The null subspace of $G_{4,1}$ as source of the main physical theories

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The relationship between geometry and physics is probably stronger in General Relativity (GR) than in any other physics field. It is the author’s belief that a perfect theory will eventually be formulated, where geometry and physics become indistinguishable, so that the complete understanding of space properties, together with proper assignments between geometric and physical entities, will provide all necessary predictions.

We don’t have such perfect theory yet, however the author intends to show that GR and Quantum Mechanics (QM) can be seen as originating from properties of the null subspace of 5-dimensional space with signature $(-+++)$, together with its associated geometric algebra $G_{4,1}$. The space so defined is really 4-dimensional because the null condition effectively reduces the dimensionality by one. Besides generating GR and QM, the same space generates also 4-dimensional Euclidean space where dynamics can be formulated and is quite often equivalent to the relativistic counterpart. Euclidean relativistic dynamics resembles Fermat’s principle extended to 4 dimensions and is thus designated as 4-Dimensional Optics (4DO).

In this presentation the author starts with the geometric algebra $G_{4,1}$ with imposition of the null displacement length condition and derives the method to transpose between the metrics of GR and 4DO; this transition is proven viable for stationary metrics. It is hopeless to apply Einstein type equations in 4DO, for the simple reason that a null Ricci tensor always leads to a metric diverging to infinity. The author uses geometric arguments to establish alternative equations which are solved for the case of a stationary mass and produce a solution equivalent to Schwarzschild’s metric in terms of PPN parameters.

As a further development, the author analyses the case of a monogenic function in $G_{4,1}$. The monogenic condition produces an equation that can be conveniently converted into Dirac’s, with the added advantage that it has built in standard model gauge group symmetry.

1 Introduction

According to general consensus any physics theory is based on a set of principles from which predictions are derived using established mathematical derivations; the validity of such theory depends on agreement between predictions and observed physical reality. In that sense this paper does not formulate physical theories because it does not presume any physical principles; for instance it does not assume speed of light constancy or equivalence between frame acceleration and gravity.

This is a paper about geometry; all along the paper, in several occasions, a parallel is made with the physical world by assigning a physical meaning to geometric entities and this allows predictions to be made. However the validity of derivations and overall consistency of the exposition is independent of prediction correctness.

The only postulates in this paper are of a geometrical nature and can be summarised in the definition of the space we are going to work with; this is the 4-dimensional null subspace of the 5-dimensional space with signature $(-+++)$.
metric space does not imply any assumption for physical space up to the point where geometric entities like coordinates and geodesics start being assigned to physical quantities like distances and trajectories. Some of those assignments will be made very soon in the exposition and will be kept consistently until the end in order to allow the reader some assessment of the proposed geometric model as a tool for the prediction of physical phenomena. Mapping between geometry and physics is facilitated if one chooses to work always with non-dimensional quantities; this is done with a suitable choice for standards of the fundamental units. From this point onwards all problems of dimensional homogeneity are avoided through the use of normalising factors listed below for all units, defined with recourse to the fundamental constants: \( h \rightarrow \text{Planck constant divided by } 2\pi \), \( G \rightarrow \text{gravitational constant} \), \( c \rightarrow \text{speed of light} \) and \( e \rightarrow \text{proton charge} \).

| Length        | Time          | Mass            | Charge |
|---------------|---------------|-----------------|--------|
| \( \sqrt{\frac{Gh}{c^3}} \) | \( \sqrt{\frac{Gh}{c^5}} \) | \( \sqrt{\frac{hc}{G}} \) | \( e \) |

This normalisation defines a system of non-dimensional units (Planck units) with important consequences, namely: 1) All the fundamental constants, \( h, G, c, e \), become unity; 2) a particle’s Compton frequency, defined by \( \nu = \frac{mc^2}{\hbar} \), becomes equal to the particle’s mass; 3) the frequent term \( GM/(c^2r) \) is simplified to \( M/r \).

