Abstract - A difference analogue of the logistic equation, which preserves integrability, is derived from Hirota’s bilinear difference equation. The integrability of the map is shown to result from the large symmetry associated with the Bäcklund transformation of the KP hierarchy. We introduce a scheme which interpolates between this map and the standard logistic map and enables us to study integrable and nonintegrable systems on an equal basis. In particular we study the behavior of Julia set at the point where the nonintegrable map passes to the integrable map.

1. INTRODUCTION

The completely integrable systems have been revealed to play a fundamental role in the study of mathematical physics[1,2]. They provide a common basis to understand phenomena in various fields of mathematical physics. If we try to deform the systems but preserving their integrability we must generalize the symmetry possessed by the systems. The quantum group is such an example[3]. The essential feature of the new symmetry is to replace the differential operators by their difference analogue[4]. On the other hand it has been known[5] that many of the
completely integrable systems have their discrete counterparts which are also integrable. A typical example of this sort is the Toda lattice which is a discrete analogue of the KdV equation along the spatial direction[6].

It is, however, not obvious that equations obtained by making discretization of variables of an integrable differential equation are integrable. The KdV equation can be derived from the Toda lattice by taking an appropriate continuous limit of the spatial variable. But, if we start from the KdV equation and discretize the spatial variable we obtain a variety of difference equations besides the Toda lattice. Those equations obtained by discretizing the variables in different ways do not preserve their integrability but create chaos in general. Our main concern in this article is to characterize the discretization of variables in integrable systems, which maintain their integrability.

Starting from the (2+1) dimensional Toda lattice and discretizing the two time variables such that all three variables in the equation appear symmetric, we arrive at Hirota’s bilinear difference equation (HBDE)[7]. This equation is known completely integrable and is equivalent to the infinite number of soliton equations belonging to the KP hierarchy[8]. In other words, Hirota’s difference equation is a simultaneous discrete counterpart of soliton equations in the KP hierarchy. As a matter of fact we can derive all of the soliton equations in the KP hierarchy from HBDE by taking appropriate continuous limits of the variables. Therefore HBDE is a desirable object to study for our purpose.

A system which evolves along discrete time can be considered as a dynamical system if we regard it as an infinite sequence of maps. In this sense our problem might be thought equivalent to characterize dynamical systems which are integrable. Since the dependent variable of HBDE is defined on an arbitrary Riemann surface, we are especially interested in the complex dynamical systems.

From physical point of view the existence of discretization of integrable systems which preserve integrability is important by itself. The intimate correspondence between the solvable statistical models on two dimensional lattice and the two dimensional conformal field theories has
been noticed from the beginning of the study of the integrable systems[9]. In the relativistic quantum field theory there arises infinity owing to the fact that the space-time is a continuous media of the fields. The lattice gauge theory[10] provides a way to avoid this difficulty. The space-time is replaced by a set of lattice points. Only at the end of all calculation of physical quantities the lattice constant is set zero. This approach causes many problems specific to the way the lattice constant is introduced into the theory. The relativistic invariance, for instance, is lost in general. If the theory on the lattice is integrable, however, we are certain that the results obtained on the lattice are correct in the continuum limit. The systems which preserve integrability after discretization of variables are also peculiar in the sense that all results calculated on computer are exact without errors.

In order to clarify the notion of integrability preserving discretization of integrable systems we like to study, in this paper, the simplest case of Hirota’s bilinear difference equation. We first show that if we consider the dependence on one variable and ignore the other variables, HBDE reduces to a difference analogue of the logistic equation, which preserves integrability. Secondly we argue that the discrete time evolution of this integrable map can be considered as a special case of the Bäcklund transformation. The existence of Bäcklund transformation characterizes integrable systems[11]. It enables one to derive a sequence of new solutions starting from a known solution. Hence it is quite natural to interpret that the Bäcklund transformation generates a time evolution of the integrable system. In the last part of this paper we introduce a scheme which enables us to discuss the completely integrable systems and nonintegrable systems on the same ground. Specifically we consider analytic properties of a map which interpolates two different types of discretization of the logistic equation. Namely we investigate a map which combines one of the standard logistic map, which presents chaos, and the one derived from HBDE through the reduction, which is completely integrable.
2. DISCRETIZATION OF LOGISTIC EQUATION

