Bounded linear operators
between $C^*$-algebras

by

Uffe Haagerup
and
Gilles Pisier*

Plan
Introduction
§ 1. Operators between $C^*$-algebras.
§ 2. Description of $E_n^k$.
§ 3. Random series in non-commutative $L_1$-spaces.
§ 4. Complements.

Introduction

Let $u: A \to B$ be a bounded linear operator between two $C^*$-algebras $A, B$. The following result was proved in [P1].

**Theorem 0.1.** There is a numerical constant $K_1$ such that for all finite sequences $x_1, \ldots, x_n$ in $A$ we have

$$
(0.1)_1 \max \left\{ \left\| \left( \sum x_i^* u(x_i) \right)^{1/2} \right\|_B, \left\| \left( \sum u(x_i) u(x_i)^* \right)^{1/2} \right\|_B \right\} \leq K_1 \|u\| \max \left\{ \left\| \left( \sum x_i^* x_i \right)^{1/2} \right\|_A, \left\| \left( \sum x_i x_i^* \right)^{1/2} \right\|_A \right\}.
$$

A simpler proof was given in [H1]. More recently an other alternate proof appeared in [LPP]. In this paper we give a sequence of generalizations of this inequality.

The above inequality $(0.1)_1$ appears as the case of “degree one” in this sequence. The next case of degree 2 seems particularly interesting, we now formulate it explicitly.

* Partially supported by the N.S.F.
Let us assume that $A \subset B(H)$ (embedded as a $C^*$-subalgebra) for some Hilbert space $H$, and similarly that $B \subset B(K)$. Let $(a_{ij})$ be an $n \times n$ matrix of elements of $A$. We define

$$[(a_{ij})]_{(2)} = \max \left\{ \left\| (a_{ij}) \right\|_{M_n(A)} , \left\| (a_{ij}^*) \right\|_{M_n(A)} , \left\| \left( \sum_{ij} a_{ij}^* a_{ij} \right)^{1/2} \right\|_A , \left\| \left( \sum_{ij} a_{ij} a_{ij}^* \right)^{1/2} \right\|_A \right\}.$$  

Then we have

**Theorem 0.2.** There is a numerical constant $K_2$ such that for all $n$ and for all $(a_{ij})$ in $M_n(A)$ we have

$$(0.1)_2 \quad [(u(a_{ij}))]_{(2)} \leq K_2 \|u\|[(a_{ij})]_{(2)}.$$  

We recall in passing the following identities for $a_{ij} \in A$ and $a_i \in A$

$$\|a_{ij}\|_{M_n(A)} = \sup \left\{ \left| \sum_{ij} \langle y_i, a_{ij} x_j \rangle \right| , \ x_j, y_i \in H \ \sum_{j} \|x_j\|^2 \leq 1 , \ \sum_{i} \|y_i\|^2 \leq 1 \right\},$$  

and

$$\left\| \left( \sum a_i^* a_i \right)^{1/2} \right\|_A = \sup \left\{ \left| \sum \langle y_i, a_i x_0 \rangle \right| , \ x_0 \in H , \ y_i \in H \ \|x_0\| \leq 1 , \ \sum \|y_i\|^2 \leq 1 \right\}.$$  

We will denote

$$(0.2) \quad [(a_i)]_{(1)} = \max \left\{ \left\| \left( \sum a_i^* a_i \right)^{1/2} \right\|_A , \ \left\| \left( \sum a_i a_i^* \right)^{1/2} \right\|_A \right\}.$$  

More generally, let us explain the general case of "degree $k$" of our main result. Let $k \geq 1$. Let $n$ be a fixed integer. We will denote $[n] = \{1, 2, \ldots, n\}$. Let $\{a_J \mid J \in [n]^k\}$ be a family of elements of $A$ indexed by $[n]^k$. Let us denote by $P_k$ the set of all the $2^k$ subsets (including the void set) of $\{1, 2, \ldots, k\}$.

For any $\alpha \subset \{1, \ldots, k\}$ we denote by $\alpha^c$ the complement of $\alpha$ and by

$$\pi_\alpha : [n]^k \to [n]^\alpha$$

the canonical projection, i.e.

$$\forall \ J = (j_1, \ldots, j_k) \in [n]^k \quad \pi(J) = (j_i)_{i \in \alpha}.$$  

2
For any $\alpha$ with $\alpha \neq \emptyset$ and $\alpha^c \neq \emptyset$ we define

\begin{equation}
\| (a_J) \|_\alpha = \sup \left\{ \left\| \sum_{J \in [n]^k} \langle a_Jx_{\pi_{\alpha}(J)}, y_{\pi_{\alpha^c}(J)} \rangle \right\| : \sum_{J \in [n]^k} \langle a_Jx_{\pi_{\alpha}(J)}, y_{\pi_{\alpha^c}(J)} \rangle \right\}
\end{equation}

where the supremum runs over all families

$$\{x_\ell \mid \ell \in [n]^\alpha\} \text{ and } \{y_m \mid m \in [n]^\alpha^c\}$$

of elements of $H$ such that $\sum \|x_\ell\|^2 \leq 1$ and $\sum \|y_m\|^2 \leq 1$. There is an alternate description, we can identify $[n]^k$ with $[n]^{\alpha^c} \times [n]^\alpha$ so that $J \in [n]^k$ is identified with $(i, j)$ with $i = \pi_{\alpha^c}(J), j = \pi_\alpha(J)$. Then $\| (a_J) \|_\alpha$ is nothing but the norm of the matrix $(a_{ij})$ acting from $\ell_2([n]^\alpha, H)$ into $\ell_2([n]^\alpha^c, H)$. For $\alpha = \emptyset$, this definition extends naturally to

$$\| (a_J) \|_\emptyset = \sup \left\{ \left\| \sum_{J \in [n]^k} \langle a_Jx_0, y_J \rangle \right\| : \sum_{J \in [n]^k} \langle a_Jx_0, y_J \rangle \right\} = \left\| \left( \sum_{J \in [n]^k} a_J^*a_J \right)^{1/2} \right\|_A$$

where the supremum runs over all $x_0 \in H, y_J \in H$ such that $\|x_0\| \leq 1$ and $\sum \|y_J\|^2 \leq 1$. Similarly, for $\alpha = \{1, \ldots, k\}$ we set

$$\| (a_J) \|_\alpha = \left\| \left( \sum_{J \in [n]^k} a_J^*a_J \right)^{1/2} \right\|_A.$$

We then define

\begin{equation}
[(a_J)](k) = \max_{\alpha \in P_k} \{ \| (a_J) \|_\alpha \}.
\end{equation}

We can now state one of our main results.

**Theorem 0.k.** For each $k \geq 1$, there is a constant $K_k$ such that for any bounded linear operator $u: A \to B$, for any $n \geq 1$ and for any family $\{a_J \mid J \in [n]^k\}$ in $A$ we have

\begin{equation}
[(u(a_J))]_{(k)} \leq K_k \|u\|[[(a_J)](k)].
\end{equation}

Moreover, we have $K_k \leq 2^{(3k/2)-1}$.

The proof is essentially in section 1 (it is completed in section 2).
We now reformulate this result in a fashion which emphasizes the connection with the notion of complete boundedness for which we refer to [Pa].

Let \( A \subset B(\mathcal{H}) \) be a \( C^* \)-algebra embedded as a \( C^* \)-subalgebra. (\( H \) a Hilbert space.) We denote as usual by \( M_n \) the set of all \( n \times n \) complex matrices (equipped with the norm of the space \( B(\ell_2^n) \)) and by \( M_n(A) \) the space \( M_n \otimes A \) equipped with its natural \( C^* \)-norm, induced by \( B(\ell_2^n(H)) \). More generally, let \( S \subset B(\mathcal{H}) \) be any closed linear subspace of \( B(\mathcal{H}) \) (\( \mathcal{H} \) is a Hilbert space). We call \( S \) an “operator space”.

We denote by \( S \otimes A \) the completion of the linear space \( S \otimes A \) equipped with the norm induced by \( B(\mathcal{H} \otimes_2 H) \) (here \( \mathcal{H} \otimes_2 H \) denotes the Hilbert space tensor product of \( \mathcal{H} \) and \( H \)). We will repeatedly use the following fact (for a proof see Lemma 1.5 in [DCH]). Let \( K \) be an arbitrary Hilbert space. Whenever \( u : S \to B(K) \) is completely bounded, the map \( I_A \otimes u : A \otimes S \to A \otimes B(K) \) is bounded and we have

\[
\|I_A \otimes u\|_{A \otimes S \to A \otimes B(K)} \leq \|u\|_{cb}.
\]

Clearly \( S \otimes A \) is again an operator space embedded into \( B(\mathcal{H} \otimes_2 H) \).

For example, we will need to consider a particular embedding of the Euclidean space \( \ell_2^n \) into \( M_n \oplus M_n \) as follows. (We equip \( M_n \oplus M_n \) with the norm \( \| (x, y) \| = \max \{ \| x \|, \| y \| \} \), for which it clearly is an operator space embedded – say – into \( M_{2n} \) in a block diagonal way.) We denote by \( E_n \) the subspace of \( M_n \oplus M_n \) formed by all the elements of the form

\[
\begin{pmatrix}
x_1 \\
\vdots \\
x_n
\end{pmatrix}
\oplus
\begin{pmatrix}
x_1 & \ldots & x_n
\end{pmatrix}
\]

with \( x_1, \ldots, x_n \in \mathbb{C} \). Let \( (e_{ij}) \) be the usual basis of \( M_n \). We denote by

\[
\delta_i = e_{i1} \oplus e_{1i}
\]

the natural basis of \( E_n \), (so that the above element can be written as \( \sum x_i \delta_i \).) As a Banach space, \( E_n \) is clearly isometric to \( \ell_2^n \). More precisely, for any \( C^* \)-algebra \( A \) and for any \( a_1, \ldots, a_n \) in \( A \) we have (this known fact is easy to check)

\[
\| \sum \delta_i \otimes a_i \|_{E_n \otimes A} = \max \left\{ \left\| \left( \sum a_i^* a_i \right)^{1/2} \right\|, \left\| \left( \sum a_i a_i^* \right)^{1/2} \right\| \right\}
\]

or equivalently

4
= [(a_1)]_{(1)}

in the preceding notation.

Let us denote by $E_n^k$ the tensor product

$$E_n \otimes \cdots \otimes E_n \quad (k \text{ times}).$$

Then, Theorem 0.k implies (and is actually equivalent to) the following.

**Proposition 0.k.** For any $u: A \to B$

$$\|I_{E_n^k} \otimes u\|_{E_n^k \otimes A \to E_n^k \otimes B} \leq 2^{(3k/2) - 1} \|u\|.$$

This proposition is proved in section 1. In section 2 we extend (0.6) and compute the norm of an element of $E_n^k \otimes A$ for $k > 1$ to deduce Theorem 0.k from Proposition 0.k.

In section 3, we develop the viewpoint of [LPP] which dualizes inequalities such as (0.1)$_1$ or (0.1)$_k$ to compute (an equivalent of) the norm of certain random series with coefficients in a non-commutative $L_1$-space. Let $(\varepsilon_j)_{j \in \mathbb{N}}$ be an i.i.d. sequence of random variables each distributed uniformly over the unimodular complex numbers. (Such variables are sometimes called Steinhaus variables.) Let $A_*$ be a non-commutative $L_1$-space. Roughly, while [LPP] treats the case of $A_*$-valued random variables which depend linearly on the sequence $(\varepsilon_j)$, we can treat variables which depend bilinearly or multilinearly in the variables $(\varepsilon_j)$. For a precise statement see Theorem 3.6 below.

It might be useful for some readers to emphasize that the variables $(\varepsilon_j)$ can be replaced by independent choices of signs or more importantly by i.i.d. Gaussian variables. All our results remain true in this setting, but with different numerical constant, this follows from the fact (due to N. Tomczak-Jaegermann) that $A_*$ is of cotype 2, see e.g. [P3] p. 36 for more details. We also would like to draw the reader’s attention to Kwapień’s paper [K] which contains “decoupling inequalities” quite relevant to the situation considered in Theorem 3.6 below. Using [K] one can deduce from (3.1) below some “non-decoupled” inequalities. For instance, we can find an equivalent of integrals of the form $\int \left\| \sum_{1 \leq i < j \leq n} \varepsilon_i \varepsilon_j x_{ij} \right\|_{A_*} dP$ where $x_{ij} \in A_*$ and $(\varepsilon_j)_{j \geq 1}$ is an i.i.d. sequence of symmetric $\pm 1$ valued random variables on a
probability space \((\Omega, P)\), and similarly in the multilinear case. We will not spell out the
details.