4-dimensional space can have amazing structure, providing countless parallels to the physical world; this paper is just a limited introductory look at such structure and parallels. The exposition makes full use of an extraordinary and little known mathematical tool called geometric algebra (GA), a.k.a. Clifford algebra, which received an important thrust with the works of David Hestenes \[1\]. A good introduction to GA can be found in Gull et al. \[2\] and the following paragraphs use basically the notation and conventions therein. A complete course on physical applications of GA can be downloaded from the internet \[3\]; the same authors published a more comprehensive version in book form \[4\]. An accessible presentation of mechanics in GA formalism is provided by Hestenes \[5\].

## 2 Introduction to geometric algebra

We will use Greek characters for the indices that span 1 to 4 and Latin characters for those that exclude the 4 value; in rare cases we will have to use indices spanning 0 to 3 and these will be denoted with Greek characters with an over bar. Einstein’s summation convention will be adopted as well as the compact notation for partial derivatives \( \partial_{\mu} = \partial/\partial x^\mu \). The geometric algebra of the hyperbolic 5-dimensional space we want to consider \( \mathcal{G}_{4,1} \) is generated by the frame of orthonormal vectors \( \{i, \sigma_\mu\}, \mu = 1 \ldots 4 \), verifying the relations

\[
\begin{align*}
    i^2 &= -1, \\
    i\sigma_\mu + \sigma_\mu i &= 0, \\
    \sigma_\mu \sigma_\nu + \sigma_\nu \sigma_\mu &= 2\delta_{\mu\nu}.
\end{align*}
\]

We will simplify the notation for basis vector products using multiple indices, i.e. \( \sigma_\mu \sigma_\nu \equiv \sigma_{\mu\nu} \). The algebra is 32-dimensional and is spanned by the basis

- 1 scalar, 1
- 5 vectors, \( \{i, \sigma_\mu\} \)
- 10 bivectors (area), \( \{i\sigma_\mu, \sigma_{\mu\nu}\} \)
- 10 trivectors (volume), \( \{i\sigma_{\mu\nu}, \sigma_{\mu\nu\lambda}\} \)
- 5 tetravectors (4-volume), \( \{I, \sigma_\mu I\} \)
- 1 pseudoscalar (5-volume), \( I \equiv i\sigma_1\sigma_2\sigma_3\sigma_4 \)

Several elements of this basis square to unity:

\[
\begin{align*}
    (\sigma_\mu)^2 &= (i\sigma_\mu)^2 = (i\sigma_{\mu\nu})^2 = (iI)^2 = 1; \\
    (\sigma_{\mu\nu})^2 &= (\sigma_{\mu\nu\lambda})^2 = (\sigma_\mu I)^2 = I^2 = -1.
\end{align*}
\]

and the remaining square to \(-1\):

\[
\begin{align*}
    i^2 &= (i\sigma_\mu)^2 = (i\sigma_{\mu\nu})^2 = (i\sigma_\mu I)^2 = 1.
\end{align*}
\]

Note that the symbol \( i \) is used here to represent a vector with norm \(-1\) and must not be confused with the scalar imaginary, which we don’t usually need. Note also that the pseudoscalar \( I \) commutes with all the other basis elements while being a square root of \(-1\) and plays the role of the scalar imaginary in complex algebra.

The geometric product of any two vectors \( a = a^0 i + a^\mu \sigma_\mu \) and \( b = b^0 i + b^\nu \sigma_\nu \) can be decomposed into a symmetric part, a scalar called the inner product, and an anti-symmetric part, a bivector called the exterior product.

\[
ab = a \cdot b + a \wedge b, \quad ba = a \cdot b - a \wedge b.
\]

Reversing the definition one can write interior and exterior products as

\[
a \cdot b = \frac{1}{2} (ab + ba), \quad a \wedge b = \frac{1}{2} (ab - ba).
\]
When a vector is operated with a multivector the inner product reduces the grade of each element by one unit and the outer product increases the grade by one. There are two exceptions; when operated with a scalar the inner product does not produce grade $-1$ but grade $1$ instead, and the outer product with a pseudoscalar is disallowed.

