FAST TRACK COMMUNICATION

General formulation of general-relativistic higher-order gauge-invariant perturbation theory

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Abstract
A gauge-invariant treatment of general-relativistic higher-order perturbations on generic background spacetime is proposed. After reviewing a general framework of the second-order gauge-invariant perturbation theory, we show the fact that the linear-order metric perturbation is decomposed into gauge-invariant and gauge-variant parts, which was the important premise of this general framework. This means that the development of the higher-order gauge-invariant perturbation theory on generic background spacetime is possible. A remaining issue to be resolved is also discussed.

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1. Introduction

Perturbation theories are powerful techniques in many areas of physics and lead to physically fruitful results. In particular, in general relativity, the construction of exact solutions is not so easy and known exact solutions are often too idealized, though there are many known exact solutions to the Einstein equation [1]. Of course, some exact solutions to the Einstein equation well describe our universe, gravitational field of stars and black holes. However, in natural phenomena, there always exist ‘fluctuations’. To describe these fluctuations, linear perturbation theories around some background spacetime are developed [2], and are used to describe fluctuations of our universe, gravitational field of stars and gravitational waves from strongly gravitating sources.

Besides the development of the general-relativistic linear-order perturbation theory, higher-order general-relativistic perturbations also have very wide applications, for example, cosmological perturbations [3–7], black hole perturbations [8] and perturbations of a neutron star [9]. In spite of these wide applications, there is a delicate issue in the treatment of general-relativistic perturbations, which is called gauge issue. General relativity is based on general covariance. Due to this general covariance, the gauge degree of freedom, which
is unphysical degree of freedom in perturbations, arises in general-relativistic perturbations. To obtain physical results, we have to fix this gauge degree of freedom or to treat some invariant quantities in perturbations. This situation becomes more complicated in higher-order perturbations. Therefore, it is worthwhile to investigate higher-order gauge-invariant perturbation theory from a general point of view.

According to this motivation, the general framework of a higher-order general-relativistic gauge-invariant perturbation theory has been discussed \[10, 11\] and applied to cosmological perturbation theory from a general point of view. Therefore, it is worthwhile to investigate higher-order gauge-invariant quantities in perturbations. This situation becomes more complicated in higher-order general-relativistic gauge-invariant perturbation theory is almost completed on general background spacetimes. The details of the ingredients of this communication are explained in \[12\].

2. General framework of higher-order gauge-invariant perturbation theory

In this section, we review the framework of the gauge-invariant perturbation theory \[10, 11\]. In any perturbation theory, we always treat two spacetime manifolds. One is the physical spacetime \((\mathcal{M}, g_{ab})\), which is our nature itself, and we want to describe \((\mathcal{M}, g_{ab})\) by perturbations. The other is the background spacetime \((\mathcal{M}_0, g_{ab})\), which is prepared as a reference by hand. We note that these two spacetimes are distinct. Further, in any perturbation theory, we always write equations for the perturbation of the variable \(Q\) by perturbations. The other is the background spacetime \((\mathcal{M}_0, g_{ab})\) which roughly states that

Equation (1) gives a relation between variables on different manifolds. Actually, \(Q("p")\) in equation (1) is a variable on \(\mathcal{M}\), while \(Q_0(p)\) and \(\delta Q(p)\) are the variables on \(\mathcal{M}_0\). Since we regard equation (1) as a field equation, equation (1) includes an implicit assumption of the existence of a point identification map \(\mathcal{M}_0 \rightarrow \mathcal{M}: p \in \mathcal{M}_0 \mapsto "p" \in \mathcal{M}\). This identification map is a gauge choice in perturbation theories \[13\].

To develop this understanding of ‘gauge’, we introduce an infinitesimal parameter \(\lambda\) for perturbations and \((n+1)\)-dimensional manifold \(\mathcal{N} = \mathcal{M} \times \mathbb{R}\) so that \(\mathcal{M}_0 = \mathcal{N}|_{\lambda=0}\) and \(\mathcal{M} = \mathcal{M}_1 = \mathcal{N}|_{\lambda=1}\). On \(\mathcal{N}\), the gauge choice is regarded as a diffeomorphism \(\lambda_\mathcal{N} : \mathcal{N} \rightarrow \mathcal{N}\) such that \(\lambda_\mathcal{N} : \mathcal{M}_0 \rightarrow \mathcal{M}_1\).