The results of the first three sections of this paper rely heavily on the following factorization
result proved in section 1: The identity map \(I_{E_n}\) on the operator space \(E_n\) has a completely
bounded factorization through the von Neumann algebra \(VN(F_n)\) associated with the left regular
representation of the free group with \(n\) generators, i.e. there are \(w_n : E_n \to VN(F_n)\) and \(v_n : VN(F_n) \to E_n\) such that
\[
I_{E_n} = v_n w_n \quad \text{and} \quad \|v_n\|_{cb} \|w_n\|_{cb} \leq 2.
\]
In section 4, we show that for any sequence of factorizations \(I_{E_n} = v_n w_n\) \((n = 1, 2, \ldots)\) of
the identity maps \(I_{E_n}\) through injective von Neumann algebras we have
\[
\lim_{n \to \infty} \|v_n\|_{cb} \|w_n\|_{cb} = +\infty.
\]
Combining these two facts about the factorization of \(I_{E_n}\) with Voiculescu’s recent result
([V1]) that the algebra of all \(n \times n\) matrices over \(VN(F_\infty)\) is isomorphic (as a von Neumann
algebra) to \(VN(F_\infty)\), we show at the end of section 4 that the von Neumann algebra
\(VN(F_n)\) is not a complemented subspace of \(B(H)\) for any \(n \geq 2\). (For very recent results
on similar questions, see [P4,CS].) We also include several general remarks about the
relation between the existence of a -completely bounded linear projection from \(B(H)\) onto
a subspace \(S\) and that of a bounded linear projection from \(B(\ell_2) \otimes B(H)\) onto \(B(\ell_2) \otimes S\).
For instance, if \(S\) is weak-\(*\) closed and if \(B(\ell_2)\overline{\otimes} S\) denotes the weak-\(*\) closure of \(B(\ell_2) \otimes S\) in
\(B(\ell_2 \otimes H)\), we show that there is a bounded linear projection from \(B(\ell_2 \otimes H)\) onto
\(B(\ell_2)\overline{\otimes} S\) if and only if there is a completely bounded one from \(B(H)\) onto \(S\).

Finally, we compare the space \(E_n\) with the linear span \(S_n\) of a free system of random
variables \(\{x_1, \ldots, x_n\}\) in a \(C^*\)-probability space \((A, \varphi)\) in the sense of Voiculescu [V1,2]. In
particular, in the case of a semicircular (or circular) system in Voiculescu’s sense, we show
that there is an isomorphism \(u\) from \(E_n\) onto the operator space \(S_n\) such that
\[
\|u\|_{cb} \|u^{-1}\|_{cb} \leq 2.
\]
§1. Operators between \( C^\ast \)-algebras.

We will use repeatedly the following fact which has been known to the first author for some time. The main point ((1.2) below) is a refinement of one of the inequalities of [H2]. (We remind the reader that we denote simply by \( C^\ast_n(F_n) \otimes A \) the minimal or spatial tensor product which is often denoted by \( C^\ast_n(F_n) \otimes \text{min} \ A \).

**Proposition 1.1.** Let \( F_n \) denote the free group on \( n \) generators \( g_1, \ldots, g_n \), and let \( C^\ast_n(F_n) \) be the reduced \( C^\ast \)-algebra of \( F_n \), i.e. the \( C^\ast \)-algebra generated by the left regular representation \( \lambda: F_n \to B(\ell^2(F_n)) \). Then

1. For any \( C^\ast \)-algebra \( A \) and for any set \( (a_g)_{g \in S} \) of elements of \( A \) indexed by a finite subset \( S \) of \( F_n \):

   \[
   \left\| \sum_{g \in S} \lambda(g) \otimes a_g \right\|_{C^\ast_n(F_n) \otimes A} \geq \max \left\{ \left\| \sum_{g \in S} a_g^* a_g \right\|^{1/2}, \left\| \sum_{g \in S} a_g a_g^* \right\|^{1/2} \right\}.
   \]

2. For any \( C^\ast \)-algebra \( A \) and for any set \( (a_g)_{g \in G} \) of elements of \( A \) indexed by a subset \( S \) of \( \{g_1, \ldots, g_n, g_1^{-1}, \ldots, g_n^{-1}\} \):

   \[
   \left\| \sum_{g \in S} \lambda(g) \otimes a_g \right\|_{C^\ast_n(F_n) \otimes A} \leq 2 \max \left\{ \left\| \sum_{g \in S} a_g^* a_g \right\|^{1/2}, \left\| \sum_{g \in S} a_g a_g^* \right\|^{1/2} \right\}.
   \]

**Proof.** (1) let \( (\delta_g)_{g \in G} \) be the standard basis of \( \ell^2(F_n) \). We may assume that \( A \subset B(K) \) for some Hilbert space \( K \). Since the min-tensor product coincide with the spatial tensor product, we have for all unit vectors \( \xi \in K \):

\[
\left\| \sum_{g \in S} \lambda(g) \otimes a_g \right\|_{C^\ast_n(F_n) \otimes A} \geq \left\| \sum_{g \in S} (\lambda(g) \otimes a_g)(\delta_e \otimes \xi) \right\|
= \left\| \sum_{g \in S} \delta_g \otimes a_g \xi \right\|
= \left( \sum_{g \in G} \|a_g \xi\|^2 \right)^{1/2}
= \left( \left( \sum_{g \in G} a_g^* a_g \right) \xi, \xi \right)^{1/2}
\]

7
Taking supremum over all unit vectors $\xi \in K$ we get

$$\left\| \sum_{g \in S} \lambda(g) \otimes a_g \right\|_{C^*_\lambda(F_n) \otimes A} \geq \left\| \sum_{g \in S} a^*_g a_g \right\|^{1/2}.$$ 

The same argument applied to the norm of $(\lambda(g) \otimes a_g)^* = \lambda(g^{-1}) \otimes a^*_g$ gives

$$\left\| \sum_{g \in G} \lambda(g) \otimes a_g \right\|_{C^*_\lambda(F_n) \otimes A} \geq \left\| \sum_{g \in S} a^*_g a_g \right\|^{1/2}.$$ 

This proves (1). Note that the statement (1) actually holds in $C^*_\lambda(\Gamma) \otimes A$ for any discrete group $\Gamma$.

(2) Consider first the case $S = \{g_1, \ldots, g_n, g_1^{-1}, \ldots, g_n^{-1}\}$. We can write $F_n$ as a disjoint union:

$$F_n = \{e\} \cup \left( \bigcup_{i=1}^n \Gamma_i^+ \right) \cup \left( \bigcup_{i=1}^n \Gamma_i^- \right)$$

where

$$\Gamma_i^+ = \text{set of reduced words starting with a positive power of } g_i,$$

$$\Gamma_i^- = \text{set of reduced words starting with a negative power of } g_i.$$ 

Let $e_0, e_i^+$ and $e_i^-$ denote the orthogonal projection of $\ell^2(F_n)$ onto the subspaces $C\delta_e$, $\ell^2(\Gamma_i^+)$ and $\ell^2(\Gamma_i^-)$ respectively. Then these projections are pairwise orthogonal and

$$e_0 + \sum_{i=1}^n e_i^+ + \sum_{i=1}^n e_i^- = I_{\ell^2(F_n)}.$$ 

For any $g \in G$ and for any generator $g_i$, the length of the reduced word for $g_i g$ is either

$$|g_i g| = |g| + 1 \quad \text{or} \quad |g_i g| = |g| - 1.$$ 

The first case exactly occurs when $g_i g$ starts with an element of $\Gamma_i^+$ and the second case when $g$ starts with an element of $\Gamma_i^-$. Hence for all $g \in G$:

$$\lambda(g_i) \delta_g = \begin{cases} 
  e_i^+ \lambda(g_i) \delta_g & \text{if } |g_i g| = |g| + 1 \\
  \lambda(g_i) e_i^- \delta_g & \text{if } |g_i g| = |g| - 1 \\
  e_i^+ \lambda(g_i) \delta_g + \lambda(g_i) e_i^- \delta_g & \text{(all cases).}
\end{cases}$$
Therefore

\[ \lambda(g_i) = e_i^+ \lambda(g_i) + \lambda(g_i)e_i^- \]

and by taking adjoints:

\[ \lambda(g_i^{-1}) = e_i^- \lambda(g_i^{-1}) + \lambda(g_i^{-1})e_i^+ . \]

Set

\[ \begin{align*}
  u_i &= e_i^+ \lambda(g_i), \\
  u_{n+i} &= e_i^- \lambda(g_i^{-1}), \\
  v_i &= \lambda(g_i^{-1})e_i^- , \\
  v_{n+i} &= \lambda(g_i^{-1})e_i^+ .
\end{align*} \]

and for simplicity of notation, set also \( g_{n+i} = g_i^{-1} \), \( i = 1, \ldots, n \). Then

\[ \lambda(g_i) = u_i + v_i, \quad i = 1, \ldots, 2n. \]

Since \( \sum_{i=1}^{n} (e_i^+ + e_i^-) = 1 - e_0 \) we have

\[ \sum_{i=1}^{2n} u_i u_i^* = \sum_{i=1}^{2n} v_i^* v_i = 1 - e_0 \leq 1. \]

So

\[ \left\| \sum_{i=1}^{2n} u_i u_i^* \right\| \leq 1 \quad \text{and} \quad \left\| \sum_{i=1}^{2n} v_i^* v_i \right\| \leq 1. \]

For elements \( c_1, \ldots, c_m, d_1, \ldots, d_m \) of a \( C^* \)-algebra \( B \) one has easily that

\[ \left\| \sum_{i=1}^{m} c_i d_i \right\| \leq \left\| \sum_{i=1}^{m} c_i^* c_i \right\|^{1/2} \left\| \sum_{i=1}^{m} d_i^* d_i \right\|^{1/2} . \]

Hence, with \( u_1, \ldots, u_{2n}, v_1, \ldots, v_{2n} \) as above and \( a_1, \ldots, a_{2n} \in A \),

\[ \left\| \sum_{i=1}^{2n} u_i \otimes a_i \right\|_{C^*_r(F_n) \otimes A} = \left\| \sum_{i=1}^{n} (u_i \otimes 1)(1 \otimes a_i) \right\|_{C^*_r(F_n) \otimes A} \]

\[ \leq \left\| \sum_{i=1}^{n} u_i u_i^* \right\|^{1/2} \left\| \sum_{i=1}^{2n} a_i a_i^* \right\|^{1/2} \]

\[ \leq \left\| \sum_{i=1}^{n} a_i a_i^* \right\|^{1/2} . \]
and similarly

$$\left\| \sum v_i \otimes a_i \right\|_{C^*_r(F_n) \otimes A} \leq \left\| \sum_{i=1}^n (1 \otimes a_i) (v_i \otimes 1) \right\|_{C^*_r(F_n) \otimes A}$$

$$\leq \left\| \sum a_i a_i^* \right\|^{1/2} \left\| \sum_{i=1}^n v_i^* v_i \right\|$$

$$\leq \left\| \sum a_i a_i^* \right\|^{1/2}$$

so altogether

$$\left\| \sum_{i=1}^{2n} \lambda (g_i) \otimes a_i \right\| = \left\| \sum_{i=1}^{2n} u_i \otimes a_i + \sum_{i=1}^{2n} v_i \otimes a_i \right\|$$

$$\leq \left\| \sum_{i=1}^{2n} a_i^* a_i \right\|^{1/2} + \left\| \sum_{i=1}^{2n} a_i a_i^* \right\|^{1/2}$$

$$\leq 2 \max \left\{ \left\| \sum_{i=1}^{2n} a_i^* a_i \right\|^{1/2}, \left\| \sum_{i=1}^{2n} a_i a_i^* \right\|^{1/2} \right\} .$$

This proves (2) in the case $S = \{g_1, \ldots, g_n, g_1^{-1}, \ldots, g_n^{-1}\}$, and the remaining cases follows from this by setting some of the $a_i$'s equal to 0.

\[\Box\]

**Remark.** The preceding statement remains true (with the obvious modifications) for the free group on infinitely many generators. See also Proposition 4.9 below for a generalization of (1.1) and (1.2).

**Remark 1.2.** The proof of (2) is an illustration of the following general principle. Let $T_1, \ldots, T_n$ be operators on a Hilbert space $H$ and let $c$ be a constant. The following properties are essentially equivalent:

(i) For any $C^*$-algebra $A$ and any set $(a_i)_{i \leq n}$ in $A$ we have

$$\left\| \sum T_i \otimes a_i \right\| \leq c \max \left\{ \left\| \left( \sum a_i^* a_i \right)^{1/2} \right\|, \left\| \left( \sum a_i a_i^* \right)^{1/2} \right\| \right\} .$$

(ii) There are operators $u_i, v_i$ in $B(H)$ such that $T_i = u_i + v_i$ and

$$\left\| \left( \sum u_i^* u_i \right)^{1/2} \right\| + \left\| \left( \sum v_i v_i^* \right)^{1/2} \right\| \leq c.$$
More precisely, we have \((ii)_c \Rightarrow (i)_c\) and \((i)_c \Rightarrow (ii)_c\). The implication \((ii)_c \Rightarrow (i)_c\) follows as above from the triangle inequality. To prove the converse, note that \((i)_c\) equivalently means that the operator \(u: E_n \to B(H)\) which maps \(\delta_i\) to \(T_i\) satisfies \(\|u\|_{cb} \leq 1\). By the extension property of c.b. maps (cf. [Pa, p.100]) there is an extension \(\tilde{u}: M_n \oplus M_n \to B(H)\) such that \(\tilde{u}(\delta_i) = T_i\) and \(\|\tilde{u}\|_{cb} \leq 1\). Letting \(u_i = \tilde{u}(e_{i1} \oplus 0)\) and \(v_i = \tilde{u}(0 \oplus e_{1i})\) we obtain a decomposition satisfying \((ii)_c\). This shows that \((i)_c\) implies \((ii)_c\).

**Proposition 1.3.** Let \(E_n \subset M_n \oplus M_n\) be the operator space

\[
E_n = \left\{ \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \oplus \begin{pmatrix} c_1 & \ldots & c_n \\ \vdots \\ \circ \end{pmatrix} \mid c_1, \ldots, c_n \in C \right\}.
\]

Then there are linear mappings

\[
w: E_n \to C^*_\lambda(F_n) \quad \text{and} \quad v: C^*_\lambda(F_n) \to E_n
\]

such that

\[
wv = I_{E_n} \quad \text{and} \quad \|v\|_{cb} \|w\|_{cb} \leq 2.
\]

Similarly, for the von Neumann algebra \(VN(F_n)\) generated \(\lambda\), there are linear mappings

\[
w_1: E_n \to VN(F_n) \quad \text{and} \quad v_1: VN(F_n) \to E_n
\]

such that

\[
v_1w_1 = I_{E_n} \quad \text{and} \quad \|v_1\|_{cb} \|w_1\|_{cb} \leq 2.
\]

In particular \(E_n\) is cb-isomorphic to a cb-complemented subspace of \(C^*_\lambda(F_n)\) (resp. of \(VN(F_n)\)).

