Newton’s law from quantum mechanics: macroscopic bodies in the vacuum

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Abstract

Newton’s force law \( \frac{dP}{dt} = F \) is derived from the Schrödinger equation for isolated macroscopic bodies in the vacuum. First we identify three elements that ensure well-defined unique center-of-mass position and momentum for a macroscopic body at each instant of time, i.e., a classical trajectory. They are (i) Heisenberg’s uncertainty relations, (ii) absence of the diffusion of the wave packet for the center of mass of a macroscopic body, due to its large mass, and (iii) a finite body-temperature which implies a radiating, metastable state of the body - a mixed state, with decoherence caused by entanglement with the photons it emits and which carry away information. Newton’s equation follows from the Ehrenfest theorem, as we explicitly verify for a macroscopic body in weak gravitational forces, in a harmonic potential, and under constant external electromagnetic fields slowly varying in space. Corrections due to its finite size such as the gravitational tidal forces known in classical physics, also appear correctly, as can be checked by application of perturbation theory to the Ehrenfest theorem. The present work in several ways complements and strengthens the well-known view that the emergence of classical physics in quantum mechanics is due to environment-induced decoherence, but sharpens further our understanding of the problem, by emphasizing the (body) temperature as a key element and demonstrating an explicit derivation of classical equations of motion for the CM of a macroscopic body in the vacuum.
1 Introduction: formulation of the problem

The mechanism for the emergence of the classical behavior of a macroscopic body has been largely clarified as due to the entanglement of the body studied with the environment (any kind of fluxes, light, air molecules, cosmic ray background, etc.) and the consequent decoherence. Quantitative analysis of these effects and characteristic features of the decoherence have been discussed in many papers [1–4].

It appears reasonable to assume that such a mechanism, applied to the experimental devices, entangled in general with the environment (the rest of the world), does explain the classical state of the measurement apparatus after each measurement - the uniqueness of each measurement outcome -, providing us with one of the indispensable ingredients for properly understanding the measurement processes.

However, this is but one of the many distinct issues about the interpretation and the fundamental aspects of quantum mechanics, which go generally under the name of the “quantum measurement problems”. The latter include different interpretations of quantum mechanical predictions, the mechanisms or models of the so-called wave-function collapse, the relation between Born’s rule and the Schrödinger equation, as well as various apparent puzzles, paradoxes and conceptual difficulties. Earlier references and discussions on those issues can be found in [5–7].

The solution to most of these problems has been discussed recently [8] from a new perspective. It is proposed that the fundamental law of quantum mechanics is various expectation values (the fluctuation averages) of the dynamical variables, which describe the relative frequencies of various outcomes in case experiment is done, and which replace Born’s rule in the conventional approach. One of the key observations in [8] is that the fundamental building blocks of our world are various (elementary or composite) particles, and hence that all elementary interactions and accordingly all measurement processes at their heart, are spacetime pointlike events. This leads to an effective spacetime localization of each measurement event, a fact perceived by us as a wave-function collapse. Another key observation is the interplay and crucial roles factorization and entanglement play in the conceptual basis of quantum mechanics in general, and in particular, in the actual time evolution of the measurement processes. The new perspective proposed in [8] that the quantum fluctuations described by the wave function are real, and that they are there, independently of any experiments or of human presence, seems to dispel the final doubt about the fundamental theory of physics being unable to make a unique prediction of each single experiment necessarily. Our point of view seems to be consistent with what we know from cosmology today, that all the structures of the universe, the galaxies, stars and planets, and ourselves, have grown out of an uncontrollable quantum density fluctuations at some stage of the inflationary universe.

The central issue of the present work, to understand how the classical mechanics, and more generally classical physics, emerges in quantum mechanics, for which systems, for which variables, and under which conditions or limits, is actually a question largely un-
related, logically, to the solution of these quantum measurement problems and various subtle issues accompanying them. Thus it is important to disentangle the two classes of the problems as clearly as possible and try to understand the way Newton’s mechanics and classical physics emerge for macroscopic bodies in quantum mechanics, without necessarily bringing into the discussion many unrelated (especially, philosophical) aspects of the quantum measurement problems.

Still, there is one key issue linking the two problems. It is the very definition of the position and momentum of the centre-of-mass of a macroscopic body, which necessarily involves a measurement, both in the quantum-mechanical and classical contexts. There, the well known difference between quantum mechanics and classical mechanics makes appearance in a significant way.

It is the purpose of this work to improve our understanding of the mechanism for the emergence of classical mechanics in quantum mechanics. In particular, we ask a simple question: does a macroscopic body in the vacuum, i.e., without a pre-existent environment, behave classically or quantum mechanically? However, we must first better define the various terms and conditions, and formulate the problem itself more precisely.

What is a macroscopic body? Is a large molecule such as a DNA macroscopic? Does it need to contain a large macroscopic number of atoms or molecules? Is \( N = 1000 \) sufficiently macroscopic, or should it be of the order of the Avogadro number \( N_A \sim 6 \cdot 10^{23} \)? Also, what does a “body” mean? Should it be a solid, such as a piece of metal, having a crystalline structure inside? Or could it be a drop of liquid, or even a volume of gas? What about a dust particle, or any piece of a biological body? Also, should the body have some particular mass, energy, density or temperature \( T \), to behave classically?

This last question brings us actually to the very starting point of our analysis. As we shall be studying the behavior of a “body in the vacuum”, from here on we mean by \( T \) the temperature of the body, and not of the environment (which is absent). Because the temperature of the surrounding space (the vacuum) is zero, clearly the problem is not a static one, unless \( T = 0 \). It will be seen below that the (large) mass and body temperatures will play essential roles for a macroscopic body to behave classically.

Now at \( T = 0 \), whatever body, microscopic or macroscopic, and independently of the state of matter, gas, liquid, solid or Bose-Einstein condensed (BEC) gas or liquid, is in a quantum mechanical state. This is because, by definition, at \( T = 0 \) all the atoms and molecules composing the body, as well as their collective motions, are in their quantum-mechanical ground state \(^1\). The whole system is in a unique ground state, which is the content of the third law of thermodynamics: the entropy is zero at \( T = 0 \) (which is known as Nernst-Planck’s law) \(^2\). The specific heat of any matter approaches to 0 as \( T \to 0 \).

\(^1\)This includes the cases the body consists of many identical fermions, where the many-body ground state must obey the Fermi-Dirac statistics, a condition which leads to various kinds of so-called degenerate Fermi gas. Examples in Nature abound, they go from ordinary metal conductors, to the neutron stars, and the white dwarfs.

\(^2\)Actually, small quantum mechanical systems may have degenerate ground states; however it is well
We must also make a qualification on the concept of the vacuum, before we proceed. In quantum field theory the “vacuum” is not an empty, sterile space. It represents the ground state with respect to all possible field excitations in the theory. Under normal conditions such a zero-point energy uniformly distributed all over the space is not observable. However, under certain local disturbance such as a strong gravity and spacetime curvature such as those near a blackhole, or extremely strong electromagnetic fields applied to a restricted region, particle pairs are produced from the vacuum, leading to observable effects (the vacuum polarization, the Unruh effects, Hawking radiation, etc.). Another well-known manifestation of the quantum fluctuations of the vacuum is the so-called Casimir effect, which exerts attractive forces between two macroscopic parallel conducting plates. We assume that it is a good approximation to disregard, for the purpose of the following discussion, these nontrivial quantum properties of the vacuum (see for instance [9]) surrounding our macroscopic body.

This said, our starting point is this. Any matter, at \( T = 0 \), or at temperatures sufficiently close to it, is quantum mechanical. The problem is to understand what “sufficiently close to \( T = 0 \)” means. In general, the matter will maintain quantum-mechanical behavior at \( T > 0 \), until \( T \) reaches some critical value. How high can \( T \) be above the absolute zero temperature, before the system loses the characteristic quantum mechanical behaviors, depends on the type of atoms and molecules, the kind and state of matter, and their density. The details depend also on the type of interactions among the constituent atoms and molecules and on the nature of their collective excitation modes. See Sec. 4 for a discussion on these questions.

In the case of a gas composed of identical bosonic atoms, the manifestation of their (collective) quantum-mechanical properties has been shown through the so-called Bose-Einstein condensation (BEC), a phenomenon in which a macroscopic number of atoms occupy the same lowest-energy quantum states, and as a result is described by a macroscopic wave function, \( \Psi(\mathbf{r}) \). BEC was first realized for the gas of \( \text{Rb}^{87} \) atoms at \( T \leq 1.7 \cdot 10^{-7} \) K [10], and for \( \text{Na}^{23} \) at \( T \leq 2 \cdot 10^{-6} \) K [11]. The dilute gas of \( \text{He}^4 \) was laser-cooled down to \( 4.7 \cdot 10^{-6} \) K, to form a BEC involving \( \sim 5 \cdot 10^5 \) \( \text{He} \) atoms [12].

In high-energy physics context (i.e., in the physics of the fundamental interactions), the same phenomenon (BEC) has been well understood, since the sixties. The condensation of identical (elementary or composite) bosons at \( T = 0 \) and the consequent spontaneous symmetry breaking of global [13,14] or local symmetries [15,16], have been some of the main known that due to tunnelling effects, the true vacuum (the lowest-energy eigenstate) is unique. In systems with an infinite number of degrees of freedom, however, this argument does not work. The system may have more than one degenerate ground states, without tunnelling among them. In these systems Nernst-Planck’s law still applies, because the system necessarily chooses, at \( T = 0 \), one of the lowest-energy states as its physical, unique ground state. Spontaneous symmetry breaking is one of the manifestations of this phenomenon.

\(^3\)A good illustration of this point is Planck’s formula for the black-body radiations. It expresses the energy distributions among various electromagnetic excitations above the vacuum, at temperature \( T \). It is described by the numbers of photons each carrying the excitation energy quantum \( h\nu = \omega \hbar \). The zero-point energy \( \frac{1}{2} \omega \hbar \) of each oscillator filling the whole space, is not observable under normal conditions.
theoretical ingredients which made possible the construction of the standard $SU(3)_{\text{QCD}} \times ((SU(2) \times U_Y(1)))_{\text{GWS}}$ theory of the strong and electroweak interactions, around 1970-1974 [17–19]. This is one of the most precisely tested theories in physics known today.

Detailed studies by using lattice simulations of QCD show that the quark-antiquark chiral-symmetry breaking condensates, $\langle \bar{q}q \rangle$, drops to 0 at a critical temperatures $T_c \simeq 190$ MeV ($\simeq 10^{12}$ K), signalling a chiral-symmetry-restoration phase transition.

Coming back to atomic physics, it is an active research area to study the properties of Fermi gas (atomic) of various types, e.g., $He^3$, at low temperatures.

Superconductivity occurring in some metal at temperature below a critical temperature, e.g., below $T_c = 9.7^0K$ for Nb, is another, extremely interesting phenomenon [20].

On the other hand, the electrons inside the normal metals behave collectively approximately as a degenerate quantum Fermi gas even at room temperatures. For instance, electric conductivity is a typical quantum-mechanical phenomenon involving the tunnel effects.

Another, special example is light. Light is a gas of noninteracting, free bosons (the photons). It is quantum mechanical at any temperature. Even though Maxwell’s theory is able to describe successfully myriad of macroscopic electromagnetic phenomena, it met with unsurmountable difficulties in explaining the behavior of the blackbody radiation spectra and its temperature dependence. Alongside the related problems of explaining the behaviour of the specific heat of various matter, which turned out to depend on the detailed nature of the microscopic quantum-mechanical excitation modes inside materials, clarifying the nature and solution of the blackbody problem has led Planck to the breakthrough discovery of quantum physics in 1900.

Various kinds of liquid also show extraordinary quantum behaviors at very low temperatures. Liquid $He^4$ exhibits superfluidity at temperatures below $\sim 1$ K, a phenomenon already known around 1930 [20]. Superfluidity phase in liquid $He^3$ occurring around $\sim 2 \cdot 10^{-3}$ K was discovered only in 1972 [21].

Even the solids exhibit quantum-mechanical behavior at sufficiently low temperatures. A recent fabrication of a mechanical resonator [22] kept at $T \sim 2 \cdot 10^{-2}$ K was regarded by some as a promising start for realizing in the laboratory an analogue of a linear superposition of the dead and alive cat states.

Remark: Actually, this might have been an excess of enthusiasm, not to be taken literally. On the one hand, a quantum superposition of macroscopic bodies require their body temperatures close to the absolute zero. On the other, a living cat needs nonzero body temperatures for its biological functions, thus it is necessarily a mixed state (see below). The conclusion is that a “quantum superposition of dead and alive cat” is a self-contradictory notion, neither realizable in practice nor in principle.

This brief survey should be sufficient to illustrate that “how does classical mechanics

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4QCD and GWS stand respectively for Quantum Chromodynamics and Glashow-Weinberg-Salam.
emerge from quantum mechanics?” is, generally speaking, not a very well-formulated question. Also, a fact that clearly came into focus in the overview is the prominent role the temperature plays in the whole discussion.

Keeping all these subtleties in mind, we set ourselves to study in this work the time-evolution of the center-of-mass (CM) position and momentum of a generic macroscopic body in the vacuum and at nonzero body temperatures, by using the Schrödinger equation (the Ehrenfest theorem).

After identifying the elements which ensure the center of mass of a macroscopic body to have a classical trajectory (Sec. 2), we proceed to find that the CM motion of such a body is indeed described by Newton’s equations (Sec. 3). We will show this for the body in weak external gravitational forces, under a harmonic potential, as well as under weak, slowly varying external electromagnetic fields. Effects due to the finite size of the body, such as the tidal effect, well known in classical physics, also appear correctly from the Ehrenfest theorem, as will be verified by applying perturbation theory.

Sec. 4 discusses briefly the temperatures characterizing the boundaries between the quantum and classical descriptions, drawing lessons from some experimental work in the literature.

