Kazhdan–Lusztig and $R$–polynomials of generalized Temperley–Lieb algebras

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Abstract
We study two families of polynomials that play the same role, in the generalized Temperley–Lieb algebra of a Coxeter group, as the Kazhdan–Lusztig and $R$–polynomials in the Hecke algebra of the group. Our results include recursions, closed formulas, and other combinatorial properties for these polynomials. We focus mainly on non–branching Coxeter graphs.

Keywords:
Temperley–Lieb algebras, Hecke algebras, Kazhdan–Lusztig basis, Coxeter groups

Introduction
The Temperley–Lieb algebra $\mathcal{T}_L(X)$ is a quotient of the Hecke algebra $\mathcal{H}(X)$ associated to a Coxeter group $W(X)$, $X$ being an arbitrary Coxeter graph. It first appeared in [20], in the context of statistical mechanics (see, e.g., [12]). The case $X = A$ was studied by Jones (see [13]) in connection to knot theory. For an arbitrary Coxeter graph, the Temperley–Lieb algebra was studied by Graham. More precisely, in [6] Graham showed that $\mathcal{T}_L(X)$ is finite dimensional whenever $X$ is of type $A, B, D, E, F, H$ and $I$. If $X \neq A$ then $\mathcal{T}_L(X)$ is usually referred to as the generalized Temperley–Lieb algebra. The algebra $\mathcal{T}_L(X)$ has many properties similar to the Hecke algebra $\mathcal{H}(X)$. In particular, in [8] Green and Losonczy show that $\mathcal{T}_L(X)$ always admits an IC basis (see [4] and [8] for definitions and

✩This work is part of the author’s doctoral dissertation, written under the direction of Prof. F. Brenti at the University of Rome “Tor Vergata”.

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further details). These bases have properties similar to the well–known Kazhdan–Lusztig basis of the Hecke algebra \( \mathcal{H}(X) \). Algebraic properties of these bases have been studied in [9] and [10]. In this work, which is a continuation of the paper [15], we investigate some combinatorial properties of them. More precisely, we look at the coefficients of the IC basis of \( TL(X) \) with respect to the standard basis, and obtain some recursive formulas for them. To do this, we find necessary to first study some auxiliary polynomials (which have no analogue in \( \mathcal{H}(X) \), and which in some sense express the relationship between \( \mathcal{H}(X) \) and \( TL(X) \)) which were first defined in [8]. As a consequence of these results we also obtain closed formulas for the polynomials expressing the inverse of an element of the standard basis as a linear combination of elements of the standard basis (or equivalently, for the coordinates of the canonical involution with respect to the standard basis). Most of our results hold for non–branching Coxeter graphs, although some hold in full generality. Our results emphasize the close relationship between Kazhdan–Lusztig and \( R\)–polynomials and their analogues in \( TL(X) \).

The organization of the paper is as follows. In the next section we recall some generalities on the Hecke algebra, Kazhdan–Lusztig polynomials and the Kazhdan–Lusztig basis of \( \mathcal{H}(X) \). Moreover, we recall the Temperley–Lieb algebras and the families of the polynomials \( \{ a_{x,w} \} \) and \( \{ L_{x,w} \} \) that we study in this work. In Sections 2, 3 we prove our results on polynomials \( \{ a_{x,w} \} \) and \( \{ L_{x,w} \} \), which hold for all finite irreducible and affine non–branching Coxeter graphs \( X \) such that \( X \neq \tilde{F}_4 \), and we obtain an explicit formula for the polynomials \( \{ a_{x,w} \} \) in type \( A \).

1. Preliminaries

In this section we recall some basic facts about Hecke algebras \( \mathcal{H}(X) \) and Temperley–Lieb algebras \( TL(X) \), \( X \) being any Coxeter graph. Let \( W(X) \) be the Coxeter group having \( X \) as Coxeter graph and \( S(X) \) as set of generators. Let \( \mathcal{A} \) be the ring of Laurent polynomials \( \mathbb{Z}[q^{1/2}, q^{-1/2}] \). The Hecke algebra \( \mathcal{H}(X) \) associated to \( W(X) \) is an \( \mathcal{A} \)–algebra with linear basis \( \{ T_w : w \in W(X) \} \) (see, e.g., [2, §6.1] and [11, §7]). For all \( w \in W(X) \) and \( s \in S(X) \) the multiplication law is determined by

\[
T_w T_s = \begin{cases} 
T_{ws} & \text{if } \ell(ws) > \ell(w), \\
qT_{ws} + (q - 1)T_w & \text{if } \ell(ws) < \ell(w),
\end{cases}
\]

where \( \ell \) denotes the usual length function of \( W(X) \). We refer to \( \{ T_w : w \in W(X) \} \) as the \( T \)–basis for \( \mathcal{H}(X) \).
Let \( e \) be the identity element of \( W(X) \). One easily checks that \( T_s^2 = (q - 1)T_s + qT_e \), being \( T_e \) the identity element, and so \( T_s^{-1} = q^{-1}(T_s - (q - 1)T_e) \). It follows that all the elements \( T_w \) are invertible, since, if \( w = s_1 \cdots s_r \) and \( \ell(w) = r \), then \( T_w = T_{s_1} \cdots T_{s_r} \). To express \( T_w^{-1} \) as a linear combination of elements in the basis, one obtains the so-called \( R \)-polynomials. For a proof of the following result we refer to [11, §7.4].

