Emergent gravity in superplastic crystals and cosmological constant problem

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We discuss emergent gravity in the superplastic crystals. We restrict ourselves by the consideration of the gapped (massive) fermions coupled to gravity. In this approach the stress-energy tensor may be defined in such a way, that being integrated over the whole volume it is a topological invariant. It is not changed, when the system is modified smoothly. The cosmological constant in this pattern is a topological invariant as well.

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I. INTRODUCTION

In the scenario of emergent gravity the quantum theory of gravity appears as the low energy approximation to a certain microscopic theory. Such a theory is not invariant under the general coordinate transformations. This invariance, therefore, is to appear dynamically at low energies. One of the first papers on emergent gravity has been written by A.D. Sakharov [1]. Even earlier it was known that the defects of the crystal lattices (disclinations and dislocations) give rise to emergent Riemann-Cartan geometry, which serves as the background for the motion of various excitations [2,3]. However, in the real materials the dynamical theory of disclinations and dislocations does not have the form of the gravitational theory. Invariance under the general coordinate transformations would appear [4] if the elastic deformations do not require energy. Such systems may be called the superplastic crystals, and they were used recently to model the quantum gravity [5] (see also [6,7]).

There exist many other scenarios of emergent gravity. Relation of the AdS/CFT correspondence to the concept of emergent gravity has been discussed, for example, in [8]. Emergence of Einstein equations from thermodynamics has been discussed in [9,10]. In [11] gravity has been linked to entropy. The possible emergence of Einstein equations from the quantum entanglement has been [12] and references therein). However, so far the complete solution has not been found.

Several scenarios that may, possibly, solve the cosmological constant problem have been proposed in the past (see, for example, [13,14] and references therein). In the present paper we propose an alternative solution of the cosmological constant problem. We consider the emergent gravity scenario in the model of the superplastic crystal [15]. This scenario allows us to look at the Standard Model fermions as at the fermionic excitations in a superplastic crystal. The emergent torsion and curvature are caused by the dislocations and the disclinations. Their dynamics gives rise to the gravitational action. At low energies it may receive the form of the Einstein action of general relativity. The cosmological constant in this pattern appears as the trace component of vacuum stress-energy tensor. We demonstrate that if the background metric weakly depends on coordinates, the fermionic stress energy tensor may be defined in such a way that being integrated over the whole volume it is a topological invariant. It is not changed, when the system is modified smoothly. The cosmological constant, therefore, is robust to the smooth variation of the system. Assuming, that the observed world is a smooth modification of the world with vanishing vacuum energy, we obtain vanishing cosmological constant.

The paper is organized as follows. In Section II we recall the description of emergent gravity in the superplastic crystals. In Section III we describe the Wigner-Weyl formalism adapted in [18,19] to the quantum field theory. In Sect. IV we end with the conclusions.

II. ELASTIC DEFORMATIONS AND EMERGENT GRAVITY

A. Elastic deformations and stress tensor

Let us parameterize the unperturbed lattice of the given crystal by coordinates \(x_k, k = 1, \ldots, D\). We will consider here the three \((D = 3)\) dimensional system. The
displacement of each point has three components \( u_a(x) \), where \( a = 1, 2, 3 \). The resulting coordinates of the crystal lattice sites are
\[
y_a(x) = x_k + u_k(x), \quad k = 1, 2, 3
\] (1)
In the absence of the displacements, when \( u_a = 0 \), the crystal is unperturbed. Metric of elasticity theory is given by
\[
g_{ik} = \delta_{ik} + 2u_{ik}, \quad u_{ik} = \frac{1}{2} \left( \partial_i u_k + \partial_k u_i + \partial_i u_k \partial_k u_i \right) \quad a = 1, 2, 3, \quad i, k = 1, ..., D (2)
\]
The stress tensor \( T^{ij} \) is defined as the response of the thermodynamical potential to elastic deformation.
\[
T^{ij}(x) = -\frac{\delta}{\delta u_{ij}(x)} \log Z
\]

B. Elasticity tetrads

The alternative way to describe elastic deformations is given in terms of the elasticity tetrads \( E_{\mu}^a(x) \), which represent the hydrodynamical variables of the elasticity theory [4, 9, 10].

