September 10, 2014

FINITE VOLUME CORRECTIONS AND DECAY OF CORRELATIONS IN THE CANONICAL ENSEMBLE

ELENA PULVIRENTI AND DIMITRIOS TSAGKAROGIANNIS

ABSTRACT. We consider a classical system of \(N\) particles confined in a box \(\Lambda \subset \mathbb{R}^d\) interacting via a finite range pair potential. Given the validity of the cluster expansion in the canonical ensemble we compute the error between the finite and the infinite volume free energy and estimate it to be bounded by the area of the surface of the box’s boundary over its volume. We also compute the truncated two-point correlation function and find that the contribution from the ideal gas case is of the order \(1/N\) while the contribution of the interactions is exponentially small with the distance.

1. Introduction

A common practise of mathematical methods in physics is to consider idealized situations by taking limits of the number of particles and/or the volume of the system to infinity to study the limiting thermodynamic quantities. However, in many situations of both theoretical (e.g. in the construction of coarse-grained Hamiltonians as in the Lebowitz - Penrose theory) and practical use (when dealing with realistic systems and computer simulations) one is also interested in obtaining exact estimates of the error between the infinite and the finite volume version of these quantities. It is a longstanding problem to compute the error terms between the logarithm of the (canonical or grand canonical) partition function and the limiting pressure or free energy. In the special case of the validity of the cluster expansion such questions can be answered; see for example [4], [2], [10] for some cases valid mainly for lattice systems or the grand canonical ensemble. In the present paper we work in the context of continuous systems in the canonical ensemble and we calculate the finite volume corrections to the free energy for both cases of periodic and zero (or general) boundary conditions and estimate the relevant error. The main technical tool is the cluster expansion whose validity has been established in a previous work [9]. However, as it will be explained, its implementation for answering the above question still requires to overcome several technical issues. Moreover, as another application of the validity of the cluster expansion we also investigate the decay of the two-point correlation function as the distance between two particles increases.

The structure of the paper is as follows: in Section 2 we present the model and the results. Then, for completeness of the presentation, in Section 3 we give the basic ideas of the cluster expansion in the canonical ensemble. In the same section we also give the outline of the proof of the finite volume corrections explaining why it is not a direct application of the existing cluster expansion result and instead a new more involved expansion has to be devised. We
present the proof in the subsequent three sections. In Section 4 we develop the new version of the cluster expansion considering as polymers rooted subsets of the set of labels of the particles carrying the extra information whether the root is close to the boundary or not. We apply the general theorem of cluster expansion for the new polymers and conclude the proof in two steps. We first show in Section 5 that the finite volume error for the polymers not vanishing in the thermodynamic limit is of the order of the boundary divided by the volume. Second, in Section 6, we prove that the remaining (thermodynamically irrelevant) terms give a lower order contribution. Last, in Section 7 we give the proof of the decay of the two-point correlation function.

2. THE MODEL AND THE RESULTS

We consider a configuration $q \equiv \{q_1, \ldots, q_N\}$ of $N$ particles (where $q_i$ is the position of the $i^{th}$ particle) confined in a box $\Lambda(\ell) := (-\frac{\ell}{2}, \frac{\ell}{2})^d \subset \mathbb{R}^d$ (for some $\ell > 0$), which we will also denote by $\Lambda$ when we do not need to explicit the dependence on $\ell$. The particles interact via a pair potential $V : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$, which is stable and of finite range. Stability means that there exists $B \geq 0$ such that:

$$\sum_{1 \leq i < j \leq N} V(q_i - q_j) \geq -BN,$$

for all $N$ and all $q_1, \ldots, q_N$. Finite range ($R > 0$) means that $V(q_i - q_j) = 0$ if $|q_i - q_j| > R$, where $|q_i - q_j|$ denotes the euclidean distance between two particles at positions $q_i$ and $q_j$. The requirement of finite range is only for technical reasons and will be clear in the sequel. However, a similar result should be true under the hypothesis of temperedness:

$$C(\beta) := \int_{\mathbb{R}^d} |e^{-\beta V(q)} - 1|dq < \infty,$$

but the proof is more involved and it is beyond the scope of the present paper. For our case of a potential with finite range $R > 0$ we ask for the following condition:

$$C(\beta, R) := \int_{B_R(0)} |e^{-\beta V(q)} - 1|dq < \infty,$$

where $B_R(0)$ is the $d$-dimensional ball with center 0 and radius $R$. A typical example of pair potential with the above features is the hard-core interaction given by:

$$V^{hc}(q_i - q_j) = \begin{cases} +\infty, & \text{if } |q_i - q_j| \leq R \\ 0, & \text{if } |q_i - q_j| > R \end{cases}$$

Note that in this case $C(\beta, R)$ would be the volume of $B_R(0)$.

In the case of periodic boundary conditions, the canonical partition function of the system is given by

$$Z_{\beta,\Lambda,N}^{\text{per}} := \frac{1}{N!} \int_{\Lambda^N} dq_1 \ldots dq_N e^{-\beta H_\Lambda(q)}.$$


where $H_\Lambda$ is the energy of the system:

$$H_\Lambda(q) = \sum_{1 \leq i < j \leq N} V^{per}(q_i, q_j).$$

(2.6)

The potential $V^{per}$ captures the periodic boundary conditions and for the case of finite range interaction it is given by

$$V^{per}(q_i, q_j) := \sum_{n=(n_1, \ldots, n_d)} V(q_i - q_j + n \ell).$$

(2.7)

We will also consider the case of zero boundary conditions, i.e., outside the domain $\Lambda$ there are no particles, which is described by the Hamiltonian: $H_\Lambda(q) = \sum_{1 \leq i < j \leq N} V(q_i, q_j)$. We denote by $Z^{0}_{\beta,\Lambda,N}$ the corresponding partition function. Given $\rho > 0$ we define the thermodynamic free energy by

$$f_{\beta}(\rho) := \lim_{|\Lambda|, N \to \infty, N = [\rho |\Lambda|]} f_{\beta,\Lambda}(N), \text{ where } f_{\beta,\Lambda}(N) := -\frac{1}{\beta |\Lambda|} \log Z_{\beta,\Lambda,N},$$

(2.8)

where $|\Lambda| = \ell^d$ is the volume of $\Lambda$. It is a general result that the above limit is independent of the boundary conditions, hence (2.8) holds for both $Z^{per}_{\beta,\Lambda,N}$ and $Z^{0}_{\beta,\Lambda,N}$. We will use the notation $Z_{\beta,\Lambda,N}$ whenever the choice of boundary conditions is not relevant and we avoid to specify it. In [9], for the case of periodic boundary conditions, it has been proved that:

**Theorem 2.1.** There exists a constant $c_0 \equiv c_0(\beta, B) > 0$ independent of $N$ and $\Lambda$ such that if $\rho C(\beta) < c_0$ then

$$\frac{1}{|\Lambda|} \log Z^{per}_{\beta,\Lambda,N} = \frac{1}{|\Lambda|} \log \frac{|\Lambda|^N}{N!} + N \sum_{n \geq 1} F_{\beta,N,\Lambda}(n),$$

(2.9)

with $N = [\rho |\Lambda|]$ and where $Z^{per}_{\beta,\Lambda,N}$ is the canonical partition function with periodic boundary conditions. In the thermodynamic limit

$$\lim_{N, |\Lambda| \to \infty, N = [\rho |\Lambda|]} F_{\beta,N,\Lambda}(n) = \frac{1}{n + 1} \beta_n \rho^n,$$

(2.10)

for all $n \geq 1$. Here $\beta_n$ is given by

$$\beta_n := \frac{1}{n!} \sum_{g \in \mathcal{B}_{n+1}} \int_{\mathbb{R}^{dn}} \prod_{(i,j) \in E(g)} (e^{-\beta V(q_i-q_j)} - 1) dq_1 \ldots dq_{n+1}, \quad q_1 \equiv 0,$$

(2.11)

where $\mathcal{B}_{n+1}$ is the set of 2-connected graphs $g$ on $(n+1)$ vertices and $E(g)$ is the set of edges of the graph $g$. We define a 2-connected graph to be a connected graph which by removing any single vertex and all related edges remains connected.
Furthermore, there exist constants $C, c > 0$ such that, for every $N$ and $\Lambda$, the coefficients $F_{\beta,N,\Lambda}(n), n \geq 1$, which are given by

$$F_{\beta,N,\Lambda}(n) = \frac{1}{n+1} \left(\frac{N-1}{n}\right) \sum_{I : A(I) = [n+1]} c_I \zeta_I,$$

with

$$P_{N,\Lambda}(n) := \frac{(N-1) \ldots (N-n)}{|\Lambda|^n} \quad \text{and} \quad B_{\beta,\Lambda}(n) := \frac{|\Lambda|^n}{n!} \sum_{I : A(I) = [n+1]} c_I \zeta_I,$$

satisfy

$$|F_{\beta,N,\Lambda}(n)| \leq C e^{-cn}. \quad (2.14)$$

From the above theorem we can calculate the limit in (2.8) by applying the Dominated Convergence Theorem combining the limit (2.10) and the bound (2.14). We obtain that

$$\beta f_\beta(\rho) = \rho (\log \rho - 1) - \sum_{n \geq 1} \frac{1}{n+1} \beta_n \rho^{n+1}, \quad (2.15)$$

where $\beta_n$ is given in (2.11). In the present paper we want to prove a more delicate estimate, namely to calculate the terms which contribute to the finite volume corrections of the free energy and estimate them by the area of the surface $\partial \Lambda$ of $\Lambda$ divided by the volume of $\Lambda$. We denote by $| \cdot |$ the Lebesgue measure of either the surface or of the volume of $\Lambda$.

