Interference of Quantum Channels

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We show how interferometry can be used to characterise certain aspects of general quantum processes, and in particular, the coherence of completely positive maps. We derive a measure of coherent fidelity, the maximum interference visibility, and the closest unitary operator to a given physical process under this measure.

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INTRODUCTION

A key requirement of quantum information processing is the ability to transform states coherently [1]. In general, quantum processes will be described by quantum channels, or Completely Positive (CP) maps [2]. A relevant question is what happens when different processes act simultaneously on a system. Surprisingly, knowledge of the individual quantum channels alone is insufficient to specify the action of the simultaneous operation of both maps [3]. An experimental determination of the interference of the two maps reveals additional information about the maps which is not taken into account in their individual descriptions and is a measure of their coherent properties. From this, we may define an operational definition of coherent fidelity between CP maps. Thus, interferometry can be used a tool to extract information inaccessible to conventional process tomography [4].

INTERFEROMETRY

Single particle interference (Fig. 1) displays the key elements of quantum mechanics: the superposition of indistinguishable paths, and the complementarity of certain observables. Interference is a consequence of the possibility of the particle taking both paths, and any process which tends to label the path of the particle will reduce the magnitude of the interference [5].

In general, the perfect interference pattern will be modified by the presence of quantum processes occurring in the upper and lower arms (Fig. 2). For unitary processes (Fig. 3), the evolution of a particle with initial internal state $|\psi\rangle \in \mathcal{H}_d$ is,

$$|\Psi_{in}\rangle = |0\rangle |\psi\rangle \quad \implies \quad \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) |\psi\rangle \quad \implies \quad \frac{1}{\sqrt{2}}(|0\rangle V |\psi\rangle + e^{i\phi} |1\rangle U |\psi\rangle)$$

FIG. 1: Mach-Zender Interferometer. We allow the possibility of different quantum processes occurring in the upper and lower arms.

FIG. 2: Interference pattern showing a phase shift and reduction in visibility. The shift is a measure of the relative phase of the two quantum processes, and the reduction of visibility is a consequence of the leakage of path information into other degrees of freedom.

FIG. 3: Quantum Network for Interfering Unitaries. The actions of the unitary operations on the internal state of the particle (lower line) are controlled by the path of the particle (upper line).
\[ \frac{1}{2} \left[ |0\rangle \langle V + e^{i\phi} U | \psi \rangle + |1\rangle \langle V - e^{i\phi} U | \psi \rangle \right]. \]

The probability of finding the particle in the \(|0\rangle\) state, corresponding to it exiting the interferometer from the horizontal output port, is given by
\[ P_0(\phi) = \frac{1}{4} \left( \langle \psi | (V + e^{i\phi} U) \right) \left( V + e^{i\phi} U | \psi \rangle \right) \]
\[ = \frac{1}{2} \left( 1 + v \cos(\phi - \alpha) \right) \] (2)

where \(|\psi\rangle \equiv \langle \psi | U\rangle | \psi \rangle \) is the new visibility of the interference pattern, and \(\alpha \equiv \arg \left( \langle \psi | U\rangle | \psi \rangle \right) \) is the shift of the interference fringes \(\hat{r}\). The magnitude of the visibility is the fidelity of the states \(U|\psi\rangle\) and \(V|\psi\rangle\), i.e. the overlap of the states exiting the upper and lower arms of the interferometer \(\hat{I}\). The higher their fidelity – hence the lower their distinguishability – the greater the interference effect. Conversely, if the states exiting the upper and lower arms were perfectly distinguishable (orthogonal), there would be no interference.

