Comments on spin-orbit interaction of anyons

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Abstract

The coupling of non-relativistic anyons (called exotic particles) to an electromagnetic field is considered. Anomalous coupling is introduced by adding a spin-orbit term to the Lagrangian. Alternatively, one has two Hamiltonian structures, obtained by either adding the anomalous term to the Hamiltonian, or by redefining the mass and the NC parameter. The model can also be derived from its relativistic counterpart.

1 Introduction

Anyons (by which we mean here a particle in the plane which carries fractional spin) [1, 2] with anomalous gyromagnetic ratio have recently been considered [3, 4] either in Souriau’s [5] symplectic, or in a novel, “enlarged Galilean” framework. Both approaches are somewhat unfamiliar to most physicists. In this Letter we continue our investigations using more conventional methods, close to the spirit of Ref. [6].

2 Exotic particles with minimal electromagnetic interaction

A curious fact known for thirty years but only investigated in more recent times is that the planar Galilei group admits a two-fold “exotic” central extension, labeled with \( m \) (the mass) and a second, “exotic” parameter \( \kappa \) [7]. Physical realizations of this symmetry have been presented, independently [8, 9]; both can be obtained from their relativistic anyons as “Jackiw-Nair” (JN) limits [10, 11]. The first of these models, referred to as the “extended exotic particle”, uses an
acceleration dependent Lagrangian \[8\]. In terms of (external) momenta, \(P_i\), and suitably defined external and internal coordinates \(X_i\) and \(Q_i\) and \((i = 1, 2)\), \[11, 12\], the model is conveniently described by the first-order Lagrangian

\[
L^0 = L^0_{\text{ext}} + L^0_{\text{int}} = \left\{ P_i \cdot \dot{X}_i + \frac{\theta}{2} \epsilon_{ij} P_i \dot{P}_j - \frac{\bar{P}^2}{2m} \right\} + \left\{ \frac{1}{2\theta} \epsilon_{ij} Q_i \dot{Q}_j + \frac{1}{2m\theta^2} \bar{Q}^2 \right\}, \quad (2.1)
\]

where we introduced the non-commutative parameter \(\theta = \kappa/m^2\). \(\bar{Q}^2\) is a constant of the motion. When \(\bar{Q}^2 = 0\), the internal space reduces to a point, and we recover the “minimal” exotic particle in \[9\]. We first consider the extended case. \(Q_i = 0\). The nontrivial Poisson-bracket relations are

\[
\{X_i, X_j\} = -\{Q_i, Q_j\} = \theta \epsilon_{ij}, \quad \{X_i, P_j\} = \delta_{ij}. \quad (2.2)
\]

Such a particle can be coupled minimally to an electromagnetic field in various ways.

(i) One possibility \[11, 12\] is to couple to the external part only by adding the usual expression

\[
L^\text{gauge}_{\text{ext}} = e (A_i \dot{X}_i + A_0), \quad (2.3)
\]

which amounts to gauging the global symmetry associated with the electric charge. This amounts to modifying the symplectic structure which determines the non-commutative geometry of the phase space, cf. (3.5) below.

(ii) In another scheme \[12\] the Hamiltonian is

\[
H_0 = \bar{P}^2 \rightarrow \frac{1}{2m} (\bar{P} - e \bar{A})^2 - \bar{e} A_0 \quad (2.4)
\]

while the non-commutative geometry is unchanged. In such a way the interaction changes Abelian gauge transformations \[12\]. \(^1\) The two schemes are equivalent in the absence of the exotic structure, \(\theta = 0\), but not for \(\theta \neq 0\).

Both schemes leave the internal motions uncoupled. They can be also coupled, however, by gauging the additional “internal” global SO(2) symmetry, \(\delta Q_i = \varphi \epsilon_{ij} Q_j, \varphi \in \mathbb{R} \[11\]. In scheme (i) the interaction of an “extended exotic particle” with an electromagnetic field is described by the Lagrangian

\[
L = L^0 + L^\text{gauge}_{\text{ext}} + L^\text{gauge}_{\text{int}}, \quad L^\text{gauge}_{\text{int}} = \frac{\bar{Q}^2}{2\theta} (A_i \dot{X}_i + A_0). \quad (2.5)
\]

