On Finsler manifolds with hyperbolic geodesic flows

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Abstract. Let \((M, F)\) be a closed \(C^\infty\) Finsler manifold and \(\varphi\) its geodesic flow. In the case that \(\varphi\) is Anosov, we extend to the Finsler setting a Riemannian vanishing result of M. Gromov about the \(L^\infty\)-cohomology.

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1. Introduction. Let \(M\) be a closed \(C^\infty\) manifold of dimension \(n\) and \(TM_0 = TM \setminus \{0\}\). A Finsler metric on \(M\) is a function \(F : TM \to \mathbb{R}^+\) satisfying:

\[
\text{(a) } F(tv) = tF(v) \quad \text{for any } v \in TM \text{ and } t > 0;
\]

\[
\text{(b) } F \text{ is strictly positive and } C^\infty \text{ on } TM_0; \text{ and}
\]

\[
\text{(c) in standard local coordinates } (x_i, y_i) \text{ on } TM, \text{ the matrix } \left( \frac{\partial^2 F^2}{\partial y_i \partial y_j} \right)_{1 \leq i, j \leq n}
\]

is positively definite.

For simplicity, we write \(F(v)\) instead of \(F(x, v)\), where \((x, v) \in TM\). A Finsler metric \(F\) on \(M\) is said to be Riemannian if for any \(x \in M\), there exists an inner product \(g_x\) on \(T_xM\) such that \(F^2_x(v) = g_x(v, v) \quad \forall v \in T_xM\). There exist lots of non-Riemannian Finsler metrics for example the so-called Randers metrics (see [5]). Given a Finsler manifold \((M, F)\), let \(c : [a, b] \to M\) be a piecewise \(C^\infty\) curve in \(M\). Since \(F\) is positively homogeneous, the length of \(c\) is well-defined as \(L(c) = \int_a^b F(c'(t)) dt\). For any \(p, q \in M\), we define \(d(p, q) = \inf_c L(c)\), where the infimum is taken over all piecewise \(C^\infty\) curves \(c\) issuing from \(p\) to \(q\). The geodesics of \((M, F)\) are characterized as the constant speed curves locally minimizing the length, which are then necessarily \(C^\infty\) curves. The Finsler metric \(F\) is said to be reversible if \(F(-v) = F(v)\) for any \(v \in TM\). In this article, we shall consider general Finsler metrics without the reversibility assumption. So generally speaking, for any \(p, q \in M\), we do not necessarily have \(d(p, q) = d(q, p)\), i.e. the distance function \(d\) is asymmetric.
This kind of asymmetry occurs naturally in many applications and offers one of the most interesting aspects in Finsler geometry (see [1]).

Gromov showed in [10] that if $M$ is a closed Riemannian manifold of negative sectional curvature, then every closed bounded form of degree $\geq 2$ on $\tilde{M}$ has a bounded primitive. This means that the $L^\infty$-cohomology of $\tilde{M}$ vanishes in degree $\geq 2$.

Given a closed Riemannian manifold $(M, g)$ of negative curvature, it is well-known that its geometric hyperbolicity resulting from the negative curvature assumption implies that its geodesic flow defined on the sphere bundle $SM$ is Anosov, i.e. dynamically hyperbolic. Therefore, the geometric hyperbolicity can be considered as stronger than the dynamical hyperbolicity. In this perspective, Cheng extended in [4] the vanishing result above to closed Riemannian manifolds with Anosov geodesic flows. In [2], Burns and Paternain obtained the same vanishing result for Riemannian manifolds with Anosov magnetic flows. In this paper, we generalize this Riemannian result to the Finsler setting:

**Theorem 1.1.** If $M$ is a closed Finsler manifold with Anosov geodesic flow, then every closed bounded form of degree $\geq 2$ on $\tilde{M}$ has a bounded primitive.

The vanishing of the $L^\infty$-cohomology in degree $\geq 2$ is closely related to the following Riemannian question of S.S. Chern: let $M$ be a $2m$-dimensional closed Riemannian manifold of negative sectional curvature. Is it true that the sign of the Euler number $\chi(M)$ and $(-1)^m$ are the same, i.e. $(-1)^m \cdot \chi(M) > 0$? Gromov proved in [10] this conjecture in the Kähler hyperbolic case.

