Conformal blocks related to the R-R states in the $\hat{c} = 1$ SCFT

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Abstract

We derive an explicit form of a family of four-point Neveu-Schwarz blocks with $\hat{c} = 1$, external weights $\Delta_i = \frac{1}{8}$ and arbitrary intermediate weight $\Delta$. The derivation is based on a set of identities obeyed in the free superscalar theory by correlation functions of fields satisfying Ramond condition with respect to the bosonic (dimension 1) and the fermionic (dimension $\frac{1}{2}$) currents.

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1 Introduction

Conformal field theory proved to be very efficient in describing second order phase transitions in two-dimensional system and is accepted as a language of string theory. Correlation functions in CFT can be expressed as sums (or integrals) of three-point coupling constants and the conformal blocks, fully determined by the symmetry alone. The basic role played by the blocks has been recognized since the appearance of the ground-breaking BPZ work [1].

In spite of the progress achieved over the years the analytic form of the general block is unknown and the explicit examples are mostly limited to the blocks corresponding to degenerate representations of the underlying Virasoro algebra or the blocks appearing in the correlation functions of free fields. An interesting example of the latter kind is a family of conformal blocks related to the Ramond states of a free scalar fields [2, 3]. The exact analytic results of [2, 3] play an essential rôle in developing the so called elliptic recursion representation of the general conformal block [4, 5].

Recently the recursion representations have been also worked out for the super-conformal blocks related to the Neveu-Schwarz algebra [6–8]. In particular the elliptic recursion has been conjectured and used for the numerical verifications of the consistency of $N = 1$ super-Liouville theory [8]. One of the missing steps of a possible rigorous proof of this conjecture is a derivation of the large intermediate weight asymptotic of the general NS superconformal block. It can be obtained by a method parallel to the one used in [4] in the Virasoro case. This however requires explicit analytic formulae of certain superconformal blocks and was the main motivation behind the present work.

The organization of the paper is as follows. In Section 2 we briefly describe the structure of $\hat{c} = 1$ free superscalar theory extended by the Ramond states both in the bosonic and the fermionic sector. In Section 3 we derive a set of relations for the 4-point functions which are used in Section 4 to derive a closed system of 6 equations for superconformal blocks. The solutions to these equations provide new nontrivial examples of NS superconformal blocks and are the main result of the present work.

2 Holomorphic currents

The free bosonic current $j(z)$ (with conformal weights $\Delta = 1$, $\bar{\Delta} = 0$) satisfies the relation:

$$j(z)j(z') \sim \frac{1}{(z - z')^2}.$$
Following [3] one may consider two types of states of a free scalar field: the NS states $|\xi\rangle_{NS}$ for which
\[ j(z) |\xi\rangle_{NS} = \sum_{n \in \mathbb{Z}} z^{-n-\frac{1}{2}} j_n |\xi\rangle_{NS}, \quad [j_n, j_m] = m\delta_{n+m}, \quad (1) \]
and the R states $|\xi\rangle_R$ characterized by
\[ j(z) |\xi\rangle_R = \sum_{k \in \mathbb{Z}+\frac{1}{2}} z^{-k-1} j_k |\xi\rangle_R, \quad [j_k, j_l] = k\delta_{k+l}. \quad (2) \]
The space of states $\mathcal{B}$ is (by construction) a direct sum
\[ \mathcal{B} = \left( \bigoplus_p \mathcal{B}_{p}^{NS} \right) \oplus \mathcal{B}^R \]
where $\mathcal{B}_{p}^{NS}$ are the NS current modules defined as a highest weight representations of the algebra (1) with the highest weight state
\[ j_0 |p\rangle_{NS} = p |p\rangle_{NS}, \quad j_n |p\rangle_{NS} = 0, \quad n \in \mathbb{N}, \quad (3) \]
and $\mathcal{B}^R$ is the R current module defined as a highest weight representation of the algebra (2) with the highest weight state
\[ j_k |0\rangle_R = 0, \quad k \in \mathbb{N} - \frac{1}{2}. \quad (4) \]
We shall use a similar construction for the free fermion current, defined by the OPE
\[ \psi(z)\psi(z') \sim \frac{1}{z-z'}. \]
In this case
\[ \psi(z) |\zeta\rangle_{NS} = \sum_{k \in \mathbb{Z}+\frac{1}{2}} z^{-k-\frac{1}{2}} \psi_k |\zeta\rangle_{NS}, \quad \{\psi_k, \psi_l\} = \delta_{k+l}, \quad (5) \]
\[ \psi(z) |\zeta\rangle_R = \sum_{n \in \mathbb{Z}} z^{-n-\frac{1}{2}} \psi_n |\zeta\rangle_R, \quad \{\psi_n, \psi_m\} = \delta_{n+m}. \quad (6) \]
The space of states $\mathcal{F}$ is a direct sum of the fermionic NS current module $\mathcal{F}^{NS}$ and the fermionic R current module $\mathcal{F}^R$ built on the highest weight states $|0\rangle_{NS}$ and $|+\rangle_R$ of the algebras (5) and (6), respectively, defined by the relations:
\[ \psi_k |0\rangle_{NS} = 0, \quad k \in \mathbb{N} - \frac{1}{2}, \]
\[ \psi_0 |+\rangle_R = \frac{1}{\sqrt{2}} |\right\rangle_R, \quad \psi_n |+\rangle_R = 0, \quad n \in \mathbb{N}. \]
The tensor product $\mathcal{B} \otimes \mathcal{F}$ decomposes into the direct sum
\[ \mathcal{B} \otimes \mathcal{F} = \left( \bigoplus_p \mathcal{B}_{p}^{NS} \otimes \mathcal{F}^{NS} \right) \oplus \mathcal{B}^R \otimes \mathcal{F}^R \oplus \left( \bigoplus_p \mathcal{B}_{p}^{NS} \otimes \mathcal{F}^R \right) \oplus \mathcal{B}^R \otimes \mathcal{F}^{NS} \]


