Localised eigenfunctions in Šeba billiards

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Abstract

We describe some new families of quasimodes for the Laplacian perturbed by the addition of a potential formally described by a Dirac delta function. As an application we find, under some additional hypotheses on the spectrum, subsequences of eigenfunctions of Šeba billiards that localise around a pair of unperturbed eigenfunctions.

1 Introduction

One of the unsolved questions in the analysis of quantum eigenfunctions concerns possible limiting distributions as the eigenvalue tends to infinity. For eigenfunctions of the Laplace operator on certain surfaces with arithmetical properties, it has been proved \cite{1, 2} that there is only one possible limit; all sequences of eigenfunctions become uniformly distributed. On the other hand, Hassell \cite{3} has proved the existence of chaotic billiard domains in $\mathbb{R}^2$ for which zero-density subsequences of eigenfunctions fail to equi-distribute in the limit.

We consider the Laplace operator plus potential supported at a single point. Such a potential has been variously referred to as delta-interaction potential, Fermi pseudo-potential or zero-range potential in different parts of the literature. Mathematically this operator can be constructed using the tools of self-adjoint extension theory.

We will prove our results for the case where the underlying space is a compact 2-dimensional manifold, for which the Laplace operator has eigenfunctions and eigenvalues denoted by $\Phi_j$ and $E_j$ respectively. We perturb this operator with a delta potential supported at the point $p$, which will remain fixed throughout, and suppressed from notations. This perturbation can be realised by a 1-parameter family of self-adjoint operators, $H_\Theta$, indexed by an angle $\Theta$ which controls the strength of the perturbation.

We fix a finite interval $I \subseteq \mathbb{R}$ containing at least one $E_j$, and define, for notational convenience,

$$\zeta_I(s, \lambda) := \sum_{E_j \in I} \frac{|\Phi_j(p)|^2}{(E_j - \lambda)^s}. \quad (1)$$
Let $\sigma \in [0, 1]$. We define

$$
\psi(x) := \sum_{E_j \in I} \frac{\Phi_j(p)}{E_j - \mu} \Phi_j(x) + \sigma \sum_{E_j \notin I} \left( E_j - \frac{\sin \Theta}{1 - \cos \Theta} \right) \frac{\Phi_j(p)}{1 + E_j^2} \Phi_j(x),
$$

for $\mu$ a solution to

$$
\zeta_I(1, \mu) = \sigma \sum_{E_j \in I} \left( E_j - \frac{\sin \Theta}{1 - \cos \Theta} \right) \frac{|\Phi_j(p)|^2}{1 + E_j^2}.
$$

Our main results are as follows:

**Theorem 1.1.** The pair $(\psi, \mu)$ is a quasimode for $H_\Theta$ with discrepancy $d$, where

$$
d^2 = \frac{(1 - \sigma)^2 \zeta_I(0, \mu) + \sigma^2 \sum_{E_j \notin I} \left( 1 + E_j \mu + \frac{\sin \Theta}{1 - \cos \Theta} (E_j - \mu) \right)^2 \frac{|\Phi_j(p)|^2}{(1 + E_j^2)^2}}{\zeta_I(2, \mu) + \sigma^2 \sum_{E_j \notin I} \left( E_j - \frac{\sin \Theta}{1 - \cos \Theta} \right)^2 \frac{|\Phi_j(p)|^2}{(1 + E_j^2)^2}}.
$$

Furthermore, if $\psi_1, \psi_2$ are defined by (2) for $\mu_1 \neq \mu_2$, two solutions of (3) then

$$
\langle \psi_1, \psi_2 \rangle = \sigma^2 \sum_{E_j \notin I} \left( E_j - \frac{\sin \Theta}{1 - \cos \Theta} \right)^2 \frac{|\Phi_j(p)|^2}{(1 + E_j^2)^2}.
$$

The construction of families of quasimodes is a key step in Hassell’s proof [3], as well as the proofs of many recent results on localisation of quantum eigenfunctions [4, 5, 6, 7, 8, 9]. One reason for this is that quasimodes can often be used to approximate eigenfunctions. In general (see the introduction to section 3 for precise statements) the smaller the discrepancy, the closer quasimodes are to true eigenfunctions. For this reason it is important to know when the discrepancy can be made small. In this direction we have the following corollary to theorem 1.1:

**Corollary 1.2.** Let $\sigma = 1$ and let $I = [0, T]$ where $T > E_1$. Then the discrepancy $d$ of the quasimode $\psi$ satisfies

$$
d \ll \frac{\mu}{\sqrt{T}}.
$$

Let $\sigma = 0$ and $I$ be any interval containing at least two $E_j$. If $\mu \in I$ then the discrepancy $d$ of $\psi$ satisfies

$$
d \leq \frac{1}{\sqrt{2}} \ell(I),
$$

where $\ell(I)$ is the length of $I$. If, additionally, $I$ contains precisely two $E_j$, then we have

$$
d \leq \frac{1}{2} \ell(I).
$$

In particular, the quasimodes with $\sigma = 1$ and $\mu$ held fixed or slowly growing, can be made arbitrarily precise by choosing $T$ as large as desired.

We are interested in ascertaining when true eigenfunctions of $H_\Theta$ have mass supported on our quasimodes. Without any assumptions on the spectrum of the Laplacian we can prove the following.
Proposition 1.3. For any consecutive eigenvalues $E_a < E_b < E_c < E_d$ from the sequence $(E_j)_{j=1}^{\infty}$, let $I = [E_b, E_c]$ and take $\sigma = 0$. Choose $\mu$ so that $\mu \in I$. Then there is an eigenfunction $\phi$ of $H_\Theta$ with eigenvalue in the interval $(E_a, E_d)$ such that

$$|\langle \phi, \psi \rangle| \geq \|\psi\| \sqrt{3} \left(1 - \frac{(E_c - E_b)^2}{4 \min\{E_d - E_c, E_b - E_a\}^2}\right)^{1/2}.$$  \hfill (9)

Proposition 1.3 is most interesting when the sequence of eigenfunctions $\Phi_j$ do not equidistribute. (For example if they are solutions to a PDE which is subject to separation of variables; see below.) Then, by considering an infinite subset of the spectrum $\{E_j\}$ along which the right-hand side of (9) is bounded away from zero, proposition 1.3 proves the existence of a sequence of eigenfunctions of $H_\Theta$ which fail to equidistribute. Such a subset of $\{E_j\}$ does exist since the mean level spacing is constant.

Clearly the best that proposition 1.3 can achieve is to prove that a sequence of quasimodes has an overlap of up to $1/\sqrt{3}$ with a subsequence of true eigenfunctions. In order to prove that a sequence of quasimodes converges fully towards a sequence of eigenvalues of $H_\Theta$ we need to make some assumptions on the spectrum of the Laplacian. Sufficient conditions for this and a precise statement of the result (theorem 4.4) are given in section 4.

The history of the study of the spectral properties of differential operators perturbed by the addition of a delta scatterer goes back at least to [10], in which a one-dimensional lattice of delta interactions was used to model an electron moving in a crystal lattice. A comprehensive historical review is given in the appendix to the book [11].

Part of our interest in the subject comes from the Šeba billiard which was introduced in [12]. In this work, a hard-walled rectangular billiard with a potential supported at a single point was considered. In terms of classical dynamics, the motion is integrable, since only a zero-measure set of trajectories meet the point at which the potential is supported. However, diffraction effects are introduced when one considers the quantum spectrum of the corresponding Schrödinger operator.

