STOCHASTIC RELATIVISTIC SHOCK-SURFING ACCELERATION

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ABSTRACT

We study relativistic particles undergoing surfing acceleration at perpendicular shocks. We assume that particles undergo diffusion in the component of momentum perpendicular to the shock plane as a result of moderate fluctuations in the shock electric and magnetic fields. We show that \( dN/dE \), the number of surfing-accelerated particles per unit energy, attains a power-law form, \( dN/dE \propto E^{-b} \). We calculate \( b \) analytically in the limit of weak momentum diffusion, and use Monte Carlo test-particle calculations to evaluate \( b \) in the weak, moderate, and strong momentum-diffusion limits.

Subject headings: acceleration of particles — MHD

1. INTRODUCTION

The acceleration of high-energy particles is an important problem in astrophysics. This paper examines one acceleration mechanism, “surfing acceleration,” as it applies to relativistic particles at perpendicular shocks. An illustration of an ion undergoing surfing acceleration at a perpendicular shock is given in Figure 1. In this picture an ion of velocity \( \mathbf{v} \) arrives just upstream of the shock with small \( |v_y| \) and is reflected by the jump in the electrostatic potential \( \Phi(x) \) at the shock, which is illustrated schematically in Figure 2. The Lorentz force then causes the ion to return to the shock, where it is again reflected and brought back to the shock front by the Lorentz force. This process continues, confining the particle to the vicinity of the shock, where it is accelerated in the \(-\hat{y}\) direction by the motional electric field, \(-\mathbf{u} \times \mathbf{B}/c\), where \( \mathbf{u} \) is the plasma velocity.

If one thinks of \( q \Phi(x) \) in Figure 2 as analogous to the profile of an ocean wave, the accelerating ion in Figure 1 is analogous to a surfer. In this paper, we focus on ions. However, shock-surfing acceleration is believed to be important for electrons as well (McClements et al. 2001; Hoshino & Shimada 2002), and it is trivial to modify the analysis of §2 and 3 to treat the electron case.

An ion’s \( x \)-oscillations in front of the shock can be approximately described in terms of an effective potential energy (Lee et al. 1996; Ucer & Shapiro 2001), which can be derived as follows. We start with the \( x \)-component of the equation of motion,

\[
\frac{d}{dt} (\gamma v_x) = q \left( E_x + \frac{v_y B_z - v_z B_y}{c} \right),
\]

where \( \gamma \) is the relativistic Lorentz factor, and we work in shock wave frame. We assume that \( E_x = -d\Phi/dx \), with \( \Phi \) a function of \( x \) alone, that \( |v_y B_z| \ll |v_z B_y| \), that \( |v_x| \ll |v_y| \), and that the particle oscillates in \( x \) on a timescale that is much shorter than the time required for \( \gamma \) or \( v_y \) to change appreciably. We also treat \( B_x \) as spatially uniform. We then multiply equation (1) by \( v_x \) and integrate in time, treating \( \gamma \) and \( v_y \) as constant, to obtain

\[
\frac{m \gamma v_x^2}{2} + U_{\text{eff}}(x) = \text{constant} \equiv w + U_{\text{min}},
\]

where

\[
U_{\text{eff}}(x) = q \left[ \Phi(x) - \frac{x v_y B_z}{c} \right],
\]

\( w \) is the energy of \( x \)-oscillations, and \( U_{\text{min}} \) is the value of \( U_{\text{eff}} \) at its local minimum at \( x = x_{\text{min}} \). We define

\[
w_{\text{max}} = U_{\text{max}} - U_{\text{min}},
\]

where \( U_{\text{max}} \) is the value of \( U_{\text{eff}}(x) \) at its local maximum. Surfing ions, which are trapped in the effective potential well, have \( 0 < w < w_{\text{max}} \). The effective potential energy is plotted in Figure 2 for \( v_y < 0 \). The turning points of an ion of given \( w \), obtained by solving equation (2) with \( v_y = 0 \), are denoted \( x_L(w) \) and \( x_R(w) \). Over many bounces, \( \gamma \) and \( v_y \) vary slowly, causing \( w \) to change in a way that can be calculated by noting that the adiabatic invariant

\[
J = \int_{x_L(w)}^{x_R(w)} p_x \, dx
\]

is approximately conserved (Ucer & Shapiro 2001).

Surfing can only occur if there is a local minimum in \( U_{\text{eff}} \), which requires that

\[
|E_x| > \frac{v_y B_z}{c}
\]

