SMOOTH CUSPIDAL AUTOMORPHIC FORMS AND INTEGRABLE DISCRETE SERIES

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Abstract. In this paper we construct smooth cuspidal automorphic forms related to integrable discrete series of a connected semisimple Lie group with finite center for classical and adelic situation as an application of the theory of Schwartz spaces for automorphic forms developed by Casselman. In the classical situation, smooth cuspidal automorphic forms are constructed via an explicit continuous map from the Frechét space of smooth vectors of a Banach realization inside $L^1(G)$ of an integrable discrete series into the space of smooth vectors of a strong topological dual of an appropriate Schwartz space.

1. Introduction

The usual definition of an automorphic form includes the assumption that the function is $K$–finite on the Archimedean part. Smooth automorphic forms are used extensively in the theory of automorphic forms (see for example [8], [12], [15], [16], [17]). Although important, there is no explicit general construction of smooth cuspidal forms which are not $K$–finite.

In our papers [18], [19], and [22] we have studied the classical construction [4] of $K$–finite cuspidal automorphic forms via Poincaré series from $K$–finite matrix coefficients of integrable discrete series. In this paper we complete these investigations by considering the smooth case using results of Casselman [7]. We explain our results and content of the paper by sections.

In this introduction, $G$ is a group of $\mathbb{R}$–points of a semisimple algebraic group $\mathcal{G}$ defined over $\mathbb{Q}$. We assume that $G$ is not compact and it is connected. Then, $G$ is a connected semisimple Lie group with finite center. In some sections of the paper we just assume the latter. Important groups such as $Sp_{2n}(\mathbb{R})$ or its double cover belong to this class. In any case, we let $K$ be a choice of a maximal compact subgroup of $G$, and $\mathcal{Z}(\mathfrak{g}_C)$ be the center of the universal enveloping algebra of the complexified Lie algebra of $G$.

We let $\Gamma \subset G$ be a congruence subgroup with respect to the arithmetic structure given by the fact that $\mathcal{G}$ defined over $\mathbb{Q}$ (see [5]). Then, $\Gamma$ is a discrete subgroup of $G$ and it has a finite covolume.

In Section 2 we recall the notion of the norm on $G$ and state some properties of the norm. This is essential to all other investigations in the paper. In Section 3 we recall the definition of $K$–finite and smooth automorphic and cuspidal forms. We recall some basic properties of automorphic forms, and, in particular, the result that claims that a $\mathcal{Z}(\mathfrak{g}_C)$–finite and $K$–finite function in $L^p(\Gamma\backslash G)$ for some $p \geq 1$ is an automorphic form (see Lemma 3-1). In Section 4 we give simple and natural proof of this result as an application of results of Casselman [7] (see Proposition 4-7). Besides, in Section 4 we prove Lemma 4-6 in which

1991 Mathematics Subject Classification. 11E70, 22E50.

The author acknowledges Croatian Science Foundation grant no. 9364.
we prepare results of Casselman for application to the construction of smooth automorphic forms. Aforementioned result about automorphic forms is also a consequence of that lemma.

In Section 5, we give a complete description of irreducible closed admissible subrepresentations of $L^1(G)$ under the right translations. It is quite likely that this is well–known but we could not find a convenient reference. Such irreducible representations are integrable discrete series. A complete description of them can be found in [13]. The main results of Section 5 are contained in Lemma 5-5 and Theorem 5-12. In Lemma 5-3 we prove that all smooth matrix coefficients of a discrete series representation belong to $L^p(G)$, for some $p \in [1, 2]$, whenever there exists a non–zero $K$–finite matrix coefficient which satisfies the same. The proof is based on Casselman–Wallach theory of globalization of $(g, K)$—modules [25], [6]. It is the key ingredient in the description of smooth vectors of realizations of integrable discrete series in $L^1(G)$ (see Lemma 5-5 (vii)).

In Section 6, after all aforementioned preparations, we prove the main result of the present paper (see Theorem 6-4). There, we just assume that $G$ is a connected semisimple Lie group with finite center and $\Gamma$ is any discrete subgroup. We assume that $G$ admits discrete series, and let $(\pi, \mathcal{H})$ be an integrable discrete series of $G$. We fix a closed irreducible subrepresentation $\mathcal{B}_{h'}$ of $L^1(G)$ infinitesimally equivalent to $(\pi, \mathcal{H})$ which is attached to a $K$–finite vector $h' \in \mathcal{H}$ via formation of certain matrix coefficients (see Lemma 5-5). In Theorem 6-4, we look at the usual formation of Poincaré series $P_\Gamma$ attached to $\Gamma$ (see the first paragraph of Section 6), in two ways.

Firstly, we prove that the map $\varphi \mapsto P_\Gamma(\varphi)$ is a continuous $G$–equivariant map from the Banach representation $\mathcal{B}_{h'}$ into the unitary representation $L^2(\Gamma \backslash G)$. When $\Gamma$ is a congruence subgroup, as an immediate consequence of asymptotic results of Wallach [23], the image is contained in the cuspidal subspace $L^2_{cusp}(\Gamma \backslash G)$.

Secondly, we may consider $P_\Gamma$ as the continuous map $P_\Gamma : \mathcal{B}_{h'} \rightarrow \mathcal{S}(\Gamma \backslash G)'$, where $\mathcal{S}(\Gamma \backslash G)'$ is the strong topological dual of the Schwartz space $\mathcal{S}(\Gamma \backslash G)$. In [7] (see also Lemma 1-5 in this paper), it was proved that the Garding space of $\mathcal{S}(\Gamma \backslash G)'$ is the space of functions of uniform moderate growth $A_{umg}(\Gamma \backslash G)$. In Theorem 6-4 (ii), we prove that the space of smooth vectors $\mathcal{B}_{h'}^\infty$ gets mapped into the subspace of $Z(g_C)$–finite vectors in $A_{umg}(\Gamma \backslash G)$. We remind the reader that when $\Gamma$ is a congruence subgroup, $Z(g_C)$–finite vectors in $A_{umg}(\Gamma \backslash G)$ are smooth automorphic forms by definition. In this way, we achieve a construction of smooth cuspidal automorphic forms (see Theorem 7-6 for details). We remark that we have a canonical isomorphism of Frechêt representations $\mathcal{B}_{h'}^\infty \simeq \mathcal{H}^\infty$ (see Lemma 5-5 (vii)).

Thirdly, the map $P_\Gamma$ could be identically zero, but in Theorem 6-4 (iv) we give a sufficient condition that the map is not zero. It is based on our usual non–vanishing criterion ([18], Theorem 4-1).

In Section 7, we give applications of Theorem 6-4. In Theorem 7-6 we construct smooth cuspidal automorphic forms, and we study when the map $P_\Gamma$ is not zero for principal congruence subgroups. The corollary of these investigations is the result for smooth adelic cuspidal automorphic forms which we state and prove in Corollary 7-9.
The first draft of the paper was written while the author visited the Hong Kong University of Science and Technology in May of 2016. The author would like to thank A. Moy and the Hong Kong University of Science and Technology for their hospitality.

2. Norms on The Group

In this section we assume that $G$ is a connected semisimple Lie group with finite center, and recall the notion of the norm on $G$. It is essential for all what follows.

We fix a minimal parabolic subgroup $P = MAN$ of $G$ in the usual way (see [24], Section 2). We have the Iwasawa decomposition $G = NAK$.

We recall the notion of a norm on the group following [24], 2.A.2. A norm $|| ||$ is a function $G \to [1, \infty[$ satisfying the following properties:

1. $||x^{-1}|| = ||x||$, for all $x \in G$;
2. $||x \cdot y|| \leq ||x|| \cdot ||y||$, for all $x, y \in G$;
3. the sets $\{x \in G; ||x|| \leq r\}$ are compact for all $r \geq 1$;
4. $||k_1 \exp (tX)k_2|| = ||\exp (X)||^t$, for all $k_1, k_2 \in K, X \in \mathfrak{p}, \ t \geq 0$.

Any two norms $|| ||_i, i = 1, 2$, are equivalent: there exist $C, r > 0$ such that $||x||_1 \leq C||x||_2$, for all $x \in G$.

Let $\Phi(\mathfrak{g}, \mathfrak{a})$ be the set of all roots of $\mathfrak{a}$ in $\mathfrak{g}$. Let $\Phi^+(\mathfrak{g}, \mathfrak{a})$ be the set of positive roots with respect to $\mathfrak{n} = \text{Lie}(N)$. Set

$$\rho(H) = \frac{1}{2} \text{tr}(\text{ad}(H)|_{\mathfrak{a}}), \ H \in \mathfrak{a}.$$ 

We set

$$m(\alpha) = \dim \mathfrak{g}_\alpha, \ \alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a}).$$

For $\mu \in \mathfrak{a}^*$, we let

$$a^\mu = \exp (\mu(H)), \ a = \exp (H).$$

We define $A^+$ to be the set of all $a \in A$ such that $a^\alpha > 1$ for all $\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a})$. Finally, we let

$$D(a) = \prod_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a})} \sinh (\alpha(H))^{m(\alpha)}, \ a = \exp (H).$$

Then, we may define a Haar measure on $G$ by the following formula:

$$\int_G f(g)dg = \int_K \int_{A^+} \int_K D(a) f(k_1ak_2)dk_1dadk_2, \ f \in C_c^\infty (G).$$

Let $\{\alpha_1, \ldots, \alpha_r\}$ be the set of simple roots in $\Phi^+(\mathfrak{g}, \mathfrak{a})$. Since $G$ is semisimple, we have that this set spans $\mathfrak{a}^*$. We define the dual basis $\{H_1, \ldots, H_r\}$ of $\mathfrak{a}$ in the standard way: $\alpha_i(H_j) = \delta_{ij}$. By ([24], Lemma 2.A.2.3), there exists, $\mu, \eta \in \mathfrak{a}^*$ such that $\mu(H_i), \eta(H_j) > 0$, for all $j$, and constants $C, D > 0$ such that

$$Ca^\mu \leq ||a|| \leq Da^\eta, \ a \in Cl(A^+).$$

We remark that $\rho(H_j) > 0$ for all $j$. So, we can find $c, d > 0$ such that

$$a^{c\rho} \leq a^\mu \leq a^\eta \leq a^{d\rho}, \ a \in Cl(A^+).$$
We record this in the next lemma:

**Lemma 2-1.** There exists real constants $c, C, d, D > 0$ such that

$$Ca^c \leq \|a\| \leq Da^d, \quad a \in Cl(A^+).$$

A consequence of above integration formula and Lemma 2-1 is the following lemma (see [24], Lemma 2.A.2.4):

**Lemma 2-2.** Maintaining above assumptions, we have

$$\int_G \|g\|^{-m} dg < \infty \text{ for } m > \max_{1 \leq i \leq r} \frac{1}{c \rho(H_i)}.$$ 

**Proof.** Let $m > 0$. Then, by above properties of the norm and the integration formula

$$\int_G \|g\|^{-m} dg = \int_{A^+} D(a)\|a\|^{-m} da = \int_{\alpha_1(H) > 0, \ldots, \alpha_r(H) > 0} D(\exp H) \|\exp H\|^{-m} dH,$$

where $dH$ is any Euclidean measure on $a$. We fix a basis $H_1, \ldots, H_r$ described above. In this basis, the right-hand side becomes

$$\int_{t_1 > 0, \ldots, t_r > 0} D \left( \exp \left( \sum_{i=1}^r t_i H_i \right) \right) \|\exp \left( \sum_{i=1}^r t_i H_i \right)\|^{-m} dt_1 \cdots dt_r.$$

Obviously, the definition of $D$ implies

$$D \left( \exp \left( \sum_{i=1}^r t_i H_i \right) \right) \leq \exp \left( \sum_{i=1}^r \rho(H_i) t_i \right),$$

for $t_i > 0, \ i = 1, \ldots, r$. By Lemma 2-1, there exists real constants $c, C$ such that

$$C \exp \left( \sum_{i=1}^r c \rho(H_i) t_i \right) \leq \|\exp \left( \sum_{i=1}^r t_i H_i \right)\|,$$

for $t_i > 0, \ i = 1, \ldots, r$. Hence, the integral is

$$\leq C^{-m} \int_{t_1 > 0, \ldots, t_r > 0} \exp \left( \sum_{i=1}^r \left( 1 - m c \rho(H_i) \right) t_i \right) dt_1 \cdots dt_r.$$ 

By elementary calculus, the integral is finite for

$$m > \max_{1 \leq i \leq r} \frac{1}{c \rho(H_i)}.$$

□

### 3. Preliminaries on Automorphic Forms

In this section we assume that $G$ is a group of $\mathbb{R}$-points of a semisimple algebraic group $G$ defined over $\mathbb{Q}$. Assume that $G$ is not compact and connected. Let $\Gamma \subset G$ be congruence subgroup with respect to the arithmetic structure given by the fact that $G$ defined over $\mathbb{Q}$ (see [5]). Then, $\Gamma$ is a discrete subgroup of $G$ and it has a finite covolume.

