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Target Space Duality III: Potentials

Orlando Alvarez\textsuperscript{1}
Blazej Ruszczycki\textsuperscript{2}

Department of Physics
University of Miami
P.O. Box 248046
Coral Gables, FL 33124 USA

Abstract

We generalize previous results on target space duality to the case where there are background fields and the sigma model lagrangian has a potential function.

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\textsuperscript{1}email: oalvarez@miami.edu
\textsuperscript{2}email: ruszczycki@physics.miami.edu
1 Introduction

In two articles [1, 2] henceforth referred to as Paper I and Paper II respectively, a general theory was developed for irreducible target space duality for classical sigma models characterized by a target space $M$, a riemannian metric $g$ and an antisymmetric tensor field $B$. By irreducible duality we mean that all fields participate in the duality transformation and that there are no spectator fields. This rules out the derivation of the important Buscher formulas [3] and also duality in WZW models [4] where the duality transformation is performed by gauging an anomaly free subgroup, e.g., see the discussions in [5 6 7 8 9 10 11 12]. The latter remark requires some explanation. For example, consider a WZW model on a compact simple Lie group $G$ where the diagonal subgroup $G_D$ of the symmetry group $G \times G$ is the anomaly free subgroup that is gauged. Schematically, the prescription to construct the dual model is that the original model with fields $g$ is augmented to an equivalent $G_D$ gauge invariant model with fields $g, A, \lambda$ where $A$ are $G_D$ gauge fields and $\lambda$ are Lagrange multipliers that enforce the vanishing of the field strengths. In principle the idea is to eliminate the variables $g$ and $A$ in favor of the Lagrange multipliers $\lambda$. Naively the original model with variables $g$ had dim$G$ degrees of freedom. The dual model with variables $\lambda$ would also have dim$G$ degrees of freedom. Unfortunately this procedure does not work for a variety of reasons. If by brute force we attempt to eliminate the variables $g, A$ then the action for the $\lambda$ fields is nonlocal. We can try to finesse things by using the gauge invariance of the theory $g \rightarrow hgh^{-1}$. Unfortunately this does not allow us to gauge $g$ to the identity element. The best we can do is gauge $g$ to an element $t$ of a maximal torus $T$ and we have a residual $T$ gauge invariance. This residual gauge invariance can be used to gauge rank$G$ of the Lagrange multipliers to zero [8]. The $A$ variables can be eliminated and we are left with a local action involving only $t$ and the remaining Lagrange multipliers. We note that the “$t$” variables are spectators and thus the methods of Papers I and II do not apply. See the worked out example in [9]. Finally we mention why the results of Papers I and II suggest that it is impossible to eliminate the variables $g, A$ in favor of a local action involving only the Lagrange multipliers $\lambda$. If $\mathfrak{g}$ denotes the Lie algebra of $G$ then the Lagrange multipliers take values in $\mathfrak{g}^*$, the dual vector space. This strongly suggests that the duality transformation here is related to the cotangent bundle $T^*G = G \times \mathfrak{g}^*$. In Papers I and II we showed that duality associated with any cotangent bundle $T^*M$ implied that $M$ is a compact Lie group with the 3-form $H$ being exact in a very specific way. The 3-form in the WZW model is topologically nontrivial.

\footnote{For a list of references see Papers I and II.}
2 Framework

The sigma model with target space $M$, metric $g$, 2-form $B$ and potential function $U$ will be denoted by $(M,g,B,U)$ and has lagrangian density

$$
\mathcal{L} = \frac{1}{2} g_{ij}(x) \left( \frac{\partial x^i}{\partial \tau} \frac{\partial x^j}{\partial \tau} - \frac{\partial x^i}{\partial \sigma} \frac{\partial x^j}{\partial \sigma} \right) + B_{ij}(x) \frac{\partial x^i}{\partial \tau} \frac{\partial x^j}{\partial \sigma} - U(x)
+ A_i(x) \frac{\partial x^i}{\partial \tau} + C_i(x) \frac{\partial x^i}{\partial \sigma}.
$$

(2.1)

