Shaken dynamics for the 2d Ising model

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Abstract

We define a Markovian parallel dynamics for a class of nearest neighbors spin systems. In the dynamics, beside the two usual parameters $J$, the strength of the interaction, and $\lambda$, the external field, it appears an inertial parameter $q$, measuring the tendency of the system to remain locally in the same state. The dynamics can be defined with arbitrary boundary conditions. We prove that the parameter $q$ allows to interpolate between spin systems defined on different lattices. Moreover we prove that for certain regions of the values of the parameters the dynamics has a stationary measure very close to the Ising Gibbs measure, and it has a mixing time much smaller than the one of the standard Glauber dynamics. We present then a geological application of this dynamics, suggesting that the presence of tangent stress in the astenosphere due to the tides tends to ease the occurrence of earthquakes.
1 Introduction

In this paper we introduce a new stochastic dynamics for the nearest neighbor Ising model in 2 dimensions. A remarkable property of this dynamics is the presence of a parameter $q$ that tunes the geometry of the system. Namely for $q$ very large the geometry is the usual square lattice, for finite $q$ the system naturally lives on the hexagonal lattice, while in the limit $q \to 0$ the system becomes the product of independent $1-d$ Ising systems. The dynamics introduced in this paper is a reversible Probabilistic Cellular Automata (PCA), i.e., it updates at each step all the spins. We prove that in some regimes of the parameters it converges relatively fast to equilibrium, and its stationary measure tends to the Gibbs measure in the thermodynamic limit.

In the past two decades, in the context of equilibrium statistical mechanics, some effort has been spent in order to study the possibility to sample the Gibbs measure of spin systems by means of PCA. Gibbs samplings can be realized by introducing Markov chains on the configuration space, i.e., given a spin configuration, say $\sigma$, one defines a transition probability matrix and samples a new configuration, say $\tau$, according to such transition probabilities. In the most popular examples of this approach the dynamics are sequential, in the sense that the allowed transitions are between pairs of configurations differing at most in one site. The main feature of PCA is the fact that the transition probabilities for the extraction of the new spins $\tau$ in each site are independent. This factorization property is interesting from a numerical point of view because the PCA are natural candidates for parallel algorithms in statistical mechanics. Moreover the “mobility” of a random dynamics in which all spins may be updated in a parallel way is expected to be higher than in the case of a sequential dynamics, and then it is possible that the mixing time of the parallel chain is smaller than the mixing times of sequential ones. For both these reasons we expect PCA to be effective algorithms for Gibbs sampling.

Despite its simplicity, however, this idea is not completely straightforward.

The first attempts at studying PCA in the context of Equilibrium Statistical Mechanics dates back to [17], where various features of the infinite-volume limit have been investigated, in particular its space-time Gibbsian nature. On the other hand, invariant measures for infinite-volume PCA may be non-Gibbsian, as shown in [12]. The definition of sequen-
tial dynamics with given invariant measure $\mu$ is easy, whereas for PCA this task is not trivial at all. Counterexamples are given in [11], while [19] provides explicit conditions on $\mu$ for the existence of a PCA reversible with respect to $\mu$. In particular, if $\mu$ is a Gibbs measure for a short range interaction, one expects that the transition probabilities of the PCA can be chosen to be local, i.e. $\pi(\tau_i|\sigma)$ depends only on $\sigma_j$ with $j$ “close”, in some sense, to $i$. In [7] it has been proven that, in general, the stationary measure defined by a local PCA may have nothing to share with the Gibbs measure, giving rise to stable checkerboard configurations.

In the past years some of the authors of the present paper have investigated the possibility of defining a class of local PCA in which the stationary measure tends to the Gibbs measure $\mu$ in the thermodynamic limit. In the context of the nearest-neighbors Ising model it has been introduced a class of PCA in which an “inertial” term appears in the dynamics, preventing the simultaneous update of a too large set of spins. The idea is the following: the spin at each site is updated with an heat-bath rule according to the local field given by the configuration at all its nearest neighbors and a self-interaction term tuned by an inertial parameter $q$. For this class of PCA the ergodic measure tends to the Gibbs measure in total variation distance in the thermodynamic limit, provided the temperature is sufficiently far form the critical one (see [9], [10], [23]). Moreover it has been realized in [20] and in [10] that it is possible to define irreversible PCA having the same features, and that for a suitable choice of parameters their mixing time is considerably smaller than that of sequential and reversible dynamics. In particular, also in the low-temperature regimes, the mixing time becomes polynomial in the volume of the system, whereas, in the same regimes, it is exponential for sequential and reversible dynamics. The basic idea of this irreversible dynamics is the following: the local field felt by each spin while updating depends on half of its neighbors, namely the one close to a fixed direction in the space, and the self-interaction term. For instance in the $2d$ Ising model, that is the case discussed in the aforementioned papers, the local field of a given site is determined by the configuration of the left and downwards spins.

The irreversible PCA mentioned above are quite sensitive to boundary conditions. In particular their stationary measure can be easily computed in the translational invariant case, i.e., with periodic boundary conditions. However for different boundary conditions it has been proven in [24] that the stationary measure, already in a very simplified context,
is really difficult to compute explicitly, and it is definitely non Gibbsian.

In this paper we define a $2d$–dynamics that is inspired by the irreversible PCA mentioned above. In this dynamics the local fields driving the transitions are alternatively given by the left and downwards spins, in the even steps, and by the right and upwards spins in the odd steps. We call this PCA a “shaken” dynamics. The homogeneous dynamics obtained by the product of an even and an odd step of the shaken process turns out to be reversible, and in the case of periodic boundary conditions it has several features of the irreversible dynamics mentioned above. Moreover by reversibility it is possible to study the stationary measure of the dynamics also with different boundary conditions.

The main tool to define the shaken dynamics is a pair Hamiltonian $H(\sigma, \tau)$ with left and downwards interactions, plus the self-interaction controlled by the parameter $q$ and possibly the presence of an external magnetic field. On the space of pair of configurations the pair Hamiltonian corresponds to an Ising model defined on a hexagonal lattice. Two of the three edges exiting from each site correspond to the left and downwards interactions of strength $J$, while the third corresponds to the self-interaction $q$. The limit $q \to 0$ corresponds to erasing these $q$-edges obtaining, from the hexagonal lattice, independent copies of $1$-$d$ Ising model. The opposite limit, $q \to \infty$, corresponds to the collapse of the hexagonal lattice into the square one, by identifying the sites connected by the $q$-edges.

Beside the possibility of sampling in a parallel and fast way the Gibbs measure of the nearest-neighbor Ising model, an extra motivation of our shaken dynamics is that the alternate interaction that defines it turns out to be a good model to take into account the effects of Earth’s tides in geodynamics. We propose in this paper a toy model in which the shaken dynamics is used to describe the friction between the Earth’s lithosphere and mantle. While the model is quite simplified from a geological point of view, it seems to catch some basic feature of the continental drift.

The paper is organized as follows. In the following subsection of this introduction we discuss the geological interpretation of the model. In section 2 we define our shaken dynamics and we present our main results about it. We prove that the stationary distribution of the shaken dynamics is close to the Gibbs distribution. In the regime of low temperature we present also a metastability argument, and we show that our dynamics is sensibly faster than a sequential Metropolis algorithm. In section 3 we give the proofs of the theorems
presented in section 2, while section 4 is devoted to the discussion of some numerical results, also related to the geological interpretation of our model, to the conclusions about our present results and to several open problems.

1.1 Geological motivation

The possibility of a tidal component in the drift of the tectonic plates is a debated argument in geodynamics. Since the discovery of the net westwards drift of the lithosphere, Earth’s rotation effects have been supposed in various models (see [27], [25] and references therein). One of the main objections to this kind of arguments is the following: assuming that the liquid astenosphere can be described in terms of a Navier-Stokes fluid with very high viscosity, the friction between lithosphere and mantle is too high to be influenced by the weak forces giving rise to oceanic and terrestrial tides. In [27] it has been suggested that the description of the liquid astenosphere in terms of Navier-Stokes equation can be reasonable on very long time scale, while the short time displacement between litosphere and mantle involves typically discrete events, i.e. the sudden random fractures of rigid constraints between the two. It is reasonable to think that the main effect of such sudden breakages are the earthquakes. In [27] it has been suggested a simplified mean field model in which the energy involved in these random fractures is also random, and the single breakage occurs on a memoryless basis (exponential distribution) with a rate depending on the energy of each constraint and on the number of constraints still active. The model is a mean field approximation in the sense that there is no geometry, no dependence on the position of the constraints. A similar model is studied here, but we want to take into account the geometry of the system. It is reasonable to think that the probability of a breaking is influenced by the state of the neighbor constraints. A constraint surrounded by broken neighbors breaks with a slightly higher probability. On the other hand a constraint with intact neighbors tends to remain intact. This suggests a ferromagnetic interaction between constraints. We want to take into account in the model also the fact that the stress between litosphere and mantle due to the gradient of the tidal forces changes its direction twice a day. This forces are very small, but they are very rapidly changing, implying the possibility to observe a mechanical fatigue on the constraints, and then an increased probability of fracture. In our model, then, the breaking probability of a constraint has a weak dependence on the state of the neighbors, first in one direction and
then in the opposite one: a shaken dynamics. The presence of the self-interacting term and an external magnetic field models the tendency of the constraints to remain in the same state, independent of the state of the vicinity. As it will be discussed in the last section, the small interaction with the neighbors has dramatic effects on the dynamics. The tectonic faults are incorporated in the model by means of a boundary condition of sites that are never connected to the underlying mantle. We describe with $\sigma_i = -1$ an intact constraint, and with $\sigma_i = +1$ a broken one. Hence the fault is modelled by a line of +1 spins. Typically the instant of a spin flip is the beginning of an earthquake, while the seismic swarm has a duration given by the interval in which the constraint remains broken. We will define in section 4 a set of values of the constants appearing in the model that is able to catch some important features of the distribution of earthquakes in space and time. Clearly the hypothesis of an underlying regular lattice of possible constraints is added here for the sake of simplicity. However this geological interpretation of our results is really very encouraging, and further researches, mainly in order to make the model more realistic, will follow. We conclude this introduction mentioning the fact that the literature about the origin of the friction between litosphere and mantle is really poor, and, as outlined above, it is mainly devoted to an application of the Navier-Stokes solutions with very low Reynolds number, corresponding to the assumption of a laminar flow.

2 Definition and main results

2.1 The model

Let $\Lambda$ be a two-dimensional $L \times L$ square lattice in $\mathbb{Z}^2$ and $\mathcal{B}_\Lambda$ denotes the set of all nearest neighbors in $\Lambda$, i.e. $\{(x, y) : x, y \in \Lambda, |x - y| = 1\}$ with $|x - y|$ being the usual lattice distance in $\mathbb{Z}^d$, plus the pairs of sites at opposite faces of the square $\Lambda$, so that the pair $(\Lambda, \mathcal{B}_\Lambda)$ is homeomorphic to the two-dimensional discrete torus $(\mathbb{Z}/L\mathbb{Z})^2$. We denote by $\mathcal{X}_\Lambda$ the set of spin configurations in $\Lambda$, i.e., $\mathcal{X}_\Lambda = \{-1, 1\}^\Lambda$.

