Ultrasensitive Atomic clock with single-mode number-squeezing

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We show that the sensitivity of an atomic clock can be enhanced below the shot-noise level by initially squeezing, and then measuring in output, the population of a single atomic level. This can simplify current experimental protocols which requires squeezing of the relative number of particles of the two populated states. We finally study, as a specific application, the clock sensitivity obtained with a single mode quantum non-demolition measurement.

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Introduction. Atomic clocks and interferometers are among the most sensitive measurement devices available within the current technology [1]. Their precision is bounded by the fundamental noise imposed by quantum mechanics uncertainties. With uncorrelated atoms, the phase sensitivity of the Ramsey sequence scales as the inverse square root of the total number of particles, the so-called shot noise (or standard quantum) limit, first observed experimentally in [2]. The possibility to overcome the shot noise limit by quantum engineering specific atomic correlations is a break ground prediction which is under intense experimental investigation. Most current efforts focus on the creation of spin-squeezed states [3,4]. This can be achieved, for instance, by manipulating a cloud of cold atoms via the back-reaction of quantum non-demolition (QND) measurements [2,5] or with interaction-induced nonlinearity using Bose Einstein Condensates (BECs) [3,6]. With BEC, entanglement-enhanced Ramsey phase sensitivity has been recently demonstrated [3].

In the current literature, it is shown that sub shot noise phase sensitivity in an atomic clock is generally associated to spin squeezing. In this manuscript we demonstrate a sub shot noise phase sensitivity up to the Heisenberg limit with the initial squeezing and the measurement in output of the particle population of a single atomic level. This can simplify current interferometric protocols. A non-destructive atom-light interaction of a single clock level has been experimentally demonstrated in [3,6]. Therefore, our prediction can be readily tested experimentally and find application for precision atomic sensors within the present state-of-the-art technology.

Sub shot-noise with single mode squeezing. We consider two atomic clock levels a and b of energy \( \hbar \omega_a \) and \( \hbar \omega_b \), respectively. The goal is to estimate the frequency difference \( \Delta \omega = \omega_a - \omega_b \) with the highest possible sensitivity. The Ramsey interferometric sequence consists of four steps: a Rabi \( \pi/2 \) pulse of constant power and frequency \( \omega \) (as close as possible in resonance with the atomic transition) applied for a time \( \tau = \pi/2\Omega_R \), being \( \Omega_R \) the Rabi frequency. Then the system freely evolves for a period of time \( T \). Finally, after a second \( \pi/2 \) pulse, the number of particles is measured in a single output level. The quantum mechanical expectation value of the number of particles in the output a mode is given by 

\[
\langle n_a \rangle_{\text{out}} = \langle \hat{n}_a \rangle_{\text{in}} \cos^2(\frac{\theta}{2}) + \langle \hat{n}_b \rangle_{\text{in}} \sin^2(\frac{\theta}{2}) - \frac{1}{2} (\langle \hat{a}^\dagger b + b^\dagger a \rangle_{\text{in}} \sin \theta \),
\]

where \( \hat{a} \) (\( \hat{a}^\dagger \)) and \( \hat{b} \) (\( \hat{b}^\dagger \)) are particle annihilation (creation) operators of the a and b mode, respectively, and \( \hat{n}_a \equiv \hat{a}^\dagger \hat{a} \) (\( \hat{n}_b \equiv \hat{b}^\dagger \hat{b} \)) is the number of particles operator. The quantity \( \langle \hat{n}_a \rangle_{\text{out}} \) depends on the phase \( \theta = \delta \times T \) accumulated during the free precession, being \( \delta = \Delta \omega - \omega \) the detuning of the Rabi pulse from the atomic transition. By collecting m measurements with results \( n_a^{(1)}, ..., n_a^{(m)} \), we can calculate the average number of particles in the a output mode \( \bar{n}_a = \sum_{m=1}^{m} n_a^{(i)}/m \). The phase is inferred by approximating the expectation value \( \langle \hat{a}^\dagger \hat{a} \rangle_{\text{out}} \) with the classical average \( \bar{n}_a \) and inverting the equation 

\[
\Delta \theta = \frac{\Delta \bar{n}_a}{\sqrt{m}} \left| \frac{d}{d\theta} \langle \hat{a}^\dagger \hat{a} \rangle_{\text{out}} \right|_{\theta = \theta_{\text{est}}},
\]

