Abstract

Compact canonical quantization on the light cone (DLCQ) is examined in the limit of infinite periodicity length $L$. Pauli Jordan commutators are found to approach continuum expressions with marginal non-causal terms of order $L^{-3/4}$ traced back to the handling of IR divergence through the elimination of zero modes. In contrast, direct quantization in the continuum (CLCQ) in terms of field operators valued distributions is shown to provide the standard causal result while at the same time ensuring consistent IR and UV renormalization.
1 Introduction

Light front quantization has emerged as an important tool in the study of non-perturbative aspects of field theories [1]. However, a major problem in this approach resides in the infrared behaviour of the continuum theory. Recently, this issue was clarified on the basis of a mathematically well-defined procedure [2]. In the early attempts to deal with these infrared problems, discretized light front quantization (DLCQ) [3] has played an important role. The popularity of DLCQ resides in the easy and conceptually simple treatment of the infrared regularisation: zero modes in the expansion of the fields were simply eliminated and later on understood as the LC-counterpart of the non-trivial ground state of equal-time (ET) quantization. The study of critical phenomena in the framework of effective theories requires using a continuum version of the quantum field theory on the light front. Indeed, critical points, critical exponents etc... are accessible only from a complete knowledge of the cut-off dependence of the critical mass, which can only be given by the continuum theory. In DLCQ the limit of infinite periodicity length $L$ cannot be achieved in a straightforward manner without further insights both on the handling of zero modes and restoration of covariance and causality in the limiting process [4]. Our approach [2] was to propose a genuine continuum treatment (CLCQ) in which fields are treated as operator-valued distributions, thereby leading to a well-defined handling of ultraviolet and light cone induced infrared divergences and of their renormalization. We focussed in [2] on the comparison of the critical coupling in the LC and ET-framework, showing that the continuum non-perturbative LC-approach is no more complex than usual perturbation theory in lowest order. The LC-critical coupling is in essential agreement with the RG-improved perturbative result at fourth order. Here we want to report on a detailed comparison between DLCQ and CLCQ treatments of important quantities like Pauli-Jordan commutator functions, which, due to necessary concision and lack of space, could not be treated therein.

In Section 2 we recall the DLCQ and CLCQ Fock expansion of the field operators and the resulting Pauli-Jordan field commutators. A detailed comparison of their behaviour in terms of the periodicity length $L$ (e.g. intrinsic cut-off $\Lambda$) is made in section 3 where the issue of covariance and causality is also discussed in the limiting process $L \to \infty$. Some conclusions and perspectives are presented in section 4.

2 DLCQ and CLCQ field operators and commutators

DLCQ was introduced [3] to resolve the zero mode problem. This mode is clearly isolated from other modes and its explicit treatment results in a "zero-mode constraint", the solution of which carries the non-perturbative aspects of the theory. In the particle sector, periodic boundary conditions are imposed, $L$ being the periodicity length, leading to the usual Fock expansion. Restricting to $1+1$ dimension the particle sector field writes

$$\phi(x) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{4\pi n}} [a_n e^{-ik_n x} + a_n^{+} e^{ik_n x}]$$

(2.1)

with

$$[a_n, a_m^{+}] = \delta_{n,m}, \quad n, m \geq 1;$$

and

$$k_n = \frac{n\pi}{L}, \quad n \in \mathbb{Z}.$$
The CLCQ approach relies on the introduction of field operator-valued distributions defined with respect to $C^\infty$-test functions with compact support [5]. Apart from formal considerations there exists a fundamental physical argument which demonstrates that it is compelling to treat the field amplitudes in the distributional sense in order to guarantee that the LC quantization procedure by itself is correct. Due to the hyperbolic form of the LC-Laplacian, initial field values have to be prescribed on characteristics, i.e. on $x^+ = 0$ and $x^- = 0$. In order to be able to transform this characteristic value problem into a problem with periodic boundary conditions, test functions $f(p^+, p^-)$ have to be introduced with the property [6]

$$\lim_{p^+ \to 0} \frac{1}{p^+} f(p^+, \frac{m^2}{p^+}) = 0$$ (2.2)

(see eq. (3.20) of ref. [6]).