3 Displacement and velocity

Any displacement in this 5-dimensional hyperbolic space can be defined by the displacement vector

$$dx = idx^0 + \sigma_\mu dx^\mu;$$

(8)

and the null space condition implies that $dx$ has zero length

$$dx^2 = dx \cdot dx = 0;$$

(9)

which is easily seen equivalent to either of the relations

$$(dx^0)^2 = \sum (dx^\mu)^2;$$

(10)

$$(dx^4)^2 = (dx^0)^2 - \sum (dx^\mu)^2.$$

These equations define the metrics of two alternative 4-dimensional spaces, one Euclidean the other one Minkowskian, both derived from the null 5-dimensional subspace.

A path on null space does not have any affine parameter but we can use Eqs. (10) to express 4 coordinates in terms of the fifth one. We will assign the letter $t$ and physical time to coordinate $x^0$ while the letter $\tau$ and physical proper time are assigned to coordinate $x^4$; total derivatives with respect to $t$ will be denoted by an over dot while total derivatives with respect to $\tau$ will be denoted by a "check", as in $\check{f}$. Dividing both members of Eq. (8) by $d\tau$ we get

$$\dot{x} = i + \sigma_\mu \dot{x}^\mu = i + v.$$  

(11)

This is the definition for the velocity vector $v$; it is important to stress again that the velocity vector defined here is a geometrical entity and its possible relation to physical velocity is a direct result of the coordinate assignments made above; if later we were to find that the velocity vector bears no relation to physical velocity only the assignments would have to be reviewed but the mathematical deductions would retain their validity.

The velocity has unit norm because $\dot{x}^2 = 0$; evaluation of $v \cdot v$ yields the relation

$$v \cdot v = \sum (\dot{x}^\mu)^2 = 1.$$  

(12)

The velocity vector can be obtained by a suitable rotation of any of the $\sigma_\mu$ frame vectors, in particular it can always be expressed as a rotation of the $\sigma_4$ vector; we will make use of this possibility later on.

At this point we are going to make a small detour for the first parallel with physics. In the previous equation we replace $x^0$ by the Greek letter $\tau$ and rewrite with $\dot{\tau}^2$ in the first member

$$\dot{\tau}^2 = 1 - \sum (\check{x}^j)^2.$$  

(13)

The relation above is well known in special relativity, see for instance Martin [4]; see also Almeida [7] and Montanus [8] for parallels between special relativity and its Euclidean space counterpart. We note that the operation performed between Eqs. (12) and (13) is a perfectly legitimate algebraic operation since all the elements involved are scalars. Obviously we could also divide both members of Eq. (8) by $d\tau$

$$\check{x} = i \dot{x}^0 + \sigma_j \dot{x}^j + \sigma_4.$$  

(14)

Squaring the second member and noting that it must be null we obtain $(\dot{x}^0)^2 - \sum (\check{x}^j)^2 = 1$. This means that we can relate the vector $i \dot{x}^0 + \sigma_j \dot{x}^j$ to relativistic 4-velocity, although the norm of this vector is symmetric to what is usual in SR. The relativistic 4-velocity is more conveniently assigned to the 5D bivector $\sigma_4 i \dot{x}^0 + \sigma_4 \check{x}^j$, which has the necessary properties. The method we have used to make the transition between 4D Euclidean space and Minkowski spacetime involved the transformation of a 5D vector into scalar plus bivector through product with $\sigma_4$; this method will later be extended to curved spaces.

We will now define a new vector $ds$ related to displacement by the scale factor $n$

$$ds = idt + n \sigma_\mu dx^\mu.$$  

(15)

In this way we are including the 4-dimensional analogue of a refractive index; the previous equation is a generalisation of the 3-dimensional definition of refractive index for an optical medium, which relates the optical path of light in that medium to the geometric path. The factor $n$ used here scales the 4D displacement vector $\sigma_\mu dx^\mu$ and so it deserves the designation of 4-dimensional refractive index; from now on we will drop the "4-dimensional" qualification because the confusion with the 3-dimensional case can always be resolved easily. The material presented in this
paper is, in many respects, a logical generalisation of optics to 4-dimensional space; so, even if the paper is only about geometry, we will frequently use the designation 4-dimensional optics (4DO) when dealing with Euclidean 4-space.

