Total mean curvature of the boundary and nonnegative scalar curvature fill-ins

Yuguang Shi, Wenlong Wang, and Guodong Wei

Abstract. In the first part of the paper, we get some estimates for the supremum of the total mean curvature of boundaries of domains with nonnegative scalar curvature, and discuss its relationship with the positive mass theorem of asymptotically flat (hyperbolic) manifolds. In the second part of the paper, we prove the extensibility of an arbitrary boundary metric to a positive scalar curvature (PSC) metric inside for a compact manifold with nonempty boundaries which completely solve an open problem due to Gromov (see Question 1.1). The results in this paper also provide some partially affirmative answers to Gromov’s conjectures formulated in [15] (see Conjecture 1.1, Conjecture 1.2 below).

1. Introduction

One of fundamental problems in Riemannian geometry is to study the relationship between the geometry of boundaries with that of interior part for compact manifolds with nonempty boundaries and with nonnegative scalar curvature. However, people even do not know the extensibility of an arbitrary boundary metric to a positive scalar curvature (PSC) metric inside. Indeed, M.Gromov raised the following question.

Question 1.1. [15], p.31] Let $X$ be a compact manifold with non-empty boundary $Y = \partial X$ and let $\gamma$ be a smooth Riemannin metric on $Y$. Does $\gamma$ always extend to a Riemannin metric $g$ on $X$ with positive scalar curvature?

Inspired by the arguments in the proof of Theorem 1.4 in [30], we are able to solve Question 1.1 completely. Namely,

Theorem 1.1. Let $X$ be a compact manifold with non-empty boundary $Y = \partial X$. Then any metric $\gamma$ on $Y$ can be extended to a Riemannin metric $g$ on $X$ with positive scalar curvature.

2010 Mathematics Subject Classification. Primary 53C20; Secondary 83C99.
Yuguang Shi, Guodong Wei are partially supported by NSFC 11671015 and 11731001. Wenlong Wang is partially supported by NSFC 11671015, 11701326.
Note that there is no mean curvature requirements in Theorem 1.1, we find that pointwisely large mean curvature rules out such extensions for certain topological types of boundaries. To state our result, we need

**Definition 1.1.** Let $C_n (n \geq 2)$ be the class of $n$-dimensional closed manifolds that can be smoothly embedded into $\mathbb{R}^{n+1}$.

By definition, any manifold in $C_n$ is orientable and null-cobordant. Actually, $C_n$ contains certain nontrivial differential topological types of closed manifolds. For example, $S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k} \in C_{n_1+n_2+\cdots+n_k}$.

Bartnik data $(\Sigma^{n-1}, \gamma, H)$ consists of an orientable closed null-cobordant Riemannian manifold $(\Sigma^{n-1}, \gamma)$ and a given smooth function $H$ on $\Sigma^{n-1}$, we say it admits an nonnegative scalar curvature (NNSC) fill-in if there is a compact Riemannian manifold $(\Omega^n, g)$ with boundary of scalar curvature $R \geq 0$, and an isometry $X: (\Sigma^{n-1}, \gamma) \mapsto (\partial\Omega^n, g|_{\partial\Omega^n})$ so that $H = H_g \circ X$ on $\Sigma$, where $H_g$ is the mean curvature of $\partial\Omega^n$ in $(\Omega^n, g)$ with respect to the outward unit normal vector. One can define fill-ins with scalar curvature $R \geq \sigma > -\infty$ similarly.

**Theorem 1.2.** Suppose $\Sigma^{n-1} \in C_{n-1} (3 \leq n \leq 7)$. For any metric $\gamma$ on $\Sigma^{n-1}$, there is a constant $h_0(\Sigma^{n-1}, \gamma)$ which depends only on $(\Sigma^{n-1}, \gamma)$, such that for any smooth function $H$ on $\Sigma^n$ satisfying $H > h_0(\Sigma^{n-1}, \gamma)$, $(\Sigma^{n-1}, \gamma, H)$ admits no fill-in of nonnegative scalar curvature.

Total mean curvature plays important roles in such fill-ins. The following conjecture due to M. Gromov (see p.23 in [15]) which is on the upper bounds of the total mean curvature of a fill-in with scalar curvature having a lower bound, i.e.

**Conjecture 1.1.** Let $\sigma > -\infty$ be a given constant, $X^n$ be a $n$-dimensional compact Riemannian manifold with scalar curvature $R \geq \sigma$, Then the integral of mean curvature $H$ (w.r.t the outward unit normal vector here and in the sequel) of the boundary $Y = \partial X$ is bounded by

$$\int_Y H d\mu \leq C,$$

where $C$ is a constant depending only on $(Y, \gamma)$ and $\sigma$, and $\gamma$ is the induced metric on $Y$ from ambient Riemannian manifold $X$.

There are several interesting work provided affirmative supporting to this conjecture. For instance, in [17, 18, 20, 21], various boundedness of mean curvature of the boundaries of compact manifolds with non negative scalar curvature (NNSC) were obtained when $n = 3$, especially, Conjecture 1.1 was verified when $Y$ is a topological $S^2$ and $H > 0$ in [20]; in [16], Gromov himself got an upper bound estimate for the infimum of the mean curvature of boundary of spin compact manifolds with NNSC; in [30], under some conditions, we got upper bounds for the total mean curvature of boundaries of high dimensional
compact manifolds with NNSC which relates to Conjecture\[\text{[11]}\]. We are particularly interested in nonnegative scalar curvature (NNSC) fill-ins in this paper.

To do that, we need the following notions, one of which was introduced in \[\text{[20]}\] when \(n = 3\).

**Definition 1.2.** For an orientable closed null-cobordant Riemannian manifold \((\Sigma^{n-1}, \gamma)\). Define \(\Lambda_+ (\Sigma^{n-1}, \gamma)\) by

\[
\Lambda_+ (\Sigma^{n-1}, \gamma) = \sup \left\{ \int \Sigma H \, d\mu_\gamma \mid H > 0, (\Sigma^{n-1}, \gamma, H) \text{ admits a NNSC fill-in} \right\}.
\]

Or more generally, for \(\kappa > -\infty\), we define

\[
\Lambda_+ (\Sigma^{n-1}, \gamma, \kappa) = \sup \left\{ \int \Sigma H \, d\mu_\gamma \mid (\Sigma^{n-1}, \gamma, H) \text{ admits a fill-in with} \right\}
\]

\[
R(g) \geq n(n-1)\kappa,
\]

and then by the definition, we see that \(\Lambda_+ (\Sigma^{n-1}, \gamma) = \Lambda_+ (\Sigma^{n-1}, \gamma, 0)\), and for simplicity, we just denote it as \(\Lambda_+ (\Sigma^{n-1}, \gamma)\). It is easy to see that for any \(\kappa \leq 0\),

\[
\Lambda_+(\Sigma^{n-1}, \gamma, \kappa) \geq \Lambda_+(\Sigma^{n-1}, \gamma) .
\] (1)

We find that \(\Lambda_+(\Sigma^{n-1}, \gamma, \kappa)\) has a deep relationship with the positive mass theorem (PMT) on asymptotically hyperbolic (AH) manifolds or asymptotically flat (AF) manifolds. Namely, Let \((S^{n-1}, \gamma_{std})\) be the standard unit sphere, we are able to show

**THEOREM 1.3.** Let \((M^n, g)\) be an asymptotically flat manifold with nonnegative scalar curvature, then \(m_{\text{ADM}}(M^n, g) \geq 0\) if and only if \(\Lambda_+ (S^{n-1}, \gamma_{std}) = (n-1)\omega_{n-1}\), and \(m_{\text{ADM}}(M^n, g) = 0\) if and only if \((M^n, g)\) is isometric to \(R^n\), here and in the sequel \(\omega_{n-1} = \text{Vol}(S^{n-1})\).

Combine with the above Theorem \[\text{1.3}\], Corollary 2.1 in \[\text{[22]}\], and Theorem 1.2 in \[\text{[23]}\], it is interesting to see that

**COROLLARY 1.1.** \(\Lambda_+ (S^{n-1}, \gamma_{std})\) can be attained, which must be the unit ball \(B^n\) in \(R^n\) in this case, if and only if the PMT holds on AF manifolds.

For the definition of AF manifolds and ADM mass \(m_{\text{ADM}}(M^n, g)\), please see Definition \[\text{5.1}\] and Definition \[\text{5.2}\] respectively. We also have the following:

**THEOREM 1.4.** Suppose PMT holds on AH manifolds, then PMT is true for AF manifolds, i.e., let \((M^n, g)\) be an AF manifold with nonnegative scalar curvature, then \(m_{\text{ADM}}(M^n, g) \geq 0\) and \(m_{\text{ADM}}(M^n, g) = 0\) if and only if \((M^n, g)\) is isometric to \(R^n\).
For the definition of AH manifolds, see Definition 5.3 and the exact meaning of “PMT holds on AH manifolds” thereafter.

A byproduct of the proof of Theorem 1.4 is the more delicate relation, i.e.,

$$\Lambda_{+, -1}(S^{n-1}, \kappa^2\gamma_{std}) = (n - 1)\sqrt{1 + \kappa^2}\omega_{n-1},$$

for some $$\kappa_0 > 0$$, then the same equality is true for all $$\kappa \leq \kappa_0$$, and implies $$\Lambda_{+}(S^{n-1}, \gamma_{std}) = (n - 1)\omega_{n-1}$$ which is equivalent to PMT on AF manifolds in turn. If PMT on AH manifolds is true, then

$$\Lambda_{+, -1}(S^{n-1}, \kappa^2\gamma_{std}) = (n - 1)\sqrt{1 + \kappa^2}\omega_{n-1},$$

for all $$\kappa > 0$$. We will investigate the opposite direct relation elsewhere.

With these things in mind, it may be important to find an explicit estimate for $$\Lambda_{+}(\Sigma^{n-1}, \gamma)$$ or $$\Lambda(\Sigma^{n-1}, \gamma)$$. Generally, it should be not easy. To us, an natural problem is the following conjecture due to Gromov (see p.31 in [15]).

Conjecture 1.2. Let $$Y = \partial X$$ of a compact manifold with NNSC, and $$Y$$ be $$\lambda$$-bi-Lipschitz homeomorphic to the unit sphere $$S^{n-1}$$, then

$$\int_Y H d\mu \leq C(\lambda)(n - 1)\omega_{n-1}, \quad (2)$$

where $$C(\lambda) \to 1$$, for $$\lambda \to 1$$, here $$H$$ is the mean curvature of $$Y$$ with respect to the outward unit normal vector.

We begin to explore the above conjecture with the surface case. Under the condition of non-negativity of Gaussian curvature of surfaces, we are able to show that $$\Lambda_{+}(S^2, \gamma)$$ is close to $$\Lambda_{+}(S^2, \gamma_0)$$ provided these two surfaces are close enough in some sense. For the explicit statement, see Theorem 2.2.

