Lyapunov Based Analysis of Continuously Observed Quantum Systems

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Abstract
This paper presents a Lyapunov based controller to stabilize and manipulate an observed quantum system. The proposed control is applied to the stochastic Schrodinger equation. In order to ensure the stability of the system at the desired final state, the conventional Ito formula is further extended to the un-differentiable random processes. Using this extended Ito formula, a novel stochastic stability theorem is developed. Continued by another convergence theorem, which ensured the convergence of the state trajectory to the desired final state, a complete Lyapunov based controller design scheme is developed for the observed open quantum systems.

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1 introduction

In recent decades, quantum technology has proposed wide applications in engineering. Quantum information theory, quantum communication and teleportation and also quantum computing are the most recent areas influenced by quantum technology. In 1990s feedback control and measurement of quantum systems has been emerged in order to aid quantum technology in achieving their aims in design, analyze and controlling quantum based systems.

Several models have been developed to describe the dynamical behavior of a quantum mechanical system since their emergence. Closed quantum systems, i.e. quantum systems not interacting with external sources, can be modeled by quantum Schrodinger equation, Quantum Liouville equation, Feynman path integral technique and quantum stochastic differential equation. Also several research has been done on controlling and stability analysis of quantum systems in some of these models.

In quantum mechanics, measurement is a controversial issue. This phenomenon highly affects the dynamical behavior of the system and it is notable that many of quantum systems are forced to be influenced by measurement because of their nature. These facts motivated the authors to propose a novel method to control and stabilize these class of frequently appearing and useful stochastic quantum systems.

Observed quantum systems are one of the main classes of open quantum systems, i.e. the systems interacting with external sources. These systems are described by an extended form of Schrodinger wave equation known as Stochastic Schrodinger Equation (SSE).

Quantum measurement and control was founded by Belavkin in early 1980s and later in 1990s, optimal control and filtering techniques was developed for open quantum systems by him. Wiseman, Mabuchi, James, Milburn, Rabitz and Shapiro are the pioneers of quantum optical measurement and feedback theory and molecular process in 1990s. In 2000s many researchers were attracted to Lyapunov based control of quantum systems. Cong, Mirrahimi, Rouche and Banieh have utilized Lyapunov theory in order to control and stabilize the bilinear form of Schrodinger equation.

Control and Stabilization of these stochastic systems have not been studied despite their particular importance in applied physics. SSE is not a linear form despite the conventional Schrodinger equation. In this paper the general class of quantum mechanical systems under measurement, which are modeled by SSE, are considered. The proposed Lyapunov candidate in this paper is based on Hilbert-Schmidt distance of quantum states. This complex Lyapunov function is not differentiable on the domain of normalized quantum states. So the conventional Lyapunov stochastic stability theory is not capable to stabilize the SSE. To this end to definitions are proposed on the first and second order directional differentiability of complex functions. Furthermore the conventional Ito formula has been extended to directionally differentiable random processes. In order to analyze the stability of the SSE at the desired final state, the stochastically stable points are defined, afterward a stabilizer control is proposed. These definitions, continued by a novel proposed stochastically stability theorem, ensures the stochastically stability of the desired point under some conditions and using the proposed control. Alongside this novel stability theorem, another novel control theorem is presented which ensures that under a mild assumption on the initial condition, the system is driven stochastically to the desired final state.

This paper consists of the following sections: In II, Stochastic Schrodinger equation (SSE) is introduced and dynamical behavior of a quantum system under Hermitian observation is developed. In III, the quantum system under consideration is introduced and the control scheme is developed under some assumptions on the system and the desired final state. Also the SSE is converted to the standard stochastic state equation. In IV an introductory stochastically stability theorem is developed in several steps, first in IV.A the complex Lyapunov function is proposed, also the common features of this Lyapunov function are studied and introduced, continued by IV.B, definitions on
the directionally differentiable complex functions are presented and an extension for Ito formula is
developed. In IV.C the previous definitions and theorems are used in order to develop the main
stochastically stability theorem. In V the convergence of the systems dynamic is studied as a novel
theorem. The simulation results and their properness in the controlling of a real quantum system
(the applications of the proposed control in manipulating a quantum NOT-gate) is presented in VI.
One may find some useful Lemmas for the proofs of the theorems in VII. At the end the references
are presented in.

2 Preliminaries and Problem formulation

2.1 Quantum dynamical systems

According to the postulates of quantum mechanics, our state of knowledge about a finite dimensional
(n-level) quantum system can be described by a normalized vector $|\psi\rangle \in S^{2^{n-1}}$, where
$S^{2^{n-1}} = H$ is the unit hypersphere in $C^n$ and together with the Euclidean inner product $\langle . | . \rangle$, construct the
underlying separable Hilbert space. Also the time evolution of a closed quantum system is governed
by the Schrodinger wave equation:

$$\frac{d}{dt} |\psi(t)\rangle = -i\hbar H |\psi(t)\rangle, \quad |\psi(0)\rangle = |\psi_0\rangle$$  \hspace{1cm} (1)

where $H (iH \in su(n))$ is a bounded (and equivalently compact) self-adjoint operator, called system
Hamiltonian. The corresponding density operator $\rho = |\psi\rangle \langle \psi|$ is always a pure (rank-one) quantum
state.

The relative phase invariance in quantum mechanics has led to some difficulties in modeling
quantum control systems. By this invariance, all the states $e^\theta |\psi\rangle$ with $\theta \in \mathbb{R}$ correspond to a same
quantum state. In order to conquer this difficulty, the following definition is developed:

D 1. $\{ |\psi\rangle \} = \{ e^\theta |\psi\rangle | \theta \in \mathbb{R} \}$ is the equivalence class of $|\psi\rangle$ and two different quantum states $|\psi_1\rangle$
and $|\psi_2\rangle$ are said to be equivalent if $|\psi_1\rangle \in \{ |\psi_2\rangle \}$.

This equivalence partitioning of quantum states will help us in the rest of this article to model
the stabilization process. Obviously, the quotient space corresponding to this partitioning is infinite
dimensional even if we consider a finite dimensional quantum system.

The following lemma will be numerously employed in the rest of this article:

Lemma 1. Consider two non-equivalent quantum states, i.e. $|\psi_1\rangle \notin \{ |\psi_2\rangle \}$, then there exists $0 < \epsilon$
and self adjoint $H' (\|H'\|_{HS} = 1, H' \neq I)$ such that:

$$|\psi_1\rangle = e^{i\epsilon H'} |\psi_2\rangle$$  \hspace{1cm} (2)

Also, neither of $|\psi_1\rangle$ and $|\psi_2\rangle$ are eigenkets of $H'$.

