Study of Graded Algebras and General Linear Group with Lie Superalgebras and R-Algebra

Khondokar M. Ahmed1, S. K. Rasel2, Jyoti Das3, Saraban Tahura4 and Salma Nasrin5

1, 2Department of Mathematics, University of Dhaka, Dhaka-1000, Bangladesh.
3Department of General Educational Development, Daffodil International University, Dhaka-1207, Bangladesh.
4Department of Mathematics, Comilla University, Comilla, Bangladesh.
5Department of Natural Sciences, University of Information Technology and Sciences, Dhaka 1212, Bangladesh.

(Received : 22 July 2018 ; Accepted : 7 January 2020 )

Abstract

Some elements of theory of \( \mathbb{Z}_2 \)-graded rings, modules and algebras. \( \mathbb{Z}_2 \)-graded tensor algebra, Lie superalgebras and matrices with entries in a \( \mathbb{Z}_2 \)-graded commutative ring are treated in our present paper. At last a Theorem 4.4 on the set of square matrices in the graded R-algebra \( M_n[\mathfrak{m}[m]|n] \) is established.

Keywords: \( \mathbb{Z}_2 \)-graded rings, modules, commutative ring and graded algebras, tensor calculus, general graded linear group GL[\( m|n] \), the set of graded matrices \( M_n((p + q) \times (m + n)) \) and graded R-algebra.

I. Introduction

Nowadays a large body of literature is available concerning graded algebras, mainly over the real or complex numbers (usually called superalgebras), their representations, etc. Classical references are [3], [6], [7], [8], [10]. The most common notations and basic results are treated in this article.

II. Graded Algebraic Structures

In general, given an arbitrary group \( G \), we can introduce \( G \)-graded algebraic objects [5], [10]. Since in order to develop a 'supergeometry' only \( \mathbb{Z}_2 \)-graded structures are needed, we shall only consider here that particular case. We shall assume as a rule that

\[ \text{graded} \equiv \mathbb{Z}_2 - \text{graded} \]

Definition 2.1. A ring \((R, +, \cdot)\) is said to be graded if \((R, +)\) has two subgroups \( R_0 \) and \( R_1 \) such that \( R = R_0 \oplus R_1 \) and \( R_\alpha R_\beta \subset R_{\alpha + \beta} \) for all \( \alpha, \beta \in \mathbb{Z}_2 \).

An element \( a \in R \) is said to be homogeneous if either \( a \in R_0 \) or \( a \in R_1 \). On the set \( h(R) \) of homogeneous elements an application \( | | \) is defined by

\[ | | : h(R) \to \mathbb{Z}_2 \]
\[ a \mapsto a \iff a \in R_\alpha. \]

The elements of degree 0 and 1 are called even and odd respectively.

Obviously, any ring \( R \) can be trivially graded: \( R_0 = R, R_1 = \{0\} \).

Example 2.2. Let \( R \) be a \( \mathbb{Z} \)-graded ring, namely, \( R = \bigoplus_{p \in \mathbb{Z}} R_p \) and \( R_p \subset R_{p+q} \) then \( R \) can be graded by taking \( R_0 \) as the sum of the even components and \( R_1 \) as the sum of the odd ones.

For any graded ring \( R \), a graded commutator \( <, > : R \times R \to R \) is defined by letting

\[ <a, b >= ab - (-1)^{|a||b|}ba \forall a, b \in h(R) \] (2.1)

The centre of \( R \) is defined as the set

\[ C(R) = \{ a \in R | <a, b> = 0 \forall b \in R \} \]

i.e. \( C(R) \) is the set of the elements of \( R \) which graded – commute with any other elements.

A graded ring \( R \) is said to be graded-commutative if \( <a, b> = 0 \forall a, b \in R \), that is, if \( C(R) = R \).

Let \( R \) be a graded ring and \( M \) be a left(right) \( R \)-module.

Definition 2.3. \( M \) is a left (right) graded \( R \)-module if it has two subgroups \( M_0 \) and \( M_1 \) such that \( M = M_0 \oplus M_1 \) and for all \( \alpha, \beta \in \mathbb{Z}_2 \), one has \( R_\alpha M_\beta \subset M_{\alpha + \beta} \subset R_\alpha M_\beta \).