**Theorem 1.1.** There is a unique family of polynomials \( \{ R_{x,w}(q) \}_{x,w \in W(X)} \subseteq \mathbb{Z}[q] \) such that

\[
T_w^{-1} = \varepsilon_w q^{-\ell(w)} \sum_{x \leq w} \varepsilon_x R_{x,w}(q) T_x,
\]

and \( R_{x,w}(q) = 0 \) if \( x \not\leq w \), where \( \varepsilon_x \) def = \((-1)^{\ell(x)}\). Furthermore, \( R_{x,w}(q) = 1 \) if \( x = w \).

Define a map \( \iota : \mathcal{H} \to \mathcal{H} \) such that \( \iota(T_w) = (T_w^{-1})^{-1} \), \( \iota(q) = q^{-1} \) and extend by linear extension. We refer the reader to [11, §7.7] for the proof of the following result.

**Proposition 1.2.** The map \( \iota \) is a ring homomorphism of order 2 on \( \mathcal{H}(X) \).

In [14], Kazhdan and Lusztig prove this basic theorem:

**Theorem 1.3.** There exists a unique basis \( \{ C_w : w \in W(X) \} \) for \( \mathcal{H}(X) \) such that the following properties hold:

(i) \( \iota(C_w) = C_w \),

(ii) \( C_w = \varepsilon_w q^{-\ell(w)} \sum_{x \leq w} \varepsilon_x q^{-\ell(x)} P_{x,w}(q^{-1}) T_x \),

where \( \{ P_{x,w}(q) \} \subseteq \mathbb{Z}[q] \), \( P_{w,w}(q) = 1 \) and \( \deg(P_{x,w}(q)) \leq \frac{1}{2}(\ell(w) - \ell(x) - 1) \) if \( x < w \).

The polynomials \( \{ P_{x,w}(q) \}_{x,w \in W(X)} \) are the so-called Kazhdan–Lusztig polynomials of \( W(X) \). In [11, §7.9] it is shown that one can substitute the basis \( \{ C_w : w \in W(X) \} \) with the equivalent basis \( \{ C'_w : w \in W(X) \} \), where

\[
C'_w = q^{-\frac{\ell(w)}{2}} \sum_{x \leq w} P_{x,w}(q) T_x.
\]

For the rest of this paper we will refer to the latter basis as the Kazhdan–Lusztig basis for \( \mathcal{H}(X) \).
Let $s_i, s_j \in S(X)$ and denote by $\langle s_i, s_j \rangle$ the parabolic subgroup of $W(X)$ generated by $s_i$ and $s_j$. Following [6], we consider the two–sided ideal $J(X)$ generated by all elements of $H(X)$ of the form

\[
\sum_{w \in \langle s_i, s_j \rangle} T_w,
\]

where $(s_i, s_j)$ runs over all pairs of non–commuting generators in $S(X)$ such that the order of $s_i s_j$ is finite.

**Definition 1.4.** The generalized Temperley–Lieb algebra is $T L(X) \overset{\text{def}}{=} H(X) / J(X)$.

When $X$ is of type $A$, we refer to $T L(X)$ as the Temperley–Lieb algebra. In order to describe a basis for $T L(X)$, we recall the notion of a fully commutative element for $W(X)$ (see [18]).

**Definition 1.5.** An element $w \in W(X)$ is fully commutative if any reduced expression for $w$ can be obtained from any other by applying Coxeter relations that involve only commuting generators. We let

\[
W_c(X) \overset{\text{def}}{=} \{ w \in W(X) : w \text{ is a fully commutative element} \}.
\]

If $X = A_{n-1}$ then $W(X) = S_n$ (see [2, Example 1.2.3]) and $W_c(A_{n-1})$ may be described as the set of elements of $W(A_{n-1})$ whose reduced expressions avoid substrings of the form $s_i s_{i \pm 1} s_i$ for all $s_i \in S$ (see [18, Proposition 1.1]). Another description of $W_c(A_{n-1})$ may be given in terms of pattern avoidance: namely, in [1, Theorem 2.1] Billey, Jockusch and Stanley show that $W_c(A_{n-1})$ coincides with the set of permutations avoiding the pattern 321. Moreover $|W_c(A_{n-1})| = C_n$, where $C_n = \frac{1}{n+1} \binom{2n}{n}$ denotes the $n$–th Catalan number (see [5, Proposition 3] for further details). A similar characterization can be given in type $B$. If $X = B_n$ then $W_c(X)$ can be described as the group of signed permutations $S_n^B$ (see [2, Example 1.2.4]). In [19, Theorem 5.1] Stembridge showed that the set of the signed permutations avoiding the patterns in $\{12, 321, 321, 231, 231\}$ and $W_c(B_n)$ coincide. Moreover $|W_c(B_n)| = (n + 2) C_n - 1$ (see [19, Proposition 5.9]).

Let $t_w = \sigma(T_w)$, where $\sigma : H \to H / J$ is the canonical projection. A proof of the following can be found in [4].

**Theorem 1.6.** $T L(X)$ admits an $A$–basis of the form $\{ t_w : w \in W_c(X) \}$. 

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We call \( \{ t_w : w \in W_c(X) \} \) the \( t \)-basis of \( TL(X) \). By (1), it satisfies
\[
t_w t_s = \begin{cases} 
  t_{w s} & \text{if } \ell(w s) > \ell(w), \\
  q t_{w s} + (q - 1) t_w & \text{if } \ell(w s) < \ell(w).
\end{cases}
\]  
(3)

Observe that if \( w s \notin W_c(X) \), then \( t_{w s} \) can be expressed as linear combination of the \( t \)-basis elements by means of the following result (see [8, Lemma 1.5]).

