A Brief Note on Single Source Fault Tolerant Reachability

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Abstract

Let $G$ be a directed graph with $n$ vertices and $m$ edges, and let $s \in V(G)$ be a designated source vertex. We consider the problem of single source reachability (SSR) from $s$ in presence of failures of edges (or vertices). Formally, a spanning subgraph $H$ of $G$ is a $k$-Fault Tolerant Reachability Subgraph ($k$-FTRS) if it has the following property. For any set $F$ of at most $k$ edges (or vertices) in $G$, and for any vertex $v \in V(G)$, the vertex $v$ is reachable from $s$ in $G - F$ if and only if it is reachable from $s$ in $H - F$. Baswana et.al. [STOC 2016, SICOMP 2018] showed that in the setting above, for any positive integer $k$, we can compute a $k$-FTRS with $2^k n$ edges. In this paper, we give a much simpler algorithm for computing a $k$-FTRS, and observe that it extends to higher connectivity as well. Our results follow from a simple application of important separators, a well known technique in Parameterized Complexity.

1 Introduction

Fault tolerant data structures aim to capture properties of real world networks, which are often prone to a small number of failures. Such data structures allow us to test various properties of the network after failures have occurred, and the repairs are awaited. The problem is modeled as a directed graph (digraph) $G$ where a small number of edges (or vertices) have failed, and a parameter $k$ is used as a bound on the maximum number of failures that may occur at a time. In this paper, we consider the problem of deciding the reachability of all vertices $v \in V(G)$ from a designated source vertex $s \in V(G)$ upon the failure of any $k$ edges (or vertices) in the input graph $G$. Specifically, our objective is to construct a sparse spanning subgraph $H$ of $G$ that preserves all reachability relationships from the source vertex $s$ upon the failure of any $k$ edges (or vertices) in $G$. More formally, we seek a spanning subgraph $H$ of $G$ with the following property: For any set $F$ of at most $k$ edges (or vertices) in $G$, and for any vertex $v \in V(G)$, there is a path from $s$ to $v$ in $G - F$ if and only if there is a path from $s$ to $v$ in $H - F$. Such a graph $H$ is called a $k$-Fault Tolerant Reachability Subgraph ($k$-FTRS). Observe that, beyond the question of deciding the reachability of a vertex $v$, the graph $H$ may also be used to find an alternate route from $s$ to $v$, if one exists, upon the failure of the edges in $F$. Now, the problem is formally defined as follows. Given as input a digraph $G$, a designated source vertex $s \in V(G)$ and an integer $k$, we must output a spanning subgraph $H$ of $G$ that is a $k$-FTRS.

Recently, Baswana et al. [3] presented an algorithm for computing a $k$-FTRS. Specifically, their algorithm runs in time $O(2^k mn)$ for a digraph $G$ of $n$ vertices and $m$ edges, and produces a $k$-FTRS where the in-degree of any vertex is upper bounded by $2^k$. Their algorithm is based on the notion of farthest min-cut that was introduced by Ford and Fulkerson [14]. They suggest that their methods may be of independent interest in other problems. This is indeed so, for the notion of important separators, which generalizes the notion of furthest cuts, is a well-known technique in Parameterized Complexity [9].

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The notion of important separators was introduced by Marx [20] to give an FPT algorithm for Multiway Cut. Subsequently, important separators and techniques based on them have been used to resolve the complexity of several important problems such as Directed Feedback Vertex Set [6], Multicut [21], Directed Multiway Cut [7], Almost 2-SAT [24], Parity Multiway Cut [18] and a linear-time FPT algorithm for Directed Feedback Vertex Set [19]. Informally speaking, important separators capture the entire collection of furthest cuts in a graph that have a bounded cardinality. We refer the reader to the textbook of Cygan et al. [9] for an introduction to important separators, and more generally to the various tools and techniques in Parameterized Complexity.

