Abstract
In this note we consider a certain class of convolution operators acting on the $L_1$ space of the one dimensional torus. We prove that the identity minus such an operator is nicely invertible on the subspace of functions with mean zero. We relate our result to the Poincaré inequality for Markov chains.

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1 Introduction
Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the one dimensional torus viewed as a compact group with the addition modulo 1, $x \oplus y = (x + y) \mod 1$, $x, y \in \mathbb{R}$ equipped with the Haar measure — the unique invariant probability measure (the Lebesgue measure). To begin with, consider the averaging operator $U_t$ acting on $L_1(\mathbb{T})$ with the norm $\|f\| = \int_{\mathbb{T}} |f|$, 

$$(U_t f)(x) = \frac{1}{2t} \int_{-t}^{+t} f(x \oplus s) \, ds, \quad t \in (0, 1).$$

(1)

If $t$ is small, is the operator $I - U_t$ invertible, or, in other words, how much does $U_t f$ differ from $f$? Of course, averaging a constant function does not change it, but excluding such a trivial case, we get a quantitative answer.
Theorem 1. Consider $t \in (0, 1)$ and $f \in L_1(\mathbb{T})$ with $\int_\mathbb{T} f = 0$. Then we have

$$\|f - U_t f\| \geq c t^2 \|f\|, \quad (2)$$

where $c > 0$ is a universal constant.

If we further estimate the left-hand side of (2) using the Sobolev inequality, see [GT], we obtain the following corollary.

Corollary 1. Let us consider $t \in (0, 1)$ and assume that $f$ belongs to the Sobolev space $W^{1,1}(\mathbb{T})$ with $\int_\mathbb{T} f = 0$. Then we have

$$\int_\mathbb{T} \left| f'(x) - \frac{f(x + t) - f(x - t)}{2t} \right| \, dx \geq c t^2 \int_\mathbb{T} |f(x)| \, dx, \quad (3)$$

where $c > 0$ is a universal constant.

Remark. Setting $t = 1/2$, inequality (3) becomes the usual Sobolev inequality, so (3) can be viewed as a certain generalization of the Sobolev inequality.

Remark. Set $f(x) = \cos(2\pi x)$. Then $\|f - U_t f\| = \frac{2}{\pi} \left(1 - \frac{1}{2\pi t} \sin(2\pi t)\right) \approx t^2$, for small $t$. Therefore, the inequality in Theorem 1 is sharp in a sense. This example also shows that Corollary 1 is sharp.

In this note we give a proof of a generalization of Theorem 1. We say that a $\mathbb{T}$-valued random variable $Z$ is $c$-good with some positive constant $c$ if $P(Z \in A) \geq c|A|$ for all measurable $A \subset \mathbb{T}$. Equivalently, by Lebesgue’s decomposition theorem it means that the absolutely continuous part of $Z$ (with respect to the Lebesgue measure) has a density bounded below by a positive constant. We say that a real random variable $Y$ is $\ell$-decent if $Y_1 + \ldots + Y_\ell$ has a nontrivial absolutely continuous part, where $Y_1, Y_2, \ldots$ are i.i.d. copies of $Y$. Our main result reads

Theorem 2. Given $t \in (0, 1)$ and an $\ell$-decent real random variable $Y$, consider the operator $A_t : L_1(\mathbb{T}) \to L_1(\mathbb{T})$ given by

$$(A_t f)(x) = \mathbb{E}f(x \oplus tY).$$

Then for all $f : \mathbb{T} \to \mathbb{T}$ with $\int_\mathbb{T} f = 0$ we have

$$\|f - A_t f\| \geq c t^2 \|f\|,$$

with a positive constant $c$ which depends only on the distribution of the random variable $Y$.  

