EIGENSYSTEM MULTISCALE ANALYSIS FOR ANDERSON LOCALIZATION IN ENERGY INTERVALS

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Abstract. We present an eigensystem multiscale analysis for proving localization (pure point spectrum with exponentially decaying eigenfunctions, dynamical localization) for the Anderson model in an energy interval. In particular, it yields localization for the Anderson model in a nonempty interval at the bottom of the spectrum. This eigensystem multiscale analysis in an energy interval treats all energies of the finite volume operator at the same time, establishing level spacing and localization of eigenfunctions with eigenvalues in the energy interval in a fixed box with high probability. In contrast to the usual strategy, we do not study finite volume Green’s functions. Instead, we perform a multiscale analysis based on finite volume eigensystems (eigenvalues and eigenfunctions). In any given scale we only have decay for eigenfunctions with eigenvalues in the energy interval, and no information about the other eigenfunctions. For this reason, going to a larger scale requires new arguments that were not necessary in our previous eigensystem multiscale analysis for the Anderson model at high disorder, where in a given scale we have decay for all eigenfunctions.

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Introduction

We present an eigensystem multiscale analysis for proving localization (pure point spectrum with exponentially decaying eigenfunctions, dynamical localization) for the Anderson model in an energy interval. In particular, it yields localization for the Anderson model in a nonempty interval at the bottom of the spectrum.

The well known methods developed for proving localization for random Schrödinger operators, the multiscale analysis \cite{ProS, FroMSS, Dr, DrK, S, CoH, FK2, GK1, Kl, BoK, GK4} and the fractional moment method \cite{AM, A, ASFH, AENSS, AiW}, are based on the study of finite volume Green’s functions. Multiscale analyses based on Green’s functions are performed either at a fixed energy in a single box, or for all energies but with two boxes with an ‘either or’ statement for each energy.

In \cite{EK} we provided an implementation of a multiscale analysis for the Anderson model at high disorder based on finite volume eigensystems (eigenvalues and eigenfunctions). In contrast to the usual strategy, we did not study finite volume Green’s functions. Information about eigensystems at a given scale was used to derive information about eigensystems at larger scales. This eigensystem multiscale analysis treats all energies of the finite volume operator at the same time, giving a complete picture in a fixed box. For this reason it does not use a Wegner estimate as in a Green’s functions multiscale analysis, it uses instead a probability estimate for level spacing derived by Klein and Molchanov from Minami’s estimate \cite{KlM, Lemma 2}. This eigensystem multiscale analysis for the Anderson model at high disorder has been enhanced in \cite{KlT} by a bootstrap argument as in \cite{GK1, Kl}.

The motivation for developing an alternative approach to localization is related to a new focus among the mathematical physics community in disordered systems with an infinite number of particles, for which Green’s function methods break down. The direct study of the structure of eigenfunctions for such systems has been advocated by Imbrie \cite{I1, I2} in a context of both single and many-body localization.

The Green’s function methods allow for proving localization in energy intervals, and hence localization has also been proved at fixed disorder in an interval at the edge of the spectrum (or, more generally, in the vicinity of a spectral gap), and for a fixed interval of energies at the bottom of the spectrum for sufficiently high disorder. (See, for example, \cite{HM, KSS, FK1, ASFH, GK2, K, GK4, AiW}. These methods do not differentiate between energy intervals and the whole spectrum; they can be used whenever the initial step can be established.

The results in \cite{EK} yield localization for the Anderson model in the whole spectrum, which in practice requires high disorder. This eigensystem multiscale analysis treats all energies of the finite volume operator at the same time, at a given scale we have decay for all eigenfunctions, and the induction step uses information about all eigenvalues and eigenfunctions. The method does not have a straightforward extension for proving localization in an energy interval, since at any give scale we would only have information (decay) about eigenfunctions corresponding to eigenvalues in the given interval. For this reason, when performing an eigenfunction multiscale analysis in an energy interval, going to a larger scale requires new arguments that were not necessary in our previous eigensystem multiscale analysis for the Anderson model at high disorder, where in a given scale we have decay for all eigenfunctions.

In this paper we develop a version of the eigensystem multiscale analysis tailored to the establishment of localization for the Anderson model in an energy interval. This version yields localization at fixed disorder on an interval at the edge of the
spectrum (or in the vicinity of a spectral gap), and at a fixed interval at the bottom of the spectrum for sufficiently high disorder.

The Anderson model is a random Schrödinger operator $H_\omega$ on $\ell^2(\mathbb{Z}^d)$ (see Definition 1.5). Multiscale analyses prove statements about finite volume operators $H_{\omega,\Lambda}$, the restrictions of $H_\omega$ to finite boxes $\Lambda$. The eigensystem multiscale analysis developed in this article establishes eigensystem localization in a bounded energy interval with good probability at large scales, as we will now explain.

An eigensystem $\{(\varphi_j, \lambda_j)\}_{j \in J}$ for $H_{\omega,\Lambda}$ consists of eigenpairs $(\varphi_j, \lambda_j)$, where $\lambda_j$ is an eigenvalue for $H_{\omega,\Lambda}$ and $\varphi_j$ is a corresponding normalized eigenfunction, such that $\{(\varphi_j)\}_{j \in J}$ is an orthonormal basis for the finite dimensional Hilbert space $\ell^2(\Lambda)$. If all eigenvalues of $H_{\omega,\Lambda}$ are simple, we can rewrite the eigensystem as $\{(\varphi_\lambda, \lambda)\}_{\lambda \in \sigma(H_{\omega,\Lambda})}$.

We define eigensystem localization in a bounded energy interval $I$ in the following way. We fix appropriate exponents $\beta, \tau \in (0,1)$ (see (1.11)), take $m > 0$, and say that a box $\Lambda$ of side $L$ is $(m, I)$-localizing for $H_\omega$ (see Definition 1.3) if $\Lambda$ is level spacing (i.e., the eigenvalues of $H_{\omega,\Lambda}$ are simple and separated by at least $e^{-L^\beta}$), and eigenfunctions corresponding to eigenvalues in the interval $I$ decay exponentially as follows: if $\lambda \in \sigma(H_{\omega,\Lambda}) \cap I$, then there exists $x_\lambda \in \Lambda$ such that the corresponding eigenfunction $\varphi_\lambda$ satisfies

$$|\varphi_\lambda(y)| \leq e^{-mh_j(\lambda)}\|y - x_\lambda\| \quad \text{for all } y \in \Lambda \text{ with } \|y - x_\lambda\| \geq L^\tau,$$

where $h_j$ (defined in (1.12)) is a concave function on $I$, taking the value one at the center of the interval and the value zero at the endpoints. The modulation of the decay of the eigenfunctions by the function $h_j$ is a new feature of our method.

Our multiscale analysis shows that eigenfunction localization in an energy interval with good probability at some large enough scale implies eigenfunction localization with good (scale dependent and improving as the scale grows) probability for all sufficiently large scales, in a slightly smaller energy interval. The key step shows that localization at a large scale $\ell$ yields localization at a much larger scale $L$. The proof proceeds by covering a box $\Lambda_L$ of side $L$ by boxes of side $\ell$, which are mostly $(m, I)$-localizing, and showing this implies that $\Lambda_L$ is $(m', I')$-localizing. There are always some losses, $m' < m$ and $I' \subset I$, but these losses are controllable, and continuing this procedure we converge to some rate of decay $m_\infty > 0$ and interval $I_\infty \neq \emptyset$.

The eigensystem multiscale analysis in an energy interval $I$ requires a new ingredient, absent in the treatment of the system at high disorder given in [E-K], where $I = \mathbb{R}$ and $h_j = 1$. In broad terms, the reason is that our energy interval multiscale scheme only carries information about eigenfunctions with eigenvalues in the interval $I$, and contains no information whatsoever concerning eigenfunctions with eigenvalues that lie outside the interval $I$. Given boxes $\Lambda_\ell \subseteq \Lambda_L$, with $\ell \ll L$, a crucial step in our analysis shows that if $(\psi, \lambda)$ is an eigenpair for $H_{\omega,\Lambda_L}$, with $\lambda \in I$ not too close to the eigenvalues of $H_{\omega,\Lambda_\ell}$ corresponding to eigenfunctions localized deep inside $\Lambda_\ell$, and the box $\Lambda_\ell$ is $(m, I)$-localizing for $H_\omega$, then $\psi$ is exponentially small deep inside $\Lambda_\ell$ (see Lemma 3.3(ii)). This is proven by expanding the values of $\psi$ in $\Lambda_\ell$ in terms of the $(m, I)$-localizing eigensystem $\{(\varphi_\nu, \nu)\}_{\nu \in \sigma(H_{\omega,\Lambda_\ell})}$ for $H_{\omega,\Lambda_\ell}$. The difficulty is that we only have decay for the eigenfunctions $\varphi_\nu$ with $\nu \in I$; we know nothing about $\varphi_\nu$ if $\nu \notin I$. We overcame this difficulty by showing that the decay of the term containing the latter eigenfunctions comes from the distance from
the eigenvalue $\lambda$ to the complement of the interval $I$, using Lemmas 3.2 and 3.3. As a result, it is natural to expect that the decay rate for the localization of eigenfunctions goes to zero as the eigenvalues approach the edges of the interval $I$. The introduction of the modulating function $h_I$ in the decay models this phenomenon.

The same difficulty appears if, given an $(m, I)$-localizing box $\Lambda$ for $H_\omega$, we try to recover the decay of the Green’s function at an energy $\lambda \in I$ not too close to the eigenvalues of $H_{\omega, \Lambda}$. The simplest approach is to decompose the Green’s function in terms of an $(m, I)$-localizing eigensystem $\{ (\varphi_\nu, \nu) \}_{\nu \in \sigma(H_{\omega, \Lambda})}$ for $H_{\omega, \Lambda}$:

$$\langle \delta_x, (H_{\omega, \Lambda} - \lambda)^{-1} \delta_y \rangle = \sum_{\nu \in \sigma(H_{\omega, \Lambda})} (\nu - \lambda)^{-1} \varphi_\nu(x) \varphi_\nu(y).$$

The sum over the eigenvalues inside the interval $I$ can be estimated using the decay of the corresponding eigenfunctions, but we have a problem estimating the sum over eigenvalues outside $I$ since we have no information concerning the spatial decay properties of the corresponding eigenfunctions. To overcome this difficulty, we use a more delicate argument (see Lemma 6.4) that decomposes the Green’s function into a sum of two analytic functions of $H_{\omega, \Lambda}$ with appropriate decay properties (see Lemmas 3.2 and 3.3 for details), obtaining the desired decay of the Green function:

$$\left| \langle \delta_x, (H_{\omega, \Lambda} - \lambda)^{-1} \delta_y \rangle \right| \leq e^{-m'} h_I(\lambda) \|x-y\|.$$

Readers familiar with the Green’s function multiscale analysis may notice that the modulation by the function $h_I$ is not required there. This has to do with the fact the Green’s function approach essentially considers each energy value separately, while the eigensystem approach treats the whole energy interval simultaneously. A Green’s function multiscale analysis is performed at a fixed energy; the modulation of the decay may appear in the starting condition, but not in the multiscale analysis proper. (The starting condition near an spectral edge is usually obtained from the Combes–Thomas estimate, which modulates the decay rate by the distance to the spectral edge.)

A version of our main result, Theorem 1.6, can be stated as follows. (The exponents $\zeta, \xi \in (0, 1)$ and $\gamma > 1$ are as in (1.1). $A_L(x)$ denotes the box in $\mathbb{Z}^d$ of side $L$ centered at $x \in \mathbb{R}^d$ as in (1.10).)

**Theorem** (Eigensystem multiscale analysis). Let $H_\omega$ be an Anderson model. Let $I_0 = (E - A_0, E + A_0) \subset \mathbb{R}$, with $E \in \mathbb{R}$ and $A_0 > 0$, and $0 < m_0 \leq \frac{1}{2} \log \left( 1 + \frac{A_0}{\kappa^2} \right)$. Suppose for some scale $L_0$ we have

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{ A_{L_0}(x) \text{ is (} m_0, I_0 \text{)-localizing for } H_\omega \} \geq 1 - e^{-L_0^\xi}.$$

Then, if $L_0$ is sufficiently large, there exist $m_\infty = m_\infty(L_0) > 0$ and $A_\infty = A_\infty(L_0) \in (0, A_0)$, with $\lim_{L_0 \to \infty} A_\infty(L_0) = A_0$ and $\lim_{L_0 \to \infty} m_\infty(L_0) = m_0$, such that, setting $I_\infty = (E - A_\infty, E + A_\infty)$, we have

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{ A_L(x) \text{ is (} m_\infty, I_\infty \text{)-localizing for } H_\omega \} \geq 1 - e^{-L^\xi},$$

for all $L \geq L_0^\eta$.

The theorem yields all the usual forms of Anderson localization on the interval $I_\infty$. In particular we obtain the following version of Corollary 1.8.

**Corollary** (Localization in an energy interval). Suppose the theorem holds for an Anderson model $H_\omega$. Then the following holds with probability one:
We also take model, and fix $0 < \xi < \zeta < \beta < \gamma < \sqrt{\frac{2}{\xi}}$ and note that

$$0 < \xi < \zeta < \beta < \frac{1}{\gamma} < 1 < \gamma < \sqrt{\frac{2}{\xi}} \quad \text{and} \quad \max \left\{ \gamma \beta, \frac{(\gamma-1)\beta+1}{\gamma} \right\} < \tau < 1,$$

and note that

$$0 < \xi < \xi \gamma^2 < \zeta < \beta < \frac{\tau}{\gamma} < \frac{1}{\gamma} < \gamma < \frac{1 - \beta}{\tau - \beta} < \gamma < \frac{\tau}{\beta}.$$  

We also take

$$\tilde{\zeta} = \frac{\zeta + \beta}{2} \in (\zeta, \beta) \quad \text{and} \quad \tilde{\tau} = \frac{1 + \tau}{2} \in (\tau, 1).$$

The following is a version of Theorem 2.2.

In particular, the conclusions of the Corollary hold in the interval $J_{\zeta, \infty}$. We establish localization in a fixed interval at the bottom of the spectrum, for sufficiently large disorder (Theorem 2.3).

Our main results and definitions are stated in Section 1. Theorem 1.6 is our main result, which we prove in Section 4. Theorem 1.7, derived from Theorem 1.6, encapsulates localization in an energy interval for the Anderson model and yields Corollary 1.8, which contains typical statements of Anderson localization and dynamical localization in an energy interval. Theorem 1.7 and Corollary 1.8 are proven in Section 5. In Section 2 we show how to fulfill the starting condition for Theorem 1.7 and establish localization in an interval at the bottom of the spectrum, for fixed disorder (Theorem 2.2) and in a fixed interval for sufficiently large disorder (Theorem 2.3). Section 6 contains notations, definitions and lemmas required for the proof of the eigensystem multiscale analysis given in Section 4. The connection with the Green’s functions multiscale analysis is established in Section 6.

1. MAIN RESULTS

In this article we will use many positive exponents, which will be required to satisfy certain relations. We consider $\xi, \zeta, \beta, \tau \in (0, 1)$ and $\gamma > 1$ such that

$$0 < \xi < \zeta < \beta < \frac{1}{\gamma} < 1 < \gamma < \sqrt{\frac{2}{\xi}} \quad \text{and} \quad \max \left\{ \gamma \beta, \frac{(\gamma-1)\beta+1}{\gamma} \right\} < \tau < 1,$$

and note that

$$0 < \xi < \xi \gamma^2 < \zeta < \beta < \frac{\tau}{\gamma} < \frac{1}{\gamma} < \gamma < \frac{1 - \beta}{\tau - \beta} < \gamma < \frac{\tau}{\beta}.$$  

We also take

$$\tilde{\zeta} = \frac{\zeta + \beta}{2} \in (\zeta, \beta) \quad \text{and} \quad \tilde{\tau} = \frac{1 + \tau}{2} \in (\tau, 1),$$
so

\[(\gamma - 1)\zeta + 1 < (\gamma - 1)\beta + 1 < \gamma \tau.\]  

(1.4)

We also consider \(\kappa \in (0, 1)\) and \(\kappa' \in [0, 1)\) such that

\[\kappa + \kappa' < \tau - \gamma \beta.\]  

(1.5)

We set

\[\varrho = \min \left\{ \kappa, \frac{1}{\gamma \beta}, \gamma \tau - (\gamma - 1)\zeta - 1 \right\}, \quad \text{note} \quad 0 < \kappa \leq \varrho < 1,\]  

and choose

\[\zeta \in (0, 1 - \varrho], \quad \text{so} \quad \varrho < 1 - \zeta.\]  

(1.7)

We consider these exponents fixed and do not make explicit the dependence of constants on them. We write \(\chi_\Lambda\) for the characteristic function of the set \(\Lambda\). By a constant we always mean a finite constant. We will use \(C_{a, b, \ldots}, C'_{a, b, \ldots}, C(a, b, \ldots)\), etc., to denote a constant depending on the parameters \(a, b, \ldots\). Note that \(C_{a, b, \ldots}\) may denote different constants in different equations, and even in the same equation.

Given a scale \(L \geq 1\), we sets

\[L = \ell^\dagger \text{ (i.e., } \ell = L^\dagger\text{)}, \quad L\tau = |L\tau|, \quad \text{and} \quad L\gamma = |L\gamma|.\]

If \(x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d\), we set \(|x| = |x|_2 = \left(\sum_{j=1}^d x_j^2\right)^{\frac{1}{2}}\), and \(|x| = |x|_{\infty} = \max_{j=1,2,\ldots,d}|x_j|\). If \(x \in \mathbb{R}^d\) and \(\Xi \subset \mathbb{R}^d\), we set \(\text{dist}(x, \Xi) = \inf_{y \in \Xi} |y - x|\).

The diameter of a set \(\Xi \subset \mathbb{R}^d\) is given by \(\text{diam} \Xi = \sup_{x,y \in \Xi} |y - x|\).

\(H\) we will always denote a discrete Schrödinger operator, that is, an operator \(H = -\Delta + V\) on \(\ell^2(\mathbb{Z}^d)\), where where \(\Delta\) is the (centered) discrete Laplacian:

\[\Delta \varphi(x) := \sum_{y \in \mathbb{Z}^d, |y - x| = 1} \varphi(y) \quad \text{for} \quad \varphi \in \ell^2(\mathbb{Z}^d),\]  

(1.8)

and \(V\) is a bounded potential. Given \(\Phi \subset \Theta \subset \mathbb{Z}^d\), we consider \(\ell^2(\Phi) \subset \ell^2(\Theta)\) by extending functions on \(\Phi\) to functions on \(\Theta\) that are identically zero on \(\Theta \setminus \Phi\). If \(\Theta \subset \mathbb{Z}^d\) and \(\varphi \in \ell^2(\Theta)\), we let \(|\varphi| = ||\varphi||_2\) and \(||\varphi||_{\infty} = \max_{y \in \Theta} |\varphi(y)|\).

Given \(\Theta \subset \mathbb{Z}^d\), we let \(H_{\Theta}\) be the restriction of \(\chi_\Theta H \chi_\Theta\) to \(\ell^2(\Theta)\). We call \((\varphi, \lambda)\) an eigenpair for \(H_{\Theta}\) if \(\varphi \in \ell^2(\Theta)\) with \(||\varphi|| = 1\), \(\lambda \in \mathbb{R}\), and \(H_{\Theta}\varphi = \lambda \varphi\). (In other words, \(\lambda\) is an eigenvalue for \(H_{\Theta}\) and \(\varphi\) is a corresponding normalized eigenfunction.) A collection \(\{(\varphi_j, \lambda_j)\}_{j \in \mathcal{J}}\) of eigenpairs for \(H_{\Theta}\) will be called an eigenbasis. If all eigenvalues of \(H_{\Theta}\) are simple, we can rewrite the eigenbasis as \(\{(\varphi_{\lambda}, \lambda)\}_{\lambda \in \sigma(H_{\Theta})}\).

Given \(\Theta \subset \mathbb{Z}^d\), a function \(\psi: \Theta \rightarrow \mathbb{C}\) is called a generalized eigenfunction for \(H_{\Theta}\) with generalized eigenvalue \(\lambda \in \mathbb{R}\) if \(\psi\) is not identically zero and

\[\langle (H_{\Theta} - \lambda) \varphi, \psi \rangle = 0 \quad \text{for all} \quad \varphi \in \ell^2(\Theta) \quad \text{with finite support}.\]  

(1.9)

In this case we call \((\varphi, \lambda)\) a generalized eigenpair for \(H_{\Theta}\). (Eigenfunctions are generalized eigenfunctions, but we do not require generalized eigenfunctions to be in \(\ell^2(\Theta)\).)

For convenience we consider boxes in \(\mathbb{Z}^d\) centered at points of \(\mathbb{R}^d\). The box in \(\mathbb{Z}^d\) of side \(L > 0\) centered at \(x \in \mathbb{R}^d\) is given by

\[\Lambda_L(x) = \Lambda^B_L(x) \cap \mathbb{Z}^d, \quad \text{where} \quad \Lambda^B_L(x) = \{y \in \mathbb{R}^d; ||y - x|| \leq \frac{L}{2}\}.\]  

(1.10)
By a box $\Lambda_L$ we will mean a box $\Lambda_L(x)$ for some $x \in \mathbb{R}^d$. It is easy to see that for all $L \geq 2$ and $x \in \mathbb{R}^d$ we have $(L-2)^d < |\Lambda_L(x)| \leq (L+1)^d$.