We are interested in characterizing integrable systems which preserve their integrability after discretization of dependent variables. Most of the integrable systems discussed in the literature of theoretical physics have infinite number of freedom and thus too much complicated to be considered as dynamical systems. Instead of studying them directly we will start our discussion in this paper from the simplest case, i.e., one dimensional ordinary differential equations. One of such equations is given by

\[ \frac{dx}{dt} = f(x). \] (1)

This is equivalent to one dimensional Newton equation, which can be solved as

\[ t = \int_{x(t)}^{x} \frac{dx}{f(x)}. \] (2)

There exists a unique solution as long as \( 1/f(x) \) is integrable. If we discretize the variable \( t \) according to Euler’s prescription, we obtain, writing \( x(t) \rightarrow z_l \):

\[ z_{l+1} = z_l + h f(z_l). \] (3)

When \( h = 1 \), this equation turns to be an infinite sequence of a map, i.e., a one dimensional dynamical system. \( h \rightarrow 0 \) reproduces the original equation. If we consider \( z_l \) on the complex plane, and \( f(z) \) is a rational function of \( z \) with degree larger than 2, this map is known[12] to have a Julia set (the closure of repelling periodic points). Therefore this map shows chaotic behaviour somewhere on the complex plane, and is not integrable. When \( f \) is a polynomial of \( z \) of degree 2, the map is called a logistic map and is studied in detail[12]. In fact if we put

\[ f(z) = z(\mu - h - \mu z) \] (4)

and \( h = 1 \) we get the standard form of the logistic map:

\[ z_{l+1} = \mu z_l(1 - z_l). \] (5)
In the limit of $h \to 0$ it turns to the logistic equation:

$$\frac{dx}{dt} = \mu x(1 - x).$$

The discretization of the logistic equation is not unique. Hence there arises a question; what is the integrable discretization of the logistic equation? To answer this question let us recall some properties possessed by integrable discrete systems. Hirota’s bilinear difference equation (HBDE) is one of such equations whose solutions are completely known[8,11]. It is given by

$$\alpha f(p + 1, q, r)f(p, q + 1, r + 1) + \beta f(p, q + 1, r)f(p + 1, q, r + 1)$$

$$+ \gamma f(p, q, r + 1)f(p + 1, q + 1, r) = 0.$$  

(7)

Here $p, q, r$ are discrete variables and $\alpha, \beta, \gamma$ are arbitrary parameters subject to the constraint $\alpha + \beta + \gamma = 0$. Infinitely many equations of the KP hierarchy can be reduced from this single equation by taking continuous limits of various combinations of the variables and the parameters[7]. To be more specific we introduce a new set of variables $\{t_n\} = \{t_1, t_2, \cdots\}$ by[8]

$$t_n = \frac{1}{n} \sum_j k_j z_j^n, \quad n \geq 1.$$

(8)

Here we assume $\sum_j k_j = 0$, which is nothing but the energy-momentum conservation law in the language of the string model[2]. This new set of variables $\{t_n\}$ are the independent variables of the soliton equations in the KP hierarchy. The variables $p, q, r$ of HBDE are any three of $k_j$’s of (8). Therefore we consider infinite number of copies of HBDE. To obtain a particular soliton equation we change, in one of the HBDE, the variables from $\{k_j\}$ to $\{t_n\}$ and reduce the number of $t$ variables by neglecting their dependence. From this argument it is apparent that a large symmetry of the KP hierarchy can be seen if we use the variables $k_j$’s instead of $t_n$’s of the soliton equations.

The solutions of HBDE are known explicitly and given by analytic functions defined on Riemann surface with arbitrary genus[13]. $\{z_j\}$ are
local coordinates of the Riemann surface. The space of the solutions is
called universal Grassmannian which is a Grassmann manifold of infinite
dimension[14]. The equation which characterizes such a large space of
solution is rather simple if we consider HBDE instead of the set of soliton
equations in the KP hierarchy.

