Isomonodromic Deformations and Supersymmetric Gauge Theories

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ABSTRACT

Seiberg-Witten solutions of four-dimensional supersymmetric gauge theories possess rich but involved integrable structures. The goal of this paper is to show that an isomonodromy problem provides a unified framework for understanding those various features of integrability. The Seiberg-Witten solution itself can be interpreted as a WKB limit of this isomonodromy problem. The origin of underlying Whitham dynamics (adiabatic deformation of an isospectral problem), too, can be similarly explained by a more refined asymptotic method (multiscale analysis). The case of $N=2$ $SU(s)$ supersymmetric Yang-Mills theory without matter is considered in detail for illustration. The isomonodromy problem in this case is closely related to the third Painlevé equation and its multicomponent analogues. An implicit relation to $tt$ fusion of topological sigma models is thereby expected.

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1 Introduction

The so called “Seiberg-Witten solutions” of four-dimensional supersymmetric
gauge theories have common building blocks — a family of complex algebraic
curves (Riemann surfaces), a meromorphic differential $dS$ on these curves,
period integrals of $dS$, and a prepotential $F$ in the sense of special geometry.
The original Seiberg-Witten theory [1] deals with SU(2) theories and uses
elliptic curves of a special form.

Recent studies on Seiberg-Witten solutions are deeply connected
with classical integrable systems. Gorsky et al. [2] first discovered this fact in
the case of $N = 2$ SU(2) supersymmetric Yang-Mills theory without matter.
According to their interpretation, the elliptic curves are the “spectral curves”
of elliptic solutions to the KdV equation, and $dS$ is a differential that emerges
in the analysis of “adiabatic deformations” (or “modulation”) of those elliptic
solutions by the so called “Whitham averaging method” [3].

The observation of Gorsky et al. was soon extended [4, 5] to the solu-
tions for other classical gauge groups [6, 7, 8, 9]. In these cases, hyperelliptic
curves of rather special types are used in place of the Seiberg-Witten elliptic
curves. Martinec and Warner [4] noticed that these hyperelliptic curves are
nothing but spectral curves of affine (periodic) Toda chain systems. The
present authors [5], meanwhile, argued that affine Toda field equation lies
in the heart of the generalized Seiberg-Witten solutions and plays the role
of the KdV equation in the work of Gorsky et al. These two observations
are not contradictory, but rather complementary. The fact is that the affine
Toda chain system corresponds to “finite-band solutions” of the affine Toda
field equation, and that adiabatic deformations of these finite band solutions
give rise to a system of Whitham-type. The prepotential $F$, in this respect,
is just a “quasi-classical $\tau$ function” [5]. This characterization of the prepo-
tential yielded an interesting application [10] to the problem of broken scale
invariance of $N = 2$ supersymmetric gauge theories with matters [11, 12].

Recently, $N = 4$ supersymmetric Yang-Mils theories, too, have been
treated in the framework of classical integrable systems (typically, the el-
liptic Calogero-Moser systems) [13, 14, 15, 16].

These results on integrable structures of supersymmetric gauge theories
show that each gauge theory is accompanied with three different types of
classical integrable systems:
1. A finite dimensional integrable dynamical system (affine Toda chain, elliptic Calogero-Moser system, etc.)

2. A soliton equation (KdV equation, affine Toda fields, etc.)

3. A Whitham system (of KdV equation, affine Toda fields, etc.)

These integrable systems are linked with each other as follows: The first integrable system is solvable by algebro-geometric methods, and each solution is associated with an algebraic curve (spectral curve). The moduli space of those spectral curves is identified with the quantum moduli space of a supersymmetric gauge theory. To be able to undergo modulation, these solutions of the first integrable system are embedded into the second integrable system as finite-band solutions. The slow dynamics on the moduli space can be separated by averaging over the fast (quasi-periodic) dynamics of the finite-band solutions. The Whitham system is the equations of motion of this slow dynamics.

The present understanding of integrable structures in Seiberg-Witten solutions is thus considerably messy. In particular, the derivation of the Whitham dynamics looks very artificial. The least persuasive point is that it cannot explain the origin of adiabatic deformations; it simply assumes that deformation takes place. The origin of the Whitham dynamics is discussed by several authors in the literature \[2, 15, 16\], who mostly seek the origin in (semi-classical) quantization of the classical integrable systems arising here. We are not satisfied with those explanations. The goal of this paper is to present a different approach based on the concept of isomonodromic deformations. This approach might be eventually absorbed into the idea of renormalization groups, but for the moment, this seems to be the most clear and tractable approach.

Our approach is inspired by an idea of Flaschka and Newell [17]. They noted, with a fairly general (though speculative) reasoning, that isomonodromic deformations in “WKB approximation” look like modulation of isospectral deformations. As they added, similar observations were already made by Boutroux [18] and Garnier [19] early in this century. According to Boutroux, for instance, solutions of the first (PI) and second (PII) Painlevé equations

\[
(\text{PI}) \quad \frac{d^2 u}{dx^2} = u^2 - x, \tag{1}
\]
behave like a (modulated) elliptic function as \( x \to \infty \). The same (but more refined) idea is also applied to the string equations of two-dimensional quantum gravity [20, 21, 22]. We now attempt to apply this idea to Seiberg-Witten solutions.