Summary and conclusion are in Sec. 5.

2 Center of mass of a macroscopic body

There are four elements which enter the derivation of Newton’s equations for a macroscopic body in quantum mechanics. Three of them are

(i) Heisenberg’s uncertainty relations for the CM of the macroscopic body;
(ii) Absence of the diffusion of the wave packet; and
(iii) The mixed-state nature of a macroscopic body at nonzero temperatures, and decoherence.

These three elements, taken together, imply that the center-of-mass position and momentum of (the narrow wave packet representing) a macroscopic body is well defined at each instant of time.

Note that this is so even if, in general, the microscopic degrees of freedom inside a macroscopic body under consideration (such as the electrons, photons, atoms and molecules inside a metal ball, a rock, planets and stars) may continue to maintain characteristic quantum-mechanical behaviors.

Once the classical, unique position and momentum of a macroscopic body is identified with the vacuum expectation values of the center-of-mass position and momentum operators in a narrow, diffusionless wave packet, it is in principle a straightforward exercise to check that
Ehrenfest’s theorem for such a wave packet coincides with Newton’s force law, in appropriate limits the macroscopic body can be regarded as pointlike.

Due to the composite, extended nature of a macroscopic body, however, classically known effects such as polarization, the tidal effects, etc., also appear naturally from the multipole expansion of the external potential. See Sec. 3.

2.1 Heisenberg’s uncertainties

For a microscopic particle, an atom or molecule, the derivation of Heisenberg’s uncertainty relation is well known. The canonical position and momentum operators of a particle, $x, p$, satisfy

$$[x, p] = i\hbar .$$

For compactness of expressions we write the formula for one-dimensional particles, but the generalization is understood. The uncertainty of $x$ or of $p$ is defined as their dispersions

$$\Delta x \equiv \sqrt{\langle (x - x_0)^2 \rangle}, \quad \Delta p \equiv \sqrt{\langle (p - p_0)^2 \rangle},$$

(2.2)

where

$$x_0 = \langle x \rangle = \langle \psi | x | \psi \rangle , \quad p_0 = \langle p \rangle = \langle \psi | p | \psi \rangle$$

(2.3)

are their mean values in the state $\psi$. One has

$$\langle \psi | A^\dagger A | \psi \rangle \geq 0$$

(2.4)

for any operator of the form,

$$A = x - x_0 + i\alpha (p - p_0) ,$$

(2.5)

where $\alpha$ is an arbitrary real number, and for any state $\psi$. That is

$$\langle (x - x_0)^2 \rangle - \alpha \hbar + \alpha^2 \langle (p - p_0)^2 \rangle \geq 0 .$$

(2.6)

It means that the discriminant of the quadratic form above is semi-negative definite, that is

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2} .$$

(2.7)

The case of a particle in three-dimensional space is a straightforward extension of these discussions.

Let us now consider the case of a macroscopic body, made of a large number $N$ of atoms and molecules, e.g., $N \simeq 10^{23}, N \simeq 10^{57},$ etc. The center-of-mass position, the total mass
and the total momentum are defined as
\[ R = \frac{1}{M} \sum_{i=1}^{N} m_i r_i ; \quad M = \sum_{i=1}^{N} m_i , \quad P = \sum_i p_i . \] (2.8)

It follows that \((R = (X_1, X_2, X_3))\)
\[ [X_k, P_\ell] = \frac{1}{M} \sum_{i=1}^{N} m_i (i\hbar \delta_{k\ell}) = i\hbar \delta_{k\ell} , \quad k, \ell = 1, 2, 3 . \] (2.9)

For simplicity let us consider the total Hamiltonian of the form,
\[ H = H_{CM} + H_{int} , \] (2.10)
\[ H_{CM} = \frac{P^2}{2M} + V(R) ; \quad H_{int} = \sum_{i=1}^{N-1} \frac{\tilde{P}_i^2}{2\mu_i} + V_{int}(\tilde{r}_1, \tilde{r}_2, \ldots \tilde{r}_{N-1}) , \] (2.11)
and the interaction potential depending only on the internal, relative positions
\[ V(r_1, r_2, \ldots r_N) = V_{int}(\tilde{r}_1, \tilde{r}_2, \ldots \tilde{r}_{N-1}) \] (2.12)
\[ r_i = R + \tilde{r}_i , \quad i = 1, 2, \ldots, N ; \quad \sum_{i=1}^{N} m_i \tilde{r}_i = 0 , \] (2.13)
and \(\mu_i\) are the reduced masses \(^5\)
\[ \frac{1}{\mu_i} = \frac{1}{m_i} - \frac{1}{M} , \quad i = 1, 2, \ldots, N - 1 . \] (2.14)

Such a Hamiltonian allows us to write the wave function in a factorized form,
\[ \Psi(r_1, r_2, \ldots r_N) = \Psi_{CM}(R)\psi(\tilde{r}_1, \tilde{r}_2, \ldots \tilde{r}_{N-1}) . \] (2.15)

Following the same procedure as for a particle, (2.1) - (2.7), by using this wave function \(\Psi\) instead of \(\psi\), one finds straightforwardly
\[ \Delta X_k \cdot \Delta P_\ell \geq \frac{\hbar}{2} \delta_{k\ell} \] (2.16)
for a macroscopic body.

Needless to say, we are using the nonrelativistic approximation here, for simplicity. Take for instance the hydrogen atom. The mass of a hydrogen atom in the ground state, is

\(^5\)In two-body \((N = 2)\) systems, it is customary to take as the internal position operator as \(r \equiv r_1 - r_2\), rather than in (2.13) here, i.e., \(\tilde{r}_1 = r_1 - R = (m_2/M)(r_1 - r_2)\). This explains the apparently different expression for the reduced mass from that found usually in the textbooks.
not equal to $m_e + m_P$, but $m_H = m_e + m_P - 14\text{eV}/c^2$, by taking into account the binding energy. However the error we make by approximating the total mass by the sum of the electron and proton mass is of the order of

$$\frac{14\text{eV}/c^2}{m_H} = \frac{14\text{eV}}{0.94\text{GeV}} \sim O(10^{-8}).$$

The same kind of errors will be even smaller in the case of the molecules, as their binding energies are in general smaller than the case of atoms.

Clearly this kind of argument cannot be applied to the level of the atomic nuclei, where the binding energy is relatively much larger. The problem might look even more serious if a similar consideration is extended to a more fundamental level. It is known today that the protons and neutrons which make up the atomic nuclei are actually bound states made of the quarks and gluons, and described by Quantum Chromodynamics (QCD). The mass of the nucleons, $\sim 940\text{MeV}/c^2 \sim 1.67 \cdot 10^{-24}\text{g}$, is mostly the effect of the strong gauge interaction dynamics, with the contribution of the up and down quark masses ($m_u \sim 5\text{MeV}/c^2, m_d \sim 10\text{MeV}/c^2$) being only a small part of it.

Clearly, the problem is that the protons and neutrons are highly relativistic, quantum mechanical bound states, for which the simple classical discussion such as (2.8), (2.9) cannot be applied.

Nevertheless, nothing prevents us from considering a macroscopic body as made up of $N \sim 10^{23}$ or $N \sim 10^{51}$ atoms or molecules, which are in turn bound states of the electrons and the atomic nuclei (protons and neutrons), and for the purpose of the discussion here, from disregarding how the protons and neutrons are themselves made of the quarks and gluons. The understanding of how a macroscopic body is made out of atoms and molecules does not require - say, most of time - the knowledge of how the physics at smaller scales explain the properties of nuclei, atoms and molecules: it is a general character of physics laws.

With this understanding, (2.16) is the correct Heisenberg’s uncertainty relation for the center-of-mass position and momentum of a macroscopic body.

What is most remarkable is the fact that a macroscopic body, e.g., made of $N = 10^{23}$ or $N = 10^{51}$ atoms and molecules, satisfies the same Heisenberg’s uncertainty relations for its CM position and momentum, as that for each of the constituent atoms. *Heisenberg’s uncertainties - the effects of quantum fluctuations - do not pile up.*

But this means that for an experimental determination of the position and momentum of the CM of a macroscopic body, Heisenberg’s uncertainty constraints may be regarded unimportant. To have some concrete ideas, one may take for instance, somewhat arbitrarily,

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6. Today we know that quantum chromodynamics (QCD) describes correctly the properties of the proton, neutron, and other strongly-interacting particles such as pions and $\rho$ mesons.

7. This fact is often simply assumed. It leads to the limits in the resolution in various precision measurements involving macroscopic bodies and optical instruments, known as the standard quantum limit (SQL) [23].
the experimental uncertainties $\Delta X, \Delta P$ of the order of

$$\Delta P \sim 10^{-2} (g) \times \frac{10^{-4} \text{cm}}{\text{sec}} \sim 10^{-6} \frac{g}{\text{cm sec}} ; \quad \Delta X \sim 10^{-4} \text{cm} .$$

(2.18)

Their product is

$$\Delta P \cdot \Delta X \sim 10^{-10} \text{erg sec} ,$$

(2.19)

many orders of magnitude larger than the quantum mechanical lower bound, $\hbar \sim 1.05 \times 10^{-27} \text{erg sec}$. Thus even allowing for much better precision by several orders of magnitudes than the rule-of-thumb (2.18), both for the position and for the momentum, one can still regard the Heisenberg uncertainty relations insignificant. In other words one can effectively take

$$\Delta P \approx 0 , \quad \Delta X \approx 0$$

(2.20)

for macroscopic bodies.

In other words, the position and momentum of a macroscopic body can be measured and determined simultaneously with an “arbitrary” precision, at macroscopic scales.

2.1.1 Observation of the gravitational waves

An alert reader might wonder how such an (obviously very simple-minded) consideration can resist a scrutiny in view of the extraordinarily precise observation of the gravitational waves at LIGO and VIRGO laboratories made recently [24]. In the first observation of GW150914 performed at the two LIGO detectors, to fix the idea, $\sim 40$ Kg mirrors used in the Michelson-Morley interferometer oscillate about $\pm 10^{-16} \text{cm}$, with frequency around 100 Hz. Thus the momentum and position uncertainties of the mirrors can be taken to be of the order of

$$\Delta P \sim 4 \cdot 10 \cdot 10^3 \cdot 10^{-16} \cdot 100 \sim 4 \cdot 10^{-10} \frac{g}{\text{cm sec}} , \quad \Delta X \sim 10^{-16} \text{cm} ,$$

(2.21)

such that their product is

$$\Delta X \cdot \Delta P \sim 4 \cdot 10^{-26} \text{erg sec} ,$$

(2.22)

a factor $\sim 10^2$ above the quantum-mechanical lower bound (2.16), i.e., $\hbar/2 \sim 5 \cdot 10^{-28} \text{erg sec}$. See Ref. [25] for a more careful analysis, where it is shown that the main source of the sensitivity limit for the gravitational wave detection comes from the effects related to the quantum fluctuations of the photons in the Fabry-Perot cavity, related to the radiation pressure and laser shot noise, rather than the Heisenberg uncertainty of the mirror CM position and momentum. In other words, the observed gravitational-wave-induced oscillation of the mirrors is a classical motion.
2.2 Diffusion of a wave packet

Another textbook result in quantum mechanics is that the wave function of a free particle (e.g., an electron, an atom) diffuses in time. For instance, a Gaussian wave packet of size $\sim a$ at time $t = 0$, 

$$
\psi(x, 0) = \text{const } e^{-x^2/a^2} \quad (2.23)
$$
evolves, according to the Schrödinger equation,

$$
\frac{ih}{m} \frac{d}{dt} \psi(x, t) = H \psi(x, t) , \quad H = \frac{p^2}{2m} , \quad p = -ih \frac{\partial}{\partial x} \quad (2.24)
$$
to a wider distribution,

$$
|\psi(x, t)| = \text{const } e^{-x^2/a(t)^2} , \quad a(t) = \sqrt{a^2 + \frac{4\hbar^2 t^2}{m^2 a^2}} \sim \frac{2\hbar}{ma} \quad (2.25)
$$
after time $t(\gg \frac{ma^2}{2\hbar})$. Note that the initial momentum uncertainty

$$
\Delta p \sim \frac{\hbar}{\Delta x} \sim \frac{\hbar}{a} \quad (2.26)
$$
is responsible for the spreading of the wave packet.

*The mass plays an essential role in the diffusion of a wave packet.* To have a concrete idea, a free electron with mass $m \sim 10^{-27}$ g described by a wave packet of size $\sim 1 \mu$m, will double its wave packet size in $t = 10^{-8}$ sec. It will spread to the size of $\sim 100$ m in one second. For a hydrogen atom, of mass $m \sim 10^{-24}$ g, described by the wave packet of size $\sim 1 \mu$m, the time needed to double its wave function spread is about $t = 10^{-5}$ sec.

In comparison, the center of mass of a macroscopic body of 1 g, described by the free Schrödinger equation

$$
\frac{ih}{M} \frac{d}{dt} \Psi = -\frac{\hbar^2 \nabla^2}{2M} \Psi \quad (2.27)
$$
and satisfying the same Heisenberg’s relations, (2.16), as for the electron, initially described by a wave packet of size $\sim 1 \mu$m for its center-of-mass wave function, will require

$$
T \sim 10^{19}\text{sec} \sim 10^{11}\text{yrs} , \quad (2.28)
$$
to double its size, which is longer than the age of our universe.

The conclusion is that the diffusion of a free wave packet - quantum fluctuations - of the center-of-mass position is negligible for a macroscopic object, simply due to its large mass.

