Twistors and Holography*

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We extend Bousso’s notion of a lightsheet - a surface where entropy can be defined in a way so that the entropy bound is satisfied - to more general surfaces. Intuitively, these surfaces may be regarded as deformations of the Bousso choice; in general, these deformations will be timelike and so we refer to them as ‘timesheets’. We show that a timesheet corresponds to a section of a certain twistor bundle over a given spacelike two-surface \( B \). We further argue that increasing the entropy flux through a given region of spacetime corresponds to increasing the volume of certain regions in twistor space. Put another way, it would seem that entropy in spacetime corresponds to volume in twistor space. We argue that this formulation may point a way towards a version of the covariant entropy bound which allows for quantum fluctuations of the lightsheet. We also point out that in twistor space, it might be possible to give a purely topological characterization of a lightsheet, at least for suitably simple spacetimes.

I. INTRODUCTION

Throughout history [1], one of the central problems faced by philosophers and scientists has been the simple query: How does one define the concepts of space and time? Are they merely abstractions which we have introduced in order to facilitate a description of the interrelationship between things which actually ‘exist’ (e.g., such as material bodies)? Or do space and time ‘exist’ in and of themselves, without any reference to observable consequences? Although such Machian musings may seem esoteric, they actually take on a new and exciting life when viewed in the light of modern ideas coming from theoretical physics, especially quantum gravity. Indeed, most theoretical physicists today would probably agree that ‘space’ and ‘time’ will be effective concepts which only emerge at low energies. At scales past the Planck energy, ‘space’ and ‘time’ will simply cease to have any operational meaning, and some more fundamental ideas (comprising quantum gravity) will have to take over.

Of course, at first it seems nonsensical to assert that the ‘ultimate theory’ should be constructed without invoking the use of the words ‘space’ or ‘time’. After all, from our earliest days of undergraduate physics, we were all weaned on physical theories which simply would not make any sense without reference to these concepts. More precisely, theories such as classical or quantum mechanics are useful precisely because they are bodies of knowledge which allow us to predict, with at least some probability, the nature of future events given some knowledge about present or past events. One of the conceptual obstructions to constructing a quantum theory of gravity is that it is unclear what the theory will have to do with prediction; although certain quantum gravity models do yield predictions (such as the No Boundary Proposal), much more work on connecting quantum gravity with the low energy world around us is needed.

On the aesthetic level, however, the construction of a quantum theory of gravity may be the most beautiful way of solving the puzzle of space and time by showing that the concepts can be removed altogether from scientific discourse (since they are low energy manifestations of some more fundamental concepts); surely, Ernst Mach would be proud of this approach. On the other hand, even if we do discover such a ‘pre-geometric’ form of quantum gravity, we will still need to understand how general relativity can emerge at low energies.

Motivated by this problem, various authors have put forward the idea that the ‘Holographic Principle’ should somehow be incorporated into any attempt to construct a quantum theory of gravity. This principle, which was first developed in papers by ’t Hooft [2] and Susskind [3], is on the surface a radical statement about how many degrees of freedom there are in Nature. In essence, the principle asserts that a physical system can be completely described by information which is stored at the boundary of the system, without exceeding one bit of information per unit Planck area.

For some time, there was no precise covariant statement of the Holographic Principle; however, this situa-

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tion was rectified in a series of elegant papers by Fischler and Susskind [4] and Bousso ([5], [6], [7]). In particular, by carefully choosing the lightlike surfaces (called ‘light-sheets’) where the entropy of a given system can reside, Bousso was able to develop a mathematically precise covariant entropy conjecture.

Soon after the work of Fischler, Susskind and Bousso (FSB), a proof of various classical versions of Bousso’s bound was provided by Flanagan, Marolf and Wald (FMW) [8]. In order to make a mathematically precise statement, which could consequently be proven, they had to take the crucial step of introducing the notion of an entropy flux vector, usually denoted $s^a$. The total entropy through a given light-sheet is then defined to be the integral of $s^a$ over the surface of the light-sheet. They showed that FSB-type bounds could be proven, provided the entropy flux vector satisfied the following two criteria:

A: $(s^a k^a)^2 \leq T_{ab} k^a k^b / (16\pi) + \sigma_{ab} \sigma^{ab} / 128\pi^2$

B: $|k^a k^b \nabla_a s_b| \leq \pi T_{ab} k^a k^b / 4 + \sigma_{ab} \sigma^{ab} / 32$

where $k^a$ denotes the tangent vector to a given null geodesic $\lambda$ generating the light sheet in question. $T_{ab}$ denotes the stress-energy tensor, and $\sigma_{ab}$ denotes the shear tensor of the null congruence [18]. In essence, one may regard conditions (1) and (2) as the ‘definition’ of an acceptable entropy flux vector $^1$. Crudely, the stress-energy part of the entropy flux is generated by ‘matter’ degrees of freedom, whereas the shear part corresponds to purely gravitational degrees of freedom [10]. Finally, we point out that there have been some criticisms of the covariant entropy bound and of the holographic principle more generally, see e.g., [11], [12], [17] and [14] as examples.

