IMPROVED $L^2$ AND $H^1$ ERROR ESTIMATES FOR THE
HESSIAN DISCRETISATION METHOD

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ABSTRACT. The Hessian discretisation method (HDM) for fourth order linear elliptic equations provides a unified convergence analysis framework based on three properties namely coercivity, consistency, and limit-conformity. Some examples that fit in this approach include conforming and nonconforming finite element methods, finite volume methods and methods based on gradient recovery operators. A generic error estimate has been established in $L^2$, $H^1$ and $H^2$-like norms in literature. In this paper, we establish improved $L^2$ and $H^1$ error estimates in the framework of HDM and illustrate it on various schemes. Since an improved $L^2$ estimate is not expected in general for finite volume method (FVM), a modified FVM is designed by changing the quadrature of the source term and a superconvergence result is proved for this modified FVM. In addition to the Adini nonconforming finite element method (ncFEM), in this paper, we show that the Morley ncFEM is an example of HDM. Numerical results that justify the theoretical results are also presented.

Keywords: fourth order elliptic equations, numerical schemes, error estimates, Hessian discretisation method, Hessian schemes, finite element method, finite volume method, gradient recovery method.

AMS subject classifications: 65N08, 65N12, 65N15, 65N30.

1. INTRODUCTION

There are many applications where fourth order elliptic partial differential equations appear, for example, thin plate theories of elasticity [6], thin beams and the Stokes problem in stream function and vorticity formulation [21]. Consider the following fourth order model problem with homogeneous clamped boundary conditions.

\[
\sum_{i,j,k,l=1}^{d} \partial_{kl}(a_{ijkl} \partial_{ij} u) = f \quad \text{in } \Omega, \tag{1.1a}
\]

\[
u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial \Omega, \tag{1.1b}
\]

where $\Omega \subset \mathbb{R}^d (d \geq 1)$ is a bounded domain with boundary $\partial \Omega$, $f \in L^2(\Omega)$ and $n$ is the unit outward normal to $\Omega$. Furthermore, the coefficients $a_{ijkl}$ are measurable bounded functions which satisfy the conditions $a_{ijkl} = a_{jikl} = a_{ijlk} = a_{klij}$ for $i, j, k, l = 1, \cdots, d$.

The Hessian discretisation method (HDM) for fourth order linear elliptic equations is a unified convergence analysis framework based on the choice of a set of discrete space and operators called altogether a Hessian discretisation (HD). The idea of the HDM is to construct a scheme by replacing the continuous space, function, gradient, and Hessian in the weak formulation with the discrete elements provided by a HD. The numerical scheme thus obtained is called a Hessian scheme. The concept of HDM is motivated by the Gradient discretisation method (GDM) [9] for second
order problems. The framework of HDM enables us to develop one study that
encompasses several numerical methods such as conforming and nonconforming
finite element methods, finite volume methods and methods based on gradient
recovery operators. It has been shown in [10] that only three properties, namely
coercivity, consistency, and limit-conformity, are sufficient to prove the convergence
of a HDM.

The finite element method (FEM) is one of the most well-known tools for solv-
ing fourth-order elliptic problems. Conforming finite element (for e.g., the Argyris
triangle, the Bogner–Fox–Schmit rectangle) methods for (1.1) requires the approx-
imation space to be a subspace of $H^2_0(\Omega)$, which results in $C^1$ finite elements that
is cumbersome for implementations [5, 8, 22]. The nonconforming Morley elements
which are based on piecewise quadratic polynomials are simpler to use and have
fewer degrees of freedom (6 degrees of freedom in a triangle). The Adini element is
a well-known nonconforming finite element on rectangular meshes with 12 degrees
of freedom in a rectangle. For an analysis of finite element approximation by a
mixed method, see [4, 13].

In [19], a finite element approximation based on gradient recovery (GR) operator
for a biharmonic problem using biorthogonal system has been studied, where the
approximation properties of the GR operator ensure the optimality of the finite
element approach. The GR operator maps an $L^2$ function to a piecewise linear
globally continuous $H^1$ function and this enables to define a Hessian matrix starting
from $P_1$ functions, see [17–19] for more details. A cell centered finite volume method
(FVM) for the approximation of a biharmonic problem has been proposed and
analyzed in [12], first on grids which satisfy an orthogonality condition, and then
on general meshes. This scheme consists of approximation by piecewise constant
functions and hence it is easy to implement and computationally cheap.

A generic error estimate has been established for the HDM applied to (1.1) in [10].
This estimate only gives linear order of convergence in $L^2$, $H^1$ and $H^2$ norms for
low-order conforming FEMs, Adini nonconforming FEM and methods based on
GR operators, provided $u \in H^4(\Omega) \cap H^2_0(\Omega)$. Also, the error estimate provides an
$O(h^{1/4} \ln(h))$ (in $d = 2$) or $O(h^{3/13})$ (in $d = 3$) convergence rate for the FVM in
the HDM framework, where $h$ denotes the mesh parameter. However, an $O(h^2)$
superconvergence rate in $L^2$ norm has been numerically observed in [10] on two
dimensional triangular and square meshes. Note that the FVM only works for
the biharmonic problem with the approximation of the Laplacian of the functions
while the other methods work for more generic fourth-order problems in the HDM
setting.

The goal of this paper is to obtain an improved error estimate in $L^2$ and $H^1$-like
norms compared to the estimate in the energy norm for the HDM applied to (1.1).
The Aubin–Nitsche duality arguments apply to establish $L^2$ and $H^1$ estimates in
the abstract framework which involve an interpolant of the solution to (1.1) in
the weak sense. However, for the $H^1$ error estimate, this is not straightforward.
Under the assumption that there exists a companion operator that lifts the discrete
space to the continuous space with certain property, an improved $H^1$ error estimate
is proved in the abstract setting. These estimates are then illustrated for some
schemes contained in the HDM framework. Since such an improved $L^2$ estimate is
not true in general for FVM even in the case of second order problems ([11] and
The weak formulation (1.2) corresponding to (1.1) can be rewritten as

\[ A\xi = B\phi \]

Assume the existence of a fourth order tensor

\[ P \]

\[ \text{understood from the context.} \]

\[ R^2(\Omega; L^2(\Omega)) \]

\[ \phi \]

\[ A \subset P \]

\[ \text{transpose} \]

\[ P \]

\[ S \]

\[ \text{basis of} \]

\[ \text{of} \]

\[ \text{and the modified FVM are presented in Section 4. Section 5 deals with the proof of the main results. Section 6 is an Appendix, that gathers various results: some technical results and the proof of the application of improved error estimates to various schemes stated in Section 3.} \]

Notations. Let \( d \) be the dimension and \( S_d(\mathbb{R}) \) be the space of symmetric matrices. A fourth order symmetric tensor \( P \) is interpreted as a linear map from \( S_d(\mathbb{R}) \) to \( S_d(\mathbb{R}) \) and let \( p_{ijkl} \) denote the indices of the fourth order tensor \( P \) in the canonical basis of \( S_d(\mathbb{R}) \). For simplicity, we follow the Einstein summation convention unless otherwise stated. The scalar product on \( S_d(\mathbb{R}) \) is defined by \( \xi \cdot \phi = \xi_{ij}\phi_{ij} \). For a function \( \xi : \Omega \rightarrow S_d(\mathbb{R}) \), denoting the Hessian matrix by \( H \) we set \( H : \xi = \partial_{ij}\xi_{ij} \). The transpose \( P^T \) of \( P \) is given by \( P^T = (p_{ijkl}) \), if \( P = (p_{ijkl}) \). Note that \( (P\phi)_{ij} = p_{ijkl}\phi_{kl} \) and \( P^T \xi : \phi = \xi : P\phi \). The tensor product \( a \otimes b \) of two vectors \( a, b \in \mathbb{R}^d \) is the 2-tensor with coefficients \( a_i b_j \). The Lebesgue measure of a measurable set \( E \subset \mathbb{R}^d \) is denoted by \( |E| \). The norm in \( L^2(\Omega) \), \( L^2(\Omega)^d \) for vector-valued functions, and \( L^2(\Omega; \mathbb{R}^{d \times d}) \) for matrix-valued functions, is denoted by \( \| \cdot \| \). We denote by \( (\cdot, \cdot) \) the \( L^2 \) inner product or duality pairing between \( H^{-1}(\Omega) \) and \( H^1_0(\Omega) \), this could be understood from the context.

1.1. Weak formulation. The weak formulation corresponding to (1.1) reads:

Find \( u \in V := H^1_0(\Omega) \) such that \( \forall v \in V, \int_{\Omega} A H u : H v \, dx = \int_{\Omega} f v \, dx \),

\[ \int_{\Omega} A H u : H v \, dx = \int_{\Omega} f v \, dx, \tag{1.2} \]

where \( A \) is the fourth order tensor with indices \( a_{ijkl} \) and \( x = (x_1, x_2, ..., x_d) \in \Omega \). Assume the existence of a fourth order tensor \( B \) such that for all \( \xi, \phi \in S_d(\mathbb{R}) \),

\[ A \xi : \phi = B \xi : B \phi. \]

Since \( B^T \xi : \phi = \xi : B^T \phi \), we obtain \( A = B^T B \).

The weak formulation (1.2) corresponding to (1.1) can be rewritten as

Find \( u \in V \) such that \( \forall v \in V, \int_{\Omega} a(u, v) = \int_{\Omega} f v \, dx \),

\[ \int_{\Omega} a(u, v) = \int_{\Omega} H^B u : H^B v \, dx \text{ and } H^B v = B H v. \tag{1.3} \]

where

\[ a(u, v) = \int_{\Omega} H^B u : H^B v \, dx \]

We assume in the following that \( B \) is constant over \( \Omega \), and that the following coercivity property holds:

\[ \exists \rho > 0 \text{ such that } \| H^B v \| \geq \rho \| v \|_{H^2(\Omega)} \forall v \in H^2_0(\Omega). \tag{1.4} \]
Hence, the weak formulation (1.3) has a unique solution by the Lax–Milgram lemma. Note that we do not necessarily discretise the full Hessian matrix and this is the purpose of the introduction of the tensors $A$ and $B$. Even for the biharmonic problem, which could be dealt with using just $B$ the identity tensor ($B\xi = \xi$), there is an interest in introducing other possible tensors that lead to the same model. Precisely because the weak formulation with $B\xi = \xi$ requires to use and discretise the entire Hessian matrix, whereas other choices of $B$, such as $B\xi = \frac{\text{tr}(\xi)}{\sqrt{d}}\text{Id}$ (where $\text{tr}(\xi)$ is the trace of $\xi$ and $\text{Id}$ is the identity matrix), lead to a weak formulation that only involves the Laplacian, and thus whose numerical approximation only requires to approximate this particular operator (not each and every second order derivative and with the full Hessian). In this paper, the FVM is built on an approximation of the Laplacian of the functions whereas the FEMs work with a generic $A$ that is independent of the model. An overview of the choice of $B$ for biharmonic and plate problems can be found in [10].

2. The Hessian discretisation method

The HDM [10] for fourth order linear elliptic partial differential equations is briefly presented in this section. The HDM consists in writing a scheme, known as a Hessian scheme (HS), by replacing the space and the continuous operators in the weak formulation (1.3) with discrete components. These discrete components are provided by a Hessian discretisation (HD).

**Definition 2.1 (B–Hessian discretisation).** A B–Hessian discretisation for homogeneous clamped boundary conditions is a quadruplet $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}}, \mathcal{H}^B_{\mathcal{D}})$ such that

- $X_{\mathcal{D},0}$ is a finite-dimensional space encoding the unknowns of the method,
- $\Pi_{\mathcal{D}} : X_{\mathcal{D},0} \to L^2(\Omega)$ is a linear mapping that reconstructs a function from the unknowns,
- $\nabla_{\mathcal{D}} : X_{\mathcal{D},0} \to L^2(\Omega)^d$ is a linear mapping that reconstructs a gradient from the unknowns,
- $\mathcal{H}^B_{\mathcal{D}} : X_{\mathcal{D},0} \to L^2(\Omega; \mathbb{R}^{d\times d})$ is a linear mapping that reconstructs a discrete version of $\mathcal{H}^B(=BH)$ from the unknowns. It must be chosen such that $\|\cdot\| := \|\mathcal{H}^B_{\mathcal{D}}\cdot\|$ is a norm on $X_{\mathcal{D},0}$.

Let $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}}, \mathcal{H}^B_{\mathcal{D}})$ be a B–Hessian discretisation. Then the related HS for (1.3) is given by

Find $u_{\mathcal{D}} \in X_{\mathcal{D},0}$ such that for any $v_{\mathcal{D}} \in X_{\mathcal{D},0}$,

$$a_{\mathcal{D}}(u_{\mathcal{D}}, v_{\mathcal{D}}) = \int_\Omega f_{\mathcal{D}} v_{\mathcal{D}} \, dx,$$

where $a_{\mathcal{D}}(u_{\mathcal{D}}, v_{\mathcal{D}}) = \int_\Omega \mathcal{H}^B_{\mathcal{D}} u_{\mathcal{D}} : \mathcal{H}^B_{\mathcal{D}} v_{\mathcal{D}} \, dx$.

