R\(^2\) gravity effects on the kinetic axion phase space

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received 18 July 2022; accepted in final form 6 September 2022
published online 22 September 2022

Abstract – In this work we consider the effect of an \(R^2\) term on the kinetic misalignment axion theory. By using the slow-roll assumptions during inflation and the field equations, we construct an autonomous dynamical system for the kinetic axion, including the effects of the \(R^2\) term and we solve numerically the dynamical system. As we demonstrate, the pure kinetic axion attractor is transposed to the right in the field phase space, and it is no longer \((\phi, \dot{\phi}) = (\langle \phi \rangle, 0)\), but it is \((\phi, \dot{\phi}) = (\langle \phi' \rangle, 0)\), with \(\langle \phi' \rangle \neq 0\) some non-zero value of the scalar field with \(\langle \phi' \rangle > \langle \phi \rangle\). This feature indicates that the kinetic axion mechanism is enhanced, and the axion oscillations are further delayed, compared with the pure kinetic axion case. The phenomenological implications on the duration of the inflationary era, on the commencing of the reheating era and the reheating temperature, are also discussed.

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Introduction. – Particle dark matter is possibly the answer to the dark matter problem, however, to date, no dark matter particle has ever been observed, see refs. [1–6] for various particle dark matter theoretical searches. This is possibly due to the fact that the dark matter particle has very small mass. One theoretically appealing small mass candidate for particle dark matter is the axion [7–66], which is elusive and it is theorized that its mass may be smaller than \(m_a \leq 10^{-12}\) eV, a fact that is impressive. Unless the LHC has large mass surprises for dark matter candidates, the axion seems to be the last resort of particle dark matter. The axion is a light scalar field, which naturally arises in string theory as the string moduli. In the recent literature, the terminology axion refers to an axion-like particle, but not to the QCD axion. The axion scalar has a primordial pre-inflationary \(U(1)\) Peccei-Quinn symmetry, which is broken during inflation in the most popular axion models, which are the canonical misalignment axion [10] and the kinetic misalignment axion [14–16]. In both the models the \(U(1)\) Peccei-Quinn symmetry is broken during the inflationary era, and the axion obtains a large vacuum expectation and rolls to its vacuum expectation value which is the minimum of the potential. The major difference between the two models is that in the case of the canonical misalignment axion model, the axion has zero kinetic energy initially, so when the axion reaches the minimum of the potential, which is its vacuum expectation value, the axion commences oscillations and thereafter redshifts as cold dark matter. On the contrary, in the kinetic misalignment axion model, the axion initially has a large kinetic energy, which actually dominates over its potential. In effect, the axion rolls down to its vacuum expectation value, but does not stop at the potential minimum, which is also its non-zero vacuum expectation value, but continues uphill deviating from its potential minimum. This feature has a dramatic effect on the reheating era, since basically the axion oscillations are significantly delayed compared to canonical misalignment axion model.

In this paper we aim to investigate the effects of modified gravity on the kinetic misalignment axion model. Modified gravity [67–71] offers an appealing theoretical framework in the context of which the inflationary and the dark energy eras can be described in an observationally viable and unified way [65,72–79], and furthermore without having the shortcomings of the general relativistic description of the dark energy era. Our aim is to investigate the effects of a popular modified gravity model, that of \(R^2\) gravity, on the kinetic misalignment model. For our study we shall adopt the dynamical systems approach, constructing an autonomous dynamical system from the field equations of the kinetic misalignment axion and we shall investigate in a quantitative way which the final attractor of the dynamical system is.

By comparing the ordinary kinetic misalignment model with the \(R^2\)-corrected kinetic misalignment axion model,
we come to the conclusion that the final attractor of the theory is different from the pure kinetic misalignment axion case. Particularly, the final attractor of the pure kinetic misalignment axion is \((\phi, \dot{\phi}) = ((\langle \phi \rangle), 0), \) with \((\langle \phi \rangle)\) being the axion’s vacuum expectation value during inflation, but further continues uphill until it reaches the value \(\langle \phi \rangle \neq 0\) some non-zero value of the scalar field with \((\phi') > (\phi)\). We show this feature numerically by studying the dynamical system, and qualitatively this means that the axion does not settle to its minimum of the potential, which is its vacuum expectation value during inflation, but further continues its trajectory uphill until it reaches the value \(\langle \phi \rangle \neq 0\).

After that it rolls down to the minimum of the potential, and the axion commences its oscillations, when \(\dot{\phi} \simeq V(\phi)\), and it starts to redshift as cold dark matter.

Thus, the \(R^2\) term further enhances the kinetic axion physics, causing a larger delay for the start of the reheating era, a feature that is phenomenologically important, since this delay is basically an enhancement of the duration of the inflationary era.

