REGULARITY RESULTS FOR WEAK SOLUTIONS OF ELLIPTIC PDES BELOW THE NATURAL EXPONENT

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Abstract. We prove a priori estimates for strong solutions to the Dirichlet problem for a divergence form elliptic operator. We give $L^p$ estimates for the second derivative for $p < 2$. Our work generalizes results due to Miranda [24].

1. Introduction

In this paper we consider the regularity of solutions to the divergence form elliptic equation

(1.1) \[
\begin{aligned}
Lu &= -\text{div} A \nabla u = f \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\]

where $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded open set whose boundary $\partial \Omega$ is $C^1$, and $A = A(x) = (a_{ij}(x))$ is an $n \times n$ matrix of real-valued, measurable functions that satisfies the ellipticity condition

(1.2) \[
\lambda |\xi|^2 \leq \langle A \xi , \xi \rangle \leq \Lambda |\xi|^2, \quad 0 < \lambda < \Lambda, \quad \xi \in \mathbb{R}^n.
\]

We derive $L^p$ estimates, $p < 2$, for solutions of this equation when $A$ has discontinuous coefficients and $f \in L^p(\Omega)$.

This and related problems have a long history. If $A$ is continuous and $\partial \Omega$ is $C^{2,\alpha}$, then these results are classical: see Gilbarg and Trudinger [17]. Miranda [24] showed that if $n \geq 3$, $\partial \Omega$ is $C^2$, and $A \in W^{1,n}(\Omega)$, then any weak solution of $Lu = f$, $f \in L^q(\Omega)$, $q \geq 2$, is a strong solution and $\|D^2u\|_{L^2(\Omega)} \leq \|f\|_{L^q(\Omega)} + \|u\|_{L^1(\Omega)}$. This result is false when $n = 2$: for a counter-example, see Example 1.4 below.

A similar problem for non-divergence form elliptic operators was considered by Chiarenza and Franciosi [4]. They proved that if $n \geq 3$, $\Omega$ is bounded and $\partial \Omega$ is $C^2$, $\ldots$
then the non-divergence form equation $\text{tr}(AD^2u) = f$, with $f \in L^2(\Omega)$ and $A$ in a certain vanishing Morrey class (a generalization of $VMO$), has a unique solution $u$ satisfying $\|u\|_{W^{2,2}(\Omega)} \leq C\|f\|_{L^2(\Omega)}$. This was generalized by Chiarenza, Frasca and Longo \cite{5}, who showed that if $f \in L^p$, $1 < p < \infty$, then the same equation has a unique solution satisfying $\|u\|_{W^{2,p}(\Omega)} \leq C\|f\|_{L^p(\Omega)}$. These results in turn were further generalized by Vitanza \cite{28, 29, 30}.

Divergence form equations of the form $\text{div} A\nabla u = \text{div} F$ were considered by Di Fazio \cite{13} on bounded domains with $\partial \Omega \in C^{1,1}$ and Iwaniec and Sbordone \cite{22} on $\mathbb{R}^n$; they showed that if and $A \in VMO$, then there exists a unique weak solution that satisfies $\|\nabla u\|_{L^p(\Omega)} \leq C\|F\|_{L^p(\Omega)}$, $1 < p < \infty$. The results for bounded domains were improved by Auscher and Qafsaoui \cite{3}, who showed that it suffices to assume $\partial \Omega$ is $C^1$ and that $A$ does not need to be real symmetric. For a generalization to nonlinear equations, see \cite{16}.

Our main theorem is a generalization of the result of Miranda to $p < 2$ and $n \geq 2$.

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open set such that $\partial \Omega$ is $C^1$. Let $A$ be an $n \times n$ real-valued matrix that satisfies (1.2). If $A \in W^{1,n}(\Omega)$, then there exists $p_0 \in (1, 2)$ so that for all $p \in (p_0, 2)$ and $f \in L^p(\Omega)$, there exists a unique solution $u$ of (1.1) that satisfies

$$ \|D^2u\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}, $$

where $C$ is independent of both $u$ and $f$.

**Remark 1.2.** To compare Theorem 1.1 to the work of Di Fazio et al. described above, note that if $A \in W^{1,n}$ then $A \in VMO$: see, for instance, \cite{8}.

The lower bound $p_0$ in Theorem 1.1 is intrinsic to the proof, and comes from our use of the Hodge decomposition. Since the gradient estimates hold for all $p > 1$, it is an open question whether our results can be extended to this range.

When $n \geq 3$, an examination of the constants shows that we can take $p = 2$ in our proof. This lets us give a new proof of the result of Miranda mentioned above, one which improves on his hypotheses since we only assume that the boundary is $C^1$.

**Corollary 1.3.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded open set such that $\partial \Omega$ is $C^1$. Let $A$ be an $n \times n$ real-valued matrix that satisfies (1.2). If $A \in W^{1,n}(\Omega)$, then for all $f \in L^2(\Omega)$, there exists a unique solution $u$ of (1.1) that satisfies

$$ \|D^2u\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}, $$

where $C$ is independent of both $u$ and $f$.

When $n = 2$, Corollary 1.3 is false, as the next result shows.
Example 1.4. Let \( B = B_{1/2}(0) \subset \mathbb{R}^2 \) be the open ball of radius \( 1/2 \) centered at the origin. Then there exists a matrix \( A \in W^{1,2}(B) \) satisfying (1.2) and a solution to
\[
- \text{div}(A\nabla u) = 0
\]
such that \( u \in W^{2,p}(B) \) for all \( p < 2 \), but \( u \notin W^{2,2}(B) \).

In dimension \( n = 2 \) we can adapt the proof of Theorem 1.1 to prove two weaker results that require higher integrability assumptions on the matrix \( A \). In both cases we need to introduce Orlicz and Orlicz-Morrey spaces. For precise definitions, see Section 2 below.