Proof. (Here we just sketch the proof) The transitivity property of $SU(n)$ as a Lie transformation
group besides its Homeomorphism to $su(n)$ by the exponential map provides the main idea for
the proof. Also both conditions $H' = I$ and $|\psi_1\rangle$ or $|\psi_2\rangle$ beeing an eigenket for $H'$ contradicts the
non-equivalence property.

\footnote{$\| . \|_{HS}$ denotes the Hilbert-Schmidt norm}
2.2 Continuously observed quantum systems

Observation in quantum systems is a very controversial issue. By the early postulates of quantum mechanics, the observation process projects the system into the eigenspace corresponding to the resulting eigenket of the observable. This viewpoint is known as projective measurement or also strong Von Neumann measurement. By this postulate, the premeasurement quantum state is completely missed. Developing POVM led to a more general model for quantum observation, known as weak measurement\[?]\,\[?\]. By weak measurement, the premeasured state, does not necessarily collapse to the eigenstates of the observable but deviates from its initial state. This deviation depends on the strength of the measurement and the distribution spread of the Gaussian measurement . On the other hand, the obtained information is unsharped. This phenomena obeys the Busch’s theorem which states: "no information without disturbance"\[?\].

A continuously observed quantum system is the one being continuously weakly measured. Denote the measurement observable by $X$ (which is necessarily self-adjoint). In\[?\], a Gaussian approach to weak measurement is followed. In this approach, the deviation of the premeasurement state depends on the strength of the measurement and the energy shifts between the eigenkets of $X$ weighted by the projection of premeasurement state into the eigenbasis generated by $X$. In the case of continuous measurement, this approach leads to the following stochastic dynamical behaviour, known as Stochastic Schrodinger Equation (SSE):

$$d|\psi\rangle = \left(\left(\begin{array}{c} -\frac{k}{\hbar}(X - \langle X \rangle)^2 \end{array}\right)dt + \sqrt{2k}(X - \langle X \rangle)dW \right)|\psi\rangle$$

(3)

where $k$ is the strength of the measurement, $dW$ is the standard Wiener process and $\langle X \rangle$ is the expected value of the observable $X$. Also it is shown that the contribution of the free evolution term in $\mathbf{1}$ add together with $\mathbf{2}$, so the complete evolution is given by:

$$d|\psi\rangle = \left(\left(\begin{array}{c} -i\hbar H - k(X - \langle X \rangle)^2 \end{array}\right)dt + \sqrt{2k}(X - \langle X \rangle)dW \right)|\psi\rangle$$

(4)

$$|\psi(0)\rangle = |\psi_0\rangle; \text{ a.s.}$$

This stochastic unitary evolution is a more general form of strong measurement. One may deduce from $\mathbf{4}$ that in the case of great measurement strength, the quantum system is projected to the eigenkets of $X$ in a short time. For convenience let us use the following notation:

$$\hat{f}(|\psi\rangle) \doteq \left(\begin{array}{c} -i\hbar H - k(X - \langle X \rangle)^2 \end{array}\right)|\psi\rangle, \hat{g}(|\psi\rangle) \doteq \left(\sqrt{2k}(X - \langle X \rangle)\right)|\psi\rangle$$

(5)

where $\hat{f}$ and $\hat{g}$ are the drift and diffusion terms respectively. Obviously $\hat{f}$ and $\hat{g}$ admit the lipschits continuity and growth condition, thus the existances and uniqueness of the strong solution of $\mathbf{4}$ and the quantum trajectories are guaranteed.

2.3 Problem formulation

Dynamical behavior of an open quantum system with Hamiltonian $H$ which is continuously measured by $X$ was studied in $\mathbf{2.2}$. In order to address the control issue, consider the following assumptions:

A 1. Consider the set of control signals $U \doteq \{u_k(t) \mid k = 1, ..., m; u_k \in L^2(\mathbb{R})\}$. Each of the Lebesgue measurable control signals $u_k(t)$ is associated with a compact control Hamiltonian $H_k$ ($iH_k \in su(n)$) as its coefficient and they appear in the Hamiltonian in an affine manner:

$$H(U) = H_0 + \sum_k u_k(t)H_k.$$  

(6)
The free evolution Hamiltonian $H(0) = H_0$ describes the system in the absence of control field.

The previous assumption is not restrictive. In application, the manipulation signals usually appear in affine form, e.g. the effect of interacting magnetic field on a spin system. Although the observable $X$ participates in the manipulation, in this article, we just address the role of control signals in the stabilization procedure. In the rest of this article, our aim is to analyze the stability properties of the quantum trajectories, driven by the nonlinear stochastic differential equation. The quantum trajectories start from initial state $|\psi_0\rangle$ almost surely and the aim is to manipulate them to the equivalent class of the desired final state $|[\psi_f]\rangle$. For convenience, the point spectrum of $H_0$ is denoted by $\sigma(H_0) \doteq \{\lambda_i\}_i^2$ and the eigenspace conjugate to $\lambda_i$ is denoted by $\sigma_{\lambda_i}(H_0)$. The following assumptions will be employed for further results:

A 2. The desired final state $|\psi_f\rangle$ (and also $|[\psi_f]\rangle$) is an eigenstate of $H_0$ with eigenvalue $\lambda_{Hf}$ (i.e. $|[\psi_f]\rangle \in \sigma_{\lambda_{Hf}}(H_0)$) and the corresponding eigenspace is degenerate.

A 3. The desired final state $|\psi_f\rangle$ is not an eigenstate of $H_k$ for at least one $k$ i.e. $\exists k; \langle[\psi_f]\rangle \notin \bigcup_j \sigma_{\lambda_j}(H_k)$.

In the upcoming sections, it will be shown that the control Hamiltonians containing $|\psi_f\rangle$ as their eigenstate, do not contribute in the manipulation process.

A 4. The desired final state $|\psi_f\rangle$ is an eigenstate of $X$ with eigenvalue $\lambda_{Xf}$. i.e. $|[\psi_f]\rangle \in \sigma_{\lambda_{Xf}}(X)$.

A 5. The control Hamiltonian set $\{H_k\}$ has at least $n-1$ elements, any of which doesn't have $|\psi_f\rangle$ as its eigenket and the Hamiltonians $\{H_0, H_1, ..., H_{n-1}\}$ constitute an $n$ element linearly independent set.

In the rest of this article, each of these assumptions will help to derive the desired results.

In the next section, the stabilization process and the stochastic boundedness analysis are studied for the formulated problem by the use of extended Lyapunov theory.

### 3 Extended stochastic Lyapunov theory

The process of designing control signals $U$ to achieve the desired goals, is inspired from stochastic Lyapunov theory. This theory is highly dependent on the Ito formula, which gives the increment of the Lyapunov function according to a continuous Feller (and thus, strong Markov) stochastic process. In the problem of manipulating a quantum system, we are faced to a difficulty in using the conventional Ito formula. In this section, it is shown that the proposed Lyapunov function is not differentiable when we have employed the complex numbers as the field for our Hilbert space.