2.1. **Basic error estimates.** Given a Hessian discretisation $\mathcal{D}$, the accuracy of a Hessian scheme is measured by three quantities. The first one is a constant, a measure of coercivity, which controls the norm of $\Pi_{\mathcal{D}}$ and $\nabla_{\mathcal{D}}$.

$$C^B_{\mathcal{D}} = \max_{w_{\mathcal{D}} \in X_{\mathcal{D},0} \setminus \{0\}} \left( \frac{\|\Pi_{\mathcal{D}} w_{\mathcal{D}}\|}{\|\mathcal{H}^B_{\mathcal{D}} w_{\mathcal{D}}\|} \cdot \frac{\|\nabla_{\mathcal{D}} w_{\mathcal{D}}\|}{\|\mathcal{H}^B_{\mathcal{D}} w_{\mathcal{D}}\|} \right).$$  (2.2)
The second measure involves an estimate of the interpolation error in the finite element framework, called the consistency in the framework of the HDM.

\[
\forall \varphi \in H^2_0(\Omega), \quad S^B_D(\varphi) = \min_{w_D \in X_D, \varphi} \left( \|\Pi_D w_D - \varphi\| + \|\nabla_D w_D - \nabla \varphi\| \right. \\
+ \left. \|\mathcal{H}^B_D w_D - \mathcal{H}^B \varphi\| \right).
\] (2.3)

Finally, the third quantity measures the error in the discrete integration by parts that known as the limit–conformity and is defined by

\[
\forall \xi \in H^B(\Omega), \quad W^B_D(\xi) = \max_{w_D \in X_D, \xi \neq 0} \frac{|\mathcal{W}^B_D(\xi, w_D)|}{\|\mathcal{H}^B_D w_D\|},
\] (2.4)

where \(H^B(\Omega) = \{\xi \in L^2(\Omega)^{d \times d} ; \mathcal{H} : B^T B \xi \in L^2(\Omega)\}\) and

\[
\mathcal{W}^B_D(\xi, w_D) = \int_{\Omega} \left( (\mathcal{H} : B^T B \xi) \Pi_D w_D - B \xi : \mathcal{H}^B_D w_D \right) \, dx.
\] (2.5)

The notation \(X \lesssim Y\) means that \(X \leq CY\) for some \(C\) depending only on \(\Omega\) and an upper bound of \(C^B_D\).

**Theorem 2.2** (Error estimate for Hessian schemes). [10, Theorem 3.6] Let \(D\) be a \(B\)-Hessian discretisation in the sense of Definition 2.1, \(u\) be the solution to (1.3) and \(w_D\) be the solution to (2.1). Then

\[
\|\Pi_D w_D - u\| + \|\nabla_D w_D - \nabla u\| + \|\mathcal{H}^B_D w_D - \mathcal{H}^B u\| \lesssim W^B_D(u),
\] (2.6)

where

\[
W^B_D(u) = W^B_D(\mathcal{H} u) + S^B_D(u).
\] (2.7)

**Remark 2.3** (Convergence of the HS). Along a sequence \((D_m)_{m \in \mathbb{N}}\) of \(B\)-Hessian discretisations, it is expected that \(C^B_{D_m}\) remains bounded, \(S^B_{D_m}(\varphi) \to 0\) for all \(\varphi \in H^2_0(\Omega)\) and \(W^B_{D_m}(\xi) \to 0\) for all \(\xi \in H^B(\Omega)\) as \(m \to \infty\) (see for example Theorem 3.12). Then Theorem 2.2 gives the convergence of the HS along sequences of such HDs.

### 2.2. Examples of HD.

A few examples of \(B\)-HD are presented in this section. We refer to [10] for a detailed analysis of these methods. In addition, it is established that the Morley ncFEM is an example of HDM. Let us first set some notations related to meshes.

**Definition 2.4** (Polysedral mesh [9, Definition 7.2]). Let \(\Omega\) be a bounded polytopal open subset of \(\mathbb{R}^d\) (\(d \geq 1\)). A polysedral mesh of \(\Omega\) is \(\mathcal{T} = (\mathcal{M}, \mathcal{F}, \mathcal{P})\), where:

1. \(\mathcal{M}\) is a finite family of non empty connected polysedral open disjoint subsets of \(\Omega\) (the cells) such that \(\overline{\Omega} = \bigcup_{K \in \mathcal{M}} \overline{\Omega}_K\). For any \(K \in \mathcal{M}\), \(|K| > 0\) is the measure of \(K\), \(h_K\) denotes the diameter of \(K\), \(\mathbf{x}_K\) is the center of mass of \(K\), and \(\mathbf{n}_K\) is the outer unit normal to \(K\).

2. \(\mathcal{F}\) is a finite family of disjoint subsets of \(\overline{\Omega}\) (the edges of the mesh in 2D, the faces in 3D), such that any \(\sigma \in \mathcal{F}\) is a non empty open subset of a hyperplane of \(\mathbb{R}^d\) and \(\sigma \subset \overline{\Omega}\). Assume that for all \(K \in \mathcal{M}\) there exists a subset \(\mathcal{F}_K\) of \(\mathcal{F}\) such that the boundary of \(K\) is \(\bigcup_{\sigma \in \mathcal{F}_K} \overline{\sigma}\). We then set \(\mathcal{M}_g = \{K \in \mathcal{M} ; \sigma \in \mathcal{F}_K\}\) and assume that, for all \(\sigma \in \mathcal{F}\), \(\mathcal{M}_g\) has exactly one element and \(\sigma \subset \partial \Omega\), or \(\mathcal{M}_g\) has two elements and \(\sigma \subset \Omega\). Let \(\mathcal{F}_{int}\) be the set of all interior faces, i.e. \(\sigma \in \mathcal{F}\) such that \(\sigma \subset \Omega\), and \(\mathcal{F}_{ext}\) the set of all exterior faces, i.e. \(\sigma \in \mathcal{F}\) such that \(\sigma \supset \partial \Omega\).
of boundary faces, i.e. $\sigma \in F$ such that $\sigma \subset \partial \Omega$. The $(d - 1)$-dimensional measure of $\sigma \in F$ is $|\sigma|$, and its centre of mass is $\mathbf{x}_\sigma$.

(3) $\mathcal{P} = \{\mathbf{x}_K\}_{K \in \mathcal{M}}$ is a family of points of $\Omega$ indexed by $\mathcal{M}$ and such that, for all $K \in \mathcal{M}$, $\mathbf{x}_K \in K$. Assume that any cell $K \in \mathcal{M}$ is strictly $\mathbf{x}_K$-star-shaped, meaning that if $\mathbf{x} \in K$ then the line segment $[\mathbf{x}_K, \mathbf{x}]$ is included in $K$.

The diameter of such a polytopal mesh is $h = \max_{K \in \mathcal{M}} h_K$. The set of internal vertices of $\mathcal{M}$ (resp. vertices on the boundary) is denoted by $V_{\text{int}}$ (resp. $V_{\text{ext}}$).

We assume that $\mathcal{M} = \mathcal{M}_h$ satisfies minimal regularity assumptions. That is, if $\rho_K = \max \{r > 0 : B(\mathbf{x}_K, r) \subset K\}$, then there exists $\eta > 0$, independent of $h$, such that $\forall K \in \mathcal{M}$, $\frac{h_K}{\rho_K} \leq \eta$.

2.2.1. Conforming finite elements. The $B$–HD $\mathcal{D} = (X_{D,0}, \Pi_D, \nabla_D, \mathcal{H}_B^D)$ for conforming FEM is defined by: $X_{D,0}$ is a finite dimensional subspace of $H_0^2(\Omega)$ and, for $v_D \in X_{D,0}$, $\Pi_D v_D = v_D$, $\nabla_D v_D = \nabla v_D$ and $\mathcal{H}_B^D v_D = \mathcal{H}_B^D v_D$. Examples of conforming finite elements include the Argyris and Bogner–Fox–Schmit (BFS) finite elements, see [5] for details.

2.2.2. Non-conforming finite elements.

- The Adini rectangle [5]: Assume that $\Omega \subset \mathbb{R}^2$ can be covered by a mesh $\mathcal{M}$ made up of rectangles. Figure 1 (left) represents an Adini rectangle $K \in \mathcal{M}$ with vertices $a_1$, $a_2$, $a_3$, and $a_4$ respectively. Each $v_D \in X_{D,0}$ is a vector of three values at each vertex of the mesh (with zero values at boundary vertices), corresponding to function and gradient values, $\Pi_D v_D$ is the function such that the values of $(\Pi_D v_D)|_K \in \mathbb{R}^3 \oplus \{x_1 x_2^2\} \oplus \{x_1^2 x_2\}$ and its gradients at the vertices are dictated by $v_D$, $\nabla_D v_D = \nabla(\Pi_D v_D)$ and $\mathcal{H}_B^D v_D = \mathcal{H}_B^D v_D$ is the broken $\mathcal{H}_B$ of $\Pi_D v_D$.

- The Morley element [5]: We recast here the classical nonconforming FEM, the Morley ncFEM, in the Hessian discretisation method with $d = 2$. Let $\mathcal{M}$ be a regular conforming triangulation of $\overline{\Omega}$ into closed triangles (see Figure 1, right). The Morley finite element is a triplet $(K, \mathbb{P}_K, \Sigma_K)$ where $K$ is a triangle, $\mathbb{P}_K = \mathbb{P}_2(K)$, space of all polynomials of degree $\leq 2$ in two variables defined on $K$ (dim $\mathbb{P}_K = 6$) and $\Sigma_K$ denote the degrees of freedom consist of the values at the vertices of the mesh and normal derivatives at the midpoints of the edges opposite to these vertices.
Let $\mathbb{P}_2(\mathcal{M})$ denote the space of all piecewise polynomials of degree at most equal to 2 defined on $\mathcal{M}$. Then the nonconforming Morley element space associated with $\mathcal{M}$ is defined by

$$V_h =: \left\{ \phi \in \mathbb{P}_2(\mathcal{M}) | \phi \text{ is continuous at } \mathcal{V}_{int} \text{ and vanishes at } \mathcal{V}_{ext}, \right.$$  
$$\forall \mathcal{\sigma} \in \mathcal{F}_{int}, \int_{\mathcal{\sigma}} \left[ \frac{\partial \phi}{\partial n} \right] \, ds = 0; \forall \mathcal{\sigma} \in \mathcal{F}_{ext}, \int_{\mathcal{\sigma}} \frac{\partial \phi}{\partial n} \, ds = 0 \right\},$$

where $[\phi]$ denote the jump of the function $\phi$ along the edges.

**Definition 2.5** (Hessian discretisation for the Morley element). Each $v_{\mathcal{D}} \in X_{\mathcal{D},0}$ is a vector of degrees of freedom at the vertices of the mesh (with zero values at boundary vertices) and at the midpoint of the edges opposite to these vertices (with zero values at midpoint of the boundary edges). $\Pi_{\mathcal{D}}v_{\mathcal{D}}$ is the function such that $(\Pi_{\mathcal{D}}v_{\mathcal{D}})|_{K} \in \mathbb{P}_{K}$ (resp. its normal derivatives) takes the values at the vertices (resp. at the edge midpoints) dictated by $v_{\mathcal{D}}$, $\nabla_{\mathcal{D}}v_{\mathcal{D}} = \nabla_{\mathcal{M}}(\Pi_{\mathcal{D}}v_{\mathcal{D}})$ is the broken gradient of $\Pi_{\mathcal{D}}v_{\mathcal{D}}$ and $\mathcal{H}^B_{\mathcal{D}}v_{\mathcal{D}} = \mathcal{H}^B_{\mathcal{M}}(\Pi_{\mathcal{D}}v_{\mathcal{D}})$ is the broken $\mathcal{H}^B$ of $\Pi_{\mathcal{D}}v_{\mathcal{D}}$.

### 2.2.3. Method based on Gradient Recovery Operators

In this method, the finite element space $V_h$ consists of piecewise linear polynomials, which are continuous over $\Omega$ and have a zero value on $\partial\Omega$. Let $u_h \in V_h$ and let $Q_h : L^2(\Omega) \rightarrow V_h$ be a gradient recovery projection operator (see, e.g., [10], Section 4.2) for a GR operator based on biorthogonal systems). This gives $Q_h \nabla u_h \in \mathbb{P}_1$, which is differentiable and hence a sort of second derivative of $u_h$ is expressed in terms of $\nabla Q_h \nabla u_h$. In order to ensure the coercivity property of this reconstructed Hessian, we consider a stabilisation function $\mathcal{S}_h \in L^\infty(\Omega)^d$ with specific design properties [10]. Then the $B$-Hessian discretisation based on a triplet $(V_h, Q_h, \mathcal{S}_h)$ is defined by: $X_{\mathcal{D},0} = V_h$ and, for $u_{\mathcal{D}} \in X_{\mathcal{D},0}$, $\Pi_{\mathcal{D}}u_{\mathcal{D}} = u_{\mathcal{D}}$, $\nabla_{\mathcal{D}}u_{\mathcal{D}} = Q_h \nabla u_{\mathcal{D}}$ and $\mathcal{H}^B_{\mathcal{D}}u_{\mathcal{D}} = B \left[ \nabla(Q_h \nabla u_{\mathcal{D}}) + \mathcal{S}_h \otimes (Q_h \nabla u_{\mathcal{D}} - \nabla u_{\mathcal{D}}) \right]$.

