GLOBAL ATTRACTORS FOR $p$-LAPLACIAN DIFFERENTIAL INCLUSIONS IN UNBOUNDED DOMAINS

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Abstract. In this work we consider a differential inclusion governed by a $p$-Laplacian operator with a diffusion coefficient depending on a parameter in which the space variable belongs to an unbounded domain. We prove the existence of a global attractor and show that the family of attractors behaves upper semicontinuously with respect to the diffusion parameter. Both autonomous and nonautonomous cases are studied.

1. Introduction. During the last ten years, many researchers have spent much effort in obtaining results on global attractors for $p$-Laplacian problems (see for example [1, 7, 12, 15, 16, 20, 24, 25, 37, 41, 42, 46, 48, 49, 50, 56, 58, 59]). It is worth noting that $p$-laplacian equations have applications in a variety of phenomena, such as nonlinear elasticity, flows in porous media, non-Newtonian fluids and many others (see [35], [36], [38] and the references therein). We also observe that such equations are often perturbed by a discontinuous nonlinear term, which leads to study a differential inclusion rather than a differential equation. Such models appear for example when studying processes of combustion in porous media [19] or conduction of electrical impulses in nerve axons [53], [54]. On the other hand, many parabolic problems in unbounded domains have been studied over the last years [2, 5, 13, 17, 18, 24, 25, 34, 45].

In this paper we consider a differential inclusion governed by a $p$-laplacian operator in an unbounded domain. We observe that, due to the absence of uniqueness of the Cauchy problem, to prove the existence of a global attractor for such partial differential inclusions it is necessary to use the theory of multivalued semiflows (or...
generalized semiflows as well) [6, 11, 29, 39, 47]. In this work, we use the theory developed in [39].

It is important to point out that one of the main difficulties when working with unbounded domains is the fact that the usual compact embeddings for Sobolev spaces fail. Another one is that $p > q$ does not imply $L^p(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$. The first problem is solved in many papers by obtaining suitable estimates of the tails of solutions (see e.g. [40], [45]), and the second one by adding a suitable linear dissipative term. In this paper we consider another approach. Namely, under some assumptions on the $p$-laplacian operator we can define a related space in which a suitable embedding is true for the unbounded domain. Moreover, this condition allows us to prove also the existence of a bounded absorbing set, a crucial step in obtaining a global attractor.

Let us consider the problem

$$\frac{\partial u_\lambda}{\partial t} - \text{div}(D^\lambda(x)|\nabla u_\lambda|^{p-2}\nabla u_\lambda) + a(x)|u_\lambda|^{p-2}u_\lambda \in f(x, u_\lambda) + g(t),$$

where $p > 2$, $u_\lambda(0) = u_{0,\lambda} \in H := L^2(\mathbb{R}^n)$, $n \geq 1$, $D^\lambda \in L^\infty(\mathbb{R}^n)$, $\infty > M \geq D^\lambda(x) \geq \sigma > 0$ a.e. in $\mathbb{R}^n$, $\lambda \in [0, \infty)$ and $D^\lambda \to D^{\lambda_1}$ in $L^\infty(\mathbb{R}^n)$ as $\lambda \to \lambda_1$, $f : \mathbb{R}^n \times \mathbb{R} \to 2^\mathbb{R}$ is a multivalued function with compact convex values and $a(x) \geq 1$ in $\mathbb{R}^n$ is a continuous function satisfying the following condition:

$$(C) \int_{\mathbb{R}^n} \frac{1}{a(x)^{2/(p-2)}} \, dx < +\infty.$$  

The authors in [51] considered problem (1) on bounded domains when $f$ is Lipschitz in the multivalued sense and $a(x) \equiv 1$ (see [31, 32] for differential equations and inclusions generated by pseudo-monotone operators in the case where $f \equiv 0$).

In this paper we will extend the results in [51] by considering unbounded domains first in the case where $f$ is Lipschitz in the multivalued sense in both autonomous and nonautonomous cases, and after that a more general situation where $f$ just satisfies a suitable growth condition.

It is also worth mentioning that for differential inclusions and reaction-diffusion equations of a similar type the regularity of all weak solutions and the global attractors have been studied in [23, 26, 27, 33], whereas the existence of a Lyapunov function was established in [21]. Related results concerning attractors for nonautonomous differential equations and inclusions can be found in [22, 28, 60].

The paper is organized as follows. In Section 2 we prove the existence of the global attractor for problem (1) in the case of a Lipschitz nonlinearity $f$ in the multivalued sense with $g(t) \equiv 0$ and the upper semicontinuity with respect to the parameter $\lambda$, as well. In Section 3 we extend the results of the previous section by proving the existence of a pullback attractor when $g(t) \neq 0$. In Section 4 we consider a more general nonlinear term.

2. Existence of the global attractor in the autonomous Lipschitz case.

First, we will consider problem (1) for a multivalued nonlinear term $f : \mathbb{R} \to 2^\mathbb{R}$ of Lipschitz type and with $g(t) \equiv 0$, that is, let us study the problem

$$\left\{ \begin{array}{l}
\frac{\partial u_\lambda}{\partial t} - \text{div}(D^\lambda(x)|\nabla u_\lambda|^{p-2}\nabla u_\lambda) + a(x)|u_\lambda|^{p-2}u_\lambda \in f(u_\lambda), \\
\quad \quad \quad \quad \quad \quad \quad \quad u_\lambda(0) = u_{0,\lambda}, \end{array} \right.$$  

where $f$ will satisfy conditions (3), (4) given below.
For a space $X$ denote by $\mathcal{P}(X)$ the set of all non-empty subsets of $X$, and by $\mathcal{C}_c(X)$ the set of all nonempty, bounded, closed, convex subsets of $X$. If $X$ is a metric space with metric $d$, we define the Hausdorff semidistance between two sets by $\text{dist}(A, B) = \sup_{y \in A} \inf_{x \in B} \rho(y, x)$, and the Hausdorff distance by $\text{dist}_H(A, B) = \max\{\text{dist}(A, B), \text{dist}(B, A)\}$.

Let $H$ be the space $L^2(\mathbb{R}^n)$ with norm $\| \cdot \|$ and scalar product $(\cdot, \cdot)$. Also, we will denote the norms in the spaces $L^p(\mathbb{R}^n)$, $2 < p \leq \infty$ by $\| \cdot \|_p$.

Let $f : \mathbb{R} \to \mathcal{C}_c(\mathbb{R})$ be a multivalued map which is Lipschitz, that is, there exists $C > 0$ such that

$$\text{dist}_H(f(x), f(z)) \leq C|x - z| \quad \forall x, z \in \mathbb{R}. \quad (3)$$

Moreover, suppose there exists $D > 0$ such that

$$\sup_{y \in f(s)} |y| \leq D|s|, \quad \forall s \in \mathbb{R}. \quad (4)$$

As a particular example, we could consider $f : \mathbb{R} \to \mathcal{C}_c(\mathbb{R})$ defined by $f(s) := [0, |s|]$.

Then we can define the associated map $F : \mathcal{D}(F) \subset H \to \mathcal{P}(H)$, given by

$$F(y(\cdot)) = \{\xi(\cdot) \in H : \xi(x) \in f(y(x)) \quad \text{a.e. in } \mathbb{R}^n\}. \quad \text{Consequently,}$$

$$\sup_{v \in F(u)} \|v\| \leq D\|u\|, \quad \forall u \in \mathcal{D}(F). \quad (5)$$

The author in [46] proved that the operator

$$A^{D^\lambda}(u) := -\text{div}(D^\lambda|\nabla u|^{p-2}\nabla u) + a|u|^{p-2}u$$

is maximal monotone in $H$ and is the subdifferential of a proper, convex and lower semicontinuous function $\varphi^{D^\lambda} : H \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\varphi^{D^\lambda}(u) = \begin{cases} \frac{1}{p} \int_{\mathbb{R}^n} D^\lambda(x)|\nabla u|^pdx + \int_{\mathbb{R}^n} a(x)|u|^pdx, & u \in E, \\ +\infty, & \text{otherwise} \end{cases}$$

where $E := \{u \in W^{1,p}(\mathbb{R}^n); \int_{\mathbb{R}^n} a(x)|u(x)|^pdx < +\infty\}$ is a reflexive Banach space with the norm given by $\|u\|_E := \left(\int_{\mathbb{R}^n} |\nabla u(x)|^p + a(x)|u(x)|^pdx\right)^{1/p}$. Moreover, the author proved that there are constants $w_1 = w_1(\sigma) > 0$, $w_2 = w_2(p, M) > 0$ such that for all $u \in E$ the following two conditions hold:

$$\langle A^{D^\lambda}u, u \rangle_{E^*, E} \geq w_1 \|u\|_E^p, \quad (6)$$

$$\|A^{D^\lambda}u\|_{E^*} \leq w_2 \|u\|_E^{p-1} < w_2(\|u\|_E^{p-1} + 1). \quad (7)$$

As a consequence, $\overline{\mathcal{D}(A^{D^\lambda})} = H$ and the operator $A^{D^\lambda} : \mathcal{D}(A^{D^\lambda}) \subset H \to H$ generates a compact semigroup $S^{D^{\lambda}}$ [14].

We consider our equation (2) in the abstract form

$$\begin{cases} \frac{d\lambda}{dt} + A^{D^\lambda}(u) \in F(u), \\ u_{\lambda}(0) = u_{0,\lambda}. \end{cases} \quad (8)$$

It can be proved, with some adjustments, in an analogous way as in [39, Lemmas 11, 12], that the operator $F$ has values in $\mathcal{C}_c(H)$ (in particular, $\mathcal{D}(F) = H$) and
that it is Lipschitz (in the multivalued sense) with the same constant $C$ from (3),
that is,
\[ \text{dist}_H(F(u), F(v)) \leq C \|u - v\| \text{ for all } u, v \in H. \] (9)

**Remark 1.** In order to consider at once the multivalued and the single-valued cases
we could consider the operator $U : H \to H$, which is a globally Lipschitz map with
Lipschitz constant $L \geq 0$, and the problem
\[
\begin{cases}
\frac{du_\lambda}{dt} + AD^\lambda(u_\lambda) \in F(u_\lambda) + U(u_\lambda), \\
u_\lambda(0) = u_{0,\lambda}.
\end{cases}
\]

It is easy to show then that the map $\tilde{F} : H \to \mathcal{P}(H)$ defined by $\tilde{F}(u) := F(u) + U(u)$
has values in $C(\mathbb{R})$ and that it is Lipschitz.

The main trouble to deal with an unbounded domain problem is the fact that $p > q$
do not imply $L^p(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$, and also that the usual compact embeddings
for Sobolev spaces in bounded domains fail. However, the author in [46] solved this
problem in a very simple way considering the space $E$ and a compact embedding
theorem for this space. We extend the result in [46] in the following lemma.

**Lemma 2.1.** If $n > p$, then $E \subset L^s(\mathbb{R}^n)$ for $2 \leq s \leq p^* = \frac{np}{n-2}$ and the inclusion
is compact for $s < p^*$. If $n \leq p$, then $E \subset L^s(\mathbb{R}^n)$ for $2 \leq s < +\infty$ with compact
inclusion.

**Proof.** The first statement is proved in [46].

Let $n \leq p$, $2 \leq s < \infty$ and choose some $\eta > \max\{p, s\}$. We note that $E \subset W^{1, p}(\mathbb{R}^n)$ and from the proof of Lemma 1 in [46] we know that $E \subset L^2(\mathbb{R}^n)$. For
$n = p$ it follows (from the embedding theorems of Sobolev spaces) that $W^{1, p}(\mathbb{R}^n) \subset L^\eta(\mathbb{R}^n)$
and then by the interpolation inequality $W^{1, p}(\mathbb{R}^n) \subset L^s(\mathbb{R}^n)$. When $n < p$
we obtain as a consequence of Morrey’s inequality that $W^{1, p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$, and then
(again using the interpolation inequality) $W^{1, p}(\mathbb{R}^n) \subset L^s(\mathbb{R}^n)$.

Let $\{u_n\}$ be a bounded sequence in $E$. Then up to a subsequence $u_n \to v$ weakly
in $L^2(\mathbb{R}^n)$. We can show that $v_n := u_n - v \to 0$ in $L^2(\mathbb{R}^n)$. Indeed, denote by $B_R$
a ball of radius $R$ centered at 0 in $L^2(\mathbb{R}^n)$, and by $B_R^c$ its complementary. Condition
(C) and $v \in L^2(\mathbb{R}^n)$ imply that for any $\epsilon > 0$ there exists $R = R(\epsilon)$ such that

\[
\left( \int_{B_R^c} \frac{1}{a(x)^{2/(p-2)}} \, dx \right)^{\frac{p-2}{p}} \leq \epsilon, \quad \int_{B_R^c} |v(x)|^2 \, dx < \epsilon.
\]

Then by the Hölder inequality we have

\[
\int_{B_R^c} |u_n(x)|^2 \, dx \leq \left( \int_{B_R^c} a(x) |u_n(x)|^p \, dx \right)^{\frac{2}{p}} \left( \int_{B_R^c} \frac{1}{a(x)^{2/(p-2)}} \, dx \right)^{\frac{p-2}{p}} < C_1 \epsilon.
\]

Hence,

\[
\int_{B_R^c} |v_n(x)|^2 \, dx \leq 2 \left( \int_{B_R^c} |u_n(x)|^2 \, dx + \int_{B_R^c} |v(x)|^2 \, dx \right) \leq 2 (C_1 + 1) \epsilon.
\]

Combining this with the compact embedding $W^{1, p}(B_R) \subset L^2(B_R)$, it is easy to see
that $v_n \to 0$ in $L^2(\mathbb{R}^n)$. 
If \( n = p \), as \( \{u_n\} \) is a bounded sequence in \( E \), then \( u_n \to v \) weakly in \( L^7(\mathbb{R}^n) \). Thus, using the interpolation inequality we can get
\[
\|v_n\|_s \leq \|v_n\|^\alpha \|v_n\|^{1-\alpha}_s \leq C_2 \|v_n\|^\alpha \to 0,
\]
where \( \frac{1}{s} = \frac{\alpha}{2} + \frac{1-\alpha}{2}, 0 < \alpha \leq 1 \). If \( n < p \), then \( u_n \to v \) weakly star in \( L^\infty(\mathbb{R}^n) \). Again by the interpolation inequality we have
\[
\|v_n\|_s \leq \|v_n\|^\alpha \|v_n\|^{1-\alpha}_s \leq C_3 \|v_n\|^\alpha \to 0,
\]
where \( \alpha = \frac{2}{s} \). Therefore, \( u_n \to v \) strongly in \( L^s(\mathbb{R}^n) \) and the embedding \( E \subset L^s(\mathbb{R}^n) \) is compact.

