Scale transformations, tree-level perturbation theory and the cosmological matter bispectrum

Jun Pan, Peter Coles and István Szapudi
1 Purple Mountain Observatory, Chinese Academy of Sciences, Nanjing 210008, China
2 School of Physics and Astronomy, University of Nottingham, Nottingham NG7 2RD
3 School of Physics and Astronomy, Cardiff University, Queenes Buildings, The Parade, Cardiff CF24 3AA
4 Institute for Astronomy, University of Hawaii, Honolulu, HI 96822, USA

Accepted 2007 September 18. Received 2007 September 17; in original form 2007 July 11

ABSTRACT
Scale transformations have played an extremely successful role in studies of cosmological large-scale structure by relating the non-linear spectrum of cosmological density fluctuations to the linear primordial power at longer wavelengths. Here, we generalize this approach to investigate the usefulness of scale transformations for non-linear higher order statistics, specifically the bispectrum. We find that the bispectrum predicted by perturbation theory at tree level can be rescaled to match the results of full numerical simulations in the weakly and intermediately non-linear regimes, especially at high redshifts, with an accuracy that is surprising given the simplicity of the procedure used. This discovery not only offers a simple practical way of calculating the matter bispectrum but also suggests that scale transformations may yet yield even deeper insights into the physics of hierarchical clustering.

Key words: Cosmology: theory – large-scale structure of Universe

1 INTRODUCTION
In the linear stage of gravitational instability, cosmological density perturbations evolve in such a way that each Fourier mode grows at the same rate as the others, preserving the shape of the primordial power spectrum. In the later stages, things become much more complicated, with mode–mode interactions changing the relative growth rates of different harmonics and coupling their phases together to generate non-zero higher order spectra (Coles & Chiang 2000). Nevertheless, inspired by the creative work of Hamilton et al. (1991), empirical formulae have been established that can accurately predict the two-point correlation function $\xi(r)$ or power spectrum $P(k)$ of cosmological density fluctuations in the non-linear regime given only the initial linear power spectrum and the background cosmological model (Jain, Mo & White 1995; Peacock & Dodds 1996). The key element of these formulae is a scale transformation which expresses the conservation of particle pairs which, in turn, is one of the infinite set of equations that forms the Born–Bogoliubov–Green–Kirkwood-Yvon (BBGKY) hierarchy (c.f. Peebles 1980). The discovery of this unexpectedly successful transformation for the non-linear $\xi(r)$ or $P(k)$ was a great leap forward in the understanding of cosmological clustering evolution, but with the renaissance of halo model (Cooray & Sheth 2002), and the emergence of various high-level perturbative techniques (e.g. Szapudi & Kaiser 2003; McDonald 2007), interest in the classic form introduced by Hamilton et al. (1991) has to some extent faded away. Nowadays it seems much more fashionable to calibrate gravitational non-linearity by tuning parameters of the halo model or using empirical fitting formulas based on it (Smith et al. 2003).

After great successes with two-point correlation functions, it was widely anticipated that the halo model would be able to generate precise non-linear three-point statistics over a full range of length-scales (Scoccimarro et al. 2001; Takada & Jain 2003). Detailed examination against simulations, however, has proved somewhat disappointing. New elements have had to be incorporated to improve the current halo model even after introducing ad hoc free parameters accounting for mass cut-off and halo boundary (Fosalba, Pan & Szapudi 2005) or replacing spherical haloes with triaxial ones (Smith, Watts & Sheth 2006).

In this paper, we revive the scale transformation idea to derive a model that approximates the non-linear contributions to the bispectrum of cosmological matter perturbations. The bispectrum is the lowest order (and therefore the simplest) diagnostic of non-Gaussianity because it vanishes identically for any Gaussian random field regardless of its power spectrum. If the initial density fluctuations are Gaussian, as is expected in the simplest inflationary models, the bispectrum is therefore completely induced by non-linear gravitational evolution. However, these higher order effects on the bispectrum are difficult to calculate with reasonable accuracy even on large scales where the evolution is quasi-linear. Differences between full numerical simulations and the predictions of Eulerian perturbation theory at tree level exceed $\sim 10–20$ per cent at a scale $k \sim 0.1\,h\,\text{Mpc}^{-1}$. The perturbation theory at one-loop level, the next possible improvement on the tree-level theory, requires heavy

*E-mail: jpan@pmo.ac.cn
numerical integration and has limited success (Scoccimarro et al. 1998). Lagrangian perturbation theory at second- and third-order provides a better template than Eulerian theory, but its dynamic range is heavily restricted by the shell-crossing scale at around \( k \sim 0.4 \, h \, \text{Mpc}^{-1} \) (Scoccimarro 2000). A further inconvenience of adopting Lagrangian perturbation theory is that the bispectrum it predicts is not computed, but measured from \( N \)-body simulations of which the particle motions are controlled by Lagrangian theory.

