A New Methodology for Generalizing Unweighted Network Measures

Sherief Abdallah\textsuperscript{1,2,*}

\textsuperscript{1}Faculty of Informatics, the British University in Dubai, UAE
\textsuperscript{2}School of Informatics, University of Edinburgh, UK

*To whom correspondence should be addressed; E-mail: shario@ieee.org.

Abstract

Several important complex network measures that helped discovering common patterns across real-world networks ignore edge weights, an important information in real-world networks. We propose a new methodology for generalizing measures of unweighted networks through a generalization of the cardinality concept of a set of weights. The key observation here is that many measures of unweighted networks use the cardinality (the size) of some subset of edges in their computation. For example, the node degree is the number of edges incident to a node. We define the effective cardinality, a new metric that quantifies how many edges are effectively being used, assuming that an edge’s weight reflects the amount of interaction across that edge.

We prove that a generalized measure, using our method, reduces to the original unweighted measure if there is no disparity between weights, which ensures that the laws that govern the original unweighted measure will also govern the generalized measure when the weights are equal. We also prove that our generalization ensures a partial ordering (among sets of weighted edges) that is consistent with the original unweighted measure, unlike previously developed generalizations. We illustrate the applicability of our method by generalizing four unweighted network measures. As a case study, we analyze four real-world weighted networks using our generalized degree and clustering coefficient. The analysis shows that the generalized degree distribution is consistent with the power-law hypothesis but with steeper decline and that there is a common pattern governing the ratio between the generalized degree and the traditional degree. The analysis also shows that nodes with more uniform weights tend to cluster with nodes that also have more uniform weights among themselves.

1 Introduction

Mining and analyzing complex networks have received significant attention in recent years due to the explosive growth of social networks and the discovery of common patterns that govern wide-range of real world networks \cite{22,23,11,8,19,10,21}. The core of mining complex networks is network measures, which are functions that summarize
the network structure to simpler numeric values. These measures are generally classified into two main classes: measures that ignore edge weights and focus primarily on the structure of the graph, which we call unweighted measures, and measures that take edge weights into account (in addition to the structure), which we call weighted measures.

Unweighted measures received the bulk of researchers’ attention, due to their simplicity, intuitiveness, and the relative ease of computation. Such an attention resulted in several influential findings such as the small world (relied on the clustering coefficient) [22] and the power-law (relied on the degree distribution) [4, 10]. Despite their popularity, unweighted measures ignore important network information: the weights. Consequently, several measures were developed in order to take weights into account. The use of weighted measures, however, is still dwarfed by the use of unweighted measures in analyzing complex networks [7, 13, 10, 21].

The widespread usage of unweighted network measures motivated the search for generalizations of unweighted measures that take weights into account [5, 2, 20]. We propose here a new methodology for generalizing measures of unweighted networks through a generalization of the cardinality concept of a set of weighted edges. The key observation here is that many measures of unweighted networks use the cardinality (the size) of some subset of edges in their computation. For example, the node degree is the number of edges incident to a node. The clustering coefficient of a node is the ratio between the number of edges between its neighbors and the number of all possible edges among the neighbors.

We propose the effective cardinality metric, a novel extension of the traditional set cardinality metric that quantifies the number of edges effectively being used among a set of weighted edges. By simply replacing the traditional cardinality with the effective cardinality, one can generalize unweighted network measures to take weights into account. The central assumption here is that an edge’s weight reflects the amount of interaction across that edge, which is a reasonable assumption in many real domains. For example, an edge weight can represent the number of times a person calls a friend, the number of packets transmitted on an Internet link, or the number of papers co-authored by two scientists.

Unlike most of the previous work, which focused on generalizing individual measures [5, 20], we provide a generalization methodology that applies to a large number of unweighted measures. More importantly, we prove that our generalized measures become identical to the original unweighted measures if there is no disparity between weights (i.e., all the weights are equal). This property ensures that all the results that apply for an unweighted measure directly follows for our generalization of the same measure if the weights are equal. We also prove the effective cardinality, the heart of our generalization, imposes a partial ordering among sets of weighted edges that is consistent with the traditional cardinality. Intuitively this means that the smaller the disparity between weights, the closer the generalized measure is to the unweighted measure. This point will become clearer in Section 2 where we discuss in detail the different properties of the effective cardinality. None of the previous attempts in generalizing unweighted measures [5, 2, 20] provided similar guarantees.

Because our generalization of unweighted measures takes weights into account and

1Other examples include heterophilicity and dyadicity. We describe these measures in further detail later.
yet upholds properties of the original unweighted measures, our generalization bridges
the gap between the extensive research made using the unweighted network measures
and the research on weighted networks. Furthermore, it allows more accurate analysis
of the networks that were previously analyzed using unweighted network measures.
For example, it is known that the degree distribution of several real-world networks is
consistent with the power law hypothesis, while other networks do not [10]. But if one
takes the disparity of weights into account, will the effective degree distribution reveal
similar observations?

