The General Structure of Eigenvalues of Non-linear Oscillators

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Abstract

Hilbert Spaces of bounded one dimensional non-linear oscillators are studied. It is shown that the eigenvalue structure of all such oscillators have the same general form. They are dependent only on the ground state energy of the system and a single functional \( \lambda(H) \) of the Hamiltonian \( H \) whose form depends explicitly on \( H \). It is also found that the Hilbert Space of the non-linear oscillator is unitarily inequivalent to the Hilbert Space of the simple harmonic oscillator, providing an explicit example of Haag’s Theorem. A number operator for the nonlinear oscillator is constructed and the general form of the partition function and average energy of an non-linear oscillator in contact with a heat bath is determined. Connection with the WKB result in the semi-classical limit is made. This analysis is then applied to the specific case of the \( x^4 \) anharmonic oscillator.
I. INTRODUCTION

In this paper we shall study the general structure of the energy eigenvalues for one dimensional non-linear oscillators. To be specific, we are interested in Hamiltonians which have the form,

$$H = \epsilon_0 \left( a^\dagger a + \frac{1}{2} \right) + V(a, a^\dagger),$$

(1)

where $V(a, a^\dagger)$ is the interaction Hamiltonian and is a functional of $a$ and $a^\dagger$, the creation and annihilation operators for the simple harmonic oscillator (SHO), and $\epsilon_0$ is the SHO energy scale. We shall restrict ourselves to bounding potentials for which $V(x) \to \infty$ when $|x| \to \infty$. When $V$ is a polynomial in $a$ and $a^\dagger$ consisting of terms $(a^\dagger)^p a^q$ the degree $l$ of $V$ is the maximum value of $l = p + q$ for the polynomial. As is well known, when $l \leq 2$ the Hamiltonian is easily diagonalizable by either shifting the operator by a constant (for $l = 1$) or by a Bogoluibov transformation (for $l = 2$). Nonetheless, the results of these analyses have had far reaching applications, including coherent and squeezed quantum states \[1\] in quantum optics, and the theories of superfluidity and superconductivity \[2\].

When $l > 2$ the oscillator is usually called anharmonic with the classical example being the oscillator with an $x^4$ or $(a + a^\dagger)^4$ interaction potential. This particular non-linear oscillator has been extensively studied since the early 1970’s (\[3\]-\[6\]; see \[7\] for a review of the literature), due mainly to the equivalence between it and the $\phi^4$ quantum field theory in one-dimension. It is hoped that a detail study of this simplified system will shed some light on the structure of $\phi^4$ theory in higher dimensions. Research on this oscillator continues today, mainly because it provides a natural test bed for such approximation schemes as the strong coupling expansion \[8\], modified perturbations schemes, \[9\], \[10\], variational modified perturbation theories \[11\], lattice methods \[12\], etc. More recently, Bender and Bettencourt \[13\] have provided a deeper understanding of the system by using multiple-scale perturbation theory showing that the frequency of oscillation depends on the energy $H$ of the state. This was interpreted by them as an operator form of mass renormalization.
The main purpose of this paper is not to present a new method of calculating the energy eigenvalues of non-linear oscillators, although we shall end up doing so. Rather, it is to study the general structure of both the Hilbert Space and the energy eigenvalues of non-linear oscillators with arbitrary binding $V(a, a^\dagger)$. The approach we shall take follows most closely the analysis done for the $l \leq 2$ oscillators. Namely, we shall attempt to construct, in much the same way, operators $\tilde{a}$ and $\tilde{a}^\dagger$ from $a$ and $a^\dagger$ which diagonalizes the Hamiltonian. We find that unlike the SHO operators, $\tilde{a}$ and $\tilde{a}^\dagger$ obey the commutation relation $[\tilde{a}, \tilde{a}^\dagger] = \lambda(H)$ where in general $\lambda(H)$ is a functional of $H$. Its precise form depends on the specific choice of $H$ and is a constant only when $l \leq 2$. The study of any non-linear oscillator thereby reduces to the study of operators having this commutation relation along with the determination of $\lambda(H)$ and the groundstate energy of the system.

Because $\lambda(H)$ is not a constant function in general, we find that $\tilde{a}$ and $\tilde{a}^\dagger$ cannot be unitarily equivalent to $a$ and $a^\dagger$. Only in the special case when $l \leq 2$ does such a transformation exists. Consequently, the Hilbert space of the non-linear oscillator is generally unitarily inequivalent to that of the simple harmonic oscillator. This is an explicit example of Haag’s Theorem, first proposed by Haag in 1955 [14] (see also [15]) for quantum field theories. In this paper Haag actually proved a weaker version of the theorem by showing that the unitary transformation between the non-interacting and interacting quantum field theories via the interaction picture does not exist. Later, this result was extended by Hall and Wightman [16] (see also [17]) who showed that based on the Wightman axioms the expectation values of the product of four or fewer fields of an interacting theory are unitarily inequivalent to those of the free theory.

There have been other attempts at using algebraic methods to analyze non-linear oscillators, of course, such as the action angle or time operator methods (see for example [18]- [22]). Both of these methods, however, are generalization of classical analytical techniques to quantum mechanical systems. They rely on the existence of the phase $\phi$ and time $T$ operators which are canonical to the number and Hamiltonian operators: $[N, \phi] = 1$, $[H, T] = 1$. Because of the positivity of the spectrum of both $N$ and $H$ for bound systems,
such operators do not exist in the usual quantum mechanical system [23] (see, however, [24]- [26] for the existence of such operators in extended quantum mechanical systems). In this sense, these methods of solution are “formal”. The approach we have taken in this paper does not suffer from these problems. It is not a generalization of classical techniques but is instead a generalization of the Bogoliubov transformation and is inherently quantum mechanical in nature. Classical solution techniques such as the action angle are used only in the semi-classical limit where they are expected to be valid.

The rest of this paper is organized as follows. In Sec II the general Hilbert Space and energy eigenvalue structure of non-linear oscillators are analyzed. It is found that both depend on a functional $\lambda(H)$ of the Hamiltonian. A number operator is constructed and the Heisenberg equations of motion are solved. Then in Sec III thermal or KMS states are analyzed and it is shown that both the partition function $Z$ and and average energy $\langle H \rangle_T$ for non-linear oscillators are similar in form to those of the SHO. In Sec IV a method of determining $\lambda(H)$ is outlined and in Sec V the connection between this method and the semi-classical WKB result is shown. Application of this analysis to the $x^4$ potential can then be found in Sec VI. Concluding remarks are given in Sec VII.

