ON THE $p$ SPECTRUM OF LAPLACIANS ON GRAPHS

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ABSTRACT. We study the $p$-independence of spectra of Laplace operators on graphs arising from regular Dirichlet forms on discrete spaces. Here, a sufficient criterion is given solely by a uniform subexponential growth condition. Moreover, under a mild assumption on the measure we show a one-sided spectral inclusion without any further assumptions. We study applications to normalized Laplacians including symmetries of the spectrum and a characterization for positivity of the Cheeger constant. Furthermore, we consider Laplacians on planar tessellations for which we relate the spectral $p$-independence to assumptions on the curvature.

1. INTRODUCTION

In [Sim80, Sim82] Simon conjectured that the spectrum of a Schrödinger operator acting on $L^p(\mathbb{R}^N)$ is $p$-independent. Hempel and Voigt gave an affirmative answer in [HV86] for a large class of potentials. Later this result was generalized in various ways. Sturm [Stu93] showed $p$-independence of the spectra for uniformly elliptic operators on a complete Riemannian manifold with uniform subexponential volume growth and a lower bound on the Ricci curvature. Moreover, Arendt [Are94] proved $p$-independence of the spectra of uniformly elliptic operators in $\mathbb{R}^N$ with Dirichlet or Neumann boundary conditions under the assumption of upper Gaussian estimates for the corresponding semigroups. While the proof strategies of Sturm and Arendt are rather similar to the one used by Hempel and Voigt, Davies [Dav95b] gave a simpler proof of the $p$-independence of the spectrum under the stronger assumption of polynomial volume growth and Gaussian upper bounds using the functional calculus developed in [Dav95a]. In recent works $p$-independence of spectral bounds are proven in the context of conservative Markov processes [KST1, Tak07] and Feynman-Kac semigroups [Che12, DLKK10, IT09]. See also [GHK+13] for Laplace operators on graphs with finite measure.

In this paper, we prove $p$-independence of spectra for Laplace operators on graphs under the assumption of uniform subexponential volume growth. This question was brought up in [Dav07, page 378] by Davies. Our framework are regular Dirichlet forms on discrete sets as introduced in [KL12]. While our result is similar to the one of Sturm [Stu93] for elliptic operators on manifolds, we do not need to assume any type of lower curvature bounds nor any type of bounded geometry. However, in various classical examples, such as Laplacians with standard weights, this disparity is resolved by the fact that uniform subexponential growth implies bounded geometry in some cases. In further contrast to [Stu93], we do not assume any uniformity of the coefficients in the divergence part of the operator such as

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uniform ellipticity and, additionally, we allow for positive potentials (in general potentials bounded from below).

We overcome the difficulties resulting from unbounded geometry, by the use of intrinsic metrics. While this concept is well established for strongly local Dirichlet forms, [Stu94] it was only recently introduced for general regular Dirichlet forms by Frank/Lenz/Wingert in [FLW10]. Since then, this concept already proved to be very effective for the analysis on graphs, see [BKW12, Fol11, Fol12, GHM11, HKW12, Hua11, Hua12, HKMW12], where it also appears under the name adapted metrics. Moreover, we employ rather weak heat kernel estimates (with the log term instead of a square) by Folz, [Fol11], which is a generalization of [Dav93] by Davies. These weak estimates turn out to be sufficient to prove the $p$-independence. Of course, the condition on uniform subexponential growth is always expressed with respect to an intrinsic metric.

Another result of this paper is the inclusion of the $\ell^2$-spectrum in the $\ell^p$-spectrum under the assumption of lower bounds on the measure only.

As applications we discuss the normalized Laplace operator, for which we prove several basic properties of the $\ell^p$ spectra such as certain symmetries of the spectrum. Moreover, we discuss consequences of $p$-independence on the Cheeger constant and give an example of $p$-independence and superexponential volume growth. Finally, we consider the case of planar tessellations which relates curvature bounds to the volume growth. In particular, we use such curvature conditions to recover results of Sturm, [Stu93] in the setting of planar tessellations.

The paper is organized as follows. In the next section we introduce the set up and present the main results. In Section 3 we show several auxiliary results in order to prove the main results in Sections 4 and 5. Applications to normalized Laplacians are considered in Section 6. The final section, Section 7, is devoted to planar tessellations and consequences of curvature bounds on the volume growth and $p$-independence.

2. Set up and main results

2.1. Graphs. Assume that $X$ is a countable set equipped with the discrete topology. A strictly positive function $m : X \to (0, \infty)$ gives a Radon measure on $X$ of full support via $m(A) = \sum_{x \in A} m(x)$ for $A \subseteq X$, so that $(X, m)$ becomes a discrete measure space.

A graph over $(X, m)$ is a pair $(b, c)$. Here, $c : X \to [0, \infty)$ and $b : X \times X \to [0, \infty)$ is a symmetric function with zero diagonal that satisfies

$$\sum_{y \in X} b(x, y) < \infty \quad \text{for } x \in X.$$  

We say $x$ and $y$ are neighbors or connected by an edge if $b(x, y) > 0$ and we write $x \sim y$. For convenience we assume that there are no isolated vertices, i.e., every vertex has a neighbor. We call $b$ locally finite if each vertex has only finitely many neighbors. The function $c$ can be interpreted either as one-way-edges to infinity or a potential or a killing term.

The normalizing measure $n : X \to (0, \infty)$ given by

$$n(x) = \sum_{y \in X} b(x, y), \quad \text{for } x \in X.$$  

often plays a distinguished role. In the case where $b : X \times X \to \{0, 1\}$, $n(x)$ gives the number of neighbors of a vertex $x$. If $n/m \leq M$ for some fixed $M > 0$ we say the graph has bounded geometry.

2.2. Intrinsic metrics and uniform subexponential growth. By a pseudo metric we understand a function $d : X \times X \to [0, \infty)$ that is symmetric, has zero diagonal and satisfies the triangle inequality. Following [FLW10], we call a pseudo metric $d$ an intrinsic metric for a graph $b$ on $(X, m)$ if

$$\sum_{y \in X} b(x, y)d(x, y)^2 \leq m(x) \quad \text{for all } x \in X.$$ 

For example one can always choose the path metric induced by the edge weights $w(x, y) = ((m/n)(x) \wedge (m/n)(y))^{\frac{1}{2}}$, for $x \sim y$, cf. e.g. [Hua11]. Moreover, we call

$$s := \sup\{d(x, y) \mid x \sim y, x, y \in X\}$$

the jump size of $d$. Note that the natural graph metric $d_n$ (i.e., the path metric with weights $w(x, y) = 1$ for $x \sim y$) is intrinsic if and only if $m \geq n$. However, in the case of bounded geometry, i.e., $n/m \leq M$ for some fixed $M > 0$, the metric $d_n/\sqrt{M}$ (which is equivalent to $d_n$) is an intrinsic metric.

Throughout the paper we assume that $d$ is an intrinsic metric with finite jump size. For the remainder of the paper, we refer to the quintuple $(X, b, c, m, d)$ whenever we speak of the graph.

We denote the distance balls centered at a vertex $x \in X$ with radius $r \geq 0$ by $B_r(x) := \{y \in X \mid d(x, y) \leq r\}$. Similar to [Stu93], we say the graph has uniform subexponential growth if for all $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that

$$m(B_{\varepsilon r}(x)) \leq C_\varepsilon e^{\varepsilon r}m(x), \quad \text{for all } x \in X, r \geq 0.$$ 

In Section 3.1 we discuss some implications of this assumption.