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Remark. One cannot hope to prove a statement similar to Theorem 2 for purely atomic measures. Indeed, let $Y$ be distributed according to the law $\mu_Y = \sum_{i=1}^{\infty} p_i \delta_{x_i}$. Then for every $\varepsilon > 0$ and every $t \in (0, 1)$ there exists $f \in L_1(\mathbb{T})$ such that $\|f - A_t(f)\| < \varepsilon$ and $\|f\| = 1$. To see this take $N$ such that $\sum_{i=N+1}^{\infty} p_i < \varepsilon/4$ and let $f_n(x) = \frac{\pi}{2} \sin(2\pi nx)$. Then $\|f_n\| = 1$. Let $n_0 \geq 8\pi/\varepsilon$. Consider a sequence $((\pi ntx_1 \mod 2\pi, \ldots, \pi ntx_N \mod 2\pi))_n$ for $n = 0, 1, 2, \ldots, n_0^N$ and observe that by the pigeonhole principle there exist $0 \leq n_1 < n_2 \leq n_0^N$ such that for all $1 \leq i \leq N$ we have $\text{dist}(\pi nx_i(n_1 - n_2), 2\pi \mathbb{Z}) \leq \frac{2\pi}{n_0}$. Taking $n = n_2 - n_1$ we obtain

$$
\|f_n - A_t(f_n)\| \leq \frac{\pi}{2} \sum_{i=1}^{N} p_i \|\sin(2\pi nx) - \sin(2\pi n(x + tx_i))\| + \frac{\varepsilon}{2} = \pi \sum_{i=1}^{N} p_i |\sin(\pi ntx_i)| \cdot \|\cos(2\pi n \oplus \pi ntx_i)\| + \frac{\varepsilon}{2} \leq 2 \sum_{i=1}^{N} p_i |\sin(\pi ntx_i)| + \frac{\varepsilon}{2} \leq \frac{4\pi}{n_0} \sum_{i=1}^{N} p_i + \frac{\varepsilon}{2} \leq \varepsilon. \quad \square
$$

Our result gives the bound for the norm of an operator of the form $(I - A_t)^{-1}$. The main difficulty is that this operator is not globally invertible. Of course, boundedness of a resolvent operator $R_\lambda(A) = (A - zI)^{-1}$ has been thoroughly studied (see e.g. [G], [Ba] which feature Hilbert space setting for Hilbert-Schmidt and Schatten-von Neumann operators). Let us also mention that the first part of the book [CE] is a set of related articles concerning mainly the problem of finding the inverse formula for certain Toeplitz-type operators. The paper [GS] contains a famous Gohberg-Semencul formula for the inverse of a non-Hermitian Toeplitz matrix. In [GH] the authors generalized the results of [CE] to the case of Toeplitz matrices whose entries are taken from some noncommutative algebra with a unit. The operators of the form $I - K$ (acting e.g. on $L_1([0, 1])$), where $K$ is a certain operator with a kernel $k(t - s)$, are continuous versions of the operators given by Toeplitz matrices. Paper [GS] deals also with this kind of operators, namely

$$(I - K)(f)(x) = f(x) - \int_{0}^{1} k(t - s)f(s) \, ds,$$

where $k \in L_1([-1, 1])$. In the case of $I - K$ being invertible, the authors give a formula for the inverse operator $(I - K)^{-1}$ in terms of solutions of certain
four integral equations. See also Article 3 in [CE] for generalizations of these formulas.

Section 2 contains the proof of the main result, Theorem 2. In Section 3 we link the ideas from Section 2 to Markov chains.

2 Proof of Theorem 2

We begin with two lemmas.

**Lemma 1.** Suppose $Y$ is an $ℓ$-decent random variable. Let $Y_1, Y_2, \ldots$ be independent copies of $Y$. Then there exists a positive integer $N = N(Y)$ and numbers $c = c(Y) > 0$, $C_0 = C_0(Y) \geq 1$ such that for all $C \geq C_0$ and $n \geq N$ the random variable

$$X_n^{(C)} = \left( C \cdot \frac{Y_1 + \cdots + Y_n}{\sqrt{n}} \right) \mod 1$$

is $c$-good.