**The \(R^2\)-corrected kinetic misalignment axion model and its phase space.** In this section we shall consider in a quantitative way the effects of the \(R^2\) term on the kinetic misalignment axion, by using the phase space approach. Specifically we shall study the dynamical system of the kinetic axion and by taking into account the changes of the \(R^2\) term on the dynamical system, we shall quantitatively study the final attractor of the theory. A direct comparison of the resulting phase space with the \(R^2\)-free model shall also be taken into account. Before we proceed to our analysis, let us briefly present the theoretical framework we shall use, the field equations and let us describe the kinetic misalignment axion mechanism.

We shall consider the following gravitational action:

\[
S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} F(R) - \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi) \right],
\]

with \(\kappa^2 = \frac{8\pi G}{c^4} = \frac{1}{M_P^2}\), and \(G\) as usual denotes Newton’s gravitational constant. Also, \(M_P\) denotes the reduced Planck mass. For the purposes of this article, we shall assume that the \(F(R)\) gravity has the following form:

\[
F(R) \approx R + \frac{1}{M^2} R^2,
\]

so it is basically the \(R^2\) model. The parameter \(M\) takes the value \(M = 1.5 \times 10^{-5} \left(\frac{c}{M_P}\right)^{-1} M_P\) for inflationary phenomenological reasons [80], with \(N\) denoting the e-foldings number as usual, but for the study of the phase space of the cosmological system we shall use the Planck units physical system. Considering a flat Friedmann-Robertson-Walker (FRW) geometric background, we have

\[
ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} (dx^i)^2,
\]

the field equations corresponding to the action (2) are

\[
3H^2 F_R = \frac{RF_R - F}{2} - 3H \ddot{F}_R + \kappa^2 \left( \rho_r + \frac{1}{2} \dot{\phi}^2 + V(\phi) \right),
\]

\[
- 2H \dot{F} = \kappa^2 \dot{\phi}^2 + \ddot{F}_R - H \dot{\phi}^2 + \frac{4\kappa^2}{3} \rho_r,
\]

\[
\ddot{\phi} + 3H \dot{\phi} + V'(\phi) = 0,
\]

with \(F_R = \frac{\partial F}{\partial R}\), while the “dot” denotes as usual differentiation with respect to the cosmic time, while the “prime” differentiation with respect to the scalar field, in our case the axion scalar field.

Let us now describe in brief the kinetic axion mechanism in order to better understand the new quantitative effects that the \(R^2\) term brings along in the theory. For details on the kinetic axion mechanism see for example [14–16].

In the context of the kinetic axion mechanism, the axion primordially has an unbroken \(U(1)\) Peccei-Quinn symmetry, which is broken during the inflationary era. Due to the broken symmetry, the axion acquires a large vacuum expectation value \(\langle \phi \rangle = \theta_a f_a\), with \(\theta_a\) being the initial misalignment angle, while \(f_a\), stands for the axion decay constant. During inflation, in the context of the kinetic axion mechanism, the axion has a non-zero and large kinetic energy. The kinetic axion mechanism is pictorially described in fig. 1. Initially the axion has a small displacement from its vacuum expectation value and a large kinetic energy. Due to the large kinetic energy the axion does not stop to the minimum of the potential, which is also its vacuum expectation value, but continues uphill until it stops. Eventually the kinetic axion rolls down again and when it reaches the minimum of the potential, the axion oscillations occur. These oscillations make the axion energy density redshift as cold dark matter, and the axion oscillations start when the reheating era commences. Primordially, the axion has the following potential:

\[
V(\phi) = m_a^2 f_a^2 \left( 1 - \cos \left( \frac{\phi}{f_a} \right) \right),
\]

however, during inflation, the axion potential for small displacements around its vacuum expectation value is:

\[
V(\phi) \simeq \frac{1}{2} m_a^2 \dot{\phi}^2,
\]
an approximation which holds true for $\phi \ll f_a$ or equivalently for $\phi \ll \langle \phi \rangle$. Thus essentially, from a dynamical point of view, the final attractor of the kinetic axion is its vacuum expectation value, at which point the axion oscillations commence.

Now let us form an autonomous dynamical system for the $R^2$-corrected kinetic axion in order to see quantitatively the effects of the $R^2$ term on the phase space of the axion. For the study of the dynamical system, we shall adopt the Planck units physical system in which $\hbar = c = \kappa = 1$, recall $\kappa = 1/M_p$ so basically $M_p = 1$.

