Geodesic Iteration Number of g-contour of a Fuzzy Graph

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ABSTRACT

Purpose: In this paper, we intend to introduce the concept of g-contour nodes in fuzzy graphs and demonstrate an application of g-contour nodes in land line telecommunication Network.

Design/approach: The concept of g-contour nodes in Fuzzy graphs is explained and is supported by several examples. The geodesic iteration number of g-contour nodes in fuzzy graphs is obtained. An application of geodesic iteration number of g-contour nodes for determining the central persons in the land-line telecommunication network (LTN) is demonstrated, by means of which churning in the land-line telecommunication system can be reduced by providing a step-wise method for canvassing each and every customer in the network.

Findings: It is proved that every extreme node of G is a g-contour node of G but not conversely. The g-contour of g-self-centred fuzzy graphs and of fuzzy trees is found to be a geodesic cover. The geodesic iteration number of the g-contour of a fuzzy graph is obtained. The geodesic iteration number of complete fuzzy graphs and of fuzzy trees is proved to coincide with the geodesic iteration number of their g-contours.

1. Introduction

Zadeh in 1965 [1] developed a mathematical phenomenon for describing the uncertainties prevailing in day-to-day life situations by introducing the concept of fuzzy sets. The theory of fuzzy graphs was later on developed by Rosenfeld in the year 1975 [2] along with Yeh and Bang [3]. Rosenfeld also obtained the fuzzy analogue of several graph theoretic concepts like paths, cycles, trees and connectedness along with some of their properties [2] and the concept of fuzzy trees [4], automorphism of fuzzy graphs [5], fuzzy interval graphs [6], cycles and co-cycles of fuzzy graphs [7] etc have been established by several authors during the course of time. Fuzzy groups and the notion of a metric in fuzzy graphs were introduced by Bhattacharya [8]. The concept of strong arcs [9] was introduced by Bhutani and Rosenfeld in the year 2003. The definition of fuzzy end nodes and some of their properties were established by the same authors in [10]. The concept of geodesic distance was introduced by Bhutani and Rosenfeld in [11]. Further studies based on geodesic distance were carried out by Sameena and Sunitha in [12–14]. Using this geodesic distance, Suvarna and Sunitha
in [15] brought the concept of geodesic iteration number and geodesic number of a fuzzy graph into existence and studied some of the properties satisfied by them. The same concepts using μ-distance were introduced by Linda and Sunitha in [16]. Linda and Sunitha in [16] also introduced the concepts of g-periphery, g-eccentric fuzzy graph, g-boundary node and g-interior node in a fuzzy graph and studied some of their properties. Caceres et al. in 2005 [17] defined a node \( v \in V \) to be a contour node of a connected graph \( G: (V, E) \) if no neighbouring node of \( v \) has an eccentricity greater than eccentricity of \( v \). In this paper, this concept is extended to fuzzy graphs using geodesic distance and is named as g-contour nodes and the set of all g-contour nodes of the fuzzy graph \( G \) is called the g-contour of \( G \). It is proved that every extreme node of \( G \) is a g-contour node of \( G \) but not conversely. The geodesic iteration number of the g-contour of a fuzzy graph is discussed. It is also proved that the g-contour of g-self centred fuzzy graphs and of fuzzy trees is geodesic covers. Also, for certain categories of fuzzy graphs, their geodesic iteration number is found to coincide with that of their g-contours.

2. Preliminaries

In this section, a brief summary of some basic definitions in fuzzy graphs is given.

**Definition 2.1:** [18] A fuzzy graph is a triplet \( G: (V, \sigma, \mu) \) where \( \sigma \) is a fuzzy subset of a set \( V \) of nodes and \( \mu \) is a fuzzy relation on \( \sigma \). That is, \( \mu(u, v) \leq \sigma(u) \) and \( \sigma(v) \), \( \forall u, v \in V \). We assume that \( V \) is finite and non-empty, \( \mu \) is reflexive (i.e. \( \mu(x, x) = \sigma(x) \), \( \forall x \)) and symmetric (i.e. \( \mu(x, y) = \mu(y, x) \), \( \forall (x, y) \)). Also we denote the underlying crisp graph [19] by \( G^* : (\sigma^*, \mu^*) \) where \(\sigma^* = \{u \in V : \sigma(u) > 0\} \) and \(\mu^* = \{(u, v) \in V \times V : \mu(u, v) > 0\} \). Here we assume \(\sigma^* = V\).

**Definition 2.2:** [18] A fuzzy graph \( H: (V, \tau, \nu) \) is called a partial fuzzy subgraph of \( G: (V, \sigma, \mu) \) if \( \tau(u) \leq \sigma(u) \forall u \in \tau \) and \( \nu(u, v) \leq \mu(u, v) \forall (u, v) \in \nu \).

**Definition 2.3:** [18] The fuzzy graph \( H: (V, \tau, \nu) \) is called a fuzzy subgraph of \( G: (V, \sigma, \mu) \) if \( \tau(u) = \sigma(u) \forall u \in \tau \) and \( \nu(u, v) = \mu(u, v) \forall (u, v) \in \nu \) and if in addition \( \tau^* = \sigma^* \), then \( H \) is called a spanning fuzzy subgraph of \( G \).

**Definition 2.4:** [18] A fuzzy graph \( H: (P, \tau, \nu) \) is called a fuzzy subgraph of \( G: (V, \sigma, \mu) \) induced by \( P \) if \( P \subseteq V \), \( \tau(u) = \sigma(u) \forall u \) in \( P \) and \( \nu(u, v) = \mu(u, v) \forall u, v \) in \( P \).

**Definition 2.5:** [18] A fuzzy graph \( G: (V, \sigma, \mu) \) is a complete fuzzy graph if \( \mu(u, v) = \sigma(u) \) and \( \sigma(v) \forall u, v \in \sigma^* \).

**Definition 2.6:** [18] A sequence of distinct nodes \( u_0, u_1, \ldots, u_n \) such that \( \mu(u_{i-1}, u_i) > 0, i = 1, 2, 3, \ldots, n \) is called a path \( P_n \) of length \( n \).

