Research Article

Numerical Solutions of Certain New Models of the Time-Fractional Gray-Scott

Sami Aljhani,1 Mohd Salmi Md Noorani,1 Khaled M. Saad,2,3 and A. K. Alomari4

1Department of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia (UKM), 43600 Bangi Selangor, Malaysia
2Department of Mathematics, College of Sciences and Arts, Najran University, POB 1988, Najran 11001, Saudi Arabia
3Department of Mathematics, Faculty of Applied Science, Taiz University, Taiz, Yemen
4Department of Mathematics, Faculty of Science, Yarmouk University, 211-63 Irbid, Jordan

Correspondence should be addressed to Sami Aljhani; sami.aljhani@gmail.com

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A reaction-diffusion system can be represented by the Gray-Scott model. In this study, we discuss a one-dimensional time-fractional Gray-Scott model with Liouville-Caputo, Caputo-Fabrizio-Caputo, and Atangana-Baleanu-Caputo fractional derivatives. We utilize the fractional homotopy analysis transformation method to obtain approximate solutions for the time-fractional Gray-Scott model. This method gives a more realistic series of solutions that converge rapidly to the exact solution. We can ensure convergence by solving the series resultant. We study the convergence analysis of fractional homotopy analysis transformation method by determining the interval of convergence employing the ℏu,v-curves and the average residual error. We also test the accuracy and the efficiency of this method by comparing our results numerically with the exact solution. Moreover, the effect of the fractionally obtained derivatives on the reaction-diffusion is analyzed. The fractional homotopy analysis transformation method algorithm can be easily applied for singular and nonsingular fractional derivative with partial differential equations, where a few terms of series solution are good enough to give an accurate solution.

1. Introduction

Differential equations play a significant role within the field of finance, engineering, physics, and biology. Therefore, these applications can be modelled through differential equations [1, 2].

Reaction-diffusion system (RDS) is known as a set of partial differential equations, which correspond to many physical phenomena. RDS can be applied in physics, biology, chemistry, epidemiology, etc. (see, for example, [3, 4]).

An RDS can be represented by the Gray-Scott model (GSM). The classical (integer derivative) GSM has been studied by several numerical techniques [5, 6]. Moreover, the existence and stability of the solution to this model in one dimension are discussed in [7]. In recent years, solutions to the fractional (noninteger) GSM have been spread at the same rate with the classical (integer derivative) GSM [8, 9].

Fractional calculus (FC) deals with integrals and derivatives of noninteger order. Scholars have shown an increasing interest in FC since it can study all phenomena accurately than what has been modelled through integer differential equations [10–21].

Three are many definitions of FC, such as the Riemann-Liouville and the Liouville-Caputo [22]. Recently, Caputo and Fabrizio (CF) proposed a new concept of fractional differentiation using the exponential decay as the kernel instead of the power law [23, 24]. Thereafter, Atangana and Baleanu (AB) developed a new concept of differentiation with nonsingular [25, 26], based on the general Mittag-Leffler function. These two concepts with fractional order in Riemann-Liouville and Liouville-Caputo sense have a nonlocal kernel.

Despite the difficulty of finding exact solutions in FC’s case, the numerical and approximate technique to obtain approximate solutions is needed. Several methods have been
applied for solving fractional differential equations, such as the fractional natural decomposition method [27, 28], \( q \)-homotopy analysis transform method [28–30], and Adams Bashforth and the Fourier spectral methods [31]. Khan et al. [32] and Kumar et al. [33, 34] coupled the homotopy analysis method (HAM) [35–37] with the Laplace transform to solve a nonlinear differential equation. This method is called the fractional homotopy analysis transform method (FHATM). The main advantage of this method is its ability to combine two powerful methods to obtain a rapidly convergent series for fractional differential equations. The FHATM provides us with a convenient way to control the convergence of the series solution.

In this paper, RDS can be represented by GSM. In order to find an approximate solution to the proposed model, the FHATM is applied. To the best of our knowledge, this paper is the first one that introduced the approximate analytic solution for the time-fractional Gray-Scott system using a non-singular fractional derivative.

2. Preliminaries and Notations

2.1. The Model. We consider the reaction-diffusion system for the cubic autocatalysis. This system contains two chemical species \( \mathcal{U} \) and \( \mathcal{V} \), whose concentration is referred by variables \( u \) and \( v \), respectively. Cubic autocatalysis is given by two reactions, which occur at a different rate:

\[
\begin{align*}
\frac{\partial u}{\partial \rho} &= \Delta u - uv^2 + A(1 - u), \\
\frac{\partial v}{\partial \rho} &= \Delta v + uv^2 - Bv.
\end{align*}
\]

(1)

where \( \mathcal{P} \) is some inert product of reaction.

