Reduction of the Elliptic $SL(N, \mathbb{C})$ top

G. Aminov  
e-mail: aminov@itep.ru  
Institute for Theoretical and Experimental Physics, Moscow, Russia  
Moscow Institute of Physics and Technology, Moscow, Russia

S. Arthamonov  
e-mail: artamonov@itep.ru  
Institute for Theoretical and Experimental Physics, Moscow, Russia  
Moscow Institute of Physics and Technology, Moscow, Russia

Abstract

We propose a relation between the elliptic $SL(N, \mathbb{C})$ top and Toda systems and obtain a new class of integrable systems in a specific limit of the elliptic $SL(N, \mathbb{C})$ top. The relation is based on the Inozemtsev limit (IL) and a symplectic map from the elliptic Calogero-Moser system to the elliptic $SL(N, \mathbb{C})$ top. In the case when $N = 2$ we use an explicit form of a symplectic map from the phase space of the elliptic Calogero-Moser system to the phase space of the elliptic $SL(2, \mathbb{C})$ top and show that the limiting tops are equivalent to the Toda chains. In the case when $N > 2$ we generalize the above procedure using only the limiting behavior of Lax matrices. In a specific limit we also obtain a more general class of systems and prove the integrability in the Liouville sense of a certain subclass of these systems. This class is described by a classical $r$-matrix obtained from an elliptic $r$-matrix.

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1 Introduction

We study four integrable systems whose equations of motion have a Lax representation with spectral parameter \(|\lambda\), \(|\mu\), \(|\nu|\). We consider periodic and non-periodic Toda chains, the elliptic Calogero-Moser model, and the elliptic \(SL(N, \mathbb{C})\) top. It was established earlier by several authors that the systems are related to each other.

In [4] Inozemtsev has proposed a procedure (IL) giving a limit relation between the Toda chains and the elliptic Calogero-Moser model. Later, the IL was generalized and used to establish connections between other integrable systems. Chernyakov and Zotov have shown in [5] that the IL applied to the \(SL(N, \mathbb{C})\) elliptic Euler-Calogero model and the elliptic Gaudin model produces new Toda-like systems endowed with additional degrees of freedom corresponding to a coadjoint orbit in \(SL(N, \mathbb{C})\). Levin, Olshanetsky, and Zotov [6] have constructed a singular symplectic transformation from the elliptic Calogero-Moser system to the elliptic \(SL(N, \mathbb{C})\) top. Using this transformation Smirnov has shown in [7] that integrable tops on the algebra \(sl(N, \mathbb{C})\) are equivalent to \(N\)-particle trigonometric and rational Calogero-Moser systems. The relations between the systems can be described by the following diagrams:

\[
\begin{align*}
\text{ECM model} & \quad \leftrightarrow \quad SL(N, \mathbb{C}) \text{ top} \\
\text{Toda system} & \quad \downarrow \text{IL} \\
\text{Elliptic CM model} & \quad \leftrightarrow \quad SL(N, \mathbb{C}) \text{ top} \\
\quad & \quad \downarrow \\
\text{Trigonometric/Rational CM model} & \quad \leftrightarrow \quad \text{Limiting top}
\end{align*}
\]

First goal of this paper is to obtain the Toda chain from the elliptic top. This complements the diagram \((1.1)\) in the following way:

\[
\begin{align*}
\text{ECM model} & \quad \leftrightarrow \quad SL(N, \mathbb{C}) \text{ top} \\
\text{Toda system} & \quad \downarrow \text{IL} \\
\end{align*}
\]

In order to obtain a new relation we will use a procedure similar to the Inozemtsev limit. The Inozemtsev limit is a combination of the trigonometric limit, infinite shifts of particle coordinates, and rescalings of the coupling constants. To obtain a limiting system equivalent to the Toda chain it is necessary to combine the Inozemtsev limit and the infinite shift of the spectral parameter. Since the spectral parameter of the elliptic \(SL(N, \mathbb{C})\) top is given on a complex torus \(T^2\) with moduli \(\tau\), under the trigonometric limit \(\text{Im}(\tau) \rightarrow +\infty\) we obtain systems with spectral parameter given on an infinite complex cylinder \(\mathbb{C}/\mathbb{Z}\).

In the case of the elliptic \(SL(2, \mathbb{C})\) top it is convenient to use an explicit form \((1.13)\) of a symplectic map from the phase space of the elliptic Calogero-Moser system to the phase space of the top (Subsection \(1.3)\). Then the Inozemtsev shifts of the elliptic Calogero-Moser system coordinates induce the rescalings of the elliptic \(SL(2, \mathbb{C})\) top coordinates. The equivalence between the limiting systems and the Toda chains is due to the bosonizations formulas which follow from the limit of \((1.13)\).

To derive an explicit form of the map between the phase spaces of the Calogero-Moser system and the elliptic top in the case of the elliptic \(SL(N > 2, \mathbb{C})\) top is not as simple as when \(N = 2\). That is why we use the scalings of coordinates induced by the limiting behavior of Lax matrices and thus generalize \(N = 2\) case. Also, the scalings of coordinates satisfy an important requirement, that is the limit of the Poisson algebra of the elliptic \(SL(N, \mathbb{C})\) top must define a Poisson structure on the phase space of the limiting system. This Poisson structure along with the values of the Casimir functions define the symplectic submanifold, for which there is the symplectic map to the phase space of the Toda chain. Equations of motion of the limiting system have Lax representation and are equivalent to the equations of motion of the Toda chain.

Second goal of this paper is to obtain in the limit a more general class of systems and prove the integrability in the Liouville sense of a certain subclass of these systems. This class appears under specific conditions on the parameters of the limit and contains Toda chains as a special case. It is possible that further study of this class will lead to establishing a connection between integrable systems mentioned above and gauge theories. Such an approach was developed earlier. For example, in [5], [9] Toda-like systems corresponding to the multi-component magnets were studied in the context of the low-energy effective \(N = 2\) SUSY gauge theories.

Now we will review general facts and notation about the integrable systems under consideration.
1.1 Elliptic $SL(N, \mathbb{C})$ top

The elliptic $SL(N, \mathbb{C})$ top is an example of Euler-Arnold top. The elliptic $SL(N, \mathbb{C})$ top is defined on a coadjoint orbit of the group $SL(N, \mathbb{C})$:

$$R^{rot} = \{ S \in \mathfrak{sl}(N, \mathbb{C}), \ S = a^{-1} S^{(0)} a \},$$

(1.2)

where $a \in SL(N, \mathbb{C})$ is defined up to the left multiplication on the stationary subgroup $G_0$ of $S^{(0)}$. The phase space $R^{rot}$ is equipped with the Kirillov-Kostant symplectic form

$$\omega^{rot} = \text{Tr} \left( S^{(0)} d a^{-1} \wedge d a^{-1} \right).$$

The Hamiltonian is defined as

$$H^{rot} = -\frac{1}{2} \text{Tr} S J(S).$$

(1.3)

Here we consider a special form of a linear operator $J$ that provides the integrability of the system

$$J(S) = \sum_{m,n} J_{mn} s_{mn} T_{mn}, \quad J_{mn} = E_2 \left( \frac{m + n \tau}{N}, \tau \right),$$

$$m, n \in \{0, \ldots, N-1\}, \quad m^2 + n^2 \neq 0,$$

where $E_2(z, \tau)$ is the second Eisenstein function (see [11]) defined on the complex torus $T^2 : \mathbb{C}/(2\omega_1 \mathbb{Z} + 2\omega_2 \mathbb{Z})$ with $\omega_1 = \frac{1}{2}, \ \tau = \omega_2/\omega_1$, and $s_{mn}$ are coordinates in the sin-algebra basis $T_{mn}$ (see [X]).

The equations of motion can be written in the Lax form [12]:

$$\frac{dL^{rot}}{dt} = N [L^{rot}, M^{rot}].$$

(1.4)

Factor $N$ in (1.4) comes from the definition of Lax matrices in the sin-algebra basis from [X].

$$L^{rot} = \sum_{m,n} s_{mn} \varphi \left[ \frac{m}{n} \right] (z) T_{mn}, \quad \varphi \left[ \frac{m}{n} \right] (z) = e \left( \frac{nz}{N} \right) \phi \left( \frac{m + n \tau}{N}, z \right),$$

$$M^{rot} = \sum_{m,n} s_{mn} f \left[ \frac{m}{n} \right] (z) T_{mn}, \quad f \left[ \frac{m}{n} \right] (z) = e \left( -\frac{nz}{N} \right) \partial_u \phi(u, z) |_{u=-\frac{m+n \tau}{N}}.$$

(1.5)

where $e(z) \equiv \exp(2\pi iz), \ i \equiv \sqrt{-1}$, and $\phi$ is a combination of theta-functions (see [1]). The Lax matrix satisfies the properties of quasi-periodicity:

$$L^{rot} (z + 1) = T_{10} L^{rot} (z) T_{10}^{-1}, \quad L^{rot} (z + \tau) = T_{01} L^{rot} (z) T_{01}^{-1}.$$

(1.6)

Consequently, $\text{Tr} (L^{rot}(z))^k$ are doubly periodic functions with the poles of order up to $k$, and thus they can be expanded in the basis consisting of the second Eisenstein function and its derivatives:

$$\text{Tr} (L^{rot} (z))^k = H_{k,0} + E_2 (z) H_{k,2} + E_2^2 (z) H_{k,3} + \cdots + E_2^{(k-2)} (z) H_{k,k}.$$

In this way we obtain the Hamiltonian (1.3)

$$H^{rot} = \frac{1}{2} H_{2,0} = \frac{1}{2} \text{Tr}(L^{rot})^2 - \frac{1}{2} \text{Tr} S^2 E_2(z, \tau).$$

(1.7)

Poisson brackets for variables $s_{mn}$ are defined by the commutator $[T_{ab}, T_{cd}]$ [A.1] of basis elements $T_{ab}$ and $T_{cd}$

$$\{ s_{ab}, s_{cd} \} = 2i \sin \left[ \frac{\pi}{N} (bc - ad) \right] s_{a+c, b+d}.$$

(1.8)

Then transition to the standard basis [A.2] gives us

$$\{ S_{ij}, S_{kl} \} = N (S_{kj} \delta_{il} - S_{il} \delta_{kj}).$$

(1.9)

Linear brackets [13], [14] can be written in terms of the Belavin-Drinfeld classical elliptic $r$-matrix $r(z)$ [13], [14], [15]. Namely,

$$\{ L_1^{rot} (z_1), L_2^{rot} (z_2) \} = [r (z_1 - z_2), L_1^{rot} (z_1) + L_2^{rot} (z_2)],$$

(1.10)
where
\[ L_1 (z) = L (z) \otimes \text{Id}, \quad L_2 (z) = \text{Id} \otimes L (z). \]

The classical \( r \)-matrix is defined by
\[ r (z) = - \sum_{m,n} \varphi \left[ \frac{m}{n} \right] (z) T_{mn} \otimes T_{-m,-n}. \tag{1.11} \]

Equation (1.10) implies the involutivity of the independent coefficients \( H_{k,z}. \) Therefore, there are \( N(N + 1)/2 - 1 \) independent integrals of motion. Note that \( H_{k,k}, \quad k \in \{2, \ldots, N\}, \) are the Casimirs corresponding to the coadjoint orbit (1.2).

### 1.2 Elliptic Calogero-Moser system

The elliptic Calogero-Moser system (CM) was first introduced in quantum version \([16], [17]\). The elliptic CM system is defined on the phase space as follows
\[ \mathcal{R}^{CM} = \left\{ (u, v), \quad \sum_{i=1}^{N} u_i = 0, \quad \sum_{i=1}^{N} v_i = 0 \right\} \]
with the canonical symplectic form
\[ \omega^{CM} = (dv \wedge du). \]

The corresponding Hamiltonian is defined via
\[ H^{CM} = \sum_{i=1}^{N} \frac{v_i^2}{2} + m^2 \sum_{i>j} E_2 (u_i - u_j, z). \]

The equations of motion defined by the Hamiltonian have the Lax representation
\[ \frac{dL^{CM}}{dt} = [L^{CM}, M^{CM}], \]
where the Lax pair can be chosen in the holomorphic form in order to construct the connection between the Calogero-Moser system and the elliptic \( SL(N, \mathbb{C}) \) top [6]
\[ L^{CM}_{ij} = \delta_{ij} v_i + m (1 - \delta_{ij}) \phi (u_i - u_j, z), \]
\[ M^{CM}_{ij} = -\delta_{ij} \sum_{k \neq j} E_2 (u_j - u_k) + \frac{\partial \phi (u, z)}{\partial u} \bigg|_{u = u_i - u_j}. \]

### 1.3 Connection between the Calogero-Moser system and the elliptic \( SL(N, \mathbb{C}) \) top

The connection mentioned in the headline was established in [6] in the form of a singular gauge transformation
\[ L^{rot} (z) = \Xi (z) L^{CM} (z) \Xi^{-1} (z). \]

This transformation leads to the symplectic map
\[ \mathcal{R}^{CM} \rightarrow \mathcal{R}^{rot}, \quad (u, v) \mapsto S. \tag{1.12} \]

In the case when \( N = 2 \) this map has the form
\[ \begin{align*}
    s_{01} &= -i v \frac{\theta_{01}(0)\theta_{01}(2u)}{\theta'(0)\theta(2u)} - m \frac{\theta_{10}(0)\theta_{00}(2u)}{\theta'(0)\theta(2u)}, \\
    s_{10} &= v \frac{\theta_{10}(0)\theta_{10}(2u)}{\theta'(0)\theta(2u)} + m \frac{\theta_{00}(2u)}{\theta'(0)\theta(2u)} \frac{\theta_{10}(0)\theta_{01}(2u)}{\theta'(0)\theta(2u)}, \\
    s_{11} &= -i v \frac{\theta_{00}(0)\theta_{00}(2u)}{\theta'(0)\theta(2u)} - m \frac{\theta_{00}(0)\theta_{10}(0)}{\theta'(0)\theta(2u)} \frac{\theta_{10}(2u)\theta_{01}(2u)}{\theta'(0)\theta(2u)}. \tag{1.13}
\end{align*} \]
1.4 Toda systems

Periodic and nonperiodic Toda systems of \( N \) interacting particles in the center of a mass frame are defined on the phase space

\[
\mathcal{R}^T = \left\{ (u, v), \sum_{i=1}^{N} u_i = 0, \sum_{i=1}^{N} v_i = 0 \right\},
\]

with the canonical symplectic form

\[
\omega^T = (dv \wedge du).
\]

The Hamiltonian of the nonperiodic system is

\[
H^{AT} = \frac{1}{2} \sum_{i=1}^{N} v_i^2 + 4\pi^2 M^2 \sum_{i=1}^{N-1} e(u_{i+1} - u_i),
\]

and of the periodic system has the form

\[
H^{PT} = \frac{1}{2} \sum_{i=1}^{N} v_i^2 + 4\pi^2 M^2 \sum_{i=1}^{N} e(u_{i+1} - u_i), \quad u_{N+1} = u_1.
\]

The equations of motion for both nonperiodic and periodic Toda systems can be written in the Lax form (18, 19, 20)

\[
\frac{d}{dt} L^{AT} = [L^{AT}, M^{AT}], \quad \frac{d}{dt} L^{PT} = [L^{PT}, M^{PT}].
\]

One can obtain Toda systems by applying the Inozemtsev limit to the Calogero-Moser system [4].

