PARABOLIC DIMENSIONAL REDUCTIONS OF 11D SUPERGRAVITY

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Abstract

Ansatze are constructed under which the solutions of 11D supergravity must be stationary points of a parabolic flow on a Riemannian manifold $M^{10-p}$. This parabolic flow turns out to be the Ricci flow coupled to a scalar field, a $(3-p)$-form, and a 4-form. This allows the introduction of techniques from parabolic partial differential equations to the search of solutions to 11D supergravity. As a first step, Shi-type estimates and criteria for the long-time existence of the flow are established.

1 Introduction

Ever since 11D supergravity was constructed by Cremmer, Julia, and Scherk [5], and even more so after the realization that it is a low-energy effective action of M Theory [24, 39, 41], there has been considerable interest in its solutions. Many have been found through a variety of ansatze (see e.g. [9, 6, 35, 36, 7, 1, 28, 18, 11] and references therein). Notable supersymmetric examples include compactifications on Einstein 7-manifolds [15], on manifolds with special holonomy [30], and multi-membrane solutions [10]. A construction of a class of solutions, starting from a Ricci-flat 8-manifold which is either compact or complete with faster than quadratic volume growth was proposed recently in [14].

Mathematically, the low-energy effective actions and compactifications of string theory have led to the discovery of many deep and unexpected phenomena, beginning with the Kähler Ricci-flat compactifications of the heterotic string proposed by Candelas, Horowitz, Strominger, and Witten [4], and the subsequent discovery of mirror symmetry. The generalization by Hull [25] and Strominger [38] of the proposal of [4] has now been found to have a very rich mathematical structure as well. The first non-Kähler solution of the Hull-Strominger system was found by Fu and Yau [16] using a geometric construction going back to Calabi and Eckmann [3, 19], and many more solutions have since been found, including an infinite number of topologically distinct types by Fei, Huang, and Picard [12], generalizing a geometric construction of Calabi [2] and Gray [20] (see e.g. [17] for more references). The Hull-Strominger system has also motivated the introduction of many new analytic methods, including flows of $(2,2)$-forms [32, 33, 34, 13].

The main goal of this paper is to begin a more systematic, analytic study of the field equations of 11D supergravity than has been available so far. The field equations of 11D

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supergravity are a system of partial differential equations, more specifically Einstein’s equation for a metric coupled to a closed 4-form $F$ (see (1.1) below), so we expect that a full analytic understanding will ultimately require the theory of non-linear partial differential equations. Since space-time is Lorentzian, the equations are hyperbolic. Hyperbolic equations are notoriously difficult in general, and many more tools seem available for elliptic and parabolic systems, such as the Calabi-Yau equation [42], the Hermitian-Yang-Mills equation [8, 40], or the Ricci flow [22, 31]. As a practical first step in the long-term program of finding solutions of 11D supergravity by partial differential equations methods, we would like then to identify ansatze by which they can be reduced to an elliptic system.

More specifically, the bosonic fields in the theory are a metric $g_{AB}$ and a closed 4-form on an 11-dimensional Lorentzian manifold $M^{11}$, and the field equations are given by

$$
\begin{align*}
\delta F &= \frac{1}{2} F \wedge F \\
R_{AB} &= \frac{1}{2} F_{AB}^2 - \frac{1}{6} |F|^2 g_{AB}
\end{align*}
$$

(1.1) (1.2)

While the problem of finding dimensional reductions which are parabolic is only non-trivial when $M^{11}$ is Lorentzian, the resulting reduction process works as well for $M^{11}$ Euclidian, and we can treat both cases simultaneously by introducing a parameter $\sigma$ which is defined to be $+1$ when $M^{11}$ is Lorentzian and $-1$ when $M^{11}$ is Euclidian.

Our starting point is to view the field equations (1.1) as stationary points of the following dynamical system

$$
\begin{align*}
\frac{\partial F}{\partial t} &= -\Box_g F - \frac{\sigma}{2} \delta (F \wedge F) \\
\frac{\partial g_{AB}}{\partial t} &= -2 R_{AB} + F_{AB}^2 - \frac{1}{3} |F|^2 g_{AB}
\end{align*}
$$

(1.3) (1.4)

where $\Box_g = dd^\dagger + d^\dagger d$ is the Hodge-Laplacian. When the metric $g_{AB}$ is Lorentzian, this flow is not parabolic, and even its short-time existence is not guaranteed in general. However, we shall find ansatze preserved by the flow, under which the most difficult Lorentz components of the metric are static, and the other components evolve by a parabolic flow. Thus assume that the space time $M^{11}$ is a warped product $M^{11} = M^{1,p} \times M^{10-p}$ with metric $g$ and 4-form $F$ of the form

$$
g = e^f \tilde{g} + \hat{g}, \quad F = d\text{vol}_{\tilde{g}} \wedge \beta + \Psi.
$$

(1.5)

Here the metric $\tilde{g}$ is a Lorentzian metric on $M^{1,p}$ if $g_{AB}$ is Lorentzian, $\hat{g}$ is a Riemannian matric on $M^{10-p}$, $f$ is a scalar function on $M^{10-p}$, $d\text{vol}_{\tilde{g}}$ is the volume form of $\tilde{g}$ on $M^{1,p}$.

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2 A derivation of these equations and our conventions for forms are provided in the appendix.

3 A priori the set of stationary points may be larger than the set of solutions of the field equations. This is an issue to be examined separately from the considerations of the present paper. A brief discussion is included in Appendix C.
and $\beta$ and $\Psi$ are closed $(3 - p)$-forms and 4-forms on $M^{10-p}$ respectively. The metric $\tilde{g}$ is Riemannian if $g_{AB}$ is Riemannian. The dimension $p$ can take any integer value between 0 and 10 (when $p \geq 4$, the form $\beta$ is 0). Assume that the metric $\tilde{g}$ is Einstein with scalar curvature $(1 + p)\tilde{\lambda}$, i.e.,

$$
\text{Ric}(\tilde{g}) = \tilde{\lambda}\tilde{g}.
$$

Then we have the following theorem:

**Theorem 1** Consider the following flow of the tuple $(\hat{g}, f, \beta, \Psi)$ on $M^{10-p}$ with the initial values $\beta_0$ and $\Psi_0$ being closed forms

\[
\begin{align*}
\frac{\partial \hat{g}_{ij}}{\partial t} &= -2\text{Ric}(\hat{g})_{ij} + (p + 1) \left( (\nabla^2_\hat{g} f)_{ij} + \frac{1}{2} f_i f_j \right) \\
&\quad - \sigma e^{(p+1)f} \beta_{ij}^2 + \Psi_{ij}^2 - \frac{1}{3} (|\Psi|^2 - \sigma e^{-(p+1)f} |\beta|^2) \hat{g}_{ij} \\
\frac{\partial f}{\partial t} &= \Delta_\hat{g} f + \frac{p + 1}{2} |\nabla_\hat{g} f|_{\hat{g}}^2 - \frac{2}{3} \sigma e^{-(p+1)f} |\beta|_{\hat{g}}^2 - \frac{1}{3} |\Psi|_{\hat{g}}^2 - 2\tilde{\lambda} e^{-f} \\
\frac{\partial \beta}{\partial t} &= -\Box_\hat{g} \beta + (-1)^{p+1} \frac{p + 1}{2} d \ast_{\hat{g}} (df \wedge \ast_{\hat{g}} \beta) - \sigma (-1)^p \frac{p + 1}{4} e^{-\frac{p+1}{2} f} df \wedge \ast_{\hat{g}} (\Psi \wedge \Psi) \\
&\quad - \sigma (-1)^p \frac{1}{2} e^{-\frac{p+1}{2} f} d \ast_{\hat{g}} (\Psi \wedge \Psi) \\
\frac{\partial \Psi}{\partial t} &= -\Box_\hat{g} \Psi + (-1)^p \frac{p + 1}{2} d \ast_{\hat{g}} (df \wedge \ast_{\hat{g}} \Psi) - \frac{p + 1}{2} e^{-\frac{p+1}{2} f} df \wedge \ast_{\hat{g}} (\beta \wedge \Psi) \\
&\quad + e^{-\frac{p+1}{2} f} d \ast_{\hat{g}} (\beta \wedge \Psi)
\end{align*}
\]

Then the following hold:

(a) The forms $\beta$ and $\Psi$ remain closed along the flow, and the pair $(g, F)$ defined by (1.5) satisfies the flow (1.3) and (1.4) on the 11-dimensional Lorentzian/Riemannian manifold $M^{11}$.

(b) Assume that $M^{10-p}$ is compact. Then the above flow (1.6) - (1.9) is weakly parabolic, and admits a smooth solution at least on some interval $[0, T_0)$ for $T_0 > 0$ depending only on the initial values.

(c) If $T < \infty$ is the maximum existence time of the above flow, then

$$
\limsup_{t \to T^-} \sup_{M^{10-p}} (|Rm| + |f| + |\beta| + |\Psi|) = \infty.
$$

The property (c) implies that the flow is well-behaved, in the sense that, in order for it to terminate, one of only 4 quantities must blow up. Its proof requires estimates of higher order for all fields $g, f, \beta, \Psi$, which are described in section §4 below. For the sake of simplicity, we have restricted ourselves here to the case of $M^{10-p}$ compact. But in view of the considerations explained in [14], the case of $M^{10-p}$ non-compact is also of interest, and we shall return to it elsewhere.
Supergravity in 11D is formulated with a Lorentz signature, but it is likely that a version with Euclidean signature would be of mathematical interest as well, just as Yang-Mills theory with Euclidean signature achieved a prominent position in both geometry and particle physics. For this Euclidean version, we can obtain criteria for long-time existence of the corresponding flow, in analogy with Theorem 1:

**Theorem 2** Consider the flow (1.3) and (1.4) with $\sigma = -1$ where $g_{AB}$ and $F$ are assumed to be a metric and a closed 4-form on a compact Riemannian 11-dimensional manifold $M^{11}$. Then the following hold

(a) The 4-form $F$ remains closed as long as the flow exists.
(b) The flow exists at least for a time interval $[0, T_0)$ for some $T_0 > 0$;
(c) If $T < \infty$ is the maximum existence time of the flow (1.3) and (1.4) then

$$\limsup_{t \to T^-} \sup_{M^{11}} (|Rm| + |F|) = \infty.$$  

The estimates for the higher order derivatives of $Rm$ and $F$ are described in §3.

Eleven-dimensional supergravity is a highly constrained physical theory, and arguably geometrically the simplest among the low-energy effective limits of M Theory. As such, it is natural to expect that its solutions may lead to some new canonical structure on eleven-dimensional manifolds and their compactifications.

Perhaps not surprisingly, the flows from the dimensional reductions of 11D supergravity and of its Euclidean version are all Ricci flows, coupled to tensor fields of different ranks. Coupled Ricci flows have been considered before in the mathematics literature [26, 27, 29, 21], in the simplest case of coupling to a single scalar field. A Perelman pseudo-locality theorem in the case of couplings to a single scalar field was obtained in [21]. The coupled flows introduced here will probably require many new techniques, which may also be interesting in their own right from the point of view of the theory of non-linear partial differential equations.

## 2 Dimensional Reductions

The main goal of this section is to establish part (a) of Theorem 1. Parts (b,c) of Theorem 1 as well as Theorem 2 require estimates and will be established in subsequent sections.

We use indices $A, B, C, \ldots$ for coordinates on $M^{11}$, indices $a, b, c, \ldots$ for coordinates on $M^{10-p}$, and indices $i, j, k, \ldots$ for coordinates on $M^{10-p}$. Then the curvature of the warped product (1.5) is given by

$$R_{ijkl} = \hat{R}_{ijkl}, \quad R_{iajb} = -e_f \left( \frac{1}{2} (\nabla^2 g f)_{ij} + \frac{1}{4} f_i f_j \right) \tilde{g}_{ab}$$

$$R_{abcd} = e_f \tilde{R}_{abcd} + \frac{1}{4} e^{2f} |\nabla_{\tilde{g}} f|^2 (\tilde{g}_{ad} \tilde{g}_{bc} - \tilde{g}_{ac} \tilde{g}_{bd}).$$
Contracting gives the relation between the Ricci curvatures,

\[ \text{Ric}(g)_{ab} = \text{Ric}(\tilde{g})_{ab} - \frac{1}{2} e^f (\Delta g f + \frac{p+1}{2} |\nabla g f|_{\tilde{g}}^2) \tilde{g}_{ab} \]
\[ \text{Ric}(g)_{ij} = \text{Ric}(\tilde{g})_{ij} - \frac{p+1}{2} \left( (\nabla^2 g f)_{ij} + \frac{1}{2} f_i f_j \right) \]

All the other Ricci curvature components vanish, i.e. \( \text{Ric}(g)_{ia} = 0 \), for all \( i, a \).

Using these formulas, part (a) of Theorem 1 can now be proved by a direct calculation. However, it may be more instructive to proceed in a way that shows how the dimensionally reduced flows (1.6) - (1.9) arise. While the space-time \( M^{11} \) is always taken to be the warped product (1.5), we shall consider different ansatze for the 4-form \( F \).

### 2.1 Possible ansatze

Ideally we would like our ansatze for \( F \) to be as general as possible. Taking the product structure into account, a very general ansatz for \( F \) is the following

\[ F = \alpha_4 + \alpha_3 \wedge \beta_1 + \alpha_2 \wedge \beta_2 + \alpha_1 \wedge \beta_3 + \Psi, \]

where \( \alpha_j \) are \( j \)-forms on \( M^{1,p} \), \( \beta_j \) are \( j \)-forms on \( M^{10-p} \), and \( \Psi \) a 4-form on \( M^{10-p} \). As \( F \) is closed, we want all these forms to be closed. Under such an ansatz and if \( \tilde{g} \) is Einstein, the \( ab \)-component of the equation (1.2) becomes

\[ \left( \tilde{\lambda} - \frac{1}{2} e^f (\Delta g f + \frac{p+1}{2} |\nabla g f|_{\tilde{g}}^2) + \frac{e^f}{6} |F|_{\tilde{g}}^2 \right) \tilde{g}_{ab} \]
\[ = \frac{1}{2} \left( e^{-3f} (\alpha_4^2)_{ab} + e^{-2f} |\beta_1|^2 (\alpha_3^2)_{ab} + e^{-f} |\beta_2|^2 (\alpha_2^2)_{ab} + |\beta_3|^2 (\alpha_1^2)_{ab} \right). \]

Therefore, a natural assumption for the flow to reduce is that all \( \alpha_j \)'s satisfy

\[ (\alpha_j^2)_{ab} := \langle t_\partial_{\alpha_j}, t_\partial_{\alpha_j} \rangle = c_j \cdot \tilde{g}_{ab} \]

for some constant \( c_j \).