### 2 WKB analysis of isomonodromy problem

Let us recall a universal formulation of Seiberg-Witten solutions due to Itoyama and Morozov [16]. They write the algebraic curves \( C \) over the quantum moduli spaces as:

\[
\det \left( w - L(z) \right) = 0,
\]

where \( z \) is a parameter (“spectral parameter”) on another algebraic curve \( C_0 \) (of genus 0 or 1), and \( L(z) \) is a finite dimensional matrix depending on \( z \) that gives the Lax operator of an associated integrable dynamical system. This “eigenvalue equation” gives a finite covering \( C \to C_0 \) (“spectral covering”). The meromorphic differential \( dS \), too, has the simple expression

\[
dS = wdz.
\]

In the case of the \( N = 2 \) SU\((s)\) Yang-Mills theory without matter, for instance, \( C_0 \) is a Riemann sphere \( \mathbb{C}P^1 \), and the spectral parameter \( z \) is given by the logarithm of the coordinate \( h \) on \( \mathbb{C}P^1 \setminus \{0, \infty\} \):

\[
z = \log h.
\]

The matrix \( L(z) \) is the Lax operator of the affine SU\((s)\) Toda chain:

\[
L(z) = \begin{pmatrix}
b_1 & 1 & & & c_2 h^{-1} \\
c_1 & b_2 & & & \\
& \ddots & \ddots & \ddots & \\
& & \ddots & \ddots & 1 \\
h & & & c_{s-1} & b_s
\end{pmatrix},
\]

\[
b_j = \frac{dq_j}{dt}, \quad c_j = \exp(q_j - q_{j+1}). \tag{6}
\]
The eigenvalue equation of $L(z)$ boils down to the simple equation

$$h - P(w) + \Lambda^2 h^{-1} = 0,$$

(7)

where $P(w)$ is a polynomial in $w$ of the form $w^s + \sum_{j=2}^s u_j w^{s-j}$. This equation can be solved for $h$ as

$$h = \frac{P(w) + y}{2}, \quad y^2 = P(w) - \Lambda^2,$$

(8)

thus yielding the familiar hyperelliptic spectral curve of a finite periodic Toda chain. This kind of integrable systems generally have a commuting set of isospectral flows with a Lax representation of the form

$$\frac{\partial L(z)}{\partial t_n} = [P_n(z), L(z)], \quad \left[ \frac{\partial}{\partial t_n} - P_n(z), \frac{\partial}{\partial t_m} - P_m(z) \right] = 0,$$

(9)

where $P_n(z)$, like $L(z)$ itself, are suitably selected matrix-valued meromorphic functions on $C_0$. The associated linear problem

$$w \psi(z) = L(z) \psi(z), \quad \frac{\partial \psi(z)}{\partial t_n} = P_n(z) \psi(z)$$

(10)

is integrable in the sense of Frobenius, and the vector- (or matrix-)valued solution $\psi(z)$ can be characterized as a “Baker-Akhiezer function” on $C$. This, conversely, gives an algebro-geometric method for solving (and constructing) this kind of integrable dynamical systems as well as more general “Hitchin systems”.

Our idea is to replace the above isospectral problem by the following isomonodromy problem:

$$\epsilon \frac{\partial \Psi(z)}{\partial z} = Q(z) \Psi(z), \quad \frac{\partial \Psi(z)}{\partial t_n} = P_n(z) \Psi(z).$$

(11)

Here $\epsilon$ is a small parameter that plays the role of the Planck constant in the subsequent (formal) “WKB analysis”. $\Psi(z)$ is not the same as $\psi(z)$, though connected with $\psi(z)$ by a simple relation as we shall show below. $Q(z)$ is conceptually the same as $L(z)$, and reduces to $L(z)$ in a limit. To give a nontrivial isomonodromy problem, however, $Q(z)$ has to be a more general matrix (see subsequent sections). According to the general theory of
isomonodromic deformations \[14, 25\], the $t$-flows leave the monodromy data of the first ODE in $z$ if and only if the above linear system is integrable in the sense of Frobenius, i.e., the zero-curvature equations

\[
\begin{bmatrix}
\frac{\partial}{\partial t_n} - P_n(z), \epsilon \frac{\partial}{\partial z} - Q(z)
\end{bmatrix} = 0, \quad \begin{bmatrix}
\frac{\partial}{\partial t_n} - P_n(z), \frac{\partial}{\partial t_m} - P_m(z)
\end{bmatrix} = 0 \quad (12)
\]

are satisfied. These zero-curvature equations, thus, gives a Lax representation of an isospectral problem.

The idea of Flaschka and Newell \[17\] (and of Novikov \[21\]) is to put $\Psi(z)$ in a WKB form (as $\epsilon \to 0$):

\[
\Psi(z) = \left(\phi(z) + \sum_{n=1}^{\infty} \epsilon^n \phi_n(z)\right) \left(\frac{\partial^2 S(z)}{\partial z^2}\right)^{-1/2} \exp\left(\epsilon^{-1} S(z)\right). \quad (13)
\]

In the leading order of this $\epsilon$-expansion, the ODE in $z$ of the above isomonodromy problem gives the algebraic equation

\[
\frac{\partial S(z)}{\partial z} \phi(z) = Q(z) \phi(z) \quad (14)
\]

for the “amplitude” $\phi(z)$. If we identify

\[
w = \frac{s(z)}{\partial S(z)/\partial z}, \quad (15)
\]

this algebraic equation gives essentially the same eigenvalue problem as in the formulation of Itoyama and Morozov. (The relation between $\phi(z)$ and $\psi(z)$ will be clarified in the next section.) The last relation can be rewritten

\[
dS(z) = w dz. \quad (16)
\]

Thus the Seiberg-Witten differential $dS$ can be reproduced from the above isomonodromy problem.

