A CHARACTERISTIC MAP FOR THE SYMMETRIC SPACE OF SYMPLECTIC FORMS OVER A FINITE FIELD

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Abstract. The characteristic map for the symmetric group is an iso-
morphism relating the representation theory of the symmetric group to
symmetric functions. An analogous isomorphism is constructed for the
symmetric space of symplectic forms over a finite field, with the spherical
functions being sent to Macdonald polynomials with parameters \((q, q^2)\).
The multiplicative structure on the bi-invariant functions is given by
an analogue of parabolic induction. As an application, the positivity
and vanishing of certain Macdonald Littlewood-Richardson coefficients
is proven.

1. Introduction

1.1. Motivation. There is a close connection between the representation
theory of certain groups and symmetric functions, of which the most well-
known and classical is for the symmetric group. In particular, an isomor-
phism between the class functions on the symmetric groups and the ring
of symmetric functions can be constructed. Here the graded multiplication
is given by Young induction, and the irreducible characters are sent to the
Schur functions.

In Macdonald’s classic book on symmetric functions \([14]\), two extensions of
this are given, one for \(GL_n(F_q)\) (originally due to Green \([8]\)), and one for the
Gelfand pair \(S_{2n}/B_n\), where \(B_n\) denotes the hyperoctahedral group (origi-
nally due to Stembridge \([24]\), although a similar connection to the symmetric
space \(GL(n, \mathbb{R})/O(n, \mathbb{R})\) was noticed by James \([13]\)). A characteristic map
can be constructed in both cases and they have applications to computing
character and spherical function values.

This paper develops an analogous theory for \(GL_{2n}(F_q)/Sp_{2n}(F_q)\) and
some applications to Macdonald polynomials are given. The symmetric space
\(GL_{2n}(F_q)/Sp_{2n}(F_q)\) is a natural \(q\)-analogue of the Gelfand pair \(S_{2n}/B_n\); the
former can be seen as the Weyl group version of the latter. Work of Bannai,
Kawanaka and Song \([1]\) gave a formula for the spherical functions in terms of
so-called basic functions, which are the analogues of the Deligne-Lusztig
characters in this setting. Already, coefficients related to the Macdonald
polynomials with parameters \((q, q^2)\) appeared, and so it is natural to seek
an analogue of the characteristic map for \(GL_{2n}(F_q)/Sp_{2n}(F_q)\).

The original motivation to seek this construction was to analyze a random
walk on the symmetric space by seeking a combinatorial formula for the
spherical functions. A more probabilistic proof was subsequently found and the analysis of the Markov chain can be found in [10].

1.2. Main results. The main contribution is to construct a characteristic map
\[ \text{ch} : \bigoplus_n \mathbb{C}[\text{Sp}_{2n}(F_q)\setminus \text{GL}_{2n}(F_q)/\text{Sp}_{2n}(F_q)] \rightarrow \bigotimes \Lambda \]
and establish some basic properties (it is assumed that \( q \) is odd). In particular, the spherical functions are shown to map to Macdonald polynomials with parameter \((q,q^2)\) and the multiplication given by a bi-invariant parabolic induction originally due to Grojnowski [9] is shown to be preserved by \( \text{ch} \).

This result should be seen as a reformulation of the work done in [1], [9] and [11] into a framework which makes it easy to transfer results between the representation theory of \( \text{GL}_{2n}(F_q)/\text{Sp}_{2n}(F_q) \) and the theory of symmetric functions. As examples, two applications are given showing both directions of this transfer.

The first is an application to the Littlewood-Richardson coefficients for Macdonald polynomials. If \( J_\lambda(x;q,t) \) denotes the integral form of the Macdonald polynomials, let \( f_{\mu\nu}(q,t) \) be the \((q,t)\) Littlewood-Richardson coefficients defined by
\[ J_\mu(x;q,t)J_\nu(x;q,t) = \sum_{\lambda} f_{\mu\nu}(q,t)J_\lambda(x;q,t). \]
These coefficients are the structure constants of \( \Lambda \) in terms of the basis given by the Macdonald polynomials, and for the Schur functions they have many interesting combinatorial, representation-theoretic and geometric interpretations.

The following two theorems are proven using the characteristic map.

**Theorem 1.1.** Let \( q = p^r \) denote an odd prime power. Then for any partitions \( \mu, \nu, \lambda \)
\[ f_{\mu\nu}(q^2) \geq 0. \]

**Theorem 1.2.** For any \( q \), if \( c_{\mu\nu}^\lambda \) (the classical Littlewood-Richardson coefficients) vanishes, then \( f_{\mu\nu}(q,q^2) \) vanishes as well.

Theorem 1.2 answers a special case of a question raised by Macdonald on whether \( c_{\mu\nu}^\lambda = 0 \) implies \( f_{\mu\nu}^\lambda(q,t) = 0 \) [14, VI, §7]. A conjecture is also made that these coefficients are ratios of polynomials in \( q \) with positive coefficients.

The second application is to combinatorial formulas for the values of spherical functions on certain double cosets. Although there are formulas in [1] already for all values of the spherical functions, they are alternating and so are unsuitable for asymptotic analysis. Asymptotic analysis of spherical functions appears in the study of Markov chains on Gelfand pairs, see [15] for examples.
1.3. Previous work. In addition to the examples already mentioned for \(S_n\), \(\text{GL}_n(\mathbb{F}_q)\) and \(S_{2n}/B_n\), there are further examples of characteristic maps appearing in the literature. In particular, Thiem and Vinroot constructed such a map for \(U_n(\mathbb{F}_{q^2})\) [25].

The connections between the representation theory of \(\text{GL}_{2n}(\mathbb{F}_q)/\text{Sp}_{2n}(\mathbb{F}_q)\) and Macdonald polynomials are not new and were already noticed in [1]. A further connection was made by Shoji and Sorlin between a related space and the modified Kostka polynomials which are the change of basis from the Hall-Littlewood polynomials to the Schur functions. This work also owes much to the reformulation of the results in [1] in terms of character sheaves, due to Grojnowski [9] and Henderson [11], some of which was later extended in the work of Shoji and Sorlin [19–21] which can also be found in the survey [22].

The computation of the image of the spherical functions under the characteristic map relies on the formulas of [1] and is in a similar spirit to the work of [25]. The multiplication comes from a bi-invariant parabolic induction originally due to Grojnowski [9].

As there is a rational parabolic for the Levi subgroup defining the multiplication operation, a simple formula can be written down and so it should be possible to give a construction closer to that of Green’s original one for \(\text{GL}_n(\mathbb{F}_q)\) (i.e. one that doesn’t use character sheaves). However, the construction given has the benefit of being more transparent, albeit at the cost of some more machinery. The construction also has the benefit of being more easily generalized, and in particular the extension to \(U_{2n}(\mathbb{F}_{q^2})/\text{Sp}_{2n}(\mathbb{F}_q)\) is straightforward.

The Pieri rule for Macdonald polynomials gives a complete understanding of the \((q,t)\) Littlewood-Richardson coefficients when \(\mu\) has one part. Various formulas have been found for the \((q,t)\) Littlewood-Richardson coefficients, including in terms of alcove walks [26] and (generalized) binomial coefficients [18].

The numerical positivity of the \((q,t)\) Littlewood-Richardson coefficients has certainly been noted before, see for example [15]. The numerical positivity of the \((q,t)\) Littlewood-Richardson coefficients for all parameters would also imply the positivity of the Jack Littlewood-Richardson coefficients, which is a consequence of a conjecture of Stanley [23].

The \(q \rightarrow 1\) case of the positivity and vanishing results was previously studied by Bergeron and Garsia [2] (see also [14]), where the Gelfand pair \(S_{2n}/B_n\) played a similar role.

Macdonald polynomials with parameters \((q,q^2)\) also appeared previously in the study of quantum symmetric spaces [17], and it would be interesting to see if there is a connection.

1.4. Outline. The paper is organized as follows. In Section 2, notation and preliminary background is reviewed. Section 3 reviews the necessary results from [9,11] on parabolic induction of character sheaves on symmetric spaces. In Section 4 the characteristic map for \(\text{GL}_{2n}(\mathbb{F}_q)/\text{Sp}_{2n}(\mathbb{F}_q)\) is constructed.
Section 5 gives an application to positivity and vanishing of the Littlewood-Richardson coefficients for Macdonald polynomials with parameters $(q, q^2)$ and Section 6 gives an application to computing spherical function values.

Section 3 is the only section which requires any familiarity with character sheaves and Proposition 3.1 gives a summary of the necessary consequences that can be understood with no knowledge of character sheaves.

2. Preliminaries

In this section, some needed background is reviewed and the notation and conventions that are used are explained. The notation and background on character sheaves needed is left for Section 3.

2.1. Notation. If $f(q)$ is a rational function in $q$, define $f(q)_{q ightarrow q^2} := f(q^2)$. For example,

$$|GL_n(F_q)|_{q ightarrow q^2} = \prod_{i=0}^{n-1} (q^{2n} - q^{2i}).$$

For a symmetric function $f$ with rational coefficients in $q$ when written in terms of $p_{\mu}$, write $f_{q ightarrow q^2}$ to denote the symmetric function obtained by replacing $q$ with $q^2$ in each coefficient.

Let $G$ be a finite group. If $S \subseteq G$ is some subset, $I_S$ will denote the indicator function for that set. If $H \subseteq G$ is a subgroup, let $C[G]^{H}$ denote the set of class functions on $G$ and let $C[H \backslash G / H]$ denote the set of $H$ bi-invariant functions on $G$.

If $S, H$ are subgroups of any group $G$, let $H_S = H \cap S$. Given an element $x$ of some finite field extension of $F_q$, let $f_x$ denote its minimal polynomial.

2.2. Macdonald polynomials. For details on Macdonald polynomials including a construction and proofs, see [14]. Consider the inner product on the ring of symmetric functions defined by

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda \prod \frac{q^{\lambda_i} - 1}{t^{\lambda_i} - 1},$$

where $z_\lambda = \prod m_i(\lambda)! t^{m_i(\lambda)}$ and $m_i(\lambda)$ denotes the number of parts of size $i$ in $\lambda$. This specializes to the Hall inner product when $q = t$ and setting $q = t^{\alpha}$ and taking a limit gives the Jack polynomial inner product. The Macdonald polynomials $P_\lambda(x; q, t)$ (indexed by partitions $\lambda$) are defined by the fact that they are orthogonal with respect to this inner product, and the change of basis to the monomial basis is upper triangular with 1 along the diagonal. When $q = 0$, the Macdonald polynomials are known as Hall-Littlewood polynomials, and are written $P_\lambda(x; 0, t) = P_\lambda(x; 0, t)$.

The Macdonald polynomials have an integral form

$$J_\lambda(x; q, t) = c_\lambda(q, t) P_\lambda(x; q, t)$$
A Characteristic Map for $GL_{2n}(\mathbb{F}_q)/Sp_{2n}(\mathbb{F}_q)$

with

$$c_\lambda(q, t) := \prod_{s \in \lambda} (1 - q^{a(s)}t^{l(s)+1}),$$

where $a(s)$ and $l(s)$ denote the arm and leg lengths respectively (so $a(s) + l(s) + 1 = h(s)$). Similarly define

$$c'_\lambda(q, t) := \prod_{s \in \lambda} (1 - q^{a(s)+1}t^{l(s)}).$$

Then $\langle J_\lambda, J_\lambda \rangle = c_\lambda(q, t)c'_\lambda(q, t)$.

The symmetric functions $J_\lambda$ can be thought of as a deformation of the Jack polynomials, which are given by taking $q = t^a$ and sending $t \to 1$ after dividing by $(1 - t)^n$.

2.3. Linear Algebraic Groups. The point of view taken is to view the finite groups of interest as rational points of an algebraic group over $\overline{\mathbb{F}_q}$. Thus, the conventions and notation may differ slightly from more classical sources such as Carter [3] which work directly over the finite field. The definitions and conventions taken mostly follow Digne and Michel [6].

In general, $G$ will denote a linear algebraic group over $\overline{\mathbb{F}_q}$. Let $F$ denote the Frobenius endomorphism, which will always be the one taking the matrix $(x_{ij})$ to $(x_{ij}^q)$. Then $G(\mathbb{F}_q)$ or $G^F$ will denote the $\mathbb{F}_q$ points, with the former being used when talking about the finite group and the latter used to emphasize the algebraic group structure.

A torus is a group which is isomorphic to $(\overline{\mathbb{F}_q})^n$ for some $n$. Note that the maximal tori contained in $GL_n$ are precisely the subgroups which are conjugate to the standard maximal torus, which consists of diagonal matrices.

A Levi subgroup $L \subseteq G$ is a subgroup that is the centralizer of some torus $T$. The Levi subgroups of $GL_n$ are all of the form $\prod GL_{n_i}$ for $\sum n_i = n$. A Borel subgroup in $GL_n$ is some subgroup conjugate to the subgroup of upper triangular matrices which is the standard Borel subgroup.

A parabolic subgroup is a subgroup containing a Borel subgroup. In $GL_n$ these subgroups are conjugate to some subgroup of block upper-triangular matrices which are the standard parabolics. All parabolic subgroups have a canonical decomposition $P = LU$ where $U$ is the unipotent radical of $P$ and $L$ is a Levi subgroup, called the Levi factor of $P$. In $GL_n$, for a standard parabolic consisting of block upper-triangular matrices, the unipotent radical consists of matrices which are the identity in each block and arbitrary above the diagonal blocks. As an example, in $GL_4$

$$P = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \right\}, \quad U = \left\{ \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}, \quad L = \left\{ \begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \right\}$$

are an example of a parabolic subgroup and the unipotent radical and Levi subgroup respectively (the *’s denote arbitrary entries although the matrices
must be invertible). As the unipotent radical is normal, every parabolic subgroup has a canonical projection \( P \to L \) which for a standard parabolic in \( GL_n \) amounts to replacing the blocks on the diagonal of a matrix in \( P \) with identity matrices. In general the projection of \( x \in P \) will be denoted by \( \pi \). As Borel subgroups are parabolic, the same applies with \( L \) being a maximal torus.