#### 3.1 Lyapunov function

In this article, a Lyapunov function is employed which is based on maximizing the transition probability to the desired final state:

$$V(|\psi\rangle) = \frac{1}{2} \left(1 - |\langle \psi_f | \psi \rangle|^2\right)$$

$^2$The compactness of $H_0$ and the separability of the underlying Hilbert space imply that the continuous and residual spectrum of $H_0$ are empty.

$^3$All of the equivalent states of $|\psi_f\rangle$ are in the same eigenspace of $H_0$ conjugate to $\lambda_{Hf}$.
This Lyapunov function (which is obviously positive) is inspired from the Hilbert-Schmidt norm of an operator. In what follows, the benefitting properties of this Lyapunov candidate will be apparent. First, the mathematical properties of \( \mathcal{V} \) is discussed within some lemmas:

**Lemma 2.** Consider the Lyapunov function in \( \mathcal{V} \) the following statements are equivalent:

(a) \( \mathcal{V}(|\psi\rangle) = 0 \)

(b) \( |\psi\rangle = [|\psi_f\rangle] \)

*Proof.* \((b) \implies (a)\) is obvious. For \((a) \implies (b)\), the sufficient condition for \((a)\) is \(|\langle \psi_f | \psi \rangle|^2 = 1\). Using the fact that both \(|\psi\rangle\) and \(|\psi_f\rangle\) are normalized, by CBS inequality, the necessary and sufficient condition is \(|\psi\rangle = c |\psi_f\rangle\) with \(|c| = 1\) which means \((b)\) by D1. \[\square\]

This lemma plays an important role in guaranteeing asymptotic stability properties in the rest of this article. By this lemma, vanishing Lyapunov function, exclusively describes the convergence of the quantum trajectories to the equivalence class of the desired final state. The following lemma will play an important role in asymptotic stability of the quantum trajectories in upcoming sections.

**Lemma 3.** Consider the Lyapunov function in \( \mathcal{V} \), suppose that \(|\psi\rangle \notin [|\psi_f\rangle]\) and \(0 < R \leq |||\psi\rangle - [|\psi_f\rangle]|| < 2\). then \( \mathcal{V}(|\psi\rangle) \) is bounded away from zero i.e. there exists \(0 < \nu(R) \leq \mathcal{V}(|\psi\rangle)\).

*Proof.* The assumption reads \(R^2 \leq \langle \psi - e^{-i\epsilon} | \psi_f \rangle \psi - e^{i\epsilon} | \psi_f \rangle \rangle \) for all real \(\epsilon\). So:

\[
R^2 \leq 2 - 2 \text{Re}(e^{-i\epsilon} \langle \psi_f | \psi \rangle).
\]

Take \(\langle \psi_f | \psi \rangle = re^{i\theta}\) where \(r \) and \(\theta\) depend on \(|\psi\rangle\), and choose \(\epsilon = -\theta\). So, for all admissible \(|\psi\rangle\) one may write:

\[
r \leq 1 - \frac{R^2}{2}.
\]

The above inequality reads:

\[
\sup_{\text{admissible } |\psi\rangle} |\langle \psi_f | \psi \rangle| \leq 1 - \frac{R^2}{2}
\]

Also it can be shown that the supremum takes the RHS value on the boundry of admissible closed set of \(|\psi\rangle\). Thus, one may define \(\nu(R) \equiv R^2 - \frac{R^2}{4}\). \[\square\]

### 3.2 Extended Ito formula

The common well-known Ito formula, gives the increment of a scalar function, of a strong Markov process driven by Ito form of stochastic differential equations. It is necessary that the function be twice-differentiable. When we treat a complex-field Hilbert space, our definition on the differentiability changes (The complex function must necessarily admit Cauchy-Reimann condition). The sufficient condition for extending the common Ito formalism to the complex case, is the holomorphy of the complex function, which is highly restrictive. The preposed Lyapunov function is neither holomorphic, nor even differentiable it its domain (ofcourse employing a holomorphic Lyapunov function is very restrictive and does not necessarily admit our conditions). In this section, the common Ito formula is extended to undifferentiable complex functions. In order to proceed, the following definitions (which are induced from Gateaux differentiation) are presented:

1. \(|\psi\rangle\) is not in an open \(R\)-neighbourhood of the set \([|\psi_f\rangle]\).
2. This terminology is used in place of "Function of a complex variable" in this article.
D 2. Consider the complex (not necessarily differentiable) function \( V(\psi) : S^{2n-1} \mapsto \mathbb{R} \). If:

(i) there exists a functional \( \nabla V(\psi) : \mathbb{C}^n \mapsto \mathbb{C} \), independent of \( \partial \psi \) such that the following limit exists for any fixed \( \partial \psi \in \mathbb{C}^n \):

\[
\lim_{h \downarrow 0} \frac{V(\psi + h \partial \psi) - V(\psi) - h (\nabla V(\psi)) \partial \psi}{h} = 0
\]  

then, \( \nabla V(\psi) \) is said the directional gradient in the direction \( \partial \psi \).

(ii) (if \( \nabla V(\psi) \) exist) there exists an operator \( \nabla^2 V(\psi) : \mathbb{C}^n \mapsto \mathbb{C}^{n \times n} \), independent of \( \partial \psi \) such that the following limit exists:

\[
\lim_{h \downarrow 0} \frac{V(\psi + h \partial \psi) - V(\psi) - h (\nabla V(\psi)) \partial \psi - \frac{h^2}{2} \langle \partial \psi | (\nabla^2 V(\psi)) \partial \psi \rangle}{h^2} = 0
\]  

then, \( \nabla^2 V(\psi) \) is said the second order directional gradient in the direction \( \partial \psi \).

The previous definition on the directional gradients, opens up a great ability for us to analyze the functional properties of \( V \). Using the limit’s definition and \( \nabla V \) will show us some benefitial properties in approximating the Lyapunov function in some neighbourhood the \( |\psi \rangle \). But in the case of \( V \) some exclusive properties will appear. In the next proposition, it is shown that first and second order gradients will help us to exactly evaluate the perturbed Lyapunov function, first the directional gradients for \( V \) are evaluated. One may easily check that the following forms are the directional gradients for \( V \):

\[
\nabla V(\psi) = -\text{Re}(\langle \psi | \psi_f \rangle (\psi_f))
\]

\[
\nabla^2 V(\psi) = -\langle \psi_f | \psi_f \rangle
\]

where \( \text{Re}(\langle , | ) = \frac{1}{2} (\langle , | + \langle | , ) \) when acts on \( |\psi \rangle \).