Following [38], when quantity depends on one spatial coordinate \((\xi)\), the GSM in one space dimension is equivalent to the following two equations:

\[
\begin{align*}
\frac{\partial u}{\partial \rho} &= \Delta u - uv^2 + A(1 - u), \\
\frac{\partial v}{\partial \rho} &= \Delta v + uv^2 - Bv.
\end{align*}
\]

(2)

(3)

The left-hand side of the above equations represents the change in concentration of \( \mathcal{U} \) (upper equation) and the concentration of \( \mathcal{V} \) (lower equation) over time. Moreover, \( \Delta u \) and \( \Delta v \) represent the Laplacian operator on 1-D. The second term in both equations (the concentration of \( \mathcal{U} \) times the square of the concentration of \( \mathcal{V} \)) represents the reaction term. As shown by the minus \( uv^2 \) in \( u \) (upper equation) and the positive \( uv^2 \) in \( v \) (lower equation), the decrease in \( u \) equals the increase in \( v \). This term shows that \( \mathcal{U} \) is converted to \( \mathcal{V} \). As a result, this amount \( uv^2 \) is subtracted from the first equation and added to the second equation. The third term in the upper equation represents the replenishment term, while the third term in the lower equation represents the diminished term. The chemical \( \mathcal{U} \) is added to a given rate (+A), scaled by \((1 - u)\), so \( u \) does not exceed 1. On the other hand, the chemical \( \mathcal{V} \) is removed to a given removal rate (−B), scaled by the concentration of \( \mathcal{V} \), so \( v \) does not go below zero.

As a result, 1 would be the maximum value for \( u \), and 0 would be the minimum value of \( v \). In the context of this model, \( A < B \). The Gray-Scott model’s parameters and functions (2) and (3) are given in Table 1.

In this study, we extend the classical GS model to the following time-fractional Gray-Scott model (TFGSM) of the orders \( \delta \) and \( \eta \). Let \( u(\xi, \rho) = u \) and \( v(\xi, \rho) = v \); then

\[
\begin{align*}
\left( \frac{D_0^\delta}{} \right) u &= \Delta u - uv^2 + A(1 - u), 0 < \delta \leq 1, \\
\left( \frac{D_0^\delta}{\eta} \right) v &= \Delta v + uv^2 - Bv, 0 < \eta \leq 1,
\end{align*}
\]

(4)

(5)

with initial conditions

\[
\begin{align*}
u(0, \xi, 0) &= u_0(\xi, 0), \\
v(0, \xi, 0) &= v_0(\xi, 0),
\end{align*}
\]

(6)

and homogeneous Neumann boundary conditions, where \((\xi, \rho) \in [0, \rho] \times [0, L], \rho > 0, L > 0, \) and the operators \( \left( \frac{D_0^\delta}{\eta} \right) u \) can be of type Liouville-Caputo \( \left( \frac{D_0^\delta}{\eta} \right) u \), Caputo-Fabrizio-Caputo \( \left( \frac{D_0^\delta}{\eta} \right) u \), and Atangana-Baleanu-Caputo \( \left( \frac{D_0^\delta}{\eta} \right) u \) time-fractional derivatives with orders \( \delta \) and \( \eta \).

2.2. Fractional Calculus. The Liouville-Caputo fractional derivative [22], the Caputo-Fabrizio fractional derivative [23], and the Atangana-Baleanu fractional derivative [26] in the Caputo sense are defined, respectively, as

\[
\begin{align*}
\left( \frac{D_0^\delta}{\eta} \right) u(\xi, \tau) &= \frac{1}{\Gamma(1 - \gamma)} \int_0^\tau (\tau - \rho)^{\gamma - 1} \frac{\partial u}{\partial \rho} d\rho, 0 < \gamma < 1, \\
\left( \frac{D_0^\gamma}{\eta} \right) u(\xi, \tau) &= \frac{F(y)}{\Gamma(1 - \gamma)} \int_0^\tau \exp\left(-\frac{y}{1 - y} (\tau - \rho)\right) \frac{\partial u}{\partial \rho} d\rho, 0 < \gamma < 1, \\
\left( \frac{D_0^\delta}{\eta} \right) u(\xi, \tau) &= \frac{F(y)}{\Gamma(1 - \gamma)} \int_0^\tau \frac{\partial u}{\partial \rho} d\rho, 0 < \gamma < 1,
\end{align*}
\]