We recall that the continuous function \( u_\lambda : [0, T] \to H \) is called an integral solution of problem (8) if \( u_\lambda(0) = u_{0,\lambda} \) and
\[
\|u_\lambda(t) - v\|^2 \leq \|u_\lambda(s) - v\|^2 + 2 \int_s^t \left( f_\lambda(\tau) - A^{D^\lambda}(v), u_\lambda(\tau) - v \right) d\tau,
\]
for any \( v \in \mathcal{D}(A^{D^\lambda}) \) and \( 0 \leq s \leq t \leq T \), where \( f_\lambda(\cdot) \in L^1(0, T; H) \) is a selection of the map \( t \mapsto F(u_\lambda(t)) \), that is, \( f_\lambda(\cdot) \in F(u_\lambda(\tau)) \) for a.a. \( \tau \in (0, T) \).

The continuous function \( u_\lambda : [0, T] \to H \) is called a strong solution of problem (8) if \( u_\lambda(0) = u_{0,\lambda}, u_\lambda(\cdot) \) is absolutely continuous on any compact subset of \( (0, T) \), \( u_\lambda(t) \in \mathcal{D}(A^{D^\lambda}) \) for a.a. \( t \in (0, T) \) and there exists a function \( f_\lambda(\cdot) \in L^2(0, T; H) \), \( f_\lambda(t) \in F(u_\lambda(t)) \) a.e. on \( [0, T] \), such that
\[
\frac{du_\lambda}{dt} + A^{D^\lambda}(u_\lambda(t)) = f_\lambda(t) \text{ a.e. on } [0, T].
\]

Let us consider the auxiliary problem
\[
\begin{align*}
\frac{du}{dt} + \partial\varphi(u(t)) &\ni f(t), \\
\varphi(u(0)) &= u_0,
\end{align*}
\]
(12)
where \( \partial\varphi \) is the subdifferential of a proper convex lower semicontinuous function \( \varphi : H \to (-\infty, +\infty] \). It is well known that for any \( f(\cdot) \in L^1(0, T; H) \), \( u_0 \in \overline{\mathcal{D}(\varphi)} \) there exists a unique integral solution of problem (12) [8].

The continuous function \( u : [0, T] \to H \) is a strong solution of (12) if \( u(\cdot) \) is absolutely continuous on any compact subset of \( (0, T) \), \( u(t) \in \mathcal{D}(\partial\varphi) \) for a.a. \( t \in (0, T) \) and
\[
\frac{du}{dt} + \partial\varphi(u(t)) \ni f(t) \text{ for a.a. } t \in (0, T).
\]

**Proposition 1.** ([9, Theorem 3.6] or [8, p.189]) For any \( f(\cdot) \in L^2(0, T; H) \), \( u_0 \in \overline{\mathcal{D}(\varphi)} \), there exists a unique strong solution of inclusion (12) such that
\[
\sqrt{\frac{du}{dt}} \in L^2(0, T; H), \quad \varphi(u(\cdot)) \in L^1(0, T),
\]
(13)
and \( t \mapsto \varphi(u(t)) \) is absolutely continuous on \( [\delta, T] \), for all \( \delta > 0 \).

If \( u_0 \in \mathcal{D}(\varphi) \), then \( \frac{du}{dt} \in L^2(0, T; H) \) and \( t \mapsto \varphi(u(t)) \) is absolutely continuous on \( [0, T] \). If \( u_0, v_0 \in \overline{\mathcal{D}(\varphi)} \), then
\[
\|u(t) - v(t)\| \leq \|u_0 - v_0\|, \text{ for all } t \in [0, T].
\]
Any integral solution \( u_\lambda (\cdot) \) of problem (8) is the unique integral solution of (12) with \( \varphi = \varphi^{D_\lambda} \) and \( f = f_\lambda \). The properties of the map \( F \) imply that for every integral solution of (8) the selection \( f_\lambda (\cdot) \) belongs to \( L^2(0,T; H) \). Then, Proposition 1 implies that it is in fact the unique strong solution of problem (12), as a strong solution of (12) is also an integral one [8]. Therefore, \( u_\lambda (\cdot) \) is also a strong solution of (8) and the sets of integral and strong solutions of (8) coincide.

From Lemmas 5, 6 in [39] the following result follows.

**Proposition 2.** Let (3)-(4) hold. Then the inclusion (8) defines a strict multivalued semiflow (or m-semiflow) \( G_\lambda (t, \cdot) : H \to \mathcal{P}(H) \) where \( G_\lambda (t, u_0) \) is the set of all integral solutions of (8) beginning at \( u_0 \in H \).

**Remark 2.** A multivalued map \( G : \mathbb{R}^+ \times H \to \mathcal{P}(H) \) is called a multivalued semiflow if \( G(t+s, u_0) \subset G(t, G(s, u_0)) \) for all \( t, s \geq 0, u_0 \in H \). It is called strict if \( G(t+s, u_0) = G(t, G(s, u_0)) \).

**Lemma 2.2.** The following property is satisfied:

\[(H) \quad \text{The sets } M_K := \{ u \in \mathcal{D}(\varphi^{D_\lambda}) : \| u \| \leq K, \| \varphi^{D_\lambda}(u) \| \leq K \} \text{ are compact in } H \text{ for any } K > 0.\]

**Proof.** Since \( E \subset H \) by Lemma 2.1 and

\[ M_K := \{ u \in \mathcal{D}(\varphi^{D_\lambda}) : \| u \| \leq K, \| \varphi^{D_\lambda}(u) \| \leq K \} = \overline{M_K}, \]

it is sufficient to show that for each \( K > 0 \), \( M_K \) is a bounded set in \( E \). Let \( K > 0 \) and \( u \in M_K \). Then, \( \langle A^{D_\lambda} u, u \rangle_{E^*, E} \leq Kp \). From (6), \( \| u \|_E \leq \frac{[Kp]}{[\frac{Kp}{w_1}]^p} \). So, the condition \( (H) \) is satisfied.

We recall that \( \mathcal{A}_\lambda \) is a global attractor for the multivalued semiflow \( G_\lambda \) if

\[ \text{dist}(G_\lambda(t, B), \mathcal{A}_\lambda) \to 0, \text{ as } t \to +\infty, \]

for all bounded set \( B \), and \( \mathcal{A}_\lambda \subset G_\lambda(t, \mathcal{A}_\lambda) \) for all \( t \geq 0 \) (negatively semi-invariance). It is invariant if \( \mathcal{A}_\lambda = G_\lambda(t, \mathcal{A}_\lambda) \) for all \( t \geq 0 \).

Now, we prove the existence of the global attractor \( \mathcal{A}_\lambda \) for problem (8).

**Theorem 2.3.** Let (3)-(4) hold. The multivalued semigroup associated with problem (8) has the global attractor \( \mathcal{A}_\lambda \). It is the minimal closed set attracting each bounded set. It is compact, invariant and maximal among all negatively semi-invariant bounded subsets in \( H \).

**Proof.** We know from Lemma 2.2 that the condition \( (H) \) is satisfied. Then the result follows from [39, Theorem 9] if we prove that there exist \( \delta > 0, M > 0 \) such that for any \( u \in \mathcal{D}(A^{D_\lambda}) \) such that \( \| u \| \geq M \) and for all \( y \in -A^{D_\lambda}(u) + F(u) \),

\[ (y, u) \leq -\delta. \quad (14) \]

Indeed, let \( u \in \mathcal{D}(A^{D_\lambda}), \xi \in F(u) \). Using the embedding \( E \subset H \) in Lemma 2.1 we have that \( \| u \|_H \leq \gamma \| u \|_E \) for some \( \gamma > 0 \). Using (5), (6), the Cauchy-Schwarz and the Young inequalities we get

\[ \langle -A^{D_\lambda}u + \xi, u \rangle_{E^*, E} \leq -\frac{w_1}{2\gamma^p} \| u \|^p + C_1, \quad (15) \]
where \( C_1 \) is a positive constant. Considering \( M := \left[ \frac{2\pi^p}{w_1}(1 + C_1) \right]^{1/p} > 0 \) and \( \delta := 1 \) we have that \( \left\langle -\Delta^\lambda u + \xi, u \right\rangle \leq -\delta \) for all \( u \in \mathcal{D}(A) \) with \( \|u\| > M \). So, condition (14) is satisfied.

\[ \square \]

### 2.1. Uniform estimates

In this section we obtain estimates in the spaces \( H \) and \( E \) for the solutions \( u_\lambda \) of problem (8) which are uniformly on \( \lambda \in [0, \infty) \). As a consequence of (5), there exists \( D > 0 \) such that

\[ \sup_{\xi \in \mathcal{F}(u_\lambda(t))} \|\xi\| \leq D\|u_\lambda(t)\|, \quad \forall \lambda \in [0, \infty). \quad (16) \]

As commented before, each integral solution \( u_\lambda(\cdot) \) of problem (8) is a strong solution of this problem. Since \( \infty > M \geq D^\lambda(x) \geq \sigma > 0 \) a.e. in \( \mathbb{R}^n \), \( \lambda \in [0, \infty) \), working with selections we can repeat the same arguments used in [46, 48, 49] to obtain the desired estimates. What essentially changes is the control on the right hand side, i.e., if \( u_\lambda \) is a solution of (8), then there exists \( \xi_\lambda \in L^1(0, T; H) \), \( \xi(t) \in F(u_\lambda(t)) \) \( t - a.e. \) in \( (0, T) \) such that

\[ \frac{\partial u_\lambda}{\partial t}(t) - \text{div}(D^\lambda \nabla u_\lambda(t)) + u_\lambda(t) = \xi(t). \]

Multiplying the equation by \( u_\lambda(t) \) we control the right hand side using (16):

\[ (\xi_\lambda(t), u_\lambda(t)) \leq D\|u_\lambda(t)\|^2, \]

for all \( \lambda \in [0, \infty) \). Thus, we obtain the following results, which are proved for example as in [49, Lemmas 2.1, 2.2].

**Lemma 2.4.** There are positive constants \( r_0, t_0 \) such that \( \|u_\lambda(t)\| \leq r_0 \), for each \( t \geq t_0 \), \( u_{0, \lambda} \in H \) and \( \lambda \in [0, \infty) \), where \( u_\lambda \) is any integral solution of (8) in \( [0, \infty) \) with initial data \( u_{0, \lambda} \in H \).

**Remark 3.** We observe that the constants \( r_0, t_0 \) in Lemma 2.4 depend neither on the initial data nor on \( \lambda \).

**Remark 4.** For any \( u_{0, \lambda} \in H \), \( t_0 > 0 \) there exists a positive constant \( \tilde{r}_0(u_{0, \lambda}, t_0) \) such that \( \|u_\lambda(t)\| \leq \tilde{r}_0(u_{0, \lambda}, t_0) \), for all \( t \in [0, t_0] \), \( \lambda \in [0, \infty) \), where \( u_\lambda \) is any integral solution of (8) with initial data \( u_{0, \lambda} \in H \). For any bounded subset \( B \) of \( H \) and \( t_0 > 0 \) there exists \( \tilde{r}_1(B, t_0) \) such that \( \|u_\lambda(t)\|_{H} < \tilde{r}_1 \), for all \( u_{0, \lambda} \in B \), \( \lambda \in [0, \infty) \), \( t \in [0, t_0] \), where \( u_\lambda \) is any integral solution of (8) with initial data \( u_{0, \lambda} \).

**Corollary 1.** There is a bounded set \( B_0 \) in \( H \) such that \( \mathcal{A}_\lambda \subset B_0 \), for any \( \lambda \in [0, \infty) \).

**Lemma 2.5.** There exist positive constants \( r_2 > 0 \) and \( t_1 > t_0 \) such that \( \|u_\lambda(t)\|_{E} \leq r_2 \), for each \( t \geq t_1 \) and \( \lambda \in [0, \infty) \), where \( t_0 \) is as in Lemma 2.4 and \( u_\lambda \) is any integral solution of (8) in \( [0, \infty) \).

**Remark 5.** For any bounded set \( B \) in \( E \) and \( t_1 > 0 \) there is a constant \( \tilde{r}_3(B, t_1) > 0 \) such that \( \|u_\lambda(t)\|_{E} < \tilde{r}_3 \), for all \( u_{0, \lambda} \in B \), \( \lambda \in [0, \infty) \) and \( t \in [0, t_1] \), where \( u_\lambda \) is any integral solution of (8) with initial condition \( u_{0, \lambda} \).