Note that $z$ and $w$ now play the role of a coordinate and its conjugate momentum. The passage from the isospectral problem to the isomonodromy problem is achieved by the substitution

\[
w \to \epsilon \frac{\partial}{\partial z}. \quad (17)
\]
This is a kind of quantization! Here $\epsilon$ corresponds to the Planck constant, and $\Psi(z)$ to a quantum mechanical wave function. Thus, we are now considering a quantum mechanical problem in the (one-dimensional) space of spectral parameter. This is in contrast with other proposals on the origin of Whitham dynamics \[2, 15, 16\], which consider quantization in an ordinary coordinate space of interacting particle systems.

3 Multiscale analysis of isomonodromy problem

The next task is to separate the above isomonodromy problem into a combination of fast (isospectral) and slow (Whitham) dynamics. Conceptually, this is similar to the Born-Oppenheimer approximation in quantum mechanics. “Multiscale analysis” is a powerful perturbative method for studying this kind of problems, and particularly popular among researchers of nonlinear waves and pattern formations \[26\]. Let us apply this method to our isomonodromy problem.

Following the idea of multiscale analysis, we introduce two sets of variables $t$ and $T$ connected by the relation

$$\epsilon t_n = T_n, \quad (18)$$

and assume that that all fields $\{u_\alpha\}$ in the system are functions of variables $t$ and $T$ as $u_\alpha = u_\alpha(t, T)$. The two sets of time variables represent fast and slow time scales. Derivatives of the fields can be written as a sum of contributions from these two scales:

$$\frac{\partial}{\partial t_n} u_\alpha(t, \epsilon t) = \frac{\partial u_\alpha(t, T)}{\partial t_n} + \epsilon \frac{\partial u_\alpha(t, T)}{\partial T_n} \bigg|_{T=\epsilon t}. \quad (19)$$

Thus the coefficient matrices $P_n(z)$ and $Q(z)$ are assumed to be functions of $(t, T)$,

$$P_n(z) = P_n(t, T, z), \quad Q(z) = Q(t, T, z). \quad (20)$$

(To emphasize the roles of $t$ and $T$, we write all independent variables explicitly.) We now look for a “wave function” of the form

$$\Psi(z) = (\phi(t, T, z) + \sum_{n=1}^{\infty} \epsilon^n \phi_n(t, T, z)) \left(\frac{\partial^2 S(T, z)}{\partial z^2}\right)^{-1/2} \exp\left(\epsilon^{-1} S(T, z)\right). \quad (21)$$
Note that $S(T, z)$ is assumed to be $t$-independent. This is an essential ansatz in the following calculations.

Now, from the leading order of $\epsilon$-expansion, $\phi(t, T, z)$ and $S(t, T, z)$ turn out to satisfy the equations

$$
\frac{\partial S(T, z)}{\partial z} \phi(t, T, z) = Q(t, T, z)\phi(t, T, z), \quad (22)
$$

$$
\frac{\partial \phi(t, T, z)}{\partial t} + \frac{\partial S(T, z)}{\partial T} \phi(t, T, z) = P_n(t, T, z)\phi(t, T, z). \quad (23)
$$

These equations can be further converted into a more familiar form

$$
w\psi(t, T, z) = Q(t, T, z)\psi(t, T, z), \quad \frac{\partial \psi(t, T, z)}{\partial t} = P_n(t, T, z)\psi(t, T, z). \quad (24)
$$

where we have defined

$$
w := \frac{\partial S(T, z)}{\partial z}, \quad \psi(z) := \phi(z)\exp\left(\sum t_n \frac{\partial S(z)}{\partial T_n}\right). \quad (25)
$$

This is exactly an isospectral problem! The associated spectral curve $C$ is defined by the eigenvalue equation

$$
det\left(w - Q(t, T, z)\right) = 0, \quad (26)
$$

which is $t$-independent, but now depends on $T$. Thus $T$ enters into the isospectral problem as “adiabatic parameters”. $w$ is now a meromorphic function on $C$, and the phase function $S(T, z)$ is to be reproduced as

$$
S(T, z) = \int^z w dz \quad (+ \text{function of } T \text{ only}). \quad (27)
$$