Using the notion of important separators, we give a very simple and conceptually appealing algorithm for computing a \( k \)-FTRS. Indeed, we generalize the problem slightly. Given a digraph \( G \), a designated source vertex \( s \in V(G) \), an integer \( \lambda \) and an integer \( k \), we output a spanning subgraph \( H \) of \( G \) such that for any set \( F \) of at most \( k \) edges (or vertices) and for any vertex \( v \in V(G) \), there are \( \lambda \) edge-disjoint paths from \( s \) to \( v \) in \( G - F \) if and only if there are \( \lambda \) such paths in \( H - F \). The graph \( H \) is called a \((\lambda, k)\)-Fault Tolerant Reachability Subgraph (\((\lambda, k)\)-FTRS). As before, \( H \) may be used for both testing the \( \lambda \)-connectivity of a vertex \( v \) from \( s \), as well as for obtaining an alternate collection of \( \lambda \) edge-disjoint paths from \( s \) to \( v \), if they exist, after the failure of up to \( k \) edges (or vertices). In particular, we obtain the following theorem.

**Theorem 1.** There is a \((\lambda, k)\)-FTRS where each vertex has in-degree at most \((k + \lambda)4^{k+\lambda}\). Further, this graph can be constructed in time \( O(4^{k+\lambda}(k + \lambda)^2(m + n)m) \).

Our bound and running time, obtained by a direct application of known results on important separators, are slightly worse than those given by Baswana et al. [3]. Our intent here is to provide a conceptual exposition of the results of Baswana et al. [3] and further, introduce important separators as an algorithmic tool for problems on fault tolerant networks, and more generally for distributed network problems. Due to space constraints the discussion of other related works has been postponed to the appendix.

**Related Work.** While a number of results are known about graph reachability for undirected graphs starting from the work of Nagamochi and Ibaraki [22], and more generally in the dynamic model [6] [12], relatively little is known in the case of digraphs. Patrascu and Thorup [23] considered the problem of edge failures in undirected graphs. They constructed a data structure that processes a batch of \( k \) edge failures in \( O(k \log^2 n \log \log n) \) time, and then answers connectivity queries for any pair of vertices in \( O(\log \log n) \) time. Subsequent results have led to a randomized data structure that uses almost-linear space (in the number of vertices) and correctly answers the queries with high probability [10]. For vertex failures in undirected graphs, Duan and Pettie [11] gave a data structure that can process a batch of \( k \) vertex failures in \( O(k^{2c+4} \log^2 n \log \log n) \) time and thereafter answer connectivity queries in \( O(k) \) time. Here \( c \) is a parameter of the algorithm offering a tradeoff between the running time and the space used. Recently, they improved this algorithm to process the \( k \) failures in \( O(k^3 \log^3 n) \) time and with the same query time as before, while using \( O(km \log n) \) space [12] [13]. We refer to [12] for further details.

In digraphs, other than [3], an optimal oracle for dual fault tolerant reachability was proposed by Choudhary [5]. Furthermore, fault tolerance in digraphs for shortest paths [10] [3] [17] [5] [4] and strongly connected components [15] [2] were also studied. The various problems in the fault tolerant model for digraphs are subject to extensive ongoing research. We refer to [3] [8] [12] [2] [4] for a detailed discussion and further details.

\(^{1}\)Here we have a sequence of edge insertions and deletions, as well as reachability queries, and we must efficiently update the data structure and answer the queries.
2 Preliminaries