As an important consequence of Lemma 2.5 it follows that \( \bigcup_{\lambda \in [0, \infty)} \mathcal{A}_\lambda \) is a bounded subset of \( E \) and once \( E \subset H \), we can conclude:

**Corollary 2.** \( \mathcal{A} := \bigcup_{\lambda \in [0, \infty)} \mathcal{A}_\lambda \) is a compact subset of \( H \).
2.2. Upper semicontinuity of the global attractors. A multivalued map \( U : \mathcal{D}(U) \subset H \to \mathcal{P}(H) \) is said to be w-upper semicontinuous if for any \( x_0 \in \mathcal{D}(U) \), \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( U(x) \subset O_\epsilon(U(x_0)) \), for any \( x \in O_\delta(x_0) \), where \( O_\epsilon(B) \) denotes an \( \epsilon \)-neighborhood of the set \( B \). It is said to be upper semicontinuous if for any \( x_0 \in \mathcal{D}(U) \) and any neighborhood \( O(U(x_0)) \) there exists \( \delta > 0 \) such that \( U(x) \subset O(U(x_0)) \), for all \( x \in O_\delta(x_0) \). Obviously, any upper semicontinuous map is w-upper semicontinuous, the converse being valid if \( U \) has compact values [3, p.45].

In this section we prove that \( \{A_\lambda\}_{\lambda \in [0,\infty)} \) is upper semicontinuous at any \( \lambda_1 \).

Since \( \lambda \mapsto A_\lambda \) has compact values, this is equivalent to the property

\[
\text{dist}(A_\lambda, A_{\lambda_1}) = \sup_{a_\lambda \in A_\lambda} \text{dist}(a_\lambda, A_{\lambda_1}) \to 0 \quad \text{as} \quad \lambda \to \lambda_1.
\]

We first prove the w-upper semicontinuity of the map \( \lambda \mapsto G_\lambda(t, A) \) with respect to \( \lambda \).

**Theorem 2.6.** The map \( \lambda \mapsto G_\lambda(t, A) \) is w-upper semicontinuous at \( \lambda_1 \) for each \( t > 0 \).

**Proof.** For simplicity, we consider \( \lambda_1 = 0 \). Suppose, on the contrary, that there exists a number \( t_0 > 0 \) such that the map \( \lambda \mapsto G_\lambda(t_0, A) \) is not w-upper semicontinuous at \( \lambda_1 \). So, there exists a \( \gamma \)-neighborhood \( O_\gamma(G_0(t_0, A)) \) such that for each \( n \in \mathbb{N} \) there exists \( 0 < \lambda_n < \frac{1}{n} \) and \( \xi_{\lambda_n} \in G_{\lambda_n}(t_0, A) \) with \( \xi_{\lambda_n} \notin O_\gamma(G_0(t_0, A)) \) (note that \( \lambda_n \to \lambda_1 = 0 \) as \( n \to +\infty \)). Then, \( \xi_{\lambda_n} = u_{\lambda_n}(t_0) \), \( u_{\lambda_n}(0) \in A \). It is enough to show that there is a subsequence \( \{\xi_{\lambda_{n_k}}\} \) of \( \{\xi_{\lambda_n}\} \) with \( \xi_{\lambda_{n_k}} \to \xi_0 \in G_0(t_0, A) \), and so we obtain a contradiction. Indeed, we have that \( u_{\lambda_n} \) is an integral solution of (8) with \( u_{\lambda_n}(0) \in A \).

So, there exists \( f_{\lambda_n} \in L^1(0, T; H) \), with \( f_{\lambda_n}(t) \in F(u_{\lambda_n}(t)) \), a.e. in \( (0, T) \), and such that \( u_{\lambda_n} \) is an integral solution over \( (0, T) \) of the problem (17):

\[
\begin{aligned}
\frac{\partial u_{\lambda_n}}{\partial t} - \text{div}(D^{\lambda_n}\nabla u_{\lambda_n}) + a|u_{\lambda_n}|^{p-2}u_{\lambda_n} = f_{\lambda_n} & \quad \text{in} \quad (0, T), \\
u_{\lambda_n}(0) = u_{0,\lambda_n},
\end{aligned}
\]

denoted by \( u_{\lambda_n}(\cdot) = I(u_{0,\lambda_n})f_{\lambda_n}(\cdot) \). We can suppose that \( t_0 \in (0, T) \). As \( A \) is compact, \( u_{\lambda_n}(0) \to u_0 \in A \). Let \( z_{\lambda_n}(\cdot) = I(u_0)f_{\lambda_n}(\cdot) \) be the integral solution of the problem

\[
\begin{aligned}
\frac{\partial z_{\lambda_n}}{\partial t} - \text{div}(D^{\lambda_n}\nabla z_{\lambda_n}) + a|z_{\lambda_n}|^{p-2}z_{\lambda_n} = f_{\lambda_n}, & \quad \text{for each} \quad t \in (0, T),
\end{aligned}
\]

By (16) and Remark 4, there exists \( L > 0 \) such that \( \|f_{\lambda_n}(t)\| \leq L \) for a.a. \( t \in [0, T] \), and for all \( n \in \mathbb{N} \). Let \( K = \{f_{\lambda_n}(\cdot) : n \in \mathbb{N}\} \) and \( M(K) = \{z_{\lambda_n}(\cdot) : n \in \mathbb{N}\} \). Once \( K \) is a bounded set in \( L^\infty(0, T; H) \), it is easy to see that it is a uniformly integrable subset. Given \( t \in (0, T) \) and \( h > 0 \) such that \( t-h \in (0, T) \), consider the operator \( T_h : M(K)(t) \to H \) defined by \( T_h z_{\lambda_n}(t) = S^{D^{\lambda_n}}(h)z_{\lambda_n}(t-h) \). In the same way as in Statement 1 in [48], but now using the compact embedding of \( E \) in \( H \) (see Lemma 2.1), one can show that the operator \( T_h : M(K)(t) \to H \) is compact. Then, from Theorem 3.2 in [48], the set \( M(K) \) is relatively compact in \( C([0, T]; H) \) and so there exist \( z \in C([0, T]; H) \) and a subsequence \( \{z_{\lambda_{n_k}}(\cdot)\} \) such that \( z_{\lambda_{n_k}} \to z \) in \( C([0, T], H) \).

Let \( v \in E \) be arbitrary. Since \( z_{\lambda_n}(\cdot) \) are strong solutions of problem (18) we can obtain that

\[
\frac{1}{2} \frac{d}{dt} \|z_{\lambda_n}(t) - v\|^2 + \langle A^{\lambda_n}(z_{\lambda_n}), z_{\lambda_n}(t) - v \rangle
\]
\[
(\lambda_n f)(v, z_n(t) - v) \leq (f_{\lambda_n}, z_{\lambda_n}(t) - v) \quad \text{for a.a. } t \in (0, T).
\]

Note that
\[
(A_{\lambda_n}^{D^\lambda_n}(z_{\lambda_n}), z_{\lambda_n}(t) - v) = \left\langle A_{\lambda_n}^{D^\lambda_n}(z_{\lambda_n}), z_{\lambda_n}(t) - v \right\rangle_{E^*, E}
\]
\[
\geq \left\langle A_{\lambda_n}^{D^\lambda_n}(v), z_{\lambda_n}(t) - v \right\rangle_{E^*, E}.
\]

Hence, after integration we obtain
\[
\frac{1}{2} \parallel z_{\lambda_n}(t) - v \parallel^2 \leq \frac{1}{2} \parallel z_{\lambda_n}(s) - v \parallel^2 + \int_s^t \left\langle f_{\lambda_n}(\tau) - A_{\lambda_n}^{D^\lambda_n}(v), z_{\lambda_n}(\tau) - v \right\rangle_{E^*, E} d\tau,
\]
for all \( 0 < s \leq t \). In fact, the continuity of \( s \mapsto z_{\lambda_n}(s) \) implies that the inequality holds for \( 0 \leq s \leq t \).

As \( \| f_{\lambda_n}(\tau) \| \leq L \), for a.a. \( 0 \leq \tau \leq T \) and for all \( n \in \mathbb{N} \), we conclude that there exists a positive constant \( \tilde{L} \) such that \( \| f_{\lambda_n} \|_{L^2(0, T; H)} \leq \tilde{L} \) for all \( n \in \mathbb{N} \). As \( L^2(0, T; H) \) is a reflexive Banach space, there is \( f \in L^2(0, T; H) \) and a subsequence \( \{ f_{\lambda_n} \} \), which we do not relabel, such that \( f_{\lambda_n} \to f \) weakly in \( L^2(0, T; H) \). Consequently \( f_{\lambda_n} \to f \) weakly in \( L^1(0, T; H) \). Moreover,
\[
\sup_{t \in [0, T]} \| u_{\lambda_n}(t) - z(t) \| \leq \sup_{t \in [0, T]} \| I(u_{0, \lambda}) f_{\lambda_n}(t) - I(u_0) f_{\lambda_n}(t) \| + \sup_{t \in [0, T]} \| z_{\lambda_n}(t) - z(t) \| \to 0 \text{ as } n \to +\infty,
\]
where we have used the inequality [8]:
\[
\| I(u_{0, \lambda}) f_{\lambda}(t) - I(v_{0, \lambda}) g_{\lambda}(t) \| \leq \| u_{0, \lambda} - v_{0, \lambda} \| + \int_0^t \| f_{\lambda}(s) - g_{\lambda}(s) \| ds.
\]

Therefore \( u_{\lambda_n} \to z \) in \( C([0, T]; H) \).

In view of [55, Proposition 1.1] for a.a. \( \tau \in (0, t) \),
\[
f(\tau) \in \bigcap_{n=1}^{\infty} \overline{\mathcal{C}}_{\lambda_k \geq n} f_{\lambda_k}(\tau),
\]
where \( \overline{\mathcal{C}} \) denotes the closure of the convex hull in \( H \). Fix \( \tau \in (0, t) \). Since \( F \) is Lipschitz, we obtain that for any \( \delta > 0 \) there exists \( n > 0 \) such that for any \( \lambda_k \geq n \),
\[
dist(F(u_{\lambda_k}(\tau)), F(z(\tau))) < \delta.
\]

As \( F(z(\tau)) \) is convex and closed, this implies that \( \overline{\mathcal{C}}_{\lambda_k \geq n} f_{\lambda_k}(\tau) \subset O_\delta(F(z(\tau))) \), and then \( f(\tau) \in F(z(\tau)) \), for a.a. \( \tau \in (0, t) \).

It is easy to see that \( A_{\lambda_n}^{D^\lambda_n}(v) \to A_{\lambda}^{D^\lambda}(v) \) weakly in \( E^* \) for any \( v \in E \). Thus, passing to the limit in (19) and taking into account that \( D(A_{\lambda}^{D^\lambda}) \subset E \), we obtain that \( z(\cdot) \) is an integral solution of problem (8) with \( \lambda = 0 \) and initial data \( u_0 \). Hence, \( z(t) \in G_0(t, A), \forall t \geq 0 \).

Thus, defining \( \xi_0 = z(t_0) \in G_0(t_0, A) \), we obtain
\[
\| \xi_n - \xi_0 - u_{\lambda_n}(t_0) - z(t_0) \| \leq \sup_{\tau \in [0, T]} \| u_{\lambda_n}(\tau) - z(\tau) \| \to 0 \text{ as } n \to +\infty,
\]
which is a contradiction, and so we conclude that the map
\[
[0, \infty) \ni \lambda \mapsto G_{\lambda}(t, A)
\]
is \( w \)-upper semicontinuous on \( \lambda_1 \) for each \( t > 0 \).
Therefore, using Theorem 2.6 and Corollary 2, we obtain immediately from Theorem 1.2 in [30] the following result.

**Theorem 2.7.** The family of global attractors \( \{A_\lambda : \lambda \in [0, \infty)\} \) of problem (8) is upper semicontinuous at any \( \lambda \).

3. **The nonautonomous Lipschitz case.** Let us consider now the following nonautonomous problem:

\[
\begin{aligned}
\frac{du_\lambda}{dt} - \text{div}(D_\lambda(x)|\nabla u_\lambda|^{p-2}\nabla u_\lambda) + a(x)|u_\lambda|^{p-2}u_\lambda &\in f(u_\lambda) + g(t), \\
u_\lambda(s) &= u_{s, \lambda},
\end{aligned}
\]

where \( f : \mathbb{R} \to C_v(\mathbb{R}) \) satisfies as before (3) and (4) and \( g \in L^2_{loc}(\mathbb{R}, H) \) is such that for some \( \beta > 0 \) it holds

\[
\int_{-\infty}^{t} e^{\beta r} \|g(r)\|^2 \, dr < \infty.
\]

We define then the map \( F : \mathbb{R} \times H \to \mathcal{P}(H) \), given by

\[
F(t, u) = F(u) + g(t).
\]

It follows from the results in Section 2 that

\[
F(t, u) \in C_v(H),
\]

\[
\sup_{y \in F(t, u)} \|y\| \leq D\|u\| + g(t),
\]

\[
\text{dist}_H(F(t, u), F(t, v)) \leq C\|u - v\|,
\]

for all \( u, v \in H, \ t \in \mathbb{R} \). Also, it is obvious that for any \( u \in H \) the map \( t \mapsto F(t, u) \) is measurable, which means that for any open set \( U \) the inverse image for \( u \) fixed given by

\[
F_u^{-1}(U) = \{t \in \mathbb{R} : F(t, u) \cap U \neq \emptyset\}
\]

is measurable.

