Complete intersection theorem and complete nontrivial-intersection theorem for systems of set partitions

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Abstract

We prove the complete intersection theorem and the complete nontrivial-intersection theorem for systems of set partitions. This means that for all positive integers \( n \) and \( t \) we find the maximum size of a family of partitions of \( n \)-element set such that any two partitions from the family have at least \( t \) common parts and we also find the maximal size under the additional condition that no \( t \) parts appear in all members of the family.

I Introduction

Let \( \Pi(n) \) be the set of partitions of \([n]\). Define the intersection of two partitions \( p_1 \cap p_2, \ p_1, p_2 \in \Pi(n) \) to be the set of common parts (blocks). We say that two partitions \( p_1, p_2 \in \Pi(n) \) are \( t \)-intersecting if the size of their intersection is at least \( t \). A family of partitions is a \( t \)-intersecting family if every two members of it are \( t \)-intersecting. The collection of \( t \)-intersecting families of partitions of \([n]\) is denoted by \( \Omega(n,t) \). We say that the family of partitions is nontrivially \( t \)-intersecting family if it is \( t \)-intersecting and fewer than \( t \) parts are common to all its members. The collection of nontrivially \( t \)-intersecting families of partitions we denote by \( \tilde{\Omega}(n,t) \).

We say that \( i \) is fixed in a partition \( p \in \Pi(n) \) if \( \{i\} \) is a singleton \( \{i\} \) block in \( p \). For \( p \in \Pi(n) \) let \( f(p), \ p \in \Pi(n) \) denote the set of points fixed by \( p \).
Define
\[ M(n, t) = \max\{|A| : A \in \Omega(n, t)\}, \]
\[ \tilde{M}(n, t) = \max\{|A| : A \in \tilde{\Omega}(n, t)\}. \]

The main result of the present work is obtaining explicit expression for \( M(n, t) \) (Theorems 1 or Theorem 2) and \( \tilde{M}(n, t) \) (Theorem 6) for all \( n \) and \( t \). The word complete in the phrase Complete Intersection Theorem underline the fact that the problem of determining values \( M(n, k) \) and \( \tilde{M}(n, k) \) is solved completely for all \( n, t \). We also say that the solution of the above problems are complete.

Let \( B(n) \) be the number of partitions of the set \([n]\), which is called the Bell number. Let also \( \tilde{B}(n) \) be the number of partitions of the set \([n]\) that do not have singletons. The Bell numbers satisfy the following relations
\[
B(n) = \frac{1}{e} \sum_{i=0}^{\infty} \frac{i^n}{i!},
\]
\[
\tilde{B}(n) = \frac{1}{e} \sum_{i=0}^{\infty} \frac{i^n}{(i+1)!},
\]
\[
B(n+1) = \sum_{i=0}^{n} \binom{n}{i} B(i)
\]
whereas \( \tilde{B}(n) \) satisfies the following relation
\[
\tilde{B}(n) = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} B(i)
\]

Define
\[
\gamma(\ell) = \frac{\sum_{i=0}^{n-\ell+1} \tilde{B} \left( n - \frac{\ell + t}{2} + 1 - i \right) \binom{n-\ell+1}{i}}{\sum_{i=0}^{n-\ell} \tilde{B} \left( n - \frac{\ell + t}{2} - i \right) \binom{n-\ell}{i}}.
\]
Note that, when \( \ell \) is fixed,
\[
\gamma(\ell) \rightarrow \infty, \text{ as } n \rightarrow \infty.
\]

Our first main result is the following theorem.

**Theorem 1**

\[
M(n, t) = \max_{r \in \left[0, \left\lfloor \frac{n-t}{2} \right\rfloor \right]} \{|p \in \Pi(n) : |[t + 2r] \cap f(p)| \geq t + r\}|.
\]

It follows from the proof of Theorem 1 that it can be reformulated as follows:
Theorem 2 Let $\ell = t + 2r$ be the largest number not greater than $n$ satisfying the relation
\[
\frac{\ell - t}{2(\ell - 1)} \gamma(\ell) \leq 1.
\] (4)

For this value of $\ell$ we have
\[
M(n, t) = \sum_{i=t+r}^{t+2r} \binom{t+2r}{i} \sum_{j=0}^{n-t-2r} \binom{n-t-2r}{j} \tilde{B}(n-i-j).
\] (5)

Our proof of this theorem is an extension of the ideas from [10], where the complete intersection theorem was proved for a family of $t$-cycle-intersecting permutations.

Remark. Each permutation of $[n]$ is determined by the set of cyclic permutations. Cycle-intersection of two permutations is the set of their common cycles. We say that two permutations are $t$-cycle-intersecting if the size of their intersection is at least $t$.

It is proved in [1] that $M(n, 1) = B(n-1)$ and for sufficiently large $n$ in terms of $t$ that $M(n, t) = B(n - t)$.

Our theorem completes the solution of the problem of determination of the value $M(n, t)$ for all $n$ and $t > 1$.

Let $2^{[n]}$ be the family of subsets of $[n]$ and $\binom{[n]}{k}$ be the family of $k$-element subsets of $[n]$. We say that a family $\mathcal{A} \subset 2^{[n]}$ is a $t$-intersecting family if for the arbitrary elements $a_1, a_2 \in \mathcal{A}$ the size of their intersection $|a_1 \cap a_2| \geq t$. Let $I(n, t)$ be the collection of $t$-intersecting families $\mathcal{A}$ of $[n]$, $I(n, k, t)$ be the collection of $t$-intersecting $k$-element families from $[n]$ and $\tilde{I}(n, t)$, $\tilde{I}(n, k, t)$ the collection of nontrivially $t$-intersecting families ($\bigcap_{A \in \mathcal{A}} A < t$). Define
\[
\tilde{M}(n, k, t) = \max_{\mathcal{A} \in \tilde{I}(n, k, t)} |\mathcal{A}|.
\]

Hilton and Milner proved the next theorem in [7].

Theorem 3 If $n > 2k$, then
\[
\tilde{M}(n, k, t) = \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1.
\]
This theorem was proved by Frankl [8] for $t > 1$.

**Theorem 4** There exists $n_0(n, k)$ such that if $n > n_0(n, k)$, then

- If $t + 1 \leq k \leq 2t + 1$, then $\tilde{M}(n, k, t) = |\nu_1(n, k, t)|$, where
  \[
  \nu_1(n, k, t) = \left\{ V \in \binom{[n]}{k} : \left| [t + 2] \cap V \right| \geq t + 1 \right\},
  \]

- If $k > 2t + 1$, then $\tilde{M}(n, k, t) = |\nu_2(n, k, t)|$, where
  \[
  \nu_2(n, k, t) = \left\{ v \in \binom{[n]}{k} : [t] \subset V, \ V \cap [t + 1, k + 1] \neq \emptyset \right\}
  \cup \left\{ [k + 1] \setminus \{i\} : i \in [t] \right\}.
  \]

In [5], the problem of determining $\tilde{M}(n, k, t)$ was solved completely for all $n, k, t$:

**Theorem 5**

- If $2k - t < n \leq (t + 1)(k - t + 1)$, then
  \[
  \tilde{M}(n, k, t) = M(n, k, t);
  \]

- If $(t + 1)(k - t + 1) < n$ and $k \leq 2t + 1$, then
  \[
  \tilde{M}(n, k, t) = |\nu_1(n, k, t)|;
  \]

- If $(t + 1)(k - t + 1) < n$ and $k > 2t + 1$, then
  \[
  \tilde{M} = \max\{|\nu_1(n, k, t)|, |\nu_2(n, k, t)|\}.
  \]

Note also that the value $M(n, k, t)$ was determined for all $n, k, t$ by Ahlswede and Khachatrian in the paper [6].

Before formulating our second main result, let’s make some additional definitions.

\[
\mathcal{H}_i = \left\{ H \in \binom{[t + i]}{t + 1} : [t] \subset H \right\}
\]

\[
\bigcup \left\{ H \in \binom{[t + i]}{t + i - 1} : [t + 1, t + i] \subset H \right\}.
\]
For $C \subset 2^n$, denote by $W(C)$ the minimal upset containing $C$ and by $M(C)$ the set of its minimal elements. Denote by $U(C)$ the set of partitions that has $W(C)$ as the family of sets of fixed elements.

Our second main result of this work is the following Theorem which completely determines $\tilde{M}(n, t)$ for all $n, t$.

**Theorem 6**

- If
  
  $\max \left\{ \ell = t + 2r : \frac{\ell - t}{2(\ell - 1)} \gamma(\ell) \leq 1 \right\} > t,$

  then
  
  $\tilde{M}(n, t) = M(n, t);$

- If
  
  $\max \left\{ \ell = t + 2r : \frac{\ell - t}{2(\ell - 1)} \gamma(\ell) \leq 1 \right\} = t,$

  then
  
  $\tilde{M}(n, t) = \max \{\nu_1(n, t), \nu_2(n, t)\},$

  where
  
  $\nu_i(n, t) = \sum_{S \in W(H_i)} \tilde{B}(n - |S|).$

**II Proof of Theorem 6**

Define the fixing procedure $F(i, j, p)$ for $i \neq j$ over the set of partitions $p \in \mathcal{P}(n)$:

$$F(i, j, p) = \begin{cases} (p \setminus p_i) \cup \{\{i\}, p_i \setminus \{i\}\}, & j \in p_i, \\ p, & \text{otherwise} \end{cases}$$

where $p_i$ is the part of $p$ that contains $i$.

The fixing operator on the family $\mathcal{A} \subset \Omega(n, t)$ is defined as follows ($p \in \mathcal{A}$)

$$F(i, j, p, \mathcal{A}) = \begin{cases} F(i, j, p), & F(i, j, p) \notin \mathcal{A}, \\ p, & F(i, j, p) \in \mathcal{A}. \end{cases}$$

Finally define the operator

$$\mathcal{F}(i, j, \mathcal{A}) = \{F(i, j, p, \mathcal{A}); p \in \mathcal{A}\}.$$
It is easy to see that the fixing operator \( F(i, j, \mathcal{A}) \) preserves the size of \( \mathcal{A} \) and its \( t \)-intersecting property. At last note that making shifting operations a finite number of times for different values of \( i \) and \( j \) allows us to obtain the compressed set \( \mathcal{A} \) with the following property: for all \( i \neq j \in [n] \),

\[
F(i, j, \mathcal{A}) = \mathcal{A}.
\]

It also has the property, that an arbitrary pair of partitions \( p_1, p_2 \) from the compressed set \( \mathcal{A} \) intersected by at least \( t \) fixed points.

