Super-rigid Donaldson-Thomas Invariants

Kai Behrend and Jim Bryan

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Abstract

We solve the part of the Donaldson-Thomas theory of Calabi-Yau threefolds which comes from super-rigid rational curves. As an application, we prove a version of the conjectural Gromov-Witten/Donaldson-Thomas correspondence of [MNOP] for contributions from super-rigid rational curves. In particular, we prove the full GW/DT correspondence for the quintic threefold in degrees one and two.

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1 Introduction

Let \( Y \) be a smooth complex projective Calabi-Yau threefold. Let \( I_n(\mathcal{O}_Y, \beta) \) be the moduli space of ideal sheaves \( I_Z \subset \mathcal{O}_Y \), where the associated subscheme \( Z \) has maximal dimension equal to one, the holomorphic Euler characteristic \( \chi(\mathcal{O}_Z) \) is equal to \( n \), and the associated 1-cycle has class \( \beta \in H_2(Y) \).

Recall that \( I_n(\mathcal{O}_Y, \beta) \) has a natural symmetric obstruction theory \cite{Th00, BF05}. Hence we have the (degree zero) virtual fundamental class of \( I_n(\mathcal{O}_Y, \beta) \), whose degree \( N_n(\mathcal{O}_Y, \beta) \in \mathbb{Z} \) is the associated Donaldson-Thomas invariant.

Let \( C = \sum_{i=1}^{s} d_i C_i \) be an effective cycle on \( Y \), and assume that the \( C_i \) are pairwise disjoint, smoothly embedded rational curves with normal bundle \( N_{C_i}/Y \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1) \). Such curves are called super-rigid rational curves in \( Y \) \cite{Pa99, BP01}. Assume that the class of \( C \) is \( \beta \).

Let \( J_n(Y, C) \subset I_n(\mathcal{O}_Y, \beta) \) be the locus corresponding to subschemes \( Z \subset Y \) whose associated cycle under the Hilbert-Chow morphism is equal to \( C \) (see Definition 2.1). Since \( J_n(Y, C) \subset I_n(\mathcal{O}_Y, \beta) \) is open and closed (see Remark 2.2), we get an induced virtual fundamental class on \( J_n(Y, C) \) by restriction. We call \( N_n(\mathcal{O}_Y, C) = \text{deg}[J_n(Y, C)]^\text{vir} \) the contribution of \( C \) to the Donaldson-Thomas invariant \( N_n(\mathcal{O}_Y, \beta) \).

The goal of this paper is to compute the invariants \( N_n(\mathcal{O}_Y, C) \).

To formulate our results, we define a series \( P_d(q) \in \mathbb{Z}[[q]] \), for all integers \( d \geq 0 \) by

\[
\prod_{m=1}^{\infty} (1 + q^m v)^d = \sum_{d=0}^{\infty} P_d(q) v^d.
\]

Moreover, recall the McMahon function

\[
M(q) = \prod_{m=1}^{\infty} \frac{1}{(1 - q^m)^m}.
\]

Then we prove (Theorem 2.4) that

\[
\sum_{n=0}^{\infty} N_n(\mathcal{O}_Y, C) q^n = M(-q)^{\chi(Y)} \prod_{i=1}^{s} (-1)^{d_i} P_{d_i}(-q).
\]

Maulik, Nekrasov, Okounkov, and Pandharipande have conjectured a beautiful correspondence between Gromov-Witten theory and Donaldson-Thomas theory which we call the GW/DT correspondence.

As an application of the above formula we prove the GW/DT correspondence for the contributions from super-rigid rational curves (Theorem 3.1). In particular, we prove the full degree \( \beta \) GW/DT correspondence (Conjecture 3 of \cite{MNOP}) for any \( \beta \) for which it is known that all
cycle representatives are supported on super-rigid rational curves (Corollary 3.2). For example, our results yield the GW/DT correspondence for the quintic threefold in degrees one and two (Corollary 3.3). As far as we know, these are the first instances of the GW/DT conjecture to be proved for compact Calabi-Yau threefolds.

The local GW/DT correspondence for super-rigid rational curves follows from the results of [MNOP] as a special case of the correspondence for toric Calabi-Yau threefolds. In contrast to Gromov-Witten theory, passing from the local invariants of super-rigid curves to global invariants is non-trivial in Donaldson-Thomas theory, and can be regarded as the main contribution of this paper.

