COMPACT QUANTUM GROUP $C^*$-ALGEBRAS AS
HOPF ALGEBRAS WITH APPROXIMATE UNIT

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Abstract. In this paper, we construct and study the representation theory of a Hopf $C^*$-
algebra with approximate unit, which constitutes quantum analogue of a compact group $C^*$-algebra. The construction is done by first introducing a convolution-product on an arbitrary Hopf algebra $H$ with integral, and then constructing the $L_2$ and $C^*$-envelopes of $H$ (with the new convolution-product) when $H$ is a compact Hopf $*$-algebra.

1. Introduction

Compact quantum groups were introduced by Woronowicz [10] and studied by several other authors [1, 3, 9], as non-commutative analogues of the function algebras on compact groups. The aim of this work is to construct some quantum analogues of the compact group $C^*$-algebras, which turn out to be non-cocommutative Hopf $C^*$-algebras with approximate unit.

More precisely, starting from a compact Hopf $*$-algebra, we construct a Hopf $C^*$-algebra with approximate unit in such a way that when the original compact Hopf $*$-algebra is the function algebra on certain compact group $G$, the resulting Hopf $C^*$-algebra reduces to the group $C^*$-algebra $C^*(G)$.

Recall that for a compact group $G$, the group $C^*$-algebra $C^*(G)$ is the completion by operator norm of the algebra $L^1(G)$ of absolutely integrable complex-valued functions on $G$. The product on $L^1(G)$ is the convolution-product, given by

$$(g * f)(x) = \int_G f(y)g(y^{-1}x)dy.$$  

With respect to this product, $C^*(G)$ is a non-unital $C^*$-algebra. In that case, there is an important notion of $\delta$-type sequences in $C^*(G)$, that approximate the unity. Such a sequence is called an approximate unit in $C^*(G)$. Moreover, the action of the group $G$ on any representation can be recovered from the corresponding action of the algebra $C^*(G)$, by using $\delta$-type sequences. In our case, however, we do not know what the quantum group defining our $C^*$-algebra is, hence our defining a Hopf algebra structure to exhibit the “group property” of our algebra.

Our construction is done in several steps. First we construct a convolution product on an arbitrary Hopf algebras with integrals, being motivated by the classical convolution product for $L_1(G)$, the completion of which (under the operator norm) defines $C^*(G)$ (see [2]). Denoting by $\tilde{H}$ the vector space $H$ with the new convolution product, we show that $\tilde{H}$ is a non-unital algebra that is an ideal of $H^*$ (the dual of $H$); that the category of completely reducible $H$-comodules is equivalent to the category of completely reducible $\tilde{H}$-comodules (2.4, 2.5) and that when $H$ is co-semi-simple then $\tilde{H}$ is isomorphic to a direct sum of full endomorphism rings of simple $H$-comodules (2.6).

In section 4, we now focus on the case when $H$ is a compact Hopf $*$-algebra. Through the works of Woronowicz, Koorwinder and Djikhuizen, [10, 2], compact Hopf $*$-algebra are
known to play a role analogous to that of the algebra of functions on compact groups in the classical theory. Since by definition, $H$ has a complex scalar product, we complete it to a Hilbert space $H_{L^2}$, which is the quantum analogue of the algebra $L^2(G)$. We show in 3.2 that the convolution product on $\hat{H}$ extends to $H_{L^2}$ and that $H_{L^2}$ possesses a (topological) coproduct and an approximate antipode and hence is a Hopf algebra with approximate unit [3.4]. Since the antipode on $H$ is not involutive, the $*$-structure defined on $H$ by $h^* := S(h^*)$ cannot be extended to $H_{L^2}$ because it is not continuous with respect to the given norm. To overcome this problem, we pass to the $C^*$-enveloping of $H_{L^2}$ to obtain $H_{C^*}$ that is the required Hopf $C^*$-algebra with approximate unit (see 4.5). We show that there is a one-one correspondence between the irreducible unitary representations of $H_{C^*}$ and those of $\hat{H}$ (see 4.4).

2. THE CONVOLUTION PRODUCT AND CO-SEMI-SIMPLICITY

We work over an algebraically closed field $K$ of characteristic zero. Let $(H, m, \eta)$ be an algebra over a field $K$, where $m$ denotes the product, $\eta$ denotes the map $K \rightarrow H, 1_K \rightarrow 1_H$. A bialgebra structure on $H$ is a pair of linear maps $\Delta : H \rightarrow H \otimes H$, $\varepsilon : H \rightarrow K$, satisfying

- $\Delta$ and $\varepsilon$ are homomorphism of algebras.
- $(\varepsilon \otimes \text{id}_H)\Delta = (\text{id}_H \otimes \varepsilon)\Delta = \text{id}_H$.
- $(\Delta \otimes \text{id}_H)\Delta = (\text{id}_H \otimes \Delta)\Delta$.

An antipode on $H$ is a linear map $H \rightarrow H$, satisfying

- $m \circ (S \otimes \text{id}_H)\Delta = m \circ (\text{id}_H \otimes S)\Delta = \eta \varepsilon$.

A bialgebra equipped with an antipode is called Hopf algebra. The antipode is then uniquely determined.

Let $(H, m, \eta, \Delta, \varepsilon)$ be a Hopf algebra. A right coaction of $H$ on a vector space $V$ is a linear map $\delta : V \rightarrow V \otimes H$, satisfying

1. $(\text{id}_V \otimes \varepsilon)\delta = \text{id}_V$,
2. $(\delta \otimes \text{id}_H)\delta = (\text{id}_V \otimes \Delta)\delta$,

in the first identity we identify $V$ with $V \otimes K$. $V$ is then called a right $H$-co-module.

We shall frequently use Sweedler’s notation, in particular $\Delta(x) := \sum_{(x)} x_{(1)} \otimes x_{(2)} = x_1 \otimes x_2$, $\delta(v) = \sum_{(v)} v_{(0)} \otimes v_{(1)} = v_0 \otimes v_1$.