This difference explains an apparent (and loosely defined) puzzle often discussed in the literature [2], why microscopic particles (e.g., electrons) are “usually” in momentum eigenstates, whereas a macroscopic body is “always” in a position eigenstate.
Note that the (negligible) spread of its CM wave packet, has absolutely nothing to do with the size (i.e. a diameter) of a macroscopic body, $L_0$. Let us recall also that the diffusion of the wave function recalled above refer to a free particle. Thus even though the atoms and the electrons inside the atoms composing a macroscopic body, remain quantum mechanical systems, their wave functions do not diffuse: they are described by stationary bound-state wave functions. Their size $L_0$ (of a macro molecule, a crystal ball, a piece of a stone, the earth, the sun, etc.) is determined by the dynamics which bind the constituent electrons, atoms and molecules in a bound state, the lattice structure, etc., in other words by the extension (i.e., spatial support) of the wave functions describing the bound state.

In the discussions below, it is indeed fundamental not to confuse the size of a macroscopic body $L_0$ as defined above, with the extension of the wave packet of its center of mass, $\Delta_{CM}$. Clearly, any experimentalist capable of determining the size of the body $\sim L_0$ should be able to measure its CM position with precision, 

$$\Delta_{CM} \ll L_0.$$  \hspace{1cm} (2.29)

This inequality will be understood throughout.

To conclude, once the position and momentum of the CM of a macroscopic body are given by a measurement done with arbitrary precision (see Sec. 2.1), the diffusion of its wave packet is negligible. Its position and momentum will be always well defined. A macroscopic body (its CM, not its internal degrees of freedom) traces a classical trajectory.

### 2.2.1 An extended wave packet is a quantum concept

The fact that a macroscopic body has a negligible quantum-mechanical diffusion, means that an extended wave packet (e.g., with $\Delta_{CM} \geq L_0$) for the center of mass of a macroscopic body, is a meaningless concept.

Both in quantum mechanics and in classical mechanics, the concept of the position and/or momentum is defined as the result of a measurement. In the quantum case, that is, for a microscopic system, Heisenberg’s uncertainty relations do not admit the simultaneous determination of the measured values of the canonically conjugate variables, $x, p$. If one desires to prepare the particle with a reasonably well-defined position, for instance in the direction perpendicular to its motion, it is sufficient to let the beam impinge upon a film with a hole of diameter $d$ in it. If it passes its position has been determined with the precision (uncertainty), accompanied by the uncertainty in its momentum \cite{26},

$$\Delta x \sim d; \quad \therefore \quad \Delta p \geq \frac{\hbar}{d}. \hspace{1cm} (2.30)$$

The particle will then diffuse (Fig. 1), and if there are many holes in the sheet, the particle waves diffusing from different holes interfere.

The image of the electron wave ripples, shown in Fig. 2, which look identical to water
ripples on a puddle surface produced by rain drops, but are smaller by a factor $\sim 10^{-6}$ in scale, have been obtained by A. Tonomura, by letting 50 kV electron beams go through a thin collodion film with tiny holes.

As the law of diffusion of a free particle wave is known, an initial wave packet of an arbitrary size can be experimentally prepared, and used as the initial condition.

In sharp contrast, a macroscopic particle position determination requires a hole of diameter $D$ larger than its size, $L_0$. If it passes its transverse position will have been measured with the precision, $\Delta X = D$, but it does not coincide with the size of its center-of-mass wave packet, even though it will provide an upper limit for the latter, see (2.29). Also, as the latter does not diffuse, there is no way to prepare its wave function with a desired, extended wave packet. Note that the diameter $D$ of the hole must be larger than $L_0$ (otherwise the body will not pass); after the passage its position is determined with a (classical) uncertainty

$$\Delta X = D > L_0,$$  \hspace{5cm} (2.31)

but it is simply due to the lack of information which (classical) path the body has taken inside the hole (Fig. 3). The momentum uncertainty caused by such a position measurement would be given by

$$\Delta P \sim \frac{\hbar}{D} < \frac{\hbar}{L_0}. \hspace{5cm} (2.32)$$

But this is much less than the original momentum uncertainty of the CM wave packet,

$$\Delta P_{CM} \sim \frac{\hbar}{\Delta_{CM}} (\gg \frac{\hbar}{L_0}). \hspace{5cm} (2.33)$$
Figure 2: Diffusion and interferences of the electron waves. They have been produced by letting a 50 kV electron beam fall on a thin collodion (celluloid) film. Courtesy by A. Tonomura.

In other words, the position measurement does not affect the original momentum uncertainty, in contrast to the case of a microscopic particle. The passage through a hole does not induce a diffusion of the wave packet for a macroscopic particle.

2.3 Decoherence: mixed-state nature of a macroscopic body at finite temperatures

The third condition for a macroscopic body to behave classically, is decoherence. It is here that the finite body-temperature plays the essential role, which makes the body in the vacuum under consideration a metastable, radiating, mixed state.

The absence of the diffusion of the CM of a macroscopic body (Sec. 2.2) was regarded by Born as a sufficient reason for such an object to behave classically. Einstein disputed it by observing that a superposition of the state

$$|\Psi\rangle = c_1|\Psi_1\rangle + c_2|\Psi_2\rangle,$$

where $\Psi_1$ and $\Psi_2$ are two narrow wave packets centered at $R = R_1$ and at $R = R_2$, separated by a macroscopic distance, is allowed by quantum mechanics, i.e., as a solution of the Schrödinger equation. A state of this kind certainly contradicts the notion of a classical particle, with its unique trajectory, at a definite position at each instant of time.

One way to explain away this apparent paradox is to note that a macroscopic body, at nonzero temperature, is a mixed state, as it radiates. The nature, and the solution, of the

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8This debate is reported, e.g., in [2].
The passage of a macroscopic particle of size $L_0 \gg \Delta_{CM}$ through a hole of size $D > L_0$. The passage does not increase momentum uncertainty: there is no diffusion.

Figure 3: The passage of a macroscopic particle of size $L_0 \gg \Delta_{CM}$, through a hole of size $D > L_0$. The passage does not increase momentum uncertainty: there is no diffusion.

A macroscopic body at finite temperatures is a mixed state: it is, at best, a metastable state,

$$|\Psi_1\rangle = |\Psi^{(0)}_1\rangle + |\Psi^{(1)}_1, \gamma\rangle + |\Psi^{(2)}_1, \gamma\gamma\rangle + \ldots$$

where $|\Psi^{(0)}_1\rangle$ is the effective "wave function" of the metastable state describing systems with atoms and molecules in excited states; $|\Psi^{(1)}_1, \gamma\rangle$ with one emitted photon flying away, with the rest of the system still containing atoms and molecules in excited states, $|\Psi^{(2)}_1, \gamma\gamma\rangle$ with two emitted photons, etc. Similarly for $|\Psi_2\rangle$,

$$|\Psi_2\rangle = |\Psi^{(0)}_2\rangle + |\Psi^{(1)}_2, \gamma\rangle + |\Psi^{(2)}_2, \gamma\gamma\rangle + \ldots$$

The coefficients of different terms in (2.35) and in (2.36) are omitted, as the norm of each term depends on time anyway. In the case of a single atom in a metastable, excited state, it is still possible to write a reasonably precise expression for its (effective) wave function (see for instance [27]); in the case of a macroscopic body at finite temperature, with many atoms and molecules in the metastable, excited states, such a description is neither available, nor really needed. The only thing needed is the fact that each of the "states", $|\Psi_1\rangle$ and $|\Psi_2\rangle$, is actually a mixed state, so that their quantum interference is negligible. The density matrix corresponding to $\Psi$ of (2.34) in the coordinate representation is diagonal, in good approximation.

It is interesting to compare the situation with the wave function of an $Ag$ atom, in the Stern-Gerlach experiment,

$$|\psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle,$$

where $\psi_1$ and $\psi_2$ represent the spin up and down states, respectively. In an inhomogeneous
magnetic field in the \( \hat{z} \) direction, the two components \( \psi_1 \) and \( \psi_2 \) get separated in the vertical direction, as the atom proceeds towards (say) the \( \hat{x} \) direction. The two wave packets might well be separated by a macroscopic distance, before the atom impinges upon the photographic plate. The wave function (2.37) still represents a pure quantum state, and the phase coherence between \( \psi_1 \) and \( \psi_2 \) can be verified and measured, if the two beams are reconverged by another magnetic field, before they hit the photographic plate, in an experimental setup known as the quantum eraser.

The contrast between the classical “wave packets” (2.34) and the quantum-mechanical pure state (2.37) (with a macroscopic separation between the two peaks of the wave packets), may be made manifest, by assuming that such a superposition has been somehow prepared at time \( t = 0 \), and by studying the time evolution of the density matrix in the position representation. The ideas of decoherence is that because of the entanglement of the particle with the environment, nondiagonal elements of the density matrix in the case of a macroscopic body, get rapidly damped in time, leaving the diagonal components, showing that the system is indeed a mixed state [1–3].

In the case of large molecules, such as \( C_{70} \), as a model for (2.34), a related study has been performed by Arndt et. al. [28], where the environment is replaced by a laser beam bombarding the \( C_{70} \) molecules, before the latter enter a double slit region (for detecting their quantum-mechanical feature, such as the diffraction and interference). This experiment verifies clearly, quantitatively, the decoherence due to the finite-temperature mixed state nature of the body.

The main observation in the present work is that, actually, it is sufficient that the body radiates, i.e., the body temperature is nonzero, for it to be a mixture, and to decohere. There is no need for a pre-existent environment. Concrete calculations of the radiation spectrum from highly excited, isolated small particles have been performed by Hansen and Campbell [29]. The result of this analysis was then used by Arndt et.al. [28] to derive the decoherence à la [1–3] for \( C_{70} \) beams, i.e., calculating the effects of the entanglement with the environment (in this case, the photons it emitted) on the density matrix in the position representation, and showing that the two (or more) nondiagonal peaks quickly die out in time. See Sec. 4 for more comments on this \( C_{70} \) experiment.

To do a quantitative analysis of decoherence of this sort for a macroscopic body made of \( N \sim 10^{23} \) atoms, in the vacuum and at the temperature \( T \neq 0 \), might look quite a nontrivial challenge.

### 2.3.1 Split wave packets for a (finite \( T \)) macroscopic body: an impossible notion

Actually, there is a simpler answer, logically sharper than such an argument, to the question why a pure state as (2.34) does not occur in nature, for a macroscopic body at nonzero

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9This expression is not quite exact. Here the laser beam is used to excite the \( C_{70} \) molecules which then radiate, a process distinct from the scattering by the environment particles, as considered in [1–3].
temperatures. The answer is, as it follows from the discussions of Sec. 2.1 ∼ Sec. 2.3, that it is not possible to prepare a doubly (or multiply) split wave packet such as (2.34), experimentally. Such a “state” represents an impossible, thus meaningless, initial condition for a macroscopic body with finite body-temperatures.

Let us note that both in quantum and classical physics, the initial condition to be used as the input of the dynamical evolution of the system, is the result of a precedent measurement. And it is here that quantum and classical mechanics differ significantly.

We can now paraphrase the discussion made already, for the impossibility of preparing a macroscopic wave function with an arbitrary extension, in Sec. 2.2.1, to see that it is not possible to prepare a split wave packet of the form, (2.34).

Let us recall, for comparison, that a beam of microscopic particles such as the electron or an atom, can be sent to a double slit, to yield a wave function of the form, (2.37), after the passage. Famous experiments such as [30] showing beautiful interference fringes à la Young, demonstrate clearly the pure-state nature of |ψ⟩. Or if the particle carries a spin, a wave packet can be split in two or more (two if s = \frac{1}{2}, three if s = 1, etc.) components in an inhomogeneous magnetic field, as in the familiar Stern-Gerlach experiment. There is no problem in preparing experimentally an initial pure state having the form (2.37); it is a perfectly good initial condition.

Vice versa, for a classical particle (a macroscopic body at finite temperatures) there is no way to prepare experimentally a coherent superposition of the form, (2.34). The fact that both |Ψ_1⟩ and |Ψ_2⟩ represent mixed states, see (2.35), (2.36), means that one is here dealing with a particle either near \( R = R_1 \) or near \( R = R_2 \). Such a state is represented as a diagonal density matrix, and not by a wave function of the form, (2.34).

A double slit wide enough for a macroscopic body to pass, will simply leave the particle either behind the one slit or the other. See Fig. 4. It may pass through one or the other if the experiment is repeated \(^{10}\). It would be a statistical mixture, not a pure state.

The Stern-Gerlach setting would not help either, to “prepare” a doubly (or multiply) split wave packet for a macroscopic body. As such an object has a well-defined position and momentum for its center of mass (see Sec. 2.1 and Sec. 2.2) always, it would simply follow a classical trajectory in a magnetic field, in a definite, generic direction (and not up or down), depending on the orientation of its magnetic moment and on the gradient of the magnetic field. This is what we know from classical physics, what is known as an empirical fact, and what indeed follows from quantum mechanics: see (3.90) below.

### 2.3.2 Settling the Born - Einstein dispute

This discussion on the impossibility of a split wave packet for a macroscopic body, concerns radiating, mixed-state, thus classical, body, at finite temperatures. As recalled in the Introduction, however, at temperatures sufficiently close to the absolute zero, any matter,\(^{10}\) Obviously, here the original transverse momentum uncertainty is assumed to be such that one cannot predict which path the particle takes.
microscopic or macroscopic, is quantum mechanical. A macroscopic coherent superposition (2.34) (a pure state of this form) is certainly possible, for such systems. The pair of the SQUID states with macroscopic fluxes of opposite signs, with possible tunnelling between them [31], are a perfect analogue of the split wave packet (2.34). Apart from technical issues there should not be any problem in constructing explicitly a doubly split wave packet by using e.g., the macroscopic wave functions of the Bose-Einstein condensed gas of some atoms.

