LINEAR AND ORBITAL STABILITY ANALYSIS FOR SOLITARY-WAVE SOLUTIONS OF VARIABLE-COEFFICIENT SCALAR FIELD EQUATIONS

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ABSTRACT. We study general semilinear scalar-field equations on the real line with variable coefficients in the linear terms. These coefficients are uniformly small, but slowly decaying, perturbations of a constant-coefficient operator. We are motivated by the question of how these perturbations of the equation may change the stability properties of kink solutions (one-dimensional topological solitons). We prove existence of a stationary kink solution in our setting, and perform a detailed spectral analysis of the corresponding linearized operator, based on perturbing the linearized operator around the constant-coefficient kink. We derive a formula that allows us to check whether a discrete eigenvalue emerges from the essential spectrum under this perturbation. Known examples suggest that this extra eigenvalue may have an important influence on the long-time dynamics in a neighborhood of the kink. We also establish orbital stability of solitary-wave solutions in the variable-coefficient regime, despite the possible presence of negative eigenvalues in the linearization.

1. INTRODUCTION

We consider a semilinear, variable-coefficient scalar field equation of the form

\[ \partial_t^2 u - \left[ a(x) \partial_x^2 u + b(x) \partial_x u + c(x) u \right] + F'(u) = 0, \quad x \in \mathbb{R}. \] (1.1)

Our assumptions on the potential \( F \) are

\[ F \in C^3(\mathbb{R}), \quad F(a_-) = F(a_+) = 0 \text{ for some } a_- < a_+, \]
\[ F'(a_\pm) = 0, \quad F''(a_\pm) = m^2 > 0, \quad F(s) > 0, \quad s \in (a_-, a_+). \] (1.2)

The linear operator \( a(x) \partial_x^2 + b(x) \partial_x + c(x) \) is assumed to be a perturbation of the 1D Laplacian \( \partial_x^2 \). More precisely, for a small parameter \( \delta > 0 \), we assume

\[ \| |a - 1| + |\partial_x a| + |b| + |c\| \|L^1(\mathbb{R})\| + \| |a - 1| + |\partial_x a| + |b| + |c\| \|L^\infty(\mathbb{R})\| \leq \delta. \] (1.3)

With \( \omega(x) = \exp(\int_{-\infty}^x b(z)/a(z) \, dz) \), the energy functional

\[ E(u) := \int_{\mathbb{R}} \frac{\omega(x)}{a(x)} \left( \frac{1}{2} (\partial_x u)^2 + \frac{1}{2} a(\partial_x u)^2 - \frac{1}{2} cu^2 + F(u) \right) \, dx, \]

is formally conserved under the flow of (1.1). We note that equation (1.1) is not invariant under translations, and that we make no parity assumptions on \( F \) or the coefficients \( a, b, \) and \( c \).

We are interested in the long-time behavior of solutions to (1.1). Our first result (Theorem 1.1) is the existence of a stationary solution \( T \) of kink or solitary-wave type, i.e. an increasing stationary solution with \( T(x) \to a_+ \) as \( x \to \pm \infty \). Standard arguments then show that (1.1) is locally well-posed for initial data \((u, \partial_x u)|_{t=0} \in H^1_T(\mathbb{R}) \times L^2(\mathbb{R})\), where \( H^1_T(\mathbb{R}) = \{ \varphi : \varphi - T \in H^1(\mathbb{R}) \} \). In this context, \( H^1_T(\mathbb{R}) \times L^2(\mathbb{R}) \) is referred to as the energy space, and indeed, it is not hard to see (using in particular that \( |\omega/a - 1| \lesssim \delta \)) that functions in this space have finite energy. Our goal is to study the stability of \( T \) with respect to small perturbations in the energy space.

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1.1. Motivation. One-dimensional kinks such as $T(x)$ are the simplest examples of topological solitons, and thus are an important model for physical phenomena arising in areas such as quantum field theory, condensed matter physics, and cosmology, among others. (See [48, 37, 28, 38] for some physics-oriented discussions.) Understanding their stability has proven to be a difficult mathematical challenge. The majority of work focuses on the constant-coefficient version of (1.1),

$$\partial_t^2 u - \partial_x^2 u + F'(u) = 0.$$  

In this constant-coefficient regime, it is standard that the assumptions (1.2) imply the existence of a kink solution connecting $a_-$ and $a_+$. (Convenient proofs of this fact may be found in [33] Lemma 1.1 or [23] Proposition 2.1.) This constant-coefficient stationary kink, which we denote by $S$, satisfies

$$-S'' + F''(S) = 0, \quad \lim_{x \to -\infty} S(x) = a_-, \quad \lim_{x \to \infty} S(x) = a_+.$$  

We find in Theorem 1.1 that $T$ and $S$ are close in an appropriate norm.

Orbital stability of $S$ in the constant-coefficient setting has been known for some time [20], and we extend this to our setting in Theorem 1.3. Asymptotic stability of kinks is more subtle, and we extend this to our setting in Theorem 1.3. Asymptotic stability of kinks is more subtle, and

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the sine-Gordon equation with nonlinearity $F(u) = \sin(u)$, which is not asymptotically stable, at least with respect to perturbations in the energy space. (See Section 1.3 for more on these examples.)

Our motivation is to understand the effect of linear perturbations of the equation (1.4) on the stability properties of kink solutions. On the one hand, given that (1.4) is in some sense an idealized model, it is important on physical grounds to understand whether stability properties of kink solutions persist under perturbations of the equation. There is also reason to expect such perturbations to have a nontrivial qualitative impact on the stability analysis (rather than simply adding a small error term) in some situations. As we explain below, this is connected with the possibility that a discrete eigenvalue may emerge from the essential spectrum of the linearized operator around the kink.

1.2. Main results. Before stating our main theorems, we make a technically convenient change of variables in (1.1) that will be in effect throughout this article. Letting $y = \int_0^x a^{-1/2}(z) \, dz$, and abusing notation by writing $u(t, y) = u(t, x(y))$ and $b(y) = b(x(y)) = a^{-1/2}(x(y)) \frac{d}{dy} a^{1/2}(x(y))$, we have

$$\partial_t^2 u - [\partial_y^2 u + b(y) \partial_y u + c(y) u] + F'(u) = 0.$$  

The hypotheses (1.3) imply

$$\|b\| + \|c\|_{L^1(\mathbb{R})} + \|b\| + \|c\|_{L^\infty(\mathbb{R})} \leq C_0 \delta,$$

for some $C_0 > 0$.

Our first result is the existence of a stationary kink:

**Theorem 1.1.** Assume that 0 is not an $L^2(\mathbb{R})$-eigenvalue of the operator $-\partial_y^2 - b \partial_y - (c - F''(S))$. Then, for $\delta > 0$ sufficiently small, there exists a solution $T$ to

$$-T'' - b(y) T' - c(y) T + F'(T) = 0, \quad \lim_{y \to -\infty} T(y) = a_-, \quad \lim_{y \to \infty} T(y) = a_+.$$  

This solution can be written $T(y) = S(y) + S_b(y)$, where $S$ solves (1.6) and

$$\|S_b\|_{W^{1,1}(\mathbb{R})} + \|S_b\|_{W^{1,\infty}(\mathbb{R})} \leq C \delta.$$
Unlike $S$, which satisfies $|S(x) - a_\pm| \lesssim e^{\mp mx}$ and $|S'(x)| \lesssim e^{-m|x|}$, our static kink $T$ does not necessarily possess exponential tails. This behavior is reminiscent of some higher-order, constant-coefficient field theories that do not fit into the assumptions \(1.2\) (see e.g. \[29\]). Under additional exponential decay assumptions on $b$ and $c$, it is possible to show $T$ has exponential asymptotics at $\pm \infty$ as in \[46\], but we do not explore the details here.

Our next result concerns the linearized operator around $T$. Writing $u(t,y) = T(y) + \varphi(t,y)$, the perturbation $\varphi$ satisfies

$$\partial_t^2 \varphi - \partial_y^2 \varphi - b \partial_y \varphi - c \varphi = F''(T) - F'(T + \varphi).$$

Adding $F''(T)\varphi$ to both sides, and defining the linear operator $L_T = -\partial_y^2 - b \partial_y - c + F''(T)$ and the nonlinearity $N(T, \varphi) = F'(T) - F'(T + \varphi) + F''(T)\varphi = O(\varphi^2)$, the equation for $\varphi$ can be written as a nonlinear Klein-Gordon equation:

\begin{equation}
\partial_t^2 \varphi + L_T \varphi = N(T, \varphi).
\end{equation}

We are most interested in situations where the spectrum of $L_S = -\partial_y^2 + F''(S)$, the operator corresponding to the constant-coefficient kink, is known exactly. We then have $L_T = L_S - b \partial_y - c + F''(T) - F''(S)$, and we ask how the perturbation $-b \partial_y - c + F''(T) - F''(S)$ changes the spectral properties of $L_S$.

The $L^2(\mathbb{R})$ spectrum of $L_S$ is given by

$$\sigma(L_S) = \{0, \lambda_1, \ldots, \lambda_n\} \cup [m^2, \infty),$$

where $\lambda_1, \ldots, \lambda_n$ is a possibly empty, increasing collection of positive, simple eigenvalues. The eigenfunction corresponding to 0 is exactly $S'$, the translation invariance mode.

As expected, discrete eigenvalues $\lambda_i$ will drift to nearby discrete eigenvalues $\lambda'_i$ of $L_T$ under the perturbation. A more delicate question is whether an extra discrete eigenvalue emerges from the essential spectrum. This aspect is especially relevant when $L_S$ has a threshold resonance, i.e. a function $R \in L^\infty(\mathbb{R}) \setminus L^2(\mathbb{R})$ satisfying $\langle L_SR \rangle = m^2 R$, as is the case for both the $\phi^4$ and sine-Gordon equations.

We derive a criterion in terms of $R$ and the coefficients $b$ and $c$ that governs whether the resonance drifts into a discrete eigenvalue.

Our results on the spectrum of $L_T$ are collected in the following theorem:

**Theorem 1.2.** Let $L_T$ and $L_S$ be as defined above. There exists a universal $c_0 > 0$ such that:

(a) The spectrum $\sigma(L_T)$ is real, the essential spectrum $\sigma_{ess}(L_T) = \sigma_{ess}(L_S) = [m^2, \infty)$, and $\sigma(L_T)$ lies in the $c_0 L$-neighborhood of $\sigma(L_S)$.

(b) For every eigenvalue $\lambda \in \sigma_d(L_S)$ with eigenvector $Y_\lambda$, there is a corresponding $\lambda' \in \sigma_d(L_T)$. The eigenvalue $\lambda'$ is real, simple, and satisfies $|\lambda - \lambda'| \leq c_0 \delta$. Also, if

$$A := \int_{\mathbb{R}} Y_\lambda([F''(T) - F''(S) - c]Y_\lambda - b \partial_y Y_\lambda) dy \neq 0,$$

then $\lambda' - \lambda$ has the same sign as $A$. The eigenfunction $Y_{\lambda'}$ of $L_T$ corresponding to $\lambda'$ satisfies $||Y_{\lambda'}(y)|| + ||Y_{\lambda'}'(y)|| \lesssim e^{-\sqrt{m^2 - \lambda}|y|}$. Furthermore, for suitable normalizations of $Y_\lambda$ and $Y_{\lambda'}$, we have

$$\|e^{\sqrt{m^2 - \lambda}|y|}Y_{\lambda'}(y) - e^{\sqrt{m^2 - \lambda}|y|}Y_\lambda(y)\|_{L^\infty(\mathbb{R})} \leq C\delta,$$

for a universal constant $C > 0$.