In Riemannian geometry, there are many such differential forms, such as all the calibration forms [23], which include powers of the Kähler form and the fundamental 3 and 4 forms in \( G_2 \) geometry. However, such forms are rare in Lorentzian geometry due to the following linear algebra fact

**Proposition 1** Let \( (V, \tilde{g}) \) be a finite-dimensional Lorentzian vector space. Suppose \( \alpha \in \wedge^j V^* \) is a \( j \)-form such that \( \alpha^2_{ab} = c \cdot \tilde{g}_{ab} \) for some constant \( c \). Then \( \alpha \) must be a constant or a constant multiple of volume form.
Proof. By choosing an orthonormal basis, we may write
\[ \tilde{g} = -(dx^0)^2 + (dx^1)^2 + \ldots + (dx^p)^2, \]
where \( \dim V = p + 1 \). Write
\[ \alpha = \frac{1}{j!} \alpha_{i_1 \ldots i_j} dx^{i_1} \wedge \ldots \wedge dx^{i_j} = \sum_{i_1 < \ldots < i_j} \alpha_{i_1 \ldots i_j} dx^{i_1} \wedge \ldots \wedge dx^{i_j}, \]
we only need to show that \( j = 0 \) or \( j = p + 1 \). It is easy to see that
\[ \alpha_{00}^2 = \sum_{1 \leq i_2 < \ldots < i_j \leq p} \alpha_{0i_2 \ldots i_j}^2 = -c \] (2.1)
and
\[ \alpha_{kk}^2 = \sum_{1 \leq i_2 < \ldots < i_j \leq p} \alpha_{ki_2 \ldots i_j}^2 - \sum_{1 \leq i_3 < \ldots < i_j \leq p} \alpha_{0ki_3 \ldots i_j}^2 = c \] (2.2)
for \( k = 1, 2, \ldots, p \). Summing (2.2) over \( k \) and add \( p \) times of (2.1), we get
\[ j \sum_{1 \leq i_1 < \ldots < i_j \leq p} \alpha_{i_1 \ldots i_j}^2 + (p + 1 - j) \sum_{1 \leq i_2 < \ldots < i_j \leq p} \alpha_{0i_2 \ldots i_j}^2 = 0. \]
If \( 0 < j < p + 1 \), we conclude from above equation that \( \alpha = 0 \), thus the proposition is proved. \( \square \)

Due to Proposition 1, the most general ansatz we shall consider takes the form
\[ F = dvol_{\tilde{g}} \wedge \beta + \Psi, \]
where \( \beta \) is a closed \((3 - p)\)-form and \( \Psi \) a closed 4-form on \( M^{10-p} \).

### 2.2 The case \( F = dvol_{\tilde{g}} \wedge \beta \)

Here for simplicity and as a warm-up, we take
\[ F = dvol_{\tilde{g}} \wedge \beta, \]
where \( \beta \) is a closed \((3 - p)\)-form on \( M^{10-p} \). We have the following formulae.
\[
*_{\tilde{g}} F = e^{-\frac{p+1}{2}f} *_{\tilde{g}} dvol_{\tilde{g}} \wedge *_{\tilde{g}} \beta = -\sigma e^{-\frac{p+1}{2}f} *_{\tilde{g}} \beta,
\]
\[
d *_{\tilde{g}} F = \sigma \frac{p + 1}{2} e^{-\frac{p+1}{2}f} df \wedge *_{\tilde{g}} \beta - \sigma e^{-\frac{p+1}{2}f} d *_{\tilde{g}} \beta,
\]
\[
*_{\tilde{g}} d *_{\tilde{g}} F = -\sigma dvol_{\tilde{g}} \wedge *_{\tilde{g}} d *_{\tilde{g}} \beta + \frac{p + 1}{2} \sigma dvol_{\tilde{g}} \wedge *_{\tilde{g}} (df \wedge *_{\tilde{g}} \beta),
\]
and

\[ d \ast_g d \ast_g F = -\sigma(-1)^{p+1}d\text{vol}_g \wedge d \ast_g d \ast_g \beta + \sigma(-1)^{p+1}\frac{p+1}{2}d\text{vol}_g \wedge d \ast_g (df \wedge \ast_g \beta), \]

\[ F \wedge F = 0. \]

From

\[ \frac{\partial F}{\partial t} = d\text{vol}_g \wedge \frac{\partial \beta}{\partial t} \]

and the equation (1.3) we get

\[ \frac{\partial \beta}{\partial t} = -(-1)^{p+1}d \ast_g d \ast_g \beta + (-1)^{p+1}\frac{p+1}{2}d \ast_g (df \wedge \ast_g \beta), \]

since \( d \ast_g \beta = (-1)^{p+1} \ast_g d \ast_g \beta \), it follows that

\[ \frac{\partial \beta}{\partial t} = -dd \ast_g \beta + (-1)^{p+1}\frac{p+1}{2}d \ast_g (df \wedge \ast_g \beta), \quad (2.3) \]

By direct calculations, we have

\[ (F^2)_{ab} = -\sigma e^{-p\int |\beta|_g^2 \tilde{g}_{ab}}, \]

and

\[ (F^2)_{ij} = -\sigma e^{-(p+1)f} (\beta^2)_{ij}, \quad |F|_g^2 = -\sigma e^{-(p+1)f} |\beta|_g^2. \]

Thus the equations (1.3) and (1.4) are reduced to ones of \((\hat{g}, f, \beta)\) on \(M^{10-p}\) as follows

\[ \frac{\partial \hat{g}_{ij}}{\partial t} = -2\text{Ric}(\hat{g})_{ij} + (p + 1)((\nabla_{\hat{g}} f)_{ij} + \frac{1}{2}f_i f_j) - \sigma e^{-(p+1)f} (\beta^2)_{ij} + \frac{1}{3} \sigma e^{-(p+1)f} |\beta|_g^2 \hat{g}_{ij}, \quad (2.4) \]

\[ \frac{\partial f}{\partial t} = \Delta_{\hat{g}} f + \frac{p+1}{2} \nabla_{\hat{g}} f |\beta|_g^2 - \frac{2}{3} \sigma e^{-(p+1)f} |\beta|_g^2 - 2 \lambda e^{-f}. \quad (2.5) \]

\[ \frac{\partial \beta}{\partial t} = -\Box_{\hat{g}} \beta + (-1)^{p+1}\frac{p+1}{2}d \ast_{\hat{g}} (df \wedge \ast_{\hat{g}} \beta), \quad (2.6) \]

where \( \Box_{\hat{g}} = dd_{\hat{g}}^! + d_{\hat{g}}^! d \) is the Hodge Laplacian with respect to \( \hat{g} \).

### 2.3 The case \( F = \Psi \)

Next we take

\[ F = \Psi, \quad dF = 0, \quad \Psi \in \wedge^4 M^{10-p}. \]

\[ \ast_g F = \sigma e^{\frac{p+1}{2}f} d\text{vol}_g \wedge \ast_g \Psi \]
\[ d \ast_g F = \sigma(-1)^{p+1} \frac{p + 1}{2} e^{\frac{p+1}{2} f} d\text{vol}_{\tilde{g}} \wedge df \wedge \ast_{\tilde{g}} \Psi + \sigma(-1)^{p+1} e^{\frac{p+1}{2} f} d\text{vol}_{\tilde{g}} \wedge d \ast_{\tilde{g}} \Psi. \]

\[ \ast_g d \ast_g F = \sigma(-1)^{p+1} \frac{p + 1}{2} \ast_{\tilde{g}} (df \wedge \ast_{\tilde{g}} \Psi) + \sigma(-1)^{p} \ast_{\tilde{g}} d \ast_{\tilde{g}} \Psi. \]

\[ d \ast_g d \ast_g F = \sigma(-1)^{p+1} \frac{p + 1}{2} d \ast_{\tilde{g}} (df \wedge \ast_{\tilde{g}} \Psi) + \sigma(-1)^{p} d \ast_{\tilde{g}} d \ast_{\tilde{g}} \Psi. \]

If \( 10 - p \leq 7 \), then \( F \wedge F = 0 \); if \( 10 - p \geq 8 \), then

\[ \ast_g (F \wedge F) = e^{\frac{p+1}{2} f} d\text{vol}_{\tilde{g}} \wedge \ast_{\tilde{g}} (\Psi \wedge \Psi), \]

\[ d \ast_{\tilde{g}} (F \wedge F) = (-1)^{p+1} \frac{p + 1}{2} e^{\frac{p+1}{2} f} d\text{vol}_{\tilde{g}} \wedge df \wedge \ast_{\tilde{g}} (\Psi \wedge \Psi) + (-1)^{p+1} e^{\frac{p+1}{2} f} d\text{vol}_{\tilde{g}} \wedge d \ast_{\tilde{g}} (\Psi \wedge \Psi) \]

Therefore, in case \( 10 - p \geq 8 \), the condition that \( F \) remains a closed 4-form \( \Psi \) on \( M^{10-p} \) in general may not be preserved along the flow. So here we assume \( 10 - p \leq 7 \) and for dimension reason \( F \wedge F = 0 \), and the equation (1.3) for \( F \) becomes

\[ \frac{\partial \Psi}{\partial t} = (-1)^p d \ast_{\tilde{g}} d \ast_{\tilde{g}} \Psi + (-1)^p \frac{p + 1}{2} d \ast_{\tilde{g}} (df \wedge \ast_{\tilde{g}} \Psi), \]

since \( d^4_{\tilde{g}} \Psi = (-1)^p \ast_{\tilde{g}} d \ast_{\tilde{g}} \Psi \), the equation above is

\[ \frac{\partial \Psi}{\partial t} = -dd^4_{\tilde{g}} \Psi + (-1)^p \frac{p + 1}{2} d \ast_{\tilde{g}} (df \wedge \ast_{\tilde{g}} \Psi), \]

Straightforward calculations show that

\[ (F^2)_{ab} = 0, \quad (F^2)_{ij} = (\Psi^2)_{ij}, \quad |F|_{\tilde{g}}^2 = |\Psi|_{\tilde{g}}^2 \]

Hence the flow equations (1.3) and (1.4) are reduced to the following equations on the Riemannian part \( M^{10-p} \) with \( 10 - p \leq 7 \)

\[ \frac{\partial \hat{g}_{ij}}{\partial t} = -2\text{Ric}(\hat{g})_{ij} + (p + 1)((\nabla_{\hat{g}}^2 f)_{ij} + \frac{1}{2} f_i f_j) + (\Psi^2)_{ij} - \frac{1}{3} |\Psi|_{\tilde{g}}^2 \hat{g}_{ij}. \quad (2.7) \]

\[ \frac{\partial f}{\partial t} = \Delta_{\tilde{g}} f + \frac{p + 1}{2} |\nabla_{\tilde{g}} f|_{\tilde{g}}^2 - \frac{1}{3} |\Psi|_{\tilde{g}}^2 - 2\bar{\lambda} e^{-f}, \quad (2.8) \]

\[ \frac{\partial \Psi}{\partial t} = -\Box_{\tilde{g}} \Psi + (-1)^p \frac{p + 1}{2} d \ast_{\tilde{g}} (df \wedge \ast_{\tilde{g}} \Psi). \quad (2.9) \]
2.4 The case \( F = d\text{vol}_g \wedge \beta + \Psi \)

In this case, we assume \( \beta = 0 \) if \( \dim M^{1-p} = 1 + p \geq 5 \). For dimensional reasons we also have \( \Psi \wedge \Psi = 0 \) in the equations below if \( \dim M^{10-p} = 10 - p \leq 7 \).

