A TOPOLOGICAL DEGREE APPROACH TO SUBLINEAR SYSTEMS OF SECOND ORDER DIFFERENTIAL EQUATIONS

ANNA CAPIETTO and WALTER DAMBROSIO
Dipartimento di Matematica, Università di Torino
Via Carlo Alberto 10, 10123 Torino - Italia
(Communicated by Jean Mawhin)

Abstract. In this paper we study the existence of radial solutions to sublinear systems of elliptic equations.
We first give a multiplicity result on solutions with prescribed nodal properties; then, we show the existence of positive solutions. The proofs are based on topological degree arguments.

1. Introduction. In this paper we are concerned with the study of radial solutions to Dirichlet problems for systems of elliptic equations; more precisely, we consider
\[
\begin{cases}
\Delta u + \gamma q(|\xi|)\partial_u G(u, v) = 0 & \text{in } \Omega \\
\Delta v + \gamma q(|\xi|)\partial_v G(u, v) = 0 & \text{in } \Omega \\
u = 0 = v & \text{on } \partial \Omega,
\end{cases}
\] (1.1)
where \( \Omega = \{\xi \in \mathbb{R}^N : 0 < a < |\xi| < b\} \) is an annulus in \( \mathbb{R}^N, \) \( N \geq 1, \) and \( \gamma > 0; \) for \( \epsilon_0 > 0, \) we set \( I_0 := [-\epsilon_0, \epsilon_0]^2 \) and we assume that \( G : I_0 \to \mathbb{R} \) and \( q : [a, b] \to \mathbb{R} \) are \( C^1 \)-functions. For simplicity, all the results are stated and proved in the case of two equations, but they are valid, by means of suitable changes, in the general case of \( K \) equations (\( K \geq 1 \)).

It is well-known that the search of radial solutions to (1.1) leads to the study of a system of ordinary differential equations of the form
\[
\begin{cases}
x''(t) + m(t, x) = 0 \\
x(0) = 0 = x(\pi),
\end{cases}
\] (1.2)
for a suitable function \( m \in C^1([0, \pi] \times \mathbb{R}^2, \mathbb{R}^2), \) with \( x = (x_1, x_2) \) (see [27]). We remark that, when one attacks systems, many classical techniques used in the search of solutions to a single equation (as, for instance, the shooting method and phase-plane analysis) cannot be applied. For results on second order differential systems we refer to the books [21, 22]: in many cases, the approach is of variational type, due to the natural fitness of this method for the study of problems involving more than one equation; however, it is quite difficult to find, by means of these techniques and in absence of restrictive conditions, multiplicity results. As far as a topological degree method is concerned, we recall the works [9, 12, 18], where the so-called
"weakly-coupled" systems are considered, and the recent paper by R. Manuel and J. Mawhin [19], where it is given an important outburst towards the study of systems with nonlinear differential operators.

Existence of radial solutions in annular domains in the case of one equation has been proved by many authors (see e.g. [11, 13, 27]), under various conditions both at infinity and near the origin. The aim of this paper is to treat nonlinearities \( G \) having only a subquadratic behaviour near the origin; on these lines, in [4] we gave a multiplicity theorem in the case of one equation. Though some techniques contained in that paper are useful, when one tries to prove an analogous result for systems some new ideas are needed.

We assume:

\( (H1) \) the following relations hold:

\[
\lim_{u \to 0} \frac{\partial_u G(u, v)}{u} = +\infty, \quad \text{uniformly in } v \in [-\epsilon_0, \epsilon_0],
\]

and

\[
\lim_{v \to 0} \frac{\partial_v G(u, v)}{v} = +\infty, \quad \text{uniformly in } u \in [-\epsilon_0, \epsilon_0];
\]

\( (H2) \) there exists a positive constant \( q_0 > 0 \) such that

\[
q(r) \geq q_0 \quad \text{for every } r \in [a, b].
\]

Moreover, setting

\[
G^u(u, v) = G(u, v) - G(0, v), \quad \forall (u, v) \in I_0,
\]

and

\[
G^v(u, v) = G(u, v) - G(u, 0), \quad \forall (u, v) \in I_0,
\]

we assume that:

\( (H3) \) there exists a positive constant \( \Upsilon \) such that

\[
|\partial_u G^u(u, v)| \leq \Upsilon G^u(u, v), \quad \forall (u, v) \in I_0,
\]

and

\[
|\partial_v G^v(u, v)| \leq \Upsilon G^v(u, v), \quad \forall (u, v) \in I_0.
\]

It is well-known that assumption (H1) represents a subquadratic condition near the origin for \( G \).

As far as hypothesis (H2) is concerned, it guarantees the uniqueness of solutions to some Cauchy problems related to (1.1) (cf. also the papers [11], [25, Th. 4-(\( \delta \)-ii)] and [17]).

Finally, assumption (H3) is a technical condition which arises when dealing with systems (it is trivially fulfilled in the case of one equation). It is satisfied e.g. by "subquadratic" polynomials and it enables us to develop some crucial energy estimates (cf. Example 2.4 and the proof of Lemma 2.7). We point out that (H3) could be replaced by a formally more complicated hypothesis which enables to study the case when a more general function of the form \( F(|\xi|, u, v) \) is considered in (1.1).

In Section 2 we show the existence of infinitely many radial solutions (having arbitrarily small norm), with prescribed nodal properties, to (1.1) for every \( \gamma > 0 \); in this way, we generalize the result in [4]. More precisely, setting for simplicity \( \gamma = 1 \) and defining the set \( \tau \) by

\[
\tau = \{(s_1, s_2) : s_i = +1 \text{ or } s_i = -1 \text{ for every } i = 1, 2\},
\]

we are able to prove the following (see Theorem 2.2):
**Theorem A** Assume that hypotheses (H1)-(H2)-(H3) hold. Then, there exists \( n^* \in \mathbb{N}^2 \) such that for every \((n_1, n_2) \in \mathbb{N}^2 \) with \( n_i \geq n_i^* \) \((i = 1, 2)\) problem (1.1) has at least four radial solutions \((u^k, v^k)\) \((k = 1, \ldots, 4)\), with \( u^k \) having exactly \( n_1 \) zeros in \([a, b)\) and \( v^k \) having exactly \( n_2 \) zeros in \([a, b)\), for every \( k = 1, \ldots, 4\). Moreover, for every \((s_1, s_2) \in \tau\) there exists \( k \in \{1, \ldots, 4\} \) such that \( \text{sgn}(u^k)'(0) = s_1 \) and \( \text{sgn}(v^k)'(0) = s_2 \).

