ODD-GRACEFUL LABELINGS OF TREES OF DIAMETER 5

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ABSTRACT. A difference vertex labeling of a graph $G$ is an assignment $f$ of labels to the vertices of $G$ that induces for each edge $xy$ the weight $|f(x) - f(y)|$. A difference vertex labeling $f$ of a graph $G$ of size $n$ is odd-graceful if $f$ is an injection from $V(G)$ to $\{0, 1, ..., 2n - 1\}$ such that the induced weights are $\{1, 3, ..., 2n - 1\}$. We show here that any forest whose components are caterpillars is odd-graceful. We also show that every tree of diameter up to five is odd-graceful.

1. Introduction

Let $G$ be a graph of order $m$ and size $n$, a difference vertex labeling of $G$ is an assignment $f$ of labels to the vertices of $G$ that induces for each edge $xy$ a label or weight given by the absolute value of the difference of its vertex labels. Graceful labelings are a well-known type of difference vertex labeling; a function $f$ is a graceful labeling of a graph $G$ of size $n$ if $f$ is an injection from $V(G)$ to the set $\{0, 1, ..., n\}$ such that, when each edge $xy$ of $G$ has assigned the weight $|f(x) - f(y)|$, the resulting weights are distinct; in other words, the set of weights is $\{1, 2, ..., n\}$. A graph that admits a graceful labeling is said to be graceful.

When a graceful labeling $f$ of a graph $G$ has the property that there exists an integer $\lambda$ such that for each edge $xy$ of $G$ either $f(x) \leq \lambda < f(y)$ or $f(y) \leq \lambda < f(x)$, $f$ is named an $\alpha$-labeling and $G$ is said to be an $\alpha$-graph. From the definition it is possible to deduce that an $\alpha$-graph is necessarily bipartite and that the number $\lambda$ (called the boundary value of $f$) is the smaller of the two vertex labels that yield the edge with weight 1. Some examples of $\alpha$-graphs are the cycle $C_n$ when $n \equiv 0 (\text{mod } 4)$, the complete bipartite graph $K_{m,n}$, and caterpillars (i.e., any tree with the property that the removal of its end vertices leaves a path).

A little less restrictive than $\alpha$-labelings are the odd-graceful labelings introduced by Gnanajothi in 1991 [4]. A graph $G$ of size $n$ is odd-graceful if there is an injection $f : V(G) \rightarrow \{0, 1, 2, ..., 2n - 1\}$ such that the set of induced weights is $\{1, 3, ..., 2n - 1\}$. In this case, $f$ is said to be an odd-graceful labeling of $G$. One of the applications of these labelings is that trees of size $n$, with a suitable odd-graceful labeling, can be used to generate cyclic decompositions of the complete bipartite graph $K_{n,n}$. In Figure 1 we show an odd-graceful tree of size 6 together with its embedding in the circular arrangement used to produce the cyclic decomposition of $K_{6,6}$. Once the labeled tree has been embedded, succesives $30^\circ$ (counterclockwise) rotations produce the desired cyclic decomposition of $K_{6,6}$.  

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Gnanajothi [4] proved that the class of odd-graceful graphs lies between the class of \(\alpha\)-graphs and the class of bipartite graphs; she proved that every \(\alpha\)-graph is also odd-graceful. The reverse case does not work, for example the odd-graceful tree shown in Figure 1 is the smallest tree without an \(\alpha\)-labeling. Since many families of \(\alpha\)-graphs are known, the most attractive examples of odd-graceful graphs are those without an \(\alpha\)-labeling or where an \(\alpha\)-labeling is unknown; for instance, Gnanajothi [4] proved that the following are odd-graceful graphs: \(C_n\) when \(n \equiv 2(\text{mod}4)\), the disjoint union of \(C_4\), the prism \(C_n \times K_2\) if and only if \(n\) is even, and trees of diameter 4 among others. Eldergil [2] proved that the one-point union of any number of copies of \(C_6\) is odd-graceful. Seoud, Diab, and Elsakhawi [5] showed that a connected \(n\)-partite graph is odd-graceful if and only if \(n = 2\) and that the join of any two connected graphs is not odd-graceful.

