Bound State Solutions of the Schrödinger Equation for a More General Woods–Saxon Potential with Arbitrary l-State∗

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The energy spectra and the wave function depending on the c-factor are investigated for a more general Woods–Saxon potential (MGWSP) with an arbitrary l-state. The wave functions are expressed in terms of the Jacobi polynomials. Two potentials are obtained from this MGWSP as the special cases. These special potentials are Hulthen and the standard Woods–Saxon potentials. We also discuss the energy spectrum and wave function for the special cases.

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In recent years, different methods have been adopted for the solution of the Schrödinger Equation (SE) with various potentials[1,1−5] using methods including numerical and analytical techniques,[6,7] supersymmetry (SUSSY),[8] and Pekeris approximation[9] as well as the Nikiforov–Uvarov method.[10] An exact solution of the SE is of high importance in non-relativistic quantum mechanics.

However, there are very few potentials for which the radial SE can be solved exactly for all n and l values. Levai and Williams developed a simple method for constructing potential for which the SE can be solved exactly in terms of special functions.[11] The exact solutions to the SE for the Woods–Saxon potential (WSP)[12] cannot be solved exactly for l ≠ 0, though Flugge[13] gave an exact expression for the wave function using the graphical method to obtain the energy eigen values for l = 0.[14]

The WSP is one of the short-range potentials well-known in physics and plays a vital role in nuclear and particle physics, atomic physics, condensed matter and chemical physics.[15] The WSP and many of its modifications have received much attention in recent years in describing metallic clusters. It is used as one of the interaction neutrons with one heavy-ion nucleus for the optical potential model.[22]

In this Letter, we attempt to solve the radial SE for this more general Woods–Saxon potential (MGWSP) using the Nikiforov–Uvarov method, and obtain the wave function and the corresponding eigenvalues for arbitrary l-states, then deduce some well-known potentials from our results.

The standard form of the Nikiforov–Uvarov (NU) equation is written as[10]

$$\psi''(s) + \frac{\gamma(s)}{\sigma(s)} \psi'(s) + \frac{\sigma(s)}{\sigma^2(s)} \psi(s) = 0,$$

where \(\sigma(s)\) and \(\bar{\sigma}(s)\) are polynomials of the second degree at most, and \(\bar{\tau}(s)\) is a first degree polynomial. Thus, from Eq. (1), the Schrödinger equation and the Schrödinger-like equations can be solved analytically by means of the special potentials with this method. Hence, in order to obtain the particular solution of Eq. (1), one can use the transformation for the wave function as

$$\psi(s) = \varphi(s) \chi_n(s).$$  (2)

This reduces Eq. (1) to an equation of hypergeometric type as

$$\sigma(s) \chi_n''(s) + \bar{\tau}(s) \chi_n'(s) + \lambda \chi_n(s) = 0,$$

and \(\varphi(s)\) is defined as a logarithmic derivative in the form

$$\varphi'(s)/\varphi(s) = \frac{\pi(s)}{\sigma(s)}.$$  (4)

The function \(\pi(s)\) and the parameter \(\lambda(s)\) are required for the Nikiforov–Uvarov method,

$$\pi(s) = \frac{\sigma' - \bar{\tau}(s)}{2} \pm \sqrt{\left(\frac{\sigma' - \bar{\tau}(s)}{2}\right)^2 - \bar{\sigma}(s) + k \sigma(s)},$$

$$\lambda(s) = k + \pi'(s).$$  (5)

Conversely, in order to find the values of \(k\) in Eq. (5), the expression under the square root must be the square of the polynomials.

Consequently, the eigenvalue equation for the Schrödinger equation becomes

$$\lambda_n = -n \tau'(s) - \frac{n(n-1)}{2} \sigma''(s),$$  (7)

where

$$\tau(s) = \bar{\tau}(s) + 2\pi(s),$$  (8)

and its derivative is negative. The other wave function can be determined using the Rodrigues Relation.
The standard WSP is defined as\[^{[10]}\]
\[
V(r) = \frac{V_0}{1 + \exp[(r - R_0)/a]}, \quad a \ll R_0.
\] (9)