Next define the usual shifting procedure \( L(v, w, p) \) for \( 1 \leq v < w \leq n \) as follows. Let \( p = \{\{j_1, \ldots, j_q-1, v, j_q+1, \ldots, j_s\}, \ldots, \{w\}, \pi_1, \ldots, \pi_c\} \in \mathcal{A} \), then

\[
L(v, w, p) = \{\{j_1, \ldots, j_q-1, w, j_q+1, \ldots, j_s\}, \ldots, \{v\}, \pi_1, \ldots, \pi_s\}.
\]

If \( p \in \mathcal{A} \) does not fix \( w \), then we set

\[
L(v, w, p) = p.
\]

Now define the shifting operator \( L(v, w, p, \mathcal{A}) \) as follows

\[
L(v, w, p, \mathcal{A}) = \begin{cases} 
L(v, w, p), & L(v, w, p) \notin \mathcal{A}, \\
 p, & L(v, w, p) \in \mathcal{A}.
\end{cases}
\]

At last define the operator \( \mathcal{L}(v, w, \mathcal{A}) : \)

\[
\mathcal{L}(v, w, \mathcal{A}) = \{L(v, w, \mathcal{A}); \; p \in \mathcal{A}\}.
\]

It is easy to see that the operator \( \mathcal{L}(v, w, \mathcal{A}) \) does not change the size of \( \mathcal{A} \) and it preserve the \( t \)-intersecting property. Later we will show, proving the Statement 1, that this operator also preserves the nontrivially \( t \)-intersecting property. Also it is easy to see that after a finite number of operations we come to the compressed \( t \)-intersecting set \( \mathcal{A} \) of the size \( M(n, t) \) for which

\[
L(v, w, \mathcal{A}) = \mathcal{A} \; \text{for} \; 1 \leq v < w \leq n
\]

and to the property that each pair of partitions of \( \mathcal{A} \) is \( t \)-intersected by fixed elements. Next we consider only such sets \( \mathcal{A} \). We denote the collection of fixed compressed \( t \)-intersecting families of partitions by \( L\Omega(n, t) \) and the collection of fixed compressed nontrivially \( t \)-intersecting families of partitions by \( (L\Omega(n, t)) \). Note that such family \( \mathcal{A} \) have the property that all partitions of \( \mathcal{A} \) have \( s \) common parts if and only if \( \left| \bigcap_{p \in \mathcal{A}} f(p) \right| = s \). We assume that all families of partitions considered next are left compressed.

Let \( \mathcal{D}(v, w, \mathcal{A}) \) be the same operator as \( \mathcal{L}(v, w, \mathcal{A}) \) but only with the condition \( v \neq w \).

We need the following
Lemma 1 If $|A| = M(n, t)$,

$$D(v, w, A) = A, \text{ for all } v, w \in [\ell]$$

and for $\ell = t + 2r$

$$\frac{\ell + 1}{r - t + 1} > \gamma(\ell + 2),$$

then $D(v, w, A) = A$, for all $v, w \in [\ell + 2]$.

Suppose that $|A| = M(n, t)$ and $\mathcal{A}$ is invariant under shifting and fixing operators. Assume also that $\mathcal{D}(v, w, A) = A$ for all $v, w \in [\ell]$, but $A \neq \mathcal{D}(v, \ell + 1, A)$ for some $v \in [\ell]$.

We set

$$\mathcal{A}' = \{ p \in A : D(v, \ell + 1, p) \notin A, v \in [\ell] \}$$

We identify the set of binary $n$-tuples with the family of subsets of $[n]$.

Define

$$B(\mathcal{A}) = \{ f(p) ; p \in A \} \subset 2^{[n]}.$$ 

It is easy to see that the set $B(\mathcal{A})$ is an upper ideal under the inclusion order. Denote by $M(\mathcal{A})$ the set of minimal elements of $B(\mathcal{A})$.

Let

$$s^+(M(\mathcal{A})) = \max_{M \in M(\mathcal{A})} s^+(M),$$

where

$$s^+(M) = \max_{i \in \mathcal{M}} i.$$ 

We also need one more lemma.

Lemma 2 If $|A| = M(n, t)$ and $s^+(M(\mathcal{A})) = \ell$, then

$$\frac{\ell - t}{2(\ell - 1)} \gamma(\ell) \leq 1.$$ (7)
Later we will show that there exists a unique \( \ell \) that satisfies inequalities (6) and (7). From this follows the statement of Theorem 1.

First we will prove 1. Assume that \( \mathcal{A}' \neq \emptyset \).

Let

\[
\mathcal{A}(i) = \left\{ p \in \mathcal{A}' : f(p) \cap [\ell] = i \right\}.
\]

It follows that \( \mathcal{A}(i) \neq \emptyset \) for some \( i \in [\ell] \).

Let

\[
\mathcal{A}'(i) = \left\{ f(p) \cap [\ell + 2, n] : p \in \mathcal{A}(i) \right\}.
\]

Define

\[
\mathcal{B}(i) = \left\{ p \in \Pi(n) : f(p) \cap [\ell] = i - 1, \ell + 1 \in p, f(p) \cap [\ell + 2, n] \in \mathcal{A}'(i) \right\}.
\]

We have

\[
|\mathcal{A}(i)| = \binom{\ell}{i} \sum_{m \in \mathcal{A}'(i)} \tilde{B}(n - i - 1 - |m|),
\]

\[
|\mathcal{B}(i)| = \binom{\ell - 1}{i - 1} \sum_{m \in \mathcal{A}'(i)} \tilde{B}(n - i - 1 - |m|)
\]

and, for any \( i \in [\ell] \), we have

\[
\mathcal{C}(i) = (\mathcal{A} \setminus \mathcal{A}(i)) \cup \mathcal{B}(\ell + t - i) \in \Omega(n, t).
\]

Next we will demonstrate that if \( \mathcal{A}(i) \neq \emptyset \) and \( i \neq \frac{\ell + t}{2} \), then

\[
\max\{|\mathcal{C}(i)|, |\mathcal{C}(\ell + t - i)|\} > |\mathcal{A}|
\]

which contradicts the maximality of \( \mathcal{A} \).

If (8) is not valid, then

\[
\left( \ell \right)_{i - 1} \sum_{m \in \mathcal{A}'(i)} \tilde{B}(n - i - 1 - |m|)
\]

\[
\leq \left( \ell \right)_{\ell + t - i} \sum_{r \in \mathcal{A}'(\ell + t - i)} \tilde{B}(n - (\ell + t - i) - 1 - |m|),
\]

\[
\left( \ell \right)_{\ell + t - i - 1} \sum_{m \in \mathcal{A}'(\ell + t - i)} \tilde{B}(n - (\ell + t - i) - 1 - |m|)
\]

\[
\leq \left( \ell \right)_{i} \sum_{m \in \mathcal{A}'(i)} \tilde{B}(n - i - 1 - |m|).
\]
We have $A(i) \neq \emptyset$ hence $A(\ell + t - i) \neq \emptyset$ and
\[
i(\ell + t - i) \leq (\ell - i + 1)(i + 1 - t).
\]
Since $t \geq 2$, the last inequality is false. This contradiction shows that $A(i) = \emptyset$ for $i \neq \frac{\ell + t}{2}$.

Now suppose $2| (\ell + t)$. We will demonstrate that if \(6\) is true, then $A\left(\frac{\ell + t}{2}\right) = \emptyset$.

We have
\[
|A\left(\frac{\ell + t}{2}\right)| = \left(\frac{\ell}{\frac{\ell + t}{2}}\right) \sum_{m \in A'} \tilde{B}\left(n - \frac{\ell + t}{2} - 1 - |m|\right).
\]

Now we will introduce the family $C \subset \Pi(n)$. Its elements are permutations $p$ which satisfy the following conditions
\[
|f(p) \cap [\ell]| = \frac{\ell + t}{2} - 1,
\]
\[
\{\ell + 1, n\} \subset f(p), \quad f(p) \cap [\ell + 2, n] \in A'\left(\frac{\ell + t}{2}\right).
\]

Define
\[
\mathcal{G} = \left\{ \left( A \setminus \left\{ p \in A\left(\frac{\ell + t}{2}\right) : \{n\} \not\subset p \right\} \right) \cup C \right\}.
\]

It is easy to see that
\[
\mathcal{G} \subset \Omega(n, t).
\]

Next we will demonstrate that if $\mathcal{A}\left(\frac{\ell + t}{2}\right) \neq \emptyset$ and \(6\) is true, then the maximality of $\mathcal{A}$ is contradicted because
\[
|\mathcal{G}| > |A|.
\] (9)

We have
\[
|\mathcal{C}| = \left(\frac{\ell}{\frac{\ell + t}{2}} - 1\right) \sum_{m \in A'(\frac{\ell + t}{2}), \{n\} \in m} \tilde{B}\left(n - \frac{\ell + t}{2} - 1 - |m|\right).
\]

Inequality \(7\) is equivalent to
\[
\left(\frac{\ell}{\frac{\ell + t}{2}} - 1\right) \sum_{m \in A'(\frac{\ell + t}{2}), \{n\} \in m} \tilde{B}\left(n - \frac{\ell + t}{2} - 1 - |m|\right)
\]
\[
> \left(\frac{\ell}{\frac{\ell + t}{2}} - 1\right) \sum_{m \in A'(\frac{\ell + t}{2}), \{n\} \not\in m} \tilde{B}\left(n - \frac{\ell + t}{2} - 1 - |m|\right).
\]
\[
\left( \frac{\ell + t}{2} \right) m \in A' \left( \frac{\ell + t}{2} \right) B \left( n - \frac{\ell + t}{2} - 1 - |m| \right) \\
- \sum_{m \in A' \left( \frac{\ell + t}{2} \right), \{n\} \in m} B \left( n - \frac{\ell + t}{2} - 1 - |m| \right). 
\]

From here we have
\[
\left( \frac{\ell + 1}{2} \right) \sum_{m \in A' \left( \frac{\ell + t}{2} \right), \{n\} \in m} B \left( n - \frac{\ell + t}{2} - 1 - |m| \right) > \left( \frac{\ell + t}{2} \right) \sum_{m \in A' \left( \frac{\ell + t}{2} \right)} B \left( n - \frac{\ell + t}{2} - 1 - |m| \right). 
\]

Hence
\[
\frac{\ell + 1}{\ell - t + 1} > \beta_1(\ell) \triangleq \frac{\sum_{m \in A' \left( \frac{\ell + t}{2} \right)} B \left( n - \frac{\ell + t}{2} - 1 - |m| \right)}{\sum_{m \in A' \left( \frac{\ell + t}{2} \right), \{n\} \in m} B \left( n - \frac{\ell + t}{2} - 1 - |m| \right). 
\]

Let’s prove that
\[
\gamma(\ell + 2) \geq \beta_1(\ell). 
\] (10)

From here it follows that (11) is true. Taking into account the condition from Lemma 1 that |A| is maximal we come to contradiction of this maximality. Thus to complete the proof of Lemma 1 we need to prove equality (10).