### 2.2.4. Finite volume scheme based on $\Delta$-adapted discretisations

Consider the finite volume scheme from [12] for the biharmonic problem on $\Delta$-adapted meshes (see Figure 2). For all $\mathcal{\sigma} \in \mathcal{F}_{int}$ with $\mathcal{M}_{\mathcal{\sigma}} = \{K, L\}$, the straight line $(x_K, x_L)$ intersects and is orthogonal to $\mathcal{\sigma}$, and for all $\mathcal{\sigma} \in \mathcal{F}_{ext}$ with $\mathcal{M}_{\mathcal{\sigma}} = \{K\}$, the line orthogonal to $\mathcal{\sigma}$ going through $x_K$ intersects $\mathcal{\sigma}$. Since $\mathcal{H}^B = \Delta$ in this method, one possible choice of $B$ is therefore to set $B\xi = \frac{\text{tr}(\xi)}{\sqrt{d}} \text{Id}$ for $\xi \in \mathcal{S}_d(\mathbb{R})$ where $\text{Id}$ is the identity matrix. This method requires only one unknown per cell.

**Figure 2.** Notations for $\Delta$-adapted discretisation
$X_{D,0}$ is the space of all real families $v_D = (v_K)_{K \in \mathcal{M}}$ such that $v_K = 0$ if $K$ touches $\partial \Omega$. The operator $\Pi_D$ reconstructs a piecewise constant function given by: for any cell $K$, $\Pi_D v_D = v_K$ on $K$. For $K \in \mathcal{M}$ and $\sigma \in F_K$, let $n_K, \sigma$ be the unit vector normal to $\sigma$ outward to $K$. For all $\sigma \in \mathcal{F}$, we choose an orientation (that is, a cell $K$ such that $\sigma \in F_K$) and set $n_\sigma = n_K, \sigma$. For each $\sigma \in \mathcal{F}_{\text{int}}$, denote by $K_\sigma^-$ and $K_\sigma^+$ the two adjacent control volumes such that the unit normal vector $n_\sigma$ is oriented from $K_\sigma^-$ to $K_\sigma^+$. For all $\sigma \in \mathcal{F}_{\text{ext}}$, denote the control volume $K \in \mathcal{M}$ such that $\sigma \in F_K$ by $K_\sigma$ and define $n_\sigma$ by $n_K, \sigma$. Let

$$d_\sigma = \begin{cases} \text{dist}(x_{K_\sigma^-}, \sigma) + \text{dist}(x_{K_\sigma^+}, \sigma) & \forall \sigma \in F_{\text{int}} \\ \text{dist}(x_{K_\sigma}, \sigma) & \forall \sigma \in F_{\text{ext}} \end{cases}$$

where $\text{dist}(x_K, \sigma)$ denotes the orthogonal distance between $x_K$ and $\sigma$. The discrete gradient $\nabla_D$ and the Laplace operator $\Delta_D$ are defined by their constant values on the cells,

$$\nabla_K v_D = \frac{1}{|K|} \sum_{\sigma \in F_K} |\sigma| (\delta_{K, \sigma} v_D) (\mathbf{v}_\sigma - x_K), \quad \Delta_K v_D = \frac{1}{|K|} \sum_{\sigma \in F_K} |\sigma| \delta_{K, \sigma} v_D,$$

and set $H^B_D v_D = \frac{\Delta_D v_D}{\sqrt{d_\sigma}} \text{Id}$, where

$$\delta_{K, \sigma} v_D = \begin{cases} v_L - v_K & \forall \sigma \in F_K \cap F_{\text{int}}, M_\sigma = \{K, L\} \\ 0 & \forall \sigma \in F_K \cap F_{\text{ext}}. \end{cases}$$

**Remark 2.6** (Rates of convergence [10]). Under regularity assumption $u \in H^4(\Omega) \cap H^3_0(\Omega)$, for low-order conforming FEMs, Adini ncFEM and gradient recovery methods based on meshes with mesh parameter “h”, $O(h)$ estimates can be obtained for $W^B_D(\nabla u)$ and $S^B_D(u)$. Theorem 2.2 then gives a linear rate of convergence for these methods. For FVM based on $\Delta$-adapted discretisations, Theorem 2.2 provides an $O(h^{1/4} |\ln(h)|)$ (in $d = 2$) or $O(h^{3/13})$ (in $d = 3$) error estimate for the Hessian scheme based on the Hessian discretisation. In addition to these results from [10], in this paper, we show that the HDM framework enables us to recover a linear rate of convergence for Morley ncFEM (see Theorem 3.12).

### 3. Main Results

The improved $L^2$ and $H^1$ error estimates for HDM are stated in this section. Also, an estimate on the accuracy measures $C^B_D$, $S^B_D$ and $W^B_D$ associated with an HD $\mathcal{D}$ using Morley ncFEM is stated at the end of this section. The proofs of the results are presented in Section 5. The improved error estimates are then applied to the methods listed in Section 2, that is, FEMs, method based on GR operators and slightly modified FVM (see Definition 3.4). The modified FVM has the same matrix as the original FVM, since only the quadrature of the source term is modified, but enjoys a super-convergence result while the standard FVM fails to super-converge.

#### 3.1. Improved $L^2$ error estimate.

For establishing the lower order $L^2$ estimates, consider the adjoint problem corresponding to (1.3), and its Hessian scheme approximation.

The weak formulation for the dual problem with source term $g \in L^2(\Omega)$ seeks $\varphi_g \in V$ such that

$$a(w, \varphi_g) = (g, w) \text{ for all } w \in V.$$  (3.1)
The Hessian scheme corresponding to (3.1) seeks \( \varphi_{g, D} \in X_{D,0} \) such that
\[
\alpha_{D}(w_D, \varphi_{g,D}) = (g, \Pi_D w_D) \quad \text{for all } w_D \in X_{D,0}.
\] (3.2)

**Theorem 3.1** (Improved \( L^2 \) error estimate for Hessian schemes).

Let \( u \) be the solution to (1.3). Let \( D \) be a \( B- \)Hessian discretisation in the sense of Definition 2.1, and let \( u_D \) be the solution to the Hessian scheme (2.1). Define
\[
g = \frac{u - \Pi_D u_D}{\| u - \Pi_D u_D \|} \in L^2(\Omega)
\]
and let \( \varphi_g \) be the solution to (3.1). Choose \( \mathcal{P}_D u, \mathcal{P}_D \varphi_g \in X_{D,0} \), where \( \mathcal{P}_D \) is a mapping from \( H^1_0(\Omega) \) to \( X_{D,0} \). Then
\[
\| \Pi_D u_D - u \| \lesssim (\| \mathcal{H}_B^B \mathcal{P}_D u - \mathcal{H}^B u \| + W_d^B(u)) (\| \mathcal{H}_B^B \mathcal{P}_D \varphi_g - \mathcal{H}^B \varphi_g \| + W_d^B(\varphi_g)) + \| \Pi_D \mathcal{P}_D u - u \| + f \| \Pi_D \mathcal{P}_D \varphi_g - \varphi_g \| + \| W_d^B(\mathcal{H}_B^B u, \mathcal{P}_D \varphi_g) \| + \| W_d^B(\mathcal{H}_B^B \varphi_g, \mathcal{P}_D u) \|,
\]
where \( W_d^B \) is defined by (2.7), and \( W_d^B \) is defined by (2.5).

**Remark 3.2** (Dominating terms). Following Remark 2.6, for FEMs and methods based on GR operators, it is expected that \( WS^B(u) = O(h) \) if \( u \in H^4(\Omega) \cap H^2_0(\Omega) \). Hence, for a given HS, Theorem 3.1 provides an improved result if we can find a mapping \( \mathcal{P}_D \) (usually an interpolant) such that \( \| \mathcal{H}_B^B \mathcal{P}_D \phi - \mathcal{H}^B \phi \| = O(h) \), \( \| \Pi_D \mathcal{P}_D \phi - \phi \| = O(h^2) \), \( W_d^B(\xi, \mathcal{P}_D \phi) = O(h^2) \) for all \( \phi \in H^4(\Omega) \cap H^2_0(\Omega) \) and all \( \xi \in H^2(\Omega)^{d \times d} \).

The proof of Theorem 3.1 is presented in Section 5.1. We now turn to the application of the above theorem to various schemes described in Section 2.2. The proof of Propositions 3.3 and 3.5 are given in Section 6, Appendix. Proposition 3.3 justifies the rates numerically observed for the method based on GR operator in [10].

**Proposition 3.3.** Let \( u \in H^4(\Omega) \cap H^2_0(\Omega) \) be the solution to (1.3) and \( u_D \) be the solution to the Hessian scheme (2.1). Then, for low-order conforming FEMs, Adini and Morley nFEMs, and gradient recovery methods, there exists a constant \( C > 0 \), not depending on \( h \), such that
\[
\| \Pi_D u_D - u \| \leq Ch^2.
\]

Since the super-convergence is not known in general for two point flux approximation (TPFA) for second order problems, it is expected that the same issue occurs for the FVM mentioned in Section 2.2.4. In order to obtain an improved result, ideas developed in [11, Section 4] for GDM is appropriately modified for the HDM.

For that, set
\[
v_\sigma = \begin{cases} \frac{\text{dist}(x_K, \sigma) v_L + \text{dist}(x_L, \sigma) v_K}{d\sigma} & \forall \sigma \in \mathcal{F}_\text{int}, \mathcal{M}_\sigma = \{K, L\} \\ 0 & \forall \sigma \in \mathcal{F}_\text{ext} \end{cases}
\] (3.3)

We now define a slightly modified HDM for FVM based on \( \Delta \)-adapted discretisations.

**Definition 3.4** (Modified FVM \( B- \)HD). Let \( D = (X_{D,0}, \Pi_D, \nabla_D, \mathcal{H}_D^B) \) be a FVM \( B- \)Hessian discretisation given in Section 2.2.4. The modified FVM \( B- \)Hessian discretisation is \( D^* = (X_{D,0}, \Pi_D^*, \nabla_D, \mathcal{H}_D^B) \), where the reconstruction function \( \Pi_D^* \) is defined by
\[
\forall v_D \in X_{D,0}, \forall K \in \mathcal{M}, \forall x \in K, \Pi_D^* v_D(x) = \Pi_D v_D(x) + \nabla_K v_D \cdot (x - x_K)
\] (3.4)
Proposition 3.5 (Superconvergence for modified FVM HD). Let \( u \in H^4(\Omega) \cap H^2_0(\Omega) \) be the solution to (1.3). Let \( u_D \) be the solution of the Hessian scheme (2.1) for the modified FVM \( B-HD \) in the sense of Definition 3.4 on a super-admissible mesh. Then for the modified FVM based on \( \Delta \)-adapted discretisations, there exist a constant \( C > 0 \), independent of \( h \), such that
\[
\| \Pi_D \cdot u_D - u \| \leq C \begin{cases} \frac{h^{1/2}}{h^{6/13}} \ln(h)^2 & \text{if } d = 2 \\ \frac{h^{1/2}}{h^{6/13}} & \text{if } d = 3 \end{cases}
\]

Recalling Remark 2.6, we see that these rates are an improvement over the rates in \( H^2 \) norm. Precisely, \( L^2 \) error estimate decays as the square of the \( H^2 \) error estimate.

3.2. Improved \( H^1 \) error estimate. To establish an improved \( H^1 \) error estimate, consider the following dual problem of (1.3).

The weak formulation for the dual problem with source term \( q \in H^{-1}(\Omega) \) seeks \( \varphi_q \in V \) such that
\[
a(w, \varphi_q) = (q, w) \quad \text{for all } w \in V.
\]
Moreover, when \( \Omega \) is convex, \( \varphi_q \in H^3(\Omega) \cap H^2_0(\Omega) \) with a priori bound \( \| \varphi_q \|_{H^3(\Omega)} \leq \| q \|_{H^{-1}(\Omega)} \) [1]. In order to state the \( H^1 \) error estimate, we need to consider the limit-conformity measure between the reconstructed Hessian \( H_D^B \) and reconstructed gradient \( \nabla D \). Define
\[
\forall \chi \in H^1_{\text{div}}(\Omega)^d, \quad \widetilde{W}_D^B(\chi) = \max_{w_D \in X_{D,0}\setminus\{0\}} \frac{\| \widetilde{W}_D^B(\chi, w_D) \|}{\| H_D^B w_D \|}.
\]
where \( H^1_{\text{div}}(\Omega)^d = \{ \chi \in L^2(\Omega)^{d \times d} : \text{div}(B^\top B \chi) \in L^2(\Omega)^d \} \) and
\[
\widetilde{W}_D^B(\chi, w_D) := \int_{\Omega} \left( B \chi : H_D^B w_D + \text{div}(B^\top B \chi) \cdot \nabla_D w_D \right) d\mathbf{x}.
\]

Assume the existence of an operator \( E_D \) which maps the discrete unknowns to the continuous space of functions. This operator plays a central role in the \( H^1 \) error estimate analysis for HDM.

