Ergodicity of principal algebraic group actions

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Dedicated to Shrikrishna Gopalrao Dani on the occasion of his 65th birthday

Abstract. An algebraic action of a discrete group $\Gamma$ is a homomorphism from $\Gamma$ to the group of continuous automorphisms of a compact abelian group $X$. By duality, such an action of $\Gamma$ is determined by a module $M = \hat{X}$ over the integer group ring $\mathbb{Z}\Gamma$ of $\Gamma$. The simplest examples of such modules are of the form $M = \mathbb{Z}\Gamma/\mathbb{Z}\Gamma f$ with $f \in \mathbb{Z}\Gamma$; the corresponding algebraic action is the principal algebraic $\Gamma$-action $\alpha_f$ defined by $f$.

In this note we prove the following extensions of results by Hayes [2] on ergodicity of principal algebraic actions: If $\Gamma$ is a countably infinite discrete group which is not virtually cyclic, and if $f \in \mathbb{Z}\Gamma$ satisfies that right multiplication by $f$ on $\ell^2(\Gamma, \mathbb{R})$ is injective, then the principal $\Gamma$-action $\alpha_f$ is ergodic (Theorem 1.3). If $\Gamma$ contains a finitely generated subgroup with a single end (e.g. a finitely generated amenable subgroup which is not virtually cyclic), or an infinite nonamenable subgroup with vanishing first $\ell^2$-Betti number (e.g., an infinite property $T$ subgroup), the injectivity condition on $f$ can be replaced by the weaker hypothesis that $f$ is not a right zero-divisor in $\mathbb{Z}\Gamma$ (Theorem 1.2). Finally, if $\Gamma$ is torsion-free, not virtually cyclic, and satisfies Linnell’s analytic zero-divisor conjecture, then $\alpha_f$ is ergodic for every $f \in \mathbb{Z}\Gamma$ (Remark 1.5).

1. Principal Algebraic Group Actions

Let $\Gamma$ be a countably infinite discrete group with integral group ring $\mathbb{Z}\Gamma$. Every $g \in \mathbb{Z}\Gamma$ is written as a formal sum $g = \sum_{\gamma} g_\gamma \cdot \gamma$, where $g_\gamma \in \mathbb{Z}$ for every $\gamma \in \Gamma$ and $\sum_{\gamma \in \Gamma} |g_\gamma| < \infty$. The set $\text{supp}(g) = \{ \gamma \in \Gamma : g_\gamma \neq 0 \}$ is called the
support of $g$. For $g = \sum_{\gamma \in \Gamma} g_{\gamma} \cdot \gamma \in \mathbb{Z}\Gamma$ we denote by $g^* = \sum_{\gamma \in \Gamma} g_{\gamma} \cdot \gamma^{-1}$ the adjoint of $g$. The map $g \mapsto g^*$ is an involution on $\mathbb{Z}\Gamma$, i.e., $(gh)^* = h^*g^*$ for all $g, h \in \mathbb{Z}\Gamma$, where the product $fg$ of two elements $f = \sum_{\gamma \in \Gamma} f_{\gamma} \cdot \gamma$ and $g = \sum_{\gamma \in \Gamma} g_{\gamma} \cdot \gamma$ in $\mathbb{Z}\Gamma$ is given by $fg = \sum_{\gamma, \gamma' \in \Gamma} f_{\gamma} g_{\gamma'} \cdot \gamma \gamma'$.

An algebraic $\Gamma$-action is a homomorphism $\alpha : \Gamma \to \text{Aut}(X)$ from $\Gamma$ to the group of (continuous) automorphisms of a compact metrizable abelian group $X$. If $\alpha$ is an algebraic $\Gamma$-action, then $\hat{\alpha} \in \text{Aut}(X)$ denotes the image of $\gamma \in \Gamma$, and $\alpha^{\gamma \gamma'} = \alpha^{\gamma} \alpha^{\gamma'}$ for every $\gamma, \gamma' \in \Gamma$. The $\Gamma$-action $\alpha$ induces an action of $\mathbb{Z}\Gamma$ on $X$ by group homomorphisms $\alpha^f : X \to X$, where $\alpha^f = \sum_{\gamma \in \Gamma} f_{\gamma} \hat{\alpha}^{\gamma}$ for every $f = \sum_{\gamma \in \Gamma} f_{\gamma} \cdot \gamma \in \mathbb{Z}\Gamma$. Clearly, if $f, g \in \mathbb{Z}\Gamma$, then $\alpha^{fg} = \alpha^f \alpha^g$.

