THE EKOR-STRATIFICATION ON THE SIEGEL MODULAR VARIETY WITH PARAHORIC LEVEL STRUCTURE

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Abstract. We study the arithmetic geometry of the reduction modulo \( p \) of the Siegel modular variety with parahoric level structure. We realize the EKOR-stratification on this variety as the fibers of a smooth morphism into an algebraic stack parametrizing homogeneously polarized chains of certain truncated displays.

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1. Introduction

Fix a rational prime \( p \), a positive integer \( g \), an auxiliary integer \( N \geq 3 \) that is not divisible by \( p \) and a non-empty subset \( J \subseteq \mathbb{Z} \) with \( J + 2g\mathbb{Z} = J \) and \( -J = J \). Then we are interested in studying the Siegel modular variety with parahoric level structure

\[ \mathcal{A}_{g,J,N} \]

over \( \mathbb{Z}_p \). This is a quasiprojective scheme that parametrizes certain polarized chains of type \( J \) of \( g \)-dimensional Abelian varieties with full level-\( N \) structure; its moduli description was first given by de Jong \cite{Jon93} and then in full generality by Rapoport and Zink \cite{RZ96}. The scheme \( \mathcal{A}_{g,J,N} \) is the prime example of an integral model of a Shimura variety at a place of parahoric bad reduction. While the generic fiber of \( \mathcal{A}_{g,J,N} \) is smooth, its special fiber typically is singular. This is related to the fact that the \( p \)-torsion of an Abelian variety in characteristic \( p \) is not étale.

Now fix an algebraic closure \( \overline{\mathbb{F}}_p \) of \( \mathbb{F}_p \), let \( \mathbb{Z}_p := W(\mathbb{F}_p) \) be the ring of \( p \)-typical Witt vectors of \( \mathbb{F}_p \) and write \( \mathbb{Q}_p := \mathbb{Z}_p[1/p] \) for its fraction field. Denote by \( \sigma \) the Frobenius on \( \mathbb{Q}_p \). Also fix a symplectic \( \mathbb{Q}_p \)-vector space \( V \) of dimension \( 2g \) and a self-dual lattice chain \( (\Lambda_i)_{i \in J} \) in \( V \). Then there exists a natural map, called the central leaves map,

\[ \Upsilon: \mathcal{A}_{g,J,N}(\overline{\mathbb{F}}_p) \to \tilde{K}_\sigma \backslash \text{GSp}(V)(\mathbb{Q}_p); \]

here \( \tilde{K} \subseteq \text{GSp}(V)(\mathbb{Q}_p) \) is the stabilizer of the lattice chain \( (\Lambda_i)_i \) and modding out by \( \tilde{K}_\sigma \) is notation for taking the quotient by the \( \sigma \)-conjugation action \( g : x \mapsto g\times\sigma(g)^{-1} \). It is roughly given by sending a polarized chain of Abelian varieties to the twisted conjugacy class corresponding to the Frobenius \( \Phi \) of the associated
rational Dieudonné module, see the work of Oort [Oor02] and also He and Rapoport [HR15]. The image of this map is given by $\tilde{K}_0 \backslash X$ where

$$X = \bigcup_{w \in \text{Adm}_{g,J}} \tilde{K}_w \tilde{K} \subseteq \text{GSp}(V)(\bar{\mathbb{Q}}_p)$$

is a finite union of double cosets that is indexed over the so called admissible set $\text{Adm}_{g,J}$.

The fibers of the composition

$$\lambda: A_{g,J,N}(\mathbb{F}_p) \xrightarrow{\Upsilon} \tilde{K}_0 \backslash X \to \tilde{K}_0 \backslash X / \tilde{K} \cong \text{Adm}_{g,J}$$

define a decomposition of $(A_{g,J,N})_{\mathbb{F}_p}$ into smooth locally closed subschemes, the so-called Kottwitz-Rapoport (KR) stratification. In fact Rapoport and Zink construct the following data.

- A flat projective scheme $M_{\text{loc}}$ over $\mathbb{Z}_p$ with an action of the parahoric $\mathbb{Z}_p$-group scheme $\mathcal{G}$ for GSp$(V)$ given by the lattice chain $(\Lambda_i)_i$. This scheme is called the local model; it parametrizes isotropic chains of $g$-dimensional subspaces $C_i \subseteq \Lambda_i$ and satisfies $M_{\text{loc}}(\mathbb{F}_p) = \tilde{K} \backslash X$.
- A smooth morphism

$$A_{g,J,N} \to [G_{\text{GM}}]$$

that parametrizes the Hodge filtration in the de Rham cohomology of a polarized chain of Abelian varieties. One recovers the map $\lambda$ from above by evaluating this morphism on $\mathbb{F}_p$-points, see [HR15, Section 7.(ii)].

He and Rapoport also consider the composition

$$\upsilon: A_{g,J,N}(\mathbb{F}_p) \xrightarrow{\Upsilon} \tilde{K}_0 \backslash X \to \tilde{K}_0 \backslash X / \tilde{K}_1 \subseteq \tilde{K} \backslash X$$

where $\tilde{K}_1 \subseteq \tilde{K}$ is the pro-unipotent radical. They observe that the codomain of $\upsilon$ is a finite set and call its fibers Ekedahl-Kottwitz-Oort-Rapoport (EKOR) strata.

In the hyperspecial case $J = 2g\mathbb{Z}$ the EKOR stratification is also called Ekedahl-Oort (EO) stratification and was first considered by Oort [Oor01]. Two points in $A_{g,2g\mathbb{Z},N}(\mathbb{F}_p)$ given by two (principally polarized) Abelian varieties lie in the same EO stratum if and only if their $p$-torsion group schemes are isomorphic. Viehmann and Wedhorn [VW10] realize the EO stratification as the fibers of a smooth morphism from $(A_{g,2g\mathbb{Z},N})_{\mathbb{F}_p}$ into an algebraic stack that parametrizes $F$-zips in the sense of Moonen and Wedhorn [MW04] with symplectic structure.

**Question.** Is it possible for general $J$ to realize the map $\upsilon$, or maybe even $\Upsilon$, as a smooth morphism from $A_{g,J,N}$ into some naturally defined algebraic stack?

The existence of such a smooth morphism would in particular give a new proof of the smoothness of the EKOR strata and the closure relations between them. More importantly, it also provides a tool that may be applied to further study the geometry of $A_{g,J,N}$.

Let us give an overview of results that have been obtained so far (and that we are aware of), also considering more general Shimura varieties than the Siegel modular variety:

- Moonen and Wedhorn [MW04] introduce the notion of an $F$-zip that is a characteristic $p$ analogue of the notion of a Hodge structure. Given an Abelian variety $A$ over some $\mathbb{F}_p$-algebra $R$ the de Rham cohomology $H_{\text{dR}}^1(A/R)$ naturally carries the structure of an $F$-zip. If $R$ is perfect then the datum of this $F$-zip is equivalent to the datum of the Dieudonné module of the $p$-torsion $A[p]$. Pink, Wedhorn and Ziegler [PWZ15] define a group theoretic version of the notion of an $F$-zip.
- Viehmann and Wedhorn [VW10] define a moduli stack of $F$-zips with polarization and endomorphism structure in a PEL-type situation with hyperspecial level structure. They construct a
morphism from the special fiber of the associated Shimura variety to this stack, parametrizing the EO stratification. They then show that this morphism is flat and use this to show that the EO strata are non-empty and quasi-affine and to compute their dimension. The smoothness of the EO strata was already shown by Vasiu [Vas08].

Zhang [Zha18] constructs a morphism from the special fiber of the Kisin integral model, see [Kis10], of a Hodge type Shimura variety to a stack of group-theoretic zips. They then show that this morphism is smooth and thus gives an EO stratification with the desired properties. Shen and Zhang [SZ17] later generalize this to Shimura varieties of Abelian type.

Shen, Yu and Zhang [SYZ21] further generalize this to Shimura varieties of Abelian type at parahoric level, where the construction of integral models is due to Kisin and Pappas, see [KP18]. However, they only construct a morphism from each KR stratum of the special fiber of the Shimura variety into a certain stack of group-theoretic zips, parametrizing the EKOR strata contained in this KR stratum. Still, they show that this morphism is smooth, thus establishing the smoothness of the EKOR strata.

Hesse [Hes20] considers an explicit moduli stack of polarized chains of \( F \)-zips and constructs a morphism from the Siegel modular variety into this stack. However it appears that such a morphism is not well-behaved in the general parahoric situation, see Remark 3.17.

- Xiao and Zhu [XZ17] consider a perfect moduli stack of mixed characteristic local shtukas as well as restricted versions in a hyperspecial situation. For a Shimura variety of Hodge type they construct a morphism from the perfection of its special fiber into the stack of local shtukas and show that it realizes the central leaves map. They claim that the induced morphism to the moduli stack of restricted local shtukas is perfectly smooth, see [XZ17, Proposition 7.2.4]; however the author thinks that their proof contains an error that cannot be easily resolved (namely the diagram appearing in the proof is not commutative).

Shen, Yu and Zhang [SYZ21] again generalize this to the parahoric situation. Similarly as Xiao and Zhu they construct a morphism from the perfected special fiber of the Shimura variety to their stack of local shtukas and claim that the induced morphism to the moduli stack of restricted local shtukas is perfectly smooth, but their proof [SYZ21, Theorem 4.4.3] contains the same error.

- Zink [Zin02] introduces the notion of a display that is a non-perfect version of the notion of a Dieudonné module. Büttel and Pappas [BP18] define a group-theoretic version; this gives a deperfection of the notion of a local shtuka from [XZ17] in the hyperspecial situation. In a parahoric Hodge type situation Pappas [Pap22] also gives a definition of group-theoretic displays, however only over \( p \)-torsionfree \( p \)-adic rings. Pappas also constructs a group-theoretic display on the \( p \)-completion of a Kisin-Pappas integral model.

1.1. Overview. Let us explain the content of the present work. The letter \( R \) denotes a \( p \)-nilpotent ring.

We start by recalling the definition of a (3n-)display in the sense of [Zin02].

**Definition 1.1 (2.9).** Let \( 0 \leq d \leq h \) be integers. Then a **display of type \((h,d)\)** over \( R \) is a tuple \((M, M_1, \Psi)\) that is given as follows.

- \( M \) is a finite projective \( W(R) \)-module of rank \( h \).
- \( M_1 \subseteq M \) is a \( W(R) \)-submodule containing \( I_R M \) and such that \( M_1/I_R M \subseteq M/I_R M \) is a direct summand of rank \( d \).
- \( \Psi : \tilde{M}_1 \to M \) is an isomorphism of \( W(R) \)-modules that we call the **divided Frobenius**.

Here \( W(R) \) denotes the ring of Witt vectors of \( R \), \( I_R \) denotes the kernel of the projection \( W(R) \to R \) and the object \( \tilde{M}_1 \) is a certain finite projective \( W(R) \)-module attached to \((M, M_1)\), see Definition 2.7 for more details. When \( R \) is a perfect ring of characteristic \( p \) then \( \tilde{M}_1 \) agrees with the Frobenius-twist \( M_1^p \).
The category of displays carries a natural duality and an action of the symmetric monoidal category of tuples \((I, i)\) consisting of an invertible \(W(R)\)-module \(I\) and an isomorphism \(i : I^\sigma \to I\).

For positive integers \(m \geq n + 1\) we define the notion of an \((m, n)\)-truncated displays that is inspired by the restricted local shnikas from \([XZ17]\) Definition 5.3.1. The word “truncated” refers to the use of truncated Witt vectors, the numbers \(m\) and \(n\) roughly measures how truncated the module \(M\) and the divided Frobenius \(Ψ\) is. One should note that this notion of an \((m, n)\)-truncated display is different from the notion of an \(n\)-truncated display as defined by Lau and Zink in \([LZ18]\), but see Remark 2.14 for a comparison. For us it will be crucial to work with the \((m, n)\)-truncated objects, see Remark 5.17.

The following theorem gives a relation between displays and \(p\)-divisible groups.

**Theorem 1.2** ([Lau10] Theorem 5.1, 2.24). There is a natural functor

\[
D : \{\text{\(p\)-divisible groups of height } h \text{ and dimension } d \text{ over } R\}^{\text{op}} \to \{\text{displays of type } (h, d) \text{ over } R\}.
\]

that restricts to an equivalence between formal \(p\)-divisible groups and \(F\)-nilpotent displays.

As we are interested in studying the moduli space \(A_{g,J,N}\) of polarized chains of Abelian varieties it makes sense to make the following definition.

**Definition 1.3** ([Lau10]). A **homogeneously polarized chain of displays of type** \((g, J)\) **over** \(R\) is a tuple

\[
((M_i)_i, (ρ_{i,j})_{i,j}, (θ_{i})_i, (M_{i,1})_i, (Ψ_i)_i, I, i, (λ_i)_i)
\]

that is given as follows.

- \(((M_i, M_{i,1}, Ψ_i)_i, (ρ_{i,j})_{i,j})\) is a diagram of shape \(J\) in the category of displays of type \((2g, J)\) such that the homomorphism of \(R\)-modules
  \[
  ρ_{i,j} : R ⊗_{W(R)} M_i \to R ⊗_{W(R)} M_j
  \]
  is of constant rank \(2g - (j - i)\) for all \(i \leq j \leq i + 2g\).
- \(θ_{i} : (M_i, M_{i,1}, Ψ_i) \to (M_{i+2g}, M_{i+2g,1}, Ψ_{i+2g})\) is an isomorphism such that we have the compatibilities \(θ_j \circ ρ_{i,j} = ρ_{i+2g,j+2g} \circ θ_i\) and \(ρ_{i,i+2g} = pθ_i\).
- \((I, i)\) is as in Definition 1.1.
- \(λ_{i} : (M_i, M_{i,1}, Ψ_i) \to (I, i) \otimes (M_{-i}, M_{-i,1}, Ψ_{-i})\) is an antisymmetric isomorphism such that we have \(λ_j \circ ρ_{i,j} = (id_{(I, i)} \otimes ρ_{x,j,-i}) \circ λ_{i}\).

We again also give an \((m, n)\)-truncated version of this definition. When \(R\) is of characteristic \(p\) then we allow \(n\) to take the additional value \(1\)-rdt that can be thought of as being slightly smaller than \(1\); “rdt” refers to the term “reductive quotient”. Roughly the case \(n = 1\)-rdt corresponds to only having a divided Frobenius on the graded pieces of the \(1\)-truncated chain of modules, see Definition 5.1.

We then show that the stack \(\text{HPolChDisp}_{g,J}^{(m,n)}\) of \((m, n)\)-truncated homogeneously polarized chains of displays over \(\text{Spf}(Z_p)\) admits a quotient stack description.

**Proposition 1.4** ([Lau10]). There exists an equivalence

\[
\text{HPolChDisp}_{g,J}^{(m,n)} \to \left((L^{(m)}G)_\Delta \backslash M^{\text{loc.}(n)}\right),
\]

where we use the following notation.

- \(L^{(m)}G\) denotes the \(m\)-truncated Witt vector positive loop group of \(G\), see Section 2.7.
- \(M^{\text{loc.}}\) is the \(p\)-completion of the local model \(M^{\text{loc.}}\) and \(M^{\text{loc.}(n)} \to M^{\text{loc.}}\) is a certain \(L^{(m)}G\)-equivariant \(L^{(m)}G\)-torsor, see Definition 2.20.
- The subscript \(\Delta\) indicates that we take the quotient by the diagonal action.
Thus our definition of \( \text{HPolChDisp}_{g,J}^{(m,n)} \) gives (up to a slight difference in normalization) a deperfection of the stack of parahoric restricted local shtukas from \cite[Section 4]{SYZ21}, see Remark \ref{rem:deperfection}. In particular we obtain bijections

\[
\text{HPolChDisp}_{g,J}^{(m)}(\bar{\mathbb{F}}_p) \to \tilde{K}_p \backslash X \quad \text{and} \quad \text{HPolChDisp}_{g,J}^{(m,1-\text{rdt})}(\bar{\mathbb{F}}_p) \to \tilde{K}_p \backslash (\tilde{K}_1 \backslash X).
\]

Applying Theorem \ref{thm:moduli-description} to the moduli description of \( A_{g,J,N}^{\wedge} \) we thus obtain a natural morphism

\[
\Upsilon: A_{g,J,N}^{\wedge} \to \text{HPolChDisp}_{g,J}^{(m,n)}
\]

that realizes the central leaves map, see Definition \ref{def:central-leaves-map} here \( A_{g,J,N}^{\wedge} \) denotes the \( p \)-completion of \( A_{g,J,N} \). For any \( m \geq 2 \) the composition

\[
(A_{g,J,N})_{\bar{\mathbb{F}}_p} \Upsilon \to \text{HPolChDisp}_{g,J}^{(m,n)} \to \text{HPolChDisp}_{g,J}^{(m,1-\text{rdt})}
\]

then realizes the map \( \nu \) parametrizing the EKOR stratification. Our main result is now the following.

**Theorem 1.5** (4.10). For every \((m,n)\) with \( n \neq 1-\text{rdt} \) the natural morphism \( A_{g,J,N}^{\wedge} \to \text{HPolChDisp}_{g,J}^{(m,n)} \) is smooth. Similarly for every \( m \geq 2 \) also the morphism \( (A_{g,J,N})_{\bar{\mathbb{F}}_p} \to \text{HPolChDisp}_{g,J}^{(m,1-\text{rdt})} \) is smooth.