II. GENERAL STRUCTURE

Given a Hamiltonian $H$ constructed from $a$, and $a^\dagger$, we seek solutions of the operator equation

$$[\hat{a}, H] = \epsilon_0 \lambda(H) \hat{a},$$

(2)

where $\hat{a}$ is understood to be a functional of $a$ and $a^\dagger$. This is an eigenvalue equation with $\hat{a}$ being the “eigenoperator” of $H$ and $\lambda(H)$ its corresponding “left eigenvalue”, although unlike the standard eigenvalue equation $\lambda(H)$ is a functional of $H$ and the ordering in eq. (2) is important. Eq. (2) does not determine $\hat{a}$ uniquely since if $\hat{a}$ satisfies eq. (2), then so does $g(H)\hat{a}$ and $\hat{a}g(H)$ where $g$ is any functional of $H$. A normalization for $\hat{a}$ is needed which we choose to be
\[ H = \epsilon_0 \left( \tilde{a}^\dagger \tilde{a} + e_g \right), \]  
\[ (3) \]

since it diagonalizes the Hamiltonian explicitly. \( \epsilon_0 e_g \) is the ground state energy of the system and is a constant. This is very similar to the way one determines the Bogolubiov transformation which diagonalizes the \( l = 2 \) Hamiltonian \( H = \epsilon_0 a^\dagger a + i\epsilon_1 (a^2 - (a^\dagger)^2)/2 \), but now \( \lambda(H) \) is a functional of \( H \). With this normalization, eq. (2) reduces to

\[ [\tilde{a}, \tilde{a}^\dagger] = \lambda(H). \]

\[ (4) \]

To show that \( \tilde{a} \) and \( \tilde{a}^\dagger \) creates and annihilates eigenstates of \( H \), we make use of the identity,

\[ [\tilde{a}, H^n] = \left\{ (\lambda(H) + H/\epsilon_0)^n - (H/\epsilon_0)^n \right\} \tilde{a}. \]

\[ (5) \]

obtained using eq. (4). Then for any given functional \( M(H) \) which is expandable in a Taylor series,

\[ [\tilde{a}, M(H)] = \left\{ M(\lambda(H) + H/\epsilon_0) - M(H) \right\} \tilde{a}. \]

\[ (6) \]

With this we see that if \( \tilde{a} \) is an eigenoperator of \( H \) with left eigenvalue \( \lambda(H) \), then so is \( \tilde{a}^n \),

\[ [\tilde{a}^n, H] = \epsilon_0 \left\{ \lambda(H) + \lambda(\lambda(H) + H/\epsilon_0) + \cdots + \lambda(\lambda(\cdots \lambda(H + H/\epsilon_0) + H/\epsilon_0) + H/\epsilon_0) \right\} \tilde{a}^n. \]

\[ (7) \]

Given eq. (4), the Hilbert Space \( \mathcal{H}_{nl} \) for non-linear oscillators and their energy eigenvalues are easily constructed in much the same way as the SHO Hilbert Space \( \mathcal{H}_{SHO} \). Namely, if \( |\tilde{\phi}\rangle_{nl} \) is an eigenstate of \( H \), then so is \( \tilde{a}^n |\tilde{\phi}\rangle_{nl} \) as well as \( (\tilde{a}^\dagger)^n |\tilde{\phi}\rangle_{nl} \). Since the spectrum of the operator \( \tilde{a}^\dagger \tilde{a} \) must be non-negative it is straightforward to show (see Appendix A) that if \( \lambda \) is a positive definite function then there exists a state (the groundstate) \( |\Omega\rangle_{nl} \) in \( \mathcal{H}_{SHO} \) for which \( \tilde{a} |\Omega\rangle_{nl} = 0 \) and has energy \( \epsilon_0 e_g \). \( \mathcal{H}_{nl} \) is therefore spanned by the states

\[ |n\rangle_{nl} = \frac{(\tilde{a}^\dagger)^n |\Omega\rangle_{nl}}{\sqrt{A_n}}, \]

\[ (8) \]
where
\[ A_n = \lambda(e_g) \cdot \left( \lambda(e_g) + \lambda(\lambda(e_g) + e_g) \right) \cdot \cdots \cdot \left( \lambda(e_g) + \lambda(\lambda(e_g) + e_g) + \cdots + \lambda(\lambda(\cdots \lambda(\lambda(e_g) + e_g) \cdots + e_g) + e_g) \right). \] (9)

They are eigenstates of \( H \) with eigenvalues \( \epsilon_0 e_n \) where
\[ e_n = e_{n-1} + \lambda(e_{n-1}) = e_g + \lambda(e_g) + \lambda(\lambda(e_g) + e_g) + \cdots + \lambda(\cdots \lambda(\lambda(e_g) + e_g) \cdots + e_g) + e_g. \] (10)

\( \lambda \) thereby determines the splitting between successive energy levels.

If \( \lambda(H) \) is a constant, then from eq. (10) we see that the energy levels of the oscillator are equally spaced. As we shall show in the next section, this is only possible for \( l \leq 2 \), which is well known. When \( l > 2 \), \( \lambda(H) \) is a functional of \( H \) and this equal spacing no longer occurs.

Notice, however, that both eq. (3) and the commutation relation eq. (4) are invariant under unitary transformations: \( \tilde{a} \rightarrow U\tilde{a}U^\dagger \). As usual, unitary transformations are canonical transformations which preserves the commutation relation. For the SHO, \( \lambda = 1 \), while for a non-linear oscillator \( \lambda(H) \) is a functional of \( H \). Since a unitary transformation cannot change the functional form of \( \lambda \), \( \tilde{a} \) and \( a \) are unitarily inequivalent. Consequently, the Hilbert Spaces \( H_{nl} \) and \( H_{SHO} \) are unitarily inequivalent Hilbert Spaces.

It is well known that under certain conditions any two solutions of the canonical commutation relation \( [a, a^\dagger] = 1 \) is connected by an unitary transformation. Our result does not contradict this. The operators \( \tilde{a} \) and \( \tilde{a}^\dagger \) which create and annihilate eigenstates of a general one dimensional oscillator do not in general obey the canonical commutation relation except in the special case of \( l \leq 2 \). Instead, \( [\tilde{a}, \tilde{a}^\dagger] = \lambda(H) \). Indeed, it is precisely for this reason that \( \tilde{a} \) and \( a \) cannot be related to one another by an unitary transformation.

If we consider now Hamiltonians of the form eq. (1) in which \( V(a, a^\dagger) \) is controlled by a single coupling constant \( \epsilon_1 \), we can label the Hilbert Space for each \( \epsilon_1 \) as \( H_{\epsilon_1} \). Then \( H_{\epsilon_1} \) is unitarily inequivalent to \( H_{SHO} \). Moreover, there cannot be a unitary transformation which maps \( H_{\epsilon_1} \rightarrow H_{\epsilon'_1} \) when \( \epsilon_1 \neq \epsilon'_1 \). If there were, then using a succession of these
transformations we can construct a unitary transformation mapping $\mathcal{H}_{\epsilon_1}$ to $\mathcal{H}_{SHO}$ and the two Hilbert Spaces would be unitarily equivalent. Thus for different values of $\epsilon_1$ the Hilbert Spaces $\mathcal{H}_{\epsilon_1}$ are inequivalent to one another. This is a concrete example of Haag’s Theorem (see [13]), first proved for quantum field theories using Lorentz invariance.