We then rewrite our problem in the abstract form

\[
\begin{aligned}
\frac{du_\lambda}{dt} + AD_\lambda(u_\lambda) &\in F(t, u_\lambda), \\
u_\lambda(s) &= u_{s, \lambda},
\end{aligned}
\]

where \( AD_\lambda \) is the same as before. Integral and strong solutions of (24) on an interval \([s, T]\) are defined in the same way as in (10), (11). It is well known [55, Theorem 3.1] that for any \( T > s \) and \( u_{s, \lambda} \in D(AD_\lambda) = H \) there exists at least one integral solution \( u_\lambda(\cdot) \) to problem (24). Also, the set of all integral and strong solutions coincide. We define then the multivalued map \( U_\lambda : \mathbb{R}_+^2 \times H \to \mathcal{P}(H) \) by

\[
U_\lambda(t, s, u_{s, \lambda}) = \{u_\lambda(t) : u_\lambda \text{ is an integral solution of (24)}\}.
\]

This map is a strict multivalued process, that is:

1. \( U_\lambda(t, t, \cdot) \) is the identity map for any \( t \in \mathbb{R} \);
2. \( U_\lambda(t, s, x) = U_\lambda(t, r, U_\lambda(r, s, x)) \) for all \( s \leq r \leq t, \ x \in H \).
This fact can be proved in the same way as in [61, Proposition 4.6]. We note that it also follows from this proof that the concatenation of two integral solutions is a new integral solution, which implies that every integral solution is global, that is, it can be continued for any forward time till \(+\infty\).

In order to study the asymptotic behaviour of solutions of problem (24) we will consider the pullback attraction of parametrized families of sets rather than the forward attraction of bounded sets as in the autonomous case. Namely, let $\mathcal{D}_\beta$ be the class of all families $\hat{D} = \{D(t) : t \in \mathbb{R}\}$ such that $D(t)$ are bounded sets and
\[
\lim_{t \rightarrow -\infty} e^{\beta t} \|D(t)\|_+ = 0,
\]
where $\|D(t)\|_+ = \sup_{y \in D(t)} \|y\|$. The class $\mathcal{D}_\beta$ is inclusion-closed, that is, if $\hat{D}_1 \in \mathcal{D}_\beta$ and $\hat{D}_2$ is such that $D_2(t) \subset D_1(t)$, for any $t \in \mathbb{R}$, then $\hat{D}_2 \in \mathcal{D}_\beta$.

We recall that a family $\hat{A} = \{A(t) : t \in \mathbb{R}\}$ is called a global pullback $\mathcal{D}_\beta$-attractor if:

1. $A(t)$ is compact for any $t \in \mathbb{R}$;
2. $\hat{A}$ is pullback $\mathcal{D}_\beta$-attracting, that is, for any $\hat{D} \in \mathcal{D}_\beta$ one has
   \[
   \lim_{s \rightarrow -\infty} \text{dist}(U_{\lambda}(t, s, D(s)), A(t)) = 0 \text{ for any } t \in \mathbb{R},
   \]
3. $\hat{A}$ is negatively semi-invariant, which means that
   \[
   A(t) \subset U_{\lambda}(t, s, A(s)) \text{ for any } s \leq t.
   \]

If in the third property we have an equality, then $\hat{A}$ is said to be invariant.

We shall establish some previous statements.

**Lemma 3.1.** The family $\hat{B}_0 = \{B_0(t) : t \in \mathbb{R}\}$ defined by
\[
B_0(t) = \{y \in H : \|y\| \leq r(t)\},
\]
\[
r^2(t) = R + e^{-\beta t} \int_{-\infty}^{t} e^{\beta r} \|g(r)\|^2 dr,
\]
where $R > 0$ is an universal constant, is pullback $\mathcal{D}_\beta$-absorbing for $U_{\lambda}$. This means that for any $t \in \mathbb{R}$ and $\hat{D} \in \mathcal{D}_\beta$ there exists $T(t, \hat{D})$ such that
\[
U_{\lambda}(t, s, D(s)) \subset B_0(t) \text{ for any } s \leq T.
\]

Moreover, $\hat{B}_0 \in \mathcal{D}_\beta$.

**Proof.** Let $\hat{D} \in \mathcal{D}_\beta$, $u_{s, \lambda} \in D(s)$ and $u_{\lambda}(\cdot)$ be an arbitrary integral solution with $u_{\lambda}(s) = u_{s, \lambda}$. Since $u_{\lambda}$ is a strong solution as well, we have
\[
\frac{du_{\lambda}}{dt} + A^{\lambda}(u_{\lambda}(t)) = f_{\lambda}(t) + g(t) \text{ for a.e. } t > s, \tag{25}
\]
where $f_{\lambda} \in L^2_{\text{loc}}(s, +\infty; H)$ and $f_{\lambda}(t) \in F(u_{\lambda}(t))$ for a.a. $t > s$. Multiplying (25) by $u_{\lambda}$ and using (15) we obtain
\[
\frac{1}{2} \frac{d}{dt} \|u_{\lambda}\|^2 + \frac{w_1}{2\gamma^p} \|u_{\lambda}\|^p \leq \frac{1}{2} \|g(t)\|^2 + \frac{1}{2} \|u_{\lambda}\|^2 + R_1,
\]
where $R_1$ is a positive constant. Young’s inequality gives
\[
\frac{\beta+1}{2} \|u_{\lambda}\|^2 \leq \frac{w_1}{2\gamma^p} \|u_{\lambda}\|^p + R_2,
\]
so

\[ \frac{d}{dt} \| u_\lambda \|^2 + \beta \| u_\lambda \|^2 \leq R_3 + \| g(t) \|^2. \]

By Gronwall's lemma we obtain

\[ \| u_\lambda (t) \|^2 \leq e^{-\beta(t-s)} \| u_{s,\lambda} \|^2 + \frac{R_3}{\beta} + e^{-\beta t} \int_s^t e^{\beta r} \| g(r) \|^2 \, dr \]

which implies by choosing \( R = 1 + \frac{R_3}{\beta} \) that \( \hat{B}_0 \) is a pullback \( \mathcal{D}_\beta \)-absorbing family.

Also, it is clear that \( \hat{B}_0 \in \mathcal{D}_\beta \). \( \square \)

**Remark 6.** The constants in inequality (26) are independent of \( \lambda \), so that the estimate is uniform with respect to this parameter.

**Lemma 3.2.** For any \( t \in \mathbb{R}, r > 0 \) the map \( U_\lambda (t+r, t, \cdot) \) maps bounded set of \( H \) into precompact ones.

**Proof.** Let \( u_\lambda (\cdot) \) be an arbitrary integral solution with \( \| u_\lambda (t) \| \leq R \), which satisfies equality (25). By the definition of subdifferential map, (22) and \( \varphi^{D_\lambda}(0) = 0 \) we have

\[ \varphi^{D_\lambda}(u_\lambda(s)) \leq \left( -\frac{du_\lambda}{dt} + f_\lambda(s) + g(s), u_\lambda(s) \right) \]

\[ \leq -\frac{1}{2} \frac{d}{dt} \| u_\lambda \|^2 + (D \| u_\lambda(s) \| + \| g(s) \|) \| u_\lambda(s) \| \]

\[ \leq -\frac{1}{2} \frac{d}{dt} \| u_\lambda \|^2 + D_1 \| g(s) \|^2 + D_2 \| u_\lambda(s) \|^2. \]

Hence, by (26) we have

\[ \int_t^{t+r} \varphi^{D_\lambda}(u_\lambda(s)) \, ds \leq \frac{1}{2} \| u_\lambda(t) \|^2 + D_1 \int_t^{t+r} \| g(s) \|^2 \, ds + D_2 \int_t^{t+r} \| u_\lambda(s) \|^2 \, ds \]

\[ \leq \frac{1}{2} R^2 + D_1 \int_t^{t+r} \| g(s) \|^2 \, ds + r D_2 \left( R^2 + \frac{R_3}{\beta} \right) \]

\[ + D_2 \int_t^{t+r} e^{-\beta s} \int_t^s e^{\beta x} \| g(x) \|^2 \, dx \, ds \]

\[ = \frac{1}{2} R^2 + D_1 \int_t^{t+r} \| g(s) \|^2 \, ds + r D_2 \left( R^2 + \frac{R_3}{\beta} \right) \]

\[ + D_2 \int_t^{t+r} \| g(x) \|^2 \int_x^{t+r} e^{\beta(x-s)} \, ds \, dx \]

\[ \leq \frac{1}{2} R^2 + \left( D_1 + \frac{D_2}{\beta} \right) \int_t^{t+r} \| g(s) \|^2 \, ds + r D_2 \left( R^2 + \frac{R_3}{\beta} \right) \]

\[ = C(t, t+r). \]

We multiply (25) by \( \frac{du_\lambda}{dt} \). Using (22) and \( \frac{d}{dt} \varphi^{D_\lambda}(u_\lambda(t)) = \left( A^{D_\lambda}(u_\lambda(t)), \frac{du_\lambda}{dt} \right) \)

for a.a. \( t \) [8, p.189] we have

\[ \frac{1}{2} \left\| \frac{du_\lambda}{dt} \right\|^2 + \frac{d}{dt} \varphi^{D_\lambda}(u_\lambda) \leq \| f_\lambda(s) \|^2 + \| g(s) \|^2 \]
\[ \leq D^2 \|u_\lambda(s)\|^2 + \|g(s)\|^2 \]
\[ \leq D^2 R^2 + \frac{D^2 R^3}{\beta} + D^2 e^{-\beta s} \int_{-\infty}^{\infty} e^{\beta x} \|g(x)\|^2 \, dx + \|g(s)\|^2 \]
\[ = l(s). \]

Denote \( C_2(t, t + r) = \int_{t}^{t+r} l(s) \, ds \). Then the uniform Gronwall’s lemma [52] implies
\[ \kappa_1 \|u_\lambda(t + r)\|_E^p \leq \varphi^{D^\lambda}(u_\lambda(t + r)) \leq \frac{C_1(t, t + r)}{r} + C_2(t, t + r). \]

Therefore, the statement follows from Lemma 2.1.

**Corollary 3.** \( U_\lambda \) is pullback \( D_\beta \)-asymptotically compact, which means that for any \( \hat{D} \in D_\beta \), \( t \in \mathbb{R} \) and every sequence of times \( s_n \to -\infty \), any sequence \( y_n \in U_\lambda(t, s_n, D(s_n)) \) is precompact.

**Proof.** In view of Lemma 3.1 there exists \( T\left(t - 1, \hat{D}\right) \) such that
\[ U_\lambda(t - 1, s_n, D(s_n)) \subset B_0(t - 1) \]
for all \( s_n \leq T \).

Then
\[ y_n \in U_\lambda(t, s_n, D(s_n)) = U_\lambda(t, t - 1, u_\lambda(t - 1, s_n, D(s_n))) \]
\[ \subset U_\lambda(t, t - 1, B_0(t - 1)) \]
and the result is a consequence of Lemma 3.2.

Further, we shall prove the upper semicontinuity of the map \( x \mapsto U_\lambda(t, s, x) \).

**Lemma 3.3.** For any \( t \geq s \), \( x, y \in H \) one has
\[ \text{dist}_H(U_\lambda(t, s, x), U_\lambda(t, s, y)) \leq e^{2C(t-s)} \|x - y\|, \tag{27} \]
where \( C \) is the constant from (9).

**Proof.** Let \( z \in U_\lambda(t, s, x) \) and let \( u_\lambda(\cdot) \) be an integral solution of problem (24) such that \( u_\lambda(s) = x \) and \( u_\lambda(t) = z \). The corresponding selection \( f_\lambda(t) \) from (25) satisfies \( \text{dist}(f_\lambda(t), F(u_\lambda(t))) = 0 \) for a.a. \( t > s \), thus \( \text{dist}(f_\lambda(t) + g(t), \bar{F}(t, u_\lambda(t))) = 0 \) for a.a. \( t > s \). In view of [55, Theorem 3.1] there exists an integral solution \( v_\lambda(\cdot) \) of problem (24) such that \( v_\lambda(s) = y \) and
\[ \|u_\lambda(t) - v_\lambda(t)\| \leq e^{2C(t-s)} \|x - y\|. \]
Hence, since \( z \in U_\lambda(t, s, x) \) is arbitrary, we have
\[ \text{dist}(U_\lambda(t, s, x), U_\lambda(t, s, y)) \leq e^{2C(t-s)} \|x - y\|. \]

Arguing in the same way we can prove the converse inequality
\[ \text{dist}(U_\lambda(t, s, y), U_\lambda(t, s, x)) \leq e^{2C(t-s)} \|x - y\|, \]
so (27) holds.

Since the operator \( A^{D^\lambda} \) generates a compact semigroup \( S^{D^\lambda} \), it follows from [55, Theorem 3.4] that the set of all integral solutions of problem (24) with an initial data \( u_{\lambda x} \) in \( H \) is compact in the space \( C([s, t], H) \) for any \( t > s \). Thus, the set \( U_\lambda(t, s, x) \) is compact in \( H \) for any \( x \in H, t \geq s \).

**Corollary 4.** The multivalued process \( U_\lambda \) has compact values and the map \( x \mapsto U_\lambda(t, s, x) \) is upper semicontinuous.
Proof. As in view of \((27)\) \(x \mapsto U_\lambda(t, s, x)\) is continuous with respect to the Hausdorff metric, it is \(w\)-upper semicontinuous. Therefore, since \(U_\lambda\) has compact values, it follows that \(x \mapsto U_\lambda(t, s, x)\) is upper semicontinuous (see the beginning of Section 2.2).

Now we are ready to state the main result in this section.

**Theorem 3.4.** The multivalued process \(U_\lambda\) possesses a global pullback \(D_\beta\)-attractor \(\hat{A} = \{A(t) : t \in \mathbb{R}\}\) defined by

\[
A(t) = \bigcap_{s \leq t} \bigcup_{\tau \leq s} U_\lambda(t, \tau, \hat{B}_0(\tau)),
\]

where \(\hat{B}_0\) is the absorbing family given in Lemma 3.1. Moreover, \(\hat{A} \in D_\beta\), it is invariant and unique.

**Proof.** Putting together the results of this section we have obtain that:
1. There exists a pullback \(D_\beta\)-absorbing family \(\hat{B}_0 \in D_\beta\) such that the sets \(B_0(t)\) are closed;
2. \(U_\lambda\) is pullback \(D_\beta\)-asymptotically compact;
3. \(U_\lambda\) has compact values;
4. The map \(x \mapsto U_\lambda(t, s, x)\) is upper semicontinuous;
5. \(U_\lambda\) is a strict multivalued process.