We conclude that Born and Einstein were both partly right: at finite temperatures a macroscopic body is a mixed state and the absence of the diffusion (due to their large mass) is indeed the basic reason for it to possess a classical trajectory; at the same time, coherent superposition of macroscopically distinct states is simply not possible. On the other hand, at temperatures sufficiently close to absolute zero, any system is quantum mechanical, and a macroscopic, a split-wave-packet state (2.34) is certainly possible. Such a state might look counter-intuitive from our daily experiences, nonetheless is nothing special in quantum mechanics. The only problem is that, for a macroscopic body, it requires extraordinarily low temperatures, and a highly sophisticated experimental work to realize one (such as those in [31] - [43]; Sec. 4 for a further discussion).

3 The Ehrenfest theorem and Newton’s equation

Once the concept of a dispersionless center-of-mass wave packet of a macroscopic body is established, with the initial condition \((\mathbf{R}_{CM}, \mathbf{P}_{CM})\) as a well-defined and experimentally realizable input (Sec. 2.1 - Sec. 2.3), the derivation of Newton’s equation itself is a
straightforward (in principle) application of Ehrenfest’s theorem.

The Ehrenfest theorem for a microscopic particle (electron, atom, molecule) reads

$$\frac{d}{dt} \langle \psi | r | \psi \rangle = \frac{1}{i\hbar} \langle \psi | [r, H] | \psi \rangle = \langle \psi | \frac{p}{m} | \psi \rangle ;$$  \hfill (3.1)

$$\frac{d}{dt} \langle \psi | p | \psi \rangle = \frac{1}{i\hbar} \langle \psi | [p, H] | \psi \rangle = -\langle \psi | \nabla V | \psi \rangle .$$  \hfill (3.2)

If $|\psi\rangle$ represents a wave packet of more or less well-defined position and momentum, this looks almost like Newton’s equation. As is well known, a straightforward interpretation of the wave packet as something representing a sort of matter distribution inside a classical particle, and the conclusion that (3.1) and (3.2) constitute the derivation of Newton’s equation in quantum mechanics, however, are not tenable. For instance, the wave packet of an electron hitting a potential barrier easily breaks into two, one transmitted and the other reflected; on the other hand, the electron itself never breaks up. A classical particle does not break either: it gets either transmitted or reflected, depending on its energy and the barrier height.

This kind of consideration forces upon us the standard interpretation of the wave function as an amplitude for relative frequencies [8] (probability amplitude), not as something representing a real matter density.

### 3.1 Ehrenfest’s theorem for a macroscopic body

Let us now generalize (3.1) and (3.2) to a multiparticle bound state described by the wave function,

$$\psi(r) \rightarrow \Psi(r_1, r_2, \ldots, r_N) ,$$  \hfill (3.3)

where $N$ is some large, macroscopic number, typically $N \sim 10^{23}$ for a piece of material of $\sim 1g$, $N \sim 10^{51}$ for the earth. The CM position, momentum and the masses are given by

$$R = \frac{1}{M} \sum_{i=1}^{N} m_i r_i ; \quad P = \sum_{i=1}^{N} p_i , \quad M = \sum_{i=1}^{N} m_i ,$$  \hfill (3.4)

The Hamiltonian is

$$H \rightarrow H = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + V(r_1, r_2, \ldots, r_N) .$$  \hfill (3.5)

All nuclei, atoms, molecules, crystals and condensed matter states are described by the Hamiltonian of this form. We assume that also a general macroscopic body under consideration (a metal piece, a dust particle, a stone, a rock, planets and stars, galaxies) is also described by the Hamiltonian, (3.5). For a neutral atom (Comlombic bound states, with a definite size of the order of Bohr’s radius, $r_B \sim 10^{-8} cm$), all interactions are internal interactions, the Coulomb interactions between the electrons and the nucleus, and pairwise
interactions among the electrons. For simplicity, we neglect writing explicitly relativistic corrections and spin-dependent terms in $H$.

In the study of the atomic spectra (the internal excitation levels), one usually neglects the dependence on the center of mass $\mathbf{R}$ in the potential, and the atom as a whole is assumed to move as a free particle. A similar comment can be made about the nuclear and molecular spectra: the main interest is in the internal excitation modes and energy spectra.

Here, we are interested in the motion of the center of mass, and the forces the composite body feels as a whole.

The first Ehrenfest’s theorem (3.1) easily generalizes to the multiparticle composite body. As

$$
\frac{d}{dt} \langle \Psi | \mathbf{R} | \Psi \rangle = \sum_i \frac{d}{dt} \langle \Psi | m_i \tilde{r}_i | \Psi \rangle = \frac{1}{i\hbar} \sum_i \langle \Psi | \frac{m_i}{M} [r_i, H] | \Psi \rangle = \langle \Psi | \frac{1}{M} \sum_i p_i | \Psi \rangle ,
$$

one finds

$$
\frac{d}{dt} \langle \Psi | \mathbf{R} | \Psi \rangle = \langle \Psi | \mathbf{P}_M | \Psi \rangle ; \quad \mathbf{P} = \sum_i p_i .
$$

If we define the classical CM position and momentum of the body as

$$
\mathbf{R}_{CM}(t) \equiv \langle \Psi | \mathbf{R} | \Psi \rangle , \quad \mathbf{P}_{CM}(t) \equiv \langle \Psi | \mathbf{P} | \Psi \rangle .
$$

then (3.7) reads

$$
\frac{d}{dt} \mathbf{R}_{CM}(t) = \mathbf{P}_{CM}(t) = \frac{\mathbf{P}_{CM}(t)}{M} ,
$$

which is the familiar relation between the position and momentum for a classical point particle.

To consider the second Ehrenfest equation (3.2), let us write the potential as

$$
V(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) = V(\tilde{\mathbf{r}}_1, \tilde{\mathbf{r}}_2, \ldots, \tilde{\mathbf{r}}_{N-1}, \mathbf{R})
$$

by introducing the relative coordinates $\tilde{\mathbf{r}}_i$,

$$
\mathbf{r}_1 = \mathbf{R} + \tilde{\mathbf{r}}_1 ; \quad \mathbf{r}_2 = \mathbf{R} + \tilde{\mathbf{r}}_2 ; \quad \ldots ; \quad \mathbf{r}_N = \mathbf{R} + \tilde{\mathbf{r}}_N ;
$$

$$
\sum_i m_i \tilde{r}_i = 0 .
$$

Any $N - 1$ of $\tilde{\mathbf{r}}_i$’s can be taken to be linearly independent, e.g., $\tilde{\mathbf{r}}_1, \tilde{\mathbf{r}}_2, \ldots, \tilde{\mathbf{r}}_{N-1}$.

We need to compute

$$
[\mathbf{P}, H] = \sum_i [p_i, H] = -i\hbar \sum_i \nabla_i
$$

21
Inverting (3.11), one has
\[ R = \frac{1}{M} \sum_i m_i r_i ; \quad \hat{r}_1 = r_1 - R ; \quad \ldots \quad \hat{r}_{N-1} = r_{N-1} - R. \]  
(3.14)
and a simple calculation shows that
\[ \sum_{i=1}^N \nabla_i = \nabla_R. \]  
(3.15)
as is well known from classical mechanics. (3.2) then also simplifies:
\[ \frac{d}{dt} \langle \Psi | \mathbf{P} | \Psi \rangle = \frac{1}{i \hbar} \sum_i \langle \Psi | [p_i, H] | \Psi \rangle = -\langle \Psi | \sum_i \nabla_i V | \Psi \rangle = -\langle \Psi | \nabla_R V | \Psi \rangle. \]  
(3.16)
(3.9) and (3.16) taken together look almost like Newton’s equation. As the left hand side of equation (3.16) is
\[ \frac{d}{dt} \mathbf{P}_{CM}(t); \]  
(3.17)
one would have to show
\[ \langle \Psi | \nabla_R V | \Psi \rangle = \nabla_{R_{CM}} V_{\text{class}}(R_{CM}) , \]  
(3.18)
with an appropriately defined classical potential \( V_{\text{class}}(R_{CM}) \), in order to derive Newton’s equation for a pointlike particle of mass \( M \) at \( R_{CM} \). But such a relation cannot be taken for granted, actually, as will be seen below.

A natural definition for \( V_{\text{class}}(R_{CM}) \) is
\[ V_{\text{class}}(R_{CM}) = \langle \Psi | V(\hat{r}_1, \hat{r}_2, \ldots \hat{r}_{N-1}, R) | \Psi \rangle. \]  
(3.19)
Now the wave function for the center of mass of a macroscopic body can be taken as an arbitrarily narrow wave packet (Sec. 2). The width of the CM wave function packet indeed corresponds to an experimental measurement (preparation) of the CM position made with an “arbitrary” precision, see (2.20), with \( \Delta X_{CM} \) certainly much smaller than body’s size, \( L_0 \):
\[ \Delta X_{CM} \ll L_0. \]  
(3.20)
Actually, the finite size of the body \( L_0 \) affects significantly the calculation of the right hand side of the second Ehrenfest theorem, (3.16), except in few special cases which will be discussed below, in Sec. 3.2 and Sec. 3.4. In more general cases, such as those discussed in Sec. 3.3, the corrections from the Newton equation for a pointlike mass \( M \), (3.18), arise from the study of Ehrenfest’s equations. These deviations reproduce the effects well known in classical physics for macroscopic bodies, such as the gravitational tidal forces. The
related effects, for microscopic systems, are well known and discussed in many quantum mechanics textbooks for microscopic systems (e.g., deriving the polarization effects for atoms and molecules). Our discussion below in Sec. 3.3 adds absolutely nothing new, in principle, from this point of view\textsuperscript{11}. It is nonetheless satisfactory that our efforts to identify Newton’s equations for a macroscopic body in quantum mechanics (Sec. 2.1 - Sec. 2.3 and Sec. 3) naturally leads to appropriate corrections due to the finite-size of macroscopic bodies, which are also well known in classical physics.

3.1.1 Mixed-state nature of a macroscopic body and Ehrenfest’s theorem

Before embarking on the actual application of the Ehrenfest theorem for macroscopic bodies in some concrete cases, let us make a brief pause, to discuss an apparently minor, but actually a fundamental, question

One of the essential ingredients for the emergence of the classical physics is decoherence [1]- [4], discussed in Sec. 2.3 in our context. It means that the macroscopic body under consideration has $T > 0$ and is radiating: it is a mixed state, which looks like (2.35), with many infrared photons emanating from it. How can a naïve treatment such as in Sec. 3.1 in which the state of the macroscopic body is represented by a pure-state wave function of $N$ atoms and molecules, justified?

There are at least two good reasons why such an approach is indeed correct. The first is the fact that the angular distribution of the emitted photons can be considered random statistically, and therefore regarded as spherically symmetric. The average recoil from the different emitted photons on the CM motion of the macroscopic body will cancel out, to become negligible.

But perhaps more significantly, the effects of recoil are negligibly small, for an emission from excited atoms and molecules bound in a solid (crystal), due to the fact that the recoil kinematics involve whole body’s large mass, $M$. This phenomenon is well known: it is the essence of the famous Mössbauer effect.

Because of (at least) these two properties of the infrared emissions from a macroscopic body, the actual motion of the CM of a macroscopic body is not significantly affected by them. The CM motion of the body described by each term of (2.35) is identical.

Keeping this understanding in mind, we proceed below to derive Newton’s equations by using the Ehrenfest theorem for a macroscopic body, treating it as if it were a pure state described by a wave function.

Remark: A moment of reflection shows that this apparently minor observation is actually at the heart of the whole discussion of the present work, on how the classical physics

\textsuperscript{11}There is however some novelty from the formal point of view. Usually, perturbation theory in quantum mechanics is used to compute either the shift in the energy eigenvalues (time-independent perturbation theory), or the transition amplitudes (time-dependent perturbation theory), caused by a perturbative potential, $V'$. Here we apply perturbation theory to the Ehrenfest theorem, to study how the expression for the “forces” gets modified with respect to the classical expectation for a pointlike particle. This is perhaps not very familiar: we discuss it in Appendix A.
emerges from quantum mechanics. What is not identical and, actually, intractably different in various terms in (2.35) is the phases and the internal wave functions describing the quantum states of the microscopic particles composing the body and of the infrared photons flying away. This is what makes a macroscopic body under consideration a mixed state. The interference terms among the different terms in (2.35), among those in (2.36), and a fortiori, between $|\Psi_1\rangle$ and $|\Psi_2\rangle$, are absent. No coherent quantum superposition of macroscopically distinct states is possible at finite temperatures.

### 3.2 Gravitational potential on the earth

As the first concrete example let us consider the case of external gravitational force on a massive body on the earth. The total potential is

$$V(r_1, r_2, \ldots, r_N) = V_{int}(\mathbf{\tilde{r}}_1, \mathbf{\tilde{r}}_2, \ldots, \mathbf{\tilde{r}}_{N-1}) + V_{grav}(R)$$

(3.21)

where the first term is independent of the CM position $R$. The Hamiltonian is the sum

$$H = H_{CM} + H_{int} ,$$

(3.22)

$$H_{CM} = \frac{P^2}{2M} + V_{grav}(R) ;$$

$$H_{int} = \sum_i \frac{\tilde{p}_i^2}{2\mu_i} + V_{int}(\mathbf{\tilde{r}}_1, \mathbf{\tilde{r}}_2, \ldots, \mathbf{\tilde{r}}_{N-1}) ,$$

(3.23)

where

$$V_{grav}(R) = g \sum_i m_i z_i = MgZ ,$$

(3.24)

where $g \simeq 980 \text{ cm/sec}^2$ is the acceleration on the surface of the earth.

