Noise-Induced Order in Extended Systems: A Tutorial

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Abstract. External fluctuations have a wide variety of constructive effects on the dynamical behavior of spatially extended systems, as described by stochastic partial differential equations. A set of paradigmatic situations exhibiting such effects are briefly reviewed in this paper, in an attempt to provide a concise but thorough introduction to this active field of research, and at the same time an overview of its current status. This work is dedicated to Lutz Schimansky–Geier on the occasion of his 50th anniversary. Through the years, Prof. Schimansky–Geier has made important contributions to the field of spatiotemporal stochastic dynamics, including seminal investigations in the early 1990’s on noise effects in front propagation, and studies of noise-induced phase transitions and noise-sustained structures in excitable media, among others.

1 Introduction

It is well accepted nowadays that noise can have rather surprising and counterintuitive effects. There are many physical situations in which noise exhibits a constructive, rather than destructive, role in the behavior of a nonlinear system. A relevant example of this fact, still under active research nowadays, is the phenomenon of stochastic resonance, in which the response of a nonlinear system to an external signal under the presence of fluctuations can be enhanced by tuning the noise intensity to an optimal (non-zero) value [1].

In another direction, studies of stochastic zero-dimensional systems (i.e., systems with only temporal dependence), reviewed in [2], showed that noise is able to induce transitions in such systems. Spatial degrees of freedom provide further interesting scenarios where the non-trivial role of external noise arises, such as phase transitions and critical phenomena [3], and pattern formation out of equilibrium [4].

The present review aims to be a brief introduction to the influence of external noise on spatially extended, d-dimensional systems (an extensive monograph on the topic can be found in [5]). As far as we know, seminal works in this direction were already carried out in the late 1970’s by Mikhailov

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Owing to the existence of spatial degrees of freedom, the behavior of extended systems is described by phases in a thermodynamic sense. Under these conditions, the presence of certain types of external noise affects, in a non-obvious way, the behavior of the corresponding deterministic (noise-less) system, or even that of the associated equilibrium system (which has internal noise terms obeying a fluctuation-dissipation relation). In particular, we expect not only quantitative changes with respect to the deterministic results, but also qualitatively different, even new, features induced by the presence of external fluctuations. In these situations, naïve predictions based on a deterministic analysis are very far from giving reliable results. Several examples can be enumerated, some of which will be reviewed here: noise-induced spatial patterns [8,9,10,11], noise-induced ordering phase transitions (both of second order [12,13,14,15], and of first order [16,17]), noise-induced disordering phase transitions [13,15,18], noise-induced phase dynamics [19], noise-induced fronts [20], and noise-sustained structures in excitable media [21,22,23].

Most of the noise-induced phenomena enumerated above act in the direction of enhancing the order in the system. This surprising fact contrasts with the noise-induced disorder that one could intuitively expect from statistical mechanics. In the following pages, we review all these ordering effects. In each of the different physical situations, the notion of “order” is defined, the mechanisms through which noise produces order are discussed, and a minimal model displaying the phenomenon is given.

2 Noise-Induced Phase Transitions

The most basic organizing phenomenon in spatially extended systems is the transition between two macroscopic phases as a certain control parameter is varied. Such phase transitions can be characterized by standard tools in Statistical Mechanics, such as scaling functions, critical exponents, and renormalization-group transformations [5].

From a dynamical perspective, the system can be described in a continuous way by a coarse-grained field \( \phi(x,t) \), representing the local density of a relevant physical variable (e.g., magnetization in a magnetic system, or relative concentration in a binary alloy). From this viewpoint, a disordered phase corresponds to the state \( \phi(x,t) = 0 \) (random distribution of up- and down-spins in magnetic systems, or homogeneous mixture in alloys), and an ordered phase is given by a non-zero field. As we will see in what follows, external noise is able to induce phase transitions from disorder to order. We consider models obeying a stochastic differential equation of the general form:

\[
\frac{\partial \phi(x,t)}{\partial t} = f(\phi) + g(\phi) \eta(x,t) + \nabla^2 \phi + \xi(x,t),
\]  

(1)
with $x$ defined in a $d$-dimensional space. The additive and multiplicative gaussian noises have zero mean and correlations:

$$\langle \xi(x, t)\xi(x', t') \rangle = 2\varepsilon \delta(x-x') \delta(t-t'),$$  \hfill (2a)

$$\langle \eta(x, t)\eta(x', t') \rangle = 2c(x-x') \delta(t-t'),$$  \hfill (2b)

where $c(x-x')$ is the spatial correlation function of the multiplicative noise $\eta(x, t)$ [$c(0)$ is proportional to the noise intensity]. The white additive noise $\xi(x, t)$ is taken to represent internal fluctuations.