**Proof.** We prove the lemma in a few steps considering more and more general assumptions about $Y$.

**Step I.** Suppose that the characteristic function of $Y$ belongs to $L_p(\mathbb{R})$ for some $p \geq 1$. In this case, by a certain version of the Local Central Limit Theorem, e.g. Theorem 19.1 in [BR], p. 189, we know that the density $q_n$ of $(Y_1 + \ldots + Y_n - n\mathbb{E}Y)/\sqrt{n}$ exists for sufficiently large $n$, and satisfies

$$\sup_{x \in \mathbb{R}} \left| q_n(x) - \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} \right| \xrightarrow{n \to \infty} 0,$$

where $\sigma^2 = \text{Var}(Y)$. Observe that the density $g_n^{(C)}$ of $X_n^{(C)}$ equals

$$g_n^{(C)}(x) = \sum_{k \in \mathbb{Z}} \frac{1}{C} q_n \left( \frac{1}{C} (x + k + \sqrt{n\mathbb{E}Y}) \right), \quad x \in [0, 1].$$

Using (5), for $\delta = \frac{e^{-2/\sigma^2} - \sqrt{2\pi\sigma}}{\sqrt{2\pi\sigma}}$ we can find $N = N(Y)$ such that

$$q_n(x) > \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} - \delta/8, \quad x \in \mathbb{R}, \ n \geq N.$$
Therefore, to be close to the maximum of the Gaussian density we sum over only those \( k \)'s for which \( x + k + \sqrt{nEY} \in (-2C, 2C) \) for all \( x \in [0, 1] \). Since there are at least \( C \) and at most \( 4C \) such \( k \)'s, we get that
\[
g_n^{(C)}(x) > \frac{1}{C} \frac{1}{\sqrt{2\pi\sigma}} e^{-2/\sigma^2} \cdot C - \frac{1}{C} \frac{\delta}{8} \cdot 4C = \frac{1}{2\sqrt{2\pi\sigma}} e^{-2/\sigma^2}.
\]
In particular, it implies that \( X_n^{(C)} \) is \( c \)-good with \( c = \frac{1}{2\sqrt{2\pi\sigma}} e^{-2/\sigma^2} \). Thus, in this case, it suffices to set \( C_0 = 1 \).

Step II. Suppose that the law of \( Y \) is of the form \( q\mu + (1 - q)\nu \) for some \( q \in (0, 1] \) and some Borel probability measures \( \mu, \nu \) on \( \mathbb{R} \) such that the characteristic function of \( \mu \) belongs to \( L_p(\mathbb{R}) \) for some \( p \geq 1 \). Notice that
\[
\mu_{Y_1 + \ldots + Y_N} = \mu_Y^N = (q\mu + (1 - q)\nu)^* \sum_{k=0}^N \binom{N}{k} q^k (1 - q)^{N-k} \mu^{*k} \nu^{*(N-k)} \geq \sum_{k=N_0}^N \binom{N}{k} q^k (1 - q)^{N-k} \mu^{*k} \nu^{*(N-k)} = c_{N,N_0} (\mu^{*N_0} * \rho_{N,N_0}),
\]
where
\[
\rho_{N,N_0} = \frac{1}{c_{N,N_0}} \sum_{k=N_0}^N \binom{N}{k} q^k (1 - q)^{N-k} \mu^{*k} - N_0 \nu^{*(N-k)}
\]
is a probability measure, and
\[
c_{N,N_0} = \sum_{k=N_0}^N \binom{N}{k} q^k (1 - q)^{N-k}
\]
is a normalisation constant. Choosing \( N_0 = \lfloor qN - C_1 \sqrt{q(1-q)/N} \rfloor \) we can guarantee that \( c_{N,N_0} \geq 1/2 \) eventually, say for \( N \geq \tilde{N} \). Denoting by \( \tilde{Y}, Z \) a random variable with the law \( \mu, \rho_{N,N_0} \) respectively and by \( \tilde{Y}_i \) i.i.d. copies of \( \tilde{Y} \), we get
\[
P \left( X_N^{(C)} \in A \right) \geq c_{N,N_0} P \left( \left( \sum_{i=1}^{N_0} \frac{\tilde{Y}_i}{\sqrt{N}} + \frac{Z_{N,N_0}}{\sqrt{N}} \right) \mod 1 \in A \right).
\]
By Step I, the first bit \( C(\tilde{Y}_1 + \ldots + \tilde{Y}_{N_0})/\sqrt{N} \) is \( c \)-good for some \( c > 0 \) and \( C \geq C_0^{(II)} = \sup_{N \geq \tilde{N}} \sqrt{N/N_0} \). Moreover, note that if \( U \) is a \( c \)-good \( \mathbb{T} \)-valued
r.v., then so is \( U \oplus V \) for every \( \mathbb{T} \)-valued r.v. \( V \) which is independent of \( U \).