The form of the Hamiltonian (3.21) allows us to take the wave function in the factorized form,

$$\Psi = \Psi_{CM}(R)\psi(\mathbf{\tilde{r}}_1, \mathbf{\tilde{r}}_2, \ldots, \mathbf{\tilde{r}}_{N-1}) .$$

(3.25)

The time independent Schrödinger equation

$$H\Psi = (H_{tot} + H_{int})\Psi = E_{tot}\Psi ,$$

(3.26)

gives

$$H_{int}\psi(\mathbf{\tilde{r}}_1, \mathbf{\tilde{r}}_2, \ldots, \mathbf{\tilde{r}}_{N-1}) = E_{int}\psi(\mathbf{\tilde{r}}_1, \mathbf{\tilde{r}}_2, \ldots, \mathbf{\tilde{r}}_{N-1}) ;$$

(3.27)

$$H_{CM}\Psi_{CM}(R) = E_{CM}\Psi_{CM}(R) ,$$

(3.28)

and

$$E_{tot} = E_{CM} + E_{int} ,$$

(3.29)

where $H_{int}$, $\psi(\mathbf{\tilde{r}}_1, \mathbf{\tilde{r}}_2, \ldots, \mathbf{\tilde{r}}_{N-1})$ and $E_{int}$ refer to the internal excitation modes of the body under consideration \textsuperscript{12}.

\textsuperscript{12}In the case of certain rigid macroscopic bodies (e.g., crystals), $H_{int}$ and $E_{int}$ contain the total angular
(3.16) can be replaced by
\[ \frac{d}{dt} \langle \Psi_{CM} | P | \Psi_{CM} \rangle = -\langle \Psi_{CM} | \nabla_R V_{grav}(R) | \Psi_{CM} \rangle . \] (3.30)

and (3.8) and (3.19) by
\[ P_{CM}(t) \equiv \langle \Psi_{CM} | P | \Psi_{CM} \rangle ; \quad R_{CM}(t) \equiv \langle \Psi_{CM} | R | \Psi_{CM} \rangle , \] (3.31)
and
\[ V_{class}(R_{CM}) = \langle \Psi_{CM} | V_{grav}(R) | \Psi_{CM} \rangle , \] (3.32)

where \( \Psi_{CM} \) is an arbitrarily narrow (see (2.20)) wave packet describing the CM wave function, centered at \( R_{CM}(t) \) and \( P_{CM}(t) \).

For such a macroscopic body to obey Newton’s law,
\[ \frac{d}{dt} P_{CM}(t) = -\nabla_{R_{CM}} V_{class}(R_{CM}) . \] (3.33)

we need an approximate equality,
\[ \langle \Psi | \nabla_R V_{grav}(R) | \Psi \rangle \simeq \nabla_{R_{CM}} V_{class}(R_{CM}) \langle \Psi | \Psi \rangle = \nabla_{R_{CM}} V_{class}(R_{CM}) . \] (3.34)

Actually, for the gravitational potential on the earth, (3.24), one has
\[ -\nabla_R V_{grav}(R) = -\nabla_R MgZ = -Mg \hat{k} , \] (3.35)
a constant (where \( \hat{k} \) is the unit vector in the positive z direction). Also,
\[ V_{class}(R_{CM}) = V_{grav}(R_{CM}) = MgZ_{CM} , \] (3.36)

where the dispersion of \( R_{CM} \) has been neglected. The derivation of Newton’s equation from quantum mechanics, (3.33), is therefore exact. No details about the internal structures of the body \( (H_{int} \text{ and } \psi(\tilde{r}_1, \tilde{r}_2, \ldots \tilde{r}_{N-1})) \) are needed to get this result.

### 3.3 Motion in a gravitational potential due to a distant mass

Let us now consider the gravitational force on a body with mass \( M = \sum_i m_i \), exerted by a distant mass \( M_0 \) at \( R_0 \). The gravitation potential is in this case
\[ V_{grav}(r_1, r_2, \ldots r_N) = \sum_i \frac{G_N m_i M_0}{|r_i - R_0|} , \] (3.37)

momentum contribution of the rotational modes of the massive body as a whole (and the related energy), which should not be considered as the internal excitation modes. In the present work, however, we will not separate these three degrees of freedom from the internal excitations.
where $G_N$ is Newton’s gravitational constant,

$$G_N \simeq 6.674 \times 10^{-11} \text{ m}^3\text{Kg}^{-1}\text{s}^{-2}.$$  \hfill (3.38)

We assume that the gravitational force is weak as compared to the internal (electromagnetic) forces which bind the system into the macroscopic body. In other words, in the total Hamiltonian,

$$H = H_{CM} + H_{int} + V_{grav}(\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_N),$$  \hfill (3.39)

$$H_{CM} = \frac{\mathbf{P}^2}{2M}, \quad H_{int} = \sum_i \frac{\tilde{p}_i^2}{2\mu_i} + V_{int}(\tilde{\mathbf{r}}_1, \tilde{\mathbf{r}}_2, \ldots \tilde{\mathbf{r}}_{N-1}),$$  \hfill (3.40)

the gravitational potential $V_{grav}$ can be regarded as a perturbation. That means that the macroscopic body under consideration is a bound state described by the time-independent Schrödinger equation,

$$H_{int} \psi^{(0)}(\tilde{\mathbf{r}}_1, \tilde{\mathbf{r}}_2, \ldots \tilde{\mathbf{r}}_{N-1}) = E_{int} \psi^{(0)}(\tilde{\mathbf{r}}_1, \tilde{\mathbf{r}}_2, \ldots \tilde{\mathbf{r}}_{N-1}).$$  \hfill (3.41)

Before taking into account the gravitational potential, the wave function thus has a factorized form,

$$\Psi = \Psi_{CM}(\mathbf{R})\psi^{(0)}(\tilde{\mathbf{r}}_1, \tilde{\mathbf{r}}_2, \ldots \tilde{\mathbf{r}}_{N-1}).$$  \hfill (3.42)

In (3.41) and (3.51) a suffix (0) (“unperturbed”) is used to indicate that the wave function $\psi^{(0)}$ stands for the macroscopic body before the gravitational potential is taken into account.

We assume also that the distance from the body to the mass $M_0$ is much larger than the linear size of the body (the spatial extension of $\psi^{(0)}$), $L_0$. We may then assume that

$$\forall i, \quad |\tilde{\mathbf{r}}_i| = |\mathbf{r}_i - \mathbf{R}| \sim L_0, \quad \frac{L_0}{|\mathbf{R} - \mathbf{R}_0|} \ll 1.$$  \hfill (3.43)

Actually, some care is needed in such an approximation. The reason is that $\mathbf{r}_i$’s and $\mathbf{R}$ in (3.43) are position operators, and as such, have no a priori definite values. However, we remember that these relations are to be used between the bra and ket of the state vector of the macroscopic body, $\Psi$. As for $\tilde{\mathbf{r}}_i$’s, their extensions are effectively limited by the support of the internal wave function $\psi^{(0)}(\tilde{\mathbf{r}}_1, \tilde{\mathbf{r}}_2, \ldots \tilde{\mathbf{r}}_{N-1})$, hence by the size $L_0$ of the body. In the case of the center of mass position $\mathbf{R}$ it is a bit subtler, as $\Psi_{CM}(\mathbf{R})$ satisfies a free Schrödinger equation. The consideration in Sec. 2.1 and Sec. 2.2, however allows us to use for the CM wave function $\Psi_{CM}(\mathbf{R})$ a well-defined, narrow wave packet, with an extension much smaller than $L_0$ (see (3.20)). The approximation (3.43) is therefore justified.
The gravitational potential can now be multipole expanded in powers of $1/|\mathbf{R} - \mathbf{R}_0|$:  

$$V_{\text{grav}}(r_1, r_2, \ldots, r_N) = \sum_i \frac{G m_i M_0}{|\mathbf{r}_i + \mathbf{R} - \mathbf{R}_0|}$$

$$= \frac{GMM_0}{|\mathbf{R} - \mathbf{R}_0|} - G \sum_i \frac{m_i \tilde{r}_i^2}{|\mathbf{R} - \mathbf{R}_0|^3} + 3G \sum_i \frac{m_i (\tilde{r}_i \cdot (\mathbf{R} - \mathbf{R}_0))^2}{|\mathbf{R} - \mathbf{R}_0|^5} + \ldots \quad (3.44)$$

The first term corresponds to the gravitational potential for the pointlike mass $M$ at $\mathbf{R}$. Note that the first, order $O((1/|\mathbf{R} - \mathbf{R}_0|)^2)$ correction term, the “dipole term”, 

$$- \sum_i \frac{G m_i \tilde{r}_i \cdot (\mathbf{R} - \mathbf{R}_0)}{|\mathbf{R} - \mathbf{R}_0|^3} \quad (3.45)$$

is actually absent in (3.44), as is well known for the gravity, as $\sum_i m_i \tilde{r}_i = 0$.

The second and third term of (3.44) represent the quadrupole corrections, the magnitude of which depends on the internal wave function, $\psi(\tilde{r}_1, \tilde{r}_2, \ldots \tilde{r}_{N-1})$, i.e., the non-spherically-symmetric form of the body.

An important point to note however is that the quadrupole (and higher) corrections will appear, even if the unperturbed wave function $\psi^{(0)}$ (the body without the influence of gravity), is perfectly spherically symmetric. In such a case, the contribution of the second and the third term of (3.44) on the right hand side of the second Ehrenfest’s theorem,

$$\frac{d\mathbf{P}_{CM}}{dt} = -\langle \Psi \rangle \sum_i \nabla_i V |\Psi\rangle \quad (3.46)$$

might appear to vanish, if the factorized wave function (3.51) is used, with a spherically-symmetric internal wave function $\psi^{(0)}(\tilde{r}_1, \tilde{r}_2, \ldots \tilde{r}_{N-1})$.

Actually, a nonvanishing quadrupole appears as the result of the perturbation to the wave function: the deformation of the body due to the external gravitational force. In order to see these effects, it is necessary to study the consequence of the perturbation due to gravity systematically in the context of the Ehrenfest theorem. In Appendix A a standard perturbation theory is applied to Ehrenfest’s theorem, to derive modifications to the force systematically, order by order in perturbation.

In order to apply perturbation theory to Ehrenfest’s theorem, as sketched in Appendix A, it is however more convenient to split the Hamiltonian differently. Namely, we write

$$V_{\text{grav}}(r_1, r_2, \ldots, r_N) = \frac{GMM_0}{|\mathbf{R} - \mathbf{R}_0|} + V_{mp}(r_1, r_2, \ldots, r_N), \quad (3.47)$$

and consider the multipole expansion terms

$$V_{mp}(r_1, r_2, \ldots, r_N) = -G \sum_i \frac{m_i \tilde{r}_i^2}{|\mathbf{R} - \mathbf{R}_0|^3} + 3G \sum_i \frac{m_i (\tilde{r}_i \cdot (\mathbf{R} - \mathbf{R}_0))^2}{|\mathbf{R} - \mathbf{R}_0|^5} + \ldots \quad (3.48)$$


as perturbation. The total Hamiltonian is written as

$$H = H_0 + V_{mp}(\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_N), \quad H_0 = \tilde{H}_{CM} + H_{int},$$

(3.49)

$$\tilde{H}_{CM} = \frac{P^2}{2M} + \frac{GMM_0}{|\mathbf{R} - \mathbf{R}_0|}, \quad H_{int} = \sum_i \frac{\tilde{p}_i^2}{2\mu_i} + V_{int}(\tilde{r}_1, \tilde{r}_2, \ldots \tilde{r}_{N-1}),$$

(3.50)

and now only the multipole expansion terms $V_{mp}$ are treated as perturbation.

Unperturbed system $H_0$ is still described by a factorized wave function,

$$\Psi = \tilde{\Psi}_{CM}(\mathbf{R}) \psi(0)(\tilde{\mathbf{r}}_1, \tilde{\mathbf{r}}_2, \ldots \tilde{\mathbf{r}}_{N-1}),$$

(3.51)

where the CM wave function satisfies now the Schrödinger equation for a pointlike mass $M$

$$\tilde{H}_{CM} \tilde{\Psi}_{CM}(\mathbf{R}) = E_{CM} \tilde{\Psi}_{CM}(\mathbf{R})$$

(3.52)

(which is Newton’s law, see below, (3.59)), whereas the internal wave function is given by (see (3.41)),

$$H_{int} \psi(0)(\tilde{\mathbf{r}}_1, \tilde{\mathbf{r}}_2, \ldots \tilde{\mathbf{r}}_{N-1}) = E_{int} \psi(0)(\tilde{\mathbf{r}}_1, \tilde{\mathbf{r}}_2, \ldots \tilde{\mathbf{r}}_{N-1}),$$

(3.53)

which describes the bound state without the effects of gravity.

The first Ehrenfest theorem takes the classic form,

$$\frac{d}{dt} \langle \Psi \mid \mathbf{R} \mid \Psi \rangle = \langle \Psi \mid \mathbf{P} \mid \Psi \rangle ; \quad \mathbf{P} = \sum_i \mathbf{p}_i,$$

(3.54)

that is,

$$\frac{d}{dt} \mathbf{R}_{CM} = \frac{\mathbf{P}_{CM}}{M}, \quad \mathbf{R}_{CM} \equiv \langle \Psi \mid \mathbf{R} \mid \Psi \rangle ; \quad \mathbf{P}_{CM} \equiv \langle \Psi \mid \mathbf{P} \mid \Psi \rangle ;$$

(3.55)

to all orders of perturbation.

The left hand side of the second Ehrenfest’s theorem also contains $\frac{d}{dt} \mathbf{P}_{CM}$ to all orders of perturbation, see Appendix A. The right hand side, which can thus be interpreted as the force, has an expansion in powers of perturbation.

Let us assume, for definiteness, that, before taking into account $V_{mp}$, our macroscopic body is described by a spherically symmetric internal wave function, $\psi(0)$.

