THE SPECTRAL THEOREM FOR BIMODULES IN HIGHER RANK GRAPH C*-ALGEBRAS

ALAN HOPENWASSER

Abstract. In this note we extend the spectral theorem for bimodules to the higher rank graph C*-algebra context. Under the assumption that the graph is row finite and has no sources, we show that a bimodule over a natural abelian subalgebra is determined by its spectrum iff it is generated by the Cuntz-Krieger partial isometries which it contains iff the bimodule is invariant under the gauge automorphisms. We also show that the natural abelian subalgebra is a masa iff the higher rank graph satisfies an aperiodicity condition.

1. Introduction

Many C*-algebras can be coordinitized – a property that proves very useful both in the study of the C*-algebra and also of its subalgebras. Coordinitization is achieved by presenting the C*-algebra as a groupoid C*-algebra. The unit space of the groupoid is associated with an abelian subalgebra which is often, though not always, a masa. (The abelian subalgebra depends on the choice of coordinates and need not be intrinsic.) A great many of the (non-self-adjoint) subalgebras of a groupoid C*-algebra either contain the “diagonal” abelian algebra or are a bimodule over it. When the groupoid is r-discrete and principal, one of the most fundamental tools used in the study of subalgebras is the spectral theorem for bimodules of Muhly and Solel [4]. Roughly speaking, this says that a bimodule is determined by the coordinates on which it is supported.

When the groupoid is not principal, it is no longer true that a bimodule is determined by its spectrum. For graph C*-algebras, [2] contains a characterization of those bimodules which are determined by their spectra: these are the bimodules which are invariant under the gauge automorphisms. (Another equivalent condition is that the bimodule be generated by the Cuntz-Krieger partial isometries which it contains.) Graph C*-algebras have been extensively studied in the last decade; see [5] for an excellent summary and a bibliography of relevant papers. More recently, considerable attention has turned to a multi-dimensional analog, the higher rank graph C*-algebras.

Date: March 29, 2022.
2000 Mathematics Subject Classification. Primary 47L40.
Key words and phrases. C*-algebras, groupoids, bimodules.
In the paper in which higher rank graph C*-algebras were first formalized [3], Kumjian and Pask modified the path groupoid model for graph C*-algebras to produce a model for higher rank graph C*-algebras. The purpose of this note is to extend the spectral theorem for bimodules as it appears in [2] for graph C*-algebras to the higher rank context. Section 2 will provide a brief review of the notation and construction of higher rank graph C*-algebras and their associated path groupoids. Section 3 is devoted to the spectral theorem for bimodules in the higher rank context. It also contains a characterization of when the “diagonal” is a masa.

2. Higher rank C*-algebras and the path groupoid

A k-graph (Λ, d) is a small category Λ together with a functor d: Λ → N^k which satisfies the following factorization property: if λ ∈ Λ and d(λ) = m + n with m, n ∈ N^k, then there exist unique µ, ν ∈ Λ such that λ = µν, d(µ) = m, and d(ν) = n. For n ∈ N^k, we let Λ^n = d^{−1}(n) and note that Λ^0 can be identified with the objects in Λ.

When k = 1, Λ is the category of finite paths from a directed graph; Λ^0 is the set of vertices; Λ^1 is the set of directed edges; and Λ^n is the set of paths of length n. A higher rank graph is a multi-dimensional analog of an ordinary directed graph.

The category Λ has range and source maps r and s (so λ is a morphism from s(λ) to r(λ)). For each object v, and each n ∈ N^k, let Λ^n(v) = {λ ∈ Λ | d(λ) = n, r(λ) = v}. We assume throughout this paper that each Λ^n(v) is a finite, non-empty set. (This is usually expressed by saying that Λ is row finite and has no sources.)

A higher rank graph C*-algebra, C*(Λ), is the universal C*-algebra generated by a family of partial isometries {s_λ | λ ∈ Λ} satisfying:

1. {s_v | v ∈ Λ^0} is a family of mutually orthogonal projections,
2. s_λ s_μ = s_λ s_μ, for all composable λ, μ ∈ Λ (i.e., for all λ, μ with r(μ) = s(λ)),
3. s_v^* s_λ = s_v, where v = s(λ),
4. for all v ∈ Λ^0 and all n ∈ N^k, s_v = ∑_{λ ∈ Λ^n(v)} s_λ s_λ^*.