Say that a subgroup \( H \subseteq G \) is rational (or \( F_q \)-rational) if it is stable under \( F \). Any rational subgroup \( H \) defines a subgroup \( H^F \subseteq G^F \). The standard Levi and parabolic subgroups are rational.

2.4. Representation theory of \( GL_n(F_q) \). To fix notation, the representation theory of \( GL_n(F_q) \) is briefly reviewed. The representation theory of \( GL_n(F_q) \) was originally developed by Green in \( [8] \) but this section follows the conventions in \( [14] \). Let \( M \) denote the group of units of \( F_q \) and let \( M_n \) denote the fixed points of \( \text{Frobenius endomorphism} \) \( F(x) = x^q \). Note that \( M_n \) may be identified with \( F^*_q \). Let \( L \) be the character group of the inverse limit of the \( M_n \) with norm maps between them. The Frobenius endomorphism \( F \) acts on \( L \) in a natural manner so let \( L_n \) denote the \( F_n \) fixed points in \( L \), and note there is a natural pairing of \( L_n \) with \( M_n \) for each \( n \) (but these pairings are not consistent).

The \( F \)-orbits of \( M \) can be viewed as irreducible polynomials over \( F_q \) under \( O \to \prod_{\alpha \in O} (x - \alpha) \). Denote by \( O(M) \) and \( O(L) \) the \( F \)-orbits in \( M \) and \( L \) respectively. Use \( \mathcal{P} \) to denote the set of partitions. Then the conjugacy classes of \( GL_n(F_q) \), denoted \( C_\mu \), are indexed by partition-valued functions \( \mu : O(M) \to \mathcal{P} \) such that

\[
\| \mu \| := \sum_{f \in O(M)} d(f) |\mu(f)| = n,
\]

where \( d(f) \) denotes the degree of \( f \). This is because \( \mu \) contains the information necessary to construct the Jordan canonical form. That is, given \( \mu \), construct a matrix in \( GL_n(F_q) \) in Jordan form by taking for each orbit \( f \in O(M) \), \( l(\mu(f)) \) blocks, of sizes \( \mu(f_i) \), for each root of \( f \). The resulting matrix has \( d(f) \) blocks of size \( \mu(f_i) \) for each \( f \) and \( i \), and adding this all up gives \( \| \mu \| = n \).

For example, the partition-valued function corresponding to the set of transvections, which have Jordan form

\[
\begin{pmatrix}
1 & 1 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1
\end{pmatrix},
\]

correspond to the partition-valued function \( \mu \) with \( \mu(f_1) = (21^{n-2}) \) and \( \mu(f) = 0 \) for \( f \neq f_1 \). Use \( q_f \) to denote \( q^{\ell(f)} \). There is a formula for the sizes
of conjugacy classes given by

$$|C_\mu| = \frac{|\text{GL}_n(F_q)|}{a_\mu(q)}$$

where

$$a_\mu(q) = q^n \prod_{f \in O(M)} q^{2n(\mu(f))} \prod_{i \geq 1} \prod_{j=1}^{m_i(\mu(f))} (1 - q^{-j})$$

with $n(\lambda) = \sum (i - 1)\lambda_i$.

Similarly, the irreducible characters of $\text{GL}_n(F_q)$, denoted $\chi_\lambda$, are indexed by functions $\lambda: O(L) \to P$ such that

$$\|\lambda\| := \sum_{\varphi \in O(L)} d(\varphi)|\lambda(\varphi)| = n,$$

where $d(\varphi)$ denotes the size of the orbit $\varphi$. The dimension of the irreducible representation corresponding to $\lambda$ is given by

$$d_\lambda = \psi_n(q) \prod_{\varphi \in O(L)} q^{n(\lambda(\varphi))} H_{\lambda(\varphi)}(q^{d(\varphi)})^{-1},$$

where $\psi_n(q) = \prod_{i=1}^n (q^i - 1)$, $q_{\varphi} = q^{d(\varphi)}$ and $H_{\lambda}(t) = \prod_{x \in \lambda} (t^{h(x)} - 1)$, $h(x)$ denoting the hook length. Note that with this convention, the trivial representation corresponds to the partition-valued function $\lambda(\chi_1) = (1^n)$ (\chi_1 being the trivial character) and 0 otherwise (this differs from the usual convention for $S_n$, where the trivial representation corresponds to the partition $(n)$).

The Deligne-Lusztig characters (also called basic characters for $\text{GL}_n(F_q)$) give another basis for the space of class functions on $\text{GL}_n(F_q)$. For a more detailed overview of Deligne-Lusztig characters, including their construction and properties, see the book of Carter [3].

Given any rational maximal torus $T \subseteq \text{GL}_n$, a (virtual) character $\zeta_T^{\text{GL}_n}(\cdot|\theta)$ of $\text{GL}_n(F_q)$ associated to some character $\theta$ of $T^F$ can be constructed. The character constructed depends only on the Weyl group orbit of $\theta$.

Deligne-Lusztig characters are rational functions in $q$ in the following sense. Given an element $g \in \text{GL}_n(F_q)$ with Jordan decomposition $g = su$ (s semisimple and u unipotent), the Deligne-Lusztig characters can be computed as

$$\zeta_T^{\text{GL}_n}(g|\theta) = \sum_{x \in \text{GL}_n(Z(s))^F} \theta(xs{x}^{-1})Q^{Z(s)}_{x^{-1}T_x}(u),$$

where $Q^{Z(s)}_{x^{-1}T_x}(u)$ is a rational function of $q$, known as the Green function. The Green functions can be computed as

$$Q^{Z(s)}_{x^{-1}T_x}(u) = \prod_{f \in O(M)} Q^f_{\gamma(f)}(q_f),$$
where $Q^\mu_n(q)$ denotes the Green polynomials, $\mu$ is a partition valued function indexing the conjugacy class of $g$ and $\gamma(f)$ is the partition given by taking $s \in x^{-1}T^Fx \cong \prod M_{k_i}$ and including as parts the $k_i/d(f)$ for which $f$ kills $s$ restricted to $M_{k_i}$. Since isomorphic maximal tori in $GL_n(F_q)$ are always conjugates, the tori $x^{-1}T^Fx$ for $x \in (GL_n/Z(s))^F$ are exactly those containing $s$ up to conjugation by $Z(s)^F$, which in turn are of the form $\prod M_{\gamma(f)}d(f)$ for $\gamma$ a partition valued function such that $|\gamma(f)| = |\mu(f)|$ for all $f \in O(M)$. This means that the Deligne-Lusztig character can be written

$$\zeta_T^{GL_n}(g|\theta) = \sum_{t \in T; |\gamma(t)| = |\mu(f)|} \theta(t) \prod_{f \in O(M)} Q^\mu_n(f(q_f)),$$

where $\gamma_t$ denotes the partition valued function corresponding to the torus $x^{-1}T^Fx$ for $t = xsx^{-1}$ and $s$ is the semisimple element as described above.

The Green polynomials give the change of basis from power sum to Hall-Littlewood polynomials and so satisfy

$$p_\rho(x) = \sum_{\mu} Q^\mu_n(t) t^{-n(\mu)} P_\mu(x; t^{-1})$$

for a formal parameter $t$.

The Deligne-Lusztig characters are invariant under conjugation of $(T, \theta)$ by $G^F$. There is a correspondence between $G^F$ orbits of pairs $(T, \theta)$ and functions $\lambda : O(L) \to P$ with $||\lambda|| = n$. Call the function $\lambda$ the combinatorial data associated to $(T, \theta)$.

The correspondence is as follows. Given a torus $T$ with rational points isomorphic to $\prod M_{k_i}$, and a character $\theta$ of $T^F$, consider the partition-valued function sending $\varphi$ to the partition with parts $k_i/d(\varphi)$ for all $i$ such that $\theta$ restricted to $M_{k_i}$ lies in the orbit $\varphi$. Conversely, given $\lambda$, $T$ can be constructed as a torus with rational points isomorphic to $\prod_{\varphi,i} M_{\lambda(\varphi),d(\varphi)}$ and $\theta$ is given by picking for each factor $M_{\lambda(\varphi),d(\varphi)}$ an element of the orbit $\varphi$ (which determines $T$ and $\theta$ up to conjugacy by $G^F$).

Given some $\lambda : O(L) \to P$, consider the Levi subgroup $L_\lambda$ whose rational points are given by

$$\prod_{\varphi \in O(L)} GL_{|\lambda(\varphi)|}(F_{q^d(\varphi)}) \subseteq GL_n(F_q),$$

with Weyl group $W(\lambda) = \prod_{\varphi \in O(L)} S_{|\lambda(\varphi)|}$. To each Weyl group element $w$ there is an associated partition-valued function sending $\varphi$ to the cycle type of $w(\varphi)$. Let $(T_w, \theta_w)$ denote the torus and character associated to the combinatorial data defined by $w$. Then there is a formula relating the Deligne-Lusztig characters to the irreducible characters of $GL_n(F_q)$ given by

$$\chi_\lambda(g) = \frac{1}{|W(\lambda)|} \sum_{w \in W(\lambda)} \prod_{\varphi \in O(L)} \chi^{Sym}_{\lambda(\varphi)}(w(\varphi)) \zeta_{T_w}^{GL_n}(g|\theta_w),$$

where the $\chi^{Sym}_{\lambda}$ are the irreducible characters of the symmetric group.
2.5. The symmetric space $\text{GL}_{2n}(F_q)/\text{Sp}_{2n}(F_q)$. Let

$$J = \begin{pmatrix} 0 & 1 & \ldots & 0 & 0 \\ -1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & -1 & 0 \end{pmatrix}$$

define the standard symplectic form and define the involution

$$\iota(X) = -J(X^T)^{-1}J$$

of $\text{GL}_{2n}(F_q)$. Then $\text{Sp}_{2n}(F_q)$ denotes the subgroup of $\text{GL}_{2n}(F_q)$ fixed by $\iota$.

Now for any $\iota$-stable subgroup $S$, let $S^\iota$ denote the subgroup of $\iota$ fixed points and $S^{-\iota}$ denote the set of $\iota$-split elements, which are elements $s \in S$ such that $\iota(s) = s^{-1}$.

A Gelfand pair is a (finite) group $G$, with a subgroup $H \subseteq G$ such that inducing the trivial representation from $H$ to $G$ gives a multiplicity-free representation. For a Gelfand pair $G/H$, any representation $\rho$ of $G$ has either no non-zero $H$-fixed vectors, or a 1-dimensional subspace fixed pointwise by $H$. Say that $\rho$ is a spherical representation, then this average is $\phi(g) = \langle v_\rho, \rho(g)v_\rho \rangle$, where $v_\rho$ is a unit $H$-fixed vector.

The spherical functions can also be computed by averaging characters over $H$. That is, $\phi(g) = |H|^{-1} \sum_{h \in H} \chi(hg)$ for $\chi$ the character of $\rho$ (if $\rho$ is not a spherical representation, then this average is 0). The spherical functions are the replacement for characters of a group and in particular, they form a basis for the space of bi-invariant functions on $G$.

Now $\text{GL}_n(F_q)/\text{Sp}_n(F_q)$ is a Gelfand pair, and the spherical functions, denoted $\phi_\lambda$, are indexed by partition-valued function $\lambda : O(L) \rightarrow \mathcal{P}$ with $\|\lambda\| = n$ (see [1], although note a different convention is used in this paper so all partitions labeling representations are transposed). For a partition $\lambda$, let $\lambda \cup \lambda$ denote the partition which contains every part of $\lambda$ twice. Then $\phi_{\lambda \cup \lambda}$ is the spherical function corresponding to the representation with character $\chi_{\lambda \cup \lambda}$ of $\text{GL}_{2n}(F_q)$.

Let $M_{\mu} \in \text{GL}_n(F_q)$ denote a conjugacy class representative of $C_{\mu}$. Let $g_{\mu} \in \text{GL}_{2n}(F_q)$ denote the matrix acting only on the odd coordinates by $M_{\mu}$. The double cosets of $\text{Sp}_{2n}(F_q)$ are indexed by $\mu : O(M) \rightarrow \mathcal{P}$, with $\|\mu\| = n$, with the $g_{\mu}$ being double coset representatives.

Two key results from [1] are reproduced below. The first relates the sizes of double cosets in $\text{GL}_{2n}(F_q)/\text{Sp}_{2n}(F_q)$ to the sizes of conjugacy classes in $\text{GL}_n(F_q)$.

**Proposition 2.1** ([1 Proposition 2.3.6]). Let $\mu : O(M) \rightarrow \mathcal{P}$ with $\|\mu\| = n$. Then

$$|H^F g_{\mu} H^F| = |H^F||C_{\mu}|_{q \rightarrow q^2},$$
where $H^F g_\mu H^F$ denotes the double coset indexed by $\mu$ in $\text{GL}_{2n}(\mathbb{F}_q)$ and $C_\mu$ denotes the conjugacy class indexed by $\mu$ in $\text{GL}_n(\mathbb{F}_q)$.