In $\Lambda$ we fix a set $B$ with fixed spins, playing the role of boundary conditions, this means that we will consider the state space $\mathcal{X}_{\Lambda,B} = \{\sigma \in \mathcal{X}_\Lambda : \sigma_x = +1 \ \forall x \in B\}$.

To introduce the Markov chain defining the dynamics, following the same ideas used in [9], [23], [20] we consider the pair Hamiltonian with asymmetric interaction
\[ H(\sigma, \tau) = - \sum_{x \in \Lambda} [J\sigma_x (\tau_x^+ + \tau_x^-) + q\sigma_x \tau_x - \lambda(\sigma_x + \tau_x)] = \]
\[ - \sum_{x \in \Lambda} [J\tau_x (\sigma_x^- + \sigma_x^+) + q\tau_x \sigma_x - \lambda(\sigma_x + \tau_x)] \quad (1) \]

where \( x^+, x^-, x^+, x^- \) are respectively the up, right, down, left neighbors of the site \( x \) on the torus \( (\Lambda, B_\Lambda) \), \( J > 0 \) is the ferromagnetic interaction, \( q > 0 \) is the inertial constant and \( \lambda > 0 \) represents an external field. We have \( H(\sigma, \sigma) = H(\sigma) - q|\Lambda| + \lambda \sum_{x \in \Lambda} \sigma_x \) where \( H(\sigma) \) is the usual Ising Hamiltonian with magnetic field \(-\lambda\)

\[ H(\sigma) = - \sum_{(x,y) \in B_\Lambda} J\sigma_x \sigma_y + \lambda \sum_{x \in \Lambda} \sigma_x \quad (2) \]

Note also that \( H(\sigma, \tau) \neq H(\tau, \sigma) \).

Define the two functions

\[ \mathcal{Z}_\sigma = \sum_{\sigma' \in X_\Lambda, B} e^{-H(\sigma, \sigma')} \quad \mathcal{Z}_\sigma = \sum_{\sigma' \in X_\Lambda, B} e^{-H(\sigma', \sigma)} \quad (3) \]

and the two asymmetric updating

\[ P_{dl}(\sigma, \sigma') := \frac{e^{-H(\sigma, \sigma')}}{\mathcal{Z}_\sigma}, \quad P_{ur}(\sigma, \sigma') := \frac{e^{-H(\sigma', \sigma)}}{\mathcal{Z}_\sigma} \quad (4) \]

Due to the definition of the pair Hamiltonian the updating given by the transition probability \( P_{dl}(\sigma, \sigma') \) is parallel: given a configuration \( \sigma \) in each site \( x \in \Lambda \) a new spin configuration \( \sigma'_x \) is chosen with a probability proportional to \( e^{h_{dl}^x(\sigma)} \sigma'_x \) where the local down-left field \( h_{dl}^x(\sigma) \) due to the configuration \( \sigma \) is given by

\[ h_{dl}^x(\sigma) = \left[ J(\sigma_{x^+} + \sigma_{x^-}) + q\sigma_x - \lambda \right], \]

so that

\[ P_{dl}(\sigma, \sigma') := \frac{e^{-H(\sigma, \sigma')}}{\mathcal{Z}_\sigma} = \prod_{x \in \Lambda} \frac{e^{h_{dl}^x(\sigma)} \sigma'_x}{2 \cosh h_{dl}^x(\sigma)} \]

Similarly for \( P_{ur}(\sigma, \sigma') \) with a up-right field

\[ h_{ur}^x(\sigma) = \left[ J(\sigma_{x^+} + \sigma_{x^-}) + q\sigma_x - \lambda \right] \]
we get
\[ P_{ur}(\sigma, \sigma') := \frac{e^{-H(\sigma', \sigma)}}{Z_\sigma} = \prod_{x \in \Lambda} \frac{e^{h_{ur}^{(\sigma)}(\sigma'_x)}}{2 \cosh h_{ur}^{(\sigma)}(\sigma'_x)}. \]

With these asymmetric transition probabilities we define on \( \mathcal{X}_{\Lambda, B} \) the shaken dynamics given by the Markov chain with transition probabilities
\[ P_{sh}(\sigma, \tau) = \sum_{\sigma' \in \mathcal{X}_{\Lambda, B}} P_{dl}(\sigma, \sigma') P_{ur}(\sigma', \tau) = \sum_{\sigma' \in \mathcal{X}_{\Lambda, B}} \frac{e^{-H(\sigma, \sigma')}}{Z_\sigma} \frac{e^{-H(\tau, \sigma')}}{Z_{\sigma'}}. \tag{5} \]

So the shaken dynamics is the composition of two asymmetric steps, with interactions in opposite directions. Note that similar definition of the dynamics could be obtained, of course, by considering
\[ P_{sh'}(\sigma, \tau) = \sum_{\sigma' \in \mathcal{X}_C} P_{ur}(\sigma, \sigma') P_{dl}(\sigma', \tau). \]

### 2.2 Results

We first collect preliminary results on the shaken dynamics in the following Theorem. In the proof we also write explicitly the invariant measure of the shaken dynamics in terms of Peierls contours.

**Theorem 2.1** The stationary measure of the shaken dynamics is
\[ \pi_{\Lambda, B}(\sigma) = \frac{Z_\sigma}{Z} \tag{6} \]
and reversibility holds. This stationary measure is the marginal of the measure on \( \mathcal{X}_{\Lambda, B} \times \mathcal{X}_{\Lambda, B} \) defined by:
\[ \pi_2(\sigma, \tau) := \frac{1}{Z} e^{-H(\sigma, \tau)}. \tag{7} \]

The space of pairs of configurations with interaction given by \( H(\sigma, \tau) \) can be represented as the configuration space \( \mathcal{X}_\mathbb{H} \) for the Ising model on an hexagonal lattice \( \mathbb{H} \). Since \( \mathbb{H} \) is a bipartite graph, the vertex set \( V \) of \( \mathbb{H} \) can be decomposed into two sets \( V = V^1 \cup V^2 \), with \( |V^i| = |\Lambda| \), \( i = 1, 2 \) and each \( \sigma \in \mathcal{X}_\mathbb{H} \) can be written as \( \sigma = (\sigma^1, \sigma^2) \) with \( \sigma^i \in \mathcal{X}_{V^i, B}, \) \( i = 1, 2 \). The shaken dynamics on \( \mathcal{X}_{\Lambda, B} \) corresponds to an alternate dynamics on \( \mathcal{X}_\mathbb{H} \) in
the following sense

\[ P^{sh}(\sigma^1, \tau^1) = \sum_{\tau^2 \in \{-1, +1\}^V} P^{alt}(\sigma, \tau) \]  

(8)

with

\[ P^{alt}(\sigma, \tau) = \frac{e^{-H(\sigma^1, \tau^2)} e^{-H(\tau^1, \tau^2)}}{Z_{\sigma^1} Z_{\tau^2}} = \prod_{x \in V^2} \frac{e^{h_x(\sigma^1) \tau^2_x}}{2 \cosh(h_x(\sigma^1))} \prod_{x \in V^1} \frac{e^{h_x(\tau^2) \tau^1_x}}{2 \cosh(h_x(\tau^2))} \]  

(9)

with

\[ h_x(\sigma^1) = J(\sigma_{z1} + \sigma_{z2}) + q\sigma_{z3} - \lambda, \]

where \(z_1, z_2, z_3 \in V_i\) nearest neighbors of \(x\) (see Fig. 2, pag. 20), and the measure \(\pi_2(\sigma^1, \sigma^2)\) is the non reversible stationary measure of \(P^{alt}\).

The results concerning the alternate dynamics introduced in the previous theorem can be expressed in a more general context. In particular the following theorem holds.

**Theorem 2.2**

Let \(H(\sigma, \tau)\) be a pair Hamiltonian with \(\sigma, \tau \in \mathcal{X}_\Lambda\) and

\[ H(\sigma, \tau) = \sum_{x \in \Lambda} \sigma_x h^1_x(\tau) = \sum_{x \in \Lambda} \tau_x h^2_x(\sigma) \]

for a family of linear functions \(h^1_x(\tau)\) and \(h^2_x(\sigma)\), \(x \in \Lambda\). The space of pairs of configurations is \(\mathcal{X} = \mathcal{X}_\Lambda \times \mathcal{X}_\Lambda = \{+1, -1\}^V\) where \(V = V^1 \cup V^2\), with \(|V^i| = |\Lambda|, i = 1, 2\) and \(\sigma, \tau \in \mathcal{X} = (\sigma^1, \sigma^2)\), with \(\sigma^i \in \mathcal{X}_{V^i}, i = 1, 2\). The pair Hamiltonian defines therefore a bipartite graph \(G = (V, E)\) with edge set \(E\) induced by the explicit form of the linear functions \(h^1_x(\tau)\) and \(h^2_x(\sigma)\).

Consider the alternate, parallel dynamics \(P^{alt}\) on \(\mathcal{X}\) with transition probabilities

\[ P^{alt}(\sigma, \tau) = \frac{e^{-H(\sigma^1, \tau^2)} e^{-H(\tau^1, \tau^2)}}{Z_{\sigma^1} Z_{\tau^2}}. \]  

(10)

Then the stationary measure of \(P^{alt}\) is

\[ \pi_2(\sigma, \tau) := \frac{1}{Z} e^{-H(\sigma, \tau)} \]  

(11)

that is the Gibbs measure on the space \(\mathcal{X}\). This dynamics is in general non reversible.

Coming back to our shaken dynamics, by equation (8) and noting that \(P^{alt}(\sigma, \tau)\) does
not depend on $\sigma^2$, we can define the evolution of $X_t^{sh}$ as a marginal of the evolution of the alternate process $X_t^{alt}$ given by the Markov chain with transition probabilities $P^{alt}(\sigma, \tau)$. This means that if we consider a path $\omega$ for the process $X^{alt}$, i.e., a sequence of configurations

$$\omega : \omega(0), \omega(1), ..., \omega(t)$$

and the associated path for the process $X^{sh}$ $\omega$

$$\omega : \omega^1(0), \omega^1(1), ..., \omega^1(t)$$

we have

$$P^{sh}(\omega) = \sum_{\omega^2(1), ..., \omega^2(t)} P^{alt}(\omega).$$

From now on let $B = \emptyset$, i.e., we consider standard periodic boundary conditions.

We denote by $\pi^G_\Lambda$ the Gibbs measure

$$\pi^G_\Lambda(\sigma) = \frac{e^{-H(\sigma)}}{Z_G} \quad \text{with} \quad Z_G = \sum_{\sigma \in \mathcal{X}_\Lambda} e^{-H(\sigma)}$$

with $H(\sigma)$ defined in (2).