over some interval of \( x \) (see, e.g., Sagdeev & Shapiro 1973; Katsouleas & Dawson 1983; Ucer & Shapiro 2001). In order for particles with \( v_y \sim c \) to surf, \( |E_x| \) must locally exceed \( B_z \), a condition that may arise at large Alfvénic Mach number (Hoshino & Shimada 2002).
The escape of a surfing particle from the effective potential well determines how far the particle can travel along the \(-\hat{y}\) direction, which in turn determines the amount of energy the particle gains. Several escape mechanisms have been considered in the literature. First, if the maximum value of \(E_x\), denoted \(E_{x,\text{max}}\), is less than \(B_z\) (assumed constant) and a surfing particle’s \(\nu_y\) increases to the point that \(\nu_y > c E_{x,\text{max}}/B_z\), then the local minimum of \(U_{\text{eff}}\) disappears and the particle escapes downstream (Sagdeev & Shapiro 1973). Second, as a nonrelativistic particle’s \(\nu_y\) increases, the slope of \(U_{\text{eff}}\) at large \(x\) increases, as illustrated in Figure 2, causing the width \(\Delta x\) of a particle’s bounce oscillations to decrease. Since \(J \sim \Delta x \nu_y\) is approximately conserved, where \(\nu_y\) is a typical value of \(|\nu_y|\), decreasing \(\Delta x\) increases \(\nu_y\), thereby increasing \(w\) (Ucer & Shapiro 2001). If \(w\) increases above \(w_{\text{max}}\), then the particle escapes downstream (Lee et al. 1996; Ucer & Shapiro 2001). Third, in an oblique shock the velocity vector of a surfing particle gradually rotates in the \(y-z\) plane because of the Lorentz force associated with \(B_x\), as depicted in Figure 3, which is an adaptation of Figure 4 of Lee et al. (1996). As this happens, \(|\nu_y|\) is eventually reduced to zero and the particle escapes upstream, since the \(\nu_y B_z/c\) component of the Lorentz force ceases to turn the particle back toward the shock (Lee et al. 1996). Fourth, wave-particle interactions can reduce a surfing particle’s \(\nu_y\) to zero, again allowing the particle to escape upstream (Shapiro et al. 2001). Fifth, as noted by Lee et al. (1996) and Hoshino & Shimada (2002), some shocks are intrinsically nonstationary and can periodically “break,” allowing surfing particles to escape downstream.

For perpendicular shocks and relativistic particles with \(|\nu_y| \sim c\), the first three mechanisms described above are absent. This has led previous authors to consider scenarios of “unlimited acceleration” when \(E_x > B_z\) over some interval in \(x\) (Katsouleas & Dawson 1983; Ucer & Shapiro 2001).

The purpose of this paper is to show that shock surfing of relativistic particles at perpendicular shocks leads to a power-law energy spectrum of accelerated particles, provided fluctuations in the shock electric and magnetic fields are not too large, and to calculate the power-law index of this spectrum. For example, suppose the typical electric and/or magnetic field structures have a length \(l_y\) along the \(\hat{y}\)-direction and a typical amplitude \(\delta E_x\), that an ion moves primarily in the \(-\hat{y}\) direction encounters a series of random forces in the \(x\)-direction (denoted \(\delta F_x\)) as a result of fluctuations in \(E_x\) and \(B_z\), as illustrated in Figure 4. These forces induce a series of random increments in \(p_x\), causing diffusion in \(p_x\).

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direction at a speed \( \sim c \), and that these structures fluctuate on a timescale that is longer than the time

\[
\Delta t \sim \frac{l}{c}
\]

(7)

for a surfing particle to traverse one of the structures. The random increment in \( p_x \) when an ion moves through one structure then has a magnitude

\[
\Delta p_x \sim \Delta t q E_y .
\]

(8)

If the field structures are space filling, the ion undergoes one such momentum kick during each time \( \Delta t \), leading to a coefficient of diffusion in \( p_x \),

\[
D_{p_x} \sim \frac{q^2 \Delta t q E_y^2 l_y}{c},
\]

(9)

that is independent of particle energy. The energy spectrum of accelerated particles can be understood in terms of two competing effects. On the one hand, diffusion in \( p_x \) stochastically de-traps particles by increasing \( w \) above \( w_{\text{max}} \), thereby allowing particles to escape downstream. On the other hand, the gradual increase in \( \gamma \) due to acceleration by \( E_y \) focuses a surfing particle with \( |v_y| \sim c \) into the bottom of the effective potential well, as follows from conservation of \( J \) (Ucer & Shapiro 2001). This combination of factors leads to a power-law energy spectrum of accelerated particles, \( dN/d\gamma \propto \gamma^{-\alpha} \). The value of \( \alpha \) depends on

\[
\alpha = \frac{2D_{p_x} c}{q|E_y| w_{\text{max}}} \sim \frac{t_{\text{acc}}}{t_{\text{diff}}},
\]

(10)

where \( t_{\text{acc}} = mc\gamma/q|E_y| \) is the time to double a surfing particle’s energy and \( t_{\text{diff}} = m\gamma w_{\text{max}}/D_{p_x} \) is the momentum-diffusion timescale. As \( \alpha \) is decreased, the acceleration time is reduced relative to the momentum-diffusion timescale, particles experience larger energy increases before escaping the surfing mechanism, and one expects the energy spectrum \( dN/d\gamma \) to become harder. This expectation is born out by the calculations and simulations to be presented in \( \S \S 2 \) and 3, which show that in the strong-scattering limit (\( \alpha \gg 1 \)) \( b \approx \alpha \), in the moderate-scattering limit (\( \alpha \approx 1 \)) \( b \approx 1-3 \), and in the weak-scattering limit (\( \alpha \ll 1 \)) \( b \approx 1 \).

