Charged Particle System in Uniform Magnetic and Electric Fields: The Role of Galilean Transformation

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Abstract

Galilean transformation relates a physical system under mutually perpendicular uniform magnetic and electric fields to that under uniform magnetic field only. This allows a complete specification of quantum states in the former case in terms of those for the latter. Based on this observation, we consider the Hall effect and the behavior of a neutral composite system in the presence of uniform electromagnetic fields.

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1. INTRODUCTION

Charged particle systems in the presence of uniform electromagnetic fields are known to exhibit many remarkable features. Even in the simplest quantum mechanical problem of a nonrelativistic charged particle moving in a uniform magnetic field, we find the famous Landau energy levels [1] with infinite degeneracy. Actually the infinite degeneracy, being a direct consequence of the noncommutativity of the translation generators [2] in the presence of a uniform magnetic field, is also to be found in many-particle systems with translation-invariant interactions. When the external magnetic field is sufficiently strong, the behavior of a physical system can be quite extraordinary. The most spectacular related development in recent years is the observation of the quantum Hall effect [3], which manifests itself as a series of plateaus in the Hall resistance of materials containing two-dimensional electron systems. A large number of literatures devoted to the theoretical explanations of this effect has appeared since [4,5].

Analytical solutions to many body problems in the presence of background electromagnetic fields are notoriously difficult. In this paper we describe certain general consequences which stem solely from the Galilean transformation property of underlying dynamical equations in the presence of a uniform magnetic field. Specifically it will be shown that a complete specification of any Galilean-invariant physical system in the presence of mutually perpendicular uniform magnetic and electric fields can be made in terms of those appropriate to the same system subject to a uniform magnetic field only. The map connecting the two cases is the Galilean transformation, aside from certain complications involving gauge transformation. Thanks to this map, energy eigenvalues and energy eigenfunctions of the two cases are simply related irrespectively of the details of interparticle interactions. The Hall effect is
in fact its most obvious realization in many electron systems, and it has other application, e.g., on the behavior of a neutral composite system such as a hydrogen atom or a neutron in the presence of uniform electromagnetic fields. In the latter example, Galilean transformation accounts for the intermixing between the center of mass motion and the Stark effect (in the background of a uniform magnetic field). It also applies to the system of non-relativistic anyons [6] with arbitrary Galilean invariant mutual interaction.

In Ref. 2, Fubini also derived the formula for the change in the Landau energy levels as an additional uniform electric field is introduced. But we find his derivation physically less transparent since, in that paper, the Galilean transformation behavior of the wave function is not addressed at all. Ours not only clarifies the physical origin of the formula but also tells explicitly how the wave functions change. The contents of this paper is quite elementary, but, to our knowledge, not exploited much in the literature. We here hope to convince readers that some of remarkable phenomena exhibited by matter in the presence of uniform electromagnetic fields are just consequences of the Galilean relativity principle.

This paper is organized as follows. Section II is a general exposition on the realization of Galilean transformation in non-relativistic classical or quantum systems, when a uniform background magnetic field is present. The result is applied to simple one-body and two-body systems in the ensuing two sections—classically in Section III and quantum mechanically in Section IV. Especially the case of a net-neutral composite system, which has a rather non-trivial dynamics, is dealt with in some detail. Section V contains a summary and discussion. In the Appendix, the behavior of particle Green’s function under Galilean transformation is described.
2. GALILEAN BOOST IN THE PRESENCE OF A UNIFORM MAGNETIC FIELD

First, consider the case of classical mechanics. Given a uniform external magnetic field $\mathbf{B} = B\hat{z}$ in some inertial frame, the equation of motion for a system of $N$ Newtonian particles (with masses $m_i$ and charges $q_i$) will read

$$m_i \frac{d^2 \mathbf{r}_i}{dt^2} = \frac{q_i}{c} \frac{d\mathbf{r}_i}{dt} \times \mathbf{B} - \frac{\partial}{\partial \mathbf{r}_i} V \quad (i = 1, \ldots, N). \quad (2.1)$$

Here, $V$ can be any Galilean-invariant interaction; but, for definiteness, we set $V = \sum_{i<j} V(\mathbf{r}_i - \mathbf{r}_j)$, i.e., equal to the sum of two-body interactions involving relative positions only. Then, in a primed coordinate system obtained by the Galilean boost

$$\mathbf{r}' = \mathbf{r} - u t, \quad t' = t, \quad (2.2)$$

we will have the equations of motion

$$m_i \frac{d^2 \mathbf{r}'_i}{dt'^2} = \frac{q_i}{c} \frac{d\mathbf{r}'_i}{dt'} \times \mathbf{B} + q_i \mathbf{E} - \frac{\partial}{\partial \mathbf{r}'_i} \left( \sum_{i<j} V(\mathbf{r}'_i - \mathbf{r}'_j) \right), \quad (2.3)$$

where $\mathbf{E} \equiv \frac{u}{c} \times \mathbf{B}$. Notice that we see also an external electric field in the primed system. This shows that two problems—classical dynamics in the presence of a uniform $\mathbf{B}$-field and that in the presence of uniform, mutually perpendicular, $\mathbf{B}$- and $\mathbf{E}$-fields—are simply related. Explicitly, if $\mathbf{r}_i = \mathbf{f}_i(t)$ is any specific solution to (2.1), then

$$\mathbf{r}'_i = \mathbf{f}_i(t) - u t, \quad \left( u \equiv -c \frac{\mathbf{E} \times \mathbf{B}}{|\mathbf{B}|^2} \right) \quad (2.4)$$

solves the system defined by (2.3), i.e., the problem with an additional uniform electric field $\mathbf{E}$ (which is assumed to be perpendicular to $\mathbf{B}$). The piece $-ut$ in (2.4) describes the famous Hall drift motion, which is derived from the Galilean transformation behavior of the system alone. Details of the interparticle interactions
do not enter in this discussion. Also, on the basis of (2.4), one might think that
the entire effect of the additional electric field is to cause the constant drift of the
whole system only. That is not always so, for the separation of the center of mass
coordinates is not always trivial if a uniform magnetic field is present [7]. See Sec. III on this.

We now turn to the corresponding quantum mechanical problem. The Hamiltonian appropriate to the equations of motion (2.1) is

\[ H = \sum_i \frac{1}{2m_i} \left[ p_i - \frac{q_i}{c} A(r_i) \right]^2 + \sum_{i<j} V(r_i - r_j), \]

(2.5)

where the vector potential \( A(r) \) should be chosen such that \( \nabla \times A(r) = B \hat{z} \). Then the time development of a quantum state is described by the wave function \( \Psi(r_1, \ldots, r_N; t) \) satisfying the Schrödinger equation

\[
i \hbar \frac{\partial}{\partial t} \Psi(r_1, \ldots, r_N; t) = (H\Psi)(r_1, \ldots, r_N; t) = \left\{ -\frac{\hbar^2}{2m_i} \left[ \frac{\partial}{\partial r_i} - i \frac{q_i}{\hbar c} A(r_i) \right]^2 + \sum_{i<j} V(r_i - r_j) \right\} \Psi(r_1, \ldots, r_N; t).
\]

(2.6)

On the other hand, in connection with the equations of motion (2.3), we can consider another Schrödinger equation

\[
i \hbar \frac{\partial}{\partial t} \Psi'(r_1, \ldots, r_N; t) = \left\{ -\sum_i \frac{\hbar^2}{2m_i} \left[ \frac{\partial}{\partial r_i} - i \frac{q_i}{\hbar c} A(r_i) \right]^2 + \sum_{i<j} V(r_i - r_j) - \sum_i q_i E \cdot r_i \right\} \times \Psi'(r_1, \ldots, r_N; t),
\]

(2.7)

where \( E \) can be any constant (electric field) vector subject to the condition \( E \cdot B = 0 \). As in the classical motion, we assert that solutions of these two problems are related by a suitable Galilean transformation modulo gauge transformation. Explicitly, for any given solution \( \Psi(r_1, \ldots, r_N; t) \) to (2.3), we have a corresponding solution to (2.7) which has the form
\[
\Psi'(r_1, \ldots, r_N; t) = e^{\frac{i}{\hbar} \alpha(r_1, \ldots, r_N; t; u)} \Psi(r_1 + ut, \ldots, r_N + ut; t) \tag{2.8}
\]

with \( u \equiv -\frac{eE \times B}{B^2} \) and the (gauge-dependent) phase \( \alpha(r_1, \ldots, r_N; t; u) \) to be specified below.