An automorphic form (or a $K$-finite automorphic form; see [8]) for $\Gamma$ is a function $f \in C^\infty(G)$ satisfying the following three conditions ([26] or [5]):
(A-1) $f$ is $Z(g_C)$--finite and $K$--finite on the right;
(A-2) $f$ is left--invariant under $\Gamma$ i.e., $f(\gamma x) = f(x)$ for all $\gamma \in \Gamma$, $x \in G$;
(A-3) there exists $r \in \mathbb{R}$, $r > 0$ such that for each $u \in U(g_C)$ there exists a constant $C_u > 0$ such that $|u.f(x)| \leq C_u \cdot ||x||^r$, for all $x \in G$.

A smooth automorphic form (see [7], [8]) for $\Gamma$ is a function $f \in C^\infty(G)$ satisfying (A1)--(A3) except possibly $K$--finiteness. We discuss smooth automorphic forms in more detail the next section.

We write $A(\Gamma \backslash G)$ (resp., $A^\infty(\Gamma \backslash G)$) for the vector space of all automorphic forms (resp., smooth automorphic forms). Obviously, $A(\Gamma \backslash G) \subset A^\infty(\Gamma \backslash G)$. It is easy to see that $A(\Gamma \backslash G)$ is a $(g,K)$--module (using [10], Theorem 1 and an argument similar to the one used in the proof of Lemma 3-2), and since $G$ is connected, the space $A^\infty(\Gamma \backslash G)$ is $G$--invariant. An automorphic form $f \in A^\infty(\Gamma \backslash G)$ is a $\Gamma$--cuspidal automorphic form if for every proper $Q$--proper parabolic subgroup $P \subset G$ we have

$$\int_{U \cap \Gamma \backslash U} f(ux) \, dx = 0, \quad x \in G,$$

where $U$ is the group of $\mathbb{R}$--points of the unipotent radical of $P$. We remark that the quotient $U \cap \Gamma \backslash U$ is compact. We use normalized $U$--invariant measure on $U \cap \Gamma \backslash U$. The space of all $\Gamma$--cuspidal automorphic forms (resp., $\Gamma$--cuspidal smooth automorphic forms) for $\Gamma$ is denoted by $A_{cusp}(\Gamma \backslash G)$ (resp., $A^\infty_{cusp}(\Gamma \backslash G)$). The space $A^\infty_{cusp}(\Gamma \backslash G)$ is a $(g,K)$--submodule of $A(\Gamma \backslash G)$. The space $A^{\infty}_{cusp}(\Gamma \backslash G)$ is $G$--invariant.

Following Casselman [7], we define

$$||g||_{\Gamma \backslash G} = \inf_{\gamma \in \Gamma} ||\gamma g||, \quad g \in G.$$ 

It is obvious that $|| \cdot ||_{\Gamma \backslash G}$ is $\Gamma$--invariant on the right, and that $||g||_{\Gamma \backslash G} \leq ||g||$ for all $g \in G$. The condition (A-3) is equivalent to

(A-3') there exists $r \in \mathbb{R}$, $r > 0$ such that for each $u \in U(g_C)$ there exists a constant $C_u > 0$ such that $|u.f(x)| \leq C_u \cdot ||x||^r_{\Gamma \backslash G}$, for all $x \in G$.

We recall the following standard result:

**Lemma 3-1.** Under above assumptions, we have the following:

(a) If $f \in C^\infty(G)$ satisfies (A-1), (A-2), and there exists $p \geq 1$ such that $f \in L^p(\Gamma \backslash G)$, then $f$ satisfies (A-3), and it is therefore an automorphic form. We speak about $p$--integrable automorphic form, for $p = 1$ (resp., $p = 2$) we speak about integrable (resp., square--integrable) automorphic form.

(b) Let $p \geq 1$. Every $p$--integrable automorphic form is integrable.

(c) Bounded integrable automorphic form is square--integrable.

(d) If $f$ is square integrable automorphic form, then the minimal $G$--invariant closed subspace of $L^2(\Gamma \backslash G)$ is a direct is of finitely many irreducible unitary representations.

(e) Every $\Gamma$--cuspidal automorphic form is square--integrable.
Proof. For the claims (a) and (e) we refer to [5] and reference there. Since the volume of \( \Gamma \setminus G \) is finite, the claim (b) follows from Hölder inequality (as in [18], Section 3). The claim (c) is obvious. The claim (d) follows from ([24], Corollary 3.4.7 and Theorem 4.2.1).

Proposition 4.7 gives simple proof of Lemma 3.1 (a) using some results of Casselman [7].

We include the proof of the following standard result since it will be useful in clarification of various issues in the next section.

**Lemma 3.2.** If \( f \in C^\infty(G) \) satisfies (A-1), (A-2) and there exists constants \( r > 0, C > 0 \) such that \(|f(x)| \leq C \cdot ||x||^r\), for all \( x \in G \), then (A-3) also holds.

**Proof.** By the assumption (A-1) and a theorem of Harish–Chandra (see [10], Theorem 1) there exists \( \alpha \in C^\infty_c(G) \) such that \( f = f \ast \alpha \). This implies that for \( u \in \mathcal{U}(g_C) \) we have

\[
uf(x) = \int_G f(y)u\alpha(y^{-1}x)dy = \int_G f(xy^{-1})u\alpha(y)dy.
\]

Hence, by using the assumption and properties (1) and (2) of the norm (see Section 2)

\[
|uf(x)| \leq \int_G |f(xy^{-1})| \cdot |u\alpha(y)|dy \leq C \int_G ||xy^{-1}||^r \cdot |u\alpha(y)|dy
\]

\[
\leq C||x||^r \int_G ||y||^r \cdot |u\alpha(y)|dy
\]

which proves the claim. \(\square\)

4. Some Results of Casselman

In this section we assume that \( G \) is a semisimple connected Lie group with finite center. We assume that \( \Gamma \) is a discrete subgroup of \( G \). For example, \( \Gamma \) could be a congruence subgroup or just a trivial group. Main result of this section is observation in Lemma 4.6 used in Section 6 in the construction of smooth automorphic forms.

We recall the definition of the Schwartz space \( \mathcal{S}(\Gamma \setminus G) \) defined by Casselman ([7], page 292). It consists of all functions \( f \in C^\infty(G) \) satisfying the following conditions:

(CS-1) \( f \) is left–invariant under \( \Gamma \) i.e., \( f(\gamma x) = f(x) \) for all \( \gamma \in \Gamma, x \in G \);

(CS-2) \( ||f||_{u,-n} < \infty \) for all \( u \in \mathcal{U}(g_C) \), and all natural numbers \( n \geq 1 \).

In above definition, for \( u \in \mathcal{U}(g_C) \), and a real number \( s \), we let

\[
||f||_{u,s} \overset{\text{def}}{=} \sup_{x \in G} ||x||_{\Gamma \setminus G}^{-s} |u.f(x)|.
\]

Since \( ||x||_{\Gamma \setminus G} \geq 1 \), we have

\[
||f||_{u,s'} \leq ||f||_{u,s},
\]

for \( s' > s \).

We recall the following result (see [7], 1.8 Proposition):

**Proposition 4.1.** Using above notation, we have the following:
(i) The Schwartz space $S(\Gamma \backslash G)$ is a Fréchet space under the seminorms: $||u||_{u,-n}, u \in \mathcal{U}(g_C), n \in \mathbb{Z}_{\geq 1}$.

(ii) The right regular representation of $G$ on $S(\Gamma \backslash G)$ is a smooth Fréchet representation of moderate growth.

We recall the definition of representation of moderate growth. Let $(\pi, V)$ be a continuous representation on the Fréchet space $V$. We say that $(\pi, V)$ is of moderate growth if it is smooth, and if for any semi-norm $\rho$ there exists an integer $n$, a constant $C > 0$, and another semi-norm $\nu$ such that

$$||\pi(g)v||_{\rho} \leq C||g||^n||v||_{\nu}, \quad g \in G, \quad v \in V.$$ 

We recall that the semi-norms on a locally convex vector space (for example, a Frechét space) $V$ are constructed via Minkowski functionals.

The following definition is from ([7], page 295).

**Definition 4-2.** The space $S(\Gamma \backslash G)'$ of tempered distributions or distributions of moderate growth on $\Gamma \backslash G$ is the strong topological dual of $S(\Gamma \backslash G)$.

For convenience of the reader, we recall the definition of a strong topological dual in our particular case. By general theory, the subset $B \subset S(\Gamma \backslash G)$ is bounded if for every neighborhood $V$ of 0 there exists $s > 0$ such that $B \subset tV$, for $t > s$. This definition is not very practical to use. Again from the general theory (and easy to see directly), $B \subset S(\Gamma \backslash G)$ is bounded if and only if it is bounded in every semi-norm defining topology on $S(\Gamma \backslash G)$ i.e.,

$$\sup_{f \in B} ||f||_{u,-n} < \infty, \quad u \in \mathcal{U}(g_C), \quad n \in \mathbb{Z}_{\geq 1}.$$ 

The strong topological dual $S(\Gamma \backslash G)'$ of $S(\Gamma \backslash G)$ is the space of continuous functionals on $X$ equipped with strong topology i.e. topology of uniform convergence on bounded sets in $S(\Gamma \backslash G)$ i.e. topology given by semi–norms

$$||\alpha||_B = \sup_{f \in B} |\alpha(f)|, \quad \text{where} \ B \text{ ranges over bounded sets of } S(\Gamma \backslash G).$$

By general theory of topological vector spaces, the space $S(\Gamma \backslash G)'$ is complete locally convex (defined by above semi–norms) vector space. The natural action of $G$ on $S(\Gamma \backslash G)'$ is continuous. The usual representation–theoretic argument are valid there ([10], Section 2).

The following lemma can be used to deal with the limits in $S(\Gamma \backslash G)'$ but of course it is not sufficient to deal with the topology on $S(\Gamma \backslash G)'$. The proof is left as an exercise to the reader.

**Lemma 4-3.** Let $\alpha_n, \ n \geq 1,$ be a sequence in $S(\Gamma \backslash G)'$ and let $\alpha \in S(\Gamma \backslash G)'$. Then, $\alpha_n \to \alpha$ if for sufficiently large numbers $M > 0$ and $m \in \mathbb{Z}_{\geq 1}$ we have

$$\lim_{n \to \infty} \sup_{||f||_{1,-m} \leq M} |\alpha_n(f) - \alpha(f)| \to 0.$$
Following Casselman, we consider the two spaces of functions: the functions of moderate growth $A_{mg}(\Gamma\backslash G)$, and the functions of uniform moderate growth $A_{umg}(\Gamma\backslash G)$. The space $A_{mg}(\Gamma\backslash G)$ consists of the functions $f \in C^\infty(G)$ satisfying the following conditions:

(MG-1) $f$ is left–invariant under $\Gamma$ i.e., $f(\gamma x) = f(x)$ for all $\gamma \in \Gamma$, $x \in G$;
(MG-2) for each $u \in U(g_C)$ there exists a constant $C_u > 0$, $r_u \in \mathbb{R}$, $r_u > 0$ such that $|u.f(x)| \leq C_u \cdot ||x||^{r_u}$, for all $x \in G$.

The space $A_{umg}(\Gamma\backslash G)$ consists of the functions $f \in C^\infty(G)$ satisfying the following conditions:

(UMG-1) $f$ is left–invariant under $\Gamma$ i.e., $f(\gamma x) = f(x)$ for all $\gamma \in \Gamma$, $x \in G$;
(UMG-2) there exists $r \in \mathbb{R}$, $r > 0$ such that for each $u \in U(g_C)$ there exists a constant $C_u > 0$ such that $|u.f(x)| \leq C_u \cdot ||x||^{r_u}$, for all $x \in G$.

We note that in the second definition $r$ is independent of $u \in U(g_C)$.

**Lemma 4-4.** We maintain the assumptions of the first paragraph of Section 3. Then, the spaces of functions which are $Z(g_C)$–finite and $K$–finite on the right in $A_{mg}(\Gamma\backslash G)$, and in $A_{umg}(\Gamma\backslash G)$ coincide, and are equal to the space $A(\Gamma\backslash G)$ of automorphic forms for $\Gamma$. Next, the space of smooth automorphic forms $A^\infty(\Gamma\backslash G)$ is a subspace of $Z(g_C)$–finite functions in $A_{umg}(\Gamma\backslash G)$. Furthermore, we have

$$A(\Gamma\backslash G) \subset A^\infty(\Gamma\backslash G) \subset A_{umg}(\Gamma\backslash G) \subset A_{mg}(\Gamma\backslash G).$$

**Proof.** The first and second claims follow from (A1)–(A3), and Lemma 3-2. The third claim is obvious. \hfill \Box

**Lemma 4-5.** The Garding space in $S(\Gamma\backslash G)'$ is equal to the space $A_{umg}(\Gamma\backslash G)$.

**Proof.** This ([7], Theorem 1.16). \hfill \Box

We remark that $S(\Gamma\backslash G)'$ is not a Fréchet space so [9] can not be applied to prove that the space of smooth vectors is the same as the Garding space. Therefore, for example, in the settings of Lemma 4-4, $A^\infty(\Gamma\backslash G)$ is just subspace of the space of all $Z(g_C)$–finite vectors in $S(\Gamma\backslash G)'$.