Because the spectrum of $\hat{a}^\dagger\hat{a}$ no longer consists of the non-negative integers, but instead depends on the energy of the state, $\hat{a}^\dagger\hat{a}$ cannot in general be interpreted as the number operator $N$ for non-linear oscillators. We shall construct this operator explicitly by first noting that $H$ does not change the occupation number $n$ of the states $|n\rangle_{nl}$. Consequently, $[N,H]=0$ with the subsidiary condition that $N|\Omega\rangle_{nl}=0$. $N(H)$ can be a functional of $H$ only. From eq. (8), it is straightforward to see that it must satisfy the commutation relation

$$[\hat{a}, N(H)] = \hat{a}.$$  \hfill (11)

Expanding $N(H)$ in a power series and once again using eq. (6), the solution to the operator equation eq. (11) reduces to finding the solution of the algebraic equation

$$N(\lambda(e) + e) - N(e) = 1 ,$$  \hfill (12)

for a given $\lambda$ with the “boundary condition” $N(e_g) = 0$ (see Appendix B). $e$ is a real number in eq. (12) and $N(H)$ is obtained by replacing $\epsilon_0 e \rightarrow H$. (Or, equivalently, eq. (12) is the resultant equation after applying the corresponding operator equation to an eigenstate of $H$ with energy $\epsilon_0 e$.)

Like the differential equation it resembles, the general solution of eq. (12) consists of the linear combination $N_p(H) + N_h(H)$ where $N_p(e)$ is the “particular” solution to eq. (12) while $N_h(e)$ is the solution to the “homogeneous” equation

$$N_h(\lambda(e) + e) - N_h(e) = 0 .$$  \hfill (13)

Whether or not such a solution exists depends on the particular form of $\lambda(e)$, although the above equations can be solved for general $\lambda$ in the semi-classical limit as we shall see in Sec. V. Unlike a differential equation, however, the single boundary condition $N(e_g) = 0$
is not sufficient to determine \(N(H)\) uniquely in general. Consider the case of the SHO.
Then \(\lambda = 1\) and the particular solution of eq. (12) gives \(N_p(H) = H/\epsilon_0 - I/2\) and is, in fact, the usual number operator. However, the solution of the homogeneous eq. (13) is any periodic function with period 1 which vanishes at \(e = e_g\). There are an infinite number of such functions, such as

\[
N(H) = H/\epsilon_0 - e_g I + C \sin(\pi(H/\epsilon_0 - e_g I)) .
\] (14)

for any real \(C\).

From eqs. (3) and (4) the study of non-linear oscillators reduces to the determination of the groundstate energy \(e_g\) and the functional \(\lambda(H)\). This is non-trivial and a method for doing so will be given in the Sec IV. For now we shall limit ourselves to a qualitative description of the energy levels by looking at different possible behaviors of \(\lambda(e)\).

For a ground state to exist, \(\lambda(e_g) > 0\) and we shall restrict our considerations to such \(\lambda\). Representing the eigenvalues of \(H/\epsilon_0\) generically by \(e\), if \(\lambda(e)\) is a monotonically increasing function which is unbounded from above, then the energy spacings between successive energy levels becomes wider as \(n\) increases and \(e_n\) grows rapidly with \(n\). If, on the other hand, \(\lambda(e) \to \text{constant as } e \to \infty\), then eventually the energy levels become equally spaced and we would once again obtain SHO type of energy levels. Notice also that if we consider eq. (10) as a non-linear transformation of \(e_n\) generated by \(\lambda(e)\), then the fix point of this transformation \(\lambda(\lambda(e) + e) = \lambda(e)\) occurs precisely when \(\lambda(e)\) goes to a constant (see Appendix B). Finally, if \(\lambda(e)\) is a monotonically decreasing function of \(e\) which decreases sufficiently rapidly, there will be an upper bound to the energy levels \(e_{\text{max}}\).

Finally, let us consider time evolution. If \(H\) does not explicitly depend on \(t\), time evolution is generated by a unitary transformation

\[
\tilde{a}(t) = e^{itH/\hbar} \tilde{a}(0)e^{-itH/\hbar},
\] (15)

which preserves the commutation relation eq. (4). Using eq. (2), the solution to the Heisenberg equation of motion is

\[
N_{\text{Heis}}(t) = N_{\text{Heis}}(0) e^{itH/\hbar},
\] (16)
\( \tilde{a}(t) = e^{-i\epsilon_0 \lambda(H)t/\hbar} \tilde{a}(0) \). \hspace{1cm} (16)

The frequency of oscillation of \( a(t) \), \( \epsilon_0 \lambda(H)/\hbar \), now depends on the Hamiltonian \( H \). This agrees with the recent result of Bender and Bettencourt \cite{13} and was interpreted by them as an operator form of mass renormalization. \( A_n \)'s dependence in eq. (9) on the energy of the state would then be wavefunction renormalization.

### III. KMS STATES

We now put the non-linear oscillator in contact with a thermal reservoir at a temperature \( T \) and consider the average energy and number density of the system. We shall denote these thermal averages by \( \langle \cdots \rangle_T \) which we shall take to be a KMS state \cite{30} - \cite{32}. Namely, if \( A(t) \) and \( B(t) \) are two operators in the Heisenberg representation, then

\[
\langle A(t) B(t) \rangle_T = \langle B(t) A(t + i\hbar \beta) \rangle_T ,
\]

where \( 1/\beta = k_B T \). Since thermal equilibrium states are stationary, we can without a loss of generality take \( t = 0 \) in eq. (17).

Applying this condition to eq. (4),

\[
\langle \lambda(H) \rangle_T = \langle \tilde{a}(0) \tilde{a}^\dagger(0) \rangle_T - \langle \tilde{a}(0) \tilde{a}^\dagger(i\hbar \beta) \rangle_T .
\]

Then using the solution eq. (16) of the Heisenberg equation of motion, eq. (2) and the commutation relation eq. (4),

\[
\langle \lambda(H) \rangle_T = \left( \langle H/\epsilon_0 - \epsilon_g \rangle \left( 1 - e^{-\epsilon_0 \lambda(H)/\hbar} \right) \right)_T ,
\]

which reduces to the usual Bose-Einstein distribution for the SHO when \( \lambda = 1 \).