Let $\hat{X}$ be the dual group of $X$. If $\hat{\alpha}$ is the automorphism of $\hat{X}$ dual to $\alpha$, then the map $\hat{\alpha} : \Gamma \to \text{Aut}(\hat{X})$ satisfies that $\hat{\alpha}^{\gamma \gamma'} = \hat{\alpha}^{\gamma} \hat{\alpha}^{\gamma'}$ for all $\gamma, \gamma' \in \Gamma$. We write $\hat{\alpha}^f : \hat{X} \to \hat{X}$ for the group homomorphism dual to $\alpha^f$ and set $f \cdot a = \hat{\alpha}^f a$ for every $f \in \mathbb{Z}\Gamma$ and $a \in \hat{X}$. The resulting map $(f, a) \mapsto f \cdot a$ from $\mathbb{Z}\Gamma \times \hat{X}$ to $\hat{X}$ satisfies that $(fg) \cdot a = f \cdot (g \cdot a)$ for all $f, g \in \mathbb{Z}\Gamma$ and turns $\hat{X}$ into a module over the group ring $\mathbb{Z}\Gamma$. Conversely, if $M$ is a countable module over $\mathbb{Z}\Gamma$, we set $X = \hat{M}$ and put $\hat{\alpha}^f a = f^* \cdot a$ for $f \in \mathbb{Z}\Gamma$ and $a \in M$. The maps $\alpha^f : \hat{M} \to \hat{M}$ dual to $\hat{\alpha}^f$, $f \in \mathbb{Z}\Gamma$, define an action of $\mathbb{Z}\Gamma$ by homomorphisms of $\hat{M}$, which in turn induces an algebraic action $\alpha$ of $\Gamma$ on $X = \hat{M}$.

The simplest examples of algebraic $\Gamma$-actions arise from $\mathbb{Z}\Gamma$-modules of the form $M = \mathbb{Z}\Gamma/\mathbb{Z}\Gamma f$ with $f \in \mathbb{Z}\Gamma$. Since these actions are determined by principal left ideals of $\mathbb{Z}\Gamma$ they are called principal algebraic $\Gamma$-actions. In order to describe these actions more explicitly we put $T = \mathbb{R}/\mathbb{Z}$ and define the left and right shift-actions $\lambda$ and $\rho$ of $\Gamma$ on $T^\Gamma$ by setting

$$ (\lambda^\gamma x)_{\gamma'} = x_{\gamma^{-1} \gamma'}, \quad (\rho^\gamma x)_{\gamma'} = x_{\gamma \gamma'}, \quad (1.1) $$

for every $\gamma \in \Gamma$ and $x = (x_{\gamma})_{\gamma' \in \Gamma} \in T^\Gamma$. The $\Gamma$-actions $\lambda$ and $\rho$ extend to actions of $\mathbb{Z}\Gamma$ on $T^\Gamma$ given by

$$ \lambda^f = \sum_{\gamma \in \Gamma} f_{\gamma} \lambda^\gamma, \quad \rho^f = \sum_{\gamma \in \Gamma} f_{\gamma} \rho^\gamma \quad (1.2) $$

for every $f = \sum_{\gamma \in \Gamma} f_{\gamma} \cdot \gamma \in \mathbb{Z}\Gamma$.

The pairing $\langle f, x \rangle = e^{2\pi i \sum_{\gamma \in \Gamma} f_{\gamma} x_{\gamma}}, f = \sum_{\gamma \in \Gamma} f_{\gamma} \cdot \gamma \in \mathbb{Z}\Gamma$, $x = (x_{\gamma}) \in T^\Gamma$, identifies $\mathbb{Z}\Gamma$ with the dual group $\hat{\mathbb{T}}^\Gamma$ of $T^\Gamma$. We claim that, under this identification,

$$ X_f := \ker \rho^f = \{ x \in T^\Gamma : \rho^f x = \sum_{\gamma \in \Gamma} f_{\gamma} \rho^\gamma x = 0 \} = (\mathbb{Z}\Gamma f)^\perp \subset \hat{\mathbb{T}}^\Gamma = T^\Gamma. \quad (1.3) $$

Indeed,

$$ \langle h, \rho^f x \rangle = \langle h, \sum_{\gamma' \in \Gamma} f_{\gamma'} \rho^{\gamma'} x \rangle = e^{2\pi i \sum_{\gamma \in \Gamma} h_{\gamma} \sum_{\gamma' \in \Gamma} f_{\gamma'} x_{\gamma' \gamma'}} \quad (h \in T^\Gamma, x \in T^\Gamma). $$

In particular, $X_f$ is a closed subspace of $T^\Gamma$.
These definitions correspond to the usual convolutions for every $h \in \mathbb{Z}\Gamma$ and $x \in \mathbb{T}\Gamma$, so that $x \in \ker \rho^f$ if and only if $x \in (\mathbb{Z}\Gamma f)\perp$.

Since the $\Gamma$-actions $\lambda$ and $\rho$ on $\mathbb{T}\Gamma$ commute, the group $X_f = \ker \rho^f \subset \mathbb{T}\Gamma$ is invariant under $\lambda$, and we denote by $\alpha_f$ the restriction of $\lambda$ to $X_f$. In view of this we adopt the following terminology.

**Definition 1.1.** $(X_f, \alpha_f)$ is the principal algebraic $\Gamma$-action defined by $f \in \mathbb{Z}\Gamma$.

In [2] the author calls a countably infinite discrete group $\Gamma$ *principally ergodic* if every principal algebraic $\Gamma$-action $\alpha_f$, $f \in \mathbb{Z}\Gamma$, is ergodic w.r.t. Haar measure on $X_f$ and proves that the following classes of groups are principally ergodic: torsion-free nilpotent groups which are not virtually cyclic, free groups on more than one generator, and groups which are not finitely generated.