### 2.1 A short-time dynamical instability

We now review the most common mechanism of noise-induced ordering transitions. To do so, we locally analyse the initial evolution of the system. Averaging (1) with respect to the probability density of the two noise terms, and neglecting the diffusive term of the equation:

$$\langle \partial_t \phi \rangle = \langle f(\phi) \rangle + \langle g(\phi) \eta(x, t) \rangle.$$  \hfill (3)

We now interpret the multiplicative noise $\eta(x, t)$ in the Stratonovich sense [24]. This choice is neither arbitrary nor interested: we simply aim to describe realistic fluctuations, temporally correlated but with a very small characteristic time. Under this interpretation, the average of the noise term appearing in (3) can be computed to be

$$\langle g(\phi) \eta(x, t) \rangle = c(0) \langle g(\phi) g'(\phi) \rangle,$$  \hfill (4)

where the prime indicates differentiation with respect to the argument $\phi$. Coming back to (3) one can say that, at the initial instants of the evolution, fluctuations around the average value of the field $\langle \phi \rangle$ can be neglected, so that $\langle h(\phi) \rangle \approx h(\langle \phi \rangle)$ for any function $h$, and one can write approximately

$$\partial_t \langle \phi \rangle \approx f(\langle \phi \rangle) + c(0)g(\langle \phi \rangle)g'(\langle \phi \rangle) \equiv f_{\text{eff}}(\langle \phi \rangle).$$  \hfill (5)

In the zero-dimensional case (no spatial coupling), this approximation is strictly valid only at short times. At long times, the system heads towards the steady state, in which the noise effect is in fact opposite to that of (5) (see [25]). In the spatially extended case, on the other hand, diffusive coupling between neighbors is able to “trap” the system in the short-time dynamics given by the effective force $f_{\text{eff}}$ in (5). Hence, a simple analysis of the zeroes of this function and their stability reveals, in a naïve but very efficient way, the transition scenario of the system [26]. We will now give some choices of $f$ and $g$, leading to different noise-induced phase transitions.
2.2 Noise-Induced Second-Order Phase Transitions

Let us first consider the following representative model:

$$\partial_t \phi = -\phi(1 + \phi^2) + \phi \eta(x, t) + \nabla^2 \phi + \xi(x, t),$$  

(6)

which corresponds, in the absence of multiplicative noise, to the well-known time-dependent Ginzburg-Landau model, frequently used in studies of critical dynamics \[3\], and whose universality class is that of the Ising model. For the deterministic parameters chosen, the system resides in the stable disordered state $\phi(x, t) = 0$. In the presence of the multiplicative noise, and according to the discussion of the previous paragraphs, the effective force (5) is equal to $f_{\text{eff}}(\phi) = [c(0) - 1] - 1 - \phi^3$, which shows that a non-zero (ordered) value of the field can appear for large enough noise intensities (for $c(0) > 1$ in this approximate analysis). This noise-induced ordering phase transition has been observed both numerically \[12\] and theoretically, through linear stability \[9\] and mean field \[14\] analysis. The change of the spatially averaged field $\frac{1}{V} \int \phi(x, t) dx$ at the bifurcation is continuous, which means that the transition is of second order. The system also displays a subsequent phase transition back to disorder, also of second order, for larger noise intensities \[18\]. In spite of the non-equilibrium character of the two transitions, a finite-size scaling analysis shows that both of them belong to the equilibrium universality class of the Ising model \[5\]. The reason for this can be traced back to the existence of the additive-noise term. In its absence (for which the disordered phase becomes an absorbing state), a new universality class appears \[27\].

