Two-sites quantum island in the quasi-ballistic regime

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Quantum Hall edge channels can be combined with metallic regions to fractionalize electrons and form correlated impurity models. We study a minimal device, that has been experimentally achieved quite recently, with two floating islands connected to three edge channels via quantum point contacts in the integer quantum Hall regime. At high transparency of the quantum point contacts, we establish a mapping to the boundary sine-Gordon model and thereby reveal the nature of the quantum critical point. We deduce from this mapping universal expressions for the conductance and noise, in agreement with the experimental findings, and discuss the competition between Kondo-like screening of each individual island and the cooperative transfer of electrons between them. We further predict that the device operated at finite voltage bias produces fractional charges $e^* = e/(N + 1)$ and propose a generalization to $N$ islands with the fractional charge $e^* = e/(N + 1)$.

I. INTRODUCTION

Models of quantum impurities embedded in fermionic environments entail a rich zoology of phase transitions with non-Fermi liquid scaling [1–4]. These models underpin many properties of correlated materials such as heavy fermions [5]. They can also be realized in superconducting [6–8] or semiconducting nanostructures with a versatile control on the parameters driving the transitions. For example, the Kondo screening of local spin degrees of freedom is routinely observed in semiconducting quantum dots in two-dimensional electron gases [9, 10], carbon nanotubes [11] or nanowire devices [12]. The screening of the impurity spin by a second channel [13, 14] is however difficult, but not impossible, to achieve experimentally with quantum dots [15–18].

An alternative route for Kondo screening with more than one channel is offered by the charge Kondo effect [19–24]. In the charge version, two degenerate and discrete charge states of a metallic island are screened by the tunnel coupling to a lead. It has been realized in an experiment [25–29] coupling a micron-size floating island to the edge channels of a two-dimensional electron gas tuned in the integer quantum Hall regime. The transmission to the metallic island is controlled by nearby quantum point contacts (QPC) playing the role of tunnel junctions. Kondo scalings, crossovers and fixed points have thus been measured with unprecedented control and detail in the two- and three-channel cases, together with the renormalization-group relevant channel asymmetries [25, 27]. Interestingly, the gate-controlled quantum point contacts can be tuned to be almost ballistic, i.e., with a large transparency, in which case the temperature or voltage scalings no longer emulate a Kondo model since many charge states are involved. Despite the absence of a Kondo mapping in the quasi-ballistic limit, the zero-temperature quantum critical point is continuously connected to the one at small transparencies where Kondo scaling holds. This continuity argument preserves the properties at zero energy: the fractional entropy, the leading temperature/voltage scaling and more generally the operator content of the zero-temperature quantum critical point are the same regardless of the transparencies. Only the temperature/voltage evolution of observables, such as the differential conductance, are different and depend on the transparencies, with a Kondo mapping restricted to low transparency.

Recently, a two-site version of the charge Kondo effect has been realized with two coupled metallic islands in a two-dimensional electron gas in a GaAs/AlGaAs heterostructure, and a novel zero-temperature quantum critical point has been identified by comparing experimental data with numerical renormalization group (NRG) calculations [30]. The competition between the screening of each individual island charge and the mediated charge coupling between the two islands may offer an insight on the competition between Kondo screening and collective magnetic ordering in correlated materials. In the case of an exchange coupling between two spins, the model has been coined as the two-impurity Kondo model [31–34] and discussed generally in the context of quantum dots [35–38].

In this paper, we consider the geometry of this experiment in the quasi-ballistic regime where each QPC is set close to full transparency. We recover analytically the properties of the zero-temperature quantum critical point observed experimentally and in NRG calculations which, as argued above, are universal in the sense that they do not depend on the transparencies of the QPC. We find in particular the same universality class as the weak tunneling between fractional quantum Hall edge states at filling $\nu = 1/3$ described by a boundary sine-Gordon model [39–42]. This is readily understood by considering the simplified case where only one QPC is weakly reflecting electrons, the other two QPC being completely open. The two open QPC in series define a dynamical Coulomb blockade environment with impedance $R_s = 2R_q$ ($R_q = h/e^2$ is the quantum unit of resistance) for the third, a model that has a known mapping [28, 43] to the physics of quasiparticle tunneling in a fractional $\nu = 1/(1 + R_s/R_q) = 1/3$ state, and predict a decreasing conductance as the temperature is lowered. The same model further holds when all three
QPC weakly reflect.

With the analytical description of the quantum critical point, we retrieve many features discussed in Ref. [30] such as: the residual fractional entropy, the scaling exponents close to the triple points and the shape of the conductance as function of the plunger gate voltages. We also predict the emission of fractional charges [44–48] in the absence of backscattering at the QPC, to be extracted from shot noise measurements [49, 50], despite being in the integer quantum Hall regime [51–53].

The organization of paper is as follows. In Sec. II we discuss the Hamiltonian formulation of two-sites quantum island in the language of bosonization. The effective Hamiltonian at weak backscattering regime is addressed in Sec. III in terms of a boundary sine-Gordon model. We outline the calculations for charge current, noise and the evolution of the triple points in Sec. IV. In Sec. V, we present details on the Bethe ansatz solution for two-sites quantum island and discuss the comparison of our results with the recent experiment [30]. Sec. VI contains a brief outline on the generalization to the multi-sites cases. We summarize our findings in Sec. VII. Mathematical details of our calculations are deferred to the Appendices.

