Heisenberg Spins on a Cylinder Section

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Classical Heisenberg spins in the continuum limit (i.e. the nonlinear $\sigma$-model) are studied on an elastic cylinder section with homogeneous boundary conditions. The latter may serve as a physical realization of magnetically coated microtubules and cylindrical membranes. The corresponding rigid cylinder model exhibits topological soliton configurations with geometrical frustration due to the finite length of the cylinder section. Assuming small and smooth deformations allows to find shapes of the elastic support by relaxing the rigidity constraint: an inhomogeneous Lamé equation arises. Finally, this leads to a novel geometric effect: a global shrinking of the cylinder section with swellings.

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I. INTRODUCTION

Microtubules and cylindrical membranes abound in nature. They are important both in biological context and in industrial applications. The coupling of magnetism with the deformability of these soft materials opens up new, exciting avenues of investigation. We can envision these microtubules to be synthesized from either magnetic organic materials or polymers, or to enclose various magnetorheological fluids in such a way that the magnetic and elastic properties are coupled. Specifically, the shape deformation can be interpreted in terms of bending elasticity of the membrane. To explore magnetoelastic properties we will approximate the surfaces of microtubules and cylindrical membranes as a continuum of classical Heisenberg-coupled spins.

Our motivation is to amplify on the concept of geometrical frustration, i.e. mismatch of different length scales in a system leading to deformation of the geometry. The length scales of interest are the characteristic width of the spin (magnetic) soliton and the underlying geometric length of the cylinder (i.e., radius). The main idea is that the mismatch of length scales is reflected in the violation of the so called self-duality equations and the increase in magnetic energy of the system from that of the minimum energy in the corresponding homotopy class. Restoring self-duality or the minimum energy is achieved by deforming the manifold in such a way that geometrical frustration is relieved.

The continuum limit of the Heisenberg Hamiltonian for classical ferromagnets or antiferromagnets, for isotropic spin-spin coupling is the nonlinear $\sigma$-model. The total Hamiltonian for a deformable, magnetoelastically coupled manifold is given by $H = H_{\text{magn}} + H_{\text{cl}} + H_{m-\text{cl}}$,

where $H_{\text{magn}}$, $H_{\text{cl}}$ and $H_{m-\text{cl}}$ represent the magnetic, elastic and magnetoelastic energy, respectively. In the present paper we will focus on the magnetic part and the elastic part only since, for quasi-one-dimensional spin solutions on the cylinder, $H_{m-\text{cl}}$ merely renormalizes $H_{\text{magn}}$. For the nonlinear $\sigma$-model, the magnetic energy on a curved surface $S$, in curvilinear coordinates, is given by

$$H_{\text{magn}} = J \int \sqrt{g} d\Omega \, g^{ij} h_{\alpha\beta} \partial_i n^\alpha \partial_j n^\beta; \quad (1)$$

where $J$ denotes the coupling energy between neighbouring spins. The order parameter $\hat{n}$ is the local magnetization unit vector specified by a point on the sphere $S^2$. The metric tensors $(g_{ij})$ and $(h_{\alpha\beta})$ describe respectively the support surface $S$ and the order parameter manifold: as customary, $d\Omega$ represents the surface area and $g$ the determinant $\det(g_{ij})$.

II. RIGID CYLINDER SECTION: NON-SELF DUAL SOLITONS

First, let us consider the nonlinear $\sigma$-model on a rigid cylinder section.

A. Formulation

For our purposes a suitable representation, in cylindrical coordinates $(\rho, \varphi, z)$, is given by

$$\rho = r, \quad (2)$$
where the constant parameter $r$ is real and positive, while the angle $\varphi$ varies from $-\pi$ to $\pi$. One can easily check that the metric is given by
\begin{equation}
g = r^2 \, d\varphi \otimes d\varphi + dz \otimes dz, \tag{3}\end{equation}
therefore $g^{zz} = g^{\varphi\varphi} = 0$ and we have
\begin{equation}g^{\varphi\varphi} \sqrt{g} = \frac{1}{r}, \quad g^{zz} \sqrt{g} = r. \tag{4}\end{equation}
From now on, we restrict ourselves to a section of the cylinder: the $z$ variable will vary from $-\Delta z$ to $\Delta z$ where $0 \leq \Delta z < \infty$.