Unlike the case of the SHO it is not possible to determine \( \langle H \rangle_T \) any further using solely the KMS condition. We must make use of a partition function and shall restrict ourselves to states which can be represented by a trace over a density matrix,

\[
\langle H \rangle_T = \frac{1}{Z} \text{Tr}_n H e^{-\beta H} ,
\]

\( 9 \)
where $Z \equiv \text{Tr}_{\mathcal{H}_{nl}} e^{-\beta H}$ is the usual partition function. Then using the identity

$$
\langle [H + \epsilon_0 \lambda(H)] e^{-\epsilon_0 \beta \lambda(H)} \rangle_T = \langle e^{-\epsilon_0 \beta \lambda(H)} \rangle_T \langle H \rangle_T - \frac{\partial}{\partial \beta} \langle e^{-\epsilon_0 \beta \lambda(H)} \rangle_T ,
$$

(21)

we obtain

$$
\langle H \rangle_T = \epsilon_0 e_g - \frac{\frac{\partial}{\partial \beta} \langle e^{-\epsilon_0 \beta \lambda(H)} \rangle_T}{1 - \langle e^{-\epsilon_0 \beta \lambda(H)} \rangle_T} ,
$$

(22)

and we see once again the importance of $\lambda(H)$. Indeed, from eq. (20) we find that

$$
Z = \frac{e^{-\beta_0 e_g}}{1 - \langle e^{-\epsilon_0 \beta \lambda(H)} \rangle_T} .
$$

(23)

As for the number operator, from eq. (12),

$$
\langle e^{-\epsilon_0 \beta \lambda(H)} \rangle_T = \langle N(H + \epsilon_0 \lambda(H)) e^{-\epsilon_0 \beta \lambda(H)} \rangle_T - \langle N(H) e^{-\epsilon_0 \beta \lambda(H)} \rangle_T .
$$

(24)

Then using

$$
\langle N(H + \epsilon_0 \lambda(H)) e^{-\epsilon_0 \beta \lambda(H)} \rangle_T = \frac{1}{Z} \text{Tr}_{\mathcal{H}_{nl}} N(H + \epsilon_0 \lambda(H)) e^{-\beta H - \epsilon_0 \beta \lambda(H)} ,
$$

(25)

and eq. (10), we find that

$$
\langle e^{-\epsilon_0 \beta \lambda(H)} \rangle_T = \langle N(H) \rangle_T
$$

and

$$
\langle N(H + \epsilon_0 \lambda(H)) e^{-\epsilon_0 \beta \lambda(H)} \rangle_T = \langle N(H) \rangle_T ,
$$

so that

$$
\langle e^{-\epsilon_0 \beta \lambda(H)} \rangle_T = \left\langle N(H) \left(1 - e^{-\epsilon_0 \beta \lambda(H)}\right) \right\rangle_T .
$$

(26)

This once again agrees with the SHO result for $\lambda = 1$.

**IV. SOLUTION OF THE EIGENVALUE PROBLEM**

$\tilde{a}$ and $\lambda(H)$ can be determined in the following manner. Since $H$ is given in terms of $a$ and $a^\dagger$, in general $\tilde{a} = \tilde{a}(a, a^\dagger)$, which is understood in terms of a power series,

$$
\tilde{a} = \sum_{r,s=0}^{\infty} b_{rs} (a^\dagger)^r a^s .
$$

(27)

By using the commutation relation $[a, a^\dagger] = 1$, we can always reduce any expansion of $\tilde{a}$ to this form. Eq. (27) is well defined only if the corresponding function
\[ f(z, \bar{z}) = \sum_{r,s=0}^{\infty} b_{rs}z^r \bar{z}^s, \quad (28) \]

is absolutely convergent on \( \mathbb{R}^2 \).

At this point we should also express \( \lambda(H) \) as a power series in \( H/\epsilon_0 \), insert this series and as well as eq. (27) in eq. (2) and obtain an infinite set of coupled equations between various \( b_{rs} \) and the coefficients of the \( \lambda \) expansion. The problem would quickly become intractable, however. We shall therefore first make the following drastic simplification. Instead of eq. (2) we shall solve the simpler equation

\[ : [\tilde{a}, H] := \epsilon_0 \lambda(H) \tilde{a}, \quad (29) \]

where \( : \) denotes normal ordering. Correspondingly, we shall take the normalization condition as

\[ : H := \epsilon_0 \left( : \tilde{a}^+ \tilde{a} + \frac{1}{2} \right). \quad (30) \]

We shall then use the solution of this equation as a guide to reconstructing the solution to eq. (2). Notice that corrections to the groundstate energy cannot be determined under this simplification and can only be determined when the full operators are reconstructed from the solution to eq. (29).

Denoting the solution to eq. (29) by the superscript \( sc \), we find that for

\[ \tilde{a}^{(sc)} = \sum_{r,s=0}^{\infty} b^{(sc)}_{rs}(a^+)^r a^s, \quad (31) \]

we have

\[
\epsilon_0 \sum_{r,s=0}^{\infty} b^{(sc)}_{rs} \lambda(H)(a^+)^r a^s = \sum_{r,s=0}^{\infty} b^{(sc)}_{rs} \left\{ \epsilon_0 (s-r)(a^+)^r a^s + s : (a^+)^r a^{s-1} [a, V(a, a^+)] : + \right. \\
\left. r : (a^+)^{r-1} [a^+, V(a, a^+)] a^s : \right\}. \quad (32)
\]

Under this normal ordering, solving eq. (32) is equivalent to solving the differential equation,

\[
\lambda^{(sc)} \left( e^{(sc)} \right) f^{(sc)}(z, \bar{z}) = \{ e^{(sc)}, f^{(sc)} \}_{PB} \equiv \frac{\partial e^{(sc)}}{\partial \bar{z}} \frac{\partial f^{(sc)}}{\partial z} - \frac{\partial e^{(sc)}}{\partial z} \frac{\partial f^{(sc)}}{\partial \bar{z}}, \quad (33)
\]
where $\tilde{a}^{(sc)} = f^{(sc)}(a, a^\dagger)$. $e^{(sc)}$ is obtained from $H/\epsilon_0$ by replacing everywhere $a \rightarrow z$ and $a^\dagger \rightarrow \bar{z}$ and the normalization condition (30) is now $e^{(sc)} = |f^{(sc)}|^2 + 1/2$. The right hand side of eq. (33) is just the classical Poisson bracket but with the generalized coordinates

$$z = \left(\frac{m\epsilon_0}{2\hbar^2}\right)^{1/2} x + i \left(\frac{1}{2m\epsilon_0}\right)^{1/2} p, \quad \bar{z} = \left(\frac{m\epsilon_0}{2\hbar^2}\right)^{1/2} x - i \left(\frac{1}{2m\epsilon_0}\right)^{1/2} p. \quad (34)$$

where $m$ is the mass of the particle. We are therefore looking for a semi-classical solution to eq. (3). Indeed, we shall see explicitly in Sec. V that the solution of eq. (33) is equivalent to the WKB approximation.