We will generally follow the notation and formalism introduced in [1, 2]. In the above we have introduced two background fields $A_i$ and $C_i$ that break worldsheet Lorentz invariance for the following reason. Assume we have a Lorentz invariant field theory with fields $(x^i, y^a)$ but where we are not interested in irreducible duality. Namely, only the fields $x^i$ participate in the duality transformation and the $y^a$ fields are spectators. In this case we can regard the fields $y^a$ as parameters and $g$, $B$ and $U$ depend on the $y^a$ parametrically. In the full lagrangian density there may be a term of type $K_{ai}(x,y) \frac{\partial y^a}{\partial x^i}$ and this will become a contribution to the second line of (2.1) when the fields $y^a$ are held fixed. The canonical momentum density is

$$
\pi_i = \frac{\partial \mathcal{L}}{\partial \dot{x}^i} = g_{ij} \dot{x}^j + B_{ij} \dot{x}^j + A_i(x).
$$

(2.2)

The hamiltonian density may be written as

$$
\mathcal{H} = \frac{1}{2} g^{ij} \left( \pi_i - B_{ik} \frac{dx^k}{d\sigma} \right) \left( \pi_j - B_{jl} \frac{dx^l}{d\sigma} \right) + \frac{1}{2} g_{ij} \frac{dx^i}{d\sigma} \frac{dx^j}{d\sigma}
+ U(x) + \frac{1}{2} g^{ij} A_i A_j - g^{ij} A_i \left( \pi_j - B_{jl} \frac{dx^l}{d\sigma} \right) - C_i \frac{dx^i}{d\sigma}.
$$

(2.3)

We are interested in studying duality between sigma models with lagrangian densities of the type (2.1). Here we consider a generalization of the canonical transformations considered in Paper I that still leads to a linear relationship between the respective $\pi$ and $dx/d\sigma$ in the two models. The generating function $F$ for the duality canonical transformation will be of the form

$$
F[x(\sigma), \tilde{x}(\sigma)] = \int \alpha + \int u(x, \tilde{x}) d\sigma,
$$

(2.4)

where $\alpha$ is a 1-form on $M \times \tilde{M}$, see Paper I, and $u$ a function on $M \times \tilde{M}$. The canonical transformation is given by

$$
(\pi - Bx^i)_i = m_{ij} \frac{d\tilde{x}^j}{d\sigma} + n_{ij} \frac{dx^j}{d\sigma} - \frac{\partial u}{\partial x^i},
$$

$$
(\tilde{\pi} - \tilde{B}\tilde{x}^i)_i = \tilde{m}_{ij} \frac{dx^j}{d\sigma} + \tilde{n}_{ij} \frac{d\tilde{x}^j}{d\sigma} + \frac{\partial u}{\partial \tilde{x}^i},
$$

where $m_{ij}$ and $n_{kl}$ will be discussed shortly.
It is best to now go to orthonormal coframes on $M$ and $\tilde{M}$. Let $(\theta^1, \ldots, \theta^n)$ be a local orthonormal coframe for $M$. The Cartan structural equations are

\[ d\theta^i = -\omega_{ij} \wedge \theta^j, \]
\[ d\omega_{ij} = -\omega_{ik} \wedge \omega_{kj} + \frac{1}{2} R_{ijkl} \theta^k \wedge \theta^l, \]

where $\omega_{ij} = -\omega_{ji}$ is the unique torsion free riemannian connection associated with the metric $g$. We remind the reader that the analog of $dx^i/d\sigma$ in an orthonormal coframe is $x^i_\sigma$ defined by $\theta^i = x^i_\sigma d\sigma$. There are similar definitions pertaining to $\tilde{M}$.

Following the discussion in Paper I, the canonical transformation may be expressed in terms of a 2-form $\gamma$ closely related to $d\alpha$ on $M \times \tilde{M}$ and given by

\[ \gamma = -\frac{1}{2} n_{ij}(x, \tilde{x}) \theta^i \wedge \theta^j + m_{ij}(x, \tilde{x}) \tilde{\theta}^i \wedge \tilde{\theta}^j + \frac{1}{2} \tilde{n}_{ij}(x, \tilde{x}) \tilde{\theta}^i \wedge \tilde{\theta}^j. \quad (2.5) \]

The 2-form $\gamma$ is not closed but satisfies

\[ d\gamma = H - \tilde{H} \quad (2.6) \]

where $H = dB$ and $\tilde{H} = d\tilde{B}$. The derivatives of the function $u$ are given in the orthonormal frame by

\[ du = u_i \theta^i - \tilde{u}_i \tilde{\theta}^i. \quad (2.7) \]

In terms of the orthonormal frame the canonical transformation may be written as

\[ (\pi - Bx_\sigma)_i = m_{ij} \tilde{x}^j_\sigma + n_{ij} x^j_\sigma - u_i, \quad (2.8) \]
\[ (\tilde{\pi} - \tilde{B} \tilde{x}_\sigma)_i = m_{ij} x^j_\sigma + \tilde{n}_{ij} \tilde{x}^j_\sigma - \tilde{u}_i, \quad (2.9) \]

where we now interpret the components of $\pi, \tilde{\pi}, B, \tilde{B}, m, n, \tilde{n}$ to be given with respect to the orthonormal frames.