Assumption 3.6 (Companion operator). Let \( D \) be a \( B \)-Hessian discretisation in the sense of Definition 2.1. There exists a linear map \( E_D : X_{D,0} \rightarrow H^2_0(\Omega) \) called the companion operator. We define
\[
\omega(E_D) := \sup_{\psi_D \in X_{D,0}\setminus\{0\}} \frac{\| \nabla_D \psi_D - \nabla E_D \psi_D \|}{\| H_D^B \psi_D \|}.
\]
Along a sequence of Hessian discretisations \((D_m)_{m\in\mathbb{N}}\), it is expected that the companion operators are defined such that \(\omega(E_{D_m}) \to 0\) as \(m \to \infty\). For example, an explicit companion operator is well-known for the Morley element with \(\omega(E_D) = O(h)\) [3].

**Theorem 3.7** (Improved \(H^1\) error estimate for Hessian schemes).

Let \(u\) be the solution to (1.3). Let \(D\) be a Hessian discretisation in the sense of Definition 2.1 and \(u_D\) be the solution to the Hessian scheme (2.1). Assume that there exists a companion operator \(E_D\) in the sense of Assumption 3.6 and define
\[
q = -\Delta_{E_D}(u_D - P_D u) = \frac{\|\nabla E_D(u_D - P_D u)\|}{H^{-1}(\Omega)}.
\]

Assume that the solution to (3.6) satisfies \(\varphi_q \in H^3(\Omega) \cap H^2_0(\Omega)\) and choose \(P_D u\), \(P_D \varphi_q \in X_{D,0}\), where \(P_D : H^2_0(\Omega) \to X_{D,0}\). Then
\[
\|\nabla u_D - \nabla u\| \leq \omega(E_D) + \|\nabla u - \nabla P_D u\| + \|\nabla \varphi_q - \nabla P_D \varphi_q\| + \|W_B^D(\varphi_q) - \varphi_q\|,
\]
where \(\omega(E_D)\) is defined by (3.9), \(W_B^D\) is defined by (2.7), and \(\tilde{W}_B^D\) is defined by (3.7).

The proof of Theorem 3.7 is given in Section 5.2.

**Remark 3.8.** Following Remark 3.2, for FEMs and methods based on GR operators, Theorem 3.7 gives an improved error estimate in \(H^1\) norm if \(\|\nabla \phi - \nabla P_D \phi\| = O(h^2), \tilde{W}_B^D(\chi, P_D \phi) = O(h^2), \omega(E_D) = O(h)\) and \(\tilde{W}_B^D(\chi) = O(h)\) for all \(\phi \in H^4(\Omega) \cap H^2_0(\Omega)\) and all \(\chi \in H^4(\Omega)^{d \times d}\).

**Remark 3.9.** The companion operators actually come with estimates on function, gradient given by (3.9) and Hessian (see e.g., [3]). The estimates on function and Hessian are not needed in the error analysis and hence we leave them undefined.

The following proposition talks about the discrete \(H^1\) error estimate for lower order conforming and non-conforming FEMs and the proof is given in Section 6, Appendix.

**Proposition 3.10.** Let \(u \in H^4(\Omega) \cap H^2_0(\Omega)\) be the solution to (1.3) and \(u_D\) be the solution to the Hessian scheme (2.1). Then, for low-order conforming FEMs, and Adini and Morley ncFEMs, there exists a constant \(C\), not depending on \(h\), such that
\[
\|\nabla u_D - \nabla u\| \leq C h^2.
\]

**Remark 3.11.** The construction of a companion operator \(E_D\) for the method based on gradient recovery operators with \(\omega(E_D)\) small enough is an open problem. Though there is a difficulty of constructing a proper companion operator and hence improved \(H^1\) theoretical rate of convergence are not obtained, we observe that the numerical rates in \(H^1\) norm are better (see Table 1). In numerical test for FVM, the \(H^2\) and \(H^1\) estimated rates of convergences appear to be both of order 1 ([10, Section 6]). This seems to indicate that we cannot expect an improved estimate in \(H^1\) norm compared to the estimate in energy norm. Hence, the FVM method is probably not amenable to an application of Theorem 3.7 (which is an indication that there might not exist, for this method, a proper companion operator).
3.3. Estimates for Morley HDM. The following theorem (proof provided in Section 5.3) establishes practical estimates on the quantities (2.2)–(2.4). This helps in establishing the convergence of the scheme.

**Theorem 3.12.** Let $\mathcal{D}$ be a $B$-Hessian discretisation for the Morley element in the sense of Definition 2.5. Then, there exists a constant $C$, not depending on $\mathcal{D}$, such that

- $C_B^B \leq C,$
- $\forall \varphi \in H^1(\Omega) \cap H_0^2(\Omega)$ $S^B_\varphi \leq C \|\varphi\|_{H^3(\Omega)},$
- $\forall \xi \in H^2(\Omega)^{d\times d}$ $W^B_\xi \leq C h \|\xi\|_{H^2(\Omega)^{d\times d}}.$

The following result is a straightforward consequence of Theorems 3.12 and 2.2.

**Corollary 3.13 (Convergence).** Let $(\mathcal{D}_m)_{m \in \mathbb{N}}$ be a sequence of $B$–Hessian discretisations for the Morley element associated with a mesh $\mathcal{M}_m$ such that $h_m \to 0$ as $m \to \infty$, with $B$ satisfying estimate (1.5). Then $\Pi_{\mathcal{D}_m} u_{\mathcal{D}_m} \to u$, $\nabla_{\mathcal{D}_m} u_{\mathcal{D}_m} \to \nabla u$ and $H^B_{\mathcal{D}_m} u_{\mathcal{D}_m} \to H_u$ as $m \to \infty$.

4. **Numerical Results**

The results of the numerical experiments for the GR method and the modified FVM are presented in this section. Consider the biharmonic problem $\Delta^2 u = f$ on $\Omega$ with homogeneous clamped boundary conditions.

4.1. **Gradient Recovery Method.** Let the relative errors in $L^2(\Omega)$, $H^1(\Omega)$ and $H^2(\Omega)$ norms be denoted by

$$
\text{err}_D(u) := \frac{\|\Pi_{\mathcal{D}} u - u\|}{\|u\|},
\qquad
\text{err}_D(\nabla u) := \frac{\|\nabla_{\mathcal{D}} u - \nabla u\|}{\|\nabla u\|},
$$

$$
\text{err}_D(H u) := \frac{\|\nabla Q_h \nabla_{\mathcal{D}} u - H u\|}{\|H u\|},
$$

where $u_{\mathcal{D}}$ is the solution to the Hessian scheme (2.1). We refer the reader to [18] for implementation procedure. To determine the effect of the stabilisation function $\mathcal{S}_h$ on the results, we multiply it by a factor $\rho$ that takes the values 0.001, 1, and 10.

4.1.1. **Example 1.** Let $\Omega = (0,1)^2$. Figure 3 shows the initial triangulation of a square domain and its uniform refinement. In this example, we choose the right-hand side load function $f$ such that the exact solution is given by $u(x,y) = \sin^2(\pi x) \sin^2(\pi y)$. The computed errors and orders of convergence in the energy, $H^1$ and $L^2$ norms with $\rho = 1$ are shown in Table 1. As seen in the table, we obtain linear order of convergence in the energy norm and quadratic order of convergence in $L^2$ norm, which agrees with the theoretical result in Proposition 3.3. Using gradient recovery operator, a quadratic rate of convergence is obtained in the $H^1$ norm (see Remark 3.11 for that).

4.1.2. **Example 2.** In this example, we consider the non-convex L-shaped domain given by $\Omega = (-1,1)^2 \setminus ((0,1) \times (-1,0))$. Figure 4 shows the initial triangulation of a L-shaped domain and its uniform refinement. The source term $f$ is chosen such that the model problem has the following exact singular solution [15]:

$$
u = (r^2 \cos^2 \theta - 1)^2 (r^2 \sin^2 \theta - 1)^2 r^{1+\gamma} g_{\gamma,\omega}(\theta),$$
**Figure 3.** Initial triangulation and uniform refinement of square domain

**Table 1.** (GR) Convergence results for the relative errors, Example 1, $\rho = 1$

| $h$     | $\text{err}_T(u)$ | Order | $\text{err}_T(\nabla u)$ | Order | $\text{err}_T(Hu)$ | Order |
|---------|--------------------|-------|---------------------------|-------|---------------------|-------|
| $0.353553$ | $3.124409$ | -     | $0.721457$ | -     | $0.855054$ | -     |
| $0.176777$ | $0.145381$ | 4.4257 | $0.099974$ | 2.8513 | $0.246640$ | 1.7936 |
| $0.088388$ | $0.036224$ | 2.0048 | $0.023098$ | 2.1138 | $0.116470$ | 1.0824 |
| $0.044194$ | $0.009068$ | 1.9982 | $0.005552$ | 2.0566 | $0.057308$ | 1.0232 |
| $0.022097$ | $0.002261$ | 2.0037 | $0.001363$ | 2.0266 | $0.028470$ | 1.0093 |
| $0.011049$ | $0.000564$ | 2.0032 | $0.000398$ | 2.0116 | $0.014198$ | 1.0037 |

where $(r, \theta)$ denote the polar coordinates, $\gamma \approx 0.5444837367$ is a non-characteristic root of $\sin^2(\gamma \omega) = \gamma^2 \sin^2(\omega)$, $\omega = \frac{3\pi}{2}$, and $g_{\gamma,\omega}(\theta) = (\frac{1}{\gamma - 1} \sin((\gamma - 1)\omega) - \frac{1}{\gamma + 1} \sin((\gamma + 1)\omega))(\cos((\gamma - 1)\theta) - \cos((\gamma + 1)\theta)) - (\frac{1}{\gamma - 1} \sin((\gamma - 1)\omega) - \frac{1}{\gamma + 1} \sin((\gamma + 1)\omega))(\cos((\gamma - 1)\theta) - \cos((\gamma + 1)\omega))$. The errors and rates of convergence are reported in Tables 2–4 respectively. This example is particularly interesting since the solution is less regular due to the corner singularity. The domain $\Omega$ being nonconvex, we expect only suboptimal orders of convergence in the energy, $H^1$ and $L^2$ norms, and this can be clearly seen from the tables. For instance, the convergence rate in $L^2$ norm is 1.5, which is suboptimal. As in Example 1, the numerical rates in $H^1$ norm

**Figure 4.** Initial triangulation and uniform refinement of L-shaped domain
are similar to those in $L^2$ norm. This improved order of convergence in $H^1$ norm is obtained with the help of gradient recovery operator (see Proposition 3.3 and Remark 3.11). It can be seen that the stabilisation parameter $\rho$ has a very small impact on the numerical results.