Let us now take
\[
F = d\text{vol}_g \wedge \beta + \Psi,
\]
where \( \beta \) and \( \Psi \) are chosen as in the previous two cases. Similar to the previous equations we have
\[
d \ast_g d \ast_g F = -\sigma(-1)^{p+1} d\text{vol}_g \wedge d \ast_g d \ast_g \beta + \sigma(-1)^{p+1} \frac{p+1}{2} d\text{vol}_g \wedge d \ast_g (df \wedge \ast_g \beta)
\]
\[
+ \sigma(-1)^p \frac{p+1}{2} d \ast_g (df \wedge \ast_g \Psi) + \sigma(-1)^p d \ast_g d \ast_g \Psi.
\]

From
\[
F \wedge F = 2d\text{vol}_g \wedge \beta \wedge \Psi + \Psi \wedge \Psi
\]
we get
\[
d \ast_g (F \wedge F) = \sigma(p+1)e^{-\frac{p+1}{2}f} df \wedge \ast_g \Psi - 2\sigma e^{-\frac{p+1}{2}f} d \ast_g \Psi
\]
\[
+ (-1)^p \frac{p+1}{2} e^{\frac{p+1}{2}f} d\text{vol}_g \wedge df \wedge \ast_g (\Psi \wedge \Psi) + (-1)^p e^{\frac{p+1}{2}f} d\text{vol}_g \wedge d \ast_g (\Psi \wedge \Psi)
\]

From the equation (1.3) on \( F \), we can derive that
\[
\frac{\partial \beta}{\partial t} = -(-1)^p d \ast_g d \ast_g \beta + (-1)^{p+1} \frac{p+1}{2} d \ast_g (df \wedge \ast_g \beta) \quad (2.10)
\]
\[
-\sigma(-1)^{p+1} \frac{p+1}{4} e^{\frac{p+1}{2}f} df \wedge \ast_g (\Psi \wedge \Psi) - \sigma(-1)^p \frac{p+1}{2} e^{\frac{p+1}{2}f} d \ast_g (\Psi \wedge \Psi),
\]
and
\[
\frac{\partial \Psi}{\partial t} = (-1)^p d \ast_g d \ast_g \Psi + (-1)^{p+1} \frac{p+1}{2} d \ast_g (df \wedge \ast_g \Psi) \quad (2.11)
\]
\[
-\frac{p+1}{2} e^{\frac{p+1}{2}f} df \wedge \ast_g \Psi + e^{\frac{p+1}{2}f} d \ast_g \Psi.
\]

We also have
\[
(F^2)_{ab} = -\sigma e^{-f} |\beta|_g^2 g_{ab}, \quad (F^2)_{ij} = -\sigma e^{-f} (\beta^2)_{ij} + (\Psi^2)_{ij}
\]
\[
|F|^2 = -\sigma e^{-f} |\beta|_g^2 + |\Psi|_g^2.
\]

Then the equations (1.3) and (1.4) are reduced to \( M^{10-p} \) as follows:
\[
\frac{\partial \hat{g}_{ij}}{\partial t} = -2\text{Ric}(\hat{g})_{ij} + (p+1)((\nabla^2 \hat{g})_{ij} + \frac{1}{2} f_i f_j) - \sigma e^{-f} (\beta^2)_{ij} \quad (2.12)
\]
\[
+ (\Psi^2)_{ij} - \frac{1}{3} (-\sigma e^{-f} |\beta|_g^2 + |\Psi|_g^2) \hat{g}_{ij}
\]
\[ \frac{\partial f}{\partial t} = \Delta_g f + \frac{p+1}{2} |\nabla_g f|^2 - \frac{2}{3} \sigma e^{-(p+1)f} (\beta^2)_{ij} - \frac{1}{3} |\Psi|^2 \hat{g} - 2\lambda e^{-f}. \]  

(2.13)

and combining with equations (2.10) and (2.11) and the assumption \( d\beta = 0 \) and \( d\Psi = 0 \) we derive the evolution equations for \( \beta \) and \( \Psi \), which are the main equations (1.8) and (1.9) we are going to study:

\[ \frac{\partial \beta}{\partial t} = -\Box_g \beta + (-1)^{p+1} \frac{p+1}{2} d \ast_g (df \wedge \ast_g \beta) \]  

(2.14)

\[ -\sigma(-1)^{p} \frac{p+1}{4} e^{\frac{p+1}{2}f} df \wedge \ast_g (\Psi \wedge \Psi) - \frac{(-1)^{p} \sigma}{2} e^{\frac{p+1}{2}f} d \ast_g (\Psi \wedge \Psi), \]

and

\[ \frac{\partial \Psi}{\partial t} = -\Box_g \Psi + (-1)^{p} \frac{p+1}{2} d \ast_g (df \wedge \ast_g \Psi) \]  

(2.15)

\[ -\frac{p+1}{2} e^{-\frac{p+1}{2}f} df \wedge \ast_g \Psi + e^{-\frac{p+1}{2}f} d \ast_g \Psi. \]

This case in section 2.4 is the most general of all three cases in sections 2.2, 2.3 and 2.4, and it is the one considered in Theorem 1.

3 Estimates for the 11D Euclidean Flow

In this section, we take \( \sigma = -1 \) in (1.3). We prove first Theorem 2, as the flow (1.3) and (1.4) is technically simpler than the dimensionally reduced flow (1.6) - (1.9) considered in Theorem 1. Since in this section we consider general compact Riemannian manifolds \( M^{11} \) and not any particular warped product, we shall lighten the notation and use \( i, j, k, \ldots \) to index coordinates on \( M^{11} \) instead of \( A, B, C, \ldots \) as in the other sections.

Part (a) of Theorem 2 follows easily from the fact that \( d\Box_g = \Box_g d \) and hence \( \partial_t (dF) = -\Box_g (dF) \). In particular, if \( dF \) is 0 at time \( t = 0 \), then \( dF(t) \) is 0 for all time \( t > 0 \) as long as the solution exists.

3.1 The short-time existence of the flow

Next we establish part (b). We adapt DeTurck’s trick to make the equations (1.3) and (1.4) strictly parabolic after re-parameterizations. We consider the following flow for a metric \( \hat{g}_{ij} \) (not to be confused with the metric \( \hat{g}_{ij} \) in warped products considered earlier) and a closed 4-form \( \hat{F} \) on \( M^{11} \),

\[ \frac{\partial \hat{F}}{\partial t} = -\Box_g \hat{F} + \frac{1}{2} d(\ast(\hat{F} \wedge \hat{F}))+ L_V \hat{F} \]

\[ \frac{\partial \hat{g}_{ij}}{\partial t} = -2 \text{Ric}(\hat{g})_{ij} + \nabla_i V_j + \nabla_j V_i + (\hat{F}^2)_{ij} - \frac{1}{3} |\hat{F}|^2 \hat{g}_{ij}, \]  

(3.1)
where the vector field \( V = V_i \frac{\partial}{\partial x^i} \) is given by
\[
V_i = \hat{g}_{ij} \hat{g}^{kl} (\hat{\Gamma}_{kl} \Gamma(g_0)_{ij} - \Gamma(g_0)_{ij})
\]
\[
\hat{\Gamma}_{ij} = \frac{1}{2} \hat{g}^{kp} (\partial_k \hat{g}_{pl} + \partial_l \hat{g}_{pk} - \partial_p \hat{g}_{kl})
\]
is the Christoffel symbol of \( \hat{g} \), and similar definition for \( \Gamma(g_0) \) and \( g_0 \) is the initial metric. \( L_V \hat{F} \) denotes the Lie derivative of the 4-form \( \hat{F} \) in the direction of \( V \).

**Lemma 1** The operators \( \hat{g}_{ij} \mapsto -2\text{Ric}(\hat{g})_{ij} + \hat{\nabla}_i V_j + \hat{\nabla}_j V_i \) and \( \hat{F} \mapsto -\Box_{\hat{g}} \hat{F} \) are both strictly elliptic.

**Proof.** This lemma is well-known as the DeTurck’s trick in the study of Ricci flow. For the convenience of readers, we provide a proof below.

The ellipticity of the operator on \( \hat{g}_{ij} \) follows from a straightforward calculation. At any fixed point \( p \in M \), we choose normal coordinates for \( g_0 \) so that \( (g_0(p))_{ij} = \delta_{ij} \) and \( dg_0(p) = 0 \). Then at \( p \) we have
\[
\hat{\nabla}_i V_j + \hat{\nabla}_j V_i = \partial_{\hat{g}_{jk}} \hat{g}^{pq} \hat{\Gamma}_{pk} + \partial_j \hat{g}_{ik} \hat{g}^{pq} \hat{\Gamma}_{k}^{\ pq} + \hat{g}_{jk} \partial_{\hat{g}}^{pq} \hat{\Gamma}_{pk}^{\ pq} + \hat{g}_{ik} \partial_j \hat{g}^{pq} \hat{\Gamma}_{pk}^{\ pq} + \hat{g}_{jk} \hat{g}^{pq} \partial_p \Gamma(g_0)^k_{pq} - \hat{g}_{jk} \hat{g}^{pq} \partial_j \Gamma(g_0)^k_{pq} - 2 \hat{g}_{pk} \hat{g}^{st} \hat{\Gamma}_{ps} \hat{\Gamma}_{ij},
\]
and
\[
\text{Ric}(\hat{g})_{ij} = \partial_{\hat{g}} \hat{\Gamma}_{ij} - \partial_j \hat{\Gamma}_{ki} + \hat{\Gamma}_{jk} \hat{\Gamma}_{pi} - \hat{\Gamma}_{ki} \hat{\Gamma}_{pj},
\]
so the leading (i.e. second) order terms in \(-2\text{Ric}(\hat{g})_{ij} + \hat{\nabla}_i V_j + \hat{\nabla}_j V_i \) are given by
\[
-2 \partial_{\hat{g}} \hat{\Gamma}_{ij}^{\ k} + 2 \partial_j \hat{\Gamma}_{ki} + \hat{\Gamma}_{jk} \hat{\Gamma}_{pi} + \hat{\Gamma}_{ki} \hat{\Gamma}_{pj}
\]
\[
= \hat{g}^{kl} \frac{\partial^2 \hat{g}_{ij}}{\partial x^k \partial x^l} - 2 \partial_k \hat{g}^{kl} \hat{\Gamma}_{ij} \hat{\Gamma}_{kl} + 2 \partial_j \hat{g}^{kl} \hat{\Gamma}_{ki} \hat{\Gamma}_{pl} + \hat{g}_{jk} \hat{g}^{pq} \partial_p \Gamma(g_0)^k_{pq} + \hat{g}_{ik} \hat{g}^{pq} \partial_j \Gamma(g_0)^k_{pq} - 2 \hat{g}_{pk} \hat{g}^{st} \hat{\Gamma}_{ps} \hat{\Gamma}_{ij},
\]
establishing our claim.

As for the operator on \( \hat{F} \), it is the negative Hodge-Laplacian, and its strict ellipticity is well-known. More explicitly, by the Lichnerowicz-Weitzenböck formula, we have
\[
-\Box_{\hat{g}} \hat{F} = \hat{g}^{pq} \nabla_p \nabla_q \hat{F} + \hat{F} \ast \hat{Rm}
\]
where \( \hat{F} \ast \hat{Rm} \) denotes the terms which are pointwise linear combination of the product of \( \hat{F} \) and \( \hat{Rm} \). The strict ellipticity of \( -\Box_{\hat{g}} \) follows.

Lemma 1 implies that the system (3.1) is a system of strictly parabolic differential equations, so the short time existence of \((\hat{g}(t), \hat{F}(t))\) for some time interval \([0, T_0)\) with \( T_0 > 0 \) follows from standard theory of parabolic partial differential equations.

Let \( \varphi_t : M \to M \) be the one-parameter subgroup of diffeomorphisms generated by \(-V\), and define
\[
g(t) = \varphi_t^* \hat{g}(t), \quad F(t) = \varphi_t^* \hat{F}(t).
\]
It is straightforward to check that \((g(t), F(t))\) solves the equations (1.3) and (1.4). This proves part (b) of Theorem 2.
3.2 The long-time existence of the flow

We now establish part (c) of Theorem 2, which is that the flow exists as long as the $|Rm|$ and $|F|^2$ remain bounded. For this, we will prove that boundedness of $|Rm|$ and $|F|^2$ implies that all the higher order derivatives of $(g, F)$ remain bounded along the flow (1.3) and (1.4). This type of estimates have been studied extensively for other geometric flows like Ricci flow or Anomaly flow [for example [22, 37, 26, 33]].

We will use the constant $C$ to denote a uniform constant depending only on the dimension $n = 11$, which may vary from line to line. We list the evolution equations of some key quantities. They can be obtained by a direct calculation, so we omit the proof.

Lemma 2 Along the flow (1.3) and (1.4), $|F|^2$ satisfies

$$
\frac{\partial}{\partial t}|F|^2 = \Delta_g |F|^2 - 2|\nabla F|^2 + Rm * F * F + \frac{1}{2} \cdot 4! F^{ijkl}(d \cdot (F \wedge F))_{ijkl}
$$

$$
+ \left(2R_{ij} - (F^2)_{ij} + \frac{1}{3}|F|^2 g_{ij}\right)(F^2)_{ij}
$$

$$
\leq \Delta |F|^2 - 2|\nabla F|^2 + C|Rm||F|^2 + c_0|\nabla F||F|^2 - \frac{4}{33}|F|^4,
$$

for some uniform constants $C > 0$ and $c_0 \geq 0$.

In the last inequality we apply the Cauchy-Schwarz inequality $F^2_{ij} F^2_{ij} \geq \frac{(g^2 F^2)_{ij}}{11} = \frac{16}{11}(|F|^2)^2$. We remark that the term involving $c_0$ comes from $F^{ijkl}(d \cdot (F \wedge F))_{ijkl}$, and if this number $c_0$ is small, say $c_0^2 \leq \frac{32}{33}$, then the long time existence of the flow can be weakened to the condition that $|Rm|$ remains bounded along the flow, since the bound on $|F|^2$ can be derive from the equation above by maximum principle. In particular, if for some choice of $F$, $F \wedge F = 0$ along the flow (e.g. in the dimension reduction of the flow), this observation may be applied.

Lemma 3 The evolution equations for the curvatures are given by

$$
\frac{\partial}{\partial t} R_{ijkl} = \Delta R_{ijkl} + 2R_{mlbk} R_{mjni} - 2R_{mkmj} R_{mnli} + 2R_{nkmi} R_{mlnj} - 2R_{mlni} R_{mkjn} - R_{ml} R_{mjk} - R_{mj} R_{imkl} - R_{mk} R_{ijml} - R_{ml} R_{ijkm} + \frac{1}{2} (F^2)_{ml} R_{mkji} + \frac{3}{2} (F^2)_{km} R_{mlji} - \frac{1}{3} |F|^2 R_{lkji} + \frac{1}{2} ((F^2)_{kj,i} - (F^2)_{jl,ki} - (F^2)_{kl,ij} - (F^2)_{ki,lj}) + \frac{1}{6} \left( - g_{kj}(|F|^2)_{ji} + g_{jl}(|F|^2)_{ki} + g_{ij}(|F|^2)_{kl} + g_{ki}(|F|^2)_{lj} \right) = \Delta R_{ijkl} + Rm * Rm + F * F * Rm + \nabla^2 F * F + \nabla F * \nabla F,
$$

where for a tensor $\alpha$ for simplicity we denote $\alpha_i$ to be the covariant derivative $\nabla_i \alpha$. 

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From now on, for a given \( T > 0 \), we assume that there is a finite constant \( K > 0 \) such that
\[
\sup_{M \times [0,T]} (|Rm| + |F|^2) \leq K.
\]

**Lemma 4** There exists a constant \( C_{K,T} \) depending only on \( K, T \) such that
\[
\sup_M \| \nabla F \|^2 \leq \frac{C_{K,T}}{t}, \quad \forall t \in (0, T).
\]

To prove Lemma 4, we need the evolution equation for the quantity \( |\nabla F|^2 = \frac{1}{4!} R_{ijkl,p} R^{ijkl,p} \).

