Harnack Inequalities of Hitting Distributions of Projections of Planar Symmetric Random Walks on the Lattice Torus

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Abstract

We give Harnack inequalities for the hitting distributions of a large family of symmetric random walks on $\mathbb{Z}^2$, and their projections onto the lattice torus $\mathbb{Z}^2_K$. This extends a framework for the simple random walk in [6], and generalizes the results in [1] to the toral projection.

1 Introduction

Consider a random walk $S_t = S_0 + \sum_{j=1}^t X_j$, on $\mathbb{Z}^2$, with $X = \{X_j\}_{j \in \mathbb{N}}$ having the following properties: $X_1$ is symmetric, has finite covariance matrix equal to a scalar times the identity, i.e., $\Gamma := \text{cov}(X_1) = cI$, $c > 0$, and $X$ is strongly aperiodic.

As usual in the literature, let

$$p_1(x, y) = p_1(y - x) = P^x(X_1 = y)$$

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be the one-step transition probability of $X$. We say $X$ satisfies **Condition A** if either $p_1$ has bounded support, or, from any point “just outside” a disc, we will enter the disc with positive probability; *i.e.*, for any $s \leq n$, for large enough $n$,

$$\inf_{y: n \leq |y| < n + s} \sum_{z \in D(x, n)} p_1(y, z) = \inf_{y \in \partial D(x, n)} P^y(X_1 \in D(x, n)) \geq ce^{-\beta n^{1/4}}, \quad (1.1)$$

where the (Euclidean) $s$-annulus around the disc $D(x, n)$ is defined as

$$\partial D(x, n)_s := D(x, n + s) \setminus D(x, n).$$

In particular, if $X_1$ has infinite range, then for any $y \in \partial D(0, n)_s$, there exists $x \in D(0, n)$ such that $p_1(y, x) > 0$.

Starting at a point $x \in A^c$, we define the hitting distribution of $A$ to be

$$H_A(x, y) := P^x(S_{T_A} = y),$$

where $T_A$ is the first hitting time of the set $A$ by the random walk:

$$T_A = \inf\{t \geq 0 : S_t \in A\}.$$

The last exit decomposition of a hitting distribution is based on the Green’s function: for $A$ a proper subset of $\mathbb{Z}^2$, $x \in A^c$, and $y \in A$,

$$H_A(x, y) = \sum_{z \in A^c} G_{A^c}(x, z)p_1(z, y), \quad (1.2)$$

where the (truncated) Green’s function, up to escaping a set $B$, is defined, for $x, y \in B$, as the total expected number of visits to $y$, starting from $x$, before escaping $B$:

$$G_B(x, y) := \mathbb{E}^x\left[\sum_{j=0}^{\infty} 1\{S_j = y; j < T_{B^c}\}\right] = \sum_{j=0}^{\infty} P^x(S_j = y; j < T_{B^c}) \quad (1.3)$$

and 0 if $x$ or $y \not\in B$. A useful identity relates the Green’s function of a set to its expected escape time: if $x \in B$,

$$\mathbb{E}^x(T_{B^c}) = \sum_{z \in B} G_B(x, z). \quad (1.4)$$

The goal of this paper is to establish Harnack inequalities of hitting distributions of discs and disc complements on the planar and toral lattices. **Condition A** is sufficient to allow the error terms in our Harnack inequalities induced by the toral projection to drop out of sight, as long as there is a moment condition in place on the size of our jumps: we assume

$$\mathbb{E}|X_1|^M = \sum_{x \in \mathbb{Z}^2} |x|^M p_1(x) < \infty \quad (1.5)$$

for some $M > 4$ (and write $M = 4 + 2\beta$ for some $\beta > 0$). While $M > 2$ suffices for interior
Harnack inequality results, $M = 3 + 2\beta$ is needed for results on the plane, and one more moment is used in our arguments in the transition to the torus.

We will switch between the planar and toral representations of the random walk and corresponding stopping times, hitting distributions, etc. Define the projections, for $x = (x_1, x_2) \in \mathbb{Z}^2$, by
\[
\pi_K : \mathbb{Z}^2 \to [-K/2, K/2)^2 \cap \mathbb{Z}^2, \\
\pi_K(x) = ((x_1 + \lfloor \frac{K}{2} \rfloor) \pmod{K}) - \lfloor \frac{K}{2} \rfloor, (x_2 + \lfloor \frac{K}{2} \rfloor) \pmod{K} - \lfloor \frac{K}{2} \rfloor) ;
\]

\[
\hat{\pi}_K : \mathbb{Z}^2 \to \mathbb{Z}_K^2, \quad \hat{\pi}_K(x) = (\pi_K(x) + (K\mathbb{Z}))^2.
\]

(For example, if $x = (-12, 6)$ and $K = 11$, then $\pi_{11}(\mathbb{Z}^2) = \{-5, \ldots, 5\}^2$, $\pi_{11}(x) = (-1, -5)$, and $\hat{\pi}_{11}(x) = (-1, -5) + (11\mathbb{Z})^2$.)

We call the set of lattice points $\pi_K(\mathbb{Z}^2) = [-K/2, K/2)^2 \cap \mathbb{Z}^2$ the primary copy in $\mathbb{Z}^2$, and for $x \in \pi_K(\mathbb{Z}^2)$, $\hat{x} := \hat{\pi}_Kx$ is its corresponding element in $\mathbb{Z}_K^2$. Any $z \in \pi_K^{-1}x$, $z \neq \pi_Kx$, is called a copy of $x$. Likewise, for a set $A \subset \mathbb{Z}^2$, $\hat{A} := \hat{\pi}_KA$ is the toral projection of $A$, and the set of all copies of $A$ is
\[
\pi_K^{-1}\pi_KA = \hat{\pi}_K^{-1}\hat{A} := \{z \in \mathbb{Z}^2 : z = x + (iK, jK), \ i, j \in \mathbb{Z}, x \in A\}.
\]

For a given $\hat{x} \in \mathbb{Z}_K^2$, we define $x$ to be the (planar) primary copy of that element; $x := \pi_K\hat{\pi}_K^{-1}\hat{x}$.

While $X_j$ is the $j$th step of the planar walk and $S_j$ its position at time $j$, we use $\hat{S}_j := \hat{\pi}_KS_j$ to denote the position of the toral walk at time $j$. The distance between two points $x, y \in \mathbb{Z}^2$ will be the Euclidean distance $|x - y|$; on the torus, the distance between two points $\hat{x}, \hat{y} \in \mathbb{Z}_K^2$ will be the minimum Euclidean distance $|\hat{x} - \hat{y}| \leq K\sqrt{2}/2$. To limit the issues regarding this distance, we will restrict any discs on $\mathbb{Z}_K^2$ to have radius $n < K/4$ (sometimes written as a diameter constraint: $2n < K/2$).

To bound our functions, we need a precise notion of bounding distance on the lattice torus $\mathbb{Z}_K^2$. As in [6], a function $f(x)$ is said to be $O(x)$ if $f(x)/x$ is bounded, uniformly in all implicit geometry-related quantities (such as $K$). That is, $f(x) = O(x)$ if there exists a universal constant $C$ (not depending on $K$) such that $|f(x)| \leq Cx$. Thus $x = O(x)$ but $Kx$ is not $O(x)$. A similar convention applies to $o(x)$.

Next, we will define a few terms describing the distance of a random walk step, relative to a reference disc of radius $n$ and an s-annulus around the disc. A small jump refers to a step that is short enough to possibly (but not necessarily) stay inside a disc of radius $n$ (i.e., $|X_1| < 2n$). A baby jump refers to a small jump that is too short to hop over an s-annulus from inside a disc (i.e., $|X_1| < s$). A medium jump refers to a step that is sufficiently large to hop out of a disc and past an s-annulus, but with magnitude strictly less than $K$, and cannot land near a toral copy of its launching point (i.e., $s < |X_1| < K - 2n$). A large jump is a step which, in the toral setting, would be considered “wrapping around” in one step (i.e., $|X_1| > K - 2n$). A targeted jump is a large jump which lands directly in a copy of the disc or annulus just launched from (i.e., $j(K - 2n) \leq |X_1| \leq j(K + 2n)/\sqrt{2}$.
for some \( j \). These terms will aid in dealing with differences between planar and toral hitting and escape times.