We digress for a second and make a general observation. Assume we have equal dimensional vector spaces $V$ and $\tilde{V}$ where we use the notation $(\cdot, \cdot)$ for the inner product on either space. Let $L : V \rightarrow \tilde{V}$ be an invertible linear transformation and let $Q(v) = \frac{1}{2}(v, v) + (a, v) + b$ be a real value quadratic function on $V$. There is a corresponding quadratic function $\tilde{Q}$ on $\tilde{V}$. Assume we are told that the affine transformation $\tilde{v} = Lv + \tilde{w}$ maps $Q$ into $\tilde{Q}$. A short computation shows that

\[ \frac{1}{2}(Lv, Lv) + (\tilde{a} + \tilde{w}, Lv) + \tilde{b} + (\tilde{a}, \tilde{w}) + \frac{1}{2}(\tilde{w}, \tilde{w}) = \frac{1}{2}(v, v) + (a, v) + b. \]

Comparing both sides we would conclude that $L$ is an isometry, $\tilde{a} + \tilde{w} = La$ and $b = \tilde{b} + (\tilde{a}, \tilde{w}) + \frac{1}{2}(\tilde{w}, \tilde{w})$.

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4 Because we will be working in orthonormal frames we do not distinguish an upper index from a lower index in a tensor.

5 Note the unconventional negative sign in the definition above. This is introduced to make subsequent equations more symmetric.
In our case we require the canonical transformation to preserve the hamiltonian densities up to a total derivative

\[ \tilde{H} = H + \frac{dh}{d\sigma}, \]  

(2.10)

where \( h \) is a function on \( M \times \tilde{M} \). Our canonical transformation, given by (2.8) and (2.9), is an affine mapping of \( (x_\sigma, \pi) \) into \( (\tilde{x}_\sigma, \tilde{\pi}) \). We can use parts of the general argument about the transformation of quadratic functions given in the previous paragraph. We have to be careful because the derivative in (2.10) modifies some of the conclusions of the previous paragraph. The linear part of the transformation must be an isometry, a restriction studied in Paper I, where we found

\[ m'm = I - \tilde{n}^2, \]  

(2.11)

\[ mm^t = I - n^2, \]  

(2.12)

\[ -mn = \tilde{m}n. \]  

(2.13)

If we write

\[ dh = h_i \theta^i - \tilde{h}_i \tilde{\theta}^i \]  

then \( dh/d\sigma = h_i x^i_\sigma - \tilde{h}_i \tilde{x}^i_\sigma \). From this we learn that

\[ h_i = C_i - m_{ji}(\ddot{u}_j + \ddot{A}_j) - n_{ij}(u_j + A_j), \]  

\[ \tilde{h}_i = \tilde{C}_i - m_{ij}(u_j + A_j) - \tilde{n}_{ij}(\ddot{u}_j + \ddot{A}_j). \]

The problem we have to solve is to find functions \( u, h : M \times \tilde{M} \to \mathbb{R} \) such that

\[ du = u_i \theta^i - \ddot{u}_i \tilde{\theta}^i, \]  

(2.14)

\[ dh = \left( C_i - m_{ji}(\ddot{u}_j + \ddot{A}_j) - n_{ij}(u_j + A_j) \right) \theta^i - \left( \tilde{C}_i - m_{ij}(u_j + A_j) - \tilde{n}_{ij}(\ddot{u}_j + \ddot{A}_j) \right) \tilde{\theta}^i. \]  

(2.15)

The integrability equations for the system given above, \( d^2u = 0 \) and \( d^2h = 0 \), lead to hyperbolic PDEs for \( u \). Finally we observe that there is one more relation that must be satisfied for (2.10) to hold:

\[ \tilde{U} + \frac{1}{2}(\ddot{u}_j + \ddot{A}_j)(\ddot{u}_j + \ddot{A}_j) = U + \frac{1}{2}(u_j + A_j)(u_j + A_j). \]  

(2.16)

Remember that \( U \) is a function on \( M \) and \( \tilde{U} \) is a functions on \( \tilde{M} \) so \( dU = U_i \theta^i \) and \( d\tilde{U} = \tilde{U}_i \tilde{\theta}^i \).