The three crystallographic planes may be chosen in the three dimensional \( (D = 3) \) crystals\(^1\). The coordinates \( X^a \) \( (a = 1, ..., D) \) are introduced in such a way, that the planes given by equations \( X^a(x) = 2\pi n^a \), \( n^a \in \mathbb{Z} \) with \( a = 1, ..., D \) represent the mentioned crystallographic planes. In the presence of elastic deformations the chosen crystallographic planes become curved. The corresponding displacement vectors satisfy the following equation
\[
X^a(x + u) = 2\pi n^a
\]
The intersections of the \( D \) surfaces
\[
X^1(r, t) = 2\pi n^1, \ldots, X^D(r, t) = 2\pi n^D, (4)
\]
coincide with the positions of certain crystal lattice sites (not necessarily all of them may be found in this way). Those intersection points constitute the rectangular lattice
\[
L = \{ r = R(n_1, \ldots, n_D) | r \in \mathbb{R}^D, n^a \in \mathbb{Z}^D \}. (5)
\]
In order to describe dynamics we define all quantities as dependent on imaginary time \( t \). (We assume from the very beginning that the Wick rotation to space - time of Euclidean signature is performed.) The extra dimensionless coordinate \( X^{D+1} = t/\Delta t \) is defined as the imaginary time in the units of \( \Delta t \). The elasticity tetrads are gradients of the phase function
\[
E_{\nu}^a(y) = \partial_{\nu} X^a(y), \quad a = 1, ..., D + 1 (6)
\]
The inverse matrix to \( E_{\mu}^a \) is denoted by \( e^a_{\mu} \). For \( a = 1, ..., D \) those quantities have units of crystal momentum, \( \{ E_{\mu}^a \} = 1/|t| \), \( |e^a_{\mu}| = |t| \) while for \( a = D + 1 \) the quantity \( E^a \) has the dimension \( 1/|t| \).

In the absence of dislocations, the tetrads \( E_{\mu}^a = E_{\mu}^a dx^\mu \) are exact differentials. They can be expressed in general form in terms of a system of deformed crystallographic coordinate planes, i.e. surfaces of constant phase \( X^a(y) = 2\pi n^a \), \( n^a \in \mathbb{Z} \) with \( a = 1, ..., D \).

In the absence of dislocations, tensor \( E_{\mu}^a(y) \) satisfies the integrability condition:
\[
dE^a = \frac{1}{2}(\partial_{\mu} E_{\nu}^a(y) - \partial_{\nu} E_{\mu}^a(y))dx^\mu \wedge dy^\nu = 0. (7)
\]
Matrix \( e^a_{\mu} \) inverse to \( E_{\mu}^a \) represents the inverse vielbein. It determines the distance along the curve \( C \) that connects the two points in space - time corresponding to the two values of \( X(X_1, X_2) \) as follows:
\[
\rho C(X_1, X_2) = \int_C \sqrt{e^\mu_\alpha e^\nu_\beta \delta_{\mu\nu} dX^\alpha dX^\beta} (8)
\]
where \( \delta_{\mu\nu} = \text{diag}(1, 1, 1, 1) \). In dimensionless coordinates \( X \) metric is defined as
\[
g_{ab}(X) = e^a_{\mu} e^b_{\nu} \delta_{\mu\nu} = \frac{\partial y^\mu}{\partial X^a} \frac{\partial y^\nu}{\partial X^b} (9)
\]
It is useful to introduce the time displacement \( u^{D+1} = 0 \). This allows to express metric as
\[
g_{ab}(X) = \left( [e^{(0)}]^\mu_\alpha \frac{\partial x^\mu}{\partial X^a} + \frac{\partial u^\alpha}{\partial X^a} \right) ( [e^{(0)}]_\beta^\nu \frac{\partial x^\nu}{\partial X^b} ) \delta_{\mu\nu} (10)
\]
where
\[
[e^{(0)}]^\mu_\alpha = \frac{\partial x^\mu}{\partial X^a}, \quad [e^{(0)}]_\beta^\nu = ( [E^{(0)}]^{-1})^\mu_\alpha (11)
\]
and
\[
[E^{(0)}]_\mu^a = \frac{\partial X^a}{\partial x^\mu} (12)
\]
Metric in coordinates \( X \) is related to metric in coordinates \( x \) as follows:
\[
\frac{\partial X^a}{\partial x^{\mu\nu}} g_{ab}(X) = g_{\mu\nu}(x) = \delta_{\mu\nu} + u_{\mu\nu} (13)
\]
that is
\[
\left[ E^{(0)} \right]^a_\mu \left[ E^{(0)} \right]^b_\nu g_{ab}(X) = g_{\mu\nu}(x) = \delta_{\mu\nu} + u_{\mu\nu} (14)
\]
Recall also, that in coordinates \( y \) metric is given simply by
\[
g_{\mu\nu}^{(0)}(y) = \delta_{\mu\nu} (15)
\]
\(^1\) Those crystallographic planes are not necessarily orthogonal for the unperturbed crystals.
Similar to the stress tensor $T^{ij}$ the response of the thermodynamical potential to the elasticity tetrads gives rise to the tensor $\Theta^i_a$:

$$\Theta^i_a(x) = -\frac{\delta}{\delta E^i_a(x)} \log Z$$

In the similar way

$$\Theta^i_\sigma(x) = -\frac{\delta}{\delta \epsilon^i_\sigma(x)} \log Z$$

Those tensors are related as follows:

$$T^{ij} = \frac{1}{2} \epsilon^j_b (\Theta^i_a) \delta^{ab}, \quad T_{ij} = \frac{1}{2} \delta^{ij} \Theta^a_\sigma e^\sigma_a$$

### C. Teleparallel gravity

In the presence of the single dislocation (placed at the line $\xi$) the coordinates $X$ become ambiguous. The vielbein $E^a_\mu$ being integrated along the closed path $C$ surrounding $\xi$ gives the Burgers vector specific for the given lattice and the given type of the dislocation $^2$:

$$\int_C dy^\mu E^a_\mu = B^a$$

In the differential form this equation is given by

$$\partial_\mu E^a_\nu(y) - \partial_\nu E^a_\mu(y) = B^a \epsilon_\mu\nu \int d\epsilon^a_\nu(\tau) \delta^{(3)}(y - \xi(\tau)), \quad D = 3$$

The dislocations distributed continuously give the vielbein $E^a_\mu$ that (being averaged) corresponds to torsion

$$T_{\mu
u}^a(y) = \partial_\mu E^a_\nu(y) - \partial_\nu E^a_\mu(y)$$

We may speak of the field $T^a_{\mu\nu}$ as of the continuous function of coordinates.

The curved geometry given by the vielbein $E^a_\mu$ is, actually, the geometry of the teleparallel gravity. This is the version of Riemann - Cartan gravity with vanishing curvature (and spin connection), and nonzero torsion. The corresponding space is parallelizable Weitzenbock space. The possible terms in the action of the dynamical theory of this type of gravity may be found in [37]. If we restrict ourselves by the terms with the derivatives up to the second, and if there is no selected direction in $D + 1$ dimensional space - time, while the P and T symmetries hold, then we are left with

$$S = \int d^{D+1}y \det E \left( \lambda + \alpha v^2 + \beta a^2 + \gamma t^2 \right)$$

Here $a, v, t$ are the irreducible components of torsion:

$$a^\kappa = \frac{1}{4 \det E} e^{\kappa\alpha\beta\gamma} T_{\alpha\beta\gamma}$$

^2 Its value is quantized according to the lattice type

$$v_\kappa = T_{\kappa\nu}$$

$$t^{\lambda}_{\mu\nu} = T^{\lambda}_{\mu\nu} + \delta^\lambda_{\mu} T^\rho_{\nu\rho} - g^\lambda_{\mu} T_{\rho\nu\rho}$$

For certain particular values of $\alpha, \beta, \gamma$ we will obtain the so called teleparallel equivalent of general relativity.

We consider the quantum vacuum as the spacetime superplastic crystal, where $X^a(y) = 2\pi an^a, \ n^a \in \mathbb{Z}$ with $a = 1, 2, 3, 4$ is the system of the four deformed crystallographic coordinate hyperplanes in $3 + 1$ D. One may assume that in the superplastic vacuum, the effective action for $E^a_\mu$ is the gravitational one given by Eq. (3). Metric $g_{\mu\nu}$, which originates from the elasticity tetrads, is:

$$g_{\mu\nu} = \delta_{ab} E^a_\mu E^b_\nu .$$

### D. Riemann - Cartan geometry

Above we discussed the case, when the torsion is present while the curvature is absent. This occurs in the presence of dislocations. The disclinations give rise to the curvature. This occurs as follows. The disclination corresponds to the removal of a sector ended at the given line. The remaining parts of the lattice are glued together. Therefore, the path around the given line captures the angle different from $2\pi$. This means that the curvature is nonzero. In the unperturbed lattice each path corresponds to vanishing parallel transporter. In the presence of a disclination the parallel transporter along the closed path, which encloses the disclination, is equal to the matrix of rotation. The distribution of disclinations may be described by continuous $SO(3) \subset SO(4)$ connection. This connection modifies derivative as usual: it becomes covariant. In the absence of dislocations we deal, therefore, with Riemannian geometry.