One practical way forward is to modify the kernel of perturbation theory at tree level to develop empirical fitting formulae such as extended and the hyper-extended perturbation theory (Colombi et al. 1997; Scoccimarro & Frieman 1999). However, the performance of these techniques known to be quite poor, even in cases where the initial power spectrum is completely scale-free (Hou et al. 2005). The best results so far in this vein relate to the empirical model of Scoccimarro & Couchman (2001, hereafter SC 2001), using hyper-extended perturbation theory, but its average deviation from \( N \)-body results is at the level of 15 per cent, still way beyond the accuracy levels expected in the era of precision cosmology. On the other hand, even this produces some dubious features, such as the apparently trough in the bispectrum at scales \( \sim 0.03 < k < \sim 0.12 \, h \, \text{Mpc}^{-1} \), where baryon acoustic oscillations leave their footprint on large-scale structure statistics.

An accurate template for the bispectrum on large scales is extremely important in the epoch where large galaxy redshift surveys are available to unveil details of large-scale structure and to constrain cosmological parameters with higher order statistics. It is possible that an accurate model can be achieved based on the formula of SC 2001 facilitated with high-precision simulations in huge box, or perhaps with a theory established by the renormalization technique of McDonald (2007). However, in this paper we will show that, at least in the quasi-linear and intermediate non-linear regime, it is possible to recover the bispectrum quite accurately using the scale transformation described above, an approach which is free from the laborious calibration of fitting parameters and formidable multidimensional integrations.

2 SCALE TRANSFORMATIONS AND THE BISPECTRUM

2.1 Scale transformations of second-order statistics

Let \( \xi(r) \) be the two-point correlation function in real space at scale \( r \). The number of neighbours of a central point within a spherical top-hat window of radius \( r \) is given by \( r^3 [1 + \bar{\xi}(r)] \), where \( \bar{\xi}(r) = 3 \int_0^r s^2 \xi(s)ds/r^3 \) is the volume-averaged two-point correlation function. Hamilton et al. (1991) proposed that by pair conservation at any stage of gravitational evolution, one can quote a Lagrangian separation \( r_L \) defined by

\[
    r_L^3 = r_N^3 [1 + \bar{\xi}(r_N)]
\]

at which the non-linear function \( \bar{\xi}_N \) is an universal function of the linear one: \( \bar{\xi}_N(r_N) = f[\bar{\xi}(r_N)] \). The effectiveness of this paradigm is at least partly explained by Nityananda & Padmanabhan (1994): such a non-linear scaling relation can arise if the pairwise velocity demonstrates certain scaling features. In fact, based on the universal properties of the pairwise velocity distribution, the pair conservation equation can be solved with an iterative technique to produce the correct non-linear \( \xi \) from the input linear function (Caldwell et al. 2001).

In Fourier space, the same sort of scaling can be applied to the dimensionless power spectrum \( \Delta^2(k) = P(k)k^3/(2\pi^2) \) by a relation of the form

\[
    k_L = [1 + \Delta_N^2(k)NL]^3/8 k_{NL},
\]

so that the non-linear power spectrum \( P_N(k_{NL}) \) is obtained through \( \Delta_N^2(k_{NL}) = f[\Delta_L^2(k)] \). The scale transformation of equation (2) does not have such a firm theoretical motivation as its real space version, although there is no question that works very well in practice (Peacock & Dodds 1996).