Let \( G \) be a digraph on \( n \) vertices and \( m \) edges. For a subset of edges \( X \subseteq E(G) \), we let \( G - X \) denote the subgraph of \( G \) with vertex set \( V(G) \) and edge set \( E(G) \setminus X \). We omit the braces when the set contains only a single edge, i.e. \( G - \{e\} \) is denoted by \( G - e \). For an edge \( e = (u, v) \), \( u \) is called the tail of \( e \) and \( v \) is called the head of \( e \). We denote these vertices by \( \text{tail}(e) \) and \( \text{head}(e) \), respectively. For \( R \subseteq V(G) \), \( \delta^-(R) \) denotes the set of in-coming edges to \( R \), i.e. the set of edges of \( G \) such that \( \text{tail}(e) \in V(G) \setminus R \) and \( \text{head}(e) \in R \). Similarly, \( \delta^+(R) \) denotes the set of out-going edges from \( R \). Let \( G \) be a digraph and \( S \) and \( T \) be two disjoint subsets of \( V(G) \). A (directed) \((S, T)\)-cut is a subset \( X \) of edges of \( G \) such that there is no path from a vertex in \( S \) to a vertex in \( T \) in \( G - X \). Any minimal \((S, T)\)-cut can be expressed as \( \delta^+(R) = X \) where \( S \subseteq R \subseteq V(G) \setminus T \) is the set of vertices that are reachable from some vertex of \( S \) in \( G - X \). Hence, for any \((S, T)\)-cut \( X \), let \( R_X \) denote the set of vertices that are reachable from \( S \) in \( G - X \). The set \( R_X \) is called the \( \text{reachability set} \) of \( X \), and if \( X \) is minimal, then \( X = \delta^+(R_X) \).

An \((S, T)\)-cut \( X \) is an \( \text{important} \) \((S, T)\)-separator if there is no other \((S, T)\)-cut \( X' \) such that \( |X'| \leq |X| \) and \( R_X \subseteq R_{X'} \). (Observe that if \( X \) is an important \((S, T)\)-separator, then it is a minimal \((S, T)\)-cut.) Let \( \mathbb{X}_k(S, T) \) denote the collection of all important \((S, T)\)- separators in \( G \) of size at most \( k \). The following observation follows immediately from the definition of important separators.

**Observation 1.** Let \( Y \) be any \((S, T)\)-cut of size at most \( k \) in \( G \). Then, there is an important \((S, T)\)-separator \( X \in \mathbb{X}_k(S, T) \) such that \( |X| \leq |Y| \) and \( R_Y \subseteq R_X \).

We have the following result on the collection \( \mathbb{X}_k(S, T) \).

**Lemma 2** (Theorem 8.36 [9]). The cardinality of \( \mathbb{X}_k(S, T) \) is upper bounded by \( 4^k \). Furthermore, \( \mathbb{X}_k(S, T) \) can be computed in time \( \mathcal{O}(|\mathbb{X}_k(S, T)|k^2(m + n)) \).

In this paper, we describe the construction of a \((\lambda, k)\)-FTRS with respect to edge failures only, because any vertex failure can be modeled by an edge failure: Split every vertex \( v \) into an edge \((w_{in}, v_{out})\), where the incoming and outgoing edges of \( v \) are respectively directed into \( v_{in} \) and directed out of \( v_{out} \) [3].

3 A simple algorithm for \((\lambda, k)\)-FTRS

Let us first consider the case where \( \lambda = 1 \), i.e. the construction of a \( k\)-FTRS. Let \( v \in V(G) \setminus \{s\} \) be an arbitrary vertex, and let \( \mathbb{X}(v) \) denote the collection of all important \((s, v)\)-separators in \( G \) of size at most \( k + 1 \). By Lemma 2 there are at most \((k + 1)4^{k+1} \) edges in the union of all such important separators. We have the following claim.

**Lemma 3.** Let \( e \in \delta^-(v) \setminus (\bigcup_{X \in \mathbb{X}(v)} X) \). Then, \( G - e \) is a \( k\)-FTRS of \( G \).

