We propose a method for determining ultra-violet divergences in the vacuum energy for systems whose spectrum of perturbations is defined through a non-linear spectrum problem, i.e., when the fluctuation operator itself depends on the frequency. The method is applied to the plasma shell model, which describes some properties of the interaction of electromagnetic field with fullerenes. We formulate a scalar model, which simplifies the matrix structure, but keeps the frequency dependence of the plasma shell, and calculate the ultra-violet divergences in the case when the plasma sheet is slightly curved. The divergent terms are expressed in terms of surface integrals of corresponding invariants.

**I. INTRODUCTION**

Recent advances in experimental techniques and in the nano-sciences have caused an increase in the interest to the theory of the Casimir effect \[1,2\]. The quantum field theory (QFT) methods proved to be very efficient in the study of the Casimir effect. However, real materials behave rather different from what we have got used to in conventional QFT. The properties of these materials change drastically at short distances and/or high frequencies. Numerically, the details of this behavior may give a dominant contribution to the Casimir energy \[3\]. The main focus of the present paper will be on the plasma sheet (or plasma shell) model introduced a few years ago by Barton \[4,5,6\] to describe certain features of interaction of giant carbon molecules (building blocks of fullerenes) with the electromagnetic field. The matching conditions for the electromagnetic field become in this model frequency-dependent, and this changes drastically the spectrum of the fluctuations.

The first problem one faces when applying the QFT methods to the Casimir energy calculations is the evaluation of divergent terms and renormalization. Though calculations for selected simple configurations may help to understand the behavior of the finite part of the Casimir energy, the requirement of selfconsistency (renormalizability) demands precise form of the counterterms on considerably more general backgrounds. This can be obtained relatively easy if the interaction with the background does not depend on the frequency, but becomes a tough problem if it does.

To understand the problem better let us start with a frequency-independent case. Consider a system where the spectrum of the fluctuations is defined by a linear equation

\[
D \phi_\omega = \omega^2 \phi_\omega,
\]

(1)

where \(\omega\) is the frequency of the fluctuations, and \(D\) is a Laplace type operator depending on background fields or other external conditions, as, e.g., boundaries. Let \(\rho(\omega^2)\) be the spectral density of the eigenvalue problem (1). Then one can define the vacuum energy as

\[
\mathcal{E} = \frac{1}{2} \int_\mu^\infty d(\omega^2) (\omega^2)^{\frac{n}{2}} \rho(\omega^2).
\]

(2)

(We use the units such that \(\hbar = c = 1\). \(\mu\) is the lower bound of the spectrum. The eigenvalues \(\omega^2\) are supposed to be real. The integral (2) is typically divergent at the upper limit and has to be regularized and renormalized. In many regularization schemes the divergences are defined by the heat kernel coefficients \(a_k(D)\) with \(k \leq n + 1\), with \(n\) being the number of the spatial dimensions. The heat kernel coefficients are the coefficients in a small \(t\) asymptotic expansion of the heat trace of the operator \(D\),

\[
K(D,t) = \text{Tr} \left( e^{-tD} \right) \simeq \sum_{k=0} t^{(k-n)/2} a_k(D).
\]

(3)

The coefficients \(a_k\) are local invariants, which ensures locality of the counterterms, and are known up to a rather high number \(k\) for a very large class of spectral problems \[7,8,9\]. Note, that the dependence of \(D\) on the background fields may be practically arbitrary, as well as the shape of the boundaries. The heat kernel expansion defines therefore the ultra violet divergences at the one-loop order for most of the relevant physical problems.

The situation becomes more complicated if the operator \(D\) itself depends on the frequency, i.e., if we are dealing with a non-linear spectral problem. If the dependence of \(D\) on \(\omega\) is rather mild, as happens on stationary but non-static backgrounds \[10,11,12\], the divergences can still be expressed through the asymptotics of the heat kernel for \(D\).
where the dependence on $\omega$ is frozen. In time-space noncommutative theories, which also lead to non-linear spectral problems, the divergences of the vacuum energy are also expressible through the polynomials of the background field, but the structure of these polynomials becomes more complicated \cite{13, 14}. However, it is unclear whether this property holds in a more general case. For example, one can calculate the divergences of the vacuum energy in a dielectric with a frequency-dependent permittivity, but only for a dielectric body having the form of a ball, or of a cylinder, or of a cube \cite{15, 16, 17}.

Can one nevertheless find the asymptotic of spectral density and the ultra-violet divergences for a fairly general class of geometries and background fields? The answer is positive, in principle (i.e., modulo technical difficulties). The method we propose consists in the following. First one has to freeze the dependence of the operator $D$ on frequency and construct the corresponding heat kernel, which may have the form of an expansion in small variations $u_A$ of the background fields, form of the boundary or boundary conditions, but has to be exact for all values of $t$ and for all values of the parameters in $D$ which will later depend on the frequency. From this heat kernel one calculates the spectral density for the frequency-independent operator, which, in turn, defines the spectral density for the original problem. The high-frequency behavior of the latter defines the ultra-violet divergences of the vacuum energy.

An important remark is in order. All expansions we shall actually consider are asymptotic expansions in local invariants constructed from $u_A$ and their derivatives ordered according to the canonical mass dimension. In the framework of such an expansion it does not make much sense to compare actual numerical values of the invariants or too look for non-local terms. The usefulness of such an expansion is based on the observation that the counterterms in QFT are local. On the other hand, application of the same expansion to calculations of the finite part of the vacuum energy is risky, even though the expansions for the spectral densities may formally be integrated as in (2).

We test our method at the example of a slightly curved plasma sheet (more precisely, at a scalar model which keeps essential features of the TM model of the electromagnetic field interacting with the plasma sheet). At the end, we find the part of the spectral density which has the form of an integral of local quantities quadratic in the extrinsic curvature, and which generates logarithmic divergences of the vacuum energy. In the conventional QFT approach this divergent part defines local counterterms which must be added to the action. In a more phenomenological approach it defines the behavior of the model near the cut off.

This paper is organized as follows. In the next section we describe the method and study a simple example of a frequency-dependent mass term. In section III we discuss the plasma shell model, including the corresponding matching conditions, the heat kernel for a frequency-independent auxiliary problem, and the spectral densities with their ultra-violet behavior. The last section contains a brief discussion of the results and some future prospects.

