Nonlinear Schrödinger type tetrahedron maps

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Abstract

This paper is concerned with the construction of new solutions in terms of birational maps to the functional tetrahedron equation and parametric tetrahedron equation. We present a method for constructing solutions to the parametric tetrahedron equation via Darboux transformations. In particular, we study matrix refactorisation problems for Darboux transformations associated with the nonlinear Schrödinger (NLS) and the derivative nonlinear Schrödinger (DNLS) equation, and we construct novel nine-dimensional tetrahedron maps. We show that the latter can be restricted to six-dimensional parametric tetrahedron maps on invariant leaves. Finally, we construct parametric tetrahedron maps employing degenerated Darboux transformations of NLS and DNLS type.

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1 Introduction

The functional tetrahedron equation is a higher-dimensional generalisation of the Yang–Baxter equation, one of the most fundamental equations of Mathematical Physics, and it was first studied by Zamolodchikov in [28, 29]. Ever since, it has attracted the interest of many scientists in the area of Mathematical Physics (indicatively, we refer to [1, 2, 3, 4, 5, 8, 9, 13, 21, 25, 26]) who studied its solutions from various aspects.

From the point of view of classical integrable systems, an important result is Sergeev’s classification of tetrahedron maps in [26] where he studied the relation between matrix trifactorisation problems and the tetrahedron equation. Another important result is the connection between the tetrahedron equation and integrable systems on the three-dimensional lattice. In particular, this connection was established in [13], as a generalisation of the ideas presented in [24], and tetrahedron maps together with their vector generalisations were constructed using the invariants of symmetry groups of three-dimensional lattice systems. Moreover, noncommutative versions of integrable systems have been of great interest over the past few decades, due to the plethora of applications in Physics. In fact, fully noncommutative versions of solutions to the tetrahedron equation have already been found (see, for example, [2]). Of course, the results

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related to the tetrahedron equation are not limited to the aforementioned, and there are plenty of other results that highlight the importance of the functional tetrahedron equation. However, the functional tetrahedron equation has not yet reached the same level of attention as its lower-dimensional analogue, the Yang–Baxter equation. This is probably due to the fact that it is more difficult to find solutions to it, and not many methods are yet available for constructing such solutions.

In this paper, we present a method for constructing solutions to the parametric functional tetrahedron equation based on the ideas presented in [17]. In particular, we study matrix refactorisation problems for certain Darboux matrices and we derive solutions to the classical (without parameters) functional tetrahedron equation. The entries of each Darboux matrix satisfy a system of differential equations, the so-called Bäcklund transformation. The latter system admits a first integral which indicates the existence of invariant leaves. On these invariant leaves our derived solutions are expressed in terms of parametric birational maps which satisfy the parametric functional tetrahedron equation.

1.1 Organisation of the paper

The paper is organised as follows.

In the next section, we present all the necessary definitions for the text to be self-contained. In particular, we give the definitions of the functional tetrahedron and the parametric tetrahedron equation, and we explain their relation with matrix refactorisation problems. Furthermore, we prove a statement regarding the invariants of the solutions to these equations which indicate the integrability of these solutions.

In section 3, we present the basic steps of a simple scheme for constructing solutions to the parametric tetrahedron equation. Also, we list all the Darboux transformations that we use throughout the text, without giving details on their derivation. For their construction one can refer to [16].

In section 4, we study matrix refactorisation problems for Darboux matrices associated with the NLS equation, and we derive novel solutions to the tetrahedron equation and the parametric tetrahedron equation. In particular, using an NLS type Darboux transformation, we derive a novel nine-dimensional birational tetrahedron map which can be restricted to a novel six-dimensional, birational, parametric tetrahedron map on invariant leaves. Then, considering the matrix refactorisation problem for a degenerated Darboux matrix of NLS type, we construct another birational, six-dimensional, parametric tetrahedron map which can be restricted to a three-dimensional one on the level sets of its invariants. The latter map at a certain limit gives a map from Sergeev’s classification [26].

In section 5, we employ Darboux transformations associated with the DNLS equation in order to derive novel solutions to the tetrahedron equation. We first consider a Darboux transformation associated with the DNLS equation, and we obtain a novel nine-dimensional tetrahedron map the restriction of which on invariant leaves is represented by a novel six-dimensional parametric tetrahedron map. Moreover, we construct another six-dimensional parametric tetrahedron map using a degenerated version of the Darboux transformation for the DNLS equation. The derived six-dimensional map can also be restricted to a three-dimensional one which gives a map from Sergeev’s classification at a certain limit. All the maps derived in this section are noninvolutive and birational.

Finally, in section 6, we present a brief summary of the results of the paper and discuss some ideas for potential extensions of the obtained results.
2 Preliminaries

2.1 Tetrahedron maps

Let $\mathcal{X}$ be an algebraic variety in $\mathbb{C}^N$. A map $T \in \text{End}(\mathcal{X}^3)$, namely

$$T : (x, y, z) \mapsto (u(x, y, z), v(x, y, z), w(x, y, z)),$$

is called a tetrahedron map if it satisfies the functional tetrahedron (or Zamolodchikov) equation

$$T_{123} \circ T_{145} \circ T_{246} \circ T_{356} = T_{356} \circ T_{246} \circ T_{145} \circ T_{123}.$$  

(2)

Functions $T_{ijk} \in \text{End}(\mathcal{X}^6)$, $i, j = 1, 2, 3$, $i \neq j$, in (2) are maps that act as map $T$ on the $ijk$ terms of the Cartesian product $\mathcal{X}^6$ and trivially on the others. For instance,

$$T_{145}(x, y, z, r, s, t) = (u(x, r, s), y, z, v(x, r, s), w(x, r, s), t).$$

A tetrahedron map can be represented on the faces of the cube, as in Figure 1, mapping three neighbour faces of the cube to the rest three on the opposite side. In particular, a tetrahedron map can be thought as a map mapping the values $x$, $y$ and $z$, assigned to the back, bottom and left sides of the cube (i.e. faces 1, 2 and 3 in the left half-cube of Figure 1), to values $u$, $v$ and $w$ assigned to the front, top and right faces of the cube, respectively.

![Figure 1: Tetrahedron map. Schematic representation.](image)

The most, probably, celebrated Tetrahedron map is [9]

$$(x, y, z) \mapsto \left( \frac{xy}{x + z + xyz}, x + z + xyz, \frac{yz}{x + z + xyz} \right),$$

following from the ‘star-triangle’ transformation in electric circuits.

Now, if we assign the complex parameters $a$, $b$ and $c$ to the variables $x$, $y$ and $z$, respectively, we define a map $T \in \text{End}((\mathcal{X} \times \mathbb{C})^3)$, namely $T : ((x, a), (y, b), (z, c)) \mapsto ((u(x, y, z), a), (v(x, y, z), b), (w(x, y, z), c))$ which we denote for simplicity as

$$T_{a,b,c} : (x, y, z) \mapsto (u_{a,b,c}(x, y, z), v_{a,b,c}(x, y, z), w_{a,b,c}(x, y, z)).$$  

(3)

Map (3) is called a parametric tetrahedron map if it satisfies the parametric functional tetrahedron equation

$$T_{a,b,c}^{123} \circ T_{a,d,e}^{145} \circ T_{b,d,f}^{246} \circ T_{c,e,f}^{356} = T_{c,e,f}^{356} \circ T_{b,d,f}^{246} \circ T_{a,d,e}^{145} \circ T_{a,b,c}^{123}. $$

(4)

Schematically, a parametric tetrahedron map can be understood as in Figure 1 where the parameters $a$, $b$ and $c$ are placed on the faces of the cube together with the variables $x$, $y$ and $z$, respectively. Moreover, a four-dimensional representation of the parametric tetrahedron equation can be seen in Figure 2.