II. NEARLY BALLISTIC MODEL

We consider the geometry realized in the experiment of Ref. [30] and illustrated in Fig. 1. It comprises three pairs of counter-propagating quantum Hall edges partially covered by two metallic islands. The chiral edge states are best described with bosonization [54, 55] with the total Hamiltonian \( H = H_0 + H_C + H_{BS} \). The Hamiltonian \( H_0 \) governing the propagation of edge states writes

\[
H_0 = \frac{v_F}{4\pi} \sum_{j=1}^{3} \int_{-\infty}^{\infty} dx \left[ (\partial_x \phi_{j,L})^2 + (\partial_x \phi_{j,R})^2 \right],
\]

where \( \phi_{j,L/R} \) represents the bosonic field corresponding to the incoming/outgoing chiral fermions and \( v_F \) stands for the Fermi-velocity. \( H_C \) describes the charging energy of the floating islands

\[
H_C = E_c \left( \hat{N}_1^2 + \hat{N}_2^2 \right),
\]

where for simplicity, the island capacitances \( C \) are chosen equal [56] and \( E_c = e^2/2C \). The number densities in the left and right island \( \hat{N}_{1/2} \) are expressed in terms of corresponding gate voltages \( N_{1/2} \), controlled by plunger gates, and charge densities \( \rho_{j,\alpha} = \frac{1}{2\pi} \partial_x \phi_{j,\alpha} \) such that

\[
\hat{N}_{1/2} = \int_0^\infty dx \left[ \rho_{1/2,R}(x) - \rho_{1/2,L}(x) \right] + \int_{-\infty}^0 dx \left[ \rho_{2/3,R}(x) - \rho_{2/3,L}(x) \right] - N_{1/2}.
\]

The above expressions can be simplified as

\[
\hat{N}_{1/2} = \frac{1}{2\pi} \left( -\delta \phi_{1/2} + \delta \phi_{2/3} - 2\pi N_{1/2} \right),
\]

where we introduce new symbols

\[
\delta \phi_j \equiv \phi_{j,R} - \phi_{j,L}.
\]

Replacing the finite portions of quantum Hall edges beneath the islands as semi-infinite lines is a standard description first introduced in Refs. [21, 22]. It is justified in the experiments by the very long dwell time of electrons in the metallic islands due to the strong mismatch between high metallic density of states and few outgoing edge channels [57]. In contrast, we assume that the two middle chiral edges are fully covered by the left and right islands and neglect the fact that there is some region of space where they are uncovered. It works as long as this uncovered region has a size corresponding to energies well above all other energy scales.

In addition to \( H_0 \) and \( H_C \), the Hamiltonian also has a part describing the weak reflection of the three interspaced QPCs,

\[
H_{BS} = \sum_{j=1}^{3} \frac{D |r_j|}{\pi} \cos \delta \phi_j,
\]

nonlinear in the bosonic fields, \( |r_j| \) are the amplitudes of the corresponding dimensionless reflection coefficients. \( D \) is a high-energy scale, or bandwidth, introduced in the bosonization framework. It is necessary at an intermediate step in the formalism but eventually disappears from all practical observable.

III. CHARGE AVERAGE AND THE SINE-GORDON MODEL

In the absence of backscattering at the QPC, \( |r_j| = 0 \), the Coulomb blockade induced by the charging energy term \( H_C \) is entirely suppressed as the flow of electrons becomes continuous and there is no charge granularity.
Electrons are continuously entering the island regions before being evenly distributed among the output channels. In the bosonization language, this uninterrupted flow of electrons is described by the quadratic Hamiltonian $H_0 + H_C$ with plasmonic excitations traveling along the chiral edges and scattered at the entrances of the island [58, 59]. It is also represented by an equivalent quantum circuit connecting transmission lines with impedance $R_q$ by capacitors $C$ [53].

Following Refs. [53, 60, 61], we use Heisenberg equations of motion to discuss the scattering of plasmonic modes. Their explicit form is detailed in appendix A together with their solution. Solving these equations, we find the following expression for the bosonic modes $\delta \phi_j$,

$$\begin{pmatrix} \delta \phi_1 \\ \delta \phi_2 \\ \delta \phi_3 \end{pmatrix} = M \begin{pmatrix} \delta \phi_A^0 \\ \delta \phi_B^0 \\ \delta \phi_C^0 \end{pmatrix} + \frac{2\pi}{3} \begin{pmatrix} -2N_{1,g} - N_{2,g} \\ N_{1,g} - N_{2,g} \\ N_{1,g} + 2N_{2,g} \end{pmatrix}. \quad (7)$$

The $3 \times 3$ scattering matrix $M$ is given in appendix A and $\tau_c = \pi / E_c$ is the Heisenberg time associated with the charging energy $E_c$. Eq. (7) is written in terms of three incoming fields $\delta \phi_A^0$, $\delta \phi_B^0$, $\delta \phi_C^0$. These three fields are themselves linear combinations of the incoming fields $\delta \phi_{1,2,3}^0$ in the three quantum Hall regions. They have standard commutation relations $[\phi_{\alpha,R/L}(t), \phi_{\beta,R/L}(t')] = -i\pi \delta(t-t')$, with $\alpha = A,B,C$, and, as incoming fields, they originate from a thermalized source with the mean occupancy

$$\langle \delta \phi_A^0(\omega) \delta \phi_A^0(\omega') \rangle = \frac{4\pi^2}{\omega^2} n_B \left( \frac{\hbar \omega}{k_B T} \right) \delta(\omega + \omega'), \quad (8)$$

involving the Bose distribution $n_B(x) = (e^x - 1)^{-1}$ with temperature $T$.

For energies well below the charging energy $\omega \tau_c \ll 1$, the scattering matrix takes a simple form

$$\lim_{\omega \tau_c \ll 1} M = -\frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (9)$$

discarding effectively the two fields $\delta \phi_B^0$, $\delta \phi_C^0$ in Eq. (7). These two fields are thus identified as charge modes that are gapped by the electrostatic charging energy on the two islands. Quite generally, we expect that a series of $N$ capacitive islands would induce a set of $N$ gapped bosonic modes. Hence, we are left with a single gapless mode $\delta \phi_A^0$ at low energy that will determine the transport properties throughout the device.

