Abstract. Valdivia’s lifting theorem of (pre) compact sets and convergent (respectively, Cauchy) sequences from a quasi-(LB) space to a metrizable, strictly barrelled space is extended to a strictly larger collection of range spaces. Specifically, we assume that the range space has a sequential web structure and do not require that it be metrizable, nor strictly barrelled, and the range space need not even be barrelled. Distinguishing examples are provided that include natural constructions of range spaces connected with applications, such as $D'_\Gamma$, the space of distributions having their wavefront sets in a specified closed cone $\Gamma$. The same and other examples could also serve as domain spaces for Valdivia’s closed graph theorem, revealing a much wider collection of domain spaces that can be used in that result.

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1. Introduction.

Nearly a decade after DeWilde’s [13] popular work on the closed graph theorem led to webbed spaces being a common version of that theorem, Valdivia [36] used webbed space structures to prove distinct closed graph and lifting theorems for locally convex spaces that were
defined as quasi- (LB) spaces. Valdivia’s results seemingly attracted less attention than might have been expected, perhaps as ‘yet another closed graph theorem’. In fact, Valdivia’s methods and results of 36 still lead an active life in research. In particular, the general structure of quasi- (LB) spaces is used quite successfully in areas such as compact resolutions in topological groups 20, descriptive topology 17, the Fréchet - Urysohn property in various contexts 10, and K analyticity in spaces of continuous functions, 31, to name a few. In this paper, we appeal to W. Robertson’s 29 original definition of webs that facilitates convergence of sequences in non-metrizable locally convex spaces so as to extend the lifting theorem of 36 from metrizable strictly barrelled locally convex spaces as the range, to those having sequential webs of 23. The collection of locally convex spaces with sequential webs includes those that are metrizable, and also includes some collections of locally convex spaces that are not necessarily metrizable, such as any sequentially retractive locally convex inductive limit; c.f. 23. Thus, for example, the space $D$ of test functions from the classical Schwartz distribution theory is allowable here. In addition, the sequential web condition does not require that the spaces be strictly barrelled or even barrelled. Examples of range spaces are provided, including that of the space $D_\Gamma'$ of distributions having their wavefront sets in a specified closed cone $\Gamma$, equipped with a locally convex topology, (denoted as the normal topology, 11, p. 1350]). This space, and its dual space, $E_\Lambda'$ have become important in mathematical physics, due to their role in the formulation of quantum field theory in curved spacetime 11, p.1345]. We observe that Valdivia’s closed graph theorem is also applicable to domain spaces that need not be strictly barrelled, and could be applied to locally convex spaces that need not even be barrelled. Thus, Valdivia’s results from 36 are deeper and more applicable than at first glance.

Topological vector spaces having completing webs are commonly used for proving closed graph and open mapping theorems, 1, 3, 7, 8, 9, 12, 16, 21, 22, 28, 29, 35, 36, 38, among others. In 36, strict webs, originally defined as strict réseau absorbants (see 13), are equipped with an ordered structure and used to prove substantial generalizations of the closed graph and lifting theorems of DeWilde 13 using techniques that connect back to those of Banach’s original closed graph theorem proof, 2. The approach taken in 36 is to assume that the linear function and strands of a web satisfy properties for closed graph, open mapping, and lifting theorems, rather than assume the domain and range spaces be of a particular type of locally
convex space. See [26, 9.1.44, p. 346] for another example of this approach. By replacing metrizability with the presence of sequential webs, Valdivia’s lifting theorem [36, 6.4, p. 162] can be generalized to spaces that need be neither strictly barrelled nor metrizable.

Valdivia’s original Lifting theorem is stated here:

Valdivia’s Lifting theorem. ([36, 6.4, p. 162]). Suppose $E$ and $F$ are Hausdorff locally convex spaces such that $F$ is a quasi-(LB) space and $E$ is metrizable and strictly barrelled. Let $T$ be a linear map from subspace $H$ of $F$ onto $E$ for which the graph of $T$ is fast closed in $F \times E$. Suppose $\{A_\alpha : \alpha \in \mathbb{N}^k\}$ is a quasi-(LB) representation in $F$. Define

$$V_{m_1 \cdots m_k} = \bigcup \{T(A_\alpha \cap H) : \alpha \in \mathbb{N}^N, \alpha = (a_n), a_n = m_n, n \in k\}.$$ 

Assume there exists a sequence $(r_k)$ of positive integers such that $V_{r_1 \cdots r_k}$ is a neighborhood of the origin in $E$ for $k = 1, 2, \ldots$. Then the following properties hold:

(a) If $(x_n)$ is a sequence in $E$ which converges to the origin, there exists a sequence $(u_n)$ in $H$, fast convergent to the origin in $F$ and such that $Tu_n = x_n, n = 1, 2, \ldots$.

(b) If $A$ is a precompact set in $E$, then there exists a subset $B$ in $H$, fast precompact in $F$ and such that $T(B) = A$.

(c) If $(y_n)$ is a Cauchy sequence in $E$, then there exists a sequence $(v_n)$ in $H$, fast convergent in $F$ and such that $Tv_n = y_n, n = 1, 2, \ldots$.

