Stability of the LCD Model

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Abstract: In this paper, first-passage probability of Markov chains is used to get a strict proof of the existence of degree distribution of the LCD model presented by Bollobás (Random Structures and Algorithms 18(2001)). Also, a precise expression of degree distribution is presented.

Keywords: LCD model; Stability; First-passage probability

Introduction. Networks exist in every expects of nature and society. Most of the systems, e.g. WWW, movie actor collaboration network, transportation network and so on, can be described as complex networks. Thus, complex networks becomes one of the important research areas in recent years.

Two important discoveries about complex networks is the small-world property[11] and the scale-free property, proposed by Barabási and Albert[1]. They had a research on World-Wide Web and found that the degree distribution of the WWW is not Poisson, as random networks and small-world networks shows, but a power-law distribution. In[1], they presented the approximate expression of network degree and used computer experiments to get $\gamma = 2.9 \pm 0.1$. It is stated that after many steps the proportion of vertices with degree $d$ should obey a power law $P(d) \propto d^{-\gamma}$. In[4] they stated an evolving network(BA model) with growth and preferential attachment. They used the mean-field method to compute for the degree distribution, and deduced $\gamma = 3$ heuristically. They thought that the probability of one vertex to get edges equals $m\Pi(k_i)$. That is to say, it is in direct proportion to degree of the vertex and the number of edges added at a unit time simultaneously. They also pointed out that most real networks are scale-free[2].

In order to make a better understanding of the phenomenon in real networks, many scholars did researches on BA model, they also had extensions to BA model and got a lot of useful results.

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Krapivsky\cite{12} let $N_k(t)$ be the number of vertices in network with degree $k$ at time $t$. They deduced the rate equations: \[
\frac{dN_k(t)}{dt} = m \frac{(k-1)N_{k-1}(t)-kN_k(t)}{\sum_k kN_k(t)} + \delta_{km}\] and used \(\lim_{t \to \infty} E[N_k(t)/t] = P(k)\) to show that $P(k) = \frac{4}{k(k+1)(k+2)}$ for the BA model with $m = 1$.

Dorogovtsev\cite{8} considered $k_i(t)$, the degree of the vertex added at time $i$ evolved at time $t$. They used $P(k, i, t)$ to express the probability of vertex $i$ having $k$ edges at time $t$. And let the average degree $P(k, t) = \frac{1}{t} \sum_{i=1}^{t} P(k, i, t)$ to be the network degree at time $t$. They presented the attraction model, where each new vertex has an initial attraction degree $a$. Thus, $k = a + q$ is the total degree with $q$ being the in-degree of the vertex. They got the master equation: \(P(k, i, t+1) = k \frac{1}{2t} P(k-1, i, t) + (1 - \frac{k}{2t}) P(k, i, t)\) and the stationary distribution: \(P(k) = \frac{2m(m+1)}{k(k+1)(k+2)}\) with $a = m$.

These two definitions of network degree is equivalent, for we have $E[N_k(t)] = \sum_{i=1}^{t} P(k, i, t)$.

As to BA model, Bollobás\cite{6} once raised the following questions: *The first is getting started: how do we take probabilities proportional to the degrees when these are all zero? The second problem is with the preferential attachment rule itself. For the BA model, or for any other BA-like model, how to find a rigorous preferential attachment scheme for $m \geq 2$?*

Aimed at this questions, Bollobás\cite{5} presented a modified BA model, called LCD model.

As Bollobás\cite{5} stated: *We start with the case $m = 1$. Consider a fixed sequence of vertices $v_1, v_2, \ldots$. We shall inductively define a random graph process $(G'_t)_{t \geq 0}$ so that $G'_1$ is a directed graph on $\{v_i : 1 \leq i \leq t\}$, as follows. Start with $G'_0$ the graph with no vertices, or with $G'_1$ the graph with one vertex and one loop. Given $G'_t$, form $G'_{t+1}$ by adding the vertex $v_t$ together with a single edge directed from $v_t$ to $v_i$, where $i$ is chosen randomly with \[
P(i = s) = \begin{cases} \frac{dG'_{t-1}(v_s)}{2t-1} & 1 \leq s \leq t-1, \\ \frac{1}{2t-1} & s = t. \end{cases}
\]

