$Λ \to 0$ limit of 2+1 Quantum Gravity for arbitrary genus

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Abstract

The abstract quantum algebra of observables for 2+1 gravity is analysed in the limit of small cosmological constant. The algebra splits into two sets with an explicit phase space representation; one set consists of $6g - 6$ commuting elements which form a basis for an algebraic manifold defined by the trace and rank identities; the other set consists of $6g - 6$ tangent vectors to this manifold. The action of the quantum mapping class group leaves the algebra and algebraic manifold invariant. The previously presented representation for $g = 2$ is analysed in this limit and reduced to a very simple form. The symplectic form for $g = 2$ is computed.

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1. Introduction.

For some time we have been analysing the abstract quantum commutator algebra for $2+1$ gravity with negative cosmological constant $\Lambda$ [1,2]:

\[
[a_{mk},a_{jl}] = [a_{mj},a_{kl}] = 0 \quad (1.1)
\]

\[
[a_{jk},a_{km}] = \left(\frac{1}{K} - 1\right)(a_{jm} - a_{jk}a_{km}) \quad (1.2)
\]

\[
[a_{jk},a_{kl}] = \left(1 - \frac{1}{K}\right)(a_{jl} - a_{kl}a_{jk}) \quad (1.3)
\]

\[
[a_{jk},a_{lm}] = \left(K - \frac{1}{K}\right)(a_{jl}a_{km} - a_{kl}a_{jm}) \quad (1.4)
\]

where $K = e^{-2i\theta}$, $\tan \theta = -\frac{\hbar}{8\alpha}$, $\Lambda = -\frac{1}{\alpha^2}$ is the cosmological constant and $\hbar$ is Planck’s constant. In (1.1-4) $m, j, l, k$ are 4 anticlockwise points of a $n$-gon (see [1]) labelled anticlockwise $m, j, l, k = 1 \cdots n$, and the time independent quantum operators $a_{lk}$ correspond to the classical $\frac{n(n-1)}{2}$ gauge invariant trace elements

\[
\alpha_{ij} = \alpha_{ji} = \frac{1}{2} \text{Tr}\left(S(t_it_{i+1}\cdots t_{j-1})\right), \quad S \in SL(2,R) \quad (1.5)
\]

The map $S: \pi_1(\Sigma) \overset{s}{\rightarrow} SL(2,R)$ is defined by the integrated anti-De Sitter connection in the initial data Riemann surface $\Sigma$ of genus $g$ when $n = 2g + 2$, and refers to one of the two spinor components, say the upper component, of the spinor group $SL(2,R) \otimes SL(2,R)$ of the gauge group $SO(2,2)$ of $2+1$ gravity with negative cosmological constant [2]. The lower component yields an independent algebra of traces $b_{ij}$ identical to (1.1-4) but with $K \rightarrow 1/K$. Moreover $[a_{ij},b_{kl}] = 0 \forall i,j,k,l$.

The algebra (1.1-4) is invariant under the quantum action of the mapping class group on traces [1], provided that the operators in (1.1-4) are ordered with the convention that $d(a_{ij})$ is increasing from left to right where $d(a_{ij}) = \frac{(i-1)(2n-2-i)}{2} + j - 1$.

In previous articles we only considered the upper component, which we now summarise. The homotopy group $\pi_1(\Sigma)$ of the surface is defined [1] by generators
$t_i, i = 1 \cdots 2g + 2$ and presentation:

$$t_1 t_3 \cdots t_{2g+1} = 1, \ t_2 t_4 \cdots t_{2g+2} = 1, t_1 t_2 \cdots t_{2g+2} = 1 \quad (1.6)$$

The last relator in (1.6) implies that $\Sigma$ is closed.

The algebra (1.1-4) seems overprescribed since the number of elements $a_{ij}$ is $n(n-1)/2 = (g+1)(2g+1)$. In [3] we determined for $n \leq 6$, i.e. $g \leq 2$ a set of $p$ linearly independent central elements $A_{nm}, m = 1 \cdots p$ where $n = 2p$ or $n = 2p + 1$, and analysed the trace identities which follow from the presentation (1.6) of the homotopy group $\pi_1(\Sigma)$ and a set of rank identities. These identities together generate a two-sided ideal, from which it was deduced that for generic $g$ there are precisely $6g-6$ independent elements for each ($\pm$) algebra. The reduction from $n(n-1)/2 = (g+1)(2g+1)$ to $6g-6$ results from the use of the above mentioned identities but is highly non unique. In [4] this was implemented explicitly for $g = 2$ reducing the number of independent variables for the (+) algebra from 15 to 6, and the representations discussed.