**Proposition 1.7.** Let \( w \in W(X) \). Then there exists a unique family of polynomials \( \{ D_x(w)(q) \} \subseteq Z[q] \) such that
\[
t_w = \sum_{x \in W_c(X) \atop x \leq w} D_x(w)(q) t_x,
\]
where \( D_{w, w}(q) = 1 \) if \( w \in W_c(X) \). Furthermore, \( D_{x, w}(q) = 0 \) if \( x \nleq w \).

From the fact that the involution \( t \) fixes the ideal \( J(X) \) (see [8, Lemma 1.4]), it follows that \( t \) induces an involution on \( TL(X) \), which we still denote by \( t \), if there is no danger of confusion. More precisely, we have the following result.

**Proposition 1.8.** The map \( t \) is a ring homomorphism of order 2 such that \( t(t_w) = (t_{w^{-1}})^{-1} \) and \( t(q) = q^{-1} \).

To express the image of \( t_w \) under \( t \) as a linear combination of elements of the \( t \)-basis, one defines a new family of polynomials (see [8, §2]).

**Proposition 1.9.** Let \( w \in W_c(X) \). Then there exists a unique family of polynomials \( \{ a_{y, w}(q) \} \subseteq Z[q] \) such that
\[
(t_{w^{-1}})^{-1} = q^{-\ell(w)} \sum_{y \in W_c(X) \atop y \leq w} a_{y, w}(q) t_y,
\]
where \( a_{w, w}(q) = 1 \).
Theorem 1.10. There exists a unique basis \( \{c_w : w \in W_c\} \) of \( TL(X) \) such that

(i) \( \iota(c_w) = c_w \),

(ii) \( c_w = \sum_{x \leq w} q^{-\frac{\ell(x)}{2}} L_{x,w}(q^{-\frac{1}{2}}) t_x \),

where \( \{L_{x,w}(q^{-\frac{1}{2}})\} \subseteq q^{-\frac{1}{2}} \mathbb{Z}[q^{\frac{1}{2}}] \), \( L_{x,x}(q^{-\frac{1}{2}}) = 1 \), and \( L_{x,w}(q^{-\frac{1}{2}}) = 0 \) if \( x \not\leq w \).

This basis is often called an IC basis (see [8, §2]). Combining Theorem 1.10 with Proposition 1.9 we get

\[
L_{x,w}(q^{-\frac{1}{2}}) = \sum_{y \in [x,w]_c} q^{\frac{\ell(x)-\ell(y)}{2}} a_{x,y}(q)L_{y,w}(q^{\frac{1}{2}}),
\]

for every \( x, w \in W_c(X) \), with \([x,w]_c = \{y \in [x,w] : y \in W_c(X)\}\).

Comparing the definition of \( c_w \) with that of \( C'_w \), we notice that the polynomials \( L_{x,w}(q^{-\frac{1}{2}}) \) play the same role as \( q^{-\frac{1}{2}} L_{x,w}(q^{-\frac{1}{2}}) \), where \( P_{x,w}(q) \) are the Kazhdan–Lusztig polynomials defined in Theorem 1.3. Since the Kazhdan–Lusztig basis and the IC basis are both \( \iota \)-invariant and since \( \iota(J) = J \), it is natural to ask to what extent \( \{\sigma(C'_w) : w \in W(X)\} \) coincides with \( \{c_w : w \in W_c(X)\} \).

In particular, one may wonder whether the canonical projection \( \sigma \) satisfies

\[
\sigma(C'_w) = \begin{cases} 
  c_w & \text{if } w \in W_c(X), \\
  0 & \text{if } w \not\in W_c(X). 
\end{cases}
\]

If \( X \) is a finite irreducible or affine Coxeter group, then relation (5) holds if and only if \( W_c(X) \) is a union of two-sided Kazhdan-Lusztig cells (see [17, Lemma 2.4] and [10, Theorem 2.2.3]). On the other hand, in [16, §3] Shi shows that \( W_c(X) \) is a union of two-sided Kazhdan-Lusztig cells if and only if \( X \) is non-branching and \( X \neq \tilde{F}_4 \). We sum up these properties in the following.

Theorem 1.11. Let \( X \) be a finite irreducible or affine Coxeter graph. Then, relation (5) holds if and only if \( X \) is non-branching and \( X \neq \tilde{F}_4 \).

2. Combinatorial properties of polynomials \( a_{x,w} \)

The first part of this section deals with the study of the \( D \)-polynomials defined in Proposition 1.7. We recall a recurrence relation for \( \{D_{x,w} \}_{x \in W_c(X), w \in W(X)} \), where \( X \) denotes an arbitrary Coxeter graph. Then we focus on the Coxeter graphs...
satisfying equation (5) and obtain an explicit formula for the $D$–polynomials indexed by elements which satisfy particular properties.

In the second part of the section we study the family of polynomials \{a_{x,w}\}_{x,w \in W_c(X)} which express the involution $\iota$ in terms of the $t$–basis, as explained in Proposition 1.9. First, we obtain a recurrence relation for $a_{x,w}$, $X$ being an arbitrary Coxeter graph. Then we derive an explicit formula for $a_{x,w}$, with $x,w \in W_c(X)$ satisfying particular properties and $X$ such that equation (5) holds.

We begin with the following recursion for the $D$–polynomials (see [15, Theorem 3.1]).

**Theorem 2.1.** Let $X$ be an arbitrary Coxeter graph. Let $w \not\in W_c(X)$ and $s \in S(X)$ be such that $ws \not\in W_c(X)$, with $ws < w$. Then, for all $x \in W_c(X)$, $x \leq w$, we have

$$D_{x,w} = \widetilde{D}_{x,w} + \sum_{y \in W_c(X), ys \not\in W_c(X)} D_{x,y} D_{y,ws},$$

where

$$\widetilde{D}_{x,w} \overset{\text{def}}{=} \begin{cases} D_{xs,ws} + (q - 1)D_{x,ws} & \text{if } xs < x, \\ qD_{xs,ws} & \text{if } x < xs \in W_c(X), \\ 0 & \text{if } x < xs \not\in W_c(X). \end{cases}$$

From here to the end of this section we will denote by $X$ a Coxeter graph satisfying (5). Observe that $D_{x,w} = \delta_{x,w}$ if $x,w \in W_c(X)$.