**Definition 1.1.** Given $R > 0$, a finite set $\Theta \subset \mathbb{Z}^d$ will be called $R$-level spacing for $H$ if all eigenvalues of $H_{\Theta}$ are simple and $|\lambda - \lambda'| \geq e^{-R^d}$ for all $\lambda, \lambda' \in \sigma(H_{\Theta})$, $\lambda \neq \lambda'$.

If $\Theta$ is a box $\Lambda_L$ and $R = L$, we will simply say that $\Lambda_L$ is level spacing for $H$.

**Definition 1.2.** Let $\Lambda_L$ be a box, $x \in \Lambda_L$, and $m \geq 0$. Then $\varphi \in \ell^2(\Lambda_L)$ is said to be $(x, m)$-localized if $\|\varphi\| = 1$ and

$$|\varphi(y)| \leq e^{-m\|y-x\|} \quad \text{for all } y \in \Lambda_L \quad \text{with } \|y-x\| \geq L_r. \quad (1.11)$$

Note that $m = 0$ is allowed in Definition 1.2.

**Definition 1.3.** Let $J = (E - B, E + B) \subset I = (E - A, E + A)$, where $E \in \mathbb{R}$ and $0 < B \leq A$, be bounded open intervals with the same center, and let $m > 0$. A box $\Lambda_L$ will be called $(m, J)$-localizing for $H$ if the following holds:

(i) $\Lambda_L$ is level spacing for $H$.

(ii) There exists an $(m, J)$-localized eigensystem for $H_{\Lambda_L}$, that is, an eigensystem $\{(\varphi_\nu, \nu)\}_{\nu \in \sigma(H_{\Lambda_L})}$ for $H_{\Lambda_L}$ such that for all $\nu \in \sigma(H_{\Lambda_L})$ there is $x_\nu \in \Lambda_L$ such that $\varphi_\nu$ is $(x_\nu, m \chi_J(\nu) h_1(\nu))$-localized, where the modulating function $h_1$ is defined by

$$h_1(t) = h \left( \frac{t-1}{A} \right) \quad \text{for } t \in \mathbb{R}, \quad \text{where } h(s) = \begin{cases} 1 - s^2 & \text{if } s \in [0,1) \smallskip \vspace{1pt} \\ 0 & \text{otherwise} \end{cases}. \quad (1.12)$$

We will say that $\Lambda_L$ is $(m, J)$-localizing for $H$ if $\Lambda_L$ is $(m, J, I)$-localizing for $H$.

Note that $h_1(t) > 0 \iff t \in I$, in particular $h_I = \chi_I h_1$. Since $\chi_I h_1 \geq h_J$, if $\Lambda_L$ is $(m, J, I)$-localizing for $H$ it is also $(m, J)$-localizing for $H$.

**Remark 1.4.** In [EK] we had $I = \mathbb{R}$ and $h_\mathbb{R} = 1$, and called a box $\Lambda_L$ $m$-localizing if it was level spacing for $H$ and for all $\nu \in \sigma(H_{\Lambda_L})$ there is $x_\nu \in \Lambda_L$ such that $\varphi_\nu$ is $(x_\nu, m)$-localized.

Given an interval $I = (E - A, E + A)$ and scales $\ell, L > 1$, we use the notation

$$I_\ell = (E - A(1 - \ell^{-\kappa}), E + A(1 - \ell^{-\kappa})), \quad (1.13)$$

$$I^L = (E - A(1 - \ell^{-\kappa})^{-1}, E + A(1 - \ell^{-\kappa})^{-1}).$$

We write $I_{\ell}^L = (I_\ell)^L = (I^L)_{\ell}$, note that $I_{\ell}^L = I$, and observe that

$$\chi_I h_I \geq \ell^{-\kappa} \chi_{I_\ell}, \quad \text{i.e., } h_I(t) \geq 1 - (1 - \ell^{-\kappa})^2 \geq \ell^{-\kappa} \quad \text{for all } t \in I_\ell. \quad (1.14)$$

**Definition 1.5.** The Anderson model is the random discrete Schrödinger operator

$$H_\omega := -\Delta + V_\omega \quad \text{on } \ell^2(\mathbb{Z}^d), \quad (1.15)$$

where $V_\omega$ is a random potential: $V_\omega(x) = \omega_x$ for $x \in \mathbb{Z}^d$, where $\omega = \{\omega_x\}_{x \in \mathbb{Z}^d}$ is a family of independent identically distributed random variables, whose common probability distribution $\mu$ is non-degenerate with bounded support. We assume $\mu$ is Hölder continuous of order $\alpha \in (\frac{1}{2}, 1]$:

$$S_\mu(t) \leq K t^\alpha \quad \text{for all } t \in [0,1], \quad (1.16)$$

where $K$ is a constant and $S_\mu(t) := \sup_{a \in \mathbb{R}} \mu \{[a, a + t]\}$ is the concentration function of the measure $\mu$. 

It follows from ergodicity (e.g., [K Theorem 3.9]) that
\[ \sigma(H_\omega) = \Sigma := \sigma(-\Delta) + \text{supp } \mu = [-2d, 2d] + \text{supp } \mu \] with probability one. (1.17)

The eigensystem multiscale analysis in an energy interval yields the following theorem.

**Theorem 1.6.** Let \( H_\omega \) be an Anderson model. Given \( m_- > 0 \), there exists a a finite scale \( L = L(d, m_-) \) and a constant \( C_{d,m_-} > 0 \) with the following property: Suppose for some scale \( L_0 \geq L \) we have
\[ \inf_{x \in \mathbb{R}^d} \mathbb{P} \{ \Lambda_{L_0}(x) \text{ is } (m_0, I_0, L^{\frac{d}{4}})\text{-localizing for } H_\omega \} \geq 1 - e^{-L_0^d}, \] (1.18)
where \( I_0 = (E - A_0, E + A_0) \subset \mathbb{R} \), with \( E \in \mathbb{R} \) and \( A_0 > 0 \), and
\[ m_- L_0^{\kappa'} \leq m_0 \leq \frac{1}{2} \log \left( 1 + \frac{1}{4d} \right). \] (1.19)
Then for all \( L \geq L_0^\gamma \) we have
\[ \inf_{x \in \mathbb{R}^d} \mathbb{P} \left\{ \Lambda_L(x) \text{ is } (m_\infty, I_\infty, L^{\frac{d}{4}})\text{-localizing for } H_\omega \right\} \geq 1 - e^{-L^d}, \] (1.20)
where, with \( \varrho \) as in (1.6),
\[ A_\infty = A_\infty(L_0) = A_0 \prod_{k=0}^{\infty} \left( 1 - L_0^{-\kappa k} \right), \quad I_\infty = (E - A_\infty, E + A_\infty), \] (1.21)
\[ m_\infty = m_\infty(L_0) = m_0 \prod_{k=0}^{\infty} \left( 1 - C_{d,m_-} L_0^{-\varrho_k} \right) < \frac{1}{2} \log \left( 1 + \frac{A_\infty}{4d} \right). \]
In particular, \( \lim_{L_0 \to \infty} A_\infty(L_0) = A_0 \) and \( \lim_{L_0 \to \infty} m_\infty(L_0) = m_0. \)

Theorem 1.6 yields all the usual forms of localization on the interval \( I_\infty \). To state these results, we fix \( \nu > \frac{d}{2} \), and for \( a \in \mathbb{Z}^d \) we let \( T_a \) be the operator on \( L^2(\mathbb{Z}^d) \) given by multiplication by the function \( T_a(x) := (x - a)^\nu \), where \( \langle x \rangle = \sqrt{1 + \|x\|^2} \).

Since \( (a + b) \leq \sqrt{2} \langle a \rangle \langle b \rangle \), we have \( \| T_a T_b^{-1} \| \leq 2^{\frac{d}{2}} \langle a \rangle \langle b \rangle \). A function \( \psi : \mathbb{Z}^d \to \mathbb{C} \) will be called a \( \nu \)-generalized eigenfunction for the discrete Schrödinger operator \( H \) if \( \psi \) is a generalized eigenfunction and \( \| T_a^{-1} \psi \| < \infty \). (\( \| T_a^{-1} \psi \| < \infty \) if and only if \( \| T_a \psi \| < \infty \) for all \( a \in \mathbb{Z}^d \).) We let \( \mathcal{V}(\lambda) \) denote the collection of \( \nu \)-generalized eigenfunctions for \( H \) with generalized eigenvalue \( \lambda \in \mathbb{R} \). Given \( \lambda \in \mathbb{R} \) and \( a, b \in \mathbb{Z}^d \), we set
\[ W_\lambda^{(a)}(b) := \begin{cases} \sup_{\psi \in \mathcal{V}(\lambda)} \frac{\| \psi(b) \|}{\| T_a^{-1} \psi \|} & \text{if } \mathcal{V}(\lambda) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases} \] (1.22)

It is easy to see that for all \( a, b, c \in \mathbb{Z}^d \) we have
\[ W_\lambda^{(a)}(a) \leq 1, \quad W_\lambda^{(a)}(b) \leq \langle b - a \rangle^\nu, \quad \text{and} \quad W_\lambda^{(a)}(c) \leq 2^{\frac{d}{2}} \langle b - a \rangle^\nu W_\lambda^{(b)}(c). \] (1.23)

**Theorem 1.7.** Suppose the conclusions of Theorem 1.6 hold for an Anderson model \( H_\omega \), and let \( I = I_\infty, m = m_\infty \). There exists a finite scale \( L = L_{d,\nu,m_-} \) such that, given \( L \leq L \in 2\mathbb{N} \) and \( a \in \mathbb{Z}^d \), there exists an event \( \mathcal{Y}_{L,a} \) with the following properties:

(i) \( \mathcal{Y}_{L,a} \) depends only on the random variables \( \{ \omega_x \}_{x \in \Lambda_{L,a}(a)} \) and
\[ \mathbb{P} \{ \mathcal{Y}_{L,a} \} \geq 1 - Ce^{-L^d}. \] (1.24)
localized eigenfunctions; see [DJLS1, DJLS2]).

decay of eigenfunction correlations) in [GK3]), and (v) is SULE (semi-uniformly
are statements of dynamical localization ((iv) is called SUDEC (summarizable uniform
model as shown in [GK3, GK4, EK]. In particular, we get the following corollary.

Theorem 1.7 implies Anderson localization and dynamical localization, and more,
and all $x, y \in \mathbb{Z}^d$ we have

\[
W_{\omega, \lambda}^{(x)}(x)W_{\omega, \lambda}^{(y)}(y) \leq C_{m, \omega, \nu} (h_1(\lambda))^{-\nu} e^{(\frac{1}{132} + \nu) mh_1(\lambda)(2d \log(x))} \frac{1}{x} e^{-\frac{7}{132} mh_1(\lambda)\|y-x\|}.
\]  

(1.29)

(iii) If $\lambda \in I$, then for all $x, y \in \mathbb{Z}^d$ we have

\[
|\psi(x)| \leq C_{m, \omega, \nu} (h_1(\lambda))^{-\nu} \left| T^{-1}_x \right| e^{(\frac{1}{132} + \nu) mh_1(\lambda)(2d \log(x))} \frac{1}{x} e^{-\frac{7}{132} mh_1(\lambda)\|y-x\|}.
\]  

(1.30)

(iv) If $\lambda \in I$, then for $\psi \in \chi_{\lambda}(H_\omega)$ and all $x, y \in \mathbb{Z}^d$ we have

\[
|\psi(x)| \leq C_{m, \omega, \nu} (h_1(\lambda))^{-\nu} \left| T^{-1}_x \right| e^{(\frac{1}{132} + \nu) mh_1(\lambda)(2d \log(x))} \frac{1}{x} e^{-\frac{7}{132} mh_1(\lambda)\|y-x\|}.
\]  

(1.31)

In Corollary 1.8 (i) and (ii) are statements of Anderson localization, (iii) and (iv) are statements of dynamical localization ((iv) is called SUDEC (summarizable uniform decay of eigenfunction correlations) in [GK3]), and (v) is SULE (semi-uniformly localized eigenfunctions; see [DJLS1, DJLS2]).

We can also derive statements of localization in expectation, as in [GK3, GK4].
2. Localization at the bottom of the spectrum

We now discuss how to obtain the initial step for the multiscale analysis at the bottom of the spectrum and prove localization. Let $H_\omega$ be an Anderson model, and set $E_0 = \inf \Sigma$ (see (1.1)), the bottom of the almost sure spectrum of $H_\omega$. We will consider intervals at the bottom of the spectrum, more precisely, intervals of the form $J = [E_0, E_0 + A]$ with $A > 0$. We set $\tilde{J} = (E_0 - A, E_0 + A)$, so $J \cap \Sigma = \tilde{J} \cap \Sigma$, call a box $(m, J)$-localizing if it is $(m, \tilde{J})$-localizing as in Definition 1.3 etc. We also set $J_L$ and $J^L$ so $\tilde{J}_L = \tilde{J}_L$ and $\tilde{J}^L = \tilde{J}^L$.

2.1. Fixed disorder.

**Proposition 2.1.** Let $H_\omega$ be an Anderson model, and set $E_0 = \inf \Sigma$. There exists a constant $C_{d,\mu} > 0$ such that, given $\zeta \in (0, 1)$, for sufficiently large $L$ we have

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\left\{ H_{\Lambda_L(x)} > E_0 + C_{d,\mu} L^{-\frac{2\zeta}{\tau}} \right\} \geq 1 - e^{-L^\zeta}. \quad (2.1)$$

In particular, for all intervals $J_\zeta(L) = [E_0, E_0 + C_{d,\mu} L^{-\frac{2\zeta}{\tau}}]$ and all $m > 0$ we have

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\left\{ \lambda_L(x) \text{ is } (m, J_\zeta(L))\text{-localizing for } H_\omega \right\} \geq 1 - e^{-L^\zeta}. \quad (2.2)$$

The estimate (2.1) follows from a Lifshitz tails estimate. It can be derived from [K] Proof of Theorem 11.4]. Although the boxes in [K] are all centered at points in $\mathbb{Z}^d$, the arguments, including the crucial [K] Lemma 6.4, can be extended to boxes centered at points in $\mathbb{R}^d$. Note that (2.2) follows trivially from (2.1). Since the probability distribution $\mu$ is a continuous measure (see (1.16)), it follows from (1.17) that $J_\zeta(L) \subset \Sigma$ for all sufficiently large $L$.

We will now combine Proposition 2.1 with Theorem 1.6, taking $I_0 = \tilde{J}_L(L_0)$, i.e., $E = E_0$ and $A_0 = C_{d,\mu} L_0^{-\frac{2\zeta}{\tau}}$ in Theorem 1.6. To satisfy (1.19) for $L$ large, we take $m_0 = \frac{1}{18} C_{d,\mu} L^{-\frac{2\zeta}{\tau}}$, $m_- = \frac{1}{18} C_{d,\mu}$ and $\kappa = \frac{2\zeta}{\tau}$. To satisfy (1.20) we require $\frac{2\zeta}{\tau} < \tau - \gamma \beta$, and then choose $0 < \kappa < \tau - \gamma \beta - \kappa'$. Since for a fixed $\xi$ we can take $\tau$ and $\gamma$ close to 1 and $\beta$ close to $\zeta$, respecting (1.1), we find we can choose the parameters in (1.1) as long as

$$\frac{2\zeta}{\tau} < 1 - \zeta \iff \zeta < \frac{d}{d+2}. \quad (2.3)$$

We obtain the following theorem.

**Theorem 2.2.** Let $H_\omega$ be an Anderson model, and fix $0 < \xi < \zeta < \frac{d}{d+2}$. Then there exists $\gamma > 1$ such that, if $L_0$ is sufficiently large, for all $L \geq L_0$ we have

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\left\{ \lambda_L(x) \text{ is } (m_{\xi,\infty}, J_{\xi,\infty}, J_{\xi,\infty}^L)\text{-localizing for } H_\omega \right\} \geq 1 - e^{-L^\xi}. \quad (2.4)$$

where

$$A_{\xi,\infty} = A_{\xi,\infty}(L_0) = C_{d,\mu} L_0^{-\frac{2\zeta}{\tau}} \prod_{k=0}^{\infty} \left( 1 - L_0^{-\gamma \xi} \right) \geq \frac{1}{2} C_{d,\mu} L_0^{-\frac{2\zeta}{\tau}}, \quad (2.5)$$

$$J_{\xi,\infty} = [E_0, E_0 + A_{\xi,\infty}] \supset [E_0, E_0 + \frac{1}{2} C_{d,\mu} L_0^{-\frac{2\zeta}{\tau}}],$$

$$m_{\xi,\infty} = m_{\xi,\infty}(L_0) = \frac{1}{18} C_{d,\mu} L^{-\frac{2\zeta}{\tau}} \prod_{k=0}^{\infty} \left( 1 - C_{d,\mu} L^{-\gamma \xi} \right) \geq \frac{1}{18} C_{d,\mu} L^{-\frac{2\zeta}{\tau}}.$$
In particular, the conclusions of Theorem 1.7 and Corollary 1.8 hold in the interval $J_{\zeta, \infty}$.

2.2. Fixed interval. We may also use disorder to start the eigensystem multiscale analysis in a fixed interval at the bottom of the spectrum. To do so we introduce a disorder parameter $g > 0$, and set $H_{g, \omega} = -\Delta + gV_\omega$. We assume $\{0\} \in \text{supp} \, \mu \subset [0, \infty)$, so it follows from (1.17) that $E_0 = -2d$. Then, given $B > 0$ and $\zeta \in (0, 1)$,

$$\inf_{x \in \mathbb{R}^d} P \{ H_{g, \Lambda_L(x)} \geq -2d + B \} \geq \inf_{x \in \mathbb{R}^d} P \{ g \omega_x \geq B \text{ for all } x \in \Lambda_L(x) \} \geq (1 - \mu([0, g^{-1}B]))^{(L+1)^d} \geq (1 - K(g^{-1}B)^\alpha)^{(L+1)^d} \geq 1 - (L + 1)^d K(g^{-1}B)^\alpha \geq 1 - L^{-\zeta} \text{ for } g \geq g_\zeta(L).$$

It follows that, given $\zeta \in (0, 1)$, for $g \geq g_\zeta(L)$ and all $m > 0$ we have

$$\inf_{x \in \mathbb{R}^d} P \{ \Lambda_L(x) \text{ is } (m, [-2d, -2d + B])\text{-localizing for } H_{g, \omega} \} \geq 1 - e^{-L^\zeta}. \quad (2.7)$$

Combining with Theorem 1.6 we obtain the following theorem.

**Theorem 2.3.** Let $H_{g, \omega}$ be an Anderson model with disorder as above, and choose exponents as in (1.3)–(1.7). Then, given $B > 0$, let $J(B) = [-2d, -2d + B]$ and pick $0 < m \leq \frac{1}{4} \log(1 + \frac{L^d}{B})$. Then, if $L_0$ is sufficiently large, for all $L \geq L_0$ and $g \geq g_\zeta(L_0)$ we have

$$\inf_{x \in \mathbb{R}^d} \mathbb{P} \left\{ \Lambda_L(x) \text{ is } (m, J_\infty(B), J_{\infty}(B))^{L^d}-\text{localizing for } H_{g, \omega} \right\} \geq 1 - e^{-L^\zeta}, \quad (2.8)$$

where

$$A_\infty = A_\infty(L_0) = B \prod_{k=0}^{\infty} \left(1 - L_0^{-\kappa_k}\right), \quad J_\infty = J_\infty(L_0) = [-2d, -2d + A_\infty), \quad m_\infty = m_\infty(L_0) = m \prod_{k=0}^{\infty} \left(1 - C_{d,m,L_0}^{-\nu_k}\right). \quad (2.9)$$

In particular, the conclusions of Theorem 1.7 and Corollary 1.8 hold in the interval $J_{\infty}$. Moreover, $\lim_{L_0 \to \infty} A_\infty(L_0) = B$ and $\lim_{L_0 \to \infty} m_\infty(L_0) = m$.

3. Preamble to the eigensystem multiscale analysis

In the sections we introduce notation and prove lemmas that play an important role in the eigensystem multiscale analysis. $H$ will always denote a discrete Schrödinger operator $H = -\Delta + V$ on $\ell^2(\mathbb{Z}^d)$.

3.1. Subsets, boundaries, etc. Let $\Phi \subset \Theta \subset \mathbb{Z}^d$. We set the boundary, exterior boundary, and interior boundary of $\Phi$ relative to $\Theta$, respectively, by

$$\partial^\Theta \Phi = \{ (u, v) \in \Phi \times (\Theta \setminus \Phi) : |u - v| = 1 \}, \quad (3.1)$$

$$\partial^\Theta_{ex} \Phi = \{ v \in (\Theta \setminus \Phi) : (u, v) \in \partial^\Theta \Phi \text{ for some } u \in \Phi \},$$

$$\partial^\Theta_{in} \Phi = \{ u \in \Phi : (u, v) \in \partial^\Theta \Phi \text{ for some } v \in (\Theta \setminus \Phi) \}.$$ 

We let

$$R_y^{\partial^\Theta \Phi} = \text{dist} (y, \partial^\Theta_{in} \Phi) \text{ for } y \in \Phi. \quad (3.2)$$
Given $t \geq 1$, we set
\[ \Phi_{\Theta,t} = \{ y \in \Phi; \dist (y, \Theta \setminus \Phi) > |t| \}, \quad \partial_{\Theta,t} \Phi = \Phi \setminus \Phi_{\Theta,t}, \quad (3.3) \]
\[ \partial_{\Theta,t} \Phi = \partial_{\infty,t} \Phi \cup \partial_{\infty} \Phi. \]

If $\Theta = Z^d$ we omit it from the notation, i.e., $\Phi^t = \Phi^{\Theta,t}$. If $\Phi = \Lambda_L(x)$, we write $\Lambda^{\Theta,t}(x) = (\Lambda_L(x))^{\Theta,t}$.