**Definition 2.7:** [4] An arc of \( G: (V, \sigma, \mu) \) with least non-zero membership value is a weakest arc of \( G \). The degree of membership of a weakest arc in the path is defined as the strength of the path. The path becomes a cycle if \( u_0 = u_n, n \geq 3 \) and a cycle is called a fuzzy cycle if it contains more than one weakest arc.
**Definition 2.8:** [18] The strength of connectedness between two nodes $u$ and $v$ is the maximum of the strengths of all paths between $u$ and $v$ and is denoted by $\text{CONN}_G(u, v)$. The fuzzy graph $G: (V, \sigma, \mu)$ is said to be connected if $\text{CONN}_G(u, v) > 0$ for every $u, v$ in $\sigma^*$. 

**Definition 2.9:** [9] An arc $(u, v)$ of a fuzzy graph is called strong if its weight is at least as great as the strength of connectedness of its end nodes $u, v$ when the arc $(u, v)$ is deleted and a $u - v$ path $P$ is called a strong path if $P$ contains only strong arcs.

**Definition 2.10:** [2] A connected fuzzy graph $G: (V, \sigma, \mu)$ is called a fuzzy tree if it has a spanning fuzzy subgraph $F: (V, \sigma, v)$, which is a tree such that for all arcs $(u, v)$ not in $F$, $\text{CONN}_F(u, v) > \mu(u, v)$.

**Definition 2.11:** [10] Two nodes $u$ and $v$ in $G$ are neighbours (adjacent) if $\mu(u, v) > 0$ and $v$ is called a strong neighbour of $u$ if the arc $(u, v)$ is strong. The set of all neighbours of $u$ is denoted by $N(u)$ and the set of all strong neighbours of $u$ is denoted by $N_s(u)$. A node $v$ is called a fuzzy end node of $G$ if it has at most one strong neighbour in $G$.

**Definition 2.12:** [5] Two fuzzy graphs $G: (V, \sigma, \mu)$ and $G': (V', \sigma', \mu')$ are isomorphic, denoted by $G \cong G'$, if there is a bijective map $h: G \rightarrow G'$ such that $\sigma(u) = \sigma'(h(u))$ for all $u \in \sigma^*$ and $\mu(u, v) = \mu'(h(u), h(v))$ for all $u, v \in \sigma^*$.

**Definition 2.13:** [11] A strong path $P$ from $x$ to $y$ is called geodesic if there is no shorter strong path from $x$ to $y$ and the length of a $u - v$ geodesic is the geodesic distance from $u$ to $v$ denoted by $d_g(u, v)$.

**Definition 2.14:** The geodesic eccentricity ($g$-eccentricity) $e_g(u)$ of a node $u$ in a connected fuzzy graph $G: (V, \sigma, \mu)$ is given by $e_g(u) = \max_{v \in V} d_g(u, v)$. The maximum $g$-eccentricity among the nodes of $G$ is its $g$-diameter denoted by $d_g(G)$. The minimum $g$-eccentricity among the nodes of $G$ is its $g$-radius denoted by $r_g(G)$.

**Definition 2.15:** [11] A node with minimum $e_g$ is called a $g$-central node and a node with maximum $e_g$ is called a $g$-peripheral node. Let $C_g(G)$ be the set of $g$-central nodes of $G$. Then the fuzzy subgraph induced by $C_g(G)$, denoted by $(C_g(G))$, is called the $g$-center of $G$.

**Definition 2.16:** [12] A connected fuzzy graph $G: (V, \sigma, \mu)$ is $g$-self centred if $(C_g(G)) \cong G$.

The following definitions and results have been taken from [11,15].

**Definition 2.17:** [11] Let $S$ be a set of nodes of a connected fuzzy graph $G: (V, \sigma, \mu)$. The geodesic closure ($S$) of $S$ is the set of all nodes in $S$ together with the nodes that lie on geodesics between nodes of $S$. $S$ is said to be convex if $S$ contains all nodes of every $u - v$ geodesic for all $u, v$ in $S$, i.e. if $S = \text{CONN}_G(u, v)$ and any cover of $G$ with minimum number of nodes is called a geodesic basis for $G$. Order of a geodesic basis is the number of nodes in it.

**Definition 2.18:** [15] Let $S$ be a set of nodes of a connected fuzzy graph $G: (V, \sigma, \mu)$ and let $S^0, S^1, S^2, \ldots$ are geodesic closures, where $S^0 = S, S^1 = (S), S^2 = (S^1) = ((S)), \ldots$ The smallest
value of \( n \) so that \( S^n = S^{n+1} \) is called the geodesic iteration number, denoted by \( \text{gin}(S) \) and the resulting set is called the convex hull of \( S \) in \( G \), denoted by \([S]\). Also, \( \text{gin}(G) = \text{Max}_{S \subseteq V} (\text{gin}(S)) \).

**Proposition 2.19:** [11] A fuzzy tree has a unique geodesic basis consisting of its fuzzy end nodes.

### 3. The \( g \)-contour of a Fuzzy Graph

Caceres et al. in [17] defined a node \( v \in V \) to be a contour node of a connected graph \( G : (V, E) \) if no neighbouring node of \( v \) has an eccentricity greater than eccentricity of \( v \). The contour of \( G \), \( Ct(G) \), is the set formed by all contour nodes of \( G \). In this section, this concept is extended to fuzzy graphs using geodesic distance and is renamed as \( g \)-contour of a fuzzy graph, \( Ctg(G) \). It is proved that every extreme node of \( G \) is a \( g \)-contour node of \( G \) but not conversely. It is also proved that the \( g \)-contour of \( g \)-self centred fuzzy graphs and of fuzzy trees is geodesic covers.

**Definition 3.1:** A \( g \)-contour node of a fuzzy graph \( G : (V, \sigma, \mu) \) is a node \( v \) whose \( g \)-eccentricity is greater or equal to that of every strong neighbour of \( v \). The \( g \)-contour of \( G \) is the set of \( g \)-contour nodes in \( V \) \((G) \) and is denoted as \( Ctg(G) \). That is, \( Ctg(G) = \{ v \in V \mid e_g(u) \leq e_g(v), \forall u \in N_s(v) \} \).

