Topology and Energy of Time Dependent Unitons

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Abstract

We consider a class of time dependent finite energy multi–soliton solutions of the $U(N)$ integrable chiral model in $(2 + 1)$ dimensions. The corresponding extended solutions of the associated linear problem have a pole with arbitrary multiplicity in the complex plane of the spectral parameter. Restrictions of these extended solutions to any spacelike plane in $\mathbb{R}^{2,1}$ have trivial monodromy and give rise to maps from a three sphere to $U(N)$. We demonstrate that the total energy of each multi–soliton is quantised at the classical level and given by the third homotopy class of the extended solution. This is the first example of a topological mechanism explaining classical energy quantisation of moving solitons.

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1 Introduction

The fact that the allowed energy levels of some physical systems can take only discrete values has been well known since the early days of quantum theory. The hydrogen atom and the harmonic oscillator are two well known examples. In these two cases the boundary conditions imposed on the wave function imply discrete spectra of the Hamiltonians. The reasons are therefore global.

The quantisation of energy can also occur at the classical level in nonlinear field theories if the energy of a smooth field configuration is finite. The reasons are again global, but one needs more subtle ideas from topology to understand what is going on. The potential energy of static soliton solutions in the Bogomolny limit of certain field theories must be proportional to integer homotopy classes of smooth maps. The details depend on the model: In pure gauge theories the energy of solitons satisfying the Bogomolny equations is given by one of the Chern numbers of the curvature. In scalar 2+1 dimensional sigma models, allowed energies of Bogomolny solitons are given by elements of \( \pi_2(\Sigma) \), where the manifold \( \Sigma \) is the target space. In both cases the boundary condition are used to show that the finite energy configurations extend to the compactification of space. See [11] for a detailed exposition of these constructions.

The situation is different for moving solitons: The total energy is the sum of kinetic and potential terms, and the Bogomolny bound is not saturated. One expects that the moving (non-periodic) solitons will have continuous energy. Attempts to construct theories with quantised total energy based on compactifying the time direction are physically unacceptable, as they lead to paradoxes related to the existence of closed time–like curves. A soliton moving along such curve could eventually reach its own past thus opening possibilities to sinister scenarios usually involving a death of somebody’s great grandparents.

In a recent publication [8] Ioannidou and Manton made the surprising observation that the total energy of the time–dependent \( SU(2) \) two–unitor solution of Ward’s 2 + 1 dimensional chiral model [15, 18] is quantised in the units of \( 8\pi \) when the pole of the corresponding extended solution is at \( \pm i \). They have shown that the two–unitor energy density calculated at any instant of time \( t \) is the same as the energy density of a static \( \mathbb{C}P^3 \) multi–lump with a parameter \( t \). The total (potential) energy of the latter model is quantised [20] which leads to the total (kinetic+potential) energy quantisation of the time–dependent unitons. The quantisation was also obtained by Lechtenfeld and Popov [9, 10] whose method was based on large time asymptotic analysis.
One expects that there are deeper topological reasons for this quantisation, and the purpose of this paper is to show that this is indeed the case.

The Ward chiral model is

\[(J^{-1}J_t)_t - (J^{-1}J_x)_x - (J^{-1}J_y)_y - [J^{-1}J_t, J^{-1}J_y] = 0,\]  

(1.1)

where \( J : \mathbb{R}^3 \to U(N), \) and \( x^\mu = (t, x, y) \) are coordinates on \( \mathbb{R}^3 \) such that the line element is \( \eta = -dt^2 + dx^2 + dy^2. \) Here we use notation \( J_\mu := \partial_\mu J. \) The equations are not fully Lorentz invariant, as the commutator term fixes a space–like direction. A positive definite conserved energy functional for (1.1) is

\[ E = \int_{\mathbb{R}^2} \mathcal{E} dx dy, \]  

(1.2)

where the energy density is given by

\[ \mathcal{E} = -\frac{1}{2} \text{Tr}((J^{-1}J_t)^2 + (J^{-1}J_x)^2 + (J^{-1}J_y)^2). \]  

(1.3)

The integrability of (1.1) allows a construction of explicit static and also time–dependent solutions by twistor or inverse–scattering methods [15, 17]. There are time–dependent solutions with non–scattering solitons [15], and also solitons that scatter [18]. A class of scattering solutions to (1.1) is given by so called time–dependent unitons

\[ J(x, y, t) = M_1 M_2 \ldots M_n, \]  

(1.4)

where the unitary matrices \( M_k, k = 1, \ldots, n \) are given by

\[ M_k = i \left( 1 - \frac{\mu}{\bar{\mu}} \right) R_k, \quad R_k = \frac{q_k^* \otimes q_k}{||q_k||^2}. \]  

(1.5)

Here \( \mu \in \mathbb{C} \setminus \mathbb{R} \) is a non-real constant and \( q_k = (1, f_{k1}, \ldots, f_{k(N-1)}) \in \mathbb{C}^N, \) with \( k = 1, \ldots, n, \) are vectors whose components \( f_{kj} = f_{kj}(x^\mu) \in \mathbb{C} \) are smooth functions which tend to a constant at spatial infinity\(^1\).

If \( n = 1 \) then \( q_1 \) is holomorphic and rational in \( \omega = x + \frac{1}{2} \mu(t + y) + \frac{1}{2} \mu^{-1}(t - y) \) [15]. Note that if \( \mu = \pm i, \) \( q_1 \) does not depend on \( t, \) and the corresponding 1–uniton is static. If \( n > 1 \) \( q_1 \) is

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\(^1\)The matrix \( R_k \) is a hermitian projection satisfying \( (R_k)^2 = R_k, \) and the corresponding \( M_k \) is a Grassmanian embedding of \( \mathbb{C}P^{N-1} \) into \( U(N). \) The results in this paper apply to the more general class of unitons obtained from the complex Grassmanian embeddings of \( Gr(K, N) \) into the unitary group. For \( \mu \) pure imaginary, a complex \( K \)–plane \( V \subset \mathbb{C}^N \) corresponds to a unitary transformation \( i(\pi V - \pi V^\perp), \) where \( \pi V \) denotes the hermitian orthogonal projection onto \( V. \) The formula (1.5) with \( \mu = i \) corresponds to \( K = 1 \) where \( Gr(1, N) = \mathbb{C}P^{N-1}. \)
still holomorphic and rational in $\omega$, but $q_2, q_3, \ldots$ are not holomorphic. The exact form of this functions is known explicitly for $n = 2, 3$ [18, 6] for the case $N = 2$. For $n > 3$ the Bäcklund transformations [7, 3] can be used to determine the $f$s recursively. The total energy (1.2) of $n$–uniton solutions is finite.