We now include the reflections of the QPCs given by Eq. (6). In the interaction picture, we replace the fields in Eq. (6) using Eq. (7). At low energy ($\ll E_c$), it is further justified [62] to average $H_{BS}$ over the gapped fields $\delta \phi_B^0$ and $\delta \phi_C^0$. For instance, the first term writes

$$\cos \delta \phi_1 = \frac{1}{2} e^{-i(\delta \phi_1^2(t))_{HE} \left( \frac{\epsilon}{2\pi} \left( \frac{2N_{1,g} + N_{2,g}}{2\pi} \right) + \text{h.c.} \right)}, \quad (10)$$

where the average $\langle \delta \phi_1^2(t) \rangle_{HE}$ only contains the $\delta \phi_{B/C}^0$ fields. We leave the details of the average calculation to appendix B and quote only the final result,

$$H_{BS} = |r| D^{1/3} E_c^{2/3} \cos \left( \frac{\delta \phi_A^0}{\sqrt{3}} + \Theta \right), \quad (11)$$

where the effective reflection coefficient $r$ involves a coherent sum over the reflections at the three different QPCs,

$$r = \frac{3 \epsilon^2 e^2}{\pi \delta^2} \left[ \left( r_1 |e^{\frac{2\pi i}{3}(2N_{1,g} + N_{2,g})} + \sqrt{3}\right)|r_2| e^{-\frac{2\pi i}{3}(N_{1,g} - N_{2,g})} + |r_3| e^{\frac{2\pi i}{3}(N_{1,g} + 2N_{2,g})} \right] = |r| e^{i\Theta}. \quad (12)$$

$\delta \phi_A^0$ is the only chargeless mode, corresponding to a specific plasmonic motion that is not influenced by the charging energy of the islands. It survives alone for energies below $E_c$ and the effective low-energy Hamiltonian that emerges after charge averaging is

$$H_{eff} = \frac{\epsilon F}{4\pi} \int_{-\infty}^{\infty} dx \left[ \left( \partial_x \phi_A^0(x) \right)^2 + \left( \partial_x \phi_B^0(x) \right)^2 \right] + |r| D^{1/3} E_c^{2/3} \cos \left( \frac{\delta \phi_A^0}{\sqrt{3}} + \Theta \right), \quad (13)$$

whereas the current operator at the right output of the device (as detailed in appendix) reads

$$\tilde{I} \equiv \tilde{I}_3(t) = -\frac{e}{2\pi} \partial_t \delta \phi_3(t) = \frac{e}{2\pi} \frac{1}{\sqrt{3}} \partial_t \delta \phi_A^0(t). \quad (14)$$

Remarkably, the model is a boundary sine-Gordon model, similar to an impurity [39] in a one-dimensional electron liquid with the Tomonaga-Luttinger parameter $K = 1/3$ or the weak quasiparticle tunneling between two edges of a fractional quantum Hall state [40–42] at filling $\nu = 1/3$.

Given a set of QPC transmissions, triple points are defined by specific values of the left and right gate voltages for which the coherent sum in Eq. (12) vanishes, $r = 0$. At those points, backscattering processes interfere destructively and an uninterrupted noiseless flow of electron is recovered with maximum conductance. In this case, the effective Hamiltonian (13) is in fact quadratic and readily solvable.

**IV. CURRENT, NOISE AND TRIPLE POINTS**

We consider a source-drain geometry where we apply a voltage on the edge channel 1 and measure the current $I \equiv \tilde{I}_3$ in channel 3. Solving the Heisenberg equations, we obtain that the input field $\delta \phi_1^0(t)$ is essentially shifted by the number of emitted temporal wavepackets $eVt/\hbar$ during the time interval $t$, which translates into a shift
of $eVt/\sqrt{3}h$ for $\delta \phi_A(t)$, and an average current $I = \langle \hat{I} \rangle$

$$I = \frac{e^2V}{3h} + \frac{e}{2\pi} \frac{\partial_t \langle \delta \phi_A(t) \rangle}{\sqrt{3}}.$$  

(15)

At the triple point $r = 0$, the Hamiltonian Eq. (13) is quadratic so that $\langle \delta \phi_A(t) \rangle = 0$, yielding from Eq. (15) the maximum conductance $G = I/V = G_{\text{max}} = e^2/3h$ as also predicted from NRG calculations [30] in the weak tunneling/strong backscattering regime. This conductance can be interpreted as the series addition of three quantum resistances $R_q$, each QPC being effectively ballistic. The triple point corresponds in fact to the quantum critical point identified in the experiment and NRG calculations of Ref. [30].

Away from the triple point, $r \neq 0$, the non-linear cosine term in Eq. (13) shifts the average bosonic field $\langle \delta \phi_A(t) \rangle$ to a non-zero value such that conductance becomes non-linear and departs from $e^2/3h$. This perturbation drives the system away from the quantum critical point and the conductance decreases with decreasing voltage/temperature towards a low-energy fixed point with zero conductance.

A. Fractional charges at weak reflection

At intermediate energies, the reflection term in Eq. (13) can be treated by perturbation theory. As discussed in appendix, the current expands as $I = I_0 + \hat{I}_1 + \hat{I}_2$ with

$$\hat{I}_0 = \hat{I}_3 = \frac{e}{2\pi} \int_t^0 dt' \left[ H_{\text{BS}}(t'), \hat{\delta}(t') \right].$$
$$\hat{I}_1 = \frac{e^2V}{3h} - \frac{e}{2\pi} \int_{-\infty}^0 dt'' \left[ H_{\text{BS}}(t''), \hat{\delta}(t) \right].$$
$$\hat{I}_2 = -\frac{e}{2\pi} \int_{-\infty}^0 dt' \int_{-\infty}^{t'} \int_{-\infty}^{t''} dt'' \left[ H_{\text{BS}}(t''), \left[ H_{\text{BS}}(t'), \hat{\delta}(t) \right] \right].$$

with $\langle \hat{I}_0 \rangle = e^2V/3h$. After taking the quantum average at zero temperature and finite bias voltage, we obtain the leading terms for the current

$$I = \frac{e^2V}{3} \left[ 1 - r^2 \left( \frac{E_c}{eV} \right)^{4/3} \frac{\pi^2}{\Gamma(2/3)} \right].$$