Theorem 1.5. Let \( \Omega \subset \mathbb{R}^2 \) be a bounded open set such that \( \partial \Omega \) is \( C^1 \). Let \( A \) be a \( 2 \times 2 \) real-valued matrix that satisfies (1.2). Suppose further that for some \( \delta > 0 \),
\[
\| \nabla A \|_{L^2((\log L)^{1+\delta}(\Omega))} < \infty.
\]
If \( f \in L^2(\Omega) \) then there exists a unique solution \( u \) of (1.1) that satisfies
\[
\| D^2 u \|_{L^2(\Omega)} \leq C \| \nabla A \|_{L^2((\log L)^{1+\delta}(\Omega))} \| f \|_{L^2(\Omega)}.
\]

Our second result gives information in the end point case when \( \delta = 0 \). To state it we use Orlicz-Morrey spaces, as defined by Sawano, Sugano and Tanaka [26].

Theorem 1.6. Let \( \Omega \subset \mathbb{R}^2 \) be a bounded open set such that \( \partial \Omega \) is \( C^1 \). Let \( A \) be a \( 2 \times 2 \) real-valued matrix that satisfies (1.2). Suppose further that for some \( 1 < r < 2 \),
\[
\nabla A \in L^{\Psi,1/r'}(\Omega), \text{ where } \Psi(t) = t^2 \log(e+t).
\]
If \( f \in L^2(\Omega) \) then there exists a unique solution \( u \) of (1.1) that satisfies
\[
\| D^2 u \|_{L^2(\Omega)} \leq C(r, \Omega) \| \nabla A \|_{L^{\Psi,1/r'}(\Omega)} \| f \|_{L^2(\Omega)}.
\]

In two dimensions, (1.4) implies that \( \nabla A \) is continuous: see Cianchi [6, 7]. Similarly, if we assume that \( \nabla A \in L^{\Psi,1/r'}(\Omega) \), then by Hölder’s inequality we have that \( \nabla A \) is in the classical Morrey space \( L^{2,1/r-1/2}(\Omega) \), which also implies that \( A \) is Hölder continuous: see [17, p. 298]. Thus both of these results follow from classical Schauder estimates: see [17]. However, these results require greater boundary regularity and so our weaker assumption of a \( C^1 \) boundary gives an improvement of these results.

It remains open whether anything can be said when \( p = n = 2 \) and \( A \in W^{1,2}(\Omega) \). We conjecture that \( D^2 u \in L^2(\Omega) \), where \( L^2 \) denotes the grand Lebesgue space with norm
\[
\| f \|_{L^2(\Omega)} = \sup_{0 < \epsilon < 1} \left( \epsilon \int_{\Omega} |f(x)|^{2-\epsilon} \, dx \right)^{\frac{1}{2-\epsilon}}.
\]
These spaces were introduced in [20] and have proved useful in the study of endpoint estimates in PDEs [18, 19]. As evidence for this conjecture, we note that the solution \( u \) given in Example 1.4 is in \( L^2(B) \). A stronger conjecture, also satisfied by our example, is that \( D^2 u \) lies in the Orlicz space \( L^2((\log L)^{-1}(\Omega)) \). (This space is a proper
subset of $L^2$: see [18].) In both cases our proof techniques are not sharp enough to produce these estimates and a different approach will be required. Another possibility is that Theorem 1.6 can be improved, and that when $n = p = 2$ it is enough to assume that $\nabla A \in L^2(\log L)$. But again, a different approach would be required, as the weighted norm inequalities we use do not provide enough information.

The remainder of this paper is organized as follows. In Section 2 we state some preliminary definitions and weighted Fefferman-Phong type inequalities that are central to our proofs. These results depend on recent work on two-weight norm inequalities for the Riesz potential [12]. In Section 3 we prove Theorem 1.1. Our proof uses ideas from [4]. In Section 4 we consider the special case when $n = 2$: we construct Example 1.4 and sketch the proofs of Theorems 1.5 and 1.6. Throughout our notation will be standard or defined as needed. Given a vector matrix function, if say that it belongs to a scalar function space (e.g. $A \in W^{1,n}(\Omega)$) we mean that each component function is an element of the function space; to compute the norm we first take the $\ell^2$ norm of the components. Constants $C$, $C(n)$, etc. may change in value at each appearance.

2. Preliminary Results

In this section we give conditions on a weight $w$ for the two-weight Sobolev inequality

$$\|fw\|_{L^p(\Omega)} \leq C\|\nabla f\|_{L^p(\Omega)}$$

to hold. Such inequalities are sometimes referred to as Fefferman-Phong inequalities: see [15]. Given the classical pointwise inequality

$$|f(x)| \leq C(n)I_1(|\nabla f|)(x), \quad f \in C^\infty_0,$$

it suffices to prove two weight estimates for the Riesz potential of order one:

$$I_1f(x) = \Delta^{-\frac{1}{2}}f(x) = c \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-1}} \, dy.$$

Sufficient conditions for such inequalities were proved by Pérez [25], but we will apply a sharper condition from [12, Theorem 3.6] that gives better information about the dependence of constants. To state these results, we need to make some definitions; for additional information on Orlicz spaces and two-weight inequalities, see [11, 12]. A convex, strictly increasing function $\Phi : [0, \infty] \to [0, \infty]$ is said to be a Young function if $\Phi(0) = 0$ and $\Phi(\infty) = \infty$. Given a Young function there exists another Young function, $\bar{\Phi}$, called the associate, such that $\Phi^{-1}(t)\bar{\Phi}^{-1}(t) \simeq t$. For our purposes there are two particularly important examples of Young functions that we will use. First, if $\Phi(t) = t^r$, $r > 1$, then $\bar{\Phi}(t) = t^{r'}$. If $\Phi(t) = t^r \log(e+t)^a$, then $\bar{\Phi}(t) \simeq t^{r'} \log(e+t)^{-\frac{a}{r-r'}}$. 
Given \(1 < p < q < \infty\) and a Young function \(\Phi\), define

\[
\alpha_{p,q,\Phi} = \left( \int_1^{\infty} \frac{\Phi(t)^{q/p}}{t} dt \right)^{1/q}.
\]

