Reduced-order observer-based interval estimation for discrete-time linear systems

Yantao Chen\textsuperscript{a}, Junqi Yang\textsuperscript{a}, Zhenhua Wang\textsuperscript{b} and Hongwei Zhang\textsuperscript{a}

\textsuperscript{a}School of Electrical Engineering and Automation, Henan Polytechnic University, Jiaozuo, People’s Republic of China; \textsuperscript{b}School of Astronautics, Harbin Institute of Technology, Harbin, People’s Republic of China

\textbf{ABSTRACT}

This paper proposes an interval estimation method for discrete-time linear systems by designing robust reduced-order observer with reachability analysis. First, the discrete-time linear system is transformed into a reduced-order system via a special equivalent transformation, which depends on the orthogonal procedure on the output matrix. Then, a kind of robust reduced-order observer is developed such that the states of the original discrete-time system can be indirectly estimated, where the $H_{\infty}$ technique is used to attenuate the effects of disturbances. Based on the estimated states provided by the reduced-order observer, the interval estimation can be obtained by reachability analysis. A simulation example is given to illustrate that the proposed method of interval estimation has better performance than the full-order observer-based interval estimation.

\textbf{ARTICLE HISTORY}

Received 3 July 2020
Accepted 24 October 2020

\textbf{KEYWORDS}

Interval estimation; reduced-order observer; reachability analysis; discrete-time system

1. Introduction

The issue of state estimation has been extensively investigated in the last decade and became a very active topic in the modern control theory, which can be widely used to various applications such as state feedback controller design and fault diagnosis. The traditional observer including of unknown input observer can estimate the system states even if there are some unknown disturbances in the control system (Bejarano & Pisano, 2011; Darouach et al., 1994; Mohamed et al., 2013; Soleymani et al., 2020; Trinh et al., 2008; F. Yang & Wilde, 1988; J. Yang et al., 2013), while the results in them cannot provide all of the possible state values, but only the point estimation of system states at each sample time. In order to obtain the admissible state values, the interval estimation methods are developed, and one of them is the set-membership method based on set intersections (Bertsekas & Rhodes, 1971; Durieu et al., 2001; Le et al., 2013; Meslem & Martinez, 2019; Shamma & Tu, 1999), which can characterize the system states at each sample time as a compact set containing all possible state values (Le et al., 2013). There are some popular geometrical forms used to represent this set, for example, ellipsoids (Bertsekas & Rhodes, 1971; Durieu et al., 2001; Meslem & Martinez, 2019), polytopes (Shamma & Tu, 1999), parallelotopes (Chisci et al., 1996; Vicino & Zappa, 1996) and zonotopes (Alamo et al., 2005; Combastel, 2015). In these methods, the complexity increases exponentially with the increasing of the number of its vertices, which will cause heavier computational burden.

The other way to obtain the admissible state values is the interval observer based on the monotone system theory. In the recent years, interval observers have received considerable attention because of the high computational efficiency (Degue et al., 2018; Efimov et al., 2013; Ichalal et al., 2018; Mazenc & Bernard, 2010; Zheng et al., 2016). It was shown that in the particular case of asymptotically stable and cooperative systems, interval observers can be designed directly. This assumption of cooperativity is the main limitation of interval observers as most of the systems are not cooperative. However, in the case of linear time-invariant systems this hypothesis can be relaxed by using a time-varying change of coordinates (Mazenc & Bernard, 2011; Raissi et al., 2012). But the coordinate transformation may cause additional conservatism, and a new interval observer was proposed to overcome this limitation and improve the estimation accuracy in Wang et al. (2018). Recently, a two-step interval estimation method (Tang, Wang, & Shen, 2019; Tang, Wang, Wang, et al., 2019) was developed by integrating robust observer design with reachability analysis technique (Girard et al., 2006), where the drawbacks of both the zonotope-based method and interval observers can be overcome. In general,
the aforementioned interval observers were full-order observers (Degue et al., 2018; Efimov et al., 2013; Ichalal et al., 2018; Mazenc & Bernard, 2010, 2011; Raissi et al., 2012; Tang, Wang, & Shen, 2019; Tang, Wang, Wang, et al., 2019; Wang et al., 2018; Zheng et al., 2016). Nevertheless, a reduce-order observer only estimates partial state that is independent of the system output. Hence, it has a lower dimension than that of the full-order observer. This implies that the reduce-order observer can be constructed with simpler structure. Besides, the partial states directly derived from the system output have no effects on the error dynamics of the reduced-order observer. Therefore, compared with the full-order observer-based interval estimation, the performance of interval estimation will be improved by developing a kind of interval estimation method based on reduced-order observer design. This stimulates us to discuss this topic in this paper.

Motivated by the above observations, this paper formulates a kind of robust reduced-order observer for discrete-time linear system, and then an interval estimation method is proposed by integrating robust reduced-order observer with reachability analysis technique (Girard et al., 2006). The contributions and advantages of this paper are summarized as follows. Frist, a new discrete-time linear system is constructed by state equivalent transformation and applying an orthogonal procedure to the output matrix, where the partial states of it can be derived from the output of the original system. Second, a reduced-order system is formulated via a special coordinate transformation, and a robust reduced-order observer that can indirectly estimate the states of the original system is developed. Third, we obtain the interval estimation by combining robust reduced-order observer design with reachability analysis, which can further optimize the state estimation interval. Compared with the method based on full-order observer, the interval estimation based on reduced-order observer with reachability analysis has better performance.

The remainder of this paper is organized as follows. In Section 2, some preliminaries are provided for later developments. The issue of reduced-order observer-based interval estimation is formulated in Section 3. Section 4 presents the robust reduced-order observer-based interval estimation, and an iterative algorithm is provided to implement the proposed method. In Section 5, simulation results are given to illustrate the performance of the proposed method. Some conclusions are summarized in Section 6.