Importantly, eq. (33) has the same symmetry properties as eq. (2). Namely, if $f^{(sc)}(z, \bar{z})$ is a solution to eq. (33), then so is $f^{(sc)}(z, \bar{z})g(e^{(sc)})$ where $g(e^{(sc)})$ is any function of $e^{(sc)}$ (although they do not satisfy the normalization condition eq. (30)). Making use of this symmetry, we change coordinates to $e^{(sc)}$ and $\theta = -i \log(z/\bar{z})/2$ from $|z|$ and $\theta$. Then eq. (33) reduces to

$$i\lambda^{(sc)} (e^{(sc)}) f^{(sc)} (e^{(sc)}, \theta) = \frac{\partial e^{(sc)}}{\partial |z|^2} \frac{\partial f^{(sc)}}{\partial \theta}, \quad (35)$$

whose solution is

$$f^{(sc)}(e^{(sc)}, \theta) = \sqrt{e^{(sc)} - 1/2} \exp \left\{ i\lambda^{(sc)} \int_0^\theta \left( \frac{\partial e^{(sc)}}{\partial |z|^2} \right)^{-1} d\phi \right\} \quad (36)$$

and satisfies the normalization condition explicitly. Determination of $\tilde{a}^{(sc)}$ is then reduced to performing the above integral, which requires inverting the equation $e^{(sc)} = e^{(sc)}(|z|, \theta)$ and solving for $|z|$ in terms of $e^{(sc)}$ and $\theta$. Next, for $f^{(sc)}$ to be analytic on $\mathbb{R}^2$, $f^{(sc)}(e^{(sc)}, 0) = f^{(sc)}(e^{(sc)}, 2\pi)$, giving

$$\frac{1}{\lambda^{(sc)} (e^{(sc)})} = \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{\partial e^{(sc)}}{\partial |z|^2} \right)^{-1} d\phi, \quad (37)$$

which determines $\lambda^{(sc)}$. Notice that in contrast to phase angle techniques which require the construction of a phase operator (see, for example [22]) and its concomitant difficulties, our analysis uses the phase only in the semi-classical limit where it is well defined.

Reconstruction of $\tilde{a}$ and $\lambda(H)$ from $f^{(sc)}(z, \bar{z})$ is now straightforward, although tedious. $\tilde{a}^{(sc)}$ can be obtained by first expanding $f^{(sc)}$ in eq. (36) in a power series in $z$ and $\bar{z}$, then
taking $\tilde{a}^{(sc)} = f^{(sc)}(a, a^\dagger)$. Since $\tilde{a}^{(sc)}$ was obtained via normal ordering, there is an ordering ambiguity when we reconstruct $\tilde{a}$ from it. Fundamentally, this arises when we replace $z \rightarrow a$, $\bar{z} \rightarrow a^\dagger$ in $f^{(sc)}(a, a^\dagger)$ since the term $\bar{z}z$ in the expansion can be replaced by either $a^\dagger a$ or $aa^\dagger = 1 + a^\dagger a$. Therefore, to determine $\tilde{a}$ we shall take $\tilde{a} = \{f^{(sc)}(a, a^\dagger)\}_{\text{order}}$, but we now replace $\bar{z}z \rightarrow a^\dagger a + A$ where the $A$'s are constants. These are determined by requiring that the resulting expansions for $\tilde{a}$ and $\lambda(H)$ satisfy both eq. (2) and eq. (3) (or equivalently eq. (3)) term by term in the expansion. This uniquely determines not only $\tilde{a}$ and $\lambda(H)$, but $e_g$ as well.

From eq. (37) we see that for $\lambda(H)$ to be independent of $H$, $\partial e^{(sc)}/\partial |z|^2 = k$, where $k$ is a function of $\theta$ only. This limits $l \leq 2$. Correspondingly, if $l > 2$, $\lambda(H)$ is necessarily a functional of $H$.

V. THE WKB APPROXIMATION

We now make the connection between the solution of eq. (33) and the semi-classical limit. From the correspondence principle, in the large $n$ limit $e_n$ goes over to the classical result. The spacings between energy levels $e_n - e_{n-1}$ are small in comparison to $e_{n-1}$ and the levels are essentially continuous. In this limit, we can then approximate

$$e_n - e_{n-1} \approx \frac{de}{dn},$$

(38)

where $e(n)$ considered as a continuous function of $n$. Then from eq. (10)

$$\frac{de}{dn} \approx \lambda^{(sc)}(e),$$

(39)

where we have replaced $\lambda \rightarrow \lambda^{(sc)}$ in this limit. Integrating and using eqs. (34) and (37),

$$n + n_\infty \approx \frac{1}{2\pi} \int_0^e \int_0^{2\pi} \left( \frac{\partial e^{(sc)}}{\partial |z|^2} \right)^{-1} de^{(sc)} d\theta,$$

(40)

where $n_\infty$ is an integration constant which can be neglected in the limit $n \rightarrow \infty$. Changing variables back to $z$ and $\bar{z}$ in the integrand of eq. (40) and using eq. (34), we find that
\begin{equation}
\int \int_{D_{\epsilon_0}} dx dp. \tag{41}
\end{equation}

The integration is now over a disk $D_{\epsilon_0}$ centered about the origin in the classical phase space. This is just the semi-classical Bohr-Sommerfeld quantization rule obtained from the WKB approximation.

We next consider the solution of the algebraic eq. (12) in the large $n$ limit. Since $\lambda(\epsilon)$ measures the energy splitting between energy levels, in this limit $\lambda(\epsilon) \ll \epsilon$ (see Appendix A) and eq. (9) can be approximated by the differential equation

\begin{equation}
\lambda^{(sc)}(\epsilon) \frac{dN}{de} \approx 1, \tag{42}
\end{equation}

where once again we have replaced $\lambda \to \lambda^{(sc)}$. The solution to this equation is trivial and we once again obtain the WKB result,

\begin{equation}
N(\epsilon) + n_\infty \approx \frac{1}{2\pi \hbar} \int \int_{D_{\epsilon_0}} dx dp. \tag{43}
\end{equation}

Notice, however, that now the operator $N(H)$ can now be obtained directly from eq. (43) by expanding the integral in powers of $\epsilon$ and replacing $\epsilon \to H/\epsilon_0$.

Finally, we consider the quantum partition function

\begin{equation}
Z \equiv \text{Tr}_{H_{nl}} e^{-\beta H} = \sum_{n=0}^{\infty} e^{-\beta \epsilon_n}, \tag{44}
\end{equation}

in the large temperature limit. Making use of the Euler-Maclaurin formula,

\begin{equation}
Z \approx \int_{0}^{\infty} e^{-\beta \epsilon_0} dn + \frac{1}{2} e^{-\beta \epsilon_0} + O(e^{-\beta \epsilon_0}). \tag{45}
\end{equation}

In the large temperature limit $k_B T \gg \epsilon_0 e_g$ we can neglect the terms $\sim e^{-\beta \epsilon_0}$. Moreover, at this energy scale $k_B T$, $\epsilon_n \gg \lambda(\epsilon_n)$. Then using eq. (39), we convert the integral over $n$ to one over $\epsilon$. Making use once again of eq. (34), we find that

\begin{equation}
Z \approx \frac{1}{2\pi \hbar} \int \int e^{-\beta \epsilon_0} dx dp, \tag{46}
\end{equation}

where the integral is over the classical phase space. This is precisely the classical result with the requisite factor of the fundamental phase space volume $2\pi \hbar$. 