We transcribe the formulas (A.15) \~ (A.21) in Appendix A to our three-dimensional macroscopic body with $N$-atoms (or molecules),

$$F \rightarrow \mathbf{F}, \quad \frac{d}{dx} \rightarrow \nabla, \quad V_0 \rightarrow \frac{GMM_0}{|\mathbf{R} - \mathbf{R}_0|} + V_{int}(\tilde{\mathbf{r}}_1, \tilde{\mathbf{r}}_2, \ldots \tilde{\mathbf{r}}_{N-1});$$

(3.56)

$$H^{(0)} \rightarrow \tilde{H}_{CM} + H_{int}, \quad V' \rightarrow V_{mp}(\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_N),$$

(3.57)

and

$$\psi^{(0)} \rightarrow \Psi = \tilde{\Psi}_{CM}(\mathbf{R}) \psi^{(0)}(\tilde{\mathbf{r}}_1, \tilde{\mathbf{r}}_2, \ldots \tilde{\mathbf{r}}_{N-1}).$$

(3.58)
To the zeroth order in $V_{np}$, (A.18) says that

$$\frac{d}{dt} P_{CM} = F^{(0)} = -\langle \tilde{\Psi}_{CM} | \nabla \frac{GM_0}{|R - R_0|} | \tilde{\Psi}_{CM} \rangle \simeq -\nabla_{R_{CM}} \frac{GM_0}{|R_{CM} - R_0|},$$  \hspace{1cm} (3.59)

where we have taken into account the fact that for a macroscopic body the CM wave function can be taken as a wave packet, as narrow as we wish. The equation (3.59), together with (3.55), is just the Newton equation for a pointlike mass of $M$ at $R_{CM}$.

A quick inspection of (A.19) in our context (3.56)-(3.58) shows that the effect of $V_{np}$ on the motion of the macroscopic body vanishes to first order, $F^{(1)} = 0$, because of the angular momentum conservation (or by use of the Wigner-Eckart theorem).

The first nonvanishing effects of $V_{np}$ on the motion of the body arise at the second order, $F^{(2)}$, see (A.21). The operator $\frac{dV}{dx_0}$ with the translation (3.56) is a zero-rank spherical tensor with respect to the internal degrees of freedom $\tilde{r}_i$. On the other hand, the first-order correction to the wave function $|\psi^{(1)}\rangle$, (A.22), contains states of angular momentum 2 or higher (see (3.48)). $|\psi^{(1)}\rangle$ would be exactly $l = 2$ state if only the quadrupole term (explicit in (3.48)) were kept, but in general contains all higher angular momenta $l = 2, 4, 6, \ldots$. The second-order wave function correction $|\psi^{(2)}\rangle$ can be seen from (A.23) to contain angular momentum states $l = 0, 2, 4, 6, \ldots$.

Physically, the nonvanishing effect of quadrupole (and higher) moments on the motion of the CM can be interpreted as due to the first order deformation of the body (first order correction to the internal wave function), and the resulting nonvanishing elements in the second-order correction to the force, (A.21), due to the now non-spherically-symmetric distribution of the atoms and molecules inside the body.

### 3.3.1 Moon and the ocean tides on the earth

A familiar phenomenon related to these considerations is the ocean tides on the earth, due to the gravitational force of the moon. A caricature situation of the earth and moon, to the zeroth-order in $V_{np}$, is shown in Fig. 5. The earth, including the ocean, is assumed to have a spherically symmetric form, before $V_{np}$ is taken into account. The earth and moon move according to Newton’s equations between the pointlike masses $M_{earth}$ and $M_{moon}$, see (3.59), (3.55). When the multipole perturbation $V_{np}$ is taken into account, the earth gets deformed. Note that the deformation of the earth’s wave function is mainly due to the shifts of the water molecules in the ocean: their bindings are much weaker than the binding of the molecules and atoms in the solid earth crust. The earth and moon now look like Fig. 6. This gives rise to the daily high and low ocean tides (roughly twice each) on the earth\textsuperscript{13}.

Though done with a crude, simplifying assumptions, this discussion may be regarded as the first derivation of a familiar daily experience (ocean tides), by a direct use of the

\textsuperscript{13}According to Feynman [44] it was Newton who has first correctly identified the cause of the daily ocean tides.
Schrödinger equation for a macroscopic body (the earth) described by a wave function with \( N \sim 10^{51} \) atoms!

Figure 5: The earth and moon, in the approximation in which the multipole expansion terms \( V_{mp} \) of (3.47), (3.48) are neglected. The earth is assumed to be perfectly spherically symmetric, including its ocean.

### 3.4 Harmonic oscillator potential

Next consider the case of a harmonic potential acting uniformly on all the components of the body. The potential is given by

\[
V = \sum_{i=1}^{N} \frac{m_i \omega^2 r_i^2}{2}.
\]  

(3.60)

In this case the Hamiltonian does not simply factorize as a sum of the CM part and the internal Hamiltonian, as in (3.23). Nevertheless, Ehrenfest’s theorem simplifies again.

The first Ehrenfest theorem reads,

\[
\frac{d}{dt} \langle \Psi \vert R \vert \Psi \rangle = \langle \Psi \vert \frac{P}{M} \vert \Psi \rangle ; \quad P = \sum_i p_i ,
\]  

(3.61)

that is,

\[
\frac{d}{dt} R_{CM} = \frac{P_{CM}}{M} .
\]  

(3.62)
The ocean

The earth

The moon

Figure 6: The earth’s wave function is now deformed by the perturbation, $V_{np}$. The main effect is in the change of distribution of the water molecules which are more loosely bound than the atoms and molecules in the earth’s crust.

The second simplifies as well:

$$\frac{d\mathbf{P}_{CM}}{dt} = \frac{d}{dt} \langle \Psi | \mathbf{P} | \Psi \rangle = \frac{1}{\hbar} \sum_i \langle \Psi | [\mathbf{p}_i, H] | \Psi \rangle = -\langle \Psi | \sum_i \nabla_i V | \Psi \rangle = -\langle \Psi | \sum_{i=1}^N m_i \omega^2 \mathbf{r}_i | \Psi \rangle = -M \omega^2 \langle \Psi | \mathbf{R} | \Psi \rangle = -M \omega^2 \mathbf{R}_{CM} .$$

The center of mass of the whole body obeys Newton’s equation with the harmonic potential.

Again, the knowledge about the internal interactions, which bind and determine the wave functions of the body is not required.

3.5 External electromagnetic fields

It is in principle straightforward to generalize these discussions to the case of a macroscopic body in an external electromagnetic fields, especially in the limit the body can be considered pointlike, and under the uniform and constant and slowly varying electromagnetic fields.

For a macroscopic particle of total charge $Q$ the Hamiltonian is given by

$$H \rightarrow H = \sum_i \left[ \left( \mathbf{p}_i - \frac{e_i}{c} \mathbf{A}(\mathbf{r}_i) \right)^2 + e_i \Phi(\mathbf{r}_i) - \mu_i \cdot \mathbf{B}(\mathbf{r}_i) \right] + V_{int}(\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_{N-1}) ,$$

$$\sum_i e_i = Q; \quad \mathbf{B} = \nabla \times \mathbf{A}; \quad \mathbf{E} = \nabla \Phi(\mathbf{r}) ,$$
and $\mu_i$ is the intrinsic magnetic moment of the $i$-th constituent. The interactions with the external electromagnetic fields are included in the expression inside the square brackets; the last term of (3.64) represents all other interactions among the atoms and molecules describing the bound state. $V_{int}$ is assumed to have no dependence on the CM coordinate,

$$\nabla_R V_{int} = 0 :$$  \hfill (3.66)

the deviation from the free motion for the CM of the body under study, is due only to the external electromagnetic fields. For simplicity, we have left the dependence of $V_{int}$ on the spin variables and on the momenta $\tilde{p}_i$ (such as in the relativistic corrections and spin-orbit interactions) implicit.

### 3.5.1 Motion in a constant electric field

Let us start with the simplest case of a constant electric field, $E$: we take

$$\Phi(\mathbf{r}) = \mathbf{E} \cdot \mathbf{r}, \quad \mathbf{A} = \mathbf{B} = 0 .$$  \hfill (3.67)

The first Ehrenfest equation for the CM position reads simply

$$\frac{d}{dt} \langle \Psi | \mathbf{R} | \Psi \rangle = \sum_i \frac{d}{dt} \langle \Psi | \frac{m_i \mathbf{r}_i}{M} | \Psi \rangle = \frac{1}{i \hbar} \sum_i \langle \Psi | \frac{m_i}{M} [\mathbf{r}_i, H] | \Psi \rangle$$

$$= \langle \Psi | \frac{1}{M} \sum_i \mathbf{p}_i | \Psi \rangle = \frac{1}{M} \langle \Psi | \mathbf{P} | \Psi \rangle .$$  \hfill (3.68)

The second equation is

$$\frac{d}{dt} \langle \Psi | \mathbf{P} | \Psi \rangle = \frac{1}{i \hbar} \sum_i \langle \Psi | [\mathbf{p}_i, H] | \Psi \rangle ,$$  \hfill (3.69)

but

$$\frac{1}{i \hbar} \sum_i \langle \Psi | [\mathbf{p}_i, \sum_j e_j \Phi(\mathbf{r}_j)] | \Psi \rangle = - \sum_i \langle \Psi | e_i \mathbf{E} | \Psi \rangle = - Q \mathbf{E} ,$$  \hfill (3.70)

where $Q = \sum_i e_i$ is the total charge of the body, whereas

$$\frac{1}{i \hbar} \sum_i \langle \Psi | [\mathbf{p}_i, V_{int}] | \Psi \rangle = - \sum_i \langle \Psi | [\nabla_i, V_{int}] | \Psi \rangle = - \langle \Psi | [\nabla_R, V_{int}] | \Psi \rangle$$

$$= - \langle \Psi | \nabla_R V_{int} | \Psi \rangle = 0$$  \hfill (3.71)

(see (3.66)). Therefore a narrow wave packet for the CM, centered at $\mathbf{R}_{CM}, \mathbf{P}_{CM}$, obeys

$$\frac{d}{dt} \mathbf{R}_{CM} = \frac{1}{M} \mathbf{P}_{CM} , \quad \frac{d}{dt} \mathbf{P}_{CM} = - Q \mathbf{E} ,$$  \hfill (3.72)
as expected. If the electric field is due to a charge \( Q_0 \) at \( R_0 \) at a distance from \( R_{CM} \) much greater than the body's size, \( L_0 \), one finds Coulomb's law,

\[
\frac{d}{dt} P_{CM} = F = -\nabla_{R_{CM}} \frac{QQ_0}{|R_{CM} - R_0|}
\]  

(3.73)
in the first approximation (in which the body can be regarded as pointlike), by repeating an argument similar to the one used to get (3.59).

### 3.5.2 Motion in a constant magnetic field

Let us next consider an external, uniform constant magneti field \( B \). The vector potential can be taken as

\[
A(r) = \frac{1}{2} B \times r .
\]  

(3.74)
The first Ehrenfest theorem now reads

\[
\frac{d}{dt} \langle \Psi | R | \Psi \rangle = \langle \Psi | \frac{1}{M} \sum_i (p_i - \frac{e_i}{c} A(r_i)) | \Psi \rangle = \langle \Psi | \frac{1}{M} \sum_i \left( p_i - \frac{e_i}{2c} (B \times r_i) \right) | \Psi \rangle
\]  

(3.75)
The expectation value of the terms containing \( r_i \) vanish due to parity, and one gets

\[
\frac{d}{dt} \langle \Psi | R | \Psi \rangle = \frac{1}{M} \langle \Psi | P | \Psi \rangle ; \quad \frac{d}{dt} R_{CM} = \frac{1}{M} P_{CM} .
\]  

(3.76)
The second Ehrenfest equation is

\[
\frac{d}{dt} \langle \Psi | P | \Psi \rangle = \frac{1}{\hbar} \sum_i \langle \Psi | [p_i, H] | \Psi \rangle
\]

\[
= \frac{1}{\hbar} \sum_i \langle \Psi | [p_i, \sum_j \left( p_j - \frac{e_j}{c} A(r_j) \right)^2] | \Psi \rangle .
\]  

(3.77)
By using (3.74), one gets after some calculation,

\[
\frac{1}{\hbar} \sum_i [p_i, \sum_j \left( p_j - \frac{e_j}{c} A(r_j) \right)^2] = \sum_i \frac{e_i}{m_i c} (p_i - \frac{e_i}{c} A(r_i)) \times B ,
\]  

(3.78)
where a vector notation is used throughout, in order to avoid confusion between the space indices and the particle indices, \( i, j \). We now recognize

\[
j_i = \frac{e_i}{m_i} (p_i - \frac{e_i}{c} A(r_i))
\]  

(3.79)
as the familiar gauge-invariant (and conserved) electric current operator of the $i$-th constituent atom or molecule in an external magnetic field. Writing now the total current as

$$J(R_{CM}) = \langle \Psi | \sum_i j_i | \Psi \rangle,$$

one finds for the CM motion,

$$\frac{d}{dt} P_{CM} = \frac{1}{c} J \times B,$$

which is the Lorentz force.

### 3.5.3 A macroscopic particle with a magnetic moment

Let us now consider a magnetic field slowly varying in space. More precisely, we assume that the space variation of $B(r)$ within the body under consideration, is negligible. We shall take

$$\nabla_r B = 0,$$

as long as $r_i$ is within the support of the internal wave function (which is a region of linear size $L_0$). Note that under these circumstances the derivation of the Lorentz force above continues to be valid: the formula (3.81) holds also for non constant magnetic field $B$.

To analyze this problem, we first modify the kinetic terms in $H$ in (3.64) as

$$\sum_i \left( \frac{p_i - e_i A(r_i)}{2m_i} \right)^2 \approx \sum_i \left[ \frac{p^2_i}{2m_i} - \frac{e_i}{4m_ic} \{p_i \cdot (B \times r_i) + (B \times r_i) \cdot p_i \} \right],$$

where we dropped the terms proportional to the square of $A$, and we have written the vector potential as

$$A(r_i) = \frac{1}{2} B \times r_i, \quad (i = 1, 2, \ldots, N),$$

even though now $B$ depends on the CM position $R$. To see how this step is justified, consider

$$\nabla_r \times \left( \frac{1}{2} (B \times r) \right) = \frac{1}{2} (r \cdot \nabla) B + B,$$

but the first term is negligible, as long as this expression is used inside the expectation value,

$$\langle \Psi | \cdots | \Psi \rangle$$

(see (3.82)).