Then the statement follows from [10, Theorem 3.3].

4. **The case of a non-Lipschitz nonlinearity.** Let us consider now the problem

\[
\left\{ \begin{array}{l}
\frac{du_\lambda}{dt} - \text{div}(D^\lambda(x)|\nabla u_\lambda|^{p-2}\nabla u_\lambda) + a(x)|u_\lambda|^{p-2}u_\lambda \in f(x, u_\lambda), \\
\end{array} \right.
\]

\[
u_\lambda(0) = u_{0, \lambda},
\]

(28)

where the (possibly) multivalued map \(f : \mathbb{R}^n \times \mathbb{R} \to 2\mathbb{R}\) satisfies the following assumptions:

\((F1)\) \(f(x, u) \in C_v(\mathbb{R})\) for a.a. \(x \in \mathbb{R}^n\) and any \(u \in \mathbb{R}\).

\((F2)\) For some \(\alpha > 0\), \(c(\cdot) \in H = L^2(\mathbb{R}^n)\) we have

\[
\sup_{y \in f(x, u)} |y| \leq \alpha |u|^q + c(x), \text{ for a.a. } x \in \mathbb{R}^n \text{ and any } u \in \mathbb{R},
\]

where \(1 \leq q < \frac{2}{p'}\).

\((F3)\) \(f\) is Caratheodory, that is, measurable in \(x\) and continuous in \(u\) (in the set-valued sense).

We recall briefly the definition of continuity of set-valued maps. The definition of upper semicontinuity is given in Section 2.2. The map \(g : \mathbb{R} \to 2\mathbb{R}\) is lower semicontinuous if for any \(x \in D(g)\), \(y \in g(x)\) and any sequence \(x_n \in D(g)\) such that \(x_n \to x\), there exists a sequence \(y_n \in g(x_n)\) such that \(y_n \to y\). It is continuous if it is upper and lower semicontinuous.

We define the multivalued operator \(B : H \to 2^H\) by

\[
B(u(\cdot)) = \{\xi(\cdot) \in H : \xi(x) \in -f(x, u(x)) \text{ x-a.e. in } \mathbb{R}^n\}.
\]

Denote by \(D(B)\) the domain of the operator \(B\).

We consider then problem (28) in the abstract form

\[
\left\{ \begin{array}{l}
\frac{du_\lambda}{dt} + A^\lambda(u_\lambda(t)) + B(u_\lambda(t)) \ni 0, \\
u_\lambda(0) = u_{0, \lambda}.
\end{array} \right.
\]

(29)
We shall obtain some properties of the operator $B$.

**Lemma 4.1.** Let $f$ satisfy (F1), (F3) and the following:

(F2) For some $\alpha > 0$, $c(\cdot) \in H$ we have

\[
\sup_{y \in f(x,u)} |y| \leq \alpha |u|^q + c(x), \text{ for a.a. } x \in \mathbb{R}^n \text{ and any } u \in \mathbb{R},
\]

where

\[
1 \leq q < p \text{ if } p \geq \frac{n}{2},
\]

\[
1 \leq q < \frac{np}{2(n-p)} \text{ if } p < \frac{n}{2}.
\]

Then $\mathcal{D}(\varphi^{D^\lambda}) \subset \mathcal{D}(B)$ and there exist $C > 0$, $0 < \gamma < 1$ such that

\[
\sup_{y \in B(u)} \|y\| \leq C(1 + (\varphi^{D^\lambda}(u))^{1-\gamma}) \text{ for any } u \in \mathcal{D}(\varphi^{D^\lambda}). \tag{30}
\]

Moreover, the constants $C, \gamma$ do not depend on $\lambda$.

If (F2) also holds, then $\frac{1}{2} < \gamma < 1$ and there exists $\beta > 0$ (not depending on $\lambda$) such that for any $u \in \mathcal{D}(\partial \varphi^{D^\lambda})$, $y \in \partial \varphi^{D^\lambda}(u) + B(u)$ we have

\[
(-y, u) + 2\varphi^{D^\lambda}(u) \leq \beta(1 - \|u\|^2). \tag{31}
\]

**Remark 7.** It is clear that (F2) implies (F̃2). Also, (30) implies

\[
\sup_{y \in B(u)} \|y\| \leq \tilde{C}(1 + \varphi^{D^\lambda}(u)) \text{ for any } u \in \mathcal{D}(\varphi^{D^\lambda}),
\]

provided $B(u)$ is measurable.

**Proof.** Let $u \in \mathcal{D}(\varphi^{D^\lambda})$. Since the map $f$ is Caratheodory in the multivalued sense, by [4, Theorem 8.2.8] we obtain that the composition $x \mapsto f(x, u(x))$ is measurable. Hence, there exists a measurable selection, that is, a measurable map $\xi(\cdot)$ such that $\xi(x) \in f(x, u(x))$ for a.a. $x \in \Omega$ (see [4, Theorem 8.1.3]).

First, let $n > p$. Let us prove that $\xi(\cdot) \in H$, which implies that $u \in \mathcal{D}(B)$. Indeed, by (F2) we have

\[
|\xi(x)|^2 \leq 2 \left( \alpha^2 |u(x)|^{2q} + c^2(x) \right) \tag{32}
\]

and the result follows from $u \in E \subset L^{\frac{np}{n-p}}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, $2q \leq \frac{np}{n-p}$ and the interpolation inequality. Moreover, (32) is true for any $y \in B(u)$, so that

\[
\sup_{y \in B(u)} \|y\|^2 \leq C_1(1 + \|u\|^{2q}_{2q}).
\]

Further, using $D^\lambda(x) \geq \sigma > 0$ we obtain

\[
\sup_{y \in B(u)} \|y\|^2 \leq C_2(1 + \|u\|^{2q}_{2q})
\]

\[
= C_2 \left[ 1 + \left( \int_{\mathbb{R}^n} |
abla u|^p dx + \int_{\mathbb{R}^n} a(x)|u|^p dx \right)^{\frac{2p}{p}} \right]
\]

\[
\leq C_3 \left[ 1 + \frac{1}{p} \left( \int_{\mathbb{R}^n} \sigma|
abla u|^p dx + \int_{\mathbb{R}^n} a(x)|u|^p dx \right)^{\frac{2p}{p}} \right]
\]

\[
\leq C_3(1 + (\varphi^{D^\lambda}(u))^{\frac{2q}{2q}}),
\]

and then

\[
\sup_{y \in B(u)} \|y\| \leq C(1 + (\varphi^{D^\lambda}(u))^{\frac{p}{p}}). \tag{33}
\]

Hence, (30) holds with $\gamma = \frac{p}{p}$. We note that $0 < \gamma < 1$ as $q < p$. 

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Now, let \( n = p \). In the same way, using the embedding \( E \subset L^q(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \),
for all \( q \geq p \), we prove \( \xi(\cdot) \in H \) and (33).

Finally, if \( n < p \), then we use the embedding \( E \subset L^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \).

If (F2) holds, then

\[
\gamma = \frac{p-q}{p} = 1 - \frac q p > 1 - \frac 1 2 = \frac 1 2.
\]

Also, let \( u \in \mathcal{D}(\partial \varphi^{D^\lambda}) \), \( y = y_1 + y_2 \in \partial \varphi^{D^\lambda}(u) + B(u) \). Then

\[
(-y, u) + 2 \varphi^{D^\lambda}(u) = \left(-A^{D^\lambda}(u), u \right) - (y_2, u) + 2 \varphi^{D^\lambda}(u)
= (-p + 2) \varphi^{D^\lambda}(u) - (y_2, u).
\]

Using \( q + 1 < \frac{p}{2} + 1 < p \) and the same embeddings as before we obtain

\[
\|(y_2, u)\| \leq \int_{\mathbb{R}^n} (\alpha |u(x)|^q + |c(x)|)|u(x)| \, dx
\leq \left(\alpha \|u\|_{2q}^q + \|c\|\right) \|u\| 
\leq \frac{C_4}{\|u\|_E^{q+1}} \left(\int_{\mathbb{R}^n} a(x)|u|^p \, dx \right)^{\frac{q+1}{p}}
\leq C_5 \left[1 + \left(\frac{1}{p} \int_{\mathbb{R}^n} \sigma|\nabla u|^p \, dx + \int_{\mathbb{R}^n} a(x)|u|^p \, dx \right)^{\frac{q+1}{p}} \right]
\leq C_5(1 + (\varphi^{D^\lambda}(u))^{\frac{q+1}{p}}).
\]

Thus, considering \( \delta = p - 2 > 0 \), we get

\[
(-y, u) + 2 \varphi^{D^\lambda}(u) \leq -\delta \varphi^{D^\lambda}(u) + C_5(1 + (\varphi^{D^\lambda}(u))^{\frac{q+1}{p}})
\leq -\frac{\delta}{2} \varphi^{D^\lambda}(u) + C_6 \leq -\frac{\delta}{2} C_7 \|u\|_E^p + C_6
\leq C_8 (1 - \|u\|_E^2),
\]

so that (31) follows from the embedding \( E \subset L^2(\mathbb{R}) \).

\[\square\]

We prove also some additional properties of the operator \( B \).

**Lemma 4.2.** Let \( f \) satisfy (F1), (F2), (F3). Then \( B(u) \) is non-empty and bounded for any \( u \in \mathcal{D}(\varphi^{D^\lambda}) \), and closed and convex for any \( u \in \mathcal{D}(B) \).

**Proof.** It follows from Lemma 4.1 that the set \( B(u) \) is non-empty and bounded if \( u \in \mathcal{D}(\varphi^{D^\lambda}) \).

Let \( y_n \rightarrow y \) in \( H \) and \( y_n \in B(u) \). Then \( y_n(x) \rightarrow y(x) \) for a.a. \( x \in \mathbb{R}^n \). Since the set \( f(x, u(x)) \) is closed by (F1), we obtain that \( y(x) \in f(x, u(x)) \) for a.a. \( x \in \Omega \).

Hence, \( y \in B(u) \) and \( B(u) \) is closed.

Finally, let \( y, z \in B(u) \) and \( 0 \leq \alpha \leq 1 \). We have to check that \( \xi = ay + (1 - \alpha)z \in B(u) \). By (F1) the set \( f(x, u(x)) \) is convex, and then \( \xi(x) \in f(x, u(x)) \) for a.a. \( x \in \Omega \).

Hence, \( \xi \in B(u) \) and \( B(u) \) is convex. \(\square\)

**Lemma 4.3.** Let \( f \) satisfy (F1), (F2), (F3). Then \( B \) satisfies the properties:

- (B1) \( B \) is measurable in the following sense: For any \( u \in L^2(a, b; H) \) such that \( u(t) \in \mathcal{D}(\varphi^{D^\lambda}) \) for a.a. \( t \in (a, b) \), there exists an \( H \)-valued measurable function \( b(t) \) such that \( b(t) \in B(u(t)) \) for a.a. \( t \in (a, b) \).
(B2) $B$ is demiclosed in the following sense: If $u_n \to u$ in $L^2(a, b; H)$ and $b_n \to b$ weakly in $L^2(a, b; H)$ with $b_n(t) \in B(u_n(t))$ for a.a. $t \in (a, b)$, then $b(t) \in B(u(t))$ for a.a. $t \in (a, b)$.

**Proof.** First, we prove (B1). Arguing as in the proof of Lemma 4.1 we obtain that the composition $(t, x) \mapsto f(x, u(t, x))$ is measurable, the existence of a measurable map $b(\cdot, \cdot)$ such that $b(t, x) \in f(x, u(t, x))$, for a.a. $(t, x) \in (a, b) \times \Omega$, and the inequality

$$|b(t, x)|^2 \leq 2 \left( \alpha^2 |u(t, x)|^2 + c^2(x) \right). \quad (34)$$

For a fixed $t$ the map $x \mapsto b(t, x)$ is measurable. Hence, since for a.a. $t$ we have $u(t) \in E \subset L^2(\mathbb{R}^n)$ (see again the proof of Lemma 4.1), we obtain that $b(t) = b(t, \cdot) \in H$ for a.a. $t \in (a, b)$. Now we define the sequence of measurable sets $\Omega_n^t$ by

$$\Omega_n^t = \{ x \in \mathbb{R}^n : |x| > n \text{ or } |b(t, x)| > n \}$$

and the sequence of measurable approximations

$$b_n(t, x) = \begin{cases} 0 & \text{if } t \in (a, b), \ x \in \Omega_n^t, \\ b(t, x) & \text{if } t \in (a, b), \ x \in \mathbb{R}^n \setminus \Omega_n^t. \end{cases}$$

It is clear that $b_n \in L^2(a, b; L^2(\mathbb{R}^n))$. Let $B_M = \{ x \in \mathbb{R}^n : |x| < M \}$. We fix $t \in (a, b)$ and restrict the functions $b_n, b$ to the ball $B_M$. It is clear that $b_n(t) \to b(t)$ in measure and

$$|b_n(t, x)| \leq |b(t, x)| \text{ for a.a. } x \in B_M.$$ 

Then the dominated convergence theorem implies that $b_n(t) \to b(t)$ in $L^2(B_M)$. Since for any $\varepsilon > 0$ there exists $M(\varepsilon)$ such that

$$\int_{\mathbb{R}^n \setminus B_M} |b_n(t, x)|^2 \, dx \leq \int_{\mathbb{R}^n \setminus B_M} |b(t, x)|^2 \, dx < \varepsilon,$$

it follows easily that $b_n(t) \to b(t)$ in $L^2(\mathbb{R}^n)$. Then $t \mapsto b(t) \in H$ is the pointwise limit of a sequence of measurable functions, so that it is measurable.