(c) If $m^2$ is a simple resonance of $L_S$, and

$$\int_{\mathbb{R}} R([F''(T) - F''(S) - c]R - b \partial_y R) dy < 0,$$

then there exists a discrete eigenvalue $\lambda$ of $L_T$ with $0 < m^2 - \lambda < c_0 \delta$. The eigenfunction $Y_\lambda$ also satisfies $||Y_\lambda(y)|| + ||Y_\lambda'(y)|| \lesssim e^{-\sqrt{m^2 - \lambda}|y|}$ and

$$\|Y_\lambda(y) - e^{-\sqrt{m^2 - \lambda}|y|}R(y)\|_{L^\infty(\mathbb{R})} \leq C(k + \delta)e^{-\sqrt{m^2 - \lambda}|y|},$$
The odd eigenfunction $Y_{\lambda}$ and $R$.

If

$$\int_{\mathbb{R}} R[(F''(T) - F''(S) - c)R - b\partial_y R] \, dy > 0,$$

then there are no eigenvalues of $\mathcal{L}_T$ in $[m^2 - c_0\delta, m^2]$, and $m^2$ is non-resonant.

(d) If $\lambda = m^2$ is not a resonance or an embedded eigenvalue of $\mathcal{L}_S$, then the same is true of $\mathcal{L}_T$, and there are no eigenvalues of $\mathcal{L}_T$ in $[m^2 - c_0\delta, m^2]$.

Part (a) of this theorem is standard, and included for clarity of exposition. Part (b) is arguably not surprising, but its proof (see Section 3) is a useful warm-up for parts (c) and (d). We also remark that the formulas in this theorem may be replaced with (more cumbersome, but in some sense more elementary) formulas that depend only on $F$, $S$, $b$, and $c$, via a first-order approximation for $F''(T) - F''(S)$. (See (3.7) and (3.11).)

The possible extra eigenvalue as in Theorem 1.2(c) is one of our primary motivations for performing this perturbation analysis. In general, eigenvalues lying in between 0 and $m^2$ have a profound impact on the stability properties of the kink. At the very least, any proof of asymptotic stability or instability for $T$ would likely need to account for this extra eigenvalue in some way.

It should be noted that we are outside the realm of analytic perturbation theory, since we do not assume any continuity of the coefficients $b, c$ with respect to $\delta$. Our spectral analysis is based on the well-known method of finding solutions $U_{\lambda}^{L_\infty}$ to the eigenvalue equation $\mathcal{L}_T U^{\lambda} = \lambda U^{\lambda}$ which decay at $\pm \infty$, and studying the Evans function (see e.g. [15, 25, 39, 26, 27]) which is related to the Wronskian of $U_{\lambda}^{L_\infty}$ and $U_{\lambda_{\infty}}^{L_\infty}$. The key property is that the Wronskian is zero when $\lambda$ is an eigenvalue or resonance of $\mathcal{L}_T$. The slow decay of our coefficients $b$ and $c$ (as well as $F''(T) - F''(S)$) rules out tools such as the Gap Lemma (see [2, 17]) which would allow one to analytically continue the Evans function past the threshold $\lambda = m^2$, but which requires exponential decay of the coefficients.

Our last main result establishes the orbital stability of $T$:

**Theorem 1.3.** There exists an $\varepsilon > 0$, depending on $\delta$, such that for any initial data $(u, \partial_t u)|_{t=0} = (T + v_1, v_2)$ for $(v_1, v_2) \in L^2 \times H^1$ with

$$\|(v_1, v_2)\|_{H^1(\mathbb{R}) \times L^2(\mathbb{R})} < \varepsilon,$$

the corresponding solution $u$ to (1.6) exists globally in time, and satisfies

$$\|u - T\|_{H^1(\mathbb{R})} + \|\partial_t u\|_{L^2(\mathbb{R})} \leq C\varepsilon,$$

for some $C$ depending on $\delta$ and $F$.

The proof is based on classical energy arguments, but must contend with the lack of translation invariance.

### 1.3. Examples.

#### 1.3.1. $\phi^4$ model.

The choice of a double-well potential $F(u) = \frac{1}{4}(1 - u^2)^2$ leads to the $\phi^4$ model (1.10)

$$\partial_x^4 u - \partial_x^2 u = u - u^3.$$

Standard references on this equation include [44, 6, 32, 41]. In this case, the kink solution $S(x) = \tanh(x/\sqrt{2})$ is known explicitly, and the linearization $\mathcal{L}_S = -\partial_x^4 + (3S^2 - 1)$ has spectrum equal to

$$\sigma(\mathcal{L}_S) = \left\{0, \frac{3}{2}\right\} \cup [2, \infty).$$

The odd eigenfunction $Y_{3/2} = \tanh(x/\sqrt{2}) \, \text{sech}(x/\sqrt{2})$ corresponding to $\lambda = \frac{3}{2}$ is known as the *internal oscillation mode*. The operator $\mathcal{L}_S$ also possesses an even resonance $R = 2\tanh^2(x/\sqrt{2}) - \text{sech}^2(x/\sqrt{2})$ at the threshold $\lambda = 2$. 
The $\phi^4$ kink is asymptotically stable with respect to odd perturbations in the energy space, by the important work of Kowalczyk-Martel-Muñoz \cite{32}. When working in the odd energy space, the even translation invariance mode at $\lambda = 0$ and the even resonance do not play any role, but the internal oscillation mode has a dramatic effect on the dynamics. The method of \cite{32} involved projecting $\varphi$ onto $Y_{3/2}$ and the continuous spectrum, and carefully tracking the interaction between these two parts induced by the nonlinear terms of (1.9). A delicate coupling between the internal oscillation mode and the continuous part leads to the dissipation of energy away from a neighborhood of the kink.

Asymptotic stability with respect to odd perturbations was extended to a variable-coefficient version of (1.10) by the second named author in \cite{46}, though the coefficients were less general than those considered here (only a second-order perturbation, which was taken to be even and exponentially decaying). The symmetry assumption means that any eigenvalue emerging from the essential spectrum would be even, and therefore can be ignored.

It remains an important open question whether this kink is asymptotically stable with respect to general perturbations. Our Theorem 1.2 implies that for certain choices of $b,c$ in (1.6), the bottom of the continuous spectrum is non-resonant and there are no extra discrete eigenvalues. Such a version of (1.6) could serve as an interesting test case for the $\phi^4$ asymptotic stability problem, especially if one is convinced that the threshold resonance is an important source of difficulties.

1.3.2. Sine-Gordon equation. The choice $F(u) = 1 - \cos(u)$ results in the sine-Gordon equation:

$$\partial_t^2 u - \partial_x^2 u = -\sin(u).$$

This equation arises in the study of superconductivity as well as of surfaces with constant negative curvature, among other areas. (See e.g. \cite{22, 7, 8} for background on this equation.)

The explicit static kink is given by $S(x) = 4 \arctan(e^x)$. The equation, which is completely integrable, possesses other special solutions including breathers and wobbling kinks \cite{21, 33}. The presence of these wobbling kinks (periodic-in-time, spatially localized perturbations of the kink) implies that $S$ is not asymptotically stable in the energy space. (However, see \cite{5} for an asymptotic stability result in a different topology, and \cite{1}, which identified an infinite-codimensional manifold of initial data near the kink for which asymptotic stability in the energy space does hold.) With $L_S = -\partial_x^2 + \cos(S)$ the linearization around $S$, it is known that

$$\sigma(L_S) = \{0\} \cup [1, \infty),$$

The failure of asymptotic stability in the energy space is consistent with the absence of an internal oscillation mode, which rules out the mechanism of stability observed for the $\phi^4$ model in \cite{32}. However, there is an odd resonance $R(x) = \tanh(x)$ at the bottom of the continuous spectrum. Our Theorem 1.2 gives conditions under which the variable-coefficient version of sine-Gordon possesses a discrete eigenvalue $\lambda$ with $0 < 1 - \lambda \ll 1$. In this case, one may ask whether the new odd eigenfunction behaves sufficiently like an internal oscillation mode that a stability mechanism like the one mentioned above comes into force. We plan to explore this question in a future article.

Somewhat different perturbed forms of the sine-Gordon equation have been considered in, e.g., \cite{13, 14, 9, 16}. The general belief is that breathers and wobbles are non-generic phenomena, so one may conjecture that some dense set of coefficients satisfying (1.7) lead to asymptotic stability.

1.3.3. Other examples. Let us briefly mention some other models whose variable-coefficient counterparts are included in our setting: the $P(\phi)_{2}$ theory \cite{37}, the double-sine-Gordon equation \cite{3}, and certain higher-order field theories \cite{28}, i.e. potentials equal to a polynomial of even degree, which in some cases satisfies the assumptions (1.2) and other cases not.

1.4. Related work. The asymptotic stability of kinks in scalar field equations such as (1.4) is an active area of inquiry. In addition to the results mentioned above, we should mention the recent work of Kowalczyk-Martel-Muñoz-Van Den Bosch \cite{33}, which proved asymptotic stability for a
general class of scalar-field models satisfying a condition on the potential $F$ that, in particular, rules out internal oscillation modes and threshold resonances. In the setting of odd perturbations, Delort-Masmoudi \cite{12} established explicit decay rates for odd perturbations of the $\phi^4$ kink on time scales of order $\varepsilon^{-4}$, where $\varepsilon$ is the size of the initial perturbation. Let us also mention asymptotic stability results by Komech-Kopylova \cite{30, 31} for kink solutions of relativistic Ginzburg-Landau equations, which are of the form (1.4) with additional assumptions of the flatness of $F$ at $a_{\pm}$.

This class of questions is a partial motivation for the closely related subject of scattering theory for NLKG equations similar to (1.9). See \cite{10, 11, 36, 47, 35, 34, 18} and the references therein.

The operator $L_S$ is (up to subtraction by $m^2I$) a Schrödinger operator with rapidly decaying potential. There is a well-established theory of spectral perturbation of Schrödinger and related operators, see e.g. the review \cite{15} for an overview. Works that specifically address perturbation of threshold resonances include \cite{24, 3, 19, 40}. As mentioned above, aspects such as the slow decay of coefficients and lack of continuous dependence on $\delta$ make it convenient to perform the perturbation “by hand” in our setting, rather than apply an abstract theorem or existing result.

1.5. Outline of the paper. In Section 2 we prove the existence of the stationary solution $T$. In Section 3 we perform a spectral perturbation analysis of the linearized operator around the kink, and in Section 4 we establish orbital stability of $T$. Appendix A contains some useful lemmas on the global solvability of second-order ODE systems.

