Relative dispersion in fully developed turbulence: from Eulerian to Lagrangian statistics in synthetic flows

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The effect of Eulerian intermittency on the Lagrangian statistics of relative dispersion in fully developed turbulence is investigated. A scaling range spanning many decades is achieved by generating a multi-affine synthetic velocity field with prescribed intermittency features. The scaling laws for the Lagrangian statistics are found to depend on Eulerian intermittency in agreement with a multifractal description. As a consequence of the Kolmogorov’s law, the Richardson’s law for the variance of pair separation is not affected by intermittency corrections.

\begin{equation}
\langle R^2(t) \rangle \sim t^3
\end{equation}

where \( R \) is the separation of a particle pair and the average is performed over many dispersion experiments or over many particle pairs. The scaling \( \langle R^2(t) \rangle \) can be obtained by a simple dimensional argument starting from Kolmogorov’s similarity law for velocity increments in fully developed turbulence

\begin{equation}
\langle |\delta \mathbf{v}(E)(R)| \rangle = \langle |\mathbf{v}(\mathbf{x} + \mathbf{R}) - \mathbf{v}(\mathbf{x})| \rangle \sim R^{1/3}
\end{equation}

with \( R = |\mathbf{R}| \). The pair of particles separates according to

\begin{equation}
\frac{d\mathbf{R}}{dt} = \delta \mathbf{v}^{(L)}(\mathbf{R})
\end{equation}

where \( \delta \mathbf{v}^{(L)} \) represents the velocity difference evaluated along the Lagrangian trajectories. Assuming \( \delta \mathbf{v}^{(E)}(R) \simeq |\delta \mathbf{v}^{(E)}(R)| \) from (2) one obtains \( dR^2/dt \sim R^{2/3} \) and hence the Richardson’s law (3).

To investigate the role of Eulerian intermittency we have performed extensive numerical investigations of the relative dispersion at very large Reynolds numbers. To accomplish this purpose we have developed a Lagrangian numerical code for particle pairs whose separation evolves according to (3) with a realistic turbulent velocity difference. We consider the Quasi-Lagrangian reference frame moving with a reference particle. The second particle is advected by the relative velocity \( \delta \mathbf{v}(\mathbf{r}, t) \) which possesses the same single-time statistics of the Eulerian velocity, whenever one considers statistically stationary flows. A realistic velocity field in this reference frame is generating by extending a recently introduced stochastic algorithm for the generation of multifractal processes.

For sake of simplicity we consider, as in (3), a two dimensional velocity field. The reason is that the
relevant aspect for the statistic of particle pairs reparation are the scaling laws for the relative velocity \( \delta v(r,t) \), which we take equal to those of three dimensional turbulence. The extension to a three dimensional velocity field is not difficult, but more expensive in terms of numerical resources.

We introduce the stream function \( \psi(r,t) \) which, in isotropic conditions, can be decomposed using polar coordinates as

\[
\psi(r,\theta,t) = \sum_{i=1}^{N} \sum_{j=1}^{n} \phi_{i,j}(t) \frac{F(k_i r)}{k_i} G_{i,j}(\theta).
\]

Being interested in velocity fields possessing scaling laws on a large number of decades we use \( k_i = 2^i k_0 \). The width of the “inertial range” is thus of order \( 2^N \). The \( \phi_{i,j}(t) \) are stochastic processes with characteristic times \( \tau_i = 2^{-2i/3} \tau_0 \), zero mean and \( \langle |\phi_{i,j}|^p \rangle \sim k_i^{-\zeta_p} \). An efficient way of to generate \( \phi_{i,j} \) is

\[
\phi_{i,j}(t) = g_{i,j}(t) z_{i,j}(t) z_{2,j}(t) \cdots z_{i,j}(t)
\]

where the \( z_{k,j} \) are independent, positive definite, identically distributed random processes with characteristic time \( \tau_k \), while the \( g_{i,j} \) are independent stochastic processes with zero mean, \( \langle g_{i,j}^2 \rangle \sim k_i^{-2/3} \) and characteristic time \( \tau_i \).