1.1 Weighted Euler characteristics

Our main tool will be the weighted Euler characteristics introduced in [Be05]. Every scheme $X$ has a canonical $\mathbb{Z}$-valued constructible function $\nu_X$ on it. The weighted Euler characteristic $\tilde{\chi}(X)$ of $X$ is defined as

$$\tilde{\chi}(X) = \chi(X, \nu_X) = \sum_{n \in \mathbb{Z}} n \chi(\nu_X^{-1}(n)).$$

More generally, we use relative weighted Euler characteristics $\tilde{\chi}(Z, X)$ defined as

$$\tilde{\chi}(Z, X) = \chi(Z, f^*\nu_X),$$

for any morphism $f : Z \to X$. Three fundamental properties are

(i) if $X \to Y$ is étale, then $\tilde{\chi}(Z, X) = \tilde{\chi}(Z, Y)$,

(ii) if $Z = Z_1 \sqcup Z_2$ is a disjoint union, $\tilde{\chi}(Z, X) = \tilde{\chi}(Z_1, X) + \tilde{\chi}(Z_2, X)$,

(iii) $\tilde{\chi}(Z_1, X_1) \tilde{\chi}(Z_2, X_2) = \tilde{\chi}(Z_1 \times Z_2, X_1 \times X_2)$.

The main result of [Be05], Theorem 4.18, asserts that if $X$ is a projective scheme with a symmetric obstruction theory on it, then

$$\deg[X]^{\text{vir}} = \tilde{\chi}(X).$$

Thus we can calculate $N_n(Y, C)$ as $\tilde{\chi}(J_n(Y, C))$.

We will also need the following fact. If $X$ is an affine scheme with an action of an algebraic torus $T$ and an isolated fixed point $p \in X$, and $X$ admits a symmetric obstruction theory compatible with the $T$-action, then

$$\nu_X(p) = (-1)^{\dim T_p X},$$

where $T_p X$ is the Zariski tangent space of $X$ at $p$. This is the main technical result of [BF05], Theorem 3.4.

Finally, we will use the following result from [BF05]. If $X$ is a smooth threefold (not necessarily proper), then

$$\sum_{m=0}^{\infty} \tilde{\chi}({\text{Hilb}}^m X)q^m = M(-q)^{\chi(X)}.$$
2 The Calculation

2.1 The open subscheme $J_n(Y, C)$

**Definition 2.1** Let $C_1, \ldots, C_s$ be pairwise distinct, super-rigid rational curves on $Y$ and let $(d_1, \ldots, d_s)$ be an $s$-tuple of non-negative integers. Let $C = \sum_i d_i C_i$ be the associated 1-cycle on $Y$ and let $\beta$ be the class of $C$ in homology. Define

$$J_n(Y, C) \subset I_n(Y, \beta)$$

to be the open and closed subscheme consisting of subschemes $Z \subset Y$ whose associated 1-cycle is equal to $C$.

**Remark 2.2** To see that $J_n(Y, C)$ is, indeed, open and closed, consider the Hilbert-Chow morphism, see [Ko96], Chapter I, Theorem 6.3, which is a morphism

$$f : I_n(Y, \beta)^{sn} \longrightarrow \text{Chow}(Y, d),$$

where $\text{Chow}(Y, d)$ is the Chow scheme of 1-dimensional cycles of degree $d = \deg \beta$ on $Y$. It is a projective scheme. Moreover, $I_n(Y, \beta)^{sn}$ is the semi-normalization of $I_n(Y, \beta)$. The structure morphism $I_n(Y, \beta)^{sn} \to I_n(Y, \beta)$ is a homeomorphism of underlying Zariski topological spaces. Therefore the Hilbert-Chow morphism descends to a continuous map of Zariski topological spaces

$$|f| : |I_n(Y, \beta)| \longrightarrow |\text{Chow}(Y, d)|.$$

Because the $C_i$ are super-rigid, the cycle $C$ corresponds to an isolated point of $|\text{Chow}(Y, d)|$. So the preimage of this point under $|f|$ is open and closed in $|I_n(Y, \beta)|$. The open subscheme of $I_n(Y, \beta)$ defined by this open subset is $J_n(Y, C)$.

**Definition 2.3** As $J_n(Y, C)$ is open in $I_n(Y, \beta)$, it has an induced (symmetric) obstruction theory and hence a virtual fundamental class of degree zero. Since $J_n(Y, C)$ is closed in $I_n(Y, \beta)$ it is projective, and so we can consider the degree of the virtual fundamental class

$$N_n(Y, C) = \deg[J_n(Y, C)]^{\text{vir}},$$

and call it the *contribution of* $C$ to the Donaldson-Thomas invariant $N_n(Y, \beta)$.