Then easy calculation shows that the Lagrangian (2.5) is quasi-invariant with respect to local internal rotations supplemented by a gauge transformation, \(\delta Q_i = \varphi(X, t) \epsilon_{ij} Q_j, \delta A_\mu = \partial_\mu \varphi\). The coefficient in the interaction term (2.5) is fixed by gauge invariance: it generates internal rotations, \(\{\bar{Q}^2, Q_i\} = 2\theta \epsilon_{ij} Q_j\). The Euler-Lagrange equations are

\[
m^* \ddot{X}_i = P_i - e m \theta \epsilon_{ij} E_j, \quad (2.6)
\]

\[
\ddot{P}_i = e B \epsilon_{ij} \dot{X}_j + e E_i, \quad (2.7)
\]

\[
\ddot{Q}_i = \epsilon_{ij} Q_j (A_k \dot{X}_k + A_0 + \frac{1}{m\theta}), \quad (2.8)
\]

where \(E_i\) and \(B\) are the electric and magnetic field, and \(e\) denotes the shifted charge \(e + \bar{Q}^2/2\theta\). \(m^* = m(1 - e\theta B)\), is the effective mass introduced in \[9\]. Equation (2.8) implies at once that the [squared] length of the internal vector, \(\bar{Q}^2\) (and hence also the shifted charge) are constants of the motion.

\(^1\)Yet another coupling scheme is put forward in \[13\].
In the general case, the “internal” variable is parallel transported, just like for a particle with nonabelian internal structure [14]. This motion is, however, a mere gauge artifact that could be eliminated by a gauge transformation with \( \varphi(t) = -t/m\theta \), which would also remove the \((m\theta)^{-1}\) in (2.8). The only physical quantity is \( \vec{Q}^2 \). Being unphysical, the motion of the internal variable \( \vec{Q} \) will, therefore not be considered in what follows. We only consider the equations (2.6-2.7).

When \( \vec{Q} = 0 \), we recover the “minimal” exotic particle of [9], coupled to an e.m. field.

In the second scheme (ii), the electromagnetic interaction including the internal motion can be obtained, as described in [12], from (2.4) by means of a noncanonical transformation of the phase space variables, supplemented with a classical Seiberg-Witten map between the corresponding gauge potential.

Therefore, in both cases, the additional coupling to internal motion amounts to replacing the original, “bare” charge by the total charge, \( e \to e + \vec{Q}^2/2 \), whose two parts can’t be measured separately.

### 3 Anomalous coupling

Anomalous coupling to the electromagnetic field has been studied before [3, 6, 15, 16, 17, 18]. The traditional rule of nonrelativistic physics, translated into the plane, says that magnetic moment interactions should be introduced by adding a term \( \mu B \) to the Hamiltonian, where \( \mu = e g s_0/2m \) is the magnetic moment. Here \( g \) is the gyromagnetic ratio and we denoted nonrelativistic spin by \( s_0 \). Here we propose to generalise this rule by also including an electric term, namely by adding to (2.5)

\[
L_{\text{anom}} = \mu B - \frac{g}{2} e \theta \epsilon_{ij} P_i E_j, \quad \mu = \frac{g e}{2m} s_0. \tag{3.1}
\]

The equations of motion look rather complicated,

\[
m^* \dot{X}_i = P_i - \left(1 - \frac{g}{2}\right) e m \theta \epsilon_{ij} E_j - m \theta e \epsilon_{ij} \partial_j B + \frac{e m g \theta^2}{2} (P_i \partial_k E_k - P_k \partial_k E_i), \tag{3.2}
\]

\[
\dot{P}_i = e (E_i + B \epsilon_{ij} \dot{X}_j) + \mu \partial_i B - \frac{e g \theta}{2} \epsilon_{kj} P_k \partial_i E_j. \tag{3.3}
\]

- for \( g = 0 \) we plainly recover the previous equations of motion (2.6-2.7-2.8).
- By (3.2) the velocity and the momentum, \( \dot{X}_i \) and \( P_i \), respectively, are not parallel in general, except for \( g = 2 \) and for constant magnetic and linear and central electric field.
- When the fields are not only weak but also constant, eqns. (3.2-3.3) reduce to the weak-field, non-relativistic equations, \# (7.1) of [9], i.e.,

\[
m^* \dot{X}_i = P_i - \left(1 - \frac{g}{2}\right) m \theta \epsilon_{ij} E_j, \tag{3.4}
\]

\[
\dot{P}_i = e (E^i + B \epsilon_{ij} \dot{X}_j)
\]