**Theorem 2.2.** There exists a constant $c'_0 \equiv c'_0(\beta, B) > 0$ independent of $N$ and $\Lambda$ such that if $\rho C(\beta, R) < c'_0$ there exist constants $\tilde{C}(\rho), \hat{C}(\rho) > 0$ such that

$$\left| \frac{1}{|\Lambda|} \log Z^\text{per}_{\beta,\Lambda,N} - \beta f_\beta(\rho) \right| \leq \tilde{C}(\rho) \frac{1}{|\Lambda|}, \quad (2.16)$$

for the case of periodic boundary conditions and

$$\left| \frac{1}{|\Lambda|} \log Z^0_{\beta,\Lambda,N} - \beta f_\beta(\rho) \right| \leq \hat{C}(\rho) \frac{|\partial \Lambda|}{|\Lambda|}, \quad (2.17)$$

for the case of zero (general) boundary conditions. Note also that $\rho = \frac{N}{|\Lambda|}$ and $f_\beta(\rho)$ is given in (2.15).

The proof of Theorem 2.2 will be outlined in Subsection 3.2 and detailed in Sections 4, 5 and 6.

Another byproduct of the cluster expansion is an expression for the truncated correlation functions.

**Definition 2.3.** Given a point $q_1 \in \Lambda$, the one-point correlation function is given by

$$\rho^{(1)}_{\Lambda,N}(q_1) := \frac{1}{(N-1)!} \int_{\Lambda_{N-1}} dq_2 \ldots dq_N \frac{1}{Z_{\beta,\Lambda,N}} e^{-\beta H_{\Lambda}(q)}, \quad (2.18)$$
Similarly, for \( q_1, q_2 \in \Lambda \), the two-point correlation function is given by
\[
\rho^{(2)}_{\Lambda,N}(q_1, q_2) := \frac{1}{(N-2)!} \int_{\Lambda^{N-2}} dq_3 \ldots dq_N \frac{1}{Z_{\beta,\Lambda,N}} e^{-\beta H_{\Lambda}(q)}.
\] (2.19)

Note that \( \frac{1}{N} \rho^{(1)}_{\Lambda,N}(q_1) dq_1 \) is the probability of having any particle in a volume \( dq_1 \) at position \( q_1 \) (among \( N \) particles). Indeed, if we consider periodic boundary conditions we have that
\[
\int_{\Lambda} \rho^{(1)}_{\Lambda,N}(q_1) dq_1 = N.
\]

Similarly, defining \( g^{(2)}_{\Lambda,N}(q) := \frac{1}{N} \rho^{(2)}_{\Lambda,N}(0, q) \) we can interpret the quantity
\[
\frac{1}{N-1} \rho \rho^{(2)}_{\Lambda,N}(0, q) dq
\]
as the probability of observing a second particle in a volume \( dq \) at position \( q \) given that there is already a particle at the origin 0. Note that using periodic boundary conditions we have that
\[
\int_{\Lambda} \rho g^{(2)}_{\Lambda,N}(q) dq = N(N-1) \frac{1}{\rho N} \int_{\Lambda} dq \int_{\Lambda^{N-2}} dq_3 \ldots dq_N \frac{1}{Z_{\beta,\Lambda,N}} e^{-\beta H_{\Lambda}(q)} = N - 1.
\]

**Remark 2.4.** In the case of the canonical ensemble the two-point correlation function does not factorize into the product of the one-point correlation functions, not even in the case of non-interacting particles. In fact, for the ideal gas we have that
\[
\rho^{(2)}_{\Lambda,N}(q_1, q_2) - \rho^{(1)}_{\Lambda,N}(q_1) \rho^{(1)}_{\Lambda,N}(q_2) = \frac{N(N-1)}{|\Lambda|^2} - \left( \frac{N}{|\Lambda|} \right)^2 = \frac{N}{|\Lambda|^2},
\] (2.20)
which indicates that we can not do better than \( 1/|\Lambda| \). However, this constraint gets relaxed if we label the particles and ask what is the probability that we find particle 2 at some position if we know that particle 1 is already somewhere. Indeed, if we label the particles one sees that since we want to put particle 2 in some position (and not any particle), it does not compete with the presence of particle 1 somewhere else and to first order (in the case of non-interaction) it will be \( 1/|\Lambda| \) for both of them. Thus, for the case of labelled interacting particles we expect to get that the error in the difference
\[
\rho^{(2),\text{lab}}_{\Lambda,N}(q_1, q_2) - \rho^{(1),\text{lab}}_{\Lambda,N}(q_1) \rho^{(1),\text{lab}}_{\Lambda,N}(q_2)
\] (2.21)
is exponentially small with respect to the distance \( |q_1 - q_2| \).

Following Remark 2.4, we define the labelled \( k \)-point correlation functions as follows (note that for convenience we have also normalized the volume integrals):

**Definition 2.5.** For any \( k = 1, 2, \ldots, N \) define
\[
\rho^{(k),\text{lab}}_{\Lambda,N}(q_1, \ldots, q_k) := \int_{\Lambda^{N-k}} \frac{dq_{k+1}}{|\Lambda|} \ldots \frac{dq_N}{|\Lambda|} \frac{1}{Z_{\beta,\Lambda,N}^{\text{int}}} e^{-\beta H_{\Lambda}(q)},
\] (2.22)
where
\[
Z_{\beta,\Lambda,N}^{\text{int}} := \int_{\Lambda^N} \frac{dq_1}{|\Lambda|} \ldots \frac{dq_N}{|\Lambda|} e^{-\beta H_{\Lambda}(q)},
\] (2.23)
For the case of the labelled correlation functions we obtain the following theorem:

**Theorem 2.6.** Let $q_1$ and $q_2$ be two fixed positions in the domain $\Lambda$. Then there exist two constants $C_1, C_2 > 0$ such that

$$|\rho_{\Lambda,N}^{(1),\text{lab}}(q_1)| \leq C_1 \quad (2.24)$$

and

$$|\rho_{\Lambda,N}^{(2),\text{lab}}(q_1, q_2) - \rho_{\Lambda,N}^{(1),\text{lab}}(q_1)\rho_{\Lambda,N}^{(1),\text{lab}}(q_2)| \leq C_2 e^{-R^{-1}|q_1-q_2|}, \quad (2.25)$$

where $\rho_{\Lambda,N}^{(2),\text{lab}}$ and $\rho_{\Lambda,N}^{(1),\text{lab}}$ are given in (2.22).

The proof will be given in Section 7. As a corollary we also obtain the case of the unlabelled correlation functions:

**Corollary 2.7.** Let $q_1$ and $q_2$ be two fixed positions in the domain $\Lambda$. Then there exist positive constants $C$ and $C'$ such that

$$|\rho_{\Lambda,N}^{(2)}(q_1, q_2) - \rho_{\Lambda,N}^{(1)}(q_1)\rho_{\Lambda,N}^{(1)}(q_2)| \leq \left( \frac{N}{|\Lambda|} \right)^2 C e^{-R^{-1}|q_1-q_2|} + C' \frac{1}{N} \left( \frac{N}{|\Lambda|} \right)^2, \quad (2.26)$$

where $\rho_{\Lambda,N}^{(2)}$ and $\rho_{\Lambda,N}^{(1)}$ are given in (2.19) and (2.18).

**Proof.** We have:

$$\rho_{\Lambda,N}^{(2)}(q_1, q_2) - \rho_{\Lambda,N}^{(1)}(q_1)\rho_{\Lambda,N}^{(1)}(q_2) = \frac{N(N-1)}{|\Lambda|^2} \rho_{\Lambda,N}^{(2),\text{lab}}(q_1, q_2) - \left( \frac{N}{|\Lambda|} \right)^2 \rho_{\Lambda,N}^{(1),\text{lab}}(q_1)\rho_{\Lambda,N}^{(1),\text{lab}}(q_2)$$

$$= \left( \frac{N}{|\Lambda|} \right)^2 \rho_{\Lambda,N}^{(2),\text{lab}}(q_1, q_2) - \rho_{\Lambda,N}^{(1),\text{lab}}(q_1)\rho_{\Lambda,N}^{(1),\text{lab}}(q_2) = -\frac{N}{|\Lambda|^2} \rho_{\Lambda,N}^{(2),\text{lab}}(q_1, q_2). \quad (2.27)$$

The first term is bounded as in Theorem 2.6, while for the second combining (2.24) and (2.25) we have:

$$|\rho_{\Lambda,N}^{(2),\text{lab}}(q_1, q_2)| \leq |\rho_{\Lambda,N}^{(2),\text{lab}}(q_1, q_2) - \rho_{\Lambda,N}^{(1),\text{lab}}(q_1)\rho_{\Lambda,N}^{(1),\text{lab}}(q_2)| + |\rho_{\Lambda,N}^{(1),\text{lab}}(q_1)\rho_{\Lambda,N}^{(1),\text{lab}}(q_2)| \leq C_2 + C_1^2, \quad (2.28)$$

which concludes the proof by choosing $C = C_2$ and $C' = C_2 + C_1^2$. \qed

3. Cluster expansion and strategy of proof of Theorem 2.2

In this section we briefly recall the cluster expansion in the canonical ensemble as proved in [9]. Elements of it will be used in the proofs of both Theorem 2.2 and Theorem 2.6. In the second part of this section, we give the strategy for the proof of Theorem 2.2. The full proof will be given in Sections 4, 5 and 6.
3.1. Cluster expansion in the canonical ensemble. We view the canonical partition function $Z_{\beta,A,N}$ as a perturbation around the ideal case, hence normalizing the measure by multiplying and dividing by $|\Lambda|^N$ in (2.5) we write

$$Z_{\beta,A,N} = Z_{\Lambda,N}^{\text{ideal}} Z_{\beta,A,N}^{\text{int}},$$

where

$$Z_{\Lambda,N}^{\text{ideal}} := \frac{|\Lambda|^N}{N!} \quad \text{and} \quad Z_{\beta,A,N}^{\text{int}} := \int_{\Lambda_N} \frac{dq_1}{|\Lambda|} \ldots \frac{dq_N}{|\Lambda|} e^{-\beta H_{\Lambda}(q)}.$$

Note that what follows holds for both periodic and zero boundary conditions, so for simplicity we will not distinguish between the two cases. We will distinguish them once this becomes relevant. For $Z_{\beta,A,N}^{\text{int}}$ we use the idea of Mayer in [6] which consists of developing $e^{-\beta H_{\Lambda}(q)}$ in the following way

$$e^{-\beta H_{\Lambda}(q)} = \prod_{1 \leq i < j \leq N} (1 + f_{i,j}) = \sum_{E \in \mathcal{E}(N)} \prod_{\{i,j\} \in E} f_{i,j},$$

where $\mathcal{E}(N) := \{\{i,j\} : i,j \in [N], i \neq j\}$, $[N] := \{1, \ldots, N\}$ and

$$f_{i,j} := e^{-\beta V(q_i - q_j)} - 1$$

for the case of zero boundary conditions. For periodic boundary conditions the cluster expansion will be the same, replacing $f_{i,j}$ by $f_{i,j} := e^{-\beta V_{\text{per}}(q_i - q_j)} - 1$. Note that in the last sum in equation (3.3) we have also the term with $E = \emptyset$ which gives 1.

