Abstract. Gram’s Law refers to the observation that the zeros of the Riemann zeta function typically alternate with certain prescribed points, known as Gram points. Although this pattern doesn’t hold for every zero, numerical results suggest that, as the height up the critical line increases, the proportion of zeros that obey Gram’s Law converges to a non-trivial limit. It is well-known that the eigenvalues of random unitary matrices provide a good statistical model for the distribution zeros of the zeta function, so one could try to determine the value of this limit by analyzing an analogous model for Gram’s Law in the framework of random matrix theory. In this paper, we will review an existing model based on random unitary matrices, for which the limit can be computed analytically, but has the wrong rate of convergence. We will then present an alternative model that uses random special unitary matrices, which gives the correct convergence rate, and discuss the large-height limit of this model.

1. Gram’s Law for the Riemann zeta function

It is well-known that the zeros of the Riemann zeta function \( \zeta(s) \) are of fundamental importance in number theory. The numerical calculation of their exact location has a long history, dating back to Riemann himself who (unpublished) computed the first few zeros, and continues up to the present day, with all the first \( 10^{13} \) zeros being currently known, as well as a couple of billion zeros after the \( 10^n \)-th zero, for each \( 13 \leq n \leq 24 \) [9]. There are two other functions associated with the zeta function that are needed when computing its zeros.

First, the Riemann-Siegel theta function is defined as

\[
\theta(t) := \arg \left[ \pi^{-\frac{1}{4}} \Gamma \left( \frac{1}{4} + \frac{it}{2} \right) \right] = \text{Im} \log \Gamma \left( \frac{1}{4} + \frac{it}{2} \right) - \frac{\log \pi}{2} t
\]

where \( \Gamma(s) \) is the gamma function, and the branch of the logarithm is determined by continuous variation up the vertical line, starting from \( \theta(0) = 0 \). The theta function takes real arguments \( t \in \mathbb{R} \), and it can be seen from the definition that it is a real-valued function. It can also be shown that at \( t_{\min} \approx 7 \) it has a minimum value of \( \theta(t_{\min}) \approx -3.5 \), while for \( t > t_{\min} \) it is strictly increasing. And by applying Stirling’s formula for the gamma function, we can obtain an asymptotic expansion

\[
\theta(t) \sim \frac{t}{2} \log \frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + \frac{1}{48t} + \frac{7}{5760t^3} + O \left( \frac{1}{t^5} \right)
\]

which is not convergent, but whose first few terms give a good approximation for \( t \gg 1 \).

The other function related to zeta is called the Hardy Z function, given by

\[
Z(t) := e^{i\theta(t)} \zeta \left( \frac{1}{2} + it \right)
\]

Like the theta function, \( Z(t) \) also takes real arguments \( t \in \mathbb{R} \), and (as a consequence of the functional equation for the zeta function) it is a real-valued function. We remark that the zeros of \( \zeta \left( \frac{1}{2} + it \right) \) coincide with the zeros of \( Z(t) \). But because the \( Z \) function is real, these zeros can be found by studying its sign changes. And in turn, the changes
in the sign of \(Z(t)\) can be determined from evaluating \(\zeta\left(\frac{1}{2} + it\right)\) at certain points, called Gram points.

**Definition 1.** For any integer \(M \geq -1\), we define the \(M\)-th Gram point \(g_M\) as the unique solution in the range \(t > 7\) of the equation

\[
\theta(g_M) = M\pi
\]

and we call a Gram interval any interval between two consecutive Gram points \([g_M, g_{M+1})\).

The definition for \(Z(t)\) [2], together with Euler’s formula, imply that at every Gram point \(g_M\) we have

\[
\zeta\left(\frac{1}{2} + ig_M\right) = (-1)^M Z(g_M)
\]  

(3)

Keeping in mind that \(Z(t)\) is a real-valued function, we obtain an alternative definition of Gram points, namely as points on the critical line at which the Riemann zeta function \(\zeta\left(\frac{1}{2} + it\right)\) takes real (non-zero) values.

In particular, if \(\zeta\left(\frac{1}{2} + it\right)\) has the same sign at two successive Gram points \(g_M\) and \(g_{M+1}\), then by (3) \(Z(t)\) must have opposite signs at these points. This means that \(Z(t)\) has at least one root in the Gram interval \([g_M, g_{M+1})\).

This technique was used by the Danish mathematician Jørgen P. Gram in 1903 to compute the first 15 zeros of the \(\zeta\left(\frac{1}{2} + it\right)\) in the range \(0 \leq t \leq 66\) [10]. He remarked that \(\zeta\left(\frac{1}{2} + ig_M\right) > 0\) for all \(-1 \leq M \leq 14\) and that each of these Gram intervals contained exactly one zeta zero or, in other words, that the Gram points alternated with the zeta zeros. Gram believed that the pattern would continue beyond the first 15 intervals, but also that it would not necessarily hold in every case. When it does hold, this phenomenon is named Gram’s Law.

**Definition 2.** Given two consecutive Gram points \(g_M\) and \(g_{M+1}\), we say that Gram’s Law holds for \([g_M, g_{M+1})\) if this Gram interval contains exactly one zero of \(\zeta\left(\frac{1}{2} + it\right)\).

The original definition of Gram’s Law was proposed by J. I. Hutchison in 1925 [12], and was given in terms of the zeros of the \(Z\) function

“Gram calculated the first fifteen roots [of \(Z(t)\)] and called attention to the fact that the [roots] and the [Gram points] separate each other. I will refer to this property of the roots as Gram’s Law. Gram expressed the belief that this law is not a general one.”

Hutchison also extended Gram’s computations to the first 138 zeros of \(\zeta\left(\frac{1}{2} + it\right)\), and discovered the first exceptions to Gram’s Law: the interval \([g_{125}, g_{126})\) doesn’t contain any zeros, while the next one \([g_{126}, g_{127})\) has two. Subsequently, in 1935 it was proved by E. C. Titchmarsh that Gram’s Law fails infinitely many times [24].

There is another, less restrictive version of Gram’s Law, called the Weak Gram Law, which states that for \(M \in \mathbb{N}\)

\[
(-1)^M Z(g_M) > 0 \quad \text{and} \quad (-1)^{M+1} Z(g_{M+1}) > 0
\]

This is equivalent to claiming that the Gram interval \((g_M, g_{M+1})\) contains an odd number of simple zeros (or a zero with odd multiplicity). Although this version also has exceptions (as can be seen from Hutchison’s results) Titchmarsh managed to prove that the Weak Gram Law is true infinitely many times [25]. And more recently, in 2009, T. S. Trudgian
has shown that for sufficiently large $T$, there exists a positive proportion of Gram intervals between $T$ and $2T$ that contain at least one zero of $\zeta(\frac{1}{2} + it)$ \[26], \[27]; in particular, this implies that the Weak Gram Law is true for a positive proportion of the time.

In the case of the (original) Gram’s Law, it has not yet been proven if it is also true infinitely many times, or whether it has a positive proportion of successes, although extensive numerical computations for regions high up the critical line suggest that it does hold for a large fraction of cases. In a series of four papers published during the 80’s, R. P. Brent, J. van de Lune and other have analyzed the first 1.5 billion Gram intervals and reported that approximately 72.61% of them obey Gram’s Law \[11], \[12], \[15], \[16]. A summary of their results can be found in table 1 expressed in terms of $G_{L,M}(k)$, which is defined as follows.

**Definition 3.** For any $k \in \mathbb{N} \cup \{0\}$ and $0 \leq L < M$, we define $G_{L,M}(k) \in [0,1]$ to be the proportion of Gram intervals between $g_L$ and $g_M$ that contain exactly $k$ zeros (in particular, $G_{L,M}(1)$ represents the fraction of intervals that obey Gram’s Law).

**Remark.** In everything that follows from here on, the first zeta zero $\gamma_1 = 14.1347\ldots$ and the corresponding Gram interval $[g_{-1}, g_0)$ are going to be excluded.

| $M$     | $M \cdot G_{0,M}(0)$ | $M \cdot G_{0,M}(1)$ | $M \cdot G_{0,M}(2)$ | $M \cdot G_{0,M}(3)$ | $M \cdot G_{0,M}(4)$ |
|---------|------------------------|------------------------|------------------------|------------------------|------------------------|
| 100     | 100                    | 916                    | 42                     |                        |                        |
| 1,000   | 42                     | 808                    | 8,390                  | 796                    | 6                      |
| 10,000  | 10,330                 | 79,427                 | 10,157                 | 86                     |                        |
| 100,000 | 116,055                | 769,179                | 113,477                | 1,289                  |                        |
| 1,000,000 | 1,253,556               | 7,507,820              | 1,223,692              | 14,932                 |                        |
| 10,000,000 | 13,197,331             | 73,771,910             | 12,864,188             | 166,570                | 1                      |
| 100,000,000 | 137,078,283            | 727,627,708            | 121,490,923            | 1,784,225              | 23                     |

Table 1. Summary of results by R. Brent, J van de Lune, et al.

In addition to these, A. M. Odlyzko has obtained that Gram’s Law holds for 70.82% among $10^6$ Gram intervals near the $10^{12}$-th zero, and 68% of the time among $10^6$ Gram intervals near the $10^{20}$-th zero \[22]. Van de Lune et al. have concluded the fourth paper in their series with the following remark:

“Our statistical material suggests that the zeros of $Z(t)$ are distributed among the Gram intervals according to some hitherto unknown probabilistic law. ( . . . ) It would be interesting to have a probabilistic model which could explain or at least support this phenomenon.”

