Critical Temperature and Tunneling Spectroscopy of Superconductor-Ferromagnet Hybrids with Intrinsic Rashba–Dresselhaus Spin-Orbit Coupling

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We investigate theoretically how the proximity effect in superconductor-ferromagnet hybrid structures with intrinsic spin-orbit coupling is manifested in two measurable quantities, namely the density of states and critical temperature. To describe a general scenario, we allow for both Rashba and Dresselhaus type spin-orbit coupling. Our results are obtained via the quasiclassical theory of superconductivity, extended to include intrinsic spin-orbit coupling in the Usadel equation and the Kupriyanov–Lukichev boundary conditions. Unlike previous works, we have derived a Riccati parametrization of the Usadel equation including spin-orbit coupling which allows us to address the full proximity effect regime and not only the linearized weak proximity regime. First, we consider the density of states of both SF bilayers and SFS trilayers, where the spectroscopic features in the latter case are sensitive to the phase difference between the two superconductors. We find that the presence of spin-orbit coupling leaves clear spectroscopic fingerprints in the density of states due to its role in creating spin-triplet Cooper pairs. Unlike SF and SFS structures without spin-orbit coupling, the density of states in the present case depends strongly on the direction of the magnetization. We also determine how the critical temperature \( T_c \) of an SF bilayer is affected by spin-orbit coupling and, interestingly, demonstrate that one can achieve a spin-valve effect with a single ferromagnet. We find that \( T_c \) displays highly non-monotonic behavior both as a function of the magnetization direction and the type and direction of the spin-orbit coupling field, offering a new way to exert control of the superconductivity in proximity structures.

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I. INTRODUCTION

Material interfaces in hybrid structures give rise to proximity effects, whereby the properties of one material can “leak” into the adjacent material, creating a region with properties derived from both materials. In superconductor-ferromagnet (SF) hybrid structures, the proximity effect causes superconducting correlations to penetrate into the ferromagnetic region and vice versa. These correlations typically decay over short distances, which in diffusive systems is of the order \( \sqrt{D}/h \), where \( D \) is the diffusion coefficient of the ferromagnet and \( h \) is the exchange field. However, for certain configurations of the exchange field, the singlet correlations from the superconductor may be converted into so-called long-range triplets (LRTs). These triplet components have spin projections parallel to \( h \) and decay over a much longer distance, of the order \( \sqrt{D/2\pi T} \), where \( T \) is the temperature. This distance is independent of \( h \), and at low temperatures it becomes increasingly large, which allows the condensate to penetrate deep into the ferromagnet. The isolation and enhancement of this feature has attracted much attention in recent years as it gives rise to novel physics and possible low-temperature applications by merging spintronics and superconductivity.

It is by now well-known that the conversion from singlet to long-range triplet components of the superconducting state can happen in the presence of magnetic inhomogeneities, i.e. a spatially varying exchange field, and until recently such inhomogeneities were believed to be the primary source of this conversion although other proposals using e.g. non-equilibrium distribution functions and intrinsic triplet superconductors also exist.

However, it has recently been established that another possible source of LRT correlations is the presence of a finite spin-orbit (SO) coupling, either in the superconducting region or on the ferromagnetic side. It can be shown that an SF structure where the magnetic inhomogeneity is due to a Bloch domain wall is gauge equivalent to one where the ferromagnet has a homogeneous exchange field and intrinsic SO coupling. It is known that SO coupling can be caused by impurities, but this results in purely isotropic spin-relaxation, and so does not permit the desired singlet-LRT conversion. To achieve such a conversion, one needs a rotation of the spin pair into the direction of the exchange field. This can be achieved by using materials with an intrinsic SO coupling, either due to the crystal structure in the case of noncentrosymmetric materials, or due to interfaces in thin-film hybrids, which cause an inhomogeneously varying spin density in the direction perpendicular to these interfaces and which also modifies the fundamental process of Andreev reflection.

In this paper, we establish how the presence of spin-orbit coupling in SF structures manifests itself in two important experimental observables: the density of states probed via tunneling spectroscopy (or conductance measurements) and the critical temperature \( T_c \). A common consequence for both of these quantities is that neither becomes independent of the magnetization direction. This is in contrast to the case without SO coupling, where the results are invariant with respect to rotations of the magnetic exchange field. This symmetry is now lifted due to SO coupling: depending on the magnetization direction, LRT Cooper pairs are created in the system which leave clear fingerprints both spectroscopically and in terms of the \( T_c \) behavior. On the technical side, we will present in this work for the first time a Riccati parametrization of the Us-
adel equation and its corresponding boundary conditions that include SO coupling. This is an important advance in terms of exploring the full physics of triplet pairing due to SO coupling as it allows for a solution of the quasiclassical equations without any assumption of a weak proximity effect.

The remainder of the article will be organised as follows: In Section II, we introduce the relevant theory and notation, starting from the quasiclassical Usadel equation, which describes the diffusion of the superconducting condensate into the ferromagnet. We also motivate our choice of intrinsic SO coupling in this section, and propose a new notation for describing Rashba–Dresselhaus couplings. The section goes on to discuss key analytic features of the equations in the limit of weak proximity; symmetries of the density of states at zero energy; and analytical results needed to calculate the critical temperature of hybrid systems. We then present detailed numerical results in Section III: we analyze the density of states of an SF bilayer in III A (see Fig. 1), and of an SFS Josephson junction in III B (see Fig. 2). We study different orientations and strengths of the exchange field and SO coupling, and in the case of the Josephson junction, the effect of altering the phase difference between the condensates. Then, in Section III C, we continue our treatment of the SF bilayer in the full proximity regime by including a self-consistent solution in the superconducting layer (see Fig. 3), and focus on how the presence of SO coupling affects the critical temperature of the system. We discover that the SO coupling allows for spin-valve functionality with a single ferromagnetic layer, meaning that rotating the magnetization direction by $\pi/2$ induces a very large change in $T_c$. Finally, we conclude in Section IV with a summary of the main results, a discussion of some additional consequences of the choices made in-text, as well as possibilities for further work.

**FIG. 1:** The SF bilayer in III A. We take the thin-film layering direction to be along the $z$-axis, use the BCS bulk solution for the superconductor, and assume a Rashba–Dresselhaus coupling in the $xy$-plane in the ferromagnetic layer.

**FIG. 2:** The SFS trilayer in III B. We take the junction direction to lie in the $z$-direction, use the BCS bulk solution for the superconductors, and assume a Rashba–Dresselhaus coupling in the $xy$-plane in the ferromagnetic layer.

**FIG. 3:** The SF bilayer in III C. It is identical to the bilayer in Fig. 1, except that we replace the BCS solution $S_{BCS}$ in the superconductor with a fully self-consistent solution $S_{SC}$.

### II. THEORY

#### A. Fundamental concepts

The diffusion of the superconducting condensate into the ferromagnet can be described by the Usadel equation, which is a second-order partial differential equation for the Green’s function of the system. Together with appropriate boundary conditions, the Usadel equation establishes a system of coupled differential equations that can be solved in one dimension. We will consider the case of diffusive equilibrium, where the retarded component $\hat{g}^R$ of the Green’s function is sufficient to describe the behaviour of the system. We start by examining the superconducting correlations in the ferromagnet, and use the standard Bardeen–Cooper–Schrieffer (BCS) bulk solution for the superconductors. In particular, we will clarify the spectroscopic consequences of having SO coupling in the ferromagnetic layer.

In the absence of SO coupling, the Usadel equation in the ferromagnet reads:

$$D_F \nabla(\hat{g}^R \nabla \hat{g}^R) + i [\epsilon \hat{\rho}_3 + \hat{M}, \hat{g}^R] = 0,$nabla(\hat{g}^R \nabla \hat{g}^R) + i [\epsilon \hat{\rho}_3 + \hat{M}, \hat{g}^R] = 0,$$

where the matrix $\hat{\rho}_3 = \text{diag}(1, -1)$, and $\epsilon$ is the quasiparticle energy. The magnetization matrix $\hat{M}$ in the above equation is

$$\hat{M} = \left(\begin{array}{cc} h \sigma & 0 \\ 0 & (h \sigma)^* \end{array}\right),$$

where $h = (h_x, h_y, h_z)$ is the ferromagnetic exchange field, $(\cdot)^*$ denotes complex conjugation, $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ is the Pauli vector, and $\sigma_i$ are the usual Pauli matrices. The corresponding Kupriyanov–Lukichev boundary conditions are:

$$2L_j \hat{\zeta}_j \hat{g}^R_j \nabla \hat{g}^R_j = [\hat{g}^R_1, \hat{g}^R_2],$$

where the subscripts refer to the different regions of the hybrid structure; in the case of an SF bilayer as depicted in Fig. 1, $j = 1$ denotes the superconductor, and $j = 2$ the ferromagnet, while $\nabla$ denotes the derivative along the junction direction $1 \rightarrow 2$. The respective lengths of the materials are denoted $L_j$, and the interface parameters $\hat{\zeta}_j = R_B / R_j$ describe the ratio of the barrier resistance $R_B$ to the bulk resistance $R_j$ of each material.

We will use the Riccati parameterisation for the quasiclassical Green’s function $\hat{g}^R$:

$$\hat{g}^R = \left(\begin{array}{cc} N(1 + \gamma) & 2N\gamma \\ -2N\gamma & -N(1 + \gamma) \end{array}\right),$$

where $N = 2$ and $\gamma = h / \hbar$. This parameterisation allows for a solution of the quasiclassical equations without any assumption of a weak proximity effect.
where the normalisation matrices are \( N = (1 - \gamma y)^{-1} \) and \( \tilde{N} = (1 - \gamma y)^{-1} \). The tilde operation denotes a combination of complex conjugation \( i \rightarrow -i \) and energy \( \varepsilon \rightarrow -\varepsilon \), with \( \gamma \rightarrow \gamma \). \( N \rightarrow \tilde{N} \).

To include intrinsic SO coupling in the Usadel equation to first order, we simply have to replace all the derivatives in Eq. (1) with their gauge covariant counterparts:\(^{21}\)

\[
\nabla(\cdot) \rightarrow \tilde{\nabla}(\cdot) \equiv \nabla(\cdot) - i[\tilde{A}, \cdot].
\]

The object \( \tilde{A} \) has both a vector structure in geometric space, and a 4 × 4 matrix structure in Spin–Nambu space, and can be written as \( \tilde{A} = \text{diag}(A_x - A_y) \) in terms of the SO field \( A = (A_x, A_y, A_z) \), which will be discussed in more detail in the next subsection. When we include the SO coupling as shown above, we derive the following form for the Usadel equation (see Appendix A):

\[
D_F \left( \partial_y^2 \gamma + 2(\partial_y \gamma) \tilde{N}(\partial_y \gamma) \right) = -2i\varepsilon\gamma - ih(-\gamma y + \gamma x) + D_F \left[ A_y \gamma + \partial_y \tilde{N}(A_x + \gamma A_y) \right] + 2iD_F \left[ (\partial_y \gamma) \tilde{N}(A_y + \gamma A_y) + (A_k + \gamma A_y) N (\partial_y \gamma) \right],
\]

where the index \( k \) indicates an arbitrary choice of direction in Cartesian coordinates. The corresponding equation for \( \gamma \) is found by taking the tilde conjugate of Eq. (5).

