EXISTENCE RESULTS FOR NON-LOCAL ELLIPTIC SYSTEMS WITH HARDY-LITTLEWOOD-SOBOLEV CRITICAL NONLINEARITIES

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Abstract. In this article, we study the following nonlinear doubly nonlocal problem involving the fractional Laplacian in the sense of Hardy-Littlewood-Sobolev inequality

\[
\begin{aligned}
(-\Delta)^s u &= au + bv + \frac{2p}{p+q} \int_{\Omega} \frac{|v(y)|^q}{|x-y|^\mu} dy |u|^{p-2} u + 2\xi_1 \int_{\Omega} \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu} dy |u|^{2^*_\mu-2} u, \quad \text{in } \Omega; \\
(-\Delta)^s v &= bu + cv + \frac{2q}{p+q} \int_{\Omega} \frac{|u(y)|^p}{|x-y|^\mu} dy |v|^{q-2} v + 2\xi_2 \int_{\Omega} \frac{|v(y)|^{2^*_\mu}}{|x-y|^\mu} dy |v|^{2^*_\mu-2} v, \quad \text{in } \Omega; \\
u = v = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,
\end{aligned}
\]

where \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^N, N > 2s, s \in (0,1), \xi_1, \xi_2 \geq 0, (-\Delta)^s \) is the well known fractional Laplacian, \( \mu \in (0,N), 1 < p, q \leq 2^*_\mu \), where \( 2^*_\mu = \frac{2N}{N-2s} \) is the upper critical exponent in the Hardy-Littlewood-Sobolev inequality. Under suitable assumptions on different parameters \( p, q, \xi_1, \xi_2 \), we are able to prove some existence and multiplicity results for the above equation by variational methods.

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1. Introduction and main results

Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain with smooth boundary \( \partial \Omega \) (at least \( C^2 \)), \( N > 2s \) and \( s \in (0,1) \). We consider the following nonlinear doubly nonlocal system involving the fractional Laplacian:

\[
(-\Delta)^s u = au + bv + \frac{2p}{p+q} \int_\Omega \frac{|v(y)|^q}{|x-y|^\mu} dy |u|^{p-2} u + 2\xi_1 \int_\Omega \frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu} dy |u|^{2_\mu^*-2} u, \quad \text{in } \Omega;
\]

\[
(-\Delta)^s v = bu + cv + \frac{2q}{p+q} \int_\Omega \frac{|u(y)|^p}{|x-y|^\mu} dy |v|^{q-2} v + 2\xi_2 \int_\Omega \frac{|v(y)|^{2_\mu^*}}{|x-y|^\mu} dy |v|^{2_\mu^*-2} v, \quad \text{in } \Omega;
\]

\[
u = v = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,
\]

\( \mu \in (0, N) \), \( \xi_1, \xi_2 \geq 0 \) and \( 1 < p, q \leq 2_\mu^* \) where \( 2_\mu^* = \frac{2N-\mu}{N-2s} \) is the upper critical exponent in the Hardy-Littlewood-Sobolev inequality. \((-\Delta)^s\) is the fractional Laplacian operator defined as

\[
(-\Delta)^s u(x) = -P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy
\]

where \( P.V. \) denotes the Cauchy principal value. This type of operators arise in many different contexts, such as, among the others, physical phenomena, stochastic processes, fluid dynamics, dynamical systems, elasticity, obstacle problems, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, multiple scattering, materials science and water waves. For more details, we refer to [1, 14]. For any measurable function \( u : \mathbb{R}^N \rightarrow \mathbb{R} \), we define the Gagliardo seminorm by setting

\[
[u]_s := \left( \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dxdy \right)^{\frac{1}{2}}
\]

Now, we introduce the fractional Sobolev space (which is a Hilbert space)

\[
H^s(\mathbb{R}^N) = \{ u \in L^2(\mathbb{R}^N) : [u]_s < \infty \},
\]

with the norm \( \| u \|_{H^s} = \left( \| u \|_{L^2}^2 + [u]_s^2 \right)^{\frac{1}{2}} \). Let the closed subspace

\[
X(\Omega) := \{ u \in H^s(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \}.
\]

It holds that \( X(\Omega) \hookrightarrow L^r(\Omega) \) continuously for \( r \in [1, 2^*_s] \) and compactly for \( r \in [1, 2^*] \), where \( 2^*_s = \frac{2N}{N-2s} \). Due to the fractional Sobolev inequality, \( X(\Omega) \) is a Hilbert space with the inner product given by

\[
\langle u, v \rangle_X := \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} dxdy,
\]

which induces the norm \( \| \cdot \|_X = [\cdot]_s \). We shall denote by \( \mu_1 \) and \( \mu_2 \) the real eigenvalues of the matrix

\[
A := \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad a, b, c \in \mathbb{R}.
\]

Without loss of generality, we will assume \( \mu_1 \leq \mu_2 \). The spectrum of \((-\Delta)^s\), with boundary condition \( u = 0 \) in \( \mathbb{R}^N \setminus \Omega \), will be denoted by \( \sigma((-\Delta)^s) \), which consists of the sequence of the eigenvalues \( \{ \lambda_{k,s} \} \) satisfying

\[
0 < \lambda_{1,s} < \lambda_{2,s} \leq \lambda_{3,s} \leq \ldots \leq \lambda_{j,s} \leq \lambda_{j+1,s} \leq \ldots \lambda_{k,s} \rightarrow \infty, \quad \text{as } k \rightarrow \infty,
\]

\[
\mu \in (0, N) \quad \xi_1, \xi_2 \geq 0 \quad 1 < p, q \leq 2^*_s \quad 2^*_s = \frac{2N-\mu}{N-2s}
\]

\[
\| u \|_{H^s} = \left( \| u \|_{L^2}^2 + [u]_s^2 \right)^{\frac{1}{2}} \quad \text{X(\Omega) \hookrightarrow L^r(\Omega) \text{ continuously for } r \in [1, 2^*_s]} \quad \text{X(\Omega) \text{ compactly for } r \in [1, 2^*]}
\]

\[
\langle u, v \rangle_X := \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} dxdy
\]

\[
\| \cdot \|_X = [\cdot]_s \quad \mu_1 \leq \mu_2 \quad \sigma((-\Delta)^s) \quad \{ \lambda_{k,s} \}
\]

\[
0 < \lambda_{1,s} < \lambda_{2,s} \leq \lambda_{3,s} \leq \ldots \leq \lambda_{j,s} \leq \lambda_{j+1,s} \leq \ldots \lambda_{k,s} \rightarrow \infty, \quad \text{as } k \rightarrow \infty,
\]
Remark 1.1. For fixed \( \lambda \) and we denote the following results are true (see \([27, 28, 30]\)).

1. \( \lambda_{k+1,s} \) is an orthonormal basis in both \( L^2(\Omega) \) and \( X(\Omega) \).

2. \( \{ \varphi_{k,s} \} \) is obtained by the eigenfunction \( k \)-spectrum

3. \( \{ \varphi_{k,s} \} \) is obtained by the eigenfunction \( k \)-spectrum

\[
\lambda_{k+1,s} = \inf_{u \in \mathbb{P}_{k+1} \setminus \{0\}} \frac{\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy}{\int_{\mathbb{R}^N} |u(x)|^2 \, dx},
\]

where

\[
\mathbb{P}_{k+1} = \{ u \in X(\Omega) : \langle u, \varphi_{j,s} \rangle_X = 0, j = 1, 2, \ldots, k \},
\]

and \( \varphi_{k,s} \) denotes the eigenfunction associated to the eigenvalue \( \lambda_{k,s} \), for each \( k \in \mathbb{N} \). The following results are true (see \([27, 28, 30]\)).

(i) If \( u \in X(\Omega) \) is a \( \lambda_{k,s} \)-eigenfunction (\( u \) is an eigenfunction corresponding to \( \lambda_{k,s} \)), then either \( u(x) > 0 \) a.e. in \( \Omega \) or \( u(x) < 0 \) a.e. in \( \Omega \);

(ii) If \( \lambda \in \sigma((-\Delta)^s) \setminus \{ \lambda_{k,s} \} \) and \( u \) is a \( \lambda \)-eigenfunction, then \( u \) changes sign in \( \Omega \), and \( \lambda \) has finite multiplicity.

(iii) \( \varphi_{k,s} \in C^{0,\sigma}(\Omega) \) for some \( \sigma \in (0,1) \) and the sequence \( \{ \varphi_{k,s} \} \) is an orthonormal basis in both \( L^2(\Omega) \) and \( X(\Omega) \).

In a pioneering paper \([3]\), Brézis and Nirenberg studied the problems of the type

\[
-\Delta u = |u|^{2^*-2} u + \lambda u \quad \text{in } \Omega; u = 0 \text{ on } \partial \Omega,
\]

where \( 2^* = \frac{N+2}{N-2} \). They proved the existence of nontrivial solutions for \( \lambda > 0, N > 4 \) by developing some skillful techniques in estimating the Minimax level. This kind of Brézis-Nirenberg problems have been extensively studied (see, e.g., \([4-7, 9, 16-20, 33, 34, 36]\) and references therein). Recently, many well-known Brézis-Nirenberg results in critical local equations have been extended to semilinear equations with fractional Laplacian. Specially, we refer to \([28, 29, 31, 32]\), where authors studied the following critical fractional Laplacian problem

\[
(-\Delta)^s u = |u|^{2^*-2} u + \lambda u \quad \text{in } \Omega; u = 0 \text{ in } \mathbb{R}^N \setminus \Omega,
\]

and shown that problem \((1.3)\) has a nontrivial weak solution under the following circumstances:

(i) \( 2s < N < 4s \) and \( \lambda \) is sufficiently large;

(ii) \( N = 4s \) and \( \lambda \) is not an eigenvalue of \((\Delta)^s u \text{ in } \Omega;\)

(iii) \( N \geq 4 \).

In \([12]\), Gao and Yang studied the Brézis-Nirenberg type problem involving the Choquard nonlinearity, that is

\[
-\Delta u = \lambda u + \left( \int_{\Omega} \frac{|u|^{2^*}_{\mu}}{|x-y|^{\mu}} \, dy \right) |u|^{2^*-2} u, \quad \text{in } \Omega, u = 0, \text{ in } \mathbb{R}^N \setminus \Omega.
\]
where \( \Omega \) is bounded domain in \( \mathbb{R}^N \). They proved the existence, multiplicity and nonexistence results for a range of \( \lambda \). Moreover, in [13] authors studied a class of critical Choquard equations

\[
- \Delta u = \left( \int_\Omega \frac{|u|^{2_\mu^*}}{|x-y|^{\mu}} \, dy \right) |u|^{2_\mu^*} - 2u + \lambda f(u), \quad \text{in } \Omega.
\]

They proved some existence and multiplicity results for the equation (1.5) under suitable assumptions on different types of nonlinearities \( f(u) \). In the nonlocal case, Mukherjee and Sreenadh in [25] considered nonlocal counterpart of problem (1.4) and obtained existence, multiplicity and nonexistence results for solutions.

Coming to the system of equations, elliptic systems involving fractional Laplacian and critical growth nonlinearities have been studied in [10, 11, 21, 24], extending the Brézis and Nirenberg results for variational systems. Particularly, in [24], Miyagaki and Pereira studied the following fractional elliptic system

\[
\begin{aligned}
(-\Delta)^s u &= au + bv + \frac{2p}{p+q}|u|^{p-2}u|v|^q + 2\xi_1 u|u|^{p+q-2}, \quad \text{in } \Omega; \\
(-\Delta)^s v &= bu + cv + \frac{2q}{p+q}|v|^{q-2}v + 2\xi_2 v|v|^{p+q-2}, \quad \text{in } \Omega; \\
u &= v = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,
\end{aligned}
\]

extending [10] by means of the Linking Theorem when

\[\lambda_{k-1,s} \leq \mu_1 < \lambda_{k,s} \leq \mu_2 < \lambda_{k+1,s}, \quad \text{if } k \geq 1.\]

Under these circumstances, resonance and double resonance phenomena \( \lambda_{k-1,s} = \mu_1 \) and \( \lambda_{k,s} = \mu_2 \) can occur. In [15], Giacomoni, Mukherjee and Sreenadh discussed the existence and multiplicity of weak solutions for the following fractional elliptic system involving Choquard type nonlinearities,

\[
\begin{aligned}
(-\Delta)^s u &= \lambda|u|^{q-2}u + \left( \int_\Omega \frac{|v(y)|^{2_\mu^*}}{|x-y|^{\mu}} \, dy \right) |u|^{2_\mu^*} - 2u, \quad \text{in } \Omega; \\
(-\Delta)^s v &= \delta|v|^{q-2}v + \left( \int_\Omega \frac{|u(x)|^{2_\mu^*}}{|x-y|^{\mu}} \, dy \right) |v|^{2_\mu^*} - 2v, \quad \text{in } \Omega; \\
u &= v = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,
\end{aligned}
\]

where \( \lambda, \delta > 0 \) are real parameters and \( 1 < q < 2 \).

Motivated by paper [11, 24], we discuss the existence and multiplicity results for problem (1.1) under the conditions that (i) \( \xi_1 = \xi_2 = 0 \), \( 1 < p, q < 2_\mu^* \), (ii) \( \xi_1 = \xi_2 = 0 \), \( p = q = 2_\mu^* \), (iii) \( \xi_1, \xi_2 > 0 \), \( p = q = 2_\mu^* \) respectively. The following are the main results.

**Theorem 1.2.** (Existence I) Assume that \( \xi_1 = \xi_2 = 0 \), \( 1 < p, q < 2_\mu^* \), \( b \geq 0 \) and \( \mu_2 < \lambda_{1,s} \). Then problem (1.1) admits a positive solution.

**Theorem 1.3.** (Existence II) Assume that \( \xi_1 = \xi_2 = 0 \), \( p = q = 2_\mu^* \), \( b \geq 0 \) and \( 0 < \mu_1 \leq \mu_2 < \lambda_{1,s} \). Then problem (1.1) admits a nonnegative solution, provided that either

(i) \( N \geq 4s \) and \( \mu_1 > 0 \), or
(ii) \( 2s < N < 4s \) and \( \mu_1 \) is large enough.
**Theorem 1.4.** (Existence III) Assume that $\xi_1, \xi_2 > 0$, $p = q = 2^*_\mu$ and $\lambda_{k-1,s} < \mu_1 < \lambda_{k,s} \leq \mu_2 < \lambda_{k+1,s}$, for some $k \in \mathbb{N}$. Then problem (1.1) admits a nontrivial solution, if one of the following conditions holds,

(i) $N \geq 4s$ and $\mu_1 > 0$,

(ii) $2s < N < 4s$ and $\mu_1$ is large enough.