The purpose of the current paper will be to provide such a probabilistic model.

1.1. **Random matrix models for zeta zeros.** Following ground-breaking work of Dyson \[3] and Montgomery \[20], a conjecture was developed — backed up by theoretical, heuristic, and numerical results — that the statistical distribution of zeros of the zeta function is the same as the distribution of eigenvalues of random unitary matrices (historically the conjecture was developed for hermitian matrices, the so-called GUE ensemble, but the statistics of eigenvalues of GUE matrices is the same as eigenvalues of Haar-distributed unitary matrices).
In order to compare zeros at different heights, they need to be rescaled so the average gap between them is unity. It follows from the count of zeros of zeta, \(N(T)\), that the density of zeros at height \(T\) is \(\frac{1}{2\pi} \log \frac{T}{2\pi}\), so this is the required scaling. For a unitary matrix of size \(N \times N\), there are \(N\) eigenphases between \([-\pi, \pi]\), thus the appropriate scaling is \(\frac{N}{2\pi}\). That is, if \(\theta_n (n = 1, \ldots, N)\) are the eigenvalues of an \(N \times N\) unitary matrix, then \(\phi_n = \frac{N}{2\pi} \theta_n\) will have an average gap of one.

Identifying these two scalings, we see that the matrix size and height up the critical line should be identified via

\[ N \approx \log \frac{T}{2\pi} \]

These matrices are chosen with Haar measure, which is the unique measure invariant under unitary transformations. If we define the Dyson product to be

\[ D_{U(N)}(\theta_1, \ldots, \theta_N) := \prod_{1 \leq m < n \leq N} |e^{i\theta_m} - e^{i\theta_n}|^2 \]

then Weyl [28] showed that Haar measure on the unitary group leads to the following probability density function for the eigenvalues

\[ P_{U(N)}(\theta_1, \ldots, \theta_N) := \frac{1}{N!(2\pi)^N} D_{U(N)}(\theta_1, \ldots, \theta_N) \]
\[ = \frac{1}{N!(2\pi)^N} \prod_{1 \leq m < n \leq N} |e^{i\theta_m} - e^{i\theta_n}|^2 \]

If we let \(J \subset [\pi, \pi]\) be an arbitrary (fixed) interval of length \(\frac{2\pi}{N}\), then the probability that \(J\) contains exactly \(k\) eigenvalues of a random \(U(N)\) matrix is

\[ E_{U(N)}(k, J) := \binom{N}{k} \int_{J^k} \int_{((-\pi, \pi) \setminus J)^{N-k}} P_{U(N)}(\theta_1, \ldots, \theta_N) d\theta_1 \ldots d\theta_N \]

(4)

Note that by rotation invariance of Haar measure, \(E_{U(N)}(k, J)\) is insensitive to the actual starting position of \(J\), only to its length.

Based on the connection between eigenvalues of random unitary matrices and zeros of the zeta function, Fujii [8] made the following conjecture:

**Conjecture 1** (Fujii, 1987). For any \(k \in \mathbb{N} \cup \{0\}\) we have

\[ \lim_{N \to \infty} E_{U(N)}(k, J) = \lim_{M \to \infty} G_{0,M}(k) \]

Although this conjecture is for the large \(N\), large \(T\) limit, we can numerically compare the two sides for low-lying zeros. Table 2 gives the values of \(E_{U(N)}(k, J)\) for small \(N\), and table 3 gives the corresponding values for the proportion of Gram intervals containing 0, 1, or 2 zeros around the appropriate height.

Since matrix size and height up the critical line are identified by \(N \approx \log \frac{2\pi}{M}\), for each matrix size we can identify a height up the critical line that roughly matches it, namely \(T = 2\pi e^N\). And for any height up the critical line, we can identify the index of the closest Gram point, by calculating

\[ \frac{1}{\pi} \theta(T) \approx \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} - \frac{1}{8} \]
Therefore, let $M_N$ be the index of the Gram point $g_{M_N}$ that lies at the height on the critical line corresponding to unitary matrices of size $N \times N$. The preceding argument shows that

$$M_N = \pi \theta(T) \approx e^N(N - 1) - \frac{1}{8}$$

One would expect that the large interval of the critical line between $g_{M_N}$ and $g_{M_N+1}$ should roughly correspond to matrices of size $N$. This is what is shown in Table 3. The values of the limits of $E_{U(N)}(k, J)$ for $k = 0, 1, 2$, are also known to be

$$\lim_{N \to \infty} E_{U(N)}(0, J) \approx 0.17022$$

$$\lim_{N \to \infty} E_{U(N)}(1, J) \approx 0.66143$$

$$\lim_{N \to \infty} E_{U(N)}(2, J) \approx 0.16649$$

As we can see for finite $N$, $E_{U(N)}$ does not provide a good approximation for $G_{M_N, M_{N+1}}$. We posit that this is caused by the wrong random matrix calculation being used. There is nothing special about the interval $J$ in $E_{U(N)}(k, J)$, other than it has the right length. However, Gram intervals aren’t just intervals of the right length, but their end points are special (namely Gram points, which are points on the critical line where the zeta function is real and non-zero). The remainder of this paper will construct random matrix “Gram points”, and then create a more believable conjecture for the rate of success of Gram’s Law, including a rate of convergence which appears to fit the data much better.

### Table 2. $E_{U(N)}(k, J)$ for $k = 0, 1, 2$ and $N = 2, \ldots, 21$

| $N$ | $E_{U(N)}(0, J)$ | $E_{U(N)}(1, J)$ | $E_{U(N)}(2, J)$ |
|-----|----------------|----------------|----------------|
| 2   | 0.148679       | 0.702642       | 0.148679       |
| 3   | 0.161362       | 0.678268       | 0.159378       |
| 4   | 0.165362       | 0.670641       | 0.162630       |
| 5   | 0.167146       | 0.667251       | 0.164060       |
| 6   | 0.168098       | 0.665445       | 0.164817       |
| 7   | 0.168666       | 0.664367       | 0.165268       |
| 8   | 0.169032       | 0.663673       | 0.165558       |
| 9   | 0.169283       | 0.663199       | 0.165755       |
| 10  | 0.169461       | 0.662860       | 0.165896       |
| 11  | 0.169593       | 0.662611       | 0.166000       |
| 12  | 0.169693       | 0.662421       | 0.166079       |
| 13  | 0.169771       | 0.662274       | 0.166140       |
| 14  | 0.169833       | 0.662157       | 0.166188       |
| 15  | 0.169882       | 0.662063       | 0.166227       |
| 16  | 0.169923       | 0.661986       | 0.166259       |
| 17  | 0.169957       | 0.661922       | 0.166286       |
| 18  | 0.169985       | 0.661869       | 0.166308       |
| 19  | 0.170009       | 0.661824       | 0.166327       |
| 20  | 0.170029       | 0.661785       | 0.166343       |
| 21  | 0.170047       | 0.661752       | 0.166357       |
1.2. Random special unitary matrices. A special unitary matrix is a unitary matrix with determinant equal to 1. It also has a Haar measure which effectively comes from the Haar measure for unitary matrices, but with one eigenangles forced to equal the value that makes the sum of all $N$ eigenangles congruent to 0 (mod 2π) since that would make the determinant equal to 1, \[11\].

That is, the probability density function for the $N$ eigenangles of a Haar distributed SU($N$) matrix is

$$P_{SU(N)}(\theta_1, \ldots, \theta_N) := \frac{1}{N!(2\pi)^{N-1}} D_{U(N)}(\theta_1, \ldots, \theta_N) \cdot \delta(\theta_1 + \ldots + \theta_N \mod 2\pi)$$

and integrating over the last variable we find

$$\int_{[-\pi, \pi]} P_{SU(N)}(\theta_1, \ldots, \theta_N) d\theta_N = \frac{1}{N!(2\pi)^{N-1}} D_{SU(N)}(\theta_1, \ldots, \theta_{N-1})$$

where

$$D_{SU(N)}(\theta_1, \ldots, \theta_{N-1}) := \int_{[-\pi, \pi]} D_{U(N)}(\theta_1, \ldots, \theta_N) \cdot \delta(\theta_1 + \ldots + \theta_N \mod 2\pi) d\theta_N$$

$$= D_{U(N)}(\theta_1, \ldots, \theta_{N-1}, -\theta_1 - \ldots - \theta_{N-1})$$

$$= \prod_{1 \leq m < n \leq N-1} |e^{i\theta_m} - e^{i\theta_n}|^2 \prod_{1 \leq n \leq N-1} |e^{i\theta_n} - e^{-i(\theta_1 + \ldots + \theta_{N-1})}|^2$$