In this paper we analyse this quantum theory (the traces $a_{ij}$ and the commutator algebra (1.1-4)) in the limit of small cosmological constant through an expansion around $\sqrt{-\Lambda} = 0$, for arbitrary genus $g$. Note that the theory was first formulated for $\Lambda = 0$ [5], and then generalised. In [6] similar results were obtained for $g = 1$ by first taking the limit of the classical (Poisson bracket) algebra and then quantising. Only traces corresponding to paths with intersections $0, \pm 1$ (eq.1.1-3) were considered. The limit as $\Lambda \to 0$ of the exact classical solution for $g = 1$ was also studied in [7]. Here we show that for arbitrary genus $g$ the two sets (the $a_{ij}$ and $b_{ij}$ or the $\pm$) of $6g-6$ variables split into another two $6g-6$ sets with an explicit phase space representation. One of these consists of $6g-6$ commuting elements which form a basis for an algebraic manifold defined by the trace and rank identities, the other set consists of $6g-6$ tangent vectors. These and the action of the quantum mapping
class group which leaves the algebra and algebraic manifold invariant are presented in Section 2. A representation is presented in Section 3. In Section 4 the previously presented representation for \( g = 2 \) is analysed in this limit and reduced to a very simple form, and the symplectic form for \( g = 2 \) is computed.

2. Separation of variables for small \( \Lambda \)

To discuss the limit of small cosmological constant it is necessary, as for \( g=1 \) [6,7], to consider simultaneously the upper and lower spinor components. From now on denote \( a_{ik} \) by \( a^+_{ik} \) and \( b_{ik} \) by \( a^-_{ik} \) and set

\[
a^\pm_{ij}(\theta) = s_{ij} \pm \theta \ t_{ij} + O(\theta^2) \tag{2.1}
\]

with \( s_{ij} = s_{ji}, \ t_{ij} = t_{ji} \) independent of \( \theta \) and \( s_{ii} = 1, t_{ii} = 0 \). The expression (2.1) therefore corresponds to an expansion of the \( a_{ij} \) to \( O(\Lambda) \) around \( \Lambda = 0, \ K = 1, \ \theta = 0 \).

The algebra of these variables can be calculated directly from (1.1-4) or by noting that they can be expressed as

\[
a^\pm_{ij}(0) = s_{ij}, \quad \frac{da^\pm_{ij}}{d\theta}(0) = \pm t_{ij} \tag{2.2}
\]

By repeated differentiation with respect to \( \theta \) in the limit \( \theta \to 0 \) it follows from (1.1-4) that all the \( s_{ij} \) commute amongst themselves whereas the \( s \) and \( t \) variables satisfy the commutators

\[
[s_{jk}, t_{km}] = -[s_{km}, t_{jk}] = i \ (s_{mj} - s_{jk} s_{km}) \tag{2.3}
\]

\[
[t_{mj}, t_{jl}] = i \ (t_{ml} - t_{jl} s_{mj} - t_{mj} s_{jl})
\]

\[
= i \ (t_{ml} - s_{mj} t_{jl} - s_{jl} t_{mj}) \tag{2.4}
\]

\[
[s_{ml}, t_{jk}] = -[s_{jk}, t_{ml}] = 2i \ (s_{jl} s_{mk} - s_{mj} s_{kl}) \tag{2.5}
\]

\[
[t_{ml}, t_{jk}] = 2i \ (s_{jl} t_{mk} - s_{mj} t_{kl} - s_{kl} t_{mj} + s_{mk} t_{jl}) \tag{2.6}
\]
with all other commutators zero. As for the algebra (1.1-4), m,j,l,k are any 4 anticlockwise points (e.g., 1,2,3,4 or 2g + 2,1,4,5 etc) of the polygon representation (see [1]). Some comments are in order.