For the proof we use a topological degree approach; more precisely, we combine the application of a continuation theorem (see e.g. [7]) with some estimates on the number of zeros of each component of a radial solution to (1.1). Moreover, we use a time-map technique for autonomous equations. We are also able to make an homotopy which carries our original problem into an uncoupled problem, for which we can use phase-plane analysis (on the lines of [4, 8]).

To the best of our knowledge, there are very few results in the literature concerning subquadratic nonlinearities; beside the papers [4, 6], we recall the early work by G. J. Butler [3], where a periodic problem is considered. Recent contributions on this subject have been given, among others, by A. Ambrosetti, J. Garcia-Azorero and I. Peral [1, 2], L. H. Erbe, S. Hu and H. Wang [15], V. Moroz [23] and H. Wang [27].

We stress the fact that, apart from [2, Sect. 2], in all these papers, which deal with a single equation, a combination of assumptions in zero and at infinity is required, while (H1)-(H3) are merely conditions of local nature.

We remark that Theorem A guarantees only the existence of radial solutions whose components have a sufficiently large number of zeros. Indeed, the subquadratic behaviour of the nonlinearity \( G \) is not sufficient to ensure, for all \( \gamma > 0 \), the existence of radial solutions with an arbitrary number of zeros (cf. Remark 3.6). However, provided that \( \gamma \) is sufficiently small, it is possible to show that such solutions do exist (see Proposition 3.5).

In Section 3 we investigate the particular case of positive solutions, i.e. solutions with one zero in \([0, \pi)\). More precisely, we prove (see Theorem 3.1):

**Theorem B** Assume (H1)-(H2)-(H3) as above. Then, there exists \( \gamma_0 > 0 \) such that for every \( \gamma \in (0, \gamma_0) \) problem (1.1) has a positive radial solution.

The proof of the above theorem follows from the estimates on the number of zeros proved in Section 2 and a detailed phase-plane analysis on the solutions of autonomous equations (see Lemma 3.4).

The study of positive solutions, in the case of a single equation, has been faced by means of different methods and, in general, existence results have been given depending on the values of the parameter \( \gamma > 0 \). We recall the papers by H. Dang, R. Manásevich and K. Schmitt [13] and H. Wang [27], where it has been proved, under conditions on the behaviour of the nonlinearity both in zero and at infinity, that positive solutions exist for small values of \( \gamma \). We also quote the paper by A. Ambrosetti, J. Garcia-Azorero and I. Peral [2], which deals (using bifurcation techniques) with sublinear functions which, in a neighbourhood of zero, are asymptotic to a power. For other related results, see the work of R. Manásevich, F. I. Njoku and F. Zanolin [20]. Among the few results on systems, we also refer to [10], where the existence of positive solutions is proved by using the method of sub-super solutions.

As a final remark, we observe that when studying Dirichlet problems associated to one equation the sublinear and the superlinear conditions, near zero and at infinity, respectively, represent dual situations, in the sense that in the former case we are able to find the existence of infinitely many solutions with small norm, while in the
latter we have infinitely many solutions with large norm. The generalization of this fact to systems is shown by a comparison between Theorem 2.2 and the results in the forthcoming paper [5].

In what follows, we will denote by $X = C^1_0([0, \pi], \mathbb{R}^2)$ the space of the functions $x \in C^1([0, \pi], \mathbb{R}^2)$ such that $x(0) = 0 = x(\pi)$ and we will indicate with $\|x\|_1$ the corresponding $C^1$-norm.

Moreover, $\partial_i$ will denote the partial derivative with respect to the variable $x_i$, for $i = 1, 2$, and $\partial_u$, $\partial_v$ the partial derivatives with respect to the variables $u$ and $v$.

Finally, by deg we mean the Leray-Schauder degree.

2. A multiplicity result. Let $0 < a < b$ and let $\Omega = \{\xi \in \mathbb{R}^N : a < |\xi| < b\}$ be an annulus in $\mathbb{R}^N$. Let us consider a system of the form

$$
\begin{align*}
\begin{cases}
\Delta u + g(|\xi|)\partial_u G(u, v) = 0 & \text{in } \Omega \\
\Delta v + g(|\xi|)\partial_v G(u, v) = 0 & \text{in } \Omega \\
u = 0 = v & \text{on } \partial \Omega.
\end{cases}
\end{align*}
$$

(2.4)

For $\epsilon_0 > 0$, we set $I_0 := [-\epsilon_0, \epsilon_0]^2$ and we assume that $G : I_0 \to \mathbb{R}$ and $q : [a, b] \to \mathbb{R}$ are $C^1$-functions; without loss of generality, we can take $G(0, 0) = 0$.

We suppose:

(H1) the following relations hold:

$$
\lim_{u \to 0} \frac{\partial_u G(u, \eta)}{u} = +\infty, \quad \text{uniformly in } v \in [-\epsilon_0, \epsilon_0],
$$

and

$$
\lim_{v \to 0} \frac{\partial_v G(u, \eta)}{v} = +\infty, \quad \text{uniformly in } u \in [-\epsilon_0, \epsilon_0];
$$

(H2) there exists a positive constant $q_0 > 0$ such that $q(r) \geq q_0$ for every $r \in [a, b]$.

Remark 2.1. We observe that assumption (H1) implies some information on the sign of the function $G$: more precisely, since we have

$$
G(u, v) - G(0, v) = \int_0^u \partial_\xi G(\xi, v) \, d\xi, \quad \forall (u, v) \in I_0,
$$

and

$$
G(u, v) - G(u, 0) = \int_0^v \partial_\eta G(u, \eta) \, d\eta, \quad \forall (u, v) \in I_0,
$$

from (H1) we deduce that there exists $\tilde{I}_0 \subset I_0$ s.t. $G(u, v) - G(0, v) \geq 0$ and $G(u, v) - G(u, 0) \geq 0$ for every $(u, v) \in \tilde{I}_0$. Moreover, being

$$
G(u, v) = \int_0^u \partial_\xi G(\xi, v) \, d\xi + \int_0^v \partial_\eta G(\xi, \eta) \, d\eta, \quad \forall (u, v) \in I_0,
$$

from (H1) we also deduce that $G$ is always positive in $\tilde{I}_0$. In what follows, for simplicity $\tilde{I}_0$ will be identified with $I_0$.

