ON NONLINEAR RUDIN-CARLESON TYPE THEOREMS

ALEXANDER BRUDNYI

Abstract. In this paper we study nonlinear interpolation problems for interpolation and peak-interpolation sets of function algebras. The subject goes back to the classical Rudin-Carleson interpolation theorem. In particular, we prove the following nonlinear version of this theorem: Let $\bar{D} \subset \mathbb{C}$ be the closed unit disk, $T \subset \bar{D}$ the unit circle, $S \subset T$ a closed subset of Lebesgue measure zero and $M$ a connected complex manifold. Then for every continuous $M$-valued map $f$ on $S$ there exists a continuous $M$-valued map $g$ on $\bar{D}$ holomorphic on its interior such that $g|_S = f$. We also consider similar interpolation problems for continuous maps $f: S \rightarrow \bar{M}$, where $\bar{M}$ is a complex manifold with boundary $\partial M$ and interior $M$. Assuming that $f(S) \cap \partial M \neq \emptyset$ we are looking for holomorphic extensions $g$ of $f$ such that $g(\bar{D} \setminus S) \subset M$.

1. Formulation of Main Results

1.1. Let $A$ be a uniform algebra on a compact Hausdorff space $X$, i.e., a closed unital subalgebra of the Banach algebra $C(X)$ of complex continuous functions on $X$ equipped with the norm $\|f\|_{C(X)} := \max_X |f|$ separating points of $X$. (For the theory of uniform algebras see, e.g., the book [G].)

A compact subset $S \subset X$ is said to be interpolation set for $A$ if the restriction to $S$ maps $A$ onto $C(S)$. The number

\begin{equation}
(1.1) \quad c_A(S) := \sup_{f \in C(S), \|f\|_{C(S)} = 1} \inf\{\|F\|_{C(X)} : F \in A, F|_S = f\}
\end{equation}

(finite by the Banach open mapping theorem) is called the interpolation constant for $S$.

In this paper we consider interpolation problems for continuous maps of $S$ in complex manifolds. To formulate our results we require several definitions.

The maximal ideal space $\mathfrak{M}(A)$ is the set of all nontrivial complex homomorphisms of $A$. It is a compact subset of the closed unit ball of the dual space $A^*$ equipped with the weak* topology. The Gelfand transform $^\vee: A \rightarrow C(\mathfrak{M}(A))$, $a(\varphi) := \varphi(a)$, maps $A$ isometrically onto a uniform subalgebra on $\mathfrak{M}(A)$ and its transpose embeds $X$ into $\mathfrak{M}(A)$. Without loss of generality we will identify $A$ with its image under $^\vee$ and $X$ with its image under the embedding.

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A complex manifold $M$ is said to be Oka if every holomorphic map $f : K \to M$ from a neighbourhood of a compact convex set $K \subset \mathbb{C}^n$, $n \in \mathbb{N}$, can be approximated uniformly on $K$ by entire maps $\mathbb{C}^n \to M$.

The class of Oka manifolds includes, in particular, complex homogeneous manifolds, complements in $\mathbb{C}^n$, $n > 1$, of complex algebraic subvarieties of codimension $\geq 2$ and of compact polynomially convex sets, Hopf manifolds (i.e., nonramified holomorphic quotients of $\mathbb{C}^n \setminus \{0\}$). Also, holomorphic fibre bundles whose bases and fibres are Oka manifolds are Oka manifolds as well. (We refer to the book [F1] and the paper [K] for other examples and basic results of the theory of Oka manifolds.)

In what follows, $C(X,Y)$ stands for the set of continuous maps between topological spaces $X$ and $Y$. For a uniform algebra $A$ on $X$ and a subspace $M \subset \mathbb{C}^n$, we denote by $A(X,M) \subset C(X,M)$ the set of maps with coordinates in $A$. For $V \subset C(X,Y)$ and $S \subset X$ the trace space $V|_S (\subset C(S,Y))$ consists of restrictions of maps in $V$ to $S$.

The following result is a particular case of [Br, Thm. 1.4].

**Theorem 1.1.** Let $M \subset \mathbb{C}^n$ be a complex regular submanifold\(^1\) and an Oka manifold and $S \subset \mathfrak{M}(A)$ be an interpolation set for the uniform algebra $A$. Then

$$A(\mathfrak{M}(A), M)|_S = C(\mathfrak{M}(A), M)|_S.$$  

In other words, under the above conditions a map $f \in C(S,M)$ extends to a $g \in A(\mathfrak{M}(A), M)$ if and only if it extends to a map from $C(\mathfrak{M}(A), M)$. In the next result, we show that for totally disconnected interpolation sets (such as in the Rudin-Carleson theorem) similar interpolation problems are always solvable in a more general setting.

Let $A$ be a uniform algebra on $X$. For a family $\mathcal{F} = \{f_1, \ldots, f_n\} \subset A$ we denote by $A_\mathcal{F} \subset A$ the closed unital subalgebra generated by $f_1, \ldots, f_n$. The maximal ideal space $\mathfrak{M}(A_\mathcal{F})$ can be naturally identified with the polynomially convex hull of the compact set $\{F(x) := (f_1(x), \ldots, f_n(x)) \in \mathbb{C}^n : x \in X\} \subset \mathbb{C}^n$, the joint spectrum of $\mathcal{F}$.

Let $M$ be a complex manifold and $g$ be a holomorphic map into $M$ defined on a neighbourhood of $\mathfrak{M}(A_\mathcal{F})$. The continuous map $F^*g := g \circ F : \mathfrak{M}(A) \to M$ is said to be holomorphic; the set of such maps is denoted by $\mathcal{O}_\mathcal{F}(\mathfrak{M}(A), M)$. Note that if $M \subset \mathbb{C}^N$, $N \in \mathbb{N}$, then $\mathcal{O}_\mathcal{F}(\mathfrak{M}(A), M) \subset A(\mathfrak{M}(A), M)$.

**Theorem 1.2.** Let $S \subset X$ be a totally disconnected interpolation set for $A$ and $M$ be a connected complex manifold.

(a) If $c_A(S) = 1$, then for every $f \in C(S,M)$ there exists a map $g \in \mathcal{O}_\mathcal{F}(\mathfrak{M}(A), M)$, where $|\mathcal{F}| = \dim_{\mathbb{C}} M$, such that $g|_S = f$.