In order to show this, we state the validity of the following inequality
\[
\gamma(\ell + 2) = \frac{\sum_{i=0}^{n-\ell-1} B \left( n - \frac{\ell + t}{2} - i \right) \left( n - \ell - 1 \right)}{\sum_{i=0}^{n-\ell-2} B \left( n - \frac{\ell + t}{2} - i - 1 \right) \left( n - \ell - 2 \right)} \geq \frac{\sum_{m \in A' \left( \frac{\ell + t}{2} \right)} B \left( n - \frac{\ell + t}{2} - 1 - |m| \right)}{\sum_{m \in A' \left( \frac{\ell + t}{2} \right), \{n\} \in m} B \left( n - \frac{\ell + t}{2} - 1 - |m| \right) = \beta_1 \\
= \frac{\sum_{i=0}^{n-\ell-2} B \left( n - \frac{\ell + t}{2} - i \right) \left( n - \ell - 2 \right)}{\sum_{i=0}^{n-\ell-2} B \left( n - \frac{\ell + t}{2} - i - 1 \right) \left( n - \ell - 2 \right)} \geq \frac{\sum_{m \in A' \left( \frac{\ell + t}{2} \right), \{n\} \notin m} B \left( n - \frac{\ell + t}{2} - 1 - |m| \right)}{\sum_{m \in A' \left( \frac{\ell + t}{2} \right), \{n\} \in m} B \left( n - \frac{\ell + t}{2} - 1 - |m| \right). 
\] (11)

Validity of this inequality follows from the next consideration. Family \( \mathcal{A} \) is compressed under fixing operator. Set of minimal elements \( M(A) \subset 2^{[\ell]} \) in the set \( B(A) \) has t-intersection property.
Hence \( \{ f(p) \cap [\ell + 2, n]; p \in A \} = 2^{[\ell + 2, n]} \) and inequality \((11)\) can be written as follows:

\[
\frac{\sum_{i=0}^{n-\ell-2} \tilde{B} \left( n - \frac{\ell + t}{2} - i \right) \binom{n-\ell-2}{i}}{\sum_{i=0}^{n-\ell-2} \tilde{B} \left( n - \frac{\ell + t}{2} - 1 \right) \binom{n-\ell-2}{i}} \geq \frac{\sum_{i=0}^{n-\ell-2} \tilde{B} \left( n - \frac{\ell + t}{2} - i - 1 \right) \binom{n-\ell-2}{i}}{\sum_{i=0}^{n-\ell-2} \tilde{B} \left( n - \frac{\ell + t}{2} - 2 \right) \binom{n-\ell-2}{i}}.
\]

or

\[
\frac{\sum_{i=0}^{n-\ell-2} \tilde{B} \left( n - \frac{\ell + t}{2} - i - 2 \right) \binom{n-\ell-2}{i}}{\sum_{i=0}^{n-\ell-2} \tilde{B} \left( n - \frac{\ell + t}{2} - i - 1 \right) \binom{n-\ell-2}{i}} \geq \frac{\sum_{i=0}^{n-\ell-2} \tilde{B} \left( n - \frac{\ell + t}{2} - i - 1 \right) \binom{n-\ell-2}{i}}{\sum_{i=0}^{n-\ell-2} \tilde{B} \left( n - \frac{\ell + t}{2} - i \right) \binom{n-\ell-2}{i}}.\tag{12}
\]

or, using identity \( \tilde{B}(m) = B(m) - \tilde{B}(m+1) \), we obtain from the inequality \((12)\) inequality

\[
\frac{\sum_{i=0}^{n-\ell-2} B \left( n - \frac{\ell + t}{2} - i - 2 \right) \binom{n-\ell-2}{i}}{\sum_{i=0}^{n-\ell-2} B \left( n - \frac{\ell + t}{2} - i - 1 \right) \binom{n-\ell-2}{i}} \geq \frac{\sum_{i=0}^{n-\ell-2} B \left( n - \frac{\ell + t}{2} - i - 1 \right) \binom{n-\ell-2}{i}}{\sum_{i=0}^{n-\ell-2} B \left( n - \frac{\ell + t}{2} - i \right) \binom{n-\ell-2}{i}}.\tag{13}
\]

Validity of last inequality follows from Holley’s correlation inequality. Let’s \( \Delta \subseteq 2^{[n]} \) be finite distributive lattice and measures \( \mu_1, \mu_2 : \Delta \to R_+ \cup \{0\}, \sum_{A \in \Delta} \mu_1(A) = \sum_{A \in \Delta} \mu_2(A) \) satisfy FKG condition

\[
\mu_1(A) \mu_2(B) \leq \mu_1 \left( A \cup B \right) \mu_2 \left( A \cap B \right), \ A, B \in \Delta. \tag{14}
\]

Then for an arbitrary nondecreasing nonnegative function \( f : \Delta \to R_+ \cup \{0\}, A \subset B \to f(A) \geq f(B) \), the Holley’s inequality stand

\[
\sum_{A \in \Delta} \mu_1(A) f(A) \geq \sum_{A \in \Delta} \mu_2(A) f(A).
\]

We choose

\[
\Omega = \{ [k], \ k \in [n - \ell - 2] \}, \ f([k]) = \frac{B \left( n - \frac{\ell + t}{2} - k - 2 \right)}{B \left( n - \frac{\ell + t}{2} - k - 1 \right)},
\]

\[
\mu_1([k]) = \frac{\tilde{B} \left( n - \frac{\ell + t}{2} - k - 1 \right) \binom{n-\ell-2}{k}}{\sum_{i=0}^{n-\ell-2} \tilde{B} \left( n - \frac{\ell + t}{2} - i - 1 \right) \binom{n-\ell-2}{i}},
\]

\[
\mu_2([k]) = \frac{\tilde{B} \left( n - \frac{\ell + t}{2} - k \right) \binom{n-\ell-2}{k}}{\sum_{i=0}^{n-\ell-2} \tilde{B} \left( n - \frac{\ell + t}{2} - i \right) \binom{n-\ell-2}{i}}.
\]

FKG condition for \( \mu_1, \mu_2 \) follows from the validity if FKG condition for Bell numbers \( B(k) \) proved in \((13)\).
We need to check the monotonicity of \( f([k]) \). If \( f([k+1]) \geq f([k]) \), then

\[
\frac{B \left( n - \frac{\ell + t}{2} - k - 2 \right)}{\tilde{B} \left( n - \frac{\ell + t}{2} - k - 1 \right)} \geq \frac{B \left( n - \frac{\ell + t}{2} - k - 1 \right)}{\tilde{B} \left( n - \frac{\ell + t}{2} - k \right)}
\]

or identity \( B(m) = \tilde{B}(m) + \tilde{B}(m+1) \), we reduce last inequality to the so-called log-convexity condition of \( \tilde{B}(m) \):

\[
\tilde{B}^2 \left( n - \frac{\ell + t}{2} - k - 1 \right) \leq \tilde{B} \left( n - \frac{\ell + t}{2} - k - 2 \right) \tilde{B} \left( n - \frac{\ell + t}{2} - k \right).
\]

In Appendix we prove that \( \tilde{B} \left( n - \frac{\ell + t}{2} - k - 1 \right) \) satisfies this inequality when \( n - \frac{\ell + t}{2} - k - 1 > 3 \). This inequality is true for all \( k \in [n - \ell - 2] \) and \( r > 2 \).

Assume at first that \( r > 2 \). Then, using Holley’s inequality we have

\[
\sum_{k=0}^{n-\ell-2} \mu_1([k]) f([k+1]) = \frac{\sum_{k=0}^{n-\ell-2} B \left( n - \frac{\ell + t}{2} - k - 2 \right) \binom{n-\ell-2}{i}}{\sum_{i=0}^{n-\ell-2} \tilde{B} \left( n - \frac{\ell + t}{2} - i - 1 \right) \binom{n-\ell-2}{i}} 
\]

\[
\geq \sum_{k=0}^{n-\ell-2} \mu_2([k]) f([k]) \geq \frac{\sum_{i=0}^{n-\ell-2} B \left( n - \frac{\ell + t}{2} - i - 1 \right) \binom{n-\ell-2}{i}}{\sum_{i=0}^{n-\ell-2} \tilde{B} \left( n - \frac{\ell + t}{2} - i \right) \binom{n-\ell-2}{i}}
\]

This proves inequality (12) when \( r > 2 \). When \( r \leq 3 \), we use identities (14) (both identities are actually the same, we just write them in form in which they we will use them later):

\[
\begin{align*}
\binom{n-t-2r-2}{i} &= \sum_{k=0}^{q} (-1)^k \binom{q}{k} \binom{n-t-2r-2+q-k}{i+q}, \quad q \geq 0, \\
n-t-2r-2 \choose a+i+r-2 &= \sum_{k=0}^{r+2-a} (-1)^k \binom{r+2-a}{k} \binom{n-t-r-a-k}{i}.
\end{align*}
\]

We have \( a \leq 2 + r \):

\[
\begin{align*}
\sum_{i=0}^{n-t-2r-2} \tilde{B} \left( n - t - r - i - a \right) \binom{n-t-2r-2}{i} &= \sum_{i=0}^{n-t-2r-2} \tilde{B} \left( r + 2 - a + i \right) \binom{n-t-2r-2}{i} \\
= \sum_{k=0}^{r+2-a} (-1)^k \binom{r+2-a}{k} \sum_{i=0}^{n-t-r-a-k} \tilde{B} \left( r + 2 - a + i \right) \binom{n-t-r-a-k}{i+r+2-a} \\
= \sum_{k=0}^{r+2-a} (-1)^k \binom{r+2-a}{k} \sum_{i=r+2-a}^{n-t-r-a-k} \tilde{B} \left( r + 2 - a + i \right) \binom{n-t-r-a-k}{i}
\end{align*}
\]

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We write this inequality for \( r \),

Using last identity we can rewrite inequality (12) as follows

Next we will prove Lemma 2.

Denoting \( S = \sum_{i=0}^{r+1} \bar{B}(i) \sum_{k=0}^{r+1} (-1)^k \binom{r+1}{k} \binom{n-t-r-a-k}{i} \), \( B = \sum_{k=0}^{r} \sum_{i=0}^{r} \bar{B}(i) \sum_{k=0}^{r} (-1)^k \binom{r+2-a}{k} \binom{n-t-r-a-k}{i} \).

