The isometry group of $n$-dimensional Einstein gyrogroup

Teerapong Suksumran
Department of Mathematics
Faculty of Science, Chiang Mai University
Chiang Mai 50200, Thailand
teerapong.suksumran@cmu.ac.th

Abstract
The space of $n$-dimensional relativistic velocities normalized to $c = 1$,

$$\mathcal{B} = \{v \in \mathbb{R}^n : \|v\| < 1\},$$

is naturally associated with Einstein velocity addition $\oplus_E$, which induces the rapidity metric $d_E$ on $\mathcal{B}$ given by $d_E(u, v) = \tanh^{-1} \| -u \oplus_E v \|$. This metric is also known as the Cayley–Klein metric. We give a complete description of the isometry group of $(\mathcal{B}, d_E)$, along with its composition law.

Keywords. Einstein velocity addition, Einstein gyrogroup, Cayley–Klein metric, gyrometric, isometry group.

2010 MSC. Primary 51A05; Secondary 83A05, 51F25, 20N05.

1 Introduction
The space of $n$-dimensional relativistic velocities normalized to $c = 1$,

$$\mathcal{B} = \{v \in \mathbb{R}^n : \|v\| < 1\},$$

has various underlying mathematical structures, including a bounded symmetric space structure [3, 4] and a gyrovector space structure [12]. Further, it is a primary object in special relativity in the case when $n = 3$ [2, 6]. Of particular importance is reflected in the composition law of Lorentz boosts:

$$L(u) \circ L(v) = L(u \oplus_E v) \circ \text{Gyr}[u, v],$$
The isometry group of $n$-dimensional Einstein gyrogroup

where $L(u)$ and $L(v)$ are Lorentz boosts parametrized by $u$ and $v$ respectively, $\oplus_E$ is Einstein velocity addition (defined below), and $\text{Gyr}[u, v]$ is a rotation of spacetime coordinates induced by an Einstein addition preserving map (namely an Einstein gyroautomorphism) [12, p. 448]. Moreover, the unit ball $B$ gives rise to a model for $n$-dimensional hyperbolic geometry when it is endowed with the Cayley–Klein metric as well as the Poincaré metric [5, 7].

The open unit ball of $\mathbb{R}^n$ admits a group-like structure when it is endowed with Einstein addition $\oplus_E$, defined by

$$u \oplus_E v = \frac{1}{1 + \langle u, v \rangle} \left( u + \frac{1}{\gamma_u} v + \frac{\gamma_u}{1 + \gamma_u} \langle u, v \rangle u \right), \quad (1.1)$$

where $\langle \cdot, \cdot \rangle$ denotes the usual Euclidean inner product and $\gamma_u$ is the Lorentz factor normalized to $c = 1$ given by $\gamma_u = \frac{1}{\sqrt{1 - \|u\|^2}}$. In fact, the space $(B, \oplus_E)$ satisfies the following properties [10, 12]:

I. (IDENTITY) The zero vector $0$ satisfies $0 \oplus_E v = v = v \oplus_E 0$ for all $v \in B$.

II. (INVERSE) For each $v \in B$, the negative vector $-v$ belongs to $B$ and satisfies

$$(-v) \oplus_E v = 0 = v \oplus_E (-v).$$

III. (THE GYROASSOCIATIVE LAW) For all $u, v \in B$, there are Einstein addition preserving bijective self-maps $\text{gyr}[u, v]$ and $\text{gyr}[v, u]$ of $B$ such that

$$u \oplus_E (v \oplus_E w) = (u \oplus_E v) \oplus_E \text{gyr}[u, v] w$$

and

$$(u \oplus_E v) \oplus_E w = u \oplus_E (v \oplus_E \text{gyr}[v, u] w)$$

for all $w \in B$.

IV. (THE LOOP PROPERTY) For all $u, v \in B$,

$$\text{gyr}[u \oplus_E v, v] = \text{gyr}[u, v] \quad \text{and} \quad \text{gyr}[u, v \oplus_E u] = \text{gyr}[u, v].$$

V. (THE GYROCOMMUTATIVE LAW) For all $u, v \in B$,

$$u \oplus_E v = \text{gyr}[u, v](v \oplus_E u).$$
The isometry group of $n$-dimensional Einstein gyrogroup

From properties I through V, it follows that $(\mathbb{B}, \oplus_E)$ forms a gyrocommutative gyrogroup (also called a $K$-loop or Bruck loop), which shares several properties with groups [9, 12]. However, Einstein addition is a nonassociative operation so that $(\mathbb{B}, \oplus_E)$ fails to form a group. Property III resembles the associative law in groups and property V resembles the commutative law in abelian groups. The map $\text{gyr}[u, v]$ in property III is called an Einstein gyroautomorphism, which turns out to be a rotation of the unit ball. Henceforward, $(\mathbb{B}, \oplus_E)$ is referred to as the $(n$-dimensional) Einstein gyrogroup.