(17)

with $E_c = 3^{1/4}E_c$, and the shot noise

$$S = \frac{e^2V}{9h} \left| r \right|^2 \left( \frac{E_c}{eV} \right)^{4/3} \frac{\pi^2}{\Gamma(2/3)}.$$  

(18)

The Fano factor for the reflected current is then

$$F = \frac{S}{I(|r| = 0) - I} = \frac{1}{3}.$$  

(19)

corresponding to the backscattering of fractional charges $e^* = e/3$. Eq. (17) can be written in a more suggestive form

$$I = \frac{1}{3} \left[ 1 - a \left( \frac{k_B T_s}{eV} \right)^{4/3} \right],$$  

(20)

with the coefficient $a = (\sqrt{3}/6)(\Gamma(1/3)/\Gamma(2/3))$, by introducing the temperature scale $[63]

$$T_s = b |r|^{3/2} E_c / k_B.$$  

(21)

with $b = [3\sqrt{3}\Gamma(5/6)]^{3/4}$. Eq. (20), valid in the perturbative regime $eV \gg k_BT_s$, gives the onset of the crossover to the low-voltage regime. While $T_s$ itself is a non-universal scale depending on microscopic details (e.g. high-energy details of the electronic dispersion relation, irrelevant processes,...), the current is found to be a universal function of $eV/k_BT_s$ and $T/T_s$. The full crossover function of $I(eV/k_BT_s, T/T_s)$ is determined from the Bethe ansatz solution of the Sine-Gordon model defined by Eq. (13), as later discussed in Sec. V A.

B. Triple points

The temperature $T_s$ from Eq. (21) sets the energy scale for the crossover between the high and low temperature or bias voltage limits. It is governed by the effective reflection coefficient $r$ and therefore vanishes at the triple points where the conductance is $e^2/3h$, independent of the temperature or bias voltage. Fig. 2 maps out the positions of the triple points for the symmetric configuration $|r_1| = |r_3|$ as $|r_2|/|r_1|$ evolves. The triple points form a periodic lattice with two sites per unit cell when the left and right gate voltages are varied. They only exist when $|r_2|/|r_1| < 2\sqrt{3}$. For $|r_2|/|r_1| > 2\sqrt{3}$, the triple points disappear: the electron-mediated coupling between the two islands becomes too weak and the two islands couple preferentially to the source and drain leads which suppresses transport at low energy. In the asymmetric case $|r_1| \neq |r_3|$, triple points only exist within upper and lower threshold of $|r_2|/|r_1|$ determined by the ratio $|r_3|/|r_1|$, 

$$\left| \frac{r_2}{r_1} \right|_{\text{upper}} = \left( 1 \frac{|r_2|}{|r_1|} + 1 \frac{|r_3|}{|r_1|} \right),$$

(23)

Beyond these thresholds, pair of triple points eventually meet and disappear. The positions of triple points in the plane defined by gate voltages is depicted in Fig. 3. From Fig 2, it is seen that the evolution of triple point with varying $|r_2|/|r_1|$ for the symmetrical configuration
is entirely on the diagonal plane \((N_{1,g}, N_{2,g}) = (N_g, N_g)\) such that

\[
N_{1,g} = \frac{1}{2\pi} \arccos \left( -\sqrt{\frac{3}{2}} \frac{|r_2|}{|r_1|} \right) = N_{2,g} = N_g. \tag{24}
\]

Then the expansion of Eq. (12) around the triple points in the symmetric case writes

\[
\left| \frac{r}{r_1} \right|^2 = 4\pi^2 \left( \frac{3e^\gamma/\pi)^2}{\pi\sqrt{3}} \right)^2 \left( (\delta N_1)^2 + (\delta N_2)^2 \right) + \left[ 2 - 3 \left( \frac{|r_2|}{|r_1|} \right)^2 \right] \delta N_1 \delta N_2, \tag{25}
\]

where we define two small parameters \(\delta N_{1/2} = N_{g,1/2} - N_g\). The Eq. (25) draws an ellipse squashed along the \(N_{1,g} = N_{2,g} = N_g\) direction, or a circle for the special case \(|r_2|/|r_1| = \sqrt{2}/3\). The two triple points are in fact symmetrically positioned with respect to the central configuration of gate voltages \(N_{1,g} = N_{2,g} = 1/2\). Along the line \(N_{1,g} = N_{2,g}\), \(r\) thus develops a double well form with zeros at the triple points and a local maximum at the center \(N_{1,g} = N_{2,g} = 1/2\).

V. CROSSOVER FUNCTION

A. Bethe ansatz solution

The boundary sine-Gordon model of Eq. (13), together with the definition of the current observable Eq. (14) and the coupling to the bias voltage described by the time-dependent shift in the boson of Eq.(15) defines the canonical out-of-equilibrium boundary sine-Gordon model. This model also faithfully describes the tunneling between \(\nu = 1/3\) fractional quantum Hall states, or between Luttinger liquids with parameter \(K = 1/3\), and in this context it has been shown that it allows for an exact out-of-equilibrium solution using Bethe Ansatz [41, 42]. The main features of the Bethe Ansatz solution are summarized in Appendix C. It is characterized by a single temperature (energy) scale \(T_\star\) entering all universal functions for physical observables such as the current, the noise, etc. Quite generally, \(T_\star\) depends on the combination of the sine-Gordon coupling constant \(r\) and the ultraviolet cutoff of the model. In our case, it is readily obtained by matching the Bethe Ansatz high-voltage \(eV \gg k_B T_\star\) expansion of the current with its microscopic perturbative evaluation in Eq. (17). The resulting expression for the temperature scale \(T_\star\) has been anticipated and given in Eq. (21). As expected, the different powers of \(D\) cancel out and the intermediate cutoff energy \(D\) finally drops out from the expression of \(T_\star\), and thus of all observables.