Let us consider the field equations (4) and (5), and in the slow-roll approximation for the $R^2$ model, the Friedmann equation takes the form

$$3H^2 \simeq -3H^2 \frac{\dot{H}}{M^2} + \kappa^2 V + \kappa^2 \dot{\phi}^2, \quad (8)$$

while the Raychaudhuri equation takes the following form at leading order:

$$-2\dot{H} - 2H^2 \frac{\ddot{H}}{M^2} \simeq \kappa^2 \dot{\phi}^2. \quad (9)$$

Upon solving the Raychaudhuri equation algebraically in terms of $\dot{H}$, we obtain

$$\dot{H} = \frac{1}{2} \left( M \sqrt{M^2 - 2\dot{\phi}^2\kappa^2 - M^2} \right). \quad (10)$$

hence, upon substituting $\dot{H}$ from eq. (10) in the Friedmann equation (8), the Hubble rate reads

$$H = \frac{\kappa \sqrt{\dot{\phi}^2 + V}}{\sqrt{3\sqrt{M^2 - 2\dot{\phi}^2\kappa^2} + 3/2}}. \quad (11)$$

Upon substituting the Hubble rate from eq. (11) into eq. (5), and by introducing the variable $\varphi = \phi$ and using the potential of eq. (7) which is valid during inflation, we obtain the following dynamical system:

$$\frac{d\varphi}{d\phi} = -3\kappa \varphi \sqrt{\frac{\dot{\varphi}^2 + V}{\sqrt{3\sqrt{M^2 - 2\varphi^2\kappa^2} + 3/2}}} - m^2 \varphi. \quad (12)$$

We can solve numerically the dynamical system (12) by using various initial conditions for $\dot{\phi}$ at $t = 0$, making sure though that $\dot{\phi}(t = 0) \neq 0$. The results of our numerical analysis can be found in fig. 2. In the left plot of fig. 2 we present the kinetic axion phase space attractor in the absence of the $R^2$ term, while in the right plot we present the $R^2$-corrected kinetic axion phase space attractor. In both plots, the two axes of symmetry meet at the kinetic axion phase space attractor in the absence of the $R^2$ term. As can be seen in the left plot, the new attractor of the theory is not $(\phi, \dot{\phi}) = (\langle \phi \rangle, 0)$, but it is $(\phi, \dot{\phi}) = (\langle \phi' \rangle, 0)$, with $\langle \phi' \rangle \neq 0$ some non-zero value of the scalar field with $\langle \phi' \rangle > \langle \phi \rangle$. Thus in the $R^2$-corrected kinetic axion case, the kinetic axion mechanism is enhanced, and the reheating era starts later than in the pure kinetic axion theory. This is due to the fact that the cosmological system is attracted to the attractor $(\phi, \dot{\phi}) = (\langle \phi' \rangle, 0)$, thus the kinetic axion further delays its downhill motion to the vacuum expectation value, and it basically starts the downhill motion to its vacuum expectation value much more later than the pure kinetic axion. This effect indicates that the reheating era in the $R^2$-corrected kinetic axion case starts at a much more later time compared to the pure kinetic axion case. In effect, in the combined $R^2$-corrected kinetic axion theory, the inflationary era is somewhat prolonged. This feature is also pointed out in ref. [81], and has some quantitative phenomenological implications. We need to note that the various phase space plots of the two panels in fig. 2 correspond to different initial conditions on the parameter $\varphi(t)$ at $t = 0$, and recall that $\varphi(t) = \phi$. Thus giving various initial velocities we get different curves in each panel, but for the $R^2$ kinetic axion curves, which are the blue ones, the attractor point is shifted to the right.

Since solving analytically the dynamical system (12) is a formidable task, we limit ourselves to a qualitative approach, which shows the shift caused by the $R^2$ gravity term. We do not discuss the actual numerical values of the shifted plot, since we are working in Planck units.

**Conclusions.** – In this work we investigated quantitatively by using a phase space approach the effects of an

![Fig. 2: The phase space attractors in the case of the pure kinetic axion theory (left panel) and the $R^2$-corrected kinetic axion case (right panel) for various initial conditions quantified in terms of a non-zero initial value of $\dot{\phi}$.](image-url)
$R^2$ term on the dynamical evolution of the kinetic axion. Specifically, by using solely the slow-roll assumptions, we constructed an autonomous dynamical system for the kinetic axion, including the $R^2$ effects. By using several appropriate initial conditions for the initial velocity of the scalar field $\phi$ we solved numerically the dynamical system and we studied the phase space attractors for both the $R^2$-corrected kinetic axion model and for the pure kinetic axion model. As we demonstrated, the pure kinetic axion attractor is transposed to the right in the phase space plot, and it is no longer $(\phi, \dot{\phi}) = ((\phi'), 0)$, but it is $(\phi, \dot{\phi}) = ((\phi'), 0)$, with $(\phi') \neq 0$ some non-zero value of the scalar field with $(\phi') > (\phi)$. This effect causes a further delay on the start of the axion oscillations in the case of the $R^2$-corrected axion case, thus the reheating era is somewhat postponed. The same conclusion is found in ref. [81], and in fact, as is pointed out in ref. [81], the combined effect of the kinetic axion model. As we demonstrated, the pure kinetic axion. Thus the combined effect of the inflationary era is prolonged due to the presence of the $R^2$ term on the dynamical evolution of the kinetic axion. 2

This research is funded by the Committee of Science of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP14869238).

Data availability statement: No new data were created or analysed in this study.

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