In order to state our extensions of these results we denote by $\ell^\infty(\Gamma, \mathbb{R}) \subset \mathbb{R}^\Gamma$, the space of bounded real-valued maps $v = (v_\gamma)$ on $\Gamma$, where $v_\gamma$ is the value of $v$ at $\gamma$, and we write $\|v\|_{\ell^\infty} = \sup_{\gamma \in \Gamma} |v_\gamma|$ for the supremum norm on $\ell^\infty(\Gamma, \mathbb{R})$. For $1 \leq p < \infty$ we set $\ell^p(\Gamma, \mathbb{R}) = \{v = (v_\gamma) \in \ell^\infty(\Gamma, \mathbb{R}) : \|v\|_p = (\sum_{\gamma \in \Gamma} |v_\gamma|^p)^{1/p} < \infty\}$. By $\ell^p(\Gamma, \mathbb{Z}) = \ell^p(\Gamma, \mathbb{R}) \cap \mathbb{Z}^\Gamma$ we denote the additive subgroup of integer-valued elements of $\ell^p(\Gamma, \mathbb{R})$; for $1 \leq p < \infty$, $\ell^p(\Gamma, \mathbb{Z}) = \ell^1(\Gamma, \mathbb{Z})$ is identified with $\mathbb{Z}\Gamma$ by viewing each $g = \sum_{\gamma} g_\gamma \cdot \gamma \in \mathbb{Z}\Gamma$ as the element $(g_\gamma)_{\gamma \in \Gamma} \in \ell^1(\Gamma, \mathbb{Z})$.

The group $\Gamma$ acts on $\ell^p(\Gamma, \mathbb{R})$ isometrically by left and right translations: for every $v \in \ell^p(\Gamma, \mathbb{R})$ and $\gamma \in \Gamma$ we denote by $\lambda^\gamma v$ and $\rho^\gamma v$ the elements of $\ell^p(\Gamma, \mathbb{R})$ satisfying $(\lambda^\gamma v)_\gamma' = v_{\gamma' \cdot \gamma}$ and $(\rho^\gamma v)_\gamma' = v_\gamma \cdot \gamma'$, respectively, for every $\gamma' \in \Gamma$. Note that $\lambda^\gamma \lambda' = \lambda^{\gamma \cdot \gamma'}$ and $\rho^\gamma \rho' = \rho^\gamma \rho' = \rho^{\gamma \cdot \gamma'}$ for every $\gamma, \gamma' \in \Gamma$.

The $\Gamma$-actions $\lambda$ and $\rho$ extend to actions of $\ell^1(\Gamma, \mathbb{R})$ on $\ell^p(\Gamma, \mathbb{R})$ which will again be denoted by $\lambda$ and $\rho$: for $h = (h_\gamma) \in \ell^1(\Gamma, \mathbb{R})$ and $v \in \ell^p(\Gamma, \mathbb{R})$ we set

$$\lambda^h v = \sum_{\gamma \in \Gamma} h_\gamma \lambda^\gamma v, \quad \rho^h v = \sum_{\gamma \in \Gamma} h_\gamma \rho^\gamma v. \quad (1.4)$$

These definitions correspond to the usual convolutions

$$\lambda^h v = h \cdot v, \quad \rho^h v = v \cdot h^*, \quad (1.5)$$

where $h \mapsto h^*$ is the involution on $\ell^1(\Gamma, \mathbb{C})$ defined as for $\mathbb{Z}\Gamma$: $h^*_\gamma = \overline{h_{\gamma^{-1}}}$, $\gamma \in \Gamma$, for every $h = (h_\gamma) \in \ell^1(\Gamma, \mathbb{C})$. For $p = 2$, the bounded linear operators $\lambda^h, \rho^h : \ell^2(\Gamma, \mathbb{R}) \rightarrow \ell^2(\Gamma, \mathbb{R})$ in (1.4) can be viewed as elements of the right (resp. left) equivariant group von Neumann algebra of $\Gamma$.

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1A discrete group $\Gamma$ is *virtually cyclic* if it has a cyclic finite-index subgroup. Virtually cyclic groups can obviously not be principally ergodic: if $\Gamma = \mathbb{Z}$, and if $\mathbb{Z}\Gamma$ is identified with the ring of Laurent polynomials $\mathbb{Z}[u \pm 1]$ in the obvious manner, then the principal algebraic $\mathbb{Z}$-action $\alpha_f$ defined by $f = 1 - u$ is trivial — and hence nonergodic — on $X_f = \mathbb{T}^\Gamma$. 
THEOREM 1.2. Let $\Gamma$ be a countably infinite discrete group which satisfies one of the following conditions:

1. $\Gamma$ contains a finitely generated amenable subgroup which is not virtually cyclic, or more generally, a finitely generated subgroup with a single end,
2. $\Gamma$ is not finitely generated,
3. $\Gamma$ contains an infinite property T subgroup, or more generally, a nonamenable subgroup $\Gamma_0$ with vanishing first $\ell^2$-Betti number $\beta_1(\Gamma_0) = 0$.

If $f \in \mathbb{Z} \Gamma$ is not a right zero-divisor, then the principal $\Gamma$-action $\alpha_f$ on $X_f$ is ergodic (with respect to the normalized Haar measure of $X_f$).

THEOREM 1.3. Let $\Gamma$ be a countably infinite discrete group which is not virtually cyclic. If $f \in \mathbb{Z} \Gamma$ satisfies that $\ker \tilde{\rho}^f = \{ v \in \ell^2(\Gamma, \mathbb{R}) : \tilde{\rho}^f(v) = v \cdot f = 0 \} = \{0\}$, then the principal $\Gamma$-action $\alpha_f$ on $X_f$ is ergodic.