**Proof.** Suppose not, and consider a set \( F \) of at most \( k \) edges and a vertex \( w \in V(G) \) such that \( w \) is unreachable from \( s \) in \((G - e) - F \), but there is a path from \( s \) to \( w \) in \( G - F \). Since \( e \in \delta^-(v) \), it follows that \( v \) is unreachable from \( s \) in \((G - e) - F \), but there is a path from \( s \) to \( v \) in \( G - F \). Thus, \( Y = F \cup \{e\} \) is an \((s, v)\)-cut in \( G \). Since \( G \) and \( G - e \) only differ on \( e \), this implies that \( e \) lies on every path from \( s \) to \( v \) in \( G - F \). We may conclude the following: (a) \( Y \) is an \((s, v)\)-cut in \( G \) of size at most \( k + 1 \); (b) the vertex \( u := \text{tail}(e) \) belongs to the reachability set of \( Y \), i.e. \( u \in R_Y \).

Now, consider the \((s, v)\)-cut \( Y \) in the graph \( G \) and the collection \( \mathbb{X}(v) \). Since \( e \notin \bigcup_{X \in \mathbb{X}(v)} X \), i.e. \( e \notin X \) for any \( X \in \mathbb{X}(v) \), and \( e = (u, v) \), it follows that \( u \notin R_X \) for any \( X \in \mathbb{X}(v) \). However, \( u \in R_Y \) and \( Y \) is \((s, v)\)-cut of cardinality \( k + 1 \) in \( G \). This is a contradiction to Observation 1. Hence, \( G - e \) is a \( k\)-FTRS of \( G \).
The above lemma can be turned into an iterative algorithm that gradually bounds the indegree of each vertex in the graph. Let \( \alpha = (k + 1)4^{k+1} \) denote the upper-bound on \( |\delta^-(v) \cap (\bigcup_{X \in \mathcal{X}(v)} X)| \) for any vertex \( v \in V(G) \setminus \{s\} \).

**Algorithm 1** An algorithm to compute a \( k \)-FTRS of a digraph \( G \) with a source vertex \( s \).

1: procedure FTRS\( (G, s, k) \)
2: Delete all incoming edges of the source vertex \( s \) in \( G \).
3: while there exists \( v \in V(G) \setminus \{s\} \) such that \( |\delta^-(v)| > \alpha \) do
4: Compute \( \mathcal{X}(v) \) via Lemma 2.
5: Pick an edge \( e \in \delta^-(v) \setminus (\bigcup_{X \in \mathcal{X}(v)} X) \) and delete it.
6: end while
7: end procedure

The correctness of the above algorithm follows from Lemma 3. Furthermore, it is clear that this algorithm terminates once the in-degree of every vertex is upper bounded by \( \alpha \). Hence, it runs in time \( O(4^k k^2 m(m+n)) \). This gives us the following corollary.

**Corollary 4.** There is a \( k \)-FTRS where each vertex has in-degree at most \( (k + 1)4^{k+1} \). Further, this graph can be constructed in time \( O(4^k k^2 (m+n) m) \).

The following simple lemma extends the above construction (and also the construction of Baswana et al. [3]) to any value of \( \lambda \).

**Lemma 5.** Let \( H \) be a \((k + \lambda - 1)\)-FTRS of \( G \). Then, \( H \) is also a \((\lambda, k)\)-FTRS of \( G \).

*Proof.* Suppose not. Consider a vertex \( v \in V(G) \setminus \{s\} \) and a set of \( F \) of at most \( k \) edges such that the following holds: there are \( \lambda \) edge-disjoint paths from \( s \) to \( v \) in \( G - F \), but there is no such collection of paths in \( H - F \). Consider a minimum \((s,v)\)-cut \( X \) in \( H - F \), and observe that \( |X| \leq \lambda - 1 \). It follows that \( Y = F \cup X \) is an \((s,v)\)-cut in \( H \) of size at most \( k + \lambda - 1 \). However, \( Y \) is not an \((s,v)\)-cut in \( G \). This is a contradiction to the fact that \( H \) is a \((k + \lambda - 1)\)-FTRS of \( G \). Therefore, \( H \) must be a \((\lambda,k)\)-FTRS of \( G \). \( \square \)

The proof of Theorem 1 follows from Corollary 4 and Lemma 5.

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