HEAT KERNELS ON SMOOTH METRIC MEASURE SPACES WITH NONNEGATIVE CURVATURE

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Abstract. We derive a local Gaussian upper bound for the $f$-heat kernel on complete smooth metric measure space $(M, g, e^{-f}dv)$ with nonnegative Bakry-Émery Ricci curvature, which generalizes the classic Li-Yau estimate. As applications, we obtain a sharp $L^1_f$-Liouville theorem for $f$-subharmonic functions and an $L^1_f$-uniqueness property for nonnegative solutions of the $f$-heat equation, assuming $f$ is of at most quadratic growth. In particular, any $L^1_f$-integrable $f$-subharmonic function on gradient shrinking and steady Ricci solitons must be constant. We also provide explicit $f$-heat kernels on Gaussian solitons.

1. Introduction and main results

In this paper we study Gaussian upper estimates for the $f$-heat kernel on smooth metric measure spaces with nonnegative Bakry-Émery Ricci curvature and their applications. Recall that a complete smooth metric measure space is a triple $(M, g, e^{-f}dv)$, where $(M, g)$ is an $n$-dimensional complete Riemannian manifold, $dv$ is the volume element of $g$, $f$ is a smooth function on $M$, and $e^{-f}dv$ (for short, $d\mu$) is called the weighted volume element or the weighted measure. The $m$-Bakry-Émery Ricci curvature [1] associated to $(M, g, e^{-f}dv)$ is defined by

$$Ric^m_f := Ric + \nabla^2 f - \frac{1}{m} df \otimes df,$$

where $Ric$ is the Ricci curvature of the manifold, $\nabla^2$ is the Hessian with respect to the metric $g$ and $m$ is a positive constant. We refer the readers to [2], [24], and [25] for further details. When $m = \infty$, we write $Ric_f = Ric^\infty_f$. Smooth metric measure spaces are closely related to gradient Ricci solitons, the Ricci flow, probability theory, and optimal transport. A smooth metric measure space $(M, g, e^{-f}dv)$ is said to be quasi-Einstein if

$$Ric^m_f = \lambda g$$

for some constant $\lambda$. When $m = \infty$, it is exactly a gradient Ricci soliton. A gradient Ricci soliton is called expanding, steady or shrinking if $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$, respectively. Ricci solitons are natural extensions of...
Einstein manifolds and have drawn more and more attentions. See [5] for a nice survey and references therein.

The associated $f$-Laplacian $\Delta_f$ on a smooth metric measure space is defined as

$$\Delta_f := \Delta - \nabla f \cdot \nabla,$$

which is self-adjoint with respect to the weighted measure. On a smooth metric measure space, it is natural to consider the $f$-heat equation

$$(\partial_t - \Delta_f)u = 0$$

instead of the heat equation. If $u$ is independent of time $t$, then $u$ is a $f$-harmonic function. Throughout this paper we denote by $H(x,y,t)$ the $f$-heat kernel, that is, for each $y \in M$, $H(x,y,t) = u(x,t)$ is the minimal positive solution of the $f$-heat equation with $\lim_{t \to 0} u(x,t) = \delta_{f,y}(x)$, where $\delta_{f,y}(x)$ is defined by

$$\int_M \phi(x)\delta_{f,y}(x)e^{-f}dv = \phi(y)$$

for $\phi \in C^\infty_0(M)$. Equivalently, $H(x,y,t)$ is the kernel of the semigroup $P_t = e^{t\Delta_f}$ associated to the Dirichlet energy $\int_M |\nabla \phi|^2 e^{-f}dv$, where $\phi \in C^\infty_0(M)$. In general the $f$-heat kernel always exists on complete smooth metric measure spaces, but it may not be unique.

When $f$ is constant, then $H(x,y,t)$ is just the heat kernel for the Riemannian manifold $(M,g)$. Cheng, Li and Yau [10] obtained uniform Gaussian estimates for the heat kernel on Riemannian manifolds with sectional curvature bounded below, which was later extended by Cheeger, Gromov and Taylor [9] to manifolds with bounded geometry. In 1986, Li and Yau [22] proved sharp Gaussian upper and lower bounds on Riemannian manifolds of nonnegative Ricci curvature, using the gradient estimate and the Harnack inequality. Grigor’yan and Saloff-Coste [15, 30, 31, 32] independently proved similar estimates on Riemannian manifolds satisfying the volume doubling property and the Poincaré inequality, using the Moser iteration technique. Davies [13] further developed Gaussian upper bounds under a mean value property assumption. Recently, Li and Xu [18] also obtained some new estimates on complete Riemannian manifolds with Ricci curvature bounded from below by further improving the Li-Yau gradient estimate.

Recently, there has been several work on $f$-heat kernel estimates on smooth metric measure spaces and its applications. In [23], X.-D. Li obtained Gaussian estimates for the $f$-heat kernel, and proved an $L^1_f$-Liouville theorem, assuming $\text{Ric}_f^m (m < \infty)$ bounded below by a negative quadratic function, which generalizes a classical result of P. Li [19]. He also mentioned that we may not be able to prove an $L^1_f$-Liouville theorem only assuming a lower bound on $\text{Ric}_f$. The main difficulty is the lack of effective upper bound for the $f$-heat kernel. In [8], by analyzing the heat kernel for a family of warped product manifolds, Charalambous and Lu also gave $f$-heat kernel estimates when $\text{Ric}_f^m (m < \infty)$ is bounded below. In [31], the first author
proved $f$-heat kernel estimates assuming $Ric_f$ bounded below by a negative constant and $f$ bounded.

In this paper we prove a local Gaussian upper bound for the $f$-heat kernel on smooth metric measure spaces with $Ric_f \geq 0$, which generalizes the classical result of Li-Yau [22].

**Theorem 1.1.** Let $(M, g, e^{-f} dv)$ be an $n$-dimensional complete noncompact smooth metric measure space with $Ric_f \geq 0$. Fix a fixed point $o \in M$ and $R > 0$. For any $\epsilon > 0$, there exist constants $c_1(n, \epsilon)$ and $c_2(n)$, such that

$$H(x, y, t) \leq \frac{c_1(n, \epsilon) e^{c_2(n) A(R)}}{V_f(B_x(\sqrt{t}))^{1/2} V_f(B_y(\sqrt{t}))^{1/2}} \times \exp \left( -\frac{d^2(x, y)}{(4 + \epsilon)t} \right)$$

for all $x, y \in B_o(\frac{1}{2}R)$ and $0 < t < R^2/4$, where $\lim_{t \to 0} c_1(n, \epsilon) = \infty$. In particular, there exist constants $c_3(n, \epsilon)$ and $c_4(n)$, such that

$$H(x, y, t) \leq \frac{c_3(n, \epsilon) e^{c_4(n) A(R)}}{V_f(B_x(\sqrt{t}))} \cdot \left( \frac{d(x, y)}{\sqrt{t}} + 1 \right)^{\frac{1}{2}} \times \exp \left( -\frac{d^2(x, y)}{(4 + \epsilon)t} \right)$$

for any $x, y \in B_o(\frac{1}{4}R)$ and $0 < t < R^2/4$, where $\lim_{t \to 0} c_3(n, \epsilon) = \infty$. Here $A(R) := \sup_{x \in B_a(3R)} |f(x)|$.

As pointed out by Munteanu-Wang [28], only assuming $Ric_f \geq 0$ may not be sufficient to derive $f$-heat kernel estimates by classical Li-Yau gradient estimate procedure [22]. But we can derive a Gaussian upper bound using the De Giorgi-Nash-Moser theory and the weighted version of Davies’s integral estimate [12].

For 1-dimensional Gaussian solitons, the $f$-heat kernel can be written explicitly.

**Example 1.2.** $f$-heat kernel for steady Gaussian soliton.

Let $(\mathbb{R}, g_0, e^{-f} dx)$ be a 1-dimensional steady Gaussian soliton, where $g_0$ is the Euclidean metric and $f(x) = \pm x$. Then $Ric_f = 0$. The heat kernel of the operator $\Delta_f = \frac{d^2}{dx^2} \mp \frac{d}{dx}$ is given by

$$H(x, y, t) = \frac{e^{\pm \frac{4t}{(4\pi t)^{1/2}}} \cdot e^{-t/4}}{(4\pi t)^{1/2}} \times \exp \left( -\frac{|x - y|^2}{4t} \right).$$

This $f$-heat kernel is solved using the separation of variables method, see the appendix for details.

**Example 1.3.** Mehler heat kernel [16] for shrinking Gaussian soliton.

Let $(\mathbb{R}, g_0, e^{-f} dx)$ be a 1-dimensional shrinking Gaussian soliton, where $g_0$ is the Euclidean metric and $f(x) = x^2$. Then $Ric_f = 2$. The heat kernel of the operator $\Delta_f = \frac{d^2}{dx^2} - 2x \frac{d}{dx}$ is given by

$$H(x, y, t) = \frac{1}{(2\pi \sinh 2t)^{1/2}} \times \exp \left( \frac{2xye^{-2t} - (x^2 + y^2)e^{-4t}}{1 - e^{-4t}} + t \right).$$
Example 1.4. Mehler heat kernel [16] for expanding Gaussian soliton.