In general, one can always express the vector potential leading to a constant magnetic field \( B \) by the form

\[
A(r) = B \times [r - (r \cdot \hat{u}) \hat{u}] - \nabla \tilde{\Lambda}(r), \tag{2.9}
\]

where \( \hat{u} \equiv u/|u| \) is assumed to be perpendicular to \( B \) and \( \tilde{\Lambda}(r) \) is an arbitrary gauge function. Note that, with \( \tilde{\Lambda}(r) = 0 \), the given vector potential becomes invariant under the Galilean transformation \( r \rightarrow r - ut \). If the vector potential has been chosen such that \( \tilde{\Lambda}(r) \) may vanish, the phase \( \alpha(r_1, \ldots, r_N; t; u) \) in (2.8) can in fact be identified with the usual 1-cocycle [8] appearing in the quantum mechanical realization of Galilean transformations, namely,

\[
\alpha(r_1, \ldots, r_N, t; u) = -\sum_i \left\{ m_i u \cdot r_i + \frac{1}{2} m_i u^2 t \right\} \quad \text{(for } \tilde{\Lambda}(r) = 0). \tag{2.10}
\]

This may be checked by a direct computation; viz., if \( \Psi \) satisfies (2.6), the wave function \( \Psi' \) given by (2.8) satisfies the Schrödinger equation of the form (2.7) with \( E = u/c \times B \). [Note that, with \( E \cdot B = u \cdot B = 0 \), \( E = u/c \times B \) is equivalent to \( u = -\frac{eE \times B}{|B|^2} \)]. When one has the vector potential of the general form (2.9), the way to proceed is now obvious—consider the Galilean boost for suitably gauge-transformed wave functions \( \Psi \) and \( \Psi' \) (which satisfy the appropriate Schrödinger equations with the vector potential given by the form invariant under the Galilean boost in question). This gives rise to the following phase function:

\[
\alpha(r_1, \ldots, r_N, t; u) = -\sum_i \left\{ m_i u \cdot r_i + \frac{1}{2} m_i u^2 t \right\} - \sum_i \frac{qi}{c} \left\{ \tilde{\Lambda}(r_i) - \tilde{\Lambda}(r_i + ut) \right\} \quad \text{(general case).} \tag{2.11}
\]
In the formula (2.8) with the phase given by (2.11), we have a precise relation connecting general time-dependent wave functions of two distinct systems, i.e., one in a uniform magnetic field \( \mathbf{B} \) and the other subject to an additional constant electric field in the direction perpendicular to \( \mathbf{B} \). This allows us to derive all pertinent informations concerning the system under the external fields, say, \( \mathbf{B} = B\hat{z} \) and \( \mathbf{E} = E\hat{y} \) from those for the system under \( \mathbf{B} = B\hat{z} \) only, just by considering the Galilean transformation with \( \mathbf{u} = -c\frac{\mathbf{E}}{\mathbf{B}}\hat{x} \). For instance, a single-particle Green’s function in the presence of both \( \mathbf{B} \)- and \( \mathbf{E} \)-fields can be written down using the Green’s function defined in the presence of \( \mathbf{B} \) only. This discussion, which further illuminates the meaning of the phase factor \( e^{\pm\alpha} \), is relegated to the Appendix. Also, for the above choice of field directions (as we shall assume below), the vector potential invariant under the Galilean boost is assumed by the familiar Landau gauge, i.e.,

\[
\mathbf{A}(\mathbf{r}) = -By\hat{x} \quad \text{(Landau gauge).} \tag{2.12}
\]

Of special importance will be the implications on the energy spectra and the respective energy eigenfunctions. To study such, let

\[
\Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N; t) = e^{-\frac{i}{\hbar}E_\mathcal{E}(\mathbf{r}_1, \ldots, \mathbf{r}_N)} \tag{2.13}
\]

represent a stationary solution to (2.6). Then, in the Landau gauge, we know on the basis of our formula (2.8) that the wave function

\[
\Psi'(\mathbf{r}_1, \ldots, \mathbf{r}_N; t) = e^{-\frac{i}{\hbar}(E + \frac{1}{2}M^2\mathbf{B}^2/\hbar^2)e^{\frac{cE}{B}\frac{\hbar}{\hbar}(m_1 + \cdots + m_N)}(\mathbf{r}_1 - \frac{cEt}{B}\hat{x}, \ldots, \mathbf{r}_N - \frac{cEt}{B}\hat{x})} \tag{2.14}
\]

(here \( M = m_1 + m_2 + \cdots + m_N \)) should be a solution to the Schrödinger equation (2.7). We now utilize the fact that, in the Landau gauge, we have \([P_x, H] = 0\) where \( P_x = \sum_i \frac{\hbar}{i}\frac{\partial}{\partial r_i} \) (i.e. equal to the translation generator in the direction of the
Galilean boost we make). Hence we may require \( \varphi_{E}(\mathbf{r}_1, \ldots, \mathbf{r}_N) \) to be a simultaneous eigenstate of \( H \) and \( P_x \), with eigenvalues \( E \) and \( P_x \). This means that, in (2.13), we may substitute

\[
\varphi_{E}(\mathbf{r}_1, \ldots, \mathbf{r}_N) = e^{\frac{i}{\hbar}P_{x}X}\varphi(Y, Z, l_1, \ldots, l_{N-1}; P_x)
\]  

(2.15)

where \( \mathbf{R} \equiv (X, Y, Z) = \frac{1}{M} \sum_i m_i \mathbf{r}_i \) represents the center of mass, and \( l_1, \ldots, l_{N-1} \) various relative coordinates whose values do not depend on the choice of the spatial origin. We make the substitution also in (2.14), to obtain the wave function of the form

\[
\Psi' = e^{-i \frac{c}{\hbar}E' t} \left[ e^{\frac{i}{\hbar}P_{x}X}\varphi(Y, Z, l_1, \ldots, l_{N-1}; P_x) \right],
\]  

(2.16)

\[
P_x' = P_x + \frac{cE}{B} M,
\]  

(2.17a)

\[
E' = E + \frac{cE}{B} P_x + \frac{1}{2} M \frac{c^2E^2}{B^2} = E + \frac{cE}{B} P_x' - \frac{1}{2} M \frac{c^2E^2}{B^2}.
\]  

(2.17b)

These provide the connection between the energy eigenvalue spectra and energy eigenfunctions of the two systems, and we emphasize that these relations apply irrespectively of the details of interparticle interactions. Also, when the center of mass dynamics cannot be trivially separated from the rest, the simple change introduced in (2.16) can alter the nature of the stationary state significantly. We will illustrate this phenomenon with the example of a two-body neutral atom in Sec. IV.