In view of the hypotheses on $f$ in the Theorems 1.2 and 1.3 it is useful to recall the following result.

PROPOSITION 1.4. Let $\Gamma$ be a countably infinite discrete amenable group. For every $f \in \mathbb{Z} \Gamma$ the following conditions are equivalent.

1. $f$ is a right zero-divisor in $\mathbb{Z} \Gamma$,
2. $\{ v \in \ell^2(\Gamma, \mathbb{R}) : f^* \cdot v = 0 \} \neq \{0\}$,
3. $f$ is a left zero-divisor in $\mathbb{Z} \Gamma$,
4. $\{ v \in \ell^2(\Gamma, \mathbb{R}) : f \cdot v = 0 \} \neq \{0\}$,
5. $\ker \tilde{\rho}^f = \{ v \in \ell^2(\Gamma, \mathbb{R}) : \tilde{\rho}^f(v) = 0 \} \neq \{0\}$.

PROOF. (4) $\iff$ (5): This follows from $(f \cdot v)^* = v^* \cdot f^*$ for all $v \in \ell^2(\Gamma, \mathbb{R})$.

(2) $\iff$ (3) $\Rightarrow$ (4): This is part of [5, Proposition 4.16].

(1) $\iff$ (2): Taking $*$ we see that (1) holds if and only if $f^*$ is a left zero-divisor in $\mathbb{Z} \Gamma$. Applying (3) $\iff$ (4) to $f^*$, we see that the latter condition is equivalent to (2).

REMARK 1.5. Linnell’s analytic zero-divisor conjecture is the conjectural statement that for any torsion-free discrete group $\Gamma$ and any nonzero $f \in \mathbb{C} \Gamma$, $\ker \tilde{\rho}^f = \{0\}$ [6, Conjecture 1]. Linnell has shown that this conjecture holds for $\Gamma$ if $G_1$ is a normal subgroup of $\Gamma$, $G_2$ is a normal subgroup of $G_1$, $\Gamma$ is torsion-free, $G_2$ is free, $G_1/G_2$ is elementary amenable, and $\Gamma/G_1$ is right orderable [7, Proposition 1.4].

If a countably infinite, torsion-free, and not virtually cyclic group $\Gamma$ satisfies Linnell’s analytic zero-divisor conjecture, then the principal $\Gamma$-action $\alpha_f$ on $X_f$ is ergodic for every $f \in \mathbb{Z} \Gamma$ by Theorem 1.3.
As a corollary to the Theorems 1.2 – 1.3 and Remark 1.5 we obtain the following results by Hayes.

**Corollary 1.6 ([2, Theorem 2.3.6 and Corollary 2.5.5]).** Suppose that \( \Gamma \) satisfies either of the following conditions.

1. \( \Gamma \) is an infinite, torsion-free, nilpotent group not isomorphic to the integers,
2. \( \Gamma \) is the free group with \( k \geq 2 \) generators.

Then the principal \( \Gamma \)-action \((X_f, \alpha_f)\) is ergodic for every \( f \in \mathbb{Z}(\Gamma) \).

**Proof.** If \( f = 0 \), then \( \alpha_f \) is the left shift-action by \( \Gamma \) on \( X_f = \mathbb{T}^\Gamma \), which is obviously ergodic. Suppose therefore that \( f \neq 0 \). Since \( \Gamma \) is either free or torsion-free nilpotent, \( \ker \tilde{\rho}^f = \{0\} \) by Remark 1.5, so that \( \alpha_f \) is ergodic by either Theorem 1.2 or 1.3. \( \square \)

Whereas the proofs of these results in [2] use structure theory of \( \Gamma \), the proofs in this paper employ cohomological methods.

### 2. Cohomological results

Let \( \Gamma \) be a countably infinite discrete group and \( M \) a left \( \mathbb{Z}\Gamma \)-module. A map \( c: \Gamma \to M \) is a 1-cocycle (or, for our purposes here, simply a cocycle) if

\[
c(\gamma \gamma') = c(\gamma) + \gamma c(\gamma')
\]  

(2.1)

for all \( \gamma, \gamma' \in \Gamma \). A cocycle \( c: \Gamma \to M \) is a coboundary (or trivial) if there exists a \( b \in M \) such that

\[
c(\gamma) = b - \gamma b
\]

(2.2)

for every \( \gamma \in \Gamma \).

A finitely generated group \( G \) has two ends if and only if it is infinite and virtually cyclic, i.e., if and only if it contains a finite-index subgroup \( G' \cong \mathbb{Z} \). Stallings’ theorem ([13]) implies that a finitely generated group \( G \) has a single end whenever it is amenable and not virtually cyclic (see [8] for a short proof).

**Proposition 2.1.** Let \( \Gamma \) be a countably infinite discrete group and \( \Delta \subseteq \Gamma \) a finitely generated subgroup with a single end. Then every cocycle \( c: \Delta \to \mathbb{Z}\Gamma \) is a coboundary.

**Proof.** By [3, Theorem 4.6] if \( \Delta \) has a single end, then every 1-cocycle \( \Delta \to \mathbb{Z}\Delta \) is a coboundary.\(^2\) It follows that for each \( \gamma \in \Gamma \) there is some \( b_\gamma \in \mathbb{Z}[\Delta \gamma] \) such that the restriction of \( c(\delta) \) on \( \Delta \gamma \) is equal to \( b_\gamma - \delta b_\gamma \) for all \( \delta \in \Delta \).