Recall that the rapidity of a vector $v$ in $\mathbb{B}$ (cf. [5, p. 1229]) is defined by

$$\phi(v) = \tanh^{-1} \|v\|. \quad (1.2)$$

**Theorem 1.1.** The rapidity $\phi$ satisfies the following properties:

1. $\phi(v) \geq 0$ and $\phi(v) = 0$ if and only if $v = 0$;
2. $\phi(-v) = \phi(v)$;
3. $\phi(u \oplus_E v) \leq \phi(u) + \phi(v)$;
4. $\phi(\text{gyr}[u, v]w) = \phi(w)$

for all $u, v, w \in \mathbb{B}$.

**Proof.** Items 1 and 2 are clear. To prove item 3, let $u, v \in \mathbb{B}$. By Proposition 3.3 of [5], $\|u \oplus_E v\| \leq \frac{\|u\| + \|v\|}{1 + \|u\| \|v\|}$. Set $u = \tanh^{-1} \|u\|$ and $v = \tanh^{-1} \|v\|$. Then

$$\|u \oplus_E v\| \leq \frac{\tanh u + \tanh v}{1 + (\tanh u)(\tanh v)} = \tanh (u + v),$$

which implies $\phi(u \oplus_E v) = \tanh^{-1} \|u \oplus_E v\| \leq u + v = \phi(u) + \phi(v)$. Item 4 follows from the fact that any gyroautomorphism of the Einstein gyrogroup is indeed the restriction of an orthogonal transformation of $\mathbb{R}^n$ to $\mathbb{B}$ so that it preserves the Euclidean norm (and hence also the rapidity); see, for instance, Theorem 3 of [10] and Proposition 2.4 of [5]. □

Theorem 1.1 implies that $d_E$ defined by

$$d_E(u, v) = \phi(-u \oplus_E v) = \tanh^{-1} \|-u \oplus_E v\| \quad (1.3)$$

for all $u, v \in \mathbb{B}$ is indeed a metric (or a distance function) on $\mathbb{B}$, called the rapidity metric of the Einstein gyrogroup. In Theorem 3.9 of [5], Kim and Lawson prove that $d_E$ agrees with the Cayley–Klein metric, defined from cross-ratios, in the...
The isometry group of $n$-dimensional Einstein gyrogroup

Beltrami–Klein model of $n$-dimensional hyperbolic geometry. Equation (1.3) includes what Ungar refers to as the (Einstein) gyrometric, which is defined by

$$\varrho_E(u, v) = \|u \oplus_E v\|$$ (1.4)

for all $u, v \in \mathbb{B}$. Using Proposition 3.3 of [5], we obtain that

$$\|u \oplus_E v\| \leq \|u\| + \|v\| \leq \|u\| + \|v\|$$

for all $u, v \in \mathbb{B}$ and so the gyrometric $\varrho_E$ is indeed a metric on $\mathbb{B}$. In fact, this is a consequence of Theorem 3.2 of [8]. Since $\tanh^{-1}$ is an injective function, it follows that a self-map of $\mathbb{B}$ preserves $d_E$ if and only if it preserves $\varrho_E$. Hence, $(\mathbb{B}, d_E)$ and $(\mathbb{B}, \varrho_E)$ have the same isometry group.

The next theorem lists some useful algebraic properties of the Einstein gyrogroup, which will be essential in studying the geometric structure of the unit ball in Section 2.

**Theorem 1.2 (See [9, 12]).** The following properties are true in $(\mathbb{B}, \oplus_E)$:

1. $-u \oplus_E (u \oplus_E v) = v$; (LEFT CANCELLATION LAW)
2. $-(u \oplus_E v) = \text{gyr}[u, v](-v \oplus_E -u)$;
3. $(-u \oplus_E v) \oplus_E \text{gyr}[-u, v](-v \oplus_E w) = -u \oplus_E w$;
4. $\text{gyr}[-u, -v] = \text{gyr}[u, v]$; (EVEN PROPERTY)
5. $\text{gyr}[v, u] = \text{gyr}^{-1}[u, v]$, where $\text{gyr}^{-1}[u, v]$ denotes the inverse of $\text{gyr}[u, v]$ with respect to composition of functions; (INVERSIVE SYMMETRY)
6. $L_u : v \mapsto u \oplus_E v$ defines a bijective self-map of $\mathbb{B}$ and $L_u^{-1} = L_{-u}$.