Define the total variation distance, or $L_1$ distance, between $\pi_\Lambda$ and $\pi^G_\Lambda$ as

$$\|\pi_\Lambda - \pi^G_\Lambda\|_{TV} = \frac{1}{2} \sum_{\sigma \in \mathcal{X}_\Lambda} |\pi_\Lambda(\sigma) - \pi^G_\Lambda(\sigma)|. \quad (12)$$

Set $\delta = e^{-2q}$, and let $\delta$ be such that

$$\lim_{|\Lambda| \to \infty} \delta^2 |\Lambda| = 0, \quad (13)$$

In the following theorem 2.3 we control the distance between the invariant measure of the shaken dynamics and the Gibbs measure at low temperature and for $q$ positive and large. Actually this is an extension of Theorem 1.2 in [23] where the case $\lambda = 0$ was considered.
Theorem 2.3 Under the assumption (13), there exist $\bar{J}$ such that for any $J > \bar{J}$

$$\lim_{|\Lambda| \to \infty} \| \pi_\Lambda - \pi^G_\Lambda \|_{TV} = 0 \quad (14)$$

Note that, following the same strategy, we can easily prove that in the case of finite volume, $|\Lambda|$ fixed, there exists $J_0$ sufficiently large, $\eta \in (0, 1)$ and $C = C(J_0, \eta, \Lambda)$ such that for any $J$ and $q$ with $J_0 < J < q(1 - \eta)$ we have

$$\| \pi_\Lambda - \pi^G_\Lambda \|_{TV} \leq C\delta^2$$

Our next and last result is on the convergence to equilibrium of the shaken dynamics in the low temperature regime at fixed volume $|\Lambda|$. We will use the following parametrization:

$$J = \frac{\beta}{2}, \quad q = \beta, \quad \lambda = \frac{\varepsilon \beta}{2}$$

and suppose $\varepsilon$ small but fixed, $\beta \to \infty$ and $\Lambda$ large ($|\Lambda| > \frac{1}{\varepsilon^2}$) fixed, i.e., independent of $\beta$. For this choice of the parameters the invariant measure of the shaken dynamics concentrates on the configuration with all negative spins, we call it $-1$, parallel to the external magnetic field $-\lambda$. However the convergence to equilibrium is delayed by the metastable behaviour of the system. Suppose to start with all positive spins, call this configuration $+1$. This state is a metastable state in the sense that, to leave it, the process has to overcome a high potential barrier. Indeed in this low temperature regime

$$\pi_\Lambda(\sigma) \propto \sum_\tau e^{-H(\sigma, \tau)} \approx e^{-\mathcal{H}(\sigma)}$$

with $\mathcal{H}(\sigma) = \min_\tau H(\sigma, \tau)$, where the configuration $\tau$, realizing the minimum, is such that $\tau_x h_x(\sigma) > 0$ for any $x \in \Lambda$. The configuration $+1$ corresponds to a local minimum of the energy $\mathcal{H}(\sigma)$ and there is a local drift to this minimum given by typical transitions of probability of order one. To leave $+1$ the process has to go “against” this drift, with transitions of exponentially small probability. Indeed small clusters of minus spin in a sea of positive spins, have the tendency to shrink, and there is a critical size of cluster of minus spin to overcome in order to prefer to grow. We want to show that in this regime of parameters, the shaken dynamics has a smaller typical time to reach the stable
configuration $-1$ w.r.t. standard single spin flip dynamics, e.g. Glauber dynamics. The advantage is not due to parallelization. Indeed even if the shaken dynamics is parallel, so that

$$P^{sh}(+1, -1) > 0,$$

at low temperature ($\beta$ large) it behaves with large probability like a single spin flip dynamics starting from $+1$. Parallelization has positive effects when the dynamics is going along and not against the drift.

The gain is due to geometrical reasons: the shaken dynamics is equivalent to the alternate dynamics on the hexagonal lattice, and the potential barrier on this lattice is lower than the corresponding barrier on the square lattice.

Denote by $\tau_{-1}^{sh}$ the first hitting time to the configuration $-1$ for the shaken dynamics and by $P_{+1}(\tau_{-1}^{sh} > t)$ its distribution starting from the configuration $+1$.

**Theorem 2.4** For $\beta$ sufficiently large and for any $\alpha > 0$ arbitrarily small we have

$$P_{+1}(\tau_{-1}^{sh} > T^{sh} e^{\alpha \beta}) < \exp\left\{-e^{a\beta}\right\}$$

for some $a > 0$, with $T^{sh} = e^{E_c^{sh} / \beta}$ and

$$E_c^{sh} = 4l_c - 2\varepsilon(l_c - 1)l_c - \varepsilon, \quad \text{where} \quad l_c = \frac{1}{\varepsilon}$$

Note that with the same parameters $J$ and $\lambda$, for the usual Glauber single spin flip dynamics we have that for $\beta$ large and $\alpha > 0$ arbitrarily small (see e.g. [22], [3])

$$P_{+1}\left(T^{Gl} e^{-\alpha \beta} < \tau_{-1}^{Gl} < T^{Gl} e^{\alpha \beta}\right) \sim 1 \quad \text{for large } \beta$$

with

$$T^{Gl} = e^{E_c^{Gl} / \beta}, \quad \text{where} \quad E_c^{Gl} = 4l_c^{Gl} - \varepsilon(l_c^{Gl} - 1)l_c^{Gl} - \varepsilon \quad \text{with} \quad l_c^{Gl} = \frac{2}{\varepsilon}$$

so that $E_c^{sh} \sim \frac{2}{\varepsilon} = \frac{1}{2} \cdot \frac{4}{\varepsilon} \sim \frac{1}{2} E_c^{Gl}$ and therefore

$$T^{sh} \sim \sqrt{T^{Gl}}.$$  (16)
By the remark after Theorem 2.3, equation (16) enables us to conclude that in this small temperature regime the shaken dynamics is an efficient tool for Gibbs sampling within a given error depending on $\beta$.

The advantages of the shaken dynamics can be summarized as follows:

- the dynamics is actually on a different lattice so there is a gain of a square root when moving “against the drift”;

- the dynamics is parallel so there is a gain in the efficiency proportional to $|\Lambda|$ when moving “along the drift”.

3 Proofs of the results

3.1 Proof of theorem 2.1

We have immediately the detailed balance condition w.r.t. the measure $\pi_{\Lambda, B}(\sigma)$ indeed

$$\sum_{\sigma' \in \mathcal{X}_{\Lambda, B}} \frac{e^{-(H(\sigma, \sigma') + H(\tau, \sigma'))}}{Z_{\sigma'}} = \frac{Z_{\sigma} P_{sh}(\sigma, \tau)}{Z_{\tau} P_{sh}(\tau, \sigma)} = \sum_{\sigma' \in \mathcal{X}_{\Lambda, B}} \frac{e^{-(H(\tau, \sigma') + H(\sigma, \sigma'))}}{Z_{\sigma'}}$$

(17)

Given a configuration $\sigma \in \mathcal{X}_{\Lambda, B}$ we denote by $\gamma(\sigma)$ its Peierls contour in the dual $\mathcal{B}_\Lambda^* = \cup_{(x, y) \in \mathcal{B}_\Lambda} (x, y)^*$

$$\gamma(\sigma) := \{ (x, y)^* \in \mathcal{B}_\Lambda^* : \sigma_x \sigma_y = -1 \}$$

(18)

Starting from the Peierls contour of $\sigma$, due to the symmetry of the interaction, we also define two subsets of $\Lambda$:

$$n_{\wedge, B}(\sigma) := \{ x \in \Lambda \setminus B : (x, x^-) \in \gamma(\sigma), (x, x^+) \in \gamma(\sigma) \}$$

(19)

and

$$n_{\vee, B}(\sigma) := \{ x \in \Lambda \setminus B : (x, x^+) \in \gamma(\sigma), \text{ or } (x, x^-) \in \gamma(\sigma) \}$$

(20)
Note that
\[ |\gamma(\sigma)| \geq |n_{\vee, B}| + 2|n_{\wedge, B}(\sigma)| \]
indeed
\[ |\gamma(\sigma)| = |n_{\vee, B}| + 2|n_{\wedge, B}(\sigma)| + \left| \{ x \in \partial \text{int} B : (x, x^+) \in \gamma(\sigma), \text{ or } (x, x^-) \in \gamma(\sigma) \} \right| + \]
\[ + 2\left| \{ x \in \partial \text{int} B : (x, x^-) \in \gamma(\sigma), (x, x^+) \in \gamma(\sigma) \} \right| . \]

Since \( h_{2l}(\sigma) \) takes values
\[ \pm(2J + q) \quad \text{for } x \in \Lambda \setminus (B \cup n_{dl} \cup \gamma_B) \]
\[ \pm(2J - q) \quad \text{for } x \in n_{dl} \]
\[ \pm q \quad \text{for } x \in \gamma_B, \]
with this notation we easily get
\[ \tilde{Z}_{\sigma} = \sum_{\tau} e^{\sum_{x} h_{2l}(\sigma) \tau_{x}} = \left[ 2 \cosh(2J + q) \right]^{|\Lambda \setminus B|} \left[ \frac{\cosh(2J - q)}{\cosh(2J + q)} \right]^{n_{\wedge, B}(\sigma)} \left[ \frac{\cosh q}{\cosh(2J + q)} \right]^{|n_{\vee, B}(\sigma)|} \]
so that
\[ \pi_{\Lambda, B}(\sigma) = \frac{\left[ \cosh(2J - q) \right]^{n_{\wedge, B}(\sigma)} \left[ \frac{\cosh q}{\cosh(2J + q)} \right]^{|n_{\vee, B}(\sigma)|}}{\sum_{\sigma'} \left[ \cosh(2J - q) \right]^{n_{\wedge, B}(\sigma')} \left[ \frac{\cosh q}{\cosh(2J + q)} \right]^{|n_{\vee, B}(\sigma')|}} \]

Representing the pair of configurations on the graph in Fig. 1, the equivalence of the spaces \( \mathcal{X}_{\Lambda, B} \times \mathcal{X}_{\Lambda, B} \) and \( \mathcal{X}_{\mathbb{H}} \) and equation (8) follow immediately.

By writing
\[ \sum_{\sigma^1, \sigma^2} \pi_2(\sigma^1, \sigma^2) P_{\text{alt}}(\sigma, \tau) = \sum_{\sigma^1, \sigma^2} \frac{e^{-H(\sigma^1, \sigma^2)}}{Z} \frac{e^{-H(\sigma^1, \tau^2)}}{Z} \frac{e^{-H(\tau^1, \tau^2)}}{Z} = \frac{e^{-H(\tau^1, \tau^2)}}{Z} = \pi_2(\tau^1, \tau^2) \]
even if in general
\[ \pi_2(\sigma^1, \sigma^2) P_{\text{alt}}(\sigma, \tau) \neq \pi_2(\tau^1, \tau^2) P_{\text{alt}}(\tau, \sigma) . \]

The final part of this proof clearly holds in the more general context considered in theorem
3.2 Proof of theorem 2.3

To prove Theorem 2.3 it is possible to argue as in the proof of Theorem 1.2 in [23].