Our conditions on weights are defined using a normalized Orlicz norm: given Young function \(\Phi\) and a cube \(Q\), let

\[
\|f\|_{\Phi,Q} = \inf \left\{ \lambda > 0 : \int_Q \Phi\left( \frac{|f(x)|}{\lambda} \right) dx \leq 1 \right\}.
\]

Given a pair of weights \((u, v)\) (i.e., non-negative, locally integrable functions) define

\[
[u, v]_{A^{1}_{p,q,\Phi}} = \sup_Q \left| Q \right|^{1/p-1/q} \left( \int_Q |u| dx \right)^{1/q} \|v^{-1/p}\|_{\Phi,Q}.
\]

where the supremum is taken over all cubes in \(\mathbb{R}^n\) with sides parallel to the coordinate axes. When \(p = q\) we define a stronger condition. Let \(\Phi\) and \(\Psi\) be Young functions, and let

\[
[u, v]_{A^{1}_{p,\Psi,\Phi}} = \sup_Q \left| Q \right|^{1/p} \|u^{1/p}\|_{\Psi,Q} \|v^{-1/p}\|_{\Phi,Q}.
\]

**Theorem 2.1.** [12, Theorem 3.6] Given \(1 < p < q < \infty\), a pair of weights \((u, v)\), and Young functions \(\Phi\) and \(\Psi\), we have that

\[
\|I_1\|_{L^p(v)\rightarrow L^q(u)} \leq C(n, p, q) \left( [u, v]_{A^{1}_{p,q,\Phi}} \alpha_{p,q,\Phi} + [u^{1-p'}, v^{1-q'}]_{A^{1}_{q',p',\Psi}} \alpha_{q',p',\Psi} \right).
\]

If \(p = q\), then

\[
\|I_1\|_{L^p(v)\rightarrow L^p(u)} \leq C(n, p) [u, v]_{A^{1}_{p,\Psi,\Phi}} \alpha_{p,\Phi} \alpha_{p',\Psi}.
\]

**Remark 2.2.** In Theorem 2.1 we need to apply the integral condition in (2.1) to the associate functions \(\Phi, \Psi\). If \(\Phi\) and \(\Psi\) are doubling (i.e., \(\Phi(2t) \leq C \Phi(t), t > 0\), and similarly for \(\Psi\)), then by a change of variables this condition can be restated in terms of \(\Phi\) and \(\Psi\). See [11, Prop. 5.10] for further information.

We can now give the Sobolev inequalities needed for our results.

**Lemma 2.3.** Fix \(n \geq 2\) and \(1 < p < n\). Let \(\Omega \subset \mathbb{R}^n\). Then, for any \(f \in W^{1,p}_0(\Omega)\) and \(w \in L^n(\Omega)\),

\[
\|fw\|_{L^p(\Omega)} \leq C(n)(p'-n')^{-1/p'} \|w\|_{L^n(\Omega)} \|
abla f\|_{L^p(\Omega)}.
\]

**Proof.** Extend \(w\) to a function on all of \(\mathbb{R}^n\) by setting it equal to 0 outside of \(\Omega\). Let \(\Psi(t) = t^n\) and \(\Phi(t) = t^r\), \(1 < r < p\); the exact value of \(r\) is not significant. Then

\[
\alpha_{p',p',\Psi} = (p'-n')^{-1/p'}, \quad \alpha_{p,\Phi} = (p-r)^{-1/p},
\]

and so we have that
\[ [w^p, 1]_{A^p_{2,\Phi}} \alpha_{p,p,\Phi} \alpha_{p',p',\Phi} \]

\[ = (p' - n')^{-1}(p - r)^{-1} \sup_Q |Q|^{1/n} \left( \int_Q w^n \, dx \right)^{1/n} \leq (p' - n')^{-1}(p - r)^{-1} \|w\|_{L^n(\Omega)}. \]

Therefore, by Lemma 2.1 we have that for all \( f \in C^0_0(\Omega), \)

\[ \|fw\|_{L^p(\mathbb{R}^n)} \leq \|I_1(\nabla f)w\|_{L^p(\mathbb{R}^n)} \leq C(n, p, r)(p' - n')^{-1/p'} \|w\|_{L^n(\Omega)} \|\nabla f\|_{L^p(\mathbb{R}^n)}. \]

The desired inequality follows for all \( f \) by a standard approximation argument. \( \square \)

When \( n \geq 3 \) we see that \( w \in L^n(\Omega) \) implies the Sobolev inequality for \( p = 2 \). When \( n = 2 \) we only get the Sobolev inequality for \( 1 < p < 2 \), and the constant blows up as \( p \) tends to 2 (and also as it tends to 1). In general \( w \in L^2(\Omega) \) will not be a sufficient condition for the Sobolev inequality in dimension 2. Indeed, fix \( x \in \Omega \) and let \( f \) be a Lipschitz function supported on \( B_{2r}(x) \), \( 0 < r < \text{dist}(x, \partial \Omega) \), equal to 1 on \( B_r(x) \) and with \( |\nabla f| \leq 1/r \). Then by the Lebesgue differentiation theorem we see that a necessary condition for the Sobolev inequality is that \( w \in L^\infty(\Omega). \)

We have two substitute results at the critical exponent when \( p = n = 2 \). To state the first, we define the non-normalized Orlicz norm: given an open set \( \Omega \) and an Orlicz function \( \Phi \),

\[ \|f\|_{L^\Phi(\Omega)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \Phi\left( \frac{|f(x)|}{\lambda} \right) \, dx \leq 1 \right\}. \]

When \( \Phi(t) = t^2 \log(e + t)^{1+\delta} \), then we write \( L^\Phi(\Omega) = L^2(\log L)^{1+\delta}(\Omega) \).