If the initial state of the particle is \(\rho_{in} = |0\rangle \langle 0| \otimes \rho\), the modified visibility is
\[ v e^{i\alpha} = \text{Tr} \left[ \rho U\rangle V \right], \] (3)
which is the expectation value \(\langle U\rangle | V \rangle \rho\). If we use the input \(\rho = \frac{I}{2}\), the maximally mixed state (equivalent to randomly sampling over a uniform distribution of pure input states), then the visibility pattern gives us the quantity
\[ \text{Tr} \left[ U\rangle V \right] = dv e^{i\alpha}, \] (4)
from which the Hilbert-Schmidt distance between two operators on \(\mathcal{H}_d\) can be derived,
\[ D^2(U, V) = \text{Tr}((U - V)\rangle (U - V)) \]
\[ = 2 \left( d - Re \left\{ \text{Tr} \left[ U\rangle V \right] \right\} \right) \]
\[ = 2d(1 - v \cos \alpha). \] (5)

Hence, we have a direct estimate of the distance between unitary processes \(\hat{r}\).

**CP MAPS**

The interference pattern can reveal important information in the case where the operations in the upper and lower arms are not unitary, but are CP maps \(U\) and \(V\). We will assume these are trace preserving and have the same input and output (finite dimensional) spaces. We can model this case by extending the state space of the entire system by appending two ancillas \(F\) and \(E\) (assuming that \(U\) and \(V\) are independent processes), which are coupled to the top and bottom paths by overall unitaries, \(U\) and \(V\), which implement the CP maps, \(U\) and \(V\) respectively \(\hat{r}\) (Figs. 4 & 5),
\[ U(\rho) = \text{Tr}_F \left[ U(\rho \otimes |f_0\rangle \langle f_0|) U\right], \] (6)
\[ V(\rho) = \text{Tr}_E \left[ V(\rho \otimes |e_0\rangle \langle e_0|) V\right], \] (7)

\[ \Lambda \equiv \rho_{\Lambda} \equiv \Pi \otimes \Lambda(|\Psi_+\rangle \langle \Psi_+|), \] (8)
where \(|\psi_{\rho}\rangle\) and \(|\sigma_{\rho}\rangle\) are orthonormal bases and \(|e_0\rangle\) and \(|f_0\rangle\) are initial states of \(E\) and \(F\) respectively.

Note that for any CP map \(\Lambda\), there exists many unitaries which implement \(\Lambda\) and that we cannot distinguish by quantum process tomography the different instantiations. We may instead uniquely specify \(\Lambda\) via the Jamiołkowski isomorphism \(\hat{r}\).

The action of the interferometer on an initially pure state of the particle is now given by
\[ |\Psi\rangle = |0\rangle \langle \psi | e_0\rangle \langle f_0| \] (11)
\[ \rightarrow \sqrt{2} \left( |0\rangle \langle \tilde{V} + e^{i\phi} \tilde{U} | \psi_e_0 f_0 \rangle + e^{i\phi} |1\rangle \langle \tilde{V} - e^{i\phi} \tilde{U} | \psi_e_0 f_0 \rangle \right) \] (12)

Thus the probability of the particle exiting from the horizontal output port is given by
\[ P_0(\phi) = \frac{1}{4} \left| \langle \psi | e_0 f_0 \rangle \right|^2. \] (13)

In general, if the particle has internal state \(\rho\), the probability is
\[ P_0(\phi) = \frac{1}{2} \left( 1 + Re \left\{ e^{i\phi} \text{Tr} \left[ \tilde{U}^\dagger \tilde{V} \otimes |e_0 f_0\rangle \langle e_0 f_0| \right] \right\} \right). \] (14)
The relevant quantity in the above can be expressed as
\[
\text{Tr} \left[ \hat{\mathcal{U}}^\dagger \hat{\mathcal{V}} \otimes |e_0 f_0\rangle \langle e_0 f_0| \right] = \text{Tr} \left[ \nu_0^\dagger \nu_0 \right],
\]
where the Kraus operators for \( \mathcal{U} \) and \( \mathcal{V} \) are given by
\[
\{ \nu_i \} = \{ \langle e_i | \mathcal{U} | e_0 \rangle \} \quad \text{and} \quad \{ \nu_j \} = \{ \langle f_j | \mathcal{V} | f_0 \rangle \}.
\]
Again, if the input is the maximally mixed state, the interference pattern depends on \( \frac{1}{d} \text{Tr}[\nu_0^\dagger \nu_0] \) only. This reduces to the previously considered case if the operations in the upper and lower paths are unitary on the internal state of the particle.

