Critical jammed phase of the linear perceptron above saturation

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We consider the spherical perceptron problem beyond the limit of capacity and we study the landscape of local minima for the linear cost function. We show that in the replica symmetry broken region of the phase diagram, local minima are critical even far away from jamming. The gap distribution contains an isostatic delta peak of marginally satisfied gaps surrounded by two power laws for the absolute value of both small satisfied (SAT) and unsatisfied (UNSAT) gaps. We propose a scaling theory for the problem and we compute the values of the critical exponents showing that they coincide and are equal to the one that characterizes the distribution of small positive gaps at the jamming transition.

Introduction – The jamming transition of spheres is a critical point [1]. At jamming, spheres form an isostatic contact network [2] while the distributions of small forces and gaps between them display power laws [3, 4]. In [5, 6] the critical exponents have been computed from the solution of the hard sphere model in the limit of infinite dimension. In this mean field analysis the jamming transition is thought of as the infinite pressure limit of hard spheres glassy states. Upon compression, it is found that hard spheres glasses undergo a Gardner transition [7] where the glass basin of configurations splits into a fractal landscape. Beyond the Gardner point, this landscape is described by full replica symmetry breaking (RSB) [8] which predicts that the Gardner phase is marginal. In the jamming limit this landscape marginality gives rise to the power laws observed in the gaps and forces distributions. Surprisingly, the mean field values of the critical exponents computed in this way agree well with the numerical simulations in three dimensions.

Subsequently, it has been argued in [9–11] that the physics of the jamming transition can be also revealed by considering neural networks models. The simplest one, the perceptron, at the limit of capacity in the non-convex regime, displays a power law distribution of gaps and forces whose critical exponents coincide with the one of hard spheres. Simple generalizations [12, 13] display the same scaling theory with the same critical exponents. Also in this context, jamming criticality appears to be inherited from full replica symmetry breaking.

Packing of spheres as well as solutions of the perceptron problem can be obtained also through decompensation protocols. One considers a cost function that penalizes unsatisfied constraints and then uses gradient descent algorithms to minimize the associated energy. In this case one typically uses harmonic or Hertzian potentials [14] that guarantee local convexity in the cost and analyticity in the first derivatives. The landscape of local minima can be studied at the mean field level (see [10] for the case of the perceptron) and the density of states, in the harmonic case, displays abundance of soft excitations related again to the emergence of the Gardner full RSB phase.

However, to the best of our knowledge, the properties of the landscape of local minima when the cost function is non-convex has not been studied much [15–17]. Non-convex cost functions push gaps to be marginally satisfied but lead to non-analiticities that are hard to study even numerically with standard gradient descent algorithm.

In this work we fill this gap. We consider the jammed (UNSAT) phase of the simplest perceptron model when the cost function is linear in the unsatisfied gaps. This modification leads to new physical properties in the UNSAT-RSB phase. The gap distribution is critical, positive and negative gaps display power law divergences for small argument. Surprisingly the critical exponent coincides with the one on the non-convex jamming line. Furthermore we find that RSB is associated to isostaticity in the sense that even if the model is taken in the UNSAT phase, there is an isostatic number of marginally satisfied gaps. While in the convex cost case jamming criticality
only holds on a line, here we have a whole phase which is jamming-critical. Therefore here landscape marginality and jamming marginality appear to coincide.

The spherical perceptron with a linear cost function has been studied back in the nineties. In [15–17] the phase diagram of the model was obtained at the replica symmetric level. While it was known that beyond the limit of capacity, or in the jammed phase, RSB is needed, systematic studies were not undertaken.

The model – The spherical linear perceptron model is defined through an $N$-dimensional vector $\mathbf{w}$ constrained to be on the $N$-dimensional hypersphere $|\mathbf{w}|^2 = N$ and by a set of $\alpha N$ $N$-dimensional random vectors $\{\mathbf{e}_\mu\}$ whose components are i.i.d. Gaussian random variables with zero mean and unit variance. For each of these vectors one can define a gap variable

$$ h_\mu = \frac{1}{\sqrt{N}} \mathbf{e}_\mu \cdot \mathbf{w} - \sigma $$

being $\sigma$ a control parameter in the problem. Given the gaps, one can define a cost function given by

$$ H[\mathbf{w}] = \sum_{\mu=1}^{\alpha N} |h_\mu| \theta(-h_\mu) $$

The phase diagram obtained in [15–17] is replotted for convenience in Fig. 1 and is given in terms of the control parameters $\sigma$ and $\alpha$.