Regarding smooth vectors in $S(\Gamma\backslash G)'$, the following remarkable lemma will be used later:

**Lemma 4-6.** Assume that $f \in L^p(\Gamma \backslash G)$, for some $p \geq 1$, and $\alpha \in C_c^\infty(G)$. Then, $f \ast \alpha$ is equal almost everywhere to a function in $A_{umg}(\Gamma\backslash G)$.

**Proof.** It follows easily from above description of topology of $S(\Gamma\backslash G)$ that $\varphi \mapsto \int_{\Gamma\backslash G} F(x)\varphi(x)dx$ belongs to $S(\Gamma\backslash G)'$ for any $F \in L^p(\Gamma \backslash G)$.
The action of convolution algebra \( C_c^\infty(G) \) on \( S(\Gamma \setminus G)' \) is given by
\[
r'(\alpha)\Lambda = \int_G \alpha(x) \ r'(x)\Lambda \ dx, \quad \Lambda \in S(\Gamma \setminus G)', \ \alpha \in C_c^\infty(G),
\]
where \( r'(x) \) is the contragredient of the right translation \( r(x) \) for \( x \in G \). By definition, we have the following:
\[
r'(\alpha)\Lambda(\varphi) = \int_G \alpha(x) \ r'(x)\Lambda(\varphi) \ dx = \int_G \alpha(x)\Lambda(r(x^{-1})\varphi) \ dx = \Lambda(r(\alpha^\vee)\varphi).
\]

We remark that
\[
r(\alpha^\vee)\varphi(x) = \int_G \alpha(y^{-1})\varphi(xy) \ dy = \varphi \ast \alpha(x).
\]

Now, let the functional \( \Lambda \) be the functional given by the integration against \( f \) and \( \Sigma \) be the functional given by the integration against \( f \ast \alpha \). Then we have the following computation:
\[
\Sigma(\varphi) = \int_{\Gamma \setminus G} (f \ast \alpha)(x)\varphi(x) \ dx
\]
\[
= \int_{\Gamma \setminus G} \left( \int_G f(xy^{-1})\alpha(y) \ dy \right) \varphi(x) \ dx
\]
\[
= \int_{\Gamma \setminus G} f(x) \left( \int_G \varphi(xy)\alpha(y) \ dy \right) \ dx
\]
\[
= \int_{\Gamma \setminus G} f(x)\varphi \ast \alpha(x) \ dx
\]
\[
= \Lambda(r(\alpha)\varphi)
\]
\[
= r'(\alpha)\Lambda(\varphi), \ \varphi \in S(\Gamma \setminus G).
\]

Thus, \( \Sigma \) belongs to the Garding space of \( S(\Gamma \setminus G)' \). Hence, by Lemma \[4-5\], there exists a \( F \in \mathcal{A}_{mg}(\Gamma \setminus G) \) such that
\[
\int_{\Gamma \setminus G} (f \ast \alpha)(x)\varphi(x) \ dx = \Sigma(\varphi) = \int_{\Gamma \setminus G} F(x)\varphi(x) \ dx,
\]
for all \( \varphi \in S(\Gamma \setminus G) \). Since \( C_c^\infty(\Gamma \setminus G) \subset S(\Gamma \setminus G) \), the claim follows. \( \square \)

This result can be used to give a new and simple proof of important result stated in Lemma \[3-1\] (a).

**Proposition 4-7.** If \( f \in C^\infty(G) \) satisfies (A-1), (A-2) of Section \[5\] and there exists \( p \geq 1 \) such that \( f \in L^p(\Gamma \setminus G) \), then \( f \) satisfies (A-3) of Section \[5\] and it is therefore an automorphic form in \( \mathcal{A}(\Gamma \setminus G) \).

**Proof.** By (A-1), \( f \) is \( Z(\mathfrak{g}_C) \)-finite and \( K \)-finite on the right. Them by a theorem of Harish–Chandra (\[10\], Theorem 1), there exists \( \alpha \in C_c^\infty(G) \) such that \( f = f \ast \alpha \). Hence, Lemma \[4-6\] implies that \( f \in \mathcal{A}_{mg}(\Gamma \setminus G) \). Now, (A-1) and Lemma \[4-3\] complete the proof. \( \square \)
5. On a Description of Certain Irreducible Subrepresentations in $L^1(G)$

In this section we assume that $G$ is a semisimple connected Lie group with finite center. We give a complete description of irreducible closed admissible subrepresentations of $L^1(G)$ under the right translations.

Let $(\pi, B)$ be a continuous representation of $G$ on the Banach space $B$. We denote by $B^\infty$ the subspace of smooth vectors in $H$. It is a complete Fréchet space under the family of semi–norms:

$$||b||_u = ||d\pi(u)b||, \quad u \in \mathcal{U}(\mathfrak{g}_C), \ b \in B^\infty,$$

where $|| \ |$ is the norm on $B$. The natural representation $\pi^\infty$ of $B^\infty$ is a smooth Fréchet representation of moderate growth ([25], Lemma 11.5.1).

Let $\hat{K}$ be the set of equivalence of irreducible representations of $K$. Let $\delta \in \hat{K}$, then we write $d(\delta)$ and $\xi_\delta$ the degree and character of $\delta$, respectively. We fix the normalized Haar measure $dk$ on $K$. For $h \in H$, we let

$$E_\delta(b) = \int_K d(\delta)\xi_\delta(k) \pi(k)b \ dk, \ B.$$

It belongs to the $\delta$–isotypic component $B(\delta)$ of $B$. We have

$$E_\delta E_\delta = E_\delta$$

$$E_\delta E_\gamma = 0 \text{ if } \delta \neq \gamma.$$

We state the following lemma that we need later:

**Lemma 5-1.** Let $b \in B^\infty$. Then, we have the following:

(i) There exists $b_1, \ldots, b_l \in B^\infty$, and $\alpha_1, \ldots, \alpha_l \in C_c^\infty(G)$ such that

$$b = \sum_{i=1}^l \pi(\alpha_i)b_i.$$

(ii) We have the following expansion

$$b = \sum_{\delta \in \hat{K}} E_\delta(b)$$

in above described Fréchet topology where the convergence is absolute:

$$\sum_{\delta \in \hat{K}} ||d\pi(u)E_\delta(b)|| < \infty, \text{ for all } u \in \mathcal{U}(\mathfrak{g}_C).$$
Proof. (i) is of course a well–known result of Dixmier–Malliavin applied to the Banach representation \((\pi, \mathcal{B})\). By the way, this implies that \((\pi^\infty, \mathcal{B}^\infty)\) is a smooth representation in its natural topology. One just needs to apply \((10), \text{Lemma } 2\). Now, having this remark, (i) is just \((10), \text{Lemma } 5\) applied \((\pi^\infty, \mathcal{B}^\infty)\). □

We consider \(L^1(G)\) as a Banach representation of \(G\) under the right–translations \(r\). Let \(\alpha \in C^\infty_c(G)\). It acts on \(L^1(G)\) as follows:

\[
 r(\alpha)f(x) = \int_G \alpha(y)f(xy)dy = \int_G f(xy^{-1})\alpha^\vee(y)dy = \int_G f(y)\alpha^\vee(y^{-1}x)dy, \quad f \in L^1(G).
\]

The function \(r(\alpha)f\) belongs to \(C^\infty(G)\) and for each \(u \in \mathcal{U}(\mathfrak{g}_C)\) we have the following:

\[
 u.r(\alpha)f(x) = \int_G f(y)u.\alpha^\vee(y^{-1}x)dy.
\]

By definition, \(r(\alpha)f\) belongs to a Garding space of the right–regular representation of \(G\) on \(L^1(G)\). Thus, the vector \(r(\alpha)f\) is smooth for that representation. Thus, \(u \in \mathcal{U}(\mathfrak{g}_C)\) acts on \(r(\alpha)f\). It is easy to see that the action is described by above formula. For example, if \(X \in \mathfrak{g}\) and \(\alpha\) is real–valued, then we compute

\[
 \int_G \left| \frac{r(\alpha)f(\exp(tX)) - r(\alpha)f(x)}{t} - (f * X.\alpha^\vee)(x) \right| dx \\
 \leq \int_G \int_G |f(y)| \left| \frac{\alpha^\vee(y^{-1}x \exp(tX)) - \alpha^\vee(y^{-1}x)}{t} - X.\alpha^\vee(y^{-1}x) \right| dx dy \\
 = \int_G |f(y)| dy \int_G \left| \frac{\alpha^\vee(x \exp(tX)) - \alpha^\vee(x)}{t} - X.\alpha^\vee(x) \right| dx \\
 = \int_G |f(y)| dy \int_G |X.\alpha^\vee(x \exp(t_xX)) - X.\alpha^\vee(x)| dx,
\]

for some \(t_x \in ]0, t[\) by the Mean value theorem. Letting \(t \rightarrow 0\), by the Dominated convergence theorem, we obtain the claim. Similar considerations hold for the left regular representation of \(G\) on \(L^1(G)\) denoted by \(l\). Let \(\beta \in C^\infty_c(G)\). It acts on \(L^1(G)\) as follows:

\[
l(\beta)f(x) = \int_G \beta(y)f(y^{-1}x)dy = \int_G \beta(xy^{-1})f(y)dy = \beta * f(x).
\]

This function belongs to \(C^\infty(G)\) and for each \(u \in \mathcal{U}(\mathfrak{g}_C)\) we have the following:

\[(5-2) \quad l(u)l(\beta)f = (l(u)\beta) * f.\]

Let \(\delta \in \hat{K}\). As before, if \(dk\) is the normalized measure on \(K\), then we let

\[
 E^l_\delta(\cdot) = \int_K d(\delta)\overline{\xi_\delta(k)l(k)} \, dk
\]

\[
 E^r_\delta(\cdot) = \int_K d(\delta)\overline{\xi_\delta(k)r(k)} \, dk.
\]
For a finite set $S \subset \hat{K}$, we let
\[ E_S^f = \sum_{\delta \in S} E_\delta^f \]
\[ E_S^c = \sum_{\delta \in S} E_\delta^c \]

Let $(\pi, \mathcal{H})$ be an irreducible unitary representation of $G$. We write $\langle \ , \ \rangle$ for the $G$-invariant inner product on $\mathcal{H}$. It is a well-known result of Harish–Chandra that it is admissible. Let $\mathcal{H}_K$ be the space of $K$-finite vectors. It is a $(\mathfrak{g}, K)$-module, and it is dense in $\mathcal{H}^\infty$ in the Fréchet topology. The space $\mathcal{H}^\infty$ is a smooth Fréchet representation of moderate growth ([25], Lemma 11.5.1).

A matrix coefficient of $\pi$ is a function on $G$ of the form $x \mapsto \langle \pi(x) h, h' \rangle$, where $h, h' \in \mathcal{H}$. Obviously, $\varphi \neq 0$ if and only if $h, h' \neq 0$. The matrix coefficient is $K$-finite on the right (resp., on the left and on both sides) if and only if $h \in \mathcal{H}_K$ (resp., $h' \in \mathcal{H}_K$ and $h, h' \in \mathcal{H}_K$). The matrix coefficient is smooth if $h, h' \in \mathcal{H}^\infty$.

From now on we assume that $G$ admits discrete series. By the well-known classification of discrete series due to Harish–Chandra, this is the case if and only if $\text{rank}(G) = \text{rank}(K)$. A unitary representation $(\pi, \mathcal{H})$ is in discrete series if it has a non-zero matrix coefficient which belongs to $L^2(G)$. Due to results of Miličić [13], most of discrete series poses a non-zero $K$-finite matrix in $L^1(G)$. The precise description of such representations in terms of Harish–Chandra parameters is contained in [13]. We say that $(\pi, \mathcal{H})$ is integrable if it is in discrete series and it has a non-zero $K$-finite matrix coefficient in $L^1(G)$ (then all $K$-finite matrix coefficients are in $L^1(G)$). In fact, it is an exercise to prove that if $(\pi, \mathcal{H})$ has a non-zero $K$-finite matrix coefficient in $L^1(G)$, then this matrix coefficient is in $L^2(G)$, and consequently $(\pi, \mathcal{H})$ is in the discrete series (see the argument in the proof of Lemma 5-8).

