Memory effects in quantum metrology

Yuxiang Yang
Institute for Theoretical Physics, ETH Zürich, Switzerland

Quantum metrology concerns estimating a parameter from multiple identical uses of a quantum channel. We extend quantum metrology beyond this standard setting and consider estimation of a physical process with quantum memory, here referred to as a parametrized quantum comb. We present a theoretic framework of metrology of quantum combs, and derive a general upper bound of the comb quantum Fisher information. The bound can be operationally interpreted as the quantum Fisher information of a memoryless channel times a dimensional factor. We then show an example where the bound can be attained up to a factor of two. With the example and the bound, we show that memory in quantum sensors plays an even more crucial role in the estimation of combs than in the standard setting of quantum metrology.

**Introduction.** Steady developments in quantum communication and quantum memory allow us to measure a physical quantity with higher precisions [1][2]. By harnessing quantum control and ancillary memory qubits, adaptive metrological strategies can improve the performance of sensing even in the presence of noise [3][4].

In standard quantum metrology, the goal is often to estimate identical copies of the same quantum gate, possibly subject to noise [7][9], that are available either in parallel [10] or in arbitrary order [11][12]. With the advance of quantum technologies, however, the focus is transitioning to more complex and realistic settings, where the parameter to estimate is contained in a network or an adaptive physical process [13][16]. In such settings, one would have to deal with a circuit with a complex underlying structure, or in a network equipped with communication channels and memories, which requires a model with a higher order structure than parametrized quantum gates.

In this work, we extend quantum metrology beyond the standard setting of estimating parametrized quantum channels. We consider estimating an unknown parameter from a physical process with quantum memory, here referred to as a parametrized (quantum) comb. We present a theoretic framework of quantum metrology with such parametrized combs, and derive a general upper bound of the comb quantum Fisher information (QFI). The QFI bound can be operationally interpreted as the quantum Fisher information of a memoryless channel times a dimensional factor. We then show an example where the bound can be attained up to a factor of two. With the example and the bound, we show that memory in quantum sensors plays an even more crucial role in quantum comb metrology than in the standard setting of quantum metrology.

**Quantum metrology in the presence of memory.** In this work, our goal is to estimate an unknown parameter $\theta$ given access to a quantum machine that has its own memory, which belongs to a family of parameterized quantum machines $\{N_\theta\}_{\theta \in \Theta}$. This quantum machine could be, for instance, a quantum circuit with ancillary qubits or a noisy physical process with an inaccessible environment. To show memory effects in quantum metrology in the most straightforward way, we focus on single parameter estimation and $\Theta \subset \mathbb{R}$. It is convenient to characterize such a quantum machine $N_\theta$ by a parametrized quantum comb [18][20], abbreviated as a comb hereafter. As illustrated in Fig. 1, the action of the comb $N_\theta$ is naturally divided into $K$ consecutive phases $P_1, P_2, \ldots, P_K$ by the use of quantum memory. For each phase $P_n$, the comb takes a quantum state from an input port $P_n^{(in)}$ and produces a quantum state from an output port $P_n^{(out)}$. Each phase is connected to the next phase by a quantum memory (or quantum communication, if the phases are spatially separate).

In the standard context of quantum gate estimation, one prepares a quantum state and sends it through unknown gates. Here, to estimate a whole comb $N_\theta$, we need to connect it to a quantum sensor $S$ with memory (hereafter referred to as a sensor), which could be a complex composition of quantum states, gates, memory, and communication channels. Mathematically, the sensor $S$ is also modeled by a quantum comb, which takes

![FIG. 1. Estimating a quantum comb with a sensor. In quantum comb metrology, the goal is to estimate a parameter $\theta$ from a comb $N_\theta$ consisting of $K$ phases (plotted here as black boxes). For this purpose, one uses a sensor $S$ (plotted in blue), which consists of preparing an input state and applying quantum gates in between the phases.](Image)
\( P_{\text{in}}^{(\text{out})} \) (for \( m \in [K-1] \)) as its input ports and \( P_{\text{out}}^{(\text{in})} \) (for \( n \in [K] \)) as well as an ancillary system \( R \) as its output ports. This means that a sensor is such a quantum comb that it “eats” the (parametrized) comb \( \mathcal{N}_\theta \) and “spits” a quantum state. Formally speaking, the link product \([15, 20] \{ \mathcal{N}_\theta \otimes \mathcal{S} \}_{\theta \in \Theta} \), which is essentially the concatenation of \( \{ \mathcal{N}_\theta \}_{\theta \in \Theta} \) and \( \mathcal{S} \), is a family of quantum states on \( \mathcal{H}_K^{(\text{out})} \otimes \mathcal{H}_R \) (see Fig. 1). Here \( \mathcal{H}_n^{(\text{out}/\text{in})} \) denotes the Hilbert space of \( P_{\text{out}/\text{in}}^{(\text{out}/\text{in})} \).