(7)
where $\rho > 0$ and $F(y) > 0$ is a normalization function satisfying

$$F(y) = (1 - y) + \frac{y}{\Gamma(y)},$$

where $F(0) = F(1) = 1$ and $E_\gamma(.)$ denotes the Mittag-Leffler function, defined by

$$E_\gamma(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\gamma k + 1)}.$$  

The Liouville-Caputo fractional integral [22], the Caputo-Fabrizio fractional integral [39], and the Atangana-Baleanu fractional integral [40] in the Caputo sense are given, respectively, as follows:

$$(^{LC}\cal{D}_r u)(\xi, r) = \frac{1}{\Gamma(\gamma)} \int_0^r (r - \rho)^{r-1} u(\xi, \rho) d\rho, \quad 0 < \gamma,$$

$$(^{CFC}\cal{D}_r u)(\xi, r) = \frac{2(1 - \gamma)}{(2 - \gamma)F(\gamma)} u(\xi, r) + \frac{2\gamma}{(2 - \gamma)F(\gamma)} \int_0^r u(\xi, \rho) d\rho, \quad 0 < \gamma < 1,$$

$$(^{ABC}\cal{D}_r u)(\xi, r) = \frac{1 - \gamma}{F(\gamma)} u(\xi, r) + \frac{\gamma}{F(\gamma)\Gamma(\gamma)} \int_0^r u(\xi, \rho) \cdot (r - \rho)^{r-1} d\rho, \quad 0 < \gamma < 1.$$  

(10)

Here, when $\gamma$ equals zero, the initial function is recovered, and when $\gamma$ equals unity, the classical ordinary integral is obtained.

The Laplace transformation of the Liouville-Caputo fractional derivative [22], the Caputo-Fabrizio fractional derivative [23], and the Atangana-Baleanu fractional derivative [26] in the Caputo sense are given, respectively, as follows:

$$\mathcal{L}\left\{^{LC}\cal{D}_r^m u\right\}(\xi, s) = s^m \mathcal{L}\left\{u\right\}(\xi, s) - \sum_{k=0}^{m-1} u^{(k)}(\xi, 0 + )s^{r-k-1},$$

$$\mathcal{L}\left\{^{CFC}\cal{D}_r^m u\right\}(\xi, s) = \frac{F(\gamma)}{s^m} \mathcal{L}\left\{u\right\}(\xi, s) - \frac{u^{(k)}(\xi, 0 + )}{s} + \frac{m}{s^{1+1}},$$

$$\mathcal{L}\left\{^{ABC}\cal{D}_r^m u\right\}(\xi, s) = \frac{F(\gamma)}{s^m} \mathcal{L}\left\{u\right\}(\xi, s) - \frac{u^{(k)}(\xi, 0 + )}{s} + \frac{m}{s^{1+1}}.$$  

(11)

2.3. Homotopy Series. The following properties can be found in [41]. Let $\varphi_1$ and $\varphi_2$ be a homotopy series of a homotopy parameter $q$ given by

$$\varphi_1 = \sum_{t=0}^{\infty} u_t q^t,$$

$$\varphi_2 = \sum_{t=0}^{\infty} v_t q^t.$$  

Then, the $n$th-order homotopy derivative is given as

$$D_n(\varphi_1) = \left. \frac{\partial^n \varphi_1}{\partial q^n} \right|_{q=0},$$

which holds the following:

$$D_n(\varphi_1) = \frac{u_n}{},$$

(14)

(a) $D_n(q^n \varphi_2) = \sum_{k=0}^{n} D_n(q^k \varphi_2) = \sum_{k=0}^{n} D_n(q^k) D_n(q^{n-k}(\varphi_2^n)), \quad \text{where} \quad n \geq 0, m \geq 0, l \geq 0, \quad \text{and} \quad 0 \leq k \leq n \text{ are integers}$

(b) If $\mathcal{L}$ is a linear operator independent of the auxiliary parameter $q$, then for homotopy series, (12) holds

$$D_n(\mathcal{L}\varphi_1) = \mathcal{L} D_n(\varphi_1).$$

(c) If $\mathcal{G}$ and $\mathcal{F}$ are functions independent of the auxiliary parameter $q$, then for homotopy series, (12) holds

$$D_n(\mathcal{G}\varphi_1 \pm \mathcal{F}\varphi_2) = \mathcal{G} D_n(\varphi_1) \pm \mathcal{F} D_n(\varphi_2).$$