The first- and the second-order perturbations of the variable \(Q\) on \(\mathcal{M}_1\) are defined by the pulled-back \(\lambda_\mathcal{M}^* Q\) on \(\mathcal{M}_0\) induced by \(\lambda_\mathcal{N}\). The pulled-back \(\lambda_\mathcal{M}^* Q\) is expanded as

\[
\lambda_\mathcal{M}^* Q = Q_0 + \lambda^{(1)} Q \bigg|_{\mathcal{M}_0} + \frac{1}{2} \lambda^{(2)} Q \bigg|_{\mathcal{M}_0} + O(\lambda^3),
\]

where \(Q_0\) is the background value of \(Q\), and all terms in equation (2) are evaluated on \(\mathcal{M}_0\). The perturbative expansion (2) is regarded as the definition of the first- and the second-order perturbations \((^{(1)} Q)\) and \((^{(2)} Q)\) of \(Q\).

When we have two different gauge choices \(\lambda_\mathcal{N}\) and \(\gamma_\mathcal{N}\), the gauge transformation is regarded as the change of the gauge choice \(\lambda_\mathcal{N} \rightarrow \gamma_\mathcal{N}\), which is given by the diffeomorphism \(\Phi_\lambda := (\lambda_\mathcal{N})^{-1} \circ \gamma_\mathcal{N} : \mathcal{M}_0 \rightarrow \mathcal{M}_0\). The diffeomorphism \(\Phi_\lambda\) does change the point identification. \(\Phi_\lambda\) induces a pull-back from the representation \(\lambda_\mathcal{M}^* Q\) to the representation \(\gamma_\mathcal{M}^* Q\), as \(\gamma_\mathcal{M}^* Q = \Phi_\lambda^* \lambda_\mathcal{M}^* Q\). From general arguments of the Taylor expansion \[5\], the pull-back \(\Phi_\lambda^*\) is expanded as

\[
\gamma_\mathcal{M}^* Q = \lambda_\mathcal{M}^* Q + \lambda \mathcal{X}_0 + \lambda^2 \mathcal{X}_1 + \mathcal{O}(\lambda^3),
\]
where $\xi_1^a$ and $\xi_2^a$ are the generators of $\Phi$. From equations (2) and (3), each order gauge transformation is given by
\begin{align}
(1) Q^\prime - Q &= \xi_{\gamma_1} Q, \\
(2) Q^\prime - Q &= 2\xi_{\gamma_2} (1) Q + \{\xi_{\gamma_1} + \xi_{\gamma_2}^2\} Q, \\
(3) Q^\prime - Q &= 2\xi_{\gamma_2} (1) Q + \{\xi_{\gamma_1} + \xi_{\gamma_2}^2\} Q, \\
(4) Q^\prime - Q &= 2\xi_{\gamma_2} (1) Q + \{\xi_{\gamma_1} + \xi_{\gamma_2}^2\} Q.
\end{align}

We also employ the order-by-order gauge invariance as a concept of gauge invariance [7]. We call the $k$th-order perturbation $(k) Q$ is gauge invariant iff $(k) Q = (k) Q$ for any gauge choice $\xi_1^a$ and $\xi_2^a$.

