Field theory capable of guaranteeing the initial conditions needed for inflation

Alexander B. Kaganovich
Physics Department, Ben Gurion University of the Negev
and Sami Shamoon College of Engineering,
Beer Sheva 84105, Israel
(Dated: October 12, 2022)

Inflatonary model of a single scalar field with primordial potential \( V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4 \) (\( m^2 > 0 \)) non-minimally coupled to gravity is studied in two-measures theory (TMT) in the Palatini formalism. In the equations of motion presented in the Einstein frame and rewritten in terms of the canonically normalized scalar field \( \phi \), there arises a TMT effective potential, which differs from the potential of the \( T \)-model in that it has a plateau of finite length: for \( \phi \) greater a certain value \( \phi_0 \) the TMT effective potential becomes exponentially steep. The length of the plateau, and hence the duration of a quasi-de Sitter inflation, is controlled by a model parameter. The appearance of this parameter, as well as the form of the TMT effective potential, are a direct consequence of the features inherent only in TMT. A detailed analysis shows that there is a rather narrow interval of initial values of \( \phi \), bounded from above by \( \phi_0 \), in which the initial kinetic \( p_{\text{kin, in}} \) and gradient \( p_{\text{grad, in}} \) energy densities turn out to be less than the potential energy density: this requires the only additional condition, which is that \( p_{\text{kin, in}} > p_{\text{grad, in}} \). Therefore, in the space-time domain where these restrictions are satisfied, the initial conditions necessary for inflation are guaranteed.

I. INTRODUCTION

Data of recent cosmological observations\[^1\]-[^4\] favor inflationary models with plateau potentials, and with the height of the plateau \( V_{pl} \sim 10^{-10} M_p^4 \). There exist a number of field theory models in which the plateau-like potentials arise due to the implementation of various original ideas, and these potentials satisfy the CMB constraints. These include the Starobinsky model\[^5\], the Goncharov-Linde model\[^6\], the Higgs inflation models\[^7\]. Of particular interest are \( \alpha \)-attractor models, which were initiated by the pioneering work\[^8\] and which have been intensively studied in recent years. To date, there is the broad class of the cosmological attractor models\[^9\]-[^17\], which generalize most of the previously proposed models with plateau potentials.

However, despite such an impressive success of plateau-like models compared to all other models, a lively discussion ensued, during which even the very idea of inflation has been called into question\[^18\], \[^19\], \[^20\], \[^21\]. The main problem formulated in paper\[^18\] is related to the height of the potential energy density plateau. Indeed, if the height of the plateau is \( V_{pl} \sim 10^{-10} M_p^4 \), then there is a huge range of possible values of the initial kinetic and gradient energy densities greater than \( V_{pl} \) up to the Planck density. Therefore, in contrast to the understanding developed in first models of chaotic inflation\[^22\], \[^23\] of how initial conditions for inflation arise, there is no reason to believe that the initial kinetic and gradient energy densities do not exceed the potential energy density, - the condition necessary for starting inflation. The purpose of this paper is to show that the consistent implementation of the ideas embodied in the two-measures theory (TMT) makes it possible to rid the inflationary theory of this problem. As will be shown, this is possible even within the framework of the model of a single scalar field, non-minimally coupled to gravity, with the standard potential

\[
V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4 \quad \text{with} \quad m^2 > 0. 
\]

The organization of paper is as follows. Sec.II gives a very brief introduction to the basics of TMT. The most important point is the presence of a metric-independent 4-form \( \Upsilon d^4x \) on the space-time manifold as a volume measure in the primordial (original) action: this is how TMT differs from traditional field theory models. Then, in Sec.III, the simplest TMT model of the scalar field \( \phi \) with potential\[^11\] non-minimally coupled to gravity is formulated and studied. The use of the volume measure \( \Upsilon d^4x \) in the primordial action\[^9\] naturally leads to the need, along with the usual vacuum-like term, to include in the primordial action a new kind of vacuum-like term with the corresponding model parameter \( V_2 \) of dimensionality (mass\)^4. It is shown that in the case of a zero effective cosmological constant, the TMT effective potential has a form similar to the potential of the \( T \)-model, but with a plateau of finite length: for \( \phi \) greater than some value \( \varphi_2 \), the potential becomes exponentially steep. The length of the plateau, and hence the duration of the quasi-de Sitter inflation, is controlled by the parameter \( V_2 \). Moreover, the principle of least action implies the requirement that the sign of \( \Upsilon \) does not change during the entire evolution of the Universe. We show that, due to the sign-definiteness of \( \Upsilon \), only the values \( \varphi < \varphi_2 \) are possible in our Universe. In Sec.IV, we study a more general model that takes into account the arbitrariness in the choice of coefficients in a linear combination of volume elements \( dV_\alpha = \sqrt{-g}d^4x \) and \( dV_\Upsilon = \Upsilon d^4x \). It turns out that in such a model, the TMT effective potential has
two plateaus. With an appropriate choice of parameters, for \( \varphi < \varphi_0 \) (\( \varphi_0 \) is close to \( \varphi_* \)), the shape of the potential is almost the same as in the model of Sec.III, but for \( \varphi > \varphi_0 \), instead of an almost exponential growth, the potential has a second, higher plateau of infinite length. In addition, in this model, when studying the possible initial conditions for inflation, it is found that the above requirement that \( \Upsilon \) be sign-definite imposes restrictions on the admissible values of the kinetic and gradient energy densities. A detailed analysis given in Appendix shows that there is a rather narrow interval of initial values \( \varphi_{\text{in}} \), bounded from above by \( \varphi_0 \), in which the initial conditions necessary for inflation are guaranteed to be satisfied; this requires the only additional condition, which is that the initial kinetic energy density is greater than the initial gradient energy density. Sec.V is devoted to a discussion of the role of TMT principles in achieving the results of this work.

II. TWO VOLUME ELEMENTS.

BRIEF DESCRIPTION OF THE PROCEDURE OF TWO-MEASURES THEORY

The main idea of TMT is that in the four-dimensional space-time, in addition to the terms included in the primordial action of the theory with the volume element \( dV_g = \sqrt{-g}d^4x \), there are terms that appear in the primordial action with a metric-independent volume element. The latter can be defined, for example, using 4 scalar functions \( \varphi_a \) on the space-time manifold

\[
dV_T = \Upsilon d^4x \equiv \varepsilon_{abcd}\varepsilon_{\mu\nu\gamma\beta}\varphi_a\partial_\mu\varphi_b\partial_\nu\varphi_c\partial_\gamma\varphi_\beta d^4x \equiv 4!d\varphi_1 \wedge d\varphi_2 \wedge d\varphi_3 \wedge d\varphi_4,
\]

where \( \Upsilon \) is a scalar density, that is under general coordinate transformations with positive Jacobian it has the same transformation law as \( \sqrt{-g} \).

The use of the volume element \( dV_T \) in the action integral along with \( dV_g \) leads to very interesting results studied in the series of papers (see e.g. [24]-[32]). Now I am going to formulate the main ideas of TMT and describe in general terms the algorithm, listing all the steps to obtain the results of the theory. The primordial action in TMT in the Palatini formalism includes the following set of variables: 4 functions \( \varphi_a \) from which \( \Upsilon \) is built, affine connection \( \Gamma^\lambda_{\mu\nu} \), primordial (original) metric tensor \( g_{\mu\nu} \), and matter fields. All these variables are assumed to be independent, and the relationship between them is the result of applying the least action principle. As far as the relationship between connection and metric is concerned, this is reminiscent of Palatini’s formalism, which are well studied in models of the usual theory containing only the volume element, \( dV_g = \sqrt{-g}d^4x \). However, the inclusion of one more volume element, \( dV_T \), significantly expands the consequences of the Palatini formalism.

The TMT procedure consists of the following steps:

- **1.** Variation with respect to scalar functions \( \varphi_a \) (\( a = 1, ..., 4 \)) yealds the equation solution of which exists if \( \Upsilon \neq 0 \) and it contains an integration constant \( \mathcal{M} \) of dimentionality (mass)\(^4\).

- **2.** Variation with respect to the affine connection leads to a differential equation, the solution of which gives the connection coefficients. Besides the original metric and its derivatives, this solution for the connection coefficients may also contain matter fields and their first derivatives. In the context of TMT, when applying the Palatini formalism, a new aspect arises: namely, the differential equation for the connection coefficients and its solution also include the following scalar \( \zeta(x) \)

\[
\zeta(x) \equiv \frac{dV_T}{dV_g} \equiv \frac{\Upsilon}{\sqrt{-g}}
\]

and its first derivatives. Obviously, the connection coefficients thus obtained are non-Riemannian (the covariant derivative of the primordial metric tensor with such connection coefficients is nonzero).

- **3.** Varying the primordial metric results in the equations similar to what occurs in the usual Palatini formalism, but now they also contain the scalar \( \zeta(x) \).

- **4.** The condition for the compatibility of the equations obtained at steps 1 and 3 has the form of an algebraic equation describing the scalar \( \zeta(x) \) as a local function of matter fields; it also contains an arbitrary integration constant \( \mathcal{M} \). In Refs. [24]-[32], this equation is called ’constraint’, which we will also use in this paper.

- **5.** All matter field equations obtained by variation of the primordial action contain the scalar \( \zeta(x) \) and its derivatives.
• 6. By a Weyl transformation of the primordial metric $g_{\mu\nu}$, one can simultaneously ensure that the new metric $\tilde{g}_{\mu\nu}$ is Riemannian, and the connection coefficients found in step 2 are converted into the Christoffel coefficients of $\tilde{g}_{\mu\nu}$. As a result, the equations obtained in step 3 take exactly the form of Einstein’s equations with the same Newton constant. Therefore $\tilde{g}_{\mu\nu}$ is the metric in the Einstein frame. The energy-momentum tensor arising in the Einstein frame, which we call ‘the TMT effective energy-momentum tensor’, contains the scalar $\zeta(x)$, which, in turn, due to the constraint mentioned in step 4, is expressed in terms of matter fields. The potential of scalar field entering to the TMT effective energy-momentum tensor will be called ‘the TMT effective potential’. A similar thing happens in the matter field equations. This feature of TMT is a source of very interesting and fundamentally new effects, which will also be demonstrated in this paper.

• 7. The sequential execution of the steps described above will be referred to as the ”TMT procedure”. Finally, to check if the TMT procedure was performed correctly, as well as for the convenience of working with the obtained equations in the Einstein frame, one can construct an action whose variation leads to these equations. For this action, we will use the term ‘TMT effective action’. The terms ‘TMT effective energy-momentum tensor’, ‘TMT effective potential’ and ‘TMT effective action’ are introduced in order not to confuse them with similar terms used when quantum corrections are taken into account.

### III. A SIMPLEST FIELD THEORY MODEL

#### A. The primordial action and equations of motion in the Einstein frame

Consider a model that includes gravity, the inflaton field with the canonical kinetic term, a nonminimal inflaton-to-scalar curvature coupling and vacuum-like terms. In this section, the primordial action is chosen as follows

$$S = \int d^4x (\sqrt{-g + \Upsilon}) [L_{\text{gr}} + L_{\phi} + L_{\text{nonmin}}] + S_{\text{vac}},$$  \hfill (4)

where

$$L_{\text{gr}} = -\frac{M_P^2}{2} R(\Gamma, g); \quad L_{\phi} = \frac{1}{2} g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} - V(\phi); \quad L_{\text{nonmin}} = -\frac{1}{2} \xi R(\Gamma, g) \phi^2$$  \hfill (5)

Here $M_P$ is the reduced Planck mass; $\Gamma$ stands for affine connection; $R(\Gamma, g) = g^{\mu\nu} R_{\mu\nu}(\Gamma)$, $R_{\mu\nu}(\Gamma) = R^\lambda_{\mu\nu\lambda}(\Gamma)$ and $R^\lambda_{\mu\nu\lambda}(\Gamma) = \Gamma^\lambda_{\mu\nu,\sigma} + \Gamma^\lambda_{\gamma\nu,\mu} - (\nu \leftrightarrow \sigma)$. In our notations the parameter $\xi$ of non-minimal coupling in the case of a conformal coupling in a theory with only the volume element $\sqrt{-gd^4x}$ would be equal to $\xi = -\frac{1}{2}$.