The elements $v_1$’s in the presentation $\delta(v) = v_0 \otimes v_1$ are called coefficients of the coaction $\delta$, and the space they span is called coefficient space. This space is a subcoalgebra of $H$. To see this, fix a basis $x_1, x_2, \ldots, x_d$. Then, the coaction is given by $\delta(x_i) = x_j \otimes a_i^j$ and the $\{a_i^j\}$ span the coefficient space. On the other hand, from (1) it follows $\Delta(a_i^j) = a_k^j \otimes a_i^k$, $\varepsilon(a_i^k) = \delta_i^k$. The comodule $V$ is simple iff $\{a_i^j\}$ is a basis for the coefficient space, see [4].

A left (right) integral on a Hopf algebra is a (non-trivial) linear functional $\int : H \rightarrow K$, which is a left (right) $H$-comodule homomorphism, where $H$ is a left (right) $H$-comodule by means of the coproduct $\Delta$ and $K$ is a left (right) $H$-comodule by means of the unity map $\eta$. Explicitly, a left (resp. right) integral $\int_l (\text{resp.} \int_r)$ on $H$ satisfies

1. $\int_l (x) = x_1 \int_l (x_2)$ (resp. $\int_r (x) = \int_r (x_1) \cdot x_2$).

It was shown by Sullivan [4] that the integral on a Hopf algebra, if it exists, is defined uniquely up to a constant. Further, we have
Lemma 2.1. Let $\int$ be a left integral on $H$. Then the bilinear form $b(g, h) = \int (gS(h))$ is non-degenerate on $H$, that is

$$\int (gS(h)) = 0, \forall h \implies g = 0.$$  \hspace{1cm} (3)

$$\int (gS(h)) = 0, \forall g \implies h = 0.$$  \hspace{1cm} (4)

In the rest of this work, we fix a left integral $\int$ on $H$.

We mention an identity, due originally to Sweedler [7], which plays a crucial role in our computations:

$$\int (gS(h_1)) \cdot h_2 = g_1 \int (g_2S(h)),$$  \hspace{1cm} (5)

if $\Delta(g) = g_1 \otimes g_2$ and $\Delta(h) = h_1 \otimes h_2$ in the notations of Sweedler.

The coalgebra structure on $H$ induces an algebra structure on its dual $H^*$ – the space of linear forms on $H$. The product of $\phi, \psi$ in $H^*$ is given by

$$\phi \ast \psi(h) = \phi(h_1)\psi(h_2), \forall h \in H.$$  \hspace{1cm} (6)

The unit element in $H^*$ is the counit $\varepsilon$ of $H$.

**Convolution product on $H$.** We first recall the classical structure. Let $G$ be a compact group. Then there exists a unique normalized Haar measure on $G$ which induces the Haar integral on $L^1(G)$. The $\ast$-product is defined on $L^1(G)$ as follows:

$$(g \ast f)(x) := \int_G f(y)g(g^{-1}x)dy.$$  \hspace{1cm} (5)

Now, let $H$ be a Hopf algebra with an integral. Being motivated by (5) we define the convolution product on $H$ by:

$$g \ast f := \int (fS(g_1)) \cdot g_2.$$  \hspace{1cm} (6)

According to (6), we also have $g \ast f = f_1 \int (f_2S(g))$.

**Lemma 2.2.** $H$ equipped with the $\ast$-product defined above is a (non-unital) algebra.
Proof. We only have to check the associativity:

\[(h * g) * f = \int (fS((h * g)_1)) \cdot (h * g)_2\]
\[= \int \left( fS \left( \int (gS(h_1) \cdot h_2) \right) \right) \cdot h_3\]
\[= \int \left( f \int (gS(h_1)) \cdot S(h_2) \right) \cdot h_3 \text{(using (4))}\]
\[= \int \left( fS(g_1) \int (g_2S(h_1)) \right) \cdot h_2\]
\[= \int \left( fS(g_1) \cdot \int (g_2S(h_1)) \right) \cdot h_2\]
\[= \int \left( \int (fS(g_1)) \cdot g_2S(h_1) \right) \cdot h_2\]
\[= \int ((g * f)S(h_1)) \cdot h_2\]
\[= h * (g * f).\]

Lemma 2.2 is proved. ■

We denote by $\tilde{H}$ the vector space $H$, equipped with the convolution product $\ast$.

**$H$-comodules and $\tilde{H}$-modules.** We now study the correspondence between $H$-comodules and $\tilde{H}$-modules. Let $V$ be an $H$-comodule. We define an action of $\tilde{H}$ on $V$ as follows:

\[h \ast v := v_0 \int (v_1S(h))\]  

We check the associativity:

\[g \ast (h \ast v) = v_0 \int (v_1S(g)) \cdot \int (v_2S(h))\]
\[\text{(using (4))} = v_0 \int \left( v_1S(h_1) \int (h_2S(g)) \right)\]
\[= v_0 \int (v_1S(g \ast h))\]
\[= (g \ast h) \ast v.\]

Let $\phi : V \rightarrow W$ be a morphism of $H$-comodules, i.e., $\phi(v)_0 \otimes \phi(v)_1 = \phi(v_0) \otimes v_1$. Then, for $h \in H, v \in v$,

\[\phi(h \ast v) = \phi(v_0) \int (v_1S(h))\]
\[= \phi(v_0) \int (v_1S(h))\]
\[= \phi(v)_0 \int (\phi(v)_1S(h))\]
\[= h \ast \phi(v).\]

Thus, $\phi$ is a morphism of $\tilde{H}$ modules. We therefore have a functor $F$ from the category of $H$-comodules into the category of $\tilde{H}$-modules, which is the identity functor on the underlying category of vector spaces.

**Proposition 2.3.** The functor $F$ defined above is full, faithful and exact.
Proof. From the definition of $F$, we see that, as vector spaces, $F(V) = V$ and $F(\phi) = \phi$. Thus, the functor $F$ is faithful and exact. It remains to show that $F$ is full, which amounts to showing that if $\phi : V \to W$ is a morphism of $\hat{H}$-module then it is a morphism of $H$-comodule. By assumption, $\phi$ satisfies

$$h \ast (\phi(v)) = \phi(h \ast v), \quad \forall h \in \hat{H}$$

or, equivalently,

$$\phi(v)_0 \int (\phi(v_1)S(h)) = \phi(v_0) \int ((v_1S(h))), \quad \forall h \in \hat{H}.$$ 

Since $\int$ is faithful (see (3)), we conclude that

$$\phi(v_0) \otimes \phi(v_1) = \phi(v_0) \otimes v_1.$$ 

In other words, $\phi$ is a homomorphism of $H$-comodules.