## 2 Main Results

Let $O(\mathbb{R}^n)$ be the orthogonal group of $n$-dimensional Euclidean space $\mathbb{R}^n$; that is, $O(\mathbb{R}^n)$ consists precisely of (bijective) Euclidean inner product preserving transformations of $\mathbb{R}^n$ (also called orthogonal transformations of $\mathbb{R}^n$). Since the unit ball $\mathbb{B}$ is invariant under orthogonal transformations of $\mathbb{R}^n$, it follows that the set

$$O(\mathbb{B}) = \{\tau|_{\mathbb{B}} : \tau \in O(\mathbb{R}^n)\},$$ (2.1)

where $\tau|_{\mathbb{B}}$ denotes the restriction of $\tau$ to $\mathbb{B}$, forms a group under composition of functions. Note that Einstein addition is defined entirely in terms of vector addition,
The isometry group of $n$-dimensional Einstein gyrogroup

scalar multiplication, and the Euclidean inner product. Hence, every orthogonal transformation of $\mathbb{R}^n$ restricts to an automorphism of $B$ that leaves the Euclidean norm invariant. In particular, the map $i: v \mapsto -v$ defines an automorphism of $B$. Note that $O(B)$ is a subgroup of the (algebraic) automorphism group of $(B, d_E)$, denoted by $\text{Aut}(B, d_E)$.

Let $u, v \in B$. It is not difficult to check that $\text{gyr}[u, v]$ satisfies the following properties:

1. $\text{gyr}[u, v]0 = 0$;
2. $\text{gyr}[u, v]$ is an automorphism of $(B, \oplus_E)$;
3. $\text{gyr}[u, v]$ preserves the gyrometric $\varrho_E$.

Hence, by Theorem 3.1 of [1], there is an orthogonal transformation $\phi$ of $\mathbb{R}^n$ for which $\phi|_B = \text{gyr}[u, v]$. This proves the following inclusion:

$$\{\text{gyr}[u, v]: u, v \in B\} \subseteq O(B).$$

**Theorem 2.1.** For all $u \in B$, the left gyrotranslation $L_u$ defined by $L_u(v) = u \oplus_E v$ is an isometry of $B$ with respect to $d_E$.

**Proof.** Note that $L_u$ is a bijective self-map of $B$ since $L_{-u}$ acts as its inverse (see, for instance, Theorem 10 (1) of [11]). From Theorem 1.2, we have by inspection that

$$\| - (u \oplus_E x) \oplus_E (u \oplus_E y) \| = \| \text{gyr}[u, x](-x \oplus_E -u) \oplus_E (u \oplus_E y) \|$$

$$= \| (-x \oplus_E -u) \oplus_E \text{gyr}[x, u](u \oplus_E y) \|$$

$$= \| (-x \oplus_E -u) \oplus_E \text{gyr}[-x, -u](u \oplus_E y) \|$$

$$= \| -x \oplus_E y \|.$$

It follows that

$$d_E(L_u(x), L_u(y)) = \tanh^{-1} \| - L_u(x) \oplus_E L_u(y) \| = \tanh^{-1} \| - x \oplus_E y \| = d_E(x, y).$$

This proves that $L_u$ is an isometry of $(B, d_E)$.

**Corollary 2.2.** The gyroautomorphisms of the Einstein gyrogroup are isometries of $B$ with respect to $d_E$.

**Proof.** Let $u, v \in B$. According to Theorem 10 (3) of [11], we have

$$L_u \circ L_v = L_u \otimes_E v \circ \text{gyr}[u, v].$$

Hence, $\text{gyr}[u, v] = L_{-u \oplus_E v}^{-1} \circ L_u \circ L_v = L_{-(u \oplus_E v)} \circ L_u \circ L_v$. This implies that $\text{gyr}[u, v]$ is an isometry of $(B, d_E)$, being the composite of isometries. 

\[ \square \]
In fact, Corollary 2.2 is a special case of the following theorem.

**Theorem 2.3.** Every automorphism of \((\mathbb{B}, \oplus_E)\) that preserves the Euclidean norm is an isometry of \(\mathbb{B}\) with respect to \(d_E\). Therefore, every transformation in \(O(\mathbb{B})\) is an isometry of \(\mathbb{B}\).

