Team-Optimal MMSE Combining for Cell-Free Massive MIMO Systems

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Abstract—Cell-free (CF) massive multiple-input-multiple-output (MIMO) systems are expected to implement advanced cooperative communication techniques to let geographically distributed access points jointly serve user equipments. Building on the Team Theory, we design the uplink team minimum mean-squared error (TMMSE) combining under limited data and flexible channel state information (CSI) sharing. Taking into account the effect of both channel estimation errors and pilot contamination, a minimum MSE problem is formulated to derive unidirectional TMMSE, centralized TMMSE and statistical TMMSE combining functions, where CF massive MIMO systems operate in unidirectional CSI, centralized CSI and statistical CSI sharing schemes, respectively. We then derive the uplink spectral efficiency (SE) of the considered system. The results show that, compared to centralized TMMSE, the unidirectional TMMSE only needs nearly half the cost of CSI sharing burden with neglectable SE performance loss. Moreover, the performance gap between unidirectional and centralized TMMSE combining schemes can be effectively reduced by increasing the number of APs and antennas per AP.

I. INTRODUCTION

Cell-free (CF) massive multiple-input-multiple-output (MIMO), where a large number of access points (APs) are geographically distributed over a large area and coherently serve all user equipments (UEs) on the same time-frequency resource, has gained plenty of attention over the past years [1]–[3]. Compared with traditional cellular networks, the main characteristics of CF massive MIMO systems is the operating regime with no cell boundaries and many more APs than UEs [4], [5]. Moreover, CF massive MIMO can also reap all benefits of massive MIMO, such as favorable propagation and channel hardening by using multiple antennas at APs [6]. Besides, both large macro-diversity and high coverage probability can be achieved in CF massive MIMO systems, which makes it a promising wireless access technology for beyond fifth-generation (B5G) networks [7].

The notion of cooperation has been extensively studied in the context of CF massive MIMO systems as a tool to extend coverage, improve spectral efficiency, and manage inter-user interference [8]. For instance, based on different implementations of CF massive MIMO systems, authors in [9] proposed different levels of cooperation among the APs, including centralized network and local processing network. Besides, signal processing methods, e.g., uplink minimum mean-squared error (MMSE) and zero-forcing (ZF) combining, can be used individually at each AP to suppress inter-user interference [10]. Results in [11] show that CF massive MIMO systems with centralized MMSE combining achieve nearly three times higher 90%-likely SE than local MMSE combining. In addition, authors in [12] observe both local partial MMSE (LP-MMSE) combining and centralized P-MMSE combining under the dynamic cooperation clustering, and demonstrate P-MMSE outperforms LP-MMSE. Moreover, closed-form expressions for the uplink SE of scalable CF massive MIMO systems under partial ZF combining are derived in [13]. However, the uplink combiner design of CF massive MIMO systems with practical cooperation regimes, limited data and flexible channel state information (CSI) sharing scheme is still a big challenge.

In order to solve this challenge, we resort to the well-known Team Theory, which developed out of the need for a mathematical model of cooperating teams within an organization in which all team members have the same object but with different information [14]. In principle, based on Team Theory, the cooperation among neighboring devices can improve the overall performance in automation and economics areas. Interestingly, Team Theory can also be used in wireless communications to help the devices select the most efficient parameters, such as power control, beamforming design and time-frequency resource utilization. For instance, the problem of cooperation in the multicell MIMO downlink precoding under distributed CSI was formulated as a team decision problem in [15]. In addition, utilizing Team Theory to model robust coordination, the authors of [16] designed the decentralized MIMO precoding in wireless networks. Very recently, the authors of [17] used uplink-downlink duality to derive downlink precoding of CF massive MIMO systems based on the Team Theory, which gives us great inspiration. However, more analysis and insights need to be provided for the uplink combiner design of CF massive MIMO with the Team Theory.