Similarly, the boundary conditions in Eq. (2) become:

\[
\partial_x \gamma_1 = \frac{1}{L_1 \xi_1} (1 - \gamma_1 \tilde{x}) N_2 (\gamma_2 - \gamma_1) + i A_k \gamma_1 + i \tilde{x} A_k^*,
\]

\[
\partial_x \gamma_2 = \frac{1}{L_2 \xi_2} (1 - \gamma_2 \tilde{x}) N_1 (\gamma_2 - \gamma_1) + i A_k \gamma_2 + i \tilde{x} A_k^*,
\]

and the \( \gamma \) counterparts are found in the same way as before. For the details of these derivations, see Appendix A.

B. Spin-orbit field

The precise form of the generic SO field \( \tilde{A} \) is imposed by the experimental requirements and limitations. As the name suggests, spin-orbit coupling couples a particle’s spin with its motion, and more specifically its momentum. As mentioned in the Introduction, the SO coupling in solids derives from asymmetries in the crystal structure. Intrinsic inversion asymmetry arises naturally due to interfaces between materials in thin-film hybrid structures such as the ones considered herein. Noncentrosymmetric crystalline structures provide an alternative source for intrinsic asymmetry, and are considered in\(^{33}\). In thin-film hybrids, the Rashba spin splitting derives from the cross product of the Pauli vector \( \sigma \) with the momentum \( \hat{\mathbf{k}} \).

\[
H_R = -\frac{\alpha}{m}(\sigma \times \hat{\mathbf{k}}) \cdot \hat{\mathbf{z}},
\]

where \( \alpha \) is called the Rashba coefficient, and we have chosen a coordinate system with \( \hat{\mathbf{z}} \) as the layering direction. Another well-known type of SO coupling is the Dresselhaus spin splitting, which can occur when the crystal structure lacks an inversion centre. If the confining direction is \( \hat{\mathbf{z}} \), then to first order \( \langle k_x \rangle = 0 \), so the Dresselhaus splitting becomes

\[
H_D = \frac{\beta}{m}(\sigma_y k_y - \sigma_x k_x),
\]

where \( \beta \) is called the Dresselhaus coefficient. When we combine both interactions, we obtain the Hamiltonian for a general Rashba–Dresselhaus SO coupling.

\[
H_{RD} = \frac{k_z}{m}(\alpha \sigma_y - \beta \sigma_x) - \frac{k_y}{m}(\alpha \sigma_x - \beta \sigma_y).
\]

In this work, we will restrict ourselves to this experimentally common form of SO coupling. As explained in Ref. 21, the SO coupling acts as a background SU(2) field, i.e. an object with both a vector structure in geometric space, and a 2 × 2 matrix structure in spin space. We can therefore identify the interaction above with an effective vector potential \( \tilde{A} \) which we will call the SO field,

\[
H_{RD} = -\frac{k_z}{m} \cdot \tilde{A},
\]

from which we derive that

\[
\tilde{A} = (\beta \sigma_x - \alpha \sigma_y, \alpha \sigma_x - \beta \sigma_y, 0).
\]

At this point, it is convenient to introduce a new notation for describing Rashba–Dresselhaus couplings, which will let us distinguish between the physical effects that derive from the strength of the coupling, and those that derive from the geometry. For this purpose, we employ polar notation defined by the relations

\[
\alpha \equiv -a \sin \chi,
\]

\[
\beta \equiv a \cos \chi,
\]

where we will refer to \( a \) as the SO strength, and \( \chi \) as the SO angle. Rewritten in the polar notation, Eq. (11) takes the form:

\[
\tilde{A} = a(\sigma_x \cos \chi + \sigma_y \sin \chi) \hat{\mathbf{z}} - a(\sigma_x \sin \chi + \sigma_y \cos \chi) \hat{\mathbf{y}}.
\]

Note that \( A_x^2 = A_y^2 = a^2 \), which means that \( A_z^2 = 2a^2 \). Another useful property is that we can switch the components \( A_y \leftrightarrow A_x \) by letting \( \chi \rightarrow 3\pi/2 - \chi \), so it is often sufficient to focus on \( A_x \) in calculations, and obtain the corresponding expressions for \( A_y \) by symmetry. From the definition above, we can immediately conclude that \( \chi = 0 \) corresponds to a pure Dresselhaus coupling, while \( \chi = \pm \pi/2 \) results in a pure Rashba coupling, with the geometric interpretation of \( \chi \) illustrated in Fig. 4.
FIG. 4: Geometric interpretation of the SO field (13) in polar coordinates: the Hamiltonian couples the momentum component \( k_x \) to the spin component \( \sigma_x \cos \chi + \sigma_y \sin \chi \) with a coefficient \(+a/m\), and the momentum component \( k_y \) to the spin component \( \sigma_x \sin \chi + \sigma_y \cos \chi \) with a coefficient \(-a/m\). Thus, \( a \) determines the magnitude of the energy splitting, and \( \chi \) the angle between the coupled momentum and spin components.

The appearance of LRT in the system depends on the interplay between SO coupling and the direction of the exchange field. For instance, if we have an SO field component along the layering direction, e.g., if we had \( A_z \neq 0 \) in Figs. 1 and 2, achievable with a noncentrosymmetric crystal or in a nanowire setup, then a non-vanishing commutator \([A, \vec{h} \cdot \vec{\sigma}]\) creates the LRT. However, we will from now only consider systems where \( A_z = 0 \), in which case the criterion for LRT is23 that \([A, [A, \vec{h} \cdot \vec{\sigma}]]\) must not be parallel to the exchange field \( \vec{h} \cdot \vec{\sigma} \). Expanding, we have

\[
[A, [A, \vec{h} \cdot \vec{\sigma}]] = 4a^2(\vec{h} \cdot \vec{\sigma} + h_x \vec{\sigma}_z) - 4a^2(h_x \sigma_y + h_y \sigma_z) \sin 2\chi ,
\]

from which it is clear that no LRTs can be generated for a pure Dresselhaus coupling \( \chi = 0 \) or Rashba coupling \( \chi = \pm \pi/2 \) when the exchange field is in-plane. However, as we will discuss in section IIIA, the effect of SO coupling becomes increasingly significant for angles close to \( \pm \pi/4 \). We also see that no LRTs can be generated for in-plane magnetization in the special case \( h_x = h_y \) and \( h_z = 0 \), since \( h_x \sigma_y + h_y \sigma_z \) can then be rewritten as \( h_{x} \sigma_{y} + h_{y} \sigma_{z} \), which is parallel to \( \vec{h} \). There is no LRT generation for the case \( h_x = h_y = 0 \) and \( h_z \neq 0 \) for similar reasons. In general, the LRT will however appear for an in-plane magnetization as long as \( h_x \neq h_y \) and the SO coupling is not of pure Dresselhaus or pure Rashba type. Once the condition for long-range triplet generation is satisfied, increasing the corresponding exchange field will also increase the proportion of long-range triplets compared with short-range triplets (SRTs). Whether or not the presence of long-range triplets can be observed in the system, even when the criteria for their existence is fulfilled, depends on other aspects such as the strength of the spin-orbit coupling and will be discussed later in this paper.

C. Weak proximity effect

In order to establish a better analytical understanding of the role played by SO coupling in the system before presenting the spectroscopy and \( T_c \) results, we will now consider the limit of weak proximity effect, which means that \( |\eta_{ij}| \ll 1, N \approx 1 \) in the ferromagnet. The anomalous Green’s function in general is given by the upper-right block of Eq. (3), \( f = 2N\gamma \), which we see reduces to \( f = 2\gamma \) in this limit. It will also prove prudent to express the anomalous Green’s function using a singlet/triplet decomposition, where the singlet component is described by a scalar function \( f_s \), and the triplet components encapsulated in the so-called \( d \)-vector \(^{34,35} \).

\[
f = (f_s + \hat{d} \cdot \hat{\sigma}) f_y , \tag{15}
\]

Combining the above with the weak proximity identity \( f = 2\gamma \), we see that the components of \( \gamma \) can be rewritten as:

\[
\gamma = \frac{1}{2} \begin{pmatrix}
2i d_x - d_z & d_z + f_s \\
\bar{d}_z - d_x & id_y + d_y
\end{pmatrix} . \tag{16}
\]

Under spin rotations, the singlet component \( f_s \) will then transform as a scalar, while the triplet component \( d = (d_x, d_y, d_z) \) transforms as an ordinary vector. Another useful feature of this notation is that it becomes almost trivial to distinguish between short-range and long-range triplet components; the projection \( d_{\parallel} = \hat{d} \cdot \hat{h} \) along the exchange field corresponds to the SRTs, while the perpendicular part \( d_{\perp} = |d \times \hat{h}| \) describes the LRTs, where \( \hat{h} \) here denotes the unit vector of the exchange field. For a concrete example, if the exchange field is oriented along the \( z \)-axis, then \( d_{\perp} \) will be the short-range component, while both \( d_x \) and \( d_y \) are long-ranged components. In the coming sections, we will demonstrate that the LRT component can be identified from its density of states signature, as measurable by tunneling spectroscopy.

In the limit of weak proximity effect, we may linearize both the Usadel equation and Kupriyanov–Lukichev boundary conditions. Using the singlet/triplet decomposition in Eq. (16), and the Rashba–Dresselhaus coupling in Eq. (11), the linearized version of the Usadel equation can be written:

\[
\frac{i}{2} D_F \partial^2_x f_x = \epsilon f_x + \hat{h} \cdot \hat{d} , \tag{17}
\]

\[
\frac{i}{2} D_F \partial^2_y d = \epsilon d + \hat{h} f_s + 2i D_F a^2 \Omega(\chi) \hat{d} , \tag{18}
\]

where we for brevity have defined an SO interaction matrix

\[
\Omega(\chi) = \begin{pmatrix}
1 & 0 & 0 \\
-\sin 2\chi & 1 & 0 \\
0 & 0 & 2
\end{pmatrix} . \tag{19}
\]

We have now managed to condense the Usadel equation down to just two coupled differential equations for \( f_x \) and \( d \), where the coupling of course is proportional to the exchange field. Furthermore, the SO interaction term has been written as a product of a factor \( 2i D_F a^2 \), which only depends on the strength \( a \); and a factor \( \Omega(\chi)d \), which only depends on the
angle $\chi$. In other words, we have successfully separated the effects of SO coupling into a strength factor and a geometric factor. The matrix $\Omega(\chi)$ becomes diagonal for a Dresselhaus coupling with $\chi = 0$ or a Rashba coupling with $\chi = \pm \pi/2$, which implies that there is no triplet mixing for such systems. In contrast, the off-diagonal terms are maximal for $\chi = \pm \pi/4$, which suggests that the triplet mixing is maximal when the Rashba and Dresselhaus coefficients have the same magnitude. In addition to the off-diagonal triplet mixing terms, we see that the diagonal terms of $\Omega(\chi)$ essentially result in imaginary energy contributions $2iD_F\tilde{a}^2$. As we will see later, this can in some cases result in a suppression of all the triplet components in the ferromagnet.