These equations are Hamiltonian. The commutation relations are those of an “ordinary” exotic particle, [9], and the spin-orbit term is added to the Hamiltonian,

\[
\{X_i, X_j\} = \frac{\theta}{1 - e \theta B} \epsilon_{ij}, \quad \{X_i, P_j\} = \frac{1}{1 - e \theta B} \delta_{ij}, \quad \{P_i, P_j\} = \frac{e B}{1 - e \theta B} \epsilon_{ij} \tag{3.5}
\]

\[
\tilde{H} = \left(\frac{\vec{p}^2}{2m} + A_0 + \mu B\right) + \frac{g}{2} e \theta \epsilon_{ij} P_i E_j. \tag{3.6}
\]
\( B \neq B_c \) where \( A_0 = -eE_i X_i \). The generic solutions of the equations of motion (3.4) are of the familiar cycloidal form describing the Hall drift of the guiding center combined with uniform rotations with frequency

\[
\Omega = \frac{eB}{m^*}.
\]  

(3.7)

Unlike in [4], the “corrected” Larmor frequency only depends on the non-commutative parameter \( \theta \) but is independent of the gyromagnetic ratio \( g \).

Remarkably, the same equations (3.4) can be derived also from another Hamiltonian structure, namely from

\[
\{\{X_i, X_j\}\} = \frac{1 - (g/2)e\theta B}{1 - e\theta B} \delta_{ij},
\]  

(3.8)

\[
\{\{X_i, P_j\}\} = \frac{1 - (g/2)e\theta B}{1 - e\theta B} \epsilon_{ij},
\]  

(3.9)

\[
\{\{P_i, P_j\}\} = \frac{1 - (g/2)e\theta B}{1 - e\theta B} \epsilon_{ij}
\]  

(3.10)

\[
H = \frac{\vec{P}^2}{2m(1 - (g/2)e\theta B)} + A_0 + \mu B.
\]  

(3.11)

These are indeed the usual “exotic” relations, but with redefined NC parameter and mass,

\[
\theta \rightarrow \frac{1 - (g/2)}{1 - (g/2)e\theta B} \theta \quad m \rightarrow m(1 - (g/2)e\theta B),
\]  

(3.12)

respectively. Thus, for constant external fields, the anomalous electric coupling term in (3.1) (or (3.6)) can be suppressed by redefining the parameters, yielding the same equations (2.6-2.7) as in the minimal model. The constant term \( \mu B \) can actually be dropped from both (3.6) and (3.11).

### 4 Relation to relativistic anyons

The anomalous theory of Ref. [3] was based on replacing the (relativistic) “bare” mass by a field-dependent expression, \( m \rightarrow M = M(eF \cdot S) \), where \( S_{\alpha\beta} \) is the spin tensor, and \( F \cdot S = -S_{\alpha\beta} F^{\alpha\beta} \) [15, 16] \(^{2}\). Now in the plane the usual requirement \( S_{\alpha\beta} P^\beta = 0 \) implies that spin is given by the momentum,

\[
S_{\alpha\beta} = \frac{8}{M} \epsilon_{\alpha\beta\gamma} P^\gamma.
\]  

(4.1)

In [3] the choice was

\[
\tilde{M}^2 = m^2 + \frac{ge}{2c^2} F \cdot S.
\]  

(4.2)

It should be stressed, however, that (4.2) is a mere Ansatz, and does not follow from any first principle. In fact, any function \( M = M(eF \cdot S) \) would yield a consistent theory [3, 15, 16]. For example,

\[
M = m + \frac{ge}{4mc^2} F \cdot S
\]  

(4.3)

could be (and has been [17]) used. In the weak-field-limit, (4.3) yields the same equations as (4.2), since \( \tilde{M} \approx M \) if \( \epsilon g F \cdot S / m^2 c^2 \ll 1 \). In what follows, we shall use the simpler expression (4.3). Then the procedure followed in [3] is readily seen to be equivalent, in the weak-field limit,
to adding to Cartan’s variational 1-form (whose integral is the classical action \([5]\)) the anomalous spin-field term

\[
\Delta \alpha = -\frac{ges}{4mM} \varepsilon_{\alpha\beta\gamma} P^\alpha F_{\beta\gamma} \left( \frac{P_\alpha dX^\alpha}{Mc^2} \right). \tag{4.4}
\]