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VI. THE $x^4$ INTERACTION

In this section we shall apply the above analysis to a non-trivial system: the $x^4$ anharmonic oscillator,

$$H = \epsilon_0 \left( a^4 + \frac{1}{2} \right) + \frac{\epsilon_1}{4} (a + a^4)^4,$$

(47)

which corresponds to

$$e^{(sc)} = \frac{1}{2} + |z|^2 + \frac{4\epsilon_1}{\epsilon_0} |z|^4 \cos^4 \theta.$$

(48)

Then

$$f^{(sc)}(e^{(sc)}, \theta) = \sqrt{e^{(sc)} - 1/2} \exp \left\{ \frac{\pi i}{2} \frac{I(\theta)}{I(\pi/2)} \right\},$$

(49)

where

$$I(\theta) = \int_0^\theta \frac{d\phi}{\sqrt{1 + \xi \cos^4 \phi}},$$

(50)

and $\xi = 16(e^{(sc)} - 1/2)\epsilon_1/\epsilon_0$. When $\epsilon_1 > 0$, this integral can be reduced to

$$I(\theta) = \frac{1}{2(1 + \xi)^{1/4}} F(\alpha|q),$$

(51)

where $F(\alpha|q)$ is the elliptical integral of the first kind and

$$\alpha = \arccos \left( \frac{\sqrt{1 + \xi} - \tan^2 \theta}{\sqrt{1 + \xi + \tan^2 \theta}} \right),$$

(52)

while

$$q = \frac{\sqrt{1 + \xi} - 1}{2\sqrt{1 + \xi}},$$

(53)

is its modulus. The analyticity of $f^{(sc)}(e^{(sc)}, \theta)$ gives

$$\lambda^{(sc)}(e^{(sc)}) = \frac{\pi}{2} \frac{(1 + \xi)^4}{K(\sqrt{q})},$$

(54)

where $K(\sqrt{q})$ is the quarter period of $F(\alpha|q)$. 

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Although our analysis is strictly valid only when $\epsilon_1 > 0$, it is instructive to see what happens for $\epsilon_1 < 0$. For $|\xi| \leq 1$,

$$I(\theta) = \frac{1}{\sqrt{1 + \sqrt{|\xi|}}} F(\alpha' | q') , \quad (55)$$

where now

$$\alpha' = \arctan \left( \frac{\tan \theta}{\sqrt{1 + \sqrt{|\xi|}}} \right) , \quad (56)$$

and

$$q' = \frac{2\sqrt{|\xi|}}{1 + \sqrt{|\xi|}} . \quad (57)$$

Then

$$\lambda^{(sc)}(e) = \frac{\pi}{2} \sqrt{1 + \sqrt{|\xi|}} K(\sqrt{q'}) , \quad (58)$$

which vanishes when $|\xi| = 1$.

When $|\xi| > 1$, $I(\theta)$ is complex and $f^{(sc)}$ no longer satisfies the normalization condition eq. (30). $\lambda^{(sc)}$ is ill-defined. Consequently, the energy states are bounded, as is well known, by $e < 1/2 + \epsilon_0/(16|\epsilon_1|)$.

To determine $\tilde{a}$, we expand eq. (49) to third order in $|z|^3$,

$$f^{(sc)}(z, \bar{z}) = z + \frac{1}{4\epsilon_0} \left\{ -3(z^2 - \bar{z}^2)z + (z + \bar{z})^3 \right\} + \frac{1}{2} \left( \frac{\epsilon_1}{\epsilon_0} \right)^2 \left\{ \frac{3}{2} z^5 + \frac{39}{4} \bar{z} z^4 - \frac{25}{8} \bar{z}^2 z^3 - 12 \bar{z}^3 z^2 - \frac{3}{8} \bar{z}^4 z + \frac{1}{4} \bar{z}^5 \right\} . \quad (59)$$

We then replace $z \rightarrow a$ and $\bar{z} \rightarrow a^\dagger$ in the above and take

$$\tilde{a} = a + \frac{1}{4\epsilon_0} \frac{\epsilon_1}{\epsilon_0} F + \frac{1}{2} \left( \frac{\epsilon_1}{\epsilon_0} \right)^2 G , \quad (60)$$

where

$$F = -3(a^2 - (a^\dagger)^2)a + (a + a^\dagger)^3 + f_1 a + f_2 a^\dagger ,$$

$$G = \frac{3}{2} a^5 + \frac{39}{4} a^\dagger a^4 - \frac{25}{8} (a^\dagger)^2 a^3 - 12(a^\dagger)^3 a^2 - \frac{3}{8} (a^\dagger)^4 a + \frac{1}{4} (a^\dagger)^5 + g_1 a^3 + g_2 a^\dagger a^2 + g_3 (a^\dagger)^2 a + g_4 (a^\dagger)^3 + g_5 a + g_6 a^\dagger . \quad (61)$$
The constants $f_1$, $f_2$, $g_1 - g_6$ are present due to the ordering ambiguity. Requiring that eq. (60) satisfies eq. (2) gives $f_1 = f_2 = 3$, while

$$g_1 + g_3 = -15, \quad g_2 = -\frac{135}{8}, \quad g_4 = -\frac{3}{8}, \quad g_5 = -\frac{153}{8}, \quad g_6 = -\frac{27}{2}. \quad (62)$$

The groundstate energy of the oscillator is also determined to this order,

$$e_g = \frac{1}{2} + \frac{3 \epsilon_1}{4 \epsilon_0} - \frac{21}{8} \left(\frac{\epsilon_1}{\epsilon_0}\right)^2. \quad (63)$$

Using now the commutation relation eq. (4), we obtain

$$\lambda(H) = I + 3 \left(\frac{\epsilon_1}{\epsilon_0}\right) \left(\frac{H}{\epsilon_0} + \frac{I}{2}\right) - \left(\frac{\epsilon_1}{\epsilon_0}\right)^2 \left\{ \frac{69}{4} \left(\frac{H}{\epsilon_0} + \frac{I}{2}\right)^2 - \frac{9}{2} \left(\frac{H}{\epsilon_0} + \frac{I}{2}\right) + \frac{15}{2} \right\}, \quad (64)$$

while

$$3g_1 + g_3 = \frac{45}{2}, \quad (65)$$

giving $g_1 = 75/4$ and $g_3 = -135/4$. This last relationship was obtained by requiring that $\lambda$ is a function of $H$ only. To this order then,

$$\tilde{a} = a + \frac{1}{4} \frac{\epsilon_1}{\epsilon_0} \left\{ -3(a^2 - (a^\dagger)^2)a + (a + a^\dagger)^3 + 3(a + a^\dagger) \right\} +$$

$$\frac{1}{2} \left(\frac{\epsilon_1}{\epsilon_0}\right)^2 \left\{ \frac{3}{2} a^5 + \frac{39}{4} a^4 a^2 - \frac{25}{8} (a^\dagger)^2 a^3 - 12(a^\dagger)^3 a^2 - \frac{3}{8} (a^\dagger)^4 a + \frac{1}{4} (a^\dagger)^5 + \right.$$  

$$\left. \frac{75}{4} a^3 - \frac{135}{8} a^2 - \frac{135}{4} (a^\dagger)^2 a - \frac{3}{8} (a^\dagger)^3 - \frac{153}{8} a - \frac{27}{2} a^\dagger \right\}. \quad (66)$$