In general the finiteness of $E$ is ensured by imposing the boundary condition (valid for all $t$)

$$J = J_0 + J_1(\varphi)r^{-1} + O(r^{-2}) \quad \text{as} \quad r \to \infty, \quad x + iy = re^{i\varphi} \quad (1.6)$$

and so for a fixed value of $t$ the matrix $J$ extends to a map from $S^2$ (conformal compactification of $\mathbb{R}^2$) to $U(N)$. The homotopy group $\pi_2(U(N)) = 0$, so there is no topological information in $J$ defined on $\mathbb{R} \times S^2$ which could be related to the total energy. We shall nevertheless show that the energy of (1.4) is quantised, and given by the third homotopy class of the extended solution to (1.1). The existence of this extended solution is linked to the complete integrability of (1.1) and the associated Lax equations with the spectral parameter. The extended solution also depends on this parameter, and hence is defined on $\mathbb{R}^3 \times \mathbb{CP}^1$. Restricting it to a space–like plane in $\mathbb{R}^3$ and an equator in a Riemann sphere of the spectral parameter gives a map $\psi$, whose domain is $\mathbb{R}^2 \times S^1$. If $J$ is an $n$–uniton solution (1.4), the corresponding extended solution satisfies stronger boundary conditions which promote $\psi$ to a map $S^3 \to U(N)$. In the next Section we shall introduce the extended solution, and impose boundary conditions on $J$ which are stronger than (1.6) and in fact provide a coordinate–free characterisation of the uniton solutions (1.4). In Section 3 we shall establish the following result:

**Theorem 1.1** The total energy of the $n$–uniton solution (1.4) with complex number $\mu = me^{i\phi}$ is quantised and equal to

$$E(n) = 4\pi \left( \frac{1 + m^2}{m} \right) |\sin(\phi)| [\psi], \quad (1.7)$$

where for any fixed value of $t$ the map $\psi : S^3 \to U(N)$ is given by

$$\psi = \prod_{k=n}^{1} \left( 1 + \frac{\pi - \mu}{\mu + \cot \left( \frac{\theta}{2} \right)} R_k \right), \quad \theta \in [0, 2\pi], \quad (1.8)$$

and

$$[\psi] = \frac{1}{24\pi^2} \int_{S^3} Tr ((\psi^{-1}d\psi)^3) \quad (1.9)$$

is an integer taking values in $\pi_3(U(N)) = \mathbb{Z}$.
The model (1.1) is $SO(1, 1)$ invariant, and in Section 3 it will be shown that the Lorentz boosts correspond to rescaling $\mu$ by a real number. The rest frame corresponds to $|\mu| = 1$, when the $y$-component of the momentum vanishes. The $SO(1, 1)$ invariant generalisation of (1.7) will be given by Theorem 3.3. Energies of soliton solutions more general than (1.4) are briefly discussed in Section 4.

2 Extended solution and its homotopy

2.1 Lax pair and trivial scattering

The proof of Theorem (1.1) relies on integrability of (1.1) and its Lax formulation, which we set up next. Let $A = A_\mu dx^\mu$ and $\Phi$ be a one–form and a function on $\mathbb{R}^{2,1}$ with values in a Lie algebra of $U(N)$ determined up to gauge transformations

$$A \longrightarrow bAb^{-1} - db b^{-1}, \quad \Phi \longrightarrow b\Phi b^{-1}, \quad b = b(x^\mu) \in U(N).$$

The system of first order equations

$$D\Phi = *F,$$

where $D\Phi = d\Phi + [A, \Phi]$ and $F = dA + A \wedge A$, gives the integrability conditions $[L_0, L_1] = 0$ for an overdetermined system of linear equations

$$L_0 \Psi := (D_y + D_t - \lambda(D_x + \Phi))\Psi = 0, \quad L_1 \Psi := (D_x - \Phi - \lambda(D_t - D_y))\Psi = 0, \quad (2.10)$$

where $\Psi$ is an $GL(N, \mathbb{C})$-valued function of $x^\mu$ and a complex parameter $\lambda \in \mathbb{C}P^1$, which satisfies the unitary reality condition

$$\Psi(x^\mu, \overline{\lambda})^* \Psi(x^\mu, \lambda) = 1.$$

The matrix $\Psi$ is also subject to gauge transformation $\Psi \longrightarrow b\Psi$. The integrability conditions for (2.10) imply the existence of a gauge $A_t = A_y$, and $A_x = -\Phi$, and a matrix $J : \mathbb{R}^3 \longrightarrow U(N)$ such that

$$A_t = A_y = \frac{1}{2}J^{-1}(J_t + J_y), \quad A_x = -\Phi = \frac{1}{2}J^{-1}J_x,$$

and equations (1.1) hold. Given a solution $\Psi$ to the linear system (2.10) one can construct a solution to (1.1) by

$$J(x^\mu) = \Psi^{-1}(x^\mu, \lambda = 0) \quad (2.11)$$
and all solutions to (1.1) arise from some $\Psi$’s. The detailed exposition of this is presented, for example, in [5].