**Lemma 5** The quantity \( |\nabla F|^2 \) satisfies
\[
\frac{\partial}{\partial t} |\nabla F|^2 \leq \Delta |\nabla F|^2 - 2|\nabla^2 F|^2 + C \left( |Rm||\nabla F|^2 + |\nabla Rm||F||\nabla F| + |\nabla F|^3 + |F||\nabla F||\nabla^2 F| + |F|^2|\nabla F|^2 \right),
\]
for some constant \( C = C(K) > 0 \).

**Proof:** By definition
\[
R_{ijkl,p} = \partial_p R_{ijkl} - F_{qijkl} \Gamma^q_{pi} - F_{ijkl} \Gamma^q_{pj} - F_{ijql} \Gamma^q_{pk} - F_{ijkq} \Gamma^q_{pl},
\]
where \( \Gamma^k_{ij} = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) \) is the Christoffel symbol of the metric \( g \) which satisfies the evolution equation
\[
\frac{\partial}{\partial t} \Gamma^k_{ij} = \frac{1}{2} g^{kl} \left( \nabla_i ( - 2R_{jl} + (F^2)_{jl} - \frac{1}{3} |F|^2 g_{jl}) + \nabla_j ( - 2R_{il} + (F^2)_{il} - \frac{1}{3} |F|^2 g_{il}) \right)
- \nabla_l ( - 2R_{ij} + (F^2)_{ij} - \frac{1}{3} |F|^2 g_{ij})
= \nabla \text{Ric} + F \ast \nabla F.
\]

Hence it follows that
\[
\frac{\partial}{\partial t} F_{ijkl,p} = \nabla_p \frac{\partial F_{ijkl}}{\partial t} - F \ast \frac{\partial \Gamma}{\partial t}
= \nabla_p (\Delta F_{ijkl} + Rm \ast F + F \ast F) + F \ast \nabla \text{Ric} + F \ast F \ast \nabla F
= \nabla_p \Delta F_{ijkl} + \nabla Rm \ast F + Rm \ast \nabla F + F \ast \nabla F + F \ast F \ast \nabla F.
\]

Differentiating the formula (3.2) and commuting derivatives, we obtain
\[
\nabla_p (-\Box g F_{ijkl}) = \Delta F_{ijkl,p} + \nabla Rm \ast F + Rm \ast \nabla F
\]
thus
\[
\frac{\partial}{\partial t} \nabla F = \Delta \nabla F + \nabla Rm \ast F + Rm \ast \nabla F + F \ast \nabla F + F \ast F \ast \nabla F.
\]
The lemma follows from this equation and the definition $|\nabla F|^2 = \frac{1}{4} F_{ijkl,p} F^{ijkl,p}$.

**Proof of Lemma 4:** We will use a constant $C_K$ to denote a constant depending only on $K$ and $n = 11$, which may vary from line to line. Set

$$G_1 := t \left( (|F|^2 + A)|\nabla F|^2 + |Rm|^2 \right) + A_1 |F|^2,$$

for some constants $A > 0$, $A_1 > 0$ to be determined. From the equations in Lemma 3 and Lemma 5, we have

$$\left( \frac{\partial}{\partial t} - \Delta \right) ( (|F|^2 + A)|\nabla F|^2 + |Rm|^2 )$$

$$\leq -|\nabla F|^4 + C_K |\nabla F|^2 - 2(A + |F|^2)|\nabla^2 F|^2 + C_K (A + |F|^2)|\nabla F|^2$$

$$+ C_K (A + |F|^2)|\nabla F|^2 |\nabla Rm| + C_K (A + |F|^2)|\nabla F|^3 - 2(\nabla |F|^2, \nabla |\nabla F|^2)$$

$$- 2|\nabla Rm|^2 + C_K |\nabla^2 F|^2 + C_K.$$

We observe that by Kato’s inequality

$$|2(\nabla |F|^2, \nabla |\nabla F|^2)| \leq C_K |\nabla F|^2 |\nabla^2 F| \leq \frac{1}{10} |\nabla F|^4 + C_K |\nabla^2 F|^2$$

and

$$C_K (A + |F|^2)|\nabla^2 F| |\nabla F| \leq C_K^2 (A + |F|^2)^2 |\nabla F|^2 + \frac{1}{10} |\nabla^2 F|^2,$$

$$C_K (A + |F|^2)|\nabla Rm| |\nabla F| \leq C_K^2 (A + |F|^2)^2 |\nabla F|^2 + \frac{1}{10} |\nabla Rm|^2,$$

and by Young’s inequality

$$C_K (A + |F|^2)|\nabla F|^3 \leq \frac{1}{10} |\nabla F|^4 + C_K (A + |F|^2)^4.$$

Combining the above inequalities and choosing $A$ sufficiently large (but depending only on $K$) so that the terms involving $|\nabla^2 F|^2$ and $|F|^4$ can be absorbed, we get

$$\left( \frac{\partial}{\partial t} - \Delta \right) ( (|F|^2 + A)|\nabla F|^2 + |Rm|^2 ) \leq -A |\nabla^2 F|^2 - \frac{1}{2} |\nabla F|^4 - |\nabla Rm|^2 + C_K. \quad (3.3)$$

Therefore if $A_1$ is chosen to be large enough, then

$$\left( \frac{\partial}{\partial t} - \Delta \right) G_1 = t \left( \frac{\partial}{\partial t} - \Delta \right) ( (|F|^2 + A)|\nabla F|^2 + |Rm|^2 ) + ( (|F|^2 + A)|\nabla F|^2 + |Rm|^2 )$$

$$+ A_1 (\frac{\partial}{\partial t} - \Delta)|F|^2$$

$$\leq t C_K + (|F|^2 + A)|\nabla F|^2 + |Rm|^2 - A_1 |\nabla F|^2 + A_1 C_K$$

$$\leq C_{K,T}$$
The estimate that $G_1 \leq C_{K,T}$ follows from the maximum principle. Lemma 4 is proved from the definition of $G_1$. \qed

Next we have the following higher order derivative estimates.

**Lemma 6** There exists a constant $C_{K,T} > 0$ such that

$$\sup_M (|\nabla Rm| + |\nabla^2 F|) \leq \frac{C_{K,T}}{t}, \quad \forall t \in (0, T).$$

To prove Lemma 6, we need the following equations which follow from standard calculations so we omit the proof.

**Lemma 7** We have the following equations:

$$\frac{\partial}{\partial t} \nabla^2 F = \Delta \nabla^2 F + \nabla^2 F \ast Rm + \nabla F \ast \nabla Rm + F \ast \nabla^2 Rm + \nabla^2 F \ast \nabla F$$

$$+ \nabla^3 F \ast F + F \ast \nabla F \ast \nabla F + F \ast F \ast \nabla^2 F.$$

$$\frac{\partial}{\partial t} \nabla Rm = \Delta \nabla Rm + \nabla Rm \ast Rm + \nabla Rm \ast F + Rm \ast F \ast \nabla F$$

$$+ \nabla^3 F \ast F + \nabla F \ast \nabla^2 F.$$

**Proof of Lemma 6:** From Lemma 7 we have the following inequalities:

$$\frac{\partial}{\partial t} |\nabla^2 F|^2 \leq \Delta |\nabla^2 F|^2 - 2|\nabla^3 F|^2 + C_{K,T}\left(|\nabla^2 F|^2 + |\nabla F||\nabla^2 F||\nabla Rm| + |\nabla^2 Rm||\nabla^2 F|\right.$$  

$$+ |\nabla F||\nabla^2 F| + |\nabla^2 F||\nabla^3 F| + |\nabla F|^2|\nabla^2 F|^2\right),$$

and

$$\frac{\partial}{\partial t} |\nabla Rm|^2 \leq \Delta |\nabla Rm|^2 - 2|\nabla^2 Rm|^2 + C_K|\nabla Rm|^2 + C_K|\nabla Rm||\nabla F|$$

$$+ C_K|\nabla Rm||\nabla^3 F| + C|\nabla F||\nabla^2 F||\nabla Rm|.$$  

By Cauchy-Schwarz inequality, we have

$$(\frac{\partial}{\partial t} - \Delta)(|\nabla^2 F|^2 + |\nabla Rm|^2)$$

$$\leq -|\nabla^3 F|^2 + C_K(1 + |\nabla F| + |\nabla F|^2)|\nabla^2 F|^2 + C_K|\nabla F|^2$$

$$- |\nabla^2 Rm|^2 + C_K|\nabla Rm|^2$$

We define a quantity

$$G_2 := t^2(|\nabla^2 F|^2 + |\nabla Rm|^2) + A_1 t((A + |F|^2)|\nabla F|^2 + |Rm|^2) + A_2 |F|^2,$$
for the constants $A > 0$, $A_1 > 0$, $A_2 > 0$ to be determined. We calculate using the equation (3.3) that
\[
\left(\frac{\partial}{\partial t} - \Delta\right)G_2 \leq 2t(|\nabla^2 F|^2 + |\nabla Rm|^2) - t^2|\nabla^3 F|^2 + t^2C_K\left(1 + |\nabla F| + |\nabla Rm|^2\right) + t^2C_K|\nabla F|^2
\]
\[
- t^2|\nabla^2 Rm|^2 + C_Kt^2|\nabla Rm|^2 + A_1((A + |F|^2)|\nabla F|^2 + |Rm|^2) + C_K A_1 t
\]
\[
- A A_1 t |\nabla^2 F|^2 - \frac{1}{2} A_1 t |\nabla F|^4 - A_1 t |\nabla Rm|^2 - A_2 |\nabla F|^2 + C_K
\]
\[
\leq C_{K,T},
\]
here the last inequality is obtained by first choosing $A_1$ large which depends only on $K$ and $T$, then picking $A_2$ big enough. Applying maximum principle to $G_2$, we get $\sup_{M \times [0,T)} G_2 \leq C_{K,T}$, and this proves Lemma 6.

In general the following higher order estimates can be proved similarly by induction.

**Theorem 3** There exists a constant $C_{K,T,m}$ for any $m \in \mathbb{Z}_+$ such that
\[
\sup_M (|\nabla^{m-1} Rm| + |\nabla^m F|) \leq \frac{C_{K,T,m}}{t^{m/2}}, \quad \forall \ t \in (0, T).
\]

**Proof:** In the cases $m = 1$ or $m = 2$, this inequality is proved in Lemma 4 and Lemma 6. For general $m$, we write
\[
G_m := t^m(|\nabla^m F|^2 + |\nabla^{m-1} Rm|^2)
\]
\[
+ \sum_{i=2}^{m-1} A_i t^i(|\nabla^i F|^2 + |\nabla^{i-1} Rm|^2) + A_1 t((A_0 + |F|^2)|\nabla F|^2 + |Rm|^2) + B|F|^2,
\]
for suitable choices of constants $A_i > 0$ and $B > 0$, by an induction argument similar to (but simpler than) the proof of Theorem 4 in next section, we can prove that
\[
\left(\frac{\partial}{\partial t} - \Delta\right)G_m \leq C_{K,T,m},
\]
from which we get $\sup_{M \times [0,T)} G_m \leq C_{K,T,m}$, which implies the desired inequality. \hfill \Box

We can now prove part (c) of Theorem 2. We argue by contradiction. If
\[
\lim_{t \to T^-} \sup_M (|Rm| + |F|) =: K < \infty.
\]
Then Theorem 3 shows that all the higher order derivatives of $(g, F)$ are bounded uniformly. Thus they converge smoothly to some tuple $(g_T, F_T)$. Applying the short time existence of the flow starting at $(g_T, F_T)$, we see that the flow can be continued through $T$, contradicting the maximal existence time $T$. \hfill \Box
4 Estimates for the Dimensionally Reduced Flow

We come now to the proof of Theorem 1. Once again, part (a) is easy because \( \partial_t \alpha = -d \Box \alpha = -\Delta \alpha \) and \( \partial_t \alpha = -d \Box \alpha = -\Delta \alpha \). Thus \( \alpha \) and \( \Delta \alpha \) continue to vanish for all time if they vanish at time \( t = 0 \). Part (b) is established in the same way as part (b) of Theorem 2: the leading terms in the flow for \( f, \beta, \Psi \) are Laplacians with respect to the Riemannian metric \( \hat{g} \), and hence strictly elliptic, while the leading term in the flow for \( \hat{g} \) can also be made elliptic by DeTurck’s trick as in Lemma 1. The short-time existence of the flow follows again from the standard theory of parabolic partial differential equations.

The main step is to establish part (c). For this, we apply diffeomorphisms to simplify the evolution equations as follows: let \( \varphi_t : M^{10-p} \to M^{10-p} \) be the one-parameter diffeomorphisms generated by \( \xi_{\varphi_t} \) after pulled back by \( \varphi_t \), e.g. \( \varphi_t = \varphi_t \). We assume the initial values are \( (g, f, \beta, \Sigma) \). We may normalize \( g, f, \beta, \Sigma \) to be 0 or \( \pm 1 \) from now on.