Consider a box $\Lambda_L \subset \Theta \subset Z^d$. Given $v \in \Theta$, we let $\hat{v} \in \partial_{\infty} \Lambda_L$ be the unique $u \in \partial_{\infty} \Lambda_L$ such that $(u,v) \in \partial_{\Theta} \Lambda_L$ if $v \in \partial_{\infty} \Lambda_L$, and set $\hat{v} = 0$ otherwise. For $L \geq 2$ we have
\[ |\partial_{\infty} \Lambda_L| \leq |\partial_{\Theta} \Lambda_L| = |\partial_{\Theta} \Lambda_L| \leq s_d L^{d-1}, \text{ where } s_d = 2^d d. \quad (3.4) \]

If $\Phi \subset \Theta \subset Z^d$, $H_\Theta = H_\Phi \oplus H_{\Theta \setminus \Phi} + \Gamma_{\Theta \Phi}$ on $\ell^2(\Theta) = \ell^2(\Phi) \oplus \ell^2(\Theta \setminus \Phi)$, where
\[ \Gamma_{\Theta \Phi}(u,v) = \begin{cases} -1 & \text{if either } (u,v) \text{ or } (v,u) \in \partial_{\Theta} \Phi, \\ 0 & \text{otherwise} \end{cases}. \quad (3.5) \]

### 3.2. Lemmas for energy intervals.

#### Lemma 3.1.
Given $t > 0$ and $\lambda \in \mathbb{R}$, let $F_{t,\lambda}(z)$ be the entire function given by
\[ F_{t,\lambda}(z) = \frac{1 - e^{-((z^2 - \lambda^2)}}{z - \lambda} \quad \text{for } z \in \mathbb{C} \setminus \{\lambda\} \quad \text{and} \quad F_{t,\lambda}(\lambda) = 2t\lambda. \quad (3.6) \]
Then, given $\Phi \subset Z^d$, for all $x,y \in \Phi$ we have
\[ |\langle \delta_x, F_{t,\lambda}(H_{\Phi}) \delta_y \rangle| \leq \inf_{\eta > 0} \frac{70}{\sqrt{\eta^2 + \lambda^2}} e^{r(\eta^2 + \lambda^2)} e^{-\log(1 + 1/\eta)} |x-y|. \quad (3.7) \]
In particular, if $\lambda \in I = (E - A, E + A)$, where $A > 0$ and $E \in \mathbb{R}$, and
\[ 0 < m \leq \frac{1}{2} \log (1 + \frac{A}{4d}), \quad (3.8) \]

it follows that for all $x,y \in \Phi$, $x \neq y$, we have
\[ |\langle \delta_x, F_{m|x-y|/2,\lambda-E}(H_{\Phi} - E) \delta_y \rangle| \leq 70 A^{-1} e^{-m h_3(\lambda) |x-y|}. \quad (3.9) \]

#### Proof.
Given $t > 0$ and $\lambda \in \mathbb{R}$, the function $F_{t,\lambda}(z)$ defined in (3.6) is clearly an entire function. Moreover, given $\eta > 0$, if $|\text{Im } z| \leq \eta$ and $c > 0$ we have,
\[ |F_{t,\lambda}(z)| \leq \frac{e^{t(\eta^2 + \lambda^2) + 1/2}}{e^{\sqrt{\eta^2 + \lambda^2}} + \lambda^2} \leq \frac{(e^{t(\eta^2 + \lambda^2) + 1/2})}{(c+2)e^{t(\eta^2 + \lambda^2)}} \leq \frac{(c+2)e^{t(\eta^2 + \lambda^2)}}{e^{t(\eta^2 + \lambda^2)}} \leq \frac{(c+2)e^{t(\eta^2 + \lambda^2)}}{\sqrt{\eta^2 + \lambda^2}} \quad \text{if } |z - \lambda| < c \sqrt{\eta^2 + \lambda^2}, \quad (3.10) \]
so we conclude that, taking $c = \sqrt{3} - 1$,
\[ F_{t,\lambda,\eta} = \sup_{|\text{Im } z| \leq \eta} |F_{t,\lambda}(z)| \leq (\sqrt{3} + 1) e^{t(\eta^2 + \lambda^2)}/\sqrt{\eta^2 + \lambda^2}. \quad (3.11) \]

Given $\Phi \subset Z^d$, it follows from [AG, Theorem 3] (note that it applies also for $H_{\Phi}$ on $\ell^2(\Phi)$), that for all $x,y \in \Phi$ we have
\[ |\langle \delta_x, F_{t,\lambda}(H_{\Phi}) \delta_y \rangle| \leq 18 \sqrt{2} F_{t,\lambda,\eta} e^{-\log(1 + 1/\eta)} |x-y| \quad (3.12) \]
\[ \leq \frac{70}{\sqrt{\eta^2 + \lambda^2}} e^{t(\eta^2 + \lambda^2)} e^{-\log(1 + 1/\eta)} |x-y| \quad \text{for all } \eta > 0. \]
To prove (3.9), we take $E = 0$ by replacing the potential $V$ by $V - E$, and note that (3.7) holds for any discrete Schrödinger operator $H$. Now let $\lambda \in I = (-A, A)$, where $A > 0$, and $m$ as in (3.8), and fix $x, y \in \Phi, x \neq y$. Since

$$\log \left(1 + \frac{d}{dx^2}\right) = \frac{d}{dx^2} \left(\frac{\eta^2 + \lambda^2}{\lambda}\right) = \log \left(1 + \frac{d}{dx^2}\right) - m \left(\frac{\lambda^2}{\lambda^2} + 1\right) = mh_1(\lambda),$$

choosing $\eta = A$, and using (3.8), we obtain

$$\log \left(1 + \frac{d}{dx^2}\right) - \frac{d}{dx^2} \left(A^2 + \lambda^2\right) = \log \left(1 + \frac{d}{dx^2}\right) - 2m + mh_1(\lambda) \geq mh_1(\lambda),$$

so (3.9) follows from (3.7) by taking $t = \frac{m|x-y|}{A^2}$ and $\eta = A$. \hfill $\square$

**Lemma 3.2.** Let $\Theta \subset \mathbb{Z}^d$, and let $\psi : \Theta \to \mathbb{C}$ be a generalized eigenfunction for $H_\Theta$ with generalized eigenvalue $\lambda \in \mathbb{R}$. Let $\Phi \subset \Theta$ be a finite set such that $\lambda \notin \sigma(H_\Phi)$. Let $A > 0$, $E \in \mathbb{R}$, $I = (E - A, E + A)$. The following holds for all $y \in \Phi$:

(i) For all $t > 0$ we have

$$\psi(y) = \left\langle e^{-t((H_\Theta - E)^2 - (\lambda - E)^2)} \delta_y, \psi \right\rangle - \left\langle F_{t, \lambda}(H_\Theta - E) \delta_y, \Gamma_{\Theta^\lambda} \psi \right\rangle,$$

where $\Gamma_{\Theta^\lambda}$ is defined in (3.8) and $F_{t, \lambda}(z)$ is the function defined in (3.9).

(ii) Let $0 < R \leq R_0^{\Theta^\lambda}$ and $m$ as in (3.8). For $\lambda \in I$ it follows that

$$\left| F_{\frac{m^2}{A^2}, \lambda}(H_\Theta - E) \delta_y, \Gamma_{\Theta^\lambda} \psi \right| \leq 70 \left| \Phi \right| A^{-1} e^{-mh_1(\lambda)R} |\psi(y)|,$$

for some $v \in \partial_{ex}^\Theta \Phi$.

**Proof.** We take $E = 0$ by replacing the potential $V$ by $V - E$. By hypothesis we have $\lambda \notin \sigma(H_\Phi)$ and

$$\langle (H_\Theta - \lambda) \varphi, \psi \rangle = 0 \quad \text{for all} \quad \varphi \in L^2(\Phi),$$

so

$$\langle (H_\Phi - H_\Theta)(H_\Phi - \lambda)^{-1} \varphi, \psi \rangle = \langle \varphi, \psi \rangle \quad \text{for all} \quad \varphi \in L^2(\Phi).$$

It follows that for all $y \in \Phi$ and $t > 0$ we have

$$\psi(y) = \langle \delta_y, \psi \rangle$$

$$= \left\langle e^{-t((H_\Phi - \lambda)^2)} \delta_y, \psi \right\rangle + \langle (H_\Phi - H_\Theta)(H_\Phi - \lambda)^{-1} \left(1 - e^{-t((H_\Phi - \lambda)^2)}\right) \delta_y, \psi \rangle$$

$$= \left\langle e^{-t((H_\Phi - \lambda)^2)} \delta_y, \psi \right\rangle - \left\langle F_{t, \lambda}(H_\Phi) \delta_y, \Gamma_{\Theta^\lambda} \psi \right\rangle,$$

where $\Gamma_{\Theta^\lambda}$ is defined in (3.8) and the function $F_{t, \lambda}(z)$ is defined in (3.9).

Let $0 < R \leq R_0^{\Theta^\lambda}$, $m$ as in (3.8), and assume $\lambda \in I = (-A, A), A > 0$. Recalling (3.9), (3.10) follows from (3.8). \hfill $\square$

**Lemma 3.3.** Let $\Phi \subset \mathbb{Z}^d, I = (E - A, E + A)$, where $A > 0$ and $E \in \mathbb{R}$, and $\lambda \in I$. Then for all $t > 0$ we have

$$\left\| e^{-t((H_\Theta - E)^2 - (\lambda - E)^2)} \chi_{\mathbb{R} \setminus I}(H_\Phi) \right\| \leq e^{-tA^2h_1(\lambda)}.$$

**Proof.** We have

$$\left\| e^{-t((H_\Theta - E)^2 - (\lambda - E)^2)} \chi_{\mathbb{R} \setminus I}(H_\Phi) \right\| \leq e^{-t(A^2 - (\lambda - E)^2)} = e^{-tA^2h_1(\lambda)}.$$

$\square$
3.3. **Lemmas for the multiscale analysis.** Let $I = (E - A, E + A)$ with $E \in \mathbb{R}$ and $A > 0$, and fix a constant $m_+ > 0$. When we state that a box $\Lambda_t$ is $(m, I)$-localizing we always assume

$$0 < m_+ \xi^{-\kappa} \leq m \leq \frac{1}{2}\log \left(1 + \frac{A}{t}\right).$$

(3.22)

We also introduce the following notation:

- Given $\Theta \subset \mathbb{Z}^d$ and $J \subset \mathbb{R}$, we set $\sigma_J(H_\Theta) = \sigma(H_\Theta) \cap J$.
- Let $\Lambda_t \subset \Theta \subset \mathbb{Z}^d$ be an $(m, I)$-localized eigensystem $\{\varphi_\nu, \nu\} \nu \in \sigma(H_{\Lambda_t})$, and let $t > 0$. Then, for $J \subset I$ we set

$$\sigma^{\Theta, t}_J(H_{\Lambda_t}) = \left\{ \nu \in \sigma_J(H_{\Lambda_t}); \ x_\nu \in \Lambda^t_{\Theta, \ell}\right\}.$$  

(3.23)

The following lemmas plays an important role in our multiscale analysis. In particular, the role of the modulating function $h_\ell$ becomes transparent in the proof of Lemma 3.4.

3.3.1. **Localizing boxes.**

**Lemma 3.4.** Let $\psi: \Theta \subset \mathbb{Z}^d \to \mathbb{C}$ be a generalized eigenfunction for $H_\Theta$ with generalized eigenvalue $\lambda \in I_\ell$. Consider a box $\Lambda_t \subset \Theta$ such that $\Lambda_t$ is $(m, I)$-localized with an $(m, I)$-localized eigensystem $\{\varphi_\nu, \nu\} \nu \in \sigma(H_{\Lambda_t})$. Suppose

$$|\lambda - \nu| \geq \frac{1}{2}e^{-\xi^t} \quad \text{for all} \quad \nu \in \sigma^{\Theta, t}_I(H_{\Lambda_t}).$$

Then for $\ell$ sufficiently large we have:

(i) If $y \in \Lambda^t_{\Theta, 2\ell}$, we have

$$|\psi(y)| \leq e^{-m_2h_1(\lambda)\xi^t} |\psi(v)| \quad \text{for some} \quad \nu \in \sigma^{\Theta, 2\ell}_I \Lambda_t,$$

(3.25)

where

$$m_2 = m_2(\ell) \geq m \left(1 - C_d, m_\lambda^{-t(\tau - \gamma \beta - \kappa)}\right).$$

(3.26)

(ii) If $y \in \Lambda^t_{\Theta, \ell}$, we have

$$|\psi(y)| \leq e^{-m_3h_1(\lambda)R_v \sigma^{\Theta, \ell}_I \Lambda_t} |\psi(v)| \quad \text{for some} \quad \nu \in \sigma^{\Theta, \ell}_I \Lambda_t,$$

(3.27)

where

$$m_3 = m_3(\ell) \geq m \left(1 - C_d, m_\lambda^{-t(\tau - \gamma \beta)}\right).$$

(3.28)

Lemma 3.4 resembles [EK, Lemma 3.5], but there are important differences. The box $\Lambda_t \subset \Theta$ is $(m, I)$-localizing, and hence we only have decay for eigenfunctions with eigenvalues in $I$. Thus we can only use (3.24) for $\nu \in \sigma^{\Theta, t}_I(H_{\Lambda_t})$. To compensate, we take $\lambda \in I_\ell$, and use Lemmas 3.2 and 3.3.

**Proof of Lemma 3.4.** We take $E = 0$ by replacing the potential $V$ by $V - E$. Given $y \in \Lambda$, we write $\psi(y)$ as in (3.15).

Setting $P_I = 1 - P_I$, we have

$$\left(\epsilon^{-t(\xi^2 - \lambda^2)}\delta_y, \psi \right) = \left(\epsilon^{-t(\xi^2 - \lambda^2)}P_I \delta_y, \psi \right) + \left(\epsilon^{-t(\xi^2 - \lambda^2)}P_I \delta_y, \psi \right).$$

(3.29)

It follows from Lemma 3.3 that

$$\left|\left(\epsilon^{-t(\xi^2 - \lambda^2)}P_I \delta_y, \psi \right)\right| \leq \|\chi_\Lambda \psi\| \|\epsilon^{-t(\xi^2 - \lambda^2)}P_I\| \leq (\ell + 1)\frac{2}{t}e^{-tA^2h_1(\lambda)} |\psi(v)|,$$

(3.30)
for some \( v \in \Lambda \).

We have
\[
\langle e^{-t(H_\lambda^2 - \lambda^2)} P_I \delta_y, \psi \rangle = \sum_{\mu \in \sigma_I(H_\Lambda)} e^{-t(\mu^2 - \lambda^2)} \varphi_\mu(y) \langle \varphi_\mu, \psi \rangle.
\] (3.31)

Let \( y \in \Lambda^{\Theta,2\ell}_\epsilon \). For \( \mu \in \sigma_I(H_\Lambda) \) we have, as shown in \[EK\] Eqs. (3.37) and (3.39),
\[
|\varphi_\mu(y) \langle \varphi_\mu, \psi \rangle| \leq 2 \ell \epsilon^{\ell} e^{L_\beta} e^{-mh_1(\mu)\ell_\sigma} |\psi(v_1)| \quad \text{for some } v_1 \in \Lambda_\ell \cup \partial_{ex}^{\Theta} \Lambda_\ell.
\] (3.32)

It follows that
\[
e^{-t(\mu^2 - \lambda^2)} |\varphi_\mu(y) \langle \varphi_\mu, \psi \rangle| \leq 2 \ell \epsilon^{\ell} e^{L_\beta} e^{-t(\mu^2 - \lambda^2)} e^{-mh_1(\mu)\ell_\sigma} |\psi(v_1)|.
\] (3.33)

We now take
\[
t = \frac{m_\ell}{2} \implies e^{-t(\mu^2 - \lambda^2)} e^{-mh_1(\mu)\ell_\sigma} = e^{-mh_1(\lambda)\ell_\sigma} \quad \text{for } \mu \in I,
\] (3.34)

obtaining
\[
\left| \left\langle e^{-\frac{m_\ell}{2} (H_\lambda^2 - \lambda^2)} P_I \delta_y, \psi \right\rangle \right| \leq 2 \ell \epsilon^{\ell + 1} e^{L_\beta} e^{-mh_1(\lambda)\ell_\sigma} |\psi(v)|
\leq e^{2L_\beta} e^{-mh_1(\lambda)\ell_\sigma} |\psi(v)|,
\] for some \( v \in \Lambda_\ell \cup \partial_{ex}^{\Theta} \Lambda_\ell \). Combining (3.29), (3.30) and (3.35) yields
\[
\left| \left\langle e^{-\frac{m_\ell}{2} (H_\lambda^2 - \lambda^2)} \delta_y, \psi \right\rangle \right| \leq 2 e^{2L_\beta} e^{-mh_1(\lambda)\ell_\sigma} |\psi(v)|,
\] (3.36)

for some \( v \in \Lambda_\ell \cup \partial_{ex}^{\Theta} \Lambda_\ell \).

Using (3.16), noting \( y \in \Lambda^{\Theta,2\ell}_\epsilon \) implies \( R_{\epsilon}^{m_\ell} \Lambda_\ell \geq 2 \ell_\sigma - 1 > \ell_\sigma \), we get
\[
\left| \left\langle F_{\epsilon,\ell} \delta_y, \varphi_{\lambda,\epsilon} \Psi \right\rangle \right| \leq 70 s d \ell^{d-1} A^{-1} e^{-mh_1(\lambda)\ell_\sigma} |\psi(v)|,
\] (3.37)

for some \( v \in \partial_{ex}^{\Theta} \Lambda_\ell \).

Combining (3.36) and (3.37), and using (3.22), we conclude that
\[
|\psi(v)| \leq C_{d,m_\ell} \epsilon^{\ell' \beta} e^{2L_\beta} e^{-mh_1(\lambda)\ell_\sigma} |\psi(v)| \leq e^{-m_2 h_1(\lambda)\ell_\sigma} |\psi(v)|,
\] (3.38)

for some \( v \in \Lambda_\ell \cup \partial_{ex}^{\Theta} \Lambda_\ell \) where, using \( \lambda \in I_\ell \),
\[
m_2 \geq m \left( 1 - C_{d,m_\ell} \epsilon^{\ell' \gamma \beta - \kappa - \kappa'} \right).
\] (3.39)

By repeating the argument as many times a necessary we can get \( v \in \partial_{ex}^{\Theta,2\ell} \Lambda_\ell \). This proves part (i).

To prove part (ii), let \( y \in \Lambda^{\Theta,\ell}_\epsilon \), so \( R_{\epsilon}^{m_\ell} \Lambda_\ell \geq \ell_\sigma \). We proceed as before, but replace (3.32) by the following estimate. For \( \mu \in \sigma_I(H_\Lambda) \) and \( v' \in \partial_{ex}^{\Theta} \Lambda_\ell \), we have, as in \[EK\] Eq. (3.41),
\[
|\varphi_\mu(y) \langle \varphi_\mu, \psi \rangle| \leq e^{-m_1 h_1(\mu) R_{\epsilon}^{m_\ell} \Lambda_\ell} \quad \text{with } m_1' \geq m(1 - \frac{s d \ell^{d-1}}{2})
\] (3.40)

so, as in \[EK\] Eq. (3.44),
\[
|\varphi_\mu(y) \langle \varphi_{\mu}, \psi \rangle| \leq 2 e^{L_\beta} e^{-m_1 h_1(\mu) R_{\epsilon}^{m_\ell} \Lambda_\ell} |\psi(v_1)| \leq e^{2L_\beta} e^{-m_1 h_1(\mu) R_{\epsilon}^{m_\ell} \Lambda_\ell} |\psi(v_1)|,
\] (3.41)
for some \( v_1 \in \partial^{\Theta}_{ex} \Lambda_t \). If \( \mu \in \sigma_I(H_{\Lambda_t}) \) with \( x_{\mu} \in \partial^{\Theta}_{in} \Lambda_t \), we have

\[
\|x_{\mu} - y\| \geq R_y^{p_{\Theta,\Lambda_t}} - \ell_{\tau} \geq R_y^{p_{\Theta,\Lambda_t}} \left( 1 - 2\ell_{\tau}^{-\gamma} \right) = R_y^{p_{\Theta,\Lambda_t}} \left( 1 - 2\ell_{\tau}^{-\frac{\gamma}{1 + \gamma}} \right),
\]

so

\[
|\varphi_{\nu}(y) \langle \varphi_{\nu}, \psi \rangle| \leq e^{-m_{h_{1}(\mu)}\|x_{\mu} - y\|} \|X_{\Lambda} \psi\|
\]

\[
\leq e^{-m_{h_{1}(\mu)}R_y^{p_{\Theta,\Lambda_t}}} \left( 1 - 2\ell_{\tau}^{-\frac{1}{\gamma}} \right) \left( \ell + 1 \right) e^{-m_{h_{1}(\mu)}R_y^{p_{\Theta,\Lambda_t}}} |\varphi_{\nu}(v_2)|,
\]

for some \( v_2 \in \Lambda \), where \( m_{1}' \) is given in (3.40). It follows that for all \( \mu \in \sigma_I(H_{\Lambda}) \) we have

\[
e^{-t(\mu^2 - \lambda^2)} |\varphi_{\mu}(y) \langle \varphi_{\mu}, \psi \rangle| \leq e^{2L_{\beta}} e^{-t(\mu^2 - \lambda^2)} e^{-m_{h_{1}(\mu)}R_y^{p_{\Theta,\Lambda_t}}} |\psi(v)|,
\]

for some \( v \in \Lambda \cup \partial^{\Theta}_{ex} \Lambda \).