\section{General Strategy}

Let us consider a non-linear spectral problem of the type

$$D(Q(\omega), u_A)\varphi = \omega^2 \varphi,$$

where $D$ is a second order partial differential operator which depends on the frequency $\omega$ through a parameter $Q(\omega)$. The parameter $Q$ may be present in the symbol of $D$ (as, e.g., a mass term), or enter the spectral problem through boundary or matching conditions (cf. sec. III). The parameters $u_A$ are supposed to be small. A good example for $u_A$ is a background field. We suppose that the spectrum of $\omega^2$ is bounded from the below for reasonably small $u_A$. Next, we assume that for all relevant values of $Q$ there exists the heat trace

$$K(D(Q, u_A), t) = \text{Tr} \left( e^{-tD(Q, u_A)} \right).$$

Let us expand $K(D(Q, u_A), t)$ in asymptotic series,

$$K(D(Q, u_A), t) = K(0)(D(Q, u_A), t) + K(1)(D(Q, u_A), t) + K(2)(D(Q, u_A), t) + \ldots$$

We shall consider exclusively the case when $u_A$ are fields having a positive mass dimension. The expansion \(6\) can then be arranged according to the total mass dimension of the fields entering each term, counting also the derivatives. I.e., $K(0)$ does not depend on $u_A$, $K(1)$ is linear in $u_A$, $K(2)$ contains the terms quadratic in $u_A$ and linear in the derivatives of $u_A$, etc. The structure of the expansion \(6\) is similar to the standard heat kernel expansion \(3\), but the numerical coefficients in front of local invariants constructed from $u_A$ become functions of $Q$ and $t$. The existence of the asymptotic expansion \(6\), and the actual construction of such an expansion, are the most non-trivial and demanding parts of our procedure.

The heat trace can be represented through a Laplace transform of the spectral density $\rho(\lambda, Q; u_A)$,

$$K(D(Q, u_A), t) = \int_{-\infty}^{\infty} d\lambda \, e^{-\lambda t} \rho(\lambda, Q; u_A).$$
where \( \mu \) is the lower bound of the spectrum. The expansion (11) defines through the inverse Laplace transformation an expansion of the spectral density

\[
\rho(\lambda, Q, u_A) = \rho(0)(\lambda, Q, u_A) + \rho(1)(\lambda, Q, u_A) + \rho(2)(\lambda, Q, u_A) + \ldots
\]

Next, we have to construct a spectral density corresponding to the initial eigenvalue problem (4). Consider the counting function for the non-linear spectral problem (4), and then

\[
D(Q, u_A)\varphi_{Q, \lambda} = \lambda \varphi_{Q, \lambda}
\]

with a fixed \( Q \). The density \( \rho(Q, \lambda; u_A) \) corresponds to this eigenvalue problem, i.e., \( \rho(Q, \lambda; u_A)d\lambda \) is the number of solutions of (4) with the eigenvalues between \( \lambda \) and \( \lambda + d\lambda \). Of course, the functions \( \varphi_{Q(\omega), \omega^2} \) do solve the equation (4), but the density is not simply \( \rho(Q(\omega), \omega^2; u_A) \) (with respect to \( d\omega^2 \)) since the spacing between the eigenvalues changes. However, the number of the eigenvalues up to certain \( \omega^2 = \lambda \) remains the same. Namely, if we define a counting function

\[
N(\lambda, Q; u_A) = \int_{\mu}^{\lambda} \rho(\nu, Q; u_A)d\nu
\]

then

\[
N(\omega^2; u_A) = N(\omega^2, Q(\omega); u_A)
\]

is the counting function for the non-linear spectral problem (4), and

\[
\rho(\omega^2; u_A) = \frac{d}{d\omega^2}N(\omega^2; u_A)
\]

is the corresponding spectral density (with respect to \( d\omega^2 \)). This result also follows from a rather complete analysis of non-linear spectral problems presented in Refs. [10, 11, 12].

The expansion (8) defines an asymptotic expansion of the spectral density of the non-linear spectral problem

\[
\rho(\omega^2; u_A) = \rho(0)(\omega^2; u_A) + \rho(1)(\omega^2; u_A) + \rho(2)(\omega^2; u_A) + \ldots
\]

Consider now the limit \( \omega^2 \to \infty \). According to (2) the part of the spectral density which decays as \( \omega^{-3} \) or slower generates ultra-violet divergences of the vacuum energy. Corresponding counterterms can be isolated in each order of the expansion in \( u_A \), and the renormalization procedure can be carried out. As in the normal case, the counterterms are expected to be local invariants constructed from \( u_A \).

Note, that \( \rho(i) \) are only parts of the full spectral density. Therefore, \( \rho(i) \) must not be positive.

Let us give here a short summary of the proposed method. If one likes to evaluate divergences of the vacuum energy for a system those fluctuations are described by the equation (4) up to a certain order in \( u_A \), one has to make the following steps.

- Consider an auxiliary problem with the dependence of \( Q \) on \( \omega \) frozen and calculate the heat kernel for \( D \) up to the desired order in \( u_A \) but exactly in \( Q \) and \( t \).
- Calculate the corresponding spectral density and the counting function \( N(\lambda, Q; u_A) \).
- The function \( N(\omega^2, Q(\omega); u_A) \) is then the counting function for our initial problem, and the derivative of \( N(\omega^2, Q(\omega); u_A) \) with respect to \( \omega^2 \) gives the corresponding spectral density. The behavior of this spectral density for large \( \omega \) defines ultra-violet divergences of the vacuum energy.

Note, that the first two steps above are most difficult from the technical point of view, and these steps should be done for the auxiliary problem. Therefore, what really counts for the complexity of the method is how \( D \) depends on \( Q \) rather than how \( Q \) depends on \( \omega \).

If we are considering only local terms (in \( u_A \)) in the expansions above, we would miss non-local divergences it they do exist. Non-local divergences are, however, to exotic to consider this is a serious drawback of the method.
A. Frequency-dependent mass term

As a toy model, let us consider a spectral problem in \( n = 3 \) spatial dimensions with the operator \( D \) of the form

\[
D = -\partial_t^2 + m^2 + V(x),
\]

where \( V(x) \) is a smooth potential which falls off at the infinity of \( \mathbb{R}^3 \). \( m^2 \) is a mass term, which will be made frequency-dependent later. We shall be interested in the terms depending linearly on the potential. The corresponding part of the heat kernel can easily be found (see, e.g., [9]).

\[
K_{(1)}(m^2, V, t) = -\frac{t^{-1/2}}{(4\pi)^{3/2}} \int d^3x V(x)e^{-m^2t}.
\]

By using (7) and (10) one immediately finds the corresponding spectral density and the counting function

\[
\rho_{(1)}(\lambda, m^2; V) = -\frac{1}{8\pi^2}(\lambda - m^2)^{-1/2}\theta(\lambda - m^2) \int d^3x V(x),
\]

\[
N_{(1)}(\lambda, m^2; V) = -\frac{1}{4\pi^2}(\lambda - m^2)^{1/2}\theta(\lambda - m^2) \int d^3x V(x),
\]

where \( \theta \) is the step function which simply tells that the lower bound of the integration in (7) and (10) is \( \mu = m^2 \). Let us now consider examples of the dependence of \( m^2 \) on \( \omega \).

a. \( m^2 \) does not depend on the frequency. By expanding \( (\lambda - m^2)^{-1/2} \) in (10) for large \( \lambda = \omega^2 \), \( (\lambda - m^2)^{-1/2} \approx \omega^{-1} + \frac{1}{2}m^2\omega^{-3} + \ldots \) we find two divergent contributions to the vacuum energy. The term \( \int d^3x V \) diverges linearly, and the term with \( \int d^3x m^2 V \) diverges logarithmically. This is a well-known result.