The Schrödinger equation for the potential \(V(r)\) is of the form\[^{[10]}\]
\[
\left( -\frac{\hbar^2}{2m} \nabla^2 + V(r) \right)\psi(r) = E\psi(r),
\] (10)
where \(\psi(r)\) is the wave function, \(E\) represents the energy eigenvalues, \(\hbar\) is Planck’s constant, \(m\) is the mass and \(\nabla^2\) is the Laplacian operator. The radial Schrödinger equation\[^{[17]}\] of Eq. (10) with the WSP is given by
\[
\frac{d^2\psi(r)}{dr^2} + \frac{2m}{\hbar^2} \left[ E - \frac{V_0}{1 + \exp[2\alpha(r - R_0)]} \right] \psi(r) - \frac{l(l+1)}{r^2} \psi(r) = 0,
\] (11)
where \(l\) is the angular momentum quantum number and \(\alpha = 1/2a\). Writing the wave function \(\psi(r) = R(r)/r\) reduces Eq. (11) into the form
\[
\frac{d^2R(r)}{dr^2} + \left[ E - \frac{V_0}{1 + \exp[2\alpha(r - R_0)]} \right] R(r) = 0.
\] (12)

Equation (12) can be expressed in terms of the effective potential as
\[
\frac{d^2R(r)}{dr^2} + \frac{2m}{\hbar^2} [E - V_{\text{eff}}(r)] R(r) = 0,
\] (13)
where the effective potential \(V_{\text{eff}}(r)\) is defined as
\[
V_{\text{eff}}(r) = V(r) + \frac{l(l+1)\hbar^2}{2\mu r^2},
\] (14)
with the second term corresponding to the centrifugal term.

Following the new improved approximation scheme\[^{[13]}\] for the centrifugal term, we can express the effective potential as
\[
V_{\text{eff}}(r) = V(r) + \frac{2l(l+1)\hbar^2}{\mu \alpha^2} \exp(-2\alpha r),
\] (15)
Taking a new coordinate
\[
s = c[\exp(2\alpha r - 1)],
\] (16)
where \(c = \exp(-2\alpha R_0)\) and it is the \(c\)-factor that characterizes the behavior of the potential as will be seen later. Equation (16) transforms Eq. (13) into the form
\[
(c + s)^2 \frac{d^2R}{ds^2} + (c + s) \frac{dR}{ds} + \frac{4m}{\hbar^2 \alpha^2} \left[ E - \frac{V_0}{c + s} - \delta_0 - \frac{\delta_0(c + s)^2}{s^2} \right] R(s) = 0,
\] (17)
where \(\delta_0 = 2\hbar^2 l(l+1)/\mu \alpha^2\).

Simplifying Eq. (17) yields
\[
\frac{d^2R}{ds^2} + \frac{s}{(c+s)} \frac{dR}{ds} + \frac{1}{s^2(c+s)} \cdot \left[ (\varepsilon^2 + \gamma^2 + \xi_1^2 - \xi_4^2)s + (\beta^2 + \xi_4^2)s + \xi_1^2 \right] R(s) = 0,
\] (18)
where the following dimensionless parameters have been used in obtaining Eq. (18):
\[
\varepsilon^2 = \frac{-mE}{2\hbar^2 \alpha^2}, \quad \gamma^2 = \frac{mV_0}{2\hbar^2 \alpha^2(1+c)}, \quad \xi_1 = \frac{m\delta_0}{2\hbar^2 \alpha^2}, \quad \xi_2 = \frac{mc\delta_0}{2\hbar^2 \alpha^2}, \quad \xi_3 = \frac{m\delta_0}{2\hbar^2 \alpha^2}, \quad \xi_4 = \frac{m}{2\hbar^2 \alpha^2}.
\] (19)

It is pertinent at this point to note that in obtaining Eq. (18), we have Taylor expanded the term \((c + s)^{-1}\) up to the second order, using the expression\[^{[17]}\]
\[
\frac{1}{c + s} = \frac{1}{c + 1} \sum_{k=1}^{\infty} (-1)^k s^k (c + 1)^k.
\] (20)

Now comparing Eq. (18) with Eq. (1), we obtain the polynomials:
\[
\tilde{r} = s, \quad \sigma(s) = s(c + s), \quad \sigma(s) = -as^2 + bs + d,
\] (21)
where
\[
a = \varepsilon^2 + \gamma^2 + \xi_1^2 + \xi_4^2, \quad b = \beta^2 + \xi_4^2, \quad c = \xi_2^2.
\] (22)

In the NU method, the new \(\pi(s)\) is defined as
\[
\pi(s) = \frac{c + s}{2} \pm \frac{1}{2} \sqrt{(4k - b_1)s^2 + (b_2 + 4kc)s + b_3},
\] (23)
where \(b_1 = -4a + 1, b_2 = 2c - 4b\) and \(b_3 = c^2 - 4d\).