2.3. Dirichlet forms and Graph Laplacians. Denote by $C_c(X)$ the space of complex valued functions on $X$ with compact support. Denote the $\ell^p$-spaces by

$$\ell^p := \ell^p(X, m) := \{f : X \to \mathbb{C} \mid \|f\|_p < \infty\}, \quad p \in [1, \infty],$$

where

$$\|f\|_{\infty} := \sup_{x \in X} |f(x)|, \quad \text{and} \quad \|f\|_p := \left(\sum_{x \in X} |f(x)|^p m(x)\right)^{\frac{1}{p}}, \quad p \in [1, \infty).$$

Note that $\ell^\infty(X, m)$ does not depend on $m$.

For $p \in [1, \infty]$, let the Hölder conjugate be denoted by $p^*$, that is $\frac{1}{p} + \frac{1}{p^*} = 1$. We denote the dual pairing of $f \in \ell^p(X, m)$, $g \in \ell^{p^*}(X, m)$ by

$$\langle f, g \rangle := \sum_{x \in X} f(x)\overline{g(x)}m(x),$$

which becomes a scalar product for $p = 2$. We define the sesqui-linear form $Q$ with domain $D(Q) \subseteq \ell^2$ by

$$Q(f, g) = \frac{1}{2} \sum_{x, y \in X} b(x, y)(f(x) - f(y))(\overline{g(x)} - \overline{g(y)}) + \sum_{x \in X} c(x)f(x)\overline{g(x)},$$

$$D(Q) = \overline{C_c(X)}^{\|\cdot\|_Q},$$

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where \( \| \cdot \|_Q = (Q(\cdot) + \| \cdot \|^2_2)^{\frac{1}{2}} \) and \( Q(f) = Q(f, f) \). The form \( Q \) is a regular Dirichlet form on \( \ell^2(X, m) \), see [FO94, KL12] and for complexification of the forms, see [HKW12, Appendix B]. The corresponding positive self-adjoint operator \( L = L_2 \) on \( \ell^2(X, m) \) acts as
\[
L f(x) = \frac{1}{m(x)} \sum_{y \in X} b(x, y)(f(x) - f(y)) + \frac{c(x)}{m(x)} f(x).
\]

Let \( \bar{L} \) be the extension of \( L \) to
\[
\tilde{F} = \{ f : X \to \mathbb{C} \mid \sum_{y \in X} b(x, y) |f(y)| < \infty \text{ for all } x \in X \}.
\]

We have \( C_c(X) \subseteq D(L) \) if (and only if) \( \bar{L} C_c(X) \subseteq \ell^2(X, m) \), see [KL12, Theorem 6]. In particular, this can easily be seen to be the case if the graph is locally finite or if \( \inf_{x \in X} m(x) > 0 \). If \( m = n \) and \( c \equiv 0 \), then \( L \) is referred to as the normalized Laplacian.

Moreover, \( L = L_2 \) gives rise to the resolvents \( G_\alpha = (L - \alpha)^{-1}, \alpha < 0 \) and the semigroups \( T_t = e^{-tL}, t \geq 0 \). These operators are positivity preserving and contractive (as \( Q \) is a Dirichlet form), and therefore extend consistently to operators on \( \ell^p(X, m) \), \( p \in [1, \infty] \) (either by monotone convergence or by density of \( \ell^2 \cap \ell^p \) in \( \ell^p, p \in [1, \infty] \) and taking the dual operator on \( \ell^1 \) to get the operator on \( \ell^\infty \)). The semigroups are strongly continuous for \( p < \infty \). See [Dav89, Theorem 1.4.1] for a proof of these facts.

We denote the positive generators of \( G_\alpha \) or \( T_t \) on \( \ell^p \) by \( L_p, p \in [1, \infty] \), and the dual operator \( L_1^* \) of \( L_1 \) on \( \ell^\infty \) by \( L_\infty \) (which normally does not have dense domain in \( \ell^\infty \)). By [KL12, Theorem 9] we have that \( L_p, p \in [1, \infty] \), are restrictions of \( \tilde{L} \). Moreover, \( L_p \) are bounded operators with norm bound \( 2C \) if \( (n + c)/m \leq C \), see e.g. [KL10, Theorem 11] or [HKW12, Theorem 9.3]. Hence, bounded geometry is equivalent to boundedness of the operators for \( c \equiv 0 \). We denote the spectrum of the operator \( L_p \) by \( \sigma(L_p) \) and the resolvent set by \( \rho(L_p) = \mathbb{C} \setminus \sigma(L_p), p \in [1, \infty] \).

By duality \( \sigma(L_p) = \sigma(L_p^*), p \in [1, \infty] \).

Throughout this paper \( C \) always denotes a constant that might change from line to line.

### 2.4. Main Results.

In this section, we state the main theorems of this paper. The first is the discrete version of Sturm’s theorem, [Stu93], whose proof is given in Section 4.

**Theorem 2.1.** Assume the graph has uniform subexponential growth with respect to an intrinsic metric with finite jump size. Then for any \( p \in [1, \infty] \)
\[
\sigma(L_p) = \sigma(L_2).
\]

**Remark.** (a) Other than [Stu93] we do not assume any type of bounded geometry or any types of lower bounds on the curvature. For a discussion and examples see Section 3.1.

(b) Sometimes loops in the graph are modeled by non-vanishing diagonal of \( b \). However, the assumption that \( b \) has zero diagonal has no influence on our main results above as possible non-vanishing diagonal terms do not enter the operators. Such loops only have an effect on \( n \) and, thus, one would have to be careful if one chooses \( m = n \).
Clearly, we can also allow for potentials $c$ such that $c/m$ is only bounded from below (as adding a positive constant shifts the $\ell^p$ spectra of the operators simultaneously).

The following theorem shows that under an assumption on the measure one spectral inclusion holds without any volume growth assumptions.

**Theorem 2.2.** If $m$ is such that $\inf_{x \in X} m(x) > 0$, then for any $p \in [1, \infty]$

$$\sigma(L_2) \subseteq \sigma(L_p).$$

The proof of Theorem 2.2 is given in Section 5.

3. Preliminaries

In this section we collect some results and facts that will be used for the proof of Theorem 2.1. Moreover, in the first subsection we discuss the relation of uniform subexponential growth and bounded geometry.

3.1. Consequences of uniform subexponential growth.

**Lemma 3.1.** Assume the graph has uniform subexponential growth. Then, for all $\varepsilon > 0$ there is $C > 0$ such that

1. $m(x) \leq C e^{\varepsilon d(x,y)} m(y)$ for all $x, y \in X$,
2. $\#B_r(x) \leq C e^{\varepsilon r}$ for all $r \geq 0$, where $\#B_r(x)$ denotes the number of vertices in $B_r(x)$,
3. $\sum_{y \in X} e^{-\varepsilon d(x,y)} \leq C$ for all $x \in X$.