For $m > 1$ add $m$ edges from $v_t$ one at a time. We define the process $(G'_t)_{t \geq 0}$ by running the process $(G'_1)$ on a sequence $v'_1, v'_2, \ldots$; the graph $G'_m$ is formed from $G'_m$ by identifying the vertices $v'_1, v'_2, \ldots, v'_m$ to form $v_1$, identifying $v'_{m+1}, v'_{m+2}, \ldots, v'_{2m}$ to form $v_2$, and so on.*

Bollobás\cite{5} found that, although the graph process \(\{G'_t\}\) is dynamic, there is a static description (Linearized Chord Diagram) of the distribution of the graph at a particular time. In\cite{7}, they studied $\pi^m_n(d)$, the number of vertices with indegree $d$ and strictly proved the following equation

\[
E(\pi^m_n(d)) \sim \frac{2m(m+1)n}{(d+m)(d+m+1)(d+m+2)}.
\]

Then, with Azuma-Hoeffding inequality\cite{10}, they got the following theorem.

*Theorem Let $m \geq 1$ be fixed, and consider the random graph process $(G'_m)_{m \geq 0}$, writing $\pi^m_n(d)$ for the number of vertices of $G'_m$ with indegree equal to $d$, i.e., with total degree $m + d$. Then After a finite number of steps...*
Let
\[ \alpha_{m,d} = \frac{2m(m+1)}{(d+m)(d+m+1)(d+m+2)} \]
and let \( \varepsilon \) be fixed. Then with probability tending to 1 as \( n \to \infty \) we have
\[ (1 - \varepsilon)\alpha_{m,d} \leq \frac{\#_m(n)}{n} \leq (1 + \varepsilon)\alpha_{m,d} \]
for every \( d \) in the range \( 0 \leq d \leq \frac{n}{115} \).

In fact, (2) already proved that \( \frac{\#_m(n)}{n} \) convergent to \( \alpha_{m,d} \) with first order. The theorem is used to prove that \( \frac{\#_m(n)}{n} \) tends to \( \alpha_{m,d} \) with probability 1. If one is only interested in the degree distribution of the LCD model, (2) is enough. Also, Bollobás[7] changed the BA model and the method used in his paper is applicable only to particular models and it is difficult to have extensions.

In this paper, we apply first-passage probability[9] to LCD model. We used the Markov theory and got a rigorous proof the existence of the steady-state degree distribution of the LCD model. And finally presented the precise expression of the degree distribution.

**Stability.** Instead of studying martingale \( X_I = E[I_m^d(G_m^m)|G_0] \) (where \( 0 \leq I \leq T \)), we consider \( k_I(T) \), the degree of vertex \( I \) at time \( T \). Following Dorogovtsev[8], \( k_I(T) \) is a random variable, and let \( P(k,I,T) = P\{k_I(T) = k\} \) the probability of vertex \( I \) having \( k \) edges at time \( T \), also take the average degree \( P(k,T) = \frac{1}{T} \sum_{I=1}^{T} P(k,I,T) \) to be the definition of the network degree at time \( T \). For fixed \( T \), \( k_I(T) \) is a random variable and for variable \( T \), \( k_I(T) \) is a nonhomogeneous Markov chain[13]. The adding of vertex \( I \) can be divided into \( m \) steps, from \((I-1)m+1\) to \((I-1)m+m\). The state-transition probability of this Markov chain is given by:

\[
P_T^{T+1} k,j = \begin{cases} 
P_{T+1} k,0 & j = 0 \text{ otherwise.}
\end{cases}
\]

where \( k = m, m+1, \cdots, (T-I+2)m \) and \( P_T^{T+1} k,j \) is the probability of vertex \( I \) with degree \( k \) at \( T \) increase by \( j \) at \( T+1 \). With (1), it follows that

\[
P_T^{T+1} k,0 = \prod_{r=1}^{m} (1 - \frac{k}{2Tm + 2(r-1) + 1})
\]

\[
P_T^{T+1} k,1 = \sum_{b=1}^{m} \prod_{q=1}^{b-1} (1 - \frac{k}{2Tm + 2(q-1) + 1}) \prod_{r=b+1}^{m} (1 - \frac{k+1}{2Tm + 2(r-1) + 1})
\]

\[
\vdots
\]

\[
P_T^{T+1} k,m = \prod_{r=1}^{m} \frac{k+r-1}{2Tm + 2(r-1) + 1}
\]
Consider the first-passage probability \( f(k, I, S) \) of the Markov chain \( k_I(T) \). \( f(k, I, S) = P\{k_I(S) = k, k_I(L) \neq k, L = 1, 2, \cdots, S - 1\} \). Based on the concept and techniques of Markov chain, the relationship between the first-passage probability and the probability of vertex degrees are as follows:

**Lemma 1** When \( 1 \leq I < S, k > m \), we have

\[
f(k, I, S) = \sum_{j=1}^{m} P(k-j, I, S-1)P_{k-j,j}^S.
\] (4)

When \( 1 \leq I < T, k > m \), we have

\[
P(k, I, T) = \sum_{S=I}^{T} f(k, I, S) \prod_{A=S+1}^{T} \prod_{r=1}^{m} \left(1 - \frac{1}{2(I-1)m + 2(r-1)+1}\right).
\] (5)

**Proof:** From the construction of LCD model, the degree of vertex \( I \) is always nondecreasing. At \( S \), the degree of vertex \( I \) can increase by \( 1, 2, \cdots, m \). With the Markov property, we have

\[
f(k, I, S) = P\{k_I(S) = k, k_I(L) \neq k, L = 1, 2, \cdots, S - 1\}
\]
\[
= \sum_{j=1}^{m} P\{k_I(S) = k, k_I(S-1) = k-j\}P_{k-j,j}^S
\]
\[
= \sum_{j=1}^{m} P(k-j, I, S-1)P_{k-j,j}^S.
\]

Thus (4) is established.

Second, observe that the earliest time for the degree of vertex \( I \) to reach \( k \) is at step \( I \), and the latest time to do so is at step \( T \). After this vertex degree becomes \( k \), it will not increase any more.

This completes the proof. \( \square \)

**Lemma 2** When \( 1 < I = S, k = m \), we have

\[
f(m, I, I) = \prod_{r=1}^{m} \left(1 - \frac{1}{2(I-1)m + 2(r-1)+1}\right).
\] (6)

When \( 1 < I = S, m < k < 2m \), we have

\[
f(k, I, I) = \sum_{r=k-m}^{m} \hat{P}(k - (m-r) - 2, I, (I-1)m + r - 1) \prod_{q=r+1}^{m} \left(1 - \frac{1}{2(I-1)m + 2(q-1)+1}\right),
\] (7)
Where $\hat{P}(k, I, (I - 1)m + r)$ means the probability of the degree of vertex $I$ to be $k$ at the $r$th step of adding in.

When $1 < I = S, k = 2m$, we have

$$f(2m, I, I) = \prod_{r=1}^{m} \left( \frac{2(r - 1) + 1}{2(I - 1)m + 2(r - 1) + 1} \right).$$

(8)

**Proof:** For $1 < I = S, k = m$, it means vertex $I$ have no loops after the first $m$ steps. For $1 < I = S, k = 2m$, it means vertex $I$ have $m$ loops after the first $m$ steps. Thus (6) and (8) are obtained.

In order to prove (7), note that from step $(I - 1)m + 1$ to $(I - 1)m + m$ are all the adding of vertex $I$. At these $m$ steps, the degree of vertex $I$ will at least increase by 1. In order for the degree of vertex $I$ first reach $k$ at $(I - 1)m + m$, then at $(I - 1)m + r$ the degree of vertex $I$ is at most $k - (m - r)$. Thus (7) is obtained.

This completes the proof. □

**Stolz theorem:** In sequence $\{\frac{x_n}{y_n}\}$, assume that $\{y_n\}$ is monotone increasing sequence with $y_n \to \infty$, if $\lim_{n \to \infty} \frac{x_n+1-x_n}{y_n+1-y_n} = l$ exists, where $-\infty \leq l \leq \infty$, then $\lim_{n \to \infty} \frac{x_n}{y_n} = l$.