Šeba billiards have become important since the observation [13, 14] that the quantum spectral statistics belong to a new universality class, different from the classes from random matrix theory conjecturally associated to chaotic dynamical systems [15, 10] or the statistics of a Poisson process conjecturally associated to fully-integrable dynamical systems [17]. It is now known that general integrable systems perturbed by the addition of such a localised scatterer also belong to the same universality class [18], as do quantum Neumann star graphs [19, 20]. Characteristic features of the spectral statistics of this universality class are an exponential decay of large level spacings, together with level repulsion.

Several analytical studies of these spectral statistics have been made [18, 21, 22, 23, 24, 25]. Typically, a key feature of these arguments is the assumption of Poissonian behaviour for the eigenvalues of the billiard table without scatterer, a conjectured consequence of the integrable dynamics (the Berry-Tabor conjecture) [17].

In the final section of this article we apply theorem 4.4 to the original Šeba billiard. Our final result is a proof that there exists a subsequence of eigenfunctions of the Šeba billiard that become localised on a pair of consecutive eigenfunctions of the unperturbed billiard, if the spectrum of the unperturbed billiard satisfies an assumption which is consistent with the Berry-Tabor conjecture.
This result is a rigorous derivation of a formal argument first proposed in [26] and mirrors a related result proved for quantum graphs with a star-shaped connectivity [27]. These so-called quantum star graph can be considered as a singular perturbation of a disconnected set of one-dimensional bonds, each supporting a wave-function. In [27] the existence of subsequences of eigenfunctions that become localised on a pair of bonds was proved. This is exactly analogous to the localisation onto a pair of unperturbed billiard eigenfunctions in theorem 4.4. In both [27] and theorem 4.4 the main idea of the proof is to show localisation in an eigenfunction with eigenvalue lying between two closely-spaced eigenvalues of the unperturbed problem.

2 Realisation of the perturbed operator

Let \( M \) be a compact 2-dimensional Riemannian manifold, possibly with piecewise-smooth boundary, and let \( \Delta \) be a self-adjoint Laplacian on \( M \).

The realisation of the operator formally defined by
\[
H = -\Delta + c\delta(x - p),
\]
where \( p \in M \) and \( \delta \) is the Dirac delta function, using the theory of self-adjoint extensions is given in many places in the literature. We refer the reader to [28, 29] for the details. Here we recapitulate only that which is necessary to fix notations. We denote by \( \| \cdot \| \) and \( \langle \cdot, \cdot \rangle \) the norm and inner product of \( L^2(M) \).

Since \( M \) is compact, \(-\Delta\) has a complete basis of eigenfunctions, \( \Phi_j \), with corresponding eigenvalues \( E_j \) which we write in non-decreasing order.

We will remove from the list of eigenvalues any \( E_j \) for which \( \Phi_j(p) = 0 \). Such eigenfunctions are not affected by a delta-scatterer at \( p \), and so it is convenient to exclude them from the spectrum. This further allows us to assume that the spectrum \( \{ E_j \} \) is simple, without losing generality.

To see this, consider an eigenspace of dimension \( r > 1 \) spanned by the eigenfunctions \( \{ \tilde{\phi}_1, \ldots, \tilde{\phi}_r \} \). Then the vectors \(( \tilde{\phi}_1(p), \ldots, \tilde{\phi}_r(p) )^T \) and \(( R, 0, \ldots, 0 )^T \) in \( \mathbb{C}^r \), where
\[
| R |^2 = \sum_{i=1}^r | \tilde{\phi}_i(p) |^2,
\]
have identical norm. This means that we can find a unitary \( r \times r \) matrix mapping the first vector to the second. Multiplying \( U \) by the vector of eigenfunctions \(( \tilde{\phi}_1, \ldots, \tilde{\phi}_r )^T \) leads to a new basis for the eigenspace, in which all but the first eigenfunction vanishes at the point \( p \), and the corresponding eigenvalue is counted with multiplicity one.

The resulting spectrum is therefore ordered so that
\[
E_1 < E_2 < E_3 \cdots
\]

We will frequently use Weyl’s law with remainder estimate [30]:
\[
N(E) := \sum_{E_j \leq E} | \Phi_j(p) |^2 = \frac{E}{4\pi} + O(E^{1/2}),
\]
where the implied constant may depend on the position of the point \( p \in \mathcal{M} \).

Define
\[
g_{\pm i}(x) := \sum_{j=1}^{\infty} \frac{\varphi_j(x)\overline{\varphi_j(p)}}{E_j \pm i}.
\] (14)

Then \( g_{\pm i} \in L^2(\mathcal{M}) \) and in fact they are the Green functions for the resolvent of \(-\Delta\) at the imaginary energies \( \pm i \), satisfying
\[
\langle f, g_{\pm i} \rangle = \frac{(-\Delta \pm i)^{-1} f(p)}{E_j \pm i}.
\] (15)

In particular,
\[
\langle \varphi_j, g_{\pm i} \rangle = \frac{(-\Delta \pm i)^{-1} \varphi_j(p)}{E_j \pm i},
\] (16)
which will be useful to know later.

Let
\[
\mathcal{D}_p := \{ f \in \text{Dom}(\Delta) : \langle f, \delta_p \rangle = 0 \},
\] (17)
and define the operator \( H_0 \) with domain \( \mathcal{D}_p \) by
\[
H_0 : f \mapsto -\Delta f.
\] (18)

\( H_0 \) is a symmetric, but not self-adjoint operator. In fact its deficiency subspaces are spanned by \( g_{\pm i} \).

It follows from the von Neumann theory that,
\[
\text{Dom}(H_0^*) = \mathcal{D}_p \oplus \text{span}\{g_i, g_{-i}\}.
\] (19)

Since the deficiency indices are equal, \( H_0 \) possesses self-adjoint extensions, constructed as follows.

First of all, note that we can write for \( \psi \in \text{Dom}(H_0^*) \),
\[
\psi = \hat{\psi} + a_+(\psi)g_i + a_-(\psi)g_{-i},
\] (20)
where \( \hat{\psi} \in \mathcal{D}_p \) and \( a_\pm(\psi) \in \mathbb{C} \). In fact we have
\[
H_0^*\psi = H_0\hat{\psi} + ia_+(\psi)g_i - ia_-(\psi)g_{-i}.
\] (21)

Since the deficiency indices of \( H_0 \) are both equal to 1, there is a 1-parameter family of self-adjoint extensions, \( H_\Theta \), \( 0 < \Theta \leq 2\pi \) with
\[
\text{Dom}(H_\Theta) = \{ \psi \in \text{Dom}(H_0^*) : a_-(\psi) = -e^{i\Theta}a_+(\psi) \}.
\] (22)

We take the self-adjoint operator \( H_\Theta \) to be the realisation of the formal operator \( H_0 \).

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\[1\] The notations \( f = O(g) \) and \( f \ll g \) both mean that there exists a positive constant \( C \) (the “implied constant”) such that \( f \leq Cg \).
3 Quasimodes

3.1 Definitions and basic properties

Let $H$ be a self-adjoint operator in a Hilbert space, without continuous spectrum.