If, contrary to what we assume, the field fluctuations were so large that the minimum of the effective potential energy disappeared at locations separated by a typical distance \( \Delta y \) along a surfing particle’s trajectory, one would not obtain a simple power-law energy spectrum; the spectrum would instead steepen above an energy \( \sim q|E_y| \Delta y \) corresponding to the energy a particle gains by surfing a distance \( \Delta y \) (Hoshino & Shimada 2002).

As mentioned previously, if a particle is assumed to undergo diffusion in \( p_y \) with diffusion coefficient \( D_{p_y} \), as well as “advection” in \( p_y \) due to \( E_y \), then a particle can in principle diffuse to \( p_y = 0 \) and escape into the upstream region. The condition under which this escape mechanism can be neglected can be understood by considering an analogy to first-order Fermi acceleration at a nonrelativistic shock, as follows.

Suppose a particle is a distance \( x \) downstream of a shock, that the downstream fluid velocity is \( u_x \), and that the particle diffuses in space with diffusion coefficient \( D_x \). The time for a particle to return to the shock in the absence of advection is \( t_{\text{diff}, x} \sim x^2/D_x \), and the time for the particle’s distance from the shock to be doubled by advection in the absence of diffusion is \( t_{\text{adv}} = x/u_x \). The probability that the particle returns to the shock is \( e^{-t_{\text{diff}, x}/t_{\text{adv}}} \). Advection-diffusion problem analogous to the advection and diffusion in \( p_y \) that would occur in shock surfing with nonzero \( D_{p_y} \). The probability that a particle escapes the surfing mechanism by diffusing in momentum space to \( p_y = 0 \) is again \( e^{-t_{\text{diff}, p_y}/t_{\text{adv}, p_y}} \), where \( t_{\text{diff}, p_y} = p_y^2/D_{p_y} \) is the time to diffuse to \( p_y = 0 \) in the absence of advection in \( p_y \), and \( t_{\text{adv}, p_y} = t_{\text{acc}} = mc\gamma/q|E_y| \) is the time for a particle’s \( y \)-momentum to double as a result of \( E_y \). Escape through \( p_y \) diffusion is thus negligible, provided that \( t_{\text{acc}} \ll t_{\text{diff}, p_y} \), or \( D_{p_y} \ll p_y^2|E_y|/mc\gamma \sim q|p_y E_y| \).

We note that it is not clear how efficiently particles are injected into the surfing acceleration mechanism. Injection appears to be natural for pickup ions accelerated by the heliospheric termination shock. In the frame of the solar wind, pickup ions are born with particle speeds comparable to the solar-wind velocity, and thus pickup ions that arrive at the heliospheric termination shock at the right point in their gyromotion have very small \( v_y \), which allows them to be trapped in front of the shock (Lee et al. 1996; Zank et al. 1996; Lever et al. 2001). Surfing acceleration in other cases, however, may be more difficult to initiate. It may be that particles can be injected at the time that a shock is formed if the shock originates from an explosive event. It is also possible that turbulent conditions in the shock transition region can feed particles into the surfing mechanism. Hoshino & Shimada (2002) have found electron surfing in particle-in-cell simulations of perpendicular shocks, demonstrating that injection does occur, at least for electrons. Additional work on injection, however, is needed.
Although our analysis is restricted to the case of perpendicular shocks, some of the results may apply to the more general case of superluminal shocks, for which the point of intersection between a field line and the shock plane moves faster than \( c \). For such shocks, there exists a “perpendicular shock frame” in which the shock is stationary and the upstream electric and magnetic fields lie in the shock plane (Begelman & Kirk 1990). There are, however, two issues that would have to be addressed before our results could be generalized to this case. First, the condition for \( E_x \) to exceed \( B_z \) in more general superluminal shocks must be found. Second, the upstream flow velocity in the perpendicular shock frame has a component along the upstream magnetic field. If particles start to surf with a significant \( v_x \) component along the shock frame has a component along the upstream magnetic field. If particles start to surf with a significant \( v_y \) component along the upstream magnetic field, they would only approach the state that we analyze with \( v \approx c \) after significant acceleration in the \( \hat{y} \) direction. As a particle approaches this state, \( |v_y| \) gradually increases, and the effects of such a gradual increase are not included in our analysis.

The remainder of this paper is organized as follows. In §2 we show analytically that surfing ions that diffuse in \( p_x \) are described by a power-law energy spectrum. We calculate analytically the power-law index for the case of weak momentum diffusion (\( \alpha \ll 1 \)), and derive a rough estimate for the power-law index in the case of strong momentum diffusion (\( \alpha \gg 1 \)). In §3 we present Monte Carlo numerical calculations of surfing in the presence of diffusion in \( p_x \), for the weak, moderate, and strong momentum-diffusion cases. We summarize our results in §4.