Let \( S \) be a set of nodes in a fuzzy graph \( G \) and let \( \langle S \rangle \) be the fuzzy subgraph of \( G \) induced by \( S \). A vertex \( u \in S \) is said to be a \( g \)-contour node of \( S \) if \( e_g(u) \geq e_g(v) \) for every strong neighbour \( v \) of \( u \) in \( \langle S \rangle \). The set of all \( g \)-contour nodes of \( S \) is called the \( g \)-contour of \( S \) and is denoted by \( Ctg(S) \).

**Example 3.2:** Consider the fuzzy graph \( G \) given in Figure 1(a).

The \( g \)-eccentricities of each node is given in brackets. Here \( x \), \( y \) and \( z \) are the \( g \)-contour nodes of \( G \) and so \( Ctg(G) = \{ x, y, z \} \). If \( S = \{ x, y, z \} \), then the fuzzy subgraph of \( G \) induced by \( S \) \( \langle S \rangle \), is as given in Figure 1(b).

The nodes \( x \) and \( z \) in \( \langle S \rangle \) are the \( g \)-contour nodes having \( g \)-eccentricity 2 whereas \( y \) has \( g \)-eccentricity 1 and so is not a \( g \)-contour node. Therefore \( Ctg(S) = \{ x, z \} \).

![Figure 1](image-url)
Remark 3.3: If $G: (V, \sigma, \mu)$ is a fuzzy graph and $x \in V(G)$, then there is a $g$-eccentric node for $x$ that is a $g$-contour node. However note that not all the $g$-eccentric nodes of a node $x$ in $G$ is a $g$-contour node of $G$.

Example 3.4: Consider the fuzzy graph $G$ given in Figure 2, where the $g$-eccentricities of each node are given in brackets.

Here, $Ct_g(G) = \{p, r\}$. Note that the node $x$ has $g$-eccentricity 1 and $q$ is a $g$-eccentric node of $x$ which is not a $g$-contour node.

Proposition 3.5: Let $G: (V, \sigma, \mu)$ be a $g$-self centred fuzzy graph. Then the $g$-contour of $G$, $Ct_g(G)$, is a geodesic cover of $G$.

**Proof:** Since $G$ is a $g$-self centred fuzzy graph, all nodes of $G$ have the same $g$-eccentricity and so by Definition 3.1, each node is a $g$-contour node. Therefore $Ct_g(G) = V(G)$ and so $(Ct_g(G)) = (V(G)) = V(G)$. Thus the $g$-contour of $G$ is a geodesic cover of $G$. 

Remark 3.6: The converse of Proposition 3.5 is not true. That is, if $Ct_g(G)$ is a geodesic cover of a fuzzy graph $G$, then $G$ need not be $g$-self centred.

Example 3.7: Consider the fuzzy tree $G: (V, \sigma, \mu)$ given in Figure 3. The $g$-eccentricities of each node are given in brackets.

Note that the fuzzy tree $G$ is not $g$-self centred. Here $Ct_g(G) = \{u, v, w\}$ and since the node $x$ lies on every geodesic joining pair of nodes of $Ct_g(G)$, we get $(Ct_g(G)) = (u, v, w, x) = V(G)$ and so $Ct_g(G)$ is a geodesic cover of $G$.

Proposition 3.8: For a fuzzy tree $G: (V, \sigma, \mu)$, the $g$-eccentric nodes of each node $x \in V(G)$ are $g$-contour nodes of $G$.

**Figure 2.** A fuzzy graph on 4 nodes.

**Figure 3.** A fuzzy tree.
Proof: A node in a fuzzy tree $G$ is a $g$-eccentric node of $G$ if and only if it is a $g$-peripheral node of $G$ [13]. Therefore, all $g$-eccentric nodes of $x$ are $g$-peripheral nodes and hence by Definition 3.1, they are all $g$-contour nodes of $G$. ■

Proposition 3.9: Let $G: (V, \sigma, \mu)$ be a fuzzy tree. Then the $g$-contour of $G$, $Ct_g(G)$, is a geodesic cover of $G$.

Proof: It is enough to show that if $v \in V(G) - Ct_g(G)$, then $v \in (Ct_g(G))$. Since $v$ is not a $g$-contour node of $G$, there is some strong neighbour $u_1$ of $G$ such that $e_g(u_1) > e_g(v)$. If $u_1$ is not a $g$-contour node of $G$, then $u_1$ has a strong neighbour $u_2$ such that $e_g(u_2) > e_g(u_1)$. Continuing in this way, we get a sequence $u_1, u_2, \ldots$ of nodes such that $e_g(u_1) < e_g(u_2) < \ldots$. Since the fuzzy graph is finite, the sequence terminates with some node $u_t$ which has the property that its $g$-eccentricity is at least as large as that of its strong neighbours. Such a node must necessarily be a $g$-contour node. By Proposition 3.8, we know that $u_t$ has a $g$-eccentric node $u_t \ast$ that belongs to the $g$-contour of $G$. Since $e_g(u_t) > e_g(u_{t-1})$ and since $G$ being a fuzzy tree has exactly one geodesic path between two nodes of $G$, it follows that $u_t \ast$ is also a $g$-eccentric node for $u_{t-1}$ and thus the strong arc $(u_t, u_{t-1})$ followed by the $u_{t-1} - u_t \ast$ geodesic is a $u_t - u_t \ast$ geodesic that contains $u_{t-1}$. Continuing in this manner, we see that the path $u_t, u_{t-1}, \ldots, u_1, v$ followed by a $v - u_t \ast$ geodesic is a $u_t - u_t \ast$ geodesic that contains $v$. Since $u_t$ and $u_t \ast$ are both $g$-contour nodes, the result follows. ■

Definition 3.10: A node $v$ in a fuzzy graph $G: (V, \sigma, \mu)$ is called an extreme node of $G$ if the fuzzy subgraph induced by its neighbours is a complete fuzzy graph. Also, the number of extreme nodes in $G$ is denoted by $\text{ex}(G)$.

Proposition 3.11: An extreme node of a fuzzy graph $G: (V, \sigma, \mu)$ is a $g$-contour node of $G$.

