Anderson localization is a general wave phenomenon that results from interferences of multiply scattered waves in random media and leads to exponential suppression of wave propagation beyond the localization length $\xi$. Recently, important advances have been made towards the observation and the theoretical description of Anderson localization of classical and, in particular, electromagnetic waves. Signatures of localization can be found in the thickness dependence of the average transmission coefficient of a random sample, in the shape of the coherent backscattering cone, in the statistical distribution of transmitted intensity, or in the time-of-flight profiles of transmitted waves. In a recent experiment, Schwarz et al. have studied the transverse distribution of intensity of a Gaussian light beam in a medium that is translationally invariant along the direction of beam propagation, but disordered in the perpendicular $xy$ plane. As was predicted by de Raedt et al., the variable $z$ plays the role of time in this case and the distribution of intensity in the $xy$ plane behaves very much as the intensity of an expanding wavepacket in a two-dimensional (2D) disordered medium. Because all waves are localized in 2D, the expansion of the beam is halted after one localization length $\xi$. This phenomenon was called “transverse localization”.

Even though the concept of transverse localization implies 2D disorder which is translationally invariant along the direction of beam propagation, one can formally attempt the same experiment in a medium with 3D disorder. As has been shown very recently, the transverse, 2D profile of intensity of a wave transmitted through a disordered slab shows clear signatures of genuine Anderson localization in 3D. The purpose of this Letter is to present a detailed theoretical study of transverse confinement of waves due to Anderson localization in 3D. This provides a guide for future experiments to exploit this interesting new phenomenon to access localization of various types of waves (light, sound, etc.).

A beam focused to a point $\rho = 0$ at the surface $z = 0$ of a disordered slab produces a bell-shaped intensity profile $T(\rho)$ at the opposite face $z = L$ of the slab, as we schematically show in Fig. 1. We analyze $T(\rho)$ in the framework on the self-consistent (SC) theory of localization, extended to account for finite-size effects.
the diffusion equation then yields
\[
D(\ell, z) = D(0,0) \exp(-2\bar{z}/\ell) \text{ with } \bar{z} = \min(z, L - z),
\]
which is inspired by the observation that this \( D(z, 0) \) represents an asymptotically exact solution in a semi-infinite medium for \( z \gg \xi \) [18]. Comparison with a numerical solution shows that this expression constitutes a rather good approximation everywhere, except in the central part of the slab, where the real value of \( D \) is larger. The expression for \( \sigma^2 \) that follows from this ansatz is
\[
\sigma_{\text{loc}}^2 = 2L\xi \left[ 1 + \frac{1}{c_{\text{loc}}/\xi - 1} - \frac{\xi}{2L} \left( 1 - e^{-L/\xi} \right) \right] - \frac{\ell}{L} \text{Coth} \left( \frac{\ell}{2L} \right). \tag{3}
\]

At the mobility edge \((k\ell = 1)\) an approximate expression for \( D \) is \( D(z, 0) \approx D(0,0)/(1 + \bar{z}/z_c) \). Just like in the case of strong disorder, this approximation for \( D \) is inspired by the solution for semi-infinite medium [18] and deviates from the numerical solution in the central part of the slab only; \( D(0,0) \) and \( z_c \) depend on the exact value of \( z_0 \) in the boundary conditions. For our purposes, it will be sufficient to set \( z_0 = 0 \) and use the corresponding \( D(0,0) = D_B \) and \( z_c \simeq 3\ell \). This gives
\[
\sigma_{\text{ME}}^2 = \frac{3L^2}{8} \left( 1 + \frac{4z_c}{L} \right). \tag{4}
\]

We note that this result is not in contradiction with the multifractal statistics of wavefunctions at the mobility edge (see Ref. [22] for a review), even though a link between \( \sigma_{\text{ME}}^2 \) and multifractality is beyond the scope of this paper.

For comparison, we summarize the analytical results for \( \sigma^2 \) following from Eqs. (2-4) in the limit of large \( L \) in Table I. To test our analytical results, we solve the SC equations of localization numerically and show the resulting mean square widths \( \sigma^2 \) as functions of slab thickness \( L \) in Fig. 2. The agreement between analytical and numerical results is satisfactory, confirming the validity of our analytical analysis.