2. ANALYTIC CALCULATION OF ENERGY SPECTRUM OF ACCELERATED IONS

We begin with the relativistic Vlasov equation with a term added to model diffusion in \( p_x \),

\[
\frac{\partial f}{\partial t} + v \cdot \nabla f + q \left( E + \frac{v \times B}{c} \right) \cdot \nabla_p f = D_{p_x} \frac{\partial^2 f}{\partial p_x^2},
\]

where \( D_{p_x} \) is taken to be independent of particle energy, as in equation (9). We consider a perpendicular shock and ignore spatial variations in the magnetic field and motional electric field:

\[
B = B_z \hat{z},
\]

\[
B_z = \text{constant} > 0,
\]

and

\[
E_y = \text{constant} < 0.
\]

We also take

\[
E_z = 0,
\]

\[
E_x = -\frac{d\Phi}{dx},
\]

and

\[
e^{-1} \frac{\partial \Phi}{\partial t} = \frac{\partial \Phi}{\partial y} = \frac{\partial \Phi}{\partial z} = 0.
\]

We confine our analysis to relativistic surfing ions, for which we assume

\[
v \approx -c,
\]

\[
|v_y| \ll c,
\]

\[
v_x = 0,
\]

\[
\frac{\partial f}{\partial y} = 0,
\]

\[
\frac{\partial f}{\partial z} = 0,
\]

\[
|v_z B_z| / c \ll |E_y|,
\]

and

\[
\gamma = \frac{|p_y|}{mc}.
\]

Equation (11) can then be rewritten as

\[
\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} + q(E_x - B_z) \frac{\partial f}{\partial p_x} + qE_y \frac{\partial f}{\partial p_y} = D_{p_x} \frac{\partial^2 f}{\partial p_x^2}.
\]

We now change variables, defining

\[
t' = t,
\]

\[
x' = x,
\]

\[
p_{y0} = p_y - qE_y t,
\]

and

\[
w = \frac{m_r v_x^2}{2} + U_{\text{eff}}(x) - U_{\text{min}},
\]

where

\[
U_{\text{eff}}(x) = q(\Phi + xB_z)
\]

is the effective potential energy for the particle’s motion in \( x \), illustrated in Figure 2. \( w \) is the energy associated with this

\[
\frac{\partial f^+}{\partial t'} \pm |v_x| \frac{\partial f^+}{\partial x'} - \frac{q|E_y| v_x}{2c} \frac{\partial f^+}{\frac{\partial w}{\partial w}} = D_{p_x} v_x \frac{\partial}{\partial w} \left( v_x \frac{\partial f^+}{\partial w} \right),
\]

where \( f^+ (f^-) \) is the distribution function for particles with \( v_x > 0 \) (\( v_x < 0 \)).

We assume that the time \( \tau_B \) for a particle to bounce in the effective potential well is much shorter than the acceleration
time or the diffusion in \( p_x \). We thus take

\[
\nu_\parallel \frac{\partial f^\pm}{\partial x} \sim \frac{f^\pm}{\tau_B}, \quad (33)
\]

and

\[
\frac{\partial f^\pm}{\partial t'} \sim \frac{q |E_x| v_x^2 \partial f^\pm}{2c \partial w} \sim D_{p_x} v_x \frac{\partial}{\partial w} \left( v_x \frac{\partial f^\pm}{\partial w} \right) \sim \frac{f^\pm}{\tau_0}, \quad (34)
\]

with

\[
\epsilon = \frac{\tau_B}{\tau_0} \ll 1. \quad (35)
\]

We then set

\[
f^\pm = f_0^\pm + \epsilon f_1^\pm + \ldots \quad (36)
\]

and expand equation (32) in powers of \( \epsilon \). Upon collecting all terms of order \( \epsilon^0 f_0 / \tau_B \) (there is only one) and dividing by \( |v_x| \), we find that

\[
\frac{\partial f_0^\pm}{\partial x} = 0. \quad (37)
\]

As in \( \S \ 1 \), we take \( x_L(w) \) and \( x_R(w) \) to be the bounce points of a particle of “energy” \( w \), with \( x_L < x_R \). Since \( f^+(x_L(w), w) \) and \( f^-(x_R(w), w) \) give the distribution function at the same point in phase space, one has \( f^+(x_L(w), w) = f^-(x_R(w), w) \). Similarly, \( f^+(x_R(w), w) = f^-(x_L(w), w) \). Equation (37) thus implies that

\[
f_0^+ = f_0^- \equiv f_0. \quad (38)
\]

Upon collecting all terms in equation (32) of order \( \epsilon^0 / \tau_B \) and dividing by \( |v_x| \), we find that

\[
\frac{1}{|v_x|} \frac{\partial f_0}{\partial t'} - \frac{q |E_x|}{2c} \frac{\partial f_0}{\partial w} = D_{p_x} \frac{\partial}{\partial w} \left( |v_x| \frac{\partial f_0}{\partial w} \right). \quad (39)
\]

We integrate equation (39) from \( x_L(w) \) to \( x_R(w) \) for \( f^+ \), then integrate equation (39) from \( x_L(w) \) to \( x_R(w) \) for \( f^- \), and then add the two resulting equations to annihilate the terms involving \( f_1^\pm \), thereby obtaining

\[
\frac{\tau_B}{\tau_0} \frac{\partial f_0}{\partial t'} - \frac{q |E_x| g f_0}{c \sqrt{2m\gamma}} \frac{\partial f_0}{\partial w} = D_{p_x} \frac{\partial}{\partial w} \left( q \sqrt{\frac{2}{m\gamma}} f_0 \right), \quad (40)
\]

where

\[
g = \int_{x_L(w)}^{x_R(w)} dx \sqrt{w - U_{\text{eff}}(x) + U_{\min}}, \quad (41)
\]

and

\[
\tau_B = \int_{x_L(w)}^{x_R(w)} \frac{dx}{|v_x|} = \sqrt{\frac{2m\gamma}{d g / dw}}. \quad (42)
\]