2.2 The closed subset $\tilde{J}_n(Y, C)$

**Definition 2.4** Let $C = \sum_i d_i C_i$ be as above and denote by supp$C$ the reduced closed subscheme of $Y$ underlying $C$. Let

$$\tilde{J}_n(Y, C) \subset J_n(Y, C) \subset I_n(Y, \beta)$$

be the closed subset consisting of subschemes $Z \subset Y$ whose underlying closed subset $Z^{\text{red}} \subset Y$ is contained in supp$C$. 
Remark 2.5 To see that \( \tilde{J}_n(Y, C) \) is closed in \( I_n(Y, \beta) \), let \( W_m \subset Y \) be the \( m \)-th infinitesimal neighborhood of \( \text{supp} C \subset Y \). For any subscheme \( Z \subset Y \), with fixed numerical invariants \( n \) and \( \beta \), and such that \( Z^{\text{red}} \subset \text{supp} C \), there exists a sufficiently large \( m \) so that \( Z \subset W_m \). For such an \( m \), consider the Hilbert scheme \( I_n(W_m, \beta) \), which is a closed subscheme of \( I_n(Y, \beta) \), as \( W_m \) is a closed subscheme of \( Y \). The underlying closed subset of \( I_n(Y, \beta) \) is equal to \( \tilde{J}_n(Y, C) \).

Remark 2.6 Informally speaking, \( J_n(Y, C) \) parameterizes subschemes whose one dimensional components are confined to \( C \), but may have embedded points anywhere in \( Y \), whereas \( \tilde{J}_n(Y, C) \) parameterizes subschemes where both the one dimensional components and the embedded points are supported on \( C \).

2.3 The open Calabi Yau \( N = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \)

We consider the open Calabi-Yau \( N \), which is the total space of the vector bundle \( \mathcal{O}(-1) \oplus \mathcal{O}(-1) \) on \( P^1 \). We denote by \( C_0 \subset N \) the zero section.

We consider the Hilbert scheme \( I_n(N, [dC_0]) \).

Let \( \mathcal{N} \) denote \( \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1)) \) and let \( D_\infty = \mathcal{N} \setminus N \). Since \( 3D_\infty \) is an anti-canonical divisor of \( \mathcal{N} \), the corresponding section defines a trivialization of \( K_N \). \( \mathcal{N} \) is naturally a toric variety, \( D_\infty \) is an invariant divisor, and we let \( T_0 \) be the subtorus whose elements act trivially on \( K_N \). Then \( T_0 \) induces a \( T_0 \)-equivariant symmetric obstruction theory on \( I_n(N, [dC_0]) \), by Proposition 2.4 of [BF05]. Moreover, the \( T_0 \) fixed points in \( I_n(N, [dC_0]) \) are isolated points whose Zariski tangent spaces have no trivial factors as \( T_0 \) representations (the proof of Lemma 4.1, Part (a) and (b) in [BF05] is easily adapted to prove this).

As in [MOP], the \( T_0 \) fixed points in \( I_n(N, [dC_0]) \) correspond to subschemes which are given by monomial ideals on the restriction to the two affine charts of \( N \). The number of such fixed points is given by \( p(n, d) \) described below.

Let \( p(n, d) \) be the number of triples \( (\pi_0, \lambda, \pi_\infty) \), where \( \pi_0 \) and \( \pi_\infty \) are 3-dimensional partitions and \( \lambda \) a 2-dimensional partition. The 3-dimensional partitions each have one infinite leg with asymptotics \( \lambda \), and no other infinite legs. Moreover, \( d = |\lambda| \) and \( n \) is given by (MOP Lemma 5)

\[
n = |\pi_0| + |\pi_\infty| + \sum_{(i,j) \in \lambda} (i + j + 1),
\]

where the size of a three dimensional partition with an infinite leg of shape \( \lambda \) along the \( z \) axis is defined by

\[
|\pi| = \# \{(i, j, k) \in \mathbb{Z}_{\geq 0}^3 : (i, j, k) \in \pi, (i, j) \not\in \lambda \}.
\]

Proposition 2.7 We have

\[
\tilde{\chi}(I_n(N, [dC_0])) = (-1)^{n-d} p(n, d).
\]
Proof. By Corollary 3.5 of [BF05], we have
\[ \tilde{\chi}(I_n(N, [dC_0])) = \sum_p (-1)^{\dim T_p}, \]
where the sum is over all $T_0$-fixed points on $I_n(N, [dC_0])$ and $T_p$ is the Zariski tangent space of $I_n(N, [dC_0])$ at $p$. The parity of $\dim T_p$ can be easily deduced from Theorem 2 of [MNOP] (just as in the proof of Lemma 4.1 (c) in [BF05]). The result is $n - d$. So all we have to notice is that $p(n, d)$ is the number of fixed points of $T_0$ on $I_n(N, [dC_0])$. □

