A priori bounds and multiplicity results for slightly superlinear and sublinear elliptic $p$-Laplacian equations

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Abstract

We consider the following problem

$$-\Delta_p u = h(x, u) \quad \text{in } \Omega, \quad u \in W^{1,p}_0(\Omega),$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$, $1 < p < N$, with a smooth boundary. In this paper we assume that $h(x, u) = a(x)f(u) + b(x)g(u)$ such that $f$ is regularly varying of index $p-1$ and superlinear at infinity. The function $g$ is a $p$-sublinear function at zero. The coefficients $a$ and $b$ belong to $L^k(\Omega)$ for some $k > \frac{N}{p}$ and they are without sign condition. Firstly, we show a priori bound on solutions, then by using variational arguments, we prove the existence of at least two nonnegative solutions. One of the main difficulties is that the nonlinearity term $h(x, u)$ does not satisfy the standard Ambrosetti and Rabinowitz condition.

1 Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ ($N \geq 2$) with a smooth boundary $\partial \Omega$. In this paper, we are concerned by the following boundary value problem:

$$\begin{cases}
-\Delta_p u = h(x, u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}$$

where $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ is the $p$-Laplacian operator, $1 < p < N$ and $h : \Omega \times [0, \infty) \rightarrow [0, \infty)$ is a Carathéodory function that satisfies some suitable conditions.

We start by stating the existing works on this elliptic problem. Then we will motivate the condition choices we made on the parameters of this problem, which culminate in the model we introduce in the next section. Indeed, in the literature, several works studied different instance of problem $(P)$. For the case $p = 2$, in [21] De Figueiredo et al. proved the existence of nontrivial solutions to problem $(P)$ in $\mathbb{R}^2$, where $h(x, u)$ has exponential growth. In [5], Bartsch and Wang proved the multiplicity of nontrivial solutions in the case where $h(x, u) = h(u) \in C^1(\Omega)$ grows superlinearly but subcritically at infinity and $h'(0) < \lambda_2$, and where $\lambda_2$ is the second small eigenvalue of $-\Delta$ on $\Omega$. Recôva and Rumbos proved in [37] the multiplicity of solutions in the case where $h(x, u)$ has a polynomial growth and satisfies the nonquadraticity condition introduced by Costa and Magalhães in [10]. In [18], De Figueiredo et al. showed, by using the variational and sub and
super solutions methods, the existence and the multiplicity of positive solutions in the case where \( h(x, u) \) is locally superlinear and sublinear. Moreover, these results were extended to the p-Laplacian case by the same authors in [19]. Recently, De Figueiredo et al. showed in [20] the existence of at least two solutions in the case where \( h(x, u) \) is locally p-sublinear at zero.

It is worth mentioning that, in all the papers mentioned above, the authors assumed, among other conditions, that the nonlinearity term \( h(x, s) \) satisfies the well known Ambrosetti-Rabinowitz (AR) condition that was introduced for the first time in [4]. The condition requires the following in the case of the p-Laplacian operator:

\[
\text{there exist } \theta > p \text{ and } s_0 > 0 \text{ such that } 0 < \theta H(x, s) \leq sh(x, s), \quad \forall s > s_0.
\]

It implies the existence of two positive constants \( C_1 \) and \( C_2 \) such that

\[
H(x, s) \geq C_1 s^\theta - C_2, \quad \forall s \geq 0.
\]

In the literature, this condition is the main tool to prove the existence of solutions to elliptic problems with variational structure. It serves in particular to prove the boundedness of Palais-Smale sequence of the energy functional associated with such problems. Nevertheless, this condition is somewhat restrictive and not being satisfied by many nonlinearities.

We can mention for instance the nonlinearity \( h(x, s) = s^{p-1} \ln(1+s)^q \), where \( q > 0 \), that was taken from [14] [15], which instead verify that, for any \( \theta > p \), \( H(x, s)/s^\theta \to 0 \) as \( s \to +\infty \). In our present setting as well, the nonlinearity \( h(x, s) \) that we are considering, does not satisfy the (AR) condition.

Many recent research have been made to drop the (AR) condition, we refer for instance to [18] [26] [28]. In these works, the authors studied different boundary value problems. In [18] De Figueiredo et al. studied the problem \( (P) \) in the case where \( p = 2 \) and the nonlinearity term \( h(x, u) = a(x)u + b(x)u^r \), where the coefficients \( a(x), b(x) \) belong to \( L^\infty(\Omega) \) with \( a(x) \) is nonnegative and \( b(x) \) is allowed to change sign. They proved that if \( a(x) \) and \( \|b\|_{L^\infty(\Omega)} - b(x) \) are suitably small, then the problem \( (P) \) has at least two positive solutions. In a related task, in [26] Hsu deals with the same nonlinearity term but in the p-Laplacian case. Moreover, he assumes that \( r = \frac{Np}{N-p} \), the coefficients \( a(x) \) and \( b(x) \) are continuous on \( \Omega \) and are positive somewhere but may change sign. He proved that, if \( 0 < q < p - 1 < N - 1 \) and \( a(x) \) is suitably small, then the problem \( (P) \) has at least two positive solutions. On the other hand, in [28] Iturriaga et al. considered more general setting. Notably, they supposed that \( h(x, u) \) grows as \( u^{p-1} \) near zero and has a \( p \)-superlinear growth at infinity. They proved under other conditions that \( (P) \) has at least two positive solutions. We point out, that the class of nonlinearities considered in these works belong to pure-power or power-like cases which is a quite restrictive class of functions. Moreover, the assumptions considered on the coefficients \( a(x) \) and \( b(x) \) are very strong. In this paper, the nonlinearity \( h(x, s) \) includes a larger class of functions. Namely, part of our nonlinearity belongs to the class of functions that is regularly varying of index \( p - 1 \) (at infinity), (see section 2). This type of functions were introduced for the first time in [32], (see also [40]).

Recently, in the case \( p = 2 \), [24] [12] considered the case where the nonlinearity term is regularly varying. More specifically, in [24] García-Melián et al. studied the problem \( (P) \) in the case where \( h(x, u) = a(x)f(u) \), such that \( f \) is regularly varying of index 1 and slightly superlinear and \( a(x) \) may change of sign. They proved the existence of at least one positive solution. Moreover, in [12] Costa et al. proved the existence of at least one positive solution. They studied as well the bifurcation in the case where \( h(x, u) = \lambda a(x)(f(u) - l) \) where \( f \) is subcritical, superlinear at infinity, and regularly varying of index \( q > 1 \), \( \lambda \) and \( l \) are positive parameters. In these two papers, it was supposed among other conditions that the coefficient \( a(x) \) is continuous on \( \Omega \), which is a strong assumption. Furthermore, in [30] Jeanjean and Quoirin consider the case where \( h(x, u) = a(x)f(u) + b(x)g(u) \) such that \( f(u) = (1 + u)\ln(1 + u) \) and \( g(u) = 1 + u \). The coefficient \( a(x) \) is without sign condition and the coefficient \( b(x) \) is nonnegative function and both belong to \( L^k(\Omega) \) with \( k > \frac{N}{2} \). They proved
that if \( a \) is suitably small, then the problem \((P)\) has at least two solutions. In [14] De Coster and Fernández proved the same results where \( b \) may change of sign. Moreover, in [7] Chaouai and Maatouk generalized the result given in [30] to the \( p \)-Laplacian operator, where \( a(x) \) is nonnegative and \( b(x) \) changes its sign. We highlight that \( f(u) \) is an example of functions that is regularly varying of index 1. In addition, the arguments used by the authors in [7, 14, 30] to prove the multiplicity of solutions are based on the explicit form of \( f \) and \( g \).

Our aim in this paper is to show the existence of at least two different nonnegative weak solutions to the problem \((P)\). It is worth mentioning that the form of our model (see section 2) is inspired by [7, 8, 9, 14, 30]. An important feature of our study lies in the incorporation of potentials and weighted nonlinearities, thereby embracing classes of the stationary nonlinear Schrödinger equations. Moreover, our model corresponds in particular cases to the time-independent case of the nonlinear Hamilton-Jacobi equations studied in [22] (see Section 5). On the other hand, the improvements given in this work are as follows; The first improvement, the nonlinearity term that we are considering belongs to a large class of functions. Notably, it composes of two general functions; i) One of them is regularly varying of index \( p - 1 \), which is inspired by [23]. ii) Second term that we consider a general \( p \)-sublinear function at zero as in [19, 20]. The second improvement, lies on the fact that we assume weak conditions on the coefficients of the nonlinearity \( h(x, s) \). More specifically, instead of assuming that the coefficients are continuous like in the recent works [23, 12], we are only assuming that our coefficients belong to \( L^k(\Omega) \), \( k > \frac{N}{p} \), and with no sign conditions. The last but not least improvement to mention is that our problem involves the \( p \)-Laplacian operator, with \( 1 < p < N \), unlike [23, 14, 30]. Furthermore, to the best of our knowledge, our results are new even in the Laplacian case i.e., \( p = 2 \).

The rest of the paper is organized as follows. In Section 2, we explicitly give the model that we will deal with, then we state the assumptions and our main results. In Section 3, we recall and state some preliminary results that will be useful for the rest of the paper. Section 4 is devoted to prove of our main results. Finally, Section 5 is concerned to some applications of the main results given in Section 2.

**Notations**

1) The Lebesgue norm \((\int_\Omega |u|^p)^{\frac{1}{p}}\) in \(L^p(\Omega)\) is denoted by \(\|u\|_p\) for \(p \in [1, +\infty[\). The norm in \(L^\infty(\Omega)\) is denoted by \(\|u\|_{L^\infty(\Omega)} := \text{ess} \sup_{x \in \Omega} |u(x)|\). The Hölder conjugate of \(p\) is denoted by \(p'\).