\(^2\)The authors are grateful to Andreas Thom for alerting us to this reference.
For each \( \delta \in \Delta \), there is a finite set \( W_\delta \) of right cosets of \( \Delta \) in \( \Gamma \) such that the support of \( c(\delta) \) is contained in \( \bigcup_{\Delta \gamma \in W_\delta} \Delta \gamma \). If \( F \) is a finite symmetric set of generators of \( \Delta \), then for any \( \Delta \gamma \not\in \bigcup_{\delta' \in F} W_{\delta'} \), one has \( (1 - \delta) \cdot b_\gamma = 0 \) for every \( \delta \in F \) and hence for every \( \delta \in \Delta \). Therefore \( c(\delta) \) is equal to 0 on \( \Delta \gamma \) for all \( \delta \in F \) and hence for every \( \delta \in \Delta \). Therefore \( c(\delta) \) is equal to 0 on \( \Delta \gamma \) for all \( \delta \in F \) and hence for every \( \delta \in \Delta \). Therefore \( c(\delta) = (1 - \delta) \cdot b \) for all \( \delta \in \Delta \).

Next we prove an analogous result for nonamenable groups with vanishing first \( \ell^2 \)-Betti number, e.g., infinite property T groups [1, Corollary 6].

**Proposition 2.2.** Let \( \Gamma \) be a countably infinite discrete group and \( \Delta \subset \Gamma \) a nonamenable subgroup with \( \beta_1^{(2)}(\Delta) = 0 \). Then every cocycle \( c: \Delta \to \mathbb{Z} \Gamma \) is a coboundary.

For the proof of Proposition 2.2 we have to discuss cocycles of \( \Gamma \) which take values in a Hilbert space \( \mathcal{H} \) carrying a unitary action \( U: \gamma \mapsto U^\gamma \) of \( \Gamma \).

A map \( c: \Gamma \to \mathcal{H} \) is a 1-cocycle for \( U \) if

\[
c(\gamma \gamma') = c(\gamma) + U^\gamma c(\gamma')
\]

for all \( \gamma, \gamma' \in \Gamma \), and such a cocycle is a coboundary if and only if there exists a \( b \in \mathcal{H} \) with

\[
c(\gamma) = b - U^\gamma b
\]

for every \( \gamma \in \Gamma \). The cocycle \( c \) is an approximate coboundary if there exists a sequence \( (c_n)_{n \geq 1} \) of coboundaries \( c_n: \Gamma \to \mathcal{H} \) such that

\[
\lim_{n \to \infty} \|c_n(\gamma) - c(\gamma)\| = 0
\]

for every \( \gamma \in \Gamma \).

The following lemma is well-known (cf. [11, Proposition 1.6]). For convenience of the reader, we give a proof here.

**Lemma 2.3.** Let \( U \) be a unitary representation of \( \Gamma \) on \( \mathcal{H} \) which does not contain the trivial representation weakly. Then every approximate coboundary \( c: \Gamma \to \mathcal{H} \) for \( U \) is a coboundary.

**Proof.** Since \( U \) does not weakly contain the trivial representation of \( \Gamma \), we can find a finite subset \( F \subset \Gamma \) and some \( \varepsilon > 0 \) such that

\[
\sum_{\delta \in F} \|v - U^\delta v\| \geq \varepsilon \|v\|
\]

for all \( v \in \mathcal{H} \).

Let \( c \) be an approximate coboundary of \( \Gamma \) taking values in \( \mathcal{H} \). Let \( (b_n)_{n \geq 1} \) be a sequence in \( \mathcal{H} \) such that the coboundaries \( c_n(\gamma) = b_n - U^\gamma b_n, \gamma \in \Gamma \), approximate \( c \) in the sense of (2.5). Then

\[
\sum_{\delta \in F} \|c(\delta)\| = \lim_{n \to \infty} \sum_{\delta \in F} \|c_n(\delta)\| \geq \varepsilon \lim_{n \to \infty} \sup \|b_n\|
\]
and hence
\[ \|c(\gamma)\| = \lim_{n \to \infty} \|c_n(\gamma)\| \leq 2 \limsup_{n \to \infty} \|b_n\| \leq 2 \varepsilon^{-1} \sum_{\delta \in F} \|c(\delta)\| \]
for all \( \gamma \in \Gamma \).

For a bounded subset \( Y \) of \( \mathcal{H} \) and \( v \in \mathcal{H} \), set \( d(v, Y) = \sup_{y \in Y} \|v-y\| \).
Since \( \mathcal{H} \) is a Hilbert space, the function \( v \mapsto d(v, Y) \) on \( \mathcal{H} \) takes a minimal value at exactly one point, namely the Chebyshev center of \( Y \), which we denote by \( \text{center}(Y) \).