**Lemma 2.2.** For all $x \in W_c(X)$ and $w \not\in W_c(X)$, we have

$$\sum_{x \leq y \leq w} D_{x,y} P_{y,w} = 0.$$

A proof of the preceding lemma appears in [15, Lemma 3.6]. It is worth noting that Lemma 2.2 implies

$$D_{x,w} = -P_{x,w} - \sum_{t \not\in W_c(X)} D_{x,t} P_{t,w},$$

for all $x \in W_c(X)$ and $w \not\in W_c(X)$ such that $x < w$.

**Lemma 2.3.** Let $x \in W_c(X)$ be such that $xs \not\in W_c(X)$ and let $w \not\in W_c(X)$ be such that $w > ws \in W_c(X)$. Then

$$D_{x,w} = -\delta_{x,ws}.$$
PROOF. We proceed by induction on $\ell(x,w)$. If $\ell(x,w) = 1$, then $D_{x,w} = D_{x,xs} = -1 = -\delta_{x,ws}$. Suppose $\ell(x,w) > 1$. Recall that $P_{x,w}(q) = P_{x,ws}(q)$, for every $x \leq w$ such that $ws < w$ (see, e.g., [2, Proposition 5.1.8]). Then, from (6) we get

$$D_{x,w} = -P_{x,w} - \sum_{t \in W_c(X)} D_{x,t} P_{t,w}$$

$$= -P_{x,w} - D_{x,xs} P_{x,ws} - \sum_{t \in W_c(X), t \neq xs} D_{x,t} P_{t,w}$$

$$= -P_{x,w} - D_{x,xs} P_{x,ws} - \sum_{t \in W_c(X), t \neq xs} D_{x,t} P_{t,w}$$

By induction hypothesis, the term $D_{x,t}$ in the first sum is equal to $-\delta_{x,ts}$, since $\ell(x,t) < \ell(x,w)$. Therefore, the first sum is zero. On the other hand, the second and the third sums can be written as

$$- \sum_{ts \in W_c(X), t < ts} D_{x,t} P_{t,w} - \sum_{t,ts \in W_c(X), t < ts} D_{x,t} P_{t,w} - \sum_{t < ts} D_{x,t} P_{t,w}.$$ 

(7)

since $t \notin W_c(X), t < ts$ implies $ts \notin W_c(X)$. To prove the statement we have to show that the term (7) is zero. First, observe that $\ell(x,z) < \ell(x,w)$. Moreover, by Proposition 2.1 and by induction hypothesis, we achieve

$$D_{x,zs} = \sum_{u \in W_c(X)} D_{x,us} D_{u,z} = \sum_{u \in W_c(X)} (-\delta_{x,u}) D_{u,z} = -D_{x,z}.$$ 

(8)

We conclude that $D_{x,zs} + D_{x,z} = 0$, for all $z \notin W_c(X)$ such that $x < z < zs < w$, and so the sum in (7) is zero. \qed

The next property for $D$–polynomials will be needed at the end of this section.

**Proposition 2.4.** Let $w \in W(X)$. Then

$$\sum_{x \in W_c(X), x \leq w} \varepsilon_x D_{x,w} = \varepsilon_w.$$ 

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PROOF. We proceed by induction on \( \ell(w) \). The proposition is trivial if \( w \in W_c(X) \), which covers the case \( \ell(w) \leq 2 \). Suppose that \( w \not\in W_c(X) \). Then, by (6) we have

\[
\sum_{x \in W_c(X)} \varepsilon_x D_{x,w} = \sum_{x \in W_c(X), x < w} \varepsilon_x (-P_{x,w}) + \sum_{x \in W_c(X), x < w} \varepsilon_x \left( - \sum_{t \not\in W_c(X), x < t} D_{x,t} P_{t,w} \right)
\]

\[
= - \sum_{x \in W_c(X), x < w} \varepsilon_x P_{x,w} - \sum_{t \not\in W_c(X), x < t} P_{t,w} \left( \sum_{x \in W_c(X), x < t} \varepsilon_x D_{x,t} \right)
\]

\[
= - \sum_{x \in W_c(X), x < w} \varepsilon_x P_{x,w} - \sum_{t \not\in W_c(X), x < t} P_{t,w} \varepsilon_t
\]

and the statement follows from the fact that \( \sum_{x \leq w} \varepsilon_x P_{x,w} = 0 \), for every \( w \in W(X) \setminus \{e\} \) (see [2, §5, Exercise 17]). \qed

Now, let us turn our attention to the study of the polynomials \( \{a_{x,w}\}_{x,w \in W_c(X)} \).