where \( \Delta \theta \) is the variance of \( \theta_{\text{est}} \), calculated with error propagation, assuming that \( \bar{n}_a \) fluctuates with variance \( \Delta \bar{n}_a^2 = \langle \hat{n}_a^2 \rangle_{\text{in}} - (\langle \hat{n}_a \rangle_{\text{in}})^2 \). In particular, when \( \theta \rightarrow 0 \) and with initial symmetric populations \( \langle \hat{n}_a \rangle_{\text{in}} = \langle \hat{n}_b \rangle_{\text{in}} \), Eq. (1) becomes

\[
\Delta \theta = \frac{2(\Delta \bar{n}_a)_{\text{in}}}{\sqrt{m} \langle \hat{a}^\dagger b + b^\dagger a \rangle_{\text{in}}} \left| \frac{d}{d\theta} \langle \hat{a}^\dagger b + b^\dagger a \rangle_{\text{in}} \right|_{\theta = \theta_{\text{est}}}. \tag{2}
\]

Equations (1) and (2) relate the interferometric phase sensitivity \( \Delta \theta \) to the fluctuations of the number of particles in a single input atomic level, and capture the main results of this manuscript. In particular, according to Eq. (2), sub shot-noise phase sensitivity can be obtained by squeezing the population fluctuations in the input a, \( \langle \hat{n}_a^2 \rangle_{\text{in}} < \langle \hat{n} \rangle/2 \), while keeping the coherence between the two modes \( \langle \hat{a}^\dagger b \rangle_{\text{in}} \sim \langle \hat{n} \rangle/2 \). Here \( \langle \hat{n} \rangle = \langle \hat{n}_a \rangle_{\text{in}} + \langle \hat{n}_b \rangle_{\text{in}} \) is the average number of particles in the input state. As an example, we consider the product of two Gaussian random variables \( \langle \psi \rangle_{a,b} \propto \sum_{n=0}^{\infty} \exp(-n(n-\langle \hat{n} \rangle/2)^2/4\sigma^2) |n\rangle_{a,b} \) as input of the interferometer, where \( \sigma_a = \sqrt{\langle \hat{n} \rangle}/2 \) and \( \sigma_b = \kappa \sqrt{\langle \hat{n} \rangle}/2 \). Number-squeezing in the a mode [11] is obtained for \( \kappa < 1 \), and equation (2) predicts a sub shot noise sensitivity \( \Delta \theta = \kappa/\sqrt{m} \langle \hat{n} \rangle \). In the Fock limit \( \sigma_a \rightarrow 0 \), Eq. (2) ...
predicts the ultimate Heisenberg scaling
\[ \Delta \theta = \frac{\sqrt{2}}{\langle \hat{n} \rangle \sqrt{m}}. \] (3)

The Fock state limit is particularly interesting and will be further discussed below.

Notice that Eq. (2) resembles the familiar relation between interferometric sensitivity and spin squeezing [12–14]. However, since the total number of particles is not fixed, number squeezing in a single clock level does not necessarily imply the squeezing of the relative population between the two input modes. Only when the total number of particles is fixed, the two become equivalent. Our analysis includes this special case.

Entanglement and Fock state limit. The relation between multiparticle-entanglement and sub shot noise phase sensitivity in a linear interferometer has been recently discussed for states of fixed [12] and fluctuating [17] number of particles (qubits). In particular, states satisfying the inequality \( F_Q[\hat{\rho}, \hat{J}_y] > \langle \hat{n} \rangle \), being \( F_Q[\hat{\rho}, \hat{J}_y] \) the quantum Fisher information [13, 17] and \( \hat{J}_y = (\hat{a} \hat{b} + \hat{b}^\dagger \hat{a})/\sqrt{2} \) [16], are entangled and provide a sub shot-noise phase sensitivity in a Ramsey interferometer. The optimal phase sensitivity, after \( m \) independent measurements, is \( \Delta \theta = 1/\sqrt{m} F_Q[\hat{\rho}, \hat{J}_y] \). In particular, for an input state
\[ \hat{\rho}_{\text{inp}} = |N\rangle_a \langle N| \otimes \hat{\rho}_b, \] (4)
being \( \hat{\rho}_b = \sum_n \rho_n |n\rangle_b \langle n| \) a generic density matrix of \( \langle \hat{n}_b \rangle_{\text{inp}} = \sum_{n=0}^{\infty} \rho_n n \) average number of particles, we find
\[ \Delta \theta = \frac{1}{\sqrt{m}} \sqrt{2N m \langle \hat{n}_b \rangle_{\text{inp}} + N + \langle \hat{n}_b \rangle_{\text{inp}}} \] (5)