Let us restrict $\Psi$ from $\mathbb{R}^{2,1} \times \mathbb{CP}^1$ to the space-like plane $t = 0$. We shall also restrict the spectral parameter to lie in the real equator $S^1 \subset \mathbb{CP}^1$ parameterised by $\theta$:

$$\Psi(t, x, y, \lambda) \rightarrow \psi(x, y, \theta) := \Psi(x, y, 0, -\cot \frac{\theta}{2}), \quad (2.12)$$

where now $\psi : \mathbb{R}^2 \times S^1 \rightarrow U(N)$ and we made change of variable for real $\lambda = -\cot (\frac{\theta}{2})$. Note that $\psi$ automatically satisfies

$$(u^\mu D_\mu - \Phi)\psi = 0, \quad (2.13)$$

where the operator anihilating $\psi$ is the spatial part of the Lax pair (2.10), given by

$$\frac{\lambda L_0 + L_1}{1 + \lambda^2} = u^\mu D_\mu - \Phi,$$

where $u = \left(0, \frac{1 - \lambda^2}{1 + \lambda^2}, \frac{2\lambda}{1 + \lambda^2}\right) = (0, -\cos \theta, -\sin \theta)$.

We impose the ‘trivial scattering’ boundary condition [1, 19]

$$\psi(x, y, \theta) \rightarrow \psi_0(\theta) \quad \text{as} \quad r \rightarrow \infty, \quad (2.14)$$

where $\psi_0(\theta)$ is an $U(N)$-valued function on $S^1$. We shall now demonstrate that this enables us to extend $\psi$ to a map from $S^3$ to $U(N)$.

First note that (2.14) implies the existence of the limit of $\psi$ at spatial infinity for all values of $\theta$, while the finite energy boundary condition (1.6) only implies the limit at $\theta = \pi$. Thus the condition (2.14) extends the domain of $\psi$ to $S^2 \times S^1$. However, it turns out that (2.14) is also a sufficient condition for $\psi$ to extend to the suspension $SS^2 = S^3$ of $S^2$. This can be seen as follows. The domain $S^2 \times S^1$ can be considered as $S^2 \times [0, 1]$ with $\{0\}$ and $\{1\}$ identified. Recall that a suspension $SX$ of a manifold $X$ is the quotient space [2]

$$SX = ([0, 1] \times X)/((\{0\} \times X) \cup (\{1\} \times X)),$$

This definition is compatible with spheres in the sense that $SS^d = S^{d+1}$.

Now the only condition $\psi$ needs to fulfils for the suspension is an equivalence relation between all the points in $S^2 \times \{0\}$, since such relation for $S^2 \times \{1\}$ will follow from the identification of $\{0\}$ and $\{1\}$. This equivalence can be achieved by choosing a gauge

$$\psi(x, y, 0) = 1. \quad (2.15)$$

Therefore $\psi$ extends to a map from $SS^2 = S^3$ to $U(N)$ if it satisfies the zero scattering boundary condition.
In addition, after fixing the gauge (2.15), there is still some residual freedom in \( \psi \) given by

\[
\psi \rightarrow \psi K,
\]

where \( K = K(\theta, x, y) \in U(N) \) is annihilated by \( u^\mu \partial_\mu \). Setting \( K = (\psi_0(\theta))^{-1} \) results in

\[
\psi(\{\infty\}, \theta) = 1.
\] (2.17)

The gauge (2.17) picks a base point \( \{x_0 = \infty\} \in \mathbb{S}^2 \), and this implies that the zero scattering condition is also sufficient for \( \psi \) to extend to the reduced suspension of \( \mathbb{S}^2 \), given by

\[
S_{\text{red}} \mathbb{S}^2 = ([0, 1] \times \mathbb{S}^2)/((\{0\} \times \mathbb{S}^2) \cup (\{1\} \times \mathbb{S}^2) \cup ([0, 1] \times \{x_0\})).
\]

This is also homeomorphic to \( \mathbb{S}^3 \). The idea of (reduced) suspension is illustrated in (Fig. 1).

Now let us justify the term ‘trivial scattering’ in (2.14). Consider equation (2.13) and restrict it to a line \((x, y) = (x_0 - \sigma \cos \theta, y_0 - \sigma \sin \theta), \sigma \in \mathbb{R}\). Now (2.13) becomes an ODE describing the propagation of

\[
\psi = \psi(x_0 - \sigma \cos \theta, y_0 - \sigma \sin \theta, \theta)
\]

along the oriented line through \((x_0, y_0)\) in \( \mathbb{R}^2 \). We can choose a gauge such that

\[
\lim_{\sigma \to -\infty} \psi = 1,
\]

and define the scattering matrix \( S : TS^1 \to U(N) \) on the space of oriented lines in \( \mathbb{R}^2 \) as

\[
S = \lim_{\sigma \to \infty} \psi.
\] (2.18)
The trivial scattering condition (2.14) then implies this matrix is trivial,

\[ S = 1. \] (2.19)

As we have explained, the boundary condition (1.6) and (2.14) imply that for each value of \( \theta \) the function \( \psi \) extends to a one-point compactification \( S^2 \) of \( \mathbb{R}^2 \). The straight lines on the plane are then replaced by the great circles, and in this context the trivial scattering condition implies that the differential operator \( u^\mu D_\mu - \Phi \) has trivial monodromy along the compactification \( S^1 = \mathbb{R} \cup \{ \infty \} \) of a straight line parametrised by \( \sigma \).

### 2.2 Topology of extended solution

In the last subsection we have explained that we can regard \( \psi \) as a map from \( S^3 \) to \( U(N) \). All such maps are characterised by their homotopy type \([2]\)

\[ [\psi] = \frac{1}{24\pi^2} \int_{S^3} \text{Tr}((\psi^{-1}d\psi)^3). \] (2.20)

The element \([\psi]\) is an integer taking values in \( \pi_3(U(N)) = \mathbb{Z} \), and is invariant under continuous deformations of \( \psi \).