Contribution of the vacuum-like terms to the primordial action is defined by

$$S_{\text{vac}} = \int d^4x \left( -\sqrt{-g} V_1 - \frac{\Upsilon^2}{\sqrt{-g}} V_2 \right).$$  \hfill (6)

If the first term in Eq.(4) was present in Einstein’s GR, $V_1$ would be a cosmological constant. The term with $V_2$ was first introduced in Ref.[22]. The first reason for adding the term with $V_2$ is that the term $\propto \Upsilon = \zeta(x) \sqrt{-g}$, which can be expected as the contribution of quantum-gravitational effects to a vacuum-like action, does not contribute to the equations of motion. Therefore, the next in powers of $\zeta$ vacuum-like term can be of the form $\propto \zeta^2 \sqrt{-g} = \frac{\Upsilon^2}{\sqrt{-g}}$. There are also more pragmatic reasons. It turns out that thanks to this term, the corresponding TMT effective potential and the TMT effective action acquire fundamentally new important properties without which the goals of this paper could not be achieved.

In general, there is no reason why the coefficients in linear combinations of $\Upsilon$ and $\sqrt{-g}$ in volume elements of different terms in the action should be the same. In the gravitational term with $L_{\text{gr}}$, the common factor of the linear combination can be absorbed by redefining Newton’s constant. After that, by rescaling the fields $\varphi_\sigma$ in Eq.(2), the volume element in the gravitational term with $L_{\text{gr}}$ becomes as in Eq.(4). Then in all other contributions to the primordial action there is only the freedom to absorb the common factor by rescaling $\phi$, the parameters in the primordial potential $V(\phi)$ and $\xi$. As a result, the corresponding volume elements can have the form $(b_i \sqrt{-g} + s_i \Upsilon)^d x$, where $b_i$ and $s_i = \pm 1$ are arbitrary model parameters. The appearance of the parameter $s$ is due to the fact that in the primordial action $\Upsilon$ can equally be both positive and negative. Therefore, in general, $\Upsilon$ can enter the primordial action with different signs. In this paper, we will restrict ourselves to the case $s = 1$. The choice of the same volume element $(\sqrt{-g} + \Upsilon)^d x$ for all terms in Eq.(4) made in this section means that we have deal with the simplest version of the model.
For the primordial inflaton potential $V(\phi)$ we choose a simple model of massive scalar field with quartic self-interaction, Eq. (11).

Now, following the prescription of the TMT procedure, we consider the equations of motion following from the primordial action $\mathcal{I}$. Varying the action with respect to scalar functions $\varphi_a$ of which $\Upsilon$ is built we get

$$ B^\mu_a \partial_\mu [L_{gr} + L_\phi + L_{nonmin} - 2\zeta V_2] = 0 \quad \text{where} \quad B^\mu_a = \varepsilon^{\mu\nu\alpha\beta} \varepsilon_{abcd} \partial_\nu \varphi_b \partial_\alpha \varphi_c \partial_\beta \varphi_d. \quad (7) $$

Since $\text{Det}(B^\mu_a) = \frac{4}{3\pi} \Upsilon^3$ it follows that if

$$ \Upsilon(x) \neq 0, $$

the equality

$$ - \frac{M_p^2}{2} \left(1 + \xi \frac{\phi^2}{M_p^2}\right) R(\Gamma, g) + \frac{1}{2} g^{\mu\nu} \phi,\phi_{,\mu}\phi,\phi_{,\nu} - V(\phi) - 2\zeta V_2 = \mathcal{M} \quad (9) $$

must be satisfied, where $\mathcal{M}$ is a constant of integration with the dimension of (mass)$^4$.

Variation with respect to $g^{\mu\nu}$ yields the equation

$$ (1 + \zeta) \left[ - \frac{M_p^2}{2} \left(1 + \xi \frac{\phi^2}{M_p^2}\right) R(\Gamma, g) + \frac{1}{2} g^{\mu\nu} \phi,\phi_{,\mu}\phi,\phi_{,\nu} \right] - \frac{1}{2} g^{\mu\nu} \left[ - \frac{M_p^2}{2} \left(1 + \xi \frac{\phi^2}{M_p^2}\right) R(\Gamma, g) + \frac{1}{2} g^{\alpha\beta} \phi,\phi_{,\alpha}\phi,\phi_{,\beta} - V(\phi) - V_1 + \zeta^2 V_2 \right] = 0, \quad (10) $$

trace of which is the following

$$ (\zeta - 1) \left[ - \frac{M_p^2}{2} \left(1 + \xi \frac{\phi^2}{M_p^2}\right) R(\Gamma, g) + \frac{1}{2} g^{\mu\nu} \phi,\phi_{,\mu}\phi,\phi_{,\nu} \right] + 2V(\phi) + 2V_1 - 2\zeta^2 V_2 = 0 \quad (11) $$

Eliminating the expression $-\frac{1}{4} M_p^2 \left(1 + \xi \frac{\phi^2}{M_p^2}\right) R(\Gamma, g)$ from Eqs. (9) and (11) one can see that the term $2\zeta^2 V_2$ is canceled, and the consistency of these equations requires that the scalar function $\zeta(x)$ satisfies the following relation

$$ \zeta = \zeta(\phi(x)) = \frac{\mathcal{M} - 2V_1 - V(\phi)}{\mathcal{M} - 2V_2 + V(\phi)} \quad (12) $$

describing $\zeta(x)$ as the local function of the inflaton $\phi(x)$. Following the terminology of earlier works [24]-[32], we will call this a constraint. It should be noted that in the Palatini formulation $\zeta(x)$ is not a physical degree of freedom. Therefore, when we call Eq. (12) a constraint, we must keep in mind that it is different in meaning from the usual constraint in the field theory models, where it describes the relationship between dynamical degrees of freedom.

The inflaton equation reads

$$ \frac{1}{\sqrt{-g}} \partial_\mu \left[ (1 + \zeta) \sqrt{-g} g^{\mu\nu} \partial_\nu \phi \right] + (1 + \zeta) \left[ V'(\phi) + \xi R(\Gamma, g) \phi \right] = 0. \quad (13) $$

Variation of the affine connection yields the equations we solved earlier for the simpler case [25]. For the case of the action $\mathcal{I}$ containing a non-minimal coupling the result is

$$ \Gamma^\lambda_{\mu\nu} = \{^\lambda_{\mu\nu} \} + (\delta^\lambda_{\mu} \sigma_{,\nu} + \delta^\lambda_{\nu} \sigma_{,\mu} - \sigma_{,\nu} \sigma_{,\mu} g_{\mu\nu} g^{\lambda\beta}), \quad (14) $$

where $\{^\lambda_{\mu\nu} \}$ are the Christoffel’s connection coefficients of the metric $g_{\mu\nu}$ and

$$ \sigma(x) = \ln \left[ \left(1 + \zeta(\phi(x)) \right) \left(1 + \xi \frac{\phi^2(x)}{M_p^2}\right) \right]. \quad (15) $$

If $\sigma(x) \neq \text{const.}$ the metricity condition does not hold and consequently geometry of the space-time with the metric $g_{\mu\nu}$ is generically non-Riemannian. In this paper, I will totally ignore a possibility to incorporate the torsion tensor, which could be an additional source for the space-time to be different from Riemannian.

It is easy to see that the transformation of the metric

$$ g_{\mu\nu} = (1 + \zeta(x)) \left(1 + \xi \frac{\phi^2(x)}{M_p^2}\right) g_{\mu\nu} , \quad (16) $$
turns the connection $\Gamma^\lambda_{\mu\nu}$ into the Christoffel connection coefficients of the metric $\tilde{g}_{\mu\nu}$ and the space-time turns into (pseudo) Riemannian.

Gravitational equations (10) expressed in terms of the metric $\tilde{g}_{\mu\nu}$ take the canonical GR form

$$R_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} R(\tilde{g}) = \frac{1}{M_P^2} T^{(eff)}_{\mu\nu}$$

with the same Newton constant as in the original frame. Here $R_{\mu\nu}(\tilde{g})$ and $R(\tilde{g})$ are the Ricci tensor and the scalar curvature of the metric $\tilde{g}_{\mu\nu}$, respectively. Therefore the set of dynamical variables using the metric $\tilde{g}_{\mu\nu}$ can be called the Einstein frame. $T^{(eff)}_{\mu\nu}$ on the right side of the Einstein equations (17) is the TMT effective energy-momentum tensor, the occurrence of which was described in the previous subsection as step 6 in the TMT procedure. In the model under study, $T^{(eff)}_{\mu\nu}$ has the form

$$T^{(eff)}_{\mu\nu} = \frac{1}{1 + \xi \phi^2} \left[ \phi_{,\mu} \phi_{,\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \phi_{,\alpha} \phi_{,\beta} \right] + \tilde{g}_{\mu\nu} U(\phi, \zeta(\phi); \mathcal{M});$$

the TMT effective potential $U(\phi, \zeta(\phi); \mathcal{M})$ appears as the following function of $\phi$ and the integration constant $\mathcal{M}$

$$U(\phi, \zeta(\phi); \mathcal{M}) = \frac{1}{1 + \xi \phi^2} \left[ V_2 + \frac{\mathcal{M} - V_1 - V_2}{1 + \zeta(\phi)^2} \right],$$

where $\zeta(\phi)$ is determined by the constraint (22).

The role of the non-minimal coupling in the flattening of the inflationary potential and in the attractor for inflation at strong coupling is well known [1], [33]. It is noteworthy that in addition to the factor $\left(1 + \xi \phi^2 \right)^{-2}$ generated by a non-minimal coupling, $U(\phi, \zeta(\phi); \mathcal{M})$ contains also the factor $(1 + \zeta(\phi))^{-2}$, the occurrence of which is due to the inherent properties of TMT. As we shall see, the fact that the plateau of the TMT effective potential has a finite length is one of the most important effects of the scalar $\zeta(\phi)$.

If $\mathcal{M} \neq V_1 + V_2$, after making use the constraint (22) the TMT effective potential $U(\phi, \zeta(\phi); \mathcal{M})$ can be represented in the $\zeta$-independent form

$$U(\phi; \mathcal{M}) = \frac{1}{1 + \xi \phi^2} \left[ V_2 + \frac{\mathcal{M} - 2V_2 + V(\phi)^2}{4(\mathcal{M} - V_1 - V_2)} \right].$$

Recall that $V(\phi)$ is defined by Eq. (11). The presence of the factor $(1 + \zeta)^{-2}$ in $U(\phi, \zeta(\phi); \mathcal{M})$ causes $V^2(\phi)/(1 + \xi \phi^2/\mathcal{M}_P^2)^2$ to appear in $U(\phi; \mathcal{M})$, which for sufficiently large $\phi$ leads to a change in the shape of the TMT effective potential from plateau to steep rise.

