DETERMINATION OF CRITICAL EXPONENTS AND EQUATION OF STATE BY FIELD THEORY METHODS

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Path integrals have played a fundamental role in emphasizing the profound analogies between Quantum Field Theory (QFT), and Classical as well as Quantum Statistical Physics. Ideas coming from Statistical Physics have then led to a deeper understanding of Quantum Field Theory and open the way for a wealth of non-perturbative methods. Conversely QFT methods are becoming essential for the description of the phase transitions and critical phenomena beyond mean field theory. This is the point we want to illustrate here. We therefore review the methods, based on renormalized $\phi^4_3$ quantum field theory and renormalization group, which have led to an accurate determination of critical exponents of the $N$-vector model, and more recently of the equation of state of the 3D Ising model. The starting point is the perturbative expansion for RG functions or the effective potential to the order presently available. Perturbation theory is known to be divergent and its divergence has been related to instanton contributions. This has allowed to characterize the large order behaviour of perturbation series, an information that can be used to efficiently “sum” them. Practical summation methods based on Borel transformation and conformal mapping have been developed, leading to the most accurate results available probing field theory in a non-perturbative regime. We illustrate the methods with a short discussion of the scaling equation of state of the 3D Ising model [1]. Compared to exponents its determination involves a few additional (non-trivial) technical steps, like the use of the parametric representation, and the order dependent mapping method. A general reference on the subject is

J. Zinn-Justin, 1989, Quantum Field Theory and Critical Phenomena, in particular chap. 28 of third ed., Clarendon Press (Oxford 1989, third ed. 1996).

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1 Introduction

Second order phase transitions are continuous phase transitions where the correlation length diverges. Renormalization group (RG) arguments [2], as well as an analysis, near dimension four, of the most divergent terms appearing in the expansion around mean field theory [3], indicate that such transitions present universal features, i.e. features independent to a large extend from the details of the microscopic dynamics. Moreover all universal quantities can be calculated from renormalizable or super-renormalizable quantum field theories. For an important class of physical systems and models (with short range interactions) one is led to a $\phi^4$-like euclidean field theory with $O(N)$ symmetry. Among those let us mention statistical properties of polymers, liquid–vapour and binary mixtures transitions, superfluid Helium, ferromagnets... We explain here how critical exponents and other universal quantities have been calculated with field theory techniques. To simplify notation we concentrate on the universality class of the Ising model (models with $Z_2$ symmetry).

The effective quantum field theory. The relevant field theory action $\mathcal{H}(\phi)$ is

$$\mathcal{H}(\phi) = \int d^dx \left\{ \frac{1}{2} |\nabla \phi(x)|^2 + \frac{1}{2} r \phi^2(x) + \frac{1}{4!} g_0 \Lambda^{4-d} \phi^4(x) \right\}, \quad (1.1)$$

where $r$ plays the role of the temperature. The mass parameter $\Lambda$ corresponds to the inverse microscopic scale and also appears as a cut-off in the Feynman diagrams of the perturbative expansion.

For some value $r_c$ the correlation length diverges (the physical mass vanishes), and the critical domain corresponds to $|t = r - r_c| \ll \Lambda^2$. The study of the critical domain reduces to the study of the large cut-off behaviour, i.e. to renormalization theory and the corresponding renormalization group. However one essential feature of the action distinguishes it from field theory in the form it was traditionally presented in particle physics: the dependence on $\Lambda$ of the coefficient of $\phi^4$ is given a priori. In particular in the dimensions of interest, $d < 4$, the “bare” coupling constant diverges, though the field theory, being super-renormalizable, requires only a mass renormalization.

To circumvent the problem of the large coupling constant the famous Wilson–Fisher $\varepsilon$-expansion [4] has been invented. The dimension $d$ is considered as a continuous variable. Setting $d = 4 - \varepsilon$ one expands both in $g_0$ and $\varepsilon$. Divergences then behave like in four dimensions, they are only logarithmic and can be dealt with. Moreover it is possible to study directly the massless (or critical) theory.