2 Elliptic \( SL(2, \mathbb{C}) \) top via the Inozemtsev limit

The main idea of the technique under consideration is to treat elliptic \( SL(2, \mathbb{C}) \) top coordinates as functions of coordinates \((u, v)\) of the elliptic Calogero-Moser model and apply Inozemtsev shift of \((u, v)\).

2.1 Periodic Toda system from the elliptic top

2.1.1 Limit of the Lax matrix and Poisson algebra

To obtain the periodic Toda system we combine the shift of coordinates \( u = U + \tau/4 \), scaling of the coupling constant \( M = mq^{1/4} \) \((q \equiv e(\tau))\), the shift of the spectral parameter \( z = \tilde{z} + \tau/2 \), and the trigonometric limit \( q \to 0 \). Then from (1.13) we derive

\[
\begin{align*}
    s_{10} &= -\frac{iv}{\pi} + O(q^{1/4}), \\
    s_{01} &= \frac{M \cos(2\pi U)}{q^{1/2}} - \frac{v \sin(2\pi U)}{\pi} + O(q^{1/4}), \\
    s_{11} &= -\frac{M \sin(2\pi U)}{q^{1/2}} - \frac{v \cos(2\pi U)}{\pi} + O(q^{1/4}).
\end{align*}
\]

Coordinates of the limiting top are scaled coordinates of the elliptic \( SL(2, \mathbb{C}) \) top

\[
\begin{align*}
    \tilde{s}_{10} &= \lim_{q \to 0} s_{10} = -\frac{iv}{\pi}, \\
    \tilde{s}_{01} &= \lim_{q \to 0} s_{01} q^{1/4} = M \cos(2\pi U), \\
    \tilde{s}_{11} &= \lim_{q \to 0} s_{11} q^{1/4} = -M \sin(2\pi U).
\end{align*}
\]

Scaled coordinates (2.1a) – (2.1c) form an algebra which arises via contraction of \( \mathfrak{sl}(2, \mathbb{C}) \) algebra

\[
\{\tilde{s}_{10}, \tilde{s}_{11}\} = 2i\tilde{s}_{01}, \quad \{\tilde{s}_{11}, \tilde{s}_{01}\} = 0, \quad \{\tilde{s}_{01}, \tilde{s}_{10}\} = 2i\tilde{s}_{11}.
\]
Also, formulas (2.1a) – (2.1c) define a symplectic map from canonical coordinates \((U, v)\) to the coordinates of the limiting top. Such formulas are known as bosonization formulas.

The following condition defines the symplectic leaf:

\[
\tilde{s}_{01}^2 + \tilde{s}_{12}^2 = \text{const} = M^2,
\]

and originates from the Casimir function of the elliptic \(SL(2, \mathbb{C})\) top

\[
s_{01}^2 + s_{10}^2 + s_{11}^2 = \text{const} = m^2.
\]

Taking into account the behavior of the function \(\varphi [m \atop n] (z)\) (B.8) (see [3]) we can write down the limiting Lax matrix

\[
\widetilde{L}^{\text{rot}} = 4\pi \begin{pmatrix}
\frac{i}{4} s_{10} & \tilde{s}_{01} \sin(\pi \tilde{z}) - \tilde{s}_{11} \cos(\pi \tilde{z}) \\
\tilde{s}_{01} \sin(\pi \tilde{z}) + \tilde{s}_{11} \cos(\pi \tilde{z}) & -\frac{i}{4} s_{10}
\end{pmatrix},
\]

where \(\widetilde{L}^{\text{rot}} = \lim_{q \to 0} L^{\text{rot}}\).

2.1.2 Limiting equations of motion and bosonization

We will use formula (1.3) for computing the limit of Hamiltonian

\[
\tilde{H}^{\text{rot}} = - \left( \tilde{s}_{01} \tilde{j}_{01} + \tilde{s}_{10} \tilde{j}_{10} + \tilde{s}_{11} \tilde{j}_{11} \right),
\]

where

\[
\tilde{j}_{10} = \lim_{q \to 0} J_{10} = \pi^2,
\]

\[
\tilde{j}_{01} = \lim_{q \to 0} J_{01} q^{-\frac{1}{2}} = -8\pi^2,
\]

\[
\tilde{j}_{11} = \lim_{q \to 0} J_{11} q^{-\frac{1}{2}} = 8\pi^2.
\]

The series expansion of \(f [m \atop n] (z)\) (B.9) (see [3]) leads to the second Lax matrix

\[
\tilde{M}^{\text{rot}} = \pi^2 \begin{pmatrix}
\tilde{s}_{10} & 4 \left( \tilde{s}_{01} + i \tilde{s}_{11} \right) e \left( \frac{-\tilde{z}}{2} \right) \\
4 \left( \tilde{s}_{01} - i \tilde{s}_{11} \right) e \left( \frac{-\tilde{z}}{2} \right) & -\tilde{s}_{10}
\end{pmatrix}.
\]

Equations of motion (1.4) preserve the same form in the limit:

\[
\frac{d\widetilde{L}^{\text{rot}}}{dt} = \left\{ \tilde{H}^{\text{rot}}, \tilde{L}^{\text{rot}} \right\} = 2 \left[ \tilde{L}^{\text{rot}}, \tilde{M}^{\text{rot}} \right].
\]

Using bosonization formulas (2.1a), (2.1b), and (2.1c) one can obtain the periodic Toda system

\[
\tilde{H}^{\text{rot}} \to H^{PT} = v^2 + 8M^2\pi^2 \cos \left( 4\pi U \right),
\]

\[
\tilde{L}^{\text{rot}} \to L^{PT} = \begin{pmatrix}
v & 4\pi M \sin \left( \pi \left( 2U + \tilde{z} \right) \right) \\
-4\pi M \sin \left( \pi \left( 2U - \tilde{z} \right) \right) & -v
\end{pmatrix},
\]

\[
\tilde{M}^{\text{rot}} \to M^{PT} = \begin{pmatrix}
-i\pi v & 4\pi^2 Me \left( -U - \frac{\tilde{z}}{2} \right) \\
4\pi^2 Me \left( U - \frac{\tilde{z}}{2} \right) & i\pi v
\end{pmatrix}.
\]
2.2 Nonperiodic Toda system from the elliptic top

2.2.1 Limit of the Lax matrix and Poisson algebra

Here we will use another shift of coordinates \( u = U + \tau/8 \), another scaling of the coupling constant \( M = m q^{1/8} \), the same shift of the spectral parameter \( z = \tilde{z} + \tau/2 \) and apply the trigonometric limit \( q \to 0 \). That gives us the following formulas for the elliptic \( SL(2, \mathbb{C}) \) top coordinates

\[
\begin{align*}
    s_{10} &= -i v + O(q^{3/8}), \\
    s_{01} &= \frac{M e(U)}{2 q^{3/8}} + i v e(U) + O(1), \\
    s_{11} &= i M e(U) - v e(U) + O(1),
\end{align*}
\]

and for the coordinates of the limiting top

\[
\begin{align*}
    \tilde{s}_{10} &= \lim_{q \to 0} s_{10} = -i v, \\
    \tilde{s}_{01} &= \lim_{q \to 0} s_{01} q^{3/8} = \frac{1}{2} M e(U), \\
    \tilde{s}_{11} &= \lim_{q \to 0} s_{11} q^{3/8} = i \frac{1}{2} M e(U).
\end{align*}
\]

These coordinates form the same algebra (2.2) as in the case of the periodic Toda system.

The limiting Lax matrix has the form

\[
\tilde{L}^{\text{rot}} = 4 \pi \begin{pmatrix}
    i \frac{1}{4} \tilde{s}_{10} & \tilde{s}_{01} \sin(\pi \tilde{z}) - \tilde{s}_{11} \cos(\pi \tilde{z}) \\
    \tilde{s}_{01} \sin(\pi \tilde{z}) + \tilde{s}_{11} \cos(\pi \tilde{z}) & -i \frac{1}{4} \tilde{s}_{10}
\end{pmatrix}.
\]

In this case the symplectic leaf of the elliptic \( SL(2, \mathbb{C}) \) top turns into

\[
\tilde{s}_{01}^2 + \tilde{s}_{11}^2 = 0. \tag{2.3}
\]

2.2.2 Limiting equations of motion and bosonization

Since we have the same shift of the spectral parameter and the same scaling of coordinates \( s_{mn} \) as in the periodic case, \( J_{mn} \) have equivalent limits

\[
\tilde{J}_{10} = \pi^2, \quad \tilde{J}_{01} = -\tilde{J}_{11} = -8 \pi^2.
\]

Also, limiting Hamiltonian and the second Lax matrix acquire the same forms

\[
\begin{align*}
    \tilde{H}^{\text{rot}} &= -\left( \tilde{s}_{01} \tilde{J}_{01} + \tilde{s}_{10} \tilde{J}_{10} + \tilde{s}_{11} \tilde{J}_{11} \right), \\
    \tilde{M}^{\text{rot}} &= \pi^2 \begin{pmatrix}
    \tilde{s}_{10} & 4 \left( \tilde{s}_{01} + i \tilde{s}_{11} \right) e \left( -\frac{\tilde{z}}{2} \right) \\
    4 \left( \tilde{s}_{01} - i \tilde{s}_{11} \right) e \left( -\frac{\tilde{z}}{2} \right) & -\tilde{s}_{10}
\end{pmatrix}.
\end{align*}
\]

The limiting Hamiltonian can be simplified on the symplectic leaf (2.3) as follows

\[
\tilde{H}^{\text{rot}} = -\left( 2 \tilde{s}_{01} \tilde{J}_{01} + \tilde{s}_{10}^2 \tilde{J}_{10} \right).
\]

The equations of motion have the Lax representation

\[
\frac{d\tilde{L}^{\text{rot}}}{dt} = \left\{ \tilde{H}^{\text{rot}}, \tilde{L}^{\text{rot}} \right\} = 2 \left[ \tilde{L}^{\text{rot}}, \tilde{M}^{\text{rot}} \right].
\]
Bosonization formulas transform the limiting top into the nonperiodic Toda system

\[ \tilde{H}^{\text{rot}} \to H^{\text{AT}} = \nu^2 + 4M^2 \pi^2 e(2U), \]
\[ \tilde{L}^{\text{rot}} \to L^{\text{AT}} = \left( \begin{array}{cc} \nu & -2i\pi Me(U + \frac{\pi}{2}) \\ 2i\pi Me(U - \frac{\pi}{2}) & -\nu \end{array} \right), \]
\[ \tilde{M}^{\text{rot}} \to M^{\text{AT}} = \left( \begin{array}{cc} -i\pi \nu & 0 \\ 4\pi^2 Me(U - \frac{\pi}{2}) & i\pi \nu \end{array} \right). \]

3 Elliptic $SL(N > 2, \mathbb{C})$ top via the Inozemtsev limit

In this section we consider a limit that is a combination of the shift of the spectral parameter $z = \tilde{z} + \tau/2$, the scalings of coordinates, and the trigonometric limit $Im(\tau) \to +\infty$. The scalings of coordinates are defined by the limiting behavior of the Lax matrix and are not derived from the symplectic map (1.12) as in the case $N = 2$. Also, the scalings of coordinates satisfy an important requirement, that is the limit of the Poisson algebra of the elliptic $SL(N, \mathbb{C})$ top must define a Poisson structure on the phase space of the limiting system. This Poisson structure along with the values of Casimir functions define the symplectic submanifold for which there is the symplectic map to the phase space of the Toda chain. Equations of motion of the limiting system have Lax representation and are equivalent to the equations of motion of the Toda chain.