Let $V$ now be a cyclic $\hat{H}$-module, i.e., there exists an element $\bar{v} \in V$, such that all $v \in V$ is obtained from $\bar{v}$ by the action of some $f \in H$. We then define the coaction of $H$ on $V$ by

$$(8) \quad \delta(v) := f_1 \ast \bar{v} \otimes S(f_2),$$

where $f$ is such that $f \ast \bar{v} = v$.

First, we have to show that this coaction is well defined, which means that it does not depend on the choice of the representative elements $\bar{v}$ and $f$. To show that the definition does not depend on $\bar{v}$ we let $\bar{v} = g \ast \tilde{v}$ and show that the definition does not change when $\bar{v}$ is replaced by $\tilde{v}$, which means

$$f_1 \ast \bar{v} \otimes S(f_2) = (f \ast g)_1 \ast \tilde{v} \otimes S((f \ast g)_2).$$

Replacing $\bar{v}$ on the left-hand side of this equation by $g \ast \tilde{v}$ and canceling $\tilde{v}$ in both sides, one is led to the following equation

$$f_1 \ast g \otimes f_2 = (f \ast g)_1 \otimes (f \ast g)_2,$$

which follows immediately from the definition.

To show the independence on the choice of $f$ we assume $f \ast \bar{v} = 0$ and show that $f_1 \ast \bar{v} \otimes S(f_2) = 0$. Indeed, we have

$$\int (f_2S(g)) \cdot f_1 \ast \bar{v} = (g \ast f) \ast \bar{v} = g \ast (f \ast v) = 0,$$

for all $g$. Hence, according to Lemma 2.1 $f_1 \ast \bar{v} \otimes f_2 = 0$.

We now proceed to check the co-associativity and co-unitary:

$$(\text{id}_H \otimes \delta)\delta(v) = \delta(f_1 \ast \bar{v} \otimes S(f_2)) = f_1 \ast \bar{v} \otimes f_2 \otimes f_3 = f_1 \ast \bar{v} \otimes \Delta(f_2) = (\text{id} \otimes \Delta_H)\delta(v).$$

$$(\text{id}_V \otimes \varepsilon)\delta(f(v)) = f_1 \ast \bar{v} \otimes \varepsilon(f_2) = f \ast \bar{v} = v.$$ 

Since simple modules are cyclic, we obtain a functor $G$ from the category of completely reducible $\hat{H}$-modules to the category of $H$-comodules.

**Theorem 2.4.** The category of completely reducible $H$-comodules is equivalent to the category of completely reducible $\hat{H}$-modules.
Proof. We show that the functors $\mathcal{F} \circ \mathcal{G}$ and $\mathcal{G} \circ \mathcal{F}$ are identity-functors on the categories of completely reducible $H$-comodules and $\hat{H}$-comodules, respectively.

Let $V$ be a simple $H$-comodule. Then $\hat{V} = F(V)$ is a simple $\hat{H}$-module. Let us fix $\bar{v} \in \hat{V}$ and for $v \in \hat{V}$, let $f \in \hat{H}$ be such that $f \ast \bar{v} = v$. By definition, we have

$$v = f \ast \bar{v} = \bar{v}_0 \int (\bar{v}_1 S(f)).$$

The coaction of $H$ on $\mathcal{G} \circ \mathcal{F}(V)$ is

$$\delta_{\mathcal{G} \circ \mathcal{F}(V)}(v) = f_1 \ast \bar{v} \otimes f_2$$

$$= \bar{v}_0 \int (\bar{v}_1 S(f_1)) \otimes f_2$$

$$= \bar{v}_0 \otimes \bar{v}_1 \int (v_2 S(f))$$

$$= \delta_V(v).$$

Thus, $\mathcal{G} \circ \mathcal{F}$ is the identity functor. The assertion for $\mathcal{F} \circ \mathcal{G}$ is proved analogously.

$\hat{H}$ is an ideal of $H^*$. In the previous section we have seen that there exists a correspondence between $H$-comodules and $\hat{H}$-modules. On the other hand, there exits a one-to-one correspondence between $H$-comodules and rational $H^*$-modules. It is then natural to ask about the relationship between $H^*$ and $\hat{H}$. We now show that $\hat{H}$ is isomorphic to the left ideal generated by the integrals in $H^*$.

It is well-known, that the rational submodule $H^\square$ of $H^*$, considered as a left module on itself, i.e. the sum of all left ideals of $H^*$, which are finite dimensional (over $\mathbb{K}$), is an $H$-Hopf module (with an appropriate $H$-action) and hence isomorphic to the tensor product of $H$ with the space spanned by the integrals [8, Thm 5.1.3]. Since the space of integrals is one-dimensional, we have an isomorphism between the two vector spaces $H$ and $H^\square$. This isomorphism can be given explicitly as follows.

By means of the integral, every element of $H$ can by considered as a linear functional on $H$ itself: $H \ni h \mapsto \int^h \in H^* : \int^h (g) := \int (g S(h))$.

**Proposition 2.5.** The map $H \ni h \mapsto \int^h \in H^*$ defined above is an isomorphism of algebras $\hat{H} \rightarrow H^\square \subset H^*$.

**Proof.** We have

$$\int^{f \ast h} (g) = \left( g S(h_1) \int \left( h_2 S(f) \right) \right)$$

$$= \int \left( g S(h_1) \right) \int \left( h_2 S(f) \right)$$

$$= \int \left( g S(f) \right) \int \left( g_2 S(h) \right)$$

$$= (\int^f \ast \int^h)(g).$$

In the strictly algebraic sense, $\hat{H}$ is not a Hopf algebra, unless $H$ is finite-dimensional. In the next section we will show that for compact Hopf $*$-algebra, there exists a natural topology in $H$ such that the completion of $H$ with respect to this topology is a topological Hopf algebra with approximate unit.
Co-semi-simple Hopf algebras. A Hopf algebra $H$ is called co-semisimple if any finite dimensional $H$-comodule decomposes into a direct sum of simple comodules. A Hopf algebra is co-semisimple if and only if it possesses an integral whose value at its unit element is nonzero. In this case, left and right integrals are equal \[8\].