Motivated by the aforementioned analysis, we utilize the
Finally, \( x \) of definitions are denoted by \(|\cdot|\), absolute value, the Euclidean norm, the trace operator and the conjugate, transpose and conjugate transpose, respectively. The \( K \) with single antennas [18], [19]. As illustrated in Fig. 1, the APs and UEs on the same time-frequency resource. We assume each AP uses and the uplink data transmission phase occupies \( \tau \). In each block, we model the channel between AP \( l \) and UE \( k \) channel uses. In each block, we model the channel between AP \( l \) and UE \( k \) as the Rayleigh fading

\[
\mathbf{h}_{kl} \sim \mathcal{CN}(0, \mathbf{R}_{kl}),
\]

where \( \mathbf{R}_{kl} \in \mathbb{C}^{N \times N} \) is the spatial correlation matrix, and \( \beta_{kl} \triangleq \text{tr}(\mathbf{R}_{kl}) / N \) is the large-scale fading coefficient.

### A. Uplink Channel Estimation

We assume that \( \tau_p \) mutually orthogonal time-multiplexed pilot sequence \( \phi_1, \ldots, \phi_{\tau_p} \) with \( ||\phi_i||_2 = \tau_p \) are utilized, and a large network with \( K > \tau_p \) is considered so that the same pilot sequence is used by different UEs. Let \( t_k \in \{1, \ldots, \tau_p\} \) denote the index of the pilot used by UE \( k \), and the other UEs assigned to the same pilot as UE \( k \) is denoted by \( \mathcal{P}_k = \{ i : t_i = t_k \} \subset \{1, \ldots, K\} \). Then, we can obtain the received signal of AP \( l \) from all UEs as

\[
\mathbf{z}_l = \sum_{i=1}^{K} \sqrt{p} \mathbf{h}_{il} \phi_{t_i}^T + \mathbf{n}_l,
\]

where \( p > 0 \) is the transmit power, \( \mathbf{n}_l \in \mathbb{C}^{N \times \tau_p} \) is the receiver noise with independent \( \mathcal{CN}(0, \sigma^2) \) entries, and \( \sigma^2 \) is the noise power. For estimating \( \mathbf{h}_{kl} \), the received signal at APs is correlated with the associated normalized pilot signal \( \phi_{t_k}/\sqrt{\tau_p} \) to obtain

\[
\mathbf{z}_{kl} = \frac{1}{\sqrt{\tau_p}} \mathbf{z}_l \phi_{t_k}^T = \sum_{i=1}^{K} \frac{\sqrt{p}}{\sqrt{\tau_p}} \mathbf{h}_{il} \phi_{t_i}^T + \frac{1}{\sqrt{\tau_p}} \mathbf{n}_l \phi_{t_k}^T = \sum_{i \in \mathcal{P}_k} \sqrt{p} \mathbf{h}_{il} + \mathbf{n}_{l,k}.
\]

Then, using standard MMSE estimation, the AP \( l \) computes the MMSE estimate as

\[
\hat{\mathbf{h}}_{kl} = \sqrt{\tau_p} \mathbf{R}_{kl} \Psi_{kl} \mathbf{z}_{kl},
\]

where

\[
\Psi_{kl} = \left( \sum_{i \in \mathcal{P}_k} \tau_p \mathbf{R}_{il} + \sigma^2 \mathbf{I}_N \right)^{-1}.
\]

In addition, the estimate \( \hat{\mathbf{h}}_{kl} \) and estimation errors \( \tilde{\mathbf{h}}_{kl} \) are independent vectors distributed as \( \mathbf{h}_{kl} \sim \mathcal{CN}(0, \mathbf{Q}_{kl}) \) and \( \tilde{\mathbf{h}}_{kl} = (\mathbf{h}_{kl} - \hat{\mathbf{h}}_{kl}) \sim \mathcal{CN}(0, \mathbf{Q}_{kl} - \mathbf{Q}_{kl}) \) with

\[
\mathbf{Q}_{kl} = \tau_p \mathbf{R}_{kl} \Psi_{kl} \mathbf{R}_{kl}.
\]