A distinctive difference between quantum comb metrology and quantum gate estimation that makes the former more tricky to deal with is the memory effect of both the sensor and the comb. Unlike quantum states that are used to probe parametrized gates, sensors are capable of memorizing the information on \( \theta \) (possibly in the quantum form) at the end of each phase and send refined information back into the comb \( \mathcal{N}_\theta \). Similarly, \( \mathcal{N}_\theta \) is also capable of storing the input from the sensor in its underlying structure and making it interact with future inputs. Our goal is to see the impact of such effects on the precision of estimating \( \theta \) from the comb \( \mathcal{N}_\theta \).

Physically, the role of the sensor is to extract \( \theta \) from the comb and to encode it into a quantum state. The quantum Fisher information (QFI) of quantum combs can be defined via the Fisher information of quantum states, by optimizing over all sensors of the comb:

**Definition 1.** The quantum Fisher information of \( \{ \mathcal{N}_\theta \} \) is defined as

\[
F_Q[\mathcal{N}_\theta] := \max_{\mathcal{S}} F_Q[\mathcal{N}_\theta \otimes \mathcal{S}],
\]

where \( F_Q[\rho] \) denotes the QFI of a quantum state \( \rho \) and the maximum is taken over all sensors \( \mathcal{S} \) such that \( \mathcal{N}_\theta \otimes \mathcal{S} \) is a quantum state.

With this definition of the comb QFI, we can apply the quantum Cramér-Rao bound [21, 22] and extend it to quantum combs: Denoting by \( \delta \theta[\mathcal{N}_\theta] \) the root-mean-square error of estimating \( \theta \) from \( \mathcal{N}_\theta \), we have

\[
\delta \theta[\mathcal{N}_\theta] = \min_{\mathcal{S}} \left( \min_{\mathcal{M}_\theta} \left[ \mathcal{N}_\theta \otimes \mathcal{S} \otimes \mathcal{M}_\theta \right] \right)
\geq \min_{\mathcal{S}} \left( \frac{1}{\sqrt{\nu F_Q[\mathcal{N}_\theta \otimes \mathcal{S}]}} \right) = \frac{1}{\sqrt{\nu F_Q[\mathcal{N}_\theta]}}
\]

where in the first step the minimum is taken over all quantum estimators \( \mathcal{M}_\theta \) that measure the state \( \mathcal{N}_\theta \otimes \mathcal{S} \) and output an unbiased estimate \( \hat{\theta} \) of \( \theta \), and \( \nu \) is the number of repetitions of the experiment.

A general upper bound on the comb QFI. From the above discussion, we can see that the precision of estimating \( \theta \) is determined by the QFI of a quantum comb. However, the derivation of the QFI is not easy even for the case when there is only one phase \( (K = 1) \) and the quantum comb is reduced to a quantum channel. A closed-form expression of the QFI was derived only for particular types of quantum channels (see, for instance, Refs. [25, 26]). Therefore, it is more sensible to look for an upper bound of the comb QFI so as to see the power and the limitation of metrology with a quantum comb.

In the following, we derive such an upper bound of the comb QFI, which implies a lower bound on the error of parameter estimation from the comb. We will use the abbreviation \( \Psi := |\Psi\rangle\langle\Psi| \) of pure state density matrices and denote by \( |\Phi^+\rangle \) a maximally entangled state \( \frac{1}{\sqrt{2}} \sum_j |j\rangle |j\rangle \).