(b) Suppose $M$ is an Oka manifold and $c_A(S) > 1$. Let $K \subset M$ be an open relatively compact subset. There is a subset $L \subset M$ containing $K$ such that for every $f \in C(S, K)$ there exists a map $g \in \mathcal{O}_\mathcal{F}(\mathfrak{M}(A), M)$, where $|\mathcal{F}| = \dim_{\mathbb{C}} M$, such that $g(\mathfrak{M}(A)) \subset L$ and $g|_S = f$.

Here $|\mathcal{F}|$ stands for the cardinality of $\mathcal{F}$.

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\(^1\)I.e., $M$ is equipped with the induced topology.
Example 1.3. (1) Let $A(\mathbb{D}^n) \subset C(\mathbb{D}^n)$ be the uniform algebra of continuous functions on the closure $\mathbb{D}^n$ of the open unit polydisk $\mathbb{D}^n \subset \mathbb{C}^n$ holomorphic on $\mathbb{D}^n$. Let $\mathbb{T}^n \subset \mathbb{D}^n$ be the boundary torus. It is proved in [RS] that every compact subset $S \subset \mathbb{T}^n$ of zero 1-dimensional Hausdorff measure is an interpolation set for $A(\mathbb{D}^n)$ with the interpolation constant 1. Since such $S$ is totally disconnected, Theorem 1.2 implies the following extension of the Rudin-Carleson theorem (see [R], [S]):

Let $M$ be a connected complex manifold. For every $f \in C(S, M)$ there exists a $g \in C(\mathbb{D}^n, M)$ holomorphic on $\mathbb{D}^n$ such that $g|_S = f$.

(2) Let $Z$ be a connected complex manifold such that the algebra $H^\infty(Z)$ of bounded holomorphic functions on $Z$ separates points. A sequence $S = \{s_n\}_{n \in \mathbb{N}} \subset Z$ is called \textit{interpolating} for $H^\infty(Z)$ if $H^\infty(Z)|_S$ coincides with the Banach space of bounded complex-valued functions on $S$ equipped with supremum norm. The interpolation constant $c(S)$ is defined similarly to (1.1). Let $X$ be the closure of $Z$ in the maximal ideal space $\mathfrak{M}(H^\infty(Z))$. We identify $H^\infty(Z)$ with its image in $C(\mathfrak{M}(H^\infty(Z)))$ under the Gelfand transform. Then the closure $\overline{S} \subset X$ of $S$ is an interpolation set for $H^\infty(Z)$ with the interpolation constant $c(S)$. Moreover, $\overline{S}$ is homeomorphic to the Stone-Čech compactification of $\mathbb{N}$ and, hence, is totally disconnected. Now, Theorem 1.2 implies that $S$ is also an interpolating sequence for bounded holomorphic maps into connected Oka manifolds:

Let $M$ be a connected Oka manifold. Let $f : S \to M$ be a bounded map with image in a compact subset $K \subset M$. Then there exist a compact subset $L \subset M$ depending on $M, K$ and $c(S)$ only and a holomorphic map $g : Z \to M$ with image in $L$ such that $g|_S = f$.

1.2. In this part we consider nonlinear interpolation problems for peak-interpolation sets. (E.g., interpolation sets in the Rudin-Carleson theorem and in [RS] are peak-interpolation.)

Recall that a compact subset $S \subset X$ is said to be \textit{peak-interpolation} for a uniform algebra $A$ on $X$ if every nonidentically zero function $f \in C(S)$ extends to a $g \in A$ that satisfies

\begin{equation}
|g(x)| < \max_S |f| \quad \forall x \in S^c := X \setminus S.
\end{equation}

Equivalently, every $f \in C(S, \overline{\mathbb{D}})$ with $f(S) \cap \mathbb{T} \neq \emptyset$ extends to a $g \in A$ that satisfies $g(S^c) \subset \mathbb{D}$. Thus, for peak-interpolation sets it is naturally to consider interpolation problems for maps into complex manifolds with boundaries.

Let $B$ be a complex Banach space and $Y \subset B$ be a complex Banach submanifold. For a uniform algebra $A$ on $X$ we denote by $A(X, Y)$ the set of maps $f \in C(X, Y)$ such that $\varphi(f) \in A$ for every $\varphi \in B^*$. We are interesting in submanifolds subject to the following definition:

A complex Banach submanifold $M \subset B$ with boundary $\partial M$ and interior $M$ is said to be \textit{universal} if for every compact Hausdorff space $X$, a uniform algebra $A \subset C(X)$ and a peak-interpolation set $S \subset X$ for $A$ the following holds:

Every $f \in C(S, M)$ with $f(S) \cap \partial M \neq \emptyset$ extends to a map $g \in A(X, M)$ such that $g(S^c) \subset M$.

The class of universal manifolds has the following properties.
Proposition 1.4. (1) Direct product of universal manifolds is a universal manifold.
(2) The set of universal submanifolds of a complex Banach space $B$ is invariant with respect to the action of the group of invertible affine transformations of $B$.
(3) If $M \subset \mathbb{C}^n$ is a universal submanifold and $F : \mathbb{C}^n \to \mathbb{C}^m$ is a holomorphic embedding, then $F(M)$ is a universal submanifold of $\mathbb{C}^m$.
(4) Every paracompact universal manifold is contractible.

It is known that a closed ball of a complex Banach space $B$ is universal, see [S]. Our next result generalizes this fact.

Let $p_M : B \to [0, \infty)$ be the Minkowski functional of an open absorbing subset $M$ of a complex Banach space $B$, i.e.,

\begin{equation}
(1.3)\quad p_M(v) := \inf_{tv \in M, t > 0} \frac{1}{t}.
\end{equation}

Then $p_M$ is homogeneous, i.e., $p_M(rv) = rp_M(v)$ for all $r \in \mathbb{R}^+$, $v \in B$.

Proposition 1.5. Suppose $M$ satisfies the condition:

\begin{equation}
(1.4)\quad \text{If } v \in \overline{M}, \text{ the closure of } M, \text{ then the entire segment } [0, v) \text{ lies in } M.
\end{equation}

Then $p_M$ is a continuous function.

Conversely, if $p : B \to [0, \infty)$ is a continuous homogeneous function, then $M := \{v \in B : p(v) < 1\}$ is an open absorbing set satisfying (1.4) and $p = p_M$.

Note that if $M$ satisfies (1.4), then $M = \{v \in B : p_M(v) < 1\}$ and $\partial M = \{v \in B : p_M(v) = 1\}$ (the boundary of $M$). Moreover, $M$ is a complex Banach manifold with boundary modelled on $B$ (in fact, $\overline{M} \setminus p_M^{-1}(0)$ is homeomorphic to $(0,1] \times \partial M$).

