Renormalization group improvement of truncated perturbative series in QCD. 
Decays of $\tau$-lepton and $\eta_c$-charmonium

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Abstract

We formulate a general scheme to improve the truncated perturbative expansion in $\alpha_s$ by means of the renormalization group in QCD for the single-scale quantities. The procedure is used for the evaluation of hadronic decay rates of $\tau$-lepton and $\eta_c$-charmonium. The scale dependence of result for $\eta_c$ is studied in the scheme of fixed value for the $\overline{\text{MS}}$-mass of charmed quark.

1 Introduction

For many physical cases in QCD, an observable quantity is usually expressed in terms of truncated series in the coupling constant $\alpha_s$ with given coefficients, so that in the next-to-next-to-leading order (N$^3$LO) we get

$$R = 1 + c_1 \frac{\alpha_s(\Lambda)}{\pi} + c_2 \left( \frac{\alpha_s(\Lambda)}{\pi} \right)^2 + c_3 \left( \frac{\alpha_s(\Lambda)}{\pi} \right)^3,$$

where $c_{1,2,3}$ are some numbers, and $\Lambda$ is a fixed scale. So, the value $R$ is the single-scale quantity. The exhausted examples are the followings:

1. The hadronic fraction of $\tau$-decay width [1, 2]

$$R_\tau = \frac{\Gamma[\tau \rightarrow \nu_\tau \text{hadrons}]}{\Gamma[\tau \rightarrow \nu_\tau e^+\nu_e]} =$$

$$R_\tau^{[0]} \left( 1 + c_1 \frac{\alpha_s(m_\tau)}{\pi} + c_2 \left( \frac{\alpha_s(m_\tau)}{\pi} \right)^2 + c_3 \left( \frac{\alpha_s(m_\tau)}{\pi} \right)^3 + \Delta r_{\text{NP}} \right),$$

(2)
where $R_{\tau}^{[0]} = 3.058$, the coefficients are given by

$$c_1^\tau = 1, \quad c_2^\tau = 5.2, \quad c_3^\tau = 26.4,$$

and $\Delta r_{NP} = -0.014 \pm 0.005$ is a nonperturbative contribution.

2. The hadronic fraction of $\eta_c$-decay width [3]

$$R_{\eta_c} = \frac{\Gamma[\eta_c \to \text{hadrons}]}{\Gamma[\eta_c \to \gamma\gamma]} = R_{\eta_c}^{[0]} \left( 1 + d_1 \frac{\alpha_s(2m_c)}{\pi} \right),$$

where

$${R}_{\eta_c}^{[0]} = \frac{C_F}{2N_c} \frac{1}{e_c^4} \frac{\alpha_s^2(2m_c)}{\alpha_{em}^2},$$

with $C_F = (N_c^2 - 1)/2N_c$, $N_c = 3$ is the number of colors, $e_c = 2/3$ is the electric charge of charmed quark, and the coefficient $d_1$ is given by

$$d_1 = \frac{199}{6} - \frac{13\pi^2}{8} - \frac{8}{9} n_f - \frac{2}{3} \ln 2,$$

where $n_f = 3$ is the number of ‘active’ flavors, and $m_c$ is the pole mass of charmed quark.

The above formulae can be used for the extraction of $\alpha_s$ at the appropriate scale. The value of $\alpha_s$-corrections is numerically significant. So, the problem is how the truncated series can be improved. The well-established approach to the solution of such the problem is a resummation of some significant terms. We mention two of such techniques. The first is the summation of $(\beta_0 \alpha_s)^n$ contributions, where $\beta_0 = 11 - \frac{2}{3} n_f$ is the first coefficient of $\beta$-function in QCD [4, 5]. The second procedure is based on an appropriate change of renormalization scheme by $\bar{\alpha}_s = \alpha_s(1 + b_1 \alpha_s + \ldots)$ to the given order in the coupling constant, which allows one to decrease a role of higher-order corrections or even to minimize it with the modification of $\bar{\beta}(\bar{\alpha}_s)$-function resulting in a different running of $\bar{\alpha}_s$ [6]. The disadvantage of above methods is twofold. First, the next-order correction while computed exactly can essentially differ from the approximation of $\beta_0 \alpha_s$-dominance. Second, the redefinition of renormalization scheme leads to the scale or normalization-point dependence of matching procedure.