If \( R \) is graded-commutative, which we shall henceforth assume, we shall use the term 'graded \( R \)-module' without ambiguity.

Having fixed two graded \( R \)-modules \( M \) and \( N \), we say that a morphism \( f : M \to N \) is \( R \)-linear on the right if \( f(xa) = f(x)a \) for all \( x \in M \) and \( a \in R \). Unless otherwise stated, by 'linear' we mean 'linear on the right'. Moreover, we say that \( f \) has degree \( |f| = \alpha \in \mathbb{Z}_2 \), if \( f(M_\alpha) \subset N_{\alpha + \beta} \) for all \( \alpha \in \mathbb{Z}_2 \). The set \( \text{Hom}(M, N) \) of \( R \)-linear morphisms \( M \to N \) (that will be denoted simply by \( \text{Hom}(M, N) \)) has a natural grading, with \( f \in \text{Hom}(M, N)_{|f|} \) whenever \( |f| = \alpha \). If \( R \) is graded-commutative, \( \text{Hom}(M, N) \) is a graded \( R \)-module, with the multiplication rule \( (af)(x) = af(x) \).

One of the most basic results in commutative ring theory, namely the Nakayama lemma, can be generalized to the graded setting. Let us define the radical of a graded-commutative ring \( R \) as the graded ideal \( \mathfrak{R} \) obtained by intersecting all maximal graded ideals of \( R \).

Proposition 2.4.(Graded Nakayama Lemma) Let \( R \) be a graded-commutative ring \( R \), \( I \) be a graded ideal contained in the radical \( \mathfrak{R} \) of \( R \) and \( M \) be a graded finitely generated \( R \)-module.

\*Author for correspondence. e-mail: meznang@yahoo.co.uk
(a) If $IM = M$, then $M = 0$.
(b) If $N$ is a graded submodule of $M$ and $M = IM + N$, then $M = N$.
(c) If $x^1, \ldots, x^m$ are even elements and $y^1, \ldots, y^n$ are odd elements in $M$ such that the images $(\tilde{x}^1, \ldots, \tilde{x}^m, \tilde{y}^1, \ldots, \tilde{y}^n)$ are generators of $M/IM\text{over } R/I$, then $(\tilde{x}^1, \ldots, \tilde{x}^m, \tilde{y}^1, \ldots, \tilde{y}^n)$ are generators of $M/\text{over } R$.

**Definition 2.5.** A graded $R$-module $F$ is said to be free if it has a basis formed by homogeneous elements.

A basis of $F$ of finite cardinality is of type $(m, n)$, if it is formed by $m$ even elements $\{f^i_0 \in F_0 \mid i = 1, \ldots, m\}$ and $n$ odd elements $\{f^i_1 \in F_1 \mid \alpha = 1, \ldots, n\}$. We have a canonical isomorphism

$$F \cong \bigoplus_{i = 1}^{m} Rf^0_i \oplus \bigoplus_{\alpha = 1}^{n} Rf^1_\alpha.$$ 

For each pair of natural numbers $m, n$ such that $m + n = p$, the $R$-module $R^p$ can be regarded as a free graded $R$-module endowed with a basis of type $(m, n)$, by letting

$$(r_{m+n})_0 \equiv r_{m+n} = R^m \oplus R^n;$$

$$(r_{m+n})_\alpha \equiv R^m_{\alpha} \oplus R^n;$$

$$(r_{m+n}) = R^m_{\alpha} \oplus R^n;$$

$$(r_{m+n}) = R_{m+n}.$$ 

$R_{m+n}$ equipped with this gradation will be denoted by $R^{m,n}$.