Only a handful of solutions to the parametric tetrahedron equation (4) are known to date; an interesting parametric tetrahedron map can be found in [1].
2.2 Tetrahedron maps and matrix refactorisation problems

Let $L = L(x, a; \lambda)$ be a matrix depending on a variable $x \in \mathcal{X}$, a parameter $a \in \mathbb{C}$ and a spectral parameter $\lambda \in \mathbb{C}$ of the form

$$L(x, a; \lambda) = \begin{pmatrix} A(x, a; \lambda) & B(x, a; \lambda) \\ C(x, a; \lambda) & D(x, a; \lambda) \end{pmatrix},$$

where its entries $A, B, C, D$ are scalar functions of $x, a$ and $\lambda$. Moreover, we define the following matrices

$$L_{12} = \begin{pmatrix} A(x, a; \lambda) & B(x, a; \lambda) \\ C(x, a; \lambda) & D(x, a; \lambda) \end{pmatrix}, \quad L_{13} = \begin{pmatrix} A(x, a; \lambda) & 0 & B(x, a; \lambda) \\ 0 & 1 & 0 \\ C(x, a; \lambda) & 0 & D(x, a; \lambda) \end{pmatrix}, \quad L_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A(x, a; \lambda) & B(x, a; \lambda) \\ 0 & 0 & D(x, a; \lambda) \end{pmatrix},$$

(6)

where $L_{ij} = L_{ij}(x, a; \lambda), i, j = 1, 2, 3$.

Following the work of Sergeev [26] and the work of Kashaev, Korepanov and Sergeev [9] we study the solutions of the following matrix trifactorisation problem

$$L_{12}(u, a; \lambda)L_{13}(v, b; \lambda)L_{23}(w, c; \lambda) = L_{23}(z, c; \lambda)L_{13}(y, b; \lambda)L_{12}(x, a; \lambda).$$

(7)

If the above matrix trifactorisation problem defines a map, we will call equation (7) its Lax representation. Equation (7) was also studied by Korepanov in [20] in the more general case where functions $A, B, C$ and $D$ in (6) are matrices.

Unlike the case of matrix refactorisation problems associated with solutions to the Yang–Baxter equation [27], the matrix refactorisation problem (7) does not admit the symmetry $(u, v, w; a, b, c) \rightarrow (x, y, z; c, b, a)$. That means that the rationality of a map $(x, y, z) \rightarrow (u(x, y; a, b), v(x, y; a, b), w(x, y; a, b))$ defined by (7) does not necessarily imply its birationality. Moreover, the trace of the right-hand side of (7) does not necessarily generate invariants for the former map. However, we have the following.

**Proposition 2.1.** If $L = L(x, a; \lambda)$ is a matrix of the form (5) with $B(x, a; \lambda) = C(x, a; \lambda)$, then the quantity

$$\text{tr} \left( L_{23}(z, c; \lambda)L_{13}(y, b; \lambda)L_{12}(x, a; \lambda) \right)$$

is a generator of invariants of the map $(x, y, z) \rightarrow (u(x, y; a, b), v(x, y; a, b), w(x, y; a, b))$ defined by (7).

**Proof.** Indeed, if $B(x, a; \lambda) = C(x, a; \lambda)$, then the matrices $L_{ij}, i, j = 1, 2, 3, i < j$, are symmetric, namely $L_{ij} = L_{ij}^T$. Therefore, the trace of the left-hand side of (7):

$$\text{tr} \left( L_{12}(u, a; \lambda)L_{13}(v, b; \lambda)L_{23}(w, c; \lambda) \right) \overset{\text{L}}{=} \text{tr} \left( L_{23}(z, c; \lambda)L_{13}(y, b; \lambda)L_{12}(x, a; \lambda) \right)$$

4
\[ L_{ij} L_{ij} = L_{ij} (x, a; \lambda) L_{ij} (y, b; \lambda) L_{ij} (z, c; \lambda), \quad (8) \]

since \( L_{ij} = L_{ij}^T, i, j = 1, 2, 3, i < j \).

Now, if we expand in \( \lambda \):

\[ \text{tr} \left( L_{23} (z, c; \lambda) L_{13} (y, b; \lambda) L_{12} (x, a; \lambda) \right) = \sum_k I_k (x, y, z) \lambda^k, \]

then from (8) follows that

\[ I_k (x, y, z) = I_k (u, v, w), \]

i.e. \( I_k (x, y, z) \) are invariants of the map \( (x, y, z) \rightarrow (u(x, y; a, b), v(x, y; a, b), w(x, y; a, b)) \) defined by (7).

### 3 Derivation of parametric tetrahedron maps: a Darboux construction scheme

In this section, following the ideas presented in [17] for the case of parametric Yang–Baxter maps, we demonstrate a scheme for constructing solutions to the parametric tetrahedron equation using Darboux transformations. The scheme can be summarised in Figure 3.

\[
M_{12} M_{13} M_{23} = M_{23} M_{13} M_{12} \quad \text{for} \quad T_{a,b,c} \circ T_{a,d,e} \circ T_{b,d,f} \circ T_{c,d,f} = T_{c,e,f} \circ T_{b,d,f} \circ T_{a,d,e} \circ T_{a,b,c}
\]

Figure 3: Darboux construction scheme.

**I.** Consider matrix trifactorisation problems (7) for Darboux matrices associated with Lax operators of the form \( L = D_x + U(u(x, t), \lambda) \), where \( U(u(x, t), \lambda) \) belongs to the Lie algebra \( \mathfrak{sl}_2 \) and depends on a (vector) potential function \( u \), a spectral parameter \( \lambda \) and two variables \( x \) and \( t \), implicitly through the potential function. From such matrix refactorisation problems we derive tetrahedron maps.

**II.** For such Darboux transformations as in step I, the associated Bäcklund transformations admit first integrals (see [16]). These first integrals indicate the existence of invariants of the tetrahedron maps derived in step I. On the level sets of these integrals, the tetrahedron maps derived in step I are represented as parametric maps.

**III.** These parametric maps are solutions to the parametric tetrahedron equation.

In the next section, we apply the above scheme to particular Darboux transformations related to the NLS and the DNLS equation.

### 3.1 Nonlinear Schrödinger type Darboux transformations

Here, we list all the Darboux transformations that we use in this text. For details regarding their derivation, one can refer to [16]. In particular, we use Darboux transformations associated with the Lax operator...
\[ \mathcal{L}_{\text{NLS}} \in \mathfrak{sl}_2[\lambda][D_x] \text{ and } \mathcal{L}_{\text{DNLS}} \in \mathfrak{sl}_2[\lambda][s_1][D_x], \text{ namely} \]

\[
\begin{align*}
\mathcal{L}_{\text{NLS}} &= D_x + \lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 2p \\ 0 & 0 \end{pmatrix}, \\
\mathcal{L}_{\text{DNLS}} &= D_x + \lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 2p \\ 0 & 2q \end{pmatrix}
\end{align*}
\]

i.e. the spatial parts of the Lax pairs for the NLS and the DNLS equation, respectively.