3. Homotopy and Laplace Transform for FHTAM

Applying the Laplace transformation on Equations (4) and (5), using the Laplace transformation formula of LC, CFC, and ABC, and then simplifying these equations, we obtain

$$\mathcal{L}\left\{(u(\xi, \rho))\right\}(s) = \frac{u(\xi, 0)}{s} + \frac{A}{s} Y_{1,s}(\cdot) Y_{1,s}(\mathcal{L}\left\{(u(\xi, \rho))\right\}_{\xi} + A(1 - u(\xi, \rho) - u(\xi, \rho) v(\xi, \rho))(s),$$

$$\mathcal{L}\left\{(v(\xi, \rho))\right\}(s) = \frac{v(\xi, 0)}{s} - Y_{1,s}(\cdot) Y_{1,s}(\mathcal{L}\left\{(v(\xi, \rho))\right\}_{\xi} - B v(\xi, \rho) + u(\xi, \rho) v(\xi, \rho))(s),$$

(15)

where $Y_{1,s}(\cdot)$ and $Y_{1,s}(\cdot)$ are defined in Table 2. It is difficult to evaluate the Laplace transformation of unknown solutions $u$ and $v$ specifically when combined in a nonlinear form.

We define the homotopy maps as follows:

$$\mathcal{H}_{(u, v)}(\tilde{u}(\xi, \rho; q), \tilde{v}(\xi, \rho; q)) = (1 - q) \mathcal{L}[u(\xi, \rho; q) - u(\xi, \rho)](s) - q h u N_u \tilde{u}(\xi, \rho; q), \tilde{v}(\xi, \rho; q)],$$

(16)
Table 2: Values of $Y_{s,\delta,\eta}(\cdot)$, $s = 1, \cdots, 4$.

|       | LC       | CFC                                      | ABC                                      |
|-------|----------|------------------------------------------|------------------------------------------|
| $Y_{1,\delta}(\cdot)$ | $\frac{1}{s^\delta}$ | $\frac{\delta + (1 - \delta)s}{sF(\delta)}$ | $\frac{\delta + (1 - \delta)s^\delta}{s^\delta F(\delta)}$ |
| $Y_{1,\eta}(\cdot)$  | $\frac{1}{s^\eta}$ | $\frac{\eta + (1 - \eta)s}{sF(\eta)}$ | $\frac{\eta + (1 - \eta)s^\eta}{s^\eta F(\eta)}$ |
| $Y_{2,\delta}(\cdot)$ | $\frac{\rho^\delta}{\Gamma(\delta + 1)}$ | $\frac{1}{F(\delta)} \left( (1 - \delta) + \delta \rho \right)$ | $\frac{1}{F(\delta)} \left( (1 - \delta) + \frac{\delta \rho^\delta}{\Gamma(\delta + 1)} \right)$ |
| $Y_{2,\eta}(\cdot)$  | $\frac{\rho^\eta}{\Gamma(\eta + 1)}$ | $\frac{1}{F(\eta)} \left( (1 - \eta) + \eta \rho \right)$ | $\frac{1}{F(\eta)} \left( (1 - \eta) + \frac{\eta \rho^\eta}{\Gamma(\eta + 1)} \right)$)
| $Y_{3,\delta}(\cdot)$ | $\frac{\rho^\delta}{\Gamma(2\delta + 1)}$ | $\left( \frac{1}{F(\delta)} \right)^2 \left( (1 - \delta)^2 + 2(1 - \delta)\delta \rho + \frac{(\delta \rho)^2}{2} \right)$ | $\left( \frac{1}{F(\delta)} \right)^2 \left( (1 - \delta)^2 + \frac{2(1 - \delta)\delta \rho^\delta}{\Gamma(1 + \delta)} + \frac{(\delta \rho^\delta)^2}{\Gamma(2\delta + 1)} \right)$ |
| $Y_{3,\eta}(\cdot)$  | $\frac{\rho^\eta}{\Gamma(2\eta + 1)}$ | $\left( \frac{1}{F(\eta)} \right)^2 \left( (1 - \eta)^2 + 2(1 - \eta)\eta \rho + \frac{(\eta \rho)^2}{2} \right)$ | $\left( \frac{1}{F(\eta)} \right)^2 \left( (1 - \eta)^2 + \frac{2(1 - \eta)\eta \rho^\eta}{\Gamma(1 + \eta)} + \frac{(\eta \rho^\eta)^2}{\Gamma(2\eta + 1)} \right)$ |
| $Y_{4}(\cdot)$       | $\frac{\rho^{\delta \eta}}{\Gamma(\delta \eta + 1)}$ | $\left( \frac{1}{F(\delta) F(\eta)} \right) \left( (1 - \delta)(1 - \eta) + (1 - \eta)\delta \rho + (1 - \delta)\eta \rho^2 + \frac{\delta \eta \rho^2}{2} \right)$ | $\left( \frac{1}{F(\delta) F(\eta)} \right) \left( (1 - \delta)(1 - \eta) + \frac{(1 - \eta)\delta \rho^\delta}{\Gamma(1 + \delta)} + \frac{(1 - \eta)\eta \rho^\eta}{\Gamma(1 + \eta)} + \frac{\delta \eta \rho^{\delta + \eta}}{\Gamma(\delta + \eta + 1)} \right)$ |
\[
\mathcal{H}_v(\bar{u}(\xi, \rho; q), \bar{v}(\xi, \rho; q)) = (1 - q) \mathcal{L}[\bar{v}(\xi, \rho; q) - v_0(\xi, \rho)](s) - q h_v N_v[\bar{u}(\xi, \rho; q), \bar{v}(\xi, \rho; q)],
\]