Note that the substitution (2.15) can be made only in the Landau gauge. If one wishes to work with the vector potential given by the general expression (2.9) (with \( \hat{\mathbf{u}} = \hat{\mathbf{x}} \)), one may instead make the substitution

\[
\varphi_{E}(\mathbf{r}_1, \ldots, \mathbf{r}_N) = e^{-i \sum_i \hat{\mathbf{u}} \cdot \mathbf{A}(r_i)} e^{\frac{i}{\hbar}P_{x}X}\varphi(Y, Z, l_1, \ldots, l_{N-1}; P_x).
\]  

(2.18)

What we have in the right hand side here is the eigenstate of the conserved translation generator in the \( x \) direction, \( \Pi_x \). In a general gauge, \( \Pi_x \) is specified as
\[ \Pi_x = \sum_i \left\{ \frac{\hbar}{i} \frac{\partial}{\partial x_i} - \frac{q_i}{c} A_x(r_i) - \frac{q_i}{c} (r_i \times B)_x \right\} , \quad (2.19) \]

and the eigenvalue of this operator is \( \mathcal{P}_x \). Using this substitution with our general formulas (2.8) and (2.11) then yields the expression
\[
\Psi' = e^{-\frac{\mathcal{E}'}{\hbar} t} e^{-i \sum_i \mathcal{A}(r_i) c \mathcal{P}'_x x} \tilde{\varphi}(Y, Z, l_1, \ldots, l_{N-1}; \mathcal{P}_x) \quad (2.20)
\]

with \( \mathcal{E}' \) and \( \mathcal{P}'_x \) specified as before. Thus the only difference from the Landau gauge case is the gauge transformation factor multiplying the energy eigenfunctions, which is of course natural.

We may summarize our observation in the following way: a quantum system under mutually perpendicular \( \mathbf{B} \)- and \( \mathbf{E} \)-fields is just the Galilean transformation of the same system subject to the \( \mathbf{B} \)-field only, and the needed boost velocity \( \mathbf{u} = -c \frac{\mathbf{E} \times \mathbf{B}}{B^2} \) is precisely that of the Hall drift motion. This connection can also be exhibited for the charge and current densities. Assuming that the wave function \( \Psi \) has been properly normalized, we may define the charge and current densities by
\[
\rho(r, t) = \int d^3 r_1 \cdots d^3 r_N \Psi^*(r_1, \ldots, r_N; t) \left( \sum_i q_i \delta^3(r - r_i) \right) \Psi(r_1, \ldots, r_N; t) , \quad (2.21a)
\]
\[
\mathbf{j}(r, t) = \int d^3 r_1 \cdots d^3 r_N \Psi^*(r_1, \ldots, r_N; t) \left( \sum_i \left( -i \hbar q_i \right) \frac{1}{2m_i} \left[ \frac{\partial}{\partial r_i} - \frac{\partial}{\partial r_i} - \frac{2iq_i}{\hbar c} \mathbf{A}(r) \right] \right) \times \Psi(r_1, \ldots, r_N; t) , \quad (2.21b)
\]

so that the current conservation, \( \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \), may hold. Let \( \rho(r) \) and \( \mathbf{j}(r) \) represent the appropriate quantities for the stationary state given by (2.13) (with the substitution (2.15)), i.e., the quantities when the system is subject to the \( \mathbf{B} \)-field only. Now, if we evaluate the charge and current densities \((\rho'(r), \mathbf{j}'(r))\) for the wave function given by (2.16), we immediately find the result
\[
\rho'(r) = \rho(r) , \quad \mathbf{j}'(r) = \mathbf{j}(r) + \rho(r) \frac{cE}{B} \hat{x} . \quad (2.22)
\]
Here the contribution $\rho(r) \frac{\partial E}{\partial \hat{B}} \hat{x}$ is the Hall current density which is seen in the presence of an additional electric field $E = E\hat{y}$. Note that this is an exact result as long as interparticle interactions are Galilean invariant.

3. APPLICATION TO SIMPLE CLASSICAL SYSTEMS

We shall here illustrate the observation made in the previous section with classical one-body and two-body systems. Field directions are taken as $B = B\hat{z}$ and $E = E\hat{y}$. For a single charged particle, the equation of motion

$$m\frac{d^2\mathbf{r}}{dt^2} = \frac{q}{c} \frac{d\mathbf{r}}{dt} \times \mathbf{B} + q\mathbf{E}$$  \hspace{1cm} (3.1)

can be solved easily. With $E = 0$, it is a well-known cyclotron motion: viz.,

$$\mathbf{r}(t) = (v_z t + z_0)\hat{z} + r_{0\perp} + K[\hat{x}\cos(\omega_c t - \theta_0) + \hat{y}\sin(\omega_c t - \theta_0)]$$ \hspace{1cm} (3.2)

where $v_z, z_0, r_{0\perp}, K (> 0), \theta_0$ are time-independent parameters and $\omega_c = \frac{qB}{mc}$ is the cyclotron frequency. Then, with $E \neq 0$, our recipe tells us that the general solution is simply

$$\mathbf{r}(t) = (v_z t + z_0)\hat{z} + r_{0\perp} + K[\hat{x}\cos(\omega_c t - \theta_0) + \hat{y}\sin(\omega_c t - \theta_0)] + \frac{cE}{B} t \hat{x}. \hspace{1cm} (3.3)$$

For a two-body system, we may write the equations of motion as

$$m_1 \frac{d^2\mathbf{r}_1}{dt^2} = \frac{q_1}{c} \frac{d\mathbf{r}_1}{dt} \times \mathbf{B} + q_1\mathbf{E} - \frac{\partial}{\partial \mathbf{r}_1} V, \hspace{1cm} (3.4)$$

$$m_2 \frac{d^2\mathbf{r}_2}{dt^2} = \frac{q_2}{c} \frac{d\mathbf{r}_2}{dt} \times \mathbf{B} + q_2\mathbf{E} - \frac{\partial}{\partial \mathbf{r}_2} V,$$

where $V$ denotes some interparticle potential. Again the case of $E = 0$ may be considered first. We here restrict our attention to the cases for which the center of mass coordinates $\mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}$ can be separated from the dynamics involving the relative position $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. There are two such situations [7]—viz., either
With $\frac{q_1}{m_1} = \frac{q_2}{m_2} = \frac{Q}{M}$ (with $Q = q_1 + q_2$ and $M = m_1 + m_2$) or (ii) $q_1 = -q_2 = q$ (i.e. a net neutral system). We also take the interparticle potential to be $V = \frac{k}{2} |\mathbf{r}_1 - \mathbf{r}_2|^2$, so that we can present closed-form solutions.

With $\frac{q_1}{m_1} = \frac{q_2}{m_2} = \frac{Q}{M}$, (3.4) can be rewritten as

\[
M \frac{d^2 \mathbf{R}}{dt^2} = \frac{Q}{c} \frac{d \mathbf{R}}{dt} \times \mathbf{B}, \quad (3.5a)
\]
\[
\frac{d^2 \mathbf{r}}{dt^2} = \frac{Q}{cM} \frac{d \mathbf{r}}{dt} \times \mathbf{B} - \frac{k}{\mu} \mathbf{r}, \quad (3.5b)
\]

where $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass. The center-of-mass dynamics is not different from the one particle motion discussed already. On the other hand, using the variables $w = x + iy$ and $z$, (3.5b) becomes

\[
\frac{d^2 z}{dt^2} + \frac{k}{\mu} z = 0,
\]
\[
\frac{d^2 w}{dt^2} + \frac{iQB}{Mc} \frac{dw}{dt} + \frac{k}{\mu} w = 0. \quad (3.6)
\]