1. A Darboux transformation for \( \mathcal{L}_{\text{NLS}} \) in (9) is:

\[
M = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} f & p \\ \bar{q} & 1 \end{pmatrix},
\]

where its entries obey the system of equations \( \partial_x f = 2(pq - \bar{p}\bar{q}), \quad \partial_x p = 2(pf - \bar{p}), \quad \partial_x \bar{q} = 2(q - \bar{q}f) \), i.e. the so-called Bäcklund transformation. A first integral of this system of differential equations is

\[
\partial_x (f - p\bar{q}) = 0.
\]

2. A ‘degenerated’ Darboux Matrix for \( \mathcal{L}_{\text{NLS}} \) reads

\[
M_a(p, f) = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} f & p \\ \bar{q} & 0 \end{pmatrix}, \quad f = \frac{px}{2p}.
\]

3. A Darboux transformation associated with operator \( \mathcal{L}_{\text{DNLS}} \) in (9) is

\[
M(p, \bar{q}, f) := \lambda^2 \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & fp \\ \bar{f}\bar{q} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
\]

where \( p \) and \( q \) satisfy a system of differential equations which possesses the following first integral,

\[
\partial_x (f^2\bar{p}\bar{q} - f) = 0.
\]

4. A ‘degenerated’ Darboux transformation related to \( \mathcal{L}_{\text{DNLS}} \) is

\[
M(p, f; a) := \lambda^2 \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & \frac{fp}{a} \\ \frac{\bar{f}\bar{q}}{p} & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\]

In the following sections, we study matrix trifactorisation problems associated with the above Darboux matrices in order to derive tetrahedron maps. As we shall see, the existence of first integrals (11) and (14) play an important role, since they indicate the invariant leaves on which the derived tetrahedron maps are expressed as parametric tetrahedron maps, namely solutions to (4).

4 Nonlinear Schrödinger type tetrahedron maps

In this section, we study matrix trifactorisation problems (7) for the Darboux transformations (10) and (12) associated to the NLS equation, and we construct novel solutions to the tetrahedron equation and the parametric tetrahedron equation. All the derived maps are noninvolutive which are in general more interesting comparing to involutive ones, since involutive maps have trivial dynamics.
4.1 A novel nine-dimensional tetrahedron map

Changing \((p, \bar{q}, f + \lambda) \to (x_1, x_2, X)\) in \((10)\), we define the following matrix

\[
M(x_1, x_2, X) = \begin{pmatrix}
X \\ x_2 \\ 1
\end{pmatrix}.
\] (16)

For this matrix \((10)\) we consider the matrix trifactorisation problem \((7)\), namely

\[
M_{12}(u_1, u_2, U)M_{13}(v_1, v_2, V)M_{23}(w_1, w_2, W) = M_{23}(z_1, z_2, Z)M_{13}(y_1, y_2, Y)M_{12}(x_1, x_2, X).
\] (17)

The above matrix refactorisation problem implies the following:

\[
u_1 = \frac{x_1(y_1y_2 - Y) + y_1z_2}{z_1z_2 - Z}, \quad u_2 = \frac{x_2Z + y_2z_1X}{XY} \]

\[
v_1 = \frac{y_1Z + x_1z_1(y_1y_2 - Y)}{y_1y_2z_1(x_1y_2 + z_2)X - (x_1y_2z_1 + z_1z_2 - Z)XY + x_2[y_1z_2 + x_1(y_1y_2 - Y)]Z}{U} \] \quad (18a)

\[
v_2 = x_2z_2 + y_2X, \quad V = \frac{XY}{U} \]

\[
w_1 = \frac{[x_2y_1Z + (y_1y_2 - Y)z_1X](z_1z_2 - Z)}{y_1y_2z_1(x_1y_2 + z_2)X - (x_1y_2z_1 + z_1z_2 - Z)XY + x_2[y_1z_2 + x_1(y_1y_2 - Y)]Z} \] \quad (18b)

\[
w_2 = x_1y_2 + z_2, \quad \]

\[
W = \frac{(x_1x_2 - X)(z_1z_2 - Z)YZ}{y_1y_2z_1(x_1y_2 + z_2)X - (x_1y_2z_1 + z_1z_2 - Z)XY + x_2[y_1z_2 + x_1(y_1y_2 - Y)]Z} \] \quad (18c)

which is a correspondence rather than a map; functions \(u_2, v_1\) and \(V\) are defined in terms of \(U\). The above correspondence does not satisfy the tetrahedron equation for any choice of \(U\). However, as we shall see below, there is at least a choice of \(U\) for which \((18)\) defines a tetrahedron map.

In particular, the determinant of equation \((17)\) implies the equation

\[(U - u_1u_2)(V - v_1v_2)(W - w_1w_2) = (X_1 - x_1x_2)(Y - y_1y_2)(Z - z_1z_2).\]

We choose \(U - u_1u_2 = X - x_1x_2\), \(V - v_1v_2 = Y - y_1y_2\), \(W - w_1w_2 = Z - z_1z_2\). Then, the following holds.

**Proposition 4.1.** The system consisting of equation \((17)\) together with \(U - u_1u_2 = X - x_1x_2\) has a unique solution, namely a map \((x_1, x_2, X, y_1, y_2, Y, z_1, z_2, Z) \to (u_1, u_2, U, v_1, v_2, V, w_1, w_2, W)\), given by

\[
x_1 \to u_1 = \frac{x_1(y_1y_2 - Y) + y_1z_2}{z_1z_2 - Z}, \quad (19a)
\]

\[
x_2 \to u_2 = \frac{(x_1x_2 - X)(y_2z_1X + x_2Z)(z_1z_2 - Z)}{y_1y_2z_1(x_1y_2 + z_2)X - (x_1y_2z_1 + z_1z_2 - Z)XY + x_2[y_1z_2 + x_1(y_1y_2 - Y)]Z}, \quad (19b)
\]

\[
X \to U = \frac{(x_1x_2 - X)(y_1y_2 - Y)X}{y_1y_2z_1(x_1y_2 + z_2)X - (x_1y_2z_1 + z_1z_2 - Z)XY + x_2[y_1z_2 + x_1(y_1y_2 - Y)]Z}, \quad (19c)
\]

\[
y_1 \to v_1 = \frac{x_1z_1(y_1y_2 - Y) + y_1Z}{(x_1x_2 - X)(z_1z_2 - Z)}, \quad (19d)
\]

\[
y_2 \to v_2 = x_2z_2 + y_2X, \quad (19e)
\]

\[
Y \to V = \frac{y_1y_2z_1(x_1y_2 + z_2)X - (x_1y_2z_1 + z_1z_2 - Z)XY + x_2[y_1z_2 + x_1(y_1y_2 - Y)]Z}{(x_1x_2 - X)(z_1z_2 - Z)}, \quad (19f)
\]
The tetrahedron property can be readily verified by substitution Theorem 4.2.

4.2 Restriction on invariant leaves: A novel six-dimensional Tetrahedron map

The system consisting of equations (18a) and equation (19) is a nine-dimensional noninvolutive tetrahedron map.