**Example 2.6.** (cf. [5]) Let $R$ be a commutative ring, and $M$ be an $R$-module. The exterior algebra of $R\text{over } R$, denoted by $\Lambda_R M$, is a $\mathbb{Z}$-graded algebra, namely $\bigoplus_{p \in \mathbb{Z}} \Lambda^p_R M$, and is alternating, i.e. $x^2 = 0$ for all $x \in \Lambda_R^{2p+1} M$. If $M$ is free and finitely generated, with a basis $\{e_i \mid i = 1, \ldots, N\}$, then $\Lambda_R M$ is a free finitely generated $R$-module, with a canonical basis relative to the basis $\{e_i\}$ which can be described as follows. Let $\mathbb{E}_N = \{\mu \mid \mu = \{1, \ldots, r\} \text{strictly increasing; } 1 \leq r \leq N\}$, where $\mu_0$ is the empty sequence, and let

$$\beta_\mu = e_{\mu(1)} \wedge \ldots \wedge e_{\mu(r)} \text{ for } \mu \neq \mu_0, \quad \beta_{\mu_0} = 1.$$ 

Then $\{\beta_\mu \mid \mu \in \mathbb{E}_N\}$ is the canonical basis of $\Lambda_R M$.

The cases $R = \mathbb{R}$ and $R = \mathbb{C}$ have a particular interest and deserve ad hoc notations:

$$\Lambda_R \mathbb{R}^L \equiv B_L; \quad \Lambda_C \mathbb{C}^L \equiv C_L.$$ 

$B_L$ is a vector space, with a canonical basis obtained from the canonical basis of $\mathbb{R}^L$ according to the above described procedure. If $m_L$ is the ideal of nilpotents of $B_L$, the vector space direct sum decomposition $B_L = \mathbb{R} \oplus m_L$ defines two projections

$$\sigma : B_L \to \mathbb{R}; \quad \sigma : B_L \to m_L$$

which are sometimes called body and soul maps.

**Tensor Products:** Let us recall that we are considering a graded-commutative ring $R$. The graded tensor product of two graded $R$-modules $M, N$ is by definition the usual tensor product $M \otimes_R N$, obtained by regarding $M$ as a right module, and $N$ as a left module, equipped with the gradation

$$(M \otimes_R N)_y = \oplus_{\alpha + \beta = y} \{m_i \otimes n_j \mid m_i \in M_\alpha, n_j \in N_\beta\}.$$ 

Evidently, $M \otimes_R N$ has a natural structure of graded $R$-module:

$$a(x \otimes y) = ax \otimes y = (-1)^{|a||x|} xa \otimes y = (-1)^{|a||x|} x \otimes ay = (-1)^{|a||x+y|} (x \otimes y)a.$$ 

(2.5)

The graded tensor product can be characterized as a ‘universal object’. To this end, given graded $R$-modules $M, N$ and $Q$, we introduce the set $L(M, N; Q)$ of the graded $R$-bilinear morphisms $f: M \times N \to Q$, homogeneous of degree $\alpha$: if $f \in L(M, N; Q)_\alpha$, then $f$ is a morphism of degree $\alpha$ such that $f(xa, y) = f(x, ay) = (-1)^{|a||x+y|} f(x, y)a$ for all $a \in R$. The set

$$L(M, N; Q) \equiv L(M, N; Q)_0 \oplus L(M, N; Q)_1$$

is endowed with a structure of graded $R$-module by enforcing the multiplication rule $(fa)(x, y) = f(ax, y)$. In the same way, if $M_1, \ldots, M_n, Q$ are graded $R$-modules, we define the graded $R$-module $L(M_1, \ldots, M_n; Q)$ formed by the graded $R$-multilinear morphisms $M_1 \times \cdots \times M_n \to Q$.

**Proposition 2.7.** There are natural isomorphisms in the category $R \text{- } \text{G Module}$

$$L(M, N; Q) \cong \text{Hom}_R(M \otimes_R N, Q) \cong \text{Hom}_R(M, \text{Hom}_R(N, Q)).$$

**Proposition 2.8.** Let $M, M', M''$ be graded $R$-modules; the following natural isomorphisms of graded $R$-modules hold:

(a) $M \otimes_R M' \cong M' \otimes_R M$, achieved by the morphism $x \otimes x' \mapsto (-1)^{|x||x'|} x' \otimes x$;

(b) $(M \otimes_R M') \otimes_R M'' \cong M' \otimes_R (M' \otimes_R M'')$, achieved by the morphism $(x \otimes x') \otimes x'' \mapsto x \otimes (x' \otimes x'')$;

(c) $R \otimes_R M = M = M \otimes_R R$.