These have the general solutions

\[
z(t) = A_z \sin \left( \sqrt{\frac{k}{\mu}} t + \theta_0 \right), \quad (3.7a)
\]
\[
w(t) = A_+ e^{-i\alpha_+ t} + A_- e^{-i\alpha_- t} \quad \left( \alpha_\pm = \frac{QB}{2Mc} \pm \sqrt{\frac{k}{\mu} + \left( \frac{QB}{2Mc} \right)^2} \right), \quad (3.7b)
\]

where $A_z$ and $\theta_0$ are real while $A_\pm$ can be arbitrary complex numbers. Using $\mathbf{r}_\pm \equiv (x, y)$ and setting $\sqrt{\frac{k}{\mu} + \left( \frac{QB}{2Mc} \right)^2} \equiv \tilde{\omega}_0$, (3.7b) can also be expressed by the form

\[
\mathbf{r}_\pm(t) = a \hat{e}_1(t) \cos(\tilde{\omega}_0 t - \theta_0) + b \hat{e}_2(t) \sin(\tilde{\omega}_0 t - \theta_0) \quad (3.8)
\]

with

\[
\hat{e}_1(t) = \cos \left( \frac{QB}{2Mc} t - \theta_1 \right) \hat{x} - \sin \left( \frac{QB}{2Mc} t - \theta_1 \right) \hat{y}, \quad \hat{e}_2(t) = \hat{z} \times \hat{e}_1(t), \quad (3.9)
\]

where $a$, $b$, $\theta_0$ and $\theta_1$ are some real constants. Note that, for $B = 0$, $\hat{e}_1$ and $\hat{e}_2$ become space-fixed and the orbit becomes elliptical. But, for $B \neq 0$, the axes of the
ellipse also undergo uniform rotation with frequency \( \frac{Q}{2Mc} \). [This is in accord with the classical Larmor theorem.] Having the additional electric field \( \mathbf{E} = E\hat{y} \) does not affect the relative dynamics; its entire effect is to add the drift motion term \( \frac{eE}{c} t \hat{x} \) in \( \mathbf{R}(t) \) (just as in (3.3)).

More interesting is the case of \( q_1 = -q_2 = q \). Here, employing the center of mass and relative coordinates, we can cast (3.4) for \( E = 0 \) as

\[
M \frac{d^2 \mathbf{R}}{dt^2} = \frac{q}{c} \frac{d\mathbf{r}}{dt} \times \mathbf{B},
\]

(3.10a)

\[
\mu \frac{d^2 \mathbf{r}}{dt^2} = \frac{q}{m_1 + m_2} \frac{m_2 - m_1}{m_1 + m_2} \frac{d\mathbf{r}}{dt} \times \mathbf{B} + \frac{q}{c} \frac{d\mathbf{R}}{dt} \times \mathbf{B} - k \mathbf{r}.
\]

(3.10b)

From (3.10a), we notice the existence of the first integrals

\[
M \frac{dZ}{dt} = \mathcal{P}_z \ (\text{const.}),
\]

(3.11a)

\[
M \frac{d\mathbf{R}_\perp}{dt} - \frac{q}{c} \mathbf{r}_\perp \times \mathbf{B} = \mathcal{P}_\perp \ (\text{const.}),
\]

(3.11b)

and, using these relations, (3.10b) can be made as equations for \( \mathbf{r} \) only:

\[
\frac{d^2 z}{dt^2} = -\frac{k}{\mu} z,
\]

(3.12a)

\[
\frac{d^2 \mathbf{r}_\perp}{dt^2} = \frac{q}{c} \frac{m_2 - m_1}{m_1 + m_2} \frac{d\mathbf{r}_\perp}{dt} \times \mathbf{B} - \left( \frac{k}{\mu} + \frac{q^2 B^2}{c^2 m_1 m_2} \right) \mathbf{r}_\perp + \frac{q}{c m_1 m_2} \mathcal{P}_\perp \times \mathbf{B}.
\]

(3.12b)

[In this paper, the subscript \( \perp \) is attached to indicate any vector which is defined to be perpendicular to the direction of \( \mathbf{B} \). Due to (3.11b), the center of mass motion is not completely independent from the relative motion in the present case. Motion in the direction of \( \mathbf{B} \) requires no explanation. On the other hand, to analyze the transverse motion, we find it convenient to introduce the displaced relative position

\[
\mathbf{r}'_\perp = \mathbf{r}_\perp - \frac{q}{cM} \mathbf{\alpha}_\perp \times \mathbf{B}, \quad \left( \mathbf{\alpha} \equiv \frac{\mathcal{P}_\perp}{k + (q^2 B^2/c^2 M)} \right).
\]

(3.13)

then, (3.11b) and (3.12b) read
\[
M \frac{d \mathbf{R}_\perp}{dt} = \frac{q}{c} \mathbf{r}'_\perp \times \mathbf{B} + k \mathbf{\alpha}_\perp , \quad (3.14a)
\]
\[
\mu \frac{d^2 \mathbf{r}'_\perp}{dt^2} = \frac{q}{c} \frac{m_2 - m_1}{m_1 m_2} \frac{d \mathbf{r}'_\perp}{dt} \times \mathbf{B} - \frac{1}{\mu} \left( k + \frac{q^2 B^2}{c^2 M} \right) \mathbf{r}'_\perp . \quad (3.14b)
\]

Equation (3.14b) has the same form as (3.5b), and hence its general solution is provided by the expression (3.8) except for the fact that we here have

\[
\tilde{\omega}_0 = \sqrt{\frac{1}{\mu} \left( k + \frac{q^2 B^2}{c^2 M} \right) + \frac{q^2 B^2 (m_2 - m_1)^2}{4 c^2 m_1^2 m_2^2}} \text{ and the axes } \hat{\mathbf{e}}_1(t), \hat{\mathbf{e}}_2(t) \text{ rotate with frequency } \frac{qB(m_2 - m_1)}{2m_1 m_2} . \]

Using this solution for \( \mathbf{r}'_\perp(t) \) in (3.14a) one can determine the center of mass motion as well and the result will clearly have the form

\[
\mathbf{R}_\perp(t) = \mathbf{R}_0 + \frac{k}{M} \mathbf{\alpha}_\perp t + \left( \text{piece oscillating about the zero vector} \right) . \quad (3.15)
\]

Here, \( \mathbf{\alpha}_\perp \) can be any constant vector.

There is a certain noteworthy point with the above solution. If \( \mathbf{\alpha}_\perp = 0 \), both \( \mathbf{r}_\perp(t) \) and \( \mathbf{R}_\perp(t) \) will oscillate about zero. But, when \( \mathbf{\alpha}_\perp \neq 0 \) or equivalently the center of mass has non-zero average velocity \( \langle \dot{\mathbf{R}}_\perp \rangle = \frac{k}{M} \mathbf{\alpha}_\perp \), the (transverse) relative vector \( \mathbf{r}_\perp(t) \) will oscillate about the average value

\[
\langle \mathbf{r}_\perp \rangle = \frac{q}{ck} \langle \dot{\mathbf{R}}_\perp \rangle \times \mathbf{B} . \quad (3.16)
\]

This shows that the composite acquires an average electric dipole moment \( \frac{q^2}{ck} |\langle \dot{\mathbf{R}}_\perp \rangle \times \mathbf{B}| \) in the direction perpendicular to both \( \mathbf{B} \) and \( \langle \dot{\mathbf{R}}_\perp \rangle \). This behavior is easily understood for the special (non-oscillating) solution given by \( \mathbf{r}_\perp(t) = \frac{q}{ck} \mathbf{\alpha}_\perp \times \mathbf{B} \) and \( \mathbf{R}_\perp(t) = \mathbf{R}_0 + \frac{k}{M} \mathbf{\alpha}_\perp t \) —then, (3.16) is nothing but the condition that the Lorentz forces acting on individual charges should balance the attractive interparticle forces.