Let \((\mathbb{R}, g_0, e^{-f}dx)\) be a 1-dimensional expanding Gaussian soliton, where \(g_0\) is the Euclidean metric and \(f(x) = -x^2\). Then \(\text{Ric}_f = -2\). The heat kernel of the operator \(\Delta_f = \frac{d^2}{dx^2} + 2x \frac{d}{dx}\) is given by

\[
H(x, y, t) = \frac{1}{(2\pi \sinh 2t)^{1/2}} \times \exp \left( \frac{2xy e^{-2t} - (x^2 + y^2)}{1 - e^{-4t}} - t \right).
\]

As applications, we apply \(f\)-heat kernel upper estimates to obtain an \(L^1_f\)-Liouville theorem on complete smooth metric measure spaces with \(\text{Ric}_f \geq 0\) and \(f\) to be of at most quadratic growth. We say \(u \in L^p_f\), if

\[
\int_M |u| p e^{-f} dv < \infty.
\]

Theorem 1.5. Let \((M, g, e^{-f}dv)\) be an \(n\)-dimensional complete noncompact smooth metric measure space with \(\text{Ric}_f \geq 0\). Assume there exist nonnegative constants \(a\) and \(b\) such that

\[
|f|(x) \leq ar^2(x) + b \text{ for all } x \in M,
\]

where \(r(x)\) is the geodesic distance function to a fixed point \(o \in M\). Then any nonnegative \(L^1_f\)-integrable \(f\)-subharmonic function must be identically constant. In particular, any \(L^1_f\)-integrable \(f\)-harmonic function must be identically constant.

From [6] and [17] any complete noncompact shrinking or steady gradient Ricci soliton satisfies the assumptions in Theorem 1.5. Hence

Corollary 1.6. Let \((M, g, e^{-f}dv)\) be a complete noncompact gradient shrinking or steady Ricci soliton. Then any nonnegative \(L^1_f\)-integrable \(f\)-subharmonic function must be identically constant.

Remark 1.7. Pigola, Rimoldi and Setti (see Corollary 23 in [29]) proved that on a complete gradient shrinking Ricci soliton, any locally lipschitz \(f\)-subharmonic function \(u \in L^p_f, 1 < p < \infty\), is constant. Our result shows that this is true in the case \(p = 1\). Brighton [3], Cao-Zhou [6], Munteanu-Sesum [27], Munteanu-Wang [28], Wei-Wylie [33] have proved several similar results.

The growth condition of \(f\) in Theorem 1.5 is sharp as explained by the following simple example.

Example 1.8. Consider the 1-dimensional smooth metric measure space \((\mathbb{R}, g_0, e^{-f}dx)\), where \(g_0\) is the Euclidean metric and \(f(x) = x^{2+2\delta}\), \(\delta = \frac{1}{2m+1}\) for \(m \in \mathbb{N}\). By direct computation, \(\text{Ric}_f \geq 0\). Let

\[
u(x) := \int_0^{\frac{|x|}{2\delta}} e^{t^{2+2\delta}} dt.
\]
Then $u$ is $f$-harmonic. Moreover we claim $u \in L^1(\mu)$. Indeed, the integration by parts implies the identity
\[
\int_1^x e^{t^2 + 2\delta} dt = \frac{1}{2 + 2\delta} \left[ e^{x^2 + 2\delta} - e + (1 + 2\delta) \int_1^x \frac{e^{t^2 + 2\delta}}{t^{1+2\delta}} dt \right].
\]
Then by L'Hospital rule, when $x$ is large enough,
\[
\int_1^x e^{t^2 + 2\delta} dt = \frac{1}{2 + 2\delta} e^{x^2 + 2\delta} (x^{1+2\delta}) (1 + o(1)).
\]
Therefore
\[
\int_{\mathbb{R}} u e^{-f} dx = \int_{-\infty}^{\infty} \left( \int_0^{\|x\|} e^{t^2 + 2\delta} dt \right) e^{-x^2 + 2\delta} dx < \infty,
\]
i.e., $u \in L^1_f$, but $u \notin L^p_f$ for any $p > 1$. On the other hand, if $\delta = 0$ then $u \not\in L^1_f$.

By Theorem 1.5, we prove a uniqueness theorem for $L^1_f$-solutions of the $f$-heat equation, which generalizes the classical result of P. Li [19].

**Theorem 1.9.** Let $(M, g, e^{-f} dv)$ be an $n$-dimensional complete noncompact smooth metric measure space with $\text{Ric}_f \geq 0$. Assume there exist nonnegative constants $a$ and $b$ such that
\[
|f|(x) \leq ar^2(x) + b \quad \text{for all } x \in M,
\]
where $r(x)$ is the geodesic distance function to a fixed point $o \in M$. If $u(x, t)$ is a nonnegative function defined on $M \times [0, +\infty)$ satisfying
\[
(\partial_t - \Delta_f)u(x, t) \leq 0, \quad \int_M u(x, t)e^{-f} dv < +\infty
\]
for all $t > 0$, and
\[
\lim_{t \to 0} \int_M u(x, t)e^{-f} dv = 0,
\]
then $u(x, t) \equiv 0$ for all $x \in M$ and $t \in (0, +\infty)$. In particular, any $L^1_f$-solution of the $f$-heat equation is uniquely determined by its initial data in $L^1_f$.

The rest of the paper is organized as follows. In Section 2 we give a relative volume comparison theorem for nonnegative Bakry-Émery Ricci curvature. Using this comparison theorem, we give a local $f$-volume doubling property, a local $f$-Neumann Poincaré inequality, a local Sobolev inequality, and a $f$-mean value inequality. In Section 3 we apply the mean value inequality to prove local Gaussian upper bounds of the $f$-heat kernel. In Sections 4 and 5 we follow the idea in [19], and establish an $L^1_f$-Liouville theorem for $f$-subharmonic functions and an $L^1_f$-uniqueness property for nonnegative solutions of the $f$-heat equation. In the appendix, we compute the $f$-heat kernel of 1-dimensional steady Gaussian soliton.
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2. Poincaré, Sobolev and mean value inequalities

Let $\Delta_f = \Delta - \nabla f \cdot \nabla$ be the $f$-Laplacian on a complete smooth metric measure space $(M, g, e^{-f} dv)$. Throughout this section, we will assume $\text{Ric}_f \geq 0$.

For a fixed point $o \in M$ and $R > 0$, we define

$$A(R) := \sup_{x \in B_o(3R)} |f|(x).$$

We often write $A$ for short. First we have the relative $f$-volume comparison results proved by Wei and Wylie [33].

**Lemma 2.1.** Let $(M, g, e^{-f} dv)$ be an $n$-dimensional complete noncompact smooth metric measure space. If $\text{Ric}_f \geq 0$, then for any $x \in B_o(R)$,

$$V_f(B_x(r_1, R_2)) \leq e^{4A R_n^2 - R_1^n} V_f(B_x(r_1, r_2))$$

for any $0 < r_1 < r_2$, $0 < R_1 < R_2 < R$, $r_1 \leq R_1$, $r_2 \leq R_2 < R$, where $B_x(R_1, R_2) := B_x(R_2) \setminus B_x(R_1)$.

**Proof of Lemma 2.1.** Wei and Wylie (see (3.19) in [33]) proved the following $f$-mean curvature comparison theorem. Recall that the weighted mean curvature $m_f(r)$ is defined as $m_f(r) := m(r) - \nabla f \cdot \nabla r = \Delta_f r$. For any $x \in B_o(R) \subset M$, if $\text{Ric}_f \geq 0$, then by the ODE comparison argument, we get

$$m_f(r) \leq \frac{n-1}{r} - \frac{2}{r} f(r) + \frac{2}{r^2} \int_0^r f(t) dt \leq \frac{n-1}{r} + 4A(r),$$

along any minimal geodesic segment from $x$.

In geodesic polar coordinates, the volume element is written as $dv = A(r, \theta) dr \wedge d\theta_{n-1}$, where $d\theta_{n-1}$ is the standard volume element of the unit sphere $S^{n-1}$. Let $A_f(r, \theta) = e^{-f}A(r, \theta)$. By the first variation of the area,

$$\frac{A'_f}{A_f} = (\ln(A_f(r, \theta)))' = m_f(r, \theta).$$

Therefore

$$\frac{A'_f}{A_f} = (\ln(A_f(r, \theta)))' = m_f(r, \theta).$$

And for $r \geq r_0 > 0$, we have

$$\frac{A_f(r, \theta)}{A_f(r_0, \theta)} = \exp \left( \int_{r_0}^r m_f(s, \theta) ds \right).$$
Let $A^{4A}_R(r) = e^{4Ar}A_R(r)$, where $A_R$ is the Riemannian volume element in $\mathbb{R}^n$. By the mean curvature comparison, $(\ln(A_f(r, \theta)))' \leq (\ln(A^{4A}_R(r, \theta)))'$. So for $r < R$,

$$A_f(R, \theta) \leq A^{4A}_R(R, \theta).$$

That is $\frac{A_f(r, \theta)}{A^{4A}_R(r, \theta)}$ is nonincreasing in $r$. Applying Lemma 3.2 in [35], we get

$$\frac{\int_{R_1}^{R_2} A_f(R, \theta)dt}{\int_{r_1}^{r_2} A_f(r, \theta)dt} \leq \frac{\int_{R_1}^{R_2} A^{4A}_R(R, \theta)dt}{\int_{r_1}^{r_2} A^{4A}_R(r, \theta)dt}$$

for $0 < r_1 < r_2$, $0 < R_1 < R_2$, $r_1 \leq R_1$ and $r_2 \leq R_2 < R$. Integrating along the sphere direction gives

$$\frac{V_f(B_x(R_1, R_2))}{V_f(B_x(r_1, r_2))} \leq e^{4A} \frac{R^n_2 - R^n_1}{r^n_2 - r^n_1},$$

for any $0 < r_1 < r_2$, $0 < R_1 < R_2 < R$, $r_1 \leq R_1$, $r_2 \leq R_2 < R$, where $B_x(R_1, R_2) := B_x(R_2) \setminus B_x(R_1)$. \hfill \Box

From (2.1), letting $r_1 = R_1 = 0$, $r_2 = r$ and $R_2 = 2r$, we get

$$V_f(B_x(2r)) \leq 2^n e^{4A} \cdot V_f(B_x(r))$$

for any $0 < r < R/2$. This inequality implies that the local $f$-volume doubling property holds. This property will play a crucial role in our paper. We say that a complete smooth metric measure space $(M, g, e^{-f}dv)$ satisfies a local $f$-volume doubling property if for any $0 < R < \infty$, there exists a constant $C(R)$ such that

$$V_f(B_x(2r)) \leq C(R) \cdot V_f(B_x(r))$$

for any $0 < r < R$ and $x \in M$. Note that the above inequality holds with $R = +\infty$, and it called the global $f$-volume doubling property.

