DIRECTED TREES IN A STRING, REAL POLYNOMIALS WITH TRIPLE ROOTS, AND CHAIN MAILS

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Abstract. This paper starts with an observation that two infinite series of simplicial complexes, which a priori do not seem to have anything to do with each other, have the same homotopy type. One series consists of the complexes of directed forests on a double directed string, while the other one consists of Shapiro-Welker models for the spaces of hyperbolic polynomials with a triple root.

We explain this coincidence in the more general context by finding an explicit homotopy equivalence between complexes of directed forests on a double directed tree, and doubly disconnecting complexes of a tree.

1. Observation

When the explicit determination of the homotopy type of families of combinatorially defined simplicial complexes is performed, it sometimes happens that the answers coincide. In such a case two natural questions arise immediately. First, one would like to have an explicit map between these complexes, which would provide a homotopy equivalence. Second, one may ask for new terminology and a new theorem, which would put this particular result in a more general context. This paper is an example of an analysis of a situation like this.

1.1. Complexes of directed forests.

Let us start by describing the two families of simplicial complexes, for which the computations yield the same answer.

Definition 1.1. ([4, 9]). Let $G$ be an arbitrary directed graph. $\Delta(G)$ is the simplicial complex $\Delta(G)$ constructed as follows: the vertices of $\Delta(G)$ are given by the edges of $G$ and faces are all directed forests which are subgraphs of $G$.

The complexes of directed forests in a given directed graph were introduced in [4] following the suggestion of R. Stanley. It was proved in [4] that $\Delta(G)$ is shellable (hence homotopy equivalent to a wedge of spheres) when $G$ has a complete sink. Furthermore, the homotopy types of complexes $\Delta(G)$ were computed for several natural families of graphs $G$. The special case, which is of particular importance for this paper is when $G$ is a double directed string.

Date: January 31, 2022.

Keywords: topological combinatorics, complexes of trees, stratifications, partitions, order complexes.

Mathematics Subject Classification (2000): Primary 57Q05, Secondary 05C05, 58K15.

This research was supported by the Research Grant of the Swiss Natural Science Foundation.
Definition 1.2. ([4]). Let \( n \) be a nonnegative integer. A double directed string on \( n + 1 \) vertices is the directed graph, denoted \( L_n \), which is defined by \( V(L_n) = [n + 1] \), and \( E(L_n) = \{(i \to i + 1), (i + 1 \to i) \mid i \in [n]\} \).

The following proposition was also proved in [4].

Proposition 1.3.

\[
\Delta(L_n) \simeq \begin{cases} 
S^{2k-1}, & \text{if } n = 3k; \\
S^{2k}, & \text{if } n = 3k + 1; \\
a \text{ point}, & \text{if } n = 3k + 2.
\end{cases}
\]

1.2. Spaces of monic hyperbolic polynomials with multiple roots.

Let us now turn to the second family of simplicial complexes. The space of monic hyperbolic polynomials in one variable of degree \( n \), which we denote by \( \text{Hyp}^n \), is naturally stratified by fixing the multiplicities of roots (in our terminology a polynomial is called hyperbolic if all of its roots are real).

These strata are indexed by number partitions which refine \( n \), and for a partition \( \lambda = (\pi_1, \ldots, \pi_t) \) we let \( \text{Hyp}^n_\lambda \) denote the one-point compactification of the set of all polynomials of the form \((x-r_1)^{\pi_1} \cdots (x-r_t)^{\pi_t}\) (note that we do not require that \( r_1, \ldots, r_t \) are distinct numbers).

To fix the multiplicities of roots is the same as to fix a number partition \( \lambda \vdash n \). While one can think of a number partition \( \lambda = (\pi_1, \ldots, \pi_t) \) simply as a set of numbers, such that \( \pi_1 + \cdots + \pi_t = n \), we use the notation \([\pi_1, \ldots, \pi_t]\) to denote the ordered \( t \)-tuple of these numbers. In such a situation, \([\pi_1, \ldots, \pi_t]\) is called a composition of \( n \). By forgetting the order of the numbers in the composition we get a number partition which is called the type of this composition. Both for a partition and for a composition, we call the number of its parts the length, and denote it by \( l(\lambda) \), resp. \( l([\pi_1, \ldots, \pi_t]) \).