Any set of partial isometries in a C*-algebra which satisfies these four conditions is known as a Cuntz-Krieger family; if {t_λ | λ ∈ Λ} is a Cuntz-Krieger family, then the map s_λ ↦ t_λ extends to a homomorphism of C*(Λ) to the C*-algebra generated by the t_λ.

The description above is take largely from [3], where the reader can find more detail and a number of examples. The same source provides more complete information about the path groupoid, G, which we now summarize.

Let Ω_k denote the following k-graph:

- Obj Ω_k = N^k.
- Ω_k = {(m, n) | (m, n) ∈ N^k × N^k and m ≤ n}.
- r(m, n) = m; s(m, n) = n.
- d: Ω_k → N^k by d(m, n) = n − m.
Infinite path space in Λ is then defined to be
\[ \Lambda^\infty = \{ x : \Omega_k \to \Lambda \mid x \text{ is a } k\text{-graph morphism} \}. \]
For \( v \in \Lambda^0 \), let \( \Lambda^\infty(v) = \{ x \in \Lambda^\infty \mid x(0) = v \} \). For each \( p \in \mathbb{N}^k \), define a shift map, \( \sigma^p : \Lambda^\infty \to \Lambda^\infty \), by \( \sigma^p(x)(m,n) = x(m+p,n+p) \).

Using the factorization property, Kumjian and Pask show that \( x \in \Lambda^\infty \) is determined by the values \( x(0,m), m \in \mathbb{N}^k \). They also show that if \( \lambda \in \Lambda \) and \( x \in \Lambda^\infty \) with \( x(0) = s(\lambda) \), then we can concatenate \( \lambda \) to \( x \): there is a unique \( y \in \Lambda^\infty \) such that \( x = \sigma^d(\lambda)y \) and \( \lambda = y(0,d(\lambda)) \). Naturally, we write \( y = \lambda x \). This leads immediately to the factorization of any infinite path \( x \in \Lambda^\infty \) as a product of a finite path (an element of \( \Lambda \)) and an infinite tail: \( x = x(0,p)\sigma^px \), for any \( p \in \mathbb{N}^k \).

For any \( \lambda \in \Lambda \), let \( Z(\lambda) = \{ \lambda x \in \Lambda^\infty \mid s(\lambda) = x(0) \} \) \[ = \{ y \in \Lambda^\infty \mid y(0,d(\lambda)) = \lambda \} \]

The collection \( \{ Z(\lambda) \mid \lambda \in \Lambda \} \) generates a topology on path space \( \Lambda^\infty \); in this topology each \( Z(\lambda) \) is a compact, open set. The map \( \lambda x \mapsto x \) is a homeomorphism of \( Z(\lambda) \) onto \( Z(s(\lambda)) \) and each map \( \sigma^p \) is a local homeomorphism.

\( \Lambda^\infty \) will be identified with the set of units in the groupoid \( \mathcal{G}_\Lambda \), which is defined by
\[ \mathcal{G}_\Lambda = \{(x,n,y) \in \Lambda^\infty \times \mathbb{Z}^k \times \Lambda^\infty \mid \sigma^p x = \sigma^q y \text{ and } n = p - q \}. \]