One can now use the algebro-geometric method [23] to construct $\psi(z)$ as the (vector-valued) Baker-Akhiezer function of a finite-band solution. In general, such a Baker-Akhiezer function takes the form

$$
\psi(z) = \phi(z)\exp\left(\sum t_n \Omega_n(z)\right), \quad (28)
$$

where $\Omega_n$ are primitive functions of Abelian differentials $d\Omega_n$ on $C,$

$$
\Omega_n(z) = \int^z d\Omega_n. \quad (29)
$$
The amplitude part $\phi(z)$ are made from several theta functions of the form $\theta(\sum t_n \sigma_n + \ldots)$, where $\sigma_n$ is a vector $\sigma_n = (\sigma_{n,1}, \ldots, \sigma_{n,g})$ of period integrals

$$\sigma_{n,j} = \frac{1}{2\pi i} \oint_{\beta_j} d\Omega_n$$

of $d\Omega_n$. These theta functions are responsible for the quasi-periodic (fast) dynamics of the isospectral problem, which is eventually “averaged out” and does not contribute to the Whitham (slow) dynamics. The main contribution to the Whitham dynamics stems from the exponential part. By matching the above algebro-geometric expression of $\psi(z)$ with the previous multiscale expression, we find that $S(z)$ and $\Omega_n(z)$ have to be connected by the equations

$$\frac{\partial S(z)}{\partial T_n} = \Omega_n(z).$$

In fact, these are the most fundamental equations (or, rather, the very definition) of a Whitham system.

One may also interpret this construction of an asymptotic solution of the isomonodromy problem in the language of averaging methods. The original Whitham averaging method is not very convenient for this purpose; Krichever’s averaging method is more suited because it is formulated in terms of Baker-Akhiezer functions (A similar analysis of the string equations of two-dimensional quantum gravity is done by Fucito et al. [22]).

This is, actually, not the end of the story. The above multiscale analysis is more or less formal, and have to be justified on a rigorous mathematical foundation. Furthermore, this kind of analysis of isomonodromy problems is faced with a new difficulty not met in isospectral problems — Stokes phenomena. Stokes phenomena take place in WKB approximation in such a way that the approximation is not uniformly valid over the whole $z$ plane (or the Riemann surface $C_0$). The $z$ plane has to be divided into subdomains in which the WKB approximation is uniformly valid; such domains are determined by the configuration of “(anti)Stokes curves”. Because of this new situation, as emphasized by Moore in the case of two-dimensional quantum gravity [28], the above analysis has to be done more carefully. Rigorous mathematical treatment of all these issues is left for future researches.
4 Example: \( N = 2 \) SU(s) supersymmetric Yang-Mills theory

We now illustrate the isomonodromy problem in more detail in the case of \( N = 2 \) SU(s) supersymmetric Yang-Mills theory without matter. In particular, the relation between \( L(z) \) and \( Q(z) \), which is left unclear in the previous sections, will be clarified.

The underlying system of soliton equation is the affine SU(s) Toda field hierarchy. This is a periodic reduction of the full Toda hierarchy \([29]\), and can also be treated in the language of infinite matrices or difference operators, but we here use a more standard \( s \times s \) matrix formulation. Just like the full Toda hierarchy, the SU(s) version has two infinite series of time variables \( t = (t_1, t_2, \ldots) \) and \( \bar{t} = (\bar{t}_1, \bar{t}_2, \ldots) \); a difference is that the flows of \( t_s, t_{2s}, \ldots \) and \( \bar{t}_s, \bar{t}_{2s}, \ldots \) are trivial. All the flows are generated by \( s \times s \) matrices \( A_n(h) \) and \( \bar{A}_n(h) \) and obey the zero-curvature equations

\[
\left. \frac{\partial}{\partial t_n} - A_n(h), \frac{\partial}{\partial \bar{t}_m} - \bar{A}_m(h) \right] = 0, \\
\left. \frac{\partial}{\partial \bar{t}_n} - \bar{A}_n(h), \frac{\partial}{\partial t_m} - \bar{A}_m(h) \right] = 0, \\
\left. \frac{\partial}{\partial t_n} - A_n(h), \frac{\partial}{\partial \bar{t}_m} - \bar{A}_m(h) \right] = 0. \tag{32}
\]

The generators \( A_1(h) \) and \( \bar{A}_1(h) \) are directly related with the field variables \( u_j \) of the SU(s) Toda field equations

\[
\frac{\partial^2 u_{j}}{\partial t_1 \partial \bar{t}_1} + \exp(u_{j+1} - u_j) - \exp(u_j - u_{j-1}) = 0 \quad \left( \sum_{j=1}^{s} u_j = 0, \ u_{j+s} = u_j \right) \tag{33}
\]

as:

\[
A_1 = \begin{pmatrix} b_1 & 1 & 0 \\ \vdots & \ddots & \vdots \\ h & \cdots & b_s \end{pmatrix}, \quad \bar{A}_1 = \begin{pmatrix} 0 & \cdots & c_s h^{-1} \\ c_1 & \ddots & \vdots \\ 0 & \cdots & c_{s-1} \end{pmatrix}, \quad b_j = \frac{\partial u_j}{\partial t_1}, \quad c_j = \exp(u_{j+1} - u_j). \tag{34}
\]
The generators $A_n(h)$ and $\bar{A}_n(h)$ of higher flows can be written

\[ A_n(h) = \Lambda(h)^n + [\text{diag}]\Lambda(h)^{s-1} + \cdots + [\text{diag}], \]
\[ \bar{A}(h) = [\text{diag}]\Lambda(h)^{-s} + \cdots + [\text{diag}]\Lambda(h)^{-1}, \quad (35) \]

where “[diag]” stands for a diagonal matrix independent of $h$, and $\Lambda(h)^\pm$ are $s \times s$ matrices of the form

\[
\begin{pmatrix}
0 & 1 & 0 \\
\vdots & \ddots & \vdots \\
h & 0 & 0
\end{pmatrix},
\quad \Lambda(h)^{-1} =
\begin{pmatrix}
1 & \cdots & \cdots \\
\vdots & \ddots & \vdots \\
0 & 1 & 0
\end{pmatrix}.
\quad (36)
\]