The discriminant in the expression under the square root has to be zero. Thus, the expression becomes the square of a polynomial of the first degree,
\[
16c^2 k^2 + (8b_2c - 16b_2)k + 4n_1 b_2 + n_2^2 = 0.
\] (24)

When we impose the basic requirement with respect to the constant \(k\), we obtain
\[
k_\pm = \frac{b_2(2 - c)}{4c^2} \pm \frac{1}{2c^2} \sqrt{b_2(1 - c)b_2 + b_1 c^2}.
\] (25)
Substituting \(k_\pm\) into Eq. (23), the following possible
Comparing Eqs. (30) and (31), we obtain the exact solution for $\pi(s)$,

$$\pi(s) = \frac{c + s}{2} \pm \frac{1}{2c^2} \sqrt{\eta_2 s - \sqrt{(1 - c)\eta_2 + b_1c^2}},$$

for $k = \frac{b_2(2 - c)}{4c^2} - \frac{1}{2c^2} \sqrt{b_2(1 - c)\eta_2 + b_1c^2};$

$$\pi(s) = \frac{c + s}{2} \pm \frac{1}{2c^2} \sqrt{\eta_2 s + \sqrt{(1 - c)\eta_2 + b_1c^2}},$$

for $k = \frac{b_2(2 - c)}{4c^2} + \frac{1}{2c^2} \sqrt{b_2(1 - c)\eta_2 + b_1c^2}.$

(26)

For the polynomial of $\tau = \tau + 2\pi$ which has a negative derivative, we select

$$k(s) = \frac{b_2(2 - c)}{4c^2} \pm \frac{1}{2c^2} \sqrt{b_2(1 - c)b_2 + b_1c^2},$$

(27)

$$\pi(s) = \frac{c + s}{2} - \frac{1}{2c^2} \sqrt{\eta_2 s - \sqrt{(1 - c)\eta_2 + b_1c^2}}.$$  

(28)

Therefore, with this selection and using $\lambda = k + \pi'$, we obtain the $\tau$ and $\lambda$ values as

$$\tau(s) = \left(-\frac{\sqrt{b_2}}{c^2} + c + 2s\right) + \frac{2}{c^2} \sqrt{(1 - c)b_2 + b_1c^2},$$

$$\lambda = \frac{\eta_2(2 - c)}{4c^2} - \frac{1}{2c^2} \sqrt{\eta_2(1 - c)b_2 + b_1c^2} + \frac{c}{2} \frac{b_2}{2c^2}.$$  

(29)

(30)

Now using Eq. (7), we have

$$\lambda_n = \frac{n\sqrt{\eta_2}}{c^2} - c - 2n(n - 1).$$

(31)

Comparing Eqs. (30) and (31), we obtain the exact energy eigenvalues as

$$E_n = 1 + \delta_0 + \frac{V_0}{(1 + c)^2} - \frac{1}{2} \left( \frac{h^2\alpha^2}{m} \right) - \frac{1}{4} \left( \frac{2\hbar\alpha}{m} \right)^2 \left( \frac{b_2}{2c^2} - \frac{4mV_0}{h^2\alpha^2} \left( \frac{1}{1 + c} + c\delta_0 \right) \right)(c - 1)$$

$$- \frac{2h^2\alpha^2}{m} \left( \frac{2n}{\sqrt{\frac{2c - 2mV_0}{h^2\alpha^2} + c\delta_0}} \right) - \frac{4 + c - 3n(n - 1)}{2[2c - \frac{2mV_0}{h^2\alpha^2} + c\delta_0]} - \frac{1}{c},$$

for $c > 1.$

(32)