3.1 Periodic Toda system from the elliptic top

3.1.1 Limit of the Lax matrix and Poisson algebra

In order to determine the exact scaling of coordinates we need to expand the function $\varphi \left[ \frac{m}{n} \right] (z)$ as series in $q$, where $q = e^{2\pi i r}$ (see [3]). We obtain

\[ \varphi \left[ \frac{m}{n} \right] (\tilde{z} + \frac{\tau}{2}) = \begin{cases} -\pi e \left( \frac{m}{2N} \right) \sin^{-1} \left( \frac{\pi m}{N} \right) + o(1), & n = 0, \\ 2\pi i e \left( -\frac{n\tilde{z}}{N} + \frac{m}{N} \right) q^{\tilde{z}m} + o \left( q^{\tilde{z}m} \right), & 0 < n < \frac{N}{2}, \\ O \left( q^{\frac{\tau}{2}} \right), & n = \frac{N}{2}, \\ -2\pi i e \left( \frac{N - n\tilde{z}}{N} \right) q^{\tilde{z}m} + o \left( q^{\tilde{z}m} \right), & \frac{N}{2} < n < N. \end{cases} \] (3.1)

Since the Lax matrix for the periodic Toda system can be written in tridiagonal form, the following substitution is reasonable

\[ s_{mn} = \tilde{s}_{mn} q^{-g(n)}, \quad m, n \in \{ 0, \ldots, N - 1 \}, \quad m^2 + n^2 \neq 0, \]
\[ g(n) = \frac{1 - \delta(n)}{2N}. \] (3.2)

This gives us the limiting matrix $\tilde{L}^{\text{rot}}$

\[ \tilde{L}^{\text{rot}} = -\pi \sum_{m=1}^{N-1} e \left( \frac{m}{2N} \right) \sin^{-1} \left( \frac{\pi m}{N} \right) \tilde{s}_{m0} T_{m0} + \\ + 2\pi i \sum_{m=0}^{N-1} e \left( -\frac{\tilde{z}}{N} + \frac{m}{N} \right) \tilde{s}_{m1} T_{m1} - e \left( \frac{\tilde{z}}{N} \right) \tilde{s}_{m,N-1} T_{m,N-1} \], (3.3)

where

\[ \tilde{\delta}(n) = \begin{cases} 1 & n \equiv 0 \mod N, \\ 0 & n \not\equiv 0 \mod N. \end{cases} \]

As one can see, coordinates $\tilde{s}_{mn}$, $1 < n < N - 1$, are not present in the adduced matrix and hence in the Hamiltonian. We will show later that Hamilton equations for these variables can be integrated in spite of the fact that their dynamics are separated from the Lax representation.
After scaling (3.2), we obtain the contraction of Poisson algebra (1.8) in the limit \( q \to 0 \)

\[
\{ \tilde{s}_{ab}, \tilde{s}_{cd} \} = 2i \sin \left( \frac{\pi}{N} (bc - ad) \right) q^{g(b)+g(d)-g(b+d)} \tilde{s}_{a+c,b+d}.
\] (3.4)

Therefore, scaled coordinates \( \tilde{s}_{mn} \) with the Poisson brackets form an algebra in the limit of \( q \to 0 \) provided that

\[
\forall k, n : \quad g(k) + g(n) - g(k + n) \geq 0.
\] (3.5)

If \( g(n) = \left( 1 - \bar{\delta}(n) \right) / (2N) \), then (3.5) is trivial and we can write down all nonzero brackets corresponding to the equality in (3.5)

\[
\{ \tilde{s}_{a0}, \tilde{s}_{cd} \} = -2i \sin \left( \frac{\pi}{N} ad \right) \tilde{s}_{a+c,d}.
\]

It is convenient to use the standard basis further. In this basis substitution (3.2) and the Lax matrix turn into

\[
S_{ij} = \tilde{S}_{ij} q^{-g(i,j)}, \quad g(i,j) = \frac{1 - \delta_{ij}}{2N}, \quad i, j \in \{1, \ldots, N\},
\]

\[
\tilde{L}_{i,j}^{\text{rot}} = \frac{2\pi i}{N} \sum_{m=1}^{N} \sum_{k=1}^{N-1} \tilde{S}_{mm} e \left( \frac{k(i - m)}{N} \right) \left( e \left( -\frac{k}{N} \right) - 1 \right)^{-1} \delta_{ij} +
\]

\[
+2\pi i \tilde{S}_{i+1,i+2} e \left( \frac{2}{N} \right) \delta (j - i - 1) - 2\pi i \tilde{S}_{i-1,i} e \left( \frac{2}{N} \right) \delta (j - i + 1).
\] (3.6)

From now on, indexes of \( \tilde{S} \) belong to \( \{1, \ldots, N\} \) and satisfy the properties of periodicity

\[
\tilde{S}_{N+i,j} = \tilde{S}_{i,N+j} = \tilde{S}_{ij} \quad i, j \in \mathbb{Z}.
\]

From (1.9), we obtain the following nonzero Poisson brackets for coordinates in the standard basis:

\[
\{ \tilde{S}_{ij}, \tilde{S}_{jk} \} = N (\tilde{S}_{ij} \delta_{ik} - \tilde{S}_{ik} \delta_{ij}).
\] (3.7)

Now, it can be easily seen that Casimir functions are

\[
\sum_{i=1}^{N} \tilde{S}_{ii}, \quad \sum_{i \neq j} \tilde{S}_{i,j} \tilde{S}_{i+1,i+1}, \ldots \tilde{S}_{i+i,i} \quad \forall j \neq l \quad i_1 \neq i_j, \quad 2 \leq k \leq N.
\] (3.8)

In (3.8) there are \( N + 2 \) independent functions depending only on variables that form the Lax matrix (3.6)

\[
\sum_{i=1}^{N} \tilde{S}_{i,i}, \quad \prod_{i=1}^{N} \tilde{S}_{i,i-1}, \quad \tilde{S}_{i,i+1} \tilde{S}_{i+1,i} \quad i \in \{1, \ldots, N\},
\]

Also, in (3.8) there are \( N (N - 3) \) Casimir functions independent as a functions of variables which are not included in the Lax matrix

\[
\left( \prod_{j=1}^{k} \tilde{S}_{i+j-1,i+j} \right) \tilde{S}_{i+k,i} \quad 1 \leq i \leq N, \quad 2 \leq k \leq N - 2.
\]

Thus, on the symplectic submanifold with nonzero values of quadratic Casimir functions \( \tilde{S}_{i,i+1} \tilde{S}_{i+1,i} \) variables which are not included in the Lax matrix are the following functions of variables included in the Lax matrix

\[
\tilde{S}_{i+k,i} = \text{const} \prod_{j=1}^{k} \tilde{S}_{i+j,i+j-1} \quad 1 \leq i \leq N, \quad 2 \leq k \leq N - 2.
\]
3.1.2 Limiting equations of motion and bosonization

The limit of Hamiltonian, as it follows from (1.7) and the fact that $\text{Tr} S^2 E_2(z, \tau) \to 0$, depends only on variables contributed in the Lax matrix $\mathcal{L}$:

$$
\tilde{H}^{\text{rot}} = \frac{1}{2} \text{Tr}(\tilde{L}^{\text{rot}})^2 = -\pi^2 \frac{N}{N} \sum_{m,n=1}^{N-1} \sum_{k=1}^{N} \tilde{S}_{mm} \tilde{S}_{nn} \left( \frac{k(n-m)}{N} \right) \left( 1 - \cos \left( \frac{2\pi k}{N} \right) \right)^{-1} + 4\pi^2 \sum_{i=1}^{N} \tilde{S}_{i,i+1} \tilde{S}_{i,i-1}.
$$

(3.9)

Using the series expansion of $f \left[ \frac{m}{n} \right] (z + \tau/2)$ (see B)

$$
f \left[ \frac{m}{n} \right] (z + \tau/2) = \begin{cases} 
-\pi^2 \sin^{-2} \left( \pi \frac{m}{N} \right) + o(1), & n = 0, \\
4\pi^2 e \left( \frac{m}{N} \right) e \left( -\frac{n \pi}{N} \right) q^\frac{m}{N} + o \left( q^{\frac{m}{N}} \right), & 0 < n < \frac{3N}{4}, \\
4\pi^2 \left[ e \left( \frac{m}{N} \right) - e \left( -\frac{n \pi}{N} + \pi \right) \right] e \left( -\frac{3}{4} \pi \right) q^\frac{N}{2} + o \left( q^{\frac{N}{2}} \right), & n = \frac{3N}{4}, \\
-4\pi^2 e \left( -\frac{m}{N} + \pi \right) e \left( -\frac{n \pi}{N} \right) q^\frac{N}{2} + o \left( q^{\frac{N}{2}} \right), & \frac{3N}{4} < n < N,
\end{cases}
$$

one can obtain the second Lax matrix

$$
\tilde{M}_{ij}^{\text{rot}} = -\pi^2 \delta_{ij} \sum_{m=1}^{N-1} \sum_{k=1}^{N} \sin^{-2} \left( \pi \frac{m}{N} \right) e \left( \frac{m(i-k)}{N} \right) \tilde{S}_{kk} + 4\pi^2 \delta_{ij} \tilde{S}_{i+1,i+2} e \left( -\frac{\pi}{N} \right),
$$

(3.10)

and ensure that the equations of motion can be written in the Lax form

$$
\frac{d}{dt} \tilde{L}^{\text{rot}} = \{ \tilde{H}^{\text{rot}}, \tilde{L}^{\text{rot}} \} = N \left[ \tilde{L}^{\text{rot}}, \tilde{M}^{\text{rot}} \right].
$$

(3.11)

Those variables that are not included in the Lax matrix have simple dynamics

$$
\frac{d}{dt} \tilde{S}_{ij} = 4\pi^2 \tilde{S}_{ij} \sum_{m=1}^{N} \sum_{k=1}^{N-1} \tilde{S}_{mm} \sin \left( \frac{\pi k(j-i)}{N} \right) \sin \left( \frac{\pi (i+j-2m)}{N} \right) \times \left( 1 - \cos \left( \frac{2\pi k}{N} \right) \right)^{-1}, \quad 1 < (j-i) \text{ mod } N < N - 1,
$$

(3.12)

and with other coordinates allow bosonization formulas

$$
\begin{align*}
\tilde{S}_{ii} &= \frac{N}{2\pi i} (v_{i-1} - v_i), \\
\tilde{S}_{i,i+1} &= M N e(u_i), \\
\tilde{S}_{i+1,i} &= M N e(-u_i), \\
\tilde{S}_{i,i+k} &= c_{i,i+k} e \left( \sum_{n=i}^{i+k-1} u_n \right) \quad 2 \leq k \leq N - 2, \quad c_{i,i+k} = \text{const},
\end{align*}
$$

(3.13)

where $u, v$ are canonical coordinates

$$
\{ v_i, u_j \} = \delta_{ij} \quad i, j \in \{ 1, \ldots, N \},
$$

and

$$
\sum_{i=1}^{N} u_i = 0, \quad \sum_{i=1}^{N} v_i = 0.
$$
Let us show that for \( u, v \) we have dynamics of the periodic Toda system in the center of a mass frame. Substituting (3.13) into Hamiltonian (3.9) and Lax matrices (3.6), (3.10) we obtain

\[
\tilde{H}^\text{rot} = N^2 \sum_{i=1}^{N} \frac{v_i^2}{2} + 4\pi^2 M^2 N^2 \sum_{i=1}^{N} e(u_{i+1} - u_i) = N^2 H^\text{PT},
\]

where \( H^\text{PT} \) has the form of periodic Toda Hamiltonian,

\[
\tilde{L}^\text{rot} = 2\pi i M_N \begin{pmatrix}
\frac{v_1}{2\pi M} & e(u_2 - \frac{z}{N}) & 0 & \ldots & 0 & -e(-u_N + \frac{z}{N}) \\
-e(-u_1 + \frac{z}{N}) & \frac{v_2}{2\pi M} & e(u_3 - \frac{z}{N}) & \ldots & 0 & 0 \\
0 & -e(-u_2 + \frac{z}{N}) & \frac{v_3}{2\pi M} & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & e(-u_{N-1} + \frac{z}{N}) & \frac{v_{N}}{2\pi M} \\
e(u_1 - \frac{z}{N}) & 0 & 0 & \ldots & -e(-u_N + \frac{z}{N}) & \frac{v_{N}}{2\pi M}
\end{pmatrix},
\]

\[
\tilde{M}_{ij} = \frac{i}{2} \sum_{m=1}^{N-1} \sum_{k=1}^{N} \sin^{-2} \left( \frac{\pi m}{N} \right) e \left( \frac{m(i - k)}{N} \right) (v_{k-1} - v_k) \delta_{ij} + 4\pi^2 M Ne \left( u_{i+1} - \frac{z}{N} \right) \bar{v} (j - i - 1).
\]

After the gauge transformation we have

\[
\tilde{L}^\text{rot} \rightarrow g^{-1} \tilde{L}^\text{rot} g, \quad \tilde{M}^\text{rot} \rightarrow g^{-1} \tilde{M}^\text{rot} g + \frac{1}{N} g^{-1} \hat{g},
\]

\[
g_{ij} = \delta_{ij} e \left( \frac{i \bar{v}}{N} \right) \prod_{k=1}^{i-1} e (-u_k),
\]

and Lax matrices take the standard form

\[
\tilde{L}^\text{rot} = 2\pi i M_N \begin{pmatrix}
\frac{v_1}{2\pi M} & e(u_2 - u_1) & 0 & \ldots & 0 & -e(\bar{z}) \\
-1 & \frac{v_2}{2\pi M} & e(u_3 - u_2) & \ldots & 0 & 0 \\
0 & -1 & \frac{v_3}{2\pi M} & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & e(u_{N} - u_{N-1}) & \frac{v_{N}}{2\pi M} \\
e(u_1 - u_N - \bar{z}) & 0 & 0 & \ldots & -1 & \frac{v_{N}}{2\pi M}
\end{pmatrix},
\]

\[
\tilde{M}^\text{rot} = 4\pi^2 M N \begin{pmatrix}
0 & e(u_2 - u_1) & 0 & \ldots & 0 & 0 \\
0 & 0 & e(u_3 - u_2) & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
e(u_1 - u_N - \bar{z}) & 0 & 0 & \ldots & 0 & e(u_{N} - u_{N-1})
\end{pmatrix}.
\]

### 3.2 Nonperiodic Toda system from the elliptic top

#### 3.2.1 Limit of Lax matrices and Poisson algebra

To obtain the Lax matrix of the nonperiodic Toda chain we are going to consider another substitution

\[
S_{ij} = \bar{s}_{ij} q^{-g(i,j)}, \quad g(i,j) = \frac{1 - \delta_{ij} - \frac{1}{2} \delta_{i1} \delta_{jN}}{2N} \quad i, j \in \{1, \ldots, N\}.
\]

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Hence, the contraction of Poisson algebra (1.9) in the limit \( q \to 0 \) takes the form
\[
\{ \tilde{S}_{ij}, \tilde{S}_{kl} \} = Nq^{g(i,j)+g(k,l)} \left( \delta_{il} \tilde{S}_{kj}q^{-g(k,j)} - \delta_{kj} \tilde{S}_{il}q^{-g(i,l)} \right),
\]
and the scaled coordinates form an algebra in the limit \( q \to 0 \) if
\[
\forall i, j, k \quad g(i, j) + g(k, i) - g(k, j) \geq 0.
\]
As it can be easily seen the above inequality is valid for \( g(i, j) \) defined in (3.16). Upon the limit \( q \to 0 \) we have nonzero brackets (3.7) and the following Lax matrices:

\[
\tilde{L}_{ij}^\text{rot} = \frac{2\pi i}{N} \sum_{m=1}^{N} \sum_{k=1}^{N-1} \tilde{S}_{mm} e \left( \frac{k(i - m)}{N} \right) \left( e \left( -\frac{k}{N} \right) - 1 \right)^{-1} \delta_{ij} + 2\pi i \tilde{S}_{i+1,i+2} e \left( -\frac{\pi}{N} \right) \tilde{S}_{ij} + 2\pi i \tilde{S}_{i,i-1} e \left( \frac{\pi}{N} \right) \delta_{i,j+1},
\]

\[
\tilde{M}_{ij}^\text{rot} = -\frac{\pi^2}{N} \delta_{ij} \sum_{m=1}^{N} \sum_{k=1}^{N-1} \sin^{-2} \left( \frac{\pi m}{N} \right) e \left( \frac{m(i - k)}{N} \right) \tilde{S}_{kk} + 4\pi^2 \delta_{i,j} \tilde{S}_{i+1,i+2} e \left( -\frac{\pi}{N} \right).
\]