Now, let us set $G^u(u, v) = G(u, v) - G(0, v), \quad \forall (u, v) \in I_0$, and

$$
G^v(u, v) = G(u, v) - G(u, 0), \quad \forall (u, v) \in I_0;
$$
Assume that hypotheses 

\begin{equation}
\text{It is easy to prove, on the lines of}
\end{equation}

(\text{H3}) there exists a positive constant \( \Upsilon \) such that
\[ |\partial_u G^u(u, v)| \leq \Upsilon G^u(u, v), \quad \forall (u, v) \in \mathcal{I}_0, \]
and
\[ |\partial_v G^u(u, v)| \leq \Upsilon G^u(u, v), \quad \forall (u, v) \in \mathcal{I}_0. \]

It is well-known that assumption (H1) represents a sublinear condition near zero for \( \nabla G \). As far as hypothesis (H2) is concerned, it guarantees the uniqueness of solutions to some Cauchy problems related to (2.4) (cf. Proposition 2.4 in [4]). It is immediate to see that hypotheses (H1) and (H2) are precisely the conditions we assumed in [4] when dealing with one equation. Assumption (H3) is a technical condition which arises when dealing with systems (it is trivially fulfilled in the case of one equation). It is satisfied by "sublinear" polynomials and it enables us to develop some crucial energy estimates (cf. Example 2.4 and the proof of Lemma 2.7).

Under conditions (H1)-(H2)-(H3) we are able to prove the following result, which generalizes to the case of systems the results contained in [4] (recall the definition of \( \tau \) given in (1.3) in the Introduction):

**Theorem 2.2.** Assume that hypotheses (H1)-(H2)-(H3) hold. Then, there exists \( n^* \in \mathbb{N}^2 \) such that for every \( (n_1, n_2) \in \mathbb{N}^2 \) with \( n_i \geq n_i^* \) (\( i = 1, 2 \)) problem (2.4) has at least four radial solutions \( (u^k, v^k) \) \( (k = 1, \ldots, 4) \), with \( u^k \) having exactly \( n_1 \) zeros in \( [a, b] \) and \( v^k \) having exactly \( n_2 \) zeros in \( [a, b] \), for every \( k = 1, \ldots, 4 \). Moreover, for every \((s_1, s_2) \in \tau \) there exists \( k \in \{1, \ldots, 4\} \) such that \( \text{sgn}(u^k)'(0) = s_1 \) and \( \text{sgn}(v^k)'(0) = s_2 \).

**Remark 2.3.** It is easy to prove, on the lines of [4, 6], that the \( C^1 \)-norm of the solutions \( u^k, v^k \), which depends on \((n_1, n_2)\), tends to zero as \( n_1, n_2 \) tend to infinity.

As it is well-known, radial solutions to (2.4) correspond, for \( r = |\xi| \), to functions \( u = u(r), v = v(r) \) satisfying the Dirichlet problem

\begin{equation}
\begin{cases}
(r^{N-1}u')' + r^{N-1}q(r)\partial_u G(u, v) = 0 \\
(r^{N-1}v')' + r^{N-1}q(r)\partial_v G(u, v) = 0 \\
u(a) = v(a) = 0 = u(b) = v(b).
\end{cases}
\end{equation}

(2.5)

Now, by means of a change of variable (see [27]), it is easy to see that there exists a positive \( C^1 \)-function \( p : [0, \pi] \rightarrow \mathbb{R} \), with \( p(t) \geq p_0^* > 0 \) for every \( t \in [0, \pi] \), such that \((u(r), v(r))\) is a solution of (2.5) if and only if \( x(t) = (x_1(t), x_2(t)) \) is a solution of

\begin{equation}
\begin{cases}
x_1'' + p(t)\partial_1 G(x_1, x_2) = 0 \\
x_2'' + p(t)\partial_2 G(x_1, x_2) = 0 \\
x_1(0) = x_2(0) = 0 = x_1(\pi) = x_2(\pi).
\end{cases}
\end{equation}

(2.6)

Therefore, we shall concentrate on the nodal properties of the solutions to (2.6).
Before giving the proof of Theorem 2.2, we show a class of systems of the form (2.6) satisfying (H1)-(H2)-(H3).

Example 2.4. Let us consider \(G_i \in C^2([-\epsilon_0, \epsilon_0], \mathbb{R})\), with \(G_i(0) = 0\), \(L_i \in C^1([-\epsilon_0, \epsilon_0], \mathbb{R})\), \(i = 1, 2\), and \(P, Q \in C^1([-\epsilon_0, \epsilon_0], \mathbb{R})\); let us study the system

\[
\begin{align*}
x_1'' + G_1'(x_1)L_1(x_2) + G_2(x_2)L_2'(x_1) + P'(x_1)Q(x_2) &= 0 \\
x_2'' + G_2'(x_2)L_2(x_1) + G_1(x_1)L_1'(x_2) + P(x_1)Q'(x_2) &= 0 \\
x_1(0) &= 0 = x_2(0), \quad x_1(\pi) = 0 = x_2(\pi).
\end{align*}
\]

Obviously, system (2.7) is of the form (2.6), with \(G(x_1, x_2) = G_1(x_1)L_1(x_2) + G_2(x_2)L_2(x_1) + P(x_1)Q(x_2)\) for every \((x_1, x_2) \in I^2\) and \(P(t) \equiv 1\) in \([0, \pi]\). According to the change of variables in [27], this means that we are treating the case when, in the system of PDEs (2.4), we have \(q(s) = s^{2(K-1)}\). Then, it is easy to see that Theorem 2.2 applies to (2.7) when we assume:

1. the functions \(G_1'\) and \(G_2'\) are sublinear near zero, i.e.
   \[
   \lim_{s \to 0} \frac{G_i'(s)}{s} = +\infty = \lim_{s \to 0} \frac{G_j'(s)}{s};
   \]
2. the following relations hold:
   \[
   L_i(s) \geq L_i(0) > 0, \quad \forall s \in [-\epsilon_0, \epsilon_0], \quad i = 1, 2,
   \]
   \[
   P(s) \geq P(0) > 0, \quad \forall s \in [-\epsilon_0, \epsilon_0] \quad \text{and} \quad Q(s) \geq Q(0) > 0, \quad \forall s \in [-\epsilon_0, \epsilon_0].
   \]

Note that in the particular case \(L_1(s) \equiv l_1 > 0, L_2(s) \equiv l_2 > 0\) we are in presence of a so-called "weakly-coupled" system.

For a more concrete illustration of a potential \(G\) satisfying (1)-(2), one can take

\[
G(x_1, x_2) = |x_1|^\alpha + (1 + x_2^2)^\beta + (1 + x_1^2)(1 + x_2^4)Q(x_2),
\]

with \(\alpha, \beta \in (0, 1)\). \(\square\)

The proof of Theorem 2.2 follows from the application of a continuation theorem given in [6] for an abstract equation of the form

\[
u = \mathcal{N}(u, \lambda),
\]

where \(X\) is a Banach space and \(\mathcal{N} : \text{dom } \mathcal{N} \subset X \times [0, 1] \to X\) is a completely continuous operator. For the statement of this theorem, we shall consider two open sets \(A\) and \(B\) such that \(A \subset \bar{A} \subset B \subset \bar{B}\) and \((B \setminus A) \subset \text{dom } \mathcal{N}\).