**Definition 3.1.** A quasimode of $H$ with discrepancy $d$ is a pair $(\psi, \mu) \in \text{Dom}(H) \times \mathbb{R}$ such that

$$\| (H - \mu) \psi \| \leq d \| \psi \|. \quad (23)$$

We are interested in the situation when the quasi-eigenvalue $\mu$ and quasi-eigenfunction $\psi$ approximate true eigenvalues $\lambda_j$ and eigenfunctions $\phi_j$ of $H$. In this direction, the following classical results apply (see, e.g. [32, 34]).

For a quasimode with discrepancy $d$, the interval $[\mu - d, \mu + d]$ contains at least one eigenvalue of $H$.

If we consider instead, the interval $[\mu - M, \mu + M]$ where $M > 0$, then

$$\sum_{\lambda_j \notin [\mu - M, \mu + M]} |\langle \psi, \phi_j \rangle|^2 \leq \frac{d^2}{M^2} \| \psi \|^2. \quad (24)$$

In particular, if $\psi$ is normalised, and the interval $[\mu - M, \mu + M]$ contains only a single eigenvalue with eigenfunction $\phi$, then there is a phase $\chi \in [0, 2\pi)$ such that

$$\| \phi - e^{i\chi} \psi \| \leq \frac{2d}{M}. \quad (25)$$

These results will be the main tools by which we relate the quasimodes constructed in the next subsection to the eigenfunctions and eigenvalues of $H_\Theta$.

3.2 Quasimodes of delta perturbations

Let $I \subseteq \mathbb{R}$ be a finite interval containing at least one point $E_j$ of the spectrum of $-\Delta$. Let $\sigma \in [0, 1]$. We will associate to the interval $I$ a family of quasimodes parametrised by $\sigma$.

We first define

$$\psi_{\sigma, I, z} := \sum_{E_j \in I} \frac{\Phi_j(p)}{E_j - z} \phi_j + \frac{\sigma}{1 - e^{i\Theta}} P_{\sigma}(g_i - e^{i\Theta} g_{-i}), \quad (26)$$

where $P_S$ is the spectral projection operator onto the set $S$,

$$P_S f := \sum_{E_j \in S} \langle f, \phi_j \rangle \phi_j, \quad (27)$$

and $I^c$ is the complement to $I$. We have the following:

**Lemma 3.2.** For $z \neq E_j$ for any $E_j \in I$, the function $\psi_{\sigma, I, z}$ satisfies

$$\| \psi_{\sigma, I, z} \|^2 = \zeta_I(2, z) + \sigma^2 \sum_{E_j \notin I} \left( E_j - \frac{\sin \Theta}{1 - \cos \Theta} \right)^2 \frac{|\Phi_j(p)|^2}{(1 + E_j^2)^2}, \quad (28)$$

with the second term being bounded by a constant independent of $I, z$ and $\sigma \in [0, 1]$. 

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Proposition 3.3. The function $\psi_{\sigma, I}$ belongs to $\text{Dom}(H_\Theta)$ and satisfies
\[
\|(H_\Theta - \mu)\psi_{\sigma,I}\|^2 = (1 - \sigma)^2 \zeta_I(0, \mu) + \sigma^2 \sum_{E_j \not\in I} \left(1 + E_j \mu + \frac{\sin \Theta}{1 - \cos \Theta} (E_j - \mu)\right)^2 \frac{|\Phi_j(p)|^2}{(1 + E_j)^2}.
\]
Using (30) we can express this as

\[ \psi_{\sigma,I} = \sum_{E_j \in I} \frac{\Phi_j(p)}{E_j - \mu} \Phi_j - \frac{\sigma}{1 - e^{i\Theta}} P_I(g_i - e^{i\Theta} g_{-i}) + \frac{\sigma}{1 - e^{i\Theta}} (g_i - e^{i\Theta} g_{-i}). \] (37)

Using (30) we can express this as

\[ \psi_{\sigma,I} = \hat{\psi}_{\sigma,I} + \frac{\sigma}{1 - e^{i\Theta}} (g_i - e^{i\Theta} g_{-i}). \] (38)

where

\[ \hat{\psi}_{\sigma,I}(x) := \sum_{E_j \in I} \left( \frac{1}{E_j - \mu} - \frac{\sigma E_j}{1 + E_j^2} + \frac{\sigma \sin \Theta}{1 - \cos \Theta} \right) \frac{\Phi_j(p)}{E_j^2} \Phi_j(x). \] (39)

Now observe that due to the definition (35) of \( \mu \), \( \hat{\psi}_{\sigma,I}(p) = 0 \), so \( \psi_{\sigma,I} \in D_p \). Thus (38) justifies the assertion \( \psi_{\sigma,I} \in \text{Dom}(H_\Theta) \).

Since \( H_0^* \) is an extension of \( H_\Theta \), we have

\[ (H_\Theta - \mu) \psi_{\sigma,I} = (H_0 - \mu) \hat{\psi}_{\sigma,I} + \frac{\sigma}{1 - e^{i\Theta}} ((i - \mu)g_i + e^{i\Theta}(i + \mu)g_{-i}). \] (40)

Now,

\[ (H_0 - \mu) \hat{\psi}_{\sigma,I} = \sum_{E_j \in I} \left( 1 - \frac{\sigma E_j (E_j - \mu)}{1 + E_j^2} + \frac{\sigma \sin \Theta}{1 - \cos \Theta} \right) \frac{\Phi_j(p)}{E_j} \Phi_j \\
= \sum_{E_j \in I} \left( 1 + (1 - \sigma)E_j^2 + \sigma E_j \mu + \frac{\sigma \sin \Theta}{1 - \cos \Theta} (E_j - \mu) \right) \frac{\Phi_j(p)}{E_j} \Phi_j. \] (41)

Using (16) we find

\[ \langle \Phi_j, (i - \mu)g_i + e^{i\Theta}(i + \mu)g_{-i} \rangle = \left( -\frac{i + \mu}{E_j + 1} \Phi_j(p) + e^{-i\Theta} \frac{-i + \mu}{E_j - 1} \Phi_j(p) \right) \]
\[ = \frac{1 - e^{-i\Theta}}{1 + E_j^2} \left( -(1 + E_j \mu) - \frac{\sin \Theta}{1 - \cos \Theta} (E_j - \mu) \right) \Phi_j(p), \] (42)

again using (31). This leads to

\[ \frac{\sigma}{1 - e^{i\Theta}} (H_\Theta - \mu) (g_i - e^{i\Theta} g_{-i}) = -\sigma \sum_{j=1}^{\infty} \left( 1 + E_j \mu + \frac{\sin \Theta}{1 - \cos \Theta} (E_j - \mu) \right) \frac{\Phi_j(p)}{1 + E_j^2} \Phi_j, \] (43)

and combining this with (41), we get

\[ (H_\Theta - \mu) \psi_{\sigma,I} = (1 - \sigma) \sum_{E_j \in I} \frac{\Phi_j(p)}{E_j} \Phi_j \\
- \sigma \sum_{E_j \in I} \left( 1 + E_j \mu + \frac{\sin \Theta}{1 - \cos \Theta} (E_j - \mu) \right) \frac{\Phi_j(p)}{1 + E_j^2} \Phi_j. \] (44)
Since the summations in (44) are over disjoint sets it is easy to calculate the norm:

\[
\| (H_\Theta - \mu) \psi_{\sigma,I} \|^2 = (1 - \sigma)^2 \sum_{E_j \in I} |\Phi_j(p)|^2
+ \sigma^2 \sum_{E_j \notin I} \left( 1 + E_j \mu + \frac{\sin \Theta}{1 - \cos \Theta} (E_j - \mu) \right)^2 \frac{|\Phi_j(p)|^2}{(1 + E_j^2)^2}.
\]

(45)

\[\square\]

3.2.1 Proof of theorem 1.1

The first part of the theorem follows from lemma 3.2 and proposition 3.3 and the definition of a quasimode.