For further study, it is convenient to rewrite $U(\phi; \mathcal{M})$ by extracting a term equal to the value of $U(\phi; \mathcal{M})$ at $\phi = 0$

$$\Lambda(\mathcal{M}) = U(\phi = 0; \mathcal{M}) = \frac{\mathcal{M}^2 - 4V_1 V_2}{4(\mathcal{M} - V_1 - V_2)}.$$  

Then we get

$$U(\phi; \mathcal{M}) = \Lambda(\mathcal{M}) + V_{eff}(\phi; \mathcal{M}), \quad (22)$$

where the $\phi$-dependent part $V_{eff}(\phi; \mathcal{M})$ of the TMT effective potential $U(\phi; \mathcal{M})$ is as follows

$$V_{eff}(\phi; \mathcal{M}) = \frac{2(\mathcal{M} - 2V_2) V(\phi) + [V(\phi)]^2}{4(\mathcal{M} - V_1 - V_2)} - \Lambda(\mathcal{M}) \left( 1 - \frac{1}{\left(1 + \xi \phi^2 \mathcal{M}_P^2 \right)^2} \right).$$

After passing to the Einstein frame in the inflaton equation (13) one must substitute the expression for the scalar curvature derived from the Einstein equations (17). And finally, using the constraint (22), we get the inflaton equation in the following form

$$\frac{1}{\sqrt{\tilde{g}}} \partial_{\mu} \left( \frac{1}{1 + \xi \phi^2 \mathcal{M}_P^2} g^{\mu\nu} \partial_{\nu} \phi \right) + \frac{\xi \phi}{\mathcal{M}_P^2} \frac{1}{1 + \xi \phi^2 \mathcal{M}_P^2} g^{\alpha\beta} \partial_{\alpha} \phi \partial_{\beta} \phi$$

$$+ \frac{\mathcal{M} - 2V_2 + V(\phi)}{2(\mathcal{M} - V_1 - V_2)(1 + \xi \phi^2 \mathcal{M}_P^2)} \left[ V'(\phi) - \frac{\xi \phi}{\mathcal{M}_P^2 (1 + \xi \phi^2 \mathcal{M}_P^2)^2} \frac{[\mathcal{M} - 2V_2 - V(\phi)]^2}{\mathcal{M} - V_1 - V_2 - 2V_2} \right] = 0.$$  

(24)
B. The zero cosmological constant case

The arbitrariness in the value of the integration constant $\mathcal{M}$ allows one to study cosmological models with an arbitrary cosmological constant. In this paper, we will restrict ourselves to a model with zero vacuum energy density, which is a fairly good approximation when studying the inflationary epoch. If $\phi = 0$ is the position of the global minimum of the TMT effective potential $U(\phi; \mathcal{M})$ described by Eqs. (20)-(23), i.e. $\phi = 0$ is the vacuum state of the classical scalar field $\phi(t)$, then $\Lambda(\mathcal{M})$ is the cosmological constant. As can be seen from Eqs. (12) and (11), in the vacuum the expression for the scalar $\zeta$ has the form

$$\zeta_v = \frac{\mathcal{M} - 2V_1}{\mathcal{M} - 2V_2}. \quad (25)$$

The zero value of the cosmological constant is reached if the integration constant $\mathcal{M}$ satisfies the relation

$$\mathcal{M}_0^2 = 4V_1 V_2, \quad (26)$$

where the subscript 0 indicates that $\Lambda(\mathcal{M}_0) = 0$. To ensure the possibility of $\Lambda(\mathcal{M}_0) = 0$, the model parameters $V_1$ and $V_2$ must have the same sign. It can be shown by direct detailed verification that results of interest are obtained if the parameters $V_1$, $V_2$ and the integration constant $\mathcal{M}_0$ are chosen so that

$$V_1 < 0; \quad V_2 < 0; \quad \mathcal{M}_0 = 2\sqrt{V_1 V_2}. \quad (27)$$

Then it follows from Eq. (26) that the value of $\zeta$ in the vacuum with zero energy density is

$$\zeta_v = \sqrt{\frac{V_1}{V_2}} > 0. \quad (28)$$

In the chosen case of a zero cosmological constant, the TMT effective potential $U(\phi; \mathcal{M})$ defined by Eqs. (20)-(23) reduces to $V_{\text{eff}}^{(0)}(\phi) \overset{\text{def}}{=} U(\phi; \mathcal{M}_0) = V_{\text{eff}}(\phi; \mathcal{M}_0)$, which has the form

$$V_{\text{eff}}^{(0)}(\phi) = \frac{1}{1 + \zeta_v} \left[ V(\phi) + \frac{(V(\phi))^2}{4|V_2|(1 + \zeta_v)} \right]. \quad (29)$$

Accordingly, when choosing (27), the inflaton equation (24) reduces to

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x} \left( \frac{1}{1 + \zeta_v} \sqrt{|g|} g^{\mu\nu} \partial_\mu \phi \right) + \frac{\xi \phi}{M_p^2 \left( 1 + \zeta_v \right)^2} \delta^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + \frac{2|V_2|(1 + \zeta_v) + V(\phi)}{2|V_2|(1 + \zeta_v)^2 \left( 1 + \zeta_v \right)} V'(\phi) = \frac{\xi \phi (V(\phi))^2}{|V_2|M_p^2(1 + \zeta_v)^2 \left( 1 + \zeta_v \right)^2} = 0. \quad (30)$$

Following the TMT procedure, we are left with the 7th step. Namely, it is necessary to make sure that Eq. (30) for the inflaton field and the Einstein equations (17) with the energy-momentum tensor $T_{\mu\nu}^{(\text{eff})}$ described by Eqs. (18), (22) with $\Lambda(\mathcal{M}_0) = 0$ and (24) are self-consistent. This can be done if these equations can be obtained from some effective action. As usual, if $\bar{g}^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} > 0$, the energy-momentum tensor $T_{\mu\nu}^{(\text{eff})}$ can be rewritten in the form of a perfect fluid. Then the pressure density plays the role of the Lagrangian in the effective action. Thus we come to the following TMT effective action

$$S_{\text{eff}} = \int \left( -\frac{M_p^2}{2} R(\bar{g}) + \frac{1}{2 \left( 1 + \zeta_v \right)^2} \bar{g}^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} - V_{\text{eff}}^{(0)}(\phi) \right) \sqrt{-\bar{g}} \, d^4 x. \quad (31)$$

It can be checked by direct calculations that the variation of $S_{\text{eff}}$ with respect to the fields $\bar{g}^{\mu\nu}$ and $\phi$ actually leads to Eqs. (30) and (17) with $T_{\mu\nu}^{(\text{eff})}$ described by Eqs. (18), (22) and (24) (with $\Lambda = 0$).

To compare the predictions of the model under study with numerous models developed to describe inflationary cosmology, it is necessary to pass to the canonically normalized scalar field. It turns out that the most interesting
results are obtained if the non-minimal coupling constant $\xi$ is positive. Therefore, in addition to choosing negative values of the model parameters $V_1$ and $V_2$ and the integration constant $M_0 = 2 \sqrt{V_1 V_2}$, we will study the results of the model obtained for $\xi > 0$.

Then the canonically normalized scalar field $\phi$ can be easily found by solving the equation $\frac{d \phi}{d \varphi} = \sqrt{1 + \xi \varphi^2}$, which gives

$$\frac{\phi}{M_P} = \frac{1}{\sqrt{\xi}} \sinh \left( \sqrt{\xi} \frac{\varphi}{M_P} \right).$$

(32)

The action (31) is then given by

$$S_{eff} = \int \left( -\frac{M_P^2}{2} R(\tilde{g}) + \tilde{g}^{\alpha\beta} \varphi,_{\alpha}\varphi,_{\beta} - V_{eff}(\phi(\varphi)) \right) \sqrt{-\tilde{g}} d^4 x,$$

(33)

where, after inserting the primordial potential (1), the TMT effective potential expressed in terms of the canonically normalized field $\varphi$, $V_{eff}(\varphi) = V_{eff}(\phi(\varphi))$, has the form

$$V_{eff}(\varphi) = \frac{M_P^4}{(1 + \zeta_v)} \sinh^4 z \left[ \frac{\lambda}{4 \xi} + \frac{m^4}{16 \xi^2 |V_2| (1 + \zeta_v)} + \frac{m^2}{2 M_P^2 \sinh^2 z} \right. + \frac{\lambda m^2 M_P^2}{16 \xi^2 |V_2| (1 + \zeta_v)} \sinh^2 z + \frac{\lambda^2 M_P^4}{64 \xi^3 |V_2| (1 + \zeta_v)} \sinh^4 z \left. \right],$$

(34)

where $z = \sqrt{\xi} \frac{\varphi}{M_P}$.

In the next subsection, we will see the need to compare the model under study with the $\alpha$-atttactor models. Here it is worth paying attention to the difference between the change of variables $\phi \to \varphi$ described by Eq. (32) and the change $\phi \to \varphi$ in the $\alpha$-atttactor models. In this regard, the models are fundamentally different. In $\alpha$-atttactor models, the kinetic term of the non-canonical scalar field $\phi$ has a pole at the boundary of the moduli space. On the contrary, in the model under study, when $\xi > 0$ is chosen, the noncanonical scalar field $\phi$ in the TMT effective action does not contain a pole. This difference is insignificant in the limit of $\varphi \to 0$: in both models $\phi \to 0$ as $\varphi \to 0$. But for $\varphi \to \infty$, the corresponding behavior of $\phi$ in these models is completely different. In the TMT model under study, $\varphi$ also tends to infinity. In $\alpha$-atttactor models, the canonical scalar field tends to infinity when the original non-canonical scalar field tends to the boundary of the moduli space.

In the zero cosmological constant case, the constraint (12) presented in terms of the canonically normalized field $\varphi$ takes the form

$$\zeta = \frac{2 |V_2| \zeta_v (1 + \zeta_v) - \frac{m^2 M_P^2}{2 \xi} \sinh^2 z - \frac{1}{2 \xi} M_P^2 \sinh^4 z}{2 |V_2| (1 + \zeta_v) + \frac{m^2 M_P^2}{2 \xi} \sinh^2 z + \frac{1}{2 \xi} M_P^2 \sinh^4 z} \quad \text{where} \quad z = \sqrt{\xi} \frac{\varphi}{M_P}.$$

(35)

C. Model parameters and preliminary discussion of the obtained modification of the T-model potential

The model contains 5 parameters: $m^2 > 0$, $\lambda > 0$, $\xi > 0$, $V_1 < 0$ and $V_2 < 0$. This allows us to hope that by fitting the parameters it will be possible to obtain agreement between the predictions of the model and the existing observational data (and, possibly, future ones). But in this paper I will limit myself to studying the possibilities of solving the problem of initial conditions for inflation.

As we have seen, if $\phi = 0$ is a vacuum state with zero energy density, then, instead of the parameter $V_1$, it is convenient to use the parameter $\zeta_v$ defined by Eq. (28). As noted above, the canonical variable $\varphi = 0$ when $\phi = 0$. Expanding the TMT effective potential (31) near $\varphi = 0$ we obtain the following expression for the square of the inflaton effective mass

$$m_{eff}^2 = \frac{m^2}{1 + \zeta_v}.$$

(36)

Assuming that the parameter $|V_1|$ does not exceed the parameter $|V_2|$, we obtain from Eq. (28)

$$0 < \zeta_v \leq 1$$

(37)

As a numerical estimate, I will take the mass of the inflaton $m_{eff} \approx 2 \cdot 10^{13} GeV$. Then for the mass parameter $m$ we have

$$m \sim 10^{13} GeV.$$
The first two terms in square brackets in (33) are responsible for the start of the plateau when $\frac{m^4}{|V_{\text{eff}}(\phi)| V_{\text{eff}}(0)} \geq \frac{1}{1 + \zeta_v}$. The last two terms in (33) are responsible for the end of the plateau and the beginning of the almost exponential growth of $V_{\text{eff}}(\phi)$. By adjusting the model parameters, one can move the value of $\phi$ at which the latter occurs to provide constraints on the initial conditions for a quasi-dS inflation.