We note the behavior of \( g(w) \) near \( w = 0 \) for future reference. Near \( x = x_{\min} \),

\[
U_{\text{eff}}(x) = U_{\min} + a(x - x_{\min})^2 + \beta(x - x_{\min})^3 + \kappa(x - x_{\min})^4 + \ldots . \quad (43)
\]

Using the technique of asymptotic matching, one can show that as \( w \to 0 \), \( g(w) \) has the asymptotic expansion

\[
g(w) = \frac{\pi w}{2\sqrt{\alpha}} \left[ 1 + w \left( \frac{15\beta^2}{32\alpha^2} - \frac{3\kappa}{8\alpha^2} \right) + \ldots \right]. \quad (44)
\]

Since \( \partial g / \partial w \) exists and is integrable, the asymptotic expansion of \( \partial g / \partial w \) near \( w = 0 \) is given by termwise differentiation of equation (44):

\[
\frac{\partial g}{\partial w} = \frac{\pi w}{2\sqrt{\alpha}} \left[ 1 + 2w \left( \frac{15\beta^2}{32\alpha^2} - \frac{3\kappa}{8\alpha^2} \right) + \ldots \right]. \quad (45)
\]

We now recast equation (40) into the standard Fokker-Planck form for the number of particles per unit \( p_{xy} \) per unit

\[
P = \int_{x_L(w)}^{x_R(w)} \left( \frac{f^+ + f^-}{2} \right) f_0 dx \approx 2\tau_B f_0. \quad (46)
\]

We start by noting from equations (23), (25), and (27) that

\[
\gamma = \frac{-p_{x0} - qE_x t'}{mc}, \quad (47)
\]

where both \( p_{x0} \) and \( E_x \) are negative. Equations (42) and (47) give

\[
\frac{\partial \tau_B}{\partial t'} = \frac{\tau_B q |E_x|}{2\gamma mc}. \quad (48)
\]

Next, we define

\[
h = \frac{w}{d g / dw}, \quad (49)
\]

and note that

\[
g \frac{\partial}{\partial w} \left( \frac{P}{2\tau_B} \right) = \frac{1}{2\sqrt{2m\gamma}} \left[ \frac{\partial}{\partial w} \left( \frac{Pw}{h} \right) - P \right]. \quad (50)
\]

Throughout the rest of this section we drop the primes, setting \( t' \to t \). Substituting \( f_0 = P / 2\tau_B \) into equation (40) and making use of equations (48) and (50), we find

\[
\frac{\partial P}{\partial t} = -\frac{\partial \Gamma}{\partial w}, \quad (51)
\]

where

\[
\Gamma = \left( \frac{D_{p_x}}{m\gamma} \frac{q |E_x| w}{2mc\gamma h} P - \frac{\partial}{\partial w} \left( \frac{D_{p_x} Pw}{2mc\gamma h} \right) \right). \quad (52)
\]

is the flux of particles in \( w \)-space. The quantity in the first set of parentheses on the right-hand side of equation (52) is \( \langle \Delta w \rangle / \Delta t \), while the coefficient of \( P \) within the second set of parentheses is \( \langle \Delta w \rangle^2 / 2\Delta t \), where \( \Delta w \) is the change in \( w \) during a small time interval \( \Delta t \), and the angle brackets denote an average over the stochastic scattering that gives rise to diffusion in \( p_x \). The term \(-q |E_x| w / 2mc\gamma h\) is the rate of change of \( w \) caused by the approximate conservation of the adiabatic invariant

\[
J = \int_{x_L(w)}^{x_R(w)} p_x dx = g \sqrt{2m\gamma}. \quad (53)
\]
That is, setting $dJ/dt = 0$ yields $dw/dt = -q|E|w/2mc\gamma\hbar$, which confirms the statement made in the introduction that $J$-conservation drives a relativistic surfering particle toward the bottom of the effective potential well (Picer & Shapiro 2001). The terms on the right-hand side of equation (52) proportional to $D_p$, give the advection and diffusion in $w$-space arising from a time average of a particle’s diffusion in $p_i$.