In the next section we will need the following result: Let \( g_1 \) and \( g_2 \) be maps from \( S^3 \) to \( U(N) \) and let \( g_1g_2 : S^3 \longrightarrow U(N) \) be given by

\[ g_1g_2(x) := g_1(x)g_2(x), \quad x \in S^3, \]

where the product on the RHS is the point-wise group multiplication. Then

\[ [g_1g_2] = [g_1] + [g_2]. \] (2.21)

This is because

\[ \text{Tr}[(g_1g_2)^{-1}d(g_1g_2)^3] = \text{Tr}[(g_1^{-1}dg_1)^3 + (g_2^{-1}dg_2)^3] + d\beta, \]

where \( \Omega \) is a two–form, and so \( d\beta \) integrates to 0 by Stokes’ theorem. This was explicitly demonstrated by Skyrme [13] in the case of \( SU(2) \).

Rather than exhibiting the exact form of \( \beta \) we shall use the following general argument. The higher homotopy groups \( \pi_d(G) \) of a Lie group \( G \) are abelian, and the group multiplication in \( G \) induces the addition in the homotopy groups: if \( g_1 \) and \( g_2 \) are maps from \( S^d \) to \( G \) then the
homotopy class of the map $g_1g_2: S^d \to G$ defined by the group multiplication is the sum of homotopy classes of $g_1$ and $g_2$. The proof of this is presented for example in [2] and essentially follows the proof that the fundamental group of a topological group is abelian. Now $\pi_3(G) = \mathbb{Z}$ for any compact simple Lie group. If $G = SU(2)$ this result just reproduces the calculation done by Skyrme as two continuous maps from $S^3$ to itself are homotopic iff they have the same topological degree. Theorem 1.1 holds for unitons with value in $G = U(N)$, where $[\psi]$ in (1.7) is the sum of homotopy classes which arise from integrals of elements of $H^3(G)$. To find out a homotopy class of a map $\psi$ we can use the formula (2.20), where the integrand is a left–invariant three–form on the group manifold pulled back to $S^3$. This is because $\pi_3(G)$ is isomorphic to the integral homology group $H_3(G, \mathbb{Z})$, and the RHS of (2.20) coincides with the homology class of the cycle $\psi(S^3) \subset G$.

We remark that some part of this topological data is encoded in the Ward equation (1.1), which can be regarded as an ordinary chiral model with torsion [16]. Any compact semi–simple group $G$ admits a connection which parallel propagates left–invariant vector fields. This connection is flat, but necessarily has torsion $T$. The torsion is totally anti–symmetric, thus giving a preferred three–form in the third cohomology group which can then be pulled back to $S^3$. The first order commutator term in (1.1) can be rewritten as

$$\epsilon^{\mu\nu}[J^{-1}\partial_\mu J, J^{-1}\partial_\nu J],$$

where $\epsilon^{\mu\nu}$ is a totally antisymmetric constant matrix. In our case $\epsilon^{\mu\nu} = \varepsilon^{\mu\nu\alpha}V_\alpha$, where $V = (0, 1, 0)$ is the space–like unit vector and (1.1) takes the form

$$(\eta^{\mu\nu} + \epsilon^{\mu\nu})(\partial_\mu (J^{-1}\partial_\nu J)) = 0.$$  

The choice of $V$ reduces the symmetry group of (1.1) down to $SO(1, 1)$. The momenta $P_t = E$, and $P_y$ are well defined, and conserved for (1.1).

The commutator term can be obtained from a Lagrangian density $\epsilon^{\mu\nu}(\partial_\mu \xi^i)(\partial_\nu \xi^j)e_{ij}(\xi)$, where the two–form $e$ is a local potential for the torsion $de = T$, and $\xi^j$ are local coordinates on $G$. The two–form $e$ is defined only locally in $G$.

3 Time dependent unitons and energy quantisation

A class of extended solutions which satisfy the trivial scattering condition (2.14) give rise to the $n$-uniton solutions defined in (1.4). These extended solutions factorise into the so called
$n$-uniton factors \[18\]

$$
\Psi = G_n G_{n-1} \ldots G_1, \quad \text{where} \quad G_k = \left(1 - \frac{\bar{\mu} - \mu}{\lambda - \mu} R_k\right) \in GL(N, \mathbb{C}), \quad R_k = \frac{q_k^* \otimes q_k}{||q_k||^2}. \quad (3.22)
$$

Here $q_k = q_k(x, y, t) \in \mathbb{C}^N, k = 1, \ldots, n$, and $\mu$ is a non-real constant. The terminology here is rather confusing, as the maxima of the energy density of the corresponding soliton solutions of (1.1) do physically scatter. The exact form of $q_k$s is determined from (2.10) by demanding that the expressions

$$(\partial_x \Psi - \lambda (\partial_t - \partial_y) \Psi) \Psi^{-1}, \quad \text{and} \quad ((\partial_t + \partial_y) \Psi - \lambda \partial_x \Psi) \Psi^{-1} \quad (3.23)$$

are independent of $\lambda$. In practice one determines the $q_k$s by a limiting procedure from solutions of a Riemann problem with simple poles \[15\].

The restricted map $\psi$ (2.12) corresponding to (3.22) is given by

$$
\psi = g_n g_{n-1} \ldots g_1, \quad \text{where} \quad g_k = 1 + \frac{\bar{\mu} - \mu}{\mu + \cot \left(\frac{\theta}{2}\right)} R_k \in U(N), \quad (3.24)
$$

where $\lambda = -\cot \left(\frac{\theta}{2}\right) \in S^1 \subset \mathbb{C}P^1$ as before and all the maps are restricted to the $t = 0$ plane. Each element $g_k$ has the limit at spatial infinity for all values of $\theta$.

$$
g_k(x, y, \theta) \longrightarrow g_{0k}(\theta) = 1 + \frac{\bar{\mu} - \mu}{\mu + \cot \left(\frac{\theta}{2}\right)} R_{0k} \quad \text{as} \quad x^2 + y^2 \longrightarrow \infty.
$$

The existence of the limit at spatial infinity $R_{0k} = \lim_{r \to \infty} R_k(x, y) = const$ is guaranteed by the finite energy condition (1.6). Hence $\psi$ (3.24) satisfies the trivial scattering condition (2.14) and extends to a map from $S^3$ to $U(N)$. The scattering matrix\footnote{Novikov \[12\] has demonstrated that given a scattering matrix on the space of oriented lines in $\mathbb{R}^D$ with $D > 2$ it is always possible to reconstruct the gauge potential and the Higgs field on $\mathbb{R}^D$ by means of a non–Abelian inverse Radon transform. The non–trivial initial data for the time dependent $n$–unitons (3.22) has trivial scattering matrix which shows that the inversion is not in general possible if $D = 2$.} (2.18) is $S = 1$.