We will now consider the case of a general exchange field in the $xy$-plane,

$$h = h\cos\theta \hat{x} + h\sin\theta \hat{y}. \quad (20)$$

Since the linearized Usadel equations show that the presence of a singlet component $f_s$ only results in the generation of triplet components along $h$, and the SO interaction term only mixes the triplet components in the $xy$-plane, the only nonzero triplet components will in this case be $d_{\parallel}$ and $d_{\perp}$. The SRT amplitude $d_{\parallel}$ and LRT amplitude $d_{\perp}$ can therefore be written:

$$d_{\parallel} = d_x \cos\theta + d_y \sin\theta, \quad (21)$$

$$d_{\perp} = -d_x \sin\theta + d_y \cos\theta. \quad (22)$$

By projecting the linearized Usadel equation for $d_{\parallel}$ along the unit vectors $x = (\cos\theta, \sin\theta, 0)$ and $y = (-\sin\theta, \cos\theta, 0)$, respectively, we then obtain separate equations for the SRTs and LRTs:

$$\frac{i}{2}D_F \partial_t^2 f_s = \varepsilon f_s + hd_{\parallel}, \quad (23)$$

$$\frac{i}{2}D_F \partial_t^2 d_{\parallel} = [\varepsilon + 2iD_F \alpha^2(1 - \sin 2\theta \sin 2\chi)]d_{\parallel} - 2iD_F \alpha^2 \cos 2\theta \sin 2\chi d_{\perp} + hf_s, \quad (24)$$

$$\frac{i}{2}D_F \partial_t^2 d_{\perp} = [\varepsilon + 2iD_F \alpha^2(1 + \sin 2\theta \sin 2\chi)]d_{\perp} - 2iD_F \alpha^2 \cos 2\theta \sin 2\chi d_{\parallel}. \quad (25)$$

These equations clearly show the interplay between the singlet component $f_s$, SRT component $d_{\parallel}$, and LRT component $d_{\perp}$. If we start with only a singlet component $f_s$, then the presence of an exchange field $h$ results in the generation of the SRT component $d_{\parallel}$. The presence of an SO field can then result in the generation of the LRT component $d_{\perp}$, where the mixing term is proportional to $\alpha^2 \cos 2\theta \sin 2\chi$. This implies that in the weak proximity limit, LRT mixing is absent for an exchange field direction $\theta = \pi/4$, corresponding to $h_x = h_y$, while it is maximized if $\theta = 0$ or $\theta = \pi$, and at the same time $\chi = \pm \pi/4$. In other words, the requirement for maximal LRT mixing is therefore that the exchange field is aligned along either the $x$-axis or $y$-axis, while the Rashba and Dresselhaus coefficients should have the same magnitude.

Moreover, these equations show another interesting consequence of having an SO field in the ferromagnet, which is unrelated to the LRT generation. Note that the effective quasiparticle energies of the SRTs and LRTs become

$$E_{\parallel} = \varepsilon + 2iD_F \alpha^2(1 - \sin 2\theta \sin 2\chi), \quad (26)$$

$$E_{\perp} = \varepsilon + 2iD_F \alpha^2(1 + \sin 2\theta \sin 2\chi). \quad (27)$$

When $\chi = \pm \pi/4$, then the SRTs are entirely unaffected by the presence of an SO coupling; the triplet mixing term vanishes for these parameters, and $E_{\parallel}$ is also clearly independent of $a$. However, when $\theta = -\chi = \pm \pi/4$, the situation is drastically different. There is still no possibility for LRT generation, however the SRT energy $E_{\parallel} = \varepsilon + 4iD_F \alpha^2$ will now obtain an imaginary energy contribution which destabilizes the SRTs. In fact, numerical simulations show that this energy shift destroys the SRT components as $a$ increases. As we will see in Section III C, this effect results in an increase in the critical temperature of the bilayer. Thus, switching between $\theta = \pm \pi/4$ in a system with $\chi \approx \pm \pi/4$ may suggest a novel method for creating a triplet spin valve.

When $\chi = \pm \pi/4$ but $\theta \neq \pm \pi/4$, the triplet mixing term proportional to $\cos 2\theta \sin 2\chi$ will no longer vanish, so we get LRT generation in the system. We can then see from the effective triplet energies that as $\theta \to \pm \arg(\chi)\pi/4$, the imaginary part of $E_{\parallel}$ vanishes, while the imaginary part of $E_{\perp}$ increases. This leads to a relative increase in the amount of SRTs compared to the amount of LRTs in the system. In contrast, as $\theta \to -\arg(\chi)\pi/4$, the imaginary part of $E_{\parallel}$ vanishes, and the imaginary part of $E_{\perp}$ increases. This means that we would expect a larger LRT generation for these parameters, up until the point where the triplet mixing term $\cos 2\theta \sin 2\chi$ becomes so small that almost no LRTs are generated at all. The ratio of effective triplet energies at the center of the gap $\varepsilon = 0$ can be written

$$\frac{E_{\perp}(0)}{E_{\parallel}(0)} = \frac{1 + \sin 2\theta \sin 2\chi}{1 - \sin 2\theta \sin 2\chi}. \quad (28)$$

D. Density of states

The density of states $D(\varepsilon)$ containing all spin components can be written in terms of the Riccati matrices as

$$D(\varepsilon) = \text{Tr}[(N(1 + \gamma \tilde{\Pi})/2, \quad (29)$$

which for the case of zero energy can be written concisely in terms of the singlet component $f_s$ and triplet components $d_{\parallel}$, $d_{\perp}$.

$$D(0) = 1 - |f_s(0)|^2 + |d_{\parallel}(0)|^2 + |d_{\perp}(0)|^2/2. \quad (30)$$

So the singlet and triplet components are directly competing to lower and raise the density of states. Furthermore, since we are primarily interested in the proximity effect in the ferromagnetic film, we will begin by using the known BCS bulk solution in the superconductor,

$$\tilde{\tilde{g}}_{\text{BCS}} = \begin{pmatrix} \cosh(\theta) & 0 & \sinh(\theta) & 0 \\ 0 & \cosh(\theta) & 0 & \sinh(\theta) \\ \sinh(\theta) & 0 & -\cosh(\theta) & 0 \\ 0 & \sinh(\theta) & 0 & -\cosh(\theta) \end{pmatrix}, \quad (31)$$

where $\theta = \text{atanh}(\Delta/\varepsilon)$, and $\phi$ is the superconducting phase. Using Eq. (16) and the definition of the tilde operation, and
comparing $g^R$ in Eq. (3) with its standard expression in a bulk superconductor Eq. (31), we can see that at zero energy the singlet component $f_{s}(0)$ must be purely imaginary and the asymmetric triplet $d_{c}(0)$ must be purely real if the superconducting phase $\phi = 0$.

By inspection of Eq. (18), we can see that a transformation $h_x \leftrightarrow h_y$, along with $d_x \leftrightarrow d_y$, leaves the equations invariant, as would be expected for a one-dimensional system aligned along the $z$-axis. The density of states, which is given by Eq. (30), will therefore be unaffected by such permutations,

$$D[h = (a, b, 0)] = D[h = (b, a, 0)] ,$$

while in general

$$D[h = (a, 0, b)] \neq D[h = (b, 0, a)] .$$

However, whenever one component of the planar field is exactly twice the value of the other component, one can confirm that the linearized equations remain invariant under a rotation of the exchange field

$$h = (a, 2a, 0) \rightarrow h = (a, 0, 2a) ,$$

with associated invariance in the density of states.

E. Critical temperature

When superconducting correlations leak from a superconductor and into a ferromagnet in a hybrid structure, there will also be an inverse effect, where the ferromagnet effectively drains the superconductor of its special properties. Physically, this effect is observable in the form of a reduction in the superconducting gap $\Delta(z)$ near the interface at all temperatures. Furthermore, if the temperature of the hybrid structure is somewhat close to the critical temperature $T_c$ of the superconductor, this inverse proximity effect can be strong enough to make the superconducting correlations vanish entirely throughout the system. Thus, proximity-coupled hybrid structures will in practice always have a critical temperature $T_c$ of a bulk superconductor. Depending on the exact parameters of the hybrid system, $T_c$ can sometimes be significantly smaller than $T_{cs}$, and in some cases it may even vanish ($T_c \rightarrow 0$).

To quantify this effect, it is no longer sufficient to solve the Usadel equation in the ferromagnet only. We will now also have to solve the Usadel equation in the superconductor,

$$D_0 \partial_0^2 \gamma = -2ie\gamma + \Delta(\sigma_y - \gamma \sigma_x \gamma) - 2(\partial_\gamma \gamma) N_\gamma(\partial_\gamma \gamma) ,$$

along with a self-consistency equation for the gap $\Delta(z)$,

$$\Delta(z) = N_0 \lambda \int_0^{\Delta_c \cosh(1/N_0 \lambda)} d\epsilon \Re \{ f_s(z, \epsilon) \} \tanh \left( \frac{\pi \epsilon / \Delta_0}{2 \gamma \epsilon T/T_{cs}} \right) ,$$

where $N_0$ is the density of states per spin at the Fermi level, and $\lambda < 0$ is the electron-electron coupling constant in the BCS theory of superconductivity. For a derivation of the gap equation, see Appendix B.

To study the effects of the SO coupling on the critical temperature of an SF structure, we therefore have to solve Eq. (5) self-consistently in the ferromagnet, Eq. (6) at the interface, Eqs. (35) and (36) in the superconductor. In practice, this is done by successively solving one of the equations at a time numerically, and continuing the procedure until the system converges towards a self-consistent solution. To obtain accurate results, we typically have to solve the Usadel equation for 100–150 positions in each material, around 500 energies in the range $(0, 2\Delta_0)$, and 100 more energies in the range $(2\Delta_0, \alpha_k)$, where the Debye cutoff $\alpha_k \approx 76\Delta_0$ for the superconductors considered herein. This procedure will then have to be repeated up to several hundred times before we obtain a self-consistent solution for any given temperature of the system. Furthermore, if we perform a conventional linear search for the critical temperature $T_c/T_{cs}$ in the range $(0, 1)$ with a precision of 0.002, it may require up to 500 such iterations to complete, which may take several days depending on the available hardware and efficiency of the implementation. The speed of this procedure may, however, be significantly increased by performing a binary search instead. Using this strategy, the critical temperature can be determined to a precision of $1/2^{k+1} \approx 0.002$ after only 8 iterations, which is a significant improvement. The convergence can be further accelerated by exploiting the fact that $\Delta(z)$ from iteration to iteration decreases monotonically to zero if $T > T_c$; however, the details will not be further discussed in this paper.

III. RESULTS

We consider the proximity effect in an SF bilayer in III A, and the corresponding SFS Josephson junction in III B, using the BCS bulk solution for the superconductors. We take the thin-film layering direction to be oriented in the $z$-direction and fix the spin-orbit coupling to Rashba–Dresselhaus type in the $xy$-plane as given by Eq. (11). For these calculations, we choose the superconductor coherence length to be $\xi_0 = 30$ nm, and use a ferromagnet of length $L_F = 15$ nm. We solve the equations using MATLAB with the boundary value differential equation package bvp6c and examine the density of states $D(\epsilon)$ for energies normalised to the superconducting gap $\Delta$.