b. \( m^2 = \alpha \omega^2 \). Taking \( \lambda = \omega^2 \) we obtain \( (\lambda - m^2) = (1 - \alpha)\omega^2 \). Therefore, for \( \alpha < 1 \) all spectral functions are as in the case above with \( m^2 = 0 \) up to a multiplier of \( \sqrt{1 - \alpha} \). For \( \alpha > 1 \) the spectral density is non-zero for negative \( \omega^2 \) meaning that the system becomes unstable. The same conclusion may be reached by considering the eigenvalue problem \( \omega^2 \varphi_{\omega} = (\alpha \omega^2 - \partial_t^2 + V)\varphi_{\omega} \).

c. \( m^2 = \gamma \omega^4 \). For \( \gamma > 0 \) the spectral density is non-zero for \( 0 \leq \omega^2 \leq 1/\gamma \), i.e. there are no high frequencies in the spectrum at all. Consequently, there are no ultraviolet divergences proportional to the potential. For \( \gamma < 0 \), instabilities appear due to the imaginary frequencies with \( \omega^2 < -1/|\gamma| \).

III. CURVED PLASMA SHEET

A. Matching conditions

The plasma shell (or plasma sheet in the noncompact case) model was introduced by Barton in Refs. [4, 5] where also the Casimir energies were explored. The Casimir force between the plasma sheet and other surfaces or molecules was calculated in [18] by using the Lifshitz formulae. Further spectral properties of the plasma shell model were studied for flat [19] and spherical [20] geometries. In both cases, the TM mode of the electromagnetic field causes more difficulties than the TE mode. The TE problem is isomorphic to the scattering on a repulsive frequency-independent delta-potential, which is a rather well understood problem. The study of the heat kernel expansion and thus of the divergences of the vacuum energy for the delta-potential on various surfaces started in Ref. [21] with the case of a sphere. Then the heat kernel for an arbitrary shape of the surface in an arbitrarily curved space, with generic bulk potential and gauge fields was obtained [22]. The spectral problem for TM mode is known to be not elliptic [18] [20]. Thus even the definition of basic spectral functions become problematic.

Here we consider exclusively the TM mode, which, for a spherical shell placed at \( r = r_0 \) may be reduced to a scalar field \( \varphi \) with the following matching conditions [4]

\[
\text{discont} (r\varphi_{\omega}) = -\frac{2q}{\omega^2}(\partial_r r\varphi_{\omega}),
\]

\[
\text{discont} (\partial_r r\varphi_{\omega}) = 0,
\]

where \( \text{discont} (f) \equiv f(r = r_0 + 0) - f(r = r_0 - 0) \). \( q \) is a constant depending on the properties of the material. Outside the shell the modes \( \varphi_{\omega} \) satisfy the Helmholtz equation

\[
(\nabla^2 + \omega^2)\varphi_{\omega} = 0.
\]

\[
(\nabla^2 + \omega^2)\varphi_{\omega} = 0.
\]
In the limit $r_0 \to \infty$ the matching conditions (18) reproduce that of the flat plasma sheet [3, 19].

To prepare for the discussion of the next subsection we need a bit more elaborate notations. Denote by $M^+$ the exterior of the shell, and by $M^-$ the interior of the shell. Let $\Sigma$ be the (spherical) interface of $M^+$ and $M^-$, and let $n^\pm$ be a unit normal to $\Sigma$ pointing inward of $M^\pm$. Denote corresponding covariant derivatives by $\nabla^\pm_n$. When acting on scalars, $\nabla^+_n = \partial_r$, and $\nabla^-_n = -\partial_r$. Then we can rewrite (18) as

$$
- \begin{pmatrix}
\nabla^+_n \varphi^+
\nabla^-_n \varphi^-
\end{pmatrix} = \begin{pmatrix}
Q + \sigma & -Q \\
-Q & Q - \sigma
\end{pmatrix}
\begin{pmatrix}
\varphi^+
\varphi^-
\end{pmatrix},
$$

where

$$Q(\omega) = \omega^2/(2q), \quad \sigma = 1/r_0. \tag{21}$$

The dependence of $\varphi$ on $\omega$ is suppressed.

We shall consider the case when the surface $\Sigma$ has the topology of a plane but is slightly curved. $M^+$ and $M^-$ are two parts of the bulk manifold separated by $\Sigma$. The non-flatness of $\Sigma$ is measured by the extrinsic curvature tensor $L_{ab}$ which has two indices corresponding to the directions tangential to $\Sigma$. Obviously, the extrinsic curvature of $\Sigma$ considered as a part of $M^+$ is minus the extrinsic curvature of $\Sigma$ as a part of $M^-$, $L^+_{ab} = -L^-_{ab}$. We shall consider a scalar field with the matching conditions (20) on $\Sigma$, where, according to (21), $Q$ is independent of the extrinsic curvature, while $\sigma$ is supposed to be of the order of $|L_{ab}|$, but independent of the frequency. Therefore, $Q$ will be treated non-perturbatively, while only the leading orders of $\sigma$ and $L_{ab}$ will be retained in all spectral functions.

Note, that for an arbitrary curved boundary the electromagnetic field cannot be separated in the TE and TM parts. Therefore, the model we are going to analyze is an approximation to real physics. We hope that the most essential features of the problem are nevertheless preserved by this simplification.

**B. Heat kernel for the auxiliary model**

The auxiliary model is obtained by simply freezing the frequency dependence in the matching conditions (20), i.e. by assuming that $Q$ is some constant. We are going to evaluate the localized heat kernel

$$K(f,D,t) = \text{Tr} \left( f \exp(-tD) \right) \tag{22}$$

where $f$ is some smooth localizing (or smearing) function, and $D$ is the standard Laplacian, $D = -\nabla^2$.

The method we are going to use is (an extension of) the method of Gilkey [23, 24], see [7, 8, 9] for an extensive overview with further explanations. The method is based on a somewhat paradoxical observation that solving a more general problem may be much easier. We introduce an arbitrary curved metric $g_{ij}$ on $M$ and generalize $D$ to be an arbitrary scalar Laplacian

$$D = -(g^{ij} \nabla_i \nabla_j + E), \tag{23}$$

where $E(x)$ is a potential. We suppose that the metric is smooth across $\Sigma$, so that $L^+_{ab} = -L^-_{ab}$. At some stage, when discussing the doubling trick, see Eq. (28) below, we shall need more general matching conditions

$$- \begin{pmatrix}
\nabla^+_n \varphi^+
\nabla^-_n \varphi^-
\end{pmatrix} = \begin{pmatrix}
S^{++} & S^{+-} \\
S^{-+} & S^{--}
\end{pmatrix}
\begin{pmatrix}
\varphi^+
\varphi^-
\end{pmatrix} \tag{24}$$