As usual, the local magnetization $\mathbf{n}$ is described by its spherical coordinates $(\Theta, \Phi)$, then the metric on the Heisenberg sphere is given by
\begin{equation}h = d\Theta \otimes d\Theta + \sin^2 \Theta \, d\Phi \otimes d\Phi. \tag{5}\end{equation}

Assuming homogeneous boundary conditions at both ends ($\Theta = 0[\pi]$ as $z \to \pm \Delta z$) allows to map each boundary of the section to a point: thus we compactify our cylinder section into the sphere $S^2$. Consequently, the mapping of our support to the order parameter manifold is classified by the homotopy group $\Pi_2(S^2)$ which is isomorphic to $\mathbb{Z}$: spin configurations may be classified according to their homotopy class.

### B. Derivation of non-self dual solitons

Henceforth, without loss of generality, only cylindrical symmetric configurations ($\partial_z \Theta = \partial_z \Phi = 0$) will be considered. Thus the magnetic Hamiltonian (1) becomes
\begin{equation}H_{magn} = \int_{-\Delta z}^{+\Delta z} \int_{-\pi}^{+\pi} \left[r \Theta^2 + \frac{\sin^2 \Theta}{r} \Phi^2\right], \tag{6}\end{equation}
where a subscript stands for differentiation. Rescaling the $z$ variable in equation (6) gives:
\begin{equation}H_{magn} = \int d\zeta \int d\varphi \left[\Theta^2 + \sin^2 \Theta \Phi^2\right], \tag{7}\end{equation}
where $\zeta \equiv z/r$ and $\Delta \zeta \equiv \Delta z/r$. The Euler-Lagrange equations corresponding to (7) are:
\begin{align} \Phi_{\varphi\varphi} &= 0, \tag{8a} \\
\Theta_{\zeta\zeta} &= \Phi_{\varphi}^2 \sin \Theta \cos \Theta. \tag{8b} \end{align}
From (8a), it follows that
\begin{equation}\Phi_{\varphi} = q_{\varphi} \quad q_{\varphi} \in \mathbb{Z}. \tag{9a}\end{equation}
Substituting this into (8a) and rescaling again the rotating angle, we get the sine-Gordon (sg) equation
\begin{equation}\Theta_{\varphi\varphi} = \sin \Theta \cos \Theta, \tag{9b}\end{equation}
where $\varphi \equiv q_{\varphi} \zeta$.

Equation (9b) may be integrated once to yield
\begin{equation}\Theta^2 = \sin^2 \Theta + \bar{m} \quad \bar{m} \in [0, +\infty). \tag{10}\end{equation}
Performing the change of variable $\sin \Theta = dn(u | 1 + \bar{m})$ where $dn$ is a Jacobi elliptic function, the differential equation becomes $u_{\varphi}^2 = 1$. Let us denote by $2q_{\zeta}$ the increasing solution of (11) specified by the parameter $\bar{m}$ and subject to the boundary condition $\Theta(0 | \bar{m}) = \frac{\pi}{2}$. Readily, we get:
\begin{equation}\sin \alpha(q | \bar{m}) = dn(q | 1 + \bar{m}). \tag{11}\end{equation}
Thus, the general sg solution, in natural coordinates, is
\begin{equation}\Theta(q | \bar{m}) = \varepsilon \alpha(q | \bar{m}) + c \quad \varepsilon = \pm 1. \tag{12}\end{equation}
Further the function $\alpha(q | \bar{m})$ satisfies
\begin{equation}\alpha(q + 2q_{\zeta} K_{\alpha} | \bar{m}) = \alpha(q | \bar{m}) + q \pi \quad q \in \mathbb{Z}, \tag{13}\end{equation}
where the quasi quarter-period $K_{\alpha}$ is related to the complete elliptic integral of the first kind $K$ by
\begin{align} K_{\alpha}(\bar{m}) &= K(1 + \bar{m}), \tag{14a} \\
&= \frac{1}{\sqrt{1 + \bar{m}}} K\left(\frac{1}{1 + \bar{m}}\right). \tag{14b} \end{align}
Clearly, the parameter $\bar{m}$ tunes the quasi quarter-period $K_{\alpha}$.