We will abbreviate the above assumption that there exists a sequence $(r_k)$ of positive integers such that $V_{r_1 \cdots r_k}$ is a neighborhood of the origin in $E$ for $k = 1, 2, \ldots$ with the term strand-neighborhood condition and replace the assumptions that $E$ is strictly barrelled and metrizable with the less restrictive assumption that $E$ has a sequential web. The main result of this paper is:

Theorem 1. (Lifting Theorem). Suppose $E$ and $F$ are Hausdorff locally convex spaces such that $F$ is a quasi-(LB) space and $E$ has a sequential web. Let $T$ be a linear map from a fast closed linear subspace $H$ of $F$ onto $E$ for which the graph of $T$ is closed in $H \times E$. If the images under $T$ of the quasi- (LB) representation of $F$ restricted to $H$ satisfy the strand - neighborhood condition, then:
(a) Every convergent sequence in $E$ is the image under $T$ of a sequence in $H$ that is fast convergent in $F$.
(b) If in $E$ each precompact set is contained in the absolutely convex hull of a null sequence, then every precompact subset of $E$ is the image under $T$ of a subset of $H$ that is fast precompact in $F$.
(c) Suppose that in addition $E$ is metrizable. Then every Cauchy sequence in $E$ is the image under $T$ of a sequence in $H$ that is fast convergent in $F$.

Theorem 1 generalizes [36, 6.4, p. 162] to range spaces $E$ that possess sequential webs but need not be metrizable and need not be strictly barrelled, as will be seen in the examples presented in section 4. The outline of this paper is as follows: The next section will consist of basic definitions, including those of webs, réseau, quasi (LB) spaces, strictly barrelled spaces, sequential webs, as well as statements of relevant results that will be used. Section three consists of the proof of Theorem 1 and a couple of consequences thereof. The penultimate section consists of distinguishing examples and basic properties of the spaces involved. Of note is that many natural examples of such spaces turn out to be non-barrelled, and this implies potential applications to such spaces. This paper concludes with a few indications regarding potential further developments.

2. Preliminaries.

2.1. General. Throughout this paper, $E = (E, \tau)$ denotes a Hausdorff locally convex topological vector space, or space, over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A convex, balanced set is absolutely convex, which we will call a disk. The expression $\text{absconv}(A)$ denotes the absolutely convex hull of a set $A$. A system of zero neighborhoods in a space $E$ with topology $\tau$ will be denoted by $N_{0,E}$. When distinction of topologies is needed, we use the notation $\tau_E$ for a space $E$. The set $\{1, 2, \cdots, k\}$ is denoted by $\underline{k}$. The closure of a set $A$ in $(E, \tau)$ will be denoted by $\overline{A}$, and by $\overline{A}^{\tau_E}$ whenever it is necessary to clarify the topology involved. Likewise, the interior of a set $A$ is $A^\circ$, and $A^\circ_{\tau_E}$ to clarify. A linear subspace $H$ of a space $F$ will be denoted by $H \leq F$. We make typical use of Banach disks and related ideas described next, which incidentally, are originally due to Grothendieck [24]:

**Definition 1.** (1) Given an absolutely convex subset $B$ of a space $E$, the linear span of $B$ is denoted by $E_B$, and we equip $E_B$
with the linear topology given by the Minkowski functional of $B$, namely, for any $x \in E_B$, $\mu_B(x) = \inf\{t > 0 : x \in tB\}$.

If additionally the disk $B$ is bounded, then with this topology $(E_B, \mu_B)$ is a normed space and we write $(E_B, || \cdot ||_B)$. It is easy to see that a base of zero neighborhoods for $(E_B, || \cdot ||_B)$ is given by $\{n^{-1}B : n \in \mathbb{N}\}$. The boundedness of $B$ implies that the injection $(E_B, || \cdot ||_B) \hookrightarrow E$ is continuous. When this normed space is complete, $B$ is called a Banach disk.

(2) A sequence $(y_n)$ in $E$ is Mackey convergent to $y$ if converges to $y$ in $(E_B, || \cdot ||_B)$, for some closed bounded disk $B$, [26, 5.1.29, p. 158]. In particular, $(y_n)$ is fast convergent, if the disk $B$ is a Banach disk, [26, 6.1.20, p. 171]. Equivalently a sequence $(y_n)$ in $E$ is Mackey convergent to $y$ [26, 5.1.1, p. 151] if there exists a sequence $(d_n)$ of positive numbers such that $d_n \to \infty$ as $n \to \infty$, and $d_n(y_n - y) \to 0$ in $E$. A space $E$ satisfies the Mackey convergence condition, if any null sequence is Mackey null [26, 5.1.29, p. 158].

(3) A linear subspace $H$ of a space $F$ is fast closed if, given a Banach disk $B$ of $F$, $H \cap F_B$ is a closed subspace of $F$, cf [36, p. 150].

(4) A subset $A$ of $E$ is fast (pre) compact if for some Banach disk $B$, $A$ is (pre) compact in $(E_B, || \cdot ||_B)$.

Definition 2. An inductive limit $E = \text{ind}_n E_n$ is sequentially retractive [26, 8.5.32, p. 295], if given any null sequence in $E$, there exists $m \in \mathbb{N}$ such that the sequence is contained in and null in $E_m$.

Definition 3. (See [15, Def 1.1, p. 52].) A projective limit of a sequence of strong duals of Fréchet-Schwartz spaces (i.e., DFS spaces) is called a PLS-space.

So as to distinguish the results here from other concepts of “lifting”, we include (c.f. [12, VI.3.5, p. 117]):

Definition 4. Consider a linear map $T : F \to E$, for locally convex spaces $E$ and $F$. Then $T$ is a lifting if either of the following hold:

(1) Every (pre-) compact subset of $E$ is the image under $T$ of a (pre-) compact subset of $F$.

(2) Every (fast) (pre-) compact set in $E$ is the image under $T$ of a fast (pre-) compact subset of $F$.

General background information on topological vector spaces can be found in [26] or [30].
2.2. Webs and Réseaux webs (absorbents) and a comparison.