From Lemma 2.1 we have

**Lemma 2.2.** Let $(M, g, e^{-f}dv)$ be an $n$-dimensional complete noncompact smooth metric measure space. If $\text{Ric}_f \geq 0$, then

$$\frac{V_f(B_x(s))}{V_f(B_y(r))} \leq 4^n e^{8A} \left(\frac{s}{r}\right)^\kappa,$$

where $\kappa = \log_2(2^n e^{4A})$, for any $0 < r \leq s < R/4$ and all $x \in B_0(s)$ and $y \in B_x(s)$. Moreover, we have

$$V_f(B_x(r)) \leq e^{4A} \left(\frac{d(x, y)}{r} + 1\right)^n V_f(B_y(r))$$

for any $x, y \in B_0(\frac{R}{4})$ and $0 < r < R/2$. 

Proof. Choose a real number $k$ such that $2^k < s/r < 2^{k+1}$. Since $y \in B(x,s)$,
$$B_x(s) \subset B_y(2s) \subset B_y(2^{k+2}r),$$
and so $V_f(B_x(s)) \leq V_f(B_y(2^{k+2}r))$. Moreover, the assumption of lemma implies the local $f$-volume doubling property (2.2). Using this, we have
$$V_f(B_x(s)) \leq (2^n e^{4A})^{k+2} V_f(B_y(r)) \leq (2^n e^{4A})^2 (s/r) \kappa V_f(B_y(r)),$$
where $\kappa = \log_2(2^n e^{4A})$. This completes the first part of lemma.

For the second part, letting $r_1 = 0$, $r_2 = r$, $R_1 = d(x,y) - r$ and $R_2 = d(x,y) + r$ in Lemma 2.1, we have
$$V_f(B_x(d(x,y) + r)) - V_f(B_x(d(x,y) - r)) \leq e^{4A} \left( \frac{d(x,y)}{r} + 1 \right)^n V_f(B_y(r))$$
for any $x, y \in B_0(1/4 R)$ and $0 < r < R/2$. Therefore we get
$$V_f(B_x(r)) \leq V_f(B_y(d(x,y) + r)) - V_f(B_y(d(x,y) - r))$$
$$\leq e^{4A} \left( \frac{d(x,y)}{r} + 1 \right)^n V_f(B_y(r))$$
for any $x, y \in B_0(1/4 R)$ and $0 < r < R/2$. The proof of the second part follows.

By Lemma 2.1 following Buser’s proof [4] or Saloff-Coste’s alternate proof (Theorem 5.6.5 in [32]), we get a local Neumann Poincaré inequality on smooth metric measure spaces, see also Munteanu and Wang (see Lemma 3.1 in [28]).

Lemma 2.3. Let $(M,g,e^{-f} dv)$ be an $n$-dimensional complete noncompact smooth metric measure space with $\text{Ric}_f \geq 0$. Denote by $r(x)$ the geodesic distance function from a fixed origin $o \in M$. Then for any $x \in B_0(R)$,
$$\int_{B_x(r)} |\varphi - \varphi_{B_x(r)}|^2 e^{-f} dv \leq c_1 e^{c_2 A} \cdot r^2 \int_{B_x(r)} |\nabla \varphi|^2 e^{-f} dv$$
for all $0 < r < R$ and $\varphi \in C^\infty(B_x(r))$, where $\varphi_{B_x(r)} := V_f^{-1}(B_x(r)) \int_{B_x(r)} \varphi e^{-f} dv$. The constants $c_1$ and $c_2$ depend only on $n$.

Remark 2.4. When $f$ is constant, this was classical result of Saloff-Coste (see (6) in [31] or Theorem 5.6.5 in [32]).

Combining Lemma 2.1 Lemma 2.2 Lemma 2.3 and the argument in [30], we obtain a local Sobolev inequality.

Lemma 2.5. Let $(M,g,e^{-f} dv)$ be an $n$-dimensional complete noncompact smooth metric measure space with $\text{Ric}_f \geq 0$. Then there exist constants $p > 2$, $c_3$ and $c_4$, all depending only on $n$ such that
$$\left( \int_{B_o(r)} |\varphi|^{\frac{2p}{p-2}} e^{-f} dv \right)^{\frac{p-2}{p}} \leq \frac{c_3 e^{c_4 A} \cdot r^2}{V_f(B_o(r))} \int_{B_o(r)} |\nabla \varphi|^2 + r^{-2} |\varphi|^2 e^{-f} dv$$
for any $x \in M$ such that $0 < r(x) < R$ and $\varphi \in C^\infty(B_o(r))$.

**Sketch proof of Lemma 2.5.** The proof is essentially a weighted version of Theorem 2.1 in [30] (see also Theorem 3.1 in [31]).

Besides, we have an alternate proof by applying the local Neumann Sobolev inequality of Munteanu and Wang (see Lemma 3.2 in [28])

$$
||\varphi - \varphi_{B_o}(r)||_{L^p_{\mu}} \leq \frac{c_3 e^{c_4 A} \cdot r}{V_f(B_o(r))^{1/p}} ||\nabla \varphi||_2,
$$

where $||f||_m := (\int_{B_o(r)} |f|^m d\mu)^{1/m}$. Munteanu and Wang proved this inequality holds without the weighted measure, but it is still true by checking their proof when integrals are with respect to the measure $e^{-f} dv$. Combining this with the Minkowski inequality

$$
||\varphi||_{L^p_{\mu}} \leq ||\varphi - \varphi_{B_o}(r)||_{L^p_{\mu}} + ||\varphi_{B_o}(r)||_{L^p_{\mu}},
$$

it is sufficient to prove

$$
||\varphi_{B_o}(r)||_{L^p_{\mu}} \leq \frac{c_3 e^{c_4 A}}{V_f(B_o(r))^{1/p}} ||\varphi||_2,
$$

which follows from Cauchy-Schwarz inequality. Hence the lemma follows. □

Lemma 2.5 is a critical step in proving the Harnack inequality by the Moser iteration technique [26]. Here we apply it to prove a local mean value inequality for the f-heat equation, which is similar to the case when $f$ is constant, obtained by Saloff-Coste [30] and Grigor’yan [15].

**Proposition 2.6.** Let $(M, g, e^{-f} dv)$ be an $n$-dimensional complete noncompact smooth metric measure space. Fix $R > 0$. Assume that (2.4) is satisfied up to this $R$. Then there exist constants $c_5(n, p)$ and $c_6(n, p)$ such that, for any real $s$, for any $0 < \delta < \delta' \leq 1$, and for any smooth positive solution $u$ of the $f$-heat equation in the cylinder $Q = B_o(r) \times (s - r^2, s)$, $r < R$, we have

$$
\sup_{Q_\delta} u \leq \frac{c_5 e^{c_4 A}}{V_f(B_o(r))^{1/p}} \int_{Q_{\delta'}} u \ d\mu \ dt,
$$

where $Q_\delta := B_o(\delta r) \times (s - \delta r^2, s)$ and $Q_{\delta'} := B_o(\delta' r) \times (s - \delta' r^2, s)$.

**Proof.** The proof is the weighted case of the argument of Theorem 5.2.9 in [32]. For the convenience of the readers, we give a detailed proof. We need to carefully compute the explicit and accurate coefficients of mean value inequality in terms of the Sobolev constants in (2.4).

Without loss of generality we assume that $\delta' = 1$. For any nonnegative function $\phi \in C^\infty_0(B)$, $B := B_o(r)$, we have

$$
\int_B (\phi u_t + \nabla \phi \nabla u) d\mu = 0.
$$
If $\phi = \psi^2 u$, $\psi \in C_0^\infty(B)$, then
\[
\int_B (\psi^2 u u_t + \psi^2 |\nabla u|^2) d\mu \leq 2 \int_B |\psi \psi \nabla u \nabla \psi| d\mu \\
\leq 3 \int_B |\nabla \psi|^2 u^2 d\mu + \frac{1}{3} \int_B \psi^2 |\nabla u|^2 d\mu.
\]
From this, we derive that
\[
\int_B (2\psi^2 u u_t + |\nabla (\psi u)|^2) d\mu \leq 10 ||\nabla \psi||^2_{\infty} \int_{\supp(\psi)} u^2 d\mu.
\]
Multiplying a smooth function $\lambda(t)$, which will be determined later, from the above inequality,
\[
\frac{\partial}{\partial t} \left( \int_B (\lambda \psi u)^2 d\mu \right) + \lambda^2 \int_B |\nabla (\psi u)|^2 d\mu \leq C \lambda (\lambda ||\nabla \psi||^2_{\infty} + |\lambda| \sup \psi^2) \int_{\supp(\psi)} u^2 d\mu,
\]
where $C$ is a finite constant which will change from line to line in the following inequalities.