A graph is a pair $g \equiv (V(g), E(g))$, where $V(g)$ is the set of vertices and $E(g)$ is the set of edges, with $E(g) \subset \{U \subset V(g) : |U| = 2\}$, $|\cdot|$ denoting the cardinality of a set. A graph $g = (V(g), E(g))$ is said to be connected if for any pair $A, B \subset V(g)$ such that $A \cup B = V(g)$ and $A \cap B = \emptyset$, there is a link $e \in E(g)$ such that $e \cap A \neq \emptyset$ and $e \cap B \neq \emptyset$. Singletons are considered to be connected. We use $\mathcal{C}_V$ to denote the set of connected graphs on the set of vertices $V \subset [N]$, where we use the notation $[N] := \{1, \ldots, N\}$.

Two sets $V, V' \subset [N]$ are called compatible (denoted by $V \sim V'$) if $V \cap V' = \emptyset$; otherwise we call them incompatible ($\sim$). This definition induces in a natural way the notion of compatibility between graphs with set of vertices $V(g), V(g') \subset [N]$, i.e., $g \sim g'$ if $V(g) \cap V(g') = \emptyset$.

With these definitions, to any set $E$ in equation (3.3) we can associate a graph, i.e., a pair $g \equiv (V(g), E(g))$, where $V(g) := \{i : \exists e \in E \text{ with } i \in e\} \subset [N]$ and $E(g) = E$. Note that the resulting graph does not contain isolated vertices. It can be viewed as the pairwise compatible (non-ordered) collection of its connected components, i.e., $g \equiv \{g_1, \ldots, g_k\}_\sim$ for some $k$, where each $g_l$, $l = 1, \ldots, k$, belongs to the set of all connected graphs on at most $N$ vertices and it contains at least two vertices. Hence,

$$e^{-\beta H_{\Lambda}(q)} = \sum_{\{g_1, \ldots, g_k\}_\sim} \prod_{l=1}^k \prod_{\{i,j\} \in E(g_l)} f_{i,j},$$

where again the empty collection $\{g_1, \ldots, g_k\}_\sim = \emptyset$ contributes the term 1 in the sum. Therefore, observing that integrals over compatible components factorize, we get
Furthermore, the sum in (3.9) is over all connected subgraphs $G$.

**Theorem 3.1**

We state the general theorem as a slightly simplified version of [1], [7].

An abstract polymer model $(\Gamma, \mathcal{G}_\Gamma, \omega)$ consists of (i) a set of polymers $\Gamma := \{\gamma_1, ..., \gamma_1(\Gamma)\}$, (ii) a binary symmetric relation $\sim$ of compatibility between the polymers (i.e., on $\Gamma \times \Gamma$) which is recorded into the compatibility graph $\mathcal{G}_\Gamma$ (the graph with vertex set $\Gamma$ and with an edge between two polymers $\gamma_i, \gamma_j$ if and only if they are an incompatible pair) and (iii) a weight function $\omega : \Gamma \rightarrow \mathbb{C}$. Then, we have the following formal relation which will become rigorous by Theorem 3.1 below (see [5], [1] and [7]):

$$Z_{\Gamma,\omega} := \sum_{\gamma_1, ..., \gamma_n} \prod_{i=1}^n \omega(\gamma_i) = \exp \left\{ \sum_{I \in \mathcal{I}} c_I \omega^I \right\},$$

where

$$c_I = \frac{1}{I!} \sum_{G \in \mathcal{G}_I} (-1)^{|E(G)|},$$

or equivalently ([1],[3])

$$c_I = \frac{1}{I!} \frac{\partial^{\sum I(\gamma)} \log Z_{\Gamma,\omega}}{\partial^{|E(G)|} \omega(\gamma_1) \cdots \partial^{|E(G)|} \omega(\gamma_n)} \bigg|_{\omega(\gamma)=0}. \tag{3.10}$$

The sum in (3.8) is over the set $\mathcal{I}$ of all multi-indices $I : \Gamma \rightarrow \{0, 1, \ldots\}$, $\omega^I = \prod_i \omega(\gamma_i)^{I(\gamma)}$, and, denoting $\text{supp} \ I := \{\gamma \in \Gamma : I(\gamma) > 0\}$, $\mathcal{G}_I$ is the graph with $\sum_{\gamma \in \text{supp} \ I} I(\gamma)$ vertices induced from $\mathcal{G}_\text{supp} \ I \subset \mathcal{G}_\Gamma$ by replacing each vertex $\gamma$ by the complete graph on $I(\gamma)$ vertices. Furthermore, the sum in (3.9) is over all connected subgraphs $G$ of $\mathcal{G}_I$ spanning the whole set of vertices of $\mathcal{G}_I$ and $I! = \prod_{\gamma \in \text{supp} \ I} I(\gamma)!$. Note that if $I$ is such that $\mathcal{G}_\text{supp} \ I$ is not connected (i.e., $I$ is not a cluster) then $c_I = 0$.

We state the general theorem as a slightly simplified version of [1], [7].

**Theorem 3.1** (Cluster Expansion). Assume that there are two non-negative functions $a, c : \Gamma \rightarrow \mathbb{R}$ such that for any $\gamma \in \Gamma$, $|\omega(\gamma)| e^{a(\gamma) + c(\gamma)} \leq \delta$ holds, for some $\delta \in (0, 1)$. Moreover, assume that for any polymer $\gamma'$

$$\sum_{\gamma \sim \gamma'} |\omega(\gamma)| e^{a(\gamma) + c(\gamma)} \leq a(\gamma'). \tag{3.11}$$

Then, for any polymer $\gamma' \in \Gamma$ we obtain that

$$\sum_{I : I(\gamma) \geq 1} |c_I \omega^I| e^{\sum_{\gamma \in \text{supp} \ I} I(\gamma) c(\gamma)} \leq |\omega(\gamma')| e^{a(\gamma') + c(\gamma')}, \tag{3.12}$$
where the coefficients $c_I$ are given in (3.10).

In view of (3.6) we represent the partition function $Z_{\beta,\Lambda,N}^{\text{int}}$ as a polymer model on $\mathcal{V}(N) := \{V : V \subset \{1, \ldots, N\}, |V| \geq 2\}$ with weights $\omega_\Lambda$ as in (3.7) and compatibility graph $G_{\mathcal{V}(N)}$. Hence the abstract polymer formulation is given by the space $(\mathcal{V}(N), G_{\mathcal{V}(N)}, \omega_\Lambda)$ and we are able to check the convergence condition (3.11) (see [9]) and thus write the following expansion:

$$Z_{\beta,\Lambda,N}^{\text{int}} := \sum_{\{V_1, \ldots, V_n\} \in I} \prod_{i=1}^n \omega_\Lambda(V_i) = \exp \left\{ \sum_{I \in I} c_I \omega_I^\Lambda \right\},$$

(3.13)

where $I$ is the set of all multi-indices $I : \mathcal{V}(N) \to \{0, 1, \ldots\}$ and the series is absolutely convergent.

### 3.2. Strategy of the proof of Theorem 2.2

In this subsection we prove (2.16) and explain the strategy of the proof of (2.17). Given (3.13) we recall from [9] (see also (2.12) and (2.13)) that

$$\frac{1}{|\Lambda|} \log Z_{\beta,\Lambda,N}^{\text{int}} = \frac{1}{|\Lambda|} \sum_{I \in I} c_I \omega_I^\Lambda = N \frac{1}{|\Lambda|} \sum_{n \geq 1} \frac{1}{n+1} P_{N,|\Lambda|(n)} B_{\beta,\Lambda}(n),$$

(3.14)

where

$$P_{N,|\Lambda|(n)} := \frac{(N-1) \ldots (N-n)}{|\Lambda|^n}$$

and

$$B_{\beta,\Lambda}(n) := \frac{|\Lambda|^n}{n!} \sum_{I: A(I) = [n+1]} c_I \omega_I^\Lambda$$

(3.15)

with $A(I) := \bigcup_{V \in \text{supp} I} V \subset [N]$. We define by

$$B_{\beta,\Lambda}^*(n) := \frac{|\Lambda|^n}{n!} \sum_{I: A(I) = [n+1]} c_I \omega_I^\Lambda$$

(3.16)

the part of the sum $B_{\beta,\Lambda}(n)$ restricted to multi-indices satisfying the following conditions:

$$I(V) = 1, \; \forall V \in \text{supp} I, \; \text{and}$$

$$n + 1 = \sum_{V \in \text{supp} I} (|V| - 1) + 1.$$ 

(3.17)

(3.18)