Further generalisation of Eq. (11) makes use of a tensor, similar to the non-isotropic refractive index of optical media

$$ds = idt + n^\mu \nu \sigma_\mu dx^\nu.$$  (16)

The velocity is accordingly defined by $v = n^\mu \nu \sigma_\mu$. The same expression can be used with any orthonormal frame, including for instance spherical coordinates, but for the moment we will restrict our attention to those cases where the frame does not rotate in a displacement in order to avoid having to derive frame vectors when taking derivatives. This restriction poses no limitation on the problems to be addressed but it is obviously inconvenient when symmetries are involved and shall later be relaxed.

The velocity can be given the more familiar form $v = g_\nu x^\nu$ if we define the refractive index frame

$$g_\nu = n^\mu \nu \sigma_\mu.$$  (17)

Obviously Eq. (16) implies that the velocity is still a unitary vector and we can express this fact with through the internal product with itself

$$v \cdot v = n^\alpha \mu \nu \sigma_\beta \nu \sigma_\beta = 1.$$  (18)

Using Eq. (16) to evaluate $ds^2 = 0$, considering the definition (17) and denoting $g_{\mu \nu} = g_\mu \cdot g_\nu$

$$(dt)^2 = g_{\mu \nu} dx^\mu dx^\nu.$$  (19)

This equation defines the metric of 4D space with signature (+ + + +), where $t$ is the geodesic arc length; this will be designated as 4DO metric because it applies to 4-dimensional optics space.

In a similar way to what allowed us to derive 4DO and Minkowski spaces from the null subspace condition, we will now show that general relativity (GR) metric can also be derived from the same condition when the refractive index is considered. In order to do this we define the reciprocal frame frame $\{-i, g^\mu\}$ such that

$$g^{\mu} \cdot g_\nu = \delta^\mu_\nu.$$  (20)

Following the procedure outlined in [4] to determine the reciprocal frame vectors we define the 4-volume tetravector

$$V = \bigwedge_\mu g_\mu = |V| \sigma_{1234},$$  (21)

where the big wedge symbol is used to make the exterior product of the $g_\mu$. The reciprocal frame vectors can then be found using the formula [4]

$$g^{\nu} = (-1)^{\nu} \bigwedge_{\nu \neq \mu} g_\mu V^{-1}.$$  (22)

Use the reciprocal frame to multiply both members of Eq. (16) first on the right then on the left by $g^\mu$, simultaneously replacing $x^\mu$ by $\tau$

$$ds g^4 = i g^4 dt + g_3 g^4 dx^3 + g_4 g^4 dx^4;$$  (23)

$$g^4 ds = g^4 dt + g_3 g_4 dx^3 + g_4 g_3 dx^4.$$  (24)

Performing the inner product between the two equations and setting $ds^2$ to zero we get

$$(d\tau)^2 = \frac{1}{g_{44}} (dt)^2 - g_{jk} dx^j dx^k;$$  (25)

which is recognizably a GR metric. Equation (16) can then generate both 4DO and GR metrics, provided some conditions are met; naturally the $g_\mu$ must be independent of $t$ if Eq. (19) is to be taken as 4DO metric definition and conversely they must not depend on $\tau$ if Eq. (25) defines a GR metric. However we can say that for static metrics at least we can convert between GR and 4DO.

4 The sources of space curvature

Equations (19) and (25) define two alternative 4-dimensional spaces 4DO and GR respectively; in the former $t$ is an affine parameter while in the latter it is $\tau$ that takes such role. Provided the metric is static the geodesics of one space can be mapped one to one with those of the other and we can choose to work on the space that best suits us.