If we wish to discuss integrable systems in comparison with noninte-
grable systems such as logistic map, however, HBDE is still complicated.
Therefore we try to reduce the dependence on the variables. As we sim-
ply neglect dependence on two independent variables, say $q$ and $r$, HBDE
becomes trivial. In order to obtain the most simple nontrivial reduction
we introduce new set of variables

$$l = p + q + r + \frac{3}{2}, \quad m = -q - \frac{1}{2}, \quad n = p + q + 1$$

and denote by $g$ the dependent variable written by new variables as

$$f(p, q, r) = g_n(l, m).$$

We then obtain

$$\alpha g_n(l + 1, m) g_n(l, m + 1) + \beta g_n(l, m) g_n(l + 1, m + 1)$$

$$+ \gamma g_{n+1}(l + 1, m) g_{n-1}(l, m + 1) = 0. \quad (10)$$

The first nontrivial relation is given if we choose

$$g_1(l, 1) = g_0(l, 2) = g_2(l, 1) \equiv z_l$$

$$g_1(l, 2) \equiv c \ (\text{const.}) \quad (11)$$

$$g_n(l, m) = 0, \quad \text{otherwise}$$

and also $\beta = -\mu \alpha$, $\gamma = (\mu - 1) \alpha$, $c = \frac{\mu - 1}{\mu}$, from which follows

$$z_{l+1} = \mu z_l (1 - z_{l+1}). \quad (12)$$

This is quite similar to the standard logistic map (5) except for the second
term on the right hand side. Since this equation is derived by a reduction
starting from an integrable difference equation, the behaviour of solutions
must be completely different from those of the standard map. In fact writing (12) as

\[ z_{l+1} = \frac{\mu z_l}{1 + \mu z_l} \]  

(13)

the solution is given by

\[ z_l = \frac{\mu^l z_0}{1 + \mu^{1-\mu} z_0} \]  

(14)

for arbitrary initial value \( z_0 \). This solution is exactly the one which coincides with one of the logistic equation (6) in the continuous limit. It is easy to check that this equation itself reduces to the logistic equation in the same limit. In this sense we call (12) the integrable discretization of the logistic equation.

The equation (12) itself was studied by Morisita\[15\] as a difference analogue of the logistic map and revived by Yamaguti\[16\] and Hirota\[17\]. We like to emphasize here that the integrability of the map (12) has the same origin as the integrability of the KP hierarchy.

3. BÄCKLUND TRANSFORMATION

As we have seen in the previous section a slight difference of the way of discretization of an integrable equation leads to a big difference of the behaviour of the system. The question we ask in this section is how a regular map is possible and does not fall into chaos. To this end we will show that the map which maintains regularity is a particular sequence of the Bäcklund transformations of the KP hierarchy. In the case of the KP hierarchy the Bäcklund transformations are auto Bäcklund transformations\[7,18\]. In other words the transformation generates a sequence of solutions of the same equation. Starting from a point of the universal Grassmannian a sequence of the Bäcklund transformations sweeps a trajectory on the space of solutions. From this result we can glance the deep connection of the integrable dynamical systems and the large symmetry associated with the Bäcklund transformation.

To begin with we first recall the fact that the time variable \( l \) of the
map (13) is a certain combination of \( k_j \)'s. We like to show that the Bäcklund transformations of the KP hierarchy are the operations which change values of \( k_j \)'s by one. It then follows that there exists a certain transformation which shifts \( l \) into \( l + 1 \) without changing other variables.