**Proof.** To prove (a) let $x, y \in X$. Using the uniform subexponential growth assumption and $x \in B_{d(x,y)}(y)$ yields $m(x) \leq m(B_{d(x,y)}(y)) \leq C e^{\varepsilon d(x,y)} m(y)$. Turning to (b) let $x \in X$ and $r \geq 0$. We obtain using (a) and the uniform subexponential growth assumption

$$\#B_r(x) = \sum_{y \in B_r(x)} m(y)/m(y) \leq C e^{\varepsilon r} m(B_r(x))/m(x) \leq C^2 e^{2\varepsilon r}.$$ 

The final statement follows also by direct calculation using (b) (with $\varepsilon_1$)

$$\sum_{y \in X} e^{-\varepsilon d(x,y)} = \sum_{r=1}^{\infty} \sum_{y \in B_r(x) \setminus B_{r-1}(x)} e^{-\varepsilon d(x,y)} \leq \sum_{r=1}^{\infty} e^{-\varepsilon r} \#B_r(x) \leq C \sum_{r=1}^{\infty} e^{(\varepsilon_1 - \varepsilon)r}$$

Hence choosing $\varepsilon_1 = \varepsilon/2$ yields the statement. \qed

**Remark.** (a) Lemma 3.1(b) implies finiteness of distance balls. On the other hand, finite jump size $s$ implies that for each vertex $x$ all neighbors of $x$ are contained in $B_s(x)$. Hence, graphs with uniform subexponential growth and finite jump size are locally finite.

(b) Finiteness of distance balls has strong consequences on the uniqueness of selfadjoint extensions. In particular, by [HKMW12, Corollary 1] implies that $Q$ is the maximal form on $\ell^2$ and that the restriction of $L_2$ to $C_0(X)$ (whenever $C_0(X) \subseteq D(L_2)$) is essentially selfadjoint.

In the following we discuss examples to clarify the relation between uniform subexponential growth and bounded geometry in the discrete setting.

Recall that we speak of bounded geometry if $n/m$ is a bounded function which is a natural adaption to the situation of weighted graphs. In Example 3.2 below,
we show that there are uniform subexponentially growing graphs with unbounded geometry. For completeness we also give a example of bounded geometry and exponential growth which is certainly well-known.

**Example 3.2.** (a) Uniform subexponential growth and unbounded geometry. Let $X = \mathbb{N}$, $m \equiv 1$, $c \equiv 0$ and consider $b$ such that $b(x,y) = 0$ for $|x-y| \neq 1$, $b(x,x+1) = x$ for $x \in 4\mathbb{N}$ and $b(x,x+1) = 1$ otherwise. Clearly $(n/m)(x) = n(x) = x+1$ for $x \in 4\mathbb{N}$ and, thus, $L_p$ is unbounded for all $p \in [1,\infty]$. Moreover, let $d$ be the path metric induced by the edge weights $w(x,x+1) = (n(x)\wedge n(x+1))^{-\frac{1}{2}}$. We obtain that $d(x,y) \geq (|x-y|-3)/4\sqrt{2}$ for all $x,y \in X$. Hence, $m(B_r(x)) = \#B_r(x) \leq \sqrt{2}(8r+6)$ which implies uniform subexponential growth.

(b) Exponential growth and bounded geometry. Take a regular tree, $c \equiv 0$, set $b$ to be one on the edges and zero otherwise and let $m \equiv 1$. This graph has bounded geometry but is clearly of exponential growth.

It is apparent that the graph in Example 3.2(a) above has bounded combinatorial vertex degree while the unbounded geometry is induced by the edge weights. So, one might wonder whether one can also present examples with unbounded combinatorial vertex degree which is the criterion for unbounded geometry in the classical setting. The proposition below shows that this is impossible under the assumptions of uniform subexponential growth and finite jump size. Recall that by the remark below Lemma 3.4 we already know that the graph must be locally finite.

The combinatorial vertex degree $\deg$ is the function that assigns to each vertex the number of neighbors, that is $\deg(x) = \#\{y \in X \mid b(x,y) > 0\}$, $x \in X$.

**Proposition 3.3.** If the graph has uniform subexponential growth with respect to a metric with finite jump size $s$, then the combinatorial vertex degree is bounded.

**Proof.** Suppose the graph has unbounded vertex degree, i.e., there is a sequence of vertices $(x_n)$ such that $\deg(x_n) \geq n^2$ for all $n \geq 1$. We show that there is a sequence of vertices $z_n$ such that $m(B_s(z_n))/m(z_n)$ is unbounded and thus the graph does not have uniform subexponential growth.

If, for $n \geq 1$, there is a neighbor $y_n$ of $x_n$ such that $m(y_n) \leq m(x_n)/\sqrt{\deg(x_n)}$, then we estimate using $x_n \in B_s(y_n)$

$$\frac{m(B_s(y_n))}{m(y_n)} \geq \frac{m(x_n)}{m(y_n)} \geq \sqrt{\deg(x_n)} \geq n$$

We set $z_n = y_n$ in this case. If, on the other hand, $m(y) \geq m(x_n)/\sqrt{\deg(x_n)}$ for all neighbors $y$ of $x_n$, then

$$\frac{m(B_s(x_n))}{m(x_n)} \geq \frac{1}{m(x_n)} \frac{m(x_n)}{\sqrt{\deg(x_n)}} \geq \sqrt{\deg(x_n)} \geq n$$

and set $z_n = x_n$ in this case. Hence, we have proven the claim. □

**Corollary 1.** Assume there is $D > 0$ such that $b \leq D$ and $m \geq 1/D$. Then, uniform subexponential growth with respect to a metric with finite jump size implies bounded geometry.

**Proof.** One simply observes that $n/m \leq D^2 \deg$ and the statement follows from the proposition above. □
This means for the standard Laplacians \( \Delta_p \varphi(x) = \sum_{y \sim x} (\varphi(x) - \varphi(y)) \) on \( \ell^p(X,1) \) and \( \Delta_p^{(n)} \varphi(x) = \frac{1}{\deg(x)} \sum_{y \sim x} (\varphi(x) - \varphi(y)) \) on \( \ell^p(X,\deg) \), that uniform subexponential growth implies bounded geometry, both in the sense of bounded \( n/m \) being bounded and also in the sense of deg being bounded.

### 3.2. Lipschitz continuous functions

We denote by \( \text{Lip}_\varepsilon^\infty \) the real valued bounded Lipschitz continuous functions with Lipschitz constant \( \varepsilon > 0 \), i.e.,

\[
\text{Lip}_\varepsilon^\infty = \{ \psi : X \to \mathbb{R} | \psi(x) - \psi(y) \leq \varepsilon d(x,y), x,y \in X \} \cap \ell^\infty(X,m).
\]

**Lemma 3.4.** Let \( \varepsilon > 0 \) and let \( s \) be the jump size of \( d \). Then, for all \( \psi \in \text{Lip}_\varepsilon^\infty \),

(a) \( e^\psi \) is a bounded Lipschitz continuous function, in particular, \( e^\psi D(Q) = D(Q) \).

(b) \( |1 - e^{\psi(x) - \psi(y)}| \leq \varepsilon e^{\varepsilon s} d(x,y), \) for \( x \sim y \).

(c) \( |(e^{\psi(x)} - e^{\psi(y)})(e^{\psi(x)} - e^{\psi(y)})| \leq 2e^{2\varepsilon s} d(x,y)^2 \), for \( x \sim y \).