DeWilde’s réseaux absorbents are commonly denoted with W. Robertson’s original terminology of webs [29]. The two definitions give rise to distinct representations, and because we will use both constructions, a slightly closer look at each is relevant. Both definitions follow.

Definition 5. ([29, p. 714]) A (Robertson) web \( W \) on a space \( F \) is a countable collection \( W \) of balanced sets, arranged in layers, where the first layer is given by \( \{ W_{n_1} : n_1 \in \mathbb{N} \} \) and subsequent layers, denoted by \( \{ W_{n_1 n_2 \cdots n_k} : k, n_1, n_2, \cdots, n_k \in \mathbb{N} \} \), that satisfy the following properties:

1. For each \( k \in \mathbb{N} \),
   \[ W_{n_1 n_2 \cdots n_{k+1}} + W_{n_1 n_2 \cdots n_{k+1}} \subset W_{n_1 n_2 \cdots n_k} \]

2. \( \bigcup \{ W_{n_1} : n_1 \in \mathbb{N} \} \) is absorbing in \( F \),

3. For each \( k \in \mathbb{N} \),
   \[ \bigcup \{ W_{n_1 n_2 \cdots n_k} : k, n_1, n_2, \cdots, n_k \in \mathbb{N} \} \]

4. Given \( (n_1, n_2, \cdots) = (n_k) \in \mathbb{N}^\infty \), the sequence \( (W_{n_1}, W_{n_1 n_2}, \cdots) \) of sets from \( W \) is called a strand, and is denoted by \( (W_k) \), that is,
   \[ (W_k)_{k \in \mathbb{N}} = (W_k) = (W_{n_1 n_2 \cdots n_k}) \].

In particular, when all subsets of a web are absolutely convex then for any strand \( (W_k) \) of \( W \), item (1) becomes:

\[ (*) \quad (\forall k \in \mathbb{N}) \quad W_{k+1} \subset \frac{1}{2} W_k. \]

Definition 6. ([13, p. 14], [12]) A réseaux (absorbents) web on a space \( F \) is a countable collection \( R \) of sets, denoted by \( \{ A_{n_1 n_2 \cdots n_k} : k, n_1, n_2, \cdots, n_k \in \mathbb{N} \} \), satisfying the following properties:

1. \( \bigcup \{ A_{n_1} : n_1 \in \mathbb{N} \} = F \),

2. For each \( k \in \mathbb{N} \),
   \[ \bigcup \{ A_{n_1 n_2 \cdots n_{k+1}} : n_{k+1} \in \mathbb{N} \} = A_{n_1 n_2 \cdots n_k} \].

The following is the completeness property usually assumed for webs or réseaux webs, on a space \( F \).
Definition 7. (\cite[p.715]{29}) A web $W$ in $F$ is of type (c) if, given any strand $(W_k)$ of $W$ and any sequence $(x_k)$ with $x_k \in W_k$ for each $k \in \mathbb{N}$, the series $\sum_{k=1}^{\infty} x_k$ converges in $F$. If in addition the convergent series satisfy $\sum_{r=k+1}^{\infty} x_r \in W_k-1$ for each $k \geq 2$, then $W$ is tight.

(2) \cite[p. 14 - 15, 12 p. 48 - 49]{13} A réseaux web $R$ in $F$ is of type (c) if given any sequence $(n_k) \in \mathbb{N}^\mathbb{N}$, and any $x_k \in A_{n_1 n_2 \cdots n_k}$ for each $k \in \mathbb{N}$, there exists a sequence $(\lambda_k)$ of positive numbers, such that the series $\sum_{k=1}^{\infty} \mu_k x_k$ converges in $F$ for all $\mu_k$ for which $0 \leq |\mu_k| \leq \lambda_k$, $k \in \mathbb{N}$. If in addition the convergent series satisfy $\sum_{k=p}^{\infty} \mu_k x_k \in A_{n_1 n_2 \cdots n_p}$, for any $p \in \mathbb{N}$, then $R$ is strict.

It is assumed here that all sets in a web or réseaux web are absolutely convex. As observed in \cite[p. 725]{29}, a web of type (c) is equivalent to a réseaux of type $C$, and a tight web is equivalent to a strict web. A space with a completing web or réseaux web is typically referred to as a webbed space, and spaces with tight webs or strict réseaux webs are referred to as strictly webbed spaces or having a strict web. This is how we will generally refer to webs in the sequel. We need a few more properties of webs:

Definition 8. A web $W$ on a space $E$ is:

(1) **Compatible** \cite[p. 156]{30} if, given any zero neighborhood $V$ in $E$, and any strand $(W_k)$ of $W$, there exists $K \in \mathbb{N}$ such that $W_K \subset V$. Hence, in this case, $(\forall k \geq K) W_k \subset V$.

(2) **Sequential** \cite[Def. 6, p. 475]{23} if it is compatible, and given any null sequence $(x_n)$ in $E$, there is a strand $(W_k)$ of $W$ for which, given any $k \in \mathbb{N}$, there is $N_k \in \mathbb{N}$ such that $x_n \in W_k$, for all $n \geq N_k$. In this construction, one assumes the property of strands $(W_k)$ of $W$. Robertson’s webs c.f. \cite[p. 156]{30} as expressed in (*) of Definition 5.

If $E$ is any metrizable space, and $\{V_k : k \in \mathbb{N}\}$ is a system of absolutely convex zero neighborhoods such that $(\forall k \in \mathbb{N}) 2V_{k+1} \subset V_k$, then $W = \{V_k : k \in \mathbb{N}\}$ is a sequential web. In particular, the $k$-th layer of $W$ consists of the set $V_k$. General references for webs on topological vector spaces include \cite, [12], and [29].