For the second part, let \( \mu_1 \neq \mu_2 \) be two solutions of (35). We have

\[
\langle \psi_{\sigma,I,\mu_1}, \psi_{\sigma,I,\mu_2} \rangle = \left\langle \sum_{E_j \in I} \frac{\Phi_j(p)}{E_j - \mu_1}, \sum_{E_j \in I} \frac{\Phi_j(p)}{E_j - \mu_2} \right\rangle + \sigma^2 \sum_{E_j \notin I} \frac{\sin \Theta}{1 - \cos \Theta} (E_j - \mu)^2 \frac{|\Phi_j(p)|^2}{(1 + E_j^2)^2},
\]

(46)

using (32). By elementary algebra,

\[
\frac{1}{(E_j - \mu_1)(E_j - \mu_2)} = \frac{1}{\mu_1 - \mu_2} \left( \frac{1}{E_j - \mu_1} - \frac{1}{E_j - \mu_2} \right),
\]

(47)

so, since by (35)

\[
\sum_{E_j \in I} \frac{|\Phi_j(p)|^2}{E_j - \mu_1} = \sum_{E_j \in I} \frac{|\Phi_j(p)|^2}{E_j - \mu_2},
\]

(48)

we get

\[
\sum_{E_j \in I} \frac{|\Phi_j(p)|^2}{(E_j - \mu_1)(E_j - \mu_2)} = 0.
\]

(49)

\[\square\]

3.2.2 Controlling the discrepancy of quasimodes

By tuning the parameter \( \sigma \), and choosing the interval \( I \) accordingly, we can find fix quasimodes with particular properties. In the previous section we have seen that sets of quasimodes with \( \sigma = 0 \) are orthogonal. We are particularly interested in when the discrepancy is small. In this subsubsection we prove corollary 1.2 that quasimodes with \( \sigma = 1 \) can be made arbitrarily precise, and that quasimodes with \( \sigma = 0 \) also can have a simple bound for the discrepancy.
Proof of corollary 1.2. Choosing $I = [0, T]$ for $T > E_1$ with $\sigma = 1$ gives, by theorem 1.1, that the discrepancy of $\psi_{1,I}$ satisfies

$$d^2\|\psi_{1,I}\|^2 = \sum_{E_j \geq T} \left( 1 + E_j \mu + \frac{\sin \Theta}{1 - \cos \Theta} (E_j - \mu) \right)^2 \frac{|\Phi_j(p)|^2}{(1 + E_j^2)^2}. \quad (50)$$

By lemma 3.2 we see that the norm of $\psi_{1,I}$ is bounded away from 0 by a constant, so that the asymptotics for $d$ are given by the term on the right-hand side of (50). Using Weyl’s law, we can estimate

$$\sum_{E_j \geq T} \left( 1 + E_j \mu + \frac{\sin \Theta}{1 - \cos \Theta} (E_j - \mu) \right)^2 \frac{|\Phi_j(p)|^2}{(1 + E_j^2)^2} \ll \mu^2 T, \quad (51)$$

which can be made arbitrarily small by increasing $T$.

For the second part with $\sigma = 0$, we have

$$d^2\|\psi_{0,I}\|^2 = \sum_{E_j \in I} |\Phi_j(p)|^2. \quad (52)$$

We observe that splitting the sum in (35) leads to

$$\sum_{E_j \in I \atop E_j > \mu} |\Phi_j(p)|^2 = \sum_{E_j \in I \atop E_j < \mu} |\Phi_j(p)|^2 \mu - E_j. \quad (53)$$

Denote by $E_+$ and $E_-$ the largest and smallest points of the spectrum $(E_j)_{j=1}^\infty$ lying in the interval $I$. Then

$$\sum_{E_j \in I \atop E_j > \mu} |\Phi_j(p)|^2 \leq (E_+ - \mu) \sum_{E_j \in I \atop E_j > \mu} \frac{|\Phi_j(p)|^2}{E_j - \mu} \quad (54)$$

and

$$\sum_{E_j \in I \atop E_j < \mu} |\Phi_j(p)|^2 \leq (\mu - E_-) \sum_{E_j \in I \atop E_j < \mu} \frac{|\Phi_j(p)|^2}{E_j - \mu} \quad (55)$$

Adding these inequalities, and using (53), we get

$$\sum_{E_j \in I} |\Phi_j(p)|^2 \leq (E_+ - E_-) \sum_{E_j \in I \atop E_j > \mu} \frac{|\Phi_j(p)|^2}{E_j - \mu} = (E_+ - E_-) \sum_{E_j \in I \atop E_j < \mu} \frac{|\Phi_j(p)|^2}{\mu - E_j}. \quad (56)$$

Since $E_+ - E_- \leq \ell(I)$ we get

$$2d^2\|\psi_{0,I}\|^2 \leq \ell(I) \sum_{E_j \in I} \frac{|\Phi_j(p)|^2}{|E_j - \mu|}. \quad (57)$$

Finally,

$$\ell(I) \sum_{E_j \in I} \frac{|\Phi_j(p)|^2}{|E_j - \mu|} \leq \ell(I)^2 \sum_{E_j \in I} \frac{|\Phi_j(p)|^2}{(E_j - \mu)^2} = \ell(I)^2 \|\psi_{0,I}\|^2, \quad (58)$$

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noting that $|E_j - \mu| \leq \ell(I)$ for $\mu \in I$.

We now consider the case with $\sigma = 0$, and $I$ containing only the two levels $E_j, E_{j+1}$. We can solve (35) directly to get

$$\mu = \frac{\left|\Phi_{j+1}(p)\right|^2 E_j + \left|\Phi_j(p)\right|^2 E_{j+1}}{\left|\Phi_{j+1}(p)\right|^2 + \left|\Phi_j(p)\right|^2}. \quad (59)$$

Substituting this value of $\mu$ into the definition of $\psi_{0,I}$ we get

$$\psi_{0,I} = \Phi_j(p)\frac{E_j - \mu}{E_{j+1} - E_j} \left(\frac{-1}{\Phi_j(p)} \Phi_j + \frac{1}{\Phi_{j+1}(p)} \Phi_{j+1}\right). \quad (60)$$

So,

$$\|\psi_{0,I}\|^2 = d^2 \|\psi_{0,I}\|^2 \frac{|\Phi_{j+1}(p)|^2 + |\Phi_j(p)|^2}{(E_{j+1} - E_j)^2} \left(\frac{1}{|\Phi_j(p)|^2} + \frac{1}{|\Phi_{j+1}(p)|^2}\right) \quad (62)$$

$$\geq \frac{4d^2 \|\psi_{0,I}\|^2}{\ell(I)^2}, \quad (63)$$

using the fact that

$$|\Phi_{j+1}(p)|^2 + |\Phi_j(p)|^2 \left(\frac{1}{|\Phi_j(p)|^2} + \frac{1}{|\Phi_{j+1}(p)|^2}\right) = 2 + \frac{|\Phi_{j+1}(p)|^2}{|\Phi_j(p)|^2} + \frac{|\Phi_j(p)|^2}{|\Phi_{j+1}(p)|^2} \geq 4. \quad (64)$$

The existence of arbitrarily precise quasimodes can be used to give a new proof of the often-used representation for eigenvalues and eigenfunctions of rank-one perturbations (see e.g. [18, 28, 24, 33]).