of highest weight supercurrent modules. The Sugawara construction

\[ T(z) = \frac{1}{2} : j(z) j(z) : - \frac{1}{2} : \psi(z) \partial \psi(z) :, \]

\[ S(z) = j(z) \psi(z), \]

defines on the first summand a free field representation of the NS superconformal algebra with the central charge \( \hat{c} = \frac{2}{3} c = 1 \). In this sector

\[ T(z) = \sum_{n \in \mathbb{Z}} z^{-n-2} L_n, \quad S(z) = \sum_{k \in \mathbb{Z} + \frac{1}{2}} z^{-k-\frac{3}{2}} S_k, \]

where

\[ L_0 = \frac{1}{2} j_0^2 + \sum_{n \in \mathbb{N}} j_{-n} j_n + \sum_{k \in \mathbb{N} - \frac{1}{2}} (k + \frac{1}{2}) \psi_{-k} \psi_k, \]

\[ L_m = \frac{1}{2} \sum_{n \in \mathbb{Z}} j_{m-n} j_n + \frac{1}{4} \sum_{k \in \mathbb{Z} + \frac{1}{2}} (2k - m) \psi_{m-k} \psi_k, \quad m \neq 0, \]

\[ S_k = \sum_{n \in \mathbb{Z}} j_n \psi_{k-n}, \]

on the subspace \( \bigoplus_p B^N_S \otimes \mathcal{F}^N_S \) and

\[ L_0 = \sum_{k \in \mathbb{N} - \frac{1}{2}} j_{-k} j_k + \sum_{n \in \mathbb{N}} (n + \frac{1}{2}) \psi_{-n} \psi_n + \frac{1}{8}, \]

\[ L_m = \frac{1}{2} \sum_{k \in \mathbb{Z} + \frac{1}{2}} j_{m-k} j_k + \frac{1}{4} \sum_{n \in \mathbb{Z}} (2n - m) \psi_{m-n} \psi_n, \quad m \neq 0, \]

\[ S_k = \sum_{n \in \mathbb{Z}} \psi_n j_{k-n} \]

on \( B^R \otimes \mathcal{F}^R \).

One easily verifies that the NS-NS supercurrent module \( B^N_S \otimes \mathcal{F}^N_S \) is an NS superconformal Verma module with the conformal weight \( \Delta_p = \frac{p^2}{2} \). We shall denote the corresponding superprimary field by \( \varphi_p(z) \):

\[ \varphi_p(0) \left| 0 \right> = \left| p \right>_{NS} \otimes \left| 0 \right>_{NS} \equiv \nu_p, \]

where \( \left| 0 \right> = \left| 0 \right>_{NS} \otimes \left| 0 \right>_{NS} \in B^N_S \otimes \mathcal{F}^N_S \) is the “true” vacuum.

On the other hand in the R-R supercurrent module \( B^R \otimes \mathcal{F}^R \) one has two superprimary states at each \( \frac{n(n+1)}{2} \) level. Indeed, since all \( \hat{c} = 1, \Delta_n = \frac{1}{2} (n + \frac{1}{2})^2 \) NS superconformal Verma modules are not degenerate the superconformal content of the R-R module can be inferred from the ratio

\[ \frac{\chi_{RR}(t)}{\chi_{cc}(t)} = 2 \sum_{n=0}^{\infty} t^{\frac{n(n+1)}{2}} \]
where $\chi_{RR}(t)$ is the character of $B^R \otimes F^R$,
\[
\chi_{RR}(t) = 2t^\frac{1}{2} \prod_{k=1}^{\infty} \frac{1 + t^k}{1 - t^{2k-1}},
\]
and
\[
\chi_{c}(t) = t^\frac{1}{2} \prod_{k=1}^{\infty} \frac{1 + t^k}{1 - t^{2k-1}}
\]
is the character of the superconformal NS module.

The R-R module is thus a direct sum of irreducible NS superconformal Verma modules with conformal weights
\[
\Delta_n = \frac{1}{2} \left( n + \frac{1}{2} \right)^2, \quad n = 0, 1, \ldots ,
\]
each weight appearing twice in the sum. We shall denote the corresponding superprimary fields by $\chi^\pm_n(z)$. In particular
\[
\chi^+_0(0) |0\rangle = |0\rangle_R \otimes |\pm\rangle_R \equiv \chi^+_0.
\]
It is also easy to check that the super-primary field with the (left) weight $\Delta_1$ can be expressed as:
\[
\chi^+_1(z) = \frac{1}{2} \left( j^2_{-\frac{1}{2}} - \psi_{-1}\psi_0 \right) \chi^+_0(z).
\] (8)

All the notions related to the superconformal algebra like primary and descended fields, families, blocks, operators-states correspondence etc, have their counterparts in the case of the supercurrent algebra. One can show in particular that the 3-point function of fields from arbitrary supercurrent families factorizes into a product of 3-point supercurrent blocks:
\[
\langle \phi_3(\xi_3, \bar{\xi}_3|z_3, \bar{z}_3)\phi_2(\xi_2, \bar{\xi}_2|z_2, \bar{z}_2)\phi_1(\xi_1, \bar{\xi}_1|z_1, \bar{z}_1) \rangle = \eta_{z_3 \bar{z}_3 z_2 \bar{z}_2}(\xi_3, \bar{\xi}_3, \xi_2, \bar{\xi}_2, \xi_1, \bar{\xi}_1).
\] (9)
The form $\eta$ is a nontrivial extension of the 3-point superconformal block in the case of one arbitrary NS-NS and two R-R supercurrent modules:
\[
\eta_{z_3 z_2 z_1}(\xi, \zeta, \zeta^{'}) , \quad \xi, \zeta^{'}, \zeta^{'}) \in B^R \otimes F^R.
\]
It is uniquely determined by Ward identities for currents $j(z)$ and $\psi(z)$. Since in the free superscalar theory the left and the right fermionic parities\(^4\) are independently preserved the form $\eta$ is necessarily even, i.e. it vanishes identically if total parity of all arguments is odd.

