Unification of gravity and Yang-Mills theory in (2+1)-dimensions.

Peter Peldán

Institute of Theoretical Physics
S-412 96 Göteborg, Sweden
Internet: tfepp@fy.chalmers.se

Abstract

A gauge and diffeomorphism invariant theory in (2+1)-dimensions is presented in both first and second order Lagrangian form as well as in a Hamiltonian form. For gauge group $SO(1, 2)$, the theory is shown to describe ordinary Einstein gravity with a cosmological constant. With gauge group $G^{tot} = SO(1, 2) \otimes G^{YM}$, it is shown that the equations of motion for the $G^{YM}$ fields are the Yang-Mills equations. It is also shown that for weak $G^{YM}$ Yang-Mills fields, this theory agrees with the conventional Einstein-Yang-Mills theory to lowest order in Yang-Mills fields.

Explicit static and rotation symmetric solutions to the Einstein-Maxwell theory are studied both for the conventional coupling and for this unified theory. In the electric solution to the unified theory, point charges are not allowed, the charges must have spatial extensions.

PACS: 04.50.+h, 04.20.Fy
1 Introduction

The quest for a unified theory of gravity and Yang-Mills theory is even older than the
theory of general relativity and Yang-Mills theory themselves. It started already in 1914
with Nordström’s work of unifying his scalar theory of gravitation with Maxwell’s theory
of electro-magnetism [1]. This attempt was in 1921 followed by Kaluza’s five dimensional
Einstein equation, which was shown to describe the coupled Einstein-Maxwell theory
[2]. This formulation is normally called the Kaluza-Klein theory, since Klein rediscovered
Kaluza’s theory in 1926 [3]. Much later, after the invention of Yang-Mills theory in 1954,
a Kaluza-Klein unification of gravity and Yang-Mills theories was considered by DeWitt,
Trautman, Kerner and others [4].

The common idea behind all these attempts is that space-time has some extra space-
like dimensions besides the normal (3+1) observable ones. These extra dimensions should
then for some reason be compactified on a very small length-scale, and therefore be non-
observable at “normal” energy-scales.

What I will present here in this paper, is a somewhat different idea of how to find a
unified theory of gravity and Yang-Mills theory. Instead of enlarging the space-time, I
consider an enlarged internal symmetry group. The normal ”internal” symmetry group
for gravity in (2+1)-dimensions is the Lorentz group $SO(1,2)$. I study a theory valid for
an arbitrary gauge group, which reduces to the conventional Einstein theory for gravity
if one chooses the gauge group to be $SO(1,2)$. For other gauge groups, like $G_{\text{tot}} =
SO(1,2) \otimes G_{YM}$, the theory has an interpretation of gravity coupled to Yang-Mills theory.
(Weinberg [5] has considered a related generalization of gravity, in which he enlarges both
the space-time dimensions and the internal symmetry group.)

In section 2, the first order Lagrangian for the unified theory is given, and it is shown
that for $SO(1,2)$ the theory reduces to Einstein gravity with a cosmological constant. The
problem with the first order formulation is that there exist no obvious metric-definition,
meaning that the physical interpretation is unclear.

In section 3, the Hamiltonian formulation is given, and a constraint analysis is per-
formed. For a canonical formulation there exist a prescription, based on purely geometrical
considerations, of how to identify the metric from the constraint algebra in any diffeo-
morphism invariant theory. I use this prescription, and the metric is identified. Then, I
compare the unified theory to the conventional Einstein-Yang-Mills theory, and show that
in the weak field limit, the two formulations agree. It is also shown that the equations
of motion governing the Yang-Mills fields is the normal Yang-Mills equations even for an
arbitrary strength of the Yang-Mills field.

In section 4, the second order pure connection formulation of the unified theory is
presented, as well as the metric formulas for the metric in terms of the connection.

Finally in section 5, explicit static and rotation symmetric solutions to both the con-
ventional and the unified Einstein-Maxwell theory, is found and compared. It is shown
that for weak Maxwell fields the solutions to the conventional theory can be found as the
lowest order terms in the solutions to the unified theory. One new interesting feature is
found in the electric solution to the unified theory: Point charges are not allowed, there
exist no solutions inside a radius $r = q$ in Schwarschild coordinates.
2 First order Lagrangian formulation

In this section, I will present a first order Lagrangian, which is a function of a gauge connection and a trio of Lie-algebra valued vector fields. For the special choice of $SO(1,2)$ as the gauge group, this Lagrangian describes ordinary Einstein gravity with a cosmological constant. In that case, the gauge connection is the spin-connection, and the Lie-algebra valued vector fields equals the triad field. For other gauge groups, there is no obvious interpretation of the theory, here in the first order Lagrangian form. However, in the Hamiltonian formulation of the theory, which is given in section 3, there exists a geometrical prescription of how to read off the metric in the constraint algebra, and it will then become clear that the metric in this theory is still the square of the vector fields.

The Lagrangian, valid for an arbitrary gauge group is

$$\mathcal{L} = \epsilon^{\alpha\beta\gamma} e_{\alpha I} F^I_{\beta\gamma} + \lambda \sqrt{-g}$$  \hspace{1cm} (1)

where $e_{\alpha I}$ and $A^I_{\alpha}$ are the basic fields. $F^I_{\alpha\beta} = \partial_{\alpha} A^I_{\beta} - \partial_{\beta} A^I_{\alpha} + f^{IJK}_{\alpha} A_{\alpha J} A_{\beta K}$. $f^{IJK}_{\alpha}$ are the structure constants of the gauge group, and the "gauge-in-dices" are raised and lowered with a bilinear invariant form of the Lie-algebra. For $SO(1,2)$ I will always choose this "group-metric" to be $\eta_{IJ} = \text{diag}(-1,1,1)$. The space-time indices are denoted $\alpha, \beta, \gamma$ and take values 0, 1, 2. $g_{\alpha\beta} := e^{*}_{\alpha I} e_{\beta I}$ and $g = \frac{1}{6} \epsilon^{\alpha\beta\gamma} \epsilon_{\delta\epsilon\sigma} g_{\alpha \delta} g_{\beta \epsilon} g_{\gamma \sigma}$. $\epsilon^{\alpha\beta\gamma}$ is totally anti-symmetric and $\epsilon^{012} = 1$ in every coordinate system, which means that $\epsilon^{\alpha\beta\gamma}$ is a tensor density of weight +1.

Now, for gauge group $SO(1,2)$ it is true that $g = -e^2$, where $e = \frac{1}{6} \epsilon^{\alpha\beta\gamma} f^{IJK}_{\alpha} e_{\alpha I} e_{\beta J} e_{\gamma K}$, and the Lagrangian (1) can be rewritten as

$$\mathcal{L}_1 = \epsilon^{\alpha\beta\gamma} e_{\alpha I} F^I_{\beta\gamma} + \lambda e$$ \hspace{1cm} (2)

which is a Lagrangian for Einstein gravity with a cosmological constant. See Witten [7]. Using another formula valid for $SO(1,2)$: $e_{\alpha I} \epsilon^{\alpha\beta\gamma} = e f^{IJK}_{\alpha} e_{\beta J} e_{\gamma K}$, where $e^{\beta J}$ is the inverse to $e_{\alpha I}$, the Lagrangian can be changed into:

$$\mathcal{L}_2 = e f^{IJK}_{\beta} e_{\gamma K} F^I_{\beta\gamma} + \lambda e$$ \hspace{1cm} (3)

which is the ordinary Hilbert-Palatini Lagrangian. This shows that the Lagrangian (1) is a natural generalization of the Hilbert-Palatini Lagrangian, to other gauge groups. It is also possible to generalize (2) and (3) directly to other gauge groups. This will however not lead to any interesting Hamiltonians. $\mathcal{L}_2$ will lead to second class constraints, if the gauge group has dimension greater than three, and $\mathcal{L}_1$ will lead to a trivial Hamiltonian describing a theory without any local degrees of freedom. So, it is only (1) that will give an interesting Hamiltonian.

Returning to (1), the equations of motion is

$$\frac{\delta S}{\delta A^I_{\alpha I}} = -2 \mathcal{D}_\beta e^{\beta\alpha\gamma} e^I_{\gamma} = 0$$ \hspace{1cm} (4)

$$\frac{\delta S}{\delta e_{\alpha I}} = \epsilon^{\alpha\beta\gamma} F^I_{\beta\gamma} + \lambda \sqrt{-g} e_{\alpha I} = 0$$ \hspace{1cm} (5)

where $e^I_{\alpha} := g^{\alpha\beta} e^I_{\beta}$ and $g^{\alpha\beta}$ is the inverse to $g_{\alpha\beta}$. Note that it is true that $e^I_{\alpha} e_{\beta I} = \delta^I_{\beta}$ for an arbitrary gauge group, but it is not true in general that $e^I_{\alpha} e_{\alpha J} = \delta^I_{J}$. It is only for
three dimensional gauge groups that both relations hold. For higher dimensional groups, 
\(e^a_I e_{aJ}\) is just a degenerate matrix of rank three.