In this paper, we adopt the philosophy that ultimately a quantum theory of gravity will be some structure which relates information - in some primal form - with the geometry of space and time. A natural question is therefore: ‘Where’ is information stored, and how much information can we store there? The covariant entropy bound gives us a proposal: Information is stored on those surfaces where the entropy bound holds. It therefore behooves us to classify the most general set of such surfaces.

II. TWISTORS, TIMESHEETS AND ENTROPY

According to one definition, a ‘null’ twistor is a null geodesic in Minkowski space [15], [16]. Consider a point $x$ in spacetime. Then there is a full ‘lightcone’ worth of null rays through that point. Put another way, a point in spacetime corresponds to an $S^2$ in twistor space. Similarly, consider some spacelike 2-surface $B$. Then the space of twistors over $B$ is an $S^2$ bundle over $B$. We denote this bundle of null rays over $B$ by $T_B$:

$$T_B = S^2 \rightarrow B$$

Given the bundle of null rays over $B$, we can consider smooth sections of this bundle. Given a section $s$ of this bundle, we will define certain surfaces, called ‘timesheets’, $^3$ associated with the section $s$ as follows: Given a point $x$ in $B$, $s$ corresponds to a null ray or vector. Since we are assuming the spacetime is time orientable, this can be decomposed into a future directed geodesic (from $x$), and similarly a past directed geodesic from $x$. The future timesheet associated with $s$, denoted $T^+ (s)$ is obtained by terminating each future directed null generator of $s$ at any caustic. There is the obvious corresponding definition for the past timesheet, $T^- (s)$. Terminating these surfaces at caustics is of course the prescription Bousso gives for terminating lightsheets, the only difference here is that these sections are in general timelike. In order to understand this, it is useful to recall the Raychaudhuri equation [18], [19], [20]:

$$\frac{d\theta}{d\lambda} = -\frac{\theta^2}{2} - \sigma_{ab} \sigma^{ab} + \omega_{ab} \omega^{ab} - 8\pi T_{ab} k^a k^b \quad (1)$$

Bousso assumes that the null generators of lightsheets are everywhere orthogonal to $B$, which means that the rotation or twist parameter vanishes:

$$\omega_{ab} = 0$$

In general one could imagine that the congruence is not orthogonal, and that the twist is non-vanishing. Assuming causality, it follows that the surface will in general be timelike. Intuitively, the reader should think of the null generators of lightsheets as null rays through that point. Put another way, a point in spacetime corresponds to an $S^2$ in twistor space. Similarly, consider some spacelike 2-surface $B$. Then the space of twistors over $B$ is an $S^2$ bundle over $B$. We denote this bundle of null rays over $B$ by $T_B$:

$$T_B = S^2 \rightarrow B$$



1Recently, other simple sufficient conditions for the entropy bound have been derived [9]. For the purposes of this note, and of these conditions will suffice.

2For the sake of simplicity, we will use the term ‘twistor’ and ‘null twistor’ interchangeably here.

3I thank R. Bousso for suggesting the name ‘timesheet’.