In this paper we present a procedure to improve the truncated series in the framework of renormalization group by introducing an auxiliary scale and taking a single-scale limit. A general formalism is given in Section 2. The numerical estimates are presented in Section 3. The analysis of scale dependence for the $\eta_c$-decay rate is performed, since the normalization at the pole mass involves the additional problem caused by the residual change of $m_c$ by the variation of normalization point in the $\overline{\text{MS}}$-mass $\bar{m}_c(\mu)$ [7]. Our results are summarized in Conclusion.
2 Renormalization group improvement

For the sake of clarity, let us start with the consideration of first-order correction.

\[ K = \frac{\mathcal{R}}{\mathcal{R}^{[0]}} = 1 + c_1 \frac{\alpha_s(\Lambda)}{\pi}. \]  
(7)

Introduce an auxiliary scale \( \Lambda' = \kappa \Lambda \), so that

\[ K = 1 + \frac{c_1}{\ln \kappa} \frac{\alpha_s(\Lambda)}{\ln \kappa}. \]  
(8)

Making use of the renormalization group relation to the first order in \( \alpha_s \),

\[ \frac{\alpha_s(\Lambda)}{\alpha_s(\kappa \Lambda)} = 1 + \frac{\beta_0}{2\pi} \alpha_s(\Lambda) \ln \kappa, \]  
(9)

we clearly get

\[ K(\kappa) = \left[ \frac{\alpha_s(\Lambda)}{\alpha_s(\kappa \Lambda)} \right]^{\frac{2c_1}{\beta_0 \ln \kappa}}, \]  
(10)

which gives the ordinary presentation improved by the renormalization group. Note, that one finds the limit

\[ \lim_{\ln \kappa \to 0} \frac{d}{d \ln \kappa} K(\kappa) \equiv 0, \]  
(11)

which will be correct for the further consideration at a fixed order in \( \alpha_s \).

The single-scale limit of \( \ln \kappa \to 0 \) can be easily evaluated

\[ K^{\text{RGI}} = \exp \left[ c_1 \frac{\alpha_s(\Lambda)}{\pi} \right], \]  
(12)

which is our result for the case of first-order correction.

In order to proceed with the higher-order corrections, let me perform the derivation in another way. So, the \( \beta \)-function has the form

\[ \beta(a) = \frac{d \ln a(\mu)}{d \ln \mu^2} = -\beta_0 a - \beta_1 a^2 - \beta_2 a^3 \]  
(13)

with \( a = \frac{\alpha_s}{4\pi} \). To the first order it gives

\[ \frac{\alpha_s(\Lambda)}{\alpha_s(\kappa \Lambda)} \approx \exp \left[ \frac{\beta_0}{2\pi} \alpha_s(\Lambda) \ln \kappa \right], \]  
(14)

at \( \ln \kappa \to 0 \). Then,

\[ \left[ \frac{\alpha_s(\Lambda)}{\alpha_s(\kappa \Lambda)} \right]^{\frac{2c_1}{\beta_0 \ln \kappa}} \approx \exp \left[ c_1 \frac{\alpha_s(\Lambda)}{\pi} \right], \]  
(15)
and expanding in $\alpha_s$, we rederive the renormalization group improvement (RGI) for the first-order correction.

Further, we can easily find the RGI for the third order in $\alpha_s$ ($N^3\text{LO}$). Indeed, since

$$\alpha_s(\Lambda) \approx \exp \left[ c_1 \frac{\alpha_s(\Lambda)}{\pi} + \bar{c}_2 \left( \frac{\alpha_s(\Lambda)}{\pi} \right)^2 + \bar{c}_3 \left( \frac{\alpha_s(\Lambda)}{\pi} \right)^3 \right], \quad (16)$$

we get

$$\left[ \frac{\alpha_s(\Lambda)}{\alpha_s(\kappa \Lambda)} \right] c_1 + 4\bar{c}_2 a + 16\bar{c}_3 a^2 \frac{4}{\ln \kappa^2} = \exp \left[ c_1 \frac{\alpha_s(\Lambda)}{\pi} + \bar{c}_2 \left( \frac{\alpha_s(\Lambda)}{\pi} \right) + \bar{c}_3 \left( \frac{\alpha_s(\Lambda)}{\pi} \right) \right], \quad (17)$$

where we put

$$\bar{c}_2 = c_1 - \frac{1}{2} c_1^2, \quad \bar{c}_3 = c_3 - \frac{1}{6} c_1^3 - c_1 \bar{c}_2. \quad (18)$$