The procedure to write the geodesic equations is the same in any curved space; if we choose to work in 4DO this involves consideration of the Lagrangian

$$L = \frac{g_{\mu \nu} \dot{x}^\mu \dot{x}^\nu}{2} = \frac{1}{2}.$$  (26)

The justification for this choice of Lagrangian can be found in several reference books but see for instance Martin [6]. From the Lagrangian one defines immediately the conjugate momenta

$$v_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = g_{\mu \nu} \dot{x}^\nu.$$  (27)
Notice the use of the lower index \((v_{\mu})\) to represent momenta while velocity components have an upper index \((v^\nu)\). The conjugate momenta are the components of the conjugate momentum vector \(v = g^{\mu} v_{\mu}\) and from Eq. (20)

\[
v = g^{\mu} v_{\mu} = g^{\mu} g_{\mu\nu} \dot{x}^\nu = g_{\nu} \dot{x}^\nu.
\] (28)

The conjugate momentum and velocity are the same but their components are referred to the reciprocal and refractive index frames, respectively.

The geodesic equations can now be written in the form of Euler-Lagrange equations

\[
v_{\mu} = \partial_{\nu} L; \tag{29}
\]

these equations define those paths that minimise \(t\) when displacements are made with velocity given by Eq. (16). Considering the parallel already made with general relativity we can safely say that geodesics of 4DO spaces have a one to one correspondence to those of GR in the majority of situations.

We are going to need geometric calculus which was introduced by Hestenes and Sobczyk as said earlier; another good reference is provided by Doran and Lasenby. The existence of such references allows us to introduce the vector derivative without further explanation; the reader should search the cited books for full justification of the definition we give below. We define two vector derivatives: the first one is represented by the symbol \(\Box\) and is referred to the reciprocal frame \(g^\mu\) while the second one uses the symbol \(\nabla\) and is referred to the Euclidean frame \(\sigma^\mu = \sigma_\mu\)

\[
\Box = g^{\mu} \partial_\mu; \tag{30}
\]

\[
\nabla = \sigma^\mu \partial_\mu; \tag{31}
\]

the two vector derivatives are obviously related since \(g^\mu\) can be expressed in terms of \(\sigma^\mu\).

The vector derivatives are vectors and as such they can be operated with any multivector using the established rules; in particular the geometric product of \(\Box\) with a multivector can be decomposed into inner and outer products. When applied to vector \(a\) the result is \((\Box a = \Box \cdot a + \Box \wedge a)\); the inner product term is the divergence of vector \(a\) and the outer product term is the exterior derivative, related to the curl but contrary to the latter it is usable in spaces of arbitrary dimension and is expressed as a bivector. We also define curved and Euclidean Laplacian as a result of multiplying each vector derivative with itself; the result is necessarily a scalar

\[
\Box^2 = \Box \cdot \Box, \quad \nabla^2 = \nabla \cdot \nabla. \tag{32}
\]

Velocity is a vector with very special significance in 4DO space because it is the unitary vector tangent to a geodesic; we therefore attribute high significance to velocity derivatives, since they express the characteristics of the particular space we are considering. When the Laplacian is applied to the velocity vector this corresponds to the product of a scalar and a vector and the result is necessarily a vector

\[
\Box^2 v = T. \tag{33}
\]

Vector \(T\) is called the sources vector and can be expanded into sixteen terms as

\[
T = (\Box^2 n^\mu) \sigma_\mu \dot{x}^\nu = T^\mu_\nu \sigma_\mu \dot{x}^\nu. \tag{34}
\]

The tensor \(T^\mu_\nu\) contains the coefficients of the sources vector and we call it the sources tensor; it is very similar to the stress tensor of GR, although its relation to geometry is different. The sources tensor influences the shape of geodesics as we shall see in one particularly important situation.

Before we begin searching solutions for Eq. (33) we note that it can be decomposed into a set of equations similar to Maxwell’s. Consider first the velocity derivative \(\Box v = \Box \cdot v + \Box \wedge v\); the result is a multivector with scalar and bivector part \(G = \Box v\). Now derive again: \(\Box G = \Box \cdot G + \Box \wedge G\); we know that the exterior derivative of \(G\) vanishes and the divergence equals the sources vector. Maxwell’s equations can be written in a similar form, as was shown in Almeida; here the velocity was replaced by the vector potential and multivector \(G\) was replaced by the Faraday bivector \(F\); Doran and Lasenby offer similar formulation for spacetime.