**Proposition 2.5.** Let \( X \) be an arbitrary Coxeter graph. Let \( w \in W_c(X) \) and \( s \in S(X) \) be such that \( w > ws \in W_c(X) \). Then, for all \( x \in W_c(X), x \leq w \), we have

\[
a_{x,w} = \widetilde{a}_{x,w} + \sum_{y \in W_c(X), y \not\in W_c(X) \setminus \{e\}} D_{x,y} a_{y,w},
\]

where

\[
\widetilde{a}_{x,w} \overset{\text{def}}{=} \begin{cases} a_{x,ws} & \text{if } x > xs, \\ qa_{x,ws} + (1-q)a_{x,ws} & \text{if } x < xs \in W_c(X), \\ (1-q)a_{x,ws} & \text{if } x < xs \not\in W_c(X). \end{cases}
\]

PROOF. On the one hand, by Proposition 1.9 we have

\[
(t_{w^{-1}})^{-1} = q^{-\ell(w)} \sum_{y \in W_c(X), y \leq w} a_{y,w} t_y,
\]
On the other hand, letting $\psi \overset{\text{def}}{=} ws$, we get

\[
(t_w^{-1})^{-1} = (t_{v-1})^{-1} (t_v)^{-1}.
\]

\[
= q^{-\ell(v)} \sum_{y \in W_c(X)} a_{y,v} t_y \cdot q^{-1} (t_s - (q-1)t_c)
\]

\[
= q^{-\ell(w)} \left( \sum_{y \in W_c(X), y \leq v} a_{y,v} t_y + \sum_{y \in W_c(X), y \leq v} a_{y,v} t_y - (q-1) \sum_{y \in W_c(X), y \leq v} a_{y,v} \right)
\]

\[
+ q^{-\ell(w)} \left( \sum_{y \in W_c(X), y \leq v, y_s > y} a_{y,v} t_y + \sum_{y \in W_c(X), y \leq v, y_s > y} a_{y,v} \right)
\]

\[
= q^{-\ell(w)} \left( \sum_{y \in W_c(X), y \leq v, y_s < y} a_{y,v} t_y + \sum_{y \in W_c(X), y \leq v, y_s < y} a_{y,v} \right)
\]

\[
+ q^{-\ell(w)} \left( \sum_{y \in W_c(X), y \leq v, y_s < y} a_{y,v} \right)
\]

\[
= q^{-\ell(w)} \left( \sum_{z \in W_c(X)} a_{z,v} t_z + \sum_{z \in W_c(X)} D_{z,sy} a_{y,v} t_z \right)
\]

\[
+ q^{-\ell(w)} \left( \sum_{y \in W_c(X), y \leq v, y_s < y} a_{y,v} q t_y - (q-1) t_s - (q-1) \sum_{y \in W_c(X), y \leq v} a_{y,v} \right),
\]

and the statement follows by extracting the coefficient of $t_s$.  

From now on, we will assume $X$ to be any Coxeter graph satisfying equation (5).  

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Corollary 2.6. Let \( x, w \in W_c(X) \). If there exists \( s \in S(X) \) such that \( ws < w \) and \( x < xs \not\in W_c(X) \), then
\[
a_{x,w} = -qa_{x,ws}.
\]

PROOF. By Proposition 2.5 we have
\[
a_{x,w} = (1 - q)a_{x,ws} + \sum_{y \in W_c(X), y < w} D_{x,ys}a_{y,ws}.
\]
On the other hand, by Lemma 2.3 \( D_{x,ys} = -\delta_{x,y} \). Therefore
\[
a_{x,w} = (1 - q)a_{x,ws} - \sum_{y \in W_c(X), y < w} \delta_{x,y}a_{y,ws} = (1 - q)a_{x,ws} - a_{x,ws},
\]
and the statement follows. \( \square \)

In the sequel we will need the following result (see \([15, Proposition 4.1]\)).

Proposition 2.7. Let \( x, w \in W_c(X) \) be such that \( x \leq w \). Then
\[
a_{x,w}(q) = \varepsilon_x\varepsilon_wR_{x,w}(q) + \sum_{y \in W_c(X), x < y < w} \varepsilon_y\varepsilon_wR_{y,w}(q)D_{x,y}(q). \tag{9}
\]

The recursion given in Corollary 2.6 can sometimes be solved explicitly.

Proposition 2.8. Let \( s_i s_{i+1} \cdots s_i + k s_i - j s_{i-j} - 1 \cdots s_i - k - 1 \) be a reduced expression for \( w \in W(A_n) \) and let \( s_i s_{i+1} \cdots s_i + k \) be a reduced expression for \( x \in W(A_n) \), with \( i \in [2, n], k \in [1, n - i], j \in [1, i - 1] \). Then
\[
a_{x,w}(q) = (-q)^k(1 - q)^j.
\]

PROOF. Observe that \( x < xs_{i+j} \not\in W_c(X) \), for every \( h \in [0, k - 1] \) and that \( ws_{i+j-k-1} < w \). By applying Corollary 2.6 to the triple \((x, w, s_{i+j-k-1})\) we get \( a_{x,w} = -qa_{x,ws_{i+j-k-1}} \). Repeat the same process with the triple \((x, ws_{i+j-k-1}, s_{i+j-k-2})\), and so on. After \( k \) iteration of the process we get \( a_{x,w}(q) = (-q)^k a_{x,ws_i \cdots s_{i+k-1}}(q) = (-q)^k a_{x,w'}(q) \), where we set
\[
w' = s_i s_{i+1} \cdots s_{i+k} s_{i-j} s_{i-j+1} \cdots s_{i-1}.
\]
To conclude the proof, we will show that \( a_{x,w'}(q) = (1 - q)^j \). Observe that \([x, w'] \simeq B_{\ell(w') - \ell(x)} \) and so \( R_{x,w'}(q) = (q - 1)^{\ell(w') - \ell(x)} \) (see \([3, Corollary 4.10]\)). On the other hand, Proposition 2.7 implies \( a_{x,w'}(q) = \varepsilon_x \varepsilon_w R_{x,w'}(q), \) since \( \{ y \in [x, w] : y \not\in W_c(X) \} = \emptyset \). Therefore \( a_{x,w'}(q) = \varepsilon_x \varepsilon_w R_{x,w'}(q) = (1 - q)^j \), as desired. \( \square \)
Next, we obtain a property for the polynomials \( \{a_{x,w}\} \) which will be used in Section 3.