The right hand side of (3.83) can be seen to be equal to

$$\sum_i \left[ \frac{p^2_i}{2m_i} - \frac{e_i}{2m_ic} B \cdot (r_i \times p_i) \right] = \sum_i \left[ \frac{p^2_i}{2m_i} - \frac{e_i}{2m_ic} B \cdot \ell_i \right],$$

14See for instance Sec. 14.1.2 of [27].
where $\ell_i$ is the orbital angular momentum operator associated with the $i$-th constituent of the body. We recognize
\[
e_i\ell_i \over 2m_i c \quad (3.88)
\]
as the magnetic moment produced by the orbital motion of the charged constituent (Bohr’s magneton). By denoting the total magnetic moment of the body as
\[
\sum_i (\langle \Psi | (\mu_i + e_i\ell_i) | \Psi \rangle = \mu, \quad (3.89)
\]
we find
\[
\frac{d}{dt} P_{\text{CM}} = -\nabla_{\text{R}_{\text{CM}}} (\mu \cdot B), \quad (3.90)
\]
a familiar classical equation.

### 3.5.4 General consideration

One might think that the generalization of these results to include other effects, such as polarization, charge conduction, etc., is quite a straightforward task. However, a moment of reflection convinces us that to try to reproduce the whole richness of electromagnetic phenomena of condensed matter (see for instance, [45]) and classical electromagnetism, starting from the Ehrenfest theorem for a macroscopic body with $N \sim 10^{26}$ atoms and molecules, is quite another matter. The problem is that the reaction of the body to the external electromagnetic fields depends on the nature of the body, whether it is a conductor or a dielectric, or something else; in other words, the full details of the internal Hamiltonian and the internal wave function $\psi(\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_{N-1})$ describing the body, come into play. In simple terms, what happens is that each charged particle, electron, atomic nuclei, molecules, ions, react differently to the external electromagnetic field, and deform the internal structure of the body, according to the way and strength with which they are bound in the body. All of the detailed spin-dependent effects also will come into play.

Under such circumstance the analysis will quickly become very difficult; the task requires a systematic classification of different physics cases, and an elaborated work organization and preraration.

One has the feeling that what has been done in Sec. 3.5.1 - Sec. 3.5.3 amounts to the (though necessary!) very first few steps towards climbing up Mt. Everest.

This is in striking contrast to the case of the external gravitational forces considered in Sec. 3.2 and Sec. 3.3. Due to the universal attractive nature, proportional to the mass, of the gravitational forces, most of the details of the internal interactions and wave functions drop out of the equations. Each constituent, small or large, inside the body reacts and accelerates in an identical manner under the external gravitational force. This explains the fact that a macroscopic body, e.g., a metal piece, which contain in its interior the electron gas which exhibits quantum-mechanical collective behavior even at room temperatures, or
even stars and planets with $N \sim 10^{60}$ atoms and molecules, still obey Newton’s law for its CM motion, as it is known empirically and as it has been shown in Sec. 3.2, Sec. 3.3.

4 Boundaries between quantum and classical

Throughout the discussion of this work, there is clearly no sharp boundary between the quantum-mechanical and classical behavior of any given system as its temperature is varied. A rough estimate of the temperatures $T \sim T_0$ under which the microscopic, mesoscopic or macroscopic system with a characteristic (lowest-excitation-) frequency $\nu$ is in its quantum-mechanical lowest states, is given by

$$T \leq T_0 \ , \quad T_0 = \frac{h\nu}{k_B} = \nu \cdot 4.8 \cdot 10^{-11} \text{ K} \ .$$

The characteristic frequency $\nu$ depends on many factors of the system, the size, the mass, $N$, the atomic, molecular or collective excitation energies, etc.

| Systems          | size         | $N$          | $\nu$         | $T_0$       | Ref. |
|------------------|--------------|--------------|---------------|-------------|-----|
| Macro pendulum (LIGO) | 10.8 Kg   | $\sim 10^{27}$ | 100           | $\sim 1.4 \cdot 10^{-6} \text{ K}$ | [42] |
| Nanomech. resonator    | $10^{-13}$ cm$^3$ | $\sim 10^{11}$ | $\sim 2 \cdot 10^{9}$ | 1 mK       | [36] |
| Micromech. drum       | $\sim \mu$m  | $\sim 10^8 - 10^{10}$ | $10^9$       | 0.1K       | [22] |
| $C_{70}$             | 70 Å        | 70           | $10^{14}$     | $\sim 3000 \text{ K}$ | [28] |

Table 1: A small sample of the experiments aimed at demonstrating the quantum mechanical behavior of a mesoscopic or macroscopic system at sufficiently low temperatures. The size (the volume, linear dimension or weight) and the number of atoms/molecules cited are a crude estimate made from the lines of the articles by the present author: any lack of precision is to be attributed to him, not to the authors of the works cited.

Some mesoscopic or macroscopic systems have been studied recently for the purpose of revealing their quantum-mechanical behaviors at low temperatures [28], [31], [32]- [43]. A small sample of these experiments is summarized in Table 1, together with a few of the characteristics of the system considered and the relevant temperatures. For $N$ (the number of atoms/molecules) spanning over many orders of magnitude, the rule-of-thumb estimate (4.1) for the typical temperature which devides quantum-mechanical and classical behaviors of the system, seems to be basically valid.

In the case of the LIGO pendulum of reduced mass 2.7 Kg [42], the quoted temperature of $1.4 \mu \text{K}$ refers to the occupation numbers of about 200. In the case of a nanomechanical resonator of [36], the quoted temperature of 56 $m\text{K}$ refers to the quantum excitation levels up to $\sim 58$. In a most interesting case of the work by O’Connel et. al. [22], the achieved temperature is roughly consistent with the system being in its quantum ground state (no phonon excitations), as verified experimentally by coupling the system to a squid qubit.

For Buckminsterfullerene molecule $C_{70}$, the frequency $\nu$ is estimated from the known
molecular excitation energies, and $T_0$ given by Arndt et. al. [28] is the temperature of the $C_{70}$ beam at which the interference fringes à la Young disappear. The temperature is raised by increasing the laser power bombarding the $C_{70}$'s before they enter the double slit region. It is remarkable that by varying continuously the laser power the authors of [28] have been able to interpolate smoothly between the pure-quantum-state $C_{70}$ (with interference fringes à la Young after the passage of the double slit) and radiating, mixed-state “classical” $C_{70}$, i.e., with no interference fringes.

The relatively higher temperatures needed in the case of $C_{70}$ as compared to the other macroscopic quantum-classical states in Table 1 reflects the fact that here the excitations relevant are the $O(eV)$ molecular excitations, whereas in other examples the characteristic frequency (and the body temperature) refers to the collective, phonon excitation modes, or even to macroscopic oscillation modes.

5 Summary and Discussion

In this work we have derived Newton’s (force-) law from the Schrödinger equation 15. Classical mechanics, initiated by Kepler, Copernicus and Galilei, was completed by Newton in the form of the mathematical principles and three fundamental laws [46]. After three and a half centuries classical mechanics was replaced by quantum mechanics, so as to be able to describe correctly the microscopic world of electrons, atoms and molecules. Now, to describe appropriately the macroscopic world, we have come back all the way out, to “explain” Newton’s equations from the more fundamental quantum mechanics.

Concretely, we first identified the basic requirements for the CM of a macroscopic body in the vacuum to possess a classical trajectory, i.e., a well-defined position and momentum at each instant of time. The first is the Heisenberg uncertainty relation. A macroscopic body of $N \sim 10^{51}$ atoms satisfies, for its center of mass (CM), the same Heisenberg lower limit for the product of the canonical position and momentum, $\Delta X_i \cdot \Delta P_j \geq \delta_{ij} \hbar/2$ as for an atom. Namely the quantum uncertainties do not pile up. This means that for a macroscopic body, and for the measurements within the appropriate range of macroscopic scale and precision, Heisenberg’s relations can be considered irrelevant. The position and momentum can be measured simultaneously, to an “arbitrary” precision, $\Delta X \approx 0$, $\Delta P \approx 0$.

The second is the absence of the diffusion for free wave packets. Here the essential factor is the mass. While for an electron with mass $\sim 10^{-27}$ g, the time needed for doubling its wave packet size (starting from a 1 $\mu$m) is $10^{-8}$ sec, the corresponding time for a macroscopic body of 1 g, with the initial CM wave packet of the same size 1 $\mu$m, exceeds the age of the universe. This means that once its position and momentum are experimentally measured with certain precision $\Delta X$, $\Delta P$, they do not grow in time: a macroscopic body follows a classical (well-defined) path: a trajectory. An extended wave

15Newton’s other law’s, that of inertia (the first law) and that of action-reaction (the third law) are incorporated in quantum mechanics from the beginning.
packet is a uniquely quantum concept, which applies only to microscopic systems.

The third, and the crucial, factor is the nonzero temperature of the body, which makes it a radiating, metastable state. It is a mixed state. Decoherence caused by the entanglement with the “lost” (unobserved) photons it emitted and which carry away the information, means that a split wave packet for a macroscopic body - a coherent superposition of two macroscopically distinct state vectors - is a meaningless notion. It cannot be prepared experimentally.

How high the body temperature must be, to lose the quantum mechanical characteristics, depends on many factors, the type of material, the states, the density, size, etc., and varies vastly. We have discussed briefly this question in Sec. 4, drawing lessons from several existing experimental studies, which show that the rough estimate of the boundary temperature is given by a simple formula, (4.1). It captures essentially the nature (and the energy) of the lowest excitation modes, characteristic of the system considered, such as an atomic or molecular excitation, phonon excitations in a solid, or simply a macroscopic oscillation of the body. The formula (4.1) was found basically valid for mesoscopic-macroscopic bodies with number of atoms (or of molecules) in a vast range spanning from $N = 70$ to $N \sim 10^{27}$.

Once these basic elements, with various subtleties, are grasped (Sec. 2), the derivation of Newton’s equation for a macroscopic body is a straightforward application of Ehrenfest’s theorem, as we have illustrated in the case of weak gravitational forces (Sec. 3.2 and Sec. 3.3), a harmonic potential (Sec. 3.4), and under weak and almost space-independent external electromagnetic fields (Sec. 3.5). As remarked in Sec. 3.5.4, our work is only at the very initial stage of a highly nontrivial task, involving classical electromagnetism with macroscopic bodies and more general discussion of electromagnetism of continuous media.

Even though we have not yet worked it out in great detail, it appears to be basically straightforward to generalize the results of the present work to the case of many macro-body problems, and to establish the Hamilton equations and other aspects of classical mechanics. Appendix B gives a sketch of the derivation of the canonical equations.

Our way of understanding how the classical mechanics emerges from quantum mechanics for an isolated macroscopic body, which (we hope) complements and completes the previous works on the environment-induced decoherence mechanism [1]-[4], seems to point to a new perspective on the relation between quantum and classical physics. Namely, for a macroscopic body at finite temperatures (hence a mixed state), classical mechanics for its center-of-mass motion emerges as the lowest-order approximation of the quantum mechanical equations.

This seems to be a novel point of view, as compared to the traditional way of thinking

\[16\] The content of Sec. 2 and Sec. 3 defines the approximations needed to derive the classical equations. The nature of the approximations involved there suggests extremely small corrections in general; their magnitude will however depend on the kind of materials, mass, and temperatures, on the presence of the environment and so on. The study of “quantum corrections to Newton’s equations” is a subject of future studies.
about this problem, for instance, that classical physics emerges in the so-called semi-
classical approximation of quantum mechanics (i.e., in the large-action limit), or that the
so-called coherent states are the closest analogue of classical objects in quantum mechanics.
Other textbook examples include the formal similarity between the commutation relations
in quantum mechanics and the Poisson brackets in classical mechanics, and the fact that
in semiclassical limit the Schrödinger equation for the (exponent of) the wave function
reduces to the Hamilton-Jacobi equation. Still further examples are the Bohr-Sommerfeld
quantization conditions and Bohr’s correspondence principle.

Though these well-known ideas certainly constitute a necessary condition for the quan-
tum mechanics to be compatible with classical mechanics, and in that sense, are manifesta-
tions of deep consistencies within the quantum theory, they mostly concern pure quantum
states. None of these represents therefore the true explanation of the emergence of classical
physics from quantum mechanics, which hinges upon the mixed-state, large mass
nature of a macroscopic body at finite temperatures, as shown in the present work.

Before closing, it is perhaps appropriate to discuss, to avoid any misunderstanding,
the nature of the quantum-classical transition characterized by a temperature, (4.1), as
compared to the so-called critical temperatures associated with various kinds of critical
phenomena and phase transitions. Although some mention of various kinds of phase tran-
sitions was made in Sec. 1, as a way of illustrating the fact that any matter behaves
quantum mechanically at temperatures sufficiently close to 0, and accordingly, in order to
pinpoint the role of the temperature as a primary factor in our problem, we hope that the
careful reader has not been misled to the idea that, or that we are claiming that, these two
questions are substantially related.

The quantum-classical “transition” discussed in this work concerns a change in the
way we describe the world; it does not deal with real discontinuities in physics. The finite
temperatures of a body are just signs of the body being in an excited, metastable quantum
state. Any such system radiates spontaneously, and the entanglement with the photons it
emitted (and which carry away information) renders the system a mixture. Decoherence
which results is one of the main causes of the body to behave classically, as we saw.