### B. Uplink Data Transmission and Combining

Under the uplink data transmission phase, the received complex baseband signal at AP \( l \) is

\[
\mathbf{y}_l = \sum_{i=1}^{K} \mathbf{h}_{il} \sqrt{p} s_i + \mathbf{n}_l,
\]

where \( s_i \sim \mathcal{CN}(0,1) \) is the transmit signal with the data transmit power \( p \), and \( \mathbf{n}_l \sim \mathcal{CN}(0, \sigma^2 \mathbf{I}_N) \) is the receiver noise. To detect the symbol transmitted from the \( k \)th UE, the received signal \( \mathbf{y}_l \) is multiplied with the conjugate of combining vector. Sending the obtained quantity \( \hat{s}_k = \mathbf{v}_{kl}^H \mathbf{y}_l \) to the CPU via the fronthaul [21], then \( \hat{s}_k \) is obtained as

\[
\hat{s}_k = \frac{1}{\sqrt{\tau_p}} \sum_{i=1}^{L} \mathbf{v}_{kl}^H \mathbf{y}_l = \frac{1}{\sqrt{\tau_p}} \sum_{i=1}^{L} \sqrt{p} \mathbf{v}_{kl}^H \mathbf{h}_{il} s_i + \frac{1}{\sqrt{\tau_p}} \sum_{i=1}^{L} \mathbf{v}_{kl}^H \mathbf{n}_l.
\]

Based on the obtained CSI with estimation errors and pilot contamination at the \( l \)th AP, we utilize the Team Theory to design the TMMSE combining in the following.

![Fig. 1: Cooperation among APs in CF massive MIMO systems.](image-url)
III. PERFORMANCE ANALYSIS

We use the standard information inequalities to obtain [22]:

\[ I(s_k; \hat{s}_k) \geq - \log \left( E \left\{ |s_k - \alpha \hat{s}_k|^2 \right\} \right). \]  

(9)

Choosing \( \alpha = \alpha^* \) with \( \alpha^* = \frac{E \{ s_k \hat{s}_k \}}{E \{ |\hat{s}_k|^2 \}} \) being the solution of \( \min_{\alpha} E \left\{ |s_k - \alpha \hat{s}_k|^2 \right\} \), leads to the well-known UaF bound as

\[ SE_k^{\text{UaF}} = \frac{-\tau_e - \tau_p}{\tau_e} \log \left( E \left\{ |s_k - \alpha^* \hat{s}_k|^2 \right\} \right). \]  

(10)

**Remark 1.** Because \( \alpha^* \) is a scalar that depends only on channel statistics, we have that if \( v_{kl} \) is the optimal solution of the following problem:

\[ \text{minimize MSE}_k(v_{kl}) = E \left\{ |s_k - \hat{s}_k(v_{kl})|^2 \right\}, \]  

(11)

it is also an optimal solution to the problem:

\[ \text{maximize } SE_k^{\text{UaF}}(v_{kl}). \]  

(12)

**Theorem 1.** Based on Remark 1, we study the following novel MMSE combining design criterion [23]:

\[ \text{minimize } MSE_k(v_{kl}) = E \left\{ \left| \sum_{i=1}^{L} v_{kl}^H h_{il} - 1 \right|^2 \right\} \]

\[ + \sum_{i \neq k} E \left\{ \left| \sum_{i=1}^{L} v_{kl}^H h_{il} \right|^2 \right\} + \frac{\sigma^2}{\beta} \sum_{i=1}^{L} E \left\{ \| v_{kl}^H \|^2 \right\}, \]  

(13)

where \( v_{kl} = f_{kl} \left( \hat{H}_l \right), \) with

\[ \hat{H}_l = [\hat{H}_{l,1}, \ldots, \hat{H}_{l,L}], l = 1, \ldots, L. \]  

(14)

Besides, \( \hat{H}_{l,j} = [\hat{h}_{lj}, \ldots, \hat{h}_{Kj}]^T \in \mathbb{C}^{K \times N} \) denotes the channel estimate of \( j \)-th AP available at AP \( l \). The corresponding channel gain is represented as \( H_l = [h_{l1}, \ldots, h_{KJ}]^T \in \mathbb{C}^{K \times N}. \)

**Proof:** Please refer to Appendix A.

Problem (13) is the known family of team decision problems, which are generally difficult to solve for general information constraints. However, we recognize that Problem (13) belongs to the class of quadratic teams. This class exhibits strong structural properties, in particular related to the following solution concept:

The combining function \( f_{kl}^* \) is a stationary solution for Problem (13) if \( MSE_k(f_{kl}^*) < \infty \) and if the following set of equalities hold

\[ \nabla f_{kl}^*(\hat{H}_l) E \left\{ \text{MSE}_k \left| \hat{H}_l \right. \right\} = 0, l = 1, \ldots, L. \]  

(15)

Submitting (13) into (15), the conditions in (15) can be evaluated as (16) at the top of next page. Making use of the law of total expectation and the available CSI, we then compute the third term of (16) as

\[ \text{E} \left\{ h_{il}^H f_{kj}^* \left( \hat{H}_j \right) \left| \hat{H}_l \right. \right\} = \hat{h}_{il} \text{E} \left\{ h_{il}^H f_{kj}^* \left( \hat{H}_j \right) \left| \hat{H}_l \right. \right\}. \]  

(17)

Due to \( \hat{h}_{il} \) and \( \hat{h}_{il} \) are independent in MMSE estimation, we compute the second term of (16) as

\[ \text{E} \left\{ h_{il}^H \hat{h}_{il}^H \left| \hat{H}_l \right. \right\} = \text{E} \left\{ h_{il}^H \hat{h}_{il}^H \left| \hat{H}_l \right. \right\} = \text{E} \left\{ \hat{h}_{il}^H + \hat{h}_{il}^H \right\} \hat{H}_l^H = \hat{h}_{il}^H + \hat{C}_{il}. \]  

(18)

We also have \( \text{E} \left\{ h_{il}^H \hat{H}_l \right\} = \hat{h}_{il}. \) Furthermore, we can derive the combining functions in the form of the set of equalities as

\[ f_{kl}^* \left( \hat{H}_l \right) = A_l \left( e_k - \sum_{j \neq l} E \left\{ \hat{H}_{j,l} f_{kj}^* \left( \hat{H}_j \right) \left| \hat{H}_l \right. \right\}, \forall l, \]  

(19)

where

\[ A_l = \left( \sum_{i=1}^{K} \left( \hat{h}_{il}^H \hat{C}_{il} + \frac{\sigma^2}{\beta} I_N \right) \right)^{-1}. \]  

(20)

**Remark 2.** Equation (19) can be divided into two parts. The matrix \( A_l \) can be recognized as a local MMSE combining stage, and the rest can be then interpreted as a corrective stage which takes into account the effect of the other APs based on the available CSI and long-term statistical information [17].

### A. Unidirectional CSI Sharing

We first consider the local channel measurements to be shared unidirectionally along a serial fronthaul\(^2\). Specifically, this particular information structure can be expressed as

\[ \hat{H}_{l,j} = \begin{cases} \hat{H}_{l,j}, & j \leq l, \\ \text{E} \left\{ \hat{H}_j \right\}, & j > l. \end{cases} \]  

(21)

**Theorem 2.** The unidirectional TMMSE combining solving (19) under the unidirectional CSI sharing (21) is given by

\[ f_{kl}^* \left( \hat{H}_l \right) = A_l S_l \prod_{s=1}^{l-1} \hat{S}_s e_k, \]  

(22)

where

\[ \hat{S}_s = I_K - A_s S_l, \]

(23)

\[ S_l = (I_K - \Pi_l A_l)^{-1} (I_K - \Pi_l), \]

(24)

\[ \Pi_l = \text{E} \left\{ A_{l+1} S_{l+1} \right\} + \Pi_{l+1} \text{E} \left\{ S_{l+1} \right\}, \Pi_L = 0, \]  

(25)

\[ A_l = \hat{H}_{l,l} A_l. \]  

(26)

**Proof:** Please refer to Appendix B.

**Corollary 1.** Under the unidirectional CSI sharing scheme in Theorem 2, we rearrange the order of APs according to the sorting rule: If \( \sum_{i=1}^{K} \beta_{ik} > \sum_{k=1}^{K} \beta_{jk} \), we set the CSI sharing order of AP \( l \) and AP \( j \) as \( l < j \). That is, the AP with poor CSI needs to know more CSI of other APs.