The first two commutators (2.3-4) correspond to paths on \( \Sigma \) with single intersections, already reported in [5] *. The last two commutators (2.5-6) correspond to paths on \( \Sigma \) with double intersections, which could be determined from the Jacobi identity which is satisfied for all triples. The first equalities of (2.3) and (2.5) follow from

\[
[a^+_{jk}, a^-_{km}] = [a^+_{ml}, a^-_{jk}] = 0
\]

whereas the commutators (2.4) and (2.6) between two \( t \)'s can be deduced from the Jacobi identity on a \( (t, t, s) \) triple.

The commuting \( s \) variables satisfy the same trace and rank identities as the \( a_{ij} \) variables [3] with \( \theta = 0 \). Given just one identity all others can be obtained by repeated commutation with the elements \( t_{ij} \) of the algebra (2.3-6), or equivalently, by repeated application of

\[
[t_{uv}, s_{jk}] \frac{\partial I(s)}{\partial s_{uv}}
\]

(to be summed over \( u, v \)). The set of commuting \( s \) variables can be used as a basis for the algebraic manifold defined by \( I(s) = 0 \).

* The commutators (2.3-4) were first computed for \( g = 1 \) in [5]. Equations (2.3-4) here correspond to equations (5.4-5) of [5] with the following identifications; the variables \( q, \nu \) of [5] correspond here to \( s, t \) respectively and the labelling of the paths \( jk, km \) (or \( mj, jl \)) on \( \Sigma \) to \( u, v \). To get precisely equations (2.3-4) it is necessary to use the \( SL(2, R) \) identities

\[
\nu(uv^{-1}) + \nu(uv) = 2(q(v)\nu(u) + q(u)\nu(v)), \quad q(uv^{-1}) + q(uv) = 2q(u)q(v).
\]
Similar identities for the $t$ variables are obtained from the $s$ identities $I(s)$ as

$$I(t) = \frac{\partial I(s)}{\partial s_{uv}} t_{uv}$$

(again, to be summed over all $u, v$). It follows that the number of $t$ identities $I(t)$ is equal to the number of $s$ identities $I(s)$ so that there are equal numbers of independent $s$ and $t$ variables. These identities are certainly not all independent. In fact they form an ideal under commutation.

Under the action of the quantum Dehn group with elements $D_{ij} = D_{ji}, i, j = 1 \cdots 2g + 2$ the $s$ and $t$ variables transform as follows:

$$D_{ml} : t_{ml} \rightarrow t'_{ml} = t_{ml}$$

$$t_{kl} \rightarrow t'_{kl} = t_{mk}$$

$$t_{mj} \rightarrow t'_{mj} = t_{jl}$$

$$t_{mk} \rightarrow t'_{mk} = 2(s_{mk}t_{ml} + s_{ml}t_{mk}) - t_{lk}$$

$$t_{jl} \rightarrow t'_{jl} = 2(s_{jl}t_{ml} + s_{ml}t_{jl}) - t_{mj}$$

$$t_{jk} \rightarrow t'_{jk} = -2(s_{mj}t_{mk} + s_{mk}t_{mj} + s_{jl}t_{kl} + s_{kl}t_{jl})$$

$$+ 4(s_{ml}s_{mk}t_{jl} + s_{ml}s_{jl}t_{mk} + s_{ji}s_{mk}t_{ml}) + t_{jk}$$

$$s_{ml} \rightarrow s'_{ml} = s_{ml}$$

$$s_{kl} \rightarrow s'_{kl} = s_{mk}$$

$$s_{mj} \rightarrow s'_{mj} = s_{jl}$$

$$s_{mk} \rightarrow s'_{mk} = 2s_{ml}s_{mk} - s_{kl}$$

$$s_{jl} \rightarrow s'_{jl} = 2s_{ml}s_{jl} - s_{mj}$$

$$s_{jk} \rightarrow s'_{jk} = -2(s_{mj}s_{mk} + s_{kl}t_{jl}) + 4s_{ml}s_{jl}s_{mk} + s_{jk}$$

and leave the algebra (2.3-6) and the ideal of $s$ and $t$ identities invariant. The
transformations (2.7) can be simplified by noting that

\[ t_{pq} = \frac{\partial s_{pq}}{\partial s_{uv}} t_{uv} \quad \text{(sum over } u, v) \]

for generic indices (points of the polygon) \( p, q, u, v \). The \( D_{ml} \) as expressed in (2.7) satisfy the identities of the Braid group \( B(2g + 2) \) [8,9].