We notice that by choosing the value of the non-minimal coupling parameter $\xi = \frac{1}{6}$ (opposite in sign to the parameter of the conformal coupling), we obtain an effective potential constructed from the hyperbolic functions like $\tanh\sqrt{\frac{\lambda}{M_P}}$, depending on the same combination $\frac{\lambda}{M_P}$ as in the simplest T-Model obtained in the conformal theory [8]. A more general case of the single field $\alpha$-attractor models [10], where the hyperbolic functions depend on the combination $\sqrt{\frac{\xi}{\lambda M_P}}$, corresponds to the choice $\xi = \frac{1}{6}$ in the model of this paper. In further evaluations I will use $\xi = \frac{1}{6}$.

For the plateau height of $V_{\text{eff}}(\phi)$ to satisfy the constraint on the inflationary energy scale [1], the choice of model parameters must ensure that

$$\frac{1}{4(1 + \zeta_v)\xi^2} \left[ \lambda + \frac{m^4}{4|V_{\text{eff}}(0)(1 + \zeta_v)|} \right] \sim 10^{-10}. \tag{39}$$

For further evaluation, we must consider the possible range of the parameters $|V_2|$ and $\lambda$. It is natural to assume that the vacuum-like parameters $V_1$ and $V_2$ can be of the order close or at least not much less than $M_P^4$. Using (38) one can conclude that

- if $\lambda < \frac{m^4}{4(1 + \zeta_v)|V_2|}$ then (39) can be satisfied if $|V_2| \lesssim (\frac{1}{2} \cdot 10^{16}\text{GeV})^4$;
- if we prefer $|V_2|$ to be closer to the Planck scale, i.e. $|V_2| > (10^{16}\text{GeV})^4$, then there must be

$$\lambda \frac{1}{4\xi^2(1 + \zeta_v)} \sim 10^{-10}, \quad \text{that is} \quad \lambda \sim 10^{-11}. \tag{40}$$

In what follows, the last choice for $\lambda$ will be assumed.

Now there is a need to discuss the parameters of the model in more detail. Note that with the chosen parameters $m$ and $\lambda$, the height of the potential energy plateau of the TMT effective potential $V_{\text{eff}}(\phi)$, Eq. (34), is controlled mainly by $\lambda$, and more precise tuning can be performed by $\xi$ and $\zeta_v$. The set of these four parameters, together with $|V_2|$ discussed below, provides a wide range of possibilities for adjusting the shape of $V_{\text{eff}}(\phi)$ to the constraints imposed by current and future cosmological observations. By the number of model parameters, the effective TMT potential $V_{\text{eff}}(\phi)$ is very different from the potential in the $\alpha$-attractor models [10], where only two parameters ($m$ and $\alpha$) are present. However, this is not the only difference. Due to the presence of an almost exponentially growing "tail", $V_{\text{eff}}(\phi)$ should be compared with the potential proposed by Linde in Ref. [34] for solving the problem of initial conditions for inflation. A potential whose shape is very similar to $V_{\text{eff}}(\phi)$ is presented in [34] as an example of a singular $\alpha$-attractor model.

After we have chosen estimates for the parameters $m$, $\lambda$, $\xi$, $\zeta_v$, the only remaining free parameter is $V_2 < 0$. It is convenient to use the parametrization $|V_2| = (q \cdot M_P)^4$. In order to get an idea of the effect of a parameter $|V_2|$, let us consider the following three cases: 1) $q = 1$, which corresponds to $|V_2| = M_P^4 \approx (2.44 \cdot 10^{18}\text{GeV})^4$; 2) $q = \frac{3}{4}$, which corresponds to $|V_2| \sim 10^{-2}M_P^4$; 3) $q = 0.1$, which corresponds to $|V_2| = 10^{-4}M_P^4 \approx (2.44 \cdot 10^{17}\text{GeV})^4$. For these three values of $|V_2|$, the plots of the effective TMT potential (34) are shown in Fig.1. As can be seen, the larger $|V_2|$, the greater the plateau length, and this dependence is very sensitive to changes in $|V_2|$.

Note that if instead of the scalar field model with the primordial potential [11] we choose a free massive scalar field $\phi$, i.e. $\lambda = 0$, then, as can be seen from Eq. (34), the TMT effective potential would have an infinite plateau.

In the field interval in which the potential has a plateau, we can restrict ourselves to the first degree of expansion in $e^{-\sqrt{\frac{\xi}{\lambda M_P}} \phi}$ and find, using the chosen estimates of the model parameters

$$V_{\text{eff}}(\phi) \approx \frac{\lambda M_P^4}{4\xi^2(1 + \zeta_v)} \left[ 1 - 8 \left( 1 + O\left(\frac{10^{-10}}{q^4}\right) \right) e^{-\sqrt{\frac{\xi}{\lambda M_P}} \phi} + O\left(e^{-2\sqrt{\frac{\xi}{\lambda M_P}} \phi}\right) \right]. \tag{41}$$

This is a representation of the potential, which, with a negligible correction, coincides with the corresponding expansion in $e^{-\sqrt{\frac{\xi}{\lambda M_P}} \phi}$ of the potential $V(\phi) \propto \tanh^{2n} \left( \frac{\phi}{\sqrt{6} M_P} \right)$ for $n=2$ found in the paper [8] where it was named "the T-model". Thus, we can state that in the model we are studying, all inflationary predictions coincide with the corresponding predictions of the T-model and, therefore, are consistent with Planck’s data to the same extent.
FIG. 1. Plots of the effective TMT potential $V_{\text{eff}}^{(0)}(\phi)$ defined by Eq. (34) show that the plateau length is controlled by the parameter $|V_2| = (qM_P)^4$. As an example, three values $q = 1$, $q = \frac{1}{3}$, $q = 0.1$ are chosen; each curve is labeled with the corresponding value of $q$. The other parameters are the same for all curves: $\xi = \frac{1}{6}$, $\eta = 1$, $\lambda = 2.4 \times 10^{-11}$, $m = 1.9 \times 10^{13}$ GeV. The points of intersection of the dashed line with the function curves correspond to $\phi^* = \phi_1$ defined by Eq. (43). $q$-dependent quantities $\phi^*$ determine the maximum possible duration of inflation. But the values of $V_{\text{eff}}^{(0)}(\phi^*)$ do not depend on $q$, and for the chosen parameters $V_{\text{eff}}^{(0)}(\phi^*)$ exceeds the plateau height by approximately 1.5 times. The latter means that inflation can be driven from the very beginning by a scalar field with an almost exponentially growing potential.

So, we found that when studying the simplest field theory model (1) with nonminimal coupling to gravity in the framework of the TMT in the Palatini formalism, an effective potential of the canonically normalized inflaton arises, which 1) has a plateau, the length of which is controlled by the parameter $V_2$; 2) when $\phi$ exceeds a certain value depending on $V_2$, the potential passes from a flat shape to an almost exponential growth. At first glance, it seems that the modification of the T-model potential obtained in the simplest model (4) is reduced only to the appearance of an exponentially growing "tail". In fact, this is only partly true. Here TMT presents us with another surprise, which will be discovered and explored in the next subsections.

D. The key role of the condition $\Upsilon(x) \neq 0$ as a TMT unique attribute

A solution (9) of Eq. (7) exists under the condition $\Upsilon(x) \neq 0$, Eq. (8), i.e., only if $\Upsilon(x) > 0$ or only if $\Upsilon(x) < 0$ (see also a footnote[35]). Therefore, only those solutions of the system of equations obtained in the previous subsections are valid for which the corresponding $\Upsilon(x)$ is sign-definite, since Eq. (7) is one of the equations of the system. The validity of the solution with respect to the sign of $\Upsilon$ can be controlled by checking the sign of the scalar $\zeta = \Upsilon/\sqrt{-g}$. Since $\zeta$ is present in all equations, checking the sign of $\zeta$ instead of the sign of $\Upsilon$ is more convenient. With such a replacement of $\Upsilon$ by $\zeta$, information is not lost, provided that the metric tensor in the original frame $g_{\mu\nu}$ is regular. If this is the case, then only those solutions of the equations are valid for which $\zeta(x)$, determined by Eq. (35), is sign-definite. In particular, in the model under consideration, with our choice of parameters and constant of integration, the value of $\zeta$ in vacuum, Eq. (28), is positive: $\zeta_v > 0$. Thus, only those cosmological solutions (together with their initial conditions) are valid for which $\zeta(x)$ is positive throughout the evolution of the universe. However, these solutions lose their validity when we try to extend them to the region of variables where $\zeta$ crosses zero and becomes negative.

Although this conclusion is correct, it requires a more accurate mathematical justification. To understand the essence of the problem, we can restrict ourselves to the simplest model that we are currently studying, where $\zeta$ determined by the constraint (35) depends only on $\phi$. But the reasoning given below is also valid in more general models, e.g., studied in Sec.IV. Let us assume that the solution of the field equations contains such a value of the field $\phi = \phi_*$ for which a formal substitution into the constraint (35) gives $\zeta = 0$. Strictly speaking, we have no right to say
that this is equivalent to $\Upsilon = 0$, because in this case the solution of the system of equations, including the solution of gravitational equations, turns out to be inapplicable. This means that on the hypersurface $\Upsilon(x) = 0$ we do not know anything about the metric tensor and therefore the equality $\zeta(\varphi_\ast) = (\Upsilon/\sqrt{-g})|_{\varphi=\varphi_\ast} = 0$ itself does not make sense. To avoid this contradiction, we must analyze what happens when the values of $\varphi$ are arbitrarily close to $\varphi_\ast$, which corresponds to the analysis at any proximity to the hypersurface $\Upsilon(x) = 0$. If the primordial metric tensor $g_{\mu\nu}$ is regular in any proximity to this hypersurface, then in the space-time region, which is very close to the hypersurface $\Upsilon(x) = 0$, we can describe the latter with unlimited accuracy by the equation $\zeta(x) = 0$. In this sense, the requirement $\Upsilon(x) \neq 0$ can be considered equivalent to the requirement $\zeta(x) \neq 0$. To verify that the primordial metric tensor $g_{\mu\nu}$ is regular in any proximity to the hypersurface $\Upsilon(x) = 0$, consider the metric tensor in the Einstein frame $\tilde{g}_{\mu\nu}$ obtained as a solution to the Einstein equations (17). If it is regular in any proximity to this hypersurface, then, according to transformation (16), the primordial metric tensor $g_{\mu\nu}$ is also regular and, therefore, control over the sign of $\Upsilon(x)$ is equivalent to control over the sign of $\zeta(x)$. It follows from this analysis that if in a homogeneous isotropic universe there exists a solution $\varphi(t)$ that admits the possibility of the limit $\zeta \to 0$ in such a way that the metric tensor in the Einstein frame $\tilde{g}_{\mu\nu}$ remains regular, then such a solution contains the limiting admissible value $\varphi_\ast$: shifting $\varphi(t)$ towards $\varphi_\ast$ brings $\zeta(\varphi(t))$ closer to 0, that is equivalent to tending to the spacelike hypersurface $\Upsilon(x) = 0$.

In this paper, we will focus on possible initial conditions for inflationary solutions to the cosmological equations. It follows from the above that those initial conditions for which $\zeta \leq 0$ should be excluded from consideration as an artifact in our Universe. As we will see in the next subsection, this is exactly what happens in the model under study.

### E. Strict bound on maximum initial value of $\varphi$ imposed by the TMT unique attribute in the simplest model

Let us start with Ref. [34], where A. Linde showed that a potential of the shape similar to Fig.1 can be obtained in a singular $\alpha$-attractor model. If $\alpha > 1/3$ and the initial value of $\varphi$ is sufficiently large, then due to the almost exponential form of the potential, the expansion of the Universe can begin with a power-law inflation. Therefore, according to Linde’s idea, inflation “may begin already at the Planck density, which solves the problem of initial conditions in this class of models along the lines of [23].”