Equation (5) is characterized by interesting limits. If the mode \( b \) is left empty, \( \rho_b = \delta_{0,0} \), the sensitivity of the clock is given by the standard quantum limit (SQL), \( \Delta \theta = 1/\sqrt{m \langle \hat{n}_b \rangle_{\text{inp}}} \), where \( \langle \hat{n}_b \rangle_{\text{inp}} \) is the average number of particles in the input state (in this case \( \langle \hat{n} \rangle = N \)). Conversely, if the mode \( b \) is left almost (but not completely) empty, with \( \langle \hat{n}_b \rangle_{\text{inp}} \ll N \), the sensitivity \( \Delta \theta \approx 1/\sqrt{2 \langle \hat{n}_b \rangle_{\text{inp}}} + 1/\sqrt{m \langle \hat{n}_b \rangle_{\text{inp}}} \) is below the SQL by a factor \( 2 \langle \hat{n}_b \rangle_{\text{inp}} / (\langle \hat{n}_b \rangle_{\text{inp}} + 1) \). Moreover, at the optimal condition \( N = \langle \hat{n}_b \rangle_{\text{inp}} \), Eq. (5) predicts \( \Delta \theta \approx \frac{2}{\langle \hat{n} \rangle_{\text{inp}}} \), independently of the input states \( \hat{\rho}_b \), and, in particular, we recover Eq. (3). Measuring the number of particles in a single output port not only provides an optimal estimation strategy but also saturates the Heisenberg limit [17]. As anticipated above, we note that Eq. (3) is not a spin squeezed state. The spin squeezing parameter \( \xi^2 = (\Delta J_z)_{\text{inp}}^2 / (\langle J_z \rangle_{\text{inp}}^2 + (J_y)_{\text{inp}}^2) \) [13, 15] diverges in general when calculated for the input state Eq. (3), and is undetermined, \( \xi^2 = 0/0 \), when also the input \( b \) is in a Fock state \( \rho_b = \delta_{0,0} \) [19]. In the latter case, corresponding to the twin Fock state, Eq. (5) recovers the Heisenberg limit sensitivity first predicted in [20].

QND state preparation. We now discuss a possible experimental implementation of the protocol discussed above. The single-mode number squeezing is produced by a QND interaction between the atoms cloud and a light field, recently demonstrated in [8, 2]. The initial atomic cloud, obtained by optically pumping the atoms to a single level, is described by the density matrix \( \hat{\rho} = \sum_{\alpha=0}^{\infty} P_N |N,0\rangle |N,0\rangle, \) where \( |N,0\rangle \equiv |N\rangle_\alpha |0\rangle_b \) is the state with \( N \) atoms in the levels \( \alpha \) and \( 0 \) atoms in level \( b \), which occurs with probability \( P_N \) \( (P_N > 0 \) and \( \sum_N P_N = 1) \). In order to equally populate in average the two input modes, we apply a \( \pi/2 \) pulse. The density matrix becomes \( \hat{\rho}_A^\parallel = \sum_{\alpha=0}^{\infty} P_N |\psi_N\rangle |\psi_N\rangle \) where \( |\psi_N\rangle = \sum_n P_N^n |n\rangle_{\alpha} |n\rangle_b \). The number of particles distribution in each mode has \( \langle \hat{n}_a \rangle = \langle \hat{n}_b \rangle = \langle \hat{n} \rangle /2 \), \( (\Delta \hat{n}_a)^2 = (\Delta \hat{n}_b)^2 = (\sigma^2 + \langle \hat{n} \rangle)/4 \) where \( \langle \hat{n} \rangle = \sum N P_N \) and \( \sigma^2 = \sum N P_N (N - \langle \hat{n} \rangle)^2 \). With this state as input of the clock, Eq. (1) predicts
\[ \Delta \theta = \frac{1}{\sqrt{\langle \hat{n} \rangle}} \sqrt{1 + \frac{\sigma^2}{\langle \hat{n} \rangle^2} (1 - \sin^2 \theta)^2 \cos^2 \theta}, \] (6)