\(^4\)In all supercurrent modules the parity is defined by the number of fermionic excitations.
If the states $\zeta, \zeta'$ belong to definite superconformal Verma modules the form $\eta$ can be also calculated using the superconformal Ward identities. For instance, for even vectors

$$\nu_{p,KM} = S_{-K} L_{-M} \nu_p \equiv S_{-k_1} \ldots S_{-k_l} L_{-m_1} \ldots L_{-m_j} \nu_p, \quad |K| \in \mathbb{N} \cup \{0\},$$

one has:

$$\eta_{z_3 z_2 z_1} (\nu_{p,KM}, \chi_{m}^\pm, \chi_{n}^\pm) = \eta_{\infty} \ 1 0 (\nu_p, \chi_{m}^\pm, \chi_{n}^\pm) \rho_{z_3 z_2 z_1} (\nu_{p,KM}, \chi_{m}, \chi_{n}),$$

and for odd ones ($|K| \in \mathbb{N} - \frac{1}{2}$):

$$\eta_{z_3 z_2 z_1} (\nu_{p,KM}, \chi_{m}^\pm, \chi_{n}^\pm) = \eta_{\infty} \ 1 0 (\nu_p, \chi_{m}^\pm, \chi_{n}^\pm) \rho_{z_3 z_2 z_1} (\nu_{p,KM}, \chi_{m}, \chi_{n}),$$

The form $\rho$ in the formulae above is the normalized 3-point superconformal block introduced in [6] and $\chi_m$ stands for the highest weight state in the superconformal Verma module with the central charge $c = \frac{3}{2}$ and the conformal weight $\Delta_m = \frac{1}{2} \left( m + \frac{1}{2} \right)^2$.

### 3 Relations for the correlation functions of R-R fields

We derive now equations for some 4-point correlation functions which will be used in the next section to obtain equations for the conformal blocks $\mathcal{F}_{\Delta_p}^1 \left[ -\frac{\Delta_1}{\Delta_4}, \frac{\Delta_2}{\Delta_4} \right] (z)$ and $\mathcal{F}_{\Delta_p}^2 \left[ -\frac{\Delta_3}{\Delta_4}, \frac{\Delta_4}{\Delta_4} \right] (z)$. The derivation is based on a supersymmetric extension of the technique [3].

Consider the correlation function with an arbitrary pattern of upper signs

$$\left\langle \chi_0^+(z_4) \chi_0^+(z_3) j_0 \chi_0^+(z_2) \chi_0^+(z_1) \right\rangle \overset{\text{def}}{=} \oint_{c_{[z_2,z_1]}} \frac{d\zeta}{2\pi i} \left\langle j(\xi) \chi_0^+(z_4) \chi_0^+(z_3) \chi_0^+(z_2) \chi_0^+(z_1) \right\rangle,$$

where the positively oriented integration contour encloses points $z_1$ and $z_2$. Equations (2) and (4) give the OPE of the primary field $\chi_0^+(z)$ and the current $j(\xi)$:

$$j(\xi) \chi_0^+(z) \sim \frac{1}{\sqrt{\xi - z}} j_\frac{1}{2} \chi_0^+(z).$$

The function

$$\left\langle j(\xi) \chi_0^+(z_4) \chi_0^+(z_3) \chi_0^+(z_2) \chi_0^+(z_1) \right\rangle \sqrt{\xi - z_4} (\xi - z_3) (\xi - z_2) (\xi - z_1)$$
is therefore a single valued, holomorphic function of $\xi$. Since any correlator of $j(\xi)$ with no operator insertion at infinity falls like $\xi^{-2}$ for large $\xi$ this function is a constant, hence

$$\langle j(\xi)\chi_0^+(z_4)\chi_0^+(z_3)\chi_0^+(z_2)\chi_0^-(z_1) \rangle = \frac{A(z_i)}{\sqrt{(\xi - z_4)(\xi - z_3)(\xi - z_2)(\xi - z_1)}}.$$  

Expanding the r.h.s. of this equation around $\xi = z_2$ and comparing the result with the OPE (13) we get

$$A(z_i) = \sqrt{z_21z_23z_24} \langle \chi_0^+(z_4)\chi_0^+(z_3)j_{-\frac{1}{2}}\chi_0^+(z_2)\chi_0^-(z_1) \rangle.$$  

The integral on the r.h.s. of (12) can be now performed explicitly and (12) takes the form:

$$\langle \chi_0^+(z_4)\chi_0^+(z_3)j_0\chi_0^+(z_2)\chi_0^+(z_1) \rangle = \sqrt{z_21z_23z_24} K(z_i) \langle \chi_0^+(z_4)\chi_0^+(z_3)j_{-\frac{1}{2}}\chi_0^+(z_2)\chi_0^+(z_1) \rangle,$$

where

$$z = \frac{z_21z_4}{z_31z_24}$$

is the four-point projective invariant and

$$K(z_i) = \oint_{c_{[z_2,z_1]}} \frac{d\xi}{2\pi i} \frac{1}{\sqrt{(\xi - z_1)(\xi - z_3)(\xi - z_2)(\xi - z_4)}} = \frac{2K(z)}{\pi \sqrt{z_31z_24}},$$

with

$$K(z) = \int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - tz)}}$$

being the complete elliptic integral of the first kind.