In [9] it has been proved that under periodic boundary conditions

$$B_{\beta,\Lambda}^*(n) = \frac{1}{|\Lambda|} \frac{1}{n!} \sum_{g \in \mathcal{B}_{n+1}} \int_{\Lambda^{n+1}} dq_1 \ldots dq_{n+1} \prod_{(i,j) \in E(g)} f_{i,j}$$

(3.19)

and consequently that

$$\lim_{\Lambda \to \mathbb{R}^d} B_{\beta,\Lambda}(n) = \lim_{\Lambda \to \mathbb{R}^d} B_{\beta,\Lambda}^*(n) = \beta_n.$$ 

(3.20)
Furthermore, note that in our case of interactions with compact support and periodic boundary conditions it is easy to see that actually $B^*_{\beta,\Lambda}(n) = \beta_n$ for all $\Lambda$. Then to prove (2.16) and (2.17) one would split as follows:

$$
\left| \frac{1}{|\Lambda|} \log Z_{\beta,\Lambda,N} - \beta f_\beta(\rho) \right| \leq \left| \frac{1}{|\Lambda|} \log \frac{|\Lambda|^N}{N!} - \rho(\log \rho - 1) \right| 
+ \left| \frac{1}{|\Lambda|} \sum_{I}^* c_I \omega_{\Lambda}^I - \sum_{n\geq 1} \frac{1}{n+1} \beta_n \rho^{n+1} \right| + \left| \frac{1}{|\Lambda|} \sum_{I}^{**} c_I \omega_{\Lambda}^I \right| 
$$

(3.21)

where by ** we denote the remaining terms. For the first term we use Stirling’s approximation:

$$
\left| \frac{1}{|\Lambda|} \log \frac{|\Lambda|^N}{N!} - \rho(\log \rho - 1) \right| = \frac{1}{|\Lambda|} \left( \log \sqrt{2\pi N} + \frac{1}{12N} + O(N^{-3}) \right). 
$$

(3.22)

For the third contribution a counting of the powers of $|\Lambda|$ appearing in the numerator and denominator (see [9]) shows that each term is of order $\frac{1}{|\Lambda|}$ (or higher). However, we still have an infinite sum and this will be addressed in Section 6, see (6.1). On the other hand, for the second term we have to estimate the error between the finite volume integrals appearing in $\omega_{\Lambda}^I$ and their infinite volume version in $f_\beta$. Under periodic boundary conditions relation (3.19) holds and such a comparison is straightforward (as it was mentioned after (3.20)). Hence, we only need to estimate the difference between $P_{\Sigma,|\Lambda|}$ given in (3.15) and $\rho^n$:

$$
\sum_{n\geq 1} \frac{1}{n+1} \rho^{n+1} \left| \frac{N(N-1)\ldots(N-n)}{N^{n+1}} \right| - 1 |\beta_n|. 
$$

(3.24)

We have:

$$
N(N-1)\ldots(N-n) = N^{n+1} \left[ \left(1 - \frac{1}{N}\right) \left(1 - \frac{2}{N}\right) \cdots \left(1 - \frac{n}{N}\right) \right]. 
$$

(3.25)

We first consider the case $n \leq N^{1/2}$. Since $\log(1 - \frac{x}{N}) < 0$ and for $x < 0$ it is implied that $x \leq e^x - 1 < 0$ we obtain

$$
\left| \prod_{i=1}^{n} \left(1 - \frac{i}{N}\right) - 1 \right| \leq \sum_{i=1}^{n} |\log(1 - \frac{i}{N})| \leq c \sum_{i=1}^{n} \frac{i}{N} = c \frac{n(n+1)}{2N}. 
$$

(3.26)

In the last inequality we use the fact that for $x \leq \frac{1}{\sqrt{2}}$ (since $n \leq N^{1/2}$) there exists $c \geq \sqrt{2}$ such that $0 > \log(1 - x) > -cx$ (the latter is true for $c > \frac{1}{1-x}$). Then for the case $n \leq N^{1/2}$ we obtain the bound

$$
\frac{c}{N} \sum_{n \leq N^{1/2}} \frac{1}{n+1} \rho^{n+1} \frac{n(n+1)}{2} |\beta_n| \leq \frac{c'}{N} 
$$

(3.27)

for some $c' > 0$. On the other hand, for the case $n > N^{1/2}$, we bound (3.24) by

$$
2 \sum_{n \geq N^{1/2}} \frac{1}{n+1} \rho^{n+1} |\beta_n| \leq 2\rho \sum_{n \geq N^{1/2}} e^{-cn} \leq 2\rho e^{-cN^{1/2}}, 
$$

(3.28)
since (2.14) and (2.10) imply that $|\frac{1}{n+1}\rho^n\beta_n| \leq Ce^{-cn}$. This, together with (6.1), proves (2.16).

On the other hand, for zero (or general) boundary conditions one would need to split each integral in $\omega^I_A$ into an interior and a boundary part. Then the collection of all interior parts should be compared with the infinite volume free energy $f_\beta(\rho)$ (as it happens in the case of periodic boundary conditions). But, to collect all these interior parts we need to rewrite the sum in (3.16) as a sum over graphs (using (3.7)) which unfortunately is not convergent.

The remedy comes from a new cluster expansion where the information about whether we have an interior or a boundary integral is included in the definition of polymers. Thus, the new polymers will consist of the sets of labels like before, plus some additional information on whether all involved particles are far from the boundary of the box or not. In this way, the desired bound for the contribution of polymers “touching” the boundary of the box will come for free as a corollary of the cluster expansion theorem (see Proposition 4.2).

4. Cluster expansion on the space of rooted sets

To implement the new cluster expansion we follow Section 3 until eq. (3.6) and then use the following splitting:

$$1 = \sum_{i \in V} \frac{1}{|V|} \left( 1_{d(q_i, \Lambda^c) < R|V|} + 1_{d(q_i, \Lambda^c) \geq R|V|} \right)$$

(4.1)

for any $V \subset \{1, \ldots, N\}$ and where $d(\cdot, \cdot)$ is the Euclidean distance. Inserting it in (3.6) we obtain:

$$Z_{\beta, \Lambda, N}^{int} = \sum_{\{V_1, \ldots, V_n\} \sim \{V_1, \ldots, V_n\}} \prod_{l=1}^n \sum_{g \in \mathcal{G}_{V_l}} \int_{\Lambda |V_l|} d \xi_l \cdots \int_{\Lambda |V_l|} d \xi_{V_l} \prod_{(j,k) \in E(g)} f_{j,k} \sum_{i \in V_l} \frac{1}{|V_l|} \times$$

$$\left( 1_{d(q_i, \Lambda^c) < R|V_l|} + 1_{d(q_i, \Lambda^c) \geq R|V_l|} \right),$$

(4.2)

where again $f_{i,j}$ has two different possible definitions in the case of zero or periodic boundary conditions (see the discussion after (3.4)). Given $V_l$ and $i \in V_l$ the quantity in the parenthesis represents the two cases: either the particle with label $i$ is closer to the boundary than $R|V_l|$ giving a boundary contribution or not. We introduce a parameter $\epsilon_i \in \{0, 1\}$ to distinguish between these two cases. We consider a real function $F$ defined as follows:

$$F(\epsilon_i) := (1 - \epsilon_i) 1_{d(q_i, \Lambda^c) < R|V_l|} + \epsilon_i 1_{d(q_i, \Lambda^c) \geq R|V_l|}.$$  

(4.3)

Then,

$$F(0) = 1_{d(q_i, \Lambda^c) < R|V_l|}$$  

(4.4)

$$F(1) = 1_{d(q_i, \Lambda^c) \geq R|V_l|}$$  

(4.5)
and hence:

\[ Z^{\text{int}}_{\beta,\Lambda,N} = \sum_{\{V_1, \ldots, V_n\}_{\sim} \atop |V_i| \geq 2, V_i} \prod_{l=1}^{n} \prod_{i \in V_l} \prod_{k \in V_l} \int_{\Lambda} dq_k |\Lambda| \prod_{\{j,k\} \in E(g)} f_{j,k} \frac{1}{|V_l|} F(\epsilon_i). \] (4.6)

We unify the sums over \( V, i \) and \( \epsilon_i \) by defining the polymers of the new expansion to be the triplets \( (V, i, \epsilon_i) \), where \( V \in \mathcal{V}(N), \mathcal{V}(N) := \{ V : V \subset [N] \}, i \in V \) and \( \epsilon_i \in \{0, 1\} \). One may think of many copies of a set of vertices \( V \), as many as the number of its elements (choice of \( i \)) each one taken two times (as \( \epsilon_i \) can take the values 0 or 1). The new polymers differ from the old ones as they are rooted sets, with the label \( i \) being the root and coloured, where \( \epsilon_i \in \{0, 1\} \) are the two colours. We use the notation \( V \) to indicate the triplet \( (V, i, \epsilon_i) \) and we will refer to \( V \) as the support of \( V \): \( V = \text{supp} V \). Two polymers \( V_1 \) and \( V_2 \) are compatible if their supports are compatible (see Section 3).