Based on the above setup, we proposed a procedure to construct gauge-invariant variables of higher-order perturbations [10]. First, we expand the metric on the physical spacetime $M_\lambda$, which is pulled back to the background spacetime $M_0$ through a gauge choice $\xi_1^a$ and $\xi_2^a$ as
\begin{align}
X_1^a g_{ab} &= g_{ab} + \lambda X_{ab} + \frac{\lambda^2}{2} X_{ab} + O(\lambda).
\end{align}

Although expression (6) depends entirely on the gauge choice $\xi_1^a$, henceforth, we do not explicitly express the index of the gauge choice $\xi_1^a$ in the expression if there is no possibility of confusion. The important premise of our proposal was the following conjecture [10]:

**Conjecture 1.** For a second-rank tensor $h_{ab}$, whose gauge transformation is given by
\begin{align}
y h_{ab} - x h_{ab} &= \xi_{\gamma_1} g_{ab},
\end{align}
there exist a tensor field $H_{ab}$ and a vector field $X^a$ such that $h_{ab}$ is decomposed as
\begin{align}
h_{ab} =: H_{ab} + \xi_{\gamma_1} g_{ab},
\end{align}

where $H_{ab}$ and $X^a$ are transformed as
\begin{align}
y H_{ab} - x H_{ab} &= 0, \\
y X^a - x X^a &= \xi_{\gamma_1}^a
\end{align}
under the gauge transformation $X \rightarrow Y$, respectively.

We call $H_{ab}$ and $X^a$ the gauge-invariant part and the gauge-variant part of $h_{ab}$, respectively.

Although conjecture 1 is nontrivial on general background spacetimes, once we accept this conjecture, we can always find gauge-invariant variables for higher-order perturbations [10]. Actually, using conjecture 1, the second-order metric perturbation $l_{ab}$ is decomposed as
\begin{align}
l_{ab} =: L_{ab} + 2\xi_{\gamma_1} h_{ab} + \frac{\lambda^2}{2} X_{ab} + O(\lambda),
\end{align}

where $y L_{ab} - x L_{ab} = 0$ and $y Y^a - x Y^a = \xi_{\gamma_1}^a + [\xi_{\gamma_1}, X]^a$. Furthermore, using the first- and second-order gauge-variant parts $(X^a$ and $Y^a$) of the metric perturbations, gauge-invariant variables for an arbitrary tensor field $Q$ other than the metric is defined by
\begin{align}
(1) Q = (1) Q + \xi_{\gamma_1} Q, \\
(2) Q = (2) Q + 2\xi_{\gamma_1} (1) Q + \{\xi_{\gamma_1} + \xi_{\gamma_2}^2\} Q, \\
(3) Q = (3) Q + 2\xi_{\gamma_1} (1) Q + \{\xi_{\gamma_1} + \xi_{\gamma_2}^2\} Q, \\
(4) Q = (4) Q + 2\xi_{\gamma_1} (1) Q + \{\xi_{\gamma_1} + \xi_{\gamma_2}^2\} Q.
\end{align}

Definitions (11) and (12) also imply that any perturbation of the first and second order is always decomposed into gauge-invariant and gauge-variant parts as
\begin{align}
(1) Q &= (1) Q + \xi_{\gamma_1} Q, \\
(2) Q &= (2) Q + 2\xi_{\gamma_1} (1) Q + \{\xi_{\gamma_1} + \xi_{\gamma_2}^2\} Q, \\
(3) Q &= (3) Q + 2\xi_{\gamma_1} (1) Q + \{\xi_{\gamma_1} + \xi_{\gamma_2}^2\} Q, \\
(4) Q &= (4) Q + 2\xi_{\gamma_1} (1) Q + \{\xi_{\gamma_1} + \xi_{\gamma_2}^2\} Q.
\end{align}
The decomposition formulae (13) and (14) are important consequences in our higher-order gauge-invariant perturbation theory. Actually, in [11], we have derived the formulae for the perturbations of the spacetime curvatures and showed that all of these are decomposed as equations (13) and (14). In addition to the spacetime curvatures, in [7], we also summarized the formulae for the perturbations of the energy–momentum tensors for matter fields, and showed that the energy–momentum tensors and the equations of motion for the matter fields are decomposed into its gauge-invariant and gauge-variant parts as equations (13) and (14). As a result, we can easily show that order-by-order perturbative equations for any equation are automatically given in gauge-invariant form [11].

Thus, based only on conjecture 1, we have developed a general framework of second-order general-relativistic perturbation theory in a gauge-invariant manner. We also note that this general framework of the second-order gauge-invariant perturbation theory is independent of the explicit form of the background metric $g_{ab}$, except for conjecture 1.