2. Stationary solution

First, we recall the existence of the static kink in the constant-coefficient case, which can be found by explicitly integrating the equation $S'' = F''(S)$. We quote from \cite{33} Lemma 1.1:

\textbf{Lemma 2.1.} Under the assumptions \cite{12} on $F$, there is a solution $S \in C^4(\mathbb{R})$ to the stationary equation

\[-S'' + F'(S) = 0,\]

with $S' > 0$ and $S \to a_{\pm}$ as $y \to \pm\infty$. Furthermore, $S$ and $S'$ satisfy

\[|S(x) - a_{\pm}| \leq Ce^{\mp my}, \quad |S'(x)| \leq Ce^{-m|x|},\]

and the energy of $S$ is finite:

\[\int_{\mathbb{R}} [S'(x)^2 + F(S(x))] \, dx < \infty.\]

We now prove the existence of a static kink $T(y)$ for our equation (1.1):

\textbf{Proof of Theorem 1.1.} Let $S$ be the stationary solution to $-S'' + F'(S) = 0$ guaranteed by Lemma 2.1. Making the ansatz $T = S + S_b$, we have the following equation for $S_b$:

\[-S_b'' - bS_b' - cS_b = bS' + cS - F'(S + S_b) + F'(S) = bS' + cS - F''(S)S_b - N(S, S_b),\]

where $N(S, S_b) = F'(S + S_b) - F'(S) - F''(S)S_b$. Defining

\[L_b = -\partial_y^2 - b(y)\partial_y - c(y) + F''(S)(y) = L_S - b(y)\partial_y - c(y),\]

equation (2.1) becomes

\[L_b S_b = bS' + cS - N(S, S_b).\]

We can find solutions $Y_{-\infty}, Y_{+\infty}$ both satisfying $L_b Y_{\pm\infty} = 0$, with $\lim_{y \to -\infty} Y_{-\infty} = 0$ and $\lim_{y \to \infty} Y_{+\infty} = 0$. In more detail, $L_b Y = 0$ may be written as the linear system $Y' = (M_1 + M_2(y))Y$, with $Y = (Y, Y')$, and

\[M_1 = \begin{pmatrix} 0 & 1 \\ m^2 & 0 \end{pmatrix}, \quad M_2(y) = \begin{pmatrix} 0 & 0 \\ -c(y) + F''(S)(y) - m^2 & -b(y) \end{pmatrix}.\]
Lemma 2.2 below implies existence of $Y_\infty$ and $Y_{-\infty}$. In particular, $Y_\infty$ and $Y_{-\infty}$ are linearly independent, since otherwise there would be a nontrivial solution in $L^2$ to $L S = 0$, contradicting our assumption that 0 is not an eigenvalue.

Define the Green’s function

$$
G(y, w) := \frac{1}{W_Y(y)} \begin{cases} 
Y_{-\infty}(y)Y_\infty(w), & y < w, \\
Y_\infty(y)Y_{-\infty}(w), & w \leq y,
\end{cases}
$$

where $W_Y(y) = \det(Y_{-\infty}, Y_\infty)$. Abel’s formula implies $W_Y(y) = W_Y(0) \exp(\int_0^y b(z) \, dz)$, which for $\delta > 0$ sufficiently small, is bounded uniformly away from 0.

For the inverse operator $\eta \mapsto \int_R G(\cdot, w)\eta(w) \, dw$, we have the following useful bounds. First,

$$
\left| \int_R G(y, w)\eta(w) \, dw \right| = \left| Y_\infty(y) \int_{-\infty}^y \frac{Y_{-\infty}(w)}{W_Y(w)} \eta(w) \, dw + Y_{-\infty}(y) \int_y^{\infty} \frac{Y_\infty(w)}{W_Y(w)} \eta(w) \, dw \right|
\leq C\|\eta\|_{L^\infty(R)} \left( e^{-my} \int_{-\infty}^y \frac{e^{mw}}{w} \, dw + e^{my} \int_y^{\infty} e^{-mw} \, dw \right)
\leq C\|\eta\|_{L^\infty(R)},
$$

for all $y \in \mathbb{R}$. We also have

$$
\left| \int_R \int_R G(y, w)\eta(w) \, dw \, dy \right| \leq C \left( \int_R e^{-my} \int_{-\infty}^y e^{mw} |\eta(w)| \, dw \, dy + \int_R e^{my} \int_y^{\infty} e^{-mw} |\eta(w)| \, dw \, dy \right).
$$

For the first term on the right, we integrate by parts to obtain

$$
\int_R e^{-my} \int_{-\infty}^y e^{mw} |\eta(w)| \, dw \, dy = \int_{-\infty}^y \frac{e^{mw-y}}{-m} |\eta(w)| \, dw \bigg|_{y=-\infty}^{y=\infty} - \int_{-\infty}^y e^{-my} e^{my} |\eta(y)| \, dy.
$$

If $\eta \in L^1(\mathbb{R})$, then since $e^{m(w-y)} \leq 1$, the boundary term at $-\infty$ vanishes, and the boundary term at $\infty$ is bounded by $\frac{1}{m} \|\eta\|_{L^1(\mathbb{R})}$. After applying a similar calculation to the last term in (2.4), we conclude

$$
\left\| \int_R G(\cdot, w)\eta(w) \, dw \right\|_{L^1(\mathbb{R})} \leq C\|\eta\|_{L^1(\mathbb{R})},
$$

for a constant depending on $m$ and the coefficients $b, c$. The estimates (2.3) and (2.5) clearly hold also if we replace $G(y, w)$ with $|G(y, w)|$.

In addition, using $|Y'_\pm(\infty)| \leq e^{\mp my}$, estimates similar to (2.3) and (2.5) imply

$$
\left\| \partial_y \int_R G(y, w)\eta(w) \, dw \right\|_{L^1(\mathbb{R})}, \quad \left\| \partial_y \int_R G(y, w)\eta(w) \, dw \right\|_{L^\infty(\mathbb{R})} \leq C\|\eta\|_{L^\infty(\mathbb{R})}.
$$

Now we write the equation (2.2) for $S_\delta$ as

$$
S_\delta(y) = (TS_\delta)(y) := g(y) - \int_R G(y, w)\mathcal{N}(S, S_\delta) \, dw,
$$

where

$$
g(y) = \int_R G(y, w)[b(w)S'(w) + c(w)S(w)] \, dw.
$$

We want to find a fixed point for $T$ in the space $X := L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with norm $\| \cdot \|_X := \| \cdot \|_{L^1(\mathbb{R})} + \| \cdot \|_{L^\infty(\mathbb{R})}$. From (2.3) and (2.5), we have

$$
\|g\|_X \leq C (\|bS' + cS\|_{L^\infty(\mathbb{R})} + \|bS' + cS\|_{L^1(\mathbb{R})}) \leq C_0\delta,
$$

since $S$ and $S'$ are bounded and $\|b + c\|_X \lesssim \delta$. For the nonlinear term, since $F$ is $C^3$ on $[a_-, a_+]$, there is some $K > 0$ such that

$$
\|\mathcal{N}(S, \eta)\| = |F'(S + \eta) - F'(S) - F''(S)\eta| \leq K\eta^2,
$$
globally in $y$. This gives

\begin{equation}
\left| \int_{\mathbb{R}} G(y, w) N(S, \eta)(w) \, dw \right| \leq K \int_{\mathbb{R}} |G(y, w)| \eta^2(w) \, dw
\end{equation}

and

\begin{equation}
\left| \int_{\mathbb{R}} \int_{\mathbb{R}} G(y, w) N(S, \eta)(w) \, dw \, dy \right| \leq K \int_{\mathbb{R}} \left( e^{-m y} \int_{-\infty}^{y} e^{m w} \eta^2(w) \, dw + e^{m y} \int_{y}^{\infty} e^{-m w} \eta^2(w) \, dw \right) \, dy,
\end{equation}

so that the estimates (2.3) and (2.5) imply

\[ \left\| \int_{\mathbb{R}} G(y, w) N(S, \eta)(w) \, dw \right\|_X \leq CK \| \eta^2 \|_X \leq CK \| \eta \|_X, \]

after applying the standard interpolation $\| \cdot \|_{L^2} \leq \| \cdot \|_{L^\infty} \| \cdot \|_{L^1}$.

With $C_0$ such that $\| g \|_X \leq C_0 \delta$, define $A := \{ \eta \in X, \| \eta \|_X \leq 2C_0 \delta \}$. For any $\eta \in A$, the above estimates imply

\[ \| T \eta \|_X = \left\| g - \int_{\mathbb{R}} G(\cdot, w) N(S, \eta)(w) \, dw \right\|_X \leq \| g \|_X + CK \| \eta \|_X^2 \leq C_0 \delta + C \delta^2, \]

so for $\delta < C_0/C$, we have $T \eta \in A$. Next, for $\eta_1, \eta_2 \in A$, we have from Taylor’s Theorem that

\[ F'(S + \eta_1) = F'(S + \eta_2) + F''(S + \eta_1)(\eta_1 - \eta_2) + \frac{1}{2} F'''(\xi_y)(\eta_1 - \eta_2)^2, \]

for some $\xi_y \in [a_-, a_+]$ depending on $y$. Using this in $N(S, \eta_1) - N(S, \eta_2)$, we have

\[ |N(S, \eta_1) - N(S, \eta_2)| = |F'(S + \eta_1) - F'(S + \eta_2) - F''(S)(\eta_1 - \eta_2)| = \left| F''(S + \eta_1) - F''(S) \right| (\eta_1 - \eta_2) + \frac{1}{2} F'''(\xi_y)(\eta_1 - \eta_2)^2 \]

\[ \leq \max_{[a_-, a_+]} |F''(s)| |\eta_1| |\eta_1 - \eta_2| + \frac{1}{2} |F'''(\xi_y)| (\eta_1 - \eta_2)^2 \]

\[ \leq K \delta |\eta_1 - \eta_2|, \]

for some $K > 0$. By (2.3) and (2.5) we have

\begin{equation}
\left\| (T \eta_1)(y) - (T \eta_2)(y) \right\|_X = \left\| \int_{\mathbb{R}} G(y, w) [N(S, \eta_1) - N(S, \eta_2)](w) \, dw \right\|_X \leq CK \delta \| \eta_1 - \eta_2 \|_X, \end{equation}

as above. The constant $CK > 0$ depends on $m$ and the $C^3$ norm of $F$. For $\delta$ sufficiently small, we conclude $T$ is a contraction on $A$, and a unique solution $S_b$ to (2.7) exists in $A$.

To derive the bounds on $S'_b$, we differentiate equation (2.7) and use the derivative bounds (2.6) and the Taylor estimate (2.8):

\[ \| S'_b \|_X = \left\| \partial_y \int_{\mathbb{R}} G(y, w) [b S' + c S - N(S, S_b)] \, dw \right\|_X \leq \| b S' + c S - N(S, S_b) \|_X \lesssim \delta + K \| S_b^2 \|_X \lesssim \delta. \]

The proof of Theorem 1.1 also provides the following approximation for $S_b = T - S$: since $S_b = g + \int_{\mathbb{R}} G(y, w) N(S, S_b)(w) \, dw$, the estimates (2.3), (2.5) imply

\begin{equation}
\left\| S_b - \int_{\mathbb{R}} \tilde{G}(\cdot, w) [b S' + c S](w) \, dw \right\|_X \leq \left\| \int_{\mathbb{R}} \tilde{G}(\cdot, w) N(S, S_b)(w) \, dw \right\|_X \lesssim \| S_b \|_X^2 \lesssim \delta^2, \end{equation}

with $\tilde{G} = G$ if 0 is not an eigenvalue of $L_b = L_S - b \partial_y - c$, and $\tilde{G} = G_\lambda$ otherwise, with $\lambda$ chosen such that $|\lambda| \lesssim \delta$, so that $\lambda S_b - \lambda \int_{\mathbb{R}} G_\lambda(y, w) S_b(w) \, dw$ are $O(\delta^2)$.
3. Perturbation of the spectrum

We consider the spectrum of

\[ \mathcal{L}_T := -\partial_y^2 - b\partial_y - c + F''(T), \]

where \( T \) is the stationary solution guaranteed by Theorem 1.1. Defining \( d = -c + F''(T) - F''(S) \), we have \( \mathcal{L}_T = \mathcal{L}_S - b\partial_y + d \). By the \( C^3 \) regularity of \( F \), we have \( |F''(T) - F''(S)| \leq K|S_0| \), and Theorem 1.1 implies \( ||d||_{L^1(\mathbb{R})} + ||d||_{L^\infty(\mathbb{R})} \lesssim \delta \). With (2.11), we can also write a first-order approximation for \( d \) as follows:

\[ d(y) = -c(y) + F''(S)(y) \int_{\mathbb{R}} \tilde{G}(y, w)[bS' + cS](w) dw + \varepsilon(y), \]

with \( \tilde{G} \) as in (2.11) and \( \varepsilon(y) = o(S_b(y)) \).