For brevity of notation, we include the normalization factor, which is taken to be the length of the ferromagnetic region $L_F$, of the SO coupling in the $\alpha$ and $\beta$ coefficients themselves, so that for instance $\alpha = 1$ in the figure legends means $\alpha L_F = 1$. Finally, in Section III C, we calculate the dependence of the critical temperature of an SF structure as a function of the different system parameters.

A. SF Bilayer

Consider the SF bilayer depicted in Fig. 1. From (14) it is clear that no LRTs will be generated if the finite exchange field is aligned with the layering direction, i.e. $h = h^z \neq 0$, since the
vector (14) will be parallel to the exchange field. In this case, increasing the SO coupling of either or both the Rashba and Dresselhaus terms will suppress the triplet components of the density of states, reproducing the well-known spectroscopic singlet minigap proximity effect, as shown in Fig. 6.

With either \( h = h_x \neq 0 \), or equivalently \( h = h_y \neq 0 \), LRTs are generated provided \( \alpha \beta \neq 0 \), and in Fig. 7 we can see that the addition of SO coupling introduces a peak in the density of states at zero energy, which saturates for a certain coupling strength. By analysing the real components of the triplets, for a gauge where the superconducting phase is zero, we see that this zero-energy peak is due to the LRT component, in this case \( d_z \), also depicted in Fig. 7. It is also evident from comparing \( h = (0, \Delta, 0) \) with \( h = (0, 3\Delta, 0) \) that increasing the field strength rapidly suppresses the density of states towards that of the normal metal.

Recall that the LRT component is defined as the component of \( d_\parallel \) that is perpendicular to \( \hat{h} \). When \( \hat{h} \) has finite values in two directions, for example produced by a rotated field, the generation of LRTs is thus strongly dependent on the angle to the field, which is a tunable parameter for sufficiently weak magnetic anisotropy. The perpendicular component \( d_\perp \) can be found easily from the cross product of the two vectors:

\[
d_\perp = [d \times \hat{h}],
\]

and in all cases we see that the LRT component generates a peak in the density of states at zero energy provided \( \alpha \beta \neq 0 \) and \( h_x \neq h_y \), with an example of the correlation shown in Fig. 8.

Comparing the cases \( h = (6\Delta, 3\Delta, 0) \) in Fig. 8, with \( h = (0, 3\Delta, 6\Delta) \) in Fig. 9, we see that the two cases are identical, as predicted in the limit of weak proximity effect. However, by comparing the effect of increasing the exchange field \( \hat{h} = \Delta \hat{y} \rightarrow 3\Delta \hat{y} \) in Fig. 7 with the effect of increasing the \( \gamma \)-component of \( \hat{h} = (6\Delta, 3\Delta, 0) \rightarrow (6\Delta, 5\Delta, 0) \), as shown in Fig. 8, we see that with the field in a single direction, increasing the field strength rapidly suppresses the SO-induced peak at zero energy, whereas we appear to see the opposite with a rotated field. In this case increasing one component of the field leads to an enhanced peak at zero energy beyond the peak saturation point for lower fields. In fact, it is interesting to consider how the density of states depends on the angle \( \theta \) between \( h_x \) and \( h_y \) at zero energy. With \( \theta = 0 \) the field is aligned with \( h_x \), and with \( \theta = \pi/2 \) it is aligned with \( h_y \). In Fig. 5 we see that the inclusion of SO coupling introduces a nonmonotonic angular dependence in the density of states, with increasingly sharp features as the SO coupling strength increases, although the optimal angle at approximately \( \theta = 7\pi/32 \) and \( \theta = 9\pi/32 \) varies minimally with increasing SO coupling. Clearly the ability to extract maximum LRT conversion from the inclusion of SO coupling is highly sensitive to the rotation angle, with near step-function behaviour delineating the regions of optimal peak in the density of states and an energy gap for strong SO coupling. It is remarkable to see how \( D(0) \) vs. \( \theta \) formally bears a strong resemblance to the evolution of a fully gapped BCS density of states \( D(\epsilon) \) vs. \( \epsilon \) to a flat density of states as the SO coupling decreases.
\( h = (0, 0, 3\Delta) \)

**FIG. 6:** Density of states \( D(\varepsilon) \) for the SF bilayer with energies normalised to the superconducting gap \( \Delta \) and SO coupling normalised to the inverse ferromagnet length \( 1/L_F \). The table shows the spectroscopic effect of increasing SO with \( \alpha = \beta \) and increasing difference between Rashba–Dresselhaus coefficients when the magnetisation \( h = (0, 0, 3\Delta) \), i.e. with the field perpendicular to the interface. In this case no long-range triplets are present.

| \( h \) | \( D(\varepsilon) \) | \( |\text{Re}(d_x)| \) | \( |\text{Re}(d_y)| \) |
| --- | --- | --- | --- |
| \( (0,\Delta,0) \) | ![Graph](image1.png) | ![Graph](image2.png) | ![Graph](image3.png) |
| \( (0,3\Delta,0) \) | ![Graph](image4.png) | ![Graph](image5.png) | ![Graph](image6.png) |

**FIG. 7:** Density of states \( D(\varepsilon) \) for the SF bilayer with energies normalised to the superconducting gap \( \Delta \) and SO coupling normalised to the inverse ferromagnet length \( 1/L_F \). The table shows the spectroscopic effect of equal Rashba–Dresselhaus coefficients when the magnetisation is oriented entirely in the \( y \)-direction, and also the correlation between the SO-induced zero-energy peak with the long-range triplet component \( |\text{Re}(d_x)| \equiv |\text{Re}(d_\perp)| \). It is clear that the predominant effect of the LRT component, which appears only when the SO coupling is included, is to increase the peak at zero energies. Increasing the field strength rapidly suppresses the density of states towards that of the normal metal.
FIG. 8: Density of states $D(\varepsilon)$ in the SF bilayer for energies normalised to the superconducting gap $\Delta$ and SO coupling normalised to the inverse ferromagnet length $1/L_F$. The table shows the spectroscopic features of the SF bilayer with rotated exchange field in the $xy$-plane. Again we see a peak in the density of states at zero energy due to the LRT component, i.e. the component of $\hat{d}$ perpendicular to $\hat{h}$, $d_\perp$. The height of this zero-energy peak is strongly dependent on the angle of the field vector in the plane, as shown in Fig. 5. For near-optimal field orientations increasing the SO coupling leads to a dramatic increase in the peak of the density of states at zero energy.

| $\hat{h}$ | $\Re(d_\perp)$ for $\hat{h} = (6\Delta, 5\Delta, 0)$ |
|-----------|-------------------------------------------------|
| $h = (6\Delta, 3\Delta, 0)$ | ![Graph](image1)
| $h = (6\Delta, 5\Delta, 0)$ | ![Graph](image2)
| $\alpha = \beta = 0$ | ![Graph](image3)
| $\alpha = \beta = 0.5$ | ![Graph](image4)
| $\alpha = \beta = 1$ | ![Graph](image5)
| $\alpha = \beta = 2$ | ![Graph](image6)
| $\alpha = \beta = 5$ |

FIG. 9: Density of states $D(\varepsilon)$ in the SF bilayer for energies normalised to the superconducting gap $\Delta$ and SO coupling normalised to the inverse ferromagnet length $1/L_F$. The table shows the spectroscopic features of the SF bilayer with a rotated exchange field in the $xz \equiv yz$-plane. Note that when the field component along the junction is twice the component in the $y$-direction, here $\hat{h} = (0, 3\Delta, 6\Delta)$, the density of states is equivalent to the case $\hat{h} = (6\Delta, 3\Delta, 0)$ illustrated in Fig. 8, as predicted in the limit of weak proximity effect.

| $\hat{h}$ | $\Re(d_\perp)$ for $\hat{h} = (0, 6\Delta, 3\Delta)$ |
|-----------|-------------------------------------------------|
| $h = (0, 3\Delta, 3\Delta)$ | ![Graph](image7)
| $h = (0, 3\Delta, 6\Delta)$ | ![Graph](image8)
| $h = (0, 6\Delta, 3\Delta)$ | ![Graph](image9)
| $\alpha = \beta = 0$ | ![Graph](image10)
| $\alpha = \beta = 0.1$ | ![Graph](image11)
| $\alpha = \beta = 0.5$ | ![Graph](image12)
| $\alpha = \beta = 1$ | ![Graph](image13)
| $\alpha = \beta = 2$ |

Where $\varepsilon/\Delta$, $D(\varepsilon)$, and $\Re(d_\perp)$ refer to the energy normalised to the superconducting gap $\Delta$, the density of states, and the real part of $d_\perp$, respectively.
B. Josephson junction

By adding a superconducting region to the SF bilayer we form an SFS Josephson junction, and it is well known that the phase difference between the superconducting regions governs how much current can flow through the junction, and the density of states for a diffusive SNS junction has been measured experimentally with extremely high precision. Here we consider such a transversal junction structure as depicted in Fig. 2, again with intrinsic SO coupling in the \( xy \)-plane (11) in the ferromagnet and with BCS bulk values for each superconductor. In III B 1 we consider single orientations of the uniform exchange field and in III B 2 we consider a rotated field.

Let us first recapitulate some known results. Without an exchange field the density of states is unaffected by the SO coupling. This is because without an exchange field the equations governing the singlet and triplet components are decoupled and thus no singlet-triplet conversion can occur. Moreover, we know that with a phase difference of \( \pi \) between the two superconductors the density of states should be that of a normal metal; identically one. Without SO coupling and as long as the exchange field remains weak, changing the phase difference can qualitatively alter the density of states from minigap to peak at zero energy (see Fig. 10), a highly useful feature permitting external control of the quasiparticle current flowing through the junction. Beyond a system specific threshold strength of exchange field the minigap can no longer be sustained and increasing the phase difference simply lowers the density of states towards that of the normal metal. Amongst the features we outline below, one of the effects of adding SO coupling is to make this useful gap-to-peak effect accessible with stronger exchange fields, i.e. for a greater range of materials. However, the gain is restricted, since increasing the gap with strong SO coupling also destroys the capability for qualitative change.

1. Josephson junction with uniform exchange field in single direction

Consider first the case in which the exchange field is aligned in a single direction. If we again restrict the form of the SO-vector to (11), aligning \( \mathbf{h} \) in the \( z \)-direction will not result in any LRTs. In this case the spectroscopic effect of the SO coupling is dictated by the singlet and short-range features of creating and widening the minigap, up to a threshold at which the well is square, much as in the SF bilayer case (Fig. 6). We will now examine the effect of increasing the exchange field aligned in the \( x \)- or, equivalently, the \( y \)-direction. If \( \mathbf{h} \) is sufficiently weak to sustain a gap independently of SO coupling, introducing weak SO coupling will increase the gap at zero phase difference while maintaining a peak at zero energy for a phase difference of \( 0.75\pi \) (see Fig. 10). Increasing the SO coupling increases this peak at zero energy. As the exchange field increases sufficiently to keep the gap closed, increasing the SO coupling increases the zero-energy peak at all phases, again due to the LRT component, reaching a saturation point before short-range features dominate. In this case, increasing the phase difference \( \phi = 0 \rightarrow \pi \) reduces the height of the density of states towards that of the normal metal and increasingly closes the double well, closing entirely at \( \phi = \pi \). As the value of the density of states at zero energy saturates for increasing SO coupling, fixed phase differences yield the same drop at zero energy regardless of the strength of SO coupling.