The following lemma is important in our investigations below. It is a consequence of deep results of Casselman and Wallach on the globalization of $(\mathfrak{g}, K)$-modules ([25], Chapter 11, or [6]).

**Lemma 5-3.** Assume that $(\pi, \mathcal{H})$ is representation in the discrete series such that it poses a non-zero $K$-finite matrix coefficient which belongs to $L^p(G)$ for some $p \in [1, 2]$. Then, all smooth matrix coefficients belong to $L^p(G)$ as well.

**Proof.** The space $\mathcal{H}^\infty$ is a smooth Fréchet representation of moderate growth ([25], Lemma 11.5.1). On that space the Fréchet algebra $S(G)$ acts (see Section 4 for the definition of the space $S(G)$; the action is described in [25], 11.8). By ([25], Theorem 11.8.2), $\mathcal{H}^\infty$ is irreducible as an algebraic module for $S(G)$. In particular, for $h'' \in \mathcal{H}^\infty$, $h'' \neq 0$, we have $\mathcal{H}^\infty = S(G)h''$. Now, let us consider the matrix coefficient $c_{h, h'}$, where $h, h' \in \mathcal{H}^\infty$. Select $h'' \in \mathcal{H}_K$ such that $h'' \neq 0$. Then, since there exists $K$-finite matrix coefficients in $L^p(G)$,
then all of them are in $L^p(G)$. In particular, we have $c_{h''} \in L^p(G)$. Next, we select $f, f' \in S(G)$ such that $h = \pi(f)h''$ and $h' = \pi(f')h''$. Now, since $\pi$ is unitary, we have the following:

$$c_{h,h'}(x) = \langle \pi(x)h, h' \rangle = \langle \pi(x)\pi(f)h'', h\pi(f')h'' \rangle$$

$$= \int_G \int_G f(g)\overline{f'(g')} \langle \pi((g')^{-1}xg)h', h'' \rangle \, dg \, dg'.$$

Hence, using Remark 5-4, we have

$$\int_G |c_{h,h'}(x)|^p \, dx \leq \int_G \int_G |f(g)| \cdot |\overline{f'(g')}| \cdot |\langle \pi((g')^{-1}xg)h', h'' \rangle|^p \, dx \, dg \, dg'.$$

Then

$$||f||_1 ||f'||_1 ||c_{h'',h''}||^p < \infty.$$  

\[\square\]

**Remark 5-4.** Let $p \in [1, \infty]$. Then the function function $x \mapsto x^p$ is convex for $x > 0$. This means that for $x_1, \ldots, x_n > 0$, and $\lambda_1, \ldots, \lambda_n \geq 0$, $\lambda_1 + \cdots + \lambda_n = 1$, we have $(\lambda_1 \cdot x_1 + \cdots + \lambda_n \cdot x_n)^p \leq \lambda_1 \cdot x_1^p + \cdots + \lambda_n \cdot x_n^p$. Now, if $H$ is a measurable space, and $\alpha \geq 0$ is a measurable function on $H$ such that $\int_H \alpha(h) \, dh = 1$. Then for every measurable function $f : H \to \mathbb{C}$, we have the following inequality:

$$\left( \int_H |f(h)| \cdot \alpha(h) \, dh \right)^p \leq \int_H |f(h)|^p \cdot \alpha(h) \, dh.$$  

This follows from the inequality above considering integral sums and taking the limit. We leave details to the reader.

The following lemma is one of the main technical results of the present section:

**Lemma 5-5.** Assume that $(\pi, \mathcal{H})$ is integrable. Put $c_{h,h'}(g) = \langle \pi(g)h, h' \rangle$, $h, h' \in \mathcal{H}_K$. Then, we have the following:

(i) $c_{h,h'}$ is a smooth vector in the Banach representation $L^1(G)$, where $G$ acts by right-translations.

(ii) Let us fix $h' \in \mathcal{H}_K$, $h' \neq 0$. Then, the map the map $c_{h'} : h \mapsto c_{h,h'}$ is an infinitesimal equivalence of $(\pi, \mathcal{H})$ with a closed admissible irreducible subrepresentation $\mathcal{B}_{h'}$ of $G$ in the Banach representation on $L^1(G)$ given by the right-translations. In particular, we have the following:

(a) $c_{h'}(\mathcal{H}_K)$ is the space of $K$-finite vectors in $\mathcal{B}_{h'}$. It is contained in $\mathcal{B}_{h'}^\infty$.

(b) We have $c_{h'}(\mathcal{H}_K(\delta)) = \mathcal{B}_{h'}^\infty(\delta) = \mathcal{B}_{h'}(\delta)$, for all $\delta \in \hat{K}$.

(c) If $\chi_{\pi}$ is the infinitesimal character of $\pi$, then all $f \in \mathcal{B}_{h'}^\infty$ transforms according to $\chi_{\pi}$: $z \cdot f = \chi_{\pi}(z) f$, $f \in \mathcal{B}_{h'}^\infty$, $z \in Z(\mathfrak{g}_{\mathbb{C}})$. In particular, the vectors in $\mathcal{B}_{h'}^\infty$ are $Z(\mathfrak{g}_{\mathbb{C}})$-finite.

(iii) For each neighborhood $V$ of $1$ and $h' \in \mathcal{H}_K$, there exists $\beta \in C_c^\infty(G)$, $\text{supp}(\beta) \subset V$, such that $l(\beta)f = f$, $f \in \mathcal{B}_{h'}$. 

In particular, $\mathcal{B}_{h'}$ consists of smooth vectors in the left–regular representation of $G$ on $L^1(G)$, and $\mathcal{B}_{h'} \subset C^\infty(G)$.

(iv) There exists a finite set $S \subset \hat{K}$, such that

$$E^l_S(f) = f, \quad f \in \mathcal{B}_{h'}.$$ 

In particular, we have the following:

$$E^l_S(\mathcal{B}_{h'}) = \mathcal{B}_{h'}.$$ 

(v) For $u \in \mathcal{U}(\mathfrak{g}_C)$ and $h' \in \mathcal{H}_K$, we have $l(u)\mathcal{B}_{h'} \subset \mathcal{B}_{d\sigma(u)h'}$.

(vi) Let us fix $h' \in \mathcal{H}_K$, $h' \neq 0$. Select $h'' \in \mathcal{H}$ such that $\langle h'', h' \rangle = d(\pi)$, where $d(\pi)$ is the formal degree of $\pi$. Then, the map $d_{h''} : \mathcal{B}_{h'} \rightarrow \mathcal{H}$ defined by

$$f \in \mathcal{B}_{h'} \mapsto \pi(f')h'' = \int_G f'(x)\pi(x)h''dx$$

is a continuous $G$–invariant embedding with the dense image. It satisfies:

$$||d_{h''}(f)|| \leq ||h''|| \cdot ||f||_1, \quad f \in \mathcal{B}_{h'}.$$ 

Moreover, we have the following:

$$d_{h''}(c_{h', h}) = h, \quad h \in \mathcal{H}_K.$$ 

(vii) Finally, the restriction of $d_{h''}$ is a continuous isomorphism of Fréchet representations $\mathcal{B}_{h'}^\infty \simeq \mathcal{H}_K^\infty$. Its inverse is the map $h \in \mathcal{H}_K^\infty \rightarrow c_{h,h'}$. In particular, $\mathcal{B}_{h'}^\infty = \{c_{h,h'}; \quad h \in \mathcal{H}_K^\infty\}.$

Proof. First, we prove (i). The function $c_{h,h'}$ is $Z(\mathfrak{g}_C)$–finite and $K$–finite on the right. Hence, by ([10], Theorem 1), there exists $\alpha \in C^\infty_c(G)$ (depending of $h$) such that

$$c_{h,h'} = c_{h,h'} \ast \alpha = r(\alpha')c_{h,h'}.$$ 

In view of the discussion before the statement of the theorem, this proves (i). Now, we prove (ii). Clearly, the space all $c_{h,h'}$, $h \in \mathcal{H}_K$ is an irreducible $(\mathfrak{g}, K)$ isomorphic to $\mathcal{H}_K$. The closure $\mathcal{B} = \mathcal{B}_{h'}$ of such functions in $L^1(G)$ is clearly $G$–invariant. For each $\delta \in \hat{K}$, we have $E_\delta(\mathcal{B})$ is spanned by the functions $c_{h,h'}$, $h \in \mathcal{H}_K(\delta)$. Indeed, for $f \in \mathcal{B}$, there exists a sequence of vectors $h_n$, $n \geq 1$ such that

$$\lim_n \int_G |f(x) - c_{h_n,h'}(x)| dx = 0.$$ 

Hence, for $\delta \in \hat{K}$, we have the following:

$$E_\delta(c_{h_n,h'})(x) = \int_G d(\delta)\xi_{\delta}(k)c_{h_n,h'}(xk)dk = \langle \pi(x)E_\delta(h_n), h' \rangle = c_{E_\delta(h_n),h'}(x), \quad x \in G.$$ 

Now, we have the following:
\[
\int_G \left| E_\delta(f)(x) - c_{E_\delta(h_n,h')}(x) \right| \, dx = \int_G \left| E_\delta(f)(x) - E_\delta(c_{h_n,h'}(x)) \right| \, dx \\
= \int_G \left\| \int_K d(\delta) \xi_\delta(k) (f(xk) - c_{h_n,h'}(xk)) \, dk \right\| \, dx \\
\leq \int_K d(\delta) |\xi_\delta(k)| \, dk \int_G |f(x) - c_{h_n,h'}(x)| \, dx.
\]

This shows

\[c_{E_\delta(h_n,h')} \xrightarrow{L^1} E_\delta(f).\]

But the sequence belongs to a finite–dimensional subspace of \( \mathcal{B} \). Which is because of that closed. Hence the claim. In particular, \( \mathcal{B} \) is admissible. Moreover, this also proves the claims in (b), and consequently in (a) since by standard argument: the vectors in \( c_{h'}(H_K) \) are \( Z(\mathfrak{g}_C) \)–finite and \( K \)–finite. Hence, real analytic, and in particular, smooth. The claim in (c) is also standard. It is true for \( f \in c_{h'}(H_K) \) which is the space of \( K \)–finite vectors in \( \mathcal{B}_{h'} \). But this space is dense in the Fréchet space \( \mathcal{B}_{h'}^\infty \) (see Lemma 5-1 (ii)). The description of topology on \( \mathcal{B}_{h'}^\infty \) given in the second paragraph of this section immediately implies the claim.

Finally, we show that \( \mathcal{B} \) is irreducible. If \( \mathcal{B}' \) is non–zero closed subrepresentation of \( \mathcal{B} \). Then, for each \( \delta \in \hat{K} \), we have

\[\mathcal{B}'(\delta) = E_\delta(\mathcal{B}') \subset E_\delta(\mathcal{B}) = \mathcal{B}(\delta) = c_{h'}(H_K(\delta)).\]

But then by (i) and the irreducibility of \( (\mathfrak{g},K) \)–module \( H_K \), we have

\[c_{h'}(H_K) \subset \mathcal{B}_K'.\]

Hence, by definition of \( \mathcal{B} \), we have \( \mathcal{B}' = \mathcal{B} \).

Let us prove (iii). We may consider \( \mathcal{H} \) to be a subrepresentation of \( L^2(G) \) under the right–translations. Since \( h' \in H_K \), \( h' \) is a \( Z(\mathfrak{g}_C) \)–finite and \( K \)–finite on the right. So, by (10), Theorem 1), for each neighborhood \( V \) of \( 1 \in G \), there exists \( \beta_1 \in \mathcal{C}_c^\infty(G) \), \( \text{supp}(\beta_1) \subset V \) such that

\[\pi(\beta_1)h' = h' \star \beta_1^\vee = h'.\]

Now, we compute

\[c_{h,h'}(x) = \int_G \langle \pi(x)h, h' \rangle \, dx\]
\[= \int_G \langle \pi(x)h, \pi(\beta_1)h' \rangle \, dx\]
\[= \int_G \int_G \overline{\beta_1(y)} \langle \pi(x)h, \pi(y)h' \rangle \, dxdy\]
\[= \int_G \beta_1(y) \left( \int_G \langle \pi(y^{-1}x)h, h' \rangle \, dx \right) \, dy\]
\[= \beta_1 \ast c_{h,h'}(x).\]
Now, let $\beta = \overline{\beta}$, and apply the Dominated convergence theorem to get (iii).

The proof of (iv) is similar to the proof of (iii). Since $h'$ is $K$–finite, there exists a finite
set $T \subset \hat{K}$ such that $E_T(h') = h'$, $E_T = \sum_{\delta \in \hat{K}} E_\delta$. Now, the reader can easily adapt the
argument in (iii) to get (iv).

