(y) = \sqrt{1 + |y|^2} u_\infty (x) \) satisfies that

\[
\det(D^2 v) = (1 + |y|^2)^{\frac{n+1+n}{2}} v^{p-1}, \quad \forall y \in \mathbb{R}^n.
\]

After applying a famous \( C^{2,\alpha} \)-estimate by Caffarelli (\cite{5}, page 135, Theorem 2), one concludes that \( v \in C^{2,\alpha}(B_1) \) for each \( \alpha \in (0, 1) \). Iterating on higher derivatives of \( v \) and using the Schauder’s estimates for linear elliptic equations with Hölder coefficients show that \( v \) is smooth in \( B_1 \). Pulling back to \( u_\infty \) and rotating the sphere \( \mathbb{S}^n \), the desired smoothness result has been shown. \( \square \)

7. **Instability of constant solution for \( p < P_n \)**

In this section, we discuss the stability of constant function \( u \equiv 1 \). A critical point \( u \) of \( I(\cdot) \) is called to be stable, if the second variation of the functional at \( u \) is non-positive. Otherwise, we will call it to be unstable. We have the following theorem.

**Theorem 7.1.** When \( p < P_n \) for

\[
P_n \equiv -2n - 5,
\]

the constant solution \( u \equiv 1 \) is a critical point of the functional \( I \) which is unstable. As a result, \( u \equiv 1 \) is not a minimizer of \( I \) in the family \( \mathcal{B} \).

To show the conclusion, let us calculate the second variation of \( I \). At beginning, we need to construct a normalized variation by

\[
\varphi_\varepsilon \equiv \varphi(\varepsilon, \theta) \equiv \frac{\alpha_n^{1/p} (1 + \varepsilon \xi)}{\left( \int_{\mathbb{S}^n} (1 + \varepsilon \xi)^p \right)^{1/p}}
\]
for a smooth function $\xi \in \tilde{S}_{n+2}(q)$. Direct computation shows that

$$\varphi' \equiv \frac{d}{de} \varphi(e, \theta) = \xi - \alpha_n^{-1} \int_{\mathbb{S}^n} \xi$$

(7.3) $$\varphi'' \equiv \frac{d^2}{de^2} \varphi(e, \theta) = -2\alpha_n^{-1} \xi \int_{\mathbb{S}^n} \xi$$

Calculating the second variation to

$$-I(\varphi_e) = \int_{\mathbb{S}^n} \varphi_e \det(\nabla^2 \varphi_e + \varphi_e I),$$

we derive

$$-\left. \frac{d^2}{de^2} \right|_{e=0} I(\varphi_e) = \int_{\mathbb{S}^n} (\delta_{ij}\delta_{rs} - \delta_{ir}\delta_{js})(\nabla^2_{ij} \varphi_e' + \varphi_e' \delta_{ij})(\nabla^2_{rs} \varphi_e' + \varphi_e' \delta_{rs})$$

(7.4) $$+ \int_{\mathbb{S}^n} \varphi_e'' + \int_{\mathbb{S}^n} \delta_{ij}(\nabla^2_{ij} \varphi_e' + \varphi_e' \delta_{ij}) + 2 \int_{\mathbb{S}^n} \varphi_e' \delta_{ij}(\nabla^2_{ij} \varphi_e' + \varphi_e' \delta_{ij}).$$

Substituting (7.3) into (7.4) and performing integration by parts, one concludes that

$$-\left. \frac{d^2}{de^2} \right|_{e=0} I(\varphi_e) = \int_{\mathbb{S}^n} |\Delta \xi|^2 - \int_{\mathbb{S}^n} |\nabla^2 \xi|^2 - 2n \int_{\mathbb{S}^n} |\nabla \xi|^2$$

$$- (n + 2)(p - n - 1) \int_{\mathbb{S}^n} \xi^2 + (n + 2)(p - n - 1)\alpha_n^{-1} \left( \int_{\mathbb{S}^n} \xi \right)^2$$

or equivalent to

(7.5) $$\left. \frac{d^2}{de^2} \right|_{e=0} I(\varphi_e) = (n + 2) \int_{\mathbb{S}^n} |\nabla \xi|^2 - (n + 2)(n + 1 - p) \int_{\mathbb{S}^n} |\xi - \bar{\xi}|^2,$$

where

$$\bar{\xi} \equiv \alpha_n^{-1} \int_{\mathbb{S}^n} \xi.$$

Now, the conclusion of Theorem 7.1 follows from the following proposition and Theorem 8.1 in next section.