For a fully developed turbulent velocity field we expect the scaling \( \langle |\psi(r,\theta)|^p \rangle \sim r^{\sigma_p + p} \) which can be simply achieved by asking that the radial function \( F(x) \) has support only for \( x \approx 1 \) and choosing the scaling of the random processes \( \phi_{i,j} \) as above described. With this choice the exponents \( \zeta_p \) are determined by the probability distribution of \( z_{i,j} \) via

\[
\zeta_p = \frac{p}{3} - \log_2(x^p).
\]

A simple way for constructing the synthetic turbulent field is with the following choice:

\[
F(x) = x^2(1-x) \quad \text{for } 0 \leq x \leq 1
\]

and zero otherwise,

\[
G_{i,1}(\theta) = 1, \quad G_{i,2}(\theta) = \cos(2\theta + \phi_i)
\]

and \( G_{i,j} = 0 \) for \( j > 2 \) (\( \phi_i \) is a quenched random phase).

It is worth remarking that this choice is rather general because it can be derived from the lowest order expansion for small \( r \) of a generic streamfunction in Quasi-Lagrangian coordinates.

The intermittency in the velocity field can be tuned by the set of parameters entering into the construction of \( \phi_{i,j}(t) \). In this Letter we shall consider synthetic turbulent fields whose intermittency corrections to the Kolmogorov scaling, i.e., nonlinear \( \zeta_p \), are close to the true three dimensional turbulence exponents [15], i.e. \( \zeta_1 = 0.39, \zeta_2 = 0.72 \) and so on.

In Figure 1 we report the Lagrangian longitudinal structure functions \( S_p^L(r) = \langle (|\delta v_r|^L(r))^p \rangle \), which are computed recording the Lagrangian velocity difference whenever the pair separation equals \( r \). Observe the wide inertial range over more than 10 decades, corresponding to an integral Reynolds number \( Re \approx 10^{10} \). Because of incompressibility the particles separate on average in time \( t_0 \) and thus also the first order Lagrangian structure function is non-zero. The most interesting and non-trivial result is that scaling exponents for the Lagrangian structure functions show up to be exactly the same \( \zeta_p \) of the Eulerian case. In terms of the multifractal formalism, this result is restated by saying that the fractal dimension \( D(h) \) for the Lagrangian statistics is the same of the Eulerian one.

With this preliminary results in mind, we can extend the dimensional argument for the Richardson’s law to the intermittent case by using the multifractal representation. Following, with a few changes, Novikov [16] we assume, in the spirit of the refined similarity hypothesis (RSH) of Kolmogorov, that \( \delta v_r(L) (R(t)) \sim (\epsilon_R t) \frac{1}{\nu} \) and \( R(t) \sim (\epsilon_R t^3)^{1/2} \) where \( \epsilon_R \) is the energy density dissipation at scale \( R \). Assuming that \( \epsilon_R \sim \delta v_r(E) (R^3)/R \) and remembering that in the multifractal description of fully developed turbulence \( \delta v_r(E) (R) \sim R^p \) with probability \( P_R(h) \sim R^{3-D(h)} \) [17], a simple calculation leads to

\[
\langle R^p(t) \rangle \sim \int dh t^{3+p-D(h)[1-h]}.
\]

In the limit of time \( t \) much smaller than the eddy turnover time at large scale the integral can be performed by steepest descent method, and we obtain the scaling laws \( \langle R(t)^p \rangle \sim t^{\alpha_p} \) where the exponents are given by

\[
\alpha_p = \inf_h \left[ \frac{p + 3 - D(h)}{1 - h} \right]
\]

From the above argument we thus expect that, in general, relative dispersion displays anomalous scaling in time (non linear \( \alpha_p \)). However there is an interesting result, already obtained in [16], for the case \( p = 2 \). From the general multifractal formalism one has that \( 3 - D(h) \geq 1 - 3h \) and the equality is satisfied for the scaling exponent \( h_3 \) which realizes the third order structure function \( \zeta_3 = 1 \). From [16] follows that \( \alpha_3 = 3 \) and thus we have that the Richardson’s law \( \langle R^3 \rangle \sim t^3 \) is not affected by intermittency corrections, while the other moments in general are. We note that the Lagrangian RSH argument leading to eq. (4) is just one dimensional reasonable assumption which can be justified only a posteriori by numerical simulations. Other different assumptions are possible leading to different predictions.