Let \(\Sigma\) be the set of the solutions of (2.8), i.e.

\[
\Sigma = \{(u, \lambda) : u = \mathcal{N}(u, \lambda)\}
\]

and, for any subset \(D \subset X \times [0, 1]\), let us denote the section of \(D\) at \(\lambda \in [0, 1]\) by \(D_\lambda = \{x \in X : (x, \lambda) \in D\}\); we also set \(\mathcal{N}_\lambda = \mathcal{N}(\cdot, \lambda)\). We have the following:

Theorem 2.5. [6, Th. 2.1] Let \(k : \Sigma \cap (\bar{B} \setminus A) \to \mathbb{N}^2\) be a continuous function; suppose that there exists \(n \in \mathbb{N}^2\) satisfying the following conditions:

\[
n \notin k(\partial(\bar{B} \setminus A))
\]

(2.9)
where $\Upsilon$ is given in assumption (H3) and 
\[ \|\| = \|\| \quad \text{uniformly in} \quad x, \] 
more precisely, we have 
\[ \text{deg}(I - N_0, U_0^n) \neq 0, \quad \text{uniformly in} \quad x. \] 
Finally, we have 
\[ \text{deg}(I - N_0, U_0^n) \neq 0, \] 
then there is a continuum $C_n \subset \Sigma$ with 
\[ \{ \lambda \in [0, 1] : \exists u \in X : (u, \lambda) \in C_n \} = [0, 1] \] 
and such that 
\[ (u, \lambda) \in C_n \implies (u, \lambda) \in (B \setminus \bar{A}) \quad \text{and} \quad k(u, \lambda) = n. \] 
In particular there is at least one solution $\bar{u} \in (B \setminus \bar{A})_1$ of the operator equation 
\[ u = N(u, 1) \] 
with 
\[ k(\bar{u}, 1) = n. \] 

In view of the application of Theorem 2.5, we make an homotopy by introducing the function $G_0 : \mathbb{R}^2 \to \mathbb{R}$ given by 
\[ G_0(x) = \frac{3}{4} \left( \sqrt{x_1^2} + \sqrt{x_2^2} \right), \quad \forall \ x \in \mathbb{R}^2; \] 
much more, we set $G(x, \lambda) = \lambda G(x) + (1 - \lambda)G_0(x)$ and $p(t, \lambda) = \lambda p(t) + (1 - \lambda)$, for every $t \in [0, \pi]$, $x \in I_0$ and $\lambda \in [0, 1]$. In what follows, we will use the notation 
\[ G_\lambda(x) = G(x, \lambda) \] 
and $p_\lambda(t) = p(t, \lambda)$ for every $t \in [0, \pi]$, $x \in I_0$ and $\lambda \in [0, 1]$. We then consider the Dirichlet problem 
\[ \begin{cases} x'' + p_\lambda(t)\nabla G_\lambda(x) = 0, \\
 x(0) = 0 = x(\pi), \end{cases} \] 
which can be written (see [21]) in the form (2.8) with respect to the Banach space 
\[ X = C_0^0([0, \pi], \mathbb{R}^2). \] 
Moreover, we denote by $\Sigma \subset X \times [0, 1]$ the set of the solutions to (2.12).

Now, we observe that assumptions (H1)-(H2)-(H3) are preserved along the homotopy; more precisely, we have 
\[ \lim_{x_1 \to 0} \frac{\partial_1 G_\lambda(x_1, x_2)}{x_1} = +\infty, \] 
uniformly in $x_2 \in [-\epsilon_0, \epsilon_0]$ and $\lambda \in [0, 1]$, and 
\[ \lim_{x_2 \to 0} \frac{\partial_2 G_\lambda(x_1, x_2)}{x_2} = +\infty, \] 
uniformly in $x_1 \in [-\epsilon_0, \epsilon_0]$ and $\lambda \in [0, 1]$. Moreover, it is easy to check that there exists a positive constant $p_0 > 0$ such that 
\[ p_\lambda(t) \geq p_0, \quad \forall \ t \in [0, \pi], \ \lambda \in [0, 1]. \] 
Finally, we have 
\[ |\partial_1 G_\lambda^2(x_1, x_2)| \leq \Upsilon G_\lambda^2(x_1, x_2), \quad \forall \ (x_1, x_2) \in I_0, \ \lambda \in [0, 1], \] 
and 
\[ |\partial_2 G_\lambda^2(x_1, x_2)| \leq \Upsilon G_\lambda^2(x_1, x_2), \quad \forall \ (x_1, x_2) \in I_0, \ \lambda \in [0, 1], \] 
where $\Upsilon$ is given in assumption (H3) and $G_\lambda^2(x_1, x_2) = G_\lambda(x_1, x_2) - G_\lambda(0, x_2)$, $G_\lambda^2(x_1, x_2) = G_\lambda(x_1, x_2) - G_\lambda(x_1, 0)$ for every $(x_1, x_2) \in I_0$ and $\lambda \in [0, 1]$. Recall
that, according to Remark 2.1, \( G_\lambda^i(x_1, x_2) \geq 0 \) for every \((x_1, x_2) \in I_0, \lambda \in [0,1], i = 1,2.\)

We are now ready to give some preliminary lemmas.

**Lemma 2.6.** Assume (H1)-(H2)-(H3); then, for every \( \epsilon \in (0, \epsilon_0) \) there exists \( \mu_\epsilon > 0 \) such that each solution of

\[
\begin{align*}
x'' + p_\lambda(t) \nabla G_\lambda(x) &= 0 \\
x(0) &= 0, \quad x'(0) = (\xi_1, \xi_2),
\end{align*}
\]

with \(|(\xi_1, \xi_2)| \leq \mu_\epsilon\), is defined in \([0, \pi]\) and satisfies

\[
x_i(t)^2 + x'_i(t)^2 \leq \epsilon^2, \quad \forall \ t \in [0, \pi], \ i = 1,2.
\]

The proof of this lemma is based on some estimates on the energy function \( E_\lambda(t) = \frac{1}{2} x'_1(t)^2 + \frac{1}{2} x'_2(t)^2 + p_\lambda(t) G_\lambda(x_1(t), x_2(t)) \); the details, being similar to the case of a single equation treated in [4], are omitted for brevity.

We now give a lemma which describes some qualitative properties of the solutions to Cauchy problems associated to the system in \((2.12)\) and which is crucial for the validity of Proposition 2.8. For its proof, it has been necessary to introduce a new "energy-like" function and to use (H3). We denote by \( \mu_0 \) the number \( \mu_{\epsilon_0} \) given in Lemma 2.6.