Alternatively, one may contemplate on making the Galilean boost to the frame in which the very center of mass is at rest. The above phenomenon can then be seen as the Stark-type effect due to the electric field thus generated. The latter view is also closely related to the discussion that follows.
When this neutral system is subject to an additional electric field \( \mathbf{E} = E\hat{y} \), the corresponding general solution can be now written down with no effort. On the basis of (2.4), all that the \( \mathbf{E} \)-field does is to introduce an extra uniform motion term \( c\frac{\mathbf{E} \times \mathbf{B}}{B^2} t \) in \( \mathbf{R}(t) \); namely in the expression (3.13), the piece \( \frac{k}{M}\mathbf{\alpha}_\perp t \) gets effectively replaced by \( \frac{k}{M}\mathbf{\alpha}_\perp + c\frac{\mathbf{E} \times \mathbf{B}}{B^2} t \), while the expression for \( \mathbf{r}_\perp(t) \) is unaffected by the presence of the \( \mathbf{E} \)-field. We now have the formula \( \langle \dot{\mathbf{R}}_\perp \rangle = \frac{k}{M}\mathbf{\alpha}_\perp + c\frac{\mathbf{E} \times \mathbf{B}}{B^2} \) for the average center of mass velocity, which implies that the relationship between \( \langle \mathbf{r}_\perp \rangle \) and \( \langle \dot{\mathbf{R}}_\perp \rangle \) in the presence of the \( \mathbf{E} \)-field also reads

\[
\langle \mathbf{r}_\perp \rangle = \frac{q}{ck}\langle \dot{\mathbf{R}}_\perp \rangle \times \mathbf{B} + \frac{q}{k}\mathbf{E}. \tag{3.17}
\]

Evidently, this has an obvious explanation in terms of the force balance again. The average electric dipole moment \( \langle \mathbf{p}_e \rangle = q\langle \mathbf{r}_\perp \rangle \) points in the direction of the \( \mathbf{E} \)-field (the usual Stark effect) only when \( \langle \dot{\mathbf{R}}_\perp \rangle = 0 \); but with a non-zero center of mass velocity, it will assume the direction of \( q\mathbf{E}' \) with \( \mathbf{E}' = \mathbf{E} + \frac{1}{c}\langle \dot{\mathbf{R}}_\perp \rangle \times \mathbf{B} \). In particular, if the average center of mass velocity is equal to \( c\frac{\mathbf{E} \times \mathbf{B}}{B^2} \) (this amounts to \( \mathbf{\alpha}_\perp = 0 \)), we find \( \langle \mathbf{r}_\perp \rangle = 0 \), i.e., the system has zero electric dipole moment; this is the result of cancellation between the force due to the \( \mathbf{E} \)-field and the Lorentz force. Although we have only considered a specific, explicitly solvable, example here, it is clear that analogous phenomena should be exhibited by any neutral composite system.

4. APPLICATION TO SIMPLE QUANTUM SYSTEMS

In this section we will give quantum mechanical discussions for the same one-body and two-body systems that we considered classically in Section III. Here our main concern will be directed to the effects on the energy eigenstates, as an additional electric field is introduced into the system already subject to a uniform magnetic field. Conventions for the field directions are as in Section III.
The Hamiltonian for a single charged particle is
\[
H = \frac{1}{2m} \left( \mathbf{p} - \frac{q}{c} \mathbf{A}(\mathbf{r}) \right)^2 - qE_y, \quad (\nabla \times \mathbf{A} = B\hat{z}). \tag{4.1}
\]

For \( E = 0 \) (i.e., zero electric field), the energy eigenstates are familiar Landau levels [1]. Especially in the Landau gauge with \( \mathbf{A} = -By\hat{x} \), \( p_x \) and \( p_z \) commute with \( H \). So it suffices to diagonalize the Hamiltonian with \( p_x(p_z) \) replaced by its eigenvalue \( P_x(P_z) \), and one finds essentially a one-dimensional harmonic oscillator problem with
\[
H = \frac{1}{2m} p_y^2 + \frac{1}{2} \frac{m\omega_c^2}{2m} \left( y + \frac{cP_x}{qB} \right)^2 + \frac{1}{2m} P_z^2, \quad \left( \omega_c \equiv \frac{|qB|}{mc} \right). \tag{4.2}
\]

From this, it follows that the energy eigenfunction corresponding to the eigenvalue \( \mathcal{E}_{n,P_x,P_z} = \hbar\omega_c(n + \frac{1}{2}) + \frac{P_x^2}{2m} \) \((n = 0, 1, 2, \ldots)\) is
\[
\phi_{n,P_x,P_z}^L(\mathbf{r}) = (\text{const.}) e^{\frac{i}{\hbar}(P_x x + P_z z)} e^{-\frac{m\omega_c}{\hbar^2}(y + \frac{cP_x}{qB})^2} H_n \left( \sqrt{\frac{m\omega_c}{\hbar^2}} \left( y + \frac{cP_x}{qB} \right) \right), \tag{4.3}
\]
where \( H_n \) is the \( n \)-th degree Hermite polynomial. Infinite degeneracy of the Landau level is manifest in the fact that the energy eigenvalue has no dependence on the quantum number \( P_x \) at all. When there is a non-zero electric field \( \mathbf{E} = E\hat{y} \), we then make use of (2.16), (2.17a) and (2.17b) to obtain the corresponding exact energy eigenstates:
\[
\mathcal{E}_{n,P_x,P_z} = \hbar\omega_c \left( n + \frac{1}{2} \right) + \frac{cE}{B} P_x - \frac{mc^2E^2}{2B^2} + \frac{P_z^2}{2m},
\]
\[
\phi_{n,P_x,P_z}^L = (\text{const.}) e^{\frac{i}{\hbar}(P_x x + P_z z)} e^{-\frac{m\omega_c}{\hbar^2}(y + \frac{cP_x}{qB} - \frac{mc^2E}{qB^2})^2} H_n \left( \sqrt{\frac{m\omega_c}{\hbar^2}} \left( y + \frac{cP_x}{qB} - \frac{mc^2E}{qB^2} \right) \right). \tag{4.4}
\]

Infinite degeneracy of the Landau level is lifted by the electric field.

As we explained in Section II, one may work in other gauges as well although, given the electric field \( \mathbf{E} = E\hat{y} \), the above Landau gauge treatment is somewhat
simpler. Take, for instance, the symmetric gauge with \( A = -\frac{1}{2} \mathbf{r} \times \mathbf{B} \). For \( \mathbf{E} = 0 \), one normally considers the simultaneous eigenstates of \( H, p_z \) and \( L_z = \frac{\hbar}{i} (x p_y - y p_x) \) in this gauge. But, when one has in mind subjecting the system to the electric field \( \mathbf{E} = E \hat{\mathbf{y}} \), the more appropriate are the eigenstates of \( H, p_z \) and \( \Pi_x = p_x - \frac{q B}{2 c y} \). [See (2.19) for the definition of \( \Pi_x \) in an arbitrary gauge.] Now identifying the eigenvalue of \( \Pi_x \) with \( P_x \), those eigenstates are simply

\[
\varphi^S_{n,P_x,P_z}(\mathbf{r}) = e^{i \frac{q B}{2 c} x y} \varphi^L_{n,P_x,P_z}(\mathbf{r}),
\]

where, for \( \varphi^L_{n,P_x,P_z}(\mathbf{r}) \), one may substitute (4.3) (if \( \mathbf{E} = 0 \)) or the expression in (4.4) (if \( \mathbf{E} \neq 0 \)). Also, readers interested in Green’s function should consult the Appendix.

We now turn to the two-body system which is described by the equations of motion (3.4). With the harmonic interparticle potential, the appropriate Hamiltonian is

\[
H = \frac{1}{2m_1} \left[ \mathbf{p}_1 - \frac{q_1}{c} \mathbf{A}(\mathbf{r}_1) \right]^2 + \frac{1}{2m_2} \left[ \mathbf{p}_2 - \frac{q_2}{c} \mathbf{A}(\mathbf{r}_2) \right]^2 + \frac{k}{2} |\mathbf{r}_1 - \mathbf{r}_2|^2 - q_1 E y_1 - q_2 E y_2.
\]

(4.6)

The charges \( q_1 \) and \( q_2 \) are assumed to satisfy the same conditions as in Section III. First, consider the case of \( \frac{q_1}{m_1} = \frac{q_2}{m_2} = \frac{Q}{M} \). Then, working in the symmetric gauge and setting the electric field to zero temporarily, the Hamiltonian (4.6) can be rewritten using the center of mass and relative coordinates as

\[
H = \frac{1}{2M} \left[ \mathbf{P} + \frac{Q}{2c} \mathbf{R} \times \mathbf{B} \right]^2 + \frac{1}{2\mu} \left[ \mathbf{p} + \frac{m_1 m_2 Q}{2 c M^2} \mathbf{r} \times \mathbf{B} \right]^2 + \frac{1}{2} k \mathbf{r}^2,
\]

(4.7)

where \( \mathbf{P} \) and \( \mathbf{p} \) are appropriate conjugate momenta:

\[
\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2, \quad \mathbf{p} = \frac{m_2}{m_1 + m_2} \mathbf{p}_1 - \frac{m_1}{m_1 + m_2} \mathbf{p}_2.
\]

(4.8)

We can construct the energy eigenfunctions in terms of the product states \( \varphi(\mathbf{R}, \mathbf{r}) = \varphi_1(\mathbf{R}) \varphi_2(\mathbf{r}) \), and that with energy eigenvalue \( \mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 \) is obtained if \( \varphi_1(\mathbf{R}) \) and \( \varphi_2(\mathbf{r}) \) satisfy the eigenvalue equations.