### 2. Preliminaries

We denote an interval subset of $\mathbb{R}$ by $[x, \bar{x}]$ where $x$ and $\bar{x}$ stand, respectively, for its lower and upper bounds, and $\mathbb{R}$ is the set of all real numbers. $\mathbb{R}^n$ and $\mathbb{R}^{n \times m}$ denote the $n$ and $n \times m$ dimensional Euchidean spaces, respectively. The symbols $>$, $\geq$, $<$ and $\leq$ should be understood elementwise. The notation $P > 0$ $(P < 0)$ is used to denote a real positive (negative) definite symmetric matrix. For a discrete transfer function $G(s)$, $\|G(s)\|_\infty$ denotes its $H_\infty$ norm.

The following definitions and properties are essential in this paper.

**Definition 2.1:** The Minkowski sum of two sets $X$ and $Y$ is defined by

$$X \oplus Y = \{x + y : x \in X, y \in Y\}$$

and the Minkowski sum of a series of sets can be denoted as

$$\bigoplus_{i=1}^{m} S_i = S_1 \oplus S_2 \oplus \cdots \oplus S_m.$$ 

**Definition 2.2:** For a set $S \in \mathbb{R}^n$, its interval hull $\text{Box}(S)$ is defined as the smallest interval vector containing it, which is denoted as

$$S \subseteq \text{Box}(S) = [\underline{s}, \bar{s}],$$

where $[\underline{s}, \bar{s}] = \{s : s \in S, \underline{s} \leq s \leq \bar{s}\}$ is an interval vector, $\bar{s}$ and $\underline{s}$ denote the minimum upper and maximum lower bounds of $s$.

**Property 2.3:** For a series of sets $S_i \in \mathbb{R}^n, i = 1, 2, \ldots, m$, one can derive from Definition 2.1 and the Minkowski sum of a series of sets that the following equation

$$\text{Box} \left( \bigoplus_{i=1}^{m} S_i \right) = \bigoplus_{i=1}^{m} \text{Box}(S_i)$$

holds.

**Definition 2.4:** An $m$-order zonotope $Z \in \mathbb{R}^n$ is an affine transformation of a hypercube $B^m$, which is defined as

$$Z = \langle p, H \rangle = \{p + Hz : z \in B^m\},$$

where $p \in \mathbb{R}^n$ is the centre of $Z$, $H \in \mathbb{R}^{n \times m}$ is called the generator matrix of $Z$, which defines the shaper of $Z$. $m$ and $n$ are the order and dimension number of $Z$, respectively.

**Property 2.5 (Combastel, 2015):** For zonotopes, the following equations

$$\langle p_1, H_1 \rangle \oplus \langle p_2, H_2 \rangle = \langle p_1 + p_2, [H_1, H_2] \rangle$$

$$L\langle p, H \rangle = \langle Lp, LH \rangle$$

hold, where $p, p_1, p_2 \in \mathbb{R}^n$ and $H, H_1, H_2 \in \mathbb{R}^{n \times m}$. 

---

**Box**
Property 2.6 (Tang, Wang, Wang, et al., 2019): For an $m$-order zonotope $Z = \langle p, H \rangle \in \mathbb{R}^m$, the components of $\text{Box}(Z) = [\underline{z}, \bar{z}]$ can be obtained from
\[
\begin{align*}
\underline{z}_i &= p_i - \sum_{j=1}^m |H_{ij}|, & i = 1, 2, \ldots, n, \\
\bar{z}_i &= p_i + \sum_{j=1}^m |H_{ij}|, & i = 1, 2, \ldots, n,
\end{align*}
\] (4)
where $\underline{z}_i, \bar{z}_i$ and $p_i$ denote the $i$th components of $\underline{z}, \bar{z}$ and $p$, respectively. $H_{ij}$ denotes the element of $H$ in the $i$th row and the $j$th column.

3. Problem formulation

In this paper, we consider the following discrete-time linear system with disturbance
\[
\begin{align*}
x_{k+1} &= Ax_k + Bu_k + Dw_k, \\
y_k &= Cx_k,
\end{align*}
\] (5)
where $x_k \in \mathbb{R}^{nx}$ is the state vector and bounded for all $k \geq 0$. $u_k \in \mathbb{R}^{nu}$ and $y_k \in \mathbb{R}^{ny}$ denote the vectors of control input and measurement output, respectively. $w_k \in \mathbb{R}^{nw}$ is the unknown disturbance vector. $A \in \mathbb{R}^{nx \times nx}$, $B \in \mathbb{R}^{nx \times nu}$, $C \in \mathbb{R}^{ny \times nx}$ and $D \in \mathbb{R}^{ny \times nw}$ are known constant matrices. Without loss of generality, the following assumptions are given.

Assumption 3.1: The dimensions of both state and output vectors satisfy $ny \leq nx$, and the output matrix $C$ is row full rank.

Assumption 3.2: The initial state value $x_0$ and unknown disturbance $w_0$ are unknown but bounded as follows
\[
x_0 \in \langle p_0, H_0 \rangle, \quad w_0 \in \langle 0, H_w \rangle,
\]
where $p_0 \in \mathbb{R}^{nx}$, $H_0 \in \mathbb{R}^{nx \times nx}$ and $H_w \in \mathbb{R}^{nw \times nw}$ are known vector and matrices.

The objective of the interval estimation method is to obtain an interval vector $[\underline{x}_k, \bar{x}_k]$, which contains the real state $x_k$, i.e.
\[
x_k \leq x_k \leq \bar{x}_k, \quad k \geq 0.
\]
Similar to the method of Tang, Wang, Wang, et al. (2019), the full-order observer-based interval estimation is proposed by using the reachability analysis technique. Here, we briefly give the results as follow.

Lemma 3.3: Given a scalar $\gamma > 0$, if there exist a positive definite symmetric matrix $P \in \mathbb{R}^{nx \times nx}$ and a matrix $Y \in \mathbb{R}^{nx \times ny}$ such that
\[
\begin{bmatrix}
-P & PA - YC & PD \\
A^T P - C^T Y^T & -P + I_{nx} & 0 \\
D^T P & 0 & -\gamma^2 I_{nw}
\end{bmatrix} < 0
\] (6)
holds. Then, the following dynamic system with gain matrix $K = p^{-1}Y$,
\[
\dot{\hat{x}}_{k+1} = Ax_k + Bu_k + K (y_k - C\hat{x}_k)
\] (7)
is a robust full-order observer of system (5). And the transfer function from $w_k$ to $e_k$, $G_{we}(s) = (sI_n - A + KC)^{-1}D$, satisfies $\|G_{we}(s)\|_\infty < \gamma$, where the state estimation error $e_k \triangleq x_k - \hat{x}_k$ is bounded such that $e_k \leq \bar{e}_k$.