When \( k = 1 \), \( \Lambda^\infty \) reduces to the usual infinite path space and \( \mathcal{G}_\Lambda \) is the usual groupoid based on shift equivalence on path space. Inversion in \( \mathcal{G}_\Lambda \) is given by \( (x,n,y)^{-1} = (y,-n,x) \). Composable elements consist of those with matching third and first coordinates, in which case multiplication is given by \( (x,n,y)(y,m,z) = (x,n+m,z) \). \( \Lambda^\infty \) is identified with the space of units, \( \mathcal{G}_\Lambda^0 \), via \( x \mapsto (x,0,x) \). A basis for a topology on \( \mathcal{G}_\Lambda \) is given by the family
\[ Z(\lambda,\mu) = \{ (\lambda z,d(\lambda) - d(\mu),\mu z) \mid z \in \Lambda^\infty(v) \}, \]
where \( \lambda,\mu \in \Lambda \) and \( s(\lambda) = s(\mu) = v \). The topology generated by this basis is locally compact and Hausdorff. \( \mathcal{G}_\Lambda \) is then a second countable, \( r \)-discrete, locally compact groupoid; each basic open set \( Z(\lambda,\mu) \) is compact. The identification of \( \Lambda^\infty \) with \( \mathcal{G}_\Lambda^0 \) is a homeomorphism. The groupoid C*-algebra, \( \mathcal{C}^*(\mathcal{G}_\Lambda) \), is isomorphic to the higher rank graph C*-algebra, \( \mathcal{C}^*(\Lambda) \).

The gauge action which appears in the spectral theorem for bimodules is an action of the \( k \)-torus \( \mathbb{T}^k \) on \( \mathcal{C}^*(\Lambda) \). First, a bit of notation: if \( t \in \mathbb{T}^k \) and \( n \in \mathbb{N}^k \) then \( t^n = t_1^{n_1}t_2^{n_2}\cdots t_k^{n_k} \). If \( \{ s_\lambda \mid \lambda \in \Lambda \} \) is a generating Cuntz-Krieger family, then so is \( \{ t^{d(\lambda)}s_\lambda \mid \lambda \in \Lambda \} \); the universal property then yields an automorphism \( \gamma_t \) of \( \mathcal{C}^*(\Lambda) \) such that \( \gamma_t(s_\lambda) = t^{d(\lambda)}s_\lambda \), for all \( \lambda \).
The fixed point algebra of the gauge action is an AF subalgebra of $C^*(\Lambda)$; it is generated by all $s_\lambda s_\mu^*$ with $d(\lambda) = d(\mu)$. The map $\Phi_0$ of $C^*(\Lambda)$ onto the fixed point algebra given by $\Phi_0(f) = \int_{\pi_k} \gamma_t(f) \, dt$ is a faithful conditional expectation. For details concerning this, see [3].

It is shown in [3] that $G_\Lambda$ is amenable; consequently, $C^*(G_\Lambda) = C^*_{red}(G_\Lambda)$. Proposition II.4.2 in [6] allows us to identify the elements of $C^*(G_\Lambda)$ with (some of the) elements of $C_0(G_\Lambda)$, the continuous functions on $G_\Lambda$ vanishing at infinity. (Note, however, that all continuous functions on $G_\Lambda$ with compact support are elements of $C^*(G_\Lambda)$.)

For each $m \in \mathbb{Z}^k$, let $G_m$ be the set of those elements $(x,n,y)$ in $G_\Lambda$ with $n = m$. The conditional expectation $\Phi_0$ is just restriction map to $G_0$. Restriction to $G_m$ is also a map of $C^*(G_\Lambda)$ into itself; this is seen by observing that it is given by the norm decreasing map $\Phi_m$ defined by $\Phi_m(f) = \int_{\pi_k} t^{-m} \gamma_t(f) \, dt$. If $B$ is a closed linear subspace of $C^*(\Lambda)$ which is left invariant by the gauge automorphisms, then $\Phi_m(B) \subseteq B$, for each $m$.

3. THE SPECTRAL THEOREM FOR BIMODULES

Throughout this section, $\Lambda$ is a $k$-graph for which each $\Lambda^m(v)$ is finite and non-empty and $G$ is the associated $r$-discrete locally compact groupoid. Elements of the groupoid $C^*$-algebra (= higher rank graph $C^*$-algebra) are viewed as continuous functions on $G$. (Since $k$ does not vary, we drop the subscript from the notation for the groupoid.) As above, we identify path space $\Lambda^\infty$ with the space of units of $G$; with this identification $C_0(\Lambda^\infty)$ becomes an abelian subalgebra of $C^*(G)$. $\Lambda^\infty$ is not compact except when $\Lambda$ has finitely many objects (“vertices”), hence the use of $C_0$.