Later it has been proposed by Parisi [5], to work at fixed dimension $d < 4$, in the massive theory (the massless theory is IR divergent). One motivation
for trying such an approach is of practical nature: it is easier to calculate Feynman diagrams in dimension three than in generic dimensions, and thus more perturbative orders are available.

The large Λ limit then is taken first at $u_0 = g_0 \Lambda^{4-d}$ fixed. This implies that one first tunes the initial parameters of the model to remain artificially close to the unstable $u_0 = 0$ gaussian fixed point: When the correlation length $\xi$ increases near $T_c$ one decreases the dimensionless bare quantity $g_0$ as $g_0^{1/(4-d)} \propto 1/\xi \Lambda$. Finally one takes the infinite $u_0$ limit. One then is confronted with a serious technical problem: perturbation theory is finite but one is interested in the infinite coupling limit.

One thus introduces a field renormalization, $\phi = Z^{1/2} \phi_R$, and a renormalized dimensionless coupling constant $g$ as in four dimensions. They are implicitly defined by the renormalization conditions for the $\phi_R$ 1PI correlation functions:

$$
\Gamma_{R}^{(2)}(p; m, g) = m^2 + p^2 + O(p^4), \quad \Gamma_{R}^{(4)}(p_i = 0; m, g) = m^{4-d} g.
$$

The role of renormalizations, however, is here different. When the corresponding Callan–Symanzik $\beta$-function has an IR stable zero, $\beta(g^*) = 0$ with $\omega \equiv \beta'(g^*) > 0$, then the new coupling $g$ has a finite limit $g = g^*$ when the initial coupling constant $u_0$ becomes large. Thus the renormalized coupling $g$ is a more suitable expansion parameter than $u_0$.

Note that the mass parameter $m$, which is proportional to the physical mass, or inverse correlation length, of the high temperature phase, behaves for $t = r - r_c \to 0_+$ as $m \propto t^\nu$, where $\nu$ is the correlation length exponent.

In contrast with the $\varepsilon$-expansion however, at fixed dimension three or two one has no small parameter. Therefore accurate determinations of $g^*$ and all other physical quantities depends on the analytic properties of the series, in addition to the number of terms available. A semi-classical analysis, based on instanton calculus, unfortunately indicates that perturbation theory in $\phi^4$ field theory is divergent. Therefore to extract any information from perturbation theory a summation method is required.

2 Critical exponents. Borel transformation and mapping

The most studied quantities are critical exponents, because they are easier to calculate. They have been extensively used to compare RG predictions with other results (experiments, high or low temperature series expansion, Monte-Carlo simulations). The first accurate values of the exponents of the $O(N)$ symmetric $N$-vector model have been reported in ref. [6] using six-loop series for RG functions [7]. Perturbative series have been summed using Borel transformation and conformal mapping. The same ideas have later been applied to
the \( \varepsilon \)-expansion when five loop series have become available, and recently to the equation of state. With time the method has been refined and the efficiency improved by various tricks but the basic principles have not changed.

**Borel transformation and conformal mapping.** Let \( R(g) \) be a quantity given by a perturbation series
\[
R(g) = \sum R_k g^k .
\] (2.1)

Large order behaviour analysis (instantons) \[8\] teaches us that, in the \( \phi^4 \) field theory, \( R_k \) behaves like \( k^a(-a)^k k! \) for \( k \) large. The value of \( a > 0 \) has been determined numerically. One thus introduces \( B(g) \), the Borel transform of \( R(g) \), which is defined by
\[
B(g) = \sum (R_k/k!) g^k .
\] (2.2)

The function \( B(g) \) is analytic at least in a circle with the singularity closest to the origin located at \( z = -1/a \). Unlike \( R(g) \), \( B(g) \) is determined by its series expansion. In the sense of formal series, \( R(g) \) can be recovered from
\[
R(g) = \int_0^\infty e^{-t} B(gt) dt ,
\] (2.3)