2) The spaces \(W_0^{1,p}(\Omega)\) and \(W^{-1,p'}(\Omega)\) are equipped with Poincaré norm and the dual norm, \(\|u\| := (\int_\Omega |\nabla u|^p)^{\frac{1}{p}}\) and \(\|\cdot\|_* := \|\cdot\|_{W^{-1,p'}(\Omega)}\) respectively.

3) We denote by \(B_R(x_0)\) the ball of radius \(R\) centered at \(x_0\) and \(\partial B_R(x_0)\) its boundary.

4) for \(u \in L^1(\Omega)\), we define \(u^+ = \max(u, 0)\) and \(u^- = \min(-u, 0)\).

5) We denote by \(C, C_i\) any positive constants which are not essential in the arguments and which may vary from one line to another.

**2 Models, assumptions and main results**

In this section, we will give the explicit form of our nonlinearity term \(h(x, u)\) of the problem \((P)\), afterwards we state our assumptions and our main results. Next, we will give some examples of the considered nonlinearities functions that satisfy our hypothesis in each result.
In this paper, our goal is to show the existence of at least two nonnegative nontrivial solutions of the following problem

\[
(Q) \begin{cases}
-\Delta_p u = a(x)f(u) + b(x)g(u) & \text{in } \Omega, \\
u \in W_0^{1,p}(\Omega),
\end{cases}
\]

where the functions \(a\) and \(b\) satisfy the following conditions

\[
\begin{align*}
a &= a^+ - a^-, \quad b = b^+ - b^- \text{ such that } a^+, a^-, b^+, b^- \in L^k(\Omega) \text{ for some } k > \frac{N}{p} \text{ and } b^- \in L^\infty(\Omega), \\
a^+(x)a^-(x) &= 0 \text{ a.e. in } \Omega, \\
\Omega^+ := \text{supp}(a^+) \text{ such that mes}(\Omega^+) > 0, \\
\text{there exists an } \epsilon > 0 \text{ such that } a^- = 0 \text{ in } \{x \in \Omega : d(x, \Omega^+) < \epsilon\}. \tag{A_{a,b}}
\end{align*}
\]

We point out, the same assumptions have been considered in the recent paper [13]. We note that the condition \(a^- = 0 \text{ in } \{x \in \Omega : d(x, \Omega^+) < \epsilon\}\) for some \(\epsilon > 0\), it’s called "thick zero set" which was introduced for the first time in [2].

Through this paper we assume that \(f, g : [0, \infty) \to [0, \infty)\) to be continuous functions and satisfy the following assumptions;

- \(f\) is regularly varying of index \(p - 1\) at infinity, i.e.

\[
\lim_{s \to +\infty} \frac{f(\lambda s)}{f(s)} = \lambda^{p-1}, \text{ for every } \lambda > 0. \tag{A^1_f}
\]

- \(f\) is \(p\)-superlinear at infinity, i.e.

\[
\lim_{s \to +\infty} \frac{f(s)}{s^{p-1}} = +\infty. \tag{A^2_f}
\]

- \(g\) is \(p\)-asymptotically linear at infinity, i.e.

\[
\lim_{s \to +\infty} \frac{g(s)}{s^{p-1}} = l_1, \text{ where } l_1 \in [0, \infty). \tag{A^1_g}
\]

For more information and developments on regularly varying of index \(p - 1\) functions we refer the reader to [23, 32, 40].

In order to prove the multiplicity of nonnegative solutions for \((Q)\), we will use variational arguments. As in [23] the main ingredients of our arguments are some a priori bounds on any nonnegative weak solution of a slightly more general version of \((Q)\), which is as follow,

\[
(Q_\lambda) \begin{cases}
-\Delta_p u = (\lambda a^+(x) - a^-(x))f(u) + b(x)g(u) & \text{in } \Omega, \\
u \in W_0^{1,p}(\Omega),
\end{cases}
\]

where \(\lambda \in [1/2, 1]\). As mentioned in the introduction, in [23] it was assumed that the coefficient \(a(x)\) is continuous on \(\Omega\) and \(b \equiv 0\). The method that was applied is the well known Gidas-Spruck rescaling argument which was introduced in [25]. We recall that this method have been extensively used when the nonlinearities term is asymptotic to a power near \(\infty\), see for instance [3, 27, 38] and references therein. In our case, we will exploit the new method used in the recent work [13] where the authors show a priori estimates for the special case \(p = 2\). Especially, this method is based on using the boundary weak Harnack inequality. However, to use this method we shall extend some results to the \(p\)-Laplacian setting, see Section 3.

Now, let us state our first result which is concerning a priori estimates of all nonnegative solutions of \((Q_\lambda)\).
Theorem 2.1 (A priori bounds). Under the assumptions \((A_{a,b}), (A_1^f), (A_2^f),\) and \((A_1^g), (A_2^g)\), there exists a positive constant \(C\) such that for any \(\lambda \in [1/2, 1]\) and every nonnegative weak solution \(u\) of \((Q_\lambda)\) satisfies
\[
\|u\|_{L^\infty(\Omega)} \leq C. 
\]

Concerning the existence of the multiplicity result, other hypothesis have to be added to the one of Theorem 2.1 for \(f\) and \(g\) which are as follow;

- \(f\) is \(p\)-superlinear at zero, i.e.
  \[
  \lim_{s \to 0^+} \frac{f(s)}{s^{p-1}} = 0, \quad (A_3^f)
  \]

- \(g\) is \(p\)-sublinear at zero, i.e.
  \[
  \lim_{s \to 0^+} \frac{g(s)}{s^{p-1}} = +\infty, \quad (A_3^g)
  \]

The assumption \((A_3^f)\) will be used to prove the geometrical structure of the functional \(I_\lambda\), defined in Section 4, associated to the problem \((Q_\lambda)\) and this condition implies that \(f(0) = 0\). On the other hand, the assumption \((A_3^g)\) will be used to prove the existence of the second weak solution. In addition, this condition implies that \(g(0) \geq 0\). In our results we will distinguish between the case where \(g(0) = 0\) and the case \(g(0) > 0\).

Due to Theorem 2.1 and to above assumptions, we can have the following multiplicity results of the problem \((Q)\).

Theorem 2.2 (First multiplicity result). Under the assumptions \((A_{a,b}), (A_1^f), (A_2^f), (A_3^f), (A_1^g)\) and \((A_2^g)\), we also assume that \(\|b^+\|_{L^k(\Omega)}\) is suitably small.

1. If \(g(0) = 0\), then the problem \((Q)\) has at least two nonnegative nontrivial weak solutions in \(W^{1,p}_0(\Omega) \cap L^\infty(\Omega)\).

2. If \(g(0) > 0\) and \(b \geq 0\), then the problem \((Q)\) has at least two positive weak solutions in \(W^{1,p}_0(\Omega) \cap L^\infty(\Omega)\).

Various functions occurring in various works are included as models for the boundary value problem \((Q)\) as one can see from next examples.

Example 2.1. Some functions that satisfy \((A_1^f) - (A_3^f)\) are as follow,

- (i) \(f(s) = s^{p-1}(\log(1 + s))^l\), \[20\]
- (ii) \(f(s) = s^{p-1}(\exp(\log(1 + s)^l) - 1)\), \[23\]
where \(l > 0\).

Example 2.2. Some functions that satisfy \((A_1^g)\) and \((A_3^g)\) are as follow

- (i) \(g(s) = 1 + s^{p-1}\), \[11\]
- (ii) \(g(s) = \exp(\frac{(n-1)s}{\epsilon + s})\), \[1\]
where \(\epsilon > 0\).
We stress that the example 2.2 (ii) appears in combustion theory or, more generally, describes reactions of Arrhenius type in the case $p = 2$, (for more details on this model, see \[6\]).

For the next result, we will add another assumption on the coefficient $a^+$. However, we will make weaker assumptions on $f$ and $g$. Specifically, instead of assuming the assumptions (\[A^3_f\]) and (\[A^2_g\]). We assume the following assumptions.

- There exists $r \in (0, p - 1)$ such that
  \[
  \lim_{s \to 0^+} \frac{f(s)}{s^r} = l_2 \text{ where } l_2 \in (0, \infty).
  \]  
  (\[A^3'_f\])

- $g$ is either $p$-sublinear or $p$-asymptotically linear at zero, i.e.
  \[
  \lim_{s \to 0^+} \frac{g(s)}{s^{p-1}} = l_3 \text{ where } l_3 \in (0, \infty).
  \]
  (\[A^2'_g\])

**Theorem 2.3** (Second multiplicity result). Under the assumptions (\[A_{a,b}\]), (\[A^1_f\]), (\[A^2_f\]), (\[A^3'_f\]), (\[A^1_g\]) and (\[A^2'_g\]), we also assume that $\|a^+\|_{L^k(\Omega)}$ and $\|b^+\|_{L^k(\Omega)}$ are suitably small. Then the conclusions of Theorem 2.2 hold.

**Example 2.3.** Some functions that satisfy (\[A^1_f\]), (\[A^2_f\]) and (\[A^3'_f\]) are as follow,

(i) $f(s) = (1 + s)^{p-1}(\log(1 + s))^l$, [7]
(ii) $f(s) = (1 + s)^{p-1}(\log(1 + \log(1 + s))^l$, [23]

where $l > 0$.

We point out that the above examples, satisfy the property (\[A^3'_f\]), for both cases, when $r = l$.

**Example 2.4.** Some functions that satisfy (\[A^1_g\]), (\[A^2'_g\]) are as follow,

(i) $g(s) = s^q$, [17]
(ii) $g(s) = \frac{s^q}{(s + \epsilon)^{p-1}}$, [16]

where $0 \leq q \leq p - 1$ and $\epsilon, \alpha > 0$.

**Remark 2.1.** If $u$ is a weak solution of (Q$_\lambda$) then from Lemma 3.1 below, $u \in C^{0,\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$. Beside, if the coefficients $a$ and $b$ belong to $L^\infty(\Omega)$ we can use [34, Theorem 1] and we deduce directly from Theorem 2.2 or Theorem 2.3, that $u \in C^{1,\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$.