The Bäcklund transformation of the KP hierarchy is generated by the operator\[ B(z, w) = \Lambda(z)\bar{\Lambda}(w) \]

where
\[
\Lambda(z) = \exp[-\sum_{n=1}^{\infty} t_n z^{-n}] \exp[\sum_{n=1}^{\infty} \frac{1}{n} z^n \partial_n],
\]
\[
\bar{\Lambda}(z) = \exp[\sum_{n=1}^{\infty} t_n z^{-n}] \exp[-\sum_{n=1}^{\infty} \frac{1}{n} z^n \partial_n].
\]

In this expression \( \partial_n \) means \( \partial/\partial t_n \), where \( t_n \)'s are those variables of the soliton equations in the KP hierarchy. Now we consider the operation of \( \Lambda(z_j) \) to \( f(k_1, k_2, \cdots) \). The right hand side factor of \( \Lambda(z_j) \) transforms \( t_n \) according to
\[
t_n \to t_n + \frac{1}{n} z_j^n, \quad n \geq 1.
\]

We observe from the expression of \( t_n \) (8) that this change of variables amounts to replace \( k_j \) by \( k_j + 1 \). Similarly the left hand side factor of \( \Lambda(z_j) \) can be rewritten in terms of \( k_j \)'s. In doing this we must be a bit careful because \( \Lambda(z) \) is not well defined when \( z \) approaches to one of \( z_j \)'s. Thus we define a new regularized operator by
\[
\Lambda_q(z_j) \equiv \prod_{l} (z_j - qz_l)^{k_l} e^{\partial_k j}.
\]

\( \Lambda(z_j) \) is recovered in the \( q \to 1 \) limit. From this expression we see that the operation of \( B_q(z_i, z_j) \equiv \Lambda_q(z_i)\bar{\Lambda}_q(z_j) \) to one of solutions of HBDE, say \( f(k_1, k_2, \cdots) \), yields
\[
B_q(z_i, z_j) f(k_1, \cdots, k_i, \cdots, k_j, \cdots) \equiv b(z_i, z_j) f(k_1, \cdots, k_i+1, \cdots, k_j-1, \cdots),
\]

where
\[
b(z_i, z_j) = \prod_{l} \left( \frac{z_i - qz_l}{z_j - qz_l} \right)^{k_l} \frac{1}{z_j - qz_i}.
\]
Therefore the operation of $B$ of the Bäcklund transformation (15) causes to $f(k_1, k_2, \cdots)$ the shifts of the variables $k_i \rightarrow k_i + 1$, $k_j \rightarrow k_j - 1$. Let $f^{(\alpha)}(k_1, k_2, \cdots)$ be the function obtained from $f(k_1, k_2, \cdots)$ by the Bäcklund transformation, i.e.,

$$e^{\alpha B_q(z_i, z_j)} f(k_1, k_2, \cdots) = f^{(\alpha)}(k_1, k_2, \cdots). \quad (20)$$

It is well known that the generator of the Bäcklund transformation $B$ has a fermionic nature, i.e., $B^2(z_i, z_j) = 0$ holds[11]. Hence we can write

$$\alpha B_q(z_i, z_j) f(k_1, k_2, \cdots) = f^{(\alpha)}(k_1, k_2, \cdots) - f(k_1, k_2, \cdots).$$

To be more specific let us choose four out of $\{k_j\}$ and call them $p, q, r, s$. Further we define $l, m, n$ from them according to (9). Then we see that the increase of $l, m, n$ by one are equivalent to the change of $p, q, r$ as

$$l \rightarrow l + 1 \Leftrightarrow r \rightarrow r + 1$$

$$m \rightarrow m + 1 \Leftrightarrow p \rightarrow p + 1, \; q \rightarrow q - 1$$

$$n \rightarrow n + 1 \Leftrightarrow p \rightarrow p + 1, \; r \rightarrow r - 1. \quad (21)$$