Equation (32) is the energy spectrum for the Schrödinger equation with the MGWSP. However, Eq. (32) can be rewritten for a case $c \ll 1$ as

$$E_n = 1 + \delta_0 + \frac{1}{2} \left( \frac{h^2\alpha^2}{m} \right) + V_0(1 - 2c)$$

$$+ \frac{1}{4m} \left( h\alpha \right)^2 \left[ 2c - \frac{4mV_0}{h^2\alpha^2} \left( \frac{1}{1 + c} + c\delta_0 \right) \right](c - 1)$$

$$- \frac{2h^2\alpha^2}{m} \left( \frac{2n}{\sqrt{2c - \frac{2mV_0}{h^2\alpha^2} + c\delta_0}} \right) - \frac{4 + c - 3n(n - 1)}{2[2c - \frac{2mV_0}{h^2\alpha^2} + c\delta_0]} - \frac{1}{c},$$

(33)

In order to find the wave function we first evaluate for the weight function from Eq. (18) as

$$\rho(s) = \frac{(c - \nu)(c + s)^{\frac{1 - \nu}{2}}}{c},$$

(34)

where

$$\mu = \left( -\frac{\sqrt{b_2}}{c^2} + c + 2 \right),$$

$$\nu = \frac{2}{c^2} \sqrt{(1 - c)\eta_2 + \eta_1 + c^2}.$$

Now using the Rodrigue relation of Eq. (18), we have

$$\chi_n(s) = N_n s^{-\frac{\nu}{\mu}}(c + s)^{\frac{\nu}{\mu}} \cdot \frac{d^n}{ds^n} \left[ (s + \frac{\nu}{\mu})(c + s)^{\frac{\nu - \mu}{\mu}} \right],$$

(35)

where $N_n$ is the normalization constant. The other wave function is obtained from Eq. (4) as

$$\varphi(s) = s^{\frac{\nu}{\mu}}(c + s)^{\frac{\nu - \mu}{\mu}}.$$  

(36)

Therefore, the radial wave function of the Schrödinger equation with the MGWSP can be written as

$$R(s) = \varphi(s)\chi_n$$

$$= \left( \frac{c^2}{\mu}c^2 \right) s^{\frac{\nu}{\mu}}(c + s)^{\frac{\nu - \mu}{\mu}} \cdot \frac{d^n}{ds^n} \left[ s^{\frac{\nu}{\mu}}(c + s)^{\frac{\nu - \mu}{\mu}} \right],$$

(37)

where $A = 2\mu/c - 1$ and $B = 2\nu/c^2$. The wave function can be expressed in terms of Jacobi polynomials as

$$R(s) = C_n s^{A/2}(c + s)^{\frac{1}{\mu} + B - \sqrt{\eta_2}}$$

$$\cdot \frac{d^n}{ds^n} \left[ s^{n + A/2}(c + s)^{n + B - \mu} \right],$$

(38)

where $N_n$ is the new normalization constant and obeys the normalization condition $\int_1^1 (R(s))^2 ds = 1$.

Finally, we write the total radial wave function as

$$R(r) = \frac{1}{r} [c(2\pi r - 1)]^{A/2}$$

$$\cdot \frac{d^n}{dr} [c(2\pi r - 1)]^{B - \sqrt{\eta_2}} P_n^{A/2,B}(r).$$

(39)

In the Schrödinger equation with an MGWSP, we obtain two energy spectra ($c \ll 1$ and $c \gg 1$) and the unnormalized wave function expressed in terms of Jacobi polynomials. Two special potentials are deduced: the Hulthen potential [19] is obtained from the MGWSP by setting $c = -1$ and the corresponding energy
eigenvalues and the wavefunction are obtained from Eqs. (33) and (39) as
\[
E_n = 1 + \delta_0 - \frac{1}{2} \frac{\hbar^2 \alpha^2}{m} + 3V
- \frac{1}{2m} (\alpha \omega)^2 \left[ -2 - \frac{4m}{\hbar^2 \alpha^2} (V_0 - \delta_0) \right]
+ \frac{2\hbar^2 \alpha^2}{m} \left[ \sqrt{-2 - \frac{2m}{\hbar^2 \alpha^2} (2V_0 - \delta_0)} \right]
+ \frac{3(1 - n(n - 1))}{2} \frac{1}{\sqrt{-2 - \frac{2m}{\hbar^2 \alpha^2} (2V_0 - \delta_0)}} + 1).
\]