Table 2. (GR) Convergence results for the relative errors, Example 2, $\rho = 0.001$

| $h$     | $err_D(u)$ | Order | $err_D(\nabla u)$ | Order | $err_D(Hu)$ | Order |
|---------|------------|-------|------------------|-------|-------------|-------|
| 0.353553 | 1.488937   | -     | 0.394870         | -     | 0.504144    | -     |
| 0.176777 | 1.85753    | 3.0028| 0.139904         | 1.4969| 0.218736    | 2.046 |
| 0.083888 | 0.58874    | 1.6577| 0.045530         | 1.6196| 0.116520    | 0.9086|
| 0.041944 | 0.18039    | 1.7065| 0.013756         | 1.7267| 0.065220    | 0.8372|
| 0.022097 | 0.00540    | 1.7401| 0.004197         | 1.7128| 0.038827    | 0.7483|
| 0.011049 | 0.001681   | 1.6835| 0.001396         | 1.5882| 0.024390    | 0.6707|
| 0.00524  | 0.000570   | 1.5617| 0.000526         | 1.4085| 0.015899    | 0.6174|

Table 3. (GR) Convergence results for the relative errors, Example 2, $\rho = 1$

| $h$     | $err_D(u)$ | Order | $err_D(\nabla u)$ | Order | $err_D(Hu)$ | Order |
|---------|------------|-------|------------------|-------|-------------|-------|
| 0.353553 | 0.447227   | -     | 0.377554         | -     | 0.441034    | -     |
| 0.176777 | 0.177626   | 1.3322| 0.142208         | 1.4087| 0.217792    | 1.0180|
| 0.083888 | 0.059387   | 1.5806| 0.046087         | 1.6256| 0.115943    | 0.9095|
| 0.041944 | 0.018023   | 1.7203| 0.013886         | 1.7307| 0.064187    | 0.8390|
| 0.022097 | 0.005360   | 1.7496| 0.004231         | 1.7147| 0.038615    | 0.7472|
| 0.011049 | 0.001661   | 1.6897| 0.001406         | 1.5894| 0.024290    | 0.6688|
| 0.00524  | 0.000562   | 1.5629| 0.000529         | 1.4100| 0.015854    | 0.6156|

Table 4. (GR) Convergence results for the relative errors, Example 2, $\rho = 10$

| $h$     | $err_D(u)$ | Order | $err_D(\nabla u)$ | Order | $err_D(Hu)$ | Order |
|---------|------------|-------|------------------|-------|-------------|-------|
| 0.353553 | 0.488271   | -     | 0.423933         | -     | 0.472514    | -     |
| 0.176777 | 0.197355   | 1.3069| 0.162455         | 1.3785| 0.220725    | 1.0594|
| 0.083888 | 0.064165   | 1.6209| 0.050639         | 1.6817| 0.116820    | 0.9567|
| 0.041944 | 0.019077   | 1.7500| 0.014842         | 1.7706| 0.064360    | 0.8601|
| 0.022097 | 0.005598   | 1.7688| 0.004440         | 1.7408| 0.038226    | 0.7516|
| 0.011049 | 0.001718   | 1.7041| 0.001455         | 1.6102| 0.024090    | 0.6662|
| 0.00524  | 0.000576   | 1.5759| 0.000541         | 1.4277| 0.015763    | 0.6119|

4.2. Modified Finite Volume Method. The numerical tests for FVM discussed in Section 2.2.4 are performed in [10, Section 6]. In this section, three numerical experiments that justify the theoretical result in Proposition 3.5 for modified FVM are presented. We conduct the test on a series of regular triangular meshes ($mesh1$ family) taken from [16] over the unit square $\Omega = (0, 1)^2$. The orthogonality property is satisfied with the point $x_K \in K$ chosen as the circumcenter of $K$. Let the relative errors in $L^2(\Omega)$, $H^1(\Omega)$ and $H^2(\Omega)$ norms be denoted by

$$err_D(u) := \frac{\|\Pi_D u - u\|}{\|u\|}, \quad err_D(\nabla u) := \frac{\|\nabla \Pi_D \nabla u - \nabla u\|}{\|\nabla u\|},$$

$$err_D(\Delta u) := \frac{\|\Delta \Pi_D \nabla u - \Delta u\|}{\|\Delta u\|}.$$
where \( u_{D^*} \) is the solution to the Hessian scheme (2.1) corresponding to the HD \( D^* \) given by Definition 3.4.

4.2.1. Example 1. In the first example, choose the right hand side function such that the exact solution is given by \( u(x, y) = x^2 y^2 (1 - x)^2 (1 - y)^2 \). The error estimates and convergence rates in the energy, \( H^1 \) and \( H^2 \) norms are presented in Table 5. We obtain a quadratic (or slightly better) rate of convergence in \( L^2 \) norm, linear rate of convergence in \( H^1 \) norm and sub-linear rate of convergence in \( H^2 \) norm. Note that the numerical test provides better result compared to the theoretical result, see Proposition 3.5. The numerical results for modified FVM are similar to those for the FVM.

![Table 5](image)

4.2.2. Example 2. In this case, we consider \( u(x, y) = x^2 y^2 (1 - x)^2 (1 - y)^2 (\cos(2\pi x) + \sin(2\pi y)) \). The numerical results, presented in Table 6, are similar to those obtained for Example 1.

![Table 6](image)

4.2.3. Example 3. The exact solution is chosen to be \( u(x, y) = x^3 y^3 (1 - x)^3 (1 - y)^3 (\exp(x) \sin(2\pi x) + \cos(2\pi x)) \). The convergence results are presented in Table 7. In this example, an \( O(h) \) convergence rate is obtained in \( H^2 \) norm. Since there is no improvement of the rates from \( H^2 \) to \( H^1 \), as mentioned in Remark 3.11, we cannot expect an improved \( H^1 \) estimate for FVM.

Remark 4.1. For rectangular meshes, in order to satisfy the orthogonality property, \( x_K \in K \) is chosen as the centre of mass of \( K \). From [11, Theorem 5.3], it follows that the difference between the source term of modified FVM and original FVM is of \( O(h^2) \). Therefore similar rate of convergence is obtained for modified FVM, since we see an \( O(h^2) \) convergence rate in \( L^2 \) and \( H^1 \) norms for FVM in [10, Section 6].
TABLE 7. (Modified FV) Convergence results, Example 3

| h   | $\text{err}_{TV}(u)$ Order | $\text{err}_{TV}(\nabla u)$ Order | $\text{err}_{TV}(\Delta u)$ Order |
|------|--------------------------|----------------------------------|----------------------------------|
| 0.250000 | 0.410550             | -                                | 0.704301                         | -                                |
| 0.125000 | 0.029103             | 3.8183                           | 0.212960                         | 1.7256                           |
| 0.062500 | 0.008773             | 7.1301                           | 0.096846                         | 1.1368                           |
| 0.031250 | 0.002041             | 1.3052                           | 0.047833                         | 1.0177                           |
| 0.015625 | 0.000503             | 2.0203                           | 0.023843                         | 1.0044                           |
| 0.007813 | 0.000125             | 2.0048                           | 0.011913                         | 1.0011                           |

5. PROOF OF THE MAIN RESULTS

The proof of the main results stated in Section 3 are provided in this section. Subsection 5.1 deals with the proof of improved $L^2$ estimate (Theorem 3.1) and the proof of improved $H^1$ estimate (Theorem 3.7) is presented in Subsection 5.2. In Subsection 5.3, the estimates associated with the Morley HDM (Theorem 3.12) are derived.

5.1. Proof of the improved $L^2$ estimate. To prove Theorem 3.1, we shall make use of the following Lemma, which estimates the error associated with the continuous bilinear form $a(\cdot, \cdot)$ and discrete bilinear form $a_D(\cdot, \cdot)$.

**Lemma 5.1.** Let $\psi, \phi \in H^1_0(\Omega)$ be such that $H : A\psi \in L^2(\Omega)$ and $H : A\phi \in L^2(\Omega)$. Then, for any $\psi_D, \phi_D \in X_D, 0$, the following holds:

$$|a(\psi, \phi) - a_D(\psi_D, \phi_D)| \leq E_D(\psi, \phi, \psi_D, \phi_D),$$  \hspace{1cm} (5.1)

where

$$E_D(\psi, \phi, \psi_D, \phi_D) = |W^B_D(H\psi, \phi_D)| + |W^B_D(H\phi, \psi_D)| + \|\Pi_D\psi_D - \psi\|\|H : A\phi\| + \|\Pi_D\phi_D - \phi\|\|H : A\psi\| + \|H^B_D\psi_D - H^B\psi\|\|H^B_D\phi_D - H^B\phi\|. \hspace{1cm} (5.2)$$

**Proof.** Use the definitions of $a(\cdot, \cdot)$ and $a_D(\cdot, \cdot)$ and perform elementary manipulations to obtain

$$a(\psi, \phi) - a_D(\psi_D, \phi_D) = \int_\Omega H^B \psi : H^B \phi \, dx - \int_\Omega H^B_D \psi_D : H^B_D \phi_D \, dx$$

$$= \int_\Omega (H^B \psi - H^B_D \psi_D) : H^B \phi \, dx$$

$$+ \int_\Omega (H^B_D \psi_D - H^B \psi) : (H^B \phi - H^B_D \phi_D) \, dx$$

$$+ \int_\Omega H^B \psi : (H^B \phi - H^B_D \phi_D) \, dx =: T_1 + T_2 + T_3. \hspace{1cm} (5.3)$$

$T_1$ can be estimated using integration by parts twice and (2.5).

$$T_1 = \int_\Omega \psi(H : A\phi) \, dx + W^B_D(H\phi, \psi_D) - \int_\Omega (H : A\phi)\Pi_D\psi_D \, dx.$$  \hspace{1cm} (5.4)

Hence, by the Cauchy–Schwarz inequality, this gives

$$|T_1| \leq |W^B_D(H\phi, \psi_D)| + \|H : A\phi\|\|\psi - \Pi_D\psi_D\|. \hspace{1cm} (5.4)$$

A use of the Cauchy–Schwarz inequality leads to an upper bound for the term $T_2$ as

$$|T_2| \leq \|H^B \psi - H^B_D \psi_D\|\|H^B \phi - H^B_D \phi_D\|. \hspace{1cm} (5.5)$$
The term $T_3$ is estimated exactly as $T_1$ interchanging the roles of $(\psi, \psi_D)$ and $(\phi, \phi_D)$, which leads to

$$|T_3| \leq |\mathcal{W}_D^B(\mathcal{H}\psi, \phi_D)| + \|\mathcal{H} : A\mathcal{H}\psi\|\|\phi - \Pi_D\phi_D\|.$$  

(5.6)

A substitution of the estimates (5.4)–(5.6) into (5.3) leads to (5.1).

We now prove the main result given by Theorem 3.1. Note that the proof is obtained by modification of the arguments of [11, Theorem 3.1] in the GDM framework to that of HDM.

**Proof of Theorem 3.1.** Choose $w = u$ in (3.1) and $w_D = u_D$ in (3.2),

$$\|u - \Pi_D u_D\| = \|g, u - \Pi_D u_D\| = a(u, \varphi_g) - a_D(u_D, \varphi_{g,D}).$$  

(5.7)

Since $u$ and $\varphi_g$ both belong to $H_0^2(\Omega)$ with $\mathcal{H} : A\mathcal{H}u = f \in L^2(\Omega)$ and $\mathcal{H} : A\mathcal{H}\varphi_g = g \in L^2(\Omega)$, a use of (5.1) in (5.7) with some manipulations lead to

$$\|u - \Pi_D u_D\| = a(u, \varphi_g) - a_D(P_Du, P_D\varphi_g) + a_D(P_Du, P_D\varphi_g) - a_D(u_D, \varphi_{g,D})$$

$$\leq E_D(u, \varphi_g - \varphi_{g,D}) + a_D(P_Du, P_D\varphi_g) - a_D(u_D, \varphi_{g,D})$$

$$= a_D(P_Du, P_D\varphi_g - \varphi_{g,D}) + a_D(P_Du - u_D, \varphi_{g,D})$$

$$+ E_D(u, \varphi_g, P_Du, P_D\varphi_g) =: T_1 + T_2 + E_D(u, \varphi_g, P_Du, P_D\varphi_g).$$  

(5.8)

An introduction of $a_D(P_Du, P_D\varphi_g)$, $a_D(u_D, P_D\varphi_g)$ and choose $v_D = P_D\varphi_g - \varphi_{g,D}$ in (2.1) to deduce

$$T_2 = -a_D(P_Du, P_D\varphi_g - \varphi_{g,D}) + a_D(u_D, P_D\varphi_g - \varphi_{g,D}) + a_D(P_Du - u_D, P_D\varphi_g)$$

$$= -[a_D(P_Du, P_D\varphi_g - \varphi_{g,D}) - (f, \Pi_D(P_D\varphi_g - \varphi_{g,D}))] + a_D(P_Du - u_D, P_D\varphi_g)$$

$$= -T_{2.1} + T_{2.2}.$$  

(5.9)

We now turn to $T_3$. Introduce the terms $a_D(P_Du, P_D\varphi_g)$, $a_D(u_D, P_D\varphi_g)$ and choose $v_D = P_D\varphi_g - \varphi_{g,D}$ in (2.5) to deduce

$$T_{2.1} = \int_\Omega (H_B^u : H_B^D(P_D\varphi_g - \varphi_{g,D}) - f\Pi_D(P_D\varphi_g - \varphi_{g,D})) \, dx$$

$$+ \int_\Omega (H_B^D(P_Du - H_B^u) : H_B^D(P_D\varphi_g - \varphi_{g,D}) \, dx$$

$$= \int_\Omega (H_B^D(P_Du - H_B^u) : (H_B^D(P_D\varphi_g - H_B^D\varphi_{g,D}) \, dx - W_D^B(Hu, PD\varphi_g - \varphi_{g,D}).$$  

Therefore, apply (2.4), the Cauchy–Schwarz inequality, (2.7), a triangle inequality and (2.6) to obtain

$$|T_{2.1}| \leq W_D^B(Hu)||H_B^D(P_D\varphi_g - H_B^D\varphi_{g,D})| + ||H_B^D(P_Du - H_B^u)||_B||H_B^D(P_D\varphi_g - H_B^D\varphi_{g,D})|$$

$$\lesssim ||H_B^D(P_D\varphi_g - H_B^D\varphi_{g,D})| + W_D^B(\varphi_g)|\|H_B^D(P_Du - H_B^u)||$$

$$\lesssim (||H_B^D(P_D\varphi_g - H_B^D\varphi_{g,D})| + W_D^B(\varphi_g)|\|H_B^D(P_Du - H_B^u)| + W_D^B(u).$$  