**Proof.** Let \((\delta_1, \ldots, \delta_n)\) be the basis of \(E_n\) determined by

\[
\sum_{i=1}^{n} c_i \delta_i = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \oplus \begin{pmatrix} c_1 & \ldots & c_n \\ \vdots \\ \circ \end{pmatrix}
\]

for \(c_1, \ldots, c_n \in C\). Define \(w: E_n \to C^*_\lambda(F_n)\) by

\[
w \left( \sum_{i=1}^{n} c_i \delta_i \right) = \sum_{i=1}^{n} c_i \lambda(g_i)
\]
and \( v: C^*_\lambda(F_n) \to E_n \) by
\[
v(x) = \sum_{i=1}^{n} \tau(\lambda(g_i)^* x) \delta_i
\]

where \( \tau \) is the trace on \( C^*_\lambda(F_n) \) given by
\[
\tau(y) = (y \delta_e, \delta_e), \quad y \in C^*_\lambda(F_n). \quad (\text{cf. [KR, p. 433]})
\]

For any set \( a_1, \ldots, a_n \) of \( n \) elements in a \( C^* \)-algebra \( A \)
\[
(w \otimes I_A) \left( \sum_{i=1}^{n} \delta_i \otimes a_i \right) = \sum_{i=1}^{n} \lambda(g_i) \otimes a_i.
\]

Since
\[
\left\| \sum_{i=1}^{n} e_i \otimes a_i \right\| = \left\| \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \otimes \begin{pmatrix} a_1 & \ldots & a_n \end{pmatrix} \right\|^{1/2} = \max \left\{ \left\| \sum_{i=1}^{n} a_i^* a_i \right\|^{1/2}, \left\| \sum_{i=1}^{n} a_i a_i^* \right\|^{1/2} \right\}
\]

it follows from Theorem 1.1 (2), that \( \| w \otimes I_A \| \leq 2 \). Hence \( \| w \|_{cb} \leq 2 \). Since
\[
\tau(\lambda(g)^* \lambda(h)) = \begin{cases} 1 & g = h \\ 0 & g \neq h \end{cases}
\]

we get for any finite subset \( S \subset F_n \) and scalars \( (c_g)_{g \in S} \)
\[
v \left( \sum_{g \in S} c_g \lambda(g) \right) = \sum_{i=1}^{n} c_{g_i} \delta_i
\]

and hence
\[
(v \otimes I_A) \left( \sum_{g \in S} \lambda(g) \otimes a(g) \right) = \sum_{i=1}^{n} \delta_i \otimes a(g_i).
\]

Let \( S^1 = S \cap \{ g_1, \ldots, g_n \} \). Then
\[
\left\| \sum_{i=1}^{n} \delta_i \otimes a(g_i) \right\| = \max \left\{ \left\| \sum_{g \in S^1} a(g)^* a(g) \right\|^{1/2}, \left\| \sum_{g \in S^1} a(g) a(g)^* \right\|^{1/2} \right\}
\]
\[
\leq \max \left\{ \left\| \sum_{g \in S} a(g)^* a(g) \right\|^{1/2}, \left\| \sum_{g \in S} a(g) a(g)^* \right\|^{1/2} \right\}.
\]

12
which by Theorem 1.1(1) is smaller than or equal to
\[
\left\| \sum_{g \in S} \lambda(g) \otimes a(g) \right\|_{C^*_\lambda(F_n) \otimes A}.
\]
Hence \( \| v \otimes I_A \| \leq 1 \) and thus \( \| v \|_{cb} \leq 1 \). Therefore
\[
\| v \|_{cb} \| w \|_{cb} \leq 2
\]
and by construction \( vw = I_{E_n} \). This implies that \( w \) is a \( cb \)-isomorphism of \( E_n \) onto its range
\[
w(E_n) = \text{span}\{\lambda(g_i) \mid i = 1, \ldots, n\}
\]
and
\[
\| w \|_{cb} \| w^{-1} \|_{cb} \leq 2.
\]
Moreover \( P = vw \) is a completely bounded projection of \( C^*_\lambda(F_n) \) onto \( w(E_n) \) and \( \| P \|_{cb} \leq 2 \). The proof with \( VN(F_n) \) in the place of \( C^*_\lambda(F_n) \) is easy since \( v \) admits an extension \( v_1 : VN(F_n) \to E_n \) with \( \| v_1 \|_{cb} \leq 1 \). We leave the details to the reader. \( \square \)

**Lemma 1.4.** ([P1, H1, LPP]). Let \( u : A \to B \) be a bounded linear operator between two \( C^* \)-algebras \( A \) and \( B \). Then for every \( n \in \mathbb{N} \)
\[
\| I_{E_n} \otimes u \|_{E_n \otimes A \to E_n \otimes B} \leq \sqrt{2} \| u \|.
\]

**Proof.** The statement of the lemma is equivalent to: For all \( a_1, \ldots, a_n \in A \)
\[
\max \left\{ \left\| \sum u(a_i)^*u(a_i) \right\|, \left\| \sum u(a_i)u(a_i)^* \right\| \right\} \leq 2 \| u \|^2 \max \left\{ \left\| \sum a_i^*a_i \right\|, \left\| \sum a_i^*a_i \right\| \right\}.
\]
This is essentially [P1], (see also [H1,LPP]). However to get the constant 2 in (1.3) one has to modify the proof of [H1, Cor. 3.4] slightly:
Let \( T : A \to H \) be a bounded linear operator from the \( C^* \)-algebra \( A \) with values in a Hilbert space. By [H1, Thm. 3.2],
\[
\sum \| T(a_k) \|^2 \leq \| T \|^2 \left( \left\| \sum a_k^*a_k \right\| + \left\| \sum a_k^*a_k \right\| \right).
\]
We can assume, that $B \subseteq B(K)$ for some Hilbert space $K$. By the above inequality (1.4) we get for any $\xi \in K$, that
\[
\sum \|u(a_k)\xi\|^2 \leq \|\xi\|^2\|u\|^2 \left( \left\| \sum a_k^*a_k \right\| + \left\| \sum a_k a_k^* \right\| \right).
\]
Clearly (1.4) also holds for conjugate linear maps, so
\[
\sum \|u(a_k)^*\xi\|^2 \leq \|\xi\|^2\|u\|^2 \left( \left\| \sum a_k^*a_k \right\| + \left\| \sum a_k a_k^* \right\| \right).
\]
Thus
\[
\max \left\{ \left\| \sum u(a_k)^*u(a_k) \right\|, \left\| \sum u(a_k)u(a_k)^* \right\| \right\} \leq \|u\|^2 \left( \left\| \sum a_k^*a_k \right\| + \left\| \sum a_k a_k^* \right\| \right)
\]
which implies (1.3).

\begin{proof}
The theorem is proved by induction on $k$. By Lemma 1.4 the theorem holds for $k = 1$. Assume next that the theorem is true for a particular $k \in \mathbb{N}$. Let
\[
w: E_n \to C^*_\lambda(F_n) \quad \text{and} \quad v: C^*_\lambda(F_n) \to E_n
\]
be as in Proposition 1.2, and let $u: A \to B$ be a linear map between two $C^*$-algebras $A$ and $B$. Clearly
\[
I_{E_n} \otimes u = (v \otimes u)(w \otimes I_A)
\]
where
\[
\|v \otimes u\| = \| (v \otimes I_B)(I_{E_n} \otimes u) \|
\leq \|v\|_{cb}\|I_{E_n} \otimes u\|
\leq \sqrt{2} \|u\|\|v\|_{cb}
\]

\end{proof}
by Lemma 1.4. Moreover $v \otimes u$ maps the $C^*$-algebra $C^*_\lambda(F_n) \otimes A$ into the $C^*$-algebra $M_n(B) \oplus M_n(B)$, so by the induction hypothesis

$$\|I_{E^n_k} \otimes v \otimes u\| \leq 2^{\frac{3}{2}k-1} \|v \otimes u\| \leq 2^{\frac{3}{2}k-\frac{1}{2}} \|u\| \|v\|_{cb}.$$  

On the other hand by (0.5)

$$\|I_{E^n_k} \otimes w \otimes I_A\| = \|I_{E^n_k} \otimes w\| \leq \|w\|_{cb}$$

Now by (1.5)

$$I_{E^n_{k+1}} \otimes u = (I_{E^n_k} \otimes v \otimes u)(I_{E^n_k} \otimes w \otimes I_A).$$

Thus, by Proposition 1.3

$$\|I_{E^n_{k+1}} \otimes u\| \leq 2^{\frac{3}{2}k-\frac{1}{2}} \|u\| \|v\|_{cb} \|w\|_{cb}$$

$$\leq 2^{\frac{3}{2}k+\frac{1}{2}} \|u\|$$

$$= 2^{\frac{3}{2}(k+1)-1} \|u\|.$$  

Hence Theorem 1.5 follows by induction on $k$.  

\[\square\]
§2. Description of $E^k_n$.

In this section, we will identify the norm in the space $E^k_n \otimes A$ with the norm previously introduced in (0.3) and (0.4) as $[(\cdot)](k)$.

**Proposition 2.1.** Let $A$ be any C*-algebra. Let $n \geq 1$, $k \geq 1$ and let $\{a_J|J \in [n]^k\}$ be elements of $A$. Then

\[(2.1) \quad [(a_J)](k) = \left\| \sum_{J \in [n]^k} \delta_J \otimes a_J \right\|_{E^k_n \otimes A}
\]

where we denote if $J = (j_1, ..., j_k)$

$$\delta_J = \delta_{j_1} \otimes ... \otimes \delta_{j_k}$$

The proof below is easy but the notation is a bit painful. Using Proposition 2.1 we can complete the proof of the results announced in the introduction.

**Proof of Theorem 0.k:** Consider an operator $u : A \to B$ between C*-algebras. By Theorem 1.5 we have for all $(a_J)$ in $A$

\[
\left\| \sum_{J \in [n]^k} \delta_J \otimes u(a_J) \right\|_{E^k_n \otimes B} \leq 2^{(3k/2) - 1} \|u\| \left\| \sum_{J \in [n]^k} \delta_J \otimes a_J \right\|_{E^k_n \otimes A}.
\]

Taking (2.1) into account this immediately implies $(0.1)_k$ and completes the proof of Theorem 0.k.

We now check (2.1). We will need the following elementary fact

**Lemma 2.2.** Let $H, H_1, H_2, H_3, H_4$ be Hilbert spaces. Let $e \in H_1$, $f \in H_4$ be norm one vectors. Let $(\varphi_j)_{j \in J}$ and $(\psi_i)_{i \in I}$ be orthonormal finite sequences in $H_2$ and $H_3$ respectively. Let $a_{ij}$ be elements of a C*-algebra $A$ embedded into $B(H)$. Then we have

\[(2.2) \quad \left\| \sum_{i \in I} \sum_{j \in J} (e \otimes \varphi_j) \otimes (\psi_i \otimes f) \otimes a_{ij} \right\| = \sup_{y_i \in H_1, x_j \in H_2} \left\{ \sum_{i,j} \langle y_i, a_{ij} - x_j \rangle \right\} , \sum_{i,j} \|x_j\|^2 \leq 1, \sum \|y_i\|^2 \leq 1 \}
\]

Here the norm on the left hand side means the norm in the space of all bounded operators from $H_1 \otimes_2 H_2 \otimes_2 H$ into $H_3 \otimes_2 H_4 \otimes_2 H$.

**Proof.** We may clearly assume without loss of generality that $H_1 = C e, H_4 = C f$ and that $(\varphi_j)$ (resp. $(\psi_i)$) is a basis of $H_2$ (resp. $H_3$). Then the norm we want to compute is
clearly equal to the norm of the operator

$$\tilde{T} = \sum_{ij} \varphi_j \otimes \psi_i \otimes a_{ij}$$

as an operator from $H_2 \otimes_2 H$ to $H_3 \otimes_2 H$. But then the general form of an element in the unit ball of $H_2 \otimes_2 H$ (resp. $H_3 \otimes_2 H$) is given by $\sum \varphi_j \otimes x_j$ (resp. $\sum \psi_i \otimes y_i$) with $x_j \in H_2$ (resp. $y_i \in H_3$) such that $\sum \|x_j\|^2 \leq 1$ (resp. $\sum \|y_i\|^2 \leq 1$). Hence the norm of $\tilde{T}$ (or of $T$) is equal to the right hand side of (2.2).

We need to introduce more notation.

Recall that $E_n \subset M_n \oplus M_n$ and $\delta_i = e_{i1} - \oplus e_{1i}$. We consider of course $M_n \oplus M_n$ as a subset of the set of all operators on $\ell^2_n \oplus \ell^2_n$. It will be convenient to denote $e^0_{ij} = e_{ij} \oplus 0$ and $e^1_{ij} = 0 \oplus e_{ij}$ in $M_n \oplus M_n$. Also $e^0_i = e_i \oplus 0$ and $e^1_i = 0 \oplus e_i$ in $\ell^2_n \oplus \ell^2_n$. As is usual, for $e, f$ in $H$, we will identify the tensor $e \otimes f$ with the operator $x \rightarrow <e,x>f$ (defined on $H$). Hence in tensor product notation we have (with the usual matricial conventions) $e_{ij} = e_j \otimes e_i$ and $\delta_i = e^0_i \otimes e^0_i + e^1_i \otimes e^1_i$. Let us denote by $H_0$ the span of $\{e_{i1}|i = 1, ..., n\}$ in $M_n$ and by $H_1$ the span of $\{e_{1i}|i = 1, ..., n\}$ in $M_n$, so that $E_n \subset H_0 \oplus H_1$. Let $P_0 : H_0 \oplus H_1 \rightarrow H_0$ (resp. $P_1 : H_0 \oplus H_1 \rightarrow H_1$) denote the canonical projection. We have $E_n \otimes^k \subset (H_0 \oplus H_1)^{\otimes k}$.