In [20], $$\Lambda_{+}(\Sigma^2, \gamma)$$ was used to define Brown-York mass for a compact surface. we generalize it to the higher dimensional cases and analyze its behavior at the infinity of higher dimensional asymptotically Schwarzschild (AS) manifolds. Namely, for a triple of (generalized) Bartnik data $$(\Sigma^{n-1}, \gamma, H)$$, we define its generalized Brown-York mass by

$$m_{BY}(\Sigma^{n-1}, \gamma, H) = \frac{1}{(n - 1)\omega_{n-1}} \left( \Lambda_{+}(\Sigma, \gamma) - \int_{\Sigma^{n-1}} H d\mu_{\gamma} \right).$$

For an oriented bounded Riemannian manifold $$(\Omega^n, g)$$ with boundary, the corresponding (generalized) Brown-York mass is defined by

$$m_{BY}(\partial\Omega, g) = \frac{1}{(n - 1)\omega_{n-1}} \left( \Lambda_{+}(\partial\Omega, g|_{\partial\Omega}) - \int_{\partial\Omega} H_g d\mu_{g|_{\partial\Omega}} \right).$$

From the results in [24, 25], we see that $$m_{BY}(\partial\Omega, g)$$ is just the same as the original definition in [11, 12] when the Gaussian curvature of
the boundary \((\partial \Omega, g)\) is nonnegative. As a corollary of Theorem 2.2 we have:

**Corollary 1.2.** Let \((\Omega, g_i)\) be a sequence of 3-dimensional compact manifolds with boundaries and nonnegative scalar curvature. If the mean curvature and Gaussian curvature of the induced metric from \(g_i\) of the boundary are positive, and \(g_i\) converges to a smooth metric \(g\) whose Gaussian curvature is positive in \(C^1\)-topology, then \(m_{\text{BY}}(\partial \Omega, g_i)\) converges to \(m_{\text{BY}}(\partial \Omega, g)\) as \(i \to \infty\).

Note that the mean curvature \(H_0\) of isometric embedding image of \((\partial \Omega, g)\) in \(\mathbb{R}^3\) is involved in the original definition of Brown-York mass \([11, 12]\). Therefore, in order to estimate the Brown-York mass, we have to estimate \(H_0\), and higher order convergence of \(g_i\) is needed in this context. In above corollary, by clever using some monotonicity of \(\Lambda_+(S^2, \gamma)\), we only need \(C^1\)-convergence of metrics.

We have similar results in higher dimensional case. To state them precisely we need some notations. Let \(M(\Sigma^{n-1})\) be the space of all smooth Riemannian metrics on \(\Sigma^{n-1}\),

\[M_{\text{psc}}(\Sigma^{n-1}) = \{\gamma \in M(\Sigma^{n-1}) \mid R(\gamma) > 0\}.\]

**Theorem 1.5.** Given any constant \(G > 0\), let \(\{\gamma_i\}\) be a sequence in \(M_{\text{psc}}(S^{n-1})\) with \(\|\gamma_i\|_{W^{1,p}(S^{n-1})} \leq G\) for some \(p > n - 1\), if \(\|\gamma_i - \gamma_{\text{std}}\|_{C^0(S^{n-1})} \to 0\), then \(\Lambda_+(S^{n-1}, \gamma_i) \to \Lambda_+(S^{n-1}, \gamma_{\text{std}})\) as \(i \to \infty\).

One of crucial step to prove Theorem 1.5 is to make use of the local connectedness of space of positive scalar curvature (Proposition 3.2) and quasi-spherical metrics. Note that \(W^{1,p}\)-boundedness of metrics is satisfied in certain compactness of a family of Riemannian manifolds, as an application of above Theorem 1.5, we have the following:

**Theorem 1.6.** For any \(\epsilon > 0, K > 0, \text{and } i_0 > 0\), there exists a constant \(\delta = C(\epsilon, n, K, i_0) > 1\) such that for any \(\gamma \in \mathcal{N}_n(K, i_0)\) with \(\text{dil}(\gamma) \leq \delta\), we have

\[(1 - \epsilon)^{n-2} \Lambda_+(S^{n-1}, \gamma_{\text{std}}) \leq \Lambda_+(S^{n-1}, \gamma) \leq (1 + \epsilon)^{n-2} \Lambda_+(S^{n-1}, \gamma_{\text{std}}).\]

Notations \(\text{dil}(\gamma)\) and \(\mathcal{N}_n(K, i_0)\) are given in p.16 below.

We also consider total mean curvature of spin and NNSC fill-ins and improve the result in [30], for specific statement, see Theorem 4.1.

One of key points to prove Theorem 1.3 and Theorem 1.4 is to show the limit of generalized Brown-York mass of the large coordinate spheres is the total mass of the ambient manifold. In surface case, i.e., \(n = 3\), that can be achieved by arguments from isometric embedding theory. In our current case, such techniques fail, we make use of monotonicity lemma (Lemma 2.1) to overcome this difficulty.

The rest of the paper run as follows: in Section 2 we present proof of estimates of \(\Lambda_+(S^2, \gamma)\) and give a proof of Conjecture 1.2 in the
case of surfaces with positive Gaussian curvature (Theorem 2.2); in Section 3 we prove similar estimates of of $\Lambda_+(S^n, \gamma)$ when $n \geq 3$; in Section 4 we show the boundedness of the total mean curvature of the boundaries of spin and NNSC fill-ins; in Section 5 we prove Theorem 1.3 and Theorem 1.4.

2. Estimates of $\Lambda_+(S^2, \gamma)$

In this section, we investigate some properties of the $\Lambda_+(S^2, \gamma)$ for two dimensional spheres with nonnegative Gauss curvature.

We first give an estimate for $\Lambda_+(S^2, \gamma)$ of two dimensional spheres with nonnegative Gaussian curvature in terms of their diameters.

**Theorem 2.1.** Suppose $\gamma$ is a smooth metric on $S^2$ with $K_\gamma \geq 0$. Then

$$2 \, \text{diam}(S^2, \gamma) < \Lambda_+(S^2, \gamma) < 6\pi \, \text{diam}(S^2, \gamma).$$

Next we prove the Hölder continuity of $\Lambda_+(S^2, \gamma)$ with respect to $\gamma$ in the class of metrics with positive Gauss curvature. To state the result, we introduce the following notion.

**Definition 2.1.** Let $\Sigma^{n-1}$ be a smooth manifold and $\gamma_1, \gamma_2$ be two metrics on $\Sigma^{n-1}$. Define the dilation between $\gamma_1$ and $\gamma_2$ by

$$\text{dil}(\gamma_1, \gamma_2) = \inf \{ \lambda \mid \text{there is a diffeomorphism } \phi \text{ such that } \lambda^{-1}\gamma_2 \leq \phi^*(\gamma_1) \leq \lambda\gamma_2 \}.$$  

When $\Sigma^{n-1}$ is diffeomorphic to $S^{n-1}$ with its canonical differential structure, $\text{dil}(\gamma; \gamma_{\text{std}})$ is abbreviated as $\text{dil}(\gamma)$.

It is clear that $\text{dil}(\cdot, \cdot)$ is symmetric and invariant under diffeomorphisms.

**Theorem 2.2.** Let $\gamma_0$ be a smooth metric on $S^2$ with $K_{\gamma_0} > 0$. For any $\lambda_0 > 1$, there is a constant $C(\gamma_0, \lambda_0)$ such that for any metric $\gamma$ with $K_\gamma \geq 0$ and $\text{dil}(\gamma, \gamma_0) \leq \lambda_0$,

$$|\Lambda_+(S^2, \gamma) - \Lambda_+(S^2, \gamma_0)| \leq C(\gamma_0, \lambda_0) (\text{dil}(\gamma, \gamma_0) - 1)^{\alpha},$$

where $\alpha$ is a universal constant not less than $1/24$.

Note that Theorem 2.1 and Theorem 2.2 jointly implies that Conjecture 1.2 is true for closed surface with positive Gauss curvature under the positive mean curvature restriction.

For $(S^2, \gamma)$ with $K_\gamma > 0$, by 2.1, $\Lambda_+(S^2, \gamma)$ is achieved only by filling in $(S^2, \gamma)$ with the region enclosed by the image of $(S^2, \gamma)$ when isometric embedded in $\mathbb{R}^3$ as a strictly convex surface. This fact enables us to exploit some ideas and techniques from convex geometry. We first prove Theorem 2.1 and Theorem 2.2 for closed surfaces with positive Gauss curvature. Then due to the following approximation lemma, the positivity condition on Gauss curvature can be relaxed to nonnegativity. Let’s begin with the following monotonicity lemma.
LEMMA 2.1. Suppose \( \{ \gamma_t \}_{t \in [0,1]} \) is a path of NNSC metrics on \( \Sigma^n \). Assume \( \gamma_t \) monotonically increases, namely \( \gamma_{t_2} \geq \gamma_{t_1} \) for \( t_2 \geq t_1 \). Then

\[
\Lambda_+(\Sigma^n, \gamma_0) \leq \Lambda_+(\Sigma^n, \gamma_1).
\]

**Proof.** For \( \varepsilon > 0 \), let \( \widetilde{\gamma}_t = e^{2\varepsilon t} \gamma_t \), and \( \tilde{g} = dt^2 + \widetilde{\gamma}_t \) on \( \Sigma \times [0,1] \). Let \( \Sigma_t \) denote \( \Sigma \times \{ t \} \), \( A_t \) and \( H_t \) denote the second fundamental form and the mean curvature of \( \Sigma_t \) induced from metric \( \tilde{g} \). It is not hard to see that \( A_t > 0 \). It follows that \( \tilde{H}_t > 0 \) and \( \tilde{H}^2_t - \| A_t \|^2 > 0 \). Assume \( (\Omega^{n+1}, \tilde{g}) \) is a NNSC fill-in of \( (\Sigma, \gamma, H) \) for some \( H > 0 \). Set \( g = u^2 dt^2 + \widetilde{\gamma}_t \), we consider the quasi-spherical metric equation for \( u(x,t) \) on \( \Sigma^n \times [0,1] \), i.e.

\[
\begin{align*}
\bar{H}_t \frac{\partial u}{\partial t} &= u^2 \Delta \widetilde{\gamma}_t u + \frac{1}{2} (u - u^3) R_{\widetilde{\gamma}_t} - \frac{1}{2} R_{\tilde{g}} u \\
u(0) &= \frac{H_0}{H} > 0.
\end{align*}
\]

Let \( A_t \) and \( H_t \) denote the second fundamental form and the mean curvature of \( \Sigma_t \) induced from metric \( g \). It is not hard to see

\[
A_t = u^{-1} \bar{A}_t, \quad H_t = u^{-1} \bar{H}_t.
\]

By the Jacobi equation, Gauss equation and relation [23], we have

\[
\frac{d}{dt} \int_{\Sigma_t} H_t \, d\mu_{\gamma_t} = \frac{1}{2} \int_{\Sigma_t} \left( \bar{H}^2_t - \| A_t \|^2 \right) u^{-1} \, d\mu_{\gamma_t} + \frac{1}{2} \int_{\Sigma_t} R_{\gamma_t} u \, d\mu_{\gamma_t} > 0.
\]

So

\[
\int_{\Sigma} H \, d\mu_{\gamma_0} = \int_{\Sigma} H_0 \, d\mu_{\gamma_0} < \int_{\Sigma} H_1 \, d\mu_{\gamma_1}.
\]

It is not hard to see that \( (\Omega^{n+1}, \tilde{g}) \cup (\Sigma \times [0,1], g) \) is a NNSC fill-in (with coners along \( \Sigma \times \{ 0 \} \), but we can make it smooth by the method in [19]) of \( (\Sigma, \gamma_1, H_1) \). It follows that

\[
\int_{\Sigma} H \, d\mu_{\gamma_0} \leq \Lambda_+(\Sigma^n, \bar{\gamma}_1).
\]

By the scaling property of \( \Lambda_+ \)-invariant, we have

\[
\Lambda_+(\Sigma^n, \bar{\gamma}_1) = e^{(n-1)\varepsilon} \Lambda_+(\Sigma^n, \gamma_1).
\]

