Ground State Degeneracy of a Class of Interacting Fermion Models

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We prove rigorously that the ground state of a class of interacting Majorana fermion models and its equivalent spinless fermion models is either unique or doubly degenerate if the lattice size $N$ is even, and is always doubly degenerate if $N$ is odd. Our proof is based on the reflection positivity of the system in the eigen-operator representation and holds in all dimensions with arbitrary lattice structures.

Understanding of physical properties of strongly correlated quantum lattice models has long been a challenging problem. Because of the subtlety of interactions, it is generally difficult to solve these models. Rigorous results and exact solutions are clearly useful benchmarks, but these are rare. In 1989, Lieb proved that the ground state of the Hubbard model is unique by utilizing a Perron-Frobenius-type argument and the spin-reflection positivity of the system\textsuperscript{[1]}. This extended the proof for the uniqueness of the ground state from an antiferromagnetic Heisenberg model\textsuperscript{[2]} to an interacting fermion system and led to a more general application of reflection positivity in quantum flux phases\textsuperscript{[3]}. Frustrated Heisenberg antiferromagnets\textsuperscript{[4]}, and other quantum lattice models has long been a challenge\textsuperscript{[5]}. The spin-reflection positivity is intimately connected with the sign rule of the ground-state wave function, which can be used to understand the minus sign problem caused by the fermion characteristics of itinerant electrons in quantum Monte Carlo simulations\textsuperscript{[6]}. In this letter, we present a rigorous proof for two theorems on the degeneracy of ground states for a class of interacting quantum lattice models of two-species Majorana fermions. We will show that the ground state is doubly degenerate when the lattice size $N$ is odd and at most doubly degenerate when $N$ is even. Our proof utilizes a new kind of reflection positivity between the two-species Majorana fermions. These theorems can be applied to a number of interacting spinless fermion models which are equivalent to the Majorana fermion model. It sheds light on the understanding of topological degeneracy and symmetry-protected topological phases of Majorana fermions\textsuperscript{[7,8]}, and the minus sign problem in an interacting spinless fermion model that was recently studied by a number of groups\textsuperscript{[9,10]}.

The model studied here is defined on a lattice with $N$ sites,

\begin{equation}
H = H_K + H_V,
\end{equation}

\begin{equation}
H_K = \sum_{ij} t_{ij} \left( \gamma_i^{(1)} \gamma_j^{(1)} - \gamma_i^{(2)} \gamma_j^{(2)} \right),
\end{equation}

\begin{equation}
H_V = -\sum_{ij} V_{ij} \left( \gamma_i^{(1)} \gamma_j^{(1)} \right) \left( -i \gamma_i^{(2)} \gamma_j^{(2)} \right),
\end{equation}

where $\gamma_i^{(\sigma)}$ ($\sigma = 1, 2$) are Majorana fermion operators at site $i$. $t_{ij}$ and $V_{ij} \geq 0$ are real hopping and interaction constants between sites $i$ and $j$, respectively. If $V_{ij} \neq 0$, we say that there is a bond between $i$ and $j$. We assume the lattice to be connected in which there is a connected path of bonds between every pair of sites.

Majorana fermion operators are self-conjugate operators. To discuss the degeneracy of ground states, it is more convenient to use the representation of eigen-operators rather than that of eigen-vectors. This is because at each lattice site $i$, $(1, \gamma_i^{(1)} \gamma_i^{(2)} , \gamma_i^{(1)} \gamma_i^{(2)} )$ form a complete orthonormal basis set for any operators defined at this site. The direct product of these operators over all lattice sites form a complete basis set for all operators defined on the lattice.