Consider the affine isometric action \( V \) of \( \Gamma \) on \( \mathcal{H} \) defined by \( V^\gamma v = U^\gamma v + c(\gamma) \) for all \( \gamma \in \Gamma \) and \( v \in \mathcal{H} \). Set \( Y = \{c(\gamma') : \gamma' \in \Gamma\} \), and let \( \gamma \in \Gamma \).
Since \( V^\gamma(Y) = Y \), we obtain that \( V^\gamma(\text{center}(Y)) = \text{center}(Y) \) and hence that \( U^\gamma(\text{center}(Y)) + c(\gamma) = \text{center}(Y) \).
Thus \( c(\gamma) = \text{center}(Y) - U^\gamma(\text{center}(Y)) \) for all \( \gamma \in \Gamma \), so that \( c \) is a coboundary. \( \square \)

**Proof of Proposition 2.2.** By [1] in the finitely generated case, and [9, Corollary 2.4] in general, if \( \Delta \) is nonamenable and \( \beta_1^{(2)}(\Delta) = 0 \), then every 1-cocycle \( \Delta \to \ell^2(\Delta, \mathbb{R}) \) for the left regular representation is a coboundary.
It follows that for each \( \gamma \in \Gamma \) there is some \( b_\gamma \in \ell^2(\Delta \gamma, \mathbb{R}) \) such that the restriction of \( c(\delta) \) on \( \Delta \gamma \) is equal to \( b_\gamma - \delta b_\gamma \) for all \( \delta \in \Delta \). Since \( c(\delta) \) has finite support for each \( \delta \in \Delta \), we conclude that the cocycle \( c : \Delta \to \ell^2(\Gamma, \mathbb{R}) \) is an approximate coboundary.

Because \( \Delta \) is nonamenable, its left regular representation on \( \ell^2(\Delta, \mathbb{R}) \) does not contain the trivial representation weakly. Since the restriction of the left regular representation of \( \Gamma \) on \( \ell^2(\Gamma, \mathbb{R}) \) to \( \Delta \) is a direct sum of copies of the left regular representation of \( \Delta \), it does not contain the trivial representation of \( \Delta \) weakly either. By Lemma 2.3 there exists \( v \in \ell^2(\Gamma, \mathbb{R}) \) satisfying
\[ c(\delta) = v - \tilde{\lambda}^\delta v = (1 - \delta)v \] (2.6)
for every \( \delta \in \Delta \).

Since \( \Delta \) is nonamenable, it is infinite. It follows that that \( v \in \ell^2(\Gamma, \mathbb{Z}) = \mathbb{Z}\Gamma \). \( \square \)

If a subgroup \( \Delta \subset \Gamma \) has more than one end then there exist nontrivial cocycles \( c : \Delta \to \mathbb{Z}\Delta \) (cf. [12, 5.2. Satz IV] or [14, Lemma 3.5]), which immediately implies the existence of nontrivial cocycles \( c : \Delta \to \mathbb{Z}\Gamma \). For example, if \( \Delta \) is the free group on \( k \geq 2 \) generators, it has nontrivial cocycles. However, Proposition 2.4 below guarantees triviality of cocycles which become trivial under right multiplication by an element \( f \in \mathbb{Z}\Gamma \) satisfying (1.6) (cf. Remark 1.5).
Proposition 2.4. Let $\Gamma$ be a countably infinite discrete group, $\Delta \subset \Gamma$ a nonamenable subgroup, and let $f \in Z\Gamma$ satisfy that $\ker \tilde{\rho}^f = \{0\}$. If $c: \Delta \rightarrow Z\Gamma$ is a cocycle such that $cf$ is a coboundary, then $c$ is a coboundary.

Lemma 2.5. Let $\Gamma$ be a countably infinite discrete group, $\Delta \subset \Gamma$ a nonamenable subgroup, and let $f \in Z\Gamma$. We write $\tilde{\lambda}_\Delta$ for the unitary representation of $\Delta$ obtained by restricting the left regular representation $\tilde{\lambda}$ of $\Gamma$ on $\ell^2(\Gamma, \mathbb{C})$ to $\Delta$.

If $c: \Delta \rightarrow \ell^2(\Gamma, \mathbb{C})$ is a cocycle for $\tilde{\lambda}_\Delta$ such that $c \cdot f = \tilde{\rho}^f c$ is a coboundary and $c(\Delta)$ is contained in the orthogonal complement $V$ of $\ker \tilde{\rho}^f$ in $\ell^2(\Gamma, \mathbb{C})$ (cf. (1.6)), then $c$ is a coboundary.

Proof. By assumption there exists a $b \in \ell^2(\Gamma, \mathbb{C})$ such that $(1 - \delta) \cdot b = c(\delta) \cdot f$ for every $\delta \in \Delta$. Let $\tilde{\rho}^f = UH$ be the polar decomposition [4, Theorem 6.1.2] of $\tilde{\rho}^f$, where $U$ is a partial isometry on $\ell^2(\Gamma, \mathbb{C})$, $H = (\tilde{\rho}^f)^{1/2} = (\tilde{\rho}^f)^{1/2}$, and both $U$ and $H$ lie in the left-equivariant group von Neumann algebra $\mathcal{N}\Gamma$.

Note that $\ker H = \ker \tilde{\rho}^f$. We write $H = \int_0^{\|\tilde{\rho}^f\|} \lambda dE_\lambda$ for the spectral decomposition of the positive self-adjoint operator $H$ and consider, for each $0 < \varepsilon < \|\tilde{\rho}^f\|$, the projection operator $P_\varepsilon = P - E_\varepsilon$, where $P$ is the orthogonal projection $\ell^2(\Gamma, \mathbb{C}) \rightarrow V$. Then one has $P_\varepsilon \rightarrow P$ in the strong operator topology as $\varepsilon \searrow 0$.