Based on reachability analysis of the estimation error, the interval estimation of $x_k$ can be obtained by the following Lemma (Tang, Wang, Wang, et al., 2019).

Lemma 3.4: For the system (5) and the observer (7), given $\hat{x}_0 = p_0$, then the system state $x_k$ is bounded by the interval estimation, $[\underline{x}_k, \bar{x}_k]$ obtained from
\[
\begin{align*}
\underline{x}_k &= \hat{x}_k + \underline{e}_k, \\
\bar{x}_k &= \hat{x}_k + \bar{e}_k,
\end{align*}
\] (8)
where $\underline{e}_k$ and $\bar{e}_k$ are determined by
\[
\begin{bmatrix}
\underline{e}_k \\ \bar{e}_k
\end{bmatrix} = \text{Box} \left( (A - KC)^k (0, H_0) \right)
\] (9)
\[
\oplus \bigoplus_{i=0}^{k-1} \text{Box} \left( (A - KC)^i D (0, H_w) \right)
\] (10)
with
\[
\begin{bmatrix}
\underline{e}_0 \\ \bar{e}_0
\end{bmatrix} = \text{Box} \left( (0, H_0) \right).
\]

The purpose of this paper is to develop a new interval estimation method for discrete-time linear systems by designing robust reduced-order observer with reachability analysis. A kind of robust reduced-order observer is first proposed, and then the interval estimation of $x_k$ is obtained by the reachability analysis of estimation error. Finally, an algorithm is also given to summarize the proposed method, and the comparisons of interval estimation results are discussed in simulation.

4. Robust reduced-order observer-based interval estimation

In this section, we propose a kind of robust reduced-order observer for system (5), where the orthogonal procedure (Zhu & Han, 2002) is employed. Then, the interval estimation is provided by integrating the robust reduced-order observer with reachability analysis technique (Girard et al., 2006).
4.1. Robust reduced-order observer design

For the row full rank matrix \( C \in \mathbb{R}^{ny \times n_y} \), we can find an invertible matrix \( \Phi \in \mathbb{R}^{ny \times ny} \) and matrix \( C' \in \mathbb{R}^{ny \times n_x} \) by Smith orthogonal procedure such that \( C = \Phi C' \) and \( C'C^T = I_{ny} \). Then, we extend the matrix \( C' \) to an orthogonal matrix \( S = [C' \Sigma] \), where \( \Sigma \in \mathbb{R}^{(n_y - n_y) \times n_y} \) and \( S \in \mathbb{R}^{ny \times ny} \). If we apply an equivalent transformation \( \chi_k = S \chi_k \) to the system (5), we will obtain the following equivalent system

\[
\begin{align*}
\chi_{k+1} &= \hat{A}\chi_k + \hat{B}u_k + \hat{D}w_k, \\
\chi_k &= \hat{C}\chi_k,
\end{align*}
\]

(11)

where \( \hat{A} = SAS^{-1}, \hat{B} = SB, \hat{D} = SD \) and \( \hat{C} = CS^{-1} \).

Remark 4.1: It is worth pointing out that for the output matrix with a special form \( C = [I_{ny} \ 0_{ny \times (n_x - n_y)}] \) or \( C = [0_{ny \times (n_x - n_y)} \ I_{ny}] \), we do not need to apply an orthogonal transformation to \( C \). Under this special output matrix, the method presented in the following part will be easy to be directly implemented. Therefore, this paper considers a more general case.

Decomposing matrices \( \hat{A}, \hat{B}, \hat{D} \) and state vector \( \chi_k \) into block matrices or vectors as follows

\[
\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix},
\]

\[
\hat{D} = \begin{bmatrix} \hat{D}_1 \\ \hat{D}_2 \end{bmatrix}, \quad \chi_k = \begin{bmatrix} \chi_{1,k} \\ \chi_{2,k} \end{bmatrix}
\]

where \( \hat{A}_{11} \in \mathbb{R}^{ny \times ny}, \hat{A}_{22} \in \mathbb{R}^{(n_x - n_y) \times (n_x - n_y)}, \hat{B}_1 \in \mathbb{R}^{ny \times n_y}, \hat{D}_1 \in \mathbb{R}^{ny \times n_x} \) and \( \chi_{1,k} \in \mathbb{R}^{ny} \). Naturally, it follows from the orthogonal matrix \( S \) that there is \( CS^{-1} = S(C'C) = [\Phi 0] \) which gives \( y_k = \Phi \chi_{1,k} \). So, the system state \( \chi_{1,k} \) can be directly obtained from measurement output \( y_k \) and given as

\[
\chi_{1,k} = \Phi^{-1}y_k.
\]

(12)

Thus, it is easy to derive from (11) and (12) that the state equation of \( \chi_{2,k} \) in system (11) is equivalent to

\[
\begin{align*}
\chi_{1,k+1} &= \hat{A}_{12}\chi_{2,k} + \hat{A}_{11}\Phi^{-1}y_k + \hat{B}_1u_k + \hat{D}_1w_k, \\
\chi_{2,k+1} &= \hat{A}_{22}\chi_{2,k} + \hat{A}_{21}\Phi^{-1}y_k + \hat{B}_2u_k + \hat{D}_2w_k.
\end{align*}
\]

(13)

Take a coordinate transformation of

\[
\varsigma_k = \begin{bmatrix} \varsigma_{1,k} \\ \varsigma_{2,k} \end{bmatrix} = \begin{bmatrix} I_{ny} & 0 \\ L & I_{n_y - n_y} \end{bmatrix} \chi_k,
\]

(14)

where \( \varsigma_{1,k} \in \mathbb{R}^{ny} \) and \( \varsigma_{2,k} \in \mathbb{R}^{n_x - n_y} \). Then, the system (13) together with (14) implies that the state \( \varsigma_{2,k} \) satisfies the following equation

\[
\begin{align*}
\varsigma_{2,k+1} &= (\hat{L}\hat{A}_{12} + \hat{A}_{22}) \varsigma_{2,k} + (\hat{L}\hat{A}_{11} + \hat{A}_{21}) \Phi^{-1}y_k \\
&\quad + (\hat{L}\hat{B}_1 + \hat{B}_2) u_k + (\hat{L}\hat{D}_1 + \hat{D}_2) w_k.
\end{align*}
\]