We illustrate the applicability of our method by generalizing four well-known un-
weighted measures: the node degree, the clustering coefficient, the dyadicity, and the
heterophilicity. Furthermore, as a case study, we analyze four real-world, weighted, so-
cial networks using two of our generalized measures: the C-degree and the C-clustering
coefficient (the letter C stands for continuous and denotes our generalization of an un-
weighted measure). The analysis shows that the C-degree distribution is consistent with
the power-law hypothesis, similar to the traditional degree distribution, but with steeper
decline (larger exponent). Furthermore, there is a common pattern governing the ratio
between the C-degree and the traditional degree. The analysis, using the C-clustering
coefficient, shows that nodes with more uniform weights tend to cluster with nodes
that also have more uniform weights among themselves. These findings confirm that
although our generalization methodology takes weights into account, the generalized
measures still behave in a manner similar to the corresponding unweighted measures,
but with more information revealed through the incorporation of weights.

The paper is organized as follows. Section 2 describes our proposed effective car-
dinality metric and provides proofs to some of its interesting properties. Section 3 il-
lustrates how our approach can be used in generalizing unweighted network measures,
with more focus on the degree and the clustering coefficient measures (an analysis of
four real-world weighted networks is also provided). Section 4 reviews the related
work. We conclude in Section 5.

2 The Effective Cardinality

Let \( E' = \{e_1, ..., e_n\} \subseteq E \) be the subset of edges that are used in computing a par-
ticular network measure, where \( n \) is the number of edges in \( E' \) and \( E \) is the set of all
network edges. For weighted networks, each edge \( e \in E \) has a corresponding weight
\( w(e) \), where \( \forall e \in E : w(e) \geq \epsilon > 0 \) (this directly follows from our assumption that
an edge’s weight quantifies the amount of interaction over the edge).

As we briefly mentioned in the previous section, and illustrate in more detail in
Section 3, many unweighted network measures rely in their computation on the car-
dinality of some subset of edges, or \( n = |E'| \). For example, for a node \( i \), the degree
\( k(i) = |E_i| \), where \( E_i \) is the set of edges incident to node \( i \). Also for node \( i \), the
clustering coefficient \( z(i) = \frac{E_i^N}{M_{AX_i}^N} \), where \( E_i^N \) is the set of edges between node \( i \)'s
neighbors and \( M_{AX_i}^N \) is the maximum number of edges that can be between these
neighbors (i.e. if node \( i \)'s neighbors formed a clique).

When weighted networks were analyzed using unweighted network measures, weights
came into play through a defined cutoff threshold: an edge is included in the graph if its weight is above a threshold, otherwise the edge is excluded [9, 12]. The computation of any unweighted measure then took place naturally. Such an approach, however, did not properly handle the disparity of interaction among neighbors, but rather approximated a weighted network with an unweighted network.

The main limitation of the traditional cardinality function (and consequently all the unweighted network measures that use it) is that it ignores edge weights. In other words, the cardinality implicitly assumes uniform weight over the edges, which can result in giving an incorrect perception of the effective use of edges. For example, a person may have 10 or more acquaintances but mainly interacts with only two of them (friends). Should that person be considered 2 times more connected than a person with only 5 acquaintances but also interacting primarily with two of them?

For concreteness, let us consider a specific numeric example. Suppose there are four sets of edges with corresponding sets of weights

\[ W_1 = \{5, 5, 5\}, W_2 = \{9, 5, 5, 1\}, W_3 = \{9, 8, 2, 1\} \text{ and } W_4 = \{20, 0, 0, 0\}. \]

The cardinalities of these weight sets are all the same and equals 4. Intuitively, however, if the weights reflect the interaction over edges, then not all the edges are being used equally and the traditional cardinality becomes a crude approximation. Instead, we want a function that summarizes a set of weighted edges into a single real number and has two properties. If edge weights are equal, then function we are looking for should assign a value equal to the cardinality. When the weights are not equal, the function should assign a value between 1 and the cardinality (maximum) such that the more equal the weights are, the higher the function. The first property ensures that the important results and the intuitiveness obtained through the use of the traditional cardinality in unweighted measures carry over to the generalized cardinality and the corresponding generalized measure. The second property insures that the generalized measure offer more valuable information than the traditional cardinality. Using the above example, we are looking for a function that assigns 4 to \(W_1\), 1 to \(W_4\), and values between 1 and 4 for \(W_2\) and \(W_3\), with the value assigned to \(W_2\) greater than the value assigned to \(W_3\) (because the two inner weights are equal in case of \(W_2\)).