Equation (8) is normally, for \(SO(1,2)\), called the torsion-free condition, which can be solved to give the spin-connection as a function of the triad. For higher dimensional gauge groups, it is still possible to solve (8) to get an expression for \(A_{aI}\) in terms of \(e_{aI}\). This will however not totally fix the connection. The solution of (8) will only give the parts of the connection that are non-orthogonal to \(e_{aI}\) in the “gauge-indices”. Equation (8) is for \(SO(1,2)\) just Einstein’s equations with a cosmological constant, but for other gauge groups, I see no obvious interpretation. It’s interpretation will become much more transparent in the Hamiltonian formulation, in next section.

3 Hamiltonian formulation

Starting from the Lagrangian (1), valid for an arbitrary gauge group, I will now perform the Legendre transform to a Hamiltonian formulation, find all constraints and calculate the constraint algebra. Then, this Hamiltonian formulation will be compared to the conventional Ashtekar and Witten Hamiltonian formulations of (2+1)-dimensional Einstein gravity. See Bengtsson [6], and Witten [7]. Using the constraint algebra, I will then read off the space-time metric in this theory, and finally I will compare this generalized theory to the normal Einstein-Yang-Mills theory, to see whether it is possible to find the normal theory in some limit of the unified theory.

3.1 Legendre transform

Inspection of the Lagrangian (1) gives that the velocities of \(e_0^I\) and \(A_0^I\) are absent in the Lagrangian, meaning that these fields will become Lagrange multiplier fields. So the only field that will have a non-vanishing momenta is \(A_{aI}\). Defining the momenta

\[
\Pi^{aI} := \frac{\partial L}{\partial A_{aI}} = \epsilon_{ab}^I e_b^I
\]

where \(a, b, c\) are spatial indices, taking values 1 and 2. \(\epsilon_{ab}^I = \epsilon_{0ab}^I\). I also define \(\epsilon_{ab}\) to be anti-symmetric and \(\epsilon_{12} = 1\). This means that \(\epsilon_{ab}^I \epsilon_{cd}^J = \delta^a_c \delta^b_d - \delta^a_d \delta^b_c\). Now, (8) can be easily inverted, to get

\[
\epsilon_{aI} = \epsilon_{ba} \Pi_I^b
\]

Putting this into the Lagrangian (1) gives

\[
L = \Pi^a_I F^I_{a0} + e_{0I} \Psi_I + \lambda \sqrt{- e_{0I} e_0^I det(\Pi^{aI} \Pi^I_{aI})} + \epsilon_{ab} \epsilon_{cd} \Pi^{bl} I \Pi^b_{aJ} e_{0j} \Pi^{cK} e_{0K}
\]

where \(\Psi^I := \epsilon_{ab}^I F_{ab}^I\) and \(det(\Pi^{aI} \Pi^I_{aI}) = \frac{1}{2} \epsilon_{ab} \epsilon_{cd} \Pi^{bl} I \Pi^b_{aJ} e_{0j} \Pi^{cK} e_{0K}\).

The Lagrangian has an inhomogeneous dependence of the Lagrange multiplier field \(e_{0I}\), which will lead to constraints that depend on the Lagrange multiplier field. This is normally not wanted, so I eliminate this inhomogeneous dependence first. To do this, I project out two components of \(e_{0I}\):

\[
e_{0I} = M_a \Pi_a^I + V_I, \quad V_I \Pi^a_I = 0 \Rightarrow M_a = \epsilon_{0I} \Pi^b_I q_{ba}, \quad V_I = e_{0I} - M_a \Pi_a^I
\]
where \( q_{ab} \) is the inverse to \( q^{ab} := \Pi^a_I \Pi^b_I \).

This gives the Lagrangian

\[
\mathcal{L} = \Pi^a_I F^I_{0a} + M_a \Pi^a_I \Psi^I + V_I \Psi^I + \lambda \sqrt{-V_I V_I \det(q^{ab})} \tag{10}
\]

The Lagrangian is still inhomogeneous in \( V_I \), but the variation of \( V_I \) gives:

\[
\frac{\delta S}{\delta V_I} = \Psi^I - \lambda \frac{1}{\sqrt{-V_I V_I \det(q^{ab})}} \det(q^{ab}) V_I = 0 \tag{11}
\]

and taking the square of (11),

\[
\Psi^I \Psi^I = -\lambda^2 \det(q^{ab}) \tag{12}
\]

This means that the variation of \( V_I \) imposes the constraint (12), and I can remove all terms in the Lagrangian containing \( V_I \), and put in the equivalent constraint (12) with a Lagrange multiplier, \( N \). Then finally, I redefine the Lagrange multiplier \( N^a := 2 \epsilon^{ba} M_b \) for later comparison with the normal formulations of gravity. The Hamiltonian is

\[
\mathcal{H} = \Pi^a_I \dot{A}^I_a - \mathcal{L}
\]

where \( N, N^a, A_{0I} \) are Lagrange multiplier fields, and \( \mathcal{H}, \mathcal{H}_a \) and \( \mathcal{G}_I \) are constraints following from variation of the Lagrange multiplier fields. The constraints are normally called the Hamiltonian constraint, the vector constraint and Gauss’ law.

This Hamiltonian will for now on be referred to as ”the unified Hamiltonian” and the theory it describes for an arbitrary gauge group will be called ”the unified theory”. In subsection 3.4, it will be shown how this unified Hamiltonian is related to the Ashtekar and Witten Hamiltonians for pure gravity.

### 3.2 Constraint analysis

Now, to make sure that there are no more constraints in the theory, one must check that the time evolution of the constraints vanishes weakly. And to check that, one needs the constraint algebra. To derive the constraint algebra, I start by deriving how the basic canonical fields transforms under transformations generated by \( \mathcal{G}_I \) and \( \mathcal{H}_a := \mathcal{H} + A_{0I} \mathcal{G}_I \):

\[
\delta \mathcal{G}^I[\Lambda^I] \Pi^a_I(x) = \{ \Pi^a_I(x), \mathcal{G}_I[\Lambda^I] \} = \Lambda^J(x) \Pi^a_K f_{JKI} \tag{14}
\]

\[
\delta \mathcal{G}^I[\Lambda^I] A_{0I}(x) = \{ A_{0I}(x), \mathcal{G}_I[\Lambda^I] \} = -\partial_a \Lambda^I(x) \tag{15}
\]

\[
\delta \mathcal{R}_a[N^a] \Pi^a_I(x) = \{ \Pi^a_I(x), \mathcal{H}_a[N^a] \} = -\mathcal{L}_{N^a} \Pi^a_I(x) \tag{16}
\]

\[
\delta \mathcal{R}_a[N^a] A_{0I}(x) = \{ A_{0I}(x), \mathcal{H}_a[N^a] \} = -\mathcal{L}_{N^a} A_{0I}(x) \tag{17}
\]
where \( \mathcal{G}^I[\Lambda_I] = \int d^2 x \mathcal{G}^I(x) \Lambda_I(x) \), and \( \mathcal{L}_{N^a} \) is the Lie-derivative in the direction \( N^a \). This means that \( \mathcal{G}^I \) is the generator of gauge transformations, and \( \mathcal{H}_a \) is the generator of spatial diffeomorphisms. It is then clear how an arbitrary, gauge and diffeomorphism covariant, function \( \Phi_{aI}(\Pi^a_I, A^j_b) \) of the basic fields will transform under the transformations generated by \( \mathcal{G}^I \) and \( \mathcal{H}_a \).

\[
\delta \mathcal{G}^I[\Lambda_I] \Phi_{aI}(\Pi^a_I, A^j_b) = \Lambda^j \Phi^K_a(\Pi^a_I, A^j_b) f_{JKI} \quad (18)
\]

\[
\delta \mathcal{H}_a[N^a] \Phi_{aI}(\Pi^a_I, A^j_b) = -\mathcal{L}_{N^a} \Phi_{aI}(\Pi^a_I, A^j_b) \quad (19)
\]

This makes it simple to calculate the constraint algebra

\[
\{ \mathcal{G}^I[\Lambda_I], \mathcal{G}^J[\Gamma_J] \} = \mathcal{G}^K[f_{KIJ}\Gamma^J] \quad (20)
\]

\[
\{ \mathcal{G}^I[\Lambda_I], \mathcal{H}_a[N^a] \} = \mathcal{G}^I[\mathcal{L}_{N^a} \Lambda^I] \quad (21)
\]

\[
\{ \mathcal{G}^I[\Lambda_I], \mathcal{H}[N] \} = 0 \quad (22)
\]

\[
\{ \mathcal{H}_a[N^a], \mathcal{H}_b[M^b] \} = \mathcal{H}_a[\mathcal{L}_{M^b}N^a] \quad (23)
\]

\[
\{ \mathcal{H}[N], \mathcal{H}_a[N^a] \} = \mathcal{H}[\mathcal{L}_{N^a}N] \quad (24)
\]

The reason why the Poisson-bracket between \( \mathcal{G}^I \) and \( \mathcal{H}_a \) is not zero is that \( \mathcal{H}_a \) has a non-gauge covariant dependence on \( A^j_a \). Now, it is only one Poisson-bracket left to calculate:

\[
\{ \mathcal{H}[N], \mathcal{H}[M] \} = \mathcal{H}_a[\Pi^a_I \Pi^j_b (N \partial^b_M - M \partial^b_N)] \quad (25)
\]

Equations (20)-(25) then give the complete constraint algebra, and the constraints form a first class set, and therefore the time-evolution of the constraints is automatically vanishing on the constraint surface. So the Hamiltonian (13) is complete, and defines a consistent theory, in this sense.