**Proposition 1.** For the proposed Lyapunov function \( V \), define \( d_{\partial \psi} V(\psi) = V(\psi) + \langle \partial \psi | V(\psi) \rangle \) for sufficiently small \( |\partial \psi \rangle \), then with the directional gradients in \( \nabla V \) the following equality holds:

\[
d_{\partial \psi} V(\psi) = \langle \nabla V(\psi) | \partial \psi \rangle + \frac{1}{2} \langle \partial \psi | (\nabla^2 V(\psi)) \partial \psi \rangle
\]

**Proof.** The proof is straightforward.

This proposition presents the directional counterpart of the Taylor seris for an undiffentiable real valued complex function (this is why the directional gradients were defined. Although this Lyapunov function is not analytic, a Taylor-like serie is presented). Also this proposition shows why this Lyapunov candidate suits the problem of manipulating the SSE. The perturbed Lyapunov value is exactly evaluated by two perturbation steps.

Based on the presented definitions, in the next theorem the extended version of Ito formula for undiffentiable complex functions of a strong Markov process is presented. This extended version of Ito formula is the starting point to use the stochastic Lyapunov theory.

**Theorem 1.** Let \( V(\psi) : S^{2n-1} \mapsto \mathbb{R} \) be a complex function with well-defined directional gradients in \( \nabla V \). Also assume that \( \nabla^2 V \) holds and \( |\psi \rangle \) be an adapted stochastic process driven by the following Ito drift-diffusion stochastic differential equation:

\[
d |\psi \rangle = f(|\psi \rangle)dt + g(|\psi \rangle)dW.
\]
Then, the following equality holds for the stochastic increment of \( V \):

\[
d_{\delta(V)} V(|\psi\rangle) = \left( \nabla V(|\psi\rangle) f(|\psi\rangle) + \frac{1}{2} \frac{d}{d^2} V(|\psi\rangle) (g(|\psi\rangle)) \right) dt + (\nabla V(|\psi\rangle) g(|\psi\rangle)) dW
\]  

(13)

**Proof.** Substituting [12] into [11] in the place of \( |\partial \psi\rangle \) gives:

\[
d_{\delta(V)} V(|\psi\rangle) = \nabla V(|\psi\rangle) f(|\psi\rangle) dt + \nabla V(|\psi\rangle) g(|\psi\rangle) dW + \left( \frac{1}{2} \frac{d}{d^2} V(|\psi\rangle) (f(|\psi\rangle)) (dt)^2 + (g(|\psi\rangle)) \frac{d}{d^2} V(|\psi\rangle) (g(|\psi\rangle)) (dW)^2 + (f(|\psi\rangle)) \frac{d}{d^2} V(|\psi\rangle) (f(|\psi\rangle)) (dt) dW \right)
\]

(14)

Keeping terms up to \((dW)^2\) and integrating in the sense of Ito, gives:

\[
V(|\psi(t)\rangle) = \int_0^t \nabla V(|\psi\rangle) f(|\psi\rangle) dt + \int_0^t \nabla V(|\psi\rangle) g(|\psi\rangle) dW + \frac{1}{2} \int_0^t (g(|\psi\rangle)) \frac{d}{d^2} V(|\psi\rangle) (g(|\psi\rangle)) (dW)^2
\]

(15)

Using the Ito multiplication rule for the third integral gives[6]:

\[
V(|\psi(t)\rangle) = \int_0^t \left( \nabla V(|\psi\rangle) f(|\psi\rangle) + \frac{1}{2} (g(|\psi\rangle)) \frac{d}{d^2} V(|\psi\rangle) (g(|\psi\rangle)) \right) dt + \int_0^t \nabla V(|\psi\rangle) g(|\psi\rangle) dW.
\]

(16)

Therefore the Ito form of stochastic differential equation for \( V(|\psi\rangle) \) is:

\[
d_{\delta(V)} V(|\psi\rangle) = \left( \nabla V(|\psi\rangle) f(|\psi\rangle) + \frac{1}{2} (g(|\psi\rangle)) \frac{d}{d^2} V(|\psi\rangle) (g(|\psi\rangle)) \right) dt + (\nabla V(|\psi\rangle) g(|\psi\rangle)) dW.
\]

(17)

In the rest of this paper the coefficient of \( dt \) will be denoted by \( L_{\delta(V)} V(|\psi\rangle) \) or \( LV(|\psi\rangle) \).

**Remark 1.** The previous theorem prescribes the stochastic increment of \( V(|\psi\rangle) \). This increment is evaluated by keeping terms up to \( O(dt) \) which is acceptable since \( dt \to 0 \). Also, for the complex functions with directional gradients not satisfying [11] one can derive in the second order approximation and is applicable as \( dt \to 0 \).

For the considering Lyapunov function [4] with directional gradients in [11], one may deduce that the stochastic increment of \( V(|\psi\rangle) \) regardin to the SSE defined by [4] and driven by control field in [4] has the following form:

\[
d_{\delta(V)} V(|\psi\rangle) = \left( -\frac{1}{\hbar} \text{Im} (\langle \psi | \psi \rangle (\psi | H(U) | \psi \rangle) + k \Re \left( \langle \psi | \psi \rangle \langle \psi | (X - \langle X \rangle)^2 | \psi \rangle \right) \right) dt + \left( -k (|\psi\rangle)^2 X - \langle X \rangle | \psi \rangle \right) dW.
\]

(18)

Based on the proposed mathematical background, in the next sections, main results on the stability of quantum trajectories are presented.
4 Main results on the stability of the SSE

4.1 Definitions

The stochastic properties, studied in this article are constructed on the complete probability space \((\Omega, \Xi, P)\), where \(\Xi\) is the \(\sigma\)-algebra generated by \(\Omega\) and \(P\) is a probability measure defined on \(\Xi\). So let us denote a quantum stochastic process, which is the solution of SSE (4) by \(|\psi(\omega, t)\rangle\) : \((\Omega \times T, \Xi \otimes B(T)) \rightarrow (\mathcal{H}, B(\mathcal{H}))\) and a quantum trajectory (a sample path of \(|\psi(\omega, t)\rangle\)) started from \(|\psi_0\rangle\) at \(t = 0\) by \(|\psi_0(t)\rangle\) : \(T \rightarrow \mathcal{H}\) where \(T \equiv [0, \infty)\) is the time index and \(B(.)\) denotes the Borel \(\sigma\)-algebra generated by the corresponding set.

Lemma 2 provides that \(\text{supp}(V|\psi\rangle)\) is compact in \(\mathbb{C}^n\), thus \(LV(|\psi\rangle)\) is the infinitesimal generator of \(V(|\psi\rangle)\) i.e.

\[
LV\{|\psi^{(\psi_0)}(t)\rangle\} = \lim_{h \downarrow 0} \frac{E[V(|\psi(\omega, t + h))\rangle] - V\{|\psi^{(\psi_0)}(t)\rangle\}}{h}.
\]

(19)

Regarding to previous sections, we propose two definitions on the stability of quantum trajectories.