Such a generalization of the cardinality measure will allow straightforward generalization of many unweighted network measures (by simply substituting the traditional cardinality with the generalized cardinality function). Simple functions for summarizing sets (such as the average, the variance, and the summation) can be very useful in summarizing weights, but they do not satisfy the two desired properties mentioned above. The heart of our generalization is a novel definition of the cardinality of a set of edges \(E'\) that takes weights into account, which we call the effective cardinality, or \(c(E')\):

\[
c(E') = \begin{cases} 
0 & \text{if } E' \text{ is empty} \\
\frac{1}{2} \left( \sum_{e \in E'} \frac{w(e)}{\sum_{o \in E'} w(o)} \log \frac{\sum_{o \in E'} w(o)}{w(e)} \right) & \text{otherwise}
\end{cases}
\]

Intuitively, the quantity \(\frac{w(e)}{\sum_{o \in E'} w(o)}\) represents the probability of an interaction over an edge \(e\) among all the edges in \(E'\). The set \(\left\{ \frac{w(e)}{\sum_{o \in E'} w(o)} : e \in E' \right\}\) is a probabil-
ity distribution and the quantity $H(E') = \sum_{e \in E'} \left[ \frac{w(e)}{\sum_{e \in E'} w(e)} \log_2 \frac{\sum_{e \in E'} w(e)}{w(e)} \right]$ is the entropy of this probability distribution, which measures the disparity between the weights: the more uniform the weights are, the higher the entropy and vice versa.\footnote{Note that the quantity $x \log_2 \frac{1}{x} \to 0$ as $x \to 0$ or $x = 1$.} The purpose of the power 2 is to convert the entropy back to the number of edges that are effectively being used.

Before discussing the important properties of the effective cardinality, let us first consider few numeric examples that illustrate the intuition behind the effective cardinality. Consider the set of weights $W = \{10, 0.01\}$. The traditional cardinality of this set is 2. However, if weights quantify the amount of interaction over edges, then the edge with weight 10 is significantly more important than the other edge in this set and the cardinality should be closer to 1 than 2. The effective cardinality, as will be clear from Lemma 4 captures this by returning the number of edges of equal weights that has the same effective cardinality. For $W = \{10, 0.01\}$, $c(\{10, 0.01\}) = 1.008$, so even though the set $W$ has two edges, the effective cardinality is equivalent to only 1.008 edges with uniform weights.

For the numeric example we have mentioned earlier, $c(W_1 = \{5, 5, 5, 5\}) = 4$, $c(W_2 = \{9, 5, 5, 1\}) = 3.3276$, $c(W_3 = \{9, 8, 2, 1\}) = 3.0219$ and $W_4 = \{20, 0, 0, 0\} = 1$, which satisfy the intuitive ordering we described earlier in this section. This ordering is not just by chance or due to a special case, but is actually guaranteed by our proposed effective cardinality. The effective cardinality satisfies three intuitive properties (proofs are given shortly after):

1. **Preserving maximum cardinality**: $\forall E' : c(E') \leq |E'|$. Furthermore, $c(E') = |E'|$ iff $\forall e \in E' : w(e) = C$, where $C$ is some constant. In other words, the effective cardinality is maximum and equals the original cardinality when there is no disparity between weights.

2. **Preserving minimum cardinality**: $c(E') = 0$ iff $E'$ is an empty set. Furthermore, $c(E') = 1$ iff $\exists u \in E' : w(u) > 0$ and $\forall v \neq u : w(v) = 0$. In other words, the effective cardinality is one when all edges, except one edge, have zero weights.

3. **Consistent partial order over weighted sets**: any function that maps a set of real numbers (weights) to a single real number imposes an implicit partial order. The effective cardinality imposes, arguably, the simplest partial order that is consistent with the above two properties. If the two sets of weighted edges have the same size, the same summation of weights, and their individual weights are the same except for two edges, then the set with more uniform weights has higher effective cardinality. A formal definition of this property is given in Lemma 4.

The intuition of the three properties can be clarified through the numeric example mentioned earlier. The three properties require the effective cardinality measure to impose the following ordering: $|W_1| = c(W_1) > c(W_2) > c(W_3) > c(W_4) = 1$. We prove each of these properties in the remainder of this section. Note that the ensemble approach \cite{2} (which is described in more detail in Section 4) made no guarantees with
regard to the partial order over the set of weighted edges. For example, the two set of weights $W_2 = \{9, 5, 5, 1\}$ and $W_3 = \{9, 8, 2, 1\}$ will have the same generalized degree under the ensemble method. Using the effective cardinality, the generalized degree (described in detail in Section 3.1) of $W_2$ is strictly higher than the generalized degree of $W_3$.

**Theorem 1** The effective cardinality satisfies the three properties described above: the maximum cardinality, the minimum cardinality, and the consistent partial ordering.

**Proof** The proof follows from the following three lemmas.

**Lemma 2** The effective cardinality satisfies the maximum cardinality property.

**Proof** When all the weights are equal to a constant $C$ we have

$$\forall e \in E': \frac{w(e)}{\sum_{o \in E'} w(o)} = \frac{C}{C|E'|} = \frac{1}{|E'|}$$

We then have

$$c(E') = 2^{\sum_{e \in E'} \frac{1}{|E'|} \log_2(|E'|)} = 2^{Y} = 2^{\log_2(|E'|)} = |E'|$$

In other words, both the cardinality and the effective cardinality of a weighted set of edges become equivalent when the weights are uniform. The effective cardinality is also maximum in this case, because the exponent is the entropy of the weight probability distribution, which is maximum when weights are uniform over edges.