It is interesting to note that the visibility is dependent on a particular decomposition of the two CP maps, in particular the overlap of the first Kraus operators of \( \mathcal{U} \) and \( \mathcal{V} \). We may interpret this in the framework of quantum jumps. The visibility is a consequence of the indistinguishability of the two possible paths of the particle through the interferometer, anything that serves to tag the passage of the particle serves to reduce the interference pattern. This may not just be an internal change in the state of the particle (created by differing unitary operations \( \mathcal{U} \) and \( \mathcal{V} \)) but also any changes in the environment which may mark the particle’s passage. Thus, the first Kraus operators of both \( \mathcal{U} \) and \( \mathcal{V} \) denote the action of the operation when there is no quantum jump of the environment “watching” the particle. The residual overlap concerns the “unitary” action of the CP map under this condition of no jump.

Note that even though a CP map may seem to be unitary when only acting on the internal degrees of freedom, i.e.
\[
\Lambda(\rho) = U \rho U^\dagger,
\]
the first Kraus operator may be the zero operator, \( \{ \lambda_0 = 0, \lambda_1 = U \} \), and hence give zero visibility, e.g.
\[
\mathcal{U}(|\psi\rangle_1 | f_0\rangle) = (U |\psi\rangle_1) | f_1\rangle \forall |\psi\rangle \in \mathcal{H}_d.
\]
In this case, the map serves as an indicator of path, entangling the fact of the passage of the particle with the environment without altering the internal state. In the interferometer, this results in the destruction of all interference. In general, a CP map will necessarily entangle the passage of the particle with environmental degrees of freedom (a non-product \( \mathcal{U} \)), hence reducing the interference beyond the effect of altering the internal state of the particle.

We may thus define a coherent fidelity between two sets of Kraus operators implementing different CP maps as,
\[
\mathcal{F} (\{ \nu_i \}, \{ \nu_j \}) = \frac{1}{d} \left| \text{Tr} \left[ \nu_0^\dagger \nu_0 \right] \right|, \tag{19}
\]
and their relative phase,
\[
\mathcal{P} (\{ \nu_i \}, \{ \nu_j \}) = \arg \left( \text{Tr} \left[ \nu_0^\dagger \nu_0 \right] \right). \tag{20}
\]
We may note that another fidelity measure on the set of CP maps has been defined via the Uhlmann fidelity\[13\] between the density operators defined in Eq. (8)\[14\],
\[
\mathcal{F}' (\mathcal{U}, \mathcal{V}) = \text{Tr} \left[ \sqrt{\sqrt{\rho_0} \mathcal{U} \rho_0 \mathcal{V} \sqrt{\rho_0}} \right]. \tag{21}
\]

MAXIMALLY COHERENT CP MAPS

For a set of Kraus operators \( \{ \lambda_i \} \) defining a CP map \( \Lambda \), we may define a measure of its self-coherence by inserting two independent instances of \( \Lambda \) into both arms of an interferometer. If the CP map is unitary, then the interference pattern will have unit visibility. However, if there are more than one Kraus operator, we can take this to measure the distance of \( \Lambda \) to the set of unitaries.