### 3.3 Reading off the metric

Given a diffeomorphism invariant theory with no obvious physical interpretation, it is not a trivial task to identify the metric-field in the theory. A basic requirement of the metric-definition should be that a given test-particle should propagate on geodesics to that metric. And if the theory is physically sensible, the propagation should be causal. Hojman Kuchař and Teitelboim [8] has examined a related question. They considered a general canonical formulation of a diffeomorphism invariant theory, in which they defined the generator of parallel deformations \( \mathcal{H}_\parallel \), and the generator of normal deformations \( \mathcal{H}_\perp \) with respect to the hypersurface. Then, by requiring path-independence of deformations, they derived a specific deformation algebra: \( \{ \mathcal{H}_\parallel, \mathcal{H}_\parallel \} \sim \mathcal{H}_\parallel \), \( \{ \mathcal{H}_\parallel, \mathcal{H}_\perp \} \sim \mathcal{H}_\perp \) and \( \{ \mathcal{H}_\perp, \mathcal{H}_\perp \} \sim \mathcal{H}_\parallel \), where the crucial part is the last bracket. These authors showed that in order to have path-independence of deformations, the structure function in the last bracket must equal the spatial metric on the hypersurface. This means that, if one knows the generator of parallel and normal deformations, it is possible to read-off the spatial metric in the constraint algebra. (In a canonical field theory, the generators of local symmetries are represented by first class constraints.) The parallel generator is singled out by the
requirement that its action on the fundamental canonical fields should be that of the Lie-derivative. This means that here is: $H^\parallel = \tilde{H}_a$. And since the pair $H, \tilde{H}_a$ satisfies the needed algebra, I conclude that $H^\perp = H$. That is because, if that would not be the case, the true generator of normal deformations could be written as: $H^\perp = H + f^a(\Pi^aI, A_{aI})\tilde{H}_a$. Checking the constraint algebra for this generator shows however that the required algebra is only found for $f^a(\Pi^aI, A_{aI}) = 0$. So, here $H^\parallel = \tilde{H}_a$ and $H^\perp = H$, meaning that the spatial metric can be read off in (25). The spatial metric is $q^{ab} = \Pi^aI\Pi^bI$. Then, to find the time-time and time-space parts of the space-time metric, one should calculate the time-evolution of the spatial metric. According to Hojman, Kuchař and Teitelboim, it is required that:

$$\{q_{ab}(x), H_{tot}\} = K_{ab}(x)$$  \hspace{1cm} (26)

where $K_{ab}$ is the extrinsic curvature of the hypersurface embedded in space-time. Together with the normal definition of the extrinsic curvature in terms of the space-time metric, this gives, with the help of the equations of motion, the following form of the space-time metric.

$$\tilde{g}^{\alpha\beta} = \sqrt{-g}g^{\alpha\beta} = \left( -\frac{1}{N^a} N^a, N^a N^b q^{ab} - \frac{N^a N^b}{N} \right)$$  \hspace{1cm} (27)

That is, given any solution to the equations of motion following from (13), the space-time metric is given by (27). I do not know if this definition of the metric, which follows from purely geometrical considerations, always will fulfill the other requirements about geodesic, causal propagation of matter fields. Now, following the fields in (27) backwards in the Legendre transform will give that the object called $g_{\alpha\beta}$ in section 2, really is the metric.

### 3.4 Comparison to the Ashtekar and Witten Hamiltonians for gravity

As shown in section 2, with the choice of $SO(1,2)$ as the gauge group, the unified theory describes ordinary Einstein gravity with a cosmological constant. I will now show that the unified Hamiltonian, with gauge group $SO(1,2)$, is related to the Witten [7], and to the Ashtekar [6] Hamiltonians via simple redefinitions of the constraints.

The Hamiltonian given by Witten, is

$$H_{tot}^W = N^I H_I - A_{0I}G^I$$

$$H_I := \Psi_I + \lambda f_{IJK} \Pi^{aJ} \Pi^{bK} \epsilon_{ab} \approx 0$$

$$G^I := D_a \Pi^{aI} \approx 0$$

which is just the Hamiltonian following from a Legendre transform from (13). Now, it is easy to write the constraints in (13) as linear combinations of those in (28):

$$H = g^I(A_{aI}, \Pi^{bJ})H_I = \frac{1}{4\lambda}(\Psi^I - \lambda f_{IJK} \Pi^{aJ} \Pi^{bK} \epsilon_{ab})H_I$$  \hspace{1cm} (29)

$$H_a = h^I_a(A_{aI}, \Pi^{bJ})H_I = \frac{1}{2}\epsilon_{ab} \Pi^{bI}H_I$$
And, the Ashtekar Hamiltonian given by Bengtsson [6], is

\begin{align*}
    H^{Ash}_{tot} & = N H^{Ash} + N^a H^{Ash}_a - A_0 \mathcal{G} I \\
    H^{Ash} & := \frac{1}{4} \varepsilon_{ab} \Pi^{aj} \Pi^{bj} \Psi^K f_{jJK} + \frac{\lambda}{2} det(\Pi^{aj} \Pi^{bj}) \approx 0 \\
    H^{Ash}_a & := \frac{1}{2} \varepsilon_{ab} \Pi^{bj} \Psi_I \approx 0
\end{align*}

(30)

This Hamiltonian can be found by a Legendre transform from (3), for $SO(1,2)$. It has the same form as the Ashtekar Hamiltonian for gravity in (3+1)-dimensions [9]. Again, the constraints in (13) can be written as simple combinations of the constraints in (30):

\begin{align*}
    \mathcal{H} & = \Phi(A^{aI}, \Pi^{bI}) H^{Ash} = \frac{-1}{\lambda det(\Pi^{aj} \Pi^{bj})} (\frac{1}{4} \varepsilon_{ab} \Pi^{aj} \Pi^{bj} \Psi^K f_{jJK} - \frac{\lambda}{2} det(\Pi^{aj} \Pi^{bj})) H^{Ash} \\
    \mathcal{H}_a & = H^{Ash}_a
\end{align*}

(31)

This shows the equivalence between the three Hamiltonians (13), (28) and (30), for $SO(1,2)$ and non-degenerate metric. Note however that, from (29) and (31) it is clear that the unified Hamiltonian (13) is not totally equivalent to the Witten, and Ashtekar Hamiltonians. That is because the generalized Hamiltonian has solutions to the constraints corresponding to $g^{ij}(A^{aI}, \Pi^{bI}) = 0$ or $\Phi(A^{aI}, \Pi^{bI}) = 0$. But this is a rather innocent extension of the original Einstein theory, it just means that the unified theory has solutions corresponding to both signs of the cosmological constant for a given cosmological constant. Classically this is of no importance while quantum mechanically this could lead to interesting effects, since the two different parts of phase-space are connected via degenerate metric solutions. What is more important, is that the unified theory really needs a cosmological constant. It can be arbitrarily small, but it must be non-zero.

This all together shows the near-equivalence of the Hamiltonians (13), (28) and (30) for gauge group $SO(1,2)$ and $\lambda$ non-zero. What about other gauge groups?

For (28) we still have a consistent theory for an arbitrary gauge group, the constraint algebra is closed. The constraint algebra will though not satisfy the general constraint algebra, given by Hojman, Kuchař and Teitelboim [8], due to the fact that the constraints are not split in parallel and normal generators of deformation. There is however another reason that makes it less interesting to generalize the Hamiltonian (28) to other gauge groups. This Hamiltonian will for an arbitrary gauge group always lack local degrees of freedom. (The number of first class constraints are half the number of phase-space variables.)

How about (30). This is the normal Ashtekar form of the constraints, and it is known that a crucial ingredients in the closure of the constraints algebra is the structure constants identity for $SO(1,2)$. For other gauge groups, which lack this identity, the constraint algebra will fail to close, and the theory will produce second class constraints.

So, it is only (13) that can be generalized to arbitrary gauge groups with a closed constraint algebra and local degrees of freedom. The interesting aspect of this generalization is that it may be possible to find a new interacting theory of gravity and Yang-Mills type of matter.
3.5 Comparison to conventional Einstein-Yang-Mills theory.