**Lemma 3** The effective cardinality satisfies the minimum cardinality property.

**Proof** When the set of edges is empty, then the effective cardinality is zero by definition. When all weights are zero except only one weight that is greater than zero, then weight probability distribution is deterministic and the entropy is zero, therefore the effective cardinality will be 1.

**Lemma 4** The effective cardinality satisfies the consistent partial order property.

**Proof** Let $E_1$ and $E_2$ be two (edge) sets such that $|E_1| = |E_2|$ (both have the same cardinality). Let $W_1$ and $W_2$ be the corresponding sets of weights, where $\sum_{e \in E_1} w(e) = \sum_{e \in E_2} w(e) = S$ (the total weights are equal). Furthermore, let $|W_1 \setminus W_2| = n - 2$, $\{|w_1, w_{12}\} = W_1 - W_2$, $\{|w_21, w_{22}\} = W_2 - W_1$, where the ”$\setminus$” operator is the ”set difference” operator (the two sets share the same weights except for two elements in each set), and $|w_1 - w_{12}| < |w_21 - w_{22}|$ (the weights of $W_1$ are more uniform than the weights of $W_2$). To prove that the effective cardinality satisfies the consistent partial ordering property, we need to prove that $c(E_1') > c(E_2')$. 

6
Without loss of generality, we can assume that \( w_{11} \geq w_{12} \) and \( w_{21} \geq w_{22} \), therefore \( w_{11} - w_{12} < w_{21} - w_{22} \). We then have

\[
 w_{11} + w_{12} = S - \sum_{w \in W_1 \cap W_2} w = w_{21} + w_{22}
\]

or

\[
 \frac{w_{11} + w_{12}}{S} = 1 - \sum_{w \in W_1 \cap W_2} \frac{w}{S} = \frac{w_{21} + w_{22}}{S} = L
\]

therefore

\[
 L \geq \frac{w_{21}}{S} > \frac{w_{11}}{S} \geq \frac{L}{2} \geq L - \frac{w_{11}}{S} > L - \frac{w_{21}}{S}
\]

where \( \frac{w_{12}}{S} = L - \frac{w_{11}}{S} \) and \( \frac{w_{22}}{S} = L - \frac{w_{21}}{S} \). Then from Lemma 5 we have \( h(L, \frac{w_{11}}{S}) > h(L, \frac{w_{21}}{S}) \), or

\[
 -\frac{w_{11}}{S} \lg \left( \frac{w_{11}}{S} \right) - (L - \frac{w_{11}}{S}) \lg (C - \frac{w_{11}}{S}) > \\
 -\frac{w_{21}}{S} \lg \left( \frac{w_{21}}{S} \right) - (L - \frac{w_{21}}{S}) \lg (C - \frac{w_{21}}{S})
\]

Therefore \( H(E'_1) > H(E'_2) \), because the rest of the entropy terms (corresponding to \( W_1 \cap W_2 \)) are equal, and consequently \( c(E'_1) > c(E'_2) \).

**Lemma 5** The quantity \( h(C, x) = -x \lg(x) - (C - x) \lg(C - x) \) is symmetric around and maximized at \( x = \frac{C}{2} \) for \( C \geq x \geq 0 \).

**Proof**

\[
 h(C, \frac{C}{2} + \delta) = -(\frac{C}{2} + \delta) \lg(\frac{C}{2} + \delta) - (\frac{C}{2} - \delta) \lg(\frac{C}{2} - \delta) = h(C, \frac{C}{2} - \delta)
\]

Therefore \( h(C, x) \) is symmetric around \( c/2 \). Furthermore, \( h(C, x) \) is maximized when

\[
 \frac{\partial h(C, x)}{\partial x} = 0 = -1 - \lg x + 1 + \lg(C - x)
\]

or

\[
 \lg x = \lg(C - x)
\]

Therefore \( h(C, x) \) is maximized at \( x = C - x = \frac{C}{2} \).

### 3 Generalizing Unweighted Network Measures Using Effective Cardinality

In principal, any unweighted network measure which uses the cardinality of some subset of edges can be generalized using the effective cardinality. In fact, while we limited the discussion so far to sets of weighted edges, all the proofs in Section 2 applies to any sets of weights, even if elements in the sets represent subgraphs, not edges. So for example, if we are interested in counting triangles of three connected vertices (which
are used in some definitions of the clustering coefficient), we can use the effective cardinality to replace the discrete count with a continuous spectrum.