At the optimal value of the phase shift \( \theta = \pi/2 \), the phase sensitivity is at the standard quantum limit \( \Delta \theta = 1/\sqrt{\langle \hat{n} \rangle} \). It is possible to overcome this limit by squeezing the number of particles fluctuations in one mode. This is done here by letting the atoms in \( a \) to interact with a coherent light field \( |a \rangle \) (we take, without loss of generality, the amplitude \( a \) to be real) via the QND Hamiltonian
\[ \hat{H}_{\text{QND}} = \hbar g (\hat{c}^\dagger \hat{a}) (\hat{a}^\dagger \hat{c}), \] (7)
where \( \hat{c} \) (\( \hat{c}^\dagger \) created (annihilate) a photon of light field mode and \( g \) is the coupling parameter. The Hamiltonian Eq. (7) describes an out of resonance interaction of the light field with the atoms in the level \( a \) [14]. We assume here that the detuning of the light is large enough to adiabatically eliminate the excited state population and neglect possible decoherence effects [21]. The coupling strength is normally weak but can be enhanced by placing the atoms inside an optical cavity [22]. The entangled atom-light system is described by the density matrix \( \hat{\rho}_{AL} = e^{i\hbar \Omega t/\hbar} \hat{\rho}_A^\parallel \otimes |\alpha\rangle \langle \alpha| e^{-i\hbar \Omega t/\hbar} \), where \( t \) is the interaction time between the light and the atomic cloud and \( \alpha \) is the amplitude of the coherent state. The effect of the QND interaction is to shift the phase of the light by a quantity proportional to the number of atoms. There are different experimental possibilities to estimate the phase shift [8, 10]. In the following we consider the homodyne measurement of the \( p \) quadrature, \( \hat{p} = (\hat{c} - \hat{c}^\dagger)/2i \). The probability to measure the eigenvalue \( p \) is given by \( P_0(p) = \text{Tr} \left[ \hat{\rho}_L^{(0)} |p\rangle \langle p| \hat{\rho}_A^\parallel \right] \), where \( \hat{\rho}_L^{(0)} = \text{Tr}_{\bar{A}}[\hat{\rho}_A^{(0)}] \) is traced over the atomic degrees of freedom [23]. After the measurement, with result \( p_0 \), the density matrix of the atomic cloud becomes \( \hat{\rho}_A^{(1)} = \text{Tr}_L[\hat{\rho}_{AL} |p_0\rangle \langle p_0| / P_0(p_0)] \). The back reaction effect induced by the measurement [24] reduces the fluctuations of the number of atoms in the \( a \) mode. In the limit \( \Omega \langle \hat{n} \rangle \ll 1 \), with \( \Omega = gt \), the quadrature probability is a sum of Gaussian functions [25] and we can distinguish two regimes. When \( \alpha \Omega \gg 1 \), the measurement result is only compatible with a precise number
of atoms and therefore the QND interaction projects the atomic population of the a mode to a Fock state. In this case the sensitivity is at the Heisenberg limit Eq.(6). Conversely, when $\alpha \Omega \lesssim 1$, which is the most realistic scenario from the experimental point of view, the result of the quadrature measurement is compatible with several values of the number of particles in the a mode, and the back reaction only produces a moderate squeezing. Therefore, in order to increase the squeezing, it is necessary to repeat $M \gg 1$ times the QND protocol [26]. Analytical results can be obtained in the limit $\gamma = \alpha^2 \Omega^2 M \lesssim 1$. In this case we obtain $(\Delta \hat{a}^\dagger \hat{a})^2 = (\sigma^2 + \langle \hat{n} \rangle)/4$ and $(\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a})_{\text{opt}} \approx 2\langle \hat{n} \rangle \cos(M\alpha^2\Omega)$. Conversely, the input $b$ retains, in average, the initial number fluctuations. By tuning $M\alpha^2g$ to values close to an integer multiple of $\pi$, Eq.(2) gives

$$\Delta \theta = \frac{1}{\langle \hat{n} \rangle \sqrt{m}} \sqrt{\frac{\sigma^2 + \langle \hat{n} \rangle}{1 + \gamma (\sigma^2 + \langle \hat{n} \rangle)}}.$$  