depending on four functions $S^{\pm\pm}$. The functions $S^{\pm\pm}$ have a positive mass dimensions and are, therefore, natural parameters of a perturbative expansion. However, $S^{\pm\pm} = 0$ does not correspond to a free propagation, but rather to two disjoint regions with Neumann boundary conditions on the surface which separates them. On a side note, we remark that this explains the failure of the multiple reflection expansion in the case of these matching conditions [25]. Throughout this subsection we shall suppose that the dimension of $M$ is arbitrary, $n = \dim M$. The last (n'th) coordinate is supposed to be normal to $\Sigma$, so that the notations introduced above for the normal and tangential coordinates remain consistent. The spectral geometry of Laplace type operators with such matching conditions has been analyzed in [26]. Basing on the analysis of that paper we can write the following expression for the heat kernel for the conditions (20) which is non-perturbative in $Q$ and contains other invariants up to the canonical mass dimension two:

$$K(f,D,t) \simeq \frac{1}{(4\pi t)^{n/2}} \int_{\Sigma} d^{n-1} x \sqrt{\nu} \left[ \beta_0(z)f + t(\beta_1(z)(L^+_a - L^-_a)(L^+_b - L^-_b)f + \beta_2(z)(L^+_a - L^-_a)^2f \\
+ \beta_3(z)(f^+_n - f^-_n)(L^+_a - L^-_a) + \beta_4(z)\frac{1}{2}(f^+_{nn} + f^-_{nn}) + \beta_5(z)\sigma(L^+_a - L^-_a)f + \beta_6(z)\sigma(f^+_n - f^-_n) \\
+ \beta_7(z)Ef + \beta_8(z)Rf + \beta_9(z)R_{anfn}f + \beta_{10}(z)\sigma^2f) \right] + K_{\text{vol}}(f,D,t). \tag{25}$$
Here $\beta_i(z)$ are unknown function depending on $Q$ through the dimensionless combination
\[ z = 2Qt^{1/2}. \tag{26} \]
(The results of the calculations are summarized at the end of this subsection). $Q$ is supposed to be constant, but $\sigma$ is allowed to depend on the coordinates on $\Sigma$. The volume part $c$ annot depend on $Q$ or $\sigma$ and is thus given by the standard expression
\[ K_{\text{vol}}(f, D, t) = \frac{1}{(4\pi t)^{n/2}} \int_M d^n x \sqrt{g} f(1 + t(E + R/6)). \tag{27} \]
We stress again that only the invariants of canonical mass dimension two or less are being kept. Only a few functions $\beta_i$ will be needed to analyze the curved plasma sheet model. Others are included for technical reasons.

Some explanations regarding the formula (25) are in order. $R_{ijkl}$ is the Riemann tensor for the metric $g_{ij}$, and $R$ is the corresponding scalar curvature. $h$ is the determinant of the metric induced on $\Sigma$. All tensor indices are taken in a local orthonormal frame, so that there is no distinction between upper and lower indices. $n^\pm$ is a unit vector along a normal geodesics pointing inward $M^\pm$. The indices $a, b$ correspond to the vectors tangential to $\Sigma$. Summation over the repeated indices is understood. Semicolon denotes the covariant derivative, e.g., $f^+_n \equiv \nabla_n f^+$. The smearing function $f$ and the metric are smooth across $\Sigma$, consequently, $f^+_n = -f^-_n$, $L^+_a b = -L^-_a b$, $f^+_{nn} = f^-_{nn}$. Therefore, the notations used in (25) are redundant, but they facilitate the comparison to Ref. [26] and allow to keep track of the symmetries more easily. Eq. (25) contains all possible invariants up to the mass dimension two. Note, that the heat kernel must be invariant under interchanging the roles of $M^+$ and $M^-$. Since $\sigma$ is antisymmetric under $M^+ \leftrightarrow M^-$, this excludes the surface integral of $\sigma$ as well as all other (non-vanishing) invariants of the mass dimension one. Since $\sigma$ is defined on the interface only, one is not allowed to differentiate it with respect to the normal vector. Finally, $R_{abab}$ is not an independent invariant.

Here we like to reiterate the remark made already a couple of times above. The expression (25) is an asymptotic expansion in local invariant ordered according to the canonical mass dimension. It has nothing to do with actual (numerical) smallness of corresponding terms.

Let us start the calculations of the unknown functions $\beta_i(z)$. By standard arguments (see, e.g., section 4.1 of Ref. [3]) one can show that $\beta_i(z)$ do not depend on $n$, i.e. the whole dependence on the dimensionality of the manifold resides in the pre-factor $(4\pi t)^{-(n-1)/2}$. This fact is extremely important for the methods we use here. As we shall see below, the so called conformal relations yield some relations on $\beta_i$ with coefficients which are linear in $n$. Since $\beta_i$ themselves do not depend on $n$, each of the conformal relations in fact produces two independent conditions on $\beta_i$’s.

The functions $\beta_0$, $\beta_1$, $\beta_6$ and $\beta_{10}$ can be computed by using Lemma 4.1 from [20] which relates the heat trace on a manifold with boundary with Robin boundary conditions to the heat trace on a “doubled” manifold with transfer conditions on the interface. This Lemma states the following. Let $M^\pm = M^0$ be a Riemannian manifold, and let $D^\pm = D^0$ be a scalar operator of Laplace type. Fix an angle $0 < \theta < \pi/2$. Let $S^{++}$ and $S^{+-}$ be arbitrary. Set
\[ S^{+-} := S^{++}, \]
\[ S^{--} := S^{++} + (\tan \theta - \cot \theta)S^{+-}, \]
\[ S_\phi := S^{++} + \tan \theta S^{+-}, \]
\[ S_\psi := S^{++} - \cot \theta S^{+-}. \tag{28} \]

Then
\[ K(f, D, t) = K(\cos^2 \theta f^+ + \sin^2 \theta f^-, D_\phi^0, t) + K(\sin^2 \theta f^+ + \cos^2 \theta f^-, D_\psi^0, t). \tag{29} \]

On the left hand side of the equation above we have the heat kernel for the operator $D$ on $M$ subject to transfer conditions on the interface, while on the right hand side there are two heat kernels on $M^\pm$ for the operator $D^0$ with Robin boundary conditions
\[ (\nabla_n \phi + S_\phi)\phi|_{\partial M^0} = 0 \quad \text{and} \quad (\nabla_n \psi + S_\psi)\psi|_{\partial M^0} = 0, \tag{30} \]
respectively. $f^\pm$ is a restriction of $f$ to $M^\pm$.