Using last identity we can rewrite inequality (12) as follows

Denoting \( S(r) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} B(n-t-r-2-k) \), we need to show the validity of inequality

\[ S^2(r + 1) \leq S(r + 2) S(r). \]

We write this inequality for \( r = 0, 1, 2, 3 \) making some transformations:

\[
\begin{align*}
\frac{B(n-t-1)}{B(n-t-2)} &\leq \frac{B(n-t)}{B(n-t-1)}, \quad r = 0; \\
\frac{B(n-t-2) - B(n-t-3)}{B(n-t-3) - B(n-t-4)} &\leq \frac{B(n-t-1) - B(n-t-2)}{B(n-t-2) - B(n-t-3)}, \quad r = 1; \\
\frac{B(n-t-3) - 2B(n-t-4) + B(n-t-5)}{B(n-t-4) - 2B(n-t-5) + B(n-t-6)} &\leq \frac{B(n-t-2) - 3B(n-t-3) + 3B(n-t-4) - B(n-t-5)}{B(n-t-3) - 3B(n-t-4) + 3B(n-t-5) - B(n-t-6)}, \quad r = 2; \\
\frac{B(n-t-4) - 3B(n-t-5) + 3B(n-t-6) - B(n-t-7)}{B(n-t-5) - 3B(n-t-6) + 3B(n-t-7) - B(n-t-8)} &\leq \frac{B(n-t-3) - 4B(n-t-4) + 6B(n-t-5) + 4B(n-t-6) - B(n-t-7)}{B(n-t-4) - 4B(n-t-5) + 6B(n-t-6) - 4B(n-t-7) + B(n-t-8)}, \quad r = 3. 
\end{align*}
\]

Next we will prove Lemma 2.

Define

\[ M_0(A) = \{ E \in M(A) ; s^+ (E) = s^+ (M(A)) = \ell \} \]

and

\[ M_1(A) = M(A) \setminus M_0(A). \]
It is easy to see that, for \( E_1 \in M_0(A) \) and \( E_2 \in M_1(A) \),

\[
\left| (E_1 \setminus \{\ell\}) \cap E_2 \right| \geq t
\]

and for \( E_1, E_2 \in M_0(A) \) and \( |E_1 \cap E_2| = t \),

\[
|E_1| + |E_2| = \ell + t.
\]

Set

\[
M_0(A) = \bigcup_i R(i),
\]

where

\[
R(i) = M_0(A) \cap \left( \left[ n \right] \right).\]

Define

\[
R'(i) = \{ E \setminus \{\ell\}; \ E \in R(i) \}.
\]

Next we are going to prove that if \((6)\) is not true, then \( R(i) = \emptyset \).

Suppose that \( R(i) \neq \emptyset \) for some \( i \). At first, assume that \( i \neq \frac{\ell + t}{2} \).

Define

\[
F_1 = M_1(A) \cup (M_0(A) \setminus (R(i) \cup R(\ell + t - i))) \cup R'(i),
\]

\[
F_2 = M_1(A) \cup (M_0(A) \setminus (R(i) \cup R(\ell + t - i))) \cup R'(\ell + t - i).
\]

It is easy to see that for \( E_1, E_2 \in F_i \) we have \( \left| E_1 \cap E_2 \right| \geq t \) and thus \( U(F_1), U(F_2) \in \Omega(n, t) \). We are going to show that if \( R(i) \neq \emptyset \), then

\[
\max\{|U(F_1)|, |U(F_2)|\} > |A|
\]

which gives us a contradiction.

We have

\[
|A \setminus U(F_1)| = |R(\ell + t - i)| \sum_{j=0}^{n-t} \binom{n-\ell}{j} \tilde{B}(n - \ell - t + i - j)
\]

(16)

and

\[
|U(F_1) \setminus A| = |R(i)| \sum_{j=0}^{n-t} \binom{n-\ell}{j} \tilde{B}(n - i - j + 1).
\]

(17)
Also
\[ |A \setminus U(F_2)| = |R(i)| \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \tilde{B}(n-i-j), \tag{18} \]
\[ |U(F_2) \setminus A| = |R(\ell+t-i)| \sum_{j=0}^{n-\ell} \tilde{B}(n-\ell-t+i-j+1). \tag{19} \]

If (15) is not true, then from (16)-(19) it follows that
\[
|R(i)| \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \tilde{B}(n-i-j+1)
\leq |R(\ell+t-i)| \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \tilde{B}(n-\ell-t+i-j),
\]
\[
|R(\ell+t-i)| \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \tilde{B}(n-\ell-t+i-j+1)
\leq |R(i)| \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \tilde{B}(n-i-j).
\]

These inequalities couldn't be valid together due to monotonicity of \( \tilde{B}(n) \).

Now consider the case \( i = \ell + \frac{t}{2} \). We are going to prove that if inequality \( (7) \) is not true, then
\( R \left( \frac{\ell+t}{2} \right) = \emptyset \). Simple averaging argument shows that there exists \( i \in [\ell-1] \) and \( Z \subset R' \left( \frac{\ell+t}{2} \right) \) such that \( i \in E \) for all \( E \in Z \) and
\[ |Z| \geq \frac{\ell-t}{2(\ell-1)} \left| R' \left( \frac{\ell+t}{2} \right) \right|. \tag{20} \]

Because \( |E_1 \cap E_2| \geq t \) when \( E_1, E_2 \in Z \) and \( R(i) = \emptyset \) when \( i \neq \frac{\ell+t}{2} \) we have for all \( E_1, E_2 \in D \), where
\[ D = \left( M(A) \setminus R \left( \frac{\ell+t}{2} \right) \right) \cup Z, \]
we have \( |E_1 \cap E_2| \geq t \). Hence \( W(D) \in \Omega(n,t) \) and now we have to show that, if \( (7) \) is not true, then
\[ |W(D)| > |A|. \tag{21} \]

Consider the partition
\[ A = W(M(A)) = S_1 \cup S_2, \]
\[ S_1 = W \left( M(\mathcal{A}) \setminus R \left( \frac{\ell + t}{2} \right) \right), \]
\[ S_2 = W \left( R \left( \frac{\ell + t}{2} \right) \right) \setminus S_1 \]

and the partition
\[ W(D) = S_1 \cup S_3, \]
\[ S_3 = W(D) \setminus S_1. \]

One can see that (35) is equivalent to
\[ |S_3| > |S_2|. \]

It is easy to show that
\[ |S_2| = \left| R \left( \frac{\ell + t}{2} \right) \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \tilde{B} \left( n - \frac{\ell + t}{2} - j \right) \right|, \]
\[ |S_3| = |Z| \sum_{j=0}^{n-\ell+1} \binom{n-\ell+1}{j} \tilde{B} \left( n - \frac{\ell + t}{2} + 1 - j \right). \]

Using (20) and (22) we conclude that
\[ \frac{\ell - t}{2(\ell - 1)} |R\left(\frac{\ell + t}{2}\right)| \sum_{j=0}^{n-\ell+1} \binom{n-\ell+1}{j} \tilde{B} \left( n - \frac{\ell + t}{2} + 1 - j \right) \]
\[ > |R\left(\frac{\ell + t}{2}\right)| \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \tilde{B} \left( n - \frac{\ell + t}{2} - j \right). \]

But from here follows the contradiction of the maximality of \( \mathcal{A} \).

Thus (7) holds.

Now we rewrite inequality (6) as follows
\[ \ell + 2 < t + 2 - \frac{t - 1}{\gamma(\ell + 2) - 2}. \]

and inequality (7) as
\[ \ell \leq t + 2 - \frac{t - 1}{\gamma(\ell) - 2}. \]

It is left for us to show that the function
\[ \varphi(\ell) = t - \ell + 2 - \frac{t - 1}{\gamma(\ell) - 2} \]

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does not change its sign in the interval \([t, n]\) more than one time. To prove this we will first show that \(\varphi\) is \(\cap\)-convex on interval \([t, n]\). Obviously \(\varphi(t) > 0\). From these facts will follow the statement of the Theorem 1.

We have

\[
\gamma(\ell) = \frac{\sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \tilde{B} \left( n - \frac{\ell + t}{2} + 1 - j \right) + \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \tilde{B} \left( n - \frac{\ell + t}{2} - j \right)}{\sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \tilde{B} \left( n - \frac{\ell + t}{2} - j \right)} = 1 + \frac{\sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \tilde{B} \left( n - \frac{\ell + t}{2} + 1 - j \right)}{\sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \tilde{B} \left( n - \frac{\ell + t}{2} - j \right)}.
\]

Now using identity (2) we derive the relations

\[
\gamma(n, \ell, t) = \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \tilde{B} \left( n - \frac{\ell + t}{2} - j \right) = \frac{1}{e} \sum_{i=1}^{\infty} (i-1)^{n-\ell} (i-1)^{n-\ell} \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} (i-1)^{j} (i-1)^{-j} = \frac{1}{e} \sum_{i=2}^{\infty} (i-1)^{\frac{\ell + t + 1}{2}} (i-2)^{n-\ell}.
\]

Similar calculations show the validity of the following identity

\[
\gamma(n + 2, \ell + 2, t) = \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \tilde{B} \left( n - \frac{\ell + t}{2} + 1 - j \right) = \frac{1}{e} \sum_{i=2}^{\infty} (i-1)^{\frac{\ell + t + 1}{2}} (i-2)^{n-\ell}.
\]

Hence, for \(\gamma(\ell) - 2\), we have the expression

\[
\gamma(\ell) - 2 = \frac{\gamma(n + 2, \ell + 2, t)}{\gamma(n, \ell, t)} = \frac{\sum_{i=2}^{\infty} (i-1)^{\frac{\ell + t + 1}{2}} (i-2)^{n-\ell}}{\sum_{i=2}^{\infty} (i-1)^{\frac{\ell + t}{2}} (i-2)^{n-\ell}} - 1 = \frac{\sum_{i=2}^{\infty} (i-1)^{\frac{\ell + t + 1}{2}} (i-2)^{n-\ell}}{\sum_{i=2}^{\infty} (i-1)^{\frac{\ell + t}{2}} (i-2)^{n-\ell}}.
\]

We obtain the following expression for the function \(\varphi(\ell)\):

\[
\varphi(\ell) = t - \ell + 2(t - 1) \frac{\sum_{i=2}^{\infty} (i-1)^{\frac{\ell + t + 1}{2}} (i-2)^{n-\ell}}{\sum_{i=2}^{\infty} (i-1)^{\frac{\ell + t}{2}} (i-2)^{n-\ell}}.
\]

It is easy to show that the second derivative of this function is negative. This completes the proof of Theorem 1.

II Proof of Theorem 6.