For instance, an open convex neighbourhood of $0 \in B$ satisfies (1.4). In this case the function $p_M$ is subadditive (i.e., $p_M(v + v') \leq p_M(v) + p_M(v')$). Also, every star body $M \subset \mathbb{C}^n$ containing $0$ satisfies (1.4). If, in addition, such $M$ is bounded and $0$ is an interior point of its kernel, then $p_M$ is a Lipschitz function, see, e.g., [1].

Theorem 1.6. Suppose $M \subset B$ is an open absorbing subset satisfying (1.4). Then $\overline{M}$ is a universal manifold.

Remark 1.7. Since $p_M$ is a homogeneous function, the theorem can be restated as follows:

Given a uniform algebra $A$ on $X$ and a peak-interpolation set $S \subset X$ for $A$ every $f \in C(S, \overline{M})$ such that $p_M \circ f \neq 0$ has an extension $g \in A(X, \overline{M})$ that satisfies

\[ p_M(g(x)) < \max_{y \in S} p_M(f(y)) \quad \forall x \in S^c. \]

For a subset $K \subset B$ we denote by $\co(K)$ the convex hull of $K$ (the minimal convex subset of $B$ containing $K$). Also, by $[K]_\varepsilon \subset B$ we denote the open $\varepsilon$-neighbourhood of $K$:

\[ [K]_\varepsilon := \{v \in B : \inf_{v' \in K} \|v - v'\|_B < \varepsilon\}. \]
Corollary 1.8. Let $A$ be a uniform algebra on $X$ and $S \subset X$ be a peak-interpolation set for $A$. Let $f \in C(S, B)$. For every $\varepsilon > 0$ there is a $g_{\varepsilon} \in A(X, [\text{co}(f(S))]_{\varepsilon})$ such that $g_{\varepsilon}|S = f$. Moreover, if $B = \mathbb{C}^n$ and co$(f(S))$ has a nonempty interior (co$(f(S)))^\circ$, then there is a $g \in A(X, \text{co}(f(S)))$ extending $f$ such that $g|S = f$ and $g(S^\circ) \subset (\text{co}(f(S)))^\circ$.

For the last statement, note that since $f(S) \subset \mathbb{C}^n$ is compact, co$(f(S))$ is compact as well by the Caratheodory theorem, and $(\text{co}(f(S)))^\circ \neq \emptyset$ provided that $f(S)$ contains $2n + 1$ linearly independent vectors over $\mathbb{R}$.

Remark 1.9. Inspired by Theorem 1.2, one can consider an analogous interpolation problem for totally disconnected peak-interpolation sets.

Problem 1.10. Let $\tilde{M}$ be a domain with boundary in a complex manifold $N$. For what $\tilde{M}$ the following holds:

(*) For every uniform algebra $A$ on $X$, a totally disconnected peak-interpolation set $S \subset X$ for $A$ and a map $f \in C(S, \tilde{M})$ with $f(S) \cap \partial M \neq \emptyset$ there are a subset $F \subset A$ with $|F| = \dim N$ and a map $g \in \mathcal{O}_F(\mathcal{M}(A), N)$ such that $g|S = f$ and $g(S^\circ) \subset M$?

Theorem 1.2 (a) asserts that every $\tilde{M}$ is near-optimal meaning that for every open neighbourhood $O \subset N$ of $\tilde{M}$ there exists a map $g \in \mathcal{O}_F(\mathcal{M}(A), O)$ with $g|S = f$. We conjecture that for $\tilde{M}$ with a 'nice' boundary (e.g., for strongly pseudoconvex domains $\tilde{M} \subset \mathbb{C}^n$) such near-optimal $g$ can be deformed to obtain the one satisfying condition (*).

Clearly, if $\tilde{M}_i \subset N_i$, $i = 1, 2$, satisfy (*), then $\tilde{M}_1 \times \tilde{M}_2 \subset N_1 \times N_2$ satisfies (*) as well. Also, universal submanifolds $\tilde{M} \subset \mathbb{C}^n$ satisfy (*). The following result gives an example of nonuniversal $\tilde{M}$ satisfying (*) (cf. Proposition 1.4(4)).

Theorem 1.11. Let $\tilde{M}$ be a connected Riemann surface with boundary embedded in a Riemann surface $N$ such that inclusion $\tilde{M} \hookrightarrow N$ is homotopy equivalence. Then $\tilde{M}$ satisfies condition (*).

Note that every connected Riemann surface with boundary $\tilde{M}$ can be embedded in its double $W$. Then there is an open neighbourhood $N \subset W$ of $\tilde{M}$ that satisfies the hypothesis of the theorem.

2. PROOFS OF THEOREMS 1.2 AND 1.11

2.1. Proof of Theorem 1.2. (a) Due to the main theorem of [FS] there is a finite locally biholomorphic surjective map $h : \mathbb{D}^n \to M$, where $n = \dim_{\mathbb{C}} M$. Since $f(S) \subset M$ is compact and $h$ is locally biholomorphic, there exist a finite open cover $(U_i)_{1 \leq i \leq k}$ of $f(S)$ and holomorphic maps $\tilde{h}_i : U_i \to \mathbb{D}^n$ such that $h \circ \tilde{h}_i = \text{id}_{U_i}$, $1 \leq i \leq k$. Consider the finite open cover $\mathcal{V} = (f^{-1}(U_i))_{1 \leq i \leq k}$ of $S$. Since $S$ is compact and totally disconnected, its covering dimension is zero (for basic results of the dimension theory, see, e.g., [N]). In particular, there is a refinement $(W_s)_{1 \leq s \leq m}$ of $\mathcal{V}$ by clopen pairwise disjoint subsets. Let $\tau : \{1, \ldots, m\} \to \{1, \ldots, k\}$ be the refinement map, i.e., $W_s \subset f^{-1}(U_{\tau(s)})$, $1 \leq s \leq m$. Let us define a map $\tilde{f} : S \to \mathbb{D}^n$ by the formula

$$\tilde{f}(x) := \tilde{h}_{\tau(s)}(f(x)), \quad x \in W_s, \quad 1 \leq s \leq m.$$
Then, \( \tilde{f} \in C(S, \mathbb{D}^n) \) and \( h \circ \tilde{f} = f \). Since \( S \) is an interpolation set with \( c_A(S) = 1 \), we can extend coordinates of \( \tilde{f} \) to get a continuous map \( \tilde{g} : \mathcal{M}(A) \to \mathbb{D}^n \) with coordinates in \( A \) such that \( \tilde{g}|_S = \tilde{f} \). Let \( \mathcal{F} \) be the family of coordinates of \( \tilde{g} \). Then \( g := h \circ \tilde{g} \in \mathcal{O}_f(\mathcal{M}(A), M) \) is the required map interpolating \( f \) on \( S \).