2.3. Ordered webs and quasi-(LB) spaces.

**Definition 9.** \cite[p. 150]{36} A web $\{W_{n_1 n_2 \cdots n_k} : k, n_1, n_2, \cdots, n_k \in \mathbb{N}\}$ is ordered if, given arbitrary natural numbers $k, r_1, r_2, \cdots, r_k$, and $s_1, s_2, \cdots, s_k$, if $r_j \leq s_j$ for all $j \in k$, then
Definition 10. \[36, \text{p. 151}\] Given \(\alpha = (a_n) \in \mathbb{N}^\mathbb{N}\) and \(\beta = (b_n) \in \mathbb{N}^\mathbb{N}\), one puts \(\alpha \leq \beta\) if \(a_n \leq b_n\) for every \(n \in \mathbb{N}\). A quasi-(LB) representation on a space \(F\) is a family of Banach disks \(\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}\) such that:

1. \(\bigcup\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\} = F\);
2. For any \(\alpha, \beta \in \mathbb{N}^\mathbb{N}\), if \(\alpha \leq \beta\), then \(A_\alpha \subset A_\beta\).

For short, we call such an \(F\) a quasi- (LB) space. For a given \(\alpha \in \mathbb{N}^\mathbb{N}\), we denote by \(F_{A_\alpha}\) the corresponding Banach space \((F_{A_\alpha}, \| \cdot \|_{A_\alpha})\).

The connection between quasi- (LB) spaces and webs that we will make use of is given by the following.

Theorem 2. \[36, \text{4.1, p. 153}\] A locally convex space is a quasi-(LB) space if, and only if, it has an absolutely convex, ordered, strict web. In that case, the strands are described by:

1. \(U_k \equiv U_{m_1m_2 \cdots m_k} = \bigcup\{A_\alpha : \alpha = (a_n) \in \mathbb{N}^\mathbb{N}, a_n = m_n, n \in [k]\}\).

Observe that \(U_{m_1} \supset U_{m_1m_2} \supset \cdots\), for each \((m_n) \in \mathbb{N}^\mathbb{N}\).

The proofs in \[36\] use ordered (strict) réseaux web structures as given in Definitions 6 and 9. The parts of proofs here that correspond to sequential webs are based on the (tight) web structures of Definition 5, where sequential convergence is easy to manage. The following simple example illustrates the difference between an ordered, tight web and ordered strict réseaux web, as being equivalent, but having distinct representations on a space.

Example 1. Tight ordered webs and strict ordered réseaux webs on any nontrivial (LB) - space.

Details. Let \(E = \text{indlim}_n(E_n)\), where, for each \(n \in \mathbb{N}\), \(B_n\) denotes the closed unit ball of the Banach space \(E_n\). Multiplying by appropriate scalars if necessary, we can assume that for each \(n \in \mathbb{N}\), one has \(B_n \subset B_{n+1}\). An ordered, tight web on \(E\) is described by setting the \(k\)th layer to be \(\{2^{-k}B_n : n \in \mathbb{N}\}\), for each \(k \in \mathbb{N}\). On the other hand, an ordered, strict réseaux web is described as follows: Certainly, \(E = \bigcup_{n=1}^\infty nB_n\), for the first layer. For the second layer, one writes each \(nB_n\) as an increasing countable union of Banach disks. Subsequent layers are obtained by repeating this process. Using values of the form \(2^j\) for appropriate values of \(j \in \mathbb{Z}\), one can start with an ordered tight web on \(E\) and construct a strict réseaux web representation, and visa versa.
Definition 11. [36, p. 160] A space is strictly barrelled if for every ordered, absolutely convex web on the space, there exists at least one strand for which the closure of each member is a zero neighborhood.

It is easy to see that every strictly barrelled space is barrelled. A barrelled space that is not strictly barrelled will be given in section 4.

The following definition is intended to express the general assumption that linear images or linear inverse images of strands of a certain ordered strict web structure satisfy a condition of closures being zero neighborhoods. See [36, p. 155, 157].

Definition 12. Consider spaces $E$ and $F$, and a linear map $T : E \to F$. Assume $\{A_\alpha : \alpha \in \mathbb{N}^N\}$ is a quasi-(LB) representation in $F$. Given $k, m_1, m_2, \ldots, m_k \in \mathbb{N}$, put

$$(2) \quad U_{m_1 m_2 \cdots m_k} = T^{-1}\left(\bigcup\{A_\alpha : \alpha = (a_n) \in \mathbb{N}^N, a_n = m_n, n \in \mathbb{N}\}\right).$$

The triple $(E, F, T)$ satisfies a strand - neighborhood condition if there exist a sequence $(r_k) \in \mathbb{N}^N$ such that for $U_k \equiv U_{r_1 r_2 \cdots r_k}$,

$$U_k = U_{r_1 r_2 \cdots r_k} \in \mathcal{N}_{0,E},$$

for each $k \in \mathbb{N}$.

For the case of a linear $T : F \to E$ such that $T$ maps a subspace $H$ of $F$ onto $E$, one defines [36, p. 157]: Given $n, m_1, m_2, \ldots, m_n \in \mathbb{N}$, put

$$(3) \quad V_{m_1 m_2 \cdots m_n} = \bigcup\{T(A_\alpha \cap H) : \alpha = (a_n) \in \mathbb{N}^N, a_n = m_n, n \in \mathbb{N}\}.$$

We assume there exists $(r_k) \in \mathbb{N}^N$ such that for $V_k \equiv V_{r_1 r_2 \cdots r_k}$,

$$V_k = V_{r_1 r_2 \cdots r_k} \in \mathcal{N}_{0,E},$$

for each $k \in \mathbb{N}$.