Since \( (\Omega^{n+1}, \tilde{g}) \) is an arbitrary suitable fill-in of \( (\Sigma^n, \gamma_0) \) with \( H_\tilde{g} > 0 \), we have

\[
\Lambda_+(\Sigma^n, \gamma_0) \leq e^{(n-1)\varepsilon} \Lambda_+(\Sigma^n, \gamma_1).
\]

Since \( \varepsilon \) is an arbitrary positive constant, in fact we get

\[
\Lambda_+(\Sigma^n, \gamma_0) \leq \Lambda_+(\Sigma^n, \gamma_1).
\]
Lemma 2.2. Let $\gamma$ be a smooth metric on $S^2$ with $K_\gamma \geq 0$. For any $\varepsilon > 0$, there is a smooth metric $\tilde{\gamma}$ on $S^2$ with $K_{\tilde{\gamma}} > 0$ such that
\[
dil(\tilde{\gamma}, \gamma) < 1 + \varepsilon, \quad \text{and} \quad |\Lambda_+(S^2, \tilde{\gamma}) - \Lambda_+(S^2, \gamma)| < \varepsilon.
\]

Proof. Let $\gamma(t)$ be the Ricci flow starts from $\gamma$ and assume the existence time is $[0, T)$. By the strong maximum principle, for $t \in (0, T)$, $K_{\gamma(t)} > 0$. Then it follows that
\[
\frac{d}{dt} \gamma(t) = -2K_{\gamma(t)}\gamma(t) \leq 0.
\]
Set $M = \max K_{\gamma}$. There is $t_1 \in (0, T)$ depending on $M$ such that for $t \in [0, t_1]$, $K_{\gamma(t)} \leq 2M$. Set $\gamma(t) = e^{2Mt}\gamma(t)$. Then
\[
\frac{d}{dt} \tilde{\gamma}(t) = 2\big(2M - K_{\gamma(t)}\big) \tilde{\gamma}(t) \geq 0.
\]
Thus we have
\[
\gamma(t) \leq \gamma \leq e^{4Mt}\gamma(t) \quad \text{for } t \in [0, t_1].
\]
Since $K_{\gamma(t)} \geq 0$, by Lemma 2.1 we see that
\[
\Lambda_+(S^2, \gamma(t)) \leq \Lambda_+(S^2, \gamma) \leq e^{2Mt}\Lambda_+(S^2, \gamma(t))
\]
for all $t \in [0, t_1]$. Due to (5) and (6), for any $\varepsilon > 0$, we can take a sufficiently small $t_2 \in (0, t_1]$ such that $\tilde{\gamma} = \gamma(t_2)$ meets the requirements. \hfill \Box

One important property for closed convex surfaces that we will exploit in the proof of the two theorems is the following monotonicity of total mean curvature (quermassintegral more generally) which should be well-known, but we cannot find its proof in literature, hence give a proof here.

Lemma 2.3. Let $\Sigma_i$ ($i = 1, 2$) be a $n$-dimensional closed convex hypersurface in $\mathbb{R}^{n+1}$. Suppose $\Sigma_1$ is contained in $\Sigma_2$. Then
\[
\int_{\Sigma_1} \sigma_{n-1}^1 d\mu \leq \int_{\Sigma_2} \sigma_{n-1}^2 d\mu,
\]
where $\sigma_{n-1}^i$ is the $n-1$ sum of principle curvatures of $\Sigma^i$.

Proof. Let $\Sigma^t_i$ be the surface with outerwise distance $t$ ($t \geq 0$) to $\Sigma_i$. We have the Steiner’s formula for the volume of $\Sigma^t_i$, namely
\[
|\Sigma^t_i| = |\Sigma_i| + \left(\int_{\Sigma_i} H \ d\mu\right) t + \cdots + \left(\int_{\Sigma_i} \sigma_k \ d\mu\right) t^k + \cdots + \left(\int_{\Sigma_i} \sigma_{n-1} \ d\mu\right) t^{n-1} + \omega_n t^n,
\]
where $\omega_n$ is the volume of $n$-dimensional unit sphere.

Since $\Sigma_1$ is enclosed by $\Sigma_2$, $\Sigma_1$ is enclosed by $\Sigma^t_2$, and being convexity of those surfaces, we have $|\Sigma^t_1| \leq |\Sigma^t_2|$ for all $t \geq 0$. It follows that
\[
\int_{\Sigma_1} \sigma_{n-1}^1 d\mu \leq \int_{\Sigma_2} \sigma_{n-1}^2 d\mu.
\]
Now we begin to prove Theorem 2.1.

**Proof of Theorem 2.1.** We begin with the positive Gauss curvature case. We first prove the upper bound estimate. Denote the image of \((S^2, \gamma)\) when isometric embedded in \(\mathbb{R}^3\) by \(\Sigma\). Suppose that \(p, q \in \Sigma\) realize \(\text{diam}(\gamma)\). Let \(pq\) be the segment connecting \(p\) and \(q\) in \(\mathbb{R}^3\) and \(|pq|\) be its length. Let \(o\) denote the midpoint of \(pq\). For any \(x \in \Sigma\), we must have 

\[
|xo| < |xp| + |po| \leq d_\gamma(x, p) + |po| \\
\leq \text{diam}(\gamma) + \frac{\text{diam}(\gamma)}{2} = \frac{3}{2} \text{diam}(\gamma).
\]

This means that \(\Sigma\) is strictly contained in the ball centered at \(o\) with radius \(R = \frac{3}{2} \text{diam}(\gamma)\). By Lemma 2.3, the total mean curvature of \(\Sigma\) is strictly less than the total mean curvature of the sphere of radius \(R\), namely

\[
\Lambda_+ (S^2, \gamma) < \Lambda_+ (S^2, R^2 \gamma_{std}) = 6\pi \text{diam}(\gamma).
\]

Next, we prove the lower bound estimate. Suppose \(p', q' \in \Sigma\) realize the extrinsic diameter of \(\Sigma\), which is denoted by \(l\). The first step is proving

\[
l > \frac{\text{diam}(\gamma)}{\pi}.
\]

Let \(o'\) denote the midpoint of \(p'q'\). Since \(\Sigma\) is convex, \(o'\) lies in the interior of \(\Sigma\). Then for any \(x \in \Sigma\), we must have \(|xo'| < l\). Otherwise, the extrinsic diameter of \(\Sigma\) is strictly greater than \(l\). So \(\Sigma\) is strictly contained in the ball centered at \(o'\) with radius \(l\). Take a plane \(P\) that passes through \(pq\). Since \(\Sigma\) is strictly convex, \(\Sigma\) and \(P\) intersect transversely. Denote the intersecting curve by \(\Gamma\). For \(\Gamma\) passes \(p\) and \(q\) and \(\Gamma \subset \Sigma\), its length \(|\Gamma| \geq 2 \text{diam}(\gamma)\). Since \(\Gamma\) is a planar convex curve that enclosed by a circle of radius \(l\), by Lemma 2.3, \(|\Gamma| < 2\pi l\). Thus we have proved \(l > \text{diam}(\gamma)/\pi\). For small \(\varepsilon_1 > 0\), take \(s, t \in p'q'\) such that \(|sp'| = |tq'| = \varepsilon_1\). Take small \(\varepsilon_2 < \varepsilon_1\) such that the cylinder with axis \(st\) and radius \(\varepsilon_2\) is enclosed by \(\Sigma\). Cap the cylinder with two hemispheres with radius \(\varepsilon_2\). Denote the combined surface by \(C\). Apparently, \(C\) is a \(C^{1,1}\) convex surface. We can choose \(\varepsilon_2\) small enough so that \(C\) is enclosed by \(\Sigma\). The total mean curvature of \(C\) is

\[
\int_C H \, d\mu = \frac{1}{\varepsilon_2} \times 2\pi \varepsilon_2 \times (l - 2\varepsilon_1) + 2 \times 4\pi \varepsilon_2 > 2\pi (l - 2\varepsilon_1).
\]

Because \(C\) is enclosed by \(\Sigma\), by Lemma 2.3, the total mean curvature of \(\Sigma\) is not less than \(2\pi (l - 2\varepsilon_1)\). Since \(\varepsilon_1\) can be arbitrarily small, in fact the total mean curvature is not less than \(2\pi l\). Finally, due to (7), we get the lower bound estimate.
By Lemma 2.2, for any $i \in \mathbb{N}$, we can find a smooth metric $\gamma_i$ with $K_{\gamma_i} > 0$ such that
\[
dil(\gamma_i, \gamma) < 1 + \frac{1}{i}, \quad \text{and} \quad \left| \Lambda_+ (S^2, \gamma_i) - \Lambda_+ (S^2, \gamma) \right| < \frac{1}{i}.
\]
Since $K_{\gamma_i} > 0$, we have
\[
2 \, \text{diam}(S^2, \gamma_i) < \Lambda_+ (S^2, \gamma_i) < 6\pi \, \text{diam}(S^2, \gamma_i).
\]
By $\dil(\gamma_i, \gamma) < 1 + \frac{1}{i}$, we have
\[
\frac{i}{i+1} \, \text{diam}(\gamma) \leq \text{diam}(\gamma_i) \leq \frac{i+1}{i} \, \text{diam}(\gamma).
\]
Letting $i \to \infty$, we draw the conclusion for the nonnegatively curved case.

\textbf{Remark 2.1.} In [34], Topping proved that the total $(n-1)$-th power mean curvature of a closed hypersurface in $\mathbb{R}^{n+1}$ is bounded below by a constant multiple of the diameter via a different approach. In two dimension, the constant there is $\pi/32$, smaller than ours here. From the proof of the lower bound estimate, we can see that the conjecture raised in [33] is true for closed convex surface.

To prove Theorem 2.2 besides the monotonicity of total mean curvature for convex surface, we need a qualitative version of the statement closeness of intrinsic metrics of two closed convex surfaces implies the closeness of their spatial forms. This is stated and explained by I. Belegradek in [6]. For the convenience of the readers, we give more details here. The statement is a combination of Volkov’s stability theorem and an isometric approximation theorem by Alestalo-Trotsenko-Väisälä (see Theorem 3.3 in [4]).

In 1967, Volkov [35] proved the following exceptionally strong result. Volkov’s original article was written in Russian, see Section 12.1 in [3] for a translation and Section 5.2 in [10] for an exposition.

\textbf{Lemma 2.4 (Volkov’s Stability Theorem).} Let $\Sigma_1$ and $\Sigma_2$ be two closed convex surfaces in $\mathbb{R}^3$ with intrinsic distance $d_1$ and $d_2$. Let $\varphi$ be a homeomorphism $\varphi : \Sigma_1 \to \Sigma_2$. Then for any $x$, $y \in \Sigma_1$, there holds
\[
\left| |\varphi(x) - \varphi(y)| - |x - y| \right| \leq C \sup_{z, w \in \Sigma_1} \left| d_2 (\varphi(z), \varphi(w)) - d_1 (z, w) \right|^{\alpha},
\]
where $| \cdot |$ denotes the Euclidean norm, $\alpha$ is an absolute constant not less than $1/24$, and $C$ is a constant depending only on the intrinsic diameters of $\Sigma_1$ and $\Sigma_2$.