The phase transitions and critical phenomena, instead, describe a genuine change in
physics occurring at a precise (critical) temperature. They describe a sudden collective
rearrangement of atoms and molecules in a large system\textsuperscript{17}, let it be a liquid, gas, solid, BEC
or a plasma. Even keeping in mind that the whole of statistical mechanics requires quantum
mechanics for its consistency at its basis, some phase transitions may still be understood
to various degrees within the context of classical physics, such as solid-liquid, liquid-vapor
transitions. In some others, such as spontaneous magnetization, superconductivity, BEC,
and superfluidity transitions, instead, quantum aspects of the systems appear so much
prominent. Nonetheless, all of these phase transitions are sometimes referred to as classical
phase transitions.

\textsuperscript{17}Strictly speaking, a true discontinuity or a mathematical singularity in thermodynamical quantities,
occur only in the systems with an infinite degrees of freedom, as is well known.
In the so-called quantum phase transitions, occurring at $T = 0$, which are a very active research field today, instead, phase transitions occur between two fully quantum-mechanical ground-state phases of the system.

These considerations illustrate clearly that the question of the emergence of the classical physics from quantum mechanics examined in the present work is conceptually distinct from, and should not be confused with, the interesting physics of various phase transitions, classical or quantum.

Acknowledgments

An inspiration for the present work came from an experiment performed in Pisa some $\sim 15$ years ago, by C. Bemporad, C. Bradaschia and F. Fidecaro, in a celebrative event for the public including school children. The setup consisted of two $\sim 3m$ tall glass tubes with diameter $\sim 30$ cm, with their interior kept in a good vacuum. A few-centimeter steel ball in the one and a light plastic sponge in the other, were let to fall simultaneously, demonstrating that they indeed fall together (!), evoking Galileo’s (actual or imaginary) experiment done from the top of the Leaning Tower around the year $\sim 1590$. The device is conserved in the entrance hall of the VIRGO/EGO gravitational-wave observatory at Cascina, near Pisa. The author is deeply indebted to F. Fidecaro for rearranging the reactivation of the apparatus and for demonstrating this glass-tube “Galileian” experiment anew for him. He also thanks F. Fidecaro and G. Cella for discussions on the physics of the gravitational wave detections at LIGO and VIRGO. He thanks P. Menotti and S. Bolognesi for comments on the manuscript and for discussions. A special thanks of the author is reserved for Hans Thomas Elze for many interesting discussions, suggestions, and for useful information about the literature. This work is supported by the INFN special research initiative grant, “GAST” (Gauge and String Theories).

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A  Perturbation theory for the Ehrenfest theorem

In the discussions of Sec. 3.3 a question arose how the perturbation theory can be applied to find out the way Ehrenfest’s theorem gets affected by a perturbation. In this Appendix, we use the standard (time-independent) perturbation theory for a 1D particle, to work out the question. For simplicity, we consider a system with a nondegenerate, discrete spectrum, and with the perturbative potential depending only on \(x\).

The Hamiltonian has the form,

\[
H = H_0 + \lambda V'(x),
\]

where \(H_0\) is the unperturbed Hamiltonian,

\[
H_0 = \frac{p^2}{2m} + V_0;
\]

\(V'\) represents the perturbation. An arbitrary parameter \(\lambda\) (which can be set to 1 at the end) has been introduced to keep track of the power of \(V'\) in various expressions.

The problem is to find the power-series solution for \(E(\lambda)\) and \(|\psi(\lambda)\rangle\)

\[
|\psi\rangle = |\psi(0)\rangle + \lambda |\psi(1)\rangle + \lambda^2 |\psi(2)\rangle + \ldots + \lambda^L |\psi(L)\rangle + \ldots;
\]

\[
E = E(0) + \lambda \varepsilon_1 + \lambda^2 \varepsilon_2 + \ldots.
\]

such that

\[
H(\lambda) |\psi(\lambda)\rangle = E(\lambda) |\psi(\lambda)\rangle,
\]

order by order in \(\lambda\), and with the boundary condition

\[
\lim_{\lambda \to 0} E(\lambda) = E(0) \quad \lim_{\lambda \to 0} |\psi(\lambda)\rangle = |\psi(0)\rangle.
\]

By denoting the unperturbed solutions of

\[
H_0 |\psi^{(0)}\rangle = E^{(0)} |\psi^{(0)}\rangle,
\]

as

\[
H_0 |k\rangle = E^{(0)}_k |k\rangle, \quad k = 0, 1, 2, \ldots
\]

the problem of finding the corrections to the \(n\)-th energy eigenvalue and associated eigenstate can be formulated by taking

\[
|\psi^{(0)}\rangle = |n\rangle, \quad E^{(0)} = E^{(0)}_n
\]

in (A.4) and (A.6) and solving for \(|\psi^{(L)}\rangle\) and \(\epsilon^{(L)}\). \(\lambda\) can be set to 1 after the calculation.

\[\text{\textsuperscript{18}}\text{prime} \text{ is there to indicate a perturbation, not a derivative.}\]
Because of the boundary condition, (A.9), all the expressions for the corrections to the energy eigenvalue and wave function below should carry the index “(n)”; it is however omitted as usual, for not cluttering the formulae.

The results can be found in any quantum mechanics textbook: to first order they are given by:

\[
\varepsilon_1 = \langle \psi^{(0)} | V' | \psi^{(0)} \rangle = \langle n | V' | n \rangle = V'_{nn}, \tag{A.10}
\]

\[
|\psi^{(1)}\rangle = \sum_{k}^{' \prime} |k\rangle \frac{1}{E_n^{(0)} - E_k^{(0)}} \langle k | V' | \psi^{(0)} \rangle = \sum_{k}^{' \prime} |k\rangle \frac{V'_{kn}}{E_n^{(0)} - E_k^{(0)}}. \tag{A.11}
\]

where the prime over the summation symbol means \( k \neq n \). The higher-order results can be found easily by an iterative procedure, see for instance [27]. The results are

\[
\varepsilon_L = \langle \psi^{(0)} | V' | \psi^{(L-1)} \rangle, \quad L = 1, 2, 3, \ldots \tag{A.12}
\]

which determines the correction to the energy to order \( \lambda^L \), and

\[
\langle k | \psi^{(L)} \rangle = \frac{1}{E_n^{(0)} - E_k^{(0)}} \langle k | V' | \psi^{(L-1)} \rangle - \frac{1}{E_n^{(0)} - E_k^{(0)}} \sum_{K=1}^{L-1} \varepsilon_K \langle k | \psi^{(L-K)} \rangle. \tag{A.13}
\]

for the wave function.

Now let us consider the effects of perturbation (A.1) in Ehrenfest’s theorem:

\[
\frac{d}{dt} \langle \psi | x | \psi \rangle = \frac{1}{i\hbar} \langle \psi | [x, H] | \psi \rangle = \langle \psi | \frac{p}{m} | \psi \rangle; \tag{A.14}
\]

\[
\frac{d}{dt} \langle \psi | p | \psi \rangle = \frac{1}{i\hbar} \langle \psi | [p, H] | \psi \rangle = -\langle \psi | \frac{dV_0}{dx} + \lambda \frac{dV'}{dx} | \psi \rangle. \tag{A.15}
\]

We note that for perturbations depending only on \( x \), (A.1), the classical relation

\[
\frac{dx_0(t)}{dt} = \frac{p_0(t)}{m}, \quad x_0(t) \equiv \langle \psi | x | \psi \rangle; \quad p_0(t) \equiv \langle \psi | p | \psi \rangle, \tag{A.16}
\]

holds to all orders of perturbation. The second Ehrenfest’s relation, (A.15), also contains on its left hand side \( dp_0(t)/dt \) to all orders. The right hand side of (A.15) thus may be considered as an expression for the force, \( F \). \( F \) may be expanded as

\[
F = F^{(0)} + \lambda F^{(1)} + \ldots + \lambda^L F^{(L)} + \ldots \tag{A.17}
\]

where

\[
F^{(0)} = -\langle \psi^{(0)} | \frac{dV_0}{dx} | \psi^{(0)} \rangle; \tag{A.18}
\]

\[
F^{(1)} = -\langle \psi^{(0)} | \frac{dV'}{dx} | \psi^{(0)} \rangle - \langle \psi^{(0)} | \frac{dV_0}{dx} | \psi^{(1)} \rangle - \langle \psi^{(1)} | \frac{dV_0}{dx} | \psi^{(0)} \rangle, \tag{A.19}
\]

45
etc. The corrections to the force to order \( L \) are given by

\[
F^{(L)} = - \sum_{K=0}^{L} \langle \psi^{(K)} | \frac{dV_0}{dx} | \psi^{(L-K)} \rangle - \sum_{K=0}^{L-1} \langle \psi^{(K)} | \frac{dV'}{dx} | \psi^{(L-K-1)} \rangle ,
\]

(A.20)

where the wave function corrections are given in (A.13).

In view of the application in Sec. 3.3, let us work out the second-order result fully.

\[
F^{(2)} = -\langle \psi^{(0)} | \frac{dV_0}{dx} | \psi^{(2)} \rangle - \langle \psi^{(1)} | \frac{dV_0}{dx} | \psi^{(1)} \rangle - \langle \psi^{(2)} | \frac{dV_0}{dx} | \psi^{(0)} \rangle \\
- \langle \psi^{(0)} | \frac{dV'}{dx} | \psi^{(1)} \rangle - \langle \psi^{(1)} | \frac{dV'}{dx} | \psi^{(0)} \rangle ,
\]

(A.21)

where

\[
\langle \psi^{(1)} \rangle = \sum_{k \neq n} \langle k | \frac{V_{kn}'}{E_{n}^{(0)} - E_{k}^{(0)}} \rangle , \\
\varepsilon_{1} = V_{nn}',
\]

(A.22)

\[
\langle k | \psi^{(2)} \rangle = \frac{1}{E_{n}^{(0)} - E_{k}^{(0)}} \langle k | V' | \psi^{(1)} \rangle - \frac{1}{E_{n}^{(0)} - E_{k}^{(0)}} \varepsilon_{1} \langle k | \psi^{(1)} \rangle.
\]

(A.23)

## B Canonical equations for many-(macro-)body systems

We sketch here the derivation (really, a strategy for it) of the canonical equations for a many-(macro-)body system made of \( S \) macroscopic bodies in the vacuum and at finite temperatures, as described in Sec. 2. We assume that the total Hamiltonian has the form,

\[
H = H_{CM} + \sum_{I} H_{int} ,
\]

(B.1)

where their centers of mass are described by

\[
H_{CM} = \sum_{I=1}^{S} \frac{\tilde{P}_{I}^{2}}{2M} + V(R_{1}, P_{1}, R_{2}, P_{2}, \ldots, R_{S}, P_{S}) ; \quad P_{I} = -i\hbar \nabla R_{I}
\]

(B.2)

and the internal Hamiltonians,

\[
H_{int} = \sum_{I=1}^{N_{I}} \frac{\tilde{P}_{I,1}^{2}}{2\mu_{I,1}} + V_{int}(\tilde{r}_{I,1}, \tilde{r}_{I,2}, \ldots, \tilde{r}_{I,N_{I}-1}) .
\]

(B.3)

are responsible for binding the \( I \)-th macroscopic body. The potential \( V \) might contain gravitational or Coulomb forces among the bodies (as discussed in Sec. 2), as well as higher-order effects such as the tidal effects, dipole-dipole interactions, van-der-Waals forces, etc., among the bodies.
Clearly the Hamiltonian (B.1) is to be understood as a sort of effective Hamiltonian, whose derivation is left for separate work; here it is just a (reasonable) assumption we adopt.

Such a Hamiltonian would allow for the wave function of a factorized form,

$$|\Psi\rangle = |\Psi_{CM}\rangle \prod_I |\psi_I\rangle$$  \hspace{1cm} (B.4)$$

where

$$H_{I\text{int}}\psi_I(\tilde{r}_{I,1}, \tilde{r}_{I,2}, \ldots \tilde{r}_{I,N_I-1}) = E_{I\text{int}}\psi_I(\tilde{r}_{I,1}, \tilde{r}_{I,2}, \ldots \tilde{r}_{I,N_I-1}) ; \hspace{1cm} (B.5)$$

$$i\hbar \frac{d}{dt} |\Psi_{CM}\rangle = H_{CM} |\Psi_{CM}\rangle = E_{CM} |\Psi_{CM}\rangle . \hspace{1cm} (B.6)$$

The Ehrenfest theorem for $R_I$ reads

$$\frac{d}{dt} R_{CMI} = \frac{d}{dt} \langle \Psi | R_I | \Psi \rangle = \frac{1}{i\hbar} \sum_i \langle \Psi_{CM} | [R_I, H_{CM}] | \Psi_{CM} \rangle = \langle \Psi_{CM} | (\nabla_{P_I} H_{CM}) | \Psi_{CM} \rangle$$

$$= \nabla_{P_{CMI}} H_{CM}(R_{CM1}, P_{CM1}, R_{CM2}, \ldots, P_{CMS}) ; \hspace{1cm} (B.7)$$

whereas the one for $P_I$ gives

$$\frac{d}{dt} P_{CMI} = \frac{d}{dt} \langle \Psi | P_I | \Psi \rangle = \frac{1}{i\hbar} \sum_i \langle \Psi_{CM} | [P_I, H_{CM}] | \Psi_{CM} \rangle = -\langle \Psi_{CM} | (\nabla_{R_I} H_{CM}) | \Psi_{CM} \rangle$$

$$= -\nabla_{R_{CMI}} H_{CM}(R_{CM1}, P_{CM1}, R_{CM2}, \ldots, P_{CMS}) , \hspace{1cm} (B.8)$$

for $I = 1, 2, \ldots, S$. In both (B.7) and (B.8), in the first and last steps the center of mass position and momentum operators inside the expectation values are replaced by the corresponding position and momentum centers of the (arbitrarily) narrow wave packet of the $I$-th body (Sec. 2). Also, the expectation value of the Hamiltonian operator in such a narrow wave packet is approximated by a classical Hamiltonian, with an arbitrary precision.

(B.7) and (B.8) are the familiar canonical Hamilton equations for the $S$-body classical system.