In our notation $\pi_{\Lambda}$ and $\pi_{\Lambda}^G$ correspond, respectively, to $\pi_{PCA}$ and $\pi_G$ used in [23]. Further let $g_x(\sigma) := J(\sigma_{x_+} + \sigma_{x_-})$ be the analogue of $h_i(\sigma)$ in [23]. Note that here we dropped all references to boundary conditions.

Recalling that that $\delta = e^{-2q}$, it is possible to write $Z_\sigma$ in the following way:
Using Jensen’s inequality the total variation distance between \( \pi \) as follows

\[
Z_\sigma = \sum_\tau e^{-H(\sigma, \tau)} = \sum_\tau e^{-H(\sigma, \tau)} e^{-[\lambda - H(\sigma, \tau)]}
\]

\[
= e^{\eta|\Lambda|} e^{-\lambda} \sum_\sigma e^{-\sum_{x: \tau_x \neq \sigma_x} -2g_x(\sigma)\sigma_x -2q + 2\lambda \sigma_x}
\]

\[
= e^{\eta|\Lambda|} e^{-\lambda} \sum_\sigma \sum_{I \subset \Lambda} \delta_{|I|} \prod_{x \in I} e^{-2g_x(\sigma)\sigma_x + 2\lambda \sigma_x}
\]

\[
= e^{\eta|\Lambda|} e^{-\lambda|\Lambda|} e^{-2|\Lambda|G(\sigma)} \prod_{x \in \Lambda} (1 + \delta e^{-2g_x(\sigma)\sigma_x + 2\lambda \sigma_x}) 
\] (23)

where the sum over \( \tau \) has been rewritten as the sum over all subsets \( I \subset \Lambda \) such that \( \tau_x = -\sigma_x \) if \( x \in I \) and \( \tau_x = \sigma_x \) otherwise and \( |V_+(\sigma)| = \sum_{x \in \Lambda} 1_{\{\sigma_x = +1\}} \) is the number of plus spins in \( \Lambda \). The factors \( e^{\eta|\Lambda|} \) and \( e^{\lambda|\Lambda|} \) do not depend on \( \sigma \) and cancel out in the ratio \( \frac{Z_{\sigma}}{Z} \).

Call \( f(\sigma) := e^{-2|\Lambda|G(\sigma)} \prod_{x \in \Lambda} (1 + \delta e^{-2g_x(\sigma)\sigma_x + 2\lambda \sigma_x}) \), \( w(\sigma) := e^{-H(\sigma)} f(\sigma) = w^{G(\sigma)} f(\sigma) \). Then (23) can be rewritten as

\[
\pi_\Lambda(\sigma) = \frac{w(\sigma)}{\sum_{\tau} w(\tau)} = \frac{w^{G(\sigma)} f(\sigma)}{\sum_{\tau} w^{G(\tau)} f(\tau)} = \frac{w^{G(\sigma)} f(\sigma)}{Z^{G(\sigma)}} \frac{Z^{G(\sigma)}}{Z^{G(f)}} = \pi^{G(\sigma)}_\Lambda \frac{\pi^{G(\sigma)}_\Lambda}{\pi^{G(f)}_\Lambda} 
\]

with \( \pi^{G(\sigma)}_\Lambda = \sum_{\tau} \pi^{G(\sigma)}_\Lambda(\tau) f(\tau) \).

Using Jensen’s inequality the total variation distance between \( \pi_\Lambda \) and \( \pi^{G(\sigma)}_\Lambda \) can be bounded as follows

\[
\|\pi_\Lambda - \pi^{G(\sigma)}_\Lambda\|_{TV} = \sum_{\sigma} \pi^{G(\sigma)}_\Lambda \left| \frac{f(\sigma)}{\pi^{G(f)}_\Lambda} - 1 \right| = \pi^{G(\sigma)}_\Lambda \left( \frac{f(\sigma)}{\pi^{G(f)}_\Lambda} - 1 \right)
\]

\[
\leq \frac{\text{var}_{\pi_\Lambda}^{G(f))}}{\pi^{G(f)}_\Lambda} = \sqrt{\frac{\pi^{G(f)}_\Lambda}{(\pi^{G(f)}_\Lambda)}^2 - 1} =: \sqrt{(\Delta(\delta))}. 
\]

To prove the theorem, it will be shown that \( \Delta(\delta) = O(\delta^2|\Lambda|) \).

By writing

\[
\Delta(\delta) = e^{\log(\pi^{G(f)}_\Lambda) - 2\log(\pi^{G(f)}_\Lambda)} - 1,
\]
the claim follows, as in the proof of Theorem 1.2 in [23], by showing that the argument of the exponential divided by $|\Lambda|$ is analytic in $\delta$ and that the first order term of its expansion in $\delta$ cancels out.

In other words the claim follows thanks to the following lemma.

**Lemma 3.1** There exists $J_c$ such that, for all $J > J_c$

1. $\frac{\log(\pi_G^\Lambda(f^2))}{|\Lambda|}$ and $\frac{\log(\pi_G^\Lambda(f))}{|\Lambda|}$ are analytic in $\delta$ for $|\delta| < J$

2. $\frac{\log(\pi_G^\Lambda(f^2))}{|\Lambda|} - 2\frac{\log(\pi_G^\Lambda(f))}{|\Lambda|} = O(\delta^2)$

**Proof:** The proof of this follows closely the proof of Lemma 2.3 in [23]. The analyticity of $\frac{\log(\pi_G^\Lambda(f^2))}{|\Lambda|}$ and $\frac{\log(\pi_G^\Lambda(f))}{|\Lambda|}$ is proven by showing that these quantities can be written as partition functions of an abstract polymer gas. The analyticity is obtained using standard cluster expansion.

To carry over this task, we will rewrite $\pi_A^G(f^k)$ in terms of standard Peierls contours (see (18)). Divide the sites in $\Lambda$ according to the value of the spins and number of edges of the Peierls contour left and below the site in the following way:

- $\Lambda_{-}$: $\{ x \in \Lambda : \sigma_x = -1 \land (\sigma_{x^-} = -1, \sigma_{x^+} = -1) \}$;
- $\Lambda_{+}$: $\{ x \in \Lambda : \sigma_x = -1 \land (\sigma_{x^-} = +1, \sigma_{x^+} = -1) \lor (\sigma_{x^-} = -1, \sigma_{x^+} = +1) \}$;
- $\Lambda_{+}$: $\{ x \in \Lambda : \sigma_x = +1 \land \sigma_{x^-} = +1, \sigma_{x^+} = +1 \}$;
- $\Lambda_{++}$: $\{ x \in \Lambda : \sigma_x = +1 \land (\sigma_{x^-} = +1, \sigma_{x^+} = +1) \}$;
- $\Lambda_{-+}$: $\{ x \in \Lambda : \sigma_x = +1 \land ((\sigma_{x^-} = +1, \sigma_{x^+} = -1) \lor (\sigma_{x^-} = -1, \sigma_{x^+} = +1)) \}$;
- $\Lambda_{-}$: $\{ x \in \Lambda : \sigma_x = +1 \land (\sigma_{x^-} = +1, \sigma_{x^+} = -1) \lor (\sigma_{x^-} = -1, \sigma_{x^+} = +1) \}$;
- $\Lambda_{-}$: $\{ x \in \Lambda : \sigma_x = +1 \land (\sigma_{x^-} = -1, \sigma_{x^+} = -1) \}$;

With this notation, $f(\sigma)$ can be written as
We have
\[ f(\sigma) = (1 + \delta e^{-4J-2\lambda})^{|\Lambda|} \]

\[
\prod_{x \in \Lambda_{-}} \frac{(1 + \delta e^{-4J-2\lambda})}{(1 + \delta e^{-4J-2\lambda})} \prod_{x \in \Lambda_{+}} \frac{(1 + \delta e^{-2\lambda})}{(1 + \delta e^{-4J-2\lambda})} \prod_{x \in \Lambda_{-}} \frac{(1 + \delta e^{4J-2\lambda})}{(1 + \delta e^{-4J-2\lambda})} \]

\[
\prod_{x \in \Lambda_{+}} \frac{e^{-2\lambda}(1 + \delta e^{-4J+2\lambda})}{(1 + \delta e^{-4J-2\lambda})} \prod_{x \in \Lambda_{-}} \frac{e^{-2\lambda}(1 + \delta e^{4J+2\lambda})}{(1 + \delta e^{-4J-2\lambda})} \prod_{x \in \Lambda_{-}} \frac{e^{-2\lambda}(1 + \delta e^{4J+2\lambda})}{(1 + \delta e^{-4J-2\lambda})}
\]

(24)

\[
= (1 + \delta e^{-4J-2\lambda})^{|\Lambda|} \xi(\sigma, \lambda)
\]

with

\[
\xi(\sigma, \lambda) = \left[ \frac{(1 + \delta e^{-2\lambda})}{(1 + \delta e^{-4J-2\lambda})} \right]^{\Lambda_{-}} \left[ \frac{(1 + \delta e^{4J-2\lambda})}{(1 + \delta e^{-4J-2\lambda})} \right]^{\Lambda_{+}}
\]

\[
\left[ \frac{e^{-2\lambda}(1 + \delta e^{-4J+2\lambda})}{(1 + \delta e^{-4J-2\lambda})} \right]^{\Lambda_{+}} \left[ \frac{e^{-2\lambda}(1 + \delta e^{4J+2\lambda})}{(1 + \delta e^{-4J-2\lambda})} \right]^{\Lambda_{-}} \left[ \frac{e^{-2\lambda}(1 + \delta e^{4J+2\lambda})}{(1 + \delta e^{-4J-2\lambda})} \right]^{\Lambda_{-}}
\]

(25)

and \(e^{-H(\sigma)} = e^{(2J+\lambda)|\Lambda|} e^{-2J|\gamma(\sigma)|-2\lambda|V_{+}(\sigma)|}\).

We have

\[
\pi^{G}_{\Lambda}(f^{k}) = \frac{1}{Z_{G}} e^{(2J+\lambda)|\Lambda| + \delta e^{-4J-2\lambda})^{k}^{|\Lambda|} \sum_{\sigma} e^{-2J|\gamma(\sigma)|-2\lambda|V_{+}(\sigma)|} \xi^{k}(\sigma, \lambda)\]

(26)

A straightforward computation yields \(\xi^{k}(\sigma, \lambda) \leq \xi^{k}(\sigma, 0)\) and then

\[
\sum_{\sigma} e^{-2J|\gamma(\sigma)|-2\lambda|V_{+}(\sigma)|} \xi^{k}(\sigma, \lambda) \leq \sum_{\sigma} e^{-2J|\gamma(\sigma)|} \xi^{k}(\sigma, 0) = 2 \sum_{\gamma} e^{-2J|\gamma|} \xi^{k}(\gamma, 0)
\]

where \(\xi^{k}(\gamma, 0)\) coincides with \(\xi^{k}(\Gamma)\) in the proof of Lemma 2.3 in \[23\], with \(|\Lambda_{-}| + |\Lambda_{+}| = |l_{1}(\Gamma)|\) and \(|\Lambda_{+}| + |\Lambda_{-}| = |l_{2}(\Gamma)|\).