As long as the algebra has the same limit as in the periodic case we have Casimir functions (3.8). But now there are \( N + 1 \) independent functions formed only by variables contributed in the Lax matrix
\[
\sum_{i=1}^{N} \tilde{S}_{ii}, \quad \prod_{i=1}^{N} \tilde{S}_{i,i+1}, \quad \tilde{S}_{i,i+1} \tilde{S}_{i+1,i} \quad i \in \{1, \ldots, N - 1\}
\]
and \( N(N - 3) + 1 \) Casimir functions independent as a functions of variables which are not included in the Lax matrix
\[
\tilde{S}_{1,N} \tilde{S}_{N,1},
\]
\[
\left( \prod_{j=1}^{k} \tilde{S}_{i+j-1,i+j} \right) \tilde{S}_{i+k,i} \quad 1 \leq i \leq N, \quad 2 \leq k \leq N - 2.
\]

### 3.2.2 Limiting equations of motion and bosonization

The limit of Hamiltonian after substitution (3.16) is
\[
\tilde{H}^\text{rot} = \frac{1}{2} \text{Tr} \left( \tilde{L}^\text{rot} \right)^2 = -\frac{\pi^2}{N} \sum_{m,n=1}^{N} \sum_{k=1}^{N-1} \tilde{S}_{mm} \tilde{S}_{nn} e \left( \frac{k(n - m)}{N} \right) \left( 1 - \cos \left( 2\pi \frac{k}{N} \right) \right)^{-1} + 4\pi^2 \sum_{i=2}^{N} \tilde{S}_{i,i+1} \tilde{S}_{i,i-1}.
\]

The equations of motion can be written in the Lax form
\[
\frac{d}{dt} \tilde{L}^\text{rot} = \{ \tilde{H}^\text{rot}, \tilde{L}^\text{rot} \} = N \left[ \tilde{L}^\text{rot}, \tilde{M}^\text{rot} \right].
\]
These equations imply simple dynamics (3.12) for variables that are not included in the Lax matrix. And for all coordinates of the limiting system there are bosonization formulas
\[
\tilde{S}_{ii} = \frac{N}{2\pi i} (v_{i-1} - v_i), \quad i \in \{1, \ldots, N\},
\]
\[
\tilde{S}_{i,i+1} = M_N e(u_i), \quad i \in \{1, \ldots, N\},
\]

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only possibility to provide the integrable systems in the limit. We will consider the following generalization:

\[ \tilde{S}_{i+1,i} = M N e(-u_i), \quad i \in \{1, \ldots, N-1\}, \]
\[ \tilde{S}_{1,N} = const \ e(-u_N), \]
\[ \tilde{S}_{i+k} = c_{i,i+k} e \left( \sum_{n=1}^{i+k-1} u_n \right), \quad 2 \leq k \leq N - 2, \quad c_{i,i+k} = const. \quad (3.20) \]

Canonical coordinates \( u, v \) have dynamics of the nonperiodic Toda chain in the center of a mass frame. After substituting (3.20) into Hamiltonian (3.19) we obtain

\[ \tilde{H}^{rot} = N^2 \sum_{i=1}^{N} \frac{v_i^4}{2} + 4 \pi^2 M^2 N^2 \sum_{i=1}^{N-1} c (u_{i+1} - u_i) = N^2 \tilde{H}^{AT}, \]

where \( \tilde{H}^{AT} \) has the form of nonperiodic Toda Hamiltonian. Lax matrices take the usual form under gauge transformation (3.15) mentioned in the periodic case

\[
\tilde{L}^{rot} = 2 \pi i M N \begin{pmatrix}
\frac{v_1}{2 \pi i M} & e(u_2 - u_1) & 0 & \ldots & 0 & 0 \\
-1 & \frac{v_2}{2 \pi i M} & e(u_3 - u_2) & \ldots & 0 & 0 \\
0 & \frac{v_3}{2 \pi i M} & -1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
e(u_1 - u_N - \tilde{z}) & 0 & 0 & \ldots & -1 & \frac{v_N}{2 \pi i M}
\end{pmatrix},
\]

\[
\tilde{M}^{rot} = 4 \pi^2 M N \begin{pmatrix}
0 & e(u_2 - u_1) & 0 & \ldots & 0 & 0 \\
0 & 0 & e(u_3 - u_2) & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
e(u_1 - u_N - \tilde{z}) & 0 & 0 & \ldots & 0 & e(u_N - u_{N-1})
\end{pmatrix}.
\]

### 3.3 More general class of limiting systems

In the previous Subsections we have considered substitutions of variables (3.2) and (3.16), which after applying the Inozemtsev limit lead to the Toda chains. It turns out that the substitutions mentioned above are not the only possibility to provide the integrable systems in the limit. We will consider the following generalization:

\[ s_{mn} = \tilde{s}_{mn} q^{-g(n)}, \quad m, n \in \{0, \ldots, N-1\}, \quad m^2 + n^2 \neq 0, \]
\[ g(i) = \begin{cases} 
\frac{k}{2N}, & 0 \leq k \leq p < \frac{N}{2}, \\
\frac{p}{2N}, & p < k < N - p, \\
\frac{N - k}{2N}, & N - p \leq k < N,
\end{cases} \quad (3.21) \]

where \( i \in \mathbb{Z}, \quad k \equiv i \mod N, \) and prove the integrability of the limiting systems in the case when \( N \) and \( p \) are relatively prime.

#### 3.3.1 Limit of Lax matrices and Poisson algebra

Scaled coordinates with Poisson brackets (3.3) form a Poisson algebra in the limit \( q \to 0 \) provided that the following condition is valid

\[ \forall k, n: \quad g(k) + g(n) - g(k + n) \geq 0. \quad (3.22) \]

For \( g(n) \) under consideration (3.22) is proved in [C].

Zero limiting brackets have the form
\[ \{ \tilde{s}_{a0}, \tilde{s}_{cd} \} = -2i \sin \left( \frac{\pi}{N} ad \right) \tilde{s}_{a+c,d}, \]
\[ \{ \tilde{s}_{ab}, \tilde{s}_{cd} \} = 2i \sin \left( \frac{\pi}{N} (bc - ad) \right) \tilde{s}_{a+c,b+d} \]
\[ (0 < a \leq p) \land (0 < b \leq p) \land (0 < a + b \leq p), \]
\[ \{ \tilde{s}_{ab}, \tilde{s}_{cd} \} = 2i \sin \left( \frac{\pi}{N} (bc - ad) \right) \tilde{s}_{a+c,b+d} \]
\[ (N - p \leq a < N) \land (N - p \leq b < N) \land (2N - p \leq a + b < 2N), \quad (3.23) \]
or in the standard basis
\[ \{ \tilde{S}_{ij}, \tilde{S}_{jk} \} = N(\tilde{S}_{ij} \delta_{ik} - \tilde{S}_{ik} \delta_{ij}), \]
\[ \{ \tilde{S}_{ij}, \tilde{S}_{kl} \} = N(\tilde{S}_{kj} \delta_{il} - \tilde{S}_{il} \delta_{kj}) \]
\[ (0 < (j - i) \text{ mod } N \leq p) \land (0 < (l - k) \text{ mod } N \leq p) \land (0 < (j + l - i - k) \text{ mod } N \leq p), \]
\[ \{ \tilde{S}_{ij}, \tilde{S}_{kl} \} = N(\tilde{S}_{kj} \delta_{il} - \tilde{S}_{il} \delta_{kj}) \]
\[ (N - p \leq (j - i) \text{ mod } N < N) \land (N - p \leq (l - k) \text{ mod } N < N) \land \]
\[ \land (N - p \leq (j + l - i - k) \text{ mod } N < N), \quad (3.24) \]
where as usual \( \land \) stands for "and".

Formulas (3.23) (or (3.24)) imply that the limiting Poisson algebra is solvable. Thus, there is no general method to construct all Casimir functions, but in the special case when \( N \) and \( p \) are relatively prime we able to present the whole set of independent Casimir functions.

At first we are interested in Casimir functions in general case when \( p < N/2 \). Since elements \( \tilde{S}_{i,i+k}, \quad k \in \{ p, \ldots , N - p \} \), have nonzero brackets only with coordinates \( \tilde{S}_{ii}, \tilde{S}_{i+k,i+k} \), we obtain the second and \( N' \)th order Casimir functions

\[ \tilde{S}_{i,i+k} \tilde{S}_{i+k,i}, \quad k \in \left\{ p, \ldots , \left\lfloor \frac{N}{2} \right\rfloor \right\}, \quad i \in \{ 1, \ldots , N \}, \]
\[ \prod_{i=1}^{N} \tilde{S}_{i,i+k}, \quad k \in \{ p, \ldots , N - p \}, \quad i \in \{ 1, \ldots , N \}, \quad (3.25) \]
where \( \lfloor x \rfloor \) is the floor function of \( x \).

In the case when \( N \) and \( p \) are relatively prime we can also construct the following \( N(N - 2p - 1) \) independent Casimir functions

\[ \left( \prod_{j=1}^{k} \tilde{S}_{i+(j-1)p,i+jp} \right) \tilde{S}_{i+kp,i}, \quad p < kp \text{ mod } N < N - p, \quad k, i \in \{ 1, \ldots , N \}. \]

If \( p < (N - 1)/2 \) it is convenient to consider two disjoint subalgebras. The first one is generated by the variables

\[ S_1 = \left\{ \tilde{S}_{ij}, \quad p < (j - i) \text{ mod } N < N - p \right\} \]

with simple dynamics, which we are going to show further. Equations of motion for the elements of the second subalgebra

\[ S_2 = \left\{ \tilde{S}_{ij}, \quad (0 \leq (j - i) \text{ mod } N \leq p) \text{ or } (N - p \leq (j - i) \text{ mod } N < N) \right\} \]

have Lax representation, so we need to obtain the number of independent Casimir functions from universal enveloping algebra of this subalgebra. Since Casimir functions lower the dimension of the symplectic submanifold, we need to majorize the degeneration factor of Poisson tensor \( \pi^{(ij)(kl)} (S_2) \):

\[ \{ F(S_2), G(S_2) \} = \pi^{(ij)(kl)} (S_2) \partial_{(ij)} F \partial_{(kl)} G, \quad \partial_{(ij)} = \frac{\partial}{\partial S_{ij}}. \]
Formulas (3.23) and (3.24) imply that Poisson tensor can be represented as a block lower matrix with respect to antidiagonal (D.1). Rank $R$ of this $(2p+1)N \times (2p+1)N$-matrix satisfies the condition $R \geq 2p(N-1)$ (D.2) (see D), which restricts the number of the independent Casimir functions up to $N+2p$ (here we treat $\sum_{i=1}^{N} \hat{S}_{ii}$ as a Casimir).

Linear brackets (3.23) and (3.24) can be written in terms of an $r$-matrix. Namely,

$$\left\{ \tilde{L}_{1}^{\text{rot}}(z_1), \tilde{L}_{2}^{\text{rot}}(z_2) \right\} = \left[ r(z_1 - z_2), \tilde{L}_{1}^{\text{rot}}(z_1) + \tilde{L}_{2}^{\text{rot}}(z_2) \right],$$

where

$$\tilde{L}_{1}(z) = \tilde{L}(z) \otimes Id, \quad \tilde{L}_{2}(z) = Id \otimes \tilde{L}(z),$$

and matrix $\tilde{r}(\bar{z})$ is the limit of elliptic $r$-matrix (4.11)

$$\tilde{r}(\bar{z}_1 - \bar{z}_2) = \lim_{\text{Im}(r) \to +\infty} r(z_1 - z_2) = \pi \sum_{m=1}^{N-1} \left( \cot \frac{m}{N} \right) \left( \pi \cot \left( \frac{\bar{z}_1 - \bar{z}_2}{2} \right) \right) T_{m0} \otimes T_{-m,0} - \pi \sin^{-1} \left( \pi \bar{z}_1 - \bar{z}_2 \right) \sum_{n=1}^{N-1} e \left( -\frac{n}{N} \right) \sum_{m=0}^{N-1} T_{mn} \otimes T_{-m,-n}.$$

Explicit expression for the elements of $\tilde{r}(\bar{z})$

$$\tilde{r}(i_{i_1},j_{j_1})(\bar{z}) = \pi \sum_{m=1}^{N-1} e \left( \frac{m(i - i_1)}{N} \right) \left( \cot \frac{m}{N} + i \right) \delta_{ij} \delta_{i_1,j_1} + \frac{2\pi i N e(\bar{z})}{1 - e(\bar{z})} \delta_{ij} \delta_{i_1,j_1},$$

where we exclude one summand proportional to $Id \otimes Id$.