We prove (v). Let $h \in \mathcal{H}_K$. Then, it is easy to see
\[ l(u)c_{h,h'} = c_{h,\pi(u)h'}. \]
Thus, we have a linear map from the space of all functions $c_{h,h'}$, $h \in \mathcal{H}_K$, into $\mathcal{B}_{\pi(u)h'}$. We
show that it extend to a bounded map $\mathcal{B}_{h'} \rightarrow \mathcal{B}_{\pi(u)h'}$. This follows from (iii) and the
expression (5.2). Let us select $\beta \in C^\infty_c(G)$ such that $f = \beta \ast f$ for all $f \in \mathcal{B}_{h'}$. Thus, the
map $f \mapsto l(u)f$ can be with the aid of (5.2) written as follows:
\[ f \mapsto l(u)f = (l(u)\beta) \ast f \]
which is clearly bounded map.

We prove (vi). It is well–known that $F \in L^1(G)$ acts on the unitary representation
$(\pi, \mathcal{H})$ as follows: $\pi(F)h = \int_G f(x)\pi(g)hdx$, $h \in \mathcal{H}$. Moreover, we have $||\pi(F)|| \leq ||h|| \cdot
||F||_1$. This immediately implies that the maps in (vi) is well–defined and continuous. Since
$(r(g)F)^\vee = l(g)F^\vee$, it is also $G$–invariant. The rest of the claims in (vi) follow from the
Schur orthogonality relation:
\[ (5.6) \quad \int_G \langle \pi(x)h''', h_1 \rangle \cdot \langle \pi(x)h', h \rangle dx = \frac{1}{d(\pi)} \langle h, h_1 \rangle \langle h'', h' \rangle = \langle h, h_1 \rangle, \quad h_1 \in \mathcal{H}, \quad h \in \mathcal{H}_K. \]
Since $\pi$ is unitary, the right–hand side can be transformed as follows:
\[ \int_G c_{h,h'}^\vee(x)\langle \pi(x)h''', h_1 \rangle dx. \]
Hence, (5.6) implies
\[ (5.7) \quad d_{h'''}(c_{h'}(h)) = \int_G c_{h,h'}^\vee(x) \pi(x)h'' dx = h, \quad h \in \mathcal{H}_K. \]
This identity implies that image of $d_{h'''}$ is dense in $\mathcal{H}$ since it contains $K$–finite vectors.

Finally, we prove (vii). It is obvious that the restriction of $d_{h'''}$ is a continuous embedding
of Fréchet spaces $\mathcal{B}_{h'}^\infty \hookrightarrow \mathcal{H}^\infty$. Using Lemma 5.3 we can extend the map $c_{h'}$ to $\mathcal{H}^\infty$ into
$L^1(G)$. Let us show that this map has image contained in $\mathcal{B}_{h'}^\infty$. Let $h \in \mathcal{H}^\infty$. Then, by
Lemma 5.1 (i), there exists $h_1, \ldots, h_l \in \mathcal{H}^\infty$, and $\alpha_1, \ldots, \alpha_l \in C^\infty_c(G)$ such that
\[ h = \sum_{i=1}^l \pi(\alpha_i)h_i. \]
Then, we have the following:
\[ c_{h,h'}(x) = \sum_{i=1}^l \int_G \alpha_i(y)\langle \pi(xy)h_i, h' \rangle dy = \sum_{i=1}^l r(\alpha_i)c_{h_i,h'}(x). \]
This implies that $c_{h,h'}$ is a smooth vector for the right action of $G$. Then, by Lemma 5.1
(ii), $c_{h,h'} \in \mathcal{B}_{h'}$. Finally, since it is smooth vector, $c_{h,h'} \in \mathcal{B}_{h'}^\infty$. Next, being based on the
Schur orthogonality relation, (5.7) is true for $h \in \mathcal{H}^\infty$. Which shows that $d_{h'}$ is bijective.
continuous map between Fréchet spaces. Hence, by the Open mapping theorem, \( d_h' \) is an isomorphism. Since \( c_h' \) is its inverse, we obtain (vii).

In the next lemma we do not assume in advance that \( G \) poses discrete series. But when favorable functions exist this must be the case. It is an analogue of ([10], Lemma 77) for \( L^2(G) \).

**Lemma 5-8.** Assume \( \varphi \in C^\infty(G) \cap L^1(G) \) that is non–zero \( \mathcal{Z}(g_C) \)–finite and \( K \)–finite on the right. Then, under the right translations, the \((g, K)\)–module generated by \( \varphi \) is a direct sum of finitely many irreducible representations each infinitesimally equivalent to an integrable discrete series of \( G \). In particular, \( G \) poses discrete series.

**Proof.** First, the standard argument shows that \( \varphi \in L^2(G) \). We recall that argument (see the proof of ([18], Theorem 3.10) and ([3], Corollary 2.22)). Since \( \varphi \) that is \( \mathcal{Z}(g_C) \) and \( K \)–finite on the right, by the result of Harish–Chandra (see [10], Theorem 1) there exists \( \alpha \in C^\infty_c(G) \) such that \( \varphi = \varphi \ast \alpha \). Since, \( \varphi \in L^1(G) \), this immediately shows that \( \varphi \) is bounded. Finally, we write \( G = 1 \) and \( G < 1 \) for the set of all \( x \in G \) satisfying \(|\varphi(x)| \geq 1 \) and \(|\varphi(x)| < 1 \), respectively. Let \( C > 0 \) be the bound of \( \varphi \). Then, we have the following:

\[
\int_G |\varphi(x)| dx = \int_{G \geq 1} |\varphi(x)| dx + \int_{G < 1} |\varphi(x)| dx \geq \int_{x \geq 1} dx.
\]

Thus, the set \( G \geq 1 \) has a finite measure. Finally,

\[
\int_G |\varphi(x)|^2 dx = \int_{G \geq 1} |\varphi(x)|^2 dx + \int_{G < 1} |\varphi(x)|^2 dx \leq C^2 \int_{G \geq 1} dx + \int_{G < 1} |\varphi(x)| dx < \infty.
\]

Of course the same is true for \( u \varphi \), where \( u \in \mathcal{U}(g_C) \). So, now the \((g, K)\)–module \( V \) generated by \( \varphi \) which is apriori in \( L^1(G) \) belongs to \( L^2(G) \) and by ([24], Corollary 3.4.7 and Theorem 4.2.1) it is direct sum of finitely many irreducible representations each of which is infinitesimally isomorphic to an discrete series:

\[
V = V_1 \oplus \cdots \oplus V_l.
\]

In particular, this forces that \( rank(K) = rank(G) \) in order to have a non–zero \( \varphi \). We note that each \( V_i \) consists of functions which are \( \varphi \in C^\infty(G) \cap L^1(G) \) that is \( \mathcal{Z}(g_C) \)–finite and \( K \)–finite on the right simply because \( V \) consists of such functions. Therefore, in what follows we may assume that \( V \) is irreducible.

We normalize Haar measure on \( K \) as follows \( \int_K dk = 1 \). Then, for an irreducible representation \( \delta \in \hat{K} \), we write \( d(\delta) \) and \( \xi_\delta \) for the degree and the character of \( \delta \), and let

\[
E^l_\delta = \int_K d(\delta) \xi_\delta(k) l(k) dk
\]

be the projector on \( \delta \)–isotypic component of the left regular representation \( l \) on \( L^2(G) \).
Since \( \varphi \in L^2(G) \), we have the following expansion which converges absolutely in \( L^2(G) \):

\[
\varphi = \sum_{\delta \in \hat{K}} E^l_\delta(\varphi).
\]

Hence there exists \( \delta \in \hat{K} \) such that

\[
(5-10) \quad E^l_\delta(\varphi) \neq 0.
\]

Obviously, this function is also in \( C^\infty(G) \subseteq L^1(G) \), and is \( Z(g_\C) \)-finite and \( K \)-finite on the right. For any \( \delta \in \hat{K} \) satisfying \( (5-10) \), \( \psi \mapsto E^l_\delta(\psi) \) is a \( (g,K) \)-intertwining operator between \( V \) and \( E^l_\delta(V) \). Since \( V \) is assumed to be irreducible, it is an isomorphism.

To complete the proof the lemma, we may assume that \( \varphi \) is \( K \)-finite on the left. But because we find it interesting not to assume that \( V \) is irreducible. We use \( (5-9) \).

Since \( \varphi \) is \( K \)-finite on the right, we can find a finite non-empty subset \( S \subseteq \hat{K} \), such that the operator \( E^l_S \) defined by \( \sum_{\delta \in S} E^l_\delta \) satisfies \( E^l_S(\varphi) = \varphi \). Since \( V \) is generated by \( \varphi \) and left and right actions commute with each other, we obtain another decomposition of \( V \) into subrepresentations:

\[
V = E^l_S(V) = E^l_S(V_1) \oplus \cdots \oplus E^l_S(V_l).
\]

For each \( i \), \( E^l_S(V_i) = \{0\} \) or isomorphic to \( V_i \) since \( V_i \) is irreducible. Hence, for each \( i \), \( E^l_S(V_i) \simeq V_i \) because we must have the same number of irreducible modules in both decomposition of \( V \).

We write

\[
\varphi = \sum_i \varphi_i, \quad \varphi_i \in E^l_S(V_i).
\]

Now, since \( \varphi \) generates \( V \), we must have \( \varphi_i \neq 0 \) for all \( i \). Next, it is obvious that each \( \varphi_i \) is also \( Z(g_\C) \) and \( K \)-finite, and in \( L^1(G) \). Let us fix \( i \in \{1, \ldots, l\} \). Since \( \varphi_i \) is also \( K \)-finite on the left, again by a result of Harish–Chandra (see [10], Theorem 1), there exists \( \beta \in C^\infty_c(G) \) such that \( \varphi_i = \beta \star \varphi_i \). This can be written as

\[
\varphi_i(x) = \int_G \beta(y) \varphi_i(yx) \, dx
\]

which easily implies that \( \varphi_i \) is a \( K \)-fine matrix coefficient of \( V_i \). Indeed, we have
\[ \varphi_i(x) = E^l_S(\varphi_i)(x) = \int_K d(\delta) \xi_\delta(k) \varphi_i(k^{-1}x) dk \]
\[ = \int_K d(\delta) \xi_\delta(k) \left( \int_G \beta^y(y) \varphi_i(yk^{-1}x) dx \right) dk \]
\[ = \int_K d(\delta) \xi_\delta(k) \left( \int_G \beta^y(y) \varphi_i(yx) dx \right) dk \]
\[ = \int_G E^r_S(\beta^y)(y) \varphi_i(yx) dx \]
\[ = \int_G E^r_\tilde{S}(\beta^y)(y) \varphi_i(yx) dx, \]

where \( E^r_\tilde{S} \) is analogously defined for the right translations, and \( \tilde{S} = \{ \tilde{\delta} : \delta \in S \} \) the set of contragredient representations. In the last equality we may replace \( E^r_\tilde{S}(\beta^y) \) with its orthogonal projection to \( Cl(V_i) \) and we are done. Having proved that \( \varphi_i \) is a \( K \)-finite matrix coefficient of \( V_i \), we immediately get each \( V_i \) is integrable. \( \square \)

The following can be seen from the last part of the proof:

**Lemma 5-11.** Assume \( \varphi \in C^\infty(G) \cap L^1(G) \) that is \( Z(g_C) \)-finite and \( K \)-finite on the left and right. Assume that under the right translations, the \((g,K)\)-module generated by \( \varphi \) is irreducible. Let \( \mathcal{H} \) and \( \mathcal{B} \) be the closures of \( V \) in \( L^2(G) \) and \( L^1(G) \), respectively. Then, the representation of \( G \) on \( \mathcal{H} \) by right translations is integrable, and there exists \( h' \in \mathcal{H}_K \) such that (see Lemma 5-5)

\[ \mathcal{B} = \mathcal{B}_{h'}. \]

Finally, we prove the following theorem:

**Theorem 5-12.** Let \( \mathcal{B} \) the an irreducible closed admissible subrepresentation of \( L^1(G) \) under the right translations. Then, there exists unique up to (unitary or infinitesimal) equivalence an integrable discrete series \((\pi, \mathcal{H})\) of \( G, \delta \in \hat{K} \), and a \( K \)-finite vector \( h' \in \mathcal{H}_K \) such that

\[( \text{the closure in } L^1(G) \text{ of } E^l_\delta(\mathcal{B}) = \mathcal{B}_{h'}). \]

In particular, irreducible closed admissible subrepresentations of \( L^1(G) \) are infinitesimally equivalent to integrable discrete series.

**Proof.** We use notation introduced in the statement and in the proof of Lemma 5-5. The fact that \( \mathcal{B} \) is admissible means that for each \( \delta \in \hat{K} \), we have

\[ \mathcal{B}(\delta) \overset{\text{def}}{=} E_\delta(\mathcal{B}). \]
is finite dimensional. Let $B^\infty$ be the space of smooth vectors in $B$. Then, ([10], Lemma 4) we have that the sum of vector spaces

$$
\sum_{\delta \in \hat{K}} B(\delta) \cap B^\infty
$$

is dense $B$.