**Remark 2.2.** In the assumption (\[A^2'_g\]) if $l_3 < +\infty$, then for $s$ small enough we obtain that $g(s) < Cs^{p-1}$ where $C$ is a positive constant. By the same arguing used in Lemma 4.2 below, we deduce directly that any nonnegative nontrivial weak solution of (Q) is positive.

**Remark 2.3.** The Theorem 2.3 is still satisfied even if $r = p - 1$ in the assumption (\[A^3'_f\]) but only in the case where $g$ is $p$-sublinear at zero, namely $l_3 = +\infty$, (see the proof of Theorem 2.3). In our application, see Subsection 5.2, this case will be included.
3 Preliminary results

In this section, we present some definitions and results which will play an important role throughout this work. We note that from now we extend \( f \) and \( g \) for \( s < 0 \) by putting \( f(s) = 0 \) and \( g(s) = g(0) \geq 0 \).

Let us begin with the following result which concerns the functions that are regularly varying of index \( p - 1 \). This result will be useful for our proofs.

**Proposition 3.1.** We assume that \( f \) is continuous on \([0, \infty)\) satisfies \((A^p)\), then for every \( \delta > 0 \), there exists a positive constant \( C \) such that

\[
    f(s) \leq C(1 + s^{p-1+\delta}), \quad s \in [0, \infty)
\]

where \( C \) depends on \( \delta \).

**Proof.** This property was done in [23] in the case \( p = 2 \), but it can be easily extended to \( p > 1 \).

In the following result we will give some properties of weak solutions of the problem \((Q_\lambda)\).

**Lemma 3.1.** Under the assumptions \((A^a,b)\), \((A^1f)\), \((A^1g)\), we have

(i) Every weak solution of \((Q_\lambda)\) belongs to \(L^\infty(\Omega)\).

(ii) Every weak solution of \((Q_\lambda)\) belongs to \(C^{0,\alpha}_0(\bar{\Omega})\) for some \( \alpha \in (0,1) \).

**Proof.** (i) We have from Proposition 3.1 that, for every \( \delta > 0 \) there exist a positive constant \( C_1 \) such that

\[
    f(s) \leq C_1(1 + s^{p-1+\delta}), \quad s \in [0, \infty)
\]

Since \( g \) is continuous on \([0, \infty)\), thus from \((A^1g)\) we deduce that, there exist positive constant \( C_2 \) such that

\[
    g(s) \leq C_2(1 + s^{p-1}).
\]

Hence, by using [33, Theorem IV-7.1] (or [36, Theorem 2.4]) we deduce the result.

(ii) By using (i) its follows directly from [33, Theorem IV-2.2].

Now, let us state the definition of super-solution and sub-solution of the following general boundary value problem

\[
    \begin{cases}
    -\Delta_p u + F(x, u) = h(x) & \text{in } \Omega, \\
    u \in W^{1,p}_0(\Omega),
    \end{cases}
\]

such that \( h \in L^1(\Omega) \) and \( F : \Omega \times \mathbb{R} \to \mathbb{R} \) is a Caratheodory function.

**Definition 3.1.** We say that \( u \in W^{1,p}(\Omega) \) is a weak super-(sub-)solution of \((3.1)\) if for all \( \varphi \in W^{1,p}_0(\Omega) \) with \( \varphi \geq 0 \), we have:

\[
    \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \varphi + \int_{\Omega} F(x, u) \varphi dx \geq (\leq) \int_{\Omega} h(x) \varphi,
\]

\[
    u \geq (\leq) 0, \quad \text{on } \partial \Omega.
\]

A function \( u \) is a weak solution of \((3.1)\) if it is a super-solution and a sub-solution.
Next, we state the following comparison principle which will have an important role for the proof of our auxiliaries lemmas and results. Let us consider the following boundary value problem

\[
\begin{aligned}
-\Delta_p u + c(x)|u|^{p-2}u &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega,
\end{aligned}
\]  

(3.2)

**Lemma 3.2.** Let \( u, v \in W^{1,p}(\Omega) \) be a super-solution and a sub-solution of (3.2) respectively. We assume \( c \in L^{\infty}(\Omega) \) is a nonnegative function. Then we have \( v \leq u \)

**Proof.** See [35, Lemma 3.1]

Now, we consider the following boundary value problem.

\[
\begin{aligned}
-\Delta_p u + c(x)|u|^{p-2}u &= h(x) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega,
\end{aligned}
\]  

(3.3)

with \( u \in W^{1,p}_0(\Omega) \). Here we suppose the coefficients \( c(x) \) and \( h(x) \) satisfy the following assumptions

\[
c, h \in L^k(\Omega) \quad \text{for some } k > \frac{N}{p}.
\]  

(3.4)

We point out, from [17, Theorem 13], that the problem (3.3) has at least one solution. In the next lemmas we state the local maximum principale and the boundary local maximum principale, which we will be needed in the next section.

**Lemma 3.3.** Under the assumption (3.3). Let \( u \in W^{1,p}(\Omega) \) be a sub-solution of (3.3). For any ball \( B_{2R}(x_0) \subset \Omega \) and any \( q \in (0,p] \), there exists \( C = C(k, q, p, N, R, \|c\|_{L^k(B_{2R}(x_0))}) > 0 \) such that

\[
\sup_{B_R(x_0)} u^+ \leq C \left[ \left( \int_{B_{2R}(x_0)} (u^+)^q \right)^{\frac{1}{q}} + \|h^+\|_{L^q(B_{2R}(x_0))} \right].
\]

**Proof.** See [35, Corollary 3.10].

**Lemma 3.4.** Under the assumption (3.3). Let \( u \in W^{1,p}(\Omega) \) be a sub-solution of (3.3) and let \( x_0 \in \partial\Omega \). For any \( R > 0 \) any \( q \in (0,p] \), there exists \( C = C(k, q, p, N, R, \|c\|_{L^k(B_{2R}(x_0) \cap \Omega)}) > 0 \) such that

\[
\sup_{B_R(x_0) \cap \Omega} u^+ \leq C \left[ \left( \int_{B_{2R}(x_0) \cap \Omega} (u^+)^q \right)^{\frac{1}{q}} + \|h^+\|_{L^q(B_{2R}(x_0) \cap \Omega)} \right].
\]

**Proof.** See [35, Corollary 3.10 and Theorem 3.11].

In the next lemma we state the well known weak Harnack inequality for \( p \)-Laplacian case.

**Lemma 3.5.** Under the assumption (3.3). Let \( u \in W^{1,p}(\Omega) \) be a nonnegative super-solution of (3.3). Then for any ball \( B_{4R}(x_0) \subset \Omega \) and any \( q \in (0, \frac{Np}{N-p}] \) there exists \( C = C(p, q, r, k, N, \|c\|_{L^k(B_{4R}(x_0))}) > 0 \) such that

\[
\left( \int_{B_{2R}(x_0)} u^q dx \right)^{\frac{1}{q}} \leq C \left[ \inf_{B_R(x_0)} u + \|h^+\|_{L^q(B_{4R}(x_0))} \right].
\]
Proof. See [35, Theorem 3.13].

For $p = 2$ in [13] it was given a new version of Brezis-Cabré lemma. Here we will generalize this new version to the $p$-Laplacian case. The proof of the following lemma is inspired from [13, Lemma 2.4] and from [11, Lemma 3.8].

Lemma 3.6. Let $u \in W^{1,p}(\Omega)$ be a super-solution of (3.3). We assume that $c \in L^\infty(\Omega)$ and $h \in L^1(\Omega)$ be a nonnegative functions. Then for any ball $B_{2R}(x_0) \subset \Omega$ there exists $C = C(R, N, p, \|c\|_\infty) > 0$ such that

$$\inf_{\Omega} \left( \frac{u(x)}{d(x, \partial \Omega)} \right)^{p-1} \geq C \int_{B_R(x_0)} h(y)dy$$

To prove the above lemma we shall use the following result.

Lemma 3.7. Let $u \in W^{1,p}(\Omega)$ be a nonnegative weak super-solution on $B_{4R}(x_0) \subset \Omega$ of (3.2). We assume that $c \in L^\infty(\Omega)$ is a nonnegative function. Let $\eta \in C_0^\infty(B_{2R}(x_0))$ be a nonnegative function equal to 1 on $B_{R}(x_0)$. Then there exists $C = C(N, p, R, \|c\|_\infty)$ such that

$$\int_{B_{2R}} |\nabla \eta||\nabla u|^{p-1}\eta^{p-1}dx \leq CR^{N-p}(\inf_{B_R} u)^{p-1}.$$  

Proof. See [35, Lemma 4.6] \qed

Next, let us start to proof the Lemma 3.6

Proof of Lemma 3.6. We assumed $h$ is as nonnegative function, then by using weak comparison principle, Lemma 3.2, we get $u \geq 0$, which imply that

$$\inf_{\Omega} \left( \frac{u(x)}{d(x, \partial \Omega)} \right)^{p-1} \geq 0.$$  

Let $B_{2R}(x_0) \subset \Omega$ be fixed, without loss of generality we can assume that

$$\int_{B_R(x_0)} h(x)dx > 0.$$  

Let $\eta \in C_0^\infty(B_{2R}(x_0))$ be a positive function such that $\eta = 1$ on $B_R(x_0)$. By using $\eta^p$ as test function of the problem (3.3), we obtain

$$p \int_\Omega \eta^{p-1}|\nabla u|^{p-2}\nabla u \nabla \eta dx + \int_\Omega c(x)u^{p-1}\eta^p dx = \int_\Omega h(x)\eta^p,$$

thus

$$\int_{B_{2R}(x_0)} h(x)dx \leq p \int_{B_{2R}(x_0)} \eta^{p-1}|\nabla u|^{p-2}\nabla u \nabla \eta dx + C_1 \int_{B_{2R}(x_0)} u^{p-1}dx.$$  