We note in passing that when the SO coupling field has a component along the junction direction (\( z \)), it can qualitatively influence the nature of the superconducting proximity effect. As very recently shown in Ref. 33, a giant triplet proximity effect develops at \( \phi = \pi \) in this case, in complete contrast to the normal scenario of a vanishing proximity effect in \( \pi \)-biased junctions.

2. Josephson junction with rotated exchange field

With two components of the field \( \mathbf{h} \), e.g. from rotation, we must again separate the cases with and without a component along the junction direction. For \( h_x \) aligned in \( x \) and \( y \), and provided we satisfy the conditions \( h_x \neq h_z \) and \( \alpha \beta \neq 0 \), increasing the SO coupling drastically increases the zero energy peak as shown in Fig. 11, again due to the LRT component. As the phase difference approaches \( \pi \), the peak reduces and the double well closes.

With one component of the exchange field along the junction and another along either \( x \) or \( y \), a phase-dictated gap-to-peak transition at zero energy is possible with stronger fields than with the field aligned in a single direction, as shown in Fig. 11. Notice that in this case increasing the phase difference \( \phi = 0 \rightarrow 0.5\pi \) gives an increase in the peak at zero energy before reducing towards the normal metal state. For higher field strengths we find once again that increasing the SO coupling increases the peak at zero energy, up to a system-specific threshold, and increasing phase difference reduces the density of states towards that of the normal metal.
| $\alpha$, $\beta$ | $h = (0, 0, \Delta)$ | $h = (0, \Delta, 0)$ | $h = (0, 0, 3\Delta)$ | $h = (0, 3\Delta, 0)$ |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0               | ![Graph](image1) | ![Graph](image2) | ![Graph](image3) | ![Graph](image4) |
| 0.1             | ![Graph](image5) | ![Graph](image6) | ![Graph](image7) | ![Graph](image8) |
| 0.5             | ![Graph](image9) | ![Graph](image10) | ![Graph](image11) | ![Graph](image12) |
| 1               | ![Graph](image13) | ![Graph](image14) | ![Graph](image15) | ![Graph](image16) |
| 2               | ![Graph](image17) | ![Graph](image18) | ![Graph](image19) | ![Graph](image20) |

**FIG. 10:** The table shows the density of states $D(\epsilon)$ in the SFS junction with increasing SO coupling and exchange field in a single direction, with $D(\epsilon)$ normalised to the superconducting gap $\Delta$ and SO coupling normalised to the ferromagnet length $L_F$. With no SO coupling and very weak exchange field we see a phase-dictated gap-to-peak qualitative change in the density of states at zero energy. When the field is strong enough to destroy this gap, increasing the phase difference simply lowers the density of states towards that of the normal metal, which is achieved at a phase difference of $\phi = \pi$. With the addition of SO coupling we see a clear difference in the density of states due to the long range triplet component, which is present when the field is oriented in $y$ but not in $z$. When LRTs are present with weak exchange fields, a phase-dictated gap-to-peak feature is retained and increased as the strength of SO coupling increases the gap, with the peak shown here at a phase difference of $0.75\pi$. For stronger exchange fields, increasing the SO coupling produces the minigap when there is no LRT component, whereas the existence of an LRT component again introduces an increasing peak at zero energy when no minigap is present.
The table shows the spectroscopic effects of increasing SO coupling in SFS with rotated exchange field. In the absence of SO coupling, the density of states is flat and featureless at low energies. Increasing the SO coupling again leads to a dramatic increase in the peak of the density of states at zero energy, while increasing the phase difference reduces the peak, diffusing into side-peaks for higher SO coupling strengths. With a component of the field in the junction direction a well-to-peak qualitative change in the density of states at zero energy can be achieved by altering the phase difference between the superconductors. This qualitative change can therefore occur in the presence of stronger exchange fields when strong SO coupling is included. Increasing the exchange field destroys the ability to maintain a gap or well in the density of states and the LRT component of the SO coupling increases the zero-energy peak as it did in the bilayer case.

**FIG. 11**: Density of states $D(\varepsilon)$ in the SFS junction for energies normalised to the superconducting gap $\Delta$ and SO coupling normalised to the ferromagnet length $L_F$. The table shows the spectroscopic effects of increasing SO coupling in SFS with rotated exchange field. In the absence of SO coupling, the density of states is flat and featureless at low energies. Increasing the SO coupling again leads to a dramatic increase in the peak of the density of states at zero energy, while increasing the phase difference reduces the peak, diffusing into side-peaks for higher SO coupling strengths. With a component of the field in the junction direction a well-to-peak qualitative change in the density of states at zero energy can be achieved by altering the phase difference between the superconductors. This qualitative change can therefore occur in the presence of stronger exchange fields when strong SO coupling is included. Increasing the exchange field destroys the ability to maintain a gap or well in the density of states and the LRT component of the SO coupling increases the zero-energy peak as it did in the bilayer case.
C. Critical temperature

In this section, we present numerical results for the critical temperature $T_c$ of an SF bilayer. The theory behind these investigations is summarized in Section II,E, and discussed in more detail in Appendix B. An overview of the physical system is given in Fig. 3. In all of the simulations we performed, we used the material parameter $N_0|\lambda|=0.2$ for the superconductor, the exchange field $h=10\Delta_0$ for the ferromagnet, and the interface parameter $\zeta=3$ for both materials. The other physical parameters are expressed in a dimensionless form, with lengths measured relative to the superconducting correlation length $\xi_S$, energies measured relative to the bulk zero-temperature gap $\Delta_0$, and temperatures measured relative to the bulk critical temperature $T_{c_0}$. This includes the SO coupling strength $a$, which is expressed in the dimensionless form $a\xi_S$. The plots presented in this subsection were generated from 12–36 data points per curve, where each data point has a numerical precision of 0.002 in $T_c/T_{c_0}$. The results were smoothed with a LOESS algorithm.

The first result that we present in this section is a plot of the critical temperature as a function of the size of the superconductor, as shown in Fig. 12.

First of all, we see that the critical temperature drops to zero when $L_S/\xi_S \leq 0.5$. This observation is hardly surprising; since the superconducting correlation length $\xi_S$ can roughly be interpreted as the size of a Cooper pair, these results mean that we need to fit at least half a Cooper pair inside the superconductor for superconductivity to manifest. After this, the critical temperature increases very rapidly, already reaching nearly 50% of the bulk value when $L_S/\xi_S = 0.6$, demonstrating that the superconductivity of the system is clearly very sensitive to small changes in parameters for this region.

The next step is then to observe how the behaviour of the system varies with the size of the ferromagnet, and these results are presented in Fig. 13.

We again observe that the critical temperature increases with the size of the superconductor, and decreases with the size of the ferromagnet. The critical temperature for a superconductor with $L_S/\xi_S = 0.525$ drops to zero at $L_F/\xi_S \approx 0.6$, and stays that way as the size of the ferromagnet increases. Thus we do not observe any nonmonotonic behaviour, such as reentrant superconductivity, for our choice of parameters. This is consistent with the results of Fominov et al., who only reported such behaviour for systems where either the interface parameter or the exchange field is drastically smaller than for the bilayers considered herein.39

Fig. 14 shows plots of the critical temperature as a function of the SO angle $\chi$ for an exchange field in the $z$-direction. The critical temperature is here independent of the SO angle $\chi$. This result is reasonable, since the SO coupling is in the $xy$-plane, which is perpendicular to the exchange field for this geometry. We also observe a noticeable increase in critical temperature for larger values of $a$. This behaviour can be explained using the linearized Usadel equation. According to Eq. (18), the effective energy $E_z$ of the triplet component in the $z$-direction becomes

$$E_z = \varepsilon + 4iD_F a^2; \quad (37)$$

so in other words, the SRTs obtain an imaginary energy shift proportional to $a^2$. However, as shown in Eq. (17), there is no corresponding shift in the energy of the singlet component. This effect reduces the triplet components relative to the singlet component in the ferromagnet, and as the triplet proximity channel is suppressed the critical temperature becomes restored to higher values.

FIG. 12: Plot of the critical temperature $T_c/T_{c_0}$ as a function of the length $L_S/\xi_S$ of the superconductor for $a=0$.

FIG. 13: Plot of the critical temperature $T_c/T_{c_0}$ as a function of the ferromagnet length $L_F/\xi_S$ for $a=0$.
The same situation for an exchange field along the x-axis is shown in Fig. 15. For this geometry, we observe a somewhat smaller critical temperature for all \( a > 0 \) and all \( \chi \) compared to Fig. 14. This can again be explained by considering the linearized Usadel equation in the ferromagnet, which suggests that the effective energy \( E_s \) of the triplet component in the x-direction should be

\[
E_s = \varepsilon + 2iD_F a^2,
\]

which has a smaller imaginary part than the corresponding equation for \( E_t \). Furthermore, note the drop in critical temperature as \( \chi \to \pm \pi/4 \). Since the linearized equations contain a triplet mixing term proportional to \( \sin 2\chi \), which is maximal precisely when \( \chi = \pm \pi/4 \), these are also the geometries for which we expect a maximal LRT generation. Thus, this decrease in critical temperature near \( \chi = \pm \pi/4 \) can be explained by a net conversion of singlet components to LRTs in the system, which has an adverse effect on the singlet amplitude in the superconductor, and therefore the critical temperature.

In Fig. 16 we present the results for a varying exchange field \( h \sim \cos \theta \hat{\mathbf{x}} + \sin \theta \hat{\mathbf{y}} \) in the xy-plane. In this case, we observe particularly interesting behaviour: the critical temperature has extrema at \(|\chi| = |\theta| = \pi/4\), where the extremum is a maximum if \( \theta \) and \( \chi \) have the same sign, and a minimum if they have opposite signs. Since \( \theta = \pm \pi/4 \) is precisely the geometries for which we do not expect any LRT generation, triplet mixing cannot be the source of this behaviour. For the choice of physical parameters chosen here, this effect results in a striking trough-to-peak increase in critical temperature of nearly 60%.

Instead, these observations may be explained using the theory developed in Section II. When we have a general exchange field and SO field in the xy-plane, Eq. (26) reveals that the effective energy of the SRT component is

\[
E_{\parallel} = \varepsilon + 2iD_F a^2(1 - \sin \theta \sin 2\chi).
\]

Since the factor \( 1 - \sin \theta \sin 2\chi \) vanishes for \( \theta = \chi = \pm \pi/4 \), we get \( E_{\parallel} = \varepsilon \) for this case. This geometry is also one where we do not expect any LRT generation, since the triplet mixing factor \( \cos \theta \sin 2\chi = 0 \), so the conclusion is that the SO coupling has no effect on the behaviour of SRTs for these parameters—at least according to the linearized equations. However, since \( 1 - \sin \theta \sin 2\chi = 2 \) for \( \theta = -\chi = \pm \pi/4 \), the situation is now dramatically different. The SRT effective energy is now \( E_{\parallel} = \varepsilon + 4iD_F a^2 \), with an imaginary contribution which again destabilizes the SRTs, and increases the critical temperature of the system.