Proof: Let $u \in V(G)$ be an extreme node of $G$. We now show that $u$ is a $g$-contour node of $G$. Let $v$ be a strong neighbour of $u$ and let $v \ast$ be a $g$-eccentric node for $v$. That is, $d_g(v, v\ast) = e_g(v)$. Now, $e_g(u) = d_g(u, u\ast) \geq d_g(u, v\ast)$. It is required to show that $e_g(u) \geq e_g(v)$.

Suppose on the contrary that $e_g(u) < e_g(v)$, i.e. $d_g(u, u\ast) < d_g(u, v\ast) < d_g(v, v\ast)$. Let $P$ be a $u - v \ast$ geodesic path not containing $v$. Then the node following $u$ on $P$, say $w$, is adjacent to $v$ since $u$ is an extreme node. However, then the arc $(v, w)$ followed by the $w - v \ast$ geodesic is a $v - v \ast$ geodesic such that $d_g(u, v\ast) = d_g(v, v\ast)$, which is a contradiction. So $e_g(u) \geq d_g(u, v\ast) \geq d_g(v, v\ast) = e_g(v)$ and therefore $u$ is a $g$-contour node of $G$. ■

Remark 3.12: Converse of Proposition 3.11 is not true as seen from Example 3.13.

Example 3.13: Consider the fuzzy graph $G$ given in Figure 4. The $g$-eccentricities of each node is given in brackets.

Here the node $x$ is a $g$-contour node of $G$. But it is not an extreme node since the fuzzy subgraph induced by its neighbours is not a complete fuzzy graph.

Proposition 3.14: In a complete fuzzy graph $G: (V, \sigma, \mu)$, each node of $G$ is a $g$-contour node as well as an extreme node.
**Proof:** By Definition 3.10, it is clear that each node in a complete fuzzy graph $G$ is an extreme node. Moreover, since all arcs in a complete fuzzy graph are strong [9], $g$-eccentricity of each node is 1. Therefore each node is a $g$-contour node.

Hence $Ctg(G) = V(G) = ex(G)$. ■

4. Geodesic iteration number of $g$-Contour of a fuzzy graph

In this section, the geodesic iteration number of the $g$-contour of a fuzzy graph is obtained. The geodesic iteration number of certain categories of fuzzy graphs is found to coincide with the geodesic iteration number of their $g$-contours.

**Proposition 4.1:** If $S$ is a convex set, then $[Ctg(S)] \subseteq S$.

**Proof:** Since $S$ is a convex set, $S^1 = (S) = S$ and so $[S] = S ... (1)$

Now by Definition 3.1, $Ctg(S) = \{u \in S: e_g(u) \geq e_g(v), \forall v \in N_l(u) \text{ and } v \in S\}$. Therefore, we get $Ctg(S) \subseteq S$. Hence, $[Ctg(S)] \subseteq [S] = S$ (from (1)) $\Rightarrow [Ctg(S)] \subseteq S$. ■

**Proposition 4.2:** Every convex set of nodes in a fuzzy graph $G: (V, \sigma, \mu)$ is the convex hull of the collection of its $g$-contour nodes. That is, if $S$ is a convex subset of $V(G)$, then $S = [Ctg(S)]$.

**Proof:** Suppose to the contrary that $S \neq [Ctg(S)]$. Since $S$ is a convex set, by Proposition 4.1, $[Ctg(S)] \subseteq S$. So, $S - [Ctg(S)] \neq \emptyset$. Let $u \in S - [Ctg(S)]$ be such that $e_g(u) \geq e_g(v)$ for all $v \in S - [Ctg(S)]$. Since $u \notin Ctg(S)$, there exists a strong neighbour $v$ of $u$ in $\langle S \rangle$ such that $e_g(v) > e_g(u)$ and by our choice of $u$, the vertex $v \in [Ctg(S)]$. Let $v^* \in S$ be a $g$-eccentric node for $v$ in $\langle S \rangle$. That is, $d_g(v, v^*) = e_g(v)$. Note that in this case, $e_g(v^*) \geq e_g(v) \geq e_g(u)$ and so $v^* \in [Ctg(S)]$.

Therefore $d_g(u, v^*) \leq e_g(u) < e_g(v) = d_g(v, v^*)$ in $\langle S \rangle$ and so $d_g(u, v^*) < d_g(v, v^*)$. Let $P$ be a $v^* - u$ geodesic path in $\langle S \rangle$. Then $P$ followed by the strong arc $(u, v)$ is a $v^* - v$ geodesic path whose length is $d_g(u, v^*) + 1 \leq d_g(v, v^*)$. So it is a geodesic path between $v^*$ and $v$ that contains $u$ where both $v, v^* \in [Ctg(S)]$. This contradicts the fact that $u \notin [Ctg(S)]$. ■

**Remark 4.3:** The above result shows that the convex hull of the $g$-contour set of a convex set of nodes in a fuzzy graph is the entire set. Thus the convex hull of the $g$-contour of a fuzzy graph coincides with the node set of the fuzzy graph. Hence $gin(Ctg(G))$ is the minimum $k$, $k \geq 0$, such that $(Ctg(G))^k = V(G)$.

**Example 4.4:** Consider the fuzzy graph given in Figure 5. The $g$-eccentricities of each node are given in brackets.
Figure 5. A fuzzy graph on 10 nodes.

Here, $Ct_g(G) = \{u, w, z\}$. Also $(Ct_g(G)) = V(G) - \{p\}$ and $(Ct_g(G))^2 = V(G)$. Therefore, $gin(Ct_g(G)) = 2$.

**Proposition 4.5:** Let $G: (V, \sigma, \mu)$ be a connected fuzzy graph. Then the geodesic iteration number of $g$-contour of $G$, $gin(Ct_g(G)) = 0$ if and only if $G$ is a $g$-self centred fuzzy graph.

**Proof:** Suppose $G$ is a $g$-self centred fuzzy graph. Then all nodes of $G$ have the same $g$-eccentricity and so by Definition 3.1, each node is a $g$-contour node.

Therefore $Ct_g(G) = V(G)$ and so $gin(Ct_g(G)) = 0$.