Corollary 2.8 We have
\[ \tilde{\chi}(\tilde{J}_n(N, dC_0), I_n(N, [dC_0])) = (-1)^{n-d} p(n, d). \]
Proof. We just have to notice that all $T_0$-fixed points are contained in $\tilde{J}_n(N, dC_0)$. □

2.4 The box counting function $p(n, d)$

Counting three dimensional partitions with given asymptotics has been shown by Okounkov, Reshetikhin, and Vafa [ORV] to be equivalent to the topological vertex formalism which occurs in Gromov-Witten theory. They give general formulas for the associated generating functions in terms of $q$ values of Schur functions which we will use to prove the following Lemma.

Lemma 2.9 The generating function for $p(n, d)$ is given by
\[ \sum_{n=0}^{\infty} p(n, d) q^n = M(q)^2 P_d(q), \]
where the power series $P_d(q)$ and $M(q)$ are defined in Equations 1 and 2.

Proof. The generating function for the number of 3-dimensional partitions with one infinite leg of shape $\lambda$ is given by equation 3.21 in [ORV]:
\[ \sum_{\text{3d partitions } \pi \text{ asymptotic to } \lambda} q^{\#} M(q)q^{-(\lambda_2 - \frac{1}{2})} s_{\lambda^t}(q^{1/2}, q^{3/2}, q^{5/2}, \ldots) \]
where $\lambda^t$ is the transpose partition, $(\lambda_2) = \sum_i \binom{\lambda_i}{2}$, $|\lambda| = \sum_i \lambda_i$, and
\[ s_{\lambda^t}(q^{1/2}, q^{3/2}, q^{5/2}, \ldots) \]
is the Schur function associated to $\lambda^t$ evaluated at $x_i = q^{(2i-1)/2}$. Using the homogeneity of Schur functions and writing
\[ s_{\lambda^t}(q) = s_{\lambda^t}(1, q, q^2, \ldots) \]
we can rewrite the right hand side of the above equation as
\[ M(q)q^{-(\lambda_2)} s_{\lambda^t}(q). \]
Observing that
\[
\sum_{(i,j) \in \lambda} (i + j + 1) = |\lambda| + \left(\frac{\lambda}{2}\right) + \left(\frac{\lambda^t}{2}\right),
\]
we get
\[
\sum_{n,d=0}^{\infty} p(n,d)q^n v^d = M(q)^2 \sum_{\lambda} s_{\lambda^t}(q)^2 q^{\lambda|+\frac{\lambda^t}{2}\mid} v^{|\lambda|}.
\]
The hook polynomial formula for \(s_{\lambda^t}(q)\) (I.3 ex pg 45, [Mac95]) is
\[
s_{\lambda^t}(q) = q^{\left(\frac{\lambda}{2}\right)-\left(\frac{\lambda^t}{2}\right)} s_{\lambda}(q).
\]
from which one easily sees that
\[
s_{\lambda^t}(q) = q^{\left(\frac{\lambda}{2}\right)-\left(\frac{\lambda^t}{2}\right)} s_{\lambda}(q).
\]
Therefore
\[
\sum_{n,d=0}^{\infty} p(n,d)q^n v^d = M(q)^2 \sum_{\lambda} s_{\lambda}(q) s_{\lambda^t}(q) q^{\lambda|+\frac{\lambda^t}{2}\mid} v^{|\lambda|}
\]
\[
= M(q)^2 \sum_{\lambda} s_{\lambda}(q, q^2, q^3, \ldots) s_{\lambda^t}(v, vq, vq^2, \ldots)
\]
\[
= M(q)^2 \prod_{i,j=1}^{\infty} (1 + q^{i+j-1} v)
\]
where the last equality comes from the orthogonality of Schur functions (I.4 equation (4.3)' of [Mac95]). By rearranging this last sum and taking the \(v^d\) term, the lemma is proved. ∎

**Remark 2.10** From the proof of the lemma we see that
\[
P_d(q) = q^d \sum_{\lambda \vdash d} s_{\lambda}(q) s_{\lambda^t}(q).
\]
From Equation (3), it is immediate that \(P_d(q)\) is a rational function in \(q\). Moreover, using the formula for total hooklength (pg 11, I.1 ex 2, [Mac95]), it is easy to check that \(P_d(q)\) is invariant under \(q \mapsto 1/q\).