However, for relation (2.3) to make sense as a relation between functions, and not only between formal series, one must know \( B(g) \) on the whole real positive axis. This implies that \( B(g) \) must be analytic near the axis, a result proven rigorously \[9\]. Moreover it is necessary to continue analytically the function from the circle to the real positive axis. Consideration of general instanton contributions suggests that \( B(g) \) actually is analytic in the cut-plane. Therefore the analytic continuation can be obtained from a conformal map of the cut-plane onto a circle:
\[
z \mapsto u(z) = az/(\sqrt{1+za}+1)^2 .
\] (2.4)

The function \( R(g) \) is then given by the new, hopefully convergent, expansion
\[
R(g) = \sum B_k \int_0^\infty e^{-t} [u(gt)]^k dt .
\] (2.5)

**Exponents.** The values of critical exponents obtained from field theory have remained after about twenty years among the most accurate determinations. Only recently have consistent, but significantly more accurate, experimental
Critical exponents of the $O(N)$ models from $d = 3$ expansion \cite{[12]}. 

| $N$ | 0     | 1     | 2     | 3     |
|-----|-------|-------|-------|-------|
| $g^*$ | 26.63 ± 0.11 | 23.64 ± 0.07 | 21.16 ± 0.05 | 19.06 ± 0.05 |
| $\gamma$ | 1.1596 ± 0.0020 | 1.2396 ± 0.0013 | 1.3169 ± 0.0020 | 1.3895 ± 0.0050 |
| $\nu$ | 0.5882 ± 0.0011 | 0.6304 ± 0.0013 | 0.6703 ± 0.0015 | 0.7073 ± 0.0035 |
| $\eta$ | 0.0284 ± 0.0025 | 0.0335 ± 0.0025 | 0.0354 ± 0.0025 | 0.0355 ± 0.0025 |
| $\beta$ | 0.3024 ± 0.0008 | 0.3258 ± 0.0014 | 0.3470 ± 0.0016 | 0.3662 ± 0.0025 |
| $\omega$ | 0.812 ± 0.016 | 0.799 ± 0.011 | 0.789 ± 0.011 | 0.782 ± 0.0013 |

results been reported in low gravity superfluid experiments \cite{[10]}. Also various numerical simulations \cite{[11]} and high temperature expansions on the lattice have claimed similar accuracies. 

The values of critical exponents have recently been updated \cite{[12]} because seven-loop terms have been obtained for two of the three RG functions. Some results are displayed above. The main improvements concern the exponent $\eta$ which was poorly determined, and the lower value of $\gamma$ for $N = 0$ (polymers).

3 3D Ising model: the scaling equation of state

Let us first recall a few properties of the equation of the state in the critical domain, in the specific case $N = 1$ (Ising-like systems), at $d = 3$.

The equation of state is the relation between magnetic field $H$, magnetization $M = \langle \phi \rangle$ (the “bare” field expectation value) and the temperature which is represented by the parameter $t = r - r_c \propto T - T_c$. It is related to the free energy per unit volume, in field theory language the generating functional $\Gamma(\phi)$ of 1PI correlation functions restricted to constant fields, i.e the effective potential $V(M) = \Gamma(M)/\text{vol}$, by $H = \partial V/\partial M$. In the critical domain the equation of state has Widom’s scaling form

$$H(M,t) = M^\delta f(t/M^{1/\beta}),$$

a form initially conjectured and which renormalization group has justified.

One property of the function $H(M,t)$ which plays an essential role in the analysis is Griffith’s analyticity: it is regular at $t = 0$ for $M > 0$ fixed, and simultaneously it is regular at $M = 0$ for $t > 0$ fixed.