2.2 The bispectrum at tree level

The bispectrum is the three-point correlation function defined in Fourier space. Denote the Fourier transformation of the cosmic density contrast with \( \delta(k) \). The bispectrum is

\[
    B(k_1, k_2, k_3) = \langle \delta(k_1) \delta(k_2) \delta(k_3) \rangle
\]

in which \( \delta_0 \) is the Dirac function. In a statistically isotropic universe, \( B(k_1, k_2, k_3) = -k_1 \cdot k_2 \cdot k_3 \) is written as \( B(k_1, k_2, -k_1) \), with \( k_3 = \sqrt{k_1^2 + k_2^2 + 2k_1k_2\mu_{12}} \) and \( \mu_{12} = k_1 \cdot k_2/(k_1k_2) \).

Because the initial distribution of dark matter is Gaussian, there is no non-zero bispectrum in the scheme of linear evolution. The first non-trivial contribution to bispectrum, at tree level, comes from the second order terms in the expansion of the cosmic density fluctuation. Explicitly,

\[
    B_{PT}(k_1, k_2, k_3) = P_L(k_1)P_L(k_2)F_2(k_1, k_2, \mu_{12}) + \text{cyc.}
\]

where cyc. refers to the other two terms obtained by making cyclic permutations of the indices of the first term, \( P_L \) is the linear power spectrum and \( F_2 \) is the kernel of second order in Eulerian perturbation theory (Goroff et al. 1986):

\[
    F_2(k_1, k_2, \mu_{12}) = \frac{10}{7} + \mu_{12} \left( \frac{k_1}{k_2} + \frac{k_2}{k_1} \right) + \frac{4}{7} \mu_{12}^2.
\]

Note the dependence on cosmological parameters of \( F_2 \) is extremely weak, so it will be ignored here.

2.3 Configuration re-arrangement

The three wavevectors defining the bispectrum form a triangular configuration, and it is well known that there is strong dependence of bispectrum on the shape of this triangle. If we assume that the transformation rule defined by equation (2) holds for arbitrary \( k \), so that a new set of scales \( \tilde{k} \) in Fourier space is given by

\[
    \tilde{k}_{1,2,3} = \left[ 1 - \Delta^2_{NL}(k_{1,2,3}) \right]^{-1/3} k_{1,2,3},
\]

with \( \Delta^2_{NL} \) computed by the formula of Smith et al. (2003), the shape of the original triangle is obviously not preserved. By the requirement of \( \sum \tilde{k}_{1,2,3} = 0 \), the transformed triangle has new set of cosines \( \tilde{\mu} \), for example,

\[
    \tilde{\mu}_{12} = -\left( \tilde{k}_1^2 + \tilde{k}_2^2 - \tilde{k}_3^2 \right)/(2\tilde{k}_1\tilde{k}_2).
\]

Now comes the crucial step in our model. We assume that the non-linear bispectrum is directly related to the tree-level bispectrum of the new configuration in Lagrangian space. In the non-linear regime, this relationship is expected to be expressed by a very complicated functional form, but here we simply conjecture that we can use the linear expression,

\[
    B(k_1, k_2, \mu_{12}) \propto B_{PT}(\tilde{k}_1, \tilde{k}_2, \tilde{\mu}_{12})
\]

The remaining ingredient we need is the factor by which the amplitude of the bispectrum is boosted during the rescaling.
2.4 Amplitude boost

We have tested many schemes to calculate the boost to be applied to
the amplitude, but the best is one involves the idea that a non-linear
fluctuation in Lagrangian space is expressible as a convolution of
the linear density contrast with some filter so that, in Fourier space,
\[ \delta_{NL}(k_L) = w(k_L) \delta_L(k_L) \]  
where \( w(k) \) is the window function (the Fourier transform of the
filter) and \( k_L \) is given by equation (2). The form of this convolution
is motivated by terms that arise in halo model from cross-correlation
terms in the power spectrum and bispectrum. In this scenario, the
non-linear growth of the fluctuation in a region is imagined to consist
of two stages: (i) aggregation of clustering power in Lagrangian
space from other regions via convolution and (ii) translation from
the Lagrangian (initial) scale to the Eulerian (final) scale.

Putting all the elements of equations (4)–(9) together, the ex-
pected non-linear bispectrum in the recipe is
\[ B(k_1, k_2, \mu_{12}) = w(\tilde{k}_1)w(\tilde{k}_2)w(\tilde{k}_3)B_{\text{PT}}(\tilde{k}_1, \tilde{k}_2, \tilde{\mu}_{12}) \]
(10)

We have to address the issue that, in the model, the possible phase
shifts induced by convolution are not taken into account. This might
have a significant effect in the strongly non-linear regime, since the
bispectrum is a phase-dependent function (Watts & Coles 2003).