In spite of the simplicity of model (6), most studies performed so far on noise-induced phase transitions have been made in a model introduced in 1994 by Van den Broeck et al. \[13\], which has an additional fifth-order saturating nonlinearity in the deterministic force, and an external noise with both additive and multiplicative contributions:

$$\partial_t \phi = -\phi(1 + \phi^2)^2 + (1 + \phi^2) \eta(x, t) + \nabla^2 \phi.$$  

(7)

The associated effective force is $f_{\text{eff}}(\phi) = [2c(0) - 1] - 1 - 2c(0) - 1 - 1 - 3 - 5$, which displays again two consecutive noise-induced phase transitions, an ordering and a disordering one, both of which belong to the Ising universality class \[25\] (which is not surprising, since the relevant terms of the deterministic hamiltonian are the same as those of the Ginzburg-Landau model). The observed phenomenology is basically equivalent to that of model (6). In particular, for both models the deterministic potential $U_{\text{det}} = -\int f(\phi) d\phi$ is monostable, so that the system is in all cases disordered in the absence of external noise. Moreover, the stochastic potential\[1\] that can be defined for the local dynamics, $U_{\text{st}} = -\int [f(\phi) - c(0)g(\phi)g'(\phi)]/[c(0)g^2(\phi)] d\phi$, is also

\[1\] The stochastic potential $U_{\text{st}}(\phi)$ is defined from the steady-state probability density $P_{\text{st}}(\phi)$ of the zero-dimensional system by means of $P_{\text{st}}(\phi) \sim \exp(-U_{\text{st}})$ \[24\].
monostable for all \(c(0)\) in the two cases, which indicates that the corresponding zero-dimensional model does not have a noise-induced transition towards order (towards non-zero field) for any of the two systems.

In its original form, model (7) cannot distinguish between the additive and multiplicative contributions of the external fluctuations to the noise-induced phenomena. Landa et al. [28] slightly modified the model in order to examine the contribution of the additive-noise term, and discovered the existence of noise-induced phase transitions (ordering and disordering) controlled by additive noise, provided a multiplicative-noise noise exists.

Another similarity between the noise-induced phase transitions exhibited by models (6) and (7) is that, in both cases, the mechanism through which noise destabilizes the disordered phase is linear, as can be easily seen by examining the effective forces in the two situations. A review of linear instability mechanisms of noise-induced phase transitions is given in [29]. On the other hand, by taking into account the discussion of Sect. 2.1, one can devise in a straightforward way models for which a noise-induced ordering phase transition is driven by a nonlinear mechanism. The system

\[
\partial_t \phi = -\phi^3 (1 + \phi^2) + \phi^2 \eta(x, t) + \nabla^2 \phi + \xi(x, t),
\]  

which has an effective force

\[
f_{\text{eff}}(\phi) = [2c(0) - 1] \phi^3 - \phi^5,
\]

is an example of this fact. In this case, the destabilization of the disordered phase \(\phi(x, t) = 0\) by noise is dynamically nonlinear.

### 2.3 Noise-Induced First-Order Phase Transitions

In all previous examples, the 0-d short-time effective force \(f_{\text{eff}}(\phi)\) displays a supercritical pitchfork bifurcation controlled by the noise intensity, corresponding to a continuous (second-order) phase transition when spatial coupling is taken into account. Suitable choices of \(f(\phi)\) and \(g(\phi)\), on the other hand, lead to discontinuous bifurcations in \(f_{\text{eff}}(\phi)\), associated to first-order noise-induced phase transitions. A first simple example is given by

\[
\partial_t \phi = -\phi + \phi^3 - \phi^5 + \phi \eta(x, t) + \nabla^2 \phi + \xi(x, t).
\]  

In this case, the effective force can be seen to be \(f_{\text{eff}}(\phi) = [c(0) - 1] \phi + \phi^3 - \phi^5\), which displays a subcritical pitchfork bifurcation controlled by noise that corresponds to a first-order phase transition in the spatially extended case. This transition is characterized by an abrupt change in the spatially averaged field and by a region of bistability between ordered and disordered states. Such features were observed by Müller et al. [19] in a model with the deterministic force of (6) and an external noise with both additive and multiplicative contributions. The separate role of additive fluctuations was investigated in a modified version of model (7), and a first-order phase transition induced by additive noise was found [19], provided multiplicative noise is present.
The noise-driven discontinuous phase transition displayed by (9) is produced by a destabilization of the linear coefficient of the 0-d effective force. Similarly to the case of second-order phase transitions, models can be defined that exhibit noise-induced first-order phase transitions driven by a nonlinear mechanism. The simplest example is:

\[ \partial_t \phi = -\phi (1 + \phi^4) + \phi^2 \eta(x, t) + \nabla^2 \phi + \xi(x, t), \] (10)

for which \( f_{\text{eff}}(\phi) = -\phi + 2c(0)\phi^3 - \phi^5 \). This function describes a subcritical pitchfork bifurcation at \( c(0) = 1 \) in 0-d, and hence a nonlinearly-driven discontinuous phase transition can be expected in the spatially extended case.