**Definition 1.2.** A closed Kähler manifold $(M, g, J)$ is said to be **Kähler hyperbolic** if the lifted Kähler form $\tilde{\omega}$ to the universal covering space $\tilde{M}$ admits a bounded primitive.

More precisely, Gromov proved the following result:

**Theorem 1.3** (Gromov [10]). Let $M$ be a $2m$-dimensional closed Kähler manifold. If $M$ is Kähler hyperbolic, then the Euler number $\chi(M)$ does not vanish and satisfies $(-1)^m \cdot \chi(M) > 0$.

Thus he obtained the following result: if $M$ is a $2m$-dimensional closed $C^\infty$ Riemannian manifold of negative sectional curvature, and $M$ is homotopy equivalent to a closed Kähler manifold, then the Euler number $\chi(M)$ does not vanish and satisfies $(-1)^m \cdot \chi(M) > 0$. We deduce from Theorems 1.1 and 1.3 the following:

**Corollary 1.** Let $M$ be a $2m$-dimensional closed $C^\infty$ Finsler manifold with Anosov geodesic flow. If $M$ is homotopy equivalent to a closed Kähler manifold, then the Euler number $\chi(M)$ does not vanish and satisfies $(-1)^m \cdot \chi(M) > 0$.

**2. Symplectic formulation of Finsler geodesic flows.** In this section, we recall some definitions concerning Finsler geodesic flows. See, for instance, [11] or [9] for more details. Let $(M, F)$ be a closed $C^\infty$ Finsler manifold and $\pi : TM_0 \to$
$M$ be the canonical projection. The potential of $(M,F)$ is defined as

$$A_v(\eta) = \frac{1}{2} \frac{d}{dt} (F^2(v + td\pi_v(\eta))) (0) \quad \forall v \in TM_0, \ \eta \in T_v(TM_0),$$

which is a $C^\infty$ 1-form on $TM_0$. In the case of a Riemannian Finsler metric $\sqrt{g}$, we have

$$A_v(\eta) = g(d\pi_v(\eta), v),$$

which is the so-called Liouville 1-form of the Riemannian metric $g$.

Let $E : TM_0 \to \mathbb{R}$ be defined by $E(v) = \frac{1}{2} (F(v))^2$. Let $\omega$ be the symplectic form on $TM_0$ obtained by pulling back the canonical symplectic form of $T^*M_0$ via the Legendre transform, then we have $\omega = dA$ (see [11, Section 2]). Let $X$ be the Hamiltonian vector field of $E$ with respect to $\omega$, i.e.

$$i_X \omega = -dE.$$

The Hamiltonian flow $\Phi$ generated by $X$ preserves the map $E$ and the symplectic form $\omega$. The projections of the orbits of $\Phi$ to $M$ are just the geodesics of $(M,F)$. The unit sphere bundle with respect to $F$ is defined as

$$S_F M = E^{-1} \left( \frac{1}{2} \right) = \{ v \in TM \mid F(v) = 1 \}.$$ 

The restriction of the Hamiltonian flow on $S_F M$ is said to be the geodesic flow of $(M,F)$, denoted by $\phi$. Since $\omega = dA$, $\phi$ integrates the Reeb field of the contact 1-form $A|_{SF M}$.

3. Anosov flows and transversality. Let us first recall some definitions: let $N$ be a closed $C^\infty$ manifold and $\psi : N \to N$ a $C^\infty$ flow generated by the vector field $X$. We say that $\psi$ is an Anosov flow if there exists a $D\psi$-invariant splitting of the tangent bundle

$$TN = E^{ss} \oplus \mathbb{R}X \oplus E^{su},$$

a Riemannian metric on $N$, and two positive numbers $a$ and $b$ such that

$$\|D\psi_t(v)\| \leq a \cdot e^{-bt} \| v \| \quad \forall v \in E^{ss}, \quad t \geq 0,$$

$$\|D\psi_{-t}(v)\| \leq a \cdot e^{-bt} \| v \| \quad \forall v \in E^{su}, \quad t \geq 0.$$ 