**Lemma 2.4.** Given a bounded open set \( \Omega \subset \mathbb{R}^2 \) and \( w \in L^2(\log L)^{1+\delta}(\Omega) \), if \( f \in W^{1,2}_0(\Omega) \), then

\[ \|fw\|_{L^2(\Omega)} \leq C\delta^{-1/2}[1 + \text{diam}(\Omega)]\|w\|_{L^2(\log L)^{1+\delta}(\Omega)} \|\nabla f\|_{L^2(\Omega)}. \]

**Proof.** We begin as in the proof of Lemma 2.3, but we now take \( \Psi(t) = t^2 \log(e + t)^{1+\delta} \) as in the hypothesis. Then

\[ \alpha_{2,2,\Phi} = \left( \int_1^{\infty} \frac{dt}{t \log(e + t)^{1+\delta}} \right)^{1/2} = C\delta^{-1/2} < \infty, \]

and

\[ [w^2, 1]_{A^1_{2,\Phi}} = \sup_Q |Q|^{1/2} \|w\|_{\Psi,Q}. \]

Since we may assume \( \text{supp}(w) \subset \Omega \), we may restrict the supremum to cubes \( Q \) such that \( |Q| \leq \text{diam}(\Omega)^2 \). Then by the definition of the norm, we have that

\[ |Q|^{1/2} \|w\|_{\Psi,Q} = \inf \left\{ \lambda > 0 : \int_Q \frac{|Q|^2 w(x)^2}{\lambda^2} \log \left( e + \frac{|Q|^{1/2} w(x)}{\lambda} \right)^{1+\delta} \, dx \leq 1 \right\}. \]
\[
\lambda \leq \inf \left\{ \lambda > 0 : \frac{\int_\Omega w(x)^2}{\lambda^2} \log \left( e + \frac{\text{diam}(\Omega)w(x)}{\lambda} \right)^{1+\delta} \ dx \leq 1 \right\}
\]
\[
\leq [1 + \text{diam}(\Omega)]\|w\|_{L^\Psi(\Omega)}.
\]

The desired inequality now follows as before. \qed

To state our next lemma we define the Orlicz-Morrey spaces following Sawano et al. [26]. Given a Young function \( \Psi \) and \( \mu > 0 \), we say a function \( u \) is in \( L^{\Psi,\mu}(\Omega) \) if
\[
\|u\|_{L^{\Psi,\mu}(\Omega)} = \sup_Q |Q|^{\mu}\|u\|_{\Psi,Q} < \infty.
\]

Remark 2.5. The Orlicz-Morrey spaces are a generalization of the classical Morrey spaces. In particular, if \( \Psi(t) = t^2 \log(e + t) \), then by Hölder’s inequality in the scale of Orlicz spaces, we have that
\[
|Q|^\mu \left( \int_Q u^2 \ dx \right)^{1/2} \leq \|u\|_{\Psi,Q},
\]
which implies that the classical Morrey space \( L^{2,1/2-\mu} \) contains \( L^{\Psi,\mu} \).

Lemma 2.6. Given an open set \( \Omega \subset \mathbb{R}^2 \), suppose that for \( 1 < r < 2 \), \( w \in L^{\Psi,1/r'}(\Omega) \), where \( \Psi(t) = t^2 \log(e + t) \). If \( f \in W^{1,2}_0(\Omega) \), then
\[
\|fw\|_{L^2(\Omega)} \leq C(r)\|w\|_{L^{\Psi,1/r'}(\Omega)}\|\nabla f\|_{L^r(\Omega)}.
\]

Proof. We again apply Theorem 2.1. Let \( \Phi(t) = t^a \), \( 1 < a < r \); then \( \Phi = t^a \), and
\[
\alpha_{r,2,\Phi} = \left( \int_1^\infty \frac{t^{2a/r} \ dt}{t^2} \right)^{1/2} = (2 - 2a/r)^{-1/2}.
\]
Moreover, since \( t^{2} \leq \Psi(t) \),
\[
[w^2, 1]_{A_{r,2,\Phi}} = \sup_Q |Q|^{\frac{1}{2} + \frac{1}{r'} - \frac{1}{2}} \left( \int_Q w^2 \ dx \right)^{1/2} \leq \sup_Q |Q|^{\frac{1}{r'}}\|w\|_{\Psi,Q} = \|u\|_{L^{\Psi,1/r'}(\Omega)}.
\]
We have that \( \bar{\Psi}(t) \approx t^2 \log(e + t)^{-1} \), and so, since \( r'/2 > 1 \),
\[
\alpha_{2,r',\Phi} = \left( \int_1^\infty \frac{dt}{t \log(e + t)^{r'/2}} \right)^{1/r'} = C(r) < \infty.
\]
Finally, we have that
\[
[1, w^{-2}]_{A_{2,r',\Phi}} = \sup_Q |Q|^{\frac{1}{2} + \frac{1}{r'} - \frac{1}{2}}\|w\|_{\Psi,Q} = \|w\|_{L^{\Psi,1/r'}(\Omega)}.
\]
Combining these estimates we get the desired inequality by Theorem 2.1 and an approximation argument. \qed
3. Proof of Theorem 1.1

We begin with a coercivity condition attributed to Meyers; a proof is given in [27].

Lemma 3.1. Given a bounded open set \( \Omega \subset \mathbb{R}^n \) with \( C^1 \) boundary, let \( A \) be an \( n \times n \) real-valued matrix that satisfies (1.2). Define the sesquilinear form

\[
a(u, v) = \int_{\Omega} A \nabla u \cdot \nabla v \, dx.
\]

Then there exists \( p_0 = p_0(n, \lambda, \Lambda, \Omega) \), \( 1 < p_0 < 2 \), such that for all \( p, p_0 < p \leq 2 \), and all \( u \in W^{1,p}_0(\Omega) \),

\[
\|u\|_{W^{1,p}_0(\Omega)} \approx \sup_{\|v\|_{W^{1,p'}_0(\Omega)} = 1} |a(u, v)|.
\]

Moreover, the constants in this equivalence depend on \( \lambda, \Lambda, n, \) and \( \Omega \). They are independent of \( p \) and of the specific matrix \( A \).