If $s$ is the variable associated to the singular point $z_s$ on the Riemann surface, the amplitude $f$ does not depend on $s$. Letting $z_j$’s associated to $p, q, r, s$ as $z_p, z_q, z_r, z_s$, the shifts of (21) are realized by the following operations:

$$l \rightarrow l + 1 \Leftrightarrow B_q(z_r, z_s)$$

$$m \rightarrow m + 1 \Leftrightarrow B_q(z_p, z_q) \quad (22)$$

$$n \rightarrow n + 1 \Leftrightarrow B_q(z_p, z_r).$$

From this we see that under the operation of $B_q(z_r, z_s)$ only the variable $l$ changes its value whereas other variables remain unchanged. According to (20) we can express $g_n(l + 1, m)$ by $g_n^{(\alpha)}(l, m)$ and $g_n(l, m)$. In other words $g_n$ at $l+1$ is determined by $g_n^{(\alpha)}$ and $g_n$ at $l$. Hence it is the generator of the Bäcklund transformation which generates the time sequence of $z_l$ under the integrable map (13).
Finally we notice about the symmetry corresponding to the Bäcklund transformation. If we expand $B(z, w)$ into double Laurent series as

$$B(z, w) = \sum_{m, n \in \mathbb{Z}} B_{mn} z^m w^{-n}$$  \hspace{1cm} (23)

we can show[11]

$$[B_{mn}, B_{m'n'}] = \delta_{m'n'} B_{mn} - \delta_{mn'} B_{m'n'}.$$  \hspace{1cm} (24)

Hence $B$ generates a group $GL(\infty)$.

4. AN INTERPOLATION OF INTEGRABLE AND NONINTEGRABLE DIFFERENCE ANALOGUES OF THE LOGISTIC EQUATION

In the previous sections we derived the integrable map associated with logistic equation starting from HBDE and showed that the integrability of the map owes to the symmetry associated to the Bäcklund transformation of the KP hierarchy. We like to clarify in this section how integrable maps can be characterized among nonintegrable ones. For this purpose we study a map which interpolates between the integrable and the nonintegrable difference analogues of the logistic equation, and argue the behaviour of the Julia set.

We have seen in §2 that we can derive two different maps (5) and (12) starting from the same logistic equation (6). Although they look similar at the level of equation the behaviour of their solutions are quite different. We like to know analytically the transition between these two systems. To proceed we generalize (5) and (12) such that it is a discretization of the logistic equation (6) and (5) and (12) are included as two particular cases. A simple generalization is given by

$$z_{l+1} = \mu z_l \{1 - \gamma z_l - (1 - \gamma)z_{l+1}\},$$  \hspace{1cm} (25)

where $\gamma$ is an arbitrary parameter. It is apparent that the standard logistic map and the integrable map are recovered if $\gamma$ is set 1 and 0 respectively. If we solve this equation for $z_{l+1}$ we find

$$z_{l+1} = \phi(z_l),$$  \hspace{1cm} (26)
where we have introduced a map $\phi(z)$;

$$
\phi(z) = \frac{\mu z (1 - \gamma z)}{1 + \mu (1 - \gamma) z}.
$$

(27)

This is what we are going to analyze in this section.

We first notice that the map $\phi(z)$ of (27) is similar to the one known as Blaschke product of degree two[19];

$$
e^{i\theta} \frac{\lambda + z}{1 + \lambda z},
$$

(28)

where $\theta$ is an arbitrary real number and $\bar{\lambda}$ is the complex conjugate of $\lambda$. This maps the unit circle on itself in the complex plane. Moreover the Julia set of this map is the unit circle. Since there is no other unstable fixed point in this map it is very convenient to concentrate our attention to this to analyze (27) for the first step.

Comparing our map with the expression of the Blaschke product, we see that the map (27) turns to (28), a Blaschke product, if the parameters satisfy the relations;

$$
\mu \gamma = -e^{i\theta}, \quad \mu = \frac{-1}{\gamma (1 - \gamma)}.
$$

(29)

We are interested in the case of small values of $|\gamma|$. As far as (29) are satisfied, the Julia set remains on the unit circle. This happens only if the value of $\mu$ diverges when $\gamma$ approaches to zero. Hence we can not draw any conclusion for the limit of $\gamma \rightarrow 0$, since the value of $\mu$ in our interest is finite.