**Proof.** Let \(\tau \in Aut(\mathbb{B}, \oplus_E)\) and suppose that \(\tau\) preserves the Euclidean norm. Then \(\tau\) is bijective. Direct computation shows that

\[
d_E(\tau(x), \tau(y)) = \tanh^{-1} ||\tau(-x \oplus_E y)|| = \tanh^{-1} ||-x \oplus_E y|| = d_E(x, y)
\]

for all \(x, y \in \mathbb{B}\). Hence, \(\tau\) is an isometry of \((\mathbb{B}, d_E)\). The remaining part of the theorem is immediate since \(O(\mathbb{B}) \subseteq Aut(\mathbb{B}, d_E)\). \(\square\)

Next, we give a complete description of the isometry group of \((\mathbb{B}, d_E)\) using Abe’s result [1].

**Theorem 2.4.** The isometry group of \((\mathbb{B}, d_E)\) is given by

\[
Iso(\mathbb{B}, d_E) = \{L_u \circ \tau : u \in \mathbb{B} \text{ and } \tau \in O(\mathbb{B})\}. \quad (2.2)
\]

**Proof.** By Theorems 2.1 and 2.3,

\[
\{L_u \circ \tau : u \in \mathbb{B} \text{ and } \tau \in O(\mathbb{B})\} \subseteq Iso(\mathbb{B}, d_E).
\]

Let \(\psi \in Iso(\mathbb{B}, d_E)\). By definition, \(\psi\) is a bijection from \(\mathbb{B}\) to itself. By Theorem 11 of [11], \(\psi = L_{\psi(0)} \circ \rho\), where \(\rho\) is a bijection from \(\mathbb{B}\) to itself that leaves \(0\) fixed. As in the proof of Theorem 18 (2) of [9], \(L_{\psi(0)}^{-1} = L_{-\psi(0)}\) and so \(\rho = L_{-\psi(0)} \circ \psi\). Therefore, \(\rho\) is an isometry of \((\mathbb{B}, d_E)\). Since \(d_E(\rho(x), \rho(y)) = d_E(x, y)\) and \(\tanh^{-1}\) is injective, it follows that

\[
||-\rho(x) \oplus_E \rho(y)|| = ||-x \oplus_E y||
\]

for all \(x, y \in \mathbb{B}\). Hence, \(\rho\) preserves the Einstein gyrometric. By Theorem 3.1 of [1], \(\rho = \tau|_{\mathbb{B}}\), where \(\tau\) is an orthogonal transformation of \(\mathbb{R}^n\). This proves the reverse inclusion. \(\square\)

By Theorem 2.4, every isometry of \((\mathbb{B}, d_E)\) has a (unique) expression as the composite of a left gyrotranslation and the restriction of an orthogonal transformation of \(\mathbb{R}^n\) to the unit ball. According to the commutation relation (55) of [9] for the case of the Einstein gyrogroup, one has the following composition law of isometries of \((\mathbb{B}, d_E)\):

\[
(L_u \circ \alpha) \circ (L_v \circ \beta) = L_{u \oplus_E \alpha(v)} \circ (gyr[u, \alpha(v)] \circ \alpha \circ \beta) \quad (2.3)
\]
for all \( u, v \in \mathbb{B}, \alpha, \beta \in O(\mathbb{B}) \). This reminds us of the composition law of Euclidean isometries. Note that \( L_u \circ \alpha = L_\tau \circ \beta \), where \( u, v \in \mathbb{B} \) and \( \alpha, \beta \in O(\mathbb{B}) \), if and only if \( u = v \) and \( \alpha = \beta \). This combined with (2.3) implies that the map \( L_\tau \circ \tau \mapsto (v, \tau) \) defines an isomorphism from the isometry group of \( \mathbb{B} \) to the gyrosemidirect product \( \mathbb{R} \times_\text{gyr} O(\mathbb{B}) \), which is a group consisting of the underlying set

\[
\{(v, \tau) : v \in \mathbb{B} \text{ and } \tau \in O(\mathbb{B})\}
\]

and group multiplication

\[
(u, \alpha)(v, \beta) = (u \oplus_E \alpha(v), \text{gyr}[u, \alpha(v)] \circ \alpha \circ \beta).
\]

(2.4)

For the relevant definition of a gyrosemidirect product, see Section 2.6 of [12]. Equation (2.4) is an analogous result in Euclidean geometry that the isometry group of \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) can be realized as the semidirect product \( \mathbb{R}^n \rtimes O(\mathbb{R}^n) \). The result that the group of holomorphic automorphisms of a bounded symmetric domain can be realized as a gyrosemidirect product is proved by Friedman and Ungar in Theorem 3.2 of [3]. Further, a characterization of continuous endomorphisms of the three-dimensional Einstein gyrogroup is obtained; see Theorem 1 of [6].