(b) According to [F2, Thm. 1.1] there is a surjective holomorphic map \( h : \mathbb{C}^n \to M \), where \( n = \dim \mathbb{C} M \), such that for every \( x \in M \) there is an open neighbourhood \( U_x \subset M \) of \( x \) and a holomorphic map \( \tilde{h}_x : U_x \to \mathbb{C}^n \) such that \( h \circ \tilde{h}_x = \text{id}_{U_x} \). Let \( (U_{x_i})_{1 \leq i \leq k} \) be a finite open cover of the compact set \( \overline{K} \) (the closure of \( K \)). Then \( V := \cup_{i=1}^k h_{x_i}(U_{x_i}) \) and \( \tilde{K} := V \cap h^{-1}(K) \) are open relatively compact subsets of \( \mathbb{C}^n \). Let \( D_{\tilde{K}} \subset \mathbb{C}^n \) be the minimal open polydisk centered at 0 containing \( \tilde{K} \) and \( c_S(A)D_{\tilde{K}} \) be the dilation of \( D_{\tilde{K}} \) with scalar factor \( c_S(A) \). We define

\[
L := h(c_S(A)D_{\tilde{K}}).
\]

Let \( f \in C(S, K) \). Then as in the proof of part (a) of the theorem we construct a map \( \tilde{f} \in C(S, D_{\tilde{K}}) \) such that \( f = h \circ \tilde{f} \). By the definition of an interpolation set, there is a map \( \tilde{g} \in C(\mathcal{M}(A), c_S(A)D_{\tilde{K}}) \) with coordinates in \( A \) such that \( \tilde{g}|_S = \tilde{f} \). Let \( \mathcal{F} \) be the family of coordinates of \( \tilde{g} \). Then \( g := h \circ \tilde{g} \in \mathcal{O}_f(\mathcal{M}(A), M) \) satisfies \( g(\mathcal{M}(A)) \subset L \) and \( g|_S = f \), as required.

2.2. Proof of Theorem [1.11] Let \( r : N_u \to N \) be the universal covering of \( M \). Then \( M_u := r^{-1}(M) \) is the universal covering of \( \tilde{M} \), and \( \partial M_u := r^{-1}(\partial M) \) and \( M_u := r^{-1}(\overline{M}) \) are the boundary and the interior of \( M_u \). By our condition, \( M_u \) and \( N_u \) are biholomorphic to \( \mathbb{D} \). Thus, without loss of generality, we assume that \( N_u \) coincides with \( \mathbb{D} \). Then \( M_u \) is a simply connected domain in \( \mathbb{D} \) and (as \( M_u \) is a manifold with boundary) \( \partial M_u \) is homeomorphic to a one-dimensional manifold and there is a neighbourhood of \( \partial M_u \) in \( M_u \) homeomorphic to \( \partial M_u \times (0, 1] \). In particular, each open arc in \( \partial M_u \) is a free boundary arc, see [P] Sec.3.1. Hence, a biholomorphic map \( h : M_u \to \mathbb{D} \) extends to an injective continuous map \( M_u \to \mathbb{D} \), see [P] Thm. 3.1.

Let \( A \) be a uniform algebra on \( X \) and \( S \subset X \) be a totally disconnected peak-interpolation set for \( A \). Let \( f \in C(S, \tilde{M}) \) be such that \( f(S) \cap \partial M \neq \emptyset \). Since \( r \) is locally biholomorphic, as in the proof of Theorem [1.12] we can construct a map \( \tilde{f} \in C(S, M_u) \) such that \( f = r \circ \tilde{f} \). Consider the map \( h \circ \tilde{f} \in C(S, \mathbb{D}) \). By our hypothesis, \( (h \circ \tilde{f})(S) \cap \mathbb{T} \neq \emptyset \). Then by the definition of the peak-interpolation set, there exists \( g' \in A \) such that \( g'|_S = h \circ \tilde{f} \) and \( g'(S^c) \subset \mathbb{D} \). In turn, \( \tilde{g} := h^{-1} \circ g' \) maps \( S^c \) in \( M_u \) and coincides with \( \tilde{f} \) on \( S \).

Let us prove that \( \tilde{g} \in A \).

In fact, let \( K := g'(X) \). Then \( K \) is a compact subset of \( L \cup \mathbb{D} \), where \( L := (h \circ \tilde{f})(S) \). Moreover, since \( \tilde{f}(S) \) is a compact subset of \( \partial M_u \), the open set \( \mathbb{T} \setminus L \subset \mathbb{T} \) is nonvoid. Let \( z \in \mathbb{T} \setminus L \). Then there is an open disk \( D \) centered at \( z \) such that \( \mathbb{D} \cap K = \emptyset \). In particular, for a point \( z' \in D \setminus \mathbb{D} \) sufficiently close to \( z \), the function \( p(z) := \frac{1}{z - z'} \), \( z \in \mathbb{D} \), lies in \( A(\mathbb{D}) \) and satisfies

\[
|p(z')| > \max_{z \in K} |p(z)|.
\]
This implies that the polynomially convex hull $\hat{K}$ of $K$ does not contain points from $\mathbb{T}\setminus L$. Thus $\hat{K}$ is a compact subset of $L \cup D$ as well. Further, the function $h^{-1}$ is continuous on $L \cup D$ and holomorphic on $D$. Thus by the Mergelyan theorem [M], $h^{-1}|_{\hat{K}}$ can be uniformly approximated by holomorphic polynomials. Hence, $\tilde{g} := h^{-1} \circ g'$ can be uniformly approximated on $X$ by holomorphic polynomials in $g'$, i.e., it lies in $A$, as claimed.

Now, $\tilde{g}(X)$ is a compact subset of $\hat{M}_u \subset D$ ($=: N_u$). Hence, $\tilde{g}(\mathfrak{M}(A))$ is a compact subset of $D$. In particular, the map $g := r \circ \tilde{g} : \mathfrak{M}(A) \to N$ lies in $\mathcal{O}_F(\mathfrak{M}(A), M)$, where $F := \{ \tilde{g} \}$, and $g|_S = f$, $g(S^c) \subset M$.