But we can parametrize our curves with proper time, \((P_\alpha dX^\alpha)/Mc^2 = d\tau\). The extra term has, therefore, the same effect as adding

\[
\Delta H = \frac{ges}{4mM} \varepsilon_{\alpha\beta\gamma} P_\alpha F_{\beta\gamma} \tag{4.5}
\]

to the Hamiltonian, since \(\int \Delta \alpha = -\int \Delta H d\tau\).

In a local Lorentz frame, putting \(s = \theta m^2 c^2 + s_0\) allows us to infer that the extra piece added to the Lagrangian is

\[
+ \frac{gem\theta B}{2M} P_0 + \frac{ges_0}{2M} B \left( \frac{P_0}{M c^2} \right) - \frac{gm}{2M} e\theta \epsilon_{ij} P_i E_j - \frac{1}{c^2} \frac{ges_0}{2mm} \epsilon_{ij} P_i E_j.
\]

\(P_0 \approx Mc^2\) and \(m/M \approx 1\) in the NR limit. Removing the first, divergent term and dropping the last one which goes to zero as \(c \to \infty\). In the JN limit, neglecting higher-order terms, we end up with \(L_{\text{anom}}\) with \(Q = 0\) in (3.1). Alternatively, the spin-orbit term \(H_{\text{anom}}\) in (3.6) is the JN limit of (4.5). The two possibilities i.e., either changing the kinetic term, or adding a spin-orbit piece to the Hamiltonian are the relativistic counterparts of the two Hamiltonian structures we found in the non-relativistic context.

### 5 Semiclassical Dirac particle

Returning to the non-relativistic setting, let us illustrate our theory on a related problem. In a recent paper [20], Bérard and Mohrbach consider a 3D Dirac particle in a constant electric field and show that, semiclassically, the particle admits, to order \(c^{-2}\), the anomalous velocity relation

\[
m \frac{dX_i}{dt} \approx P_i - \frac{1}{2} \frac{e}{mc^2} \epsilon_{ijk} \sigma_j E_k \tag{5.1}
\]

[supplemented with the Lorentz force law \(\dot{P}_i = eE_i\)], where \(\sigma\) is the spin vector. Assuming cylindrical symmetry and spin-polarized electrons, \(\sigma_i = -s\delta_{i3}\), the JN limit \(s/m^2c^2 \to \theta\) yields

\[
m \frac{dX_i}{dt} \approx P_i - \frac{1}{2} \frac{e}{cm} \theta \epsilon_{ij} P_i E_j, \tag{5.2}
\]

which is the first equation in (3.4) with \(B = 0\) and with anomalous gyromagnetic factor \(g = 1\). This value has already been found before [21]. To leading order in \(c^{-1}\), the relativistic Hamiltonian behaves as

\[
\tilde{H} \approx mc^2 + \frac{\vec{P}^2}{2m} - e\vec{E} \cdot \vec{X} \tag{5.3}
\]

\(\text{cf. (3.6)}\) Note that the naive Hamilton equation, \(\dot{X}_i = \partial H/\partial P_i\), would contain a factor \((+1/2)\) instead of \((-1/2)\) in front of the anomalous term in (5.2). The correct coefficient is recovered when the exotic part is taken into account. Either of the Hamiltonian structures

\[
\{X_i, X_j\}_\alpha = (1 - \alpha) \theta \epsilon_{ij}, \quad \{X_i, P_j\}_\alpha = \delta_{ij}, \quad \{P_i, P_j\}_\alpha = 0, \tag{5.4}
\]

\[
H_\alpha = \frac{\vec{P}^2}{2m} - e\vec{E} \cdot \vec{X} + \left( \frac{1}{2} - \alpha \right) e\theta \epsilon_{ij} P_i E_j, \tag{5.5}
\]

\(\text{3The } Q \neq 0 \text{ case could be studied starting with the “particle with torsion” [19].}\)
yields indeed the correct equations for any value of the real parameter $\alpha$. (3.5)-(3.6) corresponds to $\alpha = 0$, and (3.8)-(3.9)-(3.10)-(3.11) corresponds to $\alpha = 1/2$, respectively.