We now take

\[
t = \frac{m_{1}' R_y^{p_{\Theta,\Lambda_t}}}{A^2} \quad \Rightarrow \quad e^{-t(\mu^2 - \lambda^2)} e^{-m_{h_{1}(\mu)}\ell_{\tau}} = e^{-m_{h_{1}(\lambda)}\ell_{\tau}} \quad \text{for} \quad \mu \in I,
\]

obtaining

\[
\left| \left< e^{-\frac{m_{1}' R_y^{p_{\Theta,\Lambda_t}}}{A^2}(H_\lambda^2 - \lambda^2)}\right| \delta_y, \psi \right> \right| \leq \left( \ell + 1 \right)^{d_{1}'} \beta_{2} e^{2L_{\beta}} e^{-m_{h_{1}(\lambda)}R_y^{p_{\Theta,\Lambda_t}}} |\psi(v)|
\]

\[
\leq e^{2L_{\beta}} e^{-m_{h_{1}(\lambda)}R_y^{p_{\Theta,\Lambda_t}}} |\psi(v)|,
\]

for some \( v \in \Lambda_t \cup \partial^{\Theta}_{ex} \Lambda_t \). Combining (3.29), (3.30) and (3.46) yields

\[
\left| \left< e^{-\frac{m_{1}' R_y^{p_{\Theta,\Lambda_t}}}{A^2}(H_\lambda^2 - \lambda^2)}\right| \delta_y, \psi \right> \right| \leq 70s_d \ell^{-1} e^{-m_{h_{1}(\lambda)}R_y^{p_{\Theta,\Lambda_t}}} |\psi(v)|,
\]

for some \( v \in \partial^{\Theta}_{ex} \Lambda_t \). We conclude from (3.47) and (3.48) that

\[
|\psi(y)| \leq C_{d,m_{1}' \ell^{-1}} e^{2L_{\beta}} e^{-m_{h_{1}(\lambda)}R_y^{p_{\Theta,\Lambda_t}}} |\psi(v)| \leq e^{-m_{3}h_{1}(\lambda)R_y^{p_{\Theta,\Lambda_t}}} |\psi(v)|,
\]

for some \( v \in \Lambda_t \cup \partial^{\Theta}_{ex} \Lambda_t \) where, using \( h_{1}(\lambda) \geq \ell - \kappa \) since \( \lambda \in I_{\ell} \), we have

\[
m_{3} \geq m_{3} \left( 1 - C_{d,m_{1}' \ell^{-1}} \min \{ \ell - \gamma, -\kappa \} \right) = m \left( 1 - C_{d,m_{1}' \ell^{-1}} \right).
\]

If \( v \notin \partial^{\Theta,2\ell_{\tau}} \Lambda_t \), we can apply (3.25) repeatedly until we get (3.49) with \( v \in \partial^{\Theta,2\ell_{\tau}} \Lambda_t \).

**Lemma 3.5.** Let the finite set \( \Theta \subset \mathbb{Z}^{d} \) be \( L \)-level spacing for \( H \), and let \( \{ (\psi_{\lambda}, \lambda) \}_{\lambda \in \sigma(H_{\Theta})} \) be an eigensystem for \( H_{\Theta} \).

Then the following holds for sufficiently large \( L \):

(i) Let \( \Lambda_t \subset \Theta \) be an \( (m, I) \)-localizing box with an \( (m, I) \)-localized eigensystem \( \{ (\varphi_{\lambda}, \lambda) \}_{\lambda \in \sigma(H_{\Lambda_t})} \).
(a) There exists an injection
\[ \lambda \in \sigma_{1_{2^e}}(H_{\Lambda_\ell}) \mapsto \bar{\lambda} \in \sigma(H_{\Theta}), \]

such that for all \( \lambda \in \sigma_{1_{2^e}}(H_{\Lambda_\ell}) \) we have
\[ |\bar{\lambda} - \lambda| \leq e^{-m_{1_{2^e}}(\lambda)\ell}, \]

with \( m_1 = m_1(\ell) \geq m \left( 1 - C_{d,m} \frac{\log \ell}{\ell^{\varepsilon - \kappa_0}} \right) \),

and, redefining \( \varphi_\lambda \) so \( \langle \psi_\lambda, \varphi_\lambda \rangle > 0 \),

\[ \left\| \psi_\lambda - \varphi_\lambda \right\| \leq 2e^{-m_{1_{2^e}}(\lambda)\ell^2}. \]

(b) Let
\[ \sigma_{\{\Lambda_\ell\}}(H_{\Theta}) := \left\{ \bar{\lambda} : \lambda \in \sigma_{1_{2^e}}(H_{\Lambda_\ell}) \right\}. \]

Then for \( \nu \in \sigma_{\{\Lambda_\ell\}}(H_{\Theta}) \) we have
\[ |\psi_\nu(y)| \leq 2e^{-m_{1_{2^e}}(\nu)\ell} \]

for all \( y \in \Theta \setminus \Lambda_\ell \).

(c) If \( \nu \in \sigma_{\{\Lambda_\ell\}}(H_{\Theta}) \setminus \sigma_{\{\Lambda_\ell\}}(H_{\Theta}) \), we have
\[ |\nu - \lambda| \geq \frac{1}{2} e^{-\ell^2} \]

for all \( \lambda \in \sigma_{1_{2^e}}(H_{\Lambda_\ell}) \),

and
\[ |\psi_\nu(y)| \leq e^{-m_{2_{1_{2^e}}}(\nu)\ell}, \]

with \( m_2 = m_2(\ell) \) as in (3.26).

Moreover, if \( y \in \Lambda_{1_{2^e}} \) we have
\[ |\psi_\nu(y)| \leq e^{-m_{1_{2^e}}(\nu)\ell} \]

for some \( y_1 \in \partial \Theta_{1_{2^e}} \Lambda_\ell \).

(iii) Let \( \{\Lambda_\ell(a)\}_{a \in \mathcal{G}} \), where \( \mathcal{G} \subset \mathbb{R}^d \) and \( \Lambda_\ell(a) \subset \Theta \) for all \( a \in \mathcal{G} \), be a collection of \((m,I)\)-localizing boxes with \((m,I)\)-localized eigensystems
\[ \{ (\varphi_\lambda(a), \lambda(a)) \}_{\lambda(a) \in \sigma(H_{\Lambda_\ell(a)})}, \]

and set
\[ E^\Theta_G(\lambda) = \left\{ \lambda(a) : a \in \mathcal{G}, \lambda(a) \in \sigma_{1_{2^e}}(H_{\Lambda_\ell(a)}), \bar{\lambda}(a) = \lambda \right\} \]

for \( \lambda \in \sigma(H_{\Theta}) \),

\[ \sigma_G(H_{\Theta}) = \{ \lambda \in \sigma(H_{\Theta}) ; E^\Theta_G(\lambda) \neq \emptyset \} = \bigcup_{a \in \mathcal{G}} \sigma_{\{\Lambda_\ell(a)\}}(H_{\Theta}). \]

(a) Let \( a, b \in \mathcal{G} \), \( a \neq b \). Then, for \( \lambda(a) \in \sigma_{1_{2^e}}(H_{\Lambda_\ell(a)}) \) and \( \lambda(b) \in \sigma_{1_{2^e}}(H_{\Lambda_\ell(b)}) \),
\[ \lambda(a), \lambda(b) \in E^\Theta_G(\lambda) \implies \|x_{\lambda(a)} - x_{\lambda(b)}\| < 2\ell. \]

As a consequence,
\[ \Lambda_\ell(a) \cap \Lambda_\ell(b) = \emptyset \implies \sigma_{\{\Lambda_\ell(a)\}}(H_{\Theta}) \cap \sigma_{\{\Lambda_\ell(b)\}}(H_{\Theta}) = \emptyset. \]

(b) If \( \lambda \in \sigma_G(H_{\Theta}) \), we have
\[ |\psi_\lambda(y)| \leq 2e^{-m_{1_{2^e}}(\lambda)\ell} \]

for all \( y \in \Theta \setminus \Theta_G \), where \( \Theta_G := \bigcup_{a \in \mathcal{G}} \Lambda_\ell(a) \).

(c) If \( \lambda \in \sigma_{\{\Lambda_\ell\}}(H_{\Theta}) \setminus \sigma_G(H_{\Theta}) \), we have
\[ |\psi_\lambda(y)| \leq e^{-m_{2_{1_{2^e}}}(\lambda)\ell} \]

for all \( y \in \Theta_G := \bigcup_{a \in \mathcal{G}} \Lambda_{1_{2^e}}(a) \).
Proof. Let \( \Lambda \subset \Theta \) be be an \((m, I)\)-localizing box with an \((m, I)\)-localized eigensystem \( \{ \langle \varphi\lambda, \lambda \rangle \}_{\lambda \in \sigma(H\Lambda \ell)} \). Given \( \lambda \in \sigma_{I_{2\ell}}^{\Theta, \ell}(H\Lambda \ell) \), it follows from [EK] Eq. (3.10) in Lemma 3.2 that
\[
\text{dist} (\lambda, \sigma(H\Theta)) \leq \sqrt{8d} \frac{\ell - 1}{\ell} e^{-mh}(\lambda)\epsilon, \tag{3.64}
\]
so the existence of \( \tilde{\lambda} \in \sigma(H\Theta) \) satisfying (3.52) follows. Uniqueness follows from the fact that \( \Theta \) is \( L \)-level spacing and \( \gamma \beta < \tau \). In addition, note that \( \tilde{\lambda} \neq \tilde{\nu} \) if \( \lambda, \nu \in \sigma_{I_{2\ell}}^{\Theta, \ell}(H\Lambda \ell) \), \( \lambda \neq \nu \), because in this case we have
\[
\| \tilde{\lambda} - \tilde{\nu} \| \geq |\lambda - \nu| - |\tilde{\lambda} - \lambda| - |\tilde{\nu} - \nu| \geq e^{-\ell^2} - 2e^{-m_1(2\ell)^{-\gamma} \epsilon} \geq \frac{1}{2} e^{-\ell^2}, \tag{3.65}
\]
as \( \Lambda \subset \Theta \) is level spacing for \( H \), and \( \kappa + \beta < \tau \). Moreover, it follows from [EK] Lemma 3.3 that, after multiplying \( \varphi\lambda \) by a phase factor if necessary to get so
\[\langle \psi\lambda, \varphi\lambda \rangle \neq 0, \text{ we have (3.53)}.\]
If \( \nu \in \sigma_{I_{\ell}}(H\Theta) \), we have \( \nu = \tilde{\lambda} \) for some \( \lambda \in \sigma_{I_{2\ell}}^{\Theta, \ell}(H\Lambda \ell) \), so (3.55) follows from (3.58) as \( \varphi\lambda(y) = 0 \) for all \( y \in \Theta \setminus \Lambda \). Let \( \nu \in \sigma_{I_{\ell}}(H\Theta) \setminus \sigma(\Lambda \ell)(H\Theta) \). Then for all \( \lambda \in \sigma_{I_{2\ell}}^{\Theta, \ell}(H\Lambda \ell) \) we have
\[
|\nu - \lambda| \geq \| \nu - \tilde{\lambda} \| - |\tilde{\lambda} - \lambda| \geq e^{-\ell^2} - e^{-m_1(2\ell)^{-\gamma} \epsilon} \geq \frac{1}{2} e^{-\ell^2}, \tag{3.66}
\]
since \( \Theta \) is \( L \)-level spacing for \( H \), we have (3.55), and \( \kappa + \gamma \beta < \tau \). Thus
\[
|\nu - \lambda| \geq \frac{1}{4} e^{-\ell^2} \quad \text{for all } \lambda \in \sigma_{I_{2\ell}}^{\Theta, \ell}(H\Lambda \ell). \tag{3.67}
\]
Since \( \nu \in I_{\ell} \), we actually have (3.56). Thus (3.57) follows from Lemma 3.4(i) and \( \| \psi\lambda \| = 1 \), and (3.58) follows from Lemma 3.4(ii).
Now let \( \{ \Lambda_{\beta}(a) \}_{a \in \mathcal{G}} \), where \( \mathcal{G} \subset \mathbb{R}^d \) and \( \Lambda_{\beta}(a) \subset \Theta \) for all \( a \in \mathcal{G} \), be a collection of \((m, I)\)-localizing boxes with \((m, I)\)-localized eigensystems \( \{ \langle \varphi\lambda(a), \lambda(a) \rangle \}_{\lambda(a) \in \sigma(H\Lambda \ell(a))} \). Let \( \lambda \in \sigma(H\Theta) \), \( a, b \in \mathcal{G} \), \( a \neq b \), \( \lambda(a) \in \sigma_{I_{2\ell}}^{\Theta, \ell}(H\Lambda \ell(a)) \) and \( \lambda(b) \in \sigma_{I_{2\ell}}^{\Theta, \ell}(H\Lambda \ell(b)) \). Suppose \( \lambda(a), \lambda(b) \in \mathcal{E}_{\mathcal{G}}(\lambda) \), where \( \mathcal{E}_{\mathcal{G}}(\lambda) \) is given in (3.59). It then follows from (3.59) that
\[
\| \varphi\lambda(a) - \varphi\lambda(b) \| \leq 4 e^{-m_1(2\ell)^{-\gamma} \epsilon} e^{L^2}, \tag{3.68}
\]
so
\[
\langle \varphi\lambda(a), \varphi\lambda(b) \rangle \geq 1 - 8 e^{-2m_1(2\ell)^{-\gamma} \epsilon} e^{2L^2}. \tag{3.69}
\]
On the other hand, it follows from (1.11) that
\[
\| x_{\lambda(a)} - x_{\lambda(b)} \| \geq 2\ell \quad \implies \quad \langle \varphi_{x_{\lambda(a)}}, \varphi_{x_{\lambda(b)}} \rangle \leq (\ell + 1)^d e^{-m(2\ell)^{-\gamma} \epsilon}. \tag{3.70}
\]
Combining (3.69) and (3.70) we conclude that
\[
\lambda(a), \lambda(b) \in \mathcal{E}_{\mathcal{G}}(\lambda) \quad \implies \quad \| x_{\lambda(a)} - x_{\lambda(b)} \| < 2\ell, \tag{3.71}
\]
To prove (3.61), let \( a, b \in \mathcal{G} \), \( a \neq b \). If \( \Lambda_{\beta}(a) \cap \Lambda_{\beta}(b) = \emptyset \), we have that
\[
\lambda(a) \in \sigma_{I_{2\ell}}^{\Theta, \ell}(H\Lambda \ell(a)) \quad \text{and} \quad \lambda(b) \in \sigma_{I_{2\ell}}^{\Theta, \ell}(H\Lambda \ell(b)) \quad \implies \quad \| x_{\lambda(a)} - x_{\lambda(b)} \| \geq 2\ell, \tag{3.72}
\]
so it follows from (3.60) that \( \sigma_{I_{\ell}}(H\Theta) \cap \sigma_{I_{\ell}}(H\Theta) \) is empty.
Parts (ii)(b) and (ii)(c) are immediate consequence of parts (i)(b) and (i)(c), respectively. □
3.3.2. Buffered subsets. In the multiscale analysis we will need to consider boxes \( \Lambda_\ell \subset \Lambda_l \), that are not \((m,I)\)-localizing for \( H \). Instead of studying eigensystems for such boxes, we will surround them with a buffer of \((m,I)\)-localizing boxes and study eigensystems for the augmented subset.

**Definition 3.6.** We call \( \Upsilon \subset \Lambda_L \) an \((m,I)\)-buffered subset of the box \( \Lambda_L \) if the following holds:

(i) \( \Upsilon \) is a connected set in \( \mathbb{Z}^d \) of the form

\[
\Upsilon = \bigcup_{j=1}^J \Lambda R_j(a_j) \cap \Lambda_L, \tag{3.73}
\]

where \( J \in \mathbb{N}, a_1, a_2, \ldots, a_J \in \Lambda_L^\mathbb{R}, \text{ and } \ell \leq R_j \leq L \) for \( j = 1, 2, \ldots, J \).

(ii) \( \Upsilon \) is \( L \)-level spacing for \( H \).

(iii) There exists \( \mathcal{G}_\Upsilon \subset \Lambda_L^\mathbb{R} \) such that:

(a) For all \( a \in \mathcal{G}_\Upsilon \) we have \( \Lambda_\ell(a) \subset \Upsilon \), and \( \Lambda_\ell(a) \) is an \((m,I)\)-localizing box for \( H \).

(b) For all \( y \in \partial_{in}^{\Lambda_L} \Upsilon \) there exists \( a_y \in \mathcal{G}_\Upsilon \) such that \( y \in \Lambda_{\ell,2\ell}^\Upsilon(a_y) \).

In this case we set

\[
\bar{\Upsilon} = \bigcup_{a \in \mathcal{G}_\Upsilon} \Lambda_\ell(a), \quad \bar{\Upsilon}_\tau = \bigcup_{a \in \mathcal{G}_\Upsilon} \Lambda_{\ell,2\ell}^\Upsilon(a), \quad \bar{\Upsilon} = \Upsilon \setminus \bar{\Upsilon}, \quad \text{and} \quad \bar{\Upsilon}_\tau = \Upsilon \setminus \bar{\Upsilon}_\tau. \tag{3.74}
\]

(\( \bar{\Upsilon} = \Upsilon_{\mathcal{G}_\Upsilon} \) and \( \bar{\Upsilon}_\tau = \Upsilon_{\mathcal{G}_\Upsilon,\tau} \) in the notation of Lemma 3.5.)

The set \( \bar{\Upsilon}_\tau \supset \partial_{in}^{\Lambda_L} \Upsilon \) is a localizing buffer between \( \bar{\Upsilon} \) and \( \Lambda_L \setminus \Upsilon \), as shown in the following lemma.

**Lemma 3.7.** Let \( \Upsilon \) be an \((m,I)\)-buffered subset of \( \Lambda_L \), and let \( \{(\psi_\nu,\nu)\}_{\nu \in \sigma(H_\Upsilon)} \) be an eigensystem for \( H_\Upsilon \). Let \( \mathcal{G} = \mathcal{G}_\Upsilon \) and set

\[
\sigma_\mathcal{G}(H_\Upsilon) = \sigma_{H_\Upsilon}(H_\Upsilon) \setminus \sigma_\mathcal{G}(H_\Upsilon), \tag{3.75}
\]

where \( \sigma_\mathcal{G}(H_\Upsilon) \) is as in (3.59). Then the following holds for sufficiently large \( L \):

(i) For all \( \nu \in \sigma_\mathcal{G}(H_\Upsilon) \) we have

\[
|\psi_\nu(y)| \leq e^{-m_2 h_l(\nu)\ell} \text{ for all } y \in \bar{\Upsilon}_\tau, \text{ with } m_2 = m_2(\ell) \text{ as in (3.26)}. \tag{3.76}
\]

(ii) Let \( \Lambda_L \) be level spacing for \( H \), and let \( \{(\phi_\lambda,\lambda)\}_{\lambda \in \sigma(H_{\Lambda_L})} \) be an eigensystem for \( H_{\Lambda_L} \). There exists an injection \( \nu \in \sigma_\mathcal{G}(H_\Upsilon) \mapsto \bar{\nu} \in \sigma(H_{\Lambda_L}) \setminus \sigma_\mathcal{G}(H_{\Lambda_L}) \),

such that for \( \nu \in \sigma_\mathcal{G}(H_\Upsilon) \) we have

\[
|\bar{\nu} - \nu| \leq e^{-m_4 \ell - \kappa}, \text{ with } m_4 = m_4(\ell) \geq m \left( 1 - C_{d,m} \ell^{-\gamma \beta - \kappa} \right), \tag{3.77}
\]

and, redefining \( \psi_\nu \) so \( \langle \phi_{\bar{\nu}}, \psi_{\bar{\nu}} \rangle > 0 \),

\[
\|\phi_{\bar{\nu}} - \psi_{\bar{\nu}}\| \leq 2e^{-m_4 \ell - \kappa} e^{L \beta}. \tag{3.79}
\]
Proof. Part (i) follows immediately from Lemma 3.5(ii)(c).

Now let \( \Lambda_L \) be level spacing for \( H \), and let \( \{(\phi_{\lambda}, \lambda)\}_{\lambda \in \sigma(H_{\Lambda_L})} \) be an eigensystem for \( H_{\Lambda_L} \). It follows from [EK], Eq. (3.11) in Lemma 3.2 that for \( \nu \in \sigma_B(H_T) \) we have

\[
\| (H_{\Lambda_L} - \nu) \psi_{\nu} \| \leq (2d - 1) \left| \partial_{\lambda}^{\Lambda_L} \Upsilon \bigg| \| \psi_{\nu} \chi_{\partial_{\lambda}^{\Lambda_L} \Upsilon} \|_{\infty} \leq (2d - 1) L^{\frac{d}{2}} e^{-m_2 \lambda_{1}(\nu) \ell},
\]

where we used \( \partial_{\lambda}^{\Lambda_L} \Upsilon \subset \Upsilon \) and (3.76), and \( m_4 \) is given in (3.78). Since \( \Lambda_L \) and \( \Upsilon \) are L-level spacing for \( H \), the map in (3.77) is a well defined injection into \( \sigma(H_{\Lambda_L}) \), and (3.79) follows from (3.78) and [EK], Lemma 3.3.