The generating function for canonical transformations is only locally defined. We could ask whether it is possible to give a more global formulation. We think this is possible. Notice that of primary interest to us is not the function \( u \) but its derivatives. For this reason it is convenient to “define”

\[ v_i = u_i + A_i \quad \text{and} \quad \tilde{v}_j = \ddot{u}_j + \ddot{A}_j. \]  

(2.17)

Let \( F_A \) be the curvature \( A = A_i \theta^i \) and \( \tilde{F}_{\tilde{A}} \) be the curvature of \( \tilde{A} = \ddot{A}_i \tilde{\theta}^i \). Consider a 1-form on \( M \times \tilde{M} \) defined by

\[ \xi = v_i \theta^i - \tilde{v}_i \tilde{\theta}^i. \]  

(2.18)
The equation $d^2 u = 0$ is replaced by
\[ d\xi = F_A - \tilde{F}_A. \quad (2.19) \]

In a similar fashion the equation $d^2 h = 0$ is replaced by
\[ d\left[ -(m_{ji}\tilde{v}_j + n_{ij}v_j)\theta^i + (m_{ij}v_j + \tilde{n}_{ij}\tilde{v}_j)\tilde{\theta}^i \right] = -F_C + \tilde{F}_{\tilde{C}}, \quad (2.20) \]

where $F_C$ is the curvature of $C = C_i\theta^i$ and $\tilde{F}_{\tilde{C}}$ is the curvature of $\tilde{C} = \tilde{C}_i\tilde{\theta}^i$. Similarly the equation for the potentials becomes
\[ \tilde{U} \pm \frac{1}{2}\tilde{v}_i\tilde{v}^i = U \pm \frac{1}{2}v_i\nu^i. \quad (2.21) \]

## 3 Pseudoduality

Here we switch to the framework where we consider the map between the paths on one manifold and the paths on the other. We use directly (2.8), (2.9) and (2.11) to (2.13), having in mind the rest of the discussion as a guideline. In this way we work directly with equations of motion what makes the calculations more straightforward; we have a system of PDE’s for which we obtain the integrability conditions. Moreover, for the 2-dimensional space the Hodge duality transforms 1-forms into another 1-forms. We may use it to write the equation in geometrical, covariant fashion.

We restrict to the case $A = C = 0$. Introducing the lightcone coordinates $\sigma^\pm = \tau \pm \sigma$ the equations of motion for lagrangian (2.1) are
\[ x^k_+ = -\frac{1}{2}H_{kij}x^i_+x^j_- - \frac{1}{4}U_k, \quad (3.1) \]

where $dU = U_k\theta^k$.

We rewrite the duality transformations (2.8) and (2.9) in terms of the velocities as
\[ x^i_\tau = m_{ji}\tilde{x}^j_\sigma + n_{ij}x^j_\sigma - u_i, \quad (3.2) \]
\[ \tilde{x}^i_\tau = m_{ij}x^j_\sigma + \tilde{n}_{ij}\tilde{x}^j_\sigma - \tilde{u}_i. \quad (3.3) \]

Mimicking the computations of Section 3 of [1] we find that
\[ \begin{pmatrix} m^t & 0 \\ -\tilde{n} & I \end{pmatrix} \begin{pmatrix} \bar{x}_\sigma \\ \bar{x}_\tau + \bar{u} \end{pmatrix} = \begin{pmatrix} -n & I \\ m & 0 \end{pmatrix} \begin{pmatrix} x_\sigma \\ x_\tau + u \end{pmatrix}. \quad (3.4) \]

We can now mimic the discussion in Section 1 of [13] and restrict ourselves to the special case $T_+ = T_-$. If we lift to the frame bundle as discussed in [14] the pseudoduality equations become
\[ \bar{x}^i_\pm + \frac{1}{2}u^i = \pm \left( x^i_\pm + \frac{1}{2}u^i \right). \quad (3.5) \]

Using the notation from Section 7 of [14] we have the equations of motion may be written as
\[ d(*\xi^i) + \xi_{ij} \wedge (*\xi^j) = \frac{1}{2}h_{ijk}\xi^j \wedge \xi^k - U_i d\sigma^0 \wedge d\sigma^1, \quad (3.6) \]
and the duality equations as
\[ \tilde{\xi}^i + \tilde{u}^i \, d\sigma^0 = * (\xi^i + u^i \, d\sigma^0) \tag{3.7} \]