Let $H$ be a co-semi-simple Hopf algebra. Then it decomposes into a direct sum of simple sub-coalgebras, each of which is the coefficient space of a simple $H$-comodule \[8\]

\[ H \cong \bigoplus_{\lambda \in \Lambda} H_\lambda. \]

The set $\Lambda$ contains an element $0$ for which $H_0 \cong \mathbb{K}$. The integral computed on $H_\lambda$ is zero for $\lambda \neq 0$.

We consider the discrete topology on $\Lambda$. Let $C_0(\Lambda)$ denote the set of all compact subsets in $\Lambda$ containing 0. For any compact $K$,

\[ H_K := \bigoplus_{\lambda \in K} H_\lambda. \]

Then $H_K$ are subcoalgebra of $H$. For any $f \in H$, $f = \sum_{\lambda \in K} f_\lambda f_\lambda \in H_\lambda$, for some compact $K$.

For each $\lambda \in \Lambda$, let $V_\lambda$ be the corresponding simple $H$-comodule. Note that the $V_\lambda$ are finite dimensional for all $\lambda \in \Lambda$.

The isomorphism \[9\] becomes now an isomorphism of algebras between $\hat{H}$ and the direct sum of endomorphism ring of $V_\lambda$.

\[ \hat{H} \cong \bigoplus_{\lambda \in \Lambda} \hat{H}_\lambda \cong \bigoplus_{\lambda \in \Lambda} \text{End}_K(V_\lambda). \]

Thus we have proved

Theorem 2.6. Let $H$ be a co-semisimple Hopf algebra. Then $\hat{H}$ is isomorphic to the direct sum of full endomorphism rings of simple $H$-comodules.

The algebra $\hat{H}$ does not have a unit element. Adding a unit to this algebra is problematic when we are dealing with the norm – the unit element is something like “the Dirac delta function” which never has finite norm. Instead we have a notion of $\delta$-type sequences, which approximate the unit.

Definition 2.7. 1) Let $A$ be an algebra without unit. A system $\{e_i, i \in I\}$ of idempotents in $A$ is an approximate unit if

(i) $I$ is a partially ordered set,
(ii) For any $a \in A$, there exists $i = i(a)$, such that $e_i a = ae_i = a$, for all $j \geq i$.
For an approximate unit in a bialgebra we require further that
(iii) $\varepsilon(e_i) = 1, \forall i \in I$.
For an algebra with involution we require that
(iv) There exist $f_i$, such that $e_i = f_i f_i^*$, for all $i \in I$.

2) For a topological algebra, the condition (ii) above is replaced by
(ii') the nets $\{e_i a | i \in I\}$ and $\{ae_i | i \in I\}$ converge to $a$.

3) A Hopf algebra with approximate unit is a bialgebra with approximate unit together with a system of endomorphism $\{S_i | i \in I\}$, called an approximate antipode, satisfying

\[ m(S_i \otimes \text{id})\Delta = m(\text{id} \otimes S_i)\Delta = e_i \varepsilon. \]

The existence of such a sequence in our $\hat{H}$ is obvious. Indeed, let $e_K$ be the unit element in $\hat{H}_K$. Then $\{e_K, K \in C_0(\Lambda)\}$ is an approximate unit in $\hat{H}$. Thus $\hat{H}$ is an algebra with approximate unit.
3. Compact Hopf *-algebras

In this section we construct from a compact Hopf *-algebra a bialgebra which is a Hilbert space. Thus, \( \mathbb{K} = \mathbb{C} \). This is the first step toward the construction of our Hopf \( C^* \)-algebra. A good reference on compact Hopf *-algebra, where the algebra is referred to as CQG-algebra, is Dijkhuizen and Koornwinder [2].

By definition, a compact Hopf *-algebra over \( \mathbb{C} \) is a co-semisimple Hopf algebra with an involutive \( \mathbb{C} \)-anti-linear anti-homomorphism * such that every simple comodule is unitarizable, i.e. we can define a scalar product on this comodule such that

\[
< v_0, w > S(v_1) = < v, w_0 > S^*_1, \text{ for } v, w \in H.
\]

For any orthonormal basis of this comodule, the corresponding coefficient matrix satisfy the orthogonality condition. More precisely, let \( x_1, x_2, \ldots, x_d \) be an orthonormal basis of the comodule and \( U = (u^i_j) \) be the corresponding coefficient matrix, i.e., \( \delta(x_i) = x_k \otimes u^k_i \). Then \( U \) satisfies \( UU^* = U^*U = I \), where \( U^*_i := u^*_i j \).

**Lemma 3.1.** Let \( H \) be a co-semisimple Hopf algebra. Then the square of the antipode on \( H \) is co-inner, i.e., it can be given in terms of an invertible element \( q \) of \( H^* \) (i.e. a linear form of \( H \)):

\[
S^2(h) = q(h_1)h_2q^{-1}(h_2),
\]

where the linear form \( q \) is given by \( q(h) = \int (S^2(h_1)S(h_2)) \).

**Proof.** First we show that \( S^2(h_1)q(h_2) = q(h_1)h_2 \), or equivalently

\[
S^2(h_1)\int (S^2(h_2)S(h_3)) = \int (S^2(h_1)S(h_2)) \cdot h_3.
\]

We have

\[
\int (S^2(h_2)S(h_3)) \cdot h_4S(h_1) = S^2(h_2)\int (S^2(h_3)S(h_4)) \cdot S(h_1) \text{ (by (1))}
\]

\[
= \int (S^2(h_1)S(h_2)).
\]

Thus, the lemma will be proved if we can show that \( q \) is invertible as an element of \( H^* \).

Let \( V \) be a finite dimensional \( H \)-comodule and \( V^{**} \) be its double-dual. As vector space, \( V^{**} \) is isomorphic to \( V \) and the coaction of \( H \) on \( V^{**} \) is given by \( \delta_{v \cdot \omega}(v) = v_0 \otimes S^2(v_1) \).