We here present four example generalizations of unweighted network measures: the degree, the clustering coefficient, the dyadicity, and the heterophilicity. The resulting generalized measures inherit the three properties of the effective cardinality. Table 1 summarizes these generalizations.

| Measure                              | Unweighted | Generalized |
|--------------------------------------|------------|-------------|
| Degree of node $i$                   | $|E_i|\,\text{c}(E_i)$ |
| Clustering coefficient of node $i$   | $\frac{|E^N_i|}{\text{MAX}}\,\text{c}(E^N_i)$ |
| Dyadicity of a graph                 | $\frac{c(E\text{within})}{n\text{within}}\,\text{c}(E\text{within})$ |
| Heterophilicity of a graph           | $\frac{c(E\text{across})}{n\text{across}}\,\text{c}(E\text{across})$ |

Table 1: The summary of the generalization of four unweighted measures, where $E_i$ is the set of edges incident to node $i$, $E^N_i$ is the set of edges between neighbors of node $i$, $E\text{within}$ is the set of edges within a class of nodes, and $E\text{across}$ is the set of edges across two classes of nodes.

The dyadicity and heterophilicity were recently used to study the correlation between the types of nodes (node classes) and the network structure [21]. The dyadicity of a graph equals $\frac{|E\text{within}|}{n\text{within}}$, where $E\text{within}$ is the set of edges within a set of nodes of the same type (a class of nodes) and $n\text{within}$ is the expected number of edges within the same class of nodes if there was no correlation between the node class and the network structure. Intuitively, the dyadicity quantifies the strength of connections between nodes of the same type and whether it is above average. The heterophilicity of a graph equals $\frac{|E\text{across}|}{n\text{across}}$, where $E\text{across}$ is the set of edges across two classes of nodes and $n\text{across}$ is the expected number of edges across the two classes if there was no correlation between the node class and the network structure. The heterophilicity quantifies the strength of connections across two classes (communities) of nodes and whether it is above average. The dyadicity can be generalized, using the effective cardinality, to be $\frac{c(E\text{within})}{n\text{within}}$ and similarly the heterophilicity can be generalized to be $\frac{c(E\text{across})}{n\text{across}}$.

The degree and the clustering coefficient, of a particular node, are two of the most widely used unweighted measures, so the remainder of this section focuses on their generalization (using the effective cardinality) and illustrates their use in analyzing four real-world, weighted networks.

### 3.1 Generalizing the Degree

As mentioned earlier, the degree is a key measurement that has been used extensively in analyzing networks. A node’s degree is the number of edges incident to the node, or $|E_i|$, where $E_i$ is the set of edges incident to node $i$. The degree distribution is a common method for summarizing the degrees of all network nodes into one measure that characterizes the network. The degree measure and its distribution were used

---

3There are other network measures that also quantified the strength of connections within a class (community) of nodes, such as the modularity measure [17].
Figure 1: Example weighted network of four nodes, comparing the (discrete) degree against the C-degree. The degree distribution illustrates the benefit of taking weights into account in distinguishing nodes.

Extensively in analyzing networks and helped discovering common patterns, particularly the power law \[4, 11, 7, 10, 13\]. A degree distribution follows the power law if \( P(k) \propto k^{-\alpha} \), where \( k \) is the degree, \( \alpha \) is a constant, and \( P(k) \) is the degree distribution.

Using our definition of effective cardinality, a generalization of the degree measure, which we call the continuous degree or the C-degree, is given by the following equation:

**Definition 6** The C-degree of a node \( i \) in a network is \( r(i) \), where

\[
r(i) = c(E_i) = \begin{cases} 
0 & \text{if } i \text{ is disconnected} \\
\frac{1}{2} \left( \sum_{e \in E_i} \frac{s(e)}{w(e)} \log_2 \frac{s(e)}{w(e)} \right) & \text{otherwise}
\end{cases}
\]

Where \( E_i \) is the set of edges incident to node \( i \) and \( s(i) \) is the strength of node \( i \).

Figure 1 compares the continuous degree distribution to the (discrete) degree distribution in a simple weighted network of four nodes. A node on the boundary has an out degree of 1, while an internal node has an out degree of 2. Intuitively, however, only one of the internal nodes is fully utilizing its degree of 2 (the one to the left), while the other node (to the right) is mostly using one neighbor only. The C-degree measure captures this and shows that the internal node to the left has a C-degree of \( c(\{0.9, 0.1\}) = 2 \) while the other internal node has a C-degree of \( c(\{0.5, 0.5\}) = 2^{H(0.9, 0.1)} = 1.38 \).

The C-degree inherits the three properties we described earlier with respect to the traditional node degree. The C-degree of a node is maximum and equals the traditional discrete degree when all the weights incident to the node are equal. The C-degree of a connected node is minimum and equals one if all edges incident to the node have...
zero weights except one edge that has a weight greater than zero. And finally, everything else being equal, a node with more uniform weights incident to it has higher C-degree than a node with less uniform weights incident to it. As mentioned earlier, the three properties ensure that the four sets of weights \( W(v_1) = \{5, 5, 5, 5\} \), \( W(v_2) = \{9, 5, 5, 1\} \), \( W(v_3) = \{9, 8, 2, 1\} \) and \( W(v_4) = \{20, 0, 0, 0\} \) will have corresponding C-degree respecting the following inequality \( k(v_1) = r(v_1) > r(v_2) > r(v_3) > r(v_4) = 1 \).