## 2 Random Walk Preliminaries

In this section we give, without proof, results from [3] which are used in our Harnack inequality proofs.

First, for \( x, y \in \mathbb{Z}^2 \) such that \( |x| \ll |y| \), we have, by a Taylor expansion around \( y \),

\[
\log |y - x| = \log |y| + O \left( \frac{|x|}{|y|} \right).
\] (2.1)

In particular, if \( x \in D(0, 2r) \) and \( y \in D(0, R/2)^c \), with \( R = 4mr \), we have

\[
\log |y - x| = \log |y| + O \left( \frac{m}{m^2 - 1} \right).
\] (2.2)

Note that (2.1) and (2.2) hold in the toral case without adjustment.

### 2.1 Expected Hitting Times

Here we state some results about the expected escape and entry times of discs on the plane and torus. Our first generalizes a common disc escape argument, and improves on [10, Prop. 6.2.6].

**Lemma 2.1.** [3, Lemma 2.1] Let \( S_t = S_0 + \sum_{j=1}^{t} X_j \) be a random walk in \( \mathbb{Z}^2 \) with \( E|X_1|^2 < \infty \), and covariance matrix \( \Gamma \) such that \( tr(\Gamma) = \gamma^2 > 0 \). Then, uniformly for \( x \in D(0, n) \), and for sufficiently large \( n \),

\[
\frac{n^2 - |x|^2}{\gamma^2} \leq E^x(T_{D(0, n)^c}) \leq \frac{n^2 - |x|^2}{\gamma^2} + 2n + 1.
\] (2.3)

Computational bounds on the \( E^x(T_{s_K(D(0, n)^c)}) \), the expected toral disc escape, have a slight torally-induced error term [3 (2.29)]:

\[
\frac{n^2 - |x|^2}{\gamma^2} \leq E^x(T_{s_K(D(0, n)^c)}) \leq \frac{n^2 - |x|^2}{\gamma^2} + 2n + 1 + O(K^{-M}n^4).
\] (2.4)

### 2.2 Probability Estimates

By Markov’s inequality, large jumps are rare: if \( C_M = E(|X_1|^M) < \infty \), then since \( 2n < K/2 \),

\[
P(|X_1| > K - 2n) \leq \frac{C_M}{(K - 2n)^M} < \frac{2^MC_M}{K^M} = O(K^{-M}).
\] (2.5)
This leads to the targeted jump probability estimate, *i.e.*, the rare chance that a large jump lands back into the toral disc the walk escaped from in the planar setting \( \text{[3]} \ (2.24) \):}

\[
P^x(T_{\hat{x}_K}(\partial D(0,n)_{\delta}) > T_{D(0,n)^c}) = \sum_{z \in (\hat{x}_K^{-1}\hat{x}_K(\partial D(0,n))) \setminus D(0,n)} \sum_{y \in D(0,n)} G_{D(0,n)}(x,y)p_1(y,z) \\
\leq cK^{-M} \sum_{y \in D(0,n)} G_{D(0,n)}(x,y) = O(K^{-M}n^2). \tag{2.6}
\]

\( \text{[3]} \ (2.50) \) gives a toral gambler’s ruin probability estimate for a radius-ruin, *i.e.*, hitting the center of a disc before escaping it, in \( \mathbb{Z}_K^2 \):

\[
P^x(T_0 < T_{\hat{x}_K(D(0,n)_{\delta})}) = \frac{\log(n/|\hat{x}|) + O(|\hat{x}|^{-1/4})}{\log(n)} \left( 1 + O((\log n)^{-1}) \right) + O(K^{-M}n^2) \\
\leq \frac{\log(n/|\hat{x}|) + O(|\hat{x}|^{-1/4})}{\log(n)} \left( 1 + O((\log n)^{-1}) \right). \tag{2.7}
\]

Arguments similar to the proof of \( \text{(2.7)} \) also result in probabilities of “near ruin”, *i.e.*, entering a smaller disc (rather than hitting the center), and their “success” counterparts, in both \( \mathbb{Z}^2 \) and \( \mathbb{Z}_K^2 \): from \( \text{[3]} \ (2.51)-(2.54) \), uniformly for \( r < |x| < R \),

\[
P^x(T_{D(0,r)} > T_{D(0,R)^c}) = \frac{\log(|x|/r) + O(r^{-1/4})}{\log(R/r)} \tag{2.8}
\]

\[
P^x(T_{D(0,r)} < T_{D(0,R)^c}) = \frac{\log(R/|x|) + O(r^{-1/4})}{\log(R/r)} \tag{2.9}
\]

\[
P^x(T_{\hat{x}_K(D(0,r))} > T_{\hat{x}_K(D(0,R)_{\delta})}) = \frac{\log(|\hat{x}|/r) + O(r^{-1/4})}{\log(R/r)} + O(K^{-M}R^2) \\
= \frac{\log(|\hat{x}|/r) + O(r^{-1/4})}{\log(R/r)}. \tag{2.10}
\]

\[
P^x(T_{\hat{x}_K(D(0,r))} < T_{\hat{x}_K(D(0,R)_{\delta})}) = \frac{\log(R/|\hat{x}|) + O(r^{-1/4})}{\log(R/r)} + O(K^{-M}R^2) \\
= \frac{\log(R/|\hat{x}|) + O(r^{-1/4})}{\log(R/r)}. \tag{2.11}
\]

Next, we give bounds from \( \text{[3]} \) on the probabilities of exiting a disc when starting far inside, or entering a disc from far outside it, by jumping over an annulus.

**Lemma 2.2.** \( \text{[3] Lemmas 4.1, 4.2} \) For \( n \) sufficiently large,

\[
\sup_{\hat{x} \in \hat{x}_K(\partial D(0,n/2))} P^x(T_{\hat{x}_K(\partial D(0,n)_{\delta})} > T_{\hat{x}_K(D(0,n+s)_{\delta})}) \leq c(s^{-M+2} \lor n^{-M+2}) \tag{2.12}
\]

\[
\sup_{\hat{x} \in \hat{x}_K(D(0,n)_{\delta})} P^x(T_{\hat{x}_K(\partial D(0,n)_{\delta})} < T_{\hat{x}_K(D(0,n+s)_{\delta})}) \leq cn^2 \log(n)^2(s^{-M} + n^{-M}). \tag{2.13}
\]
2.3 Green’s Functions

To calculate Green’s functions in $\mathbb{Z}^2$, we require the potential kernel of $X$: for our walks in $\mathbb{Z}^2$, this function is defined by, for $x \in \mathbb{Z}^2$,

$$a(x) := \lim_{n \to \infty} \sum_{j=0}^{n} [p_j(0) - p_j(x)]. \quad (2.14)$$

For our class of random walks, the potential kernel can be shown to be

$$a(x) = \frac{2}{\pi \Gamma} \log |x| + C(p_1) + o(|x|^{-1}) \quad (2.15)$$

where $C(p_1)$ is a constant depending on $p_1$ but not $x$, and $\pi \Gamma = 2\pi \sqrt{\det \Gamma}$.