The proof of the theorem is complete.  

\section{Proofs of Propositions 1.4 and 1.5}

\subsection{Proof of Proposition 1.4}
Parts (1) and (2) follow directly from the definition of a universal manifold.

(3) Since $M \subset \mathbb{C}^n$ is universal and $F : \mathbb{C}^n \to \mathbb{C}^m$ is a holomorphic embedding, in order to prove that $F(M)$ is universal it suffices to check that if $g \in A(X, M)$, then $F \circ g \in A(X, F(M))$. In fact, let $\widehat{g(X)} \subset \mathbb{C}^n$ be the polynomially convex hull of the compact set $g(X) \subset \mathbb{C}^n$. By the Runge approximation theorem, see, e.g., [GR, I.F], coordinates of $F$ are uniformly approximated on a neighbourhood of $\widehat{g(X)}$ by holomorphic polynomials. Since $A$ is a uniform algebra, this implies the required statement.

(4) Let $\overline{B}^n \subset \mathbb{R}^n$ be the closed unit Euclidean ball and $S^{n-1} \subset \overline{B}^n$ be the unit sphere. Since $S^{n-1}$ is a peak-interpolation set for the algebra $C(\overline{B}^n)$, by the definition of a universal manifold every $f \in C(S^{n-1}, M)$ extends to a $g \in C(\overline{B}^n, M)$. This shows that all homotopy groups of $M$ are trivial. In turn, since $M$ is a paracompact Banach manifold, the latter implies that $M$ is contractible, see the corollary after [Pa, Thm. 15].

\subsection{Proof of Proposition 1.5}
First, we prove that under condition (1.4) the Minkowski functional $p_M : B \to [0, \infty)$ is continuous, i.e., for every $v \in B$ and every $\{v_n\}_{n \in \mathbb{N}} \subset B$ converging to $v$

$$\lim_{n \to \infty} p_M(v_n) = p_M(v).$$

To this end, for $v \in B$ we set

$$\overrightarrow{0v} := \{tv \in B : t \in \mathbb{R}_+\}.$$ 

Due to (1.4) for $v \neq 0$ there is some $t(v) \in (0, \infty]$ such that

$$[0, t(v)v) \subset M, \quad \overrightarrow{0v} \setminus [0, t(v)v) \not\subset M \quad \text{and} \quad t(v)v \in \partial M \quad \text{if} \quad t(v) < \infty.$$ 

In particular, $p_M(v) = \frac{1}{t(v)}$.

Let $\{v_n\}_{n \in \mathbb{N}} \subset B \setminus \{0\}$ be a sequence converging to $v$. If $v = 0$, then since $M$ is open, $\lim_{n \to \infty} t(v_n) = \infty$. Thus,

$$\lim_{n \to \infty} p_M(v_n) = \lim_{n \to \infty} \frac{1}{t(v_n)} = 0 = p_M(0).$$
If $v \neq 0$, then $tv \in M$ for every $t \in [0,t(v))$. Since $\{tv_n\}_{n \in \mathbb{N}}$ converges to $tv$ and $M$ is open, there is some $n(t) \in \mathbb{N}$ such that $tv_n \in M$ for all $n \geq n(t)$. This implies that

$\lim_{n \to \infty} p_M(v_n) = \lim_{n \to \infty} \frac{1}{t(v_n)} \leq \inf_{t \in [0,t(v))] \frac{1}{t} = p_M(v).$

If $p_M(v) = 0$, the latter implies that $\lim_{n \to \infty} p_M(v_n) = p_M(v)$. For otherwise, $t(v) < \infty$. Hence, $tv \in M^c$ for every $t > t(v)$. Since $\{tv_n\}_{n \in \mathbb{N}}$ converges to $tv$ and $M^c$ is open, there is some $n(t) \in \mathbb{N}$ such that $tv_n \in M^c$ for all $n \geq n(t)$. This implies that

$\lim_{n \to \infty} p_M(v_n) = \sup_{t > t(v)} \frac{1}{t} \leq \lim_{n \to \infty} \frac{1}{t(v_n)} = \lim_{n \to \infty} p_M(v_n).$

From (3.1), (3.2) we obtain that $\lim_{n \to \infty} p_M(v_n) = p_M(v)$. Thus, $p_M$ is a continuous function.

Now, assume that $p : B \to [0, \infty)$ is a continuous homogeneous function. Let $M := \{v \in B : p(v) < 1\}$. Then $M$ is an open set containing 0, i.e., $M$ is absorbing. Further, if $v \in M$, then continuity and homogeneity of $p$ imply that $p(v) \leq 1$ and $p(tv) \in M$ for all $t \in [0, 1)$. Hence, $M$ satisfies condition (1.3).

Finally, for $v \neq 0$

$p_M(v) := \inf_{tv \in M, t > 0} \frac{1}{t} = \inf_{p(v) < 1, t > 0} \frac{1}{t} = \frac{1}{p(v)} = p(v),$

as required.

\[ \square \]

4. Proofs of Theorem 1.6 and Corollary 1.8

4.1. This part contains some results used in the proof of Theorem 1.6.

Let $\theta : [0, 1] \to [0, \frac{\pi}{4}]$ be a continuous function positive on $(0, 1)$ and equal to zero at $\{0, 1\}$ and let

$\Omega := \{z = re^{i\theta} \in \mathbb{C} : 0 < \theta < \theta(r), \ r \in (0, 1)\}.$

Let $A$ be a uniform algebra on $X$ and let $S \subset X$ be a peak-interpolation set for $A$.