### 2.5 General \(Y\)

**Lemma 2.11** Let \(C\) be a super-rigid rational curve on the Calabi-Yau threefold \(Y\). Then
\[
\bar{\chi}(J_n(Y, dC), J_n(Y, dC)) = (-1)^{n-d} p(n, d),
\]
for all \(n, d\).
Proof. First of all, by Theorem 3.2 of [La81], an analytic neighborhood of $C$ in $Y$ is isomorphic to an analytic neighborhood of $C_0$ in $N$. Therefore, by the analytic theory of Hilbert schemes (or Douady spaces), see [Do66], we obtain an analytic isomorphism of $\tilde{J}_n(Y, dC)$ with $\tilde{J}_n(N, dC_0)$ which extends to an isomorphism of a tubular neighborhood of $\tilde{J}_n(Y, dC)$ in $I_n(Y, [dC])$ with a tubular neighborhood of $\tilde{J}_n(N, dC_0)$ in $I_n(N, [dC_0])$.

The formula for $\nu_X(P)$ in terms of a linking number, Proposition 4.22 of [Be05], shows that $\nu_X(P)$ is an invariant of the underlying analytic structure of a scheme $X$. Thus, we have

$$\tilde{\chi}(\tilde{J}_n(Y, dC), I_n(Y, [dC])) = \tilde{\chi}(\tilde{J}_n(N, dC_0), I_n(N, [dC_0])).$$

Finally, apply Corollary 2.8. □

Lemma 2.12 Let $f : X \to Y$ be an étale morphism of separated schemes of finite type over $\mathbb{C}$. Let $Z \subset X$ be a constructible subset. Assume that the restriction of $f$ to the closed points of $Z$, $f : Z(\mathbb{C}) \to Y(\mathbb{C})$, is injective. Then we have

$$\tilde{\chi}(f(X), Y) = \tilde{\chi}(Z, X).$$

We remark that by Chevalley’s theorem (EGA IV, Cor. 1.8.5), $f(Z)$ is constructible, so that $\tilde{\chi}(f(X), Y)$ is defined.

Proof. Without loss of generality, $Z \subset X$ is a closed subscheme and so $Z \to Y$ is unramified.

We claim that there exists a decomposition $Y = Y_1 \sqcup \ldots \sqcup Y_n$ into locally closed subsets, such that, putting the reduced structure on $Y_i$, the induced morphism $Z_i = Z \times_Y Y_i \to Y_i$ is either an isomorphism, or $Z_i$ is empty.

In fact, by generic flatness (EGA IV, Cor. 6.9.3), we may assume without loss of generality that $Z \to Y$ is flat, hence étale. By Zariski’s Main Theorem (EGA IV, Cor. 18.12.13), we may assume that $Z \to Y$ is finite, hence finite étale. Then, by our injectivity assumption, the degree of $Z \to Y$ is 1 and so $Z \to Y$ is an isomorphism.

Once we have this decomposition of $Y$, the lemma follows from additivity of the Euler characteristic over such decompositions and the étale invariance of the canonical constructible function $\nu$. □

Now we consider the case of a curve with several components. Let $C_d = \sum_{i=1}^s d_i C_i$ be an effective cycle, where the $C_i$ are pairwise disjoint super-rigid rational curves in $Y$. We assume $d_i > 0$, for all $i = 1, \ldots, s$.

For an $(s + 1)$-tuple of non-negative integers $\vec{m} = (m_0, m_1, \ldots, m_s)$, we let $|\vec{m}| = \sum_{i=0}^s m_i$. Consider, for $|\vec{m}| = n$ the open subscheme

$$U_{\vec{m}} \subset \text{Hilb}^{m_0}(Y) \times \prod_{i=1}^s J_{m_i}(Y, d_i C_i),$$

consisting of subschemes $(Z_0, (Z_i))$ with pairwise disjoint support.
Lemma 2.13  Mapping \((Z_0, (Z_i))\) to \(Z_0 \cup \bigcup_i Z_i\) defines an étale morphism
\[ f : U_{\tilde{m}} \to J_n(Y, C_{\tilde{d}}). \]

Proof. This is straightforward. See also Lemma 4.4 in [BF97]. □

Let us write \(\tilde{Y}\) for \(Y \setminus \text{supp } C\) and remark that
\[ Z_{\tilde{m}} = \text{Hilb}_{m_0}^{\tilde{m}}(\tilde{Y}) \times \prod_{i > 0} \tilde{J}_{m_i}(Y, d_i C_i) \]
is contained in \(U_{\tilde{m}}\). Moreover, the restriction \(f : Z_{\tilde{m}} \to J_n(Y, C_{\tilde{d}})\) is injective on closed points. Finally, every closed point of \(J_n(Y, C_{\tilde{d}})\) is contained in \(f(Z_{\tilde{m}})\), for a unique \(\tilde{m}\), such that \(|\tilde{m}| = n\).