Equation (13) shows that the map \( v \mapsto v_0 \otimes q(v_1) : V \to V^{**} \) is a morphism of \( H \)-comodules. If \( V \) is simple then \( V^* \) and hence \( V^{**} \) are simple. Therefore, the map above should be zero or invertible. To see that it cannot be zero we fix a basis \( x_1, x_2, \ldots, x_d \) of \( V \) and let \( U = (u^i_j) \) be the coefficient matrix. Since have \( \Delta(u^i_j) = \sum_k u^k_i \otimes u^k_j, \varepsilon(u^i_j) = \delta^j_i \),

\[
q(\sum_i u^i_j) = \int (u^k_i S(u^j_k)) = d.
\]

Thus, the map \( v \mapsto v_0 \otimes q(v_1) : V \to V^{**} \) is an isomorphism of \( H \)-comodules. Therefore the form \( q \) is invertible. 

Set \( Q^i_j = q(u^i_j) \). Then, according to Lemma [1], the matrix \( Q \) is the matrix of the isomorphism \( V \to V^{**} \) with respect to the basis \( x_1, x_2, \ldots, x_d \) of \( V \) and we have

\[
S^2(u^i_j) = Q^k_i u^k_j Q^{-1}_j.
\]

The matrix \( Q \) is very important in the study of \( H \) and will be called reflection matrix. If \( V \) is irreducible then the integral on \( u^i_j S(u^k_j) \) can be given in terms of \( Q \). In fact, from the
left-invariance of \( \int \) we have

\[
\int (u_i^j S(u_k^l)) = u_i^j S(u_k^l) \int (u_m^n S(u_n^l))
\]

Or equivalently

\[
\int (u_i^j S(u_k^l)) \cdot u_m^n = u_i^j \int (u_m^n S(u_n^l)).
\]

Since \( \{u_i^j\} \) are linearly independent, \( \int (u_i^j S(u_k^l)) = \delta_i^k C_i^l \) for some matrix \( C = (C_i^l) \).

On the other hand, according to (14), we have \( u_i^m Q_m^n S(u_n^l) = Q_i^l \). Substituting this into (15) we get \( \delta_i^m C_i^j Q_m^n = Q_i^j \). Thus \( \text{tr}(Q) \neq 0 \) and \( C_i^j = Q_i^j / \text{tr}(Q) \), and we get

\[
\int (u_i^j S(u_k^l)) = \delta_i^k Q_i^j / \text{tr}(Q).
\]

Analogously we have

\[
\int (S(u_i^j) u_k^l) = \delta_i^k (Q^{-1})_i^j / \text{tr}(Q^{-1}).
\]

From Equations (16) and (17) we get the rule for the convolution product for coefficients of simple comodules. Let \( V_\lambda \) and \( V_\mu \) be simple comodules and \( U_\lambda, U_\mu \) be the coefficient matrix with respect to some (orthogonal) bases of \( V_\lambda \) and \( V_\mu \), and \( Q_\lambda := q(U_\lambda), Q_\mu := q(U_\mu) \). An element \( f \) of \( H_\lambda \) can be represented by a matrix \( F_\lambda: f = F_\lambda u_\lambda = \text{trace}(F_\lambda U_\lambda) \). Let \( c_\lambda = \text{trace}(C_\lambda U_\lambda), d_\mu = \text{trace}(D_\mu U_\mu) \). The convolution product has the form

\[
c_\lambda * d_\mu = C_{ij} D_{kl}^k (u_\lambda)_i^j \ast (u_\mu)_k^l
\]

\[
= C_{ij} D_{kl}^k \frac{Q_{ij}^l}{\text{tr}(Q_\lambda)} (u_\lambda)_i^j \quad \text{(from (6) and (10))}
\]

\[
= C_{ij} Q_{ij} D_{kl} \frac{\text{tr}(Q_\lambda)}{\text{tr}(Q)}
\]

where in the left-hand side is the matrix product: \( (CD)_i^j = C_i^k D_k^j \). Thus, with the \( \ast \)-product, \( H_\lambda \) becomes a unital algebra, denoted by \( \hat{H}_\lambda \), and \( \hat{H}_\lambda \cong \text{End}_C(V_\lambda) \). Moreover, for any compact \( K \subset \Lambda, \hat{H}_K := \bigoplus_{\lambda \in K} \hat{H}_\lambda \) is also a unital algebra.

Let \( V_\lambda \) be a simple \( H \)-comodule, \( V_\lambda \) be the corresponding \( H \)-module. The representation of \( H \) on \( V_\lambda \) will be denoted by \( \pi_\lambda \). Then according to (7), the action of \( f = \text{trace}(F_\lambda U_\lambda) \) on \( x_i \) is given by

\[
f * x_i = x_i \int (u_i^j S(f)) = x_j F_\lambda x_i Q_\lambda^j.
\]

In other words, the matrix of \( \pi_\lambda(f) \) with respect to the basis \( \{x_i\} \) is \( F_\lambda Q_\lambda \).

For compact Hopf \( \ast \)-algebra, we can show that the matrix \( Q \), with respect to an orthonormal basis, is positive definite [8, 11]. Let \( V^\ast \) be the dual to the comodule \( V \). Let \( \xi_1, \xi_2, \ldots, \xi_d \) be the basis of \( V^\ast \), dual to the basis \( x_1, x_2, \ldots, x_d \). The corresponding coefficient matrix is then \( S(U_i) \). Since \( V^\ast \) is unitarizable, there exists a basis \( \eta_1, \eta_2, \ldots, \eta_d \) of \( V^\ast \) which is orthonormal with respect to a scalar product, such that the corresponding coefficient matrix \( W = (w_i^j) \) satisfies \( S(W) = W^\ast \). Let \( T \) be a matrix such that \( \xi_i = \eta_j T_i^j \) then

\[
w_i^j = T_{ij}^{-1} S(u_k^l) T_j^l = T_{ij}^{-1} u_k^l T_j^l,
\]

since \( (u_i^j) \) is also unitary. Therefore, by the involutivity of \( \ast \)

\[
S(w_i^j) = w_j^i = T_{ki}^{-1} u_k^l T_j^l.
\]
Substituting \( S(u^2) \) into the preceding equation and using (14) we get \( Q = \text{const} \cdot T^*T \). A direct consequence of this fact is that \( \int \) is positive definite on \( H \).