A detailed account of results in the subject of graph labelings can be found in Gallian’ survey [3].

Gnanajothi [4] conjectured that all trees are odd-graceful and verified this conjecture for all trees with order up to 10. The author has extended this up to trees with order up to 12. In this paper we prove that all trees of diameter 5 are odd-graceful and that any forest whose components are caterpillars is odd-graceful.

2. Odd-Graceful Forests

In this section we study forests that accept odd-graceful labelings. Recall that a forest with more than one component cannot be graceful because it has “too many edges”. First we prove that any graph that admits an \(\alpha\)-labeling also admits an odd-graceful labeling by transforming conveniently its \(\alpha\)-labeling.

**Theorem 1.** Any \(\alpha\)-graph is odd-graceful.

**Proof.** Let \(G\) be an \(\alpha\)-graph of size \(n\), as consequence \(G\) is bipartite with partition \((A,B)\). Suppose that \(f\) is an \(\alpha\)-labeling of \(G\) such that \(\max\{f(x) : x \in A\} < \min\{f(x) : x \in B\}\). Let \(g\) be a labeling of the vertices of \(G\) defined by

\[
g(x) = \begin{cases} 
2f(x), & x \in A \\
2f(x) - 1, & x \in B.
\end{cases}
\]

Thus, the labels assigned by \(g\) are in the set \(\{0, 1, \ldots, 2n - 1\}\), furthermore, the weight of the edge \(xy\) of \(G\) induced by the labeling \(f\), where \(x \in A\) and \(y \in B\), is \(w = f(y) - f(x)\), so its weight under the labeling \(g\) is \(g(y) - g(x) = 2f(y) - 1 - 2f(x) + 1 = 2f(y) - 2f(x) + 1 = 2f(y) - f(x)\).

Odd-graceful labelings of trees of order 11 and 12 can be found at [http://cims.clayton.edu/cbarrien/research](http://cims.clayton.edu/cbarrien/research).
$2f(x) = 2(f(y) - f(x)) - 1 = 2w - 1$. Since $1 \leq w \leq n$, we have that the weights induced by $g$ are $\{1, 3, ..., 2n - 1\}$. Therefore, $g$ is an odd-graceful labeling of $G$. □

In Figure 2 we show an example of an $\alpha$-labeling of a caterpillar, followed for the corresponding odd-graceful labeling. We use this labeling in the next theorem.

\begin{theorem}
Any forest which components are caterpillars is odd-graceful.
\end{theorem}

\begin{proof}
Let $F_i$ be a caterpillar of size $n_i \geq 1$, for $1 \leq i \leq k$. Let $u_i, v_i \in V(F_i)$ such that $d(u_i, v_i) = \text{diam}(F_i)$; so identifying $v_i$ with $u_{i+1}$, for each $1 \leq i \leq k - 1$, we have a caterpillar $F$ of size $\sum_{i=1}^{k} n_i = n$. Now we proceed to find both, the $\alpha$-labeling of $F$ and its corresponding odd-graceful labeling, using the scheme shown in Figure 2. Once the odd-graceful labeling has been obtained, we disengage each caterpillar $F_i$ from $F$, keeping their labels; in this form, the weights induced are $\{1, 3, ..., 2n - 1\}$. To eliminate the overlapping of labels we subtract 1 from each vertex label of $F_i$ when $i$ is even, in this way the weights remain the same and the labels assigned on $u_{i+1}$ and $v_i$ differ by one unit. Therefore, the labeling of the forest $\bigcup_{i=1}^{k} F_i$ is odd-graceful.

\end{proof}

In Figure 3 we show an example of this construction using the odd-graceful labeling obtained in Figure 2.

The procedure used in this proof can be extended to the disjoint union of graphs with $\alpha$-labelings. In fact, suppose that the concatenation of blocks $B_1, B_2, ..., B_k$ results in a graph $G$ whose block-cutpoint graph is a path. In [1] we proved that if each $B_i$ is an $\alpha$-graph, so it is $G$. Transforming this $\alpha$-labeling into an odd-graceful labeling and disconnecting $G$ into blocks, the disjoint union of these blocks is odd-graceful.
The disjoint union of blocks that accept $\alpha$-labelings is odd-graceful.