The second result gives a formula for the values of spherical functions in terms of Deligne-Lusztig characters on $\text{GL}_n(\mathbb{F}_q)$.

The function $\zeta_{\text{GL}_n}^\theta(\cdot | \theta)$ (or any other class function on $\text{GL}_n(\mathbb{F}_q)$) may be turned into an $H^F$-bi-invariant function on $\text{GL}_{2n}(\mathbb{F}_q)$ by defining

$$\zeta_{\text{GL}_n}^\theta(g_\mu | \theta) = \zeta_{\text{GL}_n}^\theta(M_\mu | \theta).$$

and extending to the double-coset. The following theorem relates the spherical function values to the values of the Deligne-Lusztig characters on $\text{GL}_n(\mathbb{F}_q)$.

**Theorem 2.2** ([1, Theorem 6.6.1]). Let $\lambda : O(L) \to \mathcal{P}$ with $\|\lambda\| = n$. Then

$$\phi_\lambda = \frac{1}{|W(\lambda)|} \sum_{w \in W(\lambda)} \prod_{\varphi \in O(L)} q^{-n(\lambda(\varphi))} c_{\lambda(\varphi)}(q^2, q^2) \cdot c_{\lambda(\varphi)}(w)(q^2) \times \text{sgn}(w) \frac{\zeta_{\text{GL}_n}^\theta(\cdot | \theta_w)_{q \to q^2}}{|T_w| \zeta_{\text{GL}_n}^\theta(1 | \theta_w)_{q \to q^2}},$$

where $c_{\lambda}(q, t)$ denotes the scaling from the two parameter Macdonald polynomials to their integral forms, $d_{\lambda}(w)(q)$ denotes the change of basis from power sum to two parameter Macdonald polynomials $P_{\lambda}(q^2, q)$ and $\text{sgn}$ denotes the sign character of $W(\lambda)$.

Here, $\zeta_{\text{GL}_n}^\theta(\cdot | \theta_w)$ is viewed as a rational function in $q$ as described above. Note that the convention used to label spherical representations in [1] is opposite the one used in this paper so all partitions are transposed.

### 3. Character sheaves on symmetric spaces

Here, the theory developed by Grojnowski [9] and Henderson [11] regarding character sheaves on symmetric spaces is reviewed. References are also given to the work of Shoji and Sorlin [19–21] for any cited results as both [9] and [11] are unpublished. Some familiarity with the theory of $l$-adic sheaves and algebraic groups is assumed, although beyond the results of [9] and [11], only some basic formal properties will be needed to translate the results into the language of functions.

The goal of this section is to establish the following properties of induction which are a consequence of the theory developed by Grojnowski and Henderson.

**Proposition 3.1.** Let $G$ and $H$ be groups of the form $\prod \text{GL}_{2n_i}$ and $\prod \text{Sp}_{2n_i}$ respectively, and let $L \subseteq G$ be a rational $\iota$-stable Levi subgroup, with a parabolic $P$ with Levi factor $L$ such that the projection sends $H_P$ to $H_L$. Then there is a linear function

$$\text{Ind}_{L \subseteq P}^G : \mathbb{C}[H^F_L \backslash L^F / H^F_L] \to \mathbb{C}[H^F_G \backslash G^F / H^F_G]$$
taking $H^F_L$ bi-invariant functions on $L^F$ to $H^F$ bi-invariant functions on $G^F$. Furthermore, it satisfies the following properties:

1. If $M \subseteq L$ is another rational $\iota$-stable Levi subgroup with $Q \subseteq P$ a parabolic with Levi factor $M$, then
   \[ \text{Ind}^G_{M \subseteq Q} = \text{Ind}^G_{L \subseteq P} \circ \text{Ind}^L_{M \subseteq Q \cap L}. \]

2. If $L = T$ is a maximal torus, then $H_T$ bi-invariant functions are identified with functions on $T^{-\iota}$ (which up to conjugation can be assumed to act only on odd coordinates) and if $\theta$ denotes a character on $T^{-\iota}$, then
   \[ \text{Ind}^G_{T \subseteq B}(\theta) = \frac{|T^F|}{|T^F|} \cdot \zeta_{GL_n}(\cdot \theta)^{q \rightarrow q^2}, \]

3. If $P$ is a rational parabolic with $P = LU$, then
   \[ \text{Ind}^G_{L \subseteq P}(f)(x) = |H^F \cap P^F|^{-2}(-1)^{\dim U} \sum_{h,h' \in H^F \cap P^F} f(hxh'). \]

Proposition 3.1 states all necessary results purely in the language of functions and this proposition is the only result of this section that is actually needed. It follows immediately from the results of this section. A reader that is willing to accept Proposition 3.1 on faith may safely skip the rest of the section.

3.1. Motivation. For $\text{GL}_n(F_q)$, most properties of the characteristic map are more easily seen when viewing the domain not as the ring of class functions but rather the representation ring. In the symmetric space setting, the relationship between the spherical functions and representations is less well-behaved, so representations should be replaced by sheaves instead.

All operations that can be done on representations have analogues for sheaves (for instance, tensor products, induction) but there are additional operations available. Moreover, the relationship between these sheaves and functions is well-understood and so anything that can be proven for sheaves has a function analogue.

Working with sheaves, the usual notions of parabolic induction and restriction for groups also have a natural analogue. These operations will correspond to the product and coproduct structures on $\otimes \Lambda$ through $\text{ch}$.

3.2. Preliminaries on $l$-adic sheaves. This section mostly follows [11], see there for further details and references. See also [7] for some of the basics on derived categories.

Fix some prime $l$ not equal to the characteristic of $F_q$ (nothing will depend on the choice of $l$). Since only algebraic numbers appear, all statements in this section about $\mathbb{Q}_l$ functions can be readily transferred to $\mathbb{C}$.

All varieties will be defined over $\overline{F}_q$. An $F_q$-structure on a variety $X$ is given by a Frobenius map $F : X \to X$. A morphism of varieties with
$F_q$ structures is defined over $F_q$ if it is $F$-equivariant. Let $X^F$ denote the corresponding variety over $F_q$.

If $X$ is a variety, let $D(X)$ denote the bounded derived category of constructible $\mathbb{Q}_l$-sheaves of finite rank on $X$. The objects of this category are called complexes.

Say that a complex $K$ is $F$-stable if $F^*K \cong K$. Now to any $F$-stable complex $K$, and a choice of isomorphism $\phi : F^*K \to K$, there is an associated function $\chi_K : X^F \to \mathbb{Q}_l$ called the characteristic function. It is defined by

$$\chi_K(x) := \sum_i (-1)^i \text{Tr}(\phi, H^i_x K),$$

where $H^i_x K$ denotes the stalk at $x$ of the cohomology sheaf of $K$. The dependence on $\phi$ will be dropped in the notation but a choice of such an isomorphism must always be made when taking the characteristic function.

If $f : X \to Y$ is a morphism of varieties, there are two functors, $f^* : D(Y) \to D(X)$, called the pullback and $f_! : D(X) \to D(Y)$, called the compactly supported pushforward.

A key fact is the proper base change theorem, which states that for a Cartesian square

$$\begin{array}{ccc}
A & \xrightarrow{\alpha} & B \\
\downarrow{\beta} & & \downarrow{p} \\
C & \xrightarrow{\gamma} & D
\end{array}$$

there is an isomorphism of functors

$$q^*p_! \cong \beta_! \alpha^*.$$  

These functors also interact well with taking characteristic functions (assuming that the function $f$ is defined over $F_q$), with

$$\chi_{f^*K}(x) = \chi_K(f(x))$$

and

$$\chi_{f_!K}(y) = \sum_{f(x) = y, x \in X^F} \chi_K(x).$$

There is a shift functor $[n] : D(X) \to D(X)$ taking a complex $K$ to the complex $K[n]$ with $K[n]_m = K_{n+m}$ and this functor commutes with pushforward and pullback and the corresponding effect on the characteristic function is multiplication by $(-1)^n$.

Finally, direct sums of complexes in $D(X)$ correspond to addition and the tensor product of complexes in $D(X)$ corresponds to multiplication of the corresponding characteristic functions.

If $G$ is an algebraic group acting on $X$, let $D^G(X)$ denote the $G$-equivariant derived category. All the constructions above have analogues in the equivariant setting. In particular, if the action is defined over $F_q$ then the characteristic function of an equivariant sheaf defined over $F_q$ is invariant under the action of $G^F$. 

If \( f : X \to Y \) is a principal \( G \)-bundle, then there is an equivalence of categories \( f^* : D(Y) \to D^G(X) \) (whose composition with the forgetful functor \( D^G(X) \to D(X) \) gives the usual pullback) and the inverse is denoted by \( f_* : D^G(X) \to D(Y) \). If \( f \) is defined over \( \mathbb{F}_q \) then the characteristic functions satisfy

\[
\chi_{f_* K}(y) = |G|^{-1} \sum_{f(x) = y, x \in X^F} \chi_K(x).
\]

If \( H \subseteq G \), then this equivalence also restricts to an equivalence \( D^{H \times H}(G) \cong D^H(G/H) \) between the \( H \) bi-equivariant sheaves on \( G \) and \( H \) equivariant sheaves on \( G/H \) and so when convenient bi-equivariant sheaves can be thought to live on the symmetric space \( G/H \) (this is like how there is no difference between working with bi-invariant functions on \( G \) and invariant functions on \( G/H \)).

Finally, if \( S \subseteq G \) and \( S \) acts on \( X \) then let \( X \times_S G \) denote the quotient of \( X \times G \) by the diagonal action of \( S \). There is an equivalence of categories \( D^S(X \times S G) \to D^S(X) \) induced by \( i^* \) where \( i(x) = (x, 1) \), with inverse functor \( \Gamma \) and this factors through \( D^{S \times G}(X \times G) \) with \( \Gamma \cong \beta \circ \alpha^* \) where

\[
X \xrightarrow{\alpha} X \times G \xrightarrow{\beta} X \times_S G
\]

are the obvious morphisms with \( \alpha \) a \( G \)-bundle and \( \beta \) an \( S \)-bundle (see [16, Lemma 1.4] for example).

### 3.3. Induction functors

Here, the induction functors analogous to Deligne-Lusztig induction for \( \text{GL}_n \) are introduced. Although all important results needed are in [11], for technical reasons the induction functor defined there will not suffice (the Levi subgroup that is needed has no parabolic subgroup stable under the involution \( \iota \)). Thus, the definition given by Grojnowski in [9] will be used.

It is important to note that these functors do not preserve any sort of \( \mathbb{F}_q \) structure unless the parabolic subgroup used is rational. Even if \( P \) is rational, to define induction for functions some choice of \( \mathbb{F}_q \) structure must be placed on the sheaves themselves. For now, the definition does not involve any \( \mathbb{F}_q \) structure whatsoever.

For any \( \iota \)-stable Levi subgroup \( L \) with \( P = LU \) a parabolic, not necessarily stable under \( \iota \), define a parabolic induction functor \( \text{Ind}_{L \subseteq P}^G : D^{H \times H}(L) \to D^{H \times H}(G) \) taking bi-equivariant sheaves on \( L \) to \( G \) as follows. Consider the diagram

\[
G \xleftarrow{pr} G \times G / P \xrightarrow{q} (G/U \times G/U) / L \xleftarrow{j} H \times H, \quad L \xrightarrow{i} L,
\]

where \( pr \) is the obvious projection, \( q(g, g' P) = (gg'U, g'U) / L \), \( j(h, l, h') = (hlU, h'U)L \) and \( i(l) = (1, l, 1) \). The maps are all equivariant so define the induction functor \( \text{Ind}_{L \subseteq P}^G : D^{H \times H}(L) \to D^{H \times H}(G) \) by

\[
\text{Ind}_{L \subseteq P}^G := pr \circ q^* \circ j \Gamma[\dim U + 2 \dim H / H P],
\]
where $\Gamma$ is the inverse of $i^*$. Now note that if the projection sends $H_P$ to $H_L$, and $U_P \cap H$ is connected, then $D^{H_P \times H_P}(L) \cong D^{H_L \times H_L}(L)$ and so $\text{Ind}^G_{L \subseteq P}$ may be viewed as a functor $D^{H_L \times H_L}(L) \to D^{H \times H}(G)$. Since it is assumed that $q$ is odd, $U_P \cap H$ is connected, and as only the case when $H_P$ projects into $H_L$ will be needed, induction functors will always be assumed to have domain $D^{H_L \times H_L}(L)$.

This is the induction functor defined by Grojnowski in [9]. This definition agrees with the one given by Henderson in [11] (as noted there) and the reason to use this more general notion is that Henderson’s definition requires an $\iota$-stable parabolic. Since this fact is not proven in [11], for completeness a proof is given below.

**Proposition 3.2.** Let $L$ be a rational $\iota$-stable Levi subgroup and $P$ an $\iota$-stable parabolic with $P = LU$. Then

$$\text{Ind}^G_{L \subseteq P} \cong pr^* q_0 \iota ![\dim H/H_P + \dim U/U_L - \dim H_L - \dim H]$$

where the maps are the obvious ones in the diagram

$$L \xleftarrow{pr} H \times P \times H \xrightarrow{q} H \times_{H_p} P \times_{H_p} H \xrightarrow{\iota} G.$$

**Proof.** First, note that the shifts agree and so may be ignored for the rest of the proof. Then the proof proceeds by chasing through the following diagram

$$
\begin{array}{cccccccc}
L & & & & & & & \\
\xleftarrow{pr} & H \times L \times H & \xrightarrow{q_1} & H \times_{H_p} P \times_{H_p} P \times_{H_p} H & \xrightarrow{\iota_1} & (G/U \times G/U) / L \\
& \xleftarrow{pr_1} & \downarrow & \xleftarrow{pr_2} & H \times P \times H & \xrightarrow{q_2} & H \times_{H_p} P \times_{H_p} H & \xrightarrow{\iota_2} & G \times G / P \\
& & & & & \xrightarrow{\iota_3} & \downarrow & \xrightarrow{pr} & G,
\end{array}
$$

where $pr_1(h, p, h') = (h, \bar{p}, (h')^{-1})$ (this is just because Henderson has the action on the second $H$ on the right and Grojnowski has it on the left) and $pr_2$ is induced by $pr_1$ as it is $H_P \times H_P$ equivariant, and $\iota_2(h, p, h') = (h p h', h' P)$.