D. Triplet spin-valve effect with a single ferromagnet

The results discussed in the previous section show that the critical temperature can be controlled via the magnetization direction of one single ferromagnetic layer. This is a new result originating from the presence of SO coupling. In conventional SF structures, \( T_c \) is independent of the magnetization orientation of the F layer. By using a spin-valve setup such as FSF\(^{12-16}\), it has been shown that the relative magnetization configuration between the ferromagnetic layers will tune the
$T_c$ of the system. In contrast, in our case such a spin-valve effect can be obtained with a single ferromagnet (see Fig. 16): by rotating the magnetization an angle $\pi/2$, $T_c$ goes from a maximum to a minimum. The fact that only a single ferromagnet is required to achieve this effect is of practical importance since it can be challenging to control the relative magnetization orientation in magnetic multilayered structures.

### IV. SUMMARY AND DISCUSSION

It was pointed out in Ref. 21 that for the case of transversal structures as depicted in Fig. 2, pure Rashba or pure Dresselhaus coupling and arbitrary magnetisation direction are insufficient for long range triplets to exist. Here we provide a detailed exposition of the density of states for both the SF bilayer and SFS junction with SO coupling, highlighting in particular the spectroscopic signature of long range triplets. When the conditions for LRTs are satisfied, we see that the LRT component is responsible for inducing a large increase in the peak of the density of states at zero energy. When the exchange field lies in the $xy$-plane, this effect depends nonmonotonically on the angle between the $x$ and $y$ components. In addition to the large peak at zero energy, we see that by carefully choosing the SO coupling and exchange field strengths in the Josephson junction it is again possible to control the qualitative features of the density of states by altering the phase difference between the two superconductors.

The intrinsic SO coupling present in the structures considered herein derives from their lack of inversion symmetry due to the junction interfaces, so-called interfacial asymmetry, and we restricted the form of this coupling to the experimentally common and, in some cases, tunable Rashba-Dresselhaus form. However, a lack of inversion symmetry can also derive from intrinsic noncentrosymmetry of a crystal. This could in principle be utilised to provide a component of the SO-field in the junction direction, but to date such materials have not been explored in experiments in the context of SF hybrid materials. However, analytic and numerical data suggest that these materials could have significant importance for spintronic applications making use of a large triplet Cooper pair population\(^\text{33}\).

The current analysis pertains to thin film ferromagnets. Upon increasing the length of ferromagnetic film one will increase the relative proportions of long-range to short-range triplets in the middle of the ferromagnet, but there is a corresponding penalty in sensitivity to the exchange field. For very strong ferromagnets the quasiclassical Usadel formalism may no longer describe the system behaviour appropriately, since it assumes that the impurity scattering rate is much larger than the other energy scales involved, and the Eilenberger equation should be used instead\(^\text{40}\).

In the previous section, we also observed that the presence of SO coupling will in many cases increase the critical temperature of a hybrid structure. This effect is explained through an increasing imaginary part in the quasiparticle energy, which destabilizes the triplet pairs and closes that proximity channel. However, for the special geometry $\theta = -\chi = \pm \pi/4$, the linearized equations suggest that the SRTs are unaffected by the presence of SO coupling, and this is consistent with the numerical results. We also note that for the geometries with a large LRT generation, such as $\theta = 0$ and $\chi = \pm \pi/4$, the LRT generation reduces the critical temperature again. Thus, for the physical parameters considered herein, we see that there is a very slight increase in critical temperature for these geometries, but not as large as for the geometries without LRT generation.

One particularly striking result from the critical temperature calculations is that we for the geometries $\chi = \pm \pi/4$ observe that the critical temperature changes by as much as 60% of its trough value as we change $\theta = -\pi/4$ to $\theta = +\pi/4$, i.e. by a 90° rotation of the magnetic field. This implies that it is possible to create a novel kind of triplet spin valve using an SF bilayer, where the ferromagnet has a homogeneous exchange field and Rashba–Dresselhaus coupling. This is in contrast to previous suggestions for triplet spin valves, such as the one described by Fominov et al., which have required trilayers with different homogeneous ferromagnets.\(^\text{41}\) The construction of such a device is likely to have possible applications in the emerging field of superconducting spintronics.

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Appendix A: Riccati parametrization of the Usadel equation and Kupriyanov–Lukichev boundary conditions

The $4 \times 4$ components of the retarded Green’s function $\hat{g}$ are not entirely independent, but can be expressed as

$$
\hat{g}(z, \varepsilon) = \begin{pmatrix}
N & 0 \\
0 & -\tilde{N}
\end{pmatrix} \begin{pmatrix}
1 + \gamma \tilde{\gamma} & 2\gamma \\
2\tilde{\gamma} & 1 + \gamma \tilde{\gamma}
\end{pmatrix},
$$
(A1)

which suggests that the notation can be simplified by introducing the tilde conjugation

$$
\tilde{\hat{g}}(z, +\varepsilon) = \hat{g}^*(z, -\varepsilon).
$$
(A2)

Moreover, the normalization condition $\hat{g}^2 = 1$ further constrains the possible form of $\hat{g}$ by relating the $g$ components to the $f$ components,

$$
gg - f\tilde{f} = 1, \quad gf - f\tilde{g} = 0.
$$
(A3)

Remarkably, if we pick a suitable parametrization of $\hat{g}$, which automatically satisfies the symmetry and normalization requirements above, then both the Usadel equation and the Kupriyanov–Lukichev boundary conditions can be reduced from $4 \times 4$ to $2 \times 2$ matrix equations. In this paper, we employ the so-called Riccati parametrization for this purpose, which is defined by

$$
\hat{g} = \begin{pmatrix}
N & 0 \\
0 & -\tilde{N}
\end{pmatrix} \begin{pmatrix}
1 & \gamma \tilde{\gamma} \\
\gamma \tilde{\gamma} & 1
\end{pmatrix},
$$
(A4)

where the normalization matrices are $N \equiv (1 - \gamma \tilde{\gamma})^{-1}$ and $\tilde{N} \equiv (1 - \tilde{\gamma} \gamma)^{-1}$. Solving the Riccati parametrized equations for the function $\gamma(z, \varepsilon)$ in spin space is then sufficient to uniquely construct the whole Green’s function $\hat{g}(z, \varepsilon)$. It is noteworthy that $\hat{g} \to 1$ when $\gamma \to 0$, while the elements of $\hat{g}$ diverge to infinity when $\gamma \to 1$; so we see that a finite range of variation in $\gamma$ parametrizes an infinite range of variation in $\hat{g}$.

We begin by deriving some basic identities, starting with the inverses of the two matrix products $N\gamma$ and $\gamma\tilde{N}$:

$$(N\gamma)^{-1} = \gamma^{-1}N^{-1} = \gamma^{-1}(1 - \gamma \tilde{\gamma}) = \gamma^{-1} - \gamma;$$
(A5)

$$((\gamma\tilde{N})^{-1})^-1 = \tilde{N}^{-1}\gamma^{-1} = (1 - \gamma \tilde{\gamma})\gamma^{-1} = \gamma^{-1} - \gamma.$$  
(A6)

By comparison of the results above, we see that $N\gamma = \gamma\tilde{N}$. Similar calculations for other combinations of the Riccati matrices reveal that we can always move normalization matrices past gamma matrices if we also perform a tilde conjugation in the process:

$$N\gamma = \gamma\tilde{N}, \quad \tilde{N}^{-1} = \gamma N, \quad N\gamma = \gamma\tilde{N}, \quad \tilde{N}^{-1} = \gamma N.$$  
(A7)

Since we intend to parametrize a differential equation, we should also try to relate the derivatives of the Riccati matrices. This can be done by differentiating the definition of $N$ using the matrix version of the chain rule:

$$
\partial_z N = \partial_z (1 - \gamma \tilde{\gamma})^{-1} = -\gamma \tilde{\gamma} (1 - \gamma \tilde{\gamma})^{-1} \partial_z (1 - \gamma \tilde{\gamma}) (1 - \gamma \tilde{\gamma})^{-1} = N (\partial_z \gamma \tilde{\gamma} + \gamma (\partial_z \tilde{\gamma})) (1 - \gamma \tilde{\gamma})^{-1} = N [\partial_z \gamma \tilde{\gamma} + \gamma (\partial_z \tilde{\gamma})].
$$
(A8)

Performing a tilde conjugation of the equation above, we get a similar result for $\partial_z \tilde{N}$. So the derivatives of the normalization matrices satisfy the following identities:

$$
\partial_z N = N [(\partial_z \gamma) \tilde{\gamma} + \gamma (\partial_z \tilde{\gamma})], \quad \partial_z \tilde{N} = \tilde{N} [(\partial_z \gamma) \gamma + \gamma (\partial_z \gamma)]
$$
(A9)

In addition to the identities derived above, one should note that the definition of the normalization matrix $N = (1 - \gamma \tilde{\gamma})^{-1}$ can be rewritten in many forms which may be of use when simplifying Riccati parametrized expressions; examples of this include $\gamma \tilde{\gamma} = 1 - N^{-1}$ and $1 = N - N\gamma\tilde{\gamma}$.

Now that the basic identities are in place, it is time to parametrize the Usadel equation in the ferromagnet,

$$D_F \tilde{\nabla} (g\tilde{\nabla} g) + i [\varepsilon \tilde{\sigma}_3 + \tilde{M}, g] = 0,$$
(A11)

where we for simplicity will let $D_F = 1$ in this appendix. We begin by expanding the gauge covariant derivative $\tilde{\nabla} (g\tilde{\nabla} g)$, and then simplify the result using the normalization condition $\hat{g}^2 = 1$ and its derivative $\{\hat{g}, \partial_z \hat{g}\} = 0$, which yields the result

$$\tilde{\nabla} g \cdot (g\tilde{\nabla} g) = \partial_z (\hat{g}\partial_z \hat{g}) - i\partial_z (\hat{g}\tilde{A}\hat{g})$$
$$-i[\hat{A}, \hat{g}\partial_z \hat{g}] - [\hat{A}, \hat{g}\tilde{A}\hat{g}].$$
(A12)

We then write $\hat{g}$ in component form using Eq. (A1), and also write $\tilde{A}$ on the same form using $\tilde{A} = \text{diag}(\tilde{A}_x, -\tilde{A}_z^*)$. In the rest of this appendix, we will for simplicity assume that $\tilde{A}$ is real, so that $\tilde{A} = \text{diag}(\tilde{A}_x, -\tilde{A}_z^*)$. In practice, this implies that $\tilde{A}$ can only depend on the spin projections $\sigma_x$ and $\sigma_z$. The derivation for the more general case of a complex $\tilde{A}$ is almost identical. The four terms in Eq. (A12) may then be written as follows:

$$
\partial_z (\hat{g}\partial_z \hat{g}) = \begin{pmatrix}
\partial_z (g\partial_z g - f\partial_z f) & \partial_z (g\partial_z f - f\partial_z g) \\
\partial_z (g\partial_z f - f\partial_z g) & \partial_z (g\partial_z g - f\partial_z f)
\end{pmatrix}$$
$$-\begin{pmatrix}
\tilde{A}_x (A g \partial_z g - f \partial_z f) & \tilde{A}_x (A g f + f A g) \\
\tilde{A}_x (A g f + f A g) & -\tilde{A}_x (A g \partial_z g - f \partial_z f)
\end{pmatrix};
$$
(A13)

$$
\partial_z (\hat{g}\tilde{A}\hat{g}) = \begin{pmatrix}
\partial_z (A g \partial_z g - f \partial_z f) & \partial_z (A g f + f A g) \\
\partial_z (A g f + f A g) & -\partial_z (A g \partial_z g - f \partial_z f)
\end{pmatrix};
$$
(A14)

$$[\hat{A}, \hat{g}\partial_z \hat{g}] = \begin{pmatrix}
[A, g \partial_z g - f \partial_z f] & [A, g \partial_z f - f \partial_z g] \\
[A, g \partial_z f - f \partial_z g] & [-A, g \partial_z g - f \partial_z f]
\end{pmatrix};
$$
(A15)

$$[\hat{A}, \hat{g}\tilde{A}\hat{g}] = \begin{pmatrix}
[A, g A g + f A f] & [A, A g f + f A g] \\
[A, A g f + f A g] & [-A, g A g + f A f]
\end{pmatrix}. 
$$
(A16)