**Proposition 2.9.** Let \( w \in W_c(X) \). Then

\[
\sum_{x \in W_c(X), x \leq w} \varepsilon_x \varepsilon_w a_{x,w} = q^{\ell(w)}.
\]

**Proof.** First, it is a routine exercise to prove the following property:

\[
\sum_{x \leq w} R_{x,w} = q^{\ell(w)},
\]

for every \( w \in W(X) \).

By combining (9) with Proposition 2.4 we get

\[
\sum_{x \in W_c(X), x \leq w} \varepsilon_x \varepsilon_w a_{x,w} = \sum_{x \in W_c(X), x \leq w} \varepsilon_x \varepsilon_w \left( \varepsilon_x \varepsilon_w R_{x,w} + \sum_{y \in W_c(X), x < y < w} \varepsilon_y \varepsilon_w R_{y,w} D_{x,y} \right)
\]

which can be rewritten as

\[
= \sum_{x \in W_c(X), x \leq w} R_{x,w} + \sum_{x \in W_c(X), x \leq w} \varepsilon_x \left( \sum_{y \in W_c(X), x < y < w} \varepsilon_y R_{y,w} D_{x,y} \right)
\]

\[
= \sum_{x \in W_c(X), x \leq w} R_{x,w} + \sum_{y \in W_c(X), y \leq w} \varepsilon_y R_{y,w} \left( \sum_{x \in W_c(X), x \leq y} \varepsilon_x D_{x,y} \right)
\]

\[
= \sum_{x \in W_c(X), x \leq w} R_{x,w} + \sum_{y \in W_c(X), y \leq w} \varepsilon_y R_{y,w} \varepsilon_y
\]

and the statement follows from (10). \( \square \)

### 3. Combinatorial properties of polynomials \( L_{x,w} \)

In this section we study the polynomials \( \{L_{x,w}(q^{-\frac{1}{2}})\}_{x,w \in W_c(X)} \), which play the same role, in \( TL(X) \), as the Kazhdan–Lusztig polynomials in \( \mathcal{H}(X) \). In particular, we derive a recursive formula for \( L_{x,w} \) by means of some results in [7]. Then
we obtain a recursion for \( L_{x,w} \), with \( x,w \) satisfying particular properties. Throughout this section we will assume \( X \) to be an arbitrary Coxeter graph satisfying (5). We recall that \([x,w]_c\) denotes the set \( \{ y \in [x,w] : y \in W_c(X) \} \).

It is known that the terms of maximum possible degree in the polynomials \( L_{x,w} \) and in the Kazhdan–Lusztig polynomials coincide (see [7, Theorem 5.13])

**Proposition 3.1.** For \( x,w \in W_c(X) \) let \( M(x,w) \) be the coefficient of \( q^{-\frac{1}{2}} \) in \( L_{x,w} \) and let \( \mu(x,w) \) be the coefficient of \( q^{\frac{\ell(w)-\ell(x)-1}{2}} \) in \( P_{x,w} \). Then \( M(x,w) = \mu(x,w) \).

The product of two IC basis elements can be computed by means of the following formula (see [7, Theorem 5.13]).

**Proposition 3.2.** Let \( s \in S(X) \) and \( w \in W_c(X) \). Then

\[
c_sc_w = \begin{cases} c_{sw} + \sum_{xx < x} \mu(x,w)c_x & \text{if } \ell(sw) > \ell(w); \\ 
(q^{\frac{1}{2}} + q^{-\frac{1}{2}})c_w & \text{otherwise}, 
\end{cases}
\]

where \( c_x \) def 0 for every \( x \notin W_c(X) \).

**Corollary 3.3.** Let \( s \in S(X) \) and \( w \in W_c(X) \). Then

\[
t_sc_w = \begin{cases} -c_w + q^{\frac{1}{2}} \left( c_{sw} + \sum_{xx < x} \mu(x,w)c_x \right) & \text{if } \ell(sw) > \ell(w); \\ 
qc_w & \text{otherwise}. 
\end{cases}
\]

**Proof.** Observe that \( t_s = q^{\frac{1}{2}}c_s - c_e \). So \( t_sc_w = q^{\frac{1}{2}}c_sc_w - c_w \) and the statement follows by applying Proposition 3.2. \( \square \)

**Theorem 3.4.** Let \( x,w \in W_c(X) \) be such that \( sx \in W_c(X) \) and \( sw < w \). Then

\[
L_{x,w}(q^{-\frac{1}{2}}) = L_{sx,sw}(q^{-\frac{1}{2}}) + q^{\frac{1}{2}}L_{x,sw}(q^{-\frac{1}{2}}) - \sum_{z \in [sx,sw]_c} \mu(z,sw)L_{x,z}(q^{-\frac{1}{2}}) \\
+ q^{-\frac{1}{2}} \sum_{sz \notin W_c(X)} q^{\frac{\ell(z)-\ell(s)}{2}}D_{x,z}(q)L_{z,sw}(q^{-\frac{1}{2}}),
\]

where \( c = 1 \) if \( sx < x \) and 0 otherwise.
PROOF. Let $w = sv$. By Proposition 3.2 we have that
\begin{equation}
    c_w = c_{sv} = c_sc_v - \sum_{sz < z} \mu(z, sw)c_z.
\end{equation}