Expanding in $\alpha_s$ at $\ln \kappa \rightarrow 0$, we find

$$K_{\text{rgi}} = \exp \left[ c_1 \frac{\alpha_s(\Lambda)}{\pi} + \bar{c}_2 \left( \frac{\alpha_s(\Lambda)}{\pi} \right)^2 + \bar{c}_3 \left( \frac{\alpha_s(\Lambda)}{\pi} \right)^3 \right], \quad (19)$$

Thus, the third-order improved expression has the form

We stress the renormalization group motivation used in contrast to ad hoc method of Padé approximants.

Let us show how the improvement works in a simple example. So, we consider a rather oscillating sum,

$$E = 1 - 0.5 + 0.3 = 0.8,$$

which reveals a ‘slow’ convergency, since

$$E^{[0]} = 1, \quad E^{[1]} = 0.5, \quad E^{[1]} = 0.8,$$

while

$$E_{\text{rgi}} = \exp[1 - 0.5 + (0.3 - 0.5^3)]$$

results in

$$E_{\text{rgi}}^{[0]} = 1, \quad E_{\text{rgi}}^{[1]} = 0.61, \quad E_{\text{rgi}}^{[2]} = 0.72,$$

which is ‘more stable’.

Thus, we expect that $K_{\text{rgi}}$ possesses a more numerical stability in the truncated series. Of course, if a series is essentially asymptotic, the improvement cannot cancel a ‘bad’ convergency.
Next, we have to mention the numerical problem often appearing with the $\alpha_s$-corrections to the amplitudes and the amplitudes squared if those corrections are significantly large. Indeed, the correction to the amplitude

$$A = A^{[0]}(1 + c_1\alpha_s)$$

should lead to

$$A^2 = \left(A^{[0]}\right)^2 (1 + 2c_1\alpha_s),$$

so that the ratio

$$\frac{1 + 2c_1\alpha_s}{(1 + c_1\alpha_s)^2}$$

umerically deviates from unit. The RGI has no such the problem, since the exponent does not involve the above mismatching.

Finally, we stress that the RGI does not present some kind of resummation of higher orders. In the resummation technique one certainly suggests a form of higher-order terms. In contrast, we give the exact expression produced by the renormalization group. At small $\alpha_s$ as dictated by the perturbative paradigm, the expression can be expanded till the appropriate order. Thus, one could claim that the RGI procedure looks like overfl ying the accuracy. To my opinion, one should use the RGI point as a central value of the calculated quantity, while the expansion truncated to the given order would indicate a systematic error of numerical estimate.

### 3 Numerical estimates

#### 3.1 Hadronic fraction of $\tau$-lepton width

The RGI formula for the $\tau$-lepton decays into hadrons reads off

$$\mathcal{R}_{\tau}^{\text{RGI}} = \mathcal{R}_{\tau}^{[0]} \left\{ \exp \left[ c_1 \frac{\alpha_s(m_\tau)}{\pi} + \tilde{c}_2 \left( \frac{\alpha_s(m_\tau)}{\pi} \right)^2 + \tilde{c}_3 \left( \frac{\alpha_s(m_\tau)}{\pi} \right)^3 \right] + \Delta r_{\text{NP}} \right\},$$

where

$$\tilde{c}_2 = 4.7, \quad \tilde{c}_3 = 22.53.$$  \hspace{1cm} (21)

Implementing

$$\mathcal{R}_{\tau}^{\text{exp}} = 3.635 \pm 0.014,$$

we find

$$\alpha_s(m_\tau) = 0.333 \pm 0.009,$$  \hspace{1cm} (22)

which results in

$$\alpha_s(m_Z) = 0.119 \pm 0.001,$$ \hspace{1cm} (23)

where we include the experimental uncertainty, only. For the sake of comparison, the PDG value extracted by the same measurement of $\tau$ rate reads off $\alpha_s(m_\tau) = 0.353 \pm 0.007(\text{exp}) \pm 0.030(\text{th})$, which respectively gives $\alpha_s(m_Z) = 0.121 \pm 0.003$. We point out that the theoretical
uncertainty in PDG is slightly overestimated, to our opinion, since the displacement of central value extracted in two ways equals $\Delta \alpha_s = 0.02$.