To finish the proof we must show that \( \nu \not\in \sigma_G(H_{\Lambda_L}) \) for all \( \nu \in \sigma_B(H_T) \). Suppose \( \nu \in \sigma_G(H_{\Lambda_L}) \) for some \( \nu \in \sigma_B(H_T) \). Then there is \( \alpha \in \sigma_G(H_{\Lambda_L}) \) such that \( \lambda_{(\alpha)} \in \mathcal{E}_G^{\Lambda_L}(\nu) \). On the other hand, it follows from Lemma 3.5(i)(a) that \( \lambda_{(\alpha)} \in \mathcal{E}_G^{\nu}(\lambda_{1}) \) for some \( \lambda_{1} \in \sigma_G(H_T) \). We conclude from (3.81) and (3.79) that

\[
\sqrt{2} = \| \psi_{\lambda} - \nu \| \leq \| \psi_{\lambda} - \varphi_{(\alpha)} \| + \| \varphi_{(\alpha)} - \nu \| + \| \nu - \psi_{\nu} \| \leq 4e^{-m_5 \nu \lambda_{1}(\nu) \ell} e^{L^{\beta}} + 2e^{-m_4 \nu \ell},
\]

a contradiction. \( \square \)

**Lemma 3.8.** Let \( \Lambda_L = \Lambda_L(x_0), \; x_0 \in \mathbb{R}^d \). Let \( \Upsilon \) be an \((m, I)\)-buffered subset of \( \Lambda_L \). Let \( G = G_\Upsilon \), and for \( \nu \in \sigma(H_T) \) set

\[
\mathcal{E}_G^{\Lambda_L}(\nu) = \left\{ \lambda_{(\alpha)}; \; \alpha \in \sigma_G, \lambda_{(\alpha)} \in \sigma_\Upsilon^{\Lambda_L, \ell}(H_{\Lambda_L}(\alpha)), \; \lambda_{(\alpha)} = \nu \right\} \subset \mathcal{E}_G^{\nu}(\nu),
\]

\[
\sigma_G^{\Lambda_L}(H_T) = \left\{ \nu \in \mathcal{E}_G^{\Lambda_L}(\nu); \; \mathcal{E}_G^{\Lambda_L}(\nu) \neq \emptyset \right\} \subset \sigma_G(H_T).
\]

The following holds for sufficiently large \( L \):

(i) Let \((\psi, \lambda)\) be an eigenpair for \( H_{\Lambda_L} \) such that \( \lambda \in I_\ell \) and

\[
|\lambda - \nu| \geq \frac{1}{2} e^{-L^{\beta}} \quad \text{for all} \; \nu \in \sigma_G^{\Lambda_L}(H_T) \cup \sigma_G(H_T).
\]

Then for all \( y \in \Upsilon^{\Lambda_L, 2\ell} \), we have

\[
|\psi(y)| \leq e^{-m_5 \nu \lambda_{1}(\nu) \ell} |\psi(\nu)| \quad \text{for some} \; \nu \in \partial_{\lambda}^{\Lambda_L, 2\ell} \Upsilon,
\]

where

\[
m_5 = m_5(\ell) \geq m \left( 1 - C_{d,m} e^{-\min\{\kappa, \gamma \beta - \kappa \ell} \right).
\]

(ii) Let \( \Lambda_L \) be level spacing for \( H \), let \( \{(\psi_{\lambda}, \lambda)\}_{\lambda \in \sigma(H_{\Lambda_L})} \) be an eigensystem for \( H_{\Lambda_L} \), recall (3.77), and set

\[
\sigma_{\Upsilon}(H_{\Lambda_L}) = \left\{ \nu; \; \nu \in \sigma_B(H_T) \right\} \subset \sigma(H_{\Lambda_L}) \setminus \sigma_G(H_{\Lambda_L}).
\]

Then for all

\[
\lambda \in \sigma_I(H_{\Lambda_L}) \setminus \left( \sigma_G(H_{\Lambda_L}) \cup \sigma_{\Upsilon}(H_{\Lambda_L}) \right),
\]

the condition (3.83) is satisfied, and \( \psi_{\lambda} \) satisfies (3.84).
Proof. To prove part (i), we take $E = 0$ by replacing the potential $V$ by $V - E$. Let $(\psi, \lambda)$ be an eigenpair for $H_{\Lambda \epsilon}$ satisfying (3.83). Given $y \in \Upsilon$, we write $\psi(y)$ as in (3.10). We set $P = \chi_{I} (H_{\epsilon})$ and $P = 1 - P_{I}$. We use Lemma 3.3 with $\Phi = \Upsilon$ and $J = I_{\epsilon}$.

To estimate $\langle e^{-t(H_{\epsilon}^{2} - \lambda^{2})} P_{\delta_{y}, \psi} \rangle$, let $(\psi_{\nu}, \nu) \in \sigma(H_{\epsilon})$ be an eigenpair for $H_{\epsilon}$. For each $\nu \in \sigma_{G}(H_{\epsilon})$ we fix $\lambda^{(a_{\nu})} \in \mathcal{E}^{T}_{G}(\nu)$, where $a_{\nu} \in \mathcal{G}$, $\lambda^{(a_{\nu})} \in \mathcal{A}^{L}_{G}(\nu)$, picking $\lambda^{(a_{\nu})} \in \mathcal{E}^{L}_{G}(\nu)$ if $\nu \in \sigma_{G}^{L}(H_{\epsilon})$, so $x_{\lambda^{(a_{\nu})}} \in \Lambda^{L}_{\epsilon}(a_{\nu})$. If $\nu \in \sigma_{G}(H_{\epsilon}) \setminus \sigma_{G}^{L}(H_{\epsilon})$, we have $x_{\lambda^{(a_{\nu})}} \in \Lambda^{L}_{\epsilon}(a_{\nu}) \setminus \mathcal{E}^{L}_{G}(\nu)$.

Given $J \subset \mathbb{R}$, we set $\sigma_{G, J}(H_{\epsilon}) = \sigma_{G}(H_{\epsilon}) \cap J$, $\sigma_{G, J}^{L}(H_{\epsilon}) = \sigma_{G}^{L}(H_{\epsilon}) \cap J$. We have

$$
\langle e^{-t(H_{\epsilon}^{2} - \lambda^{2})} P_{\delta_{y}, \psi} \rangle = \sum_{\nu \in \sigma_{G}(H_{\epsilon})} e^{-t(H_{\epsilon}^{2} - \lambda^{2})} \varphi_{\nu}(y) \langle \varphi_{\nu}, \psi \rangle
$$

$$
= \sum_{\nu \in \sigma_{G}^{L}(H_{\epsilon}) \cup \sigma_{B}(H_{\epsilon})} e^{-t(H_{\epsilon}^{2} - \lambda^{2})} \varphi_{\nu}(y) \langle \varphi_{\nu}, \psi \rangle
$$

$$
+ \sum_{\nu \in \sigma_{G, J}(H_{\epsilon}) \setminus \sigma_{G}^{L}(H_{\epsilon})} e^{-t(H_{\epsilon}^{2} - \lambda^{2})} \varphi_{\nu}(y) \langle \varphi_{\nu}, \psi \rangle.
$$

If $\nu \in \sigma_{G}^{L}(H_{\epsilon}) \cup \sigma_{B}(H_{\epsilon})$ we have

$$
\langle \varphi_{\nu}, \psi \rangle = (\lambda - \nu)^{-1} \langle \varphi_{\nu}, (H_{\Lambda \epsilon} - \nu) \psi \rangle = (\lambda - \nu)^{-1} \langle (H_{\Lambda \epsilon} - \nu) \varphi_{\nu}, \psi \rangle.
$$

(3.88)

It follows from (3.83) and [EK, Eq. (3.10) in Lemma 3.2] that

$$
|\varphi_{\nu}(y) \langle \varphi_{\nu}, \psi \rangle| \leq 2e^{L_{\beta}} |\varphi_{\nu}(y)| \sum_{v' \in \partial_{\epsilon}^{L} \Upsilon} \left( \sum_{v \in \partial_{\epsilon}^{L} \Upsilon} |\varphi_{\nu}(v')| \right) |\psi(v')|
$$

$$
\leq 2L^{d} e^{L_{\beta}} \left\{ 2d \max_{u \in \partial_{\epsilon}^{\Lambda \epsilon} \Upsilon} |\varphi_{\nu}(u)| \right\} |\psi(v)| \text{ for some } v_{1} \in \partial_{\epsilon}^{\Lambda \epsilon} \Upsilon.
$$

(3.89)

If $\nu \in \sigma_{B}(H_{\epsilon})$ it follows from (3.70) that

$$
\max_{u \in \partial_{\epsilon}^{\Lambda \epsilon} \Upsilon} |\varphi_{\nu}(u)| \leq e^{-m_{2}h_{1}(\nu)t_{\epsilon}}.
$$

(3.90)

If $\nu \in \sigma_{G, J}^{L}(H_{\epsilon})$, it follows from (3.53) and (1.11) that

$$
\max_{u \in \partial_{\epsilon}^{\Lambda \epsilon} \Upsilon} |\varphi_{\nu}(u)| \leq \max_{u \in \partial_{\epsilon}^{\Lambda \epsilon} \Upsilon} \left( |\varphi_{\nu}(u) - \varphi^{(a_{\nu})}(u)| + |\varphi^{(a_{\nu})}(u)| \right)
$$

$$
\leq 2e^{-m_{1}h_{1}(\lambda^{(a_{\nu})})t_{\epsilon}} e^{L_{\beta}} + e^{-m_{1}h_{1}(\lambda^{(a_{\nu})})t_{\epsilon}} e^{L_{\beta}}
$$

$$
\leq 3e^{-m_{1}h_{1}(\nu)t_{\epsilon}} e^{L_{\beta}}, \text{ where } m_{1}' \geq m_{1}(1 - e^{-C_{d, m_{1}}t_{\epsilon} - \gamma_{\epsilon} - \kappa})
$$

(3.91)

where we used (3.52). It follows that, with

$$
m_{1}'' = \min \{ m_{1}', m_{2} \} \geq m \left( 1 - C_{d, m_{1}} t_{\epsilon} - \tau - \gamma_{\epsilon} - \kappa \right),
$$

(3.92)

for all $\nu \in \sigma_{G, J}^{L}(H_{\epsilon}) \cup \sigma_{B}(H_{\epsilon})$ we have

$$
|\varphi_{\nu}(y) \langle \varphi_{\nu}, \psi \rangle| \leq 3L^{d} e^{-m_{1}h_{1}(\nu)t_{\epsilon}} |\psi(v_{1})| \text{ for some } v_{1} \in \partial_{\epsilon}^{\Lambda \epsilon} \Upsilon.
$$

(3.93)
Picking \( t = \frac{m_0^\lambda}{\lambda^2} \), for all \( \nu \in \sigma_G^L(H_T) \cup \sigma_B(H_T) \) we get
\[
\left| e^{-t(\nu^2 - \lambda^2)} \langle \vartheta_\nu(y) \rangle \langle \vartheta_\nu, \psi \rangle \right| \leq e^{\delta_L \beta} e^{-m_1^\nu h_1(\lambda) \ell_T} |\psi(v_1)| \quad \text{for some } v_1 \in \partial_{\text{ex}} \Upsilon, \quad (3.94)
\]
so
\[
\left| \sum_{\nu \in \sigma_G^L(H_T) \cup \sigma_B(H_T)} e^{-t(\nu^2 - \lambda^2)} \vartheta_\nu(y) \langle \vartheta_\nu, \psi \rangle \right| \leq (L + 1)^d e^{\delta_L \beta} e^{-m_1^\nu h_1(\lambda) \ell_T} |\psi(v_2)|, \quad (3.95)
\]
for some \( v_2 \in \partial_{\text{ex}} \Upsilon \).

Now let \( \nu \in \sigma_G(J(H_T)) \setminus \sigma_G^L(H_T) \). In this case we have \( x_{\chi(a_\nu)} \in \Lambda^{Y, \ell_T}(a_\nu) \setminus \Lambda^{\Lambda, \ell_T}(a_\nu) \), so we have
\[
\text{dist} (x_{\chi(a_\nu)}, \Upsilon \setminus \Lambda_{\nu}(a_\nu)) > \ell_T \quad \text{and} \quad \text{dist} (x_{\chi(a_\nu)}, \Lambda_L \setminus \Lambda_{\nu}(a_\nu)) \leq \ell_T, \quad (3.96)
\]
so there is \( u_0 \in \Lambda_L \setminus \Upsilon \) such that \( \| x_{\chi(a_\nu)} - u_0 \| \leq \ell_T \). We now assume \( y \in \Upsilon^{\Lambda, 2\ell_T} \), so we have \( \| y - u_0 \| > 2\ell_T \). We conclude that
\[
\| x_{\chi(a_\nu)} - y \| \geq \| y - u_0 \| - \| x_{\chi(a_\nu)} - u_0 \| > 2\ell_T - \ell_T = \ell_T. \quad (3.97)
\]
Thus
\[
\langle \vartheta_\nu(y) \rangle \leq |\vartheta_\nu(y) - \varphi_{\chi(a_\nu)}(y))| + |\varphi_{\chi(a_\nu)}(y)| \leq 2e^{-m_1 h_1(\lambda) \ell_T} e^{\delta_L \beta} + e^{-m h_1(\lambda) \ell_T} \leq 3e^{-m_1 h_1(\lambda) \ell_T} e^{\delta_L \beta} \leq 3e^{-m_1 h_1(\lambda) \ell_T} e^{\delta_L \beta},
\]
using (3.55), (3.11), and (3.52). It follows that
\[
\left| \sum_{\nu \in \sigma_G(J(H_T)) \setminus \sigma_G^L(H_T)} e^{-t(\nu^2 - \lambda^2)} \vartheta_\nu(y) \langle \vartheta_\nu, \psi \rangle \right| \leq 3(L + 1)^d e^{-m_1^\nu h_1(\lambda) \ell_T} e^{\delta_L \beta} |\psi(v_3)|, \quad (3.99)
\]
for some \( v_3 \in \Upsilon \).

Combining (3.87), (3.95) and (3.99), we get for \( y \in \Upsilon^{\Lambda, 2\ell_T} \) that
\[
\left| \left\langle e^{-t(H_T^2 - \lambda^2)} P_{\delta_T}, \psi \right\rangle \right| \leq e^{4\delta_L \beta} e^{-m_1^T h_1(\lambda) \ell_T} |\psi(v_4)|, \quad (3.100)
\]
for some \( v_4 \in \Upsilon \cup \partial_{\text{ex}} \Upsilon \).

From Lemma 3.3 we get,
\[
\left| \left\langle e^{-\frac{m_1^\nu \ell_T}{\lambda^2}(H_T^2 - \lambda^2)} P_{\delta_T}, \psi \right\rangle \right| \leq (\ell + 1)^d e^{m_1^\nu (1 - \ell_T^{-2})^2 h_1(\lambda) \ell_T} |\psi(v_5)|, \quad (3.101)
\]
for some \( v_5 \in \Upsilon \).

Combining (3.100) and (3.101), we get
\[
\left| \left\langle e^{-\frac{m_1^\nu \ell_T}{\lambda^2}(H_T^2 - \lambda^2)} \delta_T, \psi \right\rangle \right| \leq 2e^{4\delta_L \beta} e^{-m_1^\nu (1 - \ell_T^{-2})^2 h_1(\lambda) \ell_T} |\psi(v_6)|, \quad (3.102)
\]
for some \( v_6 \in \Upsilon \cup \partial_{\text{ex}} \Upsilon \).
Lemma 3.10. \[ R_y^{\ell} \geq 2\ell - 1 > \ell, \] noting \( y \in \mathcal{T}^{\Lambda_L, 2\ell} \) implies

\[
\left| \left\langle F_{\frac{\ell}{2\ell}}(H)\delta_y, \Gamma \theta^T \psi \right\rangle \right| \leq 70L^{d}A^{1/2}e^{-m'' h_{I}(\ell)\ell} |\psi(v)|, \tag{3.103}
\]

for some \( v \in \partial^{\ell} \mathcal{Y} \).

Combining (3.102) and (3.103) we get

\[
|\psi(y)| \leq C_{d,m,\ell} e^{4L^2} e^{-m'' (1-\ell^{-\epsilon})^2 h_{I}(\ell)\ell} |\psi(v)| \leq e^{-m_{5} h_{I}(\ell)\ell} |\psi(v)|, \tag{3.104}
\]

for some \( v \in \mathcal{Y} \cup \partial^{\ell} \mathcal{Y} \), where \( m_5 \) is as in (3.85). Repeating the procedure as many times as needed, we can require \( v \in \partial^{\ell} \mathcal{Y} \).

Now suppose \( \Lambda_\ell \) is level spacing for \( H \), and let \( \lambda \in \sigma_{I}(H_{\Lambda_L}) \setminus (\sigma_{G}(H_{\Lambda_L}) \cup \sigma_{T}(H_{\Lambda_L})) \). If \( \lambda \notin \sigma_{G}(H_{\Lambda_L}) \), it follows from Lemma (3.43) that (3.60) holds for all \( a \in G \). If \( \lambda \notin \sigma_{T}(H_{\Lambda_L}) \), the argument in (3.60), modified by the use of (3.78), instead of (3.62), using (3.13), gives \( |\lambda - v| \geq \frac{1}{L}e^{-L^\lambda} \) for all \( v \in \sigma_{G}(H_{\mathcal{Y}}) \). Thus we have (3.83), which implies (3.83). \( \Box \)

3.4. Suitable covers of a box. To perform the multiscale analysis in an efficient way, it is convenient to use a canonical way to cover a box of side \( L \) by boxes of side \( \ell < L \). We will use the idea of suitable covers of a box as in [GK4, Definition 3.12], adapted to the discrete case. Since we will use (3.27) to get decay of eigenfunctions in scale \( L \) from decay in scale \( \ell \), we will need to make sure \( R_y^{\ell \Lambda_L \Lambda_\ell} \approx \frac{\ell}{2} \). We will do so by ensuring that for all \( y \in \Lambda_\ell \) we can find a box \( \Lambda_\ell \) in the cover such that \( y \in \Lambda_\ell \) with \( R_y^{\ell \Lambda_L \Lambda_\ell} = \frac{\ell}{2} \geq \frac{\ell}{2} - 1 \) for a fixed \( \epsilon \in (0, 1) \). Later we will require \( \epsilon \) as in (3.13) for convenience.

Definition 3.9. Fix \( \epsilon \in (0, 1) \). Let \( \Lambda_L = \Lambda_L(x_0), \ x_0 \in \mathbb{R}^d \) be a box in \( \mathbb{Z}^d \), and let \( \ell < L \). A suitable \( \ell \)-cover of \( \Lambda_L \) is the collection of boxes

\[
\mathcal{C}_{L,\ell}(x_0) = \{ \Lambda_\ell(a) \}_{a \in \Xi_{L,\ell}} \tag{3.105}
\]

where

\[
\Xi_{L,\ell} = \Xi_{L,\ell}(x_0) := \{ x_0 + \rho\ell^\epsilon \mathbb{Z}^d \} \cap \Lambda_L^R \text{ with } \rho \in \left[ \frac{1}{2}, \frac{1}{4} \right] \cap \left\{ \frac{L}{2L} \left( \frac{k - 2}{2L} \right) : k \in \mathbb{N} \right\}. \tag{3.106}
\]

We call \( \mathcal{C}_{L,\ell} \) the suitable \( \ell \)-cover of \( \Lambda_L \) if \( \rho = \rho_{L,\ell} := \max \left[ \frac{1}{2}, 1 \right] \cap \left\{ \frac{L-\ell}{2L} : k \in \mathbb{N} \right\} \).

We adapt [GK4, Lemma 3.13] to our context.

Lemma 3.10. Let \( \ell \leq \frac{L}{2} \). Then for every box \( \Lambda_L = \Lambda_L(x_0), \ x_0 \in \mathbb{R}^d \), a suitable \( \ell \)-cover \( \mathcal{C}_{L,\ell} = \mathcal{C}_{L,\ell}(x_0) \) satisfies

\[
\Lambda_L = \bigcup_{a \in \Xi_{L,\ell}} \Lambda_\ell(a); \tag{3.107}
\]

for all \( b \in \Lambda_L \) there is \( \Lambda_\ell^{(b)} \in \mathcal{C}_{L,\ell} \) such that \( b \in \left( \Lambda_\ell^{(b)} \right)^{\Lambda_L, \ell, \ell - \epsilon} \), i.e.,

\[
\Lambda_L = \bigcup_{a \in \Xi_{L,\ell}} \Lambda_\ell^{\Lambda_L, \ell, \ell - \epsilon}(a); \tag{3.108}
\]

\[
\# \Xi_{L,\ell} = \left( \frac{L-\ell}{\rho\ell} + 1 \right)^d \leq \left( \frac{2L}{\rho\ell} \right)^d. \tag{3.109}
\]
Moreover, given \( a \in x_0 + \rho \xi Z^d \) and \( k \in \mathbb{N} \), it follows that
\[
\Lambda_{(2k \rho \xi + \ell)}(a) = \bigcup_{b \in (x_0 + \rho \xi Z^d) \cap \Lambda^\delta_{(2k \rho \xi + \ell)}(a)} \Lambda_\ell(b),
\]
and \( \{ \Lambda_\ell(b) \}_{b \in (x_0 + \rho \xi Z^d) \cap \Lambda^\delta_{(2k \rho \xi + \ell)}(a)} \) is a suitable \( \ell \)-cover of the box \( \Lambda_{(2k \rho \xi + \ell)}(a) \).