Note, however, that the $g_k$s and $\psi$ in (3.24) only extend to the ordinary suspension of $S^2$. One needs to perform the transformation (2.16) with $K = \prod_{k=1}^n g_{0k}^{-1}$ for $\psi$ to extend to the reduced suspension of $S^2$. We shall use $\psi$ as in (3.24), because (2.21) and $\pi_1(U(N)) = 0$ imply that the transformation (2.16) does not contribute to the degree and $[K(\theta) \psi] = [\psi]$. 
**Proposition 3.1** The third homotopy class of $\psi$ is given by

\[ [\psi] = \pm \frac{i}{2\pi} \int_{\mathbb{R}^2} \sum_{k=1}^{n} \text{Tr}(R_k[\partial_x R_k, \partial_y R_k])dxdy \quad \begin{cases} 0 < \phi < \pi \\ \pi < \phi < 2\pi, \end{cases} \]  

where $\mu = me^{i\phi}$.

**Proof.** The recursive application of (2.21) implies that

\[ [\psi] = \sum_{k=1}^{n} [g_k]. \]

Using (2.20), with $z = x + iy$,

\[ [g_k] = \frac{1}{8\pi^2} \int_{S^1 \times \mathbb{R}^2} \text{Tr}(g_k^{-1}\partial_\theta g_k [g_k^{-1}\partial_x g_k, g_k^{-1}\partial_y g_k])d\theta \wedge dz \wedge d\overline{z} \]

\[ = \frac{1}{16\pi^2} \Theta(\mu) \int_{\mathbb{R}^2} \text{Tr}(R_k[\partial_x R_k, \partial_y R_k])dz \wedge d\overline{z}, \]

where

\[ \Theta(\mu) = \int_0^{2\pi} \frac{(\bar{\mu} - \mu)^3 \sin^2 \left(\frac{\theta}{2}\right)}{(|\mu|^2 + (1 - |\mu|^2) \cos^2 \left(\frac{\theta}{2}\right) + (\mu + \bar{\mu}) \cos \left(\frac{\theta}{2}\right) \sin \left(\frac{\theta}{2}\right))^2} \frac{d\theta}{\pi}, \]

\[ = \pm 8\pi i \quad \begin{cases} 0 < \phi < \pi \\ \pi < \phi < 2\pi. \end{cases} \]

Hence, changing to the $(x, y)$ coordinates, we obtain

\[ [g_k] = \pm \frac{i}{2\pi} \int_{\mathbb{R}^2} \text{Tr}(R_k[\partial_x R_k, \partial_y R_k])dxdy \quad \begin{cases} 0 < \phi < \pi \\ \pi < \phi < 2\pi. \end{cases} \]

Therefore, the third homotopy class of $\psi$ is given by (3.25).

\[ \square \]

The proof of Theorem (1.1) makes use of the above Proposition and a recursive procedure of adding unitons to a given solution which we shall now explain. Let $\Psi$ be an extended solution to the Lax pair (2.10) corresponding to a solution $J$, which satisfies (1.1). Set

\[ \dot{\Psi} = G\Psi = \left(1 - \frac{\mu}{\lambda - \mu} R\right)\Psi, \quad \dot{J} = \dot{\Psi}^{-1}|_{\lambda=0} = JM, \]  

(3.27)
where $M$ is of the form (1.5), up to a constant phase factor which is irrelevant. The matrix $\hat{\Psi}$ will be an extended solution if expressions (3.23) with $\Psi$ replaced by $\hat{\Psi}$ are independent of $\lambda$. This leads to the Bäcklund relations [7, 3]. These are first order PDEs for $M$, which can be regarded as a generalisation of Uhlenbeck’s method of adding unitons for harmonic maps [14]. In terms of the hermitian projection $R$, these PDEs are

$$
R(R_t - J^{-1}J_t(1 - R)) = B
$$

$$
RR_t = C,
$$

where

$$
B = (\mu R_x - R_y + RJ^{-1}J_y)(1 - R)
$$

$$
C = \frac{1}{\mu}((\mu R_y + R_x - RJ^{-1}J_x)(1 - R)).
$$

**Proof of Theorem 1.1** We first consider a solution of the form $\hat{J} = JM$, where $J$ is an arbitrary solution of (1.1). Noting that $M$ is unitary and writing it in terms of $R$, the difference between the energy densities (1.3) of $\hat{J}$ and $J$ is given by

$$
\Delta E \equiv \hat{E} - E = \sum_a \text{Tr} \left( \kappa \bar{\kappa} R_a R_a R + \kappa (1 - \bar{\kappa} R) J^{-1} J_a R_a \right),
$$

where $a$ stands for $(t, x, y)$, $\hat{E}$ and $E$ are the energy densities of $\hat{J}$ and $J$ respectively and $\kappa = \left(1 - \frac{\mu}{\bar{\mu}} \right)$.