For any \( \lambda \in \Lambda \), let \( \lambda^* \) be such that \((V_\lambda)^* = V_{\lambda^*}\). According to Lemma 3.1, \( \ast \) is an involutive map on \( \Lambda \). From above we see that the involution \( \ast \) on \( H \) maps \( H_\lambda \) onto \( H_{\lambda^*} \).

Since the integral is zero on \( H_\lambda \) except for \( \lambda = 0 \), we see that \( \int (a) = \int (a^*) \). Consequently \( \int \) defines a scalar product \(<,>\) on \( H \), \(<a,b> := \int (ab^*)\) giving rise to a norm on \( H \) called \( L^2 \)-norm. Let \( H_{L^2} \) denote the completion of \( H \) with respect to the \( L^2 \)-norm. Then \( H_{L^2} \) is a Hilbert space.

**The Hopf Algebra \( H_{L^2} \).** Our aim is to define a Hopf algebra structure on \( H_{L^2} \), where the product is an extension of the convolution product on \( \hat{H} \). First we have to extend the convolution product to \( H_{L^2} \). This can be done provided that we showed that this product is continuous with respect to the \( L^2 \)-norm.

**Lemma 3.2.** The convolution product on \( \hat{H} \) satisfies
\[
\|f * g\|_{L^2} \leq \|f\|_{L^2} \cdot \|g\|_{L^2}.
\]

**Proof.** It is sufficient to show this inequality for \( f \) and \( g \) belonging to the same coefficient space of a simple comodule. Then, in that case, \( f \) and \( g \) can be represented in the form \( f = \text{tr}(CU) \), \( g = \text{tr}(DU) \), for a unitary multiplicative matrix \( U \) and some matrices \( C, D \) with complex scalar entries. Using (16), (18) the indicated inequality is equivalent to the following
\[
\frac{\text{tr}(CQC^*)\text{tr}(DQD^*)}{\text{tr}(Q)^2} \geq \frac{\text{tr}(DQCQ(DQC)^*)}{\text{tr}(Q)^3}.
\]

Let \( T \) be such that \( Q = TT^* \). Set \( C_1 = CT, D_1 = DT \) the above inequality has the form
\[
\text{tr}(C_1C_1^*)\text{tr}(D_1D_1^*)\text{tr}(TT^*) \geq \text{tr}(C_1T^*D_1D_1^*TC_1^*).
\]

The last inequality follows immediately from the Minkowski inequality since
\[
\text{tr}(CC^*) = \sum |c_i|^2.
\]

Therefore, the convolution product on \( \hat{H} \) can be extended to \( H_{L^2} \).

**Lemma 3.3.** The family \( \{e_K | K \in C_0(\Lambda)\} \), where \( e_K \) is the unit element of \( \hat{H}_K \), is an approximate unit in \( H_{L^2} \).

**Proof.** What we need to show is that for any \( h \in H_{L^2} \), there exists a composition sequence \( K_1 \subset K_2 \subset \ldots \) such that
\[
\lim_{n \to \infty} \|e_{K_n} * h - h\| = \lim_{n \to \infty} \|h * e_{K_n} - h\| = 0.
\]

Now, for any \( h \in H_{L^2} \), there exists at most a countable set of \( \lambda, \lambda \in \Lambda \) such that \( h_\lambda = h \ast e_\lambda \neq 0 \) and \( h = \sum \lambda h_\lambda \). The last series converges absolutely whence the assertion follows.

Then we have to define the coproduct. Notice that, the coproduct on \( H_{L^2} \), if it exists, should be dual to the original product on \( H \) by means of the integral. Thus, we consider, for an element \( h \in H \), a linear functional \( \phi_h : H \otimes H \to \mathbb{C}, \phi_h(g \otimes f) = \int (hgf) \). This is obviously continuous hence is extendable on \( H_{L^2} \otimes H_{L^2} \) (\( \otimes \) denotes the tensor product of Hilbert spaces). Hence, by Riesz theorem, there exists an element \( \Delta_s(f) \) of \( H_{L^2} \otimes H_{L^2} \), such that \( \phi_h(g \otimes f) = \int \int (\Delta_s(h), g \otimes f) \). Thus, we get a map \( H \to H_{L^2}, h \mapsto \Delta_s(h) \). Again
this is continuous and hence induces a map \( H_{L^2} \to H_{L^2} \hat{\otimes} H_{L^2} \) which is the coproduct on \( H_{L^2} \).

The counit is given by \( \varepsilon_*(f) = \int (f) \).

We check the axioms for the coproduct and counit.

**Theorem 3.4.** \( (H_{L^2}, \ast, e_K, \Delta_*, \varepsilon_*) \) is a Hopf algebra with approximate unit.

**Proof.** The proof consists of some lemmas. For convenience we shall use the notation

\[
\Delta_*(f) = \sum_{(f)} f^{(1)} \otimes f^{(2)} = f^1 \otimes f^2,
\]

where the sum on the right-hand side is an absolute convergent series in \( H_{L^2} \hat{\otimes} H_{L^2} \). \( \blacksquare \)

**Lemma 3.5.** The coproduct and the counit satisfy

\[
(\Delta_* \otimes \text{id}_{H_{L^2}})\Delta_* = (\text{id}_{H_{L^2}} \otimes \Delta_*)\Delta_* = \text{id}_{H_{L^2}},
\]

in the second equation we identify \( \mathbb{C} \hat{\otimes} H_{L^2} \) and \( H_{L^2} \hat{\otimes} \mathbb{C} \) with \( H_{L^2} \).

**Proof.** By the faithfulness of \( \int \), the first equation is equivalent to

\[
\int \int \int (\Delta_* \otimes \text{id})\Delta_*(f) \cdot g \otimes h \otimes k = \int \int \int ((\text{id} \otimes \Delta_*)\Delta_* f \cdot g \otimes h \cdot k), \forall g, h, k \in H, f \in H_{L^2}
\]

By definition, the left hand side is equal to

\[
\int \int \int (f^{11} \otimes f^{12} \otimes f^2 \cdot g \otimes h \otimes k) = \int (f^1 \cdot gh) \int (f^2 k) = \int fghk.
\]

So is also the right hand side, whence we obtain the first equation of the lemma.