Put $Q_\varepsilon = UP_\varepsilon U^*$ for every $\varepsilon$ with $0 < \varepsilon < \|\tilde{\rho}^f\|$. Then

$P_\varepsilon(c(\delta)) \cdot f = \tilde{\rho}^f P_\varepsilon(c(\delta)) = UHP_\varepsilon(c(\delta)) = UP_\varepsilon H(c(\delta)) = Q_\varepsilon UH(c(\delta)) = Q_\varepsilon c(\delta) \cdot f = Q_\varepsilon((1 - \delta) \cdot b) = (1 - \delta) \cdot Q_\varepsilon(b)$

for every $\delta \in \Delta$. Since $\|\tilde{\rho}^f v\| \geq \varepsilon \|v\|$ for every $v \in \text{range}(P_\varepsilon)$, there exists $V_\varepsilon \in \mathcal{N}\Gamma$ vanishing on the orthogonal complement of range$(\tilde{\rho}^f P_\varepsilon)$ and satisfying that $V_\varepsilon \tilde{\rho}^f v = v$ for every $v \in \text{range}(P_\varepsilon)$. Therefore

$P_\varepsilon(c(\delta)) = V_\varepsilon \tilde{\rho}^f P_\varepsilon(c(\delta)) = V_\varepsilon Q_\varepsilon((1 - \delta) \cdot b) = (1 - \delta) \cdot V_\varepsilon Q_\varepsilon(b)$.

The 1-cocycle $\delta \mapsto P_\varepsilon(c(\delta)) = (1 - \delta) \cdot V_\varepsilon Q_\varepsilon(b)$ for $\tilde{\lambda}_\Delta$ is thus a coboundary. Since $P_\varepsilon(c(\delta)) \rightarrow c(\delta)$ in $\ell^2(\Gamma, \mathbb{C})$ as $\varepsilon \searrow 0$ for every $\delta \in \Delta$, we conclude that the 1-cocycle $c: \Delta \rightarrow \ell^2(\Gamma, \mathbb{C})$ for $\tilde{\lambda}_\Delta$ is an approximate coboundary.

Since $\Delta$ is nonamenable, the left regular representation of $\Delta$ on $\ell^2(\Delta, \mathbb{C})$ does not weakly contain the trivial representation of $\Delta$. Thus, the representation $\tilde{\lambda}_\Delta$ of $\Delta$ on $\ell^2(\Gamma, \mathbb{C})$, as a direct sum of copies of the left regular representation of $\Delta$, does not weakly contain the trivial representation of $\Delta$.

From Lemma 2.3 we conclude that there is some $b \in \ell^2(\Gamma, \mathbb{C})$ satisfying $c(\delta) = (1 - \delta)b$ for every $\delta \in \Delta$.

Proof of Proposition 2.4. Suppose that $f \in Z\Gamma$ satisfies (1.6), and that $c: \Delta \rightarrow Z\Gamma$ is a 1-cocycle such that $cf$ is a coboundary. Then $cf$ is
also a coboundary when \( c \) is viewed as an \( \ell^2(\Gamma, \mathbb{C}) \)-valued cocycle for the unitary representation \( \lambda_{\Delta} \) on \( \ell^2(\Gamma, \mathbb{C}) \). Lemma 2.5 shows that there exists a \( b \in \ell^2(\Gamma, \mathbb{C}) \) such that \( c(\delta) = (1 - \delta) \cdot b \) for every \( \delta \in \Delta \). In order to prove that \( b \in Z\Gamma \) we set, for every \( \varepsilon > 0 \), \( F_\varepsilon(b) = \{ \gamma \in \Gamma : |b_\gamma| \geq \varepsilon \} \). Then \( F_\varepsilon \) is finite, and so is the set \( \{ \delta \in \Delta : |(\delta \cdot b)_\gamma| = |b_{\delta^{-1} \gamma}| \geq \varepsilon \} = \{ \delta \in \Delta : \delta^{-1} \gamma \in F_\varepsilon \} = \gamma F_\varepsilon^{-1} \cap \Delta \) for every \( \gamma \in \Gamma \). Since \( \Delta \) is nonamenable, it is infinite, and by varying \( \varepsilon \) we see that \( \lim_{\delta \to \infty} (\delta \cdot b)_\gamma = 0 \) for every \( \gamma \in \Gamma \). Since \( c(\delta)_\gamma = b_\gamma - (\delta \cdot b)_\gamma \in \mathbb{Z} \) we conclude, by letting \( \delta \to \infty \), that \( b_\gamma \in \mathbb{Z} \) for every \( \gamma \in \Gamma \). This completes the proof of the proposition. \( \square \)

3. Ergodicity of principal actions

We recall the following result from [10, Lemma 1.2 and Theorem 1.6].

**Theorem 3.1.** If \( \alpha \) is an algebraic action of a countably infinite discrete group \( \Gamma \) on a compact abelian group \( X \) with dual group \( \hat{X} \), then \( \alpha \) is ergodic if and only if the orbit \( \{ \hat{\alpha}^n a : \gamma \in \Gamma \} \) is infinite for every nontrivial \( a \in \hat{X} \).