(2.15)-(2.16)] give computational results for the internal Green’s function at zero on $\mathbb{Z}^2$ and $\mathbb{Z}^2_K$, before escaping a disc:

$$G_{D(0,n)}(0,0) = \frac{2}{\pi \Gamma} \log n + C' + O(n^{-1/4}) \quad (2.16)$$

$$\hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{0}, \hat{0}) = G_{D(0,n)}(0,0)(1 + O(K^{-M}n^2))$$

$$= \left( \frac{2}{\pi \Gamma} \log n + C' + O(n^{-1/4}) \right) (1 + O(K^{-M}n^2))$$

$$= \frac{2}{\pi \Gamma} \log n + C' + O(n^{-1/4}). \quad (2.17)$$

(2.55)-(2.58)] give calculations and bounds for $G_{D(0,n)}(x,0), \hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{x}, \hat{0}), G_{D(0,n)}(x,z)$, and $\hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{x}, \hat{z})$: for $x \in D(0,n)$ and $\hat{x} \in \hat{\pi}_K(D(0,n))$: for some $C = C(p_1) < \infty$,

$$G_{D(0,n)}(x,0) = P^x(T_0 < T_{D(0,n)}) G_{D(0,n)}(0,0)$$

$$= \frac{2}{\pi \Gamma} \log \left( \frac{n}{|x|} \right) + C + O(|x|^{-1/4}), \quad (2.18)$$

$$\hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{x}, \hat{0}) = \frac{2}{\pi \Gamma} \log \left( \frac{n}{|\hat{x}|} \right) + C + O(|\hat{x}|^{-1/4}) \quad (2.19)$$

$$G_{D(0,n)}(x,z) \leq G_{D(x,2n)}(0, z - x) \leq c \log n. \quad (2.20)$$

$$\hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{x}, \hat{z}) = G_{D(0,n)}(x,z) + O(K^{-M}n^2 \log n) \leq c \log n. \quad (2.21)$$

Lemma 2.8] gives that, for any $0 < \delta < \varepsilon < 1$, we can find $0 < c_1 < c_2 < \infty$, such that for all $\hat{x} \in \hat{\pi}_K(D(0,n)) \setminus \hat{\pi}_K(D(0,\varepsilon n)), \hat{y} \in \hat{\pi}_K(D(0,\delta n))$ and all $n$ sufficiently large such that $2n < K/2$,

$$c_1 \frac{\rho(\hat{x}) \vee 1}{n} \leq \hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{y}, \hat{x}) \leq c_2 \frac{\rho(\hat{x}) \vee 1}{n}. \quad (2.22)$$
Finally, for any $x, y \in \pi_K(D(0, n)^c_0)$ such that $|x| \leq |y|$, there are computational bounds on the external Green’s function before entering a disc:

$$G_{D(0, n)^c}(x, y) \leq c_j \log |x|, \quad (2.23)$$

$$\hat{G}_{\hat{\pi}_K(D(0, n)^c)}(\hat{x}, \hat{y}) \leq \hat{c}_j \log |\hat{x}|, \quad (2.24)$$

where $c_j, \hat{c}_j$ depend on $j > 2$, $c_j \geq \hat{c}_j$, and in the toral case, such that $|\hat{x}| < (\frac{K}{2})^{1/j}$ (there is no such restriction on the planar case).

### 3 Interior Harnack inequalities

We call our first Harnack inequality “interior”: the starting points are from the interior of a disc, and we examine the probabilities of escaping from a far larger disc around it. We find the planar version first, then move it to the torus.

**Lemma 3.1.** Uniformly for $1 \leq m \ll r$, with $s \ll \frac{r}{4m}$, $x, x' \in D(0, 2r)$, $R = 4mr$, and $y \in D(0, R)^c$,

$$H_{D(0, R)^c}(x, y) = (1 + O(m^{-1}))H_{D(0, R)^c}(x', y) + O(R^{-M} \log R), \quad (3.1)$$

where the error term is completely absorbed, i.e.,

$$H_{D(0, R)^c}(x, y) = (1 + O(m^{-1}))H_{D(0, R)^c}(x', y), \quad (3.2)$$

if $s \leq (\log R)^4$ and $y \in \partial D(0, R)^s$.

Furthermore, if $x \in \partial D(0, r)$, and $y \in D(0, R)^c$,

$$P^x\left(S_{T_{D(0, R)^c}} = y, T_{D(0, R)^c} < T_{D(0, \frac{r}{4m} + s)}\right) \quad (3.3)$$

$$= (1 + O(m^{-1}))P^x\left(T_{D(0, R)^c} < T_{D(0, \frac{r}{4m} + s)}\right)H_{D(0, R)^c}(x, y) + O(R^{-M} \log R),$$

with a similar loss of the error term if $y \in \partial D(0, R)^s$.

**Proof** (In this proof, we switch freely between $R$ and $4mr$.) First, we decompose $D(0, R)$ and examine $H_{D(0, R)^c}(x, y)$:

$$H_{D(0, R)^c}(x, y) = \left(\sum_{z \in D(0, 2mr)} + \sum_{z \in D(0, 3mr) \setminus D(0, 2mr)} + \sum_{z \in D(0, 4mr) \setminus D(0, 3mr)}\right) G_{D(0, R)}(x, z)p_1(z, y). \quad (3.4)$$

If $X$ is finite range, then for $r$ sufficiently large, the first two sums of (3.4) are zero. Otherwise, we bound the Green’s function via (2.18) and (2.20), and by Markov’s inequality,
\[ \sum_{z \in D(0,2mr)} p_1(z,y) \leq c(mr)^{-M} \leq cR^{-M}. \] Together, these yield, for some \( c < \infty \),

\[ G_{D(0,R)}(x,z) \leq G_{D(0,2R)}(0,z) \leq c \log R \]

\[ \implies \sum_{z \in D(0,2mr)} G_{D(0,R)}(x,z)p_1(z,y) \leq cR^{-M} \log R. \tag{3.5} \]

By \eqref{2.15} and \eqref{2.1}, uniformly in \( x \in D(0,2r) \) and \( y \in D(0,2mr)^c \),

\[ a(y-x) = \frac{2}{\pi} \log |y-x| + C' + O(|y-x|^{-1}) \]

\[ = \frac{2}{\pi} \log |y| + C' + O(m^{-1}) = a(y) + O(m^{-1}). \tag{3.6} \]

For \( z \in D(0,4mr) \setminus D(0,2mr) \), by the symmetry of the Green’s function, the fact that \( H \) is a probability, \[10\ (4.28)\], and \( \eqref{3.6} \), we have

\[ G_{D(0,R)}(x,z) = G_{D(0,R)}(z,x) \]

\[ = \left( \sum_{w \in D(0,R)^c} H_{D(0,R)^c}(z,w)a(w-x) \right) - a(z-x) \tag{3.7} \]

\[ = \left( \sum_{w \in D(0,R)^c} H_{D(0,R)^c}(z,w)a(w) \right) - a(z) + O(m^{-1}) \]

\[ = G_{D(0,4mr)}(z,0) + O(m^{-1}). \]

By \eqref{2.18}, \( G_{D(0,R)}(z,0) \geq c > 0 \) uniformly for \( z \in D(0,3mr) \setminus D(0,2mr) \), yielding

\[ G_{D(0,R)}(x,z) = G_{D(0,R)}(0,z)(1 + O(m^{-1})). \tag{3.8} \]

For \( z \in D(0,4mr) \setminus D(0,3mr) \), by the strong Markov property at \( T_{D(0,3mr)} \),

\[ G_{D(0,R)}(z,x) = \mathbb{E}^z(G_{D(0,R)}(S_{T_{D(0,3mr)}}(z); T_{D(0,3mr)} < T_{D(0,4mr)})) \tag{3.9} \]

\[ = \mathbb{E}^z(G_{D(0,R)}(S_{T_{D(0,3mr)}}, x); T_{D(0,3mr)} < T_{D(0,4mr)}; |X_{T_{D(0,3mr)}}| > 2mr) \]

\[ + \mathbb{E}^z(G_{D(0,R)}(S_{T_{D(0,3mr)}}, x); T_{D(0,3mr)} < T_{D(0,4mr)}; |X_{T_{D(0,3mr)}}| \leq 2mr). \]

By \eqref{2.20} and \eqref{2.13}, the last term here is bounded, for sufficiently large \( r \), by

\[ c(\log R)P^z(|X_{T_{D(0,3mr)}}| \leq 2mr) \leq c(\log R)P^z(T_{D(0,2mr)} < T_{\partial D(0,2mr)}). \]

\[ \leq c(\log R)(2mr)^2 \log(2mr)^2[(mr)^{-M} + (2mr)^{-M}] \]

\[ \leq c(\log R)^3 R^{M+2} \leq cR^{-M+2+\beta}. \]

Applying \( \eqref{3.8} \) to the first term, then switching it back to its original form, yields, for
$$z \in D(0, 4mr) \setminus D(0, 3mr),$$

$$G_{D(0,R)}(z, x) = (1 + O(m^{-1})) G_{D(0,R)}(z, 0) + O(R^{-M + 2 + \beta}). \quad (3.10)$$

The planar version of (2.22) gives us

$$G_{D(0,R)}(z, x) \geq \frac{c}{mr} \text{ for } z \in D(0, 4mr) \setminus D(0, 3mr).$$

This reduces (3.10) to

$$G_{D(0,R)}(z, x) = (1 + O(m^{-1})) G_{D(0,R)}(z, 0). \quad (3.11)$$

Combining (3.4), (3.5), (3.8), and (3.11) yields (3.1).