It turns out that in the TMT models we are studying, this idea cannot be realized. To understand the reason, we must analyze what happens when the values of $\varphi_\ast$ are marked with dots. Since the TMT effective potential $V_{\text{eff}}(\varphi)$ remains finite as $\varphi \to \varphi_\ast$, the metric tensor $g_{\mu\nu}$ is regular in any proximity to $\varphi_\ast$. Hence $\zeta > 0$ for any $\varphi < \varphi_\ast$, and $\zeta < 0$ for any $\varphi > \varphi_\ast$, i.e., $\varphi_\ast$ is the limiting upper bound of admissible values of $\varphi$. Thus, for each of the curves in Fig. 1, the part of the TMT effective potential corresponding to the interval $\varphi \geq \varphi_\ast$ is an artifact in our Universe.

Inserting (41) into the effective potential (44) we obtain

$$V_{\text{eff}}(\varphi)|_{\zeta \approx 0} \approx \lambda M_P^4 \frac{\lambda}{4\xi^2(1 + \zeta_\nu)} \left( 1 + \frac{1}{2}\zeta_\nu \right) = \left( 1 + \frac{1}{2}\zeta_\nu \right) V_{\text{eff}}(\varphi)|_{\text{plateau}},$$

where $V_{\text{eff}}(\varphi)|_{\text{plateau}}$ is the typical height of the plateau of the TMT effective potential at $1 \ll \sqrt{\xi M_P^4} < \sqrt{\xi M_P^4}$. With the chosen parameter $\zeta_\nu \leq 1$, we conclude that for the initial value $\varphi_{in}$ arbitrarily close to the limiting value $\varphi_\ast$...
the initial kinetic energy density. As an example with $\phi \approx 0$, inflation begins when the effective TMT potential $V_{eff}(\varphi)$ is achieved by choosing $\phi_{in} \approx 0$ and the value of this ratio does not depend on $\varphi_{in}$.

However, as we will see below and in Sec.IV, a TMT model can provide an alternative way to solve the problem of the initial conditions for inflation. Here, firstly, we use a simple way to avoid the complication of the inflationary scenario caused by an almost exponentially growing part of $V_{eff}(\varphi)$. To do this, it is enough to use arbitrariness in the choice of the parameter $|V_1|$ or, what is the same, in the choice of $\zeta$, see Eq. (28). For example, if $\zeta = 0.2$, then $V_{eff}(\varphi_*)$ exceeds the typical plateau height by about 10%. In this case, when $\varphi_{in}$ is close to $\varphi_*$, inflation is driven from the very beginning by a scalar field with a potential, the shape of which differs little from a flat one. As an example with $\zeta = 0.2$, the plots of the TMT effective potentials $V_{eff}(\varphi)$ for four values of $q$ are shown in Fig.2, where the points on the graphs corresponding to $q$-dependent values of $\varphi_*$ are marked. Note that the value $\zeta = 0.2$ corresponds to the choice of $|V_1| \approx 0.04|V_2| \approx (0.45qM_P)^4$.

The qualitative conclusion following from this result is as follows: in the framework of the considered simplest model, there is a region of model parameters where the theory forbids in our Universe the canonically normalized inflaton field $\varphi$ to exceed values starting from which the potential has an exponentially growing “tail”. Note that, as will be shown in Sec.IV and Appendix, this conclusion can be violated if the initial gradient energy density is greater than the initial kinetic energy density.

To complete the picture, we still need to make sure that $\zeta$ remains positive during the entire time of cosmological evolution. To control the change in $\zeta$ we need to know the sign of $\frac{d\zeta}{d\varphi}$. It follows from Eq. (32) that: 1) as $\varphi > 0$ also $\phi > 0; 2) \frac{d\varphi}{d\phi} > 0; 3) \text{sign} \left( \frac{d\zeta}{d\varphi} \right) = \text{sign} \left( \frac{d\zeta}{d\varphi} \right)$. Therefore, to simplify the calculation, instead of $\frac{d\zeta}{d\phi}$ one can find $\frac{d\zeta}{d\varphi}$.

Using Eqs. (27) and (28) we obtain from the constraint $|\zeta| < \frac{1}{2}$ the following differential equation:

$$\frac{d\zeta}{d\varphi} = -\frac{2|V_2||(1 + \zeta_0)^2}{[2|V_2|(1 + \zeta_0) + V(\varphi)]^2} V'(\varphi). \tag{46}$$
Eq. (1) shows that $V'(\phi) > 0$ for $\phi > 0$ and hence $\frac{d\zeta}{d\phi} < 0$. Therefore, in the inflation process governed by a monotonically decreasing classical scalar field $\varphi(t) > 0$, $\zeta(\varphi(t))$ increases monotonically. The initial value $\zeta_{\text{in}}$, from which inflation starts, can be arbitrarily close to $\varphi_*$, but it must be $\varphi_{\text{in}} < \varphi_*$. Accordingly, $\zeta(\varphi(t))$ can start with a positive initial value $\zeta(\varphi_{\text{in}})$, arbitrarily close to zero

$$
\zeta(\varphi_{\text{in}}) \rightarrow 0^+ \quad \text{as} \quad \varphi_{\text{in}} \rightarrow \varphi_*^-, \quad (47)
$$

and $\zeta(\varphi(t))$ increases monotonically during inflation. After the end of inflation, in the process of transition to the vacuum state, oscillations of $\varphi$ can cause oscillations of $\zeta$, but it remains positive and approaches its vacuum value $\zeta_v > 0$.

IV. A MORE GENERAL MODEL AND NATURAL TMT CONSTRAINTS ON THE MAXIMUM INITIAL KINETIC AND GRADIENT ENERGY DENSITIES

In addition to solving the problem associated with the infinite extent of the plateau, in order to initiate inflation, it is also necessary that the initial kinetic and gradient energy densities at least not exceed the height of the potential plateau

$$
\rho_{\text{kin, in}} = \frac{1}{2} \zeta_{\text{in}}^2 \lesssim O(1) \cdot 10^{-10} M_p^4, \quad \rho_{\text{grad, in}} = \frac{1}{2}(\partial^k \varphi)_{\text{in}}(\partial_k \varphi)_{\text{in}} \lesssim O(1) \cdot 10^{-10} M_p^4. \quad (48)
$$

In a slightly more general model formulated in this section, we will show that there is an interval of initial values $\varphi_{\text{in}}$, where, along with the condition $\zeta > 0$, the fulfillment of conditions (48) necessary for inflation to start is also guaranteed.

A. A more general model

As already mentioned in Sec.III.A, in the general case $\Upsilon$ and $\sqrt{-g}$ can enter the volume element with arbitrary coefficients. The choice of $(\sqrt{-g} + \Upsilon) d^4 x$ as the volume element in all gravity and matter terms, made in Sec.III.A, means that all results were obtained there with the most simplified approach to the choice of these coefficients. Therefore, it would be interesting to know what other new results can be obtained by abandoning such a simplification of the model. Bearing in mind the relevant discussion in Sec.III.A, we choose volume elements in the form $(b_1 \sqrt{-g} + \Upsilon) d^4 x$, where $b_i$ is the model parameter corresponding to the i’s term in the primordial action.

Generalizing in this way the model studied so far, the primordial action can be represented as follows

$$
S = \int d^4 x \left[ -\frac{M_p^2}{2} (\sqrt{-g} + \Upsilon) \left( 1 + \frac{\phi^2}{M_p^2} \right) R(\Gamma, g) + (b_k \sqrt{-g} + \Upsilon) \frac{1}{2} g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} - (b_p \sqrt{-g} + \Upsilon) \left( \frac{1}{2} \phi^2 + \frac{\lambda}{4} \phi^4 \right) \right]
$$

$$
- \int d^4 x \left[ \sqrt{-g} V_1 + \frac{\Upsilon^2}{\sqrt{-g}} V_2 \right], \quad (49)
$$

where $b_k$ and $b_p$ are additional model parameters. I should note that this is not the most general action, because another parameter of the same type can be added to the volume element, coming with a non-minimal coupling. One can believe that the reason for the deviation of the parameters $b_k$ and $b_p$ from unity is quantum corrections. Then it is natural to assume that these corrections are small, and for further consideration I choose $0 < b_k < 1$ and $0.5 < b_p < 1$. But it is worth noting that in the calculations, the results of which are given below, the possible smallness of $1 - b_k$ and $1 - b_p$ is not used.

It is known [28] that in TMT models with $b_k \neq 1$ and $b_p \neq 1$, the implementation of all seven steps of the TMT procedure listed in Sec.II leads to an effective TMT action with a structure typical for K-essence models [40]. Therefore, repeating again the TMT procedure in model (49), we quite expectedly come to a similar result. Here I use the same choice for all other model parameters that was made in Sec.III. The integration constant $\mathcal{M}$, as in Sec.III.B, is also chosen according to Eq. (27), which provides zero vacuum energy density. The last fact will be marked with (0) in the notation of all relevant quantities. After passing to the canonically normalized scalar field $\varphi$ by means of Eq. (32), the expression for the TMT effective action has the form

$$
S_{\text{eff}} = \int \left[ -\frac{M_p^2}{2} R(\tilde{g}) + L_{\text{eff}}^{(0)} (\phi(\varphi), \zeta(\varphi, X_\varphi)) \right] \sqrt{-\tilde{g}} d^4 x, \quad (50)
$$
where the TMT effective Lagrangian for the scalar field \( \phi \) appears as following

\[
L_{\text{eff}}^{(0)}(\phi(\varphi), \zeta(\varphi, X_\varphi)) = X_\varphi - U_{\text{eff}}^{(0)}(\phi(\varphi), \zeta(\varphi, X_\varphi)), \quad X_\varphi = \frac{1}{2} \eta^\alpha \phi \eta^\beta \phi \beta
\]  

(51)

and \( U_{\text{eff}}^{(0)}(\phi(\varphi), \zeta(\varphi, X_\varphi)) \) in terms of \( \zeta = \zeta(\varphi, X_\varphi) \) reads

\[
U_{\text{eff}}^{(0)}(\phi(\varphi), \zeta(\varphi, X_\varphi)) = \frac{1}{\cosh^4 z} \left[ \frac{|V_2|(1 + \zeta)^2 + (1 - b_p) V(\phi(\varphi))}{(1 + \zeta)^2} - |V_2| \right], \quad \text{where} \quad z = \sqrt{\xi \frac{\varphi}{M_P}}
\]  

(52)

Here \( \zeta = \zeta(\varphi, X_\varphi) \) is determined by the constraint the origin of which was discussed in detail in previous section and will be equally important for all subsequent analysis. Now, given constraint (53), that at the minimum of \( V \)

\[
\text{typical for K-essence models}[40]
\]

\[\zeta \]

(53)

A significant difference from the simplest model studied in Sec.III is that now \( \zeta \) turns out to be a function depending not only on \( \varphi \) but also on \( X_\varphi \).