**Lemma 4.1.** Given $\varepsilon \in (0, 1)$ and a compact set $E \subset S^c$ there is a function $h_\varepsilon \in A$ such that

$h_\varepsilon(X) \subset \Omega, \ h_\varepsilon|_S = 1, \ |h_\varepsilon(x)| < 1 \quad \forall x \in S^c \quad \text{and} \quad |h_\varepsilon(x)| \leq \varepsilon \quad \forall x \in E.$

**Proof.** Since $\Omega$ is a simply connected domain whose boundary is the Jordan curve

$$\gamma(t) := \begin{cases} 0 \leq t \leq \frac{1}{2} \\ 2t \quad \frac{1}{2} \leq t \leq 1 \end{cases}$$

by the Carathéodory theorem, see, e.g., [2], there is a conformal map $\mathbb{D} \to \Omega$ that extends to a homeomorphism $G : \overline{\mathbb{D}} \to \Omega$. Let $z_0 := G^{-1}(0), z_1 := G^{-1}(1) \in \mathbb{T}$. Let $\chi \in A$ be such that $\chi|_S = 1$ and $|\chi(x)| < 1$ for all $x \in S^c$ (existing by the definition of a peak-interpolation.
set. Then there exists \( r \in (0, 1) \) such that \( \chi(E) \subset \mathbb{D}_r := \{ z \in \mathbb{C} : |z| < r \} \). Consider the set of Möbius transformations of \( \mathbb{D} \):

\[
g_a(z) := \frac{z - az_0}{1 - az_0^{-1}z}, \quad z \in \mathbb{D}, \quad -1 < a < 1.\]

Then \( g_a(z_0) = z_0 \) for all \( a \) and \( \lim_{a \to -1} g_a(\mathbb{D}_r) = \{ z_0 \} \) (convergence in the Hausdorff metric). In particular, there exists \( a_0 \in (-1, 0) \) such that \( G \circ g_{a_0} \) maps \( \mathbb{D}_r \) into \( \Omega \cap \mathbb{D}_r \). This map sends \( g_{a_0}^{-1}(z_1) \) to 1. Consider the function \( \chi_{\epsilon} := g_{a_0}^{-1}(z_1) \chi \in A \). Since \( |g_{a_0}^{-1}(z_1)| = 1 \), we have \( \chi_{\epsilon}|S = g_{a_0}^{-1}(z_1), \chi_{\epsilon}(S^c) \subset \mathbb{D} \) and \( \chi_{\epsilon}(E) \subset \mathbb{D}_r \). We set \( h_{\epsilon} := G \circ g_{a_0} \circ \chi_{\epsilon} \). Since \( G \circ g_{a_0} \in A(\mathbb{D}) \), it is a uniform limit of a sequence of holomorphic polynomials. Hence, \( h_{\epsilon} \in A \) and has the required properties.

**Lemma 4.2.** Let \( \Psi : [0, 1] \to \mathbb{R}_+ \) be a continuous strictly increasing function equal to 0 at 0. There exists a sequence of functions \( \{ \psi_i \}_{i \in \mathbb{N}} \subset C([0, 1]) \) positive on \((0, 1)\) and equal to zero at \( \{ 0, 1 \} \) such that

\[
\sum_{i=1}^k \psi_i(r_i) \leq \Psi\left( \sum_{i=1}^k \frac{r_i}{2^i} \right), \quad r_i \in [0, 1], \ 1 \leq i \leq k, \ k \in \mathbb{N};
\]

here the equality holds if and only if all \( r_i = 0 \).

**Proof.** We set \( \psi_1(r_1) := (1 - r_1)\Psi\left( \frac{r_1}{4} \right) \leq \Psi\left( \frac{1}{2} \right) \), \( r_1 \in [0, 1] \). Suppose the required \( \psi_i \) are already defined for all \( i \leq k - 1 \). Then we define

\[
\psi_k(r_k) := (1 - r_k) \cdot \min_{r_1, \ldots, r_{k-1} \in [0, 1]} \left\{ \Psi\left( \sum_{i=1}^{k-1} \frac{r_i}{2^i} \right) - \sum_{i=1}^{k-1} \psi_i(r_i) \right\}, \quad r_k \in [0, 1].
\]

By the induction hypothesis, \( \psi_k \) is continuous equal to 0 at \( \{ 0, 1 \} \) and (since \( \Psi \) is strictly increasing) for \( r_k \not\in \{ 0, 1 \} \)

\[
\psi_k(r_k) > (1 - r_k) \cdot \min_{r_1, \ldots, r_{k-1} \in [0, 1]} \left\{ \Psi\left( \sum_{i=1}^{k-1} \frac{r_i}{2^i} \right) - \sum_{i=1}^{k-1} \psi_i(r_i) \right\} = 0.
\]

I.e., \( \psi_k \) is positive on \((0, 1)\).

Next, if one of \( r_i \neq 0, 1 \leq i \leq k \), then by the induction hypothesis

\[
\psi_k(r_k) \leq (1 - r_k) \left( \Psi\left( \sum_{i=1}^{k} \frac{r_i}{2^i} \right) - \sum_{i=1}^{k-1} \psi_i(r_i) \right) < \Psi\left( \sum_{i=1}^{k} \frac{r_i}{2^i} \right) - \sum_{i=1}^{k} \psi_i(r_i).
\]

This proves the required statement. \( \square \)

4.2. **Proof of Theorem 1.6.** Let \( f \in C(S, \tilde{M}) \), \( f(S) \cap \partial M \neq \emptyset \). According to [S] there is a map \( g \in A(X, B) \) such that \( g|_S = f \). Consider the compact set \( \hat{K} := g(X) \subset B \). By Mazur’s theorem, see, e.g., [Ca] Ch. VI, 4.8], the closure of the convex balanced hull of \( K \),

\[
\hat{K} := \{ \sum_{i=1}^n c_i v_i, v_i \in K, c_i \in \mathbb{D}, \sum_{i=1}^n |c_i| = 1, n \in \mathbb{N} \}\]
is compact. Let $V \subset B$ be the subspace generated by vectors in $\hat{K}$ equipped with norm $\| \cdot \|_B$ and $M|_V := M \cap V$. Then $M|_V$ is an open subset of $V$ and its Minkowski functional (defined on $V$) coincides with $p_M|_V$, i.e., it is continuous. Consider the modulus of continuity of $p_M|_{\hat{K}}$:

$$\omega_{p_M|_{\hat{K}}}(t) := \sup \{ |p_M(v) - p_M(v')| : v, v' \in \hat{K}, \|v - v'\|_B \leq t \}, \quad t \geq 0.$$  

Since $K$ is compact and convex, $\omega_{p_M|_{\hat{K}}}: [0, \text{diam } \hat{K}] \to [0, \infty)$ is a nondecreasing continuous subadditive function equal to zero at 0. We set

$$\omega(t) := t + \omega_{p_M|_{\hat{K}}}(t), \quad t \in [0, \text{diam } \hat{K}].$$  

Then $\omega$ is a strictly increasing continuous subadditive function. Moreover, since $p_M|_{\hat{K}}$ attains value 1 on $K$ and equals 0 at 0, the range of $\omega$ contains interval $[0, 1]$. Thus the inverse $\omega^{-1}: [0, T] \to [0, \text{diam } \hat{K}], T := \omega(\text{diam } \hat{K}) \geq 1$, of $\omega$ is an increasing continuous function equals 0 at 0.