With slight abuse of notation we still define the activity as a function \( \omega_\Lambda : \mathcal{V}(N) \times \{1, \ldots, N\} \times \{0, 1\} \to \mathbb{R} \) and for a polymer \( V = (V, i, \epsilon_i) \) it has the following expression

\[ \omega_\Lambda(V) = \sum_{g \in C_V} \int_{\Lambda} dq_1 |\Lambda| \cdots \int_{\Lambda} dq_{|V|} |\Lambda| \prod_{\{j,k\} \in E(g)} f_{j,k} \frac{1}{|V|} F(\epsilon_i). \] (4.7)

Then the canonical partition function (4.6) can be written as:

\[ Z^{\text{int}}_{\beta,\Lambda,N} = \sum_{\{V_1, \ldots, V_k\}_{\sim} \atop |\text{supp} V_l| \geq 2} \prod_{l=1}^{k} \omega_\Lambda(V_l). \] (4.8)

Thus, we are again in the context of Theorem 3.1 and obtain

\[ Z^{\text{int}}_{\beta,\Lambda,N} = \exp \left\{ \mathcal{S}^{(0)}_{\beta,\Lambda,N} + \mathcal{S}^{(1)}_{\beta,\Lambda,N} \right\}, \quad \text{where for } i = 0, 1, \mathcal{S}^{(i)}_{\beta,\Lambda,N} := \sum_I c_I \omega_\Lambda^I. \] (4.9)

The sum \( \sum^{(0)} \) contains all multi-indices \( I \) such that there is at least one choice of \( V \in \mathcal{V}(N) \) and \( i \in V \) with \( \epsilon_i = 0 \) (boundary contributions). On the other hand, the sum \( \sum^{(1)} \) contains only those multi-indices for which for every choice of \( V \in \text{supp} I \) the value of the corresponding \( \epsilon_i \) is equal to 1, i.e., it consists of the polymers which are localized in the interior of \( \Lambda \).

With this new cluster expansion, following the arguments of subsection 3.2, we refine (3.21) by first splitting between the terms \( \mathcal{S}^{(0)}_{\beta,\Lambda,N} \) which interact with the boundary and those \( \mathcal{S}^{(1)}_{\beta,\Lambda,N} \) which are far from it. Then, the latter we split as in (3.21) between the ones that will produce the irreducible coefficients (\( * \) terms) and the rest (\( ** \) terms). Hence we have:

\[ \log Z^{\text{int}}_{\beta,\Lambda,N} = \mathcal{S}^{(0)}_{\beta,\Lambda,N} + \mathcal{S}^{(1),*}_{\beta,\Lambda,N} + \mathcal{S}^{(1),**}_{\beta,\Lambda,N}, \] (4.10)
where $S_{\beta,\Lambda,N}^{(0)}$, $S_{\beta,\Lambda,N}^{(1),\ast}$ and $S_{\beta,\Lambda,N}^{(1),\ast\ast}$ are given in (4.9) taking into account the further restrictions $\ast$ and $\ast\ast$. Summing up we have:

$$\left|\frac{1}{|\Lambda|} \log Z_{\beta,\Lambda,N}^{0} - \beta f_\beta(\rho)\right| =$$

$$\left|\frac{1}{|\Lambda|} \log \frac{|\Lambda|^N}{N!} + \frac{1}{|\Lambda|} \left(S_{\beta,\Lambda,N}^{(0)} + S_{\beta,\Lambda,N}^{(1),\ast} + S_{\beta,\Lambda,N}^{(1),\ast\ast}\right) - \rho \log \rho - 1 - n \log n + \beta_n \rho^{n+1}\right|. \quad (4.11)$$

Using (3.23), we need to estimate the following quantities:

$I_1 := \left|\frac{1}{|\Lambda|} S_{\beta,\Lambda,N}^{(0)}\right| \quad (4.12)$

$I_2 := \left|\frac{1}{|\Lambda|} S_{\beta,\Lambda,N}^{(1),\ast} - \sum_{n \geq 1} \frac{1}{n+1} \beta_n \rho^{n+1}\right| \quad (4.13)$

$I_3 := \left|\frac{1}{|\Lambda|} S_{\beta,\Lambda,N}^{(1),\ast\ast}\right|. \quad (4.14)$

For the first term (which will be the main contribution) see Proposition 4.2 below. The terms $I_2$ and $I_3$ will be treated in Sections 5 and 6, respectively.

In what follows we first check the validity of the hypothesis of Theorem 3.1 and then based on (3.12) we derive a bound on $I_1$.

We have, exactly as in [9]:

**Lemma 4.1.** There exists a constant $c'_0 \equiv c'_0(\beta, B) > 0$ such that for $\rho C(\beta, R) < c'_0$ there exist positive constants $a$ and $\delta \in (0, 1)$ such that for every $\Lambda := (-\frac{\ell}{2}, \frac{\ell}{2})^d \subset \mathbb{R}^d$,

$$\sup_{V \equiv (V, i, \epsilon_i)} |\omega_{\Lambda}(V)| e^{a|V|} \leq \delta \quad (4.15)$$

holds, where $N = |\rho| \Lambda|$. Moreover, for any $\tilde{V}$:

$$\sum_{V \sim \tilde{V}} |\omega_{\Lambda}(V)| e^{a|V|} \leq a |\tilde{V}|. \quad (4.16)$$

**Proof.** To bound $|\omega_{\Lambda}(V)|$ we use the tree-graph inequality ([8], Proposition 6.1(a)):

$$\left|\sum_{g \in C_n} \prod_{(j,k) \in E(g)} f_{j,k}\right| \leq e^{2\beta B n} \sum_{T \in T_n} \prod_{(j,k) \in E(T)} |f_{j,k}|, \quad (4.17)$$

where $T_n$ and $C_n$ are respectively the set of trees and connected graphs with $n$ vertices. For a fixed $\tilde{V} = (V, i, \epsilon_i)$ with $|V| = |\tilde{V}| = n$, we have

$$|\omega_{\Lambda}(V)| e^{a|V|} \leq e^{(a+2\beta B)n} \sum_{T \in T_n} \int_{\Lambda^n} \frac{dq_1}{|\Lambda|} \cdots \frac{dq_n}{|\Lambda|} \prod_{(j,k) \in E(T)} |f_{j,k}| \frac{1}{n} |F(\epsilon_i)|. \quad (4.18)$$
Given a rooted tree $T$ let us call $(a_1, b_1), (a_2, b_2), \ldots, (a_{n-1}, b_{n-1})$ its edges. We consider 1 as the root of the tree and using the change of variables:

$$y_k = q_{a_k} - q_{b_k}, \quad \forall k = 2, \ldots, n.$$  \hspace{1cm} (4.19)

$$y_1 = q_1$$ \hspace{1cm} (4.20)

we obtain:

$$\int_{\Lambda^n} \frac{dq_1}{|\Lambda|} \cdots \frac{dq_n}{|\Lambda|} \prod_{\{j,k\} \in E(T)} |f_{j,k}| = \frac{1}{|\Lambda|^n} \int_{\Lambda^n} dq_1 \cdots dq_n \prod_{k=1}^{n-1} |f_{a_k, b_k}| |F(\epsilon_i)|$$

$$\leq \frac{1}{|\Lambda|^n} \int_{\Lambda} dq_1 \int_{\Lambda} dy_2 \cdots \int_{\Lambda} dy_n \prod_{k=2}^n |e^{-\beta V(y_k)} - 1| |F(\epsilon_i)|$$

$$\leq \frac{1}{|\Lambda|^n} \int_{\Lambda} dq_1 |F(\epsilon_i)| \left( \int_{\Lambda} dx |e^{-\beta V(x)} - 1| \right)^{n-1}$$

$$\leq C(\beta, R)^{n-1} \frac{1}{|\Lambda|^n} \int_{\Lambda} dq_1 |F(\epsilon_i)| \leq C(\beta, R)^{n-1} \frac{1}{|\Lambda|^{n-1}}$$ \hspace{1cm} (4.21)

if we use the bound $|F(\epsilon_i)| \leq 1$. Note that in the case $\epsilon_i = 0$ we obtain the better bound $\leq C(\beta, R)^{n-1} 2 \frac{d R n}{|\Lambda|^{\ell d-1}}$. Then, since the number of all trees in $T_n$ is $n^{n-2}$, from (4.18) we obtain (recalling that $N = \lceil \rho |\Lambda| \rceil$):

$$|\omega_\Lambda(V)| e^{n|V|} \leq e^{(2\beta B + a)n} \frac{n^{n-2}}{|\Lambda|^{n-1}} C(\beta, R)^{n-1} = e^{(2\beta B + a)} \left( e^{(2\beta B + a)} C(\beta, R) \rho \right)^{n-1},$$ \hspace{1cm} (4.22)

or the better estimate

$$|\omega_\Lambda(V)| e^{n|V|} \leq \frac{e^{(2\beta B + a)}}{n} \left( e^{(2\beta B + a)} C(\beta, R) \rho \right)^{n-1} 2 \frac{d R}{\ell d}$$ \hspace{1cm} (4.23)

in the case $\epsilon_i = 0$. If we choose $\rho C(\beta, R)$ such that:

$$\delta' := \rho e^{(2\beta B + a)} C(\beta, R) < 1,$$ \hspace{1cm} (4.24)

then for any $V$

$$|\omega_\Lambda(V)| e^{n|V|} \leq \frac{1}{2} \rho C(\beta, R) e^{2(2\beta B + a)},$$ \hspace{1cm} (4.25)

by using the bound $2 \leq n \leq N$ and the fact that $\rho e^{(2\beta B + a)} C(\beta, R) < 1$. Then, defining $\delta := \rho C(\beta, R) e^{2(2\beta B + a)}$, (4.15) is satisfied.

For any fixed $\hat{V}$ we have:

$$\sum_{V \sim \hat{V}} |\omega_\Lambda(V)| e^{n|V|} = \sup_{k \in \hat{V}} |\hat{V}| \sum_{V \ni k} |\omega_\Lambda(V)| e^{n|V|} \leq \sup_{\hat{V}} |\hat{V}| \sum_{V \ni \hat{V}} \sum_{i \in \hat{V}} \sum_{\epsilon_i = 0,1} |\omega_\Lambda(V, i, \epsilon_i)| e^{n|V|}$$ \hspace{1cm} (4.26)
and if we sum over the cardinality of the set $V$, $V \ni k$:

$$
\sum_{n \geq 2} \binom{N - 1}{n - 1} 2n e^{(2\beta B + a)n} \frac{n^{n-2}}{|\Lambda|^{n-1}} C(\beta, R)^{n-1}
\leq 2e^{(2\beta B + a)} \sum_{n \geq 2} \left( \frac{N}{|\Lambda|} \right)^{n-1} \frac{n^{n-1}}{(n-1)!} (C(\beta, R) e^{(2\beta B + a)})^{n-1} \leq 2e^{(2\beta B + a)} \frac{\delta' e}{\sqrt{\pi}} \frac{1}{1 - \delta' e},
$$

(4.27)

where in the first expression we have used (4.22) and estimated the sum over $i \in V$ by $n$.