3. Decomposition of the linear-order metric perturbation

Now, we give the outline of a proof of conjecture 1. To do this, we only consider the background spacetimes which admit ADM decomposition [14]. Therefore, the background spacetime $M_0$ considered here is $(n + 1)$-dimensional spacetime which is described by the direct product $\mathbb{R} \times \Sigma$. Here, $\mathbb{R}$ is a time direction and $\Sigma$ is a spacelike hypersurface ($\dim \Sigma = n$). The background metric $g_{ab}$ is given as

$$g_{ab} = -\alpha^2 (\frac{\partial t}{\partial t})_a (\frac{\partial t}{\partial t})_b + q_{ij} (\frac{\partial x^i}{\partial x^a}) (\frac{\partial x^j}{\partial x^b}). \quad (15)$$

Since the ADM decomposition (15) is a local decomposition of the metric, we may restrict our attention to a local patch which is covered by a single coordinate system (15). In this case, we have to keep in mind the boundary of this patch. In this sense, $\Sigma$ may have its boundary $\partial \Sigma$. This situation may occur in black hole (or singularity) formations. Furthermore, in this communication, we only consider the case where $\alpha = 1$ and $\beta^i = 0$, for simplicity. This simple case includes not only many homogeneous cosmological models but also the Lemaître–Toleman–Bondi solution [15]. The most general case where $\alpha \neq 1$ and $\beta^i \neq 0$ is discussed in [12].

To consider the decomposition (8) of $h_{ab}$, first, we consider the components of the metric $h_{ab}$ as

$$h_{ab} = h_{tt} (\frac{\partial t}{\partial t})_a (\frac{\partial t}{\partial t})_b + 2h_{ti} (\frac{\partial t}{\partial x^i})_a (\frac{\partial t}{\partial t})_b + h_{ij} (\frac{\partial t}{\partial x^i})_a (\frac{\partial t}{\partial x^j})_b. \quad (16)$$

Under the gauge transformation (4), in the case where $\alpha = 1$ and $\beta^i = 0$, these components $\{h_{tt}, h_{ti}, h_{ij}\}$ are transformed as

$$Yh_{tt} - Xh_{tt} = 2\partial_t \xi_t, \quad (17)$$

$$Yh_{ti} - Xh_{ti} = \partial_i \xi_t + D_i \xi_t + 2K^j_{i,j} \xi_j, \quad (18)$$

$$Yh_{ij} - Xh_{ij} = 2D_j \xi_j + 2K_{ij} \xi_t, \quad (19)$$

where $K_{ij}$ is the extrinsic curvature of $\Sigma$ in $M_0$ and $D_i$ is the covariant derivative associated with the metric $q_{ij}$ ($D_i q_{jk} = 0$). In our case, $K_{ij} = -\frac{1}{2} \partial_i q_{ij}$. Inspecting gauge-transformation rules (17)–(19), we introduce a new symmetric tensor $\tilde{H}_{ab}$ whose components are given by

$$\tilde{H}_{tt} := h_{tt}, \quad \tilde{H}_{ti} := h_{ti}, \quad \tilde{H}_{ij} := h_{ij} - 2K_{ij} X_t. \quad (20)$$
Here, we assume the existence of the variable $X_t$, whose gauge-transformation rule is given by $\gamma X_t - \chi X_t = \xi_t$. This assumption is confirmed later soon. Since the components $\hat{H}_i$ and $\hat{H}_{ij}$ are regarded as a vector and a symmetric tensor on $\Sigma$, respectively, $\hat{H}_i$ and $\hat{H}_{ij}$ are decomposed as [16]

\begin{equation}
\hat{H}_i = D_i h_{(V_L)} + h_{(V_V)}, \quad D^i h_{(V_V)} = 0,
\end{equation}

\begin{equation}
\hat{H}_{ij} = \frac{1}{n} g_{ij} h_{(L)} + 2 \left( D_i h_{(TV V)_j} - \frac{1}{n} g_{ij} D^i h_{(TV V)_j} \right) + h_{(TT)_ij},
\end{equation}

\begin{equation}
h_{(TV V)_i} = D_i h_{(TV VL)} + h_{(TV V)_i}, \quad D^i h_{(TV V)_i} = 0, \quad D^i h_{(TT)_ij} = 0.
\end{equation}