Assume now that the flow exists on the time interval \([0, T)\), and that there exists a finite constant \( K > 0 \), such that

\[
\sup_{M^{10-p} \times [0, T)} (|Rm| + |f| + |\beta| + |\Sigma|) \leq K. \tag{4.8}
\]

We are going to show that the higher order derivatives of \( (g, f, \beta, \Sigma) \) are bounded on the
Similarly we have

$$\sup_{M^{10-p}} |\nabla f|^2 = \frac{C}{t}, \quad \text{for any } t \in (0, T).$$

**Proof.** By straightforward calculations, we have

$$\left(\frac{\partial}{\partial t} - \Delta \right) |\nabla f|^2 = -2|\nabla^2 f|^2 - \frac{p+1}{2} |\nabla f|^4 + \sigma e^{-\langle p+1 \rangle} f^2 |\beta|^2 |\nabla \beta|^2 - \frac{1}{6} |\nabla f|^2$$

Thus for some constant $C_K > 0$

$$\left(\frac{\partial}{\partial t} - \Delta \right) |\nabla f|^2 \leq -2|\nabla^2 f|^2 - \frac{p+1}{2} |\nabla f|^4 + C_K(|\nabla f|^2 + |\nabla f||\nabla \beta| + |\nabla f||\nabla \Psi|). \quad (4.9)$$

To control the terms involving $|\nabla \beta|$ and $|\nabla \Psi|$, we calculate the equations for $|\beta|^2$ and $|\Psi|^2$. From (4.6), we have

$$\left(\frac{\partial}{\partial t} - \Delta \right) |\beta|^2 = 2 \left( R_{ij} - \frac{p+1}{4} f_i f_j + \frac{1}{2} e^{-\langle p+1 \rangle} f^2 (\beta^2)_{ij} \right) - 2|\nabla \beta|^2 + Rm \ast \beta \ast \beta + \nabla^2 f \ast \beta \ast \beta + \nabla f \ast \nabla \beta + e^{\langle p+1 \rangle} (\nabla f \ast \beta \ast \nabla \Psi \ast \Psi)$$

so

$$\left(\frac{\partial}{\partial t} - \Delta \right) |\beta|^2 \leq -2|\nabla \beta|^2 + C_K(|\nabla \beta|^2 + |\nabla f||\nabla \beta| + |\nabla f| + |\nabla \Psi| + 1). \quad (4.10)$$

Similarly we have

$$\frac{\partial}{\partial t} |\Psi|^2 = \Delta |\Psi|^2 - 2|\nabla \Psi|^2 + 2 \left( R_{ij} - \frac{p+1}{4} f_i f_j + \frac{1}{2} e^{-\langle p+1 \rangle} f^2 (\beta^2)_{ij} - \frac{1}{2} (\Psi^2)_{ij} \right)$$

$$- \frac{1}{6} (\sigma e^{-\langle p+1 \rangle} f \ast |\beta|^2)_{ij} + Rm \ast \Psi \ast \Psi + e^{\langle p+1 \rangle} (\nabla f \ast \beta \ast \nabla \Psi \ast \Psi)$$

$$\leq \Delta |\Psi|^2 - 2|\nabla \Psi|^2 + C_K(|\nabla f| + |\nabla \beta| + |\nabla \Psi| + 1). \quad (4.11)$$

and

$$\left(\frac{\partial}{\partial t} - \Delta \right) f^2 \leq -2|\nabla f|^2 + C_K. \quad (4.12)$$
Combining the equations (4.9), (4.10), (4.11) and (4.12), we get
\[
\left(\frac{\partial}{\partial t} - \Delta\right)\left(|\nabla f|^2 + |\beta|^2 + |\Psi|^2\right) \leq -|\nabla^2 f|^2 - \frac{p+1}{2}|\nabla f|^4 - |\nabla \Psi|^2 - |\nabla \beta|^2 + C_K, \tag{4.13}
\]
and for \(G_0 := t(|\nabla f|^2 + |\beta|^2 + |\Psi|^2) + A_0 f^2\)
\[
\left(\frac{\partial}{\partial t} - \Delta\right)G_0 \leq t\left(-|\nabla^2 f|^2 - \frac{p+1}{2}|\nabla f|^4 - |\nabla \Psi|^2 - |\nabla \beta|^2\right) - |\nabla f|^2 + C_{K,T},
\]
for \(A_0 > 1\) sufficiently large depending only on \(K, T\). In deriving the inequality above, we use the Cauchy-Schwarz inequalities, e.g., \(|\nabla f||\nabla \beta| \leq \frac{1}{10}|\nabla \beta|^2 + \frac{2}{3}||\nabla f||^2\). By maximum principle, it follows that \(\sup_{M^{10-\rho} \times [0,T]} G_0 \leq C_{K,T}\), and from the definition of \(G_0\), it follows that \(\sup_{M^{10-\rho}} |\nabla f|^2 \leq \frac{C_{K,T}}{t}\) for all \(t \in (0,T)\). \(\square\)

**Lemma 9** The following formula holds for the Levi-Civita connection \(\Gamma\)
\[
\frac{\partial}{\partial t}\Gamma = \nabla Rm + \nabla^2 f \ast \nabla f + e^{-(p+1)f}(\nabla f \ast \beta \ast \beta + \nabla \beta \ast \beta) + \nabla \Psi \ast \Psi. \tag{4.14}
\]

**Proof.** We write the equation \(\frac{\partial g_{ij}}{\partial t} = -2h_{ij}\), where
\[
h_{ij} := R_{ij} - \frac{p+1}{4}f_if_j + \sigma \frac{1}{2}e^{-(p+1)f}(\beta^2)_{ij} - \frac{1}{2}(\Psi^2)_{ij} - \frac{1}{6}(\sigma e^{-(p+1)f}|\beta|^2 - |\Psi|^2)g_{ij}. \tag{4.15}
\]
By definition \(\Gamma^k_{ij} = \frac{1}{2}g^{kl}(\partial_l g_{ji} + \partial_j g_{il} - \partial_i g_{lj})\), it follows that
\[
\frac{\partial}{\partial t}\Gamma^k_{ij} = -g^{kl}(\nabla_l h_{ji} + \nabla_j h_{il} - \nabla_i h_{lj}),
\]
from which we get the desired equation for \(\frac{\partial \Gamma}{\partial t}\). \(\square\)

**Lemma 10** The following equation holds for some \(C_{K,T} > 0\):
\[
\left(\frac{\partial}{\partial t} - \Delta\right)||\nabla \beta||^2 \leq -2||\nabla^2 \beta||^2 + C_{K,T}\left(||\nabla \beta||^2 + ||\nabla Rm||\nabla \beta|| + ||\nabla^3 f||\nabla f|| + ||\nabla^2 f||\nabla \beta|| + ||\nabla f||\nabla \beta|| + ||\nabla f||\nabla \Psi|| + ||\nabla^2 \Psi||\nabla \beta|| + ||\nabla \Psi||\nabla \beta|| + 1\right) \tag{4.16}
\]

**Proof.** From the formula \(\frac{\partial}{\partial t} \nabla \beta = \nabla \frac{\partial \beta}{\partial t} + \beta \ast \frac{\partial \Gamma}{\partial t}\), and the formulas (4.14) and (4.6)
\[
\frac{\partial}{\partial t} \nabla \beta = \Delta \nabla \beta + \nabla \beta \ast Rm + \beta \ast \nabla Rm + \nabla^3 f \ast \beta + \nabla^2 f \ast \nabla \beta \tag{4.17}
\]
\[
+ \nabla f \ast \nabla^2 \beta + e^{4f(f)}\left(\nabla f \ast \nabla f \ast \Psi \ast \Psi + \nabla^2 f \ast \Psi \ast \Psi + \nabla f \ast \nabla \Psi \ast \Psi \ast \Psi \right.
\]
\[
+ \nabla^2 \Psi \ast \Psi + \nabla \Psi \ast \nabla \Psi)\),
\]
where we use the formula \(\nabla \Delta \beta = \Delta \nabla \beta + \beta \ast \nabla Rm + \nabla \beta \ast Rm\). From this equation the estimate (4.16) follows. \(\square\)
Lemma 11 The following holds:

\[
\left( \frac{\partial}{\partial t} - \Delta \right) |\nabla \Psi|^2 \leq -2|\nabla^2 \Psi|^2 + C_{K,T}\left( |\nabla \Psi|^2 + |\nabla Rm||\nabla \Psi| + |\nabla \Psi|^2 \right) \\
+ |\nabla f|^2 |\nabla \Psi| + |\nabla^2 f||\nabla \Psi| + |\nabla f||\nabla \Psi|^2 \\
+ |\nabla^2 \beta||\nabla \Psi| + |\nabla \beta||\nabla \Psi|^2 + |\nabla \Psi||\nabla^2 \Psi| + 1 \right) \tag{4.18}
\]

Proof. From the equation \( \frac{\partial}{\partial t} \nabla \Psi = \nabla \frac{\partial}{\partial t} \Psi + \Psi \star \frac{\partial}{\partial t} \Gamma \) and equations (4.7), (4.14), we have

\[
\frac{\partial \nabla \Psi}{\partial t} = \Delta \nabla \Psi + \nabla Rm \star \Psi + Rm \star \nabla \Psi + e^{-\frac{p+1}{4}} f \left( \nabla f \star \nabla f \star \nabla \beta \star \nabla \beta \star \Psi + \nabla \beta \star \nabla \Psi + \nabla \beta \star \nabla \beta \star \nabla \Psi + \nabla \beta \star \nabla \Psi + \nabla \beta \star \nabla \beta \star \nabla \Psi \right) \tag{4.19}
\]

The estimate (4.18) follows from the above equation. \( \Box \)

Lemma 12 \(|Rm|^2\) satisfies the following

\[
\left( \frac{\partial}{\partial t} - \Delta \right) |Rm|^2 \leq -2|\nabla Rm|^2 + C_{K,T}\left( |\nabla f|^2 + |\nabla \beta|^2 + |\nabla^2 \beta| + |\nabla \Psi|^2 + |\nabla^2 \Psi| + 1 \right) \tag{4.20}
\]

Proof. From the evolution equation of \( Rm \) that

\[
\frac{\partial}{\partial t} R_{ijkl} = -\nabla_i \nabla_l h_{jk} - \nabla_j \nabla_k h_{il} + \nabla_i \nabla_k h_{lj} + \nabla_j \nabla_i h_{lk} - h_{km} R_{ijml} - h_{ml} R_{ijkm},
\]

where \( h_{ij} \) is given in (4.15), we get

\[
\frac{\partial Rm}{\partial t} = \Delta Rm + Rm \star Rm + \nabla^2 f \star \nabla^2 f + Rm \star \Psi \star \Psi + \nabla \Psi \star \nabla \Psi + \nabla^2 \Psi \star \Psi \\
+ e^{-\frac{p+1}{4}} f \left( Rm \star \beta \star \beta + \nabla f \star \nabla f \star \beta \star \beta + \nabla^2 f \star \beta \star \beta + \nabla f \star \nabla \beta \star \beta \\
+ \nabla \beta \star \nabla \beta + \nabla^2 \beta \star \beta \right), \tag{4.21}
\]

where we apply Riccati equations in deriving the above, in particular, \( \nabla^3 f \) terms can be cancelled in the equations. The estimate (4.20) follows from (4.21). \( \Box \)

Lemma 13 We have the following equation for \(|\nabla^2 f|^2\)

\[
\left( \frac{\partial}{\partial t} - \Delta \right) |\nabla^2 f|^2 \leq -2|\nabla^3 f|^2 + C_{K,T}\left( |\nabla^2 f|^2 + |\nabla f|^4 + 1 \right) \\
+ \frac{1}{10} |\nabla^2 \beta|^2 + \frac{1}{10} |\nabla \beta|^4 + \frac{1}{10} |\nabla^2 \Psi|^2 + \frac{1}{10} |\nabla \Psi|^4
\]

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Proof. From the equation \( f_{ij} = \partial_{ij}^2 f - \Gamma_{ij}^k f_k \) and the evolution equation \((4.5)\) for \( f \), we have

\[
\frac{\partial}{\partial t} \nabla^2 f = \nabla^2 f + \nabla f \ast \frac{\partial \Gamma}{\partial t}
\]

\[
= \Delta \nabla^2 f + \nabla^2 f \ast \nabla \psi + \nabla^2 \psi \ast \nabla \psi - \frac{p+1}{2} \nabla f \ast \nabla^2 f
\]

\[
+ \nabla f \ast \nabla \psi \ast \nabla \psi + \tilde{\lambda} e^{-f} (\nabla^2 f + \nabla f \ast \nabla f)
\]

\[
+ e^{-(p+1)f} (\nabla^2 \beta * \beta + \nabla \beta \ast \nabla \beta + \nabla f \ast \nabla \beta * \beta + \nabla^2 f * \beta * \beta)
\]

(4.22)

from (4.22) we have

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \left| \nabla^2 f \right|^2 \leq -2 |\nabla^3 f|^2 + C_{K,T} \left( (\nabla^2 f)^2 + |\nabla^2 f| |\nabla \beta|^2 + |\nabla^2 f| |\nabla \psi|^2 + |\nabla^2 f| |\nabla \psi| + 1 \right)
\]

\[
\leq -2 |\nabla^3 f|^2 + C_{K,T} \left( (\nabla^2 f)^2 + |\nabla \psi|^2 + 1 \right) + \frac{1}{10} \left| \nabla^2 \beta \right|^2
\]

\[
+ \frac{1}{10} |\nabla \beta|^4 + \frac{1}{10} \left| \nabla^2 \psi \right|^2 + \frac{1}{10} \left| \nabla \psi \right|^4
\]

by Cauchy-Schwarz inequalities. \( \square \)

With these formulas in hand, we are now ready to prove the derivatives estimates.

Lemma 14 There exists a constant \( C(K,T) > 0 \) such that

\[
\sup_{M^{10-p}} \left( |\nabla \beta|^2 + |\nabla \psi|^2 + |\nabla^2 f|^2 \right) \leq \frac{C}{t}, \quad \forall \ t \in (0, T).
\]

(4.23)

Proof. By (4.10) and (4.16), we have for any constant \( A_2 > 0 \)

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \left( |\beta|^2 + A_2 |\nabla \beta|^2 \right)
\]

\[
= \left( \frac{\partial}{\partial t} - \Delta \right) |\beta|^2 |\nabla \beta|^2 + (|\beta|^2 + A_2) \left( \frac{\partial}{\partial t} - \Delta \right) |\nabla \beta|^2 \leq -2 |\nabla \beta|^4 + 2 A_2 |\nabla^2 \beta|^2 + C_{K,T} \left( 1 + |\nabla \beta|^2 + |\nabla \beta|^2 |\nabla^2 f|^2 + |\nabla f| |\nabla \beta|^3 + |\nabla \psi| |\nabla \beta|^2
\]

\[
+ A_2 |\nabla \psi|^2 |\nabla \beta| + A_2 |\nabla^3 f| |\nabla \beta| + A_2 |\nabla^2 f| |\nabla \beta|^2 + A_2 |\nabla \beta||\nabla^2 \beta| + A_2 |\nabla f|^2 |\nabla \beta| + A_2 |\nabla \psi|^2 |\nabla \beta| + A_2 |\nabla \psi|^2 |\nabla \beta| + |\nabla \beta|^2 |\nabla^2 \beta| \right)
\]

\[
\leq - \frac{3}{2} |\nabla \beta|^4 - A_2 |\nabla^2 \beta|^2 + C_{K,T} \left( 1 + |\nabla^2 f|^2 + |\nabla f|^4 \right) + \frac{1}{10} |\nabla \psi|^2
\]

\[
+ \frac{1}{10} |\nabla^3 f|^2 + \frac{1}{10} |\nabla^2 \psi|^2 + \frac{1}{10} |\nabla \psi|^4
\]

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for $A_2 > 1$ large enough depending only on $K, T$, and in the last inequality above, we use Cauchy-Schwarz inequalities to simplify the expression.