3 COMPARISON WITH SIMULATIONS

Four outputs at \( z = 0, 0.5, 1, 2 \) of the Very Large Simulation (VLS)
data provided by the Virgo consortium are used to check our model.
The simulation runs consist of 512\(^3\) particles in a cubic box of size
479 h\(^{-1}\) Mpc, and the cosmological parameters adopted are \( \Omega_m = 0.3, \Omega_c = 0.7, \Gamma = 0.21, h = 0.7 \) and \( \sigma_8 = 0.9 \) (MacFarland et al. 1998). In order to estimate cosmic variance, the VLS simulation
is divided into eight distinct subcubes of half of the original size.
The bispectrum is measured using the method of Scoccimarro et al.
(1998) and the 1\( \sigma \) dispersion among the eight measurements is
taken as an estimate of the error bars.

To avoid any bias involved with taking ratios of two statistical
quantities and leakage of the power spectrum into the bis-
pectrum, we work with the bispectrum itself rather than the re-
duced bispectrum \( Q_3 = B / (P_1 P_2 + P_1 P_3 + P_2 P_3) \) throughout this
paper.

3.1 Equilateral Triangles

The special case of equilateral triangles is particular straightforward
to interpret because it depends only on one scale. Moreover, under
the effect of the re-arrangement by equation (7), an equilateral
triangle does not change shape, so this case should display self-
similar evolution.

In Fig. 1, the bispectrum of equilateral triangles of simulations
is plotted against tree-level perturbation theory, our model equa-
tion (10) and the SC 2001 for reference. The rescaled tree-level per-
turbation theory agrees with simulations remarkably well at scales
\( k < 0.7 h \text{ Mpc}^{-1} \) at \( z = 0 \). The range of scales in agreement also
keeps increasing at higher redshift, reaching \( k \approx 3 h \text{ Mpc}^{-1} \) at \( z = 2 \).

In the quasi-non-linear regime at large scales of \( k < 0.1 h \text{ Mpc}^{-1} \)
at \( z = 0 \), equation (10) (our model) differs little from the tree-level
perturbation theory as desired, while the suspicious lowering of

![Figure 1](https://academic.oup.com/mnras/article-abstract/382/4/1460/1142283)

© 2007 The Authors. Journal compilation © 2007 RAS, MNRAS 382, 1460–1464
Figure 2. $B(k_1, k_2, \theta)$ at $z = 0, \theta = \cos^{-1} \mu_{12}$, see Fig. 1 for legend and labels.

SC 2001 is clearly seen at low redshifts. Due to the large cosmic variance, it is not yet possible to tell which model is better.

Our model predicts too much power at scales in the strongly non-linear regime, where the bispectrum of the simulations begins to display a power-law behaviour. The breakdown can be attributed either or both of the two principal weaknesses in our consideration: either equation (8) is simply too rough or the correlation between window functions at different values of $k$ is not negligible.

3.2 Other configurations

The extra complexity of bispectrum over the power spectrum is that it possesses angular dependence as well as scale dependence. Comparison of simulations with models is drawn in Figs 2 and 3 for $z = 0, 1$, respectively.

It is very clear that the our model agrees much better with the simulations at higher redshift than the alternatives we considered. It is also observed that equation (10) is more accurate when the ratio $k_2/k_1 \leq 2$; within this range the variation with $\theta$ also matches the simulation results rather well. The major problem of our model for very tilted triangles is that if we expand $B(k_1, k_2, \theta)$ with Legendre polynomials then it has a very low quadrupole moment (Szapudi 2004).

The overall performance of equation (10), especially at high redshift, is comparable with the perturbation theory at one-loop level although it is far simpler to implement. In fact, as a quick check we realize that the reduced bispectrum $Q_3$ of our model is very similar to the one-loop results demonstrated in Scoccimarro et al. (1998), but emerges in a much more straightforward way using our rescaling ansatz.