The vector bundles $E^{ss}$ and $E^{su}$ are called the strong stable and strong unstable distributions of $\psi$. They are both integrable to Hölder continuous foliations with $C^\infty$ leaves, denoted respectively by $F^{ss}$ and $F^{su}$. The vector bundles $E^{ss} \oplus \mathbb{R}X$ and $E^{su} \oplus \mathbb{R}X$ are called the stable and unstable distributions of $\psi$. They are also integrable to Hölder continuous foliations with $C^\infty$ leaves, denoted respectively by $F^s$ and $F^u$. For any $x \in N$, the leaves of $F^{ss}$, $F^{su}$, $F^s$, and $F^u$ containing $x$ are denoted respectively by $W^{ss}(x)$, $W^{su}(x)$, $W^s(x)$, and $W^u(x)$.

Let $(M,F)$ be a closed $C^\infty$ Finsler manifold of negative flag curvature (see [5]), it is well-known that its geodesic flow defined on $S_F M$ is Anosov [8]. See, for instance, [7] for a general study of Finsler geodesic flows in negative flag curvature. According to Theorem 2 and the comments concerning the Anosov hypothesis in [12], we have the following proposition:
Proposition 1 ([12]). Let $(M, F)$ be a closed $C^\infty$ Finsler manifold of dimension $n$ and $\varphi$ its geodesic flow defined on $SFM$. If $\varphi$ is Anosov, then its stable foliation $F^s$ is transverse to the fibers of the sphere bundle $\pi : SFM \to M$.

The proof of the following proposition is given only for the convenience of the reader since its arguments are known.

Proposition 2. Consider the lifted stable foliation $\tilde{F}^s$ on the covering space $\tilde{SF}M$. If the geodesic flow $\varphi$ is Anosov, then each leaf of $\tilde{F}^s$ intersects each fiber of the sphere bundle $\tilde{\pi} : \tilde{SF}M \to \tilde{M}$ exactly once. In addition, $\tilde{M}$ is diffeomorphic to $\mathbb{R}^n$, where $n = \dim(M)$.

Proof. Let $v \in S\tilde{F}M$ and $\tilde{W}^s(v)$ the leaf of $\tilde{F}^s$ containing $v$. We deduce from the above proposition that the foliation $\tilde{F}^s$ is transverse to the fibers of the sphere bundle $\tilde{\pi} : \tilde{SF}M \to \tilde{M}$. Since the fibers are compact, a result of C. Ehresmann (see [3]) implies that the map

$$\tilde{\pi} |_{\tilde{W}^s(v)} : \tilde{W}^s(v) \to \tilde{M}$$

is a covering map. Since $\tilde{M}$ is simply connected, $\tilde{\pi} |_{\tilde{W}^s(v)}$ is a diffeomorphism. Therefore, $\tilde{W}^s(v)$ intersects each fiber of the bundle $S\tilde{F}M \to \tilde{M}$ at exactly one point.

Take $v \in S\tilde{F}M$ such that $\tilde{W}^s(v)$ contains no lifts of $\varphi$-periodic orbits. It is well-known that $\tilde{W}^s(v)$ is diffeomorphic to $\mathbb{R}^n$. We deduce that $\tilde{M}$ is diffeomorphic to $\mathbb{R}^n$. □

4. Proof of Theorem 1.1. Let $(M, F)$ be a closed $C^\infty$ Finsler manifold with Anosov geodesic flow $\varphi$. Let $v \in S\tilde{F}M$ and $\tilde{W}^s(v)$ be the leaf containing $v$ of the foliation $\tilde{F}^s$. We denote by $\tilde{X}$ the tangent vector field of the lifted flow $\tilde{\varphi}$ over $S\tilde{F}M$. By Proposition 2, $\tilde{\pi}$ sends $\tilde{W}^s(v)$ diffeomorphically onto $\tilde{M}$. We define

$$\tilde{E}^{ss} = D\tilde{\pi}(\tilde{E}^{ss} |_{\tilde{W}^s(v)}) \quad \text{and} \quad \tilde{X} = D\tilde{\pi}(\tilde{X} |_{\tilde{W}^s(v)}).$$