To compare the unified theory to the conventional Einstein-Yang-Mills theory, I will choose a gauge group \( G_{tot} = SO(1, 2) \otimes G^{YM} \), and rewrite the unified Hamiltonian (33) in a form similar to the conventional Hamiltonian. From there it is easy to show that the two formulations agree for weak Yang-Mills fields. Then, I will also show that the equations of motion for the Yang-Mills fields in the generalized theory is the normal Yang-Mills equations of motion, meaning that the changes come in the "Einstein’s equations” part.

The conventional Hamiltonian for coupled Einstein-Yang-Mills is given by Romano [10]. See also Ashtekar, Romano and Tate [11].

\[
\mathcal{H}_{tot}^{conv} = N\mathcal{H}^{conv} + N^a\mathcal{H}_a^{conv} - A_{0i}\mathcal{G}^i - A_{0i}\mathcal{G}^i \\
\mathcal{H}^{conv} := \frac{1}{4}\epsilon_{ab}\Pi^a\Pi^b\Psi^K f_{IJK} + \frac{\lambda}{2}\det(\Pi^a\Pi^b) + \frac{1}{4}(\epsilon_{ab}\epsilon_{cd}\Pi^d\Pi^c E^a E^b + B^i B_b)
\]

where \( \{A_{ai}(x), \Pi^{bj}(y)\} = \delta^b_a\delta^j_i\delta^2(x-y) \) is the conjugate pair for the \( SO(1, 2) \) gravity part. And, \( \{A_{ai}(x), E^{bj}(y)\} = \delta^b_a\delta^j_i\delta^2(x-y) \) is the conjugate pair for the \( G^{YM} \) Yang-Mills matter part. \( B^i := e^{ab}F^i_{ab} \) and the covariant derivative \( \mathcal{D}_a \) "knows how to act" on both \( SO(1, 2) \) and \( G^{YM} \) indices. Note that in this formulation, the spatial metric is the square of \( \Pi^a; \ g^{ab} = \Pi^a\Pi^b \)

The unified theory, with gauge group \( G_{tot} = SO(1, 2) \otimes G^{YM} \), has the Hamiltonian:

\[
\mathcal{H}_{tot} = N\mathcal{H} + N^a\mathcal{H}_a - A_{0i}\mathcal{G}^i - A_{0i}\mathcal{G}^i \\
\mathcal{H} := \frac{1}{4}\epsilon_{ab}\Pi^a\Pi^b\Psi^K f_{IJK} + \frac{\lambda}{2}(\det(\Pi^a\Pi^b) + \epsilon_{ab}\epsilon_{cd}\Pi^d\Pi^c E^a E^b + \det(E^a E^b))
\]

Remember that here is the spatial metric the "square" of the total momenta: \( g^{ab} = \Pi^a\Pi^b + E^a E^b \). Now, to compare these two different formulations, I will rewrite \( \mathcal{H} \) in (33) in a form similar to the form of \( \mathcal{H}^{conv} \) in (32). To do this, I first solve \( \mathcal{H}_a = 0 \) for \( \Psi^I \):

\[
\Psi^I = \Phi(A_{ai}, \Pi^{bj}, A_{ai}, E^{bj}) f_{IJK} \Pi^K \epsilon_{a} - \Pi^a q_{ab} E^b B_i
\]

where \( \Phi(A_{ai}, \Pi^{bj}, A_{ai}, E^{bj}) \) is a yet unknown function, and \( q_{ab} \) is the inverse to \( q^{ab} := \Pi^a\Pi^b \). Putting this into \( \mathcal{H} = 0 \), gives:

\[
\Phi(A_{ai}, \Pi^{bj}, A_{ai}, E^{bj}) = \pm\left(\frac{\lambda^2}{4} + \frac{1}{4\det(q^{ab})}(\lambda^2\epsilon_{a} \Pi^d E^a E^b + \lambda^2 \det(E^a E^b) + B^i B_i + B_i E^{ai} q_{ab} B_j^j)^{\frac{1}{2}}\right)
\]
Equations (34) and (35) give the total solution to the diffeomorphism constraints \( H \) and \( H_a \). That means that (34) could be used together with (35) as constraints instead of \( H \) and \( H_a \). I wanted however constraints that looked like the conventional theory, and to get that I simply multiply (34) with \( f^{IJK} \Pi^K_a \Pi^b_I \epsilon_{ab} \). This finally gives the equivalent constraints:

\[
\tilde{H} = \frac{1}{4} \epsilon_{ab} \Pi^a_I \Pi^b_J \Psi^K f_{IJK} + \pm \frac{|\lambda| \det(q_{ab})}{2} \left( B^i B_i + B_i E^{ai} q_{ab} E^{bj} B_j \right)
\]

\[
H_a := \frac{1}{2} \epsilon_{ab} (\Pi^b_I \Psi_I + E^{bi} B_i)
\]  

(36)

When comparing the conventional Hamiltonian (32) and the unified Hamiltonian (36) it is important to note that \( \Pi^a_I \) in the two different theories does not have the same interpretation. In the conventional theory, \( \Pi^a_I \) is the "square-root" of the metric, while in the generalized theory, the more complicated relation \( g^{ab} = \Pi^a_I \Pi^b_I \) holds. However, for weak Yang-Mills fields, the two definitions coincide to lowest order in Yang-Mills fields. Now, it is easy to do a series expansion of \( \tilde{H} \) for small \( E^{ai} \) and \( B^i \) around de-Sitter space. Remember that the unified theory equals Einstein’s theory with a cosmological constant, when \( G = SO(1,2) \), meaning that de-Sitter space-time and vanishing Yang-Mills fields is a solution to the unified theory. So, \( E^{ai} E^b_i \ll 1 \), \( B^i B_i \ll \lambda^2 \) and \( \Pi^a_I \Pi^b_I \sim 1 \) ⇒

\[
\tilde{H} = \frac{1}{4} \epsilon_{ab} \Pi^a_I \Pi^b_I \Psi^K f_{IJK} + \frac{|\lambda|}{2} \det(\Pi^a_I \Pi^b_I) \pm \frac{|\lambda|}{4} (\epsilon_{ab} \epsilon_{cd} \Pi^b_I \Pi^d_I E^{ai} E^c_i + \frac{B^i B_i}{\lambda^2}) + \vartheta(\langle E^{ai} E^b_i \rangle^2, \frac{(B^i B_i)^2}{\lambda^4})
\]

(37)

which shows that the unified theory agrees with the conventional theory to lowest order in weak Yang-Mills fields on approximately de-Sitter space-time, if one rescales the Yang-Mills fields: \( \tilde{E}^{ai} := \sqrt{|\lambda|} E^{ai} \), \( \tilde{B}^i := \frac{1}{\sqrt{|\lambda|}} B^i \). \( \tilde{E}^{ai} \) and \( \tilde{B}^i \) is to be interpreted as the physical Yang-Mills fields. The \( \pm \) signs in front of the Yang-Mills coupling can always be absorbed in the \( G^{YM} \) "group-metric".

The question is then: What kind of corrections to the normal theory do we get for strong Yang-Mills fields? I will now show that it is only the "Einstein’s equation" part of the theory that changes. The Yang-Mills equation of motion remains unaltered. This is easily seen by comparing the equation of motion for the Yang-Mills fields coming from (32) and (33).

Conventional theory:

\[
\dot{A}_{ai} - D_a A_{0i} - \frac{N}{2} \Pi^{cf} \Pi^d_I \epsilon_{ca} \epsilon_{db} E^b_i - \frac{1}{2} N^b \epsilon_{ba} B_i = 0
\]

\[
-\dot{E}^{ai} + f^{ijk} E^c_j A_{0k} + \epsilon_{ca} D_c (N B^i) + \epsilon_{ca} D_c (N^e \epsilon_{eb} E^{bi}) = 0
\]

(38)  

(39)

And the unified theory:
\[
\dot{A}_{ai} - D_{a}A_{0i} - \frac{N\lambda}{2}(\Pi^{cI}\Pi^{dI} + E^{cI}E^{dI})e_{ca}e_{db}E_{i}^{b} - \frac{1}{2}N^{b}e_{bb}B_{i} = 0 \tag{40}
\]

\[
-\dot{E}^{ai} + f_{ijk}E_{j}^{a}A_{0k} + e^{ca}D_{c}(\frac{N}{\lambda}B^{i}) + e^{ca}D_{c}(N^{c}e_{cb}E^{bi}) = 0 \tag{41}
\]

which together with the fact that \( q^{cd} = \Pi^{cI}\Pi^{dI} \) in the conventional theory, and \( q^{cd} = \Pi^{cI}\Pi^{dI} + E^{cI}E^{dI} \) in the generalized theory, gives that the physical Yang-Mills fields should satisfy the same equation of motion in both theories. The equation of motion for the gravity part will however be different in the two theories. This will become even clearer in the second order formulation, in the next section.

4 Second order, pure connection Lagrangian formulation.

In this section, I will derive a second order pure connection Lagrangian corresponding to the unified theory (1) and (13). This second order formulation is most easily found by eliminating the \( e_{\alpha I} \) field from the first order Lagrangian (1). It can also be found by performing a Legendre transform from the Hamiltonian (13), treating the diffeomorphism constraints as primary constraints. I will use the former method here.