Proof. The system consisting of equations (18a) and equation $U - u_1 u_2 = X - x_1 x_2$ has a unique solution given by (19a) - (19c). Moreover, substituting $U$ given by (19c) to (18b) and (18c), we obtain $v_1$ and $V$ given in (19d) and (19e), respectively. The tetrahedron property can be readily verified by substitution to the Tetrahedron equation. Finally, for the involutivity of the map we have

$$w_2(u_1, u_2, U, v_1, v_2, V, w_1, w_2, W) = x_1 y_2 + z_2 + \frac{(x_2 z_2 + y_2 X)[x_1(y_1 y_2 - Y) + y_1 z_2]}{z_1 z_2 - Z}.$$ 

That is, $T \circ T \neq \text{id}$. Thus, map (19) is noninvolutive.

The existence of first integral (11) indicates the integrals of map (19). We can restrict the latter to a novel nine-dimensional one on the level sets of these integrals. In particular we have the following.

Theorem 4.2. 1. The quantities $\Phi = X - x_1 x_2$, $\Psi = Y - y_1 y_2$ and $\Omega = Z - z_1 z_2$ are invariants of the map (19).

2. Map (19) can be restricted to a noninvolutive parametric six-dimensional tetrahedron map

$$(x_1, x_2, y_1, y_2, z_1, z_2) \xrightarrow{T_{a,b,c}} (u_1, u_2, v_1, v_2, w_1, w_2),$$

given by:

$$x_1 \mapsto u_1 = \frac{b x_1 - y_1 z_2}{c};$$ (20a)

$$x_2 \mapsto u_2 = \frac{a c x_2 (c + z_1 z_2) + y_2 z_1 (a + x_1 x_2)}{abc - [x_2 z_2 + y_2 (a + x_1 x_2)][b x_1 z_1 - y_1 (c + z_1 z_2)]];$$ (20b)

$$y_1 \mapsto v_1 = \frac{y_1 (c + z_1 z_2) - b x_1 z_1}{ac};$$ (20c)

$$y_2 \mapsto v_2 = x_2 z_2 + y_2 (a + x_1 x_2);$$ (20d)

$$z_1 \mapsto w_1 = \frac{c [b z_1 (a + x_1 x_2) - x_2 y_1 (c + z_1 z_2)]}{abc - [x_2 z_2 + y_2 (a + x_1 x_2)][b x_1 z_1 - y_1 (c + z_1 z_2)]];$$ (20e)

$$z_2 \mapsto w_2 = x_1 y_2 + z_2.$$ (20f)

on the invariant leaves

$$A_a := \{(x_1, x_2, X) \in \mathbb{C}^3 : X = a + x_1 x_2\}, \quad B_b := \{(y_1, y_2, Y) \in \mathbb{C}^3 : Y = b + y_1 y_2\},$$

$$C_c := \{(z_1, z_2, Z) \in \mathbb{C}^3 : Z = c + z_1 z_2\}.$$ (21)
3. Map (20) admits the following invariants:

\[ I_1 = (a + x_1 x_2)(b + y_1 y_2), \quad I_2 = (b + y_1 y_2)(c + z_1 z_2), \quad I_3 = ay_1 y_2 + (x_2 y_1 + z_1)(x_1 y_2 + z_2). \]  

**Proof.** Regarding equation (20). It can be verified by substitution that map (20) satisfies the parametric tetrahedron equation. Thus, the birationality of map (20) can be implied by solving equation (23) for in (24) is not symmetric, thus does not fall in the category of maps for which proposition 2.1 holds.

**Remark 4.3.** A novel ‘degenerated’ six-dimensional parametric tetrahedron map of NLS type

\[ M_{12}(u_1, u_2; a)M_{13}(v_1, v_2; b)M_{23}(w_1, w_2; c) = M_{23}(z_1, z_2; c)M_{13}(y_1, y_2; b)M_{12}(x_1, x_2; a), \]  

where the associated matrix is given by

\[ M(x_1, x_2; a) = \begin{pmatrix} a + x_1 x_2 & x_1 \\ x_2 & 1 \end{pmatrix}. \]  

**Remark 4.4.** The invariants (22) are not obtained from the trace of the right-hand side of (23). Matrix M in (24) is not symmetric, thus does not fall in the category of maps for which proposition 2.1 holds.

**Remark 4.5.** As mentioned in section 2 the symmetry break in equation (23) does not automatically imply birationality of map (20). However, solving equation (23) for \((x_1, x_2, y_1, y_2, z_1, z_2)\), one can easily see that map (20) is birational.

**4.3 A novel ‘degenerated’ six-dimensional parametric tetrahedron map of NLS type**

Here, we employ the second Darboux matrix of NLS type, namely matrix (12), to derive another parametric tetrahedron map. In particular, we change \((f + \lambda, p) \rightarrow (x_1, x_2)\) in (12), and define the following matrix

\[ M(x_1, x_2; a) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \]  

For matrix (25) we consider the following matrix trifactorisation problem:

\[ M_{12}(u_1, u_2; a)M_{13}(v_1, v_2; b)M_{23}(w_1, w_2; c) = M_{23}(z_1, z_2; c)M_{13}(y_1, y_2; b)M_{12}(x_1, x_2; a). \]

The above equation implies the system of polynomial equations

\[
\begin{align*}
    u_1 v_1 &= x_1 y_1, \\
    u_2 w_1 + c \frac{u_1 v_2}{w_2} &= x_2 y_1, \\
    u_2 w_2 &= y_2, \\
    \frac{v_1}{u_2} &= \frac{z_1 x_2}{x_1}, \\
    ac \frac{v_2}{u_2 w_2} &= \frac{x_2 z_2}{y_2}, \\
    \frac{b}{v_2} &= \frac{ac}{x_2 z_2}.
\end{align*}
\]
Theorem 4.6. The map defined by
\[ u_2 = \frac{ax_1 x_2 y_1 y_2}{u_1 (ay_2 z_1 + bx_1 x_2 z_2)}; \quad v_1 = \frac{x_1 y_1}{u_1}; \quad v_2 = \frac{bx_2 z_2}{ac}; \quad w_1 = \frac{z_1}{x_1} u_1; \quad w_2 = \frac{ay_2 z_1 + bx_1 x_2 z_2}{ax_1 x_2 y_1} u_1. \] (28)

That is, \( u_2, v_1, v_2, w_1 \) and \( w_2 \) depend on \( u_1 \).

As in the previous section, this correspondence does not define a tetrahedron map for any choice of \( u_1 \). However, for the choice \( u_1 = y_1 \) we have the following.

Theorem 4.6. The map defined by
\[ (x_1, x_2, y_1, y_2, z_1, z_2) \xrightarrow{T_{a,b,c}} \left( y_1, \frac{ax_1 x_2 y_2}{ay_2 z_1 + bx_1 x_2 z_2}, x_1, \frac{bx_2 z_2}{ac}, \frac{y_1 z_1}{x_1}, \frac{ay_2 z_1 + bx_1 x_2 z_2}{ax_1 x_2} \right) \] (29)
is a six-dimensional parametric tetrahedron map and it is noninvolutive and birational. Moreover, map (29) admits the following invariants:
\[ I_1 = x_1 y_1, \quad I_2 = x_1 + y_1, \quad I_3 = y_1 z_1. \] (30)

Proof. Map (29) follows after substitution of \( u_2 = x_2 \) to (28). The tetrahedron property can be readily verified by substitution to the tetrahedron equation.