Using the algebra of modes $j_k$ and $\psi_m$ and relations (7) and (8) one gets:

$$j(\xi)j_{-\frac{1}{2}}\chi_0^+(z) \sim \frac{1}{2(\xi - \frac{z}{\sqrt{2}})^2} \chi_0^+(z) + \frac{1}{\sqrt{\xi - z}} L_{-1}\chi_0^+(z) + \frac{1}{\sqrt{\xi - z}} \chi_1^+(z).$$  

It then follows from OPE-s (13) and (15) that

$$\langle j(\xi)\chi_0^+(z_4)\chi_0^+(z_3)j_{-\frac{1}{2}}\chi_0^+(z_2)\chi_0^+(z_1) \rangle \sqrt{(\xi - z_4)(\xi - z_3)(\xi - z_2)(\xi - z_1)},$$

considered as a function of $\xi$, is holomorphic on $\mathbb{C} \setminus \{z_2\}$, vanishes at infinity like $\xi^{-2}$ and has a simple pole at $\xi = z_2$, hence

$$\langle j(\xi)\chi_0^+(z_4)\chi_0^+(z_3)j_{-\frac{1}{2}}\chi_0^+(z_2)\chi_0^+(z_1) \rangle = \frac{1}{\sqrt{(\xi - z_4)(\xi - z_3)(\xi - z_2)(\xi - z_1)}} \left(\frac{B(z_i)}{\xi - z_2} + C(z_i)\right).$$
Expanding the r.h.s. of (16) around \( \xi = z_2 \) and comparing with (15) we get:

\[
B(z_i) = \frac{1}{2} \sqrt{z_{21}z_{23}z_{24}} \left\langle \chi_0^+(z_4)\chi_0^+(z_3)\chi_0^+(z_2)\chi_0^+(z_1) \right\rangle, 
\]

\[
C(z_i) = \sqrt{z_{21}z_{23}z_{24}} \left[ \left\langle \chi_0^+(z_4)\chi_0^+(z_3)L_{-1}\chi_0^+(z_2)\chi_0^+(z_1) \right\rangle + \left\langle \chi_0^+(z_4)\chi_0^+(z_3)\chi_1^+(z_2)\chi_0^+(z_1) \right\rangle \right. 
+ \left. \frac{1}{4} \left( \frac{1}{z_{21}} + \frac{1}{z_{23}} + \frac{1}{z_{24}} \right) \left\langle \chi_0^+(z_4)\chi_0^+(z_3)\chi_0^+(z_2)\chi_0^+(z_1) \right\rangle \right]. 
\]

Inserting this into (16) and integrating along \( C_{[z_2,z_1]} \) one obtains:

\[
\left\langle \chi_0^+(z_4)\chi_0^+(z_3)j_0j_{-\frac{1}{2}}\chi_0^+(z_2)\chi_0^+(z_1) \right\rangle 
= (z_{21}z_{32}z_{42})^{\frac{1}{2}} \frac{\partial}{\partial z_2} \left[ (z_{21}z_{32}z_{42})^{\frac{1}{2}} K(z_i) \left\langle \chi_0^+(z_4)\chi_0^+(z_3)\chi_0^+(z_2)\chi_0^+(z_1) \right\rangle \right] 
+ (z_{21}z_{32}z_{42})^{\frac{1}{2}} K(z_i) \left\langle \chi_0^+(z_4)\chi_0^+(z_3)\chi_1^+(z_2)\chi_0^+(z_1) \right\rangle 
\]

where we used the CWI

\[
L_{-1}\chi_0^+(z) = \partial_z \chi_0^+(z).
\]

Another set of equations for correlation functions can be obtained using the OPE-s:

\[
\sqrt{2}\psi(\xi) \chi_0^+(z) \sim \frac{1}{\sqrt{\xi - z}} \chi_0^+(z), 
\]

(18)

\[
\sqrt{2}\psi(\xi) \chi_1^+(z) \sim -\frac{1}{2(\xi - z)^{\frac{1}{2}}} \chi_0^+(z) + \frac{1}{\sqrt{\xi - z}} L_{-1}\chi_0^+(z), 
\]

(19)

which can be easily derived from the algebra of modes \( \psi_m \) together with the relations (7) and (8). It follows from (18) and (19) that

\[
\frac{\left\langle \sqrt{2}\psi(\xi) \chi_0^+(z_4)\chi_0^+(z_3)\chi_m^+(z_2)\chi_0^+(z_1) \right\rangle}{\sqrt{(\xi - z_1)(\xi - z_2)(\xi - z_3)(\xi - z_4)}} \]

(20)

is an analytic function of \( \xi \), with poles at the locations \( z_i \) and vanishing at infinity faster than \( \xi^{-1} \). Sum of its residues must therefore vanish and with the help of (18) we get in particular

\[
0 = \frac{\left\langle \chi_0^-(z_4)\chi_0^+(z_3)\chi_0^+(z_2)\chi_0^-(z_1) \right\rangle}{\sqrt{z_{41}z_{42}z_{43}}} + \frac{\left\langle \chi_0^+(z_4)\chi_0^-(z_3)\chi_0^+(z_2)\chi_0^-(z_1) \right\rangle}{\sqrt{z_{31}z_{32}z_{34}}} 
+ \frac{\left\langle \chi_0^+(z_4)\chi_0^+(z_3)\chi_0^-(z_2)\chi_0^-(z_1) \right\rangle}{\sqrt{z_{21}z_{23}z_{24}}} + \frac{\left\langle \chi_0^+(z_4)\chi_0^+(z_3)\chi_0^+(z_2)\chi_0^+(z_1) \right\rangle}{\sqrt{z_{12}z_{13}z_{14}}} 
\]

(21)

for \( m = 0 \) and
\[-(z_2 z_3 z_4)^{-\frac{3}{4}} \frac{\partial}{\partial z_2} \left[ (z_2 z_3 z_4)^{\frac{1}{4}} \langle \lambda_0^+ (z_4) \lambda_0^- (z_3) \lambda_1^- (z_2) \lambda_0^- (z_1) \rangle \right] = \]
\[= (z_1 z_2 z_3 z_4)^{-\frac{1}{4}} \langle \lambda_0^- (z_4) \lambda_0^+ (z_3) \lambda_1^+ (z_2) \lambda_0^- (z_1) \rangle + (z_2 z_3 z_4 z_4)^{-\frac{1}{4}} \langle \lambda_0^- (z_4) \lambda_0^- (z_3) \lambda_1^+ (z_2) \lambda_0^- (z_1) \rangle + (z_1 z_2 z_3 z_4)^{-\frac{1}{4}} \langle \lambda_0^+ (z_4) \lambda_0^+ (z_3) \lambda_1^- (z_2) \lambda_0^- (z_1) \rangle \]  \hspace{1cm} (22)

for \( m = 1 \).