Denote by \(\Omega_0(n, t) \subset \Omega(n, t)\) the collection of the families of partitions \(A\) such that \(\left| \bigcap_{p \in A} f(p) \right| = 0\).
Statement 1

\[ \tilde{M}(n, t) = \max_{A \in L \tilde{\Omega}(n, t)} |A|, \]  
\[ M_0(n, t) = \max_{A \in \Omega_0(n, t)} |A| = \tilde{M}(n, t). \]

Moreover, if \( A \in \tilde{\Omega}(n, t) \) and \( |A| = \tilde{M}(n, t) \), then \( A \in \Omega_0(n, t) \).

Proof. First we will prove (23). For \( A \in \tilde{\Omega}(n, t) \) assume that \( |A| = \tilde{M}(n, t) \). One can see that either \( L(v, w, A) \in \tilde{\Omega}(n, t) \) or \( L(v, w, A) \in \Omega(n, t) \setminus \Omega(n, t) \). In the first case we continue shifting. Assume that the second case occurs. We can assume that \( \cap_{p \in L(v, w, A)} f(p) = [t - 1] \) and also that \( \cap_{p \in L(v, w, A)} f(p) = [t] \). Because \( A \) is maximal, then

\[ \{p \in \Omega(n, t) : [t + 1] \subset f(p)\} \subset A. \]  

(24)

There are \( p_1, p_2 \in A \) such that

\[ f(p_1) \cap [t + 1] = [t] \]

and

\[ f(p_2) \cap [t + 1] = [t - 1] \cup \{t + 1\}. \]

Now we apply the shifting \( L(v, w, A) \) for \( v \neq w \in \{n\} \setminus \{t, t + 1\} \). We have \( \cap_{p \in L(v, w, A)} f(p) = [t - 1] \). Thus we can assume that \( L(v, w, A) = A \) for all \( v \neq w \in \{n\} \setminus \{t, t + 1\} \) and

\[ f(p_1) = \{a\} \setminus \{t + 1\}, a \geq t, a \neq t + 1, \]

\[ f(p_2) = \{b\} \setminus \{t\}, b > t. \]

From here and (24) it follows that

\[ C = U((\{t - 1\} \cup C : C \subset [t, \min\{a, b\}])) \subset A \]

and for all \( L(v, w, C) = C \) where \( v \neq w \in \{n\} \). Thus \( |\cap_{p \in A} f(p)| < t \).

Now we prove second part of the Statement. Assume that \( A \subset \tilde{\Omega}(n, t) \setminus \Omega_0(n, t) \) and \( |A| = \tilde{M}(n, t) \). We can suppose that \( A \) is shifted and \( \{1\} \in f(p) \) for all \( p \in A \). We can also assume that \( A \in L\tilde{\Omega}(n, t) \). Consider \( p \in \Omega(n, t) : f(p) = \{2, \ldots, n - 1\} \). Next we will show that \( p \in A \), which leads to the contradiction of the maximality of \( A \). Suppose that there exists a partition \( p_1 \in A \) such that

\[ |\{2, n - 1\} \cap f(p_1)| \leq t - 1. \]

We can assume that \( f(p_1) = [t] \cup \{n\} \). We have \( p_2 : f(p_2) = [t - 1] \cup \{n\} \) belongs to \( A \) and hence \( p_3 : f(p_3) = [t] \) also belongs to \( A \). But then \( |f(p_3) \cap f(p_2)| = t - 1 \) which contradicts the \( t \)-intersecting property of \( A \).
For further convenience we will make some changes in the definitions, which we will use next. Let $g(A)$ be the family of subsets of $[n]$ such that $A = U(g(A))$. If $A$ is maximal, then we can assume that $g(A)$ is upset and $g^*(A)$ is the set of its minimal elements. It is easy to see that $A \in \Omega(n, t)$ if and only if $g(A) \in I(n, t)$ and $A \in \tilde{\Omega}(n, t)$ if and only if $g(A) \in \tilde{I}(n, t)$. We can assume that $g(A)$ is left compressed.

Define

$$
s^+(a = (a_1 < \ldots < a_j)) = a_j,
$$

$$
s^+(g(A)) = \max_{a \in g^*(A)} s^+(a),
$$

$$
s_{\min} = \min_{A \in L\tilde{\Omega}(n, t): |A| = M(n, t)} s^+(g(A)).
$$

It is easy to see that $A \in L\Omega(n, t)$ is a disjoint union

$$
A = U_{f \in g^*(A)} Q(f),
$$

where

$$
Q(f) = \left\{ A \in 2^{[n]} : A = f \cup B, B \in [s^+(f), n] \right\},
$$

and if $f \in g(A)$ is such that $s^+(f) = s^+(g(A))$, then the set of partitions generated only by $f$ is

$$
\mathcal{A}_f = (U(f) \setminus U(g^*(A) \setminus \{f\})) = Q(f).
$$

Note also a simple fact that if $f_1, f_2 \in g^*(A)$ and $i \notin f_1 \cup f_2$, $j \in f_1 \cap f_2$ for some $i < j$, then

$$
|f_1 \cap f_2| \geq t + 1.
$$

Next lemma helps us to establish possible sets of $g^*(A)$ for maximal $A \in L\tilde{\Omega}(n, t)$ when $M(n, t)$ is not this maximum. To make the formulation more clear we repeat in Lemma all conditions which we have considered before as default.

**Lemma 3** For $A \in L\tilde{\Omega}(n, t)$ assume that $|A| = \tilde{M}(n, t)$ and $g(A) \in G(A)$ is such that $s^+(g(A)) = s_{\min}(G(A))$, then for some $i \geq 2$

$$
g^*(A) = \mathcal{H}_i.
$$

Suppose that $\ell = s^+(g(A))$, $g_0(A) = \{ g \in g^*(A) : s^+(g) = \ell \}$ and $g_1(A) = g^*(A) \setminus g_0(A)$. It is easy to see that $\ell > t + 1$. From above it follows that if $f_1, f_2 \in g_0(A)$ and $|f_1 \cap f_2| = t$, then $|f_1| + |f_2| = \ell + t$. Denote

$$
\left| \bigcap_{f \in g_1(A)} f \right| = \tau.
$$
Consider consequently two cases \( \tau < t \) and \( \tau \geq t \).

Assume at first that \( \tau < t \). Consider the partition
\[
g_0(\mathcal{A}) = \bigcup_{t<i<\ell} R_i, \quad R_i = g_0(\mathcal{A}) \cap \binom{[n]}{i}.
\]
Denote
\[
R'_i = \{ f \subset [\ell - 1] : f \cup \{\ell\} \in R_i \}.
\]
As above, because the set \( g(\mathcal{A}) \) is left compressed, it follows that for
\[
f_i \in R'_i, \quad f_j \in R'_j \text{ and } i + j \neq \ell + t, \quad |f_i \cap f_j| \geq t.
\]

Next we show that \( R_i = \emptyset \).
Assume at first that \( \forall R_i \neq \emptyset \) we have \( R_{\ell+t-i} = \emptyset \), then for
\[
g' = (g^*(\mathcal{A}) \setminus g_0(\mathcal{A})) \bigcup \bigcup_{t<i<\ell} R'_i \in I(n, k)
\]
we have
\[
|U(g')| \geq |\mathcal{A}| \quad \text{and} \quad s^+(g') < s^+(g(\mathcal{A}))
\]
which contradicts our assumptions.

Now assume that \( R_i, R_{\ell+t-i} \neq \emptyset \). At first we consider the case when \( i \neq (\ell + t)/2 \). Consider the new sets
\[
\varphi_1 = g_1(\mathcal{A}) \bigcup \left( g_0(\mathcal{A}) \setminus \left( R_i \cup R_{\ell+t-i} \right) \right) \cup R'_i;
\]
\[
\varphi_2 = g_1(\mathcal{A}) \bigcup \left( g_0(\mathcal{A}) \setminus \left( R_i \cup R_{\ell+t-i} \right) \right) \cup R'_{\ell+t-i}.
\]
We have \( \varphi_i \in I(n, k) \). Thus,
\[
\mathcal{A}_i = U(\varphi_i) \in \tilde{\Omega}(n, t).
\]
We will show that, under the last assumption,
\[
\max_{j=1,2} |\mathcal{A}_i| > |\mathcal{A}| \quad \text{(26)}
\]
and come to a contradiction. Using (25) it is easy to see that:
\[
|\mathcal{A} \setminus \mathcal{A}_i| = |R_{\ell+t-i}| \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \tilde{B}(n - \ell - t + i - j),
\]

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\[ |A_1 \setminus A| \geq |R_i| \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \tilde{B}(n - i - j + 1), \]
\[ |A \setminus A_2| = |R_i| \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \tilde{B}(n - i - j), \]
\[ |A_2 \setminus A| \geq |R_{\ell+t-i}| \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \tilde{B}(n - \ell - t + i - j + 1). \]

From these equalities it follows that, if (26) is not valid, then
\[ |R_{\ell+t-i}| \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \tilde{B}(n - \ell - t + i - j) \geq |R_i| \sum_{j=0}^{n-\ell} \tilde{B}(n - i - j + 1) \]
and
\[ |R_i| \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \tilde{B}(n - i - j) \geq |R_{\ell+t-i}| \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \tilde{B}(n - \ell - t + i - j + 1). \]

Since \( \tilde{B}(n+1) > \tilde{B}(n) \) when \( n > 0 \), the last two inequalities couldn’t be valid together. This contradiction shows that \( R_i = \emptyset \) when \( i \not= (\ell + t)/2 \).