From the property of eigenvectors of the Hamiltonian, it is simple to show that at each symmetry distinct sector, an eigen-operator $O$ satisfies the following eigen-equation

\begin{equation}
HO = OH = EO,
\end{equation}

where $E$ is the eigen-energy. In the subspace spanned by $n$ degenerate eigenstates of $H$ with an eigenvalue $E$, say $\Omega_E = \{|\Psi_a\rangle, \ a = 1, \cdots, n\}$, $O$ can be always written as a superposition of the following $n^2$ independent operators $|\Psi_a\rangle \langle \Psi_b|$, where both $|\Psi_a\rangle$ and $|\Psi_b\rangle$ belong to $\Omega_E$. Thus for each symmetry distinct energy level, the degeneracy of eigenvectors is equal to the square root of the degeneracy of the corresponding eigen-operators. Moreover, an eigen-operator can be always symmetrized to become a hermitian operator $O^\dagger = O$, no matter whether the state is degenerate or not.

The Hamiltonian conserves the parity (even or odd) of the number of $\gamma^{(\sigma)}$-fermions. Thus the eigen-operators can be diagonalized according to the parities of $\gamma^{(1)}$ and $\gamma^{(2)}$ fermions into four blocks. Operators at each parity sector can be expanded using the basis operators defined by

\begin{equation}
\Gamma^{(1)}_\alpha = i^{(m/2)} \gamma_{i_1}^{(1)} \cdots \gamma_{i_m}^{(1)},
\end{equation}

\begin{equation}
\Gamma^{(2)}_\alpha = (-i)^{(m/2)} \gamma_{i_1}^{(2)} \cdots \gamma_{i_m}^{(2)},
\end{equation}

where $\alpha = (i_1, \cdots, i_m)$ denotes the configuration of Majorana fermions. $|m/2\rangle = m/2$ or $(m - 1)/2$ if $m$ is even or odd. $\Gamma^{(\sigma)}_\alpha$ is called even or odd, if $\alpha$ contains an even or odd number of Majorana fermion. The operators such
defined are hermitian and form a complete orthonormal basis set of operators
\[
\text{Tr} \Gamma^{(\sigma)}_\alpha \Gamma^{(\sigma')}_\alpha = \delta_{\alpha, \alpha'}.
\] (5)
Moreover, the Hamiltonian commutes with the following two operators
\[
\Delta^{(1)} = \sum_{\gamma} \gamma^{(1)}_1 \ldots \gamma^{(1)}_N, \quad \Delta^{(2)} = \sum_{\gamma} (-i)^{\sum \gamma_i (2)} \gamma^{(2)}_1 \ldots \gamma^{(2)}_N.
\] (6)
\[
\Delta^{(1)} \text{ commutes or anticommutes with } \Delta^{(2)} \text{ for even or odd } N.
\]
An eigen-operator can be expanded using the basis operators as
\[
O = \sum_{\alpha, \beta} \Psi_{\alpha, \beta} \Gamma^{(1)}_\alpha \Gamma^{(2)}_\beta.
\] (8)
In the even-even sector, the wave function \( \Psi_{\alpha, \beta} \) is a real and symmetric matrix, satisfying the following eigen-equation
\[
K \Psi + \Psi K - \sum_{ij} V_{ij} L_{ij} \Psi L_{ij} = E \Psi,
\] (9)
where \( K \) and \( L_{ij} \) are real matrices defined by
\[
L_{ij, \alpha' \alpha} = \text{Tr} \left[ \Gamma^{(1)}_{\alpha} \Gamma^{(1)}_{\alpha'} \gamma^{(1)}_j \right],
\] (10)
\[
K_{\alpha' \alpha} = \sum_{ij} t_{ij} L_{ij, \alpha' \alpha},
\] (11)
The eigen-energy is given by
\[
E (\Psi) = 2 \text{Tr} \left( K \Psi^2 \right) - \sum_{ij} V_{ij} \text{Tr} (\Psi L_{ij} \Psi L_{ij}),
\] (12)
if the wave function \( \Psi \) is normalized, \( \text{Tr} (\Psi^2) = 1 \).
In the case the lattice size \( N \) is odd, \( \Delta^{(\sigma)} \) contains an odd number of Majorana fermions. It defines a transformation between two operators with opposite parities for the \( \sigma \)-fermions. For an given eigen-operator in the even-even sector \( O \), \( \Delta^{(1)} O \) generates an operator in the odd-even sector. Similarly, operators in the even-odd and odd-odd sectors can be generated by applying \( \Delta^{(2)} \) and \( \Delta^{(1)} \Delta^{(2)} \) to \( O \), respectively. Since both \( \Delta^{(1)} \) and \( \Delta^{(2)} \) commute with the Hamiltonian, it is simple to show that these four operators are all the eigenoperators of \( H \) with degenerate energy. Thus if the ground state eigenoperator is \( n \)-fold degenerate in the even-even parity sector, then the total degeneracy of the eigen-operators is \( 4n \) and the ground state degeneracy is \( 2\sqrt{n} \). This leads to the following theorem:

**Theorem I:** For the model defined by Eq. (1), the ground state is doubly degenerate if \( N \) is odd.

**Proof:** We only need to show that the ground state is non-degenerate, i.e. \( n = 1 \), in the even-even parity sector. We first show that among all degenerate ground states in the even-even parity sector there always exists an eigenfunction which is semi-positive definite.

Let us assume \( \Psi \) to be a real and symmetric eigen-function of the Hamiltonian, satisfying Eq. (1). By diagonalize it using an unitary matrix \( U \), \( \Psi = U \lambda U^\dagger \), we can define a trial wave function \( |\Psi| = U |\lambda| U^\dagger \) by setting the diagonal matrix \( \lambda \) to its absolute value. \( |\Psi| \) is semi-positive defined. From Eq. (12), it is simple to show that
\[
E (|\Psi|) \leq E (\Psi)
\] (13)
if not all \( V_{ij} \) are zero. Thus if \( \Psi \) is a ground state wave function, so is \( |\Psi| \).

With \( \Psi \) given then, now let us consider the semi-positive matrix \( R = |\Psi| - \Psi \) which is also a multiple of a ground state and satisfies (9). We denote \( \Omega \) as the ensemble of all vectors \( v \) that satisfy \( Rv = 0 \). For a given \( v \) in \( \Omega \), by acting \( v^\dagger \) and \( v \) to both sides of the eigen-equations, it can be shown that \( RL_{ij} v = RL_{ij}^\dagger v = 0 \) for all connected bonds, hence \( L_{ij} v \) and \( L_{ij}^\dagger v \) are also in \( \Omega \). Note that \( v \) corresponds to an operator \( r (v) = \sum_{\alpha = \text{even}} v_{\alpha} \Gamma^{(\alpha)}_\alpha \) in the even-parity sector. The actions of \( L_{ij} v \) and \( L_{ij}^\dagger v \) are equivalent to multiplying \( r (v) \) by \( \Gamma^{(\alpha)}_ij \) from its right and left hand side, respectively. As the lattice is connected, all basis operator \( \Gamma^{(\alpha)}_j \) in the even-parity sector can be generated from the product of a set of connected \( \Gamma^{(\alpha)}_ij \). \( \Gamma^{(\alpha)}_ij \) itself is also a basis operator. If \( r (v) \) is non-zero, one can always convert it into the unity operator by applying successively the basis operators to it, from either left or right hand side. This is because \( \{ \Gamma^{(\alpha)}_ij, \Gamma^{(\alpha')}_ij \} = 0 \) if \( i \in \alpha \) and \( j \notin \alpha \). Thus for any given \( \alpha \) with \( v_\alpha \neq 0 \) and \( (i \in \alpha, j \notin \alpha) \), one can remove this term from \( r (v) \) by the transformation \( r (v) \rightarrow \Gamma^{(\alpha)}_ij r (v) \Gamma^{(\alpha)}_ij^\dagger + r (v) \). Repeating this step by step, we can reduce \( r (v) \) to a single basis operator with a constant coefficient, which in turn can be further reduced to an unity operator by multiplying its inverse. Thus starting from any given \( v \neq 0 \) in \( \Omega \), one can generate all other vectors in \( \Omega \). This implies that \( \Psi = \pm |\Psi| \) and \( \Psi \) is positive definite.