**Strategy of proof.** By the Serre-Tate theorem the smoothness of the morphism at a point of \( |A_{g,J,N}^{\wedge}| \) corresponding to a polarized chain of Abelian varieties only depends on the associated polarized chain of \( p \)-divisible groups. Using Theorem \ref{thm:moduli-description} we then show that the morphism smooth along the formal locus, i.e. the locus of chains of Abelian varieties with formal \( p \)-divisible groups. To obtain the smoothness in general we then finally show that there are enough points in \( |A_{g,J,N}^{\wedge}| \) that specialize into the formal locus, see Corollary \ref{cor:enough-points} for a precise statement. \hfill \Box

As a natural next step it would be interesting to generalize our results to the case of a more general Shimura variety at parahoric level instead of the Siegel modular variety. It could be expected that there exists a stack of \( (\mathcal{G},\mu) \)-displays for every datum \( (\mathcal{G},\mu) \) consisting of a parahoric \( \mathbb{Z}_p \)-group scheme \( \mathcal{G} \) and a minuscule geometric conjugacy class \( \mu \) of cocharacters of the generic fiber of \( \mathcal{G} \), as well as truncated versions that recover the definition from \cite{BP18} in the hyperspecial case and give back our definition when the generic fiber of \( \mathcal{G} \) is either a general linear group or a group of symplectic similitudes. In a situation where the local group datum \( (\mathcal{G},\mu) \) comes from a Shimura datum there then should be a natural smooth morphism from the \( p \)-completion of the corresponding Shimura variety into the stack of truncated \( (\mathcal{G},\mu) \)-displays. As already mentioned above, partial results in this direction have been achieved by Pappas in \cite{Pap22}.

**1.2. Acknowledgements.** I am very grateful to Ulrich Görtz for introducing me to the subject of Shimura varieties and their special fibers and for all the support I received from him throughout this project. Furthermore, I would like to thank Jochen Heinloth, Ludvig Modin, Herman Rohrbach, Pol van Hoften and Torsten Wedhorn for helpful conversations. Finally I also want to thank the anonymous referee their valuable feedback, suggesting significant improvements to an earlier version of the manuscript.

This work was partially funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation), Graduiertenkolleg 2553 and SFB-TRR 358 - 491392403.

**2. Pairs and displays**

In this section we give recall some of the theory of (not necessarily nilpotent) displays from \cite{Zin02}. We also develop an analogous theory of \((m,n)\)-truncated displays that is inspired by the definition of \((m,n)\)-restricted local shtukas given in \cite{XZ17}.

The letter \( R \) denotes a \( p \)-nilpotent ring and \((m,n)\) denotes a tuple of positive integers with \( m \geq n + 1 \).
2.1. **Witt vectors.** We use the following notation concerning Witt vectors.

- We write $W(R)$ for the ring of *(p-typical)* Witt vectors of $R$ and $I_R \subseteq W(R)$ for its augmentation ideal, i.e. the kernel of the projection $W(R) \to R$.

  Similarly we write $W_n(R)$ for the ring of $n$-truncated Witt vectors of $R$ and $I_{n,R} \subseteq W_n(R)$ for its augmentation ideal.

- We denote the **Witt vector Frobenius** on $W(R)$ by $\sigma: W(R) \to W(R)$. Given a $W(R)$-module $M$ we write $M^\sigma := W(R) \otimes_{\sigma,W(R)} M$ for its Frobenius twist.

  Note that $\sigma$ induces a ring homomorphism $\sigma: W_n(R) \to W_n(R)$ on truncated Witt vectors (here it is crucial that $m \geq n + 1$, at least when $R$ is not of characteristic $p$).

  When $R$ is of characteristic $p$ we also write $\sigma: R \to R$ for the $p$-power Frobenius on $R$.

- We write
  
  $$W(R)^{\sigma=id} := \{ x \in W(R) \mid \sigma(x) = x \} \subseteq W(R)$$

  for the subring of $\sigma$-invariant elements. Note that we have a natural isomorphism

  $$Z_p(R) := \text{Cont}([\text{Spec}(R)], Z_p) \to W(R)^{\sigma=id};$$

  when $R$ is of characteristic $p$ this follows from the chain of identifications

  $$Z_p(R) \cong W(F_p(R)) \cong W(R^{\sigma=id}) \cong W(R)^{\sigma=id}$$

  and the general case follows because the reduction morphism $Z_p(R) \to Z_p(R/pR)$ is an isomorphism and the $\sigma$-stable ideal $\ker(W(R) \to W(R/pR)) \subseteq W(R)$ is killed by a power of $\sigma$ so that every $\sigma$-invariant element $x \in W(R/pR)$ lifts uniquely to a $\sigma$-invariant element $\bar{x} \in W(R)$.

  Similarly we write

  $$W_m(R)^{\sigma=id} := \{ x \in W_m(R) \mid \sigma(x) = x \in W_n(R) \} \subseteq W_m(R).$$

  Note that this is slightly ambiguous as the chosen $m$ does not appear in the notation; however it should always be clear from the context what $n$ is used.

- Recall that the inverse of the **Verschiebung** is a $\sigma$-linear map $I_R \to W(R)$. We denote its linearization by $\sigma^{\text{div}}: I_R^\sigma \to W(R)$ and call it the **divided Frobenius**. For $x \in I_R$ we have $p \cdot \sigma^{\text{div}}(1 \otimes x) = \sigma(x)$, justifying the name.

  We also have a truncated variant of the divided Frobenius $\sigma^{\text{div}}: W_n(R) \otimes_{\sigma,W_m(R)} I_{m,R} \to W_n(R)$.

- Let $M, N$ be finite projective $W(R)$-modules and let $f: M \to I_R N \subseteq N$ be a $W(R)$-linear map. We write $f^{\sigma,\text{div}}$ for the composition

  $$f^{\sigma,\text{div}}: M^\sigma \xrightarrow{f^\sigma} I_R^\sigma \otimes_{W(R)} N^\sigma \xrightarrow{\sigma^{\text{div}} \otimes \text{id}} N^\sigma.$$ 

  Then we have $p \cdot f^{\sigma,\text{div}} = f^{\sigma}$ and given homomorphisms of finite projective $W(R)$-modules $g: L \to M$ and $h: N \to P$ we have $(f \circ g)^{\sigma,\text{div}} = f^{\sigma,\text{div}} \circ g^{\sigma}$ and $(h \circ f)^{\sigma,\text{div}} = h^{\sigma} \circ f^{\sigma,\text{div}}$.

  Similarly, given finite projective $W_m(R)$-modules $M$ and $N$ and a homomorphism of $W_m(R)$-modules $f: M \to I_{m,R} N \subseteq N$ we write $f^{\sigma,\text{div}}$ for the composition

  $$f^{\sigma,\text{div}}: W_n(R) \otimes_{\sigma,W_m(R)} M \xrightarrow{f^\sigma} \left( W_n(R) \otimes_{\sigma,W_m(R)} I_{m,R} \right) \otimes_{W_m(R)} \left( W_n(R) \otimes_{\sigma,W_m(R)} N \right) \xrightarrow{\sigma^{\text{div}} \otimes \text{id}} W_n(R) \otimes_{\sigma,W_m(R)} N.$$ 

  This construction has similar properties as the non-truncated version.

- Given a smooth affine $\mathbb{Z}_p$-group scheme $G$ we write $L^+G$ for the *(Witt vector)* **positive loop group** of $G$, i.e. the flat affine $\mathbb{Z}_p$-group scheme given by $(L^+G)(R) = G(W(R))$ (see also [BH20, Section 2.2]).
We also write $L^{(n)}G$ for the $n$-truncated positive loop group of $G$, i.e. the smooth affine $\mathbb{Z}_p$-group scheme given by $(L^{(n)}G)(R) = G(W_n(R))$.

We also record the following technical lemma that will be used multiple times throughout the article.

**Lemma 2.1.** Suppose that we are given an admissible linearly topologized ring $A$ (see [Stacks, Tag 07E8]) such that $p$ is topologically nilpotent in $A$. Let $M, M'$ be finite projective $A$-modules of rank $h$ and let $f : M \to M'$ and $g : M' \to M$ be morphisms of $A$-modules such that $g \circ f = p \cdot \text{id}_M$ and $f \circ g = p \cdot \text{id}_{M'}$. Suppose furthermore that there exist integers $\ell$ and $\ell'$ with $\ell + \ell' = h$ such that for every continuous ring homomorphism $A \to k$ with $k$ an algebraically closed field the induced homomorphisms of $k$-vector spaces

$$f : k \otimes_A M \to k \otimes_A M' \quad \text{and} \quad g : k \otimes_A M' \to k \otimes_A M$$

are of rank $\ell$ and $\ell'$.

Then the induced homomorphisms of $A/pA$-modules

$$f : M/pM \to M'/pM' \quad \text{and} \quad g : M'/pM' \to M/pM$$

are of constant rank $\ell$ and $\ell'$ in the sense that their respective image is a direct summand (hence finite projective) of the indicated rank.

**Proof.** Let $a \subseteq A$ be an ideal of definition that contains $p$. Note that the condition that $f$ and $g$ are of constant rank $\ell$ and $\ell'$ is representable by a finitely presented locally closed subscheme $Z \subseteq \text{Spec}(A/a)$. By assumption we have $|Z| = |\text{Spec}(A/a)|$ so that $Z$ is the vanishing locus of a nilpotent ideal $b \subseteq A/a$. After replacing $a$ by the preimage of $b$ under $A \to A/a$ we thus may assume that the homomorphisms of $A/a$-modules

$$f : M/aM \to M'/aM' \quad \text{and} \quad g : M'/aM' \to M/aM$$

are of constant rank $\ell$ and $\ell'$.

Modulo $a$ the morphisms $f$ and $g$ now compose to 0 in both directions and are of complementary constant rank. Thus there exist direct sum decompositions

$$M/aM = P \oplus P' \quad \text{and} \quad M'/aM' = P \oplus P'$$

with respect to which we have $f = (\begin{smallmatrix} \text{id}_P & 0 \\ 0 & 0 \end{smallmatrix})$ and $g = (\begin{smallmatrix} 0 & 0 \\ \text{id}_{P'} & 0 \end{smallmatrix})$. Lift these decompositions to

$$M = Q \oplus Q' \quad \text{and} \quad M' = Q \oplus Q'.$$

After possibly modifying the lifts we may then assume that

$$f = \begin{pmatrix} \text{id}_Q & 0 \\ 0 & * \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} * & * \\ * & \text{id}_{Q'} \end{pmatrix}.$$ 

The assumptions $g \circ f = p \cdot \text{id}_M$ and $f \circ g = p \cdot \text{id}_{M'}$ then imply that we in fact have

$$f = \begin{pmatrix} \text{id}_Q & 0 \\ 0 & p \cdot \text{id}_{Q'} \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} p \cdot \text{id}_Q & 0 \\ 0 & \text{id}_{Q'} \end{pmatrix}$$

so that the claim follows. \qed
2.2. Pairs and displays. Let $h$ and $d$ denote integers with $0 \leq d \leq h$.

**Definition 2.2.** A pair (of type $(h, d)$) over $R$ is a tuple $(M, M_1)$ consisting of a finite projective $W(R)$-module $M$ (of rank $h$) and a $W(R)$-submodule $M_1 \subseteq M$ with $I_R M \subseteq M_1$ and such that $M_1 / I_R M \subseteq M / I_R M$ is a direct summand of rank $d$.

An $m$-truncated pair over $R$ is a tuple $(M, M_1)$ consisting of a finite projective $W_m(R)$-module $M$ and a $W_m(R)$-submodule $M_1 \subseteq M$ with $I_{m,R} M \subseteq M_1$ and such that $M_1 / I_{m,R} M \subseteq M / I_{m,R} M$ is a direct summand.

**Definition 2.3.** Let $(M, M_1)$ and $(M', M'_1)$ be two pairs over $R$. Then a morphism $f: (M, M_1) \to (M', M'_1)$ is a morphism of $W(R)$-modules $f: M \to M'$ such that $f(M_1) \subseteq M'_1$.

In the same way we also define morphisms of $m$-truncated pairs.

**Remark 2.4.** Let $(M, M_1)$ be a pair over $R$ and let $R \to R'$ be a morphism of $p$-complete rings. Then we can form the base change $(M', M'_1) = (M, M_1)_{R'}$ that is a pair over $R'$. It is characterized by

$$M' = W(R') \otimes_{W(R)} M \quad \text{and} \quad M'_1 / I_{R'} M' = R' \otimes_R (M_1 / I_R M).$$

Similarly we can also base change $m$-truncated pairs.

From the descent result [Zin02, Corollary 34] it follows that the assignment $R \mapsto \{\text{pairs over } R\}$ defines a stack of $W(\mathcal{O}_{\text{Spf}(\mathbb{Z}_p)})$-linear categories for the fppc topology; here $W(\mathcal{O}_{\text{Spf}(\mathbb{Z}_p)})$ denotes the sheaf of rings that is given by $R \mapsto W(R)$.

Similarly, using [Lau10, Lemma 3.12], we see that the assignment $R \mapsto \{m$-truncated pairs over $R\}$ defines a stack of $W_m(\mathcal{O}_{\text{Spf}(\mathbb{Z}_p)})$-linear categories.

**Remark 2.5.** There are natural truncation functors

$$\{\text{pairs over } R\} \to \{m$-truncated pairs over $R\}$$

and

$$\{m'\text{-truncated pairs over } R\} \to \{m$-truncated pairs over $R\}$$

for $m \leq m'$.

**Lemma 2.6.** Let $(M, M_1)$ be a pair over $R$. Then $(M, M_1)$ has a normal decomposition $(L, T)$, i.e. a direct sum decomposition $M = L \oplus T$ such that $M_1 = L \oplus I_R T$. Given a second pair $(M', M'_1)$ with normal decomposition $(L', T')$, every morphism of pairs $f: (M, M_1) \to (M', M'_1)$ can be written in matrix form $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a: L \to L'$, $b: T \to T'$, $c: L \to I_R T'$, $d: T \to T'$.

The same is true for $m$-truncated pairs.

**Proof.** Set $L' := M_1 / I_R M \subseteq M / I_R M$. By definition it is a direct summand so that we can choose a complement $T' \subseteq M / I_R M$. As $W(R)$ is Henselian along $I_R$ we can now lift the decomposition $M / I_R M = L' \oplus T'$ to a decomposition $M = L \oplus T$ as desired. The claim about writing morphisms as matrices with respect to chosen normal decompositions is immediate.

**Definition 2.7.** We define a natural $\sigma$-linear functor

$$\{\text{pairs (of type $(h, d)$) over } R\} \to \left\{\text{finite projective } W(R)\text{-modules} \right\},$$

$$(M, M_1) \mapsto \tilde{M}_1$$

as follows.
• Given a pair \((M, M_1)\) over \(R\) with normal decomposition \((L, T)\), we set \(\tilde{M}_1 \coloneqq L^\sigma \oplus T^\sigma\).

• Given two pairs \((M, M_1)\) and \((M', M'_1)\) with normal decompositions \((L, T)\) and \((L', T')\) and a morphism \(f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}: (M, M_1) \to (M', M'_1)\) we define

\[
\tilde{f} := \begin{pmatrix} a^\sigma & p \cdot b^\sigma \\ c^\sigma \cdot \text{div} & d^\sigma \end{pmatrix}: \tilde{M}_1 \to \tilde{M}'_1.
\]

In the same way we also define a natural functor

\[
\{m\text{-truncated pairs over } R\} \to \{\text{finite projective } W_n(R)\text{-modules}\},
\]

\[
(M, M_1) \mapsto \tilde{M}_1.
\]

**Remark 2.8.** We make the following remarks.

• Checking that the functor \((M, M_1) \to \tilde{M}_1\) from Definition 2.7 is well-defined amounts to checking that the definition of \(\tilde{f}\) is compatible with identities and composition. This can be easily verified using the properties of \((-)^{\sigma, \text{div}}\), see Section 2.1.

The definition \(\tilde{f} := \begin{pmatrix} a^\sigma & p \cdot b^\sigma \\ c^\sigma \cdot \text{div} & d^\sigma \end{pmatrix}\) imitates the expression \(\frac{1}{p} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\); the latter a priori only makes sense after base changing to \(W(R)[1/p]\) but then both terms agree.

• Let \((M, M_1)\) be a pair over \(R\). Then we can informally think of \(\tilde{M}_1\) as the “correct version” of the Frobenius twist \(M_1^\sigma\) that usually fails to be a finite projective \(W(R)\)-module.

We have a natural surjective \(W(R)\)-linear map

\[
M_1^\sigma \cong L^\sigma \oplus (I_R^\sigma \otimes_{W(R)} T^\sigma) \xrightarrow{(\text{id}, \sigma^{\text{div}} \otimes \text{id})} L^\sigma \oplus T^\sigma \cong \tilde{M}_1
\]

and this map actually is an isomorphism when \(R\) is a perfect ring of characteristic \(p\).

We also have natural \(W(R)\)-linear maps

\[
\tilde{M}_1 \cong L^\sigma \oplus T^\sigma \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} L^\sigma \oplus T^\sigma \cong M^\sigma
\]

and

\[
M^\sigma \cong L^\sigma \oplus T^\sigma \xrightarrow{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} L^\sigma \oplus T^\sigma \cong \tilde{M}_1
\]

that we can informally think of as the “inclusion” and the “multiplication-by-\(p\)-map” respectively. Similar maps also exist in the truncated situation.