Proof. The upper estimate for \( a(u, v) \) is just Hölder’s inequality; it is the lower estimate that is non-trivial. From [27] it is clear that \( p_0 \) and the constant in the lower estimate depend only on \( \Lambda, \lambda \), and a constant that comes from the Hodge decomposition. In [21] a careful estimate is given for this constant; in particular it is uniformly bounded when \( p \) is bounded away from 1 and \( \infty \). Also, in [27] the result is proved for “regular” domains, which are defined abstractly in [21]. However, regular domains include Lipschitz domains: see [19]. \( \square \)

Fix a matrix \( A \) satisfying (1.2), and fix \( p, p_0 < p < 2 \), where \( p_0 \) is as in Lemma 3.1. By Di Fazio [13] and Auscher and Qafsaoui [3], for any \( f \in L^p(\Omega) \) the equation \( Lu = f \) has a unique solution \( u \in W^{1,p}_0(\Omega) \) such that

\[
\|\nabla u\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)},
\]

with \( C \) independent of \( f \).

We first prove the desired estimate on \( D^2u \) in the special case when \( f \in C^\infty(\Omega) \) and \( A \in C^\infty(\Omega) \); afterwards we will prove the general case by an approximation argument. Let \( u \) be the solution of (1.1). Then \( u \in C^\infty(\Omega) \): see Evans [14, Th. 3, Sec. 6.3.1]. (Note that there is an implicit assumption on the regularity of the boundary because of an appeal to a Poincaré-Sobolev type inequality for functions without compact support in \( \Omega \); \( C^1 \) is more than sufficient for this purpose.) We now have the pointwise identity

\[
f = -\text{div } A \nabla u = -\sum_{i,j} \left( a_{ij} u_{x_j} \right)_{x_i}.
\]
Fix \( s, 1 \leq s \leq n \). By Lemma 3.1 there exists \( v \in C_0^2(\Omega) \), \( \|v\|_{W_0^{1,p'}} = 1 \), and 
\( \kappa = \kappa(n, \lambda, \Lambda, \Omega) > 0 \) such that 
\[
|a(u_{xs}, v)| \geq \kappa \|u_{xs}\|_{W_0^{1,p}}.
\]
If we multiply \( f \) by \( v_{xs} \) and integrate over \( \Omega \), then integrating by parts twice we get 
\[
\int_\Omega f v_{xs} \, dx = - \int_\Omega \sum_{i,j} (a_{ij} u_{x_j})_{x_i} v_{xs} \, dx = \int_\Omega \sum_{i,j} (a_{ij} u_{x_j}) v_{x_{xs}x_i} \, dx
\]
\[
= - \int_\Omega \sum_{i,j} (a_{ij} u_{x_j})_{x_i} v_{xs} \, dx = - \int_\Omega \sum_{i,j} (a_{ij})_{x_i} u_{x_j} v_{xs} \, dx - \int_\Omega \sum_{i,j} a_{ij} u_{x_j} x_{xs} v_{x_i} \, dx.
\]
Therefore, if we take absolute values, rearrange terms, and combine this with the previous estimate, we get 
\[
\kappa \|\nabla (u_{xs})\|_{L^p(\Omega)} \leq \kappa \|u_{xs}\|_{W_0^{1,p}} \leq |a(u_{xs}, v)|
\]
\[
= \left| \int_\Omega A \nabla (u_{xs}) \cdot \nabla v \, dx \right| \leq \int_\Omega \left| \sum_{i,j} (a_{ij})_{x_i} u_{x_j} v_{x_i} \right| \, dx + \int_\Omega |f v_{xs}| \, dx = I_1 + I_2.
\]
We estimate \( I_1 \) and \( I_2 \) separately. The estimate for the latter is straightforward: by Hölder’s inequality,
\[
I_2 \leq \|f\|_{L^p(\Omega)} \|v_{xs}\|_{L^{p'}(\Omega)} \leq \|f\|_{L^p(\Omega)} \|v\|_{W_0^{1,p'}(\Omega)} = \|f\|_{L^p(\Omega)}.
\]
To estimate \( I_1 \), let
\[
A_s = \left( (a_{ij})_{x_s} \right), \quad U = |\nabla A| = \left( \sum_{i,j,s} (a_{ij})_{x_s}^2 \right)^{1/2}.
\]
Fix \( \epsilon > 0 \); the exact value of \( \epsilon \) will be given below. Since \( U \in L^n(\Omega) \), there exists \( K = K(\epsilon, U) \) such that
\[
(3.3) \quad \left( \int_{\{x: U(x) > K\}} U(x)^n \, dx \right)^{1/n} < \epsilon.
\]
Let \( U_1 = U \chi_{\{x: U(x) > K\}} \) and \( U_2 = U - U_1 \). Then by Hölder’s inequality and Lemma 2.3, we can estimate as follows:
\[
I_1 = \int_\Omega |A_s \nabla u \cdot \nabla v| \, dx
\]
\[
\leq \int_\Omega U |\nabla u| |\nabla v| \, dx
\]
\[
\leq \left( \int_\Omega |\nabla u|^p \, dx \right)^{1/p} \left( \int_\Omega |\nabla v|^{p'} \, dx \right)^{1/p'}
\]
\[
\leq \left( \int_{\Omega} \left| \nabla u U_1 \right|^p \, dx \right)^{1/p} + \left( \int_{\Omega} \left| \nabla u U_2 \right|^p \, dx \right)^{1/p}
\]
\[
\leq C(n)(p' - n')^{-1/p'} \epsilon \left( \int_{\Omega} |D^2 u|^p \, dx \right)^{1/p} + K(\epsilon, U) \left( \int_{\Omega} |\nabla u|^p \, dx \right)^{1/p}.
\]

Each of the above estimates hold for all values of \( s \). Therefore, by Minkowski's inequality, if we sum over all \( s \) and combine these estimates, we get that
\[
\kappa \|D^2 u\|_{L^p(\Omega)} \leq \sum_s \kappa \|\nabla (u_{x_s})\|_{L^p(\Omega)} \leq C(n, \epsilon) \|D^2 u\|_{L^p(\Omega)} + K(\epsilon, U) \|\nabla u\|_{L^p(\Omega)} + n\|f\|_{L^p(\Omega)}.
\]