Multiplying the relations (3.28) and their hermitian conjugates yields the following identities

$$
\text{Tr}(R_t R_t R) = \text{Tr}(CC^*)
$$

$$
\text{Tr}(J^{-1}J_t R_t) = \text{Tr}(CB^* - BC^*)
$$

$$
\text{Tr}(RJ^{-1}J_t R_t) = \text{Tr}((C - B)C^*).
$$

The terms involving time derivatives in (3.29) are of the form $R_t R_t R$, $J^{-1}J_t R_t$ and $RJ^{-1}J_t R_t$, which, by (3.30), can be written in terms of the spatial derivatives only. Thus by direct substitution and some rearrangements (3.29) becomes

$$
\Delta E = -\frac{\kappa}{\mu} \text{Tr} \left( (1 + |\mu|^2) R[R_x, R_y] + \mathcal{T} \right),
$$

where $\mathcal{T} = \partial_x(RJ^{-1}J_y) - \partial_y(RJ^{-1}J_x)$ gives no contribution to the difference in the energy functionals of $\hat{J}$ and $J$. This is because

$$
\text{Tr} \int_{\mathbb{R}^2} \mathcal{T} dx \wedge dy = \lim_{r \to \infty} \int_{D_r} d(\text{Tr}(RJ^{-1}dJ))
$$

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\[
\begin{align*}
&= \lim_{r \to \infty} \oint_{C_r} \text{Tr}(RJ^{-1} \, dJ) = \text{Tr} \left( \lim_{r \to \infty} \oint_{C_r} (JR)^* \, dJ \right) \\
&\leq \lim_{r \to \infty} \left( \text{Tr} \left( \frac{(JR_0)^*}{r} (J_1(\varphi = 2\pi) - J_1(\varphi = 0)) \right) + 2\pi r \left\{ \frac{|c_2|}{r^2} + \frac{|c_3|}{r^3} + \ldots \right\} \right) = 0,
\end{align*}
\]

by Stokes’s theorem, where \( C_r \) denotes the circle enclosing the disc \( D_r \) of radius \( r \), \( \varphi \) is a coordinate on \( C_r \), and \( |c_i| \) is the bound of \( \text{Tr}((JR_i)^* \partial_\varphi J) \), \( i = 1, 2, \ldots \). We have used the boundary condition

\[
\lim_{r \to \infty} JR = (JR_0) + (JR_1(\varphi)r^{-1} + O(r^{-2})),
\]

which follows from (1.6) for \( \hat{J} = JM \), and the fact that integrands are continuous on the circle and hence bounded. Since \( (JR)_0 \) is a constant matrix, the first term in the series is a total derivative.

So far we have only used the assumption that \( J \) is a solution of (1.1), but not that it has to be a uniton solution defined by (1.4). Therefore, we have a more general result for the total energy of a Ward solution of the form \( \hat{J} = JM \), where \( J \) is an arbitrary solution to Ward equation. Let \( \hat{E} \) and \( E \) be the total energies of \( \hat{J} \) and \( J \) respectively, then

\[
\hat{E} = E + \frac{(\mu - \bar{\mu})(1 + |\mu|^2)}{|\mu|^2} \int_{\mathbb{R}^2} \text{Tr}(R[R_x, R_y]) \, dx \, dy.
\]

(3.32)

From this, the explicit expression for the total energy of an \( n \)-uniton solution (1.4) follows.

First, consider a 1-uniton solution \( J_{(1)} = M_1 \). It can be written as \( J_{(1)} = J_{(0)}M_1 \), where the constant matrix \( J_{(0)} \), which satisfies (1.1) trivially, is chosen to be the identity matrix. Then, from (3.32), the total energy of a 1-uniton solution is given by

\[
E_{(1)} = \frac{(\mu - \bar{\mu})(1 + |\mu|^2)}{|\mu|^2} \int_{\mathbb{R}^2} \text{Tr}(R_1[\partial_x R_1, \partial_y R_1]) \, dx \, dy.
\]

(3.33)

Therefore, using (3.32), we show by induction that the total energy of an \( n \)-uniton solution (1.4) is given by

\[
E_{(n)} = \frac{(\mu - \bar{\mu})(1 + |\mu|^2)}{|\mu|^2} \sum_{k=1}^{n} \int_{\mathbb{R}^2} \text{Tr}(R_k[\partial_x R_k, \partial_y R_k]) \, dx \, dy
\]

(3.34)

\[
= \pm 4\pi \left( \frac{1 + m^2}{m} \right) \sin(\phi) \left[ \psi \right] \begin{cases} 0 < \phi < \pi \\ \pi < \phi < 2\pi, \end{cases}
\]

where \( \mu = me^{i\phi} \), and we have used (3.25).
We remark that the formula (3.26) reveals another topological interpretation of the energy quantisation which is useful in practical calculations. Consider the group element (3.24) with the index $k$ dropped. The Grassmannian projector $R$ in (3.22) corresponds to a smooth map from the compactified space to the projective space $q : S^2 \rightarrow \mathbb{CP}^{N-1}$. The homotopy group $\pi_2(\mathbb{CP}^{N-1}) = \mathbb{Z}$ is non-trivial and the degree of $q$ is obtained by evaluating the homology class on a standard generator for $H^2(\mathbb{CP}^{N-1})$ represented in a map $q = (1, f_1, \ldots, f_{N-1})$ by the Kähler form

$$\Omega = -4i\partial\bar\partial \ln(1 + \sum_{j=1}^{N-1} |f_j|^2).$$

This evaluation is just the integrating, thus

$$[q] = \frac{i}{8\pi} \int_{\mathbb{R}^2} q^*(\Omega).$$

Evaluating the integrand we verify that

$$i\text{Tr}(R[R_x, R_y]) = \frac{1}{4} q^*(\Omega).$$

We conclude that the energy is proportional to the sum of the topological degrees of Grassmannian projectors involved in the definition of unitons.