To state the isometric approximation lemma, we need the following two notions.
There is an isometry $S$ where $d(\text{diam}(A), \epsilon) = \delta$. It follows that $(\lambda, \gamma)$ is small, and its value may vary from line to line. Since $\Sigma$ is a closed strictly convex surface, $\theta(\Sigma_0) > 0$. Apply the isometric approximation theorem to the setting $A = \Sigma_0$ and $f = \phi$, we can find an isometry $S : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\sup_{z \in \Sigma_0} |\phi(z) - S(z)| \leq C |\lambda + \epsilon - 1|^\alpha.$$  

Inequality (8) implies that $\Sigma$ lies in the $\delta$-neighborhood of $S(\Sigma_0)$, where $\delta = C |\lambda + \epsilon - 1|^\alpha$. Take the barycenter of $S(\Sigma_0)$ to be the origin.
When $\delta$ is sufficiently small, there are constants $k_1$ and $k_2$ depending only on the shape of $\Sigma_0$ such that $\Sigma$ is enclosed by $(1 - k_1 \delta)S(\Sigma_0)$ and $(1 + k_2 \delta)S(\Sigma_0)$. Then by the monotonicity of the total mean curvature and the invariance of the total mean curvature under isometry, we have
\[
(1 - k_1 \delta)\Lambda_+(S^2, \gamma_0) \leq \Lambda_+(S^2, \gamma) \leq (1 + k_2 \delta)\Lambda_+(S^2, \gamma_0).
\]
By letting $\varepsilon \to 0$, we complete the proof for the positive Gauss curvature case. Via a very similar approach as in the proof of Theorem 2.1, we can prove the conclusion also holds for the nonnegative Gauss curvature case. \hfill \Box

3. Estimates of $\Lambda_+(S^n, \gamma)$ when $n \geq 3$

In this section, we will give estimates of $\Lambda_+(S^n, \gamma)$ when $n \geq 3$. A key observation is that we are able to construct PSC-path connecting two metrics of positive scalar curvature if they are very close in the $C^0$ topology. Our main argument is the Ricci-DeTurck flow. Let us begin with its brief overview.

Let $M$ be an $n$-dimensional compact and closed manifold and $g_0$ be a smooth metric on $M$. The Ricci flow $\bar{g}(t)$ is a smooth, time-dependent family of Riemannian metrics solving the following equation
\[
\begin{align*}
\frac{\partial}{\partial t} \bar{g}(t) &= -2\text{Ric}(\bar{g}(t)) \quad \text{in} \quad M \times (0, T) \\
\bar{g}(0) &= g_0,
\end{align*}
\]
It is well known that a short time solution to the Ricci flow always exists and is unique (e.g., see [32]). Now we consider the Ricci-DeTurck flow with a background metric. Let $X_{\bar{g}}$ be the operator which maps metric tensors to vector fields defined by
\[
X_{\bar{g}}(g) = \sum_{i=1}^{n}(\nabla^g_{e_i} e_i - \nabla^\bar{g}_{e_i} e_i),
\]
where $\{e_i\}_{i=1}^{n}$ is any local orthonormal basis with respect to $g$. Then the Ricci-DeTurck flow equation with background metric $\bar{g}$ (here $\bar{g}$ is either a fixed metric or a Ricci flow starting from $\bar{g}(0)$) is
\[
\frac{\partial}{\partial t} g(t) = -2\text{Ric}(g(t)) - \mathcal{L}_{X_{\bar{g}(g(t))}} g(t).
\]
If $g(t)$ solves (11), then one can obtain a Ricci flow via a family of diffeomorphisms. More precisely, if $g(t)$ solves (11), letting $\chi(t)$ be a family of diffeomorphisms satisfying
\[
\begin{align*}
X_{\bar{g}}(g(t)) f &= \frac{\partial}{\partial t}(f(\chi(t))) \quad \text{for all} \quad f \in C^\infty(M), \\
\chi(0) &= \text{id},
\end{align*}
\]
then $\chi(t)^* g(t)$ is the solution of Ricci flow with initial data $g(0)$. 
Note that under the Ricci-DeTurck flow, scalar curvature satisfies the following evolution equation (see [32]),

$$\partial_t R \geq \Delta_g(t) R - \langle X, \nabla R \rangle + \frac{2}{n} R^2.$$  

If \( R^{(0)} \geq \kappa \), then

$$R(g(t)) \geq \kappa \left( 1 - \frac{2\kappa t}{n} \right).$$

Hence, we see that nonnegativity of scalar curvature is preserved under the Ricci-DeTurck flow. Simon [28] have studied the existence and higher order estimates of the Ricci-DeTurck flow on a fixed background metric. In [8], Burkhardt-Guim studied the Ricci-DeTurck flow on a Ricci flow background and obtained the following result:

**Proposition 3.1** (Lemma 3.3, Corollary 3.4 in [8]). There exists a positive constant \( c'_k = c'_k(n) \) such that the following is true:

For every continuous metric \( g_0 \in C^0(M) \) and every smooth metric \( \bar{g}_0 \) on \( M \). Let \( \bar{g}(t) \) be the Ricci flow starting from \( \bar{g}_0 \). If \( \| g_0 - \bar{g}_0 \|_{L^\infty(M)} < \epsilon' \), then there exist \( T = T(\bar{g}(t)) > 0 \) and

$$c_k = c_k \left( n, \sup_{0 \leq l \leq k} \| \nabla^l Rm(\bar{g}(t)) \|_{L^\infty(M)} \right)$$

such that (11) admits a solution \( g(t) \) which is smooth on \( M \times (0, T] \), continuous on \( M \times [0, T] \) and satisfies

$$\| \nabla^k (g(t) - \bar{g}(t)) \|_{L^\infty(M)} \leq c_k t^{-\frac{k}{2}} \| g_0 - \bar{g}_0 \|_{L^\infty(M)}, \quad (13)$$

for all \( t \in (0, T] \). In particular, if \( g^i \) is sequence of \( C^0 \) metrics on \( M \) such that

$$\lim_{i \to \infty} g^i = g \quad \text{in} \quad C^0(M),$$

and \( g^i(t), g(t) \) are the Ricci-DeTurck flows with respect to \( \bar{g}(t) \) starting from \( g^i \) and \( g \) respectively, then \( g^i(t) \) converges locally smoothly to \( g(t) \) on \( M \times (0, T] \). Here \( \nabla \) denotes the covariant derivative with respect to \( \bar{g}(t) \) and \( \| \cdot \| \) is the norm taken with respect to \( \bar{g}_0 \).

For the Ricci-DeTurck flow on a fixed metric background, if we have some further assumptions on the initial metric \( g_0 \), then we can obtain better estimate of the derivative of this flow.

**Lemma 3.1.** Let \( g(t) \) be the solution to (11) with \( \bar{g} \) is a fixed metric starting form a NNSC metric \( g_0 \). Then \( g(t) \) is also a NNSC metric for all \( t \leq T \) and

1. (see [28]) if \( g_0 \) is smooth and \( \| \nabla g_0 \| \) is bounded, then for all \( t \leq T \)

$$\| \nabla g(t) \| \leq C; \quad \text{and} \quad \| \nabla^2 g(t) \| \leq Ct^{-1/2}.$$
By employing the Ricci-DeTurck flow, one can prove the following local connectedness of $\mathcal{M}_{psc}(\Sigma^{n-1})$ which may have its own interests.

**Proposition 3.2.** Let $\gamma_0$ be a smooth PSC-metrics on $\Sigma^{n-1}$. Then there exists $\delta = \delta(\gamma_0)$ such that for any smooth PSC-metric $\gamma$ with

$$\|\gamma - \gamma_0\|_{C^0(\Sigma^{n-1})} \leq \delta,$$

there is a path $\eta_t$ in $\mathcal{M}_{psc}(\Sigma^{n-1})$, $0 \leq t \leq 1$, joining $\gamma$ and $\gamma_0$ with

$$\frac{d}{dt}\eta(t)^2 \leq C t^{-2},$$

where $C$ depends only on the bound of the sectional curvature of $\gamma_0$.

**Proof of Proposition 3.2.** Let $\gamma(t)$ be the Ricci flow starting from $\gamma_0$. By Proposition 3.1, there exists a solution $\gamma(t)$, $0 < t \leq T$ to (11) with $\gamma$ as the initial data, $\tilde{\gamma}(t)$ as the background metric satisfying

$$\|\nabla^k (\gamma(t) - \tilde{\gamma}(t))\|_{C^0(\Sigma^{n-1})} \leq c_k t^{-\frac{k}{2}} \|\gamma - \gamma_0\|_{C^0(\Sigma^{n-1})}.$$  

Now, fix an $0 < s < T$. Let $\|\gamma - \gamma_0\|_{C^0(\Sigma^{n-1})}$ be small enough so that $\|\nabla^k (\gamma(s) - \tilde{\gamma}(s))\|_{C^0(\Sigma^{n-1})}$ is sufficiently small for $k = 0, 1, 2$. Then it is easy to see that the straight line connecting $\gamma(s)$ and $\tilde{\gamma}(s)$ is a PSC-path. Now, let

$$\bar{\eta}_t = \begin{cases} 
\gamma(t) & \text{for } 0 \leq t \leq s, \\
\frac{\tilde{\gamma}(s) - \gamma(s)}{1 - 2s} (t - s) + \gamma(s) & \text{for } s < t < 1 - s, \\
\tilde{\gamma}(1 - t) & \text{for } 1 - s \leq t \leq 1.
\end{cases}$$

Then it follows easily that $\eta_t$ is a continuous path in $\mathcal{M}_{psc}(\Sigma^{n-1})$ that joints $\gamma$ and $\gamma_0$. \hfill \Box

The main result in this section is the following:

**Theorem 3.1.** Given any constant $G > 0$. Let $\{\gamma_i\}$ be a sequence in $\mathcal{M}_{psc}(\mathbb{S}^{n-1})$ with $\|\gamma_i\|_{W^{1,p}(\mathbb{S}^{n-1})} \leq G$ for some $p > n - 1$. If $\|\gamma_i - \gamma_{std}\|_{C^0(\mathbb{S}^{n-1})} \to 0$, then $\Lambda_+(\mathbb{S}^{n-1}, \gamma_i) \to \Lambda_+(\mathbb{S}^{n-1}, \gamma_{std})$ as $i \to \infty$.

**Proof of Theorem 3.1.** For any metrics $\gamma$ with $\|\gamma - \gamma_{std}\| \leq \epsilon'(n)$ where $\epsilon'(n)$ is the constant in Proposition 3.1, we know that there is a solution $g(t), t \in (0, T(n, |Rm(\gamma)|)]$ to (11) starting from $\gamma$ with $\bar{g} = \gamma_{std}(t)$ where $\gamma_{std}(t) = (1 - 2(n - 2)t)\gamma_{std}$ which solves (9) with initial data $g_0 = \gamma_{std}$. Note that it is easy to see from (10) that $X_{\gamma_{std}}(g) = \frac{d}{dt}\gamma_{std}(t).$
Thus, equation (11) with \( g = \gamma_{std} \) is equal to (11) with \( g = \gamma_{std}(t) \). The remains of proof is divided into following two steps. Here we make a convention that \( C \) is a constant which may varies from line to line and depends only on \( n, p, G \) and \( \gamma_{std} \).

**Step 1:** Since \( \|\nabla g(t)\|_{LP(S^{n-1})} \leq G \), by Lemma 3.1 we have
\[
|\nabla g(t)|^2 \leq G t^{-\sigma}; \quad \text{and} \quad |\nabla^2 g(t)|^2 \leq Ct^{-1-\sigma},
\]
where \( \sigma = (n-1)/p \) and the constant \( C \) depends only on \( n, p, G \) and \( \gamma_0 \).