This implies that the proof can be concluded following the same steps as in \[23\].

\[\square\]
3.3 Proof of theorem 2.4

We proceed by following standard arguments in the study of metastability, see eg. [22], [28].

As noted at the end of Theorem 2.1, we can define the evolution of $X_{sh}^t$ as a marginal of the evolution of the alternate process $X^{alt}$ so that $\tau_{-1}^sh \leq \tau_{-1}^{alt}$. So we have to study metastability in the hexagonal anisotropic lattice with the alternate dynamics.

We first prove that with our choice of the parameters, the alternate process is in the Freidlin Wentzell (FW) regime ([13]). This means that to each transition we can associate a non negative exponential costs $\Delta(\sigma, \tau)$, i.e., for large $\beta$ we have

$$e^{-\Delta(\sigma, \tau)\beta - \alpha\beta} \leq P^{alt}(\sigma, \tau) < e^{-\Delta(\sigma, \tau)\beta + \alpha\beta}$$

with $\alpha = \alpha(\beta) \to 0$ for $\beta \to \infty$. For shortness, with standard notation, we will write

$$P^{alt}(\sigma, \tau) \asymp e^{-\Delta(\sigma, \tau)\beta}.$$

Indeed we have that

$$\Delta(\sigma, \tau) = \sum_{x^1 \in V^1} \Delta_{x^1}(\sigma, \tau) + \sum_{x^2 \in V^2} \Delta_{x^2}(\sigma, \tau)$$

where

$$\Delta_{x^1}(\sigma, \tau) = \left[\tau_{x^1}^1 2h_{x^1}(\tau^2)\right]_-, \quad \Delta_{x^2}(\sigma, \tau) = \left[\tau_{x^2}^2 2h_{x^2}(\sigma^1)\right]_-$$

$h_{x^2}(\sigma^1) = \frac{1}{2}(\sigma^1_{z_1} + \sigma^1_{z_2} + 2\sigma^1_{z_3} - \varepsilon)$ with $z_1, z_2, z_3 \in V^1$ the nearest neighbors of $x^2$, $z_3$ being the neighbor related to $x^2$ by a $q$-bond (orthogonal to horizontal dual bond in Fig. 2) and similarly for $h_{x^1}(\tau^2)$. Here $[\cdot]_-$ denotes the negative part. These relations for the transition costs are immediately obtained, recalling our parametrization, by the definition ([10] and noting that for large $\beta$ and for any $a \in \mathbb{R}$ we have

$$\frac{e^{a\beta}}{e^{a\beta} + e^{-a\beta}} \asymp e^{-2\beta[a]}.$$

The quantities $\Delta_x(\sigma, \tau)$ take values in the set $\{0, 4 - \varepsilon, 2 - \varepsilon, \varepsilon, 2 + \varepsilon, 4 + \varepsilon\}$. Note that $\Delta(\sigma, \tau) = 0$ only if $\Delta_x(\sigma, \tau) = 0$ for any $x \in V$. Moreover $\Delta(\sigma, \tau) = 0$ implies $H(\sigma) \geq$
Figure 2: A spin configuration on portion of the hexagonal lattice with its Peierls’ contour. Black dots represent “minus” spins and white dots “plus” spins. Horizontal edges in the contour have cost $q$ whereas slanted edges have cost $J$.

$H(\tau)$. Indeed by definition (10), if $\Delta(\sigma, \tau) = 0$ we have

$$\min_{\tau} H(\sigma^{1}, \tau) = H(\sigma^{1}, \tau^{2}) \quad \text{and} \quad \min_{\tau} H(\tau, \tau^{2}) = H(\tau^{1}, \tau^{2}).$$

The study of metastability in the FW regime is a well known problem and general results are available ([22], [3], [2]). We do not perform here the complete metastability analysis but we will describe just the first step of the renormalization procedure in [28]. Even if more complete results on the behavior of the process in the escape from metastability could be obtained, we will restrict ourself to deduce the estimate (15).

In the FW regime, to each finite path $\phi: \sigma \rightarrow \tau$ from $\sigma$ to $\tau$, i.e.,

$$\phi: \sigma_{0} = \sigma, \sigma_{1}, ..., \sigma_{t} = \tau$$

with $t$ independent of $\beta$, we associate the cost

$$I(\phi) = \sum_{i=0}^{t-1} \Delta(\sigma_{i}, \sigma_{i+1})$$
since it is easy to prove (see e.g. [13], [28])

\[ \mathbb{P}(X^\text{alt}_s \text{ follows } \phi) \equiv \mathbb{P}_{\sigma}(X^\text{alt}_s = \phi_s, \ \forall s \in [0, t]) \propto \exp\{-I(\phi)\beta\} \]

and to each pair of states \( \sigma, \tau \) we associate the minimal cost

\[ V(\sigma, \tau) = \min_{\phi: \sigma \to \tau} I(\phi). \]

The states \( \sigma \) and \( \tau \) are equivalent, and we write \( \tau \sim \sigma \), if \( V(\sigma, \tau) = V(\tau, \sigma) = 0 \). The stable configurations are given by the following definition (see [13]).

**Definition 3.2**

\[ M = \{ \sigma \in \mathcal{X}_H : \forall \tau \not\sim \sigma \ \ V(\sigma, \tau) > 0 \}. \]

Given a configuration \( \sigma \in \mathcal{X}_H \) we denote by \( \gamma(\sigma) \) its Peierls contour in the dual \( \mathcal{B}_H^* = \bigcup_{(x,y) \in \mathcal{B}_H} (x,y)^* \), i.e.,

\[ \gamma(\sigma) := \{(x,y)^* \in \mathcal{B}_H^* : \sigma_x \sigma_y = -1\} \quad (27) \]

Note that Peierls contours live on the dual lattice, i.e., on the triangular lattice, see eg Fig. 2. A Peierls contour is the union of piecewise linear curves separating spins with opposite sign in \( \sigma \).

**Definition 3.3** Let \( M_0 \) be the set of configurations with the following two properties:

1) \( M_0 \supset \{+1, -1\} \);

2) for any \( \sigma \in M_0 \) with Peierls contour \( \gamma(\sigma) \neq \emptyset \) we have that \( \gamma(\sigma) \) does not contain any bond corresponding to interaction \( q \) (horizontal bonds in Fig. 2) and every spin \(+1\) of \( \sigma \) shares with \( \gamma(\sigma) \) at most a single dual bond.

Note that if \( \sigma \in M_0 \), then in each dual vertex there are 0, 2 or 4 dual bonds contained in \( \gamma(\sigma) \) (not horizontal in the figure). In the case of 4 we have two pairs of bonds incident to the dual vertex with an angle \( \frac{\pi}{3} \) between them, and each of these two pairs belongs to the boundary of a cluster of minus spins. With this rule of separation of contours incident to the same dual vertex, we can distinguish separated connected components in \( \gamma(\sigma) = \bigcup_i \gamma_i \) for \( \sigma \in M_0 \). Each \( \gamma_i \) is a piecewise linear curve, and it can be closed or winding around
the torus. More precisely Peierls contours can be classified as follows:

\[ \Gamma_w = \{ \gamma : \gamma \text{ winds around the torus } \mathbb{H} \} \] (28)

\[ \Gamma_c = \{ \gamma : \gamma \text{ does not wind around the torus } \mathbb{H} \}. \] (29)

For \( \sigma \in M_0 \), if \( \gamma_i \in \Gamma_w \) then it is a piecewise linear curve made of segments of consecutive dual edges with angles \( \frac{2\pi}{3} \). On the other hand if \( \gamma_i \in \Gamma_c \) then it contains a cluster of spins -1 with exactly two internal angles equal to \( \frac{\pi}{3} \) (the higher and the lower vertex in Fig. 2). We call sides of \( \gamma \) the four segments adjacent to these two angles. Moreover the parallelogram circumscribed to the cluster of spin -1 has perimeter \( |\gamma| \). Note also that in this triangular geometry, given a parallelogram with perimeter \( 2n \) the maximal enclosed area is \( \frac{n^2}{2} \) if \( n \) is even, and is \((n + 1)\lfloor \frac{n}{2} \rfloor\) in the odd case (the double of the area contained in the same perimeter in the square lattice).

We have

**Proposition 3.4**

\[ M_0 = M \]

and there are no equivalent configurations in \( M \).

**Proof of Proposition 3.4** We first note that for any \( \sigma \in \mathcal{X}_{\mathbb{H}} \) there is a unique configuration \( T_0(\sigma) \in \mathcal{X}_{\mathbb{H}} \) such that \( \Delta(\sigma, T_0(\sigma)) = 0 \) and for any other \( \sigma' \neq T_0(\sigma) \) we have \( \Delta(\sigma, \sigma') > 0 \). Indeed \( T_0(\sigma) \) corresponds to the evolution at zero temperature of \( \sigma \) and is the unique configuration in which in any site \( x \in V^2 \) the spin is parallel to \( h_x(\sigma^1) \) and in any site \( x \in V^1 \) the spin is parallel to \( h_x(T_0(\sigma)^2) \). For each \( \sigma \in M_0 \) we have that \( T_0(\sigma) = \sigma \) and for each \( \tau \neq \sigma \) we have \( V(\sigma, \tau) > 0 \) so that \( M_0 \subseteq M \). On the other hand if \( \sigma \notin M_0 \) there exists a \( \gamma \in \gamma(\sigma) \) containing bonds corresponding to interaction \( q \) or a spin +1 of \( \sigma \) sharing with \( \gamma \) at least two dual bonds. This means that \( T_0(\sigma) \neq \sigma \) and \( H(\sigma) > H(T_0(\sigma)) \), so that \( \sigma \notin M \), that is \( M \subseteq M_0 \) and this complete the proof.

□

We have the following recurrence property in \( M \):
**Proposition 3.5** There exists a finite $t_0$ (independent of $J$) and a constant $b$ such that for any $t > t_0$

$$
\sup_{\sigma \in X_M} P(\tau^\text{alt}_M > t) < \exp\left\{ -\frac{b}{t_0} \right\}
$$

(30)

where $\tau^\text{alt}_M$ is the first hitting time to $M$ for the process $X^\text{alt}_t$.