Note that \( \Lambda^{(b)}_\ell \) does not denote a box centered at \( b \), just some box in \( C_{L,\ell}(x_0) \) satisfying (3.109). By \( \Lambda^{(b)}_\ell \) we will always mean such a box. We will use
\[
R^{b}_{\ell} \Lambda^{(b)}_\ell \geq \ell - \frac{\epsilon}{2} - 1 \quad \text{for all} \quad b \in \Lambda_L.
\]
Note also that \( \rho \leq 1 \) yields (3.109). We specified \( \rho = \rho_{L,\ell} \) in for the suitable \( \ell \)-cover for convenience, so there is no ambiguity in the definition of \( C_{L,\ell}(x_0) \).

Suitable covers are convenient for the construction of buffered subsets (see Definition 3.6) in the multiscale analysis. We will use the following observation:

**Remark 3.11.** Let \( C_{L,\ell} \) be a suitable \( \ell \)-cover for the box \( \Lambda_L \), and set
\[
k_\ell = k_{L,\ell} = \lfloor \rho^{-1} \ell^{1-\xi} \rfloor + 1.
\]
Then for all \( a, b \in C_{L,\ell} \) we have
\[
\Lambda^a_\ell(a) \cap \Lambda^b_\ell(b) = \emptyset \quad \iff \quad \|a - b\| \geq k_\ell \rho \ell^{\xi}.
\]

### 3.5. Probability estimate for level spacing

The eigensystem multiscale analysis uses a probability estimate of Klein and Molchanov [KIM, Lemma 2], which we state as in [EK, Lemma 2.1]. If \( J \subset \mathbb{R} \), we set \( \text{diam} J = \sup_{s, t \in J} |s - t| \).

**Lemma 3.12.** Let \( H_\omega \) be an Anderson model as in Definition 1.15. Let \( \Theta \subset \mathbb{Z}^d \) and \( L > 1 \). Then
\[
\mathbb{P}\{ \Theta \text{ is } L\text{-level spacing for } H_\omega \} \geq 1 - Y_\mu \rho^{-(2\alpha - 1) L^\beta} |\Theta|^2.
\]
where
\[
Y_\mu = 2^{2\alpha - 1} \bar{K}^2 (\text{diam supp} \mu + 2d + 1),
\]
with \( \bar{K} = K \) if \( \alpha = 1 \) and \( \bar{K} = 8K \) if \( \alpha \in (\frac{1}{2}, 1) \).

In the special case of a box \( \Lambda_L \), we have
\[
\mathbb{P}\{ \Lambda_L \text{ is level spacing for } H \} \geq 1 - Y_\mu (L + 1)^{2d} \rho^{-(2\alpha - 1) L^\beta}.
\]

### 4. Eigensystem multiscale analysis

In this section we fix an Anderson model \( H_\omega \) and prove Theorem 1.6. Note that \( \varphi \) is given in (1.6).

**Proposition 4.1.** Fix \( m_- > 0 \). There exists a a finite scale \( L = L(d, m_-) \) with the following property: Suppose for some scale \( L_0 \geq L \) we have
\[
\inf_{x \in \mathbb{R}^d} \mathbb{P}\{ \Lambda_{L_0}(x) \text{ is } (m_0, I_0)\text{-localizing for } H_\omega \} \geq 1 - e^{-L_0^{\delta}},
\]
where \( I_0 = (E - A_0, E + A_0) \subset \mathbb{R} \), with \( E \in \mathbb{R} \) and \( A_0 > 0 \), and
\[
m_- L_0^{-\kappa'} \leq m_0 \leq \frac{1}{2} \log \left( 1 + \frac{4m}{m_-} \right).
\]
Set $L_{k+1} = L_k^\epsilon$, $A_{k+1} = A_k(1 - L_k^{-\epsilon})$, and $I_{k+1} = (E - A_{k+1}, E + A_{k+1})$, for $k = 0, 1, \ldots$. Then for all $k = 1, 2, \ldots$ we have
\[
\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_{L_k}(x) \text{ is } (m_k, I_k, I_{k-1})\text{-localizing for } H_\omega\} \geq 1 - e^{-L_k^\epsilon},
\] (4.3)
where
\[
m_k = (1 - C_{d,m} L_k^{-\epsilon}) \leq m_k < \frac{1}{2} \log (1 + \frac{A(1 + \epsilon^\gamma)}{4d^2}).
\] (4.4)

The proof of Proposition 4.1 relies on the following lemma, the induction step for the multiscale analysis.

**Lemma 4.2.** Fix $m_0 > 0$. Let $I = (E - A, E + A) \subset \mathbb{R}$, with $E \in \mathbb{R}$ and $A > 0$, and $m_0 > 0$. Suppose for some scale $\ell$ we have
\[
\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_{\ell}(x) \text{ is } (m, I)\text{-localizing for } H_\omega\} \geq 1 - e^{-\ell^\gamma},
\] (4.5)
where
\[
m_0 \Theta^{-\ell^\gamma} \leq m \leq \frac{1}{2} \log (1 + \frac{A(1 + \epsilon^\gamma)}{4d^2}).
\] (4.6)
Then, if $\ell$ is sufficiently large, we have (recall $L = \ell^\gamma$)
\[
\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_{L}(x) \text{ is } (M, I_\ell, I)\text{-localizing for } H_\omega\} \geq 1 - e^{-L^\gamma},
\] (4.7)
where
\[
m_0 L^{-\epsilon} < m \leq (1 - C_{d,m} L^{-\gamma}) \leq M < \frac{1}{2} \log \left(1 + \frac{A(1 - \ell^{-\gamma})}{4d^2}\right).
\] (4.8)

**Proof.** To prove the lemma we proceed as in [EK] Proof of Lemma 4.5, with some modifications. The crucial estimate (3.27) is a somewhat weaker statement than its counterpart [EK] Eq. (3.31)]. For this reason we are forced to modify the definition of an $\ell$-cover of a box, and use the version given in Definition 3.9 with $\zeta$ as in (1.3), which differs from the version given in [EK] Definition 3.10 which has $\zeta = 1$. In particular, we have (3.113), while in [EK] the corresponding statement holds with the simpler $\|a - b\| \geq 2\rho\epsilon$.

We assume (4.5) and (4.6) for a scale $\ell$. We take $\Lambda_{\ell} = \Lambda(x_0)$, where $x_0 \in \mathbb{R}^d$, and let $C_{L,\ell} = C_{L,\ell}(x_0)$ be the suitable $\ell$-cover of $\Lambda_{\ell}$ (with $\zeta$ as in (1.3)). Given $a, b \in \Xi_{L,\ell}$, we will say that the boxes $\Lambda_{\ell}(a)$ and $\Lambda_{\ell}(b)$ are disjoint if and only if $\Lambda_{\ell}(a) \cap \Lambda_{\ell}(b) = \emptyset$, that is, if and only if $\|a - b\| \geq k_{\ell}\rho\epsilon^\gamma$ (see Remark 3.11). We take $N = N_\ell = \lceil \ell^{-1}\rho \epsilon^\gamma \rceil$ (recall (4.3)), and let $B_N$ denote the event that there exist at most $N$ disjoint boxes in $C_{L,\ell}$ that are not $(m, I)$-localizing for $H_\omega$. For sufficiently large $\ell$, we have, using (3.14), (1.5), and the fact that events on disjoint boxes are independent, that
\[
\mathbb{P}\{B_N\} \leq \left(\frac{2}{M}\right)^{(N+1)d} e^{-(N+1)\ell^\gamma} = 2^{(N+1)d}(\gamma + \epsilon)(N+1)^d e^{-(N+1)\ell^\gamma} < \frac{1}{2} e^{-L^\gamma}.\] (4.9)

We now fix $\omega \in B_N$. There exists $A_N = A_N(\omega) \subset \Xi_{L,\ell} = \Xi_{L,\ell}(x_0)$ such that $|A_N| \leq N$ and $\|a - b\| \geq k_{\ell}\rho\epsilon^\gamma$ if $a, b \in A_N$ and $a \neq b$, with the following property: if $a \in \Xi_{L,\ell}$ with $\text{dist}(a, A_N) \geq k_{\ell}\rho\epsilon^\gamma$, so $\Lambda_{\ell}(a) \cap \Lambda_{\ell}(b) = \emptyset$ for all $b \in A_N$, the box $\Lambda_{\ell}(a)$ is $(m, I)$-localizing for $H_\omega$. In other words,
\[
a \in \Xi_{L,\ell} \setminus \bigcup_{b \in A_N} \Lambda_{2(k_{\ell} - 1)\rho\epsilon^\gamma}(b) \quad \implies \quad \Lambda_{\ell}(a) \text{ is } (m, I)\text{-localizing for } H_\omega.\] (4.10)
We want to embed the boxes \( \{ \Lambda_{\ell}(b) \}_{b \in A_N} \) into \( (m, I) \)-buffered subsets of \( \Lambda_L \). To do so, we consider graphs \( G_i = (\Xi_{L,\ell}, E_i), \ i = 1, 2, \) both having \( \Xi_{L,\ell} \) as the set of vertices, with sets of edges given by

\[
E_1 = \{ \{a, b\} \in \Xi_{L,\ell}^2; \ |a - b| \leq (k_\ell - 1)\rho \ell^c \} \quad (4.11)
\]

\[
E_2 = \{ \{a, b\} \in \Xi_{L,\ell}^2; \ a \neq b \text{ and } \Lambda^R_\ell(a) \cap \Lambda^R_\ell(b) \neq \emptyset \},
\]

\[
E_2 = \{ \{a, b\} \in \Xi_{L,\ell}^2; \ k_\ell \rho \ell^c \leq |a - b| \leq 3(k_\ell - 1)\rho \ell^c \}
\]

\[
= \{ \{a, b\} \in \Xi_{L,\ell}^2; \ Lambda^R_\ell(a) \cap Lambda^R_\ell(b) = \emptyset \text{ and } |a - b| \leq 3(k_\ell - 1)\rho \ell^c \}.
\]

Given \( \Psi \subset \Xi_{L,\ell} \), we let \( \overline{\Psi} = \Psi \cup \partial^\omega_{\text{ex}} \Psi \), where \( \partial^\omega_{\text{ex}} \Psi \), the exterior boundary of \( \Psi \) in the graph \( G_1 \), is defined by

\[
\partial^\omega_{\text{ex}} \Psi = \{ a \in \Xi_{L,\ell} \setminus \Psi; \ \text{dist}(a, \Psi) \leq (k_\ell - 1)\rho \ell^c \} \quad (4.12)
\]

Let \( \Phi \subset \Xi_{L,\ell} \) be \( G_2 \)-connected, so \( \text{diam} \Phi \leq 3\rho \ell (|\Phi| - 1) \). Then

\[
\Phi = \Xi_{L,\ell} \cap \bigcup_{a \in \Phi} \Lambda^{2^\ell} \big( \rho_\ell (2^\ell + 1) \big)(a) = \{ a \in \Xi_{L,\ell}; \ \text{dist}(a, \Phi) \leq \rho \ell \} \quad (4.13)
\]

is a \( G_1 \)-connected subset of \( \Xi_{L,\ell} \) such that

\[
\text{diam} \Phi \leq \text{diam} \Phi + 2\rho \ell \leq 3(k_\ell - 1)\rho \ell^c (|\Phi| - 1). \quad (4.14)
\]

We set

\[
\Upsilon^{(0)}_\Phi = \bigcup_{a \in \Phi} \Lambda_{\ell}(a) \quad \text{and} \quad \Upsilon_\Phi = \Upsilon^{(0)}_\Phi \cup \bigcup_{a \in \partial^\omega_{\text{ex}} \Phi} \Lambda_{\ell}(a) = \bigcup_{a \in \overline{\Phi}} \Lambda_{\ell}(a). \quad (4.15)
\]

Let \( \{ \Phi_r \}_{r=1}^R = \{ \Phi_r(\omega) \}_{r=1}^R \) denote the \( G_2 \)-connected components of \( A_N \) (i.e., connected in the graph \( G_2 \)); we have \( R \in \{ 1, 2, \ldots, N \} \) and \( \sum_{r=1}^R |\Phi_r| = |A_N| \leq N \).

We conclude that \( \{ \Phi_r \}_{r=1}^R \) is a collection of disjoint, \( G_1 \)-connected subsets of \( \Xi_{L,\ell} \), such that

\[
\text{dist}(\Phi_r, \Phi_s) \geq k_\ell \rho \ell^c \quad \text{if} \quad r \neq s. \quad (4.16)
\]

Moreover, it follows from \( 4.10 \) that

\[
a \in \mathcal{G} = \mathcal{G}(\omega) = \Xi_{L,\ell} \setminus \bigcup_{r=1}^R \Phi_r \quad \implies \quad \Lambda_{\ell}(a) \text{ is } (m, I)\text{-localizing for } H_\omega. \quad (4.17)
\]

In particular, we conclude that \( \Lambda_{\ell}(a) \) is \( (m, I)\)-localizing for \( H_\omega \) for all \( a \in \partial^\omega_{\text{ex}} \Phi_r \), \( r = 1, 2, \ldots, R \).

Each \( \Upsilon_r = \Upsilon_{\Phi_r}, \ r = 1, 2, \ldots, R \), clearly satisfies all the requirements to be an \( (m, I) \)-buffered subset of \( \Lambda_L \) with \( G_{\Upsilon_r} = \partial^\omega_{\text{ex}} \Phi_r \) (see Definition \( 3.6 \), except that we do not know if \( \Upsilon_r \) is \( L \)-level spacing for \( H_\omega \). (Note that the sets \( \{ \Upsilon_r^{(0)} \}_{r=1}^R \) are disjoint, but the sets \( \{ \Upsilon_r \}_{r=1}^R \) are not necessarily disjoint.) Note also that it follows from \( 4.14 \) that

\[
\text{diam} \Upsilon_r \leq \text{diam} \Phi_r + \ell \leq (k_\ell - 1)\rho \ell^c (3|\Phi_r| + 1) + \ell \leq 5\ell|\Phi_r|, \quad (4.18)
\]
so, using \( (1.4) \), we have
\[
\sum_{r=1}^{R} \text{diam } \mathcal{Y}_r \leq 5\ell N \leq 5\ell^{(\gamma-1)\zeta+1} \ll \ell^{\gamma} = L^{\gamma}.
\] (4.19)

We can arrange for \( \{ \mathcal{Y}_r \}_{r=1}^{R} \) to be a collection of \((m, I)\)-buffered subsets of \( \Lambda_L \) as follows. It follows from Lemma 3.12 that for any \( \Theta \subset \Lambda_L \) we have
\[
P \{ \Theta \text{ is } L\text{-level spacing for } H_\omega \} \geq 1 - Y_\mu e^{-(2\alpha-1)\lambda L^\beta} (L + 1)^{2d}.
\] (4.20)

Let
\[
\mathcal{F}_N = \bigcup_{r=1}^{N} \mathcal{F}(r), \text{ where } \mathcal{F}(r) = \{ \Phi \subset \Xi_{L, r}; \Phi \text{ is } \mathbb{G}_2\text{-connected and } |\Phi| = r \}.
\] (4.21)

Setting \( \mathcal{F}(r, a) = \{ \Phi \in \mathcal{F}_r; a \in \Phi \} \) for \( a \in \Xi_{L, r} \), and noting that each vertex in the graph \( \mathbb{G}_2 \) has less than \((6k_\ell - 5)d\) nearest neighbors, we get
\[
|\mathcal{F}(r, a)| \leq (r - 1)! (14\ell^{1-c})^{(r-1)d} \implies |\mathcal{F}(r)| \leq (L + 1)^d (r - 1)! (14\ell^{1-c})^{(r-1)d} \implies |\mathcal{F}_N| \leq (L + 1)^d N! (14\ell^{1-c})^{(N-1)d}.
\] (4.22)

Letting \( \mathcal{S}_N \) denote the event that the box \( \Lambda_L \) and the subsets \( \{ \mathcal{Y}_\Phi \}_{\Phi \in \mathcal{F}_N} \) are all \( L\)-level spacing for \( H_\omega \), we get from (4.20) and (4.22) that
\[
P \{ \mathcal{S}_N^c \} \leq Y_\mu \left( 1 + (L + 1)^d N! (14\ell^{1-c})^{(N-1)d} \right) (L + 1)^{2d} e^{-(2\alpha-1)\lambda L^\beta} < \frac{1}{L^{\zeta}}
\] (4.23)

for sufficiently large \( L \), since \((\gamma - 1)^\zeta < (\gamma - 1)\beta < \gamma \beta \) and \( \zeta < \beta \).

We now define the event \( \mathcal{E}_N = \mathcal{B}_N \cap \mathcal{S}_N \). It follows from (4.19) and (4.23) that
\[
P \{ \mathcal{E}_N \} > 1 - e^{-L^\zeta}.
\] To finish the proof we need to show that for all \( \omega \in \mathcal{E}_N \) the box \( \Lambda_L \) is \((M, I_r, I)\)-localizing for \( H_\omega \), where \( M \) is given in (4.18).

Let us fix \( \omega \in \mathcal{E}_N \). Then we have (4.17), \( \Lambda_L \) is level spacing for \( H_\omega \), and the subsets \( \{ \mathcal{Y}_\Phi \}_{\Phi \in \mathcal{F}_N} \) constructed in (4.13) are \((m, I)\)-buffered subsets of \( \Lambda_L \) for \( H_\omega \). It follows from (3.106) and Definition 3.6(iii) that
\[
\Lambda_L = \left\{ \bigcup_{a \in \mathcal{G}} \Lambda_L^{L^{\zeta}}(a) \right\} \cup \left\{ \bigcup_{r=1}^{R} \mathcal{Y}_r \right\}.
\] (4.24)

Since \( \omega \) is fixed, we omit it from the notation. Let \( \{ (\psi_\lambda, \lambda) \}_{\lambda \in \sigma(H_{\Lambda_L})} \) be an eigensystem for \( H_{\Lambda_L} \). Given \( a \in \mathcal{G} \), let \( \{ (\varphi_{\lambda(a)}^{(a)}), \lambda^{(a)} \}_{\lambda^{(a)} \in \sigma(H_{\Lambda_L(a)})} \) be an \((m, I)\)-localized eigensystem for \( \Lambda_L(a) \). For \( r = 1, 2, \ldots, R \), let \( \{ (\phi_{\lambda}^{(r)}, \mu_{\lambda}^{(r)}) \}_{\lambda \in \sigma(H_{\mathcal{T}_r})} \) be an eigensystem for \( H_{\mathcal{T}_r} \), and set
\[
\sigma_{\mathcal{T}_r}(H_{\Lambda_L}) = \{ (\tilde{\nu}^{(r)}, \mu_{\lambda}^{(r)}) \in \sigma(H_{\mathcal{T}_r}) \} = \sigma(H_{\Lambda_L}) \setminus \mathcal{G}(H_{\Lambda_L}),
\] (4.25)

where \( \tilde{\nu}^{(r)} \) is given in (3.17), which gives \( \sigma_{\mathcal{T}_r}(H_{\Lambda_L}) \subset \sigma(H_{\Lambda_L}) \setminus \mathcal{G}(H_{\Lambda_L}) \), but the argument actually shows \( \sigma_{\mathcal{T}_r}(H_{\Lambda_L}) \subset \sigma(H_{\Lambda_L}) \setminus \mathcal{G}(H_{\Lambda_L}) \). We also set
\[
\sigma_{\mathcal{G}}(H_{\Lambda_L}) = \bigcup_{r=1}^{R} \sigma_{\mathcal{T}_r}(H_{\Lambda_L}) \subset \sigma(H_{\Lambda_L}) \setminus \mathcal{G}(H_{\Lambda_L}).
\] (4.26)
We claim
\[ \sigma_{I_\ell}(H_{A_L}) \subset \sigma_G(H_{A_L}) \cup \sigma_B(H_{A_L}). \] (4.27)
To see this, suppose we have \( \lambda \in \sigma_{I_\ell}(H_{A_L}) \setminus (\sigma_G(H_{A_L}) \cup \sigma_B(H_{A_L})) \). Since \( A_L \) is level spacing for \( H \), it follows from Lemma 3.8(ii)(c) that
\[ |\psi_\lambda(y)| \leq e^{-m_{2h_L}(\lambda)\ell}\tau \quad \text{for all} \quad y \in \bigcup_{a \in \mathcal{G}} A^\Lambda_{I_\ell,2\ell}(a), \] (4.28)
and it follows from Lemma 3.8(ii) that
\[ |\psi_\lambda(y)| \leq e^{-m_{3h_L}(\lambda)\ell}\tau \quad \text{for all} \quad y \in \bigcup_{r=1}^{R} \Upsilon^\Lambda_{r,2\ell}. \] (4.29)

Using \( \lambda \in I_\ell, (4.24), (4.6), \) and (3.85) we conclude that (note \( m_5 \leq m_2 \))
\[ 1 = \|\psi_\lambda\| \leq e^{-m_{2h_L}(\lambda)\ell}\tau \ (L + 1) \frac{2}{2} \leq e^{-\frac{\ell}{2} + \epsilon(k+\epsilon)\ell}\tau \ (L + 1) \frac{\ell}{2} < 1, \] (4.30)
a contradiction. This establishes the claim.