2. Preliminary Stuff

2.1. Notations and setting. Now, we consider the Hilbert space given by the product space

\[ Y(\Omega) := X(\Omega) \times X(\Omega), \]

equipped with the inner product

\[ \langle (u,v), (\varphi,\psi) \rangle_Y := \langle u, \varphi \rangle_X + \langle v, \psi \rangle_X \]

and the norm

\[ \|(u,v)\|_Y := (\|u\|^2_X + \|v\|^2_X)^{\frac{1}{2}}. \]

We shall consider $L^m(\Omega) \times L^m(\Omega)(m > 1)$ equipped with the standard product norm

\[ \|(u,v)\|_{L^m \times L^m} := (\|u\|_{L^m}^2 + \|v\|_{L^m}^2)^{\frac{1}{2}}. \]

We recall that

\[ \mu_1 |U|^2 \leq (AU,U)_{R^2} \leq \mu_2 |U|^2, \quad \text{for all } U := (u,v) \in \mathbb{R}^2. \]

By a solution of (1.1) we mean a weak solution, that is, a pair of functions $(u,v) \in Y(\Omega)$ such that

\[ \langle (u,v), (\varphi,\psi) \rangle_Y - \int_{\Omega} (A(u,v), (\varphi,\psi))_{R^2} \, dx - \int_{\Omega} \frac{\partial F}{\partial u} \varphi \, dx - \int_{\Omega} \frac{\partial F}{\partial v} \psi \, dx = 0, \]

for all $(\varphi,\psi) \in Y(\Omega)$, where

\[ F(u,v) = \frac{2}{p+q} \int_{\Omega} \frac{|v(y)|^q}{|x-y|^\mu} \, dy |u|^p + \frac{1}{2^*_\mu} \left[ \xi_1 \int_{\Omega} \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu} \, dy |u|^{2^*_\mu} + \xi_2 \int_{\Omega} \frac{|v(y)|^{2^*_\mu}}{|x-y|^\mu} \, dy |v|^{2^*_\mu} \right]. \]

Now define the functional $J_s : Y(\Omega) \to \mathbb{R}$ by setting

\[ J_s(U) = J_s(u,v) = \int_{R^{2N}} \frac{|u(x) - u(y)|^2 + |v(x) - v(y)|^2}{|x-y|^{N+2s}} \, dx \, dy - \frac{1}{2} \int_{R^N} (A(u,v), (u,v))_{R^2} \, dx - \int_{\Omega} F(U) \, dx, \]
whose Fréchet derivative is given by

\[ J_s'(u,v)(\varphi,\psi) = \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y)) + (v(x) - v(y))(\psi(x) - \psi(y))}{|x-y|^{N+2s}}
- \int_{\Omega} (A(u,v), (\varphi, \psi))_{\mathbb{R}^{2N}} dx
- \frac{2p}{p+q} \int_{\Omega} \frac{|u(x)|^{p-2}u(x)|v(y)|^q}{|x-y|^\mu} \varphi dx dy
- \frac{2q}{p+q} \int_{\Omega} \frac{|v(x)|^p|v(y)|^q}{|x-y|^\mu} \psi dx dy
- 2\xi_1 \int_{\Omega} \frac{|u(x)|^{2\mu-2}u(x)|u(y)|^{2\mu}}{|x-y|^\mu} \varphi dx dy
- 2\xi_2 \int_{\Omega} \frac{|v(x)|^{2\mu}|v(y)|^{2\mu}}{|x-y|^\mu} \psi dx dy, \]

for every \((\varphi, \psi) \in Y(\Omega)\).

In this paper, we set the following notation for product space \(S \times S := S^2\) and

\[ w^+(x) := \max \{w(x), 0\}, w^-(x) := \min \{w(x), 0\}, \]

for positive and negative part of a function \(w\). Consequently we get \(w = w^+ + w^-\). During chains of inequalities, universal constants will be denoted by the same letter \(C\) even if their numerical value may change from line to line.

### 2.2. Some important conclusions

Here we list some important conclusions. The first one is the following well-known Hardy-Littlewood-Sobolev inequality.

**Proposition 2.1.** (Hardy-Littlewood-Sobolev inequality, \cite[Theorem 4.3]{23}) Let \(t, r > 1\) and \(0 < \mu < N\) with \(\frac{1}{t} + \frac{\mu}{N} + \frac{1}{r} = 2\), \(f \in L^t(\mathbb{R}^N)\) and \(h \in L^r(\mathbb{R}^N)\). There exists a sharp constant \(C(t, N, \mu, r)\), independent of \(f, h\) such that

\[ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^\mu} dx dy \leq C(t, N, \mu, r) \|f\|_{L^t(\mathbb{R}^N)} \|h\|_{L^r(\mathbb{R}^N)}. \]

If \(t = r = \frac{2N}{2N-\mu}\) then

\[ C(t, N, \mu, r) = C(N, \mu) = \frac{\pi^\frac{2}{\mu} \Gamma(\frac{N}{2} - \frac{\mu}{2})}{\Gamma(\frac{N}{2})} \left\{ \frac{\Gamma(N)}{\Gamma(N-\frac{\mu}{2})} \right\}^{-1+\frac{\mu}{N}}. \]

In this case, there is equality in (2.8) if and only if \(f \equiv (\text{constant})h\) and

\[ h(x) = A(\gamma^2 + |x-a|^2)^{-(2N-\mu)/2}. \]

for some \(A \in \mathbb{C}, 0 \neq \gamma \in \mathbb{R}\) and \(a \in \mathbb{R}^N\).

**Remark 2.2.** For \(u \in H^s(\mathbb{R}^N)\), let \(f = h = |u|^p\), by Hardy-Littlewood-Sobolev inequality,

\[ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^p|u(y)|^p}{|x-y|^\mu} dx dy \]

is well defined for all \(p\) satisfying

\[ 2\mu := \left(\frac{2N-\mu}{N}\right) \leq p \leq \left(\frac{2N-\mu}{N-2s}\right) := 2\mu^*. \]

Next result is a basic inequality, which plays a great role in the latter proof.
Proposition 2.3. (Lemma 2.3) For \( u, v \in L^{\frac{2N}{N-\mu}}(\mathbb{R}^N) \), we have
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^p |v(y)|^p}{|x-y|^\mu} \, dx \, dy \leq \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^p |u(y)|^p}{|x-y|^\mu} \, dx \, dy \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x)|^p |v(y)|^p}{|x-y|^\mu} \, dx \, dy \right)^{\frac{1}{p}},
\]
where \( \mu \in (0, N) \) and \( p \in [2\mu, 2^*_\mu] \).

2.3. Abstract critical point theorems. We will prove Theorem 1.3 and Theorem 1.4 by the following abstract critical point theorems respectively.

Theorem 2.4. (Mountain pass theorem, [37, Theorem 2.10]) Let \( X \) be a Banach space, \( J \in C^1(X, \mathbb{R}) \), \( e \in X \) and \( r > 0 \) be such that \( ||e|| > r \) and \( \frac{d}{dt} J(\gamma(t)) \bigg|_{t=0} \neq 0 \).

If \( J \) satisfies the \((PS)_c\) condition with
\[
c := \inf_{\gamma \in \Gamma, t \in [0, 1]} J(\gamma(t)),
\]
\( \Gamma := \{ \gamma \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) = e \} \).

Then \( c \) is a critical value of \( J \).

Theorem 2.5. (Linking theorem, [37, Theorem 2.12]) Let \( X \) be a real Banach space with \( X = V \oplus W \), where \( V \) is finite dimensional. Suppose \( J \in C^1(X, \mathbb{R}) \) and

(i) There are constants \( \rho, \alpha > 0 \) such that \( J|_{\partial B_\rho \cap W} \geq \alpha \), and
(ii) There is an \( e \in \partial B_\rho \cap W \) and constants \( R_1, R_2 > \rho \) such that \( J|_{\partial Q} \leq 0 \), where
\[
Q = (\overline{B_{R_1}} \cap V) \oplus \{ re, 0 < r < R_2 \}.
\]

Then \( J \) possesses a \((PS)_c\) sequence where \( c \geq \alpha \) can be characterized as
\[
c = \inf_{h \in \Gamma} \max_{u \in Q} J(h(u)),
\]
where
\[
\Gamma = \{ h \in C(\overline{Q}, X) : h = \text{id on } \partial Q \}.
\]

Remark 2.6. Here \( \partial Q \) is the boundary of \( Q \) relative to the space \( V \oplus \text{span}\{e\} \), and when \( V = \{0\} \), this theorem refers to the usual mountain pass Theorem. We recall that if \( J|_V \leq 0 \) and \( J(u) \leq 0, \forall u \in V \oplus \text{span}\{e\} \) with \( ||u|| \geq R \), then \( J \) verifies (ii) for \( R \) large enough. Fixed \( k \in \mathbb{N} \), define the following subspaces
\[
V = \text{span} \{ (0, \varphi_{1,s}), (\varphi_{1,s}, 0), (0, \varphi_{2,s}), (\varphi_{2,s}, 0), \ldots, (0, \varphi_{k-1,s}), (\varphi_{k-1,s}, 0) \}
\]
and
\[
W = V^\perp = (P_k)^2.
\]
3. Case $1: \xi_1 = \xi_2 = 0$, $1 < p, q < 2^*_\mu$

**Proof of Theorem 1.2** Let $\Omega$ be a bounded domain and suppose that

\[(3.1)\]

$$b \geq 0,$$

\[(3.2)\]

$$\mu_2 < \lambda_{1,s}.$$

Consider the function $I : Y(\Omega) \to \mathbb{R}$ defined by

\[(3.3)\]

$$I(U) := \frac{1}{2}\|U\|_Y^2 - \frac{1}{2} \int_\Omega (AU, U)_{\mathbb{R}^2} dx.$$

We shall minimize the functional $I$ restricted to the set

$$M := \{U = (u, v) \in Y(\Omega) : \int_\Omega \int_\Omega \frac{|u^+|^{p}|v^+|^{q}}{|x-y|^\mu} dxdy = 1\}.$$

By virtue of (3.2) the embedding $X(\Omega) \hookrightarrow L^2(\Omega)$ (with the sharp constant $\lambda_{1,s}$), we have

\[(3.4)\]

$$I(U) \geq \frac{1}{2}\min \left\{1, \left(1 - \frac{\mu_2}{\lambda_{1,s}}\right)\right\} \|U\|_Y^2 \geq 0.$$

Define

\[(3.5)\]

$$I_0 := \inf_{\mathcal{M}} I,$$

and let $(U_n) = (u_n, v_n) \subset \mathcal{M}$ be a minimizing sequence for $I_0$. Then $I(U_n) = I_0 + o_n(1) \leq C$, for some $C > 0$ (where $o_n(1) \to 0$, as $n \to \infty$) and consequently by (3.4), we get

\[(3.6)\]

$$[u_n]_s^2 + [v_n]_s^2 = \|u_n\|_X^2 + \|v_n\|_X^2 = \|U_n\|_Y^2 \leq C'.$$

Hence, there are two subsequences of $\{u_n\} \subset X(\Omega)$ and $\{v_n\} \subset X(\Omega)$ (that we will still label as $u_n$ and $v_n$) such that $U_n = (u_n, v_n)$ converges to some $U = (u, v)$ in $Y(\Omega)$ weakly and

\[(3.7)\]

$$[u]_s^2 \leq \liminf_n \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{N+2s}} dxdy,$$

\[(3.8)\]

$$[v]_s^2 \leq \liminf_n \int_{\mathbb{R}^{2N}} \frac{|v_n(x) - v_n(y)|^2}{|x-y|^{N+2s}} dxdy,$$

Now we will show that $U := (u, v) \in \mathcal{M}$. Indeed, since $(U_n) \subset \mathcal{M}$, we have

\[(3.9)\]

$$\int_\Omega \int_\Omega \frac{|u_n^+(x)|^{p}|v_n^+(y)|^{q}}{|x-y|^\mu} dxdy = 1.$$

In view of the compact embedding $X(\Omega) \hookrightarrow L^r(\Omega)$ for all $r < 2^*_s = \frac{2N}{N-2s}$, as $1 < p, q < 2^*_\mu$, we get

\[(3.10)\]

$$\int_\Omega \int_\Omega \frac{|u^+(x)|^{p}|v^+(y)|^{q}}{|x-y|^\mu} dxdy \to \int_\Omega \int_\Omega \frac{|u^+(x)|^{p}|v^+(y)|^{q}}{|x-y|^\mu} dxdy, \text{ as } n \to \infty,$$

thus $\int_\Omega \int_\Omega \frac{|u^+(x)|^{p}|v^+(y)|^{q}}{|x-y|^\mu} dxdy = 1$ and consequently $U := (u, v) \in \mathcal{M}$ with $u, v \neq 0$. We now show that $U = (u, v)$ is a minimizer for $I$ on $\mathcal{M}$ and both components $u, v$ are nonnegative. By passing to the limit in $I(U_n) = I_0 + o_n(1)$, where $o_n(1) \to 0$ as $n \to \infty$, using (3.7), (3.8) and the strong convergence of $(u_n, v_n)$ to $(u, v)$ in $(L^2(\Omega))^2$, as $n \to \infty$, we conclude that
\( I(U) \leq I_0. \) Moreover, since \( U \in \mathcal{M} \) and \( I_0 = \inf_{\mathcal{M}} I \leq I(U) \), we achieve that \( I(U) = I_0. \) This proves the minimality of \( U \in \mathcal{M} \). On the other hand, let

\[
G(U) = \int_{\Omega} \int_{\Omega} \frac{|u^+(x)||v^+(y)|^q}{|x-y|^\mu} dx dy - 1,
\]

where \( U(u, v) \in Y(\Omega) \). Note that \( G \in C^1 \) and since \( U \in \mathcal{M} \),

\[
G'(U)U = (p + q) \int_{\Omega} \int_{\Omega} \frac{|u^+(x)||v^+(y)|^q}{|x-y|^\mu} dx dy = p + q \neq 0,
\]

hence, by Lagrange Multiplier Theorem, there exists a multiplier \( \zeta \in \mathbb{R} \) such that

\[
I'(U)(\varphi, \psi) = \zeta G'(U)(\varphi, \psi), \forall (\varphi, \psi) \in Y(\Omega). \tag{3.11}
\]

Taking \((\varphi, \psi) = (u^-, v^-) := U^-\) in (3.11), we get

\[
\|U^-\|^2_Y = \int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^{N+2s}} dx dy + \int_{\mathbb{R}^{2N}} \frac{v^+(x)v^-(y) + v^-(x)v^+(y)}{|x-y|^{N+2s}} dx dy + \int_{\Omega} (AU^-, U^-) \, dx.
\]

Dropping this formula into the expression of \( I(U^-) \), we have

\[
I(U^-) = \frac{b}{2} \int_{\Omega} (v^+u^- + u^+v^-) dx + \frac{1}{2} \int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^{N+2s}} dx dy - \frac{1}{2} \int_{\mathbb{R}^{2N}} \frac{v^+(x)v^-(y) + v^-(x)v^+(y)}{|x-y|^{N+2s}} dx dy \leq 0,
\]

since \( b \geq 0, u^- \leq 0 \) and \( u^+ \geq 0 \). On the other hand,

\[
I(U^-) \geq \frac{1}{2} \min \left\{ 1, (1 - \frac{\mu^2}{\lambda_{1,s}}) \right\} \|U^-\|^2_Y \geq 0,
\]

we get \( U^- = (u^-, v^-) = (0, 0) \) and therefore \( u, v \geq 0 \). We now prove the existence of a positive solution to (1.1). Using again (3.11), we see that

\[
\|U\|^2_Y - \int_{\Omega} (AU, U) \, dx - \zeta (p + q) = 0
\]

and since \( U \in \mathcal{M} \), we conclude that

\[
I_0 = I(U) = \frac{\zeta (p + q)}{2} > 0,
\]
Then by (3.11), $U$ satisfies the following system, weakly,
\[
\begin{cases}
(-\Delta)^s u = au + bv + \frac{2pI_0}{p+q} \int_\Omega \int_\Omega \frac{|u|^{p-1}|v|^q}{|x-y|^\mu} dx dy, & \text{in } \Omega; \\
(-\Delta)^s v = bu + cv + \frac{2qI_0}{p+q} \int_\Omega \int_\Omega \frac{|u|^p|v|^{q-1}}{|x-y|^\mu} dx dy, & \text{in } \Omega; \\
u = v = 0, & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}
\]