**Proof.** The first statement of (a) follows from mean value theorem, that is for any \( x,y \in X \) we have

\[
|e^{\psi(x)} - e^{\psi(y)}| \leq |\psi(x) - \psi(y)| e^{|\psi|_{\infty}} \leq \varepsilon d(x,y) e^{|\psi|_{\infty}}.
\]

Now, \( e^\psi D(Q) \subseteq D(Q) \) is a consequence of \cite{LIPSCHITZ} Lemma 3.5. The other inclusion follows since \( e^{-\psi} \) is also bounded and Lipschitz continuous. Similarly, we get (b) using the Taylor expansion of the exponential function

\[
|1 - e^{\psi(x) - \psi(y)}| = \sum_{k \geq 1} \frac{(\psi(x) - \psi(y))^k}{k!} \leq \varepsilon d(x,y) \sum_{k \geq 1} \frac{(\varepsilon s)^k}{k!} \leq \varepsilon d(x,y) e^{\varepsilon s},
\]

and similarly, using \( |(e^{-\psi(x)} - e^{-\psi(y)})(e^{\psi(x)} - e^{\psi(y)})| = 2 \sum_{k \in \mathbb{N}} \frac{(\psi(x) - \psi(y))^k}{k!} \) we get (c). \( \square \)

### 3.3. Kernels

Let \( A : D(A) \subseteq \ell^p \to \ell^q \), \( p,q \in [1,\infty] \) be a densely defined linear operator. We denote by \( \|A\|_{p,q} \) the operator norm of \( A \), i.e.

\[
\|A\|_{p,q} = \sup_{f \in D(A), \|f\|_p = 1} \|Af\|_q.
\]

Note that any such operator \( A : D(A) \subseteq \ell^p \to \ell^q \), \( p < \infty \), with \( C_c(X) \subseteq D(A) \) admits a kernel \( k_A : X \times X \to \mathbb{C} \) such that

\[
Af(x) = \sum_{x \in X} k_A(x,y) f(y)m(y)
\]

for all \( f \in D(A), x \in X \), which can be obtained by

\[
k_A(x,y) = \frac{1}{m(x)m(y)} (A1_y, 1_x),
\]

where \( 1_v(w) = 1 \) if \( w = v \) and \( 1_v(w) = 0 \) otherwise.

We recall the following well known lemma which shows that the operator norm of \( A : \ell^p \to \ell^q \) can be estimated by its integral kernel.

**Lemma 3.5.** Let \( p \in [1,\infty] \) and let \( A \) be a densely defined linear operator with \( C_c(X) \subseteq D(A) \subseteq \ell^p \). Then,

(a) \( \|A\|_{p,q} \leq \left( \sum_y \|k_A(\cdot,y)\|_q^p m(y) \right)^{1/p} \) for \( q < \infty \),

(b) \( \|A\|_{p,\infty} \leq \sup_x \|k_A(x,\cdot)\|_p \), and equality holds if \( p = 1 \).
3.4. Heat kernel estimates. We denote the kernel of the semigroup $T_t$, $t \geq 0$ by $p_t$. As the semigroups are consistent on $L^p$, $p \in [1, \infty]$, i.e., they agree on their common domains, the kernel $p_t$ does not depend on $p$.

The following heat kernel estimate will be the key to $p$-independence of spectra of $L_p$. It is proven in [Pol11], based on [Day93], for locally finite graphs and $c \equiv 0$. However, on the one hand local finiteness is not used in [Pol11] for this result and, on the other hand, the remark below Lemma 3.1 shows that we are in the local finite situation anyway whenever we assume uniform subexponential growth. We conclude the statement for $c \geq 0$ by a Feynman-Kac formula.

**Lemma 3.6.** We have for all $t \geq 0$ and $x, y \in X$

$$p_t(x, y) \leq \left( m(x)m(y) \right)^{-1} e^{-\frac{1}{2}d(x,y)\log \frac{d(x,y)}{2m}}.$$  

**Proof.** Denote the semigroup of the graph $(b, 0)$ by $T^{(0)}_t$ and the kernel by $p^{(0)}_t$ and correspondingly for $(b, c)$ by $T_t$ and $p_t$.  

For $c \equiv 0$ the estimate is found in [Pol11, Theorem 2.1] for $p^{(0)}_t$. Now, by a Feynman-Kac formula, see e.g. [DvC00, GKS], we have

$$p_t(x, y) = T_t \delta_y(x) = \mathbb{E}_x \left[ e^{-\int_0^t \frac{b}{m}(X_s) ds} \delta_y(X_t) \right] \leq \mathbb{E}_x \left[ \delta_y(X_t) \right] = T^{(0)}_t \delta_y(x) = p^{(0)}_t(x, y),$$

where $\delta_y = 1_X/m$. This proves the claim. \qed

By basic calculus, we obtain the following heat kernel estimate.

**Lemma 3.7.** For all $\beta > 0$ there exists a constant $C(\beta)$ such that for all $t \geq 0$, $x, y \in X$

$$p_t(x, y) \leq \left( m(x)m(y) \right)^{-1} e^{-\frac{1}{2}d(x,y)+\beta t C(\beta)},$$

**Proof.** Let $\beta > 0$ and $x > 0$ let $f(x) = -x \log(x/2e) + \beta x$. Direct calculation shows that the function $f$ assumes its maximum on the domain $(0, \infty)$ at the point $x_0 = 2e^{\beta}$. In particular, setting $C(\beta) = 2e^{\beta}$ yields

$$-\frac{d(x,y)}{t} \log \frac{d(x,y)}{2et} \leq -\beta \frac{d(x,y)}{t} + C(\beta).$$

for all $t > 0$ and $x, y \in X$. \qed

4. Proof for uniform subexponential growth

In this section we prove Theorem 2.1 following the strategy of [Stu93]. The proof is divided into several lemmas and as always we assume that $d$ is an intrinsic metric with finite jump size $s$.

**Lemma 4.1.** For every compact set $K \subseteq \rho(L_2)$ there is $\varepsilon > 0$ and $C < \infty$ such that for all $z \in K$ and all $\psi \in \text{Lip}_\varepsilon$

$$\|e^{-\psi} (L_2 - z)^{-1} e^\psi\|_{2,2} \leq C.$$
Proof. Let $\epsilon > 0$ and $\psi \in \text{Lip}_\infty$. By Lemma 3.3 we have $e^{-\psi}D(Q) = e^\psi D(Q) = D(Q)$. Let $Q_\psi$ be the (not necessarily symmetric) form with domain $D(Q_\psi) = D(Q)$ acting as

$$Q_\psi(f,g) := Q(e^{-\psi}f,e^\psi g) - Q(f,g).$$

Application of Leibniz rule yields, for $f \in D(Q_\psi)$,

$$|Q_\psi(f,f)| = \frac{1}{2} \sum_{x,y \in X} b(x,y) |f(y)|^2 (e^{-\psi(x)} - e^{-\psi(y)}) (e^\psi(x) - e^\psi(y))$$

$$+ \frac{1}{2} \sum_{x,y \in X} b(x,y) \overline{f(y)} (f(x) - f(y)) (1 - e^{-\psi(y)-\psi(x)})$$

$$+ \frac{1}{2} \sum_{x,y \in X} b(x,y) f(y) (1 - e^{\psi(x)-\psi(y)}) (\overline{f(x)} - f(y)).$$

Applying Cauchy-Schwarz inequality, Lemma 3.4(b) and (c) and the intrinsic metric property, gives

$$\ldots \leq C \|f\|_2^2 \sum_{x,y \in X} |b(x,y)| d(x,y)^2 + 2C\epsilon \left( \sum_{x,y \in X} |f(x)|^2 b(x,y) d(x,y)^2 \right)^{\frac{1}{2}} \|Q(f)\|_2^{\frac{1}{2}}.$$

Hence, the basic inequality $2ab \leq (1/\delta) a^2 + \delta b^2$ for $\delta > 0$ and $a,b \geq 0$ (applied with $a = C\|f\|_2^2$ and $b = Q(f)^{\frac{1}{2}}$) yields

$$|Q_\psi(f,f)| \leq C \epsilon^2 (1 + \frac{1}{\delta}) \|f\|_2^2 + \delta Q(f).$$