For simplicity of notation, let $A$ denote the groupoid $C^*$-algebra and let $D$ denote $C_0(\Lambda^\infty)$. At the end of the section we will discuss when $D$ is a masa in $A$.

Since $G$ is $r$-discrete, the Haar system can be taken to be counting measure, and so is not mentioned explicitly. Since elements of $A$ are interpreted as functions on $G$, multiplication is given by a convolution type formula

$$fg(x,n,y) = \sum f(x,p,z)g(z,q,y)$$

where the sum is taken over all composable pairs $(x,p,z)$ and $(z,q,y)$ with $p + q = n$. (For functions in $A$, the series will converge.) In particular, if $f \in A$ and $g \in D$,

(1) $$gf(x,n,y) = g(x,0,x)f(x,n,y),$$

(2) $$gf(x,n,y) = f(x,n,y)g(y,0,y).$$

For each $\lambda \in \Lambda$, let $s_\lambda$ denote the characteristic function of the set $Z(\lambda, s(\lambda))$. Then \{ $s_\lambda \mid \lambda \in \Lambda$ \} forms a Cuntz-Krieger family and generates $A$ as a $C^*$-algebra. This can be checked using the definition of $Z(\lambda, s(\lambda))$ and the formula given above for multiplication. Note also that, for $\lambda, \mu \in \Lambda$ with $s(\lambda) = s(\mu)$, $s_\lambda s_\mu^*$ is the characteristic function of the set $Z(\lambda, \mu).$
If \( B \subseteq A \) is a bimodule over \( D \), we define the spectrum of \( B \) to be:

\[
\sigma(B) = \{(x, n, y) \in G \mid f(x, n, y) \neq 0 \text{ for some } f \in B\}.
\]

The spectrum \( \sigma(B) \) is an open subset of \( G \). On the other hand, any open subset \( P \) of \( G \) determines a \( D \)-module \( A(P) \) given by

\[
A(P) = \{ f \in A \mid f(x, n, y) = 0 \text{ for all } (x, n, y) \notin P \}.
\]

Since \( P \) is open, if \( (x, n, y) \in P \), then there is a basic open set \( Z(\lambda, \mu) \) such that \( (x, n, y) \in Z(\lambda, \mu) \subseteq P \). It follows that \( s_\lambda s_\mu^* \in A(P) \); since \( s_\lambda s_\mu^* \) has the value 1 at \( (x, n, y) \), we obtain \( \sigma(A(P)) = P \), for any open subset \( P \subseteq G \).

It is clear that if \( B \) is a bimodule over \( D \) then \( B \subseteq A(\sigma(B)) \); equality does not always hold. A counterexample in the special case of Cuntz algebras (algebras determined by 1-graphs with only one vertex) can be found in [1]. Also, it is shown in [2] that there is a counterexample for any graph \( C^* \)-algebra which is not AF. (For AF \( C^* \)-algebras the Muhly-Solel spectral theorem for bimodules says that \( B = A(\sigma(B)) \) always.) Thus counterexamples exist for all 1-graphs which contain a loop.

A characterization of those bimodules which are determined by their spectra – \( B = A(\sigma(B)) \) – is given in the graph \( C^* \)-algebra context in [2]. The main result in this note is the extension to the higher rank context:

**Theorem** (Spectral Theorem for Bimodules). Let \( \Lambda \) be a row finite \( k \)-graph with no sources. Let \( G \) be the associated path groupoid. Let \( A = C^*(\Lambda) = C^*(G) \) and \( D = C_0(\Lambda^\infty) \). If \( B \subseteq A \) is a bimodule over \( D \), then the following are equivalent:

1. \( B = A(\sigma(B)) \).
2. \( B \) is generated by the Cuntz-Krieger partial isometries which it contains.
3. \( B \) is invariant under the gauge automorphisms.