For $\alpha \in \{0, 1\}^k$ we denote

$$P_\alpha : (H_0 \oplus H_1)^k \rightarrow (H_0 \oplus H_1)^k$$

the projection defined by

$$P_\alpha = P_{\alpha(1)} \otimes P_{\alpha(2)} \otimes ... \otimes P_{\alpha(k)}.$$

Let us denote by $I_X$ the identity on $X$. Then we have

$$I_{(H_0 \oplus H_1)^{\otimes k}} = (I_{H_0 \oplus H_1})^{\otimes k}$$

$$= (P_0 + P_1)^{\otimes k}$$

$$= \sum_{\alpha \in \{0, 1\}^k} P_{\alpha(0)} \otimes ... \otimes P_{\alpha(k)}$$

$$= \sum_{\alpha} P_\alpha$$

(2.3)
Proof of Proposition 2.1: Let $T = \sum_{J \in [n]^k} \delta_J \otimes a_J$. By (2.3) we have

$$ T = \sum_{\alpha} T_{\alpha} $$

where

$$ T_{\alpha} = \sum_J P_{\alpha}(\delta_J) \otimes a_J. $$

We now claim that

(2.4) $\|T_{\alpha}\| = \|(a_J)\|_{\alpha}.$

To check this, we can assume for simplicity (up to a permutation of the factors in the tensor product) that $\alpha$ is the indicator function of the set \{1, 2, ..., $p$\} for some $p$ with $1 \leq p \leq k$. Then if $J = (j_1, ..., j_k)$ we have

(2.5) $P_{\alpha}(\delta_J) = e_{j_1}^1 \otimes ... \otimes e_{j_p}^1 \otimes e_{1,j_{p+1}}^0 \otimes ... \otimes e_{1,j_k}^0.$

(Recall the convention that the tensor $e \otimes f$ represents the operator $x \rightarrow <e, x>f$). Let $e^1(\alpha) = e_1^1 \otimes ... \otimes e_1^1$ -(p times) and $f^0(\alpha) = e_1^0 \otimes ... \otimes e_1^0$ (k - p times).

Then (2.5) yields

$$ P_{\alpha}(\delta_J) = (e^1(\alpha) \otimes e_{j_{p+1}}^0 \otimes ... \otimes e_{j_k}^0) \otimes (e_{j_1}^1 \otimes ... \otimes e_{j_p}^1 \otimes f^0(\alpha)). $$

If we now write $e^\varepsilon_{\{j_1, ..., j_p\}}$ instead of $e^\varepsilon_{j_1} \otimes ... \otimes e^\varepsilon_{j_p}$ for $\varepsilon = 0$ or 1, we can rewrite the last identity as

(2.6) $P_{\alpha}(\delta_J) = (e^1(\alpha) \otimes e_{\pi_{\alpha}(J)}^0) \otimes (e_{\pi_{\alpha}(J)}^1 \otimes f^0(\alpha)),$

where we recall that $\pi_{\alpha} : [n]^k \rightarrow [n]^{\alpha}$ denotes the canonical projection. Then the above lemma 2.2 gives in the present particular case

$$ \|T_{\alpha}\| = \|\sum_J (e^1(\alpha) \otimes e_{\pi_{\alpha}(J)}^0) \otimes (e_{\pi_{\alpha}(J)}^1 \otimes f^0(\alpha)) \otimes a_J\| = \|(a_J)\|_{\alpha}. $$

This proves our claim (2.4).
Now, we can conclude. Let us denote $h^0 = \ell_2^0 \oplus 0$ and $h^1 = 0 \oplus \ell_2^0$ in $\ell_2^0 \oplus \ell_2^0$. Let $K_\alpha$ be the support of $T_\alpha$ \textit{(i.e. the orthogonal of its kernel)} and let $R_\alpha$ be the range of $T_\alpha$. Then the preceding formula (2.6) shows that $K_\alpha$ is equal to the tensor product $F_1 \otimes F_2 \otimes ... \otimes F_k$ where
\[
F_j = \mathbb{C}e_1^1 \quad \text{if} \quad -j \in \alpha
\]
and
\[
F_j = h^0 \quad \text{if} \quad -j \notin \alpha.
\]

It follows that the subspaces ($K_\alpha$) are mutually orthogonal. Similarly, the family ($R_\alpha$) is mutually orthogonal. By a well known estimate it follows that
\[
\left\| \sum \alpha \right\| = \max_\alpha \left\| T_\alpha \right\|.
\]

This completes the proof.
§3. Random series in non-commutative $L_1$-spaces.

Let $A$ be a von Neumann algebra with a predual denoted by $A^*$. Let $\xi_1, \ldots, \xi_n \in A^*$ and let (recall (0.2))

$$[(\xi_i)]^*_1 = \sup \left\{ \left| \sum a_i \right| \left| a_i \in A \quad [(a_i)]_1 \leq 1 \right. \right\}.$$ 

For instance, if $A = B(H)$, $A^* = C_1(H)$ (the space of trace class operators on $H$) and we have clearly

$$[(\xi_i)]^*_1 = \inf \left\{ tr \left( \sum x_i^* x_i \right)^{1/2} + tr \left( \sum y_i y_i^* \right)^{1/2} \right\}$$

where the infimum runs over all decompositions $\xi_i = x_i + y_i$ in $C_1(H)$.

Let $T^N$ be the infinite dimensional torus equipped with its normalized Haar measure $\mu$. The following result is proved in [LPP].

For all $\xi_1, \ldots, \xi_n$ in $A^*$

$$\frac{1}{2} [(\xi_i)]^*_1 \leq \int \left\| \sum_{j=1}^n e^{it_j} \xi_j \right\|_{A^*} d\mu(t) \leq [(\xi_i)]^*_1.$$  

(See Theorem 3.3 below and its proof.)

It is easy to deduce from (3.1) a necessary and sufficient condition for a series of the form

$$S(t) = \sum_{j=1}^\infty e^{it_j} \xi_j, \quad t = (t_j)_{j \in \mathbb{N}} \in T^N$$

to converge in $L_2(T^N, \mu; A^*)$. The aim of this section is to prove a natural extension of (3.1) to double series of the form

$$S(t', t'') = \sum_{j,k=1}^\infty e^{it'_j} e^{it''_k} \xi_{jk}$$

with $\xi_{jk} \in A^*$, $t', t'' \in T^N$. More generally, we will consider for any $k \geq 1$, elements $\xi_{j_1,j_2\ldots,j_k}$ in $A^*$ and will find an equivalent for the expression

$$\int \left\| \sum_{j_1 \leq n, \ldots, j_k \leq n} e^{it_{j_1}} \ldots e^{it_{j_k}} \xi_{j_1,j_2\ldots,j_k} \right\|_{A^*} d\mu(t_1) \ldots d\mu(t^k).$$
See Theorem 3.6 below for an explicit statement.

Let $A$ be a $C^*$-algebra throughout this section. We will denote simply

$$C_n = C^*_\lambda(F_n)$$

and

$$C^k_n = C_n \otimes \cdots \otimes C_n \quad (k \text{ times}).$$

We always equip the tensor products such as $E_n \otimes A$, $C_n \otimes A$, $C^k_n \otimes A$ with the spatial (or minimal) tensor product. More precisely, whenever $S \subset B(K)$ is an operator space and $A \subset B(H)$ is a $C^*$-algebra, we will denote by $S \otimes A$ the linear tensor product equipped with the norm induced by $B(K \otimes_2 H)$.

Let $G$ be a discrete group. For $t \in G$, let $\lambda_*(t)$ denote the element of $C^*_\lambda(G)^*$ given by

$$\forall \ a \in C^*_\lambda(G) \quad \langle \lambda_*(t), a \rangle = \langle a \delta_e, \delta_t \rangle.$$

Clearly

$$\langle \lambda_*(s), \lambda_*(t) \rangle = \begin{cases} 1 & \text{if } s = t \\ 0 & \text{otherwise.} \end{cases}$$

Note that if $C^*_\lambda(G)^*$ is identified with $B_\lambda(G)$ in the usual way (see for instance [E]) then $\lambda_*(t)$ simply corresponds to the function $\delta_t$.

For any $J = (j_1, \ldots, j_k) \in [n]^k$ we denote by $g_J$ the element of $(F_n)^k$ defined by

$$g_J = (g_{j_1}, \ldots, g_{j_k}).$$

Then with the obvious identification

$$C^*_\lambda((F_n)^k) = C^k_n$$

we have $\lambda(g_J) = \lambda(g_{j_1}) \otimes \cdots \otimes \lambda(g_{j_k})$. We will also consider the dual $E_n^*$ of the space $E_n$ considered in section 1 and will denote by $\{\delta^*_j\}$ the basis of $E_n^*$ which is biorthogonal to $\{\delta_j\}$. We will also consider $E^k_n = E_n \otimes \cdots \otimes E_n \ (k \text{ times})$ and its dual $(E^k_n)^*$. We will denote for any $J = (j_1, \ldots, j_k)$ in $[n]^k$

$$\delta^*_J = \delta^*_{j_1} \otimes \cdots \otimes \delta^*_{j_k} \in (E^k_n)^*$$

21
and
\[ \lambda_*(g_J) = \lambda_*(g_{j_1}) \otimes \cdots \otimes \lambda_*(g_{j_k}) \in (C_n^k)^*. \]

We will denote by \( \Omega \) the infinite dimensional torus i.e. we set
\[ \Omega = \mathbb{T}^N \]
and we equip \( \Omega \) with the normalized Haar measure \( \mu \). (In most of what follows, it would be more appropriate to replace \( \Omega \) by \( \Omega_n = \mathbb{T}^n \), but we try to simplify the notation.) We will denote by
\[ \varepsilon_j : \Omega \to \mathbb{T} \]
the sequence of the coordinate functions on \( \Omega \). Moreover, we will consider the product \( \Omega^k \) equipped with the product measure \( \mu^k \). For any \( J = (j_1, \ldots, j_k) \in [n]^k \), let \( \varepsilon_J : \Omega^k \to \mathbb{T} \) be the function defined by
\[ \forall (t_1, \ldots, t_k) \in \Omega^k \quad \varepsilon_J(t_1, \ldots, t_k) = \varepsilon_{j_1}(t_1) \cdots \varepsilon_{j_k}(t_k). \]
Equivalently \( \varepsilon_J = \varepsilon_{j_1} \otimes \cdots \otimes \varepsilon_{j_k} \). We first record a simple consequence of Proposition 1.3.

**Lemma 3.1.** For any \( \{\varepsilon_j \mid j \leq n\} \) in \( A^* \) we have
\[ \frac{1}{2} \left\| \sum \delta_j^* \otimes \xi_j \right\|_{(F_n \otimes A)^*} \leq \left\| \sum \lambda_*(g_j) \otimes \xi_j \right\|_{(C_n \otimes A)^*} \leq \left\| \sum \delta_j^* \otimes \xi_j \right\|_{(F_n \otimes A)^*}. \]

**Proof.** Let \( v, w \) be as in Proposition 1.3. Since \( w \delta_j = \lambda(g_j) \) and \( v(\lambda(g_j)) = \delta_j \) we have
\[ (w \otimes I_A)^*(\lambda_*(g_j) \otimes \xi) = \delta_j^* \otimes \xi_j \quad \text{and} \quad (v \otimes I_A)^*(\delta_j^* \otimes \xi_j) = \lambda_*(g_j) \otimes \xi_j. \]
Hence, recalling (0.5), Lemma 3.1 follows from \( \|w\|_{cb} \leq 2 \) and \( \|v\|_{cb} \leq 1 \). \( \square \)

The next lemma is rather elementary.

**Lemma 3.2.**

(i) Consider \( \{\xi_{ij} \mid i, j = 1, \ldots, n\} \) in \( A^* \). For any orthonormal systems \( \varphi_1, \ldots, \varphi_n \) and \( \psi_1, \ldots, \psi_n \) in \( L_2(\mu) \) (where \( \mu \) is a probability as above) we have
\[ \int \left\| \sum \varphi_i(t)\psi_j(s)\xi_{ij} \right\|_{A^*} d\mu(t) d\mu(s) \leq \|\xi_{ij}\|_{M_n(A)^*}. \]
(ii) For any \( k \geq 1 \) and any \( (\xi_j) \) in \( A^* \) we have

\[
(3.3) \quad \left\| \sum_{J \in [n]^k} \varepsilon_J \xi_J \right\|_{L_1(\mu; A^*)} \leq \left\| \sum_{J \in [n]^k} \delta_J^* \otimes \xi_J \right\|_{(E_n^k \otimes A)^*}.
\]

**Proof:** (i) To prove this, it clearly suffices to assume that \( A \) is a von Neumann algebra and that \( \xi_{ij} \in A_* \). Since \( M_n(A) \) is a subspace of \( M_n(B(H)) \) for some Hilbert space \( H \), by duality its predual \( M_n(A)^* \) is a quotient of \( M_n(B(H))^* \). This shows that it suffices to prove (i) for \( A = B(H) \) and \( \xi_{ij} \in B(H)^* \). Then we can identify \( M_n(B(H))^* \) with the projective tensor product \( \ell_2^n(H)^* \otimes \ell_2^n(H) \). Consider an element \( x \) (resp. \( y \)) in the unit ball of \( \ell_2^n(H) \) (resp. \( \ell_2^n(H)^* \)). Let \( \xi \) be the element of \( M_n(B(H))^* \) defined by \( \xi = y \otimes x \) or equivalently, \( \xi = (\xi_{ij}) \) with \( \xi_{ij} = y_j \otimes x_i \). For such a \( \xi \) we have

\[
\left( \int \left\| \sum \varphi_i(t)\psi_j(s)\xi_{ij} \right\|_{A^*}^2 d\mu(t)d\mu(s) \right)^{1/2} = \left( \int \left\| \sum \varphi_i(t)x_i \right\|_{A^*}^2 d\mu(t) \right) \left\| \sum \psi_j(s)y_j \right\|_{(\ell_2^n(H)^*)^k}^{1/2}
= \|x\| \|y\| \leq 1.
\]

Since the unit ball of \( M_n(B(H))^* \) is the closed convex hull of elements of this form, we obtain (3.2).