As an application of Theorem 2.4, we show that the space \( \mathbb{B}, d_E \) is homogeneous; that is, there is an isometry of \( \mathbb{B}, d_E \) that sends \( u \) to \( v \) for all arbitrary points \( u \) and \( v \) in \( \mathbb{B} \). We also give an easy way to construct point-reflection symmetries of the unit ball.

**Theorem 2.5 (Homogeneity).** If \( u \) and \( v \) are arbitrary points in \( \mathbb{B} \), then there is an isometry \( \psi \) of \( \mathbb{B}, d_E \) such that \( \psi(u) = v \). In other words, \( \mathbb{B}, d_E \) is homogeneous.

**Proof.** Let \( u, v \in \mathbb{B} \). Define \( \psi = L_\alpha \circ L_{-u} \). Then \( \psi \) is an isometry of \( \mathbb{B}, d_E \), being the composite of isometries. It is clear that \( \psi(u) = v \oplus_E (-u \oplus_E u) = v \).

**Theorem 2.6 (Symmetry).** For each point \( v \in \mathbb{B} \), there is a point-reflection \( \sigma_v \) of \( \mathbb{B}, d_E \) corresponding to \( v \); that is, \( \sigma_v \) is an isometry of \( \mathbb{B}, d_E \) such that \( \sigma_v^2 \) is the identity transformation of \( \mathbb{B} \) and \( v \) is the unique fixed point of \( \sigma_v \).

**Proof.** Let \( t \) be the negative map of \( \mathbb{B} \); that is, \( t(w) = -w \) for all \( w \in \mathbb{B} \). In view of (1.1), it is clear that \( t \) is an automorphism of \( \mathbb{B} \) with respect to \( \oplus_E \). By Theorem 2.3, \( t \) is an isometry of \( \mathbb{B}, d_E \). Define \( \sigma_v = L_{\alpha \circ t \circ L_{-v}} \). Then \( \sigma_v \) is an isometry of \( \mathbb{B}, d_E \) that is a point-reflection of \( \mathbb{B} \) corresponding to \( v \). The uniqueness of the fixed point of \( \sigma_v \) follows from the fact that \( \theta \) is the unique fixed point of \( t \).

**Acknowledgements.** The author would like to thank Themistocles M. Rassias for his generous collaboration. He also thanks anonymous referees for useful comments.
The isometry group of $n$-dimensional Einstein gyrogroup

References

[1] T. Abe, *Gyrometric preserving maps on Einstein gyrogroups, Möbius gyrogroups and Proper Velocity gyrogroups*, Nonlinear Funct. Anal. Appl. 19 (2014), 1–17.

[2] Y. Friedman and T. Scarr, *Physical applications of homogeneous balls*, Progress in Mathematical Physics, vol. 40, Birkhäuser, Boston, 2005.

[3] Y. Friedman and A. Ungar, *Gyrosemidirect product structure of bounded symmetric domains*, Results Math. 26 (1994), no. 1-2, 28–38.

[4] S. Kim and J. Lawson, *Smooth Bruck loops, symmetric spaces, and non-associative vector spaces*, Demonstr. Math. 44 (2011), no. 4, 755–779.

[5] , *Unit balls, Lorentz boosts, and hyperbolic geometry*, Results Math. 63 (2013), 1225–1242.

[6] L. Molnár and D. Virosztek, *On algebraic endomorphisms of the Einstein gyrogroup*, J. Math. Phys. 56 (2015), no. 8, 082302 (5 pages).

[7] J. Ratcliffe, *Foundations of hyperbolic manifolds*, 2nd ed., Graduate Texts in Mathematics, vol. 149, Springer, New York, 2006.

[8] T. Suksumran, *On metric structures of normed gyrogroups*, arXiv:1810.10491.

[9] , *Essays in mathematics and its applications: In honor of Vladimir Arnold*, Th. M. Rassias and P. M. Pardalos (eds.), ch. The Algebra of Gyrogroups: Cayley’s Theorem, Lagrange’s Theorem, and Isomorphism Theorems, pp. 369–437, Springer, Switzerland, 2016.

[10] T. Suksumran and K. Wiboonton, *Einstein gyrogroup as a B-loop*, Rep. Math. Phys. 76 (2015), 63–74.

[11] , *Isomorphism theorems for gyrogroups and L-subgyrogroups*, J. Geom. Symmetry Phys. 37 (2015), 67–83.

[12] A. Ungar, *Analytic hyperbolic geometry and Albert Einstein’s Special Theory of Relativity*, World Scientific, Hackensack, NJ, 2008.