**Proof.** (1) \( \Rightarrow \) (2). Assume \( P \) is an open subset of \( G \). Let \( B \) be the bimodule generated by the Cuntz-Krieger partial isometries in \( A(P) \). Each such partial isometry has its support in \( P \), so \( \sigma(B) \subseteq P \) and \( B \subseteq A(P) \). We need to show that any function \( f \) in \( A(P) \) is actually in \( B \). We claim that it is sufficient to do this for functions which are supported on some \( Z(\lambda, \mu) \subseteq P \). Indeed, it then follows readily that functions supported on compact subsets of \( P \) are in \( B \) (every compact subset of \( P \) is contained in a finite union of subsets of the form \( Z(\lambda, \mu) \)) and the compactly supported functions in \( A(P) \) are dense in \( A(P) \).

If \( f \) has support in \( Z(\lambda, \mu) \), with the aid of convolution formulas (1) and (2) it is easy to find a function \( g \) supported in \( \Lambda^\infty \) such that \( f = gs_\lambda s_\mu^* \). Since \( s_\lambda s_\mu^* \in B \) and \( g \in D \), \( f \in B \) also.

(2) \( \Rightarrow \) (3). Since a gauge automorphism maps a Cuntz-Krieger partial isometry to a scalar multiple of itself, \( B \) is trivially left invariant when it is generated by its Cuntz-Krieger partial isometries.
(3) ⇒ (1). Let \( \mathcal{B} \) be a gauge invariant bimodule and let \( P = \sigma(\mathcal{B}) \). Since \( \mathcal{B} \subseteq A(P) \) is automatic, we just need to show that \( A(P) \subseteq \mathcal{B} \). For each \( m \in \mathbb{Z}^k \), let \( P_m = P \cap \mathcal{G}_m \), so that \( P = \bigcup_m P_m \). Since \( \Phi_m \) maps \( A(P) \) onto \( A(P_m) \) and, for each \( f, f \) is in the closed linear span of the \( \Phi_m(f) \), we need merely show that \( A(P_m) \subseteq \mathcal{B} \), for each \( m \).

Fix \( m \). Suppose that \( \alpha, \beta \in \Lambda \) satisfy \( s(\alpha) = s(\beta) \) and \( d(\alpha) - d(\beta) = m \). Denote \( \mathcal{G}_{\alpha, \beta} = \{(\alpha z, m, \beta w) \mid z, w \in \Lambda^\infty(s(\alpha))\} \) and \( P_{\alpha, \beta} = P_m \cap \mathcal{G}_{\alpha, \beta} \). Now, by what we have just proven \( A(P_m) \) is the closed linear span of the Cuntz-Krieger partial isometries which it contains. But if \( s_\alpha s_\beta^* \) is one of these, then \( s_\alpha s_\beta^* \in A(P_{\alpha, \beta}) \), so \( A(P_m) \) is the closed linear span of the \( A(P_{\alpha, \beta}) \). This reduces the task to showing that \( A(P_{\alpha, \beta}) \subseteq \mathcal{B} \) for each suitable pair \( \alpha, \beta \).

We can finish the proof by transferring the problem to (a subset of) \( \mathcal{G}_0 \); the latter is a principal groupoid so the Muhly-Solel spectral theorem for bimodules is available. Let

\[
\mathcal{G}_0(s(\alpha)) = \{(z, 0, w) \mid z, w \in \Lambda^\infty(s(\alpha))\}.
\]

The map \( \psi: \mathcal{G}_0(s(\alpha)) \to \mathcal{G}_{\alpha, \beta} \) given by \( (z, 0, w) \mapsto (\alpha z, m, \beta w) \) is a homeomorphism. Let \( \mathcal{Q} \) be the inverse image of \( P_{\alpha, \beta} \) under this map. Note that \( f \mapsto s_\alpha f s_\beta^* \) carries \( \mathcal{Q} \) onto \( A(P_{\alpha, \beta}) \).