In the framework of the $\varepsilon$-expansion, the function $f(x)$ has been determined up to order $\varepsilon^2$ for the general $O(N)$ model, \cite{[13]} and order $\varepsilon^3$ for $N = 1$ \cite{[14]}.
The calculations presented here are performed within the framework of the $\phi^4_3$ massive field theory renormalized at zero momentum (eq. (1.2)). Five loop series for the effective potential have been reported $[15]$. The conditions (1.2) imply that the effective potential $V$ expressed in terms of the expectation value of the renormalized field $\varphi = \langle \phi_R \rangle$, has a small $\varphi$ expansion of the form

$$V(\varphi) - V(0) = \frac{1}{2}m^2 \varphi^2 + \frac{1}{4}mg\varphi^4 + O(\varphi^6) = (m^3/g)V(z, g),$$

(3.2)

where $z$ is a dimensionless variable $z = \varphi \sqrt{g/m}$. For $g = g^*$

$$z \propto M/\sqrt{mZ} \propto M/m^{(1+\eta)/2} \propto Mt^{-\beta}.$$ (3.3)

The equation of state is related to the derivative $F$ of the reduced effective potential $V$ with respect to $z$

$$H \propto t^{\beta\delta} F(z), \quad F(z) \equiv F(z, g^*) = \frac{\partial V(z, g^*)}{\partial z}.$$ (3.4)

*The problem of the low temperature phase.* To determine the equation of state in the whole physical range a new problem arises. In this framework it is difficult to calculate physical quantities in the ordered phase because the theory is parametrized in terms of the disordered phase correlation length $\xi = m^{-1} \propto t^{-\nu}$ which is singular at $T_c$ (as well as all correlation functions normalized as in (1.2)). In the limit $m \to 0$, at $\varphi$ fixed, $z \to \infty$ as seen in eq. (3.3). In this limit from eq. (3.1) one finds

$$H(M, t = 0) \propto M^\delta \Rightarrow F(z) \propto z^\delta.$$ (3.5)

The perturbative expansion of the scaling equation of state leads to an expression only adequate for the description of the disordered phase.

In the case of the $\varepsilon$-expansion the scaling relations (and thus the limiting behaviour (3.3)) are exactly satisfied order by order. Moreover the change to the variable $x \propto z^{-1/\beta}$ (more appropriate for the regime $t \to 0$) gives an expression for $f(x) \propto F(x^{-\beta})x^{\beta\delta}$ that is explicitly regular in $x = 0$ (Griffith’s analyticity). Still, even there a numerical problem arises when $\varepsilon = 1$ is set.

In the case of fixed dimension perturbation theory, instead, because IR scaling is obtained only for $g = g^*$ and not for generic values of $g$, scaling properties are not satisfied at any finite order in $g$.

Several approaches can be used to deal with the problem of continuation to the ordered phase. A rather powerful method, motivated by the results obtained within the $\varepsilon$-expansion scheme, is based on the parametric representation.
4 Parametric representation of the equation of state and ODM

Both the scaling and regularity properties of the equation of state can be more easily expressed by parametrizing it in terms of two new variables $R$ and $\theta$, setting:

$$M = m_0 R^\beta \theta, \quad t = R (1 - \theta^2), \quad H = h_0 R^\beta \delta h(\theta),$$

(4.1)

where $h_0, m_0$ are normalization constants. Then the function $h(\theta)$ is an odd function of $\theta$ which from Griffith’s analyticity is regular near $\theta = 1$, which is $z$ large, and near $\theta = 0$ which is $z$ small. It vanishes for $\theta = \theta_0$ which corresponds to the coexistence curve $H = 0, T < T_c$. In terms of the scaling variable $z$ used previously one finds

$$z = \rho \theta / (1 - \theta^2)^\beta, \quad h(\theta) = (1 - \theta^2)^\beta \delta F(z(\theta)),$$

(4.2)

where $\rho$ is an arbitrary parameter. Note if we know a few terms of the expansion of $F(z)$ in powers of $z$ we know the same number of terms in the expansion of $h(\theta)$ in powers of $\theta$. But because $h$ is a more regular function, the latter expansion has a much larger domain of validity.