The set of all compositions of \( n \) is partially ordered by refinement. Namely, let \( x = [\alpha_1, \ldots, \alpha_{l(x)}] \) and \( y = [\beta_1, \ldots, \beta_{l(y)}] \) be two compositions of \( n \), we say that \( x \leq y \) if and only if \( \alpha_j = \beta_{j-1} + 1 + \cdots + \beta_j \), for \( 1 \leq j \leq l(x) \), and some \( 0 = i_0 < i_1 < \cdots < i_{l(x)} = l(y) \). Since \( \beta_i > 0 \), for \( i = 1, \ldots, l(y) \), the indices \( i_1, \ldots, i_{l(x)-1} \) are uniquely defined.

Given a number partition \( \lambda = (\pi_1, \ldots, \pi_t) \) of \( n \), we define \( D_\lambda \) to be the poset consisting of all compositions of \( n \) which are less or equal of some composition of \( n \) of type \( \lambda \). Thus, the number of maximal elements of \( D_\lambda \) is equal to the number of different ways to impose an order on the numbers \( \pi_1, \ldots, \pi_t \). Note, that \( D_\lambda \) has a minimal element, the composition consisting of just the number \( n \), and it is easy to see that \( D_\lambda \cup \{\hat{1}\} \) is a lattice, where \( \hat{1} \) is an artificially added maximal element.

Since the lower intervals of \( D_\lambda \) are Boolean algebras, and \( D_\lambda \) itself is a meet-semilattice, there exists a unique simplicial complex, which we denote by \( \delta_\lambda \), such that \( D_\lambda \) is the face poset of \( \delta_\lambda \), i.e., the elements of \( D_\lambda \) and the simplices of \( \delta_\lambda \) are in bijection, and the partial order relation on \( D_\lambda \) corresponds under this bijection to the inclusions of simplices of \( \delta_\lambda \).

As the following theorem, proved in [3], shows, the simplicial complex \( \delta_\lambda \) provides a combinatorial model for the stratum \( \text{Hyp}^n_\lambda \).
**Theorem 1.4.** ([5, Theorem 3.5(a)]). Let $\lambda$ be a number partition of $n$, then the one-point compactification of the strata indexed by $\lambda$, $\overline{\Hyp^n_\lambda}$, is homeomorphic to the double suspension of the simplicial complex $\delta_\lambda$.

Complete description of the homotopy type of the simplicial complexes $\delta_\lambda$ were obtained for several classes of number partitions $\lambda$, see [5, 8]. In particular the following is well-known.

**Proposition 1.5.** ([5, Corollary 3.10], [8, Proposition 3.4(a)]). For $\lambda = (k, 1^t)$, where $k \geq 2$, $t \geq 0$, we have

$$\delta_\lambda \simeq \begin{cases} S^{2m-1}, & \text{if } t = km, \text{ for some } m \in \mathbb{Z}; \\ S^{2m}, & \text{if } t = km + 1, \text{ for some } m \in \mathbb{Z}; \\ \text{point}, & \text{otherwise.} \end{cases}$$

By comparing Propositions 1.3 and 1.5, we get that

$$\delta_{(3, 1^t)} \simeq \Delta(L_t).$$

In the next section we will provide an explicit homotopy equivalence between $\delta_{(3, 1^t)}$ and $\Delta(L_t)$, while we shall state and prove a more general theorem in Section 3.

### 2. Explicit homotopy equivalence

#### 2.1. Further descriptions of the simplicial complexes $\delta_{(3, 1^t)}$ and $\Delta(L_t)$.

Let us note an alternative description of the simplicial complex $\delta_{(3, 1^t)}$. We take as simplices all subsets $\sigma$ of $\{1, \ldots, t + 2\}$, for which there exist $1 \leq i \leq t + 1$, such that $i, i + 1 \notin \sigma$; in other words, the maximal simplices are precisely all sets $\{1, \ldots, t + 2\} \setminus \{i, i + 1\}$, for $i = 1, \ldots, t + 1$. Please note, that we step a bit away from the usual conventions of the simplicial complexes, in that in our description it is not necessary that all 1-element subsets of $\{1, \ldots, t + 2\}$ are simplices. For example, $\delta_{(3)}$ is simply empty, while $\delta_{(3, 1)}$ has only 2 vertices.