Since the solution of the problem (3.3) is a nonnegative super-solution of the following problem

$$\begin{cases}
-\Delta_p u + c(x)u^{p-1} = 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}$$


Therefore, by using lemma \[\text{(3.7)}\] and lemma \[\text{(3.5)}\] we deduce from \[\text{(3.6)}\] that

\[
\int_{B_R(x_0)} h(x)dx \leq C_2 \left( \inf_{B_R(x_0)} u \right)^{p-1}.
\]  

(3.6)

Since \( B_{2R}(x_0) \subseteq \Omega \), then for all \( x \in \overline{B_R(x_0)} \) there exist \( C_3, C_4 \) two positive constants such that \( C_3 \leq d(x, \partial \Omega) \leq C_4 \). Hence,

\[
\left( \frac{u(x)}{d(x, \partial \Omega)} \right)^{p-1} \geq C_5 \int_{B_R(x_0)} h(x)dx, \text{ for all } x \in \overline{B_R(x_0)}.
\]  

(3.7)

Next, let \( z \) be the unique solution to the problem

\[
\left\{
\begin{aligned}
-\Delta_p z + \|c\|_{\infty} z^{p-1} &= 0 & &\text{in } \Omega \setminus \overline{B_R(x_0)}, \\
z &= 0 & &\text{on } \partial \Omega, \\
z &= 1 & &\text{on } \partial B_R(x_0).
\end{aligned}
\right.
\]  

(3.8)

Therefore from the strong maximum principle \[\text{(4.1)}\] \( z \) is positive on \( \Omega \setminus \overline{B_R(x_0)} \) and due to Hopf’s lemma \[\text{(4.1)}\] we deduce that there exists a positive constant \( C_6 \) such that

\[
z(x) \geq C_6 d(x, \Omega), \; \forall x \in \Omega \setminus \overline{B_R(x_0)}.
\]  

(3.9)

Now, we define

\[
v(x) = \frac{C_2^{1/(p-1)} u(x)}{\left( \int_{B_R(x_0)} h(y)dy \right)^{1/(p-1)}}.
\]

From \( \text{(3.9)} \) we can easily see that \( v(x) \geq 1 \) for all \( x \in \overline{B_R(x_0)} \) and \(-\Delta_p v + \|c\|_{\infty} v^{p-1} \geq 0\), which imply that \( v \) is a super-solution of \( \text{(3.8)} \). Thus by weak comparison principle, Lemma \[\text{(5.2)}\] we get that \( v(x) \geq z(x) \) for all \( x \in \Omega \setminus \overline{B_R(x_0)} \). Thus, from \( \text{(3.9)} \) we obtain

\[
\left( \frac{u(x)}{d(x, \partial \Omega)} \right)^{p-1} \geq C_7 \int_{B_R(x_0)} h(x)dx, \text{ for all } x \in \Omega \setminus \overline{B_R(x_0)}.
\]  

(3.10)

By combining \[\text{(3.7)}\] and \[\text{(3.10)}\] we deduce the result. \(\square\)

Let us state the boundary weak Harnack inequality which is established recently in \[\text{(13)}\]. However this inequality was established in the case of the problem \[\text{(3.2)}\] and under stronger condition then \[\text{(3.4)}\].

**Lemma 3.8.** Let \( u \in W^{1,p}(\Omega) \) be a nonnegative super-solution of \[\text{(3.2)}\]. We assume \( c \in L^\infty(\Omega) \), is a nonnegative function. Let \( x_0 \in \partial \Omega \). Then there exist \( R > 0, \; \epsilon = \epsilon(p, R, \|c\|_{L^\infty(B_R(x_0) \cap \Omega)}, \Omega) > 0 \) and \( C = C(p, R, \epsilon, \|c\|_{L^\infty(B_R(x_0) \cap \Omega)}, \Omega) > 0 \) such that for all \( R \in (0, R) \),

\[
\left( \int_{B_R(x_0) \cap \Omega} \left( \frac{u(x)}{d(x, \partial \Omega)} \right)^\epsilon dx \right)^{\frac{1}{\epsilon}} \leq C \inf_{B_R(x_0) \cap \Omega} \frac{u(x)}{d(x, \partial \Omega)}.
\]

**Proof.** See \[\text{(13)}\] Theorem 3.1. \(\square\)

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Since we will use the variational argument to prove the multiplicity of solutions of the problem \((Q)\), let us recall firstly the standard definition of Palais-Smale sequence at level \(c\), Palais-Smale condition at level \(c\) of a functional \(I \in C^1(E, \mathbb{R})\) where \(E\) is a Banach space and some results.

**Definition 3.2.** Let \(E\) be a Banach space with dual space \(E^*\) and \(\{u_n\}\) is a sequence of \(E\). We say that \(\{u_n\}\) is a Palais-Smale sequence at the level \(c\) if

\[
I(u_n) \to c, \quad \text{and} \quad \|I'(u_n)\|_{E^*} \to 0.
\]

**Definition 3.3.** Let \(\{u_n\}\) be a Palais-Smale sequence at the level \(c\) of \(E\). We say that \(\{u_n\}\) satisfies the Palais-Smale condition at the level \(c\) if \(\{u_n\}\) possesses a convergent subsequence.

Finally, let us recall some results from [29] which will be very helpful for us.

**Theorem 3.1.** Let \(E\) be a Banach space with dual space \(E^*\) endowed with the norm \(\|\cdot\|_E\). We consider a family \(\{I_\lambda\} \subset C^1(E, \mathbb{R})\) with the form

\[
I_\lambda = A(u) - \lambda B(u), \quad \forall \lambda \in [1/2, 1],
\]

where \(B(u) \geq 0\) for every \(u \in E\) and such that either \(A(u) \to +\infty\) or \(B(u) \to +\infty\) as \(\|u\|_E \to +\infty\).

We assume that there exists \(v_0 \in E\) such that with \(\Gamma = \{ \gamma \in C([0, 1], E) : \gamma(0) = 0, \gamma(1) = v_0 \}\), we have for all \(\lambda \in [1/2, 1]\)

\[
c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} I_\lambda(\gamma(t)) \geq \max\{ I_\lambda(0), I_\lambda(v_0) \}.
\]

Then, for almost every \(\lambda \in [1/2, 1]\), there exists a bounded Palais-Smale sequence \(v_n \subset E\) at level \(c_\lambda\) such that

1. \(v_n\) is bounded,
2. \(I_\lambda(v_n) \to c_\lambda\),
3. \(I'_\lambda(v_n) \to 0\) in \(E^*\).

**Corollary 3.1.** Let \(E, I_\lambda\) be as in Theorem 3.1 and we assume that both \(B\) and \(B'\) take bounded sets to bounded sets. Suppose in addition that for all \(\lambda \in [1/2, 1]\), all bounded Palais-Smale sequences admit a convergence subsequence. Then there exists a subsequence \(\{(\lambda_n, u_n)\} \subset [1/2, 1] \times E\) with \(\lambda_n \to 1\), \(I_{\lambda_n}(u_n) = c_{\lambda_n}\), \(I'_{\lambda_n}(u_n) = 0\). Moreover, if \(\{u_n\}\) is bounded then

\[
I_{\lambda_n}(u_n) \to \lim_{n \to \infty} c_{\lambda_n} = c_1 \quad I'_{\lambda_n}(u_n) \to 0 \quad \text{in} \ \ E^*.
\]

### 4 Proof of the main results

In this section we will give the proofs of our main results.
4.1 Proof of a priori bounds

Proof of Theorem 2.1. As we have seen in Lemma 3.1, every weak solution of \((Q_\lambda)\) belong to \(L^\infty(\Omega)\). Now, let us firstly show that there exists a positive constant \(C\) such that
\[
\|u\|_{L^\infty(\Omega_+)} < C. \tag{4.1}
\]
For that we arguing by contradiction. We assume that there exist a sequences \(\{\lambda_n\} \subset [1/2,1]\) and \(\{u_n\}\) nonnegative solutions of \((Q_\lambda)\) for \(\lambda = \lambda_n\) such that \(\|u_n\|_{L^\infty(\Omega_+)} \to +\infty\) as \(n \to +\infty\). Let \(\{x_n\} \subset \Omega_+\) be such that \(u_n(x_n) = \|u_n\|_{L^\infty(\Omega_+)}\). Passing to a sub-sequences we may assume that \(\lambda_n \to \lambda \in [1/2,1]\) and \(x_n \to \bar{x} \in \Omega_+\). Therefore, from now we will distinguish the following two cases:

- **Case 1**: \(\bar{x} \in \Omega_+ \cap \Omega\).
- **Case 2**: \(\bar{x} \in \Omega_+ \cap \partial \Omega\).

Through this proof we denote \(B_{kR} = B_{kR}(\bar{x})\) where \(k \in \mathbb{N}\).

**Case 1**: From the assumption \((A_{a,b})\) there exists a \(R > 0\) such that \(a^-(-x) = 0\) in \(B_{4R} \subset \Omega\) and \(a^+(x) \geq 0\) in \(B_R\).

If \(u_n < 1\) on \(B_{4R}\), there is nothing to prove because we get directly a contradiction. Let us assume that \(u_n \geq 1\) on \(B_{4R}\).