(15)

Besides, we can obtain \( \chi_{2,k} = \varsigma_{2,k} - L \Phi^{-1}y_k \) from (12) and (14). Inserting it into (15), there is

\[
\begin{align*}
\varsigma_{2,k+1} &= (\hat{L}\hat{A}_{12} + \hat{A}_{22}) \varsigma_{2,k} \\
&\quad + ((\hat{L}\hat{A}_{11} + \hat{A}_{21}) - (\hat{L}\hat{A}_{12} + \hat{A}_{22}) L) \Phi^{-1}y_k \\
&\quad + (\hat{L}\hat{B}_1 + \hat{B}_2) u_k + (\hat{L}\hat{D}_1 + \hat{D}_2) w_k.
\end{align*}
\]

(16)

Theorem 4.1: Under Assumption 3.1, if for a given scalar \( \rho > 0 \), there exist a positive definite symmetric matrix \( Q \in \mathbb{R}^{(n_x - n_y) \times (n_x - n_y)} \) and matrix \( X \in \mathbb{R}^{(n_x - n_y) \times ny} \) such that the following linear matrix inequality

\[
\begin{bmatrix}
-Q & X\hat{A}_{12} + Q\hat{A}_{22} & XD_1 + QD_2 \\
(\hat{X}\hat{A}_{12} + Q\hat{A}_{22})^T & -Q + I_{n_y - n_y} & 0 \\
(XD_1 + QD_2)^T & 0 & -\rho^2 I_n \end{bmatrix} < 0
\]

(17)

has a feasible solution, then the following system

\[
\begin{align*}
\hat{\varsigma}_{2,k+1} &= (\hat{L}\hat{A}_{12} + \hat{A}_{22}) \hat{\varsigma}_{2,k} \\
&\quad + ((\hat{L}\hat{A}_{11} + \hat{A}_{21}) - (\hat{L}\hat{A}_{12} + \hat{A}_{22}) L) \Phi^{-1}y_k \\
&\quad + (\hat{L}\hat{B}_1 + \hat{B}_2) u_k \\
\hat{\varsigma}_{k} &= S^{-1} I_p \left[ \begin{bmatrix} L & 0 \end{bmatrix} \begin{bmatrix} \Phi^{-1}y_k \end{bmatrix} \right]
\end{align*}
\]

(18)

is a robust reduced-order observer of the system (5) with gain matrix \( L = Q^{-1}X \). The transfer function from \( w_k \) to \( \hat{\varsigma}_{2,k} \), \( G_{w,\hat{\varsigma}}(s) = (sI_p - L\hat{A}_{12} - \hat{A}_{22})^{-1}(L\hat{D}_1 + \hat{D}_2) \), satisfies \( \|G_{w,\hat{\varsigma}}(s)\|_\infty < \rho \), where \( \hat{\varsigma}_{2,k} \) is the state estimation error between the system (16) and reduced-order observer (18), and \( \hat{\varsigma}_{k} \) is the estimation of system state \( \varsigma_{k} \) in the case of reduced-order observer.

Proof: It is easy to obtain the state estimation error system from (16) and (18) as

\[
\hat{\varsigma}_{2,k+1} = \varsigma_{2,k+1} - \hat{\varsigma}_{2,k+1} = (\hat{L}\hat{A}_{12} + \hat{A}_{22}) \varsigma_{2,k} + (\hat{L}\hat{D}_1 + \hat{D}_2) w_k.
\]

(19)

Firstly, we define a Lyapunov function \( V(\hat{\varsigma}_{2,k}) = \hat{\varsigma}_{2,k}^TQ\hat{\varsigma}_{2,k} \). Taking the difference of the Lyapunov function
yields
\[
\Delta V(\tilde{s}_{2,k}) = \tilde{s}_{2,k+1}^T Q \tilde{s}_{2,k+1} - \tilde{s}_{2,k}^T Q \tilde{s}_{2,k} \\
= \tilde{s}_{2,k}^T (L\tilde{A}_{12} + \tilde{A}_{22})^T Q (L\tilde{A}_{12} + \tilde{A}_{22}) \tilde{s}_{2,k} \\
+ \tilde{s}_{2,k}^T (L\tilde{A}_{12} + \tilde{A}_{22})^T Q (L\tilde{D}_1 + \tilde{D}_2) w_k \\
+ w_k^T (L\tilde{D}_1 + \tilde{D}_2)^T Q (L\tilde{A}_{12} + \tilde{A}_{22}) \tilde{s}_{2,k} \\
+ w_k^T (L\tilde{D}_1 + \tilde{D}_2)^T Q (L\tilde{D}_1 + \tilde{D}_2) w_k - \tilde{s}_{2,k}^T Q \tilde{s}_{2,k}
\]

Let \( J = \Delta V(\tilde{s}_{2,k}) + \tilde{s}_{2,k}^T \tilde{s}_{2,k} - \rho^2 w_k^T w_k \), which together with (20) gives that
\[
J = \Delta V(\tilde{s}_{2,k}) + \tilde{s}_{2,k}^T \tilde{s}_{2,k} - \rho^2 w_k^T w_k
\]

where \( \varphi_{11} = (L\tilde{A}_{12} + \tilde{A}_{22})^T Q (L\tilde{A}_{12} + \tilde{A}_{22}) - Q + I_{n_x-n_y} \), \( \varphi_{12} = (L\tilde{A}_{12} + \tilde{A}_{22})^T Q (L\tilde{D}_1 + \tilde{D}_2) \), \( \varphi_{21} = (L\tilde{D}_1 + \tilde{D}_2)^T Q (L\tilde{A}_{12} + \tilde{A}_{22}) \), \( \varphi_{22} = (L\tilde{D}_1 + \tilde{D}_2)^T Q (L\tilde{D}_1 + \tilde{D}_2) - \rho^2 I_{n_u} \).