Let us now concentrate on isotropic space, characterised by orthogonal refractive index vectors \(g_\mu\) whose norm can change with coordinates but is the same for all vectors. Normally we relax this condition by accepting that the three \(g_j\) must have equal norm but \(g_4\) can be different. The reason for this relaxed isotropy is found in the parallel usually made with physics by assigning dimensions 1 to 3 to physical space. Isotropy in a physical sense need only be concerned with these dimensions and ignores what happens with dimension 4. We will therefore characterise an isotropic space by the refractive index frame \(g_j = n_\nu \sigma_j,\ g_4 = n_4 \sigma_4\). Indeed we could also accept a non-orthogonal \(g_4\) within the relaxed isotropy concept but we will not do so in this work.

We will only investigate spherically symmetric solutions independent of \(x^4\); this means that the refractive index can be expressed as functions of \(r\) if we adopt
spherical coordinates. The vector derivative in spherical coordinates is of course

\[
\Box = \frac{1}{n_r} \left( \sigma_r \partial_r + \frac{1}{r} \sigma_\theta \partial_\theta + \frac{1}{r \sin \theta} \sigma_\varphi \partial_\varphi \right) + \frac{1}{n_4} \sigma_4 \partial_4. \tag{35}
\]

The Laplacian is the inner product of \( \Box \) with itself but the frame vectors’ derivatives must be considered; all the derivatives with respect to \( r \) are zero and the others are

\[
\begin{align*}
\partial_\theta \sigma_r &= \sigma_\theta, & \partial_\varphi \sigma_r &= \sin \theta \sigma_\varphi, \\
\partial_\theta \sigma_\theta &= -\sigma_r, & \partial_\varphi \sigma_\theta &= \cos \theta \sigma_\varphi, \\
\partial_\theta \sigma_\varphi &= 0, & \partial_\varphi \sigma_\varphi &= -\sin \theta \sigma_r - \cos \theta \sigma_\theta.
\end{align*} \tag{36}
\]

After evaluation the curved Laplacian becomes

\[
\Box^2 = \frac{1}{(n_r)^2} \left( \partial_r r^2 - \frac{2}{r} \partial_r - \frac{n_r'}{n_r} \partial_r + \frac{1}{r^2} \partial_\theta \theta \right) + \frac{\cot \theta}{r^2} \partial_\theta + \frac{\csc^2 \theta}{r^2} \partial_\varphi + \frac{1}{(n_4)^2} \partial_4. \tag{37}
\]

The search for solutions of Eq. (33) must necessarily start with vanishing second member, a zero sources situation, which one would implicitly assign to vacuum; this is a wrong assumption as we will show. Zeroing the second member implies that the Laplacian of both \( n_r \) and \( n_4 \) must be zero; considering that they are functions of \( r \) we get the following equation for \( n_r \)

\[
n_r'' + \frac{2n_r'}{r} = \frac{(n_r')^2}{n_r} = 0, \tag{38}
\]

with general solution \( n_r = b \exp(a/r) \). It is legitimate to make \( b = 1 \) because the refractive index must be unity at infinity. Using this solution in Eq. (37) the Laplacian becomes

\[
\Box^2 = e^{-a/r} \left( d_r^2 + \frac{2}{r} d_r + \frac{a}{r^2} d_r \right). \tag{39}
\]

When applied to \( n_4 \) and equated to zero we obtain solutions which impose \( n_4 = n_r \) and so the space must be truly isotropic and not relaxed isotropic as we had allowed. The solution we have found for the refractive index components in isotropic space can correctly model Newton dynamics, which led the author to adhere to it for some time [10]. However if inserted into Eq. (25) this solution produces a GR metric which is verifiably in disagreement with observations; consequently it has purely geometric significance.