Next we choose $\psi$ and $\lambda$ such that, for any $0 < \sigma' < \sigma < 1$, and $\kappa = \sigma - \sigma'$,
\begin{enumerate}
  \item $0 \leq \psi \leq 1$, $\supp(\psi) \subset \sigma B$, $\psi = 1$ in $\sigma' B$ and $|\nabla \psi| \leq 2(\kappa r)^{-1}$;
  \item $0 \leq \lambda \leq 1$, $\lambda = 0$ in $(-\infty, s - \sigma r^2)$, $\lambda = 1$ in $(s - \sigma' r^2, +\infty)$, and $|\lambda(t)| \leq 2(\kappa r)^{-2}$.
\end{enumerate}
Let $I_{\sigma} := (s - \sigma r^2, s)$ and $I'_{\sigma} := (s - \sigma' r^2, s)$. For any $t \in I_{\sigma'}$ integrating the above inequality over $(s - r^2, t)$,
\[
\sup_{I'_{\sigma}} \left\{ \int_B \psi^2 u^2 d\mu \right\} + \int_{B \times I_{\sigma'}} |\nabla (\psi u)|^2 d\mu dt \leq C (r \kappa)^{-2} \int_{Q_{\sigma}} u^2 d\mu dt.
\]
On the other hand, by the Hölder inequality and the assumption of proposition, for some $p > 2$, we have
\[
\int_B \phi^{2(1 + \frac{2}{p})} d\mu \leq \left( \int_B |\phi|^{\frac{2p}{p-2}} d\mu \right)^{\frac{p-2}{p}} \cdot \left( \int_B \phi^2 d\mu \right)^{\frac{2}{p}}
\]
\[
\leq \left( \int_B \phi^2 d\mu \right)^{\frac{2}{p}} \cdot \left( E(B) \int_B (|\nabla \phi|^2 + r^{-2} |\phi|^2) d\mu \right)
\]
for all $\phi \in C_0^\infty(B)$, where $E(B) := c_3 e^{-A} : r^2 \cdot V_f(B_{\rho}(r))^{-2/p}$. Combining (2.6) and (2.7) yields
\[
\int_{Q_{\sigma'}} u^{2\theta} d\mu dt \leq E(B) \left[ C(r \kappa)^{-2} \int_{Q_{\sigma}} u^2 d\mu dt \right]^\theta
\]
with $\theta = 1 + 2/p$. For any $m \geq 1$, $u^m$ is also a smooth positive solution of $(\partial_t - \Delta_f) u(x, t) \leq 0$. Hence the above inequality indeed implies
\[
\int_{Q_{\sigma'}} u^{2m\theta} d\mu dt \leq E(B) \left[ C(r \kappa)^{-2} \int_{Q_{\sigma}} u^{2m} d\mu dt \right]^\theta
\]
for $m \geq 1$. 

Now we will apply (2.8) to produce the iterated formula. We set \( \kappa_i = (1 - \delta)^{2^{-i}} \), which satisfies \( \sum \kappa_i = 1 - \delta \). Also set \( \sigma_0 = 1, \sigma_{i+1} = \sigma_i - \kappa_i = 1 - \sum \kappa_j \). Applying (2.8) for \( m = \theta^i, \sigma = \sigma_i, \sigma' = \sigma_{i+1} \), we have
\[
\int_{Q_{\sigma_{i+1}}} u^{2\theta^i+1} d\mu dt \leq E(B) \left[ C^{i+1} ((1 - \delta) r)^{-2} \int_{Q_{\sigma_i}} u^{2\theta^i} d\mu dt \right]^{\theta^i}.
\]
Therefore
\[
\left( \int_{Q_{\sigma_{i+1}}} u^{2\theta^i+1} d\mu dt \right)^{\theta^i-1} \leq C^{i \theta^i} (1 - \delta)^{-2} \int_{Q_{\sigma_i}} u^{2\theta^i} d\mu dt,
\]
where \( \Sigma \) denotes the summations from 1 to \( i + 1 \). Letting \( i \to \infty \)
\[
(\text{2.9}) \quad \sup_{Q_{\sigma}} \left\{ u^2 \right\} \leq C \cdot E(B)^{p/2} \cdot [(1 - \delta) r]^{-2-p} \| u \|_{2, Q}^2
\]
for some \( p > 2 \).

Formula (2.9) in fact is an \( L^2 \)-mean value inequality. Next step, we apply (2.8) to prove (2.5) by a different iterative argument. Let \( \sigma \in (0, 1) \) and \( \rho = \sigma + (1 - \sigma)/4 \). Then (2.9) indeed implies
\[
\sup_{Q_{\sigma}} \{ u \} \leq F(B) \cdot (1 - \sigma)^{-1-1/2-p} \| u \|_{2, Q_{\sigma}},
\]
where \( F(B) := c_3 e^{c4A} \cdot r^{-1} \cdot V_f(B_o(r))^{-1/2} \). Since
\[
\| u \|_{2, Q} \leq \| u \|_{\infty, Q}^{1/2} \cdot \| u \|_{1, Q}^{1/2}
\]
for any \( Q \), we further have
\[
(\text{2.10}) \quad \| u \|_{\infty, Q_{\sigma_i}} \leq F(B) \cdot \| u \|_{1, Q}^{1/2} \cdot (1 - \sigma)^{-1-1/2-p} \| u \|_{\infty, Q_{\sigma_{i+1}}}^{1/2}
\]
Now we will apply (2.10) to produce the iterated formula. Fix \( \delta \in (0, 1) \) and set \( \sigma_0 = \delta, \sigma_{i+1} = \sigma_i + (1 - \sigma_i)/4 \), which satisfy \( 1 - \sigma_i = (3/4) \cdot (1 - \delta) \). Applying (2.10) to \( \sigma = \sigma_i \) and \( \rho = \sigma_{i+1} \), we have
\[
\| u \|_{\infty, Q_{\sigma_i}} \leq (4/3)^{(1+p/2)j} F(B) \cdot \| u \|_{1, Q}^{1/2} \cdot (1 - \delta)^{-1-1/2-p} \| u \|_{\infty, Q_{\sigma_{i+1}}}^{1/2}
\]
Therefore, for any \( i \),
\[
\| u \|_{\infty, Q_{\delta}} \leq (4/3)^{(1+p/2)\sum \kappa_j} \left( F(B) \cdot \| u \|_{1, Q}^{1/2} \cdot (1 - \delta)^{-1-1/2-p} \right) \| u \|_{\infty, Q_{\sigma_i}}^{\sum \kappa_j},
\]
where \( \Sigma \) denotes the summations from 0 to \( i - 1 \). At last, letting \( i \to \infty \) gives
\[
\| u \|_{\infty, Q_{\delta}} \leq (4/3)^{(2+p)j} \left( F(B) \cdot \| u \|_{1, Q}^{1/2} \cdot (1 - \delta)^{-1-1/2-p} \right)^2,
\]
that is,
\[
\| u \|_{\infty, Q_{\delta}} \leq (4/3)^{(2+p)j} c_5 e^{c6A} (1 - \delta)^{-2-p} \cdot r^{-2} \cdot V_f(B_o(r))^{-1} \cdot \| u \|_{1, Q}
\]
and the proposition follows. \( \square \)
3. Gaussian upper bounds of the $f$-heat kernel

In this section, we will apply Proposition 2.6 and Lemma 2.2 to give Gaussian upper bounds of the $f$-heat kernel on complete smooth metric measure spaces with nonnegative Bakry-Émery Ricci curvature. To prove Theorem 1.1, we need a weighted version of Davies’ integral estimate [12].

Lemma 3.1. Let $(M, g, \mu)$ be an $n$-dimensional complete smooth metric measure space. Let $\lambda_1(M) \geq 0$ be the bottom of the $L^2$-spectrum of the $f$-Laplacian on $M$. Assume that $B_1$ and $B_2$ are bounded subsets of $M$. Then (3.1)

\[ \int_{B_1} \int_{B_2} H(x, y, t) d\mu(y) d\mu(x) \leq V_f(B_1)^{1/2} V_f(B_2)^{1/2} \exp \left( -\lambda_1(M) t - \frac{d^2(B_1, B_2)}{4t} \right), \]

where $d(B_1, B_2)$ denotes the distance between the sets $B_1$ and $B_2$.

Proof of Lemma 3.1. By the approximation argument, it suffices to prove (3.1) for the $f$-heat kernel $H_{\Omega}$ of any compact set with boundary $\Omega$, containing $B_1$ and $B_2$. Indeed, let $\Omega_i$ be a sequence of compact exhaustion of $M$ such that $\Omega_i \subset \Omega_{i+1}$ and $\cup_i \Omega_i = M$. If we prove (3.1) for the $f$-heat kernel $H_{\Omega_i}$ for any $i$, then the lemma follows by letting $i \to \infty$ and observing that $\lambda_1(\Omega_i) \to \lambda_1(M)$, where $\lambda_1(\Omega_i) > 0$ denotes the first eigenvalue of the Dirichlet $f$-Laplacian on $\Omega_i$, and $\lambda_1(M) := \inf_{\Omega \subset M} \lambda_1(\Omega)$.