Substitution (3.21) preserves in the limit coordinates of the Lax matrix with respect to the following subalgebra basis elements:

$$T_{mn}, \quad n \in \{-p, \ldots, p\},$$

which gives

$$\tilde{L}_{\text{rot}} = -\pi \sum_{m=1}^{N-1} e \left( \frac{m}{2N} \right) \sin^{-1} \left( \frac{m}{N} \right) \tilde{s}_{mn} T_{m0} + 2\pi i \sum_{m=0}^{P} \sum_{m=1}^{N-1} e \left( -\frac{m}{N} + \frac{m}{N} \right) \tilde{s}_{ml} T_{ml} - e \left( \frac{m}{N} \right) \tilde{s}_{m,-m} T_{m,-m}.$$

In the standard basis the Lax matrix acquires the form

$$\tilde{L}_{ij}^{\text{rot}} = 2\pi i \sum_{l=1}^{P} \left( \hat{S}_{i+1,i+1} e \left( -\frac{m}{N} \right) \delta(j - i - l) - \hat{S}_{i+1,i} e \left( \frac{m}{N} \right) \delta(j - i + l) \right) + \frac{2\pi i}{N} \sum_{m=1}^{N} \sum_{k=1}^{N-1} \tilde{s}_{m,n} e \left( \frac{k(i - m)}{N} \right) \left( e \left( \frac{k}{N} \right) - 1 \right)^{-1} \delta_{ij}.$$

It is convenient to use both the above and gauge transformed form of the Lax matrix

$$\tilde{L}_{ij}^{\text{rot}} = g \tilde{L}_{ij}^{\text{rot}} g^{-1}, \quad g_{ij} = \delta_{ij} e \left( \frac{m}{N} \right).$$

Denoting $w = e(\bar{z})$ we obtain
When \( p \) and consequently is equivalent to the periodic Toda chain (see Subsection 3.1), namely

\[
\text{Limiting Hamiltonian and the second Lax matrix have similar structures as in the case when the limiting system}
\]

3.3.2 Lax representation of limiting equations of motion

Limiting Hamiltonian and the second Lax matrix have similar structures as in the case when the limiting system is equivalent to the periodic Toda chain (see Subsection 3.1), namely

\[
\tilde{L}_g = 2\pi i \begin{pmatrix}
  l_1 & \tilde{S}_{23} & \ldots & \tilde{S}_{2,2+p} & 0 & \ldots & 0 & -w\tilde{S}_{1,N-p+1} & \ldots & -w\tilde{S}_{1,N} \\
  -\tilde{S}_{21} & l_2 & \tilde{S}_{34} & \ldots & \tilde{S}_{3,3+p} & 0 & \ldots & 0 & \ddots & \vdots \\
  \vdots & -\tilde{S}_{32} & \ddots & \ddots & \ddots & \ddots & 0 & \ldots & 0 & -w\tilde{S}_{pN} \\
  -\tilde{S}_{p+1,1} & \ldots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots \\
  0 & -\tilde{S}_{p+2,2} & \ldots & -\tilde{S}_{p+2,p+1} & l_{p+2} & \ddots & \ddots & \ddots & 0 & \vdots \\
  \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
  0 & \ldots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & l_{N-p} & \tilde{S}_{N-p+1,N-p+2} & \ldots & \tilde{S}_{N-p+1,1} \\
  \tilde{S}_{N-p+2,2} \quad w & 0 & \ldots & 0 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots \\
  \vdots & 0 & \ldots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & l_{N-1} & \tilde{S}_{N,1} \\
  \tilde{S}_{N+1,2} \quad w & \ldots & \tilde{S}_{2p+1} \quad w & 0 & \ldots & 0 & -\tilde{S}_{N,N-p} & \ldots & -\tilde{S}_{N,N-1} & l_N
\end{pmatrix}
\]

where

\[
l_i = \frac{1}{2\pi i} L_{ii} = \frac{1}{N} \sum_{m=1}^{N} \sum_{k=1}^{N-1} \tilde{S}_{mm} e \left( \frac{k(i-m)}{N} \right) \left( e \left( \frac{k}{N} \right) - 1 \right)^{-1},
\]

and consequently

\[
\tilde{S}_{ii} = l_{i-1} - l_i. \tag{3.29}
\]

3.3.2 Lax representation of limiting equations of motion

Limiting Hamiltonian and the second Lax matrix have similar structures as in the case when the limiting system is equivalent to the periodic Toda chain (see Subsection 3.1), namely

\[
\tilde{H}^\text{rot} = \frac{1}{2} \text{Tr} \left( \tilde{L}^\text{rot} \right)^2 = -\frac{\pi^2}{N} \sum_{m=1}^{N} \sum_{n=1}^{N-1} \tilde{S}_{mm} \tilde{S}_{nn} e \left( \frac{k(n-m)}{N} \right) \left( 1 - \cos \left( \frac{2\pi k}{N} \right) \right)^{-1} + \\
+4\pi^2 \sum_{i=1}^{N} \sum_{j=1}^{N} \tilde{S}_{i+1,j} \tilde{S}_{i+1,j+1}.
\tag{3.30}
\]

\[
\tilde{M}_{ij}^\text{rot} = -\frac{\pi^2}{N} \sum_{m=1}^{N} \sum_{n=1}^{N-1} \tilde{S}_{mm} \sin^{-2} \left( \frac{\pi k}{N} \right) e \left( \frac{k(i-m)}{N} \right) \delta_{ij} + \\
+4\pi^2 \sum_{i=1}^{N} \tilde{S}_{i+1,j} e \left( -\frac{i}{N} \right) \tilde{S}_{i+1,j} \delta(j-i).\tag{3.31}
\]

When \( p = 1 \) these formulae turn into (3.39) and (3.44), respectively.

After the limit equations of motion also have Lax representation

\[
\frac{d}{dt} \tilde{L}^\text{rot} = \{ \tilde{H}^\text{rot}, \tilde{L}^\text{rot} \} = N \left[ \tilde{L}^\text{rot}, \tilde{M}^\text{rot} \right]. \tag{3.31}
\]
In the case when \( p < (N - 1)/2 \) there are variables \( S_1 \) which are not included in the Lax pair. Hamilton equations for these variables are

\[
\frac{d}{dt} \tilde{S}_{ij} = 4\pi^2 \tilde{S}_{ij} \sum_{m=1}^{N} \sum_{k=1}^{N-1} \tilde{S}_{mm} \sin \left( \frac{\pi k(j - i)}{N} \right) \sin \left( \frac{\pi k(i + j - 2m)}{N} \right) \times \\
\times \left( 1 - \cos \left( \frac{2\pi k}{N} \right) \right)^{-1}, \quad p < (j - i) \mod N < N - p.
\]

On the symplectic submanifold with nonzero values of the second order Casimir functions \( \tilde{S}_{i,i+p}\tilde{S}_{i+p,i}, \quad i \in \{1, \ldots, N\} \) in the case when \( N \) and \( p \) are relatively prime variables \( S_1 \) are the following functions of variables \( S_2 \)

\[
\tilde{S}_{i+kp,i} = \text{const} \prod_{i=1}^{k} \tilde{S}_{i+jp,i+(j-1)p}, \quad p < kp \mod N < N - p, \quad k, i \in \{1, \ldots, N\}.
\]

Thus, if we solve the equations of motion (3.31), we immediately obtain the solutions of the equations of motion for variables \( S_1 \).

3.3.3 Integrability

The Lax operator of the elliptic \( SL(N, \mathbb{C}) \) top satisfies properties of quasi-periodicity (16). Namely,

\[
L^{rot}(z + \tau) = T_{10}L^{rot}(z)T_{10}^{-1}, \quad L^{rot}(z + \tau) = T_{01}L^{rot}(z)T_{01}^{-1}.
\]

After taking the trigonometric limit \( \text{Im}(\tau) \to +\infty \) the Lax operator has only one quasi-period

\[
\tilde{L}^{rot}(\tilde{z} + 1) = T_{10}\tilde{L}^{rot}(\tilde{z})T_{10}^{-1}.
\]

Since \( \text{Tr} \left( \tilde{L}^{rot}(\tilde{z}) \right) \) are periodic functions in \( \tilde{z} \), they can be expanded in Fourier basis \( \{e(j\tilde{z}) \equiv w^j, \quad j \in \mathbb{Z}\} \).

From the gauge transformed Lax matrix \( \tilde{L}_g^{rot} \) it follows that there are finite number of nonzero coefficients in this expansion.

**Proposition 3.1.** The trace of the \( k \)-th power of the Lax matrix has the form

\[
\text{Tr} \left( \tilde{L}^{rot}(\tilde{z}) \right)^k = \sum_{j=-M}^{M} H_{kj}w^j, \quad \text{where} \quad M = \left\lfloor \frac{kp}{N} \right\rfloor, \quad w \equiv e(\tilde{z}). \tag{3.32}
\]

**Proof.** Replacing \( e(\tilde{z}) \) by \( w \) in formula (5.27) we obtain a convenient form of the Lax matrix

\[
\tilde{L}^{rot} = -\pi \sum_{m=1}^{N-1} e \left( \frac{m}{2N} \right) \sin^{-1} \left( \frac{m}{N} \right) \tilde{s}_{m0}T_{m0} + 2\pi i \sum_{n=1}^{N-1} \sum_{m=0}^{N-1} e \left( \frac{m}{N} \right) w^{-\frac{m}{N}} \tilde{s}_{mn}T_{mn} - w^{\frac{m}{N}} \tilde{s}_{m,-n}T_{m,-n} = \\
= \sum_{n=-p}^{p} \sum_{m=0}^{N-1} e(m,n) \tilde{s}_{mn}w^{-\frac{m}{N}}T_{mn}.
\]

Then

\[
\left( \tilde{L}^{rot}(\tilde{z}) \right)^k = \sum_{m_{1,n_{1}}} \ldots \sum_{m_{k,n_{k}}} w^{-\sum_{i=1}^{k} m_{i,n_{i}}} \prod_{i=1}^{k} e(m_{i,n_{i}}) \tilde{s}_{m_{i,n_{i}}}T_{m_{i,n_{i}}}. \tag{3.33}
\]

By the properties of \( T_{mn} \) (see A) the following condition holds

\[
\text{Tr} \left( \prod_{i=1}^{k} T_{m_{i,n_{i}}} \right) \neq 0 \quad \Rightarrow \quad \sum_{i=1}^{k} n_{i} \equiv 0 \mod N \quad \Rightarrow \quad \sum_{i=1}^{k} \frac{n_{i}}{N} \in \mathbb{Z}. \tag{3.34}
\]

As \( n_{i} \in \{-p, \ldots, p\} \) for any \( i \), we derive the second condition \( \sum_{i=1}^{k} n_{i} \leq kp \), which along with (3.34) implies (3.32). \( \square \)
Proposition 3.2. The coefficients $H_{kj}$ are in involution, i.e.,

$$\{H_{k1j1}, H_{k2j2}\} = 0.$$  \hspace{1cm} (3.35)

Proof. Exactly as in the case of the elliptic top we have

$$\left\{ \text{Tr} \left( L_{\text{rot}}(\tilde{z}_1) \right)^{k_1}, \text{Tr} \left( L_{\text{rot}}(\tilde{z}_2) \right)^{k_2} \right\} = \text{Tr} \left\{ \left( L_{\text{rot}}(\tilde{z}_1) \right)^{k_1}, \left( L_{\text{rot}}(\tilde{z}_2) \right)^{k_2} \right\}.$$  

Then, it follows from (3.32) that these functions Poisson commute. Using the expansion (3.32) we get the involutivity of the coefficients (3.35).

\[ \blacksquare \]

In particular, all functions $H_{kj}$ Poisson commute with the Hamiltonian (3.30). Moreover, among coefficients $H_{kj}$ we have Casimir functions of the form $H_{k(j),\pm j}$, $k(j) = \left\lfloor j \frac{N}{p} \right\rfloor$, $j \in \{1, \ldots, p\}$ ($\left\lfloor x \right\rfloor$ is the ceiling function of $x$). The latter statement is proved in Proposition 3.3 below. It is possible to visualize these Casimirs and integrals of motion in the form of the following triangle

$$H_{20}$$

$$\vdots \quad \vdots \quad \vdots$$

$$H_{\left\lfloor \frac{N}{p} \right\rfloor - 1, -1} \quad H_{\frac{N}{p}, 0} \quad H_{\left\lfloor \frac{N}{p} \right\rfloor, 1} \quad \vdots \quad \vdots \quad \vdots$$

$$H_{\left\lfloor \frac{2N}{p} \right\rfloor - 2, -1} \quad H_{\frac{2N}{p}, 0} \quad H_{\left\lfloor \frac{2N}{p} \right\rfloor, 1} \quad H_{\left\lfloor \frac{2N}{p} \right\rfloor, 2}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$H_{N-p} \quad : \quad : \quad : \quad : \quad : \quad : \quad \vdots$$

If $N$ and $p$ are relatively prime, Casimir functions $H_{N, \pm p}$ are proportional to those introduced in (3.25)

$$H_{N, \pm p} \propto \prod_{i=1}^{N} S_{i, i+p}.$$

Also, one can note that $H_{N,p}$, $H_{N,-p}$ and the second order Casimir functions $S_{i, i+p}S_{i+p, i}$, $i \in \{1, \ldots, N\}$, are not independent.

Proposition 3.3. The coefficients $H_{k(j),\pm j}$, $k(j) = \left\lfloor j \frac{N}{p} \right\rfloor$, $j \in \{1, \ldots, p\}$, of expansion (3.32) are Casimir functions.

Proof. Functions $H_{k(j),\pm j}$ are the coefficients of the terms $w^{\pm j}_1 \equiv e(\pm j \tilde{z}_1)$ of the expansion of $\text{Tr} \left( L_{\text{rot}}(\tilde{z}_1) \right)^{k(j)}$ in (3.32). To prove the statement of the proposition we will show that terms $w^{\pm j}_1$ are not present in the expansion of $\left\{ \text{Tr} \left( L_{\text{rot}}(\tilde{z}_1) \right)^{k(j)}, L_{\text{rot}}(\tilde{z}_2) \right\}$ as a series in $w_1$. Using representation (3.26) of linear brackets (3.23), (3.24) and the following form of $r$-matrix

$$\tilde{r}(\tilde{z}) = \sum_{m,n} \tilde{\varphi}^m_n(\tilde{z}) T_{mn} \otimes T_{-m,-n},$$

where

$$\tilde{\varphi}^m_n(\tilde{z}) = \begin{cases} \pi \cot \left( \frac{\pi m}{N} \right) - \cot (\pi \tilde{z}), & n = 0, \quad 0 < m < N, \\ -\frac{\pi}{\sin (\pi \tilde{z})} e\left( \frac{\tilde{z}}{2} - \frac{n \tilde{z}}{N} \right), & 0 < n < N, \quad 0 \leq m < N, \end{cases}$$

we obtain

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\[
\left\{ \left( L_{rot}(\tilde{z}_1) \right)^k, L_{rot}(\tilde{z}_2) \right\} = \sum_{m,n} \tilde{\varphi} \left[ \frac{m}{n} \right] (\tilde{z}_1 - \tilde{z}_2) \left[ T_{mn}, \left( L_{rot}(\tilde{z}_1) \right)^k \right] \otimes T_{-m,-n} + 
+ \sum_{m,n} \tilde{\varphi} \left[ \frac{m}{n} \right] (\tilde{z}_1 - \tilde{z}_2) \left( \sum_{i=1}^{k} \left( L_{rot}(\tilde{z}_1) \right)^{i-1} T_{mn} \left( L_{rot}(\tilde{z}_1) \right)^{k-i} \right) \otimes \left[ T_{-m,-n}, L_{rot}(\tilde{z}_2) \right].
\]