Now, let’s fix \( \alpha > 0 \) such that \( \alpha(1-\sigma) - 2 > 0 \). Let \( s_1 > 0 \) be a constant which will be determined later with \( s_1^2 < T \). Then we define
\[
t(s) = s^\alpha, \quad s \in [0, s_1],
\]
and
\[
\varphi(s) = 1 + \frac{\epsilon}{2s_1}t, \quad s \in [0, s_1].
\]

Let
\[
\bar{g} = ds^2 + \varphi^2(s) g(t(s)), \quad s \in [0, s_1].
\]

Obviously, \( g(t(s)) \) is a path of PSC connecting \( \gamma \) and \( g(s_1^2) \). Let \( \bar{H}_s \) be the mean curvature of \( S^{n-1} \) with respect to \( \partial_s \)-direction and \( \bar{A}_s \) be its associated second fundamental form. It is easy to see that
\[
\bar{H}_s = \frac{(n-1)\varphi'}{\varphi} + \frac{1}{2} t'(s) tr_{s'} \gamma'
\]
\[
\geq \frac{(n-1)\epsilon}{2s_1} - Cs_{s_1'^{\alpha(1-\sigma)-2}},
\]
and
\[
\bar{H}_s^2 - \|\bar{A}_s\|^2 = (n-1)(n-2) \left( \frac{\varphi'}{\varphi} \right)^2 + (n-2)t'(s) \frac{\varphi'}{\varphi} tr_{s'} \gamma'
\]
\[
+ \frac{1}{4} |t'(s)|^2 \left( |tr_{s'} \gamma'|^2 - |\gamma'(t)|^2 \right)
\]
\[
\geq (n-1)(n-2) \left( \frac{\epsilon}{2s_1} \right)^2 - C \epsilon \frac{s_{s_1'^{\alpha(1-\sigma)-2}}} - Cs_{s_1'^{\alpha(1-\sigma)-2}}.
\]

Thus, we can choose \( s_1 = s_1(\epsilon, n, p, G) \) small such that
\[
\bar{H}_s > 0, \quad \bar{H}_s^2 - \|\bar{A}_s\|^2 > 0,
\]
and
\[
\|\gamma_{std}(s_1^2) - \gamma_{std}\| \leq \frac{\epsilon}{16n}.
\]

By the standard parabolic equation theory we see that the following quasi-spherical metric equation
\[
\begin{cases}
\bar{H}_s \frac{\partial u}{\partial s} = u^2 \Delta_{s^2 g(t(s))} u + \frac{1}{2} (u - u^3) R_{s^2 g(t(s))} - \frac{1}{2} R_{g} u, \\
u(0) = \frac{\bar{H}_0}{H} > 0.
\end{cases}
\]
always has solution.
Now, let
\[ g = u^2 ds^2 + \varphi^2(s) g(t(s)), \quad s \in [0, s_1] \]
and let \( H_s \) be the mean curvature of \( S^{n-1} \) with respect to \( g \). Then
\[
\frac{d}{ds} \int_{S^{n-1}} H_s d\mu_s = \frac{1}{2} \int_{S^{n-1}} \left( \tilde{H}_s - \| \tilde{A}_s \|^2 \right) u^{-1} d\mu_s + \frac{1}{2} \int_{S^{n-1}} R_{\gamma_s} u d\mu_s > 0.
\]
Note that
\[
\int_{S^{n-1}} H_0 d\mu_0 = \int_{S^{n-1}} u(0)^{-1} \tilde{H}_0 d\mu_{\gamma} = \int_{S^{n-1}} H d\mu_{\gamma}.
\]
Thus
\[
\Lambda(S^{n-1}, \gamma) \leq \Lambda(S^{n-1}, (1 + \epsilon/2)^2 g(s_1^0)). \tag{15}
\]

**Step 2:** Note that, it follows from Proposition 3.1 that
\[
\| \nabla^k (g(s_1^0) - \gamma_{std}(s_1^0)) \|_{L^\infty(S^{n-1})} \leq c_k s_1^{-k} \| \gamma - \gamma_{std} \|_{L^\infty(S^{n-1})}.
\]
Thus, there exist a \( \delta = \delta(n, \epsilon, G, \gamma_{std}, \gamma) \) such that if \( \| \gamma - \gamma_{std} \|_{L^\infty(S^{n-1})} \leq \delta \), there holds
\[
\| g(s_1^0) - \gamma_{std} \|_{C^2} \leq \frac{\epsilon}{8n}. \tag{16}
\]
Let
\[
\gamma_t = (\gamma_{std} - g(s_1^0)) t + g(s_1^0), \quad t \in [0, 1];
\]
and
\[
\phi(t) = \frac{\epsilon}{2} t + 1, \quad t \in [0, 1].
\]
Then it is easy to see from (16) that \( \gamma_t \) is a smooth path of PSC connecting \( g(s_1^0) \) and \( \gamma_{std} \). Now, set
\[
\tilde{g} = dt^2 + \phi^2(t) \gamma_t, \quad t \in [0, 1]
\]
and denote by \( \tilde{H}_t \) and \( \tilde{A}_t \) the mean curvature and second fundamental form of the slice at time \( t \) in \((S^{n-1} \times [0, 1], \tilde{g})\) with respect to \( \partial_t \) direction. Then, a simple computation yields
\[
\tilde{H}_t > 0 \quad \text{and} \quad \tilde{H}_t^2 - \| \tilde{A}_t \|^2 > 0.
\]
By the same arguments as in the **Step 1**, we obtain
\[
\Lambda_+ \left( S^{n-1}, (1 + \epsilon/2)^2 g(s_1^0) \right) \leq \Lambda_+ \left( S^{n-1}, (1 + \epsilon)^2 g_{std} \right) = (1 + \epsilon)^{n-2} \Lambda_+ \left( S^{n-1}, \gamma_{std} \right). \tag{17}
\]
Combining (15) and (17), we complete the proof. \( \square \)
From the definition of fill-in, we see that for any diffeomorphism \( \psi \) on \( S^{n-1} \), \( \Lambda_+(S^{n-1}, \gamma) = \Lambda_+(S^{n-1}, \psi^*\gamma) \). By the convergence theory of Riemannian manifolds, we know that the condition \( W^{1,p} \)-boundedness of metrics can be obtained by some assumptions of curvature. The following Theorem 3.2 can be regarded as an application of Theorem 3.1. Let \( \text{dil}(\gamma) = \text{dil}(\gamma, \gamma_{\text{std}}) \), it is clear that \( \text{dil}(\gamma) \) is invariant under diffeomorphisms. In the following, all norms are taken with respect to \( \gamma_{\text{std}} \). Define

\[
N_n(K, i_0) = \{ \gamma \in \mathcal{M}_{\text{psc}}(S^{n-1}) \mid \text{Ric}_\gamma \geq -K, \text{inj}_\gamma \geq i_0 \},
\]

where \( K \) and \( i_0 \) are some positive constants.

**Theorem 3.2.** For any \( \epsilon > 0, K > 0, \) and \( i_0 > 0 \) there exists a constant \( \delta = C(\epsilon, n, K, i_0) > 1 \) such that for any \( \gamma \in N_n(K, i_0) \) with \( \text{dil}(\gamma) \leq \delta \),

\[
(1 - \epsilon)^n \Lambda_+(S^{n-1}, \gamma_{\text{std}}) \leq \Lambda_+(S^{n-1}, \gamma) \leq (1 + \epsilon)^n \Lambda_+(S^{n-1}, \gamma_{\text{std}}).
\]

To prove this result, we need the following lemma.

**Lemma 3.2.** Let \( \gamma \) be a \( W^{1,p} (p > n) \) metric on \( S^n \) with \( \text{dil}(\gamma) = 1 \). Then we can find a \( W^{2,p} \) homeomorphism \( \psi \) such that

\[
\gamma = \psi^*(\gamma_{\text{std}}), \quad \text{and} \quad \|D\psi\|_{W^{1,p}} \leq C(n, \|\gamma\|_{W^{1,p}}).
\]

**Proof.** By definition, there exists a family of diffeomorphisms \( \psi_i \) such that

\[
\frac{i}{i+1} \psi_i^*(\gamma_{\text{std}}) \leq \gamma \leq \frac{i+1}{i} \psi_i^*(\gamma_{\text{std}}).
\]

(20)

Denote the inverse of \( \psi_i \) by \( \phi_i \). By (20), we have

\[
\|D\psi_i\|_{L^\infty} \leq \frac{i+1}{i} \|\gamma\|_{L^\infty}, \quad \text{and} \quad \|D\phi_i\|_{L^\infty} \leq \frac{i+1}{i} \|\gamma^{-1}\|_{L^\infty}.
\]

Since \( \psi_i \) and \( \phi_i \) are uniformly Lipschitz, by taking subsequence if necessary, we may assume \( \psi_i \to \psi \) and \( \phi_i \to \phi \). Then \( \phi \) and \( \psi \) are Lipschitz and \( \phi \circ \psi = \text{id} \). So \( \phi \) and \( \psi \) are Lipschitz homeomorphisms. Equation (20) implies

\[
\frac{i}{i+1} d_{\gamma_{\text{std}}}((\psi_i(x), \psi_i(y)) \leq d_\gamma(x, y) \leq \frac{i+1}{i} d_{\gamma_{\text{std}}}((\psi_i(x), \psi_i(y)).
\]

Letting \( i \to \infty \), we arrive at

\[
d_\gamma(x, y) = d_{\gamma_{\text{std}}}((\psi(x), \psi(y)).
\]

Then by the proof of Theorem 2.1 in [31], it is not hard to see that \( \psi \in W^{2,p} \) and \( \|D\psi\|_{W^{1,p}} \leq C(n, \|\gamma\|_{W^{1,p}}) \).

\[\square\]

Now, we are ready to give the proof of Theorem 3.2.
Proof of Theorem 3.2. We take a contradiction argument. If the conclusion is not true, then there exist a constant $\epsilon_0 > 0$ and a sequence of metrics $\{\gamma_i\} \in \mathcal{N}_n(K, i_0)$ with
$$\lim_{i \to \infty} \text{dil}(\gamma_i) \to 1,$$
such that
$$\Lambda_+(S^{n-1}, \gamma_i) \geq (1 + \epsilon_0)^{n-2} \Lambda_+(S^{n-1}, \gamma_{\text{std}}).$$

By Anderson-Cheeger’s compactness result (see [1]), there is a subsequence of $\{\gamma_i\}$ (we still denote it by $\{\gamma_i\}$) and corresponding diffeomorphisms $\phi_i$ such that $\phi_i^*(\gamma_i)$ converges to a $W^{1,p}$ metric $\gamma_\infty$ in the $W^{1,p}$ sense.

By the assumption $\lim_{i \to \infty} \text{dil}(\gamma_i) \to 1$, we can find diffeomorphisms $\psi_i$ and constants $\lambda_i \to 1$ such that
$$\lambda_i^{-1} \psi_i^*(\gamma_{\text{std}}) \leq \phi_i^*(\gamma_i) \leq \lambda_i \psi_i^*(\gamma_{\text{std}}). \quad (21)$$

Since $\phi_i^*(\gamma_i)$ converges to $\gamma_\infty$ in the $W^{1,p}$ sense, there exist constants $\mu_i \to 1$ such that
$$\mu_i^{-1} \phi_i^*(\gamma_{\text{std}}) \leq \psi_i^{-1}(\gamma_\infty) \leq \mu_i \psi_i^{-1}(\gamma_{\text{std}}). \quad (22)$$

It follows from (21) and (22) that
$$(\lambda_i \mu_i)^{-1} \phi_i^*(\gamma_{\text{std}}) \leq \psi_i^{-1}(\gamma_\infty) \leq (\lambda_i \mu_i)^{-1} \phi_i^*(\gamma_{\text{std}}).$$

Since $\lambda_i \mu_i \to 1$, we conclude that $\text{dil}(\gamma_\infty) = 1$.