If $f: M \to P$, $g: N \to Q$ are morphisms of graded modules over a graded ring $R$, the tensor product $f \otimes g: M \otimes_R N \to P \otimes_R Q$ is the morphism defined by the condition

$$(f \otimes g)(m \otimes n) = (-1)^{|m||n|} f(m) \otimes g(n).$$

(2.6)

**III. Graded Algebras and Graded Tensor Calculus**

Let $R$ be a graded-commutative ring.

**Definition 3.1.** A graded $R$-algebra $P$ is a graded $R$-module endowed with a graded $R$-bilinear multiplication

$$P \otimes P \to P,$$

$$x \otimes y \mapsto x \cdot y.$$
A graded $R$-algebra $P$ is said to be graded-commutative if all graded commutators

$$< x, y > = x \cdot y - (-1)^{|x||y|} y \cdot x,$$

defined on the analogy of equation (2.1), vanish.

**Example 3.2.** The graded module $B_2(C_1)$, as advertised in Example 2.6, equipped with the exterior product, is a graded-commutative $\mathbb{R}$-algebra (C-algebra).

The graded tensor product $P \otimes_R Q$ of two graded $R$-algebras $P$ and $Q$ is defined as the tensor product of the underlying $R$-modules equipped with the multiplication naturally induced by those of $P$ and $Q$

$$Q: (x_1 \otimes y_1) \cdot (x_2 \otimes y_2) = (-1)^{|y_1||x_2|}(x_1 \cdot x_2) \otimes (y_1 \cdot y_2).$$

**Definition 3.3.** A graded Lie $R$-algebra (or Lie $R$-superalgebra) $\mathfrak{B}$ is a graded $R$-algebra, whose multiplication, called graded Lie bracket and denoted by $[\cdot, \cdot]$, satisfies the following identities:

$$[x, y] = -(-1)^{|x||y|}[y, x]; \quad (3.1)$$

$$(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||z|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0. \quad (3.2)$$

**Remark 3.4.** Given a graded Lie algebra $\mathfrak{B}$, its even part $\mathfrak{B}_0$ is a Lie algebra over the ring $R_0$.

An important class of graded Lie algebras can be constructed in terms of the notion of graded derivation.

Let $P$ be a graded-commutative $R$-algebra.

**Definition 3.5.** A homogeneous morphism $D \in \text{End}_{\mathfrak{g}}P$ is a graded derivation of $P$ over $R$ if it fulfills the following condition (called the graded Leibnitz rule)

$$D(x \cdot y) = D(x) \cdot y + (-1)^{|x||D|} x \cdot D(y). \quad (3.3)$$

The graded $R$-submodule of $\text{End}_{\mathfrak{g}}P$ generated by the derivations of $P$ will be denoted by $\text{Der}_{\mathfrak{g}}P$, or simply $\text{Der}_P$.

**Proposition 3.6.** $\text{Der}_P$, equipped with the graded Lie bracket

$$[D_1, D_2] \equiv D_1 \circ D_2 - (-1)^{|D_1||D_2|} D_2 \cdot D_1, \quad (3.4)$$

is a graded Lie $R$-algebra.

By identifying $R$ with the submodule $R$, $1 \subset P$, condition (3.4) implies that, for all $D \in \text{Der}_P, D(R) = 0$. We notice that $\text{Der}_P$ is a (left) graded $P$-module in a natural way, by letting $(xD)(y) = x \cdot D(y)$.

**Definition 3.7.** A graded derivation of $P$ over $R$ with values in $M$ is a homogeneous element $D \in \text{Hom}_{\mathfrak{g}}P(M)$ which fulfills a graded Leibnitz rule formally identical with equation (3.3).

The graded $P$-submodule of $\text{Hom}_{\mathfrak{g}}P(M)$ generated by the graded derivations of $P$ with values in $M$ will be denoted by $\text{Der}_R(P, M)$.