Using the solutions of equation (13), the $q_{\zeta} \pi$-soliton configuration consistent with the boundary conditions can be obtained easily. Up to an irrelevant additive multiple of $\pi$, we have
\begin{equation}\Theta(\zeta) = \varepsilon \alpha(q_{\varphi} \zeta | \bar{m}) - \delta_{\text{even},q_{\zeta}} \frac{\pi}{2}. \tag{15a}\end{equation}
where the parameter $\bar{m}$ is given by
\begin{equation}\bar{m} = K_{\alpha}^{-1}\left(\frac{q_{\varphi} \Delta \zeta}{q_{\zeta}}\right). \tag{15b}\end{equation}
If we consider an infinite cylinder, the solution (15) represents a soliton lattice with a period $2K_{\alpha}$. When $\bar{m}$ tends to $0^+$, this period tends to $\infty$ and we recover the self-dual equations.
C. Geometric frustration

The magnetic energy $E_{magn}$ of the above configuration (13) may be compared with the corresponding topological minimum energy $E$, which does not depend on the geometry of the support manifold. Performing the Bogomol’nyi’s decomposition (4) reads

$$E = 8\pi J|Q|,$$  

where the topological charge (i.e., the winding number) $Q$ equals to $q_\ell q_\varphi$. Let $E_{magn}$ denote the ratio $E_{magn}/E$. A straightforward calculation shows that $E_{magn}$ depends on the parameter $\tilde{m}$ only; we have

$$E_{magn} = E(1 + \tilde{m}) + \frac{1}{2} \tilde{m} \left[ E\left(\frac{1}{1+\tilde{m}}\right) - \frac{1}{2} \frac{\tilde{m}}{1+\tilde{m}} \right] \left(\frac{1}{1+\tilde{m}}\right),$$  

which increases strictly from 1 to $\infty$ with respect to $\tilde{m}$: $E$ is the complete elliptic integral of the second kind. Therefore, according to (15), $E_{magn}$ is increasing function of the radius $r$ and decreasing function of $\Delta z$.

III. ELASTIC CYLINDER SECTION: FRUSTRATION RELEASE

Next, let us relax the rigidity constraint and consider the nontrivial spin configuration on an elastic cylinder section. Accordingly, the soliton will try to minimize its magnetic energy $E_{magn}$ to the minimum energy $E$ by deforming the elastic support (13).

A. Deformable metric

Since the radius $r$ appears as the relevant geometric parameter and $z$ variable as the relevant curvilinear coordinate, we relax $r$ with respect to $z$ and write

$$r(z) = r_0 \left[ 1 + A(z) \right],$$  

where $r_0$ represents the spontaneous radius and the function $A$ describes local deformations. The metric on the deformable manifold in this case remains orthogonal, and

$$g^{\varphi\varphi} = \frac{1 + r_0 A_z^2}{r}, \quad g^{zz} = \frac{r}{\sqrt{1 + r_0 A_z^2}}.$$  

B. Derivation of small and smooth deformations

As the problem is quasi-one-dimensional, the magnetoelastic energy $H_{m-cl}$ merely renormalizes the spin coupling energy $J$ in the magnetic energy $H_{magn}$. Therefore, we add to the nonlinear $\sigma$-model Hamiltonian (14) only the elastic energy which is essentially stored in the bending of the deformable support (14):

$$H_{cl} = \frac{1}{2} k_\ell \int S \sqrt{g} \, d\Omega \left( H - H_0 \right)^2.$$  