It results that the non-linear differential conductance exhibits the scaling form

\[
G(V,T) = \frac{1}{3} \frac{e^2}{h} g \left( \frac{eV}{k_B T_\star}, \frac{T}{T_\star} \right), \tag{26}
\]

with \(g(x,y)\) a universal function. At zero temperature, an analytical expression for \(g\) is derived from the Wiener-
Ref. [67]: high-temperature asymptotic behaviors were derived in
ing linear conductance is shown in Fig. 4. Its low- and
functions of the temperature ratio
appendix C. Observables are then given by universal
the solution at finite temperature and is detailed in
appendix C.

The corresponding trace is shown in Fig. 8 in the ap-
pendix C.

The thermodynamic Bethe ansatz [41, 42] provides
the solution at finite temperature and is detailed in
appendix C. Observables are then given by universal
functions of the temperature ratio
T/T∗.

The linear conductance is shown in Fig. 4. Its low- and
high-temperature asymptotic behaviors were derived in
Ref. [67]:

\begin{equation}
g(0, y < 1) \simeq \frac{32}{5} (2\pi y)^4,
\end{equation}

\begin{equation}
g(0, y > 1) \simeq 1 - \frac{\sqrt{\pi} \Gamma \left( \frac{1}{4} \right)^{3/2}}{6 \Gamma \left( \frac{3}{4} \right)^2} \frac{1}{(6\pi y)^{3/2}}.
\end{equation}

The universal function \( g(0, y) \) describes a monotonous
crossover between the high-temperature unitary conduct-
tance \( e^2/(3h) \) and the vanishing zero-temperature con-
ductance. The \( T^4 \) scaling at low temperature in Eq. (28)
can be readily understood from an inelastic co-tunneling
perspective [68–71]. The transfer of one electron from the
to the right leads occurs via virtual processes where
each island is excited with an electron-hole pair. The
phase space for each pair is proportional to \( T^2 \) yielding an
overall \( T^4 \) scaling for the conductance. In the bosoniza-
tion language, the operator responsible for the \( T^4 \) scaling
is \( e^{i\sqrt{3}\delta \theta_A} \), the dual to the cosine backscatter-
ting term of Eq. (11), where \( \delta \theta_A \) is the canonical conju-
gate to the field \( \delta \phi_A \). It can also be written as

\begin{equation}
e^{i\sqrt{3}\delta \theta_A} = e^{i\delta \phi_1} e^{i\delta \phi_2} e^{i\delta \phi_3}.
\end{equation}

The vertex operator \( e^{i\delta \phi_1} \) corresponds to the transfer
of one electron across the first (left) QPC and can be
fermionized back as \( \psi_{R,1}^\dagger \psi_{L,1} \). The field \( \psi_{L,R,1} \) annihilates an electron on the left (resp. right) side of the first
QPC. Using similar refermionization at each QPC, one
obtains the leading irrelevant low-energy operator

\begin{equation}
e^{i\sqrt{3}\delta \theta_A} \propto \psi_{R,1}^\dagger \psi_{L,1} \psi_{R,2}^\dagger \psi_{L,2}^\dagger \psi_{R,3} \psi_{L,3}.
\end{equation}

and describes the universal crossover from low to high
voltage. As already noticed [39, 42, 64–66], it exhibits a
duality between its low- and high-energy expansions.
Interestingly, the differential conductance is super-ballistic
at large voltage as it exceeds [42] the ballistic limit
\( e^2/(3h) \), before falling off down to zero at vanishing bias.
The corresponding trace is shown in Fig. 8 in the ap-
pendix C.

The thermodynamic Bethe ansatz [41, 42] provides
the solution at finite temperature and is detailed in
appendix C. Observables are then given by universal
functions of the temperature ratio
T/T∗. The resulting
linear conductance is shown in Fig. 4. Its low- and
high-temperature asymptotic behaviors were derived in
Ref. [67]:

\begin{equation}
g(0, y = 1) \simeq \frac{32}{5} (2\pi y)^4,
\end{equation}

\begin{equation}
g(0, y > 1) \simeq 1 - \frac{\sqrt{\pi} \Gamma \left( \frac{1}{4} \right)^{3/2}}{6 \Gamma \left( \frac{3}{4} \right)^2} \frac{1}{(6\pi y)^{3/2}}.
\end{equation}

The vertex operator \( e^{i\delta \phi_1} \) corresponds to the transfer
of one electron across the two-island structure together electron-hole ex-
citations in both islands.

The operator of Eq. (30) involves the fermionic reser-
voirs around the three QPC symmetrically. This symme-
try originates in fact from the symmetric expression of
\( \delta \phi_A \) in terms of \( \delta \phi_{1,2,3} \) which is preserved along the inte-
grable crossover, yielding eventually the six operators in
Eq. (30) and the \( T^4 \) scaling. Such a symmetry is a phys-
ical one in the case of tunneling between two fractional
quantum Hall states at \( \nu = 1/3 \) as only electrons, re-
formed as triplet of quasiparticles, are physically allowed
to tunnel at low energy. For our system, this threefold
symmetry is not generally preserved. The Bethe-ansatz
solution predicts a Fermi liquid state at low temperature
but the obtained scaling \( T^4 \) is not symmetry-protected
and indeed a \( T^2 \) scaling was identified in the NRG cal-
culation of Ref. [30]. In contrast, the high-temperature
region of Fig. 4 is robust and in agreement with the mea-
surements and NRG calculations of Ref. [30].

B. Comparison with Ref. [30]

The geometry of two connected quantum islands dis-
cussed in this paper has been realized in the experiment of
Ref. [30] together with NRG calculations adapted to the
limit of weakly transparent QPC - the opposite limit
is the focus of our work. Nevertheless, as already men-
tioned in the introduction, the properties of the quantum
critical point are expected to be unique, independent of
the bare QPC transmissions, and our work can shed light
on many findings of Ref. [30].