From the parametric representation of the equation of state it is then possible to derive a representation for the singular part of the free energy per unit volume as well as various universal ratios of amplitudes.

Order dependent mapping (ODM). In the framework outlined before, the approximate $h(\theta)$ that one obtains by summing perturbation theory at fixed dimension, is still not regular. The terms singular at $\theta = 1$, generated by the mapping (4.2), do not cancel exactly due to summation errors. The last step thus is to Taylor expand the approximate expression for $h(\theta)$ around $\theta = 0$ and to truncate the expansion, enforcing in this way regularity. A question then arises, to which order in $\theta$ should one expand? Since the coefficients of the $\theta$ expansion are in one to one correspondence with the coefficients of the small $z$ expansion of the function $F(z)$, the maximal power of $\theta$ in $h(\theta)$, should be equal to the maximal power of $z$ whose coefficient can be determined with reasonable accuracy. As noted before, although the small $z$ expansion of $F(z)$ at each finite loop order in $g$ contains an infinite number of terms, the evaluation of the coefficients of the higher powers of $z$ is increasingly difficult.

Therefore one has to ensure the fastest possible convergence of the small $\theta$ expansion. For this purpose one uses the freedom in the choice of the parameter $\rho$ in eq. (4.2): one determines $\rho$ by minimizing the last term in the truncated small $\theta$ expansion, thus increasing the importance of small powers of $\theta$ which are more accurately calculated. This is nothing but the application to this particular example of the series summation method based on ODM [17].
5 Numerical results

One first determines the first coefficients $F_{2l+1}$ of the small $z$ (small field) expansion of the function $F(z)$ as accurately as possible, using the same method as for exponents, i.e. Borel–Leroy transformation and conformal mapping. One finds

$$F_5 = 0.01711 \pm 0.00007, \quad F_7 \times 10^4 = 4.9 \pm 0.5, \quad F_9 \times 10^5 = -7 \pm 5.$$

One then determines by the ODM method the parameter $\rho$ and the function $h(\theta)$ of the parametric representation, as explained before. One obtains successive approximations in the form of polynomials of increasing degree. At leading order one obtains a polynomial of degree five. It is not possible to go beyond $h_9(\rho)$ because already $F_9$ is poorly determined. Note that one has here a simple test of the relevance of the ODM method. Indeed, once $h(\theta)$ is determined, assuming the values of the critical exponents $\gamma$ and $\beta$ one can recover a function $F(z)$ which has an expansion to all orders in $z$. As a result one obtains a prediction for the coefficients $F_{2l+1}$ which have not yet been taken into account to determine $h(\theta)$. The relative difference between the predicted values and the ones directly calculated gives an idea about the accuracy of the ODM method. The simplest representation of the equation of state, consistent with all data, is given by

$$h(\theta) = \theta - 0.76201(36) \, \theta^3 + 8.04(11) \times 10^{-3} \, \theta^5,$$

(5.1)

(errors on the last digits in parentheses) that is obtained from $\rho^2 = 2.8667$. This expression of $h(\theta)$ has a zero at $\theta_0 = 1.154$, which corresponds to the coexistence curve. The coefficient of $\theta^7$ in eq. (5.1) is smaller than $10^{-3}$. Note that for the largest value of $\theta^2$ which corresponds to $\theta_0^2$, the $\theta^5$ term is still a small correction.

Concluding remarks. Within the framework of renormalized quantum field theory and renormalization group, the presently available series allow, after proper summation, to determine accurately critical exponents for the $N$-vector model and the complete scaling equation of state for 3D Ising-like ($N = 1$) systems. In the latter example additional technical tools, beyond Borel summation methods, are required in which the parametric representation plays a central role. From the equation of state new estimates of some amplitude ratios have been deduced which seem reasonably consistent with all other available data.

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