Recall that $c_s = q^{-\frac{1}{2}}(t_s + t_e)$. Hence
\begin{align*}
    c_sc_v &= q^{-\frac{1}{2}}c_v + q^{-\frac{1}{2}}t_sc_v \\
    &= q^{-\frac{1}{2}}c_v + \sum_{x \in W_c(X) \atop x \leq sw} q^{-\frac{\ell(x)}{2}}L_{x, sw}t_xt_x \\
    &= q^{-\frac{1}{2}}c_v + \sum_{xx \in W_c(X) \atop x < xx} q^{-\frac{\ell(x)}{2}}L_{x, sw}(t_{xx} + \sum_{x < xx} q^{-\frac{\ell(x)}{2}}L_{x, sw}(q_{xx} + (q - 1)t_x)) \\
    &+ q^{-\frac{1}{2}}\left(\sum_{xx \in W_c(X) \atop x < xx} q^{-\frac{\ell(x)}{2}}L_{x, sw}\left(\sum_{y \in W_c \atop y < xx} D_{y, xx}t_y\right)\right) \\
    &= q^{-\frac{1}{2}}c_v + \sum_{xx \in W_c(X) \atop x < xx} q^{-\frac{\ell(x)}{2}}L_{x, sw}(t_{xx} + \sum_{x < xx} q^{-\frac{\ell(x)}{2}}L_{x, sw}(q_{xx} + (q - 1)t_x)) \\
    &+ q^{-\frac{1}{2}}\left(\sum_{y \in W_c(X) \atop y < w} \left(\sum_{xx \in W_c(X) \atop x < xx} q^{-\frac{\ell(x)}{2}}D_{x, xx}L_{x, sw}\right)t_y\right).
\end{align*}

Suppose that $su > u$ and extract the coefficient of $t_{su}$ on both sides of (11). It follows that
\begin{equation*}
    L_{su, w} = L_{u, sw} + q^\frac{1}{2}L_{su, sw} + \sum_{sz \in W_c(X) \atop z < sz} q^{-\frac{\ell(u)}{2}}D_{su, sz}L_{z, sw} - \sum_{z \in [u, w] \atop sz < z} \mu(z, sw)L_{su, z}.
\end{equation*}

Otherwise, if $su < u$ then
\begin{equation*}
    L_{su, w} = L_{u, sw} + q^{-\frac{1}{2}}L_{su, sw} + q^{-1}\sum_{sz \in W_c(X) \atop z < sz} q^{-\frac{\ell(u)}{2}}D_{su, sz}L_{z, sw} - \sum_{z \in [u, w] \atop sz < z} \mu(z, sw)L_{su, z}.
\end{equation*}

The statement follows by applying the substitution $x = su$. □
In [15, Theorem 5.1] the following result is proved.

**Theorem 3.5.** Let $X$ be such that equation (5) holds. For all elements $x, w \in W_c(X)$ such that $x < w$ we have

$$L_{x,w} = q^{\frac{\ell(x) - \ell(w)}{2}} \left( P_{x,w} + \sum_{y \in W_c(X)} D_{x,y} P_{y,w} \right). \quad (12)$$

**Lemma 3.6.** Let $x, w \in W_c(X)$. If there exists $s \in S(X)$ such that $sw < w$ and $x < sx \not\in W_c(X)$, then $L_{x,w} = 0$.

**Proof.** By (12) we get

$$L_{x,w} = q^{\frac{\ell(x) - \ell(w)}{2}} \left( P_{x,w} + \sum_{y \in W_c(X)} D_{x,y} P_{y,w} \right)$$

$$= q^{\frac{\ell(x) - \ell(w)}{2}} \left( P_{x,w} + D_{x,xx} P_{xx,w} + \sum_{y \in W_c(X), y \neq xx} D_{x,y} P_{y,w} \right)$$

$$= q^{\frac{\ell(x) - \ell(w)}{2}} \left( \sum_{y \in W_c(X), y \neq xx} D_{x,y} P_{y,w} \right). \quad (13)$$

Denote by (*) the expression in round brackets in (13). Then (*) is zero and the statement follows. In fact, by applying relation (8) and Lemma 2.3, we get

$$(* \quad = \sum_{y \notin W_c(X)} D_{x,y} P_{y,w} + \sum_{y \in W_c(X), y \neq xx} D_{x,y} P_{y,w}$$

$$= \sum_{y \in W_c(X), y \neq xx} D_{x,y} P_{y,w} + \sum_{y \in W_c(X)} D_{x,y} P_{y,w} + \sum_{y \in W_c(X), y \neq xx} D_{x,y} P_{y,w}$$

$$= \sum_{y \notin W_c(X), y \neq xx} \left( D_{x,xy} + D_{x,y} \right) P_{y,w} + \sum_{y \neq xx} D_{x,y} P_{y,w}$$

$$= \sum_{y \notin W_c(X), y \neq xx} (-\delta_{x,xy}) P_{y,w} = 0,$$
The next result is the analogue of a well-known property of the Kazhdan–Lusztig polynomials (see, e.g., [2, Proposition 5.1.8]).

**Theorem 3.7.** Let $x, w \in W_c(X)$ be such that $x < w$. If there exists $s \in S(X)$ such that $sw < w$ and $x < sx \in W_c(X)$, then

$$L_{x,w} = q^{-\frac{1}{2}}L_{sx,w}.$$

**Proof.** By Corollary [3.3] we get $t_s c_w = q c_w$, since $\ell(sw) < \ell(w)$ by hypothesis.

Furthermore, by Theorem 1.10 if $x < sx \in W_c(X)$ then $[t_{sx}](q c_w) = q \cdot q^{-\frac{\ell(sx)}{2}}L_{sx,w}$.