4 DISCUSSION

Following the path pioneered by Hamilton et al. (1991) and Peacock & Dodds (1996), we deploy a scale transformation argument to construct a well-behaved model of the bispectrum of dark matter fluctuations which is valid in the quasi-linear and intermediate non-linear regimes. On the basis of the tests, we have been able to perform, the resulting approximation seems to work exceptionally well, although we are somewhat hampered by numerical limitations. A full assessment of the precision and reliability of this idea will have to wait until simulations of even larger size and higher resolution than the VLS are available. Nevertheless, the important point may lie even deeper than the pragmatic usefulness of this as a simplifying ansatz. The scale transformation of equation (1) or (2) may contain more physical meaning than has been previously thought, and it may not after all be the case that higher order correlation functions in the non-linear regime necessarily pose such fierce analytical and numerical challenges as has generally been assumed.

The tree-level bispectrum is fully determined by the linear power spectrum, while the success of our model seems to tell us that the non-linear bispectrum is governed by the non-linear power spectrum. It therefore seems possible that having the power spectrum at hand, one can accurately derive the bispectrum of a broad range of configurations using only the knowledge that it was generated by gravitational physics.

It is well known that a Gaussian random field is fully described by its power spectrum, but does a non-Gaussian field evolved by gravity from Gaussian initial conditions also possess this feature at some level? It will be very interesting to test this idea by seeing if the trispectrum can also be obtained by rescaling tree-level pertur-
bation theory. More fundamentally, if gravitational interactions do manage to act in such a way then why is it that the simple scale transformation, equation (2), manages to capture so much complicated physics?

ACKNOWLEDGMENT

JP acknowledges the fellowship of the One Hundred Talents program of CAS, this work is also partly support by NSFC under grant 10643002 and by PPARC through grant PPA/G/S/2000/00057. IS acknowledges support from grants NSF AST02-06423, NSF AM S04-04 34 413 and NASA NNG 06G E71G. The simulations in this paper were carried out by the Virgo Supercomputing Consortium using computers based at Computing Centre of the Max-Planck Society in Garching and at the Edinburgh Parallel Computing Centre.

REFERENCES

Caldwell R. R., Juszkiewicz R., Steinhardt P. J., Bouchet F. R., 2001, ApJ, 547, L93
Coles P., Chiang L.-Y., 2000, Nat., 406, 376
Colombi S., Bernardeau F., Bouchet F. R., Hernquist L., 1997, MNRAS, 287, 241
Cooray A., Sheth R., 2002, Phys. Rep., 371, 1
Fosalba P., Pan J., Szapudi I., 2005, ApJ, 632, 29
Goroff M. H., Grinstein B., Rey S.-J., Wise M. B., 1986, ApJ, 311, 6
Hamilton A. J. S., Kumar P., Lu E., Matthews A., 1991, ApJ, 374, L1
Hou Y. H., Jing Y. P., Zhao D. H., Börner G., 2005, ApJ, 619, 667
Jain B., Mo H. J., White S. D. M., 1995, MNRAS, 276, L25
MacFarland T., Couchman H. M. P., Pearce F. R., Pichlmeier J., 1998, New Astron., 3, 687
McDonald P., 2007, Phys. Rev. D., 75, 043514
Nityananda R., Padmanabhan T., 1994, MNRAS, 271, 976
Peacock J. A., Dodds S. J., 1996, MNRAS, 280, L19
Peebles P. J. E., 1980, The Large-Scale Structure of the Universe. Princeton University Press, Princeton, New Jersey
Scoccimarro R., 2000, ApJ, 544, 597
Scoccimarro R., Colombi S., Fry J. N., Frieman J. A., Hivon E., Melott A. L., 1998, ApJ, 496, 586
Scoccimarro R., Frieman J. A., 1999, ApJ, 520, 35
Scoccimarro R., Couchman H. M. P., 2001, MNRAS, 325, 1312 (SC 2001)
Scoccimarro R., Sheth R. K., Hui L., Jain B., 2001, ApJ, 546, 20
Smith R. E. et al., 2003, MNRAS, 341, 1311
Smith R. E., Watts P. I. R., Sheth R. K., 2006, MNRAS, 365, 214
Szapudi I., 2004, ApJ, 605, L89
Szapudi I., Kaiser N., 2003, ApJ, 583, L1
Takada M., Jain B., 2003, MNRAS, 340, 580
Watts P. I. R., Coles P., 2003, MNRAS, 338, 806

This paper has been typeset from a \TeX/\LaTeX\ file prepared by the author.