D 3. The quantum stochastic process \(|\psi(\omega, t)\rangle \equiv |\psi_f\rangle\) driven by 4 is said to be:

(i) **stochastically stable** if for any \(0 < \epsilon\):

\[
\lim_{\|\partial\psi\| \to 0} P \left\{ \sup_{0 \leq t \leq T} \| |\psi(\psi_f) + |\partial\psi\rangle\rangle - |\psi_f\rangle\rangle \| \geq \epsilon \right\} = 0
\]

(20)

(ii) **stochastically asymptotically stable** if it is stochastically stable and also:

\[
\lim_{\|\partial\psi\| \to 0} P \left\{ \sup_{0 \leq t \leq T} \| |\psi(\psi_f) + |\partial\psi\rangle\rangle - |\psi_f\rangle\rangle \| = 0 \right\} = 1.
\]

(21)

The previous definitions present two notions of stability for the considered quantum trajectories. An stable quantum trajectory, remains in a neighbourhood of the equivalence class of the desired final state almost surely as the initial value tends to that equivalence class. The second notion guarantees that the quantum trajectory does not escape the equivalence class almost surely.

4.2 Stochastic stability of SSE

In order to study the stability of quantum trajectories, the conditions \(\hat{f}(|\psi_f\rangle) = \hat{g}(|\psi_f\rangle) = 0\) must necessarily hold. These conditions are satisfied if \(A2\) and \(A4\) are satisfied in addition to the condition that control signals \(U\) vanish at \(|\psi_f\rangle\). Thus \(A2\) and \(A4\) are the basic necessary assumptions in the rest of this section. Also one may deduce that in the shadow of \(A2\) and \(A4\) the infinitesimal generator \(LV(|\psi\rangle)\) in (18) takes the following form:

\[
LV(|\psi\rangle) = -\frac{1}{\hbar} \sum_{k=1}^{n} u_k(t) \text{Im} \left( \langle \psi | \psi_f \rangle \langle \psi_f | H_k | \psi \rangle \right).
\]

(22)

Now the necessity of \(A4\) is more obvious. In the absence of \(A4\) the SSE (4) would be uncontrollable since all the coefficients of control signals in (22) would vanish everywhere in \(S^{2n-1}\). In the rest of this paper, the following choice for the control signals will be employed:

\[
u_k(t) = \alpha_k \text{Im} \left( e^{i\xi \langle \psi | \psi_f \rangle} \langle \psi_f | H_k | \psi \rangle \right)
\]

(23)
where $\alpha_k \in \mathbb{R}^+$. This control signal has also been employed in stabilizing the deterministic Schrodinger equation [2]. Substituting the proposed control signals yield to:

$$LV(|\psi|) = -\frac{1}{\hbar} \sum_{k=1}^{n} \alpha_k \langle |\psi | \psi_f \rangle \left( \lim_{t \to \infty} e^{iE(t)} \langle \psi_f | H_k | \psi \rangle \right)^2 \leq 0. \quad (24)$$

Now, the following lemma can be stated:

**Lemma 4.** Consider the Lyapunov function $4$ where the quantum dynamic is driven by $2$ with control signals $27$. Then the process $V(|\psi|)$ is a supermartingale. Also $\lim_{t \to \infty} E[V(|\psi|)]$ exists and is equal to $E[V(\lim_{t \to \infty} |\psi|^2(\omega, t))]$.

**Proof.** Assume that $R > 2$ and define $N_R \doteq \{ |\psi| \ | |\psi| - |\psi_f| | \leq R \}$. Assume that $|\psi_0| = |\psi_f| + |\partial \psi| \in N_R$ almost surely. If $\tau_{N_R} \doteq \inf \{ t \ | \langle |\psi|^2(\omega) \rangle \not\in N_R \}$ be the first exit time from $N_R$, by Lemma 4 one may deduce:

$$E[V(|\psi| |\omega, \tau_{N_R} \wedge t|)] \leq E[V(|\psi|)] \quad (27)$$

which expresses that the stopped process $V(|\psi| |\omega, \tau_{N_R} \wedge t|)$ is also a supermartingale. Based on these derivations, the following theorem is presented which plays an important role in stabilization procedure and shows the stochastic stability of $|\psi_f|$.

**Theorem 2.** Consider the SSE $3$ with assumptions discussed in $27$. Assume that $1$ to $3$ hold. With control signals $27$, the quantum stochastic process $|\psi^2(\omega, t)\rangle \equiv |\psi_f\rangle$ is stochastically stable.

**Proof.** Assume that $|\psi_0| = |\psi_f| + |\partial \psi| \in N_R$ for some $0 < R < 2$. By Lemma 4 and $27$, one writes:

$$E[\sup_{0<t} V \left( |\psi| (\tau_{N_R} \wedge t) \right)] \leq V(|\psi_0|) \quad (28)$$

Now define $y(u) \doteq \sup_{0<u} ||\psi^2(\tau_{N_R} \wedge t) - |\psi_f| ||$ and $\Omega' \doteq \{ \omega | R \leq y(\omega) \}$, $28$ can be rewritten as:

$$V(|\psi_0|) \geq \int_{\Omega'} \sup_{0<t} V \left( |\psi|^2(\tau_{N_R} \wedge t) \right) dP(\omega) \geq \int_{\Omega'} \sup_{0<t} V \left( |\psi| (\tau_{N_R} \wedge t) + |\partial \psi| \right) dP(\omega) \geq \left( \inf_{y \leq R} \sup_{0<t} V \left( \langle |\psi|^2(\tau_{N_R} \wedge t) + |\partial \psi| \rangle \right) \right) \cdot P \{ y > R \} \quad (29)$$
By Lemma 3 one may deduce:

\[
P \left\{ \sup_t \left\| \psi_r^{(\psi_f) + |\psi_f\rangle} (t) - |\psi_f\rangle \right\| > R \right\} \leq \frac{V(|\psi_0\rangle)}{\nu(R)}. \tag{30}\]

Now, using Lemma 2 and the continuity of \(V(|\psi\rangle)\) gives:

\[
\lim_{|||\psi\rangle|| \to 0} P \left\{ \sup_t \left\| \psi_r^{(\psi_f) + |\psi_f\rangle} (t) - |\psi_f\rangle \right\| > R \right\} \leq \lim_{|||\psi\rangle|| \to 0} \frac{V(|\psi_0\rangle)}{\nu(R)} = 0 \tag{31}\]

Theorem 2 reveals that the desired final state \(|\psi_f\rangle\) is stochastically stable and the trajectories remain in any prescribed neighbourhood of \(|\psi_f\rangle\) with probability 1. Also \(|\psi_0\rangle \in |\psi_f\rangle\) is a special case of this result.