We claim that there exist \(R > 0\) and \(C > 0\) such that
\[
\inf_{\partial B_R} u_n < C. \tag{4.2}
\]
Otherwise, it is easy to see that
\[
-\Delta_p u_n \geq \lambda a^+(x) f(u_n) - b^-(x) g(u_n) \text{ in } B_{4R}. \tag{4.3}
\]
However, from \((A^+_{g,1})\) there exists a positive constant \(C_1\) such that
\[
g(u_n) \leq C_1 u_n^{p-1} \text{ in } B_{4R}, \tag{4.4}
\]
it follows from (4.3) that
\[
-\Delta_p u_n + C_1 b^-(x) u_n^{p-1} \geq \lambda a^+(x) f(u_n) \text{ in } B_{4R}. \tag{4.5}
\]
Hence, from Lemma 3.6 there exists a positive constant \(C_2\) such that
\[
\left(\inf_{B_R} \frac{u_n(x)}{d(x, \partial B_{4R})}\right)^{p-1} \geq \frac{C_2}{2} \int_{B_R} a^+(y) f(u_n) dy,
\]
which imply
\[
1 \geq C_3 \int_{B_R} a^+(y) \frac{f(u_n)}{\inf_{B_R} u_n(x)} u_n^{p-1}(y) dy,
\]
then
\[
1 \geq C_3 \int_{B_R} a^+(y) \frac{f(u_n)}{u_n^{p-1}(y)} dy. \tag{4.6}
\]
Now, if \( \inf_{B_R} u_n(x) \to +\infty \) as \( n \to \infty \). Hence from (4.6) we obtain
\[
\int_{A_R^+} a^+(y) \frac{f(u_n)}{u_n^{p-1}(x)} dy \to \infty \quad \text{as} \quad n \to \infty,
\]
which contradict with (4.6). Therefore, we get the claim.

In one hand, we obtain easily from (4.5) that
\[
-\Delta_p u_n + C_1 b^-(x) u_n^{p-1} \geq 0 \quad \text{in} \quad B_{4R},
\]
then from Lemma 3.5 that, for any \( q \in (0, \frac{N(p-1)}{N-p}) \) there exists a positive constant \( C_4 \) such that
\[
\left( \int_{B_{2R}} u_n^q dx \right)^{\frac{1}{q}} \leq C_4 \inf_{\partial B_r} u_n.
\]

On the other hand, we have
\[
-\Delta_p u_n \leq a^+(x) f(u_n) + b^+(x) g(u_n) \quad \text{in} \quad B_{4R}.
\]
Since \( f \) is regular varying of index \( p-1 \), then, from Proposition 3.1, we have for all \( \eta > 0 \) there exists a positive constant \( C_5 \), which depends on \( \eta \), such that
\[
f(u_n) \leq C_5 u_n^{p-1+\eta}.
\]
Furthermore, by (4.4) and (4.9) we get that
\[
-\Delta_p u_n \leq d_n(x) u_n^{p-1} \quad \text{in} \quad B_{4R},
\]
where \( d_n(x) = C_5 a^+(x) u_n^\eta + C_1 b^+(x) \). Next, let show that \( d_n \) is bounded in \( L^\beta(B_{2R}) \), where \( \beta > \frac{N}{p} \). For that we choose \( \eta \in (0, \frac{q}{q+p}(\frac{p}{N} - 1)) \) and we set \( \beta = \frac{kq}{q+p} \). Hence by using Hölder inequality we get
\[
\int_{B_{2R}} (d_n(x))^\beta dx \leq C_6 \left( \int_{B_{2R}} (a^+(x) u_n^\eta)^\beta dx + \int_{B_{2R}} (b^+(x))^\beta dx \right)^{\frac{1}{\beta/k}}.
\]
By using (4.2) and (4.8) we obtain that
\[
\int_{B_{2R}} (d_n(x))^\beta dx \leq C_8.
\]
Therefore, from Lemma 3.3 for any \( \gamma \in (0, p) \) there exists \( C_9 \) a positive constant such that
\[
\sup_{B_R} u_n \leq C_9 \left( \int_{B_{2R}} u_n^\gamma dx \right)^{\frac{1}{\gamma}}.
\]
By choosing \( \gamma = q = p - 1 \) and by combining (4.2), (4.8) and (4.11) we deduce that
\[
\sup_{B_R} u_n \leq C_{10},
\]
which imply a contradiction.
**Case 2**: From the assumption \(A_{a,b}\) there exists a \(R \in (0, R'/2]\), where \(R'\) is given from Lemma 3.8 and \(\omega = \Omega\) with \(\partial \omega\) of class \(C^{1,1}\) such that \(B_{2R} \cap \omega \subset \omega\), \(a^-(x) = 0\) and \(a^+(x) \geq 0\) in \(\omega\).

As in the case 1, if \(u_n < 1\) on \(\omega\), there is nothing to prove because we get directly a contradiction. We assume that \(u_n \geq 1\) on \(\omega\).

Next, let us prove that there exists a positive constant \(C\) such that

\[
\inf_{B_{2R} \cap \omega} \frac{u_n(x)}{d(x, \partial \omega)} < C. \tag{4.12}
\]

Let \(\bar{R}\) and \(z \in \Omega\) be such that \(B_{4\bar{R}}(z) \subset B_{2R} \cap \omega\) and \(a^+(x) \geq 0\) in \(B_{\bar{R}}\), by using the same arguing in the claim of the case 1, there exists \(C_1\) a positive constant such that

\[
\inf_{B_{\bar{R}}} u_n(x) < C_1. \tag{4.13}
\]

Moreover, since \(B_{4\bar{R}}(z) \subset B_{2R} \cap \omega\), we have

\[
\inf_{B_{2R} \cap \omega} \frac{u_n(x)}{d(x, \partial \omega)} \leq \inf_{B_{\bar{R}}} \frac{u_n(x)}{d(x, \partial \omega)} \leq \frac{1}{\bar{R}} \inf_{B_{\bar{R}}} u_n(x),
\]

then from (4.13) we deduce directly (4.12). On the other hand, from (4.12) we have that

\[-\Delta_p u_n + C_2 b^{-}(x) u_n^{p-1} \geq 0 \text{ in } \omega,\]

thus from Lemma 3.8 and the inequality (4.12) there exist \(\epsilon > 0\) and \(C_3 > 0\) such that

\[
\left( \int_{B_{2R} \cap \omega} \left( \frac{u(x)}{d(x, \partial \omega)} \right)^\epsilon \, dx \right)^{\frac{1}{\epsilon}} \leq C_3, \tag{4.14}
\]

since \(d(x, \partial \omega) < \text{diam}(\Omega)\), it follows that

\[
\left( \int_{B_{2R} \cap \omega} (u(x))^\epsilon \, dx \right)^{\frac{1}{\epsilon}} \leq C_3 \text{ diam}(\Omega). \tag{4.15}
\]

However, we can have as in case 1 that

\[-\Delta_p u_n \leq d_\eta(x) u_n^{p-1} \text{ in } \omega,\]

where \(d_\eta(x) = C_4 a^+(x) u_n^\eta + C_5 b^+(x)\), for any \(\delta > 0\). In addition, for this case we choose \(\delta \in (0, \frac{\eta}{\eta + \delta})\) and we set \(s_\gamma = \frac{k \epsilon}{\epsilon + 5\gamma} > \frac{N}{p}\). Then we can prove as above that \(d_\eta\) is bounded in \(L^{s_\gamma}(B_{2R} \cap \omega)\). Therefore from Lemma 3.4 for any \(\beta \in (0, p]\) there exists a positive constant \(C_6\), such that

\[
\sup_{B_{2R} \cap \omega} u_n \leq C_6 \left( \int_{B_{2R} \cap \omega} u_n^\beta \, dx \right)^{\frac{1}{\beta}}. \tag{4.16}
\]

Hence by choosing \(\beta = \epsilon\). We combine (4.15) and (4.16) then we get

\[
\sup_{B_{2R} \cap \omega} u_n \leq C_7,
\]

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which imply the contradiction that we are looking for in this case.

Next, let us show that there is a positive $C > 0$ such that

$$\|u\|_{L^\infty(\Omega \setminus \overline{\Omega}_+)} < C. \quad (4.17)$$

We set $K = \Omega \setminus \overline{\Omega}_+$. We have that

$$-\Delta_p u = -a^-(x)f(u) + b(x)g(u) \quad \text{in } K.$$ 

We define $z = u - \sup_{\partial K} u$, then we get that

$$-\Delta_p z \leq b^+(x)g(z + \sup_{\partial K} u) \quad \text{in } K.$$ 

From the assumption $(A_1)$, we can easily have that

$$-\Delta_p z \leq C_1 b^+(x)(1 + z^{p-1}) \quad \text{in } K,$$

where $C_1$ is a positive constant. Therefore $z$ is a sub-solution of the following problem

$$\begin{cases}
-\Delta_p v = C_1 b^+(x)(1 + v^{p-1}) & \text{in } K, \\
v = 0 & \text{on } \partial K,
\end{cases} \tag{4.18}$$

where $v \in W^{1,p}_0(K) \cap L^\infty(K)$. From [33, Theorem IX-22] we have $v \in C^{0,\alpha}(\overline{K})$. Thus, from the weak comparison principle in [15, Theorem 3.1] we get that

$$u \leq v + \sup_{\partial K} u \quad \text{in } K. \quad (4.19)$$

However, from [24, Theorem 1.2] there exists a positive constant $C_2$ such that

$$\|v\|_{L^p(K)} \leq C_2.$$ 

Then, by using Lemma 3.3 we deduce directly that there exists a positive constant $C_3$ such that

$$\sup_K v \leq C_3.$$ 

Hence, from (4.19), we obtain that

$$u \leq C_3 + \sup_{\Omega_+} u \quad \text{in } K,$$

which imply (4.17), hence the end of the proof.

4.2 Proof of the multiplicity of solutions

One of the most fruitful ways to deal with problem $(Q_\lambda)$, for any $\lambda \in [1/2, 1]$, is the variational method, which takes into account that the weak solutions of $(Q_\lambda)$ are critical points in $W^{1,p}_0(\Omega)$ of the following functional

$$I_\lambda(u) = \frac{1}{p} \int_\Omega |\nabla u|^p - \int_\Omega (\lambda a^+(x) - a^-(x))F(u) - \int_\Omega b(x)G(u), \quad (4.20)$$
such that $F(s) = \int_0^s \tilde{f}(s) ds$ and $G(s) = \int_0^s \tilde{g}(s) ds$, where

$$
\tilde{f}(s) = \begin{cases} f(s) & \text{if } s \geq 0, \\ 0 & \text{if } s < 0, \end{cases} \quad \tilde{g}(s) = \begin{cases} g(s) & \text{if } s \geq 0, \\ g(0) & \text{if } s < 0. \end{cases}
$$

Before we start to proof our main results. It is worth to mention that we will use the same method used in $\text{[23]}$ where the proof depend on an abstract result due to $\text{[29]}$. This method is based on embedding the functional $I$ into a one-parameter family of functionals $I_\lambda$, with $\lambda \in [1/2, 1]$, such that $I_1 = I$. On the other hand, the a priori estimates obtained in the last section will help us to get easily the boundness of the Palais-Smale sequence for $I_\lambda$, for all $\lambda \in [1/2, 1]$.