**Lemma 2.7.** Assume (H1)-(H2)-(H3); then, the following statements hold:

1. Let \( x \) be a solution of the initial value problem

\[
\begin{align*}
x'' + p_\lambda(t) \nabla G_\lambda(x) &= 0 \\
x(t_0) &= x_0, \quad x'(t_0) = x'_0, \quad t_0 \in [0, \pi],
\end{align*}
\]

with \(|(x_0, x'_0)| \leq \mu_0\); let us suppose that \((x_0)_i = 0 = (x'_0)_i\) for \(i = 1\) or \(i = 2\). Then, \( x \) is defined in \([0, \pi]\) and we have \(x(t) \equiv 0\) in \([0, \pi]\).

2. For every \( \mu \in \mathbb{R}^2 \) with \( \mu_i \neq 0 \) for \(i = 1,2\) and \(|\mu| \leq \mu_0\), there exists \( \eta_\mu > 0 \) such that every solution \( x \) of

\[
\begin{align*}
x'' + p_\lambda(t) \nabla G_\lambda(x) &= 0 \\
x(0) &= 0, \quad x'(0) = \mu
\end{align*}
\]

is defined in \([0, \pi]\) and satisfies

\[
x_i(t)^2 + x'_i(t)^2 \geq \eta_\mu, \quad \forall \ t \in [0, \pi], \ i = 1,2.
\]

**Proof.** The results are easy consequences of some estimates on suitable energy functions associated to the system \( x'' + p_\lambda(t) \nabla G_\lambda(x) = 0 \). Indeed, as far as Statement 1 is concerned, let \( x \) be a solution of \((2.20)\) and let us suppose, for simplicity, that \((x_0)_1 = (x'_0)_1 = 0\) (the other case is completely analogous). From Lemma 2.6, we already know that \( x \) is defined on \([0, \pi]\); moreover, from the choice of \( \mu_0 \), we also have \(|x'_2(t)| \leq \epsilon_0\) for every \( t \in [0, \pi]\).

Let us consider

\[
E_{1,\lambda}(t) = \frac{1}{2} x'_1(t)^2 + p_\lambda(t) G_\lambda^1(x_1(t), x_2(t)), \quad \forall \ t \in [0, \pi],
\]
For every \( x \), that for every solution from Lemma 2.7, it is easy to see that, for any \((x_1(t), x_2(t))\) \( x_1'(t) + p_\lambda(t) G_1^\lambda (x_1(t), x_2(t)) + p_\lambda(t) \partial_1 G_\lambda (x_1(t), x_2(t)) x_1'(t) + p_\lambda(t) \partial_2 G_\lambda (x_1(t), x_2(t)) x_2'(t) , \quad \forall t \in [0 , \pi]. \)

Using (2.15) and (2.17) we deduce that there exists a constant \( Z > 0 \) such that

\[
|E_1'(t)| \leq Z E_1(t), \quad \forall t \in [0 , \pi],
\]

and so

\[
E_1(t) \leq E_1(t_0) e^{Zt}, \quad \forall t \in [0 , \pi].
\]

Since \( E_1(t_0) = 0 \), we have that \( E_1(t) \equiv 0 \) in \([0 , \pi]\) and, by simple arguments, \( x_1(t) \equiv 0 \) in \([0 , \pi]\).

Finally, as for Statement 2, from (2.23) we deduce that for any solution \( x \) of (2.21) we have

\[
E_1(t) \geq K_1 \mu_1^2, \quad \forall t \in [0 , \pi],
\]

for some \( K_1 > 0 \). Since

\[
\lim_{(x_1, y_1) \to (0,0)} \left\{ \frac{1}{2} y_1^2 + p_\lambda(t) G_1^\lambda (x_1, x_2) \right\} = 0, \]

uniformly in \( t \), \( \lambda \) and \( x_2 \), from (2.24) we see that there exists \( \eta_1 > 0 \) such that

\[
E_1(t) \geq \frac{1}{2 K_1} \mu_1^2 \quad \Rightarrow \quad x_1(t)^2 + x_1'(t)^2 \geq \eta_1, \quad \forall t \in [0 , \pi].
\]

In a similar way, using analogous estimates on the function \( E_2, \lambda \) defined by

\[
E_2, \lambda(t) = \frac{1}{2} x_2'(t)^2 + p_\lambda(t) G_\lambda^\lambda (x_1(t), x_2(t)) , \quad \forall t \in [0 , \pi],
\]

we obtain the existence of \( \eta_2 > 0 \) such that \( x_2(t)^2 + x_2'(t)^2 \geq \eta_2 \) for every \( t \in [0 , \pi] \); taking \( \eta_\mu = \min\{\eta_1, \eta_2\} \), we get the result.

Now, for every \( \mu \in \mathbb{R}^2 \), with \( \mu_i \neq 0 \) for \( i = 1, 2 \) and \( |\mu| \leq \mu_0 \), let us set

\[
\Sigma_\mu := \{(x, \lambda) \in \Sigma : x'(0) = \mu \}.
\]

From Lemma 2.7, it is easy to see that, for any \( (x, \lambda) \in \Sigma_\mu \), every component of \( x \) has only simple zeros in \([0 , \pi]\). Hence, for \( i = 1, 2 \), the number \( n_i(x, \lambda) \) of zeros of \( x_i \) in \([0 , \pi]\) is well defined. Recalling [16, Lemma 3.1], we deduce that the map \( k : \Sigma_\mu \rightarrow \mathbb{N}^2, (x, \lambda) \mapsto (n_1(x, \lambda), n_2(x, \lambda)) \) is continuous. Moreover, arguing as in \([4, 6]\), it is possible to prove the following:

**Proposition 2.8.** There exists \( n^* \in \mathbb{N}^2 \) such that for every solution \( (x, \lambda) \) of (2.12) and for every \( i = 1, 2 \) we have:

\[
| x_i'(0) | = \mu_0 \quad \Rightarrow \quad n_i(x, \lambda) < n^*_i.
\]

**Proposition 2.9.** For every \( m = (n_1, n_2) \in \mathbb{N}^2 \) there exists \( \mu_\mu \in (0 , \mu_0) \) such that for every solution \( (x, \lambda) \) of (2.12) we have:

\[
| x_i'(0) | \leq \mu_\mu \quad \Rightarrow \quad n_i(x, \lambda) > m_i.
\]
In this section, we shall be concerned with the following 