Inserting \( \zeta(\varphi, X_\varphi) \) to Eq. (52) and using Eqs. (1) and (32) we obtain the final expression for the TMT effective Lagrangian of scalar field \( \varphi \), \( L_{\text{eff}}^{(0)}(\phi(\varphi), \zeta(\varphi, X_\varphi)) \), which can be represented in the following form, typical for K-essence models[10]

\[
L_{\text{eff}}^{(0)}(\varphi, X_\varphi) = X_\varphi - V_{\text{eff}}^{(0)}(\varphi) - K_1(\varphi) X_\varphi - K_2(\varphi) \frac{X_\varphi^2}{M_P^4},
\]  

(54)

where

\[
V_{\text{eff}}^{(0)}(\varphi) = \frac{M_P^4}{4 \xi^2} \tanh^4 z \frac{\lambda q^4(\zeta_\varphi + b_p) + \frac{m^2}{4 M_P^2} \sinh^2 z + \frac{\lambda^2}{16 q^4} \sinh^4 z + 2 \xi q^4(\zeta_\varphi + b_p) \frac{m^2}{M_P^2} \cdot \sinh^2 z}{(1 + \zeta_\varphi)^2 q^4 + (1 - b_p) \left( \frac{1}{2 q^2} \frac{m^2}{M_P^2} \sinh^2 z + \frac{\lambda}{4 q^4} \sinh^4 z \right)},
\]  

(55)

\[
K_1(\varphi) = \frac{1 - b_p}{2 \cosh^2 z} \cdot \frac{8 q^4(1 + \zeta_\varphi) + \frac{1}{4 q^2} \frac{m^2}{M_P^2} \cdot \sinh^2 z + \lambda \sinh^4 z}{4 q^4(1 + \zeta_\varphi)^2 + 2(1 - b_p) \zeta_\varphi \frac{m^2}{M_P^2} \cdot \sinh^2 z + \lambda(1 - b_p) \sinh^4 z},
\]  

(56)

\[
K_2(\varphi) = \frac{(1 - b_p)^2}{4 q^4(1 + \zeta_\varphi)^2 + \frac{1 - b_p}{2 q^2} \frac{m^2}{M_P^2} \cdot \sin^2 z + \frac{(1 - b_p)}{4 q^4} \sinh^4 z}.
\]  

(57)

As can be seen from Eq. (55) and Fig. 3, the choice of \( b_p \neq 1 \) radically changes the behavior of \( V_{\text{eff}}^{(0)}(\varphi) \) at \( \varphi \gg M_P \); instead of the almost exponential unlimited growth that was at \( b_p = 1 \), a second plateau appears. However, if \( X_\varphi = 0 \), then, as in the case of \( b_p = 1 \) in Sec.III, this range of \( \varphi \) is also an artifact. To show this, we first note, using the constraint (53), that at the minimum of \( V_{\text{eff}}^{(0)}(\varphi) \), i.e. for \( \varphi(x) = 0 \), scalar \( \zeta \) has the same value \( \zeta_\varphi > 0 \) as defined by Eq. (28). Therefore, throughout the entire process of cosmological evolution, \( \zeta \) must be positive. This was of paramount importance in the previous section and will be equally important for all subsequent analysis. Now, given the constraint (53) when \( X_\varphi = 0 \), we see that \( \zeta \rightarrow 0^+ \) when \( \varphi \rightarrow \varphi_0^- \) where \( \varphi_0 \) is defined by the relation

\[
\sinh^4 \frac{\sqrt{\xi \zeta_\varphi}}{M_P} = \frac{8 q^4(1 + \zeta_\varphi)}{\lambda(2 b_p - 1)}
\]  

(58)

obtained after neglecting a very small correction from the term \( \propto \sinh^2 \sqrt{\xi \zeta_\varphi} M_P \). This is a generalization of the definition of \( \varphi_\ast \), Eq. (14), to the case \( b_p \neq 1 \). For the parameters used in the plot in Fig. 3, relation (58) gives \( \varphi_0 \approx 12.9 M_P \). The corresponding point on the curve is marked with a dot. It can be seen that again, as in Sec.III, the value \( V_{\text{eff}}^{(0)}(\varphi_0) \) exceeds the typical plateau height by about 10%.
B. TMT constraints on the maximum initial kinetic and gradient energy densities and initial conditions for inflation

The standard, non-TMT, formulation of the initial conditions for inflation driven by a scalar field includes specifying or at least estimating the initial value of the field $\varphi_{\text{in}}$ and its first derivatives or, equivalently, the initial kinetic $\rho_{\text{kin,in}}$ and gradient $\rho_{\text{grad,in}}$ energy densities. It is fundamentally important and taken for granted that there is no dependence in any form between $\varphi_{\text{in}}$ and $\rho_{\text{kin,in}}$, as well as between $\varphi_{\text{in}}$ and $\rho_{\text{grad,in}}$. In the simplest model of section III, although there are restrictions on the initial value $\varphi_{\text{in}}$, no restrictions on the $\rho_{\text{kin,in}}$ and $\rho_{\text{grad,in}}$ arise. But in the more general model with action (49), the scalar $\zeta$ given by the constraint (53) turns out to depend not only on $\varphi_{\text{in}}$, but also on $X_{\varphi}=\frac{1}{2}g^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta}$. As a result, the condition $\zeta > 0$ imposes restrictions in the form of inequalities on the admissible ranges of $\varphi$ and $X_{\varphi}$.

The results of the requirement that $\zeta(\varphi, X_{\varphi})$ be positive can be obtained from a detailed study of the inequality

$$\zeta = \frac{2|V_2|\zeta_v(1 + \zeta_v) - (2b_p - 1)\left[\frac{m^2M_p^2}{4}\sinh^2 z + \frac{\lambda}{4\xi^2}M_p^4\sin^4 z\right] - (1 - b_k)\cosh^4 z \cdot X_{\varphi}}{2|V_2|(1 + \zeta_v) + \frac{m^2M_p^2}{4}\sinh^2 z + \frac{\lambda}{4\xi^2}M_p^4\sin^4 z + (1 - b_k)\cosh^4 z \cdot X_{\varphi}} > 0$$

(59)

We are interested in the restrictions on the initial values of $\varphi_{\text{in}}$ and $X_{\varphi}^{(\text{in})}$ imposed by the condition $\zeta(\varphi_{\text{in}}, X_{\varphi}^{(\text{in})}) > 0$ and upper constraints on $\rho_{\text{kin,in}}$ and $\rho_{\text{grad,in}}$, Eq. (48). Inflation is possible only if all these conditions are met together. A discussion of some related fundamental issues, as well as elementary calculations of the corresponding analysis, are collected in the Appendix. The main results obtained with the model parameters used in the previous subsection can be represented as the following conclusions.

- If $X_{\varphi}^{(\text{in})} > 0$, then on the interval

$$12.6M_P \leq \varphi_{\text{in}} < \varphi_0 \approx 12.9M_P$$

(60)

$\zeta(\varphi_{\text{in}}, X_{\varphi}^{(\text{in})}) > 0$ and the necessary conditions (48) for the beginning of inflation are guaranteed to be fulfilled.

- For $\varphi_{\text{in}} < 12.6M_P$ and $12.9M_P < \varphi_{\text{in}} < 13.5M_P$, $\zeta(\varphi_{\text{in}}, X_{\varphi}^{(\text{in})})$ is also positive and it is possible that the conditions necessary for inflation to start are met. But unlike the interval (60), the latter is not guaranteed.
• For $\varphi > 13.5 M_P$ the condition $\zeta(\varphi, X_\varphi) > 0$ is satisfied only if $X_\varphi < 0$ and $|X_\varphi| > V_{eff}^{(0)}(\varphi)$. The latter means that the part of the graph of $V_{eff}^{(0)}(\varphi)$ in Fig.3 in the region $\varphi_{in} > 13.5 M_P$ may not be an artifact only in the case of a sufficiently strong inhomogeneity.

It is noteworthy that there is a fairly wide range of model parameters in which they can be changed without a significant impact on the order of the width of the found interval $[54]$. Reasoning in the spirit of chaotic inflation, we can consider $\varphi_{in}$ as a random variable. Bearing in mind that a significant part of the potential is flat, without claiming good accuracy of estimates, we can assume that the probability has a flat prior distribution. As can be seen from Eqs.(51)-(57), the TMT effective Lagrangian and a significant part of the potential is flat, without claiming good accuracy of estimates, we can assume that the onset of inflation can be found as $(12.9 - 12.6)/12.9 \approx 0.023$.

As can be seen from the shape of $V_{eff}^{(0)}(\varphi)$ in Fig.3, the resulting upper bound on $\varphi_{in}$ means that, with the chosen model parameters, inflation can be only due to the first (lower) plateau. A more detailed numerical analysis shows that this qualitative conclusion does not require fine tuning and can be obtained in a quite wide range of model parameters. But it cannot be ruled out that there is another acceptable choice of model parameters, under which the necessary conditions for the beginning of inflation are also realized on the second (higher) plateau. In this case, two consecutive inflations are possible. However, the study of this issue is beyond the scope of this paper.

We must keep in mind that the described conclusions follow from the analysis of the model $\mathcal{L}_4$ if the integration measure density $\Upsilon$ can be used for investigation of the role of the volume element $dV_4$ as a smooth differential 4-form defined on the space-time manifold $M_4$. Here it is appropriate to recall the following well-known mathematical propositions, which hold for a differentiable manifolds, see e.g. Sec.15 in Ref.[41]. Applying them to $M_4$ these propositions are Any nonvanishing 4-form $\omega$ on $M_4$ determines a unique orientation of $M_4$ for which $\omega$ is positively oriented at each point. And conversely, if $M_4$ is given an orientation, then there is a smooth nonvanishing 4-form on $M_4$ that is positively oriented at each point. Similar propositions hold for a negatively oriented 4-form $\omega$ on $M_4$.

The inclusion of the volume 4-form $\Upsilon(x)d^4x$ in the principle of least action and the natural requirement of self-consistency of the field equations led us to the conclusion that the measure density $\Upsilon$ must be nonvanishing and sign-definite. But if the properties of the volume measure $\Upsilon(x)d^4x$ on $M_4$ that follow from the field dynamics completely coincide with the conditions imposed on the 4-form in the above mathematical propositions, then we have the right to identify the sign of $\Upsilon$ with the sign of the $M_4$ orientation. It follows from this that in the context of cosmology the sign-definiteness of $\Upsilon$ can be interpreted as the preservation of the orientation of the space-time manifold $M_4$ of our Universe during the entire process of its evolution. Therefore, the existence of an upper bound on the admissible values of the inflaton field $\varphi$ (for example, the value $\varphi_*$ discovered in Sec.III.E) acquires an interesting interpretation: if $\varphi$ crossed $\varphi_*$, this would be accompanied by a change in the orientation of $M_4$.

Finally, it is appropriate to recall here that the volume measure $\sqrt{-g}d^4x$ can also be used only if $M_4$ is orientable[41],[42].

• The second aspect is related to the fact that the use of the 4-form allows the appearance in the primordial action of a fundamentally new vacuum-like term $-\int d^4x \sqrt{-g}V_2$ along with the standard one $-\int d^4x \sqrt{-g}V_1$. As we saw in this paper, without the corresponding parameter $V_2 \neq 0$, all the results of the work would be impossible.

V. DISCUSSION

In the framework of TMT in the Palatini formalism, we explored the possibility of using a scalar field model with a standard potential $V(\varphi) = \frac{1}{2}m^2\varphi^2 + \frac{1}{4}\lambda\varphi^4$ and non-minimal coupling to gravity to describe inflation, focusing on the new opportunities provided by TMT for solving the problem of initial conditions for inflation. Based on the analysis of this particular model, the existence of a unique attribute of TMT has been demonstrated that is crucial to enabling the new opportunities. This unique attribute consists in the need to ensure that $\Upsilon(\varphi, X_\varphi, X_\varphi)$ do not change under the replacement $\varphi \rightarrow -\varphi$. Therefore, we can restrict ourselves to positive $\varphi$ only. Then a rough estimate of the probability of occurrence of initial conditions guaranteeing the onset of inflation can be found as $(12.9 - 12.6)/12.9 \approx 0.023$.

As can be seen from the shape of $V_{eff}^{(0)}(\varphi)$ in Fig.3, the resulting upper bound on $\varphi_{in}$ means that, with the chosen model parameters, inflation can be only due to the first (lower) plateau. A more detailed numerical analysis shows that this qualitative conclusion does not require fine tuning and can be obtained in a quite wide range of model parameters. But it cannot be ruled out that there is another acceptable choice of model parameters, under which the necessary conditions for the beginning of inflation are also realized on the second (higher) plateau. In this case, two consecutive inflations are possible. However, the study of this issue is beyond the scope of this paper.