The proof of our main results is as follows. In the first step, we show the existence of the first critical point for the $I_1$ by using the Corollary $3.4$. Precisely, we show that the functional $I_1$ satisfies the Palais-Smale condition at the level $c_1 > 0$. In the second step, we show the existence of the second critical point of $I_1$ on $B_r(0)$ (which is a local minimum) by using the lower semicontinuity argument. Finally, we will prove that these critical points are not the same.

In the following lemma we prove the strongly convergence of Palais-Smale sequence for $I_\lambda$.

**Lemma 4.1.** Under the assumptions $\{A_{\text{un}}, A_1\}, \{A_{\text{g}}, A_2\}$, then, for all $\lambda \in [1/2, 1]$, any bounded Palais-Smale sequence for $I_\lambda$ defined by $\text{[16,23]}$ admits a convergent subsequence.

**Proof.** Let $\{u_n\} \subset W_0^{1,p}(\Omega)$ be a bounded Palais-Smale sequence for $I_\lambda$ at level $c > 0$. Then there exists a subsequence denoted again by $\{u_n\}$ such that $u_n$ converges weakly to $u$ in $W_0^{1,p}(\Omega)$, strongly in $L^r(\Omega)$ with $r \in [1,p^*)$ and almost everywhere in $\Omega$. We have that

$$
\langle I'(u_n), u_n - u \rangle = \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla (u_n - u) dx - \int_\Omega a_{\lambda}(x) \tilde{f}(u_n)(u_n - u) dx - \int_\Omega b(x) \tilde{g}(u_n)(u_n - u) dx.
$$

Let $\delta \in (0,\left(\frac{p}{N} - \frac{1}{k}\right)p^*)$, $s \in (1,\frac{p}{\frac{p}{p-1}+\delta})$ and $r \in (1,p^*)$ such that $\frac{1}{k} + \frac{1}{s} + \frac{1}{r} = 1$. From Proposition $3.1$, Hölder’s inequality and Sobolev’s embedding, we have

$$
|\int_\Omega a_{\lambda}(x) \tilde{f}(u_n)(u_n - u) dx| \leq \int_\Omega |a_{\lambda}(x) \tilde{f}(u_n)| |u_n - u| dx \\
\leq \|a\|_k \|\tilde{f}(u_n)\|_s \|u_n - u\|_r \\
\leq C_1 \|a\|_k \left(1 + \|u_n\|^{p-1+\delta}\right) \|u_n - u\|_r \\
\leq C_2 \|a\|_k \left(1 + \|u_n\|^{p-1+\delta}\right) \|u_n - u\|_r
$$

Hence,

$$
\int_\Omega a_{\lambda}(x) \tilde{f}(u_n)(u_n - u) dx \to 0 \quad \text{as } n \to \infty.
$$

Form the assumption $\{A_{\text{g}}\}$, there $C_3$ a positive constant such that

$$
|\tilde{g}(u_n)| \leq C_3(1 + |u_n|^{p-1}).
$$

By using the same argument we show easily that

$$
\int_\Omega b(x) \tilde{g}(u_n)(u_n - u) dx \to 0 \quad \text{as } n \to \infty.
$$
Since \( \langle I'(u_n), u_n - u \rangle \to 0 \) as \( n \to \infty \). Thus, we deduce that
\[
\int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \nabla (u_n - u) \, dx \to 0 \text{ as } n \to \infty.
\]
Therefore, by using [17, Theorem 10], we conclude that \( u_n \) converges strongly to \( u \) in \( W^{1,p}_0(\Omega) \).

In the following lemma we show the nonnegativity of weak solutions of \( (Q_{\lambda}) \).

**Lemma 4.2.** Under the assumptions \((A_{a,b})\), \((A_{1}^f)\), \((A_{3}^f)\) (or \((A_{3}^{f'})\)), and \((A_{1}^g)\).

1. If \( g(0) = 0 \). Then, every weak solution of \( (Q_{\lambda}) \) is nonnegative.
2. If \( g(0) > 0 \) and \( b \geq 0 \). Then, every weak solution of \( (Q_{\lambda}) \) is positive.

**Proof.**

1. By multiplying the first equation of the problem \( (Q_{\lambda}) \) by the test function \( -u^- \) and from the definition of \( \tilde{f} \) and \( \tilde{g} \), we get that
\[
\int_{\Omega} |\nabla u^-|^p \, dx = -g(0) \int_{\Omega} b(x) u^- \, dx.
\] (4.21)

If \( g(0) = 0 \). Thus, \( \int_{\Omega} |\nabla u^-|^p \, dx = 0 \) which imply that \( u \geq 0 \).

2. Since \( g(0) > 0 \) and \( b \geq 0 \), then, from (4.21) we obtain that \( \int_{\Omega} |\nabla u^-|^p \, dx = 0 \) which mean \( u \geq 0 \).

By using the Proposition 3.1 and the assumption \((A_{3}^f)\) (or \((A_{3}^{f'})\)), we have for all \( \delta > 0 \) there exist a positive constant \( C_1 \) such that
\[
-\Delta_p u \geq d(x)u^{p-1} \text{ in } \Omega,
\]
where \( d(x) = -C_1 a^-(x)(1 + u^\delta) \). Thus, by using Theorem 2.1 we obtain that \( d \) is bounded in \( L^k(\Omega) \).

Hence, from Lemma 3.5 we deduce either \( u = 0 \) or \( u > 0 \). Since \( g(0) > 0 \) and \( b \geq 0 \) we conclude that \( u > 0 \).

**4.2.1 Proof of the first multiplicity result**

**Proof of Theorem 2.2.** Firstly, we obtain from \((A_{1}^f)\) and \((A_{3}^{f'})\) that, there exists \( C_1 \) a positive constant and for all small positive \( \epsilon \) we have
\[
f(t) \leq \epsilon t^{p-1} + C_1 t^{p-1+\eta} \quad \forall (t, \eta) \in \mathbb{R}_+ \times (0, \infty),
\] (4.22)
and from \((A_{3}^g)\) that, there exist a positive constant \( C_2 \) such that
\[
g(t) \leq C_2 (1 + t^{p-1}) \quad \forall t \in \mathbb{R}_+.
\] (4.23)

Then from (4.22) and (4.23) we get that \( I_{\lambda} \in C^1(W^{1,p}_0(\Omega), \mathbb{R}) \), for any \( \lambda \in [1/2, 1] \), (see for example [17] page 356). On the other hand, to use the Theorem 3.1 we shall set
\[
A(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p + \int_{\Omega} a^-(x) F(u) - \int_{\Omega} b(x) G(u),
\]
and
\[ B(u) = \lambda \int_{\Omega} a^+(x)F(u). \]
We have \( F \) is a nonnegative function, thus by using (4.23), Hölder’s inequality and Sobolev’s embedding we get
\[
A(u) \geq \frac{1}{p} \int_{\Omega} |\nabla u|^p - \int_{\Omega} b(x)G(u) \\
\geq \frac{1}{p} ||u||^p - C_2(\|b^+\|_{L^k(\Omega)} \|u\|_{L^{k'}(\Omega)} + \|b^+\|_{L^k(\Omega)} \|u\|^{p}_{L^{k'}(\Omega)}) \\
\geq ||u||^p \left( \frac{1}{p} - C_3 \|b^+\|_{L^k(\Omega)} (||u||^{1-p} + 1) \right).
\]
Hence by choosing
\[ \|b^+\|_{L^k(\Omega)} < \frac{1}{C_3p}, \]
we obtain directly that \( A(u) \to +\infty \) as \( ||u|| \to +\infty \). Next, it is obvious that for all \( u \in W^{1,p}_0(\Omega) \) we have \( B(u) \geq 0 \). In addition, \( B \) and \( B' \) map bounded sets into bounded sets (see for example [17] page 355).