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\[
\frac{1}{2M} \left[ \frac{\hbar}{i} \frac{\partial}{\partial R} + \frac{Q}{2c} R \times B \right]^2 \varphi_1(R) = \mathcal{E}_1 \varphi_1(R), \tag{4.9a}
\]
\[
\left\{ \frac{1}{2\mu} \left[ \frac{\hbar}{i} \frac{\partial}{\partial r} + \frac{m_1 m_2 Q}{2c M^2} r \times B \right]^2 + \frac{1}{2} kr^2 \right\} \varphi_2(r) = \mathcal{E}_2 \varphi_2(r). \tag{4.9b}
\]

Solving \((4.9b)\) in the same way as the one-particle Landau-level problem is solved in the symmetric gauge, one finds the eigenvalue spectrum

\[
\mathcal{E}_2 = \hbar \sqrt{\frac{k}{\mu}} \left( n_2 + \frac{1}{2} \right) - \frac{\hbar Q B}{2Mc} (n - s) + \hbar \tilde{\omega}_0 (n + s + 1) \tag{4.10}
\]

(here \(\tilde{\omega}_0 = \sqrt{\frac{k}{\mu} + \left( \frac{Q B}{2Mc} \right)^2} \), and \(n_2, n\) and \(s\) can be arbitrary non-negative integers), with the corresponding eigenfunction given as \((w = x + iy)\)

\[
\varphi_2(r) = \text{(const.)} e^{-i \frac{\hbar \tilde{\omega}_0}{k} w} \left( \frac{\sqrt{k}}{\hbar} \right)^{\frac{1}{2}} H_n \left( \frac{\sqrt{k}}{\hbar} \right) e^{i \frac{\tilde{\omega}_0}{k} w} \left( \frac{\partial}{\partial w} \right)^n \left( \frac{\partial}{\partial w} \right)^s e^{-i \frac{\hbar \tilde{\omega}_0}{k} w}. \tag{4.11}
\]

In the case of identical particles with \(q_1 = q_2\) and \(m_1 = m_2\), appropriate symmetry conditions should be further satisfied by \(\varphi_2(r)\). On the other hand, the eigenvalue equation \((4.9a)\) is precisely the one relevant for the one-particle Landau-level problem in the symmetric gauge and hence we know the solutions already, i.e., \((4.5)\) but for the reidentification of the variables involved. Turning on the electric field \(\mathbf{E} = E\hat{y}\) is now trivial—only the center of mass dynamics, described by the function \(\varphi_1(R)\) and eigenvalue \(\mathcal{E}_1\), gets affected and the change is precisely in the same way as in the one-particle problem discussed above. Analogous analysis can be carried out adopting the Landau gauge also.

We now move on to the case of \(q_1 = -q_2 = q\). For this net neutral two-body system, the situation becomes more complex. We shall work in the symmetric gauge, and then the Hamiltonian \((4.6)\) (with \(\mathbf{E} = 0\)) can be expressed as

\[
H = \frac{1}{2m_1} \left[ p + \frac{m_1}{M} \mathbf{P} + \frac{q}{2c} \left( \mathbf{R} + \frac{m_2}{M} \mathbf{r} \right) \times \mathbf{B} \right]^2
+ \frac{1}{2m_2} \left[ -p + \frac{m_2}{M} \mathbf{P} - \frac{q}{2c} \left( \mathbf{R} - \frac{m_1}{M} \mathbf{r} \right) \times \mathbf{B} \right]^2 + \frac{1}{2} kr^2 \tag{4.12}
\]
Here we further introduce the operators

$$\Pi \equiv P - \frac{q}{2c} r \times B, \quad l \equiv p + \frac{q}{2c} R \times B,$$

(4.13)

and then it is easy to show that \((\Pi, R)\) and \((l, r)\) satisfy the canonical commutation relations:

$$[\Pi_i, X_j] = [l_i, x_j] = -i\hbar\delta_{ij}, \quad [\Pi_i, x_j] = [l_i, X_j] = 0,$$

$$[\Pi_i, l_j] = [\Pi_i, \Pi_j] = [l_i, l_j] = 0.$$ (4.14)

Employing these new variables, the Hamiltonian (4.12) reads

$$H = \frac{1}{2m_1} \left[ l + \frac{q}{2c} r \times B + \frac{m_1}{M} \Pi \right]^2 + \frac{1}{2m_2} \left[ l - \frac{q}{2c} r \times B - \frac{m_2}{M} \Pi \right]^2 + \frac{1}{2} kr^2.$$ (4.15)

Evidently, we have \([\Pi_i, H] = 0; \Pi_i\) are the conserved translation generators, and all three components are simultaneously diagonalizable (for the present net neutral system) together with \(H\). Adopting the differential operator realizations \(\Pi = \frac{h}{i} \frac{\partial}{\partial R}\) and \(l = \frac{h}{i} \frac{\partial}{\partial r}\) (which are unitarily equivalent to the realizations based on \(P = \frac{h}{i} \frac{\partial}{\partial R}\) and \(p = \frac{h}{i} \frac{\partial}{\partial r}\)), we may thus look for the energy eigenfunctions having the form

$$\varphi(R, r) = e^{i\frac{p \cdot R}{\hbar}} \varphi(r; P)$$

(4.16)

with \(\varphi(r; P)\) satisfying the Schrödinger equation appropriate to a one-body problem:

$$\left\{ \frac{1}{2m_1} \left[ \frac{h}{i} \frac{\partial}{\partial r} + \frac{q}{2c} r \times B + \frac{m_1}{M} \mathcal{P} \right]^2 + \frac{1}{2m_2} \left[ \frac{h}{i} \frac{\partial}{\partial r} - \frac{q}{2c} r \times B - \frac{m_2}{M} \mathcal{P} \right]^2 \right\} \varphi(r; P) = \mathcal{E} \varphi(r; P).$$ (4.17)