Since \( \epsilon > 0 \) is arbitrary, we can fix \( \epsilon = \kappa/2C(n, p) \) and then rearrange terms to get
\[
\|D^2 u\|_{L^p(\Omega)} \leq C(\kappa, n)K(\epsilon, U) \|\nabla u\|_{L^p(\Omega)} + C(\kappa, n)\|f\|_{L^p(\Omega)} \leq C_0\|f\|_{L^p(\Omega)},
\]

where the last inequality follows from (3.2). This completes the proof of inequality (1.3) when \( f \) and \( A \) are sufficiently smooth. Note that the constant \( C_0 \) depends on \( p, n, \lambda, \Lambda, \Omega \) and the constant \( K \) from inequality (3.3).

We will now show that we can take an arbitrary \( f \). Fix \( f \in L^p(\Omega) \), and fix a sequence of functions \( \{f_j\} \) in \( C^\infty(\Omega) \) that converge to \( f \) in \( L^p(\Omega) \). Fix \( A \in C^\infty(\Omega) \) and let \( u_j \) be the solution to \( Lu_j = f_j \), and let \( u \in W^{1,p}_0(\Omega) \) be the solution to \( Lu = f \). By inequality (3.2) and the Sobolev inequality, we have that
\[
\|u - u_j\|_{L^p(\Omega)} \leq C\|\nabla (u - u_j)\|_{L^p(\Omega)} \leq C\|f - f_j\|_{L^p(\Omega)}.
\]

Therefore, \( u_j \to u \) in \( W^{1,p}_0(\Omega) \).

Thus, by (3.4) \( \int \Omega u_{x_s} \phi dx = \lim \int \Omega (u_j)_{x_s} \phi dx = \lim \int \Omega (u)_{x_s} \phi dx = \int \Omega v_{r,s} \phi dx \). Therefore, \( u \in W^{2,p}_0(\Omega) \) and \( u_j \to u \) in \( W^{2,p}_0(\Omega) \). Inequality (1.3) for \( u \) now follows immediately.

Finally, we prove that we can take arbitrary \( A \in W^{1,n}(\Omega) \). Fix such an \( A \), and let \( \{A_j\} \) be a sequence of matrices in \( C^\infty(\Omega) \) that converges to \( A \) in \( W^{1,n}(\Omega) \). It follows at once from the standard construction of the \( A_j \) (cf. Adams and Fournier [1]) that we may assume that the \( A_j \) are elliptic with the same ellipticity constants as \( A \). Finally, let \( U_j = |\nabla A_j| \); then \( U_j \to U = |\nabla A| \) in \( L^2(\Omega) \). By the converse to the dominated
convergence theorem (see Lieb and Loss [23, Th. 2.7]), if we pass to a subsequence, then we may assume that $U_j \to U$ pointwise a.e., and there exists $g \in L^2(\Omega)$ such that $U_j(x) \leq g(x)$ a.e. Therefore, by the dominated convergence theorem (again passing to a subsequence) we may assume that (3.3) holds (with fixed $\epsilon$) for each $U_j$ with a constant $K$ independent of $j$.

Fix $f \in L^p(\Omega)$ and let $u_j$ be the solution of $-\text{div} A_j \nabla u_j = f$ and let $u$ be the solution of $Lu = -\text{div} A \nabla u = f$. Then for any $\phi \in C^\infty_0(\Omega)$,

$$\int_\Omega A_j \nabla u_j \cdot \nabla \phi \, dx = - \int_\Omega f \phi \, dx = \int_\Omega A \nabla u \cdot \nabla \phi \, dx.$$  

Therefore, 

$$\int_\Omega (A \nabla u - A \nabla u_j + A \nabla u - A_j \nabla u_j) \nabla \phi \, dx = 0,$$

and so by rearranging terms we have that 

$$|a(\nabla u - \nabla u_j, \phi)| = \left| \int_\Omega A(\nabla u - \nabla u_j) \cdot \nabla \phi \, dx \right| \leq \int_\Omega |(A - A_j) \nabla u_j \cdot \nabla \phi| \, dx.$$  

By Lemma 3.1 there exists $\phi$ such that $\|\phi\|_{W^{1,p'}(\Omega)} = 1$ and $\kappa > 0$ such that 

$$(3.5) \quad \kappa \|u - u_j\|_{W^{1,p}(\Omega)} \leq \int_\Omega |(A - A_j) \nabla u_j \cdot \nabla \phi| \, dx \leq \|A - A_j\|_{L^n(\Omega)} \|\nabla u_j\|_{L^{np}(\Omega)} \|\nabla \phi\|_{L^{p'}(\Omega)}.$$  

The last estimate follows by Hölder’s inequality, since 

$$\frac{1}{n} + \frac{n - p}{np} + \frac{1}{p'} = 1.$$  

The last term on the righthand side of (3.5) is at most 1. By our choice of the $A_j$, the first term tends to 0 as $j \to \infty$. And by the Sobolev inequality, 

$$\|\nabla u_j\|_{L^{np}(\Omega)} \leq C \|D^2 u_j\|_{L^p(\Omega)} \leq C \|f\|_{L^p(\Omega)};$$  

the final inequality holds since by our choice of the $A_j$, inequality (1.3) holds for each $u_j$ with a constant independent of $j$. Therefore, the middle term on the righthand side of (3.5) is uniformly bounded. Hence, $u_j \to u$ in $W^{1,p}_0(\Omega)$.