By the previous discussion, this leads to the existence of a radial nodal solution of case of two equations. We will prove the following result: analysis (see also Remark 3.6). As above, for simplicity we will only consider the supported, in the case of an autonomous equation, by some elementary phase-plane

assume that \( \epsilon_1 > 0, \) \( \epsilon_2 > 0, \) and \( \epsilon_3 > 0 \) for \( i = 1, 2 \) and, consistently with the notation of Theorem 2.5, let us choose

\[
B = \{(x, \lambda) \in X \times [0, 1] : x_i'(0) < \mu_0, \quad \forall i = 1, 2\},
\]

\[
A = \{(x, \lambda) \in X \times [0, 1] : x_i'(0) < \mu_n, \quad \forall i = 1, 2\}
\]

and set \( C = \overline{B} \setminus A. \)

Now, let us define

\[
\Sigma^0 = \{(x, \lambda) \in \Sigma \cap C : n_i(x, \lambda) = n_i, \quad i = 1, 2\}.
\]

If \( (x, \lambda) \in \Sigma \cap (\partial C), \) then there exists \( i \in \{1, 2\} \) such that \( x_i'(0) = \mu_0 \) or \( x_i'(0) = \mu_n; \) therefore, by Proposition 2.8 and Proposition 2.9, we have \( n_i(x, \lambda) < n_i^* \) or \( n_i(x, \lambda) > n_i. \) In any case, this is absurd and so (2.9) holds.

Finally, we refer to the paper [4] (see the Proof of Theorem 3.2 therein) to show that, defining

\[
\Sigma = \{(x, \lambda) \in \Sigma \cap C : n_i(x, \lambda) = n_i, \quad i = 1, 2\}.
\]

As far as (2.10) is concerned, we observe that if \( (x, \lambda) \in \Sigma \), then we obtain the existence of a solution \( x \) of (2.6) with \( x_i'(0) > 0 \) for \( i = 1, 2 \) and such that \( x_1 \) has exactly \( n_1 \) zeros in \([0, \pi]\). By the previous discussion, this leads to the existence of a radial nodal solution of (2.4).

In order to prove the existence of the other solutions to (2.6), it is sufficient to modify the choice of the set \( A \) defined above: let us fix \( (s_1, s_2) \in \tau \) and suppose for instance that \( s_1 = -1 \) and \( s_2 = +1 \) (the other cases are similar). It is now easy to see that, defining

\[
A' = \{(x, \lambda) \in X \times [0, 1] : x_i'(0) > -\mu_n, \quad x_i'(0) < \mu_n\},
\]

then we obtain the existence of a solution \( x \) with \( \text{sgn} \ (x_i')(0) = s_i \) for \( i = 1, 2 \). \( \square \)

3. Positive solutions. In this section, we shall be concerned with the following system

\[
\begin{aligned}
\Delta u + \gamma|\xi|\partial_u G(u, v) &= 0 \quad \text{in} \ O \\
\Delta v + \gamma|\xi|\partial_v G(u, v) &= 0 \quad \text{in} \ O \\
u = 0 &= v \quad \text{on} \ \partial O.
\end{aligned}
\]

(3.25)

With the same notation of Section 2, we take \( \Omega = \{\xi \in \mathbb{R}^N : 0 < a < |\xi| < b\}, \)

\( \epsilon_0 > 0, \) \( q : [a, b] \rightarrow \mathbb{R}, \) \( G : [0, \epsilon_0] \times [0, \epsilon_0] \rightarrow \mathbb{R} \) with \( G(0, 0) = 0. \) Moreover, we assume that \( \gamma \) is a positive real number. Under the same assumptions on \( q \) and \( G \) we made in Section 2, it is immediate to see that (3.25) has infinitely many radial solutions for every \( \gamma > 0. \) However, Theorem 2.2 guarantees only the existence of solutions with a "sufficiently large" number of zeros (see also Proposition 3.5). It is our goal in this section to show the existence of positive radial solutions to (3.25), in the case when the parameter \( \gamma \) is sufficiently small. The validity of this fact is supported, in the case of an autonomous equation, by some elementary phase-plane analysis (see also Remark 3.6). As above, for simplicity we will only consider the case of two equations. We will prove the following result:
\textbf{Theorem 3.1.} Assume (H1)-(H2)-(H3) as in Section 2. Then, there exists \( \gamma_0 > 0 \) such that for every \( \gamma \in (0, \gamma_0) \) problem (3.25) has a positive radial solution.

The proof of Theorem 3.1 will be performed by means of the abstract continuation theorem 2.5. For this reason, and since we shall be dealing with sublinear nonlinearities (as in Section 2), in what follows we argument on the lines of Section 2: we skip all the proofs that do not significantly differ from those we already performed, and we give the details of the ones that have required new ideas.

As usual, positive radial solutions to (3.25) correspond to positive solutions of

\[
\begin{cases}
x''_1 + \gamma p(t) \partial_1 G(x_1, x_2) = 0 \\
x''_2 + \gamma p(t) \partial_2 G(x_1, x_2) = 0 \\
x_1(0) = x_2(0) = 0 = x_1(\pi) = x_2(\pi),
\end{cases}
\]  

(3.26)

where \( p \) is a positive \( C^1 \)-function on \([0, \pi] \). Now, we still denote by \( G \) the extension of \( G \) to \([-\epsilon_0, 0] \times [-\epsilon_0, 0] \) such that \( G(-x_1, x_2) = G(x_1, -x_2) = G(x_1, x_2) \) for every \((x_1, x_2) \in I_0 := [-\epsilon_0, \epsilon_0] \times [-\epsilon_0, \epsilon_0] \). Then, we are led to study the homotopized problem

\[
\begin{cases}
x'' + \gamma p_{\lambda}(t) \nabla G_{\lambda}(x) = 0 \\
x(0) = 0 = x(\pi),
\end{cases}
\]  

(3.27)

where \( p_{\lambda}(t) = \lambda p(t) + (1 - \lambda) \) and \( G_{\lambda}(x_1, x_2) = \lambda G(x_1, x_2) + (1 - \lambda)(3/4(\sqrt{x_1^2 + x_2^2})) \), for every \( t \in [0, \pi], \ (x_1, x_2) \in I_0 \) and \( \lambda \in [0, 1] \). From (H1)-(H2)-(H3), arguing as in Section 2, it is easy to check that there is \( p_0 > 0 \) such that

\[
p_{\lambda}(t) \geq p_0, \quad \forall \ t \in [0, \pi], \ \lambda \in [0, 1].
\]  

(3.28)

Moreover, with the same notation as in (2.16)-(2.17), we have

\[
|\partial_1 G_{\lambda}^2(x_1, x_2)| \leq \Upsilon G_{\lambda}^2(x_1, x_2), \quad \forall \ (x_1, x_2) \in I_0, \ \lambda \in [0, 1],
\]  

(3.29)

and

\[
|\partial_2 G_{\lambda}^1(x_1, x_2)| \leq \Upsilon G_{\lambda}^1(x_1, x_2), \quad \forall \ (x_1, x_2) \in I_0, \ \lambda \in [0, 1],
\]  

(3.30)

where \( \Upsilon \) is given in assumption (H3).