In particular, the generators of the trivial flows are given by powers of $\Lambda(h)$:

\[
A_s(h) = \Lambda(h), \quad A_{2s}(h) = \Lambda(h)^2, \ldots,
\]
\[
\bar{A}_s(h) = \Lambda(h)^{-1}, \quad \bar{A}_{s}(h) = \Lambda(h)^{-2}, \ldots.
\quad (37)
\]

As in the full Toda hierarchy, $A_n(h)$ and $\bar{A}_n(h)$ can be expressed more explicitly in terms of two extra Lax operators. (Details are omitted here, because we do not need them in the following.) The associated linear problem is given by

\[
\frac{\partial \Psi}{\partial t_n} = A_n(h)\Psi, \quad \frac{\partial \Psi}{\partial \bar{t}_n} = \bar{A}_n(h)\Psi, \quad \Psi = ^t(\psi_1, \ldots, \psi_s).
\quad (38)
\]

The vector elements $\psi_1, \ldots, \psi_s$ are, in fact, part of the Baker-Akhiezer functions $\psi_j(t, \bar{t}, \lambda)$ ($\lambda = h^{1/s}$) of the full Toda hierarchy. In the $s$-periodic reduction, they are related as

\[
\psi_{j+s} = \lambda^s \psi_j = h \psi_j.
\quad (39)
\]

(In this sense, $h$ is nothing but the Floquet multiplier in the spectral analysis of periodic Toda chains.)

We now impose the following homogeneity condition to $\psi_j$'s:

\[
\lambda \frac{\partial \psi_j}{\partial \lambda} = \left( \sum_{n=1}^{\infty} n t_n \frac{\partial}{\partial t_n} + j - \sum_{n=1}^{\infty} n \bar{t}_n \frac{\partial}{\partial \bar{t}_n} \right) \psi_j.
\quad (40)
\]
This amounts to assigning the following dimensions (or “weights”) to the time and spectral variables, and requiring $\psi_j$ to be a homogeneous function of degree $-j$:

$$t_n \rightarrow n, \quad \bar{t}_n \rightarrow -n, \quad \lambda \rightarrow -1.$$ \hspace{1cm} (41)

This forces the matrix elements of $A_n(h)$ and $\bar{A}_n(h)$ to be also homogeneous. In particular, $b_j, c_j$ and $u_j$ become homogeneous functions of degree $-1$, 0 and 0.

It is well known that this kind of homogeneity constraints convert an isospectral problem (soliton equation) to an isomonodromy problem [17, 25]. To see how this mechanism works in the present case, let us write the homogeneity conditions in the following matrix form:

$$\lambda \frac{\partial \Psi}{\partial \lambda} = \left( \sum_{n=1}^{\infty} n t_n \frac{\partial}{\partial t_n} + \Delta - \sum_{n=1}^{\infty} n \bar{t}_n \frac{\partial}{\partial \bar{t}_n} \right) \Psi, \quad \Delta := (i\delta_{ij}).$$ \hspace{1cm} (42)

One can use the aforementioned linear differential equations to rewrite the $t$- and $\bar{t}$-derivatives on the right hand side. This eventually leads to a linear ODE of the form

$$\epsilon h \frac{\partial \Psi}{\partial h} = \epsilon \frac{\partial \Psi}{\partial z} = Q(z) \Psi,$$ \hspace{1cm} (43)

where

$$Q(z) := \frac{\epsilon}{s} \left( \sum_{n=1}^{\infty} n t_n A_n(h) + \Delta - \sum_{n=1}^{\infty} n \bar{t}_n \bar{A}_n(h) \right).$$ \hspace{1cm} (44)

This is exactly the monodromy problem that we have sought for!

More precisely, this “constrained” affine Toda field hierarchy is not yet an isomonodromy problem. A true isomonodromy problem arises when only a finite number of time variables are left nonzero. For instance, given two positive integers $r$ and $\bar{r}$, the constrained hierarchy restricted to the finite dimensional submanifold

$$t_r = \frac{s}{\epsilon r}, \quad t_{r+1} = t_{r+2} = \ldots = 0,$$

$$\bar{t}_{\bar{r}} = -\frac{s}{\epsilon \bar{r}}, \quad \bar{t}_{\bar{r}+1} = \bar{t}_{\bar{r}+2} = \ldots = 0,$$ \hspace{1cm} (45)

gives an isomonodromy problem. The discrete parameters $(r, \bar{r})$ resemble the index of a “critical point” or a “universality class” in two-dimensional quantum gravity. Hopefully, our isomonodromy problems, too, might have some “stringy” interpretation.
The case of \((r, \bar{r}) = (1, 1)\) is special, because no free time variables are left here. It is, however, exactly in this case that the Lax operator \(L(z)\) of the Seiberg-Witten solution is reproduced:

\[
\lim_{\epsilon \to 0} Q(z)\big|_{t_1 = -t_1 = s/\epsilon, t_2 = \ldots = 0} = A_1(h) + \bar{A}_1(h) = L(z). \tag{46}
\]

(Note that the term proportional to \(\Delta\) simply disappears in the limit as \(\epsilon\).) Thus, in a strict sense, the Seiberg-Witten solution itself corresponds to no isospectral problem. The remaining flows in the Seiberg-Witten solution are flows in the Seiberg-Witten periods

\[
a_j = \oint_{\alpha_j} dS \tag{47}
\]

or in their dual periods \(a_{DJ}\); they are however not isomonodromic. The fact is that a tower of isomonodromic families with different indices \((r, \bar{r})\) are “hidden” behind the Seiberg-Witten solution, and they become visible only when embedded into the affine Toda hierarchy.