I start with the first order Lagrangian:

\[
\mathcal{L} = \epsilon^{\alpha\beta\gamma}e_{\alpha I}F_{I}^{\gamma} + \lambda\sqrt{-g} \tag{42}
\]

Now, I want to eliminate the \( e_{\alpha I} \) field from the Lagrangian, by solving its equation of motion.

\[
\frac{\delta S}{\delta e_{\alpha I}} = F^{*\alpha I} + \lambda\sqrt{-g}e^{\alpha I} = 0 \tag{43}
\]

where \( F^{*\alpha I} := \epsilon^{\alpha\beta\gamma}F_{I}^{\beta\gamma} \). Remember that \( g_{\alpha\beta} := e_{\alpha I}e_{\beta I} \), \( g^{\alpha\beta}g_{\beta\gamma} = \delta_{\gamma}^{\alpha} \), \( g = \frac{1}{6}\epsilon^{\alpha\beta\gamma}\epsilon^{\delta\epsilon\sigma}g_{\alpha\delta}g_{\beta\epsilon}g_{\gamma\sigma} \), \( e^{\alpha I} := g^{\alpha\beta}e_{\beta I} \). Equation (43) gives that

\[
F^{*\alpha I}F_{I}^{*\beta} = -\lambda^{2}gg^{\alpha\beta} \tag{44}
\]

Taking the determinant of both sides, yields

\[
\text{det}(F^{*\alpha I}F_{I}^{*\beta}) = -\lambda^{6}g^{2} \tag{45}
\]

Now, (13) and (13) give

\[
e^{\alpha I} = \text{sign}(-\lambda)(-\frac{1}{\lambda^{2}}\text{det}(F^{*\alpha I}F_{I}^{*\beta}))^{-\frac{1}{4}}F^{*\alpha I} \tag{46}
\]

Putting this into (12), gives

\[
\mathcal{L} = -2\text{sign}(\lambda)(-\frac{1}{\lambda^{2}}\text{det}(F^{*\alpha I}F_{I}^{*\beta}))^{\frac{3}{4}} \tag{47}
\]
which is the wanted pure connection, second order Lagrangian for the unified theory. For gauge group $SO(1,2)$ it is possible to simplify the Lagrangian slightly by using the relation: $\det(F^{*\alpha I}F^I) = -\det(F^{*\alpha I})^2$, $\det(F^{*\alpha I}) = \frac{1}{6}\epsilon_{\alpha\beta\gamma}\tilde{f}_{IJK}F^{*\alpha I}F^{*\beta J}F^{*\gamma K}$. This gives the Lagrangian

$$\mathcal{L}^{SO(1,2)} = -\frac{2}{\lambda}\sqrt{\det(F^{*\alpha I})}$$

(48)

which is the Lagrangian that was found in [12], using a Legendre transform from Ashtekar’s Hamiltonian for gravity with a cosmological constant.

### 4.1 Metric formulas.

To find the metric formulas in this pure connection formulation, I start with the expression for the space-time metric (27) that followed from Hojman, Kuchař and Teitelboim’s “metric definition”. Then one can just follow the fields through the Legendre transform. The metric formula is

$$\tilde{g}^{\alpha\beta} = \sqrt{-g}g^{\alpha\beta} = \left(\frac{-1}{N}N^a N^b - \frac{N^a N^b}{N}\right)$$

(49)

given in terms of the phase-space coordinates and Lagrange multiplier fields. Now, performing the Legendre transform, and following the fields through, gives:

$$\tilde{g}^{\alpha\beta} = \frac{1}{\lambda^2}\sqrt{\det(F^{*\alpha I}F^I)}$$

(50)

$$\sqrt{-g} = \frac{1}{|\lambda|}(-\frac{1}{\lambda^2}\det(F^{*\alpha I}F^I))^{\frac{1}{4}}$$

(51)

$$g^{\alpha\beta} = (-\frac{1}{\lambda^2}\det(F^{*\alpha I}F^I))^{-\frac{1}{2}}F^{*\alpha I}F^I$$

(52)

$$g_{\alpha\beta} = -\frac{2}{\lambda^2}((-\frac{1}{\lambda^2}\det(F^{*\alpha I}F^I))^{-\frac{1}{2}}F^{*\gamma I}F^I F^{*\sigma} F_{\sigma\beta I})$$

(53)

### 4.2 Equations of motion

Finally in this section, I will show that the equations of motion following from the Lagrangian (17) is just the ordinary Yang-Mills equations. Varying the Lagrangian (17) with respect to the connection $A_{\alpha I}$, and using the metric-formulas (50)-(53) gives

$$\frac{\delta S}{\delta A_{\alpha I}} = \mathcal{D}_\beta(\sqrt{-g}g^{\beta\sigma}g^{\alpha\sigma}F^I_{\gamma\sigma}) = 0$$

(54)

the Yang-Mills equations of motion. Note however that here is the metric-field not a non-dynamically background field. The metric is a function of the connection. Since this Lagrangian also is valid for pure gravity with a cosmological constant this means that Einstein gravity with a cosmological constant in (2+1)-dimensions really is an $SO(1,2)$ Yang-Mills theory, on its own dynamical background space-time! For other gauge groups, like $G^{tot} = SO(1,2) \otimes G^{YM}$, this means that the equation of motion for the Yang-Mills field is the normal one, while Einstein’s equation is changed.
5 Static, rotation symmetric solutions to Einstein-Maxwell theory.

To compare the unified theory, presented in section 2-4, with the conventional Einstein-Yang-Mills theory, I will here study some simple explicit solutions to both theories. I will study the Einstein-Maxwell theory, which means choosing gauge group $G_{\text{tot}} = SO(1, 2) \otimes U(1)$ for the unified theory. To simplify the theories as much as possible, I will only look for static and rotation symmetric solutions. These kind of solutions has previously been studied for Einstein-Maxwell without a cosmological constant, by Deser and Mazur [13] and Melvin [14]. However, for a non-vanishing cosmological constant, I have not been able to find any work done on explicit solutions to this theory, so I will start by deriving solutions to the conventional theory. Then, the unified theory will be solved under the same assumptions of symmetries, and the explicit solutions will be compared. In the solutions it will be clear that the solution to the conventional theory is the lowest order term, in Maxwell fields, in the solution to the unified theory. Otherwise, the only interesting new feature for the solution to the generalized theory, is that, for a static electric charge, there exist no solution to the unified theory, inside a circle of radius $r = q$ in Schwardschild coordinates.

5.1 Conventional theory.

The conventional Einstein-Maxwell theory with a cosmological constant was given in (32).

\begin{align}
\mathcal{H}^{\text{conv}}_{\text{tot}} &= N \mathcal{H}^{\text{conv}} + N^a \mathcal{H}^{\text{conv}}_a - A_{0I} \mathcal{G}^I - A_0 \mathcal{G} \\
\mathcal{H}^{\text{conv}}_a &= \frac{1}{4} \epsilon_{ab} \Pi^a \Pi^b \Psi^K f_{IJK} + \frac{\lambda}{2} det \{ \Pi^a \Pi^b \} + \frac{1}{4} (\epsilon_{ab} \epsilon_{cd} \Pi^d \Pi^c E^a E^c + B^2) \approx 0 \\
\mathcal{G}^I_{\text{conv}} &= \mathcal{D}_a \Pi^a_I \approx 0 \\
\mathcal{G} &= \mathcal{D}_a E^a \approx 0
\end{align}

and the space-time metric is

\begin{equation}
\tilde{g}^{\alpha\beta} = \sqrt{-g} g^{\alpha\beta} = \left( \begin{array}{cc} -\frac{1}{N} & \frac{N^a}{N} \\ \frac{N^a}{N} & -\frac{N^a N^b}{N} \end{array} \right) \end{equation}

Now, a static and rotation symmetric metric can always be put in a form

\begin{equation}
g_{\alpha\beta} = \left( \begin{array}{ccc} -\xi^2(r) & 0 & 0 \\ 0 & \chi^2(r) & 0 \\ 0 & 0 & \psi^2(r) \end{array} \right)
\end{equation}

where $\xi(r), \chi(r)$ and $\psi(r)$ are three arbitrary functions of some radial coordinate $r$. One could also fix the $r$-coordinate gauge by e.g choosing $\psi(r) = r$, which corresponds to Schwardschild coordinates, but I will not fix that gauge yet. This static, rotation symmetric metric then means

\begin{equation}
N^a = 0, \quad \Pi^r \Pi^0_I = 0
\end{equation}
But I still have to fix the $SO(1, 2)$ and $U(1)$ gauge. The $U(1)$ gauge should be fixed so that the gauge choice one makes always can be reached by a $U(1)$ gauge transformation from an arbitrary static and rotation symmetric field configuration. This singles out $A_r = 0$ as a good gauge choice. For the $SO(1, 2)$ gauge, one has the same requirements plus that one wants the spatial metric to be positive definite. Therefore, I make the choices: $\Pi^I = (0, \psi(r), 0)$, $\Pi^{\theta} = (0, 0, \chi(r))$. This is three $SO(1, 2)$ gauge choices together with the orthogonality condition (58). Note that this gauge choice will agree with the static and rotation symmetric metric (57), if $N = \frac{\xi(r)}{\chi(r)\psi(r)}$. Now, it is a straightforward task to put in this static and rotation symmetric Ansatz together with the gauge choices, into the constraints and equations of motion following from the Hamiltonian (55). This will rather immediately give that

\begin{align*}
E^r &= q, \quad E^\theta = 0, \quad B = \frac{k}{N(r)}, \quad qk = 0
\end{align*}  

where $q$ and $k$ are constants. This means that one cannot have both electric and magnetic static, rotation symmetric fields in the same solution. I start with the electric solution.