This is exactly what happens automatically with the test functions defined below. Condition (2.2) ensures, as discussed in detail in ref. [6], that the field values on the characteristic $x^- = 0$ become dependent quantities and, as a consequence, the quantization can be performed prescribing boundary values for $x^+ = 0$ at $x^- = -L$ and $x^- = L$, where $L \to \infty$. The field can be expressed in a chart independent way as a surface integral over a manifold, thereby showing that the ultraviolet (UV) behaviour on the Minkowski manifold dictates the UV and IR behaviour on the LC manifold. This is due to the regularisation properties of the test function which are automatically transfered from the first to the second case.

In this context the field writes

$$\phi_{LC}(x) = \int_0^\infty \frac{dp^+}{4\pi p^+} [a(p^+) e^{-ip^-x} + a^+(p^+) e^{ip^-x}] f_{LC}(p^+, \hat{p}^-(p^+))$$ (2.3)

with

$$[a(p^+), a^+(p^+)] = 4\pi p^+ \delta(p^+ - p'^+).$$

In (2.2) $\hat{p}^-(p^+)$ stands for the on-shell condition $m^2/p^+$ and $f_{LC}$ is the test function in momentum space which falls off with all its derivatives sufficiently fast as a function of the Minkowski arguments $p_0, p_z$ ($p^+ = \frac{1}{2}(p^0 + p^3)$, $p^- = \frac{1}{2}(p^0 - p^3)$). Its behaviour as a function of $p^+$ is discussed in [2]: the singular behaviour of $\frac{1}{p^+}$ in (2.2) is completely damped out by the behaviour of $f_{LC}$ for $p^+ \to 0$, eliminating $p^+ = 0$ as an accumulation point. The ensuing renormalization is independent of the particular choice of $f_{LC}$.

We examine first the Pauli Jordan commutator $\Delta(x) = [\phi(x), \phi(0)]$ evaluated at $x^+ = 0$. In the DLCQ case on finds

$$\Delta_{DLCQ}(x^+ = 0, x^-) = \sum_{n=-\infty, \neq 0}^{\infty} \frac{1}{4\pi n} e^{-i \frac{n\pi x^-}{L}}$$

$$= -\frac{i}{4}[\text{sign}(x^-) - \frac{x^-}{L}],$$ (2.4)

where $\text{sign}(x) = \pm 1$ if $x \geq 0$, $\text{sign}(0) = 0$.

Within CLCQ, with $\hat{f}(p^+) \equiv f_{LC}(p^+, \hat{p}^-(p^+))$, the corresponding expression writes

$$\Delta_{DLCQ}(x^+ = 0, x^-) = -\frac{i}{2\pi} \int_0^{\infty} \frac{dp^+}{p^+} \hat{f}^2(p^+) \sin(p^+ x^-).$$ (2.5)

The test function $\hat{f}$ is strictly one in the interval $[\frac{1}{2}, \Lambda - \frac{1}{2}]$, varies between 0 and 1 in the intervals $[0, \frac{1}{2}]$ and $[\Lambda - \frac{1}{2}, \Lambda]$, and is zero outside.
3 Comparison of the DLCQ and CLCQ Pauli Jordan commutators

The behaviour of $g_A(x^-) = 4i\Delta_{DLCQ}(x^+ = 0, x^-)$ is sketched in Figure 1.