To confirm the one-to-one correspondence between $\{\hat{H}_i, \hat{H}_{ij}\}$ and $\{h_{(V L)}, h_{(V V)}, h_{(L)}, h_{(TV V)_i}, h_{(TV V)_i}, h_{(TT)_ij}\}$, we have to discuss the boundary conditions for the variables $\{h_{(V L)}, h_{(V V)}, h_{(L)}, h_{(TV V)_i}, h_{(TV V)_i}, h_{(TT)_ij}\}$ at the boundaries $\partial \Sigma$ on the background hypersurface $(\Sigma, q_{ab})$. However, in this communication, we do not discuss these boundary conditions in detail. Instead, we assume the existence of the Green functions of two elliptic derivative operators $\Delta_1 = D_i D^i$ and $\Delta^{ij} := q^{ij} \Delta + (1 - \frac{2}{n}) D^i D^j + (n) R^{ij}$, where $(n) R^{ij}$ is the Ricci curvature on $\Sigma$. The boundary conditions for the variables are implicitly included in the Green functions of these derivative operators and the one-to-one correspondence of the sets $\{\hat{H}_i, \hat{H}_{ij}\}$ and $\{h_{(V L)}, h_{(V V)}, h_{(L)}, h_{(TV V)_i}, h_{(TV V)_i}, h_{(TT)_ij}\}$ are guaranteed by these two Green functions.

Further, we also decompose the component $\xi_i$ of a generator of gauge transformation as $\xi_i =: D_i \xi(L) + \xi(V)_i$. Gauge-transformation rules for $\{h_{(V L)}, h_{(V V)}, h_{(L)}, h_{(TV V)_i}, h_{(TV V)_i}, h_{(TT)_ij}\}$ are summarized as

\begin{equation}
\gamma h_{(V L)} - \chi h_{(V L)} = 2 \partial_t \xi(L),
\end{equation}

\begin{equation}
\gamma h_{(V V)} - \chi h_{(V V)} = 2 \partial_t \xi(V)_i + 2 D^j (K^{ij} D_j \xi(L)) + D^k K \xi(V)_k,
\end{equation}

\begin{equation}
\gamma h(L) - \chi h(L) = 2 D^i \xi(L),
\end{equation}

\begin{equation}
\gamma h_{(TV V)_i} - \chi h_{(TV V)_i} = 2 D^j (K^{ij} D_j \xi(L)) + D^k K \xi(V)_k,
\end{equation}

\begin{equation}
\gamma h_{(TT)_ij} - \chi h_{(TT)_ij} = 0.
\end{equation}

We first find the variable $X_t$ in equation (20). From the above gauge-transformation rules, we easily see that the combination

\begin{equation}
X_t := h_{(V L)} - \partial_t h_{(TV V)_i} - \Delta^{-1} [2 D_k (K^{kj} D_j h_{(TV V)_i}) + D^k K h_{(TV V)_i}],
\end{equation}

satisfy $\gamma X_t - \chi X_t = \xi_t$. Thus, we have confirmed the existence of the variable $X_t$, which is assumed in the starting point. Furthermore, we also find the variable $X_i$

\begin{equation}
X_i := h_{(TV V)_i} = D_i h_{(TV V)_i} + h_{(TV V)_i},
\end{equation}

which satisfy the gauge-transformation rule $\gamma X_i - \chi X_i = \xi_i$.