On the other hand, Theorem 1.10 implies that

$$t_s c_w = \sum_{x \in W_c(X) \atop x \leq w} q^{-\frac{\ell(x)}{2}}L_{x,w} t_s t_x$$

$$= \sum_{sx \in W_c(X) \atop sx > x} q^{-\frac{\ell(sx)}{2}}L_{x,w} t_s t_x + \sum_{sx \not\in W_c(X) \atop sx > x} q^{-\frac{\ell(sx)}{2}}L_{x,w} t_s t_x +$$

$$+ \sum_{x \in W_c(X) \atop sx < x} q^{-\frac{\ell(x)}{2}}L_{x,w}(qt_{sx} + (q - 1)t_x)$$

$$= \sum_{sx \in W_c(X) \atop sx > x} q^{-\frac{\ell(sx)}{2}}L_{x,w} t_s t_x + \sum_{sx \not\in W_c(X) \atop sx > x} q^{-\frac{\ell(sx)}{2}}L_{x,w} \left( \sum_{y \in W_c(X) \atop y < sx} D_{y,sx} t_y \right) +$$

$$+ q \sum_{sz \in W_c(X) \atop z < sz} q^{-\frac{\ell(sz)}{2}}L_{sz,w} t_{sz} + (q - 1) \sum_{sz \in W_c(X) \atop z < sz} q^{-\frac{\ell(sz)}{2}}L_{sz,w} t_{sz}$$

$$= \sum_{sx \in W_c(X) \atop sx > x} q^{-\frac{\ell(sx)}{2}}L_{x,w} t_s t_x + q^2 q^{-\frac{\ell(sx)}{2}}L_{sx,w} t_s t_x + q^2 q^{-\frac{\ell(sx)}{2}}L_{sx,w} t_s t_x +$$

$$- q^{-\frac{1}{2}} q^{-\frac{\ell(sx)}{2}}L_{sx,w} t_s t_x + \sum_{x \in W_c(X) \atop x \leq w} \left( \sum_{sz \in W_c(X) \atop z \in (x,w)_c} q^{-\frac{\ell(sz)}{2}}D_{x,sz} L_{z,w} \right) t_x. \quad (15)$$

as desired.
By extracting the coefficient of $t_{sx}$ in (14) and (15) we obtain

$q^\frac{1}{2} q^{-\ell(s)x} L_{sx,w} = q^{\frac{\ell(x)}{2}} L_{x,w} + q^\frac{1}{2} q^{-\ell(s)x} L_{sx,w} - q^\frac{1}{2} q^{-\ell(x)} L_{sx,w} + \sum_{sz \not\in W_c(X)} \sum_{z \in (x,w)_c} q^{-\ell(z)} D_{sx,sz} L_{z,w},$

that is

$L_{x,w} = q^{-\frac{1}{2}} L_{sx,w} - \sum_{sz \not\in W_c(X)} \sum_{z \in (x,w)_c} q^{\frac{\ell(x)-\ell(z)}{2}} D_{sx,sz} L_{z,w}.$

Observe that Lemma 3.6 implies $L_{z,w} = 0$, since $z < sz \not\in W_c(X)$, and the statement follows.

We conclude this section with two results inspired by similar properties for the Kazhdan–Lusztig polynomials (see, e.g., [2, §5, Exercises 16, 17]).

**Proposition 3.8.** Let $w \in W_c(X)$ and define

$F_w(q^{-\frac{1}{2}}) \overset{\text{def}}{=} \sum_{x \in W_c(X)} \sum_{x \leq w} \varepsilon_x q^{-\frac{\ell(x)}{2}} L_{x,w}(q^{-\frac{1}{2}}).$

Then $F_w(q^{-\frac{1}{2}}) = \delta_{e,w}.$

**Proof.** The case $w = e$ is trivial. Suppose $w \neq e$. Combining (4) with Proposition
we have

\[ F_w \left( q^{-\frac{1}{2}} \right) = \sum_{x \in W(X), x \leq w} \varepsilon_x q^{-\frac{\ell(x)}{2}} \left( \sum_{u \in W_c(X), u \leq x} a_{u,x}(q) L_{x,w} \left( q^{\frac{1}{2}} \right) \right) \]

\[ = \sum_{x \in W(X), x \leq w} \left( \sum_{u \in W_c(X), u \leq x} \varepsilon_u q^{-\frac{\ell(u)}{2}} a_{u,x}(q) L_{x,w} \left( q^{\frac{1}{2}} \right) \right) \]

\[ = \sum_{x \in W(X), x \leq w} \varepsilon_x q^{-\frac{\ell(x)}{2}} L_{x,w} \left( q^{\frac{1}{2}} \right) \left( \sum_{u \in W_c(X), u \leq x} \varepsilon_u a_{u,x}(q) \right) \]

\[ = \sum_{x \in W(X), x \leq w} \varepsilon_x q^{-\frac{\ell(x)}{2}} L_{x,w} \left( q^{\frac{1}{2}} \right) q^{\ell(x)} \]

\[ = \sum_{x \in W(X), x \leq w} \varepsilon_x q^{-\frac{\ell(x)}{2}} L_{x,w} \left( q^{\frac{1}{2}} \right) \]

\[ = F_w \left( q^{\frac{1}{2}} \right). \]

This implies that \( F_w \left( q^{-\frac{1}{2}} \right) \) is constant. On the other hand, the constant term in \( F_w \left( q^{-\frac{1}{2}} \right) \) is zero since \( L_{x,w} \in q^{-\frac{1}{2}} \mathbb{Z}[q^{-\frac{1}{2}}] \) by Theorem 1.10 and the statement follows. \( \square \)
Acknowledgements

I would like to thank Prof. Francesco Brenti for introducing me to this topic and for many useful conversations.

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