Now using the homogeneity of system, we get $\tau > 0$ such that $W = (I_0)^\tau U$ is a solution of (1.1). Since $b \geq 0$ and $u, v \geq 0$ we get, in weak sense
\[
\begin{cases}
(-\Delta)^s u \geq au, & \text{in } \Omega; \\
(-\Delta)^s v \geq cv, & \text{in } \Omega; \\
u \geq 0, v \geq 0 & \text{in } \Omega; \\
u = v = 0, & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}
\]

By the strong maximum principle (cf. [22], Theorem 2.5), we conclude $u, v > 0$ in $\Omega.$

4. Case 2: $\xi_1 = \xi_2 = 0, p = q = 2^*_s$.

In this case, we have the function $J_s : Y(\Omega) \to \mathbb{R}$ by setting
\[
J_s(U) = \frac{1}{2} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2 + |v(x) - v(y)|^2}{|x-y|^{N+2s}} dx dy
- \frac{1}{2} \int_{\mathbb{R}^N} (A(u, v), (u, v))_{\mathbb{R}^2} dx - \frac{1}{2^*_s} \int_\Omega \int_\Omega \frac{|u^+(x)|^{2^*_s} |v^+(y)|^{2^*_s}}{|x-y|^\mu} dx dy,
\]
whose Fréchet derivative is given by
\[
J'_s(u, v)(\varphi, \psi) = \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y)) + (v(x) - v(y))(\psi(x) - \psi(y))}{|x-y|^{N+2s}} dx dy
- \int_\Omega (A(u, v), (\varphi, \psi))_{\mathbb{R}^2} dx - \int_\Omega \int_\Omega \frac{|u^+(x)|^{2^*_s-1} |v^+(y)|^{2^*_s}}{|x-y|^\mu} \varphi dx dy
- \int_\Omega \int_\Omega \frac{|u^+(x)|^{2^*_s} |v^+(y)|^{2^*_s-1}}{|x-y|^\mu} \psi dx dy,
\]
for every $(\varphi, \psi) \in Y(\Omega)$.

4.1. Minimizers and some estimates. Let
\[
S_s := \inf_{u \in X(\Omega) \setminus \{0\}} S_s(u),
\]
where
\[
S_s(u) := \frac{\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dx dy}{(\int_{\mathbb{R}^N} |u(x)|^{2^*_s} dx)^{\frac{2^*_s}{2^*_s}}}
\]
is the associated Rayleigh quotient. Define the following related minimizing problems as
\[
S_s^H := \inf_{u \in X(\Omega) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dx dy}{(\int_{\Omega} \int_\Omega \frac{|u(x)|^{2^*_s} |u(y)|^{2^*_s}}{|x-y|^\mu} dx dy)^{\frac{2^*_s}{2^*_s}}}
\]
and

\[
\widetilde{S}^H_s = \inf_{(u,v)\in \mathcal{Y}(\Omega)\setminus \{(0,0)\}} \frac{\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2 + \beta|v(x) - v(y)|^2}{|x-y|^{N+2s}} dx dy}{(\int_\Omega \int_\Omega \frac{|u(x)|^2}{|x-y|^{\mu}} dx dy)^{\frac{1}{\mu}}},
\]

**Proposition 4.1.**

(i) ([8, Lemma 2.15]) The constant $S^H_s$ is achieved by $u$ if and only if $u$ is of the form

\[
C(\frac{t}{t^2 + |x-x_0|^2})^{\frac{N-2s}{2}} x, \quad x \in \mathbb{R}^N,
\]

for some $x_0 \in \mathbb{R}^N, C > 0$ and $t > 0$. Also it satisfies

\[
(-\Delta)^s u = \left( \int_{\mathbb{R}^N} \frac{|u|^2}{|x-y|^{\mu}} dy \right) |u|^{2\mu-2} u \text{ in } \mathbb{R}^N.
\]

and this characterization of $u$ also provides the minimizers for $S_s$.

(ii) ([15, Lemma 2.5]) $S^H_s = \frac{S_s}{C(N,\mu)^{\frac{1}{\mu}}}$.

(iii) ([15, Lemma 2.6]) $\widetilde{S}^H_s = 2S^H_s$.

Now we will construct auxiliary functions and make some estimates with the help of Proposition 4.1. From [31], consider the family of function $\{U_\epsilon\}$ defined as

\[
U_\epsilon(x) = \epsilon^{-\frac{N-2s}{2}} u^\ast \left( \frac{x}{\epsilon} \right), \quad x \in \mathbb{R}^N,
\]

where $u^\ast(x) = \overline{u} \left( \frac{x}{\|u\|^2} \right)$, $\overline{u}(x) = \frac{\overline{u}(x)}{\|u\|^2}$ and $\overline{u} = \alpha (\beta^2 + |x|^2)^{-\frac{N-2s}{2}}$ with $\alpha \in \mathbb{R}\setminus\{0\}$ and $\beta > 0$ are fixed constants. Then for each $\epsilon > 0$, $U_\epsilon$ satisfies

\[
(-\Delta)^s u = |u|^{2s-2} u, \text{ in } \mathbb{R}^N,
\]

in addition,

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|U_\epsilon(x) - U_\epsilon(y)|^2}{|x-y|^{N+2s}} dx dy = \int_{\mathbb{R}^N} |U_\epsilon|^2 dx = S^N_s.
\]

Without loss of generality, we assume $0 \in \Omega$ and fix \(\delta > 0\) such that $B_{4\delta} \subset \Omega$. Let $\eta \in C^\infty(\mathbb{R}^N)$ be such that $0 \leq \eta \leq 1$ in $\mathbb{R}^N$, $\eta \equiv 1$ in $B_\delta$ and $\eta \equiv 0$ in $\mathbb{R}^N \setminus B_{2\delta}$. For $\epsilon > 0$, we denote by $u_\epsilon$ the following function

\[
u_\epsilon(x) = \eta(x) U_\epsilon(x),
\]

for $x \in \mathbb{R}^N$. We have the following results for $u_\epsilon$ in [31, Propositions 21, Propositions 22] and [28, Proposition 7.2].

**Proposition 4.2.** Let $s \in (0,1)$ and $N > 2s$. Then, the following estimates hold true as $\epsilon \to 0$:

(i) \(\int_{\mathbb{R}^N} \frac{|u_\epsilon(x) - u_\epsilon(y)|^2}{|x-y|^{N+2s}} dx dy \leq S^N_s + O(\epsilon^{N-2s})\),

(ii) \(\int_{\mathbb{R}^N} |u_\epsilon|^2 dx = S^N_s + O(\epsilon^N)\),
(iii) $\int_{\mathbb{R}^N} |u_\epsilon|^2 \, dx \geq \begin{cases} C_s \epsilon^{2s} + O(\epsilon^{N-2s}), & \text{if } N > 4s; \\ C_s \epsilon^{2s} \log \epsilon + O(\epsilon^{2s}), & \text{if } N = 4s; \\ C_s \epsilon^{N-2s} + O(\epsilon^{2s}), & \text{if } 2s < N < 4s; \end{cases}$

for some positive constant $C_s$ depending on $s$.

(iv) $\int_{\mathbb{R}^N} |u_\epsilon| \, dx = O(\epsilon^{N-2s})$.

**Remark 4.3.** From Proposition 4.1 (ii) and Proposition 4.2 (i), we get

$$\int_{\mathbb{R}^{2N}} \frac{|u_\epsilon(x) - u_\epsilon(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \leq S_{s,\hat{\lambda}}^H + O(\epsilon^{N-2s}) = C(N, \mu) \frac{N-2s}{2s} N \frac{N}{2s} (S_s^H)^{\frac{N}{2s}} + O(\epsilon^{N-2s}).$$

**Proposition 4.4.** ([15, Proposition 2.8]) Let $s \in (0,1)$ and $N > 2s$. Then, the following estimate holds true as $\epsilon \to 0$:

$$\int_\Omega \int_\Omega \frac{|u_\epsilon(x)|^{2\mu} |u_\epsilon(y)|^{2\mu}}{|x - y|^\mu} \, dx \, dy \geq C(N, \mu) \frac{N}{2s} (S_s^H)^{\frac{N}{2s}} - O(\epsilon)\frac{N}{2s}.$$  

Now consider the following minimization problem

$$S_{s,\lambda} = \inf_{v \in X_0(\Omega) \setminus \{0\}} S_{s,\lambda}(v),$$

where

$$S_{s,\lambda}(v) = \frac{\int_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^2}{|x - y|^{N + 2s}} \, dx \, dy - \lambda \int_{\mathbb{R}^N} |v(x)|^2 \, dx}{\left( \int_\Omega \int_\Omega \frac{|v(x)|^{2\mu} |v(y)|^{2\mu}}{|x - y|^\mu} \, dx \, dy \right)^{\frac{1}{2\mu}}}.$$ 

**Lemma 4.5.** Let $N > 2s$ and $s \in (0,1)$. Then the following facts hold.

(i) For $N \geq 4s$,

$$S_{s,\lambda}(u_\epsilon) < S_s^H, \text{ for all } \lambda > 0, \text{ provided } \epsilon > 0 \text{ is sufficiently small}.$$ 

(ii) For $2s < N < 4s$, there exists $\lambda_s > 0$ such that for all $\lambda > \lambda_s$, we have

$$S_{s,\lambda}(u_\epsilon) < S_s^H, \text{ provided } \epsilon > 0 \text{ is sufficiently small}.$$ 

**Proof.** Case 1: $N > 4s$. By Proposition 4.2 (iii), (4.7) and (4.8), we infer

$$S_{s,\lambda}(u_\epsilon) \leq \frac{C(N, \mu) \frac{N}{2s} + O(\epsilon^{N-2s}) - \lambda \epsilon^{2s} + O(\epsilon^{N-2s})}{\left( \int_\Omega \int_\Omega \frac{|v(x)|^{2\mu} |v(y)|^{2\mu}}{|x - y|^\mu} \, dx \, dy \right)^{\frac{1}{2\mu}}} \leq S_s^H - \lambda \epsilon^{2s} + O(\epsilon^{N-2s}) < S_s^H, \text{ if } \lambda > 0, \epsilon > 0 \text{ is sufficiently small}.$$ 

Case 2: $N = 4s$.

$$S_{s,\lambda}(u_\epsilon) \leq \frac{C(N, \mu) \frac{N}{2s} + O(\epsilon^{N-2s}) - \lambda \epsilon^{2s} \log \epsilon + O(\epsilon^{2s})}{\left( \int_\Omega \int_\Omega \frac{|v(x)|^{2\mu} |v(y)|^{2\mu}}{|x - y|^\mu} \, dx \, dy \right)^{\frac{1}{2\mu}}} \leq S_s^H - \lambda \epsilon^{2s} \log \epsilon + O(\epsilon^{2s}) < S_s^H, \text{ if } \lambda > 0, \epsilon > 0 \text{ is sufficiently small}.$$
Case 3: $2s < N < 4s$.

$$S_{s, \lambda}(u) \leq \frac{C(N, \mu) \frac{N-2s}{2s} \mu \mathcal{N}(S^H_2)^N}{C(N, \mu) \frac{N-2s}{2s} \mu \mathcal{N}(S^H_2)^N + O(\epsilon^{N-2s}) - \lambda C_s \epsilon^{N-2s} + O(\epsilon^2)} + O(1)$$

$$\leq S^H_2 + \epsilon^{N-2s}(O(1) - \lambda C_s) + O(\epsilon^2),$$

for all $\lambda > 0$ large enough ($\lambda \geq \lambda_s$), $\epsilon > 0$ sufficiently small. \hfill \Box

4.2. Compactness convergence.

**Proposition 4.6.** Let $s \in (0, 1), N > 2s$ and $0 < \mu < N$. If $\{u_n\}, \{v_n\}$ are bounded sequences in $L^{\frac{2N}{N-\mu}}(\Omega)$ such that $u_n \to u, v_n \to v$ almost everywhere in $\Omega$ as $n \to \infty$, we have

$$\int \int |u_n(x)|^2 \mu |v_n(y)|^2 \mu \frac{dx dy}{|x-y|^\mu} - \int \int |u_n - u(x)|^2 \mu |v_n - v(y)|^2 \mu \frac{dx dy}{|x-y|^\mu},$$

as $n \to \infty$. The Hardy-Littlewood-Sobolev inequality implies that

$$\int \int |u_n(x)|^2 \mu - |u_n - u(x)|^2 \mu \frac{dx}{|x-y|^\mu} \to \int \int |u(x)|^2 \mu \frac{dx}{|x-y|^\mu},$$

and

$$\int \int |v_n(y)|^2 \mu - |v_n - v(y)|^2 \mu \frac{dx}{|x-y|^\mu} \to \int \int |v(y)|^2 \mu \frac{dx}{|x-y|^\mu},$$

in $L^{\frac{2N}{N-\mu}}(\Omega)$ as $n \to \infty$. On the other hand, we notice that

$$\int \int |u_n(x)|^2 \mu |v_n(y)|^2 \mu \frac{dx dy}{|x-y|^\mu} - \int \int |u_n - u(x)|^2 \mu |v_n - v(y)|^2 \mu \frac{dx dy}{|x-y|^\mu}$$

$$= \int \int |u_n(x)|^2 \mu - |u_n - u(x)|^2 \mu |v_n(y)|^2 \mu - |v_n - v(y)|^2 \mu \frac{dx dy}{|x-y|^\mu}$$

$$+ \int \int |u_n(x)|^2 \mu - |u_n - u(x)|^2 \mu |v_n - v(y)|^2 \mu \frac{dx dy}{|x-y|^\mu}$$

$$+ \int \int |v_n(y)|^2 \mu - |v_n - v(y)|^2 \mu |u_n - u(y)|^2 \mu \frac{dx dy}{|x-y|^\mu}.$$
Lemma 4.7. (Boundedness) The $\text{(PS)}_c$ sequence $\{(u_n, v_n)\} \subset Y(\Omega)$ is bounded.