For the second equation of the lemma, we have, for all \( g \in H \),

\[
\int \varepsilon(f^1) f^2 \cdot g = \int f^1 \int (f^2 g) = \int f^g.
\]

The lemma is therefore proved. \( \blacksquare \)

**Lemma 3.6.** The map \( \Delta_* : H_{L^2} \to H_{L^2} \hat{\otimes} H_{L^2} \) is a homomorphism of algebras.

**Proof.** This is equivalent to the fact that for any \( h \cdot k \in H, f, g \in H_{L^2} \),

\[
\int (f \ast g)hk = \int ((f^1 \ast g^1)h) \cdot \int ((f^2 \ast g^2)k).
\]

The left hand side is equal to

\[
\int (f_{h_1}k_1) \cdot \int (gh_2k_2) = \int (f_{h_1}) \int (f_{k_1}^2) \int (g_{h_2}^1) \int (g_{k_2}^2) = \int ((f^1 \ast g^1)h) \cdot \int ((f^2 \ast g^2)k).
\]

The Lemma is proved. \( \blacksquare \)

The above two lemmas imply that \( H_{L^2} \) is a bialgebra with approximate unity. It remains to find an approximate antipode. By definition, we have to find a system \( \{S_{*K} | K \in \mathcal{C}_0(\Lambda)\} \) of linear endomorphism on \( H_{L^2} \), such that \( m_*(S_{*K} \otimes \text{id})\Delta_* \) and \( m_*(\text{id} \otimes S_{*K})\Delta_* \) are approximate units.

For any \( h \in H_{\Lambda} \), we define \( S_*(h) \in H_{\Lambda_*} \) to be such that \( \forall g \in H_{\Lambda} \),

\[
\int (hS(g)) = \int (S_*(h)g).
\]

In fact, \( S_*(h) \) can be computed explicitly

\[
S_*(h) = S(h_1)p(h_2)q(h_3),
\]
where \( q \) is the linear functional defined in Lemma 3.1, \( p \) is given by

\[
p(h) = \int S(h_2)h_1, \forall h \in H.
\]

Set \( S_\lambda|_{H_\mu} = 0, \forall \mu \neq \lambda \). For any \( K \subset C_0(\Lambda) \) set \( S_\mu = \bigoplus_{\lambda \in K} S_\lambda \).

**Lemma 3.7.** \( \{S_\mu|K \subset C_0(\Lambda)\} \) is an approximate antipode.

**Proof.** We show that

\[
S_\mu f^1 \ast f^2 = \varepsilon_s(f)e_K = \int (f)e_K, \forall f \in H, K \in C_0(\Lambda).
\]

Since \( \int S_\mu(f)g = \int fS(g_K) \), we have

\[
\int ((S_\mu(f^1) \ast f^2)g) = \int (S_\mu(f^1)g_1) \cdot \int (f^2g_2) = \int (f^1S(g_1)) \cdot \int (f^2g_2) = \int (f) \cdot \int (e_Kg),
\]

whence the assertion follows. The lemma is proved.

We have therefore finished the proof of Theorem 3.4.

**Remark.** Since the antipode on \( H \) is not involutive, the \( \ast \)-structure on \( \hat{H} \) defined by \( h^* = S(h^*) \) cannot be extended to \( H_{L^2} \), because it is not continuous with respect to the given norm. To overcome this obstruction we have to pass to the \( C^* \)-envelope of \( H_{L^2} \) which is the subject of our next section.

4. The \( C^* \)-algebra \( H_{C^*} \)

Let \( H \) be a compact Hopf \( \ast \)-algebra. Let \( \ast \) be the involutive map on \( \hat{H} \) defined in the previous section \( f^* = S(f^*) \).

**Lemma 4.1.** All simple \( \hat{H} \)-modules have the structure of \( \ast \)-modules.

**Proof.** According to Theorem 2.4, a simple \( \hat{H} \)-comodule is equivalent to some module \( \hat{V}_\lambda \) induced from a simple unitary \( \hat{H} \)-comodule \( V_\lambda \). We check that, with respect to the given scalar product on \( V_\lambda \),

\[
\langle h \ast v, w \rangle = \langle v, h^* \ast w \rangle, \forall h \in \hat{H}, v, w \in \hat{V}_\lambda.
\]

Indeed,

\[
\langle h \ast v, w \rangle = \langle v_0, w \rangle = \int (hS(v_1)) = \langle v, w_0 \rangle = \int (w_1h^*) = \langle v, w_0 \rangle = \int (S(h^*)S(w_1)) = \langle v, h^* \ast w \rangle.
\]

We introduce the following semi-norm on \( \hat{H} \)

\[
\|f\|_{C^*} := \sup_{\lambda \in \Lambda} \|\pi_\lambda(f)\|.
\]

The lemma below will show that this is a norm, i.e., it is bounded.
Let $U = U_\lambda$ be a unitary coefficient matrix corresponding to the simple unitary comodule $V_\lambda$ and $Q = Q_\lambda$ be the corresponding reflection matrix. If $0 \neq f \in \tilde{H}_\lambda$, then $f = F_j u_j^i$ for a complex matrix $F$. Since $f$ act on $V_\mu$ as zero if $\mu \neq \lambda$, then according to (19),
\begin{equation}
\|f\|_{C^*} = \|\pi_\lambda(f)\| = \|FQ\| > 0.
\end{equation}

**Lemma 4.2.** The $L^2$- and $C^*$-norms on $\tilde{H}$ satisfy
\[ \|f\|_{L^2} \geq \|f\|_{C^*}. \]

**Proof.** Since $\|f\|_{L^2}^2 = \sum \|f_\lambda\|_{L^2}^2$ and $\|f\|_{C^*} = \sup \|f_\lambda\|_{C^*}$, it is sufficient to check the above equation for $f \in \tilde{H}_\lambda$. Thus $f = F_j u_j^i$, for $(u_j^i) = (u_\lambda)_{j^i}$ a unitary multiplicative matrix.