• The theory of *higher displays* as developed in [Lau21] and [Dan] gives a way to conceptualize Definition 2.7.

Let \(W(R)^\oplus\) be the \(\mathbb{Z}\)-graded ring underlying the Witt frame (see [Dan, Definition 2.2]) and recall that it is equipped with two ring homomorphisms \(\tau, \sigma: W(R)^\oplus \to W(R)\). Now giving a pair \((M, M_1)\) over \(R\) is equivalent to giving a finite projective graded \(W(R)^\oplus\)-module whose type is concentrated in \([0, 1]\) and the functors \((M, M_1) \to M\) and \((M, M_1) \to \tilde{M}_1\) correspond to base changing along \(\tau\) and \(\sigma\) respectively.

**Definition 2.9.** A *display over \(R\)* is a tuple \((M, M_1, \Psi)\) where \((M, M_1)\) is a pair over \(R\) and \(\Psi: \tilde{M}_1 \to M\) is an isomorphism of \(W(R)\)-modules. We call \(\Psi\) the *divided Frobenius* of the display.

Similarly, an \((m, n)\text{-truncated display over } R\) is a tuple \((M, M_1, \Psi)\) where \((M, M_1)\) is an \(m\text{-truncated pair over } R\) and \(\Psi: \tilde{M}_1 \to W_n(R) \otimes_{W_m(R)} M\) is an isomorphism of \(W_n(R)\)-modules.

**Remark 2.10.** Similarly as was explained for pairs in Remark 2.4 and Remark 2.5 one can also base change and truncate displays.
The assignment $R \mapsto \{\text{displays over } R\}$ defines an fpqc-stack of $\mathbb{Z}_p$-linear categories; here $\mathbb{Z}_p$ denotes the sheaf that is given by $R \mapsto \mathbb{Z}_p(R)$.

Similarly the assignment $R \mapsto \{(m,n)\text{-truncated displays over } R\}$ defines a stack of $W_m(\mathcal{O}_{\text{Spf}(\mathbb{Z}_p)})\sigma=\text{id}$-linear categories.

**Definition 2.11.** Let $(M, M_1, \Psi)$ be a display over $R$. We define the Frobenius of $(M, M_1, \Psi)$ as the composition

$$F_M : M^\sigma \to \widetilde{M}_1 \xrightarrow{\Psi} M,$$

where the first arrow is the “multiplication-by-$p$-map” introduced in Remark 2.8. Furthermore we say that $(M, M_1, \Psi)$ is $F$-nilpotent if there exists some $N \in \mathbb{Z}_{\geq 0}$ such that the $R/pR$-module homomorphism

$$R/pR \otimes (F_M \circ \cdots \circ F_M^{\sigma-1}) : R/pR \otimes_{W(R)} M^\sigma \to R/pR \otimes_{W(R)} M$$

vanishes.

Now let $(M, M_1, \Psi)$ be an $(m,n)$-truncated display over $R$. Then we again define the Frobenius of $(M, M_1, \Psi)$ as the composition

$$F_M : W_n(R) \otimes_{\sigma, W_n(R)} M \to \widetilde{M}_1 \xrightarrow{\Psi} W_n(R) \otimes_{W_n(R)} M$$

and say that $(M, M_1, \Psi)$ is $F$-nilpotent if there exists some $N \in \mathbb{Z}_{\geq 0}$ such that the $R/pR$-module homomorphism

$$(R/pR \otimes_{W_n(R)} F_M) \circ \cdots \circ (R/pR \otimes_{W_n(R)} F_M)^{(p^{N-1})} : (R/pR \otimes_{W_n(R)} M)^{(p^N)} \to R/pR \otimes_{W_n(R)} M$$

vanishes.

**Remark 2.12.** It follows immediately from the definition that an $(m,n)$-truncated display over $R$ is $F$-nilpotent if and only if its $(2,1)$-truncated base change to $R/pR$ is $F$-nilpotent.

**Remark 2.13** (Relation with Dieudonné modules). Suppose that $R$ is perfect of characteristic $p$. Then there is the classical notion of a Dieudonné module over $R$.

Such a Dieudonné module over $R$ is a tuple $(M, F_M)$ consisting of a finite projective $W(R)$-module $M$ and an isomorphism $F_M : M^\sigma[1/p] \to M[1/p]$ that satisfies $pM \subseteq F_M(M^\sigma) \subseteq M$.

We have a natural equivalence of categories

$$\{\text{displays over } R\} \to \{\text{Dieudonné modules over } R\}$$

that sends a display $(M, M_1, \Psi)$ to $(M, F_M)$ where $F_M$ is the Frobenius from Definition 2.11. The inverse of this equivalence is given by sending $(M, F_M)$ to $(M, M_1, \Psi)$, where

$$M_1 := p \cdot F_M^{-1}(M)^{\sigma-1} \subseteq M \quad \text{and} \quad \Psi : \widetilde{M}_1 \cong p \cdot F_M^{-1}(M) \xrightarrow{\psi^{-1}, F_M} M.$$

Moreover the display corresponding to a Dieudonné module $(M, F_M)$ over $R$ is of type $(h, d)$ if and only if $M$ is a finite projective $W(R)$-module of rank $h$ and $M/F_M(M^\sigma)$ is a finite projective $R$-module of rank $d$.

**Remark 2.14** (Relation with truncated displays in the sense of Lau and Zink). Let us recall the notions of truncated pairs and displays from [LZ18], see also [Lau10].

Consider the ring

$$W_n(R) := W_{n+1}(R)/(0, \ldots, 0, R[p]),$$

where $R[p] \subseteq R$ denotes the $p$-torsion in $R$. We have $(0, \ldots, 0, R[p]) \cdot I_{n+1,R} = 0$ so that $I_{n+1,R}$ is naturally a $W_n(R)$-module and we have a Frobenius $\sigma : W_n(R) \to W_n(R)$ as well as a divided Frobenius $\sigma^{\text{div}} : W_n(R) \otimes_{\sigma, W_n(R)} I_{n+1,R} \to W_n(R)$. 


We now have the $W_n(R)$-linear category of $n$-truncated Lau-Zink-pairs that has the following description. Every $n$-truncated Lau-Zink-pair (that we informally denote by $(M, M_1)$) has a normal decomposition $(L, T)$ where $L$ and $T$ are finite projective $W_n(R)$-modules and given two such objects $(M, M_1)$ and $(M', M_1')$ with normal decompositions $(L, T)$ and $(L', T')$, morphisms $f: (M, M_1) \to (M', M_1')$ are given by matrices $f = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$ with

$$a: L \to L', \quad b: T \to T', \quad c: L \to I_{n+1,R} \otimes_{W_n(R)} T', \quad d: T \to T'.$$

There again is a $\sigma$-linear functor

$$\{ n\text{-truncated Lau-Zink-pairs over } R \} \to \{ \text{finite projective } W_n(R)\text{-modules} \}, \quad (M, M_1) \mapsto \tilde{M}_1$$

that sends an $n$-truncated Lau-Zink-pair $(M, M_1)$ with normal decomposition $(L, T)$ to

$$\tilde{M}_1 := (W_n(R) \otimes_{\sigma, W_n(R)} L) \oplus (W_n(R) \otimes_{\sigma, W_n(R)} T)$$

and a morphism $f = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$ as above to $\tilde{f} := \left( \begin{array}{cc} a' & b' \\ c' & d' \end{array} \right)$ similar as in Definition 2.7. An $n$-truncated Lau-Zink-display is then defined to be a tuple $(M, M_1, \Psi)$ where $(M, M_1)$ is an $n$-truncated Lau-Zink-pair and $\Psi: \tilde{M}_1 \to W_n(R) \otimes_{W_n(R)} M$ is an isomorphism.

From this description it is clear that there exist natural truncation functors

$$\{ (m', n')\text{-truncated displays over } R \} \to \{ n\text{-truncated Lau-Zink-displays over } R \}$$

for $n \leq n'$ and

$$\{ n'\text{-truncated Lau-Zink-displays over } R \} \to \{ (m, n)\text{-truncated displays over } R \}$$

for $m \leq n'$.

Let us remark that for $R$ of characteristic $p$ we have an equivalence

$$\{ 1\text{-truncated Lau-Zink-displays over } R \} \to \{ F\text{-zips over } R \},$$

where the right hand side denotes the the category of $F$-Zips from [MW04, Definition 1.5] with type concentrated in $[0, 1]$, see [Lau21, Example 3.6.4].

2.3. Duals and twists.

**Definition 2.15.** Let $(M, M_1)$ be a pair over $R$. Then we define its dual

$$(M, M_1)^\vee := (M^\vee, M_1^\vee)$$

as follows.

- $M^\vee = \text{Hom}_{W(R)}(M, W(R))$ is the dual of the finite projective $W(R)$-module $M$.
- $M_1^\vee \subseteq M$ is the $W(R)$-submodule of all $\omega: M \to W(R)$ such that $\omega(M_1) \subseteq I_R$. Equivalently it is the preimage under

$$M^\vee \to M^\vee/I_R M^\vee \cong (M/I_RM)^\vee$$

of the orthogonal complement $(M_1/I_RM)^\perp \subseteq (M/I_RM)^\vee$.

This endows the category of pairs over $R$ with a $W(R)$-linear duality in the sense of Definition [A, 3] and Remark [A, 3].

We similarly define duals of $m$-truncated pairs.

**Remark 2.16.** Note that if $(M, M_1)$ is an $(m$-truncated) pair of type $(h, d)$ over $R$ then its dual $(M, M_1)^\vee$ is of type $(h, h - d)$.
Lemma 2.17. The functor
\[ \{ \text{pairs over } R \} \to \{ \text{finite projective } W(R)\text{-modules} \}, \quad (M, M_1) \mapsto \tilde{M}_1 \]
is naturally compatible with dualities.

The same is true for the functor
\[ \{ m\text{-truncated pairs over } R \} \to \{ \text{finite projective } W_n(R)\text{-modules} \}, \quad (M, M_1) \mapsto \tilde{M}_1. \]

Proof. Given a pair \((M, M_1)\) over \(R\) we need to define a natural isomorphism \(\tilde{M}_1^* \to \tilde{M}_1^\vee\). Let \((L, T)\) be a normal decomposition of \((M, M_1)\). Then \((T^\vee, L^\vee)\) is a normal decomposition of \((M, M_1)^\vee\) and we can define the desired isomorphism as
\[
\begin{pmatrix}
0 & \text{id}_{L^\vee} \\
\text{id}_{T^\vee} & 0
\end{pmatrix} : \tilde{M}_1^* \cong T^\vee \oplus L^\vee \to (L^\sigma \oplus T^\sigma)^\vee \cong \tilde{M}_1^\vee.
\]

Definition 2.18. Let \((M, M_1, \Psi)\) be a display over \(R\). Then we define its dual
\[(M, M_1, \Psi)^\vee := (M^\vee, M_1^*, \Psi^\vee, -1),\]
where we implicitly use Lemma 2.17 to make sense of \(\Psi^\vee, -1\) as an isomorphism \(\tilde{M}_1^* \to M^\vee\). This endows the category of displays over \(R\) with a \(\mathbb{Z}_p(R)\)-linear duality.

We similarly define duals of \((m, n)\)-truncated displays.

Definition 2.19. Let \((M, M_1)\) be a pair over \(R\) and let \(I\) be an finite projective \(W(R)\)-module. Then we define the twist
\[ I \otimes (M, M_1) := (I \otimes_{W(R)} M, I \otimes_{W(R)} M_1). \]
This defines an action of the symmetric monoidal category of finite projective \(W(R)\)-modules on the category of pairs over \(R\) in the sense of Definition A.2.

We similarly define twists of \(m\)-truncated pairs over \(R\) by finite projective \(W_m(R)\)-modules.

Remark 2.20. In fact the symmetric monoidal category of finite projective \(W(R)\)-modules is rigid and \(W(R)\)-linear and the action defined above is naturally compatible with dualities and the \(W(R)\)-linear structure, see Remark A.4, Remark A.5, and Remark A.6.

In the same way also the action in the truncated setting is naturally compatible with dualities and the \(W_m(R)\)-linear structure.

Lemma 2.21. The functor
\[ \{ \text{pairs over } R \} \to \{ \text{finite projective } W(R)\text{-modules} \}, \quad (M, M_1) \mapsto \tilde{M}_1 \]
is naturally equivariant with respect to the symmetric monoidal functor
\[ \{ \text{finite projective } W(R)\text{-modules} \} \to \{ \text{finite projective } W(R)\text{-modules} \}, \quad I \mapsto I^\sigma \]
in the sense of Definition A.2 and Remark A.4.

Similarly the functor
\[ \{ m\text{-truncated pairs over } R \} \to \{ \text{finite projective } W_n(R)\text{-modules} \}, \quad (M, M_1) \mapsto \tilde{M}_1 \]
is naturally equivariant with respect to the symmetric monoidal functor
\[ \{ \text{finite projective } W_m(R)\text{-modules} \} \to \{ \text{finite projective } W_n(R)\text{-modules} \}, \quad I \mapsto W_n(R) \otimes_{\sigma, W_m(R)} I. \]
Theorem 2.24. Let $(I,\iota)$ be a pair consisting of a finite projective $W(R)$-module $I$ and an isomorphism $\iota: I^\sigma \to I$. Then we define the twist $(I,\iota) \otimes (M,M_1,\Psi) := (I \otimes_{W(R)} M, I \otimes_{W(R)} M_1, \iota \otimes \Psi),$

where we implicitly use Lemma 2.21 to make sense of $\iota \otimes \Psi$ as an isomorphism $(I \otimes_{W(R)} M_1)^{\sim} \to M$. This defines an action of the symmetric monoidal category of tuples $(I,\iota)$ as above on the category of displays over $R$ and in fact this action is naturally compatible with dualities and the $\mathbb{Z}_p(R)$-linear structure.

We similarly define twists of $(m,n)$-truncated displays over $R$ by tuples $(I,\iota)$ consisting of a finite projective $W_m(R)$-module $I$ and an isomorphism $\iota: W_n(R) \otimes_{\sigma, W_m(R)} I \to W_n(R) \otimes_{W_m(R)} I$.

Remark 2.23. We can conceptually understand duals and twists of pairs and displays in terms of higher displays.

To a pair $(M,M_1)$ there is a corresponding finite projective graded module over $W(R)^\oplus$ with type concentrated in $[0,1]$ (see Remark 2.23). The dual pair $(M,M_1)^{\vee}$ then corresponds to taking the dual of that module and shifting it by 1 and the twist $I \otimes (M,M_1)$ corresponds to viewing $I$ as a finite projective graded $W(R)^\oplus$-module with type concentrated in 0 and forming the tensor product over $W(R)^\oplus$. The various categorical properties of taking duals and twists then all follow from the fact that finite projective graded modules over a graded ring form a rigid symmetric monoidal category.

2.4. The display of a $p$-divisible group. We have the following result of Lau, relating $p$-divisible groups and displays.

Theorem 2.24 (Lau10). There is a natural $\mathbb{Z}_p(R)$-linear exact functor

$$
D: \{\text{p-divisible groups over } R\}^{\text{op}} \to \{\text{displays over } R\}
$$

that is compatible with dualities and satisfies the following properties.

- For a $p$-divisible group $X$ over $R$ of height $h$ and dimension $d$ the display $D(X)$ is of type $(h,d)$.
- Suppose that $R$ is a perfect ring of characteristic $p$. Then $D$ coincides with classical (contravariant) Dieudonné theory.
- $D$ restricts to an equivalence

$$
D: \{\text{formal p-divisible groups over } R\}^{\text{op}} \to \{\text{F-nilpotent displays over } R\}.
$$

- Let $X$, $X'$ be $p$-divisible groups over $R$ of height $h$ and let $f: X' \to X$ be an isogeny of height $r \in \mathbb{Z}_{\geq 0}$. Assume that there exists an isogeny $g: X \to X'$ such that $g \circ f = p \cdot \text{id}_X$, and $f \circ g = p \cdot \text{id}_X$ (note that this forces $r \leq h$). Then the homomorphism of $W(R)/pW(R)$-modules

$$
D(f): D(X)/pD(X) \to D(X')/pD(X')
$$

is of constant rank $h - r$.

Proof. See Lau10 Proposition 4.1 and Theorem 5.1. For the last claim we can use Lemma 2.1 to reduce to the situation that $R = k$ is an algebraically closed field of characteristic $p$ where the statement is a standard result from classical Dieudonné theory.
3. (Polarized) chains of pairs and displays

As in Section 2, $R$ denotes a $p$-nilpotent ring and $(m, n)$ denotes a tuple of positive integers with $m \geq n + 1$. When $R$ is of characteristic $p$ then we allow $n$ to take the additional value 1-rdt (where “rdt” refers to the term “reductive quotient”) that we think of as being slightly smaller than 1. In the case $n = 1$-rdt we require $m \geq 2$.

3.1. Chains of pairs and displays. Fix a positive integer $h$, a second integer $0 \leq d \leq h$ and a non-empty subset $J \subseteq \mathbb{Z}$ such that $J + h\mathbb{Z} = J$. Let $E$ be the set of “edges of $J$”, i.e. the set of tuples $(i, j)$ consisting of consecutive elements in $J$. For $e = (i, j) \in E$ we write $|e| := j - i$ for the “length of $e$”.