Let us check (B2). Since $u_n \to u$ in $L^2((a, b) \times \mathbb{R}^n)$, we have that $u_n(t, x) \to u(t, x)$ for a.a. $(t, x)$. On the other hand, the continuity of the map $u \mapsto f(x, u)$ implies that for a.a. $(t, x)$ we have

$$\text{dist}(b_n(t, x), f(x, u(t, x))) \leq \text{dist}(f(x, u_n(t, x), f(x, u(t, x)))) \to 0 \text{ as } n \to \infty. \quad (35)$$

Using [55, Proposition 1.1] for a.e. $t \in (a, b)$ we obtain

$$b(t) \in \bigcap_{n=1}^{\infty} \cap_{k \geq n} b_k(t) = \mathcal{A}(t).$$

Fix $t$ and denote $\mathcal{A}_n(t) = \cap_{k \geq n} b_k(t)$. Clearly, $z \in \mathcal{A}(t)$ if and only if there exist $z_n \in \mathcal{A}_n(t)$ such that $z_n \to z$ in $H$. Thus, up to a subsequence we have $z_n(x) \to z(x)$, a.e. in $\mathbb{R}^n$. On the other hand, $z_n \in \mathcal{A}_n(t)$ implies that

$$z_n(t) = \sum_{i=1}^{N} \lambda_i b_{k_i}(t),$$

where $\lambda_i \in [0, 1]$, $\sum_{i=1}^{N} \lambda_i = 1$ and $k_i \geq n$, for any $i$. Note that $f(x, u(t, x)) = [a_1(t, x), a_2(t, x)]$ as $f$ has convex, bounded, closed values. Then (35) implies that for any $\varepsilon > 0$ and a.a. $x \in \Omega$ there exists $n = n(t, x, \varepsilon)$ such that

$$b_k(t, x) \subset [a_1(t, x) - \varepsilon, a_2(t, x) + \varepsilon], \ \forall k \geq n.$$
Thus,
\[ z_n(t, x) \subset [a_1(t, x) - \varepsilon, a_2(t, x) + \varepsilon] \]
and passing to the limit we have
\[ z(t, x) \in [a_1(t, x), a_2(t, x)] \text{ for a.a. } x. \]

Therefore, \( z(t) \in B(u(t)) \), so that \( b(t) \in A(t) \subset B(u(t)) \), a.e. on \((a, b)\). \( \square \)

Using general results proved in [44] about the existence of solutions for abstract
differential inclusions we will obtain the existence of strong solutions of problem
(29) in the following sense.

**Definition 4.4.** A function \( u_\lambda(\cdot) \in C([0, T], H) \) is called a strong solution of
problem (29) if \( u_\lambda(0) = u_{0, \lambda}, u_\lambda(\cdot) \in W^{1, 2}([\delta, T], H) \) on each interval \([\delta, T]\), \( 0 < \delta < T \), \( u_\lambda(t) \in \mathcal{D}(A^{D_\lambda}) \) a.e. and there exists a function \( b_\lambda(\cdot) \in L^2(0, T; H) \),
\( b_\lambda(t) \in B(u_\lambda(t)) \) a.e. on \([0, T]\), such that
\[
\frac{du_\lambda}{dt} + A^{D_\lambda}(u_\lambda(t)) + b_\lambda(t) = 0 \text{ a.e. on } [0, T].
\]

First, we establish an auxiliary result.

**Proposition 3.** Let \((F1) - (F3)\) hold. Then for any \( \lambda \in [0, \infty) \), \( u_{0, \lambda} \in H, T > 0 \),
there exist functions \( u_\lambda, g_\lambda, b_\lambda : [0, T] \to H \) such that:
1. \( u_\lambda(\cdot) \in C([0, T], H), u_\lambda(0) = u_{0, \lambda}; \)
2. \( u_\lambda(\cdot) \) is absolutely continuous on \([\delta, T]\) for any \( 0 < \delta < T \) and \( u_\lambda(t) \in \mathcal{D}(A^{D_\lambda}) \)
a.e.;
3. \( g_\lambda(\cdot), b_\lambda(\cdot) \in L^2(0, T; H); \)
4. \( g_\lambda(t) = A^{D_\lambda}(u_\lambda(t)) \) a.e.;
5. \( b_\lambda(t) \in B(u_\lambda(t)) \) a.e.;
6. The equality
\[
\frac{du_\lambda}{dt}(t) + g_\lambda(t) + b_\lambda(t) = 0 \text{ a.e.}
\]
holds.

Moreover,
\[
\frac{d^2u_\lambda}{dt^2}(t), \frac{d^2g_\lambda(t)}{dt^2}, \frac{d^2b_\lambda(t)}{dt^2} \in L^2(0, T; H),
\]
\[
\varphi^{D_\lambda}(u_\lambda(t)) \in L^1(0, T), \quad t\varphi^{D_\lambda}(u_\lambda(t)) \in L^\infty(0, T).
\]

**Proof.** We note that by Lemma 2.2 the level sets \( H_k \) for the maps \( \varphi^{D_\lambda} \) are compact
in \( H \).

In view of Lemmas 4.1, 4.2, 4.3 and Remark 7 the conditions of Theorems III
and IV in [44] are satisfied, from which the result follows. \( \square \)

**Remark 8.** Every function \( u_\lambda \) satisfying properties 1-6 can be continued globally,
that is, to a function defined on \([0, +\infty)\) satisfying the same properties in every
interval \([0, T]\). This follows also from Theorem IV in [44].

**Theorem 4.5.** Let \((F1) - (F3)\) hold. Then for any \( \lambda \in [0, \infty) \), \( u_{0, \lambda} \in H \), there
exists a globally defined strong solution \( u_\lambda(\cdot) \) of problem (29). This solution satisfies
(37) for every \( T > 0 \).
Proof. Let $u_{\lambda}(\cdot)$ be the function from Proposition 3. It remains to prove only that $b_{\lambda}(\cdot) \in L^2(0, T; H)$ for any $T > 0$. From (30) we have
\[
\|b_{\lambda}(t)\|^2 \leq C_1(1 + \left| \varphi^{D^\lambda}(u_{\lambda}(t)) \right|^{2-2\gamma}).
\]
By (37) we obtain
\[
\left| \varphi^{D^\lambda}(u_{\lambda}(t)) \right|^{2-2\gamma} = \frac{1}{t^{2-2\gamma}} \left| t\varphi^{D^\lambda}(u_{\lambda}(t)) \right|^{2-2\gamma} \leq C_2 \frac{1}{t^{2-2\gamma}}.
\]
Since $\frac{1}{2} < \gamma < 1$, it follows that $b_{\lambda}(\cdot) \in L^2(0, T; H)$ for any $T > 0$. Hence, $u_{\lambda}(\cdot)$ is a globally defined strong solution.

We have obtained the existence of a strong solution with good regularity properties. We need to prove further that every strong solution also satisfies good properties.

Lemma 4.6. Every strong solution of (29) $u_{\lambda}(\cdot)$ with $u_{0,\lambda} \in H$ satisfies (13) and $t \mapsto \varphi^{D^\lambda}(u_{\lambda}(t))$ is absolutely continuous on $[\delta, T]$, for all $\delta > 0$. If $u_{0,\lambda} \in \mathcal{D}(\varphi^{D^\lambda})$, then $\frac{du_{\lambda}}{dt} \in L^2(0, T; H)$ and $t \mapsto \varphi^{D^\lambda}(u_{\lambda}(t))$ is absolutely continuous on $[0, T]$.

Proof. Since $b_{\lambda}(\cdot) \in L^2(0, T; H)$, by Proposition 1 we have that $u_{\lambda}(\cdot)$ is the unique strong solution of the problem
\[
\begin{cases}
\frac{du_{\lambda}}{dt} + A^{D^\lambda}(u_{\lambda}(t)) = -b_{\lambda}(t), \\
u(0) = u_0.
\end{cases}
\]
Hence, the required properties follow.

Remark 9. In view of Remark 8 and Lemma 4.6 every strong solution of (29) can be extended to the whole semiline $[0, \infty)$ and satisfies the properties given in this lemma for every $T > 0$.

We prove now that the concatenation of two global strong solutions is a new global strong solution.

Lemma 4.7. Let $u_{1,\lambda}(\cdot), u_{2,\lambda}(\cdot)$ be two strong global solutions of problem (29) such that $u_{1,\lambda}(0) = u_{0,\lambda}, u_{2,\lambda}(0) = u_{1,\lambda}(\tau)$. Then
\[
u_{\lambda}(t) = \begin{cases} u_{1,\lambda}(t) & \text{if } 0 \leq t \leq \tau, \\
u_{2,\lambda}(t-\tau) & \text{if } t \geq \tau,
\end{cases}
\]
is a global strong solution of (29) with $u_{\lambda}(0) = u_{0,\lambda}$.

Proof. Let
\[
b_{\lambda}(t) = \begin{cases} b_{1,\lambda}(t) & \text{for a.a. } 0 < t < \tau, \\
b_{2,\lambda}(t-\tau) & \text{for a.a. } t > \tau,
\end{cases}
\]
where $b_{i}$ are the functions given in Definition 4.4. Also, let $T > \tau$ and $v_{\lambda}(\cdot)$ be the unique strong solution of problem (38). It is clear that $v_{\lambda}(t) = u_{1,\lambda}(t) = u_{\lambda}(t)$ for $0 \leq t \leq \tau$. We note that $\tau_{\lambda}(\cdot) = v_{\lambda}(\cdot+\tau)$ is the unique strong solution of (38) with $\bar{b}_{\lambda}(\cdot) = b_{\lambda}(\cdot+\tau)$ and $\tau_{\lambda}(0) = u_{1,\lambda}(\tau)$. Since $\bar{b}_{\lambda}(t) = b_{\lambda}(t + \tau) = b_{2,\lambda}(t)$, by uniqueness in problem (38) we have that $\tau_{\lambda}(t) = u_{2,\lambda}(t)$, so that $v_{\lambda}(t) = \tau_{\lambda}(t-\tau) = u_{2,\lambda}(t-\tau) = u_{\lambda}(t)$ for any $t \geq \tau$. Thus, $u_{\lambda}(\cdot)$ is the unique strong solution of (38) for the function $b_{\lambda}(\cdot)$ and $u_{\lambda}(0) = u_{0,\lambda}$. Hence, the regularity properties in Definition 4.4 hold. Finally, it is easy to see that equality (36) is satisfied.
For any \( u_{0,\lambda} \in H \) denote by \( \mathcal{D}_\lambda(u_{0,\lambda}) \) the set of all global strong solutions of (29). Assuming conditions (F1) – (F3) this set is non-empty for every \( u_{0,\lambda} \in H \). Any \( u(\cdot) \in \mathcal{D}_\lambda(u_{0,\lambda}) \) satisfies the regularity properties given in Lemma 4.6. We define the multivalued operator \( G_\lambda : \mathbb{R}^+ \times H \to \mathcal{P}(H) \) by
\[
G_\lambda(t,u_{0,\lambda}) = \{ u_\lambda(t) : u_\lambda(\cdot) \in \mathcal{D}_\lambda(u_{0,\lambda}) \}. \tag{39}
\]

**Lemma 4.8.** \( G_\lambda \) is a strict multivalued semiflow, that is,
\[
G_\lambda(t+s,u_{0,\lambda}) = G_\lambda(t,G_\lambda(s,u_{0,\lambda})),
\]
for all \( t,s \geq 0, \ u_{0,\lambda} \in H \).

**Proof.** It is easy to see that the translation \( v_\lambda(\cdot) = u_\lambda(\cdot+s) \) of a strong solution is again a strong solution. Hence, the inclusion \( G_\lambda(t+s,u_{0,\lambda}) \subset G_\lambda(t,G_\lambda(s,u_{0,\lambda})) \) follows. The converse inequality is a consequence of Lemma 4.7. \( \square \)

We shall obtain now some estimates for strong solutions. In the following two lemmas the constants are independent on \( \lambda \) and the initial data.

**Lemma 4.9.** Assume (F1) – (F3). There are positive constants \( r_0, \ t_0 \) such that
\[
\| u_\lambda(t) \| \leq r_0, \text{ for each } t \geq t_0, \lambda \in [0,\infty), \ u_\lambda \in \mathcal{D}_\lambda(u_{0,\lambda}), \ u_{0,\lambda} \in H.
\]
Also, there exist positive constants \( \delta_1, \delta_2 > 0 \) such that
\[
\| u_\lambda(t) \|^2 \leq e^{-\delta_1 t} \| u_\lambda(0) \|^2 + \delta_2 \text{ for all } t \geq 0. \tag{40}
\]

**Proof.** In view of (31) we have
\[
\frac{1}{2} \frac{d}{dt} \| u_\lambda(t) \|^2 + 2 \varphi^{D_\lambda}(u_\lambda(t)) \leq \beta.
\]
Since
\[
\varphi^{D_\lambda}(u_\lambda(t)) \geq \kappa_1 \| u_\lambda(t) \|^p_E \geq \kappa_2 \| u_\lambda(t) \|^p \geq \kappa_3 \| u_\lambda(t) \|^2 - \kappa_4,
\]
for some \( \kappa_j > 0 \), we have
\[
\frac{d}{dt} \| u_\lambda(t) \|^2 + \gamma_2 \| u_\lambda(t) \|^2 - \beta_2 \leq \frac{d}{dt} \| u_\lambda(t) \|^2 + \gamma_1 \| u_\lambda(t) \|^p \leq \beta_1.
\]
Then by [52, p.164] and Gronwall’s lemma we have
\[
\| u_\lambda(t) \|^2 \leq \left( \frac{\beta_1}{\gamma_1} \right)^{\frac{1}{2}} + \left( \gamma_1 \left( \frac{p}{2} - 1 \right) t \right)^{-\frac{1}{p-2}},
\]
\[
\| u_\lambda(t) \|^2 \leq e^{-\gamma_2 t} \| u_\lambda(0) \|^2 + \beta_3.
\]
Thus, the result follows. \( \square \)

**Lemma 4.10.** Assume (F1) – (F3). There exist positive constants \( r_2 > 0 \) and \( t_1 > t_0 \) (where \( t_0 \) is taken from Lemma 4.9) such that
\[
\| u_\lambda(t) \|_E \leq r_2, \text{ for each } t \geq t_1, \lambda \in [0,\infty), \ u_\lambda \in \mathcal{D}_\lambda(u_{0,\lambda}), \ u_{0,\lambda} \in H.
\]
Also, there exist positive constants \( \alpha_j \) such that
\[
\int_t^{t+r} \varphi^{D_\lambda}(u_\lambda(s)) ds \leq \alpha_1 (1 + \| u_\lambda(0) \|^2), \tag{41}
\]
\[
\alpha_2 \| u_\lambda(t+r) \|^p_E \leq \varphi^{D_\lambda}(u_\lambda(t+r)) \leq \alpha_3 \left( \frac{(1+r)(1+\| u_\lambda(0) \|^2)}{r} + r \right) e^{\alpha_4 r}, \tag{42}
\]
\[
\int_t^T \left\| \frac{du_\lambda}{dt} \right\|^2 dt \leq \alpha_5 \left( \frac{(1+r)(1+\| u_\lambda(0) \|^2)}{r} + r \right) e^{\alpha_4 r} (1 + (T-r)), \tag{43}
\]

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for all \( t \geq 0, \ 0 < r < T, \ \lambda \in [0, \infty) \).