This shows that $Q_\psi$ is $Q$ bounded with bound $0$. According to Kat95 Theorem VI.3.9 this implies that the form $Q_\psi + Q$ is closed and sectorial. It can be checked directly that the corresponding operator is $e^\psi L_2 e^{-\psi}$ with domain $D_\psi = e^\psi D(L_2)$. Moreover, for $K \subseteq \rho(L_2)$ compact, we can choose $\epsilon,\delta > 0$ that $2 \|C(1+1/\delta) e^2 e^{2\psi} + \delta L_2 \cdot (L_2 - z)^{-1}\|_{L_2} < 1$ for all $z \in K$ since $C$ is a universal constant. Therefore, again by Kat95 Theorem VI.3.9] this implies existence of $C = C(K,\epsilon)$ such that

$$\|e^{-\psi}(L_2 - z)^{-1}e^{\psi}\|_{L_2} = \|(e^\psi L_2 e^{-\psi} - z)^{-1}\|_{L_2} \leq C$$

for all $z \in K$ and $\psi \in \text{Lip}_\infty^\infty$. □

Let us recall some well known facts about consistency of semigroups and resolvents. By Dav89 Theorem 1.4.1] the semigroups $T_t$ are consistent on $\ell^p$. By the spectral theorem the Laplace transform for the resolvent $G_z = (L_2 - z)^{-1}$ holds for $f \in \ell^2$ and $z \in \{w \in \mathbb{C} \mid \Re w < 0\}$ (the open left half plane). By density and duality arguments, this formula extends to $f$ in $\ell^p$, $p \in [1,\infty)$, in the strong sense and to $p = \infty$ in the weak sense. This shows that the resolvents $(L_p - z)^{-1}$ are consistent on $\ell^p$ for $z \in \{w \in \mathbb{C} \mid \Re w < 0\}$.

We denote by $g_\alpha$ the kernel of the resolvent $G_\alpha = (L_p - \alpha)^{-1}$ which is independent of $p \in [1,\infty]$ for $\alpha < 0$.

**Lemma 4.2.** Assume the graph has uniform subexponential volume growth. For any $\epsilon > 0$ there exists $\alpha < 0$ and $C < \infty$ such that
Lemma 4.4. If \( p \) is a theorem that it is bounded for all \( x, y \in X \),

(b) \( \| e^{\alpha G_x e^{-\psi}} \| \leq C \) for all \( \psi \in \text{Lip}_\alpha^\infty \).

(c) \( \| m e^{\alpha G_x e^{-\psi}} \| \leq C \) for all \( \psi \in \text{Lip}_\alpha^\infty \).

Proof. (a) By the Laplace transform of the resolvent and Lemma 3.7 we get

\[
g_\alpha(x, y) = \int_0^\infty e^{\alpha t} p_t(x, y) dt \leq (m(x)m(y))^{-\frac{1}{\alpha}} e^{-\epsilon d(x, y)} \int_0^\infty e^{(\alpha + C)t} dt,
\]

which yields the statement for \( \alpha < -C \).

(b) By Lemma 3.3(a), \( \psi \in \text{Lip}_\alpha^\infty \), part (a) above (with \( \epsilon_1 \geq 2\epsilon \)) and Lemma 3.1(c)

\[
\| e^{\alpha G_x e^{-\psi}} \| \leq \sup_{y \in X} \| g_\alpha(\cdot, y) e^{\psi(y)} m(y) \|_2^2
\]

\[
\leq C \sup_{y \in X} \sum_{x \in X} e^{(2\epsilon - 2\epsilon_1)d(x, y)} < \infty.
\]

The proof of (c) works similarly using Lemma 3.3(b).

Lemma 4.3. Assume the graph has uniform subexponential volume growth. Then, \( (L_2 - z)^{-2} \) extends to a bounded operator on \( L^p \) for all \( z \in \rho(L_2) \) and \( p \in [1, \infty] \). Moreover, for all compact \( K \subseteq \rho(L_2) \) there is \( C < \infty \) such that for all \( z \in K \) and \( p \in [1, \infty] \)

\[
\|(L_2 - z)^{-2}\|_{p, p} \leq C.
\]

Proof. For \( z \in \rho(L_2) \) denote by \( g_2(z) \) the kernel of the squared resolvent \( (G_2)^2 = (L_2 - z)^{-2} \). Applying the resolvent identity twice yields

\[
(G_2)^2 = (G_\alpha + (z - \alpha)G_\alpha G_\alpha)(G_\alpha + (z - \alpha)G_\alpha G_\alpha) = G_\alpha(I + (z - \alpha)G_\alpha)^2 G_\alpha,
\]

for all \( \alpha < 0 \). Therefore,

\[
m^\frac{1}{2} e^{\psi}(G_\alpha)^2 e^{-\psi} m^\frac{1}{2} = \left( m^\frac{1}{2} e^{\psi} G_\alpha e^{-\psi} \right) \left( I + (z - \alpha) e^{\psi/2} G_\alpha e^{-\psi/2} \right)^2 \left( e^{\psi} G_\alpha e^{-\psi} m^\frac{1}{2} \right),
\]

for all \( \epsilon > 0 \) and \( \psi \in \text{Lip}_\alpha^\infty \). Taking the norm \( \| \cdot \|_1 \) and factorizing \( \| \cdot \|_1 \) yields that \( \| \cdot \|_2 \) yields that \( U := m^\frac{1}{2} e^{\psi} G_\alpha^2 e^{-\psi} m^\frac{1}{2} \) is a bounded operator \( \ell^1 \rightarrow \ell^\infty \) by Lemma 4.3 and Lemma 4.2 with appropriate choice of \( \alpha < 0 \), \( \epsilon > 0 \) and all \( \psi \in \text{Lip}_\alpha^\infty \). Hence, the operator \( U \) admits a kernel \( k_U(x, y) = (m(x)m(y))^{\frac{1}{2}} e^{\psi(x) - \psi(y)} g_2(z)(x, y) \), \( x, y \in X \), and we conclude from Lemma 3.5(b) that

\[
|g_2(z)(x, y)| \leq C(m(x)m(y))^{-\frac{1}{2}} e^{\psi(y) - \psi(y)}.
\]

For chosen \( \epsilon > 0 \) and any fixed \( x, y \in X \) let \( \psi : u \mapsto \epsilon(d(u, y) \wedge d(x, y)) \) and we obtain from Lemma 3.5(a) (with \( \epsilon \))

\[
|g_2(z)(x, y)| \leq C(m(x)m(y))^{-\frac{1}{2}} e^{-\epsilon d(x, y)} \leq C(m(x)^{-1} e^{-\frac{1}{2} \epsilon d(x, y)}.
\]

Thus, using Lemma 3.5(a) and Lemma 3.1(c), we obtain

\[
\|G_2(z)\|_{1, 1} \leq \sup_{y \in X} \sum_{x \in X} \|g_2(z)(x, y)\| m(x) \leq C \sup_{y \in X} \sum_{x \in X} e^{-\frac{1}{2} \epsilon d(x, y)} < \infty.
\]

As \( G_2(z) \) is bounded for \( p = 1 \) and \( p = 2 \), it follows from the Riesz-Thorin interpolation theorem that it is bounded for \( p \in [1, 2] \) and by duality for \( p \in [1, \infty] \).