Now we consider the function $u(x, t) = e^{t\Delta_f} \mathbf{1}_{B_1}$ with Dirichlet boundary condition: $u(x, t) = 0$ on $\partial \Omega$. Then (3.2)

\[ \int_{B_2} \int_{B_1} H_{\Omega}(x, y, t) d\mu(y) d\mu(x) = \int_{B_2} \left( \int_{\Omega} H_{\Omega}(x, y, t) \mathbf{1}_{B_1} d\mu(y) \right) d\mu(x) \]

\[ = \int_{B_2} u(x, t) d\mu(x) \]

\[ \leq V_f(B_2)^{1/2} \left( \int_{B_2} u^2(x, t) d\mu(x) \right)^{1/2}. \]

For some $\alpha > 0$, we define $\xi(x, t) := \alpha d(x, B_1) - \frac{\alpha^2}{2} t$ and consider the function

\[ J(t) := \int_{\Omega} u^2(x, t) e^{\xi(x, t)} d\mu(x). \]

Claim: Function $J(t)$ satisfies

\[ J(t) \leq J(t_0) \cdot \exp(-2\lambda_1(\Omega)(t - t_0)) \]

for all $0 < t_0 \leq t$. 

This claim will be proved later. Assuming it, we now continue to prove Lemma 3.1. If \( x \in B_2 \), then \( \xi(x, t) \geq \alpha d(B_2, B_1) - \frac{\alpha^2}{2} t \). Hence

\[
J(t) \geq \int_{B_2} u^2(x, t)e^{\xi(x, t)}d\mu(x) \\
\geq \exp \left( \alpha d(B_2, B_1) - \frac{\alpha^2}{2} t \right) \int_{B_2} u^2(x, t)d\mu(x).
\]  

(3.4)

On the other hand, if \( x \in B_1 \) then \( \xi(x, 0) = 0 \). Using (3.3) and the continuity of \( J(t) \) at \( t = 0^+ \), we have

\[
J(t) \leq J(0) \cdot \exp (-2\lambda_1(\Omega)t) \\
= \int_{\Omega} e^{\xi(x, 0)} 1_{B_1} d\mu(x) \cdot \exp (-2\lambda_1(\Omega)t) \\
= V_f(B_1) \cdot \exp (-2\lambda_1(\Omega)t)
\]  

(3.5)

Combining (3.2), (3.4) and (3.5), and choosing \( \alpha = d(B_1, B_2)/t \), we get

\[
\int_{B_1} \int_{B_2} H_{\Omega}(x, y, t)d\mu(x)d\mu(y) \leq V_f(B_1)^{1/2}V_f(B_2)^{1/2} \exp \left( -\lambda_1(\Omega)t - \frac{d^2(B_1, B_2)}{4t} \right)
\]  

for any compact set \( \Omega \subset M \). Lemma 3.1 is proved.

**Proof of the claim.** Since \( \xi_t \leq \frac{1}{2}\|\nabla \xi\|^2 \) and \( u_t = \Delta f u \), we directly compute

\[
J'(t) \leq 2 \int_{\Omega} u\Delta f ue^{\xi}d\mu(x) - \frac{1}{2} \int_{\Omega} u^2 e^{\xi} \|\nabla \xi\|^2 d\mu(x)
\]

\[
= -2 \int_{\Omega} |\nabla u|^2 e^{\xi}d\mu(x) - 2 \int_{\Omega} u(\nabla u, \nabla \xi)e^{\xi}d\mu(x) - \frac{1}{2} \int_{\Omega} u^2 e^{\xi} \|\nabla \xi\|^2 d\mu(x)
\]

\[
= -2 \int_{\Omega} (u \nabla \xi + 2\nabla u)^2 e^{\xi}d\mu(x)
\]

\[
= -2 \int_{\Omega} |\nabla (ue^{\xi/2})|^2 d\mu(x).
\]

Moreover the definition of \( \lambda_1(\Omega) \) implies

\[
\int_{\Omega} |\nabla (ue^{\xi/2})|^2 d\mu(x) \geq \lambda_1(\Omega) \int_{\Omega} |ue^{\xi/2}|^2 d\mu(x) = \lambda_1(\Omega)J(t).
\]

Substituting this into (3.6) gives \( J'(t) \leq -2\lambda_1(\Omega)J(t) \) and the claim is proved. \( \square \)

Modifying the argument of [13] (see also [20]), we now prove the upper bounds of \( f \)-heat kernel.
Proof of Theorem 7.1. We let \( u : (y, s) \mapsto H(x, y, s) \) be a \( f \)-heat kernel, which is smooth. Under the assumption \( t \geq r_2^2 \), applying the smooth function \( u \) to Proposition 2.6 with the fixed \( x \in B_0(R/2) \), we have

\[
\sup_{(y, s) \in Q_\delta} H(x, y, s) \leq \frac{c_5e^{c_6A}}{r_2^2V_f(B_2)} \cdot \int_{t-1/4r_2^2}^t \int_{B_2} H(x, \zeta, s) d\mu(\zeta) ds
\]

(3.7)

\[
= \frac{c_5e^{c_6A}}{4V_f(B_2)} \cdot \int_{B_2} H(x, \zeta, s') d\mu(\zeta)
\]

for some \( s' \in (t - 1/4r_2^2, t) \), where \( Q_\delta := B_y(\delta r_2) \times (t - \delta r_2^2, t) \) with \( 0 < \delta < 1/4 \), and \( B_2 = B_y(r_2) \subseteq B_0(R) \) for \( y \in B_0(R/2) \). Applying Proposition 2.6 and the same argument to the positive solution

\[
v(x, s) = \int_{B_2} H(x, \zeta, s) d\mu(\zeta)
\]

of the \( f \)-heat equation, for the variable \( x \) with \( t \geq r_1^2 \), we also get

(3.8)

\[
\sup_{(x, s) \in Q_\delta} \int_{B_2} H(x, \zeta, s) d\mu(\zeta) \leq \frac{c_5e^{c_6A}}{r_1^2V_f(B_1)V_f(B_2)} \cdot \int_{t-1/4r_1^2}^t \int_{B_1} \int_{B_2} H(\xi, \zeta, s) d\mu(\zeta) d\mu(\xi) ds
\]

\[
= \frac{c_5e^{c_6A}}{4V_f(B_1)} \cdot \int_{B_1} \int_{B_2} H(\xi, \zeta, \zeta'' d\mu(\zeta) d\mu(\xi)
\]

for some \( \zeta'' \in (t - 1/4r_1^2, t) \), where \( \tilde{Q}_\delta := B_x(\delta r_1) \times (t - \delta r_1^2, t) \) with \( 0 < \delta < 1/4 \), and \( B_1 = B_x(r_1) \subseteq B_0(R) \) for \( x \in B_0(R/2) \). Now letting \( r_1 = r_2 = \sqrt{t} \) and combining (3.7) with (3.8), the smooth \( f \)-heat kernel satisfies

(3.9)

\[
H(x, y, t) \leq \frac{(c_5e^{c_6A})^2}{V_f(B_1)V_f(B_2)} \cdot \int_{B_1} \int_{B_2} H(\xi, \zeta, \zeta'' d\mu(\zeta) d\mu(\xi)
\]

for all \( x, y \in B_0(R/2) \) and \( 0 < t < R^2/4 \). Using Lemma 3.1 and noticing that \( \zeta'' \in (\sqrt{\frac{3}{4}} t, t) \), then (3.9) becomes

(3.10)

\[
H(x, y, t) \leq \frac{(c_5e^{c_6A})^2}{V_f(B_x(\sqrt{t}))^{1/2}V_f(B_y(\sqrt{t}))^{1/2}} \times \exp \left( -\frac{3}{4} \sqrt{\frac{3}{4}} t - \frac{d^2(B_1, B_2)}{4t} \right)
\]

for all \( x, y \in B_0(R/2) \) and \( 0 < t < R^2/4 \). Notice that if \( d(x, y) \leq 2\sqrt{t} \), then \( d(B_x(\sqrt{t}), B_y(\sqrt{t})) = 0 \) and hence

\[
-\frac{d^2(B_x(\sqrt{t}), B_y(\sqrt{t}))}{4t} = 0 \leq 1 - \frac{d^2(x, y)}{4t},
\]

and if \( d(x, y) > 2\sqrt{t} \), then \( d(B_x(\sqrt{t}), B_y(\sqrt{t})) = d(x, y) - 2\sqrt{t} \), and hence

\[
-\frac{d^2(B_x(\sqrt{t}), B_y(\sqrt{t}))}{4t} = -\frac{(d(x, y) - 2\sqrt{t})^2}{4t} \leq -\frac{d^2(x, y)}{4(1 + \epsilon)t} + C(\epsilon)
\]
for some constant $C(\epsilon)$, where $\epsilon > 0$. Here if $\epsilon \to 0$, then the constant $C(\epsilon) \to \infty$. Therefore in any case, (3.10) becomes
\begin{equation}
H(x, y, t) \leq \frac{c_7(n, \epsilon)e^{2c_6A}}{V_f(B_x(\sqrt{t}))^{1/2}V_f(B_y(\sqrt{t}))^{1/2}} \times \exp \left( -\frac{3}{4} \lambda_1 t - \frac{d^2(x, y)}{4(1 + \epsilon)t} \right)
\end{equation}
for all $x, y \in B_o(\frac{1}{2}R)$ and $0 < t < R^2/4$. Also notice that Lemma 2.2 states that
\begin{equation}
V_f(B_x(\sqrt{t})) \leq e^{4A} \left( \frac{d(x, y)}{\sqrt{t}} + 1 \right)^n V_f(B_y(\sqrt{t}))
\end{equation}
for all $x, y \in B_o(\frac{1}{4}R)$ and $0 < t < R^2/4$. Applying this formula to (3.11) yields
\begin{equation}
H(x, y, t) \leq \frac{c_7(n, \epsilon)e^{2c_6+2c_6A}}{V_f(B_x(\sqrt{t}))} \times \left( \frac{d(x, y)}{\sqrt{t}} + 1 \right)^{\frac{n}{2}} \times \exp \left( -\frac{3}{4} \lambda_1 t - \frac{d^2(x, y)}{(4 + \epsilon)t} \right)
\end{equation}
for all $x, y \in B_o(\frac{1}{4}R)$ and $0 < t < R^2/4$. \hfill \Box