Our goal is to investigate how the spectrum of \( \mathcal{L}_S \) changes under the perturbation \( -b\partial_y + d \).

Since \( \mathcal{L}_T \) is self-adjoint with respect to the inner product

\[ (f, g)_\omega := \int_{\mathbb{R}} \omega f g dy, \]

with \( \omega = \int_{-\infty}^{y} b(z) dz \), the spectrum \( \sigma(\mathcal{L}_T) \) is real. Since the perturbation is relatively \( \mathcal{L} \)-compact, we have \( \sigma_{\text{ess}}(\mathcal{L}_T) = \sigma_{\text{ess}}(\mathcal{L}_S) \). (See, e.g. [21, Chapter 14].) Given our upper bounds on \( b \) and \( d \), it is standard that \( \sigma(\mathcal{L}_T) \) lies in the \( c_0|\delta \) neighborhood of \( \sigma(\mathcal{L}_S) \), for some \( c_0 > 0 \). (An elementary argument to this effect can be found in the proof of Theorem 3.1 in [46].) This already establishes part (a) of Theorem 1.2.

To analyze the eigenvalue problem, we write the equation \( (\mathcal{L}_S - \lambda)Y^\lambda = 0 \) in vector form:

\[ (Y^\lambda)'(y) = \begin{pmatrix} 0 & 1 \\ m^2 - \lambda & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ F''(S) - m^2 & 0 \end{pmatrix} Y^\lambda(y). \]

For any \( \lambda \leq m^2 \), Lemma A.2(a) implies there exist \( Y^\lambda_\pm, Y^\lambda_{\pm\infty} \in L^\infty_{\text{loc}}(\mathbb{R}) \) satisfying \( (\mathcal{L}_S - \lambda)Y^\lambda = 0 \), and

\[ \lim_{y \to \pm\infty} e^{ky}Y^\lambda_\pm(y) = \begin{pmatrix} 1 \\ k \end{pmatrix}, \]

with \( k = \sqrt{m^2 - \lambda} \). For \( \lambda < m^2 \), we also obtain the integral representations

\[ e^{ky}Y^\lambda_\infty = \begin{pmatrix} 1 \\ -k \end{pmatrix} - \frac{1}{2} \int_{y}^{\infty} (F''(S) - m^2)Y^\lambda_\infty(w)e^{kw} \begin{pmatrix} (e^{2k(y-w)} - 1)/k \\ e^{2k(y-w)} + 1 \end{pmatrix} dw \]

\[ e^{-ky}Y^\lambda_{-\infty} = \begin{pmatrix} 1 \\ k \end{pmatrix} - \frac{1}{2} \int_{-\infty}^{y} (F''(S) - m^2)Y^\lambda_{-\infty}(w)e^{-kw} \begin{pmatrix} (e^{2k(y-w)} - 1)/k \\ e^{2k(y-w)} + 1 \end{pmatrix} dw. \]

For the operator \( \mathcal{L}_T \), we similarly apply Lemma A.2(a) with \( V = F''(S) - m^2 + d \) to obtain \( U^\lambda_\pm, V_\pm \) solving \( (\mathcal{L}_T - \lambda)U^\lambda, V^\lambda = 0 \), with the same boundary conditions (3.3), and for \( \lambda < m^2 \),

\[ e^{ky}U^\lambda_\infty = \begin{pmatrix} 1 \\ -k \end{pmatrix} - \frac{1}{2} \int_{y}^{\infty} [(F''(S) - m^2 + d)U^\lambda_\infty - b(U^\lambda_\infty)'e^{kw} \begin{pmatrix} (e^{2k(y-w)} - 1)/k \\ e^{2k(y-w)} + 1 \end{pmatrix} dw \]

\[ e^{-ky}U^\lambda_{-\infty} = \begin{pmatrix} 1 \\ k \end{pmatrix} - \frac{1}{2} \int_{-\infty}^{y} [(F''(S) - m^2 + d)U^\lambda_{-\infty} - b(U^\lambda_{-\infty})'e^{-kw} \begin{pmatrix} (e^{2k(y-w)} - 1)/k \\ e^{2k(y-w)} + 1 \end{pmatrix} dw. \]

First, we prove a suitable approximation lemma for \( Y^\lambda_\pm \) and \( U^\lambda_\pm \) for nearby values of \( \lambda \):
Lemma 3.1. Assume $\|d\|_{L^1(\mathbb{R})} + \|b\|_{L^1(\mathbb{R})} \leq 1$. For any compact subset $B$ of $(-\infty, m^2)$, there exists a constant $C > 0$ such that for any $\lambda_1, \lambda_2 \in B$, there holds

$$
\|e^{k_1y} Y_{\infty}^{\lambda_1} - e^{k_2y} U_{\infty}^{\lambda_2}\|_{L^\infty([0,\infty), \mathbb{R}^2)} \leq C(|\lambda_1 - \lambda_2| + \|d\|_{L^1(\mathbb{R})} + \|b\|_{L^1(\mathbb{R})}),
$$

$$
\|e^{-k_1y} Y_{\infty}^{\lambda_1} - e^{-k_2y} U_{\infty}^{\lambda_2}\|_{L^\infty((-\infty,0], \mathbb{R}^2)} \leq C(|\lambda_1 - \lambda_2| + \|d\|_{L^1(\mathbb{R})} + \|b\|_{L^1(\mathbb{R})}),
$$

where $k_i = \sqrt{m^2 - \lambda_i}$.

Proof. We prove only the first estimate, as the second follows by a similar argument.

From (3.4) and (3.5) we have

$$
J = \int_0^\infty \left( (F''(S) - m^2) \left( \frac{e^{2k_1(y-w)} - 1}{k_1} \right) \right) Y_{\infty}^{\lambda_1}(w)e^{k_1w} dw
$$

$$
- \left( \frac{e^{2k_2(y-w)} - 1}{k_2} \right) [(F''(S) - m^2 + d)U_{\infty}^{\lambda_2}(w) - b(U_{\infty}^{\lambda_2})']e^{k_2w} dw.
$$

where

$$
J_1(y) := \left( \begin{array}{c} 0 \\ k_2 - k_1 \end{array} \right)
$$

$$
- \frac{1}{2} \int_0^\infty (F''(S) - m^2) e^{k_2w}U_{\infty}^{\lambda_2}(w) \left( \frac{e^{2k_1(y-w)} - 1}{k_1} - \frac{e^{2k_2(y-w)} - 1}{k_2} \right) dw.
$$

$$
J_2(y) := \frac{1}{2} \int_0^\infty \left( \frac{e^{2k_1(y-w)} - 1}{k_1} - \frac{e^{2k_2(y-w)} - 1}{k_2} \right) dU_{\infty}^{\lambda_2} - b(U_{\infty}^{\lambda_2})'e^{k_2w} dw.
$$

Since $y - w \leq 0$, the mean value theorem applied to $x \mapsto e^{2x(y-w)}$ and $x \mapsto (e^{2x(y-w)} - 1)/x$ implies, after a straightforward calculation, the inequalities

$$
\left| \left( \frac{e^{2k_1(y-w)} - 1}{k_1} - \frac{e^{2k_2(y-w)} - 1}{k_2} \right) \right| \leq C(1 + |y - w|)|k_1 - k_2|,
$$

for a constant $C$ depending on $B$. Since $|F''(S) - m^2| \leq e^{-m|w|}$ and $e^{k_2w}U_{\infty}^{\lambda_2}(w)$ is uniformly bounded on $[0, \infty)$, we therefore have $\|J_1\|_{L^\infty([0,\infty), \mathbb{R}^2)} \leq C|k_1 - k_2|$.

For $J_2$, since $e^{k_2w}U_{\infty}^{\lambda_2}$ is bounded on $[0, \infty)$, we have $\|J_2\|_{L^\infty([0,\infty), \mathbb{R}^2)} \leq C\|d + b\|_{L^1(\mathbb{R})}$, for a constant depending only on $k_2$.

Define the integral kernel

$$
K(y,w) = \frac{1}{2} \left( F''(S) - m^2 \right) \left( \begin{array}{cc} \frac{e^{2k_1(y-w)} - 1}{k_1} & 0 \\ e^{2k_1(y-w)} + 1 & 0 \end{array} \right),
$$

From the exponential decay of $F''(S) - m^2$ we see that

$$
\int_0^\infty \sup_{0 < y < w} \|K(y,w)\| dw
$$

is bounded by a constant depending only on $k_1$ and $m$. Lemma [A.1] then implies

$$
\|e^{k_1y} Y_{\infty}^{\lambda_1}(y) - e^{k_2y} U_{\infty}^{\lambda_2}(y)\|_{L^\infty((0,\infty), \mathbb{R}^2)} \leq C\|J_1 + J_2\|_{L^\infty([0,\infty), \mathbb{R}^2)} \leq C(|k_1 - k_2| + \|d\|_{L^1(\mathbb{R})} + \|b\|_{L^1(\mathbb{R})}).
$$

Since $|k_1 - k_2| \leq C|\lambda_1 - \lambda_2|$ for a constant depending only on $K$, the proof is complete. \qed
Now, we are ready to derive a result that governs the direction in which eigenvalues of $L_T$ drift under the perturbation:

**Theorem 3.2.** Assume $\|d\|_{L^1(\mathbb{R})} + \|b\|_{L^1(\mathbb{R})} \leq \delta < 1$. For any eigenvalue $\lambda_0 < m^2$ of $L_T$ with eigenfunction $Y$, there exists a simple, real eigenvalue $\lambda$ of $L_T$ with $|\lambda - \lambda_0| \leq C\delta$. Furthermore, we have the following expansion for $\lambda$:

$$\lambda = \lambda_0 + \frac{\int_{\mathbb{R}} Y_0 dY_0 - bY_0^\prime}{\int_{\mathbb{R}} (Y_0)^2} + O(\delta^2).$$

In particular, if

$$A := \int_{\mathbb{R}} Y_0 dY_0 - bY_0^\prime \neq 0,$$

then $\lambda - \lambda_0$ has the same sign as $A$.

**Remark.** Using the formula (3.2), one can show that $A$ has the same sign as

$$\int_{\mathbb{R}} Y_0^2 (y) \left( -c(y) + F''(S)(y) \int_{\mathbb{R}} G(y, w) [bS' + cS'](w) dw \right) - b(y)Y_0^\prime(y)Y_0(y) \right) dy.$$

**Proof.** With $Y_0(\pm\infty)$ solving (3.4), since $\lambda_0$ is a simple eigenvalue, we have $Y_0(\pm\infty) = c_\pm Y_0$, for constants $c_\pm$. Let $k_* = \sqrt{m^2 - \lambda_0}$. From our construction, it is clear that $Y_0$ decays exponentially at a rate $Y_0(y) \lesssim e^{-k_*|y|}$.