All examples presented in the last two Sections have been founded on an analysis of the effective force that governs the short-time behavior of the zero-dimensional system, which we already argued that would become trapped by spatial coupling in the extended case. The conclusions obtained by this approach can be confirmed by other methods, such as mean-field approximations [30] and numerical simulations of the complete models.

### 2.4 Noise-Induced Phase Dynamics

We have just seen that external noise can induce non-equilibrium phase transitions, both of first and second order, between two stationary phases. Now we aim to analyze the dynamical aspects related to the appearance of a non-equilibrium phase in the system. Let us consider a stationary phase, in equilibrium or not, with well-defined properties (symmetries). If a control parameter, such as temperature, is suddenly changed, the system undergoes a dynamical process in order to reach a new steady state corresponding to the new value of the control parameter. If the initial and final states belong to the same phase, the dynamical process is simple and linear relaxation dynamics can be used to model it. On the other hand, if the final state belongs to a phase different from the initial one, then a pattern dynamics appears. The pattern is composed of topological defects which evolve towards the new steady state. This process is controlled by a small number of parameters, and exhibits dynamics of different universality classes, which are well characterized by a dynamical exponent and by the scaling properties of the structure function. Two universality classes of particular interest are those of phase ordering and phase separation.

In the following paragraphs, we examine whether external noise can induce these dynamical processes, and what is the resulting dynamical universality class. The answer to the first question is simple: since external noise can induce ordering phase transitions, it is also able to induce the corresponding dynamics. The second question is much more involved. We can advance that the dynamical universality class is not changed, because the physical growth mechanisms are the same.
Noise-induced phase ordering. This dynamics arises in ferromagnetic systems, when a sudden decrease in temperature drives the system away from a homogeneous state of zero magnetization towards a final state of finite magnetization. During this process, the system exhibits magnetic domains, some of which grow at the expense of the rest until the whole system presents a homogeneous finite magnetization. The characteristic size of these domains grows according to the Allen-Cahn law, \( R(t) \sim t^{1/2} \).

A representative model of this situation is that defined by (6). We have shown that external noise can induce an ordering phase transition in that model. Therefore, beyond the transition point, noise can be considered to drive an initially disordered phase into an ordered state, through the formation of domains of the two coexisting (positive and negative) ordered phases and their subsequent dynamics of competition as in equilibrium models. Numerical simulations confirm this noise-induced dynamics [31]. Moreover, the process has the characteristics of the Allen-Cahn dynamical universality class: a clear evidence of the scaling behavior of the structure function with a power law \( \sim t^{1/2} \) is observed. The fact that the universality class is not changed can be understood because the driving mechanism of the ordering process is the same as in the deterministic case: the local curvature of the interface.

Noise-induced phase separation. This universality class arises in homogeneous alloys of two atomic species\(^2\) initially at high temperature. Following a sudden cooling, the two components of the alloy start to segregate, producing domains rich in one of the species, whose characteristic size follows the Lifshitz-Slyozov law, \( R(t) \sim t^{1/3} \). The system evolves towards a final state consisting on two large domains separated by an interphase.

A dynamical model of phase separation can be constructed from model (6) if mass conservation is imposed. The explicit model reads:

\[
\frac{\partial \phi}{\partial t} = \nabla^2 \left( \phi + \phi^3 - \nabla^2 \phi + \phi \eta(x, t) \right) + \nabla : \xi(x, t),
\]

where \( \xi(x, t) \) is now a random vector field. This system evolves with the restriction that the spatially averaged field, \( \frac{1}{V} \int \phi(x, t) \, dx = \phi_0 \), is conserved. As in the case of model (6), the external multiplicative noise \( \eta(x, t) \) induces an ordering phase transition, followed by a disordered one. In the ordered region, domains of the two non-zero phases coexist and try to grow. However, the dynamics is different from the case of phase ordering. Owing to the conservation law, domains have to grow at the expense of other domains, which may be located far away. The driving mechanism is thus diffusion controlled by local curvature, as in the deterministic case, and hence we expect the same dynamical universality class of Lifshitz-Slyozov. Numerical simulations confirm this fact [31]. A scaling behavior of the structure function following the power law \( \sim t^{1/3} \) is observed.