(ii) Consider a subset \( \alpha \subset \{1, ..., k\} \). We denote by \( \alpha^c \) its complement. Recall that for elements \( (a_J)_{J \in [n]^k} \) in \( A \) the norm \( \| (a_J) \|_\alpha \) defined in (0.3) can be viewed as the norm of a matrix acting from \( \ell_2([n]^\alpha, H) \) into \( \ell_2([n]^{\alpha^c}, H) \). Therefore we deduce from (3.2) that for any \( (\xi_J)_{J \in [n]^k} \) in \( A^* \) we have

\[
(3.4) \quad \int \left\| \sum_{J \in [n]^k} \varepsilon_J \xi_J \right\|_{A^*}^k d\mu^k \leq \| (\xi_J) \|_{\alpha}^*.
\]

Observe that by duality (2.1) has the following consequence.

If \( \left\| \sum_{J \in [n]^k} \delta_J^* \otimes \xi_J \right\|_{(E_n^k \otimes A)^*} \leq 1 \) then there is a decomposition

\[
\xi_J = \sum_{\alpha \subset \{1, ..., k\}} \xi_j^\alpha \quad \text{with} \quad \sum_{\alpha} \| (\xi_J) \|_{\alpha}^* \leq 1.
\]

Therefore (3.3) follows from (3.4) and the triangle inequality.

We now reformulate the main result of [LPP] in our framework.
Theorem 3.3. For any \( \{\xi_j \mid j \leq n\} \) in \( A^* \) we have

\[
(3.5) \quad \left\| \sum \varepsilon_j \xi_j \right\|_{L_1(\mu; A^*)} \leq \left\| \sum \delta_j^* \otimes \xi_j \right\|_{(E_n \otimes A)^*} \leq 2 \left\| \sum \varepsilon_j \xi_j \right\|_{L_1(\mu; A^*)}.
\]

**Proof.** The left side is (3.3) above for \( k = 1 \). By our earlier analysis of \( E_n \otimes A \), the right side is clearly equivalent to the following fact.

Assume \( \left\| \sum \varepsilon_j \xi_j \right\|_{L_1(\mu; A^*)} < 1 \). Then there is a decomposition \( \xi_j = x_j + y_j \) in \( A^* \) such that

\[
\forall (a_j) \in A \quad \left| \sum \langle x_j, a_j \rangle \right| \leq \left\| \left( \sum a_j^* a_j \right)^{1/2} \right\|,
\]

and

\[
\left| \sum \langle y_j, a_j \rangle \right| \leq \left\| \left( \sum a_j a_j^* \right)^{1/2} \right\|.
\]

This is precisely what is proved in section II of [LPP], except that the sequence \( (\varepsilon_j) \) on \( \Omega \) is replaced by the sequence \( (e^{i3^j t}) \) on the one dimensional torus. By a routine averaging argument, one can then obtain the preceding fact as stated above with \( (\varepsilon_j) \). (Note actually that the approach of [LPP] can be developed directly for the functions \( (\varepsilon_j) \), this is explicit in [P2].)

We now relate certain series on \( \mathbb{Z}^n \) (formed by iterating the expressions appearing in Theorem 3.3) with the corresponding series on the free group \( F_n = \mathbb{Z} \ast \cdots \ast \mathbb{Z} \). In other words, our aim is to compare for these series the free group \( F_n \) with \( n \) generators with its commutative counterpart \( \mathbb{Z}^n \).

**Lemma 3.4.** For any \( \{\xi_J \mid J \in [n]^k\} \) in \( A^* \) we have (the summation being over all \( J \) in \( [n]^k \))

\[
2^{-k} \left\| \sum \varepsilon_J \xi_J \right\|_{L_1(\mu^k; A^*)} \leq \left\| \sum \lambda_*(g_J \otimes \xi_J) \right\|_{(C_k \otimes A)^*} \leq 2^k \left\| \sum \varepsilon_J \xi_J \right\|_{L_1(\mu^k; A^*)}.
\]

**Proof.** By the preceding three statements, we know that this holds for \( k = 1 \). We now argue by induction. Assume Lemma 3.4 proved for an integer \( k \geq 1 \), and let us prove it for \( k + 1 \).

Consider elements \( \{\xi_{J_j} \mid J \in [n]^k, j \in [n]\} \) in \( A^* \). We have

\[
\sum_{J' \in [n]^{k+1}} \lambda_*(g_{J'}) \otimes \xi_{J'} = \sum_{J \in [n]^k} \lambda_*(g_J) \otimes \left( \sum_{j \leq n} \lambda_*(g_j) \otimes \xi_{J_j} \right).
\]
By the induction hypothesis, we have

\[(3.6) \quad \left\| \sum_{J'} \lambda_*(g_{J'}) \otimes \xi_{J'} \right\|_{(C^{k+1}_N \otimes A)^*} \leq 2^k \int_{\Omega^k} \left\| \sum_{J} \varepsilon_J(t) \eta_J \right\|_{(C \otimes A)^*} \, d\mu^k(t) \]

where \( \eta_J = \sum_{j} \lambda_*(g_j) \otimes \xi_{Jj} \).

Now for each fixed \( t \) in \( \Omega^k \), we have by (3.5) and Lemma 3.1

\[ \left\| \sum_{J} \varepsilon_J(t) \eta_J \right\|_{(C \otimes A)^*} \leq 2 \int \left\| \sum_{J} \varepsilon_J(t) \left( \sum_{j} \varepsilon_j(s) \xi_{Jj} \right) \right\|_{A^*} \, d\mu(s). \]

Integrating over \( t \in \Omega^k \) this yields

\[(3.7) \quad \int \left\| \sum_{J} \varepsilon_J(t) \eta_J \right\|_{(C \otimes A)^*} \, d\mu^k(t) \leq 2 \int \left\| \sum_{J' \in [n]^{k+1}} \varepsilon_{J'} \xi_{J'} \right\|_{A^*} \, d\mu^{(k+1)}, \]

hence (3.6) and (3.7) yield the induction step for \( k + 1 \). This concludes the proof for the right side inequality in Lemma 3.4. The proof of the other inequality is entirely similar.

We now come to the main result of this section.

**Theorem 3.5.** For any \( \{ \xi_J \mid J \in [n]^k \} \) in \( A^* \), we have

\[ \left\| \sum \varepsilon_J \xi_J \right\|_{L_1(\mu^k, A^*)} \leq \sum_{J} \left\| \delta_J^* \otimes \xi_J \right\|_{(E^k_n \otimes A)^*} \leq 2^{2k} \left\| \sum \varepsilon_J \xi_J \right\|_{L_1(\mu^k, A^*)}. \]

**Proof.** With \( v \) and \( w \) as in Proposition 1.1, we have \( \|w^{\otimes k}\|_{cb} \leq 2^k \), hence by (0.5)

\[ \|w^{\otimes k} \otimes I_A\|_{E^k_n \otimes A \rightarrow C^k_n \otimes A} \leq 2^k. \]

Moreover, we have \( w^{\otimes k}(\delta_J) = \lambda(g_J) \) hence \( (w^{\otimes k} \otimes I_A)^*(\lambda_*(g_J) \otimes \xi_J) = \delta_J^* \otimes \xi_J \). This yields

\[ \left\| \sum \delta_J^* \otimes \xi_J \right\|_{(E^k_n \otimes A)^*} \leq 2^k \left\| \sum \lambda_*(g_J) \otimes \xi_J \right\|_{(C^k_n \otimes A)^*}. \]

Combined with Lemma 3.4 this gives the right side in Theorem 3.5. The left side has already been proved in Lemma 3.2.

**Remark.** A slight modification of our proof yields Theorem 3.5 with the constant \( 2^{2k-1} \) instead of \( 2^{2k} \).
Remark. Let $k$ be a fixed integer. Consider the mapping

$$Q_k: C(\Omega^k) \to E_n^k$$

defined by

$$\forall f \in C(\Omega^k) \quad Q_k(f) = \sum_{J \in [n]^k} \hat{f}(J) \delta_J,$$

where $\hat{f}$ is the Fourier transform of $f$, i.e. $\hat{f}(J) = \int f(t) \bar{\xi}_J(t) d\mu^k(t)$. Let $N_k = \text{Ker}(Q_k)$. Dualizing (3.3) we find that $\|Q_k\|_{cb} \leq 1$. Hence, considering $Q_k$ modulo its kernel and equipping $C(\Omega^k)/N_k$ with its quotient operator space structure (in the sense of [BP,ER]), we find a map

$$U_k: C(\Omega^k)/N_k \to E_n^k$$

with $\|U_k\|_{cb} \leq 1$.

Then Theorem 3.5 admits the following dual reformulation: $U_k: C(\Omega^k)/N_k \to E_n^k$ is a complete isomorphism and $\|U_k^{-1}\|_{cb} \leq 2^{2k}$. In other words, the space $C(\Omega^k)/N_k$ is, for each $k$, completely isomorphic (uniformly with respect to $n$) to $E_n^k$.

Assume now that $A$ is a von Neumann algebra and let $A_*$ be its predual. We define for any family $(x_J)_{J \in [n]^k}$ in $A_*$ the norm which is dual to the norm $\| \cdot \|_\alpha$ defined in (0.3). We set

$$\|(x_J)\|_\alpha^* = \sup \left\{ \left| \sum_{J \in [n]^k} \langle a_J, x_J \rangle \right| : a_J \in A, \quad \|a_J\|_\alpha \leq 1 \right\}.$$  

Then we define

$$[(x_J)]_{(k)}^* = \inf_{\alpha \in \{0,1\}^k} \sum_{\alpha} \|x_J^\alpha\|_{\alpha}^*$$

where the infimum runs over all $x_J^\alpha$ in $A_*$ such that $x_J = \sum_{\alpha \in \{0,1\}^k} x_J^\alpha$.

Assume that $A = (A_*)^*$ is a von Neumann subalgebra of $B(H)$ and let $q: N(H) \to A_*$ be the quotient mapping which is the preadjoint of the embedding $A \hookrightarrow B(H)$. We can also write

$$\|(x_J)\|_\alpha^* = \inf \left\{ \sum |\lambda_m| \right\}$$

26
where the infimum runs over all the possibilities to write \((x_J)\) as a series
\[
x_J = \sum_m \lambda_m h^m_{\pi,\alpha}(J) \otimes k^m_{\pi,\alpha^c}(J)
\]
where \((h^m_i)_{i \in [n]^\alpha}\) and \((k^m_j)_{j \in [n]^\alpha^c}\) are elements of \(H\) such that \(\sum_i \|h^m_i\|^2 \leq 1\) and \(\sum_j \|k^m_j\|^2 \leq 1\) for each \(m\).

The identity of (3.8) and (3.10) is clear since the dual norms are the same by (0.3). Similarly it is clear that the dual space to \((A_*)^{n^k}\) equipped with the norm \([ \cdot ]^{*}_{(k)}\) can be identified with \((A)^{n^k}\) equipped with the norm \([ \cdot ]_{(k)}\). By Proposition 2.1, this means that \((A_*)^{n^k}\) equipped with the norm \([ \cdot ]^{*}_{(k)}\) can be viewed as a predual (isometrically) of \(E_n^k \otimes A\).

Hence, we can now rewrite Theorem 3.5 a bit more explicitly. For all \((x_J)\) in \(A_*\), we have (as announced in the beginning of this section)

\[
(2^{2^k})^{-1}[(x_J)]^{*}_{(k)} \leq \left\| \sum \varepsilon_J x_J \right\|_{L_1(\Omega^k, A_*)} \leq [(x_J)]^{*}_{(k)}.
\]

In particular, we can state for emphasis.