**Theorem 3.4.** The solutions $\lambda$ to the equation

$$\sum_{j=1}^{\infty} \left(\frac{1}{E_j - \lambda} - \frac{E_j}{1 + E_j^2}\right) |\Phi_j(p)|^2 = \frac{\sin \Theta}{1 - \cos \Theta} \sum_{j=1}^{\infty} |\Phi_j(p)|^2 \frac{1}{1 + E_j^2}, \quad (65)$$

are eigenvalues of $H_\Theta$ with corresponding eigenfunctions given by

$$\phi(x) = \sum_{j=1}^{\infty} \frac{\Phi_j(p)}{E_j - \lambda} \Phi_j(x). \quad (66)$$

Note that the left-hand side of (65) converges pointwise, and (66) converges in $L^2(\Omega)$.

By analysing the resolvent, it is possible to extend theorem 3.4 to get the following [28, Theoreme 2],

**Theorem 3.5.** Apart from the solutions to (65), there are no other points of the spectrum of $H_\Theta$ in any of the intervals $(E_M, E_{M+1})$. 

11
4 Localisation results

In this section we will consider the extent to which eigenfunctions of $H_\Theta$ can be approximated by quasimodes. In particular we will focus on the quasimodes with $\sigma = 0$. First we shall prove proposition 1.3 which is straightforward. Then we shall show that strengthening the assumptions made on the spectrum of $\Delta$ leads to a proof of full convergence.

Proof of proposition 1.3. The length of the interval $I$ is $\ell(I) = E_c - E_b$. Let $M = \min\{E_d - \mu, \mu - E_a\} \geq \min\{E_d - E_c, E_b - E_a\}$. By applying (24) with this $M$ we get,

$$\sum_{\lambda_j \in [E_a, E_d]} |\langle \psi_{0,I}, \phi_j \rangle|^2 \geq \|\psi_{0,I}\|^2 \left(1 - \frac{\ell(I)^2}{4 \min\{E_d - E_c, E_b - E_a\}^2}\right).$$

(67)

From theorem 3.5 there are only three eigenvalues of $H_\Theta$ in the interval $[E_a, E_d]$. It therefore follows that for at least one of these three eigenfunctions its inner-product squared with $\psi_{0,I}$ is at least $\frac{1}{3}$ of the right-hand side of (67).

We now consider how to improve proposition 1.3 at the expense of making further assumptions about the spectrum of $-\Delta$. For simplicity we will focus henceforth on the choice of parameter $\Theta = \pi$.

Figure 1: Part of the spectrum of $-\Delta$ and $H_\pi$. Vertical bars denote eigenvalues of $H_\pi$, and circles denote eigenvalues of $-\Delta$. The triangle $\mu$ is a quasi-eigenvalue approximating $\lambda$. See main text for further explanation.

In figure 1 a cartoon of part of the spectrum of $H_\pi$ and $-\Delta$ is displayed. Highlighted are four consecutive eigenvalues of $-\Delta$, labelled $E_a, E_b, E_c$ and $E_d$, chosen so that $E_c - E_b \leq \varepsilon$. (The positions of all points depend on $\varepsilon$.)

Between $E_b$ and $E_c$ is an eigenvalue, $\lambda$, of $H_\pi$.

We find a quasimode $\psi_{0,I}$ associated to the interval $I = [E_b, E_c]$ with quasi-eigenvalue $\mu$ approximating $\lambda$. By corollary 1.2 the discrepancy of this quasimode is no greater than $\varepsilon/2$. Between $E_c$ and $E_d$ is another eigenvalue $\lambda^*$ of $H_\pi$. In order to be able to apply (25), we need to be sure that $\lambda^*$ is not too close to $E_c$. An argument to show that this is the case is given below.

The eigenvalue between $E_a$ and $E_b$ can be handled with a similar method.

We shall make the following assumption on the spectral sequence of $\Delta$:
Assumption 4.1. For some \(0 < q < 1/2\) and \(1 < \rho < 2(1 - q)\), there exists a sequence \((\varepsilon_n)_{n=1}^\infty\), \(\varepsilon_n \downarrow 0\) such that for each \(n\) there are four consecutive eigenvalues, \(E_a(n) < E_b(n) < E_c(n) < E_d(n) \ll \varepsilon_n^\rho\), satisfying
\[
E_c - E_b \ll \varepsilon_n
\]
\[
E_d - E_c \gg \varepsilon_n^q
\]
\[
E_b - E_a \gg \varepsilon_n^q.
\]
as \(n \to \infty\).

Assumption 4.1 asserts that the positions of eigenvalues of \(-\Delta\) occur with the spacings as described above, and furthermore that this does not happen too high up in the spectrum. This upper bound is necessary as a consequence of the non-uniform convergence in \(\lambda\) of the series in (65). In appendix B we show that assumption 4.1 is satisfied almost surely if the sequence \((E_j)\) comes from a Poisson process. In this sense, assumption 4.1 is consistent with the Berry-Tabor conjecture, if \(-\Delta\) is the Hamiltonian corresponding to an integrable dynamical system.

We shall also assume a lower bound for the absolute values of the eigenfunctions \(\Phi_j\) at the point \(p\).

Assumption 4.2. There exists a constant \(c_0 > 0\) independent of \(j\) such that
\[
|\Phi_j(p)| \geq c_0.
\]

Remark 4.3. In fact we require only that assumption 4.2 holds for (possibly a subsequence of) the sequence of pairs \(\Phi_b(p)\) and \(\Phi_c(p)\) for eigenfunctions associated with the sequences of energy levels \(E_b\) and \(E_c\) defined in assumption 4.1.

We recall that the spectral sequence is defined in such a way that \(\Phi_j(p) \neq 0\) for all \(j\). Thus assumption 4.2 disqualifies subsequences of eigenfunctions converging to 0 at the point \(p\).

This assumption reflects the fact that if \(|\Phi_j(p)|\) becomes small, two eigenvalues of \(H_\pi\) will approach \(E_j\). Then we would only be able to prove that the quasi-eigenfunction approximates a certain linear combination of these eigenfunctions of \(-\Delta\), rather than an actual eigenfunction. Assumption 4.2 can be relaxed slightly—see remark 4.5 below.

Theorem 4.4. Assume that the spectrum of \(-\Delta\) satisfies assumptions 4.1 and 4.2. Then the sequence of quasimodes \(\psi_{0, I}\) associated to the sequence of intervals \(I = [E_b, E_c]\) and \(\mu \in I\), with \(E_a, \ldots, E_d\) as described in assumption 4.1, after normalisation, converge in \(L^2\) to a subsequence of true eigenfunctions of \(H_\pi\).

Let us fix a point \(n\) of the sequence \((\varepsilon_n)\) with \(\varepsilon_n = \varepsilon\), and \(I\) fixed as described in the statement of theorem 4.4.