Inspecting gauge-transformation rules (24)–(30) and using the variables $X_t$ and $X_i$ defined by equations (31)–(32), we find gauge-invariant variables as follows:

\begin{equation}
-2 \Phi := h_{(V V)} - 2 \partial_t \hat{X}_t,
\end{equation}
\[ -2n\Psi := h_{(L)} - 2D^i \dot{X}_i, \] (34)

\[ \nu_i := h_{(T V V)} - \partial_i h_{(T V V)} - 2K^j_i (D_j h_{(T V L)} + h_{(T V V)}) + D_i \Delta^{-1} [2D_k (K^j_l D_j h_{(T V L)}) + D^j K h_{(T V V)}], \] (35)

\[ \chi_{ij} := h_{(T T)}_{ij}. \] (36)

These gauge-invariant variables correspond to the components of the perturbative metric in the longitudinal gauge and are conformally related to those in cosmological perturbations in [6] as \( \Phi = a^2 \Phi, \Psi = a^2 \Psi, \nu_i = a^2 \nu_i \) and \( \chi_{ij} = a^2 \chi_{ij} \), where \( a \) is the scale factor of the background spacetime in [6].

In terms of the variables \( \Phi, \Psi, \nu_i, \chi_{ij}, X_t \) and \( X_i \), the original components of \( h_{ab} \) are given by

\[ h_{tt} = -2\Phi + 2\partial_t X_t, \] (37)

\[ h_{tj} = \nu_j + D_j X_t + \partial_j X_t + 2K^j_l X_j, \] (38)

\[ h_{ij} = -2\Psi q_{ij} + \chi_{ij} + D_i X_j + D_j X_i + 2K_{ij} X_t. \] (39)

Comparing equation (8), we easily find that natural choices of \( \mathcal{H}_{ab} \) and \( X_a \) are

\[ \mathcal{H}_{ab} = -2\Phi (\text{d}t)_a (\text{d}t)_b + 2\nu_i (\text{d}t)_a (\text{d}x^i)_b + (-2\Psi q_{ij} + \chi_{ij}) (\text{d}x^i)_a (\text{d}x^j)_b, \] (40)

\[ X_a = X_a (\text{d}t)_a + X_i (\text{d}x^i), \] (41)

respectively. These choices show that the linear-order metric perturbation \( h_{ab} \) is represented in the form of equation (8).

4. Summary and discussions

In summary, we showed the outline of a proof of conjecture 1 which is the important premise of our general framework of gauge-invariant perturbation theory. Although we only consider the background spacetime with \( \alpha = 1 \) and \( \beta^i = 0 \), the above proof is extended to the general case where \( \alpha \neq 1 \) and \( \beta^i \neq 0 \) [12]. We also note that the decomposition (8) is not unique as pointed out in [7]. Therefore, we should regard that what we have shown in this communication is an explicit procedure to carry out the decomposition (8) to emphasize the existence of \( \mathcal{H}_{ab} \) and \( X_a \) in equation (8).

In our proof, we assumed the existence of the Green functions for the derivative operators \( \Delta \) and \( D^{ij} \). In this sense, we have specified the boundary conditions for the perturbative variables at the boundary \( \partial \Sigma \) of \( \Sigma \), because the explicit expression of a Green function depends on boundary conditions. This also implies that we have ignored the ‘zero-mode’ which belong to the kernel of these derivative operators. These zero-modes correspond to the degree of freedom of the boundary conditions at the boundary \( \partial \Sigma \). To include these modes into our consideration, different treatments of perturbations and careful arguments of the boundary conditions for the perturbations will be necessary. We call this problem as zero-mode problem.

Even in the cosmological perturbations, zero-mode problem exists. We leave the resolution of this zero-mode problem as a future work.

Although this zero-mode problem should be resolved, we have confirmed the important premise of our general framework of the second-order gauge-invariant perturbation theory on generic background spacetime. Due to this, we have the possibility of applications of...
our framework for the second-order gauge-invariant perturbation theory to perturbations on
generic background spacetime. Actually, in the cosmological perturbation case, we have
developed the second-order cosmological perturbations along this general framework [6, 7].
Similar developments will be also possible for any-order perturbations in a two-parameter case
[10]. Therefore, we may say that the wide applications of our gauge-invariant perturbation
theory are opened. We also leave these developments as future works.

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