For (3.2), let $$y \in \partial D(0, R)_e$$. The only thing we need to do here is show that the error terms are absorbed, i.e., for some $$c > 0$$, with $$M = 4 + 2\beta$$,

$$m^{-1}H_{D(0,R)_e}(x, y) \geq cR^{-M} \log R. \quad (3.12)$$

Wlog, we can show this for $$x = 0$$. First note that, for $$|z| \leq \frac{R}{100}$$, by (2.18) we have

$$G_{D(0,R)}(z, 0) \geq c \log \frac{R}{(R/100)} = c \log 100 \geq c \geq \frac{c}{R}$$

for some $$c > 0$$, and for $$z \in D(0, R) \setminus D(0, R/100)$$, by the planar version of (2.22),

$$G_{D(0,R)}(z, 0) \geq \frac{c}{mR} \text{ as well. Hence, by this, a last exit decomposition, and (1.1),}$$

$$m^{-1}H_{D(0,R)_e}(0, y) = m^{-1} \sum_{z \in D(0,R)} G_{D(0,R)}(0, z)p_1(z, y) \geq \frac{c}{mR} \sum_{z \in D(0,R)} p_1(z, y) \quad (3.13)$$

$$\geq \frac{c}{mR} e^{-\beta s^{1/4}} = c(mR)^{-1} e^{-\beta \log R} = cm^{-1}R^{-1-\beta} > cR^{-M} \log R.$$

To show (3.3), we start with the decomposition

$$P^x \left( S_{T_{D(0,R)_e} = y, T_{D(0,R)_e} < T_{D(0, \frac{r}{4m} + s)} } \right) \quad (3.14)$$

$$= H_{D(0,R)_e}(x, y) - P^x \left( S_{T_{D(0,R)_e} = y, T_{D(0,R)_e} > T_{D(0, \frac{r}{4m} + s)} } \right).$$

By the strong Markov property at $$T_{D(0, \frac{r}{4m} + s)}$$,

$$P^x \left( S_{T_{D(0,R)_e} = y, T_{D(0,R)_e} > T_{D(0, \frac{r}{4m} + s)} } \right)$$

$$= \mathbb{E}^x \left[ H_{D(0,R)_e} \left( S_{T_{D(0, \frac{r}{4m} + s)}, T_{D(0,R)_e} > T_{D(0, \frac{r}{4m} + s)} } \right) \right]. \quad (3.15)$$

By (3.1), uniformly in $$w \in D(0, 2r),$$

$$H_{D(0,R)_e} \left( S_{T_{D(0, \frac{r}{4m} + s)}, y} \right) = (1 + O(m^{-1})) H_{D(0,R)_e}(w, y) + O(R^{-M} \log R).$$

By (2.8) and (2.9), with $$m \gg 1$$, uniformly for $$x \in \partial D(0, r)$$, (say $$|x| = cr, 1 < c < 2$$),
\[ \exists c', c'' > 0 \text{ such that} \]
\[ P^x \left( T_{D(0,R)^c} < T_{D(0, \frac{r}{4m} + s)} \right) = \frac{\log(c'm)}{\log(c''m^2)} = \frac{1}{2} + o \left( \left( \frac{r}{m} \right)^{-1/4} \right), \]
\[ P^x \left( T_{D(0,R)^c} > T_{D(0, \frac{r}{4m} + s)} \right) = \frac{1}{2} + o \left( \left( \frac{r}{m} \right)^{-1/4} \right), \] (3.16)

so the probabilities are both bounded below by a constant. (The small \( m \) case operates similarly, but due to the small constants involved, the lower bound must be reduced; \( \frac{1}{4} \) for one of them suffices.) Combining these and (3.15) into (3.14) yields

\[ P^x \left( S_{T_{D(0,R)^c}} = y, T_{D(0,R)^c} < T_{D(0, \frac{r}{4m} + s)} \right) \]
\[ = H_{D(0,R)^c}(x, y) - \mathbb{E}^x \left[ H_{D(0,R)^c} \left( S_{T_{D(0, \frac{r}{4m} + s)}}, y \right); T_{D(0,R)^c} > T_{D(0, \frac{r}{4m} + s)} \right] \]
\[ = H_{D(0,R)^c}(x, y) \left[ P^x \left( T_{D(0,R)^c} < T_{D(0, \frac{r}{4m} + s)} \right) + P^x \left( T_{D(0,R)^c} > T_{D(0, \frac{r}{4m} + s)} \right) \right] \]
\[- (1 + O(m^{-1})) P^x \left( T_{D(0,R)^c} > T_{D(0, \frac{r}{4m} + s)} \right) + O(R^{-M} \log R) \]
\[ = H_{D(0,R)^c}(x, y) \left( 1 + O(m^{-1}) \right) P^x \left( T_{D(0,R)^c} < T_{D(0, \frac{r}{4m} + s)} \right) + O(R^{-M} \log R). \]

We now move these results to the torus.

**Proposition 3.2.** For large \( r \) and \( 1 \leq m \ll r \) such that \( R = 4mr < K/6 \) and \( s \leq (\log R)^4 \), uniformly for \( \hat{x}, \hat{x}' \in \hat{\pi}_K(D(0,2r)) \) and \( \hat{y} \in \hat{\pi}_K(D(0,R)^c) \),
\[ \hat{H}_{\hat{\pi}_K(D(0,R)^c)}(\hat{x}, \hat{y}) = \left( 1 + O(m^{-1}) \right) \hat{H}_{\hat{\pi}_K(D(0,R)^c)}(\hat{x}', \hat{y}) \]
\[ + O(R^{-M} \log R \lor K^{-M} R^2). \] (3.17)

Furthermore, uniformly in \( \hat{x} \in \hat{\pi}_K(\partial D(0,r)) \) and \( \hat{y} \in \hat{\pi}_K(D(0,R)^c) \),
\[ P^\hat{x} \left( \hat{S}_{T_{\hat{\pi}_K(D(0,R)^c)}} = \hat{y}, T_{\hat{\pi}_K(D(0,R)^c)} < T_{\hat{\pi}_K(D(0, \frac{r}{4m} + s))} \right) \]
\[ = (1 + O(m^{-1})) P^\hat{x} \left( T_{\hat{\pi}_K(D(0,R)^c)} < T_{\hat{\pi}_K(D(0, \frac{r}{4m} + s))} \right) \hat{H}_{\hat{\pi}_K(D(0,R)^c)}(\hat{x}, \hat{y}) \]
\[ + O(R^{-M} \log R \lor K^{-M} R^2). \] (3.18)

If \( \hat{y} \in \hat{\pi}_K(\partial D(0,R)^c) \), the error term is absorbed in both of these statements.

**Proof** As before, wlog, we can take \( \hat{x}' = \hat{0} \). Let \( s \) be the size of the annulus for \( \hat{y} \). For brevity, set
\[ A_p := \{ S_{T_{D(0,R)^c}} = y \}, \quad d_p := |S_{T_{D(0,R)^c}} - S_{T_{D(0,R)^c} - 1}|, \]
\[ A_t := \{ \hat{S}_{\hat{\pi}_K(D(0,R)^c)} = \hat{y} \}, \quad d_t := |\hat{S}_{\hat{\pi}_K(D(0,R)^c)} - \hat{S}_{\hat{\pi}_K(D(0,R)^c) - 1}|. \]
Note that \(x\) and \(y\) are the primary copies of \(\hat{x}\) and \(\hat{y}\), and so \(|x - y| \leq \frac{K}{\sqrt{2}}\), but that \(d_p\) is a planar distance using a planar escape time and \(d_t\) is a planar distance using a toral escape time; hence, both can exceed \(\frac{K}{\sqrt{2}}\), the maximum distance between two points in \(\mathbb{Z}_K^2\).