An interesting situation takes place if \(t_1\) and \(\bar{t}_1\) are free variables while \(t_2 = t_3 = \ldots = 0\) and \(\bar{t}_2 = \bar{t}_3 = \ldots = 0\). In this setting, the homogeneity condition implies that the Toda field variables \(u_j\) are functions of the radial coordinate

\[
x := 4\sqrt{t_1 \bar{t}_1} \tag{48}
\]

only. (The numerical factor “4” is inserted to make the final answer simpler.) The Toda field equations now becomes a system of ODE’s:

\[
4 \frac{d^2 u_j}{dx^2} + 4 \frac{1}{x} \frac{du_j}{dx} + \exp(u_{j+1} - u_j) - \exp(u_j - u_{j-1}) = 0. \tag{49}
\]

In particular, in the case of \(SU(2)\), just a single independent field \(u\) is left \((u_1 = -u_2 = u/2)\) and obeys an ODE of the form

\[
\frac{d^2 u}{dx^2} + \frac{1}{x} \frac{du}{dx} + \sinh u = 0. \tag{50}
\]

This is an expression of the third Painlevé equation (PIII). The system in the \(SU(s)\) case may be interpreted as a multicomponent analogue of PIII. Remarkably, these equations of PIII type also emerge in \(t \bar{t}\) fusion of topological sigma models \[31\]. (This kind of isomonodromy problems in \(t \bar{t}\) fusion models are considered in a more general context by Dubrovin \[31\].)
The affine Toda field hierarchy, thus, plays the same role as the KdV and modified KdV hierarchies do in two-dimensional quantum gravity. These hierarchies of soliton equations provide a universal framework in which to consider different models (isomonodromy problems) on an equal footing. The associated Whitham hierarchies, too, has a universal structure, as demonstrated in our previous paper [5]. This hierarchy (Whitham-Toda hierarchy) inherits various features of the affine Toda field hierarchy. For instance, it has two infinite series of flows, and the flows of $T_s, T_{2s}, \ldots$ and $\bar{T}_s, \bar{T}_{2s}, \ldots$ are trivial. Note that they cannot be deduced directly from the properties of the affine Toda chain system. It is not the Toda chain system but the Toda field system that is responsible for the Whitham dynamics.

5 Conclusion

We have shown that an isomonodromy problem underlies Seiberg-Witten solutions of four-dimensional supersymmetric gauge theories. As emphasized in Introduction, integrable structures of the Seiberg-Witten solutions are considerably involved, and several different classical integrable systems appear to be related to the same Seiberg-Witten solution. Our main observation is that those apparently different features of integrability can be derived from a single isomonodromy problem. This is not just of purely mathematical interest. Quantum gravity and topological conformal field theories in two dimensions are all reorganized into some isomonodromy problems [20, 28, 31, 32]. We believe that our isomonodromy problem, too, will be essentially “stringly”, and somehow related to the recent string theoretical interpretation of the Seiberg-Witten theory [1, 33].

Such a possible link with string theories is particularly plausible in the case of $N = 2$ SU(s) supersymmetric Yang-Mills theories without matter. We have considered this case in some detail to illustrate our idea. The isomonodromy problem of this case is obtained from the affine SU(s) Toda field system with a constraint (homogeneity condition). In principle, the other classical gauge groups can be treated in the same way. We have noticed that this isomonodromy problem is implicitly related, via the third Painleve equation and its multicomponent analogues, to $tt$ fusion of topological sigma models.

A technical clue of our calculations is the method of multiscale analysis (a
refined version of the ordinary WKB analysis). This method enables us, just as in the analysis of two-dimensional quantum gravity, to derive the hypothetical picture of the Seiberg-Witten solutions as “adiabatic deformation of finite-band solutions in a soliton equation”. A mathematically rigorous proof will, however, require more careful and hard analytical considerations. (A similar problem is also recently studied by mathematicians from a different point of view [34].)

The next interesting targets are $N = 4$ supersymmetric Yang-Mills theories. We expect that this will lead to an isomonodromy problem over an elliptic curve. More general Hitchin systems over a complex algebraic curve of arbitrary genus, too, will be similarly converted to an isomonodromy problem. This will be a nice way to construct nontrivial isomonodromy problems on a Riemann surface. Actually, isomonodromy problems over Riemann surfaces are already studied by mathematicians in a more general context [35]. Multiscale analysis of those generalized isomonodromy problems is mathematically a tough issue, but it deserves to be studied.