By Lemma 3.2, there is a $W^{2,p}$ homeomorphism $\psi$ with $\|D\psi\|_{W^{1,p}} \leq C(n, \|\gamma_{\text{std}}\|_{W^{1,p}})$ such that
$$\gamma_\infty = \psi^*(\gamma_{\text{std}}).$$

We can find a diffeomorphism $\tilde{\psi}$ by mollifying $\psi$ such that $\|\tilde{\psi}^*(\gamma_{\text{std}}) - \gamma_\infty\|_{W^{1,p}}$ is sufficiently small. Since $\phi_i^*(\gamma_i)$ converges to $\gamma_\infty$ in the topology of $W^{1,p}$. By Theorem 3.1 we know that
$$\limsup_{i \to \infty} \Lambda_+(S^{n-1}, \gamma_i) \leq (1 + \epsilon_0/2)^{n-2} \Lambda_+(S^{n-1}, \gamma_{\text{std}}).$$

Then we get a contradiction. By the same arguments, the lower bound estimate of $\Lambda_+(S^{n-1}, \gamma_{\text{std}})$ can also be obtained. Thus we finish the proof of Theorem 3.2. \qed

4. Total mean curvature of spin fill-ins

In this section, we will consider the upper bound of total mean curvature of the boundary of a compact spin manifold with NNSC. We call $(\Omega^n, g)$ is a spin fill-in for Bartnik data $(\Sigma^{n-1}, \gamma, H)$ if $\Omega^n$ is a spin manifold and $(\Omega^n, g)$ is a fill-in for $(\Sigma^{n-1}, \gamma, H)$.

Theorem 4.1. For any $n \geq 3$, let $\gamma$ be a Riemannian metric on $S^{n-1}$. Then there is a constant $h_0 = h_0(\gamma)$ such that if $(S^{n-1}, \gamma, H)$ admits a spin NNSC fill-in and $H > 0$, then
$$\int_{S^{n-1}} H \, d\mu_\gamma \leq h_0.$$
Remark 4.1. If we replace the assumption \( R \geq 0 \) by \( R \geq \sigma \), for any \( \sigma > -\infty \), the same conclusion is still true, and the arguments of the proof of Theorem 4.1 works too.

Proof of Theorem 4.1. We are going to show that if \((\Omega^n, \tilde{g})\) is a spin fill-in for Bartnik data \((S^{n-1}, \gamma, H)\) with nonnegative scalar curvature and \( H > 0 \), then there is a constant \( h_0 = h_0(\gamma) \) such that

\[
\int_{S^{n-1}} H d\mu_{\gamma} \leq h_0.
\]  

(23)

Let \( \tilde{\gamma}_t = t\gamma + (1-t)\gamma_{\text{std}} \) for \( t \) in \([0,1]\). Choose a constant \( a > 0 \) large enough so that \( e^{2a_t} t_2 > e^{2a_t} t_1 \) for any \( 0 \leq t_1 < t_2 \leq 1 \), and denote \( \tilde{\gamma}_t = e^{2a_t} \gamma_t \). Let \( K = \min_{S^{n-1}} R_{\gamma_t} \) which depends only on \( \gamma, \tilde{A}_t \) and \( \tilde{H}_t \) denote the second fundamental form and the mean curvature of \( \Sigma_t \) induced from the metric \( \tilde{g} = dt^2 + \tilde{\gamma}_t \). Since \( \tilde{\gamma}_t \) strictly monotonically increases, \( \tilde{A}_t > 0 \). It follows that \( \tilde{H}_t > 0 \) and \( \tilde{H}_t^2 - ||\tilde{A}_t||^2 > 0 \). Consider the quasi-spherical metric equation

\[
\begin{aligned}
\tilde{H}_t \frac{\partial u}{\partial t} &= u^2 \Delta_{\gamma_t} u + \frac{1}{2} (u - u^3) R_{\gamma_t} - \frac{1}{2} \tilde{R}_{\tilde{g}} u + \frac{1}{2} Ku^3 \\
\end{aligned}
\]

(24)

Since the coefficient of \( u^3 \) is \( K - R_{\gamma_t} \leq 0 \), above equation has solution on the whole \([0,1]\), and hence the scalar curvature \( R(g) = K \) on \( S^{n-1} \times [0,1] \), where \( g = u^2 dt^2 + \tilde{\gamma}_t \).

Note that \( R_{\gamma_t} \geq R_g \). Let \( A_t \) and \( H_t \) denote the second fundamental form and the mean curvature of \( \Sigma_t \) induced from metric \( g \). It is not hard to see

\[
A_t = u^{-1} \tilde{A}_t, \quad H_t = u^{-1} \tilde{H}_t.
\]

(25)

By the Jacobi equation, Gauss equation and relation (25), we have

\[
\begin{aligned}
\frac{d}{dt} \int_{\Sigma_t} H_t d\mu_{\gamma_t} &= \int_{\Sigma_t} \left( H_t^2 - ||A_t||^2 - \text{Ric}(\nu, \nu) \right) u d\mu_{\gamma_t} \\
&= \frac{1}{2} \int_{\Sigma_t} \left( H_t^2 - ||A_t||^2 + R_{\gamma_t} - R_g \right) u d\mu_{\gamma_t} \\
&> 0.
\end{aligned}
\]

It follows that

\[
\int_{S^{n-1}} H d\mu_{\gamma} = \int_{\Sigma_0} H_0 d\mu_{\gamma_0} < \int_{\Sigma_1} H_1 d\mu_{\gamma_1}.
\]

Glue \((\Omega, \tilde{g})\) to \((S^{n-1} \times [0,1], g)\) along \((S^{n-1}, \gamma)\) and denote the new manifold by \((\Omega', \tilde{g}')\). Then \((\Omega', \tilde{g}')\) is a spin manifold with corners where the mean curvatures from two sides match. And \((\Omega', \tilde{g}')\) is a spin fill-in of \((S^{n-1}, e^{2a_{\gamma_{\text{std}}}} \gamma, H_1)\). On \(\Omega'\) away from the corner, \(R_{\tilde{g}}' \geq \min\{0, K\} \). For simplicity, we still denote \((\Omega', \tilde{g}')\) as \((\Omega, g)\), \(H\) as the mean curvature of the boundary of the new manifold \((\Omega, g)\) with respect to the outward
unit normal vector, then clearly $H > 0$. Without loss of generality, we assume $R_{\hat{g}} \geq -n(n - 1), a = \log \sinh 1$ in the sequel.

Taking the background hyperbolic metric $g_0 = dr^2 + \sinh^2 r \gamma_{std}$. Denote $H^0(r)$ be the mean curvature of $r$-slice. Then $H^0(r) = (n - 1) \coth r$. Consider the quasi-spherical metric equation

$$
\begin{aligned}
\begin{cases}
\sinh(2r) \frac{\partial u}{\partial r} = \frac{2}{n - 1} u^2 \Delta_{\gamma_{std}} u + (n - 2 + n \sinh^2 r)(u - u^3) \\
u(r, 1) = \frac{(n - 1) \coth 1}{H} > 0.
\end{cases}
\end{aligned}
$$

(26)

Since the coefficient of $u^3$ is negative, above equation has positive solution on the whole $[1, \infty)$. Set $\hat{g} = u^2 dr^2 + \sinh^2 r \gamma_{std}$. Then $R_{\hat{g}} \equiv -n(n - 1)$. Let $\hat{A}_r$ and $\hat{H}(r)$ denote the second fundamental form and the mean curvature of $\Sigma_t$ induced from metric $\hat{g}$. Set

$$
m(r) = \frac{1}{(n - 1) \omega_{n - 1}} \int_{S^n} (H^0(r) - \hat{H}(r)) \cosh r \, d\mu_r.
$$

Then it is easy to see that,

$$
m(r) = \sinh^{n-2} r \cosh^2 r \int_{S^{n-1}} (1 - u^{-1}(\omega, r)) \, d\omega.
$$

(27)

We have

$$
m'(r) = \sinh^{n-3} r \cosh r (n - 2 + n \sinh^2 r) \int_{S^{n-1}} (1 - u^{-1}) \, d\omega \\
+ \sinh^{n-2} r \cosh^2 r \int_{S^{n-1}} u^{-1} \frac{\partial u}{\partial r} \, d\omega \\
= \sinh^{n-3} r \cosh r (n - 2 + n \sinh^2 r) \int_{S^{n-1}} (1 - u^{-1}) \, d\omega \\
+ \frac{1}{2} \sinh^{n-3} r \cosh r \int_{S^{n-1}} \frac{2}{n - 1} \Delta_{\gamma_{std}} u + (n - 2 + n \sinh^2 r)(u^{-1} - u) \, d\omega \\
= -\frac{1}{2} \sinh^{n-3} r \cosh r \int_{S^{n-1}} u^{-1}(u - 1)^2 \, d\omega \leq 0.
$$

So $m(r)$ monotonically decreases.

On the other hand, by the same computations in the proof of Theorem 2.1 in [37], we see that the solution of (26) satisfying

$$
u = 1 + e^{-nr} v(\omega) + o(e^{-(n+1)r}) \quad \text{as } r \to \infty,
$$

(28)

where $v$ is a smooth function on $S^{n-1}$. Combining this with (27), we see that

$$
\lim_{r \to \infty} m(r) = 2^{-n} \int_{S^{n-1}} v \, d\omega
$$

(29)

Let

$$
\rho = \frac{e^r}{e^r - 1}.
$$
Then
\[ \hat{g} = u^2 dr^2 + \sinh^2 r \gamma_{std} \]
\[ = u^2 \sinh^{-2} \rho (d\rho^2 + u^{-2} \gamma_{std}) \]  
(30)

Combining (28) with Lemma 6.5 in [9], we see that there is a geodesic defining function \( \hat{r} \) for \( \hat{g} \) with
\[ \hat{g} = \sinh^{-2} \rho (d\hat{\rho}^2 + \gamma_{std} + 2^{1-n} \hat{\rho}^n v \gamma_{std} + O(\hat{\rho}^{n+1})) , \]
where \( \hat{\rho} \) is defined by
\[ \hat{r} = \frac{\cosh \hat{\rho} - 1}{\sinh \hat{\rho}} , \]

Glue \( (\Omega, g) \) to \( (S^{n-1} \times [1, \infty), \hat{g}) \) along \( S^{n-1} \), the resulting manifold is spin and asymptotically hyperbolic (AH) with corners, its scalar curvature is at least \(-n(n-1)\) away from the corners, and mean curvature along the two sides of corners with respect to the outward unit normal vectors are equal. Therefore, by Theorem 1.1 in [9] we see that
\[ \int_{S^n-1} v d\omega \geq 0 . \]

Together with monotonicity of \( m(r) \) and (29) we see that (23) is true and which completes the proof of Theorem 4.1.

\[ \square \]

5. The relationship with Positive mass theorems

In this section, we are going to investigate the relationship between the quantity \( \Lambda_+ (S^{n-1}, \gamma_{std}) \) with the positive mass theorems (PMT) of asymptotically flat (AF) manifolds. Let us begin with

**Definition** 5.1. Let \( n \geq 3 \). A Riemannian manifold \( (M^n, g) \) is said to be asymptotically flat (AF) if there is a compact set \( K \subset M^n \) such that \( M^n \setminus K \) is diffeomorphic to \( \mathbb{R}^n \setminus B_1(0) \) and in this coordinates, \( g \) satisfies
\[ |g_{ij} - \delta_{ij}| + |x| |\partial g_{ij}| + |x|^2 |\partial^2 g_{ij}| + |x|^3 |\partial^3 g_{ij}| = O(|x|^{-p}) \]
for some \( p > \frac{n-2}{2} \). Furthermore, we require that
\[ \int_{M^n} |R_g| d\mu_g < \infty . \]

An AF manifold \( (M^n, g) \) is called asymptotically Schwarzschild (AS) if there is a compact set \( K \subset M^n \) such that \( M^n \setminus K \) is diffeomorphic to \( \mathbb{R}^n \setminus B_1(0) \) and in this coordinates, \( g \) satisfies
\[ g_{ij} = \left( 1 + \frac{2m}{n-2} r^{2-n} \right) \delta_{ij} + b_{ij} , \]
where \( r = |x| , m > 0 , \) and \( b_{ij} \) decays as
\[ |b_{ij}| + r |\partial b_{ij}| + r^2 |\partial^2 b_{ij}| + r^3 |\partial^3 b_{ij}| = O(r^{1-n}) . \]
For an AF manifold in Definition 5.1 we can define a conserved quantity which is called ADM mass as following.