To prove (3.17), first we re-label (3.1) as
\[
H_{D(0,R)}(x, y) = P^x(A_p) = \left(1 + O\left(\frac{r}{R}\right)\right) P^x(A_p) + O\left(R^{-M} \log R\right). \quad (3.19)
\]
We have the decomposition
\[
P^x(A_p) = P^x(A_p; d_p < K - 2R) + P^x(A_p; d_p \geq K - 2R). \quad (3.20)
\]
On the plane, the second term of (3.20) is zero for all but the furthest-away \(y\) in the primary copy (i.e., \(K - 2R \leq |x - y| \leq \frac{K}{\sqrt{2}}\); for those \(y\), we have, by (2.5), (1.4), and (2.3),
\[
P^x(A_p; d_p \geq K - 2R) = \sum_{z \in D(0,R)} G_{D(0,R)}(x, z) P^z(|X_1 - z| > K - 2R)
\leq cK^{-M} R^2. \quad (3.21)
\]
The toral version can be written using planar distances as a decomposition, but using the toral disc escape time means a further decomposition comparing the planar and toral escape times a la (2.6). We decompose \(\hat{H}_{\pi_K(D(0,R)K)}(\hat{x}, \hat{y}) = \hat{P}^\hat{x}(A_t)\) as
\[
\hat{P}^\hat{x}(A_t) = \hat{P}^\hat{x}(A_t; d_t < K - 2R; T_{D(0,R)} = T_{\pi_K(D(0,R)K)})
+ \hat{P}^\hat{x}(A_t; d_t \geq K - 2R; T_{D(0,R)} = T_{\pi_K(D(0,R)K)})
+ \hat{P}^\hat{x}(A_t; T_{D(0,R)} < T_{\pi_K(D(0,R)K)}).
\]
In the torus, the first term of (3.22) equals the first term of (3.20), plus a large jump error which contains some paths from the second term of (3.20) (if \(y\) is far): by (3.21),
\[
\hat{P}^\hat{x}(A_t; d_t < K - 2R; T_{D(0,R)} = T_{\pi_K(D(0,R)K)}) = P^x(A_p; d_p < K - 2R) + P^x(A_p; d_p \geq K - 2R; d_t < K - 2R)
= P^x(A_p) + O(K^{-M} R^2).
\]
The second term of (3.22) only occurs if the final, escaping jump is large: by (2.5), (1.4), and (2.4), just as in (3.21),
\[
\hat{P}^\hat{x}(A_t; d_t \geq K - 2R; T_{D(0,R)} = T_{\pi_K(D(0,R)K)}) \leq \hat{P}^\hat{x}(A_t; d_t \geq K - 2R)
= \sum_{\hat{z} \in \pi_K(D(0,R))} \hat{G}_{\pi_K(D(0,R))}(\hat{x}, \hat{z}) \hat{P}^\hat{x}(|X_1 - z| > K - 2R) \leq cK^{-M} R^2.
\]
The last term of (3.22) requires a large jump to have occurred. Hence, by (2.6),
\[ P^x(A_t; T_{D(0,R)}^c < T_{\hat{\pi}_K(D(0,R)_{\hat{\pi}_K}^c)} \leq cK^{-M}R^2. \]

Therefore, (3.22) reduces to
\[ P^x(A_t) = P^x(A_p) + O(K^{-M}R^2). \]

(Due to targeting, this is generalizable to any planar set \( B \subset D(0,R) \) for \( R < K/4 \).

Combining this with (3.19) gives us (3.17):
\[ P^x(A_t) = P^x(A_p) + O(K^{-M}R^2) \]
\[ = \left(1 + O\left(\frac{R}{K}\right)\right) P^x(A_p) + O(K^{-M}R^2) + O(R^{-M} \log R) \]
\[ = \left(1 + O\left(\frac{R}{K}\right)\right) P^x(A_t) + O(K^{-M}R^2) + O(R^{-M} \log R). \]

The proof of (3.18) follows from the Markov property argument for (3.3), using the appropriate toral identities: (3.17) for (3.1), and (2.10)-(2.11) for (2.8)-(2.9).

\[ \square \]

4 Exterior Harnack inequality

To aid our construction of an exterior Harnack inequality (moving from outside a disc to a much smaller disc far inside it), we first establish uniform bounds on external Green’s functions and probabilities in the torus and plane. Fix \( \delta < 1 \) and use \( r \geq c^n \) for some \( n > 13 \), \( R = 4mr \) for some \( 1 \leq m \ll r \), and \( s \leq (\log R)^4 \). First, for \( \hat{x} \in \hat{\pi}_K(\partial D(0,R)_{R/100}) \) and \( \hat{y} \in \hat{\pi}_K(D(0,R)_{\hat{\pi}_K}^c) \), we show that
\[ \hat{G}_{\hat{\pi}_K(D(0,r+s)_{\hat{\pi}_K}^c)}(\hat{x}, \hat{y}) \geq c > 0. \]

Pick some \( \hat{x}_1 \in \hat{\pi}_K(\partial D(0,R)) \), and, proceeding clockwise, choose points \( \hat{x}_2, \ldots, \hat{x}_{36} \in \hat{\pi}_K(\partial D(0,R)) \) whose rays beginning at 0 divide \( \hat{\pi}_K(\partial D(0,R)) \) into 36 approximately equal arcs. The distance between any two adjacent such \( \hat{x}_j \) is, for sufficiently large \( R \), approximately \( 2R \sin(\pi/36) \approx 0.174R \). Thus, using discs of radius \( R/5 \) (so adjacent circles contain their neighbor’s centers), and by (2.11) we have for any \( j = 1, \ldots, 36 \)
\[ \inf_{\hat{x} \in \hat{\pi}_K(D(x_j, R/5))} P^\hat{x}(T_{\hat{\pi}_K(D(x_{j+1}, R/5))} < T_{\hat{\pi}_K(D(0,r+s))}) \]
\[ \geq \inf_{\hat{x} \in \hat{\pi}_K(D(x_j, R/5))} P^\hat{x}(T_{\hat{\pi}_K(D(x_{j+1}, R/5))} < T_{\hat{\pi}_K(D(x_{j+1}, R/2)_{\hat{\pi}_K}^c)}) \geq c_1 > 0, \]

for some \( c_1 \) independent of \( n, r, R, n \) large, and \( \hat{x}_{37} = \hat{x}_1 \).

Hence, by the strong Markov property, rotating through the arcs, we have
\[ \inf_{j,k} \inf_{\hat{x} \in \hat{\pi}_K(D(x_j, R/5))} P^\hat{x}(T_{\hat{\pi}_K(D(x_k, R/5))} < T_{\hat{\pi}_K(D(0,r+s))}) \geq c_2 := c_1^{36}. \]
Furthermore, it follows from (2.7) that for any $j$,

$$\inf_{\hat{x}, \hat{x}' \in \hat{\pi}_K(D(x_j, R/5))} P^{\hat{x}}(T_{\hat{x}'} < T_{\hat{\pi}_K(D(0, r + s))}) \geq \frac{c_3}{\log R}$$

(4.4)

for some independent $c_3 > 0$.