3. Odd-Graceful Trees of Diameter Five

Every tree of diameter at most 3 is a caterpillar, therefore it is odd-graceful. Gnanajothi [4] proved that every rooted tree of height 2 (that is, diameter 4) is odd-graceful. In the next theorem we represent trees of diameter 5 as rooted trees of height 3 and prove that they are odd-graceful.

Let $T$ be a tree of diameter 5; $T$ can be represented as a rooted tree of height 3 by using any of its two central vertices as the root vertex. Note that only one of the vertices in level 1 has descendants in level 3; this vertex will be located in the right extreme of level 1. Now, within each level, the vertices are placed from left to right in such a way that their degrees are increasing. In the proof of the next theorem we use this type of representation of $T$, that is, assuming that $v$ (one of the two central vertices) is the root.

**Theorem 4.** All trees of diameter five are odd-graceful.

*Proof.* Let $T$ be a tree of diameter 5 and size $n$. Suppose that $T$ has been drawn according to the previous description. Let $v_{i,j}$ denote the $i$th vertex of level $j$, for $j = 1, 2, 3$, this vertex is placed at the right of $v_{i+1,j}$. Consider the labeling $f$ of the vertices within each level given by recurrence as follows: $f(v) = 0$, $f(v_{1,1}) = 2n - 2 \deg(v) + 1$, $f(v_{1,2}) = 2$, $f(v_{1,3}) = 3$, and $f(v_{i,j}) = f(v_{i-1,j}) + d(v_{i,j}, v_{i-1,j})$ where $i \geq 2$ and $1 \leq j \leq 3$.

We claim that $f$ is an odd-graceful labeling of $T$. In fact, let us see that there is no overlapping of labels. On level 0 the label used is 0 and on level 2 all labels are even being 2 the smallest label used here. On levels 1 and 3 the labels used are odd; on level 1 the labels used are $2n-1, 2n-3, ..., 2n-2 \deg(v) + 1$, while on level 3 the labels used are $3, 5, ..., 2 \deg(v_{1,2}) - 1$. Now we need to prove that $2n - 2 \deg(v) + 1 > 2 \deg(v_{1,2}) - 1$; since $T$ is a tree of diameter 5, at least two vertices on level 1 have descendants, so $n + 1 > \deg(v) + \deg(v_{1,2})$, which implies the desired inequality.

As a consequence of the fact that labels used in consecutive levels have different parity, each weight obtained is an odd number not exceeding $2n-1$. Suppose that $v_{i+1,j}$ and $v_{i,j}$ have the same father $x$, by definition of $f$, the edges $xv_{i+1,j}$ and $xv_{i,j}$ have consecutive weights. If $v_{i+1,j}$ and $v_{i,j}$ have different father, $x$ and $y$ respectively, then $|f(y) - f(v_{i,j})| = |(f(x) + 2) - (f(v_{i+1,j}) + 4)| = |f(x) - f(v_{i+1,j}) - 2|$. Thus, the weights are $2n - 2 \deg(v) - 1, ..., 2 \deg(v) + 1$. On level 2, the weights are $2n - 2 \deg(v) - 3, ..., 2 \deg(v_{1,2}) - 1$, and on level 3 the weights are $2 \deg(v_{1,2}) - 3, ..., 1$.

Therefore, $f$ is an odd-graceful labeling of $T$. \hfill $\Box$

In Figure 4 we present a scheme of this labeling for a tree of size 13.
Similar arguments can be used to find odd-graceful labelings of trees of diameter 6; however we do not have a general labeling scheme for this case. So it is an open problem determining whether trees of diameter 6 are odd-graceful. In Figure 5, we give an example of an odd-graceful labeling for a tree of size 17 and diameter 6.

To conclude this section, we show in Figure 6 an odd-graceful labeling for a special type of tree of diameter 6, namely the star $S(n, 3)$ with $n$ spokes of length 3.
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