Note that following the top path gives the first definition and the bottom gives the second. Then after checking that the first square and the bottom triangle commute and the second square is a Cartesian square, the result follows by the proper base change theorem and functoriality of pullback and pushforward. $\square$

A key result is that the composition of two induction functors is again an induction functor. The following proposition is given in [9] without proof so some details of the proof are supplied.

**Proposition 3.3 ([9 pg. 12]).** Let $M \subseteq L$ be $\iota$-stable Levi subgroups, and let $Q \subseteq P$ be parabolic subgroups with Levi factors $M$ and $L$ respectively. Then

$$\text{Ind}^G_{L \subseteq P} \circ \text{Ind}^H_{M \subseteq Q \cap L} \cong \text{Ind}^G_{M \subseteq Q}.$$
Proof. The shifts agree and so may be ignored. Consider the following commutative diagram

\[
\begin{array}{ccccccc}
H_L \times H_L \times Q & \to & M \times H_L \times Q & \to & H \times H_L \times H & \to & H \times H_L \times H \\
\downarrow f_1 & & \downarrow 1 & & \downarrow f_2 & & \downarrow f_3 \\
(L \times Q_L \times L) / M & \to & (H \times H_L \times L) / M & \to & (H \times H_L \times H) / M & \to & (H \times H_L \times H) / M \\
\downarrow g_1 & & \downarrow g_2 & & \downarrow g_3 & & \downarrow g_4 \\
H \times H_L \times H & \to & H \times H_L \times H & \to & H \times H_L \times H & \to & H \times H_L \times H \\
\downarrow h_1 & & \downarrow h_2 & & \downarrow h_3 & & \downarrow h_4 \\
L & \to & L & \to & L & \to & L \\
\end{array}
\]

where in the first column and bottom row the morphisms are the ones coming from the definition of parabolic induction, \( \alpha_i(x) = (1, x, 1) \) for \( x \) in the domain of \( \alpha_i \) as the second column is obtained from the first by applying the functor \( X \mapsto H \times H_L \times H \). \( f_1, g_1 \) and \( h_1 \) by this functor (so \( f_2(h, x, h') = (h, f_1(x), h') \) and so on), and the following definitions are given for the other morphisms:

\[
\begin{align*}
\beta_1(h_1, h_2, m, h_3, h_4) &= (h_1 h_2, m, h_4 h_3) \\
\beta_2((h_1, l_1 U_Q \wedge L, h_2, l_2 U_Q \wedge L) M) &= (h_1 l_1 U_Q, h_2 l_2 U_Q) M \\
\beta_3(h_1, l_1, l_2 Q \wedge L, h_2) &= (h_1 l_1 l_2 U_P, h_2 l_2 U_P) U_Q \wedge L M \\
\gamma_3(g_1, g_2 Q) &= (g_1 g_2 U_P, g_2 U_P) U_Q \wedge L M \\
f_3(h_1, m, h_2) &= (h_1 m U_Q, h_2 U_Q) M \\
g_3((g_1 U_P, g_2 U_P) U_Q \wedge L M) &= (g_1 U_Q, g_2 U_Q) M \\
h_3((g_1 U_P, g_2 U_P) U_Q \wedge L M) &= (g_1 U_P, g_2 U_P) L \\
h_4(g_1, g_2 Q) &= (g_1, g_2 P) \\
p(g_1, g_2 P) &= g_1.
\end{align*}
\]

To check the equality of functors first note that following the rightmost path along the diagram gives \( \text{Ind}_{M \subseteq Q}^G \), the left side gives \( \text{Ind}_{M \subseteq Q \wedge L}^G \) while the bottom gives \( \text{Ind}_{L \subseteq P}^G \). Then as the diagram commutes and the squares labeled 1, 2, 3, and 4 are Cartesian squares, the result follows from proper base change and functoriality of pullback and pushforward.

\[\square\]

3.4. Characteristic functions of induced complexes. The induction functors define parabolic induction of complexes but give no rational structure on the resulting complex and so more is needed to define induction of functions. The idea is to restrict to a smaller set of sheaves which are closed under induction and have a canonical \( \mathbb{F}_q \) structure. For more details on this
section, see [11, Ch. 5, 6] or [22], noting that any sheaf on \(G/H \times V\) can be restricted to \(G/H \times \{0\}\).

First, consider the group case. For any maximal torus \(T\) and a (tame) rank one local system \(L\) on \(T\), there is an associated complex \(K_{(T, L)}\) defined using intersection cohomology. This complex depends only on the \(G\) orbit of \((T, L)\).

Moreover, if \(T\) is rational and \(L\) is \(F\)-stable, there is a unique \(F\)-structure making the characteristic function an irreducible character of \(T^F\). Then \(K_{(T, L)}\) has an induced \(F\) structure coming from the structures on \(T\) and \(L\) (through the construction in terms of intersection cohomology). The corresponding characteristic functions \(\chi_{K_{(T, L)}}\) depend only on the \(G\) orbit of \((T, L)\).

These complexes \(K_{(T, L)}\) are exactly the ones obtained by inducing \(L\) from \(T\) with any choice of Borel subgroup. The benefit of the intersection cohomology definition is that it gives an \(F\) structure.

In the symmetric space setting where \(G = \text{GL}_{2n}\) and \(H = \text{Sp}_{2n}\), the above still holds with modifications, namely replacing a maximal torus with a maximal \(\iota\)-stable torus and characters replaced with spherical functions. There is only one \(H\)-conjugacy class of maximal \(\iota\)-stable torus, and so in particular every maximal \(\iota\)-stable torus \(T\) contains a maximal \(\iota\)-split torus \(T^{-\iota}\) with \(T^{-\iota} \cong T/T_H\) and is contained in an \(\iota\)-stable Borel subgroup. The \(H^F\) conjugacy classes of maximal \(\iota\)-split tori are in bijection with the \(\text{GL}_n^F\)-conjugacy classes of \(F\)-stable maximal tori of \(\text{GL}_n\), as every maximal \(\iota\)-split torus can be conjugated to act on the odd coordinates.

There are complexes \(K_{(T^{-\iota}, L)}\) associated to local systems \(L\) on maximal \(\iota\)-split tori \(T^{-\iota}\). When both have an \(F\) structure, then \(K_{(T^{-\iota}, L)}\) does as well and their characteristic functions are invariant under \(H^F\). See [11] or [22] for details on their construction and properties.

The transitivity of induction implies that the collection of complexes \(\text{Ind}^G_{T \subseteq B}(L)\) is closed under bi-invariant parabolic induction. The strategy will be to give these complexes an \(F\) structure, and then show that their characteristic functions form a basis for the bi-invariant functions, uniquely defining induction for any bi-invariant function. The next theorem relates the induced complexes to the complexes \(K_{(T^{-\iota}, L)}\) and is due to Grojnowski [9], although see also [20, Theorem 1.16].

**Theorem 3.4 ( [9, Lemma 7.4.4]).** Let \(T\) be a rational maximal \(\iota\)-stable torus and \(L\) an \(F\)-stable rank one local system on \(T^{-\iota}\). There is an isomorphism

\[
\text{Ind}^G_{T \subseteq B}(L) \cong K_{(T^{-\iota}, L)} \otimes H^\bullet_c(\mathcal{B}^{Z_H(T^{-\iota})})[\dim T^{-\iota}].
\]

where \(\mathcal{B}^{Z_H(T^{-\iota})}\) denotes the flag variety of \(Z_H(T^{-\iota})\).

This theorem means that \(\text{Ind}^G_{T \subseteq B}(L)\) has a canonical \(F\) structure (since \(H^\bullet_c(\mathcal{B}^{Z_H(T^{-\iota})})\) is a constant sheaf) and so the characteristic function of induced complexes, at least from a maximal torus, may be taken.
The characteristic functions of the complexes $K_{(T, L)}$ are related to both the Deligne-Lusztig characters and the parabolic induction of characters from a maximal $\iota$-split torus. These functions are also called basic characters and should be thought of as the analogue of Deligne-Lusztig characters in the symmetric space setting.

**Proposition 3.5.** Let $T$ be a rational maximal $\iota$-stable torus and $\theta$ an irreducible character of $(T^{-i})^F$. Let $L_\theta$ denote the corresponding local system on $T^{-i}$ or $T$. Then

$$(-1)^n \chi_{K_{(T^{-i}, L_\theta)}} = \frac{|(T^{-i})^F|}{|T^{-i})^F|_{q \to q^2}} \chi_{\text{Ind}_{T \subseteq B}^G(L_\theta)\zeta_{\text{GL}_n(T^{-i} \cdot |_{\theta})}} = \zeta_{\text{GL}_n(T^{-i})} (|\theta|)_{q \to q^2}.$$ 

The first equality follows from the previous theorem by taking characteristic functions and the second equality follows from Theorem 5.3.2 in [1], although as the notation there differs significantly, see also [11, Proposition 6.9].

**Proposition 3.6.** The functions $\chi_{(T^{-i}, L)}$ as $(T^{-i}, L)$ runs over $H^F$ orbits of rational maximal $\iota$-split tori and $F$-stable local systems on the torus form a basis for the space of $H^F$ bi-invariant functions on $G^F$.

This second proposition follows from the inner product formula given in [11, Theorem 6.7] (see also [20, Theorem 3.11]) and the fact that there are exactly enough such functions as the indexing set is the same as for the characters of $\text{GL}_n(F_q)$.

These two propositions give a way to define induction for any bi-invariant function. Define

$$\text{Ind}_{L \subseteq P}^G \chi_{\text{Ind}_{T \subseteq B}^L(L)} = \chi_{\text{Ind}_{T \subseteq B'}^L(L)}$$

where $B'$ is any Borel subgroup with Levi factor $T$ in $G$ and extend by linearity. Since the $\chi_{\text{Ind}_{T \subseteq B}^L(L)}$ form a basis for the $H^F_L$ bi-invariant functions on $L^F$, this is well-defined.

In the case that the parabolic subgroup $P$ is rational, the induction functor actually gives an $F_q$ structure on $\text{Ind}_{L \subseteq P}^G(K)$ if $K$ has one. When $K = K_{(T, L)}$, this $F_q$ structure agrees with the one coming from intersection cohomology. This gives a much simpler definition of induction of functions in this case since there is an explicit formula for this induction operation. The proof of the following proposition is straightforward.

**Proposition 3.7.** Let $L$ be a rational $\iota$-stable Levi subgroup, with a rational parabolic $P = LU$. Then if $K$ is a complex on $L$,

$$\chi_{\text{Ind}_{L \subseteq P}^G(K)} = |H^F \cap P^F|^{-2} (-1)^{\dim U} \sum_{h, h' \in H^F : hzh' \in P^F} \chi_K(hzh').$$
4. The Characteristic Map

In this section, a characteristic map
\[ \text{ch} : C[\text{Sp}_{2n}(F_q) \backslash \text{GL}_{2n}(F_q) / \text{Sp}_{2n}(F_q)] \to \bigotimes_{f \in O(M)} \Lambda \]
is constructed from the \( \text{Sp}_{2n}(F_q) \) bi-invariant functions on \( \text{GL}_{2n}(F_q) \). Here the tensor product \( \otimes \Lambda \) is taken over irreducible \( f \) and each factor is isomorphic to a copy of the symmetric functions. The notational convention will be to drop the tensor and if \( p \in \Lambda \) corresponds to the factor indexed by \( f \), then write \( p(f) = p(x_1, f, \ldots) \). Before proceeding, the theory for \( \text{GL}_n(F_q) \) is reviewed because the construction for \( \text{GL}_{2n}(F_q) / \text{Sp}_{2n}(F_q) \) is both very similar and some facts about the characteristic map for \( \text{GL}_n(F_q) \) are used in the proofs. The exposition follows [14].

4.1. The \( \text{GL}_n(F_q) \) Theory. Define
\[ \text{ch}_{\text{GL}} : \bigoplus_n C[\text{GL}_n(F_q)]^{\text{GL}_n(F_q)} \to \bigotimes_{f \in O(M)} \Lambda \]
by
\[ \text{ch}_{\text{GL}_n}(I_{C_\mu}) = \prod_{f \in O(M)} q_f^{-n(\mu(f))} P_\mu(f)(f; q_f^{-1}), \]
where \( I_{C_\mu} \) denotes the indicator function for \( C_\mu \) with \( \mu : O(M) \to \mathcal{P} \) and \( P_\mu(x; t) \) is the Hall-Littlewood symmetric function.