The desired inequality is then equivalent to
\[ \frac{\operatorname{tr}(FQF^*)}{\operatorname{tr}(Q)} \geq \frac{\|FQ\|^2}{\operatorname{tr}(Q)^2}. \]

Since $Q$ is positive definite, $Q = T^*T$. Hence
\[ \|FQ\|^2 = \|FTT^*\| \leq \|FT\|^2\|T^*\|^2 \leq \operatorname{tr}(FQF^*)\operatorname{tr}(Q). \]

Here we use the inequality
\[ \|A\|^2 \leq \operatorname{tr}(AA^*). \]

We define the algebra $H_{C^*}$ to be the completion of $\tilde{H}$ with respect to the norm $\|\cdot\|_{C^*}$. Since $\|\cdot\|_{C^*}$ is an operator norm, $H_{C^*}$ is a $C^*$-algebra. By virtue of Lemma 4.2 we have
\begin{equation}
H_{L^2} \subset H_{C^*}.
\end{equation}

From the decomposition (11) of $\tilde{H}$ and the definition of the norm $\|\cdot\|$, we can easily obtain an isomorphism of the two $C^*$-algebras
\begin{equation}
H_{C^*} \cong \prod_{\lambda \in \Lambda} \operatorname{End}_C(V_\lambda),
\end{equation}
where the product on the right-hand side of the equation is over all families $\{x_\lambda \in \operatorname{End}_C(V_\lambda)\}$ with $\|x_\lambda\| \to 0$ as $\lambda \to \infty$.

**Lemma 4.3.** All irreducible unitary representations of $H_{C^*}$ are finite-dimensional and irreducible over $\tilde{H}$.

**Proof.** Let $\pi : H_{C^*} \to \mathcal{B}(\mathcal{H})$ be an irreducible unitary representation of $H_{C^*}$ in a Hilbert space $\mathcal{H}$. Then $\mathcal{H}$ is a $\tilde{H}$-module. Let $M \subset \mathcal{H}$ be a simple $\tilde{H}$-submodule. Then it has a structure of a simple $H$-comodule, hence is finite dimensional over $C$. Thus $M$ is closed in $\mathcal{H}$.

Since $\tilde{H}$ is dense in $H_{C^*}$ and $M$ is closed in $\mathcal{H}$, $M$ is a representation of $H_{C^*}$. By the irreducibility of $\pi$, we conclude that $M = \mathcal{H}$.

As a corollary of Lemmas 4.1 and 4.3, we have

**Proposition 4.4.** There exists a 1-1 correspondence between irreducible unitary representations of $H_{C^*}$ and those of $\tilde{H}$.

Thus $H_{C^*}$ is a $C^*$-algebra of type I.

From the construction of $H_{C^*}$, it is easy to see, that if $G$ is a compact group and $H = C[G]$ – the Hopf algebra of representative functions on $G$, which is a dense subalgebra of the algebra $C^\infty(G)$ of all continuous complex valued functions on $G$, then $H_{C^*} \cong C^*(G)$ – the group $C^*$-algebra of $G$.

In the case of a compact group $G$, the action of $G$ on any representation can be recovered by the action of $C^*(G)$, by using $\delta$-type sequences. In our case, we do not know, what the quantum group defining $H$ is. Therefore we have to introduce a Hopf algebra structure on $H_{C^*}$ to exhibit the “group” property of this algebra.
Theorem 4.5. \( H_{C^*} \) is a Hopf \( C^* \)-algebra with approximate unit.

Proof. Recall that the topological tensor product \( H_{C^*} \otimes H_{C^*} \) is the completion of \( \hat{H} \otimes \hat{H} \) with respect to the norm

\[
\| f \otimes g \|_{C^*} := \sup_{\lambda, \mu \in \Lambda} \| \pi_\lambda(f) \otimes \pi_\mu(g) \|.
\]

For any \( f \in \hat{H} \subset H_{L^2}, \Delta(f) \in H_{C^*} \otimes H_{C^*} \). Thus, it is sufficient to show that

\[
\| \Delta(f) \|_{C^*} \leq \| f \|_{C^*}.
\]

But this inequality is obvious, as for any \( \lambda, \mu \in \Lambda \), \( V_\lambda \otimes V_\mu \) is a representation of \( H_{L^2} \) by means of the map \( (\pi_\lambda \otimes \pi_\mu)\Delta \), hence decomposes into a direct sum of irreducible representations, namely \( \pi_\lambda \otimes \pi_\mu \cong \oplus_{\gamma} \varepsilon_{\mu, \gamma} \pi_\gamma \), consequently

\[
\| (\pi_\lambda \otimes \pi_\mu)\Delta(f) \| \leq \sup_{\gamma \in \Lambda} \| \pi_\gamma(f) \| = \| f \|_{C^*}.
\]

To show that \( \varepsilon_* \) extends onto \( H_{C^*} \), we remark that \( \varepsilon_* : \hat{H} \rightarrow \mathbb{C} \) is multiplicative, i.e., induces a representation, for

\[
\varepsilon_*(f * g) = \int (f_1 \int (f_2 S(g))) = \int (f) \int (g) = \varepsilon_*(f) \varepsilon_*(g).
\]

Hence

\[
\| \varepsilon_*(f) \| \leq \| f \|_{C^*},
\]

thus \( \varepsilon_* \) extends onto \( H_{C^*} \).

According to Lemma 4.1, for \( f \in \hat{H} \),

\[
\pi_\lambda(f^*) = \pi_\lambda(f)^*, \forall \lambda \in \Lambda.
\]

Therefore the involutive map \( * \) extends on \( H_{C^*} \).

Using (4.4) and \( (\mathfrak{R}) \), we can show that the unit element \( e_\lambda \) in \( H_\lambda \) satisfies \( e_\lambda^* = e_\lambda \). Since \( \{ e_K | K \in C_0(\Lambda) \} \) is an approximate unit of \( \hat{H} \) and since \( \hat{H} \) is dense in \( H_{C^*} \), \( \{ e_K | K \in C_0(\Lambda) \} \) is an approximate unit on \( H_{C^*} \). Consequently, \( \{ S_K | K \in C_0(\Lambda) \} \) is an approximate antipode on \( H_{C^*} \). The proof of Theorem 4.5 is completed. \( \blacksquare \)

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