**Theorem 3.6.** Let \(A \subset B(H)\) be a von Neumann subalgebra with predual \(A_*\) and let \(q: H \hat{\otimes} H \to A_*\) be the corresponding quotient mapping. Consider \(\{x_J \mid J \in [n]^k\}\) in \(A_*\) such that
\[
\left\| \sum \varepsilon_J x_J \right\|_{L_1(\Omega^k, A_*)} < 1.
\]

Then \((x_J)\) admits a decomposition as
\[
x_J = \sum_{\alpha \in \{0, 1\}^k} x^\alpha_J
\]
with
\[
x^\alpha_J = q \left( \sum_n \lambda^\alpha_n h^m_{\pi,\alpha}(J) \otimes k^m_{\pi,\alpha^c}(J) \right)
\]
where for each \(\alpha\), \(\{h^m_i \mid i \in [n]^\alpha\}\) and \(\{k^m_j \mid j \in [n]^\alpha^c\}\) are elements of \(H\) such that \(\sum_i \|h^m_i\|^2 \leq 1\) and \(\sum_j \|k^m_j\|^2 \leq 1\) and where \(\lambda^\alpha_n\) are scalars such that
\[
\sum_{\alpha} \sum_m |\lambda^\alpha_m| < 2^{2^k}.
\]
Conversely, if $(x_J)$ admits such a decomposition, we must have
\[ \| \sum x_J \|_{L_1(\Omega^k, A_*)} < 2^{2k}. \]

**Proof.** The proof is nothing but (3.9), (3.10) and (3.11) spelt out explicitly.

**Remark.** The preceding theorem proves one of the conjectures formulated in [P2] in the case $A = B(H), A_* = H \hat{\otimes} H$. 
§4. Complements.

The following result shows that in Proposition 1.3, the algebra \((C^*_\lambda(F_n))_{n=1}^\infty\) cannot be substituted by any sequence of nuclear algebras.

**Theorem 4.1.** Let \(A\) be either a nuclear \(C^*\)-algebra or an injective von Neumann algebra, and let \(I_{E_n} = vw\) be a factorization of \(I_{E_n}\) through \(A\). Then

\[
\|v\|_{cb}\|w\|_{cb} \geq \frac{1}{2}(1 + \sqrt{n}).
\]

For the proof we need the following.

**Lemma 4.2.** Consider the subspace \(S_n\) of \(M_n \oplus M_n\) given by

\[
S_n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ \end{pmatrix} \oplus \begin{pmatrix} y_1 \\ \vdots \\ y_n \\ \end{pmatrix} \mid x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{C} \right\}
\]

and define \(R : S_n \to S_n\) by

\[
R(x + y) = y^t \oplus x^t, \quad x \oplus y \in S_n.
\]

Then

(a) \(\frac{1}{2}(I_{S_n} + R)\) is a projection of \(S_n\) onto \(E_n\) and

\[
\left\| \frac{1}{2}(I_{S_n} + R) \right\|_{cb} = \frac{1}{2}(1 + \sqrt{n}).
\]

(b) For any projection \(Q\) of \(S_n\) onto \(E_n\) (resp. \(M_n \oplus M_n\) onto \(E_n\)) one has

\[
\|Q\|_{cb} \geq \frac{1}{2}(1 + \sqrt{n}).
\]

**Proof.** a) Obviously \(R^2 = I_{S_n}\) and \(E_n = \{a \in S_n \mid Ra = a\}\). Hence \(\frac{1}{2}(I_{S_n} + R)\) is a projection of \(S_n\) onto \(E_n\). Let \(A\) be a \(C^*\)-algebra. Then

\[
S_n \otimes A = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \\ \end{pmatrix} \oplus \begin{pmatrix} b_1 \\ \vdots \\ b_n \\ \end{pmatrix} \mid a_1, \ldots, a_n, b_1, \ldots, b_n \in A \right\}
\]
and
\[
(R \otimes I_A) \left( \begin{pmatrix} a_1 & \cdots & \cdots & b_n \\ \vdots & \ddots & \cdots & \vdots \\ \cdots & \cdots & \ddots & \cdots \\ a_n & \cdots & \cdots & b_n \end{pmatrix} \right) = \begin{pmatrix} b_1 & \cdots & \cdots & a_n \\ \vdots & \ddots & \cdots & \vdots \\ \cdots & \cdots & \ddots & \cdots \\ b_n & \cdots & \cdots & a_n \end{pmatrix}.
\]

Since
\[
\max \left( \left\| \sum b_i^*b_i \right\|^{1/2}, \left\| \sum a_i^*a_i \right\|^{1/2} \right) \leq \sqrt{n} \max \{\|a_1\|, \ldots, \|a_n\|, \|b_1\|, \ldots, \|b_n\|\}
\]

it follows that \(\|R \otimes 1_A\| \leq \sqrt{n}\). Hence \(\|R\|_{cb} \leq \sqrt{n}\) and thus
\[
\left\| \frac{1}{2}(I_{S_n} + R) \right\|_{cb} \leq \frac{1}{2}(1 + \sqrt{n}).
\]

To prove the converse inequality it suffices to consider \(n \geq 2\). Let \(A\) be the Cuntz algebra \(O_n\) (cf. [C]), which is generated by \(n\) isometries \(s_1, \ldots, s_n \in B(H)\) satisfying
\[
\begin{align*}
(4.1) & \quad s_i^*s_j = \delta_{ij}I \\
(4.2) & \quad \sum_{i=1}^n s_i s_i^* = 1.
\end{align*}
\]

By (4.2) the element
\[
z = \begin{pmatrix} s_1^* \\ \vdots \\ s_n^* \end{pmatrix} \oplus \begin{pmatrix} s_1 & \cdots & s_n \end{pmatrix}
\]

in \(S_n \otimes A\) has norm \(\|z\| = 1\), while
\[
\left( \frac{1}{2}(I_{S_n} + R) \otimes I_A \right) (z) = \frac{1}{2} \begin{pmatrix} s_1 + s_1^* \\ \vdots \\ s_n + s_n^* \end{pmatrix} \oplus \begin{pmatrix} s_1 + s_1^* & \cdots & s_n + s_n^* \end{pmatrix}
\]

has norm
\[
\frac{1}{2} \left\| \sum_{i=1}^n (s_i + s_i^*)^2 \right\|^{1/2} = \frac{1}{2} \sup \left\{ \sum_{i=1}^n \|s_i^*s_i + s_i + s_i^* s_i^* + (s_i^*)^2\| \xi \in H, \|\xi\| = 1 \right\}^{1/2}
\]

\[
= \frac{1}{2} \sup \left\{ \left\| \sum_{i=1}^n s_i^*s_i + s_i + s_i^* s_i^* + (s_i^*)^2 \right\| \xi \in H, \|\xi\| = 1 \right\}^{1/2}
\]

30
By (4.1) and (4.2), $\sum_{i=1}^{n} s_i^* s_i = nI$ and $\sum_{i=1}^{n} s_i s_i^* = I$. Set

$$v = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} s_i^2.$$

By (4.1), $v^* v = I$ so $v$ is an isometry. By (4.1) the range of $v$ is orthogonal to the range of the isometry $s_1 s_2$: Indeed for $\xi, \eta \in H$

$$(v \xi, s_1 s_2 \eta) = \frac{1}{\sqrt{n}} \left( \sum_{i} (s_2 s_1^* s_i^2 \xi, \eta) \right)$$

$$= \frac{1}{\sqrt{n}} (s_2 s_1 \xi, \eta)$$

$$= 0.$$

Hence $v$ is a non-unitary isometry. Therefore the point spectrum of $v^*$ contains the open unit disk $D$ (cf. e.g. [KP], p.253). Hence also the “numerical range” of $v$

$$\left\{ (v \xi, \xi) \mid \|\xi\| = 1 \right\} = \left\{ (\xi, v^* \xi) \mid \|\xi\| = 1 \right\}$$

contains the open unit disk. In particular the number 1 is in the closure of this set. Therefore

$$\sup_{\|\xi\| = 1} \left( \left( \sum_{i} s_i^* s_i + s_i s_i^* + s_i^2 + (s_i^*)^2 \right) \xi, \xi \right)$$

$$= n + 1 + 2\sqrt{n} \sup_{\|\xi\| = 1} (\text{Re}(v \xi, \xi))$$

$$\geq n + 1 + 2\sqrt{n}$$

$$= (1 + \sqrt{n})^2.$$

Hence $\|\left( \frac{1}{2}(I_{S_n} + R) \otimes I_A \right)(z)\| \geq \frac{1}{2}(1 + \sqrt{n})\|z\|$, which proves (a).

(b) Let $Q$ be a projection from $S_n$ onto $E_n$. Set $\tilde{Q} = QR = RQR$. Then $\tilde{Q}$ is also a projection from $S_n$ to $E_n$. Let $i_m$ denote the identity on $M_m$ and $t_m$ the transposition of $M_m$. Then

$$\tilde{Q} \otimes i_m = (R \otimes t_m)(Q \otimes i_m)(R \otimes t_m).$$

Since $t_n \otimes t_m$ can be identified with transposition on $M_{nm}$, $\|t_n \otimes t_m\| = 1$. Hence by the definition of $R$,

$$\|R \otimes t_m\| \leq 1.$$
Therefore
\[ \| \hat{Q} \otimes i_m \| \leq \| Q \otimes i_m \|, \quad m \in \mathbb{N} \]
and so \( \| \hat{Q} \|_c \leq \| Q \|_c \), and therefore also
\[ \left\| \frac{1}{2}(Q + \hat{Q}) \right\|_c \leq \| Q \|_c. \]
But
\[ \frac{1}{2}(Q + \hat{Q}) = Q \left( \frac{1}{2}(I_{S_n} + R) \right) \]
and since \( Q \) is the identity on \( E_n \), which is the range of \( \frac{1}{2}(I_{S_n} + R) \), we have \( \frac{1}{2}(Q + \hat{Q}) = \frac{1}{2}(I_{S_n} + R) \). Thus
\[ \| Q \|_c \geq \left\| \frac{1}{2}(I_{S_n} + R) \right\| = \frac{1}{2}(1 + \sqrt{n}). \]
If \( \psi: M_n \oplus M_n \xrightarrow{\text{onto}} E_n \) is a projection of norm 1, then from the above
\[ \| \psi \|_c \geq \| \psi \|_{S_n} \|_c \geq \frac{1}{2}(1 + \sqrt{n}), \]
proving (b).

**Proof of Theorem 4.1.** Let \( I_{E_n} = vw \) be a factorization of \( I_{E_n} \) through an injective von Neumann algebra \( A \). By the injectivity of \( A \), \( w \) can be extended to a linear map \( \tilde{w}: M_n \oplus M_n \to A \) such that \( \| \tilde{w} \|_c \leq \| w \|_c \) (cf. [Pa] Theorem 7.2). Clearly \( Q = v\tilde{w} \) is a projection of \( M_n \oplus M_n \) onto \( E_n \). Hence by (b) in the preceding lemma
\[ \| v \|_c \| w \|_c \geq \| v \|_c \| \tilde{w} \|_c \geq \| Q \|_c \geq \frac{1}{2}(1 + \sqrt{n}). \]
This proves the announced result when \( A \) is an injective von Neumann algebra. If \( A \) is a nuclear \( C^* \)-algebra, and \( I_{E_n} = vw \) as above, we can extend \( v \) to a \( \sigma(A^{**}, A^*) \)-continuous linear map \( \tilde{v}: A^{**} \to E_n \) such that \( \| \tilde{v} \|_c = \| v \|_c \). Since \( A^{**} \) is an injective von Neumann algebra (cf. e.g. [CE]), we are now reduced to the preceding case.

**Remark 4.3.** The constant \( \frac{1}{2}(1 + \sqrt{n}) \) is best possible in Theorem 4.1 : Namely let \( A = M_n \oplus M_n \), let \( w: E_n \to M_n \oplus M_n \) be the inclusion map and define a projection \( v: M_n \oplus M_n \to E_n \) by
\[ v(x \oplus y) = \frac{1}{2}(I_{S_n} + R)(xp \oplus py), \quad x \oplus y \in M_n \oplus M_n \]
32
where $p = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix}$. Then clearly $vw = I_{E_n}$ and

$$\|v\|_{cb} = \|v\|_{cb} \|w\|_{cb} \leq \left\| \frac{1}{2} (I_{S_n} + S) \right\|_{cb} = \frac{1}{2} (1 + \sqrt{n}),$$

which indeed shows that Theorem 4.1 is sharp.

In connection with Lemma 4.2 (b), note that there is obviously a projection $P: M_n \oplus M_n \to E_n$ with (ordinary) norm $\|P\| \leq 1$ (simply take $P = v$ with $v$ as in Remark 4.3). However, we will show below that the projection constant of $E_n \otimes M_n$ in $(M_n \oplus M_n) \otimes M_n$ goes to infinity when $n \to \infty$. To see this it is clearer to place the discussion in a broader context.

Let $S \subset B(H)$ be a closed subspace. We define $\lambda(S)$ (resp. $\lambda_{cb}(S)$, $\lambda_n(S)$) to be the infimum of the constants $\lambda$ such that there is a projection $P: B(H) \to S$ satisfying $\|P\| \leq \lambda$ (resp. $\|P\|_{cb} \leq \lambda$, resp. $\|I_{M_n} \otimes P\|_{M_n(B(H)) \to M_n(S)} \leq \lambda$). Then by the extension theorem of c.b. maps (cf. [W,Pa]), these constants are invariants of the “operator space” structure of $S$. By this we mean that if $S_1 \subset B(K)$ is another operator space which is completely isometric to $S$ (resp. such that for some constant $\lambda$ there is an isomorphism $u: S \to S_1$ with $\|u\|_{cb} \|u^{-1}\|_{cb} \leq \lambda$) then $\lambda(S_1) = \lambda(S)$, $\lambda_{cb}(S_1) = \lambda_{cb}(S)$, $\lambda_n(S_1) = \lambda_n(S)$ (resp. $\frac{1}{n} \lambda(S) \leq \lambda(S_1) \leq n \lambda(S)$ and similarly for the other constants).

By a simple averaging argument we can prove

**Proposition 4.4.** Let $S \subset B(H)$ be a closed subspace. Consider $M_n(S) = M_n \otimes S \subset B(\ell^2_n(H))$. Then

(i) $\lambda_n(S) = \lambda(M_n(S))$.