Now, let us prove that \( I_{\lambda} \) has has a geometrical structure. From (4.22) and by using Hölder’s inequality we get that
\[
\int_{\Omega} a^+(x)F(u) \leq \epsilon \|a^+\|_{L^k(\Omega)} ||u||^p_{L^{k'}(\Omega)} + C_1 \|a^+\|_{L^k(\Omega)} ||u||^{p+\eta}_{L^{k'}(\Omega)}.
\]
Due to the assumption \( k > N/p \), we choose \( \eta > 0 \) small enough such that \( (p+\eta)k' < \frac{pN}{N-p} \). Hence, by using Sobolev’s embedding we obtain
\[
\int_{\Omega} a^+(x)F(u) \leq C_4 \|a^+\|_{L^k(\Omega)} ||u||^p_{L^{k'}(\Omega)} + C_5 \|a^+\|_{L^k(\Omega)} ||u||^{p+\eta}_{L^{k'}(\Omega)}.
\]
Furthermore, as above we get from (4.23) that
\[
\int_{\Omega} b^+(x)G(u) \leq C_3 \|b^+\|_{L^k(\Omega)} (||u|| + ||u||^p).
\]
Since \( F \) and \( G \) are nonnegative thus from (4.25) and (4.26) we deduce that
\[
I_{\lambda}(u) \geq \frac{1}{p} \int_{\Omega} |\nabla u|^p - \int_{\Omega} a^+(x)F(u) - \int_{\Omega} b^+(x)G(u) \\
\geq \frac{1}{p} - \|a^+\|_{L^k(\Omega)} (C_4 \epsilon + C_5 \|u\|^{\eta}) - C_3 \|b^+\|_{L^k(\Omega)} (||u||^{1-p} + 1) \|u\|^p.
\]
Therefore, let \( u \in \partial B_p(0) \) and by choosing \( \rho \) small enough we choose
\[ \|b^+\|_{L^k(\Omega)} < \frac{1}{2pC_3(p^{1-p} + 1)}, \]
we deduce \( I_{\lambda}(u) \geq \nu > 0 \), for all \( \lambda \in [1/2, 1] \).
Next, we choose \( y \in \Omega^+ \) and \( R > 0 \) such that \( B_R(y) \subset \Omega^+ \). Let \( v \) be a positive function that belongs to \( C_0^\infty(B_R(y)) \) such that \( a^+ v \geq 0 \). From the definition of \( I_\lambda \), we have

\[
I_\lambda (tv) = \int_{\Omega} |\nabla v|^p - \lambda \int_{\Omega} (\lambda a^+(x) - a^- (x)) F(tv) - b(x) G(tv) \, dx \\
= t^p \int_{B_R(y)} |\nabla v|^p - \lambda \int_{B_R(y)} a^+(x) \frac{F(tv)}{t^p v^p} v^p - b(x) \frac{G(tv)}{t^p v^p} v^p. 
\]

By using (4.3), there exists a positive constant \( C_0 \) such that

\[
\int_{B_R(y)} |b(x)\frac{G(tv)}{t^p v^p} v^p| \leq C_0 \|b\|_{L^p(B_R(y))} \quad \text{for } t \text{ large enough},
\]

and from (4.7) we obtain

\[
\int_{B_R(y)} a^+(x) \frac{F(tv)}{t^p v^p} v^p \to +\infty \quad \text{as } t \to +\infty.
\]

Hence by choosing \( v_0 = tv \), defined in Theorem 4.1, with \( t \) large enough we obtain that \( I_\lambda (v_0) < 0 \). Therefore, from Theorem 4.1 we deduce that \( I_\lambda \) admit a Palais-Smale sequence \( \{u_n\} \) at level \( c_\lambda \). Thus, by using Lemma 4.1 we have that all bounded Palais-Smale sequences admit a convergent subsequence. Then from Corollary 3.1 there exist a sequences \( \{\lambda_n\} \subset [\frac{1}{2}, 1] \) and \( \{u_n\} \subset W_0^{1,p}(\Omega) \) such that \( \lambda_n \to 1 \), \( I_{\lambda_n}(u_n) = c_{\lambda_n} \) and \( I_{\lambda_n} (u_n) = 0 \), which means that \( u_n \) is a weak solution of the following problem

\[
(Q_{\lambda_n}) \quad \begin{cases}
-\Delta_p u = a_{\lambda_n}(x) f(u) + b(x) g(u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

From Lemma 3.1 we have that \( u_n \in L^\infty(\Omega) \). Moreover, from Lemma 4.2 we obtain that \( u_n \) is nonnegative.

Beside, from Theorem 2.1 there exist a positive constant \( C \) such that \( \|u_n\|_{L^\infty(\Omega)} < C \). Then by multiplying the first equation of problem \( (Q_{\lambda_n}) \) by \( u_n \) and by using Proposition 3.1 and (4.23) we get that

\[
\int_{\Omega} |\nabla u_n|^p \, dx = \int_{\Omega} a_{\lambda_n}(x) f(u_n) u_n \, dx + \int_{\Omega} b(x) g(u_n) u_n \, dx \\
\leq C_7 \left( \int_{\Omega} |a(x)||u_n| + |u_n|^{p+\delta} \right) dx + \int_{\Omega} \left| b(x) \right| \left( |u_n| + |u_n|^p \right) \, dx \\
\leq C_8 \left( \|a\|_k + \|b\|_k \right).
\]

Therefore, \( \{u_n\} \) is bounded in \( W_0^{1,p}(\Omega) \). Hence, we can find a subsequence, still denoted \( \{u_n\} \), such that \( u_n \) converges weakly to \( u \) in \( W_0^{1,p}(\Omega) \), strongly in \( L^r(\Omega) \) with \( r \in [1, p^*) \) and almost everywhere to \( u \) in \( \Omega \). From Lemma 4.1 we have \( u_n \) converges strongly to \( u \) in \( W_0^{1,p}(\Omega) \). Therefore, we deduce directly that \( u \in W_0^{1,p}(\Omega) \) is a nonnegative weak solution of the problem \( (Q) \) and from Corollary 3.1 we have \( I_1(u) = c_1 > 0 \) which mean that \( u \) is nontrivial.

Next, let us show the existence of the second weak solution of the problem \( (Q) \). Firstly, we have \( I(0) = 0 \) then \( \inf_{v \in B_\rho(0)} I(v) \leq 0 \), where \( \rho \) is defined as above. On the other hand, let \( \phi \in C_0^\infty(\Omega) \) be a positive function such that \( a \phi > 0 \) and \( b \phi > 0 \). From the definition of \( I \), for all \( t > 0 \), we have

\[
I(t \phi) = t^p \int_{\Omega} |\nabla \phi|^p - \int_{\Omega} a(x) \frac{F(t \phi)}{t^p \phi^p} \phi^p - b(x) \frac{G(t \phi)}{t^p \phi^p} \phi^p.
\]

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By using (A.1) and (A.2), we get for $t \to 0^+$ that
\[
\int_\Omega a(x) \frac{F(t\phi)}{t^p \phi^p} \phi^{p+1} \to 0,
\]
and
\[
\int_\Omega b(x) \frac{G(t\phi)}{t^p \phi^p} \phi^p \to +\infty.
\]
Hence, $I(t\phi) < 0$ for $t$ small enough. Then $\inf_{v \in B_\rho(0)} I(v) < 0$. Moreover, it has been shown that $I(v) \geq \nu > 0$ with $\|v\| = \rho$. Then there exists a sequence $\{v_n\} \subset B_\rho(0)$ such that $I(v_n)$ converges to $\inf_{v \in B_\rho(0)} I(v)$. Since $\{v_n\}$ is bounded in $W^{1,p}_0(\Omega)$ then there exists a subsequence denoted again by $\{v_n\}$ such that $v_n$ converges to $v$ weakly in $W^{1,p}_0(\Omega)$ and converges strongly in $L^r(\Omega)$ for some $r \in [1, p^*)$ respectively. In addition, since $\|v\| \leq \inf_{n \to \infty} \|v_n\|$. Hence, we obtain $I(v) \leq \inf_{v \in B_\rho(0)} I(v)$. Therefore, $v$ is a nonnegative and nontrivial local minimum of $I$ in $B_\rho(0)$.

We conclude that $(Q)$ have one weak solution $u$ at level $c_1$, which mean that $I(u) = c_1 > 0$ and a second weak solution $v \in B_\rho(0)$ such that $I(v) < 0$. Therefore, the problem $(Q)$ has at least two distinct nonnegative and nontrivial weak solutions in $W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$. Moreover, by using Lemma 4.2 we conclude the result.  

4.2.2 Proof of the second multiplicity result

Proof of Theorem 2.3 As it was quoted before every weak solution of the problem $(Q_\lambda)$ is a critical point of the variational formula (4.20). By the same arguing used in the proof of Theorem 2.2 we have $I_\lambda \in C^1(W^{1,p}_0(\Omega), \mathbb{R})$. Let us show that $I_\lambda$ has a geometrical structure for every $\lambda \in [1/2, 1]$.

From Proposition 3.1 and due to (A.2), there exist $C_1$ and $C_2$ two positive constants such that $f(t) \leq C_1 t^r + C_2 t^{p-1+n}$, for any $\eta > 0$. Hence, by choosing $\eta$ small enough, by using Hölder’s inequality and Sobolev’s embedding we obtain that
\[
\int_\Omega a_\lambda(x) F(u) \leq C_3 (\|a^+\|_{L^p(\Omega)} \|u\|^{r+1} + \|a^+\|_{L^p(\Omega)} \|u\|^{p+\eta}).
\]
Moreover, from (4.23) there exists a positive constant $C_4$ such that $g(t) \leq C_4 (1 + t^{p-1})$. As above we have
\[
\int_\Omega b(x) G(u) \leq C_5 (\|b^+\|_{L^p(\Omega)} \|u\| + \|b^+\|_{L^p(\Omega)} \|u\|^p).
\]
From the definition of $I_\lambda$ we get that
\[
I_\lambda(u) \geq \|u\|^p \left( \frac{1}{p} - C_3 \|a^+\|_{L^p(\Omega)} (\|u\|^{r+1+p} + \|u\|\eta) - C_5 \|b^+\|_{L^p(\Omega)} (\|u\|^{1+p} + 1) \right).
\]
Therefore, let $u \in \partial B_\rho(0)$, by choosing $\rho$ small enough,
\[
\|b^+\|_{L^p(\Omega)} \leq \frac{1}{3pC_5(\rho^{1+p} + 1)},
\]
and
\[
\|a^+\|_{L^p(\Omega)} \leq \frac{1}{3pC_3(\rho^{r+1+p} + \rho^\eta)}.
\]

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We deduce that $I_\lambda(u) \geq \nu_2 > 0$ on $\partial B_\rho (0)$, for any $\lambda \in [1/2, 1]$. By using the same arguing to proof Theorem \ref{teo1} we show the existence of the first nonnegative and nontrivial weak solution $u \in W_0^{1,p}(\Omega) \cap L^\infty (\Omega)$ at level $c_1 > 0$.