Thus, the preferable value of coupling constant following from the $\tau$-lepton hadronic width is given by
\[ \alpha_s(m_Z) = 0.119 \pm 0.002 \] (24)
with the central point closer to the ‘world average’.

### 3.2 Hadronic width of $\eta_c$-charmonium

The problem with the estimate of hadronic width of $\eta_c$-charmonium is twofold. First, the scale setting in the $\alpha_s$-correction is beyond the accuracy, since its variation contributes to $\alpha_s^2$. So, we should put the arbitrary scale by
\[ \mathcal{R}_{\eta_c} = \mathcal{R}_{[0]}^{\eta_c} \left( 1 + d_1 \frac{\alpha_s}{\pi} \right). \] (25)

The second point is the prescription for the pole mass of charmed quark. In the perturbative QCD, the pole mass is strictly defined. The relation between the $\overline{\text{MS}}$-running mass $\bar{m}_c(\mu)$ and the pole mass is known to the $\alpha_s^3$-order [8]. Explicitly, to the $\alpha_s^2$-terms [9] we put
\[ m_{\text{pole}} = \bar{m}(\mu) \left( 1 + c_1(\mu) \frac{\alpha_s^{\overline{\text{MS}}}}{4\pi} + c_2(\mu) \left( \frac{\alpha_s^{\overline{\text{MS}}}}{4\pi} \right)^2 \right), \] (26)
with
\[ c_1(\mu) = C_F (4 + 3L), \] (27)
\[ c_2(\mu) = C_F C_A \left( \frac{1111}{24} - 8\zeta(2) - 4I_3(1) + \frac{185}{6} L + \frac{11}{2} L^2 \right) \]
\[ -C_F T_F n_f \left( \frac{71}{6} + 8\zeta(2) + \frac{26}{3} L + 2L^2 \right) \] (28)
\[ +C_F^2 \left( \frac{121}{8} + 30\zeta(2) + 8I_3(1) + \frac{27}{2} L + \frac{9}{2} L^2 \right) - 12C_F T_F (1 - 2\zeta(2)), \]
where $I_3(1) = \frac{3}{2} \zeta(3) - 6\zeta(2) \ln 2$, and $L = 2\ln(\mu/m_{\text{pole}})$. The value of pole mass is the renormalization invariant. However, at reasonable scales $\mu$, the residual dependence due to the truncation of perturbative series is numerically significant. The reason of such the dependence is a growth of coefficients in series as caused by the renormalon. In fact, the pole mass becomes a scale-dependent quantity. To avoid this problem, the operative procedure is to fix a short-distance mass $m_{\text{QCD}}$ free off the renormalon and to perform the calculations with the series expressed in terms of $m_{\text{QCD}}$. We exploit two schemes, which lead to results close enough to each other.

The first scheme is given by the $\overline{\text{MS}}$-running mass $\bar{m}(\mu)$. Taking
\[ \bar{m}_c(\bar{m}_c) = 1.4 \text{ GeV}, \]
we calculate the pole mass shown in Fig. 1. We have checked that the implication of RGI procedure to the relation between the pole and running masses is consistent with the above result, and the effect of RGI can be absorbed into the decrease of $\bar{m}_c(\bar{m}_c)$-value by about 50 MeV, which below the systematic accuracy of matching procedure as discussed below.

Figure 1: The pole mass of charmed quark calculated in two schemes versus the normalization scale. The dashed line gives the result of matching in the potential approach, the solid line does by the perturbative relation between the pole and running masses shifted with $-\Delta m$ in (30) at $\alpha_s(m_z) = 0.118$, the short-dashed line is the same as the solid one but at $\alpha_s(m_z) = 0.121$.