**Definition 3.1.** A chain (of type $(h, J)$) over $R$ is a tuple

$$(M_i)_{i \in J}, (\rho_{i,j})_{i,j \in J, i \leq j}, (\theta_i)_{i \in J}$$

that is given as follows.

- $((M_i), (\rho_{i,j}))$ is a diagram of finite projective $W(R)$-modules of rank $h$ of shape $J$ such that the homomorphism of $R/pR$-modules

  $$\rho_{i,j} : R/pR \otimes W(R) M_i \to R/pR \otimes W(R) M_j$$

  is of constant rank $h - (j - i)$ for all $i \leq j \leq i + h$.

- $\theta_i : M_i \to M_{i+h}$ are isomorphisms such that we have $\theta_j \circ \rho_{i,j} = \rho_{i+h,j+h} \circ \theta_i$ and $\rho_{i,i+h} = p \cdot \theta_i$.

For $n \neq 1$-rdt similarly define the notion of an $n$-truncated chain over $R$ by replacing $W(R)$ with $W_n(R)$ in the above.

Now suppose that $R$ is of characteristic $p$. Then a 1-rdt-truncated chain over $R$ is a tuple $((N_e)_{e \in E}, (\theta_e)_{e \in E})$ that is given as follows.

- $N_e$ is a finite projective $R$-module of rank $|e|$.
- $\theta_e : N_e \to N_{e+h}$ is an isomorphism.

**Remark 3.2.** There are obvious natural truncation functors

$$\{\text{chains over } R\} \to \{\text{n-truncated chains over } R\}$$

and

$$\{\text{n'-truncated chains over } R\} \to \{\text{n-truncated chains over } R\}$$

for $n \leq n'$.

When $R$ is of characteristic $p$ then we also define a truncation functor

$$\{\text{1-truncated chains over } R\} \to \{\text{1-rdt-truncated chains over } R\}$$

by sending a 1-truncated chain $((M_i), (\rho_{i,j}))$ to the 1-rdt-truncated chain $((N_e), (\theta_e))$ that is given as follows (where we write $e = (i, j)$).

- $N_e := \ker(\rho_{i,j}) \subseteq M_i$.
- $\theta_e : N_e \to N_{e+h}$ is the isomorphism induced by $\theta_i$.

Note that for $e = (i, j)$ as above we have an exact sequence

$$M_i \xrightarrow{\rho_{i,j}} M_j \xrightarrow{\rho_{i,j+h}} M_{i+h} \xrightarrow{\rho_{i+h,j+h}} M_{j+h}$$

so that we obtain an isomorphism $\ker(\rho_{i+h,j+h}) \cong \text{coker}(\rho_{i,j})$. Composing this with $\theta_e$ yields an isomorphism $N_e \cong \text{coker}(\rho_{i,j})$.
Remark 3.3. Let \( ((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i) \) be a chain over \( R \). Then actually already the morphism of \( W(R)/pW(R) \)-modules

\[
\rho_{i,j} : M_i/pM_i \to M_j/pM_j
\]

is of constant rank \( h - (j - i) \) for all \( i \leq j \leq i + h \); this follows from Lemma 2.1 applied to \( A = W(R) \), \( \rho = \rho_{i,j} \) and \( \rho' = \theta_i^{-1} \circ \rho_{j,i+h} \).

The same is true for \( n \)-truncated chains.

Definition 3.4. A chain of pairs (of type \( (h, J, d) \)) over \( R \) is a tuple

\[
((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (M_i,1)_i)
\]

that is given as follows.

- \( ((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i) \) is a chain over \( R \).
- \( M_i \subseteq M_i \) is a \( W(R) \)-submodule such that \( (M_i, M_i,1) \) is a pair of type \( (h, d) \) over \( R \) and such that we have \( \rho_{i,j}(M_i,1) \subseteq M_j,1 \) and \( \theta_i(M_i,1) = M_{i+h,1} \).

We similarly define the notion of an \( m \)-truncated chain of pairs over \( R \).

Proposition 3.5. Let \( ((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (M_i,1)_i) \) be a chain of pairs over \( R \). Then the tuple

\[
((\bar{M}_{i,1})_i, (\bar{\rho}_{i,j})_{i,j}, (\bar{\theta}_i)_i)
\]

is a chain over \( R \).

The same is true for \( m \)-truncated chains of pairs (where the construction yields an \( n \)-truncated chain).

Proof. We only need to check that the morphism of \( R/pR \)-modules

\[
\bar{\rho}_{i,j} : R/pR \otimes_{W(R)} \bar{M}_{i,1} \to R/pR \otimes_{W(R)} \bar{M}_{j,1}
\]

is of constant rank \( h - (j - i) \) for all \( i \leq j \leq i + h \). Lemma 2.1 allows us to reduce to the case where \( R = k \) is an algebraically closed field of characteristic \( p \). The morphism \( \rho_{i,j} \) then gives rise to a commutative diagram

\[
\begin{array}{ccc}
M_{i,1} & \xrightarrow{\rho_{i,j}} & M_{j,1} \\
\downarrow & & \downarrow \\
M_i & \xrightarrow{\rho_{i,j}} & M_j
\end{array}
\]

of free \( W(k) \)-modules of rank \( h \) where all the morphisms are injective. Moreover the cokernels of the vertical morphisms have length \( h - d \) and the cokernel of the lower horizontal morphism has length \( j - i \). Thus also the cokernel of the upper horizontal morphism has length \( j - i \). As the image of \( M_{i,1} \to M_{j,1} \) contains \( pM_{i,1} \) we can conclude that the morphism of \( k \)-vector spaces \( \rho_{i,j} : M_{i,1}/pM_{i,1} \to M_{j,1}/pM_{j,1} \) is of rank \( h - (j - i) \). Twisting by \( \sigma \) then gives the result (see Remark 2.3).

The result for \( m \)-truncated chains of pairs now follows because every such \( m \)-truncated chain of pairs over \( R \) can be lifted to a (non-truncated) chain of pairs over \( R \). \( \square \)

Definition 3.6. A chain of displays over \( R \) is a tuple

\[
((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (M_i,1)_i, (\Psi_i)_i)
\]

that is given as follows.

- \( ((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (M_i,1)_i) \) is a chain of pairs over \( R \).
- \( (\Psi_i)_i : ((\bar{M}_{i,1})_i, (\bar{\rho}_{i,j})_{i,j}, (\bar{\theta}_i)_i) \to ((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i) \)

is an isomorphism of chains over \( R \).
We similarly define the notion of an \((m, n)\)-truncated chain of displays over \(R\).
We write
\[
\text{ChDisp}_{h, J, d}^{(m, n)} \quad \text{and} \quad \text{ChDisp}_{h, J, d}^{(m, n)\text{-rdt}}
\]
for the stacks over \(\text{Spf}(\mathbf{Z}_p)\) of chains of displays and \((m, n)\)-truncated chains of displays (the stacks \(\text{ChDisp}_{h, J, d}^{(m, n)\text{-rdt}}\) are in fact defined over \(\text{Spec}(\mathbf{F}_p)\)).

\textbf{Remark 3.7.} Let \(((M_i)_i, (\rho_{i, j})_{i, j}, (\theta_i)_i, (\Psi_i)_i)\) be a chain of displays over \(R\). Then \((M_i, M_{i, 1}, \Psi_i)_i\) is a display of type \((h, d)\) over \(R\) and we have morphisms of displays
\[
\rho_{i, j}: (M_i, M_{i, 1}, \Psi_i) \to (M_j, M_{j, 1}, \Psi_j) \quad \text{and} \quad \theta_i: (M_i, M_{i, 1}, \Psi_i) \to (M_{i+h}, M_{i+h, 1}, \Psi_{i+h}).
\]
Conversely suppose we are given displays \((M_i, M_{i, 1}, \Psi_i)\) of type \((h, d)\) over \(R\) and morphisms of displays \(\rho_{i, j}\) and \(\theta_i\) as above, and assume that \(((M_i)_i, (\rho_{i, j})_{i, j}, (\theta_i)_i)\) is a chain over \(R\). Then the tuple \(((\tilde{M}_1)_i, (\rho_{i, j})_{i, j}, (\tilde{\theta}_i)_i, (\tilde{\Psi}_i)_i)\) is a chain of displays over \(R\).

The same is true for \((m, n)\)-truncated chains of displays.

3.2. \textbf{Polarized chains.} Let us now specialize to the situation where \(h = 2g\) is even, \(d = g\) and \(-J = J\). Given \(e = (i, j) \in \mathcal{E}\) we write \(-e := (-j, -i) \in \mathcal{E}\).

\textbf{Remark 3.8.} Recall that the categories of pairs and displays (over \(R\)) and their truncated variants carry a natural duality and a compatible action of a suitable rigid symmetric monoidal category, see Section 2.3.

Consequently we obtain a similar structure on the categories of chains, chains of pairs, chains of displays and their truncated variants. The various base change, forgetful and truncation functors are naturally compatible with this extra structure. The situation is summarized by the following table.

| category with duality | symmetric monoidal category | coefficients |
|-----------------------|-----------------------------|--------------|
| chains (over \(R\))   | invertible \(W(R)\)-modules | \(W(R)\)     |
| chains of pairs        | invertible \(W(R)\)-modules | \(W(R)\)     |
| chains of displays     | tuples \((I, i); I\) an invertible \(W(R)\)-module and \(\iota: I^* \to I\) an isomorphism | \(\mathbf{Z}_p(R)\) |
| \((m, n)\)-truncated chains of displays \((n \neq 1\text{-rdt})\) | invertible \(W_n(R)\)-modules | \(W_n(R)\) |
| \(1\text{-rdt}\)-truncated chains | invertible \(R\)-modules | \(R\) |
| \(m\)-truncated chains of pairs | invertible \(W_m(R)\)-modules | \(W_m(R)\) |
| \((m, n)\)-truncated chains of displays \((n \neq 1\text{-rdt})\) | tuples \((I, i); I\) an invertible \(W_m(R)\)-module and \(\iota: W_n(R) \otimes_{\sigma, W_m(R)} I \to W_n(R) \otimes_{W_m(R)} I\) an isomorphism | \(W_m(R)^{\sigma = \text{id}}\) |
| \((m, 1\text{-rdt})\)-truncated chains of displays | tuples \((I, i); I\) an invertible \(W_m(R)\)-module and \(\iota: R \otimes_{\sigma, W_m(R)} I \to R \otimes_{W_m(R)} I\) an isomorphism | \(W_m(R)^{\sigma = \text{id}}\) |

For example, the dual of a chain \(((M_i)_i, (\rho_{i, j})_{i, j}, (\theta_i)_i)\) over \(R\) is
\[
((M_i)_i, (\rho_{i, j})_{i, j}, (\theta_i)_i)^{\vee} = ((M_i^{\vee})_i, (\rho_{i, j}^{\vee})_{i, j}, (\theta_i^{\vee})_{i, j})
\]
and its twist by an invertible $W(R)$-module $I$ is

$$I \otimes ((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i) = ((I \otimes_{W(R)} M_i)_i, (\id_I \otimes \rho_{i,j})_{i,j}, (\id_I \otimes \theta_i)_i).$$

Let us also spell out the duality coherence datum for the truncation functor

$$\{1\text{-truncated chains over } R\} \to \{1\text{-rdt-truncated chains over } R\}$$

for $R$ of characteristic $p$. Let $((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i)$ be a 1-truncated chain over $R$, let $((N_{e,c})_e, (\theta_e)_e)$ be its 1-rdt-truncation and let $((N'_{e,c})_e, (\theta'_e)_e)$ be the 1-rdt-truncation of its dual. Then we specify the natural isomorphism

$$((N'_{e,c})_e, (\theta'_e)_e) \to ((N_{e,c})_e, (\theta_e)_e)^\vee = ((N'_{e,c})_e, (\theta'_{-e,c})_e)$$

to be given by

$$N'_{e,c} \cong \ker(\rho_{-e,c,i}) \to \coker(\rho_{-e,c,i})^\vee \cong N'_{e,c}.$$

**Definition 3.9.** A polarized chain (of type $(g, J)$) over $R$ is a polarized object in the category of chains (of type $(2g, J, g)$) over $R$, i.e. a tuple

$$(M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (\lambda_i)_i$$

that is given as follows.

- $((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i)$ is a chain over $R$.
- $(\lambda_i)_i: ((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i) \to ((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i)^\vee$ is an antisymmetric isomorphism.

A homogeneously polarized chain over $R$ is a homogeneously polarized object in the category of chains over $R$, i.e. a tuple

$$(M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, I, (\lambda_i)_i$$

that is given as follows.

- $((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i)$ is a chain over $R$.
- $I$ is an invertible $W(R)$-module.
- $(\lambda_i)_i: ((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i) \to I \otimes ((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i)^\vee$ is an antisymmetric isomorphism.

In the same way we define the notions of (homogeneously) polarized chains of pairs and displays as well as their truncated variants.

We write

$$\text{PolChDisp}_{g,J}, \quad \text{HPolChDisp}_{g,J}, \quad \text{PolChDisp}^{(m,n)}_{g,J}, \quad \text{HPolChDisp}^{(m,n)}_{g,J}$$

for the stacks over $\text{Spf}(\mathbb{Z}_p)$ of polarized chains of displays, homogeneously polarized chains of displays and their truncated variants.

**Remark 3.10.** The functor

$$\{\text{chains of pairs over } R\} \to \{\text{chains over } R\},$$

$$(M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (M_{i,1})_i \mapsto (\widetilde{M_{i,1}})_i, (\widetilde{\rho_{i,j}})_{i,j}, (\widetilde{\theta_i})_i$$

(see Proposition 3.5) is naturally compatible with dualities and equivariant with respect to the symmetric monoidal functor

$$\{\text{invertible } W(R)\text{-modules}\} \to \{\text{invertible } W(R)\text{-modules}\}, \quad I \mapsto I^\sigma;$$

it thus induces functors

$$\{(\text{homogeneously) polarized chains of pairs over } R\} \to \{(\text{homogeneously) polarized chains over } R\}.$$
A polarized chain of displays over $R$ can now be equivalently described as consisting of a polarized chain of pairs $((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (M_{i,1})_i, (\lambda_i)_i)$ over $R$ together with an isomorphism

$$(\Psi_i)_i : ((\tilde{M}_{i,1})_i, (\tilde{\rho}_{i,j})_{i,j}, (\tilde{\theta}_i)_i, (\tilde{\lambda}_i)_i) \to ((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (\lambda_i)_i)$$

and a homogeneously polarized chain of displays over $R$ can be equivalently described as consisting of a homogeneously polarized chain of pairs $((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (M_{i,1})_i, I, (\lambda_i)_i)$ over $R$ together with an isomorphism

$$(\Psi_i)_{\iota} : ((\tilde{M}_{i,1})_i, (\tilde{\rho}_{i,j})_{i,j}, (\tilde{\theta}_i)_i, (\tilde{\lambda}_i)_i) \to ((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, I, (\lambda_i)_i).$$

The same is true for $(m,n)$-truncated (homogeneously) polarized chains of displays over $R$.

**Lemma 3.11.** The morphism of stacks

$$\text{PolChDisp}_{g,J} \to \text{HPolChDisp}_{g,J}$$

is a torsor under the flat affine $\mathbb{Z}_p$-group scheme $\mathbb{Z}_p^\times$.

Similarly, the morphism of stacks

$$\text{PolChDisp}^{(m,n)}_{g,J} \to \text{HPolChDisp}^{(m,n)}_{g,J}$$

is a torsor under the smooth affine $\mathbb{Z}_p$-group scheme $(L'^{(m)} G_m)^{\sigma = \text{id}}$ that is given by $R \mapsto W_m(R)^{\times, \sigma = \text{id}}$.

**Proof.** This follows from Lemma A.8 and the fact that every tuple $(I, \iota)$ consisting of an invertible $W(R)$-module $I$ together with an isomorphism $\iota : I^\sigma \to I$ can be trivialized after base changing along a faithfully flat ring homomorphism $R \to R'$.

### 3.3. Quotient stack descriptions.

We are now interested in finding quotient stack descriptions of the stack of chains of displays $\text{ChDisp}_{h,J,d}$ and its truncated and (homogeneously) polarized variants.

Let us first return to the situation of Section 3.1 where $h$, $J$ and $d$ are arbitrary. Fix a $\mathbb{Q}_p$-vector space $V$ of dimension $h$ and a tuple $(\Lambda_i)_{i \in J}$ of $\mathbb{Z}_p$-lattices $\Lambda_i \subseteq V$ such that the following conditions are satisfied.

- For $i \leq j$ we have $\Lambda_i \subseteq \Lambda_j$ and the $\mathbb{Z}_p$-module $\Lambda_j / \Lambda_i$ is of length $j - i$.
- $\Lambda_{i+h} = p^{-1} \Lambda_i$.

For $i \leq j$ we write $\rho_{i,j} : \Lambda_i \to \Lambda_j$ for the inclusion and we also write $\theta_i : \Lambda_i \to \Lambda_{i+h}$ for the isomorphism given by multiplication with $p^{-1}$.

We then have the associated parahoric $\mathbb{Z}_p$-group scheme

$$\text{GL}((\Lambda_i)_i) := \text{Aut}((\Lambda_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i)$$

with generic fiber $\text{GL}((\Lambda_i)_i)_{\mathbb{Q}_p} = \text{GL}(V)$ and the local model

$$M^{\text{loc,GL}} : R \mapsto \left\{ (C_i) \mid \begin{array}{l} C_i \subseteq \Lambda_{i,R} \text{ a direct summand of rank } d \\ \text{such that } \rho_{i,j}(C_i) \subseteq C_j \text{ and } \theta_i(C_i) = C_{i+h} \end{array} \right\}$$

that is a projective $\mathbb{Z}_p$-scheme with a natural $\text{GL}((\Lambda_i)_i)$-action whose generic fiber is the Grassmannian of $d$-dimensional subspaces of $V$, see [RZ96, Theorem 3.11 and Definition 3.27].