Now consider the case \( i = (\ell + t)/2 \). By pigeon-hole principle, there exists \( k \in [\ell - 1] \) and \( S \subset R'_{(\ell+t)/2} \) such that \( k \not\in B \) for all \( B \in S \) and
\[
|S| \geq \frac{\ell - t}{2(\ell - 1)} |R'_{(\ell+t)/2}|. \tag{27}
\]

Hence, as before, we have \( |B_1 \cap B_2| \geq t \) for all \( B_1, B_2 \in S \) and
\[
f' = (g^*(A) \setminus R_{(\ell+t)/2}) \cup S \in \bar{I}(n, t) .
\]

Next we show that
\[
|U(f')| > |A|. \tag{28}
\]
Consider the partition
\[
A = G_1 \cup G_2 ,
\]
where
\[
G_1 = U(g^*(A) \setminus R_{(\ell+t)/2}) , \quad G_2 = U(R_{(\ell+t)/2}) \setminus U(g^*(A) \setminus R_{(\ell+t)/2}) .
\]
Consider also the partition
\[ U(f') = \mathcal{G}_1 \cup \mathcal{G}_3, \]
where
\[ \mathcal{G}_3 = U(\mathcal{S}) \setminus U(g^*(\mathcal{A}) \setminus R(\ell+\ell/2). \]
We should show that
\[ |\mathcal{G}_3| > |\mathcal{G}_2|. \quad (29) \]
We have
\[ |\mathcal{G}_2| = |R(\ell+\ell/2)| \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \bar{B} \left( n - \frac{\ell + t}{2} - j \right) \]
and
\[ |\mathcal{G}_3| \geq |\mathcal{S}| \sum_{j=0}^{n-\ell+1} \binom{n-\ell+1}{j} \bar{B} \left( n - \frac{\ell + t}{2} - j + 1 \right). \]
Hence, for (29) to be true, it is sufficient that
\[ \frac{\ell - t}{2(\ell - 1)} \gamma(\ell) > 1. \]
The last inequality is true because, otherwise, from (6) it follows that \( \bar{M}(n, k) = M(n, k). \) Hence \( R_{\ell+\ell/2} = \emptyset. \)

Now consider the case \( \tau \geq t. \) We have
\[ \bigcap_{f \in g_1(\mathcal{A})} f = [\tau], \]
\[ \ell = s^+(g(\mathcal{A})) > \tau \]
and for all \( f \in g_0(\mathcal{A}), \)
\[ \left| F \cap [\tau] \right| \geq \tau - 1, \]
if \( |f \cap [\tau]| = \tau - 1, \) then \( [\tau + 1, \ell] \in f. \)

Let's show that \( \tau \leq t + 1. \)
If \( \tau \geq t + 2, \) then, for \( f_1, f_2 \in g(\mathcal{A}), \)
\[ \left| f_1 \cap f_2 \cap [\tau] \right| \geq \tau - 2 \geq t \]
and thus, setting \( g_0(\mathcal{A}) = \{ f \subset [\ell - 1] : f \cup \ell \in g_0(\mathcal{A}) \} \), we have

\[
\varphi = (g^*(\mathcal{A}) \setminus g_0(\mathcal{A})) \cup g_0(\mathcal{A}) \in \tilde{I}(n, k)
\]

and

\[
|U(\varphi)| \geq |\mathcal{A}|, \ s^+(\varphi) < \ell.
\]

This gives us the contradiction of minimality of \( \ell \).

Assume now that \( \tau = t + 1 \). In this case it is necessary that \( \ell = t + 2 \). Otherwise, using the argument above (deleting \( \ell \) from each element of \( g_0(\mathcal{A}) \)), we end up generating the set \( \varphi \in \tilde{I}(n, k) \) for which \( |U(\varphi)| \geq |\mathcal{A}| \) and \( s^+(\varphi) < \ell \). It is clear that \( \tau = t + 1 \) and \( \ell = t + 2 \), then \( g^*(\mathcal{A}) = \mathcal{H}_2 \).

At last, consider the case \( \tau = t \). Define \( g'_0(\mathcal{A}) = \{ f \in g_0(\mathcal{A}) : |f \cap [t]| = t - 1 \} \). We have

\[
g'_0(\mathcal{A}) \subset \{ f \subset [\ell] : |f \cap [t]| = t - 1, [t + 1, \ell] \subset f \}
\]

and for \( f \in g^*(\mathcal{A}) \setminus g'_0(\mathcal{A}) \) we have \([t] \subset f\) and \(|f \cap [t + 1, \ell]| \geq 1\).

Hence

\[
U(g^*(\mathcal{A})) \subset U(\mathcal{H}_{t-1}).
\]

Since \( \mathcal{A} \) is maximal, \( g^*(\mathcal{A}) = \mathcal{H}_{t-1} \). Family \( \mathcal{H}_{n-t} \) is trivially \( t \)-intersecting, so we can assume that \( i < n - t \). Denote \( S_i = |U(\mathcal{H}_i)| \). Next we will prove that if \( S_i < S_{i+1} \), then \( S_{i+1} < S_{i+2} \). We have

\[
S_i = (n - i)! - \sum_{j=0}^{n-t-i} \binom{n-t-i}{j} \tilde{B}(n-t-j) + t \sum_{j=0}^{n-t-i} \tilde{B}(n-t-i-j+1)
\]

and we should show that from inequality

\[
\sum_{j=0}^{n-t-i-1} \binom{n-t-i-1}{j} \tilde{B}(n-t-j+1) \geq t \sum_{j=0}^{n-t-i-1} \binom{n-t-i-1}{j} \tilde{B}(n-t-j-i+1) \tag{30}
\]

follows

\[
\sum_{j=0}^{n-t-i-2} \binom{n-t-i-2}{j} \tilde{B}(n-t-j+1) \geq t \sum_{j=0}^{n-t-i-2} \binom{n-t-i-2}{j} \tilde{B}(n-t-j-i). \tag{31}
\]

We rewrite inequality \((30)\) as follows

\[
\sum_{j=0}^{n-t-i-2} \binom{n-t-i-2}{j} \tilde{B}(n-t-j+1) + \sum_{j=0}^{n-t-i-2} \binom{n-t-i-2}{j} \tilde{B}(n-t-j) \geq t \sum_{j=0}^{n-t-i-2} \binom{n-t-i-2}{j} \tilde{B}(n-t-i-j+1) + t \sum_{j=0}^{n-t-i-2} \binom{n-t-i-2}{j} \tilde{B}(n-t-i-j).
\]
From here, it is clear that if (31) is true, then (15) is also true. From here and expressions for $S_2$ and $S_{n-t-1}$ follows the statement of Theorem 6. Since, for fixed $t$,

$$\frac{\sum_{j=1}^{n-t-2} \binom{n-t-2}{j} \tilde{B}(n - t - j)}{\sum_{j=0}^{n-t-2} \binom{n-t-2}{j} \tilde{B}(n - t - 1 - j)} \to \infty, \ n \to \infty,$$

and

$$\frac{\tilde{B}(n - t - 1)}{\sum_{j=1}^{n-t-2} \binom{n-t-2}{j} \tilde{B}(n - t - j)} \to 0, \ n \to \infty,$$

it follows that for sufficiently large $n$ and fixed $t$:

$$S_2 = B(n - t) - \tilde{B}(n - t) - \tilde{B}(n - t - 1) + t > S_{n-t-1} = B(n - t)$$

$$- \sum_{j=0}^{n-t-2} \binom{n-t-2}{j} \tilde{B}(n - t - j) + t \sum_{j=0}^{n-t-2} \binom{n-t-2}{j} \tilde{B}(n - t - j - 1).$$

Therefore, for $n > n_2(t)$,

$$\tilde{M}(n, t) = B(n - t) - \tilde{B}(n - 1) - \tilde{B}(n - t - 1) + t.$$
Appendix

Let us remark that FKG inequality says that for $\mu : 2^m \rightarrow \mathbb{R}_+$ such that
\[
\mu(a)\mu(b) \leq \mu \left( a \cap b \right) \mu \left( a \cup b \right), \quad a, b \in 2^m,
\] (32)

and for a pair of nondecreasing functions $f_1, f_2 : 2^m \rightarrow \mathbb{R}$, the following inequality is valid:
\[
\sum_{Y \in 2^m} \mu(Y) f_1(Y) \geq \sum_{Y \in 2^m} \mu(Y) f_2(Y) \leq \sum_{Y \in 2^m} \mu(Y) f_1(Y) f_2(Y) \sum_{Y \in 2^m} \mu(Y).
\] (33)

Now we choose
\[
\mu(Y) = \tilde{B} \left( n - \frac{\ell + t}{2} - |Y| \right).
\]

Note that if (32) is true for this choice of $\mu$, then setting $f_1 = I_{X \in \mathcal{F} : x \in X}$ and $f_2 = I_{X \in 2^{n-(\ell+t)/2-1} : x \in X}$ in (33) proves inequality (??).

Define $\bar{a} = n - t - r - a$, $\bar{b} = n - t - r - b$, $\bar{\delta} = n - r - t - \delta$. Then inequality (32) is equivalent to inequality
\[
\tilde{B}(\bar{a}) \tilde{B}(\bar{b}) \leq \tilde{B}(\bar{a} \cap \bar{b}) \tilde{B}(\bar{a} \cup \bar{b}).
\]

Function $F(i) \geq 0$, $i \in \mathbb{Z}_+$ is called log-convex if
\[
F(i + 1)F(i - 1) \geq F^2(i).
\] (34)

FKG condition (32) is equivalent to the log-convexity property of $\tilde{B}$. We are going to demonstrate that $F(|Y|) = \mu(Y) = \tilde{B} \left( n - \frac{\ell + t}{2} - |Y| \right)$ satisfy inequality (34) for all possible $|Y|$, except $n - t - r - |Y| \neq 2, 4$.

**Lemma 4** Inequality
\[
\tilde{B}(k + 1)\tilde{B}(k - 1) \geq \tilde{B}^2(k)
\] (35)

is true for $k \in \mathbb{Z}_+ \setminus \{2, 4\}$

From above considerations it follows it is left to consider the case $k = 2m$, $m > 2$. We will prove this lemma by using asymptotic of $\tilde{B}(k)$.

Next part of text we devote to finding the asymptotic fo $\tilde{B}(n)$.

**Lemma 5** The following asymptotic of $\tilde{B}(n)$ is true
\[
\tilde{B}(n) = \frac{n! \exp(e^r - r - 1)}{r^n(4\pi B)^{1/2}}(1 \pm 11e^{-r}), \quad r \geq 12.
\] (36)

where $r$ satisfy equality $r(e^r - 1) = n$ and $B = \frac{1}{2}r((r + 1)e^r - 1)$. 

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To proof this lemma we will use The Moser - Wyman expansion of the Bell numbers [11]. We will follow the text [12] and introduce the extension of proof from [12] for extend Bell number $\tilde{B}(n)$ for completeness (it is quite similar, besides we need calculate the explicit bounds for rest term of asymptotic also).