**Proposition 3.8.** Let $M$ and $N$ be $R$-modules. There is a natural morphism of graded $R$-modules

$$\phi: N \otimes M^* \to \text{Hom}(M, N)$$
defined by $\phi(n \otimes \omega)(m) = n\omega(m)$. This induces a morphism

$$\gamma: M^* \otimes N^* \to (M \otimes N)^*$$

whose expression is

$$\gamma(\omega \otimes \eta)(m \otimes n) = (-1)^{|n||\omega|} \omega(m) \eta(n).$$

Both morphisms are bijective whenever $M$ is free and finitely generated.

Graded Exterior Algebra: Let $M$ be a graded $R$-module and let us denote by

$$T^p M = M \otimes \cdots \otimes M$$

(3.1)

The $p$-th tensor power of $M$, graded as usual. We can consider as in the non-graded setting the graded tensor algebra of $M$,

$$T(M) = \bigotimes\limits_{p=0}^{\infty} T^p M, \quad (3.5)$$

which is in a natural way a bigraded $R$-algebra (i.e. it has the usual $\mathbb{Z}$-gradation of the tensor algebra, together with the $\mathbb{Z}_2$-gradation it carries as a graded $R$-algebra).

The graded exterior algebra $\Lambda_R M$ of $M$ (denoted simply by $\Lambda M$) is defined as the quotient of $T(M)$ by the graded ideal $\mathfrak{I}(M)$ generated by elements of the form $m_1 \otimes m_2 + (-1)^{|m_1||m_2|} m_2 \otimes m_1$, with $m_1, m_2$ homogeneous. The product induced in $\Lambda M$ by this quotient is denoted by $\wedge$ and is called the (graded) wedge product, as usual. If we let $\mathfrak{I}(M) = \mathfrak{I}(M) \cap T^p M$, since $\mathfrak{I}(M)$ is generated by homogeneous elements, we obtain $\mathfrak{I}(M) = \bigotimes\limits_{p=0}^{\infty} \mathfrak{I}(M)$ and therefore,

$$\wedge M = \bigotimes\limits_{p=0}^{\infty} \wedge^p M$$

with $\wedge^p M = T^p M / \mathfrak{I}(M)$.

We wish to ascertain the relationship existing between the exterior algebra $\Lambda M$ and the modules of alternating graded multilinear forms: this will be realized by a morphism analogous to the morphism

$$\gamma: \Lambda^p M_1 \otimes \cdots \otimes \Lambda^p M_n \to (M_1 \otimes \cdots \otimes M_n)^* \approx \mathcal{L}(M_1, \ldots, M_n; R). \quad (3.6)$$

If $F_p \in \text{Hom}(T^p M, R)$ and $F_q \in \text{Hom}(T^q M, R)$ are homogeneous graded multilinear forms, $F_p \otimes F_q$ acts on a family of homogeneous elements according to the formula:
$$ (F_p \otimes F_q)(m_1, \ldots, m_{p+q}) = (-1)^{|F_q||m_1|+\cdots+|m_{p+q}|} F_p(m_1, \ldots, m_n) F_q(m_{p+1}, \ldots, m_{p+q}). $$

Let $S_p$ be the group of permutation of $p$ objects. For any $\sigma \in S_p$ and any $F_p \in \text{Hom}(T^p M, R)$, we write, for homogeneous elements $m_1, \ldots, m_p \in M$,

$$ F_p^{\sigma}(m_1, \ldots, m_p) = (-1)^{\Delta_1(\sigma, m)} F_p(m_{\sigma(1)}, \ldots, m_{\sigma(p)}), $$

where

$$ \Delta_1(\sigma, m) = \sum_{1 \leq i < j \leq p} \sum_{\sigma(i) > \sigma(j)} |m_{\sigma(i)}||m_{\sigma(j)}|. \quad (3.7) $$

**Definition 3.9.** A graded multilinear form $F_p \in \text{Hom}(T^p M, R)$ is said to be alternating if $F_p^{\sigma} = (-1)^{|\sigma|} F_p$ for every $\sigma \in S_p$, where $|\sigma|$ is the parity of the permutation $\sigma$.