$$\text{Fig.1: the DLCQ function } g_A(x^-).$$

To evaluate $g_B(x^-) = 4i\Delta_{CLCQ}(x^+ = 0, x^-)$, we choose

$$f(p) = \begin{cases} 
1 - \exp\left[\frac{1}{\Lambda^2p^2 - 1} + 1\right] & 0 \leq p < \frac{1}{\Lambda} \\
1 & \frac{1}{\Lambda} \leq p \leq \Lambda - \frac{1}{\Lambda} \\
1 - \exp\left[\frac{1}{\Lambda^2(p - \Lambda)^2 - 1} + 1\right] & \Lambda - \frac{1}{\Lambda} < p \leq \Lambda \\
0 & p > \Lambda 
\end{cases}$$

with $\Lambda = 100$, and calculate $g_B(x^-)$ numerically. The results are plotted in Figure 2 at three different spatial scales.

$$\text{Fig.2: the CLCQ function } g_B(x^-) \text{ at different spatial scales.}$$

Near the origin $g_B(x^-)$ rises to 1 over distances shorter with increasing $\Lambda$. It is followed by an oscillatory fall-off with an average slope in $\frac{1}{\Lambda}$, corresponding to the straight line of $g_A(x^-)$ in DLCQ. Finally for large values of $x^-$ ($\geq 10\Lambda$) $g_B(x^-)$ remains oscillating around zero.

Hence in both cases the decay zone and the asymptotic region where $g(x^-)$ is null or quasi-null, reflect the elimination of the zero mode, $n = 0$ for DLCQ and a halo around $p^+ = 0$ for CLCQ. However it is the presence of the UV-regularisation in CLCQ which is responsible for the smeared out rise near $x^- = 0$ and small short wave length oscillations for small $x^-$, at variance with DLCQ where no such regularisation is present. Clearly the n-summation can be arbitrarily cutt-off to deal with the UV-divergence but the approach to the continuum is not under control since the limiting procedure of infinite cutt-off and
infinite periodicity length compatible with causality is not known. To discuss these points we examine now the commutator for space or time like separation.

For DLCQ we have

\[ \Delta_{DLCQ}(x^+, x^-) = -\frac{i}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \left( \frac{n\pi x^-}{L} + \frac{1}{4} \frac{m^2 L x^+}{\pi n} \right) \] (3.2)

and for CLCQ the corresponding expression is

\[ \Delta_{CLCQ}(x^+, x^-) = -\frac{i}{2\pi} \int_0^\infty dp^+ \frac{d}{dp^+} \sin \left( \frac{1}{4} \frac{m^2 x^+}{p^+} + p^+ x^- \right) \hat{f}(p^+). \] (3.3)

The integral in (3.3) is convergent even if \( \hat{f} = 1 \) everywhere and a straightforward change of the integration variable shows that \( \Delta_{CLCQ} \) depends only on the product \( x^+ x^- \). The limit \( \Lambda \to \infty \) can be taken safely with the result

\[ \Delta_{CLCQ}(x^+, x^-) = -\frac{i}{4} \left[ \text{sign}(x^+) + \text{sign}(x^-) \right] J_0(m\sqrt{x^+ x^-}), \] (3.4)

which is the correct causal covariant expression, with \( J_0(x) \) the Bessel function of order zero.

Clearly for \( \Delta_{DLCQ} \) the limit \( L \to \infty \) cannot be taken before the sum is carried out, as the sinus becomes ill-defined. As shown in Appendix A this limit requires some care. Using eq.(A.19) one finds

\[
\Delta_{DLCQ}(x^+, x^-) \bigg|_{L \to \infty} = -\frac{1}{4} \left[ \text{sign}(x^+) + \text{sign}(x^-) \right] J_0(m\sqrt{x^+ x^-})
\]

\[ + \frac{i}{2mLx^+} \sqrt{x^+ x^-} + 2Lx^+ \text{sign}(x^+) J_1(m\sqrt{x^+ x^-} + 2Lx^+ \text{sign}(x^+)) + O(L^{-5/4}) \] (3.5)

Hence the causal covariant expression is retrieved in the limit \( L \to \infty \). However the marginal non causal term in \( J_1 \) in eq.(3.4) originates from the elimination of the zero mode in the infinite sum of eq.(3.2) (for \( x^+ = 0 \) it is just \( \frac{i}{4\pi} \), cf. eq(2.4)). Its disappearance as \( L \to \infty \) indicates that in the continuum the infrared problems would remain at variance with CLCQ. Thus in DLCQ, \( L \) has to be kept finite to achieve IR regularisation, at the expense of the appearance of a causality violating term of order \( (L^{-3/4}) \). Due to the regularisation properties of the test functions, the situation in CLCQ is far more satisfactory since the approach provides a well defined handling of UV and IR-divergences and of their renormalization.