We will apply Lemma 2.12 to the diagram

\[ Z_{\tilde{m}} = \text{Hilb}_{m_0}^{\tilde{m}}(\tilde{Y}) \times \prod_{i > 0} \tilde{J}_{m_i}(Y, d_i C_i) \]

Thus, we may calculate as follows:
\[ \tilde{\chi}(J_n(Y, C_{\tilde{d}})) = \sum_{|\tilde{m}|=n} \tilde{\chi}(f(Z_{\tilde{m}}), J_n(Y, C_{\tilde{d}})) \]
\[ = \sum_{|\tilde{m}|=n} \tilde{\chi}(Z_{\tilde{m}}, U_{\tilde{m}}) \]
\[ = \sum_{|\tilde{m}|=n} \tilde{\chi} \left( \text{Hilb}_{m_0}^{\tilde{m}}(\tilde{Y}) \times \prod_{i > 0} \tilde{J}_{m_i}(Y, d_i C_i), \text{Hilb}_{m_0}^{\tilde{m}}(Y) \times \prod_{i > 0} J_{m_i}(Y, d_i C_i) \right) \]
\[ = \sum_{|\tilde{m}|=n} \tilde{\chi} \left( \text{Hilb}_{m_0}^{\tilde{m}}(\tilde{Y}), \text{Hilb}_{m_0}^{\tilde{m}}(Y) \right) \prod_{i > 0} \tilde{\chi} \left( \tilde{J}_{m_i}(Y, d_i C_i), J_{m_i}(Y, d_i C_i) \right) \]
\[ = \sum_{|\tilde{m}|=n} \tilde{\chi} \left( \text{Hilb}_{m_0}^{\tilde{m}}(\tilde{Y}) \right) \prod_{i > 0} (-1)^{m_i - d_i} p(m_i, d_i). \]

Now we perform the summation:
\[ \sum_{n=0}^{\infty} \tilde{\chi}(J_n(Y, C_{\tilde{d}})) q^n \]
\[ = \sum_{n=0}^{\infty} \left( \sum_{|\tilde{m}|=n} \tilde{\chi} \left( \text{Hilb}_{m_0}^{\tilde{m}}(\tilde{Y}) \right) \prod_{i=1}^{\tilde{s}} (-1)^{m_i - d_i} p(m_i, d_i) \right) q^n \]
\[ = \sum_{n=0}^{\infty} \sum_{|\tilde{m}|=n} \tilde{\chi} \left( \text{Hilb}_{m_0}^{\tilde{m}}(\tilde{Y}) \right) q^n \prod_{i=1}^{\tilde{s}} (-1)^{d_i} p(m_i, d_i) (-q)^{m_i} \]
\[
\begin{align*}
\sum_{m_0=0}^{\infty} \tilde{\chi}(\text{Hilb}^{m_0}(\check{Y}))q^{m_0} & = \sum_{m_0=0}^{\infty} \tilde{\chi}(\text{Hilb}^{m_0}(\check{Y}))q^{m_0} \\
& = M(-q)^{\chi(\check{Y})} \prod_{i=1}^{s} (-1)^{d_i} P_d(-q) \\
& = M(-q)^{\chi(\check{Y})} \prod_{i=1}^{s} (-1)^{d_i} P_d(-q) \\
& = M(-q)^{\chi(\check{Y})} \prod_{i=1}^{s} (-1)^{d_i} P_d(-q) \\
\end{align*}
\]

By the main result of [BF05], Theorem 4.18, we have
\[
N_\nu(Y, C_{\vec{d}}) = \tilde{\chi}(J_\nu(Y, C_{\vec{d}})).
\]
This finishes the proof of:

**Theorem 2.14** The partition function for the contribution of $C_{\vec{d}}$ to the Donaldson-Thomas invariants of $Y$ is given by
\[
Z(Y, C_{\vec{d}}) = \sum_{n=0}^{\infty} N_n(Y, C_{\vec{d}}) q^n = M(-q)^{\chi(\check{Y})} \prod_{i=1}^{s} (-1)^{d_i} P_d(-q).
\]

**Corollary 2.15** Define the reduced partition function
\[
Z'(Y, C_{\vec{d}}) = \frac{Z(Y, C_{\vec{d}})}{Z(Y, 0)}.
\]
Then we have
\[
Z'(Y, C_{\vec{d}}) = \prod_{i=1}^{s} (-1)^{d_i} P_d(-q),
\]
a rational function in $q$, invariant under $q \mapsto 1/q$.