One more set of equations can be derived integrating (16) around \( z_3 \):

\[\langle \chi_0^+ (z_4) j_{-\frac{1}{2}} \chi_0^+ (z_3) j_{-\frac{1}{2}} \chi_0^+ (z_2) \rangle \]
\[= \int \frac{d \xi}{2 \pi i} \frac{1}{\sqrt{\xi - z_3}} \left\langle \chi_0^+ (z_4) \chi_0^+ (z_3) j_{-\frac{1}{2}} \chi_0^+ (z_2) \right\rangle \]
\[= \sqrt{(z_2 z_3 z_4 z_4)} \left\langle \frac{\partial}{\partial z_2} (\chi_0^+ (z_4) \chi_0^+ (z_3) \chi_0^+ (z_2) \chi_0^- (z_1)) + (\chi_0^- (z_4) \chi_0^+ (z_3) \chi_0^+ (z_2) \chi_0^- (z_1)) \right\rangle + \frac{1}{4} \left( \frac{1}{z_2} + \frac{1}{z_3} + \frac{1}{z_4} \right) \langle \chi_0^+ (z_4) \chi_0^+ (z_3) \chi_0^- (z_2) \rangle \] \hspace{1cm} (23)

\section{4 Superconformal blocks for R-R weights}

Any 4-point function of R-R operators factorizes on NS-NS states. Since in this case the supercurrent and the superconformal modules coincide one has

\[\langle \chi_0^+ (\infty) \chi_0^+ (1) j_0 \chi_0^+ (z) \chi_0^+ (0) \rangle \]
\[= \sum_p \sum_{K,M,L,N} \eta_{\infty \chi_0^+ \chi_0^+ \nu_{p,KM}} B^{K,M,L,N} \eta_{\infty \chi_0^+ \chi_0^+}, \]

where due to the left parity conservation the sum runs over even states, \(|K|, |L| \in \mathbb{N} \cup \{0\}\). Taking into account factorization properties (10) and the definitions of the superconformal blocks given in [6] one gets

\[\langle \chi_0^+ (\infty) \chi_0^+ (1) j_0 \chi_0^+ (z) \chi_0^+ (0) \rangle \]
\[= \sum_p \eta_{\infty \chi_0^+ \chi_0^+ \nu_{p,KM}} \eta_{\infty \chi_0^+ \chi_0^+} \times \sum_{K,M,L,N} \rho_{\infty \chi_0^+ \chi_0^+ \nu_{p,KM}} B^{K,M,L,N} \rho_{\infty \chi_0^+ \chi_0^+} \]
\[= \sum_p C_p \left[ \Delta_0 \Delta_0 \Delta_0 \right] (z), \]
where \( C_p \equiv \eta_{\infty \times 10}(\chi_0^+, \chi_0^+, \nu_p)\eta_{\infty \times 10}(\nu_p, \chi_0^+, \chi_0^+) \). On the other hand using the relation
\[
\eta_{\infty \times 10}(\nu_p, j_{-\frac{1}{2}} \chi_0^+, \chi_0^+) = p \eta_{\infty \times 10}(\nu_p, \chi_0^+, \chi_0^+)
\]
shown in the Appendix (A.2) and the formula
\[
j_{-\frac{1}{2}} \chi_0^+ = \sqrt{2} S_{-\frac{1}{2}} \chi_0^+
\]
one obtains
\[
\langle \chi_0^+(\infty) \chi_0^+ (1) j_{-\frac{1}{2}} \chi_0^+ (z) \chi_0^+(0) \rangle = \sum_p p C_p F_{\Delta p}^1 \left[ \begin{array} {c} \Delta_0 \\ \Delta_0 \Delta_0 \end{array} \right] (z).
\]

For the sake of brevity we have ignored so far the \( \bar{z} \) dependence of the correlation functions. If we choose in the right sector the fields \( \chi_0^+(\bar{z}) \), the \( \bar{z} \) dependence of the correlator is described by the anti-holomorphic factor
\[
C_p F_{\Delta p}^1 \left[ \begin{array} {c} \Delta_0 \\ \Delta_0 \Delta_0 \end{array} \right] (\bar{z})
\]
for each \( p \). These factors are the same in both correlation functions appearing in equation (14). Since they are linearly independent one gets from (14) the following equation for superconformal blocks:
\[
F_{\Delta p}^1 \left[ \begin{array} {c} \Delta_0 \\ \Delta_0 \Delta_0 \end{array} \right] (z) = \frac{2K(z)}{\pi} \sqrt{z(1-z)} F_{\Delta p}^1 \left[ \begin{array} {c} \Delta_0^* \Delta_0 \\ \Delta_0 \Delta_0 \end{array} \right] (z).
\]

The function \( \langle \chi_0^+(\infty) \chi_0^- (1) j_0 \chi_0^+(z) \chi_0^-(0) \rangle \) factorizes on odd states. Using formulae (A.1) and (A.2) one has in this case
\[
\langle \chi_0^+(\infty) \chi_0^- (1) j_0 \chi_0^+(z) \chi_0^-(0) \rangle = \sum_p p \eta_{\infty \times 10}(\chi_0^+, S_{-\frac{1}{2}} \chi_0^+, \nu_p) \eta_{\infty \times 10}(\nu_p, S_{-\frac{1}{2}} \chi_0^+, \chi_0^- + 0)
\]
\[
\times \sum_{K,M,L,N} \rho_{\infty \times 10}(\chi_0, \chi_0, \nu_p, K,M) B_{K,M,L,N}^{K,M,L,N} \rho_{\infty \times 10}(j_0 \nu_p, L,N, \chi_0, \chi_0)
\]
\[
= \sum_p p \Delta_p \Delta_p C_p F_{\Delta p}^{1/2} \left[ \begin{array} {c} \Delta_0 \\ \Delta_0 \Delta_0 \end{array} \right] (z)
\]
and
\[
\langle \chi_0^+(\infty) \chi_0^- (1) j_{-\frac{1}{2}} \chi_0^+(z) \chi_0^- (0) \rangle = \sum_p p C_p F_{\Delta p}^{1/2} \left[ \begin{array} {c} \Delta_0^* \Delta_0 \\ \Delta_0 \Delta_0 \end{array} \right] (z),
\]

which yields
\[
\Delta_p F_{\Delta p}^{1/2} \left[ \begin{array} {c} \Delta_0 \Delta_0 \\ \Delta_0 \Delta_0 \end{array} \right] (z) = \frac{2K(z)}{\pi} \sqrt{z(1-z)} F_{\Delta p}^{1/2} \left[ \begin{array} {c} \Delta_0^* \Delta_0 \\ \Delta_0 \Delta_0 \end{array} \right] (z).
\]