Proof. From (2.5) and the definition of $\lambda_{1,s}$, we have

$$C + C\| (u_n, v_n) \|_Y \geq J_s (u_n, v_n) - \frac{1}{2} \cdot \frac{1}{2^n} J'_s (u_n, v_n) (u_n, v_n)$$

$$= \left( \frac{1}{2} - \frac{1}{2^n} \right) \| (u_n, v_n) \|_Y^2$$

$$- \left( \frac{1}{2} - \frac{1}{2^n} \right) \int_{\mathbb{R}^N} (A(u_n, v_n), (u_n, v_n)) dx$$

$$\geq \left( \frac{1}{2} - \frac{1}{2^n} \right) (1 - \frac{\mu^2}{\lambda_{1,s}}) \| (u_n, v_n) \|_Y^2.$$

Since $\mu_2 < \lambda_{1,s}$, the assertion follows. \qed

Lemma 4.8. If $\{(u_n, v_n)\} \subset Y(\Omega)$ be a $(\text{PS})_c$ sequence for the functional $J_s$ with

$$c < \frac{N + 2s - \mu}{2N - \mu} (S_H^s)^{\frac{2N - \mu}{N + 2s - \mu}},$$

then $\{(u_n, v_n)\}$ has a convergent subsequence.

Proof. Let $(u_0, v_0)$ be the weak limit of $\{(u_n, v_n)\}$ and define $w_n := u_n - u_0, z_n := v_n - v_0$, then we know $w_n \rightharpoonup 0, z_n \rightharpoonup 0$ in $H^s(\mathbb{R}^N)$ and $w_n \to 0$ a.e. in $\mathbb{R}^N, z_n \to 0$ a.e. in $\mathbb{R}^N$. Moreover, by [26, Lemma 5] and Proposition 4.6, we know

$$\| u_n \|_X^2 = \| w_n \|_X^2 + \| u_0 \|_X^2 + o_n(1),$$

$$\| v_n \|_X^2 = \| z_n \|_X^2 + \| v_0 \|_X^2 + o_n(1),$$

and

$$\int_{\Omega} \int_{\Omega} \frac{|u_n^+(x)|^{2^*_p}|v_n^+(y)|^{2^*_p}}{|x-y|^\mu} dxdy = \int_{\Omega} \int_{\Omega} \frac{|w_n^+(x)|^{2^*_p}|z_n^+(y)|^{2^*_p}}{|x-y|^\mu} dxdy$$

$$+ \int_{\Omega} \int_{\Omega} \frac{|u_0^+(x)|^{2^*_p}|v_0^+(y)|^{2^*_p}}{|x-y|^\mu} dxdy + o_n(1).$$
Consequently, we have
\[
c \leftarrow J_s(u_n, v_n) = \frac{1}{2} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2 + |v(x) - v(y)|^2}{|x - y|^{N + 2s}} \, dx dy - \frac{1}{2} \int_{\mathbb{R}^N} (A(u_n, v_n), (u_n, v_n))_{\mathbb{R}^2} dx - \frac{1}{2 \mu} \int_{\Omega} \int_{\Omega^n} \frac{|u_n^+(x)|^{2\mu} |v_n^+(y)|^{2\mu}}{|x - y|^{\mu}} \, dx dy
\]
\[
\geq \frac{1}{2} \left( \int_{\mathbb{R}^N} \frac{|w_n(x) - w_n(y)|^2}{|x - y|^{N + 2s}} \, dx dy + \int_{\mathbb{R}^N} \frac{|u_0(x) - u_0(y)|^2}{|x - y|^{N + 2s}} \, dx dy \right)
+ \int_{\mathbb{R}^N} \frac{|z_n(x) - z_n(y)|^2}{|x - y|^{N + 2s}} \, dx dy + \left( \int_{\mathbb{R}^N} |z_n|^2 dx + \int_{\mathbb{R}^N} |u_0|^2 dx \right)
- \frac{\mu_2}{2} \left( \int_{\mathbb{R}^N} |w_n|^2 dx + \int_{\mathbb{R}^N} \frac{|v_n|^2}{|x - y|^{N \mu}} dx \right)
- \frac{1}{2 \mu} \left( \int_{\Omega} \int_{\Omega^n} \frac{|w_n^+(x)|^{2\mu} |z_n^+(y)|^{2\mu}}{|x - y|^{\mu}} \, dx dy + \int_{\Omega} \int_{\Omega^n} \frac{|u_0^+(x)|^{2\mu} |v_0^+(y)|^{2\mu}}{|x - y|^{\mu}} \, dx dy \right)
+ o_n(1),
\]
so
\[
c \geq J_s(u_0, v_0) + \frac{1}{2} \left( \int_{\mathbb{R}^N} \frac{|w_n(x) - w_n(y)|^2}{|x - y|^{N + 2s}} \, dx dy \right)
+ \int_{\mathbb{R}^N} \frac{|z_n(x) - z_n(y)|^2}{|x - y|^{N + 2s}} \, dx dy - \frac{\mu_2}{2} \left( \int_{\mathbb{R}^N} |w_n|^2 dx + \int_{\mathbb{R}^N} |z_n|^2 dx \right)
- \frac{1}{2 \mu} \left( \int_{\Omega} \int_{\Omega^n} \frac{|w_n^+(x)|^{2\mu} |z_n^+(y)|^{2\mu}}{|x - y|^{\mu}} \, dx dy + \int_{\Omega} \int_{\Omega^n} \frac{|u_0^+(x)|^{2\mu} |v_0^+(y)|^{2\mu}}{|x - y|^{\mu}} \, dx dy \right)
+ o_n(1).
\]

From the boundedness of Palais-Smale sequences (see Lemma 4.7) and compact embedding theorems, passing to a subsequence if necessary, there exists \((u_0, v_0) \in Y(\Omega)\), such that \((u_n, v_n) \rightharpoonup (u_0, v_0)\) weakly in \(Y(\Omega)\) as \(n \to \infty\), \((u_n, v_n) \to (u_0, v_0)\) a.e. in \(\Omega\) and strongly in \(L^r(\Omega)\) for \(1 \leq r < 2^*_s\). Since \(|u_n|^{2^*_s} \to |u_0|^{2^*_s}, |v_n|^{2^*_s} \to |v_0|^{2^*_s}| in L^{2N/N-s}(\Omega)\) as \(n \to \infty\), by the Hardy-Littlewood-Sobolev inequality, the Riesz potential defines a linear continuous map from \(L^{\frac{2N}{N-s}}(\Omega)\) to \(L^{\frac{2N}{N-s}}(\Omega)\), hence
\[
\int_{\Omega} \frac{|u_n^+(y)|^{2\mu}}{|x - y|^{N \mu}} dy \to \int_{\Omega} \frac{|u_0^+(y)|^{2\mu}}{|x - y|^{N \mu}} dy in L^{\frac{2N}{N-s}}(\Omega),
\]
\[
\int_{\Omega} \frac{|v_n^+(y)|^{2\mu}}{|x - y|^{N \mu}} dy \to \int_{\Omega} \frac{|v_0^+(y)|^{2\mu}}{|x - y|^{N \mu}} dy in L^{\frac{2N}{N-s}}(\Omega).
\]
as \(n \to \infty\). Combining these with the fact that
\[
|u_n|^{2^*_s-1} \to |u_0|^{2^*_s-1} in L^{N+2s/N-s}(\Omega),
|v_n|^{2^*_s-1} \to |v_0|^{2^*_s-1} in L^{N+2s/N-s}(\Omega).
\]
as \( n \to \infty \), we obtain

\[
\int_{\Omega} \frac{|u_n^+(y)|^{2s} |v_n^+(x)|^{2s-1} dy}{|x-y|^{\mu}} \to \int_{\Omega} \frac{|u_0^+(y)|^{2s} |v_0^+(x)|^{2s-1} dy}{|x-y|^{\mu}} \text{ in } L^{\frac{2N}{N+2s}}(\Omega),
\]

\[
\int_{\Omega} \frac{|u_n^+(y)|^{2s} |u_n^+(x)|^{2s-1} dy}{|x-y|^{\mu}} \to \int_{\Omega} \frac{|u_0^+(y)|^{2s} |u_0^+(x)|^{2s-1} dy}{|x-y|^{\mu}} \text{ in } L^{\frac{2N}{N+2s}}(\Omega),
\]

as \( n \to \infty \). Since, for any \( \varphi, \psi \subset X(\Omega) \),

\[
0 \leftarrow J_n'(u_n, v_n)(\varphi, \psi)
\]

\[
= \int_{\mathbb{R}^{2N}} \frac{(u_n(x) - u_n(y))(\varphi(x) - \varphi(y)) + (v_n(x) - v_n(y))(\psi(x) - \psi(y))}{|x-y|^{N+2s}}
\]

\[
- \int_{\Omega} (A(u_n, v_n), (\varphi, \psi))_{\mathbb{R}^2} dx - \int_{\Omega} \int_{\Omega} \frac{|v_n^+(y)|^{2s} |u_n^+(x)|^{2s-1} \varphi(x)}{|x-y|^{\mu}} dy dx
\]

\[
- \int_{\Omega} \int_{\Omega} \frac{|u_n^+(y)|^{2s} |v_n^+(x)|^{2s-1} \psi(x)}{|x-y|^{\mu}} dy dx = 0,
\]

Passing to the limit as \( n \to \infty \), we obtain

\[
\int_{\mathbb{R}^{2N}} \frac{(u_0(x) - u_0(y))(\varphi(x) - \varphi(y)) + (v_0(x) - v_0(y))(\psi(x) - \psi(y))}{|x-y|^{N+2s}}
\]

\[
- \int_{\Omega} (A(u_0, v_0), (\varphi, \psi))_{\mathbb{R}^2} dx - \int_{\Omega} \int_{\Omega} \frac{|v_0^+(y)|^{2s} |u_0^+(x)|^{2s-1} \varphi(x)}{|x-y|^{\mu}} dy dx
\]

\[
- \int_{\Omega} \int_{\Omega} \frac{|u_0^+(y)|^{2s} |v_0^+(x)|^{2s-1} \psi(x)}{|x-y|^{\mu}} dy dx = 0,
\]

which means that \((u_0, v_0)\) is a weak solution of the problem (1.1).

Taking \( \varphi = u_0, \psi = v_0 \) as a test function in equation (4.16), we have

\[
\int_{\mathbb{R}^{2N}} \frac{|u_0(x) - u_0(y)|^2 + |v_0(x) - v_0(y)|^2}{|x-y|^{N+2s}}
\]

\[
= \int_{\Omega} (A(u_0, v_0), (u_0, v_0))_{\mathbb{R}^2} dx + 2 \int_{\Omega} \int_{\Omega} \frac{|u_0^+(x)|^{2s} |v_0^+(y)|^{2s}}{|x-y|^{\mu}} dy dx,
\]

so for \( 0 < \mu < N \),

\[
J_n(u_0, v_0) = \frac{N + 2s - \mu}{2N - \mu} \int_{\Omega} \int_{\Omega} \frac{|u_0^+(x)|^{2s} |v_0^+(y)|^{2s}}{|x-y|^{\mu}} dy dx \geq 0.
\]

Using (4.14), (4.17) and \( \int_{\mathbb{R}^N} |w_n|^2 dx \to 0, \int_{\mathbb{R}^N} |z_n|^2 dx \to 0 \), as \( n \to \infty \), we get

\[
c \geq \frac{1}{2} \left( \int_{\mathbb{R}^{2N}} \frac{|w_n(x) - w_n(y)|^2}{|x-y|^{N+2s}} dx dy + \int_{\mathbb{R}^{2N}} \frac{|z_n(x) - z_n(y)|^2}{|x-y|^{N+2s}} dx dy \right)
\]

\[
- \frac{1}{2s} \int_{\Omega} \int_{\Omega} \frac{|u_n^+(x)|^{2s} |v_n^+(y)|^{2s}}{|x-y|^{\mu}} dy dx + o_n(1).
\]
Since \((u_0, v_0)\) is a weak solution of (1.1), \((u_0, v_0)\) must be a critical point of \(J_s\) which gives 
\[
\langle J'_s(u_0, v_0), (u_0, v_0) \rangle = 0,
\]
hence 
\[
o_n(1) = \langle J'_s(u_n, v_n), (u_n, v_n) \rangle
= \langle J'_s(u_0, v_0), (u_0, v_0) \rangle + \int_{\mathbb{R}^{2N}} \frac{|w_n(x) - w_n(y)|^2}{|x - y|^{N+2s}} dxdy 
+ \int_{\mathbb{R}^{2N}} \frac{|z_n(x) - z_n(y)|^2}{|x - y|^{N+2s}} dxdy - 2 \int_{\Omega} \int_{\Omega} \frac{|w_n^+(x)|^{2^*_n} |z_n^+(y)|^{2^*_n}}{|x - y|^\mu} dxdy + o_n(1)
\]
(4.19) 
\[
= \int_{\mathbb{R}^{2N}} \frac{|w_n(x) - w_n(y)|^2}{|x - y|^{N+2s}} dxdy + \int_{\mathbb{R}^{2N}} \frac{|z_n(x) - z_n(y)|^2}{|x - y|^{N+2s}} dxdy
- 2 \int_{\Omega} \int_{\Omega} \frac{|w_n^+(x)|^{2^*_n} |z_n^+(y)|^{2^*_n}}{|x - y|^\mu} dxdy + o_n(1).
\]
From (4.19), we know there exists a nonnegative constant \(m\) such that 
\[
\int_{\mathbb{R}^{2N}} \frac{|w_n(x) - w_n(y)|^2}{|x - y|^{N+2s}} dxdy + \int_{\mathbb{R}^{2N}} \frac{|z_n(x) - z_n(y)|^2}{|x - y|^{N+2s}} dxdy \to m,
\]
and 
\[
\int_{\Omega} \int_{\Omega} \frac{|w_n^+(x)|^{2^*_n} |z_n^+(y)|^{2^*_n}}{|x - y|^\mu} dxdy \to \frac{m}{2},
\]
as \(n \to \infty\). Thus from (4.18), we obtain 
\[
c \geq \frac{N + 2s - \mu}{4N - 2\mu}m,
\]
(4.20) 
By the definition of the best constant \(\tilde{S}_s^H\), we have 
\[
\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2 + |v(x) - v(y)|^2}{|x - y|^{N+2s}} dxdy \geq \tilde{S}_s^H \left( \int_{\Omega} \int_{\Omega} \frac{|u(x)|^{2^*_n} |v(y)|^{2^*_n}}{|x - y|^\mu} dxdy \right)^{\frac{N-2s}{N+2s}}
\]
which yields \(m \geq \tilde{S}_s^H \left( \frac{2}{N+2s-\mu} \right)^{\frac{N-2s}{2N+2s-\mu}}\). Thus we have either \(m = 0\) or 
\[
m \geq \frac{1}{2N+2s-\mu} \left( \tilde{S}_s^H \right)^{\frac{N-2s}{N+2s-\mu}},
\]
If \(m \geq \frac{1}{2N+2s-\mu} \left( \tilde{S}_s^H \right)^{\frac{N-2s}{N+2s-\mu}}\), by Proposition 4.1 (iii), then we obtain from (4.20) that 
\[
c \geq \frac{N + 2s - \mu}{2N - \mu} \left( \tilde{S}_s^H \right)^{\frac{N-2s}{N+2s-\mu}},
\]
which contradicts with the fact that \(c < \frac{N+2s-\mu}{2N-\mu} \left( \tilde{S}_s^H \right)^{\frac{N-2s}{N+2s-\mu}}\). Thus \(m = 0\), and 
\[
\|(u_n - u_0, v_n - v_0)\|_Y \to 0,
\]
as \(n \to \infty\). This completes the proof of Lemma 4.8. \(\square\)