Hence, there exists $\delta \in \hat{K}$ such that $B(\delta) \cap B^\infty \neq 0$. We get

$$
B^\infty(\delta) \stackrel{def}{=} E_\delta(B^\infty) = B(\delta) \cap B^\infty \neq 0.
$$

We obtained a finite dimensional space different than 0, invariant under $Z(g_C)$. Thus, $B$ contains a non–zero $\varphi \in C^\infty(G) \cap L^1(G)$ that is $Z(g_C)$–finite and $K$–finite on the right. As in the proof of Lemma 5-8 (see (5-10)), select $\delta \in \hat{K}$ such that $E^l_\delta(\varphi) \neq 0$. Then, using arguments in the proof of Lemma 5-8 we obtain the theorem. □

6. PREPARATION FOR APPLICATION TO AUTOMORPHIC FORMS

In this section we still assume that $G$ is a semisimple connected Lie group with finite center. We assume that $\Gamma$ is a discrete subgroup of $G$. Then, for $\varphi \in L^1(G)$ we can form the Poincaré series $P_\Gamma(\varphi)(g) := \sum_{\gamma \in \Gamma} \varphi(\gamma \cdot g)$. Since

$$
\int_{\Gamma \setminus G} |P_\Gamma(\varphi)(g)| \, dg \leq \int_{\Gamma \setminus G} \left( \sum_{\gamma \in \Gamma} |\varphi(\gamma \cdot g)| \right) \, dg = \int_G |\varphi(g)| \, dg < +\infty,
$$

the series $\sum_{\gamma \in \Gamma} \varphi(\gamma \cdot g)$ converges absolutely almost everywhere and $P_\Gamma(\varphi) \in L^1(\Gamma \setminus G)$. It is obvious that the map

$$
\varphi \longmapsto P_\Gamma(\varphi)
$$

is a continuous $G$–equivariant map of Banach representations

(6-1) $L^1(G) \longrightarrow L^1(\Gamma \setminus G).
$

This map is never zero. Indeed, select $\varphi \in C^\infty_c(G)$, $\varphi \neq 0$, with a support in a sufficiently small neighborhood $V$ of 1 $\in G$ which satisfies $VV^{-1} \cap \Gamma = \{1\}$. Then, we have the following:

$$
P_\Gamma(\varphi)(g) := \sum_{\gamma \in \Gamma} \varphi(\gamma \cdot g) = \varphi(g), \; g \in V.
$$

This proves the claim.

It is considerable harder to decide when $P_\Gamma(\varphi) \neq 0$. A sufficient condition is contained in the following lemma.

**Lemma 6-2.** Let $\varphi \in L^1(G)$. Then, we have the following:

(i) Assume that there exists a compact neighborhood $C$ (i.e., an open set which closure is compact) in $G$ such that

$$
\int_C |\varphi(g)| \, dg > \int_{G-C} |\varphi(g)| \, dg \quad \text{and} \quad \Gamma \cap C \cdot C^{-1} = \{1\}.
$$

Then, $P_\Gamma(\varphi) \neq 0$. 

(ii) Let \( \Gamma_1 \supset \Gamma_2 \supset \ldots \) be a sequence of discrete subgroups of \( G \) such that \( \cap_{n \geq 1} \Gamma_n = \{1\} \). Then, there exists \( n_0 \) depending on \( \varphi \) such that \( P_{\Gamma_n}(\varphi) \neq 0 \) for \( n \geq n_0 \).

Proof. The claim (i) is \([18], \text{Theorem 4-1}\). Finally, the claim (iii) is \([18], \text{Corollary 4-9}\). \( \square \)

The following lemma is a variation of a standard argument:

Lemma 6-3. Let \( V \) be an open neighborhood of \( 1 \in G \) such that \( VV^{-1} \cap \Gamma = \{1\} \). Assume that \( \beta \in C^\infty_c(G) \) such that \( \text{supp}(\beta) \subset V \), and \( \varphi \in L^1(G) \). Put \( \psi = \beta \ast \varphi \in C^\infty(G) \in L^1(G) \).

Then, \( P_{\Gamma}(\psi) \) is a bounded continuous function on \( G \) in \( L^1(\Gamma \backslash G) \). More precisely, we have the following estimate:

\[
||P_{\Gamma}(\psi)||_\infty \leq ||\beta||_\infty ||\varphi||_1,
\]

Proof. We have

\[
\psi(\gamma x) = \int_G \beta(\gamma xy^{-1})\varphi(y)dy = \int_G \beta(\gamma xy)\varphi(y)dy, \quad x \in G, \quad \gamma \in \Gamma.
\]

Note that \( \beta(\gamma xy) \neq 0 \) implies that \( y \in x^{-1}\gamma^{-1}V \).

Since \( VV^{-1} \cap \Gamma = \{1\} \), the sets \( x^{-1}\gamma^{-1}V, \quad \gamma \in \Gamma \)

are disjoint. Thus, we get

\[
|P_{\Gamma}(\psi)(x)| \leq \sum_{\gamma \in \Gamma} |\psi(\gamma x)| \leq ||\beta||_\infty \sum_{\gamma \in \Gamma} \int_{x^{-1}\gamma^{-1}V} |\varphi(y)|dy \leq ||\beta||_\infty ||\varphi||_1.
\]

\( \square \)

Now, we prove the main result of the present section. It contains a novel approach to convergence of Poincaré series. The reader now should review Lemma 5-5.

Theorem 6-4. Assume that \( G \) admits discrete series. Let \( (\pi, \mathcal{H}) \) is an integrable discrete series of \( G \). Let \( h' \in \mathcal{H}_K, \ h' \neq 0 \). Let \( \Gamma \subset G \) be a discrete subgroup. Then, we have the following:

(i) The map \( \varphi \mapsto P_{\Gamma}(\varphi) \) is a continuous \( G \)-equivariant map from the Banach representation \( \mathcal{B}_{h'} \) into the unitary representation \( L^2(\Gamma \backslash G) \). The image \( P_{\Gamma}(\mathcal{B}_{h'}) \) is either zero, or it is an embedding and its closure is an irreducible subspace unitary equivalent to \( (\pi, \mathcal{H}). \)

(ii) The smooth vectors \( \mathcal{B}_{h'}^\infty \) are mapped under \( P_{\Gamma} \) into the subspace of \( \mathcal{Z}(g_C) \)-finite vectors in \( \mathcal{A}_{\text{wmg}}(\Gamma \backslash G) \).

(iii) Furthermore, we may consider, \( P_{\Gamma} : \mathcal{B}_{h'} \rightarrow L^1(\Gamma \backslash G) \subset \mathcal{S}(\Gamma \backslash G)' \) (see Section 4). This map is a continuous map from a Banach space \( \mathcal{B}_{h'} \) into a locally convex space \( \mathcal{S}(\Gamma \backslash G)' \).

(iv) Let \( \Gamma_1 \supset \Gamma_2 \supset \ldots \) be a sequence of discrete subgroups of \( G \) such that \( \cap_{n \geq 1} \Gamma_n = \{1\} \). Then, there exists \( n_0 = n_0(h') \geq 1 \) such that \( P_{\Gamma_n} : \mathcal{B}_{h'} \rightarrow L^2(\Gamma_n \backslash G) \) is a continuous embedding for \( n \geq n_0 \).
Proof. Let us prove (i). Let $V$ be an open neighborhood of $1 \in G$ such that $VV^{-1} \cap \Gamma = \{1\}$. By Lemma 5.3 (iii), there exists $\beta \in C_c^\infty(G)$, supp $(\beta) \subset V$, such that

$$\beta * \varphi = l(\beta) \varphi = \varphi, \quad \varphi \in B_{h'}.$$

By Lemma 6.3 we have

$$||P_\Gamma(\varphi)||_\infty \leq ||\beta||_\infty ||\varphi||_1,$$

Thus, we have the following:

$$\int_{\Gamma \setminus G} |P_\Gamma(\varphi)(x)|^2 \, dx \leq ||\beta||_\infty ||\varphi||_1 \int_{\Gamma \setminus G} |P_\Gamma(\varphi)(x)| \, dx \leq ||\beta||_\infty ||\varphi||_1^2,$$

by the computation given at the beginning of this section. Hence

$$\left(\int_{\Gamma \setminus G} |P_\Gamma(\varphi)(x)|^2 \, dx\right)^{1/2} \leq ||\beta||_\infty^{1/2} ||\varphi||_1.$$

This shows that the map $\varphi \mapsto P_\Gamma(\varphi)$ is well-defined and continuous. It is obviously $G$–invariant.

Let us assume that $B_0 \in B_{h'}$ is irreducible (see Lemma 5.3 (ii)), $P_\Gamma$ must be an embedding. Let $H_0$ be its closure in $L^2(\Gamma \setminus G)$. It is obviously $G$–invariant. We show that $H_0$ is admissible. Indeed, since $B_0$ is dense in $H_0$ and the projector $E_\delta$ is continuous (see Section 5) for each $\delta \in \hat{K}$, we obtain that

$$P_\Gamma(B_0(\delta)) = B_0(\delta)$$

is dense in

$$H_0(\delta).$$

But $B_0$ is admissible (see Lemma 5.3 (ii)). So each $B_0(\delta)$ is finite dimensional. Thus, closed in $H_0(\delta)$. Hence

$$H_0(\delta) = B_0(\delta).$$

This proves that $H_0$ is admissible. Also, via the map $P_\Gamma$ we obtain that $(g, K)$ modules $(B_0)_K$ and $(H_0)_K$. Next, by Lemma 5.3 (ii), we obtain that $(g, K)$ modules $(B_0)_K$ are $H_K$ are isomorphic. Thus, we obtain (i).

Now, we prove (ii). Let $\varphi \in B_{h'}^\infty$. By Lemma 5.1 (i), there exists $\varphi_1, \ldots, \varphi_l \in B^\infty$ and $\alpha_1, \ldots, \alpha_l \in C_c^\infty(G)$ such that

$$\varphi = \sum_{i=1}^l \varphi_i * \alpha_i.$$

Then, by direct computation

$$P_\Gamma(\varphi) = \sum_{i=1}^l P_\Gamma(\varphi_i) * \alpha_i.$$

Next, Lemma 4.6 implies that

$$P_\Gamma(\varphi_i) * \alpha_i \in A_{umg}(\Gamma \setminus G),$$

for all $i$. This proves $P_\Gamma(\varphi) \in A_{umg}(\Gamma \setminus G)$. To complete the proof of (ii) we need to prove that $P_\Gamma(\varphi)$ is $Z(g_{Z})$–finite. Indeed, by (i), the map $P_\Gamma : B_{h'} \rightarrow L^2(\Gamma \setminus G)$ is continuous. Then,
Let $B.$ The reader should now refer to the description of topology on $S \Gamma \backslash G.$ Let us denote by $P_T(\varphi)$ the functional $f \mapsto \int_{\Gamma \backslash G} P_T(\varphi)(x)f(x)dx,$ $f \in S(\Gamma \backslash G).$ The reader should now refer to the description of topology on $S(\Gamma \backslash G)^\prime$ (see the description of topology after Definition \[4\]). Let $B \subset S(\Gamma \backslash G)$ be a bounded set. Then, there exists $M_B > 0$ such that

$$\sup_{f \in B} ||f||_{1,-1} \leq M_B.$$ 

This implies

$$|f(x)| \leq M_B \cdot ||x||_{\Gamma \backslash G}^{-1}, \quad f \in B, \quad g \in G.$$ 

Since $||x||_{\Gamma \backslash G} \geq 1$ for all $x \in G,$ we obtain

$$|f(x)| \leq M_B, \quad f \in B, \quad g \in G.$$ 

Now, we prove the continuity of the map

$$||P_T(\varphi)||_B = \sup_{f \in B} \left| \int_{\Gamma \backslash G} P_T(\varphi)(x)f(x)dx \right|$$

$$\leq \sup_{f \in B} \int_{\Gamma \backslash G} |P_T(\varphi)(x)||f(x)|dx$$

$$\leq M_B \int_{\Gamma \backslash G} |P_T(\varphi)(x)|dx$$

$$\leq M_B \int_{G} |\varphi(x)|dx = M_B ||\varphi||_1.$$ 

Finally, we prove (iv). We consider a $K$–finite matrix coefficient $c_{h',h'} \in L^1(G)$ defined in Lemma \[5-5\]. Then, by Lemma \[6-2\] (ii), there exists $n_0 = n_0(h') \geq 1$ such that $P_{\Gamma_n}(c_{h',h'}) \neq 0.$ Thus, for such $n,$ the map $P_{\Gamma_n}: B_{h'} \rightarrow L^2(\Gamma_n \backslash G)$ is non–zero. But, by Lemma \[6-5\] (ii), $B_{h'}$ is a closed irreducible subrepresentation of $L^1(G)$ (under the right–translations), and $P_T$ is continuous on $B_{h'}.$ Hence the claim in (iv). \qed

By Lemma \[6-2\] (i) the number $n_0 = n_0(h')$ can be computed as follows. Since $c_{h',h'} \in L^1(G),$ there exists a compact neighborhood $C$ (i.e., an open set which closure is compact) in $G$ such that

$$\int_C |c_{h',h'}(g)|dg > \int_{G-C} |c_{h',h'}(g)|dg.$$ 

Since $G$ is countable at infinity such $C$ exists. Now, since $\cap_{n \geq 1} \Gamma_n = \{1\},$ there exists $n_0 = n_0(h') \geq 1$ such that $\Gamma_n \cap C \cdot C^{-1} = \{1\}$ for $n \geq n_0.$ Now, we apply Lemma \[6-2\] (i) to see that $P_{\Gamma_n}(c_{h',h'}) \neq 0.$ When $G$ is a group of $\mathbb{R}$–points of a semisimple algebraic group defined over $\mathbb{Q}$ and $\Gamma_n$ is a sequence of congruence subgroups, that was indicated in (\[18\], Theorem 6.1). Explicit computations for $G = SL_2(\mathbb{R})$ were performed in \[19\].