5.1.1 Electric solution to conventional theory.

Electric solution means $k = 0$ and $q \neq 0$. Using this in the equations of motion and constraints gives

\begin{align*}
E^r &= q, \quad E^\theta = 0, \quad B = 0 \\
A_{01} &= 0 \quad A_{03} = 0 \\
A_{\theta 2} &= 0 \quad A_{\theta 3} = 0 \\
A_{r1} &= 0 \quad A_{r2} = 0 \quad A_{r3} = 0
\end{align*}

(60)

and defining $\xi(r) := N(r)\psi(r)\chi(r)$ gives the equations:

\begin{align*}
A'_{0} &= -\frac{q}{2} \frac{\xi(r)\chi(r)}{\psi(r)} \\
A_{02} &= \frac{\xi'(r)}{\chi(r)} \\
A'_{02} &= -\frac{\lambda}{2} \xi(r)\chi(r) + \frac{q^2}{4} \frac{\xi(r)\chi(r)}{\psi^2(r)} \\
A_{i\theta} A_{02} &= \frac{\lambda}{2} \xi(r)\psi(r) + \frac{q^2}{4} \frac{\xi(r)}{\psi(r)} \\
A'_{\theta} &= -\frac{\psi'(r)}{\chi(r)} \\
A'_{\theta} &= \frac{\lambda}{2} \psi(r)\chi(r) + \frac{q^2}{4} \frac{\chi(r)}{\psi(r)}
\end{align*}

(61-66)

where the prime denotes differentiation with respect to $r$. Remember that the $SO(1, 2)$ "group-metric" is $\eta_{IJ} = diag(-1, 1, 1)$, meaning that $A^1_0 = -A_{01}$. Now it is time to fix the $r$-coordinate gauge, and this is done here by choosing Schwardscild coordinates: $\eta_{\theta \theta} = \psi^2(r) = r^2$. Using this gauge, it is an easy task to solve all the equations (62)-(66), yielding
\[
\chi(r) = \frac{1}{\sqrt{C_1 - \frac{\lambda r^2}{2} - \frac{q^2}{2} \log(r)}} \tag{67}
\]
\[
\xi(r) = C_2 \sqrt{C_1 - \frac{\lambda r^2}{2} - \frac{q^2}{2} \log(r)} \tag{68}
\]

where \( C_1 \) and \( C_2 \) are constants of integration. This gives for the metric

\[
g_{\alpha\beta} = \begin{pmatrix}
-(C_2)^2(C_1 - \frac{\lambda r^2}{2} - \frac{q^2}{2} \log(r)) & 0 & 0 \\
0 & (C_1 - \frac{\lambda r^2}{2} - \frac{q^2}{2} \log(r))^{-1} & 0 \\
0 & 0 & \gamma^2
\end{pmatrix} \tag{69}
\]

If one puts \( \lambda = 0 \), one recovers the solution found by Deser and Mazur \[13\] and Melvin \[14\]. The solution for the connection follows from equations (61)–(66). If one want to relate this solution to the normal covariant Einstein equation, the solution (69) solves the equations:

\[
R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = \frac{\lambda}{2} g_{\alpha\beta} - T_{\alpha\beta} \tag{70}
\]

\[
T_{\alpha\beta} = 2 F_{\alpha\gamma} g^{\sigma\rho} F_{\beta\sigma} - \frac{1}{2} g_{\alpha\beta} (g^{\gamma\sigma} g^{\epsilon\delta} F_{\gamma\epsilon} F_{\sigma\delta})
\]

\[
\partial_\alpha (\sqrt{-g} g^{\alpha\beta} g^{\gamma\rho} F_{\beta\sigma}) = 0
\]

with

\[
F_{\alpha\beta} = \begin{pmatrix}
0 & -C_2 q & 0 \\
C_2 q & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \tag{71}
\]

The normal factor, \( 8\pi G \) is included in \( q \).

5.1.2 Magnetic solution to conventional theory.

Magnetic solution means that \( q = 0 \) but \( k \neq 0 \). This gives

\[
E^r = 0 \quad E^\theta = 0 \quad B(r) = \frac{k}{N(r)}
\]

\[
A_{01} = 0 \quad A_{03} = 0 \\
A_{\theta 2} = 0 \quad A_{\theta 3} = 0 \\
A_{r 1} = 0 \quad A_{r 2} = 0 \quad A_{r 3} = 0
\]

and again defining \( \xi(r) := N(r) \psi(r) \chi(r) \) gives the equations:

\[
A_0 = \text{constant}\tag{73}
\]

\[
A_{02} = \frac{\xi'(r)}{\chi(r)} \tag{74}
\]

\[
A'_{02} = -\frac{\lambda}{2} \xi(r) \chi(r) + \frac{k^2 \chi(r)}{4 \xi(r)} \tag{75}
\]
\[ A_{\theta}^1 A_{\theta}^2 = \frac{\lambda}{2} \xi(r) \psi(r) - \frac{k^2 \psi(r)}{4 \xi(r)} \] (76)

\[ A_{\theta}^1 = -\frac{\psi'(r)}{\chi(r)} \] (77)

\[ A_{\theta}^1 = \frac{\lambda}{2} \psi(r) \chi(r) + \frac{k^2}{4} \frac{\chi(r) \psi(r)}{\xi^2(r)} \] (78)

Now it is time to fix the \( r \)-coordinate gauge, and the reason why I did not choose the Schwardschild gauge from the beginning, is that in choosing that gauge, I have not been able to find an explicit solution for the metric. That is, with the gauge choice \( g_{\theta \theta} = \psi^2(r) = r^2 \), the equations (74)-(78) has no simple explicit solution for \( \xi(r) \) and \( \chi(r) \). (One gets relations like: \(-\frac{\lambda}{4} \xi^2(r) + \frac{k^2}{4} \log(\xi(r)) = Dr\) So instead I try the gauge choice \( \xi(r) = r \). with this gauge choice it is easy to find the solution to (74)-(78).

\[ \chi(r) = \sqrt{C_3 - \frac{\lambda r^2}{2} + \frac{k^2}{2} \log(r)} \] (79)

\[ \psi(r) = C_4 \sqrt{C_3 - \frac{\lambda r^2}{2} + \frac{k^2}{2} \log(r)} \] (80)

where \( C_3 \) and \( C_4 \) are constants of integration. This gives for the metric

\[ g_{\alpha \beta} = \begin{pmatrix}
-r^2 & 0 & 0 & 0 \\
0 & (C_3 - \frac{\lambda r^2}{2} + \frac{k^2}{2} \log(r))^{-1} & 0 & 0 \\
0 & 0 & (C_4)^2 & 0 \\
0 & 0 & 0 & \frac{C_4 k}{r}
\end{pmatrix} \] (81)

which shows a remarkable symmetry between the electric and magnetic solution. If one puts \( \lambda = 0 \) it is possible to change coordinates into Schwardschild or conformal coordinates, and this solution will agree with Melvin’s solution [14].