An important comment is in order. It is remarkable that in the case of weak disorder \((k\ell > 1)\), \( \sigma^2 = 2L^2/3 \) does not depend on \( D \) and even a very small, but finite and spatially uniform diffusion coefficient would lead to the same result for \( \sigma^2 \). This emphasizes the importance of the position dependence of \( D(z, 0) \) in the localized regime \( k\ell < 1 \) and at the mobility edge \( k\ell = 1 \). Note that this is the first time that the position dependence of \( D \) is absolutely vital to obtain a result that is different from the universal diffuse outcome. Scale-dependent but spatially uniform diffusion coefficients \( D \propto 1/L \) and \( D \propto \exp(-L/\xi) \) put forward by the scaling theory [13] explain the scaling of the stationary transmission coefficient \( T \propto 1/L^2 \) at the mobility edge and \( T \propto \exp(-L/\xi) \) in the localized regime, respectively. However, being uniform in space, these expressions for \( D \) result in exactly the same mean square width of the transmitted beam as

| \( k\ell \) | \( D(z, 0) \) | \( \sigma^2 \) |
|------|------|------|
| \( k\ell \gg 1 \) (diffusion) | \( D_B \left[ 1 - \frac{1}{(k\ell)^2} \right] \) | \( \frac{2}{3}L^2 - O(\ell^2) \) |
| \( k\ell = 1 \) (mobility edge) | \( D_B \left[ 1 + \bar{z}/\xi \right] \) | \( \frac{3}{8}L^2 + O(Lz_c) \) |
| \( k\ell < 1 \) (localization) | \( D_B e^{-2\bar{z}/\ell} \) | \( 2L\xi + O(\ell^2) \) |

TABLE I: Summary of analytical results for the position-dependent diffusion coefficient \( D(z, 0) \) and its associated mean square width \( \sigma^2 \) of the position-resolved transmission coefficient \( T(\rho) \) describing transmission of a tightly focused monochromatic beam through a disordered slab of thickness \( L \gg \ell \). \( \ell \) is the mean free path, \( k \) is the wavenumber, \( z_c \simeq 3\ell \), \( \xi \) is the localization length, and \( \bar{z} = \min(z, L - z) \). We assume \( z_c \ll L \) for \( k\ell = 1 \) and \( \ell \ll \xi \ll L \) for \( k\ell < 1 \).
obtained in the localized regime, with $\xi$ in the opposite case of $\sigma$ and the resulting equation for the correction i.e. absorption plays exactly the same role as localization. In contrast, in the localized regime $\sigma^2 \simeq 2L\xi$ links $\sigma$ to the localization length $\xi$, which offers an elegant way of measuring the latter experimentally.

Absorption was a serious obstacle for the unambiguous interpretation of a number of experiments on Anderson localization [5, 6, 7, 8]. It is therefore important to study the role of absorption in the context of transverse confinement of waves in 3D. A straightforward calculation in the regime of weak disorder ($k\ell \gg 1$) shows that if the macroscopic absorption length $L_a = \sqrt{L/3}\mu_a$ (where $\mu_a$ is the absorption coefficient) is much longer than the sample thickness $L$, $\sigma^2$ given in Table I only acquires a small correction $-2L^2/45L_a^2$. Hence, weak absorption is not an obstacle for observing transverse confinement. However, in the opposite case of $L_a \ll L$ we obtain $\sigma^2 \simeq 2LL_a$, i.e. absorption plays exactly the same role as localization and the resulting equation for $\sigma^2$ coincides with the one obtained in the localized regime, with $\xi$ replaced by $L_a$. This indicates that a study of stationary transverse confinement cannot distinguish localization from absorption and hence suffers from the same drawbacks as previous works [5, 6, 7, 8].

A way to overcome complications due to absorption is suggested by recent works [9, 10, 11, 12]. The idea is to study the dynamics of wave propagation rather than the stationary transport. We adopt this idea here too and replace the continuous incident beam in the experiment depicted in Fig. 1 by a short pulse. This complicates the analysis considerably because no simple analytic approximation exists for $D(z, \Omega)$ at arbitrary $\Omega \neq 0$, contrary to the stationary $\Omega = 0$ case. One way to calculate $\sigma^2$, which now acquires a time dependence, is to solve the SC equations of localization numerically. We show the results of this calculation in Fig. 3. The difference between diffuse and localized regimes is manifest in Fig. 3. The rise of $\sigma^2(t)$ with time is unbounded for $k\ell > 1$, whereas $\sigma^2(t)$ saturates at a finite value $\sigma^2_\infty$ for $k\ell < 1$. The latter result can be understood from the following approximate calculation which, however, turns out to be quite adequate. Instead of using the self-consistent equation for $D(r, \Omega)$, let us simply set $D(z, \Omega) = -i\Omega \xi^2$, which is the solution of SC model in the infinite medium in the limit $\Omega \rightarrow 0$. A straightforward calculation then yields $\sigma^2_\infty = 2L\xi(1-\xi/L)$ up to the first order in $\xi/L \ll 1$. This equation, shown by a dashed horizontal line in Fig. 3, falls fairly close to the numerical result at $L/\xi = 4$, allowing us to conjecture that it might be a good estimate of $\sigma^2_\infty$ in the limit of $L \gg \xi$. Our calculation suggests that the saturation of $\sigma^2(t)$ at a constant level takes place not only in the localized regime, but even at the mobility edge. We find the asymptotic value of $\sigma(t)$ to be $\sigma_\infty \approx L$ (see Fig. 3). It is worthwhile to note that our time-dependent $\sigma^2(t)$ is not sensitive to absorption and that all curves of Fig. 3 remain exactly the same in an absorbing medium. This provides a solution to the long-standing issue of distinguishing localization from absorption [5, 6, 7, 8], as has been recently demonstrated in Ref. [10].