For \( c = 1 \) the MGWSP changes to the standard WS potential. The corresponding energy eigenvalues and wavefunction for this potential are obtained from Eqs. (32) and (39) as
\[
E_n = 1 + \delta_0 + \frac{V_0}{4} - \frac{1}{2} \left( \frac{\hbar^2 \alpha^2}{m} \right)
- \frac{2\hbar^2 \alpha^2}{m} \left[ \frac{\sqrt{-2 - \frac{2m}{\hbar^2 \alpha^2} (\frac{V_0}{2} + \delta_0)}}{2n} \right]
+ \frac{5 - 3n(n - 1)}{2} \left( \frac{1}{\sqrt{-2 - \frac{2m}{\hbar^2 \alpha^2} (\frac{V_0}{2} + \delta_0)}} - 1 \right)^2,
\]
\[
R(r) = \frac{1}{r} \left[ \frac{\left( \exp(2\alpha r) - 1 \right)}{\sqrt{\pi}} \right]^{A/2}
\cdot \left[ \frac{\left( \exp(2\alpha r) \right)^{B - \sqrt{\pi}} \rho_n^{(\frac{A}{2}, B)} (r) \cdot \left( \left( \exp(2\alpha r) - 1 \right) \right)^{A/2}}{\sqrt{\pi}} \right]^{B - \sqrt{\pi}} \rho_n^{(\frac{A}{2}, B)} (r).
\]

In conclusion, using the NU method we have discussed the approximate solution of the Schrödinger equation for an MGWSP for arbitrary \( l \)-states, and obtained the energy eigenvalues equation and wavefunction of the Schrödinger for the MGWSP for \( c \ll 1 \) and \( c \gg 1 \). In addition, as a special case, the Hulthen potential and the standard WS potential are discussed.

We can say that the present results obtained for the special case of the MGWSP give us some interesting applications in various quantum mechanical studies and nuclear scattering problems. Finally, it can be seen that the results are consistent with those in the literature.

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References

[1] Ikot A N and Akpabio L E 2010 Appl. Phys. Res. 2 202
[2] Ikot A N, Akpabio L E and Obu J A 2011 J. Vectorial Relativ. 6 1
[3] Meyur S and Debnath S 2009 Bulg. J. Phys. 36 77
[4] Antia A D, Ikot A N and Akpabio L E 2010 Eur. J. Sci. Res. 46 107
[5] Ahmed S A S and Buragohan L 2010 Bulg. J. Phys. 37 133
[6] Ikot A N, Akpabio L E and Umoren E B 2011 J. Sci. Res. 3 25
[7] Bayrak O and Boztosun I 2006 J. Phys. A: Math. Gen. 39 6955
[8] Cooper F, Khare A and Sukhatme U 1995 Phys. Rep. 251 267
[9] Pekeris C I 1934 Phys. Rev. 45 98
[10] Nikiforov A F and Uvarov V B 1988 Special Functions of Mathematical Physics (Basel: Birkhauser)
[11] Levai G and Williams B W 1993 J. Phys. A: Math. Gen. 26 3301
[12] Woods R D and Saxon D S 1954 Phys. Rev. 95 577
[13] Flugge S 1994 Practical Quantum Mechanics (Berlin: Springer) vol 1
[14] Badalov V B H, Ahmadov H I and Ahmadov A I 2009 Int. J. Mod. Phys. E 18 631
[15] Ikhdair S M and Sever R 2008 Int. J. Mod. Phys. E 17 1107
[16] Berkdemir C, Berkdemir A and Sever R 2005 Phys. Rev. C 72 027001
[17] Janvicius A J and Jurgaitis D 2007 Siauliai Math. Semin. 2 5
[18] Xu Y, He S and Jia C S 2008 J. Phys. A 41 255302
[19] Ikot A N, Akpabio L E and Uwah E J 2011 Electron. J. Theor. Phys. 8 225
[20] Ikhdair S M 2005 arXiv:quant-ph/0507272
[21] Berkdemir A, Berkdemir C and Sever R 2006 Mod. Phys. Lett. A 21 2087
[22] Bohr A and Mottelson B R 1969 Nuclear Structure (New York: Benjamin/Cummmning)