Proof of theorem 4.4. In order to use (25) we will employ partial summation, to estimate the position of eigenvalues of \(H_\pi\). If \(g\) is a smooth function, then
\[
\sum_{X \in E_j \subset Y} g(E_j)|\Phi_j(p)|^2 = g(Y)N(Y) - g(X)N(X) - \int_X^Y g'(t)N(t) \, dt,
\]
(70)
where $N(t)$ has been defined in \[13\]. Equation \[70\] may be proved by Riemann-Stieltjes integration. Let $\lambda^*$ be the solution of \[65\] lying between $E_c$ and $E_d$. Let
\[
g(t) := \frac{1}{t - \lambda^*} - \frac{t}{1 + t^2} = \frac{1 + t\lambda^*}{(t - \lambda^*)(1 + t^2)},
\]
and observe that $g(t) > 0$ if $t > \lambda^*$ and $g(t) < 0$ if $t < \lambda^*$. By \[65\] we have
\[
0 = \sum_{j=1}^{\infty} g(E_j)|\Phi_j(p)|^2 \leq g(E_c)|\Phi_c(p)|^2 + \sum_{E_j \geq E_d} g(E_j)|\Phi_j(p)|^2.
\]
Now, by \[70\],
\[
\sum_{E_j \geq E_d} g(E_j)|\Phi_j(p)|^2 = -g(E_d)N(E_d) - \int_{E_d}^{\infty} g'(t)N(t) \, \mathrm{d}t
\]
\[
= \frac{1}{4\pi} \int_{E_d}^{\infty} g(t) \, \mathrm{d}t + O\left(g(E_d)E_d^{1/2} + \int_{E_d}^{\infty} |g'(t)|t^{1/2} \, \mathrm{d}t\right),
\]
using \[13\].

Since
\[
g'(t) = \frac{-1}{(t - \lambda^*)^2} - \frac{1}{1 + t^2} + \frac{2t^2}{(1 + t^2)^2},
\]
we get
\[
\int_{E_d}^{\infty} |g'(t)|t^{1/2} \, \mathrm{d}t \sim \int_{E_d}^{\infty} \frac{t^{1/2}}{(t - \lambda^*)^2} \, \mathrm{d}t
\]
\[
\leq \left(\frac{E_d}{E_d - \lambda^*}\right)^{1/2} \int_{E_d}^{\infty} \frac{1}{(t - \lambda^*)^{3/2}} \, \mathrm{d}t
\]
\[
\leq \left(\frac{E_d}{E_d - \lambda^*}\right)^{1/2} \frac{1}{(E_d - \lambda^*)^{1/2}}
\]
\[
= \frac{E_d^{1/2}}{E_d - \lambda^*}.
\]
We can also calculate
\[
\int_{E_d}^{\infty} g(t) \, \mathrm{d}t = \int_{E_d}^{\infty} \frac{1}{t - \lambda^*} - \frac{t}{1 + t^2} \, \mathrm{d}t = -\ln\left(\frac{E_d - \lambda^*}{\sqrt{1 + E_d^2}}\right).
\]
So we have
\[
\sum_{E_j \geq E_d} g(E_j)|\Phi_j(p)|^2 = \frac{-1}{4\pi} \ln\left(\frac{E_d - \lambda^*}{\sqrt{1 + E_d^2}}\right) + O\left(\frac{E_d^{1/2}}{E_d - \lambda^*}\right),
\]
in which the dominant term on the RHS is actually the error term. We have, from \[72\]
\[
g(E_c)|\Phi_c(p)|^2 \leq \frac{E_d^{1/2}}{E_d - \lambda^*}
\]
\[
\Rightarrow \frac{|\Phi_c(p)|^2}{E_c - \lambda^*} \leq \frac{\varepsilon^{-\rho/2}}{E_d - E_c - (\lambda^* - E_c)}
\]
\[
\leq \frac{\varepsilon^{\rho/2}}{\varepsilon^q - (\lambda^* - E_c)},
\]
implying the lower bound
\[ \lambda^* - E_c \gg \varepsilon^{p/2+q}. \] (80)
To see this, observe that if \( \lambda^* = o(\varepsilon^{q+p/2}) \) then we would have from (79)
\[ \frac{|\Phi_c(p)|^2}{E_c - \lambda^*} \ll \varepsilon^{-p/2-q}, \] (81)
a contradiction.

By the same method, we can establish the same bound for the solution to (65) between \( E_a \) and \( E_b \), and by theorem 3.5 we deduce that there is an interval of size
\[ M \approx \varepsilon^{q+p/2} \]
about \( \mu \) such that \([\mu - M, \mu + M]\) contains only one eigenvalue of \( H_\pi \). Since \( q + p/2 < 1 \), and since the discrepancy of \( \psi_{0,1} \) is \( O(\varepsilon) \), equation (25) allows us to conclude that the normalised quasimode differs from the true eigenfunction associated to \( \mu \) by an amount which converges to 0 as \( \varepsilon \to 0 \).

Remark 4.5. From the proof of theorem 4.4 we see that we can relax assumption 4.2 to demanding only that \( |\Phi_j(p)| \gg \varepsilon^r/2 \) with \( 0 < r < 1 - q - p/2 \). However, in a generic situation this is unlikely to be achieved. In appendix A we show that for a badly-approximable position of the point \( p \) in a rectangle, the best possible bound is
\[ |\Phi_j(p)| \gg E_j, \] (82)
which is not sufficiently slow.

5 Application to rectangular Šeba billiards

In this section we will apply theorem 4.4 to the original Šeba billiard \([12]\). We consider a rectangular billiard \( \Omega = (0, 2a) \times (0, 2b) \subseteq \mathbb{R}^2 \) and point \( p = (a, b) \) at the centre of the billiard. However, we remark that we could position \( p \) at any point with co-ordinates that are rational multiples of the side lengths without significant changes to the forthcoming analysis.

The eigenvalues of \(-\Delta\), the Laplacian with Dirichlet boundary conditions, are given by
\[ E_{n,m} = \frac{\pi^2}{4} \left( \frac{n^2}{a^2} + \frac{m^2}{b^2} \right), \] (83)
where \( n, m \in \mathbb{N} \), and the corresponding eigenfunctions are
\[ \Phi_{n,m}(x, y) = \frac{1}{\sqrt{ab}} \sin \left( \frac{n\pi x}{2a} \right) \sin \left( \frac{m\pi y}{2b} \right). \] (84)
If either \( n \) or \( m \) are even, then the symmetry of the problem forces \( \Phi_{n,m}(p) = 0 \). So for these values of \( n \) and \( m \), \( \Phi_{n,m} \in D_p \), and are automatically eigenfunctions of the extended operator \( H_\pi \). We exclude these eigenvalues from the spectrum, as discussed in section 2.