Then the estimate on $I_1$ follows:

**Proposition 4.2.** There exists $C > 0$ such that

$$
\left| \frac{1}{|\Lambda|} S_{\beta, \Lambda, N}^{(0)} \right| \leq C \frac{\ell^{d-1}}{\ell^d}.
$$

(4.29)

**Proof.** From (3.12) applied to the clusters over the new polymers we have that

$$
\left| \frac{1}{|\Lambda|} \sum_{V \ni \Lambda} \sum_{I: I(V) \geq 1} |c_I \omega_I^{|V|} | \right| \leq \frac{N}{|\Lambda|} \sum_{V: V \ni \Lambda} \sum_{i \in V} |\omega_A(V, i, 0)| c^{a|V|} \leq C(\rho) \frac{\ell^{d-1}}{\ell^d},
$$

(4.30)

using (4.23) and where $C(\rho)$ is a positive constant which depends on $\rho = \frac{N}{|\Lambda|}$:

$$
C(\rho) := \sum_{n \geq 2} \frac{N^{n-1}}{(n-1)!} e^{(2\beta B + a)n} \frac{n^{n-2}}{|\Lambda|^{n-1}} C(\beta, R)^{n-1}.
$$

(4.31)

The sum is convergent and hence we conclude the proof. $\square$

5. **Estimate for $I_2$**

Recalling the splitting (4.10) of the partition function with zero boundary conditions, we can repeat exactly the same steps of the new cluster expansion and produce this splitting for the partition function with periodic boundary conditions:

$$
\log Z_{\beta, \Lambda, N}^{\text{per}, \text{int}} = S_{\beta, \Lambda, N}^{(0), \text{per}} + S_{\beta, \Lambda, N}^{(1), \ast, \text{per}} + S_{\beta, \Lambda, N}^{(1), \ast \ast, \text{per}},
$$

(5.1)

where $S_{\beta, \Lambda, N}^{(0), \text{per}}$, $S_{\beta, \Lambda, N}^{(1), \ast, \text{per}}$ and $S_{\beta, \Lambda, N}^{(1), \ast \ast, \text{per}}$ are the corresponding to the splitting in (4.10) terms but computed with periodic boundary conditions. The key observation here is that

$$
S_{\beta, \Lambda, N}^{(1), \ast} = S_{\beta, \Lambda, N}^{(1), \ast, \text{per}}.
$$

(5.2)
This is true since all the terms in the sum are in the interior of $\Lambda$ (as it is indicated by the upper script (1) in the sum).

Following the proof of Proposition 4.2 for the case of periodic boundary conditions we have that there exists $C > 0$ such that

$$\left| \frac{1}{|\Lambda|} S^{(0),\text{per}}_{\beta,\Lambda,N} \right| \leq C \frac{\ell^{d-1}}{\ell^d}. \quad (5.3)$$

Furthermore, repeating the steps leading to (6.2) we obtain that

$$\left| \frac{1}{|\Lambda|} S^{(1),\ast\ast,\text{per}}_{\beta,\Lambda,N} \right| \leq \frac{C}{|\Lambda|} \quad (5.4)$$

Then, from (5.1) using (5.2), (5.3) and (5.4) we get:

$$\left| S^{(1),\ast}_{\beta,\Lambda,N} - \log Z^{\text{per, int}}_{\beta,\Lambda,N} \right| \leq C \frac{\ell^{d-1}}{\ell^d}. \quad (5.5)$$

Furthermore, comparing with (2.16) (using also (3.23)) we obtain that

$$|I_2| \leq C \frac{\ell^{d-1}}{\ell^d} \quad (5.6)$$

concluding the estimate for $I_2$.

6. Estimate for $I_3$

In this section we prove that there exists a constant $C > 0$ such that

$$\left| \frac{1}{|\Lambda|} \sum_{I}^{**} c_I \omega_{\Lambda}^I \right| \leq C \frac{1}{|\Lambda|}, \quad (6.1)$$

as a direct consequence of (6.7) and of Lemmas 6.1 and 6.2 below. The estimate for $I_3$ is analogous since the constraint (1) in (4.14) does not play any role and can be removed going to an upper bound. Then it is enough to repeat the same steps as for the proof of (6.1) since the main ingredient is again the validity of Theorem 3.1 (and in particular of (3.12)) which is true for both expansions over polymers $V$ and polymers $\overline{V}$. Thus, we also have that

$$|I_3| \leq C \frac{1}{|\Lambda|}. \quad (6.2)$$

Note also that the two estimates (6.1) and (6.2) are valid for both periodic and zero boundary conditions.

We start by rewriting the sum over multi-indices as a sum over ordered sequences $(V_1, \ldots, V_n)$. Recalling $c_I$ from (3.9) it is straightforward that

$$\left| \frac{1}{|\Lambda|} \sum_{I}^{**} c_I \omega_{\Lambda}^I \right| \leq \frac{1}{|\Lambda|} \sum_{n \geq 1} \frac{1}{n!} \sum_{(V_1, \ldots, V_n)} |\phi^T(V_1, \ldots, V_n)| \prod_{i=1}^{n} |\omega_{\Lambda}(V_i)|, \quad (6.3)$$
where
\[ \phi^T(V_1, \ldots, V_n) := \sum_{g \in C_n} \prod_{(i,j) \in E(g)} -1_{\{V_i \sim V_j\}} \] (6.4)
if \( n \geq 2 \), while \( \phi^T(V) = 1 \). We exploit the constraint **. This means that there exists a collection of, say \( k \) elements, each one in \( \mathcal{V}(N) \), for which the conditions (3.17) and (3.18) fail. Hence, given \( k \leq n \) and the labels \( i_1, \ldots, i_k \in \{1, \ldots, n\} \) of these elements (all different), we define the following sets in \( \mathcal{V}(N)^n \) : for \( k \geq 3 \) we let
\[ A_{i_1, \ldots, i_k} := \{ \{V_1, \ldots, V_n\} \text{ with } \{1, \ldots, n\} \supset \{i_1, \ldots, i_k\} : \exists v_1, \ldots, v_k \in [N], \text{ all different s.t. } V_{i_l} \cap V_{i_{l+1}} \supset \{v_l\}, \text{ for } l = 1, \ldots, k, \text{ with } i_{k+1} \equiv i_1 \} \] (6.5)
and for \( k = 2 \)
\[ A_{i_1, i_2} := \{ \{V_1, \ldots, V_n\} \text{ with } \{1, \ldots, n\} \supset \{i_1, i_2\} : |V_{i_1} \cap V_{i_2}| \geq 2 \}. \] (6.6)
We have:
\[ \left| \frac{1}{|\Lambda|} \sum_{I} c_I \phi^T(V_{i_1}, \ldots, V_{i_k}) \right| \leq \frac{1}{|\Lambda|} \sum_{n \geq 2} \frac{1}{n!} \sum_{(V_1, \ldots, V_n)} \prod_{i=1}^{n} |\phi^T(V_1, \ldots, V_n)| \prod_{l=1}^{n} |\omega_\Lambda(V_l)| \]
\[ \leq \frac{1}{|\Lambda|} \sum_{n \geq 2} \frac{1}{n!} \sum_{(V_1, \ldots, V_n)} \prod_{i=1}^{n} |\phi^T(V_1, \ldots, V_n)| \prod_{I} |\omega_\Lambda(V_l)| \]
\[ \leq \frac{1}{|\Lambda|} \sum_{(V_1, \ldots, V_k)} \prod_{l=1}^{k} |\omega_\Lambda(V_l)| \cdot \left(1 + \frac{1}{(n-k)!} \sum_{(V_{k+1}, \ldots, V_n)} |\phi^T(V_1, \ldots, V_n)| \prod_{l=k+1}^{n} |\omega_\Lambda(V_l)| \right), \] (6.7)
where in the last inequality we use the fact that each choice of \( i_1, \ldots, i_k \) is the same, hence we consider one choice (on the first \( k \) positions) and multiply with the cardinality \( \binom{n}{k} \). We first prove that the sum over \( n \geq k + 1 \) is bounded. Then the estimate that this is of order \( \frac{1}{|\Lambda|} \) will come from the first sum over the sets \( V_1, \ldots, V_k \) with the constraint \( A_{1, \ldots, k} \) (i.e., that are not satisfying (3.17) and (3.18)). We have:

**Lemma 6.1.** Given \( V_1, \ldots, V_k \) pairwise incompatible, there exists a constant \( C \) such that
\[ \sum_{n \geq k+1} \frac{1}{(n-k)!} \sum_{(V_{k+1}, \ldots, V_n)} |\phi^T(V_1, \ldots, V_n)| \prod_{l=k+1}^{n} |\omega_\Lambda(V_l)| \leq C \prod_{l=k}^{k} |V_l| e^{\|V_l\|}. \] (6.8)

**Proof.** To prove it we follow the argument given in Cammarota [2]. The main idea is to write \( \phi^T(V_1, \ldots, V_n) \) as a sum over trees with branches emerging from each \( V_i, i = 1, \ldots, k \) and then use the strategy leading to (3.12) by summing each branch independently.
We first work for $k = 2$ and then argue that the general case $k \geq 3$ is similar. Using the tree-graph inequality (see [8], Proposition 6.1 (a)) we obtain:

\[
\sum_{(V_3, \ldots, V_n)} \phi^T(V_1, \ldots, V_n) \prod_{l=3}^n |\omega_{\Lambda}(V_l)| \leq \sum_{(V_3, \ldots, V_n)} \sum_{T \in \mathcal{T}_n} \prod_{\{i,j\} \in E(T)} \prod_{l=3}^n |\omega_{\Lambda}(V_l)|
\]

\[
= \sum_{T \in \mathcal{T}_n} \sum_{G(V_3, \ldots, V_n) \supset \mathcal{T} \cup \{1,2\}} \prod_{l=3}^n |\omega_{\Lambda}(V_l)|. \tag{6.9}
\]