We must keep in mind that the described conclusions follow from the analysis of the model $\mathcal{L}_4$ if the integration constant $\mathcal{M}$ is chosen so that the cosmological constant is equal to zero (see Eqs.(21) and (27)).
Turning to the discussion of the role of the hypersurface $\Upsilon(x) = 0$ in $M_4$, whose existence was discovered while studying the equations of the model, we seem to find a special kind of cosmological singularity. Indeed, when we try to push back in time the initial conditions for solutions describing the evolution of a homogeneous isotropic Universe, we face an insuperable obstacle in the form of a spacelike hypersurface $\Upsilon(x) = 0$. For example, as it follows from the detailed analysis of the simplest model in Secs.III.D and III.E, in any proximity to hypersurface $\Upsilon(x) = 0$ all dynamical variables and curvature are regular, but this spacelike hypersurface defines a limiting upper bound constraint on $X$ to improbable and we will not consider it.

The model proposed in the paper has signs of what is commonly called “naturalness” and this, of course, is associated primarily with the standard form of the potential $V(\phi)$, Eq.(1) and the canonical kinetic term of the inflaton $\phi$ in the primordial action \[19\]. But it should also be noted that the vacuum-like parameters $|V_1|$ and $|V_2|$ in the primordial action are close to $M^4_P$, which seems to be a quite natural choice.

Finally, I would like to point out that if we really believe in the effectiveness of mathematics, then we have no reason to ignore the possibility of including the 4-form $\Upsilon$ as a volume measure in the primordial action. Moreover, this seems quite natural, since the construction of smooth manifolds, and in particular a 4D space-time manifold, begins with equipping the topological space with a differentiable structure. As a result, the volume 4-form appears even before the 4D differentiable manifold is equipped with an affine connection and metric\[11, 42\].

VI. ACKNOWLEDGEMENTS

I am grateful to E. Guendelman, E. Nissimov and S. Pacheva for a detailed discussion of the differences between densities $\Phi$ and $\Upsilon$ of the alternative volume measures.

Appendix: Analysis of the condition $\zeta > 0$ in the model of Sec.IV

1. Constraint on the maximum initial value of $X_\phi > 0$ in a model with $b_k \neq 1$ and $b_p \neq 1$

The results obtained in this Appendix follow from a detailed study of the inequality \[59\]. But before proceeding to such a study, it is necessary to note an essential difference between the way in which the kinetic and gradient energies enter into the Einstein equations and into the constraint \[53\]. Indeed, while the kinetic and gradient energy densities enter the Einstein equations in the form of a sum, they enter the constraint in the form $X_\phi$ to such a study, it is necessary to note an essential difference between the way in which the kinetic and gradient action are close to $M^4$.

It is convenient to describe the position of $\varphi_{in}$ relative to $\varphi_0$ defined by Eq.\[68\]. Since $\sqrt{\varphi_{in}/M_P} \gg 1$, Eq.\[68\] reduces to
\[32q^4\varphi_{in}(1 + \zeta_0) = (2b_p - 1) \lambda / 4\xi^2 e^{4\sqrt{\varphi_{in}/M_P}}. \]
Inserting (A.3) to Eq.(A.2) we obtain

\[ 0 \leq X_{\varphi}^{(in)} < \frac{\lambda(2b_p - 1)}{4\xi^2(1 - b_k)} M_p^4 \left[ \exp \left( 4\sqrt{\xi} \left( \frac{\varphi_0}{M_P} - \frac{\varphi_{\text{in}}}{M_P} \right) \right) - 1 \right] \overset{\text{def}}{=} X(\varphi_{\text{in}}), \]  

(A.4)

where \( X(\varphi_{\text{in}}) \) describes the limiting upper bound of admissible values of \( X_{\varphi}^{(in)} \geq 0 \) as a function of \( \varphi_{\text{in}} \). More precisely, here the term "limiting upper bound of admissible values of \( X_{\varphi}^{(in)} \geq 0 \)" means that \( \xi > 0 \) for any \( X_{\varphi}(\phi) \) satisfying \( 0 \leq X_{\varphi}(\phi) < X(\varphi_{\text{in}}) \) and \( \xi < 0 \) for any \( X_{\varphi}(\phi) > X(\varphi_{\text{in}}) \). Note that \( X(\varphi_{\text{in}}) > 0 \) if \( \varphi_{\text{in}} < \varphi_0 \).

We see that: 1) \( X(\varphi_{\text{in}}) \) tends to zero as \( \varphi_{\text{in}} \to \varphi_0^- \); 2) \( X(\varphi_{\text{in}}) \) increases as \( \varphi_{\text{in}} \) decreases. This means that by shifting \( \varphi_{\text{in}} \) in a decreasing direction, one can reach a state \( \varphi_{\text{in}}^{(\text{min})} \), where \( X(\varphi_{\text{in}}) \) becomes equal to \( V_{\text{eff}}^{(0)}(\varphi_{\text{in}}) \), and for \( \varphi_{\text{in}} < \varphi_{\text{in}}^{(\text{min})} \) we get \( X(\varphi_{\text{in}}) > V_{\text{eff}}^{(0)}(\varphi_{\text{in}}) \). Thus, in the interval \( \varphi_{\text{in}}^{(\text{min})} < \varphi_{\text{in}} < \varphi_0 \) it is guaranteed fulfillment of the condition \( X_{\varphi}^{(in)} < V_{\text{eff}}^{(0)}(\varphi_{\text{in}}) \), which is necessary for inflation to start. For \( \varphi_{\text{in}} < \varphi_{\text{in}}^{(\text{min})} \) the condition necessary for inflation is possible, but not guaranteed, since \( \xi \) remains positive over an interval of \( X_{\varphi}(\phi) \geq 0 \), which also includes \( X_{\varphi}^{(in)} > V_{\text{eff}}^{(0)}(\varphi_{\text{in}}) \).

The value of \( \varphi_{\text{in}}^{(\text{min})} \) can be estimated by the approximate equality

\[ X(\varphi_{\text{in}})|_{\text{max}} = X(\varphi_{\text{in}})|_{\varphi_{\text{in}} = \varphi_{\text{in}}^{(\text{min})}} = \frac{\lambda(2b_p - 1)}{4\xi^2(1 - b_k)} M_p^4 \left[ \exp \left( 4\sqrt{\xi} \left( \frac{\varphi_0}{M_P} - \frac{\varphi_{\text{in}}^{(\text{min})}}{M_P} \right) \right) - 1 \right] \approx V_{\text{eff}}^{(0)}(\varphi_{\text{in}}^{(\text{min})}) = \frac{k_1 \lambda(\zeta_v + b_p)}{4\xi^2(1 + \zeta_v)^2} M_p^4, \]  

(A.6)

where the dimensionless factor \( k_1 \geq 1 \) is introduced in order to take into account the change in the height of the potential when \( \varphi_{\text{in}} \) changes on the segment where the plateau turns into a rapidly growing function. Since we are interested in the existence of an interval of \( \varphi_{\text{in}} \) in which the conditions for inflation are prepared, we can restrict ourselves to a qualitative account of this change.

Thus, as it follows from Eqs.(A.4)-(A.6), if the initial value \( \varphi_{\text{in}} \) of the canonically normalized classical inflaton field is in the interval

\[ \varphi_{\text{in}}^{(\text{min})} \leq \varphi_{\text{in}} < \varphi_0, \quad \text{where} \quad \varphi_{\text{in}}^{(\text{min})} = \varphi_0 - \frac{M_p}{4\sqrt{\xi}} \ln \left( 1 + \frac{k_1(1 - b_k)(\zeta_v + b_p)}{(2b_p - 1)(1 + \zeta_v)^2} \right), \]  

(A.7)

then both the condition \( \xi > 0 \) and the condition \( X(\varphi_{\text{in}}) < V_{\text{eff}}^{(0)}(\varphi_{\text{in}}) \) are guaranteed.

We see that the value of \( k_1 \) can affect \( \varphi_{\text{in}}^{(\text{min})} \), but for any \( k_1 > 0 \) we have \( \varphi_{\text{in}}^{(\text{min})} < \varphi_0 \). Then, analyzing the graph in Fig.3, we see that \( 1 \leq k_1 \leq 1.1 \). As an example, to estimate the interval of values \( \varphi_{\text{in}} \) described by the inequalities \( (A.7) \), in addition to the parameters used in Fig.3, let us choose \( b_k = b_p = 0.7 \). Then, taking \( k_1 = 1 \), we obtain

\[ 12.6 M_p \approx \varphi_{\text{in}}^{(\text{min})} \leq \varphi_{\text{in}} < \varphi_0 \approx 12.9 M_p \]  

(A.8)

Changing \( k_1 \) from \( k_1 = 1 \) to \( k_1 = 1.1 \) has no significant effect on the width of the interval of \( \varphi_{\text{in}} \) in which the conditions for the beginning of inflation are satisfied.

Finally, we must consider the effect of the last two terms in the effective Lagrangian \( L_{\text{eff}}^{(0)}(\varphi, X_{\varphi}) \), Eq.(54), in the interval described by Eq.(A.8). Adding the values of the parameters \( q = \frac{1}{3} \) and \( \lambda \approx 2 \cdot 10^{-11} \) to the set of parameters \( b_k, b_p \) and \( \zeta \) used in (A.8), it is easy to estimate \( K_1 \) and \( K_2 \). For maximum values of \( K_1 \) and \( K_2 \) we get: \( K_{1,\text{max}} = K_1(\varphi_{\text{in}}^{(\text{min})}) \approx 3.7 \cdot 10^{-5} \) and \( K_{2,\text{max}} = K_2(\varphi_{\text{in}}^{(\text{min})}) \approx 1.3 \). So using also Eqs.(A.4) and (A.6), the contribution of the last two terms to the effective Lagrangian (54) is estimated to be

\[ K_1(\varphi_{\text{in}}) X_{\varphi}^{(in)} + K_2(\varphi_{\text{in}}) \left( \frac{X_{\varphi}^{(in)}}{M_P} \right)^2 \lesssim 3.7 \cdot 10^{-5} \cdot V_{\text{eff}}^{(0)}(\varphi_{\text{in}}), \]  

(A.9)

which is negligible if we take into account the absence of any reliable data on the beginning of inflation.
2. Constraint on the maximum initial value of $|X_\varphi|$ in a model with $b_k \neq 1$ and $b_p \neq 1$ if $X_\varphi < 0$

If $X_\varphi^{(in)} < 0$, i.e., the initial gradient energy density is larger than the initial kinetic energy density, then it is useful to rewrite the constraint \[43\] in the form

$$
\zeta = \zeta(\varphi_{in}, X_\varphi^{(in)}) = \frac{32q^2 \zeta \left(1 + \zeta_n \right) - (2b_p - 1) \frac{\lambda}{4\xi^2} e^{4z_n} - (1 - b_k) \frac{|X_\varphi^{(in)}|}{M_p^2} e^{4z_n}}{32q^2 \left(1 + \zeta_n \right) - \frac{\lambda}{4\xi^2} e^{4z_n} - (1 - b_k) \frac{|X_\varphi^{(in)}|}{M_p^2} e^{4z_n}}, \quad \text{where } z_n = \sqrt{\frac{\lambda \varphi_{in}}{M_p} \gg 1} \quad (A.10)
$$

and, as before, the negligible terms $\propto e^{2z_n}$ are omitted. Again, as in the previous subsection, it is convenient to describe the position of $\varphi_{in}$ relative to $\varphi_0$ defined by Eq. \[43\]. Then it follows from Eq. \[A.10\] that the condition $\zeta > 0$ is satisfied if

$$
\frac{(2b_p - 1) \frac{\lambda}{4\xi^2} (e^{4(z_0 - z_n)} - 1) + (1 - b_k) \frac{|X_\varphi^{(in)}|}{M_p^2}}{\frac{\lambda}{4\xi^2} \left(2b_p - 1 e^{4(z_0 - z_n)} + 1 \right) - (1 - b_k) \frac{|X_\varphi^{(in)}|}{M_p^2}} \equiv \frac{N}{D} > 0, \quad \text{where } z_0 = \sqrt{\frac{\lambda \varphi_0}{M_p}} \quad (A.11)
$$