Lemma 4.4. If \( \sigma(L_p) \subseteq [0, \infty) \) for all \( p \in [1, \infty] \), then \( \sigma(L_2) \subseteq \sigma(L_p) \).
Proof: The operators $(L_p - z)^{-1}$ and $(L_{pq} - z)^{-1}$ are consistent for $z \in \{ w \in \mathbb{C} \mid \Re w < 0 \} \subseteq \rho(L_p) = \rho(L_{pq})$ by the discussion above Lemma 4.2 for all $p \in [1, \infty]$. By the assumption $\sigma(L_p) \subseteq [0, \infty)$ the resolvent sets are connected which yields by [HV87, Corollary 1.4] that $(L_p - z)^{-1}$ and $(L_{pq} - z)^{-1}$ are consistent for $z \in \rho(L_p) \cap \rho(L_{pq})$ for $p, q \in [1, \infty]$. Moreover, by the standard theory [Dav07, Lemma 8.1.3] $(L_p - z)^{-1}$ and $(L_{pq} - z)^{-1}$ are analytic on $\rho(L_p) = \rho(L_{pq})$. By the Riesz-Thorin theorem these resolvents can be consistently extended to analytic $\ell^2$-bounded operators, see [Dav07, Lemma 1.4.8]. That is, as a $\ell^2$-bounded operator-valued function $(L_p - z)^{-1}$ is analytic on $\rho(L_p)$ which is consistent with $(L_q - z)^{-1}$ on $\rho(L_p) \cap \rho(L_q)$. Note that, $(L_q - z)^{-1}$ is analytic on $\rho(L_p)$ which is also the maximal domain of analyticity. Thus, the statement follows by unique continuation. 

Proof of Theorem 2.1. We start by showing $\sigma(L_p) \subseteq \sigma(L_q)$. For the kernel $g_z^{(2)}$ of $(L_q - z)^{-2}$ and fixed $x, y \in X$ the function $\rho(L_q) \to \mathbb{C}$, $z \mapsto g_z^{(2)}(x, y)$ is analytic. By Lemma 4.3 we know that for any compact $K \subseteq \rho(L_q)$ the operators $(L_q - z)^{-2}$, $z \in K$ are bounded on $\ell^p$, $p \in [1, \infty]$. Therefore $(L_q - z)^{-2}$ is analytic as a family of $\ell^p$-bounded operators for $z \in \rho(L_q)$. On the other hand, $(L_q - z)^{-2}$ is analytic as a family of $\ell^p$-bounded operators with domain of analyticity $\rho(L_p)$ by [HV86, Lemma 3.2]. Since $(L_q - z)^{-2}$ and $(L_p - z)^{-2}$ agree on $\{ w \in \mathbb{C} \mid \Re w < 0 \}$, by unique continuation they agree as analytic $\ell^p$ operator-valued functions on $\rho(L_q)$. As the domain of analyticity of $(L_q - z)^{-2}$ is $\rho(L_p)$, this implies $\rho(L_q) \subseteq \rho(L_p)$.

On the other hand, since $\sigma(L_p) \subseteq \sigma(L_q) \subseteq [0, \infty)$ the statement $\sigma(L_p) \subseteq \sigma(L_q)$ follows from Lemma 4.3. □

5. Proof for uniformly positive measures

In this section we consider measures that are uniformly bounded from below by a positive constant and prove Theorem 2.2. We notice that $\inf_{x \in X} m(x) > 0$ implies

$$\ell^p \subseteq \ell^q, \quad 1 \leq p \leq q \leq \infty.$$ 

Moreover, by [KL12, Theorem 5] we know the domains of the generators $L_p$ in this case explicitly, namely

$$D(L_p) := \{ f \in \ell^p \mid \tilde{L} f \in \ell^p \}.$$ 

In particular, this gives

$$D(L_p) \subseteq D(L_q), \quad 1 \leq p \leq q \leq \infty,$$

where $\tilde{L}$ was defined in Section 2.3. Furthermore, it can be checked directly that $C_c(X) \subseteq D(L_p)$.

Lemma 5.1. Assume $\inf_{x \in X} m(x) > 0$. Then, for all $1 \leq p \leq q \leq \infty$ and all $z \in \rho(L_p) \cap \rho(L_q)$ the resolvents $(L_p - z)^{-1}$ and $(L_q - z)^{-1}$ are consistent on $\ell^p = \ell^p \cap \ell^q$.

Proof. Let $1 \leq p < q \leq \infty$ and $z \in \rho(L_p) \cap \rho(L_q)$. As $D(L_p) \subseteq D(L_q)$ and $L_q = L_p$ on $D(L_p)$, we have for all $f \in \ell^p \subseteq \ell^q$

$$(L_q - z)(L_p - z)^{-1} f = (L_p - z)(L_p - z)^{-1} f = f.$$ 

Hence, $(L_p - z)^{-1}$ and $(L_q - z)^{-1}$ are consistent on $\ell^p = \ell^p \cap \ell^q$. □
Proof of Theorem 2.2. Let \( p \in [1, 2] \). By the lemma above the resolvents \((L_p - z)^{-1}\) and \((L_p' - z)^{-1}\) are consistent for \( z \in \rho(L_p) = \rho(L_p') \) on \( \ell^p \). By the Riesz-Thorin interpolation theorem \((L_p' - z)^{-1}\) is bounded on \( \ell^2 \). We will show that \((L_p - z)^{-1}\) is an inverse of \((L_2 - z)\) for \( z \in \rho(L_p) = \rho(L_p') \). So, let \( z \in \rho(L_p) \). As \( D(L_2) \subseteq D(L_p') \), \( \ell^2 \subseteq \ell^p' \) and \( L_p, L_p' \) are restrictions of \( \tilde{L} \) we have for \( f \in D(L_2) \)

\[
(L_p - z)^{-1}(L_2 - z)f = (L_p' - z)^{-1}(L_p' - z)f = f.
\]

Secondly, let \( f \in \ell^2 \) and \((f_n)\) be such that \( f_n \in \ell^p \) and \( f_n \to f \) in \( \ell^2 \). As \((L_p' - z)^{-1}\) is \( \ell^2 \)-bounded, \((L_p' - z)^{-1}f_n \to (L_p' - z)^{-1}f, n \to \infty, \) in \( \ell^2 \). By the lemma above \((L_p' - z)^{-1}f_n = (L_p - z)^{-1}f_n \in D(L_2) \), and, thus,

\[
(L_2 - z)(L_p' - z)^{-1}f_n = (L_p' - z)(L_p' - z)^{-1}f_n = f_n \to f, \quad n \to \infty,
\]

in \( \ell^2 \). Since, \( L_2 \) is closed we infer \((L_p' - z)^{-1}f \in D(L_2) \) and \((L_2 - z)(L_p' - z)^{-1}f = f \). Hence, \((L_p' - z)^{-1}\) is an inverse of \((L_2 - z)\) and, thus, \( z \in \rho(L_2) \).

\[ \Box \]

Remark. The abstract reason behind Theorem 2.2 is that the semigroup \( e^{-t\tilde{L}} \) is ultracontractive, i.e. a bounded operator from \( \ell^2 \) to \( \ell^\infty \), which is a consequence of \( e^{-t\tilde{L}} \) being a contraction on \( \ell^\infty \) (as \( Q \) is a Dirichlet form) and the uniform lower bound on the measure. Knowing this one can deduce by duality and interpolation that \( e^{-t\tilde{L}} \) is a bounded operator from \( \ell^p \) to \( \ell^q, p \leq q, \) (cf. [Sim82, Proof of Theorem B.1.1]) and employ the proof of [HV86, Proposition 2.1] or [HV87, Proposition 3.1].

6. Spectral Properties of Normalized Laplacians

In this section, we consider normalized Laplace operators that is we assume \( m = n, c \equiv 0 \) where \( n(x) = \sum_{y \in X} b(x, y), \) \( x \in X \).