**Proof:** See [14].

**Lemma 3** $\lim_{T \to \infty} P(m, T)$ exists, and

$$P(m) \triangleq \lim_{T \to \infty} P(m, T) = \frac{2}{m+2} > 0.$$  

(9)

**Proof:** From the construction of LCD model, it follows that

$$P(m, I, T) = f(m, I, I) \prod_{A=I+1}^{m} \prod_{q=1}^{m} \left( 1 - \frac{m}{2(A-1)m + 2(q-1) + 1} \right) \prod_{A=I+1}^{m} \prod_{q=1}^{m} \left( 1 - \frac{m}{2(A-1)m + 2(q-1) + 1} \right).$$

In particular, $P(m, 1, T) = 0$. So,

$$P(m, T) = \frac{1}{T} \sum_{l=1}^{T} P(m, l, T) = \frac{1}{T} \sum_{l=2}^{T} P(m, l, T)$$

$$= \frac{1}{T} \sum_{l=2}^{T} \prod_{r=1}^{m} \left( 1 - \frac{r}{2(I-1)m + 2(r-1) + 1} \right) \prod_{A=I+1}^{m} \prod_{q=1}^{m} \left( 1 - \frac{m}{2(A-1)m + 2(q-1) + 1} \right)$$

$$= \frac{1}{T} \prod_{A=3}^{m} \prod_{q=1}^{m} \left( 1 - \frac{m}{2(A-1)m + 2(q-1) + 1} \right) \left\{ \prod_{r=1}^{m} \left( 1 - \frac{r}{2m+2(r-1)+1} \right) \right\}$$

$$+ \sum_{I=3}^{m} \prod_{r=1}^{m} \left( 1 - \frac{r}{2(I-1)m + 2(r-1) + 1} \right) \prod_{B=3}^{m} \prod_{h=1}^{m} \left( 1 - \frac{m}{2(B-1)m + 2(h-1) + 1} \right)^{-1}. $$
Next, let
\[ X_N = \prod_{r=1}^{m} \left(1 - \frac{r}{2m+2(r-1)+1}\right) + \sum_{I=3}^{N} \prod_{r=1}^{m} \left(1 - \frac{r}{2(I-1)m+2(r-1)+1}\right) \prod_{B=3}^{I} \prod_{h=1}^{m} \left(1 - \frac{m}{2(B-1)m+2(h-1)+1}\right)^{-1}, \]
and
\[ Y_N = N \prod_{A=3}^{N} \prod_{q=1}^{m} \left(1 - \frac{m}{2(A-1)m+2(q-1)+1}\right)^{-1} > 0. \]

Thus, it follows that
\[ X_{N+1} - X_N = \prod_{r=1}^{m} \left(1 - \frac{r}{2Nm+2(r-1)+1}\right) \prod_{B=3}^{N+1} \prod_{h=1}^{m} \left(1 - \frac{m}{2(B-1)m+2(h-1)+1}\right)^{-1}, \]
\[ Y_{N+1} - Y_N = (N + 1 - N \prod_{r=1}^{m} \left(1 - \frac{m}{2Nm+2(r-1)+1}\right)) \prod_{A=3}^{N+1} \prod_{q=1}^{m} \left(1 - \frac{m}{2(A-1)m+2(q-1)+1}\right)^{-1} > 0. \]

Since \( Y_N > 0 \) and \( Y_{N+1} - Y_N > 0 \), \( \{Y_N\} \) is a strictly monotone increasing nonnegative sequence, hence \( Y_N \to \infty \). Also, by assumption,
\[ \frac{X_{N+1} - X_N}{Y_{N+1} - Y_N} = \frac{\prod_{r=1}^{m} (2Nm+2(r-1)+1) + o(N^m)}{\prod_{r=1}^{m} (2Nm+2(r-1)+1) + mN \sum_{i=1}^{m} \prod_{r\neq i}^{m} (2Nm+2(r-1)+1) + o(N^m)} \to \frac{2}{m+2} \mbox{,} (N \to \infty). \]

With Stolz theorem, we have
\[ P(m) \triangleq \lim_{T \to \infty} P(m, T) = \lim_{N \to \infty} \frac{X_N}{Y_N} = \lim_{N \to \infty} \frac{X_{N+1} - X_N}{Y_{N+1} - Y_N} = \frac{2}{m+2} > 0. \]

This completes the proof. \( \Box \)