If diffusion in $p_i$ causes a particle’s $w$ to increase to $w_{\text{max}}$, the particle escapes downstream. We thus seek a solution to equation (51) subject to the boundary condition

$$P(w_{\text{max}}, t) = 0. \quad (54)$$

We also require that $P$ be finite and differentiable at $w = 0$. Since there are no $p_{\gamma,0}$ derivatives in equation (51), we can treat particles with different values of $p_{\gamma,0}$ independently. We henceforth focus on particles with a single value of $p_{\gamma,0}$ and assume that a large number of such particles are injected into the surfeing mechanism at $t = 0$. We rewrite equation (47), dropping the prime on $t'$, to obtain

$$\gamma = \left(\frac{q|E|}{mc}\right)(d + t), \quad (55)$$

where $d = p_{\gamma,0}/|E|$ is the time required to double the particle’s initial $y$-momentum. Because a surfeing particle’s energy increases linearly in time, the number of particles that are accelerated to a final Lorentz factor in the interval $\gamma, \gamma + d\gamma$, denoted $(dN/d\gamma)\,d\gamma$, is equal to the number of particles that escape the effective potential well in the time interval $(t, t + dt)$, where $t = (mc\gamma/q|E|) - d$ and $dt = (mc/q|E|)\,d\gamma$ from equation (55). Thus,

$$\frac{dN}{d\gamma} = \frac{mc\Gamma(w_{\text{max}}, t)}{|q|E|} \left. \right|_{t = (mc\gamma/q|E|) - d}. \quad (56)$$

Using equation (55), we can rewrite equation (51) as

$$(d + t)\frac{\partial P}{\partial t} = \frac{\partial}{\partial u} \left[\left(\frac{u}{2h} - \frac{\alpha}{2}\right)P\right] + \frac{\partial^2}{\partial u^2} \left(\frac{\alpha u P}{2h}\right) = \mathcal{L}(P), \quad (57)$$

where

$$u = \frac{w}{w_{\text{max}}} \quad (58)$$

and

$$\alpha = \frac{2D_p c}{q|E|w_{\text{max}}}. \quad (59)$$

The eigenvalue equation

$$\mathcal{L}(P_n) = -\lambda_n P_n \quad (60)$$

can be written in Sturm-Liouville form. We consider eigenfunctions that are finite and differentiable at $u = 0$ and satisfy $P_n(1) = 0$. \quad (61)

For these boundary conditions, and since $g(0) = 0$, the eigenvalues are real and the eigenfunctions of $\mathcal{L}$ form a complete set on the interval $0 < u < 1$. The solution for $P(u, t)$ can thus be expanded as

$$P(u, t) = \sum_{n=0}^{\infty} \chi_n(t)P_n(u). \quad (62)$$

From equation (57),

$$\chi_n(t) = \chi_n(0) \left(\frac{d}{d + t}\right)^{\lambda_n} \quad (63)$$

where $\chi_n(0)$ is determined from the initial conditions on $P$. We order the eigenvalues and their corresponding eigenfunctions so that $\lambda_0 < \lambda_1 < \lambda_2 < \ldots$, etc. For $t \gg d$, $P(u, t)$ is dominated by the first term in the sum in equation (62),

$$P(u, t) \propto P_0(u)^{-\lambda_0}, \quad (64)$$

and, from equations (52), (56), and (64),

$$\frac{dN}{d\gamma} \propto \gamma^{-b}, \quad (65)$$

where

$$b = 1 + \lambda_0. \quad (66)$$

We note that

$$\mathcal{L}(P_n) = \frac{\alpha}{2} \frac{\partial}{\partial u} \left[ e^{-u/\alpha} g \frac{\partial}{\partial u} \left( e^{\alpha u} P_n \right) \right], \quad (67)$$

where

$$g' = \frac{dg}{du}. \quad (68)$$

Multiplying equation (60) by $e^{u/\alpha} P_n/g'$, integrating from $u = 0$ to $u = 1$, and integrating by parts, we find that

$$\lambda_0 > 0. \quad (69)$$

2.1. Smallest Eigenvalue in the Limit of Weak Momentum Diffusion ($\alpha \ll 1$)

We now calculate $\lambda_0$ to leading order in $\alpha$ when $\alpha < 1$ for any $U_{\text{eff}}(x)$ with a single potential well as in Figure 2. We take $P_0(0) = 1$. \quad (70)

From Sturm-Liouville theory, we know that $\lambda_0$ is the unique value of $\lambda_0$ in equation (60) for which $P_n$ satisfies the boundary conditions and has no zeros in the interval $0 < u < 1$. We use this fact to calculate $\lambda_0$ in the following four steps: (1) we assume an order of magnitude for $\lambda_0$, (2) we derive the resulting solution for $P_0$, (3) we determine the value of $\lambda_0$ by requiring that $P_0(0) = 1$ and $P_0(1) = 0$, and finally (4) we verify that $P_0$ has no zeros for $0 < u < 1$.

Starting with step 1, we set

$$\lambda_0 = \alpha^{-1} e^{-1/\alpha} \theta \quad (71)$$

and assume that $\theta$ is of order unity. This value is close to $\lambda$'s lower limit of 0. The corresponding particle energy spectrum
is hard \( (b \approx 1) \), which is expected in the small-\( \alpha \) limit, since the acceleration time is much less than the momentum-diffusion time (see eq. [10]).