6.1. Symmetries of the spectrum. Recall that a graph is called bipartite if the vertex set \( X \) can be divided into two disjoint subsets \( X_1 \) and \( X_2 \) such that every edge connects a vertex in \( X_1 \) to a vertex in \( X_2 \), i.e., \( b(x, y) > 0 \) for a pair \((x, y) \in X \times X \) implies \((x, y) \in (X_1 \times X_2) \cup (X_2 \times X_1) \).

Theorem 6.1. Assume \( m = n, c \equiv 0 \) and let \( p \in [1, \infty] \). Then, \( \sigma(L_p) \) is included in \( \{z \in \mathbb{C} \mid |z - 1| \leq 1\} \) and is symmetric with respect to \( \mathbb{R} = \{z \in \mathbb{C} \mid \Im z = 0\} \), i.e., \( \lambda \in \sigma(L_p) \) if and only if \( \overline{\lambda} \in \sigma(L_p) \). Moreover, if the graph is bipartite, then \( \sigma(L_p) \) is symmetric with respect to the line \( \{z \in \mathbb{C} \mid \Re z = 1\} \), i.e., \( \lambda \in \sigma(L_p) \) if and only if \( (2 - \lambda) \in \sigma(L_p) \).

The proof of the second part of the theorem is based on the following lemma, [Dav07, Lemma 1.2.13].

Lemma 6.2. Let \( A : \mathcal{B} \to \mathcal{B} \) be a bounded operator on a Banach space \( \mathcal{B} \). Then \( \lambda \in \sigma(A) \) if and only if at least one of the following occurs:

(i) \( \lambda \) is an eigenvalue of \( A \).

(ii) \( \lambda \) is an eigenvalue of \( A^* \), where \( A^* \) is the dual operator of \( A \).

(iii) There exists a sequence \((f_n)\) in \( \mathcal{B} \) with \( ||f_n|| = 1 \) and \( \lim_{n \to \infty} ||Af_n - \lambda f_n|| = 0 \).

Proof of Theorem 6.1. Let \( p \in [1, \infty] \). As \( n/m = 1 \) the operator \( L_p \) is bounded on \( \ell^p \) by [KL10, Theorem 11] (or [HKLW12, Theorem 9.3]) with bound 2. Moreover, the operator \( L_p \) can be represented as \( L_p = I - P_p \) where \( P_p \) is the transition matrix acting as \( P_p f(x) = \frac{1}{n(x)} \sum_y b(x, y) f(y) \). Direct calculation shows that \( ||P_p||_{p,p} \leq 1 \).
As the spectral radius is smaller than the norm, \( \sigma(P_p) \subseteq \{ z \in \mathbb{C} \mid |z| \leq 1 \} \). Now, \( I \) is a spectral shift by 1, so the first statement follows.

The first symmetry statement follows from the fact that the integral kernel of \( L_p \) is real valued.

Let now \( X_1 \) and \( X_2 \) be a bipartite partition of \( X \). For a function \( f : X \to \mathbb{C} \), we denote

\[
\tilde{f} = 1_{X_1} f - 1_{X_2} f,
\]

where \( 1_W \) is the characteristic function of \( W \subseteq X \). We separate the proof into three cases according to the preceding lemma.

Case 1: \( \lambda \) is an eigenvalue of \( L_p \) with eigenfunction \( f \). By direct calculation it can be checked that \( \tilde{f} \) is an eigenfunction of \( L_p \) for the eigenvalue \( 2 - \lambda \).

Case 2: \( \lambda \) is an eigenvalue of \( L_p^* \). By the same argument as in Case 1 we get that \( 2 - \lambda \) is an eigenvalue of \( L_p^* \). Therefore, by \( \sigma(L_p^*) = \sigma(L_p) \), we conclude the result.

Case 3: \( \lambda \) is such that there is \( f_n \) with \( \| f_n \| = 1 \), \( n \geq 1 \) and \( \lim \| (L_p - \lambda)f_n \|_p = 0 \).

It is easy to see that \( \| \tilde{f}_n \|_p = \| f_n \|_p \) and by direct calculation it follows \( \| (L_p - (2 - \lambda))f_n \|_p = \| (L_p - \lambda)f_n \|_p \). Thus, the statement \( (2 - \lambda) \in \sigma(L_p) \) follows by Lemma 6.2(iii).

6.2. Cheeger constants. Define the Cheeger constant \( \alpha \geq 0 \) to be the maximal \( \beta \geq 0 \) such that for all finite \( W \subseteq X \)

\[
\beta n(W) \leq |\partial W|,
\]

where \( |\partial W| := \sum_{(x,y) \in W \times (X \setminus W)} b(x,y) \). For recent developments concerning Cheeger constants see also [BHJ12, BKW12].

**Proposition 6.3.** Assume \( m = n \) and \( c \equiv 0 \). Then, \( \inf \sigma(L_1) = \inf \sigma(L_2) \) if and only if \( \alpha = 0 \).

**Proof.** As \( L_\infty \) is bounded it is easy to see that the constant functions are eigenfunctions to the eigenvalue 0. Hence, \( \inf \sigma(L_1) = \inf \sigma(L_\infty) = 0 \). Now, \( 0 = \inf \sigma(L_2) \) is equivalent to \( \alpha = 0 \) by a Cheeger inequality, [KL10] (cf. [DK86] and [DK88]).

**Remark.** The proposition can also be obtained as a consequence of [Tak07, Theorem 3.1]. The considerations therein show that one direction of the result is still valid in certain situations involving unbounded operators using the Cheeger constant defined in [BKWW12]. However, in this case it is usually hard to determine whether the constant functions are in the domain of \( L_\infty \). Nevertheless, this is the case if the graph is stochastically complete, (see [KL12] for characterizations of this case).

6.3. Spectral independence and superexponential growth. This previous proposition shows that \( \alpha = 0 \) is a necessary condition for the \( p \)-independence of the spectrum for \( p \in \{1, \infty \} \). However, the next proposition shows that if we exclude \( p = \{1, \infty \} \), then we can have \( p \)-independence of the spectrum even if \( \alpha > 0 \) or if the subexponential volume growth condition is not satisfied.

Define the **Cheeger constant at infinity** \( \alpha_\infty \geq 0 \) to be the maximum of all \( \beta \geq 0 \) such that for some finite \( K \subseteq X \) and all finite \( W \subseteq X \setminus K \) and \( \beta n(W) \leq |\partial W| \).

One can easily check that under the assumption of \( m(X) = \infty \), \( \alpha > 0 \) if and only if \( \alpha_\infty > 0 \). The inequality \( \alpha \leq \alpha_\infty \) is obvious. If \( \alpha = 0 \) but \( \alpha_\infty > 0 \), the Cheeger estimates, see e.g. [Fuj96b, Kel10, KL10] imply \( 0 \in \sigma(L_2) \), but \( 0 \not\in \sigma_{\text{ess}}(L_2) \). This,
Theorem 6.4. Assume \( m = n \) and \( \alpha_\infty = 1 \). Then, \( \sigma(L_p) = \sigma(L_2) \) for all \( p \in (1, \infty) \) and the graph has superexponential volume growth with respect to the natural graph metric, i.e., \( \lim_{r \to \infty} \frac{1}{r} \log n(B_r(x)) = \infty \), for all \( x \in X \).