Let $\tau$ be the $C^\infty$ flow on $\tilde{M}$ generated by $\tilde{X}$. Therefore, for any $t \in \mathbb{R}$,

$$\tau_t = \tilde{\pi} \circ \tilde{\varphi}_t \circ \tilde{\pi}^{-1}$$

and $\tilde{E}^{ss}$ is $\tau$-invariant. Since by Proposition 1 the distribution $E^s$ is transverse to the vertical bundle $V(SFM)$, there exists $C_1 > 0$ such that for any $\xi \in \tilde{E}^s(v)$,

$$\frac{1}{C_1} \tilde{F}(D\tilde{\pi}(\xi)) \leq |\xi| \leq C_1 \tilde{F}(D\tilde{\pi}(\xi)),$$

where $|\xi|$ is calculated with respect to the lift of an arbitrarily chosen Riemannian metric on $SF M$.

Proposition 3. There exist positive constants $C$ and $b$ such that for any $x \in \tilde{M}$ and any $u \in \tilde{E}^{ss}(x)$, we have that for any $t \geq 0$,

$$\tilde{F}(Dt\tau_t(u)) \leq Ce^{-bt}\tilde{F}(u).$$
Proof. Since \( \varphi \) is Anosov, there exist \( a, b > 0 \) such that for any \( \xi \in \tilde{E}^{ss} \),
\[
|D\tilde{\varphi}_t(\xi)| \leq ae^{-bt} |\xi|
\]
for any \( t \geq 0 \). Let \( \xi \in \tilde{E}^{ss} \) such that \( d\tilde{\varphi}(\xi) = u \). We have
\[
\tilde{F}(D\tau_t(u)) = \tilde{F}(D\tilde{\varphi}(D\varphi_t(\xi))) \leq C_1 |D\varphi_t(\xi)| \\
\leq C_1 e^{-bt} |\xi| \leq C_1 e^{-bt} \tilde{F}(u)
\]
for any \( t \geq 0 \). We set \( C = C_1^2 a \).

**Definition 4.1.** A \( C^\infty \) \( k \)-form \( \alpha \) on \( \tilde{M} \) is said to be **bounded** if
\[
\||\alpha||_\infty = \sup_{x \in \tilde{M}} \|\alpha\|_x < +\infty,
\]
where \( \|\alpha\|_x = \sup_{e_1, \ldots, e_k \in T_x\tilde{M} \setminus \{0\}} |\alpha(e_1, \ldots, e_k)| \).