We now carry out step 2. We set

\[
P_0 = - \frac{g'}{g(0)} \left( e^{-u/\alpha} + e^{-1/\alpha} F \right) \tag{72}
\]

and rewrite \( \mathcal{L}(P_0) = -\lambda_0 P_0 \) as

\[
\frac{\partial}{\partial u} \left[ e^{-u/\alpha} g \frac{\partial}{\partial u} \left( e^{u/\alpha} F \right) \right] = -\frac{2\theta g' e^{-u/\alpha}}{\alpha^2} \left( e^{-u/\alpha} + e^{-1/\alpha} F \right). \tag{73}
\]

Equations (70) and (61) give

\[
F(0) = 0 \tag{74}
\]

and

\[
F(1) = -1. \tag{75}
\]

We assume that \( F \) decreases monotonically from 0 to -1 as \( u \) increases from 0 to 1, an assumption that we verify at the end. Then for \( |1 - u| \gg \alpha \), we can take \( e^{-u/\alpha} + e^{-1/\alpha} F \approx e^{-u/\alpha} \) on the right-hand side of equation (73). In addition, when \( |1 - u| \ll 1 \), the entire right-hand side of equation (73) is of order \( e^{-1/\alpha} \) and can be replaced by 0 to an excellent approximation. Thus, throughout the interval \( 0 < u < 1 \), we can neglect the second term in parentheses on the right-hand side of equation (73), and write

\[
\frac{\partial}{\partial u} \left[ e^{-u/\alpha} g \frac{\partial}{\partial u} \left( e^{u/\alpha} F \right) \right] = -\frac{2\theta g' e^{-u/\alpha}}{\alpha^2}. \tag{76}
\]

Integrating twice, using \( g(0) = 0 \) from equation (44), and imposing equation (74), we find that

\[
F = -\frac{2\theta e^{-u/\alpha}}{\alpha^2} \int_{0}^{u} du_1 e^{u_1/\alpha} \int_{0}^{u_1} du_2 e^{u_2/\alpha} g(u_2). \tag{77}
\]

We now carry out step 3. Imposing equation (75) and evaluating the integral in equation (77) to lowest order in \( \alpha \), we find

\[
\theta = \frac{g(1)}{2g'(0)}. \tag{78}
\]

For step 4, we note that for \( 0 < u < 1 \), \( g \) and \( g' \) are positive. Thus, \( F \) decreases monotonically from 0 to -1 as \( u \) is increased from 0 to 1, as assumed. This implies that \( P \) has no zeros for \( 0 < u < 1 \) and that the value of \( \lambda_0 \) given by equations (71) and (78) is indeed (a good approximation of) the smallest eigenvalue.

Equations (45), (71), and (78) give

\[
\lambda_0 = \frac{\alpha^{-1} e^{-1/\alpha} g(1) \sqrt{a}}{\pi \omega_{\text{max}}}, \tag{79}
\]

where \( a \) is defined by equation (43). The power-law index of the energy spectrum of accelerated particles is then

\[
b = 1 + \frac{\alpha^{-1} e^{-1/\alpha} g(1) \sqrt{a}}{\pi \omega_{\text{max}}}. \tag{80}
\]

From equation (72), it can be seen that \( P_0 \) is strongly peaked near \( u = 0 \). This means that if a large number of particles start surfing at \( t = 0 \), the majority of those remaining at large times will be concentrated near the bottom of the effective potential well (with \( u \leq \alpha \), since \( P_0 \approx e^{-u/\alpha} \) for \( u < 1 \)).

For reference, the higher eigenvalues can be found by expanding equation (60) in powers of \( \alpha \), solving for \( \dot{P}/h \) in an inner region with \( u \ll 1 \) and an outer region with \( u \gg \alpha \), and then matching the inner and outer solutions in the region \( \alpha \ll u \ll 1 \). One finds that \( \dot{P}/h \) satisfies Kummer’s equation in the inner region regardless of the precise form of \( U_{\text{eff}}(x) \). This is because \( h \approx 1 \) in the inner region regardless of the exact form of \( U_{\text{eff}}(x) \). The inner solution can be matched to the outer solution only if

\[
\lambda_n \approx \frac{n}{2}, \tag{81}
\]

for \( n = 1, 2, \ldots \), provided \( n \) is of order unity (e.g., \( \ll \alpha^{-1} \)). The \( n \) zeros of \( P_n \), denoted \( u_1, u_2, \ldots, u_n \), occur in the inner region for \( n \) of order unity, with each of the \( u_i \) of order \( \alpha \).

2.2. Smallest Eigenvalue in the Limit of Strong Momentum Diffusion \((\alpha \gg 1)\)

When \( \alpha \gg 1 \), we can write equation (60) to lowest order in \( \alpha^{-1} \) as

\[
\frac{\partial^2}{\partial u^2} \left( \frac{uP}{2h} \right) - \frac{\partial}{\partial u} \left( \frac{P}{2} \right) = -\psi P, \tag{82}
\]

where

\[
\lambda = \alpha \psi. \tag{83}
\]

We again seek a solution for \( P(u) \) that satisfies \( P(0) = 1 \) and \( P(1) = 0 \). In going from equation (60) to equation (82) we have retained the highest order derivative. We thus have a regular perturbation problem and do not expect a boundary layer. Since there is no small parameter in equation (82), we expect the smallest value of \( \psi \) to be of order unity. Then, to lowest order in \( \alpha^{-1} \),

\[
b \approx \alpha \tag{84}
\]

to within a factor of order unity.