Because
\[ \sum_{n=0}^{\infty} \frac{\tilde{B}(n)x^n}{n!} = \exp(e^x - x - 1), \]
using Cauchy’s formula, we have
\[ \frac{\tilde{B}(n)}{n!} = \frac{1}{2\pi i} \oint_{|z|=r} \frac{\exp(e^z - z - 1)}{z^{n+1}} \, dz. \]

Contour integration yields
\[ \tilde{B}(n) = \frac{n!}{2\pi r^n} \int_{-\pi}^{\pi} \exp(\epsilon r e^{i\theta} - r e^{i\theta} - i n \theta - 1) \, d\theta. \]

Define
\[ F(\theta) = e^{\epsilon r e^{i\theta}} - r e^{i\theta} - i n \theta - (e^r - r). \]

We have
\[ \tilde{B}(n) = A \int_{-\pi}^{\pi} \exp(F(\theta)) \, d\theta \]
where
\[ A = \frac{n! \exp(e^r - r - 1)}{2\pi r^n}. \]

Define
\[ \epsilon = e^{-\frac{1}{2} \epsilon}, \quad J_1 = \int_{-\pi}^{\pi} \exp(F(\theta)) \, d\theta, \quad J_2 = \int_{\epsilon}^{\pi} \exp(F(\theta)) \, d\theta. \]

Using inequality $\cos(\theta) \leq 1 - \frac{\epsilon^2}{2} + \frac{\epsilon^4}{24}$, we have
\[ \tilde{B}(n) = A J_1 + A J_2 + A \int_{-\epsilon}^{\epsilon} \exp(F(\theta)) \, d\theta, \]
\[ | \exp(F(\theta)) | = \exp(Re(F(\theta))) = \exp \left( e^r \cos(\theta) \cos(r \sin(\theta)) - e^r + r(1 - \cos(\theta)) \right) \]
\[ \leq \exp \left( e^r - e^r + r(1 - \cos(\theta)) \right) \]
\[ \leq \exp \left( e^r \left( e^r \left( 1 - \frac{\epsilon^2}{2} + \frac{\epsilon^4}{24} \right) - 1 \right) + r \right) \]
\[ \leq \exp \left( \frac{\epsilon^2}{2} \left( 1 - \frac{\epsilon^2}{12} \right) + r \right) \]
\[ \leq \exp \left( \frac{e^2}{2} \left( 1 - \frac{\epsilon^2}{12} \right) + r \right). \]
\[
\leq \exp \left( -\frac{1}{2} e^r r^2 \left( 1 - \frac{e^2}{12} \right) \left( 1 - \frac{e^2 \left( 1 - \frac{r^2}{12} \right)}{4} \right) + r \right)
\]
\[
\leq \exp \left( -\frac{1}{2} e^{r/4} r \left( 1 - \frac{e^{-3r/4}}{12} \right) \left( 1 - \frac{e^{-3r/4} \left( 1 - \frac{r^2}{12} \right) r}{4} \right) + r \right).
\]

Because
\[
1 - \frac{e^{-3r/4}}{12} > \frac{11}{12}, \quad 1 - \frac{e^{-3r/4} \left( 1 - \frac{r^2}{12} \right) r}{4} > \frac{3}{4}, \text{ when } r > 12,
\]
we have
\[
|\exp(F(\theta))| \leq \exp \left( -r \frac{1}{3} e^{r/4} + r \right) < e^{-\frac{7}{4} e^{r/4}}
\]
and, hence
\[
J_2 \leq A \pi e^{-\frac{7}{4} e^{r/4}}.
\]
We have
\[
\tilde{B}(n) = A \left( \int_{-\epsilon}^{\epsilon} \exp(F(\theta)) d\theta \pm e^{-\frac{7}{4} e^{r/4}} \pi \right).
\]
Consider the expansion
\[
F(\theta) = (r e^r - r - n) i \theta - \frac{1}{2} (r^2 e^r + r e^r - r) \theta^2 + \sum_{k=3}^{\infty} \left( \frac{d}{dr} \right)^k \left( e^r - r \right) (i \theta)^k, \quad r(e^r - 1) = n.
\]
Hence we have
\[
F(\theta) = -\frac{1}{2} (r^2 e^r + r e^r - r) \theta^2 + \sum_{k=3}^{\infty} \left( \frac{d}{dr} \right)^k \left( e^r - r \right) (i \theta)^k.
\]
Define
\[
\phi = \left( \frac{1}{2} (r^2 e^r + r e^r - r) \right)^{1/2} \theta, \quad a_k = \frac{e^{-r} \left( r \frac{d}{dr} \right)^{k+2} \left( e^r - r e^{-r} \right) (i \phi)^{k+2}}{(k+2)! \left( \frac{1}{2} (r^2 + r - r e^{-r}) \right)^{k+2}}, \quad z = e^{-r/2},
\]
\[
f(z) = \sum_{k=1}^{\infty} a_k z^k.
\]
Then
\[
F(\theta) = -\phi^2 + f(z), \quad \tilde{B}(n) = C \left( \int_{-h}^{h} \exp(-\phi^2 + F(z)) dz \pm \pi e^{-\frac{7}{4} e^{r/4}} \left( \frac{1}{2} r(r+1)e^r - r \right)^{1/2} \right),
\]
\[
h = \left( \frac{1}{2} r((r+1)e^r - 1) \right)^{1/2} e^{-3r/8}, \quad C = \frac{A}{\left( \frac{1}{2} r((r+1)e^r - 1) \right)^{1/2}}.
\]
Consider the expansion
\[ e^f(z) = \sum_{k=0}^{\infty} b_k z^k, \quad b_0 = e^{f(0)} = 1, \quad b_1 = e^{f(0)} f'(0) = a_1, \quad b_2 = a_2 + \frac{a_1^2}{2}. \]

We have
\[ |a_k| = \left| \frac{\left( \sum_{m=1}^{k+2} S(k+2, m) r^m - r e^{-r} \right) (i \phi)^{k+2}}{(k+2)! \left( \frac{1}{2} (r^2 + r - re^{-r}) \right)^{(k+2)/2}} \right| \leq \frac{2^{k+2}}{(k+2)!} |\phi|^{k+2} B(k+2) < |2\phi|^{k+2}. \]

Here \( S(m, k) \) is Stirling number of second kind, \( B(n) \) is Bell number. We used inequalities \( B(n) \leq n!, \quad \left( r \frac{d}{dr} \right)^{k+2} (e^r) = \sum_{n=1}^{k+2} S(k+2, n) r^n e^r \leq r^{k+2} B(k+2) e^r \). We use formula for coefficients \( b_k \) in composite function \( \exp \left( \sum_{k=2}^{\infty} a_k z^k \right) = \sum_{k=1}^{\infty} b_k z^k \):

\[
\begin{align*}
 b_m &= \sum_{\sum_{p=1}^{m} p j_p = m, \sum_{p=1}^{m} j_p = k} \frac{1}{\prod_{p=1}^{m} j_p! \prod_{p=1}^{m} a_p^{j_p}} \leq \sum_{\sum_{p=1}^{m} p j_p = m, \sum_{p=1}^{m} j_p = k} \frac{1}{\prod_{p=1}^{m} (p^2) j_p!} \leq (2\phi)^m \sum_{\sum_{p=1}^{m} p j_p = m, \sum_{p=1}^{m} j_p = k} \frac{(2\phi)^{2k}}{m! (j_p)!} \\
 &= (2\phi)^m \sum_{\sum_{p=1}^{m} p j_p = m} \sum_{\sum_{p=1}^{m} j_p = k} \frac{(2\phi)^{2k}}{m! (j_p)!} = (2\phi)^m \sum_{\sum_{p=1}^{m} p j_p = m} \frac{(2\phi)^{2k}}{m! (j_p)!} \leq (2\phi)^m \sum_{\sum_{p=1}^{m} p j_p = m} \frac{(2\phi)^{2k}}{m! (j_p)!} \leq (2\phi)^m \sum_{\sum_{p=1}^{m} p j_p = m} \frac{(2\phi)^{2k}}{m! (j_p)!} \leq (2\phi)^m (1 + (2\phi)^2)^{m-1}.
\end{align*}
\]

Next we have
\[
\left| \sum_{k=s}^{\infty} b_k z^k \right| \leq \left( \left| 2\phi \right|^{s+2} (1 + |2\phi|^2)^{s-1} |z|^s \right) \sum_{i=0}^{\infty} \mu_i = \frac{|2\phi|^{s+2} (1 + |2\phi|^2)^{s-1} |z|^s}{M},
\]
\[ M = 1 - |z| \left| 2\phi (1 + |2\phi|^2) \right| \mu = |2\phi| (1 + |2\phi|^2) |z| < 1. \]

We impose conditions \( M > \frac{1}{2}, \quad |\phi| \leq h, z = e^{-r/2} \). We have
\[
\begin{align*}
\tilde{B}(n) &= C \left( \int_{-h}^{h} e^{-\phi^2} d\phi + e^{-r} \int_{-h}^{h} b_2 e^{-\phi^2} d\phi \pm \int_{-\infty}^{\infty} \sum_{k=2}^{\infty} b_{2k}(\phi) e^{-kr} \big| e^{-\phi^2} d\theta \right) \\
&\pm \pi e^{-\frac{r^2}{4}} \left( \frac{1}{2} (r (r + 1) e^r - r) \right)^{1/2} \\
&\quad \left( \frac{1}{2} (r (r + 1) e^r - r) \right)^{1/2} \quad \text{(37)}
\end{align*}
\]
Because
\[ \int_{h}^{\infty} e^{-\phi^2} d\phi = \frac{1}{2h} e^{-h^2} \left( 1 + \frac{1}{h^2} \right), \] we have
\[ \int_{-h}^{h} e^{-\phi^2} d\phi = \int_{-\infty}^{\infty} e^{-\phi^2} d\phi - 2 \int_{h}^{\infty} e^{-\phi^2} d\phi = \sqrt{\pi} - \frac{1}{h} e^{-h^2} \left( 1 + \frac{1}{h^2} \right). \]
It follows the asymptotic equation
\[ \tilde{B}(n) = C \left( e^{-r} \int_{-h}^{h} b_2 e^{-\phi^2} d\phi + \sqrt{\pi} - \frac{1}{h} e^{-h^2} \left( 1 + \frac{1}{h^2} \right) \pm e^{-2r} \int_{-\infty}^{\infty} (2\phi)^4 (1 + (2\phi)^2) e^{-\phi^2} d\theta \right) \]
\[ \pm \pi e^{-r} \left( \frac{1}{2} (r(r + 1)e^r - r) \right)^{1/2} \]
Next we have
\[ a_1 = \sqrt{2 \frac{r^3 + 3r^2 + r - re^{-r}}{3(r^2 + r - re^{-r})^{3/2}} \frac{(i\phi)^3}{\phi^4}}; \]
\[ a_2 = \frac{r^4 + 6r^3 + 7r^2 + r - re^{-r}}{6(r^2 + r - re^{-r})} \frac{\phi^4}{\phi^4} - \frac{(r^3 + 3r^2 + r - re^{-r})^2}{9(r^2 + r - re^{-r})^3} \frac{\phi^6}{\phi^6}. \]
Integrating in parts and using asymptotic (38) we have
\[ \int_{-h}^{h} \phi^4 e^{-\phi^2} d\phi = -h^3 e^{-h^2} - \frac{3}{2} h e^{-h^2} + \frac{3}{4} \left( \sqrt{\pi} - 2 \int_{h}^{\infty} e^{-\phi^2} d\phi \right) = -h^3 e^{-h^2} - \frac{3}{2} h e^{-h^2} \]
\[ + \frac{3}{4} \left( \sqrt{\pi} - 2 \int_{h}^{\infty} e^{-\phi^2} d\phi \right) = -h^3 e^{-h^2} - \frac{3}{2} h e^{-h^2} + \frac{3}{4} \left( \sqrt{\pi} - \frac{1}{h^2} e^{-h^2} \left( 1 + \frac{1}{h^2} \right) \right), \]
\[ \int_{-h}^{h} \phi^6 e^{-\phi^2} d\phi = -h^5 e^{-h^2} + \frac{5}{2} \int_{-h}^{h} \phi^4 e^{-\phi^2} d\phi \int_{-h}^{h} \phi^8 e^{-\phi^2} d\phi = -h^5 e^{-h^2} + \frac{5}{2} \int_{-h}^{h} \phi^6 e^{-\phi^2} d\phi \int_{-h}^{h} \phi^8 e^{-\phi^2} d\phi \\
= -h^5 e^{-h^2} + \frac{9}{2} \int_{-h}^{h} \phi^8 e^{-\phi^2} d\phi. \]
Because \( \frac{r^{3/8}}{\sqrt{2}} < h < re^{r/8} \sqrt{2} \), \( r^8 e^r > 100r^6 e^{3r/4} \), we have
\[ \int_{-h}^{h} (2\phi)^4 (1 + (2\phi)^2) e^{-\phi^2} d\phi = 16 \int_{-\infty}^{\infty} (\theta^4 + 12\theta^6 + 48\theta^8 + 64\theta^{10}) e^{-\phi^2} d\phi \]
\[ = -32 \left( e^{-h^2} h \left( \frac{5133}{2} + 1711h^2 + 334h^4 + 129h^6 + 32h^8 \right) - \frac{5133}{3} \left( \sqrt{\pi} - \frac{e^{-h^2}}{h} \left( 1 + \frac{1}{h^2} \right) \right) \right) \]
\[ = -32 \left( e^{-h^2} h \left( \frac{5133}{2} + 1711h^2 + 334h^4 + 129h^6 + 32h^8 \right) - \frac{5133}{3} \left( \sqrt{\pi} - \frac{e^{-h^2}}{h} \left( 1 + \frac{1}{h^2} \right) \right) \right), \]
\[ -32 \left( e^{-\frac{9}{2}e^{r'/4}} (640r^8 e^r + 1200r^6 e^{3r'/4} + 1600r^4 e^{r'/2} + 4000re^{r'/4} + 3000) + 5133 \left( \sqrt{\pi} + e^{-\frac{9}{2}e^{r'/4}} \right) \right) \]