In terms of the covariant Einstein equation, this solution solves (70), with

\[ F_{\alpha \beta} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & \frac{C_4 k}{r} & 0 \\
0 & -\frac{C_4 k}{r} & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \] (82)

5.2 The unified theory.

The unified theory with gauge group \( G^{\text{tot}} = SO(1, 2) \otimes U(1) \) is described by

\[ \mathcal{H}_{\text{tot}} = \mathcal{N} \mathcal{H}^{\text{tot}} - A_{0I} \mathcal{G}^I - A_0 \mathcal{G} \] (83)

\[ \mathcal{H} : = \frac{1}{4\lambda} (\Psi^I \Psi_I + B^2 + \lambda^2 (\det(\Pi^a_I \Pi^b_I)) + \epsilon_{ab} \epsilon_{cd} \Pi^b_I \Pi^d_I E^a E^c) \approx 0 \]

\[ \mathcal{H}_a : = \frac{1}{2} \epsilon_{ab} (\Pi^b_I \Psi_I + E^b B) \approx 0 \]

\[ \mathcal{G}_I : = D_a \Pi^a_I \approx 0 \]

\[ \mathcal{G} : = \partial_a E^a \approx 0 \]
where I have chosen the total "group-metric" to be \( \eta_{ij} = \text{diag}(-1, 1, 1, 1) \) and \( i = I \) for \( 1 \leq i \leq 3 \) and \( i = 4 \) concerns the \( U(1) \) fields. Note that there is nothing unique about this choice of "group-metric". I could equally well have chosen \( \eta_{44} = -1 \). The only requirement the "group-metric" has to fulfill is that it should be left invariant under gauge transformations. The densitized space-time metric is given by the formula

\[
\tilde{g}^{\alpha\beta} = \sqrt{-g} g^{\alpha\beta} = \left( \frac{-1}{N} \frac{N^a}{N} N g^{ab} - \frac{N^a N^b}{N} \right) \tag{84}
\]

where \( g^{ab} = \Pi^{aI} \Pi^b_I + E^a E^b \) now. The static and rotation symmetric Ansatz is the same as for the conventional theory:

\[
g_{\alpha\beta} = \begin{pmatrix}
-\xi^2(r) & 0 & 0 \\
0 & \chi^2(r) & 0 \\
0 & 0 & \psi^2(r)
\end{pmatrix} \tag{85}
\]

where \( \xi(r), \chi(r) \) and \( \psi(r) \) are three arbitrary functions of some radial coordinate \( r \). The only coordinate-gauge that is left to fix is the \( r \)-coordinate. This Ansatz then means

\[
N^a = 0, \quad \Pi^T_I \Pi_I^a = 0 \tag{86}
\]

Then it is time to fix the \( SO(1,2) \) and \( U(1) \) gauge. The \( U(1) \) gauge is fixed in the same way as for the conventional theory: \( A_r = 0 \). But, in fixing the \( SO(1,2) \) gauge one must now be more careful. What is required is again that the gauge choice could be reached by a \( SO(1,2) \) gauge transformation from an arbitrary static and rotation symmetric field configuration. And that the spatial metric \( g^{ab} \) should be positive definite. But since \( g^{ab} = \Pi^{aI} \Pi^b_I + E^a E^b \) this means that one must allow \( \Pi^{aI} \) to be time-like with respect to the \( SO(1,2) \) "group-metric". Since the solution to \( G = 0 \) and \( \dot{A}_\theta - \{A_\theta, H\} = 0 \) is \( E^r = q \) and \( E^\theta = 0 \), a good gauge choice respecting the mentioned requirements should be: \( E'^I = (q, \gamma(r), 0) \) and \( E'^I = (0, 0, \chi(r)) \). For \( \gamma(r) > q \) it is always possible to gauge-rotate (boost) \( E'^I \) into \( E'^I = (0, \psi(r) = \sqrt{\gamma^2(r) - q^2}, 0) \), and for \( \gamma(r) < q \) it is always possible to gauge-rotate (boost) \( E'^I \) into \( E'^I = (\sqrt{q^2 - \gamma^2(r)}, 0, 0) \). One can then go on and solve the constraints and equations of motion in the two different regions \( \gamma(r) > q \) and \( \gamma(r) < q \), and then glue the solutions back together. In doing this, one soon notice that there exist no real solution to the constraints \( H = 0 \) and \( H_a = 0 \) for \( \gamma(r) \) in the region \( \gamma(r) \leq q \). This means that it is in fact okay to use the same gauge choice here as in the conventional theory: \( E'^I = (0, \psi(r), 0) \) and \( E'^I = (0, 0, \chi(r)) \).

Now, using these gauge choices together with the static and rotation symmetric Ansatz in the constraints and equations of motion following from [83], gives

\[
E^r = q, \quad E^\theta = 0, \quad B = \frac{k}{N(r)}, \quad qk = 0 \tag{87}
\]

where \( q \) and \( k \) are konstants. This again means that one cannot have both electric and magnetic static, rotation symmetric fields in the same solution. I start with the electric solution.
5.2.1 Electric solution to the unified theory.

Electric solution: \( q \neq 0 \) and \( B = 0 \). Using this in the equations, gives

\[
\begin{align*}
E^r &= q \quad E^0 = 0 \quad B = 0 \\
A_{01} &= 0 \quad A_{03} = 0 \\
A_{\theta 2} &= 0 \quad A_{\theta 3} = 0 \\
A_{r 1} &= 0 \quad A_{r 2} = 0 \quad A_{r 3} = 0
\end{align*}
\]

(88)

and defining \( \xi(r) := N(r)\psi(r)\chi(r) \) gives the equations:

\[
\begin{align*}
A'_0 &= -\frac{\lambda q \xi(r)\chi(r)}{2} \\
A_{02} &= \pm \text{sign}(\lambda) \frac{\xi'(r)}{\chi(r)} \\
A'_{02} &= -\frac{\lambda}{2} \sqrt{\frac{\psi^2(r) - q^2}{\psi(r)}} \xi(r)\chi(r) \\
A^1_{\theta} A_{02} &= \frac{\lambda}{2} \xi(r)\psi(r) \\
A^1_{\theta} &= -\frac{\psi(r)}{\sqrt{\psi^2(r) - q^2}} \frac{\psi'(r)}{\chi(r)} \\
A'_{\theta} &= \pm \frac{|\lambda|}{2} \psi(r)\chi(r)
\end{align*}
\]

(89)–(94)

Now, choosing the Schwarzschild gauge \( g_{\theta\theta} = \psi^2(r) = r^2 \) gives the solution

\[
\begin{align*}
\chi(r) &= \frac{\sqrt{\psi^2(r) - q^2}}{\sqrt{\psi^2(r) - q^2} + \frac{|\lambda| q^2}{2} \log(r + \sqrt{r^2 - q^2})} \\
\xi(r) &= D_2 \sqrt{\psi^2(r) + \frac{|\lambda| r^2}{2} \sqrt{r^2 - q^2} + \frac{|\lambda| q^2}{2} \log(r + \sqrt{r^2 - q^2})}
\end{align*}
\]

(95)–(96)

where \( D_1 \) and \( D_2 \) are constants of integration. This gives for the space-time metric

\[
\begin{align*}
g_{tt} &= -(D_2)^2 \left( D_1 \mp \frac{|\lambda| r}{2} \sqrt{\psi^2(r) - q^2} \mp \frac{|\lambda| q^2}{2} \log(r + \sqrt{r^2 - q^2}) \right)
\quad (97) \\
g_{rr} &= \frac{r^2}{r^2 - q^2} \left( D_1 \mp \frac{|\lambda| r}{2} \sqrt{\psi^2(r) - q^2} \mp \frac{|\lambda| q^2}{2} \log(r + \sqrt{r^2 - q^2}) \right)^{-1} \\
g_{\theta\theta} &= r^2
\end{align*}
\]

Doing a Taylor series expansion of the metric for \( q \ll r \) gives

\[
\begin{align*}
g_{tt} &= -(D_2)^2 \left( D_1 \mp \frac{|\lambda| r^2}{2} \mp \frac{|\lambda| q^2}{2} \log(r) \right) + o\left(\frac{q^2}{r^2}\right)
\quad (98) \\
g_{rr} &= \left( D_1 \mp \frac{|\lambda| r^2}{2} \mp \frac{|\lambda| q^2}{2} \log(r) \right)^{-1} + o\left(\frac{q^2}{r^2}\right) \\
g_{\theta\theta} &= r^2
\end{align*}
\]

18
where \( \tilde{D}_1 = D_1 \pm \frac{\lambda q^2}{4} \pm \frac{\lambda q^2 \log(2)}{2} \). This shows that the solution to the generalized theory agrees with the solution to the conventional theory, to lowest order in the Maxwell fields. Note that the physical electric field is the rescaled \( E^{\alpha}_{\text{phys}} = \sqrt{\lambda} E^{\alpha} \).

Here in the explicit solution (97) it is clear that this solution is only valid outside a circle of radius \( r = q \), and this is not just a coordinate singularity. Calculating the invariant curvature scalar \( R := g^{\alpha \beta} R_{\alpha \beta} \), gives that \( R \) becomes complex inside this radius. It is also clear from the form of the constraints \( \mathcal{H} \) and \( \mathcal{H}_a \) that with \( B = 0 \) and \( \Pi^{\alpha I} \) time-like, there exist no real solution for \( \Psi^I \). (Remember that \( q_{\theta \theta} \leq q \) implies that \( \Pi^r \) is time-like.) The conclusion of this must then be that space-time does not exists inside a radius \( r = q \) around a point-charge. Or, electrical point-charges is not allowed, charged objects must have extensions. If this result would be valid in (3+1)-dimensions, we could calculate what distance \( r = q \) corresponds to for an electron: \( r \sim \frac{\sqrt{G}}{\sqrt{\epsilon_0} c^2} \sim 10^{-26} \text{m} \).

However, if this result is a purely (2+1)-dimensional effect, we do not know the constants of nature, and therefore cannot do this calculation.