We make the following observation:
Remark 6-5. Let $\Gamma \subset G$ be a discrete subgroup. Let $h' \in \mathcal{H}_K$, $h' \neq 0$. Assume that $P_\Gamma (B_{h'}) = 0$. Then, by Lemma 6-2 (i), for every compact neighborhood $C$ (i.e., an open set which closure is compact) in $G$ such that $\Gamma \cap C \cdot C^{-1} = \{1\}$ we have
\[
\int_C |\varphi(g)| \, dg \leq \frac{1}{2} \int_G |\varphi(g)| \, dg, \quad \varphi \in B_{h'}.
\]

We end this section with a comment. So far, we have studied explicitly constructed automorphic forms $P_\Gamma (\varphi)$ and when they are non–zero. On the other hand, given integrable discrete series $(\pi, \mathcal{H})$ of $G$, then we have the following classical observation which is in [14].

Lemma 6-6 (Miličić). Assume that $G$ admits discrete series. Let $(\pi, \mathcal{H})$ be an integrable discrete series representation of $G$. Then, the orthogonal complement of the algebraic sum
\[
\sum_{h' \in \mathcal{H}_K} P_\Gamma (B_{h'})
\]
in $L^2(\Gamma \backslash G)$ does not contain a $G$–invariant closed subspace equivalent to $\pi$. In other words, $\pi$–isotypic component in $L^2(\Gamma \backslash G)$ is given by the closure of the algebraic sum $\sum_{h' \in \mathcal{H}_K} P_\Gamma (B_{h'})$.

Proof. We start from the following observation. A function $f \in L^1(G)$ acts as a bounded operator on the unitary representation $L^2(\Gamma \backslash G)$:
\[
(r(f), \varphi)(x) = \int_G \varphi(xy) f(y) \, dy = \int_G \varphi(y) f^\vee(y^{-1} x) \, dy = \varphi \ast f^\vee(x).
\]
Assuming that $f$ is a smooth vector in the Banach representation $L^1(G)$ under the left translations, we obtain that $f^\vee$ is a smooth vector in the Banach representation $L^1(G)$ under the right translations. Therefore, the resulting function satisfies
\[
r(f), \varphi \in C^\infty(G) \cap L^2(\Gamma \backslash G).
\]
The details are left to the reader but a hint is given in the computation given after the proof of Lemma 5-1. Therefore, the value $r(f), \varphi(1)$ is well–defined.

Based on this, we have the following formula for the inner product:
\[
(6-7) \quad \langle P_\Gamma (f), \varphi \rangle = \int_{\Gamma \backslash G} \varphi(y) \overline{P_\Gamma (f)(y)} \, dy = \int_G \varphi(y) \overline{f(y)} \, dy = r(\overline{f}), \varphi(1)
\]
since $\varphi$ is $\Gamma$–invariant on the right.

Now, we show that in the orthogonal complement in $L^2(\Gamma \backslash G)$ of the (algebraic) sum of subspaces $\sum_{h' \in \mathcal{H}_K} P_\Gamma (B_{h'})$ there is no closed irreducible subspace equivalent to $\pi$. Indeed, if $W$ is such a subspace, then by (6-7) and Lemma 5-3 (iii), we have
\[
r(\overline{f}), \varphi(1) = \langle P_\Gamma (\overline{f}), \varphi \rangle = 0, \quad \varphi \in W, \quad f \in B_{h'}, \quad h' \in \mathcal{H}_K.
\]
We remark that $r(\overline{f}) \varphi$ is a $K$–finite function by Lemma 5-5 (iv) and the fact
\[
r(k) (r(\overline{f}) \varphi) = r \left( \overline{l(k)f} \right), \quad k \in K.
\]
Also, for \( u \in \mathcal{U}(\mathfrak{g}_C) \), using Lemma 5-5 (v), we have

\[
    r(u) \left( r(f) \varphi \right) = r \left( l(u) f \right) . \varphi.
\]

Thus, by our assumption

\[
    r(u) \left( r(f) \varphi \right) (1) = r \left( l(u) f \right) . \varphi(1) = 0.
\]

Now, we have that \( r(f) \varphi \) is real-analytic, and has all derivatives at 1 equal to zero. This implies

\[
    r(f) . \varphi = 0, \quad \varphi \in W, \quad f \in B_{h'}, \quad h' \in \mathcal{H}_K.
\]

But \( W \) is unitary equivalent to \( (\pi, \mathcal{H}) \). Let \( \varphi \in W \) be a non-zero \( K \)-finite vector. Assume that under the fixed unitary equivalence corresponds to say \( h' \in \mathcal{H}_K \). Then, by Schur orthogonality (see the proof of Lemma 5-5 (vi))

\[
    r(c_{h', h'}) . h' \neq 0.
\]

(In (5-6), we let \( h = h_1 = h'' = h' \).) This contradicts above equality. \( \square \)

7. Application to Automorphic Forms

In this section we apply the results of Section 6 to prove results for automorphic forms. We start stating hypothesis on \( G \) and \( \Gamma \). In this section we assume that \( G \) is a group of \( \mathbb{R} \)-points of a semisimple algebraic group \( G \) defined over \( \mathbb{Q} \). Assume that \( G \) is not compact and connected. Let \( \Gamma \subset G \) be congruence subgroup with respect to the arithmetic structure given by the fact that \( G \) defined over \( \mathbb{Q} \) (see [5]). Then, \( \Gamma \) is a discrete subgroup of \( G \) and it has a finite covolume. We give the details of how to construct congruence subgroups.

Let \( \mathbb{A} \) (resp., \( \mathbb{A}_f \)) be the ring of adeles (resp., finite adeles) of \( \mathbb{Q} \). For each prime \( p \), let \( \mathbb{Z}_p \) be the maximal compact subring of \( \mathbb{Q}_p \). Then, for almost all primes \( p \), the group \( G \) is unramified over \( \mathbb{Q}_p \); in particular, \( G \) is a group scheme over \( \mathbb{Z}_p \), and \( G(\mathbb{Z}_p) \) is a hyperspecial maximal compact subgroup of \( G(\mathbb{Q}_p) \) ([27], 3.9.1). Let \( G(\mathbb{A}_f) \) be the restricted product of all groups \( G(\mathbb{Q}_p) \) with respect to the groups \( G(\mathbb{Z}_p) \) where \( p \) ranges over all primes \( p \) such that \( G \) is unramified over \( \mathbb{Q}_p \):

\[
    G(\mathbb{A}_f) = \prod_p G(\mathbb{Q}_p).
\]

Note that

\[
    (7-1) \quad G(\mathbb{A}) = G(\mathbb{R}) \times G(\mathbb{A}_f).
\]

The group \( G(\mathbb{Q}) \) is embedded into \( G(\mathbb{R}) \) and \( G(\mathbb{Q}_p) \). It is embedded diagonally in \( G(\mathbb{A}_f) \) and in \( G(\mathbb{A}) \).

The congruence subgroups of \( G \) are defined as follows (see [5]). Let \( L \subset G(\mathbb{A}_f) \) be an open compact subgroup. Then, considering \( G(\mathbb{Q}) \) diagonally embedded in \( G(\mathbb{A}_f) \), we may consider the intersection

\[
    (7-2) \quad \Gamma_L = L \cap G(\mathbb{Q}).
\]
Now, we consider $G(\mathbb{Q})$ as subgroup of $G = G(\mathbb{R})$. One easily show that the group $\Gamma_L$ is discrete in $G$ and it has a finite covolume. The group $\Gamma_L$ is called a congruence subgroup attached to $L$.

We introduce a family of principal congruence groups (which depend on an embedding over $\mathbb{Q}$ of $G$ into some $SL_M$). We fix an embedding over $\mathbb{Q}$

\begin{equation}
\mathcal{G} \hookrightarrow SL_M
\end{equation}

with a image Zariski closed in $SL_M$. Then there exists $N \geq 1$ such that

\begin{equation}
\mathcal{G} \text{ is a group scheme over } \mathbb{Z}[1/N] \text{ and the embedding (7-3) is defined over } \mathbb{Z}[1/N].
\end{equation}

We fix such $N$.

As usual, we let $G_\mathbb{Z} = G(\mathbb{Q}) \cap SL_M(\mathbb{Z})$, and $G_{\mathbb{Z}_p} = G(\mathbb{Q}_p) \cap SL_M(\mathbb{Z}_p)$ for all prime numbers $p$. We remark that $G$ is a group scheme over $\mathbb{Z}_p$ and the embedding (7-3) is defined over $\mathbb{Z}_p$ when $p$ does not divide $N$. Then $G_{\mathbb{Z}_p} = G(\mathbb{Z}_p)$ but $G$ may not be unramified over such $p$. In general, $G_{\mathbb{Z}_p}$ is just an open compact subgroup of $G(\mathbb{Q}_p)$.

Now, we are ready to define the standard congruence subgroups with respect to the embedding (7-3). Let $n \geq 1$. Then, we let

\begin{equation}
\Gamma(n) = \{x = (x_{i,j}) \in G_\mathbb{Z} : x_{i,j} \equiv \delta_{i,j} \pmod{n}\}.
\end{equation}

The first result of the present section is the following theorem in which we give a construct smooth automorphic forms. The proof contains a non–standard proof of cuspidality of Poincaré series (see for example [18], Theorem 3-10 for the standard proof).

**Theorem 7-6.** Assume that $G$ admits discrete series. Let $(\pi, \mathcal{H})$ is an integrable discrete series of $G$. Let $h' \in \mathcal{H}_K$, $h' \neq 0$. Then, we have the following:

(i) For each congruence subgroup $\Gamma$, the map $P_\Gamma$ maps the space of smooth vectors $B_{h'}^\infty$ of $B_{h'}$ into the space of smooth cuspidal forms $A_{\text{cusp}}^\infty(\Gamma \setminus G)$ for $\Gamma$.

(ii) Assume that a family of principal congruence subgroups is defined with respect to the embedding (7-3). There exists $n_0$ which depends on $\pi$ and $h'$ only, such that for $n \geq n_0$, the map $P_\Gamma(n)$ is an embedding of $B_{h'}^\infty$ into $A_{\text{cusp}}^\infty(\Gamma(n) \setminus G)$.

**Proof.** We prove (i). By Theorem 6-4 (i), the image $P_\Gamma(B_{h'})$ is either zero, or it is an embedding and its closure is an irreducible subspace unitary equivalent to $(\pi, \mathcal{H})$. If the image is zero, then we clearly prove (i). But if the image is not–zero, then it is in the discrete part of $L^2(\Gamma \setminus G)$. Since the image is infinitesimally equivalent to a discrete series $(\pi, \mathcal{H})$, it must be contained in the cuspidal part of $L^2(\Gamma \setminus G)$ by a well–known result of Wallach [23]. In particular, all functions in $P_\Gamma(B_{h'}^\infty)$ are $\Gamma$–cuspidal.

Next, By Theorem 6-4 (ii), $P_\Gamma(B_{h'}^\infty)$ is contained in the subspace of $\mathcal{Z}(g_C)$–finite vectors in $A_{\text{cusp}}^\infty(\Gamma \setminus G)$ which is the space of smooth automorphic forms $A_{\text{cusp}}^\infty(\Gamma \setminus G)$ (see Lemma 4-4). Now, (i) follows.