In the rest of this section, we will show that under some conditions on the control Hamiltonians in 6 the stochastically asymptotically stability can be achieved.

### 4.3 Stochastic Asymptotic stability of SSE

Theorem 2 represented a stability condition the quantum trajectories based on 24. In this sense, the system would evolve until reaching its invariant set which is itself a subset of \(\{ |\psi\rangle | LV(|\psi\rangle) = 0 \} \). In order to characterize this set, let us denote the set of eigenvalues of control Hamiltonians by \(\sigma(H) = \bigcup_k \sigma(H_k)\). Also based on the Cartan decomposition of \(su(n)\), one can always find a basis in which \(H_0\) is diagonal. Denote this basis by \(\{ |1\rangle, |2\rangle, ..., |n\rangle \} \) where the bases are mutually orthogonal. Without loss in generality, assume that the eigenspace corresponding to \(|1\rangle\) is degenerate and \(|\psi_f\rangle = |1\rangle\). Now by the use of 24 the following theorem can be stated:

**Theorem 3.** Consider the state dynamic 3 and the Lyapunov function 7 also assume that \(A\) to \(A\) hold. The set of quantum states in which \(LV(|\psi\rangle) = 0\) can be decomposed into the following two subsets (i.e. \(\{ |\psi\rangle | LV(|\psi\rangle) = 0 \} = A \cup B\):

- \(A = \{ |\psi_f\rangle \}^\perp\)
- \(B = \{ |\psi\rangle \forall k, \exists \lambda_k \in \mathbb{R} : \langle \psi_f | H_k - \lambda_k I | \psi \rangle = 0 \}

Also, the following statements hold:

1. In the case that the control Hamiltonians \(\{ H_k \}\) have no common eigenkets, \(B\) includes at most one equivalence class of states for each choice of \(\{ \lambda_k \in \mathbb{R}, k = 1, ..., n - 1 \}\).
2. In the case that the control Hamiltonians \(\{ H_k \}\) have \(s\) independent common eigenkets (each of them is an eigenket for at least 2 of the Hamiltonians) then \(B\) includes at most \(1 + s\) different equivalence classes of quantum states.
3. \(|\psi_f\rangle\) \(\in B\) for some choice of \(\{ \lambda_k \in \mathbb{R} - \sigma(H) \}\).

**Proof.** The obvious. In this proof, first we neglect the unitarity of the quantum states, and after finding the un-normalized solution subspace, it will be intersected with the unit circle. Partition \(\mathbb{R}\) as \(\mathbb{R} = (\mathbb{R} - \sigma(H_k)) \cup \sigma(H_k)\). If \(|\psi_0\rangle \notin |\psi_f\rangle^\perp\), 24 implies that we should search for the common solutions of \(\text{Im} \left( e^{i\mathcal{L}(|\psi_f\rangle}) \langle \psi_f | H_k | \psi \rangle \right) = 0\) for all \(k\), but:

\[
\text{Im} \left( e^{i\mathcal{L}(|\psi_f\rangle}) \langle \psi_f | H_k | \psi \rangle \right) = 0 \Leftrightarrow \langle \psi_f | H_k | \psi \rangle = \lambda_k \langle \psi_f | | \psi \rangle \Leftrightarrow \langle \psi_f | H_k - \lambda_k I | \psi \rangle = 0 \tag{32}\]
for real $\lambda_k$'s.

Let us first prove $B^{[1]}$. Assume that $\lambda_k \in \mathbb{R} - \sigma(H_k)$ for all $k$. Thus $H_k - \lambda_k I$ is nonsingular. So, we may characterize the subspace in which the solutions of $(32)$ belong for each $k$ as:

\[
S_k(\lambda_k) \doteq \text{span} \left\{ (H_k - \lambda_k I)^{-1} |2\rangle, ..., (H_k - \lambda_k I)^{-1} |n\rangle \right\}
\]  

(33)

which is an $n - 1$ dimensional subspace regarding to linearly independance of $|j\rangle$'s. Thus, the solution of $(32)$ must necessarily belong to the intersection of $S_k(\lambda_k)$'s for each choice of $\{\lambda_k \in \mathbb{R} - \sigma(H_k)\}$:

\[
|\psi\rangle \in \bigcap_k S_k(\lambda_k).
\]  

(34)

By $A^5$ and Lemma $5$, we may deduce that none of the subspaces $S_k(\lambda_k)$ can exactly coincide. For further demonstrations, one may show that $\sum_{j=2}^{n} \langle j | (H_t - \lambda_t I)(H_u - \lambda_u I)^{-1} |1\rangle |j\rangle$ belongs to $S_t(\lambda_t)$ but not $S_u(\lambda_u)$ for each distinct $u$ and $t \in \{1, 2, ..., n - 1\}$. Also for each distinct $s, u$, and $t$, $\sum_{j=2}^{n} \langle j | (H_t - \lambda_t I)(H_u - \lambda_u I)^{-1} |1\rangle |j\rangle$ and $\sum_{j=2}^{n} \langle j | (H_s - \lambda_s I)(H_u - \lambda_u I)^{-1} |1\rangle |j\rangle$ can not be collinear (based on Lemma $5$). The intersection of $n - 1$ non-coincident $n - 1$ dimensional subspaces is no more than $1$-dimensional. Now, intersecting the $1$-dimensional solution subspace with the unit sphere implies that $B$ includes at most one equivalence class of quantum states for each choice of $\{\lambda_k \in \mathbb{R} - \sigma(H_k)\}$.

Based on this proof, choosing $\lambda_k = \langle \psi_f | H_k | \psi_f \rangle$ results $B^{[3]}$.

Now assume that $\lambda_k \in \sigma(H_k)$ for some $k$'s but not all of them. In this case, the solution space for $\{S_k(\lambda_k)\}$ are defined in a more general manner in order to include singular $(H_k - \lambda_k I)$'s. First define the nonhomogenous part of $S_k(\lambda_k)$ as:

\[
\text{nh}S_k(\lambda_k) \doteq \{ |\psi\rangle |(H_k - \lambda_k I) |\psi\rangle = |2\rangle \} \cup \{ |\psi\rangle |(H_k - \lambda_k I) |\psi\rangle = |3\rangle \} \cup ... \cup \{ |\psi\rangle |(H_k - \lambda_k I) |\psi\rangle = |n\rangle \}
\]  

(35)

which includes at most $n - 1 - \text{deg}(\lambda_k, H_k)$ independent vectors (deg$(\lambda_k, H_k)$ is the degeneracy of $\lambda_k$ for $H_k$). Also define the homogenous part $\text{h}S_k(\lambda_k)$ to be the kernel of $(H_k - \lambda_k I)$. Now the solution space can be defined as:

\[
S_k(\lambda_k) \doteq \text{span} \{ \text{h}S_k(\lambda_k), \text{nh}S_k(\lambda_k) \}
\]  