4.3. Mountain Pass geometry.

**Lemma 4.9.** Suppose \(\mu_2 < \lambda_{1,s}\). The functional \(J_s\) satisfies

(i) There exist \(\beta, \rho > 0\) such that \(J_s(u, v) \geq \beta\) if \(\|(u, v)\|_Y = \rho\);

(ii) there exists \((e_1, e_2) \in Y(\Omega) \setminus \{(0, 0)\}\) with \(\|(e_1, e_2)\|_Y > \rho\) such that \(J_s(e_1, e_2) \leq 0\).
Proof. (i) From the definition of $S^H_s$, we get
\begin{equation}
\int_\Omega \int_\Omega \frac{|u^+(x)|^{2\mu} |v^+(y)|^{2\mu}}{|x-y|^\mu} dxdy \leq \frac{1}{(S^H_s)^{2\mu}} \|(u,v)\|_Y^{2\mu}.
\end{equation}
Combining with (2.5) and the definition of $\lambda_{1,s}$, we get
\[ J_s(u,v) \geq \frac{1}{2} \left( 1 - \frac{\lambda_2}{\lambda_{1,s}} \right) \|(u,v)\|_Y^2 - \frac{1}{2S^H_s} \|(u,v)\|_Y^{2\mu}.
\]
Since $2 < 2\mu$ and thus, some $\beta, \rho > 0$ can be chosen such that $J_s(u,v) \geq \beta$ for $\|(u,v)\|_Y = \rho$.
(ii) Choose $(\tilde{u}_0, \tilde{v}_0) \in Y(\Omega) \setminus \{(0,0)\}$ with $\tilde{u}_0 \geq 0, \tilde{v}_0 \geq 0$ a.e. and $\tilde{u}_0 \tilde{v}_0 \neq 0$. Then
\[ J_s(t\tilde{u}_0, t\tilde{v}_0) = \frac{t^2}{2} \int_{\mathbb{R}^{2N}} \frac{|\tilde{u}_0(x) - \tilde{u}_0(y)|^2 + |\tilde{v}_0(x) - \tilde{v}_0(y)|^2}{|x-y|^{N+2s}} dxdy - \frac{t^2}{2} \int_{\mathbb{R}^N} (A(\tilde{u}_0, \tilde{v}_0, \tilde{u}_0, \tilde{v}_0)) dx - \frac{t^2}{2S^H_s} \int_{\Omega} \int_{\Omega} \frac{|\tilde{u}_0(x)|^{2\mu} |\tilde{v}_0(y)|^{2\mu}}{|x-y|^\mu} dxdy,
\]
by choosing $t > 0$ sufficiently large, the assertion follows. This concludes the proof.

Lemma 4.10. If $(u,v) \in Y(\Omega)$ is a critical point of $J_s$, then $(u^-, v^-) = (0,0)$.

Proof. By choosing $\varphi := u^- \in X(\Omega)$ and $\psi := v^- \in X(\Omega)$ as test functions in (4.1) and using the elementary inequality
\[ (w_1 - w_2)(w_1^- - w_2^-) \geq (w_1^- - w_2^-)^2, \text{ for all } w_1, w_2 \in \mathbb{R},
\]
we obtain
\[ \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(u^-(x) - u^-(y)) + (v(x) - v(y))(v^-(x) - v^-(y))}{|x-y|^{N+2s}} dxdy \geq \int_{\mathbb{R}^{2N}} \frac{(u^- - u^-)(y)^2 + (v^- - v^-)^2}{|x-y|^{N+2s}} dxdy.
\]
Now, note that, since $b \geq 0$ and $w^- \leq 0$ and $w^+ \geq 0$, it holds
\[ \int_{\mathbb{R}^N} (A(u,v),(u^-, v^-)) dx \leq \int_{\mathbb{R}^N} (A(u^- , v^-),(u^-, v^-)) dx.
\]
In fact, it follows
\[ (A(u,v),(u^-, v^-)) = (A(u^- , v^-),(u^-, v^-)) + b(v^+ u^- + u^+ v^-),
\]
\[ \leq (A(u^- , v^-),(u^-, v^-)).
\]
In turn, from the formula for $J'_s(u,v)(u^-, v^-)$, it follows that
\[ J'_s(u,v)(u^-, v^-) \geq \int_{\mathbb{R}^{2N}} \frac{(u^- - u^-)(y)^2 + (v^- - v^-)^2}{|x-y|^{N+2s}} dxdy - \int_{\Omega} (A(u^- , v^-),(u^-, v^-)) dx \geq I(u^-) + I(v^-),
\]
where we have set
\[ I(w) := \int_{\mathbb{R}^{2N}} \frac{(w(x) - w(y))^2}{|x-y|^{N+2s}} dxdy - \mu_2 \int_\Omega |w|^2 dx = \|w\|_X^2 - \mu_2 \|w\|_{L^2(\Omega)}^2.
\]
On the other hand, by definition of $\lambda_{1,s}$, we have
\[ I(w) \geq \left( 1 - \frac{\mu_2}{\lambda_{1,s}} \right) \|w\|_X^2, \]
which finally yields the inequality
\[ J_s'(u, v)(u^-, v^-) \geq \left( 1 - \frac{\mu_2}{\lambda_{1,s}} \right) (\|u^-\|_X^2 + \|v^-\|_X^2). \]
Since \( \{(u, v) \} \subset Y(\Omega) \) is a critical point of \( J_s \), we get \( J_s'(u, v)(u^-, v^-) = 0 \), from which that assertion immediately follows. \( \square \)

**Proof of Theorem 1.3** By Lemma 4.9 and the Mountain Pass Theorem, there exists a sequence \( \{(u_n, v_n)\} \subset Y(\Omega) \), so called \((PS)_c\)-Palais Smale sequence at level \( c \), such that
\[ (4.22) \quad J_s(u_n, v_n) \to c, J_s'(u_n, v_n) \to 0, \]
where \( c \) is given by
\[ c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_s(\gamma(t)), \]
with
\[ \Gamma = \{ \gamma \in C([0,1], Y(\Omega)) : \gamma(0) = (0,0) \text{ and } J_s(\gamma(1)) \leq 0 \}. \]
From (2.5), we know that there exists \( u_\epsilon \) such that
\[ J_s(tu_\epsilon, tu_\epsilon) \leq t^2 \|u_\epsilon\|^2 - \mu_1 t^2 \|u_\epsilon\|_{L^p}^2 - \frac{t^{2+\gamma_p}}{2\mu} \int_\Omega \int_\Omega \frac{|u_\epsilon(x)|^{2+\gamma_p} |u_\epsilon(y)|^{2+\gamma_p}}{|x-y|^\gamma} \, dx \, dy \]
\[ := f(t). \]
It is easy to verify that \( f(t) \) attains its maximum at \( t_* = \left[ \frac{\|u_\epsilon\|^2 - \mu_1 \|u_\epsilon\|_{L^p}^2}{\frac{1}{f_\Omega \int_\Omega \int_\Omega \frac{|u_\epsilon(x)|^{2+\gamma_p} |u_\epsilon(y)|^{2+\gamma_p}}{|x-y|^\gamma} \, dx \, dy} \right]^{2+\gamma_p}. \]
By the definition of \( S_{s,\lambda}(v) \) and Lemma 4.5, we have
\[ c \leq \sup_{t \geq 0} J_s(tu_\epsilon, tu_\epsilon) \]
\[ \leq f(t_*) = \frac{N + 2s - \mu}{2N - \mu} \left[ \left( \|u_\epsilon\|^2 - \mu_1 \|u_\epsilon\|_{L^p}^2 \right) \frac{d}{dx} \right]^{2N-\gamma_p} \]
\[ = \frac{N + 2s - \mu}{2N - \mu} \left( S_{s,\mu_1}(u_\epsilon) \right)^{2N-\gamma_p} < \frac{N + 2s - \mu}{2N - \mu} \left( S_H \right)^{2N-\gamma_p}. \]
If one of the following conditions holds,
\[(i) \quad N \geq 4s \text{ and } \mu_1 > 0, \text{ or} \]
\[(ii) \quad 2s < N < 4s \text{ and } \mu_1 \text{ is large enough.} \]
Therefore, combining with Lemma 4.8 and Lemma 4.10, we get that problem (1.1) has a nonnegative solution with critical value \( c \in (0, \frac{N+2s-\mu}{2N-\mu} (S_H)^{2N-\gamma_p}). \)
5. Case 3: $\xi_1, \xi_2 > 0$, $p = q = 2^*_\mu$

In this case, we have the function $J_s : Y(\Omega) \to \mathbb{R}$ by setting

$$J_s(U) = J_s(u, v) = \frac{1}{2} \left( \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2 + |v(x) - v(y)|^2}{|x - y|^{N+2s}} dxdy \right)$$

$$- \frac{1}{2} \int_{\mathbb{R}^{2N}} (A(u, v), (u, v))_{\mathbb{R}^2} dx - \frac{1}{2^*\mu} \left( \int_{\Omega} \int_{\Omega} \frac{|u(x)|^{2^*_\mu} |v(y)|^{2^*_\mu}}{|x-y|^\mu} dxdy \right)$$

$$+ \xi_1 \int_{\Omega} \int_{\Omega} \frac{|u(x)|^{2^*_\mu} |u(y)|^{2^*_\mu}}{|x-y|^\mu} dxdy + \xi_2 \int_{\Omega} \int_{\Omega} \frac{|v(x)|^{2^*_\mu} |v(y)|^{2^*_\mu}}{|x-y|^\mu} dxdy,$$

whose Fréchet derivative is given by

$$J'_s(u, v)(\varphi, \psi) = \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y)) + (v(x) - v(y))(\psi(x) - \psi(y))}{|x - y|^{N+2s}} dxdy$$

$$- \int_{\Omega} (A(u, v), (\varphi, \psi))_{\mathbb{R}^2} dx - \int_{\Omega} \int_{\Omega} \frac{|u(x)|^{2^*_\mu - 2} u(x)v(y)^{2^*_\mu}}{|x - y|^\mu} \varphi dxdy$$

$$- \int_{\Omega} \int_{\Omega} \frac{|u(x)|^{2^*_\mu} |v(y)|^{2^*_\mu - 2} v(y)}{|x - y|^\mu} \psi dxdy - 2\xi_1 \int_{\Omega} \int_{\Omega} \frac{|u(x)|^{2^*_\mu - 2} u(x)|u(y)|^{2^*_\mu}}{|x - y|^\mu} \varphi dxdy$$

$$- 2\xi_2 \int_{\Omega} \int_{\Omega} \frac{|v(x)|^{2^*_\mu} |v(y)|^{2^*_\mu - 2} v(y)}{|x - y|^\mu} \psi dxdy,$$

for every $(\varphi, \psi) \in Y(\Omega)$. Meanwhile,

$$F(u, v) = \frac{1}{2^*\mu} \left[ \int_{\Omega} \frac{|v(y)|^{2^*_\mu}}{|x-y|^\mu} dxdy + \xi_1 \int_{\Omega} \frac{|u(x)|^{2^*_\mu}}{|x-y|^\mu} dxdy + \xi_2 \int_{\Omega} \frac{|v(y)|^{2^*_\mu}}{|x-y|^\mu} dxdy \right].$$

5.1. Minimizers. For notational convenience, if $(u, v) \in Y(\Omega)$ we set

$$B(u, v) := \int_{\Omega} \int_{\Omega} \frac{|u(x)|^{2^*_\mu} |v(y)|^{2^*_\mu}}{|x-y|^\mu} dxdy.$$

and let

$$\tilde{S}^H_\xi = \inf_{(u, v) \in Y(\Omega) \setminus \{(0, 0)\}} \frac{\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2 + |v(x) - v(y)|^2}{|x-y|^{N+2s}} dxdy}{(B(u, v) + \xi_1 B(u, u) + \xi_2 B(v, v))^{\frac{1}{2^*_\mu}}}.$$

**Remark 5.1.** Let $T(u, v) := |u|^{2^*_\mu} |v|^{2^*_\mu} + \xi_1 |u|^{2^*_\mu} + \xi_2 |v|^{2^*_\mu}$. It is clear that $T(u, v)^{\frac{1}{2^*_\mu}}$ is 2-homogeneous, i.e.

$$T(\varphi U) = \varphi^{2^*_\mu} T(U), \forall U \in \mathbb{R}^2, \forall \varphi \geq 0.$$

there exists a constant $M > 0$ satisfying

$$T(u, v)^{\frac{1}{2^*_\mu}} \leq M(|u|^2 + |v|^2), \text{ for all } u, v \in \mathbb{R}.$$

where $M$ is the maximum of the function $T(u, v)^{\frac{1}{2^*_\mu}}$ attained in some $(s_0, t_0)$ of the compact set $\{(s, t) : s, t \in \mathbb{R}, |s|^2 + |t|^2 = 2\}$. Let $m = M^{-1}$, so we have that

$$T(s_0, t_0)^{\frac{1}{2^*_\mu}} = m^{-1}(s_0^2 + t_0^2).$$
The following result shows the relation between \( S^H_s \) and \( \tilde{S}^H_\xi \). The proof is similar to [24, Lemma 2.3].

**Lemma 5.2.** Let \( \Omega \) be a smooth bounded domain, then
\[
\tilde{S}^H_\xi = mS^H_s.
\]
Moreover, if \( g_0 \) realizes \( S^H_s \) then \( (s_0 g_0, t_0 g_0) \) realizes \( \tilde{S}^H_\xi \), for some \( s_0, t_0 > 0 \).