3. Representations.

The algebra (2.3-6) admits the following representation;

Let the set of the basis \( s_{ij} \) variables act as configuration space variables by multiplication, and let the \( t_{ij} \) variables act by differentiation

\[ t_{ij} = C_{ij,kl}(s) \frac{\partial}{\partial s_{kl}} \quad \text{(sum over } k, l) \quad (3.1) \]

where

\[ C_{ij,kl}(s) = [t_{ij}, s_{kl}] = -C_{kl,ij}(s) \]

is at most quadratic in \( s \). Therefore the \( t_{ij} \) can be considered as tangent vectors to the algebraic manifold with basis \( s_{ij} \).

4. Representation for \( g = 2 \).

For \( g = 2 \), each \((\pm)\) algebra (1.1-4) consists of 15 elements \( a_{ij}^{\pm} \) but by use of computer algebra we were able [4] to reduce to \( 6 = 6g - 6 \) independent variables for each \((\pm)\) as follows, by satisfying all the trace and rank identities. A convenient choice for the 6 independent elements is given by 6 commuting angles \( \varphi_{-b}^{\pm} = -\varphi_{b}^{\pm}, \quad b = \pm1 \cdots \pm3 \) with

\[ \varphi_{a}^{\pm}(K) = \varphi_{a}^{-}\left(\frac{1}{K}\right) \]

and

\[ a_{12}^{\pm} = \frac{\cos \varphi_{1}^{\pm}}{\cos \theta}, \quad a_{34}^{\pm} = \frac{\cos \varphi_{2}^{\pm}}{\cos \theta} \quad a_{56}^{\pm} = \frac{\cos \varphi_{3}^{\pm}}{\cos \theta} \quad (4.1) \]
with corresponding commuting momenta $\pi^\pm_a, a = 1, 2, 3$ with

$$\pi^+_a(K) = \pi^-_a\left(\frac{1}{K}\right)$$

and the only non-zero commutators

$$[\varphi^\pm_a, \pi^\pm_b] = \pm 2i\theta\delta_{ab}, \ a, b = 1, 2, 3 \quad (4.2)$$

In the following the single indices 1, 2, 3 refer to the three commuting sectors 12, 34, 56 whereas the double indices refer to two points of the hexagon.

It can be checked that the 24 remaining $a^\pm_{ik}$ can be expressed in terms of the $\varphi^\pm_a$ and their conjugate momenta $\pi^\pm_a$. For example the quantum, ordered operator $a^+_{23}$ can be expressed as

$$a^+_{23} = \cos\pi^+_1 \cos\pi^+_2 + \left(\cot\varphi^+_1 \cot\varphi^+_2 - \frac{\cos\varphi^+_3}{\cos\theta \sin\varphi^+_1 \sin\varphi^+_2}\right) \sin\pi^+_1 \sin\pi^+_2 - i \tan\theta \left(\cot\varphi^+_2 \cos\pi^+_1 \sin\pi^+_2 + \cot\varphi^+_1 \cos\pi^+_2 \sin\pi^+_1\right) \quad (4.3)$$

so that all the 30 $a^\pm_{ik}$ are functions of $K = e^{-2i\theta}$, and the six conjugate pairs $\varphi^\pm_a, \pi^\pm_a, a = 1, 2, 3$. In [4] it was shown that the requirement that this representation of all the $a^\pm_{ik}$ by hermitian operators determines the norm in the Hilbert space spanned by the $\cos\varphi_a$, and restricts the range of the $\varphi_a$. There is some evidence [10-11] that the trace holonomies (4.1) should be unbounded (hyperbolic) in agreement with the partial results of [4].