(ii) If $S$ is $\sigma(B(H), B(H)_*)$-closed in $B(H)$ then

\begin{equation}
\lambda_{cb}(S) = \sup_{n \geq 1} \lambda_n(S).
\end{equation}

For any infinite dimensional Hilbert space $K$ we have

\begin{equation}
\lambda_{cb}(S) \leq \lambda(B(K) \otimes S).
\end{equation}
Moreover let $B(K)\overline{\otimes}S$ denote the weak-* closure of $B(K)\otimes S$ in $B(K \otimes H)$. Then,

$$\lambda_{cb}(S) = \lambda(B(K)\overline{\otimes}S).$$

**Proof.** (i) The inequality $\lambda(M_n \otimes S) \leq \lambda_n(S)$ is obvious, so we turn to the converse. Assume that there is a projection

$$P: M_n \otimes B(H) \to M_n \otimes S$$

with $\|P\| \leq \lambda$.

Let $U_n$ be the group of all $n \times n$ unitary matrices. Consider then the group $G = U_n \times U_n$ equipped with its normalized Haar measure $m$. We will use the representation

$$\pi: G \to B(M_n, M_n)$$

defined by

$$\pi(u, v)x = u xv^*.$$  

We can define an operator $\tilde{P}: M_n \otimes B(H) \to M_n \otimes B(H)$ by the following formula

(4.5) \[ \tilde{P} = \int (\pi(u, v) \otimes I_{B(H)}) P(\pi(u, v) \otimes I_{B(H)})^{-1} dm(u, v). \]

Note that $\pi(u, v)$ leaves $M_n \otimes S$ invariant so that the range of $\tilde{P}$ is included in $M_n \otimes S$ and $\tilde{P}$ restricted to $M_n \otimes S$ is the identity, hence $\tilde{P}$ is a projection from $M_n \otimes B(H)$ onto $M_n \otimes S$. Moreover, by Jensen’s inequality (notice that $\pi(u, v) \otimes I_{B(H)}$ is an isometry on $M_n \otimes B(H)$) we have

$$\|\tilde{P}\| \leq \|P\| \leq \lambda.$$

Furthermore, using the translation invariance of $m$ in (4.5) we find

(4.6) \[ \forall (u_0, v_0) \in G \quad \tilde{P}(\pi(u_0, v_0) \otimes I_{B(H)}) = (\pi(u_0, v_0) \otimes I_{B(H)}) \tilde{P}, \]

so that $\tilde{P}$ commutes with $\pi(u_0, v_0) \otimes I_{B(H)}$. By well known facts this implies that $\tilde{P}$ is of the form

$$\tilde{P} = I_{M_n} \otimes Q$$
for some operator \( Q \) which has to be a projection onto \( S \). Indeed, since \( M_n \) is spanned by \( \mathcal{U}_n \), the above formula (4.6) is equivalent to: For all \( a, b \) in \( M_n \) and for all \( x \) in \( M_n \otimes B(H) \),

\[
\tilde{(a \otimes 1)x(b \otimes 1)} = (a \otimes 1)(b \otimes 1).
\]

Let \((e_{ij})_{i,j=1,...,n}\) denote the matrix units in \( M_n \). Set \( x = e_{ij} \otimes y \), where \( y \) is in \( B(H) \) and \( i, j \) are in \( \{1,...,n\} \). Applying (4.7) to \( a = 1 - e_{ii} \) and \( b = 1 - e_{jj} \), one gets \((1 - e_{ii})(e_{ij} \otimes y)(1 - e_{jj}) = 0 \) i.e. \((e_{ij} \otimes y) = e_{ij} \otimes z\) for some \( z \) in \( B(H) \) depending on \( y, i \) and \( j \). However applying (4.7) again, this time with \( a = e_{ki} \) and \( b = e_{jl} \) it follows that \( z \) is independent of \( i \) and \( j \). Hence \( \tilde{I}_{M_n} \otimes Q \), for some operator \( Q \) (which has to be a projection onto \( S \)). Finally, we conclude

\[
\|I_{M_n} \otimes Q\| = \|	ilde{P}\| \leq \lambda
\]

hence \( \lambda_n(S) \leq \lambda(M_n \otimes S) \). This proves (i).

We now check (ii). Consider an arbitrary closed subspace \( S \subset B(H) \) and let \( \overline{S} \) be the \( \sigma(B(H), B(H)_*) \)-closure of \( S \). We claim that there is an operator \( Q: B(H) \to \overline{S} \) such that \( Q|_S = I_S \) and \( \|Q\|_{cb} \leq \sup_n \lambda_n(S) \).

Let \( \varepsilon_n > 0 \) be such that \( \varepsilon_n \to 0 \). For each \( n \) there is a projection \( P_n: B(H) \to S \) such that

\[
\|I_{M_n} \otimes P_n\|_{M_n(B(H)) \to M_n(S)} \leq (1 + \varepsilon_n)\lambda_n(S).
\]

Let \( \mathcal{U} \) be a non-trivial ultrafilter on \( \mathbb{N} \). For any bounded sequence \((\alpha_n)\) of real numbers (or for any relatively compact sequence in a topological space) we will denote simply by \( \lim_{\mathcal{U}} \alpha_n \) the limit of \( \alpha_n \) when \( n \to \infty \) along \( \mathcal{U} \). For any \( x \in B(H) \) let

\[
Q(x) = \lim_{\mathcal{U}} P_n(x)
\]

where the limit is in the \( \sigma(B(H), B(H)_*) \)-sense. Observe that \( \|Q\| \leq \lim_{\mathcal{U}} \|P_n\| \leq \sup_n \lambda_n(S) \).

More generally for any integer \( m \geq 1 \) we clearly have

\[
\forall y \in M_m \otimes B(H) \quad (I_{M_m} \otimes Q)(y) = \lim_{\mathcal{U}} (I_{M_m} \otimes P_n)(y)
\]

hence \( \|I_{M_m} \otimes Q\| \leq \lim_{\mathcal{U}} \|I_{M_m} \otimes P_n\| \) but when \( n \geq m \) we have obviously

\[
\|I_{M_m} \otimes P_n\| \leq \|I_{M_n} \otimes P_n\|
\]

35
hence by (4.8) we obtain
\[ \| I_m \otimes Q \| \leq \lim_{t \to t} (1 + \varepsilon_n) \lambda_n(S) \leq \sup_n \lambda_n(S), \]
so that \( \| Q \|_{cb} \leq \sup_n \lambda_n(S) \). Clearly \( Q(B(H)) \subset S \) and \( Q|_S = I_S \). This proves our claim and in the case \( S = S \) we obtain (4.3). (Note that \( \lambda_{cb}(S) \geq \sup_n \lambda_n(S) \) is trivial.) We now turn to (4.4). We may clearly assume \( K = \ell_2 \). Recall that there is obviously a completely contractive projection \( \pi_n : B(\ell_2) \to M_n \) (here \( M_n \) is considered as a subspace of \( B(\ell_2) \) in the usual way) hence
\[ \lambda_n(S) = \lambda(M_n \otimes S) \leq \| \pi_n \| \lambda(B(\ell_2) \otimes S) = \lambda(B(\ell_2) \otimes S) \]
which implies by (4.3)
\[ \lambda_{cb}(S) \leq \lambda(B(\ell_2) \otimes S). \]
This concludes the proof of (4.4).

To prove the last assertion, note that \( M_n(S) \) is clearly contractively complemented in \( B(\ell_2) \otimes S \) hence we have
\[ \lambda_{cb}(S) \leq \sup_{n \geq 1} \lambda_n(S) = \sup_{n \geq 1} \lambda(M_n(S)) \leq \lambda(B(\ell_2) \otimes S). \]
To prove the converse inequality, note that \( B(\ell_2) \otimes B(H) \) can be identified with the space of matrices \( a = (a_{ij})_{i,j \in \mathbb{N}} \) which are bounded on \( \ell_2(H) \), and \( B(\ell_2) \otimes S \) can be identified with the subspace formed by all matrices with entries in \( S \). Then if \( P \) is a bounded projection from \( B(H) \) onto \( S \), defining
\[ \tilde{(a)} = (P(a_{ij}))_{i,j \in \mathbb{N}} \]
we obtain a projection from \( B(\ell_2) \otimes B(H) \) to \( B(\ell_2) \otimes S \) with \( \| \| \leq \| P \|_{cb} \). To check this last estimate, observe that the norm of an element \( a = (a_{ij})_{i,j \in \mathbb{N}} \) in \( B(\ell_2) \otimes B(H) \) is the supremum over \( n \) of the norms in \( M_n(B(H)) \) of the matrices \( (a_{ij})_{i,j \leq n} \). This yields the last assertion. \( \blacksquare \)

**Corollary 4.5.** Let \( H, K \) be Hilbert spaces. Consider a completely isometric embedding \( E_n \to B(H) \). Then, if \( \dim K = \infty \), for any projection \( P \) from \( B(K) \otimes B(H) \) to \( B(K) \otimes E_n \) we have
\[ \| P \| \geq \frac{1}{2}(\sqrt{n} + 1). \]

36
A fortiori the same holds for any projection \( P \) from \( B(K \otimes H) \) onto \( B(K) \otimes E_n \).

**Proof.** By the preceding statement, this follows from Theorem 4.1.

**Corollary 4.6.** Let \( M \subset B(H) \) be a von Neumann subalgebra such that \( M \) is isomorphic (as a von Neumann algebra) to \( M_n(M) \) for some integer \( n \geq 2 \). Then if there is a bounded linear projection from \( B(H) \) onto \( M \), there is also a completely bounded one.

**Proof:** Note that if \( M \) is isomorphic to \( M_n(M) \), then obviously it is isomorphic to \( M_n(M_n(M)) = M_{n^2}(M) \), and similarly to \( M_{n^3}(M) \), and so on. Hence this follows clearly from the first two parts of Proposition 4.4 and the observation preceding Proposition 4.4.

In particular we have using [V1]

**Corollary 4.7.** Let \( M \subset B(H) \) be a von Neumann subalgebra. If \( M \) is isomorphic to the von Neumann algebra \( VN(F_n) \) (resp. \( VN(F_\infty) \)) associated to the free group with \( n > 1 \) generators (resp. countably many generators) then there is no bounded linear projection from \( B(H) \) onto \( M \).

**Proof:** First note that \( VN(F_n) \) trivially embeds into \( VN(F_\infty) \) as a subalgebra which is the range of a completely contractive projection. Therefore by Proposition 1.3 and Theorem 4.1 there is no completely bounded projection from \( B(H) \) onto \( M \) if \( M \) is isomorphic to \( VN(F_\infty) \). By [V1] \( M_n(VN(F_\infty)) \) is isomorphic to \( VN(F_\infty) \) for all \( n \). Hence Corollary 4.7 for \( VN(F_\infty) \) follows from the preceding corollary. To obtain the case of finitely many generators, recall the well known fact that \( F_\infty \) can be embedded in \( F_n \) for all \( n \geq 2 \). (If \( a, b \) are two of the generators of \( F_n \), then it is easy to check, that \( b, aba^{-1}, \ldots, a^nb a^{-n}, \ldots \) are free generators of a subgroup isomorphic to \( F_\infty \).) Therefore if \( M = VN(F_n) \) for \( n > 1 \), then \( VN(F_\infty) \) is isomorphic to a von Neumann subalgebra \( N \subset M \), and since \( M \) is a finite von Neumann algebra, \( N \) is the range of a conditional expectation, hence there is a bounded projection from \( M \) onto \( N \). Since there is no bounded projection from \( B(H) \) onto \( N \) by the first part of the proof, a fortiori there cannot exist a bounded projection from \( B(H) \) onto \( M \). 

37
For two operator spaces $E$ and $F$ of the same finite dimension $n$, one can define the complete version of the Banach-Mazur distance between $E$ and $F$ by

$$d_{cb}(E, F) = \inf \{ \|u\|_{cb}, \|u^{-1}\|_{cb}\},$$

where the infimum is taken over all invertible linear maps $u$ from $E$ to $F$. By Proposition 1.3 it follows that

$$d_{cb}(E_n, \operatorname{span}\{\lambda(g_i) \mid i = 1, \ldots, n\}) \leq 2$$

for all $n \in \mathbb{N}$. The next proposition shows that the same inequality holds if the unitary operators $\lambda(g_1), \ldots, \lambda(g_n)$ are replaced by a semicircular or circular system of operators in the sense of Voiculescu [V1].

**Proposition 4.8.** Let $n \in \mathbb{N}$ and let $x_1, \ldots, x_n$ be a semicircular or circular system of operators on a Hilbert space, then the map $u: E_n \to \operatorname{span}\{x_1, \ldots, x_n\}$ given by

$$u: \sum_{k=1}^{n} c_k \delta_k \mapsto \sum_{k=1}^{n} c_k x_k, \quad c_1 \in \mathbb{C}$$

satisfies $\|u\|_{cb}\|u^{-1}\|_{cb} \leq 2$.