The set $\text{Alt}(M \times \cdots \times M; R)$ of all alternating graded multilinear forms is a submodule of $\text{Hom}(T^p M, R)$; we can introduce a projection morphism, which is no more than the graded antisymmetrization:

$$ \Delta_1(\sigma, m, \omega^q) = \Delta_1(\sigma, m) + |\omega^q| \sum_{\sigma(1) = 1}^{\sigma(p)} |m_{\sigma(i)}|. \quad (2.9) $$

where in terms of the symbol $\Delta_1(\sigma, m)$ previously defined, we get

$$ \Delta_1(\sigma, m, \omega^q) = \Delta_1(\sigma, m) + |\omega^q| \sum_{\sigma(1) = 1}^{\sigma(p)} |m_{\sigma(i)}|. \quad (2.9) $$

**IV. Matrices**

Given a graded-commutative ring $R$, an $R$-module morphism $R^{m|n} \rightarrow R^{l|q}$ can be regarded, relative to the canonical bases of $R^{m|n}$ and $R^{l|q}$, as a $(p + q) \times (m + n)$ matrix with entries in $R$.

$$ X = \begin{pmatrix} X_{11} & X_{12} \\ X_{31} & X_{32} \end{pmatrix} \quad (4.1) $$

which acts on column vectors in $R^{n|m}$ from the left. The set $M_{R}[(p + q) \times (m + n)]$ of such matrices can be graded so as to be naturally isomorphic to the graded $R$-module $\text{Hom}_R(R^{m|n}, R^{l|q})$, by decreeing that:

- $X$ is even if $X_{11}$ and $X_{44}$ have even entries, while $X_{22}$ and $X_{33}$ have odd entries;
- $X$ is odd if $X_{11}$ and $X_{44}$ have odd entries, while $X_{22}$ and $X_{33}$ have even entries;

The set of matrices of the form (4.1), equipped with this gradation, will be denoted by $M_{R}[(p + q) \times (m + n)]$. The set of square matrices $M_{R}[m|n](which are obtained by letting $p = m$, $q = n)$ is a graded $R$-algebra.

The usual notation of trace and determinant of a matrix can be expended to the matrices in $M_{R}[m|n]$, thus obtaining the concepts of graded trace and Berezinian (also called supertrace and superdeterminant respectively). For any matrix $X \in M_{R}[p|q; m|n]$, regarded as a morphism $X : R^{m|n} \rightarrow R^{m|n}$, we define the graded transpose of $X$—denoted by $X^{\dagger}$—as the matrix corresponding to the morphism $X^{\star} : (R^{l|q})^{*} \rightarrow (R^{m|n})^{*}$ dual to $X$. With reference to equation (4.1), one obtains the following relations, where the superscript $t$ denotes the usual matrix transportation:

$$ \begin{pmatrix} X_{11} & X_{12} \\ X_{31} & X_{32} \end{pmatrix}^{\dagger} = \begin{pmatrix} X_{11}^{\dagger} & X_{12}^{\dagger} \\ X_{31}^{\dagger} & X_{32}^{\dagger} \end{pmatrix} \quad (4.2) $$

The graded transportation behaves naturally with respect to matrix multiplication:

$$ (XY)^{\dagger} = (-1)^{|X||Y|} Y^{\dagger} X^{\dagger}. $$
The graded trace of $X$ is the element $\text{Str}X = \sum_i a_i^*a^i \in R$. Alternatively, one can give a direct characterization by letting, for all homogeneous $X \in M_R[m][n]$,

$$\text{Str} = \text{Tr}X_4 - (-1)^{|X|}\text{Tr}X_4$$

(4.3)

where $\text{Tr}$ designates the usual trace operation. The graded trace determines an $R$-module morphism $\text{Str}: M_R[m][n] \rightarrow R$, which is natural with respect to graded transportation and matrix multiplication:

$$\text{Str}(X^0) = \text{Str}X$$

$$\text{Str}(XY) = (-1)^{|X||Y|}\text{Str}(YX).$$

(4.4)

Let us notice that, by denoting by $I_{m\times n}$ the identity matrix, one has $\text{Str} I_{m\times n} = m - n$.