To solve (4.17), we utilize new variables, \(r' = r - \frac{q}{cM} \frac{P \times B}{k + (q^2 B^2/c^4 M)}\). Note that the shifted relative position, \(r'\), entered our classical discussion also (see (3.13)). Then, after some straightforward rearrangements, it is possible to recast (4.17) into the form
\[
\left\{ \frac{1}{2\mu} \left[ \frac{\hbar}{i} \frac{\partial}{\partial r'} + \frac{m_2 - m_1}{M} \frac{q}{2c} r' \times B - \frac{m_2 - m_1}{2M} \frac{k}{\epsilon^2 M} \mathcal{P}_\perp \right] \right. \\
+ \frac{1}{2} k z'^2 + \frac{k P_\perp^2}{2M(k + \frac{q^2 B^2}{\epsilon^2 M})} + \frac{P_z^2}{2M} \left. \right\} \varphi = \mathcal{E} \varphi , \quad (4.18)
\]

where \( r' \equiv (r'_1, z') \) and \( \mathcal{P} \equiv (\mathcal{P}_\perp, P_z) \). We further write our wave function as

\[
\varphi = e^{\frac{i}{\hbar} \bar{\hbar} m_2 - m_1 \frac{q}{2c} r' \times B - \frac{m_2 - m_1}{2M} \frac{k}{\epsilon^2 M} \mathcal{P}_\perp + \frac{1}{2} \left( k + \frac{q^2 B^2}{\epsilon^2 M} \right) P_\perp r'_\perp} \tilde{\varphi} , \quad (4.19)
\]

so that the resulting equation for \( \tilde{\varphi} \) may assume the (almost) same form as \( (4.9b) \):

\[
\left\{ \frac{1}{2\mu} \left[ \frac{\hbar}{i} \frac{\partial}{\partial r'} + \frac{m_2 - m_1}{M} \frac{q}{2c} r' \times B \right] \right. \\
+ \frac{1}{2} k z' + \frac{k P_\perp^2}{2M(k + \frac{q^2 B^2}{\epsilon^2 M})} + \frac{P_z^2}{2M} \left. \right\} \tilde{\varphi} = \tilde{\mathcal{E}} \tilde{\varphi} , \quad (4.20)
\]

The eigenfunctions of \( (4.20) \) are thus given by the expression \( (4.11) \) (with \( (x, y, z) \) taken by \( (x', y', z') \)) if the parameter \( \tilde{\omega}_0 \) there is suitably adjusted; in the present case, \( \tilde{\omega}_0 \) should be identified with

\[
\sqrt{\frac{1}{\mu} \left( k + \frac{q^2 B^2}{\epsilon^2 M} \right) + \frac{q^2 B^2 (m_1 - m_2)^2}{4 \epsilon^2 m_1 m_2^2}} . \quad [\text{Recall that this frequency also figured in our classical discussion.}] \quad \text{This eigenfunction, which is denoted as } \tilde{\varphi}_{n_z,n_s}(r'), \text{ corresponds to the eigenvalue}
\]

\[
\tilde{\mathcal{E}} = \hbar \sqrt{\frac{k}{\mu} \left( n_z + \frac{1}{2} \right) + \hbar \frac{(m_2 - m_1) q B}{2 m_1 m_2 c} (n - s) + \hbar \tilde{\omega}_0 (n + s + 1)} . \quad (4.21)
\]

Note that the functions \( \tilde{\varphi}_{n_z,n_s}(r') \) have no dependence on \( \mathcal{P} \) except through the definition of \( r' \).

The eigenfunctions found above are the ones appropriate when \( \Pi \) and \( \mathbf{l} \) are realized as \( \frac{\hbar}{i} \frac{\partial}{\partial R} \) and \( \frac{\hbar}{i} \frac{\partial}{\partial r} \). To make direct connection with our discussion in Section II, it should be desirable to have them brought to the expressions appropriate to more conventional realizations \( \mathbf{P} = \frac{\hbar}{i} \frac{\partial}{\partial R} \) and \( \mathbf{p} = \frac{\hbar}{i} \frac{\partial}{\partial r} \). As one can easily verify, this job is effected by a simple multiplicative phase, \( e^{\phi R \cdot \mathbf{R} \times \mathbf{B}} \). Including this phase factor, we may now express the complete eigenfunctions of the Hamiltonian \( (4.15) \) by
\[ \varphi_{P,n_z,n,s}(\mathbf{R}, \mathbf{r}) = e^{\frac{i}{\hbar} \mathbf{R} \times \mathbf{B}} e^{\frac{\mathbf{P} \cdot \mathbf{R}}{k + (q^2 B^2/c^2 M K) \mathcal{P}_z}} \varphi_{n_z,n,s}(\mathbf{r}^*) \bigg|_{\mathbf{r}^* = \frac{q}{cM} \mathcal{P}_z^{-1}} \] 

with the energy eigenvalue given as

\[ E_{P,n_z,n,s} = \frac{k}{k + \frac{q^2 B^2}{c^2 M K}} \mathcal{P}_z^2 + \frac{\mathcal{P}_x^2}{2M} + \hbar \sqrt{\frac{k}{\mu} \left( n_z + \frac{1}{2} \right) + \hbar \bar{\omega}_0(n + s + 1) - \hbar \left( \frac{m_2 - m_1}{2m_1m_2} \right)(n - s)} \hspace{1cm} (4.23) \]

\( (n_z, n, s: \text{non-negative integers}) \)

For these energy eigenstates, the expectation value of the relative position operator \( \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \) is consistent with the classical result in (3.16). Are the expressions (4.22) also consistent with the general form of the eigenfunctions we have in (2.18)? Yes, indeed. For that, it suffices to notice that the first phase factor in the right hand side of (4.22) can be rewritten as

\[ e^{i \frac{qB}{2c} (x_1y_1 - x_2y_2)} \hspace{1cm} (4.24) \]

We use this in (4.22) and then, except for the appropriate gauge transformation factor \( e^{i \frac{qB}{2c} (x_1y_1 - x_2y_2)} \) (needed to convert the Landau-gauge results into the symmetric-gauge results), the dependence on the variable \( X \) in the resulting expression is entirely in \( e^{i \mathcal{P}_z X} \). Hence ours are fully consistent with (2.18).

Energy eigenstates in the presence of the additional electric field \( \mathbf{E} = E\mathbf{\hat{y}} \) can readily be identified also. In the symmetric gauge, our recipe tells us that the energy eigenfunctions for this case are

\[ \varphi'_{P,n_z,n,s}(\mathbf{R}, \mathbf{r}) = e^{\frac{i}{\hbar} \mathbf{R} \times \mathbf{B}} e^{\frac{\mathbf{P} \cdot \mathbf{R}}{k + (q^2 B^2/c^2 M K) \mathcal{P}_z}} \varphi_{n_z,n,s}(\mathbf{r}^*) \bigg|_{\mathbf{r}^* = \frac{q}{cM} \mathcal{P}_z^{-1}} \] 

\( (\mathcal{P}_z^* \equiv \mathcal{P}_z - \frac{cE}{B} \mathbf{\hat{x}}) \hspace{1cm} (4.25) \)
with the energy eigenvalue given by the formula 
\[ E'_{\mathbf{p}_z,n,s} = [E_{\mathbf{p}_z,n,s}]_{\mathbf{p} \to (\mathbf{p}_z^\perp, \mathbf{p}_z)} + \frac{cE}{B} \mathbf{p}_z - \frac{1}{2} M \frac{e^2 E^2}{B^2} \]  
\[ (E_{\mathbf{p}_z,n,s} \text{ is in (4.23))}. \) Note that \( \mathbf{P} \) in (1.24) corresponds to the eigenvalue of the translation generator \( \Pi \) (see (4.13)). In this gauge the center of mass velocity operator is \( \dot{\mathbf{R}} = \frac{1}{M} \left[ \mathbf{\Pi} \right] \right\} + \left[ \mathbf{q} c \mathbf{r} \times \mathbf{B} \right] \), and hence for the eigenstate (4.25) its expectation value is found to be

\[ \langle \dot{\mathbf{R}} \rangle = \frac{1}{M} \left[ \mathbf{P} + \frac{q}{c} \langle \mathbf{r} \rangle \times \mathbf{B} \right] \]
\[ = -\frac{kc}{qB^2} \langle \mathbf{r} \rangle \times \mathbf{B} + \frac{cE}{B} \frac{p_z}{M} \hat{\mathbf{z}}, \]  
(4.26)

where, on the second line, we have dispensed with \( \mathbf{P}^\perp \) using the formula \( \langle \mathbf{r} \rangle = \frac{q}{cM} \frac{\mathbf{p}^\perp \times \mathbf{B}}{k + (q^2 B^2/c^2 M)} \). The relation (4.26) has the classical counterpart in (3.17).