Now, since \( v_2 \circ T_{a,b,c} = \frac{by_2}{ac} \neq y_2 \), it follows that \( T_{a,b,c} \circ T_{a,b,c} \neq \text{id} \), and therefore the map is noninvolutive. Additionally, the inverse of map (29) is given by
\[ (u_1, u_2, v_1, v_2, w_1, w_2) \xrightarrow{T_{a,b,c}^{-1}} \left( v_1, \frac{u_2 w_1 w_2 + cu_1 v_2}{u_1 w_2}, u_1 u_2 w_2, \frac{v_1 w_1}{u_1}, \frac{acu_1 v_2 w_2}{bcu_1 v_2 + bw_2 w_1 w_2} \right), \]

namely (29) is birational.

Finally, we have
\[ u_1 v_1 \xrightarrow{(29)} x_1 y_1, \quad u_1 + v_1 \xrightarrow{(29)} x_1 + y_1, \quad \text{and} \quad u_1 w_1 \xrightarrow{(29)} x_1 z_1, \]
i.e. map (29) admits the invariants \( I_i, i = 1, 2, 3 \), given by (30).

\[ \square \]

4.3.1 Restriction on the level sets of the invariants

Here, we restrict the map (29) on the level sets of its invariants (30). In particular, we have the following.

Proposition 4.7. Map (29) can be restricted to a three-dimensional noninvolutive parametric map given by
\[ (x, y, z) \xrightarrow{T_{a,b,c}} \left( \frac{ax y}{ay + bx z}, \frac{bx z}{ax}, \frac{ay + bx z}{ax} \right) \] (31)

Proof. Setting \( I_1 = 1, I_2 = 2 \) and \( I_3 = 1 \), where \( I_i, i = 1, 2, 3 \), are given by (30), and solving for \( x_1, y_1, z_1 \) we obtain \( x_1 = y_1 = z_1 = 1 \). We substitute to (13), and we obtain a map \( x_2 \mapsto u_2(x_2, y_2, z_2) \), \( y_2 \mapsto v_2(x_2, y_2, z_2) \) and \( z_2 \mapsto w_2(x_2, y_2, z_2) \). After relabelling \( (x_2, y_2, z_2, u_2, v_2, w_2) \rightarrow (x, y, z, u, v, w) \), we obtain map (31). Noninvolutivity of the map follows from the fact that, for instance, \( (v \circ T_{a,b,c})(x, y, z) = \frac{by}{ac} \neq y \), thus \( T_{a,b,c} \circ T_{a,b,c} \neq \text{id} \). \[ \square \]

Remark 4.8. From the above parametric tetrahedron map, one can obtain map (20) in Sergeev’s classification (25) (and also in [9]), considering the limit \( b \rightarrow a \).
4.4 A novel six-dimensional parametric tetrahedron map which does not restrict to a tetrahedron map on the level sets of its invariants

In this section, we demonstrate by an example that a restriction of a tetrahedron map on invariant leaves is not necessarily a tetrahedron map.

In section 4.3, we saw that for the choice $u_1 = y_1$, the correspondence \((32)\) defines the six-dimensional tetrahedron map \((29)\) which can be restricted to the parametric three-dimensional tetrahedron map \((31)\). As mentioned earlier, the correspondence \((27)\) does not define a parametric tetrahedron map for any choice of the free variable. And even when a six-dimensional tetrahedron map is defined, this does not necessarily imply that its three-dimensional restriction on invariant leaves has the tetrahedron property. To demonstrate this, we express $u_1, v_1, v_2, w_1$ and $w_2$ in terms of $u_2$ in \((27)\).

Specifically, the choice $u_2 = x_2$ in \((27)\) implies the following.

**Theorem 4.9.** The map defined by

\[
(x_1, x_2, y_1, y_2, z_1, z_2) \xrightarrow{T_{a,b,c}} \left( \frac{ax_1y_1y_2}{ay_2z_1 + bx_1x_2z_2}, x_2, z_1 + \frac{bx_1x_2z_2}{ay_2}, \frac{bx_2z_2}{ac}, ay_1y_2z_1, \frac{y_2}{x_2} \right) \quad (32)
\]

is a six-dimensional parametric tetrahedron map and it is noninvolutive and birational. Moreover, map \((32)\) admits the following invariants:

\[
I_1 = x_1y_1, \quad I_2 = x_2, \quad I_3 = \frac{x_1}{z_1}.
\]

**Proof.** Expressing $u_1, v_1, v_2, w_1$ and $w_2$ in terms of $u_2$ in \((27)\), we obtain

\[
u_1 = \frac{ax_1x_2y_1y_2}{u_2(ay_2z_1 + bx_1x_2z_2)}; \quad v_1 = u_2 \left( \frac{z_1}{x_2} + \frac{bx_1z_1}{ay_2} \right); \quad v_2 = \frac{bx_2z_2}{ac}; \quad w_1 = \frac{ax_2y_1y_2z_1}{u_2(ay_2z_1 + bx_1x_2z_2)}; \quad w_2 = \frac{y_2}{u_2} \quad (34)
\]

Map \((32)\) follows after substitution of $u_2 = x_2$ to \((34)\). The tetrahedron property can be readily verified by substitution to the tetrahedron equation. Now, since $v_1(u_1, u_2, v_1, v_2, w_1, w_2) = \frac{ay_1y_2(ay_1y_2z_1 + bx_1x_2z_2)}{u_2(ay_2z_1 + bx_1x_2z_2)} \neq y_1$, it follows that $T_{a,b,c} \circ T_{a,b,c} \neq \text{id}$, and therefore the map is noninvolutive. Additionally, the inverse of map \((32)\) is given by

\[
(u_1, u_2, v_1, v_2, w_1, w_2) \xrightarrow{T_{a,b,c}^{-1}} \left( \frac{u_1u_2v_1w_2}{cu_1v_2 + u_2w_1w_2}, u_2, w_1 + cu_1v_2, u_2w_1, \frac{u_2v_1w_1w_2}{cu_1v_2 + u_2w_1w_2}, \frac{acv_2}{bu_2} \right).
\]

namely \((32)\) is birational.

Finally, we have

\[
u_1 = x_1y_1, \quad u_2 = x_2, \quad \text{and} \quad \frac{u_1}{w_1} = \frac{x_1}{z_1},
\]

i.e. map \((32)\) admits the invariants $I_i, i = 1, 2, 3$, given by \((33)\).

\[\square\]

4.5 Restriction on the level sets of the invariants

Here, we restrict the map \((32)\) on the level sets of its invariants \((30)\). In particular, we have the following.

**Proposition 4.10.** Map \((32)\) can be restricted to a three-dimensional noninvolutive parametric map given by

\[
(x, y, z) \xrightarrow{T_{a,b,c}} \left( \frac{ay + bz}{axy}, \frac{bz}{ac}, y \right) \quad (35)
\]
Proof. Setting \(I_1 = I_2 = I_3 = 1\), where \(I_i, i = 1, 2, 3\), are given by (30), and solving for \(x_1, x_2, z_1\) we obtain \(x_1 = z_1 = \frac{1}{y_1}\) and \(x_2 = 1\). We substitute to (32), and we obtain a map \(y_1 \mapsto v_1(y_1, y_2, z_2), y_2 \mapsto v_2(y_1, y_2, z_2)\) and \(z_2 \mapsto w_2(y_1, y_2, z_2)\). After relabelling \((y_1, y_2, z_2, v_1, v_2, w_2) \rightarrow (x, y, z, u, v, w)\), we obtain map (33). The noninvolutivity of the maps follows from the fact that, for instance, \((w \circ T_{a,b,c})(x, y, z) = \frac{ac}{bc} \neq z\), thus \(T_{a,b,c} \circ T_{a,b,c} \neq id\). \(\square\)

Remark 4.11. Map (35) does not satisfy the parametric tetrahedron equation.