If there are two normalized ground states, \( \Psi_1 \) and \( \Psi_2 \) with \( \Psi_1 \neq \pm \Psi_2 \), then for any real constant \( p \), \( \Psi_p = \Psi_1 + p \Psi_2 \) is also a ground state. It is simple to verify that there exists a \( p \) for which \( \Psi_p \) is neither positive nor negative semidefinite. This contradicts the assertion that \( \Psi_p = \pm |\Psi_p| \). Thus the ground state is unique in the even-even parity sector. Q.E.D.

When \( N \) is odd, the ground state can be set as the eigenstate of \( \Delta = \Delta^{(1)} \Delta^{(2)} \) with the eigenvalues, \( \pm 1 \). But the eigenoperators of \( \Delta \) do not conserve the parity of Majorana fermions.

In the case \( N \) is even, both \( \Delta^{(1)} \) and \( \Delta^{(2)} \) are good quantum numbers and the eigen-states can be classified by their eigenvalues. As \( \Delta^{(\sigma)} \) (\( \sigma = 1, 2 \)) conserves the
parity of the $\sigma$-fermions and the square of its eigenvalue is always equal to 1, the eigen-operator should commute with $\Delta^{(\sigma)}$. The basis operator $\Gamma_\alpha^{(\sigma)}$ commutes or anti-commutes with $\Delta^{(\sigma)}$ for even or odd $\alpha$. The operators in the even-odd, odd-even, or odd-odd parity sectors anticommute with at least one of the operators in $\Delta^{(1)}$ and $\Delta^{(2)}$, and thus cannot be the eigen-operators of the Hamiltonian. Thus the ground states exist only in the even-even parity sector.

In the even-even parity sector, the eigen-operators can be block-diagonalized according to the eigenvalues of $\Delta^{(1)}$ and $\Delta^{(2)}$ into four blocks. The basis operators are now defined by $\Gamma_\alpha^{(\sigma, \pm)} = c_{\alpha, \pm} \left[1 \pm \Delta^{(\sigma)}\right] \Gamma^{(\sigma)}$, and the eigen-equation becomes

$$K^{\mu} \Psi_{\mu} + \Psi_{\mu} K^{\nu} - \sum_{ij} V_{ij} L_{ij}^\mu \Psi_{\mu} L_{ij}^\nu = E \Psi_{\mu},$$

(14)

where $c_{\alpha, \pm}$ is a normalization constant, $\Psi_{\mu}$, with $\mu = \pm$ and $\nu = \pm$ are the eigen-functions, and

$$L_{ij, \alpha' \alpha}^\mu = \text{Tr} \left[ \Gamma_\alpha^{(1, \mu)} \Gamma_{i j}^{(1)} \Gamma_\alpha^{(1, \mu)} \right];$$

(15)

$$K_{\alpha' \alpha}^{\mu} = \sum_{ij} t_{ij} L_{ij, \alpha' \alpha}^\mu.$$  

(16)

**Theorem II:** For the model defined by Eq. (1), there are at most two linear independent ground states labeled by $\Delta^{(1)} = \Delta^{(2)} = \pm 1$ when $N$ is even.

Proof: Again we need only to consider the even-even parity sector. Following the same steps presented in the proof of Theorem I, it can be shown that the lowest energy state is unique in each of the $\Delta^{(1)} = \Delta^{(2)} = \pm 1$ blocks.