The proof can be found in [28] (see proposition 2.2). The idea is very simple: as shown in the proof of Proposition 3.4, for each $\sigma \not\in M$ there is a path $\phi = \phi(\sigma)$ such that $\phi : \sigma \to M$ with $I(\phi) = 0$ of finite length $t(\sigma) \leq t_0$, since at each step the energy strictly decreases. This means that for each $\sigma \not\in M$ we can construct the event

$$
\mathcal{E}(\sigma) = \{ X^\text{alt}_t \text{ follows } \phi(\sigma) \}
$$

realizing the arrival to the set $M$, and such that $P(\mathcal{E}(\sigma)) \asymp 1$ uniformly on $\sigma \not\in M$. The event $\tau^\text{alt}_M > t$ implies

$$
\mathcal{E}(\sigma) \cap \mathcal{E}(X^\text{alt}_{t_0}) \cap \mathcal{E}(X^\text{alt}_{2t_0}) \cap \cdots \cap \mathcal{E}(X^\text{alt}_{nt_0})
$$

with $n = \left\lfloor \frac{t}{t_0} \right\rfloor$ and from this (30) immediately follow. Equation (30) implies in particular that the probability that the process remains outside $M$ for a time $t = e^{\alpha \beta}$ with $\alpha > 0$ arbitrarily small, is super-exponentially small (SES) in $\beta$.

□

Following again [28] we can define a renormalized Markov chain $\tilde{X}_t$ among states in $M$ corresponding to the original chain $X^\text{alt}_t$ looked on the time scale

$$
t_1 = e^{V_1 \beta} \quad \text{where} \quad V_1 = \min_{\sigma \in M, \tau \in X_M} V(\sigma, \tau)
$$

(31)

with the following construction. Let start in a configuration $\sigma \in M$ and consider the sequences of times $(o_k)$ in which the process comes out from a configuration in $M$ and the times $(i_k)$ in which the process comes in $M$ as follows:

$$
\zeta_0 = 0, \quad o_k = \min\{ t > \zeta_{k-1} : X^\text{alt}_t \neq X^\text{alt}_{\zeta_{k-1}} \}, \quad i_k = \min\{ t > o_k : X^\text{alt}_t \in M \}
$$
\[ \zeta_k = \begin{cases} 
\zeta_{k-1} + t_1 & \text{if } o_k - \zeta_{k-1} > t_1 \\
i_k & \text{if } o_k - \zeta_{k-1} \leq t_1 
\end{cases} \]

The times \( \zeta_k \) are stopping time and with this sequence we define on \( M \) the homogeneous Markov chain

\[ \bar{X}_k = X_{\zeta_k}^{alt}. \tag{32} \]

By the general theory in [28] we have the following transition probabilities for the chain \( \bar{X}_k \):

\[ \bar{P}(\sigma, \tau) \asymp e^{-(\bar{\Delta}(\sigma, \tau)-V_1)\beta} \tag{33} \]

with

\[ \bar{\Delta}(\sigma, \tau) = \inf_{t,\phi: \phi_0=\sigma, \phi_t=\tau, \phi_s \not\in M \forall s \in (0,t)} I(\phi) \tag{34} \]

taking values larger or equal to \( V_1 \). This means that the chain \( \bar{X} \) is again in the FW regime and it represents a coarse graining version of the chain \( X^{alt} \) in the sense that at each path \( \phi \) of \( X^{alt} \) we can associate a coarse grained path \( CG(\phi) \) of \( \bar{X} \) by considering the states visited by \( \phi \) at the sequence of times \( \zeta_k \) evaluated on \( \phi \). On the other direction, to each path \( \tilde{\phi} \) of the chain \( \bar{X} \) we can associate a set of paths \( \phi \) of \( X^{alt} \) given by \( \{ \phi : CG(\phi) = \tilde{\phi} \} \) having a probability

\[ \mathbb{P}(X^{alt} \text{ follows a path in } \{ \phi : CG(\phi) = \tilde{\phi} \}) \asymp \mathbb{P}(\bar{X} \text{ follows } \tilde{\phi}). \]

In this way we can study the behavior of the process \( X^{alt} \) by using a simpler chain \( \bar{X}_k \).

Note that for any hitting time \( \tau_A^{alt} \) for the process \( X_t^{alt} \), with \( A \subset M \) and for any \( T \) if we define \( \bar{T} = \frac{T}{t_1} \) and \( \bar{\tau}_A \) the hitting time to \( A \) of the process \( \bar{X}_k \) we have:

\[ \mathbb{P}(\tau_A^{alt} > T e^{\alpha\beta}) < \mathbb{P}(\bar{T} \bar{\tau}_A > \bar{T} e^{\bar{\alpha}\beta}) + SES \tag{35} \]

with \( \bar{\alpha} < \alpha \).

Indeed let \( \nu(T) = \max\{k : \zeta_k < T\} \), we have

\[ \mathbb{P}(\tau_A^{alt} > T e^{\alpha\beta}) = \mathbb{P}(\tau_A^{alt} > T e^{\alpha\beta} \cap \{ \nu(T e^{\alpha\beta}) > \frac{\bar{T}}{2} e^{\alpha\beta} \}) + \mathbb{P}(\tau_A^{alt} > T e^{\alpha\beta} \cap \{ \nu(T e^{\alpha\beta}) \leq \frac{\bar{T}}{2} e^{\alpha\beta} \}) \]

\[ \leq \mathbb{P}(\bar{\tau}_A > \frac{\bar{T}}{2} e^{\alpha\beta}) + \mathbb{P}(\nu(T e^{\alpha\beta}) \leq \frac{\bar{T}}{2} e^{\alpha\beta}) \]
Note that $\zeta_{k+1} - \zeta_k \leq 2t_1$ if, by the recurrence property in $M$ (see (30)), we can exclude in $[0, T]$ the permanence of the process outside $M$ for a time $t_1$:

$$\mathbb{P}(\nu(Te^{\alpha\beta}) \leq \frac{T}{2}e^{\alpha\beta}) \leq \mathbb{P}(\exists t \in [0, T]: X^{alt}_s \not\in M \forall s \in [t, t + t_1]) < Te^{-bt_1} = SES$$

and so (35) holds.

In our case, since $M = M_0$, we can evaluate the different transition probabilities of $\tilde{X}$.

**Proposition 3.6** Suppose $\frac{1}{\varepsilon} \not\in \mathbb{N}$.

1) (minimal cost)

$$V_1 = \min_{\sigma \in M, \tau \neq \sigma} V(\sigma, \tau) = \varepsilon = \min_{\sigma, \tau \in M, \sigma \neq \tau} \Delta(\sigma, \tau)$$

2) (starting move)

If $\sigma_{(1,1)}$ is the configuration with only a couple of minus spins forming a rhombus of side one somewhere in $\Lambda$, we have

$$\tilde{P}(+1, \sigma_{(1,1)}) \asymp e^{-(4-2\varepsilon)\beta}$$

3) (decreasing moves)

If $\sigma_n$ is the configuration with a cluster of minus with minimal side $n$ and $\sigma_n^{-}$ is the configuration obtained by removing a minimal side

$$\tilde{P}(\sigma_n, \sigma_n^{-}) \asymp e^{-2\varepsilon(n-1)\beta}$$

4) (increasing moves)

If $\sigma^+$ is a configuration in $M$ obtained by $\sigma$ adding a row of minus spins

$$\tilde{P}(\sigma, \sigma^+) \asymp e^{-(2-2\varepsilon)\beta}$$

5) (staying moves)
If $\sigma$ has not minimal sides of length 1 then

$$\bar{P}(\sigma, \sigma) \asymp 1$$

$\text{6) (other moves)}$

All the other transition probabilities are asymptotically smaller or equal to $e^{-2\beta}$.

The size $n_c = \left\lceil \frac{1}{\varepsilon} \right\rceil + 1$ turns out to be critical in the sense that clusters with minimal side $n \geq n_c$ has a probability to grow larger than the probability to shrink since $2 - 2\varepsilon < 2\varepsilon(n - 1)$ and vice versa clusters with minimal side $n < n_c$ has a probability to shrink larger than the probability to grow since $2 - 2\varepsilon > 2\varepsilon(n - 1)$.

**Proof of Proposition 3.6**

By (33) and (34) we have to estimate the minimal cost of paths going from a configuration $\sigma \in M_0$ to another configuration $\tau \in M_0$. To leave a configuration $\sigma \in M_0$ one has to flip at least a spin, say in $x$, and call $b^*(x)$ the set of dual bonds around $x$. The cost of this spin flip is

$$4 \pm \varepsilon \quad \text{if} \quad |b^*(x) \cap \gamma(\sigma)| = 0 \quad (36)$$

$$2 + \varepsilon \quad \text{if} \quad |b^*(x) \cap \gamma(\sigma)| = 1, \sigma_x = -1 \quad (37)$$

$$2 - \varepsilon \quad \text{if} \quad |b^*(x) \cap \gamma(\sigma)| = 1, \sigma_x = +1 \quad (38)$$

$$\varepsilon \quad \text{if} \quad |b^*(x) \cap \gamma(\sigma)| = 2, \sigma_x = -1 \quad (39)$$

Points 2), 5) and 6) immediately follow. Point 4) is an immediate consequence of (26) by noting that once the spin-flip from plus to minus occurred in $x$, in the next half-step of the dynamics the neighbor site $x'$, sharing with $x$ a $q$-dual bond, is also flipped to minus without an additional cost. The new row then grows only in one direction at zero cost. In order to decrease a minus cluster one has to start to flip a minus spin in $x$ with $|b^*(x) \cap \gamma(\sigma)| = 2$ in the first half-step and then in the subsequent half-steps is necessary to continue flipping the minus spins in the side, with a cost $\varepsilon$ at each spin flip but the last one. Indeed all the configurations visited in this path are not in $M_0$ with the exception of
the last one in which the complete side of the cluster has been erased. From this we have immediately point 3) and 1).

Define the resistance of a configuration $\sigma$ in $M$ as the non negative real number $r(\sigma)$ such that

$$\max_{\tau \neq \sigma} \tilde{P}(\sigma, \tau) \asymp e^{-r(\sigma)\beta}.$$ 

We have for any $\sigma \in M$:

$$\tilde{P}(\sigma, \sigma) = 1 - \sum_{\tau \neq \sigma} \tilde{P}(\sigma, \tau) \geq 1 - e^{-r(\sigma)\beta + \alpha \beta} \quad \forall \alpha > 0 \quad (40)$$

We can classify configurations in $M$ in the following way

$$M = \{+1\} \cup \{-1\} \cup M_{sup} \cup M_{red}$$

with $M_{sup}$ the set of super-critical configurations without sides or with all sides of length at least $n_c$ and $M_{red}$ the set of reducible configurations in which there exists a side of length less than $n_c$. We have $r(\sigma) = 2 - 2\epsilon$ for all $\sigma \in M_{sup}$ and $r(\sigma) = 2\epsilon(n - 1) < 2 - 2\epsilon$ for all $\sigma \in M_{red}$.

Denote by $\bar{\tau}_A$ the first hitting time to the set $A$ for the process $\bar{X}$.