Equation (8) is obtained for $\theta \sim 0$ and arbitrary fluctuations $\sigma$. In particular, when $\gamma > \sigma^2/\langle \hat{n} \rangle + \sigma^2$, we obtain a sub shot noise ($\Delta \theta < 1/\sqrt{\langle \hat{n} \rangle m}$) limit. In figure (1) we plot the phase sensitivity, calculated using Eq.(1) with a coherent state having $\sigma^2 = \langle \hat{n} \rangle = 10^5$, as a function of the phase shift $\theta$. The different lines correspond to different values of $\gamma$. In particular, the dotted black line is obtained for $\gamma = 0$ and is given by Eq.(6). The upper (lower) horizontal lines is $1/\sqrt{\langle \hat{n} \rangle}$ ($\sqrt{2}/\langle \hat{n} \rangle$). The figure shows that sub shot noise can be obtained around an optimal value of the phase shift, $\theta_{\text{opt}}$ (minimum of each curve). In figure (2a) we show $\theta_{\text{opt}}$ as a function of $\gamma$: it is close to $\theta_{\text{opt}} \sim \pi/2$ for moderate squeezing and rapidly tends to $\theta_{\text{opt}} \sim 0$ by increasing $\gamma$. In figure (2b) we show the optimal phase sensitivity as a function of $\gamma$. Numerical results (circles) agree with the solid blue line given by Eq.(6) in the limit of relatively small $\gamma$, where the optimal phase shift is $\theta_{\text{opt}} \sim 0$ [27]. For $\gamma > 1$ the QND project the state close to Fock, and the phase sensitivity converges to Eq.(4) (lower solid horizontal line). Notice that Eq.(6) applies also when the total number of particles is fixed ($\sigma = 0$). In this case, the squeezing of the number of particles of a single level is equivalent to relative number squeezing. In figure (2B) the dotted red line corresponding to the case $\sigma = 0$ (and $\langle \hat{n} \rangle = 10^5$ atoms), superposes to the dots, obtained from numerical simulations with $\sigma^2 = \langle \hat{n} \rangle$. This clearly shows that relative number squeezing does not provide any advantage with respect to single mode number squeezing: for the same number of particles and squeezing parameters we obtain the same level of sub shot noise interferometric sensitivity.

**Discussion.** As discussed above, a QND measurement on a single energy level is sufficient to prepare an input states useful to reach a sub shot noise phase sensitivity. This can potentially simplify current experimental schemes. The reduction of the fluctuations of the relative population (while preserving the coherence) with a QND atom-light interaction has been recently demonstrated. This has been done by i) addressing the two atomic levels with two carefully detuned laser beams [4] or ii) with a single laser beam tuned at a very precise frequency and detuning where the index of refraction of the gas crosses a zero value [5]. On the other hand, when addressing a single level, it is possible to perform QND with the freedom of choosing the laser frequency and detuning so to minimize the decoherence effect due to spontaneous emission. As shown in [4, 5] this is one of the main limi-
itations of the level of squeezing reached experimentally. Within our scheme, it is also possible to select the atomic transition in order to maximize the response of the light field to the atom-light interaction. A further experimental advantage of our proposal is that the clock frequency can be estimated by measuring the number of particles in a single output port of the Ramsey interferometer. This avoids the further noise introduced by inverting and measuring the population of the second mode, as currently done. We thus expect that our protocol might experimentally provide higher squeezing with a more robust apparatus.

Conclusion. We have discussed a new protocol to reach a sub shot noise sensitivity (up to the Heisenberg limit) in an atomic clock. This requires: i) reduced particles number fluctuations in a single input mode and ii) the measurement of the number of particles in a single output. Our results can be interpreted in terms of useful entanglement created by squeezing the population fluctuations in a single mode. We provide a simple analysis of the number squeezing produced by a QND interaction between a light field and the atomic sample showing, for this experimentally relevant situation, the possibility to readily verify our predictions. Since Fock states of a small number of photons are currently experimentally available [28], our predictions can be relevant also in the optical domain.

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