To control the invariants appearing with $\beta_0$, $\beta_4$, $\beta_6$ and $\beta_{10}$ it is enough to consider the case of constant $\sigma$. Then,
\[ \tan \theta = \frac{\sigma}{Q} + \sqrt{\frac{\sigma^2}{Q^2} + 1} = 1 + \frac{\sigma}{Q} + \frac{\sigma^2}{2Q^2} + O(\sigma^3) \tag{31} \]
and
\[ S_\phi = -\frac{\sigma^2}{2Q} + O(\sigma^3), \quad S_\psi = 2Q + \frac{\sigma^2}{2Q} + O(\sigma^3). \]  

(32)

Another ingredient we need, is the heat kernel for the free Laplacian \( \Delta = -\partial_t^2 \) subject to the Robin boundary conditions at a plane boundary with a constant \( S \) which was derived in [27]

\[
K(f, \Delta_S, t) = \frac{1}{(4\pi t)^{(n-1)/2}} \int_{\partial M} d^{n-1}x \sum_{p=0}^{\infty} \left( \sum_{k=1}^{\infty} f(p)(St^{1/2})^{p+k} \frac{2^{-p-1}}{\Gamma\left(\frac{k+p}{2} + 1\right)} + f(p)t^{p/2} \frac{2^{-p-2}}{\Gamma\left(\frac{k+p}{2} + 1\right)} \right),
\]

(33)

where \( f(p) \) is the \( p \)th normal derivative of \( f \). We do not write here a trivial volume part of the heat kernel. For each \( p \) one can also write a closed expression in terms of the error function

\[
\text{erf} (z) = 1 - \text{erfc} (z) = \frac{2}{\sqrt{\pi}} \int_{0}^{z} e^{-u^2} \, du.
\]

(34)

For example, for \( p = 0 \) we have

\[
K(f, \Delta_S, t) = \frac{1}{2(4\pi t)^{(n-1)/2}} \int_{\partial M} d^{n-1}x \left[ e^{St} \text{erf} (St^{1/2}) + e^{St} - \frac{1}{2} \right].
\]

(35)

In the formulae above, Eqs. (33) and (35) we corrected the \( S \)-independent terms of the corresponding formulae in [27]. Such terms, being of the order of “zero reflections” were outside of the focus of the paper [27].

Let us stress, that as noted in [27], the expression (35) is exact, i.e. it takes into account the possibility of non-analytic contributions which may not be seen at power-series expansions, see also [28, 29].

By collecting (28) - (35) together, we obtain,

\[
\beta_0(z) = \frac{1}{2} [e^{z^2} (\text{erf} (z) + 1)]
\]

\[
= \frac{1}{2} \sum_{k=0}^{\infty} z^k \frac{1}{\Gamma\left(\frac{k}{2} + 1\right)},
\]

\[
\beta_4(z) = \frac{1}{8} \sum_{k=0}^{\infty} z^k \frac{1}{\Gamma\left(\frac{k}{2} + 2\right)},
\]

\[
\beta_6(z) = 2\beta_4(z),
\]

\[
\beta_{10}(z) = 2\beta_6(z).
\]

(36)

Some terms in the expansions above can be checked against the corresponding terms in [27]. These expansions are rapidly convergent for positive \( z \). Therefore, it does not play a role whether one uses closed formulae in terms of the error function, or an expanded form. Using expanded forms is a little bit easier.

To define \( \beta_6 \) and \( \beta_7 \) we use the product Lemma (see, e.g., sec. 4.1 of Ref. [9]) which states that if \( M = M_1 \times M_2 \), and also \( D = D_1 \otimes 1 + 1 \otimes D_2 \), with \( D_1 \) and \( D_2 \) acting independently on functions over \( M_1 \) and \( M_2 \), the resulting heat kernel factorizes into a product of the heat kernel for \( D_1 \) and the heat kernel for \( D_2 \). This means that if \( f(x_1, x_2) = f_1(x_1)f_2(x_2) \), where \( x_1 \) and \( x_2 \) are coordinates on \( M_1 \) and \( M_2 \), respectively,

\[
K(f, D, t) = K(f_1, D_1, t)K(f_2, D_2, t).
\]

(37)

Let us take \( M_2 \) being a curved manifold without a singular surface, and \( M_1 \) being a flat manifold with a singular surface and define some transfer conditions on it. Let \( D_1 \) and \( D_2 \) be scalar Laplacians, and let \( E \) be independent of the coordinates on \( M_1 \). Then, to the linear order of \( E \) and the scalar curvature \( R \)

\[
K(f_2, D_2, t) = \frac{t}{(4\pi t)^{-n/2} \sqrt{g_2}} \int d^nx_2 \sqrt{g_2} \left( E + \frac{1}{6} R \right).
\]

(38)

Then, by comparing corresponding terms on both sides of (37), we obtain

\[
\beta_7(z) = \beta_0(z), \quad \beta_8(z) = \frac{1}{6} \beta_0(z).
\]

(39)
The remaining functions $\beta_1, \beta_2, \beta_3, \beta_5$ and $\beta_9$ will be computed by using conformal properties of the operator $D$. One can show that

$$\frac{d}{de} \epsilon = 0 \left[ \text{Tr} \left[ \exp \left( -e^{-2x^f D} t \right) \right] = -2t \frac{d}{dt} \text{Tr} \left[ \exp (-t D) \right] \right].$$  \hspace{1cm} (40)

It is easy to figure out (or to look up in \[7, 8, 9\]) how the geometric quantities entering the operator $D$, Eq. (23), transform the local scale transformation $D \rightarrow e^{-2\epsilon f} D$. In particular, the transformation of the metric is the standard Weyl rescaling, $g_{ij} \rightarrow e^{2\epsilon f} g_{ij}$. The transformation of other quantities, as the connection and the potential, are less standard, as they contain specific contributions which ensure the homogeneous transformation law for $D$. For our purposes, it is enough to consider the transformations vanishing on $\Sigma$, $f|_{\Sigma} = 0$. Under this assumption, the variations of relevant geometric invariants on the interface $\Sigma$ read

$$\frac{d}{de} \epsilon = 0 \left[ E = \frac{1}{4}(n-2)(f_{;nn}^+ + f_{;nn}^-) - \frac{1}{8}(n-2)(L_{;aa}^+ - L_{;aa}^-)(f_{;n}^+ - f_{;n}^-), \right.$$  \hspace{1cm} (41)

$$\frac{d}{de} \epsilon = 0 \left[ R = -(n-1)(f_{;nn}^+ + f_{;nn}^-) + \frac{1}{2}(L_{;aa}^+ - L_{;aa}^-)(f_{;n}^+ - f_{;n}^-), \right.$$  \hspace{1cm} (42)

$$\frac{d}{de} \epsilon = 0 \left[ R_{nn} = \frac{1}{2}(n-1)(f_{;nn}^+ + f_{;nn}^-) - \frac{1}{4}(L_{;aa}^+ - L_{;aa}^-)(f_{;n}^+ - f_{;n}^-), \right.$$  \hspace{1cm} (43)

$$\frac{d}{de} \epsilon = 0 \left[ L_{;ab}^\pm = -\delta_{ab} f_{;n}^\pm \right].$$

We must keep the matching conditions invariant under the local conformal (scale) transformations. This is achieved by transforming $\sigma$ to compensate the variation of the connection in $\nabla_n$,

$$\frac{d}{de} \epsilon = 0 \left[ \sigma = \frac{1}{4}(n-2)(f_{;n}^+ - f_{;n}^-) \right]$$

(cf. \[26\]). $Q$ does not transform.