Now, according to Section 2, we need to check the validity of Lemma 2.6 and 2.7 when we consider, instead of \( G \), the function \( \gamma G \); indeed, this can be done since (3.28), (3.29) and (3.30) hold. It is important to observe that, due to the fact that the constant \( p_0 \) in (3.28) does not depend on \( \gamma > 0 \), the constant \( \mu_0 = \mu_{\epsilon_0} \) given in Lemma 2.6 is also independent from \( \gamma > 0 \).

As a consequence, like in Section 2, the functional "number of zeros" is continuous. The final (and crucial) estimates of Section 2 (cf. Proposition 2.9 and Proposition 2.8) contain an lower (respectively, upper) estimate on the number of zeros of the components of the solutions to (3.27); since we are now dealing with positive solutions (i.e. solutions with only one zero in \([0, \pi]\)), those statements need to be reformulated as follows.

\textbf{Proposition 3.2.} For every \( \gamma > 0 \) there exists \( \mu_\gamma \in (0, \mu_0) \) such that problem (3.27) has no positive solutions \((x, \lambda)\) with \( |x'_i(0)| \leq \mu_\gamma, \ i = 1, 2.\)
For the proof it is sufficient to take, in Proposition 2.9, \((m_1, m_2) = (1, 1)\).
As for the upper estimate on the number of zeros, a new argument has to be developed; indeed, this is precisely the point where a restriction on the constant \(\gamma\) is needed. We have the following:

**Proposition 3.3.** There exists \(\gamma_1 > 0\) such that for every \(\gamma \in (0, \gamma_1)\) problem (3.27) has no positive solutions \((x, \lambda)\) with \(|x'_1(0)| = \mu_0\) or \(|x'_2(0)| = \mu_0\).

**Proof.** We show that there exists \(\gamma_1 > 0\) such that if problem (3.27) has a positive solution \((x, \lambda)\) with \(|x'_1(0)| = \mu_0\) or \(|x'_2(0)| = \mu_0\), then \(\gamma \geq \gamma_1\).
Let us suppose that \((x, \lambda)\) is a solution of (3.27) with \(|x'_1(0)| = \mu_0\), the other case being similar. By integrating the first equation in (3.27) we obtain

\[
x'_1(t) - x'_1(0) + \gamma \int_0^t p_{\lambda}(s) \partial_1 G_{\lambda}(x_1(s), x_2(s)) \, ds = 0, \quad \forall \, t \in [0, \pi].
\]
Since \(x_1(0) = 0 = x_1(\pi)\), there exists \(t_0 \in (0, \pi]\) such that \(x'_1(t_0) = 0\); hence

\[
\mu_0 = \gamma \left| \int_0^{t_0} p_{\lambda}(s) \partial_1 G_{\lambda}(x_1(s), x_2(s)) \, ds \right|.
\]
Now, we denote by \(S\) the maximum of the function \(\partial_1 G\) on \(I_0\), by \(S_0\) the maximum of the function \(\partial_1 G_0\) on \(I_0\) and by \(P\) the maximum of the function \(p\) on \([0, \pi]\). Then we have

\[
\left| \int_0^{t_0} p_{\lambda}(s) \partial_1 G_{\lambda}(x_1(s), x_2(s)) \, ds \right| \leq \pi(S + S_0)(P + 1), \quad \forall \, \lambda \in [0, 1].
\]
This implies that

\[
\gamma = \frac{\mu_0}{\int_0^{t_0} p_{\lambda}(s) \partial_1 G_{\lambda}(x_1(s), x_2(s)) \, ds} \geq \gamma_1 := \frac{\mu_0}{\pi(S + S_0)(P + 1)}.
\]

We have now developed all the estimates relative to the homotopized system (3.27). In order to apply Theorem 2.5 (cf. assumption (2.11)), we need to study problem (3.27) with \(\lambda = 0\); we are then led to discuss the existence of positive solutions to the Dirichlet problem

\[
\left\{ \begin{array}{l}
u'' + \gamma \sqrt{u} = 0 \\
u(0) = u(\pi) = 0
\end{array} \right.
\]
where \(u : [0, \pi] \rightarrow \mathbb{R}\) and \(\gamma\) is sufficiently small. To this aim, we develop some time-map estimates and a detailed phase-plane analysis for solutions to (3.31). Indeed, for \(\mu_\gamma\) given in Proposition 3.2, we can prove the following result:

**Lemma 3.4.** There exists \(\gamma_2 > 0\) such that for every \(\gamma \in (0, \gamma_2)\) problem (3.31) has a positive solution \(u\) with \(u'(0) \in (\mu_\gamma, \mu_0)\).

As already observed, the proof of Lemma 3.4 consists on the study of the range of the so-called time-map \(T_\gamma : (0, +\infty) \rightarrow \mathbb{R}^+\); more precisely, we use the fact that (3.31) has a positive solution \(u\) with \(u'(0) = \mu > 0\) if and only if \(T_\gamma(\mu) = \pi/2\). A more complete discussion on this approach can be found e.g. in [4, 8].

We are now in position to conclude the proof of Theorem 3.1.
Proof of Theorem 3.1. Let $\gamma_0 = \min(\gamma_1, \gamma_2)$ and let us fix $\gamma \in (0, \gamma_0)$. As usual, problem (3.27) can be written in the form (2.8) and we denote by $\Sigma \subset X \times [0, 1]$ the set of its solutions. We will apply Theorem 2.5 with $n = (1, 1)$ (recall that positive solutions correspond to solutions with only one zero in $[0, \pi]$).

We define
$$B = \{ (x, \lambda) \in X \times [0, 1] : x'_i(0) < \mu_0 \quad \forall \ i = 1, 2 \}$$
and
$$A = \{ (x, \lambda) \in X \times [0, 1] : x'_i(0) < \mu_\gamma \quad \forall \ i = 1, 2 \};$$
we also set $C = \overline{B} \setminus A$ and
$$\Sigma_1 = \{ (x, \lambda) \in \Sigma \cap C : n_i(x, \lambda) = 1, \ i = 1, 2 \}.$$

If $(x, \lambda) \in \Sigma \cap (\partial C)$, then there exists $i \in \{1, 2\}$ such that $x'_i(0) = \mu_\gamma$ or $x'_i(0) = \mu_0$; therefore, by Proposition 3.2 and Proposition 3.3, respectively, this is absurd. As a consequence, (2.9) holds.