\(^2\)It is different from the sequential linear processing algorithm of radio stripe in [24], which can achieve the same performance as centralized MMSE.
Finally, we can derive $a_{kl}$ as

$$a_{kl} + \sum_{j \neq l}^{L} \mathbb{E} \left\{ A_j a_{kj} \mid \hat{H}_l \right\} = e_k, l = 1, \ldots, L.$$  

(30)

Finally, we can derive $a_{kl}$ as

$$a_{kl} + \sum_{j \neq l}^{L} \mathbb{E} \left\{ A_j a_{kj} \mid \hat{H}_l \right\} = e_k, l = 1, \ldots, L.$$

(31)

C. Statistical CSI Sharing

For the statistical CSI sharing, each AP knows its local instantaneous CSI and statistical CSI of other APs. This is expressed by formula as

$$\hat{H}_{i,j} = \mathbb{E} \{ H_j \}, j \neq l.$$  

(32)

Therefore, the statistical TMMSE combining solving (19) under the statistical CSI sharing (32) also can be is expressed as (28). It is similar to the steps of the centralized CSI sharing scheme, we obtain the following equations:

$$a_{kl} + \sum_{j \neq l}^{L} \mathbb{E} \left\{ A_j a_{kj} \mid \hat{H}_l \right\} = e_k, l = 1, \ldots, L,$$

(33)

which can be used to derive $a_{kl}$.

IV. NUMERICAL RESULTS AND DISCUSSION

Let $L$ APs and $K$ UEs uniformly and independently distribute within a square of size 500 m x 500 m in our simulation setup. Moreover, we make use of the three-slope propagation model from [1] as

$$\beta_{kl} \text{[dB]} = \begin{cases}  
-81.2, & d_{kl} < 10m \\
-61.2 - 20 \log_{10} \left( \frac{d_{kl}}{10m} \right), & 10m \leq d_{kl} < 50m \\
-35.7 - 35 \log_{10} \left( \frac{d_{kl}}{10m} \right) + F_{kl}, & d_{kl} \geq 50m, 
\end{cases}$$

(34)

where $d_{kl}$ denotes the horizontal distance between AP $l$ and UE $k$. When the distance is larger than 50m and the shadowing terms $F_{kl} \sim N(0, \sigma^2)$ are correlated as

$$\mathbb{E} \{ F_{kl} F_{ij} \} = \frac{\sigma^2}{2} \left( 2^{-\delta_{ki}/100m} + 2^{-\upsilon_{ij}/100m} \right),$$

(35)

where $\delta_{ki}$ represents the distance between the $i$th UE and $i$th UE, $\upsilon_{ij}$ represents the distance between the $i$th AP and $j$th AP. We assume the carrier frequency is $f_c = 2$ GHz, and the bandwidth is $B = 20$ MHz. Moreover, both the pilot transmit power and the data transmission power are $p = 23$ dBm. Besides, the noise power is $\sigma^2 = -96$ dBm, the coherence block contains $\tau_c = 200$ channel uses.

Fig. 2 compares the CDF of SE for unidirectional, centralized and statistical TMMSE combining schemes, where CF massive MIMO systems operate in unidirectional, centralized and statistical CSI sharing schemes, respectively. In addition, the traditional centralized MMSE and local MMSE combining in [9] are considered for comparison. It is found that, compared
The reason is that, while increasing the length of the pilot sequence decreases with the increase of the length of the pilot sequence.

It is clear that the average SE performance of both centralized MIMO systems against the length of the pilot sequence. It can save nearly half the cost of CSI sharing. Moreover, we find more UEs under the same parameters, the SE gaps between unidirectional TMMSE and centralized TMMSE are 1.4 bit/s/Hz and 0.6 bit/s/Hz, respectively.

$L = 100$ different numbers of antennas per AP ($\tau_p = 10$).

Fig. 4: Average uplink SE for CF massive MIMO systems against the length of pilot sequence ($L = 50$, $N = 2$).

with centralized TMMSE combining, unidirectional TMMSE combining only have 5.6% median SE performance loss, but it can save nearly half the cost of CSI sharing. Moreover, we find that statistical TMMSE combining achieves larger uplink SE than local MMSE combining. The reason is that both local estimates and global statistical CSI are used to design statistical TMMSE combiner. Besides, the centralized MMSE combining in [9] has the same SE performance as our derived centralized TMMSE combining.