The main difference between our regime of high QPC
transparency and Ref. [30] is the absence of a Kondo
scaling region. For a weak QPC transparency, a Kondo
resonance forms progressively with decreasing tempera-
ture, yielding a conductance that increases when tem-
perature is lowered and eventually saturates at a value
\( G_0 = \frac{e^2}{\hbar} \) at the triple point. The resonance is characterized by the Kondo temperature scale \( T_K \). \( T_K \) increases with the QPC transparency and the Kondo effect thus disappears when the transparency exceeds a threshold for which \( T_K \sim E_c \). In the quasi-ballistic regime, i.e. almost transparent, many charge states are occupied and there is no Kondo effect. In summary, the conductance at large transparency, shown in Fig. 4, is a monotonous and increasing function of temperature, with the characteristic temperature scale \( T^* \) whereas, in the weakly transparent case of Ref. [30], the conductance also starts as an increasing function of \( T/T^* \) but then crosses over to a decreasing function of \( T/T^* \).

Interestingly, the nature of the relevant perturbation in the vicinity of the quantum critical (triple) point does not depend on the QPC bare transparency. We expect that the \( \sim (T/T_*)^{-4/3} \) correction of Eq. (28) is valid regardless of the transparency and applies as well to the experiment of Ref. [30] under the condition that \( T, T^* \ll T_K, E_c \). A similar discussion with two temperature scales can be found in Ref. [13, 14, 33] for the two-channel Kondo model slightly away from charge degeneracy [25]. The \( T^{-4/3} \) scaling is in fact clearly observed in the NRG data of Ref. [30], and our Fig. 4 coincides well with NRG from high to moderate values of \( T/T_* \).

All in all, our approach recovers other findings from Ref. [30]. It identifies the nature of the quantum critical point with its leading relevant perturbation: a boundary sine-Gordon model with the Luttinger parameter \( K = 1/3 \). The critical point is characterized [72] by the fractional residual entropy \( \Delta S = \ln(\sqrt{3}) \), also observed in the NRG calculation of Ref. [30]. Our approach also explains the value of the critical exponents in the vicinity of the quantum critical point: from Eqs. (21) and (25), \( T_* \) varies as \( \sim \delta N_g^{-3/2} \) and \( \sim |r_2|^3/2 \), in precise agreement with Ref. [30].

Since \( T_* \) depends explicitly on the left and right gate voltages through Eq. (21) and Eq. (12), we can evaluate the linear conductance as a function of the gate voltage \( N_{1,g} \) along the symmetric line \( N_{1,g} = N_{2,g} \) for a fixed temperature \( T \). The result is shown in Figs. 5 for different values of \( |r_2| \), illustrating the competition between the cooperative transport through the two islands via their edge-state mediated coupling on one hand and the current suppression of each individual island due to the relevant QPC backscattering on the other hand. The very same behaviour for the conductance has been measured and calculated via NRG in Ref. [30] (see their Figure 2) in the weakly transparent regime, and interpreted as a competition between Kondo screening of each individual island and the antiferromagnetic inter-island binding.

VI. GENERALIZATION TO MANY ISLANDS

The results presented above can be straightforwardly extended to more complex setups such as many quantum islands connected in series by single channel QPCs. We consider as an example the illustrative case of three islands depicted in Fig 7. In this case, we express the four bosonic fields \( \delta \phi_j \) in terms of the four incoming fields \( \delta \phi_0 \) via a \( 4 \times 4 \) scattering matrix \( \mathbf{M} \). In the low-energy limit, \( \mathbf{M} \) writes

\[
\lim_{\omega \tau_c \ll 1} \mathbf{M} = \frac{1}{2} \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}.
\]
This shows that, out of the four incoming fields, only one \( \delta \phi_0 \) is a gapless mode and the remaining three modes can be integrated out to obtain an effective Hamiltonian energies well below the charging energy, similarly to the previous case of two connected islands. The result of the averaging in the absence of gate voltages gives

\[
H_{\text{BS}} = r' D^{1/4} E_c^{3/4} \cos \left( \frac{\delta \phi_0}{2} \right),
\]

where the effective reflection coefficient writes

\[
r' = \frac{\sqrt{2}}{\pi} \left( \frac{e'}{\pi} \right)^{3/4} \left[ (|r_1| + |r_4|) \left( 17 - 2\sqrt{2} \right) \frac{1}{\sqrt{2}} \right.
+ \left( 17 - 2\sqrt{2} \right) \frac{1}{\sqrt{2}} (|r_2| + |r_3|) \right].
\]

We find again a boundary sine-Gordon model where the dimension of the relevant operator is 1/4. The subsequent analysis takes the same form as for two islands and the transport properties are calculated analogously. Computing the current and the noise perturbatively for \( T \gg T^* \), where \( T^* \sim |r'|^6 E_c/k_B \) is the temperature scale generated by the boundary term, we find the backscattering of fractional charges \( e^* = e/4 \).

We can readily generalize our setup to \( N_{\text{QI}} \) islands connected by interspersed individual QPCs. In that case, the theory follows the same line, one obtains a single gapless bosonic mode that survives at energies well below the charging energy. The corresponding scattering matrix \( M \) of size \((N_{\text{QI}} + 1) \times (N_{\text{QI}} + 1)\) simplifies at low energy with a single non-vanishing column with entries \( 1/\sqrt{N_{\text{QI}}} \). The resulting model for the quantum critical point and its leading relevant perturbation is still a boundary sine-Gordon model with the operator dimension \( \nu = 1/(N_{\text{QI}} + 1) \). Perturbative evaluations of the current and noise give the backscattered fractional charge \( e^* = e/(N_{\text{QI}} + 1) \).

\[\text{VII. CONCLUSION}\]

In light of a recent experiment [30], we investigate theoretically the quantum criticality associated with two floating islands connected to three edge channels via quantum point contacts in the integer quantum Hall regime. Assuming high transparency of the quantum point contacts, we unveil the nature of the quantum critical point by establishing an explicit mapping to the boundary sine-Gordon model with Luttinger parameter \( K = 1/3 \). From this mapping to the boundary sine-Gordon model, we study the features of different observables reported in the experiment [30]. We compute the residual fractional entropy, and we also find that the critical exponents for the conductance close to the triple points and the shape of the conductance as a function of the plunger gate voltages are in full agreement with experimental findings.