The energy levels can now be straightforwardly calculated from eq. (10),

$$e_n = e_{n-1} + \lambda(e_{n-1}),$$

$$= e_g + \sum_{r=0}^{n-1} \lambda(e_r). \quad (67)$$

Using eq. (64), and keeping terms to order $(\epsilon_1/\epsilon_0)^2$ only, we obtain after re-arrangement,

$$e_n = e_g + n + \frac{3}{4} \frac{\epsilon_1}{\epsilon_0} \sum_{r=0}^{n-1} (r + 1)$$

$$- \left(\frac{\epsilon_1}{\epsilon_0}\right)^2 \left( \frac{51}{4} \sum_{r=0}^{n-1} r^2 + \frac{51}{2} \sum_{r=0}^{n-1} r + 18n + \frac{21}{8} \right). \quad (68)$$
Then
\[ e_n = n + \frac{1}{2} + \frac{3\epsilon_1}{4\epsilon_0}(2n^2 + 2n + 1) - \left(\frac{\epsilon_1}{\epsilon_0}\right)^2 \left(\frac{17}{4}n^3 + \frac{51}{8}n^2 + \frac{59}{8}n + \frac{21}{8}\right), \]  
which is the standard second order perturbation result. Notice also that if we keep terms only up to \( \epsilon_1/\epsilon_0 \), then \( e_n - e_{n-1} \equiv \lambda(e_n) \approx 1 + 3n\epsilon_1/\epsilon_0 \). This is precisely the result obtained by Bender and Bettencourt [13].

It is instructive to compare eq. (64) with the expansion of eq. (54),
\[ \lambda^{(sc)}(e) = 1 + 3 \left(\frac{\epsilon_1}{\epsilon_0}\right) (e^{(sc)} - 1/2) - \frac{69}{4} \left(\frac{\epsilon_1}{\epsilon_0}\right)^2 (e^{(sc)} - 1/2)^2. \] (70)
Notice that in both expansions the coefficients of the highest power of the energy in each term are the same. This is a generic feature. Quantum mechanical corrections to \( \lambda^{(sc)} \) only results in the appearance of lower powers of \( H/\epsilon_0 \) in each term of the expansion. Moreover, if we then use \( \lambda^{(sc)} \) to calculate \( e_n \), we find
\[ e_n^{(sc)} = n + \frac{1}{2} + \frac{3\epsilon_1}{2\epsilon_0}(n^2 - n) - \left(\frac{\epsilon_1}{\epsilon_0}\right)^2 \left(\frac{17}{4}n^3 - \frac{33}{8}n^2 - \frac{1}{8}n\right), \] (71)
which also agrees with eq. (67) in the large \( n \) limit. This also is a generic feature of the expansion since the coefficient of the highest power of \( n \) in each term of the expansion is obtained from \( \lambda^{(sc)} \) only.

The above perturbative result is valid only for small \( \epsilon_1 \) and \( n \). In the large \( n \) limit, the semi-classical result is valid and
\[ \lambda(e_n) \approx \lambda^{(sc)}(e_n) \approx \frac{\pi}{K(1/\sqrt{2})} \left(\frac{e_n\epsilon_1}{\epsilon_0}\right)^{1/4}. \] (72)
To compare with the WKB result, from [4] we know that
\[ e_n^{WKB} \approx \frac{3^{4/3}\pi^2}{[\Gamma(1/4)]^{8/3}} \left(\frac{\epsilon_1}{\epsilon_0}\right)^{1/3} n^{4/3}. \] (73)
This gives the energy splitting between levels as
\[ e_n^{WKB} - e_{n+1}^{WKB} \approx \frac{4}{3} e_n^{WKB} \approx \frac{4\pi^{3/2}}{3} \left(\frac{e_n^{WKB}\epsilon_1}{\epsilon_0}\right)^{1/4}. \] (74)
Since \( K(1/\sqrt{2}) = [\Gamma(1/4)]^2/4/\sqrt{\pi} \), this is precisely the form of \( \lambda(e_n) \) for large \( n \) and we see explicitly the equivalence between \( \lambda^{(sc)} \) and the WKB approximation.
We have shown that the study of non-linear oscillators is equivalent to the study of algebras satisfying eqs. (3) and (4); the SHO being a special case of this algebra. In addition, the Hilbert Space eq. (8) and eigenvalues eq. (10) of these algebras all have the same form. The number operator for non-linear oscillators was also constructed. Results of this general analysis were used to determine the general form of the partition function and average energy for an non-linear oscillator in contact with a heat bath. Analysis of non-linear oscillators therefore reduces to determining the function $\lambda(H)$ and the groundstate $e_g$ of the oscillator. This can be done by first making a semi-classical approximation, which requires only the evaluation of a single integral, and then using it as a guide to constructing $\hat{a}$ and $\hat{a}^\dagger$ in terms of $a$ and $a^\dagger$. This analysis was applied to the $x^4$ interaction and both the standard second order perturbation result as well as the WKB result were obtained. Moreover, the recent results of Bender and Bettencourt were also obtained within this framework.

Unlike the Bogoluibov transformation, the mapping between $(\hat{a}, \hat{a}^\dagger)$ and $(a, a^\dagger)$ is non-linear and cannot be generated by a simple unitary transformation. The two Hilbert Spaces $\mathcal{H}_{\epsilon_1}$ and $\mathcal{H}_{SHO}$ are unitarily inequivalent. Indeed, each value of $\epsilon_1$ determine separate Hilbert Spaces all of whom are inequivalent to one another. This result provides a concrete example of Haag’s Theorem proved first for quantum field theories in higher dimensions. Based on the results of this theorem and the generality of our analysis, we expect a similar construction to hold for the $\phi^4$ theory in higher dimensions. Notice, however, that this construction requires a natural energy scale to define $\lambda(H)$. For Hamiltonians of the form eq. (11) we have such an energy scale: $\epsilon_0$. For quantum field theories, however, no such natural energy scale exists. An energy scale would have to be introduced, providing a natural introduction of a high (or low) energy cutoff in the theory.