The average uplink SE for CF massive MIMO systems under three different CSI sharing schemes is shown in Fig. 3, as an increasing function of the number of APs. We notice that the performance gap between unidirectional TMMSE and centralized TMMSE gradually decreases with the increase of the number of APs. For instance, at $L = 20$ and $L = 100$, the SE gaps are 1.4 bit/s/Hz and 0.6 bit/s/Hz, respectively. However, under the same parameters, the SE gaps between statistical and centralized TMMSE are 3.3 bit/s/Hz and 4.4 bit/s/Hz, respectively.

Fig. 4 illustrates the average uplink SE for CF massive MIMO systems against the length of the pilot sequence. It is clear that the average SE performance of both centralized and unidirectional TMMSE combining increases first and then decreases with the increase of the length of the pilot sequence. The reason is that, while increasing the length of the pilot sequence to reduce pilot contamination, the channel uses for transmitting data are also decreasing. Moreover, the optimal length of the pilot sequence at the point where we achieve maximum SE increases with the increase of the number of UEs. Furthermore, we also find more UEs lead to a larger performance gap between unidirectional TMMSE and centralized TMMSE combining.

Fig. 5 shows the average uplink SE for CF massive MIMO systems with different numbers of antennas per AP. It is clear that the unidirectional TMMSE based on large-scale fading sorting outperforms the original unidirectional TMMSE. The reason is that the weak coverage APs can use more information to find that, compared to centralized TMMSE combining, the unidirectional TMMSE combining saves nearly half the cost of CSI sharing while with only limited performance loss. Moreover, the performance gap between unidirectional and centralized TMMSE combiners can be further reduced by increasing the number of APs and antennas per AP.

V. CONCLUSION

In this paper, we design the uplink TMMSE combining of CF massive MIMO systems based on the so-called theory of teams. Taking into account both channel estimation errors and pilot contamination, we derive unidirectional, centralized and statistical TMMSE combining schemes, where CF massive MIMO systems operate in unidirectional, centralized and statistical CSI sharing schemes, respectively. It is interesting to find that, compared to centralized TMMSE combining, the unidirectional TMMSE combining saves nearly half the cost of CSI sharing while with only limited performance loss. Moreover, the performance gap between unidirectional and centralized TMMSE combiners can be further reduced by increasing the number of APs and antennas per AP.

APPENDIX A

PROOF OF THEOREM 1

Submitting (8) into (11), we derive
\[
\mathbb{E} \left\{ |s_k - \hat{s}_k|^2 \right\} = 1 - 2\mathbb{E} \left\{ \sum_{l=1}^{L} v_{kl}^H h_l s_i - \frac{1}{\sqrt{p}} \sum_{l=1}^{L} v_{kl}^H n_l \right\}^2.
\]

Due to $s_i \sim \mathcal{CN}(0, 1), i = 1, \ldots, K$ are independent and identically distributed. We further obtain
\[
\mathbb{E} \left\{ |s_k - \hat{s}_k|^2 \right\} = 1 - 2\mathbb{E} \left\{ \sum_{l=1}^{L} v_{kl}^H h_l \right\}^2 + \frac{\sigma^2}{p} \sum_{l=1}^{L} \mathbb{E} \left\{ \|v_{kl}\|^2 \right\}. \quad (36)
\]