**Proof.** Let \( \{g_n\} \subset X(\Omega)\setminus\{0\} \) be a minimizing sequence for \( S^H_s \) and consider the sequence \((\tilde{u}_n, \tilde{v}_n) = (s_0 g_n, t_0 g_n)\). Substituting \((\tilde{u}_n, \tilde{v}_n)\) in quotient (5.2), we get
\[
\frac{(s_0^2 + t_0^2)\|g_n\|^2}{(s_0^{2
u} t_0^{2
u} + \xi_1 s_0^{2
u} + \xi_2 t_0^{2
u})^{\frac{1}{2
u}}} \geq \tilde{S}^H_\xi.
\]
and consequently by (5.4) follows that
\[
m \cdot \frac{\|g_n\|^2}{B(g_n, g_n)^{\frac{1}{2
u}}} \geq \tilde{S}^H_\xi.
\]
Taking the limit in (5.7), we obtain
\[
m S^H_s \geq \tilde{S}^H_\xi.
\]
In order to prove the reversed inequality, let \( \{(u_n, v_n)\} \) be a minimizing sequence for \( \tilde{S}^H_\xi \). We set \( u_n = r_n v_n \) for \( r_n > 0 \). By Proposition 2.3, we obtain
\[
\frac{\|(u_n, v_n)\|^2}{(B(u_n, v_n) + \xi_1 B(u_n, u_n) + \xi_2 B(v_n, v_n))^{\frac{1}{2
u}}} \geq \frac{(1 + \frac{1}{r_n}) S^H_s}{\left(\frac{1}{r_n^{2
u}} + \xi_1 + \xi_2 \frac{1}{r_n^{2\mu}}\right)^{\frac{1}{2\mu}}}.
\]
Now, by inequality (5.3), we obtain
\[
m \left(\frac{1}{r_n^{2
u}} + \xi_1 + \xi_2 \frac{1}{r_n^{2\mu}}\right)^{\frac{1}{2\mu}} \leq 1 + \frac{1}{r_n^{2\mu}}.
\]
Hence, using the inequalities (5.8) and (5.9), we have
\[
\frac{\|(u_n, v_n)\|^2}{(B(u_n, v_n) + \xi_1 B(u_n, u_n) + \xi_2 B(v_n, v_n))^{\frac{1}{2
u}}} \geq m S^H_s.
\]
Therefore, passing to the limit in the above inequality, we have the desired reversed inequality. \( \square \)

5.2. **Compactness convergence.**

**Lemma 5.3.** (Boundedness) The \((PS)_c\) sequence \( \{(u_n, v_n)\} \subset Y(\Omega) \) is bounded.

**Proof.** Let \( U_n \in Y(\Omega) \) be a \((PS)_c\) sequence, we have
\[
J_s(U_n) - \frac{1}{2}(J_s'(U_n), U_n) = (2^*_\mu - 1) \int_\Omega F(U_n) dx \leq C_1(1 + \|U_n\|_Y).
\]
for some positive constant $\widetilde{C}_1$. From (2.5), we have

$$J_s(U_n) + \frac{1}{2}\langle J'_s(U_n), U_n \rangle = \|U_n\|^2_Y - \int_{\Omega}(A(u, v), (u, v))dx - (2^*_\mu + 1)\int_{\Omega}F(U_n)dx$$

(5.11)

$$\leq \|U_n\|^2_Y - \mu_1\|U_n\|^2_{L^2} - (2^*_\mu + 1)\int_{\Omega}F(U_n)dx$$

$$\leq \widetilde{C}_2(1 + \|U_n\|_Y).$$

for some positive constant $\widetilde{C}_2$. Recalling that $2^*_\mu > 2$, by Hölder inequality and [35, Lemma 2.2], we get

$$\|u_n\|^2_{L^2} \leq |\Omega|^{\frac{1}{2^*_\mu}}\|u_n\|^2_{L^{2^*_\mu}} \leq \widetilde{C}_3\left(\int_{\Omega}\int_{\Omega}\frac{|u_n(x)|^{2^*_\mu}|u_n(y)|^{2^*_\mu}}{|x-y|}\right)^{\frac{1}{2^*_\mu}},$$

$$\|v_n\|^2_{L^2} \leq |\Omega|^{\frac{1}{2^*_\mu}}\|v_n\|^2_{L^{2^*_\mu}} \leq \widetilde{C}_4\left(\int_{\Omega}\int_{\Omega}\frac{|v_n(x)|^{2^*_\mu}|v_n(y)|^{2^*_\mu}}{|x-y|}\right)^{\frac{1}{2^*_\mu}}.$$}

for some positive constant $\widetilde{C}_3$ and $\widetilde{C}_4$. Combining with (5.10), we get

(5.12)

$$\|U_n\|^2_{L^2} \leq \widetilde{C}_5\left(\left(\int_{\Omega}\int_{\Omega}\frac{|u_n(x)|^{2^*_\mu}|u_n(y)|^{2^*_\mu}}{|x-y|}\right)^{\frac{1}{2^*_\mu}}\right) \leq \widetilde{C}_6(1 + \|U_n\|_Y)^{\frac{1}{2^*_\mu}}.$$}

for some positive constant $\widetilde{C}_5$ and $\widetilde{C}_6$. Hence, by (5.10)-(5.12), we conclude that

$$\|U_n\|^2_Y \leq \widetilde{C}_7(1 + \|U_n\|_Y) + \widetilde{C}_8(1 + \|U_n\|_Y)^{\frac{1}{2^*_\mu}}.$$}

for some positive constant $\widetilde{C}_7$ and $\widetilde{C}_8$. Therefore, we conclude that the sequence $\{U_n\}$ is bounded.

□

Since $\{U_n\}$ is bounded in $Y(\Omega)$, up to a subsequence, still denoted by $U_n$, there exists $U = (u_0, v_0) \in Y(\Omega)$ such that

(5.13) $\quad U_n \rightharpoonup U$ in $Y(\Omega),$

(5.14) $\quad U_n \rightarrow U$ in $L^{2^*_\mu}(\Omega) \times L^{2^*_\mu}(\Omega),$

(5.15) $\quad U_n \rightharpoonup U$ a.e in $\Omega,$

(5.16) $\quad U_n \rightarrow U$ in $L^r(\Omega) \times L^r(\Omega)$, for all $r \in [1, 2^*_\mu),$

Lemma 5.4. The following relations hold true:

(i) $\quad J_s(U) = (2^*_\mu - 1)\int_{\Omega}F(U)dx \geq 0.$

(ii) $\quad J_s(U_n) = J_s(U) + \frac{1}{2}\|U_n - U\|^2_Y - \int_{\Omega}F(U_n - U)dx + o(1).$

(iii) $\quad \|U_n - U\|^2_Y = 2 \cdot 2^*_\mu \int_{\Omega}F(U_n - U)dx + o(1).$
Proof. i) Since $|u_n|^2_n \rightarrow |u_0|^2_n, |v_n|^2_n \rightarrow |v_0|^2_n$ in $L^{2N/(N-m)}(\Omega)$ as $n \rightarrow \infty$, by (4.15), we get

$$\nabla F(U_n) \rightarrow \nabla F(U) \text{ in } L^{2N/(N-m)}(\Omega) \times L^{2N/(N-m)}(\Omega).$$

(5.17)

So for any $\Theta \in \text{Y}(\Omega)$, $\int_{\Omega}(\nabla F(U_n), \Theta)_{\mathbb{R}^2}dx \rightarrow \int_{\Omega}(\nabla F(U), \Theta)_{\mathbb{R}^2}dx$, we have

$$J'(U_n)(\Theta) = o(1).$$

(5.18)

Passing to the limit in (5.18) as $n \rightarrow \infty$, and combining with the above convergences, we obtain

$$\langle U, \Theta \rangle_Y - \int_{\Omega}(AU, \Theta)_{\mathbb{R}^2}dx - \int_{\Omega}(\nabla F(U), \Theta)dx = 0, \forall \Theta \in \text{Y}(\Omega)$$

(5.19)

which means $U$ is a weak solution of (1.1).

Notice that the nonlinearity $F$ is $2 \cdot 2^*_\mu$-homogeneous, particularly, we have

$$\langle \nabla F(U), U \rangle_{\mathbb{R}^2} = uF_u(U) + vF_v(U) = 2 \cdot 2^*_\mu F(U), \forall U = (u, v) \in \mathbb{R}^2.$$  

(5.20)

Combined with $J'_s(U)U = 0$, we reach the conclusion.

ii) By Lemma 5.3, the sequence $U_n$ is bounded in $\text{Y}(\Omega) \hookrightarrow L^{2^*_\mu}(\Omega) \times L^{2^*_\mu}(\Omega)$, hence $U_n$ is bounded in $L^{2^*_\mu}(\Omega) \times L^{2^*_\mu}(\Omega)$. Since $U_n \rightarrow U$ a.e. in $\Omega$, by the Brézis-Lieb Lemma, we have

$$\|U_n\|_{Y}^2 = \|U_n - U\|_{Y}^2 + \|U\|_{Y}^2 + o(1).$$

(5.21)

$$\|U_n\|_{L^{2^*_\mu}}^2 = \|U_n - U\|_{L^{2^*_\mu}}^2 + \|U\|_{L^{2^*_\mu}}^2 + o(1).$$

(5.22)

By Proposition 4.6, we get

$$\int_{\Omega}F(U_n)dx = \int_{\Omega}F(U)dx + \int_{\Omega}F(U_n - U)dx + o(1), \text{ as } n \rightarrow \infty.$$  

(5.23)

Therefore, using that $U_n \rightarrow U$ in $L^r(\Omega) \times L^r(\Omega)$, for all $r \in [1, 2^*_\mu)$, by the definition of $J_s$, (5.21), (5.22) and (5.23), we deduce ii).

iii) By (5.14), (5.17) and (5.20), we get

$$\int_{\Omega}(\nabla F(U_n) - \nabla F(U), U_n - U)_{\mathbb{R}^2}dx$$

$$= \int_{\Omega}(\nabla F(U_n), U_n)_{\mathbb{R}^2}dx - \int_{\Omega}(\nabla F(U), U)_{\mathbb{R}^2}dx + o(1)$$

$$= 2 \cdot 2^*_\mu \int_{\Omega}F(U_n)dx - 2 \cdot 2^*_\mu \int_{\Omega}F(U)dx + o(1).$$

(5.24)

Therefore, using (5.23), we get

$$\int_{\Omega}(\nabla F(U_n) - \nabla F(U), U_n - U)_{\mathbb{R}^2}dx = 2 \cdot 2^*_\mu \int_{\Omega}F(U_n - U)dx + o(1).$$

(5.25)
On the other hand,
\[
o(1) = J'_s(U_n)(U_n - U) = J'_s(U_n)(U_n - U) - J'_s(U)(U_n - U)
\]
\[
= \langle U_n, U_n - U \rangle_Y - \int_\Omega (AU_n, U_n - U)_{\mathbb{R}^2} dx - \int_\Omega (\nabla F(U_n), U_n - U)_{\mathbb{R}^2} dx
\]
\[
- \langle U, U_n - U \rangle_Y + \int_\Omega (AU, U_n - U)_{\mathbb{R}^2} dx + \int_\Omega (\nabla F(U), U_n - U)_{\mathbb{R}^2} dx
\]
\[
= \langle U_n - U, U_n - U \rangle_Y - \int_\Omega (A(U_n - U), U_n - U)_{\mathbb{R}^2} dx
\]
\[
- \int_\Omega (\nabla F(U_n) - \nabla F(U), U_n - U)_{\mathbb{R}^2} dx.
\]

(5.26)

Hence, from (5.16) and (5.25), it follows that
\[
\|U_n - U\|_Y^2 = 2 \cdot 2^s \mu \int_\Omega F(U_n - U) dx + o(1), \quad n \to \infty.
\]

This concludes the proof. □

**Lemma 5.5.** Let \( N > 2s, 0 < \mu < N \) and \( \{U_n\} \) be a \((PS)_c\) sequence of \( J_s \) with

\[
c < \frac{N + 2s - \mu}{2N - \mu} \left( \frac{\tilde{S}_\xi^H}{2} \right)^{\frac{2N - \mu}{N + 2s - \mu}}.
\]

(5.27)

Then, \( \{U_n\} \) has a convergent subsequence.

**Proof.** We assume that

\[
\|U_n - U\|_Y^2 \to L, \quad n \to \infty.
\]

(5.28)

From Lemma 5.4 (iii),

\[
2 \cdot 2^s \mu \int_\Omega F(U_n - U) dx \to L, \quad n \to \infty
\]

(5.29)

and consequently \( L \in [0, \infty) \). By the definition of \( \tilde{S}_\xi^H \), we have

\[
L \geq \frac{\tilde{S}_\xi^H}{2} \left( \frac{L}{2} \right)^{\frac{2N - \mu}{N + 2s - \mu}}
\]

and consequently, either

\[
L = 0 \quad \text{or} \quad L \geq \left( \frac{1}{2} \right)^{\frac{N + 2s - \mu}{N + 2s - \mu}} \left( \frac{\tilde{S}_\xi^H}{2} \right)^{\frac{2N - \mu}{N + 2s - \mu}}.
\]

If \( L \geq \left( \frac{1}{2} \right)^{\frac{N + 2s - \mu}{N + 2s - \mu}} \left( \tilde{S}_\xi^H \right)^{\frac{2N - \mu}{N + 2s - \mu}} \), from Lemma 5.4 (iii), it is follows that

\[
\frac{1}{2} \|U_n - U\|_Y^2 - \int_\Omega F(U_n - U) dx = \frac{N + 2s - \mu}{2(2N - \mu)} \|U_n - U\|_Y^2 + o(1).
\]

Therefore, using Lemma 5.4 (ii) and above equality, we see that

\[
J_s(U) + \frac{N + 2s - \mu}{2(2N - \mu)} \|U_n - U\|_Y^2 = J_s(U) + \frac{1}{2} \|U_n - U\|_Y^2 - \int_\Omega F(U_n - U) dx + o(1)
\]

\[
= J_s(U_n) + o(1) = c + o(1), \quad n \to \infty.
\]

(5.30)
So\
\[
J_s(U) + \frac{N + 2s - \mu}{2(2N - \mu)} L \geq \frac{N + 2s - \mu}{2(2N - \mu)}
\]
\[
\geq \frac{N + 2s - \mu}{2(2N - \mu)} \left( \frac{1}{2} \right)^{\frac{2N - \mu}{N + 2s - \mu}} \left( S^H_{\mu} \right)^{\frac{2N - \mu}{N + 2s - \mu}} = \frac{N + 2s - \mu}{2N - \mu} \left( \frac{S^H_{\mu}}{2} \right)^{\frac{2N - \mu}{N + 2s - \mu}}
\]
which contradicts (5.27). Thus $L = 0$ and therefore, by (5.28), we have
\[
\|U_n - U\|_{Y}^2 \to 0, \text{ as } n \to \infty
\]
and so the assertion of Lemma 5.5 follows.

5.3. Linking geometry.

**Lemma 5.6.** If $F$ is a finite dimensional subspace of $Y(\Omega)$, then there exists $R > 0$ large enough such that $J_s(u, v) \leq 0$, for all $(u, v) \in F$ with $\|u, v\|_Y \geq R$ and $uv \neq 0$.

**Proof.** Choose $(\tilde{u}_0, \tilde{v}_0) \in F$ with $\tilde{u}_0 \tilde{v}_0 \neq 0$, then
\[
J_s(t\tilde{u}_0, t\tilde{v}_0) = \frac{t^2}{2} \int_{\mathbb{R}^{2N}} \frac{|\tilde{u}_0(x) - \tilde{u}_0(y)|^2 + |\tilde{v}_0(x) - \tilde{v}_0(y)|^2}{|x - y|^{N+2s}} dxdy
\]
\[
- \frac{t^2}{2} \int_{\mathbb{R}^{2N}} (A(\tilde{u}_0, \tilde{v}_0), (\tilde{u}_0, \tilde{v}_0)) d(x) - \frac{t^{2+2s}}{2^s} \left( \int_\Omega \int_\Omega \frac{|\tilde{u}_0(x)|^{2s}|\tilde{v}_0(x)|^{2s}}{|x - y|^s} dxdy \right)
\]
\[
+ \xi_1 \int_\Omega \int_\Omega \frac{|\tilde{u}_0(x)|^{2s}|\tilde{v}_0(x)|^{2s}}{|x - y|^s} dxdy + \xi_2 \int_\Omega \int_\Omega \frac{|\tilde{v}_0(x)|^{2s}|\tilde{v}_0(x)|^{2s}}{|x - y|^s} dxdy,
\]
by choosing $t > 0$ large enough, the assertion follows. This concludes the proof. \qed

**Lemma 5.7.**

(5.31) If $\lambda_{k-1,s} < \mu_1 < \lambda_{k,s} \leq \mu_2 < \lambda_{k+1,s}$, for some $k \geq 1$,

Then the functional $J_s$ satisfies:

(i) There exist $\alpha, \rho > 0$ such that $J_s(u, v) \geq \alpha$ for all $(u, v) \in W$ with $\|u, v\|_Y = \rho$.