By (4.11) and (4.18), we have for any constant $A_1 > 0$

$$\left( \frac{\partial}{\partial t} - \Delta \right) \left( |\nabla \Psi|^2 + A_1 |\nabla \Psi|^2 \right)$$

(4.25)

$$\leq -2 |\nabla \Psi|^4 - 2A_1 |\nabla \Psi|^2 + C_{K,T} \left( 1 + A_1 |\nabla \Psi|^2 + A_1 |\nabla f||\nabla \Psi|^2 \right.$$

$$+ A_1 |\nabla f|^2 |\nabla \Psi| + A_1 |\nabla^2 f||\nabla \Psi| + A_1 |\nabla \beta||\nabla \Psi| + A_1 |\nabla^2 \beta||\nabla \Psi|$$

$$+ A_1 |\nabla \beta||\nabla \Psi|^2 + |\nabla \Psi||\nabla^2 \Psi| \right)$$

(4.26)

$$\leq - |\nabla \Psi|^4 - A_1 |\nabla \Psi|^2 + C_{K,T} \left( 1 + |\nabla f|^4 + |\nabla^2 f|^2 + |\nabla \beta|^2 \right) + \frac{1}{10} |\nabla^2 \beta|^2 + \frac{1}{10} |\nabla Rm|^2,$$

where in the last inequality we take $A_1 > 1$ to be sufficiently large and apply the Cauchy-Schwarz inequalities. We denote

$$G_1 := (|\Psi|^2 + A_1 |\nabla \Psi|^2 + (|\beta|^2 + A_2) |\nabla \beta|^2 + |Rm|^2 + |\nabla f|^2).$$

By (4.20), (4.22), (4.24) and (4.25), after some cancellations we have

$$\left( \frac{\partial}{\partial t} - \Delta \right) G_1$$

$$\leq - |\nabla \beta|^4 - |\nabla^2 \beta|^2 - |\nabla \Psi|^4 - |\nabla^2 \Psi|^2 - |\nabla Rm|^2 - |\nabla^3 f|^2$$

$$+ C_{K,T} \left( 1 + |\nabla^2 \beta|^2 + |\nabla f|^4 \right).$$

Combining (4.13) we get

$$\left( \frac{\partial}{\partial t} - \Delta \right) \left( t G_1 + A_3 (|\nabla f|^2 + |\beta|^2 + |\Psi|^2) \right)$$

$$\leq G_1 + t \left( - |\nabla \beta|^4 - |\nabla^2 \beta|^2 - |\nabla \Psi|^4 - |\nabla^2 \Psi|^2 - |\nabla Rm|^2 - |\nabla^3 f|^2$$

$$+ C_{K,T} \left( 1 + |\nabla^2 \beta|^2 + |\nabla f|^4 \right) \right) - A_3 (|\nabla^2 \beta|^2 + |\nabla f|^4 + |\nabla \Psi|^2 + |\nabla \beta|^2) + A_3 C_{K,T}$$

$$\leq t \left( - |\nabla \beta|^2 |\nabla \Psi|^2 - |\nabla Rm|^2 - |\nabla^3 f|^3 \right) + C_{K,T} A_3 + C_{K,T}$$

(4.27)

$$\leq C_{K,T} A_3 C_{K,T}.$$
Lemma 15 $|\nabla Rm|^2$ satisfies the following inequality

\[
(\frac{\partial}{\partial t} - \Delta)|\nabla Rm|^2 \\
\leq -2|\nabla^2 Rm|^2 + C_{K,T}|\nabla Rm|\left(|\nabla Rm| + t^{-1/2}(|\nabla^3 f| + |\nabla^2 \beta| + |\nabla^2 \Psi|) + t^{-1/2} + t^{-3/2} + t^{-1} + |\nabla^3 f| + |\nabla^3 \beta| + |\nabla^3 \Psi|)\right).
\]

**Proof.** We calculate using the formula $\frac{\partial}{\partial t} \nabla Rm = \nabla \frac{\partial Rm}{\partial t} + Rm \ast \frac{\partial \Gamma}{\partial t}$ and (4.21)

\[
\frac{\partial}{\partial t} \nabla Rm \tag{4.29} \\
= \Delta \nabla Rm + Rm \ast \nabla Rm + \nabla^2 f \ast \nabla^3 f + \nabla Rm \ast \Psi \ast \Psi + Rm \ast \nabla \Psi \ast \Psi + Rm \ast \nabla^2 f \ast \nabla f \\
+ \nabla^2 \Psi \ast \nabla \Psi + \nabla^3 \Psi \ast \Psi + e^{-\left(p+1\right)}f\left(Rm \ast \nabla f \ast \beta \ast \beta + Rm \ast \nabla Rm \ast \beta \ast \beta + Rm \ast \nabla \beta \ast \beta \\
+ \nabla f \ast \nabla f \ast \nabla f \ast \beta \ast \beta \ast \beta + \nabla^2 f \ast \nabla f \ast \nabla f \ast \beta \ast \beta \ast \beta + \nabla^2 f \ast \nabla f \ast \beta \ast \beta \\
+ \nabla^2 f \ast \nabla^2 \beta \ast \beta + \nabla f \ast \nabla^2 \beta \ast \beta + \nabla f \ast \nabla \beta \ast \nabla \beta + \nabla^2 \beta \ast \nabla \beta + \nabla^3 \beta \ast \beta\right),
\]

therefore it follows that

\[
(\frac{\partial}{\partial t} - \Delta)|\nabla Rm|^2 \\
\leq -2|\nabla^2 Rm|^2 + C_{K,T}|\nabla Rm|\left(|\nabla Rm| + |\nabla^2 f||\nabla^3 f| + |\nabla f| + |\nabla \beta| + |\nabla \Psi| \\
+ |\nabla f|^3 + |\nabla f||\nabla^2 f| + |\nabla f|^2|\nabla \beta| + |\nabla^2 f||\nabla \beta| + |\nabla f||\nabla^2 \beta| \\
+ |\nabla f||\nabla \beta| + |\nabla \beta||\nabla^2 \beta| + |\nabla^3 \beta| + |\nabla^2 \Psi||\nabla \Psi| + |\nabla^3 \Psi|\right) \\
\leq -2|\nabla^2 Rm|^2 + C_{K,T}|\nabla Rm|\left(|\nabla Rm| + t^{-1/2}|\nabla^3 f| + t^{-1/2} + t^{-3/2} + t^{-1} + |\nabla^3 f| \\
+ t^{-1/2}|\nabla^2 \beta| + |\nabla^3 \beta| + t^{-1/2}|\nabla^2 \Psi| + |\nabla^3 \Psi|\right).
\]

\]

Lemma 16 $|\nabla^3 f|^2$ satisfies the inequality

\[
(\frac{\partial}{\partial t} - \Delta)|\nabla^3 f|^2 \\
\leq -2|\nabla^4 f|^2 + C_{K,T}|\nabla^3 f|\left(|\nabla^3 f| + t^{-1/2}(|\nabla Rm| + |\nabla^2 \beta| + |\nabla^2 \Psi|) + t^{-1} + t^{-3/2} \\
+ |\nabla^3 \beta| + |\nabla^3 \Psi| + t^{-1}|\nabla^3 f|\right).
\]

**Proof.** By (4.22) we calculate

\[
\frac{\partial}{\partial t} \nabla^3 f = \nabla \frac{\partial}{\partial t} \nabla^2 f + \nabla^2 f \ast \frac{\partial \Gamma}{\partial t}
\]

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\[ \Delta \nabla^3 f + \nabla^3 f \ast Rm + \nabla^2 f \ast \nabla Rm + e^{-(p+1)f} (\nabla^3 f \ast \beta \ast \beta + \nabla^2 f \ast \nabla \beta \ast \beta \\
+ \nabla f \ast \nabla f \ast \nabla \beta \ast \beta + \nabla f \ast \nabla^2 \beta \ast \beta + \nabla f \ast \nabla \beta \ast \nabla \beta + \nabla^3 \beta \ast \beta + \nabla^2 \beta \ast \nabla \beta \ast \beta \\
+ \nabla f \ast \nabla f \ast \nabla f \ast \nabla \beta \ast \beta + \nabla^2 f \ast \nabla f \ast \beta \ast \beta + \nabla^2 f \ast \nabla \beta \ast \beta + \nabla^3 \Psi \ast \Psi \\
+ \nabla^2 \Psi \ast \nabla \Psi + \nabla^2 f \ast \nabla f \ast \nabla^3 f \ast \nabla f \ast \nabla \Psi \ast \Psi \\
+ \nabla f \ast \nabla^2 \Psi \ast \Psi + \nabla f \ast \nabla \Psi \ast \nabla \Psi + \lambda e^{-f} (\nabla^2 f \ast \nabla f + \nabla^3 f \ast \nabla f \ast \nabla f) \],

Thus by the Cauchy-Schwarz inequality and Lemmas 8 and 14, we have

\[
\left( \frac{\partial}{\partial t} - \Delta \right) |\nabla^3 f|^2 \\
\leq -2 |\nabla^4 f|^2 + C_{K,T} |\nabla^3 f| \left( |\nabla^3 f| + |\nabla^2 f| |\nabla Rm| + |\nabla^2 f| |\nabla \beta| + |\nabla f| |\nabla^2 \beta| + |\nabla f| |\nabla^3 \beta| + |\nabla^2 f| |\nabla \beta| + |\nabla f| |\nabla^2 \Psi| + |\nabla^2 f| |\nabla f| + |\nabla f| |\nabla^3 \beta| + |\nabla^2 \beta| + |\nabla^2 \Psi| + |\nabla f| |\nabla^2 f| + |\nabla f| |\nabla \beta| + |\nabla f| |\nabla^2 \Psi| + |\nabla f| |\nabla \Psi| + |\nabla f| |\nabla^2 f| \right) \\
\leq -2 |\nabla^4 f|^2 + C_{K,T} |\nabla^3 f| \left( |\nabla^3 f| + t^{-1/2}(|\nabla Rm| + |\nabla^2 \beta| + |\nabla^2 \Psi|) + t^{-1} + t^{-3/2} + |\nabla^3 \beta| + |\nabla^3 \Psi| + t^{-1} |\nabla^3 f| \right).
\]

This finishes the proof of Lemma 16.

\textbf{Lemma 17} Along the flow $|\nabla^2 \beta|^2$ satisfies the inequality

\[
\left( \frac{\partial}{\partial t} - \Delta \right) |\nabla^2 \beta|^2 \leq -2 |\nabla^3 \beta|^2 + C_{K,T} |\nabla^2 \beta| \left( |\nabla^2 \beta| + t^{-1/2}(|\nabla Rm| + |\nabla^3 f| + |\nabla^2 \beta| + |\nabla^2 \Psi|) + |\nabla^2 Rm| + |\nabla^4 f| + t^{-1/2} |\nabla^3 \beta| + t^{-1} |\nabla^2 \beta| + t^{-1} + t^{-3/2} + |\nabla^3 f| + |\nabla^3 \Psi| \right),
\]

for some constant $C_{K,T} > 0$.

\textbf{Proof.} By (4.17), we have

\[
\frac{\partial \nabla^2 \beta}{\partial t} = \nabla \frac{\partial \nabla \beta}{\partial t} + \nabla \beta \ast \frac{\partial \Gamma}{\partial t} \\
= \Delta \nabla^2 \beta + \nabla^2 \beta \ast Rm + \nabla \beta \ast \nabla Rm + \beta \ast \nabla^2 Rm + \nabla^4 f \ast \beta + \nabla^3 f \ast \nabla \beta \\
+ \nabla^2 f \ast \nabla^2 \beta + \nabla f \ast \nabla^3 \beta + \nabla^2 f \ast \nabla \beta + \nabla \beta \ast \nabla \Psi \ast \Psi \\
+ e^{\frac{\Delta f}{2}} \left( \nabla f \ast \nabla f \ast \nabla \Psi \ast \Psi + \nabla f \ast \nabla^2 f \ast \Psi \ast \Psi + \nabla f \ast \nabla f \ast \nabla \Psi \ast \Psi \\
+ \nabla f \ast \nabla^2 \Psi \ast \Psi + \nabla f \ast \nabla \Psi \ast \nabla \Psi + \nabla^3 f \ast \Psi \ast \Psi + \nabla^2 f \ast \nabla \Psi \ast \Psi \\
+ \nabla^3 \Psi \ast \Psi + \nabla^2 \Psi \ast \nabla \Psi \right) + e^{-(p+1)f} (\nabla f \ast \nabla \beta \ast \beta + \nabla \beta \ast \nabla \beta \ast \beta) .
\]
Then by Cauchy-Schwarz inequality and Lemmas 8 and 14, we have

\[
\left(\frac{\partial}{\partial t} - \Delta\right)|\nabla^2 \beta|^2 \leq -2|\nabla^3 \beta|^2 + C_{K,T}|\nabla^2 \beta| \left( |\nabla^2 \beta| + |\nabla \beta| |\nabla Rm| + |\nabla^2 Rm| + |\nabla^4 f| + |\nabla \beta| |\nabla \beta| \\
+ |\nabla^2 \beta| |\nabla f| + |\nabla f| |\nabla^3 \beta| + |\nabla f| |\nabla \beta| |\nabla^2 \beta| + |\nabla \beta| |\nabla \Psi| + |\nabla f|^3 + |\nabla f| |\nabla^2 f| \right) \\
\leq -2|\nabla^3 \beta|^2 + C_{K,T}|\nabla^2 \beta| \left( |\nabla^2 \beta| + t^{-1/2} (|\nabla Rm| + |\nabla^2 \beta| + |\nabla^2 \Psi|) + |\nabla^2 Rm| \right) \\
+ |\nabla^2 \beta|^2 + \frac{1}{4} \nabla^2 f \nabla \Psi + \frac{1}{3} \nabla^2 f \nabla \Psi + \frac{1}{3} \nabla^2 f \nabla \Psi + |\nabla^3 \beta| + |\nabla^3 \Psi|, \\
\right)
\]

Lemma 18 Along the flow $|\nabla^2 \Psi|^2$ satisfies the inequality

\[
\left(\frac{\partial}{\partial t} - \Delta\right)|\nabla^2 \Psi|^2 \leq -2|\nabla^3 \Psi|^2 + C_{K,T}|\nabla^2 \Psi| \left( |\nabla^2 \Psi| + t^{-1/2} (|\nabla Rm| + |\nabla^2 \beta| + |\nabla^2 \Psi|) + |\nabla^2 Rm| \right) \\
+ t^{-3/2} + t^{-1} + |\nabla^3 \beta| + |\nabla^3 \Psi|,
\]

for some constant $C_{K,T} > 0$.