| $N$ | $M_N$ | $G_{M_N,M_{N+1}}(0)$ | $G_{M_N,M_{N+1}}(1)$ | $G_{M_N,M_{N+1}}(2)$ |
|---|---|---|---|---|
| 2 | 7 | 1.000000 | 0.016129 | 0.967741 |
| 3 | 40 | 0.016129 | 0.976774 | 0.016129 |
| 4 | 164 | 0.039351 | 0.921296 | 0.039351 |
| 5 | 594 | 0.069669 | 0.860661 | 0.069669 |
| 6 | 2,017 | 0.083059 | 0.834538 | 0.081744 |
| 7 | 6,580 | 0.095051 | 0.810527 | 0.093791 |
| 8 | 20,867 | 0.105168 | 0.790572 | 0.103348 |
| 9 | 64,825 | 0.111233 | 0.778687 | 0.108924 |
| 10 | 198,238 | 0.116361 | 0.768576 | 0.113764 |
| 11 | 598,741 | 0.121410 | 0.758585 | 0.118597 |
| 12 | 1,790,303 | 0.125309 | 0.750841 | 0.122389 |
| 13 | 5,308,961 | 0.128694 | 0.744212 | 0.125490 |
| 14 | 15,633,856 | 0.131542 | 0.738581 | 0.128210 |
| 15 | 45,766,243 | 0.134146 | 0.733422 | 0.130716 |
| 16 | 133,291,658 | 0.136422 | 0.728930 | 0.132871 |
| 17 | 386,479,244 | 0.138428 | 0.724956 | 0.134802 |
| 18 | 1,116,219,475 | 0.140223 | 0.721401 | 0.136526 |
| 19 | 3,212,681,417 | 0.141825 | 0.718223 | 0.138077 |
| 20 | 9,218,138,713 | 0.143277 | 0.715342 | 0.139481 |
| 21 | 26,376,314,690 | 0.144590 | 0.712736 | 0.140756 |
| 22 | 75,283,169,769 | ... | ... | ... |
2. Gram’s Law for random matrices

2.1. $U(N)$ Gram points and intervals. Since random matrix theory proved to be a good model for the zeros of zeta, Keating and Snaith [13] introduced the characteristic polynomial of a random unitary matrix as a model for the distribution of the Riemann zeta function on the critical line

$$\Lambda_A(\theta) := \det(I_N - Ae^{-i\theta})$$

In order to motivate our random matrix equivalent of Gram points, recall from the previous section the way in which the zeta function is related to the Hardy function

$$\zeta \left( \frac{1}{2} + it \right) = Z(t)e^{-it(\theta)} = Z(t)\cos \theta(t) - iZ(t)\sin \theta(t)$$

If we want to find the points on the critical line at which the zeta function is real, we have to impose the condition that its imaginary part should be zero, from which we get

$$\zeta \left( \frac{1}{2} + it \right) \in \mathbb{R} \iff \Im \zeta \left( \frac{1}{2} + it \right) = 0 \iff Z(t)\sin \theta(t) = 0$$

The last condition is equivalent to the following two possibilities

- $Z(t) = 0$, which gives all the non-trivial zeros that are on the critical line;
- $\sin(\theta(t)) = 0 \iff \theta(t) = M\pi$ for some integer $M$, from which we obtain the Gram points $g_M$.

If we denote the eigenvalues of $A$ by $e^{i\theta_1}, \ldots, e^{i\theta_N}$, then its characteristic polynomial can be re-expressed as

$$\Lambda_A(\theta) = \prod_{j=1}^{N} (1 - e^{i(\theta_j - \theta)})$$

$$= \prod_{j=1}^{N} \exp \left( \frac{i(\theta_j - \theta)}{2} \right) \left[ \exp \left( -\frac{i(\theta_j - \theta)}{2} \right) - \exp \left( \frac{i(\theta_j - \theta)}{2} \right) \right]$$

$$= (-2i)^N \prod_{j=1}^{N} \left[ \exp \left( \frac{i(\theta_j - \theta)}{2} \right) \sin \left( \frac{\theta_j - \theta}{2} \right) \right]$$

$$= (-2)^N \exp \left( iN\pi \right) \exp \left( \frac{i}{2} \sum_{j=1}^{N} \theta_j - \theta \right) \prod_{j=1}^{N} \sin \left( \frac{\theta_j - \theta}{2} \right)$$

We continue the above analogy by searching for the points $\theta \in [-\pi, \pi)$ on the unit circle at which the characteristic polynomial is real, namely the points $\theta$ where

$$\Lambda_A(\theta) \in \mathbb{R} \iff \Im \Lambda_A(\theta) = 0 \iff \sin \left( \frac{N\pi}{2} + \frac{\theta_1 + \ldots + \theta_N}{2} - \frac{N\theta}{2} \right) \prod_{j=1}^{N} \sin \left( \frac{\theta_j - \theta}{2} \right) = 0$$

As before, this leads to two possible cases

- $\prod_{j=1}^{N} \sin \left( \frac{\theta_j - \theta}{2} \right) = 0$
\[
\sin \left( \frac{N\pi}{2} + \frac{\theta_1 + \ldots + \theta_N}{2} - \frac{N\theta}{2} \right) = 0 \Leftrightarrow \frac{N\pi}{2} + \frac{\theta_1 + \ldots + \theta_N}{2} - \frac{N\theta}{2} = m\pi, \text{ for some } m \in \mathbb{Z}.
\]

From the first condition we recover the eigenangles \(\theta_1, \ldots, \theta_N\) of the unitary matrix, which are the random matrix analogue of the zeta zeros. From the second condition, we obtain another set of points, given by

\[
\theta \in \left\{ \frac{\theta_1 + \ldots + \theta_N}{N} + \pi - \frac{2m\pi}{N}, \ m = 0, 1, \ldots, N - 1 \right\}
\]

We note that only \(N\) elements of this set are distinct modulo \(2\pi\), and because they represent the points on the unit circle at which the characteristic polynomial of a \(U(N)\) matrix is real (but not necessarily zero), we will consider them to be the analogous \(U(N)\) Gram points.

**Definition 4.** If \(A\) is a \(U(N)\) matrix with eigenvalues \(e^{i\theta_1}, \ldots, e^{i\theta_N}\), we define the corresponding \(U(N)\) Gram points as

\[
\gamma_{U(N)}^m := \frac{\theta_1 + \ldots + \theta_N}{N} - \pi + \frac{2m\pi}{N}, \ m = 0, 1, \ldots, N - 1
\]

We also define a \(U(N)\) Gram interval as any interval on the unit circle between two consecutive \(U(N)\) Gram points.

We remark that the \(U(N)\) Gram points are placed along the unit circle at equal distance from each other in steps of \(\frac{2\pi}{N}\), rather than being distributed arbitrarily. Furthermore, they are not a fixed set of points, but they depend on matrix \(A \in U(N)\) under consideration, depending on \(\text{arg}(\det A) = \theta_1 + \ldots + \theta_N \mod 2\pi\).

With these definitions in mind, we can now start to investigate what is the probability of having exactly \(k\) eigenvalues of a random \(U(N)\) matrix inside one of these \(U(N)\) Gram intervals, and understand how does this differ from the probability from Fujii’s conjecture. However, it is difficult to compute this quantity in a direct way (the problem originates from the fact that this probability is essentially an integral over the eigenangles \(\theta_1, \ldots, \theta_N\) and each \(U(N)\) Gram point depends on all of them). In order to overcome this difficulty, we will relate this quantity to the corresponding probability for a particular kind of \(U(N)\) matrices, namely the special unitary matrices, and then focus on computing that probability.

2.2. \(SU(N)\) Gram points and intervals. If \(A\) is a \(SU(N)\) matrix then, by definition, \(\text{arg}(\det A) = \theta_1 + \ldots + \theta_N = 0 \mod 2\pi\), and this simplifies the definition of \(SU(N)\) Gram points.

**Definition 5.** We define the \(SU(N)\) Gram points as

\[
\gamma_{SU(N)}^m := -\pi + \frac{2m\pi}{N}, \ m = 0, 1, \ldots, N - 1
\]

We also define an \(SU(N)\) Gram interval as any interval along the unit circle between two consecutive \(SU(N)\) Gram points.

Similar to the \(U(N)\) case, these represent the points on the unit circle at which the characteristic polynomial of a \(SU(N)\) matrix is real (but not necessarily zero), and they are distributed equidistant in steps of \(\frac{2\pi}{N}\). However, unlike the \(U(N)\) case, the \(SU(N)\) Gram points do not depend in any way on the eigenangles, which implies that they are the same for all \(SU(N)\) matrices and are fixed on the unit circle. As we’ll see later, this makes it easier to compute the probability of having exactly \(k\) eigenvalues of a random \(SU(N)\)
matrix inside a SU($N$) Gram interval. For now, we will prove the following result, which relates this quantity with the corresponding probability from the previous subsection:

**Lemma.** For any $k = 0, \ldots, N$, we have that

$$P(\text{exactly } k \text{ eigenvalues of a } U(N) \text{ matrix lie in a } U(N) \text{ Gram interval})$$

$$= P(\text{exactly } k \text{ eigenvalues of a } SU(N) \text{ matrix lie in a } SU(N) \text{ Gram interval})$$

where the first probability is over Haar measure for $U(N)$ and the second probability is over Haar measure for $SU(N)$.