Since \( L = Q^{-1} X \), we have \( X = Q L \). Substituting it into (17) yields
\[
\begin{bmatrix}
- Q & (L\tilde{A}_{12} + \tilde{A}_{22})^T Q (L\tilde{D}_1 + \tilde{D}_2) \\
(L\tilde{A}_{12} + \tilde{A}_{22})^T Q & - Q + I_{n_x-n_y} & 0 \\
(L\tilde{D}_1 + \tilde{D}_2)^T Q & 0 & - \rho^2 I_{n_u}
\end{bmatrix} < 0.
\]

By using Schur complement to (21), we can obtain that
\[
\begin{bmatrix}
\varphi_{11} & \varphi_{12} \\
\varphi_{21} & \varphi_{22}
\end{bmatrix} < 0
\]
which is obviously means \( J < 0 \). Under zero initial condition, \( J < 0 \) means that
\[
V(\tilde{s}_{2,k+1}) + \sum_{i=0}^{k} (\tilde{s}_{2,i}^T \tilde{s}_{2,i} - \rho^2 w_i^T w_i) < 0.
\]

Since the positive Lyapunov function \( V(\tilde{s}_{2,k+1}) \), the above inequality gives
\[
\sum_{i=0}^{k} \tilde{s}_{2,i}^T \tilde{s}_{2,i} < \rho^2 \sum_{i=0}^{k} w_i^T w_i
\]
which implies \( \|Gw(\tilde{s}_2(s))\|_\infty < \rho \) for \( k \to \infty \). Therefore, the inequality (17) guarantees that the state estimation error system (19) is quadratically stable with an \( H_\infty \) norm bound \( \rho \).

Besides, after we obtain the state estimation \( \tilde{s}_{2,k} \) from the reduced-order observer (18), we can get the state estimation of system (5) as follows
\[
\hat{x}_k = S^{-1} \begin{bmatrix} l_{n_y} & 0 \\ 0 & l_{n_x-n_y} \end{bmatrix}^{-1} \begin{bmatrix} \Phi^{-1} y_k \\ \hat{s}_{2,k} \end{bmatrix}
\]
since the equivalent transformation \( \chi_k = S x_k \) and the coordinate transformation (14).

Note that the minimal \( \gamma \) and \( \rho \) can be obtained by solving the following optimization problems, respectively,
\[
\min \gamma^2 \text{ s.t. } (22) \quad \text{and} \quad \min \rho^2 \text{ s.t. } (17).
\]
Then, the feasible solutions of them give the gain matrices of both full-order observer and reduced-order observer by \( K = P^{-1} Y \) and \( L = Q^{-1} X \), respectively.

### 4.2. Robust reduced-order observer-based interval estimation

After the robust reduced-order observer is proposed, the interval estimation of \( x_k \) can be obtained by the reachability analysis of estimation error. For this, we give the following theorem.

**Theorem 4.2:** Under Assumption 3.1 and the given \( \hat{x}_k^0 = p_0 \), the state \( x_k \) is bounded by the interval estimation. Based on the state estimation \( \hat{x}_k \) coming from the reduced-order observer (18), \([\underline{x}_k, \overline{x}_k]\) can be obtained from
\[
\underline{x}_k = \hat{x}_k + \bar{e}_k, \quad \overline{x}_k = \hat{x}_k + \underline{e}_k,
\]
where the estimation error is defined as \( e_k = x_k - \hat{x}_k \) and its upper bound \( \bar{e}_k \) and lower bound \( \underline{e}_k \) are determined by
\[
[\underline{e}_k, \bar{e}_k] = Box \left( S^{-1} \begin{bmatrix} \Sigma^T (L\tilde{A}_{12} + \tilde{A}_{22})^T (L\tilde{D}_1 + \tilde{D}_2) & 0 \\ 0 & H_0 \end{bmatrix} \right) \oplus \bigoplus_{i=0}^{k-1} Box \left( S^{-1} \begin{bmatrix} \Sigma^T (L\tilde{A}_{12} + \tilde{A}_{22})^T (L\tilde{D}_1 + \tilde{D}_2) & (0, H_0) \end{bmatrix} \right)
\]
(25)
with
\[
[e_0, \tilde{e}_0] = \text{Box} \left( \Sigma^T \left[ \begin{array}{cc} 0 & I_{n_x - n_y} \end{array} \right] S (0, H_0) \right). \quad (26)
\]

**Proof:** One can obtain that the equivalent transformation \( x_k = S x_k \) together with the coordinate transformation (14) gives that
\[
x_k = S^{-1} x_k = S^{-1} \left[ \begin{array}{cc} I_{n_y} & 0 \\ L & I_{n_x - n_y} \end{array} \right]^{-1} \tilde{s}_k
\]
Thus, the state estimation error \( e_k = x_k - \tilde{x}_k \) can be denoted as
\[
e_k = S^{-1} \left[ \begin{array}{cc} I_{n_y} & 0 \\ L & I_{n_x - n_y} \end{array} \right]^{-1} \left[ \Phi^{-1} y_k \right]
- S^{-1} \left[ \begin{array}{cc} I_{n_y} & 0 \\ L & I_{n_x - n_y} \end{array} \right]^{-1} \left[ \Phi^{-1} y_k \right]
= S^{-1} \left[ \begin{array}{cc} I_{n_y} & 0 \\ -L & I_{n_x - n_y} \end{array} \right] \left[ \begin{array}{c} 0 \\ \tilde{s}_{2,k} \end{array} \right]
= S^{-1} \left[ \begin{array}{c} 0 \\ \tilde{s}_{2,k} \end{array} \right] (27)
\]
which follows that
\[
e_k = \Sigma^T \tilde{s}_{2,k} \quad (28)
\]