For $\lambda$ near $\lambda_0$, let $k = \sqrt{m^2 - \lambda}$ and let $U_0^\pm(\pm\infty)$ be the solutions to (3.5) as above. From Lemma 3.1 and our assumption that $\|d\|_{L^1(\mathbb{R})} + \|b\|_{L^1(\mathbb{R})} \lesssim \delta$, we can write

$$U_0^\pm(\pm\infty) = e^{(k_* - k)y} Y_0^\pm(\pm\infty) + e^{\mp ky} E_{\pm \infty}(y),$$

with $\|E_{\infty}\|_{L^\infty([-\infty, \infty]^2)} + \|E_{-\infty}\|_{L^\infty([-\infty, \infty]^2)} \lesssim \delta$. Denote the Wronskian

$$W_U(\lambda, y) = \det(U_0^\pm(\pm\infty), y) = \det(U_0^\mp(\pm\infty), y).$$

By Abel’s formula, $\exp(\int_{-\infty}^0 b(z) dz) W_U(\lambda, y)$ is independent of $y$. We focus on $y = 0$ and apply (3.8) to obtain

$$W_U(\lambda, 0) = \det(Y_{0, -\infty}, E_{-\infty}(0), Y_{-\infty, -\infty}(0), E_{-\infty, -\infty}(0))$$

$$= \det(Y_{0, -\infty}(0), E_{-\infty}(0)) + \det(E_{-\infty}(0), Y_{-\infty, -\infty}(0)) + \det(E_{-\infty}(0), E_{-\infty, -\infty}(0))$$

$$= \det(Y_{0, -\infty}(0), U_{-\infty}(0)) + \det(U_{-\infty}(0), Y_{-\infty, -\infty}(0)) + O(\delta^2).$$

In the second line, we used that $Y_{0, -\infty}$ and $Y_{-\infty, -\infty}$ are parallel, and in the last line, we used $E_{-\infty}(0) = U_{-\infty}(0) - Y_{-\infty, -\infty}(0)$ and $|E_{-\infty, -\infty}(0)| \lesssim \delta$. Since

$$\det(Y_{0, -\infty}(0), U_{-\infty}(0)) = \int_{-\infty}^0 Y_{0, -\infty}^\prime - (d + \lambda_* - \lambda)U_{-\infty}^\prime - b(U_{-\infty}^\prime)^\prime$$

we can use (3.8) again to write

$$\det(Y_{0, -\infty}(0), U_{-\infty}(0)) = \int_{-\infty}^0 Y_{0, -\infty}^\prime - (d + \lambda_* - \lambda)U_{-\infty}^\prime - b(U_{-\infty}^\prime)^\prime dw$$

$$= \int_{-\infty}^0 e^{(k_* - k)x} Y_{-\infty}^\prime - (d + \lambda_* - \lambda)Y_{-\infty}^\prime - b(Y_{-\infty}^\prime)^\prime dw$$

$$+ \int_{-\infty}^0 e^{\pm x} (d + \lambda_* - \lambda)E_{-\infty, -\infty}^\prime - bE_{-\infty, -\infty}^\prime dw,$$
where $k$ such that for any $\lambda$

useful in tracking how a threshold resonance of $L_0$. First, we prove a modified version of Lemma 3.1 for the borderline case.

Assume $\lambda$ which implies the first-order expansion for $Y(\lambda)$ defined as in the proof of Lemma 3.1, with $\lambda$ an eigenvalue of $L_T$.

Proof. For any $\lambda$, there holds

$\|e^{(k_*-k)|w|} - 1\| \leq |k_*-k||w|e^{(k_*-k)|w|}$ and the exponential decay of $Y$, we have, for $\lambda$ an eigenvalue of $L_T$,

$0 = e^{f_* \infty b(z)dz} e^{\lambda w} \int_{-\infty}^{\infty} [(d + \lambda - \lambda)(Y_w(w)) - bY_w(w)Y'_w(w)] dw + O(\delta^2),$

which implies the first-order expansion for $\lambda$ in the statement of the theorem. \qed

Now, we analyze the threshold resonance $R$, which is an $L^\infty$ function solving $L_SR - m^2 R = 0$. First, we prove a modified version of Lemma 3.1 for the borderline case $m^2$. This will be useful in tracking how a threshold resonance of $L_S$ translates to the spectrum of $L_T$. Writing $(L_S - m^2)Y = 0$ in vector form as above, Lemma (A.2) implies

$$Y^m_\infty(y) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \int_y^\infty (F''(S) - m^2)Y^m_\infty \begin{pmatrix} y-w \\ 1 \end{pmatrix} dw.$$ \hfill (3.10)

Lemma 3.3. Assume $\|d\|_{L^1(R)} + \|b\|_{L^1(R)} \leq 1$. For any $\lambda < m^2$, there exists a constant $C > 0$ such that for any $\lambda \in (\lambda_0, m^2)$, there holds

$$\|Y^m_\infty - e^{ky}U^\lambda_\infty\|_{L^\infty(0,1), R^2} \leq C(k + \|d\|_{L^1(R)} + \|b\|_{L^1(R)}),$$

$$\|Y^m_\infty - e^{-ky}U^\lambda_\infty\|_{L^\infty((-\infty,0), R^2)} \leq C(k + \|d\|_{L^1(R)} + \|b\|_{L^1(R)}),$$

where $k = \sqrt{m^2 - \lambda}$.

Proof. The proof is similar to Lemma 3.1 with the difference that $Y^m_\infty$ satisfies the modified integral equation (3.10). From (3.10) and (3.5), we have

$$Y^m_\infty(y) - e^{ky}U^\lambda_\infty(y) = \begin{pmatrix} 0 \\ k \end{pmatrix} - \frac{1}{2} \int_y^\infty \left( 2(F''(S) - m^2) \begin{pmatrix} y-w \\ 1 \end{pmatrix} Y^m_\infty(w) - \left( \frac{e^{2k(y-w)} - 1}{e^{2k(y-w)} + 1} \right) [(F''(S) - m^2 + d)U^\lambda_\infty(w) - b(U^\lambda_\infty y^{k w})] dw \right)$$

$$= J_1(y) + J_2(y) - \frac{1}{2} \int_y^\infty (F''(S) - m^2)Y^m_\infty(w) \left( \frac{e^{2k(y-w)} - 1}{e^{2k(y-w)} + 1} \right) (Y^m_\infty(w) - e^{kw}U^\lambda_\infty(w)) dw,$$ with

$$J_1(y) := \begin{pmatrix} 0 \\ k \end{pmatrix} - \frac{1}{2} \int_y^\infty (F''(S) - m^2)Y^m_\infty(w) \left( 2(y-w) - \frac{e^{2k(y-w)} - 1}{e^{2k(y-w)} + 1} \right) dw,$$

and $J_2(y)$ defined as in the proof of Lemma 3.1 with $\lambda$ replacing $\lambda_2$.

We claim that $\|J_1\|_{L^\infty(0,1), R^2} \leq C|k|$. Indeed, applying the mean value theorem to $f(x) = e^{2k(y-w)}$ gives

$$|f(k) - f(0)| \leq |k| \sup_{0 \leq x \leq k} |f'(x)| \leq |k|2|y-w|e^{2k(y-w)} \leq |k|2|y-w|,$$
or \(|e^{2k(y-w)}-1| \leq 2|k||y-w|\). Next, Taylor’s Theorem implies \(f(k) = f(0) + f'(0)k + \varepsilon\), with \(|\varepsilon| \leq \frac{1}{2}k^2\sup_{0 < x < k}|f''(x)|\). We have \(f'(0) = 2(y-w)\) and \(f''(x) = 4(y-w)^2e^{2x(y-w)} \leq 4(y-w)^2\), since \(y-w < 0\). This gives \(e^{2k(y-w)} - 1 = 2k(y-w) + \varepsilon\), with \(|\varepsilon| \leq 2k^2(y-w)^2\), or

\[
|(e^{2k(y-w)} - 1)/k - 2(y-w)| \leq 2k(y-w)^2.
\]

Plugging these inequalities into the definition of \(J_1\) and using decay of \(F''(S) - m^2\) gives the desired estimate.

The same calculation as in the proof of Lemma 3.3 implies that the boundedness property (A.1) is satisfied for this integral equation, and Lemma A.1 establishes the conclusion of the lemma. \(\square\)

In the following theorem, we make the (mild) assumption that the limits at \(\pm \infty\) of \(R(y)\) are nonzero.

**Theorem 3.4.** (a) Assume that \(m^2\) is a simple resonance for \(L_S\), i.e. that there exists \(R \in L^\infty(\mathbb{R}) \setminus L^2(\mathbb{R})\) with \(L_SR = m^2R\). Then there exists \(\delta > 0\) depending only on the function \(R\), such that if \(\|b\|_{L^1(\mathbb{R})} + \|d\|_{L^1(\mathbb{R})} \lesssim \delta\) and

\[
\int_\mathbb{R} R[dR - bR']\,dw < 0,
\]

then there exists a discrete eigenvalue \(\lambda\) of \(L_T\) with \(0 < m^2 - \lambda < C\delta\). If

\[
\int_\mathbb{R} R[dR - bR']\,dw > 0,
\]

then there is no discrete eigenvalue of \(L_T\) in a neighborhood of the essential spectrum, i.e. the discrete spectrum \(\sigma_d(L_T)\) consists of the same number of eigenvalues as \(\sigma_d(L_S)\).

(b) On the other hand, if \(m^2\) is nonresonant and not an eigenvalue of \(L_S\), then for \(\delta\) is sufficiently small, \(m^2\) cannot be a resonance or an eigenvalue of \(L_T\), and there is no eigenvalue of \(L_T\) in a neighborhood of the essential spectrum.