**Proposition 7.1.** Supposing that the Poincaré inequality

(7.6) $$\int_{\mathbb{S}^n} |\nabla \xi|^2 \geq \lambda_1(n) \int_{\mathbb{S}^n} \xi^2$$

holds for all functions

(7.7) $$\xi \in \mathcal{G}_n \equiv \left\{ \xi \in H^1(\mathbb{S}^n) \mid \int_{\mathbb{S}^n} \xi = 0, \xi \text{ is } (n + 2) - \text{ symmetric} \right\}$$
with best constant $\lambda_1(n) > 0$, we have the instability of the constant solution $u \equiv 1$ for
\begin{equation}
\text{(7.8)} \quad p < \mathcal{P}_n \equiv n + 1 - \lambda_1(n).
\end{equation}

**Remark.** The best constant of Poincaré inequality (7.6) in family (7.7) is attained by utilizing a usual variational problem and applying the method of lower semi-continuity for weakly convergence.

**Proof.** Letting $\xi$ be the eigenfunction of $-\Delta_{S^n}$ corresponding to the least eigenvalue $\lambda_1(n)$, one has
\begin{equation}
\text{(7.9)} \quad \int_{S^n} |\nabla \xi|^2 = \lambda_1(n) \int_{S^n} \xi^2.
\end{equation}
Substituting into (7.5) yields that
\[ \frac{d^2}{d\varepsilon^2} I(\varphi_\varepsilon) = (n + 2)(\lambda_1(n) - n - 1 + p) \int_{S^n} \xi^2 < 0 \]
in case $p < \mathcal{P}_n$. The instability of constant solution $u \equiv 1$ has been shown. \[\square\]

### 8. Best Constant of Poincaré Inequality on $\widetilde{S}_{n+2}(q)$

**Theorem 8.1.** For each $n \geq 1$, the best constant of Poincaré’s inequality on $\widetilde{S}_{n+2}(q)$ is given by
\begin{equation}
\text{(8.1)} \quad \lambda_1(n) = 3(n + 2)
\end{equation}

**Proof.** In the one dimensional case $n = 1$, (7.6)-(7.7) change to
\begin{equation}
\text{(8.2)} \quad \int_0^{2\pi} \xi_\theta^2 \geq \lambda_1(1) \int_0^{2\pi} \xi^2
\end{equation}
and
\begin{equation}
\text{(8.3)} \quad \xi \in \mathcal{G}_1 \equiv \left\{ \xi \in H^1(S^1) \middle| \int_{S^1} \xi = 0, \xi \text{ is a } 2\pi/3 - \text{periodic function} \right\}.
\end{equation}
Therefore, $\xi \in \mathcal{P}$ is equivalent to
\[ \xi(\theta) = \Sigma_{k=1}^\infty \left( a_k \cos(3k\theta) + b_k \sin(3k\theta) \right), \]
which implies that $\lambda_1(1) = 9$ for $n = 1$.

When $n \geq 2$, the determining of the best constant $\lambda_1(n)$ of Poincaré inequality (7.6)-(7.7) is related to the eigenvalues problem of Laplace-Beltrami operator $-\Delta_{S^n}$ on $S^n$. As well known that all eigenfunctions $u$ of $-\Delta_{S^n}$ are given by homogeneous harmonic polynomials $h(y) = |y|^\mu u(\frac{y}{|y|})$ of degree $\mu$.
on $\mathbb{R}^n$, whose eigenvalues are given exactly by $\lambda \equiv \mu(n + \mu - 1)$ for positive integer $\mu$. In a paper of Kazdan ([26], page 12), the multiplicities of the eigenvalues were also calculated explicitly. Note that the mean value of $u$ equals to zero if and only if

$$\int_{B_1} h(y)dy = 0,$$

and $u$ is $(n + 2)$–symmetric if and only if $h$ is $(n + 2)$–symmetric. Setting

$$h(y) = \Sigma_{|\alpha| = \mu} a^\alpha y^\alpha, \quad \alpha \equiv (\alpha_1, \cdots, \alpha_{n+1}) \in \mathbb{R}^{n+1}$$

for $y^\alpha \equiv \prod_{i=1}^{n+1} y_i^{\alpha_i}$ as usual, and summing the above relations between $u$ and $h$, we obtain that

$$\Sigma_{|\alpha| = \mu} a^\alpha \int_{B_1} y^\alpha dy = 0$$

and

$$\Sigma_{|\beta| = \mu} a^\beta \int_{B_1} (\beta_j + 2)(\beta_j + 1) = 0, \quad \forall \beta, |\beta| = \mu - 2,$$

where $e_j$ is a vector whose $j$–th coordinate is one and the others are zero.