Let

\[
\mathcal{C} = \{f \in A(\mathcal{G}_0(s(\alpha))) \mid s_\alpha f s_\beta^* \in \mathcal{B}\}.
\]

We claim that \( \mathcal{C} \) is a bimodule over \( \mathcal{D} \). Since \( \mathcal{D} \) is generated by projections of the form \( s_\lambda s_\lambda^* \), it suffices to show that \( \mathcal{C} \) is closed under multiplication left and right by such projections. Now if \( f \in \mathcal{C} \), then, since \( s_\alpha s_\lambda s_\lambda^* f s_\beta^* \neq 0 \) exactly when \( s_\lambda s_\lambda^* \leq s_\alpha s_\alpha^* \),

\[
s_\alpha s_\lambda s_\lambda^* f s_\beta^* = s_\alpha s_\lambda s_\lambda^* s_\alpha f s_\beta^* = s_\alpha s_\alpha^* s_\alpha f s_\beta^* \in \mathcal{B}.
\]

The last assertion uses \( s_\alpha s_\alpha^* \in \mathcal{D} \) and \( s_\alpha f s_\beta^* \in \mathcal{B} \). Thus \( \mathcal{C} \) is a left bimodule over \( \mathcal{D} \); the argument that it is a right bimodule is similar.

The definition of \( \mathcal{Q} \) implies that \( \sigma(\mathcal{C}) \subseteq \mathcal{Q} \). The gauge invariance of \( \mathcal{B} \) implies that \( \sigma(\mathcal{C}) = \mathcal{Q} \). Indeed, let \( q \in \mathcal{Q} \) and let \( p = \psi(q) \). Since \( p \in \mathcal{B} \), there is \( f \in \mathcal{B} \) such that \( f(p) \neq 0 \). Then \( \Phi_m(f)(p) \neq 0 \) and, by gauge invariance, \( \Phi_m(f) \in \mathcal{B} \). If \( g = s_\alpha^* \Phi_m(f) s_\beta \), then \( g \in \mathcal{C} \) and \( g(q) \neq 0 \).

Since \( \sigma(\mathcal{C}) = \mathcal{Q} \) and the Muhly-Solel spectrum for bimodules holds in \( A(\mathcal{G}_0) \), we have \( \mathcal{C} = A(\mathcal{Q}) \). This implies that \( A(P_{\alpha, \beta}) \subseteq \mathcal{B} \). \( \square \)

As mentioned earlier, \( \mathcal{D} = C_0(\Lambda^\infty) \) need not be a masa in \( \mathcal{A} \). For the graph \( C^* \)-algebra case, it was shown in [2] that \( \mathcal{D} \) is a masa if, and only if, every loop has an entrance. Kumjian and Pask [3] define an analogous condition, the aperiodicity condition, for higher rank graphs and use this to extend the Cuntz-Krieger uniqueness theorem. Their condition also extends the masa theorem. Here are the relevant definitions: an element \( x \in \Lambda^\infty \) is periodic with non-zero period \( p \in \mathbb{Z}^k \) if, for every \( (m, n) \in \Omega \) with \( m + p \geq 0 \), \( x(m + p, n + p) = x(m, n) \). If there is an element \( n \in \mathbb{N}^k \) such that \( \sigma^n(x) \)
is periodic, $x$ is \textit{eventually periodic}; otherwise, $x$ is \textit{aperiodic}. Finally, $\Lambda$ satisfies the \textit{aperiodicity condition} if, for every $v \in \Lambda^0$, there is an aperiodic path $x \in \Lambda^\infty(v)$.

Note that $x$ is eventually periodic with period $p$ if, and only if, $(x, p, x) \in \mathcal{G}$. Kumjian and Pask prove that $\Lambda$ satisfies the aperiodicity condition if, and only if, the points in $\mathcal{G}$ with trivial isotropy are dense in $\mathcal{G}^0$ [3, Proposition 4.5]. We will show below that the aperiodicity condition is also equivalent to the assertion that $\mathcal{G}^0$ is the interior of the isotropy group bundle $\mathcal{G}^1$. (Note: in the Kumjian-Pask proposition, $\mathcal{G}^0$ is viewed as $\Lambda^\infty$; we will view $\mathcal{G}^0$ as the open subset $\{(x, 0, x) \mid x \in \Lambda^\infty\} \subset \{x, p, x \in \mathcal{G} \mid p \in \mathbb{Z}^k\}$.) Renault [6, Proposition II.4.7] has shown that, $C^*_0(\mathcal{G}^0)$ is a masa in $\mathcal{A}^*_{\text{red}}(\mathcal{G})$ if, and only if, $\mathcal{G}^0$ is the interior of $\mathcal{G}^1$. Since the path groupoid $\mathcal{G}$ is amenable, Renault’s Proposition yields the masa theorem.