Let us collect the terms with $\frac{1}{2}(f_{;nn}^+ + f_{;nn}^-)$ in Eq. (40). This gives the equation

$$\frac{1}{2}(n-2)\beta_7(z) - 2(n-1)\beta_8(z) + (n-1)\beta_9(z) = [(n-3) - z\partial_z] \beta_4(z),$$  \hspace{1cm} (44)

which, with the help of Eq. (49), may be simplified to

$$\frac{1}{6}(n-4)\beta_0(z) + (n-1)\beta_9(z) = [(n-3) - z\partial_z] \beta_4(z).$$  \hspace{1cm} (45)

The functions $\beta_i(z)$ do not depend on $n$. By taking $n = 1$ we obtain the identity

$$\frac{1}{2} \beta_0(z) = [2 + z\partial_z] \beta_4(z),$$

which may serve as a consistency check. Taking $n = 4$ yields the value of $\beta_9$,

$$\beta_9(z) = \frac{1}{3}(1 - z\partial_z) \beta_4(z) = \frac{1}{24} \sum_{k=0}^3 z^k \frac{1}{\Gamma \left( \frac{k}{2} + 2 \right)}.$$  \hspace{1cm} (46)

Next, lets us collect the terms with $\sigma(f_{;n}^+ - f_{;n}^-)$ in Eq. (40). This yields the equation

$$(1 - n)\beta_5(z) + \frac{1}{2}(n-2)\beta_{10}(z) = [(n-3) - z\partial_z] \beta_6(z).$$

Again, $n = 1$ provides a consistency check. For $n = 2$ we obtain

$$\beta_5(z) = [1 + z\partial_z] \beta_6(z) = \frac{1}{4} \sum_{k=0}^4 z^k \frac{1}{\Gamma \left( \frac{k}{2} + 2 \right)}.$$  \hspace{1cm} (47)
The only structure which has not been used yet in the conformal relations is \((f^+_n - f^-_n)(L_{\alpha\alpha}^+ - L_{\alpha\alpha}^-)\). It gives the condition

\[
\frac{1}{8}(2-n)\beta_7(z) + \frac{1}{2}(n-1)\beta_6(z) - \frac{1}{4} \beta_5(z) + \frac{1}{4}(n-2)\beta_5(z) + 2(1-n)\beta_2(z) - 2\beta_1(z) = [(n-3) - z\partial_z]\beta_4(z)
\]  

(49)

or

\[
\frac{1}{24}(4-n)\beta_6(z) - \frac{1}{4} \beta_5(z) + \frac{1}{4}(n-2)\beta_5(z) + 2(1-n)\beta_2(z) - 2\beta_1(z) = [(n-3) - z\partial_z]\beta_3(z).
\]  

(50)

Since the functions \(\beta_i(z)\) do not depend on \(n\), this equation is equivalent to two independent conditions. However, we still have three unknown functions \(\beta_1, \beta_2, \text{ and } \beta_3\).

To get one more relation between the unknown functions we shall use the properties of the heat kernel under local scale transformations when the smearing function is also transformed. It is a purely formal computation [23, 24] to show that

\[
\frac{d}{de}\bigg|_{e=0} a_{n-2}(e^{-2\epsilon_f}F, e^{-2\epsilon_f}D) = 0,
\]  

(51)

where, similarly to (3), \(a_n(F, D)\) are the coefficients in an expansion of the smeared heat kernel \(K(F, D, t)\). One has to note that for each dimension \(n\) of the manifold the equation (51) restricts just one heat kernel coefficient, in contrast to (10). Let \(\beta^k_i\) be the coefficient in front of \(z^k\) in the Taylor expansion of \(\beta_i(z)\),

\[
\beta_i(z) = \sum_{k=0}^{\infty} z^k \beta^k_i.
\]  

(52)

Since \(z \sim t^{1/2}\), cf (20), each \(\beta^k_i\) contributes to \(a_{k+i}\). I.e., Eq. (51) gives restrictions on \(\beta^k_i\) in the dimension \(n = k + 5\). Let us collect the terms in (51) proportional to \((f^+_n - f^-_n)(F^+_n + F^-_n)\). Since now the smearing function is also transformed, such terms may also appear from conformal transformations of the second normal derivative of \(F\):

\[
\frac{d}{de}\bigg|_{e=0} (F^+_n + F^-_n) = -\frac{5}{2}(f^+_n - f^-_n)(F^+_n + F^-_n) + \ldots
\]  

(53)

(cf. Ref. [24]). Dots denote other terms, e.g. \(\simeq f_{n+1} F\), which we do not use in this calculation. We have,

\[
-\frac{5}{4}\beta^k_2 - (n-1)\beta^k_3 + \frac{1}{4}(n-2)\beta^k_6 = 0.
\]  

(54)

Next, we remember that \(n = k + 5\) and use (50) to obtain

\[
\beta^k_2 = \frac{1}{64} \frac{2k+1}{\Gamma\left(\frac{k+2}{2}\right)}.
\]  

(55)

Now we are ready to compute the remaining functions. Taking \(n = 1\) and \(n = 2\) in Eq. (50) gives \(\beta_1\) and \(\beta_2\), respectively,

\[
\beta_1(z) = \frac{1}{192} \sum_{k=0}^{\infty} z^k \frac{(k+1)(2k+1)}{\Gamma\left(\frac{k+2}{2}\right)},
\]

\[
\beta_2(z) = \frac{1}{384} \sum_{k=0}^{\infty} z^k \frac{(k+1)(5k+13)}{\Gamma\left(\frac{k+2}{2}\right)}.
\]

(56)

For the convenience of the reader we summarize the results of calculations of this subsection in Table I by providing the references to corresponding equations. One can easily check that all \(\beta_i(z)\) are consistent with the results of [26] where several leading terms of the \(z\)-expansions for each \(\beta\) were calculated.