Numerical Simulations – Local minima of the cost function defined in Eq. (2) turn out to be non analytic angular points determined by the intersection of hyperplanes. In order to treat this non-analyticity we smooth out the singularity at $h_\mu = 0$ and define a smoothed cost function as

$$ H_\varepsilon[\mathbf{w}] = \sum_{\mu=1}^{\alpha N} \left|h_\mu + \frac{\varepsilon}{2}\right| \theta(-h_\mu - \varepsilon) + \frac{1}{2\varepsilon} \sum_{\mu=1}^{\alpha N} h_\mu^2 \theta(\varepsilon + h_\mu) \theta(-h_\mu) + \frac{\mu}{2} (|\mathbf{w}|^2 - N)^2 $$

where $\varepsilon > 0$ and the last term is added to implement the spherical constraint on the vector $\mathbf{w}$. We implemented the numerical minimization of the cost function defined in Eq. (3) using the routine BFGS [18] of the SciPy library [19]. For each value of $\varepsilon$ the algorithm reaches a minimum of the cost function. In order to describe the properties of the energy landscape (2) we need to take the limit $\varepsilon \to 0^+$. We observe that when $\varepsilon$ is sufficiently small, there is a fixed number of gaps whose magnitude is in the interval $D = [-\varepsilon, 0]$. Decreasing the value of $\varepsilon$ we observe that the gaps contained in $D$ typically do not change indicating that in the $\varepsilon \to 0^+$ limit they are marginally satisfied and we call them contacts. Therefore, the distribution of gaps contains a delta function at $h = 0$. If we call $I_D$ the number of contacts we can define an isostaticity index $c = I_D/N$. In Fig. 2 we plot $c$ as a function of $\sigma$ for $\alpha = 5$. In the region where replica symmetry is unbroken, we observe that $c < 1$ and as soon as the minimization is run in the RSB region we find $c = 1$. [20].

Once the contacts are identified we can construct the statistics of strictly positive and negative gaps. In the replica symmetric phase both distributions have a finite limit when $|h| \to 0$. As soon as we enter the RSB region we start to observe power law divergences for $|h| \to 0$. In Fig. 3 we plot the cumulative distribution of both positive and negative gaps for minima with an average energy

![Figure 2](image1.png) 
![Figure 3](image2.png)
forces  \( \hat{f} \) the jamming point. The distribution of these forces has support in temperature limit linear perceptron can be analyzed by studying the zero emerging scaling behavior even in the UNSAT phase.

There are three classes of small gaps: one set of gaps that are identically zero, and two sets of positive and negative gaps that accumulate around zero. Unset of gaps that are identically zero, and two sets of positive and negative gaps that accumulate around zero. Therefore we find critical behavior in the gap distribution far away from the jamming point.

Finally, in the RSB region we can obtain the virtual forces \( \hat{f} \) associated to the isostatic delta peak of contacts. These are defined as the Lagrange multipliers needed to enforce that the corresponding gaps are identically zero [22]. The distribution of these forces has support in \([0,1]\) and, interestingly, appears to be critical close to both edges \( \hat{f} = 0 \) and \( \hat{f} = 1 \). In Fig. 4 we plot the distributions of both \( \hat{f} \) and \( 1 - \hat{f} \). Both edges display a power law pseudogap with exponents \( \theta \) and \( \theta' \), which appear to be very close to each other and \( \theta \approx \theta' \approx 0.42 \).

Therefore the numerical simulations show that when the energy minimization is carried out in the RSB-UNSAT phase, there are three classes of small gaps: one set of gaps that are identically zero, and two sets of positive and negative gaps that accumulate around zero. Unlike for the harmonic case, here we have that there is emerging scaling behavior even in the UNSAT phase.

Theory – The phase diagram and the properties of the linear perceptron can be analyzed by studying the zero temperature limit \( \beta = 1/T \to \infty \) of its free energy given by [23]

\[
\hat{f} = -\frac{1}{\beta N} \ln \int dx e^{-\beta H(x)}
\]

where the overline stands for the average over the random patterns \( \xi^\mu \). The average over disorder can be performed using the replica method and the main steps are described in [11, 16]. In the replica symmetry broken region the solution of the model is described by the full RSB Parisi PDEs [8]

\[
\frac{\partial m(q,h)}{\partial q} = -\frac{1}{2} m''(q,h) \frac{x(q)}{\lambda(q)} m(q,h) [1 + m'(q,h)]
\]

\[
\frac{\partial P(q,h)}{\partial q} = \frac{1}{2} \left[ P''(q,h) - 2 \frac{x(q)}{\lambda(q)} P(q,h) m(q,h) \right]
\]

where the primes indicate partial derivatives with respect to \( h \) and the initial conditions are given by