5 Derivative nonlinear Schrödinger type tetrahedron maps

In this section, we study matrix trifactorisation problems (7) for the Darboux transformations (13) and (15) associated to the DNLS equation.

5.1 A novel nine-dimensional tetrahedron map

Changing \((\lambda^{-1}p, \lambda^{-1}q, \lambda^2 f) \rightarrow (x_1, x_2, X)\) in (10), we define the following matrix

\[
M(x_1, x_2, X) = \begin{pmatrix} x_1 & x_1X \\ x_2X & 1 \end{pmatrix}. \tag{36}
\]

For matrix (36) we consider the matrix trifactorisation problem (7), i.e.

\[
M_{12}(u_1, u_2, U) M_{13}(v_1, v_2, V) M_{23}(w_1, w_2, W) = M_{23}(z_1, z_2, Z) M_{13}(y_1, y_2, Y) M_{12}(x_1, x_2, X). \tag{37}
\]

The above matrix equation implies the following correspondence:

\[
u_1 = \frac{x_1 X(y_1 y_2 Y - 1) + y_1 z_2 Z}{z_1 z_2 Z - 1} \frac{Y}{Z U}, \quad u_2 = (x_2 + y_2 z_1) \frac{Z}{Y}, \tag{38a}
\]

\[
v_1 = \frac{y_1 + x_1 z_1 X(y_1 y_2 Y - 1)}{1 + x_2 y_1 z_2 Z + (y_1 y_2 Y - 1)[x_1(x_2 + y_2 z_1 Y)X + z_1 z_2 Z]} \frac{1}{X}, \tag{38b}
\]

\[
v_2 = (y_2 + x_2 z_2 Z) U, \quad V = \frac{XY}{U}, \quad w_1 = \frac{x_2 y_1 + z_1 (y_1 y_2 Y - 1)}{x_1 x_2 X - 1}, \tag{38c}
\]

\[
w_2 = \frac{(x_1 y_2 X Y + z_2 Z)(1 + x_2 y_1 z_2 Z + (y_1 y_2 Y - 1)[x_1(x_2 + y_2 z_1 Y)X + z_1 z_2 Z])}{(x_1 x_2 X - 1)(z_1 z_2 Z - 1)}, \tag{38d}
\]

\[
W = \frac{(x_1 x_2 X - 1)(z_1 z_2 Z - 1)Z}{1 + x_2 y_1 z_2 Z + (y_1 y_2 Y - 1)[x_1(x_2 + y_2 z_1 Y)X + z_1 z_2 Z]} \tag{38e}.
\]

The elements \(u_1, v_2\) and \(V\) are expressed in terms of \(U\). However, there is at least a choice of \(U\) for which (18) defines a tetrahedron map.

The determinant of equation (37) implies the relation

\[
(U - u_1 u_2 U^2)(V - v_1 v_2 V^2)(W - w_1 w_2 W^2) = (X - x_1 x_2 X^2)(Y - y_1 y_2 Y^2)(Z - z_1 z_2 Z^2)
\]

We choose \(U - u_1 u_2 U^2 = X - x_1 x_2 X^2, V - v_1 v_2 V^2 = Y - y_1 y_2 Y^2\) and \(W - w_1 w_2 W^2 = Z - z_1 z_2 Z^2\). Then, the following holds.
Proposition 5.1. The system consisting of equation (37) together with \( U - u_1 u_2 U^2 = X - x_1 x_2 X^2 \) has a unique solution given by a map

\[
(x_1, x_2, X, y_1, y_2, Y, z_1, z_2, Z) \xrightarrow{T} (u_1, u_2, U, v_1, v_2, V, w_1, w_2, W)
\]

defined by

\[
x_1 \mapsto u_1 = \frac{1 + x_2 y_1 z_2 Z + (y_1 y_2 Y - 1) [x_1 (x_2 + y_2 z_1 Y) X + z_1 z_2 Z]}{(x_1 x_2 X - 1)(x_1 z_1 Z - 1)^2 Z} \cdot [x_1 X (y_1 y_2 Y - 1) + y_1 z_2 Z] Y,
\]

\[
x_2 \mapsto u_2 = (x_2 + y_2 z_1) Z,
\]

\[
X \mapsto U = \frac{(x_1 x_2 X - 1)(x_1 z_1 Z - 1)}{1 + x_2 y_1 z_2 Z + (y_1 y_2 Y - 1)[x_1 (x_2 + y_2 z_1 Y) X + z_1 z_2 Z] Y},
\]

\[
y_1 \mapsto v_1 = \frac{y_1 + x_1 z_1 X (y_1 y_2 Y - 1)}{(x_1 x_2 X - 1)(y_2 Y + x_2 z_2 Z)(z_1 z_2 Z - 1) Y},
\]

\[
y_2 \mapsto v_2 = \frac{1 + x_2 y_1 z_2 Z + (y_1 y_2 Y - 1)[x_1 (x_2 + y_2 z_1 Y) X + z_1 z_2 Z] Y}{(x_1 x_2 X - 1)(y_2 Y + x_2 z_2 Z)(z_1 z_2 Z - 1) Y},
\]

\[
Y \mapsto V = \frac{1}{1 + x_2 y_1 z_2 Z + (y_1 y_2 Y - 1)[x_1 (x_2 + y_2 z_1 Y) X + z_1 z_2 Z] Y}(x_1 x_2 X - 1)(z_1 z_2 Z - 1) Y,
\]

\[
z_1 \mapsto w_1 = \frac{x_2 y_1 z_1 (y_1 y_2 Y - 1)}{x_1 x_2 X - 1},
\]

\[
z_2 \mapsto w_2 = \frac{(x_1 y_2 X Y + z_2 Z) [1 + x_2 y_1 z_2 Z + (y_1 y_2 Y - 1)[x_1 (x_2 + y_2 z_1 Y) X + z_1 z_2 Z] Y]}{(x_1 x_2 X - 1)(z_1 z_2 Z - 1)},
\]

\[
z \mapsto W = \frac{(x_1 x_2 X - 1)(z_1 z_2 Z - 1)}{1 + x_2 y_1 z_2 Z + (y_1 y_2 Y - 1)[x_1 (x_2 + y_2 z_1 Y) X + z_1 z_2 Z] Y}.
\]

Map (39) is a nine-dimensional birational tetrahedron map.

Proof. The system consisting of equations (38a) and \((1 - u_1 u_2 U) U = (1 - x_1 x_2 X) X\) has a unique solution given by (39a), (39b) and (39c). Now substituting \(U\) from (39c) to the first two equations of (38c), we obtain \(v_2\) and \(V\) given by (39c) and (39f), respectively.