**Remark.** As above, using (3.10), the quantity \(\int_\mathbb{R} R[dR - bR']\,dw\) has the same sign as

\[
(3.11) \quad \int_\mathbb{R} \left[ R^2(y) \left( -c(y) + F''(S)(y) \int_\mathbb{R} G(y,w)[bS' + cS](w)\,dw \right) - b(y)R'(y)R(y) \right] dy
\]

**Proof.** With \(Y_{m^2}^{\pm} \) solving (3.10), we have \(Y_{m^2}^{\pm} = c_{\pm}R\), for constants \(c_{\pm}\).

Our first step is to analyze the unperturbed Wronskian \(W_Y(\lambda, y) = \det(Y_{\lambda}^\pm, Y_{\lambda}^\mp)\). With \(\lambda\) near \(m^2\) and \(k = \sqrt{m^2 - \lambda}\), by abuse of notation, we write \(W_Y(k, y) = W_Y(m^2 - k^2, y)\). Applying Lemma 3.3 with \(b = d = 0\), we may write

\[
Y_{\pm}^\lambda(y) = e^{\mp ky}(Y_{m^2}^{\pm}(y) + E_{\pm}), \quad y \in \mathbb{R},
\]

with \(\|E_{\infty}\|_{L^\infty([0,\infty),\mathbb{R})} + \|E_{-\infty}\|_{L^\infty((-\infty,0),\mathbb{R})} \lesssim k\). By the equation satisfied by \(Y_{m^2}^{\pm}\), the Wronskian \(W_Y(\lambda, y)\) is independent of \(y\). Proceeding as in the proof of Theorem 5.2, we have as before (see (3.9))

\[
(3.12) \quad W_Y(k, 0) = \det(Y_{m^2}^{\pm}(0), Y_{m^2}^{\pm}(0)) + \det(Y_{\infty}^{\pm}(0), Y_{m^2}^{\pm}(0)) + O(k^2).
\]
A direct calculation shows \( \det(Y_{\pm\infty}^2, Y_{\pm\infty}^\lambda)'(y) = (m^2 - \lambda)Y_{\pm\infty}^2Y_{\pm\infty}^\lambda \), which gives, since \( k^2 = m^2 - \lambda \),

\[
W_Y(k, 0) = k^2 \int_{-\infty}^{0} Y_{-\infty}^2Y_{-\infty}^\lambda dw + k^2 \int_{0}^{\infty} Y_{-\infty}^2Y_{-\infty}^\lambda dw + O(k^2)
\]

\[
= k^2 \int_{-\infty}^{0} Y_{-\infty}^2 e^{kw}(Y_{-\infty}^m + E_1) dw + k^2 \int_{0}^{\infty} Y_{-\infty}^2 e^{-kw}(Y_{-\infty}^m + E_1^\infty) dw + O(k^2)
\]

\[
= k^2 c_+ c_- \int_{-\infty}^{\infty} e^{-k|w|} R^2 dw
\]

\[
+ k^2 \int_{-\infty}^{\infty} R(w)e^{-k|w|} \left( 1_{\{w<0\}} c_+ E_1^\infty + 1_{\{w>0\}} c_- E_1^\infty \right) dw + O(k^2).
\]

Note that all integrals converge, since \( Y_{\pm\infty}^2 \), \( e^{\pm ky}Y_{\pm\infty}^\lambda \), and \( e^{\pm ky}E_{\pm\infty}^1 \) are all uniformly bounded.

In the last expression of (3.13), we note that the first term is proportional to \( k \). Indeed, since \( R \) has non-zero limits as \( x \rightarrow \pm\infty \), there exist \( \zeta, M > 0 \) (independent of \( k \)) such that \( R^2(y) \geq \zeta \) if \( |y| \geq M \). As a result, for any \( k \in (0, 1) \), there is \( \int_{\mathbb{R}} e^{-k|w|} R^2 dw \geq 2\zeta e^{-M/k} \). On the other hand, we have \( \int_{\mathbb{R}} R^2 e^{-k|w|} dw \leq 2\|R\|_{L^\infty(\mathbb{R})} \), it is also clear that, since \( \|\epsilon_{\pm\infty}\|_{L^\infty(\mathbb{R})} \lesssim k \), the second term on the right in (3.13) is \( O(k^2) \). To sum up, we have shown

\[
W_Y(k, 0) = c_+ c_- A(k) + O(k^2),
\]

with \( A(k) \geq A_0 k \) for some \( A_0 > 0 \) independent of \( k \).

Now we turn to the perturbed operator \( L_T \). Let \( U^\lambda_{\pm\infty} \) be the solutions to (3.3) as above. Applying Lemma 3.3 with \( \lambda_1 = \lambda_2 = \lambda \), we have

\[
U^\lambda_{\pm\infty}(y) = Y^\lambda_{\pm\infty}(y) + \tilde{E}_{\pm\infty}(y),
\]

with \( \|e^{ky}\tilde{E}_{\infty}\|_{L^\infty([0,\infty],\mathbb{R})} + \|e^{-ky}\tilde{E}_{-\infty}\|_{L^\infty((-\infty,0],\mathbb{R})} \lesssim \|d\|_{L^1(\mathbb{R})} + \|b\|_{L^1(\mathbb{R})} \lesssim \delta \).

With the Wronskian \( W_U(\lambda, y) \) defined as in the proof of Theorem 3.2 we again write \( W_U(k, y) = W_U(m^2 - k^2, y) \), and obtain

\[
W_U(\lambda, 0) = \det(Y^\lambda_{\pm\infty}(0), Y^\lambda_{-\infty}(0)) + \det(\tilde{E}_{-\infty}(0), Y^\lambda_{-\infty}(0)) + \det(Y^\lambda_{+\infty}(0), \tilde{E}_{-\infty}(0)) + \det(\tilde{E}_{-\infty}(0), \tilde{E}_{-\infty}(0)) = W_Y(\lambda, 0) + \det(\tilde{E}_{\infty}(0), Y^\lambda_{+\infty}(0)) + \det(\tilde{E}_{\infty}(0), \tilde{E}_{-\infty}(0)) + O(\delta^2).
\]

Since \( \tilde{E}_{\pm\infty} = U^\lambda_{\pm\infty} - Y^\lambda_{\pm\infty} \) satisfy \( \tilde{E}'_{\pm\infty} = (F''(S) - m^2 - \lambda)\tilde{E}_{\pm\infty} - b(U^\lambda_{\pm\infty})' + dU^\lambda_{\pm\infty} \), we have

\[
\det(\tilde{E}_{\pm\infty}, Y^\lambda_{\pm\infty})(y) = [dU^\lambda_{\pm\infty} - b(U^\lambda_{\pm\infty})]'Y^\lambda_{\pm\infty}.
\]

Because \( b, d \in L^1(\mathbb{R}) \), \( |U^\lambda_{\pm\infty}| \lesssim e^{-ky} \), and \( |Y^\lambda_{\pm\infty}| \lesssim e^{ky} \), the expression \( [dU^\lambda_{\pm\infty} - b(U^\lambda_{\pm\infty})]'Y^\lambda_{\pm\infty} \) is integrable on \([0, \infty)\), and we can use (3.13) to write

\[
\det(\tilde{E}_{-\infty}(0), Y^\lambda_{-\infty}(0)) = \int_{0}^{\infty} [dU^\lambda_{\pm\infty} - b(U^\lambda_{\pm\infty})]'Y^\lambda_{\pm\infty} dw
\]

\[
= \int_{0}^{\infty} [dY^\lambda_{\mp\infty} - b(Y^\lambda_{\pm\infty})]'Y^\lambda_{\pm\infty} dw + \int_{0}^{\infty} [d\tilde{E}_{\mp\infty} - b\tilde{E}_{\mp\infty}]Y^\lambda_{\pm\infty} dw,
\]

where the last integral converges and is \( O(\delta^2) \) since \( \|\tilde{E}_{\infty}\| \lesssim \delta e^{-ky} \) and \( Y^\lambda_{-\infty} \lesssim e^{ky} \). For the first integral on the right, we use Lemma 3.1 with \( d = b = 0 \) and obtain

\[
\int_{0}^{\infty} [dY^\lambda_{\mp\infty} - b(Y^\lambda_{\pm\infty})]'Y^\lambda_{\pm\infty} dw = \int_{0}^{\infty} [dY_{\mp\infty}^m - b(Y_{\pm\infty}^m)'Y_{\pm\infty}^m] dw + O(\delta k).
\]
After applying a similar analysis to \( \det(Y_\infty^\infty(0), \tilde{\mathbf{E}}_{-\infty}(0)) \), the expression (3.16) becomes

\[
W_U(k, 0) = W_Y(k, 0) + c_+ c_- \int_{-\infty}^{\infty} [d(R(w))^2 - bR(w)R'(w)] \, dw + O(\delta k) + O(\delta^2).
\]

With (3.14), this implies

\[
W_U(k, 0) = c_+ c_- \left( A(k) + \int_{-\infty}^{\infty} [d(R(w))^2 - bR(w)R'(w)] \, dw \right) + O(\delta k) + O(\delta^2).
\]

For \( \delta \) small enough, the expression inside the parentheses determines whether any zeroes of \( W_U(k, 0) \) are present for \( k > 0 \). The bound \( A(k) \geq A_0 k \) with \( A_0 > 0 \) implies statement (a) of the theorem.

For statement (b), the assumption that \( m^2 \) is not a resonance or eigenvalue implies \( W_Y(0, 0) \neq 0 \). The approximation (3.15) easily implies \( W_U(0, 0) = W_Y(0, 0) + O(\delta) \neq 0 \) for \( \delta \) small enough. □

4. Orbital Stability

In this section, we prove orbital stability, i.e. that solutions starting close to \( T \) are always close to some shifted version of \( T \).

Proof of Theorem 7.3. For any solution \( u \) of (1.6), the energy

\[
E(u) = \int_{\mathbb{R}} \left[ \frac{1}{2} (\partial_y u)^2 + \frac{1}{2} (\partial_y u)^2 - \frac{1}{2} cu^2 + F(u) \right] \omega(y) \, dy,
\]

is conserved, where \( \omega(y) = \exp(\int_{-\infty}^{y} b(z) \, dz) = 1 + O(\delta) \), uniformly in \( y \). We also define the potential energy

\[
E_p(u) = \int_{\mathbb{R}} \left[ \frac{1}{2} (\partial_y u)^2 - \frac{1}{2} cu^2 + F(u) \right] \omega(y) \, dy.
\]

A simple computation shows that

\[
|E_p(\psi) - \tilde{E}_p(\psi)| \leq C \delta \left( ||\psi||^2_{L^\infty(\mathbb{R})} + \tilde{E}_p(\psi) \right), \quad \psi \in H^1(\mathbb{R}),
\]

where \( \tilde{E}_p \) is the potential energy corresponding to the constant coefficient equation (1.4):

\[
\tilde{E}_p(\psi) := \int_{\mathbb{R}} \left[ \frac{1}{2} (\partial_y \psi)^2 + \frac{1}{2} (\partial_y \psi)^2 + F(\psi) \right] \, dy
\]

The idea is to use the (known) property that \( \tilde{E}_p(\psi) - \tilde{E}_p(S) \) controls the distance between \( \psi \) and \( S \), to show the corresponding fact for \( E_p \) and \( T \). In more detail, for \( q > 0 \), define

\[
d_q(\psi, T)^2 := \inf_{\xi \in \mathbb{R}} \int_{\mathbb{R}} [(\partial_y \psi(y) - T(y + \xi))^2 + q(\psi(y) - T(y + \xi))^2] \, dy,
\]

for any \( \psi \) in the energy space. We define \( d_q(\psi, S) \) in the analogous way. Proposition 1 of [20] proves the following: There exist \( C, r, q > 0 \) such that

\[
d_q(\psi, S)^2 \leq C(\tilde{E}_p(\psi) - \tilde{E}_p(S)),
\]

whenever \( d_q(\psi, S) \leq r \). Note that

\[
|\tilde{E}_p(T) - \tilde{E}_p(S)| = \int_{\mathbb{R}} \left| \frac{1}{2} (\partial_y S_0)^2 + \partial_y S \partial_y S_0 + F(S + S_0) - F(S) \right| \, dy
\]

\[
\leq \|S_0\|^2_{H^1(\mathbb{R})} + \|\partial_y S\|^2_{L^2(\mathbb{R})} + \|F\|_{C^1([a, b])} \|S_0\|_{L^1(\mathbb{R})}
\]

\[
\lesssim \delta.
\]

*Proposition 1 in [20] is stated for \( u(t, \cdot) \) where \( u \) is a solution of (1.4), but an examination of the proof shows that the conclusion holds for any \( \psi(y) \) satisfying the hypotheses stated here.
by Theorem 1.1. Using (4.1) twice, we then have
\[
d_q(\psi, S)^2 \leq C \left( E_p(\psi) - E_p(S) \right) + C\delta \left( \left\| \psi \right\|_2 L^\infty(\mathbb{R}) + \tilde{E}_p(\psi) + \tilde{E}_p(S) \right)
\]
(4.2)
\[
\leq C \left( E_p(\psi) - E_p(T) \right) + C\delta \left( 1 + \tilde{E}_p(\psi) + \tilde{E}_p(S) \right).
\]
To get to the last line, we used Sobolev embedding to write this term into the left-hand side.