(ii) If $Q = (V \cap B_R(0)) \oplus [0, R]^2$, where $e \in W \cap \partial B_1(0)$ is a fixed vector, then $J_s(u, v) < 0$ for all $(u, v) \in \partial Q$ and $R > \rho$ large enough.

**Proof.** Considering the following subspaces $W = Z_k \oplus H$, where
\[
Z_k = \text{span}\{((\varphi_{k,s,0})), (0, \varphi_{k,s})\} \text{ and } H = \text{span}\\{(0, \varphi_{k+1,s})\ldots\},
\]
we have that if $U \in W$, then $U = U^k + \overline{U}$ with $U^k \in Z_k$ and $\overline{U} \in H$. Since $\|U\|_Y^2 = \|U^k\|_Y^2 + \|\overline{U}\|_Y^2$, by (2.5) and (4.21) we have
\[
J_s(U) \geq \frac{1}{2}(\|U^k\|_Y^2 + \|\overline{U}\|_Y^2) - \frac{\mu_2}{2}(\|U^k\|_{L^2}^2 + \|\overline{U}\|_{L^2}^2) - C(\|U^k\|_Y^2 + \|\overline{U}\|_Y^2)^{2s},
\]
where $U = (u, v)$ and $C := C(\xi_1, \xi_2) > 0$ is a constant. Therefore, using that $U^k \in Z_k \subset W$ and $\overline{U} \in H$, we obtain
\[
\|U^k\|_{L^2}^2 \leq \frac{1}{\lambda_{k,s}} \|U^k\|_Y^2 \text{ and } \|\overline{U}\|_{L^2}^2 \leq \frac{1}{\lambda_{k+1,s}} \|\overline{U}\|_Y^2.
\]
Consequently,

\[
J_\alpha(U) \geq \left( \frac{1}{2} \| U \|_Y^2 - \frac{\mu_2}{2} \| U \|_{L^2}^2 \right) + \left( \frac{1}{2} \| U^k \|_Y^2 - \frac{\mu_2}{2} \| U^k \|_{L^2}^2 \right) - C(\| U^k \|_Y^2 + \| U \|_{L^2}^2)_{2^\alpha} \\
\geq \frac{1}{2} \left( 1 - \frac{\mu_2}{\lambda_{k+1,s}} \right) \| U \|_Y^2 + \frac{1}{2} \left( 1 - \frac{\mu_2}{\lambda_{k,s}} \right) \| U^k \|_Y^2 - C\| U^k \|_Y^{2^\alpha} \\
- C\| U \|_{L^2}^{2^\alpha}.
\]

(5.32)

Taking \( \| U \|_Y = \rho \) small enough, since \( \| U \|_Y^2 = \| U^k \|_Y^2 + \| U \|_{L^2}^2 \), we get that \( \| U^k \|_Y := y(\rho) \) and \( \| U \|_Y := z(\rho) \) are small enough. Now consider the function

\[
\alpha(z) = \frac{1}{2} \left( 1 - \frac{\mu_2}{\lambda_{k+1,s}} \right) z^2 + \frac{1}{2} \left( 1 - \frac{\mu_2}{\lambda_{k,s}} \right) y(\rho)^2 - C(y(\rho)_{2^\alpha} + z_{2^\alpha}) \\
= h(z) + \frac{1}{2} \left( 1 - \frac{\mu_2}{\lambda_{k,s}} \right) y(\rho)^2 - C y(\rho^{2^\alpha}),
\]

where \( h(z) = \frac{1}{2} \left( 1 - \frac{\mu_2}{\lambda_{k+1,s}} \right) z^2 - C z_{2^\alpha} \). By (5.31), the maximum value of \( h(z) \), for \( \rho \) sufficiently small, is given by

\[
\overline{h} := \frac{N + 2s - \mu}{2N - \mu} \left( \frac{1}{2\mu C} \right)^{\frac{N-2s}{N+2s-\mu}} \left( \frac{1}{2} \left( 1 - \frac{\mu_2}{\lambda_{k+1,s}} \right) \right)^{\frac{2N-\mu}{N+2s-\mu}} > 0,
\]

which is independent of \( \rho \) and it is assumed at

\[
\overline{z} := \left( \frac{1}{2\cdot 2^\alpha C} \right)^{\frac{N-2s}{2(N+2s-\mu)}} \left( 1 - \frac{\mu_2}{\lambda_{k+1,s}} \right)^{\frac{N-2s}{2(N+2s-\mu)}}.
\]

Therefore, it is possible to choose \( y(\rho) \) small enough, such that

\[
\alpha(\overline{z}) = \overline{h} - Cy(\rho)^2 - C y(\rho)^{2^\alpha} \geq \overline{h} - (c + C) y(\rho)^2 > 0,
\]

where \( c = \frac{1}{2} \left( \frac{\mu_2}{\lambda_{k,s}} - 1 \right) \geq 0 \). Hence, by the estimate (5.32) and by the above information, for \( \| U \|_Y = \rho \) small enough, there exists \( \alpha > 0 \) such that \( J_\alpha(U) \geq \alpha \). This proves the statement (i).

To prove item (ii), we take \( U = (u, v) \in V \), where \( u = \Sigma_{i=1}^{k-1} u_i \varphi_{i,s}, v = \Sigma_{i=1}^{k-1} v_i \varphi_{i,s} \). Using [27, Proposition 9], we get

\[
\int_{\mathbb{R}^N} |u|^2 dx = \Sigma_{i=1}^{k-1} u_i^2, \quad \int_{\mathbb{R}^N} |v|^2 dx = \Sigma_{i=1}^{k-1} v_i^2,
\]

also

\[
\int_{\mathbb{R}^N} |(-\Delta)^{\frac{1}{2}} u|^2 dx + \int_{\mathbb{R}^N} |(-\Delta)^{\frac{1}{2}} v|^2 dx = \Sigma_{i=1}^{k-1} (u_i^2 + v_i^2) \| \varphi_{i,s} \|_X \\
= \Sigma_{i=1}^{k-1} (u_i^2 + v_i^2) \lambda_{i,s}.
\]
Then, using (2.5), we are going to prove that \( J_s(U) < 0 \) on \( V \). Let \( U = (u, v) \in V \), since \( \lambda_{k-1,s} < \mu_1 < \lambda_{k,s} \leq \mu_2 < \lambda_{k+1,s} \), we have that
\[
J_s(u, v) \leq \frac{1}{2} \sum_{i=1}^{k-1} (u_i^2 + v_i^2) \lambda_{i,s} - \frac{\mu_1}{2} \sum_{i=1}^{k-1} (u_i^2 + v_i^2) \left( \int_{\Omega} \frac{|u(x)|^{2^*_\mu} |v(x)|^{2^*_\mu}}{|x-y|^\mu} dx \right) + \xi_1 \int_{\Omega} \frac{|u(x)|^{2^*_\mu} |u(x)|^{2^*_\mu}}{|x-y|^\mu} dx dy + \xi_2 \int_{\Omega} \frac{|v(x)|^{2^*_\mu} |v(x)|^{2^*_\mu}}{|x-y|^\mu} dx dy
\]
\[
\leq \frac{1}{2} \sum_{i=1}^{k-1} (u_i^2 + v_i^2) (\lambda_{i,s} - \mu_1) < 0.
\]
Now, to end the proof, it is enough to apply Lemma 5.6 to the finite dimensional subspace \( V + \text{span} \{e\} \) containing \( Q = (V \cap \overline{B_R(0)}) \cap [0, R]e \), for some \( e \in W \cap \partial B_1(0) \) and \( R > \rho \). □

**Remark 5.8.** Notice that, in Lemma 5.7 we can choose the finite dimensional subspace \( F \) of \( Y(\Omega) \) as
\[
F_e = V \oplus \text{span} \{e\} = V \oplus \text{span} \{(\bar{z}_e, 0)\},
\]
where \( V = \text{span} \{(0, \varphi_{1,s}), (\varphi_{1,s}, 0), (0, \varphi_{2,s}), (\varphi_{2,s}, 0), \ldots, (0, \varphi_{k-1,s}), (\varphi_{k-1,s}, 0)\}, \bar{z}_e = \frac{z_e}{\|z_e\|_X} \), with \( z_e = u_e - \sum_{j=1}^{k-1} (\int_{\Omega} u_e \varphi_{j,s} dx) \varphi_{j,s} \).

**Lemma 5.9.** Let \( s \in (0, 1) \), \( N > 2s \) and \( M_s := \max_{u \in G} S_{s, \mu_1} \), where \( G := \{u \in F_e : \int_{\Omega} \int_{\Omega} \frac{|u(x)|^{2^*_\mu} |u(y)|^{2^*_\mu}}{|x-y|^\mu} dx dy = 1\} \). Suppose \( \lambda_{k-1,s} < \mu_1 < \lambda_{k,s} \leq \mu_2 < \lambda_{k+1,s} \), for some \( k \in \mathbb{N} \), then

(i) \( M_s \) is achieved by \( u_M \in F_e \) and \( u_M \) can be written as follows
\[
u = \nu + t u_e, \quad \text{with } \nu \in \text{span} \{\varphi_{1,s}, \varphi_{2,s}, \ldots, \varphi_{k-1,s}\} \text{ and } t \geq 0;
\]

(ii) \( M_s < S_s^H \), provided
\[a) \quad N \geq 4s \text{ and } \mu_1 > 0, \text{ or}\]
\[b) \quad 2s < N < 4s \text{ and } \mu_1 \text{ is large enough.}\]

**Proof.** (i) Thanks to the Weierstrass Theorem, \( M_s \) is achieved at \( u_M \). Since \( u_M \in F_e \) and by the definition of \( F_e \), we have that \( u_M = \bar{v} + tz_e \), for some \( \bar{v} \in \text{span} \{\varphi_{1,s}, \varphi_{2,s}, \ldots, \varphi_{k-1,s}\} \) and \( t \geq 0 \). From the definition of \( z_e \) in Remark 5.8, we have that
\[
(5.33) \quad u_M = \nu + t u_e,
\]
where
\[
\nu = \bar{v} - t \sum_{i=1}^{k-1} \int_{\Omega} u_e \varphi_{i,s} dx \varphi_{i,s} \in \text{span} \{\varphi_{1,s}, \varphi_{2,s}, \ldots, \varphi_{k-1,s}\}.
\]

(ii) First let \( t = 0 \), then \( u_M = \nu \) and
\[
M_s = \|\nu\|^2 - \mu_1 \int_{\mathbb{R}^N} |\nu|^2 dx \leq (\lambda_{k-1,s} - \mu_1) \|\nu\|^2_{L^2(\Omega)} < 0 < S_s^H.
\]
Now, suppose \( t > 0 \), we find that \( \bar{v} \) and \( z_e \) are orthogonal in \( L^2(\Omega) \), then
\[
\|u_M\|^2_{L^2(\Omega)} = \|\bar{v}\|^2_{L^2(\Omega)} + \|z_e\|^2_{L^2(\Omega)}. \quad \text{Since } \int_{\Omega} \int_{\Omega} \frac{|u(x)|^{2^*_\mu} |u(y)|^{2^*_\mu}}{|x-y|^\mu} dx dy = 1, \text{ using [25, Lemma 4.7], we get a} \]
constant $C_0 > 0$ (independent of $\epsilon$) such that $\|u_M\|_{L^\mu_\nu(\Omega)} \leq C_0$. Subsequently, using Hölder inequality, we get a constant $C_1 > 0$ (also independent of $\epsilon$) such that $\|u_M\|^2_{L^2(\Omega)} \leq C_1$. Therefore, we can find $C_2 > 0$ such that $\|u_M\|^2_{L^2(\Omega)}$ and $\|\tilde{\nu}\|^2_{L^2(\Omega)}$ are both uniformly bounded in $\epsilon$. By computations, we get

$$
\|u_\epsilon\|_{N(3N - 2\mu + 2s)\frac{2N}{N}} = \left(\int_{\Omega} \frac{U_\epsilon}{x^{2N - \mu}y^{2N - \mu}} dx\right)^{\frac{2N - \mu}{N}} \leq \left(\int_{B_2} \frac{U_\epsilon}{x^{2N - \mu}y^{2N - \mu}} dx\right)^{\frac{2N - \mu}{N}} \leq C_{3\epsilon}^{\frac{N - 2s}{2}} \left(\int_{0}^{\frac{2\epsilon}{r}} \frac{r^{N - 1}}{(1 + r^2)^{\frac{N(3N - 2\mu + 2s)}{2N - \mu}}} dr\right)^{\frac{2N - \mu}{N}} \leq O(\epsilon^{\frac{N - 2s}{2}})
$$

(5.34)

where $C_3 > 0$ is a constant. Since $\varphi_{1, s}, \varphi_{2, s}, \ldots, \varphi_{k - 1, s} \in L^\infty(\Omega)$, we have $\tilde{\nu} \in L^\infty(\Omega)$. Using the fact that the map $t \mapsto t^{2\mu}\nu$ is convex, for $t > 0$ and span $\{\varphi_{1, s}, \varphi_{2, s}, \ldots, \varphi_{k - 1, s}\}$ is a finite dimensional space, all norms are equivalent, we get

$$
1 = \int_{\Omega} \int_{\Omega} \frac{|u_M(x)|^2 u_M(x)}{|x - y|^\mu} dxdy = \int_{\Omega} \int_{\Omega} \frac{|\nu + tu_\epsilon(x)|^{2\mu}}{|x - y|^\mu} dxdy \\
\geq \int_{\Omega} \int_{\Omega} \frac{|tu_\epsilon(x)|^{2\mu}}{|x - y|^\mu} dxdy + 2 \cdot 2^\mu \int_{\Omega} \int_{\Omega} \frac{|\nu(x)|^{2\mu - 1}|tu_\epsilon(x)|}{|x - y|^\mu} dxdy \\
\geq \int_{\Omega} \int_{\Omega} \frac{|tu_\epsilon(x)|^{2\mu}|tu_\epsilon(y)|^{2\mu}}{|x - y|^\mu} dxdy - 2 \cdot 2^\mu \|\nu\|_{L^\infty(\Omega)} \int_{\Omega} \int_{\Omega} \frac{|tu_\epsilon(x)|^{2\mu - 1}|tu_\epsilon(y)|^{2\mu - 1}}{|x - y|^\mu} dxdy \\
\geq \int_{\Omega} \int_{\Omega} \frac{|tu_\epsilon(x)|^{2\mu}|tu_\epsilon(y)|^{2\mu}}{|x - y|^\mu} dxdy - C_4 \|\nu\|_{L^2(\Omega)} \|u_\epsilon\|_{N(3N - 2\mu + 2s)\frac{2N}{N}}^{\frac{3N - 2\mu + 2s}{2N - \mu} \frac{2N}{N}} \|u_\epsilon\|_{\frac{N(3N - 2\mu + 2s)}{2N - \mu} \frac{2N}{N}}^{\frac{3N - 2\mu + 2s}{2N - \mu} \frac{2N}{N}}
$$