References

[1] N. Seiberg and E. Witten, Electric-magnetic duality, monopole condensation and confinement in $N = 2$ supersymmetric Yang-Mills theory, Nucl. Phys. B426 (1994), 19-52, [hep-th/9407087]; Errata, ibid. B430 (1994), 485-486; Duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD, ibid. B431 (1994), 484-550, [hep-th/9408099].

[2] A. Gorsky, I. Krichever, A. Marshakov, A. Mironov and A. Morozov, Integrability and exact Seiberg-Witten solution, Phys. Lett. 335B (1995), 466-474, [hep-th/9505033].

[3] G.B. Whitham, Linear and nonlinear waves (Wiley, 1974).
H. Flaschka, M.G. Forest and D.W. McLaughlin, Multiphase averaging and the inverse spectral solution of the Korteweg-de Vries equation, Comm. Pure Appl. Math. 33 (1980) 739-784.
S.Yu. Dobrokhotov and V.P. Maslov, Finite-zone, almost-periodic solutions in WKB approximations, J. Soviet Math. 16, No. 6 (1981), 1433-1487.
B.A. Dubrovin and S.P. Novikov, Hydrodynamics of weakly deformed
soliton lattices: Differential geometry and Hamiltonian theory, Russian Math. Surveys 44, No. 6 (1989), 35-124.

[4] E. Martinec and N. Warner, Integrable systems and supersymmetric gauge theories, Nucl. Phys. B459 (1996), 97-112, [hep-th/9509161].

[5] T. Nakatsu and K. Takasaki, Whitham-Toda hierarchy and $N = 2$ supersymmetric Yang-Mills theory, [hep-th/9509162].

[6] P. Argyres and A. Faraggi, The vacuum structure and spectrum of $N = 2$ supersymmetric SU($N$) gauge theory, Phys. Rev. Lett. 73 (1995), 3931-3934, [hep-th/9411057].

[7] A. Klemm, W. Lerche, S. Theisen and S. Yankielowicz, Simple singularities and $N = 2$ supersymmetric Yang-Mills theory, Phys. Lett. 344B (1995), 169-175, [hep-th/9411048].

[8] U.H. Danielsson and B. Sundborg, The moduli space and monodromies of $N = 2$ supersymmetric SO($2R + 1$) Yang-Mills theory, Phys. Lett. B358 (1995), 273-280, [hep-th/9504102].

[9] A. Brandhuber and K. Landsteiner, On the monodromies of $N = 2$ supersymmetric Yang-Mills theory with gauge group SO($2n$), Phys. Lett. B358 (1995), 73-80, [hep-th/9507068].

[10] T. Eguchi and S.-K. Yang, Prepotentials of $N = 2$ supersymmetric gauge theories and soliton equations, [hep-th/9510183].

[11] A. Hanany and Y. Oz, On the quantum moduli space of vacua of $N = 2$ supersymmetric SU($N_c$) gauge theories, Nucl. Phys. B452 (1995), 283-312, [hep-th/9505073].
    A. Hanany, On the quantum moduli space of vacua of $N = 2$ supersymmetric gauge theories, [hep-th/9509176].

[12] P. Argyres, M.R. Plesser and A.D. Shapere, The Coulomb phase of $N = 2$ supersymmetric QCD, Phys. Rev. Lett. 75 (1995), 1699-1702, [hep-th/9505100].
    P. Argyres and A. Shapere, The vacuum structure of $N = 2$ super-QCD with classical gauge groups, [hep-th/9509173].
[13] R. Donagi and E. Witten, Supersymmetric Yang-Mills theory and integrable systems, hep-th/9510101.

[14] A. Gorsky and A. Marshakov, Towards effective topological gauge theories on spectral curves, hep-th/9510224.

[15] E. Martinec, Integrable structures in supersymmetric gauge and string theory, Phys. Lett. B367 (1996), 91-96, hep-th/9510204. E. Martinec and N. Waner, Integrability in $\mathcal{N} = 2$ gauge theory: A proof, hep-th/9511052.

[16] H. Itoyama and A. Morozov, Integrability and Seiberg-Witten theory. Curves and periods, hep-th/9511126; Prepotential and the Seiberg-Witten theory, hep-th/9512161.

[17] H. Flaschka and A.C. Newell, Monodromy- and spectrum-preserving deformations I, Commun. Math. Phys. 76 (1980), 65-116; Multiphase similarity solutions of integrable evolution equations, Physica 3D (1981), 203-221.

[18] P. Boutroux, Recherches sur les transcendents de M. Painlevé et l’étude asymptotique des équations différentiels du second ordre, Ann. Sci. L’Ecole Norm. Super. 30 (1913), 255-376.

[19] R. Garnier, Sur une classe de systèmes différentiels Abéliens déduits de la théorie des équations linéaires, Rend. Circ. Mat. Palermo 43 (1919), 155-191.

[20] G. Moore, Geometry of the string equations, Commun. Math. Phys. 133 (1990), 261-304.

[21] S.P. Novikov, Quantization of finite-gap potentials potentials and nonlinear quasiclassical approximation in nonperturbative string theory, Funct. Anal. Appl. 24 (1990), 296-306.