(36)

which is at most $n - 1$ dimensional. Consider $s$, $t$, and $u$ such that $(H_s - \lambda_s I)$ is nonsingular and $(H_t - \lambda_t I)$ and $(H_u - \lambda_u I)$ be singular, the vector $|\psi_{ts}\rangle \doteq \sum_{j=2}^{n} \langle j | (H_t - \lambda_t I)(H_s - \lambda_s I)^{-1} |1\rangle |j\rangle$

(37)

which possibly may be the zero vector) belongs to $S_t(\lambda_t)$ but not $S_s(\lambda_s)$. Assume the same condition for $|\psi_{us}\rangle$. If $|\psi_{ts}\rangle$ is not collinear to a multiple of $|\psi_{us}\rangle$, $S_t(\lambda_t)$, $S_u(\lambda_u)$ and $S_s(\lambda_s)$ would be non-coincident. To show that $|\psi_{ts}\rangle$ and $|\psi_{us}\rangle$ are not collinear, consider the contradiction, if $|\psi_{ts}\rangle$ where collinear to a multiple of $|\psi_{us}\rangle$, then for every $\alpha$ (due to subspace properties for the nullspace):

\[
|\psi_{ts}\rangle = \alpha |\psi_{us}\rangle \Leftrightarrow \sum_{j=2}^{n} \langle j | ((H_t - \lambda_t I) - \alpha(H_s - \lambda_s I))(H_s - \lambda_s I)^{-1} |1\rangle = 0
\]  

it would be necessary that $(H_s - \lambda_s I)^{-1} |1\rangle$ be simultaneously an eigenket of $(H_t - \lambda_t I)$ and $(H_s - \lambda_s I)$. So by the assumption in $B^{[2]}$ if there were no common eigenket for control Hamiltonians, the intersection $\bigcap_k S_k(\lambda_k)$ would be at most $1$-dimensional. The case that $\lambda_k \in \sigma(H_k)$ for all $k$ is a special
case of what has been proved. So $B\{|$ has been proved.

Consider the case that there exists common eigenkets for control Hamiltonians. Thus the intersection subspaces $S_1(\lambda_k)\cap S_2(\lambda_s)$ and $S_3(\lambda_s)\cap S_4(\lambda_s)$ (which are at most $(n - 2)$-dimensional) may coincide. Thus if there where $s$ common eigenkets, with the proposed statement, the intersection $\bigcap_k S_k(\lambda_k)$ may be at most $(1 + s)$-dimensional which proves $B\{|$.

The previous theorem, revealed that for each set of $\{\lambda_k \in \mathbb{R}, k = 1, ..., n - 1\}$, in the case that the control hamiltonians do not have common eigenkets, the invariant set includes at most one quantum equivalence class. This result will enormously help to provide further useful conditions for asymptotically stochastically stability. In the rest of this paper, assume that the control Hamiltonians do not share any eigenkets, which is not very restrictive.

Now consider the case that $|\psi\rangle \in A$:

Knowing that $\langle \psi | \psi_f > = 0$, let us study the invariancy for this situation For an infinitesimal time duration, the inner product would evolve as follows:

$$E[\langle \psi_f | \psi(dt) \rangle] = \frac{-i}{\hbar} \sum_k \alpha_k (\text{Im}(\langle \psi_f | H_k | \psi \rangle))^2 \langle \psi_f | H_k | \psi \rangle dt.$$ (38)

Thus, if $\text{Im}(\langle \psi_f | H_k | \psi \rangle) \neq 0$ for at least one $k$, the quantum trajectory is expected to escape the orthogonal subspace $|[\psi_f]\rangle^\perp$. On the other hand, assume that there exists $|\psi\rangle \in |\psi_f\rangle^\perp$ that for at least one $k$, $\langle \psi_f | H_k | \psi \rangle = re^{i\theta} \neq 0$. Putting $|\hat{\psi}\rangle = e^{-i\theta} |\psi\rangle$ (which also belongs to $|\psi_f\rangle^\perp$) gives \(\text{Im}(\langle \psi_f | H_k | \hat{\psi} \rangle) = 0\). Therefore the problem in this situation reduces to find the minimal set of control Hamiltonians $\{H_k\}$ such that:

$$\{ |\psi\rangle \in |\psi_f\rangle^\perp | \exists k: \langle \psi_f | H_k | \psi \rangle = 0 \} = \emptyset.$$ (39)

The following theorem can be stated now:

**Theorem 4.** Consider the SSE $A\{|$ Assume that $A\{|$ to $A\{|$ hold. Then the quantum trajectories starting from $|[\psi_f]\rangle^\perp$, will escape it with probability 1, i.e. the set $A$ in theorem $A\{|$ is not an invariant set.

**Proof.** Based on the statement above, it suffices to show that $A\{|$ holds. For every $|\psi\rangle \in |\psi_f\rangle^\perp$, one may write $|\psi\rangle = \sum_{j=2}^{n} c_j |j\rangle$. Also in this coordinate, each of the control Hamiltonians can be written as $H_k = \sum_{h=1}^{n} \sum_{l=1}^{n} c_{khl} |h\rangle \langle l|$ (of course with some restrictions on $c_{khl}$). Thus:

$$\langle \psi_f | H_k | \psi \rangle = \sum_{j=2}^{n} c_{kj} c_j |j\rangle.$$ (40)

Also $A\{|$ is provided if the system of linear equations

$$\begin{pmatrix} c_{112} & c_{113} & \cdots & c_{11n} \\ \vdots & \ddots & \vdots \\ c_{n-112} & c_{n-113} & \cdots & c_{n-11n} \end{pmatrix} \begin{pmatrix} c_2 \\ c_3 \\ \vdots \\ c_n \end{pmatrix} = \mathbf{0}$$ (41)

has no nontrivial solution. But this condition is always provided due to linearly independance of $\{H_k\}$ in $A\{|$.
Remark 2. Based on the proof of Theorem 4, \( m = n - 1 \) is the minimal number of independent control Hamiltonians which is stated in \( A \).

Now, based on these two stated theorems, one of the striking features of this theory can be stated. By Theorem 3, it is revealed that the right invariant set of quantum states, is at most 1-dimensional for each choice of \( \{ \lambda_k \} \). On the other hand, Theorem 4 revealed that set \( A \) in Theorem 3 is not right invariant. Now let us investigate the set \( B \). Consider that the quantum system, is initiated in the quantum equivanelec class \( [\psi_0] \) almost surely and there exists a set \( \{ \lambda_k \in \mathbb{R} \} \) such that for all \( k \), 
\[
\langle \psi_f | H_k - \lambda_k I | \psi \rangle = 0.
\]
In order to investigate the right invariance property, one must inspect whether or not the dynamics inspired by 4 preserve the vanishing \( LV (|\psi^{(i)}(t)\rangle) \) or not. To this end, the following theorem shows that the invariant set is exclusively containing \( [\psi_f] \):

**Theorem 5.** Assume that 4 and 5 hold. If the control Hamiltonians do not share any common eigenkets, then the invariant set of 4 exclusively includes \([\psi_f]\).