In order to extend the notion of determinant, we must consider the subgroup $GL_R[m][n]$ of the matrices in $M_R[m][n]$ corresponding to an even invertible endomorphisms. $GL_R[m][n]$ is the natural extension of the notion of general linear group, so that it will be called the general graded linear group.

**Proposition 4.1.** A matrix $X \in M_R[m][n]$ is in $GL_R[m][n]$ if and only if $X_4 \in GL_R[m][0]$ and $X_0 \in GL_R[0][n]$, i.e. $X$ is invertible if and only if $X_1$ and $X_4$ are invertible as ordinary matrices with entries in $R_0$.

**Definition 4.2.** [1], [3], [4] Let $X \in GL_R[m][n]$, the Berezinian of $X$ is the element in $GL_R[1][0]$ given by

$$\text{Ber}X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$$

$$= \det(X_1 - X_2X_3^{-1}X_3)(\det X^{-1}).$$

(4.5)

**Proposition 4.3.** The mapping $\text{Ber}: GL_R[m][0] \rightarrow GL_R[0][n]$ is a group morphism, that coincides with the determinant whenever $n = 0$:

$$\text{Ber}(XY) = \text{Ber}X \text{Ber}Y \ \forall X, Y \in GL_R[m][n]$$

(4.6)

**Theorem 4.4.** A matrix in $X \in M_R[m][n]$ is invertible if and only if $\sigma(X) \in GL[m + n]$.

**Proof.** The ‘only if’ part is trivial, since $\sigma$ is ring morphism. To show the converse, it suffices to prove that a matrix $Z \in M_R[p][0]$ is invertible as a matrix with entries in $(B_L)_0$ if $\sigma(Z)$ is invertible. In the case $p = 1$ this is a consequence of the fact that in $B_L$ the morphism $\sigma$ is the natural projection $(B_L)_0 \rightarrow (B_L)_0/(n_L)_0$. The result is easily extended to $p > 1$ by inclusion. □

**V. Conclusion**

We start with given an arbitrary group $G$ and introducing $G$-graded algebraic objects and for a given graded-commutative ring $R$ and $R$-module morphism can be regarded, relative to the canonical bases of the canonical bases of $R^{m\times n}$ and $R^{p\times q}$, as a $(p + q) \times (m + n)$ matrix with entries in $R$, $X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$ which acts on column vectors in $R^{m\times n}$ from the left. Finally, this article induces a **Theorem 4.4** on a matrix of graded $R$-algebra. This paper will be helpful for other researchers.

**References**

1. Arnowitt, R., Nath, P., Zumino, B. 1975. Superfield densities and action principle in superspace, *Phys. Lett.* 56B, 81-84.
2. Atiyah, M. F., Macdonal, I. G. 1969. *Introduction to commutative algebra*, Addision-Wesley, Reading, MA.
3. Bartocci, C., Bruzzo, U., Hernández Ruipérez, D. 1991. *The geometry of supermanifolds*. Kluwer, Dordrecht..
4. Berezin, F. A., Leĭtes, D.A. 1975. *Supermanifolds*, Soviet Math. Dokl. 16,1218-1222.
5. Bourbaki, N. 1970. *Elément de mathématique. AlgèbreI(Chapitres 1 à 3)*, Hermann, Paris.
6. Corwin, L., Ne’eman, Y., Sternberg, S. 1975. Graded Lie algebras in mathematics and physics (Bose-Fermi symmetry), Rev. Modern Phys. 47, 573-603.
7. Kac, V.G. 1977. Lie superalgebras, *Adv. In Math.* 26, 8-96.
8. Kac, V.G. 1977, A sketch of Lie Superalgebra theory, *Commun. Math. Phys.* 53, 31-64.
9. Kobayashi, S., Nomizu, K. 1963. *Foundations of differential geometry. I*, Inter-science Publ., New York.
10. Năstăsescu, C., Van Oystaeyen, F. 1982. *Graded ring theory*, North-Holland, Amsterdam.
11. Scheunert, M. 1979. *The theory of Lie Superalgebra*, Lecture Notes Math., Springer Verlag, Berlin,716.