4 Conclusion

It has been shown that dynamical properties of LC-quantized scalar fields whose basic manifestation is in the Pauli-Jordan commutator function differ essentially if quantized in DLCQ or in the continuum. DLCQ on a finite interval yields causality violating terms being proportional to \( L^{-3/4} \) which come in addition to the frame independent result of the continuum theory. Unfortunately this does not mean that the two versions coincide in the limit \( L \to \infty \) since in this limit the infrared regularisation of DLCQ is lost.

To conclude we want to add a remark concerning the LC-lattice method introduced by Destri and de Vega [7] and elaborated by Faddeev and coworkers [8]. This approach works on
a LC-space-time lattice. The basic building blocks of field dynamics being causal transfer matrices between neighbouring points along light-like directions, problems with causality are avoided by construction in this discretization scheme. However the main argument in favour of this approach lies in the integrability properties in closest connection to those of the continuum.

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5 References

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Appendix A

In this appendix we derive the expression of the periodic Pauli-Jordan function in the limit of infinite periodicity length $L$.

Consider the periodic distribution with period $\lambda = \frac{2\pi}{K}$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inKx}$$

and the class of $C^\infty$-test function $\varphi(x)$ with the properties :

$$\{ x \in [0, 1] : \varphi(x) + \varphi(x - 1) = 1 ; \varphi(0) = 1, \varphi(1) = 0, \frac{d^p \varphi(x)}{dx^p} \bigg|_{x=1} = 0 \ \forall p \geq 1 \}. \quad (A.2)$$
This constitutes a decomposition of unity since by construction
\[ \sum_{p=-\infty}^{\infty} \varphi(x + p) = 1, \quad \forall x. \]  
(A.3)

![Diagram of \( \varphi(x) \) decomposing unity.]

The Fourier transform \( \phi(k) \) of \( \varphi(x) \) has the property
\[ \phi(0) = 1, \quad \phi(2p\pi) = 0, \quad \forall p \text{ integer } \neq 0. \]  
(A.4)

The coefficient \( C_n \) in the expansion of \( f(x) \) are then given by
\[ C_n = \frac{1}{\lambda} \int_{-\infty}^{\infty} f(x) \varphi(\frac{x}{\lambda}) e^{inKx} dx. \]  
(A.5)

If \( f(x) \) is a standard integrable function of period \( \lambda \), \( C_n \) is just the usual Fourier coefficient since \( \varphi(\frac{x}{\lambda}) + \varphi(\frac{x}{\lambda} - 1) = 1 \).

We consider now, for \((a, b) \in \mathbb{R}\), the distribution
\[ T_{ab}(x) = \frac{1}{2i} \sum_{p=-\infty}^{\infty} e^{i(ax + \frac{b}{p})} \left(1 - \frac{\sin(\pi x)}{\pi x}\right) \delta(x - p). \]  
(A.6)

With the \( C^\infty \)-test function \( \Omega(x) \) which decomposes unity we have
\[
T_{ab}(x)(\Omega) = \frac{1}{2i} \sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(ax + \frac{b}{p})} \left(1 - \frac{\sin(\pi x)}{\pi x}\right) \delta(x - p) \Omega(x) dx
\]
\[ = \frac{1}{2i} \sum_{p=-\infty}^{\infty} \frac{e^{i(ap + \frac{b}{p})}}{p} \left(1 - \frac{\sin \pi p}{\pi p}\right) \Omega(p) \]
\[ = \sum_{p=1}^{\infty} \frac{1}{p} \sin(ap + \frac{b}{p}) \]
(A.7)

since \( \Omega(p) = 1, \quad \forall p \text{ integer or zero } \) (cf Fig.A1).