It can be readily verified that map (39) is a tetrahedron map by substitution to the tetrahedron equation. Finally, we have \(T \circ T \neq \text{id}\). Thus, map (39) is noninvolutive. \(\square\)

5.2 Restriction on invariant leaves: A novel six-dimensional Tetrahedron map

The existence of first integral (14) indicates the integrals of map (39). We can restrict the latter to a novel nine-dimensional one on the level sets of these integrals. In particular, we have the following.

Theorem 5.2. 1. The quantities \(\Phi = X - x_1 x_2 X^2\), \(\Psi = Y - y_1 y_2 Y^2\) and \(\Omega = Z - z_1 z_2 Z^2\) are invariants of the map (39).

2. Map (39) can be restricted to a six-dimensional parametric tetrahedron map

\[
(x_1, X, y_1, Y, z_1, Z) \xrightarrow{T_{a,b,c}} (u_1, U, v_1, V, w_1, W),
\]

given by

\[
x_1 \mapsto u_1 = \frac{f_{b,c}(X, Y, Z) [f_{b,a}(X, Y, X) f_{b,c}(X, Y, Z) + f_{b,0}(X, Y, -1) x_1 z_1 X^2 Y Z]}{ac^2 x_1 y_1 z_1^2 X^2 Y^2 Z},
\]

(40a)
\[ X \mapsto U = \frac{acx_1y_1z_1X^2Y^2}{f_{b,a}(X,Y,X)f_{b,c}(X,Y,Z) + f_{b,0}(X,Y,-1)x_1z_1X^2YZ}, \quad (40b) \]
\[ y_1 \mapsto v_1 = \frac{x_1y_1z_1XYZf_{b,0}(X,Y,-1)}{f_{b,a}(X,Y,X)f_{b,c}(X,Y,Z) + f_{b,0}(X,Y,-1)x_1z_1X^2YZ}, \quad (40c) \]
\[ Y \mapsto V = \frac{acx_1y_1z_1XY}{f_{b,a}(X,Y,X)f_{b,c}(X,Y,Z) + f_{b,0}(X,Y,-1)x_1z_1X^2YZ}, \quad (40d) \]
\[ z_1 \mapsto w_1 = \frac{f_{b,a}(X,Y,X)}{ax_1XY}, \quad (40e) \]
\[ Z \mapsto W = \frac{acx_1y_1z_1XY^2Z}{f_{b,a}(X,Y,X)f_{b,c}(X,Y,Z) + f_{b,0}(X,Y,-1)x_1z_1X^2YZ}, \quad (40f) \]

where \( f_{b,c}(X,Y,Z) = bx_1z_1XZ + y_1Y(c - Z) \), on the invariant leaves
\[
\begin{align*}
A_a := \{(x_1, x_2, X) \in \mathbb{C}^3 : X - x_1x_2X^2 = a \}, \\
B_b := \{(y_1, y_2, Y) \in \mathbb{C}^3 : Y - y_1y_2Y^2 = b \}, \\
C_c := \{(z_1, z_2, Z) \in \mathbb{C}^3 : Z - z_1z_2Z^2 = c \}. 
\end{align*}
\]

3. Map \((40)\) admits the following invariants:
\[
I_1 = \frac{X}{Z}, \quad I_2 = XY, \quad I_3 = Z - bX - aYZ + \frac{y_1Y(a - X)(c - Z)}{x_1z_1XZ} - \frac{x_1z_1XZ(b - Y)}{y_1Y}. \quad (42)
\]

**Proof.** Concerning \(1\). The existence of these invariants is indicated by the existence of the first integral \((14)\). This can be verified by straightforward calculation.

With regards to \(2\), we set \(\Phi = a, \Psi = b\) and \(\Omega = c\). Now, using the conditions \(x_2 = \frac{x - a}{x_1X}, y_2 = \frac{y - b}{y_1Y}\) and \(z_2 = \frac{z - c}{z_1Z}\), we eliminate \(x_2, y_2\) and \(z_2\) from the nine-dimensional map \((39)\), and we obtain \((40)\). It can be verified by substitution that map \((40)\) satisfies the parametric tetrahedron equation. For the involutivity check, we have that \(w_1(u_1, U, v_1, V, w_1, W) \neq z_1\), thus \(T_{a,b,c} \circ T_{a,b,c} \neq id\), and the map is noninvolutive.

Finally, regarding \(3\), it can be readily proven that \(\frac{U}{W} \equiv \frac{X}{Z}, \quad UV \equiv XY\) and \(W - bU - aVW + \frac{1}{uv_1V(a-U)(c-W)} - \frac{w_1w_1UV(b-V)}{uv_1V} Z - bX - aYZ + \frac{y_1Y(a-x)(c-z)}{x_1z_1XZ} - \frac{x_1z_1XZ(b-y)}{y_1Y}\).

**Corollary 5.3.** Map \(40\) has the following Lax representation
\[
M_{M_{12}}(u_1, U; a)M_{M_{13}}(v_1, V; b)M_{23}(w_1, W; c) = M_{23}(z_1, Z; c)M_{13}(y_1, Y; b)M_{M_{12}}(x_1, X; a), \quad (43)
\]
where the associated matrix is given by
\[
M(x_1, X; a) = \begin{pmatrix}
X + a & x_1X \\
\frac{X - a}{x_1X} & 1
\end{pmatrix}. \quad (44)
\]

**Remark 5.4.** The invariants \((42)\) do not follow from the trace of the right-hand side of \((43)\). Matrix \(M\) in \((44)\) is not symmetric.

**Remark 5.5.** Although the birationality of map \((40)\) does not follow automatically from \((23)\), due to a symmetry break, solving equation \((43)\) for \((x_1, X, y_1, Y, z_1, Z)\), one can see that map \((40)\) is indeed birational.

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5.3 A novel ‘degenerated’ six-dimensional parametric tetrahedron map of DNLS type

Now, we make use of the second Darboux matrix of DNLS type, namely matrix (15) to derive another parametric tetrahedron map. Specifically, we change \((\lambda^2 f, \lambda^{-1}p) \rightarrow (x_1, x_2)\) in (45), and define the following matrix

\[
M(x_1, x_2; a) := \lambda^2 \left( \begin{array}{cc} x_1 \alpha & x_1 x_2 \\ x_1 x_2 & 0 \end{array} \right).
\]  

(45)

For matrix (45) we consider the following matrix trifactorisation problem:

\[
M_{12}(u_1, u_2, a)M_{13}(v_1, v_2, b)M_{23}(w_1, w_2, c) = M_{23}(z_1, z_2, c)M_{13}(y_1, y_2, b)M_{12}(x_1, x_2, a).
\]  

(46)

The above equation implies the system of polynomial equations

\[
\begin{align*}
\frac{u_1 v_1}{u_1 w_1 w_2} &= \frac{ax_1^2 x_2 y_2}{ay_2 w_1 w_2}, & v_1 &= \frac{x_1 y_1}{u_1}, & w_1 &= \frac{u_1 z_1}{x_1}, & w_2 &= \frac{y_2}{x_1 x_2} + \frac{bx_1 z_2}{ay_1}
\end{align*}
\]

(47)

which defines the following correspondence

\[
u_2 = \frac{ax_1^2 x_2 y_2}{ay_2 w_1 w_2}, \quad v_1 = \frac{x_1 y_1}{u_1}, \quad w_1 = \frac{u_1 z_1}{x_1}, \quad w_2 = \frac{y_2}{x_1 x_2} + \frac{bx_1 z_2}{ay_1}
\]

namely, \(u_2, v_1, v_2\) and \(w_1\) depend on \(u_1\).