**Proposition 8.1.** For each $n \geq 1$, the set

$$\mathcal{H}_{n+1} \equiv \left\{ h \in \tilde{S}_{n+2}(q) \setminus \{0\} \mid \int_{B_1} h(y)dy = 0, \ h \text{ is a homogeneous harmonic polynomial on } \mathbb{R}^{n+1} \right\}$$

is nonempty. Moreover, the least degree of $h \in \mathcal{H}_{n+1}$ is given by

$$\mu_1(n) = 3, \ \forall n \geq 1.$$

**Proof.** At first, one knows that the best constant $\lambda_1(n)$ of Poincaré inequality (7.6)-(7.7) is attained at some function $u \in S_n$. After expanding $u$ to be a homogeneous harmonic function $h$ of degree $\mu_1(n)$ determined by

$$\mu_1(n)(n + \mu_1(n) - 1) - \lambda_1(n) = 0,$$

Liouville property shows that $h$ must be a harmonic polynomial, as well also be $(n + 2)$–symmetric. So, the set $\mathcal{H}_{n+1}$ is nonempty, and it remains to calculate $\mu_1(n)$.

In the case $n = 1$, it is not hard to see that $\mu_1(1) = 3$ and all homogenous harmonic polynomials $h \in \mathcal{H}_2$ of degree three are given by

$$h(x, y) = A(x^3 - 3xy^2) + B(3x^2y - y^3), \ \forall (x, y) \in \mathbb{R}^2$$

for each given constants $A$ and $B$. For dimension $n \geq 2$, the proofs were divided into the following two lemmas.
Lemma 8.1. For \( n \geq 2 \), there exists at least one homogeneous harmonic polynomial \( h \) of degree three which is \((n+2)\)-symmetric and satisfies the property

\[
(8.10) \quad \int_{B_1} h(y) dy = 0
\]
of zero mean.

**Proof.** After rotating, one may assume that the frame \( \{q_i\}_{i=1}^{n+2} \) in the definition of \((n+2)\)-symmetricity satisfies

\[
(8.11) \quad q_1 = (1, 0, \cdots, 0), \quad \text{span}\{q_2, \cdots, q_{n+2}\} \perp q_1.
\]

Noting that for each \( j, k = 2, \cdots, n+2, j \neq k \), there exists an unique mapping \( \phi_{jk} \in SO(n+1) \) such that

\[
(8.12) \quad q_j = \phi_{jk}(q_1), \quad q_k = \phi_{jk}(q_j), \quad q_1 = \phi_{jk}(q_l), \quad \forall l \neq 1, j, k.
\]

Taking \( n+2 \) linear functions

\[
(8.13) \quad l_1(y) \equiv y_1, \quad l_j(y) \equiv \langle q_j, y \rangle, \quad \forall j = 2, \cdots, n+2, \quad y \in \mathbb{R}^{n+1},
\]
and setting

\[
 h_{n+2}(y) \equiv \sum_{(i-j)(j-k)\neq 0} l_i(y)l_j(y)l_k(y), \quad \forall y \in \mathbb{R}^{n+1},
\]

it is clear that \( h \) is a \((n+2)\)-symmetric polynomial of homogeneous degree 3. We claim that \( h_{n+2} \) is harmonic and has zero mean in sense of \((8.10)\) over ball \( B_1 \). Actually, the harmonicity of \( h_{n+2} \) follows from

\[
\Delta h_{n+2} = \frac{n(n+1)}{2} \langle q_1, q_2 \rangle \sum_{i=1}^{n+2} l_i(y) = \frac{n(n+1)}{2} \langle q_1, q_2 \rangle \langle \sum_{i=1}^{n+2} q_i, y \rangle = 0.
\]

To show the zero mean property, after restricting to \( S^n \), the function

\[
u(x) \equiv h_{n+2}(x), \quad \forall x \in S^n
\]
satisfies the eigen-equation

\[
(8.14) \quad - \Delta_{S^n} u = 3(n+2)u, \quad \forall x \in S^n.
\]

Integrating over the sphere, it yields that

\[
\int_{B_1} h_{n+2}(y) = \frac{1}{n+4} \int_{S^n} u = 0.
\]

So, the claim holds true and the proof of the lemma was done. \( \square \)

Lemma 8.2. For \( n \geq 2 \), the degree of polynomial \( h \in \mathcal{H}_{n+1} \) can not be less than 3.
Proof. Supposing for some \(a = \sum_{j=1}^{n+2} \varsigma_j q_j \in \mathbb{R}^{n+1}\),

\[ h(y) \equiv \langle a, y \rangle = \sum_{j=1}^{n+2} \varsigma_j \langle q_j, y \rangle \]
is a homogeneous harmonic polynomial of degree one. For fixed \(k, l, 1\) being different from each others, one has