Also $\Delta(L_t)$ has other descriptions, two of which we list next.

**Description 1 (via forbidden patterns).**

$\Delta(L_t)$ is the simplicial complex, with the set of vertices being the set $\{1, \ldots, 2t\}$, where we take as simplices all subsets $\sigma \subseteq \{1, \ldots, 2t\}$, such that $\{i, i + 1\} \not\subseteq \sigma$, for any $i = 1, \ldots, 2t - 1$. These simplicial complexes have appeared in various guises (for example, in [3] they were called sparse complexes).

With this description at hand it is obvious that the simplicial complex $\Delta(L_t)$ is isomorphic to $\Delta(L'_t)$, where $L'_t$ is the directed graph on $t + 2$ vertices, $V(L'_t) = \{1, \ldots, t + 2\}$, given by

$$E(L'_t) = \{(1 \rightarrow 2), (t + 2 \rightarrow t + 1)\} \cup \{(i \rightarrow i + 1), (i + 1 \rightarrow i) | i = 2, \ldots, t\}.$$
Definition 2.1. For an arbitrary graph \( \Gamma \) we can define the independence simplicial complex \( I(\Gamma) \) as follows. The set of vertices of \( I(\Gamma) \) is equal to the set of vertices of \( \Gamma \). The subset \( \sigma \subseteq I(\Gamma) \) is a simplex if and only if the vertices in \( \sigma \) can be colored with the same color.

Note that we use the usual requirement, that no two vertices of the same color are connected by an edge. In the graph theory literature, sets of vertices which can be colored with the same color are traditionally called independent sets, which is why we chose this name for the complexes \( I(\Gamma) \). The condition for being a simplex in the Definition 2.1 could be reformulated slightly differently, namely, if \( \sigma \) is a simplex of \( I(\Gamma) \), and \((x, y)\) is an edge of \( \Gamma \), then either \( x \notin \sigma \) or \( y \notin \sigma \).

Description 2 (via order complexes). Recall that to any partially ordered set (poset for short) one can associate a simplicial complex whose vertex set is the set of the elements of the poset, and whose set of simplices consists of all the completely ordered subsets of elements of the poset (these ordered subsets are also known as chains). This simplicial complex is called order complex of the poset. Connecting this definition to our situation, one can see that \( \Delta(L_t) \) is isomorphic to the order complex of the poset \( P_t \), given by:

- the set of the elements of \( P_t \) is \( \{1, \ldots, 2t\} \);
- the partial order on \( P_t \) is given by: \( x > y \) if and only if \( |x - y| \geq 2 \).

\[ \text{Figure 2. The poset } P_4. \]

Note that it is rather unusual for a graph \( G \) to be such that there exists a poset \( P \), such that \( \Delta(G) = \Delta(P) \). For example, a careful check should convince the reader that no such poset \( P \) exists for the graph on the Figure 3.

\[ \text{Figure 3.} \]

2.2. The explicit homotopy equivalence.

We are now in the position to define the map which shall give an explicit homotopy equivalence between \( \Delta(L_t) \) and \( \delta(3,1^t) \). Namely, let \( \phi \) be the simplicial map defined by

\[ \phi : \Delta(L_t) \rightarrow \delta(3,1^t) \]

\[ (x \rightarrow y) \mapsto x. \]
Theorem 2.2. The induced map of topological spaces \( \phi : \Delta(L_t) \to \delta(3,1^t) \) is a homotopy equivalence.

It is not difficult to show that \( \phi \) is well-defined and prove the Theorem 2.2. However, we shall not do that here, as it will follow from a more general Theorem 3.2 in the next section.

2.3. Chain mails.

To conclude this section, let us give yet another interpretation of the simplicial complexes \( \Delta(L_t) \) and \( \delta(3,1^t) \), which allows to view the map \( \phi \) in a somewhat different light. Consider \( t + 2 \) unit rings arranged as shown on Figure 4.

![Figure 4](image)

If we identify the rings with the vertices of \( \delta(3,1^t) \), then the complex \( \delta(3,1^t) \) is identified with the simplicial complex on the set of rings, in which the simplices are all collections of rings with at least two missing rings in a row.