In order to proceed further we try to see the map from different coordinate system in the complex plane. Namely we introduce a new variable $w$ which is a Möbius transformation of $z$, i.e., $w = \psi(z)$. In this new coordinate system, $\phi$ turns to $\tilde{\phi}$ following to

$$
\phi(z) \rightarrow \tilde{\phi}(z) = \psi^{-1} \circ \phi \circ \psi(z).
$$

(30)

The map $\phi(z)$ of (27) has three fixed points at 0, $\infty$ and $1 - \frac{1}{\mu}$. We want 0 and $\infty$ to remain as fixed points after the change of the variable, so
that the new map still keeps the form
\[ \tilde{\phi}(z) = e^{i\theta} z \frac{\lambda + z}{1 + \lambda' z}. \]  
(31)

One of such coordinate transformations is
\[ \psi(z) = \frac{1 - \frac{1}{\mu}}{(\gamma + \mu - \mu\gamma)e^{i\theta}z + 1}, \]  
(32)

which yields a new map of the form (31) with
\[ \lambda = \frac{1 + \gamma - \mu\gamma}{\mu + \gamma - \mu\gamma}e^{-i\theta}, \quad \lambda' = \mu e^{i\theta}. \]  
(33)

In the new coordinate the fixed points are at 0, \( \infty \) and
\[ p = -e^{-i\theta} \frac{1}{\mu + \gamma - \mu\gamma}. \]

The multipliers of the fixed points at 0, \( \infty \) and \( p \) are given, respectively, as
\[ \lambda_0 = \frac{1 + \gamma - \mu\gamma}{\mu + \gamma - \mu\gamma}, \quad \lambda_\infty = \mu, \quad \lambda_p = 1 - \frac{1}{\gamma}. \]  
(34)

Notice that \( z = p \) is a repelling fixed point irrespective to the value of \( \mu \) when \( |\gamma| \) is small. The nature of the fixed point at \( \infty \) is uniquely determined by the value of \( \mu \) alone.

If \( \mu \) varies independent on the value of \( \gamma \), the product \( \lambda_0 \cdot \lambda_\infty \) goes to 1 as \( \gamma \) tends to 0. This means that, for small values of \( |\gamma| \), the origin is attractive when \( \infty \) is repelling and vice versa. This situation can be also understood from the map itself. Letting \( \gamma \) be zero in (31) we get
\[ \tilde{\phi}(z) = \frac{1}{\mu}z. \]  
(35)

The sequence of this map attracts all points in the complex plane to the origin (resp. \( \infty \)) if \( |\mu| \) is larger (resp. smaller) than 1. Therefore the Julia set reduces to a point at \( \infty \) or 0 depending on the value of \( |\mu| \) is larger or smaller than 1, respectively.

The condition for the map (31) to be a Blaschke product is
\[ \frac{1 + \gamma - \mu\gamma}{\mu + \gamma - \mu\gamma} = \bar{\mu}, \quad \text{or} \quad \gamma = \frac{|\mu|^2 - 1}{|\mu - 1|^2}. \]  
(36)
Under the constraint (36) for the parameters we calculate the values of the multipliers (34). We then find \( \lambda_0 = \bar{\mu}, \lambda_\infty = \mu \), hence the origin as well as the \( \infty \) are attractive fixed points when \( |\mu| < 1 \). The third fixed point is at

\[ p = e^{-i\theta} \frac{\bar{\mu} - 1}{\mu - 1}. \]

Therefore \( p \) is on the unit circle and is a repelling point where \( \gamma \) takes a negative real value. It then follows that all points inside of the unit circle are attracted to the origin by the map whereas the points outside of the circle are pushed away to infinity. The behaviour of the map on the unit circle is chaotic, because the Julia set is the unit circle. It can not escape from the unit circle as long as \( \mu \) satisfies the constraint (36) for a given value of \( \gamma \). This is true also when the value of \( |\gamma| \) approaches to zero. Does this mean that the Julia set remains at \( \gamma = 0 \) ? To see this point more carefully, we restrict the value of \( \mu \) to satisfy (36) and write the map explicitly;

\[
\tilde{\phi}(z) = e^{i\theta} z \frac{\bar{\mu} e^{-i\theta} + z}{1 + \mu e^{i\theta} z} = \mu^{-1} z \frac{\bar{\mu} + e^{i\theta} z}{\mu^{-1} + e^{i\theta} z}.
\] (37)