**Lemma 4** When \( k > m \), if \( \lim_{T \to \infty} P(k-1, T) \) exists, then \( \lim_{T \to \infty} P(k, T) \) also exists and
\[ P(k) \triangleq \lim_{T \to \infty} P(k, T) = \frac{k-1}{k+2} P(k-1) > 0 \quad (10) \]
Proof: With (4) and (5) of Lemma 1, we have

\[ P(k, T) = \frac{1}{T} \sum_{I=1}^{T-1} P(k, I, T) + \frac{1}{T} P(k, T, T) \]

\[ = \frac{1}{T} \sum_{I=1}^{T-1} \sum_{S=I}^{T} f(k, I, S) \prod_{A=S+1}^{T} \prod_{q=1}^{m} \left( 1 - \frac{k}{2(A-1)m + 2(q-1) + 1} \right) + \frac{1}{T} P(k, T, T) \]

\[ = \frac{1}{T} \sum_{I=1}^{T-1} f(k, I, I) \prod_{A=I+1}^{T} \prod_{q=1}^{m} \left( 1 - \frac{k}{2(A-1)m + 2(q-1) + 1} \right) + \frac{1}{T} P(k, T, T) \]

\[ + \frac{1}{T} \sum_{I=1}^{T-1} \sum_{S=I+1}^{T} \sum_{j=1}^{m} P(k-j, I, S-1) P_{k-j,j}^{S} \prod_{A=S+1}^{T} \prod_{q=1}^{m} \left( 1 - \frac{k}{2(A-1)m + 2(q-1) + 1} \right) \]

\[ = f(T) + \frac{1}{T} \sum_{I=1}^{T-1} \sum_{S=I+1}^{T} \sum_{b=1}^{m} \frac{k-1}{2(S-1)m + 2(b-1) + 1} P(k-1, I, S-1) \]

\[ \prod_{A=S+1}^{T} \prod_{q=1}^{m} \left( 1 - \frac{k}{2(A-1)m + 2(q-1) + 1} \right) + \frac{1}{T} P(k, T, T) + O\left( \frac{1}{T} \right) \]

Consider \( f(T) \) firstly, obviously, \( f(k, 1, 1) = \delta_{k,2m} \). Also, when \( 1 \leq I = S, k > 2m \), then \( f(k, I, I) = 0 \). With (7) of Lemma 2, when \( 1 < I = S, m < k < 2m \), we have

\[ f(T) = \frac{1}{T} \sum_{I=1}^{T-1} f(k, I, I) \prod_{A=I+1}^{T} \prod_{q=1}^{m} \left( 1 - \frac{k}{2(A-1)m + 2(q-1) + 1} \right) \]

\[ < \frac{1}{T} \sum_{I=2}^{T-1} f(k, I, I) < \frac{1}{T} \sum_{I=2}^{T-1} \sum_{r=k-m}^{m} \frac{k-(m-r)-1}{2(I-1)m + 2(r-1) + 1} \]

\[ < \frac{1}{T} \sum_{I=2}^{T-1} \frac{k-1}{2(I-1)} < (k-1) \frac{\ln T + \gamma}{T}. \]

Take limits on both sides, then \( \lim_{T \to \infty} f(T) = 0 \) with \( m < k < 2m \).

Then, with (8) of Lemma 2, when \( 1 < I = S, k = 2m \), we have

\[ f(T) < \frac{1}{T} (1 + \sum_{I=2}^{T-1} f(2m, I, I)) < \frac{1}{T} (1 + \sum_{I=2}^{T-1} \frac{1}{(I-1)m}) \]

Take limits on both sides, then \( \lim_{T \to \infty} f(T) = 0 \) with \( k = 2m \).

Thus, when \( k > m \), we have \( \lim_{T \to \infty} f(T) = 0 \).
Now, consider the second item.

\[
\begin{align*}
\frac{1}{T} & \sum_{S=2}^{T} \sum_{b=1}^{m} \frac{k-1}{2(S-1)m + 2(b-1) + 1} P(k-1, I, S-1) \\
\prod_{A=S+1}^{T} \prod_{q=1}^{m} \left(1 - \frac{k}{2(A-1)m + 2(q-1) + 1}\right)
\end{align*}
\]

\[
= \frac{1}{T} \sum_{S=2}^{T} \sum_{b=1}^{m} \frac{k-1}{2(S-1)m + 2(b-1) + 1} P(k-1, I, S-1) \\
\prod_{A=S+1}^{T} \prod_{q=1}^{m} \left(1 - \frac{k}{2(A-1)m + 2(q-1) + 1}\right)
\]

\[
= \frac{1}{T} \sum_{S=2}^{T} \sum_{b=1}^{m} \frac{(k-1)(S-1)}{2(S-1)m + 2(b-1) + 1} P(k-1, S-1) \\
\prod_{A=S+1}^{T} \prod_{q=1}^{m} \left(1 - \frac{k}{2(A-1)m + 2(q-1) + 1}\right) \left(\sum_{b=1}^{m} \frac{k-1}{2m + 2(b-1) + 1} P(k-1, 1) \right)
\]

\[
+ \sum_{S=3}^{T} \sum_{b=1}^{m} \frac{(k-1)(S-1)}{2(S-1)m + 2(b-1) + 1} P(k-1, S-1) \prod_{S=3}^{S} \prod_{h=1}^{m} \left(1 - \frac{k}{2(B-1)m + 2(h-1) + 1}\right)^{-1}
\]