Instead, we concentrate on the more interesting subsequence where \( n \) and \( m \) are both odd, e.g. \( n = 2s + 1 \) and \( m = 2t + 1 \) with \( s, t = 0, 1, 2, \ldots \). Then we have
\[ \Phi_{s,t}(p) = \frac{1}{\sqrt{ab}}(-1)^{s+t}, \] (85)
so that along this sequence assumption 4.2 is satisfied. The corresponding set of eigenvalues is given by

$$E_{s,t} = \pi^2 \left( \frac{(s + \frac{1}{2})^2}{a^2} + \frac{(t + \frac{1}{2})^2}{b^2} \right), \quad s, t = 0, 1, 2, \ldots$$  (86)

For generic choices of \(a\) and \(b\) it is conjectured that the set of values given by (86) behave statistically like the event times of a Poisson process [17, 34, 35, 36]. Under assumption 4.1 for the set of values (86), theorem 4.4 asserts the existence of a subsequence \((j_n) \subseteq \mathbb{N}\) such that

$$\|\phi_{j_n} - \psi_n\| \to 0 \quad \text{as} \quad n \to \infty,$$  (87)

where \(\phi_{j_n}\) are eigenfunctions of \(H_{\pi}\), and \(\psi_n\) are of the form

$$\psi_n = \frac{1}{\sqrt{2}} \left( \Phi_{j_n} + (-1)^{\beta_n} \Phi_{j_n+1} \right),$$  (88)

where \(\beta_n\) can be 0 or 1 and depends on the relative signs of \(\Phi_{j_n}(p)\) and \(\Phi_{j_n+1}(p)\). So the subsequence \((\phi_{j_n})\) converges to a superposition of two consecutive unperturbed eigenfunctions of \(-\Delta\).

The consequences for this subsequence are most striking when one considers the momentum representation. This is given by the Fourier transform:

$$\hat{\phi}_{j}(p_x, p_y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ip_x x - ip_y y} \phi_{j}(x, y) \, dx \, dy.$$  (89)

For an ergodic system, the quantum ergodicity theorem of Šnirel’man, Zelditch and Colin de Verdière [37, 38, 39] would imply that the momentum representation of almost all eigenfunctions equi-distributes around the circle of radius \(\sqrt{E_j}\) as \(j \to \infty\):

$$|\hat{\phi}_{j}(x)|^2 \to \frac{1}{\pi} \delta(|x|^2 - E_j) \quad \text{as} \quad j \to \infty,$$  (90)

where convergence in (90) is in the weak sense. Šeba billiards are not ergodic, but we see a very different behaviour to (90) for the subsequence \((\phi_{j_n})\).

From Parseval’s theorem, it follows that

$$\hat{\phi}_{j_n} - \hat{\psi}_n \to 0 \quad \text{in} \quad L^2 \text{ norm}.$$  (91)

The momentum representation of the unperturbed eigenfunctions \(\Phi_{n,m}\) is

$$\hat{\Phi}_{n,m}(p_x, p_y) = \frac{2\pi n m \sqrt{ab}}{(4p_x^2 a^2 - n^2 \pi^2)(4p_y^2 b^2 - m^2 \pi^2)} \left( (-1)^n e^{-2ip_x} - 1 \right) \left( (-1)^m e^{-2ip_y} - 1 \right).$$  (92)

Since

$$\frac{n\pi}{4p_x^2 a^2 - n^2 \pi^2} = \frac{1}{2} \left( \frac{1}{2p_x a - n\pi} - \frac{1}{2p_x a + n\pi} \right),$$  (93)

we re-scale and write

$$\hat{\Phi}_{n,m}(np_x, mp_y) = \pi \sqrt{ab} \frac{2nm}{2nm} \left( \delta_n(2p_x a + \pi) - \delta_n(2p_x a - \pi) \right) \left( \delta_m(2p_y b - \pi) - \delta_m(2p_y b + \pi) \right),$$  (94)
where $\delta_n$ is the smoothed-delta function

$$\delta_n(t) := \frac{1 - e^{-int}}{\pi it}. \quad (95)$$

The function $\delta_n(t)$ converges weakly to $\delta(t)$ as $n \to \infty$. Furthermore, it satisfies

$$|\delta_n(t)|^2 \sim \frac{2n}{\pi} \delta(t) \quad \text{as} \quad n \to \infty. \quad (96)$$

Hence

$$nm|\tilde{\Phi}_{n,m}(np_x, mp_y)|^2 \sim ab \left( \delta(2px - \pi) + \delta(2px + \pi) \right) \left( \delta(2py - \pi) + \delta(2py + \pi) \right) \quad (97)$$

as $n, m \to \infty$. The momentum eigenfunction localises around the 4 points

$$(p_x, p_y) = \left( \pm \frac{n\pi}{2a}, \pm \frac{m\pi}{2b} \right), \quad (98)$$

which satisfy $p_x^2 + p_y^2 = E_{n,m}$. Since $\psi_n$ is a superposition of $\Phi_{jn}$ and $\Phi_{jn+1}$, the states in the subsequence $\hat{\Phi}_{jn}$ become localised around 8 points, which all lie on the circle with radius $\sqrt{E_{jn}}$, very much in contrast to the expected equi-distribution for ergodic systems. Numerical simulations illustrating this behaviour have been presented in [26].

This localisation is, in some sense, analogous to the scarring phenomenon which occurs in some chaotic systems. Since these states are not associated with an unstable periodic orbit, they do not fall into the very precise definition of a scar given in [40]. Rather they are localising around ghosts of departed tori of the unperturbed integrable system. Nevertheless they cannot be explained simply by using torus quantisation, and so they provide a further example of the already rich behaviours in systems with intermediate statistics.

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**A Non-constant unperturbed eigenfunctions at the position of the scatterer**

In order to consider what can happen when the value of the unperturbed eigenfunctions at the position of the scatterer can vary, let us consider the rectangular billiard $\Omega$, with sides of length $a$ and $b$, and Dirichlet boundary conditions.
The energy levels are given by

\[ E = E_{n,m} = \pi^2 \left( \frac{n^2}{a^2} + \frac{m^2}{b^2} \right), \]

for \( n, m \geq 1 \) integers.

**Lemma A.1.**

\[ \frac{1}{n^2m^2} \geq \frac{4\pi^4}{a^2b^2 E^2}. \]

**Proof.** We have

\[ 0 \leq \pi^4 \left( \frac{n^2}{a^2} - \frac{m^2}{b^2} \right)^2 = \frac{n^4\pi^4}{a^4} - 2 \frac{n^2m^2\pi^4}{a^2b^2} + \frac{m^4\pi^4}{b^4} = E^2 - 4 \frac{n^2m^2\pi^4}{a^2b^2}, \]

and then re-arrange to get the required estimate. \( \square \)

The eigenfunctions themselves are proportional to

\[ \sin \left( \frac{n\pi x}{a} \right) \sin \left( \frac{m\pi y}{b} \right). \]

Let us choose the point \( p = (x_p, y_p) \in \Omega \) so that \( x_p/a \) and \( y_p/b \) are badly-approximable, in the sense that

\[ \left| n \frac{x_p}{a} - r \right| \geq \frac{C}{n} \quad \forall n, r \in \mathbb{Z}, \]

(this is the best we can hope to do if we want to bound the eigenfunctions away from 0). Then

\[ n \frac{x_p}{a} = r + \vartheta(n) \]

where \( \vartheta \) can depend on \( x_p \) and \( r \) and satisfies

\[ |\vartheta(n)| \gg \frac{1}{n} \]

uniformly. Furthermore this bound is achieved if \( r/n \) is a continued fraction approximant to \( x_p/a \). We get

\[ \sin^2 \left( \frac{n\pi x_p}{a} \right) \gg \frac{1}{n^2}. \]

With a similar bound for the contribution of the \( y \)-coordinate, we find that the best bound we can obtain is

\[ |\Phi_{n,m}(p)|^2 \gg \frac{1}{n^2m^2} \gg \frac{1}{E^2} \]

and this bound is sharp.