\[ = \pm 2^{17} \left( e^{-\frac{9}{2}e^{r'/4}} r^8 e^r + 1 \right) , \]

\[ \int_{-h}^{h} b_2 e^{-\phi^2} d\phi = \int_{-h}^{h} a_2 e^{-\phi^2} d\phi + \frac{1}{2} \int_{-h}^{h} a_1^2 e^{-\phi^2} d\phi \]

\[ = \frac{r^4 + 6r^3 + 7r^2 + r - re^{-r}}{6(r^2 + r - re^{-r})^2} \left( -h^3 e^{-h^2} - \frac{3}{2} he^{-h^2} + \frac{3}{4} \left( \sqrt{\pi} - \frac{1}{h} e^{-h^2} \left( 1 \pm \frac{1}{h^2} \right) \right) \right) \]

\[ - \frac{1}{2} \frac{(r^3 + 3r^2 + r - re^{-r})^2}{9(r^2 + r - re^{-r})^2} \left( -h^3 e^{-h^2} - \frac{5}{2} h^3 e^{-h^2} - \frac{15}{4} he^{-h^2} + \frac{15}{8} \left( \sqrt{\pi} - \frac{1}{h} e^{-h^2} \left( 1 \pm \frac{1}{h^2} \right) \right) \right) \]

\[ = \left( \frac{\sqrt{\pi} r^4 + 6r^3 + 7r^2 + r - re^{-r}}{8 \left( 24 \frac{5}{2} h^3 + \frac{15}{4} \frac{15}{8h} \right)} \right) - \frac{1}{2} \frac{(r^3 + 3r^2 + r - re^{-r})^2}{9(r^2 + r - re^{-r})^2} \left( h^3 + \frac{3}{2} h + \frac{3}{4h} \right) \]

\[ = \pm e^{-h^2} \left( \frac{r^4 + 6r^3 + 7r^2 + r - re^{-r}}{6(r^2 + r - re^{-r})^2} + \frac{1}{2} \frac{(r^3 + 3r^2 + r - re^{-r})^2}{9(r^2 + r - re^{-r})^3} \right) \]

\[ = \left( \frac{\sqrt{\pi} r^4 + 6r^3 + 7r^2 + r - re^{-r}}{8 \left( 24 \frac{5}{2} h^3 + \frac{15}{4} \frac{15}{8h} \right)} \right) - \frac{1}{2} \frac{(r^3 + 3r^2 + r - re^{-r})^2}{9(r^2 + r - re^{-r})^3} \left( h^3 + \frac{3}{2} h + \frac{3}{4h} \right) \]

We are ready to write the asymptotic

\[ \tilde{B}(n) = C \sqrt{\pi} \left( 1 + \frac{1}{\sqrt{\pi}} e^{-r} \int_{-h}^{h} b_2 e^{-\phi^2} d\phi \pm \frac{1}{\sqrt{\pi}} \left( 2\sqrt{\pi} e^{-\frac{1}{2}e^{r'/4}} + e^{-2r} \int_{-\infty}^{\infty} (2\phi)^4 (1 + (2\phi)^2)^3 e^{-\phi^2} d\phi \right) \right) \]

\[ + \frac{1}{2} \int_{-h}^{h} b_2 e^{-\phi^2} d\phi \pm \frac{1}{\sqrt{\pi}} \left( 2\sqrt{\pi} e^{-\frac{1}{2}e^{r'/4}} + 2^{17} \left( e^{-\frac{1}{2}e^{r'/4}} r^8 e^r + 1 \right) e^{-2r} + \sqrt{\pi} e^{-\frac{1}{2}e^{r'/4}} r^8 e^{r/2} \right) \]

\[ = C \sqrt{\pi} \left( 1 + e^{-r} \left( \frac{1}{8} \frac{r^4 + 6r^3 + 7r^2 + r - re^{-r}}{6(r^2 + r - re^{-r})^2} + \frac{5}{24} \frac{(r^3 + 3r^2 + r - re^{-r})^2}{9(r^2 + r - re^{-r})^3} \right) \right) \]
Next, using asymptotic (42) we prove inequality (35) for sufficiently large

Inequality (41) is equivalent to the inequality

As we noticed before last inequality equivalent to log - convexity of function $\bar{f}$. Then using software "Mathematica" we show that inequality (43) is true for all other values of $a, b, \delta, a + b - \delta < n - t - r$, the inequality (41) follows from the inequality

Define $\bar{a} = n - r - t - a, \bar{b} = n - r - t - b, \bar{\delta} = n - r - t - \delta = \bar{a} \cap \bar{b}, n - r - t - (a + b) + \delta = \bar{a} \cup \bar{b}$

Inequality (41) is equivalent to the inequality

Define $\bar{B}(\bar{a}) \bar{B}(\bar{b}) \leq \bar{B}(\bar{a} \cap \bar{b}) \bar{B}(\bar{a} \cup \bar{b}).$ (42)

As we noticed before last inequality equivalent to log - convexity of function $\bar{f}(n)$:

Next, using asymptotic (42) we prove inequality (35) for sufficiently large $n > n_0$ where $n_0 < e^{12}$. Then using software "Mathematica" we show that inequality (43) is true for all other values of $n = 2m < e^{12}, m > 2$. This complete the proof of Lemma 4. Simple calculations show the validness of the inequality

Following inequality is valid:

Indeed

Hence for $r = r(n), \ln(n) > r(n) > 10$, we have

$$C(n - 1)C(n + 1) - (C(n))^2 > \frac{(n - 1)!(n + 1)!e^{\frac{n}{r} + \frac{1}{n}}}{2\pi r^{n-2} (r + \frac{1}{n})^n (n + r + \frac{1}{n})}$$
To satisfy the last inequality is sufficient to impose the inequality

\[
\frac{1}{(n - 1 + r) \left( r^2 + (n - 1)(r + 1) \right)^{1/2}} \left( \frac{r + \frac{1}{n}}{n} \right)^{1/2} - \frac{n!^2 e^{2\pi}}{2\pi r^{2(n-1)}(r + n)^2(r^2 + n(r + 1))} > \left( \frac{e^{r-2}}{(1 - 1/n)} - 1 \right) \frac{n!^2 e^{2\pi}}{2\pi r^{2n}(r + n)^2(r^2 + n(r + 1))}.
\]

Using inequalities \( \frac{n}{n-1} > 1 + \frac{1}{n}, \ e^{r-2} > 1 + r^{-2} \) and, hence \( \frac{e^{r-2}}{(1-1/n)} > 1 + r^{-2} \) we have

\[
(C(n - 1)C(n + 1) - (C(n))^2) > V = \frac{n!^2 e^{2\pi}}{2\pi r^{2n+2}(r + n)^2(r^2 + n(r + 1))}.
\]

Let’s \( \tilde{B}(n) = \sqrt{\pi}C(n)(1 + d(n)). \) We need to prove the inequality

\[
C(n - 1)(1 + d(n - 1))C(n + 1)(1 + d(n + 1)) \geq C^2(n)(1 + d(n))^2
\]

or

\[
C(n - 1)C(n + 1) - C^2(n) > V
\]

\[
> C^2(n)(2d(n) + d^2(n)) + C(n - 1)C(n + 1)(d(n - 1) + d(n + 1) + d(n - 1)d(n + 1))
\]

\[
> C^2(n)(2d(n) + d^2(n) + d(n - 1) + d(n + 1) + d(n - 1)d(n + 1)).
\]

To satisfy the last inequality is sufficient to impose the inequality

\[
C(n - 1)C(n + 1) - C^2(n) > V > C^2(n)(2|d(n)| + d^2(n) + |d(n - 1)d(n + 1)| + |d(n - 1)| + |d(n + 1)|) \quad (44)
\]

or

\[
\frac{V}{C^2(n)} = \frac{1}{r^2} > 2|d(n)| + d^2(n) + |d(n - 1)d(n + 1)| + |d(n - 1)| + |d(n + 1)|.
\]

We can assume that \( d(m) < 12e^{-r}, \ m = n, n + 1, n - 1. \) Then

\[
2|d(n)| + d^2(n) + |d(n - 1)d(n + 1)| + |d(n - 1)| + |d(n + 1)| < 48e^{-r} + 288 e^{-2r} < \frac{1}{r^2}, \ r > 10.
\]

It is left to check inequality (13) for \( r \leq 12, \ n \neq 2, 4. \) We can do this with the help of software "Mathematica".

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