On the other hand the periodic distribution
\[ f(x) = \sum_{p=-\infty}^{\infty} \delta(x - p) \]
admits also the Fourier expansion (A.1) with \( K = 2\pi \) and \( C_n = 1 \), directly from (A.5).
Hence we have the well known representation
\[ \sum_{p=-\infty}^{\infty} e^{2ip\pi x} = \sum_{p=-\infty}^{\infty} \delta(x - p). \]  
(A.8)
\( T_{ab}(x)(\Omega) \) is then also given by

\[
T_{ab}(x)(\Omega) = \sum_{p=-\infty}^{\infty} \int_{0}^{\infty} \frac{dx}{x} \Omega(x) \sin[(a + 2p\pi)x + \frac{b}{x}](1 - \frac{\sin \pi x}{\pi x}). \tag{A.9}
\]

\( \Omega(x) \) being a decomposition of unity and since the integral is well defined with \( \Omega(x) = 1 \) on the whole integration domain, we have

\[
\int_{0}^{\infty} \frac{dx}{x} \sin[(a + 2p\pi)x + \frac{b}{x}] = \frac{\pi}{2} \{ \text{sign}(a + 2p\pi) + \text{sign}(b) \} J_0(2\sqrt{(a + 2p\pi)b}), \tag{A.10}
\]

and

\[
\sqrt{(a + (2p + 1)\pi)b} \ J_1(2\sqrt{(a + (2p + 1)\pi)b}) \ = \ [\text{sign}(a + (2p + 1)\pi) + \text{sign}(b)]
\]

\[
\sqrt{(a + (2p - 1)\pi)b} \ J_1(2\sqrt{(a + (2p - 1)\pi)b}) \} \tag{A.11}
\]

Here \( \text{sign}(x) = \pm 1 \quad x \geq 0 \), \( \text{sign}(0) = 0 \) and \( J_n(x) \) is the ordinary Bessel function of order \( n \).

Specializing to the discretized light-cone variables \( a = \frac{\pi x^-}{L}, b = \frac{m^2}{4} \frac{Lx^+}{\pi}, \) we have

\[
\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \left[ \frac{n\pi x^-}{L} + \frac{m^2}{4} \frac{Lx^+}{n\pi} \right] = \sum_{p=-\infty}^{\infty} \{ \text{sign}(x^- + 2pL) + \text{sign}(x^+) \} \ J_0(\sqrt{x^2 + 2pLx^+})
\]

\[
- \frac{1}{mLx^+} \sum_{p=-\infty}^{\infty} \{ \text{sign}(x^- + (2p+1)L) + \text{sign}(x^+) \} \sqrt{x^2 + (2p+1)Lx^+} \ J_1(\sqrt{x^2 + (2p+1)Lx^+})
\]

\[
- \text{sign}(x^- + (2p - 1)L) + \text{sign}(x^+) \} \sqrt{x^2 + (2p - 1)Lx^+} \ J_1(\sqrt{x^2 + (2p - 1)Lx^+}) \} \tag{A.12}
\]

This is invariant indeed under the replacement \( x^- \rightarrow x^- + 2mL, \ \forall m \) integer. If \( x^+ = 0 \) one has, since \( -L \leq x^- \leq L, \) and \( \forall N \) integer > 0

\[
\sum_{p=-N}^{N} \text{sign}(x^- + 2pL) = \text{sign}(x^-) \tag{A.13}
\]

and

\[
\frac{1}{2L} \sum_{p=-N}^{N} \{ \text{sign}(x^- + (2p+1)L)[x^- + (2p+1)L] - \text{sign}(x^- + (2p-1)L)[x^- + (2p-1)L] \} = \frac{x^-}{L}, \tag{A.14}
\]

in agreement with eq. (2.3).