We state Valdivia’s closed graph theorem below, as its use will be important in the proof of the main result. We use the notation of Definition 12.

Valdivia’s Closed Graph Theorem ([36, 5.11, p. 157, 6.1.5, p. 163]). A linear map from a Hausdorff locally convex space to a quasi
- (LB) space having a closed graph is continuous if the strand neighborhood condition is satisfied. In particular, we have, in \( E \),

\[
\emptyset \neq U_k^{\alpha E} = \bigcup_{r_1 r_2 \ldots r_k} U_{r_1 r_2 \ldots r_k}, \ k \in \mathbb{N},
\]

\[
(2) \ T \text{ is continuous.}
\]

One more result is needed in the proof of Theorem 1:

**Lemma 1.** [Lemma 3, p. 60] Let \( T : F \to E \) be linear, for topological vector spaces \( E \) and \( F \). Consider

\[
F \overset{\varphi}{\rightarrow} F/T^{-1}(\{0\}) \overset{\tilde{T}}{\rightarrow} E,
\]

where \( \varphi \) is the canonical linear quotient map, and \( \tilde{T} \) is the linear map such that \( T = \tilde{T} \circ \varphi \). If \( G_T \) denotes the graph of \( T \), then \( G_T \) is closed if and only if, \( G_{\tilde{T}} \) is closed.

### 3. Proof of the main result.

**Proof of Theorem 1.** Our methods here are inspired by those in [8], [9], [16], and [36]. For the sake of completeness, full details are given here. It suffices to prove (a) and (b), because the proof of part (c) only requires the observation that any metrizable space has a sequential web and is otherwise exactly the same as in [36, 5.6, p. 158].

(a). The linear map \( T \) is not assumed to be continuous, and the closed graph theorem does not apply to \( T : F \to E \) (the assumptions are reversed). Nevertheless, the arrival at the conclusion can be obtained by taking a detour through the quotient space \( H/T^{-1}(\{0\}) \) in such a way as to apply Valdivia’s closed graph theorem. This is done in Step 2. After that, we apply the assumption of a sequential web to prove the existence of the desired sequence. By way of the linearity of topologies and functions, it suffices to consider null sequences in our work.

**Step 1. Setting up spaces and linear maps.** Let \( T \) be a linear map from a fast closed linear subspace \( H \) of \( F \) onto \( E \), for which the graph \( G_T \) of \( T \) is closed in \( H \times E \). Let \( \{A_\alpha : \alpha \in \mathbb{N}^N\} \) be a quasi-(LB) representation on \( F \). Suppose \( E \) has a sequential web denoted by \( Z \), for which we denote strands by \( (Z_k)_{k \in \mathbb{N}} = (Z_k) \). Consider any null sequence \( (y_n) \) in \( E \). We wish to show there exists \( \beta \in \mathbb{N}^N \) and a sequence \( (u_n) \) in \( F_{A_\beta} \cap H \) such that \( (u_n) \) is fast convergent to zero in \( F \), and \( T(u_n) = y_n \), for all \( n \in \mathbb{N} \). Using previous notation, we have
By the strand-neighborhood assumption, there exists a sequence \((r_k) \in \mathbb{N}_0^E\) such that for all \(k \in \mathbb{N}\), \(V_{r_1 \ldots r_k} \in \mathcal{N}_{0,E}\).

We proceed with a factoring through the null space \(T^{-1}(\{0\})\), as follows: Denote by \(\varphi\) the canonical quotient map from \(H\) onto \(H/T^{-1}(\{0\})\). Next, let \(S : E \to H/T^{-1}(\{0\})\) be the bijective linear map such that \(T = S^{-1} \circ \varphi\). Observe that for all \(\alpha \in \mathbb{N}^N\), \(\varphi(A_\alpha \cap H) = S(T(A_\alpha \cap H))\).

Because \(H\) is a fast closed linear subspace of \(F\), for every \(\alpha \in \mathbb{N}^N\), \(H \cap F_{a_\alpha} = H \cap F_{a_\alpha}\), hence, \(\{A_\alpha \cap H : \alpha \in \mathbb{N}^N\}\) is a quasi- (LB) representation in \(H\). By the continuity of \(\varphi\) from \(H\) onto \(H/T^{-1}(\{0\})\), an application of [36, 3.1, p. 151] guarantees that \(\{\varphi(A_\alpha \cap H) : \alpha \in \mathbb{N}^N\}\) is a quasi- (LB) representation in \(H/T^{-1}(\{0\})\).

Given
\[
C_{m_1 \ldots m_k} = \bigcup \{\varphi(A_\alpha \cap H) : \alpha \in \mathbb{N}^N, \alpha = (a_n), a_n = m_n, n \in k\}
\]

we put
\[
U_{m_1 \ldots m_k} = S^{-1}(C_{m_1 \ldots m_k}).
\]

By (3) of Definition 12, we obtain
In particular, each $V_k = V_{r_1 r_2 \cdots r_k}$ of this step, there exists for each $k \in \mathbb{N}$ such that for each $n \geq k$, $S$ is satisfied for $k$. We apply the compatibility of $Z$ as a consequence of the property $Z$ implies the existence of a positive integer $N_{j_k}$ such that

$$Z_{j_k+k} \subset 2^{-1} Z_{j_k+k-1} \subset 2^{-2} Z_{j_k+k-2} \subset \cdots \subset 2^{-k} Z_k \subset k^{-1} Z_k,$$

as a consequence of the property $Z_{k+1} \subset \frac{1}{2} Z_k$. Next, the sequential property of $Z$ implies the existence of a positive integer $N_{j_k}$ such that for all $n \geq N_{j_k}$, $y_n \in Z_{j_k+k} \subset k^{-1} Z_k$.