6 Further generalizations

A slightly modified model is obtained replacing the momentum in (3.1), $P_i$, by the velocity, $\dot{X}_i$:

$$L'_\text{anom} = \mu B - \frac{g}{2} me\theta \epsilon_{ij} \dot{X}_i E_j,$$

(6.1)

Magnetic moment interaction of such kind has been considered before [18]. Eqn. (6.1) is also reminiscent of the interaction of a magnetic moment with an electric charge [22].

Adding (6.1) to our Lagrangian (2.5) amounts indeed to changing the potentials in (2.6)-(2.7)-(2.8) according to

$$A_0 \rightarrow A'_0 = A_0 + \left(\frac{\mu}{e}\right) B,$$

$$A_i \rightarrow A'_i = A_i - \left(\frac{mg\theta}{2}\right) \epsilon_{ij} E_j.$$

Eliminating the momenta in the new equations of motion and dropping terms which contain second derivatives of the field, we obtain

$$\frac{d}{dt} (m^* \dot{X}_i) = e(B \epsilon_{ij} \dot{X}_j + E_i) + \mu \partial_t B - me\theta \epsilon_{ij} \frac{dE_j}{dt} + \frac{emg\theta}{2} \epsilon_{ij} (\dot{X}_j \partial_k E_k + \partial_t E_j).$$

(6.2)

with the new magnetic field, $B'$, replacing $B$ in the new effective mass, $m \rightarrow m^* = m(1 - e\theta B')$.

For the sake of comparison, neglecting terms which are higher-order in the fields, from (3.2-3.3), we would get instead

$$\frac{d}{dt} (m^* \dot{X}_i) = e(B \epsilon_{ij} \dot{X}_j + E_i) + \mu \partial_t B - me\theta \epsilon_{ij} \frac{dE_j}{dt} + eE_i,$$

(6.3)

This is readily transformed into the form (6.2). In a weak and slowly varying field, the two models only differ in the form of the effective mass.

It is worth remembering that anomalous velocity relations of the type studied here have been considered in the context of the Anomalous Hall Effect [23] and in the semiclassical theory of the Bloch electron [24]. Equations (2.6)-(2.7), or their “anomalous” generalization in constant external fields, (3.4), is indeed a special case of the more general system

$$\dot{X}_i + \theta(\vec{P}) \epsilon_{ij} \dot{P}_j = \partial_{P_i} \mathcal{E},$$

(6.3)

$$eB \epsilon_{ij} \dot{X}_j - \dot{P}_i = -eE_i,$$

(6.4)

where $\mathcal{E} = \mathcal{E}_0(\vec{P}) - B\mathcal{M}(\vec{P})$ is the total energy with $\mathcal{E}_0$ and $\mathcal{M}$ denoting the Bloch band energy and the magnetization, respectively. These equations can be derived, under quite general assumptions, by semiclassical calculations applied to the dynamics of wave packets in a two-dimensional crystal [24]. Note that the non-commutative parameter has been promoted to a function of the momentum [25].

The system (6.3-6.4) can actually be reduced to first order equations for the $P_i$ alone,

$$(1 - eB\theta(\vec{P})) \dot{P}_i = eB \epsilon_{ij} \partial_{P_j} \mathcal{E} + eE_i,$$

(6.5)

that can be integrated by solving with respect to $P_1$, say, using the conserved quantity

$$\mathcal{C} = \mathcal{E} - \frac{\epsilon_{ij} P_i E_j}{B}.$$

(6.6)
Thus the problem is reduced to quadratures. Note that eqn. (6.5) is actually Hamilton’s equation for $C$ as Hamiltonian and Poisson bracket (3.5c) in $P$-space alone.

In conclusion, we mention that another way of introducing anomalous coupling for constant e.m. fields has been advocated by us in [4]. There we introduced an “enlarged” planar Galilei group, which incorporates field variables besides space-time. Interestingly, the square of (6.6) is proportional to a Casimir of the enlarged symmetry algebra in [4], and anomalous coupling can then be achieved by adding this Casimir to the Hamiltonian.

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