For non zero \( x^+ \), (A.12) reduces to the continuum causal contribution of eq. (2.8) and non-causal terms :

\[
\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \left[ \frac{n\pi x^-}{L} + \frac{m^2}{4} \frac{Lx^+}{n\pi} \right] \{ \text{sign}(x^+) + \text{sign}(x^-) \} J_0(\sqrt{x^+x^-}) +
\]

8
\[
\lim_{N \to \infty} \left\{ 2\text{sign}(x^+) \sum_{p=1}^{N} J_{0}(m\sqrt{x^+x^- + 2px^+\text{sign}(x^+))} - \frac{2}{mLx^+} \sqrt{x^+x^- + (2N+1)Lx^+\text{sign}(x^+))} \right\}.
\]

(A.15)

The limit \( N \to \infty \) in (A.15) is still elusive because the compensation between the two diverging terms in \( N \) is not explicit. However the remaining sum in (A.15) can be given in an integral form using the contour integral representation of \( J_{0}(z) \)

\[
J_{0}(z) = \frac{1}{2\pi i} \int_{C} \frac{dz}{s} e^{-(\frac{z^2}{2})},
\]

where \( C \) is the contour around the negative real axis and encircling the origin in the clockwise direction. Then the geometric sum over \( p \) can be performed and, with \( \alpha = \frac{m^2}{4} x^+x^- \) and \( \beta = \frac{m^2}{2} Lx^+ \text{sign}(x^+) \), positive, we have the result

\[
\sum_{p=1}^{N} J_{0}(m\sqrt{x^+x^- + 2px^+\text{sign}(x^+))} = \frac{1}{2\pi i} \int_{C} \frac{dz}{(z - \frac{\alpha + i}{\sqrt{2}(N+1)}}) e^{z^2}
\]

\[
= \frac{1}{2\pi i} \int_{C} \frac{dz}{\sqrt{\alpha + \frac{\beta}{2}(N+1)}} \frac{\sinh\left(\frac{N\beta z}{2}\right)}{\sinh\left(\frac{\beta z}{2}\right)}
\]

\[
= \frac{1}{\beta} \left( \sqrt{\alpha + \beta(N + \frac{1}{2})} J_{1}(2\sqrt{\alpha + \beta(N + \frac{1}{2})}) - \sqrt{\alpha + \frac{\beta}{2}} J_{1}(2\sqrt{\alpha + \frac{\beta}{2}}) \right) \quad \text{(A.17)}
\]

since the hyperbolic sine in the denominator reduces to its argument in the large \( N \) limit. Collecting terms in (A.17) we have the result

\[
2\text{sign}(x^+) \sum_{p=1}^{N} J_{0}(m\sqrt{x^+x^- + 2px^+\text{sign}(x^+))} = -\frac{2}{mLx^+} \sqrt{x^+x^- + Lx^+\text{sign}(x^+)}
\]

\[
J_{1}(m\sqrt{x^+x^- + Lx^+\text{sign}(x^+))} + \frac{2}{mLx^+} \sqrt{x^+x^- + (2N+1)Lx^+\text{sign}(x^+)}
\]

\[
J_{1}(m\sqrt{x^+x^- + (2N+1)Lx^+\text{sign}(x^+))} + O(L^{-5/4}) \quad \text{(A.18)}
\]

Now the limit \( N \to \infty \) can be taken in (A.15) as the diverging term in \( N \) in (A.15) is cancelled exactly by the one in (A.18), leaving the result

\[
\frac{2}{\pi} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x^+}{L} + \frac{m^2}{4} \frac{Lx^+}{n\pi} \right) = [\text{sign}(x^+) + \text{sign}(x^-)] J_{0}(m\sqrt{x^+x^-})
\]

\[
-\frac{2}{mLx^+} \sqrt{x^+x^- + 2Lx^+\text{sign}(x^+)} J_{1}(m\sqrt{x^+x^- + 2Lx^+\text{sign}(x^+))} + O(L^{-5/4}). \quad \text{(A.19)}
\]