Conversely suppose that $gin(Ct_g(G)) = 0$. Then $Ct_g(G) = V(G)$ and so each node of $G$ is a $g$-contour node. Then by Definition 3.1, it follows that each node has the same $g$-eccentricity and so $G$ is $g$-self centred fuzzy graph. $\blacksquare$

**Corollary 4.6:** For a complete fuzzy graph $G: (V, \sigma, \mu)$, $gin(Ct_g(G)) = gin(G)$.

**Proof:** For a complete fuzzy graph $G$, each arc is the geodesic between its end nodes. So, for any $S \subseteq V(G)$, each pair of nodes in $S$ is connected by a geodesic.

i.e. no geodesic between a pair of nodes of $S$ contains another node.

So, $S^1 = (S) = S = S^0$.

This is true for any $S \subseteq V(G)$. Hence $gin(G) = 0$.

Also, since all arcs in a complete fuzzy graph are strong [9], $g$-eccentricity of each node is 1 and so $G$ is $g$-self centred. Therefore, by Proposition 4.5, $gin(Ct_g(G)) = 0$. Thus we get $gin(G) = gin(Ct_g(G))$. $\blacksquare$

**Remark 4.7:** For a fuzzy cycle $G: (V, \sigma, \mu)$, $gin(Ct_g(G)) \neq gin(G)$.

**Example 4.8:** Consider the fuzzy cycle $G$ given in Figure 6.

Since a fuzzy cycle is $g$-self centred, $gin(Ct_g(G)) = 0$. But consider the set $S = \{v_1, v_5\}$. Clearly $S^1 = (S) = \{v_1, v_2, v_5\} \neq S$. Also, $S^2 = (S^1) = S^1$ and so $gin(S) = 1$. From this, it follows that $gin(G) \neq 0$ and so $gin(G) \neq gin(Ct_g(G))$.

**Proposition 4.9:** For a fuzzy tree $G: (V, \sigma, \mu)$, $gin(G) = gin(Ct_g(G))$.

**Proof:** Since a fuzzy tree is not $g$-self centred, it follows from Proposition 4.5 that $gin(Ct_g(G)) \neq 0$ and so $Ct_g(G) \neq V(G)$. But by Proposition 3.9, $Ct_g(G)$ is a geodesic cover of $G$. Therefore $(Ct_g(G)) = V(G)$ and so by Remark 4.3, $gin(Ct_g(G)) = 1$. Also for a fuzzy tree $G,$
there is a unique strong path between any two fuzzy end nodes of $G$. Therefore, if $S \subseteq V(G)$ is the set of all fuzzy end nodes of $G$, then $S^1 = (S) = V(G)$ and so $S^2 = (S^1) = S^1$. Hence $\text{gin}(S) = 1$.

Also, for all other subsets $S$ of $V(G)$, $\text{gin}(S)$ is either 0 or 1 and so $\text{gin}(G) = \max\{0, 1\} = 1$. Therefore, $\text{gin}(G) = \text{gin}(Ct_g(G))$. 

5. Application of $g$-contour Nodes in Telecommunication Systems

Telecommunication is the exchange of information over any distance by a telecommunication path. Telecommunication occurs when the exchange of information between communication participants includes the use of technology. It is one of the most essential and unavoidable system in our daily life today. A revolution in wireless communication began in the first decade of the twentieth century with the pioneering developments in radio communications. This paved path for mobile phones that are wireless devices used to make and receive telephone calls over a radio link while moving around a wide geographic area. However, this technological device has gradually become a threat to our land line telecommunication networks. The number of land line subscribers continuously decreases due to upgrades in digital technology and the conveniences that come with switching to wireless (cellular) or Internet-based alternatives. Churn is a term used by companies to denote the loss of customers. Churn prediction is essential for businesses as it helps to detect customers who are likely to cancel a subscription or service. Churn of customers in telecommunication is a big problem for service providers and hence making a list of churning persons is an important task for service providers. Thus in order to reduce churning in the land line telecommunication system, the service providers give attractive offers and facilities for calls made within the system such as free calls on public holidays and reduced tariff on other days and so on. Also high speed net facilities such as broad band has also helped to a great extend in reducing the churning percentage among customers. A service provider gives more importance to the customer having more connected people. By canvassing these ‘connected people’ and brain washing them about the offers and facilities provided by the land line communication system, the service provider might succeed to a great extend in reducing the churn in the network and also in including new customers in the network. By repeating this process with the connections of these ‘connected people’, a wider access to the network is provided until everyone in the network has been canvassed and those about to churn are identified. This repeated process of canvassing customers can
be compared to the iterative method of taking geodesic closures (Definition 2.17) among nodes in a fuzzy graph.

5.1. Representation of Land Line Telecommunication System by a Fuzzy Graph Model

Samanta and Pal in [20] suggested that a social network can be represented by a fuzzy graph, thereby introducing the concept of fuzzy social network. A telecommunication network is a social network and the same authors in [21] discussed telecommunication network by using a fuzzy graph model. In this section, a method to represent the land line telecommunication network (LTN) by an undirected fuzzy graph is proposed by modifying the membership values of the links in the telecommunication network suggested by Samanta and Pal in [21]. The modifications include representation of telecommunication system by an undirected fuzzy graph $G: (V, \sigma, \mu)$ instead of the directed fuzzy graph $G\rightarrow (V, \sigma, \mu^{\rightarrow})$ where the membership value $\mu^{\rightarrow}(a, b)$ of the link between two customers $a$ and $b$ depends on $t$, the outgoing call time of $a$ to $b$, whereas in $G$, the membership value $\mu(a, b)$ depends on the variable $t'$, the sum of out-going call durations of $a$ to $b$ and of $b$ to $a$. And using this fuzzy graph representation $G: (V, \sigma, \mu)$ of LTN, a formula to determine the central persons in the network is obtained in Section 5.5. It is further established that the $g$-contour nodes of $G: (V, \sigma, \mu)$ are indeed the central persons in the network and that by focusing on these $g$-contour nodes, a service provider can canvass each person falling in the geodesic path between them. If all the customers get covered, he has completed his mission, or else he again canvasses those falling in the geodesic path between customers in its geodesic closure. Thus this repeated process is equivalent to the iterative method of taking geodesic closures and the geodesic iteration number of $g$-contour nodes indicates the minimum number of times a service provider has to set out canvassing each customer in LTN until all the customers get covered. Let $V_1 = \{c_1, c_2, \ldots, c_n\}$ be the set of all registered customers in the land line telecommunication network LTN and $V_2 = \{c_{n+1}, c_{n+2}, \ldots, c_m\}$ be the outside customers connected to the members of LTN. Let $V = V_1 \cup V_2$. The membership values of the customers are given by $\sigma: V \rightarrow [0, 1]$ and the membership values of the links between the customers are given by $\mu: V \times V \rightarrow [0, 1]$. Then, the telecommunication system is represented by a fuzzy graph $G: (V, \sigma, \mu)$.