In the limit $\Lambda \to 0$ the relationship of these variables to those of Section 1 is as follows. Let

$$\varphi^\pm_a = Q_a \mp \theta \ p_a, \quad \pi^\pm_b = q_b \pm \theta \ P_b \quad (4.4)$$
where all \( Q_a, q_a, P_b \) and \( p_b \) are independent of \( \theta^* \), given by

\[
Q_a = \varphi_a^+(0) = \varphi_a^-(0), \quad p_a = -\frac{\partial \varphi_a^+}{\partial \theta}(0) = \frac{\partial \varphi_a^-}{\partial \theta}(0)
\] (4.5)

Their conjugate variables are, respectively,

\[
P_b = \frac{\partial \pi^+_b}{\partial \theta}(0) = -\frac{\partial \pi^-_b}{\partial \theta}(0) \quad q_b = \pi^+_b(0) = \pi^-_b(0)
\] (4.6)

with the only non-zero commutators following from (4.2)

\[
[P_a, Q_b] = [p_b, q_a] = -i\delta_{ab}
\] (4.7)

so that from (4.1) we have directly the six independent variables**

\[
s_{12} = \cos Q_1, \quad s_{34} = \cos Q_2 \quad s_{56} = \cos Q_3
\] (4.8)

and

\[
t_{12} = p_1 \sin Q_1, \quad t_{34} = p_2 \sin Q_2, \quad t_{56} = p_3 \sin Q_3
\] (4.9)

All of the commuting \( s_{ij} \) variables are expressed in terms of the \( q_a \) and \( Q_a \), the notation has obviously been chosen because of its suggestive nature e.g. \( s_{23} \) is, from (4.3) given by

\[
s_{23} = \cos q_1 \cos q_2 + T_{12,3} \sin q_1 \sin q_2
\] (4.10)

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* A similar phenomenon occurs for \( g = 1 \) [7]. Here the combinations \( Q_a = \frac{\varphi_a^+ + \varphi_a^-}{2}, \ p_a = \frac{\varphi_a^+ - \varphi_a^-}{2g}, \ q_a = \frac{\pi_a^+ + \pi_a^-}{2} \) and \( P_a = \frac{\pi_a^+ - \pi_a^-}{2g} \) remain finite and constant in the limit \( \theta \to 0 \).

** Three of the six independent variables of [12] for \( \Lambda = 0 \) and \( g = 2 \) should be compared with (4.8). These correspond, in our notation, to the traces of the representation of elements 16, 25 and 34 of the hexagon (see [4] for details). They can be obtained from our 12, 34, and 56 variables by the Dehn transformation \( D_{15} \) (eq.(2.7)).
where
\[ T_{ij,k} = \cot Q_i \cot Q_j - \frac{\cos Q_k}{\sin Q_i \sin Q_j} \quad i, j, k \text{ cyclical.} \]

Note that all of the \( s_{ij} \) can be expressed in this form, e.g. from (4.8)
\[ s_{12} = \cos Q_1 = \cos Q_2 \cos Q_3 - T_{23,1} \sin Q_2 \sin Q_3 \]
and
\[ s_{13} = \cos(q_1 + Q_1) \cos q_2 + T_{12,3} \sin(q_1 + Q_1) \sin q_2 \]
\[ s_{24} = \cos q_1 \cos(q_2 - Q_2) + T_{12,3} \sin q_1 \sin(q_2 - Q_2) \]
\[ s_{14} = \cos(q_1 + Q_1) \cos(q_2 - Q_2) + T_{12,3} \sin(q_1 + Q_1) \sin(q_2 - Q_2) \]
\[ s_{25} = \cos(q_1 - Q_1) \cos(q_3 + Q_3) + T_{31,2} \sin(q_1 - Q_1) \sin(q_3 + Q_3) \]
\[ s_{36} = \cos(q_2 + Q_2) \cos(q_3 - Q_3) + T_{23,1} \sin(q_2 + Q_2) \sin(q_3 - Q_3) \]

These variables have striking familiarity and are clearly related to those presented in the Appendix of [13]. All of the remaining \( s_{ij} \) can be obtained from the above by cyclical rotation of the sector indices 1,2,3, which corresponds to a cyclical rotation of the hexagon by 2 points (e.g. \( s_{35} \) is obtained from \( s_{13} \) etc).