We define the required extension $h \in A(X, \hat{M})$ of $f$ by the formula

$$h(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} h_k(x) g(x), \quad x \in X,$$

for some $h_k \in A$, $|h_k| \leq 1$. Here $h_k$ maps $X$ in the closure of a domain $\Omega_k := \{ z = re^{i\theta} \in \mathbb{C} : 0 < \theta < \theta_k(r), \ r \in (0, 1) \}$, where $\theta_k: [0, 1] \to [0, \frac{\pi}{4}]$ is a continuous function positive on $(0, 1)$ and equals zero at $\{0, 1\}$.

To this end, we set

$$m := \max_{x \in X} \| g(x) \|_B, \quad m' := \max_{x \in X} p_M(g(x))$$

and choose continuous functions $\tilde{\theta}_k: [0, 1] \to \mathbb{R}_+$ positive on $(0, 1)$ and equals 0 at $\{0, 1\}$ such that for every $n \in \mathbb{N}$

$$\sum_{k=1}^{n} \tilde{\theta}_k(r_k) \leq \frac{1}{2} \omega^{-1} \left( \sum_{k=1}^{n} \frac{r_k}{2^k} \right), \quad r_k \in [0, 1], \ k \in \mathbb{N}.$$  

This is possible due to Lemma 4.2.

Next, we define

$$\theta_k(r_k) := \min \left\{ \frac{\phi_k}{m} \tilde{\theta}_k(1 - r_k), \frac{\pi}{4} \right\}, \quad r_k \in [0, 1], \ k \in \mathbb{N}.$$  

Also, we define a sequence $\{ \varepsilon_n \}_{n \in \mathbb{Z}_+}$ of positive numbers converging to 0 by the formulas

$$\varepsilon_0 := 1, \quad \varepsilon_{n+1} := \min \left\{ \varepsilon_n, \frac{2n}{m} \omega^{-1} \left( \frac{\varepsilon_n}{\max \{ m, 1 \} \omega^{-1} \left( \frac{\varepsilon_n}{m} \right)} \right), \frac{1}{2^{n+2}} \right\}, \quad n \geq 0.$$  

Fix a proper open neighbourhood $U \subset X$ of $S$. Let

$$U_n := \{ x \in U : p_M(g(x)) < 1 + \varepsilon_n \}, \quad n \in \mathbb{N}.$$  


Then \( U_n \subseteq U \) is an open neighbourhood of \( S \). We choose \( h_n \in A \) with image in \( \bar{\Omega}_n \) such that \( h_n(S) = 1, |h_n|_{S^c} < 1 \) and \( |h_n(x)| \leq \varepsilon_n \) for all \( x \in U_n \) (\( \neq \emptyset \)) (see Lemma 4.1). Note that \( U_1 \supseteq U_2 \supseteq \cdots \supseteq U_n \supseteq \cdots \) because the sequence \( \{\varepsilon_n\}_{n \in \mathbb{N}} \) is nonincreasing. We set for convenience \( U_0 := X \), \( h_0 := 0 \) and \( \theta_0 := 0 \).

Since maps \( h \) and \( h' \),
\[
h(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} h_k(x)g(x) \quad \text{and} \quad h'(x) := \sum_{k=1}^{\infty} \frac{1}{2^k} h_k(x)g(x), \quad x \in X,
\]
map \( X \) into \( \hat{K} \),
\[
\|h(x) - h'(x)\|_B = \left\| \sum_{k=1}^{\infty} \frac{1}{2^k} h_k(x) \left( e^{i\text{Arg}(h_k(x))} - 1 \right) g(x) \right\|_B \leq \sum_{k=1}^{\infty} \frac{m}{2^k} |h_k(x)| 2 \sin \left( \frac{\text{Arg}(h_k(x))}{2} \right) \leq \sum_{k=1}^{\infty} \frac{m}{2^k} \theta_k(|h_k(x)|) \leq \sum_{k=1}^{\infty} \frac{m}{2^k} \theta_k(|h_k(x)|) \leq \text{diam} \hat{K}, \quad x \in X.
\]

(4.7)

Suppose \( x \in U_n \setminus U_{n+1}, n \in \mathbb{Z}_+ \). Then (4.7), subadditivity of \( \omega_{p_M|\hat{K}} \) and (4.5) imply
\[
p_M(h(x)) \leq p_M(h'(x)) + \omega_{p_M|\hat{K}}(\|h(x) - h'(x)\|_B) \leq \left( \sum_{k=0}^{\infty} \frac{1}{2^k} |h_k(x)| \right) p_M(g(x)) + \omega_{p_M|\hat{K}} \left( \sum_{k=0}^{\infty} \frac{m}{2^k} |h_k(x)| \theta_k(|h_k(x)|) \right) \leq \left[ \sum_{k=0}^{\infty} \frac{1}{2^k} |h_k(x)| \right] (1 + \varepsilon_n) + \omega_{p_M|\hat{K}} \left( \sum_{k=0}^{\infty} \frac{m}{2^k} |h_k(x)| \theta_k(|h_k(x)|) \right) + \left[ \sum_{k=0}^{n} \frac{1}{2^k} |h_k(x)| \right] (1 + \varepsilon_n) + \omega \left( \sum_{k=0}^{n} \frac{m}{2^k} \theta_k(|h_k(x)|) \right) \leq \left[ \sum_{k=0}^{n} \frac{1}{2^k} |h_k(x)| \right] (1 + \varepsilon_n) + \omega \left( \sum_{k=0}^{n} \frac{m}{2^k} \theta_k(|h_k(x)|) \right) + \frac{m' \varepsilon_{n+1}}{2^n} + \omega_{p_M|\hat{K}} \left( \frac{m' \varepsilon_{n+1}}{2^n} \right).
\]

(4.8)

Note that due to (4.5)
\[
\frac{m' \varepsilon_{n+1}}{2^n} + \omega_{p_M|\hat{K}} \left( \frac{m \varepsilon_{n+1}}{2^n} \right) \leq \max \left\{ \frac{m'}{m}, 1 \right\} \cdot \omega \left( \frac{m \varepsilon_{n+1}}{2^n} \right) \leq \varepsilon_n.
\]

(4.9)