(5.11)
The term $T_{2,2}$ is similar to $T_1$, upon swapping the primal and dual problems, that is $(f, u, u_D, g, \varphi, \varphi_D) \leftrightarrow (g, \varphi, \varphi_D, f, u_D)$. Hence, from (5.9),
\[ |T_{2,2}| \leq \|f\| \|\varphi - \Pi_D P_D \varphi\| + E_D(u, \varphi, P_D u, P_D \varphi). \] (5.12)
Plug the estimates (5.11) and (5.12) in (5.10) to obtain
\[ |T_2| \lesssim (\|B^D_P u - B^D u\| + W_{B_P}(u))(\|\|B^D_P P_D \varphi - B^D \varphi\| + W_{B_D}(\varphi)) + \|f\| \|\varphi - \Pi_D P_D \varphi\| + E_D(u, \varphi, P_D u, P_D \varphi). \] (5.13)
A substitution of (5.9) and (5.13) in (5.8) leads to
\[ \|u - \Pi_D u_D\| \lesssim (\|B^D_P u - B^D u\| + W_{B_P}(u))(\|\|B^D_P P_D \varphi - B^D \varphi\| + W_{B_D}(\varphi)) + \|u - \Pi_D P_D u\| + \|f\| \|\varphi - \Pi_D P_D \varphi\| + E_D(u, \varphi, P_D u, P_D \varphi), \]
where we have used the fact that $\|g\| = 1$. Finally, the proof is complete by using the definition (5.2) of $E_D$ and noticing that $H : A H u = f \in L^2(\Omega)$ and $H : A H \varphi = g \in L^2(\Omega)$. \quad \square

5.2. Proof of the improved $H^1$ estimate.

Proof of Theorem 3.7. A use of the triangle inequality leads to
\[ \|\nabla D u_D - \nabla u\| \leq \|\nabla D u_D - \nabla P_D u\| + \|\nabla P_D u - \nabla u\|. \] (5.14)
Let us estimate $\|\nabla D u_D - \nabla P_D u\|$. Set $v_D = u_D - P_D u \in X_D, 0$. Introduce $\nabla E_D u_D$ and $\nabla D u_D$, and use triangle inequalities, (3.9) and (2.6) to obtain
\[ \|\nabla D v_D\| \leq \|\nabla D u_D - \nabla E_D v_D\| + \|\nabla E_D v_D\| \leq \omega(E_D) \|H^D_B v_D\| + \|\nabla E_D v_D\| \leq \omega(E_D) \|H^D_B u_D - H^B u\| + \|H^B u - H^B P_D u\| + \|\nabla E_D v_D\| \lesssim \omega(E_D) (W_{B_D}(u) + \|H^B u - H^B P_D u\|) + \|\nabla E_D v_D\|. \] (5.15)
Consider $\|\nabla E_D v_D\|$. From (3.6) with $w = E_D v_D$,
\[ \|\nabla E_D v_D\| = a(E_D v_D, \varphi_q) = \int_\Omega (H^D_B E_D v_D - H^B_D v_D) : H^B \varphi_q \, dx \]
\[ + \int_\Omega H^B_D v_D : H^B \varphi_q \, dx =: T_1 + T_2. \] (5.16)
An integration by parts and a use of (3.8), (3.7), the Cauchy–Schwarz inequality, (3.9), the triangle inequality and (2.6) yield
\[ |T_1| \leq \int_\Omega |\text{div}(A H \varphi_q) : (\nabla D v_D - \nabla E_D v_D)| \, dx + \int_\Omega |\tilde{W}_{B_D}(H \varphi_q)| \|H^B_D v_D\| \]
\[ \leq \omega(E_D) \|H^D_B v_D\| \|\text{div}(A H \varphi_q)\| + \int_\Omega |\tilde{W}_{B_D}(H \varphi_q)| \|H^B_D v_D\| \lesssim (\omega(E_D) \|\text{div}(A H \varphi_q)\| + \|\tilde{W}_{B_D}(H \varphi_q)\| \|H^B u - H^B P_D u\|). \] (5.17)
Simple manipulations leads to
\[ T_2 = \int_\Omega (H^B u - H^B P_D u) : H^B \varphi_q \, dx + \int_\Omega (H^B_D u_D - H^B u) : (H^B \varphi_q - H^B_D P_D \varphi_q) \, dx \]
\[ + \int_\Omega (H^B_D u_D - H^B u) : H^B_D P_D \varphi_q \, dx =: T_{2,1} + T_{2,2} + T_{2,3}. \] (5.18)
Integration by parts, (3.8) and the Cauchy–Schwarz inequality imply that
\[ |T_{2,1}| \leq \|\text{div}(A H \varphi_q)\| \|\nabla D u_D - \nabla u\| + |\tilde{W}_{B_D}(H \varphi_q, P_D u)|. \] (5.19)
Apply Cauchy–Schwarz inequality and (2.6) to obtain
\[
|T_{2.2}| \leq \|\mathcal{H}^B B - \mathcal{H}^B Du| \|\mathcal{H}^B \varphi - \mathcal{H}^B D \varphi| \leq WS^B_D(u)\|\mathcal{H}^B \varphi - \mathcal{H}^B D \varphi||. 
\]
(5.20)

Since \( \mathcal{H} : A H u = f \), by (2.5) and (2.1) with \( v_D = P_D \varphi \), the term \( T_{2.3} \) can be estimated as
\[
T_{2.3} \leq - \int_\Omega (\mathcal{H} : A H u) \Pi D D \varphi \, dx + W^B_D(H u, P_D \varphi) + \int_\Omega \mathcal{H}^B_D u_D : \mathcal{H}^B_D D \varphi \, dx
\]
\[
= - \int_\Omega (\mathcal{H} : A H u) \Pi D D \varphi \, dx + W^B_D(H u, P_D \varphi) + \int_\Omega f \Pi D D \varphi \, dx
\]
\[
= W^B_D(H u, P_D \varphi). 
\]
(5.21)

A substitution of (5.19)–(5.21) in (5.18) yields
\[
|T_2| \lesssim \|\text{div}(A H \varphi)\| \|\nabla u - \nabla D P_D u\| + |\tilde{W}^B_D(H \varphi, P_D u)|
\]
\[
+ WS^B_D(u)\|\mathcal{H}^B \varphi - \mathcal{H}^B D \varphi|| + |W_D^B(H u, P_D \varphi)|. 
\]
(5.22)

Plug (5.17) and (5.22) in (5.16) to obtain an estimate for \( \|\nabla D v_D\| \) as
\[
\|\nabla D v_D\| \lesssim (\omega(E_D)|\text{div}(A H \varphi)\| + \tilde{W}^B_D(H \varphi)) (WS^B_D(u) + \|\mathcal{H}^B u - \mathcal{H}^B D P_D u||)
\]
\[
+ \|\text{div}(A H \varphi)\| \|\nabla u - \nabla D P_D u\| + |\tilde{W}^B_D(H \varphi, P_D u)|
\]
\[
+ WS^B_D(u)\|\mathcal{H}^B \varphi - \mathcal{H}^B D \varphi|| + |W_D^B(H u, P_D \varphi)|. 
\]
(5.23)

A use of the apriori bound for the dual problem \( \|\varphi\|_{H^1(\Omega)} \lesssim 1 \) yields
\[
\|\nabla D v_D\| \lesssim (\omega(E_D) + \tilde{W}^B_D(H \varphi)) (WS^B_D(u) + \|\mathcal{H}^B u - \mathcal{H}^B D P_D u||)
\]
\[
+ \|\nabla u - \nabla D P_D u\| + |\tilde{W}^B_D(H \varphi, P_D u)|
\]
\[
+ WS^B_D(u)\|\mathcal{H}^B \varphi - \mathcal{H}^B D \varphi|| + |W_D^B(H u, P_D \varphi)|. 
\]
(5.24)

A substitution of (5.24) into (5.15) leads to an estimate on \( \|\nabla D v_D\| \) (with \( v_D = u_D - P_D u \in X_{D,0} \)) which when plugged on (5.14) gives
\[
\|\nabla D v_D - \nabla u\| \lesssim (\omega(E_D) + \tilde{W}^B_D(H \varphi)) (WS^B_D(u) + \|\mathcal{H}^B u - \mathcal{H}^B D P_D u||)
\]
\[
+ \|\nabla u - \nabla D P_D u\| + |\tilde{W}^B_D(H \varphi, P_D u)|
\]
\[
+ WS^B_D(u)\|\mathcal{H}^B \varphi - \mathcal{H}^B D \varphi|| + |W_D^B(H u, P_D \varphi)|
\]
and this completes the proof. \( \square \)

5.3. Proof of the HDM properties for the Morley element.

Proof of Theorem 3.12. Let \( D = (X_{D,0}, \Pi_D, \nabla_D, \mathcal{H}^B_D) \) be a \( B \)-Hessian discretisation for the Morley ncFEM in the sense of Definition 2.5. In the sequel, we will use a generic constant \( C \), which will take different values at different places but will always be independent of the mesh size \( h \).

• Coercivity: Let \( v_D \in X_{D,0} \). Since \( [\Pi_D v_D] = 0 \) at the face vertices for any \( v_D \in X_{D,0} \) and \( [\nabla_D v_D] = 0 \) at the edge midpoints, use Lemma 6.4 twice and the coercivity property of \( B \) given by (1.5) to obtain
\[
\|\Pi_D v_D\| \leq C\|\nabla_D v_D\| \leq C\|\mathcal{H}_D v_D\| \leq C\varphi^{-1}\|\mathcal{H}^B_D v_D\|.
\]
This with (2.2) concludes the estimate on \( C^B_D \).
For the methods based on gradient recovery operator,

(i) For conforming FEMs and Morley ncFEM,

\[ \inf_{w_D \in X_{D,0}} \| \Pi_D w_D - \varphi \| \leq C h^3 \| \varphi \|_{H^3(\Omega)}, \quad \inf_{w_D \in X_{D,0}} \| \nabla D w_D - \nabla \varphi \| \leq C h^2 \| \varphi \|_{H^3(\Omega)}, \]

\[ \inf_{w_D \in X_{D,0}} \| \mathcal{H}_D^B w_D - \mathcal{H}^B \varphi \| \leq C h \| \varphi \|_{H^3(\Omega)}. \]

Therefore, we obtain \( S_B^D (\varphi) \leq C h \| \varphi \|_{H^3(\Omega)}. \)

(ii) Limit–conformity: For any \( \xi \in H^2_0 (\Omega)^{d \times d} \) and \( v_D \in X_{D,0} \), cellwise integration by parts yields

\[
\int_{\Omega} (\mathcal{H} : A\xi) \Pi_D v_D \, dx = \sum_{K \in \mathcal{M}} \int_K (\mathcal{H} : A\xi) \Pi_D v_D \, dx
\]

\[
= \int_{\Omega} A\xi : \mathcal{H}_D v_D \, dx - \sum_{K \in \mathcal{M}} \int_{\partial K} (A\xi n_K) \cdot \nabla_D v_D \, ds(x)
\]

\[
+ \sum_{K \in \mathcal{M}} \int_{\partial K} (\text{div}(A\xi) \cdot n_K) \Pi_D v_D \, ds(x).
\]

This gives

\[
\int_{\Omega} (\mathcal{H} : A\xi) \Pi_D v_D \, dx - \int_{\Omega} A\xi : \mathcal{H}_D v_D \, dx = -\sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{F}_K} \int_{\sigma} (A\xi n_{K,\sigma}) \cdot \nabla_D v_D \, ds(x)
\]

\[
+ \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{F}_K} \int_{\sigma} (\text{div}(A\xi) \cdot n_{K,\sigma}) \Pi_D v_D \, ds(x).
\]

An appropriate modification to the proof of [20, Lemma 3.5] yields

\[
\left| \int_{\Omega} (\mathcal{H} : A\xi) \Pi_D v_D \, dx - \int_{\Omega} A\xi : \mathcal{H}_D v_D \, dx \right| \leq C h^{-1} \| \mathcal{H}_D^B v_D \| \| \xi \|_{H^2(\Omega)^{d \times d}}
\]

and this leads to the desired estimate on \( \mathcal{W}_D^B \).

6. Appendix

In this section, the proofs of Propositions 3.3, 3.5 and 3.10 are presented. This is followed by some technical results.

6.1. Proof of the applications of improved \( L^2 \) error estimate. We start by a preliminary result that states the approximation properties of the classical interpolant \( \mathcal{P}_D \) for various methods.