Let us now analyze the time dependence of $\sigma^2(t)$. First, in the diffuse regime $k\ell \gg 1$ one readily obtains $D = D_B$: $\sigma^2 \simeq 2L^2/3$. It is the position dependence of $D$ that accounts for the new expressions for $\sigma^2$ that we find at strong disorder. Incidentally, $\sigma^2 = 3L^2/8$ at the mobility edge differs from the result in the diffuse regime by just a numerical coefficient. In contrast, in the localized regime $\sigma^2 \simeq 2L\xi$ links $\sigma$ to the localization length $\xi$, which offers an elegant way of measuring the latter experimentally.
\(\sigma^2(t) \simeq 4Dt\), where \(D \simeq D_B[1 - (k\ell)^{-2}]\) can be assumed position-independent. For \(k\ell = 3\) this result is shown in Fig. 3 by a dashed line that is indeed very close to the result of the numerical calculation. To study the time dependence of \(\sigma^2(t)\) at the mobility edge and in the localized regime, we replotted the curves of the main plot, except the one corresponding to \(k\ell = 3\), in the inset of Fig. 3 in a log-log scale. We clearly see that the initial growth of \(\sigma^2(t)\) with \(t\) is power-law: \(\sigma^2(t) \propto t^\alpha\) with \(\alpha \simeq 0.5\). It is remarkable that this power-law growth is observed not only in the localized regime but at the mobility edge \(k\ell = 1\) as well. To understand the power exponent \(\alpha \simeq 0.5\), let us consider a point source of waves located inside a random medium of size \(L \gg \ell\), very far from its boundaries. Neglecting the position dependence of \(D\) yields the return probability \[\text{Eq. (1)}\]

\[C(r, r', \Omega) = \frac{1}{2\pi^2} \int_{q_{min}}^{q_{max}} \frac{dq}{-\Omega + D(\Omega)q^2},\]

where \(q_{min} \sim 1/L\) accounts for the finite size of the medium and \(q_{max} \sim 1/\ell\) ensures convergence of the integration. Inserting this expression into Eq. (1) and solving for \(D(\Omega)\) at \(k\ell = 1\), we find \(D(\Omega) \propto (-\Omega)^{1/3}\) for \(\Omega \gg D_B/L^2\) and \(D(\Omega) \propto (-\Omega)^{1/2}\) for \(\Omega \ll D_B/L^2\). The typical mean square radius \(\langle r^2 \rangle\) of the intensity profile in 3D is then \(\langle r^2 \rangle \propto t^{2/3}\) for \(t \ll L^2/D_B\) and \(\langle r^2 \rangle \propto t^{1/2}\) for \(t \gg L^2/D_B\). If we now assume that \(\langle r^2 \rangle\) and \(\sigma^2(t) = \langle r^2 \rangle\) in transmission through a slab behave in a similar way, we obtain a qualitative explanation for the power-law scaling \(\sigma^2(t) \propto t^{1/2}\) that we observe in Fig. 3 at \(t > t_D\). Note that the above simple argument does not explain saturation of \(\sigma^2(t)\) at a constant level at larger times.

Another interesting and unexpected feature of \(\sigma^2(t)\) that we found sufficiently deep in the localized regime is its nonmonotony with time: the curve corresponding to \(L/\xi = 4\) in Fig. 3 has a weakly pronounced but clearly visible maximum at \(t/t_D \simeq 30\). Even though this nonmonotony is predicted by the SC model (we checked that the maximum is present at least for \(L/\ell = 50–200\) and \(L/\xi = 3–6\)), it remains to be seen whether it is an artifact of the model or a real physical phenomenon.

In conclusion, we have shown that the transverse confinement of waves in 3D random media can be very useful for demonstrating Anderson localization and for measuring the localization length in an experiment. The dependence of the mean square width \(\sigma^2\) of a tightly focused beam transmitted through a slab of random medium on the slab thickness \(L\) and on time (in a pulsed experiment) is qualitatively different in the diffuse and in the localized regimes of wave propagation. This qualitative difference is due to the position dependence of the diffusion coefficient induced by interferences. In a pulsed experiment, \(\sigma^2(t)\) is independent of absorption in the disordered sample and thus measuring \(\sigma^2(t)\) allows to avoid the risk of confusing localization and absorption. We expect our results to hold independent of the microscopic structure of disorder. For correlated disorder, \(\ell\) should be replaced by the bare (i.e., unaffected by macroscopic interferences) transport mean free path.

We thank J.H. Page for stimulating discussions. The computations presented in this paper were performed on the cluster PHYNUM (CIMENT, Grenoble). S.E.S. acknowledges financial support from the French ANR (project No. 06-BLAN-0096 CAROL) and the French Ministry of Education and Research.