**Proof.** Let $e^{i\theta_1}, \ldots, e^{i\theta_N}$ be the eigenvalues of a $U(N)$ matrix. For simplicity, we will use

$$I = \left[\frac{\theta_1 + \ldots + \theta_N}{N} - \pi, \frac{\theta_1 + \ldots + \theta_N}{N} - \pi + \frac{2\pi}{N}\right] \pmod{2\pi}$$

as a generic $U(N)$ Gram interval and denote its complement by

$$[-\pi, \pi) \setminus I = \left[\frac{\theta_1 + \ldots + \theta_N}{N} - \pi + \frac{2\pi}{N}, \frac{\theta_1 + \ldots + \theta_N}{N} + \pi\right] \pmod{2\pi}$$

Because the $U(N)$ probability density is a symmetric function in all eigenangles, it can be shown that for any $k$, the probability of having $k$ eigenvalues of a $U(N)$ matrix in a $U(N)$ Gram interval is the same for any of the $U(N)$ Gram intervals. Starting with this fact, we have that

$$P(\text{exactly } k \text{ eigenvalues of a } U(N) \text{ matrix lie in a } U(N) \text{ Gram interval})$$

$$= \binom{N}{k} P(\theta_1, \ldots, \theta_k \in I \text{ and } \theta_{k+1}, \ldots, \theta_N \in [-\pi, \pi) \setminus I)$$

$$= \binom{N}{k} \int \cdots \int_{\mathcal{R}} P_{U(N)}(\theta_1, \ldots, \theta_N) \, d\theta_1 \cdots d\theta_N$$

$$= \binom{N}{k} \frac{1}{N!(2\pi)^N} \int \cdots \int_{\mathcal{R}} D_{U(N)}(\theta_1, \ldots, \theta_N) \, d\theta_1 \cdots d\theta_N \quad (5)$$

where the $N$-dimensional integral is over a region $\mathcal{R}$ described by the restrictions

$$\mathcal{R} : \begin{cases} \theta_1, \ldots, \theta_k \in I \\ \theta_{k+1}, \ldots, \theta_N \in [-\pi, \pi) \setminus I \end{cases}$$

which written out explicitly gives

$$\mathcal{R} : \begin{cases} \frac{\theta_1 + \ldots + \theta_N}{N} - \pi \leq \theta_n < \frac{\theta_1 + \ldots + \theta_N}{N} - \pi + \frac{2\pi}{N} & (n = 1, \ldots, k) \\ \frac{\theta_1 + \ldots + \theta_N}{N} - \pi + \frac{2\pi}{N} \leq \theta_n < \frac{\theta_1 + \ldots + \theta_N}{N} + \pi & (n = k + 1, \ldots, N) \end{cases}$$

which can be rearranged to give

$$\mathcal{R} : \begin{cases} -\pi \leq \theta_n - \frac{\theta_1 + \ldots + \theta_N}{N} < -\pi + \frac{2\pi}{N} & (n = 1, \ldots, k) \\ -\pi + \frac{2\pi}{N} \leq \theta_n - \frac{\theta_1 + \ldots + \theta_N}{N} < \pi & (n = k + 1, \ldots, N) \end{cases}$$
We now perform the following change of variables:

\[ \lambda_n = \theta_n - \frac{\theta_1 + \ldots + \theta_N}{N} \quad (n = 1, \ldots, N - 1) \]
\[ \lambda_N = N\theta_N \]

The determinant of the Jacobian matrix is 1. Note that modulo \(2\pi\) the restriction on \(\theta_N\) is lost when it comes to considering \(\lambda_N\). Furthermore, with this change of variables, we have that

\[ \theta_N - \frac{\theta_1 + \ldots + \theta_N}{N} = -\lambda_1 - \ldots - \lambda_{N-1} \]

and the previous region of integration \(\mathcal{R}\) is now described by the conditions

\[ \mathcal{R}' : \begin{cases} -\pi \leq \lambda_n < -\pi + \frac{2\pi}{N} & (n = 1, \ldots, k) \\ -\pi + \frac{2\pi}{N} \leq \lambda_n < \pi & (n = k + 1, \ldots, N - 1) \\ -\pi + \frac{2\pi}{N} \leq -\lambda_1 - \ldots - \lambda_{N-1} < \pi \end{cases} \]

If we denote a generic SU\((N)\) Gram interval and its complement by

\[ \mathcal{J} = \left[-\pi, -\pi + \frac{2\pi}{N}\right] \quad \text{and} \quad [-\pi, \pi) \setminus \mathcal{J} = \left[-\pi + \frac{2\pi}{N}, \pi\right) \]

then \(\mathcal{R}'\) becomes

\[ \mathcal{R}' : \begin{cases} \lambda_1, \ldots, \lambda_n \in \mathcal{J} \\ \lambda_{n+1}, \ldots, \lambda_{N-1} \in [-\pi, \pi) \setminus \mathcal{J} \\ -\lambda_1 - \ldots - \lambda_{N-1} \in [-\pi, \pi) \setminus \mathcal{J} \end{cases} \]

Since there is no restriction imposed on \(\lambda_N\), it can be taken \(\lambda_N \in [-\pi, \pi)\)

The old variables \(\theta_n\) can be expressed in terms of the new variables \(\lambda_n\) as:

\[ \theta_n = \lambda_n + (\lambda_1 + \ldots + \lambda_{N-1}) + \frac{\lambda_N}{N} \quad (n = 1, \ldots, N - 1) \]
\[ \theta_N = \frac{\lambda_N}{N} \]

We note that

\[ \theta_m - \theta_n = \lambda_m - \lambda_n \quad \text{for} \quad m, n = 1, \ldots, N - 1 \]

and

\[ \theta_n - \theta_N = \lambda_n + (\lambda_1 + \ldots + \lambda_{N-1}) \quad \text{for} \quad n = 1, \ldots, N - 1 \]
which implies that

\[
D_{U(N)}(\theta_1, \ldots, \theta_N) = \prod_{1 \leq m < n \leq N} |e^{i\theta_m} - e^{i\theta_n}|^2
\]

\[
= 2^{N(N-1)} \prod_{1 \leq m < n \leq N} \left( \sin \frac{\theta_m - \theta_n}{2} \right)^2
\]

\[
= 2^{N(N-1)} \prod_{1 \leq m < n \leq N-1} \left( \sin \frac{\theta_m - \theta_n}{2} \right)^2 \prod_{1 \leq n \leq N-1} \left( \sin \frac{\theta_n - \theta_N}{2} \right)^2
\]

\[
= 2^{N(N-1)} \prod_{1 \leq m < n \leq N-1} \left( \sin \frac{\lambda_m - \lambda_n}{2} \right)^2 \prod_{1 \leq n \leq N-1} \left( \sin \frac{\lambda_n + (\lambda_1 + \ldots + \lambda_{N-1})}{2} \right)^2
\]

\[
= \prod_{1 \leq m < n \leq N-1} |e^{i\lambda_m} - e^{i\lambda_n}|^2 \prod_{1 \leq n \leq N-1} |e^{i\lambda_n} - e^{-i(\lambda_1 + \ldots + \lambda_{N-1})}|^2
\]

\[
\mathcal{D}_{SU(N)}(\lambda_1, \ldots, \lambda_{N-1})
\]

Putting everything together, we obtain that the initial integral (5) can be expressed in the new system of variables as

\[
\left( \frac{N}{k} \right) \frac{1}{N!(2\pi)^N} \int \cdots \int_R D_{U(N)}(\theta_1, \ldots, \theta_N) \, d\theta_1 \ldots d\theta_N
\]

\[
= \left( \frac{N}{k} \right) \frac{1}{N!(2\pi)^N} \int_{-\pi}^{\pi} \left[ \int \cdots \int_{R'} D_{SU(N)}(\lambda_1, \ldots, \lambda_{N-1}) \, d\lambda_1 \ldots d\lambda_{N-1} \right] d\lambda_N
\]

\[
= \left( \frac{N}{k} \right) \frac{1}{N!(2\pi)^{N-1}} \int \cdots \int_{R'} D_{SU(N)}(\lambda_1, \ldots, \lambda_{N-1}) \, d\lambda_1 \ldots d\lambda_{N-1}
\]

Now explicitly putting the conditions in \( R' \) into the integral, this becomes

\[
\left( \frac{N}{k} \right) \frac{1}{N!(2\pi)^{N-1}} \int_{\mathbb{J}^k} \int_{[-\pi, \pi]^{N-1-k}} D_{SU(N)}(\lambda_1, \ldots, \lambda_{N-1}) \times
\]

\[
\times \chi_{[-\pi, \pi] - \mathbb{J}}(-\lambda_1 - \ldots - \lambda_{N-1} \mod 2\pi) \, d\lambda_1 \ldots d\lambda_{N-1}
\]

where \( \chi_I(y) \) denotes the characteristic function, or indicator function, of the interval \( I \).

Since

\[
\int_I \delta(x - y) \, dx = \chi_I(y)
\]

re-introducing the variable \( \lambda_N \) we have that our integral equals

\[
\left( \frac{N}{k} \right) \frac{1}{N!(2\pi)^{N-1}} \int_{\mathbb{J}^k} \int_{[-\pi, \pi]^{N-k}} D_{U(N)}(\lambda_1, \ldots, \lambda_N) \cdot \delta(\lambda_1 + \ldots + \lambda_{N \mod 2\pi}) \, d\lambda_1 \ldots d\lambda_N
\]

\[
= \left( \frac{N}{k} \right) \int_{\mathbb{J}^k} \int_{[-\pi, \pi]^{N-k}} P_{SU(N)}(\lambda_1, \ldots, \lambda_N) \, d\lambda_1 \ldots d\lambda_N
\]

\[
= \left( \frac{N}{k} \right) P(\lambda_1, \ldots, \lambda_k \in \mathbb{J} \text{ and } \lambda_{k+1}, \ldots, \lambda_N \in [-\pi, \pi] \setminus \mathbb{J} \text{ and } \lambda_1 + \ldots + \lambda_N = 0 \text{ mod } 2\pi)
\]

\[
= P(\text{exactly } k \text{ eigenvalues of a } SU(N) \text{ matrix lie in a } SU(N) \text{ Gram interval})
\]

as required. In the last step we have used, as in the beginning, the fact that for any \( k \) the probability of having \( k \) eigenvalues of a \( SU(N) \) matrix in a \( SU(N) \) Gram interval is the same for any \( SU(N) \) Gram interval. \( \square \)
In analogy with the quantity from Fujii’s conjecture, we will denote the later probability from the above lemma by $E_{SU(N)}(k, \mathcal{J})$.