In fact, by [Gör01, Theorem 4.25], $M^{\text{loc,GL}}$ is flat over $\mathbb{Z}_p$, its special fiber is reduced and the irreducible components of its special fiber are normal with rational singularities. We write $M^{\text{loc,GL}}$ for the $p$-completion of $M^{\text{loc,GL}}$.

Note also that the reductive quotient $\text{GL}((\Lambda_i)_i)^{\text{red}}_{\mathbb{F}_p}$ of the special fiber of $\text{GL}((\Lambda_i)_i)$ identifies with

$$\text{GL}((\Lambda_e)_e) := \text{Aut}((\Lambda_e)_e, (\theta_e)_e) \cong \prod_{e \in E / h \mathbb{Z}} \text{GL}(\Lambda_e),$$
where
\[ \Lambda_e := \ker(\rho_{i,j} : \Lambda_i/p\Lambda_i \to \Lambda_j/p\Lambda_j) \]
and \( \theta_e : \Lambda_e \to \Lambda_{e+h} \) is the isomorphism induced by \( \theta_i \) (for \( e = (i, j) \in \mathcal{E} \)).

**Lemma 3.12.** We have natural equivalences of groupoids
\[ \{ \text{chains over } R \} \to \{ L^+GL((\Lambda_i)_i)\text{-torsors over } R \} \]
and
\[ \{ n\text{-truncated chains over } R \} \to \{ (n)GL((\Lambda_i)_i)\text{-torsors over } R \} \]
for \( n \neq 1\text{-rdt} \). When \( R \) is of characteristic \( p \), we also have a natural equivalence of groupoids
\[ \{ 1\text{-rdt-truncated chains over } R \} \to \{ GL((\Lambda_e)_e)\text{-torsors over } R \}. \]

**Proof.** This follows from [RZ96, Theorem 3.11] and [BH20, Lemma 2.12]. \( \Box \)

**Lemma 3.13.** We have natural equivalences of groupoids
\[ \{ \text{chains of pairs over } R \} \to \left[ L^+GL((\Lambda_i)_i)\backslash M_{\text{loc,GL}} \right](R) \]
and
\[ \{ m\text{-truncated chains of pairs over } R \} \to \left[ L^+(m)GL((\Lambda_i)_i)\backslash M_{\text{loc,GL}} \right](R). \]

**Proof.** An object in \( [L^+GL((\Lambda_i)_i)\backslash M_{\text{loc,GL}}](R) \) is given by a \( GL((\Lambda_i)_i)\)-torsor \( P \over R \) over \( W(R) \) together with a \( GL((\Lambda_i)_i)\)-equivariant morphism \( q : P \to M_{\text{loc,GL}} \). By Lemma 3.12, \( P \) corresponds to a chain \( ((M_i), (\rho_{i,j})_{i,j}, (\theta_i)) \) over \( R \) and from the definition of \( M_{\text{loc,GL}} \), we see that \( q \) corresponds to a tuple \((C_i)_i\) of direct summands \( C_i \leq M_i/I_RM_i \) of rank \( d \) that are compatible under \( \rho_{i,j} \) and \( \theta_i \). Giving such a tuple \((C_i)_i\) is evidently equivalent to giving a tuple \((M_{i,1})_i\) of \( W(R)\)-submodules \( M_{i,1} \subseteq M_i \) such that \((M_i), (\rho_{i,j}), (\theta_i)\) is a chain of pairs over \( R \). \( \Box \)

**Definition 3.14.** Let
\[ M_{\text{loc,GL},+} \to M_{\text{loc,GL}} \]
be the \( L^+GL((\Lambda_i)_i)\)-equivariant \( L^+GL((\Lambda_i)_i)\)-torsor that corresponds to the functor from Proposition 3.12 under the equivalences from Lemma 3.12 and Lemma 3.13.

Explicitly, a point in \( M_{\text{loc,GL},+}(R) \) is given by a chain of pairs \((M_i), (\rho_{i,j}), (\theta_i), (M_{i,1})_i\) over \( R \) together with trivializing isomorphisms
\[ ((M_{i,1}), (\rho_{i,j}), (\theta_i)) \to ((\Lambda_i, W(R)_i), (\rho_{i,j}), (\theta_i)) \]
and
\[ ((M_i), (\rho_{i,j}), (\theta_i)) \to ((\Lambda_i, W(R)_i), (\rho_{i,j}), (\theta_i)). \]
The group \( L^+GL((\Lambda_i)_i) \times L^+GL((\Lambda_i)_i) \) acts on \( M_{\text{loc,GL},+} \) by changing the two trivializations; the projection \( M_{\text{loc,GL},+} \to M_{\text{loc}} \) is then a torsor with respect to the action of the first copy of \( L^+GL((\Lambda_i)_i) \) and equivariant with respect to the action of the second copy of \( L^+GL((\Lambda_i)_i) \).

For \( n \neq 1\text{-rdt} \) we write
\[ M_{\text{loc,GL},(n)} \to M_{\text{loc}} \]
for the reduction of \( M_{\text{loc,GL},+} \) to an \( (n)GL((\Lambda_i)_i)\)-torsor. By Proposition 3.15, the action of the second copy of \( L^+GL((\Lambda_i)_i) \) on \( M_{\text{loc,GL},(n)} \) factors through \( L^+(m)GL((\Lambda_i)_i) \).

Finally we also write
\[ M_{\text{loc,GL},(1\text{-rdt})} \to M_{\text{loc}}^F \]
for the reduction of \( M_{\text{loc,GL},+} \) to a \( GL((\Lambda_e)_e)\)-torsor.
Proposition 3.15. We have equivalences of stacks over \( \text{Spf}(\mathbb{Z}_p) \)
\[
\text{ChDisp}_{h,J,d} \rightarrow \left( L^+ \text{GL}((\Lambda_1 i)_i) \right)_\Delta \setminus M_{\text{loc,GL}^+,}\quad \text{and} \quad \text{ChDisp}_{h,J,d}^{(m,n)} \rightarrow \left( L^{(m)} \text{GL}((\Lambda_1 i)_i) \right)_\Delta \setminus M_{\text{loc,GL}^+,n},
\]
where the subscript \( \Delta \) indicates that we take the quotient by the diagonal action.

In particular \( \text{ChDisp}_{h,J,d}^{(m,n)} \) is a p-adic formal algebraic stack of finite presentation over \( \text{Spf}(\mathbb{Z}_p) \) (an algebraic stack of finite presentation over \( \text{Spec}(\mathbb{F}_p) \)) when \( n = 1 \)-rdt) and for \( (m,n) \leq (m',n') \) the morphism \( \text{ChDisp}_{h,J,d}^{(m',n')} \rightarrow \text{ChDisp}_{h,J,d}^{(m,n)} \) is smooth.

Proof. By reformulating the definition of \( M_{\text{loc,GL}^+} \) we arrive at a natural identification
\[
M_{\text{loc,GL}^+}(R) \cong \left\{ \left( (M_i)_{i}, (\Psi_i)_{i} \right) \left| \begin{array}{c}
\text{is a chain of displays over } R \\
\end{array}\right. \right\}
\]
and under this identification the action of \( L^+ \text{GL}((\Lambda_1 i)_i) \times L^+ \text{GL}((\Lambda_1 i)_i) \) is given by
\[
(k_1, k_2). \left( (M_i)_{i}, (\Psi_i)_{i} \right) = \left( k_2 \cdot (M_i)_{i}, k_1 \cdot (\Psi_i)_{i} \cdot k_2^{-1} \right),
\]
where we view \( k_2 \) as an isomorphism of chains of pairs
\[
k_2: ( (\Lambda_i, W(R)_i), (\rho_{i,j})_{i,j}, (\theta_i)_i, (M_i)_i ) \rightarrow ( (\Lambda_i, W(R)_i), (\rho_{i,j})_{i,j}, (\theta_i)_i, k_2 \cdot (M_i)_i )
\]
in order to make sense of \( k_2 \). From this the claim now readily follows. \( \square \)

Remark 3.16 (Relation with local shtukas). Let us recall the notions of (restricted) local shtukas in mixed characteristic from [XZ17], see also [SYZ21], Section 4. We warn the reader that our normalizations are slightly different than the ones in the references.

Fix a connected reductive group \( G \) over \( \mathbb{Q}_p \) and a parahoric \( \mathbb{Z}_p \)-group scheme \( G \) with generic fiber \( G \). We write \( L^+ \mathcal{G} \) for the perfection of the special fiber of \( L^+ \mathcal{G} \) and use similar notation for the truncated positive loop groups. We also define the (Witt vector) loop group \( L^+ \mathcal{G} \) as the functor
\[
L^+ \mathcal{G}: \{ \text{perfect rings of characteristic } p \} \rightarrow \{ \text{groups} \}, \quad R \mapsto G(W(R)[1/p]).
\]
We then have the affine flag variety
\[
\mathcal{F}_{\mathcal{G}} := L^+ \mathcal{G} \setminus L^+ \mathcal{G}
\]
that is an ind-projective ind-perfect scheme over \( \mathbb{F}_p \), by [BS17, Corollary 9.6]. We equip \( L^+ \mathcal{G} \) with the action
\[
(L^+ \mathcal{G} \times L^+ \mathcal{G}) \times L^+ \mathcal{G} \rightarrow L^+ \mathcal{G}, \quad ((k_1, k_2), g) \mapsto k_1 g \sigma(k_2)^{-1}
\]
and \( \mathcal{F}_{\mathcal{G}} \) with the corresponding action
\[
L^+ \mathcal{G} \times \mathcal{F}_{\mathcal{G}} \rightarrow \mathcal{F}_{\mathcal{G}}, \quad (k, L^+ \mathcal{G} \cdot g) \mapsto L^+ \mathcal{G} \cdot g \sigma(k)^{-1}.
\]

This makes the projection \( L^+ \mathcal{G} \rightarrow \mathcal{F}_{\mathcal{G}} \) into an \( L^+ \mathcal{G} \)-equivariant \( L^+ \mathcal{G} \)-torsor.

Let us also fix a minuscule conjugacy class \( \mu \) of cocharacters of \( G_{\overline{Q}_p} \) (where \( Q_p \) is a fixed algebraic closure of \( Q_p \)) and assume for simplicity that it is defined over \( Q_p \). Associated to \( \mu \) we then have the admissible locus \( A_{\mathcal{G},\mu} \subseteq \mathcal{F}_{\mathcal{G}} \) that is the descent to \( \mathbb{F}_p \) of the closed perfect subscheme
\[
\bigcup_{w \in \text{Adm}(\mu)_{\mathcal{G}}} \mathcal{F}_{\mathcal{G},w} \subseteq \mathcal{F}_{\mathcal{G},\mathbb{F}_p},
\]
where \( \text{Adm}(\mu)_{\mathcal{G}} \) denotes the \( \mu \)-admissible set for \( \mathcal{G} \), see [HR15, Section 2], and \( \mathcal{F}_{\mathcal{G},w} \subseteq \mathcal{F}_{\mathcal{G},\mathbb{F}_p} \) is the \( L^+ \mathcal{G} \)-orbit corresponding to \( w \). Write
\[
A_{\mathcal{G},\mu}^+ \subseteq L^+ \mathcal{G}
\]
for the preimage of $A_{G,\mu}$ under the projection $\mathbb{L}G \to \mathcal{F}_G$ and write

$$A_{G,\mu}^{(n)} \to A_{G,\mu}$$

for the reduction of $A_{G,\mu}^+$ to an $\mathbb{L}^{(n)}G$-torsor, where we formally set $\mathbb{L}^{(1\text{-rdt})}G$ to be the perfection of the reductive quotient $G_{F_p}^{\text{rdt}}$. By [AGLR22] proof of Theorem 3.16] the action of $\mathbb{L}^+G$ on $A_{G,\mu}$ factors through $G_{F_p}^{\text{rdt}}$ and the argument in [SYZ21] proof of Lemma 4.4.2] then shows that the $\mathbb{L}^+G$-equivariant structure on $A_{G,\mu}^{(n)} \to A_{G,\mu}$ factors through $\mathbb{L}^{(m)}G$.

The stack of local shtukas for $(G, \mu)$ is then defined to be

$$\text{Sht}^{\text{loc}}_{G,\mu} := \left( (\mathbb{L}^+G) \Delta \backslash A_{G,\mu}^+ \right)$$

and the stack of $(m, n)$-restricted local shtukas for $(G, \mu)$ is defined similarly as

$$\text{Sht}^{\text{loc},(m,n)}_{G,\mu} := \left( (\mathbb{L}^{(m)}G) \Delta \backslash A_{G,\mu}^{(n)} \right).$$

Now let us specialize to the situation where $G = \text{GL}(\Lambda_i)_i$ and $\mu = \mu_d$ is the conjugacy class of those cocharacters of $\text{GL}(V_i, \mathbb{Q}_p)$ that induce a weight decomposition $V_i = V_1 \oplus V_0$ with $V_1$ of dimension $d$. Then we have the explicit description

$$A^{+}_{\text{GL}(\Lambda_i)_i, \mu_d}(R) = \left\{ g \in \text{GL}(V) (W(R)/[1/p]) \mid p\Lambda_i, W(R) \subseteq g\Lambda_i, W(R) \subseteq \Lambda_i, W(R) \text{ and } \Lambda_i, W(R)/g\Lambda_i, W(R)\right\};$$

this follows from the equality between the $\mu$-admissible and the $\mu$-permissible set [Kr04, Theorem 3.5], see also [Gör01, Section 4.3]. Using the description of $M_{\text{loc}, \text{GL}^+, +}$ from the proof of Proposition 3.15 and applying Remark 2.13 we obtain an $(\mathbb{L}^+ \text{GL}(\Lambda_i)_i) \times \mathbb{L}^+ \text{GL}(\Lambda_i)_i)$-equivariant isomorphism

$$(M_{\text{loc}, \text{GL}^+, +})_{F_p}^{\text{pf}} \to A^{+}_{\text{GL}(\Lambda_i)_i, \mu_d}, \quad (M_{(i, i), (\Psi_{i, j}))} \to F_M,$$

where $F_M: W(R)[1/p] \otimes \mathbb{Q}_p V \cong (W(R)[1/p] \otimes \mathbb{Q}_p V)^* \to W(R)[1/p] \otimes \mathbb{Q}_p V$ is the Frobenius of the chain of displays $M = ((\Lambda_i, W(R))_i, (\rho_{i, j}, i, j), (\theta_{i, j}, i, j), (\Psi_{i, j}))$. Consequently we obtain equivalences

$$(\text{ChDisp}^{(m, n)}_{h, d, j})_{F_p}^{\text{pf}} \to \text{Sht}^{\text{loc}, (m, n)}_{\text{GL}(\Lambda_i)_i, \mu_d} \quad \text{and} \quad (\text{ChDisp}^{(m, n)}_{h, d, j})_{F_p}^{\text{pf}} \to \text{Sht}^{\text{loc}, (m, n)}_{\text{GL}(\Lambda_i)_i, \mu_d}$$

between the perfected special fibers of the stacks of chains of displays and the stacks of local shtukas for $(\text{GL}(\Lambda_i)_i, \mu_d)$.

**Remark 3.17.** One could also define a stack

$$\text{ChDisp}^{\text{LZ}, (n)}_{h, d, j,d}.$$

of $n$-truncated chains of Lau-Zink-displays over $R$ (see Remark 2.14) in the straightforward way. This is done in [Hes20] for $n = 1$.

However this definition is pathological away from the hyperspecial case $J = r + h\mathbb{Z}$, even when restricting to perfect rings of characteristic $p$. The truncation morphism

$$\text{ChDisp}_{h, d, j,d} \to \text{ChDisp}^{\text{LZ}, (n)}_{h, d, j,d}$$

is not surjective, and its topological image is not even locally closed (in particular [Hes20, Lemma 2.51] is false) as the following example shows.

Let $(h, d) = (2, \mathbb{Z}, 1)$ and define an $n$-truncated chain $M = ((M_i, M_{i, 1}, \Psi_{i}), (\rho_{i, j}, i, j), (\theta_{i, j}))$ of Lau-Zink-displays over $R = F_p[x^{1/p^\infty}, y^{1/p^\infty}]$ as follows.
We now claim that the set-theoretic locus $Z$ is the standard $n$-truncated Lau-Zink-pair of type $(2,1)$ with fixed normal decomposition $(L,T) = (W_n(R), W_n(R))$.

$\theta_i := \text{id}_{(M,M_1)}$.

$\rho_{0,1} := \begin{pmatrix} [x] & 1 \\ p + [y] \cdot p^n & 0 \end{pmatrix}, \quad \rho_{1,2} := \begin{pmatrix} 0 & 1 \\ p + [y] \cdot p^n - [x] & 0 \end{pmatrix} \in \text{End}_R((M,M_1)) \cong \begin{pmatrix} W_n(R) & W_n(R) \\ I_{n+1,R} & W_n(R) \end{pmatrix}$.

This uniquely determines $\rho_{i,j}$ for all $i, j$.