\(^2\) Phase separation in fluids is another universality class not discussed here.
3 Noise-Induced Structures

Besides phase transitions, a second and very important ordering mechanism in spatially extended systems is that of *structure formation*. The spontaneous appearance and sustenance of spatiotemporal patterns is common to many non-equilibrium extended media, including hydrodinamical, optical, chemical and biological systems [4]. The system is usually described by a stochastic partial differential equation (or a set of them), for which a self-sustained non-homogeneous solution is considered to be an ordered state. For instance, in the case of a single-field system, an ordered state is represented by a solution $\phi(x,t)$ which depends explicitly on $x$, and which may or may not depend on time. We will now review the effects of external noise on several different pattern-forming systems.

3.1 Noise-Induced Stationary Patterns

Let us begin by examining whether noise is able to induce stationary non-homogeneous (ordered) states in a pattern-forming system. The standard model displaying such a phenomenology is:

$$\partial_t \phi = -\phi(1 + \phi^2) + \phi \eta(x,t) - (\nabla^2 + k_0^2)^2 \phi + \xi(x,t),$$  \hspace{1cm} (12)

which corresponds to the well-known Swift-Hohenberg model, widely used to describe the formation of stationary structures in hydrodynamics and nonlinear optics, among other fields. For the parameter region chosen, the stationary solution of (12) in the absence of multiplicative noise is the disordered state $\phi(x,t) = 0$. When the external fluctuations are considered, the homogeneous solution becomes unstable, as can be seen in a simple way from a linear stability analysis of the first statistical moment of $\phi$. To that end, we linearise (12), transform it to Fourier space and average the resulting equation with respect to both noise distributions, which leads to:

$$\partial_t \langle \phi \rangle = \left[ c(0) - 1 - (k^2 - k_0^2)^2 \right] \langle \phi \rangle.$$  \hspace{1cm} (13)

This analysis shows that the disordered phase $\phi(x,t) = 0$ becomes unstable at $c(0) = 1$ for a non-zero wavenumber $k_0$, corresponding to the appearance of a periodic pattern of wavelength $2\pi/k_0$. A similar conclusion, at least at first order in noise intensity, can be observed when the stability of higher-order statistical moments is analysed\(^3\).

The model of Van den Broeck et al. [13] with the addition of a Swift-Hohenberg-like spatial coupling term has also been examined in search of noise-induced patterns. They have been observed coming from multiplicative [10] and additive [11] noise.

\(^3\) This fact contrasts with the case of 0-d systems, where statistical moments of different order have different instability thresholds at all orders [32].
3.2 Noise-Induced Propagation

The simplest non-stationary phenomenon in a spatially extended system is the propagation with constant velocity of a structure through the system. We will see in what follows that external noise is also able to induce such an ordering effect.

**Noise-Induced Fronts.** A front is a spatiotemporal structure linking two different homogeneous states (kink). Experiments have shown that noise supports front propagation in a chain of bistable diode resonators [33]. We now examine the effect of multiplicative noise on the one-dimensional propagation of a front in the following field equation:

\[
\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} - \phi(1 + \phi^2) + \phi \eta(x, t),
\]

(14)

In the absence of noise, the homogeneous phase \( \phi(x, t) = 0 \) is the only stable state of the system. Neither fronts nor any kind of spatial structure can be sustained at large times; any initial condition \( \phi(x, 0) \neq 0 \) decays rapidly to the above-mentioned disordered state. In the presence of noise, a systematic contribution to the deterministic dynamics arises as shown in Sect. 2.1. One can therefore write an effective model that has the same behavior as (14) on average:

\[
\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} - \phi(1 - c(0) + \phi^2) + \xi_{\text{eff}}(x, t),
\]

(15)

where the effective noise \( \xi_{\text{eff}}(x, t) \) has zero mean. According to this equation, the homogeneous zero state becomes unstable for \( c(0) > 1 \). Under this condition, an initial localised perturbation produces a kink-antikink pattern that propagates until a non-zero state \( \phi_{\text{st}} \sim \sqrt{c(0) - 1} \) invades all the system. The mean velocity of the propagating front can be computed to be \( v = 2\sqrt{c(0) - 1} \). These analytical predictions can be confirmed by numerical simulations of the exact model [20].