Lemma 6.1 (Interpolation [5, 10]). Let \( \psi \in H^3(\Omega) \cap H^2_0(\Omega) \) and \( \phi \in H^4(\Omega) \cap H^3_0(\Omega) \). The classical interpolant satisfies

(i) For conforming FEMs and Morley ncFEM,

\[ \| \Pi_D \mathcal{P}_D \psi - \psi \| \leq Ch^3, \| \nabla D \mathcal{P}_D \psi - \nabla \psi \| \leq Ch^2 \text{ and } \| \mathcal{H}_D^B \mathcal{P}_D \psi - \mathcal{H}^B \psi \| \leq Ch. \]

(ii) For Adini ncFEM,

\[ \| \Pi_D \mathcal{P}_D \phi - \phi \| \leq Ch^4, \| \nabla D \mathcal{P}_D \phi - \nabla \phi \| \leq Ch^3 \text{ and } \| \mathcal{H}_D^B \mathcal{P}_D \phi - \mathcal{H}^B \phi \| \leq Ch^2. \]

(iii) For the methods based on gradient recovery operator,

\[ \| \Pi_D \mathcal{P}_D \psi - \psi \| \leq Ch^2, \| \nabla D \mathcal{P}_D \psi - \nabla \psi \| \leq Ch^2 \text{ and } \| \mathcal{H}_D^B \mathcal{P}_D \psi - \mathcal{H}^B \psi \| \leq Ch. \]
The next lemma establishes an estimate on the limit–conformity measure $W_D^B$ given by (2.5) for various schemes.

**Lemma 6.2.** Let $\xi \in H^2(\Omega)^{d \times d}$, $\psi \in H^3(\Omega) \cap H^2_0(\Omega)$ and $\phi \in H^2(\Omega) \cap H^3_0(\Omega)$.

(i) For conforming FEMs, we have $W_D^B(\xi, \mathcal{P}_D \psi) = 0$.

(ii) For Adini ncFEM, $W_D^B(\xi, \mathcal{P}_D \phi) = O(h^2)$.

(iii) For Morley ncFEM and gradient recovery methods, $W_D^B(\xi, \mathcal{P}_D \psi) = O(h^2)$.

**Proof.** (i) **Conforming FEMs.** Since $X_D, 0 \subseteq H_0^2(\Omega)$, using integration by parts twice, the limit-conformity measure vanishes, that is, $W_D^B = 0$.

(ii) **Nonconforming FEM: the Adini rectangle.** Let $\phi \in H^2(\Omega) \cap H^3_0(\Omega)$ and $\xi \in H^2(\Omega)^{d \times d}$. Introduce the term $(\mathcal{H} : A \xi)\phi$ in (2.5), use the Cauchy–Schwarz inequality and Lemma 6.1 to obtain

$$W_D^B(\xi, \mathcal{P}_D \phi) \leq \left| \int_{\Omega} (\mathcal{H} : A \xi) \Pi_D \mathcal{P}_D \phi - (\mathcal{H} : A \xi) \phi \right| \leq \left| \int_{\Omega} (\mathcal{H} : A \xi) \phi - B \xi : \mathcal{H}_D^B \mathcal{P}_D \phi \right|$$

$$\leq \|\mathcal{H} : A \xi\| \|\Pi_D \mathcal{P}_D \phi - \phi\| + \left| \int_{\Omega} (\mathcal{H} : A \xi) \phi - B \xi : \mathcal{H}_D^B \mathcal{P}_D \phi \right|$$

$$\leq C h^4 + \left| \int_{\Omega} (\mathcal{H} : A \xi) \phi - B \xi : \mathcal{H}_D^B \mathcal{P}_D \phi \right|.$$  

Apply integration by parts twice to deduce

$$W_D^B(\xi, \mathcal{P}_D \phi) \leq C h^4 + \left| \int_{\Omega} (B \xi : \mathcal{H}_D^B \phi - B \xi : \mathcal{H}_D^B \mathcal{P}_D \phi) \right|.$$  

A use of the Cauchy–Schwarz inequality and Lemma 6.1 leads to

$$W_D^B(\xi, \mathcal{P}_D \phi) \leq C h^4 + \|B \xi\| H_2^B \mathcal{P}_D \phi - \mathcal{H}_D^B \phi \| \leq C h^2.$$  

(iii)(a) **Nonconforming FEM: the Morley triangle.** Let $\psi \in H^3(\Omega) \cap H^2_0(\Omega)$ and $\xi \in H^2(\Omega)^{d \times d}$. Proceeding as in the proof of limit conformity $W_D^B(\xi, \mathcal{P}_D \psi)$ for the Adini’s rectangle (with $\|\Pi_D \mathcal{P}_D \psi - \psi\| \leq C h^3$), from (6.1), we arrive at

$$W_D^B(\xi, \mathcal{P}_D \psi) \leq C h^4 + \left| \int_{\Omega} (B \xi : \mathcal{H}_D^B \psi - B \xi : \mathcal{H}_D^B \mathcal{P}_D \psi) \right|.$$  

Let $\xi_K$ be the average value of $\xi$ on the cell $K \in \mathcal{M}$. By the mesh regularity assumption, $\|\xi - \xi_K\|_{L^2(K)^{d \times d}} \leq C h \|\xi\|_{H^1(K)^{d \times d}}$ (see, e.g., [9, Lemma B.6]). An introduction of $B \xi_K$ in the above inequality and a use of the Cauchy–Schwarz inequality and Lemma 6.1 yield

$$W_D^B(\xi, \mathcal{P}_D \psi) \leq C h^3 + \sum_{K \in \mathcal{M}} \|B \xi - B \xi_K\|_{L^2(K)^{d \times d}} \|B \xi_K\psi - B \xi_K \mathcal{P}_D \psi\|_{L^2(K)^{d \times d}}$$

$$+ \left| \int_{K} B \xi_K : (\mathcal{H}_D^B \psi - \mathcal{H}_D^B \mathcal{P}_D \psi) \right|$$

$$\leq C h^2 + \sum_{K \in \mathcal{M}} \int_{K} B \xi_K : (\mathcal{H}_D^B \psi - \mathcal{H}_D^B \mathcal{P}_D \psi) \ dx.$$
For $K \in \mathcal{M}$, we have [14]
\[
\int_K \mathcal{H}^B_D \mathcal{P}_D \psi \, dx = \int_K \mathcal{H}^B \psi \, dx.
\]
(6.3)
Hence, $\mathcal{W}^B_D(\xi, \mathcal{P}_D \psi) = \mathcal{O}(h^2)$.

(iii)(b) GRADIENT RECOVERY METHOD. Note that for the GR method, $\Pi_D \mathcal{P}_D \psi = \mathcal{P}_D \psi \in V_h$, an $H^1_0$-conforming finite element space which contains the piecewise linear functions, and $\|\nabla \mathcal{P}_D \psi - \nabla \psi\| \leq C h$. Let us consider $\mathcal{W}^B_D(\xi, \mathcal{P}_D \psi)$. Reproducing the same steps as in the proof for Adini's rectangle (with $\|\Pi_D \mathcal{P}_D \psi - \psi\| \leq C h^2$), from (6.1) and the definition of reconstructed Hessian $\mathcal{H}^B_D$ (see Section 2.2), we obtain
\[
|\mathcal{W}^B_D(\xi, \mathcal{P}_D \psi)| \leq Ch^2 + \left|\int_{\Omega} \left( A \xi : \mathcal{H} \psi - A \xi : \nabla Q_h \nabla \mathcal{P}_D \psi \right) \, dx \right| + \left|\int_{\Omega} A \xi : (\mathcal{S}_h \otimes (Q_h \nabla \mathcal{P}_D \psi - \nabla \mathcal{P}_D \psi)) \, dx \right| =: Ch^2 + A_1 + A_2.
\]
Since $Q_h \nabla \mathcal{P}_D \psi \in H^1_0(\Omega)$, an integration by parts, the Cauchy–Schwarz inequality and the approximation property of $\mathcal{P}_D$ given by Lemma 6.1 show that
\[
|A_1| = \left| -\int_{\Omega} \nabla \psi \cdot \text{div}(A \xi) \, dx + \int_{\Omega} Q_h \nabla \mathcal{P}_D \psi \cdot \text{div}(A \xi) \, dx \right| \\
\leq \|Q_h \nabla \mathcal{P}_D \psi - \nabla \psi\| ||\text{div}(A \xi)|| = ||\nabla \mathcal{P}_D \psi - \nabla \psi\| ||\text{div}(A \xi)|| \leq Ch^2.
\]
To estimate $A_2$, we shall make use of the orthogonality property of the stabilisation function. For all $K \in \mathcal{M}$, denoting by $V_h(K) = \{v|_K : v \in V_h, K \in \mathcal{M}\}$ the local finite element space,
\[
[\mathcal{S}_h|_K \otimes (Q_h \nabla - \nabla)(V_h(K))] \perp \nabla V_h(K)^d
\]
where the orthogonality is understood in $L^2(K)^{d \times d}$ with the inner product induced by “$\cdot$”. Let $\xi_K$ denote the average of $\xi$ over $K \in \mathcal{M}$. Since the finite dimensional space $V_h$ contains the piecewise linear functions, $\nabla V_h(K)$ contains the constant vector-valued functions on $K$ and thus, by the orthogonality condition, the Cauchy–Schwarz inequality, the boundedness of $\mathcal{S}_h$, the triangle inequality and the approximation properties of the interpolant,
\[
|A_2| = \left| \sum_{K \in \mathcal{M}} \int_K (A \xi - A \xi_K) : [\mathcal{S}_h \otimes (Q_h \nabla \mathcal{P}_D \psi - \nabla \mathcal{P}_D \psi)] \right| \\
\leq C \sum_{K \in \mathcal{M}} \|\xi - \xi_K\|_{L^2(K)^{d \times d}} \|Q_h \nabla \mathcal{P}_D \psi - \nabla \mathcal{P}_D \psi\|_{L^2(K)^d} \\
\leq Ch \|\nabla \mathcal{P}_D \psi - \nabla \mathcal{P}_D \psi\| \\
\leq Ch \left( \|\nabla \mathcal{P}_D \psi - \nabla \psi\| + \|\nabla \psi - \nabla \mathcal{P}_D \psi\| \right) \leq Ch^2.
\]
Therefore, we obtain $\mathcal{W}^B_D(\xi, \mathcal{P}_D \psi) = \mathcal{O}(h^2)$.

Proof of Proposition 3.3. The proof of Proposition 3.3 follows from Theorem 2.2, Remark 2.6, Lemma 6.1 and Lemma 6.2. □
Proof of Proposition 3.5. As a consequence of Stokes’ formula, we have for \( K \in \mathcal{M} \), 
\[
\sum_{\sigma \in \mathcal{F}_K} |\sigma| \, n_{K,\sigma} = 0 \quad \text{(see the proof of [9, Lemma B.3])}.
\]
A use of (3.3) and the superadmissible mesh condition 
\[
n_{K,\sigma} = \frac{d_{K,\sigma}}{d_{K,\sigma}}
\]
leads to 
\[
\tilde{\nabla}_K v_D = \frac{1}{|K|} \sum_{\sigma \in \mathcal{F}_K} |\sigma| (v_\sigma - v_K) \, n_{K,\sigma} = \nabla_K v_D,
\]
where \((\nabla_D v_D)_K = \nabla_K v_D\) as defined in Section 2.2.4. Hence,
\[
\int_K \nabla_D v_D \, dx = \int_K \nabla_K v_D \, dx = |K| \nabla_K v_D.
\]
The definition of \( D^* \), the above relation between \( \tilde{\nabla}_K \) and \( \nabla_D \), and (2.2) imply 
\[
\forall v_D \in X_{D,0}, \quad \|\Pi_D v_D - \Pi_{D^*} v_D\|_{L^2(\Omega)} \lesssim h \|\mathcal{H}^B_D v_D\|.
\]
Therefore, following the proof of [9, Remark 7.51], we obtain the same estimates on \( C^B_D, S^B_D \), and \( W^B_D \) for \( D^* \) as that for the original FVM HD \( D \) given in Section 2.2.4. Thus, from Remark 2.6, under regularity assumption, an \( O(h^{1/4} \ln h) \) (in \( d = 2 \)) or \( O(h^{3/13}) \) (in \( d = 3 \)) error estimate can be obtained for the Hessian scheme based on modified FVM HD \( D^* \). Note that to prove the error estimates for original FVM, the interpolation \( \mathcal{P}_D \) is constructed by solving a TPFA scheme for second order problem, i.e., by considering \( |K| \Delta_K \mathcal{P}_D \phi = \int_K \Delta \phi \, dx \) for \( \phi \) smooth enough and \( K \in \mathcal{M} \). To preserve a superconvergence for this modified FVM, the idea is to construct \( \mathcal{P}_{D^*} \phi \) by solving the modified TPFA scheme, where \( \Pi_D \) is replaced by \( \Pi_{D^*} \). Since TPFA and Hybrid Mimetic Mixed (HMM) schemes are the same on superadmissible meshes, from [11, Theorem 4.6],
\[
\|\Pi_{D^*} \mathcal{P}_{D^*} \phi - \phi\| \lesssim h^2 \|\phi\|_{H^2(\Omega)}.
\]
To estimate \( W^B_D(\xi, \mathcal{P}_D \phi) \), for \( \phi \in H^4(\Omega) \cap H^2_0(\Omega) \) and \( \xi \in H^2(\Omega)^{d \times d} \), consider (2.5) with \( D = D^* \). Introduce \( (\mathcal{H} : A\xi) \phi \), use the Cauchy-Schwarz inequality, (6.4) and integration by parts twice to obtain
\[
|W^B_D(\xi, \mathcal{P}_D \phi)| \leq \left| \int_\Omega \left((\mathcal{H} : A\xi)(\Pi_{D^*} \mathcal{P}_D \phi - \phi)\right) \, dx \right| \\
+ \left| \int_\Omega \left((\mathcal{H} : A\xi)\phi - B\xi : \mathcal{H}^B_D \mathcal{P}_D \phi\right) \, dx \right| \\
\leq C h^2 + \left| \int_\Omega B\xi : (\mathcal{H}^B_D \phi - \mathcal{H}^B_D \mathcal{P}_D \phi) \, dx \right|.
\]
The second term on the right-hand side of the above inequality can be estimated by considering the projection of \( B\xi \) on piecewise constant functions on the mesh \( \mathcal{M} \). Let \( B_k \xi_k \) be the projection of \( B\xi \) on \( K \in \mathcal{M} \). Since \( \Delta_K \mathcal{P}_D \phi \) is the projection of \( \Delta \phi \) on piecewise constant functions on \( \mathcal{M} \) (that is, \( |K| \Delta_K \mathcal{P}_D \phi = \int_K \Delta \phi \, dx \)), a use of the orthogonality property of the projection operator, the Cauchy-Schwarz inequality and the approximation property leads to
\[
|W^B_D(\xi, \mathcal{P}_D \phi)| \leq C h^2 + \left| \sum_{K \in \mathcal{M}} \int_\Omega (B\xi - B_k \xi_k) : (\mathcal{H}^B_D \phi - \mathcal{H}^B_D \mathcal{P}_D \phi) \, dx \right| \leq C h^2.
\]
A substitution of the above estimate, (6.4) and estimates given by Remark 2.6 in Theorem 3.1 with \( D = D^* \) yields the desired estimate. \( \square \)
6.2. Proof of the applications of improved $H^1$ error estimate.