First, observe that any sensor can be decomposed as \( \mathcal{S} = \Psi \otimes \mathcal{V} \), where \( \Psi \) is a suitable input state to be sent into the first input port and \( \mathcal{V} \) is a quantum comb. An obstacle in determining \( F_Q[\mathcal{N}_\theta] \) is that the information on \( \theta \) can flow out of \( \mathcal{N}_\theta \) via an output port and then back into \( \mathcal{N}_\theta \) via subsequent input ports, owing to the memory effect of \( \mathcal{V} \). To overcome this obstacle, we use a teleportation trick, which works by noticing that the action of a sensor \( \mathcal{S} \) is equivalent to the following probabilistic protocol, as depicted in Fig. 2(a):

1. Send a proper state \( \Phi_1 \) into the first input port and a maximally entangled state \( \Phi^+_n \) into the \( n \)-th input port for \( n > 1 \), with \( |\Phi^+_n\rangle \) being the maximally entangled state on \( \mathcal{H}_n^{(\text{in})} \otimes \mathcal{H}_n^{(\text{in})} \) being the maximally entangled state on \( \mathcal{H}_n^{(\text{in})} \otimes \mathcal{H}_n^{(\text{in})} \) of pure state density matrices; \( \mathcal{H}_n^{(\text{in})} \).
2. feed all but the last of the output ports of \( \mathcal{N}_\theta \otimes \Psi_1 \) into a quantum comb \( \mathcal{V} \);
3. perform a Bell test on the \( n \)-th output port of \( \mathcal{V} \) and the open end of \( \Phi^+_n \) (for \( n > 1 \)); postselect the outcome \( \Phi^+_n \).

Notice that we are abusing a bit the notation here, since the quantum comb \( \mathcal{V} \) has different (but isomorphic) output Hilbert spaces from the one in the decomposition of \( \mathcal{S} \).

In view of estimating the parameter \( \theta \), this probabilistic protocol can be regarded as one step of probabilistic metrology [27, 29], where one performs a measurement and postselects on one particular outcome to enhance the performance of parameter estimation. The limitation of probabilistic quantum metrology can be made clear via the following equation [30, Eq. (30)]: For a family of parametrized quantum states \( \{ \rho_\theta \} \) and a quantum operation (i.e. a completely positive, trace-nonincreasing linear map) \( \mathcal{M}^{(\text{succ})} \), we have

\[
F_Q[\rho_\theta] \geq p_\theta^{(\text{succ})} \cdot F_Q[p_\theta^{(\text{succ})}],
\]

where \( p_\theta^{(\text{succ})} := \text{Tr} \mathcal{M}^{(\text{succ})}(\rho_\theta) \) and \( \rho_\theta^{(\text{succ})} := \mathcal{M}^{(\text{succ})}(|\rho_\theta\rangle\langle\rho_\theta|) / p_\theta^{(\text{succ})} \) is the output state of \( \mathcal{M}^{(\text{succ})} \).
In our case, the probability that the teleportation of a $d$-dimensional system succeeds without requiring a unitary correction is $d^{-2}$. The success probability of the postselection is thus

$$p^{\text{succ}}_\theta = \left(\prod_{i=2}^{K} d_i^{(\text{in})}\right)^2,$$

where $d_i^{(\text{in})}$ is the dimension of the $i$-th output port. The comb QFI is thus bounded as

$$F_Q[\Phi_{N_\theta}\ast\Psi_1, \mathcal{V}] \geq p^{\text{succ}}_\theta \cdot F_Q[\mathcal{N}_\theta \ast S],$$

where $\Phi_{N_\theta}\ast\Psi_1$ (see also Fig. 2(b)) is the Choi state of $\mathcal{N}_\theta \ast \Psi_1$, defined as

$$\Phi_{N_\theta}\ast\Psi_1 := \mathcal{N}_\theta \ast \left(\Psi_1 \otimes \Phi_2^+ \otimes \cdots \otimes \Phi_K^+\right).$$

Moreover, the output ports of $\mathcal{V}$ are now detached from the input ports of $\mathcal{N}_\theta$ thanks to the teleportation trick. Noticing that data processing does not increase distinguishability [24 Chapter 6], to maximize the comb QFI we should always take the quantum comb $\mathcal{V}$ to be a sequence of isometries. Under this condition, we have

$$F_Q[\Phi_{N_\theta}\ast\Psi_1, \mathcal{V}] = F_Q[\Phi_{N_\theta}\ast\Psi_1].$$

Combining Eq. (4), Eq. (5), and Eq. (7), we get that the QFI of $\mathcal{N}_\theta \ast S$ is bounded by \(\left(\prod_{i=2}^{K} d_i^{(\text{in})}\right)^2\) times the QFI of the quantum state $\Phi_{N_\theta}\ast\Psi_1$ (see Fig. 2(b)). To sum up with, we derived the following theorem:

**Theorem 1.** The QFI of a comb $\mathcal{N}_\theta$ is upper bounded as

$$F_Q[\mathcal{N}_\theta] \leq \left(\prod_{i=2}^{K} d_i^{(\text{in})}\right)^2 \max_{\Psi_1} F_Q[\Phi_{N_\theta}\ast\Psi_1],$$

where $\Phi_{N_\theta}\ast\Psi_1$ is defined in Eq. (6) and $\Psi_1$ is an input state to the first input port of $\mathcal{N}_\theta$.