4. $L^1_f$-LIOUVILLE THEOREM

In this section, we use the $f$-heat kernel estimates proved in Section 3 to prove $L^1_f$-Liouville theorems on complete noncompact smooth metric measure spaces. Our result extends the classical $L^1$-Liouville theorems obtained by P. Li [19] and the weighted versions proved by X.-D. Li [23] and the first author [34].

We start from a useful lemma.

**Lemma 4.1.** Under the same assumptions of Theorem 1.5, then the complete smooth metric measure space $(M, g, e^{-f}dv)$ is stochastically complete, i.e.,
\begin{equation}
\int_M H(x, y, t)e^{-f}dv(y) = 1.
\end{equation}

**Proof.** In Lemma 2.1 letting $r_1 = R_1 = 0$, $r_2 = 1$, $R_2 = R > 1$ and $x = o$, then
\begin{equation}
V_f(B_o(R)) \leq C(n, b)R^n e^{c(n, o)R^2}
\end{equation}
for all $R > 1$, where we used the assumption: $|f|(x) \leq ar^2(x) + b$. Hence
\begin{equation}
\int_1^\infty \frac{R}{\log V_f(B_o(R))}dR = \infty.
\end{equation}
By Grigor’yan’s Theorem 3.13 in [16], this implies that the metric measure space $(M, g, e^{-f}dv)$ is stochastically complete. \hfill \Box

**Remark 4.2.** The growth of $f$ in lemma 4.1 is essentially sharp when $\Ric_f \geq 0$. If we relax the growth condition of $f$, then the integral formula (4.1) is finite and the metric measure space $(M, g, e^{-f}dv)$ is stochastically incomplete. We remark that stochastically complete condition is not sufficient to imply an $L^1$-Liouville theorem for (positive) harmonic functions.
Some counterexamples are provided in [11] and [21]. However, stochastically complete condition is sufficient to imply an $L^1$-Liouville theorem for nonnegative superharmonic functions [14].

We now follow the arguments of Li [19] to prove Theorem 1.5. We first prove the following integration by parts formula.

Theorem 4.3. Under the same assumptions of Theorem 1.5, for any non-negative $L^1$-integrable $f$-subharmonic function $g$, we have

$$\int_M \Delta g H(x, y, t) \, d\mu(y) = \int_M H(x, y, t) \Delta g(y) \, d\mu(y).$$

Proof of Theorem 4.3. Similar to the proof of Theorem 1 in [19], applying the Green formula on $B_o(R)$, we have

$$\left| \int_{B_o(R)} \Delta g H(x, y, t) \, d\mu(y) - \int_{B_o(R)} H(x, y, t) \Delta g(y) \, d\mu(y) \right|$$

$$= \left| \int_{\partial B_o(R)} H(x, y, t) \frac{\partial}{\partial r} g(y) \, d\mu_{\sigma, R}(y) - \int_{\partial B_o(R)} H(x, y, t) \frac{\partial}{\partial r} g(y) \, d\mu_{\sigma, R}(y) \right|$$

$$\leq \int_{\partial B_o(R)} |\nabla H|(x, y, t) g(y) \, d\mu_{\sigma, R}(y) + \int_{\partial B_o(R)} H(x, y, t) |\nabla g|(y) \, d\mu_{\sigma, R}(y),$$

where $d\mu_{\sigma, R}$ denotes the weighted area measure induced by $d\mu$ on $\partial B_o(R)$. We shall show that the above two boundary integrals vanish as $R \to \infty$. By considering large values of $R$, we may assume $x \in B_o(R/8)$. The proof can be achieved by five steps.

Step 1. Let $g(x)$ be a nonnegative $f$-subharmonic function. Applying Lemma 2.1 and Lemma 2.3 by the Moser’s technique to yield the $f$-mean value inequality, i.e. Proposition 2.6 (see also (3.11) in [28])

$$\sup_{B_o(R)} g(x) \leq C e^{\alpha R^2} V^{-1}_f(2R) \int_{B_o(2R)} g(y) \, d\mu(y),$$

where constants $C$ and $\alpha$ depend on $n, a$ and $b$. Here we have used theorem assumption: $Ric_f \geq 0$ and $|f| \leq ar^2(x) + b$. Consider $\phi(y) = \phi(r(y))$ to be a nonnegative cut-off function satisfying $0 \leq \phi \leq 1$, $|\nabla \phi| \leq \sqrt{3}$ and

$$\phi(r(y)) = \begin{cases} 1 & \text{on } B_o(R + 1) \setminus B_o(R), \\ 0 & \text{on } B_o(R - 1) \cup (M \setminus B_o(R + 2)). \end{cases}$$
Since $g$ is $f$-subharmonic function, by the Cauchy-Schwarz inequality we have
\[ 0 \leq \int_M \phi^2 g \Delta_f g d\mu = -\int_M (\nabla^2 g) \nabla \phi \nabla g d\mu = -2 \int_M \phi g (\nabla \phi \nabla g) d\mu - \int_M \phi^2 |\nabla g|^2 d\mu \leq 2 \int_M |\nabla \phi|^2 g^2 d\mu - \frac{1}{2} \int_M \phi^2 |\nabla g|^2 d\mu. \]

Then using the definition of $\phi$ and (4.2), we have that
\[ \int_{B_o(R+1) \setminus B_o(R)} |\nabla g|^2 d\mu \leq 4 \int_M |\nabla \phi|^2 g^2 d\mu \leq 12 \int_{B_o(R+2)} g^2 d\mu \leq 12 \sup_{B_o(R+2)} g \cdot \|g\|_{L^1(\mu)} \leq \frac{C e^{\alpha(R+2)^2}}{V_f(2R+4)} \cdot \|g\|_{L^1(\mu)}^2. \]

On the other hand, the Cauchy-Schwarz inequality implies that
\[ \int_{B_o(R+1) \setminus B_o(R)} |\nabla g| d\mu \leq \left( \int_{B_o(R+1) \setminus B_o(R)} |\nabla g|^2 d\mu \right)^{1/2} \cdot \left[ V_f(R+1) \setminus V_f(R) \right]^{1/2}. \]

Combining the above two inequalities, we have
\[ (4.3) \quad \int_{B_o(R+1) \setminus B_o(R)} |\nabla g| d\mu \leq C_1 e^{\alpha R^2} \cdot \|g\|_{L^1(\mu)}, \]

where $C_1 = C_1(n, a, b)$.

**Step 2.** We first estimate the $f$-heat kernel $H(x, y, t)$. Recall that, by letting $\epsilon = 1$ in Theorem 1.1, the $f$-heat kernel $H(x, y, t)$ satisfies
\[ H(x, y, t) \leq \frac{c_3}{V_f(B_x(\sqrt{t}))} \cdot \left( \frac{d(x, y)}{\sqrt{t}} + 1 \right)^{\frac{\alpha}{2}} \times \exp \left( c_4 A(4R) - \frac{d^2(x, y)}{5t} \right) \]

for any $x, y \in B_o(R)$ and $0 < t < R^2/4$, where $c_3 = c_3(n)$, $c_4 = c_4(n)$, and $A(4R) = \sup_{x \in B_o(12R)} |f|(x)$. Combining this with theorem assumptions yields
\[ (4.4) \quad H(x, y, t) \leq \frac{c_8}{V_f(B_x(\sqrt{t}))} \cdot \left( \frac{d(x, y)}{\sqrt{t}} + 1 \right)^{\frac{\alpha}{2}} \times \exp \left( c_9 R^2 - \frac{d^2(x, y)}{5t} \right) \]
for any \( x, y \in B_o(R) \) and \( 0 < t < R^2/4 \), where \( c_8 = c_8(n, b) \) and \( c_9 = c_9(n, a) \). Together this with (4.3) gives

\[
J_1 := \int_{B_o(R+1) \setminus B_o(R)} H(x, y, t) |\nabla g(y)| d\mu(y)
\]

\[
\leq \sup_{y \in B_o(R+1) \setminus B_o(R)} H(x, y, t) \cdot \int_{B_o(R+1) \setminus B_o(R)} |\nabla g| d\mu
\]

\[
\leq \frac{C_2 \|g\|_{L^1(\mu)}}{V_f(B_x(\sqrt{t}))} \cdot \left( \frac{R + 1 + d(o, x)}{\sqrt{t}} + 1 \right)^{\frac{9}{2}} \times \exp \left( -\frac{(R - d(o, x))^2}{5t} + c_9(R + 1)^2 \right),
\]

where \( C_2 = C_2(n, a, b) \). Thus, for \( T \) sufficiently small and for all \( t \in (0, T) \) there exists some fixed constant \( \beta > 0 \) such that

\[
J_1 \leq \frac{C_3 \|g\|_{L^1(\mu)}}{V_f(B_x(\sqrt{t}))} \cdot \left( \frac{R + 1 + d(o, x)}{\sqrt{t}} + 1 \right)^{\frac{9}{2}} \times \exp \left( -\beta R^2 + c\frac{d^2(o, x)}{t} \right),
\]

where \( C_3 = C_3(n, a, b) \). Hence, for all \( t \in (0, T) \) and all \( x \in M \), \( J_1 \) tends to zero as \( R \) tends to infinity.