**Definition 5.2.** The Arnowitt-Deser-Misner (ADM) mass \([2]\) of an AF manifold \((M^n, g)\) is defined by

\[
m_{\text{ADM}}(M^n, g) = \lim_{r \to \infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} (g_{ij,i} - g_{ii,j}) \nu^j \, dS_r,
\]

where \(S_r\) is the coordinate sphere near the infinity, \(\nu\) is the Euclidean outward unit normal to \(S_r\), and \(dS_r\) is the Euclidean area element on \(S_r\).

Besides these notations, we also need the following

**Definition 5.3.** A complete noncompact Riemannian manifold \((M^n, g)\) is said to be asymptotically hyperbolic (AH) if there is a compact manifold \((X^n, g)\) with boundary \(\partial X^n\) and a smooth function \(t\) on \(X^n\) such that the following are true:

(i) \(M^n\) is diffeomorphic to \(X^n \setminus \partial X^n\) (we will identify \(M^n\) and \(X^n \setminus \partial X^n\) in the sequel).
(ii) \(t = 0\) on \(\partial X^n\), and \(t > 0\) on \(M^n\).
(iii) \(g = t^2 g\) extends to be \(C^n\) up to the boundary.
(iv) \(|dt|_g = 1\) at \(\partial X^n\).
(v) Each component \(\Sigma\) of \(\partial X^n\) is the standard \((n-1)-\) sphere \((S^{n-1}, g_0)\) and there is a collar neighborhood of \(\Sigma\) where

\[
g = \sinh^{-2} t(dt^2 + g_t)
\]

with

\[
g_t = g_0 + \frac{t^n}{n} h + O(t^{n+1})
\]

where \(h\) is a \(C^{n-1}\) symmetric two tensor on \(S^{n-1}\).

Let \((M^n, g)\) be an AH manifold with scalar curvature \(R \geq -n(n-1)\), for simplicity, we assume \(M^n\) has only one end, we say \(PMT\) holds on \(M^n\) (see Theorem 1.1 in [36]) if

\[
\int_{S^{n-1}} \text{trace}_{g_0} h dV_{g_0} \geq 0,
\]

and the equality holds if and only if \((M^n, g)\) is isometric to the hyperbolic space \(\mathbb{H}^n\).

The first result in this section is the

**Theorem 5.1.** Let \((M^n, g)\) be an asymptotically flat manifold with nonnegative scalar curvature, then \(\Lambda_+(S^{n-1}, \gamma_{std}) = (n-1)\omega_{n-1}\) if and only if \(m_{\text{ADM}}(M^n, g) \geq 0\), and \(m_{\text{ADM}}(M^n, g) = 0\) if and only if \((M^n, g)\) is isometric to \(\mathbb{R}^n\).
Proof. If PMT holds on AF manifolds, then by convergence and nonincreasing of Brown-York mass along quasi-spherical metric (see Theorem 2.1 and Lemma 4.2 in [24]), we see that \( \Lambda_+ (S^{n-1}, \gamma_{\text{std}}) \leq (n - 1) \omega_{n-1} \), on the other hand we obviously have \( \Lambda_+ (S^{n-1}, \gamma_{\text{std}}) \geq (n - 1) \omega_{n-1} \), thus conclusion is true. Hence, it is enough to show that the equality \( \Lambda_+ (S^{n-1}, \gamma_{\text{std}}) = (n - 1) \omega_{n-1} \) implies PMT on AF manifolds.

Due to Schoen-Yau’s result (the statement in the bottom of p.48 in [27]), it suffices to show Theorem 5.1 when \( g \) is AS.

The Christoffel symbol of \( g \) is calculated as
\[
\Gamma^k_{ij} = m (\delta_{ij} x^k - \delta_{ik} x^j - \delta_{jk} x^i) r^{-n} + O(r^{-n}).
\]
In particular,
\[
g^{ij} \Gamma^k_{ij} x^k = m (n - 2) r^{2-n} + O(r^{1-n}).
\]
Simple calculation gives
\[
|\nabla r| = r^{-1} \sqrt{g^{ij} x^i x^j} = 1 - \frac{m}{n - 2} r^{2-n} + O(r^{1-n}),
\]
\[
\partial_k |\nabla r| = |\nabla r|^{-1} \left( \frac{g^{ij} x^j}{r^2} - \frac{|\nabla r|^2 x^k}{r^2} + \frac{(g^{ij})_k x^i x^j}{2r^2} \right)
= \frac{m x^k}{r^n} + O(r^{-n}),
\]
and
\[
\Delta r = g^{ij} \left( \frac{\delta_{ij}}{r} - \frac{x^i x^j}{r^3} - \Gamma^k_{ij} x^k \right) r^{-n} + O(r^{-n}).
\]
Let \( \nu \) denote the outward unit normal of \( S_r \). It is not hard to see
\[
\nu = \frac{\nabla r}{|\nabla r|}.
\]
The mean curvature of \( S_r \) in \((M^n, g)\) is
\[
H = \text{div} \left( \frac{\nabla r}{|\nabla r|} \right)
= \frac{\Delta r}{|\nabla r|} - \frac{\nabla r \cdot \nabla |\nabla r|}{|\nabla r|^2}
= \frac{n - 1}{r} \left( 1 - \frac{n - 1}{n - 2} \frac{m}{r^{n-2}} \right) + O(r^{-n}).
\]
It is not hard to see that the induced metric \( \gamma_r \) on \( S_r \) satisfies
\[
\gamma_r = r^2 \left( 1 + \frac{2}{n - 2} \frac{m}{r^{n-2}} \right) (\gamma_{\text{std}} + h),
\]
where $h = O(r^{1-n})$. So the area of $S_r$ is

$$A(\gamma_r) = \omega_{n-1} r^{n-1} \left( 1 + \frac{n-1}{n-2} \frac{m}{r^{n-2}} \right) + O(1).$$

It follows that

$$\int_{S_r} H \, d\mu_{\gamma_r} = (n-1)\omega_{n-1} r^{n-2} + O(r^{-1}). \quad (32)$$

By the scaling property of the $\Lambda_+$-invariant, we have

$$\Lambda_+(S_r, \gamma_r) = r^{n-2} \left( 1 + \frac{2}{n-2} \frac{m}{r^{n-2}} \right)^{\frac{n-2}{2}} \Lambda_+(S^{n-1}, \gamma_{std} + h).$$

On the other hand, by Lemma 2.1 we have

$$\Lambda_+(S^{n-1}, \gamma_{std})(1 - |h|)^{\frac{n-2}{2}} \leq \Lambda_+(S^{n-1}, \gamma_{std} + h) \leq \Lambda_+(S^{n-1}, \gamma_{std})(1 + |h|)^{\frac{n-2}{2}}.$$

It follows that

$$\Lambda_+(S_r, \gamma_r) = (r^{n-2} + m) \Lambda_+(S^{n-1}, \gamma_{std}) + O(r^{-1}). \quad (33)$$

Then together with (32) and (33), we see that, as $r$ goes to the infinity, we have

$$m_{BY}(S_r, \gamma_r) = m + O(r^{-1}),$$

and by the definition of $\Lambda_+(S^{n-1}, \gamma, 0)$ and nonnegativity of the scalar curvature of $(M^n, g)$, we see that $m_{BY}(S_r, \gamma_r) \geq 0$, hence, we get $m \geq 0$. Once we get nonnegativity of the ADM mass, the rigidity part of Theorem 5.1 can be obtained by usual deformation arguments. For instance, if $(M^n, g)$ is not Ricci flat, then we can run Ricci flow on $M$, and get an AF metric with strictly positive scalar curvature and zero ADM mass (13), and then by conformal deformation, we get an AF metric with strictly positive scalar curvature and negative ADM mass, which is contradiction, thus, $(M^n, g)$ is Ricci flat if its ADM mass vanishes. Finally, by Corollary 6.7 in [7], we know that $(M^n, g)$ is isometric to $R^n$, which completes the proof of Theorem 5.1.

Remark 5.1. As a by-product, by combining (32) and (33), for AS metrics, we obtain

$$\lim_{r \to \infty} m_{BY}(S_r, \gamma_r) \geq m.$$

It is interesting to see that Theorem 5.1 reveals relationship between PMT on AH manifolds with that on AF manifolds. Namely,
Theorem 5.2. Suppose PMT holds on AH manifolds, then PMT is true for AF manifolds, i.e., let \((M^n, g)\) be an asymptotically flat manifold with nonnegative scalar curvature, then \(m_{ADM}(M^n, g) \geq 0\) and \(m_{ADM}(M^n, g) = 0\) if and only if \((M^n, g)\) is isometric to \(\mathbb{R}^n\).

**Proof.** By the definition, we see that for any \(\kappa < 0\),
\[
\Lambda_+, \kappa (S^{n-1}, \gamma_{std}) = |\kappa|^{2-n} \Lambda_+, -1(S^{n-1}, \kappa^2 \gamma_{std}),
\]
and
\[
\Lambda_+, -1(S^{n-1}, \kappa^2 \gamma_{std}) \geq (n - 1) |\kappa|^{n-2} \sqrt{1 + \kappa^2} \omega_{n-1}. \tag{34}
\]
On the other hand, by the same arguments in the proof of the monotonicity of \(m(r)\), (29) in the Theorem 4.1, and together with PMT on AH manifolds, we see that
\[
\Lambda_+, -1(S^{n-1}, \kappa^2 \gamma_{std}) \leq (n - 1) |\kappa|^{n-2} \sqrt{1 + \kappa^2} \omega_{n-1}. \tag{35}
\]
Together with (34) implies that
\[
\Lambda_+, \kappa (S^{n-1}, \gamma_{std}) = (n - 1) \sqrt{1 + \kappa^2} \omega_{n-1}. \tag{36}
\]
Due to (1), we obtain that for any \(\kappa < 0\)
\[
\Lambda_+(S^{n-1}, \gamma_{std}) \leq (n - 1) \sqrt{1 + \kappa^2} \omega_{n-1},
\]
hence
\[
\Lambda_+(S^{n-1}, \gamma_{std}) \leq (n - 1) \omega_{n-1},
\]
Note that \((S^{n-1}, \gamma_{std})\) is the boundary of the standard unit ball in \(\mathbb{R}^n\), we see that
\[
\Lambda_+(S^{n-1}, \gamma_{std}) \geq (n - 1) \omega_{n-1}.
\]
Thus, we finally get
\[
\Lambda_+(S^{n-1}, \gamma_{std}) = (n - 1) \omega_{n-1},
\]
and by Theorem 5.1, we know that PMT on AF manifolds holds, thus, we finish the proof of Theorem 5.2. \(\square\)

6. Extensibility of boundary metrics to PSC metrics inside and obstructions on Mean Curvature

In this section, we prove Theorem 1.1 and Theorem 1.2. To prove the results, we exploit the PSC-cobordism of Bartnik data.