Since \( \int_{\mathbb{R}} (u(t, y) - S(y + \xi))^2 \, dy \to \infty \) as \( \xi \to \pm \infty \), there is some \( \xi_0 \in \mathbb{R} \) where the optimum defining \( d_q(u(t, \cdot), S) \) is achieved. To save space, write \( T_\xi = T(y + \xi_0) \), and similarly for \( S_\xi \) and \( S_{b, \xi} \). We then have
\[
d_q(\psi, T)^2 \leq \int_{\mathbb{R}} \left[ \left( \partial_y \psi - T_\xi \right)^2 + q(\psi - T_\xi)^2 \right] \, dy
\]
(4.3)
\[
= \int_{\mathbb{R}} \left[ \left( \partial_y \psi - S_\xi \right)^2 + q(\psi - S_\xi)^2 \right] \, dy
\]
\[
+ \int_{\mathbb{R}} \left[ (S_{b, \xi})^2 + qS_{b, \xi}^2 - 2(\partial_y \psi - S_\xi)^2 \partial_y S_{b, \xi} - 2q(\psi - S_\xi)S_{b, \xi} \right] \, dy
\]
\[
\leq d_q(\psi, S)^2 + Cq\|S_b\|^2_{H^1(\mathbb{R})} + \|S_b\|_{H^1(\mathbb{R})}\|\psi - S\|_{H^1(\mathbb{R})}
\]
\[
\leq d_q(\psi, S)^2 + Cq\|S_b\|^2_{H^1(\mathbb{R})} + \|S_b\|_{H^1(\mathbb{R})} d_q(\psi, S)^2
\]
\[
\leq 2d_q(\psi, S)^2 + Cq^2.
\]
For \( \delta > 0 \) small enough compared to \( r \) and \( q \), this implies \( d_q(\psi, T)^2 \leq 2d_q(\psi, S)^2 + r/2 \). By exchanging the roles of \( T \) and \( S \) in this calculation, we also obtain \( d_q(\psi, S)^2 \leq 2d_q(\psi, T) + r/2 \).

Next, combining (4.2) and (4.3),
\[
d_q(\psi, T)^2 \leq C \left( E_p(\psi) - E_p(T) \right) + C\delta \left( 1 + \tilde{E}_p(\psi) + \tilde{E}_p(S) \right).
\]
(4.4)
This inequality holds for \( \psi \) such that \( d_q(\psi, S) \leq r \). By above, we can ensure this condition by choosing \( d_q(\psi, T) \leq r/4 \).

Now, for a solution \( u \) to (1.6) with \( d_q(u(0, \cdot), T) \leq r/4 \) and \( \partial_t u(0, \cdot) \) sufficiently small in \( L^2(\mathbb{R}) \), (4.4) implies that
\[
d_q(u(t, \cdot), T)^2 \leq C(E(u(t, \cdot)) - E(T)) + C\delta \left( 1 + \tilde{E}_p(u(t, \cdot)) + \tilde{E}_p(S) \right).
\]
The quantity \( E(u(t, \cdot)) \) is conserved in time. Calculations similar to (4.1) show that \( \tilde{E}_p(u) \leq 2E_p(u) + C\delta\|u\|^2_{L^\infty(\mathbb{R})} \leq E(u) + C\delta d_q(u, T)^2 \), and the last term may be combined into the left side.

We finally have
\[
d_q(u(t, \cdot), T)^2 \leq C(E(u_0) - E(T)) + C\delta \left( 1 + \tilde{E}_p(S) \right).
\]
This right-hand side is independent of \( t \), which implies the solution \( u \) never leaves the neighborhood of \( T \) as long as it exists. As above, for every \( t \), there is some \( \xi = \xi(t) \) at which the infimum defining \( d_q(u, T) \) is achieved. By standard arguments, this time-independent bound on \( d_q(u(t, \cdot), T) \) combined with energy conservation implies the solution \( u \) exists for all \( t \in [0, \infty) \). \( \square \)

**Appendix A. ODE Methods**

In this section, we collect some convenient facts about the solvability and asymptotics of \( 2 \times 2 \) first-order systems on \( \mathbb{R} \).

First, we have a standard lemma on vector-valued integral equations of Volterra type:

**Lemma A.1.** For \( a \in \mathbb{R} \) and \( U \in L^\infty([a, \infty), \mathbb{R}^2) \), the Volterra equation
\[
Z(y) = U(y) + \int_y^\infty K(y, w)Z(w) \, dw,
\]
where \( Z \) solves
\[
\begin{align*}
\left( \begin{array}{c}
\partial_y Z_1(y) \\
\partial_y Z_2(y)
\end{array} \right) & = \left( \begin{array}{c}
K(y, w)Z_1(w) \\
K(y, w)Z_2(w)
\end{array} \right), \\
Z_1(y) & = U_1(y), \\
Z_2(y) & = U_2(y)
\end{align*}
\]
has a unique solution in $L^\infty([a, \infty), \mathbb{R}^2)$, provided

\begin{equation}
\mu := \int_a^\infty \sup_{a < y < w} \|K(y, w)\| \, dw < \infty,
\end{equation}

where $\| \cdot \|$ is the operator norm of the matrix $K(y, w)$. This solution is given by the iteration

\begin{equation}
Z(y) = U(y) + \sum_{n=1}^\infty \int_a^\infty \cdots \int_a^\infty \prod_{i=1}^m 1_{(y_{i-1}, y_i)}K(y_{i-1}, y_i)U(y_n) \, dy_n \cdots \, dy_1,
\end{equation}

with $y_0 = y$. This solution satisfies

\[ \|Z\|_{L^\infty([a, \infty), \mathbb{R}^2)} \leq e^{\mu} \|U\|_{L^\infty([a, \infty), \mathbb{R}^2)}. \]

**Proof.** See [42, Lemma 2.4] for a proof of the corresponding fact for scalar-valued Volterra equations. The proof in the present vector-valued case is essentially the same, so we omit it. \qed

Next, we address a class of linear systems that arise from the eigenvalue problems in Sections 2 and 3.

**Lemma A.2.**

(a) For $k > 0$, consider the system

\begin{equation}
Y'(y) = (M_1 + M_2(y))Y(y),
\end{equation}

where

\[ M_1 = \begin{pmatrix} 0 & 1 \\ k^2 & 0 \end{pmatrix}, \quad M_2(y) = \begin{pmatrix} 0 & 0 \\ V(y) & -b(y) \end{pmatrix}, \]

with $V, b \in L^1(\mathbb{R})$. There exist solutions $Y_{-\infty}, Y_\infty$ defined on $\mathbb{R}$, such that

\begin{equation}
\lim_{y \to -\infty} e^{ky}Y_\infty(y) = \begin{pmatrix} 1 \\ -k \end{pmatrix}, \quad \lim_{y \to -\infty} e^{-ky}Y_{-\infty}(y) = \begin{pmatrix} 1 \\ k \end{pmatrix},
\end{equation}

and the bound $|Y_\infty(y)| \leq C e^{-ky}$ holds for all $y \in \mathbb{R}$, where the constant depends on $k$ and $\|V + b\|_{L^1(\mathbb{R})}$. These solutions also satisfy the integral equations

\begin{equation}
Y_\infty(y) = \begin{pmatrix} 1 \\ -k \end{pmatrix} e^{-ky} - \frac{1}{2} \int_y^\infty (V(w)Y_\infty - bY_\infty')(w) \left( \begin{pmatrix} e^{k(y-w)} - e^{-k(y-w)} \\ e^{k(y-w)} + e^{-k(y-w)} \end{pmatrix} \right) \, dw,
\end{equation}

\begin{equation}
Y_{-\infty}(y) = \begin{pmatrix} 1 \\ k \end{pmatrix} e^{ky} + \frac{1}{2} \int_{-\infty}^y (V(w)Y_{-\infty} - bY_{-\infty}')(w) \left( \begin{pmatrix} e^{k(y-w)} - e^{-k(y-w)} \\ e^{k(y-w)} + e^{-k(y-w)} \end{pmatrix} \right) \, dw.
\end{equation}

(b) For $k = 0$, assume in addition that $(1 + |y|^2)^{1/2} V$ and $(1 + |y|^2)^{1/2} b$ lie in $L^1(\mathbb{R})$. Then there exist solutions $Y_{-\infty}, Y_\infty$ to (A.3) satisfying

\[ \lim_{y \to -\infty} Y_{\pm\infty}(y) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \]

as well as the integral equations

\begin{equation}
Y_\infty(y) = \begin{pmatrix} 0 \\ y \end{pmatrix} - \int_y^\infty (V(w)Y_\infty - bY_\infty')(w) \left( \begin{pmatrix} y - w \\ 1 \end{pmatrix} \right) \, dw,
\end{equation}

\begin{equation}
Y_{-\infty}(y) = \begin{pmatrix} 0 \\ y \end{pmatrix} + \int_{-\infty}^y (V(w)Y_{-\infty} - bY_{-\infty}')(w) \left( \begin{pmatrix} y - w \\ 1 \end{pmatrix} \right) \, dw.
\end{equation}

**Proof.** (a) Note that the eigenvalues of $M_1$ are $\pm k$ corresponding to eigenvectors $\begin{pmatrix} 1 \\ \mp k \end{pmatrix}$. We will find a solution to the integral equation

\begin{equation}
Y_\infty(y) = \begin{pmatrix} 1 \\ -k \end{pmatrix} e^{-ky} - \int_y^\infty e^{M_1(y-w)}M_2(w)Y_\infty(w) \, dw, \quad y \in \mathbb{R},
\end{equation}
satisfying \(|Y_\infty(y)| \leq Ce^{-ky}\) and \(\lim_{y \to \infty} e^{ky}Y_\infty = \left(\frac{1}{-k}\right)\). By direct calculation, such \(Y_\infty\) also solves (A.3), as well as the first integral equation in (A.5). Letting \(Z(y) = e^{ky}Y_\infty(y)\), (A.7) is equivalent to

\[
Z(y) = \left(\frac{1}{-k}\right) - \int_y^\infty e^{M_1(y-w)}M_2(w)e^{k(y-w)}Z(w)\,dw, \quad y \in \mathbb{R}.
\]

By diagonalizing \(M_1\), we obtain

\[
e^{M_1(y-w)}M_2 = \frac{1}{2} \begin{pmatrix}
\frac{1}{y}V(e^{k(y-w)} - e^{-k(y-w)}) & -\frac{1}{y}b(e^{k(y-w)} - e^{-k(y-w)}) \\
V(e^{k(y-w)} + e^{-k(y-w)}) & -b(e^{k(y-w)} + e^{-k(y-w)})
\end{pmatrix}.
\]

With \(K(y, w) := e^{M_1(y-w)}M_2(w)e^{k(y-w)}\), we therefore have

\[
\|K(y, w)\| \leq C(1 + e^{2k(y-w)})(|V(w)| + |b(w)|),
\]

and that

\[
\int_0^\infty \sup_{0 < y < w} ||K(y, w)|| \, dw \leq C(||V||_{L^1(\mathbb{R})} + ||b||_{L^1(\mathbb{R})}).
\]

Lemma A.1 now implies a solution to (A.8) exists on \([0, \infty)\), and \(\|Z\|_{L^\infty([0, \infty), \mathbb{R}^2)}\) is bounded by a constant, which implies the boundary condition (A.4) holds for \(Y_\infty\), as well as the upper bound

\[
|Y_\infty(y)| \leq Ce^{-ky}, \quad y \geq 0,
\]

where \(Y_\infty = (Y_\infty, Y'_\infty)\). Applying a similar argument with \(-y\) replacing \(y\), we can obtain a solution \(Y_{-\infty}\) defined on \(\mathbb{R}\) with

\[
\lim_{y \to -\infty} e^{-ky}Y_{-\infty}(y) = \left(\frac{1}{k}\right) \quad \text{and} \quad |Y_{-\infty}(y)| + |Y'_{-\infty}(y)| \leq Ce^{ky}, \quad y \leq 0.
\]

For \(y < 0\), we can write

\[
Y_\infty(y) = c_0Y_{-\infty}(y) \left(\int_0^y \exp\left(-\int_\infty^w b(z) \, dz\right) \frac{\exp(b(y))}{Y^2_\infty(w)} \, dw + c_1\right),
\]

with \(c_0, c_1\) chosen so that \(Y_\infty(0)\) and \(Y'_\infty(0)\) match our previous definition. This formula implies \(Y_\infty = (Y_\infty, Y'_\infty)\) solves (A.3) and satisfies \(|Y_\infty(y)| \leq Ce^{-ky}\) for negative \(y\) also. By a similar method, we extend \(Y_{-\infty}\) to the real line and obtain \(|Y_{-\infty}(y)| \leq Ce^{ky}\) for all \(y \in \mathbb{R}\).