where

\[
N_v[\bar{u}(\xi, \rho; q), \bar{v}(\xi, \rho; q)] = \mathcal{L}[\bar{u}(\xi, \rho; q)](s) - \frac{1}{s} (\bar{u}(\xi, 0) + A Y_1(\cdot)) + Y_1(\cdot) \mathcal{L}[\bar{u}(\xi, \rho)](s) + A (1 - \bar{u}(\xi, \rho)) - \bar{u}(\xi, \rho) v_0^2(\xi, \rho) (s).
\]

Expanding \(\bar{u}(\xi, \rho; q)\) and \(\bar{v}(\xi, \rho; q)\) by the Taylor series with respect to the embedding parameter \(q\), we obtain

\[
\bar{u}(\xi, \rho; q) = u_0(\xi, \rho) + \sum_{m=1}^{\infty} u_m(\rho) q^m,
\]

\[
\bar{v}(\xi, \rho; q) = v_0(\xi, \rho) + \sum_{m=1}^{\infty} v_m(\rho) q^m,
\]

subject to the initial conditions

\[
\bar{u}(\xi, \rho; 0) = u_0(\xi, \rho),
\]

\[
\bar{v}(\xi, \rho; 0) = v_0(\xi, \rho).
\]

There are three cases of solutions depending on the parameter \(q \in [0, 1]\):

(a) If \(q = 0\) (we are on the linear operator), where

\[
\bar{u}(\xi, \rho; 0) = u_0(\xi, \rho),
\]

\[
\bar{v}(\xi, \rho; 0) = v_0(\xi, \rho)
\]

(b) If \(q = 1\) (we are on the nonlinear operator), where

\[
\bar{u}(\xi, \rho; 1) = u(\xi, \rho),
\]

\[
\bar{v}(\xi, \rho; 1) = v(\xi, \rho)
\]

(c) If \(q\) varies from zero to one, the solution of the Equations (4) and (5) vary from the initial guesses \(u_0(\xi, \rho)\) and \(v_0(\xi, \rho)\) to the exact solutions \(u(\xi, \rho)\) and \(v(\xi, \rho)\).

Expanding \(\bar{u}(\xi, \rho; q)\) and \(\bar{v}(\xi, \rho; q)\) by the Taylor series with respect to the embedding parameter \(q\), we obtain

\[
\bar{u}(\xi, \rho; q) = u_0(\xi, \rho) + \sum_{m=1}^{\infty} u_m(\rho) q^m,
\]

\[
\bar{v}(\xi, \rho; q) = v_0(\xi, \rho) + \sum_{m=1}^{\infty} v_m(\rho) q^m,
\]

where

\[
u_m(\rho) = \frac{1}{m!} \frac{\partial^m \bar{u}(\xi, \rho; q)}{\partial q^m} \bigg|_{q=0},
\]

\[
u_m(\rho) = \frac{1}{m!} \frac{\partial^m \bar{v}(\xi, \rho; q)}{\partial q^m} \bigg|_{q=0}.
\]

If \(u_0(\xi, \rho), v_0(\xi, \rho)\), the auxiliary parameter \(h_v\), and the auxiliary linear operator \(L\) are properly chosen, then according to [36], the series (25) and (26) converges at \(q = 1\), and we have

\[
n(\xi, \rho; 1) = u_0(\xi, \rho) + \sum_{m=1}^{\infty} u_m(\rho) \ i.e. u(\xi, \rho)
\]

\[
n(\xi, \rho; 1) = v_0(\xi, \rho) + \sum_{m=1}^{\infty} v_m(\rho) \ i.e. v(\xi, \rho)
\]

which must be one of the solutions of Equations (4) and (5).