In the remaining part of this Section we shall prove a Lorentz invariant generalisation of Theorem (1.1). We start off by looking at the quantisation of the momentum. Following [15], we have chosen the conserved energy functional for a solution of (1.1) to be that obtained from the energy-momentum tensor of the associated standard chiral model. However, for (1.1) only the energy and the $y$-component of momentum are conserved, while the $x$-component of momentum is not. The conserved $y$-momentum is given by

$$P = \int_{\mathbb{R}^2} \mathcal{P} dx dy, \quad (3.35)$$

where the momentum density is

$$\mathcal{P} = -\text{Tr}(J^{-1} J_x J^{-1} J_y). \quad (3.36)$$

It turns out that this is also quantised and proportional to the third homotopy class of the restricted extended solution.

**Proposition 3.2** The $y$-momentum of the $n$-uniton solution (1.4) is given by

$$P_{(n)} = -4\pi \left(\frac{1 - m^2}{m}\right) |\sin(\phi)| \langle [\psi] \rangle. \quad (3.37)$$

Thus, unless $[\psi] = 0$, $P = 0$ if and only if $m = 1$. 

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Proof. We first consider $\hat{J} = JM$ as in the the proof of Theorem 1.1. The difference between the $y$-momentum densities (3.36) of $\hat{J}$ and $J$ is given by

$$\Delta P \equiv \hat{P} - P = \kappa \text{Tr} \left( (1 - \bar{\kappa} R)(J^{-1} J_t R_y + J^{-1} J_y R_t) + \bar{\kappa} (R_y R_t) \right).$$  \hspace{1cm} (3.38)

Then the substitution

$$\text{Tr}(J^{-1} J_t R_y R) = \text{Tr} \left( (C - B) R_y \right)$$

$$\text{Tr}(J^{-1} J_t R R_y) = \text{Tr} \left( (B^* - C^*) R_y \right)$$

from the Bäcklund relations (3.28) gives

$$\Delta P = \frac{\kappa}{\mu} \text{Tr} \left( (1 - |\mu|^2) R[R_x, R_y] + \mathcal{T} \right), \quad \text{where} \quad \kappa = \left(1 - \frac{\mu}{\bar{\mu}}\right).$$

The term $\mathcal{T} = \partial_x(RJ^{-1} J_y) - \partial_y(RJ^{-1} J_x)$ gives no contribution to the difference in the $y$-momenta of $\hat{J}$ and $J$ as to the difference in the energies. Thus we have a result for Ward solution of the form in $\hat{J} = JM$, where $J$ is an arbitrary solution to Ward equation, its $y$-momentum is given by

$$\hat{P} = P - \frac{(\mu - \bar{\mu})(1 - |\mu|^2)}{|\mu|^2} \int_{\mathbb{R}^2} \text{Tr}(R[R_x, R_y]) dxdy,$$  \hspace{1cm} (3.39)

where $\hat{P}$ and $P$ are $y$-momenta of $\hat{J}$ and $J$ respectively.

We then proceed by induction to obtain the expression for the $y$-momentum of an $n$-uniton solution (1.4), in the same way as for the total energy. This gives

$$P_{(n)} = -\frac{(\mu - \bar{\mu})(1 - |\mu|^2)}{|\mu|^2} \sum_{k=1}^n \int_{\mathbb{R}^2} \text{Tr}(R_k[\partial_x R_k, \partial_y R_k]) dxdy$$

$$= \mp 4\pi \left(1 - \frac{m^2}{m} \right) \sin(\phi) [\psi] \left\{ \begin{array}{l} 0 < \phi < \pi \\ \pi < \phi < 2\pi. \end{array} \right. \hspace{1cm} (3.40)$$

We shall now exploit the $SO(1, 1)$ invariance of (1.1) to combine Theorem 1.1 and Proposition 3.2 in the Lorentz invariant form

**Theorem 3.3** For an $n$-uniton solution, the $SO(1, 1)$ invariant relation

$$E_{(n)}^2 - P_{(n)}^2 = 64\pi^2 \sin^2(\phi)[\psi]^2$$  \hspace{1cm} (3.41)

holds.
Proof. Since the equation (1.1) is invariant under $SO(1,1)$, we can generate new solutions from a given one by boosts in the $y-t$ plane. In the coordinates \( \{x, u = \frac{1}{2}(t+y), v = \frac{1}{2}(t-y)\} \), the boosts are given by \( x \rightarrow x, \ u \rightarrow su, \ v \rightarrow s^{-1}v, \ s \in \mathbb{R}^* \). We shall show that a boost of an \( n \)-uniton solution with a pole \( \mu \) in the extended solution gives rise to another \( n \)-uniton solution with the pole \( \mu' = s\mu \).

Consider the Bäcklund relations (3.28) expressed in the \( \{x,u,v\} \) coordinates,

\[
(\mu R_x - R_u + RJ^{-1}J_u)(1 - R) = 0 \quad (3.42)
\]

\[
(\mu R_v - R_x + RJ^{-1}J_x)(1 - R) = 0.
\]

Let \( J \) be an arbitrary solution of (1.1), and \( R(x,u,v) \) be the hermitian projector satisfying (3.42). Under the boost to another solution \( J \rightarrow J' \) we have \( R \rightarrow R' = R(x,su,s^{-1}v) \). Changing the coordinates, we see that \( R' \) will satisfy (3.42) with \( \mu \) and \( J \) replaced by \( \mu' \) and \( J' \), if \( \mu' = s\mu \). That is, each restricted uniton factor transforms as

\[
g_k = 1 + \frac{\bar{\mu} - \mu}{\mu + \cot \left(\frac{\theta}{2}\right)} R_k(x,u,v,\mu) \rightarrow g'_k = 1 + \frac{s\bar{\mu} - s\mu}{s\mu + \cot \left(\frac{\theta}{2}\right)} R_k(x,su,s^{-1}v,\mu).
\]

Since boost is a continuous transformation it does not change the homotopy types, and

\[
[\psi(x,u,v)] = [\psi(x,su,s^{-1}v)].
\]

Hence, under the transformation, \( E_{(n)} \) and \( P_{(n)} \) only change due to the explicit factors of \( \mu \) in (1.7) and (3.37) respectively. The boosts rescale \( \mu \) by \( m \rightarrow sm \), keeping the phase \( \phi \) fixed. This leads to the \( SO(1,1) \) invariance of \( E_{(n)}^2 - P_{(n)}^2 \). The formula (3.41) follows directly from (1.7) and (3.37).