This gives
\[
\left\{ \text{Tr} \left( L_{rot}(\tilde{z}_1) \right)^k, L_{rot}(\tilde{z}_2) \right\} = k \sum_{m,n} \tilde{\varphi} \left[ \frac{m}{n} \right] (\tilde{z}_1 - \tilde{z}_2) \text{Tr} \left( \left( L_{rot}(\tilde{z}_1) \right)^{k-1} T_{mn} \right) \times 
\times \left[ T_{-m,-n}, L_{rot}(\tilde{z}_2) \right].
\]

(3.37)

Substituting the explicit expressions of Sin-Algebra basis elements \( T_{mn} \) and \( \tilde{\varphi} \left[ \frac{m}{n} \right] (\tilde{z}) \) into formula \( (3.37) \) we get
\[
\left\{ \text{Tr} \left( L_{rot}(\tilde{z}_1) \right)^k, L_{rot}(\tilde{z}_2) \right\}_{i_1 j_1} = \pi k \sum_{m=1}^{N-1} N \sum_{i=1}^{N} \cot \left( \frac{\pi m}{N} \right) \left( \frac{im}{N} \right) \left( L_{rot}(\tilde{z}_2) \right)_{i_1 j_1} \times 
\times \left( \left( L_{rot}(\tilde{z}_1) \right)^{k-1} \right)_{ii} \left( e^{\frac{-i m_1}{N}} - e^{\frac{-j m_1}{N}} \right) + 
+ \pi k N \left( L_{rot}(\tilde{z}_2) \right)_{i_1 j_1} \sum_{i=1}^{N} \left( \left( L_{rot}(\tilde{z}_1) \right)^{k-1} \right)_{ii} (\delta_{i_1 i} - \delta_{j_1 j}) - 2 \pi k N \frac{w_1}{w_1 - w_2} K, L_{rot}(\tilde{z}_2) \right\}_{i_1 j_1}, \quad (3.38)
\]

where
\[
K_{i_1 j_1} = e^{\frac{-((i_1 - j_1) \bmod N) (\tilde{z}_1 - \tilde{z}_2)}{N}} \left( \left( L_{rot}(\tilde{z}_1) \right)^{k-1} \right)_{i_1 j_1} = 
= \left( \frac{w_2^{((i_1 - j_1) \bmod N)}}{w_1} \right)^{((i_1 - j_1) \bmod N)} \left( \left( L_{rot}(\tilde{z}_1) \right)^{k-1} \right)_{i_1 j_1}.
\]

By the properties of \( T_{mn} \)
\[
\left( \prod_i T_{m_i n_i} \right)_{i_1 j_1} \propto e^{\frac{i_1 \sum_m m_i}{N}} \tilde{\delta} \left( i_1 + \sum_l n_l - j_1 \right).
\]

Therefore, each element of matrix \( K \) is a Laurent polynomial in \( w_1 \), which can be seen from expression \( (3.38) \). More precisely,
\[
K_{i_1 j_1} = \sum_{m_{1,n_1}} \ldots \sum_{m_{k,n_k-1}} w_2^{((i_1 - j_1) \bmod N)} w_1^{-(i_1 - j_1) \bmod N} \prod_{l=1}^{k-1} \tilde{s}_{m_{l,n_l}},
\]
and the degree of \( w_1 \) is the integer
\[
-\frac{(i_1 - j_1) \bmod N}{N} - \sum_{l=1}^{k-1} \frac{n_l}{N} = -\frac{(i_1 - j_1) \bmod N}{N} - \frac{xN + (j_1 - i_1) \bmod N}{N} = 
= -\left( x + 1 - \delta_{i_1 j_1} \right), \quad x \in \mathbb{Z}.
\]

(3.39)

To derive the maximum \( x_{\text{max}} \) and minimum \( x_{\text{min}} \) of \( x \) we use the condition
\[
\sum_{i=1}^{k-1} n_i = \left| xN + (j_1 - i_1) \mod N \right| \leq (k - 1)p.
\]

Consequently,
\[
x_{\max} = \left[ \frac{(k - 1)p - (j_1 - i_1) \mod N}{N} \right], \quad x_{\min} = -\left[ \frac{(k - 1)p + (j_1 - i_1) \mod N}{N} \right].
\]

In the case when \( k = \lfloor jN/p \rfloor \) we have \( \lfloor (k - 1)p/N \rfloor = j - 1 \) and hence
\[
x_{\max} \leq j - 1, \quad x_{\min} \geq -j.
\]

Moreover, \( x_{\min} = 1 - j \) for the diagonal elements of matrix \( K \). According to formula (3.39) we obtain the maximum and minimum degrees \( d_{\max} \) and \( d_{\min} \) of \( w_1 \) in matrix \( K \), respectively,
\[
d_{\min} = -j, \quad d_{\max} = j - 1.
\]

Hence, each element of matrix \( \left[ K, \tilde{L}^{\text{rot}}(z_2) \right] \) is a Laurent polynomial in \( w_1 \) with maximum and minimum degrees \( d_{\max} \) and \( d_{\min} \). Due to the fact that
\[
K|_{w_1 = w_2} = \left( \tilde{L}^{\text{rot}}(z_2) \right)^{k-1},
\]

polynomials \( w_1^{j} \left[ K, \tilde{L}^{\text{rot}}(z_2) \right]_{i_{j_{i_{1_{j_1}}}}} \) have root \( w_1 = w_2 \) and
\[
-2\pi i N \frac{w_1}{w_1 - w_2} \left[ K, \tilde{L}^{\text{rot}}(z_2) \right]_{i_{j_{i_{1_{j_1}}}}}
\]

(the last term in (3.38) is a Laurent polynomial in \( w_1 \) with degrees from the interval \([1 - j, j - 1]\)). Also, the other two terms in (3.38) contain only the diagonal elements of matrix \( \left( \tilde{L}^{\text{rot}}(z_1) \right)^{k-1} \) and thus they are Laurent polynomials in \( w_1 \) with degrees from the same interval \([1 - j, j - 1]\). Therefore, terms \( w_1^{\pm j} \) are not present in the expansion (3.38).

Furthermore, it is convenient to treat \( H_{kj} \) as functions of the following set of variables
\[
S = \left\{ l_i, \quad i \in \{1, \ldots, N\}, \quad \sum_i l_i = 0; \quad S_{jk}, \quad j \neq k, \quad j, k \in \{1, \ldots, N\} \right\},
\]

where there are \( N - 1 \) independent coordinates \( l_i \) and \( N - 1 \) independent coordinates \( S_{jj} \), which are connected through nondegenerate transformation (3.40).

Proposition 3.4. If \( N \) and \( p \) are relatively prime, then the coefficients \( H_{kj}, k > 0, \lfloor jN/p \rfloor \leq k \leq N, \lfloor j \rfloor < p, \) the second order Casimir functions \( \tilde{S}_{i,1+p} \tilde{S}_{i,1+p}, \quad 1 \leq i \leq N \), and \( H_{Np} \) are functionally independent.

Proof. Coefficients \( H_{kj} \) are the \( k \)'th order homogeneous polynomials in variables \( \tilde{S}_{mn} \). To prove the independence we are going to consider the terms with the maximum degree of the variables \( \{l_i, \quad 1 \leq i \leq N\} \). These terms in turn contain ones with the maximum degree of the variables \( \left\{ \tilde{S}_{i,1+p}, \quad 1 \leq i \leq N \right\} \). Let us denote these terms by \( H'_{kj} \). Then the independence of \( H_{kj} \) follows from the independence of \( H'_{kj} \). Moreover, the statement of the proposition follows from the independence of \( H'_{kj}, \quad k > 0, \lfloor jN/p \rfloor \leq k \leq N, \lfloor j \rfloor < p, \) and the second order Casimir functions (the second order Casimir functions are monomials and remain the same after taking terms with the maximum degree of any set of variables). After taking the leading terms in Casimirs we obtain
\[
H'_{10} \propto \sum_{i=1}^{N} l_i,
\]

\[
H'_{k(j), \neq} \propto \sum_{i=1}^{N} \left( \prod_{m=1}^{k(j)-1} \tilde{S}_{i,(m-1)p,i,mp} \right) \tilde{S}_{i,(k(j)-1)p,i}, \quad k(j) = \left\lfloor \frac{Nj}{p} \right\rfloor, \quad 0 < j < p. \tag{3.40}
\]

By the properties of Casimirs as the coefficients in (3.32), \( H'_{kj} \) are homogeneous polynomials in \( l_i \)'s and the summands of (3.40), so it is convenient to treat these summands as new variables.
The expression

\[
X_{i \pm (k(j) - 1)p, i} = \left( \prod_{m=1}^{k(j)-1} \tilde{S}_{i \pm (m-1)p, i \pm mp} \right) \tilde{S}_{i \pm (k(j) - 1)p, i}, \quad 1 \leq i \leq N, \quad 0 < j < p.
\]

The expression

\[
\tilde{S}_{i \pm (k(j) - 1)p, i} = \tilde{S}_{i \pm \frac{k(j)}{2}p, i} = \tilde{S}_{i \pm Nj \equiv (Nj) \text{ mod } p, i} = \tilde{S}_{i \equiv (Nj) \text{ mod } p, i}
\]

implies the independence of \( X \) in the case when \( N \) and \( p \) are relatively prime and \( 0 < j < p \).

Therefore, we introduce the set of variables

\[
\mathcal{S}^1 = \left\{ l_i, 1 \leq i \leq N; \ X_{i, i \pm j}, (1 \leq i \leq N) \land (0 < j < p); \ \tilde{S}_{i, i \pm p}, 1 \leq i \leq N \right\}
\]

and expand the differentials of the functions \( H'_{kj}, \ 1 < k \leq N, \ |j| < p, \ H'_{Np}, \) and the second order Casimirs in the basis of the differentials of \( \mathcal{S}^1 \). We treat the differentials of \( l_i, \ 1 \leq i \leq N, \) as independent due to the presence of \( H'_{10}, \) which determines the connection between them. That leads to the Jacobian matrix of the map from \( \mathcal{S}^1 \) to Hamiltonians and Casimirs. The independence of functions under consideration follows directly from the fact that the Jacobian matrix has maximum rank. After appropriate ordering of variables, Hamiltonians, and Casimirs one can write the Jacobian matrix in a block lower triangular form

\[
\begin{pmatrix}
  l_1 & \ldots & l_N & \ldots & X(j) & X(-j) & \ldots & \tilde{S}_{pN} & \{ \tilde{S}_{i, i+p} \} \\
  V & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  \vdots & \vdots & \ddots & 0 & 0 & 0 & 0 & 0 & 0 \\
  H'(j) & \vdots & \ldots & A(j) & 0 & 0 & 0 & 0 & 0 \\
  H'(-j) & \vdots & \ldots & A(j) & 0 & 0 & 0 & 0 & 0 \\
  \vdots & \vdots & \ldots & \ldots & \ddots & 0 & 0 & 0 & 0 \\
  H'_{Np} & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 \\
  C_2 & 0 & 0 & 0 & 0 & 0 & \vdots & D & \\
\end{pmatrix}
\]

Since we care about the rank of the Jacobian matrix, without loss of generality we can neglect numerical common factors in each \( H'_{k,j} \) and use the ordering

\[
H'(0) = \begin{pmatrix}
  H'_{00} \\
  \vdots \\
  H_{N0}
\end{pmatrix}; \quad H'(j) = \begin{pmatrix}
  H'_{k(j),j} \\
  \vdots \\
  H'_{Nj}
\end{pmatrix}, \quad 0 < |j| < p; \quad C_2 = \begin{pmatrix}
  \tilde{S}_{1,1+p} \tilde{S}_{1+p,1} \\
  \vdots \\
  \tilde{S}_{N,N+p} \tilde{S}_{N+p,N}
\end{pmatrix}.
\]

Also, in (3.41) we denote

\[
X(j) = \{ X_{i-(Nj) \text{ mod } p, i}, i = 1 + mp, 0 \leq m \leq N - k(j) \}, \quad 0 < j < p,
\]

\[
X(-j) = \{ X_{i-(Nj) \text{ mod } p, i}, i = 1 + mp, 0 \leq m \leq N - k(j) \}, \quad 0 < j < p,
\]

and sort out variables in amount equal to the number of elements in \( H'(\pm j) \) so that \( A(j) \) are square matrices,

\[
\{ \tilde{S}_{i, i+p} \} = \{ \tilde{S}_{i, i+p}, 1 \leq i \leq N \}.
\]

Due to the fact that every diagonal block of the matrix (3.41) is square, we can calculate the determinant of (3.41). The first diagonal block is a square Vandermonde matrix

\[
V = \begin{pmatrix}
  1 & \ldots & 1 \\
  l_1 & \ldots & l_N \\
  \vdots & \vdots & \vdots \\
  l_N & \ldots & l_{N-1}
\end{pmatrix}
\]
with the well-known determinant $\det V = \prod_{1 \leq i < j \leq N} (l_j - l_i)$.

The blocks $A(j)$ are more complicated. Every element $A_{mn}(j)$ of $A(j)$ is a complete homogeneous symmetric polynomial (see [22]) $h_{m-1}(n, j)$ in $\{l_{1+(n-1)p+rp}, \ 0 \leq r \leq k(j) - 1\}$, i.e.,

$$A_{mn}(j) = h_{m-1}(n, j), \quad \sum_{m=0}^{+\infty} h_m(n, j) t^m = \prod_{r=0}^{k(j)-1} \left(1 - t l_{1+(n-1)p+rp}\right)^{-1}.$$

The determinants of these matrices are

$$\det A(j) = \prod_{i_1 > i_2} (l_{i_1} - l_{i_2}), \quad i_1 = \{1 + mp, \quad k(j) \leq m \leq 2(k(j) - 1)\}, \quad i_2 = \{1 + mp, \ 0 \leq m \leq k(j)\},$$

(E.3) (see E). The last two diagonal blocks in (3.41) are simple:

$$b = \frac{\partial H_N^p}{\partial S_{pN}} = \frac{\partial H_N^p}{\partial S_{pN}} \propto \prod_{i \neq p} S_{i, i-p},$$

$$D = \text{diag}\left\{S_{1+p,1}, S_{2+p,2}, \ldots, S_{pN}\right\}.$$  

Thus, the determinant of the matrix (3.41) is the following product:

$$b \det V \det D \prod_{j=1}^{p-1} \left(\det A(j)\right)^2$$

and the Jacobian matrix has the maximum rank.