Define a multiplication on \( \bigoplus_n C[\text{GL}_n(F_q)]^{\text{GL}_n(F_q)} \) given by parabolic induction. That is, given two class functions, \( f \) on \( \text{GL}_n(F_q) \) and \( g \) on \( \text{GL}_m(F_q) \), define a class function on \( \text{GL}_{n+m}(F_q) \) by embedding \( \text{GL}_n(F_q) \times \text{GL}_m(F_q) \) as a standard Levi subgroup, viewing \( f \times g \) as a function on this Levi subgroup, extending to a rational parabolic subgroup by the canonical quotient, and then inducing to the full group. With this multiplication, \( \text{ch} \) is an isomorphism of graded algebras on \( \bigoplus_n C[\text{GL}_n(F_q)]^{\text{GL}_n(F_q)} \). In addition, this map is an isometry when \( C[\text{GL}_n(F_q)]^{\text{GL}_n(F_q)} \) is equipped with the usual inner product and \( \Lambda \) with the inner product defined by
\[ \langle p_\mu, p_\mu \rangle_{\text{GL}_n} = \prod_{f \in O(M)} z_\mu(f) \prod_i \frac{1}{q_i^{-1}} \]
for \( \mu : O(M) \to \mathcal{P} \).

In general, if there is some expression involving a partition \( \mu, F(\mu), \) with \( F(0) = 1 \), then the same expression with \( \mu : O(M) \to \mathcal{P} \) will be defined as the product \( \prod_{f \in O(M)} F(\mu(f)) \). Thus, write
\[ z_\mu = \prod_{f \in O(M)} z_\mu(f). \]
If the expression contains \( q \), in each factor it should be replaced by \( q_f \). Symmetric functions labeled by a partition valued function are interpreted
similarly. Thus,

\[ p_\mu = \prod_{f \in O(M)} p_{\mu(f)}(f). \]

Then define

\[ \tilde{p}_n(x) := \begin{cases} 
  p_{n/d}(f_x)(f_x) & \text{if } d(f_x) \mid n \\
  0 & \text{else} 
\end{cases}, \]

\[ \tilde{p}_n(\xi) := \begin{cases} 
  (-1)^{n-1} \sum_{x \in M_n} \xi(x) \tilde{p}_n(x) & \text{if } \xi \in L_n \\
  0 & \text{else} 
\end{cases}, \]

\[ p_n(\varphi) := \tilde{p}_{n d(\varphi)}(\xi), \]

where \( \xi \in \varphi \) for \( \varphi \in O(L) \). The \( p_n(\varphi) \) may be viewed as power sum symmetric functions in "dual variables \( x_{i,\varphi} \)" (that is, they may be formally viewed as symmetric functions in these variables, and the definition of \( p_n(\varphi) = p_n(x_{i,\varphi}) \) already available can be used to define other symmetric functions, e.g. \( s_\lambda(\varphi), e_\lambda(\varphi) \)).

As a matter of convention, \( \mu \) will denote functions \( O(M) \to \mathcal{P} \) and \( \lambda \) will denote functions \( O(L) \to \mathcal{P} \), and symmetric functions indexed by \( \mu \) will always be in variables \( x_{i,f} \) for \( f \in O(M) \) and symmetric functions indexed by \( \lambda \) will always be in the variables \( x_{i,\varphi} \). For a function \( F \) on partitions, \( F(\lambda) \) should be interpreted similarly to \( F(\mu) \) and symmetric functions indexed by \( \lambda \) should be interpreted similarly to those indexed by \( \mu \).

Now the characteristic map on the characters can be computed as

\[ \text{ch}_{GL_n}(\chi_\lambda) = s_\lambda \]

where the \( s_\lambda \) denote the Schur functions. Note by orthonormality of the characters that the inner product on the dual variables is just the standard Hall inner product. That is,

\[ \langle p_\lambda, p_\lambda \rangle_{GL_n} = z_\lambda. \]

Finally, note that if \( T \) is a rational maximal torus in \( GL_n \) with \( T^F \cong M_{k_1} \times \cdots \times M_{k_r} \) and \( \theta \) an irreducible character of \( T^F \), then

\[ \text{ch}_{GL_n}(s_T^{GL_n}(\cdot|\theta)) = \prod_i p_{k_i/d(\varphi_i)}(\varphi_i) = p_\lambda, \]

where \( \varphi_i \) is the orbit of \( \theta|_{M_{k_i}} \) and \( \lambda \) is the combinatorial data associated to \((T,\theta)\). In fact, this could be used as a definition of the Deligne-Lusztig characters for \( GL_n(F_q) \).

4.2. Definition of the Characteristic Map. Now return to the case of interest. Define the inner product \( \langle f, g \rangle = \sum_{x \in GL_{2n}(F_q)} f(x)g(x) \) on \( C[Sp_{2n}(F_q) \setminus GL_{2n}(F_q) / Sp_{2n}(F_q)] \) and the inner product on \( \otimes \Lambda \) by

\[ \langle p_\mu, p_\mu \rangle = z_\mu \prod_{f \in O(M)} \prod_i \frac{1}{q_f^{2\mu(f)_i} - 1}. \]
Define the characteristic map $ch$ by
\[
ch(I_{H^Fg_{i,H^F}}) = \prod_{f \in O(M)} q_f^{-2n(\mu(f))} P_{\mu(f)}(f; q_f^{-2})
\]
and extending linearly.

Note that the coefficients of $p_\mu$ in $ch(I_{H^Fg_{i,H^F}})$ (which are rational in $q$), are the same as those in $ch_{GL_n}(I_{C_\mu})$ but with $q$ replaced with $q^2$. It is natural to expect in light of Theorem 2.2 that the same is true for the spherical functions. To establish this, the following lemma is needed.

**Lemma 4.1.** Let $t$ denote a formal variable and use $\zeta_{T}^{GL_n}(C_\mu|\theta,t)$ to denote the Deligne-Lusztig character $\zeta_{T}^{GL_n}(\cdot|\theta)$, which is a rational function in $q$, in terms of $t$. Here, $T^F \cong \prod M_{k_i}$ and $\theta = \prod \theta_i$ with $\theta_i \in \varphi_i$. Then
\[
\sum_\mu \zeta_{T}^{GL_n}(C_\mu|\theta,t) \prod_{f \in O(M)} t_f^{-n(\mu(f))} P_{\mu(f)}(f; t_f^{-1}) = \prod_i p_{k_i/d(\varphi_i)}(\varphi_i)
\]
as polynomials with coefficients in $C(t)$.

**Proof.** This proof follows that of Theorem 4.2 in [22], where it is proven for the case of $t = -q$, but the proof given works for a formal parameter.

It suffices to show that the coefficient of $P_{\mu(f)}(f; q_f^{-1})$ in $\prod p_{k_i/d(\varphi_i)}(\varphi_i)$ is $\zeta_{T}^{GL_n}(C_\mu|\theta,t)$. Write
\[
\prod_i p_{k_i/d(\varphi_i)}(\varphi_i) = \prod_{x_i \in M_{k_i}} \sum \theta_i(x_i)p_{k_i/d(\varphi_i)}(f_{x_i})
\]
\[
= \sum \theta(x) \prod_{i} p_{k_i/d(\varphi_i)}(f_{x_i}).
\]
Now rewrite $\prod p_{k_i/d(\varphi_i)}(f_{x_i}) = \prod_{f \in O(M)} P_{\gamma_x(f)}(f)$ and use the fact that the Green polynomials are the change of basis from power sum to Hall-Littlewood polynomials to obtain
\[
\prod p_{k_i/d(\varphi_i)}(\varphi_i) = \sum \theta(x) \prod_{f \in O(M)} \left( \sum_{|\mu(f)| = |\gamma_x(f)|} Q_{\gamma_x(f)}^{\mu(f)}(t_f) t_f^{-n(\mu(f))} P_{\mu(f)}(f, f_f^{-1}) \right)
\]
\[
= \sum_{\mu} \left( \sum_{x \in T, |\gamma_x(f)| = |\mu(f)|} \theta(t) Q_{\gamma_x}^{\mu}(t) \right) \prod_{f \in O(M)} t_f^{-n(\mu(f))} P_{\mu(f)}(f, f_f^{-1})
\]
\[
= \sum_{\mu} \zeta_{T}^{GL_n}(C_\mu|\theta,t) \prod_{f \in O(M)} t_f^{-n(\mu(f))} P_{\mu(f)}(f, f_f^{-1})
\]
as required. \qed

**Proposition 4.2.** Let $T$ be a rational maximal torus of $GL_n$ with $T^F \cong M_{k_1} \times \ldots \times M_{k_r}$ and $\theta$ a character of $T^F$. Consider the Deligne-Lusztig character $\zeta_{T}^{GL_n}(\cdot|\theta)$, which induces an $H^F$ bi-invariant function on $GL_{2n}(F_q)$. 
Then
\[ \text{ch}(\zeta^{GL_n}_T(\cdot|\theta)_{q \to q^2}) = \prod_i P_{k_i/d(\varphi_i)}(\varphi_i) = p_\lambda, \]
where \( \varphi_i \) is the \( F \)-orbit of \( \theta|_{M_{k_i}} \in L \) and \( \lambda \) is the combinatorial data associated to \( (T, \theta) \).

**Proof.** Write
\[ \zeta^{GL_n}_T(\cdot|\theta)_{q \to q^2} = \sum_{\mu} \zeta^{GL_n}_T(g_{\mu}|\theta)_{q \to q^2} I_{H^F g_{\mu} H^F} \]
and apply the characteristic function, giving
\[ \sum_{\mu} \zeta^{GL_n}_T(H^F g_{\mu} H^F|\theta)_{q \to q^2} \prod_{f \in O(M)} q_f^{-2n(\mu(f))} P_{\mu(f)}(f; q_f^{-2}) = \prod_i P_{k_i/d(\varphi_i)}(\varphi_i) \]
by the lemma with \( t = q^2 \).

Now \( \text{ch}(\phi_\lambda) \) may be computed.

**Proposition 4.3.** Let \( \lambda : O(L) \to P \) with \( ||\lambda|| = n \). Then
\[ \text{ch}(\phi_\lambda) = \frac{1}{\psi_n(q^2)} \prod_{\varphi \in O(L)} q_\varphi^{-n(\lambda(\varphi))} J_{\lambda(\varphi)}(q_\varphi, q_\varphi^2). \]

**Proof.** First, note that
\[ \phi_\lambda = \frac{1}{|W(\lambda)|} \sum_{w \in W(\lambda)} \left( \prod_{\varphi \in O(L)} q_\varphi^{-n(\lambda(\varphi))} c_{\lambda(\varphi)} d_{\lambda(\varphi)} (w)(q_\varphi) \right) \times \text{sgn}(w) \frac{\zeta^{GL_n}_T(\cdot|\theta_w)_{q \to q^2}}{|T^F_w| \zeta^{GL_n}_T(1|\theta_w)_{q \to q^2}} \]
from Theorem 2.2. Recall that \( c_\lambda \) denotes the scaling from \( P_\lambda \) to \( J_\lambda \), \( d_\lambda \) denotes the change of basis from power sum symmetric functions to Macdonald symmetric functions and \( \zeta^G_T \) denotes a Deligne-Lusztig character.

Now \( \zeta^{GL_n}_T(1) = \varepsilon(T)|GL_{2n}^F/T^F|_{q'} \) (see Theorem 7.5.1)], where for an integer \( n \), \( n_{q'} \) denotes the \( q' \)-free part, and \( \varepsilon(C) = (-1)^s(C) \) where \( s \) denotes the split rank. Note in this case that \( \varepsilon(T_w) = \text{sgn}(w) \). Thus, \( \zeta^{GL_n}_T(1) = \frac{\text{sgn}(w)\psi_n(q^2)}{|T^F_w| q_\varphi^{-n(\lambda(\varphi))}} \). Then apply ch, obtaining
\[ \frac{1}{|W(\lambda)|} \sum_{w \in W(\lambda)} \prod_{\varphi \in O(L)} q_\varphi^{-n(\lambda(\varphi))} c_{\lambda(\varphi)} d_{\lambda(\varphi)} (w)(q_\varphi) \prod_{\varphi \in O(L)} p_{\varphi \in O(L)}(\varphi)|T^F_w|_{q \to q^2} \]
as applying Proposition 4.2 gives
\[ \text{ch}(\zeta^{GL_n}_T(\cdot|\theta_w)_{q \to q^2}) = \prod_{\varphi \in O(L)} p_{\varphi}(\varphi). \]
Next, notice that if \( \omega_{q^2} \) denotes the automorphism of \( \otimes \Lambda \) which sends 
\( p_n(\varphi) \to (q^2 \varphi - 1)/(t^2 \varphi - 1)p_n(\varphi) \), then 
\[
\omega_{q^2} \prod_{\varphi \in O(L)} P_{w(\varphi)} = \frac{\prod_{\varphi \in O(L)} P_{w(\varphi)}(\varphi)|_{T^F_w|_{q \to q^2}}}{|T^F_w|}
\]
and so 
\[
\text{ch}(\phi) = \omega_{q^2} \frac{1}{\psi_n(q^2)} \prod_{\varphi \in O(L)} \left( q_{\varphi}^{-n(\lambda(\varphi))} c_{\lambda(\varphi)}(q_{\varphi}^2, q_{\varphi}) \sum_{w \in S_{\lambda(\varphi)}} d_{\lambda(\varphi)}(w)(q_{\varphi})p_{w(\varphi)} \right)
\]
\[
= \omega_{q^2} \frac{1}{\psi_n(q^2)} \prod_{\varphi \in O(L)} \left( q_{\varphi}^{-n(\lambda(\varphi))} J_{\lambda(\varphi)}(q_{\varphi}, q_{\varphi}^2) \right)
\]
\[
= \frac{1}{\psi_n(q^2)} \prod_{\varphi \in O(L)} q_{\varphi}^{-n(\lambda(\varphi))} J_{\lambda(\varphi)}(q_{\varphi}, q_{\varphi}^2)
\]