It remains to show $D^2 u$ exists and estimate its norm. By inequality (1.3), the sequence $\{D^2 u_j\}$ is uniformly bounded in $L^p(\Omega)$, and so has a weakly convergent subsequence. Passing to this subsequence, we can repeat the argument at (3.4) to conclude that $u \in W^{2,p}_0(\Omega)$ and $D^2 u_j$ converges weakly to $D^2 u$. But then we have that 

$$\|D^2 u\|_{L^p(\Omega)} \leq \liminf_{j \to \infty} \|D^2 u_j\|_{L^p(\Omega)} \leq C \|f\|_{L^p(\Omega)},$$  

and this completes the proof.
4. The Case $n = 2$

In this section we consider the two dimensional case. We first construct Example 1.4 and then prove Theorems 1.5 and 1.6.

Construction of Example 1.4. Our example is adapted from one given by Clop et al. [9, p. 205] and is based on the theory of quasiregular mappings. Let $B = B_{1/2}(0)$ and let $z = x + iy$. Define

$$f(z) = z(1 - 2 \log |z|).$$

Then

$$\partial f(z) = -2 \log |z| \quad \text{and} \quad \bar{\partial} f(z) = \frac{z}{\bar{z}},$$

and so $f$ satisfies the Beltrami equation $\bar{\partial} f = \mu \partial f$ with Beltrami coefficient

$$\mu(z) = \frac{z}{\bar{z} \log(\frac{|z|}{2})} = \frac{z}{|z|^2 \log(\frac{|z|}{2})}.$$

If we let $u = \text{Re } f$, that is,

$$u(x, y) = x(1 - \log(x^2 + y^2)),$$

then $u$ satisfies the equation

$$-\text{div}(A \nabla u) = 0$$

where $A$ is the symmetric, real-valued matrix

$$A = \begin{bmatrix} \frac{|1 - \mu|^2}{1 - |\mu|^2} & -2 \text{Im } \mu \\ -2 \text{Im } \mu & \frac{1 + |\mu|^2}{1 - |\mu|^2} \end{bmatrix} = \frac{1 + \sigma^2}{1 - \sigma^2} \text{Id} - \frac{2}{1 - \sigma^2} \begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix},$$

and

$$\sigma = |\mu| = -\frac{1}{\log(x^2 + y^2)}, \quad \alpha = \text{Re } \mu = \frac{x^2 - y^2}{x^2 + y^2} \sigma, \quad \beta = \text{Im } \mu = \frac{2xy}{x^2 + y^2} \sigma.$$

This follows from a straightforward calculation: for the details, see [2, p. 412].

We claim that $A$ is elliptic and in $W^{1,2}(B)$, and that $u \in W^{2,p}(B)$ for $p < 2$ but not when $p = 2$. By our choice of domain, $0 \leq \sigma \leq k = (\log 4)^{-1}$. Let $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$; then

$$\langle A\xi, \xi \rangle = \frac{1 + \sigma^2}{1 - \sigma^2} |\xi|^2 - \frac{2\alpha(\xi_1^2 - \xi_2^2) + 4\beta \xi_1 \xi_2}{1 - \sigma^2}.$$

Since

$$\alpha(\xi_1^2 - \xi_2^2) + 4\beta \xi_1 \xi_2 = 2(\alpha, \beta) \cdot (\xi_1^2 - \xi_2^2, 2\xi_1 \xi_2),$$
by the Cauchy-Schwarz inequality we have that
\[ |2\alpha(\xi_1^2 - \xi_2^2) + 4\beta \xi_1 \xi_2| \leq 2\sqrt{\alpha^2 + \beta^2} \sqrt{(\xi_1^2 - \xi_2^2)^2 + 4\xi_1^2 \xi_2^2} = 2\sigma|\xi|^2. \]

Hence,
\[ -2\sigma|\xi|^2 \leq 2\alpha(\xi_1^2 - \xi_2^2) + 4\beta \xi_1 \xi_2 \leq 2\sigma|\xi|^2, \]
and if we combine this with inequality (4.1), we get
\[ \frac{1 - k}{1 + k}|\xi|^2 \leq \frac{1 - \sigma}{1 + \sigma}|\xi|^2 \leq \langle A\xi, \xi \rangle \leq \frac{1 + \sigma}{1 - \sigma}|\xi|^2 \leq \frac{1 + k}{1 - k}|\xi|^2. \]

Thus, \( A \) is elliptic with \( \lambda = \frac{1 - k}{1 + k} \) and \( \Lambda = \frac{1 + k}{1 - k} \).

To see that \( A = (a_{ij}) \in W^{1,2}(B) \), a lengthy (and Mathematica assisted) calculation shows that
\[ \frac{\partial a_{11}}{\partial x} = 4x[x^2 - y^2 - 2y^2 \log^3(x^2 + y^2) + (x^2 - y^2) \log^2(x^2 + y^2) + 2(x^2 + 2y^2) \log(x^2 + y^2)] \]
\[ (x^2 + y^2)^2(\log^2(x^2 + y^2) - 1)^2 \]
and the derivatives \( \frac{\partial}{\partial x}a_{ij} \) and \( \frac{\partial}{\partial y}a_{ij} \) are similar. It follows that
\[ \left| \frac{\partial}{\partial x}a_{ij} \right|, \left| \frac{\partial}{\partial y}a_{ij} \right| \leq C\frac{|\log^3(x^2 + y^2)|}{(x^2 + y^2)^{\frac{3}{2}}(\log^2(x^2 + y^2) - 1)^2} \in L^2(B). \]

Finally to see that \( u \in W^{2,p}(B) \) for \( p < 2 \) but not in \( W^{2,2}(B) \), another calculation shows that
\[ u_{xx}(x, y) = \frac{-2x(x^2 + 3y^2)}{(x^2 + y^2)^2}, \quad u_{xy}(x, y) = \frac{-2y(y^2 - x^2)}{(x^2 + y^2)^2}, \quad u_{yy}(x, y) = \frac{-2x(x^2 - y^2)}{(x^2 + y^2)^2}. \]

Thus, each derivative is bounded by a constant multiple of \( (x^2 + y^2)^{-\frac{3}{2}} \in L^p(B) \), so \( u \in W^{2,p} \). On the other hand,
\[ \int_B |u_{xx}|^2 \, dx \, dy = \infty, \]
so \( u \not\in W^{2,2}(B) \). \( \square \)