**Definition 6.** We define the probability of having exactly $k$ eigenvalues of a random Haar-distributed $SU(N)$ matrix inside a $SU(N)$ Gram interval $\mathcal{J}$ as

$$E_{SU(N)}(k, \mathcal{J}) := \binom{N}{k} \int_{\mathcal{J}} \int_{(-\pi,\pi) \setminus \mathcal{J}}^{N-k} \mathcal{P}_{SU(N)}(\theta_1, \ldots, \theta_N) \, d\theta_1 \ldots d\theta_N \quad (6)$$

As we have just proved, $E_{SU(N)}(k, \mathcal{J})$ also represents the probability of finding $k$ eigenvalues of a $U(N)$ matrix in a $U(N)$ Gram interval, and by using the above formula, we can compute it numerically for the first few values of $k$ and $N$. This is shown in Table 4.

| $N$ | $E_{SU(N)}(0, \mathcal{J})$ | $E_{SU(N)}(1, \mathcal{J})$ | $E_{SU(N)}(2, \mathcal{J})$ |
|-----|----------------|----------------|----------------|
| 2   | 1.000000       |                 |                 |
| 3   | 0.023074       | 0.954844        | 0.021090        |
| 4   | 0.040362       | 0.920641        | 0.037630        |
| 5   | 0.067146       | 0.867251        | 0.064059        |
| 6   | 0.084764       | 0.832111        | 0.081483        |
| 7   | 0.097237       | 0.807224        | 0.093839        |
| 8   | 0.106532       | 0.788673        | 0.103058        |
| 9   | 0.113727       | 0.774310        | 0.110200        |
| 10  | 0.119461       | 0.762860        | 0.115896        |
| 11  | 0.124138       | 0.753520        | 0.120545        |
| 12  | 0.128026       | 0.745755        | 0.124412        |
| 13  | 0.131309       | 0.739197        | 0.127678        |
| 14  | 0.134118       | 0.733586        | 0.130474        |
| 15  | 0.136549       | 0.728730        | 0.132894        |
| 16  | 0.138673       | 0.724486        | 0.135009        |
| 17  | 0.140545       | 0.720746        | 0.136874        |
| 18  | 0.142207       | 0.717424        | 0.138530        |
| 19  | 0.143693       | 0.714455        | 0.140011        |
| 20  | 0.145029       | 0.711785        | 0.141343        |
| 21  | 0.146237       | 0.709371        | 0.142547        |

Table 4. $E_{SU(N)}(k, \mathcal{J})$ for $k = 0, 1, 2$ and $N = 2, \ldots, 21$

On the one hand, if we compare those values with the corresponding values of $E_{U(N)}(k, J)$ from Table 2, we see that $E_{SU(N)}(k, \mathcal{J})$ has a different rate of convergence, in the sense that it doesn’t converge to its large $N$ limit as fast. On the other hand, if we look at the entries in Table 3, we notice that for each $k$ and $N$, $E_{SU(N)}(k, \mathcal{J})$ does provide a good approximation for $G_{MN, MN+1}(k)$.

The observation that these two quantities appear to have the same rate of convergence hints at the possibility that they should also have the same asymptotic limit, which leads us to put forward the following alternative to Conjecture 1:

**Conjecture 2.** For any $k \in \mathbb{N} \cup \{0\}$

$$\lim_{N \to \infty} E_{SU(N)}(k, \mathcal{J}) = \lim_{M \to \infty} G_{0,M}(k)$$

Finally, we remark that although $E_{SU(N)}(k, \mathcal{J})$ converges at a slower rate compared to $E_{U(N)}(k, J)$, this does not necessarily imply that they don’t tend to the same limit; in
order to clarify whether this is or not the case, we have to obtain a more explicit formula that describes how \( E_{SU(N)}(k, J) \) depends on the matrix size for finite, but arbitrarily large \( N \). This will be the topic of the next section.

3. Explicit Formulas for \( E_{SU(N)}(k, J) \)

3.1. Eigenvalue distribution for \( U(N) \) matrices. One of the main quantities of interest needed for studying how the eigenvalues of random matrices are distributed is the \( n \)-level density, defined as follows:

**Definition 7.** The \( U(N) \) \( n \)-level density (or the \( n \)-point correlation function) of the eigenangles of a unitary matrix is defined as

\[
R^n_{U(N)}(\theta_1, \ldots, \theta_n) := \frac{N!}{(N-n)!} \int_{[-\pi,\pi]^{N-n}} \mathcal{P}_{U(N)}(\theta_1, \ldots, \theta_N) \, d\theta_{n+1} \ldots d\theta_N =
\]

\[
= \frac{1}{(N-n)! (2\pi)^N} \int_{[-\pi,\pi]^{N-n}} \mathcal{D}_{U(N)}(\theta_1, \ldots, \theta_N) \, d\theta_{n+1} \ldots d\theta_N
\]

It represents the probability of finding an eigenangle (regardless of labeling) around each of the points \( \theta_1, \ldots, \theta_n \), and with all the other eigenangles being integrated out.

In order to go from the \( n \)-level density to the number of eigenvalues in an interval, we first note that the Dyson product can be expanded in the form of a trigonometric Fourier series

\[
\mathcal{D}_{U(N)}(\theta_1, \ldots, \theta_N) = \prod_{1 \leq j < k \leq N} |e^{i\theta_j} - e^{i\theta_k}|^2
\]

\[
= \prod_{1 \leq j < k \leq N} [2 - 2 \cos(\theta_j - \theta_k)]
\]

\[
= \sum_{(j_1, \ldots, j_N) \in \mathbb{Z}^N_{j_1 + \ldots + j_N = 0}} c_{j_1,\ldots,j_N} \cos(j_1\theta_1 + \ldots + j_N\theta_N)
\]

where \( c_{j_1,\ldots,j_N} \neq 0 \) if and only if the corresponding \( j_1, \ldots, j_N \in \mathbb{Z} \) satisfy the conditions \( j_1 + \ldots + j_N = 0 \) and \( |j_l| < N \) for all \( l = 1, \ldots, N \). This implies that the \( n \)-level density becomes

\[
R^n_{U(N)}(\theta_1, \ldots, \theta_n)
\]

\[
= \frac{1}{(N-n)! (2\pi)^N} \sum_{(j_1, \ldots, j_N) \in \mathbb{Z}^N_{j_1 + \ldots + j_N = 0}} c_{j_1,\ldots,j_N} \int_{[-\pi,\pi]^{N-n}} \cos(j_1\theta_1 + \ldots + j_N\theta_N) \, d\theta_{n+1} \ldots d\theta_N
\]

The above expression can be simplified with the following observation: for each cosine term in the sum, if any of the coefficients \( j_m \) corresponding to the integration variable \( \theta_m \) (that is, for \( m = n+1, \ldots, N \)) is non-zero, then the integral over that term is zero, since the cosine is being integrated over a full period. This implies that the only the terms in the sum that give non-zero contributions to the \( n \)-level density are those terms for which \( j_{n+1} = \ldots = j_N = 0 \) or, in other words, the terms that depend only on the eigenangles \( \theta_1, \ldots, \theta_n \), together with the constant term \( c_{0,\ldots,0} \).

In the particular case of the \( U(N) \) one-level density, because there is no cosine term in \( \mathcal{D}_{U(N)}(\theta_1, \ldots, \theta_N) \) that depends only on \( \theta_1 \), the constant term \( c_{0,\ldots,0} \) is the one that makes
the only non-zero contribution. It is known that $c_0, \ldots, c_0 = N!$, which implies
\[
\mathcal{D}_{U(N)}(\theta_1, \ldots, \theta_N) = N! + \ldots
\]
which implies that
\[
R_{U(N)}^1(\theta_1) = \frac{N}{2\pi}
\]
Similarly, for the $U(N)$ two-level density, we get a contribution from the constant term, together with the terms that depend only on $\theta_1, \theta_2$, which are
\[
\mathcal{D}_{U(N)}(\theta_1, \ldots, \theta_N) = N! - (N-2)! \cdot 2 \sum_{a=1}^{N-1} (N-a) \cos(a\theta_1 - a\theta_2) + \ldots
\]
which in turn implies that
\[
R_{U(N)}^2(\theta_1, \theta_2) = \frac{1}{(2\pi)^2} \left[ (N-1)N - 2 \sum_{a=1}^{N-1} (N-a) \cos(a\theta_1 - a\theta_2) \right]
\]
With this approach, exact knowledge of the $U(N)$ $n$-level density requires exact knowledge of the coefficients of all terms in $\mathcal{D}_{U(N)}(\theta_1, \ldots, \theta_N)$ that depend only on $\theta_1, \ldots, \theta_n$, but in practice, these coefficients become more difficult to compute explicitly as $n$ increases.