To finish the proof we need to show that \( \{\psi_\lambda, \lambda \} \in \sigma(H_{A_L}) \) is an \( (M, I_\ell, I) \)-localized eigensystem for \( A_L \), where \( M \) is given in (4.8). We take \( \lambda \in \sigma_{I_\ell}(H_{A_L}) \), so \( h_{I_\ell}(\lambda) > 0 \). In view of (4.27) we consider several cases:

(i) Suppose \( \lambda \in \sigma_G(A_L) \). In this case \( \lambda \in \sigma_{\Lambda(\alpha)}(H_{A_L}) \) for some \( \alpha \in \mathcal{G} \). We pick \( x_\lambda \in \Lambda_1(\alpha) \). In view of (4.24) we consider two cases:

(a) If \( y \in A^\Lambda_{I_\ell,2\ell}(a) \) for some \( a \in \mathcal{G} \) and \( \|y - x_\lambda\| \geq \ell \), we must have \( \Lambda_{I_\ell}(\alpha) \cap \Lambda_{I_\ell}(a) = \emptyset \), so it follows from (3.61) that \( \lambda \notin \sigma(\Lambda_{I_\ell}(a))(H_{A_L}) \), and, since \( R^\Lambda_{\infty}(\lambda) \geq \left\lceil \frac{\ell}{\ell + \ell_\ell} \right\rceil \), (3.85) yields
\[ |\psi_\lambda(y)| \leq e^{-m_{3h_L}(\lambda)\ell}\tau |\psi_\lambda(y)| \quad \text{for some} \quad y \in \partial \Lambda_{I_\ell,2\ell}(a). \] (4.31)
In particular,
\[ \|y - y_1\| \leq \ell - \left\lfloor \frac{\ell + \ell_\ell}{2} \right\rfloor \leq \ell + \ell_\ell + 1 \leq \frac{\ell + 2\ell_\ell}{2}. \] (4.32)

(b) If \( y \in \Upsilon^\Lambda_{r,2\ell} \) for some \( r \in \{1, 2, \ldots, R\} \), and \( \|y - x_\lambda\| \geq \ell + \mathrm{diam} \Upsilon_r \), we must have \( \Lambda_{I_\ell}(\alpha) \cap \Upsilon_r = \emptyset \). It follows from (3.61) that \( \lambda \notin \sigma_{\Upsilon_r}(H_{A_L}) \), and clearly \( \lambda \notin \sigma_{\Upsilon_r}(H_{A_L}) \) in view of (4.25). Thus Lemma 3.8(ii) gives
\[ |\psi_\lambda(y)| \leq e^{-m_{3h_L}(\lambda)\ell}\tau |\psi_\lambda(y)| \quad \text{for some} \quad y_1 \in \partial \Lambda_{I_\ell,2\ell}(\Upsilon_r). \] (4.33)
In particular,
\[ \|y - y_1\| \leq \mathrm{diam} \Upsilon_r. \] (4.34)

(ii) Suppose \( \lambda \notin \sigma_G(A_L) \). Then it follows from (4.27) that we must have \( \lambda_s \in \sigma_{\Upsilon_s}(H_{A_L}) \) for some \( s \in \{1, 2, \ldots, R\} \). We pick \( x_\lambda \in \Upsilon_{s,2\ell} \). In view of (4.24) we consider two possibilities:

(a) If \( y \in A^\Lambda_{I_\ell,2\ell}(a) \) for some \( a \in \mathcal{G} \), and \( \|y - x_\lambda\| \geq \ell + \mathrm{diam} \Upsilon_s \), we must have \( \Lambda_{I_\ell}(a) \cap \Upsilon_s = \emptyset \), and Lemma 3.8(i)(c) yields (4.31).

(b) If \( y \in \Upsilon^\Lambda_{r,2\ell} \) for some \( r \in \{1, 2, \ldots, R\} \), and \( \|y - x_\lambda\| \geq \mathrm{diam} \Upsilon_s + \mathrm{diam} \Upsilon_r \), we must have \( r \neq s \). Thus Lemma 3.8(ii) yields (4.33).
Now consider $y \in \Lambda_L$ such that $\|y - x_\lambda\| \geq L_\tau$. Suppose $|\psi_\lambda(y)| > 0$, since otherwise there is nothing to prove. We estimate $|\psi_\lambda(y)|$ using either (4.31) or (4.33) repeatedly, as appropriate, stopping when we get too close to $x_\lambda$ so we are not in one of the cases described above. (Note that this must happen since $\psi_\lambda(y) > 0$.) We accumulate decay only when we use (4.31), and just use $e^{-m_{x_\lambda}(\Lambda)\ell_\tau}$ repeatedly, as appropriate, stopping when we get too close to $\Lambda_L$.

In view of (4.19), this can be guaranteed by requiring
\[ S = \|y - x_\lambda\| - 5\ell(\gamma - 1)\zeta + 1 + 2\ell \leq \|y - x_\lambda\|. \]

We can thus have
\[ S = \left( \frac{2}{\ell + 2\ell} \right) \left( \|y - x_\lambda\| - 5\ell(\gamma - 1)\zeta + 1 + 2\ell \right) - 1 \]
\[ \geq \frac{2}{\ell + 2\ell} \left( \|y - x_\lambda\| - 5\ell(\gamma - 1)\zeta + 1 + 2\ell \right) - 2 \]
\[ \geq \frac{2}{\ell + 2\ell} \left( \|y - x_\lambda\| - 5\ell(\gamma - 1)\zeta + 1 + 3\ell - 2\ell \right) \]
\[ \geq 2\ell \left( \|y - x_\lambda\| - 6\ell(\gamma - 1)\zeta + 1 \right) \]

Thus we conclude that
\[ |\psi_\lambda(y)| \leq e^{-m_{x_\lambda}(\lambda)} \left( \frac{2}{\ell + 2\ell} \right) \left( \|y - x_\lambda\| - 6\ell(\gamma - 1)\zeta + 1 \right) \leq e^{-Mh_1(\lambda)}\|y - x_\lambda\| \]
(4.38)

where
\[ M \geq m_3 \left( 1 - C_{d,m_\epsilon} \ell^{-\min\{1 - \zeta, \gamma - (\gamma - 1)\zeta - 1\}} \right) \]
(4.39)
\[ = m_3 \left( 1 - C_{d,m_\epsilon} \ell^{-\min\{\gamma - (\gamma - 1)\zeta - 1\}} \right) \]
\[ \geq m \left( 1 - C_{d,m_\epsilon} \ell^{-\min\{\gamma - (\gamma - 1)\zeta - 1\}} \right) = m \left( 1 - C_{d,m_\epsilon} \ell^{-\zeta} \right), \]
where we used (4.7), (4.20), and (1.6). In particular, $M$ satisfies (4.3) for sufficiently large $\ell$.

We conclude that $\{\psi_\lambda(\Lambda)\}_\lambda \in c(\Lambda_L)$ is an $(M, I_\epsilon, I)$-localized eigensystem for $\Lambda_L$, where $M$ satisfies (4.3), so the box $\Lambda_L$ is $(M, I_\epsilon, I)$-localizing for $H_\omega$. \hfill \Box

**Proof of Proposition 4.3.** We assume (4.1) and (4.2) and set $L_{k+1} = L_k$, $A_{k+1} = A_k(1 - L_k^{-\kappa'})$, and $I_{k+1} = (E - A_{k+1}, E + A_{k+1})$ for $k = 0, 1, \ldots$. Since if a box $\Lambda_L$ is $(M, I_\epsilon, I)$-localizing for $H_\omega$ it is also $(M, I_\epsilon)$-localizing, if $L_0$ is sufficiently large it follows from Lemma 4.2 by an induction argument that for all $k = 1, 2, \ldots$ we have (4.3) and (4.4). \hfill \Box

**Proposition 4.3.** Fix $m_\epsilon > 0$. There exists a a finite scale $\mathcal{L} = \mathcal{L}(d, m_\epsilon)$ with the following property: Suppose for some scale $L_0 \geq \mathcal{L}$ we have
\[ \inf_{x \in \mathbb{R}^d} \mathbb{P} \{ \Lambda_{L_0}(x) \text{ is } (m_0, I_0)\text{-localizing for } H_\omega \} \geq 1 - e^{-L_0}, \]
where $I_0 = (E - A_0, E + A_0) \subset \mathbb{R}$, with $E \in \mathbb{R}$ and $A_0 > 0$, and
\[ m_{-L_0^{-\kappa'}} \leq m_0 \leq \frac{1}{2} \log \left( 1 + \frac{\log \tau}{\tau} \right). \]
(4.41)
Set $L_{k+1} = L_k^\ast$, $A_{k+1} = A_k (1 - L_k^{-\kappa})$, and $I_{k+1} = (E - A_{k+1}, E + A_{k+1})$, for $k = 0, 1, \ldots$. Then for all $k = 1, 2, \ldots$ we have

$$\inf_{x \in \mathbb{R}^d} \mathbb{P} \{ \Lambda_L(x) is (m_k, I_k, I_{k-1})-localizing for H_\omega \} \geq 1 - e^{-L^\xi} \text{ for } L \in [L_k, L_{k+1}).$$

(4.42)

where

$$m_{-} L_k^{-\kappa'} < m_{k-1} (1 - C_{d,m_-} L_k^{-\theta}) \leq m_k < \frac{1}{2} \log \left( 1 + \frac{1}{L^{\beta}} \right),$$

(4.43)

with $C_{d,m_-}$ as in (4.1).

Proof. We can apply Proposition 4.1, so we have $L_0 \geq L$, so we have the conclusions of Proposition 4.1.

Given a scale $L \geq L_1$, let $k = k(L) \in \{1, 2, \ldots \}$ be defined by $L_k \leq L < L_{k+1}$. We have $L_k = L_k^0 \leq L < L_{k+1} = L_k^\gamma$, so $L = L_k^\gamma$ with $\gamma \leq \gamma' < \gamma^2$. We proceed as in Lemma 4.2. We take $\Lambda_L = \Lambda_L(x_0)$, where $x_0 \in \mathbb{R}^d$, and let $C_{L,L_{k-1}} = C_{L_kL_{k-1}} (x_0)$ be the suitable $L_{k-1}$-cover of $\Lambda_L$. We let $B_0$ denote the event that all boxes in $C_{L,L_{k-1}}$ are $(m_{k-1}, I_{k-1})$-localizing for $H_\omega$. It follows from (3.109) and (4.3) that

$$\mathbb{P} \{ B_0 \} \leq \left( \frac{2L}{\xi L_k^\gamma} \right)^d e^{-L_{k-1}^\gamma} = 2^d L_k^{-\kappa} e^{-L_{k-1}^\gamma} \leq 2^d L^{(1 - \gamma')d} e^{-L_{k-1}^\gamma} \leq \frac{1}{2} e^{-L_{k-1}^\gamma},$$

(4.44)

if $L_0$ is sufficiently large, since $\xi \gamma^2 < \zeta$. Moreover, letting $S_0$ denote the event that the box $\Lambda_L$ is level spacing for $H_\omega$, it follows from Lemma 3.12 that

$$\mathbb{P} \{ S_0 \} \leq Y_L e^{-(2\alpha - 1) L^\beta} (L + 1)^{2d} \leq \frac{1}{2} e^{-L_{k-1}^\gamma},$$

(4.45)

if $L_0$ is sufficiently large, since $\xi < \beta$. Thus, letting $E_0 = B_0 \cap S_0$, we have

$$\mathbb{P} \{ E_0 \} \geq 1 - e^{-L_{k-1}^\gamma}.$$

(4.46)

It only remains to prove that $\Lambda_L$ is $(m_k, I_k, I_{k-1})$-localizing for $H_\omega$ for all $\omega \in E_0$. To do so, we fix $\omega \in E_0$ and proceed as in the proof of Lemma 4.2. Since $\omega \in E_0$, we have $G = G(\omega) = \mathbb{R} \setminus L_{k-1}$. Since $\omega$ is now fixed, we omit them from the notation. As in the proof of Lemma 4.2, we get, noticing that $(I_{k-1})_{L_{k-1}} = I_k$,

$$\sigma_{I_k}(H_{\Lambda_L}) \subset \sigma_{G}(H_{\Lambda_L}),$$

(4.47)

similarly to 4.27.

Let $\{ (\psi_\lambda, \lambda) \}_{\lambda \in \sigma(H_{\Lambda_L})}$ be an eigensystem for $H_{\Lambda_L}$. To finish the proof we need to show that the eigensystem is $(m_k, I_k, I_{k-1})$-localized eigensystem for $\Lambda_L$. Let $\lambda \in \sigma_{I_k}(H_{\Lambda_L})$, then by (4.17) we have we have $\lambda \in \sigma_{G}(H_{\Lambda_L})$, and hence $\lambda \in \sigma_{\{ \Lambda_{L-1}(a) \}}(H_{\Lambda_L})$ for some $a \in G$. If $y \in \Lambda_L$ and $|y - x_\lambda| \geq 2L_{k-1}$, it follows from (3.110) that $y \in \Lambda_L^{\Lambda_{L-1}} \{ \frac{L_{k-1} - L_{k-1}^\gamma}{2} \} (a)$ for some $a \in G$, and moreover $\Lambda_{L-1}(a) \cap \Lambda_{L-1}(a) = \emptyset$, so it follows from (3.61) that $\lambda \notin \sigma_{\{ \Lambda_{L-1}(a) \}}(H_{\Lambda_L})$, and, since $R_y^{\partial H_{\Lambda_{L-1}}(a)} \geq \left[ \frac{L_{k-1} - L_{k-1}^\gamma}{2} \right]$, (3.50) yields

$$|\psi_\lambda(y)| \leq e^{-m_{k-1,3}h_{L_{k-1}}(\lambda) \left[ \frac{L_{k-1} - L_{k-1}^\gamma}{2} \right]} \| \psi_\lambda \| \text{ for some } y_1 \in \partial \Lambda_{L-1}(a),$$

(4.48)
where we need
\[ m_{k-1,3} = m_{k-1,3}(L_{k-1}) \geq m_{k-1} \left( 1 - C_{d,m} L_{k-1}^{-\frac{1}{3}} \right), \]
and we have
\[ \|y - y_1\| \leq \frac{L_{k-1} + 2L_{k-1}^5}{2}, \]
as in (4.32).

Now consider \( y \in \Lambda_L \) such that \( \|y - x_\lambda\| \geq L_\sigma \). Suppose \( |\psi_\lambda(y)| > 0 \), since otherwise there is nothing to prove. We estimate \( |\psi_\lambda(y)| \) using either (4.48) repeatedly, as appropriate, stopping when we get within \( 2L_{k-1} \) of \( x_\lambda \). In view of (4.50), we can use (4.48) \( S \) times, as long as
\[ \frac{L_{k-1} + 2L_{k-1}^5}{2} S + 2L_{k-1} \leq \|y - x_\lambda\|. \]

We can thus have
\[ S = \left[ \frac{2}{L_{k-1} + 2L_{k-1}^5} (\|y - x_\lambda\| - 2L_{k-1}) \right] - 1 \]

\[ \geq \left[ \frac{2}{L_{k-1} + 2L_{k-1}^5} (\|y - x_\lambda\| - 3L_{k-1} - 2L_{k-1}^5) \right] \geq \frac{2}{L_{k-1} + 2L_{k-1}^5} (\|y - x_\lambda\| - 4L_{k-1}). \]

Thus we conclude that
\[ |\psi_\lambda(y)| \leq e^{-m_{k-1,3}h_{k-1}(\lambda)} \left[ \frac{L_{k-1} - L_{k-1}^5}{L_{k-1} + 2L_{k-1}^5} \right] (\|y - x_\lambda\| - 4L_{k-1}) \]

\[ \leq e^{-m_{k-1,3}h_{k-1}(\lambda)} \|y - x_\lambda\| \]

where \( m_k \) can be taken the same as in (4.4).

We conclude that \( \{(|\psi_\lambda, \lambda|)_{\lambda \in \sigma(H_{\lambda_L})}\} \) is an \((m_k, I_k, I_{k-1})\)-localized eigensystem for \( \Lambda_L \), where \( m_k \) satisfies (4.4), so the box \( \Lambda_L \) is \((m_k, I_k, I_{k-1})\)-localizing for \( H_\omega \).

**Proof of Theorem 1.6** Let \( L_{k+1} = L_{k}^\gamma, A_{k+1} = A_k(1-L_k^\kappa), I_{k+1} = (E - A_{k+1}, E + A_{k+1}) \), and \( m_{k+1} = m_k \left( 1 - C_{d,m} L_k^{-\frac{1}{3}} \right) \) for \( k = 0, 1, \ldots \). Given \( L \geq L_0^\gamma = L_1 \), let \( k = k(L) \in \{1, 2, \ldots \} \) be defined by \( L_k \leq L < L_{k+1} \). Since
\[ A_{k+1} \left( 1 - L_k^{-\frac{1}{3}} \right)^{-1} < A_{k-1} \implies I_{k+1}^+ \subset I_{k-1}, \]
we conclude that (1.20) follows from (4.42).

5. Localization

In this section we consider an Anderson model \( H_\omega \) and prove Theorem 1.7 and Corollary 1.8.

**Lemma 5.1.** Fix \( m_- > 0 \), let \( A > 0 \), and \( I = (E - A, E + A) \). There exists a finite scale \( L_{d,v,m_-} \) such that for all \( L \geq L_{d,v,m_-}, \) \( a \in \mathbb{Z}^d \), letting \( L = L_0^\gamma \), given an \((m, I, I^\ell)\)-localizing box \( \Lambda_L(a) \) for the discrete Schrödinger operator \( H \), where \( m \) satisfies (4.32), then for all \( \lambda \in I \),
\[ \max_{b \in \Lambda_L(a)} W_\lambda(b) > e^{-\frac{1}{4}m_{b,\ell}(\lambda)L} \implies \min_{\theta \in \sigma_{\ell,\ell}(H_{\Lambda_L(a)})} |\lambda - \theta| < \frac{1}{2}e^{-L^\beta}. \]
Proof. Note that
\[ I \subset I_L^I \subset I_L^I \quad \text{and} \quad \inf_{\lambda \in I} h_{\ell, I} (\lambda) \geq L^{-\beta}. \] (5.2)
Now let \( \lambda \in I \subset I_L^I \), and suppose \( |\lambda - \theta| \geq \frac{\beta}{4} e^{-L^\beta} \) for all \( \theta \in \sigma_{I, I} (H_{I_L^I(a)}) \). Let \( \psi \in V(\lambda) \). Then it follows from Lemma 5.3(ii) that for large \( L \) and \( b \in \Lambda_{\ell, I}(a) \) we have
\[ |\psi(b)| \leq e^{-m_3 \beta_{\ell, I}(\lambda) (\frac{\beta}{4} - \frac{1}{2})} \| T_{a}^{-1} \psi \| \left( \frac{L}{2} + 1 \right)^{\nu} \leq e^{-m_3 \beta_{\ell, I}(\lambda) L} \| T_{a}^{-1} \psi \|. \] (5.3)
\[
\square
\]

Proof of Theorem 1.7. Assume Theorem 1.6 holds for some \( L_0 \), and let \( I = I_{\infty} \), \( m = m_{\infty} \). Consider \( L_0 \leq L \leq 2N \) and \( a \in \mathbb{Z}^2 \). We have
\[ \Lambda_{\ell, I}(a) = \bigcup_{b \in \{ a + \frac{1}{4} L \mathbb{Z}^d \}} \Lambda_{\ell, I}(b). \] (5.4)
Let \( \mathcal{Y}_{L, a} \) denote the event that \( \Lambda_{\ell, I}(a) \) is level spacing for \( H_{\omega} \) and the boxes \( \Lambda_{\ell, I}(b) \) are \( (m, I, I^I) \)-localizing for \( H_{\omega} \) for all \( b \in \{ a + \frac{1}{4} L \mathbb{Z}^d \} \) with \( \| b - a \| \leq 2L \), where \( L = \ell^7 \). It follows from \( (1.20) \) and Lemma 3.12 that
\[ \mathbb{P} \{ \mathcal{Y}_{L, a} \} \leq 5^{d-2} e^{-L^\beta} + Y_\mu (5L + 1)^{2d} e^{-(2a-1)(5L)^\beta} \leq C \mu e^{-L^\beta}. \] (5.5)
Suppose \( \omega \in \mathcal{Y}_{L, a} \), \( \lambda \in I \), and \( \max_{b \in \Lambda_{\ell, I}(a)} W^{(a)}_{\omega, \lambda}(b) > e^{-\frac{\beta}{4} \mu_{\ell, I} L} \). It follows from Lemma 5.1 that \( \min_{\theta \in \sigma_{I, I} (H_{\ell, I}(a))} |\lambda - \theta| < \frac{\beta}{4} e^{-L^\beta} \). Since \( \Lambda_{\ell, I}(a) \) is level spacing for \( H_{\omega} \), using Lemma 5.3(i) we conclude that
\[ \min_{\theta \in \sigma_{I, I} (H_{\ell, I}(b))} |\lambda - \theta| \geq e^{-(5L)^\beta} - 2e^{-m_1 \beta_{\ell, I}(\lambda) L^\beta} - \frac{\beta}{4} e^{-L^\beta} \] (5.6)
\[ \geq e^{-(5L)^\beta} - 2e^{-m_1 L^\beta} - \frac{\beta}{4} e^{-L^\beta} \geq \frac{\beta}{4} e^{-L^\beta} \]
for all \( b \in \{ a + \frac{1}{4} L \mathbb{Z}^d \} \) with \( L \leq \| b - a \| \leq 2L \). Since
\[ A_{\ell, I}(a) \subset \bigcup_{b \in \{ a + \frac{1}{4} L \mathbb{Z}^d \} \cap \{ L \leq \| b - a \| \leq 2L \}} \Lambda_{\ell, I}(b), \] (5.7)
it follows from Lemma 5.3(ii) that for all \( y \in A_{\ell, I}(a) \) we have, given \( \psi \in V_{\omega, \lambda} \),
\[ |\psi(y)| \leq e^{-m_3 \beta_{\ell, I}(\lambda) (\frac{\beta}{4} - 2)} \| T_{a}^{-1} \psi \| \left( \frac{L}{2} + 1 \right)^{\nu} \leq e^{-m_3 \beta_{\ell, I}(\lambda) L} \| T_{a}^{-1} \psi \| \] (5.8)
\[ \leq e^{-\frac{\beta}{4} \mu_{\ell, I}(\lambda) \| y - a \|} \| T_{a}^{-1} \psi \|, \]
so we get
\[ W^{(a)}_{\omega, \lambda}(y) \leq e^{-\frac{\beta}{4} \mu_{\ell, I}(\lambda) \| y - a \|} \text{ for all } y \in A_{\ell, I}(a). \] (5.9)
Since we have \( (1.23) \) we can conclude that for \( \omega \in \mathcal{Y}_{L, a} \) we always have
\[ W^{(a)}_{\omega, \lambda}(y) \leq \max \left\{ e^{-\frac{\beta}{4} \mu_{\ell, I}(\lambda) \| y - a \|}, e^{-\frac{\beta}{4} \mu_{\ell, I}(\lambda) \| y - a \|} \right\} \leq e^{-\frac{\beta}{4} \mu_{\ell, I}(\lambda) \| y - a \|} \text{ for all } y \in A_{\ell, I}(a). \] (5.10)
\[
\square
\]
Proof of Corollary [1.28] Parts (i) and (ii) are proven in the same way as [1.26], Theorem 7.1(i)-(ii), using $h_{yL} \geq h_1$ for all $L > 1$.