Theorem 1 provides an upper bound on the QFI of an arbitrary quantum comb. The only optimization required in the upper bound is the maximization of the QFI over input states to the first port. Notice that, since the quantum comb $\mathcal{N}_\theta \ast \left(\Phi_2^+ \otimes \cdots \otimes \Phi_K^+\right)$ has only one input port, the optimization is equivalent to finding the QFI of a quantum channel.

The QFI term on the right hand side of Eq. (8) is attained by a phase-parallel scheme, namely by feeding a quantum state into each of the input ports. In a phase-parallel scheme, the comb $\mathcal{N}_\theta$ is treated as a quantum channel with a multipartite input. The information on $\theta$ never flows back into the comb $\mathcal{N}_\theta$. Therefore, the comb QFI bound (8) shows that the memory effect of a sensor improves the sensitivity of parameter estimation at most by a factor exponential in $K$. Note that a phase-parallel scheme is not necessarily local, since a phase of $\mathcal{N}_\theta$ may consist of joint operations on multiple physical nodes in a realistic quantum network.

It is the main objective of quantum metrology to compare different strategies in terms of the asymptotic scalings of their QFIs with respect to the amount of required resources. In the conventional setting of parallel gate estimation [10][31], for instance, the resource is taken to be the number of calls to the parametrized quantum gate. Here in a quantum comb, the resource is quantified by the number of phases $K$. An obvious observation is that the scaling suggested by the bound (8) is distinct from what one encounters when estimating $\mathcal{N}_\theta$ that does not have a memory. For instance, if one gets back to the standard context of quantum metrology and considers each phase to be an individual quantum channel $\mathcal{C}_\theta$, the scaling of the comb QFI is the Heisenberg scaling $K^2$ if one applies a suitable adaptive strategy [32][33], whereas a phase-parallel scheme achieves the standard quantum limit scaling $K$. In Eq. (8), in contrast, if all the input dimensions are equal to $d$, we get

$$F_Q[\mathcal{N}_\theta] \leq d^{2(K-1)} \cdot F_Q[\mathcal{N}_\theta, \text{phase - parallel}],$$

where $F_Q[\mathcal{N}_\theta, \text{phase - parallel}]$ corresponds to the optimization term in Eq. (8). Next, we present a scenario of quantum metrology in the presence of memory effects, where the above bound is saturated up to a factor of two.

**Estimating a protected parameter.** We now consider a scenario where the optimal strategy is exponentially more
FIG. 3. Estimating a protected parameter. The comb \( N_\theta \) carrying a protected parameter \( \theta \) have \( K \) phases (highlighted by blue boxes). The first phase consists of performing first a parametrized gate \( V_\theta \) and then a shielded unitary \( U \). The key unitary \( U^t \) that recovers the information on \( \theta \) is distributed among the remaining \( K-1 \) phases. If the key unitary is missing, the information on \( \theta \) will be destroyed by the shielded unitary, which is selected using underlying randomness.

The output state is thus

\[
\Phi_{N_\theta \ast \psi} = \int dU \left( U_{A_{\text{out}}} V_{\theta, A_{\text{in}}} \otimes X_{A_{\text{in}}} \otimes U_{B_{\text{in}}}^\dagger \otimes I_{B_{\text{in}}} \right) \left( \Phi_{A_{\text{in}}}^{+} + \Phi_{B_{\text{in}}}^{+} \right).
\]

Using the property \((X \otimes I)|\Phi^+\rangle = (I \otimes X^T)|\Phi^+\rangle\) which holds for any matrix \(X\), the input state can be rewritten as

\[
\Phi_{N_\theta \ast \psi} = \left( T_{A_{\text{in}}} B_{\text{in}} \otimes X_{A_{\text{in}}} V_{\theta, A_{\text{in}}}^T \otimes I_{B_{\text{in}}} \right) \left( \Phi_{A_{\text{in}}}^{+} + \Phi_{B_{\text{in}}}^{+} \right),
\]

where \( T \) is a twirling channel \( T(\rho) := \int dU \left( U \otimes U^t \right) \rho \left( U^t \otimes U \right) \).