As a result, \( X_N^{(c)} \) is \( c/2 \)-good.

**Step III.** Now we consider the general case, i.e. \( Y \) is \( \ell \)-decent for some \( \ell \geq 1 \). For \( n \geq \ell \) we can write

\[
C \cdot \frac{Y_1 + \ldots + Y_n}{\sqrt{n}} = C \sqrt{\frac{\lfloor n/\ell \rfloor}{n}} \cdot \frac{e_1 + \ldots + e_{\lfloor n/\ell \rfloor}}{\sqrt{n/\ell}} + \frac{\tilde{R}}{\sqrt{n}}
\]

with \( e_j = Y_{(j-1)\ell+1} + \ldots + Y_{j\ell} \) for \( j = 1, \ldots, \lfloor n/\ell \rfloor \), and \( \tilde{R} = Y_{\lfloor n/\ell \rfloor \ell+1} + \ldots + Y_n \).

Since the absolutely continuous part of the law \( \mu \) of \( e_j \) is nontrivial, then \( \mu \) is of the form \( q\nu_1 + (1-q)\nu_2 \) with \( q \in (0, 1] \) and the characteristic function of \( \nu_1 \) belonging to some \( L_p \). Indeed, \( \mu \) has a bit which is a uniform distribution on some measurable set whose characteristic function is in \( L_2 \). Therefore, applying Step II for \( e_j \)'s we get that \( X_n^{(c)} \) is \( c \)-good when \( C \sqrt{\frac{\lfloor n/\ell \rfloor}{n}} \geq C^{(II)}_0 \).

So we can set \( C_0 = C^{(II)}_0 \sqrt{2\ell} \).

**Lemma 2.** Suppose \( Z \) is a \( \mathbb{T} \)-valued c-good random variable and \( B_Z \) is the operator defined by \((B_Z f)(x) = \mathbb{E} f(x \oplus Z)\) for \( f \in L_1(\mathbb{T}) \). Then \( \|B_Z f\| \leq (1 - c) \|f\| \) for all \( f \in L_1(\mathbb{T}) \) satisfying \( \int_{\mathbb{T}} f = 0 \).

**Proof.** Let \( \mu \) be the law of \( Z \). Define the measure \( \nu(A) = \mu(A) - c|A| \) for measurable \( A \subset \mathbb{T} \). Since \( \mu \) is c-good, \( \nu \) is a nonnegative Borel measure on \( \mathbb{T} \). Take \( f \in L_1(\mathbb{T}) \) with mean zero. Then we have

\[
\|B_Z f\| = \int_0^1 \left| \int_0^1 f(x \oplus s) \, d\mu(s) \right| \, dx = \int_0^1 \left| \int_0^1 f(x \oplus s) \, d\nu(s) \right| \, dx \\
\leq \int_0^1 \int_0^1 |f(x \oplus s)| \, d\nu(s) \, dx \\
= \|f\| \int_0^1 \, d\nu(s) = (1 - c) \|f\|.
\]

Now we are ready to give the proof of Theorem 2.