Our analytical description also demonstrates the competition between Kondo-like screening of each individual island and the cooperative transfer of electrons between them via the conductance measurements. In addition, we report the emission of fractional charges \( e^* = e/3 \) whose value can be extracted from shot noise measurements. Our work also sheds light towards extending the recent experiment to charge-Kondo clusters. For multi \((N_{\text{QI}})\)-sites quantum islands, we show that the mapping to the boundary sine-Gordon model still holds – with Luttinger parameter \( K = 1/(N_{\text{QI}} + 1) \) – and find a fractional charge emission \( e^* = e/(N_{\text{QI}} + 1) \).

\[\text{ACKNOWLEDGMENT}\]

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\[\text{Appendix A: Equations of motion for bosonic fields}\]

With number density operators \( \hat{N}_{1/2} \) presented in Eq. (4), we arrive at the equation of motions for the bosonic fields (for details see Ref. [53])

\[
\begin{align*}
\delta \phi_1 &= \delta \phi_0 - \frac{1}{i \omega \tau_c} (\delta \phi_1 + \delta \phi_2 - 2 \pi N_{1,g}), \\
\delta \phi_2 &= \delta \phi_0 + \frac{1}{i \omega \tau_c} (-\delta \phi_1 + 2 \delta \phi_2 - \delta \phi_3 + 2 \pi (N_{2,g} - N_{1,g})), \\
\delta \phi_3 &= \delta \phi_0 + \frac{1}{i \omega \tau_c} (-\delta \phi_2 + \delta \phi_3 - 2 \pi N_{2,g}).
\end{align*}
\]

By performing orthogonal transformation

\[
\begin{pmatrix}
\delta \phi_0 \\
\delta \phi_B \\
\delta \phi_C
\end{pmatrix} =
\begin{pmatrix}
-\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}}
\end{pmatrix}
\begin{pmatrix}
\delta \phi_1 \\
\delta \phi_2 \\
\delta \phi_3
\end{pmatrix},
\]

we express the Eq. (A1) into the form already presented in Eq. (7) with the scattering matrix

\[
M =
\begin{pmatrix}
-\frac{1}{\sqrt{3}} & -\omega \tau_c (2i + \omega \tau_c) & -\omega \tau_c (4i + \omega \tau_c) \\
-\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}(3i + \omega \tau_c)} & \frac{1}{\sqrt{6}(3i + \omega \tau_c)} \\
-\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}(4i + \omega \tau_c)} & \frac{1}{\sqrt{6}(4i + \omega \tau_c)}
\end{pmatrix}.
\]

(A3)

whose low energy limit \( \omega \tau_c \ll 1 \) is given in Eq. (9).
Appendix B: Effective Hamiltonian

We start form the Hamiltonian Eq (6) accounting for weak backscattering in three QPCs. From appendix A, we see that $\delta \phi_t$ entering into Eq (6) are expressed in terms of three incoming fields $\delta \phi^\alpha_0 (\alpha = A, B, C)$. While $\alpha = A$ is the gapless mode, $\alpha = B, C$ represent gapped modes. We then integrated out the gapped modes to arrive at the backscattering Hamiltonian expressed in terms of single gapless mode $\delta \phi^0_1$. In the following, we provide a quick summary on the integrating out of the high-energy modes. From the scattering matrix presented in appendix A, we have for the field $\delta \phi_1$

$$
\delta \phi_1 = \frac{\delta \phi^0_1}{\sqrt{3}} - \frac{2\pi}{3} (2N_{1,g} + N_{2,g}) \frac{\omega \tau_c (2i + \omega \tau_c) \delta \phi^0_2}{\sqrt{2} (\omega^2 \tau_c^2 + 4i \omega \tau_c - 3)} - \frac{\omega \tau_c (4i + \omega \tau_c) \delta \phi^0_3}{\sqrt{6} (\omega^2 \tau_c^2 + 4i \omega \tau_c - 3)}.
$$

(A1)

Above equation results in the cosine term in the backscattering Hamiltonian of the QPC1 as given in Eq.(10). The average over the high energy modes $(\delta \phi^2(t))_{HE}$ is obtain by using the input/output scattering formalism developed in Refs. [53, 60, 61] followed by the application of identity

$$
\langle \delta \phi^0_\alpha (\omega) (\delta \phi^0_\alpha)\rangle = 2 \left[ 1 + n_B \left( \frac{h \omega}{k_B T} \right) \right] \delta_{\alpha \alpha} \delta (\omega - \omega').
$$

At zero temperature, we arrive at the result

$$
\langle \delta \phi^2(t) \rangle_{HE} = \frac{4}{3} \int_0^\infty d\omega \frac{\omega \tau_c^2 \left[ 7 + (\omega \tau_c)^2 \right]}{[1 + (\omega \tau_c)^2] [9 + (\omega \tau_c)^2]} e^{-\frac{\omega}{T}} = -\frac{4}{3} \log \left[ 3^{3/4} e^\gamma \frac{h}{D_c} \right].
$$

(B2)

Repeating the same procedure for QPC2 and QPC3, the Eq. (6) finally writes into the form of Eq. (11) presented in the main text.

Appendix C: Bethe ansatz solution

In the main text, we showed that the two-sites quantum island in quasi-ballistic regime falls into the same universality class as the weak tunneling between fractional quantum Hall edge states at filling $\nu = 1/3$. The latter problems are described by a boundary sine-Gordon model whose Bethe ansatz solution for the charge current has been thoroughly investigated [39–42]. In particular, the Bethe Ansatz equations can be solved analytically at vanishing temperature by using the Wiener-Hopf technique [42], providing a closed series representation of the universal scaling function for the current in the low and large bias voltage regimes at zero temperature:

$$
I = \left\{ \begin{array}{ll}
2 \frac{e^2 V}{3h} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{\pi} I(3n)}{2^{2n} (n!)^{1/2} (2n + 1)} x^{2n}, & x < x_0 \\
\frac{e^2 V}{3h} \left[ 1 - \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{\pi} I(1)}{6(n!)^{1/2} (2n + 1)} x^{-2n} \right], & x > x_0,
\end{array} \right.
$$

where $x = eV/k_BT_B$ with $T_B$ being the boundary temperature and the convergence radius of the series expansion reads $x_0 = \sqrt{2}/3^{1/4}$. We then matched the above expression of current for $x > x_0$ stopping the series at $n = 1$ with that obtained perturbatively Eq. (26) providing the matching $T_B = T_*$. The resulting zero temperature differential conductance is presented in Eq. (26) and corresponding universal curve is depicted in Fig. 8.