Using standard techniques of representation theory \[^{[4]} \]

\[
\tau_{AB} \left( \Phi_{A_{\text{in}}}^{+} + \Phi_{B_{\text{in}}}^{+} \right) = \frac{1}{D^2} \Phi_{A_{\text{in}}}^{+} \otimes \Phi_{B_{\text{in}}}^{+} + \frac{1}{D^2} (I - \Phi^+)_{A_{\text{in}}} \otimes \pi_{A_{\text{in}}}^- \quad (13)
\]

where \( \pi^- := \frac{1}{D^2-1} (I - \Phi^+) \) for a bipartite system. This twirling \( \tau_{AB} \) swaps the entanglement with \( A \) from \( A \) to \( B \) with a success probability \( 1/D^2 \), by performing correlated single-qubit random unitaries.

Substituting Eq. (13) into Eq. (12), we obtain the output state as

\[
\Phi_{N_\theta \ast \psi} = \frac{1}{D^2} (\Psi_{\theta})_{B_{\text{in}}} A_{\text{in}} \otimes \Phi_{A_{\text{in}}}^{+} \otimes \Phi_{B_{\text{in}}}^{+} \notag + \frac{1}{D^2} (I - \Psi_{\theta})_{B_{\text{in}}} A_{\text{in}} \otimes \pi_{A_{\text{in}}}^- \otimes \Phi_{B_{\text{in}}}^{+} \notag \quad (14)
\]

Since \( \pi^- \) and \( \Phi^+ \) are orthogonal to each other, the QFI of the output state can be evaluated using the following lemma:

**Lemma 1.** Consider a parametric family of pure states \( \{ \psi_\theta \} \) and a Hilbert space \( \mathcal{H} \) containing the supports of the family as well as its derivative. If the QFI of \( \psi_\theta \) is
\( F_Q[\psi_\theta] \), then the QFI of \( \rho_\theta := (\dim H - 1)^{-1} (I_H - \psi_\theta) \)
is \( (\dim H - 1)^{-1} F_Q[\psi_\theta] \).

The proof can be found in Appendix. Applying Lemma 1 to the state in Eq. (14), we get

\[
F_Q[\Phi_{N_\theta}^\ast \psi] = \frac{2}{D^2} F_Q[\psi_\theta].
\]

Finally, we get the relation between the QFI of \( N_\theta \) and the QFI of \( \Phi_{N_\theta}^\ast \psi \) as

\[
F_Q[N_\theta] = \frac{1}{2} \left( \prod_{i=1}^{K-1} d_B^{(in)} \right)^2 F_Q[\Phi_{N_\theta}^\ast \psi],
\]

by substituting Eq. (15) and Eq. (10) into Eq. (11). This clearly shows that the bound (8) is tight up to a factor of two for this scenario of estimating a protected phase. Memory effects in quantum metrology is thus manifested by the fact that the optimal adaptive sensor is exponentially more powerful than the phase-parallel sensor.

**Conclusion.** We established a framework for quantum comb metrology and showed the effect of memory in quantum metrology. This could be the start of a new research direction, which deals with metrology in a fully quantum network or estimation in the presence of non-Markovian noise. We conclude with a remark that the quantum combs considered in this work have definite causal structures, while it was recently shown that indefinite causal structures, while it was recently shown that indefinite causal structures, can extend to metrology, which may be related to probing an unknown spacetime structure.

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Proof of Lemma 1

The QFI of a quantum state can be calculated via the equation $F_Q[\rho_\theta] = \text{Tr} [\rho_\theta L_\theta^2]$, where $L_\theta$ is the symmetric logarithmic derivative (SLD) operator determined via the equation $\frac{\partial \rho}{\partial \theta} = \frac{1}{2} (\rho L_\theta + L_\theta \rho)$. In particular, the SLD operator of a pure state $\psi_\theta$ is just $2 (\frac{\partial \psi}{\partial \theta})$, which can be seen by taking partial derivative of both sides of the equation $\psi_\theta = \psi_\theta^2$. With this, it is straightforward to see that $L_\theta = -2 (\frac{\partial \psi}{\partial \theta})$ is a solution to the equation

$$\frac{\partial \rho}{\partial \theta} = \frac{1}{2} (\rho L_\theta + L_\theta \rho).$$

The QFI of $\rho_\theta$ is then $\text{Tr} \frac{\partial \rho}{\partial \theta} L_\theta = \frac{F_Q[\psi_\theta]}{\dim H - 1}$. 

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