To show that the second weak nonnegative solution of $(Q)$ on $B_\rho (0)$. Let $\phi \in C_0^\infty (\Omega)$ be a positive function such that $a_\lambda \phi > 0$ and $b \phi > 0$. From the definition of $I_\lambda$ for all $t > 0$ we have

$$I_\lambda (t \phi) = t^p \left( \frac{1}{p} \int _\Omega |\nabla \phi|^p - t^{r+1-p} \int _\Omega a_\lambda (x) \frac{F(t \phi)}{t^{r+1} \phi^{r+1}} \phi^{r+1} - \int _\Omega b(x) \frac{G(t \phi)}{t^{p \phi^p}} \phi^p \right).$$

From $(A_1^p)$ and $(A_2^p)$, we obtain that

$$t^{r+1-p} \int _\Omega a_\lambda (x) \frac{F(t \phi)}{t^{r+1} \phi^{r+1}} \phi^{r+1} \to +\infty$$

and

$$\int _\Omega b(x) \frac{G(t \phi)}{t^{p \phi^p}} \phi^p \to +\infty,$$

or

$$\int _\Omega b(x) \frac{G(t \phi)}{t^{p \phi^p}} \phi^p \to c < +\infty,$$

as $t \to 0^+$. It follows that $I_\lambda (t \phi) < 0$ for $t$ small enough. For the rest of the proof is still the same as the proof of Theorem \ref{teo1}.

\section{Applications}

The aim of this section is to give a class of problems that have the same form of the problem $(Q)$, where the conditions $(A_1^p)$, $(A_2^p)$, $(A_3^p)$, $(A_4^p)$ and $(A_5^p)$ are satisfied and for which Theorems \ref{teo1} and \ref{teo2} hold. In particular, we give two applications for both Theorem \ref{teo1} and Theorem \ref{teo2}.

The first application concerns the case where $g(0) = 0$ and the second one, deals with the case where $g(0) > 0$ is satisfied.

\subsection{Application 1: Quasilinear equation with a potential}

Let us consider the following problem:

$$(P_1) \begin{cases} -\Delta_p u = a(x)(1 + u)^{p-1}(\ln(1 + u))^q + b(x)u^m & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega, \end{cases}$$

where $q > 0$ and $m > 0$. This type of problem have been considered in several works in the case where $p = 2$, see for instance, \cite{3}, \cite{9}. To the best of our knowledge there is no multiplicity result in this type of problem in the case where $a$ and $b$ change of sign and $p > 1$.

The following result is an application of Theorem \ref{teo1}.

\begin{corollary}
Under the assumption $(A_{0,b})$. Let $q > p - 1$, $0 < m \leq p - 1$, and $\|b^{+}\|_{L^{q}(\Omega)}$ is suitably small. Then, the problem $(P_1)$ has at least two nonnegative nontrivial weak solutions in $W_0^{1,p}(\Omega) \cap L^\infty (\Omega)$.
\end{corollary}

\begin{proof}
We set $f(s) = (1 + s)^{p-1}(\ln(1+s))^q$ and $g(s) = s^m$. We can easily show that $f$ satisfies the assumptions $(A_1^p)$, $(A_3^p)$ and $(A_5^p)$ if $q > p - 1$. The function $g$ satisfies $(A_3^p)$ and $(A_5^p)$ if $m \in (0,p-1)$. Hence by using Theorem \ref{teo1} we obtain directly the result.
\end{proof}

The following result is an application of Theorem \ref{teo2}. 

Corollary 5.2. Under the assumptions \(\{A_{a,b}\}\). Let \(0 < q < p - 1\), \(0 < m \leq p - 1\), \(\|a^+\|_{L^k(\Omega)}\) and \(\|b^+\|_{L^\infty(\Omega)}\) be suitably small. Then, the problem \((P_1)\) has at least two nonnegative nontrivial weak solutions in \(W^{1,p}_0(\Omega) \cap L^\infty(\Omega)\).

Proof. As in the above proof, we set \(f(s) = (1 + s)^{p-1}(\ln(1 + s))^q\) and \(g(s) = s^m\). We can easily show that \(f\) satisfies the assumptions \((A^1_f)\), \((A^3_f)\) and \((A^3'_f)\) if \(0 < q < p - 1\). The function \(g\) satisfies \((A^1_g)\) and \((A^2'_g)\) if \(m \in (0, p - 1]\). Hence by using Theorem 2.3 we obtain directly the result.

5.2 Application 2: Quasilinear equations with a \(p\)-Gradient term

We consider the following problem

\[
(P_2) \begin{cases}
-\Delta_p u = a(x)u^q + |\nabla u|^p + b(x) & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\]

where \(q > 0\). This type of problem is now a very active field of research and it was investigated in various papers, for instance, \([7, 15, 30, 31]\) and the references therein. It is worth to mention that until now there is no result concerning the multiplicity result in the case where \(a\) changes of sign and for the problem \((P_2)\) with \(p > 1\).

We observe that the problem \((P_2)\) has no relation with the problem \((Q)\) due to the presence of the \(p\)-gradient term. However, if we perform the following Kazdan-Kramer change of variable,

\[
v = e^{\frac{u}{p-1}} - 1.
\]

We obtain the following problem

\[
(Q_2) \begin{cases}
-\Delta_p v = a(x)f(v) + b(x)g(v) & \text{in } \Omega, \\
v = 0 & \text{on } \partial\Omega,
\end{cases}
\]

where

\[
f(s) = (p-1)^{q-p+1}(1 + s)^{p-1}(\ln(1 + s))^q, \quad (5.2)
\]

and

\[
g(s) = \frac{(1 + s)^{p-1}}{(p-1)^{p-1}}, \quad (5.3)
\]

for all \(s \geq 0\).

In the following lemma we will prove that, if \(u\) is a nonnegative weak solution of \((P_2)\) then \(v\) defined by \((5.1)\) is a nonnegative weak solution of \((Q_2)\). In other words, to show the existence of at least two nonnegative weak solution of \((P_2)\) it is equivalent to show the existence of at least two nonnegative weak solution of \((Q_2)\).

Lemma 5.1. Under the assumption \(\{A_{a,b}\}\), let \(b \geq 0\), then

1. Every weak solution of \((Q_2)\) is positive.
2. If \(v \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)\) is a positive weak solution of \((Q')\) if only if

\[
u = (p-1)\ln(1 + v) \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)
\]

is a positive weak solution of \((P_2)\).
Proof. 1. See Lemma 4.2

2. Let \( v \) be a positive solution of \((Q_2)\). From the expression of \( f \) and \( g \) it is seen that \( v \) is a solution of

\[
- \Delta_p v = a(x)(p-1)^{-q+1}(1+v)^{p-1} \ln(1+v)^q + b(x) \frac{(1+v)^{p-1}}{(p-1)^{p-1}}.
\]  

(5.4)

Let \( u = (p-1) \ln(1+v) \), then \( \nabla u = (p-1) \frac{\nabla v}{1+v} \), since \( v \) is positive, we get easily from (5.1) that \( u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \) and positive. Let \( \phi \in W_0^{1,p}(\Omega) \), we set \( \varphi = \frac{(p-1)^{-q+1} \phi}{(1+v)^q} \), then \( \varphi \in W_0^{1,p}(\Omega) \). Hence, from (5.4), we have.

\[
\int_{\Omega} |\nabla v|^{p-2} \nabla v \nabla \varphi = (p-1)^{q-1} \int_{\Omega} a(x)(1+v)^{p-1} \ln(1+v)^q \varphi + \int_{\Omega} b(x) \frac{(1+v)^{p-1}}{(p-1)^{p-1}} \varphi.
\]  

(5.5)

Since

\[
\nabla \varphi = \frac{(p-1)^{-q+1} \nabla \phi}{(1+v)^q} - \frac{(p-1)^{-q+1} \phi \nabla v}{(1+v)^q},
\]

and

\[
\nabla v = \frac{\nabla u}{p-1} e^{u/(p-1)},
\]

Then

\[
\int_{\Omega} |\nabla v|^{p-2} \nabla v \nabla \varphi = \int_{\Omega} e^{u/(p-1)} |\nabla u|^{p-2} \nabla u \left( \frac{(p-1)^{-q+1} \nabla \phi}{(1+v)^q} - \frac{(p-1)^{-q+1} \phi \nabla v}{1+v} \right)
= \int_{\Omega} |\nabla u|^{p-2} \nabla u \left( \nabla \phi - \frac{(p-1)^{-q+1} \phi \nabla v}{1+v} \right)
= \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \phi - \int_{\Omega} |\nabla u|^{p} \phi.
\]  

(5.6)

On the other hand we have

\[
(p-1)^{q-1} \int_{\Omega} a(x)(1+v)^{p-1} \ln(1+v)^q \varphi = \int_{\Omega} a(x) u^q \varphi,
\]  

(5.7)

and

\[
\int_{\Omega} b(x) \frac{(1+v)^{p-1}}{(p-1)^{p-1}} \varphi = \int_{\Omega} b(x) \phi.
\]  

(5.8)

By combining (5.6), (5.7) and (5.8) we deduce that

\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \phi = \int_{\Omega} a(x) u^q \phi + \int_{\Omega} |\nabla u|^{p} \phi + \int_{\Omega} b(x) \phi,
\]

which mean that \( u \) is a positive weak solution of \((P_2)\). By the same arguments we prove the reverse statement.

Finally, let us give applications of Theorem 2.2 and Theorem 2.3 in the case where \( g(0) > 0 \) and \( b \geq 0 \). The following result is an application of Theorem 2.2.
Corollary 5.3. Under the assumptions $\{A_{a,b}\}$, let $q > p - 1$ and $b \geq 0$. If $\|b\|_{L^k(\Omega)}$ is suitably small, then, the problem $\left(Q_2\right)$ has at least two positive weak solutions in $W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$.

The following result is an application of Theorem 2.3.

Corollary 5.4. Under the assumptions $\{A_{a,b}\}$, let $0 < q \leq p - 1$ and $b \geq 0$. If $\|a^+\|_{L^k(\Omega)}$ and $\|b\|_{L^k(\Omega)}$ are suitably small, then the problem $\left(Q_2\right)$ has at least two positive weak solutions in $W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$.

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