Now let \( \gamma \) go to zero along the negative real axis, so that \( |\mu| < 1 \) is satisfied. The absolute value of \( \mu \) tends to one owing to the constraint (36). The multipliers at 0 and \( \infty \) approach to 1, and thus these points cease to be attractors. Then the map is given by (35) again as we see from (37). In this case, however, \( \mu \) is pure phase and the sequence of this map forces every point on the complex plane to rotate uniformly around the origin. The angle of the rotation is determined by the phase of \( \mu \). From (36) we can calculate the value of the phase. Writing \( \mu = |\mu|e^{i2\pi \chi} \) it turns out to be

\[
2\pi \chi = \lim_{\gamma \to 0} \arccos \left( 1 + \frac{1 - |\mu|}{\gamma} \right).
\]

Therefore the manner of the rotation is uniquely determined by the value of the limit of the ratio of \( 1 - |\mu| \) and \( \gamma \).

It is known that a sequence of the rotation of an irrational number of phase causes one dimensional ergodic behaviour of zero entropy[20]. If
we choose the ratio between $1 - |\mu|$ and $\gamma$ arbitrarily the phase of $\mu$ will fall to an irrational number with the probability one as $\gamma$ approaches zero. Then the rotation of all $z$ on the complex plane caused by (37) are most probably ergodic with entropy 0, except for the neutral fixed points at 0 and $\infty$. The Julia set, which was on the unit circle when $\gamma < 0$, does not exist when $|\gamma|$ is set equal zero.

Let us summarize the above results. As the value of $\gamma$ approaches to 0 we have three cases:

1) if $|\mu| > 1$, the Julia set is at $\infty$,
2) if $|\mu| < 1$, the Julia set is at 0,
3) if $|\mu| = 1$, there exists no Julia set but all the points between 0 and $\infty$ on the complex plane rotate ergodically so long as the ratio between $1 - |\mu|$ and $\gamma$ does not go to a rational number as $\gamma$ goes to 0.

We can further proceed the analysis of our map (27) from the mathematical point of view. As a complex dynamical system the map is interesting by itself. Since further investigation requires more preparation, however, we shall report it elsewhere[21].

**DISCUSSIONS**

The completely integrable systems constitute a small portion exceptional among general nonlinear systems and behave quite different from others. Nevertheless they describe important nature of fundamental problems in various fields of theoretical physics. The large symmetry of the integrable systems enables us to determine their solutions analytically, whereas the missing of some conserved currents prevents us to predict the behaviour of solutions in the generic case. The soliton type of equations are completely integrable systems whose detail features have been studied in the last two decades. In order to proceed further and to understand the real importance of the role played by integrable systems in theoretical physics it is necessary to study them in comparison with nonintegrable systems. In other words the real understanding of the basic problems in
theoretical physics will be achieved when the complete integrable systems are fully characterized among general nonlinear systems.

There has been, however, no systematic way to investigate nonintegrable systems. The nonintegrable systems which have been studied are mainly one dimensional dynamical systems. Even the simplest one of such systems is far from totally investigated. This should be contrasted with the case of the completely integrable systems, in which the most interesting systems are infinite dimensional and solutions are given analytically. Therefore it is difficult to find a way to discuss these two different systems on a common basis.

In this paper we have shown that the simplest reduction of Hirota’s bilinear difference equation turns out to be an integrable difference analogue of the logistic equation. We provided a scheme which interpolates this equation and the standard logistic map. It enables us to study integrable and nonintegrable maps on the equal ground. In particular we concentrated our attention to the behaviour of the Julia set at the point where the system turns from nonintegrable to integrable.

The integrable part we studied here, is a small portion of HBDE. It will be the most desirable if there is a simple scheme in which HBDE itself is included as an integrable part of some nonlinear system. Although we have not found such a scheme we could interpret the integrability of the map results from the large symmetry possessed by the integrable system such as $GL(\infty)$ which generates the Bäcklund transformation of the KP hierarchy.
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