since \( S^{-1} = [C^T \quad \Sigma^T]. \) Besides, \( \tilde{s}_{2,k} \) can be obtained from the error system (19) by
\[
\tilde{s}_{2,k} = (L\tilde{A}_{12} + \tilde{A}_{22})^k \tilde{s}_{2,0}
+ \sum_{i=0}^{k-1} (L\tilde{A}_{12} + \tilde{A}_{22})^i (L\tilde{D}_1 + \tilde{D}_2) w_{k-1-i}
\]
which together with (28) gives that
\[
e_k = \Sigma^T (L\tilde{A}_{12} + \tilde{A}_{22})^k \tilde{s}_{2,0}
+ \Sigma^T \sum_{i=0}^{k-1} (L\tilde{A}_{12} + \tilde{A}_{22})^i (L\tilde{D}_1 + \tilde{D}_2) w_{k-1-i} \quad (29)
\]
Meanwhile, one can also derive from the Equation (27) that \( \tilde{s}_{2,k} = [0 \ I_{n_x - n_y}] S e_k \) which means \( \tilde{s}_{2,0} = [0 \ I_{n_x - n_y}] S e_0. \) Inserting the above equation into (29), we can obtain that
\[
e_k = \Sigma^T (L\tilde{A}_{12} + \tilde{A}_{22})^k [0 \ I_{n_x - n_y}] S e_0
+ \Sigma^T \sum_{i=0}^{k-1} (L\tilde{A}_{12} + \tilde{A}_{22})^i (L\tilde{D}_1 + \tilde{D}_2) w_{k-1-i} \quad (30)
\]
Define the reachable set of \( e_k \) as \( \Theta_k, \) then we can derive from (30) and Definition 2.1 that
\[
\Theta_k = \Sigma^T (L\tilde{A}_{12} + \tilde{A}_{22})^k [0 \ I_{n_x - n_y}] S \Theta_0
+ \bigoplus_{i=0}^{k-1} \text{Box} \left( \Sigma^T (L\tilde{A}_{12} + \tilde{A}_{22})^i (L\tilde{D}_1 + \tilde{D}_2) (0, H_{w_i}) \right).
\]
So, the interval hull of \( \Theta_k \) can be got from Property (1) and the above equation by
\[
\text{Box} (\Theta_k) = \text{Box} \left( \Sigma^T (L\tilde{A}_{12} + \tilde{A}_{22})^k [0 \ I_{n_x - n_y}] S \Theta_0 \right)
+ \bigoplus_{i=0}^{k-1} \text{Box} \left( \Sigma^T (L\tilde{A}_{12} + \tilde{A}_{22})^i (L\tilde{D}_1 + \tilde{D}_2) (0, H_{w_i}) \right).
\]
Furthermore, \( x_0 \in (p_0, H_0) \) indicates that \( e_0 \in (0, H_0) \) for \( \tilde{x}_0 = p_0, \) which implies \( \Theta_0 = (0, H_0). \) So, from (25), (26) and (31), we have \( [\tilde{e}_k, \tilde{e}_k] = \text{Box} (\Theta_k) \) for \( k \geq 0. \) Since \( e_k \in \Theta_k \subseteq \text{Box} (\Theta_k), \) it follows that \( e_k \leq e_k \leq \tilde{e}_k. \) Then, from the state estimation error \( e_k = x_k - \tilde{x}_k, \) we get \( \tilde{x}_k \leq \tilde{x}_k \leq \tilde{x}_k \leq \tilde{x}_k \leq \tilde{e}_k. \) Thus, we will get \( \tilde{x}_k \leq x_k \leq \tilde{x}_k \) from (24). ■

The bounded state \( x_k \) together with the equivalent transformation \( \chi_k = S x_k \) and coordinate transformation (14) shows that the state of the reduced-order system (18) is also bounded. There gives the following corollary to describe the initial state \( \tilde{s}_{2,0}. \)

**Corollary 4.3:** The initial state \( \tilde{s}_{2,0} \) of the reduced-order observer is unknown but bounded, and \( \tilde{s}_{2,0} \in Z_r \) for \( Z_r \) is a zonotope and defined as
\[
Z_r = \left[ [L \ I_{n_x - n_y}] S p_0, [L \ I_{n_x - n_y}] S H_0 \right].
\]

**Proof:** The coordinate transformation (14) implies \( \tilde{s}_{2,k} = [L \ I_{n_x - n_y}] \chi_k. \) Substituting the equivalent transformation \( \chi_k = S x_k \) into the above equation, we can obtain \( \tilde{s}_{2,k} = [L \ I_{n_x - n_y}] S x_k. \) So, the initial state of \( \tilde{s}_{2,0} \) should be satisfied with
\[
\tilde{s}_{2,0} = [L \ I_{n_x - n_y}] S x_0
\]
which follows from the Property (3) that
\[
\tilde{s}_{2,0} \in [L \ I_{n_x - n_y}] S (p_0, H_0)
= \left[ [L \ I_{n_x - n_y}] S p_0, [L \ I_{n_x - n_y}] S H_0 \right]
\]
since \( x_0 \in (p_0, H_0). \) ■
In order to better understand and summarize the reduced-order observer-based interval estimation, and to compare the interval estimation results with the full-order observer-based case, an iterative algorithm (Algorithm 1) is provided in detail.

**Algorithm 1 Interval estimations based on reachability analysis**

**Input:** Matrices $A$, $B$, $C$ and $D$, input $u_k$, output $y_k$; disturbance zonotope $(0, H_w)$

**Output:** Interval estimations $\bar{x}_k, \hat{x}_k$ from the full-order observer, $\bar{x}'_k$ and $\hat{x}'_k$ from the reduced-order observer