(b) In the case \(k = 0\), we have

\[
e^{M_1(y-w)}M_2(w) = \begin{pmatrix} 1 & y-w & 0 \\ 0 & 1 & 0 \\ V(y) & b(y) & 0 \end{pmatrix} = \begin{pmatrix} (y-w)V(y) & -(y-w)b(y) \\ V(y) & -b(y) \end{pmatrix},
\]

and the integral equation (A.7) reduces to

\[
Y_\infty(y) = \left(1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) - \int_y^\infty (V(w)Y_\infty - bY'_\infty(w)) \left(\begin{pmatrix} y-w \\ 1 \end{pmatrix}\right) \, dw.
\]

Defining \(K(y, w) = e^{M_1(y-w)}M_2(w)\), we have

\[
\|K(y, w)\| \leq C\sqrt{1 + (y-w)^2}(|V(w)| + |b(w)|),
\]

and

\[
\int_0^\infty \sup_{0 < y < w} ||K(y, w)|| \, dw \leq C(||(1 + |y|^2)^{1/2}V||_{L^1(\mathbb{R})} + ||(1 + |y|^2)^{1/2}b||_{L^1(\mathbb{R})}).
\]

By Lemma A.1, a solution \(Y_\infty\) exists on \([0, \infty)\), which also solves (A.3) by a direct calculation. Applying a similar method for \(Y_{-\infty}\) and extending both solutions to the real line proceeds as in the proof of (a). \(\Box\)
References

[1] M. A. Alejo, C. Muñoz, and J. M. Palacios. On the asymptotic stability of the sine-Gordon kink in the energy space. Preprint. ArXiv:2003.09358.

[2] J. Alexander, R. Gardner, and C. Jones. A topological invariant arising in the stability analysis of travelling waves. J. Reine Angew. Math., 410:167–212, 1990.

[3] C. Cacciapuoti, R. Carlone, and R. Figari. Perturbations of eigenvalues embedded at threshold: two-dimensional solvable models. J. Math. Phys., 52(8):083515, 2011.

[4] D. K. Campbell, M. Peyrard, and P. Sodano. Kink-antikink interactions in the double sine-Gordon equation. Phys. D, 19(2):165–205, 1986.

[5] G. Chen, J. Liu, and B. Lu. Long-time asymptotics and stability for the sine-Gordon equation. Preprint. arXiv:2009.04260, 2020.

[6] S. Cuccagna. On asymptotic stability in 3D of kinks for the \( \phi^4 \) model. Trans. Amer. Math. Soc., 360(5):2581–2614, 2008.

[7] S. Cuenda, N. R. Quintero, and A. Sánchez. Sine-Gordon wobbles through Bäcklund transformations. Discrete Contin. Dyn. Syst. Ser. S, 4(5):1047–1056, 2011.

[8] J. Cuevas-Maraver, P. G. Kevrekidis, and F. Williams, editors. The sine-Gordon model and its applications, volume 10 of Nonlinear Systems and Complexity. Springer, Cham, 2014. From pendula and Josephson junctions to gravity and high-energy physics.

[9] A. D’Anna, M. De Angelis, and G. Fiore. Towards soliton solutions of a perturbed sine-Gordon equation. Rend. Accad. Sci. Fis. Mat. Napoli (4), 72:95–110, 2005.

[10] J.-M. Delort. Existence globale et comportement asymptotique pour l’équation de Klein-Gordon quasi linéaire à données petites en dimension 1. Ann. Sci. École Norm. Sup. (4), 34(1):1–61, 2001.

[11] J.-M. Delort, D. Fang, and R. Xue. Global existence of small solutions for quadratic quasilinear Klein-Gordon systems in two space dimensions. J. Funct. Anal., 211(2):288–323, 2004.

[12] J.-M. Delort and N. Masmoudi. Long time dispersive estimates for perturbations of a kink solution of one dimensional cubic wave equations. Preprint: hal-02862414v2, 2020.

[13] J. Denzler. Nonpersistence of breather families for the perturbed sine Gordon equation. Comm. Math. Phys., 158(2):397–430, 1993.

[14] G. Derks, A. Doelman, S. A. van Gils, and T. Visser. Travelling waves in a singularly perturbed sine-Gordon equation. Phys. D, 180(1-2):40–70, 2003.

[15] W. Evans. Nerve axon equations. IV. The stable and the unstable impulse. Indiana Univ. Math. J., 24(12):1169–1190, 1974/75.

[16] G. Fiore, G. Guerriero, A. Maio, and E. Mazziotti. On kinks and other travelling-wave solutions of a modified sine-Gordon equation. Meccanica, 50(8):1989–2006, 2015.

[17] R. A. Gardner and K. Zumbrun. The gap lemma and geometric criteria for instability of viscous shock profiles. Comm. Pure Appl. Math., 51(7):797–855, 1998.

[18] P. Germain and F. Pusateri. Quadratic Klein-Gordon equations with a potential in one dimension. Preprint. arXiv:2006.15688, 2020.

[19] F. Gesztesy and H. Holden. A unified approach to eigenvalues and resonances of Schrödinger operators using Fredholm determinants. J. Math. Anal. Appl., 123(1):181–198, 1987.

[20] D. B. Henry, J. F. Perez, and W. F. Wreszinski. Stability theory for solitary-wave solutions of scalar field equations. Comm. Math. Phys., 85(3):351–361, 1982.

[21] P. D. Hislop and I. M. Sigal. Introduction to spectral theory, volume 113 of Applied Mathematical Sciences. Springer-Verlag, New York, 1996. With applications to Schrödinger operators.

[22] V. G. Ivancevic and T. T. Ivancevic. Sine-Gordon solitons, kinks and breathers as physical models of nonlinear field theories. Phys. Rev. E, 90:023208, Aug 2014.

[23] A. Khare and A. Saxena. Family of potentials with power law kink tails. J. Phys. A, 52(36):365401, 31, 2019.
[30] E. Kopylova and A. I. Komech. On asymptotic stability of kink for relativistic Ginzburg-Landau equations. *Arch. Ration. Mech. Anal.*, 202(1):213–245, 2011.

[31] E. A. Kopylova and A. I. Komech. On asymptotic stability of moving kink for relativistic Ginzburg-Landau equation. *Comm. Math. Phys.*, 302(1):225–252, 2011.

[32] M. Kowalczyk, Y. Martel, and C. Muñoz. Kink dynamics in the $\phi^4$ model: asymptotic stability for odd perturbations in the energy space. *J. Amer. Math. Soc.*, 30(3):769–798, 2017.

[33] M. Kowalczyk, Y. Martel, C. Muñoz, and H. V. D. Bosch. A sufficient condition for asymptotic stability of kinks in general $(1+1)$-scalar field models. *Preprint. ArXiv:2008.01276*.

[34] M. Kowalczyk, Y. Martel, C. Muñoz, and H. V. D. Bosch. A sufficient condition for asymptotic stability of kinks in general $(1+1)$-scalar field models. *Preprint. ArXiv:2008.01276*.

[35] H. Lindblad, J. Luhrmann, W. Schlag, and A. Soffer. On modified scattering for 1D quadratic Klein-Gordon equations with non-generic potentials. *Preprint. arXiv:2012.15191*, 2020.

[36] H. Lindblad, J. Luhrmann, and A. Soffer. Decay and asymptotics for the 1D Klein-Gordon equation with variable coefficient cubic nonlinearities. *Preprint. arXiv:1907.09922*, 2019.

[37] H. Lindblad and A. Soffer. Scattering for the Klein-Gordon equation with quadratic and variable coefficient cubic nonlinearities. *Trans. Amer. Math. Soc.*, 367(12):8861–8909, 2015.

[38] M. A. Lohe. Soliton structures in $P(\phi)_2$. *Phys. Rev. D*, 20:3120–3130, Dec 1979.

[39] N. Manton and P. Sutcliffe. *Topological solitons*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 2004.

[40] R. L. Pego and M. I. Weinstein. Eigenvalues, and instabilities of solitary waves. *Philos. Trans. Roy. Soc. London Ser. A*, 340(1656):47–94, 1992.

[41] J. Rauch. Perturbation theory for eigenvalues and resonances of Schrödinger Hamiltonians. *J. Functional Analysis*, 35(3):304–315, 1980.

[42] R. M. Ross, P. G. Kevrekidis, D. K. Campbell, R. Decker, and A. Demirkaya. $\phi^4$ solitary waves in a parabolic potential: existence, stability, and collisional dynamics. In *A dynamical perspective on the $\phi^4$ model*, volume 26 of *Nonlinear Syst. Complex.*, pages 213–234. Springer, Cham, 2019.

[43] W. Schlag, A. Soffer, and W. Staubach. Decay for the wave and Schrödinger evolutions on manifolds with conical ends. *I. Trans. Amer. Math. Soc.*, 362(1):19–52, 2010.

[44] H. Segur. Wobbling kinks in $\phi^4$ and sine-Gordon theory. *J. Math. Phys.*, 24(6):1439–1443, 1983.

[45] H. Segur and M. D. Kruskal. Nonexistence of small-amplitude breather solutions in $\phi^4$ theory. *Phys. Rev. Lett.*, 58(8):747–750, 1987.

[46] B. Simon. Schrödinger operators in the twentieth century. *Journal of Mathematical Physics*, 41(6):3523–3555, 2000.

[47] S. Snelson. Asymptotic stability for odd perturbations of the stationary kink in the variable-speed $\phi^4$ model. *Transactions of the American Mathematical Society*, 370(10):7437–7460, 2018.

[48] J. Sterbenz. Dispersive decay for the 1D Klein-Gordon equation with variable coefficient nonlinearities. *Trans. Amer. Math. Soc.*, 368(3):2081–2113, 2016.

[49] T. Vachaspati. *Kinks and domain walls*. Cambridge University Press, New York, 2006. An introduction to classical and quantum solitons.

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