It is interesting to ask, for a given \( \Lambda \), and for all possible compatible sets of Kraus operators, what is the maximum visibility or self-coherence? In other words, for all sets of Kraus operators \( \{ \lambda_i \} \) implementing \( \Lambda \), what is the largest value of \( \frac{1}{d} \text{Tr}[\lambda_0^\dagger \lambda_0] \)? The canonical method of constructing a set of Kraus operators of a CP map is given by Choi\[15\]. The operators \( \{ \lambda_i \} \) created are linearly independent and thus represent the minimum number of operators required to represent \( \Lambda \). If \( \{ \lambda_i \} \) also implement \( \Lambda \), they are related by
\[
\lambda_i^\dagger = \sum_k u_{ik} \lambda_k, \tag{23}
\]
where \( u \) is an isometry in general, or unitary when the number of elements in each set are equal. It can easily be shown that the largest possible visibility is obtained when \( \{ \lambda_i \} \) are orthogonal,
\[
\text{Tr} \left[ \lambda_i^\dagger \lambda_j \right] = \delta_{ij} \text{Tr} \left[ \lambda_i^\dagger \lambda_j \right], \tag{24}
\]
and the largest element is $\lambda_0$. When $\Lambda$ is placed in both arms, this upper limit is a measure of the intrinsic (de)coherence of the process. The visibility of an actual realisation of $\Lambda$ may be smaller than this maximum due to processes as in Eq. (15) but this does not represent intrinsic decoherence of the map itself.

We can also find the closest unitary operator to a given set of Kraus operators $\{\lambda_i\}$ which induce the CP map $\Lambda$ by considering an interferometer with $\Lambda$ occurring one arm, and a unitary $U$ operation in the other arm which we may alter as we like. We can maximise the interference pattern by changing $U$, and hence obtain the closest unitary to $\{\lambda_i\}$. The visibility is given by

$$v = \frac{1}{d} \left| \text{Tr} \left[ \lambda_i^* U \right] \right| = \frac{1}{d} \left| \text{Tr} \left[ \sqrt{\lambda_0 \lambda_i^* U \lambda_0 U} \right] \right|,$$

(25)

where we have used the polar decomposition of $\lambda_0 = \sqrt{\lambda_0 \lambda_0^* U \lambda_0 U}$. The visibility is thus maximised when $U = U_{\lambda_0}^\dagger$. If the eigenvalues of $\lambda_0^2 \lambda_0$ are $\{r_j\}$, then the visibility when $\Lambda$ is in both arms is simply $v_{\Lambda \Lambda} = \sum_j r_j$, whereas if we compare $\Lambda$ with its closest unitary, it is $v_{\Lambda U} = \sum \sqrt{r_j}$, and it is easy to see that $v_{\Lambda U} \geq v_{\Lambda \Lambda}$.

We can also consider what is the maximum coherent fidelity between two CP maps $\mathcal{U}$ and $\mathcal{V}$. Let $\{v_i\}$ and $\{v'_j\}$ be orthogonal Kraus sets for $\mathcal{U}$ and $\mathcal{V}$ respectively (Eq. (24)), and $(A_{ij}) = (\text{Tr}[v_i^\dagger v_j])$ be the matrix of their inner products. If $v'_i = \sum_j g_{ij} v_j$ and $v_i = \sum_j h_{ij} v_j$ are also compatible Kraus operators for $\mathcal{U}$ and $\mathcal{V}$ (Eq. (28)), then

$$\left| \text{Tr} \left[ v'_i v_j^\dagger \right] \right| = \sum_{ij} \text{Tr} \left[ g_{ij}^* v_i^\dagger h_{0j} v_j^\dagger \right] = \sum_i g_{i0}^* \sum_j A_{ij} h_{0j}^\dagger$$

(26)

is maximised when

$$\|A g_0\| = \max_{\|g\|=1} \|A g\|$$

(27)

$$\tilde{h}_{0j} = \frac{A g_0^*}{\|A g_0\|},$$

(28)

where we have used the operator norm of $A = (A_{ij})$, and $g_0$ and $h_0$ are the first column vectors of the isometries $g_{ij}$ and $h_{ij}$ relating $\{v_i\}$ and $\{v_j\}$ to $\{v'_i\}$ and $\{v'_j\}$ respectively. This reduces to the previous case where both CP maps are the same and thus $(A_{ij})$ is real diagonal.

CONCLUSION

Interferometry can be applied to the case of non-unitary processes to extract information about the underlying physical processes which implement them. In particular, we can derive a measure of the coherence of a quantum operation, its maximum for any CP map, and the closest unitary under this measure. It is to be seen whether consideration of dynamical CP maps can impose further internal structure on quantum operations.

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