\[
m(q_M, h) = (1 - q_M) \ln \gamma_{1-q_M} \ast e^{-\beta|h|\theta(-h)}
\]

\[
P(q_M, h) = \gamma_{q_M}(h + \sigma)
\]

where \( \gamma_\Delta \) is a Gaussian with zero mean and variance \( \Delta \) and \( \ast \) stands for the convolution operation. The function \( x(q) \) is defined in the interval \( q \in [q_M, q_M] \) and is related to the distribution of the overlap \( q = w_1 \cdot w_2/N \) between two solutions or local minima of the cost function through the relation \( P(q) = dx(q)/dq \). The function \( \lambda(q) \) is defined through \( x(q) \) by

\[
\lambda(q) = 1 - q_M + \int_{q}^{q_M} dx(p)
\]

The properties of Eqs. (5) coming from the SAT phase and approaching the jamming limit have been analyzed in [9, 11] and are independent on the precise form of the cost function. Here we want to analyze what happens in the RSB-UNSAT phase far away from jamming. In the zero temperature limit, one has \( q_M \to 1 \). Therefore we need to study Eqs. (5) in the double scaling limit of \( \beta \to \infty \) and \( q \to 1 \). In the following we will propose a scaling solution to make sense of the power laws observed in numerical simulations. We assume that \( 1 - q_M \sim T^\kappa \) being \( \kappa \) a non trivial critical exponent. Taking first the limit \( T \to 0 \) and analyzing the scaling region \( 1 - q_M \ll 1 - q \ll 1 \) we can show that the equation for \( m(q,h) \) in Eqs. (5) admits the following scaling solution

\[
m(q,h) = \begin{cases} 
-\sqrt{1-q} M_+ \left( \frac{h}{\sqrt{1-q}} \right) & |h| \ll \lambda(q) \\
-h + \sqrt{1-q} M_- \left( \frac{h + \lambda(q)}{\sqrt{1-q}} \right) & h \sim -\lambda(q)
\end{cases}
\]

where we have defined \( \dot{\lambda}(q) = \beta \lambda(q) \sim (1-q)^{(\kappa-1)/\kappa} \) and where the two scaling functions \( M_+ \) and \( M_- \) satisfy the two ODEs

\[
M_+(t) - t M'_+(t) = M''_+(t)
\]

\[
t M'_-(t) - M_-(t) = -M''_-(t)
\]

with the Dirichelet boundary conditions

\[
M_-(t \to -\infty) = M_+(t \to -\infty) = t
\]

\[
M_-(t \to \infty) = M_+(t \to \infty) = 0
\]
We note that the equation for $\mathcal{M}_+$ coincides with the one that one gets studying the jamming transition coming from the SAT phase [11]. Also the function $P(q, h)$ develops a scaling regime in the double scaling limit. This is described by

$$
P(q,h) = \begin{cases} 
    p_+(h) & h \gg \sqrt{1-q} \\
    (1-q)\gamma_0 p_0 \left( \frac{\sqrt{1-q}}{h} \right) & h \sim \sqrt{1-q} \\
    \tilde{\lambda}(q)^{-1} p_-(h\tilde{\lambda}(q)^{-1}) & -h \sim \tilde{\lambda}(q) \\
    (1-q)\gamma_0 \tilde{p}_0 \left( \frac{h+\tilde{\lambda}(q)}{\sqrt{1-q}} \right) & |h+\tilde{\lambda}(q)| \sim \sqrt{1-q} \\
    \tilde{p}_+ (h+\tilde{\lambda}(q)) & h+\tilde{\lambda}(q) \ll \sqrt{1-q} 
\end{cases}
$$

As for the jamming transition [6] only $p_0$ and $\tilde{p}_0$ are universal functions fixed by two scaling equations given by

$$\begin{aligned}
\frac{\alpha}{\kappa} p_0(t) + \frac{1}{2} t p_0''(t) &= \frac{1}{2} p_0''(t) \left( p_0(t, \mathcal{M}_+(t)) \right)' \\
\frac{\alpha}{\kappa} \tilde{p}_0(t) + \frac{1}{2} t \tilde{p}_0''(t) &= \frac{1}{2} \tilde{p}_0''(t) + \frac{\kappa-1}{\kappa} \left( \tilde{p}_0(t)-\mathcal{M}_-(t) \right)' .
\end{aligned}
$$