3. NUMERICAL CALCULATION OF ENERGY SPECTRUM OF ACCELERATED PARTICLES

In this section, we confirm and extend the analytic results of § 2 with the use of Monte Carlo test-particle calculations. We follow a set of test particles (ions) with \( p_x = 0 \) that obey the equations of motion

\[
\frac{dp_x}{dt} = q \left( E_x + \frac{v_x B_z}{c} \right) + \tilde{\delta} p_x, \tag{85}
\]

and

\[
\frac{dp_y}{dt} = q \left( E_y - \frac{v_y B_z}{c} \right), \tag{86}
\]

where \( \tilde{\delta} p_x \) is a stochastic function of time that increments \( p_x \) by \( \pm \Delta p_x \) (equal chance for + or −) during each time step \( \Delta t \),
where \( \Delta p_x = \left(2D_{px}\Delta t\right)^{1/2} \) and \( D_{px} \) is a constant. We work in the shock wave frame, with \( E_x = -d\Phi/dx \), where

\[
\Phi(x) = \Phi_0 e^{-(x/d)^2}.
\]  

(87)

We set \( E_y = -\gamma_{sh} v_{sh} B_0/c \), where \( v_{sh}, \gamma_{sh}, \) and \( B_0 \) are the shock speed, the Lorentz factor for the shock speed, and the upstream magnetic field in the laboratory frame, respectively. Although \( B_x \) is in general not constant, we assume a constant value, \( B_x = \gamma_{sh} B_0 \). We set \( v_{sh} = 0.25c, \gamma_{sh} = 1.0328, \) and \( \Phi_0/B_{gd} = 4.5 \). The minimum of the approximate effective potential \( \Phi + xB_x \) occurs at \( x = 1.63d \). We start the particles at \( x = 2.25d \), with \( p_x = -0.1m_ec, p_y = -10m_ec, \) and \( p_z = 0 \). By varying \( D_{px} \), we are able to consider different values of \( \alpha \) in equation (59).

To improve the statistics, we implement particle splitting as follows. We start each simulation with 1000 particles. At each time step, we update \( p \) and \( x \) for each particle. When a particle escapes from the effective potential well, the calculation for that particle is stopped and \( \gamma \) at that time is recorded. When half of the original particles have escaped, each of the remaining particles is "split"—that is, a copy of each particle is created with the same position and momentum, and both the original and the copy are subsequently tracked. When we calculate the energy spectrum, the weightings of split particles are halved. The splitting process is repeated each time that the particle number drops to 500 until the end of the simulation, when every particle’s \( \gamma \) exceeds a threshold value, \( \gamma_m = 500 \).

In the weak-scattering case, \( \alpha \ll 1 \), a difficulty for calculating the spectrum is that almost all of the particles reach the threshold \( \gamma_m \) before escaping. We therefore use another method to obtain the spectrum when \( \alpha \ll 1 \). We assume that the spectrum is given by a power law, \( dN/d\gamma \propto \gamma^{-b} \), with \( b > 1 \). The number of particles with \( \gamma \) between \( \gamma_1 \) and \( \gamma_2 \) is given by

\[
N_{1\rightarrow 2} = C \int_{\gamma_1}^{\gamma_2} \gamma^{-b} \, d\gamma = \frac{C(\gamma_2^{-1-b} - \gamma_1^{-1-b})}{1-b},
\]  

(88)

where \( C \) is a constant. The number of particles with \( \gamma > \gamma_2 \) is

\[
N_{2\rightarrow \infty} = \frac{C\gamma_2^{-1-b}}{1-b}.
\]  

(89)

From equations (88) and (89), we obtain the spectral index \( b \) as

\[
b = 1 - \frac{\ln(1 + N_{1\rightarrow 2}/N_{2\rightarrow \infty})}{\ln(\gamma_1/\gamma_2)}.
\]  

(90)

We take \( \gamma_1 = 250 \) and \( \gamma_2 = 500 \) and track 5000 particles for each value of \( \alpha \).

Our numerical results for \( \alpha \leq 1 \) are plotted in Figure 5, along with our analytic result (eq. [80]) for the same shock parameters. The numerical results and analytic result converge for \( \alpha \ll 1 \). Our numerical results for \( \alpha > 1 \) are plotted in Figure 6 and are consistent with equation (84) when \( \alpha \gg 1 \).

4. SUMMARY

In this paper we model surfing acceleration of relativistic particles at perpendicular shocks as a process in which particles undergo diffusion in \( p_x \) (\( x \) being the direction of the shock normal) while propagating in steady-state electric and magnetic field profiles. For shocks with \( E_x > B_x \) over some interval in \( x \), where \( z \) is the direction of the magnetic field, and for relativistic ions propagating primarily in the \(- \hat{y}\) direction, surfing produces a power-law energy spectrum of accelerated particles, provided the momentum-diffusion coefficient \( D_{px} \) is
independent of energy. We have calculated analytically the power-law index for the case of weak momentum diffusion, given in equation (80), and have carried out Monte Carlo test-particle calculations to determine the power-law index for the weak, moderate, and strong momentum-diffusion cases (§ 3).

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