**Proof:** Assume first that $x_1, \ldots, x_n$ is a semicircular system of selfadjoint operators in the sense of [V1]. By [V2], we can exchange $x_1, \ldots, x_n$ with the operators

$$x_k = \frac{1}{2} (s_k + s_k^*), \quad k = 1, \ldots, n$$

where $s_1, \ldots, s_n$ are the “creation operators” $\xi \mapsto e_i \otimes \xi$ on the full Fock space

$$\mathcal{H} = \mathbb{C} \otimes \left( \bigoplus_{n=1}^{\infty} H^\otimes n \right)$$

based on a Hilbert space $H$ with orthonormal basis $(e_1, \ldots, e_n)$. In particular $s_1, \ldots, s_n$ are $n$ isometries with orthogonal ranges, and therefore

$$\sum_{k=1}^{n} s_k s_k^* \leq 1.$$
Hence, as in the proof of Proposition 1.1, we get that for any \( n \)-tuple \( a_1, \ldots, a_n \) of elements in a C*-algebra \( A \),

\[
\left\| \sum_k x_k \otimes a_k \right\| \leq \frac{1}{2} \left( \left\| \sum_k s_k \otimes a_k \right\| + \left\| \sum_k s_k^* \otimes a_k \right\| \right)
\]

\[
\leq \frac{1}{2} \left( \left\| \sum_k s_k s_k^* \right\|^{1/2} \left\| \sum_k a_k^* a_k \right\|^{1/2} + \left\| \sum_k s_k s_k^* \right\|^{1/2} \left\| \sum_k a_k a_k^* \right\|^{1/2} \right)
\]

\[
\leq \max \left\{ \left\| \sum_k a_k a_k^* \right\|^{1/2}, \left\| \sum_k a_k a_k^* \right\|^{1/2} \right\}.
\]

Hence \( \|u\|_{cb} \leq 1 \). To prove that \( \|u^{-1}\|_{cb} \leq 2 \), notice that by [V1], [V2], the C*-algebra generated by \( x_1, \ldots, x_n \) and 1 has a trace

\[
\tau: C^*(x_1, \ldots, x_n, 1) \to \mathbb{C}
\]

(namely the vector-state given by a unit vector in the \( \mathbb{C} \)-part of the Fock space \( \mathcal{H} \)), with the properties:

\[
\tau(1) = 1, \quad \tau(x_k^2) = \frac{1}{4} \quad \text{and} \quad \tau(x_k x_\ell) = 0 \quad k \neq \ell.
\]

Let \( a_1, \ldots, a_n \) be \( n \) operators in a C*-algebra \( A \), and let \( S(A) \) denote the state space of \( A \). Then

\[
\left\| \sum_k x_k \otimes a_k \right\|^2 \geq \sup_{\omega \in S(A)} (\tau \otimes \omega) \left( \left( \sum_k x_k \otimes a_k \right)^* \left( \sum_\ell x_\ell \otimes a_\ell \right) \right)
\]

\[
= \frac{1}{4} \sup_{\omega \in S(A)} \sum_k a_k^* a_k
\]

\[
\geq \frac{1}{4} \left\| \sum_k a_k^* a_k \right\|.
\]

and similarly \( \left\| \sum_k x_k \otimes a_k \right\|^2 \geq \frac{1}{4} \left\| \sum_k a_k a_k^* \right\| \). Hence

\[
\left\| \sum_k x_k \otimes a_k \right\| \geq \frac{1}{2} \max \left\{ \left\| \sum_k a_k^* a_k \right\|^{1/2}, \left\| \sum_k a_k a_k^* \right\|^{1/2} \right\}
\]

39
proving \( \|u^{-1}\|_{cb} \leq 2 \).

Assume finally that \( y_1, \ldots, y_n \) is a circular system. Then

\[
y_k = \frac{1}{\sqrt{2}} (x_{2k-1} + ix_{2k}), \quad k = 1, \ldots, n,
\]

where \((x_1, \ldots, x_{2n})\) is a semicircular system of selfadjoint operators. Therefore the statement about circular systems in Proposition 4.8 follows from the one on semicircular systems by observing, that the map

\[
\sum_{k=1}^{n} c_k \delta_k \rightarrow \frac{1}{\sqrt{2}} \sum_{k=1}^{n} c_k (e_{2n-1} + e_{2k})
\]
defines a \( cb \)-isometry of \( E_n \) onto its range in \( E_{2n} \).

To conclude this paper we give a generalization of Proposition 1.1 to free products of discrete groups, or more generally free products of \( C^* \)-probability spaces in the sense of [V1] and [V2]. We refer to [V1] and [V2] for the terminology.

**Proposition 4.9.** Let \((A, \varphi)\) be a \( C^* \)-algebra equipped with a faithful state \( \varphi \). Let \((A_i)_{i \in I}\) be a free family of unital \( C^* \)-subalgebras of \( A \) in the sense of [V1] or [V2]. Consider elements \( x_i \in A_i \) such that for some \( \delta > 0 \)

\[
\forall -i \in I \quad \|x_i\| \leq 1, \quad \varphi(x_i) = 0 \quad \text{and} \quad - \min\{\varphi(x_i^* x_i), \varphi(x_i x_i^*)\} \geq \delta^2.
\]

Then, for all finitely supported families \((a_i)_{i \in I}\) in \( B(H) \) (\( H \) Hilbert) we have

\[
\delta \max\{\left\| \sum a_i^* a_i \right\|^{1/2}, \left\| \sum a_i a_i^* \right\|^{1/2}\} \leq \left\| \sum x_i \otimes a_i \right\| \leq 2 \max\{\left\| \sum a_i^* a_i \right\|^{1/2}, \left\| \sum a_i a_i^* \right\|^{1/2}\}.
\]

**Proof.** We may assume that \( I \) is finite. The lower bound in (4.9) is proved exactly as in the semicircular case. To prove the upper bound we will prove that \( A \) can be faithfully represented as a \( C^* \)-algebra of operators on a Hilbert space \( H \), such that \( x_i \) admits a decomposition \( x_i = u_i + v_i \) with \( u_i, v_i \) in \( B(H) \) and

\[
\left\| \sum u_i^* u_i \right\| \leq 1 \quad \text{and} \quad -\left\| \sum v_i v_i^* \right\| \leq 1.
\]

The upper bound in (4.9) then follows as in the semicircular case.
Following the notation of [V 2, pp. 558-559], we let \((H_i, \xi_i)\) be the space of the GNS-representation \(\pi_i = \pi_{\varphi|A_i}\). In particular \(\xi_i\) is a unit-vector in \(H_i\) and 

\[
\varphi(x) = (\pi_i(x)\xi_i, \xi_i) \text{ when } x \in A_i.
\]

Then \(A\) can be realized as the \(C^*\)-algebra of operators on the Hilbert space 

\[
(H, \xi) = \bigoplus_{i \in I} - (H_i, \xi_i)
\]

generated by \(\bigcup_{i \in I} \lambda_i \circ \pi_i(A_i)\), where \(\lambda_i : B(H_i) \to B(H)\) is the *-representation defined in [V2, sect. 1.2]. For simplicity of notation we will identify \(A_i\) with its range in \(B(H)\), i.e.

we set

\[
\lambda_i \circ \pi_i(x) = x \text{ when } x \in A_i.
\]

Let \(x \in A_i\). Corresponding to the decomposition 

\[
H_i = H_i^0 \oplus \mathbb{C}\xi_i,
\]

we can write \(\pi_i(x)\) as a \(2 \times 2\) matrix

\[
\pi_i(x) = \begin{pmatrix} b & \eta \\ \zeta^* & t \end{pmatrix}
\]

where \(b \in B(H_i^0), \eta, \zeta \in H_i^0\) and \(t \in \mathbb{C}\). (Here we identify \(\eta, \zeta\) with the corresponding linear maps from \(\mathbb{C}\) to \(H_i^0\), and we also identify \(\mathbb{C}\) with \(\mathbb{C}\xi_i\).) The action of \(x = \lambda_i \circ \pi_i(x)\) on \(\bigoplus_{i \in I}(H_i, \xi_i)\) can now be explicitly computed from [V 2, sect. 1.2]. One finds:

\[
(4.11) \quad x\xi = \eta \otimes \xi + t\xi,
\]

\[
(4.12) \quad x(h_1 \otimes \ldots \otimes h_n) = bh_1 \otimes \ldots \otimes h_n + (h_1, \zeta) h_2 \otimes \ldots \otimes h_n \text{ when } n \geq 1,
\]

\(h_k \in H_i^0, \ i = i_1 \neq i_2 \neq \ldots \neq i_n\),

\[
(4.13) \quad x(h_1 \otimes \ldots \otimes h_n) = \eta \otimes h_1 \otimes \ldots \otimes h_n + th_1 \otimes \ldots \otimes h_n, \text{ when } n \geq 1, \ h_k \in H_i^0, \ i \neq i_1 \neq i_2 \neq \ldots \neq i_n
\]

where \(h_2 \otimes \ldots \otimes h_n = \xi\) for \(n = 1\).

Let \(e_i \in B(H)\) be the orthogonal projection of \(H\) onto the subspace

\[
H_i = \bigoplus_{n=1}^{\infty} (\bigoplus (H_{i_1} \otimes \ldots \otimes H_{i_n}))
\]

41
where the second direct sum contains all $n$-tuples $(i_1, ..., i_n)$ for which $i = i_1 \neq i_2 \neq ... \neq i_n$.

From (4.11), (4.12) and (4.13) one gets for all $x$ in $A_i$

\[
(4.14) \quad (1 - e_i)x(1 - e_i) = \varphi(x)(1 - e_i)
\]

where we have used that

\[ t = (\pi_i(x)\xi_i, \xi_i) = \varphi(x). \]

Let now $x_i \in A_i$, $\|x_i\| - \leq 1$, $\varphi(x_i) = 0$. Then by (4.14)

\[
(1 - e_i)x_i(1 - e_i) = 0
\]

Thus $x_i = u_i + v_i$, where

\[ u_i = x_ie_i \quad \text{and} \quad v_i = e_ix_i(1 - e_i). \]

Since $\|x_i\| - \leq 1$, and since $(e_i)_{i \in I}$ is a set of pairwise orthogonal projections,

\[
\sum_{i \in I} u_i^*u_i \leq \sum_{i \in I} e_i \leq 1
\]

and

\[
\sum_{i \in I} v_iv_i^* \leq \sum_{i \in I} e_i \leq 1.
\]

This completes the proof of proposition 4.9. \[ \blacksquare \]
REFERENCES

[BP] Blecher D. and Paulsen V., Tensor products of operator spaces. *J. Funct. Anal.* 99 (1991) 262-292.

[C] Cuntz J., Simple $C^*$-algebras generated by isometries. *Comm. Math. Phys.* 57 (1977), 173-185.

[CE] Choi M.D. and Effros E.G., Nuclear $C^*$-algebras and injectivity. The general case. *Indiana Univ. Math. J.* 26 (1977), 443-446.

[CS] Christensen E. and Sinclair A., On von Neumann algebras which are complemented subspaces of $B(H)$. Preprint.

[DCH] de Cannière J. and Haagerup U., Multipliers of the Fourier algebras of some simple Lie groups and their discrete subgroups. *Amer. J. Math.* 107 (1985), 455-500.

[ER] Effros E. and Ruan Z.J., A new approach to operators spaces. *Canadian Math. Bull.* 34 (1991) 329-337.

[E] Eymard P., L’algèbre de Fourier d’un groupe localement compact. *Bull. Soc. Math. France* 92 (1964), 181-236.

[G] Grothendieck A., Résumé de la théorie métrique des produits tensoriels topologiques. *Boll. Soc. Mat. São-Paulo* 8 (1956), 1-79.

[H1] Haagerup U., Solution of the similarity problem for cyclic representations of $C^*$-algebras. *Annals of Math.* 118 (1983), 215-240.

[H2] Haagerup U., An example of a non-nuclear $C^*$-algebra which has the metric approximation property. *Inventiones Mat.* 50 (1979), 279-293.

[KP] Kadison R.V. and Petersen G.K., Means and convex combinations of of unitary operators, *Math. Scand.* 57 (1985), 249-266.

[KR] Kadison R. and Ringrose J., Fundamentals of the theory of operator algebras, Vol. II *Academic Press*, New-York 1986.
[K] Kwapieni S., Decoupling inequalities for polynomial chaos. *Ann. Probab.* 15 (1987) 1062-1071.

[LPP] Lust-Piquard F. and Pisier G., Non commutative Khintchine and Paley inequalities. *Arkiv fUr Mat.* 29 (1991), 241-260.

[Pa] Paulsen V., Completely bounded maps on C*-algebras and invariant operator ranges. *Proc. Amer. Math. Soc.* 86 (1982) 91-96.

[P1] Pisier G., Grothendieck’s theorem for noncommutative C*-algebras with an appendix on Grothendieck’s constants. J. Funct. Anal. 29 (1978) 397-415.

[P2] Pisier G., Random series of trace class operators. *Proceedings IV° C.L.A. P.E.M. (Mexico Sept. 90).* To appear.

[P3] Pisier G., Factorization of linear operators and the Geometry of Banach spaces. CBMS (Regional conferences of the A.M.S.) no 60, (1986), Reprinted with corrections 1987.

[P4] Pisier G., Espace de Hilbert d’opérateurs et interpolation complexe. *C. R. Acad. Sci. Paris* Série I, 316 (1993) 47-52.

[V1] Voiculescu D., Circular and semicircular systems and free product factors. In *Operator algebras, unitary representations, Enveloping algebras, and invariant theory*, (edited by A.Connes, M.Duflo, A.Joseph, et R. Rentschler) *Colloque en l’honneur de J.Dixmier. Birkhauser, Progress in Mathematics* vol. 92 (1990) 45-60.

[V2] Voiculescu D., Symmetries of some reduced free product C*-algebras, in "Operator algebras and their connections with Topology and Ergodic Theory", *Springer Lecture Notes in Math.* 1132 (1985) 556-588.

[W] Wittstock G., Ein operatorwertiger Hahn-Banach satz, *J. Funct. Anal.* 40 (1981) 127-150.

Addresses

U.Haagerup:

Odense University

DK 5230 Odense, DENMARK
G. Pisier:
Texas A. and M. University
College Station, TX 77843, U. S. A.
and
Université Paris 6
Equipe d’Analyse, Boîte 186,
75255 Paris Cedex 05, France