**Definition 6.1 (29).** Given Bartnik data \((\Sigma_i^{n-1}, \gamma_i, H_i) \ (i = 1, 2)\), we say \((\Sigma_i^{n-1}, \gamma_1, H_1)\) is PSC-cobordant to \((\Sigma_2^{n-1}, \gamma_2, H_2)\) if there is an orientable \(n\)-dimensional manifold \((\Omega^n, g)\) such that \(\partial \Omega^n = \Sigma_1^{n-1} \cup \Sigma_2^{n-1}\), \(R_g > 0\), the induced metric on \(\Sigma_i^{n-1}\) is \(\gamma_i\) and the mean curvature (with respect to the outward normal) is \(H_i\) for \(i = 1, 2\).
**Lemma 6.1.** Let $\Sigma$ be a closed manifold, and $\gamma_0$, $\gamma_1$ be two smooth metrics on $\Sigma$ with $\gamma_1 > \gamma_0$. Then for any smooth function $h$ on $\Sigma$, there are smooth functions $h_0$ and $h_1$ on $\Sigma$, such that $(\Sigma, \gamma_0, h_0)$ is PSC-cobordant to $(\Sigma, \gamma_1, h_1)$ and $h_1 > h$. Moreover, the corresponding region is diffeomorphic to $\Sigma \times [0,1]$.

**Proof of Lemma 6.1.** We construct the required PSC-cobordism via quasi-spherical type equation. Set $\gamma_t = (1 - t)\gamma_0 + t\gamma_1$. Then $\{\gamma_t\}_{t \in [0,1]}$ is a smooth path of metrics on $\Sigma$ and monotonically increases. Set $\Omega = \Sigma \times [0,1]$. On $\Omega$, define $\bar{g} = dt^2 + \gamma_t$. Denote the slice $\Sigma \times \{t\}$ by $\Sigma_t$. Let $\bar{A}_t$ and $\bar{H}_t$ be the second fundamental form and the mean curvature of $\Sigma_t$ induced from $\bar{g}$ with respect to the $\partial_t$-direction. Since $\gamma_t$ strictly monotonically increases, $\bar{A}_t > 0$. It follows that $\bar{H}_t > 0$, and $\bar{H}_t^2 - \|\bar{A}_t\|^2 > 0$.

For constants $\delta > 0$ and $\varepsilon > 0$, consider the following quasi-spherical metric equation on $\Sigma \times [0,1]$ with Cauchy data

\[
\begin{aligned}
\bar{H}_t \frac{\partial u}{\partial t} &= u^2 \Delta_{\gamma_t} u + \frac{1}{2} (\delta - R_{\gamma_t}) u^3 + \frac{1}{2} (R_{\gamma_t} - R_{\bar{g}}) u, \\
u(x,0) &\equiv \varepsilon > 0.
\end{aligned}
\tag{37}
\]

Since the reaction part of this reaction-diffusion equation has the factor $u$, by comparison with the solution to the corresponding ODE, for sufficiently small $\varepsilon$, above equation has a positive solution on the whole $[0,1]$. And there is a constant $\varepsilon_0 > 0$ depending only on $\sup_{\Omega} |R_{\gamma_t}|$, $\sup_{\Omega} |R_{\bar{g}}|$, $\sup_{\Omega} \bar{H}_1^{-1}$ and $\delta$, such that for $\varepsilon \leq \varepsilon_0$, there is a constant $C$ independent of $\varepsilon$ such that

$$0 < u \leq C\varepsilon$$

for all $(x,t) \in \Omega$.

Set $g = u^2 dt^2 + \gamma_t$. Then $R_g \equiv \delta > 0$. Let $A_t$ and $H_t$ denote the second fundamental form and the mean curvature of $\Sigma_t$ induced from metric $g$ with respect to the $\partial_t$-direction. It is not hard to see

$$H_t = u^{-1} \bar{H}_t.$$ 

It follows that

$$H_1 \geq \frac{1}{\varepsilon C} \min H_1.$$ 

Hence, we can choose sufficiently small $\varepsilon$ so that $H_1 > h$. Set $h_0 = -H_0$, $h_1 = H_1$. Then $(\Sigma, \gamma_0, h_0)$ is PSC-cobordant to $(\Sigma, \gamma_1, h_1)$, and $h_1 > h$. \qed

The proof of Theorem 1.1 is based on above lemma and the following strong result due to Gromov himself.

**Theorem 6.1 ([14], p.31).** On an open $n$-manifold $M$ ($n > 1$) there exist Riemannian metrics both with positive and with negative curvature.
Proof of Theorem 1.1. By Theorem 6.1 there is a metric \( g_1 \) of positive sectional curvature on \( X \). Denote the induced metric on \( Y \) from \( g_1 \) by \( \gamma_1 \), and the mean curvature of \( Y \) in \( (X, g_1) \) with respect to the outward normal by \( h \). We may assume \( \gamma_1 > h \) after a suitable rescaling. By Lemma 6.1 there are smooth functions \( h_0 \) and \( h_1 \) on \( Y \), such that \( (Y, \gamma, h_0) \) is PSC-cobordant to \( (Y, \gamma_1, h_1) \) and \( h_1 > -h \). Denote the corresponding region of PSC-cobordism by \( (\Omega, g_0) \). Glue \( (\Omega, g_0) \) to \( (X, g_1) \) along \( (Y, \gamma_1) \). Denote the obtained manifold by \( (\tilde{\Omega}, \tilde{g}) \). It is not hard to see that \( \tilde{g} \) is Lipschitz across \( (Y, \gamma_1) \). Away from \( (Y, \gamma_1) \), \( \tilde{g} \) is smooth and \( R_{\tilde{g}} > 0 \). Since \( h_1 > -h \), after carrying Miao’s mollifying procedure and a suitable conformal deformation, we may get a PSC metric \( g \) on \( \tilde{X} \) with \( g = \tilde{g} \) on \( \partial X \). Apparently, \( g|_{\partial X} = \gamma \). Since \( \Omega \) is diffeomorphic to \( \Sigma \times [0,1] \), \( \tilde{X} \) is diffeomorphic to \( X \). Thus \( g \) can be regarded as a metric on \( X \). This finishes the proof. \( \square \)

Proof of Theorem 1.2. By definition, \( \Sigma^n \) can be smoothly embedded into \( \mathbb{R}^{n+1} \). Choose an arbitrary embedding map \( F : \Sigma^n \to \mathbb{R}^{n+1} \). Then \( F^*(\gamma_E) \) is a smooth metric on \( \Sigma^n \), where \( \gamma_E \) denotes the canonical Euclidean metric on \( \mathbb{R}^{n+1} \). For \( \lambda > 0 \), set \( H_0 = \lambda F \). Observe, \( F_0^*(\gamma_E) = \lambda^2 F^*(\gamma_E) \). So there is a \( \lambda_0 \) such that \( F_{\lambda_0}^*(\gamma_E) > \gamma \). Denote the mean curvature of \( F_{\lambda_0} \) in \( \mathbb{R}^{n+1} \) with respect to the outward normal by \( H \). By Lemma 6.1 there are smooth functions \( H_0 \) and \( H_1 > H \) on \( \Sigma \), such that \( (\Sigma^n, \gamma, H_0) \) is PSC-cobordant to \( (\Sigma^n, F_0^*(\gamma_E), H_1) \) via a Riemannian region \( (\Sigma^n \times [0,1], \gamma) \). Denote the exterior region of \( F_{\lambda_0} \) in \( \mathbb{R}^{n+1} \) by \( \tilde{\Omega} \). Take \( h_0(\Sigma^n, \gamma) = \max H_0 \). Suppose \( \tilde{H} \) is a smooth function on \( \Sigma^n \) satisfying \( \tilde{H} > h_0(\Sigma^n, \gamma) \), and \( (\tilde{\Omega}, \tilde{g}) \) is a fill-in of \( (\Sigma^n, \gamma, H) \) with NNSC. Glue \( (\tilde{\Omega}, \tilde{g}) \) to \( (\Sigma^n \times [0,1], \gamma) \) along \( (\Sigma_0, 0) \), and glue the obtained manifold to \( (\tilde{\Omega}, \tilde{g}) \) along \( (\Sigma_1, \gamma_1) \). Denote the new manifold by \( (M^{n+1}, \tilde{g}) \). Then \( (M^{n+1}, \tilde{g}) \) is a complete manifold with corners and a flat end. The scalar curvature of \( (M^{n+1}, \tilde{g}) \) is nonnegative away from the corners. Along the corners \( \Sigma_0 \) and \( \Sigma_1 \), the inside mean curvature \( H_- \) is strictly greater than the outside mean curvature \( H_+ \). By Miao’s result, \( (M^{n+1}, \tilde{g}) \) must have positive mass, which contradicts the flatness of the end. Consequently, for pointwisely sufficient large \( H \), \( (\Sigma^n, \gamma, H) \) admits no fill-in of NNSC. From the proof, we may see the threshold \( h_0 \) depends only on \( (\Sigma^n, \gamma) \).

\( \square \)

We can generalize Theorem 1.2 to fill-ins with scalar curvature bounded below.

Theorem 6.2. Suppose \( \Sigma^n \in \mathcal{C}_n \) \((2 \leq n \leq 6)\). For any metric \( \gamma \) on \( \Sigma^n \) and any constant \( \kappa \), there is a constant \( h_0(\Sigma^n, \gamma, \kappa) \), such that for any smooth function \( H \) on \( \Sigma^n \) satisfying \( H > h_0(\Sigma^n, \gamma, \kappa) \), \( (\Sigma^n, \gamma, H) \) admits no fill-in of scalar curvature \( R \geq -n(n-1)\kappa^2 \).
Sketch of Proof. The proof is essentially the same as the proof of Theorem 1.2. The difference is embedding \((\Sigma^n, \gamma)\) into an anti-de Sitter-Schwarzschild space of scalar curvature \(-n(n - 1)\kappa^2\) and mass \(m < 0\) (\(|m|\) is very small), then using the positive mass theorem for asymptotically hyperbolic manifold with negative mass aspect function \([1]\) to draw a contradiction. \(\Box\)

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(Yuguang Shi) Key Laboratory of Pure and Applied Mathematics, School of Mathematical Sciences, Peking University, Beijing, 100871, P. R. China
E-mail address: ygshi@math.pku.edu.cn

(Wenlong Wang) School of Mathematics and LPMC, Nankai University, Tianjin, 300071, P. R. China
E-mail address: wangwl@nankai.edu.cn

(Guodong Wei) School of Mathematics (Zhuhai), Sun Yat-sen University, Zhuhai, Guangdong 519082, PR China
E-mail address: weigd3@mail.sysu.edu.cn