\[
\begin{align*}
    h(y) &= h(\phi_{kl}(y)) = \sum_{j \neq k, l} \varsigma_j \langle \phi_{kl}^{-1}(q_j), y \rangle \\
    &\quad + \varsigma_k \langle \phi_{kl}^{-1}(q_k), y \rangle + \varsigma_l \langle \phi_{kl}^{-1}(q_l), y \rangle + \varsigma_1 \langle \phi_{kl}^{-1}(q_1), y \rangle \\
    &= \sum_{j \neq k, l} \varsigma_j \langle q_j, y \rangle + \varsigma_k \langle q_k, y \rangle + \varsigma_l \langle q_l, y \rangle + \varsigma_1 \langle q_1, y \rangle,
\end{align*}
\]

where \((n + 2)\)-symmetry of \(h\) has been used and \(\phi_{kl} \in SO(n + 1)\) is the orthonormal mapping defined by (8.12). Summing the last formula for different \(k, l, 1\) which do not equal mutually, we conclude that

\[
C_{k+l-1}^2 h(y) = \frac{n^2}{2} (h(y) - \varsigma_1 \langle q_1, y \rangle) + \frac{n}{2} \left( \sum_{k \neq 1} \varsigma_k \langle q_k, y \rangle \right) \\
+ \left( \sum_{k \neq 1} \varsigma_k \langle \sum_{k \neq 1} q_k, y \rangle - \sum_{k \neq 1} \varsigma_k \langle q_k, y \rangle + \frac{n}{2} \varsigma_1 \langle \sum_{k \neq 1} q_1, y \rangle \right) \\
= \frac{n^2 - 2}{2} h(y) + \langle q_1, y \rangle \left( - \frac{(n + 2)(n - 1)}{2} \varsigma_1 + \frac{n + 2}{2} \sum_{k \neq 1} \varsigma_k \right).
\]

So, we have

\[ \varsigma_k = 0, \quad \forall k \neq 1, \]

which in turn implies that \(\varsigma_1 = 0\) as well. Hence, the degree of polynomial \(h \in \mathcal{H}_{n+1}\) can not be one. It remains to show that it can also not be two. Otherwise, let

\[ h(y) = \sum_{k, j=1}^{n+1} a_{ij} y_i y_j \in \mathcal{H}_{n+1}. \]

After rotating the coordinates, one may assume that

\[ a_{ij} = \lambda_i \delta_{ij}, \quad \forall i, j = 1, 2, \ldots, n + 1. \]

Fixing \(j, k \neq 1, j \neq k\) and taking \(\phi_{jk} \in SO(n + 1)\) coming from (8.12), one may not assume that (8.11) holds further. Using the fact \(h(y) = h(\phi_{jk}(y))\) and selecting two points \(y = q_j \equiv (q_{j1}, \ldots, q_{j(n+1)})\) and \(y = q_1 \equiv (q_{11}, \ldots, q_{1(n+1)})\), it yields that

\[ \sum_{i=1}^{n+1} \lambda_i q_{ji}^2 = \sum_{i=1}^{n+1} \lambda_i q_{ki}^2 \]

for each \(j \neq k\). Introducing an auxiliary variable \(d\), we can transform (8.16) into the following system

\[ \sum_{i=1}^{n+1} \lambda_i q_{ji}^2 + d = 0, \quad \forall j = 1, 2, \ldots, n + 2. \]
Since $\{p_j\}_{j=1}^{n+2}$ spread evenly on $\mathbb{S}^n$, the rank of the $(n+2)\times(n+2)$ coefficient matrix

$$A = \begin{pmatrix}
q_{11}^2 & q_{12}^2 & \cdots & q_{1(n+1)}^2 & 1 \\
q_{21}^2 & q_{22}^2 & \cdots & q_{2(n+1)}^2 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
q_{(n+2)1}^2 & q_{(n+2)2}^2 & \cdots & q_{(n+2)(n+1)}^2 & 1
\end{pmatrix}$$

equals to $n+1$. As a result, the dimension of the set of the solutions equals to one. Combining with the fact that

$$(\lambda_1, \lambda_2, \cdots, \lambda_{n+1}, d) = (1, 1, \cdots, 1, -1)$$

is a non-trivial solution of (8.17), one gets that

(8.18) $\lambda_i = \kappa = -d$, $\forall i = 1, 2, \cdots, n+1$

give all solutions for constant $\kappa$. However, it is inferred from harmonicity of $h$

$$\Delta_{g^{n+1}} h(y) = 2\kappa = 0.$$

So, we derived the desired conclusion $h \equiv 0$ and completed the proof of this lemma. $\square$

Now, Proposition 8.1 is exactly a corollary of Lemma 8.1 and 8.2. And so, the proof of Theorem 8.1 has been completed. $\square$

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The Department of Mathematics, Shantou University, Shantou, 515063, P. R. China.
*Email address: szdu@stu.edu.cn*