Define, for all $k \in \mathbb{N}$, $d_n = k$, $N_{j_k} \leq n < N_{j_{k+1}}$. Clearly, $\lim_{n \to \infty} d_n = \infty$. We apply the compatibility of $Z$ to the zero neighborhoods $V_l = V_{r_1 r_2 \cdots r_l}$, $l \in \mathbb{N}$, as follows. For a fixed $l \in \mathbb{N}$, there exists $K_l \in \mathbb{N}$ such that for all $k \geq K_l$, $Z_k \subset V_l$. Next, by the construction at the beginning of this step, there exists $N_{j_k} \in \mathbb{N}$ such that for all $n \geq N_{j_k}$, $d_n y_n \in V_l$. The surjectivity of $T$ allows us to find a sequence $(w_n)$ in $H$, such that for all $n \geq N_1$, $w_n \in H$ such that $T(w_n) = d_n y_n$, $N_{j_k} \leq n < N_{j_{k+1}}$, for each $k \in \mathbb{N}$. By [9] 5.21 (i) p.43] with $p = 1$, there exists $\beta \in \mathbb{N}^\mathbb{N}$ such that for each $n \in \mathbb{N}$, $w_n \in F_{A_\beta}$, where $F_{A_\beta}$ is the Banach space corresponding to the Banach disk $A_\beta$, having $\{m^{-1} : m \in \mathbb{N}\}$ as a base for a system of zero neighborhoods. Put $u_n = \frac{1}{d_n} w_n$. Then for all $n \geq N_{j_k}$, $w_n \in A_\beta$, hence, $u_n \in \frac{1}{d_n} A_\beta$, and because $\lim_{n \to \infty} d_n = \infty$, $(u_n)$ is fast convergent to zero in $F_{A_\beta}$. Finally, $u_n \in H \cap F_{A_\beta}$, and
\[ T(u_n) = S^{-1} \left( \varphi \left( \frac{1}{d_n} w_n \right) \right) = \frac{1}{d_n} d_n y_n = y_n, \]
finishing the proof of part (a).

(b). Let \( E \) be any space such that each precompact set is contained in the absolutely convex hull of a null sequence. The goal is to prove that for any precompact \( B \subset E \), there exists \( \beta \in \mathbb{N}^\mathbb{N} \) and a set \( M \subset F_{A_\beta} \cap H \) such that \( M \) is precompact in \( F_{A_\beta} \), and \( T(M) = B \).

By the assumption on precompact sets, there is a null sequence \((y_n)\) in \( E \) such that \( \text{absconv}\{y_n\}^E \supset 2B \). Let \( \beta \in \mathbb{N}^\mathbb{N} \) and \((u_n)\) in \( H \cap F_{A_\beta} \) corresponding to \( (y_n) \), be as in the proof of part (a). Define \( D = \text{absconv}\{\frac{1}{2} u_n : n \in \mathbb{N}\} \|\cdot\|_{A_\beta} \). Because the subspace \( H \) is fast closed in \( F \), and the injection \((F_{A_\beta}, \|\cdot\|_{A_\beta}) \hookrightarrow F\) is continuous, we obtain:

\[ D \subset \overline{H \cap F_{A_\beta}}^{\|\cdot\|_{A_\beta}} \subset \overline{H \cap F_{A_\beta}}^F = H \cap F_{A_\beta} \subset H. \]

In particular, intersecting with \( H \), the following is continuous:

\[ H \cap F_{A_\beta} \hookrightarrow H \xrightarrow{\varphi} H/T^{-1}(\{0\}). \]

That \((F_{A_\beta}, \|\cdot\|_{A_\beta})\) is a Banach space implies \( D \subset F_{A_\beta} \) is compact. The continuity of the linear mappings in (**) ensures that \( \varphi(D) \) is compact in \( H/T^{-1}(\{0\}) \). Next, the graph of \( T \) being closed in \( H \times E \) tells us that \( H/T^{-1}(\{0\}) \) is Hausdorff ([30, prop. V.1, p.77]). Thus, \( \varphi(D) \) is closed in \( H/T^{-1}(\{0\}) \). We use the continuity of \( S \) from part (a) to observe that the set \( S^{-1}(\varphi(D)) \) is closed in \( E \). Meanwhile, \( S^{-1}(\varphi(D)) = T(D) \supset B \), which means \( T^{-1}(B) \subset D \). In order to ensure we capture exactly \( B \), we intersect; that is, we let \( M = T^{-1}(B) \cap D \).

This finishes the proof. \( \Box \)

The assumption on \( E \) in part (b) includes metrizable spaces by way of a variant of the Banach - Dieudonné Theorem ([25, 10.(3)p. 273]). As it also follows, this collection of spaces includes any sequentially retractive inductive limit of metrizable spaces, which of course, need not be metrizable: Witness any strict ([26, 0.3.1, p. 2]) proper (LB) space ([26, 8.5.18, p. 288], or the space \( \mathcal{D} \) of test functions from distribution theory. Moreover, we will see (Example 3 in the next section) that such spaces are not strictly barrelled. At the same time, part (c) does not assume the space \( E \) is strictly barrelled. The following corollary
indicates that Theorem 1 here properly extends Valdivia’s lifting theorem to more general spaces, including those often seen in applications.