As in the case of the NLS equation, this correspondence does not define a tetrahedron map for any choice of \(u_1 = y_1\). However, for the choice \(u_1 = y_1\) we have the following.

**Theorem 5.6.** The map defined by

\[
(x_1, x_2, y_1, y_2, z_1, z_2) \xrightarrow{T_{a,b,c}} (y_1, \frac{ax_1^2 x_2 y_2}{ay_1 y_2 + bx_1 z_2}, x_1, \frac{y_1 z_1}{x_1}, \frac{ay_1 y_2 + bx_1 z_2}{ax_1 x_2 y_1})
\]

(48)

is a six-dimensional parametric tetrahedron map, it is noninvolutive and birational. Additionally, map (48) admits the following invariants:

\[
I_1 = x_1 + y_1, \quad I_2 = y_1 z_1, \quad I_3 = \frac{x_1}{z_1}.
\]

(49)

**Proof.** By substitution of \(u_1 = y_1\) to (47), we obtain map (48). The tetrahedron property can be verified by substitution to the tetrahedron equation.

We have that \(T_{a,b,c} \circ T_{a,b,c} \neq \text{id}\), since, for instance, \(v_2(u_1, u_2, v_1, v_2, w_1, w_2) = \frac{by_2}{ac} \neq y_2\). Thus, the map is noninvolutive. Moreover, the inverse of map (48) is given by

\[
(u_1, u_2, v_1, v_2, w_1, w_2) \xrightarrow{T_{a,b,c}^{-1}} \left( v_1, \frac{u_1 w_1^2 w_2 + cv_1 v_2}{v_1 w_1 w_2}, u_1, u_2 w_1 w_2, \frac{v_1 w_1}{u_1}, \frac{ac u_1 v_2 w_2}{b c v_1 v_2 + b u_2 w_1 w_2} \right),
\]

i.e. (48) is birational.

Finally, we have

\[
u_1 + v_1 \overset{(48)}{=} x_1 + y_1, \quad v_1 w_1 \overset{(48)}{=} y_1 z_1, \quad \text{and} \quad \frac{u_1}{w_1} \overset{(48)}{=} \frac{x_1}{z_1},
\]

i.e. map (29) admits the invariants \(I_i, i = 1, 2, 3\), given by (49).
5.4 Restriction on the level sets

Now, we restrict the map (48) on the level sets of its invariants (49). Specifically, we have the following.

**Proposition 5.7.** Map (48) can be restricted to a three-dimensional noninvolutive parametric tetrahedron map given by (31).

**Proof.** Setting $I_1 = 1$, $I_2 = \frac{1}{2}$ and $I_3 = \frac{1}{2}$, where $I_i$, $i = 1, 2, 3$, are given by (49), we obtain $x_1 = y_1 = \frac{1}{2}$ and $z_1 = 1$. We substitute to (48), and we obtain a map $x_2 \mapsto u_2(x_2, y_2, z_2)$, $y_2 \mapsto v_2(x_2, y_2, z_2)$ and $z_2 \mapsto w_2(x_2, y_2, z_2)$. After relabelling $(x_2, y_2, z_2, u_2, v_2, w_2) \rightarrow (x, y/2, z, u/2, v, w)$, we obtain map (31). □

6 Conclusions

In this paper, we present novel solutions to the functional tetrahedron and parametric tetrahedron equation which are noninvolutive and birational. Noninvolutivity is an important property for a map, since involutive maps have trivial dynamics.

Our approach is based on the study of matrix trifactorisation problems for Darboux transformations, and we study the cases of Darboux transformations associated with the NLS and DNLS equation. Specifically, in the NLS case, we construct the tetrahedron map (19) which can be restricted to a parametric tetrahedron map (20). Additionally, we derive map (29) which can be restricted to map (31) on the level sets of its invariants. The latter at a certain limit gives map (20) in Sergeev’s classification [26]. In DNLS case, we construct the tetrahedron map (39) which can be restricted to a parametric tetrahedron map (40). Moreover, we construct map (48) which can also be restricted on the level sets of its invariants to map (31).

Our results could be extended in the following way:

1. Study the integrability of the derived maps;
2. Study the more general matrix trifactorisation problem (7) with a spectral parameter where matrices $L_{ij}$, $i, j = 1, 2, 3$, $i < j$, depend on a spectral parameter $\lambda$;
3. Find solutions to the entwining parametric tetrahedron equation;
4. Study the corresponding 3D-lattice equations;
5. Extend the results to the case of Grassmann algebras.

Regarding 1, the existence of invariants (22), (30), (42) and (49) for maps (20), (29), (40) and (48), respectively, is already a sign of integrability of these maps. In fact, in all these cases, the derived invariants are functionally independent and they are in number half as the dimension of the corresponding maps. In order to claim complete integrability in the Liouville sense, one needs to find a Poisson bracket with respect to which the invariants are in involution.

Concerning 2, although the Darboux transformations employed in this paper depend on a spectral parameter $\lambda$, in the consideration of the associated matrix refactorisation problem (7) the spectral parameter was rescaled; see matrices (16), (25), (36) and (45). In fact, if we let the latter matrices depend explicitly on the spectral parameter, then equation (7) has no solutions for $u$ and $v$. The question arises as to whether equation (7) defines tetrahedron maps for certain matrices $A, B, C$ and $D$ which explicitly depend on a spectral parameter $\lambda$. 

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With regards to 3, we mean the following equation

\[ Q_{123} \circ R_{145} \circ S_{246} \circ T_{356} = T_{356} \circ S_{246} \circ R_{145} \circ Q_{123}, \]

and its parametric version

\[ Q_{a,b,c}^{123} \circ R_{a,d,e}^{145} \circ S_{b,d,f}^{246} \circ T_{c,e,f}^{356} = T_{c,e,f}^{356} \circ S_{b,d,f}^{246} \circ R_{a,d,e}^{145} \circ Q_{a,b,c}^{123}. \]

For \( Q \equiv R \equiv S \equiv T \), we obtain equations (2) and (4), respectively. Similarly, to [19], we believe that solutions to these equations can be obtained by studying trifactorisation problems of Darboux transformations together with their degenerated versions. For \( Q \equiv R \equiv S \), a Hirota type entwining tetrahedron map appears in [10].

Regarding 4, it makes sense to study whether the maps derived in this paper are related to certain integrable 3D-lattice equations. Some of the available methods in the literature are the following: i. The relation between tetrahedron maps and 3D-lattice equations via symmetries of the latter which was established in [13]. ii. In [22, 23] it was demonstrated how to lift lattice equations to Yang–Baxter maps and these ideas can be employed to the case of tetrahedron maps. iii. Since the invariants of the maps [29, 32, 29, 48] are in separated form, one may employ the method presented in [11, 12].

Concerning 5, the extension of integrable lattice equations to the case of Grassmann algebras via Grassmann extended Darboux transformations in [7] motivated the initiation of the extension of the theory of Yang–Baxter maps to the noncommutative (Grassmann) case [6, 15, 18, 14]. The study of matrix trifactorisation problems [5] for Grassmann extended Darboux transformations may lead to solutions of the Grassmann extended tetrahedron and parametric tetrahedron equation. Fully noncommutative versions of tetrahedron maps are found in [4].

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