\( \square \)

4.3. The Isometry Property. Next, it will be shown that the map \( \text{ch} \) is an isometry, up to a constant.

**Lemma 4.4.** The map \( \text{ch} \) satisfies 
\[
\langle \phi, \psi \rangle = q^{-n}|H^F|^2 \langle \text{ch}(\phi), \text{ch}(\psi) \rangle.
\]

**Proof.** It suffices to compute the norms of indicator functions. It is clear that 
\[
\langle I_{H^F g_\mu H^F}, I_{H^F g_\mu H^F} \rangle = |H^F g_\mu H^F|. \]

Now 
\[
\langle \text{ch}(I_{H^F g_\mu H^F}), \text{ch}(I_{H^F g_\mu H^F}) \rangle = \langle \text{ch}(I_{C_\mu}), \text{ch}(I_{C_\mu}) \rangle_{GL_n(F_q)}, q \rightarrow q^2,
\]
where \( C_\mu \) denotes the conjugacy class of \( GL_n(F_q) \) associated to \( \mu \). This can be shown by noting that 
\[
\text{ch}(I_{H^F g_\mu H^F}) = \text{ch}(I_{C_\mu})_{q \rightarrow q^2},
\]
and expanding into power sum symmetric functions, and then using the fact that 
\[
\langle p_\mu, p_\mu \rangle = \langle p_\mu, p_\mu \rangle_{GL_n(F_q)} = 1
\]
because the characteristic map for \( GL_n(F_q) \) is an isometry. Finally, 
\[
|H^F|^2 q^{-n} \frac{a_\mu(q^2)}{\alpha_\mu(q^2)} = |H^F| \frac{|GL_n(F_q)|_{q \rightarrow q^2} a_\mu(q^2)}{\alpha_\mu(q^2)}
\]
because 
\[
|\text{Sp}_{2n}(F_q)| = q^{n^2} \prod (q^{2i} - 1) \quad \text{and} \quad |\text{GL}_{2n}|_{q \rightarrow q^2} = q^{n^2-n} \prod (q^{2i} - 1),
\]
But then 
\[
|C_\mu|_{q \rightarrow q^2} = \frac{|GL_n(F_q)|_{q \rightarrow q^2} a_\mu(q^2)}{\alpha_\mu(q^2)}
\]
since \( a_\mu(q) \) is the size of the centralizer of an element in \( C_\mu \), and this gives 
\[
q^{-n}|H^F|^2 \langle \text{ch}(I_{H^F g_\mu H^F}), \text{ch}(I_{H^F g_\mu H^F}) \rangle = |H^F g_\mu H^F|
\]
using Proposition 4.1. This shows that $\text{ch}$ is an isometry up to the specified constant.

\[\square\]

**Remark 4.5.** The inner product on $\mathbf{C}[\text{Sp}_{2n}(F_q) \setminus \text{GL}_{2n}(F_q) / \text{Sp}_{2n}(F_q)]$ could be renormalized so that $\text{ch}$ is actually an isometry. This would also remove the constants appearing in Proposition 4.8 and so in some sense this inner product would be more natural although to avoid confusion with the usual one this is not done. These constants also appear in the characteristic map for the Gelfand pair $S_{2n}/B_n$, see [14, VII, §2].

Lemma 4.4 could also be proven by computing the norms of spherical functions. Strictly speaking, this is not necessary, but the computation helps illustrate the use of the dual variables so it is included.

To compute the analogue for the spherical functions, first $\langle \cdot, \cdot \rangle$ must be computed on the dual variables $x_{i,\varphi}$ (since the indicator functions are in terms of $f \in O(M)$ and the spherical functions are in terms of $\varphi \in O(L)$).

**Proposition 4.6.** For $\lambda : O(L) \rightarrow \mathcal{P}$,

\[
\langle p_\lambda, p_\lambda \rangle = z_\lambda \prod_{\varphi \in O(L)} \prod_i \frac{q_\varphi^{\lambda(\varphi)_i} - 1}{q_\varphi^{2\lambda(\varphi)_i} - 1}.
\]

**Proof.** Let $p_\lambda(\varphi)$ denote the symmetric function obtained by expanding $p_\lambda(\varphi)$ in terms of $p_\lambda(f)$ and taking the complex conjugate of all coefficients, extended multiplicatively. First note that

\[
\sum_{\varphi \in L_n} \bar{p}_n(\varphi) \otimes p_n(\varphi) = \sum_{\varphi \in L_n} \sum_{x, y \in M_n} \varphi(x)\varphi(y) p_n(x) \otimes p_n(y)
\]

\[
= (q^n - 1) \sum_{x \in M_n} \bar{p}_n(x) \otimes p_n(y).
\]

Then

\[
\sum_{\varphi \in L_n} \bar{p}_n(\varphi) \otimes p_n(\varphi) = \sum_{\varphi \in O(L), d(\varphi) | n} d(\varphi) p_n/d(\varphi) (\varphi) \otimes p_n/d(\varphi) (\varphi)
\]

and

\[
\sum_{x \in M_n} \bar{p}_n(x) \otimes p_n(x) = \sum_{f \in O(L), d(f) | n} d(f) p_n/d(f) (f) \otimes p_n/d(f) (f)
\]

and so multiplying by $\frac{q^n - 1}{n(q^n - 1)}$ and summing over all $n$ gives

\[
\sum_{n \geq 1} \frac{1}{n} \sum_{f \in O(M)} (q_\varphi^{2n} - 1) p_n(f) \otimes p_n(f) = \sum_{n \geq 1} \frac{1}{n} \sum_{\varphi \in O(L)} \frac{q_\varphi^{2n} - 1}{q_\varphi^n - 1} p_n(\varphi) \otimes p_n(\varphi).
\]

Finally, exponentiate both sides to obtain

\[
\sum_{\lambda} \frac{1}{z_\lambda} \left( \prod_{\varphi \in O(L)} \prod_i \frac{q_\varphi^{\lambda(\varphi)_i} - 1}{q_\varphi^{2\lambda(\varphi)_i} - 1} \right) p_\lambda \otimes p_\lambda = \sum_{\mu} \frac{1}{z_\mu} \left( \prod_{f \in O(M)} \prod_i (q_\varphi^{2\mu(f)_i} - 1) \right) p_\mu \otimes p_\mu
\]
and this power series identity implies that
\[
\langle p_{\lambda}, p_{\lambda} \rangle = z_{\lambda} \prod_{\varphi \in O(L)} \prod_{i} \frac{q^{\lambda(\varphi)_{i}}}{q^{2\lambda(\varphi)_{i}} - 1}.
\]

Now the norms of spherical functions may be computed as follows.

**Alternative proof of Lemma 4.4.** Note that for any spherical function, it is always the case that
\[
\langle \phi_{\lambda}, \phi_{\lambda} \rangle = \frac{|GL_{2n}(F_q)|}{d_{\lambda;\lambda}}
\]
(see e.g. [14, VII, §1]). Now compute
\[
\frac{|H^F|^2}{q^n} \langle \chi(\phi_{\lambda}), \chi(\phi_{\lambda}) \rangle = \frac{|H^F|^2}{q^n\psi_{n}(q^2)^2} \prod_{\varphi \in O(L)} q^{-2n(\lambda(\varphi)')} \langle J_{\lambda}(q, q^2), J_{\lambda}(q, q^2) \rangle.
\]
Because \( \langle J_{\lambda}(q, q^2), J_{\lambda}(q, q^2) \rangle = c_{\lambda}(q, q^2)c_{\lambda'}(q, q^2) = H_{\lambda;\lambda}(q) \), this is equal to
\[
|GL_{2n}(F_q)|\psi_{2n}(q)^{-1} \prod_{\varphi \in O(L)} q^{-n((\lambda(\varphi)\cup\lambda(\varphi))')} H_{\lambda;\lambda}(q)
\]
and finally the dimension formula
\[
d_{\lambda;\lambda} = \psi_{2n}(q) \prod_{\varphi \in O(L)} q^{-n((\lambda(\varphi)\cup\lambda(\varphi))')} H_{\lambda;\lambda}(q)^{-1}
\]
gives
\[
\frac{|H^F|^2}{q^n} \langle \chi(\phi_{\lambda}), \chi(\phi_{\lambda}) \rangle = \frac{|GL_{2n}(F_q)|}{d_{\lambda;\lambda}}
\]
as desired. \(\square\)

4.4. **Parabolic Induction.** Let \( G = GL_{2(n+m)} \) and \( H = Sp_{2(n+m)} \) and let \( L \) be the Levi subgroup \( GL_{2n} \times GL_{2m} \) embedded block diagonally. Note \( L \) is stable under \( \iota \) and is rational. Let \( P \) be the standard parabolic containing \( L \) consisting of block upper triangular matrices and let \( x \mapsto \mathcal{F} \) denote the canonical projection \( P \to L \). Importantly, note that with this choice of \( L \) and \( P \), \( H_{P} = H_{L} \). Then Proposition 3.1 applies and defines a function
\[
Ind_{\iota, P}^{\mathcal{F}} : C[H_L^F \setminus L^F / H_L^F] \to C[H^F \setminus G^F / H^F]
\]
which can be viewed as taking a \( Sp_{2n}(F_q) \) bi-invariant function \( f \) on \( GL_{2n}(F_q) \) and a \( Sp_{2n}(F_q) \) bi-invariant function on \( GL_{2m}(F_q) \) and producing a function, denoted \( f \circ g \), \( Sp_{2(n+m)}(F_q) \) bi-invariant on \( GL_{2(n+m)}(F_q) \). This can be done for any \( n, m \) and so defines a product on \( \bigoplus C[Sp_{2n}(F_q) \setminus GL_{2n}(F_q) / Sp_{2n}(F_q)] \).
Let $f \times g$ denote the function on $\text{GL}_{2n}(F) \times \text{GL}_{2m}(F)$ given by $f \times g(t_1, t_2) = f(t_1)g(t_2)$. Since $P$ is rational, Proposition 3.1 gives the formula
\[
f \times g(x) = |\text{Sp}_{2n}^F|^{-2} |\text{Sp}_{2m}^F|^{-2} \sum_{h,h' \in \text{Sp}_{2(n+m)}^F} (f \times g)(h \cdot h').
\]
This function is clearly $H^F$ bi-invariant, and the product is associative by transitivity of induction. The main theorem of this section establishes that the characteristic map is multiplicative with respect to this multiplication on $\oplus C[\text{Sp}_{2n}(F) \backslash \text{GL}_{2n}(F) / \text{Sp}_{2n}(F)]$.

**Theorem 4.7.** Let $f, g$ be $\text{Sp}_{2n}(F)$ and $\text{Sp}_{2m}(F)$ bi-invariant functions on $\text{GL}_{2n}(F)$ and $\text{GL}_{2m}(F)$ respectively. Then
\[
\text{ch}(f \times g) = \text{ch}(f) \text{ch}(g).
\]

**Proof.** It suffices to show that if $T_1$ and $T_2$ are rational maximal $\iota$-stable tori in $\text{GL}_{2n}$ and $\text{GL}_{2m}$ respectively, and $\theta_1, \theta_2$ are characters of $(T_1^{-1})^F, (T_2^{-1})^F$ then
\[
\zeta_{T_1}^{\text{GL}_n}(-|\theta_1)_{\iota\rightarrow q^2} \times \zeta_{T_2}^{\text{GL}_m}(-|\theta_2)_{\iota\rightarrow q^2} = \zeta_{(T_1^{-1})^F \times (T_2^{-1})^F}^{\text{GL}_{2(n+m)}}(-|\theta_1 \times \theta_2)_{\iota\rightarrow q^2}
\]
since the $\zeta_T^{G, \iota}(-|\theta)_{\iota\rightarrow q^2}$ form a basis.

Now let $L \subseteq P$ denote the Levi and parabolic subgroups defining the $\ast$ operation. Then the second part of Proposition 3.1 shows that it is equivalent to establish that
\[
\text{Ind}_{L \subseteq P}^{\text{GL}_{2(n+m)}} \left( \text{Ind}_{T_1 \subseteq B_1}^{\text{GL}_{2n}}(\theta_1) \times \text{Ind}_{T_2 \subseteq B_2}^{\text{GL}_{2m}}(\theta_2) \right)
\]
\[
= \text{Ind}_{T_1 \times T_2 \subseteq B_1 \times B_2}^{\text{GL}_{2(n+m)}}(\theta_1 \times \theta_2)
\]
as $|(T_1^{-1} \times T_2^{-1})^F| = |(T_1^{-1})^F||T_2^{-1})^F|$ is multiplicative. But this follows from transitivity of induction given by the first part of Proposition 3.1. \qed

### 4.5. Parabolic Restriction.

The restriction operation, which is adjoint to induction, is much simpler to define. If $f$ is $\text{Sp}_{2(n+m)}(F)$ bi-invariant on $\text{GL}_{2(n+m)}(F)$, the restriction to $L = \text{GL}_{2n} \times \text{GL}_{2m}$ with respect to the rational parabolic $P \supseteq L$ is defined by
\[
\text{Res}_{L \subseteq P}^{\text{GL}_{2(n+m)}}(f)(x) := \sum_{p \in P^F, \pi = x} f(p).
\]
This function is $H^F_L$ bi-invariant. Note that this definition coincides with the usual definition of parabolic restriction of class functions. The first observation is that this operation is the adjoint of parabolic induction.