Let us now associate a graph \( \Gamma \) to this string of rings as follows. We take vertex on each small (smaller than \( \pi \)) arc of the ring, which is cut out by another ring, except for the two small arcs on the ends. Then we connect each pair of vertices, where an edge can be drawn without intersecting the rings; that is we connect by an edge each pair of vertices which see each other without having to look through the rings. Clearly we obtain a string with \( 2t \) vertices, and it is immediate that \( I(\Gamma) = \Delta(L_t) \).

We can now see our map \( \phi : I(\Gamma) \to \delta(3,1^t) \) as the one which maps each vertex of \( \Gamma \) to the ring to which it belongs. This mental image is the main reason to introduce more general structures, which we, following the visual analogy, call chain mails.

In fact, there are various ways to generalize the situation shown on Figure 4. We choose one which will be sufficient for our purposes.

Definition 2.3. A chain mail is a collection of (not necessarily convex) polygons \( R_1, \ldots, R_p \) on the plane, such that

- for all \( i < j \), \( R_i \not\subseteq R_j \) and \( R_j \not\subseteq R_i \);
- for all \( i < j \), if \( R_i \cap R_j \) is nonempty, then \( \partial R_i \cap \partial R_j \) consists of two points and \( (R_i \setminus \partial R_i) \cap (R_j \setminus \partial R_j) \neq \emptyset \);
- each triple intersection \( R_i \cap R_j \cap R_k, i < j < k \), is empty.

Note that if \( \partial R_i \cap \partial R_j \neq \emptyset \), then the boundary of each of the PL curves \( \partial R_i \) and \( \partial R_j \) is divided into two parts: one part lies inside \( R_j \), resp. \( R_i \), while the other one lies outside.

To each chain mail one can associate a graph \( \Gamma \) as follows.

- Vertices of \( \Gamma \). For each pair \( i \neq j, i, j \in \{1, \ldots, p\} \), such that \( \partial R_i \cap \partial R_j \neq \emptyset \), we put one vertex \( v \) of the graph \( G \) on the part of \( \partial R_i \) which lies inside \( R_j \) if there exists \( k \in \{1, \ldots, p\} \), \( k \neq i, k \neq j \), such that one can connect \( v \) with \( \partial R_k \), with a PL curve, without intersecting any \( \partial R_h, h \in \{1, \ldots, p\} \); see Figure 6.
• Edges of $\Gamma$. Two vertices are connected by an edge, if they can be connected by a PL curve, which does not intersect $\partial R_h$, for all $h \in \{1, \ldots, p\}$, other than at its endpoints; again see Figure 6.

3. Homotopy equivalence in the context of doubly connected complexes

In this section we generalize the Theorem 2.2, and give a direct proof of this generalization.

3.1. $k$-fold disconnecting complexes of trees.

First we need some terminology. For any tree $T$ there is a natural way to augment it as follows: for each leaf of $T$ we add one extra vertex, which is a new leaf, and one extra edge, which connects the new leaf with the old one. We shall call the augmented tree $\hat{T}$. Alternatively, the class of all augmented trees could be described as the class of all trees where each path leading from a branch vertex to a leaf is at least of length 2.

One can also turn the tree into a directed tree, as one augments it, as follows: all the edges of $T$ are replaced by pairs of edges going in opposite directions, while the new edges are each replaced by just one edge, which is directed to point from the new leaf. We denote the thus obtained directed tree by $\tilde{T}$.

![Figure 5](image)

**Definition 3.1.** Given a tree $T$. For a positive integer $k$, let $D_k(T)$ be the simplicial complex defined as follows. The vertices of $D_k(T)$ are the vertices of $\hat{T}$, while $S \subseteq V(\hat{T})$ is a simplex if for any path $(x_1, \ldots, x_t)$ connecting two different leaves of $\hat{T}$ (note that $t \geq 4$ is necessitated), there exists $1 \leq i \leq t - k + 1$, such that $x_i, x_{i+1}, \ldots, x_{i+k-1} \notin S$.

We call $D_k(T)$ the $k$-fold disconnecting complex of $T$. When $k = 2$ we simply call $D_2(T)$ the doubly disconnecting complex of $T$. Clearly, when $T$ is a string (a tree without branching) with $t$ vertices, the complex $D_2(T)$ is isomorphic to $\delta(3,1^t)$.