We have analyzed four real world weighted networks\(^4\) that capture coauthorships between scientists. Three of which were extracted from preprints on the E-Print Archive\(^1\): condensed matter (an updated version of the original dataset that includes data between Jan 1, 1995 and March 31, 2005), astrophysics, and high-energy theory. The fourth network represents coauthorship of scientists in network theory and experiment\(^2\). The weight between two scientists \( i \) and \( j \) reflects the strength of their collaboration and is given by the equation \( w_{ij} = \sum_m \frac{\delta^m_i \delta^m_j}{n^m - 1} \), where \( \delta^m_i = 1 \) if scientist \( i \) was a co-author of paper \( m \) and \( n^m \) is the number of co-authors for paper \( m \)\(^3\).

Figure 2 displays the C-degree distribution (CDD) and the (discrete) degree distribution (DD) for the four collaboration network. The figure uses log-log scale with the power law fit based on \(^5\)\(^6\). Interestingly, the CDD follows a pattern similar to the DD, despite taking weights into account. However, the power-law fit for the CDD has steeper decline (higher \( \alpha \)) than the DD.

One would expect that as the degree of a node increases, the node will interact primarily with a smaller subset of neighbors, particularly in social networks where humans have limited communication capacity. To verify this intuition, we define the degree utilization metric as the ratio between the C-degree and the degree of a node: \( u(v) = \frac{r(v)}{k(v)} \). The degree utilization metric captures the percentage of links that a node uses effectively, therefore we expect the degree utilization to decrease as the degree increases. Figure 3 plots the degree utilization against the (discrete) degree for the four collaboration networks. A common pattern emerges in the four networks. For low degrees, the degree utilization is relatively high (a node with few links makes the best of them). For node degree greater than some constant the bias towards high degree utilization disappears. However, and to our surprise, a cone is observed, which starts wide at low degrees and gets narrower as the degree increases (the average degree utilization is plotted as a line in the figure). In other words, for degrees above some threshold, nodes varies in their utilization of their available links. However, this variation reduces as the degree increases, while the mean remains relatively stable.

### 3.2 Generalizing the Clustering Coefficient

As mentioned earlier, the clustering coefficient is a measure that quantifies the clustering or connectivity among a node’s neighbors. When averaged over all nodes, the clustering coefficient represent the connectivity of the whole network. The clustering coefficient is an important property for identifying small world networks\(^{22}\) and is

\(^4\)Available through [http://www-personal.umich.edu/~mejn/netdata/](http://www-personal.umich.edu/~mejn/netdata/)
\(^5\)Source code adopted from [http://www.santafe.edu/~aaronc/powerlaws/](http://www.santafe.edu/~aaronc/powerlaws/)
Figure 2: Comparing the discrete degree distribution (DD) with the continuous degree distribution (CDD) for the four collaboration networks. The power law fit (PL fit) is also shown with the associated power.

given by the equation \( \frac{c(E^N_i)}{\text{MAX}_i^N} \), where \( E^N_i \) is the set of edges between node \( i \)'s neighbors and \( \text{MAX}_i^N \) is the maximum number of edges that can be between these neighbors.

The generalized clustering coefficient of a node \( i \) using the effective cardinality is:

\[
o(i) = \frac{c(E^N_i)}{\text{MAX}_i^N}
\]

Figure 4 provides a simple motivating example of 3-nodes. The C-clustering coeffi-
Figure 3: Scatter plot of a node degree against its degree utilization for the four collaboration networks. The average utilization per degree is also plotted.

Figure 5 shows the scatter plot of the (discrete) clustering coefficient versus the (discrete) degree (shown in log scale) for the four collaboration networks. The main observation clear from the graph is that in general, the clustering coefficient decreases with the increase of the degree.

Figure 6 shows the scatter plot of the C-clustering coefficient versus the C-degree for the four collaboration networks. The continuous version of the scatter plot follows the general observation in the discrete case: the clustering coefficient decreases with the increase of the degree. Nevertheless, the scatter plot for the continuous measures
(a) network of three nodes. The numbers inside each node represent the clustering coefficient (top) and the C-clustering coefficient (bottom).

(b) Scatter plot of the degree against the clustering coefficient for the network in (a). Notice that all three nodes have the same degree and clustering coefficient.

(b) Scatter plot of the C-degree against the C-clustering coefficient for the network in (a). The three nodes are nicely separated in plot, because of taking weights into account.