Essentially the same method can be applied to the equation (17). This and relations (A.2), (A.5) lead to the equations:
Integrating (30) we get

\[ 2 \Delta_p F_{\Delta p}^{1} \left[ \Delta_0^* \Delta_0 \right] (z) = [z(1 - z)]^{\frac{4}{p}} \frac{\partial}{\partial z} \left[ \frac{2K(z)}{\pi} [z(1 - z)]^{\frac{1}{p}} F_{\Delta p}^{1} \left[ \Delta_0^* \Delta_0 \right] (z) \right] \]  

(26)

\[ + \Delta_p \frac{2K(z)}{\pi} [z(1 - z)]^{\frac{1}{p}} F_{\Delta p}^{1} \left[ \Delta_0^* \Delta_0 \right] (z), \]

\[ 2 F_{\Delta p}^{\frac{1}{2}} \left[ \Delta_0^* \Delta_0 \right] (z) = [z(1 - z)]^{\frac{1}{p}} \frac{\partial}{\partial z} \left[ \frac{2K(z)}{\pi} [z(1 - z)]^{\frac{1}{p}} F_{\Delta p}^{\frac{1}{2}} \left[ \Delta_0^* \Delta_0 \right] (z) \right] \]  

(27)

\[ + (\Delta_p - \frac{1}{2}) \frac{2K(z)}{\pi} [z(1 - z)]^{\frac{1}{p}} F_{\Delta p}^{\frac{1}{2}} \left[ \Delta_0^* \Delta_0 \right] (z). \]

The next two equations can be obtained from (21) and (22), respectively:

\[ \Delta_p F_{\Delta p}^{\frac{1}{2}} \left[ \Delta_0 \Delta_0 \right] (z) = (1 - \sqrt{1 - z}) z^{-\frac{1}{2}} F_{\Delta p}^{1} \left[ \Delta_0 \Delta_0 \right] (z) \]  

(28)

\[ [z(1 - z)]^{\frac{1}{2}} \frac{\partial}{\partial z} \left[ [z(1 - z)]^{\frac{1}{2}} F_{\Delta p}^{1} \left[ \Delta_0 \Delta_0 \right] (z) \right] = \]  

\[ = \Delta_p (\Delta_p - \frac{1}{2}) \sqrt{z} F_{\Delta p}^{\frac{1}{2}} \left[ \Delta_0 \Delta_0 \right] (z) + \Delta_p \frac{2K(z)}{\pi} \sqrt{1 - z} F_{\Delta p}^{1} \left[ \Delta_0 \Delta_0 \right] (z). \]  

(29)

Formulae (24) – (28) allow to express the functions \( F_{\Delta p}^{1} \left[ \Delta_0 \Delta_1 \right] (z), \) \( F_{\Delta p}^{1} \left[ \Delta_0^* \Delta_0 \right] (z) \) and \( F_{\Delta p}^{\frac{1}{2}} \left[ \Delta_0 \Delta_0 \right] (z) \) in terms of \( F_{\Delta p}^{1} \left[ \Delta_0 \Delta_0 \right] (z). \) Using (29) we then arrive at the equation

\[ \frac{dG_p(z)}{dz} = \left[ \frac{\pi^2 \Delta_p}{4z(1 - z) K^2(z)} - \frac{1 - \sqrt{1 - z}}{4z\sqrt{1 - z}} \right] G_p(z), \]  

(30)

where

\[ G_p(z) = [z(1 - z)]^{\frac{1}{p}} \left( \frac{2K(z)}{\pi} \right)^{\frac{1}{2}} F_{\Delta p}^{1} \left[ \Delta_0 \Delta_0 \right] (z). \]

Integrating (30) we get

\[ F_{\Delta p}^{1} \left[ \Delta_0 \Delta_0 \right] (z) = (16q)^{\Delta_p} \left( \frac{1 + \sqrt{1 - z}}{2} \right)^{\frac{1}{p}} [z(1 - z)]^{-\frac{1}{p}} \left( \frac{\pi}{2K(z)} \right)^{\frac{1}{2}}. \]

Using relations:

\[ \frac{2K(z)}{\pi} = \theta_3(q), \quad \left( \frac{1 + \sqrt{1 - z}}{2} \right)^{\frac{1}{p}} \theta_3(q) = \theta_3(q^2), \quad \left( \frac{1 - \sqrt{1 - z}}{2} \right)^{\frac{1}{p}} \theta_3(q) = \theta_2(q^2) \]

where theta functions are defined in the standard way:

\[ \theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \quad \theta_2(q) = \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2} \]

one finally obtains:
\[ F^{1}_{\Delta p} \left[ \frac{\Delta_0}{\Delta_0} \right] (z) = [z(1-z)]^{-\frac{3}{4}} (16q)^{\Delta_p} \frac{\Delta_p}{\Delta_p} \theta_3^{-2}(q) \theta_3(q^2), \] (31)

\[ F^{\frac{1}{2}}_{\Delta p} \left[ \frac{\Delta_0}{\Delta_0} \right] (z) = [z(1-z)]^{-\frac{3}{4}} (16q)^{\Delta_p} \frac{\Delta_p}{\Delta_p} \theta_3^{-2}(q) \theta_2(q^2), \] (32)