What kind of “singularity” is \( r = q \) then? Does the curvature invariants diverge, or is the invariant distance to this limit infinite? I have only been able to check one curvature invariant, \( R \), and that invariant is finite and well behaved in the limit \( r \to q \)

\[
R \sim \frac{2C_1}{q^2} - \lambda \log(q) + \vartheta(\sqrt{r-q})
\]  

(99)

It is also clear that the invariant length \( \int_0^q \sqrt{g_{rr}} dr \) must be finite since \( g_{rr} \sim \frac{1}{r-q} \) for \( r \sim q \), which means that the integral is finite.

### 5.2.2 Magnetic solution to the unified theory.

Magnetic solution means: \( q = 0 \) and \( B \neq 0 \). Using this together with the static and rotation symmetric Ansatz as well as the gauge choices, gives

\[
E^r = 0 \quad E^\theta = 0 \quad B(r) = \frac{k}{N(r)}
\]

\[
A_{01} = 0 \quad A_{03} = 0
\]

\[
A_{02} = 0 \quad A_{03} = 0
\]

\[
A_{r1} = 0 \quad A_{r2} = 0 \quad A_{r3} = 0
\]

and defining \( \xi(r) := N(r) \psi(r) \chi(r) \) gives the equations:

\[
A_0 = \text{konst}
\]

(101)

\[
A_{02} = \pm \text{sign}(\lambda) \frac{\partial \sqrt{\xi^2(r) + \frac{k^2}{\lambda^2}}}{} \xi(r)
\]

(102)

\[
A_{02}' = -\frac{\lambda}{2} \xi(r) \chi(r)
\]

(103)

\[
A_0^1 A_{02} = \frac{\lambda}{2} \xi(r) \psi(r)
\]

(104)

\[
A_0^1 = -\frac{\psi'(r)}{\chi(r)}
\]

(105)

\[
A_0^1 = \pm \frac{\lambda}{2} \xi(r) \psi(r) \chi(r)
\]

(106)
Here I fix the $r$-coordinate gauge like I did for the conventional theory: $\xi(r) = r$. This gives the solution

\[
\chi(r) = \frac{r}{\sqrt{r^2 + \frac{k^2}{\lambda^2}}} \frac{1}{D_3 \mp \frac{|\lambda| r}{2} \sqrt{r^2 + \frac{k^2}{\lambda^2}} \pm \frac{|\lambda| k^2}{2 \lambda^2} \log(r + \sqrt{r^2 + \frac{k^2}{\lambda^2}})}
\]  

(107)

\[
\psi(r) = D_4 \sqrt{D_3 \mp \frac{|\lambda| r}{2} \sqrt{r^2 + \frac{k^2}{\lambda^2}} \pm \frac{|\lambda| k^2}{2 \lambda^2} \log(r + \sqrt{r^2 + \frac{k^2}{\lambda^2}})}
\]  

(108)

where $D_3$ and $D_4$ are constants of integration. This gives for the space-time metric

\[
g_{tt} = -r^2
\]  

(109)

\[
g_{rr} = \frac{r^2}{r^2 + \frac{k^2}{\lambda^2}} (D_3 \mp \frac{|\lambda| r}{2} \sqrt{r^2 + \frac{k^2}{\lambda^2}} \pm \frac{|\lambda| k^2}{2 \lambda^2} \log(r + \sqrt{r^2 + \frac{k^2}{\lambda^2}}))^{-1}
\]

\[
g_{\theta\theta} = (D_4)^2 (D_3 \mp \frac{|\lambda| r}{2} \sqrt{r^2 + \frac{k^2}{\lambda^2}} \pm \frac{|\lambda| k^2}{2 \lambda^2} \log(r + \sqrt{r^2 + \frac{k^2}{\lambda^2}}))
\]

and doing a Taylor series expansion of the metric for $\frac{k}{\lambda} \ll r$ gives to lowest order in $\frac{k}{\lambda}$, the magnetic solution to the conventional theory. Note here also that the physical magnetic field is the rescaled $B$: $B_{phys} = \frac{B}{\sqrt{\lambda}}$.

6 Conclusions and outlook.

The most interesting question to ask in a new physical theory, is if the theoretical predictions given by the theory agrees with the known experimental results. In the case of theories defined in space-time dimensions different from (3+1), we have of course no direct experimental results. This makes me asking another related question: What if we lived in (2+1)-dimensions and knew that our space-time was approximately Minkowskian, and that Yang-Mills equations described our physics to very high accuracy. Could we then rule out this unified theory, based on experimental results for Yang-Mills (Maxwell) theories? I would say that we could not. And that is because, as shown in both section 3 and 4, the equations of motion governing the matter Yang-Mills fields in the unified theory are the normal Yang-Mills equations, and that is what we can measure in experiments. It was also shown that for very weak Yang-Mills fields $A_{phys}^{YM} \ll \sqrt{\lambda}$ the unified theory agrees with the conventional coupling. It is first when we can measure the back-reaction of the metric-field, that we can rule out one of the two proposed theories. All this will be of interest if the same construction, for the unified theory, will work equally well in (3+1)-dimensions. There are however other facts about the unified theory, that could discriminate it from being a good physical theory. One thing is that it could be that one gets non-causal propagation of the fields in a solution. I have not addressed that question in this paper.

If one now tries the same type of construction for a unified theory in (3+1)-dimensions, one gets problem not encountered here for (2+1)-dimensions. My starting point for finding this unified theory was the Ashtekar Hamiltonian, and the possibility of generalizing it to
arbitrary gauge groups. It was in (3+1)-dimensions that the generalization of Ashtekar’s variables first was found \[15\]. It was there shown that there exist an infinite number of apparently distinct generalizations, while here for (2+1)-dimensions the generalization seems to be unique. This is closely related to the existence of an infinite number of cosmological constants in (3+1)-dimensions \[16\], while there is only one cosmological constant in (2+1)-dimensions \[12\]. Disregarding that problem, there is another problem in (3+1)-dimensions; the Ashtekar formulation is complex, and one needs reality conditions to select the real physical solutions, which may be hard to find in a possible unified theory.

The latest years there has been an increasing interest in (2+1)-dimensional quantum gravity. It would be interesting to see whether it is possible to quantize the unified theory also, using for instance the loop-representation quantization \[17\]. In fact, my motivation to start looking for a unified theory in terms of Ashtekar’s variables, was the possibility to find a theory that is better suited for the loop-representation quantization in (3+1)-dimensions \[18\], than the conventional Einstein-Yang-Mills theory \[11\]. See however Gambini and Püllin \[19\] for a treatment of the Einstein-Maxwell theory in the loop-representation.

Finally I summarize the most interesting features of the unified theory presented in this paper. For gauge-group $G^{\text{tot}} = SO(1,2) \otimes G^{YM}$, the equations of motion for the Yang-Mills part are the normal Yang-Mills equations. And for very weak Yang-Mills fields, the unified theory agrees with the conventional Einstein-Yang-Mills theory to lowest order in the Yang-Mills fields.

Acknowledgement

I thank Ingemar Bengtsson for discussions, ideas and criticism throughout this work. I also thank Professor Abhay Ashtekar for helping me with a problem in an early stage of this work.

References

[1] G. Nordström, Phys. Zeitsch. 15 (1914) 504
[2] T. Kaluza, Sitzungsber. Preuss. Akad. Wiss. Phys. Math. Klasse 966 (1921)
[3] O. Klein, Z. F. Physik 37 (1926) 895
[4] Modern Kaluza-Klein theories, T. Appelquist (ed), Frontiers in physics,(Addison-Wesley 1987), and references therein.
[5] S. Weinberg, Phys. Lett 138B (1984) 47
[6] I. Bengtsson Int. J. Mod. Phys. 4 (1989) 5527
[7] E. Witten, Nucl. Phys. B311 (1988) 46
[8] S. Hojman, K. Kuchař and C. Teitelboim, Ann. Phys., NY 96 (1976) 88
[9] A. Ashtekar, Phys. Rev. D36 (1987) 1587
[10] J. Romano, Geometrodynamics vs. connection dynamics. (In the context of (2+1)-
and (3+1)-gravity), Syracuse Thesis 1991

[11] A. Ashtekar, J. D. Romano and R. S. Tate, Phys. Rev. D40 (1989) 2572

[12] P. Peldán, Class. Quant. Grav 9 (1992) 2079

[13] S. Deser and P. O. Mazur, Class. Quant. Grav 2 (1985) L51

[14] M. A. Melvin, Class. Quant. Grav 3 (1986) 117

[15] P. Peldán, Phys. Rev. D46 (1992) R2279

[16] I. Bengtsson, Phys. Lett. 254B (1991) 55

[17] A. Ashtekar, V. Husain, C. Rovelli, J. Samuel and L. Smolin, Class. Quant. Grav 6
(1989) L185

[18] C. Rovelli and L. Smolin, Nucl. Phys. B331 (1990) 80

[19] R. Gambini and J. Püllin, Utah preprint UU-HEP-92/9, hepth@xxx9210110