Let us define the vectors that deduce the \(m\)th-order deformation equations from the zeroth-deformation Equations (20) and (21), given as follows:

\[
n_m(\xi, \rho) = \{u_m(\xi, \rho), v_m(\xi, \rho)\}, \quad m = 1, 2, \cdots, n,
\]

\[
n_m(\xi, \rho) = \{u_m(\xi, \rho), v_m(\xi, \rho)\}, \quad m = 1, 2, \cdots, n.
\]
Upon differentiating the zeroth-deformation in Equations (20) and (21) \(m\) times with respect to the embedding parameter \(q\), setting \(q = 0\), and finally dividing them by \(m!\), we have the so-called \(m\)th-order deformation equations as follows:

\[
\mathcal{L}[u_m(\xi, \rho) - \chi_m u_{m-1}(\xi, \rho)] = h_u R_{m,u} \left( \bar{u}_{m-1}, \bar{v}_{m-1} \right), \quad m = 1, 2, \ldots, n, \tag{31}
\]

\[
\mathcal{L}[v_m(\xi, \rho) - \chi_m v_{m-1}(\xi, \rho)] = h_v R_{m,v} \left( \bar{u}_{m-1}, \bar{v}_{m-1} \right), \quad m = 1, 2, \ldots, n. \tag{32}
\]

Applying the inverse Laplace transform to Equations (31) and (32), we obtain

\[
u_m(\xi, \rho) = \chi_m v_{m-1}(\xi, \rho) + h_v \mathcal{L}^{-1} \left( R_{m,v} \left( \bar{u}_{m-1}, \bar{v}_{m-1} \right) \right), \quad m = 1, 2, \ldots, n, \tag{33}
\]

Here,

\[
R_{m} \left( \bar{u}_{m-1}, \xi, \rho \right) = \mathcal{L}[u_{m-1}(\xi, \rho)] - \left( 1 - \chi_m \right) \frac{1}{s} (u_0 + A Y_{1,0}(\cdot)) + Y_{1,0}(\cdot) \mathcal{L} \left( (u_{m-1})_\xi + A u_{m-1} \right) - \sum_{i=0}^{m-1} u_{m-1-i}(\xi, \rho) \sum_{j=0}^{i} v_j(\xi, \rho) v_{i-j}(\xi, \rho) \cdot (s) \text{red}, \tag{34}
\]

\[
R_{m} \left( \bar{v}_{m-1}, \xi, \rho \right) = \mathcal{L}[v_{m-1}(\xi, \rho)] \left( s \right) - \left( 1 - \chi_m \right) \left( \frac{v_0}{s} \right) + Y_{1,0}(\cdot) \mathcal{L} \left( (v_{m-1})_\xi - B v_{m-1} \right) + \sum_{i=0}^{m-1} u_{m-1-i}(\xi, \rho) \sum_{j=0}^{i} v_j(\xi, \rho) v_{i-j}(\xi, \rho) \cdot (s) \text{red}, \tag{35}
\]

where the superscript (.) is replaced by (LC), (CFC), and (ABC).

Consider the initial guesses \(u_0(\xi, \rho) = u(\xi, 0)\) and \(v_0(\xi, \rho) = v(\xi, 0)\); then using Equations (35) and (36), the first two terms are given as

\[
u_1(\xi, \rho) = -h_v Y_{1,0}(\cdot) \left( (u_0)_\xi - u_0 v_0^2 + A(1 - u_0) \right), \tag{36}
\]

\[
u_1(\xi, \rho) = -h_v Y_{1,0}(\cdot) \left( (v_0)_\xi + u_0 v_0^2 - B v_0 \right), \tag{37}
\]

\[
u_2(\xi, \rho) = (1 + h_v) u_1(\xi, \rho) - h_v^2 Y_{1,0}(\cdot) \left( (u_0)_\xi - u_0 v_0^2 + A(1 - u_0) \right) \left( v_0^2 + A \right) + \left( (u_0)_\xi - u_0 v_0^2 + A(1 - u_0) \right) \left( v_0^2 + A \right) - 2 h_v^2 Y_{1,0}(\cdot) \left( (u_0)_\xi - u_0 v_0^2 + A(1 - u_0) \right) \left( v_0^2 + A \right)
\]

\[
u_2(\xi, \rho) = (1 + h_v) v_1(\xi, \rho) + h_v^2 Y_{1,0}(\cdot) \left( (v_0)_\xi + u_0 v_0^2 - B v_0 \right) \left( 2 u_0 v_0 - B \right) + \left( (v_0)_\xi + u_0 v_0^2 - B v_0 \right) \left( 2 u_0 v_0 - B \right) + h_v^2 Y_{1,0}(\cdot) \left( v_0^2 + A(1 - u_0) - u_0 v_0^2, \right)
\]

where \(Y_{1,0}(\cdot), s = 2, 3, 4\) is defined in Table 2.
Figure 1: The $h_{u,v}$ curves obtained from the 3rd order of the FHATM solutions using the ABC, CFC, and LC when $\delta$ and $\eta$ tend to 1 and $\xi = 10$.