Now we are to prove the main statement of this subsection.

**Proposition 3.5.** If $N$ and $p$ are relatively prime, then the systems under consideration are completely integrable in the Liouville sense.

**Proof.** Poisson algebra of the systems can be separated into two disjoint subalgebras. As it was stated earlier in Subsection 3.3.1 the elements of the first subalgebra are the functions of the elements of the second one on the generic symplectic submanifold. The submanifold corresponding to the elements of the second subalgebra has the dimension $(2p + 1)N - 1$. There is the following condition (see D) for the dimension $R$ of the symplectic leaf of this submanifold:

$$R \geq 2p(N - 1).$$

Propositions 3.3 and 3.4 give us $2p + N - 1$ independent Casimir functions and, consequently, we have the equality

$$R = 2p(N - 1).$$

According to the Liouville theorem about integrable systems it is necessary to have $R/2$ functionally independent Hamiltonians in involution for the complete integrability. From Propositions 3.2, 3.4, and (3.36) we have $p(N - 1) = R/2$ independent Hamiltonians in involution.

4 Conclusion

We have proposed a procedure giving a limit relation between the elliptic $SL(N, \mathbb{C})$ top and the Toda chains. This procedure is similar to the Inozemtsev limit and is a combination of the shift of the spectral parameter, the scalings of coordinates and the trigonometric limit.

Also, in Subsection 3.3 we have shown, that the generalization (3.21) of the above procedure provides a new class of integrable systems in the case when $N$ and $p > 1$ are relatively prime. The open problem is to understand whether the limiting systems are integrable in general case when $p < N/2$. Some statements, such as Propositions 3.1, 3.2, 3.3, are still valid in general case, but the whole set of independent Casimir functions and Hamiltonians is not clear.
Acknowledgements

We would like to thank A. M. Levin and A. V. Zotov for many fruitful discussions and A. Smirnov for useful remarks. We are especially grateful to M. A. Olshanetsky for initiating the work and fruitful discussions. The work of both authors was supported by grants RFBR 09-02-00-393, RFBR Consortium E.I.N.S.T.E.I.N 09-01-92437-CE and by Federal Agency for Science and Innovations of Russian Federation under contract 14.740.11.0347.

A Sin-Algebra

We will use the following notation to simplify formulae:

\[ \tilde{\delta}(n) = \begin{cases} 1, & n \equiv 0 \mod N, \\ 0, & n \not\equiv 0 \mod N, \end{cases} \]

\[ e(z) = \exp(2\pi iz). \]

Elements \( T_{mn} \) of the Sin-Algebra basis in \( \mathfrak{sl}(N, \mathbb{C}) \) can be written in the form

\[ (T_{mn})_{ij} = e \left( \frac{mn - (m \mod N)(n \mod N)}{2N} \right) T_{m \mod N, n \mod N}, \]

For \( m, n \in \mathbb{Z}, \ (m \not\equiv 0 \mod N) \) or \( (n \not\equiv 0 \mod N) \), the quasi-periodic condition can be introduced

\[ T_{mn} = e \left( \frac{mn - (m \mod N)(n \mod N)}{2N} \right) T_{m \mod N, n \mod N}, \]

where \( e((mn - (m \mod N)(n \mod N)) / (2N)) = \pm 1. \)

The commutator relations in this basis are

\[ [T_{mn}, T_{kl}] = 2i \sin \left( \frac{\pi}{N} (kn - ml) \right) T_{m+k, n+l}. \]  

(B.1)

One can establish the following relations between the coordinates in the standard \( \{S_{ij}\} \) and Sin-Algebra \( \{s_{mn}\} \) bases

\[ S_{ij} = \sum_{m,n} s_{mn} (T_{mn})_{ij}, \quad s_{mn} = \frac{1}{N} \sum_{i,j} S_{ij} (T_{-m,n})_{ij}. \]  

(A.2)

B Degenerations of elliptic functions

Definitions and properties of elliptic functions are borrowed mainly from [11] and [23]. The main object is the theta function with characteristics defined via

\[ \theta \left[ \begin{array}{c} a \\ b \end{array} \right] (z, \tau) = \sum_{j \in \mathbb{Z}} q^{\frac{1}{2}j^2} e((j + a)(z + b)), \]

where \( q = e(\tau) \equiv \exp(2\pi i\tau). \)

We will also need the Eisenstein functions

\[ \varepsilon_k(z) = \lim_{M \to +\infty} \sum_{n=-M}^{M} (z + n)^{-k}, \quad E_k(z) = \lim_{M \to +\infty} \sum_{n=-M}^{M} \varepsilon_k(z + n\tau). \]

(B.1)

To determine the limits of Lax matrices we will use the series expansions of the following functions

\[ \vartheta(z) = \theta \left[ \begin{array}{c} 1/2 \\ 1/2 \end{array} \right] (z, \tau) = \sum_{j \in \mathbb{Z}} q^{\frac{1}{2}j^2} \exp \left( \left( j + \frac{1}{2} \right)^2 - \left( \frac{1}{2} \right)^2 \right). \]  

(B.2)
\[
\phi(u, z) = \frac{\vartheta'(u + z) \vartheta'(0)}{\vartheta'(u) \vartheta'(z)},
\]

\(\varphi \left[ \frac{m}{n} \right] (z) = e \left( -\frac{n z}{N} \right) \varphi \left( -\frac{m + n \tau}{N}, z \right), \quad f \left[ \frac{m}{n} \right] (z) = e \left( -\frac{n z}{N} \right) \partial_u \phi(u, z)|_{u = -\frac{m + n \tau}{N}}.\) (B.4)

The functions satisfy the following well-known identities:

\[
\phi(u, z)\phi(-u, z) = E_2(z) - E_2(u),
\]

\[
\partial_u \phi(u, z) = \phi(u, z)(E_1(u + z) - E_1(u)), \quad \text{(B.5)}
\]

parity

\[
E_k(-z) = (-1)^k E_k(z),
\]

\[
\vartheta(-z) = -\vartheta(z),
\]

\[
\phi(u, z) = \phi(z, u) = -\phi(-u, -z),
\]

and quasi-periodicity

\[
E_1(z + 1) = E_1(z), \quad E_1(z + \tau) = E_1(z) - 2\pi i,
\]

\[
E_2(z + 1) = E_2(z), \quad E_2(z + \tau) = E_2(z), \quad \text{(B.6)}
\]

\[
\vartheta(z + 1) = -\vartheta(z), \quad \vartheta(z + \tau) = -q^{-\frac{1}{2}} \vartheta(-z) \vartheta(z),
\]

\[
\phi(u + 1, z) = \phi(u, z), \quad \phi(u + \tau, z) = e(-z) \phi(u, z).
\]

We will examine degenerations of elliptic functions [B.3] in the following limit:

\[
z = \bar{z} + \frac{\tau}{2}, \quad \text{Im}(\tau) \to +\infty.
\]

Using definition (B.3) one can reduce the expansion of \(\varphi \left[ \frac{m}{n} \right] (z)\) to the expansion of theta functions. Considering the main non-vanishing terms we have

\[
\vartheta \left( -\frac{m}{N} - \frac{n \tau}{N} \right) = \begin{cases}
2q^\frac{1}{2} \sin \left( \frac{m}{N} \right) + o \left( q^\frac{1}{2} \right), & n = 0, \\
igq^\frac{1}{2} \pi \left( -\frac{m}{2N} \right) + o \left( q^\frac{1}{2} \pi \right), & 0 < n < N,
\end{cases}
\]

\[
\vartheta \left( \bar{z} + \frac{\tau}{2} - \frac{m}{N} - \frac{n \tau}{N} \right) = \begin{cases}
-2q^\frac{1}{2} \sin \left( \frac{m}{N} \pi \right) + o \left( q^\frac{1}{2} \right), & n = \frac{N}{2}, \\
igq^\frac{1}{2} \pi \left( -\frac{m}{2N} \right) + o \left( q^\frac{1}{2} \pi \right), & \frac{N}{2} < n < N,
\end{cases}
\]

which gives

\[
\phi \left( -\frac{m + n \tau}{N}; \bar{z} + \frac{\tau}{2} \right) = \begin{cases}
-\pi e \left( \frac{m}{2N} \right) \sin^{-1} \left( \frac{m}{N} \right) + o \left( 1 \right), & n = 0, \\
2\pi ig \left( \frac{m}{N} \right) + o \left( q^\frac{1}{2} \right), & 0 < n < \frac{N}{2}, \\
4\pi q^\frac{1}{2} \sin \left( \frac{m}{N} \pi \right) e \left( \frac{m}{2N} + \frac{1}{2} \bar{z} \right) + o \left( q^\frac{1}{2} \right), & n = \frac{N}{2}, \\
-2\pi ig \left( \bar{z} \right) + o \left( q^\frac{1}{2} \right), & \frac{N}{2} < n < N.
\end{cases} \quad \text{(B.7)}
\]
\[ \varphi \left[ \frac{m}{n} \right] \left( \tilde{z} + \frac{\tau}{2} \right) = \begin{cases} -\pi e \left( \frac{m}{2N} \right) \sin^{-1} \left( \frac{m}{N} \right) + o \left( 1 \right), & n = 0, \\ 2\pi i q^{\frac{m}{N}} e \left( \frac{m}{N} - \frac{n\tilde{z}}{N} \right) + o \left( q^{\frac{m}{N}} \right), & 0 < n < \frac{N}{2}, \\ 4\pi q^{\frac{m}{N}} \sin \left( \pi \left( \tilde{z} - \frac{m}{N} \right) \right) + o \left( q^{\frac{m}{N}} \right), & n = \frac{N}{2}, \\ -2\pi i q^{\frac{N-n}{N}} e \left( \frac{N-n}{N} \tilde{z} \right) + o \left( q^{\frac{N-n}{N}} \right), & \frac{N}{2} < n < N. \end{cases} \] (B.8)

To evaluate the limit of \( f \left[ \frac{m}{n} \right] \) we expand \( E_1(\tilde{x} - \sigma \tau) \) as a series in \( q \). From the definition \([B.1]\) we get

\[
E_1(\tilde{x} - \sigma \tau) = \lim_{M \to +\infty} \sum_{n=-M}^{M} \varepsilon_1 (\tilde{x} + (n - \sigma) \tau) \varepsilon_1(\tilde{x} - \sigma \tau) + \\
+ \lim_{M \to +\infty} \sum_{n=1}^{M} (\varepsilon_1 (\tilde{x} + (n - \sigma) \tau) + \varepsilon_1(\tilde{x} - (n + \sigma) \tau)).
\]

Using the explicit formula for \( \varepsilon_1(x) \) from \([1]\)

\[
\varepsilon_1(x) = \pi \cot(\pi x) = \pi i e(x) + o \left( e(x) \right), \quad \Im(x) \to +\infty, \\
\quad \Im(x) \to +\infty,
\]

one can write down the leading term of \( E_1(\tilde{x} - \sigma \tau) \) as follows:

\[
E_1(\tilde{x} - \sigma \tau) = \begin{cases} 
\pi \cot(\pi \tilde{x}) + o \left( 1 \right), & \sigma = 0, \\
\pi i + 2\pi i q^{\sigma} e(-\tilde{x}) + o \left( q^{\sigma} \right), & 0 < \sigma < \frac{1}{2}, \\
\pi i + 2\pi i q^{\frac{1}{2}} \left( e(-\tilde{x}) - e(\tilde{x}) \right) + o \left( q^{\frac{1}{2}} \right), & \sigma = \frac{1}{2}, \\
\pi i - 2\pi i q^{1-\sigma} e(\tilde{x}) + o \left( q^{1-\sigma} \right), & \frac{1}{2} < \sigma < 1,
\end{cases}
\]

and using \([B.6]\) generalize it to \( \sigma \in \mathbb{R} \):

\[
E_1(\tilde{x} - \sigma \tau) = \begin{cases} 
2\pi i \sigma + \pi \cot(\pi \tilde{x}) + o \left( 1 \right), & \{ \sigma \} = 0, \\
2\pi i \sigma + \pi i + 2\pi i q^{\sigma} e(-\tilde{x}) + o \left( q^{\sigma} \right), & 0 < \{ \sigma \} < \frac{1}{2}, \\
2\pi i \sigma + \pi i + 2\pi i q^{\frac{1}{2}} \left( e(-\tilde{x}) - e(\tilde{x}) \right) + o \left( q^{\frac{1}{2}} \right), & \{ \sigma \} = \frac{1}{2}, \\
2\pi i \sigma + \pi i - 2\pi i q^{1-\sigma} e(\tilde{x}) + o \left( q^{1-\sigma} \right), & \frac{1}{2} < \{ \sigma \} < 1,
\end{cases}
\]

where \( \{ \sigma \} \) is the fractional part of \( \sigma \).