**Proof.** Behrend and Fantechi prove [BF05] that
\[
Z(Y, 0) = \sum_{n=0}^{\infty} \tilde{\chi}(\text{Hilb}^n Y)q^n = M(-q)^{\chi(Y)};
\]
the formula for $Z'(Y, C_{\vec{d}})$ then follows immediately from Theorem 2.14.
For the proof that $Z'(Y, C_{\vec{d}})$ is a rational function invariant under $q \mapsto 1/q$, see Remark 2.10. □
3 The super-rigid GW/DT correspondence.

3.1 The usual GW/DT correspondence

The Gromov-Witten/Donaldson-Thomas correspondence of \[\text{MNOP}\] can be formulated as follows.

Let \(Y\) be a Calabi-Yau threefold and let

\[
Z_{DT}(Y, \beta) = \sum_{n \in \mathbb{Z}} N_n(Y, \beta)q^n
\]

be the partition function for the degree \(\beta\) Donaldson-Thomas invariants.

Let

\[
Z'_{DT}(Y, \beta) = \frac{Z_{DT}(Y, \beta)}{Z_{DT}(Y, 0)}
\]

be the reduced partition function.

In Gromov-Witten theory, the reduced partition function for the degree \(\beta\) Gromov-Witten invariants, \(Z'_{GW}(Y, \beta)\), is given by the coefficients of the exponential of the \(\beta \neq 0\) part of the potential function:

\[
1 + \sum_{\beta \neq 0} Z'_{GW}(Y, \beta)v^\beta = \exp \left( \sum_{\beta \neq 0} N_g^{GW}(Y, \beta)u^{2g-2}v^\beta \right).
\]

Here

\[
N_g^{GW}(Y, \beta) = \deg[M_g(Y, \beta)]^{\text{vir}}
\]

is the genus \(g\), degree \(\beta\) Gromov-Witten invariant of \(Y\).

The conjectural GW/DT correspondence states that

(i) The degree 0 partition function in Donaldson-Thomas theory is given by

\[
Z_{DT}(Y, 0) = M(-q)^{\chi(Y)},
\]

(ii) \(Z'_{DT}(Y, \beta)\) is a rational function in \(q\), invariant under \(q \mapsto 1/q\), and

(iii) the equality

\[
Z'_{GW}(Y, \beta) = Z'_{DT}(Y, \beta)
\]

holds under the change of variables \(q = -e^{iu}\).

Part (i) is proved for all \(Y\) in \[\text{BF05}\].

3.2 The super-rigid GW/DT correspondence

In an entirely parallel manner, we can formulate the GW/DT correspondence for \(N_u(Y, C_{\vec{d}})\), the contribution from a collection of super-rigid rational curves \(C_{\vec{d}} = \sum_i d_i C_i\).

Just as in Donaldson-Thomas theory, there is an open component of the moduli space of stable maps

\[
\overline{M}_g(Y, C_{\vec{d}}) \subset \overline{M}_g(Y, \beta)
\]
parameterizing maps whose image lies in the support of $C_d$. There are corresponding invariants given by the degree of the virtual class:

$$N^G_W(Y, C_d) = \deg[M_g(Y, C_d)]^{vir}$$

We define $Z'_{GW}(Y, C_d)$ by replacing $N^G_W(Y, \beta)$ on the right side of formula (4) by $N^G_W(Y, C_d)$.

Then we can formulate our results as follows.

**Theorem 3.1** The GW/DT correspondence holds for the contributions from super-rigid rational curves. Namely, let $C_d = d_1C_1 + \cdots + d_sC_s$ be a cycle supported on pairwise disjoint super-rigid rational curves $C_i$ in a Calabi-Yau threefold $Y$, and let $Z'_{DT}(Y, C_d)$ and $Z'_{GW}(Y, C_d)$ be defined as above. Then

1. $Z'_{DT}(Y, C_d)$ is a rational function of $q$, invariant under $q \mapsto 1/q$, and
2. the equality

$$Z'_{DT}(Y, C_d) = Z'_{GW}(Y, C_d)$$

holds under the change of variables $q = -e^{iu}$.