Also, we have for \( n \geq 1 \) (since \( \varepsilon_n \leq \frac{1}{2^{n+1}} \) and \( |h_k| \leq 1 \))
\[
(1 - \varepsilon_n) - \left( \sum_{k=1}^{n} \frac{1}{2^k} |h_k(x)| \right) (1 + \varepsilon_n) \geq \sum_{k=1}^{n} \left( \frac{1 - |h_k(x)|}{2^k} + \frac{1}{2^n} - \varepsilon_n \left( 2 - \frac{1}{2^n} \right) \right) > \sum_{k=1}^{n} \frac{1 - |h_k(x)|}{2^k}.
\]
This and (4.3), (4.10) imply for $n \geq 1$,

\[
\left( \sum_{k=1}^{n} \frac{1}{2^k} |h_k(x)| \right) (1 + \varepsilon_n) + \omega \left( \sum_{k=1}^{n} \frac{m}{2^k} \theta_k(|h_k(x)|) \right) < \\
1 - \varepsilon_n + \omega \left( \sum_{k=1}^{n} \frac{m}{2^k} \theta_k(|h_k(x)|) \right) - \sum_{k=1}^{n} \frac{1 - |h_k(x)|}{2^k} \leq 1 - \varepsilon_n.
\]

(4.10)

Thus, applying estimates (4.9) and (4.10) to (4.8) we obtain that for all $x \in U_n \setminus U_{n+1}$, $n \in \mathbb{Z}_+$, $p_M(h(x)) < 1$, i.e., $h(x) \in M$ in this case.

Further, if $x \in \cap_{n \in \mathbb{Z}_+} U_n$, then $p_M(g(x)) \leq 1$. Hence, as in (4.8) using continuity of $\omega$ and $\omega^{-1}$ and (4.3), (4.4) we obtain

\[
p_M(h(x)) \leq \left( \sum_{k=1}^{\infty} \frac{1}{2^k} |h_k(x)| \right) + \omega \left( \sum_{k=1}^{\infty} \frac{m}{2^k} \theta_k(|h_k(x)|) \right) = \\
- \left( \sum_{k=1}^{\infty} \frac{1 - |h_k(x)|}{2^k} \right) + 1 + \omega \left( \sum_{k=1}^{\infty} \frac{m}{2^k} \theta_k(|h_k(x)|) \right) = \\
1 + \lim_{n \to \infty} \omega \left( \sum_{k=1}^{n} \frac{m}{2^k} \theta_k(|h_k(x)|) \right) - \sum_{k=1}^{n} \frac{1 - |h_k(x)|}{2^k} \leq \\
1 + \omega \left( \frac{1}{2} \omega^{-1} \left( \sum_{k=1}^{\infty} \frac{1 - |h_k(x)|}{2^k} \right) \right) - \sum_{k=1}^{\infty} \frac{1 - |h_k(x)|}{2^k} \leq 1.
\]

(4.11)

Since $\frac{1}{2} \omega^{-1}(t) < \omega^{-1}(t)$ for $t > 0$, the equality in (4.11) holds if and only if

\[
\sum_{k=1}^{\infty} \frac{1 - |h_k(x)|}{2^k} = 0.
\]

This implies that $|h_k(x)| = 1$ for all $k \in \mathbb{N}$. In turn, according to our construction (see Lemma 4.3), the latter implies that $x \in S$. Thus, $p_M(h(x)) < 1$ for $x \in (\cap_{n \in \mathbb{Z}_+} U_n) \setminus S$, i.e., $h(x) \in M$ for such $x$ as well.

Therefore $h$ maps $X$ in $M$, $h|_S = f$ and $h(S^c) \subset M$, as required. \qed

### 4.3. Proof of Corollary 1.8

The first statement follows directly from Theorem 1.6 (see also Remark 1.7) as $[\text{co}(f(S))]_c \subset B$ is a bounded open convex set containing $f(S)$. The second one is a consequence of Theorem 1.6 as well as $(\text{co}(f(S)))^c \subset \mathbb{C}^n$ is a bounded open convex set and $f(S) \cap \partial(\text{co}(f(S))) \neq \emptyset$. \qed

### References

[Br] A. Brudnyi, Dense stable rank and Runge type approximation theorems for $H^\infty$ maps, J. Aust. Math. Soc., to appear. [arXiv:2006.04179]

[C] L. Carleson, Representations of continuous functions, Math. Z. 66 (1957), 447–451.

[Co] J. B. Conway, A course in functional analysis, Springer, New York, 1990.
[F1] F. Forstnerič, Stein manifolds and holomorphic maps, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 56, Springer, Heidelberg, 2011.

[F2] F. Forstnerič, Surjective holomorphic maps onto Oka manifolds. In: Angella D., Medori C., Tomassini A. (eds) Complex and Symplectic Geometry, pp. 73–84. Springer INdAM Series, Vol. 21. Springer, Cham, 2017.

[FS] J. E. Fornaess and E. L. Stout, Spreading polydiscs on complex manifolds, Amer. J. Math. 99 (5) (1977), 993–960.

[G] T. W. Gamelin, Uniform algebras, Prentice-Hall (1969).

[GR] R. C. Gunning and H. Rossi, Analytic functions of several complex variables, Prentice-Hall, Englewood Cliffs, N.J., 1965.

[K] Y. Kusakabe, Oka properties of complements of holomorphically convex sets, arXiv:2005.08247, 2020.

[M] S.N. Mergelyan, On the representation of functions by series of polynomials on closed sets, Transl. Amer. Math. Soc., 3 (1962), 287–293, Dokl. Akad. Nauk SSSR, 78 (3) (1951), 405–408.

[N] J. Nagata, Modern dimension theory, Wiley (Interscience), Groningen, 1965.

[Pa] R. S. Palais, Homotopy theory of infinite dimensional manifolds, Topology 5 (1966), 1–16.

[P] C. Pommerenke, Boundary behavior of conformal maps, Springer, Berlin, 1992.

[R] W. Rudin, Boundary values of continuous analytic functions, Proc. Amer. Math. Soc. 7 (1956), 808–811.

[RS] W. Rudin and E. L. Stout, Boundary properties of functions of several complex variables, J. Math. Mech. 14 (1965), 991–1005.

[S] E. L. Stout, On some restriction algebras. Function Algebras (Proc. Internat. Sympos. on Function Algebras, Tulane Univ., New Orleans, La., 1965), 6–11, Scott-Foresman, Chicago, 1966.

[T] F. A. Toranzos, Radial functions of convex and star-shaped bodies, Am. Math. Mon., 74 (3) (1967), 278–280.

Department of Mathematics and Statistics
University of Calgary
Calgary, Alberta, Canada
T2N 1N4

Email address: abrudnyi@ucalgary.ca