5.2. Membership Values of Customers in LTN

Before defining the membership values of the customers in $V_1$, Samanta and Pal in [21] defined a co-related term ‘recognition number’ as follows. In every society, recognition of each person is measured by some members of the society. In LTN, the number of those members is taken as the recognition number. It is denoted by $N$. This recognition number ($N$) may not be equal for all social networks. But it should be pre-determined for a particular social network. For example, a social network may assume 3 as recognition number, i.e. for each new member of the social network, recommendation of 3 persons is necessary. Another social network may use another value of $N$. Here, let $V_1 = \{c_1, c_2, \ldots, c_n\}$ be the set of all registered customers in the telecommunication network LTN. Let $\varphi: V_1 \rightarrow [0, 1]$ be a mapping such that $\varphi(c_k) = e_k/N$, Where $N$ is fixed integer for the network and $e_k$ is the
number of distinct connected people (i.e. distinct phone numbers) of the customer $c_k \in V_1$ per unit interval of time in the network.

5.3. Membership Values of Customers of Other Network Connected to LTN

We are interested in those customers of other networks who are connected to the customers of LTN. Here, $V_2 = \{c_{n+1}, c_{n+2}, \ldots, c_m\}$ are the outside customers connected to the members of LTN. Before introducing the membership values of customers outside LTN, a co-related term ‘satisfied time of calling’ is defined as in [21]. If an outside customer of LTN calls certain amount of time to customers of LTN, then the outside customer is also valuable for LTN. We take $T$, a real positive number, as fixed amount of time. If the call duration of an outside customer to LTN is greater than $T$, the customer is taken as valuable. This fixed amount of time is called Satisfied time of calling. Let out-going call time of $c_k \in V_2$ to any customer of LTN be $t$ and $T$ be the satisfied time of calling of a customer to LTN. Let the mapping $\sigma: V \rightarrow [0, 1]$ be such that $\sigma(c_k) = \varphi(c_k)$ for all $c_k \in V_1$ and for all $c_k \in V_2$,

$$\sigma(c_k) = \begin{cases} \frac{t}{T} & \text{if } t < T \\ 1 & \text{if } t \geq T \end{cases}$$

5.4. Membership Value of a Link Between Two Customers

In the telecom world, the more people talk to each other, the closer they get. So strength between two customers depends on how much time they call to each other by phones per unit interval of time. Note that in [21], the telecommunication system is represented by a directed fuzzy graph $G^\rightarrow = (V, \sigma, \mu^\rightarrow)$ where the membership value of a link between customers is given by

$$\mu^\rightarrow(c_i, c_j) = \begin{cases} \frac{t}{T} (\sigma(c_i) \land \sigma(c_j)) & \text{if } t \in [0, T] \\ \sigma(c_i) \land \sigma(c_j) & \text{if } t > T \end{cases}.$$

Where $t$ is the duration of out-going calls per unit interval of time from $c_i$ to $c_j$ and $T$ is the satisfied time of calling.

The underlying fuzzy graph of $G^\rightarrow$ is denoted by $G = (V, \sigma, \mu)$, where $\mu: V \times V \rightarrow [0, 1]$ is such that $\mu(a, b) = \frac{\mu^\rightarrow(a, b) + \mu^\rightarrow(b, a)}{2}$ for all $a, b \in V$.

Now in this paper, we consider only undirected fuzzy graphs and accordingly, the membership values of links between customers is given by the mapping $\mu: V \times V \rightarrow [0, 1]$ such that

$$\mu(c_i, c_j) = \begin{cases} \frac{t'}{T} (\sigma(c_i) \land \sigma(c_j)) & \text{if } t' \in [0, T] \\ \sigma(c_i) \land \sigma(c_j) & \text{if } t' > T \end{cases}.$$

Where $t'$ is the sum of out-going call durations of $c_i$ to $c_j$ and of $c_j$ to $c_i$ per unit interval of time and $T$, satisfied time of calling, is fixed positive real number for a network.

5.5. Central Person in LTN

One of the most widely studied concepts in social network analysis is that of centrality in a network. Numerous measures, including degree centrality, closeness, eigenvector centrality, information centrality, etc. have been developed to define the concept of centrality in a network. Degree centrality measures the direct linkage of a customer to others. In all kinds of networks, central persons are more valuable than others as they can send messages to more people and also can collect more information. However, degree centrality
only measures the number of direct friends (or linkages). So it does not measure the number of connected people by a path. It may be noted that friend of a friend shares or collects information to or from the person. The amount of information shared between two customers in a telecommunication network is directly proportional to the calling time between them. So importance of a linkage in a telecom network will gradually decrease from a person to another person having low calling time. The δ-arcs in the fuzzy graph representation of LTN indicate linkages having less priority due to low calling time taken by the customers. In fuzzy social network (FSN), Sovan Samanta and Madhumangal Pal defined centrality [20] of a person as the weighted sum of fuzzy distances of connected persons along certain paths. The same authors in [21] defined centrality of a customer in fuzzy telecommunication network (FTN) as the weighted sum of fuzzy distances of directly connected customers and star customers connected by a certain path. In this paper, we define centrality of a customer as the weighted sum of fuzzy distances of connected persons along geodesic paths where the weights are g-eccentricities of the customers.

In LTN, if a customer \( c_k+1 \) is directly connected with the customer \( c_k \), then we say that \( c_k+1 \) is distance-1 friend of \( c_k \). Let the set of all distance-1 friends of \( c_k \) be denoted by \( d_1(c_k) \).

That is, \( d_1(c_k) = \{c_i \in V : c_i \text{ is a distance-1 friend of } c_k\} \).