In Figure 2 we plot the result of \( \langle R^p(t) \rangle^{1/p} \) for different sizes of the inertial range. We indeed observe for
Given a set of thresholds \( \tilde{\epsilon} \) and \( \tilde{\epsilon} \) which has been successfully applied in the approach which is based on the statistics at fixed scale, numerical simulations with the standard approach. show quite clearly the difficulties that may arise in experimental apparatus. For example, references \([6,18]\) is generally limited due to the Reynolds number and the experimental apparatus. For instance in experimental data where the inertial range can be measured with higher accuracy, especially in the case of moderate inertial range simulation (figure 3). For this reason we suggest that this kind of analysis should be preferred when dealing with experimental data. Also in this case there is an exponent \( \zeta_3 - 3 = -2 \) unaffected by the intermittency.

In the construction of the synthetic turbulent field we have implicitly assumed that the Lagrangian time \( \tau_L \) and the Eulerian time \( \tau_E \) are of the same order of magnitude, an assumption consistent with the experimental data and theoretical arguments (see, e.g., \([21]\)). Nevertheless it should be interesting to study the relevance of the ratio \( \tau_L/\tau_E \) on the intermittent corrections. One could, indeed, expects that for \( \tau_L/\tau_E \gg 1 \) the Lagrangian intermittency disappears.

Finally we note that the case of non intermittent turbulence, i.e., \( \zeta_3 = \frac{p}{3} \), corresponds to keep fixed \( z_{i,j} = 1 \). In this case one may ask if our results are realistic also for the Richardson’s constant \( G_{\Delta} \) defined from the the pair dispersion law \( R^2(t) = G_{\Delta} t^{3} \) where \( \tau \) is the average energy dissipation rate. The value of \( \tau \) can be obtained from the second order Eulerian structure function, which reads \( S_2(E)(R) = \langle \delta v_i(E)(R) \rangle^2 = C_L \tau^{2/3} R^{2/3} \) where \( C_L \) is a universal constant related to the Kolmogorov constant. According to the experimental measurements we fix \( C_L = 2.0 \), leading to \( G_{\Delta} = 0.190 \pm 0.005 \) for the Richardson’s constant which is in agreement with previous values \([13]\).

In this Letter, using a synthetic turbulence model, we have the first evidence that the relative dispersion statistics for Lagrangian tracers in fully developed turbulence is affected by the intermittency of the Eulerian field. Our numerical results are in agreement with the identification of Lagrangian and Eulerian intermittency. We have suggested a new approach based on the Lagrangian doubling times which seems very promising for data analysis. The present work are a first step towards the clarification of Lagrangian-Eulerian relationship in fully developed turbulence. It would be extremely interesting to check our claims by mean of direct numerical simulations or laboratory experiments.

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\[ \langle \frac{1}{T^p(R)} \rangle \simeq \int dh R^{p(h-1)} R^{3-D(h)} \simeq R^{\kappa_p - p} \tag{11} \]

from which follows that the doubling time statistics contains the same information on the Eulerian intermittency as the relative dispersion exponents \([10]\).

As reported in Figure 2b prediction (11) is very well verified in our simulations. Note that the scaling region is wider than that of Figure 2a and the scaling exponent can be measured with higher accuracy, especially in the case of moderate inertial range simulation (figure 5). For this reason we suggest that this kind of analysis should be preferred when dealing with experimental data. Also in this case there is an exponent \( \zeta_3 - 3 = -2 \) unaffected by the intermittency.

\[ \frac{\tau}{G_{\Delta}} = \frac{2}{3} \]

\[ \langle \frac{1}{T^p(R)} \rangle \simeq \int dh R^{p(h-1)} R^{3-D(h)} \simeq R^{\kappa_p - p} \]
FIG. 1. Lagrangian longitudinal structure functions $S_p^{(L)}(r)$ for $p = 1, 2, 3, 4$ (from top to bottom) for $N = 30$ shells intermittent velocity field. The continuous lines represent the theoretical scaling with exponents $\xi_1 = 0.39$, $\xi_2 = 0.72$, $\xi_3 = 1.0$ and $\xi_4 = 1.24$.

FIG. 2. Relative dispersion $\langle R^p(t) \rangle^{1/p}$ (a) and inverse doubling times $\langle 1/T^p(R) \rangle^{1/p}$ (b) for $p = 1, 2, 3, 4$ (from top to bottom) for $N = 30$ shells averaged over $10^5$ realizations. The continuous lines represent the theoretical scaling as described in the text. The data for different $p$ are shifted for a better viewing.

FIG. 3. Relative dispersion $\langle R(t) \rangle$ for for $N = 20$ shells averaged over $10^4$ realizations. In the inset we show the corresponding average inverse doubling time $\langle 1/T \rangle$. Observe the enhancement of the scaling range in the latter case.