\[ F^{1}_{\Delta p} \left[ \frac{\Delta_0}{\Delta_0} \right] (z) = [z(1-z)]^{-\frac{3}{4}} (16q)^{\Delta_p} \frac{\Delta_p}{\Delta_p} \theta_3^{-4}(q) \theta_3(q^2), \] (33)

\[ F^{\frac{1}{2}}_{\Delta p} \left[ \frac{\Delta_0}{\Delta_0} \right] (z) = [z(1-z)]^{-\frac{3}{4}} (16q)^{\Delta_p} \frac{\Delta_p}{\Delta_p} \theta_3^{-4}(q) \theta_2(q^2), \] (34)

\[ F^{1}_{\Delta p} \left[ \frac{\Delta_0}{\Delta_0} \right] (z) = [z(1-z)]^{-\frac{3}{4}} (16q)^{\Delta_p} \frac{\Delta_p}{\Delta_p} \theta_3^{-6}(q) \left( \frac{\partial}{\partial q} \theta_3(q^2) \theta_2(q^2) \right), \] (35)

\[ F^{\frac{1}{2}}_{\Delta p} \left[ \frac{\Delta_0}{\Delta_0} \right] (z) = [z(1-z)]^{-\frac{3}{4}} (16q)^{\Delta_p} \frac{\Delta_p}{\Delta_p} \theta_3^{-6}(q) \left( \frac{\partial}{\partial q} \theta_2(q^2) \theta_2(q^2) \right). \] (36)

Equations for functions \( F^{l}_{\Delta_p} \left[ \frac{\Delta_0}{\Delta_0} \right] (z) \) can be obtained from (23) using the relations (A.2), (A.5), (A.6):

\[ \frac{2\Delta_p}{\sqrt{z}} F^{1}_{\Delta_p} \left[ \frac{\Delta_0}{\Delta_0} \right] (z) = \left( \frac{\partial}{\partial z} + \frac{1}{4z(1-z)} \right) F^{1}_{\Delta_p} \left[ \frac{\Delta_0}{\Delta_0} \right] (z) + \Delta_p F^{1}_{\Delta_p} \left[ \frac{\Delta_0}{\Delta_0} \right] (z), \]

\[ \frac{2\Delta_p}{\sqrt{z}} F^{\frac{1}{2}}_{\Delta_p} \left[ \frac{\Delta_0}{\Delta_0} \right] (z) = \left( \frac{\partial}{\partial z} + \frac{1}{4z(1-z)} \right) F^{\frac{1}{2}}_{\Delta_p} \left[ \frac{\Delta_0}{\Delta_0} \right] (z) + (\Delta_p - \frac{1}{2}) F^{\frac{1}{2}}_{\Delta_p} \left[ \frac{\Delta_0}{\Delta_0} \right] (z). \]

From the results (31), (35) and (32), (36) one gets, respectively:

\[ F^{1}_{\Delta_p} \left[ \frac{\Delta_0}{\Delta_0} \right] (z) = z^{-\frac{3}{4}}(1-z)^{-\frac{3}{4}} (16q)^{\Delta_p} \frac{\Delta_p}{\Delta_p} \theta_3(q^2) \theta_3(q^2) \left( 1 - \frac{q}{\Delta_p} \theta_3^{-1}(q) \frac{\partial \theta_3(q)}{\partial q} + \frac{\theta_3^4(q)}{4\Delta_p} \right), \] (37)

\[ F^{\frac{1}{2}}_{\Delta_p} \left[ \frac{\Delta_0}{\Delta_0} \right] (z) = -z^{-\frac{3}{4}}(1-z)^{-\frac{3}{4}} (16q)^{\Delta_p} \frac{\Delta_p}{\Delta_p} \theta_2(q^2) \theta_2(q^2) \Delta_p \left( 1 - \frac{q}{\Delta_p} \theta_3^{-1}(q) \frac{\partial \theta_3(q)}{\partial q} + \frac{\theta_2^4(q)}{4\Delta_p} \right). \]

Explicit expressions for the conformal blocks (31 – 37) constitute the main result of the present work and were used in the derivation of the elliptic recurrence representation of the NS blocks [9].

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Appendix

In the Appendix we shall derive formulae expressing some three-point correlation functions of primary fields $j_{-\frac{1}{2}}\chi_0^\pm(z)$, $\chi_1^\pm(z)$ and $S_{-\frac{1}{2}}\chi_1^+(z)$ through the “basic” three-point function $\langle \varphi_p(z_3)\chi_0^+(z_2)\chi_0^+(z_1) \rangle$. The used methods are simplified versions of those that led to the derivation of equations (14) – (22).

It follows from the OPE-s:

$$
\psi(\xi)\chi^\pm(z) \sim \frac{1}{\sqrt{\xi - z}}\psi_0\chi^\pm(z) = \frac{1}{\sqrt{2(\xi - z)}}\chi^\pm(z),
$$

$$
\psi(\xi)\varphi_p(z) \sim 1,
$$

that the function

$$
f(\xi) = \frac{1}{\sqrt{(\xi - z_2)(\xi - z_1)}} \langle \psi(\xi)\varphi_p(z_3)\chi_0^-(z_2)\chi_0^+(z_1) \rangle
$$

is analytic in the complex $\xi$ plane save the simple poles at $\xi = z_2$, $\xi = z_1$, and falls off at infinity faster than $\xi^{-1}$. We thus have

$$
0 = \oint_{z_3} \frac{d\xi}{2\pi i} f(\xi) = - \oint_{z_2} \frac{d\xi}{2\pi i} f(\xi) + \oint_{z_1} \frac{d\xi}{2\pi i} f(\xi)
$$

$$
= - \frac{1}{\sqrt{2z_2z_1}} \langle \varphi_p(z_3)\chi_0^+(z_2)\chi_0^+(z_1) \rangle + \frac{1}{\sqrt{2z_1z_2}} \langle \varphi_p(z_3)\chi_0^-(z_2)\chi_0^-(z_1) \rangle
$$

so that

$$
\langle \varphi_p(z_3)\chi_0^-(z_2)\chi_0^-(z_1) \rangle = \frac{\sqrt{z_1z_2}}{\sqrt{z_2z_1}} \langle \varphi_p(z_3)\chi_0^+(z_2)\chi_0^+(z_1) \rangle = i \langle \varphi_p(z_3)\chi_0^+(z_2)\chi_0^+(z_1) \rangle
$$

or, equivalently,

$$
\eta_{z_3z_2z_1}(\nu_p, \chi_0^-, \chi_0^-) = i \eta_{z_3z_2z_1}(\nu_p, \chi_0^+, \chi_0^+).
$$