**Corollary 3.7** For any $\alpha$ sufficiently small there exists $a > 0$ such that

i) for each $\sigma \in M_{red}$

$$\mathbb{P}_\sigma(\bar{\tau}_{\{+1\} \cup M_{sup}} > e^{2\epsilon(n_c-2)\beta + \alpha \beta}) \leq e^{-a \beta}$$

ii) for each $\sigma \in M_{sup}$

$$\mathbb{P}_\sigma(\bar{\tau}_{\{-1\}} > e^{(2-2\epsilon)\beta + \alpha \beta}) \leq e^{-a \beta}$$

iii) for each $\sigma \in M$

$$\mathbb{P}_\sigma(\bar{\tau}_{\{-1, +1\}} > e^{(2-2\epsilon)\beta + \alpha \beta}) \leq e^{-a \beta}$$

**Proof of Corollary 3.7**
i) for each $\sigma \in M_{\text{red}}$

$$\mathbb{P}_\sigma(\bar{\tau}_{\{1\}} \cup M_{\text{sup}} > e^{(n_c-2)\beta+\alpha\beta}) \leq \mathbb{P}(B_1) + \mathbb{P}(B_2)$$

with

$$B_1 = \{ \text{ in } [0, e^{2\varepsilon(n_c-2)\beta+\alpha\beta}] \text{ there are moves of } \bar{X} \text{ different from staying or decreasing} \}$$

$$B_2 = \{ \bar{\tau}_{\{1\}} \cup M_{\text{sup}} > e^{2(n_c-2)\beta+\alpha\beta} \} \cap B_1^c$$

We have

$$\mathbb{P}(B_1) \leq e^{(n_c-2)\beta+\alpha\beta} e^{-(2-2\varepsilon)\beta} < e^{-2(1-\varepsilon(n_c-1))\beta+\alpha\beta} < e^{-a\beta}$$

with $a < 2(1-\varepsilon(n_c-1)) - \alpha$. As far as $B_2$ is concerned, note that by $B_1^c$ we have only staying or decreasing moves. Moreover every decreasing move has a cost smaller or equal to $2\varepsilon(n_c-2)\beta$ so that the number of decreasing moves within $2e^{2\varepsilon(n_c-2)\beta+\alpha\beta}$ is larger than $e^{\alpha\beta/2}$ with large probability. Such a number of decreasing move, when increasing or other moves are forbidden by $B_1^c$, produces with high probability a configuration in $\{+1\} \cup M_{\text{sup}}$.

ii) By using a similar argument, for each $\sigma \in M_{\text{sup}}$

$$\mathbb{P}_\sigma(\bar{\tau}_{\{1\}} \cup M_{\text{sup}} > e^{(2-2\varepsilon)\beta+\alpha\beta}) \leq \mathbb{P}(C_1) + \mathbb{P}(C_2)$$

with

$$C_1 = \{ \text{ in } [0, e^{(2-2\varepsilon)\beta+\alpha\beta}] \text{ there are moves of } \bar{X} \text{ different from staying or increasing} \}$$

$$C_2 = \{ \bar{\tau}_{\{1\}} > e^{(2-\varepsilon)\beta+\alpha\beta} \} \cap C_1^c$$

We have

$$\mathbb{P}(C_1) \leq e^{(2-2\varepsilon)\beta+\alpha\beta} e^{-2\varepsilon(n_c-1)\beta} < e^{-2(\varepsilon(n_c-1)-1+\varepsilon)\beta+\alpha\beta} < e^{-a\beta}$$

with $a < 2(\varepsilon(n_c-1)) - \alpha$. The argument for $C_2$ is similar to that of $B_2$. 

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iii) Combining i) with ii) recurrence to \( \{-1,+1\} \) in a time \( e^{(2-2\epsilon)\beta + \alpha \beta} \) can be easily obtained for \( \bar{X} \).

\[ \square \]

With Proposition \[3.6\] and Corollary \[3.7\] we conclude the proof of the Theorem. We use again the same strategy: it is sufficient to prove

\[ \mathbb{P}^+_{+1}(\bar{\tau} - 1 > \bar{T} e^{\bar{\alpha} \beta}) < e^{-a \beta} \]

with \( \bar{T} = e^{E_{\bar{c}}^{+\beta} - \epsilon \beta} = e^{(4\epsilon_n - 2\epsilon_n(n_c - 1) - 2\epsilon) \beta} \). To this purpose we construct an event \( \mathcal{E} \) such that, \( \mathcal{E} \) implies that starting from \(+1\) the process \( \bar{X} \) reaches \(-1\) within a time \( e^{(2-2\epsilon)\beta + \alpha \beta/4} \), for a fixed positive \( \alpha < \bar{\alpha} \) and with

\[ \mathbb{P}^+_{+1}(\mathcal{E}) > e^{-\left( E_{\bar{c}}^{+\beta} - (2-\epsilon)\beta + \alpha \beta/2 \right)} \]

With such an event we can conclude the proof of the Theorem as follows. We divide the time interval \( \bar{T} e^{\bar{\alpha} \beta} \) into subintervals of length \( 2e^{(2-2\epsilon)\beta + \alpha \beta/4} \). The event \( \{\bar{\tau} - 1 > \bar{T} e^{\bar{\alpha} \beta}\} \) implies that in each of these subintervals is not realized the following event:

\[ \bar{\tau}_{\{-1,+1\}} \leq e^{(2-2\epsilon)\beta + \alpha \beta/4} \quad \text{and if} \quad \bar{X}_{\bar{\tau}_{\{-1,+1\}}} = +1 \quad \text{then} \quad \mathcal{E} \quad \text{is verified.} \]

By Corollary \[3.7\] and by \[42\] this event has a probability larger than \( (1 - e^{-a \beta}) e^{-\left( E_{\bar{c}}^{+\beta} - (2-\epsilon)\beta + \alpha \beta/2 \right)} \), so noting that for \( 0 < a < b \) we have

\[ (1 - e^{-a \beta}) e^{b \beta} \leq \exp\{-e^{(b-a)\beta}\} \]

we immediately obtain

\[ \mathbb{P}^+_{+1}(\bar{\tau} - 1 > \bar{T} e^{\bar{\alpha} \beta}) \leq \left( 1 - \frac{1}{2} e^{-\left( E_{\bar{c}}^{+\beta} - (2-\epsilon)\beta + \alpha \beta/2 \right)} \right) \left[ \frac{e^{\bar{\alpha} \beta}}{2e^{(2-\epsilon)\beta + \alpha \beta/4}} \right] \leq \exp\left\{ -\frac{1}{4} e^{\left( \bar{\alpha} - \alpha/4 \right) \beta} \right\}. \]

To construct the event \( \mathcal{E} \) we proceed as follows. Let \( \sigma_{(n,m)} \) be a configuration with a parallelogram of minus spins of sides \( n, m \) in a sea of plus spins and \( \bar{\phi} \) be the following
sequence of configurations in $M$ with an increasing cluster of minus spins which is a rhombus or a quasi-rhombus:

$$\tilde{\phi}_0 = +1, \tilde{\phi}_1 = \sigma_{(1,1)}, \tilde{\phi}_2 = \sigma_{(1,2)}, \tilde{\phi}_3 = \sigma_{(2,2)}, \ldots$$

$$\ldots, \tilde{\phi}_{2n-1} = \sigma_{(n,n)}, \tilde{\phi}_{2n} = \sigma_{(n,n+1)}, \ldots, \tilde{\phi}_{2n_c-2} = \sigma_{(n_c-1,n_c)}, \tilde{\phi}_{2n_c-1} = \sigma_{(n_c,n_c)}$$

The event $\mathcal{E}$ is constructed on the chain $\tilde{X}$ starting at $+1$ as follows:

- the first not staying move is the transition to $\sigma_{(1,1)}$ occurred within a time $e^{2(2-2\varepsilon)\beta}$;

- for each $i \in \{1, 2n_c - 2\}$ the chain $\tilde{X}$ leaves the state $\tilde{\phi}_i$ within a time $e^{r(\tilde{\phi}_i)\beta + \alpha'\beta}$, with $\alpha' < \frac{\alpha}{4n_c}$, reaching the configuration $\tilde{\phi}_{i+1}$;

- when reached the supercritical configuration $\sigma_{(n_c,n_c)}$ the chain $\tilde{X}$ reaches $-1$ in a time smaller than $e^{2(2-2\varepsilon)\beta + \alpha\beta/4}$.

We have

$$\mathbb{P}(\mathcal{E}) = \mathbb{P}(+1, \sigma_{(1,1)}) e^{2(2-2\varepsilon)\beta} \times$$

$$\prod_{i=1}^{2n_c-2} \left( \sum_{t=0}^{e^{r(\tilde{\phi}_i)\beta + \alpha\beta}} \tilde{P}(\tilde{\phi}_i, \tilde{\phi}_t) \tilde{P}(\tilde{\phi}_t, \tilde{\phi}_{t+1}) \right) \mathbb{P}(\sigma_{(n_c,n_c)})(\tau_t - 1) \leq e^{2(2-2\varepsilon)\beta + \alpha\beta/4}.$$ 

By (40) and again by using (43) we get

$$e^{r(\tilde{\phi}_i)\beta + \alpha'\beta} \sum_{t=0}^{e^{r(\tilde{\phi}_i)\beta + \alpha'\beta}} \left[ 1 - e^{-r(\tilde{\phi}_i)\beta + \alpha'\beta/2} \right] t \geq e^{r(\tilde{\phi}_i)\beta - \alpha'\beta/2} (1 - e^{-\alpha'\beta/2}) \geq e^{r(\tilde{\phi}_i)\beta - \alpha'\beta}$$

and so

$$\mathbb{P}(\mathcal{E}) \geq e^{-(4-2\varepsilon)\beta} e^{2(2-2\varepsilon)\beta} (1 - e^{-\alpha\beta}) \prod_{n=1}^{n_c-1} e^{-2(2-2\varepsilon)\beta} e^{4\varepsilon(n-1)\beta - 2\alpha'\beta} =$$

$$e^{-(4-2\varepsilon)\beta} e^{2(2-2\varepsilon)\beta} (1 - e^{-\alpha\beta}) \exp \left\{ \sum_{n=1}^{n_c-1} \left( -2(2-2\varepsilon)\beta + 4\varepsilon(n-1)\beta - 2\alpha'\beta \right) \right\} =$$

$$e^{-(4-2\varepsilon)\beta} e^{2(2-2\varepsilon)\beta} (1 - e^{-\alpha\beta}) e^{-(E_2^{\text{sh}} + \varepsilon)\beta - 2\alpha'(n_c-1)\beta} \geq e^{-(E_2^{\text{sh}} + \varepsilon - 2)\beta - \alpha\beta/2}.$$ 

□
4 Discussion of the results and open problems

4.1 Geological interpretation

In this section we discuss the geological interpretation of the shaken dynamics. We start from some phenomenological aspects that are well known in the geological literature. Tectonic plates are separated by faults, and such separations are the main sites of the geological active events: in particular, earthquakes far from faults are very rare. When an earthquake starts, it is typically followed by a seismic swarm, whose duration may vary from some days to some months.