**Corollary 3.2.** Let \( \Gamma \) be a countably infinite discrete group, \( f \in Z\Gamma \), and let \( \alpha_f \) be the principal algebraic \( \Gamma \)-action on the group \( X_f \) with Haar measure \( \mu_f \) (cf. Definition 1.1). For \( a \in Z\Gamma / Z\Gamma f = \hat{X}_f \) let \( S(a) = \{ \gamma \in \Gamma : \gamma \cdot a = a \} \) be its stabilizer.

Then \( \alpha_f \) is ergodic with respect to \( \mu_f \) if and only if \( S(a) \) has infinite index in \( \Gamma \) for every nonzero \( a \in Z\Gamma / Z\Gamma f \).

**Proof of Theorem 1.2.** Suppose that \( f \in Z\Gamma \) is not a right zero-divisor, but that \( \alpha_f \) is nonergodic. By Corollary 3.2 there exists an \( h \in Z\Gamma \) such that \( h \notin Z\Gamma f \) and the \( \Gamma \)-orbit \( D = \{ \gamma h + Z\Gamma f : \gamma \in \Gamma \} \) of \( a = h + Z\Gamma f \) in \( Z\Gamma / Z\Gamma f \) is finite. We denote by

\[
\Delta = \{ \delta \in \Gamma : \delta h - h \in Z\Gamma f \}
\]

(3.1)

the stabilizer of \( a \), which has finite index in \( \Gamma \) by hypothesis, and consider the cocycle \( c : \Delta \to Z\Gamma \) given by

\[
h - \delta h = c(\delta)f
\]

(3.2)

for every \( \delta \in \Delta \) (here we are using that \( f \) is not a right zero-divisor). If \( \Delta_0 \subset \Delta \) is an infinite subgroup on which \( c \) is a coboundary then \( c(\delta) = b - \delta b \) for some \( b \in Z\Gamma \) and every \( \delta \in \Delta_0 \). Hence \( c(\delta)f = (1 - \delta)b f = (1 - \delta)h \) for every \( \delta \in \Delta_0 \). Since \( \Delta_0 \) is infinite, this implies that \( h = bf \in Z\Gamma f \), contrary to our choice of \( h \). In other words, if \( c \) is a coboundary when restricted to any infinite subgroup, we run into a contradiction with our assumption that \( \alpha_f \) is nonergodic.
Proof of (1). If $\Gamma_0 \subset \Gamma$ is a finitely generated subgroup with a single end, then the same is true for its finite-index subgroup $\Delta \cap \Gamma_0$ where $\Delta$ is from (3.1). Proposition 2.1 shows that $c$ is a coboundary on $\Delta \cap \Gamma_0$.

As was explained at the beginning of the proof of this theorem this contradicts the non-ergodicity of $\alpha_f$.

Proof of (2). This is [2, Theorem 2.4.1]. For convenience of the reader we include the proof. Let $\Gamma_0 \subset \Gamma$ be the subgroup generated by $\text{supp}(h) \cup \text{supp}(f)$. Since $\Gamma$ is not finitely generated there exists an increasing sequence of subgroups $\Gamma_n \subset \Gamma$, $n \geq 1$, such that $\Gamma_{n+1}$ is generated over $\Gamma_n$ by a single element $\gamma_{n+1} \in \Gamma_{n+1} \setminus \Gamma_n$. Put $D = \{\gamma h + Z\Gamma f : \gamma \in \Gamma\} \subset Z\Gamma / Z\Gamma f$ and $D_n = \{\gamma h + Z\Gamma f : \gamma \in \Gamma_n\} \subset Z\Gamma / Z\Gamma f$, $n \geq 0$. Then $|D_0| \leq |D_1| \leq \cdots \leq |D_n| \leq \cdots \leq |D| < \infty$. Hence there exists an $N \geq 0$ with $\gamma_{N+1} h + Z\Gamma f = \gamma' h + Z\Gamma f$ for some $\gamma' \in \Gamma_N$. Then $(\gamma_{N+1} - \gamma') h = gf$ for some $g \in Z\Gamma$. We write $g = g_1 + g_2$ with $\text{supp}(g_1) \subset \Gamma_N$ and $\text{supp}(g_2) \cap \Gamma_N = \emptyset$. Then

\begin{equation}
\gamma_{N+1} h - g_2 f = g_1 f + \gamma' h. \tag{3.3}
\end{equation}

All the terms on the right hand side of (3.3) are supported in $\Gamma_N$, whereas the supports of the terms on the left hand side of (3.3) are disjoint from $\Gamma_N$. Hence both sides of (3.3) have to vanish, which means that $\gamma_{N+1} h = g_2 f$ and $h \in \gamma_{N+1}^{-1} g_2 f \in Z\Gamma f$, contrary to our choice of $h$. As explained above, this contradiction proves the ergodicity of $\alpha_f$.

Proof of (3). If $\Gamma_0 \subset \Gamma$ is a nonamenable subgroup with $\beta_1^{(2)}(\Gamma_0) = 0$, then the same is true for its finite-index subgroup $\Delta \cap \Gamma_0$. By Proposition 2.2, the cocycle $c : \Delta \cap \Gamma_0 \longrightarrow Z\Gamma$ is a coboundary, which leads to a contradiction as in (1). □

Proof of Theorem 1.3. If $\Gamma$ is amenable, use Theorem 1.2 (1) or (2). If $\Gamma$ is nonamenable, combine the argument at the beginning of the proof of Theorem 1.2 with Proposition 2.4. □

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