Step 3. Below we shall estimate the gradient of \( H \). Here we adapt the Li’s proof trick (see Section 18 in [20]). Consider the integral with respect to \( d\mu \):

\[
\int_M \phi^2(y)|\nabla H|^2(x, y, t) = -2 \int_M \left< H(x, y, t) \nabla \phi(y), \phi(y) \nabla H(x, y, t) \right>
\]

\[
- \int_M \phi^2(y) H(x, y, t) \Delta f H(x, y, t)
\]

\[
\leq 2 \int_M |\nabla \phi|^2 H^2(x, y, t) + \frac{1}{2} \int_M \phi^2(y)|\nabla H|^2(x, y, t)
\]

\[
- \int_M \phi^2(y) H(x, y, t) \Delta f H(x, y, t).
\]

This implies (4.5)

\[
\int_{B_o(R+1) \setminus B_o(R)} |\nabla H|^2
\]

\[
\leq \int_M \phi^2(y)|\nabla H|^2(x, y, t)
\]

\[
\leq 4 \int_M |\nabla \phi|^2 H^2 - 2 \int_M \phi^2 H \Delta f H
\]

\[
\leq 12 \int_{B_o(R+2) \setminus B_o(R-1)} H^2 + 2 \int_{B_o(R+2) \setminus B_o(R-1)} H |\Delta f H|
\]

\[
\leq 12 \int_{B_o(R+2) \setminus B_o(R-1)} H^2 + 2 \left( \int_{B_o(R+2) \setminus B_o(R-1)} H^2 \right)^{\frac{1}{2}} \left( \int_{M} (\Delta f H)^2 \right)^{\frac{1}{2}}.
\]
Moreover, by Lemma 4.1, we have
\[ \int_M H(x, y, t) d\mu(y) = 1 \]
for all \( x \in M \) and \( t > 0 \). Using this and (4.4), we can estimate
\[
\int_{B_o(R+1) \setminus B_o(R)} H^2(x, y, t) d\mu \leq \sup_{y \in B_o(R+2) \setminus B_o(R-1)} H(x, y, t) \\
\leq \frac{c_8}{V_f(B_o(\sqrt{t}))} \left( \frac{(R + 2 - d(o, x))}{\sqrt{t}} + 1 \right)^{\frac{n}{2}} \\
\times \exp \left[ - \frac{(R - 1 - d(o, x))^2}{5t} + c_9(R + 2)^2 \right].
\]
Also, we claim that there exists a constant \( C_4 > 0 \) such that
\[
\int_M (\Delta f H)^2(x, y, t) d\mu \leq \frac{C_4}{t^2} H(x, y, t).
\]
To prove the claim, we only need to check this inequality for any Dirichlet \( f \)-heat kernel defined on a compact subdomain of \( M \). Because \( f \)-heat kernel on \( M \) can be obtained by taking limits of Dirichlet \( f \)-heat kernels on a compact exhaustion of \( M \). Indeed, if \( H(x, y, t) \) is a Dirichlet \( f \)-heat kernel on a compact subdomain \( \Omega \subset M \), using the eigenfunction expansion, then
\[ H(x, y, t) = \sum_i e^{-\lambda_i t} \psi_i(x) \psi_i(y), \]
where \( \{\psi_i\} \) are orthonormal basis of the space of \( L^2_f \) functions with Dirichlet boundary value satisfying the equation
\[ \Delta f \psi_i = -\lambda_i \psi_i. \]
Differentiating with respect to the variable \( y \), we have
\[ \Delta f H(x, y, t) = -\sum_i \lambda_i e^{-\lambda_i t} \psi_i(x) \psi_i(y). \]
Noticing that \( s^2 e^{-2s} \leq C_5 e^{-s} \) for all \( 0 \leq s < \infty \), therefore
\[ \int_M (\Delta f H)^2 d\mu(y) \leq C_5 t^{-2} \sum_i e^{-\lambda_i t} \psi_i^2(x) = C_5 t^{-2} H(x, x, t) \]
and claim (4.7) follows. Now combining (4.5), (4.6) and (4.7), we obtain
\[
\int_{B_o(R+1) \setminus B_o(R)} |\nabla H|^2 d\mu \leq C_6 \left[ V_f^{-1} + t^{-1} V_f^{-\frac{3}{2}} H^\frac{1}{2}(x, x, t) \right] \\
\times \left( \frac{(R + 2 - d(o, x))}{\sqrt{t}} + 1 \right)^{\frac{n}{2}} \exp \left[ - \frac{(R - 1 - d(o, x))^2}{10t} + c_9(R + 2)^2 \right].
\]
Therefore, by (4.2) and (4.8), using Cauchy-Schwarz inequality we see that
\[
\int_{B_0(R+1)\setminus B_0(R)} |\nabla H| d\mu \leq [V_f(B_0(R+1))|V_f(B_0(R))|^{1/2} \times \left[ \int_{B_0(R+1)\setminus B_0(R)} |\nabla H|^2 d\mu \right]^{1/2} \\
\leq C_6 V_f^{1/2}(B_0(R+1)) \left[ V_f^{-1} + t^{-1} V_f^{-\frac{1}{2}} H_{x,t}^2(x,x,t) \right]^{1/2} \\
\times \left( \frac{R + 2 - d(o,x)}{\sqrt{t}} + 1 \right)^{\frac{1}{2}} \times \exp \left[ - \frac{(R - 1 - d(o,x))^2}{20t} + c_9 (R + 2)^2 \right].
\]
Thus, by (4.2) and (4.8), using Cauchy-Schwarz inequality we see that
\[
J_2 := \int_{B_0(R+1)\setminus B_0(R)} |\nabla H(x,y,t)| g(y) d\mu(y) \\
\leq \sup_{y \in B_0(R+1)\setminus B_0(R)} g(y) \int_{B_0(R+1)\setminus B_0(R)} |\nabla H(x,y,t)| d\mu(y) \\
\leq \frac{C_7 \|g\|_{L^1(\mu)}}{V_f^{1/2}(B_0(2R+2))} \left[ V_f^{-1} + t^{-1} V_f^{-\frac{1}{2}} H_{x,t}^2(x,x,t) \right]^{1/2} \\
\times \left( \frac{R + 2 - d(o,x)}{\sqrt{t}} + 1 \right)^{\frac{1}{2}} \times \exp \left[ - \frac{(R - 1 - d(o,x))^2}{20t} + c_{10} (R + 2)^2 \right],
\]
where \( V_f := V_f(B_o(\sqrt{t})) \). Similar to the case of \( J_1 \), by choosing \( T \) sufficiently small, for all \( t \in (0,T) \) and all \( x \in M \), \( J_2 \) also tends to zero when \( R \) tends to infinity.

**Step 4.** By the mean value theorem, for any \( R > 0 \) there exists \( \bar{R} \in (R,R+1) \) such that
\[
J := \int_{\partial B_0(\bar{R})} [H(x,y,t)|\nabla g|(y) + |\nabla H|(x,y,t)g(y)] d\mu_{\partial B_0(\bar{R})}(y) \\
= \int_{B_0(R+1)\setminus B_0(R)} [H(x,y,t)|\nabla g|(y) + |\nabla H|(x,y,t)g(y)] d\mu(y) \\
= J_1 + J_2.
\]
By step 2 and step 3, we know that by choosing \( T \) sufficiently small, for all \( t \in (0,T) \) and all \( x \in M \), \( J \) tends to zero as \( \bar{R} \) (and hence \( R \)) tends to infinity. Therefore we complete Theorem 4.3 for \( T \) sufficiently small.

**Step 5.** At last, using the semigroup property of the \( f \)-heat equation,
\[
\frac{\partial}{\partial(s+t)} \left( e^{(s+t)\Delta_f} g \right) = \frac{\partial}{\partial t} \left( e^{s\Delta_f} e^{t\Delta_f} g \right) = e^{s\Delta_f} \frac{\partial}{\partial t} \left( e^{t\Delta_f} g \right) \\
= e^{s\Delta_f} e^{t\Delta_f} (\Delta_f g) = e^{(s+t)\Delta_f} (\Delta_f g),
\]
which implies Theorem 4.3 for all time \( t > 0 \). \( \square \)

We are now ready to prove Theorem 1.5 by Theorem 4.3, following the idea in [19].