Substituting these results back into Eq. (A12), we can find the
upper blocks of the covariant derivative $\hat{\nabla} \cdot (g \hat{\nabla} \hat{g})$,
\begin{align}
[\hat{\nabla} \cdot (g \hat{\nabla} \hat{g})]^{(1,1)} &= \partial_z (g \partial_z g - f \partial_z f) - i \partial_z (g A_z g + f A_z f) \\
&= N [(\partial_\gamma \gamma - \gamma \partial_\gamma \gamma)] N \\
&= 2N \gamma N, \quad \text{(A17)}
\end{align}

\begin{align}
[\hat{\nabla} \cdot (g \hat{\nabla} \hat{g})]^{(1,2)} &= \partial_z (g \partial_z f - f \partial_z g) - i \partial_z (g A_z f + f A_z g) \\
&= 2N \gamma N, \quad \text{(A18)}
\end{align}

In this context, the notation $M^{(n,m)}$ refers to the $n$th row and $m$th column in Nambu space. Since the Green’s function $g$ and background field $\hat{g}$ also have a structure in spin space, the $(1, 1)$ element in Nambu space is the upper-left $2 \times 2$ block of the matrix, and the $(1, 2)$ element is the upper-right one.

There are two kinds of expressions that recur in the equations above, namely the components of $g \partial_z g$, and the components of $g \hat{A}$. After we substitute in the Riccati parametrization $g = 2N - 1$ and $f = 2N \gamma$, these components take the form:
\begin{align}
[g \partial_z g]^{(1,1)} &= g \partial_z g - f \partial_z f \\
&= 2N [(\partial_\gamma \gamma - \gamma \partial_\gamma \gamma)] N \\
&= 2N \gamma N, \quad \text{(A19)}
\end{align}

\begin{align}
[g \partial_z g]^{(1,2)} &= g \partial_z f - f \partial_z g \\
&= 2N \gamma N, \quad \text{(A20)}
\end{align}

\begin{align}
[g \hat{A}]^{(1,1)} &= g A_g + f A_{\hat{f}} \\
&= 4N \gamma N - 2 \gamma N + A \\
&= 4N \gamma N, \quad \text{(A21)}
\end{align}

\begin{align}
[g \hat{A}]^{(1,2)} &= g A_f + f A_{\hat{g}} \\
&= 4N \gamma N, \quad \text{(A22)}
\end{align}

If we explicitly calculate the commutators of $\hat{A}$ with the two matrices $g \partial_z g$ and $g \hat{A}$, then we find:
\begin{align}
[A, g \partial_z g]^{(1,1)} &= [A, g \partial_z g - f \partial_z f] \\
&= 2N \gamma N, \quad \text{(A23)}
\end{align}

\begin{align}
[A, g \partial_z g]^{(1,2)} &= [A, g \partial_z f - f \partial_z g] \\
&= 2N \gamma N, \quad \text{(A24)}
\end{align}

\begin{align}
[A, g \hat{A}]^{(1,1)} &= [A, g A_g + f A_{\hat{f}}] \\
&= 4N \gamma N, \quad \text{(A25)}
\end{align}

\begin{align}
[A, g \hat{A}]^{(1,2)} &= [A, g A_f + f A_{\hat{g}}] \\
&= 4N \gamma N, \quad \text{(A26)}
\end{align}

If we instead differentiate the aforementioned matrices with respect to $z$, we obtain:
\begin{align}
[\partial_z (g \partial_z g)]^{(1,1)} &= \partial_z (g \partial_z g - f \partial_z f) \\
&= 2N [(\partial_\gamma \gamma - \gamma \partial_\gamma \gamma)] N \\
&= 2N \gamma N, \quad \text{(A27)}
\end{align}

\begin{align}
[\partial_z (g \partial_z g)]^{(1,2)} &= \partial_z (g \partial_z f - f \partial_z g) \\
&= 2N \gamma N, \quad \text{(A28)}
\end{align}

\begin{align}
[\partial_z (g \hat{A})]^{(1,1)} &= \partial_z (g A_g + f A_{\hat{f}}) \\
&= 2N \gamma N, \quad \text{(A29)}
\end{align}

\begin{align}
[\partial_z (g \hat{A})]^{(1,2)} &= \partial_z (g A_f + f A_{\hat{g}}) \\
&= 2N \gamma N, \quad \text{(A30)}
\end{align}

Combining all of the equations above, we can express Eqs. (A17) and (A18) using Riccati matrices. In order to isolate the second-order derivative $\partial_\gamma^2 \gamma$ from these, the trick is to multiply Eq. (A17) by $\gamma$ from the right, and subsequently subtract the result from Eq. (A18):
\begin{align}
\frac{1}{2} N^{-1} \{ [\hat{\nabla} \cdot (g \hat{\nabla} \hat{g})]^{(1,2)} - [\hat{\nabla} \cdot (g \hat{\nabla} \hat{g})]^{(1,1)} \gamma \} \\
= \partial_\gamma^2 \gamma + 2(\partial_\gamma \gamma) \hat{N} (\partial_\gamma \gamma) \\
- 2i (A_z + \gamma A_z) \hat{N} (\partial_\gamma \gamma) - 2i (\partial_\gamma \gamma) \hat{N} (A_z + \gamma A_z) \\
- 2(\partial_\gamma \gamma + \gamma A_z) \hat{N} (\partial_\gamma \gamma) - \partial_\gamma^2 \gamma. \quad \text{(A31)}
\end{align}

If we finally rewrite $[\hat{\nabla} \cdot (g \hat{\nabla} \hat{g})]^{(1,1)}$ and $[\hat{\nabla} \cdot (g \hat{\nabla} \hat{g})]^{(1,2)}$ in the equation above by substituting in the Usadel equation (A11), then we obtain the following equation for the Riccati matrix $\gamma$:
\begin{align}
\partial_\gamma^2 \gamma = -2i \gamma - \partial_\gamma (\gamma A_z + \gamma A_z) - 2i (\partial_\gamma \gamma) \hat{N} (\partial_\gamma \gamma) \\
+ 2i (A_z + \gamma A_z) \hat{N} (\partial_\gamma \gamma) + 2i (\partial_\gamma \gamma) \hat{N} (A_z + \gamma A_z) \\
+ 2(\partial_\gamma \gamma + \gamma A_z) \hat{N} (\partial_\gamma \gamma) + \partial_\gamma^2 \gamma - \gamma. \quad \text{(A32)}
\end{align}

The corresponding equation for $\gamma$ can be found by tilde conjugation of the above. After restoring the diffusion coefficient $D_F$, and generalizing the derivation to a complex SO field $A$, the above result takes the form shown in Eq. (5).

After parametrizing the Usadel equation, the next step is to do the same to the Kupriyanov–Lukichev boundary conditions. The gauge covariant version of Eq. (2) may be written
\begin{align}
2 \lambda_n \zeta_0 g_n \hat{\nabla} g_n = [\hat{g}_1, \hat{g}_2], \quad \text{(A33)}
\end{align}
which upon expanding the covariant derivative $\tilde{g} \nabla \tilde{g}$ becomes
\[
\tilde{g}_n \partial_z \tilde{g}_n = \frac{1}{2} \Omega_n [\tilde{g}_1, \tilde{g}_2] + i \tilde{g}_n \{ A_z, \tilde{g}_n \} ,
\] (A34)
where we have introduced the notation $\Omega_n = 1/L_n \zeta_n$ for the interface parameter. We will now restrict our attention to the (1,1) and (1,2) components of the above,
\[
g_n \partial_z g_n - f_n \partial_z f_n = \frac{1}{2} \Omega_n (g_1 f_2 - g_2 f_1 - f_1 \tilde{f}_2 + f_2 \tilde{f}_1)
+ i g_n \{ A_z, g_n \} + i f_n \{ A_z, f_n \} ,
\] (A35)
\[
g_n \partial_z f_n - f_n \partial_z g_n = \frac{1}{2} \Omega_n (g_1 f_2 - g_2 f_1 - f_1 \tilde{f}_2 + f_2 \tilde{f}_1)
+ i g_n \{ A_z, f_n \} + i f_n \{ A_z, g_n \} .
\] (A36)
Substituting the Riccati parametrizations $g_n = 2N_n - 1$ and $f_n = 2N_n \gamma_n$ in the above, we then obtain:
\[
N_n [(\partial_z \gamma_n) y_n - \gamma_n (\partial_z y_n)] N_n = \Omega_n N_n (1 - \gamma_1 \gamma_2) N_2
- i N_n (1 - \gamma_1 \gamma_2) A N_1
- i N_n \{ A_z, \gamma_2 \} y_n N_2
+ 2 i N_n \{ A_z, \gamma_2 \} \gamma_2 y_n N_2 ,
\] (A37)
\[
N_n [(\partial_z \gamma_n) y_n - \gamma_n (\partial_z y_n)] y_n = \Omega_n N_n (1 - \gamma_1 \gamma_2) y_2 N_2
- i N_n (1 - \gamma_1 \gamma_2) y_1 N_1
+ i N_n (1 + \gamma_2 \gamma_2) \gamma_n N_2
+ i N_n \gamma_n \{ A_z, \gamma_2 \} \gamma_2 y_n N_2 .
\] (A38)
If we then multiply Eq. (A37) by $\gamma_n$ from the right, subtract this from Eq. (A38), and divide by $N_n$ from the left, then we obtain the following boundary condition for $\gamma_n$:
\[
\partial_z \gamma_n = \Omega_n (1 - \gamma_1 \gamma_2) N_2 (\gamma_2 - y_n)
+ \Omega_n (1 - \gamma_1 \gamma_2) N_1 (y_n - \gamma_1)
+ i \{ A_z, \gamma_n \} .
\] (A39)
When we evaluate the above for $n = 1$ and $n = 2$, then it simplifies to the following:
\[
\partial_z \gamma_1 = \Omega_1 (1 - \gamma_1 \gamma_2) N_2 (\gamma_2 - y_1) + i \{ A_z, \gamma_1 \} ,
\] (A40)
\[
\partial_z \gamma_2 = \Omega_2 (1 - \gamma_1 \gamma_2) N_1 (y_2 - \gamma_1) + i \{ A_z, \gamma_2 \} .
\] (A41)
The boundary conditions for $\partial_z \gamma_1$ and $\partial_z \gamma_2$ are found by tilde conjugating the above. If we generalize the derivative to a complex SO field $\hat{A}$, and substitute back $\Omega_n = 1/L_n \zeta_n$ in the result, then we arrive at Eq. (6).