**Procedure:**
1. Compute the matrices $\Phi, C', \Sigma, S$ and the system matrices $A_{11}, A_{12}, A_{21}, A_{22}, B_1, B_2, D_1$ and $D_2$.
2. Obtain matrices $K$ and $L$ by solving the optimization problems (22) and (23).
3. $x_0 = p_0$, $\hat{x}_0 = [L \ l_{n_x-n_y}]S_p0, \hat{x}'_0 = S^{-1}\left[ \begin{array}{c} l_p \ 0 \end{array} \right]$\end{equation}
   $W_0 = D(0, H_w)$ and $W_0' = (LD_1 + D_2)(0, H_w)$.
4. $S_{x_0} = (0, H_0), S_{\hat{x}_0} = [0 \ l_{n_x-n_y}]S(0, H_0), S_{\hat{x}_0'} = \emptyset$ and $S_{\hat{x}_0'} = \emptyset$.
5. for $k \geq 0$
   6. $[e_k, \hat{e}_k] = Box(S_{x_k}) \oplus S_{w_k}, [e'_k, \hat{e}'_k] = \Sigma^T(Box(S'_{x_k}) \oplus S'_{w_k})$;
   7. $\bar{x}_k = \bar{x}_k + e_k, \hat{x}_k = \hat{x}_k + \hat{e}_k, \bar{x}'_k = \bar{x}'_k + e'_k, \hat{x}'_k = \hat{x}'_k + \hat{e}'_k$;
   8. $\hat{x}_{k+1} = A\hat{x}_k + B_1u_k + K(y_k - C\hat{x}_k)$;
   9. $\bar{x}_{k+1} = (LA_{12} + \bar{A}_{22})\bar{x}_k + (LA_{12} + \bar{A}_{22})L\Phi^{-1}y_k$ + $(LB_1 + \bar{B}_2)u_k$;
   10. $\hat{x}'_{k+1} = S^{-1}\left[ \begin{array}{c} l_p \ 0 \end{array} \right]^{-1}\Phi^{-1}y_{k+1}$;
   11. $S'_{x_{k+1}} = (A - K)S_{x_k}, S'_{w_{k+1}} = S_{w_k} \oplus Box(W_{k}), W_{k+1} = (A - K)W_k$;
   12. $S'_{x_{k+1}} = (LA_{12} + \bar{A}_{22})S'_{x_k}, S'_{w_{k+1}} = S'_{w_k} \oplus Box(W'_k), W'_{k+1} = (LA_{12} + \bar{A}_{22})W'_k$.

end procedure

Note that for a discrete-time system with simultaneous unknown disturbance and measurement noise, the interval estimation will be more meaningful by designing reduced-order observer. However, this requires different methods to deal with the measurement noise, while the dimension of observer is less than the one of original system. There are some full-order interval observers to tackle the measurement noise, but the reduced-order case is still a problem to be solved, which will be our next consideration. Besides, how to extend the method proposed in this paper to the time-delay system (Qian, Li, Chen, et al., 2020; Qian, Li, Zhao, et al., 2020; Qian, Xing, et al., 2020) is also meaningful topic.

5. Simulations

In this section, an example derived from Meslem et al. (2018) is studied to show the effectiveness of the proposed method. The system considered in this paper has the form of (5) with the following parameters.

$$A = \begin{bmatrix} -0.1321 & 1.1732 & -0.1794 & 0.1522 \\ -0.8148 & 0.1689 & 0.3771 & 0.9308 \\ -0.0871 & 0.4465 & -0.0722 & -0.3125 \\ 0.0804 & 0.2929 & -0.1056 & -0.0631 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.2589 \\ -0.6342 \\ 0.3125 \\ -0.7458 \end{bmatrix},$$

$$D = \begin{bmatrix} 0.1264 & 0 & 0 & 0.1267 \\ 0 & 0.0932 & 0 & 0 \\ 0.0958 & 0 & 0.1325 & 0 \\ 0.1052 & 0 & 0 & 0.2189 \end{bmatrix},$$

$$C = [0.3502 \ 0.9298 \ 0.2389].$$

For matrix $C$, by Smith orthogonal procedure we can obtain invertible matrix $\Phi = 1.0221$ and matrix $C' = [0.3426 \ 0.9097 \ 0.2346]$ such that $C = \Phi C'$ and $C' \Sigma^T = L_n$. Then, we can find matrix

$$\Sigma = \begin{bmatrix} 0.9395 & 0 & -0.3318 & -0.0856 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.2497 & -0.9683 \end{bmatrix},$$

such that the matrix $C'$ can be extended to an orthogonal matrix

$$S = \begin{bmatrix} 0.3426 & 0 & 0.9097 & 0.2346 \\ 0.9395 & 0 & -0.3318 & -0.0856 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.2497 & -0.9683 \end{bmatrix}.$$

Thus, one can derive the matrices of system (11) as

$$\tilde{A} = \begin{bmatrix} -0.2323 & -0.0277 & 0.8769 & 0.2012 \\ -0.0992 & -0.0725 & 0.9290 & -0.2779 \\ 0.2823 & -0.9702 & 0.1689 & -0.8071 \\ 0.0385 & -0.1201 & -0.1721 & 0.0374 \end{bmatrix},$$

$$\tilde{B} = \begin{bmatrix} 0.1983 \\ 0.2042 \\ -0.6342 \\ 0.8002 \end{bmatrix},$$

$$\tilde{D} = \begin{bmatrix} 0.1551 & 0 & 0.1205 & 0.0097 \\ 0.0780 & 0 & -0.0440 & -0.1378 \\ 0 & 0.0932 & 0 & 0 \\ -0.0779 & 0 & 0.0331 & -0.2120 \end{bmatrix},$$

which give the block matrices $\tilde{A}_{11}, \tilde{A}_{12}, \tilde{A}_{21}, \tilde{A}_{22}, \tilde{B}_1, \tilde{B}_2, \tilde{D}_1$ and $\tilde{D}_2$. 

By solving the optimization problems (22) and (23), we get the gain matrices of full-order and reduced-order observers as

\[ K = \begin{bmatrix} -0.3755 \\ -0.5059 \\ -0.3167 \\ -0.0215 \end{bmatrix}, \quad L = \begin{bmatrix} -0.9948 \\ 0.2316 \\ 0.2326 \end{bmatrix} \]

with the minimal \( \gamma = 0.6582 \) and \( \rho = 0.4688 \). We assume that the parameters of the initial reachable set are set as \( p_0 = [0 \ 0 \ 0 \ 0]^T \) and \( H_0 = 3I_3 \), respectively. In the simulation, the initial state \( x_0 \in \{p_0, H_0\} \) is a random number and \( x_0 = [0.4979 \ 1.3219 \ -2.9993 \ -1.1860]^T \). The control input is set as \( u_k = 0.1 \). The disturbance \( w_k \) is bounded as |\( w_k \)| \( \leq 1 \). The simulation results are shown in Figures 1–4 for the full-order observer-based and reduced-order observer-based interval estimation. We can see from Figures 1–4 that the lower and upper bounds of the interval estimation from the full-order observer contain the case from the reduced-order observer in a finite time.