The whole dataset of the beginning of earthquakes seems not to be correlated with the tidal phase. However, a correlation appears if earthquakes are divided into two types: the earthquakes due to the separation of the two plates (normal earthquakes) and the ones due to the approaching of the plates, with consequent subduction and orogeny (thrust earthquakes). For a more detailed discussion see [25], [8].

In what follows we determine suitable values for the parameters of our shaken dynamics in order to mimic this qualitative phenomenon. To this purpose the following facts have to be taken into account. First of all, the system has a natural discrete clock given by the alternation of the two opposite directions of the stress between litosphere and mantle: the physical time related to this clock is the time between a high tide and the subsequent low tide, namely $6^h 13'$. Hence, measuring time in terms of steps of the dynamics, we observe that, at a given site $x$, both the time interval between two earthquakes and the duration of the swarms are very long, say $10^5 \div 10^6$ and $20 \div 400$ time units, respectively. This suggests that the inertial term $\rho$ has to be larger than $\lambda$ and $J$. The parameter $\lambda$ tunes the difference of inertia in the case of intact constraints (higher inertia) and broken constraints (lower inertia) and it is still quantitatively quite significant. Finally, the contribution of the (alternate) interaction with the neighbors has to be much smaller, but anyway it can not be neglected, since almost no earthquakes occur far from the faults, i.e., in a neighborhood of intact constraints.

Recalling that $\sigma_x = +1$ represents a broken constraint, we model a fault with a set of fixed spins with plus sign.
Keeping in mind these observations we proceed as follows.

In order to describe earthquakes near a fault, we want the probability

$$P(\tau_x = +1 | x \in \Lambda_{++}) = \frac{e^{2J-q-\lambda}}{2\cosh(-2J + q + \lambda)}$$

(44)

to be of the order of $10^{-5} \div 10^{-6}$.

Moreover, both probabilities

$$P(\tau_x = +1 | x \in \Lambda_{--}) = \frac{e^{-2J-q-\lambda}}{2\cosh(2J + q + \lambda)}$$

$$P(\tau_x = +1 | x \in \Lambda_{+-}) = \frac{e^{-q-\lambda}}{2\cosh(q + \lambda)}$$

must be very small, since they represent the probability of an earthquake in a site with two or one intact neighbors.

To describe the typical duration of a swarm we ask the probability

$$P(\tau_x = +1 | x \in \Lambda_{-+}) = \frac{e^{-2J+q-\lambda}}{2\cosh(-2J + q - \lambda)}$$

(45)

to be of the order of $1 - 10^{-2}$. Whereas the other two probabilities

$$P(\tau_x = +1 | x \in \Lambda_{++}) = \frac{e^{2J+q-\lambda}}{2\cosh(2J + q - \lambda)}$$

$$P(\tau_x = +1 | x \in \Lambda_{+-}) = \frac{e^{q-\lambda}}{2\cosh(q - \lambda)}$$

should be closer to 1, saying that the leading term that determines the end of a swarm, i.e. the flip of $\tau_x$ to the value $-1$, is $P(\tau_x = +1 | x \in \Lambda_{-+})$.

A suitable choice for the parameters $q$, $\lambda$ and $J$ coherent with the previous requirements is $q = 5$, $\lambda = 2$, $J = 0, 5$.

We performed some numerical simulations of the evolution of the shaken dynamics using the previous values for the parameters. In this setting:

- the torus $(\Lambda, \mathcal{B}_\Lambda)$ is interpreted as a finite region of Earth’s surface;
- the fault is modelled by setting $\sigma_x = +1$ on the diagonal of $(\Lambda, \mathcal{B}_\Lambda)$. 

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Each simulation was run for $10^6$ steps corresponding to a time frame of about 1500 years. The number of earthquakes (i.e. spin flips) at each site and the duration of the swarms (i.e. permanence of a plus sign between flips) have been tracked. The results obtained in a typical realization of a simulation are presented in Fig. 3.

![Image](a) ![Image](b)

Figure 3: Image (a) shows the number of earthquakes (spin flips) at each site. Brighter colours correspond to higher number of events. The histogram of image(b) summarizes the duration of the swarms.

Note that similar values for the leading probabilities (44), (45) could be obtained considering a more conventional dynamics where the spin at each site $x$ interacts with all its neighbors, by setting $q_a = q_s - 2J_s$ where $q_s$ and $J_s$ are the parameters of the shaken dynamics and $q_a$ is the corresponding inertial term of the dynamics which “looks in all directions”. In other words the number of events of a shaken dynamics is the same of the number of events observed in a “all directions” dynamics with a lower value of $q$. This fact can be interpreted as a weakened effective friction between litosphere and mantle due to the presence of tidal forces.

### 4.2 Other remarks

Let us conclude our paper with some other general comments and open problems.

The pair Hamiltonian and the idea used in the definition of the shaken dynamics, i.e., to consider only half of the interactions (down-left), allow to interpolate between different lattices. For example, as a consequence of theorem 2.2, we can define a suitable shaken dynamics on the triangular lattice and obtain a corresponding alternate dynamics on the
square lattice with equilibrium measure the usual Gibbs measure. Note, however, that the idea of alternate dynamics on even and odd sites is already present in the literature (see [5]).

As a second remark we want to note that, in the low temperature regime, the parameter \( q \) tunes the shape of the critical droplet. Actually a further gain in the time to reach the stable state \(-1\) could be obtained by choosing the parameter \( q = J = \frac{\beta}{2} \). In this case the critical configuration is an hexagon and the critical size can be derived by maximizing the energy of hexagons of side \( l \)

\[
E^H(l) = 6l - 6\varepsilon l^2
\]

obtaining \( l^H_c = \frac{1}{2\varepsilon} \) and so an expected critical energy

\[
E^H_c(l) \sim 6l^H_c - 6\varepsilon(l^H)^2 = \frac{3}{2\varepsilon} = \frac{3}{8}E^G_{\varepsilon}.
\]

However we have to note that the extension of Theorem 2.3 to this case is more difficult and stronger estimates in the cluster expansion argument would be necessary.

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**References**

[1] D. F. Bayley, *Counting Arrangements of 1’s and -1’s*, Mathematical Magazine, 69, 128, 131 (1996).

[2] J. Beltrán, C. Landim, *A martingale approach to metastability*, Probab. Theory Related Fields, 161, 267-307 (2015).
[3] A. BOVIER, F. DEN HOLLANDER, Metastability - a Potential-Theoretic Approach
Grundlehren der mathematischen Wissenschaften 351, Springer, Berlin (2015).

[4] T. CHOU, K. MALICK, AND R. K. P. ZI, Non-Equilibrium Statistical Mechanics:
From a Paradigmatic Model to Biological Transport, Rep. Prog. Phys., 74, 116601
(2011).

[5] E.N.M. CIRILLO A Note on the Metastability of the Ising Model: The Alternate
Updating Case, J. Stat. Phys., 106, 385-390 (2002).

[6] E.N.M. CIRILLO, F.R. NARDI, Metastability for stochastic dynamics with a parallel
heat bath updating rule, J. Stat. Phys., 110, 183-217 (2003).

[7] E.N.M. CIRILLO, F.R. NARDI, C. SPITONI, Metastability for reversible probabilistic
cellular automata with self-interaction, J. Stat. Phys., 132, 431-471 (2008).

[8] E.S. COCHRAN, J.E. VIDALE, S. TANAKA, Earth tides can trigger shallow thrust
fault earthquakes, Science, 306, 1164-1166 (2004).

[9] P. DAI PRA, B. SCOPPOLA, E. SCOPPOLA, Sampling from a Gibbs measure with pair
interaction by means of PCA, J. Statist. Phys., 149, 722-737 (2012).

[10] P. DAI PRA, B. SCOPPOLA, E. SCOPPOLA, Fast mixing for the low-temperature 2D
Ising model through irreversible parallel dynamics J. Statist. Phys., 159, 1-20 (2015).

[11] D.A. DAWSON Synchronous and asynchronous reversible Markov systems Can. Math.
Bull., 17(5), 633649 (1974/75).

[12] R. FERNÁNDEZ, A. TOOM, Non Gibbsiannes of the invariant measures of non-
reversible cellular automata with totally asymmetric noise, Astrisque, 287, 71-87
(2003).

[13] M.I. FREIDLIN, A.D.WENTZELL, Random Perturbation of Dynamical Systems,
Springer-Verlag, Berlin (1984).

[14] L. DE CARLO, D.GABRIELLI, Gibbsian stationary non equilibrium states,
arXiv:1703.02418v1.

[15] G. GALLAVOTTI, Nonequilibrium and irreversibility, Springer-Verlag, Heidelberg
(2014).
[16] A. Gaudilli`ere, C. Landim, *A Dirichlet principle for non reversible Markov chains and some recurrence theorems*, Probab. Theory Related Fields, **158**, 55-89 (2013).

[17] S. Goldstein, R. Kuik, J.L. Lebowitz, C. Maes, *From PCA’s to equilibrium systems and back*, Commun. Math. Phys., **125**, 71-79 (1989).

[18] A. Kolmogorov, *Zur Theorie der Markoffschen Ketten*, Math. Ann., **112**, 155-160 (1936).

[19] O. Kozlov, N. Vasilyev, *Reversible Markov chains with local interaction, multicomponent random systems* Adv. Probab. Related Topics, **6**, 451-469 (1980).

[20] C. Lancia, B. Scoppola, *Equilibrium and non-equilibrium Ising models by means of PCA*, J. Stat. Phys., **153**, 641-653 (2013).

[21] S. A. Ng, *Some identities and formulas involving generalized Catalan numbers*, arXiv:math/0609596v1 (2006).

[22] E. Olivieri, M.E. Vares, *Large Deviations and Metastability*, Cambridge University Press, Cambridge (2005).

[23] A. Procacci, B. Scoppola, E. Scoppola, *Probabilistic Cellular Automata for the low-temperature 2d Ising Model*, J. Statist. Phys., **165**, 991-1005 (2016).

[24] A. Procacci, B. Scoppola, E. Scoppola, *Effects of boundary conditions on irreversible dynamics*, arXiv:1703.04511v1

[25] F. Riguzzi, G. Panza, P. Varga, C. Doglioni *Can Earth’s rotation and tidal despinning drive plate tectonic?* Tectonophysics, **484**, 60-73 (2010).

[26] H. Robbins, *A Remark on Stirling’s Formula*, Amer. Math. Monthly, **62**, 26-29 (1955).

[27] B. Scoppola, D. Boccaletti, M. Bevis, E. Carminati, C. Doglioni *The westward drift of the litosphere: a rotational drag?* Bull. Geol. Soc. Am., **118**, 199-209 (2006).

[28] E. Scoppola, *Renormalization Group for Markov Chains and Application to Metastability*, J. Statist. Phys., **73**, 83-121 (1993).
[29] E. W. Weisstein, *Catalan's Triangle* MathWorld - A Wolfram Web Resource. Retrieved March 28, (2012).