**Proof of Theorem 2.** Let \( Y_1, Y_2, \ldots \) be independent copies of \( Y \). Observe that

\[
(A^n f)(x) = \mathbb{E} f \left( x \oplus tY_1 \oplus \ldots \oplus tY_n \right) \\
= \mathbb{E} f \left( x \oplus \left( t\sqrt{n} \left( \frac{Y_1 + \ldots + Y_n}{\sqrt{n}} \right) \mod 1 \right) \right).
\]
Take \( n(t) = C_0^2 \lceil 1/t^2 \rceil N \), where \( C_0 \) and \( N \) are the numbers given by Lemma 1. Therefore, with \( X_n^{(C)} \) defined by (4), we can write
\[
(A_t^{n(t)} f)(x) = \mathbb{E} f \left( x \oplus X_n^{(C)} \right),
\]
where \( C = t \sqrt{n(t)} = t C_0 \sqrt{1/t^2} N \geq C_0 \sqrt{N} \geq C_0 \). Thus \( X_n^{(C)} \) is \( c(Y) \)-good with some constant \( c(Y) \in (0, 1) \). From Lemma 2 we have
\[
\left\| A_t^{n(t)} f \right\| \leq (1 - c(Y)) \| f \|
\]
for all \( f \) satisfying \( \int_T f = 0 \).

The operator \( A_t \) is a contraction, namely \( \| A_t f \| \leq \| f \| \) for all \( f \in L_1(\mathbb{T}) \). Using this observation and the triangle inequality we obtain
\[
\| f - A_t f \| \geq \frac{1}{n} \left( \| f - A_t f \| + \| A_t f - A_t^2 f \| + \ldots + \| A_t^{n-1} f - A_t^n f \| \right) \\
\geq \frac{1}{n} \| f - A_t^n f \|.
\]
Taking \( n = n(t) \) we arrive at
\[
\frac{1}{n(t)} \left\| f - A_t^{n(t)} f \right\| \geq \frac{1}{t^2 + 1} \cdot \frac{1}{C_0^2} \cdot C_0 \sqrt{n(t)} \left( \| f \| - \left\| A_t^{n(t)} f \right\| \right) \geq \frac{c(Y)}{2 C_0^2} \cdot t^2 \cdot \| f \|.
\]
It suffices to take \( c = c(Y)/(2C_0^2 \cdot N) \).

Remark. Consider an \( \ell \)-decent random variable \( Y \). As it was noticed in the proof of Lemma 1 (Step III), the law \( Y_1 + \ldots + Y_\ell \) has a bit whose characteristic function is in \( L_2 \). Conversely, if the law of \( S_m = Y_1 + \ldots + Y_m \) has the form \( q \mu + (1 - q) \nu \) with \( q \in (0, 1) \) and the characteristic function of \( \mu \) belonging to \( L_p \) for some \( p \geq 1 \), then the characteristic function of the bit \( \mu^{\lceil p/2 \rceil} \) of the sum of \( \lceil p/2 \rceil \) i.i.d. copies of \( S_m \) is in \( L_2 \). In particular, that bit has a density function in \( L_1 \cap L_2 \). Thus \( Y \) is \( (m \lceil p/2 \rceil) \)-decent.

## 3 Markov chains

Let \( V = \{v_1, \ldots, v_N\} \) be a finite state space and let us consider a stationary Markov chain \( (X_n)_{n \geq 0} \) with values in \( V \). Let \( A \) be a transition matrix of this chain and take
\[
p_n(x, y) = \mathbb{P} (X_n = y \mid X_0 = x).
\]
Note that \( p_n(x, y) = (A^n)_{x,y} \). The measure \( p \) on \( V \) is called stationary if the vector \( P = (p(v_1), \ldots, p(v_N)) \) satisfies \( PA = P \). We have written \( p(v_i) \) instead of \( p(\{v_i\}) \) to shorten notation. Using our approach from the proof of Theorem 2 we obtain the following result.