We sum over $T \in \mathcal{T}_n$ as follows: we sum over two trees $T_1$ (respectively $T_2$) with root the label 1 (respectively 2) of cardinality $n_1 + 1$ (respectively $n_2 + 1$) and a link from any vertex of $T_1$ to the root 2 of $T_2$. Note that $n_1 + n_2 = n - 2$. To implement it we first partition the set of labels $\{3, \ldots, N\}$ into two subsets $N_1$ and $N_2$ and construct two trees $T_1$ and $T_2$ from each one with the additional root 1 and 2. The two trees are linked by $\{\xi, 2\}$ for some $\xi \in T_1$. We find an estimate by removing the extra link $\{\xi, 2\}$ as well as $\{1, 2\}$ from the constraint in (6.9) and obtaining

\[
(6.9) \leq \sum_{\{N_1, N_2\} \text{ part. of } \{3, \ldots, n\}} \sum_{T_1 \in \mathcal{T}_{N_1 \cup \{1\}}} \sum_{\xi \in V(T_1)} \sum_{j=1,2} \prod_{(V_l) \in N_j \cap (\xi \cup j) \supset T_j} \prod_{l \in N_j} |\omega_{\Lambda}(V_l)|, \tag{6.10}
\]

where we also recall that $V(T_1)$ is the set of vertices of tree $T_1$.

A similar relation is true for the general case $k \geq 3$. We decompose $T$ into a tree $T_1$ with root the label 1, then a link from some $\xi_1 \in T_1$ to the root 2 of the subtree $T_2$ and similarly another link from some $\xi_2 \in T_2$ to root 3 of the next subtree $T_3$ etc. We relax the constraint by removing the links between the subtrees and obtain:

\[
(6.9) \leq \sum_{\{N_1, \ldots, N_k\} \text{ part. of } \{3, \ldots, n\}} \sum_{T_1 \in \mathcal{T}_{N_1 \cup \{1\}}} \sum_{\xi_1 \in V(T_1)} \cdots \sum_{\xi_k \in V(T_k)} \prod_{j=1}^k \prod_{(V_l) \in N_j \cap (\xi_j \cup j) \supset T_j} \prod_{l \in N_j} |\omega_{\Lambda}(V_l)|. \tag{6.11}
\]

To calculate the right hand side of (6.11) we sum over the cardinalities of $N_1, \ldots, N_k$ obtaining:

\[
\sum_{n_1, \ldots, n_k} \frac{(n-k)!}{n_1! \cdots n_k!} (n_1 + 1) \cdots (n_k + 1) \prod_{j=1}^k \left( \sum_{T \in \mathcal{T}_{N_j+1}} \mathcal{A}_{n_j, V_j}(T) \right), \tag{6.12}
\]

where for $T \in \mathcal{T}_{N_j+1}$ with root $V_j$ we have defined the following quantity:

\[
\mathcal{A}_{n_j, V_j}(T) := \sum_{(V_l) \in T_{N_j+1}} \prod_{l=2}^{n_j+1} |\omega_{\Lambda}(V_l)|. \tag{6.13}
\]
Following [2], we have that for a tree \( T \in T_m \) whose vertices \( i \in \{1, ..., m\} \) have degrees \( d_i \) the following estimate holds:

\[
\mathcal{A}_{m, V_1}(T) \leq |V_1|^d a^{m-1} \prod_{j \in \{2, ..., m\}} (d_j - 1)!
\]  

(6.14)

where \( a \) is the constant appearing in Lemma 4.1. Moreover, the number of trees on \( m \) vertices with degrees \( d_1, \ldots, d_m \) is given by the Cayley formula:

\[
\frac{(m - 2)!}{\prod_{i=1}^{m} (d_i - 1)!}.
\]  

(6.15)

Thus, for the sums over trees in (6.12) using (6.14) and (6.15) we have:

\[
\sum_{T \in T_{n_1+1}} \mathcal{A}_{n_1, V_1}(T) \leq \sum_{d_1, \ldots, d_{n_1+1}} \frac{(n_1 - 1)!}{\prod_{i=1}^{n_1+1} (d_i - 1)!} |V_1|^{d_1} a^{n_1} \prod_{i=2}^{n_1+1} (d_i - 1)! \]

\[
= a^{n_1} (n_1 - 1)! \sum_{d_1=1}^{n_1} \frac{|V_1|^{d_1}}{(d_1 - 1)!} \Gamma_{n_1}(2n_1 - d_1),
\]

where

\[
\Gamma_k(m) := \sum_{d_1, \ldots, d_k: d_1 + \ldots + d_k = m} 1.
\]  

(6.17)

Applying an induction argument on \( k \) we obtain that

\[
\Gamma_k(m) \leq \frac{m^{k-1}}{(k-1)!},
\]  

(6.18)

thus,

\[
\Gamma_{n_1}(2n_1 - d_1) \leq \frac{(2n_1 - d_1)^{n_1-1}}{(n_1 - 1)!} \leq \frac{1}{2} (2e)^{n_1}.
\]  

(6.19)

Hence we have:

\[
\sum_{T \in T_{n_1+1}} \mathcal{A}_{n_1, V_1}(T) \leq \frac{1}{2} (2ae)^{n_1} (n_1 - 1)! |V_1| \sum_{d_1=1}^{n_1} \frac{|V_1|^{d_1-1}}{(d_1 - 1)!}
\]

(6.20)
and therefore the l.h.s. of (6.8), using (6.12) and (6.20) is bounded by

\[
\sum_{n \geq 3} \frac{1}{(n-k)!} \sum_{n_1, \ldots, n_2, n_3=n-k} \frac{(n-k)!}{n_1! \ldots n_k!} \prod_{i=1}^{k} \left[ \frac{1}{2} (2ae)^{n_i} |V_i| e^{\|V_i\|} \right]
\] (6.21)

\[
\leq \prod_{i=1}^{k} \left[ \sum_{n_i \geq 1} \frac{1}{n_i!} \frac{1}{2} (2ae)^{n_i} |V_i| e^{\|V_i\|} \right]
\]

\[
\leq \prod_{i=1}^{k} \left[ \sum_{n_i \geq 1} \frac{1}{2} (2ae)^{n_i} |V_i| e^{\|V_i\|} \right]
\]

\[
\leq C \prod_{i=1}^{k} |V_i| e^{\|V_i\|},
\]

which concludes the proof. □

To obtain a bound for (6.7) we exploit the fact that the fixed sets \( V_1, \ldots, V_k \) are in \( A_1, \ldots, k \).

This is given in the following Lemma:

**Lemma 6.2.** For any set \( A_1, \ldots, k \) defined in (6.5) we have that there exists a positive constant \( C \) such that

\[
\frac{1}{|A|} \sum_{\{V_1, \ldots, V_k\}} 1_{A_1, \ldots, k} \prod_{l=1}^{k} (|\omega_{\Lambda}(V_l)||V_l| e^{\|V_l\|}) \leq C \frac{1}{|A|},
\] (6.22)

**Proof.** Let \( n_1, \ldots, n_k \) be the cardinalities of \( V_1, \ldots, V_k \). Choosing the labels for an element \( V_1, \ldots, V_k \) of \( A_1, \ldots, k \) we look for the highest power of \( N \). This occurs when we obtain a factor of \( \left( \begin{array}{c} N \\ n_1 \end{array} \right) \sim \frac{N^{n_i}}{n_i!} \) for the first, \( \frac{N^{n_i-1}}{(n_i-1)!} \) for the second (since there is at least one common label with \( V_1 \)), similarly for the next ones and for the last \( \frac{N^{n_k-2}}{(n_k-2)!} \) since it has one common with the previous and another common with the first one (see the definition of \( A_1, \ldots, k \) in (6.5)).

So the power of \( N \) is at most \( n_1 + (n_2 - 1) + \ldots + (n_k - 2) = \sum_{i=1}^{k} n_i - k \). Any other case has a smaller exponent for \( N \) since the sets \( V_i, i = 1, \ldots, k \) may share more labels. On the other hand, from (4.22), the activities of each \( V_i \) satisfy

\[
|\omega_{\Lambda}(V_i)||V_i| e^{\|V_i\|} \leq e^{(2\beta B+a)n_i} \frac{n_i^{n_i-2}}{|A|^{n_i-1}} n_i C(\beta, R)^{n_i-1}.
\] (6.23)

Hence, the power of \( |A| \) in the denominator is again \( \sum_{i=1}^{k} n_i - k \) and combined with the numerator gives \( \rho \) to that power. Thus, overall, any term in the sum in (6.22) is of the order \( \frac{1}{|A|} \) or higher. Furthermore, the sum of all the terms is convergent because of (3.12). □
7. Correlations

In this section we first show how the validity of the cluster expansion is related to the calculation of the truncated correlation function and then conclude by proving Theorem 2.6. Let $h_1, h_2$ be measurable functions on $\Lambda$. We define:

$$\Psi_{\Lambda,N,h_1,h_2}(a_1, a_2) := \int_{\Lambda^N} \frac{dq_1}{|\Lambda|} \cdots \frac{dq_N}{|\Lambda|} \prod_{i=1,2} (1 + a_i h_i(q_i)) e^{-\beta H_{\Lambda}(q)}.$$ \hfill (7.1)

Given some fixed $q_1', q_2' \in \Lambda$ we choose the functions $h_1, h_2$ such that $h_i(q_i) := 1$ for $q_i = q_i'$ and 0 otherwise, for both $i = 1, 2$. Then with a slight abuse of notation we call $\Psi_{\Lambda,N,q_1',q_2'}(a_1, a_2)$ the corresponding function (7.1) for this special choice of $h_1$ and $h_2$. We have that:

$$\rho_{\Lambda,N}^{(2),lab}(q_1', q_2') - \rho_{\Lambda,N}^{(1),lab}(q_1') \rho_{\Lambda,N}^{(1),lab}(q_2') = \frac{\partial^2}{\partial a_1 \partial a_2} \log \Psi_{\Lambda,N,q_1',q_2'}(a_1, a_2)|_{a_1=a_2=0}. \hfill (7.2)$$

Similarly, for $a_2 = 0$ and by choosing $h_1$ as before, again by slight abuse of notation (denoting by $\Psi_{\Lambda,N,q_1}$ the resulting function) we have that

$$\rho_{\Lambda,N}^{(1),lab}(q_1') = \frac{\partial}{\partial a_1} \log \Psi_{\Lambda,N,q_1'}(a_1, 0)|_{a_1=0}. \hfill (7.3)$$

Thus, to calculate (7.3) and (7.2) we first compute the cluster expansion for the function $\Psi_{\Lambda,N,h_1,h_2}(a_1, a_2)$ and then apply it for the two particular cases appearing in (7.2) and (7.3).