**Proof.** By the definition of subdifferential map, \( \varphi^{D\lambda}(u) \geq 0, \ \frac{1}{2} < \gamma < 1 \) and (30) we have

\[
\varphi^{D\lambda}(u(t)) \leq -\left( \frac{du\lambda}{dt} + b\lambda(t), u(t) \right) \\
\leq - \frac{1}{2} \frac{d}{dt} \|u\|^{2} + C((\varphi^{D\lambda}(u(t)))^{1-\gamma} + 1) \|u(t)\| \\
\leq \frac{1}{2} \varphi^{D\lambda}(u(t)) - \frac{1}{2} \frac{d}{dt} \|u\|^{2} + C_{1} + C_{2} \|u(t)\|^{2}.
\]

Hence, by Lemma 4.9,

\[
\int_{t}^{t+r} \varphi^{D\lambda}(u(t)) ds \leq \|u(t)\|^{2} + 2C_{1}r + 2C_{2} \int_{t}^{t+r} \|u(t)\|^{2} ds \\
\leq r^{2}_{0} + 2C_{1}r + 2C_{2}rr_{0}^{2}, \text{ for } t \geq t_{0},
\]

Multiplying (36) by \( \frac{du\lambda}{dt} \) and using \( \frac{d}{dt}\varphi^{D\lambda}(u(t)) = \left( A^{D\lambda}(u(t)), \frac{du\lambda(t)}{dt} \right) \) for a.a. \( t \) [8, p.189] and (30) we have

\[
\frac{1}{2} \left\| \frac{du\lambda}{dt} \right\|^{2} + \frac{d}{dt}\varphi^{D\lambda}(u) \leq \frac{1}{2} \left\| b\lambda(t) \right\|^{2} \leq C_{3}(\left| \varphi^{D\lambda}(u(t)) \right|^{2-2\gamma} + 1) \quad (44)
\]

\[
\leq C_{4}(\varphi^{D\lambda}(u(t)) + 1).
\]

Then the uniform Gronwall’s lemma \([52]\) gives

\[
\kappa_{1} \|u\lambda(t+r)\|_{E}^{p} \leq \varphi^{D\lambda}(u\lambda(t+r)) \leq \left( \frac{r_{0}^{2} + 2C_{1}r + 2C_{2}rr_{0}^{2} + C_{4}r}{r} \right) r C_{4}r, \forall t \geq t_{0},
\]

\[
\kappa_{1} \|u\lambda(t+r)\|_{E}^{p} \leq \varphi^{D\lambda}(u\lambda(t+r)) \leq \left( C_{3}(1 + r) \left( 1 + \|u\lambda(t)\|^{2} \right) \right) \frac{C_{4}r}{r} C_{4}r, \quad (45)
\]

for all \( r \geq 0 \). Integrating in (44) over \((r, T)\) and using (45) we obtain

\[
\int_{r}^{T} \left\| \frac{du\lambda}{dt} \right\|^{2} dt \leq 2\varphi^{D\lambda}(u\lambda(r)) + C_{4} \int_{r}^{T} (\varphi^{D\lambda}(u\lambda(t)) + 1) dt \\
\leq (2 + C_{4}(T - r)) \left( C_{5}(1 + r)(1 + \|u\lambda(t)\|^{2}) \right) e^{C_{4}r} \\
+ C_{4}(T - r).
\]

Thus, the result follows. \( \Box \)

As a consequence of Lemma 4.10 we obtain that the map \( G_{\lambda} \) is compact for positive times.

**Corollary 5.** Assume \((F1) - (F3)\) and let \( \lambda \in [0, \infty) \) be fixed. Then for any \( t > 0 \) the map \( u_{0,\lambda} \mapsto G_{\lambda}(t, u_{0,\lambda}) \) is compact, that is, \( G_{\lambda}(t, B) \) is a relatively compact set for any bounded set \( B \).

**Proof.** Lemma 2.2 implies that the level sets \( H_{L} \) for the maps \( \varphi^{D\lambda} \) are compact in \( H \). Then the result follows by (40), (42). \( \Box \)
Lemma 4.11. Assume (F1) – (F3) and let λ ∈ [0, ∞) be fixed. Let \( u^{n}_{0,\lambda} \to u_{0,\lambda} \), as \( n \to +\infty \), where \( u^{n}_{0,\lambda}, u_{0,\lambda} \in \overline{D(\varphi^{D_{\lambda}})} \), and let \( u^{\lambda}_{\cdot} (\cdot) \in D_{\lambda}(u^{n}_{0,\lambda}) \). Then there exist \( u^{\lambda}_{\cdot} (\cdot) \in D_{\lambda}(u_{0,\lambda}) \) and a subsequence \( u^{n\iota}_{\lambda} (\cdot) \) such that

\[
\begin{align*}
    u^{n\iota}_{\lambda} \to u_{\lambda} \text{ in } C([\varepsilon, T], H) \text{ for all } 0 < \varepsilon < T.
\end{align*}
\]

Proof. We fix first \( T > 0 \). In view of (40) up to a subsequence we have

\[
\begin{align*}
    u^{n}_{\lambda} \to u_{\lambda} \text{ weakly in } L^{2}(0, T; H),
    \frac{d u^{n}_{\lambda}}{d t} \to \frac{d u_{\lambda}}{d t} \text{ in the sense of } H\text{-valued distributions.}
\end{align*}
\]

We shall prove that \( u_{\lambda}(\cdot) \in D_{\lambda}(u_{0,\lambda}) \).

By (30) with \( \frac{1}{2} < \gamma < 1 \), (41) and Hölder’s inequality the selections \( b^{\alpha}_{\lambda}(\cdot) \) satisfy

\[
\begin{align*}
    \int_{0}^{T} \|b^{\alpha}_{\lambda}(t)\|^{2} dt & \leq D_{1} \int_{0}^{T} \left( 1 + \left| \varphi^{D_{\lambda}}(u^{\alpha}_{\lambda}(t)) \right|^{2-2\gamma} \right) dt \\
    & \leq D_{2} \left( 1 + \int_{0}^{T} \left| \varphi^{D_{\lambda}}(u^{\alpha}_{\lambda}(t)) \right|^{2-2\gamma} dt \right) \leq D_{3}.
\end{align*}
\]

Hence, (36), (43) and (47) imply that \( g^{\alpha}_{\lambda}(\cdot) = A^{D_{\lambda}}(u^{\alpha}_{\lambda}(\cdot)) \) satisfy

\[
\begin{align*}
    \int_{\varepsilon}^{T} \| g^{\alpha}_{\lambda}(t) \|^{2} dt & \leq 2 \left[ \int_{\varepsilon}^{T} \left\| \frac{d u^{\alpha}_{\lambda}}{d t}(t) \right\|^{2} dt + \int_{\varepsilon}^{T} \|b^{\alpha}_{\lambda}(t)\|^{2} dt \right] \leq D_{4}(\varepsilon) \text{ for all } \varepsilon > 0.
\end{align*}
\]

Let \( \varepsilon_{N} = \frac{1}{N}, N \in \mathbb{N} \). Inequality (43) implies that the sequence \( u^{\alpha}_{\cdot} (\cdot) \) is equicontinuous in \([\varepsilon_{N}, T]\). Also, (40), (42) and Lemma 2.1 imply that the sequence \( \{ u^{\alpha}_{\lambda}(t) \} \) is relatively compact for all \( t \in [\varepsilon_{N}, T] \). Then from the Ascoli-Arzelà theorem and (46) we can easily deduce that

\[
\begin{align*}
    u^{\alpha}_{\lambda} \to u_{\lambda} \text{ in } C([\varepsilon_{N}, T], H),
    \frac{d u^{\alpha}_{\lambda}}{d t} \to \frac{d u_{\lambda}}{d t} \text{ weakly in } L^{2}(\varepsilon_{N}, T; H).
\end{align*}
\]

We introduce the function \( \psi_{\lambda} : L^{2}(\varepsilon_{N}, T; H) \to (-\infty, +\infty] \) given by

\[
\begin{align*}
    \psi_{\lambda}(u) = \begin{cases}
        \int_{\varepsilon_{N}}^{T} \varphi^{D_{\lambda}}(u(t)) dt, & \text{if } \varphi^{D_{\lambda}}(u) \in L^{1}(\varepsilon_{N}, T), \\
        +\infty, & \text{otherwise},
    \end{cases}
\end{align*}
\]

which is proper, convex and lower semicontinuous. It is known [9, p.47] that for any \( g \in L^{2}(\varepsilon_{N}, T; H), g \in \partial \psi_{\lambda}(u) \) if and only if \( g(t) \in \partial \varphi^{D_{\lambda}}(u(\cdot)) \), a.e. \( t \in (\varepsilon_{N}, T) \).

Also, by (47) and (48) we obtain passing to a subsequence that

\[
\begin{align*}
    b^{\lambda}_{\cdot} \to b_{\lambda} \text{ weakly in } L^{2}(0, T; H),
    g^{\lambda}_{\cdot} \to g_{\varepsilon_{N},\lambda} \text{ weakly in } L^{2}(\varepsilon_{N}, T; H).
\end{align*}
\]

Since the operator \( \partial \psi_{\lambda} \) is demiclosed (see [8]), we obtain that \( g_{\varepsilon_{N},\lambda} \in \partial \psi_{\lambda}(u_{\lambda}), i = 1, 2 \), where we are considering \( u_{\lambda} \) as a function in \( L^{2}(\varepsilon_{N}, T; H) \). Thus, \( g_{\varepsilon_{N},\lambda}(t) = \partial \varphi^{D_{\lambda}}(u_{\lambda}(t)) = A^{D_{\lambda}}(u_{\lambda}(t)) \) for a.a. \( t \in (\varepsilon_{N}, T) \). Also, by property (B2) we obtain that \( b_{\lambda}(t) \in B(u_{\lambda}(t)) \) for a.a. \( t \in (\varepsilon_{N}, T) \).

We note that by a diagonal argument we can choose a common subsequence for all \( N \).
Theorem 4.13. Assume (F1) – (F3) and let λ ∈ [0, ∞) be fixed. Then the map $u_{0,\lambda} \mapsto G_{\lambda}(t, u_{0,\lambda})$ is upper semicontinuous for all $t \geq 0$.

Proof. If not, then there exists a neighborhood $O$ of $G_{\lambda}(t, u_{0,\lambda})$ and sequences $u_{n,\lambda} \to u_{0,\lambda}$, $y_n \in G_{\lambda}(t, u_{n,\lambda})$ such that $y_n \not\in O$. But then Lemma 4.11 implies the existence of a subsequence $y_{n_j}$ such that $y_{n_j} \to y \in G_{\lambda}(t, u_{0,\lambda})$, which is a contradiction.

Corollary 6. Assume (F1) – (F3) and let λ ∈ [0, ∞) be fixed. Then the map $G_{\lambda}$ has compact values.

Proof. The case $t = 0$ is obvious. If $t > 0$, it follows from Corollary 5 that the set $G_{\lambda}(t, u_{0,\lambda})$ is relatively compact, and from Lemma 4.11 that it is closed.

Further, we establish the existence of a bounded absorbing set for $G_{\lambda}$, that is, a set $B_0$ such that for any bounded set $B \subset H$ there exists a time $T(B, \lambda)$ such that $G_{\lambda}(t, B) \subset B_0$ for all $t \geq T$.

Lemma 4.12. Assume (F1) – (F3). Then $G_{\lambda}$ possesses a bounded absorbing set $B_0$, which does not depend on λ.

Proof. In view of (40) the set $B_0 = \{v \in H : \|v\|^2 \leq \delta_2 + 1\}$ is absorbing. We note that the time $T(B)$ does not depend on λ.

Now we are ready to prove the existence of a global compact attractor.

Theorem 4.13. Assume (F1) – (F3). Then problem (29) generates a family of strict multivalued semiflows $G_{\lambda}$ in the phase space $H$ having a global compact invariant attractor $A_{\lambda}$, which is the minimal closed set attracting all bounded sets. Moreover, $A_{\lambda} \subset B_0$, for all λ ∈ [0, ∞), where $B_0$ is the absorbing set from Lemma 4.12.