**Proposition 4.8.** Let $G = \text{GL}_{2(n+m)}$, $H = \text{Sp}_{2(n+m)}$ and let $L = \text{GL}_{2n} \times \text{GL}_{2m}$. Pick some rational parabolic $P \supseteq L$. If $f$ is $H^F$ bi-invariant on $G^F$ and $g$ is $H^F_L$ bi-invariant on $L^F$, then
\[
\langle f, \text{Ind}_{L \subseteq P}^G g \rangle = |\text{Sp}_{2(n+m)}^F|^2 |\text{Sp}_{2n}^F|^{-2} |\text{Sp}_{2m}^F|^{-2} \langle \text{Res}_{L \subseteq P}^F g, f \rangle.
\]
Proof. Compute
\[
\langle f, \text{Ind}^G_{L \subseteq P}(g) \rangle = \sum_{x \in G^F} f(x) \text{Ind}^G_{L \subseteq P}(g)(x)
\]
\[
= |\text{Sp}^F_{2n}|^{-2} |\text{Sp}^F_{2m}|^{-2} \sum_{x \in G^F} \sum_{h, h' \in H^F \atop h x h' \in P^F} f(h x h') g(h x h')
\]
\[
= |H^F|^2 |\text{Sp}^F_{2n}|^{-2} |\text{Sp}^F_{2m}|^{-2} \sum_{p \in P^F} f(p) \overline{g(p)}
\]
where the fact that \( f \) is \( H^F \) bi-invariant is used. On the other hand,
\[
\langle \text{Res}^G_{L \subseteq P}(f), g \rangle = \sum_{x \in L^F} \sum_{p \in P^F, \overline{p}=x} f(p) \overline{g(x)}
\]
\[
= \sum_{p \in P^F} f(p) \overline{g(p)}
\]
\[\square\]

Remark 4.9. Parabolic restriction also arises as the function analogue of the parabolic restriction functor defined by Grojnowski [9]. Since the properties of parabolic restriction that are needed can be proven by elementary means, this fact will not be invoked but extensions to the situation where \( P \) is not a rational parabolic (for instance, to \( U_{2n}(\mathbb{F}_q) / \text{Sp}_{2n}(\mathbb{F}_q) \)) would require the use of this.

Remark 4.10. The ring of symmetric functions has a Hopf algebra structure, and it turns out that \( \text{ch} \) preserves this structure as well, with restriction defining the coproduct for \( \otimes \mathbb{C} \text{[Sp}_{2n}(\mathbb{F}_q) \setminus \text{GL}_{2n}(\mathbb{F}_q), \text{Sp}_{2n}(\mathbb{F}_q)] \). See [27] for a development of the characteristic map from the Hopf algebra perspective (for \( S_n \) and \( \text{GL}_n(\mathbb{F}_q) \)).

To see that \( \text{ch} \) preserves the coproduct, first note that restriction naturally extends to a coproduct, by summing over all \( n, m \). Then as the coproduct and product are adjoint for both \( \otimes \Lambda \) and \( \oplus \mathbb{C} \text{[Sp}_{2n}(\mathbb{F}_q) \setminus \text{GL}_{2n}(\mathbb{F}_q), \text{Sp}_{2n}(\mathbb{F}_q)] \) and \( \text{ch} \) already preserves the product and inner product structure, it automatically preserves the coproduct structure as well.

Actually, it is not hard to see that after defining the product and coproduct structure, that \( \oplus \mathbb{C} \text{[Sp}_{2n}(\mathbb{F}_q) \setminus \text{GL}_{2n}(\mathbb{F}_q), \text{Sp}_{2n}(\mathbb{F}_q)] \) is a positive self-dual Hopf algebra, and so by the structure theory for such algebras, it must be isomorphic to a tensor product of symmetric functions. A proof could be given following this spirit although developing the theory of parabolic induction seems unavoidable.

5. Macdonald Littlewood-Richardson coefficients

5.1. Littlewood-Richardson coefficients. In this section, Theorem 1.1 and Theorem 1.2 on positivity and vanishing of the Macdonald Littlewood-Richardson coefficients for parameters \( (q, q^2) \) are proven. The argument
for Theorem 1.1 follows that of Macdonald [14, VII, §2] while the proof of Theorem 1.2 follows that of Bergeron and Garsia [2]. Both proofs were for the Jack polynomials with parameter 2. Slightly more is needed as unlike in the Weyl group case, the restriction operation is not simply function restriction. In their works, the Gelfand pair $S_{2n}/B_n$ was used, and the results of this section may be viewed as a $q$-deformation of their result. The Hecke algebra of unipotent class functions on $GL_n(F_q)$ is a $q$-deformation of the group algebra for $S_n$. Similarly, the only functions considered in this section will be unipotent spherical functions (that is, the functions spanned by $\phi_\lambda$ for $\lambda$ non-trivial only on the trivial character).

**Definition 5.1.** Define the Macdonald Littlewood-Richardson coefficients or $(q,t)$ Littlewood-Richardson coefficients $f_{\mu \nu}^{\lambda}(q,t)$ by

$$J_\mu(x;q,t)J_\nu(x;q,t) = \sum_\lambda f_{\mu \nu}^{\lambda}(q,t)J_\lambda(x;q,t).$$

This definition differs slightly from that in [14] in that the integral version of the Macdonald polynomials are used, but they differ from the original definition by a positive scalar and so the positivity and vanishing of either definition is equivalent. Note that these coefficients are rational functions of $q$ and $t$ (see e.g. [26]).

The goal of this section is to prove Theorem 1.1 and Theorem 1.2.

**Remark 5.2.** Note that the Jack polynomial positivity result cannot be derived from Theorem 1.1 as the restriction that $q$ is a prime power prevents a limit $q \to 1$ from being taken.

There are some trivial extensions of Theorem 1.1 and Theorem 1.2. In particular, as the homomorphism $\omega_{q,q^2}$ satisfies $\omega_{q,q^2}J_\lambda(q,q^2) = J_{\lambda'}(q^2,q)$ and $J_\lambda(x;q^{-1},t^{-1}) = (-1)^{|\lambda|}q^{-n(\lambda')}t^{-n(\lambda')-|\lambda|}J_\lambda(x;q,t)$, both results extend to parameters $(q^2,q)$, $(q^{-1},q^2)$ and $(q^{-2},q^{-1})$.

### 5.2. Positive-definite functions

For any finite group $G$, a positive-definite function on $G$ is a function $f: G \to \mathbb{C}$ such that the matrix indexed by $G$ whose $x,y$ entry is $f(x^{-1}y)$ is a positive-definite matrix. A key fact is that any bi-invariant positive-definite function is a positive linear combination of spherical functions.

**Proposition 5.3** ([14, VII, §1]). Let $G/H$ be a Gelfand pair. Any spherical function is positive-definite, and moreover if $f$ is an $H$ bi-invariant function on $G$ that is positive-definite, then for any spherical function $\phi$ on $G$, $\langle f, \phi \rangle \geq 0$.

If $\alpha : G \to H$ is a group homomorphism, then given functions $f : G \to \mathbb{C}$ and $g : H \to \mathbb{C}$, define the functions $\alpha^* g : G \to \mathbb{C}$, or the pullback, and
\( \alpha_* f : H \to C \), or the pushforward, by
\[
\alpha^* g(x) = g(\alpha(x)) \\
\alpha_* f(x) = \sum_{\alpha(y) = x} f(y).
\]

From the definition, it’s clear that \((\alpha \circ \beta)^* = \beta^* \circ \alpha^* \) and \((\alpha \circ \beta)_* = \alpha_* \circ \beta_* \). There are two basic properties that are needed, namely that pullback and pushforward are adjoint and that pushforward and pullback preserve positive-definite functions.

**Lemma 5.4.** If \( \alpha : G \to H \), and \( f : H \to C \) and \( g : G \to C \), then \( \langle \alpha^* f, g \rangle = \langle f, \alpha_* g \rangle \).

**Proof.** Note that
\[
\langle \alpha^* f, g \rangle = \sum_{x \in G} f(\alpha(x))g(x)
\]
and
\[
\langle f, \alpha_* g \rangle = \sum_{y \in H} f(y) \sum_{\alpha(x) = y} g(x)
\]
which are equal. \( \square \)

**Lemma 5.5.** If \( f : G \to C \) is positive-definite, and \( \alpha : G \to H \) and \( \beta : H \to G \) are group homomorphisms, then \( \alpha_* f \) and \( \beta^* f \) are also positive-definite.

**Proof.** Note positive definiteness is equivalent to having
\[
\sum_{x,y \in G} f(x^{-1}y)h(x)h(y) \geq 0
\]
for all functions \( h : G \to C \). Then
\[
\sum_{x,y \in H} \beta^* f(x^{-1}y)h(x)h(y) = \sum_{x,y \in G} f(x^{-1}y)\beta_* h(x)\beta_* h(y) \geq 0
\]
so \( \beta^* f \) is positive-definite.

For \( \alpha_* f \), note that \( \alpha \) can always be factored as surjection and an injection. If \( \alpha \) is surjective, then
\[
\sum_{x,y \in H} \alpha_* f(x^{-1}y)h(x)h(y) = \ker \alpha^{-1} \sum_{x,y \in G} f(x^{-1}y)\alpha^* h(x)\alpha^* h(y) \geq 0
\]
and if \( \alpha \) is injective, then
\[
\sum_{x,y \in G} \beta_* f(x^{-1}y)h(x)h(y) = \sum_{x,y \in G} f(x^{-1}y)h(x)h(y) \\
= \sum_{z \in G/H} \sum_{x,y \in H} f(x^{-1}y)h(zx)h(zy) \geq 0
\]
where each summand is non-negative by considering the function \( x \mapsto h(zx) \) on \( H \). \( \square \)
Lemma 5.6. Let $f$ be a positive-definite $\text{Sp}_{2(n+m)}(\mathbb{F}_q)$ bi-invariant function on $\text{GL}_{2(n+m)}(\mathbb{F}_q)$, and let $L \subseteq P$ be the standard $\text{GL}_{2n} \times \text{GL}_{2m}$ Levi subgroup and parabolic. Then $\text{Res}_{L \subseteq P}^{\text{GL}_{2(n+m)}}(f)$ is a positive-definite function on $\text{GL}_{2n}(\mathbb{F}_q) \times \text{GL}_{2m}(\mathbb{F}_q)$.

Proof. Let $i : P^F \to \text{GL}_{2(n+m)}^F$ denote the inclusion map and $\text{pr} : P^F \to L^F$ denote the canonical projection. Then

$$\text{Res}_{L \subseteq P}^G(f) = \text{pr}_* i^*(f),$$

and so if $f$ is a positive-definite function, then by Proposition 5.5 so is $\text{Res}_{L \subseteq P}^G(f)$. □

Corollary 5.7. The parabolic induction of positive-definite bi-invariant functions is positive-definite.

Proof. Since parabolic induction and restriction are adjoints, if $f$ is a positive-definite bi-invariant function on $L^F$, then for any spherical function $\phi$ on $G^F$

$$\langle \text{Ind}_{L \subseteq P}^G(f), \phi \rangle = |\text{Sp}_{2(n+m)}^F|^2 |\text{Sp}_{2n}^F|^{-2} |\text{Sp}_{2m}^F|^{-2} \langle f, \text{Res}_{L \subseteq P}^G \phi \rangle$$

is non-negative by expanding both arguments in terms of spherical functions. Thus, $\text{Ind}_{L \subseteq P}^G(f)$ is positive-definite. □

5.3. Proof of Theorem 1.1 and Theorem 1.2 With the characteristic map, statements about the Macdonald polynomials may be transferred to statements about spherical functions. This machinery allows for simple proofs of Theorem 1.1 and Theorem 1.2 in the spirit of [2].

Proof of Theorem 1.1 Fix $\mu, \nu, \lambda$ and let $|\mu| = m$ and $|\nu| = n$. Assume that $|\lambda| = m + n$ as otherwise $f^\lambda_{\mu \nu} = 0$. Note that $f^\lambda_{\mu \nu}$ is a positive scalar multiple of $(J_\lambda, J_\mu, J_\nu)$, and so also a positive scalar multiple of $\langle \phi_\lambda, \phi_\mu \ast \phi_\nu \rangle$ by Proposition 4.3 and Theorem 4.7. Here $\lambda, \mu, \nu$ are viewed as functions $O(L) \to P$ with value $\lambda, \mu, \nu$ respectively at the trivial character and 0 otherwise (or in other words, identifying $A$ with the factor in $\otimes A$ associated to the trivial character which amounts to working only with unipotent spherical functions).

Now by Proposition 5.3 it is enough to show that $\text{Ind}_{L \subseteq P}^G(\phi_\mu \ast \phi_\nu)$ is a positive-definite bi-invariant function on $\text{GL}_{2(m+n)}(\mathbb{F}_q)$. But since $\phi_\mu \ast \phi_\nu$ is positive-definite, then by the corollary so is $\text{Ind}_{L \subseteq P}^G(\phi_\mu \ast \phi_\nu)$. □

Remark 5.8. This proof gives a representation-theoretic interpretation of the $(q, q^2)$ Littlewood-Richardson coefficients (with $q$ an odd prime power) in terms of multiplicities of spherical functions in the bi-invariant parabolic induction, similar to how the classical Littlewood-Richardson coefficients can be viewed as the multiplicities of irreducible representations in the Young induction.
Proof of Theorem 5.3. Assume that $c_{\mu\nu}^\lambda = 0$. Take $|\mu| = m$, $|\nu| = n$ and assume $|\lambda| = m + n$ as otherwise $f_{\mu\nu}^\lambda(q, q^2) = 0$. Then as in the proof of Theorem 1.1, it is equivalent to show that $\langle \phi_{\lambda}, \phi_{\mu} \ast \phi_{\nu} \rangle = 0$ since it’s a positive scalar multiple of $f_{\mu\nu}^\lambda(q, q^2)$ (again work with unipotent spherical functions).