Proof of Proposition 3.10. • CONFORMING FEMs. Let $\psi \in H^3(\Omega) \cap H^2_D(\Omega)$. Since $X_{D,0} \subseteq H^3_D(\Omega)$, by applying integration by parts, the measure of limit-conformity $\tilde{W}^B_D$ vanishes. Also, companion operator $E_D$ is nothing but the identity operator which implies $\omega(E_D) = 0$. Hence, under regularity assumption on $u$, combine these estimates along with Remark 2.6, Lemma 6.1 and Lemma 6.2 in Theorem 3.7 to obtain $\|\nabla u_D - \nabla u\| \leq Ch^2$.

• NON-CONFORMING FEM: THE ADINI RECTANGLE. The estimate $\omega(E_D) = O(h)$ for a companion operator which maps the Adini rectangle to the Bogner–Fox–Schmit rectangle [5] has been done in [2]. For $\chi \in H^1_{D,0}(\Omega)^d$ and $v_D \in X_{D,0}$, cellwise integration by parts yields

$$\int_\Omega (B_\chi : H_D^B v_D + \text{div}(A\chi) \cdot \nabla_D v_D) \, dx = \sum_{\sigma \in F} \int_\sigma (A\chi n_\sigma) \cdot \|\nabla_D v_D\| \, ds(x).$$

From [10, Theorem 7.2] and (3.7), we deduce that $\tilde{W}^B_D(\chi) = O(h)$. Let $\phi \in H^4(\Omega) \cap H^2_D(\Omega)$. Introduce $\text{div}(A\chi) \cdot \nabla \phi$ in (3.8), use an integration by parts, the Cauchy–Schwarz inequality and Lemma 6.1 to obtain

$$|\tilde{W}^B_D(\chi, P_D\phi)| \leq \left| \int_\Omega (B_\chi : H_D^B P_D\phi + \text{div}(A\chi) \cdot \nabla \phi) \, dx \right| \leq \left| \int_\Omega \text{div}(A\chi) \cdot (\nabla_D P_D\phi - \nabla \phi) \, dx \right| \leq C h^2.$$ 

The proof is complete by invoking Remark 2.6, Lemma 6.1, Lemma 6.2 and Theorem 3.7.

• NON-CONFORMING FEM: THE MORLEY TRIANGLE. For the Morley element, there exists a companion operator such that $\omega(E_D) = O(h)$, see [3] for more details. Let us estimate $\tilde{W}^B_D(\chi)$, where $\chi \in H^1_{D,0}(\Omega)^d$. For $v_D \in X_{D,0}$,

$$\int_\Omega (B_\chi : H_D^B v_D + \text{div}(A\chi) \cdot \nabla_D v_D) = \sum_{\sigma \in F} \int_\sigma (A\chi n_\sigma) \cdot \|\nabla_D v_D\| \, ds(x). \quad (6.5)$$

From (5.25) and (3.7), we obtain $\tilde{W}^B_D(\chi) = O(h)$. Let $\psi \in H^3(\Omega) \cap H^2_D(\Omega)$. In order to evaluate $\tilde{W}^B_D(\chi, P_D\psi)$, introduce $\text{div}(A\chi) \cdot \nabla \psi$ in (3.8), use an integration by parts and the Morley interpolation property given by Lemma 6.1. Hence,

$$|\tilde{W}^B_D(\chi, P_D\psi)| \leq C h^2 + \int_\Omega (H_D^B P_D\psi - H^B \psi) \, dx.$$

Now, reproduce the same steps as in the limit conformity $\tilde{W}^B_D(\xi, P_D\psi)$ proof for Morley triangle (with $\xi = \chi$) and thus from (6.2)–(6.3), $\tilde{W}^B_D(\chi, P_D\psi) = O(h^2)$.

As a consequence, for the Morley triangle, if $u \in H^4(\Omega) \cap H^2_D(\Omega)$, combine the above estimates, Theorem 2.2, Remark 2.6, Lemmas 6.1–6.2 and Theorem 3.7 to obtain the required result. \qed
6.3. Technical results.

**Lemma 6.3** (Poincaré inequality along an edge in $L^2$ norm). [10, Lemma A.1] Let $σ$ be an edge of a polygonal cell, $w \in H^1(σ)$ and assume that $w$ vanish at a point on the edge $σ \in F$. Then $∥w∥_{L^2(σ)} ≤ h_σ ∥∇_Mw∥_{L^2(σ)^d}$, where $h_σ$ is the length of $σ$.

**Lemma 6.4.** Let $k ≥ 0$ be an integer and $w \in P_k(M)$. If for all $σ \in F$ there exists $x_σ \in σ$ such that $∥w∥(x_σ) = 0$, then there exists $C > 0$ such that $∥w∥ ≤ C∥∇_Mw∥$.

**Proof.** Consider the $∥·∥_{dG,h}$ norm defined by: For all $w \in H^1(M)$,

$$∥w∥_{dG,h}^2 := ∥∇_Mw∥^2 + \sum_{σ \in F} \frac{1}{h_σ} ∥w∥_{L^2(σ)}^2.$$  (6.6)

Since $∥w∥(x_σ) = 0$ for all $σ \in F$, a use of Lemma 6.3 and the trace inequality (see [7, Lemma 1.46]) yields

$$∥w∥_{L^2(σ)} ≤ h_σ ∥∇_Mw∥_{L^2(σ)^d} ≤ h_σ \sum_{K \in M, σ} ∥∇_Mw|_K∥_{L^2(σ)^d} ≤ Ch_σ \sum_{K \in M, σ} h_K^{-1/2} ∥∇_Mw∥_{L^2(K)^d}$$  (6.7)

where $C > 0$ depends only on $k$ and $η$. A substitution of (6.7) in (6.6) leads to

$$∥w∥_{dG,h} ≤ ∥∇_Mw∥^2 + 2 \sum_{σ \in F} Ch_σ \sum_{K \in M, σ} h_K^{-1} ∥∇_Mw∥_{L^2(K)^d}^2 ≤ \sum_{K \in M, σ} ∥∇_Mw∥_{L^2(K)^d}^2 ≤ C∥∇_Mw∥^2.$$  

Use the fact that $∥w∥ ≤ C∥w∥_{dG,h}$ ([7, Theorem 5.3]) to deduce $∥w∥ ≤ C∥∇_Mw∥$. □

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**References**

[1] H. Blum and R. Rannacher. On the boundary value problem of the biharmonic operator on domains with angular corners. Math. Methods Appl. Sci., 2(4):556–581, 1980.
[2] S. C. Brenner. A two-level additive Schwarz preconditioner for nonconforming plate elements. Numer. Math., 72(4):419–447, 1996.
[3] S. C. Brenner, L.-y. Sung, H. Zhang, and Y. Zhang. A Morley finite element method for the displacement obstacle problem of clamped Kirchhoff plates. J. Comput. Appl. Math., 254:31–42, 2013.
[4] F. Brezzi and P.-A. Raviart. Mixed finite element methods for 4th order elliptic equations, Topics in numerical analysis, III (Proc. Roy. Irish Acad. Conf., Trinity Coll., Dublin, 1976). Academic Press, London, 1977.
[5] P. G. Ciarlet. The finite element method for elliptic problems. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1978. Studies in Mathematics and its Applications, Vol. 4.
[6] P. G. Ciarlet. Mathematical elasticity. Vol. II, volume 27 of Studies in Mathematics and its Applications. North-Holland Publishing Co., Amsterdam, 1997. Theory of plates.
[7] D. A. Di Pietro and A. Ern. *Mathematical aspects of discontinuous Galerkin methods*, volume 69 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer, Heidelberg, 2012.

[8] J. Douglas, Jr., T. Dupont, P. Percell, and R. Scott. A family of $C^1$ finite elements with optimal approximation properties for various Galerkin methods for 2nd and 4th order problems. *RAIRO Anal. Numér.*, 13(3):227–255, 1979.

[9] J. Droniou, R. Eymard, T. Gallouët, C. Guichard, and R. Herbin. *The gradient discretisation method*, volume 82 of *Mathématiques et Applications*. Springer International Publishing AG, Aug 2018. https://hal.archives-ouvertes.fr/hal-01382358.

[10] J. Droniou, B. P. Lamichhane, and D. Shylaja. The Hessian Discretisation Method for Fourth Order Linear Elliptic Equations. *J. Sci. Comput.*, 78(3):1405–1437, 2019.

[11] J. Droniou and N. Nataraj. Improved $L^2$ estimate for gradient schemes and super-convergence of the TPFA finite volume scheme. *IMA J. Numer. Anal.*, 38(3):1254–1293, 2018.

[12] R. Eymard, T. Gallouët, R. Herbin, and A. Linke. Finite volume schemes for the biharmonic problem on general meshes. *Math. Comp.*, 81(280):2019–2048, 2012.

[13] R. S. Falk. Approximation of the biharmonic equation by a mixed finite element method. *SIAM J. Numer. Anal.*, 15(3):556–567, 1978.

[14] D. Gallistl. Morley finite element method for the eigenvalues of the biharmonic operator. *IMA J. Numer. Anal.*, 35(4):1779–1811, 2015.

[15] P. Grisvard. *Singularities in boundary value problems*, volume 22 of *Recherches en Mathématiques Appliquées [Research in Applied Mathematics]*. Masson, Paris; Springer-Verlag, Berlin, 1992.

[16] R. Herbin and F. Hubert. Benchmark on discretization schemes for anisotropic diffusion problems on general grids. In *Finite volumes for complex applications V*, pages 659–692. ISTE, London, 2008.

[17] B. P. Lamichhane. A mixed finite element method for the biharmonic problem using biorthogonal or quasi-biorthogonal systems. *J. Sci. Comput.*, 46(3):379–396, 2011.

[18] B. P. Lamichhane. A stabilized mixed finite element method for the biharmonic equation based on biorthogonal systems. *J. Comput. Appl. Math.*, 235(17):5188–5197, 2011.

[19] B. P. Lamichhane. A finite element method for a biharmonic equation based on gradient recovery operators. *BIT*, 54(2):469–484, 2014.

[20] P. Lascaux and P. Lesaint. Some nonconforming finite elements for the plate bending problem. *Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge Anal. Numér.*, 9(R-1):9–53, 1975.

[21] J.-L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod; Gauthier-Villars, Paris, 1969.

[22] P. Percell. On cubic and quartic Clough-Tocher finite elements. *SIAM J. Numer. Anal.*, 13(1):100–103, 1976.