One can check now that both $((\tilde{M}_i, 1), (\tilde{\rho}_{i,j}), \tilde{\theta}_i)$ and $((M_i), (\rho_{i,j}), (\theta_i))$ are trivial $n$-truncated chains over $R$. We let $(\Psi_i)_i$ be an arbitrary isomorphism between them.

We now claim that the set-theoretic locus $Z \subseteq |\text{Spec}(R)|$ where $M$ lifts to a chain of displays is given by the constructible subset $Z = D(x) \cup V(y)$ that is not locally closed.

From the smoothness of $\text{GL}((\Lambda_i)_i)$ it follows that an $n$-truncated chain of Lau-Zink-displays lifts to a chain of displays if and only if its underlying $n$-truncated chain of Lau-Zink-pairs lifts.

Consider the $n$-truncated chain of Lau-Zink pairs $M' := ((M_i, M_{i,1}), (\rho'_{i,j})_{i,j}, (\theta_i))$ over $\mathbf{F}_p[x^{1/p^n}]$ where the $\rho'_{i,j}$ are defined by setting

$$\rho'_{0,1} := \begin{pmatrix} [x] & 1 \\ p & 0 \end{pmatrix} \quad \text{and} \quad \rho'_{1,2} := \begin{pmatrix} 0 & 1 \\ p & -[x] \end{pmatrix}.$$ 

Then $M'$ lifts to a chain of pairs that is defined by the same expression.

Now $M$ (or rather its underlying $n$-truncated chain of Lau-Zink-pairs) clearly becomes isomorphic to $M'$ after base changing along $R \rightarrow (R/yR)^{pl} \cong \mathbf{F}_p[x^{1/p^n}]$ so that we have $V(y) \subseteq Z$. Moreover we also have an isomorphism $(\alpha_i)_i: M'_{R[x^{-1}]} \rightarrow M_{R[x^{-1}]}$ that is defined by

$$\alpha_0 := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \alpha_1 := \begin{pmatrix} 1 \\ [x^{-1}y] \cdot p^n \\ 1 \end{pmatrix},$$

so that we also have $D(x) \subseteq Z$.

Now let $R \rightarrow k$ be a ring homomorphism from $R$ into a perfect field $k$ such that the image of $x$ in $k$ vanishes while the image of $y$ in $k$ is non-zero, and suppose that $M_k$ lifts to a chain of pairs $M'^{\text{lift}}$. After compatibly lifting normal decompositions $M'^{\text{lift}}$ is given by

$$\rho'^{\text{lift}}_{0,1} := \begin{pmatrix} p^n \cdot a & 1 + p^n \cdot b \\ p + [y] \cdot p^n + p^{n+1} \cdot c & p^n \cdot d \end{pmatrix} \quad \text{and} \quad \rho'^{\text{lift}}_{1,2} := \begin{pmatrix} p^n \cdot e & 1 + p^n \cdot f \\ p + [y] \cdot p^n + p^{n+1} \cdot g & p^n \cdot h \end{pmatrix}$$

for some $a, \ldots, h \in W(k)$. But then the upper left entry of the matrix representing $\rho'^{\text{lift}}_{1,2} \circ \rho'^{\text{lift}}_{0,1}$ is given by

$$(p^n \cdot e) \cdot (p^n \cdot a) + (1 + p^n \cdot f) \cdot (p + [y] \cdot p^n + p^{n+1} \cdot c) \equiv p + [y] \cdot p^n \mod p^{n+1}.$$ 

This is a contradiction as $\rho'^{\text{lift}}_{1,2} \circ \rho'^{\text{lift}}_{0,1}$ really should be multiplication by $p$.

This proves the claim.
Let us now come back to the situation of Section \ref{sec:b} where \( h = 2g \) is even, \( d = g \) and \( -J = J \). Fix a polarization \( \lambda: V \to V^* \) that restricts to isomorphisms \( \lambda_i: \Lambda_i \to \Lambda^\vee \) for all \( i \in J \). We also set \( \Gamma \colonequals \mathbb{Z}_p \) and sometimes think of \( \lambda \) as a homogeneous polarization \( \lambda: V \to \Gamma[1/p] \otimes \mathbb{Q}_p \). We also have \( \lambda: \Lambda_i \to \Gamma \otimes \mathbb{Z}_p \Lambda^\vee \).

We then again have associated parahoric \( \mathbb{Z}_p \)-group schemes
\[
\text{Sp}(\Lambda_{i,j}) := \text{Aut}(\Lambda_{i,j}, (h_{i,j})_i, (\lambda_i)_i)
\]
and
\[
\text{GSp}(\Lambda_{i,j}) := \text{Aut}(\Lambda_{i,j}, (h_{i,j})_i, (\lambda_i)_i)
\]
with generic fibers \( \text{Sp}(\Lambda_{i,j})_{\mathbb{Q}_p} = \text{Sp}(V) \) and \( \text{GSp}(\Lambda_{i,j})_{\mathbb{Q}_p} = \text{GSp}(V) \) that fit into a short exact sequence
\[
1 \to \text{Sp}(\Lambda_{i,j}) \to \text{GSp}(\Lambda_{i,j}) \to G_m \to 1,
\]
where the second arrow is given by sending a symplectic similitude to the similitude factor. We also have the local model
\[
\mathcal{M}_{\text{loc,Sp}} = \mathcal{M}_{\text{loc,GSp}}: R \mapsto \left\{ (C_i)_j \in \mathcal{M}_{\text{loc,GL}}(R) \mid \lambda_i(C_i) = C^\perp_{\lambda_i} \right\}
\]
that is a closed subscheme of \( \mathcal{M}_{\text{loc,GL}} \) stable under the action of \( \text{GSp}(\Lambda_{i,j}) \subseteq \text{GL}(\Lambda_{i,j}) \) whose generic fiber is the Grassmannian of Lagrangian subspaces of \( V \) (see \cite{Gor03}, Theorem 3.16 and Definition 3.27).

Again, by \cite{Gor03} Theorem 2.1, \( \mathcal{M}_{\text{loc,Sp}} = \mathcal{M}_{\text{loc,GSp}} \) is flat over \( \mathbb{Z}_p \), its special fiber is reduced and the irreducible components of its special fiber are normal with rational singularities. We write \( \mathcal{M}_{\text{loc,Sp}} = \mathcal{M}_{\text{loc,GSp}} \) for the \( p \)-completion of \( \mathcal{M}_{\text{loc,Sp}} = \mathcal{M}_{\text{loc,GSp}} \).

The reductive quotients \( \text{Sp}(\Lambda_{i,j})_{\mathbb{F}_p} \) and \( \text{GSp}(\Lambda_{i,j})_{\mathbb{F}_p} \) of the special fibers of \( \text{Sp}(\Lambda_{i,j}) \) and \( \text{GSp}(\Lambda_{i,j}) \) identify with
\[
\text{Sp}(\Lambda_{i,j})_{\mathbb{F}_p} := \text{Aut}(\Lambda_{i,j}, (h_{i,j})_i, (\lambda_e)_e) \quad \text{and} \quad \text{GSp}(\Lambda_{i,j})_{\mathbb{F}_p} := \text{Aut}(\Lambda_{i,j}, (h_{i,j})_i, (\lambda_e)_e)
\]
where \( \lambda_e: \Lambda_e \to \Lambda^\vee_e \cong (\Gamma/p\Gamma) \otimes \mathbb{F}_p \Lambda^\vee_e \) is the isomorphism induced by \( (\lambda_i)_i \). These groups have the following explicit description.

- Suppose that \( 0, g \in J \). Then we have
  \[
  \text{Sp}(\Lambda_{i,j})_{\mathbb{F}_p} = \prod_{e \in \mathcal{E}/(h_{i,j}) \setminus \{e_0\}} \text{GL}(\Lambda_e) \quad \text{and} \quad \text{GSp}(\Lambda_{i,j})_{\mathbb{F}_p} = \prod_{e \in \mathcal{E}/(h_{i,j}) \setminus \{e_0\}} \text{GL}(\Lambda_e).
  \]
- Suppose that \( 0 \not\in J \) and \( g \in J \) and write \( e_0 \in \mathcal{E} \) for the edge \( (i,j) \) with \( i < 0 < g \). Then we have
  \[
  \text{Sp}(\Lambda_{i,j})_{\mathbb{F}_p} = \text{Sp}(\Lambda_{e_0}) \times \prod_{e \in \mathcal{E}/(h_{i,j}) \setminus \{e_0\}} \text{GL}(\Lambda_e) \quad \text{and} \quad \text{GSp}(\Lambda_{i,j})_{\mathbb{F}_p} = \text{GSp}(\Lambda_{e_0}) \times \prod_{e \in \mathcal{E}/(h_{i,j}) \setminus \{e_0\}} \text{GL}(\Lambda_e).
  \]
- Suppose that \( 0 \in J \) and \( g \not\in J \) and write \( e_g \in \mathcal{E} \) for the edge \( (i,j) \) with \( i < g < J \). Then we similarly have
  \[
  \text{Sp}(\Lambda_{i,j})_{\mathbb{F}_p} = \text{Sp}(\Lambda_{e_g}) \times \prod_{e \in \mathcal{E}/(h_{i,j}) \setminus \{e_g\}} \text{GL}(\Lambda_e) \quad \text{and} \quad \text{GSp}(\Lambda_{i,j})_{\mathbb{F}_p} = \text{GSp}(\Lambda_{e_g}) \times \prod_{e \in \mathcal{E}/(h_{i,j}) \setminus \{e_g\}} \text{GL}(\Lambda_e).
  \]
- Suppose that \( 0, g \not\in J \) and write \( e_0, e_g \in \mathcal{E} \) as before. Then we have
  \[
  \text{Sp}(\Lambda_{i,j})_{\mathbb{F}_p} = \text{Sp}(\Lambda_{e_0}) \times \text{Sp}(\Lambda_{e_g}) \times \prod_{e \in \mathcal{E}/(h_{i,j}) \setminus \{e_0, e_g\}} \text{GL}(\Lambda_e)
  \]
and
\[
\text{GSp}(\Lambda_{i,j})_{\mathbb{F}_p} = \text{GSp}(\Lambda_{e_0}, \Lambda_{e_g}) \times \prod_{e \in \mathcal{E}/(h_{i,j}) \setminus \{e_0, e_g\}} \text{GL}(\Lambda_e).
\]
Lemma 3.18. We have natural equivalences of groupoids
\[
\{\text{polarized chains over } R\} \to \{L^+ \text{Sp}(\Lambda_i)\text{-torsors over } R\}
\]
and
\[
\{\text{n-truncated polarized chains over } R\} \to \{L^{(n)} \text{Sp}(\Lambda_i)\text{-torsors over } R\}
\]
for \( n \neq 1\text{-rdt}\). When \( R \) is of characteristic \( p \) we also have a natural equivalence of groupoids
\[
\{\text{1-rdt-truncated polarized chains over } R\} \to \{\text{Sp}(\Lambda_e)\text{-torsors over } R\}.
\]
Similarly we also have a natural equivalence between homogeneously polarized chains over \( R \) and \( L^+ \text{GSp}(\Lambda_i i)\text{-torsors over } R \) and truncated variants.

Proof. The claims for (truncated) polarized chains follow from [RZ96, Theorem 3.16] and [BH20, Lemma 2.12].

To prove the claim for homogeneously polarized chains we need to see that every such homogeneously polarized chain \(((M_i,i), (\rho_i,\theta_i), I, (\lambda_i))\) over \( R \) can be trivialized fpqc-locally on \( \text{Spec}(R) \). Now we can certainly find a local trivialization of \( I \) so that the homogeneously polarized chain lifts to a polarized chain that can be trivialized locally by the first part. \( \square \)

Lemma 3.19. We have natural equivalences of groupoids
\[
\{\text{polarized chains of pairs over } R\} \to \left[ L^+ \text{Sp}(\Lambda_i) \setminus \text{M}^{\text{loc,Sp}} \right](R).
\]
and
\[
\{\text{m-truncated polarized chains of pairs over } R\} \to \left[ L^{(m)} \text{Sp}(\Lambda_i) \setminus \text{M}^{\text{loc,Sp}} \right](R).
\]
We also have analogous equivalences for (m-truncated) homogeneously polarized chains of pairs.

Proof. This follows in the same way as Lemma 3.18. \( \square \)

Definition 3.20. Let
\[
\text{M}^{\text{loc,Sp,+}} \to \text{M}^{\text{loc,Sp}}
\]
be the \( L^+ \text{Sp}(\Lambda_i)\text{-equivariant} \) \( L^+ \text{Sp}(\Lambda_i)\text{-torsor} \) corresponding to the functor from Remark 3.10 for polarized chains under the equivalences from Lemma 3.18 and Lemma 3.19 (as in Definition 3.14).

Similarly, let
\[
\text{M}^{\text{loc,GSp,+}} \to \text{M}^{\text{loc,GSp}}
\]
be the \( L^+ \text{GSp}(\Lambda_i)\text{-equivariant} \) \( L^+ \text{GSp}(\Lambda_i)\text{-torsor} \) corresponding to the analogous functor from Remark 3.10 for homogeneously polarized chains. Note that this agrees with the base change of the \( L^+ \text{Sp}(\Lambda_i)\text{-torsor} \) \( \text{M}^{\text{loc,Sp,+}} \) along the morphism of \( \mathbb{Z}_p \)-group schemes \( L^+ \text{Sp}(\Lambda_i) \) \( \to \) \( L^+ \text{GSp}(\Lambda_i) \).

We also use the notation \( \text{M}^{\text{loc,Sp},(n)} \) and \( \text{M}^{\text{loc,GSp},(n)} \) as in Definition 3.14.

Proposition 3.21. We have equivalences of stacks over \( \text{Spf}(\mathbb{Z}_p) \)
\[
\text{PolChDisp}_{g,J} \to \left[ (L^+ \text{Sp}(\Lambda_i))_\Delta \setminus \text{M}^{\text{loc,Sp},+} \right], \quad \text{PolChDisp}^{(m,n)}_{g,J} \to \left[ (L^{(m)} \text{Sp}(\Lambda_i))_\Delta \setminus \text{M}^{\text{loc,Sp},(n)} \right]
\]
and
\[
\text{HPolChDisp}_{g,J} \to \left[ (L^+ \text{GSp}(\Lambda_i))_\Delta \setminus \text{M}^{\text{loc,GSp},+} \right], \quad \text{HPolChDisp}^{(m,n)}_{g,J} \to \left[ (L^{(m)} \text{GSp}(\Lambda_i))_\Delta \setminus \text{M}^{\text{loc,GSp},(n)} \right].
\]
In particular $\text{PolChDisp}^{(m,n)}_{g,J}$ and $\text{HPolChDisp}^{(m,n)}_{g,J}$ are $p$-adic formal algebraic stacks of finite presentation over $\text{Spf}(\mathbb{Z}_p)$ (algebraic stack of finite presentation over $\text{Spec}(\mathbb{F}_p)$ when $n = 1$-rdt) and for $(m, n) \leq (m', n')$ the morphisms $\text{PolChDisp}^{(m',n')}_{g,J} \to \text{PolChDisp}^{(m,n)}_{g,J}$ and $\text{HPolChDisp}^{(m',n')}_{g,J} \to \text{HPolChDisp}^{(m,n)}_{g,J}$ are smooth.

Proof. This is analogous to Proposition 3.15. 

Remark 3.22 (Relation with local shtukas, continued). Similarly as discussed in Remark 3.16 we have equivalences
\[ \left( \text{HPolChDisp}^{(m,n)}_{g,J} \right)^{\text{pf}}_{\mathbb{F}_p} \to \text{Sh}^{\text{loc}}_{\text{GSp}((A_i)_i), \mu_g} \quad \text{and} \quad \left( \text{HPolChDisp}^{(m,n)}_{g,J} \right)^{\text{pf}}_{\mathbb{F}_p} \to \text{Sh}^{\text{loc}}_{\text{GSp}((A_i)_i), \mu_g}, \]
where $\mu_g$ is the conjugacy class of those cocharacters of $\text{GSp}(V)$ that induce a weight decomposition $V_{\mathbb{C}_p} = V_1 \oplus V_0$ with $V_0, V_1 \subseteq V_{\mathbb{C}_p}$ Lagrangian.

The non-homogeneously polarized case however does not quite fit into the group-theoretic framework of local shtukas. The problem is essentially that given a polarized chain of displays $M$ over $R$ the corresponding Frobenius $F_M : M[1/p]^\alpha \to M[1/p]$ is not a symplectic isomorphism; it only preserves the symplectic form up to the scalar $p$. The situation can be remedied as follows.