Again, the equation for $p_0(t)$ coincides with the one found on approaching the jamming transition from the SAT phase. The boundary conditions for Eqs. (12) are

$$\begin{aligned}
p_0(t \to \infty) &\sim |t|^{-\gamma'} \\
p_0(t \to -\infty) &\sim |t|^\theta' \\
\tilde{p}_0(t \to -\infty) &\sim |t|^\theta' \\
\tilde{p}_0(t \to \infty) &\sim |t|^{-\gamma'} 
\end{aligned}
$$

which imply the matching conditions

$$
\begin{aligned}
p_+(t \to 0^+) &\sim t^{-\gamma'} \\
p_-(t \to 0^-) &\sim |t|^{-\gamma'} \\
p_- (t \to 1^+) &\sim |t-1|^\theta' .
\end{aligned}
$$

and the scaling relations

$$
\begin{aligned}
\gamma &= \frac{2a}{\kappa} \\
\theta &= \frac{1 - \kappa + a}{\kappa/2 - 1} \\
\gamma' &= \frac{2\tilde{a}}{\kappa} \\
\theta' &= \frac{1 - \kappa + \tilde{a}}{\kappa/2 - 1} .
\end{aligned}
$$

Again these scaling relations are the same as the ones found at the jamming transition point coming from the SAT phase [11]. Looking at Eqs. (12) it easy to realize that if one finds a solution for $\mathcal{M}_+$ and $p_0(t)$ then the solution for $\tilde{p}_0(t)$ and $\mathcal{M}_-$ is given by

$$\tilde{p}_0(t) = p_0(-t) \quad \mathcal{M}_-(t) = t + \mathcal{M}_+(-t) .
$$

This implies that $\tilde{a} = a$, $\gamma = \gamma'$, $\theta = \theta'$ and therefore that the exponent $\kappa \simeq 1.41 \ldots$ is the same as the one appearing at the jamming transition coming from the SAT side [5, 11]. Finally the marginal stability of the fullRSB phase implies that

$$\alpha \int_{-1}^0 dt \, p_-(t) = 1
$$

which implies that the gap distribution contains an isostatic delta peak of contacts. Using Eqs.(12) and Eq. (17) we get that the gap distribution is given by

$$\begin{aligned}
\rho(h) &= \frac{1}{\alpha N} \sum_{\mu=1}^{\alpha N} \delta(h-h_{\mu}) \\
&\sim \rho_+ h^{-\gamma}(h) + \rho_- (-h)^{-\gamma}(h) + \frac{1}{\alpha} \delta(h) \quad h \to 0
\end{aligned}
$$

in the RSB-UNSAT phase with $\rho_+$ and $\rho_-$ two positive constants. Finally, as on the jamming line, the virtual forces associated to the contacts are described by the distribution $p_-(t)$ and therefore they are characterized by power law pseudogaps with exponent $\theta = 0.42311 \ldots$, consistent with the numerical value shown in Fig. 4.

**Conclusions** – We have analyzed the properties of the RSB-UNSAT phase of the spherical linear perceptron and we have shown that the whole phase displays jamming criticality. As soon as the system enters the RSB phase, the gap distribution shows an isostatic delta peak and the gap and force distributions are characterized by a set of power laws. Both positive and negative gaps display power law divergences for small values of the gaps while the virtual forces associated to the contacts are described by two pseudogaps with an associated critical exponent. Despite the fact that we are far away from jamming the values of the critical exponents are related to the ones arising at the jamming point. This provides evidence that when an isostatic delta peak of contacts is present in the gap distribution (regardless if this happens at jamming or far away) and the model is in a replica symmetry broken phase, one has universal critical behavior, with exponents determined by the scaling regime of the fullRSB equations. There are two clear future directions. On the one hand it is crucial to understand what happens for different cost functions. In particular the so called Gardner-Derrida cost function [24] for which the energy equals the number of UNSAT gaps seems to give rise some critical behavior in the UNSAT phase (see Fig. 4 of [25]). Furthermore it is clear that performing the same analysis for zero temperature jammed packings of linear spheres may provide a new test for RSB in such systems.

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[20] When the minimization is run in the RSB region we typically find $N - 1$ gaps in $D$ so that, observing that the spherical constraint reduces the number of degrees of freedom by one, we get exact isostaticity.
[21] For each sample we target the energy using a standard decompression protocol.
[22] The virtual forces can be obtained either as the negative gaps in $D$ rescaled by $\varepsilon$ or by getting the contacts and then computing analytically the corresponding Lagrange multipliers. While the first procedure always works, the second one works only when exact isostaticity is found. This is not always the case since for some samples our minimization protocol does not find exactly $N - 1$ gaps in $D$. However this seems a finite size and numerical effect and when producing the plots in Fig. 4 we selected the samples for which perfect isostaticity was found. In any case we checked that using the $\varepsilon > 0$ procedure gives the same power laws in the cumulative distributions.
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