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Similarly, the shear tensor satisfies the equation
\[ 8\pi T_{ab}k^a k^b + \sigma_{ab}\sigma^{ab} \geq \omega_{ab}\omega^{ab} \tag{3} \]
If the inequality (3) is satisfied everywhere along a given timesheet, then we will say that the timesheet is *holographic*, for the simple reason that the entropy bound will hold on the timesheet. But there is yet another subtlety: It could be the case that the twist tensor *diverges* too quickly, so that the inequality (3) is violated. To see that this may not happen, recall the equation for the twist tensor [19]:
\[ k^c \nabla_c \omega_{ab} = \frac{d}{d\lambda}(\omega_{ab}) = -\theta \omega_{ab} \tag{4} \]
Similarly, the shear tensor satisfies the equation
\[ \frac{d}{d\lambda}(\sigma_{ab}) = -\theta \sigma_{ab} + C_{abcd}k^c k^d \tag{5} \]
where the ‘hat’ (\(\hat{C}\)) operation over the Weyl tensor \(C\) is explained in [19]. Now, suppose that the inequality (3) is initially satisfied - then \(\theta\) is initially decreasing towards minus infinity. If we assume that \(\theta\) reaches minus infinity in a finite amount of affine parameter time, we can estimate how \(\omega_{ab}\) and \(\sigma_{ab}\) must vary in order for the conjugate point to be reached. Following along the geodesic, it follows that as \(\theta\) runs to minus infinity, \(\omega_{ab}\) and \(\sigma_{ab}\) must ‘scale’ at the same rate, so that their effects are precisely cancelled. While this does not forbid violation of (3) at some points where the original covariant entropy bound holds, it does suggest that as long as there is enough shear it should be possible to allow for fluctuations away from orthogonality.

So when can holographic timesheets exist? Well, it is clear that they only exist when there is a ‘sufficient’ amount of entropy floating around in the spacetime. If both the shear tensor and the stress-energy tensor vanish, then the rotation will have to vanish and one must return to using the Bousso prescription.

Turning this around, suppose we try to ‘force’ a given amount of entropy flux through a given region. Then the criteria A and B for the covariant entropy bound imply that we must correspondingly see an increase in either the shear tensor or the stress-energy tensor through that region. But this in turn implies that we will have more freedom in our choice of holographic timesheets. Put another way, we could think of measuring the ‘size’ of the space of holographic timesheets. The space of holographic timesheets on \(B\), which we denote \(H(B)\), is a compact subset of the compact twistor bundle over \(B\) (here we are assuming that \(B\) is some compact spacelike 2-surface), so \(H(B)\) has some finite volume, \(\text{vol}H(B)\). So it would seem that there is a direct relationship between entropy flux in spacetime, and the volume of the space of holographic timesheet sections in twistor space. We emphasize that this proposal is a definition of (and indeed a conjecture about) how entropy in spacetime might be realized in twistor space.

### III. CONCLUSION: LIGHTSHEETS VIA TWISTOR TOPOLOGY?

An old dream of the twistor programme was that physics should be formulated in twistor space - that in some sense twistors are more ‘fundamental’ than spacetime points [16], [15]. Here, we have explored this possibility, at least for the case of ‘real’ twistors, which may be thought of as the bundle of null rays over a given spacetime. Since the modern, fully covariant realizations of the holographic entropy bound suggest that we should think of information as being defined on null rays, this is suggestive that twistor theory may indeed play a role in the construction of a quantum theory of gravity. At the very least, holography is a little more ‘obvious’ in twistor space: Given a holographic timesheet, the entropy integral over the timesheet is manifestly two-dimensional in twistor space. The assumptions underlying the bound then amount to the statement that you can only place a certain amount of ‘entropy’ on a given twistor or null ray. It would be interesting to have some fundamental explanation for this statement.

While this construction may seem academic, it may also have some useful applications. In particular, one may imagine scenarios where the Bousso lightsheet may be forced to fluctuate [17]. Such a fluctuation will generically produce some timesheet. As long as the fluctuation is sufficiently small, so that (3) is still satisfied, then the covariant entropy bound will still hold. In this sense, we feel that allowing for timesheets in the formulation of the bound only makes the bound more robust.

Finally, we would like to mention an exciting possibility that may allow for the construction of an entropy bound that only uses *topological* properties of twistor space. It has been known, through the work of Low and others ([21], [22]), that there is a relationship between linking in twistor space and causal structure in spacetime. To be precise, recall that a point in spacetime corresponds to an \(S^2\) in twistor space. Then it turns out that, at least for suitably ‘simple’ spacetimes, two points in spacetime are causally related if and only if the corresponding two-spheres are ‘linked’. Now assume that we focus on a small causal diamond in the spacetime, which is globally hyerbolic with some Cauchy surface \(S\), and let \(A\) be some spacelike region with boundary \(B\). Low [23] has further argued that the Bousso choice for the lightsheet of \(B\) corresponds precisely to the future horismos.
of $A^4$. Put another way, in twistor space the lightsheet is just a boundary between a region of linked spheres and a region of unlinked spheres. Thus, the lightsheet can be specified in terms of purely topological data in twistor space. We would emphasize that this can only really be true for very simple spacetimes, but it would be interesting to see how robustly these speculations can be implemented.

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\[4\]The reader will recall that the future horismos of $A$ is just the boundary of the Cauchy development.