The similitude factor morphism $L\text{GSp}(V) \to L\text{G}_m$ restricts to a faithfully flat morphism
\[ \mathcal{A}^+_{\text{GSp}((A_i)_i), \mu_g} \to p \cdot L^+ \text{G}_m \]
and we define $\mathcal{A}^+_{\text{Sp}((A_i)_i), \mu_g}$ to be the fiber of this morphism over $p \in (p \cdot L^+ \text{G}_m)(\mathbb{F}_p)$. Then the isomorphism
\[ (\mathcal{M}^{\text{loc}, \text{GSp}^+})^{\text{pf}}_{\mathbb{F}_p} \to \mathcal{A}^+_{\text{GSp}((A_i)_i), \mu_g} \]
restricts to an $(L^+ \text{Sp}((A_i)_i) \times L^+ \text{Sp}((A_i)_i))$-equivariant isomorphism
\[ (\mathcal{M}^{\text{loc}, \text{Sp}^+})^{\text{pf}}_{\mathbb{F}_p} \to \mathcal{A}^+_{\text{Sp}((A_i)_i), \mu_g} \]
so that we obtain equivalences
\[ (\text{PolChDisp}^{(m,n)}_{g,J})^{\text{pf}}_{\mathbb{F}_p} \to \text{Sh}^{\text{loc}}_{\text{Sp}((A_i)_i), \mu_g} \cong \left[ \left( L^+ \text{Sp}((A_i)_i) \right)_\Delta \backslash \mathcal{A}^+_{\text{Sp}((A_i)_i), \mu_g} \right] \]
and
\[ (\text{PolChDisp}^{(m,n)}_{g,J})^{\text{pf}}_{\mathbb{F}_p} \to \text{Sh}^{\text{loc}}_{\text{Sp}((A_i)_i), \mu_g} \cong \left[ \left( L^+ \text{Sp}((A_i)_i) \right)_\Delta \backslash \mathcal{A}^+_{\text{Sp}((A_i)_i), \mu_g} \right]. \]

3.4. (Polarized) chains of $p$-divisible groups. Let us again first return to the situation of Section 3.1 where $h, J, d$ are arbitrary.

Definition 3.23. A chain of $p$-divisible groups (of type $(h, J, d)$) over $R$ is a diagram $((X_i)_i, (\rho_{i,j})_{i,j})$ of $p$-divisible groups of height $h$ and dimension $d$ over $R$ of shape $J^{\text{op}}$ such that $\rho_{i,j} : X_j \to X_i$ is an isogeny of height $j - i$ and $\text{ker}(\rho_{i,i+h}) = X_{i+h}[p]$. We write
\[ \text{ChBT}_{h, J, d} \]
for the stack over $\text{Spf}(\mathbb{Z}_p)$ of chains of $p$-divisible groups.

In the situation of Section 3.2 we equip the category of chains of $p$-divisible groups over $R$ with a duality by setting
\[ ((X_i)_i, (\rho_{i,j})_{i,j})^\vee := ((X_{-i,j})_i, (\rho_{-i,j})_{i,j}) \]
and define a polarized chain of $p$-divisible groups (of type $(g, J)$) over $R$ to be a polarized object $((X_i)_i, (\rho_{i,j})_{i,j}, (\lambda_i)_i)$ in the category of chains of $p$-divisible groups over $R$. As before we write
\[ \text{PolChBT}_{g, J} \]
for the stack over $\text{Spf}(\mathbb{Z}_p)$ of polarized chains of $p$-divisible groups.
Proposition 3.24. The functor $D$ from Theorem 2.24 induces a morphism
\[ \text{ChBT}_{h,J,d} \to \text{ChDisp}_{h,J,d} \]
that restricts to an equivalence $\text{ChBT}_{h,J,d}^\text{fml} \to \text{ChDisp}_{h,J,d}^\text{F-nilp}$ between the substack of those chains of $p$-divisible groups $((X_i), (\rho_{i,j}), (\theta_i), (\Psi_i))$ such that the $X_i$ are formal $p$-divisible groups and the substack of those chains of displays $((M_i), (\rho_{i,j}), (\theta_i), (\Psi_i))$ such that the $(M_i, M_{i,1}, \Psi_i)$ are $F$-nilpotent.

In the situation of Section 3.2 we similarly obtain a morphism
\[ \text{PolChBT}_{g,J} \to \text{PolChDisp}_{g,J} \]
that restricts to an equivalence $\text{PolChBT}_{g,J}^\text{fml} \to \text{PolChDisp}_{g,J}^\text{F-nilp}$.

Proof. Given a chain of $p$-divisible groups $((X_i), (\rho_{i,j}), (\theta_i), (\Psi_i))$ over $R$ the condition $\ker(\rho_{i,i+h}) = X_{i+h}[p]$ implies that there exists a uniquely determined isomorphism $\theta_i: X_{i+h} \to X_i$ such that $\mu \cdot \theta_i = \rho_{i,i+h}$. We can then apply the functor $D$ to the data $X_i, \rho_{i,j}, \theta_i$ and by Remark 3.7 this yields a chain of displays; here we use the last part of Theorem 2.24 to verify the rank condition for the homomorphism $D(\rho_{i,j})$. Thus we obtain the desired morphism $\text{ChBT}_{h,J,d} \to \text{ChDisp}_{h,J,d}$.

The claim that this morphism restricts to an equivalence $\text{ChBT}_{h,J,d}^\text{fml} \to \text{ChDisp}_{h,J,d}^\text{F-nilp}$ follows immediately from the corresponding statement in Theorem 2.24. \qed

4. Application to the Siegel modular variety

As in Section 3, $(m, n)$ denotes a tuple of positive integers with $m \geq n + 1$ where we allow $n$ to take the additional value $1-rd$.

4.1. The Siegel modular variety at parahoric level. Fix a positive integer $g$ and a non-empty subset $J \subseteq \mathbb{Z}$ such that $J + 2g\mathbb{Z} = J$ and $-J = J$ as in Section 3.2. Also fix an auxiliary integer $N \geq 3$ such that $p \nmid N$ and equip $(\mathbb{Z}/N\mathbb{Z})^{2g}$ with the standard symplectic form that is represented by the matrix
\[ \begin{pmatrix} 0 & \tilde{I}_g \\ -\tilde{I}_g & 0 \end{pmatrix}, \quad \text{where} \quad \tilde{I}_g := \begin{pmatrix} 1 & \cdots \\ \cdots & 1 \end{pmatrix}. \]

Definition 4.1. We define the Siegel modular variety as the moduli problem
\[ A_{g,J,N}: \{ \mathbb{Z}_p\text{-algebras}\} \to \{ \text{sets}\} \]
by setting $A_{g,J,N}(R)$ to be the set of isomorphism classes of tuples $((A_i), (\rho_{i,j}), (\lambda_i), (\eta))$ that are given as follows.

- $((A_i), (\rho_{i,j}), (\lambda_i))$ is a diagram of projective Abelian varieties of dimension $g$ over $R$ of shape $J^{op}$ such that $\rho_{i,j}: A_j \to A_i$ is an isogeny of degree $p^{r_{i,j}}$ and $\ker(\rho_{i,i+2g}) = A_{i+2g}[p]$.
- $(\lambda_i): ((A_i), (\rho_{i,j}), (\eta)) \to ((A_i^{\sigma}), (\rho_{i,j}^{\sigma}), (\eta^{\sigma}))$ is a symmetric isomorphism such that for some (or equivalently every) $i \geq 0$ the symmetric isogeny
\[ A_i \xrightarrow{\lambda_i} A_i^{\sigma} \xrightarrow{\rho_{i,i+1}^{\sigma}} A_i \]
is a polarization.
• \( \eta: A[N] \to (\mathbb{Z}/N\mathbb{Z})^{2g} \) is an isomorphism of finite étale \( R \)-group schemes such that there exists an isomorphism \( \mu_N \to \mathbb{Z}/N\mathbb{Z} \) making the diagram

\[
\begin{array}{ccc}
A[N] \times A[N] & \xrightarrow{\eta \times \eta} & \mu_N \\
\downarrow & & \downarrow \\
(\mathbb{Z}/N\mathbb{Z})^{2g} \times (\mathbb{Z}/N\mathbb{Z})^{2g} & \xrightarrow{\eta \times \eta} & \mathbb{Z}/N\mathbb{Z}
\end{array}
\]

commutative; here \( A[N] \) is the \( N \)-torsion of any of the Abelian varieties \( A_i \) (note that \( \rho_{i,j} \) restricts to an isomorphism \( A_j[N] \to A_i[N] \) because \( p \) does not divide \( N \)) and the upper horizontal arrow is the Weil pairing with respect to \( (\lambda_i)_i \).

We also write \( A^\wedge_{g,J,N} \) for the \( p \)-completion of \( A_{g,J,N} \), i.e. its restriction to the category of \( p \)-nilpotent rings.

**Proposition 4.2.** \( A_{g,J,N} \) is representable by a quasi-projective \( \mathbb{Z}_p \)-scheme. Moreover there exists a natural smooth morphism

\[ A_{g,J,N} \to \left[ \text{GSp}((\Lambda_i)_i) \right]_{\text{Mloc,GSp}}, \]

where we use the notation from Section 3.3.

**Proof.** The representability follows from [MFK94, Theorem 7.9], see also [RZ96, Definition 6.9]. For the second claim, see [HR15, Section 7(ii)]. \( \square \)

**Definition 4.3.** We define the morphism \( \Upsilon: A^\wedge_{g,J,N} \to \text{PolChDisp}_{g,J} \) as the composition\[ A^\wedge_{g,J,N} \to \text{PolChBT}_{g,J} \to \text{PolChDisp}_{g,J}, \]

where the first arrow is given by

\[ (A_i), (\rho_{i,j}), (\lambda_i), \eta) \mapsto ((A_i[p^\infty]), (\rho_{i,j}), (\lambda_i)) \]

(this is well defined because the functor \( A \mapsto A[p^\infty] \) from projective Abelian varieties to \( p \)-divisible groups is compatible with dualities up to a sign \(-1\), see [Oda69, Proposition 1.8]) and the second arrow is the one from Proposition 3.24.

We then also have the induced morphism

\[ v^{(m,n)}: A^\wedge_{g,J,N} \to \text{PolChDisp}_{g,J}^{(m,n)} \]

for \( n \neq 1\text{-rdt} \) and

\[ v^{(m,1\text{-rdt})}: (A_{g,J,N})_{F_p} \to \text{PolChDisp}_{g,J}^{(m,1\text{-rdt})}. \]

**Remark 4.4.** Perfection of the composition

\[ (A_{g,J,N})_{F_p} \xrightarrow{v^{(m,1\text{-rdt})}} \text{PolChDisp}_{g,J}^{(m,1\text{-rdt})} \to \text{HPolChDisp}_{g,J}^{(m,1\text{-rdt})} \]

yields the morphism \( v_K \) from [SYZ21, Section 4.4], at least up to the difference in normalization (see Remark 3.16 and Remark 3.22).

In particular the fibers of this composition are precisely the EKOR strata defined in [HR15, Definition 6.4]. More precisely we have a natural bijection

\[ \text{HPolChDisp}_{g,J}^{(m,1\text{-rdt})}_{F_p} \cong \tilde{K}_O \setminus (\tilde{K}_1 \setminus X), \quad \text{where } X = \bigcup_{w \in \text{Adm}_{g,J}} \tilde{K}w\tilde{K} \subseteq \text{GSp}(V)(\mathbb{Q}_p) \]
as in the introduction. For \( x \in \tilde{K}_g \setminus (\tilde{K}_1 \setminus X) \) we thus have an associated reduced locally closed substack \( \mathcal{C}_x \subseteq (\text{HPolChDisp}^{(m,1\text{-rdt})}_{g,J})_{\mathbb{F}_p} \) that satisfies \( |\mathcal{C}_x| = \{x\} \). The locally closed subscheme \( \text{EKOR}_x \subseteq (\mathcal{A}_{g,J,N})_{\mathbb{F}_p} \) defined by the pullback square

\[
\begin{array}{ccc}
\text{EKOR}_x & \longrightarrow & (\mathcal{A}_{g,J,N})_{\mathbb{F}_p} \\
\downarrow & & \downarrow \\
\mathcal{C}_x & \longrightarrow & (\text{HPolChDisp}^{(m,1\text{-rdt})}_{g,J})_{\mathbb{F}_p}
\end{array}
\]

then is precisely the EKOR stratum corresponding to \( x \), up to possibly carrying some non-reduced structure.

Below in Theorem 4.10 we will prove that the morphism \( \psi^{(m,1\text{-rdt})} \) is smooth and by Lemma 3.11 this implies the smoothness of \( \mathcal{A}_{g,J} \rightarrow \text{HPolChDisp}^{(m,n)}_{g,J} \). Observing that \( \mathcal{C}_x \rightarrow \text{Spec}(\mathbb{F}_p) \) is a gerbe, hence smooth, we can then deduce the smoothness of \( \text{EKOR}_x \) over \( \mathbb{F}_p \).

4.2. Smoothness of the morphism \( \psi^{(m,n)} \).

**Definition 4.5.** Let \( B_g \) be the finite partially ordered set of symmetric Newton polygons starting in \((0,0)\), ending in \((2g, g)\) and with slopes in \([0, 1]\), where we declare \( \nu \leq \nu' \) if \( \nu \) lies above \( \nu' \). This is precisely the set \( B(\text{GSp}(V), \mu_g) \), see [HR13, Section 2] and it classifies isocrystals of height \( 2g \) and slopes contained in \([0, 1]\) that are furthermore equipped with a symplectic form valued in the standard simple isocrystal of slope 1, see also [RR96, Remark 3.4.(iii)].

We have a natural map of sets \( |\text{PolChBT}_{g,J}| \rightarrow B_g \) that sends \( ((X_i), (\rho_{i,j}), (\lambda_i)) \in \text{PolChBT}_{g,J}(k) \) for some algebraically closed field \( k \) of characteristic \( p \) and their preimages in \( |\mathcal{A}^\wedge_{g,J,N}| \) by \( S'_{\nu} \) and call these subsets Newton strata.

**Proposition 4.6.** We have the following properties.

- The \( S_\nu \subseteq |\mathcal{A}^\wedge_{g,J,N}| \) are locally closed subsets.
- Let \( \nu, \nu' \in B_g \). Then the intersection \( S_\nu \cap S_{\nu'} \) is non-empty if and only if \( \nu \leq \nu' \).
- \( |\text{PolChBT}_{g,J}| \) is the union of those \( S'_{\nu} \) such that \( \nu \) does not contain segments of slope 0, 1.

**Proof.** This is all contained in [HR13]. See in particular Axiom 3.5, Theorem 5.6 and Section 7. \( \square \)

**Proposition 4.7.** Let \( \nu, \nu' \in B_g \) with \( \nu \leq \nu' \) and let \( x \in S'_{\nu} \cap S_{\nu'} \). Then there exists a preimage \( y \in S_{\nu'} \) of \( x \) that specializes to a point in \( S_{\nu} \).

**Proof.** By Proposition 4.6 there exists a point in \( S_{\nu'} \) specializing to a point in \( S_{\nu} \). This specialization can be realized by a point

\[
A' = ((A'_i), (\rho'_{i,j}), (\lambda'_i), (\eta')) \in \mathcal{A}_{g,J,N}(R)
\]

for some rank one valuation ring \( R \) of characteristic \( p \) with algebraically closed fraction field \( K \). Write \( X' := A'[p^\infty] \in \text{PolChBT}_{g,J}(R) \) for the image of \( A' \). After possibly enlarging \( R \) we also find an object

\[
X = ((X_i), (\rho_{i,j}), (\lambda_i)) \in \text{PolChBT}_{g,J}(K)
\]

representing \( x \).

Now \( X'_K \) and \( X \) both lie in \( S'_{\nu} \), hence there exists an isomorphism between their associated isocrystals that is compatible with the polarizations. After multiplying such an isomorphism with \( p^M \) for a suitable big enough integer \( M \geq 0 \) we obtain an isogeny

\[
f = (f_i)_i : X'_K \rightarrow X
\]

in \( \text{ChBT}_{g,J}(K) \) that satisfies \( f^* \lambda = p^{2M} \cdot \lambda' \).
For $i \in J$, let $H_i \subseteq X_i'$ be the flat closure of $\ker(f_i) \subseteq X_i'$. This is a finite free $R$-subgroup scheme of $X_i'$ of order $p^{2gM}$. Define $A_i'' := A_i' / H_i$ and let $\rho_i'' : A_i'' \to A_i'$ be the isogeny induced by $\rho_i'$. Then there exists a unique isomorphism $\lambda_i'' : A_i'' \to (A_i'')^\vee$ that makes the diagram

\[
\begin{array}{c}
A_i' \xrightarrow{p^{2gM} \lambda_i'} (A_i')^\vee \\
\downarrow \\
A_i'' \xrightarrow{\lambda_i''} (A_i'')^\vee
\end{array}
\]

commutative. In this way we obtain a point $A'' = ((A_i'')_i, (\rho_i'', j)_{i,j}, (\lambda_i'')_i, \eta') \in A_{g,J,N}(R)$ (note that $A_i''[N] = A_i'[N]$ because $H_i$ is $p$-primary).

By construction $A''$ still realizes a specialization from $S_{\nu'}$ to $S_{\nu}$ and moreover $f$ gives rise to an isomorphism $A''[p^\infty]_K \cong X$ in $\text{PolChBT}_{g,J}(K)$. Thus setting $y \in S_{\nu'} \subseteq |A_{g,J,N}|$ to be the point corresponding to $A''_K$ finishes the proof.

**Corollary 4.8.** Every point $x \in |\text{PolChBT}_{g,J}|$ has a preimage $y \in |A_{g,J,N}|$ that specializes to a point mapping into $|\text{PolChBT}^{\text{fin}}_{g,J}|$.

**Proof.** This follows from combining Proposition 4.7 with the last part of Proposition 4.6. □

**Lemma 4.9.** Let $X$, $Y$ be locally Noetherian formal algebraic stacks, let $f : X \to Y$ be a morphism representable by algebraic stacks that is locally of finite type, and let $x \in |X|$.