Figure 4: Example weighted network of three nodes, comparing the (discrete) clustering coefficient against the C-clustering coefficient. The scatter plot of the degree against the clustering coefficient illustrates the benefit of taking weights into account in distinguishing nodes.

covers more area, because both the C-degree and the C-clustering coefficient produce continuous spectrum of values (unlike the discrete degree and the discrete clustering coefficient). More importantly, one can observe an interesting pattern in the continuous scatter plot: nodes with high C-clustering coefficient (above 0.8) tend to have more discrete C-degree. This is clear from the concentration of points with high C-clustering coefficient around the discrete degrees, which is not apparent in points with low clustering coefficient. Using Lemma 2, this observation means that nodes with incident weights that are more uniform (hence the more discrete degree) tend to cluster with nodes that have more uniform weights among themselves (hence the higher clustering coefficient).

4 Related Work

In general, one can classify weighted network measures into two classes: measures that generalize unweighted network measures to take weights into account, and measures that have no connection to unweighted measures. Surveying all weighted measures that have no connection to unweighted measures is beyond the scope of this paper and have little relevance to the contribution of this paper, which generalizes unweighted network
Figure 5: Scatter plot of a node’s discrete degree against its discrete clustering coefficient for the four collaboration networks.

measures. For completeness, we provide here a sample of these measures that are related to some unweighted measure (interested reader may refer to survey papers on the subject [16, 6, 7]). The *strength* of a node is the summation of all weights incident to a node. The strength becomes identical to the node’s degree in the very special case when all the weights are equal to 1, but it has very weak partial ordering among nodes. For example, all the nodes in Figure 1 have the same strength of 1. The *weight distribution* is similar to the degree distribution except that it measures the frequency of a particular edge weight. A more recent work [14] analyzed a graph’s total weight, \( \sum_{e \in E} w(e) \), against the graph’s total number of edges, |E|, over time. That work also analyzed the degree of a node, \( k(v) \), against the node’s strength, \( s(v) \). While useful, the above measures neither captured the disparity of weights among edges nor provided a methodology for generalizing unweighted measures.
Figure 6: Scatter plot of a node’s continuous degree against its continuous clustering coefficient for the four collaboration networks.

The network measure $Y(v) = \sum_{e \in E(v)} \left( \frac{w(e)}{d(v)} \right)^2$ successfully captured the disparity of interaction within a node $v$ [3]. However, unlike our generalization, the $Y$ measure is not a generalization of the degree measure as it fails to satisfy the first two properties in Section 2 (if the weights are equal, the $Y$ measure of a node does not become equal to the node’s degree). Furthermore, no guarantee over the partial ordering imposed by the measure was provided, unlike our methodology which provided a guarantee on the partial ordering imposed by our generalization.

Unlike our methodology, which can be used to generalize several unweighted measures, there have been several attempts to generalize specific unweighted measures. The weighted clustering coefficient [5] was an attempt to generalize the clustering coefficient. The generalization relied on an alternative definition of the clustering coefficient that used triplets [22]. A triplet connected to a node is a subgraph containing
the original node in addition to two other connected neighbors. The intuition behind the weighted clustering coefficient for node $i$ is to weigh every edge between two of its neighbors, $j$ and $k$, using the weights on edges $(i, j)$ and $(i, k)$. However, unlike our generalization, the weight on edge $j, k$ was ignored.

A recent attempt to generalize the clustering coefficient used the ratio between the total value of closed triplets and the total value of all triplets [20]. The authors proposed four functions to evaluate (summarize) weighted triplets: the arithmetic mean, the geometric mean, the minimum, and the maximum. Unfortunately, all the four proposed functions (and therefore the proposed generalization) have very poor distinguishing powers, in addition to the very week connection to the original clustering coefficient. For example, using the proposed generalization in [20] (and any of the four proposed functions) all the nodes in Figure 4 will have a generalized clustering coefficient of 1, a limitation that was previously reported [20]. On the other hand, our proposed generalized clustering coefficient successfully distinguishes all the three nodes.

Perhaps the most related work to our contribution is the ensemble approach, which provides a methodology for generalizing almost all unweighted network measures [3]. The first step of the method was to normalize edge weights to ensure all weights are between 0 and 1 (more restrictive than our approach, which only assumes weights are non-negative). The next step was to randomly generate an ensemble of unweighted networks from the original weighted network, where the weight of an edge represented the probability of generating the edge. The final step was to compute the generalized unweighted measure as the average of the unweighted measure for each network in the ensemble. Despite its relative simplicity, and the ability to generalize almost all unweighted measures, the ensemble method suffers from several limitations. Unlike our generalization, the ensemble method can only generalize unweighted measures for the whole network, not for individual nodes. So, for example, it can not be used to generate scatter plots similar to those in Figure 6 where points represent individual nodes. Another limitation is the need to generate large number of networks in the ensemble in order to provide more accurate generalized measures (in order to sample edges with very small weights). Furthermore, the ensemble method does not provide any partial ordering guarantee. For example, suppose two nodes have the following sets of incident weights $A = \{9, 8, 2, 1\}$ and $B = \{9, 5, 5, 1\}$. Under the ensemble approach, both nodes will have the same generalized degree (the exact value of the generalized degree depends on the weights of other edges in the network, which affect the normalization of edges). Using our proposed generalization, node $B$’s degree is guaranteed to be greater than node $A$’s C-degree by Lemma 4.