The last step is to sum up the \(z\)-expansions for selected functions \(\beta_i(z)\) which are going to use below in sec. III D.

\[
\beta_5(z) = \frac{1}{4} \frac{d}{dz} \left[ \frac{1}{2} e^{-z} (\text{erf}(z) + 1) \right],
\]

(57)

\[
\beta_1(z) = \frac{1}{12} \beta_5(z) - \frac{7}{192} \beta(z),
\]

(58)

\[
\beta_2(z) = \frac{5}{48} \beta_5(z) - \frac{7}{384} \beta(z),
\]

(59)
TABLE I: Summary of the calculations of $\beta_i(z)$.

| $\beta_i$: | $\beta_0, \beta_4, \beta_6, \beta_{10}$ | $\beta_7, \beta_8$ | $\beta_9$ | $\beta_5$ | $\beta_3$ | $\beta_1, \beta_2$ |
|------------|--------------------------------------|--------------------|------------|------------|-------------|------------------|
| Equation:  | (36)                                 | (39)               | (46)       | (48)       | (55)        | (56)             |

where

$$\tilde{\beta}(z) = \frac{d}{dz} \left[ \frac{1}{z^3} \left( e^{z^2} (\text{erf}(z) + 1) - 1 - \frac{2z}{\sqrt{\pi}} - z^2 \right) \right].$$

(60)

Note, that in principal the power series expansion, even though they are convergent for all positive $z$, define the functions up to non-analytic terms which are exponentially small at $z \to 0$. Nevertheless, since the expression for the Robin heat kernel is exact and does not include non-analytic parts, the functions $\beta_i$ with $i = 4, \ldots, 10$ also do not include non-analytic parts, and their expressions in terms of the error function are also exact (though not all of them are presented explicitly here). However, to obtain $\beta_3$ the use of an expanded form was essential to write down the conformal relation (54). Therefore, $\beta_3(z)$ in principle can contain a non-analytic part. Let us assume that the non-analytic part of $\beta_3(z)$ is zero, which is a very natural assumption. Then, there is no non-analyticity in $\beta_1$ and $\beta_2$ which are defined by the other functions through algebraic equations, and $\beta_1$ and $\beta_2$ are restored in a unique way as written above.

C. Spectral densities: plane case

We have completed the calculation of the heat kernel, and it remains to calculate the spectral densities. However, there is a subtlety in application of our methods in the present case, which is related to the treatment of unstable modes with negative $\omega^2$. To understand the problem we start with the case of a plane plasma sheet.

One can easily see that flat ($\sigma = 0, L_{ab}^{\pm} = 0$) auxiliary problem (with $Q$ positive and independent of $\omega$) admits surface modes which decay as $e^{-2Q|x|^n}$ for positive $x^n$ and as $-e^{-2Q|x|^n}$ for negative $x^n$. The Helmholtz equation (19) gives the dispersion equation

$$\omega^2 = p^2 - (2Q)^2,$$

(61)

where $p$ is the momentum parallel to the interface $\Sigma$. This dispersion relation corresponds to the excitations localized at the surface of the interface and having a negative mass squared $-4Q^2$. Due to the presence of such tachionic excitations, the heat kernel has a part which grows for large positive $t$. This part is defined by the growing part of $\beta_0(z)$, cf. Eqs. (25) and (36). In $n = 3$ dimensions it reads

$$K_{\text{tachion}}(f, D, t) = \frac{1}{4\pi t} e^{i4Q^2t} \int_{\Sigma} d^2 x f.$$

(62)

For $f = 1$ the heat kernel above is divergent, which is a consequence of the translation invariance of the problem. Due to this invariance, the relevant quantities are densities per unit area of the interface, as, e.g., the surface Casimir energy density. Therefore, we omit the integral of the smearing function in Eq. (62) thus obtaining a surface density of the heat trace. The corresponding spectral density and the counting function

$$\rho_{\text{tachion}}(\lambda, Q) = \frac{1}{4\pi} \theta(\lambda + 4Q^2),$$

(63)

$$N_{\text{tachion}}(\lambda, Q) = \frac{1}{4\pi} (\lambda + 4Q^2)^2 \theta(\lambda + 4Q^2),$$

(64)

also have the meaning of surface densities. By using Eqs. (11), (12) and (21), one obtains the part of the spectral density for the nonlinear spectral problem, which is generated by the tachionic part of the auxiliary problem:

$$\rho_{\text{tachion}}(\omega^2) = \frac{1}{4\pi} \left( 1 + \frac{2\omega^2}{q^2} \right) \theta \left( \omega^2 + \omega^4 \right).$$

(65)

This spectral density has two branches. One corresponds to real frequencies, $\omega^2 > 0$, while the other - to imaginary frequencies, $\omega^2 < -q^2$. 


The flat plasma sheet model can also be analyzed directly. There is a mode, the so-called surface plasmon, which falls exponentially away from Σ with the eigenfrequencies

\[ \omega_{\text{s.p.}}^2 = \frac{q}{2} \left( \sqrt{q^2 + 4p^2} - q \right). \]

(66)

The spectral density corresponding to the surface plasmon is simply the density of the momenta \( p \) parallel to Σ, i.e., \((2\pi)^{-2}d^2p\), which after integration over the directions of \( p \) becomes \((4\pi)^{-1}d|p|^2\). To obtain the spectral density in terms of \( \omega^2 \) one has to change variables in the previous expression which yields

\[ \rho_{\text{s.p.}} = \frac{1}{4\pi} \left( \frac{d\omega_{\text{s.p.}}^2}{d\omega^2} \right)^{-1}. \]

(67)

After some simple algebra the density (67) can be shown to reproduce the positive branch of the spectral density (65). One can also demonstrate that the negative branch of (65), \( \omega^2 < -q^2 \), correspond to the solutions which grow exponentially away from Σ and thus do not belong to the physical spectrum. In our approach, the negative branch must be excluded by hand. Obviously, in one perturbs the plane plasma sheet by making it slightly curved, this cannot result in the eigenvalues far away from the spectrum of non-perturbed problem.

D. Spectral densities: curved case

Let us now proceed with the case of a curved plasma sheet. Let us restrict ourselves to the case of flat ambient space with no external potentials, so that \( R_{ijkl} = E = 0 \). Next, we put the smearing function \( f = 1 \). If we suppose that the extrinsic curvature and \( \sigma \) are well localized, the surface integral in (20) remains convergent (otherwise, we can consider the surface density of the vacuum energy as described in the subsection above). Let us remind, that \( \sigma \) is supposed to be of the same order as the extrinsic curvature. Therefore, we arrive at the following second-order heat kernel

\[ K_2(D(Q), t) = \frac{1}{(4\pi t)^{11}} \int_{\Sigma} d^2x \sqrt{h} \left[ \beta_1(z)(L_{ab}^+ - L_{ab}^-)(L_{ab}^+ - L_{ab}^-) + \beta_2(z)(L_{aa}^+ - L_{aa}^-) \right. \]

\[ + \left. \beta_3(z)(L_{ab}^+ - L_{ab}^-) + \beta_4(z)\sigma^2 \right]. \]

(68)

(the 1st order heat kernel \( K_1 \) vanishes identically). The functions which remain relevant for the calculations of this subsection are \( \beta_1, \beta_2, \beta_3, \) and \( \beta_4 \). The numbering conventions do not look very natural, but, unfortunately, any change of notations would inevitably bring in misprints. The present author kindly asks the reader for understanding.