Table 4: List of variables and parameters values.

| Figure | $\rho$ | $\xi$ | $\delta, \eta$ | $h_{u,v}$ | $A$ | $B$ | $L$ | $L_1$ | $M$ | $N$ |
|--------|--------|-------|----------------|------------|-----|-----|-----|-------|-----|-----|
| 1      | 0.0001 | 10    | 0.99           | $h_{u,v} \in (-3.4,1.4)$ | 0.125 | 0.125 | 10  | 10    | –   | –   |
| 2      | $\frac{kL_1}{N}$ | $\frac{sL}{M}$ | 0.99           | $h_{u,v} \in (-0.8,0.2)$ | 0.125 | 0.125 | 10  | 10    | 10  | 10  |
| 3      | 5      | $\xi \in (-20,20)$ | Varies         | $h_{u,v}^*$ (optimal value of $h_{u,v}$) | 0.125 | 0.125 | –   | –     | –   | –   |
| 5      | $\frac{kL_1}{N}$ | $\frac{sL}{M}$ | 0.99           | Varies     | 0.125 | 0.125 | 10  | 10    | 10  | 10  |
| 4      | $\rho \in (0,80)$ | $\xi \in (0,80)$ | 0.99           | $-0.25$    | 0.125 | 0.125 | 100 | –     | –   | –   |
Table 5: Regions of convergence, optimal values of $h_{u,v}$, and minimum values.

| Operator | $h_u$ (optimal value of $h_u$) | Minimum value of $E_u(h_u)$ | $h_v$ (optimal value of $h_v$) | Minimum value of $E_v(h_v)$ |
|----------|--------------------------------|----------------------------|--------------------------------|----------------------------|
| $u(\xi, \eta)$ | $-1.9 \leq h_u \leq -0.19$ | -0.597282 | 0.0000803049 |
| ABC | $-1.5 \leq h_v \leq 0.15$ | -0.473649 | 0.000232817 |
| LC | $-1.9 \leq h_u \leq -0.19$ | -0.596935 | 0.0000223021 |

$\nu(\xi, \eta)$

| ABC | $-1.9 \leq h_v \leq -0.19$ | -0.597282 | 0.0000803049 |
| CFC | $-1.5 \leq h_v \leq 0.15$ | -0.473649 | 0.000232817 |
| LC | $-1.9 \leq h_v \leq -0.19$ | -0.596935 | 0.0000223021 |

Figure 2: The square residual function Equations (38) and (39) using the third-order approximation solution of the FHATM solutions using the ABC, CFC, and LC.
Figure 3: Different values of $\delta$ and $\eta$ using the ABC, CFC, and LC.
Table 6: The absolute error of \( u(\xi, \rho) \) and \( v(\xi, \rho) \).

| \( \xi \) | \( \rho \) | ABC | CFC | LC |
|--------|--------|-----|-----|----|
| 0      | 10     | 2.88537 \times 10^{-6} | 2.88537 \times 10^{-6} | 0  |
| 0      | 20     | 1.04222 \times 10^{-6} | 1.069 \times 10^{-6} | 1.04725 \times 10^{-6} |
| 20     | 20     | 5.01874 \times 10^{-12} | 5.86325 \times 10^{-12} | 5.14655 \times 10^{-12} |
| 60     | 60     | 2.9643 \times 10^{-14} | 3.60822 \times 10^{-14} | 1.29896 \times 10^{-14} |

4. Numerical Results

To demonstrate the efficiency of the FHATM for solving the time-fractional Gray-Scott equation, we present the solution in figures and tables for several values of fractional derivatives.