Similarly, if there is a geodesic path (i.e. shortest strong path) between \( c_k \) and \( c_{k+1} \) containing \( k \) edges or links, then \( c_{k+1} \) is a distance-\( k \) friend of \( c_k \).

That is, \( d_k(c_k) = \{c_i \in V : c_i \text{ is a distance-}k \text{ friend of } c_k\} \).

Let \( p_i = c_1, c_2, \ldots, c_k = p_j \) be the customers on the geodesic path between \( p_i \) and \( p_j \). Samanta and Pal in [20] defined fuzzy distance \( D_f(p_i, p_j) \) between \( p_i \) and \( p_j \) along this path as \( D_f(p_i, p_j) = \sum_{l=1}^{k-1} \mu(c_l, c_{l+1}) \).

In a network, it may be observed that there are multiple geodesic paths between two customers. In LTN, those geodesic paths of same length whose fuzzy distance \( D_f \) is maximum are considered. If there are \( k \) links in this path of maximum fuzzy distance, then we denote this distance by \( D_f^k \). That is, \( D_f^k(p_i, p_j) \) represents the fuzzy distance between the customers \( p_i \) and \( p_j \) along a geodesic path containing exactly \( k \) links. We calculate the \( g \)-eccentricities of each node (customer) in the fuzzy graph representation of LTN. Suppose that a customer \( c_k \) has \( g \)-eccentricity \( p \), then it means that he/she has at most distance-\( p \) friends. Then the centrality \( C(c_k) \) of a customer \( c_k \) of LTN is evaluated as follows

\[
C(c_k) = p[\sum_{c_1 \in d_1(c_k)} D_f^1(c_k, c_1) + \sum_{c_2 \in d_2(c_k)} D_f^2(c_k, c_2) + \cdots + \sum_{c_p \in d_p(c_k)} D_f^p(c_k, c_p)].
\]

### 5.6. \( g \)-Contour Nodes as Central Persons in LTN

In this paper, we show that all \( g \)-contour nodes of the fuzzy graph representation of LTN are central persons of the LTN and by focusing on these central persons, a service provider can canvass each person falling in the geodesic path between them. That is, by focusing on the \( g \)-contour nodes \( S = C_g(G) \), the service provider manages to canvass all persons falling in the geodesic closure \( S^1 = (S) \). If \( S \neq V(G) \), he again canvasses those falling in the geodesic paths between members of \( (S) \), i.e. \( S^2 = (S^1) = ((S)) \). Then again, if \( S^2 \neq V(G) \), he canvasses those falling in \( S^3 = (S^2) = ((S^1)) \) and so on until \( S^k = V(G) \).

The minimum \( k \) for which \( S^k = V(G) \), which is equal to \( gin(C_g(G)) \), indicates the minimum number of times a service provider sets out for canvassing the customers in the network until all the customers in the network are covered. Thus in this paper, we show that the
repeated process of canvassing customers can be compared to the iterative method of taking geodesic closures among the $g$-contour nodes (central customers) in the fuzzy graph representation of LTN and the geodesic iteration number of $g$-contour nodes determines the minimum number of steps required for a service provider to set out and canvass each and every customer in LTN.

5.7. An Example of the Land Line Telecommunication System LTN

A sample of customers $V_1 = \{A_1, A_2, A_3, A_4, A_5\}$ in the network LTN and a set of customers of other networks connected to the customers of LTN $V_2 = \{B_1, B_2, B_3, B_4, B_5\}$ are taken. All the connections of customers are shown in Figure 7. The $g$-eccentricities of each customer are given in squares.

Here, fix recognition number as 4 and satisfied time of calling as 30 min per interval of time. To illustrate our model, we assume that the customers are connected with some other people as per data given in Table 1. Based on this data, the $\sigma$-values of each customer is evaluated and shown in column 3 of the same table.

From Table 1, we see that $A_1 \in V_1$ has only 1 connected customer. So $\sigma(A_1) = \varphi(A_1) = 1/4 = 0.25$.

Similarly, membership values of other customers in LTN are calculated. Now to assign the membership values of the customers outside LTN, we have to collect the duration of out-going calls made by them to the customers of LTN. The amount of time is shown in Table 2.

Here, $B_1$ has out-going call duration say 100 min to $A_1$ and 80 min to $A_2$. So, total outgoing call duration of $B_1$ to LTN is 180 min. This amount of time is bigger than the satisfied calling
Table 2. List of other network customers with out-going call duration.

| Name of Customer | Total Out-going call duration to LTN | Membership Values |
|------------------|--------------------------------------|-------------------|
| B₁               | 180                                  | 1                 |
| B₂               | 85                                   | 1                 |
| B₃               | 30                                   | 1                 |
| B₄               | 24                                   | 0.8               |
| B₅               | 12                                   | 0.4               |

Table 3. Link membership values of members within LTN.

| Link          | Call duration (t') | μ-values |
|---------------|--------------------|----------|
| (A₂, A₃)      | 100                | 0.75     |
| (A₃, A₄)      | 15                 | 0.5      |
| (A₄, A₅)      | 60                 | 1        |

Table 4. Link membership values between two customers, one from LTN and the other from another network.

| Link          | Call duration (t') | μ-values |
|---------------|--------------------|----------|
| (B₁, A₁)      | 145                | 0.25     |
| (B₁, A₂)      | 20                 | 0.5      |
| (B₁, A₄)      | 15                 | 0.5      |
| (B₂, A₂)      | 40                 | 0.75     |
| (B₂, A₁)      | 30                 | 1        |
| (B₂, A₄)      | 15                 | 0.5      |
| (B₃, A₁)      | 24                 | 0.8      |
| (B₃, A₅)      | 06                 | 0.2      |
| (B₄, A₁)      | 30                 | 0.8      |
| (B₄, A₅)      | 09                 | 0.24     |
| (B₅, A₂)      | 15                 | 0.2      |

Time. Hence $\sigma(B₁) = 1$. Again, the total out-going call duration of $B₄$ to LTN is 24 min. So $\sigma(B₄) = 24/30 = 0.8$. Similarly, membership values of the customers in $V₂$ are depicted in column 3 of Table 2. Now the membership values of the link between customers within LTN are shown in Table 3.