More explicitly we have that
\[ \tilde{\xi}^i = * \xi^i + u^i \, d\sigma^1 - \tilde{u}^i \, d\sigma^0, \tag{3.8} \]
\[ \xi^i = * \tilde{\xi}^i + \tilde{u}^i \, d\sigma^1 - u^i \, d\sigma^0. \tag{3.9} \]

Next we define the covariant derivatives of \( u_i \) and \( \tilde{u}_i \) by
\[ du_i + \omega_{ij}^{} u^j = u^i_{\prime} \omega^j + u^i_{\prime\prime} \tilde{\omega}^j, \tag{3.10} \]
\[ d\tilde{u}_i + \tilde{\omega}_{ij} \tilde{u}^j = \tilde{u}^i_{\prime} \omega^j + \tilde{u}^i_{\prime\prime} \tilde{\omega}^j. \]

From \( d^2 u = 0 \) we see that
\[ u^i_{\prime} = u^i_{\prime j}, \tag{3.11} \]
\[ \tilde{u}^i_{\prime} = \tilde{u}^i_{\prime j}, \tag{3.12} \]
\[ u^i_{\prime\prime} = -\tilde{u}^i_{\prime j}. \tag{3.13} \]

The reason for the unusual negative sign in (3.13) is the unconventional definition (2.7).

To determine conditions necessitated for the duality equations we study the integrability conditions on (3.8) by taking its exterior derivative
\[ 0 = -\frac{1}{2} h^i_{\ jk} \xi^j \land \xi^k - u^i_{\prime j} \xi^j \land d\sigma^1 + \tilde{u}^i_{\prime j} \xi^j \land d\sigma^0 \]
\[ - u^i_{\prime\prime j} \xi^j \land d\sigma^1 + \tilde{u}^i_{\prime\prime j} \tilde{\xi}^j \land d\sigma^0 \]
\[ - \xi^j \land \tilde{\xi}^j \land \xi^j \land \tilde{\xi}^j \]
\[ - V^i d\sigma^1 \land d\sigma^0 - \tilde{u}^i d\sigma^0 \land \xi^j \land \tilde{\xi}^j + \tilde{u}^i d\sigma^0 \land \tilde{\xi}^j \land \tilde{\xi}^j \]

As in [14] we identify the orthogonal groups in the two frame bundles by requiring that
\[ \bar{\omega}_{ij} + \frac{1}{2} H_{ij k} \bar{\omega}^k = \omega_{ij} + \frac{1}{2} \bar{H}_{ij k} \omega^k. \tag{3.14} \]

Substituting this into the equation above leads to
\[ 0 = -\frac{1}{2} h^i_{\ jk} \xi^j \land \xi^k + \frac{1}{2} \tilde{h}^i_{\ jk} \xi^j \land \tilde{\xi}^k \]
\[ - u^i_{\prime j} \xi^j \land d\sigma^1 + \frac{1}{2} \tilde{u}^i_{\prime j} \xi^j \land d\sigma^0 \]
\[ - \frac{1}{2} \tilde{h}^i_{\ jk} \tilde{u}^j \xi^k \land d\sigma^0 - \frac{1}{2} h^i_{\ jk} \xi^j \land \tilde{\xi}^k - \frac{1}{2} u^i_{\prime j} \tilde{\xi}^j \land d\sigma^1 \]
\[ + \frac{1}{2} \tilde{u}^i_{\prime\prime j} \tilde{\xi}^j \land d\sigma^0 + \frac{1}{2} h^i_{\ jk} \tilde{u}^j \tilde{\xi}^k \land d\sigma^0 - V^i d\sigma^1 \land d\sigma^0 \]
Next we use the following Hodge duality relations

\[ \xi^i \wedge \xi^j = -(\ast \xi^i \wedge \ast \xi^j) , \]
\[ (\ast \xi^i) \wedge \partial \sigma^1 = -\xi^i \wedge \partial \sigma^0 , \]
\[ (\ast \xi^i) \wedge \partial \sigma^0 = -\xi^i \wedge \partial \sigma^1 , \]
\[ (\ast \xi^i) \wedge \xi^k = (\ast \xi^k) \wedge \xi^i , \]