The second is the potential scheme described in ref [10]. In this case, we calculate the scale-dependent matching of perturbative 2-loop scatic potential $V_{\text{pert}}(r, \mu)$ involving the 3-loop running $\alpha_s$ with the phenomenological QCD-motivated static potential $V(r)$ containing both the 2-loop short-distance coulomb-like contribution as well as the long-distance linear confining term preserving the infrared stability. Then, the potential and, hence, the $V$-masses are free off the renormalon. The heavy quark masses are fixed by the measured spin-average mass-spectra of heavy quarkonia. So,

$$m_c^\gamma = 1.468 \text{ GeV}, \quad m_b^\gamma = 4.873 \text{ GeV}. \quad (29)$$

The matching of scale-dependent perturbative potential $\delta V(\mu) = V(r) - V_{\text{pert}}(r, \mu)$ is extracted numerically as described in ref [10]. Thus, the cancellation of renormalon in the sum of $2m_{\text{pole}} + V_{\text{pert}}(r, \mu)$ gives

$$m_c^{\text{pole}}(\mu) = m_c^\gamma + \frac{1}{2} \delta V(\mu) - \Delta m, \quad (30)$$

up to a constant shift $\Delta m$ independent of the scale. The matching with the perturbative pole mass in (26) gives $\Delta m = -155 \pm 15$ MeV, depending on the variation of coupling constant.
\( \alpha_s(m_z) \) in the limits of 0.118 - 0.123. The value of \( \Delta m \) indicates the accuracy of matching procedure. The result is presented in Fig.1, which reveals a good agreement of two schemes used.

Figure 2: The fraction of hadronic width for the \( \eta_c \)-charmonium calculated with the fixed value of pole mass for the charmed quark \( m_c = 1.64 \) GeV (the short-dashed curve) and with the scale-dependent pole mass in the schemes of fixed running mass (the solid curve) and of potential approach (the dashed curve).

Then, the perturbative formula (25) with (30) results in the \( R_{\eta_c} \) shown in Fig. 2, where-from we get

\[ R_{\eta_c} = 2.6 \cdot 10^3 \] (31)

at \( \mu = 3.9 \) GeV with

\( \alpha_s(2m_c) = 0.242, \quad \alpha_s(\mu) = 0.240, \quad m_c = 1.64 \) GeV.

The estimate in (31) is slightly greater than the value \( R_{\eta_c} = 2.1 \cdot 10^3 \) given by Bodwin and Chen [5]. We stress the scale-stability of our result.

Further, at the same scale we find

\[ R_{\eta_c}^{\text{RGI}} = 3.7 \cdot 10^3. \] (32)

Then, comparing (32) with (31) we obtain the final estimate including the theoretical uncertainty due to possible contributions of higher orders and, hence, the induced scale-dependence by the variation of central values as

\[ R_{\eta_c}^{\text{th}} = (3.7 \pm 1.1) \cdot 10^3, \] (33)
which is in agreement with the experimental value

\[ R_{\eta_c}^{\text{exp}} = (3.3 \pm 1.3) \cdot 10^3, \]

to be compared with \( R_{\eta_c}^{\text{NNA}} = (3.01 \pm 0.030 \pm 0.034) \cdot 10^3 \) obtained in [5] under the resummation of \((\beta_0 \alpha_s)^n\)-terms. We point out that the improvement of the experimental accuracy combined with the calculation of \( \alpha_s^2 \)-correction would give a good opportunity to extract the mass of charmed quark. In this respect, we refer to ref. [11], where the \( \alpha_s^2 \)-corrections were taken into account in the ratio of widths for the decays of \( J/\psi \rightarrow e^+e^- \) and \( \eta_c \rightarrow \gamma\gamma \), so that the analysis suffers from the uncertainties related with the relativistic corrections entering the ratio for the different initial states. The advantage of \( R_{\eta_c} \) is the cancellation of such the initial state corrections.

4 Conclusion

We have developed a general scheme to improve the estimate of truncated perturbative series in QCD by the tool of renormalization group for the single-scale quantities. The method allows one to get more realistic central values of the quantities as well as to estimate the theoretical uncertainty of results by comparison of RGI values with the perturbatively expanded ones. The RGI receipt for the calculation of quantity (1), (7) is given by (18) and (19).

We have applied the approach to the fractions of hadronic widths for the \( \tau \)-lepton and \( \eta_c \)-charmonium, which allows us to get realistic estimates of

\[ \alpha_s(m_\tau) \quad \text{and} \quad R_{\eta_c} = \frac{\Gamma[\eta_c \rightarrow \text{hadrons}]}{\Gamma[\eta_c \rightarrow \gamma\gamma]} \]

in a reasonable agreement with the appropriately measured values.

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