Let $\text{Av}_k$ denote the averaging operation sending a function $f$ on $\text{GL}_{2k}(F_q)$ to $x \mapsto |H|^k \sum_{h \in \text{Sp}_{2k}(F_q)} f(hxh')$. Let $G = \text{GL}_{2(n+m)}$, $H = \text{Sp}_{2(n+m)}$ and $L \subseteq P$ denote the Levi and parabolic defining $. Then

\[
\langle \text{Res}_L^G \left( \text{Av}_{n+m}(\chi_{\lambda\mu\nu}), \text{Av}_n \times \text{Av}_m(\chi_{\mu\nu} \times \chi_{\nu\lambda\mu}) \right) \rangle = \langle \text{pr}_s \ast \text{Av}_{n+m}(\chi_{\lambda\mu\nu}), \chi_{\mu\nu} \times \chi_{\nu\lambda\mu} \rangle = \langle \text{Av}_{n+m}(\chi_{\lambda\mu\nu}), i_s \ast \text{pr}_s^\ast(\chi_{\mu\nu} \times \chi_{\nu\lambda\mu}) \rangle = \text{Av}_{n+m}(\chi_{\lambda\mu\nu} \cdot i_s \ast \text{pr}_s^\ast(\chi_{\mu\nu} \times \chi_{\nu\lambda\mu}))(1)
\]

where in the last line the multiplication on functions is convolution.

Now

\[
\chi_{\lambda\mu\nu} \cdot i_s \ast \text{pr}_s^\ast(\chi_{\mu\nu} \times \chi_{\nu\lambda\mu})(x) = \text{Tr} \left( \rho_{\lambda\mu\nu}(x) \sum_{y \in P^F} \rho_{\lambda\mu\nu}(y^{-1}) \text{pr}_s^\ast(\chi_{\mu\nu} \times \chi_{\nu\lambda\mu})(y) \right)
\]

where $\rho_{\lambda\mu\nu}$ denotes the corresponding representation and this is 0 unless

\[
\langle \chi_{\lambda\mu\nu}, \text{pr}_s^\ast(\chi_{\mu\nu} \times \chi_{\nu\lambda\mu}) \rangle_{P^F} \neq 0,
\]

because $\text{pr}_s^\ast(\chi_{\mu\nu} \times \chi_{\nu\lambda\mu})$ is an irreducible character. By Frobenius reciprocity

\[
\langle \chi_{\lambda\mu\nu}, \text{Ind}_L^G(\chi_{\mu\nu} \times \chi_{\nu\lambda\mu}) \rangle \neq 0,
\]

where $\text{Ind}_L^G$ denotes the standard parabolic induction for $\text{GL}_n$. This happens exactly when $c_{\mu\nu}^\lambda$ is non-zero.

Here, the theorem can be extended to hold for all $q$ because $f_{\mu\nu}^\lambda(q, q^2)$ is a rational function in $q$ (see e.g. [25] for example) and so if it vanishes at infinitely many points it vanishes everywhere. \hfill \Box

5.4. Further positivity conjectures. It is known that the $(q, t)$ Littlewood-Richardson coefficients are rational functions in $q$ and $t$ (see e.g. [26]). This suggests that not only are these coefficients positive for all positive values of $q$, but that some normalization should be a polynomial in $q$ with positive integer coefficients.

Conjecture 5.9. The $(q, q^2)$ Littlewood-Richardson coefficients are ratios of polynomials in $q$ with positive integer coefficients.

This conjecture is equivalent to the statement that either $f_{\mu\nu}^\lambda$, vanishes or is strictly positive for all $q > 0$. This has been checked using Sage for all $f_{\mu\nu}^\lambda(q, q^2)$ with $|\mu|, |\nu| \leq 4$. The results above clearly imply that as a function of $q$, $f_{\mu\nu}^\lambda(q, q^2)$ is eventually positive for large enough $q$, but showing positivity of coefficients seems to be a much harder task. Another interesting
question is whether these coefficients have some interesting combinatorial interpretation (after some suitable renormalization to clear the denominator).

There is also an extension to general parameters \((q, t)\). It has been noticed that for \(q, t \in (0, 1)\) (or equivalently \((1, \infty)\)) \(f_{\mu \nu}^\lambda(q, t)\) always seems to be non-negative (see for example [15]). It is not the case that these coefficients are non-negative if \(q < 1\) and \(t > 1\), so they cannot be written as a ratio of polynomials with positive integer coefficients. It is unclear what (if any) combinatorial meaning these coefficients encode.

6. Computation of Spherical Function Values

In this section, the values of spherical functions on the double coset of non-symplectic transvections is computed. Similar computations could be done for the double cosets generated by \(\text{diag}(a, 1, \ldots, 1)\) for \(a \in F_q^*\). This section should be seen as an application of the characteristic map to use the Pieri rule for Macdonald polynomials to compute spherical function values.

**Proposition 6.1** ([14, VI, §6]). Let

\[
\psi'_{\lambda/\mu} := \prod_{s \in C_{\lambda/\mu} \setminus R_{\lambda/\mu}} \frac{b_{\lambda}(s; q, t)}{b_{\mu}(s; q, t)},
\]

where

\[
b_{\lambda}(s; q, t) := \frac{1 - q^{a(s)}t^{l(s)+1}}{1 - q^{a(s)+1}t^{l(s)}},
\]

and where \(C_{\lambda/\mu}\) denotes the columns of \(\lambda\) intersecting \(\lambda/\mu\) and similarly \(R_{\lambda/\mu}\) but for rows. Then

\[
P_{\mu}(x; q, t)e_r(x) = \sum_\lambda \psi'_{\lambda/\mu} P_{\lambda}(x; q, t),
\]

where the sum is over partitions \(\lambda\) such that \(\lambda \setminus \mu\) is a vertical strip with \(r\) boxes.

First the value at the identity will be computed as a similar computation shows up in the transvection computation, even though the value is already known to be 1.

Define \(\delta\) as the specialization homomorphism on \(\otimes \Lambda\) given by

\[
\delta(p_n(\varphi)) = \frac{1}{q^n - 1}.
\]

The following lemma is essentially proven in [12].

**Lemma 6.2.** For any \(F \in \otimes \Lambda\), \(\langle F, e_n(f_1) \rangle = \delta(\omega_{q, q^2} F)\).

**Proof.** Since \(\langle F, e_n(f_1) \rangle = \langle \omega_{q, q^2} F, e_n(f_1) \rangle_{\text{GL}_n}\) and \(\omega_{q, q^2}\) is invertible it suffices to check that \(\langle F, e_n(f_1) \rangle_{\text{GL}_n} = \delta(F)\). Since both sides are linear in \(F\), it is enough to check on a basis, which is done in [12, Equation 2.5]. \(\square\)
The value of $\phi_\lambda(1)$ can be computed as follows. First note that $P_\lambda = e_n$ when $\lambda = (1^n)$ and so

$$\phi_\lambda(1) = |H^F|^{-1}\langle \phi_\lambda, I_{H^F} \rangle$$

can be computed by using the characteristic map and Lemma 4.4, giving

$$q^{-n}\text{Sp}_{n}(F_q)|\psi_n(q^2)^{-1}q^{-(n^2-n)} \prod_{\varphi \in O(L)} q_{\varphi^{-1}(\lambda(\varphi))}(J_\lambda(q, q^2), e_n(f_1))$$

$$= \prod_{\varphi \in O(L)} q_{\varphi^{-1}(\lambda(\varphi))}(J_\lambda(q, q^2), e_n(f_1)).$$

Using Lemma 6.2 and the fact that $\omega_{q^2} J_\lambda(q, q^2) = J_{\lambda'}(q^2, q)$ along with $\delta(J_\lambda(q^2, q)) = \prod_{s \in \lambda(\varphi)} q_{\varphi(s)}^{\omega(s)}$ [14, VI, §8], the inner product can be computed giving

$$\prod_{\varphi \in O(L)} q_{\varphi^{-1}(\lambda(\varphi))}\delta(J_{\lambda'}(q^2, q))$$

$$= \prod_{\varphi \in O(L)} q_{\varphi^{-1}(\lambda(\varphi))} \prod_{s \in \lambda(\varphi)} q_{\varphi(s)}^{\omega(s)}$$

$$= 1$$

The analogous computation of $\phi_\lambda(I_{H^F g_1 H^F})$ where $\mu(f_1) = (21^{n-2})$ and 0 otherwise requires an additional lemma.

**Lemma 6.3.** We have

$$e_{n-1}(f_1)e_1(f_1) = \sum_{||\lambda||=n} \sum_{\lambda_0} \delta(J_{\lambda_0}(q^2, q)) c_{\lambda_0}(q, q^2) c_{\lambda_0}(q, q^2)(1 - q) \psi_{\lambda/\lambda_0} \tilde{J}_\lambda(q, q^2),$$

where the second sum is over all partition-valued functions with $||\lambda_0|| = n - 1$ obtained by removing one box from some $\lambda(\varphi)$ with $d(\varphi) = 1$, and $\tilde{J}_\lambda(q, q^2)$ denotes the dual basis to $J_\lambda(q, q^2)$ under the inner product.

**Proof.** First, note that $e_k(f_1) = \sum_{||\lambda||=n} \delta(J_{\lambda_0}(q^2, q)) \tilde{J}_\lambda(q, q^2)$ which is an easy consequence of Lemma 6.2. Thus

$$e_{n-1}(f_1)e_1(f_1) = \left( \sum_{||\lambda_1||=n-1} \delta(J_{\lambda_1}(q^2, q)) \tilde{J}_{\lambda_0}(q, q^2) \right) \left( \sum_{||\lambda_2||=1} \delta(J_{\lambda_2}(q^2, q)) \tilde{J}_{\lambda_0}(q, q^2) \right).$$

There are exactly $q - 1$ partition valued functions $\lambda$ with $||\lambda|| = 1$, which give $e_1(\varphi)$ for $\varphi \in L_1$ as the polynomials $J_{\lambda_2}(q, q^2)$. Thus, apply Pieri’s rule for $r = 1$, and so $C_{\lambda_2/\mu}$ consists of the column that is added, and similarly for the row. The arm/leg lengths in $\lambda$ are exactly one more than in $\mu$ because of the added box, and so after relabeling $\lambda_1$ with $\lambda_0$

$$e_{n-1}(f_1)e_1(f_1) = \sum_{||\lambda||=n} \sum_{\lambda_0} \delta(J_{\lambda_0}(q^2, q)) c_{\lambda_0}(q, q^2) c_{\lambda_0}(q, q^2)(1 - q) \psi_{\lambda/\lambda_0} \tilde{J}_\lambda(q, q^2),$$

$$\sum_{||\lambda||=n} \sum_{\lambda_0} \delta(J_{\lambda_0}(q^2, q)) c_{\lambda_0}(q, q^2) c_{\lambda_0}(q, q^2)(1 - q) \psi_{\lambda/\lambda_0} \tilde{J}_\lambda(q, q^2),$$
Proposition 6.4. Let \( \hat{\lambda} \) from the scaling \( \varphi \) otherwise. Then
\[
\phi_\lambda(g_\mu) = q^{2n-2}(q^2 - 1)\left(\sum_{\lambda_0} \frac{c'_\lambda(q, q^2)\psi_\lambda'_{\lambda_0}}{c_{\lambda_0}(q, q^2)(1 - q)} q^n(\lambda_0) - n(\lambda') - \frac{q^{2n} - 1}{q^{2n-2}(q^2 - 1)}\right)
\]
for all spherical functions \( \phi_\lambda \) where \( \lambda_0 \) is obtained from \( \lambda \) by removing a single box from some \( \lambda(\varphi) \) with \( d(\varphi) = 1 \).

Proof. Note that
\[
\phi_\lambda(g_\mu) = \frac{1}{\langle H^F g_\mu H^F \rangle} \langle \phi_\lambda, I_{H^F g_\mu H^F} \rangle
\]
by Lemma \[6.4\] From \[12\ Eq. 2.4\],
\[
P_{\mu(f_1)}(f_1; q) = e_{n-1}(f_1)e_1(f_1) - \left(\sum_{i=0}^{n-1} q^i\right) e_n(f_1).
\]
Lemma \[6.2\] can handle the \( e_n(f_1) \) term, and Lemma \[6.3\] will give the other term. Note that although the notation \( c'_\lambda \), and other functions indexed by partitions, is used for partition-valued functions by taking a product over the domain, in almost all cases due to cancellation only one partition will be relevant.

Now evaluate the desired inner product, obtaining
\[
\langle J_\lambda(q, q^2), P_{\mu(f_1)}(f_1, q^{-2}) \rangle
\]
\[
= \sum_{\lambda_0} \frac{\delta(J_{\lambda_0}(q^2, q))c'_\lambda(q, q^2)\psi_\lambda'_{\lambda_0}}{c_{\lambda_0}(q, q^2)(1 - q)} - \frac{q^{2n} - 1}{q^{2n-2}(q^2 - 1)} \delta(J_\lambda(q^2, q))
\]
which gives the desired result with the remaining factors since
\[
\prod_{\varphi \in O(L)} q^{n(\lambda_0(\varphi))} = \delta(J_{\lambda_0}(q^2, q)).
\]

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