We define the space $V^* := \bigcup_{A \subset \{1,2\}} V_A$, where $V_A$ contains all $V$ such that $V \supset A$ and to which we attribute an activity depending on $A$. Hence, denoting by the ordered pairs $(V, A)$ its elements (for $V \in V$ with $V \supset A$ and $A \subset \{1,2\}$), we define the activities

$$\omega_{\Lambda,h_1,h_2}((V, A)) := \prod_{i \in A} a_i \sum_{g \in G_V} \int_{\Lambda^{|V|}} \prod_{i \in A} h_i(q_i) \prod_{(i,j) \in E(g)} f_{i,j} \prod_{i \in V} \frac{dq_i}{|\Lambda|}. \hfill (7.4)$$

Note that if $A = \emptyset$ we have that $\omega_{\Lambda,h_1,h_2}((V, \emptyset)) = \omega_{\Lambda}(V)$ as given in (3.7). Following the strategy of Section 3 we can write

$$\Psi_{\Lambda,N,h_1,h_2}(a_1, a_2) = \sum_{\{(V_1,A_1),...,(V_k,A_k)\} \sim \{V_l(A_l)\}_{l=1}^k} \prod_{(V_l,A_l) \in V^*} \omega_{\Lambda,h_1,h_2}((V_l,A_l)). \hfill (7.5)$$

Moreover, we say that two elements are compatible, $(V_1, A_1) \sim (V_2, A_2)$, if $V_1 \sim V_2$. N.B.: In the sequel, for simplicity of the notation, we will denote the elements of $V^*$ either as pairs $(V, A)$ or just $V$ if the explicit knowledge of $A$ is irrelevant, as for example in (7.7) and (7.10). We are in the context of the abstract polymer model in the space $(V^*, G_{V^*}, \omega_{\Lambda,h_1,h_2})$ and we apply Theorem 3.1. One can check the convergence condition exactly as in Lemma 4.1 (see also [9]), with the only modification that now a given element $V$ has to be chosen.
among the new augmented set $\mathcal{V}^*$, which will just give an extra factor 4. Then we obtain
\[
\log \Psi_{\Lambda,N,h_1,h_2}(a_1, a_2) = \sum_{I \in \mathcal{I}(\mathcal{V})} c_I \omega^I_{\Lambda,N,h_1,h_2}.
\] (7.6)

Moreover, as in (3.12), we have the bound:
\[
\sum_{I : I(V') \geq 1} |c_I \omega^I_{\Lambda,N,h_1,h_2}| e^{\sum_{V \in \text{supp} I} I(V) e(V)} \leq |\omega(V')| e^{a(V') + c(V')},
\] (7.7)
for any $V' \in \mathcal{V}^*$ and for some non-negative functions $a, c : \mathcal{V}^* \to \mathbb{R}$.

**Proof of Theorem 2.6.** For the proof of (2.25), we use (7.2) and in order to estimate its right hand side we have to identify the non vanishing terms. These are the ones of order $a_1 a_2$ and consist of those clusters in (7.6) that either have only one element in $\mathcal{V}_{\{1\}}$, only one element in $\mathcal{V}_{\{2\}}$, both with multiplicity one, and all others in $\mathcal{V}$, or it has only one in $\mathcal{V}_{\{1,2\}}$ (again with multiplicity one) and all others in $\mathcal{V}$. We denote this class by $\mathcal{I}_{a_1,a_2}$. Using (7.2), and evaluating for the particular $h_1$ and $h_2$ depending on the two fixed configurations $q_1$ and $q_2$ the left hand side of (2.25) is equal to
\[
| \sum_{I \in \mathcal{I}_{a_1,a_2}} c_I \omega^I_{\Lambda,q_1,q_2} | = e^{-R^{-1}|q_1' - q_2'|} \sum_{I \in \mathcal{I}_{a_1,a_2}} \left| c_I \omega^I_{\Lambda,q_1,q_2} \right| e^{R^{-1}|q_1' - q_2'|} 
\leq e^{-R^{-1}|q_1' - q_2'|} \sum_{I \in \mathcal{I}(\mathcal{V}^*)} \left| c_I \omega^I_{\Lambda,q_1,q_2} \right| e^{\sum_{V \in \text{supp} I} |V|},
\] (7.8)
where we use the notation $\omega_{\Lambda,q_1,q_2}$ for the activities (7.4) after plugging in the particular choice of $h_1$ and $h_2$ and we recall that $R$ is the range of interaction. The second inequality of (7.8) is true since if $I((V,A)) = 1$ for $A = \{1,2\}$ then $|V| R \geq |q_1' - q_2'|$ and if $I((V,A)) = I((V',A')) = 1$ for $A = \{1\}$ and $A' = \{2\}$ then $\sum_{V \in \text{supp} I} |V| R \geq |q_1' - q_2'|$. Then the right hand side of (7.8) can be further bounded by using (7.7) and summing over all $V \ni 1$.

To prove (2.24) we proceed in a similar fashion. We use (7.6) with $\omega_{\Lambda,h_1,h_2}$ replaced by $\omega_{\Lambda,q_1'}((V,\{1\})) := a_1 \sum_{g \in \mathcal{C}_V} \int_{\Lambda|V|} \mathbf{1}_{\{q_1 = q_1'\}} \prod_{(i,j) \in E(g)} f_{i,j} \prod_{i \in V} \frac{d q_i}{|A_i|}$. (7.9)

The terms which give contributions of first order in $a_1$ are those with only one element in $\mathcal{V}_{\{1\}}$ (with multiplicity one) and all others in $\mathcal{V}$. We denote this set of multi-indices by $\mathcal{I}_{a_1}$. Then, by applying (3.12) we obtain that
\[
| \sum_{I \in \mathcal{I}_{a_1}} c_I \omega^I_{\Lambda,q_1'} | \leq \sum_{V \in \mathcal{V}_{\{1\}}} \sum_{I : I(V) \geq 1} |c_I \omega^I_{\Lambda,q_1'} | \leq \sum_{V \ni 1} |\omega_{\Lambda,q_1'}(V)| e^{a(V)} \leq C_1,
\] (7.10)
which concludes the proof. \qed
Acknowledgments. It is a great pleasure to thank Errico Presutti for suggesting us the problem and for his continuous advising. We also acknowledge discussions with Marzio Cassandro and Sabine Jansen. The research of both authors was partially supported by the FP7-REGPOT-2009-1 project “Archimedes Center for Modeling, Analysis and Computation” (under grant agreement no 245749). E. P. is further supported by ERC Advanced Grant 267356 VARIS of Frank den Hollander.

REFERENCES

[1] A. Bovier, M. Zahradník, A simple inductive approach to the problem of convergence of cluster expansion in polymer models, J. Stat. Phys. (2000), 100, 765–777.
[2] C. Cammarota, Decay of correlations for infinite range interactions in unbounded spin systems, Comm. Math. Phys. (1982), 85, 51728.
[3] R. L. Dobrushin, Estimates of Seminvariants for the Ising Model at Low Temperatures, Topics in Statistical Physics, AMS Translation Series 2, Vol. 177, AMS, Advances in the Mathematical Sciences 32 (1995), 59–81.
[4] R. Kotecký, Cluster expansion, Encyclopedia of Mathematical Physics, 2006.
[5] R. Kotecký, D. Preiss, Cluster expansion for abstract polymer models, Comm. Math. Phys. (1986), 103, 491–498.
[6] J. E. Mayer, M. G. Mayer, Statistical Mechanics, New York, John Wiley and Sons, 1940.
[7] F. R. Nardi, E. Olivieri, M. Zahradník, On the Ising model with strongly anisotropic external field, J. Stat. Phys. (1999), 97, 87–145.
[8] S. Poghosyan, D. Ueltschi, Abstract cluster expansion with applications to statistical mechanical systems, J. Math. Phys. (2009), 50, 053509.
[9] E. Pulvirenti, D. Tsagkarogiannis, Cluster expansion in the canonical ensemble, Comm. Math. Phys. (2012), 316, 289–306.
[10] D. Ueltschi, Cluster expansions and correlation functions, Mosc. Math. J. (2004), 4, 511 0304003.

Elena Pulvirenti, Mathematical Institute, Leiden University, P.O. Box 9512, 2300 RA Leiden, The Netherlands
E-mail address: pulvirentie@math.leidenuniv.nl

Dimitrios Tsagkarogiannis, Department of Mathematics, University of Sussex, Brighton BN1 9QH, UK
E-mail address: D.Tsagkarogiannis@sussex.ac.uk