Appendix A

The proof that a ground state exists for the non-linear oscillator when $\lambda(e) > 0$ for $e \geq e_g$ follows in the same manner as that for the SHO. Let $\hat{H} = H/\epsilon_0 - e_g = \hat{a}^\dagger \hat{a}$. Then, if $|\phi_0\rangle$ is
an eigenstate of \( \hat{H} \) with eigenvalue \( \phi_0 \),

\[
\phi_0 = \langle \phi_0 | \hat{H} | \phi_0 \rangle = |\tilde{a} | \phi_0 \rangle |^2 \geq 0 ,
\]

and all the eigenvalues of \( \hat{H} \) are non-negative. Next, consider the state \( |\phi_{-1} \rangle \equiv \tilde{a} | \phi_0 \rangle \). Then

\[
\hat{H} |\phi_{-1} \rangle = \phi_0 |\phi_{-1} \rangle - \lambda(H) |\phi_{-1} \rangle .
\]

Since \( \lambda \) is a functional of \( \hat{H} + e_g \), from the above \( |\phi_{-1} \rangle \) must be an eigenvalue of \( \hat{H} \) also, which we shall label as \( \phi_{-1} \). Then eq. (75) reduces to \( \phi_{-1} = \phi_0 - \lambda(\phi_{-1} + e_g) \). Since \( \lambda(e) > 0 \), we have \( \phi_{-1} < \phi_0 \).

Similarly, the states \( |\phi_{-n} \rangle \equiv \tilde{a}^n | \phi_0 \rangle \) are also eigenstates of \( \hat{H} \) with eigenvalues \( \phi_{-n} \). Moreover, they satisfy a sequence of strict inequalities

\[
\phi_{-n} < \phi_{-(n-1)} < \cdots < \phi_{-1} < \phi_0 .
\]

Since \( \phi_{-n} \geq 0 \) for all \( n \), this sequence must end. Namely, for some \( m \), \( \phi_m = 0 \). Then

\[
0 = \langle \phi_{-m} | \hat{H} | \phi_{-m} \rangle = |\tilde{a} | \phi_{-m} \rangle |^2 \geq 0 \text{ or } \tilde{a} | \phi_{-m} \rangle = 0 .
\]

The ground state is then identified as \( |\Omega \rangle = | \phi_{-m} \rangle \) and \( H |\Omega \rangle = \epsilon_0 e_g |\Omega \rangle \).

The existence of a groundstate is only guaranteed when \( \lambda \) is positive definite. In the semi-classical limit it can be shown that this holds for wide classes of bounding potentials. For the polynomial interaction potential \( V(a, a^\dagger) = \epsilon_1 (a + a^\dagger)^l/l \),

\[
\lambda^{(sc)} \sim e^{1/2 - 1/l} \left( \frac{\epsilon_1}{\epsilon_0} \right)^{1/l} .
\]

For the exponential interaction potential \( V(a, a^\dagger) = \epsilon_1 e^{\alpha^2 (a + a^\dagger)^2} \),

\[
\lambda^{(sc)} \sim \alpha \sqrt{\frac{e}{\log (e \epsilon_0 / \epsilon_1)}} .
\]

In both cases \( \lambda^{(sc)} \) is positive definite. Notice also that \( \lambda^{(sc)}/e^{(sc)} \to 0 \) as \( e^{(sc)} \to \infty \), justifying the approximations made in SecV.

Appendix B
The form of eq. (6) as well as the algebraic equation eq. (12) suggests that we look at a finite difference form of the one dimensional Lie derivative. Given any function \( \lambda(x) \), we define the finite difference Lie operator \( \mathcal{L}_\lambda \) by

\[
\mathcal{L}_\lambda f(x) \equiv f(\lambda(x) + x) - f(x) ,
\]

(80)

where \( f(x) \) is any function of \( x \). In the limit \( \lambda(x) \to 0 \) for all \( x \), it is straightforward to see that eq. (80) reduces to the usual Lie derivative. Moreover, for any two functions \( f(x) \) and \( g(x) \) and constants \( a, b \),

\[
\mathcal{L}_\lambda (af(x) + bg(x)) = a\mathcal{L}_\lambda f(x) + b\mathcal{L}_\lambda g(x) .
\]

(81)

\( \mathcal{L}_\lambda \) is therefore a linear operator on the space of functions on \( \mathbb{R}^d \). It is not, however, a derivation since it does not satisfy the product rule,

\[
\mathcal{L}_\lambda \{f(x)g(x)\} = f(x)\mathcal{L}_\lambda g(x) + g(x)\mathcal{L}_\lambda f(x) + \mathcal{L}_\lambda f(x)\mathcal{L}_\lambda g(x) .
\]

(82)

Finally, for any two given functions \( \lambda(x) \) and \( \xi(x) \), the commutator of two finite difference Lie operators

\[
[\mathcal{L}_\lambda, \mathcal{L}_\xi]f(x) = f(x + \lambda(x) + \xi(x + \lambda(x))) - f(x + \xi(x) + \lambda(x + \xi(x))) ,
\]

(83)

vanishes if and only if \( \mathcal{L}_\lambda \xi(x) = \mathcal{L}_\xi \lambda(x) \).

Using \( \mathcal{L}_\lambda \), the particular solution of eq. (12) becomes the solution of the operator equation,

\[
\mathcal{L}_\lambda N_p = 1 .
\]

(84)

Of more interest is the homogeneous solution to eq. (12),

\[
\mathcal{L}_\lambda N_h = 0 ,
\]

(85)

which lies in the kernel of \( \mathcal{L}_\lambda \), ker \( \mathcal{L}_\lambda \), for a given \( \lambda \). Notice that when \( \lambda \) is a real constant, ker \( \mathcal{L}_\lambda \) is the space of all periodic functions with period \( \lambda \). When \( \lambda \) is a function of \( x \), ker \( \mathcal{L}_\lambda \)
will contain generalizations of periodic functions to those whose frequencies are $x$ dependent. This agrees quite well with the observation that in the semi-classical limit, $1/\lambda^{(sc)}$ reduces to the WKB result. Of particular interest is when $\lambda \in \ker \mathcal{L}_\lambda$:

$$\mathcal{L}_\lambda \lambda = 0. \quad (86)$$

From eq. (10) we see that for this $\lambda$ the energy levels are equally spaced and is determined solely by $\lambda(e_g)$. The constant function, and thus the SHO, clearly satisfies eq. (86). Whether there exists other non-trivial solutions to eq. (86) for physically realizable non-linear oscillators is still an open question.
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Aside from requiring that $\lambda$ be a positive definite function, there are no other positivity requirements for eqs. (2) and (4). In Appendix A, this positivity requirement is shown to hold for a wide variety of potentials in the semi-classical limit. This is in contrast to the commutation relations for the time and phase operators.

The case where $V$ depends on time explicitly, as through the coupling constant $\epsilon_1$, is much more subtle. Neither eq. $[15]$ nor eq. $[16]$ are valid since in general $[H(t), H(t')] \neq 0$ for $t \neq t'$.
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