In principle, it should be possible to perform an analogous analysis for a three-body system also, say, assuming charge values \((2q, -q, -q)\) and mutual harmonic interactions between the particles. This could give some rough information on the behavior of a neutron in the presence of constant electromagnetic fields.

5. DISCUSSION AND SUMMARY

In this paper we discussed the physical role of Galilean transformation in general non-relativistic charged-particle systems in the presence of a uniform magnetic field. By a Galilean transform with a judicious boost velocity, one obtains the same system but now under mutually perpendicular magnetic and electric fields. Thus any problem under the latter circumstance is completely solved in terms of a judicious Galilean transform of the same problem under a uniform magnetic field only. Applications of this observation have been made for relatively simple, classical and quantum mechanical, systems. This observation is clearly at the heart of the Hall effect, and also explains the Stark-like effect exhibited by a neutral composite system.
Galilean transformations can be applied to the anyon system in a uniform magnetic field. [See Ref. 9 for related literatures.] A particularly convenient way to deal with the unconventional statistical property of anyons is by introducing the Aharonov-Bohm-type interactions for every pair of particles. These Aharonov-Bohm interactions are still Galilean invariant, and hence our discussion given in Section II requires no essential change. Consequently, once one has solutions to the problem of anyons in the presence of a uniform magnetic field, those results may simply be Galilean-transformed (according to the procedures detailed by us in Section II) to obtain corresponding solutions in the presence of mutually perpendicular magnetic and electric fields. In this aspect, anyons are not different from ordinary particles.

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APPENDIX: GALILEAN TRANSFORMATION OF GREEN’S FUNCTION

Let $G'(\mathbf{r}_f, t_f; \mathbf{r}_i, t_i)$ denote the quantum mechanical Green’s function in the one-particle system defined by the Hamiltonian (4.1), and $G(\mathbf{r}_f, t_f; \mathbf{r}_i, t_i)$ Green’s function in the presence of the magnetic field $\mathbf{B}$ only, i.e., $G(\mathbf{r}_f, t_f; \mathbf{r}_i, t_i) = \ldots$
[\text{G}'(\mathbf{r}_f, t_f; \mathbf{r}_i, t_i)]|_{E=0}$. Then, looking at (2.8) and (2.11), astute readers will immediately make the identification

\begin{equation}
G'(\mathbf{r}_f, t_f; \mathbf{r}_i, t_i) = e^{\frac{\alpha}{\hbar}[(\mathbf{r}_f, t_f) - (\mathbf{r}_i, t_i)]}G(\mathbf{r}_f + \mathbf{u} t_f, t_f; \mathbf{r}_i + \mathbf{u} t_i, t_i),
\end{equation}

(A.1)

\begin{equation}
\alpha(\mathbf{r}, t) = -m \mathbf{u} \cdot \mathbf{r} - \frac{1}{2} m \mathbf{u}^2 t - \frac{q}{c} [\tilde{\Lambda}(\mathbf{r}) - \tilde{\Lambda}(\mathbf{r} + \mathbf{u} t)],
\end{equation}

(A.2)

where $\mathbf{u} \equiv -c \frac{\mathbf{E} \times \mathbf{B}}{B^2}$, and $\tilde{\Lambda}(\mathbf{r})$ is defined through (2.3). Hence, given the explicit form for $G(\mathbf{r}_f, t_f; \mathbf{r}_i, t_i)$, the expression for $G'(\mathbf{r}_f, t_f; \mathbf{r}_i, t_i)$ may also be written down using this relation. Moreover, if $L'(\mathbf{r}, \dot{\mathbf{r}})$ denotes the Lagrangian corresponding to the Hamiltonian (4.1) and $L(\mathbf{r}, \dot{\mathbf{r}})$ that in the presence of the $\mathbf{B}$-field only, it is possible to demonstrate by direct calculation that

\begin{equation}
L'(\mathbf{r}, \dot{\mathbf{r}}) = L(\mathbf{r} + \mathbf{u} t, \dot{\mathbf{r}} + \mathbf{u}) + \frac{d}{dt} \alpha(\mathbf{r}, t), \quad \left(\mathbf{u} = -c \frac{\mathbf{E} \times \mathbf{B}}{B^2}\right).
\end{equation}

(A.3)

By considering this relation together with the path integral representation of Green’s function, one has an alternative understanding of the above relationship between the Green’s functions.

Explicitly, in the Landau gauge, we have

\begin{equation}
G(\mathbf{r}_f, t_f; \mathbf{r}_i, t_i) = \theta(t_f - t_i) \left(\frac{m}{2 \pi i \hbar (t_f - t_i)}\right)^{\frac{3}{2}} \left(\frac{\omega_c}{\sin[\frac{\omega_c}{2} (t_f - t_i)]}\right) \times e^{\frac{im}{2 \hbar} \left[\frac{(x_f - x_i)^2}{t_f - t_i} + \frac{\omega_c}{2} \cot[\frac{\omega_c}{2} (t_f - t_i)]\right] (x_f - x_i)^2 + (y_f - y_i)^2 + \omega_c (x_f - x_i)(y_i + y_f)}
\end{equation}

(A.4)

Then, using (A.1), we can immediately write corresponding Green’s function in the presence of both $\mathbf{B} = B \mathbf{z}$ and $\mathbf{E} = E \mathbf{y}$ as (here, $\mathbf{u} \equiv -c \frac{\mathbf{E} \times \mathbf{B}}{B^2}$)

\begin{equation}
G'(\mathbf{r}_f, t_f; \mathbf{r}_i, t_i) = \theta(t_f - t_i) \left(\frac{m}{2 \pi i \hbar (t_f - t_i)}\right)^{\frac{3}{2}} \left(\frac{\omega_c}{\sin[\frac{\omega_c}{2} (t_f - t_i)]}\right) e^{-im u (x_f - x_i) - \frac{1}{2} m u^2 (t_f - t_i)}
\end{equation}

\begin{equation}
\times e^{\frac{im}{2 \hbar} \left[\frac{(x_f - x_i)^2}{t_f - t_i} + \frac{\omega_c}{2} \cot[\frac{\omega_c}{2} (t_f - t_i)]\right] (x_f - x_i + u (t_f - t_i))^2 + (y_f - y_i)^2 + \omega_c [x_f - x_i + u (t_f - t_i) + (y_i + y_f)]}.
\end{equation}

(A.5)
REFERENCES

[1] Landau L D 1930 Z. Phys. 64 629
   Landau L D and Lifschitz E M 1977 Quantum Mechanics: Non-relativistic Theory
   (New York: Pergamon)

[2] For a recent disposition on this aspect, see
   Fubini S 1990 Int. J. Mod. Phys. A 5 3533

[3] von Klitzing K, Dorda G and Pepper M 1980 Phys. Rev. Lett. 45 494

[4] Laughlin R 1983 Phys. Rev. Lett. 50 1395; 1983 Phys. Rev. B 29 3383

[5] For an excellent review, see
   Girvin S and Prange R 1990 The Quantum Hall Effect (New York: Springer)
   and references therein

[6] Leinaas J and Myrheim J 1977 Nuovo Cimento 37B 1
   Goldin G, Menikoff R and Sharp D H 1981 J. Math. Phys. 22 1664
   Wilczek F 1982 Phys. Rev. Lett. 48 1144
   Wu Y S 1984 Phys. Rev. Lett. 52 2103

[7] Avron J E, Herbst I W and Simon B 1978 Ann. Phys. (NY) 114 431

[8] See, for instance,
   Jackiw R 1985 Current Algebra and Anomalies ed S Treiman, R Jackiw, B Zumino and E Witten (Singapore: World Scientific)

[9] Johnson M D and Cantright G S 1990 Phys. Rev. B 41 6870
   Polychronakos A 1991 Phys. Lett 264B 362
   Cho K H and Rim C 1992 Ann. Phys. (NY) 213 295
   Dunne G, Lerda A, Sciuto S and Trugenberger C A 1992 Nucl. Phys. B 370 601

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