It is natural to associate a chain mail to the graph $\hat{T}$, as is shown on Figure 6.
For each vertex of \( \hat{T} \) we draw a (not necessarily convex) polygon, so that these polygons form a chain mail, i.e., satisfy conditions of the Definition 2.3, and so that two polygons intersect if and only if the corresponding vertices of \( \hat{T} \) are connected by an edge. We rely on the visual clarity of the Figure 6 for the elucidation of how such a chain mail looks like.

However, we would like to mention one formal way to construct this chain mail. For each vertex of \( \hat{T} \) take the union of the closed halves of the edges adjacent to this vertex (in the standard metric associated to any graph, this is the closed ball of radius 1/2 centered at this vertex). Take the closed \( \epsilon \)-neighborhood of this union, where \( \epsilon \) is small, and depends on the specific imbedding of \( \hat{T} \) into the plane. The polygons can now be taken as sufficiently fine (again depending on the imbedding of \( \hat{T} \)) PL approximations of these \( \epsilon \)-neighborhoods. The graph \( \Gamma \) depicted on Figure 6 is the graph associated to this chain mail, as described in the subsection 2.3. Note that \( \mathcal{I}(\Gamma) = \Delta(\hat{T}) \).

3.2. The main theorem.

**Theorem 3.2.** Let \( T \) be a tree, then the simplicial complexes \( \mathcal{D}_2(\hat{T}) \) and \( \Delta(\hat{T}) \) are homotopy equivalent.

**Proof.** The map \( \phi : \Delta(\hat{T}) \rightarrow \mathcal{D}_2(\hat{T}) \) is defined analogously to the one in the previous section. One way to describe it is to set \( \phi(x \rightarrow y) = x \). The other way, in terms of chain mails, is to say that each vertex of the graph \( \Gamma \) associated to the chain mail is mapped to the boundary of the polygon, to which it belongs. Recall that, as mentioned above, \( \mathcal{I}(\Gamma) \) is isomorphic to \( \Delta(\hat{T}) \), and we are tacitly talking here about \( \phi : \mathcal{I}(\Gamma) \rightarrow \mathcal{D}_2(\hat{T}) \).

**Claim 1.** \( \phi \) is well-defined.

Let \( S \) be a simplex of \( \Delta(\hat{T}) \), \( S \neq \emptyset \). We need to show that \( \phi(S) \) has a “2-gap” in each path connecting two leaves. Without loss of generality, we can assume that \( S \) is the maximal simplex of \( \Delta(\hat{T}) \).

Assume this is not the case, and choose a path \( P = (x_1, \ldots, x_t) \) in \( \mathcal{D}_2(\hat{T}) \) which does not have a 2-gap, again \( t \geq 4 \). If \( (x_1 \rightarrow x_2) \notin S \), and \( (x_2 \rightarrow x_3) \notin S \), then \( x_1, x_2 \notin \phi(S) \), thus \( P \) has a 2-gap, which is a contradiction. In the same way, either
(x_t \to x_{t-1}) \in S \) or \((x_{t-1} \to x_{t-2}) \in S\). So \(S\) contains two edges are directed to each other, which, by maximality of \(S\), means that there must exist \(2 \leq i \leq t - 2\), such that
\((x_{i-1} \to x_i), (x_{i+2} \to x_{i+1}) \in S\), which in turn implies that \(x_i, x_{i+1} \notin S\). This is again a contradiction to the assumption that \(P\) has no 2-gap.

**Claim 2.** \(\phi\) gives a homotopy equivalence.

We shall show that for each (closed) simplex \(S \subseteq \mathcal{D}_2(\hat{T})\), \(\phi^{-1}(S)\) is contractible. In fact, by using induction on the number of vertices of \(\hat{T}\), we shall prove that it is a cone. The theorem then follows by Quillen’s Lemma, see 

Take any path \(P = (x_1, \ldots, x_t)\) in \(\hat{T}\), such that \(P \cap S \neq \emptyset\). Since \(P \cap S\) must contain a gap consisting of at least two elements, we can always find, (possibly after reversing the indexing of \(x_i\)’s), \(2 \leq i \leq t - 1\), such that \(x_{i-1} \in S\), and \(x_i, x_{i+1} \notin S\).