To expand \( \partial_u \phi(u, z) \big|_{u=\tilde{u} - \sigma \tau} \) in the limit \( \Im(\tau) \to +\infty \) it is convenient to use formula \([B.5]\). Assuming \( z = \tilde{z} + \tau/2 \) we have to consider the following cases depending on the value of \( \sigma \):

1. \( \sigma = 0 \)

\[
\phi \left( \tilde{u}, \tilde{z} + \frac{\tau}{2} \right) = \frac{\pi e \left( -\frac{\tau}{2} \tilde{u} \right)}{\sin \pi \tilde{u}} + o \left( 1 \right), \quad E_1(\tilde{u}) = \pi \cot \pi \tilde{u} + o \left( 1 \right), \\
E_1(\tilde{u} + \tilde{z} + \frac{\tau}{2}) = -\pi i + o \left( 1 \right), \\
\downarrow \\
\partial_u \phi(u, z) \big|_{u=\tilde{u} - \sigma \tau} = -\pi^2 \sin^{-2} \pi \tilde{u} + o \left( 1 \right).
\]
2. $0 < \sigma < \frac{1}{2}$

\[ \phi \left( \bar{u} - \sigma \tau, \bar{z} + \frac{\tau}{2} \right) = 2\pi i q e(-\bar{u}) + o(q), \]

\[ E_1(\bar{u} - \sigma \tau) = \pi i + 2\pi i q e(-\bar{u}) + o(1), \]

\[ E_1 \left( \bar{u} + \bar{z} + \left( \frac{1}{2} - \sigma \right) \tau \right) = -\pi i - 2\pi i q^{\frac{1}{2} - \sigma} e(\bar{u} + \bar{z}) + o \left( q^{\frac{1}{2} - \sigma} \right), \]

\[ \partial_u \phi(u, z) |_{u=\bar{u} - \sigma \tau} = 4\pi^2 q e(-\bar{u}) + o(q). \]

3. $\sigma = \frac{1}{2}$

\[ \phi(u, z) = 4\pi q^{\frac{1}{2}} \sin(\pi (\bar{u} + \bar{z})) e \left( -\frac{1}{2} \bar{u} + \frac{1}{2} \bar{z} \right) + o \left( q^{\frac{1}{2}} \right), \]

\[ E_1(\bar{u} + \bar{z}) = \pi \cot(\pi (\bar{u} + \bar{z})) + o(1), \]

\[ E_1 \left( \bar{u} - \frac{1}{2} \tau \right) = \pi i + o(1), \]

\[ \partial_u \phi(u, z) |_{u=\bar{u} - \sigma \tau} = 4\pi^2 q^{\frac{1}{2}} e(-\bar{u}) + o \left( q^{\frac{1}{2}} \right). \]

4. $\frac{1}{2} < \sigma < 1$

\[ \phi \left( \bar{u} - \sigma \tau, \bar{z} + \frac{\tau}{2} \right) = -2\pi i q^{\frac{1}{2}} e(\bar{z}) + o \left( q^{\frac{1}{2}} \right), \]

\[ E_1(\bar{u} - \sigma \tau) = \pi i - 2\pi i q^{\frac{1}{2} - \sigma} e(\bar{u}), \]

\[ E_1 \left( \bar{u} + \bar{z} + \left( \frac{1}{2} - \sigma \right) \tau \right) = \pi i + 2\pi i q^{\sigma - \frac{1}{2}} e(-\bar{u} - \bar{z}) + o \left( q^{\sigma - \frac{1}{2}} \right), \]

\[ \partial_u \phi(u, z) |_{u=\bar{u} - \sigma \tau} = 4\pi^2 q^{\frac{1}{2}} e(-\bar{u}) + o \left( q^{\frac{1}{2}} \right) =
\begin{align*}
& \begin{cases}
4\pi^2 q e(-\bar{u}) + o(q), & \frac{1}{2} < \sigma < \frac{3}{4}, \\
4\pi^2 q^{\frac{1}{2}} \left( e(-\bar{u}) - e(\bar{u} + \bar{z}) \right), & \sigma = \frac{3}{4}, \\
-4\pi^2 q^{\sigma - \frac{1}{2}} e(\bar{u} + \bar{z}), & \frac{3}{4} < \sigma < 1.
\end{cases}
\end{align*}
\]

Summarizing all the special cases we obtain

\[ \partial_u \phi(u, z) |_{u=\bar{u} - \sigma \tau} =
\begin{align*}
& \begin{cases}
-\pi^2 \sin^{-2} \pi \bar{u} + o(1), & \sigma = 0, \\
4\pi^2 q e(-\bar{u}) + o(q), & 0 < \sigma < \frac{3}{4}, \\
4\pi^2 q^{\frac{1}{2}} \left[ e(-\bar{u}) - e(\bar{u} + \bar{z}) \right] + o \left( q^{\frac{1}{2}} \right), & \sigma = \frac{3}{4}, \\
-4\pi^2 q^{\sigma - \frac{1}{2}} e(\bar{u} + \bar{z}) + o \left( q^{\sigma - \frac{1}{2}} \right), & \frac{3}{4} < \sigma < 1.
\end{cases}
\end{align*}
\]

Finally, from (B.4) we get

\[ f \left[ \begin{array}{c}
m \\
n \end{array} \right] \left( \frac{\bar{z} + \frac{\tau}{2}}{2} \right) =
\begin{align*}
& \begin{cases}
-\pi^2 \sin^{-2} \left( \pi \frac{m}{N} \right) + o(1), & n = 0, \\
4\pi^2 \left( \frac{m}{N} \right) e \left( -\frac{n \bar{z}}{N} \right) q^{\frac{m}{N}} + o \left( q^{\frac{m}{N}} \right), & 0 < n < \frac{3N}{4}, \\
4\pi^2 \left[ e \left( \frac{m}{N} \right) - e \left( \frac{n \bar{z}}{N + \bar{z}} \right) \right] e \left( -\frac{3}{4} \bar{z} \right) q^{\frac{m}{N}} + o \left( q^{\frac{m}{N}} \right), & n = \frac{3N}{4}, \\
-4\pi^2 \left( \frac{m}{N} + \bar{z} \right) e \left( -\frac{n \bar{z}}{N} \right) q^{\frac{m}{N}(1 - \frac{1}{N})} + o \left( q^{\frac{m}{N}(1 - \frac{1}{N})} \right), & \frac{3N}{4} < n < N.
\end{cases}
\end{align*}
\]
C Scaling inequality

We are going to prove the inequality $\alpha_i + \alpha_j - \alpha_{i+j} \geq 0$, $\forall i, j \in \mathbb{Z}$, for the function

$$\alpha_i = \begin{cases} \frac{k}{2N}, & 0 \leq k < p \leq \frac{N}{2}, \\ \frac{p}{2N}, & p \leq k \leq N - p, \\ \frac{1}{2} - \frac{k}{2N}, & N - p < k < N, \end{cases}$$

where $k \equiv i \mod N$.

Taking into account that $\alpha_{i+N} = \alpha_i$, it is enough to consider the case $i, j \in [-N/2; N/2]$, where we have

$$\alpha_i = \begin{cases} \frac{|i|}{2N}, & |i| \leq p, \\ \frac{p}{2N}, & p \leq |i| \leq \frac{N}{2} \end{cases}$$

It is easy to check that there are the following two cases:

1. If $(|i| > p) \lor (|j| > p)$, then

$$\alpha_{i+j} \leq \frac{p}{2N} \leq \alpha_i + \alpha_j.$$ 

2. If $(|i| \leq p) \land (|j| \leq p)$, then

$$\alpha_{i+j} \leq \frac{|i+j|}{2N} \leq \frac{|i|}{2N} + \frac{|j|}{2N} = \alpha_i + \alpha_j.$$ 

D Dimension of the symplectic leaf

The phase space of the system from Subsection 3.3 is equipped with the following Poisson structure:

$$\{\tilde{S}_{ij}, \tilde{S}_{kl}\} = N(\tilde{S}_{ij}\delta_{ik} - \tilde{S}_{ik}\delta_{ij}),$$

$$\{\tilde{S}_{ij}, \tilde{S}_{kl}\} = N(\tilde{S}_{kj}\delta_{i\ell} - \tilde{S}_{i\ell}\delta_{kj}),$$

$$0 < (j - i) \mod N \leq p \land \frac{N}{2} < (l - k) \mod N \leq p \land (j + l - i - k) \mod N \leq p),$$

$$\{\tilde{S}_{ij}, \tilde{S}_{kl}\} = N(\tilde{S}_{kj}\delta_{i\ell} - \tilde{S}_{i\ell}\delta_{kj}),$$

$$(N - p \leq (j - i) \mod N < N) \land (N - p \leq (l - k) \mod N < N) \land$$

$$\land (N - p \leq (j + l - i - k) \mod N < N).$$

We are interested in the specific subalgebra

$$\mathcal{S}_2 = \left\{\tilde{S}_{ij}, (0 \leq (j - i) \mod N \leq p) \lor (N - p \leq (j - i) \mod N < N)\right\}$$

and the restriction of the Poisson tensor $\pi^{(ij)(kl)}(\mathcal{S}_2)$ on it

$$\{F(\mathcal{S}_2), G(\mathcal{S}_2)\} = \pi^{(ij)(kl)}(\mathcal{S}_2) \frac{\partial}{\partial S_{ij}} F \frac{\partial}{\partial S_{kl}} G, \quad \frac{\partial}{\partial S_{ij}} = \frac{\partial}{\partial S_{ij}}.$$ 

The dimension of the symplectic leaf of the Poisson submanifold $\mathcal{S}_2$ is the rank $R$ of $\pi^{(ij)(kl)}(\mathcal{S}_2)$. Our aim is to minorize the rank $R$. Note that $\pi^{(ij)(kl)}(\mathcal{S}_2)$ can be represented as a square $(2p + 1)N \times (2p + 1)N$-matrix. To write down this matrix in a block triangular form we use the following ordering:

$$Y = \{Y(k), k = \pm p, \ldots, \pm 1, 0\}, \quad Y(k) = \left\{\tilde{S}_{i,i+k}, i = 1, \ldots, N\right\}.$$
Thus, we have

\[
\begin{pmatrix}
Y(p) & Y(-p) & Y(p-1)Y(1-p) & \cdots & Y(1)Y(-1) & Y(0) \\
Y'(p) & Y'(-p) & 0 & 0 & 0 & 0 \\
Y'(p-1) & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
Y(1) & 0 & (P_+(p-1) & 0 & P_-(p-1) & 0 \\
Y(-1) & 0 & (P_+(p)P_-(p)) & \cdots & \cdots & 0 \\
Y(0) & (P_+(p)P_-(p)) & \cdots & \cdots & 0 & 0
\end{pmatrix}
\]

(D.1)

where

\[
(P_+(k))_{ij} = \begin{cases} 
\bar{S}_{i,i+p-k}, & \text{if } \bar{S}_{i,j} \neq 0 \\
0, & \text{otherwise}
\end{cases}, \\
(P_-(k))_{ij} = \begin{cases} 
\bar{S}_{i,i+k-p}, & \text{if } \bar{S}_{i,j} \neq 0 \\
0, & \text{otherwise}
\end{cases}
\]

It can be easily seen that the rank of each matrix \(P_\pm(k)\) is equal to \(N - 1\). Due to the block triangular form of the matrix (D.1) we obtain the required condition

\[ R \geq 2p(N - 1). \]  

(E) Addition to Proposition 3.4, \( \det A(j) \)

Here we are to consider square matrices \( A(j) \) with the following structure: every element \( A_{mn}(j) \) of \( A(j) \) is a complete homogeneous symmetric polynomial (see [22]) \( h_{m-1}(n, j) \) in variables \( \{l_{1+(n-1)p+r}, 0 \leq r \leq k(j) - 1\} \), i.e.,

\[ A_{mn}(j) = h_{m-1}(n, j), \quad \sum_{m=0}^{+\infty} h_m(n,j) t^m = \prod_{r=0}^{k(j)-1} \left(1 - t l_{1+(n-1)p+r}\right)^{-1}, \]  

(E.1)

where we use the notation \( k(j) = \lceil jN/p \rceil \) from Proposition 3.4.

The determinants of the matrices under consideration are homogeneous polynomials of order \((k(j)/(k(j)-1))/2\). In order to compute these determinants we will find the appropriate number of roots and calculate their values at one particular point.

As one can see from (E.1) columns in matrix \( A(j) \) depend on the following set of variables:

| Column Number | Set of Variables |
|---------------|------------------|
| 1             | \( l_1 \)        |
| 2             | \( l_1 \)        |
| \vdots        | \( l_1 \)        |
| \( k(j) \)    | \( l_1 \)        |

(E.2)

It follows from (E.2) that two adjacent columns will coincide if we put the first variable, e.g., \( l_1 \), in one column equal to the last variable, e.g., \( l_{1+k(j)-1} \), in the subsequent column. Thus, we have \([jN/p] - 1\) roots. It turns out that the result can be generalized to any \( c \) adjacent columns, where \( 2 \leq c \leq k(j) = [jN/p] \). In other words, if the first variable in the first column is equal to the last variable in the last column, then the determinant is equal to zero. Indeed, without loss of generality we can consider the first \( c \) columns. By assumption,

\[ l_{1+(k(j)+c-2)p} = l_1. \]

We will prove that in this case the rows of \( A(j) \) are linearly dependent. If we multiply the \( m \)th row of matrix by the elementary symmetric polynomial \((-1)^{k(j)-m}e_{k(j)-m}(j)\) in variables \( \{l_{1+rp}, 0 \leq r \leq 2k(j) - 2, r \neq k(j) + c - 2\} \)
\[
\sum_{m \geq 0} (-1)^m e_m(j) t^m = \prod_{r=0, r \neq k(j) + c-2}^{2k(j)-2} (1 - t l_{1+rp}) ,
\]
then after summation we get the zero row due to the fact that the elements of the row are coefficients in front of \( t^{k(j)-1} \) in the following polynomial with the maximum degree \( k(j) - 2 \):

\[
\sum_{m \geq 0} (-1)^m e_m(j) t^m \sum_{m_1 \geq 0} h_{m_1}(n,j) t^{m_1} = \\
\begin{cases}
\left( \prod_{r=0}^{n-2} (1 - t l_{1+rp}) \right) \left( \prod_{r=k(j)+n-1, r \neq k(j) + c-2}^{2k(j)-2} (1 - t l_{1+rp}) \right), & n < c, \\
\left( \prod_{r=1}^{n-2} (1 - t l_{1+rp}) \right) \left( \prod_{r=k(j)+n-1}^{2k(j)-2} (1 - t l_{1+rp}) \right), & n \geq c.
\end{cases}
\]

Thus, we have \((k(j)(k(j) - 1))/2\) roots. Now we calculate the determinant at the following point

\[
l_{i_1} = 1, \quad l_{i_2} = 0, \quad A_{mn} = \left( \frac{m+n-3}{m-1} \right) ,
\]

\[
i_1 = \{1 + mp, \quad k(j) \leq m \leq 2(k(j) - 1)\}, \quad i_2 = \{1 + mp, \quad 0 \leq m \leq k(j)\} .
\]

Subtracting columns according to the well-known formula, i.e.,

\[
\begin{pmatrix}
N+1 \\
m+1
\end{pmatrix}
= \begin{pmatrix}
N \\
m
\end{pmatrix} + \begin{pmatrix}
N \\
m+1
\end{pmatrix}
\]

\[
\downarrow
\]

\[
A_{m+1,n} - A_{m+1,n-1} = A_{mn},
\]

one can obtain a lower unit triangular matrix, so at the point selected above we have

\[
det A = 1.
\]

Finally, we get

\[
det A(j) = \prod_{i_1 > i_2} (l_{i_1} - l_{i_2}) ,
\]

\[
i_1 = \{1 + mp, \quad k(j) \leq m \leq 2(k(j) - 1)\}, \quad i_2 = \{1 + mp, \quad 0 \leq m \leq k(j)\} . \tag{E.3}
\]

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