Combining with (5.34) with above inequality, we get

$$
\int_{\Omega} \int_{\Omega} \frac{|tu_\epsilon(x)|^{2\mu}|tu_\epsilon(y)|^{2\mu}}{|x - y|^\mu} dxdy \leq 1 + C_4 \|\nu\|_{L^2(\Omega)} O(\epsilon^{\frac{N - 2s}{2}}).
$$

(5.35)
Hence, using the definition of $S_{s,\mu_1}$ and (5.33), we get
\begin{equation}
M_e = \int_{\mathbb{R}^N} |u_M(x) - u_M(y)|^2 \, dx \, dy - \mu_1 \int_{\mathbb{R}^N} |u_M(x)|^2 \, dx \\
= \int_{\mathbb{R}^N} |(\nu(x) + tu_\epsilon(x)) - (\nu(y) + tu_\epsilon(y))|^2 \, dx \, dy - \mu_1 \int_{\mathbb{R}^N} |\nu(x) + tu_\epsilon(x)|^2 \, dx \\
= \int_{\mathbb{R}^N} |\nu(x) - \nu(y)|^2 \, dx \, dy + t^2 \int_{\mathbb{R}^N} |u_\epsilon(x) - u_\epsilon(y)|^2 \, dx \, dy \\
+ 2t \int_{\mathbb{R}^N} |(\nu(x) - \nu(y))(u_\epsilon(x) - u_\epsilon(y))| \, dx \, dy - \mu_1 t^2 \int_{\mathbb{R}^N} |u_\epsilon(x)|^2 \, dx \\
- 2\mu_1 t \int_{\mathbb{R}^N} |u_\epsilon(x)\nu(x)| \, dx \\
\leq (\lambda_{k-1,s} - \mu_1)\|\nu\|^2_{L^2(\Omega)} + S_{s,\mu_1}(u_\epsilon) \left( \int_{\Omega} \int_{\Omega} \frac{|tu_\epsilon(x)|^2 |tu_\epsilon(y)|^2}{|x - u|^\mu} \, dx \, dy \right)^{\frac{N-2s}{2N-\mu}} \\
+ 2t \int_{\mathbb{R}^N} |(\nu(x) - \nu(y))(u_\epsilon(x) - u_\epsilon(y))| \, dx \, dy - 2\mu_1 t \int_{\mathbb{R}^N} |u_\epsilon(x)\nu(x)| \, dx.
\end{equation}

Now we write $\nu = \sum_{i=1}^{k-1} \nu_i \varphi_{i,s}$ for some $\nu_i \in \mathbb{R}$, so that $\|\nu\|^2_{L^2(\Omega)} = \sum_{i=1}^{k-1} \nu_i^2$. By the Hölder inequality and the equivalence of the norms in a finite dimensional space,
\begin{align*}
\|\langle u_\epsilon, \nu \rangle_X\| &= \sum_{i=1}^{k-1} \lambda_{i,s} \nu_i \int_{\Omega} u_\epsilon(x) \varphi_{i,s}(x) \, dx \\
&\leq \sum_{i=1}^{k-1} \lambda_{i,s} \nu_i \|u_\epsilon\|_{L^1(\Omega)} \|\varphi_{i,s}\|_{L^\infty(\Omega)} \\
&\leq \sum_{i=1}^{k} \lambda_{k,s} \|u_\epsilon\|_{L^1(\Omega)} \|\nu\|_{L^\infty(\Omega)} \\
&\leq \frac{1}{\kappa} \|u_\epsilon\|_{L^1(\Omega)} \|\nu\|_{L^2(\Omega)},
\end{align*}
for suitable $\tilde{k}$ and $\bar{k} > 0$. More explicitly,
\begin{equation}
\left| \int_{\mathbb{R}^N} \frac{(\nu(x) - \nu(y))(u_\epsilon(x) - u_\epsilon(y))}{|x - y|^{N+2s}} \, dx \, dy \right| \leq \frac{1}{\kappa} \|u_\epsilon\|_{L^1(\Omega)} \|\nu\|_{L^2(\Omega)}.
\end{equation}

Gathering the results in (5.35), (5.36) and (5.37), using again the Hölder inequality and Proposition 4.2 (iv), we get
\begin{align*}
M_e &\leq (\lambda_{k-1,s} - \mu_1)\|\nu\|^2_{L^2(\Omega)} + S_{s,\mu_1}(u_\epsilon) \left( 1 + C_4 \|\nu\|_{L^2(\Omega)} O(\epsilon^{-\frac{N-2s}{2}}) \right)^{\frac{N-2s}{2N-\mu}} + 2t \|u_\epsilon\|_{L^1(\Omega)} \|\nu\|_{L^2(\Omega)} \\
&\quad - 2\mu_1 t \|u_\epsilon\|_{L^1(\Omega)} \|\nu\|_{L^\infty(\Omega)} \\
&\leq (\lambda_{k-1,s} - \mu_1)\|\nu\|^2_{L^2(\Omega)} + S_{s,\mu_1}(u_\epsilon) \left( 1 + C_4 \|\nu\|_{L^2(\Omega)} O(\epsilon^{-\frac{N-2s}{2}}) \right)^{\frac{N-2s}{2N-\mu}} + \kappa \|u_\epsilon\|_{L^1(\Omega)} \|\nu\|_{L^2(\Omega)} \\
&\leq (\lambda_{k-1,s} - \mu_1)\|\nu\|^2_{L^2(\Omega)} + S_{s,\mu_1}(u_\epsilon) \left( 1 + C_4 \|\nu\|_{L^2(\Omega)} O(\epsilon^{-\frac{N-2s}{2}}) \right) + O(\epsilon^{-\frac{N-2s}{2}}) \|\nu\|_{L^2(\Omega)}.
\end{align*}
Since the parabola $(\lambda_{k-1,s} - \mu_1)\|\nu\|^2_{L^2(\Omega)} + O(\epsilon^{-\frac{N-2s}{2}}) \|\nu\|_{L^2(\Omega)}$ stays always below its vertex, that is
\begin{align*}
(\lambda_{k-1,s} - \mu_1)\|\nu\|^2_{L^2(\Omega)} + O(\epsilon^{-\frac{N-2s}{2}}) \|\nu\|_{L^2(\Omega)} \leq \frac{1}{4(\lambda_{k-1,s} - \mu_1)} O(\epsilon^{N-2s}) = O(\epsilon^{N-2s}).
\end{align*}
From Lemma 4.5, we get
Case 1: \( N > 4s \),
\[ M_c \leq \left( S^H_s - \mu_1 C_s \epsilon^{2s} + O(\epsilon^{N-2s}) \right) \left( 1 + C_4 \|\nu\|_{L^2(\Omega)} O(\epsilon^{N-2s}) \right) + (\lambda_{k-1,s} - \mu_1) \|\nu\|^2_{L^2(\Omega)} \]
+ \( O(\epsilon^{N-2s}) \|\nu\|_{L^2(\Omega)} \)
\[ \leq S^H_s - \mu_1 C_s \epsilon^{2s} + O(\epsilon^{N-2s}) \]
\[ < S^H_s , \]
for sufficiently small \( \epsilon > 0 \) and \( \mu_1 > 0 \).
Case 2: \( N = 4s \),
\[ M_c \leq \left( S^H_s - \mu_1 C_s \epsilon^{2s} |\log\epsilon| + O(\epsilon^{2s}) \right) \left( 1 + C_4 \|\nu\|_{L^2(\Omega)} O(\epsilon^{N-2s}) \right) + (\lambda_{k-1,s} - \mu_1) \|\nu\|^2_{L^2(\Omega)} \]
+ \( O(\epsilon^{N-2s}) \|\nu\|_{L^2(\Omega)} \)
\[ \leq S^H_s - \mu_1 C_s \epsilon^{2s} |\log\epsilon| + O(\epsilon^{2s}) \]
\[ < S^H_s , \]
for sufficiently small \( \epsilon > 0 \) and \( \mu_1 > 0 \).
Case 3: \( 2s < N < 4s \),
\[ M_c \leq \left( S^H_s + \epsilon^{N-2s} (O(1) - \mu_1 C_s) + O(\epsilon^{2s}) \right) \left( 1 + C_4 \|\nu\|_{L^2(\Omega)} O(\epsilon^{N-2s}) \right) + (\lambda_{k-1,s} - \mu_1) \|\nu\|^2_{L^2(\Omega)} \]
+ \( O(\epsilon^{N-2s}) \|\nu\|_{L^2(\Omega)} \)
\[ \leq S^H_s + \epsilon^{N-2s} (O(1) - \mu_1 C_s) + O(\epsilon^{2s}) \]
\[ < S^H_s , \]
for sufficiently small \( \epsilon > 0 \) and \( \mu_1 \) large enough.

\[ \square \]

**Proof of Theorem 1.4** By Lemma 5.6 and Lemma 5.7, we have \( J_s \) satisfies the geometric structure of the Linking Theorem, so the Linking critical level of \( J_s \), i.e.
\[ c = \inf_{h \in \Gamma} \max_{(u,v) \in Q} J_s(h(u,v)), \]
where
\[ \Gamma = \{ h \in C(\overline{Q}, Y) : h = id \text{ on } \partial Q \}, \]
and
\[ Q = (\overline{B}_R \cap V) \oplus \{ r(\bar{z}_e, 0) : 0 < r < R \}. \]
Notice that, for all \( h \in \Gamma \), we have
\[ c = \inf_{h \in \Gamma} \max_{(u,v) \in Q} J_s(h(u,v)) \leq \max_{(u,v) \in Q} J_s(h(u,v)). \]
We claim that
\[ \max_{(u,v)\in(F_r)^2} J_s(h(u,v)) = \max_{(u,v)\in(F_r)^2, \eta\neq 0} J_s(\eta(\frac{u}{|\eta|}, \frac{v}{|\eta|})). \]

Taking \( r = 2 \) and using \( (F_r)^2 \), we obtain
\[ c \leq \max_{(u,v)\in(F_r)^2, (u,v)\neq (0,0)} J_s(h(u,v)) = \max_{(u,v)\in(F_r)^2, \eta\neq 0} J_s(\eta(\frac{u}{|\eta|}, \frac{v}{|\eta|})). \]

We claim that
\[ \max_{(u,v)\in(F_r)^2, \eta\geq 0} J_s(\eta(u,v)) \leq \max_{(u,v)\in(F_r)^2, \eta\geq 0} J_s(\eta(u,v)). \]

To verify the Claim, fixed \( U = (u,v) \in (F_r)^2 \) such that \( uv \neq 0 \), by (2.5), for all \( r \geq 0 \), we infer
\[ J_s(rU) \leq \frac{r^2}{2} \left( \|U\|^2_{Y} - \mu_1 \|U\|^2_{L^2(\Omega)} \right) - \frac{r^2 \gamma_\mu}{2\mu} \left( \int\int_{\Omega} \frac{|u(x)|^{2^*_{\mu}} |v(x)|^{2^*_{\mu}}}{|x-y|^{\mu}} dxdy \right) + \xi_1 \int\int_{\Omega} \frac{|u(x)|^{2^*_{\mu}} |u(x)|^{2^*_{\mu}}}{|x-y|^{\mu}} dxdy + \xi_2 \int\int_{\Omega} \frac{|v(x)|^{2^*_{\mu}} |v(x)|^{2^*_{\mu}}}{|x-y|^{\mu}} dxdy \]
\[ := \frac{A r^2}{2} - \frac{r^2 \gamma_\mu B}{2\mu} := g(r). \]

Notice that \( r_0 = \left( \frac{A}{2B} \right)^{\frac{1}{2^*_{\mu}-2}} \) is the maximum point of \( g(r) \), which maximum value is given by
\[ \max_{r \geq 0} J_s(rU) \leq \frac{N + 2s - \mu}{2N - \mu} \left( \frac{A}{2B^{2^*_{\mu}}} \right)^{\frac{2^*_{\mu}}{2^*_{\mu}-1}}. \]

Then
\[ \max_{r \geq 0} J_s(rU) \leq \frac{N + 2s - \mu}{2N - \mu} \left\{ \frac{\|U\|^2_{Y} - \mu_1 \|U\|^2_{L^2(\Omega)}}{2(B(u,v) + \xi_1 B(u,u) + \xi_2 B(v,v))^{\frac{1}{\gamma_\mu}}} \right\}^{\frac{2^*_{\mu}}{2^*_{\mu}-1}}, \]

Therefore, it is sufficient to show that
\[ \widetilde{M}_\epsilon := \max_{(u,v)\in(F_r)} \frac{\|U\|^2_{Y} - \mu_1 \|U\|^2_{L^2(\Omega)}}{2(B(u,v) + \xi_1 B(u,u) + \xi_2 B(v,v))^{\frac{1}{\gamma_\mu}}} < \frac{1}{2^{\frac{1}{\gamma_\mu}}}. \]

Define
\[ M_\epsilon := \max_{u \in F_r \setminus \{0\}} \frac{\|u\|^2_{X} - \mu_1 \|u\|^2_{L^2(\Omega)}}{\left( \int\int_{\Omega} \frac{|u(x)|^{2^*_{\mu}} |u(y)|^{2^*_{\mu}}}{|x-y|^{\mu}} dxdy \right)^{\frac{1}{2^*_{\mu}}}} \]
\[ = \max_{u \in F_r \setminus \{0\}} \frac{\|u\|^2_{X} - \mu_1 \|u\|^2_{L^2(\Omega)}}{\left( \int\int_{|x-y|=1} \frac{|u(x)|^{2^*_{\mu}} |u(y)|^{2^*_{\mu}}}{|x-y|^{\mu}} dxdy \right)^{\frac{1}{2^*_{\mu}}}}. \]

Taking \( s_0, t_0 > 0 \) as in Remark 5.1 and \( u_M \) as in Lemma 5.9, then \( \widetilde{M}_\epsilon \) is achieved by function \( U_M = (s_0 u_M, t_0 u_M) \). Therefore, from \( u_M \) as in Lemma 5.9 and using (5.4), we can conclude
that
\[
\tilde{M}_e = \frac{1}{2} \left( s_0^2 + t_0^2 \right) \left( \| (u_M, u_M) \|_2^2 - \mu_1 \| (u_M, u_M) \|_{(L^2(\Omega))^2}^2 \right)
\]
\[
= \frac{1}{2} mM_\epsilon < \frac{1}{2} mS^H \frac{1}{s} \xi, \]
if one of the following conditions holds
a) \( N \geq 4s \) and \( \mu_1 > 0 \), or
b) \( 2s < N < 4s \) and \( \mu_1 \) is large enough \( (\mu_1 > \lambda_{k-1,s} > 0) \).

Now, using the Linking theorem and Lemma 5.5, we conclude that problem (1.1) has a nontrivial solution with critical value \( c \geq \alpha \).
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