Remark. For $U(N)$ it is well known that there is a much more elegant approach for finding the $n$-level densities, for example, as described in Conrey’s survey paper [3]. However, that approach does not work for $SU(N)$ matrices, essentially because there appears to be no determinant form of the probability density of eigenvalues.

In addition to the $n$-level density, the other important quantity required for the study of eigenvalue distribution is the so-called generating function.

**Definition 8.** If $J \subset [-\pi, \pi)$ is an arbitrary interval on the unit circle of any length and $E_{U(N)}(k, J)$ is defined as in (4), the $U(N)$ generating function is given by
\[
\mathcal{E}_{U(N)}(z, J) := \sum_{k=0}^{N} (1 + z)^k E_{U(N)}(k, J)
\]
From the generating function one can immediately recover the desired probabilities by
\[
E_{U(N)}(k, J) = \frac{1}{k!} \left( \frac{d^k}{dz^k} \mathcal{E}_{U(N)}(z, J) \right) \bigg|_{z=-1}
\]
It can be shown (see, for example, [3]) that the generating function can also be expressed in terms of all the $n$-level densities as
\[
\mathcal{E}_{U(N)}(z, J) = 1 + \sum_{n=1}^{N} \frac{z^n}{n!} \int_{J^n} R_{U(N)}^n(\theta_1, \ldots, \theta_n) \ d\theta_1 \ldots d\theta_n
\] (7)
However, in the case of $U(N)$ (which does not hold for $SU(N)$) an identity that is due to Gram shows that this sum is equal to a $N \times N$ determinant
\[
1 + \sum_{n=1}^{N} \frac{z^n}{n!} \int_{J^n} R_{U(N)}^n(\theta_1, \ldots, \theta_n) \ d\theta_1 \ldots d\theta_n = \det_{N \times N} \left[ I_N + \frac{z}{2\pi} \left( \int_{J} e^{i(m-n)\theta} d\theta \right) \right]_{1 \leq m, n \leq N}
\]
3.2. **Eigenvalue distribution for SU(N) matrices.** We now proceed to extend the notions defined above to SU(N) matrices, in order to derive a formula for $E_{SU(N)}(k, J)$.

**Definition 9.** We define the SU(N) $n$-level density of a special unitary matrix as

$$
R^n_{SU(N)}(\theta_1, \ldots, \theta_n) := \frac{N!}{(N-n)!} \int_{[-\pi, \pi]^N-N} \mathcal{D}_{SU(N)}(\theta_1, \ldots, \theta_N) \, d\theta_{n+1} \cdots d\theta_N
$$

This time, we can only have $n = 1, \ldots, N-1$, because if the values of $N-1$ eigenvalues are given, then the $N$-th one is already determined by the restriction $\theta_1 + \ldots + \theta_N = 0 \mod 2\pi$, so it can not also be fixed to an arbitrary value.

As in the U(N) case, knowledge of the SU(N) $n$-level density depends on knowledge of the coefficients of all terms in $\mathcal{D}_{SU(N)}(\theta_1, \ldots, \theta_{N-1})$ that depend only on $\theta_1, \ldots, \theta_n$ which, in turn, depends on knowledge of the corresponding terms in $\mathcal{D}_{U(N)}(\theta_1, \ldots, \theta_N)$ before the substitution $\theta_N = -\theta_1 - \ldots - \theta_{N-1}$ was made.

For example, to obtain the SU(N) one-level density, we first remark that

$$
\mathcal{D}_{U(N)}(\theta_1, \ldots, \theta_N) = N! + 2(-1)^{N-1}(N-1)! \cos((N-1)\theta_1 - \theta_2 - \ldots - \theta_N) + \ldots
$$

which, after the substitution $\theta_N = -\theta_1 - \ldots - \theta_{N-1}$ becomes

$$
\mathcal{D}_{SU(N)}(\theta_1, \ldots, \theta_{N-1}) = N! + 2(-1)^{N-1}(N-1)! \cos(N\theta_1) + \ldots
$$

So we see that, unlike $\mathcal{D}_{U(N)}(\theta_1, \ldots, \theta_N)$, the $\mathcal{D}_{SU(N)}(\theta_1, \ldots, \theta_{N-1})$ does contain a term that depends only on $\theta_1$. And because $\mathcal{D}_{SU(N)}(\theta_1, \ldots, \theta_{N-1})$ has the same constant term as $\mathcal{D}_{U(N)}(\theta_1, \ldots, \theta_N)$, we can express the SU(N) one-level density as the U(N) one-level density plus the integral of that additional term

$$
R^n_{SU(N)}(\theta_1) = \frac{N}{2\pi} + \frac{2(-1)^{N-1}\cos(N\theta_1)}{2\pi}
$$

Similarly, in order to compute the SU(N) two-level density, it can be shown that $\mathcal{D}_{SU(N)}(\theta_1, \ldots, \theta_{N-1})$ contains all the terms in $\mathcal{D}_{U(N)}(\theta_1, \ldots, \theta_N)$ that depended only on $\theta_1, \theta_2$ plus several additional ones, given by

$$
\mathcal{D}_{SU(N)}(\theta_1, \ldots, \theta_{N-1})
= \left[N! - (N-2)! \sum_{a=1}^{N-1} 2(N-a) \cos(a\theta_1 - a\theta_2) \right] + 4(N-2)! \cos(N\theta_1 + N\theta_2) + 2(-1)^{N-1}(N-1)! \cos(N\theta_1) + 2(-1)^{N-1}(N-1)! \cos(N\theta_2)
$$

$$
+ 4(-1)^{N-2}(N-2)! \sum_{k_1=1 \atop k_2=1}^{N-1} \sum_{k_1+k_2=N} \cos(k_1\theta_1 + k_2\theta_2) + \ldots
$$
This, again, leads to the fact that the $SU(N)$ two-level density can be written in terms of the $U(N)$ two-level density plus an additional contribution

$$R_{SU(N)}^2(\theta_1, \theta_2) = R_{U(N)}^2(\theta_1, \theta_2) + \frac{2}{(2\pi)^2}[2\cos(N\theta_1 + N\theta_2) -$$

$$- \cos((N+1)\theta_1 + (N-1)\theta_2) - \cos((N-1)\theta_1 + (N+1)\theta_2) +$$

$$+ (-1)^{N-1}(N-1)\cos(N\theta_1) + (-1)^{N-1}(N-1)\cos(N\theta_2) +$$

$$+ 2(-1)^{N-2}\sum_{k_1=1}^{N-1}\sum_{k_2=1}^{N-1}\cos(k_1\theta_1 + k_2\theta_2)]$$

The above observations can be summarized and generalized in the following way: let $n = 1, \ldots, N-1$ be fixed. All the terms in $D_{U(N)}(\theta_1, \ldots, \theta_N)$ that depend only on $\theta_1, \ldots, \theta_n$ also appear unchanged in $D_{SU(N)}(\theta_1, \ldots, \theta_{N-1})$ and this, in turn, implies that all the terms of the $U(N)$ $n$-level density are always included among the terms of the $SU(N)$ $n$-level density. Furthermore, for any $m = 1, \ldots, n$ there are other terms in $D_{U(N)}(\theta_1, \ldots, \theta_N)$ that depend on all the eigenangles, such as

$$\cos(j_1\theta_1 + \ldots + j_n\theta_n - m\theta_{n+1} - \ldots - m\theta_N), \quad j_1 + \ldots + j_n = m(N-n)$$

which, after the substitution $\theta_N = -\theta_1 - \ldots - \theta_{N-1}$, depend only on $\theta_1, \ldots, \theta_n$

$$\cos((j_1 + m)\theta_1 + \ldots + (j_n + m)\theta_n)$$

In other words, these are terms that do not appear in the $U(N)$ $n$-level density (those are the terms when $m = 0$), but contribute the $SU(N)$ $n$-level density; we will denote all these extra terms collectively by

$$X_{SU(N)}^n(\theta_1, \ldots, \theta_n) := \sum_{m=1}^{n-1} \sum_{(j_1, \ldots, j_n) \in \mathbb{Z}^n, \ j_1 + \ldots + j_n = m(N-n)} \cos((j_1 + m)\theta_1 + \ldots + (j_n + m)\theta_n) \cdot \frac{2 \cdot c_{j_1+m \cdots j_n+m}^{(N,n)}}{(2\pi)^n}$$

We will also relabel the indexes as $k_i = j_i + m$, so that

$$X_{SU(N)}^n(\theta_1, \ldots, \theta_n) = \frac{2}{(2\pi)^n} \sum_{m=1}^{n-1} \sum_{(k_1, \ldots, k_n) \in \mathbb{Z}^n, \ k_1 + \ldots + k_n = mN} \cos(k_1\theta_1 + \ldots + k_n\theta_n) \cdot c_{k_1 \ldots k_n}^{(N,n)}$$

It can be shown that for any $N > n \geq 2$, uniformly in $k_1, \ldots, k_n$ the coefficients of $X_{SU(N)}^n$ are at most

$$|c_{k_1 \ldots k_n}^{(N,n)}| \leq (N-1)(N-2) \ldots (N-n+1)$$

We conclude the discussion about $SU(N)$ $n$-level densities by restating that we have

$$R_{SU(N)}^n(\theta_1, \ldots, \theta_n) = R_{U(N)}^n(\theta_1, \ldots, \theta_n) + X_{SU(N)}^n(\theta_1, \ldots, \theta_n), \quad n = 1, \ldots, N-1$$

with explicit bounds on the coefficients in $X_{SU(N)}^n$ for all $n$.