that we substitute into the integrability conditions to obtain

\[ 0 = -u^m_{ji} \xi^j \wedge \partial \sigma^1 - \tilde{u}^m_{ji} \xi^j \wedge \partial \sigma^1 + h^i_{jk} u^j \xi^k \wedge \partial \sigma^1 \]
\[ + u^i_{ji} \xi^j \wedge \partial \sigma^0 + \tilde{u}^i_{ji} \xi^j \wedge \partial \sigma^0 + \frac{1}{2} h^i_{jk} u^j \tilde{\xi}^k \wedge \partial \sigma^0 - \frac{1}{2} h^i_{jk} \tilde{u}^j \xi^k \wedge \partial \sigma^0 \]
\[ - u_j u^{i \bar{i} j} \partial \sigma^1 \wedge \partial \sigma^0 - h^i_{jk} w^j \tilde{u}^k \partial \sigma^1 \wedge \partial \sigma^0 + \tilde{u}_j \tilde{u}^{i \bar{i} j} \partial \sigma^1 \wedge \partial \sigma^0 - V^i \partial \sigma^1 \wedge \partial \sigma^0 . \]

Extracting the information contained in the equation above and the ones obtained by taking the exterior derivative of (3.9) we see that

\[ \begin{align*}
    u''_{ij} + \tilde{u}''_{ij} + h_{ijk} u^k = 0, \\
    u'_{ij} + \tilde{u}''_{ij} - \frac{1}{2} \tilde{h}_{ijk} u^k + \frac{1}{2} h_{ijk} \tilde{u}^k = 0, \\
    U_i + u'_{ij} w^j - \tilde{u}_j \tilde{u}^j + h_{ijk} w^j \tilde{u}^k = 0, \\
    \tilde{u}''_{ij} + u''_{ij} - \frac{1}{2} h_{ijk} \tilde{u}^k + \frac{1}{2} \tilde{h}_{ijk} u^k = 0, \\
    \bar{U}_i + \tilde{u}''_{ij} \tilde{w}^j - u''_{ij} \tilde{w}^j + h_{ijk} \tilde{w}^j u^k = 0.
\end{align*} \]

From the above we immediately learn that

\[ \begin{align*}
    h_{ijk} u^k &= \tilde{h}_{ijk} \tilde{u}^k, \\
    h_{ijk} \tilde{u}^k &= \tilde{h}_{ijk} u^k, \\
    h_{ijk} w^j \tilde{u}^k &= 0.
\end{align*} \]

An important observation that follows from the above is that if we go back to \( M \times \tilde{M} \) then we expect that \( u \) should be a function of both sets of variables, \( \text{i.e.} \) a nontrivial function on \( M \times \tilde{M} \).
The above is consistent with the condition that on $M \times \tilde{M}$ we require that
\[
\left( U + \frac{1}{2} u_i u^i \right) = \left( \tilde{U} + \frac{1}{2} \tilde{u}_i \tilde{u}^i \right) + \text{constant},
\]
in agreement with (2.16)

4 Conclusions

We obtained a set of nonlinear algebraic equations in a sense they do not contain the derivatives of $x^i$ or $\tilde{x}^i$. The geometric condition (3.14) on connections on $M$ and $\tilde{M}$ is unchanged by the presence of the potentials and the conclusions from [14] hold in the case discussed here. In addition we have equations (3.24) to (3.30) involving second derivatives of the generating function $u$ and the derivatives of the potential. Using (3.24) to (3.28) we can integrate (3.30) and (3.31) obtaining (3.31) which has already appeared as a condition (2.21) for hamiltonian density to be preserved. This condition has not been used in the following derivation. Having the solution of equations of motion on $M$ the condition (3.31) is a constraint for the solution on $\tilde{M}$. There could be however a choice of generating function $u$ by which the constraint is satisfied automatically. If with an appropriate choice of coordinate system we have $r$ coordinates for which $u_i = \partial u / \partial x^i$ for $i = 1...r$ (see (2.7), the definition of $u_i$) using the following substitution for the generating function
\[
u = f_1(x^1 + \tilde{x}^1) + g_1(x^1 - \tilde{x}^1) + ... + f_r(x^r + \tilde{x}^r) + g_r(x^r - \tilde{x}^r)
\]
the condition (3.31) has the form
\[-(U(x) - \tilde{U}(\tilde{x})) = f'_1(x^1 + \tilde{x}^1)g'_1(x^1 - \tilde{x}^1) + ... + f'_r(x^r + \tilde{x}^r)g'_r(x^r - \tilde{x}^r)
\]
We are interested in functions for which the combinations such as $f'_i(x^i + \tilde{x}^i)g'_i(x^i - \tilde{x}^i)$ separate into sum of two terms, one being only a function of $x^i$ and the other only of $\tilde{x}^i$. In general case we still have (3.24) to (3.28) which give a nontrivial constraint.

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