The inadequacy of the isotropic solution found above for relativistic predictions deserves some thought, so that we can search for solutions guided by the results that are expected to have physical significance. In the physical world we are never in a situation of zero sources because the shape of space or the existence of a refractive index must always be tested with a test particle. A test particle is an abstraction corresponding to a point mass considered so small as to have no influence on the shape of space; in reality a point particle is a black hole in GR, although this fact is always overlooked. A test particle must be seen as source of refractive index itself and its influence on the shape of space should not be neglected in any circumstances. If this is the case the solutions for vanishing sources vector may have only geometric meaning, with no connection to physical reality.

The question is then how should we include the test particle in Eq. (33) in order to find physically meaningful solutions. Here we will make an \textit{ad hoc} proposal, without further justification, because the author has not yet completed the work that will provide such justification in geometric terms. The second member of Eq. (33) will not be zero and we will impose a sources vector based on the Euclidean Laplacian

\[
J = -\nabla^2 n_4 \sigma_4. \tag{40}
\]

Equation (35) becomes

\[
\Box^2 v = -\nabla^2 n_4 \sigma_4; \tag{41}
\]

as a result the equation for \( n_r \) remains unchanged but the equation for \( n_4 \) becomes

\[
n_4'' + \frac{2n_4'}{r} = \frac{(n_4')^2}{n_r} = -n_r'' + \frac{2n_r'}{r}. \tag{42}
\]

When \( n_r \) is given the exponential form found above, the solution is \( n_4 = \sqrt{n_r} \). This can now be entered into Eq. (25) and the coefficients can be expanded in series and compared to Schwarzschild’s for the determination of parameter \( a \). The final solution, for a stationary mass \( M \) is

\[
n_r = e^{2M/r}, \quad n_4 = e^{M/r}. \tag{43}
\]

Equation (41) can be interpreted in physical terms as containing the essence of gravitation. When solved for spherically symmetric solutions, as we have done, the first member provides the definition of a stationary gravitational mass as the factor \( M \) appearing in the exponent and the second member defines inertial mass as \( \nabla^2 n_4 \). Gravitational mass is defined with recourse to some particle which undergoes its influence and is animated with velocity \( v \) and inertial mass cannot be defined without some field \( n_4 \) acting upon it. Complete investigation of the sources tensor elements and their
relation to physical quantities is not yet done; it is believed that the 16 terms of this tensor have strong links with homologous elements of stress tensor in GR but this will have to be verified.

5 Wave optics in 4D

In the previous paragraphs we have seen how relativistic dynamics can be derived from an extension of geometric optics into 4D Euclidean space and the question naturally arises if a similar extension of wave optics can provide any new insight into physics; this section will give us some idea of the possibilities opened by such approach.

Any 4D wave must verify the general wave equation

\[ \Box^2 \psi = \frac{\partial^2 \psi}{\partial t^2}. \]  

We expect waves to somehow represent elementary particles; for this to be possible they must be compatible with the velocity definition \( \Box \) and Dirac’s equation. The latter is a first order differential equation, able with the velocity definition (28) and Dirac’s equation (5) for elementary particles; for this to be possible they must be compatible with homologous elements of stress tensor in GR but we feel this to be a rather restricting option. In fact there is no reason why one should not be able to use any of the 12; 23; 13 choices for the bivector index or even bivector combinations; for instance \( u = \sigma_{12} + \sigma_{23} + \sigma_{13} \) even bivector combinations; for instance \( u = \sigma_{12} + \sigma_{23} + \sigma_{13} \). Since \( \sigma_{14} \) squares to unity and \( \sigma_{14} \) squares to minus unity, so it is legitimate to make assignments to the Dirac matrices: \( \gamma^0 \equiv -\sigma_{41}, \gamma^1 \equiv \sigma_{42} \). The Last term in the previous equation must be examined with consideration for the proposed solution; deriving \( \psi \) with respect to \( x^4 \) we get \( \partial_4 \psi = p_4 \psi_0 u \exp[\mu(x^0 t + p_\mu x^\mu)] \). Since \( u \) will always commute with the exponential, this simplifies to

\[ \partial_4 \psi = -p_4 \psi u. \]