\[\square\]

Examples. Consider the \( SU(2) \) case, where the third homotopy class is equal to the topological degree and set \( \mu = i \). The uniton factors are of the form

\[
M_k = \frac{i}{1 + |f_k|^2} \begin{pmatrix}
|f_k|^2 - 1 & -2f_k \\
-2\bar{f}_k & 1 - |f_k|^2
\end{pmatrix}.
\]

\( n = 1 \). In the one–uniton case \( \partial_t M_1 = 0 \), and \( M_1 \) is given by (1.5) with \( f_1 = f_1(z) \) a rational function of some fixed degree \( Q \). The energy density is

\[
\mathcal{E}_1 = \frac{8|f_1'|^2}{(1 + |f_1|^2)^2} = -i\text{Tr} (M_1[\partial_z M_1, \partial_z^2 M_1])
\]
and \( E = 8\pi \deg(g_1) \) in agreement with (1.7). In this case \( g_1 \) is a suspension of a rational map \( f_1 : \mathbb{CP}^1 \to \mathbb{CP}^1 \) and \( \deg(g_1) = \deg(f_1) \) is a simple illustration of the Freundenthal Theorem which says that a suspension of maps of \( d \)-spheres induces an isomorphism of the homotopy groups.

**n = 2.** In the two–uniton case \( M_1 \) and \( M_2 \) are given by (1.5) with \( \mu = i \) and

\[
q_1 = (1, f), \quad q_2 = (1 + |f|^2)(1, f) - 2i(tf' + h)(\overline{f}, -1),
\]

where \( f \) and \( h \) are rational functions of \( z \). Define \( k = 2(tf' + h) \). The total energy density is

\[
E = \frac{8(|1 + |f|^2)k' - 2k|f'f''|^2 + 16|k|2|f'f''|^2 + 16(1 + |f|^2)|f'f''|^2}{(|k|^2 + (1 + |f|^2)^2)^2}
\]

and

\[
E = \int_{\mathbb{R}^2} E \, dx dy = 8\pi (\deg(g_1) + \deg(g_2))
\]

for all \( t \). The quantisation of energy in this case has first been observed in [8], where it was shown that \( E = 8\pi Q \) where generically \( Q = 2 \deg f + \deg h \). However, \( Q = \max(2 \deg f, \deg h) \) if both \( f \) and \( h \) are polynomials. Our formula (1.7) is valid for all pairs \( (f, h) \).

### 4 Conclusions

We have established the relation between the total energy of time dependent solitons (1.4) and homotopy classes of associated extended solutions. To the best of our knowledge this is the first example of a topological mechanism ensuring the classical energy quantisation of moving solitons.

The \( n \)--uniton solutions (1.4) form a subclass of all finite energy solitons which satisfy the ‘trivial scattering’ boundary condition (2.14). Dai and Terng [3] have demonstrated that the extended solution corresponding to the general ‘trivial scattering’ soliton has poles at non–real points \( \mu_1, ..., \mu_r \) with multiplicities \( n_1, ..., n_r \), and is a product of simple elements \( G_{k,\alpha} \alpha = 1, ..., r \) of the form in (3.22). Our case (3.22) corresponds to \( r = 1 \), but the method used in the proof of Theorem 1.1 applies to the general case as one can choose a different \( \mu \) at each iteration of the Bäcklund transformations (3.28). Formulae (3.26) and (3.32) lead to the general form of the total energy of ‘trivial scattering’ solitons

\[
E = 4\pi \sum_{k=1}^{n_\alpha} \sum_{\alpha=1}^{r} \frac{1 + m_\alpha^2}{m_\alpha} |\sin \phi_\alpha| |g_{k,\alpha}|, \quad \mu_\alpha = m_\alpha e^{i\phi_\alpha},
\]

(4.44)
where
\[ g_{k,\alpha} = 1 + \frac{\bar{\mu}_\alpha - \mu_\alpha}{\mu_\alpha + \cot \left( \frac{\theta}{2} \right)} R_{k,\alpha} \in U(N) \]
and \( R_{k,\alpha} \) are hermitian projections whose form is determined by the Bäcklund relations. This
agrees with the result of Lechtenfeld and Popov [10]. The formula (4.44) is not directly linked
to the homotopy type of the extended solution, and the \( SO(1,1) \) invariance can not be easily
incorporated. This is why we have focused on the special case (1.4).

In [4] the \( SU(2) \) integrable chiral model (1.1) has been analysed in the moduli space approx-
imation, when the time dependent slowly moving solitons correspond to curves in the moduli
space of static solitons which are geodesic with respect to the natural metric
\[ h(\dot{\gamma}, \dot{\gamma}) = \frac{1}{2} \dot{\gamma}^p \dot{\gamma}^q \int_{\mathbb{R}^2} \frac{\left| \partial_p f \partial_q f \right|}{(1 + |f|^2)^2} dx dy \]
on the space of rational maps. Here \( f = f(z, \gamma) \) is a rational meromorphic function of \( z = x + iy \)
which depends on real parameters (positions of zeroes and poles) \( \gamma^p \), and \( \partial_p f = \partial f / \partial \gamma^p \).

The kinetic energy of these approximate solitons is small, and their total energy is close (in
the units of \( 8\pi \)) to the degree of the associated rational map. Theorem (1.1) gives a class of
exact solutions with quantised total energy, and one may expect that the approximate solitons
of [4] arise from the time–dependent unitons by some limiting procedure.

Acknowledgements

We wish to thank Marcin Kaźmierczak, Nick Manton and Lionel Mason for valuable comments,
and Ivan Smith for clarifying some aspects of homotopy theory. We also thank the anonymous
referee for valuable comments. Prim Plansangkate is grateful to the Royal Thai Government
for funding her research.

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