For later notional simplicity, we also extend the definition of $X_{SU(N)}^n$ to the case $n = N$ as

$$X_{SU(N)}^n(\theta_1, \ldots, \theta_n) = \begin{cases} R_{SU(N)}^n(\theta_1, \ldots, \theta_n) - R_{U(N)}^n(\theta_1, \ldots, \theta_n), & \text{for } n = 1, \ldots, N-1 \\ N!P_{SU(N)}(\theta_1, \ldots, \theta_N) - R_{U(N)}^N(\theta_1, \ldots, \theta_N), & \text{for } n = N \end{cases}$$

(9)
We can continue the analogy with the U(N) quantities by studying the SU(N) generating function, first for an arbitrary interval, and then for a SU(N) Gram interval.

**Definition 10.** If $J \subset [-\pi, \pi)$ is an arbitrary interval on the unit circle of any length and $E_{SU(N)}(k, J)$ is defined as in (9), the SU(N) generating function is given by

$$E_{SU(N)}(z, J) := \sum_{k=0}^{N} (1 + z)^k E_{SU(N)}(k, J)$$

As in the U(N) case, we can obtain a formula for $E_{SU(N)}(k, J)$ if we know the generating function explicitly

$$E_{SU(N)}(z, J) = \frac{1}{k!} \left. \left( \frac{d^k}{dz^k} E_{SU(N)}(z, J) \right) \right|_{z=-1}$$

We will now use the fact that each SU(N) level density depends on the corresponding U(N) level density to prove that the SU(N) generating function can also be expressed in terms of the U(N) generating function.

**Theorem.** For an arbitrary interval $J \subset [-\pi, \pi)$ on the unit circle of any length, we have that

$$E_{SU(N)}(z, J) = E_{U(N)}(z, J) + \sum_{n=1}^{N} \frac{z^n}{n!} \int_{J^n} X_{SU(N)}^n(\theta_1, \ldots, \theta_n) d\theta_1 \ldots d\theta_n \quad (10)$$

where $X_{SU(N)}^n$ is defined in (9).

**Proof.** First, we denote the SU(N) Haar measure by $d\mu_{SU(N)} = P_{SU(N)}(\theta_1, \ldots, \theta_N) d\theta_1 \ldots d\theta_N$ and show that

$$\int_{SU(N)} (1 + z\chi_{J}(\theta_1)) \ldots (1 + z\chi_{J}(\theta_N)) d\mu_{SU(N)} = \sum_{k=0}^{N} (1 + z)^k E_{SU(N)}(k, J)$$

(where, again, $\chi_{J}(\theta)$ represents the characteristic function of the interval $J$.)

For $k = 0, \ldots, N$, let $P_k \subset SU(N)$ be the subset of all SU(N) matrices with exactly $k$ eigenangles in $J$. These sets are all pairwise disjoint and their union is SU(N), so they form a partition of SU(N). This implies that

$$\int_{SU(N)} (1 + z\chi_{J}(\theta_1)) \ldots (1 + z\chi_{J}(\theta_N)) d\mu_{SU(N)}$$

$$= \int_{P_0 \cup \ldots \cup P_N} (1 + z\chi_{J}(\theta_1)) \ldots (1 + z\chi_{J}(\theta_N)) d\mu_{SU(N)}$$

$$= \sum_{k=0}^{N} \int_{P_k} (1 + z\chi_{J}(\theta_1)) \ldots (1 + z\chi_{J}(\theta_N)) d\mu_{SU(N)}$$

$$= \int_{P_0} d\mu_{SU(N)} + (1 + z) \int_{P_1} d\mu_{SU(N)} + \ldots + (1 + z)^N \int_{P_N} d\mu_{SU(N)}$$

and, by definition, $E_{SU(N)}(k, J)$ represents the measure of the set of SU(N) matrices which have precisely $k$ eigenangles in $J$, so

$$E_{SU(N)}(k, J) = \int_{P_k} d\mu_{SU(N)}$$
That is, we have shown that

$$\mathcal{E}_{SU(N)}(z, J) = \int_{SU(N)} (1 + z \chi_J(\theta_1)) \cdots (1 + z \chi_J(\theta_N)) \, d\mu_{SU(N)}$$

$$= \int_{[-\pi, \pi]^N} (1 + z \chi_J(\theta_1)) \cdots (1 + z \chi_J(\theta_N)) \, \mathcal{P}_{SU(N)}(\theta_1, \ldots, \theta_N) \, d\theta_1 \cdots d\theta_N$$

After we open all the brackets, this becomes

$$\mathcal{E}_{SU(N)}(z, J) = 1 + \sum_{n=1}^{N} z^n \int_{[-\pi, \pi]^N} h_n(\chi_J(\theta_1), \ldots, \chi_J(\theta_N)) \, \mathcal{P}_{SU(N)}(\theta_1, \ldots, \theta_N) \, d\theta_1 \cdots d\theta_N$$

where the $h_n$'s are elementary symmetric polynomials in $\chi_J(\theta_1), \ldots, \chi_J(\theta_N)$

$$\begin{cases} h_1 = \chi_J(\theta_1) + \cdots + \chi_J(\theta_N) \\ h_2 = \sum_{1 \leq i < j \leq N} \chi_J(\theta_i) \chi_J(\theta_j) \\ \cdots \cdots \cdots \\ h_N = \chi_J(\theta_1) \cdots \chi_J(\theta_N) \end{cases}$$

Because each $h_n$ has $\binom{N}{n}$ terms, and the probability density function $\mathcal{P}_{SU(N)}(\theta_1, \ldots, \theta_N)$

is invariant under the permutation of any of its arguments, the SU($N$) generating function, $\mathcal{E}_{SU(N)}(z, J)$, becomes

$$1 + \sum_{n=1}^{N} z^n \int_{[-\pi, \pi]^N} \chi_J(\theta_1) \cdots \chi_J(\theta_n) \, \mathcal{P}_{SU(N)}(\theta_1, \ldots, \theta_N) \, d\theta_1 \cdots d\theta_N =$$

$$= 1 + \sum_{n=1}^{N} \frac{z^n}{n! (N-n)!} \int_{[-\pi, \pi]^{N-n}} \chi_J(\theta_1) \cdots \chi_J(\theta_n) \, \mathcal{P}_{SU(N)}(\theta_1, \ldots, \theta_N) \, d\theta_1 \cdots d\theta_N =$$

$$= 1 + \sum_{n=1}^{N} \frac{z^n}{n!} \int_{J^n} \left[ \frac{N!}{(N-n)!} \int_{[-\pi, \pi]^{N-n}} \mathcal{P}_{SU(N)}(\theta_1, \ldots, \theta_N) \, d\theta_{n+1} \cdots d\theta_N \right] \, d\theta_1 \cdots d\theta_n =$$

$$= 1 + \sum_{n=1}^{N-1} \frac{z^n}{n!} \int_{J^n} R^n_{SU(N)}(\theta_1, \ldots, \theta_n) \, d\theta_1 \cdots d\theta_n + \frac{z^N}{N!} \int_{J^N} N! \mathcal{P}_{SU(N)}(\theta_1, \ldots, \theta_N) \, d\theta_1 \cdots d\theta_N$$

where in the last line we use (8) to obtain $R^n_{SU(N)}$ for $n = 1, \ldots, N-1$ and leave $n = N$ alone. Now if we use (9), we find that $\mathcal{E}_{SU(N)}(z, J)$ equals

$$1 + \sum_{n=1}^{N-1} \frac{z^n}{n!} \int_{J^n} [R^n_{U(N)}(\theta_1, \ldots, \theta_n) + X^n_{SU(N)}(\theta_1, \ldots, \theta_n)] \, d\theta_1 \cdots d\theta_n +$$

$$+ \frac{z^N}{N!} \int_{J^N} [R^N_{U(N)}(\theta_1, \ldots, \theta_N) + X^N_{SU(N)}(\theta_1, \ldots, \theta_N)] \, d\theta_1 \cdots d\theta_N$$
and re-grouping the terms, this equals

\[ 1 + \sum_{n=1}^{N} \frac{z^n}{n!} \int_{J_n} R^n_{U(N)}(\theta_1, \ldots, \theta_n) \, d\theta_1 \ldots d\theta_n + \sum_{n=1}^{N} \frac{z^n}{n!} \int_{J_n} X^n_{SU(N)}(\theta_1, \ldots, \theta_n) \, d\theta_1 \ldots d\theta_n \]

and noting the first term is the \(U(N)\) generating function, we have shown that

\[ \mathcal{E}_{SU(N)}(z, J) = \mathcal{E}_{U(N)}(z, J) + \sum_{n=1}^{N} \frac{z^n}{n!} \int_{J_n} X^n_{SU(N)}(\theta_1, \ldots, \theta_n) \, d\theta_1 \ldots d\theta_n \]

which was our desired result.