Since $\hat{\pi}_K(\partial D(0, R_{100})) \subset \bigcup_{j=1}^{36} \hat{\pi}_K(D(x_j, R/5))$, combining (4.3) and (4.4) we have, for some independent $c_4 > 0$,

$$\inf_{\hat{x}, \hat{x}' \in \hat{\pi}_K(\partial D(0, R_{100}))} P^{\hat{x}}(T_{\hat{x}'} < T_{\hat{\pi}_K(D(0, r + s))}) \geq \frac{c_4}{\log R}.$$  

(4.5)

It then follows from (2.17) that

$$\inf_{\hat{x}, \hat{x}' \in \hat{\pi}_K(\partial D(0, R_{100}))} \hat{G}_{\hat{\pi}_K(D(0, r + s))}(\hat{x}, \hat{x}')$$

$$= \inf_{\hat{x}, \hat{x}' \in \hat{\pi}_K(\partial D(0, R_{100}))} P^{\hat{x}}(T_{\hat{x}'} < T_{\hat{\pi}_K(D(0, r + s))}) \hat{G}_{\hat{\pi}_K(D(0, r + s))}(\hat{x}', \hat{x}')$$

$$\geq \frac{c_4}{\log R} \hat{G}_{\hat{\pi}_K(D(x_j', R/2))}(\hat{x}', \hat{x}') \geq c_5 > 0$$

(4.6)

for some independent $c_5 > 0$. Using the strong Markov property, (4.6), and (2.13), we see
that
\[
\inf_{\hat{x} \in \hat{\pi}_K(D(0, r+s)_{\hat{y}}'_{K})} \hat{G}_{\hat{\pi}_K(D(0, r+s)_{\hat{y}}'_{K})}(\hat{z}, \hat{x}) = \tilde{G}_{\tilde{\pi}_K(D(0, r+s)_{\tilde{y}}'_{K})}(\tilde{z}, \tilde{x}) \geq E\tilde{z} \left( \tilde{G}_{\tilde{\pi}_K(D(0, r+s)_{\tilde{y}}'_{K})}(\tilde{S}_{\tilde{K}(D(0, r+s))}, \hat{x}); \tilde{S}_{\tilde{K}(D(0, r+s))} \in \tilde{\pi}_K(\partial D(0, R)_{R/100}) \right) \geq c > 0.
\] (4.7)

This gives (4.1). Applying the same argument once more,
\[
\inf_{\hat{x} \in \hat{\pi}_K(D(0, r+s)_{\hat{y}}'_{K})} \hat{G}_{\hat{\pi}_K(D(0, r+s)_{\hat{y}}'_{K})}(\hat{z}, \hat{x}) \geq E\hat{z} \left( \hat{G}_{\hat{\pi}_K(D(0, r+s)_{\hat{y}}'_{K})}(\hat{S}_{\hat{K}(D(0, r+s))}, \hat{x}); \hat{S}_{\hat{K}(D(0, r+s))} \in \hat{\pi}_K(\partial D(0, R)_{R/100}) \right) \geq c > 0.
\] (4.8)

Hence, for all \( \hat{x}, \hat{y} \in \hat{\pi}_K(D(0, R)_{K}) \),
\[
\hat{G}_{\hat{\pi}_K(D(0, r+s)_{\hat{y}}'_{K})}(\hat{x}, \hat{y}) \geq c > 0.
\] (4.9)

Next, we look at the external Green’s function near the \( r \)-disc: uniformly for \( \hat{x} \in \hat{\pi}_K(D(0, R)_{K}) \) and \( \hat{z} \in \hat{\pi}_K(D(0, 2r) \setminus \hat{\pi}_K(D(0, 5r/4)) \), we have by (4.9) and (2.10),
\[
\hat{G}_{\hat{\pi}_K(D(0, r+s)_{\hat{y}}'_{K})}(\hat{x}, \hat{z}) = \hat{G}_{\hat{\pi}_K(D(0, r+s)_{\hat{y}}'_{K})}(\hat{z}, \hat{x}) = E\hat{z} \left( \hat{G}_{\hat{\pi}_K(D(0, r+s)_{\hat{y}}'_{K})}(\hat{S}_{\hat{K}(D(0, r+s))}, \hat{x}); T_{\hat{K}(D(0, r+s))} < T_{\hat{K}(D(0, r+s))} \right) \geq c \frac{P\hat{z}(T_{\hat{K}(D(0, r+s))} < T_{\hat{K}(D(0, r+s))})}{\log m} \geq c \frac{\hat{G}_{\hat{\pi}_K(D(0, r+s)_{\hat{y}}'_{K})}(\hat{x}', \hat{z})}{\log R}.
\] (4.10)

Getting closer to the disc, for any \( \varepsilon > 0 \), uniformly in \( \hat{x}, \hat{x}' \in \hat{\pi}_K(\partial D(0, R)_{R/100}) \) and \( \hat{z} \in \hat{\pi}_K(D(0, 2r) \setminus \hat{\pi}_K(D(0, r + (1 + \varepsilon)s)) \), we have by the strong Markov property and (4.5), for any \( \hat{x}' \in \partial D(0, R)_{R/100} \),
\[
\hat{G}_{\hat{\pi}_K(D(0, r+s)_{\hat{y}}'_{K})}(\hat{x}, \hat{z}) \geq P\hat{z}(T_{\hat{K}(D(0, r+s))} < T_{\hat{K}(D(0, r+s))}) \hat{G}_{\hat{\pi}_K(D(0, r+s)_{\hat{y}}'_{K})}(\hat{x}', \hat{z}) \geq c \frac{\hat{G}_{\hat{\pi}_K(D(0, r+s)_{\hat{y}}'_{K})}(\hat{x}', \hat{z})}{\log R}.
\] (4.11)

In view of (2.22), if \( \hat{x}' \in \hat{\pi}_K(\partial D(0, R)_{R/100}) \) is chosen as close as possible to the ray from the origin which passes through \( \hat{z} \), we have
\[
\hat{G}_{\hat{\pi}_K(D(0, r+s)_{\hat{y}}'_{K})}(\hat{x}', \hat{z}) \geq \hat{G}_{\hat{\pi}_K(D(\hat{x}', |\hat{x}'|-(r+s)))}(\hat{x}', \hat{z}) \geq cR.
\] (4.12)
which, combined with (4.11), gives us

\[
\inf_{\hat{x} \in \mathfrak{K}(D(0,r+s)_{cK})} \hat{G}_{\mathfrak{K}}(D(0,r+s)_{cK})^{\hat{x}}(\hat{x}, \hat{z}) \geq \frac{c}{R \log R}. \tag{4.13}
\]

Using the strong Markov property, (4.13), and (2.12), we see that

\[
\inf_{\hat{x} \in \mathfrak{K}(D(0,1.01r)_{cK})} \hat{G}_{\mathfrak{K}}(D(0,1.01r)_{cK})^{\hat{x}}(\hat{x}) \geq \mathbb{E}^{\hat{z}} \left( \hat{G}_{\mathfrak{K}}(D(0,r+s)_{cK})^{\hat{x}}(\hat{x}, \hat{\hat{z}}; \hat{S}_{T_{\mathfrak{K}}(D(0,1.01r)_{cK})}, \hat{x}) \right) \geq \frac{c}{R \log R},
\]

Hence

\[
\inf_{\hat{x} \in \mathfrak{K}(D(0,1.01r)_{cK})} \hat{G}_{\mathfrak{K}}(D(0,1.01r)_{cK})^{\hat{x}}(\hat{x}, \hat{y}) \geq \frac{c}{R \log R}. \tag{4.15}
\]

By removing the (hidden) targeted jump error terms, the entire argument in (4.11)-(4.15) also applies to the plane. We now find a general planar Harnack inequality for entering a small disc from far outside.