According to [5], we take the initial conditions as

\[
\begin{align*}
  u(\xi, 0) &= \frac{3 - \sqrt{1}}{4} - \frac{\sqrt{2\rho}}{4} \tanh \left( \frac{\sqrt{\rho}}{4} \xi \right), \\
  v(\xi, 0) &= \frac{1 + \sqrt{1}}{4} + \frac{\sqrt{2\rho}}{4} \tanh \left( \frac{\sqrt{\rho}}{4} \xi \right),
\end{align*}
\]

and the exact solution of Equations (2) and (3) is given by

\[
\begin{align*}
  u(\xi, \rho) &= \frac{3 - \sqrt{1}}{4} - \frac{\sqrt{2\rho}}{4} \tanh \left( \frac{\sqrt{\rho}}{4} (\xi - \rho) \right), \\
  v(\xi, \rho) &= \frac{1 + \sqrt{1}}{4} + \frac{\sqrt{2\rho}}{4} \tanh \left( \frac{\sqrt{\rho}}{4} (\xi - \rho) \right).
\end{align*}
\]

We evaluated the intervals of convergence for the LC, CFC, and ABC by finding \( h_{u,v} \) curves, and the averaged residual error. Furthermore, we tested the accuracy of the results obtained by employing FHATM by comparing it with the exact solution. Figure 1 shows the \( u'(10, 0) \) and \( v'(10, 0) \) against \( h_{u,v} \) taking the values in Table 4. We plot \( h_{u,v} \) curves of the third terms of the FHATM solution for the fractional-time LC, CFC, and ABC equations (4) and (5) with the aim to observe the intervals of convergence. From this figure, we note that the straight line parallel with the \( h_{u,v} \) -axis provides the region of convergence according to [36]. These valid regions are listed in Table 5. We notice that \( h_{u,v} \) curves do not give the optimal value of the auxiliary parameter \( h_{u,v} \) that can make Equations (28) and (29) converge fast. So, according to [42], we compute the optimal values of parameter \( h_{u,v} \) from the minimum of the average residual errors. The square residual is defined as

\[
SE_{u,v}(h_{u,v}) = \frac{1}{(N+1)(M+1)} \sum_{k=0}^{N} \sum_{i=0}^{M} \left( \frac{10k}{N}, \frac{10i}{M} \right) \left[ \frac{u_{i}^{N+1}(x)}{N}, \frac{10i}{M} \right]^{2}.
\]

Figure 2 and Table 5 show the average residual error for the LC, CFC, and ABC operators. These show \( SE_{u,v}(h_{u,v}) \) and \( SE_{u,v}(h_{u,v}) \) for 3 terms obtained using FHATM. We set into Equations (45) and (46) the parameter values given in Table 4. Using the command “Minimize” in Mathematica, we plotted the residual error against \( h_{u,v} \) to get the optimal values \( h_{u,v}^{\ast} \). From Table 5, it is seen that the FHATM for LC, CFC, and ABC operators converges rapidly. Note that only three iterations are considered here. Therefore, the accuracy of the results can be improved by considering more terms, where the error converges to zero.

Figure 3 and Table 6 show the comparison of 3 terms in the FHATM solution for the LC, CFC, and ABC operators with the exact solution in Equations (43) and (44). Table 6 presents the absolute error of the FHATM solution using
parameter values given in Table 4. We noted from this table that the FHATM solution for LC, CFC, and ABC operators is in excellent agreement with the exact solutions. Moreover, Figure 3 shows the comparison between the exact solution and approximate solution obtained by 3 terms of FHATM for the LC, CFC, and ABC operators for parameter values listed in Table 4. We observe from Figure 3 that the solution obtained by FHATM increases rapidly to the exact solution following the increase in δ and η. Those tables and figures demonstrate the efficacy of the presented algorithm for solving the time-fractional Gray-Scott equation.

5. Conclusion

In this paper, the Gray-Scott equation was extended to the time-fractional Gray-Scott equation of Liouville-Caputo (LC), Caputo-Fabrizio-Caputo (CFC), and Atangana-Baleanu-Caputo (ABC) type. The fractional homotopy analysis transform technique is used to derive analytic solutions for TFGSE. This method gives the solutions in a series form that converges rapidly in nonlinear time-fractional GS equation. The interval of the convergence by \( h_{\nu, \psi} \) curves in Figure 1 and the optimal value of \( h_{\nu, \psi} \) were found by least square error as given Figure 2. Also, the solutions obtained were compared with the exact solution, which were in excellent agreement. The effect of the fractional derivative on the concentration of \( U \) increases when δ decreases while the concentration of \( V \) is decreases. Moreover, the results obtained using FHATM agree well with the numerical result presented in [5], and the absolute error less than \( 3 \times 10^{-6} \) as given in Table 6. In conclusion, the FHATM method is a powerful method to handle fractional operators of LC, CFC, and ABC type, generating highly accurate data.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors’ Contributions

All authors equally contributed to this work. All authors read and approved the final manuscript.

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