[22] F. Fucito, A. Gamba, M. Martellini and O. Ragnisco, Nonlinear WKB analysis of the string equation, Int. J. Mod. Phys. B6 (1992), 2123-2148, hep-th/9112074.
[23] B.A. Dubrovin, V.B. Matveev and S.P. Novikov, Nonlinear equations of Korteweg-de Vries type, finite band operators and Abelian varieties, Russian Math. Surveys 31, No. 1 (1976), 59-146.
I.M. Krichever, Methods of algebraic geometry in the theory of nonlinear equations, Russian Math. Surveys 32, No. 6 (1977), 185-214; Integration of nonlinear equations by the method of algebraic geometry, Funct. Anal. Appl. 11 (1977), 12-26.

[24] N.J. Hitchin, Stable bundles and integrable systems, Duke Math. J. 54 (1987), 91-114; Flat connections and geometric quantization, Commun. Math. Phys. 131 (1990), 347-380.

[25] M. Jimbo, T. Miwa and K. Ueno, Monodromy preserving deformations of linear ordinary differential equations with rational coefficients, Physica 2D (1981), 306-352.
M. Jimbo and T. Miwa, ditto II, ibid 2D (1981), 407-448.

[26] A. Newell and J. Whitehead, Finite bandwidth, finite amplitude convection, J. Fluid Mech. 38 (1969), 279-303.
J.D. Gibbon and M.J. McGuiness, Amplitude equations at the critical points of unstable dispersive physical systems, Proc. R. Soc. London A377 (1981), 185-219.
A. Newell, The dynamics of patterns: A survey, In: Propagation in Systems far from Equilibrium (ed. J.E. Wesfeld), Les Houches 1987 (Springer-Verlag, 1988).

[27] I.M. Krichever, Method of averaging for two-dimensional integrable equations, Funct. Anal. Appl. 22 (1990), 200-212; Spectral theory of two-dimensional periodic operators and its applications, Russian Math. Surveys 44, No. 2 (1989), 145-225.

[28] G. Moore, Matrix models of 2D gravity and isomonodromic deformation, Prog. Theor. Phys. Supp. 102 (1990), 255-258.

[29] K. Ueno and K. Takasaki, Toda lattice hierarchy, in: Group Representations and Systems of Differential Equations (ed. K. Okamoto), Advanced Studies in Pure Math. 4 (North-Holland/Kinokuniya, 1984).

[30] S. Cecotti and C. Vafa, Exact results for supersymmetric $\sigma$ models, Phys. Rev. Lett. 68 (1992), 903-906.

18
[31] B. Dubrovin, Geometry and integrability of topological-antitopological fusion, Commun. Math. Phys. 152 (1993), 539-564, hep-th/9206037.

[32] B.A. Dubrovin, Integrable systems in topological field theory, Nucl. Phys. B379 (1992), 627-689; Integrable systems and classification of 2-dimensional topological field theories, In: Integrable Systems (ed. O. Babelon et al.), Progress in Mathematics 115 (Birkhäuser, 1993), hep-th/9209040.

[33] S. Kachru, A. Klemm, W. Lerche, P. Mayr and C. Vafa, Nonperturbative results on the point particle limit of $N = 2$ heterotic string compactifications, Nucl. Phys. B459 (1996), 537-558, hep-th/9508155.

[34] N. Joshi and M.D. Kruskal, An asymptotic approach to the connection problem for the first and the second Painlevé equations, Phys. Lett. 130 (1988), 129-137.
T. Aoki, T. Kawai and Y. Takei, WKB analysis of Painlevé transcendents with a large parameter II — multiple-scale analysis of Painlevé transcendents, Kyoto preprint RIMS-1038 (1995).

[35] K. Okamoto, On Fuchs's problem on a torus, I. Funkcial. Ekvac. 14 (1971), 137-152; ditto II, J. Fac. Sci. Univ. Tokyo, Sect. IA, 24 (1977), 357-371; Déformation d’une équation différentielle linéaire avec une singularité irrégulière sur un tore, ibid. 26 (1979), 501-518.
K. Iwasaki, Moduli and deformation for Fuchsian projective connections on a Riemann surface, J. Fac. Sci. Univ. Tokyo, Sect. IA, 38 (1991), 431-531; Fuchsian moduli on a Riemann surface — its Poisson structure and Poincaré-Lefschetz duality, Pacific J. Math. 155 (1992), 319-340.
Corrections to the First Version

6th line in ABSTRACT:
“(adiabatic deformation of an isomonodromy problem)” → “(adiabatic deformation of an isospectral problem)”

Three items in 5th paragraph of section 1:
“1. A inite” → “1. A finite”

4th line in the 6th paragraph of section 1:
“adiatatic deformation” → “adiabatic deformations”

4th line above eq. (1):
“isomonodromic deformations.” → “isospectral deformations.”

1st line below eq. (10):
“(or matrix-valued)” → “(or matrix-)valued”

eq. (13):
Insert “\left(\frac{\partial^2 S(z)}{\partial z^2}\right)^{-1/2}” in front of “exp”

eq. (21):
Insert “\left(\frac{\partial^2 S(T, z)}{\partial z^2}\right)^{-1/2}” in front of “exp”

2nd-3rd lines in the last paragraph of section 5:
“This lead to an isomonodromy problem over an elliptic curve. More general Hitchin systems over a complex algebraic curve of arbitrary genus, too, can be . . .” → “We expect that this will lead to an isomonodromy problem over an elliptic curve. More general Hitchin systems over a complex algebraic curve of arbitrary genus, too, will be . . .”