A general formula for the operators \( t_{ij} \) in terms of the operators \( Q_a, q_a, P_a, p_a \), ordered with all momenta to the right and consistent with (3.1) is
\[ t_{ij} = i \left( \frac{\partial s_{ij}}{\partial Q_a} \frac{\partial}{\partial q_a} - \frac{\partial s_{ij}}{\partial q_a} \frac{\partial}{\partial Q_a} \right) = \frac{\partial s_{ij}}{\partial q_a} P_a - \frac{\partial s_{ij}}{\partial Q_a} p_a \]
(4.11)
when the commutators (4.7) are represented by
\[ p_a = -i \frac{\partial}{\partial q_a}, \quad P_a = -i \frac{\partial}{\partial Q_a} \]
(4.12)

One example is
\[ t_{23} = \]
\[ \left( \frac{T_{12,3} \cos q_1 \sin q_2}{\sin Q_1 \sin Q_2} - \cos q_2 \sin q_1 \right) P_1 + \left( \frac{T_{12,3} \cos q_2 \sin q_1}{\sin Q_1 \sin Q_2} - \cos q_1 \sin q_2 \right) P_2 \]
\[ - \frac{T_{13,2} \sin q_1 \sin q_2}{\sin^2 Q_1 \sin Q_2} P_1 - \frac{T_{23,1} \sin q_1 \sin q_2}{\sin Q_1 \sin^2 Q_2} P_2 - \frac{\sin q_1 \sin q_2 \sin Q_3}{\sin Q_1 \sin Q_2} P_3 \]
Clearly (4.11) gives (4.9) for $ij = 12, 34, 56$, and

$$[t_{ij}, s_{ij}] = -i\left(\frac{\partial s_{ij}}{\partial Q_a} \frac{\partial s_{ij}}{\partial q_a} - \frac{\partial s_{ij}}{\partial q_a} \frac{\partial s_{ij}}{\partial Q_a}\right) = 0$$

It can be checked that the algebra (2.3-6) and the identities $I(s)$ and $I(t)$ are satisfied by all 15 $s_{ij}$ and all 15 $t_{ij}$. The identities $I(s)$ can all be derived from, for example

$$s_{12}s_{34} + s_{23}s_{14} - s_{13}s_{24} - s_{56} = 0$$
$$s_{12}s_{46} + s_{24}s_{16} - 2s_{34}s_{45} - s_{14}s_{26} + s_{35} = 0$$
$$2(2s_{34}s_{56}s_{45} - s_{34}s_{46} - s_{56}s_{35})$$
$$+ s_{14}s_{25} - s_{12}s_{45} - s_{24}s_{15} + s_{36} = 0$$

and their images under cyclical permutations of the indices $1 \cdots 6$.

The symplectic volume form on the algebraic manifold defined by $I(s) = 0$ expressed as

$$dq_a \wedge dQ_a$$

(4.13)
can be simplified by noting that the $t_{ij}$ can be identified with the differentials $ds_{ij}$ as follows; from

$$dI(s) = \frac{\partial I(s)}{\partial s_{ij}} ds_{ij} = \frac{\partial I(s)}{\partial s_{ij}} t_{ij}$$

then (4.11) should be compared with

$$ds_{ij} = \frac{\partial s_{ij}}{\partial q_a} dq_a + \frac{\partial s_{ij}}{\partial Q_a} dQ_a$$

and implies the identifications

$$dQ_a = -p_a \quad \text{or} \quad dq_a = P_a$$

so the symplectic form (4.13) can be written simply as

$$P_a \wedge dQ_a = p_a \wedge dq_a$$

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The commuting \( q_a \) and \( Q_a, a = 1, 2, 3 \) form a basis for configuration space, whereas the commuting momentum (tangent space) variables \( p_a \) and \( P_a \) are given by (4.12).

The action of the mapping class group (2.7) for \( g = 2 \) has been computed explicitly only on the cosines of the variables \( q_a \) and \( Q_a \), on the sines it is very complicated and not useful. A simple example however is given by the transformation \( D_{2j-1,2j}, j = 1, 2, 3 \), generated classically on \( \pi_1(\Sigma) \) by

\[
t_2 \to t_1 t_2 \quad t_6 \to t_6 t_1^{-1}
\]

and on the traces of holonomies by the map (canonical transformation) which leaves invariant the volume form (4.13)

\[
D_{2j-1,2j}; q_j \to q_j + Q_j
\]

with inverse

\[
D^{-1}_{2j-1,2j}; q_j \to q_j - Q_j
\]

and can be identified with a Dehn twist [14] provided the \( Q_a \) play the role of the length variables and the \( q_a \) are the angle variables for a given closed path on \( \Sigma \). The full action of this group is under study and will be reported elsewhere.

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