Here the constant $k_\ell$ denotes the bending rigidity of the (cylinder) material, $H$ represents the mean curvature, and the spontaneous mean curvature $H_0$ tends to bias the mean curvature for recovering the spontaneous shape. Assuming small and smooth deformations and expanding to second order in $\Lambda$, $A_z$ and $\Lambda_{zz}$ lead to

$$H_{cl} = \frac{1}{2} k_\ell \int d\zeta \Lambda^2.$$  

Before deriving the Euler-Lagrange equation for the total Hamiltonian $H = H_{magn} + H_{cl}$, we calculate the magnetic energy associated with the nontrivial spin configuration (14). Expanding to second order in $\Lambda$ and $A_z$ the relations (19) enable to rewrite (7) as follows

$$H_{magn} = J \int d\zeta d\varphi \left[ \Theta_\ell^2 + \sin^2 \Theta_\varphi^2 \right] - (\Lambda + A_z^2) \left[ \Theta_\ell^2 - \sin^2 \Theta_\varphi^2 \right] + \Lambda^2 \sin^2 \Theta_\varphi^2 \left(1 + \tilde{m}^2\right).$$  

On the other hand, according to (10) and (11), the spin configuration (15) verifies

$$\Theta_\ell^2 - \sin^2 \Theta_\varphi^2 = \tilde{m} q_\varphi^2,$$

where we have set

$$m = 1 + \tilde{m},$$  

$$A = \frac{2}{mq_\varphi^2} \left[ 1 + \frac{c_2}{16 \kappa q_\varphi^2} \right],$$  

$$B = \frac{2}{mq_\varphi^2},$$  

$$j = \frac{-1}{q_\varphi^2 \sqrt{1 + m}}.$$  

3
Here we have introduced the relative coupling energy \( \kappa \equiv J/k_c \) and the variable \( \varrho \equiv q_\varphi \zeta \) (see (9b)). The linear inhomogeneous second-order differential equation (24a) is related to the well-known homogeneous (Jacobian) Lamé’s equation which occurs in several physical contexts\(^{20,21}\). We have found no direct treatment of (24a) in the literature. However, an approach based on the derivation of the Lamé functions\(^{20,21}\) allows to find a particular solution denoted by \( L(\cdot | m; A, B, j) \). Therefore, if small and smooth deformations are assumed, a suitable deformation function \( \Lambda \) is given by

\[
\Lambda(\zeta) = L(q_\varphi \zeta | 1 + \tilde{m}; A, B, j).
\]  

(25)

C. Frustration release

The geometric frustration becomes evident in considering the Euler-Lagrange equation (24a): \( \Lambda = 0 \) (i.e. the rigid cylinder section) is not a solution. Now let us turn our attention to the deformation mechanism. As we have seen, the magnetic energy \( E_{\text{magn}} \) of the configuration (15) is an increasing function of the radius \( r \); therefore for sufficiently small and smooth deformations, the soliton tries to decrease the radius \( r \). In other words, the soliton tends to collapse the cylinder section. On the other hand, the elastic Hamiltonian (20) tries to maintain the spontaneous shape of the deformable support. Consequently, the competition between the magnetic energy and the elastic energy induces a decrease of the radius \( r \). Since according to (8) the soliton energy is essentially localized in the spread zone, the deformation is less important in the spin flip region.

To understand the geometric meaning of the frustration release, let us define the relative dilatation \( \lambda \) by

\[
\lambda \equiv \frac{r}{r_0} = 1 + \Lambda.
\]  

(26)

As shown in Fig. 1, the decrease of the radius leads to a global shrinking whereas a local swelling arises where the spins twist.

IV. CONCLUSION

In conclusion, we showed that cylindrical magnetic surfaces, if approximated by classical Heisenberg spins, exhibit novel magnetoelastic deformation effects in the presence of spin solitons due to a mismatch of length scales. In particular, we showed that the Euler-Lagrange equation for the nonlinear \( \sigma \)-model on a cylinder section with homogeneous boundary conditions is the sine-Gordon equation. As announced, the frustration release leads to a novel geometric effect: the cylinder section is globally shrunk and a swelling appears in the region of the soliton.

\[\text{FIG. 1. The relative dilatation } \lambda \text{ as defined in (26) corresponding to the deformation function (25) associated with cylinder sections in presence of a } 2\pi \text{-soliton versus } \zeta/\Delta \zeta \text{ for different values of } \Delta \zeta \text{ and with the relative coupling energy } \kappa \text{ fixed to } 1/16.\]

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