(A.1)

Here and below we adopt the convention that for $j < l$:

$$
z_j = e^{i\pi z_l}.
$$

Next, integrating around $\xi = z_3$ the identity

$$
\langle j(\xi)\varphi_p(z_3)\chi(z_2)\chi(z_1) \rangle = \frac{z_2z_3\sqrt{z_2z_1}}{(\xi - z_3)\sqrt{(\xi - z_2)(\xi - z_1)}} \langle \varphi_p(z_3)j_{-\frac{1}{2}}\chi(z_2)\chi(z_1) \rangle
$$

we get

$$
p \langle \varphi_p(z_3)\chi(z_2)\chi(z_1) \rangle = \sqrt{\frac{z_1z_2^{z_2z_3}}{z_3z_1}} \langle \varphi_p(z_3)j_{-\frac{1}{2}}\chi(z_2)\chi(z_1) \rangle,
$$

12
\[ \eta_{z_3 z_2 z_1} (\nu_p, j_{-\frac{1}{2}} \lambda_0^+, \lambda_0^-) = p \sqrt{\frac{z_{31}}{z_{21} z_{32}}} \eta_{z_3 z_2 z_1} (\nu_p, \lambda_0^+, \lambda_0^+) \]  \hspace{1cm} (A.2)

Analogous computation gives

\[ \eta_{z_4 z_3 z_2} (\lambda_0^+, j_{-\frac{1}{2}} \lambda_0^+, \nu_p) = -ip \sqrt{\frac{z_{42}}{z_{43} z_{32}}} \eta_{z_4 z_3 z_2} (\lambda_0^+, \lambda_0^+, \nu_p). \]

Using the OPE

\[ j(\xi) j_{-\frac{1}{2}} \chi_0^+ (z) \sim \frac{1}{2(\xi - z)^2} \chi_0^+(z) - \frac{1}{\xi - z} j_{-\frac{1}{2}} \chi_0^+(z) \]

we next get

\[ \langle j(\xi) \varphi_\nu (z_3) j_{-\frac{1}{2}} \chi_0^+ (z_2) \chi_0^+ (z_1) \rangle = \left( \frac{a}{\xi - z_2} + b \right) \frac{1}{(\xi - z_3) \sqrt{(\xi - z_1)(\xi - z_2)}}. \]  \hspace{1cm} (A.3)

with

\[ a = \frac{1}{2} z_{23} \sqrt{z_{21}} \langle \varphi_\nu (z_3) \chi_0^+ (z_2) \chi_0^+ (z_1) \rangle, \]
\[ b = z_{23} \sqrt{z_{21}} \left[ \langle \varphi_\nu (z_3) j_{-\frac{1}{2}} \chi_0^+ (z_2) \chi_0^+ (z_1) \rangle + \frac{1}{4} \left( \frac{1}{z_{21}} + \frac{2}{z_{23}} \right) \langle \varphi_\nu (z_3) \chi_0^+ (z_2) \chi_0^+ (z_1) \rangle \right]. \]

Integrating (A.3) around \( \xi = z_3 \) we derive a relation

\[ \eta_{z_3 z_2 z_1} (\nu_p, j_{-\frac{1}{2}} \chi_0^+, \lambda_0^+) = \left( \frac{2\Delta_p - \frac{1}{4}}{z_{21}} + \frac{2\Delta_p}{z_{32}} \right) \eta_{z_3 z_2 z_1} (\nu_p, \lambda_0^+, \lambda_0^+). \]  \hspace{1cm} (A.4)

Since

\[ \eta_{z_3 z_2 z_1} (\nu_p, \lambda_0^+, \lambda_0^+) = \Delta_p \eta_{z_3 z_2 z_1} (\nu_p, j_{-\frac{1}{2}} \lambda_0^+, \lambda_0^-), \]

we get

\[ \eta_{z_3 z_2 z_1} (\nu_p, \lambda_0^+, \lambda_0^+) = \eta_{z_3 z_2 z_1} (\nu_p, j_{-\frac{1}{2}} \lambda_0^+, \lambda_0^+) - \frac{1}{2} \eta_{z_3 z_2 z_1} (\nu_p, (j_{-\frac{1}{2}} + \psi_{-1} \psi_0) \lambda_0^+, \lambda_0^-) \]
\[ = \eta_{z_3 z_2 z_1} (\nu_p, j_{-\frac{1}{2}} \lambda_0^+, \lambda_0^+) - \frac{\partial}{\partial z_2} \eta_{z_3 z_2 z_1} (\nu_p, \lambda_0^+, \lambda_0^+). \]  \hspace{1cm} (A.5)

Finally, similar calculation with the help of the relation

\[ S_{-\frac{1}{2}} \chi_1^\pm = \left( 3j_{\frac{1}{2}} \psi_0 + 2S_{-\frac{3}{2}} - 5L_{-\frac{1}{2}} \right) \lambda_0^\pm, \]

gives

\[ \eta_{z_3 z_2 z_1} (\nu_p, S_{-\frac{1}{2}} \chi_1^+, \lambda_0^-) = \frac{ip}{\sqrt{2}} \left( \frac{\Delta_p - \frac{1}{2}}{z_{21} z_{32}} \right)^{\frac{1}{2}} \eta_{z_3 z_2 z_1} (\nu_p, \lambda_0^+, \lambda_0^+). \]  \hspace{1cm} (A.6)
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