Proof. By (4.19) we have the equation

\[
\begin{align*}
\frac{\partial}{\partial t} \nabla^2 \Psi &= \nabla \frac{\partial \nabla \Psi}{\partial t} + \nabla \Psi * \frac{\partial \Gamma}{\partial t} \\
&= \Delta \nabla^2 \Psi + \nabla^2 \Psi * \nabla Rm + \nabla \Psi * \nabla Rm + \nabla^2 Rm * \nabla \Psi + \nabla \Psi * \nabla^2 f * \nabla f \\
&\quad + e^{-\frac{n+1}{2}} f \left( \nabla f * \nabla f * \nabla f * \beta * \Psi + \nabla f * \nabla^2 f * \beta * \Psi + \nabla f * \nabla f * \nabla \beta * \Psi \right) \\
&\quad + \nabla f * \nabla f * \beta * \nabla \Psi + \nabla f * \nabla^2 \beta * \Psi + \nabla f * \nabla \beta * \nabla \Psi + \nabla \beta * \nabla \Psi + \nabla \beta * \nabla \Psi + \nabla \beta * \nabla^2 \Psi + \beta * \nabla^3 \Psi \right) \\
&\quad + e^{-\frac{n+1}{2}} f \left( \nabla f * \nabla \Psi * \beta * \beta + \nabla \beta * \beta * \nabla \Psi + \nabla \Psi * \nabla \Psi \right). \\
\end{align*}
\]

By Cauchy-Schwarz inequality, Lemmas 8 and 14 we have

\[
\left(\frac{\partial}{\partial t} - \Delta\right)|\nabla^2 \Psi|^2 \leq -2|\nabla^3 \Psi|^2 + C_{K,T}|\nabla^2 \Psi| \left( |\nabla^2 \Psi| + |\nabla \Psi| |\nabla Rm| + |\nabla^2 Rm| + |\nabla \Psi| |\nabla f| |\nabla^2 f| + |\nabla^3 f| \\
\right)
\]

25
+|∇f||∇^2f| + |∇f|^2|∇β| + |∇f|^2|∇Ψ| + |∇f||∇^2β| + |∇f||∇β||∇Ψ| + |∇f||∇^2Ψ| + |∇f||∇β|\left|\nabla \psi\right| + |∇f||∇β|\left|\nabla \psi\right| + |∇f||∇β|\left|\nabla \psi\right| + |∇f||∇β|\left|\nabla \psi\right| + |∇f||∇β|\left|\nabla \psi\right| + |∇f||∇β|\left|\nabla \psi\right| + |∇f||∇β|\left|\nabla \psi\right| + |∇f||∇β|\left|\nabla \psi\right| + |∇f||∇β|\left|\nabla \psi\right| + |∇f||∇β|\left|\nabla \psi\right|
\begin{align*}
\leq -2|\nabla^3\Psi|^2 + C_{K,T}|\nabla^2\Psi|\left(|\nabla^2\Psi| + t^{-1/2}(|\nabla Rm| + |\nabla^2\beta| + |\nabla^2\Psi|) + |\nabla^2 Rm|
+ t^{-3/2} + t^{-1} + |\nabla^3 f| + |\nabla^3 \beta| + |\nabla^3 \Psi|\right).
\end{align*}

Lemma 19 There exists a constant $C = C(K, T) > 0$ such that

$$\sup_{M^{10-p}}\left(|\nabla Rm| + |\nabla^2 \beta| + |\nabla^2 \Psi| + |\nabla^3 f|\right) \leq \frac{C}{t}, \ \forall \ t \in (0, T)$$

Proof. Combining the inequalities (4.28), (4.30), (4.31) and (4.32) and applying the Cauchy-Schwarz inequality several times, it follows that for $G_2 := |\nabla Rm|^2 + |\nabla^3 f|^2 + |\nabla^2 \beta|^2 + |\nabla^2 \Psi|^2$,

$$(\frac{\partial}{\partial t} - \Delta)(t^2 G_2)$$

\begin{align*}
\leq 2tG_2 + t^2 \left\{ -|\nabla^2 Rm|^2 - |\nabla^4 f|^2 - |\nabla^3 \beta|^2
- |\nabla^3 \Psi|^2 + C_{K,T}(1 + t^{-1/2})\left(|\nabla Rm|^2 + |\nabla^3 f|^2 + |\nabla^2 \beta|^2 + |\nabla^2 \Psi|^2\right)
+ C_{K,T}t^{-3/2}(|\nabla Rm| + |\nabla^3 f| + |\nabla^2 \beta| + |\nabla^2 \Psi|) + C_{K,T}t^{-1}|\nabla^2 \beta|^2 + C_{K,T}\right\} \\
\leq C_{K,T}tG_2 + t^2 \left\{ -|\nabla^2 Rm|^2 - |\nabla^4 f|^2 - |\nabla^3 \beta|^2 - |\nabla^3 \Psi|^2\right\} + C_{K,T}.
\end{align*}

Combining with the inequality (4.27) with $G_1$ given in (4.26), we get for $A_4$ large enough

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left\{t^2 G_2 + A_4\left(tG_1 + A_3(|\nabla f|^2 + |\beta|^2 + |\Psi|^2)\right)\right\}$$

\begin{align*}
\leq t^2 \left(-|\nabla^2 Rm|^2 - |\nabla^4 f|^2 - |\nabla^3 \beta|^2 - |\nabla^3 \Psi|^2\right) + C_{K,T}.
\end{align*}

Applying maximum principle it follows that $\sup_{M^{10-p}} G_2 \leq \frac{C_{K,T}}{t^2}$. The desired estimate then follows from the definition of $G_2$.

Next we prove the higher order estimates for general $m \in \mathbb{Z}_+$.

**Theorem 4** For any $2 \leq m \in \mathbb{Z}_+$, there exists a constant $C = C(m, K, T) > 0$ such that

$$\sup_{M^{10-p}}\left(|\nabla^{m-1} Rm| + |\nabla^m \beta| + |\nabla^m \Psi| + |\nabla^{m+1} f|\right) \leq \frac{C(m, K, T)}{t^{m/2}}, \ \forall \ t \in (0, T). \ (4.33)$$
We will prove this theorem by induction. To begin with, we need equations on the higher order derivatives of $Rm, f, \beta$ and $\Psi$. For notation convenience we denote

$$\gamma = \nabla f \ast \nabla f + e^{-(p+1)f} \beta \ast \beta + \Psi \ast \Psi,$$

by (4.14), we know $\frac{\partial \Gamma}{\partial t} = \nabla Rm + \nabla \gamma.$

**Lemma 20** We have the following evolution equation for $\nabla^m \beta$

$$\frac{\partial \nabla^m \beta}{\partial t} = \Delta \nabla^m \beta + \sum_{i+j=m} \nabla^i \beta \ast \nabla^j Rm + \sum_{i+j=m+1} \nabla^{i+1} f \ast \nabla^j \beta$$

$$+ \sum_{i+j+k=m+1} \nabla^i (e^{\frac{p+1}{2}f}) \ast \nabla^j \Psi \ast \nabla^k \Psi + \sum_{i+j=m-1} \nabla^i \beta \ast \nabla^{j+1} \gamma.$$

where in the summations $i, j, k$ are nonnegative integers.

**Proof.** We will prove this formula by induction. The formula holds for $m = 1$ by (4.17). Assume it has been proved for $m-1$, then

$$\frac{\partial \nabla^m \beta}{\partial t} = \nabla \frac{\partial \nabla^{m-1} \beta}{\partial t} + \nabla^{m-1} \beta \ast \nabla Rm + \nabla^{m-1} \beta \ast \nabla \gamma$$

$$= \nabla \left( \Delta \nabla^{m-1} \beta + \sum_{i+j=m-1} \nabla^i \beta \ast \nabla^j Rm + \sum_{i+j=m} \nabla^{i+1} f \ast \nabla^j \beta + \sum_{i+j=m-2} \nabla^i \beta \ast \nabla^{j+1} \gamma \right.$$

$$+ \sum_{i+j+k=m} \nabla^i (e^{\frac{p+1}{2}f}) \ast \nabla^j \Psi \ast \nabla^k \Psi \left. + \nabla^{m-1} \beta \ast \nabla Rm + \nabla^{m-1} \beta \ast \nabla \gamma \right).$$

from which the equation (4.34) follows by expanding the terms in the bracket. Note that we need the formula $\nabla \Delta \nabla^{m-1} \beta = \Delta \nabla^m \beta + \nabla^{m-1} \beta \ast \nabla Rm + \nabla^m \beta \ast Rm$, which follows from the Riccati equations. \hfill \Box

**Lemma 21** We have the evolution equation for $\nabla^m \Psi$

$$\frac{\partial \nabla^m \Psi}{\partial t} = \Delta \nabla^m \Psi + \sum_{i+j=m} \nabla^i \Psi \ast \nabla^j Rm + \sum_{i+j=m+1} \nabla^{i+1} f \ast \nabla^j \beta$$

$$+ \sum_{i+j+k=m+1} \nabla^i (e^{\frac{p+1}{2}f}) \ast \nabla^j \Psi \ast \nabla^k \Psi + \sum_{i+j=m-1} \nabla^i \Psi \ast \nabla^{j+1} \gamma.$$

**Proof.** When $m = 1$, this equation is given by (4.19). The general equation follows by similar calculation as in deriving the equation (4.34) for $\nabla^m \beta$, so we omit the details. \hfill \Box
Lemma 22 The evolution equation for $\nabla^{m+1} f$ is given by

\[
\frac{\partial \nabla^{m+1} f}{\partial t} = \Delta \nabla^{m+1} f + \sum_{i+j=m-1} \nabla^i f \ast \nabla^j Rm + \nabla^{m+1} e^{-f} \tag{4.36}
\]

\[
+ \sum_{i+j+k=m+1} \nabla^i (e^{-(p+1)f}) \ast \nabla^j \beta \ast \nabla^k \beta + \sum_{i+j=m+1} \nabla^i \Psi \ast \nabla^j \Psi \\
+ \sum_{i+j=m-1} \nabla^{i+1} f \ast \nabla^{j+1} \gamma.
\]

Proof. When $m=1$, this equation is given by (4.22). The general equation can be proved similarly as in Lemma 20. \[\square\]

Lemma 23 The evolution equation for $\nabla^m Rm$ is

\[
\frac{\partial \nabla^m Rm}{\partial t} = \Delta \nabla^{m-1} Rm + \sum_{i+j=m-1} \nabla^i Rm \ast \nabla^j Rm + \sum_{i+j=m-1} \nabla^{i+2} f \ast \nabla^{j+2} f \tag{4.37}
\]

\[
+ \sum_{i+j+k+l=m+1} \nabla^i Rm \ast \nabla^j (e^{-(p+1)f}) \ast \nabla^k \beta \ast \nabla^l \beta + \sum_{i+j=m+1} \nabla^i \Psi \ast \nabla^j \Psi \\
+ \sum_{i+j=k=m-1} \nabla^i Rm \ast \nabla^j \Psi \ast \nabla^k \Psi + \sum_{i+j+k=m+1} \nabla^i (e^{-(p+1)f}) \ast \nabla^j \beta \ast \nabla^k \beta \\
+ \sum_{i+j=m-2} \nabla^i Rm \ast \nabla^{j+1} \gamma.
\]

Proof. When $m=2$ the equation is (4.29). The general case follows by induction similar as in Lemma 20. \[\square\]

Proof of Theorem 4. We will use induction to prove (4.33). When $m=2$, this is proved in Lemma 19. So we assume the estimate has been proved for any nonnegative integer no bigger than $m-1$. Our goal is to prove (4.33) for $m$. By induction assumption, the following hold for a constant $C = C(m, K, T) > 0$: 

\[
\sup_{M_{10}^{10-p}} |\nabla f| \leq \frac{C}{t^{k/2}}, \quad \forall k \leq m - 1, \tag{4.38}
\]

\[
\sup_{M_{10}^{10-p}} \left( |\nabla^k \beta| + |\nabla^k \Psi| + |\nabla^{k-1} Rm| + |\nabla^{k+1} f| \right) \leq \frac{C}{t^{k/2}}, \quad \forall k \leq m - 1, \tag{4.39}
\]

\[
\sup_{M_{10}^{10-p}} \left( |\nabla^i e^{p+1} f| + |\nabla^i e^{-(p+1)f}| + |\nabla^i e^{-f}| \right) \leq C \left( \frac{1}{t^{i/2}} + 1 \right), \quad \forall i \leq m. \tag{4.40}
\]