### B Assumption [4.1] for the event times of a Poisson process

The purpose of this appendix is to prove the following result. Let \( 0 < q < 1/2 \) and \( 1 < \rho < 2(1 - q) \) be fixed throughout.
Proposition B.1. Let \( P = (E_j)_{j=1}^{\infty} \) be the sequence of event times for a Poisson process with parameter 1. There is, almost surely, a sequence \((\varepsilon_n)_{n=1}^{\infty}, \varepsilon_n \downarrow 0\) such that for each \( n \) there are four consecutive members of \( P, E_a < E_b < E_c < E_d < \varepsilon_n - \rho \), satisfying
\[
E_c - E_b < \varepsilon_n
E_d - E_c > \varepsilon_q
E_b - E_d > \varepsilon_n.
\]

Thus, assumption \([1]\) is almost surely satisfied for a Poisson process.

As a model for a Poisson process, we will let \( \xi_1, \xi_2, \ldots \) be a sequence of independent exponentially distributed random variables with parameter 1. Then, defining
\[
E_1 = \xi_1
E_2 = \xi_1 + \xi_2
\]
\[
E_j = \sum_{i=1}^{j-1} \xi_i
\]
The sequence \( P = (E_j)_{j=1}^{\infty} \) so-formed is a Poisson process.

Proposition B.2. Let \( \varepsilon > 0 \). The probability that there are four consecutive members of \( P, E_a < E_b < E_c < E_d < \varepsilon - \rho \) satisfying
\[
E_c - E_b < \varepsilon
E_d - E_c > \varepsilon_q
E_b - E_d > \varepsilon_n,
\]
is \( 1 - O(\varepsilon^{1+q}) \).

The notation \( O(\varepsilon^{\infty}) \) refers to a quantity which goes to zero faster than any power of \( \varepsilon \). One can say that the event described in proposition B.2 occurs with overwhelming probability.

Let us fix \( 1 < \rho' < \rho \), and chose \( N = 3M \sim \varepsilon^{-\rho'} \) where \( M \in \mathbb{N} \). Let us define the events \( S_j, j = 0, \ldots, M - 1 \), by
\[
S_j = \{ \xi_{3j+1} > \varepsilon, \xi_{3j+2} < \varepsilon, \xi_{3j+3} > \varepsilon^q \}.
\]

Lemma B.3. The events \( S_j, j = 0, \ldots, M - 1 \) are independent, and the probability that at least one of them occurs is \( 1 - O(\varepsilon^{\infty}) \).

Proof. The independence of the events \( S_j \) clearly follows because they are defined on independent random variables. We first calculate the probability of one of them. By independence of \( \xi_1, \xi_2, \xi_3 \),
\[
P(S_0) = P(\xi_2 < \varepsilon)P(\xi_1 > \varepsilon^q)P(\xi_3 > \varepsilon^q)
= \left( \int_0^{\varepsilon} e^{-x} \, dx \right) \left( \int_{\varepsilon^q}^{\infty} e^{-x} \, dx \right)^2
= (1 - e^{-\varepsilon})(e^{-\varepsilon^q})^2
= \varepsilon + O(\varepsilon^{1+q}).
\]
Then, by independence of the $S_j$s,

$$p_1 := \mathbb{P} \text{(at least one } S_j \text{ occurs)} = 1 - (1 - \mathbb{P}(S_0))^M = 1 - (1 - \varepsilon + O(\varepsilon^{1+\eta}))^M. \quad (113)$$

So, we have

$$\log(1 - p_1) = -M\varepsilon + O(M\varepsilon^{1+\eta}) \sim -\frac{1}{3}\varepsilon^{1-\rho'}. \quad (114)$$

For $\varepsilon$ sufficiently small, this yields

$$1 - p_1 \leq \exp \left( -\frac{1}{6}\varepsilon^{-(\rho'-1)} \right) = O(\varepsilon^\infty). \quad (115)$$

The probability that the upper bound of $\varepsilon^{-\rho}$ is met is given in the following lemma

**Lemma B.4.** The probability that $E_N < \varepsilon^{-\rho}$ is $1 - O(\varepsilon^\infty)$.

**Proof.** Let $\alpha > 0$. The probability density for $E_N$ is $\Gamma(N)^{-1} x^{N-1} e^{-x}$. So

$$p_2 := \mathbb{P}(E_N < N^{1+\alpha}) = 1 - \frac{1}{\Gamma(N)} \int_{N^{1+\alpha}}^\infty x^{N-1} e^{-x} \, dx$$

$$= 1 - \frac{\exp(-N^{1+\alpha})}{\Gamma(N)} \int_0^\infty (x + N^{1+\alpha})^{N-1} e^{-x} \, dx$$

$$= 1 - \exp(-N^{1+\alpha})N^{(1+\alpha)(N-1)} \sum_{j=0}^{N-1} \frac{1}{\Gamma(N-j)N^{(1+\alpha)j}}, \quad (116)$$

expanding the binomial. Using $\frac{1}{\Gamma(N-j)N^{(1+\alpha)j}} \leq \frac{N^j}{\Gamma(N)}$ we can estimate

$$\sum_{j=0}^{N-1} \frac{1}{\Gamma(N-j)N^{(1+\alpha)j}} \leq \frac{1}{\Gamma(N)} \sum_{j=0}^{N-1} \frac{1}{N^{\alpha j}} \ll \frac{1}{\Gamma(N)}, \quad (117)$$

where the implied constant could depend on $\alpha$. This leads to

$$1 - p_2 \ll \frac{\exp(-N^{1+\alpha})N^{(1+\alpha)(N-1)}}{\Gamma(N)} \sim \frac{\exp(-N^{1+\alpha} + N)\alpha(N-1)}{\sqrt{2\pi(N-1)}}$$

$$\ll \frac{1}{N^{\infty}}, \quad (118)$$

where Stirling’s formula has been used. This last line is $O(\varepsilon^\infty)$ since $N^{-1} \sim \varepsilon^{\rho'}$. Finally setting

$$\alpha = \frac{\rho}{\rho'} - 1 > 0 \quad (119)$$

gives the required estimate. □
Proof of proposition B.2. We are interested in the events corresponding to lemmata B.3 and B.4 happening simultaneously. By the inclusion-exclusion principle, the probability that this happens is at least $p_1 + p_2 - 1 = 1 - O(\varepsilon^\infty)$.

Proof of proposition B.1. Let $E_n, n \in \mathbb{N}$ be the event that there are found four consecutive members of $P$, $E_a < E_b < E_c < E_d < n^q$ satisfying

$$
E_c - E_b < \frac{1}{n} \\
E_d - E_c > \frac{1}{n^q} \\
E_b - E_a > \frac{1}{n^q}.
$$

(120)

By proposition B.2, $P(E_n^c) \ll \frac{1}{n^2}$. Hence, by the Borel-Cantelli lemma, the probability that infinitely many $E_n^c$ occur is zero. Equivalently, only finitely many $E_n^c$ occur, almost surely. So, almost surely, there is an infinite subsequence of $n \in \mathbb{N}$ such that $E_n$ occurs.

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