**Proof.** Let us consider \( LV (|\psi^{(i)}(dt)\rangle) \). By SSE 4 one should find a set \( \{ \lambda_k \in \mathbb{R} \} \) such that:

\[
\langle \psi_f | (H_k - \lambda_k I) \left( I + \left( -\frac{i}{\hbar} H_0 - k(X - \langle X \rangle)^2 \right) dt + \sqrt{2k}(X - \langle X \rangle) dW \right) | \psi_0 \rangle = 0.
\]

Note that \( u_k(0) = 0 \) and thus the effect of \( H_k \) vanishes. The presence of Wiener process implies that both of the following inequalities should hold simultaniously:

\[
\langle \psi_f | (H_k - \lambda_k I) \left( I + \left( -\frac{i}{\hbar} H_0 dt \right) \right) | \psi_0 \rangle = 0.
\]

Also it is intuitively obvious that \( \lambda_k \) in uniformly continuous in \( t \) and if written as \( \hat{\lambda}_k = \lambda_k + \delta_k \) for real \( \delta_k \), \( \delta_k \to 0 \) as \( dt \to 0 \). Let us investigate 42. Reordering the terms reads:

\[
\langle \psi_f | (H_k - \lambda_k I) \left( \frac{-i}{\hbar} H_0 dt \right) | \psi_0 \rangle = \delta_k \langle \psi_f | \psi_0 \rangle \left( 1 + \frac{-idt}{\hbar} \hat{H} \right).
\]

But the second term of RHS is of second order of perturbation and is negligible as \( dt \to 0 \):

\[
\langle \psi_f | (H_k - \lambda_k I) \left( \frac{-i}{\hbar} H_0 dt \right) | \psi_0 \rangle = \delta_k \langle \psi_f | \psi_0 \rangle.
\]

Thus the question reduces to: if there exist the set \( \{ \delta_k \in \mathbb{R} \} \), such that for the (at most)1-dimensional members of \( B \) in Theorem 3, 4 and 5 holds for all \( k \).

Define \( \hat{H}_k \doteq (H_k - \lambda_k I) \left( \frac{-i}{\hbar} H_0 dt \right) \). So one should try to find the set \( \{ \delta_k \in \mathbb{R} \} \) such that \( \langle \psi_f | \hat{H}_k - \delta_k I | \psi_0 \rangle = 0 \) besides \( \langle \psi_f | \hat{H}_k - \lambda_k I | \psi_0 \rangle = 0 \), for all \( k \), but this is the same as what we tried in the proof of Theorem 4 with the difference that in this case, there are \( 2(n - 1) \) solution spaces one of which is at the most \( n - 1 \)-dimensional. The assumption of not sharing any eigenkets for \( H_k \)’s implies that the common solution of \( \langle \psi_f | H_k - \lambda_k I | \psi_0 \rangle = 0 \) includes at most one independent ket.

So, in order to keep the same solution to be the solution for all of \( \langle \psi_f | \hat{H}_k - \delta_k I | \psi_0 \rangle = 0 \), (or in other words, two functionals \( \langle \psi_f | (H_k - \lambda_k I) \) and \( \langle \psi_f | (\frac{-i}{\hbar} (H_k - \lambda_k I) H_0 - \delta_k I) \) share a same kernel), by Lemma 5 there are only two possibilities: 1) which are whether \( H_0 = cI \) for some scalar complex \( c \), which is impossible, or 2) \( |\psi_0\rangle \) be an eigenket for \( H_0 \) which implies \( \delta_k = 0 \). Regarding
to this explanation, the only possibilities to be included in the invariant set are the eigenkets of $H_k$ making $LV (|\psi\rangle) = 0$. On the other hand, even if $H_0$ is degenerate (and the stationary states can be a super-position of eigenstates with the same energy level) all of the stationary states apart from $[|\psi_f\rangle]$ have to lay on $[|\psi_f\rangle]^\perp$. But, on the contrary, by Theorem 4, $[|\psi_f\rangle]^\perp$ is not invariant. By the virtue that $|\psi_f\rangle = |1\rangle$ is degenerate, also no stationary states can be in the superposition of the eigenspace corresponding $|1\rangle$ and the eigenspaces in $[|\psi_f\rangle]^\perp$. In the light of the facts outlined above and by the use of B.3 the right invariant set is solely restricted to $[|\psi_f\rangle]$.

Remark 3. It is notably to remark that this proof implies that even if $H_0$ is degenerate in the eigenspaces except $|1\rangle$, if $A_1$ to $A_5$ hold and $\{H_k\}$ dont share any eigenkets, the $\Omega$-limit set merely includes $[|\psi_f\rangle]$. Although the nondegenary condition for $H_0$ was essential in all of the research on Lyapunov control of Shrodinger equation, in this paper it is waived by the virtue of proposed theory.

5 Appendix

Lemma 5. Consider the control Hamiltonians in 6. Assume that $A_1$ to $A_5$ hold. Then for each choice of $\{\lambda_k \in \mathbb{R}, k = 1, ..., n - 1\}$, the set $\{H_1 - \lambda_1 I, H_2 - \lambda_2 I, ..., H_{n-1} - \lambda_{n-1} I\}$ is linearly independent.

Proof. Without loss of generality assume that $\lambda_1 \neq 0$. By contradiction assume that $\{H_1 - \lambda_1 I, H_2 - \lambda_2 I, ..., H_{n-1} - \lambda_{n-1} I\}$ is linearly dependent. Thus for some nonzero set $\{c_2, ..., c_{n-1}\}$ one may find an scalar $\alpha$ such that:

$$\alpha (H_1 - \lambda_1 I) = \sum_{k=2}^{n-1} c_k (H_k - \lambda_k I) \Rightarrow \alpha H_1 = \sum_{k=2}^{n-1} (c_k H_k) - \sum_{k=2}^{n-1} (c_k \lambda_k) I + \alpha \lambda_1 I.$$  

By the fact that $iH_k \in su(n)$ (and thus they are traceless), one has the unique choise of $\alpha = \frac{\sum_{k=2}^{n-1} (c_k \lambda_k)}{\lambda_1}$. So one deduces that:

$$\alpha H_1 = \sum_{k=2}^{n-1} (c_k H_k)$$  \hspace{1cm} (44)

which contradicts the assumption $A_5$ in both cases $\alpha = 0$ and $\alpha \neq 0$. \qed