6. Conclusions

In this paper, we propose a novel interval estimation method for discrete-time linear systems by designing the reduced-order observer and combining it with the reachability analysis. A kind of reduced-order observer is proposed to indirectly estimate the system states, and the interval estimation of system state is presented by the reachability analysis of estimation error. An iterative algorithm is provided to illustrate the implementing process of the proposed method. The simulation results show that the reduced-order observer-based interval estimation can obtain more accurate estimation than the full-order observer-based one.
Disclosure statement
No potential conflict of interest was reported by the author(s).

Funding
This work is supported by National Nature Science Foundation of China [grant numbers 61403129 and 61973105]. This work is also supported by the Fundamental Research Funds for the Universities of Henan Province [grant number NSRF180335], the Innovative Scientists and Technicians Team of Henan Provincial High Education [grant number 2019GGJS062], and the Programme of Key Young Teacher of Henan Province Higher University [grant number 2019GGJS062].

Le, V. T. H., Stoica, C., Alamo, T., Camacho, E. F., & Dumur, D. (2013). Zonotopic guaranteed state estimation for uncertain systems. *Automatica*, 49(11), 3418–3424. https://doi.org/10.1016/j.automatica.2013.08.014

Mazenc, F., & Bernard, O. (2010). Asymptotically stable interval observers for planar systems with complex poles. *IEEE Transactions on Automatic Control*, 55(2), 523–527. https://doi.org/10.1109/TAC.2009.2037472

Mazenc, F., & Bernard, O. (2011). Interval observers for linear time-invariant systems with disturbances. *Automatica*, 47(1), 140–147. https://doi.org/10.1016/j.automatica.2010.10.019

Meslem, N., Loukkas, N., & Martinez, J. J. (2018). Using set invariance to design robust interval observers for discrete-time linear systems. *International Journal of Robust and Nonlinear Control*, 28(11), 3623–3639. https://doi.org/10.1002/rnc.v28.11

Meslem, N., & Martinez, J. J. (2019). Hybrid interval-ellipsoidal set-membership state estimations for uncertain discrete-time linear systems. In *4th Conference on control and fault tolerant systems* (pp. 348–353). IEEE.

Mohamed, M., Nahla, K., & Safya, B. (2013). Design of a nonlinear observer for mechanical systems with unknown inputs. *Systems Science & Control Engineering*, 1(1), 105–112. https://doi.org/10.1080/21642583.2013.799448

Qian, W., Li, Y., Chen, Y., & Liu, W. (2020). $L_1 - L_{\infty}$ filtering for stochastic delayed systems with randomly occurring nonlinearities and sensor saturation. *International Journal of Systems Science*, 51(13), 2360–2377. https://doi.org/10.1080/00207721.2020.1794080

Qian, W., Li, Y., Zhao, Y., & Chen, Y. (2020). New optimal method for $L_2 - L_{\infty}$ state estimation of delayed neural networks. *Neurocomputing*, 415, 258–265. https://doi.org/10.1016/j.neucom.2020.06.118

Qian, W., Xing, W., & Fei, S. (2020). $H_{\infty}$ state estimation for neural networks with general activation function and mixed time-varying delays. *IEEE Transactions on Neural Networks and Learning Systems*. Online first. https://doi.org/10.1109/TNNLS.2020.3016120

Raissi, T., Efimov, D., & Zolghadri, A. (2012). Interval state estimation for a class of nonlinear systems. *IEEE Transactions on Automatic Control*, 57(1), 260–265. https://doi.org/10.1109/TAC.2011.2164820

Shamma, J. S., & Tu, K. Y. (1999). Set-valued observers and optimal disturbance rejection. *IEEE Transactions on Automatic Control*, 44(2), 253–264. https://doi.org/10.1109/9.746252

Soleymani, R., Nekoui, M. A., & Moarefianpour, A. (2020). Robust adaptive unknown input observer design for a class of disturbed systems. *Systems Science & Control Engineering*, 8(1), 249–257. https://doi.org/10.1080/21642583.2020.1739578

Tang, W., Wang, Z., & Shen, Y. (2019). Interval estimation for discrete-time linear systems: A two-step method. *Systems & Control Letters*, 123, 69–74. https://doi.org/10.1016/j.sysconle.2018.11.001

Tang, W., Wang, Z., Wang, Y., Raissi, T., & Shen, Y. (2019). Interval estimation methods for discrete-time linear time-invariant systems. *IEEE Transactions on Automatic Control*, 64(11), 4717–4724. https://doi.org/10.1109/TAC.9

Trinh, H., Tran, T. D., & Fernando, T. (2008). Disturbance decoupled observers for systems with unknown inputs.
IEEE Transactions on Automatic Control, 53(10), 2397–2402. 
https://doi.org/10.1109/TAC.2008.2007530

Vicino, A., & Zappa, G. (1996). Sequential approximation of feasible parameter sets for identification with set-membership uncertainty. *IEEE Transactions on Automatic Control*, 41, 774–785. https://doi.org/10.1109/9.506230

Wang, Z., Lim, C. C., & Shen, Y. (2018). Interval observer design for uncertain discrete-time linear systems. *Systems & Control Letters*, 116, 41–46. https://doi.org/10.1016/j.sysconle.2018.04.003

Yang, F., & Wilde, R. W. (1988). Observer for linear systems with unknown inputs. *IEEE Transactions on Automatic Control*, 33(7), 677–681. https://doi.org/10.1109/9.1278

Yang, J., Zhu, F., & Sun, X. (2013). State estimation and simultaneous unknown input and measurement noise reconstruction based on associated observer. *International Journal of Adaptive Control and Signal Processing*, 27(10), 846–858. https://doi.org/10.1002/acs.2360

Zheng, G., Efimov, D., Bejarano, F. J., Perruquetti, W., & Wang, H. (2016). Interval observer for a class of uncertain nonlinear singular systems. *Automatica*, 71, 159–168. https://doi.org/10.1016/j.automatica.2016.04.002

Zhu, F., & Han, Z. (2002). A note on observers for Lipschitz nonlinear systems. *IEEE Transactions on Automatic Control*, 47(10), 1751–1754. https://doi.org/10.1109/TAC.2002.803552