Proof. It was shown in [KL10] (cf. [Kel10, Fuj96b] for the unweighted case) that \( \alpha_\infty = 1 \) implies \( \sigma_{\text{ess}}(L_2) = \{1\} \) which is equivalent to \( I - L_2 \) being a compact operator. Since \( I - L_p \) is a consistent family of bounded operators for \( p \in [1, \infty] \) it follows from a well-known result by Krasnosel’skii and Persson (see [Dav07, Theorem 4.2.14]) that \( I - L_p \) is compact for all \( p \in [2, \infty) \). Using a theorem of Schauder (see for instance [Dav07, Theorem 4.2.13]), we conclude that \( I - L_p \) is compact for all \( p \in (1, \infty) \). Now it follows from [Dav07, Theorem 4.2.15] that the spectrum of \( L_p \) is \( p \)-independent for all \( p \in (1, \infty) \). The second part of the proposition follows directly from [Fuj96b, Theorem 1] (for the weighted case combine [KL10, Theorem 19] and [HKW12, Theorem 4.1]). \( \square \)

Remark 6.5. (a) For examples satisfying the assumptions of the theorem above see [Kel10, Fuj96b].

(b) Under the assumption \( m = n \) and \( \alpha > 0 \) one can show that \( I - L_1 \) and \( I - L_\infty \) are not compact operators. Assume the contrary, i.e., \( I - L_1 \) and \( I - L_\infty \) are compact, then [Dav07, Theorem 4.2.15] implies that the spectrum is \( p \)-independent for all \( p \in [1, \infty] \). This, however, contradicts Proposition 6.3.

7. Tessellations and Curvature

For this final section, we restrict our attention to planar tessellations and relate curvature bounds to volume growth and \( p \)-independence. We show analogue statements to Proposition 1 and 2 of [Stu93] and discuss the case of uniformly unbounded negative curvature.

We consider graphs such that \( b \) takes values in \( \{0,1\} \) and \( c \equiv 0 \). The two prominent choices for \( m \) are either \( m = n \) or \( m = 1 \). For \( m = n \) the natural graph metric \( d_n \) is an intrinsic metric and for \( m = 1 \) the path metric \( d_1 \) given by

\[
d_1(x,y) = \inf_{x_0,\ldots,x_n = y} \sum_{i=1}^{n} (n(x_{i-1}) \lor n(x_i))^{-\frac{1}{2}}, \quad x,y \in X,
\]

is an intrinsic metric. Both metrics have the jump size at most 1, in particular, both metrics have finite jump size. We denote the Laplacian with respect to \( m = n \) by \( \Delta_p^{(n)} \) and with respect to \( m = 1 \) by \( \Delta_p \), \( p \in [1, \infty] \).

Note that by Theorem 2.2 we have for all \( p \in [1, \infty] \)

\[
\sigma(\Delta_2^{(n)}) \subseteq \sigma(\Delta_p^{(n)}) \quad \text{and} \quad \sigma(\Delta_2) \subseteq \sigma(\Delta_p).
\]

Let \( G \) be a planar tessellation, see [BP01, BP06] for background. Denote by \( F \) the set of faces and denote the degree of a face \( f \in F \), that is the number of vertices contained in face, by \( \deg(f) \).

We define the vertex curvature \( \kappa : X \to \mathbb{R} \) by

\[
\kappa(x) = 1 - \frac{n(x)}{2} + \sum_{f \in F, x \in f} \frac{1}{\deg(f)}.
\]
The following theorem is a discrete analogue of [Stu93, Proposition 1].

**Theorem 7.1.** If \( \kappa \geq 0 \), then it has quadratic volume growth both with respect to \( d_n \) and \( m = n \) and with \( d_1 \) and \( m \equiv 1 \). In particular, \( \sigma(\Delta_{p}^{(n)}) = \sigma(\Delta_{2}^{(n)}) \) and \( \sigma(\Delta_{p}) = \sigma(\Delta_{2}) \) for all \( p \in [1, \infty] \).

**Proof.** By [HJL, Theorem 3] the curvature assumption is equivalent to the measure \( m = n \) and the metric \( d_n \). Moreover, it is direct to check that \( \kappa \geq 0 \) implies that the vertex degree is bounded by 6. Hence, we have bounded geometry and thus \( d_n \) and \( d_1 \) are equivalent and for \( m \equiv 1 \) we have \( m \geq n/6 \). Thus, the volume growth of the graph is quadratically bounded also with respect to the measure \( m \equiv 1 \) and the metric \( d_1 \). The ‘in particular’ is now a consequence of Theorem 7.1.

**Remark.** The result can be easily extended to planar tessellation with finite total curvature, i.e., \( \sum_{x \in X} |\kappa(x)| < \infty \) or equivalently vanishing curvature outside of a finite set, see [CC08] and also [DM07]. This can be seen as [HJL, Theorem 1.1] easily extends to the finite total curvature case.

The next theorem is a discrete analogue of [Stu93, Proposition 2]. We say that a graph with measure \( m \) has at least exponential volume growth with respect to a metric \( d \) if for the corresponding distance balls \( B_r(x) \) about some vertex \( x \),

\[
\mu = \liminf_{r \to \infty} \inf_{x \in X} \frac{1}{r} \log m(B_r(x)) > 0.
\]

**Theorem 7.2.** If \( \kappa < 0 \), then the tessellation has at least exponential volume growth, both with respect to \( d_n \) and \( m = n \) and with \( d_1 \) and \( m \equiv 1 \). Moreover, \( \inf \sigma(\Delta_{1}^{(n)}) \neq \inf \sigma(\Delta_{2}^{(n)}) \) and if we have additionally bounded geometry, then \( \inf \sigma(\Delta_{1}) \neq \inf \sigma(\Delta_{2}) \).

**Proof.** [Hig01, Theorem C, Proposition 2.1] implies that if \( \kappa(x) < 0 \) for all \( x \in X \), then \( \sup_{x \in X} \kappa(x) \leq -1/1806 < 0 \). Hence, by [Hig01, Theorem B] we have positive Cheeger constant, \( \alpha > 0 \). Thus by a Cheeger inequality [DK86] \( \inf \sigma(\Delta_{1}^{(n)}) \geq \alpha^2/2 > 0 \) and by elementary computations (cf. [Ke10]) we infer \( \inf \sigma(\Delta_{2}) \geq \inf \sigma(\Delta_{2}^{(n)}) \cdot \inf \deg(x) > 0 \). By [HKW12, Corollary 4.2] this implies at least exponential volume growth with respect to both metrics. The second statement follows along the lines of the proof of Proposition 6.4.

We end this section by an analogue theorem of Theorem 6.4.

**Theorem 7.3.** If \( \inf_{K \subseteq X, \text{ finite}} \sup_{x \in X \setminus K} \kappa(x) = -\infty \), then \( \sigma(\Delta_{p}^{(n)}) = \sigma(\Delta_{2}^{(n)}) \) and \( \sigma(\Delta_{p}) = \sigma(\Delta_{2}) \) for all \( p \in (1, \infty) \). In particular, the spectrum is purely discrete and the eigenfunctions of \( \Delta_{2} \) are contained in \( \ell^p \) for all \( p \in (1, \infty) \).

**Proof.** By [Ke10, Theorem 3] the curvature assumption is equivalent to pure spectrum of \( \Delta_{2} \). Moreover, this is equivalent to compact resolvent and compact semigroup on \( \ell^2 \). Thus, the statement for \( \Delta_{p}^{(n)} \) follows from [Dav89, Theorem 1.6.3]. On the other hand, the curvature assumption implies \( \alpha_{\infty} = 1 \), [Ke10], and the statement about \( \Delta_{p}^{(n)} \) follows from Theorem 6.4.

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