Now let us prove Theorem 1.1: we suppose that \( k \geq 2 \), let \( \alpha \) be a closed bounded \( C^\infty \) \( k \)-form on \( \tilde{M} \). Since \( \tilde{M} \) is diffeomorphic to \( \mathbb{R}^n \), \( \alpha \) is exact. Let us prove that there exists a bounded 1-form \( \beta \) such that \( \alpha = d\beta \). Let \( T \) be an arbitrary, fixed real number, and \( \tau \) be the flow on \( \tilde{M} \) generated by the vector field \( \tilde{X} \). Then we define
\[
\beta_T = - \int_0^T \tau^*_s(i_{\tilde{X}} \alpha) ds.
\]
Since \( \alpha \) is closed, we have
\[
\alpha - \tau^*_T \alpha = - \int_0^T \frac{d}{ds} \tau^*_s \alpha ds = - \int_0^T \tau^*_s L_{\tilde{X}} \alpha ds = - \int_0^T \tau^*_s di_{\tilde{X}} \alpha ds = d\beta_T.
\]
As we have seen above, \( \tilde{\pi} : \tilde{W}^s(v) \to \tilde{M} \) is a \( C^\infty \) diffeomorphism. Moreover, \( \tilde{E}^{ss} = D\tilde{\varphi}(\tilde{E}^{ss} |_{\tilde{W}^s(v)}) \) and \( \tilde{X} = D\tilde{\varphi}(\tilde{X} |_{\tilde{W}^s(v)}) \). We deduce that
\[
T\tilde{M} = \tilde{E}^{ss} \oplus \mathbb{R} \cdot \tilde{X}.
\]
Now let \( x \in \tilde{M} \) and \( w \in T_x\tilde{M} \). Thus there exist \( w' \in \tilde{E}^{ss}(x) \) and \( w'' \) colinear with \( \tilde{X}(x) \) such that \( w = w' + w'' \). Since the distribution \( E^{ss} \) on \( SFM \) is transverse to the vector field \( \tilde{X} \), there exists a positive constant \( C_2 \) such that
\[
\tilde{F}(w') \leq C_2 \tilde{F}(w).
\]
Therefore, for any \( w_1, \ldots, w_{k-1} \in T_x\tilde{M} \), we have, by Proposition 3, the following:
\[
|\tau^*_s(i_{\tilde{X}} \alpha) x(w_1, \ldots, w_{k-1})| \\
= |\alpha_{\tau^*_s(x)}(\tilde{X}(\tau^*_s(x)), D_x\tau^*_s(w_1), \ldots, D_x\tau^*_s(w_{k-1}))| \\
= |\alpha_{\tau^*_s(x)}(\tilde{X}(\tau^*_s(x)), D_x\tau^*_s(w'_1), \ldots, D_x\tau^*_s(w'_{k-1}))| \\
\leq \|\alpha\|_\infty \tilde{F}(x)(\tau^*_s(x)) \tilde{F}(D_x\tau^*_s(w'_1)) \cdots \tilde{F}(D_x\tau^*_s(w'_{k-1})) \\
\leq \|\alpha\|_\infty C_1 \|X\|_\infty C^{k-1} e^{-(k-1)bs} \tilde{F}(w'_1) \cdots \tilde{F}(w'_{k-1}) \\
\leq \|\alpha\|_\infty C_1 \|X\|_\infty C^{k-1} e^{-(k-1)bs} C_2^{k-1} \tilde{F}(w_1) \cdots \tilde{F}(w_{k-1}),
\]
where $\|X\|_\infty$ is calculated with respect to an arbitrarily chosen Riemannian metric on $S_F M$, and we are considering, without loss of generality, that all $k - 1$ terms of type $C_2$ concerning $w'_i$ are equal, $i = 1, \ldots, k - 1$. Hence the form $\beta_T$ converges as $T \to +\infty$ to a form $\beta$ such that

$$|\beta_x(w_1, \ldots, w_{k-1})| \leq \|\alpha\|_\infty \|X\|_\infty C_1 C^{k-1} C_2^{k-1} \tilde{F}(w_1) \cdots \tilde{F}(w_{k-1}),$$

thus $\beta$ is bounded. Moreover, since $\tau^*_T \alpha$ tends to zero, $\alpha = d\beta$. The proof of Theorem 1.1 is complete.

The proof of Corollary 1 is given for the convenience of the reader (see [4]): suppose that $\Psi : M \to M_1$ is a homotopy equivalence of $M$ to a closed Kähler manifold $M_1$. By an approximation if necessary, we can assume that $\Psi$ is $C^\infty$. Let $\pi : \tilde{M} \to M$ and $\pi_1 : \tilde{M}_1 \to M_1$ be the universal covering maps. Let $\omega$ be the Kähler form of $M_1$, which is certainly bounded. Therefore, with respect to the lifted Finsler metric $\tilde{F}$, the pull back $(\Psi \circ \pi)^*(\omega) = \pi^*(\Psi^*(\omega))$ is also a bounded 2-form on $\tilde{M}$. Thus by Theorem 1.1, there exists a bounded 1-form $\beta$ on $\tilde{M}$ such that

$$(\Psi \circ \pi)^*(\omega) = d\beta.$$

Then we can deduce as in [6] that $\tau^*_1(\omega)$ also admits a bounded primitive, i.e. $M_1$ is Kähler hyperbolic. So by Theorem 1.3, we have $(-1)^m \cdot \chi(M_1) > 0$. Since the Euler number is a homotopy invariant, we get $(-1)^m \cdot \chi(M) > 0$.

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