From the symmetry between the two kinds of fermions, it is straightforward to show that the lowest energy state of $\Delta^{(1)} = -\Delta^{(2)} = 1$ is degenerate with that of $\Delta^{(2)} = -\Delta^{(1)} = 1$. If $\Psi_{++} = W$ is an eigen-operator of the lowest energy state in the $\Delta^{(1)} = -\Delta^{(2)} = 1$ block, then the corresponding eigen-operator in the $\Delta^{(1)} = -\Delta^{(2)} = 1$ block is given by $\Psi_{+-} = WT$. By taking the singular value decomposition of $W = U \Lambda V^\dagger$, we can introduce two trial wave functions $\Psi_{++} = U \Lambda U^\dagger$ and $\Psi_{--} = V \Lambda V^\dagger$ defined in the $\Delta^{(1)} = \Delta^{(2)} = \pm 1$ blocks, respectively. Here $U$ and $V$ are two unitary matrices and $\Lambda$ is a positive semidefinite diagonal matrix.

Using Eq. (14), it can be shown that

$$E(\Psi_{+-}) + E(\Psi_{-+}) \geq E(\Psi_{++}) + E(\Psi_{--}),$$

(17)

where the equality holds if and only if $\Delta U^\dagger L^{(1)}_{ij} U = \Delta V^\dagger L^{(2)}_{ij} V$ on all bonds with $V_{ij} > 0$. This means that if $\Psi_{+-}$ is a ground state, so are $\Psi_{++}$ and $\Psi_{-+}$. Since the ground state in the $\Delta^{(1)} = \Delta^{(2)} = 1$ is unique, $\Lambda$ should be strictly positive. Thus we have

$$U^\dagger L^{(1)}_{ij} U = V^\dagger L^{(2)}_{ij} V$$

(18)

for all $V_{ij} > 0$. Since $\Delta^{(\sigma)}$ can be always contracted from a product of $(i \gamma^i_1 \gamma^j_2)$ on a set of connected bonds, this equation implies that

$$U^\dagger \Delta^{(\sigma)} U = V^\dagger \Delta^{(\sigma)} V,$$

(19)

where $\Delta^{(\sigma)}$ is the matrix of $\Delta^{(\sigma)}$ in the $\Delta^{(1)} = \Delta^{(2)} = \pm 1$ representation. This equation does not hold because $\Delta^{(1)} = I$ and $\Delta^{(2)} = -I$. Thus the ground state exists only in the $\Delta^{(1)} = \Delta^{(2)} = \pm 1$ blocks and is at most two-fold degenerate. The degeneracy occurs if and only if these two states are degenerate. Q.E.D.

Thus for even $N$, the ground state is either unique or doubly degenerate. The degeneracy happens if the Hamiltonian is invariant under a symmetry transformation $T$ but $\Delta^{(\sigma)}$ is odd under this transformation, i.e. $T \Delta^{(\sigma)} T^\dagger = -\Delta^{(\sigma)}$. Some examples in which the ground state is doubly degenerate are

1. A system defined on a $L_1 \times L_2 \times \ldots \times L_n$ lattice with translational invariance (the periodic boundary condition is implicitly assumed) along the first dimension (not necessary to be invariant along any other dimension), if $L_1$ is even and $L_2 \times \ldots \times L_n$ is odd, then the ground state is two-fold degenerate.

2. A system of $N = 4m - 2$ ($m$ is a positive integer) which is invariant under a one-to-one correspondence mapping between half of the lattice sites with the other half ones, the ground state is two-fold degenerate. This includes the systems with center reflection or mirror reflection symmetries.