We denote

\[
G_m := |\nabla^{m-1} Rm|^2 + |\nabla^m \beta|^2 + |\nabla^m \Psi|^2 + |\nabla^{m+1} f|^2.
\]
We calculate making use of (4.34), (4.35), (4.36) and (4.37)

\[
(\frac{\partial}{\partial t} - \Delta)G_m \\
\leq -2|\nabla^{m+1}\beta|^2 - 2|\nabla^{m+1}\Psi|^2 - 2|\nabla^{m+2}f|^2 - 2|\nabla^m Rm|^2 \\
+ C|\nabla^m \beta| \left( \sum_{i+j=m} |\nabla^i \beta||\nabla^j Rm| + \sum_{i+j=m+1} |\nabla^{i+1} f||\nabla^j \beta| + \sum_{i+j=m-1} |\nabla^i \beta||\nabla^{j+1} \gamma| \right) \\
+ \sum_{i+j+k=m+1} |\nabla^{i+1} f||\nabla^j \beta| + C|\nabla^m \Psi| \left( \sum_{i+j=m} |\nabla^i \Psi||\nabla^j Rm| \right) \\
+ \sum_{i+j+m+1} |\nabla^{i+1} f||\nabla^j \beta| + \sum_{i+j=m-1} |\nabla^i \Psi||\nabla^{j+1} \gamma| \right) \\
+ C|\nabla^{m+1} f| \left( \sum_{i+j=m-1} |\nabla^{i+2} f||\nabla^j Rm| + |\nabla^{m+1} e^{-f}| + \sum_{i+j=m+1} |\nabla^i \Psi||\nabla^j \Psi| \\
+ \sum_{i+j+k+m+1} |\nabla^{i+1} e^{-p+1} f||\nabla^j \beta| + \sum_{i+j=m-1} |\nabla^{i+1} f||\nabla^{j+1} \gamma| \right) \\
+ C|\nabla^{m-1} Rm| \left( \sum_{i+j=m-1} |\nabla^i Rm||\nabla^j Rm| + \sum_{i+j=m-1} |\nabla^{i+2} f||\nabla^{j+2} f| + \sum_{i+j=m+1} |\nabla^i \Psi||\nabla^{j+1} \gamma| \right) \\
+ \sum_{i+j+k+m-1} |\nabla^i Rm||\nabla^{j+1} \gamma| \right) \\
\right)
\]

We will use (4.38), (4.39) and (4.40) to estimate the terms on the RHS of the above inequality.

- The terms \(C|\nabla^m \beta|(\cdots)\):

\[
C|\nabla^m \beta| \left( \sum_{i+j=m} |\nabla^i \beta||\nabla^j Rm| + |\nabla^{m+1} \beta| + |\nabla^m \Psi| + |\nabla^m \beta| \right) \\
+ |\nabla^m Rm| + |\nabla^{m+2} f| + |\nabla^{m+1} f| + |\nabla^{m+1} \Psi| + \frac{1}{t^{(m+1)/2}} + \frac{1}{t^{m/2}} \right) \\
\leq C t^{-1/2} (|\nabla^m \beta|^2 + |\nabla^{m-1} Rm|^2 + |\nabla^m \Psi|^2) + C(1 + t^{-1}) |\nabla^m \beta|^2 + C|\nabla^{m+1} f|^2 \\
+ \frac{1}{10} \left( |\nabla^m Rm|^2 + |\nabla^{m+2} f|^2 + |\nabla^{m+1} \Psi|^2 \right) + C t^{-(m+1)/2} |\nabla^m \beta| + C
\]

- The terms \(C|\nabla^m \Psi|(\cdots)\):

\[
C|\nabla^m \Psi| \left( \sum_{i+j=m} |\nabla^i \Psi||\nabla^j Rm| + |\nabla^{m+1} \beta| + |\nabla^m \beta| + |\nabla^m \Psi| \right) + |\nabla^m Rm| \\
\right)
\]
Combining the inequalities above, we get
\[
+Ct^{-1/2}(|\nabla^m \Psi|^2 + |\nabla^{m+1} f|^2 + |\nabla^{m+1} \beta|^2 + |\nabla^m \beta|^2) + C(1 + t^{-1})|\nabla^m \Psi|^2 + C|\nabla^{m+1} f|^2
\]
\[
+\frac{1}{10} \left( |\nabla^{m+1} \beta|^2 + |\nabla^{m+1} f|^2 + |\nabla^m Rm|^2 + |\nabla^{m+1} \Psi|^2 \right) + C \frac{|\nabla^m \Psi|}{t^m} + C.
\]

- The terms \( C|\nabla^{m+1} f|(\cdots) \)

\[
C|\nabla^{m+1} f|(\cdots)
\]
\[
\leq C|\nabla^{m+1} f| \left\{ t^{-1/2}(|\nabla^m \Psi| + |\nabla^m \beta|) + |\nabla^{m+1} f| + |\nabla^{m+1} \Psi| + |\nabla^{m+1} \beta| + \frac{1}{t^{m/2}} + \frac{1}{t^{(m+1)/2}} \right\}
\]
\[
\leq Ct^{-1/2}(|\nabla^{m+1} f|^2 + |\nabla^{m+1} \Psi|^2 + |\nabla^{m+1} \beta|^2) + C|\nabla^{m+1} f|^2 + \frac{1}{10} \left( |\nabla^{m+1} \Psi|^2 + |\nabla^{m+1} \beta|^2 \right) + C \frac{|\nabla^{m+1} f|}{t^{(m+1)/2}} + C.
\]

- The terms \( C|\nabla^{m-1} Rm|(\cdots) \)

\[
C|\nabla^{m-1} Rm|(\cdots)
\]
\[
\leq C|\nabla^{m-1} Rm| \left\{ t^{-1/2}(|\nabla^m \Psi| + |\nabla^m \beta|) + |\nabla^{m-1} Rm| + |\nabla^{m+1} f| + |\nabla^{m+1} \Psi| + |\nabla^{m+1} \beta| + \frac{1}{t^{m/2}} + \frac{1}{t^{(m+1)/2}} \right\}
\]
\[
\leq Ct^{-1/2}(|\nabla^{m-1} Rm|^2 + |\nabla^m \Psi|^2 + |\nabla^m \beta|^2) + C|\nabla^{m-1} Rm|^2 + C|\nabla^{m+1} f|^2 + \frac{1}{10} \left( |\nabla^{m+1} \beta|^2 + |\nabla^{m+1} \Psi|^2 \right) + C \frac{|\nabla^{m-1} Rm|}{t^{(m+1)/2}} + C.
\]

Combining the inequalities above, we get
\[
\left( \frac{\partial}{\partial t} - \Delta \right) t^m G_m
\]
\[
= mt^{m-1} G_m + t^m \left( \frac{\partial}{\partial t} - \Delta \right) G_m
\]
\[
\leq mt^{m-1} G_m + t^m \left\{ - |\nabla^{m+1} \beta|^2 - |\nabla^{m+1} \Psi|^2 - |\nabla^{m+2} f|^2 - |\nabla^m Rm|^2
\right.
\]
\[
+ Ct^{-1/2} G_m + CG_m + Ct^{-1} (|\nabla^m \beta|^2 + |\nabla^m \Psi|^2) + C \right\} + C t^{(m-1)/2} \left( |\nabla^m \beta| + |\nabla^m \Psi| + |\nabla^{m+1} f| + |\nabla^{m-1} Rm| \right)
\]
\[
\leq Ct^{m-1} G_m - t^m G_{m+1} + C, \quad (4.41)
\]

where in the last step we applied Cauchy-Schwarz inequality. Note that from the calculations above it is not hard to see that the inequality (4.41) in fact holds for \( t^i G_i \) for any
\[ 1 \leq i \leq m, \text{ maybe with different constants which depend only on } K, T, m. \] Define
\[ H := t^m G_m + \sum_{i=1}^{m-1} B_i t^i G_i + B_0 (|\nabla f|^2 + |\beta|^2 + |\Psi|^2). \]

For suitable choice of the constants \( B_i \)'s, we have
\[ (\frac{\partial}{\partial t} - \Delta) H \leq -t^m G_{m+1} + C, \]
for some constant \( C = C(K, T, m) > 0. \) From maximum principle, it follows that
\[ \sup_{M^{10-p}} H \leq C(K, T, m), \text{ thus } \sup_{M^{10-p}} G_m \leq \frac{C(K,T,m)}{t^m} \text{ for any } t \in (0, T). \] Thus we finish the proof of Theorem 4.

Once we have the estimates of Theorem 4, the proof for part (c) of Theorem 1 can be completed in the same way as for part (c) of Theorem 2.

We conclude by observing that Theorem 1 implies similar theorems for the simpler dimensional reductions described in sections 2.2 and 2.3. In some cases, under suitable assumptions, the theorem can be strengthened. For example, in the case §2.3 and if \( \lambda \leq 0, \) since \( \beta \) is automatically 0, it is easy to obtain a bound on \( |f| \) on any finite time interval \([0, T)\). Thus the maximum time \( T \) of existence is now characterized by
\[ \limsup_{t \to T^-} \sup_{M^{10-p}} (|Rm| + |\Psi|) = \infty. \] (4.42)

Also, when the dimension \( p \) satisfies \( p \geq 4, \) the forms \( \beta \) and \( \Psi \wedge \Psi \) are automatically 0, and the equations simplify a great deal.

### A Conventions

We consider metrics \( g \) on an 11-dimensional manifold \( M^{11} \), which can be either Lorentzian or Riemannian. We distinguish between the two cases by a number \( \sigma \), which is defined to be +1 if \( M^{11} \) is Lorentzian, and -1 if \( M^{11} \) is Riemannian.

The 4-form \( F \) is expressed in components as
\[ F = \frac{1}{4!} F_{ABCD} dx^A \wedge dx^B \wedge dx^C \wedge dx^D \]
and \( F_{AB}^2, |F|^2 \) and \( |\nabla F|^2 \) are defined by
\[ F_{AB}^2 = \frac{1}{3!} F_{ACDE} F_B^{CDE}, \quad |F|^2 = \frac{1}{4!} F_{ACDE} F^{ACDE}, \quad |\nabla F|^2 = \frac{1}{4!} F_{ACDE,B} F^{ACDE,B}. \]
The operator $\Box_g = d d^\dagger + d^\dagger d$ is the Hodge-Laplacian. We also need the Laplacian $\Delta_g$ defined on tensors or forms by $\Delta_g = g^{AB} \nabla_B \nabla_A$. The Lichnerowicz-Weitzenböck formula says that the two differ by curvature terms, e.g. on forms such as $F$,

$$\Box_g F = -\Delta_g F + Rm \ast F$$

where $Rm \ast F$ denotes a pointwise bilinear expression in the components of $Rm$ and $F$.

The Hodge $\ast$ is defined by

$$\alpha \wedge \ast \beta = \langle \alpha, \beta \rangle \sqrt{-\sigma g} d^{11}x$$

To emphasize the metric sometimes we write $\ast_g$ instead of $\ast$. In particular, if we set $d\text{vol}_g = \sqrt{-\sigma g} d^{11}x$, then $d\text{vol}_g = -\sigma$ and $\ast 1 = d\text{vol}_g$. The adjoint $d^\dagger$ of the exterior derivative $d$ acting on 4-forms is then given by

$$d^\dagger = -\sigma \ast d \ast.$$ (A.1)

## B  The field equations of 11D supergravity

We provide here a derivation of the field equations of 11D supergravity for the convenience of the reader. Let $g$ be a metric on $M^{11}$ and $A$ be a 3-form. Let $F = dA$. Recall that we set $\sigma = +1$ if $g$ is Lorentzian, and $\sigma = -1$ if $g$ is Riemannian. The action of 11D supergravity is given by

$$\mathcal{L}[g, A] = \int_M R \sqrt{-\sigma g} - \frac{1}{2} F \wedge \ast F + \frac{1}{6} F \wedge F \wedge A,$$ (B.1)

The Euler-Langrange equations of the functional $\mathcal{L}$ in (B.1) are readily found to be

$$d \ast F = \frac{1}{2} F \wedge F$$

$$R_{AB} - \frac{1}{2} R g_{AB} = \frac{1}{2} F_{AB}^2 - \frac{1}{4} |F|^2 g_{AB}.$$ (B.2)

Contracting the second equation in (B.2), we see that $R = \frac{1}{6} |F|^2$. Therefore we find that this equation is equivalent to

$$R_{AB} = \frac{1}{2} F_{AB}^2 - \frac{1}{6} |F|^2 g_{AB}.$$ (B.3)

This is the form of the equation used in (1.1). Moreover, if we apply $d\ast$ on both sides of the first equation in (B.2), we get

$$dd^\dagger F = -\frac{\sigma}{2} d \ast (F \wedge F),$$

combining with $dF = 0$ we see $\Box_g F = -\frac{\sigma}{2} d \ast (F \wedge F)$. 

32
C Field equations and stationary points

Clearly the solutions of the field equations must be stationary points of the dynamical system. We discuss here some simple situations when the converse is true.

Suppose that the cohomology group \( H^3(M^{11}, \mathbb{R}) \) is 0. Then at a stationary point, we have \( \ast d \ast F - \frac{1}{2} \ast (F \wedge F) = dB \) for some smooth 2-form \( B \). Applying \( d \ast \) to both sides of this equation and using the fact that \( F \) is closed gives \( \ast d \ast dB = 0 \), i.e. \( d^\dagger dB = 0 \).

If \( M^{11} \) is compact, this implies that \( |dB|^2 = 0 \), and if \( M^{11} \) is also Riemannian, this implies that \( dB = 0 \). The field equations are then satisfied. Thus the field equations (1.1) are equivalent to the stationary point condition of the flow (1.3) when \( M^{11} \) is a compact Riemannian manifold with vanishing cohomology group \( H^3(M^{11}, \mathbb{R}) \).

More generally, it follows from the stationary point condition that the form \( \alpha \)

\[
\alpha = \ast d \ast F - \frac{1}{2} \ast (F \wedge F) \tag{C.1}
\]

is both closed and co-closed, i.e., \( d\alpha = 0 \) and \( d^\dagger \alpha = 0 \). Thus \( \alpha \) must be 0 and the field equations are satisfied if \( M^{11} \) is assumed not to have any such form which is non-trivial. This is equivalent to the non-existence of non-trivial harmonic forms when \( M^{11} \) is Riemannian.

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