Part (iii) is proven similarly to [1.24] and [1.26]. We use the fact that for any $L_0 \in 2\mathbb{N}$, setting $L_{k+1} = 2L_k$ for $k = 0, 1, 2, \ldots$, we have (recall (1.26))

$$
Z^d = \Lambda_{L_k} \cup \bigcup_{j=k}^{\infty} A_{L_j} \quad \text{for} \quad k = 0, 1, 2, \ldots
$$

(5.11)

Given $k \in \mathbb{N}$, we set $L_k = 2^k$, and consider the event

$$
\mathcal{Y}_k := \bigcap_{x \in Z^d; ||x|| \leq 2^k \frac{L_k}{4}} \mathcal{Y}_{L_k, x},
$$

(5.12)

where $\mathcal{Y}_{L_k, x}$ is the event given in Theorem [1.7]. It follows from (1.24) that for sufficiently large $k$ we have

$$
P\left\{ \mathcal{Y}_k \right\} \geq 1 - C \left( 2e \frac{L_k}{4} + 1 \right)^d e^{-\frac{L_k}{4}} \geq 1 - 3^d C e^{-\frac{L_k}{4}},
$$

(5.13)

so we conclude from the Borel-Cantelli Lemma that

$$
P\left\{ \mathcal{Y}_\infty \right\} = 1, \quad \text{where} \quad \mathcal{Y}_\infty = \liminf_{k \to \infty} \mathcal{Y}_k.
$$

(5.14)

We now fix $\omega \in \mathcal{Y}_\infty$, so there exists $k_\omega \in \mathbb{N}$ such that $\omega \in \mathcal{Y}_{L_k, x}$ for all $k_\omega \leq k \in \mathbb{N}$ and $x \in Z^d$ with $||x|| \leq 2^k \frac{L_k}{4}$. We set $k_\omega' = \max \{ k_\omega, 2 \}$. Given $x \in Z^d$, we define $k_x \in \mathbb{N}$ by

$$
e^{\frac{1}{4} \frac{L_k}{2}} \leq ||x|| \leq e^{\frac{1}{4} \frac{L_k}{2}} \quad \text{if} \quad k_x \geq 2,
$$

(5.15)

and set $k_x = 1$ otherwise. We set $k_{\omega, x} = \max \{ k_\omega', k_x \}$

Let $x \in Z^d$. If $y \in B_{\omega, x} = \bigcup_{k=k_{\omega, x}} A_{L_k}(x)$, we have $y \in A_{L_{k_1}}(x)$ for some $k_1 \geq k_{\omega, x}$ and $\omega \in \mathcal{Y}_{L_{k_1}, x}$, so it follows from (1.27) that

$$
W^{(x)}_{\omega, \lambda}(x)W^{(x)}_{\omega, \lambda}(y) \leq e^{-\frac{1}{\nu} m_{h_1}(\lambda)||y-x||}
$$

(5.16)

for all $\lambda \in I$. If $y \notin B_{\omega, x}$, we must have $||y-x|| < \frac{1}{2} e^{\frac{1}{4} \frac{L_k}{2}}$, so for all $\lambda \in \mathbb{R}$, using (1.23) and (5.15),

$$
W^{(x)}_{\omega, \lambda}(x)W^{(x)}_{\omega, \lambda}(y) = W^{(x)}_{\omega, \lambda}(x)W^{(x)}_{\omega, \lambda}(y) e^{\frac{1}{\nu} m_{h_1}(\lambda)||y-x||} e^{-\frac{1}{\nu} m_{h_1}(\lambda)||y-x||}
$$

(5.17)

$$
\leq \langle y-x \rangle^{\nu} e^{\frac{1}{\nu} m_{h_1}(\lambda)||y-x||} e^{-\frac{1}{\nu} m_{h_1}(\lambda)||y-x||}
$$

$$
\leq \langle \frac{1}{2} e^{\frac{1}{4} \frac{L_k}{2}} \rangle^{\nu} e^{\frac{1}{\nu} m_{h_1}(\lambda)||y-x||} e^{-\frac{1}{\nu} m_{h_1}(\lambda)||y-x||}
$$

$$
\leq \left\{ \begin{array}{ll}
\left( \frac{1}{2 \frac{L_k}{2}} \right)^{\nu} e^{\frac{1}{\nu} m_{h_1}(\lambda)||y-x||} e^{-\frac{1}{\nu} m_{h_1}(\lambda)||y-x||} & \text{if} \ k_{\omega, x} = k_x
\end{array} \right.
$$

(5.18)
Combining (5.16) and (5.17), noting \( |x|^{2d} > e \) if \( k_x \geq 2 \), and \( h_I(\lambda) \leq 1 \), we conclude that for all \( \lambda \in I \) with \( h_I(\lambda) > 0 \) and \( x, y \in \mathbb{Z}^d \) we have
\[
W_{\omega,\lambda}(x)W_{\omega,\lambda}(y) \leq C_{m,\omega,\nu} \left( (2d \log \langle x \rangle)^{\frac{1}{2}} \right) e^{\frac{1}{132} m h_I(\lambda)(2d \log \langle x \rangle)} e^{-\frac{1}{132} m h_I(\lambda) \|y-x\|} \leq C_{\omega,\lambda} \left( m h_I(\lambda) \right)^{-\nu} e^{\frac{1}{132} m h_I(\lambda)(2d \log \langle x \rangle)} e^{-\frac{1}{132} m h_I(\lambda) \|y-x\|},
\]
which is (1.29).

Part (iv) follows from (iii), since (1.29) implies
\[
|\psi(x)| \leq C_{m,\omega,\nu} \left( h_I(\lambda) \right)^{-\nu} \left( T^{-1}_x \psi \right)^2 e^{\frac{1}{132} m h_I(\lambda)(2d \log \langle x \rangle)} e^{-\frac{1}{132} m h_I(\lambda) \|y-x\|} \leq C_{m,\omega,\nu} \left( h_I(\lambda) \right)^{-\nu} \left( T^{-1}_0 \psi \right)^2 e^{\frac{1}{132} m h_I(\lambda)(2d \log \langle x \rangle)} e^{-\frac{1}{132} m h_I(\lambda) \|y-x\|},
\]
for all \( x, y \in \mathbb{Z}^d \), which is (1.30).

Part (v) similarly follows from (iii) using the discrete equivalent of [GK3] Eq. (4.22).

6. Connection with the Green’s functions multiscale analysis

Let \( H_\Theta \) be an Anderson model. Given \( \Theta \subset \mathbb{Z}^d \) finite and \( z \notin \sigma(H_\Theta) \), we set
\[
G_\Theta(z) = (H_\Theta - z)^{-1} \quad \text{and} \quad G_\Theta(z;x,y) = \langle \delta_z, (H_\Theta - z)^{-1}\delta_y \rangle \quad \text{for} \quad x, y \in \Theta.
\]

**Definition 6.1.** Let \( E \in \mathbb{R} \) and \( m > 0 \). A box \( \Lambda_L \) is said to be \((m,E)\)-regular if \( E \notin \sigma(H_{\Lambda_L}) \) and
\[
|G_{\Lambda_L}(E;x,y)| \leq e^{-m \|x-y\|} \quad \text{for all} \quad x, y \in \Lambda_L \quad \text{with} \quad \|x-y\| \geq \frac{L}{100}.
\]

The following theorem is a typical result from the Green’s function multiscale analysis. [ProS] [ProMSS] [DrK] [GK1] [Kl].

**Theorem 6.2.** Let \( J \subset \mathbb{R} \) be a bounded open interval, \( 0 < \xi < \zeta < 1 \), and \( m_0 > 0 \). Suppose for some scale \( L_0 \) we have
\[
\inf_{x \in \mathbb{R}^d} \mathbb{P} \{ \Lambda_{L_0}(x) \text{ is } (m,\lambda)\text{-regular} \} \geq 1 - e^{-L_0^\xi} \quad \text{for all} \quad \lambda \in J.
\]
Then, given \( m \in (0,m_0) \), if \( L_0 \) is sufficiently large, we have
\[
\inf_{x \in \mathbb{R}^d} \mathbb{P} \{ \Lambda_L(x) \text{ is } (m,\lambda)\text{-regular} \} \geq 1 - e^{-L^\xi} \quad \text{for all} \quad \lambda \in J,
\]
and
\[
\inf_{x, y \in \mathbb{R}^d} \mathbb{P} \{ \text{for all } \lambda \in J \text{ either } \Lambda_L(x) \text{ or } \Lambda_L(y) \text{ is } (m,\lambda)\text{-regular} \} \geq 1 - e^{-L^\xi}.
\]

Here (6.4) are the conclusions of the single energy multiscale analysis, and (6.5) are the conclusions of the energy interval multiscale analysis.

Given a bounded open interval \( J \) and \( m > 0 \), we call a box \( \Lambda_L \) \((m,J)\)-uniformly localizing for \( H \) if \( \Lambda_L \) is level spacing for \( H \), and there exists an eigensystem
\((\varphi_\nu, \nu)\) for \(H_{\Lambda_L}\) such that for all \(\nu \in \sigma_f(H_{\Lambda_L})\) there is \(x_\nu \in \Lambda_L\) such that \(\varphi_\nu\) is \((x_\nu, m)\)-localized. Note that if \(\Lambda_L\) is \((m, J)\)-localizing for \(H\) (as in Definition \(\ref{def:localization}\)), it follows from \(\ref{def:localization}\) that \(\Lambda_L\) is \((mr^{-\kappa}, J_r)\)-uniformly localizing for \(H\) for all \(r > 1\).

**Proposition 6.3.** Let \(J \subset \mathbb{R}\) be a bounded open interval, \(0 < \xi' < \xi < 1\), and \(m > 0\). Suppose there exists \(\mathcal{L}\) such that the Anderson model \(H_\omega\) satisfies \(\ref{eq:Anderson1}\) for all \(L \geq \mathcal{L}\). Then, given \(m' \in (0, m)\), for sufficiently large \(L\) we have

\[
\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_L(x) \text{ is } (m', J)\text{-uniformly localizing for } H_\omega\} \geq 1 - e^{-L^\xi'}. \tag{6.6}
\]

Proposition 6.3 is proved exactly as the analogous result in \(\text{[EK, Proposition 6.4]}\).

We now show that the conclusions of Theorem \(\ref{thm:main}\) imply a result similar to the conclusions of Theorem \(\ref{thm:approximation}\).

**Lemma 6.4.** Fix \(m_+ > 0\). Let \(I = (E - A, E + A) \subset \mathbb{R}\), with \(E \in \mathbb{R}\) and \(A > 0\), and \(m > 0\). Suppose that \(\Lambda_L\) is \((m, I)\)-localizing for \(H\), where

\[
m_-L^{-\kappa} \leq m \leq \frac{1}{2} \log \left(1 + \frac{1}{d_m}\right). \tag{6.7}
\]

Then, for sufficiently large \(L\), \(\Lambda_L\) is \((m''h_I(\lambda), \lambda)\)-regular for all \(\lambda \in I_L\) with \(\text{dist} \{\lambda, \sigma(H_{\Lambda_L})\} \geq e^{-L^\beta}\), where

\[
m'' \geq m \left(1 - C_{d,m}L^{-(1-\tau)}\right). \tag{6.8}
\]

**Proof.** We take \(E = 0\) by replacing the potential \(V\) by \(V - E\).

Let \(\lambda \in I\) with \(\text{dist} \{\lambda, \sigma(H_{\Lambda_L})\} \geq e^{-L^\beta}\). For all \(t > 0\) we have

\[
G_{\Lambda_L}(\lambda) = (H_{\Lambda_L} - \lambda)^{-1} = F_{t,\lambda}(H_{\Lambda_L}) + (H_{\Lambda_L} - \lambda)^{-1}e^{-t(H_{\Lambda_L}^2 - \lambda^2)} \tag{6.9}
\]

where the function \(F_{t,\lambda}\) is defined in \(\ref{eq:F_t,lambda}\).

Let \(\nu \in \sigma_f(H_{\Lambda_L})\) and \(x, y \in \Lambda_L\) with \(\|x - y\| \geq 100m\). In this case either \(\|x - x_\nu\| \geq L_\tau\) or \(\|y - x_\nu\| \geq L_\tau\). Say \(\|x - x_\nu\| \geq L_\tau\), then

\[
|\varphi_\nu(x)|\varphi_\nu(y) \leq \begin{cases} e^{-mh_I(\nu)(\|x-x_\nu\|+\|y-x_\nu\|)} & \text{if } \|y - x_\nu\| \geq L_\tau \\ e^{-mh_I(\nu)(\|x-x_\nu\|)} & \text{if } \|y - x_\nu\| < L_\tau \end{cases}
\]

so we conclude that

\[
|\varphi_\nu(x)|\varphi_\nu(y) \leq e^{-m'\xi h_I(\nu)}\|x-y\|, \quad \text{where } m' \geq m(1 - 100L_\tau^{-1}). \tag{6.11}
\]

Now let \(P_t = \chi_I(H_{\Lambda_L}), \ P_t = 1 - P_t\). Since

\[
\langle \delta_x, (H_{\Lambda_L} - \lambda)^{-1}e^{-t(H_{\Lambda_L}^2 - \lambda^2)}P_t\delta_y \rangle = \sum_{\mu \in \sigma_f(H_{\Lambda_L})} (\mu - \lambda)^{-1}e^{-t(\mu^2 - \lambda^2)}\varphi_\mu(x)\varphi_\mu(y),
\]

it follows from \(\ref{eq:delta_x}\) that

\[
\left|\langle \delta_x, (H_{\Lambda_L} - \lambda)^{-1}e^{-t(H_{\Lambda_L}^2 - \lambda^2)}P_t\delta_y \rangle\right| \leq e^{L^\beta} \sum_{\mu \in \sigma_f(H_{\Lambda_L})} e^{-t(\mu^2 - \lambda^2)}|\varphi_\mu(x)|\varphi_\mu(y)| \leq e^{L^\beta} \sum_{\mu \in \sigma_f(H_{\Lambda_L})} e^{-t(\mu^2 - \lambda^2)}e^{-m'\xi h_I(\mu)}\|x-y\|. \tag{6.13}
\]
We now take
\[ t = \frac{m'[\|x-y\|]}{A^2} \implies e^{-t(\mu^2 - \lambda^2)}e^{-m' h_I(\mu)\|x-y\|} = e^{-m' h_I(\lambda)\|x-y\|} \quad \text{for } \mu \in I, \] (6.14)

obtaining
\[ \left| \left\langle \delta_x, (H_{\Lambda_L} - \lambda)^{-1}e^{-t(H^2_{\Lambda_L} - \lambda^2)}P_\delta y \right\rangle \right| \leq (L + 1)d e^{L\beta} e^{-m' h_I(\lambda)\|x-y\|}. \] (6.15)

It follows from Lemma 3.3 that
\[ \left| \left\langle \delta_x, (H_{\Lambda_L} - \lambda)^{-1}e^{-t(H^2_{\Lambda_L} - \lambda^2)}\bar{P}_\delta y \right\rangle \right| \leq e^{L\beta} e^{-m' h_I(\lambda)\|x-y\|}, \] (6.16)

so
\[ \left| \left\langle \delta_x, (H_{\Lambda_L} - \lambda)^{-1}e^{-t(H^2_{\Lambda_L} - \lambda^2)}\delta_y \right\rangle \right| \leq 2(L + 1)d e^{L\beta} e^{-m' h_I(\lambda)\|x-y\|}. \] (6.17)

It follows from (3.9), using (6.7), that
\[ \left| \left\langle \delta_x, F_{\frac{m'[x-y]}{A^2}}(H_{\Lambda_L})\delta_y \right\rangle \right| \leq 70A^{-1}e^{-m' h_I(\lambda)\|x-y\|} \leq 70m^{-1}L^{\kappa'} e^{-m' h_I(\lambda)\|x-y\|}. \] (6.18)

Combining (6.9), (6.17) and (6.18), we get
\[ |G_{\Lambda_L}(\lambda; x, y)| \leq \left( 70m^{-1}L^{\kappa'} + 2(L + 1)d e^{L\beta} \right) e^{-m' h_I(\lambda)\|x-y\|}. \] (6.19)

We now require \( \lambda \in I_L \), obtaining
\[ |G_{\Lambda_L}(\lambda; x, y)| \leq e^{-m' h_I(\lambda)\|x-y\|}, \] (6.20)

where
\[ m'' \geq m' \left( 1 - C_{d,m,-L^{-(1-\beta-\kappa')}} \right) \]
\[ \geq m \left( 1 - C_{d,m,-L^{-\min\{1-\tau,1-\beta-\kappa'\}}} \right) = m \left( 1 - C_{d,m,-L^{-(1-\tau)}} \right). \] (6.21)

\[ \square \]

**Proposition 6.5.** Suppose the conclusions of Theorem 4.4 hold for an Anderson model \( H_\omega \), and let \( I = I_\infty \), \( m = m_\infty \). Then, given \( 0 < \zeta' < \xi \), there exists a finite scale \( \zeta_1 \) such that for all \( L \geq \zeta_1 \) we have
\[ \inf_{x \in \mathbb{R}^d} \mathbb{P} \left\{ \Lambda_L(x) \text{ is (}m''h_I(\lambda), \lambda\text{)-regular} \right\} \geq 1 - e^{-L^\zeta'} \text{ for all } \lambda \in I_L, \] (6.22)

and
\[ \inf_{x,y \in \mathbb{R}^d \atop \|x-y\| > L} \mathbb{P} \left\{ \text{for } \lambda \in I_L \text{ either } \Lambda_L(x) \text{ or } \Lambda_L(y) \text{ is (}m''h_I(\lambda), \lambda\text{)-regular} \right\} \geq 1 - e^{-L^\zeta'}, \] (6.23)

where \( m'' \) is given in (6.8).
Proof. Suppose the conclusions of Theorem 1.6 hold for an Anderson model $H_\omega$, and let $I = I_\infty$, $m = m_\infty$, and let $L \geq L_0^\beta$. Since the Wegner estimate gives (see Lemma 3.12 for the notation)

$$\mathbb{P}\left\{ \|G_{A_L}(\lambda)\| \leq e^{L^\beta} \right\} \geq 1 - \tilde{K} \alpha e^{-\alpha L^\beta} (L + 1)^d \geq 1 - \frac{1}{2} e^{-L^{\beta'}}$$

for large $L$, it follows from (1.20) and Lemma 6.3 that for $L$ large we have (6.22).

Now consider two boxes $\Lambda_L(x_1)$ and $\Lambda_L(x_2)$, where $x_1, x_2 \in \mathbb{R}^d$, $\|x_1 - x_2\| > L$. Define the events

$$A = \{ \Lambda(x_1) \text{ and } \Lambda(x_2) \text{ are both } (m, I)\text{-localizing for } H_\omega \},$$

$$B = \{ \text{dist}(\sigma(\Lambda_L(x_1)), \sigma(\Lambda_L(x_2))) \geq 2e^{-L^\beta} \}$$

Since $\|x_1 - x_2\| > L$, the boxes are disjoint, so it follows from (1.20) that

$$\mathbb{P}\{A\} \geq 1 - 2e^{-L^{\beta}} \geq 1 - \frac{1}{2} e^{-L^{\beta'}}$$

and the Wegner estimate between boxes gives

$$\mathbb{P}\{B\} \geq 1 - \tilde{K} \alpha e^{-\alpha L^\beta} (L + 1)^d \geq 1 - \frac{1}{2} e^{-L^{\beta'}}$$

so we have

$$\mathbb{P}\{A \cap B\} \geq 1 - e^{-L^{\beta'}}.$$

Moreover, for $\omega \in A \cap B$ and $\lambda \in \mathbb{R}$, the boxes $\Lambda(x_1)$ and $\Lambda(x_2)$ are both $(m, I)$-localizing, and we must have either $\|G_{A_L(x_1)}(\lambda)\| \leq e^{L^\beta}$ or $\|G_{A_L(x_2)}(\lambda)\| \leq e^{L^\beta}$, so for $\lambda \in I_L$ the previous argument shows that either $\Lambda(x_1)$ or $\Lambda(x_2)$ is $(m''h_I(\lambda), \lambda)$-regular for large $L$. We proved (6.23). 

\[ \square \]

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