**Proof of Theorem 1.5.** Let \( g(x) \) be a nonnegative, \( L^1_f \)-integrable and \( f \)-subharmonic function defined on \( M \). Now we define a space-time function
\[
 g(x,t) := \int_M H(x,y,t)g(y)\,d\mu(y)
\]
with initial data \( g(x,0) = g(x) \). From Theorem 4.3 we conclude that
\[
 \frac{\partial}{\partial t} g(x,t) = \int_M \frac{\partial}{\partial t} H(x,y,t)g(y)\,d\mu(y) \]
\[
 = \int_M \Delta f_y H(x,y,t)g(y)\,d\mu(y) \]
\[
 = \int_M H(x,y,t)\Delta f_y g(y)\,d\mu(y) \geq 0,
\]
which means that \( g(x,t) \) is increasing for all \( t \). By Lemma 4.1
\[
 \int_M H(x,y,t)\,d\mu(y) = 1
\]
for all \( x \in M \) and \( t > 0 \). This implies
\[
 \int_M g(x,t)\,d\mu(x) = \int_M \int_M H(x,y,t)g(y)\,d\mu(y)\,d\mu(x) = \int_M g(y)\,d\mu(y).
\]
Since \( g(x,t) \) is increasing in \( t \) by (4.9), then \( g(x,t) = g(x) \) and hence \( g(x) \)
be a nonnegative \( f \)-harmonic function, i.e. \( \Delta f g(x) = 0 \).

On the other hand, for any positive constant \( a \), let us define a new function
\[
 h(x) := \min\{g(x), a\}. \]
Then \( h \) satisfies
\[
 0 \leq h(x) \leq g(x), \quad |\nabla h| \leq |\nabla g| \quad \text{and} \quad \Delta f h(x) \leq 0.
\]
In particular, it will satisfy the same estimates, (4.2) and (4.3), as the function \( g \). Hence we can show that
\[
 \frac{\partial}{\partial t} h(x,t) = \frac{\partial}{\partial t} \int_M H(x,y,t)h(y)\,d\mu(y) \]
\[
 = \int_M H(x,y,t)\Delta f_y h(y)\,d\mu(y) \leq 0.
\]
Notice that here \( h \) is still \( L^1_f \). Following the same argument as before,
we again have that \( \Delta f h(x) = 0 \). By the regularity theory of \( f \)-harmonic functions, this is impossible unless \( h = g \) or \( h = a \). Since \( a \) is arbitrary and \( g \) is nonnegative, this implies \( g \) must be identically constant. At last, due to the fact that the absolute value of a \( f \)-harmonic function is a nonnegative \( f \)-subharmonic, the theorem follows. \( \square \)
5. $L^1_f$-uniqueness property

For the completeness we follow the arguments of [19] to give a detailed proof of Theorem 1.9.

Proof of Theorem 1.9. Let $u(x, t) \in L^1_f$ be a nonnegative function for all $t > 0$ satisfying all assumptions of Theorem 1.9. For $\epsilon > 0$, let $u_\epsilon(x) = u(x, \epsilon)$. Define

$$e^{t\Delta_f} u_\epsilon(x) := \int_M H(x, y, t) u_\epsilon(y) d\mu(y)$$

and

$$F_\epsilon(x, t) := \max\{0, u(x, t + \epsilon) - e^{t\Delta_f} u_\epsilon(x)\}.$$

Then $F_\epsilon(x, t)$ is nonnegative and satisfies

$$\lim_{t \to 0} F_\epsilon(x, t) = 0 \quad \text{and} \quad (\partial_t - \Delta_f) F_\epsilon(x, t) \leq 0.$$

Let $T > 0$ be fixed. Define $g(x) := \int_0^T F_\epsilon(x, t) dt$, which satisfies

$$\Delta_f g(x) = \int_0^T \Delta_f F_\epsilon(x, t) dt \geq \int_0^T \partial_t F_\epsilon(x, t) dt = F_\epsilon(x, T) \geq 0.$$

Moreover,

$$\int_M g(x) d\mu = \int_0^T \int_M F_\epsilon(x, t) d\mu dt \leq \int_0^T \int_M |u(x, t + \epsilon) - e^{t\Delta_f} u_\epsilon(x)| d\mu dt$$

$$\leq \int_0^T \int_M u(x, t + \epsilon) d\mu dt + \int_0^T \int_M e^{t\Delta_f} u_\epsilon(x) d\mu dt < \infty,$$

where the first term on the right is finite by our assumption, and the second term is finite because $e^{t\Delta_f}$ is a contractive semigroup in $L^1_f$. Therefore, $g(x)$ is a nonnegative $L^1_f$-integrable $f$-subharmonic function. By Theorem 1.5, $g(x)$ must be constant. Combining this with (5.2) yields $F_\epsilon(x, t) = 0$. Hence $F_\epsilon(x, T) \equiv 0$ for all $x \in M$ and $T > 0$. This implies

$$e^{t\Delta_f} u_\epsilon(x) \geq u(x, t + \epsilon).$$

Now we estimate the function $e^{t\Delta_f} u_\epsilon(x)$ in (5.1). Applying the upper bound estimate (1.2) of the heat kernel $H(x, y, t)$ and letting $R = 2d(x, y) + 1$, we have

$$e^{t\Delta_f} u_\epsilon(x) \leq \frac{C}{V_f(B_x(\sqrt{t}))} \left( \frac{d(x, y)}{\sqrt{t}} + 1 \right)^{\frac{d}{2}}$$

$$\times \int_M \left[ \exp \left( C d^2(x, y) - \frac{d^2(x, y)}{5t} \right) u(y, \epsilon) \right] d\mu(y).$$

Thus there exists a sufficiently small $t_0 > 0$ such that for all $0 < t < t_0$, we have

$$\lim_{\epsilon \to 0} e^{t\Delta_f} u_\epsilon(x) = 0$$

since the theorem assumption:

$$\lim_{\epsilon \to 0} \int_M u(x, \epsilon) d\mu(x) = 0.$$
By the semigroup property, we further conclude \( \lim_{\varepsilon \to 0} e^{t \Delta} u_\varepsilon(x) = 0 \) for all \( x \in M \) and \( t > 0 \). Combining this with (5.3) yields \( u(x, t) \leq 0 \). Therefore \( u(x, t) \equiv 0 \), since \( u \) is nonnegative. To prove the uniquely property, we only need to consider its absolute value and apply the above result. □

6. Appendix

In the appendix we solve for the \( f \)-heat kernel of 1-dimensional steady Gaussian soliton \( (\mathbb{R}, g_0, e^{-f}dx) \), where \( g_0 \) is the Euclidean metric, and \( f = kx \) with \( k = \pm 1 \). The method is standard separation of variables. Suppose the \( f \)-heat kernel is of the form

\[
H(x, y, t) = \varphi(y)\phi(x)\psi(t) \times \exp \left( -\frac{|x - y|^2}{4t} \right).
\]

For fixed \( y \), we get

\[
H_t = \varphi \phi e^{-\frac{|x - y|^2}{4t}} \left( \psi_t + \psi \frac{|x - y|^2}{4t^2} \right),
\]

\[
H_x = \varphi \psi e^{-\frac{|x - y|^2}{4t}} \left( \phi_x - \phi \frac{x - y}{2t} \right),
\]

\[
H_{xx} = \varphi \psi e^{-\frac{|x - y|^2}{4t}} \left( \phi_{xx} + \phi \frac{x - y}{2t} \phi_x - \phi \frac{x - y}{t} - \phi \frac{1}{2t} \right).
\]

So \( H_t = H_{xx} - f_x H_x \) implies

\[
\phi \left( \psi_t + \psi \frac{|x - y|^2}{4t^2} \right) = \psi \left( \phi_{xx} + \phi \frac{|x - y|^2}{4t^2} - \phi \frac{x - y}{t} - \phi \frac{1}{2t} \right) - k\psi \left( \phi_x - \phi \frac{x - y}{2t} \right).
\]

That is,

\[
\frac{\psi_t}{\psi} = \frac{\phi_{xx} - k\phi_x}{\phi} - \frac{x - y}{2t}, \quad \frac{2\phi_x - k\phi}{\phi} - \frac{1}{2t}.
\]

Therefore

\[
\frac{\phi_{xx} - k\phi_x}{\phi} = C_1, \quad \frac{(2\phi_x - k\phi)(x - y)}{\phi} = C_2, \quad \frac{\psi_t}{\psi} = C_1 - \frac{1 + C_2}{2t},
\]

From above, their solutions are

\[
\phi = C_3e^{\frac{kx}{2}}, \quad \psi = C_4\frac{1}{\sqrt{t}}e^{-\frac{t}{4}},
\]

where \( C_1, C_2, C_3, C_4 \) are constants.

By the initial condition \( \lim_{t \to 0} u(x, t) = \delta_{f,y}(x) \) we get \( \phi(y) = e^{\frac{kx}{2}} \), and \( C_3C_4 = \frac{1}{2\sqrt{\pi}} \). Therefore the \( f \)-heat kernel is

\[
H(x, y, t) = \frac{e^{\pm\frac{kx}{2}} \cdot e^{-\frac{t}{4}}}{\sqrt{4\pi t}} \times \exp \left( -\frac{|x - y|^2}{4t} \right).
\]

It is easy to check that \( \int_{\mathbb{R}} H(x, y, t)e^{-f(x)}dx = 1 \), which confirms the stochastic completeness proved in Lemma 4.1.
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