**Proposition.** $\Lambda$ satisfies the aperiodicity condition if, and only if, $\mathcal{G}^0$ is the interior of $\mathcal{G}^1$.

**Proof.** Assume that the aperiodicity condition holds. Let $(x, p, x) \in \mathcal{G}^1$ with $p \neq 0$. We shall show that we can approximate $(x, p, x)$ by points in $\mathcal{G}$ which are not in $\mathcal{G}^1$. This shows that $(x, p, x)$ is not in the interior of $\mathcal{G}^1$. Since $\mathcal{G}^0$ is an open subset of $\mathcal{G}^1$, it follows that $\mathcal{G}^0$ is the interior.

Let $Z(\alpha, \beta)$ be a neighborhood of $(x, p, x)$. For $m$ sufficiently large (meaning for each $m_i$ sufficiently large), $m + p \geq 0$ and both $x(0, m)$ and $x(0, m + p)$ lie in $Z(\alpha)$ and in $Z(\beta)$. Since $(x, p, x) \in \mathcal{G}$, $\sigma^m(x) = \sigma^{m+p}(x)$ and $x(0, m)$ and $x(0, m + p)$ have a common source $v$. Choose $y$ aperiodic in $\Lambda^\infty(v)$. Let $z = x(0, m)y$ and $w = x(0, m + p)y$. Then $z \neq w$ and $(z, p, w) \in Z(\alpha, \beta)$. So $(z, p, w) \notin \mathcal{G}^1$ and $(z, p, w)$ approximates $(x, p, x)$.

Now suppose that $\Lambda$ does not satisfy the aperiodicity condition. By Proposition 4.5 in [3], there is $x \in \Lambda^\infty$ which cannot be approximated by aperiodic points. Since $x$ must be eventually periodic there is a non-zero element $p$ of $\mathbb{Z}^k$ such that $(x, p, x) \in \mathcal{G}$. If $(x, p, x)$ could be approximated in the topology of $\mathcal{G}$ by points outside $\mathcal{G}^1$, it would follow that $x$ is a limit of aperiodic points in $\Lambda^\infty$ – a contradiction. This shows that $(x, p, x)$ is in the interior of $\mathcal{G}^1$ and so $\mathcal{G}^0$ is not the interior. \hfill $\square$

This Proposition, Proposition II.4.7 in [6], and the amenability of $\mathcal{G}$ yield the following theorem.

**Theorem.** $\mathcal{D}$ is a masa in $\mathcal{A}$ if, and only if $\Lambda$ satisfies the aperiodicity condition.

**References**

[1] Alan Hopenwasser and Justin Peters, Subalgebras of the Cuntz C*-algebra, arXiv math.OA/0304013
[2] Alan Hopenwasser, Justin Peters, and Stephen Power, Subalgebras of graph C*-algebras, arXiv math.OA/0409075
[3] Alex Kumjian and David Pask, Higher rank graph C*-algebras, New York J. Math. 6 (2000), 1–20 (electronic). MR 2001b:46102
[4] Paul S. Muhly and Baruch Solel, *Subalgebras of groupoid C*-algebras*, J. Reine Angew. Math. 402 (1989), 41–75. MR 90m:46098

[5] Iain Raeburn, *Graph algebras*, CBMS Regional Conf. Ser. in Math., Amer. Math. Soc., Providence, to appear.

[6] Jean Renault, *A groupoid approach to C*-algebras*, Springer, Berlin, 1980. MR 82h:46075

Department of Mathematics, University of Alabama, Tuscaloosa, AL 35487

E-mail address: ahopenwa@euler.math.ua.edu