3.3. Application to \(SU(N)\) Gram intervals. As previously mentioned, up to this point \(J \subset [-\pi, \pi]\) was assumed to be an arbitrary interval on the unit circle of any length. We conclude this section with a brief discussion on how \(\mathcal{E}_{SU(N)}(z, J)\) and \(E_{SU(N)}(k, J)\) are affected in the particular case when \(J\) is taken to be a \(SU(N)\) Gram interval, for example taking \(J = \left[-\pi, -\pi + \frac{2\pi}{N}\right]\).

First, it can be easily seen that the integral of the \(X^1_{SU(N)}(\theta)\) term over \(J\) will always be zero

\[ X^1_{SU(N)}(\theta) = \frac{2(-1)^{N-1} \cos(N\theta)}{2\pi} \implies \int_{J} X^1_{SU(N)}(\theta) \, d\theta = 0 \]

so in this case of \(SU(N)\) Gram intervals, the sum in (10) starts from \(n = 2\)

\[ \mathcal{E}_{SU(N)}(z, J) = \mathcal{E}_{U(N)}(z, J) + \sum_{n=2}^{N} \frac{z^n}{n!} \int_{J_n} X^n_{SU(N)}(\theta_1, \ldots, \theta_n) \, d\theta_1 \ldots d\theta_n \]

We recall from the formula for the \(SU(N)\) two-level density that the \(X^2_{SU(N)}(\theta_1, \theta_2)\) term is explicitly given by

\[ X^2_{SU(N)}(\theta_1, \theta_2) = \frac{2}{(2\pi)^2} \left[ 2\cos(N\theta_1 + N\theta_2) - \cos((N+1)\theta_1 + (N-1)\theta_2) \right. \\
\left. - \cos((N-1)\theta_1 + (N+1)\theta_2) + (-1)^{N-1}(N-1) \cos(N\theta_1) + \\
\right. \\
\left. + (-1)^{N-1}(N-1) \cos(N\theta_2) + 2(-1)^{N-2} \sum_{k_1=1}^{N-1} \sum_{k_2=1}^{N-1} \cos(k_1\theta_1 + k_2\theta_2) \right] \]

This can be used to show that its integral over a pair of \(SU(N)\) Gram intervals is given by

\[ \int_{J^2} X^2_{SU(N)}(\theta_1, \theta_2) \, d\theta_1 d\theta_2 = \frac{4}{\pi^2} \left[ \frac{1}{N^2 - 1} \left( \sin \frac{\pi}{N} \right)^2 - \sum_{k_1=1}^{N-1} \sum_{k_2=1}^{N-1} \frac{1}{k_1 k_2} \sin \left( \frac{k_1 \pi}{N} \right) \sin \left( \frac{k_2 \pi}{N} \right) \right] \]

We remark that the double sum can be re-expressed as a single sum

\[ \sum_{k_1=1}^{N-1} \sum_{k_2=1}^{N-1} \frac{1}{k_1 k_2} \sin \left( \frac{k_1 \pi}{N} \right) \sin \left( \frac{k_2 \pi}{N} \right) = \sum_{k=1}^{N-1} \frac{1}{k(N-k)} \sin \left( \frac{k\pi}{N} \right) \sin \left( \frac{\pi - k\pi}{N} \right) \]

\[ = \frac{2}{N} \sum_{k=1}^{N} \frac{1}{k} \left( \sin \frac{k\pi}{N} \right)^2 \]
and this sum can be approximated by an integral, using Euler-Maclaurin summation

\[
\sum_{k=1}^{N} \frac{1}{k} \left( \sin \frac{k\pi}{N} \right)^2 = \int_{1}^{N} \frac{1}{x} \left( \sin \frac{x\pi}{N} \right)^2 \, dx + O \left( \frac{1}{N^2} \right) = \int_{\frac{1}{N}}^{1} \frac{\sin y\pi}{y} \, dy + O \left( \frac{1}{N^2} \right)
\]

The integral, in turn, can be expressed in terms of the cosine integral function \( \text{Ci}(x) \)

\[
\int_{\frac{1}{N}}^{1} \frac{(\sin y\pi)^2}{y} \, dy = \frac{1}{2} \left[ \text{Ci} \left( \frac{2\pi}{N} \right) - \text{Ci}(2\pi) - \log \left( \frac{1}{N} \right) \right]
\]

where \( \text{Ci}(x) \) is defined as

\[
\text{Ci}(x) := - \int_{x}^{\infty} \frac{\cos t}{t} \, dt = \gamma + \log x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2n(2n)!}
\]

This implies that the above sum is equal to

\[
\sum_{k=1}^{N} \frac{1}{k} \left( \sin \frac{k\pi}{N} \right)^2 = \frac{1}{2} [\gamma + \log(2\pi) - \text{Ci}(2\pi)] + O \left( \frac{1}{N^2} \right)
\]

while the double sum becomes

\[
\sum_{k_{1}=1}^{N-1} \sum_{k_{2}=1}^{N} \frac{1}{k_{1}k_{2}} \sin \left( \frac{k_{1}\pi}{N} \right) \sin \left( \frac{k_{2}\pi}{N} \right) = \frac{1}{N} [\gamma + \log(2\pi) - \text{Ci}(2\pi)] + O \left( \frac{1}{N^{3}} \right)
\]

In the case of the first term in the integral of \( X_{SU(N)}^{2}(\theta_{1}, \theta_{2}) \), as \( N \) increases, it has the order

\[
\frac{1}{N^2 - 1} \left( \sin \frac{\pi}{N} \right)^2 = O \left( \frac{1}{N^{4}} \right)
\]

Putting the previous results together, we obtain that the contribution coming from the integral of \( X_{SU(N)}^{2}(\theta_{1}, \theta_{2}) \) is

\[
\int_{\mathcal{J}^2} X_{SU(N)}^{2}(\theta_{1}, \theta_{2}) \, \, d\theta_{1}d\theta_{2} = -\frac{\alpha}{N} + O \left( \frac{1}{N^{3}} \right)
\]

where

\[
\alpha = \frac{4[\gamma + \log(2\pi) - \text{Ci}(2\pi)]}{\pi^2} = 0.987944 \ldots
\]

Numerical results suggest that the value of the integral of \( X_{SU(N)}^{2}(\theta_{1}, \ldots, \theta_{n}) \) over \( \mathcal{J}^n \) decreases by several orders of magnitude as \( n = 2, \ldots, N \) increases, which implies that the integral of the \( X_{SU(N)}^{2}(\theta_{1}, \theta_{2}) \) term gives the main error to the SU(\( N \)) generating function.

Note that the integral of \( X_{SU(N)}^{2}(\theta_{1}, \theta_{2}) \) appears explicitly only in \( E_{SU(N)}(k, \mathcal{J}) \), for \( k = 0, 1, 2 \). Writing these out explicitly, we have the following approximations for these probabilities

\[
E_{SU(N)}(0, \mathcal{J}) = \mathcal{E}_{SU(N)}(z, \mathcal{J}) \bigg|_{z=-1} = \left( \mathcal{E}_{U(N)}(z, \mathcal{J}) + \frac{z^2}{2} \int_{\mathcal{J}^2} X_{SU(N)}^{2} \, d\theta_{1}d\theta_{2} + \ldots \right) \bigg|_{z=-1} = E_{U(N)}(0, \mathcal{J}) + \frac{1}{2} \int_{\mathcal{J}^2} X_{SU(N)}^{2} \, d\theta_{1}d\theta_{2} + \ldots \approx E_{U(N)}(0, \mathcal{J}) - \frac{\alpha}{2N} \frac{1}{20}
\]
and

\[ E_{SU(N)}(1, J) = \frac{d}{dz} E_{SU(N)}(z, J) \bigg|_{z=-1} = \frac{d}{dz} \left( E_{U(N)}(z, J) + \frac{z^2}{2} \int_{\mathcal{J}} X_{SU(N)}^2 \, d\theta_1 d\theta_2 + \ldots \right) \bigg|_{z=-1} = E_{U(N)}(1, J) - \int_{\mathcal{J}} X_{SU(N)}^2 \, d\theta_1 d\theta_2 + \ldots \approx E_{U(N)}(1, J) + \frac{\alpha}{N} \]

and

\[ E_{SU(N)}(2, J) = \frac{1}{2} \frac{d^2}{dz^2} E_{SU(N)}(z, J) \bigg|_{z=-1} = \frac{1}{2} \frac{d^2}{dz^2} \left( E_{U(N)}(z, J) + \frac{z^2}{2} \int_{\mathcal{J}} X_{SU(N)}^2 \, d\theta_1 d\theta_2 + \ldots \right) \bigg|_{z=-1} = E_{U(N)}(2, J) + \frac{1}{2} \int_{\mathcal{J}} X_{SU(N)}^2 \, d\theta_1 d\theta_2 + \ldots \approx E_{U(N)}(2, J) - \frac{\alpha}{2N} \]

So in conclusion, we note that in the large $N$ limit, the $SU(N)$-probability of finding 0, 1, or 2 eigenvalues in a $SU(N)$ Gram interval converges to the $U(N)$-probability of finding 0, 1, or 2 eigenvalues in an arbitrary interval of length $\frac{2\pi}{N}$, and we’ve estimated the rate of convergence. This supports a belief of Odlyzko [22], who said that

“it seems reasonable to expect that at large heights the local distribution of the zeros will be independent of Gram points, which leads to the above assumption. In other words, the expectation is that at large heights, any grid of points spaced like the Gram points would exhibit similar behavior with respect to location of zeros.”

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