In order to recover Dirac’s equation from Eq. (43) we have to consider the field free situation, which amounts to replacing the curved vector derivative \( \Box \) by its Euclidean counterpart \( \nabla \); the equation then becomes

\[ (-\sigma_4 i \partial_t + \sigma_{43} \partial_3 + \partial_4) \psi = 0. \]  

Now note that \( \sigma_4 i \) squares to unity and \( \sigma_{43} \) squares to minus unity, so it is legitimate to make assignments to the Dirac matrices: \( \gamma^0 \equiv -\sigma_{41}, \gamma^1 \equiv \sigma_{42} \). The Last term in the previous equation must be examined with consideration for the proposed solution; deriving \( \psi \) with respect to \( x^4 \) we get \( \partial_4 \psi = p_4 \psi_0 u \exp[\mu(x^0 t + p_\mu x^\mu)] \). Since \( u \) will always commute with the exponential, this simplifies to

\[ \partial_4 \psi = -p_4 \psi u. \]

We can then make the further assignments \( p_0 = E \), \( p_4 = m \) and write

\[ \gamma^0 \partial_4 \psi = -m \psi u. \]

Dirac’s equation has been written in a similar form by Hestenes \cite{12, 13}, Doran et al. \cite{14} and Doran and Lasenby \cite{4}; in all cases the authors chose \( u = \sigma_{12} \) but we feel this to be a rather restricting option. In fact there is no reason why one should not be able to use any of the 12; 23; 13 choices for the bivector index or even bivector combinations; for instance \( u = \sigma_{12} + \sigma_{23} + \sigma_{13} \). Seem a perfectly reasonable choice, with the advantage that it is symmetric in the 3 spatial coordinates. Pending further studies we propose the following associations for the elementary particles:

- **down quarks**: \( u = \sigma_{32} \) and permutations;
- **up quarks**: \( u = (\sigma_{23} + \sigma_{31})/\sqrt{2} \) and permutations;
- **electron**: \( u = (\sigma_{21} + \sigma_{32} + \sigma_{13})/\sqrt{3} \).

Electric charge is obviously encoded by 1/3 the number of basis bivectors intervening in \( u \) with charge sign being given by the direct or reverse order of the vectors. Consistently with the former assignments we propose that anti-particles use the basis bivectors symmetric to those listed above. For instance \( u = \sigma_{32} \) is a down quark with charge \(-1/3\) and \( u = \sigma_{23} \) is its anti-particle with charge \(+1/3\). Spin is associated with the \( \pm \) sign in the exponent of \( \psi \), one of the signs for up spin and the other for down spin.
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6 Conclusion

Our point of departure is the assumption that physics will one day become indistinguishable from geometry, once unification and true understanding of physics has been achieved. The exposition was essentially geometric, demonstrating that the 4-dimensional space that can be obtained from 5-dimensions with signature $(-+++)$. It incorporates relations that are the same as found in the main physical theories.

It was shown that general relativity metrics can be derived in such space, as well as the metrics of 4D Euclidean space; a conversion formula between the two spaces’ metrics was derived for stationary metric cases. The equations relating space curvature to its sources were investigated for Euclidean space and solved for the case of spherically symmetric mass. The solution that was found is PPN equivalent to Schwarzschild’s, although it will produce very different predictions for large $M/r$.

The study of geodesics in 4D Euclidean space is equivalent to the extension of geometric optics to 4 dimensions, justifying the designation 4-dimensional optics. This extension would not be complete if it did not apply to wave optics. It is shown that a 4D wave equation can be obtained from the monogenic condition applied in 5D. The latter is also shown to produce Dirac equation and to be compatible with the dynamics equations established before.

A somewhat speculative encoding for the seven elementary particles of the first generation is also proposed, based on the multiple square roots of $-1$ present in the algebra. This is a subject to be developed in forthcoming publications.

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