**Corollary 1.** Let \( E = \lim_n(E_n) \) be a sequentially retractive inductive limit of metrizable spaces, with quasi-(LB) space \( F, H \), and \( T \) as in Theorem 1. If the images of the quasi-(LB) representation of \( H \) under \( T \) satisfy the strand-neighborhood condition, then every convergent sequence in \( E \) is the image under \( T \) of a sequence in \( H \) that is fast convergent in \( F \), and each precompact subset of \( E \) is the image under \( T \) of a subset of \( H \) that is fast precompact in \( F \).

**Proof:** It suffices to prove that such a space \( E \) has a sequential web. That proof is found in [23, Prop. 9, p. 476]. \( \square \).

For the corollary below, we consider the space \( D'_\Gamma \) of distributions having their wavefront sets in a specified closed cone \( \Gamma \), equipped with its normal topology, as defined in [11, p. 1350]. Apart from its importance in distribution theory, the space \( D'_\Gamma \) is interesting in its own right, due to the combination of topological vector space properties it does, and does not, possess. A lucid description of several such properties can be found in [11]. A few such properties worth mentioning here are that \( D'_\Gamma \) is: a normal space of distributions, complete, nuclear, but is neither metrizable nor bornological (its strong dual, \( E'_\Lambda \), is an incomplete inductive limit). Moreover, \( D'_\Gamma \) is not barrelled [11, Cor. 36, p. 1376]. Nonetheless, it turns out that \( D'_\Gamma \) satisfies the properties of the range space of Theorem 1:

**Corollary 2.** Suppose \( F \) is a quasi-(LB) space and \( T \) is a linear map from a fast closed subspace \( H \) of \( F \) onto \( D'_\Gamma \), for which the graph of \( T \) is closed in \( H \times D'_\Gamma \). If the images of the quasi-(LB) representation of \( H \) under \( T \) satisfy the strand-neighborhood condition, then for any sequence \( (y_n) \) converging to 0 in \( D'_\Gamma \), there exists \( \beta \in \mathbb{N}^\mathbb{N} \) and a sequence \( (u_n) \) in \( F_{A_\beta} \cap H \) such that \( (u_n) \) is fast convergent, and \( T(u_n) = y_n \), for all \( n \in \mathbb{N} \), where \( A_\beta \) is from the quasi-(LB) space representation on \( F \).

**Proof:** We only need to prove that \( D'_\Gamma \) has a sequential web. As a projective limit of a sequence of strictly webbed spaces, \( D'_\Gamma \) is strictly webbed by hereditary properties, namely, that the strong dual of a Fréchet space as well as projective limits of sequences of such spaces are strictly webbed [30]. Moreover, \( D'_\Gamma \) satisfies the Mackey convergence condition, [11, Lemma 21, p. 1363], which allows us to apply [23]
4. **Examples**

In [36 Sect. 6, p. 160], the domain space of the closed graph theorem is assumed to be strictly barrelled. Likewise, in the lifting theorem [36 5.12, p. 157, 6.4, p.162] the range space is assumed to be metrizable and strictly barrelled. The following examples reveal that the collection of domain spaces in Valdivia’s closed graph theorem (respectively, range spaces of Theorem 1), are in fact strictly larger than the collection of strictly barrelled (respectively, metrizable strictly barrelled) spaces.

**Example 2.** Some spaces having a sequential web that are not strictly barrelled:

(a) Any metrizable space that is not strictly barrelled.
(b) The space \((l_1, \sigma(l_1, l_\infty))\).

*Details:* (a): Obvious.

(b): Thanks to Shur’s theorem (c.f. [14, p.85]), weakly convergent sequences are norm convergent in \(l_1\). Thus, a sequential web is given by \(Z = \{2^{-n}B : n \in \mathbb{N}\}\), where \(B\) is the closed unit ball of \(l_1\). The space \((l_1, \sigma(l_1, l_\infty))\) is, of course, neither metrizable nor barrelled. Incidentally, for any \(1 < p, q < \infty\) as conjugate exponents, \((l_p, \sigma(l_p, l_q))\) is neither (strictly) barrelled, nor does it have a sequential web.

The next two examples show that the properties of having a sequential web and being strictly barrelled are distinct. In addition, there are barrelled spaces that have sequential webs but are not strictly barrelled.

**Example 3.** Barreled spaces having a sequential web that are not strictly barrelled.

*Details:* Let \(E\) be any proper sequentially retractive inductive limit of metrizable barrelled spaces \(E_n\). As an inductive limit of barrelled spaces, \(E\) is also barrelled. Consider the web formed by the strands consisting of sets of the form \(\{2^{-k}V_{k,n} : k \in \mathbb{N}\}\), where, for each fixed \(n \in \mathbb{N}\), \(\{V_{k,n} : k \in \mathbb{N}\}\) is a decreasing sequence of absolutely convex sets forming a system of zero neighborhoods in \(E_n\) such that for every \(k \in \mathbb{N}, V_{k+1,n} \subset 2^{-1}V_{k,n}\). Sequential retraction tells us that this
construction forms a sequential web on \( E \). This web is also absolutely convex, ordered, and has the property that the closure of every set in every strand has empty interior in \( E \). In particular, any strict (LB)-space is barrelled, is quasi- (LB), has a sequential web, and yet is not strictly barrelled.

For the next example, note that by \([23, \text{Thm 12, p. 477}]\), any space with a sequential web satisfies the Mackey convergence condition.