**Noise-Sustained Signal Propagation.** Propagating kink-antikink combinations behave as a train of traveling pulses that can act as information bits in a communication system. In a chain of asymmetrical double-well oscillators, these pulses are intrinsically unstable in the absence of noise. Multiplicative noise is able to sustain propagation of these pulses, as can be understood in a simple way by analysing the following model:

\[
\frac{\partial \phi}{\partial t} = \phi(1 - \phi)(\phi - a) + D \frac{\partial^2 \phi}{\partial x^2} + v \frac{\partial \phi}{\partial x} + \phi \eta(x, t).
\]

(16)

This equation contains a diffusive and a convective term in order to model a uni-directional coupling in this one-dimensional system. For \( 1/2 < a < 1 \), the
deterministic potential is an asymmetric double well, with $\phi = 0$ more stable than $\phi = 1$ (which are the two fixed points of the system). Therefore, in the absence of noise a kink-antikink pulse shrinks and decays as it propagates. Noise is able to prevent this decay, as can be seen by writing an effective model with a zero-mean noise, as has been made in the previous section. It is easy to see that an optimal value of noise intensity renders the effective potential of this model symmetric, which allows sustained propagation of pulses through the system.

3.3 Noise-Supported Structures in Excitable Media

Due to their special characteristics, excitable systems are especially sensitive to noise. Perturbations due to noise are able, for instance, to make the system jump from the rest to the excited state. But besides nucleating excitation pulses, external noise can also sustain their propagation in the subexcitable regime (in which such motion would not be possible under deterministic conditions). This fact has been observed numerically for the case of spiral waves, in what provided the first example, up to our knowledge, of spatio-temporal stochastic resonance. Additionally, chemical subexcitable media have also provided the first experimental evidence of noise-sustained pulse propagation. Such constructive effects of noise can be modeled by cellular automata; the obtained results lead also to predictions of noise-induced synchronization and global oscillations. Another possible modeling procedure is by means of continuous FitzHugh-Nagumo models of excitable media. Analyses in this direction have shown that external multiplicative noise is able to support spiral turbulent states in simple activator-inhibitor models, and that additive noise induces synchronization phenomena.

3.4 Other Constructive Effects of Noise

There are several other examples of noise-induced order in extended media that have not been described in the previous pages. We have already briefly mentioned the phenomenon of spatiotemporal stochastic resonance in excitable media. In that case, a certain non-zero but finite value of noise intensity exists for which the propagation of excitation waves is optimal. The extension to spatially extended systems of the phenomenon of stochastic resonance can be understood in other ways. In its purest interpretation, it corresponds to the enhanced response of a spatially bistable system (i.e., whose Lyapunov functional has two minima corresponding to two stable spatiotemporal states) to a harmonic signal.

Another constructive effect of noise, in this case not as counterintuitive, is the sustenance of drifting structures is systems with a convection term. In this case, structures would escape through the system boundaries, swept by convection, in the absence of noise. Fluctuations ensure a continuous creation of structures, which are therefore sustained by noise.
Noise has also been seen to have a constructive influence in globally coupled systems. As an example, a model of interacting Brownian particles in a periodic potential has been recently found to exhibit a noise-induced phase transition [40], in which the ordered phase can be interpreted as a ratchet-like transport mechanism, even though the underlying potential is symmetric.

4 Conclusions

We have tried to review, in a clear and pedagogical way, the ordering influence that external noise exerts in the spatiotemporal dynamics of extended media. Two main topics have been considered, namely noise-induced phase transitions and noise-induced structure formation. In each situation, a specific notion of order has been introduced. In the case of phase transitions, order is understood in a coarse-graining sense, so that it corresponds to a non-zero (even uniform) field. In the case of pattern formation, order is defined in opposition to uniformity, so that it corresponds to a non-uniform field profile, either depending or not on time. In any of the two cases, external noise can be seen to induce order. In each particular situation, the ordered phase can be characterised by standard tools used in the corresponding deterministic (or equilibrium) phenomenology. We interpret multiplicative noise in the Stratonovich sense, impelled by a search of realistic modeling of the fluctuations. Phase transitions induced by multiplicative noise in the Ito interpretation have been recently found [41], but they have a disordering character. Finally, we should also note that all the phenomena reviewed here can be explained by the short-time dynamical instability mechanism described in Sect. 2.1. But ordering transitions exist that are driven by a different mechanism, an investigation of which is currently in progress [42].

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