With the help of the perfect square trinomial, (36) becomes (13) to finish the proof. In addition, it is worth noting that we also derive
\[
\mathbb{E} \left\{ s_k \hat{s}_k^* \right\} = \mathbb{E} \left\{ \sum_{l=1}^{L} v_{kl}^H h_l \right\}, \quad (37)
\]
\[
\mathbb{E} \left\{ \hat{s}_k^* \right\} = \mathbb{E} \left\{ \sum_{l=1}^{K} \left( \sum_{i=1}^{L} v_{kl}^H h_{il} \right)^2 \right\} + \frac{\sigma^2}{p} \sum_{l=1}^{L} \mathbb{E} \left\{ \|v_{kl}\|^2 \right\}, \quad (38)
\]
to calculate the value of $\alpha^*$.
Based on the Remark 2 and the unidirectional CSI sharing scheme (21), where the AP $l$ only knows the CSI of AP $j$ ($j \leq l$). Therefore, we assume the unidirectional TMMSE combining at AP $l$ is in the form of (22). Then, by submitting (22) into (19), we can derive

$$
\left(S_l + \sum_{j=l}^{L} E \left\{ A_j S_j \prod_{s=\ell+1}^{l-1} \bar{S}_s \right\} \prod_{s=1}^{l-1} \bar{S}_s \right) \prod_{s=1}^{l-1} \bar{S}_s = I_K.
$$

(39)

Since $S_l$ and $\bar{S}_l$ are independent from $S_j$ and $\bar{S}_j$ for $i \neq j$, we have

$$
\sum_{j=l}^{L} E \left\{ A_j S_j \prod_{s=\ell+1}^{l-1} \bar{S}_s \right\} = \sum_{j=l}^{L} E \left\{ A_j S_j \right\} \prod_{s=\ell+1}^{l-1} E \left\{ \bar{S}_s \right\} = E \left\{ \Lambda_{L+1} S_{L+1} \right\} + \sum_{j=l+1}^{L} E \left\{ A_j S_j \right\} \prod_{s=\ell+1}^{l-1} E \left\{ \bar{S}_s \right\} = E \left\{ \Lambda_{L+1} S_{L+1} \right\} + \sum_{j=l+1}^{L} E \left\{ A_j S_j \right\} \prod_{s=\ell+1}^{l-1} \bar{S}_s \right\} \prod_{s=1}^{l-2} \bar{S}_s = \Pi L - 1.
$$

(40)

The second and last terms of the above chain of equalities define a recursion terminating with $E \left\{ \Lambda_{L} S_{L} \right\} + 0E \left\{ \bar{S}_{L} \right\} = \Pi L - 1$. This recursion gives precisely

$$
\sum_{j=l}^{L} E \left\{ A_j S_j \right\} \prod_{s=\ell+1}^{l-1} \bar{S}_s \right\} \prod_{s=1}^{l-2} \bar{S}_s = \Pi L.
$$

(41)

With the property $S_l + \Pi_l \bar{S}_l = I_K$, (39) can be simplified to

$$
\prod_{s=1}^{l-2} \bar{S}_s + \sum_{j<l}^{L} A_j S_j \prod_{s=\ell+1}^{l-1} \bar{S}_s = \bar{S}_{l-1} \prod_{s=1}^{l-2} \bar{S}_s + \prod_{s=1}^{l-2} \bar{S}_s = \bar{S}_{l-1} \prod_{s=1}^{l-2} \bar{S}_s + \prod_{s=1}^{l-2} \bar{S}_s + \sum_{j<l}^{L} A_j S_j \prod_{s=\ell+1}^{l-1} \bar{S}_s = \bar{S}_{l-1} + A_{l-1} S_{l-1} \prod_{s=\ell+1}^{l-1} \bar{S}_s + \sum_{s=1}^{L} A_j S_j \prod_{s=\ell+1}^{l-1} \bar{S}_s = \bar{S}_{l-1} + \prod_{s=\ell+1}^{l-1} \bar{S}_s + \sum_{j<l}^{L} A_j S_j \prod_{s=\ell+1}^{l-1} \bar{S}_s = \prod_{s=1}^{l-2} \bar{S}_s + \sum_{j<l}^{L} A_j S_j \prod_{s=\ell+1}^{l-1} \bar{S}_s.
$$

(42)

where the last equation follows the definition of $\bar{S}_l$, and we identify another recursive structure among the remaining terms. By continuing until termination, we finally obtain

$$
\prod_{s=1}^{l-1} \bar{S}_s + \sum_{j<l}^{L} A_j S_j \prod_{s=\ell+1}^{l-1} \bar{S}_s = I_K,
$$

which proves the key statement that all the matrix inverses involved exist.