**Example 4. A strictly barrelled space without a sequential web.**

*Details:* Let \( I \) be the index set consisting of all increasing, unbounded sequences of positive real numbers. Put \( E = \Pi_{\alpha \in I} \mathbb{R} \). As an uncountable product of the Banach space \( \mathbb{R} \), \( E \) is a Baire space, by \([4, \text{Chap. III, Sect. 7}]\). Thus, \( E \) is totally barrelled, hence strictly barrelled, by \([36, 6.17, \text{p. 160}]\). On the other hand, in \([26, 5.1.32, \text{p. 159}]\), it is shown that \( E \) does not satisfy the Mackey convergence condition, guaranteeing that \( E \) cannot have a sequential web.

We finish this section with two observations of hereditary properties of spaces having sequential webs, that are relevant to Theorem 1. In particular, in part (b) of the proposition below strict barrelledness is replaced by the property of having a sequential web, which as we have just seen, is a distinct collection of spaces. Compare to the strict barrelledness result of \([36, 6.16, \text{p. 160}]\).

**Proposition 1.** The property of having a sequential web is preserved under:

(a) Linear subspaces.

(b) Finitely many products.

*Proof:* For (a), let \( Z \) be a sequential web on \( E \). If \( L \) is a subspace of \( E \), define \( Z_L \) to be the set of all \( Z \cap L \), as \( Z \) runs through the sets of \( Z \). The property of having a sequential web follows immediately.

Regarding (b), it suffices to prove the case for the product of two spaces \( E \) and \( F \), the general finite product case following by induction. Denote by \( Z_E \) and \( Z_F \) sequential webs in \( E \) and \( F \), respectively. For each \( k \in \mathbb{N} \), define the \( k \)th layer in \( E \times F \) by the products of the sets in the \( k \)th layer in \( Z_E \) with the sets in the \( k \)th layer of \( Z_F \). It is easy to verify that this web is compatible with the topology of the product \( E \times F \). Now suppose \((w_n) = (x_n, y_n)\) is any null sequence in \( E \times F \). By assumption, there is a strand \( (Z_k) \) in \( Z_E \) for which, given any \( k \in \mathbb{N} \), there exists \( N_k \in \mathbb{N} \) such that \( x_n \in Z_k \) for all \( n \geq N_k \). Similarly, there
is a strand \( (S_k) \) in \( Z_F \) satisfying that, given any \( k \in \mathbb{N} \), there exists \( M_k \) in \( \mathbb{N} \) such that \( y_n \in S_k \) for all \( n \geq M_k \). Putting \( J_k = \max\{N_k, M_k\} \), for each \( k \in \mathbb{N} \), one has for all \( k \in \mathbb{N} \), and for all \( n \geq J_k \):

\[
w_n = (x_n, y_n) \in Z_k \times S_k. \]

\[ \square \]

5. Concluding remarks

Some implications for further research regarding the application of Valdivia’s closed graph theorem and the lifting theorem proved here are outlined.

Applying Valdivia’s closed graph theorem to spaces that are not barrelled. The relevant examples presented here, including non-barrelled spaces that need not have sequential webs, could appear as domain spaces for Valdivia’s closed graph theorem.

Applying Theorem 1 to normed or metrizable spaces that are not barrelled. It seems some of the easiest examples of spaces having a sequential web and are not strictly barrelled turn out to be metrizable and non-barrelled. Far from being “exotic”, normed and metrizable spaces that are not barrelled are rather common. Examples of such spaces include:

1. On every infinite dimensional Banach space there exists a finer, non-barrelled norm \([26, 4.6.7 \text{ (iv)}, p. 131]\).
2. Metrizable, non-normed non-barrelled spaces can be formed by countable products of non-barrelled normed spaces, \([26, 4.2.5, p. 103]\).
3. From \([19, \text{Chapt. 6}]\), we have: Let \( l^\infty_0(\Sigma) \) denote the space of simple functions on an algebra \( \Sigma \) of sets, endowed with a natural norm arising from the Minkowski functional. Then, every separable infinite-dimensional subspace of \( l^\infty_0(\Sigma) \) is a normed, non-barrelled space \([19, 6.3.2, p. 127]\). Furthermore, various algebras \( \Sigma \) are constructed in \([19, p. 131 - 135]\) for which \( l^\infty_0(\Sigma) \) is not barrelled.

General non-barrelled spaces. Our results here, as well as Valdivia’s closed graph theorem appear pertinent to the study of general, not necessarily metrizable, non-barrelled spaces like: Any infinite dimensional Banach space with its weak topology, and as we have seen, as well as \( D' \). Other such spaces, such as tensor products of spaces related to \( D' \), as in \([5]\), could also be worth studying. More examples of
non-barrelled spaces and results related to such spaces can be found in [6], [18], [32], [33], [34], [37], to mention a few instances.

Developing results in other contexts. The webbed and strictly webbed space results initially proved for locally convex spaces over the fields of \(\mathbb{R}\) or \(\mathbb{C}\) have since been developed for general topological vector spaces, for locally convex spaces over non-Archimedean fields, for bornological vector spaces, and for locally convex topological \(\tilde{\mathbb{C}}\) - modules, resulting in useful applications of each those versions. See [1], [21], [22], and [38], respectively. Valdivia’s closed graph, open mapping, and lifting theorems have been developed for general topological vector spaces, as in [8], [9], [16]. Beyond that, no further developments have been made. Such research regarding Valdivia’s closed graph theorem and the lifting theorem obtained here, in contexts of locally convex spaces over non-Archimedean fields, bornological vector spaces, and locally convex topological \(\tilde{\mathbb{C}}\) - modules, has potential for correspondingly useful applications.

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