Next, let

\[
X_N = \sum_{b=1}^{m} \frac{k-1}{2m + 2(b-1) + 1} P(k-1, 1) + \sum_{S=3}^{T} \sum_{b=1}^{m} \frac{(k-1)(S-1)}{2(S-1)m + 2(b-1) + 1} P(k-1, S-1)
\]

\[
\prod_{B=3}^{S} \prod_{h=1}^{m} \left(1 - \frac{k}{2(B-1)m + 2(h-1) + 1}\right)^{-1},
\]

and

\[
Y_N = N \prod_{A=3}^{N} \prod_{q=1}^{m} \left(1 - \frac{k}{2(A-1)m + 2(q-1) + 1}\right)^{-1} > 0.
\]

Thus, it follows that

\[
X_{N+1} - X_N = \sum_{b=1}^{m} \frac{(k-1)N}{2Nm + 2(b-1) + 1} P(k-1, N) \prod_{B=3}^{N+1} \prod_{h=1}^{m} \left(1 - \frac{k}{2(B-1)m + 2(h-1) + 1}\right)^{-1},
\]

\[
Y_{N+1} - Y_N = \{N + 1 - N \prod_{r=1}^{m} \left(1 - \frac{k}{2Nm + 2(r-1) + 1}\right)\}
\]

\[
\prod_{A=3}^{N+1} \prod_{q=1}^{m} \left(1 - \frac{k}{2(A-1)m + 2(q-1) + 1}\right)^{-1} > 0.
\]
Since $Y_N > 0$ and $Y_{N+1} - Y_N > 0$, \{\{Y_N\}\} is a strictly monotone increasing nonnegative sequence, hence $Y_N \to \infty$.

\[
\frac{X_{N+1} - X_N}{Y_{N+1} - Y_N} = \frac{\sum_{b=1}^{m} \frac{(k-1)N}{2Nm+b(k-1)+1} P(k-1, N)}{N + 1 - N \prod_{r=1}^{m} \left(1 - \frac{k}{2Nm+2(r-1)+1}\right)} \to \frac{k-1}{k+2} P(k-1), (N \to \infty).
\]

With Stolz theorem, we have

\[
P(k) = \lim_{T \to \infty} \left( f(T) + \frac{1}{T} P(k, T, T) + O\left(\frac{1}{T}\right)\right) = \lim_{N \to \infty} \frac{X_N}{Y_N}
\]

\[
= \lim_{N \to \infty} \frac{X_{N+1} - X_N}{Y_{N+1} - Y_N} = \frac{k-1}{k+2} P(k-1) > 0.
\]

This completes the proof. □

**Theorem 1** The steady-state degree distribution of the LCD model exists, and is given by

\[
P(k) = \frac{2m(m+1)}{k(k+1)(k+2)} \sim 2m^2k^{-3} > 0. \quad (11)
\]

**Proof:** By mathematical induction, if follows from Lemmas 3 and 4 that the steady-state degree distribution of the LCD model exists. Then, solving (10) iteratively, we obtain

\[
P(k) = \frac{k-1}{k+2} P(k-1) = \frac{k-1}{k+2} \frac{k-2}{k+1} \frac{k-3}{k} P(k-3).
\]

By continuing the process till $k = 3 + m$, we get

\[
P(k) = \frac{2m(m+1)}{k(k+1)(k+2)} \sim 2m^2k^{-3} > 0.
\]

This completes the proof. □

**Conclusion** The method of first-passage probability developed by Hou et al.[9] has great generality. We apply this method to the LCD model, and got the precise expression of degree distribution. Of course, allowing loops and multiple edges made the LCD model different from the BA model. Thus, the transition probability of Markov chains changed a lot. And the method is quite different from [9]. Moreover, the method presented in this paper can be extended to the attraction model[8] and other models with multiple edges.

**References**

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