FIG. 8. The universal zero-temperature differential conductance (in the unit of $G_0 = e^2/3h$) as a function of $eV/k_BT_*$. 

At finite temperature, the current can be computed in a standard way [41, 42] from the solution of the so-called Thermodynamical Bethe Ansatz (TBA) equations, an approach that we summarize here for completeness. The sine-Gordon model in $\nu = 1/3$ has a spectrum consisting in a pair of kink-antikink, carrying charge $q = \pm e$ in our case, and of a breather which can be viewed as a kink-antikink neutral boundstate. The corresponding quasi-particle modes are denoted $A_k (\theta)$, with the quantum number $a = +, -, 0$ labeling the kink, the antikink, and the breather respectively. Momentum $p$ is parametrized by a rapidity $\theta = \ln \frac{x}{x_0}$ as $\frac{k_B T}{2\pi e^n \cos \theta}$. Those quasiparticles are not free: while integrability results in the many-body scattering between quasiparticles factorizing as elementary two-body scattering, just as in a free theory, the interaction between two quasiparticles $A_{a_1} (\theta_1)$ and $A_{a_2} (\theta_2)$ is encoded in a non-trivial two-body scattering matrix $S_{a_1 a_2} (\theta_1 - \theta_2)$ [73].

As a result of this interaction, at finite temperature the quasiparticles distributions are not that of free particles, but rather obey a set of non-linear equations. In the thermodynamical limit $L \to \infty$ (with $L$ the system size), we write the densities of occupied quasiparticles as $\frac{k_B T}{2\pi e^n \cos \theta} \rho_a (\theta)$ with $\rho_a$ the reduced densities. The reduced densities can then be encoded by pseudo energies $\epsilon_c = \epsilon_\pm$ and $\epsilon_0$ via $\rho_a = f_a (A) \rho_a$ where $P_a = \partial_\theta \epsilon_a$ is the total (occupied+empty) density of quasiparticles, and $f_a = (1 + e^{\epsilon_a - \mu_a})^{-1}$ is the occupation function. We also introduce the functions $L_a (\theta) = \ln (1 + e^{\epsilon_a (\theta) - \mu_a})$, 

\[ L_a (\theta) = \ln (1 + e^{\epsilon_a (\theta) - \mu_a}) \]

\[ \frac{e^2 V}{3h} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{\pi} I(3n)}{2^{2n} (n!)^{1/2} (2n + 1)} x^{2n}, \]

\[ \frac{e^2 V}{3h} \left[ 1 - \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{\pi} I(1)}{6(n!)^{1/2} (2n + 1)} x^{-2n} \right], \]

\[ \left\{ \begin{array}{ll}
2 \frac{e^2 V}{3h} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{\pi} I(3n)}{2^{2n} (n!)^{1/2} (2n + 1)} x^{2n}, & x < x_0 \\
\frac{e^2 V}{3h} \left[ 1 - \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{\pi} I(1)}{6(n!)^{1/2} (2n + 1)} x^{-2n} \right], & x > x_0,
\end{array} \right. \]
which are connected to the densities via $\partial_t L_a = P_a - \rho_a$. The reduced chemical potentials read:

$$\mu_0 = 0 \quad ; \quad \mu_\pm = \pm \frac{eV}{2k_B T}.$$

The TBA equations for the pseudoenergies read

$$\epsilon_c(\theta) = \frac{1}{\pi \cosh(2\theta)} \ast L_0(\theta), \quad (C1)$$

$$\epsilon_0(\theta) = \frac{1}{\pi \cosh(2\theta)} \ast \left(L_+(\theta) + L_-(\theta)\right), \quad (C2)$$

where the convolution is defined by $f(\theta) \ast g(\theta) = \int d\theta' f(\theta') g(\theta - \theta')$. Equations (C1,C2) have to be supplemented with the boundary conditions $\epsilon_c(\theta) \approx e^\theta$ and $\epsilon_0(\theta) \approx \sqrt{2} e^\theta$. These equations are solved numerically, yielding the densities $\rho_a$. The current is then obtained through a rate equation [41, 42]

$$I = \frac{e k_B T}{2\pi \hbar} \int d\theta (\rho_+ - \rho_-)(\theta) \mathcal{T}(\theta), \quad (C3)$$

where $\mathcal{T}(\theta)$ is the probability that a kink is scattered into an anti-kink at the impurity site. This probability depends on the boundary coupling, and reads $\mathcal{T}(\theta) = \frac{(T/T_*)^4}{\cosh^4(\theta)}$ where $T_0$ is the scale at which $T = \frac{1}{2}$. It is related to the scale $T_*$ as $T_0 = \frac{(1)}{\sqrt{\pi}} T_* [42]$. We then express the linear conductance $G(V = 0, T) = \lim_{V \to 0} \frac{\partial I}{\partial V}$ as

$$G(V = 0, T) = \frac{e^2}{\hbar} \int \frac{d\theta}{1 + e^{\epsilon_c(\theta)}} \frac{1}{\cosh^2(2\theta + 2 \ln \frac{2T}{\pi c})}. \quad (C4)$$

The exact low- and high-temperature asymptotic behaviors of Eq. (C4) can be derived analytically using Keldysh perturbation theory [67] and are presented in Eq. (28). The full conductance curve together with its asymptotic behaviors is depicted in Fig. 4.

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