It turns out that the analysis of restrictions on $\varphi_{in}$ that ensure both $\zeta > 0$ and the condition $|X_\varphi^{(in)}| \lesssim V_{eff}(\varphi_{in})$ should be carried out separately for the cases $\varphi_{in} < \varphi_0$ and $\varphi_{in} > \varphi_0$.

a. The case of the interval $\varphi_{in} < \varphi_0$

Obviously, in this case the numerator $N$ in \[A.11\] is positive. The denominator $D > 0$ if

$$
|X_\varphi^{(in)}| < \frac{\lambda}{4\xi^2 (1 - b_k)} M_p^2 \left[\frac{2b_p - 1}{\zeta_n} \exp \left(4\sqrt{\xi} \frac{(\varphi_0 - \varphi_{in})}{M_p} \right) + 1 \right] \equiv |\hat{X}(\varphi_{in})|. \quad (A.12)
$$

where $|\hat{X}(\varphi_{in})|$ describes the limiting upper bound of admissible values of $|X_\varphi^{(in)}|$ as a function of $\varphi_{in}$. Trying to repeat the analysis similar to that which was made after Eq. \[A.10\], we also notice that $|\hat{X}(\varphi_{in})|$ decreases monotonically as $\varphi_{in}$ increases. But unlike the case when $X_\varphi > 0$, now when $\varphi_{in} \to \varphi_0^-$ the limiting upper bound of admissible values of $|X_\varphi^{(in)}|$, $|\hat{X}(\varphi_{in})|$, tends to not, but

$$
|\hat{X}(\varphi_{in})| \to \frac{\lambda}{4\xi^2 (1 - b_k)} \left(\frac{2b_p - 1}{\zeta_n} + 1 \right) M_p^2 \approx 1.8 \cdot 10^{-9} M_p^4, \quad (A.13)
$$

where the same parameter values were used for the numerical estimation as in the previous subsection. The result obtained shows that the minimum value of $|\hat{X}(\varphi_{in})|$ exceeds the height of the potential plateau. Therefore, in the interval $\varphi_{in} < \varphi_0$, the requirement $\zeta > 0$ is incompatible with condition $|X_\varphi^{(in)}| < V_{eff}(\varphi_{in})$ and, consequently, the initial conditions necessary for the beginning of inflation cannot arise.

b. The case of the interval $\varphi_{in} > \varphi_0$

In this case, to ensure $\zeta > 0$, we have to consider two options: (A) $N > 0, D > 0$; (B) $N < 0, D < 0$.

Let us start from the option (B). From Eq. \[A.11\], as a result of combining the conditions $N < 0$ and $D < 0$, we get a double inequality

$$
\frac{\lambda}{4\xi^2} \left(1 + \frac{2b_p - 1}{\zeta_n} e^{-4(z_n - z_0)} \right) < \frac{1 - b_k}{M_p^2} |X_\varphi^{(in)}| < (2b_p - 1) \frac{\lambda}{4\xi^2} \left(1 - e^{-4(z_n - z_0)} \right), \quad (A.14)
$$

which implies that

$$
(2b_p - 1) \left(\frac{1}{\zeta_n} + 1 \right) e^{-4(z_n - z_0)} < 2(b_p - 1). \quad (A.15)
$$
But for the chosen parameter $b_p = 0.7$, the latter is impossible, and therefore, in the case of option (B), $\zeta$ cannot be positive.

Now let us consider the option (A).

For $\varphi_{in} > \varphi_0$ the numerator $N$ in the inequality (A.11) is positive if

$$\left| \frac{X_{\varphi}^{(in)}}{M_p^2} \right| > \frac{2b_p - 1}{1 - b_k} \frac{\lambda}{4\xi^2} \left( 1 - e^{-4(z_{in} - z_0)} \right).$$

(A.16)

Then for the denominator $D$ we get

$$D > \frac{\lambda}{4\xi^2} \left[ (2b_p - 1) \left( \frac{1}{\zeta_v} + 1 \right) e^{-4(z_{in} - z_0)} + 2(1 - b_p) \right] > 0$$

(A.17)

Therefore $\zeta > 0$ for any $\varphi_{in} > \varphi_0$ if negative $X_{\varphi}^{(in)}$ satisfies the inequality (A.16). This means that, contrary to what we learned earlier, i.e. when $X_{\varphi}^{(in)} \geq 0$, if the gradient energy density prevails over the kinetic energy density, then the second (higher) plateau of $V_{eff}^{(0)}(\varphi)$ may not be an artifact and this becomes possible due to $X_{\varphi}^{(in)} < 0$. However, inflationary expansion in this range of initial values $\varphi_{in}$ and $X_{\varphi}^{(in)}$ can only start if the upper bound on $X_{\varphi}^{(in)}$ is satisfied

$$|X_{\varphi}^{(in)}| \lesssim V_{eff}^{(0)}(\varphi_{in}) \approx \frac{k_2 \lambda (\zeta_v + b_p)}{4\xi^2 (1 + \zeta_v)^2} M_p^4,$$

(A.18)

which is an additional requirement here, in contrast to what was in the case $X_{\varphi}^{(in)} > 0$ considered in subsection 1 of the Appendix. Similar to what was done in Eq.(A.6), a dimensionless factor $k_2 \geq 1$ is introduced to take into account the change in the height of the potential when $\varphi_{in}$ changes in the segment where the plateau passes into a rapidly growing function, see Fig.3. Bearing in mind the shape of the effective TMT potential $V_{eff}^{(0)}(\varphi)$, we see that for $\varphi_{in} > \varphi_0$ it is impossible to exclude in advance the scenario when inflation occurs in two stages: first due to a higher plateau, and then due to a lower plateau. Therefore, in accordance with the graph of $V_{eff}^{(0)}(\varphi)$, the range of parameter $k_2$ should be $1 < k_2 \lesssim 1.3$. The latter estimate, of course, depends on the choice of model parameters. Note also that to estimate $V_{eff}^{(0)}(\varphi_{in})$, instead of the height of the lower plateau multiplied by $k_2$, one can use the height of the higher plateau multiplied by the factor $k_3$ with the appropriate range of values. But this leads to the same result.

Combining (A.16) and (A.18), we get an interval for $\varphi_{in} > \varphi_0$ where: 1) $\zeta > 0$ and 2) there is a possibility that a negative $X_{\varphi}^{(in)}$ satisfies the condition $|X_{\varphi}^{(in)}| \lesssim V_{eff}^{(0)}(\varphi_{in})$

$$\varphi_0 < \varphi_{in} < \varphi_{in}^{(max)}, \text{ where } \varphi_{in}^{(max)} = \varphi_0 - \frac{M_p}{4\sqrt{\xi}} \ln \left( 1 - \frac{k_2 (1 - b_k)(\zeta_v + b_p)}{(2b_p - 1)(1 + \zeta_v)^2} \right).$$

(A.19)

Thus, if $X_{\varphi}^{(in)} < 0$, then inflation is possible only if $\varphi_{in}$ is in the interval (A.19), but unlike the case $X_{\varphi}^{(in)} > 0$, inflation is not guaranteed, since it is not forbidden that $|X_{\varphi}|$ exceeded $V_{eff}^{(0)}(\varphi)$. It is interesting to note that $\zeta$ can also be positive for $\varphi > \varphi_{in}^{(max)}$, but this can only happen if $|X_{\varphi}|$ bounded from below: $|X_{\varphi}| > V_{eff}^{(0)}(\varphi)$.

Using the same parameters as above and the range of $k_2$, we get the following numerical estimate for the interval (A.19)

$$12.9 M_p \approx \varphi_0 < \varphi_{in} < \varphi_0 + 0.6 M_P \approx 13.5 M_P.$$

(A.20)
Moreover, the solutions \( \Phi(x) > 0 \) and for \( \Phi(x) < 0 \) generally contain different integration constants; the case of solutions with equal integration constants is extremely improbable.

Here it is worth noting the difference between the model under study, which uses the volume form \( dV = \Upsilon d^4x \), Eq.(2), defined in terms of four scalar functions \( \varphi_a \), from models that use the generally covariant measure of integration \( dV = \Phi d^4x \) with volume measure density of the form

\[
\Phi = \frac{1}{3!} \epsilon^{\mu \nu \alpha \beta} \partial_\mu A_{\nu \alpha \beta},
\]

where \( A_{\nu \alpha \beta} \) is an auxiliary 3-index antisymmetric tensor gauge field. If the original action contains the term \( \int L \Phi d^4x \) with the corresponding Lagrangian \( L \), then the variation with respect to \( A_{\nu \alpha \beta} \) gives the solution \( L = M = \text{const.} \) without restrictions on the sign of \( \Phi \). Such an approach to choosing an alternative non-Riemannian measure of integration was used, for example, in models of Refs.[36]-[38], where the original actions contain more than one term of this kind with different 3-index antisymmetric tensor gauge fields. Another example of such an approach is the use in Ref.[39] of the alternative non-Riemannian integration measure density on the p-brane world-volume manifold.

[36] D. Benisty, E. I. Guendelman, A. Kaganovich, E. Nissimov, S. Pacheva, "Non-canonical volume-form formulation of modified gravity theories and cosmology", Eur.Phys.J.Plus 136 (2021) 1, 46 [arXiv:2006.04063 [gr-qc]].

[37] D. Benisty, E. I. Guendelman, A. Kaganovich, E. Nissimov, S. Pacheva, "Dynamically generated inflationary two-field potential via non-Riemannian volume forms", Nucl.Phys.B 951 (2020) 114907 [arXiv:1907.07625 [astro-ph.CO]].

[38] E. Guendelman, R. Herrera, D. Benisty, "Unifying inflation with early and late dark energy with multiple fields: Spontaneously broken scale invariant two measures theory", Phys.Rev.D 105 (2022) 12, 124035 [arXiv:2201.06470 [gr-qc]].

[39] E. I. Guendelman, A. Kaganovich, E. Nissimov, S. Pacheva, "Einstein-Rosen 'Bridge' Needs Lightlike Brane Source", Phys.Lett.B 681 (2009) 457-462 [arXiv:0904.3198 [hep-th]].

[40] T. Chiba, T.Okabe and M Yamaguchi, "Kinetically driven quintessence", Phys.Rev.D 62 (2000) 023511 [arXiv:9912463 [astro-ph]].

C. Armendariz-Picon, V. Mukhanov and P.J. Steinhardt, "A Dynamical solution to the problem of a small cosmological constant and late time cosmic acceleration", Phys.Rev.Lett. 85 (2000) 4438-4441 [arXiv:0004134 [astro-ph]].

C. Armendariz-Picon, V. Mukhanov and P.J. Steinhardt, "Essentials of k essence", Phys.Rev.D 63 (2001) 103510 [arXiv:0006373 [astro-ph]].

T. Chiba, "Tracking K-essence", Phys.Rev.D 66 (2002) 063514 [arXiv:0206298 [astro-ph]].

J.K. Erickson, R.R. Caldwell, P.J. Steinhardt, C. Armendariz-Picon, V.F. Mukhanov, "Measuring the speed of sound of quintessence", Phys.Rev.Lett. 88 (2002) 121301 [arXiv:0112438 [astro-ph]].

[41] John M. Lee "Introduction to Smooth Manifolds", Second Edition, Springer Science+Business Media New York , 2013.

[42] S.W. Hawking, G.F.R. Ellis, "The large scale structure of space-time", Cambridge University Press, 1973.