**Proof of Theorem 1.5.** Most of the proof is identical to the proof of Theorem 1.1, setting \( n = p = 2 \). However, in two places we need to make specific changes to the proof. The proof for \( f \) and \( A \) smooth is the same up to inequality (3.3). We again split \( U \), but now we fix \( \epsilon \) (to be determined below) and find \( K \) such that
\[ \|U\chi_{\{U > K\}}\|_{L^q(\Omega)} < \epsilon, \]
where $\Psi(t) = t^2 \log(e + t)^{1+\delta}$. (This is again possible by the dominated convergence theorem in the context of Orlicz spaces.) Let $U = U_1 + U_2 = U\chi_{\{U > K\}} + U\chi_{\{U \leq K\}}$; then by Lemma 2.4,

$$
(4.3) \left( \int_\Omega (|\nabla u| U)^2 \, dx \right)^{1/2} \leq \left( \int_\Omega (|\nabla u| U_1)^2 \, dx \right)^{1/2} + K \left( \int_\Omega |\nabla u|^2 \, dx \right)^{1/2} 
\leq \epsilon C(\delta, \Omega) \left( \int_\Omega |D^2 u|^2 \, dx \right)^{1/2} + K \|f\|_{L^2(\Omega)}.
$$

The argument now proceeds as before, yielding

$$
\|D^2 u\|_{L^2(\Omega)} \leq C_0 \|f\|_{L^2(\Omega)},
$$

where again the constant $C_0 = C_0(n, p, \lambda, \Lambda, \Omega, K)$.

The proof for arbitrary $f \in L^2(\Omega)$ goes through exactly as before. For the proof for arbitrary $\nabla A \in L^p(\Omega)$, we fix smooth $A_j \to A$ in $W^{1,p}(\Omega)$ (the Sobolev space defined with respect to the $L^p$ norm), and we may again assume that the $A_j$ have the same ellipticity constants and that we may choose $K$ such that (4.2) holds for all $U_j = |\nabla A_j|$ with a constant $K$ independent of $j$. This is possible since all the arguments for $W^{1,p}(\Omega)$ extend to $W^{1,\Psi}(\Omega)$ with almost no change. Smooth functions are dense, see [1], and the proof of density again shows that ellipticity constants are preserved. The converse of dominated convergence also holds in this setting; the proof is implicit in the literature. For a proof in a different context that readily adapts to Orlicz spaces, see [10, Prop. 2.67].

The proof now continues as before until inequality (3.5). Here we need to apply the generalized Hölder’s inequality in the scale of Orlicz spaces (see [11, Lemma 5.2]). If we let $\Phi(t) = \exp(t^{1/2}) - 1$, then

$$
\Psi^{-1}(t)\Phi^{-1}(t) \approx \frac{t^{1/2}}{\log(e + t)^{1+\delta}} \leq t^{1/2}.
$$

Therefore, we can estimate as follows:

$$
\left| \int_\Omega A(\nabla u - \nabla u_j) \cdot \nabla \phi \, dx \right| \leq \int_\Omega |(A - A_j)\nabla u_j \cdot \nabla \phi| \, dx 
\leq \|(A - A_j)\nabla u_j\|_{L^2(\Omega)} \|\nabla \phi\|_{L^2(\Omega)} \leq \|A - A_j\|_{L^\Psi(\Omega)} \|\nabla u_j\|_{L^\Psi(\Omega)} \|\nabla \phi\|_{L^2(\Omega)}.
$$

As in the previous argument, we have chosen $\phi$ so that $\|\nabla \phi\|_{L^2(\Omega)} \leq 1$. We also have that $\|A - A_j\|_{L^\Psi(\Omega)} \to 0$ as $j \to \infty$. Therefore, we could complete the proof as before if we can show that

$$
\|\nabla u_j\|_{L^\Psi(\Omega)} \leq C \|f\|_{L^2(\Omega)}
$$

with a constant independent of $j$. 
Let $\Phi_0(t) = \exp(t^2) - 1$. Then for $t \geq 1$, $\Phi(t) \leq \Phi_0(t)$, and so by the properties of Orlicz norms (see [11, Sec.5.2]) there exists a constant depending on $\delta$ and $\Omega$ such that $\|
abla u_j\|_{L^\Phi(\Omega)} \leq C \|
abla u_j\|_{L^\Phi_0(\Omega)}$. But by Trudinger’s inequality [31, Thm. 2.9.1] we have the endpoint Sobolev inequality:

$$\|
abla u_j\|_{L^\Phi_0(\Omega)} \leq C \|
abla u_j\|_{L^2(\Omega)}.$$

By the first part of the proof we have that $\|D^2u_j\|_{L^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}$ with a constant independent of $j$; combining these inequalities we get the desired estimate and this completes the proof.

Proof of Theorem 1.6. The proof is nearly identical to the proof of Theorem 1.5. Let $\Psi(t) = t^2 \log(e + t)$. The first half of the proof for smooth $f$ and $A$ is the same until (4.3). Here, we use Lemma 2.6 and Hölder’s inequality to get

$$\left(\int_\Omega (|\nabla u|^2 U) dx\right)^{1/2} \leq \left(\int_\Omega (|\nabla u| U_1)^2 dx\right)^{1/2} + K \left(\int_\Omega |\nabla u|^2 dx\right)^{1/2}$$

$$\leq \epsilon C(\delta, \Omega) \left(\int_\Omega |D^2 u|^r dx\right)^{1/r} + K \|f\|_{L^2(\Omega)}$$

$$\leq \epsilon C(\delta, \Omega) |\Omega|^{1/(2/r')} \left(\int_\Omega |D^2 u|^2 dx\right)^{1/2} + K \|f\|_{L^2(\Omega)}.$$

We can now complete the proof of the smooth case as before. The remainder of the proof goes through before, only now we apply the generalized Hölder’s inequality with $\Psi(t)$ and $\Phi(t) = \exp(t^2) - 1$ and then directly apply Trudinger’s inequality.

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