**Theorem 3.** Let \( n_0 \geq 1 \). Suppose that a Markov chain satisfies a Doeblin-type condition, namely there exists a constant \( c \in (0,1] \) such that for all \( x, y \in V \) we have \( p_{n_0}(x, y) \geq cp(y) \), where \( p \) is a stationary distribution. Then for all \( f : V \to \mathbb{R} \) with \( \sum_{x \in V} f(x)p(x) = 0 \) we have

\[
\|f - Af\|_{L_1(p)} \geq \frac{c}{n_0} \|f\|_{L_1(p)}.
\]

We would like to link the above \( L_1 \) inequality to the Poincaré inequality. Assume that a transition matrix \( A \) is reversible with respect to the stationary state \( p \), i.e. the detailed-balance condition holds, \( p(x)A_{x,y} = p(y)A_{y,x} \) for all \( x, y \in V \). If \( 1 = \lambda_1 > \lambda_2 \geq \ldots \geq \lambda_N \geq -1 \) is the spectrum of the transition matrix \( A \), then the Poincaré inequality reads

\[
(1 - \lambda_2) \text{Var}(f) \leq \langle f, (I - A)f \rangle_{L_2(p)}, \quad f \in L_2(p).
\]

The constant \( 1 - \lambda_2 \) is the best possible. For functions with mean zero, the Schwarz inequality applied to the RHS yields

\[
c_2 \|f\|_{L_2(p)} \leq \|(I - A)f\|_{L_2(p)},
\]

with \( c_2 = 1 - \lambda_2 \) being the spectral gap. Theorem 3 gives an analogous bound but in \( L_1 \),

\[
c_1 \|f\|_{L_1(p)} \leq \|(I - A)f\|_{L_1(p)}.
\]

One might ask if there is any relation between the optimal \( L_1 \) and \( L_2 \) constants, \( c_1 \) and \( c_2 \) respectively. Obviously \( c_1 \leq c_2 \) (test (7) with an eigenvector corresponding to the eigenvalue \( \lambda_2 \)).

To apply Theorem 3 we need to know that the Doeblin condition holds. It can be guaranteed in the case of reversible Markov chains. Then the constant in our \( L_1 \)-inequality (7) can be expressed explicitly in terms of the spectrum of the transition matrix. Unfortunately, the dependence on the number of states \( N \) occurs as well.

**Corollary 2.** Suppose that \( A \) is the transition matrix of an aperiodic and irreducible Markov chain. Let \( A \) be reversible with respect to the unique
stationary state $p$. Then for all $f : V \to \mathbb{R}$ with $\sum_{x \in V} f(x)p(x) = 0$ we have
\[ \|f - Af\|_{L_1(p)} \geq \frac{1}{C} \|f\|_{L_1(p)}, \]
where
\[ C = 2 \left\lceil \frac{\ln(2N)}{\ln(1/\lambda_*)} \right\rceil, \]
and
\[ \lambda_* = \max\{|\lambda|, \lambda \text{ is an eigenvalue of } A, \lambda \neq 1\} \]

Proof. From Lemma 12.2 (iii) in [LPW], estimating crudely, we obtain
\[ |p_n(x, y) - p(y)| \leq p(y) \cdot N \cdot \lambda_*^n. \]
Note that Lemma 12.1 from [LPW] implies that $\lambda_* < 1$.

If $n = n_0 = \left\lceil \frac{\ln[1/(2N)]}{\ln\lambda_*} \right\rceil$, then $N \cdot \lambda_*^n \leq 1/2$. Thus, the Doeblin condition is satisfied with $c = \frac{1}{2}$. Theorem 3 finishes the proof. \hfill \square

Corollary 2 allows us to estimate the $c_1$ constant in terms of the spectrum of $A$. Moreover, if it happens that $\lambda_* = \lambda_2$, then we can bound $c_1$ using the spectral gap,
\[ c_1 \geq \left( 2 \left\lceil \frac{\ln(2N)}{\ln(1/(1 - c_2))} \right\rceil \right)^{-1}. \]
This is the case if, e.g., $\text{spec } A \subset [0, 1]$.

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