The above two theorems can be extended to apply to a number of interacting fermion models that are equivalent to the Majorana fermion model (1). A local fermion operator can be defined from two local Majorana operators. If we define a fermion operator by

$$c_i = \frac{\gamma^{(1)}_{i} + i \gamma^{(2)}_{i}}{2},$$

(20)

the model defined by Eq. (11) then becomes a pairing model with interactions

$$H = \sum_{ij} 2t_{ij} (c_i c_j - h.c.) - \sum_{ij} V_{ij} (2n_i - 1)(2n_j - 1).$$

(21)

When the lattice size $N$ is odd, this model possesses a hidden $Z_2$ symmetry and the ground state is doubly degenerate. These two degenerate states can be taken as the eigenstate of $\Delta = \prod_i (2n_i - 1)$ with $\Delta = \pm 1$. When $N$ is even, the ground state is at most doubly degenerate, and the two lowest energy states are also an eigenstate of $\Delta$ with the eigenvalue $\Delta = (-1)^{N/2}$. If the system is a translation invariant ladder with odd number of chains, the ground state is always two-fold degenerate. If $N = 4m - 2$ with $m$ a positive integer, and Hamiltonian is invariant under a one-to-one correspondence mapping between half of the lattice sites with the other half ones, the ground state is also two-fold degenerate.
On a bipartite lattice, if we define the fermion operators on the first sublattice by Eq. (20), but the fermion operators on the second sublattice by

\[ c_j = \frac{\gamma_j^{(2)} + i\gamma_j^{(1)}}{2}, \tag{22} \]

then the Majorana fermion model becomes a spinless fermion model

\[ H = \sum_{ij} 2t_{ij}(c_i^\dagger c_j + h.c.) + \sum_{ij} V_{ij}(2n_i - 1)(2n_j - 1), \tag{23} \]

where \( \tilde{V}_{ij} = V_{ij} \geq 0 \) if \( i \) and \( j \) belong to different sublattices, and \( V_{ij} = -V_{ij} \leq 0 \) if \( i \) and \( j \) belong to the same sublattice. This model, as shown by Refs. [8–10], is minus sign free in quantum Monte Carlo simulations. Again it possesses a hidden \( Z_2 \) symmetry and the ground state is two-fold degenerate when \( N \) is odd. When \( N \) is even, there are at most two linear independent ground states and these two lowest energy states are also the eigenstates of \( \Delta = (-)^{N/2} \prod_i (2n_i - 1) \) with the eigenvalue \( \Delta = 1 \).

To summarize, we prove two theorems on the ground state degeneracy for the interacting quantum lattice model of Majorana fermions [11] in the eigen-operator formulation. The theorems are also valid for the spinless fermion models that are equivalent to this Majorana fermion model. It is interesting to generalize this work to the systems with more than two-species Majorana fermions, and to other interacting fermion models with reflection positivity, for example the model with parafermions [11].

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[1] E. H. Lieb, Phys. Rev. Lett. 62, 1201 (1989).
[2] E. Lieb and D. Mattis, J. Math. Phys. 3, 749 (1962).
[3] E. H. Lieb, Phys. Rev. Lett. 73, 2158 (1994).
[4] E. H. Lieb and P. Schupp, Phys. Rev. Lett. 83, 5362 (1999).
[5] G.-S. Tian, Journal of Statistical Physics 116, 629 (2004).
[6] A. Kitaev, in Advances in Theoretical Physics, AIP Conference Proceedings, Vol. 1134, edited by Lebedev, V and Feigelman, M (Russian Fdn Basic Res; Dynasty Fdn; Russian Acad Sci; RAO UES Russia, 2009) pp. 22-30, Landau Memorial Conference on Advances in Theoretical Physics, Chernogolovka, Russia, June 22-26, 2008.
[7] Y. Bahri and A. Vishwanath, Phys. Rev. B 89, 155135 (2014).
[8] E. F. Huffman and S. Chandrasekharan, Phys. Rev. B 89, 111101 (2014).
[9] L. Wang, P. Corboz, and M. Troyer, New Journal of Physics 16, 103008 (2014).
[10] Z.-X. Li, Y.-F. Jiang, and H. Yao, arXiv:1408.2269 (2014).
[11] A. Jaffe and F. L. Pedrocchi, arXiv:1406.1381