5 Conclusion

We proposed a new methodology for generalizing measures of unweighted networks. The heart of our generalization is the effective cardinality, a novel extension to the traditional set cardinality to take weights into account. We illustrated the applicability of our method by generalizing four unweighted network measures: the node degree, the clustering coefficient, the dyadicity, and the heterophilicity. Furthermore, we compared the generalized degree to the traditional degree using four real world net-
works and showed that the generalized degree distribution follows a similar pattern to the traditional degree distribution, but with steeper decline (larger exponent of the power-law fit). We also investigated the ratio between the generalized degree and the traditional degree and showed that on average the ratio is bounded, even for nodes with high-degree. The analysis of the generalized clustering co-efficient revealed that nodes with more uniform incident weights tend to cluster with nodes that have more uniform weights among themselves.

References

[1] S. E. Ahnert and T. M. A. Fink. Clustering signatures classify directed networks. *Phys. Rev. E*, 78(3):036112, Sept. 2008.

[2] S. E. Ahnert, D. Garlaschelli, T. M. A. Fink, and G. Caldarelli. Ensemble approach to the analysis of weighted networks. *Phys Rev E*, 76(1), 2007.

[3] E. Almaas, B. Kovacs, T. Vicsek, Z. N. Oltvai, and A. L. Barabasi. Global organization of metabolic fluxes in the bacterium, escherichia coli. *Nature*, 427:839, 2004.

[4] A. L. Barabasi and R. Albert. Emergence of scaling in random networks. *Science*, 286(5439):509–512, October 1999.

[5] A. Barrat, M. Barthélemy, R. Pastor-Satorras, and A. Vespignani. The architecture of complex weighted networks. *Proceedings of the National Academy of Science*, 101:3747–3752, Mar. 2004.

[6] M. Barthélemy, A. Barrat, R. Pastor-Satorras, and A. Vespignani. Characterization and modeling of weighted networks. *Physica A*, 346:34–43, Feb. 2005.

[7] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D.-U. Hwang. Complex networks: Structure and dynamics. *Phys. Rep.*, 424:175–308, Feb. 2006.

[8] D. Chakrabarti and C. Faloutsos. Graph mining: Laws, generators, and algorithms. *ACM Comput. Surv.*, 38(1):2, 2006.

[9] A. Chapanond, M. S. Krishnamoorthy, and B. Yener. Graph theoretic and spectral analysis of enron email data. *Comp. Math. Organ. Theory*, 11(3):265–281, 2005.

[10] A. Clauset, C. Rohilla Shalizi, and M. E. J. Newman. Power-law distributions in empirical data. *ArXiv e-prints*, June 2007.

[11] M. Faloutsos, P. Faloutsos, and C. Faloutsos. On power-law relationships of the internet topology. *Comput. Commun. Rev.*, 25:251–262, 1999.

[12] T. Kalisky, S. Sreenivasan, L. A. Braunstein, S. V. Buldyrev, S. Havlin, and H. E. Stanley. Scale-free networks emerging from weighted random graphs. *Phys. Rev. E*, 73(2):025103, 2006.
[13] J. Leskovec, J. Kleinberg, and C. Faloutsos. Graph evolution: Densification and shrinking diameters. *ACM Trans. Knowl. Discov. Data*, 1(1):2, 2007.

[14] M. McGlohon, L. Akoglu, and C. Faloutsos. Weighted graphs and disconnected components: patterns and a generator. In *SIGKDD*, pages 524–532, New York, NY, USA, 2008. ACM.

[15] M. E. Newman. Scientific collaboration networks. II. Shortest paths, weighted networks, and centrality. *Phys. Rev.*, 64(1):016132, July 2001.

[16] M. E. Newman. Analysis of weighted networks. *Phys. Rev. E*, 70(5):056131, Nov. 2004.

[17] M. E. Newman and M. Girvan. Finding and evaluating community structure in networks. *Phys. Rev. E*, 69(2):026113, Feb. 2004.

[18] M. E. J. Newman. Coauthorship networks and patterns of scientific collaboration. *Proc Natl Acad Sci*, 98:404–409, 2001.

[19] M. E. J. Newman. Finding community structure in networks using the eigenvectors of matrices. *Phys. Rev. E*, 74:036104, 2006.

[20] T. Opsahl and P. Panzarasa. Clustering in weighted networks. *Social Networks*, 31(2):155–163, May 2009.

[21] J. Park and A.-L. Barabasi. Distribution of node characteristics in complex networks. *Proceedings of the National Academy of Science*, 104:17916–17920, Nov. 2007.

[22] D. J. Watts and S. H. Strogatz. Collective dynamics of ‘small-world’ networks. *Nature*, 393:440–442, June 1998.