As far as (2.10) is concerned, we can repeat the argument in the proof of Theorem 2.2. Finally, by Lemma 3.4, we already know that there is a positive solution $(x_1, x_2)$ at level $\lambda = 0$ such that $(x_1, x_2, 0) \in C$. We refer again to the paper [4] (see Proof of Theorem 3.2 therein) to show that the local degree associated to this solution is different from zero; in this way, (2.11) is satisfied.

Hence, all the assumptions of Theorem 2.5 are fulfilled and we deduce the existence of a positive solution of (3.27) which corresponds, as already observed, to a positive radial solution of (3.25).

Finally, due to the fact that we obtained positive solutions as solutions having one zero in $[0, \pi)$, it is not difficult to prove also the following:

**Proposition 3.5.** For every $k = (k_1, k_2) \in \mathbb{N}^2$, there exists $\gamma_k > 0$ such that for every $\gamma \in (0, \gamma_k)$ problem (3.25) has a radial solution $(u, v)$ with $u$ and $v$ having $k_1$, $k_2$ zeros in $[0, \pi)$, respectively.

**Remark 3.6.** It is possible to check, with time-map arguments, that a subquadratic behaviour of the nonlinearity $G$ (without any assumption at infinity) is not sufficient to guarantee the existence of positive solutions to (3.25) for all $\gamma$.

**Remark 3.7.** 1. The results contained in Theorems 2.2 and 3.1 can be obtained also in the case when we replace the Laplace operator by the $p$-laplacian; indeed, it is sufficient to observe that, also for the $p$-laplacian, it is possible to compute the number of zeros of the radial solutions by means of a suitable integral formula (see [12]).

2. We stress the fact that we could generalize Theorem 2.2 to a system containing general boundary conditions of Sturm-Liouville type.

**Acknowledgements.** The authors are thankful to Prof. F. Zanolin for useful discussions and suggestions on the subject of this paper.

**REFERENCES**

[1] Ambrosetti A., Garcia-Azorero J. and Peral I., Multiplicity results for some nonlinear elliptic equations. *J. Funct. Anal.*, 137 (1996), 219–242.

[2] Ambrosetti A., Garcia-Azorero J. and Peral I., Quasilinear equations with a multiple bifurcation. *Differential Integral Equations*, 10 (1997), 37–50.

[3] Butler G. J., Periodic solutions of sublinear second order differential equations. *J. Math. Anal. Appl.*, 62 (1978), 676–690.
Capietto A. and Dambrosio W., Boundary value problems with sublinear conditions near zero. *NoDEA Nonlinear Differential Equations Appl.*, 6 (1999), 149–172.

Capietto A. and Dambrosio W., Multiplicity results for systems of superlinear second order equations. *Quad. Dip. Matematica, Univ. Torino*, 54 (1999), 1–17.

Capietto A., Dambrosio W. and Zanolin F., Infinitely many radial solutions to a boundary value problem in a ball. *Ann. Mat. Pura Appl.*, to appear.

Capietto A., Mawhin J. and Zanolin F., A continuation approach to superlinear periodic boundary value problems. *J. Differential Equations*, 88 (1990), 347–395.

Capietto A., Mawhin J. and Zanolin F., Boundary value problems for forced superlinear second order ordinary differential equations. In *Nonlinear Partial Differential Equations and Their Applications*, Collège de France Seminar, Vol. 12, Longman, Harlow, 1994, 55–64.

Castro A. and Lazer A. C., On periodic solutions of weakly coupled systems of differential equations. *Boll. Un. Mat. Ital. B (5)*, 18 (1981), 733–742.

Castro A., Maya C. and Shivaji R., An existence result for a class of sublinear semipositone systems. *Dynam. Contin. Discrete Impuls. Systems*, to appear.

Cheng Y., On the existence of radial solutions of a nonlinear elliptic bvp in an annulus. *Math. Nachr.*, 165 (1994), 61–77.

Dambrosio W., Multiple solutions of weakly-coupled systems with $p$-laplacian operators. *Results Math.*, 36 (1999), 34–54.

Dang H., Manásevich R. and Schmitt K., Positive radial solutions of some nonlinear partial differential equations. *Math. Nachr.*, 186 (1997), 101–113.

de Figueiredo D. G., Semilinear elliptic systems. *Proc. Autumn School on Nonlinear Analysis and Differential Equations*, Lisbon, 1998, to appear.

Erbe L. H., Hu S. and Wang H., Multiple positive solutions of some boundary value problems. *J. Math. Anal. Appl.*, 184 (1994), 640–648.

García-Huidobro M., Manásevich R. and Zanolin F., Infinitely many solutions for a Dirichlet problem with a non homogeneous $p$-Laplacian operator in a ball. *Adv. Differential Equations*, 2 (1997), 203–230.

Heidel J. W., Uniqueness, continuation and nonoscillation for a second order nonlinear differential equation. *Pacific J. Math.*, 71 (1970), 715–721.

Henrard M., Infinitely many solutions of weakly coupled superlinear systems. *Adv. Differential Equations*, 2 (1997), 753–778.

Manásevich R. and Mawhin J., Periodic solutions for nonlinear systems with $p$-laplacian-like operators. *J. Differential Equations*, 145 (1998), 367–393.

Manásevich R., Njoku F. I. and Zanolin F., Positive solutions for the one-dimensional $p$-laplacian. *Differential Integral Equations*, 8 (1995), 213–222.

Mawhin J., Topological degree methods in Nonlinear Boundary Value Problems. CBMS Series, Amer. Math. Soc., Providence, RI, 1979.

Mawhin J. and Willem M., *Critical point theory and Hamiltonian systems*. Applied Mathematical Sciences, 74, Springer-Verlag, New York-Berlin, 1989.

Moroz V., Solutions of superlinear at zero elliptic equations via Morse theory. *Top. Methods Nonlinear Anal.*, 10 (1997), 387–397.

Rabinowitz P. H., Multiple critical points of perturbed symmetric functionals. *Trans. Amer. Math. Soc.*, 272 (1982), 753–769.

Reichel W. and Walter W., Radial solutions of equations and inequalities involving the $p$-laplacian. *J. Inequal. and Appl.*, 1 (1997), 47–71.

Struwe M., Multiple solutions of anticoercive boundary value problems for a class of ordinary differential equations of second order. *J. Differential Equations*, 37 (1980), 285–295.

Wang H., On the existence of positive solutions for semilinear elliptic equations in the annulus. *J. Differential Equations*, 109 (1994), 1–7.

Received November 1999; revised April 2000.

E-mail address: capietto@dm.unito.it

E-mail address: walterd@dm.unito.it