**Appendix B: Derivation of the self-consistency equation for $\Delta$**

In this paper, we follow the convention where the Keldysh component of the anomalous Green’s function is defined as
\[
F^K_{\sigma\sigma'}(z, t'; z', t) = -i \langle \{ \psi_{\sigma} (z, t), \psi_{\sigma'} (z', t) \} \rangle ,
\] (B1)
where $\psi_{\sigma}(z, t)$ is the spin-dependent fermion annihilation operator, and the superconducting gap is defined as
\[
\Delta(z, t) = \lambda (\psi_{\sigma}(z, t) \psi_{\sigma'}(z, t)) ,
\] (B2)
where $\lambda < 0$ is the electron–electron coupling constant in the BCS theory. For the rest of this appendix, we will also assume that we work in an electromagnetic gauge where $A$ is a purely real quantity. Comparing Eqs. (B1) and (B2), and using the fermionic anticommutation relation
\[
\psi_{\sigma}(z, t) \psi_{\sigma'}(z, t) = - \psi_{\sigma'}(z, t) \psi_{\sigma}(z, t) ,
\] (B3)
we see that the superconducting gap $\Delta(z, t)$ can be expressed in terms of the Green’s functions in two different ways,
\[
\Delta(z, t) = \frac{i \lambda}{2} F^K_{\sigma\sigma'}(z, t) ,
\] (B4)
\[
\Delta(z, t) = - \frac{i \lambda}{2} F^K_{\sigma\sigma'}(z, t) .
\] (B5)
We may then perform a quasiclassical approximation by first switching to Wigner mixed coordinates, then Fourier transforming the relative coordinates, then integrating out the energy dependence, and finally averaging the result over the Fermi surface to obtain the isotropic part. The resulting equations for the superconducting gap are
\[
\Delta(z, t) = \frac{1}{4} N_0 \lambda \int d \epsilon f^K_{\sigma\sigma'}(z, t, \epsilon) ,
\] (B6)
\[
\Delta(z, t) = - \frac{1}{4} N_0 \lambda \int d \epsilon f^K_{\sigma\sigma'}(z, t, \epsilon) ,
\] (B7)
where $f^K_{\sigma\sigma'}$ is the quasiclassical counterpart to $F^K_{\sigma\sigma'}$, $\epsilon$ is the quasiparticle energy, and $N_0$ is the density of states per spin at the Fermi level.

In the equilibrium case, the Keldysh component $\tilde{g}^K$ can be expressed in terms of the retarded and advanced components of the Green’s function,
\[
\tilde{g}^K = (\tilde{g}^R - \tilde{g}^A) \tanh(\epsilon/2T) ,
\] (B8)
and the advanced Green’s function may again be expressed in terms of the retarded one,
\[
\tilde{g}^A = - \rho_3 \tilde{g}^R \rho_3 ,
\] (B9)
which implies that the Keldysh component can be expressed entirely in terms of the retarded component,
\[
\tilde{g}^K = (\tilde{g}^R - \rho_3 \tilde{g}^R \rho_3) \tanh(\epsilon/2T) .
\] (B10)
If we extract the relevant anomalous components $f^K_1$ and $f^K_3$ from the above, we obtain the results
\[
f^K_1 = [f^K_{1\uparrow}(z, +\epsilon) + f^K_{1\downarrow}(z, -\epsilon)] \tanh(\epsilon/2T) ,
\] (B11)
\[
f^K_3 = [f^K_{1\uparrow}(z, +\epsilon) + f^K_{1\downarrow}(z, -\epsilon)] \tanh(\epsilon/2T) .
\] (B12)
We then switch to a singlet/triplet-decomposition of the retarded component $f^R$, where the singlet component is described by a scalar function $f_s$, and the triplet component by
the so-called $d$-vector $(d_x, d_y, d_z)$. This parametrization is defined by the matrix equation

$$f^R = (f_s + d \cdot \sigma) i \sigma_y,$$

(B13)
or in component form,

$$\begin{pmatrix} f^R_x \\ f^R_y \\ f^R_z \end{pmatrix} = \begin{pmatrix} i d_y - d_z & d_z + f_s \\ d_z - f_s & i d_y + d_x \end{pmatrix}. \quad \text{(B14)}$$

Parametrizing Eqs. (B11) and (B12) according to Eq. (B14), we obtain

$$f^K_{\uparrow \downarrow} (r, \varepsilon) = \left[ d_z (r, \varepsilon) + f_s (r, \varepsilon) \right. + d_z (r, -\varepsilon) - f_s (r, -\varepsilon) \tanh (\varepsilon/2T), \quad \text{(B15)}$$

$$f^K_{\uparrow \downarrow} (r, \varepsilon) = \left[ d_z (r, \varepsilon) - f_s (r, \varepsilon) \right. + d_z (r, -\varepsilon) + f_s (r, -\varepsilon) \tanh (\varepsilon/2T). \quad \text{(B16)}$$

The triplet component $d_z$ can clearly be eliminated from the above equations by subtracting Eq. (B15) from Eq. (B16),

$$f^K_{\uparrow \downarrow} - f^K_{\uparrow \downarrow} = 2 \left[ f_s (r, \varepsilon) - f_s (r, -\varepsilon) \right] \tanh (\varepsilon/2T), \quad \text{(B17)}$$

and a matching expression for the superconducting gap can be acquired by adding Eqs. (B6) and (B7),

$$2 \Delta (r) = \frac{1}{4} N_0 \lambda \int \text{d} \varepsilon \left[ f^K_{\uparrow \downarrow} (r, \varepsilon) - f^K_{\uparrow \downarrow} (r, -\varepsilon) \right] \tanh (\varepsilon/2T). \quad \text{(B18)}$$

By comparing the two results above, we finally arrive at an equation for the superconducting gap which only depends on the singlet component of the quasiclassical Green’s function:

$$\Delta (r) = \frac{1}{4} N_0 \lambda \int \text{d} \varepsilon \left[ f_s (r, \varepsilon) - f_s (r, -\varepsilon) \right] \tanh (\varepsilon/2T). \quad \text{(B19)}$$

If the integral above is performed for all real values of $\varepsilon$, it turns out to be logarithmically divergent e.g. for a bulk superconductor. However, physically, the quasiparticles cannot be expected to have infinitely large energies; in practice, their energy spectrum is restricted by the energy spectra of the phonons that mediate the attractive electron–electron interactions in the superconductor. This may therefore be resolved by introducing a Debye cutoff $\omega_c$, such that we only integrate over the region where $|\varepsilon| < \omega_c$. Including the integration range, the gap equation is therefore

$$\Delta (r) = \frac{1}{4} N_0 \lambda \int_{-\omega_c}^{\omega_c} \text{d} \varepsilon \left[ f_s (r, \varepsilon) - f_s (r, -\varepsilon) \right] \tanh (\varepsilon/2T). \quad \text{(B20)}$$

The equation above can however be simplified even further. First of all, both $f_s (\varepsilon) - f_s (-\varepsilon)$ and $\tanh (\varepsilon/2T)$ are clearly antisymmetric functions of $\varepsilon$, which means that the product is a symmetric function, and so it is sufficient to perform an integral over positive values of $\varepsilon$,

$$\Delta (r) = \frac{1}{2} N_0 \lambda \int_0^{\omega_c} \text{d} \varepsilon \left[ f_s (r, \varepsilon) - f_s (r, -\varepsilon) \right] \tanh (\varepsilon/2T). \quad \text{(B21)}$$

However, because of the term $f_s (r, -\varepsilon)$, we still need to know the Green’s function for negative values of $\varepsilon$ before we can calculate the gap. On the other hand, the singlet component of the quasiclassical Green’s functions also has a symmetry when the superconducting gauge is chosen as real

$$f_s (r, \varepsilon) = -f_s^* (r, -\varepsilon), \quad \text{(B22)}$$

which implies that

$$f_s (r, \varepsilon) - f_s (r, -\varepsilon) = 2 \text{Re} \{ f_s (r, \varepsilon) \}. \quad \text{(B23)}$$

Substituting Eq. (B23) into Eq. (B21), the gap equation takes a particularly simple form, which only depends on the real part of the singlet component $f_s (r, \varepsilon)$ for positive energies $\varepsilon$:

$$\Delta (r) = N_0 \lambda \int_0^{\omega_c} \text{d} \varepsilon \text{Re} \{ f_s (r, \varepsilon) \} \tanh (\varepsilon/2T). \quad \text{(B24)}$$

Let us now consider the case of a BCS bulk superconductor, which has a singlet component given by the equation

$$f_s (r) = -\frac{\Delta}{\sqrt{\varepsilon^2 - \Delta^2}}, \quad \text{(B25)}$$

so that the gap equation may be written as

$$\Delta = N_0 |\lambda| \int_0^{\omega_c} \text{d} \varepsilon \text{Re} \left\{ -\frac{\Delta}{\sqrt{\varepsilon^2 - \Delta^2}} \right\} \tanh (\varepsilon/2T). \quad \text{(B26)}$$

The part in the curly braces is only real when $|\varepsilon| \geq \Delta$, which means that the equation can be simplified by changing the lower integration limit to $\Delta$. After also dividing the equation by $N_0 |\lambda|$, we then obtain the self-consistency equation

$$\frac{1}{N_0 |\lambda|} = \int_\Delta |\varepsilon| \tanh (\varepsilon/2T) / \sqrt{\varepsilon^2 - \Delta^2}. \quad \text{(B27)}$$

For the zero-temperature case, where $T \to 0$ and $\Delta \to \Delta_0$, performing the above integral and reordering the result yields

$$\Delta_0 = \lambda_0 \text{cosh} (1/N_0 |\lambda|). \quad \text{(B28)}$$

Using the above equation for $\Delta_0$, and the well-known result

$$\frac{\Delta_0}{T_c} = \frac{\pi}{\Delta_0} \Delta_0 \text{cosh} (1/N_0 |\lambda|),$$

where $\gamma \approx 0.57722$ is the Euler–Mascheroni constant, we can finally rewrite Eq. (B24) as:

$$\Delta (r) = N_0 \lambda \int_0^{\omega_c} \text{d} \varepsilon \text{Re} \{ f_s (r, \varepsilon) \} \tanh \left( \frac{\pi \varepsilon/\Delta_0}{2e^T / T_c} \right). \quad \text{(B30)}$$

This version of the gap equation is particularly well-suited for numerical simulations. One advantage is that we only need to know the Green’s function for positive energies, which halves
the number of energies that we need to solve the Usadel equation for. Moreover, the appropriate Debye cutoff ωc can be self-consistently calculated from the other parameters N0|λ| and Δ0 that describe the superconductor, instead of being yet another parameter that has to be provided by experiments. The equation also takes a particularly simple form if we use energy units where Δ0 = 1 and temperature units where Tc = 1, which is common practice in such simulations.

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