**Proof.** For (ii), see Corollary 2.15. To prove (iii), we reproduce a calculation well known to the experts (e.g. [Ka04]).

By the famous multiple cover formula of Faber-Pandharipande [FP00] (see also [Pa99]),

$$N^G_W(Y, C_d) = \sum_{i=1}^s c(g, d_i),$$

where $c(g, d)$ is given by

$$\sum_{g \geq 0} c(g, d) u^{2g-2} = \frac{1}{d} \left( \sin \left( \frac{du}{2} \right) \right)^{-2}.$$

We compute $Z'_{GW}(Y, C_d)$ and make the substitution $q = -e^{iu}$:

$$1 + \sum_{(d_1, \ldots, d_s) \neq 0} Z'_{GW}(Y, C_d) v_1^{d_1} \cdots v_s^{d_s}$$

$$= \exp \left( \sum_{j=1}^s \sum_{d_j=1}^\infty \sum_{g=0}^\infty c(g, d_j) u^{2g-2} v_j^{d_j} \right)$$

$$= \prod_{j=1}^s \exp \left( \sum_{d_j=1}^\infty \frac{v_j^{d_j}}{d_j} \left( 2 \sin \frac{d_j u}{2} \right)^{-2} \right)$$

$$= \prod_{j=1}^s \exp \left( \sum_{d_j=1}^\infty \frac{v_j^{d_j}}{d_j} \left( 1 - e^{i d_j u} \right)^{-2} \right)$$

$$= \prod_{j=1}^s \exp \left( \sum_{d_j=1}^\infty \sum_{m_j=1}^\infty \frac{-m_j}{d_j} e^{i m_j u} v_j^{d_j} \right)$$
\[
\prod_{j=1}^{s} \exp \left( \sum_{m_{j}=1}^{\infty} m_{j} \log \left( 1 - v_{j} e^{im_{j}u} \right) \right)
= \prod_{j=1}^{s} \prod_{m_{j}=1}^{\infty} (1 - (-q)^{m_{j}} v_{j})^{m_{j}}
= \prod_{j=1}^{s} \sum_{m_{j}=1}^{\infty} P_{d_{j}} (-q) (-v_{j})^{d_{j}}
= \sum_{(d_{1}, \ldots, d_{s})} \prod_{j=1}^{s} (-1)^{d_{j}} P_{d_{j}} (-q) v_{j}^{d_{j}}.
\]

Therefore,
\[
Z'_{GW}(Y, C_{d}) = \prod_{j=1}^{s} (-1)^{d_{j}} P_{d_{j}} (-q)
\]
and so by comparing with Corollary 2.15 the theorem is proved. □

The following corollary is immediate.

**Corollary 3.2** Let \( Y \) be a Calabi-Yau threefold and let \( \beta \in H_{2}(Y, \mathbb{Z}) \) be a curve class such that all cycle representatives of \( \beta \) are supported on a collection of pairwise disjoint, super-rigid rational curves. Then the degree \( \beta \) GW/DT correspondence holds:
\[
Z'_{DT}(Y, \beta) = Z'_{GW}(Y, \beta).
\]

For example, we have:

**Corollary 3.3** Let \( Y \subset \mathbb{P}^{4} \) be a quintic threefold, and let \( L \) be the class of the line. Then for \( \beta \) equal to \( L \) or \( 2L \), the GW/DT correspondence holds.

**Proof.** By deformation invariance of both Donaldson-Thomas and Gromov-Witten invariants, it suffices to let \( Y \) be a generic quintic threefold. It is well known that there are exactly 2875 pairwise disjoint lines on \( Y \) and they are all super-rigid. The conics on \( Y \) are all planar and hence rational, and it is known that there are exactly 609250 pairwise disjoint conics and they are super-rigid as well. For these facts and more, see [Ka86]. □

Note that we cannot prove the GW/DT conjecture by this method for the quintic in degree three (and higher) due to the presence of elliptic curves in degree three.

Explicit formulas for the reduced Donaldson-Thomas partition function of a generic quintic threefold \( Y \) in degrees one and two are given below:
\[
Z'_{DT}(Y, L) = 2875 \frac{q}{(1-q)^{2}}
\]
\[
Z'_{DT}(Y, 2L) = 609250 \frac{q}{(1-q)^{2}} \cdot 2875 \cdot \frac{-2q^{3}}{(1+q)^{3}(1-q)^{2}}
= -3503187500 \frac{1}{(q-q^{-1})^{4}}
\]
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