**Proposition 4.1.** Let \( R = 4mr \) with \( 1 \leq m \ll r \ (m = o(r^{1/4})) \) and large enough \( r \), and \( s \leq (\log R)^4 \). Then, uniformly for \( x, x' \in D(0, R)^c \) and \( y \in \partial D(0, r) \),

\[
H_{D(0,r+s)}(x, y) = (1 + O \left( m^{-1} \log m \right)) H_{D(0,r+s)}(x', y). \tag{4.16}
\]

Furthermore, for \( x, x' \in \partial D(0, R) \sqrt{R} \),

\[
P^x(S_{T_{D(0,r+s)}} = y; T_{D(0,r+s)} < T_{D(0,4mR)^c}) = (1 + O \left( m^{-1} \log m \right)) H_{D(0,r+s)}(x, y) P^x(T_{D(0,r+s)} < T_{D(0,4mR)^c})
\]

\[
= (1 + O \left( m^{-1} \log m \right)) P^x(S_{T_{D(0,r+s)}} = y; T_{D(0,r+s)} < T_{D(0,4mR)^c}).
\]

**Proof** For \( x, x' \in D(0, R)^c \) and \( y \in \partial D(0, r) \), we have the last exit decomposition

\[
H_{D(0,r+s)}(x, y) = \left( \sum_{z \in D(0,0.5r/4)} + \sum_{z \in D(0,2r)} + \sum_{z \in D(0,2r)^c} \right) G_{D(0,r+s)^c}(x, z)p_1(z, y). \tag{4.18}
\]

Let \( x, x' \in \partial D(0, R) \) and set \( N \geq 4mR \). Uniformly for \( z \in D(0, 2r) \cup D(0, N)^c \), by (2.15) and (2.2),

\[
a(x - z) = \frac{2}{\pi T} \log |x - z| + C' + O(|x - z|^{-1}) = a(x' - z) + O(m^{-1}). \tag{4.19}
\]
Using the same approach as in (3.7), (4.19) implies that, for $A(r+s,N) := D(0,N) \setminus D(0,r+s)$,

$$G_{A(r+s,N)}(x,z) = G_{A(r+s,N)}(x',z) + O(m^{-1}),$$

which, by letting $N \to \infty$ and applying the dominated convergence theorem,

$$G_{D(0,r+s)}(x,z) = G_{D(0,r+s)}(x',z) + O(m^{-1}).$$

(4.20)

Applying (4.10) to (4.20) yields, for $z \in D(0,2r) \setminus D(0,r+s)$,

$$G_{D(0,r+s)}(x,z) = (1 + O(m^{-1} \log m)) G_{D(0,r+s)}(x',z).$$

(4.21)

Next, by the symmetry of the Green’s function, the strong Markov property at $T_{D(0,5r/4)}$, (4.21) for $z \in D(0,5r/4) \setminus D(0,r+s)$, and decomposing, we have

$$G_{D(0,r+s)}(x,z) = G_{D(0,r+s)}(z,x)$$

(4.22)

$$= \mathbb{E}^z(G_{D(0,r+s)}(S_{T_{D(0,5r/4)}}); T_{D(0,5r/4)} < T_{D(0,r+s)}),$$

$$= \mathbb{E}^z(G_{D(0,r+s)}(S_{T_{D(0,5r/4)}}); T_{D(0,5r/4)} < T_{D(0,r+s)}),$$

$$+ \mathbb{E}^z(G_{D(0,r+s)}(S_{T_{D(0,5r/4)}}); T_{D(0,5r/4)} < T_{D(0,r+s)}),$$

By (4.21) on the first term and (2.23) on the second term, (4.22) is bounded above:

$$G_{D(0,r+s)}(z,x) \leq (1 + O(m^{-1} \log m))$$

$$\mathbb{E}^z(G_{D(0,r+s)}(S_{T_{D(0,5r/4)}}); T_{D(0,5r/4)} < T_{D(0,r+s)}),$$

$$+ c \log(R) \mathbb{P}^z(|S_{T_{D(0,5r/4)}}| > 2r).$$

(4.23)

Applying the first two lines of (4.22) again, the first term here is

$$(1 + O(m^{-1} \log m)) \mathbb{E}^z(G_{D(0,r+s)}(S_{T_{D(0,5r/4)}}); T_{D(0,5r/4)} < T_{D(0,r+s)}),$$

$$= (1 + O(m^{-1} \log m)) G_{D(0,r+s)}(z,x') = (1 + O(m^{-1} \log m)) G_{D(0,r+s)}(x',z).$$

A last exit decomposition of $P^z(|S_{T_{D(0,5r/4)}}| > 2r)$, then (2.19) and (1.5) yield

$$G_{D(0,r+s)}(x,z) \leq (1 + O(m^{-1} \log m)) G_{D(0,r+s)}(x',z)$$

$$+ c \log(R) \sum_{|y|<5r/4 \atop 2r<|w|} G_{D(0,5r/4)}(z,y)p_1(y,w)$$

$$\leq (1 + O(m^{-1} \log m)) G_{D(0,r+s)}(x',z) + c \log(R) \log(r) r^{-M+2}.$$

Since this argument is symmetric in $x$ and $x'$, then we have that for $z \in D(0,5r/4) \setminus D(0,r+s)$, and $c \log R = c(\log 4 + \log m + \log r) = O(\log r),$

$$G_{D(0,r+s)}(x,z) = (1 + O(m^{-1} \log m)) G_{D(0,r+s)}(x',z) + O(r^{-M+2}(\log r)^2).$$

(4.24)
Finally, by (2.23), for \( z \in D(0, 2r)^c \), \( G_{D(0,r+s)^c}(x, z) = O(\log R) \). Thus, for \( y \in \partial D(0, r)_s \), since \( \sum_{z \in D(0, 2r)^c} p_1(z, y) \leq O(r^{-M}) \) by symmetry and Markov’s inequality,

\[
\sum_{z \in D(0, 2r)^c} G_{D(0,r+s)^c}(x, z)p_1(z, y) = O(r^{-M} \log r). \tag{4.25}
\]

Combining (4.21), (4.24), and (4.25) bounds the sums in (4.18) to

\[
H_{D(0,r+s)}(x, y) = (1 + O(m^{-1} \log m))H_{D(0,r+s)}(x', y) + O(r^{-M+2}(\log r)^2). \tag{4.26}
\]

To complete the proof of (4.16) for \( x, x' \in \partial D(0, R)_R \), we must show that, uniformly for \( x \in \partial D(0, R)_R \) and \( y \in \partial D(0, r)_s \),

\[
r^{-M+2}(\log r)^2 \leq c(m^{-1} \log m)H_{D(0,r+s)}(x, y). \tag{4.27}
\]

With \( A_r := D(0, 2r) \setminus D(0, r + (1 + \varepsilon)s) \), using a last exit decomposition and bounding with the planar version of (4.15),

\[
H_{D(0,r+s)}(x, y) = \sum_{z \in D(0,r+s)^c} G_{D(0,r+s)^c}(x, z)p_1(z, y) \geq \sum_{z \in A_r} G_{D(0,r+s)^c}(x, z)p_1(z, y) \geq \frac{\varepsilon''}{R \log R} \sum_{z \in A_r} p_1(z, y)
\]

for any \( \varepsilon > 0 \). Note that the annulus \( A_r \) contains the disc \( D(v, 2(1 + \varepsilon)s) \), where \( v := (r + 3(1 + \varepsilon)s)/|y| \). Thus, \( 2(1 + \varepsilon)s \leq |y - v| \leq 3(1 + \varepsilon)s \), and (4.1) (where we consider \( y \in \partial D(v, 2(1 + \varepsilon)s)_{(1+\varepsilon)s}) \), and with \( s \leq (\log R)^4 \leq c(\log r)^4 \),

\[
\sum_{z \in A_r} p_1(z, y) \geq \sum_{z \in D(v, 2(1+\varepsilon)s)} p_1(z, y) \geq ce^{-\beta((1+\varepsilon)s)^{1/4}} \geq cr^{-(1+\varepsilon)^{1/4}\beta}. \tag{4.29}
\]

Hence, combining (4.28) and (4.29), and since \( m \leq \sqrt{r}, R < r^2 \), some \( \varepsilon' > 0 \), and \( r^\beta > (\log r)^3 \) for large enough \( r \),

\[
c(m^{-1} \log m)H_{D(0,r+s)}(x, y) \geq \frac{c m^{-1} \log m r^{-(1+\varepsilon)^{1/4}\beta}}{R \log R} \geq cr^{-(1+\varepsilon)^{1/4}\beta} m^{-1} \log m (mr)^{-1} (2 \log r)^{-1} \geq cr^{-1-(1+\varepsilon')\beta} m^{-2} (\log m)(\log r)^{-1} \geq cr^{-2-2\beta} (\log m)(\log r)^2 \geq cr^{-M+2}(\log r)^2, \tag{4.30}
\]

which proves (4.27), and hence (4.16), for \( x, x' \in \partial D(0, R)_R \).