Proof. It follows easily from Lemma 4.12 and Corollary 5 the existence of a compact set $K_\lambda$ such that

$$\text{dist}(G_{\lambda}(t, B), K_\lambda) \to 0, \text{ as } t \to +\infty,$$

for any bounded set $B$. Also, by Corollaries 6, 7 the map $u_{0,\lambda} \mapsto G_{\lambda}(t, u_{0,\lambda})$ is upper semicontinuous and has compact values. The result follows then from [39, Theorem 4 and Remark 8]. From the definition of global attractor it follows easily that $A_{\lambda} \subset B_0$. □
As an important consequence of Lemma 4.10 it follows that $\bigcup_{\lambda \in [0, \infty)} \mathcal{A}_\lambda$ is a bounded subset of $E$ and once $E \subset H$, we can conclude:

**Corollary 8.** Assume $(F1) - (F3)$. Then $A := \overline{\bigcup_{\lambda \in [0, \infty)} \mathcal{A}_\lambda}$ is a compact subset of $H$.

Finally, we shall prove the upper semicontinuity of the attractors with respect to $\lambda$.

As in the proof of Lemma 4.11 we introduce the function $\psi^{D^\lambda} : L^2 (r, T; H) \to (-\infty, +\infty]$ given by

$$
\psi^{D^\lambda} (u) = \begin{cases} 
\int_r^T \varphi^{D^\lambda} (u (t)) \, dt, & \text{if } \varphi^{D^\lambda} (u (\cdot)) \in L^1 (r, T), \\
+\infty, & \text{otherwise},
\end{cases}
$$

where $0 < r < T$.

**Lemma 4.14.** Let $u_{\lambda_n} \to u$ in $L^2 (r, T; H)$ and weakly in $L^p (r, T; W^{1, p} (\mathbb{R}^n))$ as $\lambda_n \to \lambda$. Assume also that $g_{\lambda_n} \to g$ weakly in $L^2 (r, T; H)$, where $g_{\lambda_n} (t) \in \partial \varphi^{D^\lambda_n} (u_{\lambda_n} (t))$, for a.e. $t \in (r, T)$. Then $g \in \partial \psi^{D^\lambda} (u)$.

**Proof.** Take $v \in L^p (r, T; E)$. Since $a \geq 1$ and $D^{\lambda_n} \geq 0 > 0$ we get from Tartar's inequality that

$$
\int_r^T \left\langle A^{D^\lambda_n} (u_{\lambda_n} (t)), u_{\lambda_n} (t) - v (t) \right\rangle_{E^*, E} \, dt \\
= \int_r^T \int_{\mathbb{R}^n} D^{\lambda_n} (x) \left( \left| \nabla u_{\lambda_n} \right|^{p-2} \nabla u_{\lambda_n} - \left| \nabla v \right|^{p-2} \nabla v \right) (\nabla u_{\lambda_n} - \nabla v) \, dx \, dt \\
+ \int_r^T \int_{\mathbb{R}^n} a(x) \left( \left| u_{\lambda_n} \right|^{p-2} u_{\lambda_n} - |v|^{p-2} v \right) (u_{\lambda_n} - v) \, dx \, dt \geq 0.
$$

As $g_{\lambda_n} \to g$ weakly in $L^2 (r, T; H)$ and $u_{\lambda_n} \to u$ in $L^2 (r, T; H)$, we have

$$
\int_r^T \left\langle A^{D^\lambda_n} (u_{\lambda_n} (t)), u_{\lambda_n} (t) - v (t) \right\rangle_{E^*, E} \, dt \to \int_r^T \int_{\mathbb{R}^n} g(t, x) (u(t, x) - v(t, x)) \, dx \, dt.
$$

On the other hand, $D^{\lambda_n} \to D^\lambda$ in $L^\infty (\mathbb{R}^n)$, $u_{\lambda_n} \to u$ weakly in $L^p (r, T; W^{1, p} (\mathbb{R}^n))$ imply

$$
\int_r^T \left\langle A^{D^\lambda} (v(t)), u_{\lambda_n} (t) - v (t) \right\rangle_{E^*, E} \, dt \\
= \int_r^T \int_{\mathbb{R}^n} D^\lambda (x) \left| \nabla v \right|^{p-2} \nabla v (\nabla u_{\lambda_n} - \nabla v) \, dx \, dt + \int_r^T \int_{\mathbb{R}^n} a(x) \left| v \right|^{p-2} v (u_{\lambda_n} - v) \, dx \, dt \\
\to \int_r^T \int_{\mathbb{R}^n} D^\lambda (x) \left| \nabla v \right|^{p-2} \nabla v (\nabla u - \nabla v) \, dx \, dt + \int_r^T \int_{\mathbb{R}^n} a(x) \left| u \right|^{p-2} u (u - v) \, dx \, dt \\
= \int_r^T \left\langle A^{D^\lambda} (v(t)), u(t) - v (t) \right\rangle_{E^*, E} \, dt.
$$

Thus,

$$
\int_r^T \int_{\mathbb{R}^n} g(t, x) (u(t, x) - v(x)) \, dx \, dt - \int_r^T \left\langle A^{D^\lambda} (v(t)), u(t) - v \right\rangle_{E^*, E} \, dt \geq 0.
$$

Now, if we take any $v \in D(\partial \psi^{D^\lambda}) \subset L^p (r, T; E)$, we obtain

$$
\int_r^T \int_{\mathbb{R}^n} \left( g(t, x) - (A^{D^\lambda} (v(t)))(x) \right) (u(t, x) - v(t, x)) \, dx \, dt \geq 0.
$$
Since the operator $\partial \psi^{D^\lambda}$ is maximal monotone, we obtain that $g \in \partial \psi^{D^\lambda}(u)$. \hfill $\square$

**Lemma 4.15.** Assume $(F1)-(F3)$. Let $u^n_{0,\lambda_n} \to u_{0,\lambda}$, where $u^n_{0,\lambda_n}, u_{0,\lambda} \in H$, and let $u^n_{\lambda_n}(\cdot) \in D_{\lambda_n}(u^n_{0,\lambda_n})$, $\lambda_n \to \lambda$. Then there exist $u_\lambda(\cdot) \in D_{\lambda}(u_{0,\lambda})$ and a subsequence $u^n_{\lambda_{n_j}}(\cdot)$ such that

$$u^n_{\lambda_{n_j}} \to u_\lambda \text{ in } C([\varepsilon,T], H) \text{ for all } 0 < \varepsilon < T.$$

**Proof.** We fix $T > 0$. In view of (40), (41), (43) and $E \subset W^{1,p}(\mathbb{R}^n)$ up to a subsequence we have

$$u^n_{\lambda_n} \to u_\lambda \text{ weakly in } L^2(0,T; H),$$

$$u^n_{\lambda_n} \to u_\lambda \text{ weakly in } L^p(0,T; W^{1,p}(\mathbb{R}^n)),$$

$$\frac{du^n_{\lambda_n}}{dt} \to \frac{du_\lambda}{dt} \text{ weakly in } L^2(\varepsilon,T; H), \text{ for all } \varepsilon > 0.$$

We shall prove that $u_\lambda(\cdot) \in D_{\lambda}(u_{0,\lambda})$.

Let $\varepsilon_N = \frac{1}{N}, N \in \mathbb{N}$. Inequality (43) implies that the sequence $u^n_{\lambda_n}(\cdot)$ is equicontinuous in $[\varepsilon_N,T]$. Also, by (42) and the compact embedding $E \subset H$ (see Lemma 2.1) the sequence $\{u^n_{\lambda_n}(t)\}$ is relatively compact for all $t \in [\varepsilon_N,T]$. Then from the Ascoli-Arzelà theorem and (50) we can easily deduce that

$$u^n_{\lambda_n} \to u_\lambda \text{ in } C([\varepsilon_N,T], H).$$

We introduce the function $\psi^{D^\lambda} : L^2(\varepsilon_N,T; H) \to (-\infty, +\infty]$ given by (49). By $\frac{1}{2} < \gamma < 1, (30), (40), (41), (43)$ we obtain as in Lemma 4.11 that

$$b^n_{\lambda_n} \to b_\lambda \text{ weakly in } L^2(0,T; H),$$

$$g^n_{\lambda_n} \to g_{\varepsilon_N,\lambda} \text{ weakly in } L^2(\varepsilon_N,T; H).$$

Lemma 4.14 implies that $g_{\varepsilon_N,\lambda} \in \partial \psi^{D^\lambda}(u_\lambda)$, where we are considering $u_\lambda$ as a function in $L^2(\varepsilon_N,T; H)$. Thus, $g_{\varepsilon_N,\lambda}(t) \in \partial \varphi^{D^\lambda}(u_\lambda(t))$ for a.a. $t \in (\varepsilon_N,T)$. Also, by property (B2) we have that $b_\lambda(t) \in B(u_\lambda(t))$ for a.a. $t \in (\varepsilon_N,T)$.

As in the proof of Lemma 4.11 one can prove that $u_\lambda(\cdot)$ is a strong solution of (29).

Finally, we prove that $u_\lambda(0) = u_{0,\lambda}$. We shall consider the functions $u^n_{\lambda_n}(t), z(t)$, which are strong solutions of the problems

$$\frac{du^n_{\lambda_n}(t)}{dt} + A^{D^\lambda_n}(u^n_{\lambda_n}(t)) + B(u^n_{\lambda_n}(t)) \geq 0, \quad u^n_{\lambda_n}(0) = u^n_{0,\lambda_n},$$

and

$$\frac{dz(t)}{dt} + A^{D^\lambda}(z(t)) \geq 0, \quad z(0) = u_{0,\lambda},$$

respectively. The difference $\omega^n(t) = u^n_{\lambda_n}(t) - z(t)$ satisfies

$$\frac{d\omega^n(t)}{dt} + A^{D^\lambda_n}(u^n_{\lambda_n}(t)) - A^{D^\lambda}(z(t)) + B(u^n_{\lambda_n}(t)) \geq 0.$$

Since $A^{D^\lambda_n}$ is monotone, using (30) we obtain that

$$\frac{1}{2} \int_{\mathbb{R}^n} \left| \omega^n(t) \right|^2 \leq C \left( 1 + \left| \varphi^{D^\lambda_n}(u^n_{\lambda_n}(t)) \right|^{1-\gamma} \right) \left\| \omega^n(t) \right\|$$

$$- \int_{\mathbb{R}^n} (D^{\lambda_n}(x) - D^{\lambda}(x)) |\nabla z(t)|^{p-2} \nabla z(t) \nabla \omega^n(t) dx$$
Problem (28) by adding a function in a similar way as in Section 3 we could consider the nonautonomous Remark 10. Assume the following result.

\[ \lambda \]

Integrating over \((0, t)\) and using Holder’s inequality, \(\varphi^{D^\lambda}(z(\cdot)) \in L^1(0, T)\) and (40)-(41) we obtain

\[
\|u^n(\lambda_t) - u_{0,\lambda}\|^2 \leq \left\| u^n_{0,\lambda,n} - u_{0,\lambda} \right\|^2 + 2C_0 t + 2C_7 t^\gamma \left( \int_0^t \left\| \varphi^{D^\lambda}(u^{n}_{\lambda_n}(s)) \right\| ds \right)^{1-\gamma} + C_5 \|D^\lambda - D^\lambda\| \int_0^t \left( \|z(s)\|_{W^{1,p}(\mathbb{R}^n)} + \|u^n(s)\|_{W^{1,p}(\mathbb{R}^n)} \right) ds
\]

Passing to the limit as \(n \to \infty\) we have

\[
\|u_\lambda(t) - z(t)\|^2 \leq C_8 (t + t^\gamma), \text{ for each } t \in (0, T].
\]

Since \(z(t) \to u_{0,\lambda}\) as \(t \to 0^+\), we obtain the result. Hence, \(u_\lambda(\cdot) \in D^\lambda(u_{0,\lambda})\).

The result is proved for an arbitrary \(T > 0\) using a diagonal argument.

**Corollary 9.** Assume \((F1)-(F3)\). Then the map \(\lambda \mapsto G_\lambda(t, A)\) is \(w\)-upper semicontinuous for each \(t > 0\).

**Proof.** If not, then there exist \(\lambda \in [0, \infty), t > 0, \varepsilon\)-neighborhood \(O_\varepsilon(G_\lambda(t, A))\) and sequences \(u^n_{0,\lambda_n} \in A, \lambda_n \to \lambda, y_n \in G_\lambda(t, u^n_{0,\lambda_n})\) such that \(y_n \notin O_\varepsilon(G_\lambda(t, A))\). Then \(y_n = u^n_{\lambda_n}(t)\), where \(u^n_{\lambda_n}(\cdot) \in D^\lambda(u^n_{0,\lambda_n})\). Since \(A\) is compact, passing to a subsequence \(u^n_{0,\lambda_n} \to u_{0,\lambda} \in A\). If we take \(\varepsilon < t\), it follows from Lemma 4.15 that up to a subsequence \(u^n_{\lambda_n} \to u_\lambda\) in \(C([\varepsilon, t], H)\) with \(u_\lambda(\cdot) \in D^\lambda(u_{0,\lambda})\). Hence, \(y_n = u^n_{\lambda_n}(t) \to u_\lambda(t) \in G_\lambda(t, A)\), a contradiction.

Therefore, using Corollaries 8, 9, we obtain immediately from Theorem 1.2 in [30] the following result.

**Theorem 4.16.** Assume \((F1)-(F3)\). Then the family of global attractors \(\{A_\lambda : \lambda \in [0, \infty)\}\) is upper semicontinuous at any \(\lambda\).

**Remark 10.** In a similar way as in Section 3 we could consider the nonautonomous problem (28) by adding a function \(g(t)\) in the right-hand side and prove the existence of a pullback attractor.

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