Then $f$ is smooth at $x$ if and only if every lifting problem

\[
\begin{array}{c}
\text{Spec}(B) \xrightarrow{\pi} X \\
\downarrow \\
\text{Spec}(B') \xrightarrow{f} Y
\end{array}
\]

where $B' \to B$ is a surjective homomorphism between Artinian local rings and the map $|\text{Spec}(B)| \to |X|$ has image $\{x\}$, admits a solution.

**Proof.** When $X$ and $Y$ are schemes then this is precisely [Stacks, Tag 02HX]. The general case can be reduced to this, see also [Stacks, Tag 0DNV]. □

**Theorem 4.10.** For $n \neq 1$-rdt the morphism $v^{(m,n)} : A_{g,J,N}^{(n)} \to \text{PolChDisp}_{g,J}^{(m,n)}$ is smooth. Similarly also the morphism $v^{(m,1-\text{rdt})} : (A_{g,J,N})_{F_p} \to \text{PolChDisp}_{g,J}^{(m,1-\text{rdt})}$ is smooth.

**Proof.** Assume that $n \neq 1$-rdt. The morphism $v^{(m,n)}$ factors as the composition

$A_{g,J,N}^{(n)} \xrightarrow{\pi} \text{PolChBT}_{g,J} \to \text{PolChDisp}_{g,J} \to \text{PolChDisp}_{g,J}^{(m,n)}$.

We have the following information.

- $\pi : A_{g,J,N}^{(n)} \to \text{PolChBT}_{g,J}$ is formally étale by the Serre-Tate theorem, see [Dri76, Appendix].
• \( \text{PolChBT}_{g,J} \to \text{PolChDisp}_{g,J} \) restricts to an isomorphism \( \text{PolChBT}_{g,J}^{\text{fml}} \to \text{PolChDisp}_{g,J}^{\text{F-nilp}}, \) see Proposition 3.24. As \( \text{PolChDisp}_{g,J}^{\text{F-nilp}} \subseteq \text{PolChDisp}_{g,J} \) is stable under nilpotent thickenings this implies that every lifting problem

\[
\begin{array}{ccc}
\text{Spec}(B) & \longrightarrow & \text{PolChBT}_{g,J} \\
\downarrow & & \downarrow \\
\text{Spec}(B') & \longrightarrow & \text{PolChDisp}_{g,J}
\end{array}
\]

where \( B' \to B \) is a surjective homomorphism of \( p \)-nilpotent rings with nilpotent kernel and the map \( \text{Spec}(B) \to \text{PolChBT}_{g,J} \) has image inside \( \text{PolChBT}_{g,J}^{\text{fml}} \), admits a (unique) solution.

• \( \text{PolChDisp}_{g,J} \to \text{PolChDisp}_{g,J}^{(m,n)} \) is formally smooth, see Proposition 3.21.

Let \( U \subseteq |A^\wedge_{g,J,N}| \) be the smooth locus of \( \nu^{(m,n)} \). Using Lemma 4.9 we can now deduce the following.

• Every point \( x \in |A^\wedge_{g,J,N}| \) that maps into \( |\text{PolChBT}_{g,J}^{\text{fml}}| \) is contained in \( U \).

• \( U \) is of the form \( U = \pi^{-1}(V) \) for some \( V \subseteq \text{PolChBT}_{g,J} \).

Now Corollary 4.8 together with the openness of \( U \subseteq |A^\wedge_{g,J,N}| \) implies that \( V = |\text{PolChBT}_{g,J}| \) and consequently that \( U = |A^\wedge_{g,J,N}| \) as desired.

The case \( n = 1 \text{-rdt} \) now follows from Proposition 3.15. \qed

APPENDIX A. CATEGORIES WITH ACTIONS AND DUALITIES

A.1. Commutative monoids and actions. Fix an ambient \((2,1)\)-category \( \mathbf{C} \) with finite 2-products. We write \( \{\ast\} \in \mathbf{C} \) for an empty 2-product.

**Definition A.1.** A **commutative monoid in** \( \mathbf{C} \) is an object \( \mathcal{M} \in \mathbf{C} \) that is equipped with 1-morphisms

\[
\odot: \mathcal{M} \times \mathcal{M} \to \mathcal{M}, \quad 1: \{\ast\} \to \mathcal{M}
\]

and 2-isomorphisms

\[
(a \odot b) \odot c = a \odot (b \odot c), \quad a \odot b \to b \odot a, \quad 1 \odot a \to a
\]

between 1-morphisms \( \mathcal{M}^i \to \mathcal{M} \) for \( i = 3, 2, 1 \) such that the diagrams

\[
\begin{array}{ccc}
((a \odot b) \odot c) \odot d & \longrightarrow & (a \odot (b \odot c)) \odot d \\
\downarrow & & \downarrow \\
(a \odot (b \odot c)) \odot d & \longrightarrow & a \odot ((b \odot c) \odot d) \longrightarrow a \odot (b \odot (c \odot d))
\end{array}
\]

\[
\begin{array}{ccc}
((a \odot b) \odot c) \odot d & \longrightarrow & (a \odot b) \odot (c \odot d) \\
\downarrow & & \downarrow \\
(a \odot b) \odot (c \odot d) & \longrightarrow & a \odot (b \odot (c \odot d))
\end{array}
\]

\[
\begin{array}{ccc}
a \odot b & \longrightarrow & a \odot b \\
\downarrow & & \downarrow \\
b \odot a & \longrightarrow & b \odot a
\end{array}
\]

\[
\begin{array}{ccc}
(a \odot b) \odot c & \longrightarrow & (b \odot a) \odot c \\
\downarrow & & \downarrow \\
(b \odot (a \odot c)) \longrightarrow b \odot (c \odot a)
\end{array}
\]

\[
\begin{array}{ccc}
(a \odot b) \odot c & \longrightarrow & a \odot (b \odot c) \\
\downarrow & & \downarrow \\
(b \odot c) \odot a & \longrightarrow & b \odot (c \odot a)
\end{array}
\]
are commutative.

The commutative monoids in \( \mathbf{C} \) naturally form the class of objects of a \((2,1)\)-category that is given as follows.

A 1-morphism \( F : \mathcal{M} \to \mathcal{N} \) between two commutative monoids in \( \mathbf{C} \) is a 1-morphisms \( F : \mathcal{M} \to \mathcal{N} \) in \( \mathbf{C} \) that is equipped with 2-isomorphisms \( F(a \otimes b) \to F(a) \otimes F(b), \) \( F(1) \to 1 \) such that the diagrams

\[
\begin{array}{ccc}
F((a \otimes b) \otimes c) & \longrightarrow & F(a \otimes b) \otimes F(c) \\
\downarrow & & \downarrow \\
F(a \otimes (b \otimes c)) & \longrightarrow & F(a) \otimes F(b \otimes c)
\end{array}
\]

\[
\begin{array}{ccc}
F(a \otimes b) & \longrightarrow & F(a) \otimes F(b) \\
\downarrow & & \downarrow \\
F(b \otimes a) & \longrightarrow & F(b) \otimes F(a)
\end{array}
\]

\[
\begin{array}{ccc}
F(1 \otimes a) & \longrightarrow & F(1) \otimes F(a) \\
\downarrow & & \downarrow \\
F(a) & \longrightarrow & 1 \otimes F(a)
\end{array}
\]

are commutative.

A 2-isomorphism \( \alpha : F \to G \) between two 1-morphisms \( F,G : \mathcal{M} \to \mathcal{N} \) as above is a 2-isomorphism \( \alpha : F \to G \) in \( \mathbf{C} \) such that the diagrams

\[
\begin{array}{ccc}
F(a \otimes b) & \xrightarrow{\alpha} & G(a \otimes b) \\
\downarrow & & \downarrow \\
F(a) \otimes F(b) & \xrightarrow{\alpha \otimes \alpha} & G(a) \otimes G(b)
\end{array}
\]

\[
\begin{array}{ccc}
F(1) & \xrightarrow{\alpha} & G(1) \\
\downarrow & & \downarrow \\
1 & \xrightarrow{\kappa} & 1
\end{array}
\]

are commutative.

**Definition A.2.** Let \( \mathcal{M} \) be a commutative monoid in \( \mathbf{C} \). An \( \mathcal{M} \)-object in \( \mathbf{C} \) is an object \( \mathcal{C} \in \mathbf{C} \) that is equipped with a 1-morphism

\( \otimes : \mathcal{M} \times \mathcal{C} \to \mathcal{C} \)

and 2-isomorphisms

\[
(a \otimes b) \otimes x \to a \otimes (b \otimes x), \quad 1 \otimes x \to x
\]
such that the diagrams
\[
\begin{align*}
((a \otimes b) \otimes c) \otimes x & \to (a \otimes (b \otimes c)) \otimes x & \to a \otimes ((b \otimes c) \otimes x) & \to a \otimes (b \otimes (c \otimes x)) \\
((a \otimes b) \otimes c) \otimes x & \to (a \otimes b) \otimes (c \otimes x) & \to a \otimes (b \otimes (c \otimes x))
\end{align*}
\]
are commutative.

The tuples \((\mathcal{M}, \mathcal{C})\) consisting of a commutative monoid \(\mathcal{M}\) in \(\mathbf{C}\) and an \(\mathcal{M}\)-object \(\mathcal{C}\) in \(\mathbf{C}\) naturally form the class of objects of a \((2,1)\)-category that is given as follows.

A 1-morphism \((F, F') : (\mathcal{M}, \mathcal{C}) \to (\mathcal{N}, \mathcal{D})\) consists of a 1-morphisms \(F : \mathcal{M} \to \mathcal{N}\) of commutative monoids in \(\mathbf{C}\) and a 1-morphism \(F' : \mathcal{C} \to \mathcal{D}\) in \(\mathbf{C}\) that is equipped with a 2-isomorphism \(F'(a \otimes x) \to F(a) \otimes F'(x)\) such that the diagrams
\[
\begin{align*}
F'(1 \otimes x) & \to F(1) \otimes F'(x) & \to 1 \otimes F'(x) \\
\downarrow & \downarrow & \downarrow \\
F'(a \otimes (b \otimes x)) & \to F(a) \otimes F'(b \otimes x) & \to F(a) \otimes (F(b) \otimes F'(x)) \\
\downarrow & & \\
F'((a \otimes b) \otimes x) & \to F(a \otimes b) \otimes F'(x) & \to (F(a) \otimes F(b)) \otimes F'(x)
\end{align*}
\]
are commutative.

A 2-isomorphism \((\alpha, \alpha') : (F, F') \to (G, G')\) between two 1-morphisms \((F, F'), (G, G') : (\mathcal{M}, \mathcal{C}) \to (\mathcal{N}, \mathcal{D})\) as above consists of a 2-isomorphism \(\alpha : F \to G\) between 1-morphisms of commutative monoids in \(\mathbf{C}\) and a 2-isomorphism \(\alpha' : F' \to G'\) in \(\mathbf{C}\) such that the diagram
\[
\begin{align*}
F'(a \otimes x) & \xrightarrow{\alpha'} G'(a \otimes x) \\
\downarrow & \downarrow \\
F(a) \otimes F'(x) & \xrightarrow{\alpha \otimes \alpha'} G(a) \otimes G'(x)
\end{align*}
\]
is commutative.

A.2. Categories with duality.

**Definition A.3.** A **category with duality** is a category \(\mathcal{C}\) that is equipped with an equivalence of categories \(\mathcal{C}^\text{op} \to \mathcal{C}, \ x \mapsto x^\vee\) and a natural isomorphism \(x \to x^{\vee \vee}\).
between functors $\mathcal{C} \to \mathcal{C}$ such that the diagram

$$
\begin{array}{ccc}
  x^\vee & \xrightarrow{\sim} & x^\vee \\
  \downarrow & & \downarrow \\
  x^{\vee\vee} & \xleftarrow{\sim} & x^{\vee\vee}
\end{array}
$$

is commutative (where the sloped arrow on the right is the image of the isomorphism $x \to x^{\vee\vee}$ under the duality equivalence above).

A functor $F: \mathcal{C} \to \mathcal{D}$ between categories with duality is a functor $F: \mathcal{C} \to \mathcal{D}$ that is equipped with a natural isomorphism $F(x^\vee) \to F(x)^\vee$ such that the diagram

$$
\begin{array}{ccc}
  F(x) & \xrightarrow{\sim} & F(x)^\vee \\
  \downarrow & & \downarrow \\
  F(x^{\vee\vee}) & \xleftarrow{\sim} & F(x)^{\vee\vee}
\end{array}
$$

is commutative (where the second horizontal arrow denotes the inverse of the dual of the isomorphism $F(x^\vee) \to F(x)^\vee$).

A natural isomorphism $\alpha: F \to G$ between two functors $F,G: \mathcal{C} \to \mathcal{D}$ as above is a natural isomorphism $\alpha: F \to G$ such that the diagram

$$
\begin{array}{ccc}
  F(x^\vee) & \xrightarrow{\alpha} & G(x^\vee) \\
  \downarrow & & \downarrow \\
  F(x)^\vee & \xleftarrow{\alpha^{-1}} & G(x)^\vee
\end{array}
$$

is commutative.

In this way the categories with duality naturally form the class of objects of a (2,1)-category.

**Remark A.4.** The (2,1)-category of (essentially small) categories with duality has finite 2-products and the natural forgetful 2-functor

$$\{\text{categories with duality}\} \to \{\text{categories}\}$$

preserves finite 2-products.

Thus we obtain the notions of a symmetric monoidal category with duality and an action of such a symmetric monoidal category with duality on a category with duality.

**Remark A.5.** Every rigid symmetric monoidal category naturally carries a duality. Thus we obtain a natural (2,1)-functor

$$\{\text{rigid symmetric monoidal categories}\} \to \{\text{symmetric monoidal categories with duality}\}.$$

**Remark A.6.** Let $\Lambda$ be a commutative ring. Then the notions of a category with duality, a symmetric monoidal category (with duality) and an action of a symmetric monoidal category (with duality) on a category (with duality) have obvious $\Lambda$-linear variants that are obtained by requiring all appearing functors to be $\Lambda$-(multi-)linear.

**Definition A.7.** Let $\mathcal{C}$ be a $\mathbb{Z}$-linear category with duality. Then a morphism $f: x \to x^\vee$ in $\mathcal{C}$ is called antisymmetric if the diagram

$$
\begin{array}{ccc}
  x & \xrightarrow{f} & x^{\vee\vee} \\
  \downarrow & & \downarrow \\
  x^\vee & \xleftarrow{\sim} & f^\vee
\end{array}
$$
is commutative. We define the groupoid $\text{Pol}(C)$ of polarized objects in $C$ as the groupoid of tuples $(x, \lambda)$ with $x \in C$ and $\lambda \colon x \to x^\vee$ an antisymmetric isomorphism, where isomorphisms $(x, \lambda) \to (y, \zeta)$ are given by isomorphisms $x \to y$ in $C$ such that the diagram

$$
\begin{array}{ccc}
x & \longrightarrow & y \\
\downarrow^{\lambda} & & \downarrow^{\zeta} \\
x^\vee & \longrightarrow & y^\vee
\end{array}
$$

is commutative (where the lower horizontal arrow is the inverse of the dual of $x \to y$).

Now suppose that $C$ is equipped with an action of a $\mathbb{Z}$-linear rigid symmetric monoidal category $M$ (see Remark [A.5]). Then a morphism $f \colon x \to a \otimes x^\vee$ in $C$ with $a \in M$ invertible is called antisymmetric if the diagram

$$
\begin{array}{ccc}
x & \longrightarrow & x^\vee \\
\downarrow & \nearrow^{a \otimes 1} & \downarrow^{a \otimes f^\vee} \\
\longrightarrow & a \otimes (a \otimes x^\vee)^\vee
\end{array}
$$

is commutative. In the case $a = 1$ this recovers the first definition above. We define the groupoid $\text{HPol}(C) = \text{HPol}^M(C)$ of $(M\text{-})$homogeneously polarized objects in $C$ as the groupoid of tuples $(x, a, \lambda)$ with $x \in C$, $a \in M$ invertible and $\lambda \colon x \to a \otimes x^\vee$ an antisymmetric isomorphism, where isomorphisms $(x, a, \lambda) \to (y, b, \zeta)$ are given by tuples of isomorphisms $x \to y$ in $C$ and $a \to b$ in $M$ such that the diagram

$$
\begin{array}{ccc}
x & \longrightarrow & y \\
\downarrow^{\lambda} & & \downarrow^{\zeta} \\
a \otimes x^\vee & \longrightarrow & b \otimes y^\vee
\end{array}
$$

is commutative.

**Lemma A.8.** Let $C$ be a $\mathbb{Z}$-linear category with duality that is equipped with an action of a $\mathbb{Z}$-linear rigid symmetric monoidal category $M$. Then there is a natural 2-Cartesian diagram of groupoids

$$
\begin{array}{ccc}
\text{Pol}(C) & \longrightarrow & \text{HPol}(C) \\
\downarrow & & \downarrow \\
\{\ast\} & \longrightarrow & M^\cong
\end{array}
$$

where $M^\cong$ denotes the groupoid core of $M$.

**Proof.** The upper horizontal arrow is given by $(x, \lambda) \mapsto (x, 1, \lambda)$, the right vertical arrow by $(x, a, \lambda) \mapsto a$ and the lower horizontal arrow by $\ast \mapsto 1$. The diagram is in fact strictly commutative so that we can use the identity as a commutativity constraint.

Now the diagram is strictly Cartesian and the right vertical arrow is a fibration (both claims can be checked by a direct computation) so that the diagram is also 2-Cartesian. \qed

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