Next we show (4.16) for \( x \in D(0, 2R)^c \). Decompose the hitting distribution on whether or
Thus, combining the two, we have for \( x \in D(0, 2R)^c \) and \( y \in \partial D(0, r)_s, \)

\[
H_{D(0,r+s)}(x, y) = P^x(S_{T_{D(0,r+s)}} = y, \ T_{\partial D(0,R)} > T_{D(0,R)})
+ P^x(S_{T_{D(0,r+s)}} = y, \ T_{\partial D(0,R)} < T_{D(0,R)}).
\]

We can bound the first term by (2.13):

\[
P^x(S_{T_{D(0,r+s)}} = y, \ T_{\partial D(0,R)} > T_{D(0,R)}) \leq P^x(T_{\partial D(0,R)} > T_{D(0,R)})
\]

\[
\leq cR^2(\log R)^2(R^{-M} + R^{-M}) \leq cR^{-M+2}(\log R)^2 < cr^{-M+2}(\log r)^2.
\]

By the strong Markov property at \( T_{\partial D(0,R)}, \) the second term can be bounded, uniformly for \( x' \in \partial D(0, R)_R, \) by (4.16):

\[
P^x(S_{T_{D(0,r+s)}} = y, \ T_{\partial D(0,R)} < T_{D(0,R)})
= \mathbb{E}^x(H_{D(0,r+s)}(S_{T_{\partial D(0,R)}}, y), \ T_{\partial D(0,R)} < T_{D(0,R)})
\leq (1 + O(m^{-1} \log m))H_{D(0,r+s)}(x', y).
\]

Thus, combining the two, we have for \( x \in D(0, 2R)^c \) and \( x' \in \partial D(0, R)_R, \)

\[
H_{D(0,r+s)}(x, y) = \left(1 + O\left(m^{-1} \log m\right)\right) \ H_{D(0,r+s)}(x', y) + O(r^{-M+2}(\log r)^2),
\]

which gives (4.16) for \( x \in D(0, 2R)^c \) and \( x' \in \partial D(0, R)_R. \) Applying (4.16) again for the same \( x' \) and \( x'' \in D(0, 2R)^c \) gives (4.16) for \( x, x'' \in D(0, 2R)^c. \)

To prove (4.17) for \( x, x' \in \partial D(0, R) \sqrt{R}, \) decompose \( H_{D(0,r+s)}(x, y) \) over the event \( \{T_{D(0,r+s)} > T_{D(0,4mR)}\} \) to get

\[
P^x(S_{T_{D(0,r+s)}} = y, \ T_{D(0,r+s)} < T_{D(0,4mR)^c})
= \mathbb{E}^x(H_{D(0,r+s)}(S_{T_{D(0,4mR)^c}}; y), \ T_{D(0,r+s)} > T_{D(0,4mR)^c})
\]

By the strong Markov property at \( T_{D(0,4mR)^c}, \) the last term of (4.32) can be further decomposed to

\[
P^x(S_{T_{D(0,r+s)}} = y, \ T_{D(0,r+s)} > T_{D(0,4mR)^c})
= \mathbb{E}^x(H_{D(0,r+s)}(S_{T_{D(0,4mR)^c}}; y); T_{D(0,r+s)} > T_{D(0,4mR)^c})
\]

\[
= (1 + O(m^{-1} \log m))H_{D(0,r+s)}(x, y)P^x(T_{D(0,r+s)} > T_{D(0,4mR)^c}),
\]

which gives us the first equality in (4.17). The second follows from (2.9) and (2.1), since,
for \( x, x' \in \partial D(0, R) \sqrt{R} \), if \( |x| = R \) and \( |x'| = R + \sqrt{R} \),

\[
\frac{P^x_{x'}(T_{D(0, r+s)} < T_{D(0, 4mR)^c})}{P^x_{x'}(T_{D(0, r+s)} < T_{D(0, 4mR)^c})} = \frac{\log \left( \frac{R + \sqrt{R}}{r + s} \right) + O(r^{-1/4})}{\log \left( \frac{R}{r + s} \right) + O(r^{-1/4})}
\]

(4.34)

\[
= 1 + O \left( \frac{\sqrt{R}}{R \log \left( \frac{R}{r + s} \right)} \right) + o(r^{-1/4}) = 1 + o(m^{-1} \log m). \tag*{□}
\]

When attempting to move the planar exterior Harnack inequality to the torus, we run into difficulties in dealing with walks that wander and enter far-off copies of \( D(0, r + s) \) instead of the primary copy. We modify the exterior Harnack inequality for the toral case to fit our requirements.

**Proposition 4.2.** Let \( R = 4mr \) with \( 1 \leq m = o(r^{1/4}) \) and large enough \( r \), \( 4mR < K/4 \), and \( s \leq (\log R)^4 \). Then, uniformly for \( \hat{x}, \hat{x}' \in \hat{\pi}_k(\partial D(0, R) \sqrt{R}) \) and \( \hat{y} \in \hat{\pi}_k(\partial D(0, r)_s) \),

\[
P^\hat{x}(\hat{S}_{T_{\hat{\pi}_K(D(0, r+s))}/\hat{\pi}_K} < T_{\hat{\pi}_K(D(0, 4mR)^c)_K})
= (1 + O \left( m^{-1} \log m \right)) P^\hat{x}(\hat{S}_{T_{\hat{\pi}_K(D(0, r+s))}/\hat{\pi}_K} < T_{\hat{\pi}_K(D(0, 4mR)^c)_K})
\]

(4.35)

**Proof** For brevity, set

\[
D_* := D(0, r + s) \cup D(0, 4mR)^c, \quad A_p := \{ S_{D_*} = y \},
\]

\[
\hat{D}^* := \hat{\pi}_K(D(0, r + s)) \cup \hat{\pi}_K(D(0, 4mR)^c), \quad A_t := \{ \hat{S}_{D_*} = \hat{y} \}.
\]

We start our walk at the primary copy \( y \), consider the planar landing at the primary copy \( y \), and decompose \( P^x(A_t) = \hat{H}_{D_*}(x, y) \) along the planar large disc escape time \( T_{D(0, 4mR)^c} \) and the toral annulus escape time \( T_{D_*} \):

\[
P^x(A_t) = P^x(A_t; T_{D_*} < T_{D(0, 4mR)^c}) + P^x(A_t; T_{D_*} \geq T_{D(0, 4mR)^c}). \tag*{(4.36)}
\]

Since \( \hat{\pi}_K^{-1} \hat{D}^* \subset D^* \), \( T_{D_*} \leq T_{D_*} \) a.s. The first term of (4.36) happens in the event \( \{ T_{D_*} = T_{D_*} = T_{D(0, r+s)} \} \), so the entirety of its action before the final step is inside the primary copy of \( D(0, 4mR) \). Hence,

\[
P^x(A_t; T_{D_*} < T_{D(0, 4mR)^c}) = P^x(A_t; T_{D_*} = T_{D_*}) = P^x(A_p).
\]

Note that \( P^x(A_p) \) is (4.17). The second term of (4.36) only occurs if a targeted jump lands in a non-primary copy of \( D(0, 4mR) \setminus D(0, r + s) \). Hence, by (2.6),

\[
P^x(A_t; T_{D_*} \geq T_{D(0, 4mR)^c}) \leq P^x(T_{D(0, 4mR)^c} < T_{\hat{\pi}_K(D(0, 4mR)^c)_K}) \leq O(K^{-M}(mR)^2).
\]
thus reduces to $P^x(A_t) = P^x(A_p) + O(K^{-M}(mR)^2)$, which, by (4.17), is
\[ P^x(A_t) = (1 + O(m^{-1} \log m))P^x(A_p) + O(K^{-M}(mR)^2). \] (4.37)
Since $M > 4$, the error term $O(K^{-M}(mR)^2) = o(K^{-2-\beta})$ is absorbed via (4.30) applied to the $P^x(A_p)$ term above, with (3.16) one “level” up $(D(0,4mR)^c$ as the outer bound instead of $D(0,R)^c$, $D(0,r+s)$ instead of $D(0,\frac{r}{4m}+s)$, and $x, x' \in \partial D(0,R)\sqrt{R}$ instead of $\partial D(0,r)$, which yields (4.35). \qed

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