The claim (ii), follows from (i) apply the criterion given by Lemma 6-2 (i) to any non–zero function $\varphi \in B_{h'}^\infty$. The details are left to the reader since they are similar to the ones in the proof of ([18], Theorem 6-1). \qed
Now, we consider adelic automorphic forms. An automorphic form is a function \( f : \mathcal{G}(\mathbb{A}) \rightarrow \mathbb{C} \) satisfying the following conditions (see \([5], 4.2\)):

(AA-1) \( f(\gamma x) = f(x) \), for all \( \gamma \in \mathcal{G}(\mathbb{Q}), x \in \mathcal{G}(\mathbb{A}) \),

(AA-2) There exists an open compact subgroup \( L \subset \mathcal{G}(\mathbb{A}_f) \) such that \( f \) is right-invariant under \( L \),

(AA-3) For each \( x_f \in \mathcal{G}(\mathbb{A}_f) \), the function \( x_{\infty} \mapsto f(x_{\infty}, x_f) \) satisfies the analogous conditions to those (A-1) and (A-3) of Section \([3]\).

We remark that for an open compact subgroup \( L \subset \mathcal{G}(\mathbb{A}_f) \), there exists a finite set \( \mathcal{C} \subset \mathcal{G}(\mathbb{A}_f) \) such that \( \mathcal{G}(\mathbb{A}_f) = \mathcal{G}(\mathbb{Q}) \cdot \mathcal{C} \cdot L \) (see \([1]\)). We may assume that \( \mathcal{C} \) is the set of representatives of double cosets \( \mathcal{G}(\mathbb{Q}) \mathcal{G}(\mathbb{A}_f)/L \). Then, by (AA-1) and (AA-2), \( f \) is completely determined by the functions \( f_c \) on \( \mathcal{G} \) defined by \( f_c = f|_{G \times c} \) for any \( c \in \mathcal{C} \). The function \( f_c \) is an automorphic form for \( \Gamma_{cLc^{-1}} \) (see (7-2)). Next, \( f \) is a cuspidal automorphic form if

\[
(7-7) \quad \int_{U_{\mathcal{P}}(\mathbb{Q}) \backslash U_{\mathcal{P}}(\mathbb{A})} f(nx)dn = 0 \quad \text{(for all } x \in \mathcal{G}(\mathbb{A})),
\]

for all proper \( \mathcal{Q} \)-parabolic subgroups \( \mathcal{P} \) of \( \mathcal{G} \). Here \( U_{\mathcal{P}} \) denotes the unipotent radical of \( \mathcal{P} \). It is observed in \((5), 4.4\) that \( f \) is a cuspidal automorphic form if and only if \( f_c \) is a \( \Gamma_{cLc^{-1}} \)-cuspidal for all \( c \in \mathcal{C} \). This is a consequence of a simple integration formula (see for example \([20], \text{Lemma 2.3}\) for complete account):

\[\textbf{Lemma 7-8.} \quad \text{Let } \psi : U_{\mathcal{P}}(\mathbb{Q}) \backslash U_{\mathcal{P}}(\mathbb{A}) \rightarrow \mathbb{C} \text{ be a continuous function which is right-invariant under some open compact subgroup denoted by } L_P \subset U_{\mathcal{P}}(\mathbb{A}_f). \text{ Then we have the following formula:}
\]

\[
\int_{U_{\mathcal{P}}(\mathbb{Q}) \backslash U_{\mathcal{P}}(\mathbb{A})} \psi(u)du = vol_{U_{\mathcal{P}}(\mathbb{A})}(L_P) \cdot \int_{\Gamma_{L_P} \backslash U_{\mathcal{P}}(\mathbb{R})} \psi(u_{\infty})du_{\infty},
\]

where \( \Gamma_{L_P} \) is a discrete subgroup of \( U_{\mathcal{P}}(\mathbb{A}) \) defined as before: \( \Gamma_{L_P} = U_{\mathcal{P}}(\mathbb{Q}) \cap L_P \).

The space of all automorphic forms we denote by \( \mathcal{A}(\mathcal{G}(\mathbb{Q}) \backslash \mathcal{G}(\mathbb{A})) \). It is a \((\mathfrak{g},K) \times \mathcal{G}(\mathbb{A}_F)\)-module. Its submodule is the space of all cuspidal automorphic forms \( \mathcal{A}_{cusp}(\mathcal{G}(\mathbb{Q}) \backslash \mathcal{G}(\mathbb{A})) \).

The smooth automorphic forms is defined by forgetting \( K \)-finiteness assumption in (AA-3). The space of all smooth automorphic forms and smooth cuspidal automorphic forms we denote by \( \mathcal{A}^\infty(\mathcal{G}(\mathbb{Q}) \backslash \mathcal{G}(\mathbb{A})) \) and \( \mathcal{A}_{cusp}^\infty(\mathcal{G}(\mathbb{Q}) \backslash \mathcal{G}(\mathbb{A})) \), respectively. The claims analogous to those above for relation between smooth automorphic forms on \( \mathcal{G}(\mathbb{A}) \) and \( \mathcal{G} \) are easily checked to be true.

\[\textbf{Corollary 7-9.} \quad \text{Assume that } \mathcal{G} \text{ admits discrete series. Let } (\pi, \mathcal{H}) \text{ is an integrable discrete series of } \mathcal{G}. \text{ Let } h' \in \mathcal{H}_K, h' \neq 0. \text{ For each prime number } p, \text{ we select a function } f_p \in C_c^\infty(\mathcal{G}(\mathbb{Q}_p)) \text{ such that for almost all } p \text{ where } \mathcal{G} \text{ is unramified over } \mathbb{Q}_p \text{ we have } f_p = 1_{\mathcal{G}(\mathbb{Z}_p)} \text{ (the characteristic function of a hyperspecial maximal open compact subgroup of } \mathcal{G}(\mathbb{Q}_p). \text{ Let } f_{\infty} \in C_b^\infty(K'). \text{ Then, the Poincaré series}
\]

\[
\sum_{\gamma \in \mathcal{G}(\mathbb{Q})} (f_{\infty} \otimes_p f_p)(\gamma x), \quad x \in \mathcal{G}(\mathbb{A}),
\]

converges absolutely almost everywhere to an element of \( \mathcal{A}_{cusp}^\infty(\mathcal{G}(\mathbb{Q}) \backslash \mathcal{G}(\mathbb{A})) \).
Proof. The finite part of the tensor product \( f_{\text{fin}} = \otimes'_p f_p \) belongs to \( \mathcal{O}_c(\mathcal{G}(\mathbb{A}_f)) \). Therefore, there exists an open compact subgroup \( L \subset \mathcal{G}(\mathbb{A}_f) \) such that \( f_{\text{fin}} \) is right–invariant under \( L \). As we noted above, the set, say \( C \), of representatives of double cosets \( \mathcal{G}(\mathbb{Q}) \backslash \mathcal{G}(\mathbb{A}_f) / L \) is finite.

We fix such set. Then, there exists a finite sequence \( \gamma_1, \ldots, \gamma_k \in \mathcal{G}(\mathbb{Q}) \), and \( c_1, \ldots, c_k \in C \) such that the cosets \( \gamma_i c_k L \) are disjoint for different indices and we have

\[
 f_{\text{fin}} = \sum_{i=1}^k f_{\text{fin}}(\gamma_i c_i)1_{\gamma_i c_i L}.
\]

Next, applying the decomposition (7-1), for \( x = (x_\infty, \delta c_l) \), where \( x_\infty \in \mathbb{G} \), \( \delta \in \mathcal{G}(\mathbb{Q}) \), \( c \in C \), and \( l \in L \), we can write:

\[
 \sum_{\gamma \in \mathcal{G}(\mathbb{Q})} |f_\infty \otimes f_{\text{fin}}| (\gamma \cdot (x_\infty, \delta c_l)) = \sum_{\gamma \in \mathcal{G}(\mathbb{Q})} |f_\infty \otimes f_{\text{fin}}| (\gamma x_\infty, \gamma \delta c_l))
 = \sum_{\gamma \in \mathcal{G}(\mathbb{Q})} |f_\infty(\gamma x_\infty)| |f_{\text{fin}}(\gamma \delta c)|
 = \sum_{i=1}^k |f_{\text{fin}}(\gamma_i c_i)| \sum_{\gamma \in \mathcal{G}(\mathbb{Q})} |f_\infty(\gamma x_\infty)| 1_{\gamma_i c_i L}(\gamma \delta c).
\]

The last expression is equal to zero if \( c \neq c_i \) for all \( i \), and if \( c = c_j \) (for unique \( j \), then the expression reduces to

\[
 |f_{\text{fin}}(\gamma_j c_j)| \sum_{\gamma \in \mathcal{G}(\mathbb{Q})} |f_\infty(\gamma x_\infty)| 1_{\gamma_j c_j L}(\gamma \delta c_j).
\]

Non–zero terms in above sum comes from the case

\[
 \gamma \delta c_j \in \gamma_j c_j L
\]

Equivalently,

\[
 \gamma_j^{-1} \gamma \delta \in \Gamma_{c_j L c_j^{-1}}.
\]

Thus, after changing the summation index \( \gamma \) appropriately, the sum becomes

\[
 |f_{\text{fin}}(\gamma_j c_j)| \sum_{\gamma \in \mathcal{G}(\mathbb{Q})} |f_\infty(\gamma_j \delta^{-1} x_\infty)| .
\]

The last sum converges absolutely almost everywhere by the remark at beginning of Section 6. The same remark could be directly applied to the adelic Poincaré series (see [18]) giving the short proof, but we needed this extended argument. Indeed, all above computations are still valid if we remove all absolute values. In view of Theorem 7-6 (i), there exists

\[
 F_j \in \mathcal{A}_{\text{cusp}} \left( \Gamma_{c_j L c_j^{-1}} \backslash \mathcal{G} \right)
\]

such that

\[
 F_j(\gamma_j \delta^{-1} x_\infty) = \sum_{\gamma \in \mathcal{G}(\mathbb{Q})} f_\infty(\gamma \gamma_j \delta^{-1} x_\infty)
\]
is everywhere for \( x_{\infty} \in G_{\infty} \). Next, the function

\[
x_{\infty} \mapsto F_j(\gamma_j \delta^{-1} x_{\infty})
\]

belongs to

\[
A_{\text{cusp}}^{\infty} \left( \Gamma \delta \Gamma_{c_j L_{c_j}^{-1} \delta^{-1}} \backslash G \right)
\]
as it is easy to check directly from the definition of a smooth cuspidal automorphic form (see Section 3). This implies the corollary. \( \square \)

Following methods of \([18], \text{Theorems 4.1}\) one can developed sufficient conditions that the adelic Poincaré series from Corollary 7-9 is not identically zero. For example, one could fix a prime number \( p_0 \) and then shrink the support of \( f_{p_0} \). There are other possibilities. We leave it to the interested reader as an exercise from the following result.

Before we state the proposition, using the notation of Corollary 7-9, we remark that we can select a compact neighborhood \( C_{\infty} \subset G \) (i.e., compact set which is a closure of an open set) such that

\[
\int_{C_{\infty}} |f_{\infty}(x_{\infty})| \, dx_{\infty} > \frac{1}{2} \int_{G} |f_{\infty}(x_{\infty})| \, dx_{\infty}
\]
since \( f_{\infty} \in L^1(G) \).

**Proposition 7-11.** We maintain the assumptions of Corollary 7-9. Let \( C_{\infty} \subset G \) be a compact set such that (7-10) holds. Then, the Poincaré series

\[
\sum_{\gamma \in \mathcal{G}(\mathbb{Q})} \left( f_{\infty} \otimes' f_p \right) (\gamma x), \quad x \in \mathcal{G}(\mathbb{A})
\]
is not identically zero if

\[
\mathcal{G}(\mathbb{Q}) \cap \left( \prod_p \text{supp}(f_p) \times C_{\infty} \right) \cdot \left( \prod_p \text{supp}(f_p) \times C_{\infty} \right)^{-1} = \{1\}.
\]

We note that \( f_p = 1_{\mathcal{G}(\mathbb{Z}_p)} \) for almost all \( p \). Consequently, \( \text{supp}(f_p) = \mathcal{G}(\mathbb{Z}_p) \).

**Proof.** Note that \( f_{\infty} \otimes' f_p \in L^1(\mathcal{G}(\mathbb{A})) \). Now, we apply the general criterion \([18], \text{Theorems 4.1}\) which claims that if we can find a compact neighborhood \( C \subset \mathcal{G}(\mathbb{A}) \) such that

\[
\int_C |(f_{\infty} \otimes' f_p) (x)| \, dx > \frac{1}{2} \int_G |(f_{\infty} \otimes' f_p) (x)| \, dx,
\]
and

\[
\mathcal{G}(\mathbb{Q}) \cap C \cdot C^{-1} = \{1\}.
\]
Since the integrals in above inequality are factorizable, by our assumption we can take

\[
C = \prod_p \text{supp}(f_p) \times C_{\infty}.
\]

\( \square \)
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