Let us start with the contribution of \( \beta_1 \), cf. Eq. (36). By using the identity

\[ e^{4Q^2} \text{erfc}(2Q t^{1/2}) = \frac{2}{\pi} \int_0^\infty d\lambda e^{-\lambda t} \frac{d}{d\lambda} \arctan \left( \frac{2Q}{\lambda^{1/2}} \right) \]

(69)

we arrive at the following expression for the counting function

\[ N_{10}(\lambda, Q) = \frac{1}{4\pi} \int_{\Sigma} d^2x \sigma(x)^2 \cdot \bar{N}_{10}(\lambda, Q), \]

\[ \bar{N}_{10}(\lambda, Q) = 2\theta(\lambda + 4Q^2) + \frac{2}{\pi} \arctan \left( \frac{2Q}{\lambda^{1/2}} \right). \]

(70)

By using Eqs. (12) and (21) one obtains an expression for the spectral density of the non-linear spectral problem

\[ \rho_{10}(\omega^2) = \frac{1}{4\pi} \int_{\Sigma} d^2x \sigma(x)^2 \cdot \tilde{\rho}_{10}(\omega^2), \]

\[ \tilde{\rho}_{10}(\omega^2) = 2\delta(\omega^2) + \frac{q}{\pi\omega(\omega^2 + q^2)}. \]

(71)

We dropped another delta-function, \( \delta(\omega^2 + q^2) \), since it corresponds to a non-physical part of the spectrum in accordance with the previous subsection. The part of \( \rho_{10}(\omega^2) \) which decays as \( \omega^{-3} \) at large \( \omega \), namely

\[ \rho_{10}^{\text{div}}(\omega^2) = \frac{1}{4\pi^2} \frac{q}{\omega^3} \int_{\Sigma} d^2x \sigma^2 \]

(72)
generates a logarithmic divergence in the vacuum energy \(\mathcal{E}\).

Let us now proceed with contributions of other invariants appearing in \(\mathcal{E}\). The inverse Laplace transformation of the functions \(\beta_5, \beta_1, \beta_2\) (see \([37, 60]\)) may be performed by using e.g. the textbook \([30]\). The resulting expressions for the spectral densities are becoming rather lengthy, and thus we keep track of divergent contributions only. They read

\[
\rho^{\text{div}}(\omega^2) = \frac{1}{4\pi^2} \frac{q}{\omega^3} \int d^2x \sqrt{h} \left[ \frac{1}{12} (L^+_{ab} - L^-_{ab})(L^+_{ab} - L^-_{ab}) + \frac{5}{48} (L^+_{aa} - L^-_{aa})^2 + \sigma (L^+_{aa} - L^-_{aa}) + \sigma^2 \right],
\]

where we have also included \((72)\). This is the main result of this paper in what concerns the curved plasma sheet model.

The formula \((73)\) defines all divergences of the vacuum energy in in the plasma sheet model up to the quadratic order in the extrinsic curvature. We like to remind that in sec. \(\text{III.A}\) we argued that \(\sigma\) has to be considered as being of the same order as \(L_{ab}\). We see, that there are no divergences in the linear order, and that all divergences are logarithmic. The expansion in the extrinsic curvature is an asymptotic one. Although it is enough to state the presence of divergences of certain kind and the necessity of counterterms, it is not enough to use the spectral densities for calculations of the finite part of the vacuum energy, as it is not clear how large or small the higher order corrections could be.

In principle, one cannot totally exclude that some other divergences will also appear in the case of curved plasma sheet. Here we mean higher orders of the extrinsic curvature, or with derivatives acting on \(L_{ab}\) or \(\sigma\). However, we find this very unlikely. Higher order invariants are usually less divergent, and the quadratic terms already diverge only logarithmically.

It is instructive to compare our results to the ones obtained in \([20]\) for the spherical plasma shell. Since the volume of a two-sphere of the radius \(r_0\) is proportional to \(r_0^2\), and the extrinsic curvature is inverse proportional to \(r_0\), the expression \((73)\) does not depend on the radius. The paper \([20]\) found linear divergences which do not depend on the radius and, therefore, correspond to the extrinsic curvature squared. (One should take into account that our conventions for the heat kernel coefficients differ from the ones used in \([20]\).) This means that the divergences of plasma model are sensitive to the topology of the interface surface \(\Sigma\) in contrast to (most of) the frequency-independent problems. This is an expected result, since to derive the part of the spectral densities which generates divergences we needed the heat kernel for all values of \(t\) and not only the small \(t\) asymptotics.

From the spectral densities we have calculated above one can derive the the heat kernel coefficients for the non-linear spectral problem. This step is, however, unnecessary in our approach since the divergences have already been calculated.

Let us remind that the electromagnetic field may be reduced to scalar fields only for some specific geometries (including flat and spherical ones). Therefore, the problem studied in this section is an idealization of the full problem. However, the main complication corresponding to calculate the electromagnetic field directly is a rather nontrivial matrix structure of the matching conditions, which makes the calculations more involved, but not to the point where the technical difficulties become unresolvable.

IV. CONCLUSIONS

In this paper we suggested a method to evaluate the ultra-violent behavior of spectral density and resulting divergences of the vacuum energy for the problems with generic non-linear dependence on the spectral parameter. The divergences are represented in a form similar to the heat kernel coefficients, i.e. through integrals of local invariants associated with the problem. The method, however, requires considerably more information on the spectral properties of the operator involved that just the asymptotic expansion of the heat kernel. These divergences appeared to be logarithmic, and they depend crucially on the topology of the interface surface. This fact has certain implications for the plasma sheet model. Clearly it implies that a renormalizable model (in the sense of \([20]\)) must contain some topology-dependent contributions perhaps corresponding to global degrees of freedom.

We expect that the method can be considerably improved by using recent advances in related areas. For example, to calculate the heat kernel one can employ the worldline formalism \([31]\). Recent advances in calculation of the Casimir force between bodies of various shapes may also be useful (cf. \([32, 33, 34, 35, 36, 37, 38]\) and references therein). We also like to note some similarities to the method of Ref. \([39]\) applied to one-dimensional systems with arbitrary dispersion relations. Finally, we expect that with some modifications the methods proposed above may be applied to such complicated problems as boundary conditions depending on the tangential momentum studied in Ref. \([40]\).
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[34] A. Wirzba, “The Casimir effect as scattering problem,” J. Phys. A 41, 164003 (2008) [arXiv:0711.2395 [quant-ph]].
[35] B. Dobrich, M. DeKieviet and H. Gies, “Scalar Casimir-Polder Forces For Uniaxial Corrugations,” Phys. Rev. D 78, 125022 (2008).
[36] S. Reynaud, P. A. Maia Neto and A. Lambrecht, “Casimir energy and geometry: beyond the proximity force approximation,” J. Phys. A 41, 164004 (2008).
[37] A. Lambrecht and V. N. Marachevsky, “Casimir interaction of dielectric gratings,” arXiv:0806.3142 [quant-ph].
[38] H. Ahmedov and I. H. Duru, “Cavity Shape and Casimir Energy,” arXiv:0804.4382 [hep-th].
[39] S. Bachmann and A. Kempf, “On the Casimir effect with general dispersion relations,” J. Phys. A 41, 164021 (2008).
[40] A. O. Barvinsky, A. Y. Kamenshchik, C. Kiefer and D. V. Nesterov, “Effective action and heat kernel in a toy model of brane-induced gravity,” Phys. Rev. D 75, 044010 (2007) arXiv:hep-th/0611326.