For example, $\mu(A₃, A₄) = 15/30 (1 \text{ and } 1) = 0.5$.

In Table 4, link membership values between two customers, one from LTN and the other from another network are listed below.

Using the definition of centrality given in Section 5.5, the centrality of each customer in Figure 7 can be evaluated as follows.

Centrality of A₁:

In Figure 7, $d₁(A₁) = \{B₁\}$, $d₂(A₁) = \{A₂, A₄\}$, $d₃(A₁) = \{A₅, A₃, B₂\}$ and $d₄(A₁) = \{B₄, B₅, B₃\}$. 


Now, \( \sum_{c_1 \in d_1(A_1)} D^1_f(A_1, c_1) = D^1_f(A_1, B_1) = \mu(A_1, B_1) = 0.25. \)

Also, \( \sum_{c_2 \in d_2(A_1)} D^2_f(A_1, c_2) = D^2_f(A_1, A_2) + D^2_f(A_1, A_4) \)

\[
= \{\mu(A_1, B_1) + \mu(B_1, A_2)\} \\
+ \{\mu(A_1, B_1) + \mu(B_1, A_4)\} \\
= \{0.25 + 0.5\} + \{0.25 + 0.5\} \\
= 1.5
\]

Again, \( \sum_{c_3 \in d_3(A_1)} D^3_f(A_1, c_3) = D^3_f(A_1, A_5) + D^3_f(A_1, A_3) + D^3_f(A_1, B_2) \)

\[
= \{\mu(A_1, B_1) + \mu(B_1, A_4) + \mu(A_4, A_5)\} \\
= \{\mu(A_1, B_1) + \mu(B_1, A_2) + \mu(A_2, A_3)\} \\
+ \{\mu(A_1, B_1) + \mu(B_1, A_2) + \mu(A_2, B_2)\} \\
= \{0.25 + 0.5 + 1\} + \{0.25 + 0.5 + 0.75\} \\
+ \{0.25 + 0.5 + 0.75\} \\
= 1.75 + 1.5 + 1.5 \\
= 4.75
\]

Similarly, \( \sum_{c_4 \in d_4(A_1)} D^4_f(A_1, c_4) = D^4_f(A_1, B_3) + D^4_f(A_1, B_4) + D^4_f(A_1, B_5) \)

\[
= \{\mu(A_1, B_1) + \mu(B_1, A_2) + \mu(A_2, A_3) + \mu(A_3, B_3)\} \\
+ \{\mu(A_1, B_1) + \mu(B_1, A_2) + \mu(A_2, A_3) + \mu(A_3, B_4)\} \\
+ \{\mu(A_1, B_1) + \mu(B_1, A_4) + \mu(A_4, A_5) + \mu(A_5, B_5)\} \\
= \{0.25 + 0.5 + 0.75 + 0.8\} + \{0.25 + 0.5 + 0.75 + 0.8\} \\
+ \{0.25 + 0.5 + 1 + 0.2\} = 6.55
\]

The centrality of the customer \( A_1 \) is thus calculated as follows.

\[
C(A_1) = 4 \left[ \sum_{c_1 \in d_1(A_1)} D^1_f(A_1, c_1) + \sum_{c_2 \in d_2(A_1)} D^2_f(A_1, c_2) \\
+ \sum_{c_3 \in d_3(A_1)} D^3_f(A_1, c_3) + \sum_{c_4 \in d_4(A_1)} D^4_f(A_1, c_4) \right] \\
= 4[0.25 + 1.5 + 4.75 + 6.55] \\
= 52.2
\]

Similarly, centralities of other customers in the network are as given in Table 5.

Note that centrality increases with increasing \( g \)-eccentricity. Thus all \( g \)-contour nodes of the fuzzy graph can be considered to be central persons of the telecommunication network.
Table 5. Centrality of customers.

| Customer | g-eccentricity | Centrality |
|----------|----------------|------------|
| A₁       | 4              | 52.2       |
| A₂       | 4              | 45.2       |
| A₃       | 3              | 29.4       |
| A₄       | 2              | 16.6       |
| A₅       | 3              | 42.9       |
| B₁       | 3              | 33.15      |
| B₂       | 3              | 35.4       |
| B₃       | 4              | 64.8       |
| B₄       | 4              | 64.8       |
| B₅       | 4              | 63.6       |

as they have wider access to the remaining customers of LTN. By focusing on these central customers of LTN, a service provider decides to canvass each person falling in the geodesic path between them.

In other words, in Figure 7, \( S = \{A₁, A₂, B₃, B₄, B₅\} = \text{Ctg}(G) \) is the set of central customers in the fuzzy graph \( G: (V, \sigma, \mu) \) given in Figure 7. Taking its closure, we get \((S) = V(G) - B₂\). That is, by focusing on the \( g \)-contour nodes \( A₁, A₂, B₃, B₄ \) and \( B₅ \), the service provider manages to canvass all except one person \( B₂ \) in the network. Since \((S) \neq V(G)\), he again canvasses those falling in the geodesic paths between members of \((S)\), i.e. \( S^2 = ((S)) \).

Here \( S^2 = ((S)) = V(G) \) and hence each and every customer in the selected network gets covered.

In this example, the minimum \( k \) for which \( S^k = V(G) \) is 2 (i.e. \( \text{gin} \text{Ctg}(G) = 2 \)) which indicates that a service provider has to set out twice in order to canvass each and every customer in the land line telecommunication network given in Figure 7.

6. Conclusion

In this paper, the concept of contour nodes in crisp graphs is extended to fuzzy graphs using geodesic distance and is named as \( g \)-contour nodes and the set of all \( g \)-contour nodes of the fuzzy graph \( G \) is called the \( g \)-contour of \( G \). It is proved that every extreme node of \( G \) is a \( g \)-contour node of \( G \) but not conversely. The geodesic iteration number of the \( g \)-contour of a fuzzy graph is obtained. It is also proved that the \( g \)-contour of \( g \)-self centred fuzzy graphs and of fuzzy trees is geodesic covers. Also, for certain categories of fuzzy graphs, their geodesic iteration number is found to coincide with that of their \( g \)-contours. An application of \( g \)-contour nodes in the land line telecommunication system is demonstrated.

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