Obviously, \((x_{i-1} \to x_i) \in \phi^{-1}(S)\). If \(\phi^{-1}(S)\) is not a cone with apex at \((x_{i-1} \to x_i)\), then there must exist some simplex \(\sigma\) in \(\phi^{-1}(S)\), such that \(\sigma \cup \{(x_{i-1} \to x_i)\} \notin \phi^{-1}(S)\). This means that either \((x_i \to x_{i-1}) \in \sigma\), or \((y \to x_i) \in \sigma\), for some \(y \neq x_{i-1}\). The first option is ruled out by the fact that \(x_i \notin S\), furthermore, in the second option we have \(y \neq x_{i+1}\), since \(x_{i+1} \notin S\). Thus, we conclude that the vertex \(x_i\) has to have valency at least 3.

Let \(\hat{Q}\) be the induced subtree of \(\hat{T}\) consisting of all those vertices \(v \in \hat{T}\), which can be reached from \(x_i\) without passing through \(x_{i+1}\); this includes \(x_i\) itself, see the Figure 7.

![Figure 7](image-url)

Since the valency of \(x_i\) in \(\hat{T}\) is at least 3, its valency in \(\hat{Q}\) is at least 2, hence each leaf of \(\hat{Q}\) is also a leaf of \(\hat{T}\). Let furthermore \(\tilde{Q}\) denote the directed graph associated to \(\hat{Q}\); the rule of association is the same as before: the internal edges are replaced by pairs of oriented edges directed in opposite directions, while the edges connecting leaves to the internal vertices of the tree are replaced by single directed edges pointing away from the leaves. Let \(\hat{\phi}_Q : \Delta(\hat{Q}) \to \mathcal{D}_2(\hat{Q})\) be the restriction of the map \(\phi\).

By induction assumption, \(\hat{\phi}^{-1}_Q(S \cap \hat{Q})\) is a cone. Let us denote its apex by \((x \to y)\). If \(\phi^{-1}(S)\) is not a cone with apex \((x \to y)\), then there must exist a simplex \(\sigma \in \phi^{-1}(S)\), such that either \((y \to x) \in \sigma\), or \((z \to y) \in \sigma\). By the construction of \(\hat{Q}\), we conclude that the first option is impossible, and in the second option we must have \(z = x_{i+1}\). This yields a contradiction, as it implies that \(x_{i+1} \in S\). \(\square\)
4. Further questions

As mentioned above, when $T$ is a string with $t$ vertices, $t \geq 1$, $D_2(T)$ is isomorphic to $\delta_{(3,1)}$. More generally, one can see that in this case, if $t \geq k - 1$, $D_k(T)$ is isomorphic to $\delta_{(k+1,1^t+2-k)}$ (for the case $k = 1$, $t = 0$, we use the convention that $\hat{T}$ is the graph consisting of two vertices, which are connected by an edge). Unfortunately, we do not know how to generalize the Theorem 3.2, or even the Theorem 2.2, to this case.

The family of complexes, which, because of the connections to spaces of monic hyperbolic polynomials with multiple roots, appears to us to be of great interest, is $\{\delta_\lambda\}_{\lambda \vdash n}$. It is effortless to extend the Definition 3.1 to this case, so that one can talk about $\lambda$-fold disconnecting complexes of trees. The difficulty arises from the fact that we do not know of any natural replacement for the complexes $\Delta(G)$.

One way to view the Theorems 3.2 and 2.2 is to see them as expressing a kind of duality. The complexes $\delta_\lambda$ are given by listing their maximal simplices, while the complexes $\Delta(G)$ are given by listing minimal non-simplices, which turn out to have dimension 1. Such a description would have been impossible for the complexes $\delta_\lambda$ themselves, since in fact they contain complete skeletons in the dimensions up to roughly half of the number of vertices.

In light of the above, it seems interesting to ask: can one find simplicial complexes, having some nice combinatorial description in terms of minimal non-simplices of small dimension, which are homotopy equivalent to complexes $\delta_\lambda$?

The Theorems 3.2 and 2.2 can be thought of as the first step on the path of providing a complete and satisfactory answer to that question.

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