If both the disclinations and the dislocations are present, there is the emergent Riemann - Cartan geometry. The basic variables are the vielbein $E^a_\mu$ and the spin connection $\omega_\mu \in so(D+1), \ D = 3$. The mentioned above parallel transporter around the disclination results in the rotation matrix:

$$P \exp \left( \int_C \omega_\mu dy^\mu \right) = e^{\phi \Sigma}$$

Here $\phi$ is the angle specific for the given disclination, while $l$ is the unity vector along the axis of rotation, while $\Sigma$ is the generator of rotation. As usual, curvature $R_{\mu\nu}$ is defined as

$$\lim_{|A| \to 0} P \exp \left( \int_{A} \omega_\mu dy^\mu \right) = \exp \left( \frac{1}{2} \int_{A} R_{\mu\nu} dy^\mu \wedge dy^\nu \right)$$

The covariant derivative is given by

$$D_\mu = \partial_\mu + \omega_\mu$$
while the definition of torsion is modified as follows:

\[ T^a_{\mu\nu}(y) = D_{\mu}E^a_\nu(y) - D_{\nu}E^a_\mu(y) \]

There are a lot of possible local terms in the action that contain derivatives up to the second. Those terms may be composed in different way of torsion, curvature, and various covariant derivatives of the vielbein. These terms were classified in [37].

E. The fields experienced by the fermions in the presence of dislocations and disclinations

Let us consider the tight-binding model with Hamiltonian

\[ H = \sum_{X_1, X_2} \bar{\Psi}^{A}_{X_1} t^{AB}_{X_1 X_2} \Psi^{B}_{X_2} \]  

(12)

Hopping parameters \( t^{AB}_{X_1 X_2} \) correspond to the jumps of the fermions between the sites \( X_1 \) and \( X_2 \) of the lattice. The simplest dependence of the hopping parameters on the elastic deformations is

\[ t^{AB}_{X_1 X_2} = t^{AB} + t^{AB} \beta \left( \sqrt{e^a_\mu e^b_\nu \delta_{\mu\nu} (X_1^a - X_2^a)(X_1^b - X_2^b)} \right. \]

\[ \left. - \sqrt{[e(0)]^a_\mu [e(0)]^b_\nu \delta_{\mu\nu} (X_1^a - X_2^a)(X_1^b - X_2^b)} \right) \]  

(13)

Here \( \beta \) is the so-called Gruneisen parameter. Suppose that the model admits the topologically protected Fermi points. In their vicinities the emergent Weyl fermions appear (this is the case of the so-called Weyl semimetals). In the absence of multiple dislocations and disclinations the above mentioned modification of the hopping parameters results in the appearance of emergent gauge field and emergent teleparallel gravity. The emergent gauge field appears in the first approximation to the complete theory, while the emergent gravity appears in the next approximation. The emergent gravity is characterized by torsion while the emergent gauge field is characterized by its field strength. The presence of multiple dislocations and disclinations modifies the emergent gravity experienced by the fermions. Namely, it has been shown that the dislocations carry the emergent singular torsion flux and the emergent singular \( U(1) \) flux. In the similar way one may demonstrate, that the disclinations carry the singularity of curvature.

The especially interesting case is when the emergent torsion carried by the dislocations is much larger than the emergent torsion caused by the above mentioned modification of the hopping parameters. This occurs when the Gruneisen parameter \( \beta \) is small \( \beta \ll 1 \). In this situation we deal with the Weyl fermions that experience the described above gravity with torsion. In this case those elastic deformations, which are not caused by the appearance of the disclinations and the dislocations, do not require energy. As a result, the low energy effective theory appears to be invariant under the diffeomorphisms. This is the case of the superelastic crystal.

Thus, we may suppose that the Standard Model with gravity appears as the low energy description of the superelastic crystal with the fermionic excitations inside it. The real gravity appears as the theory of the elastic vielbein. The presence of the dislocations results in torsion while the presence of disclinations results in curvature. Notice, that the mentioned here tight-binding model may be thought of as an effective model, while the true microscopic theory remains continuous. Below we will concentrate on the formalism that allows us to deal with such a microscopic theory. In this pattern of space-time the time still remains distinguished from space. The curvature and the torsion do not have nontrivial time components. The description of the space-time gravity and the Standard Model fermions as an effective theory may be thought of as the theory written in the reference frame with \( E^a_{\mu} = \delta^{a+1}_{\mu} \). This is the theory with the partially fixed gauge. (The symmetry with respect to the space-time diffeomorphisms is reduced down to the diffeomorphisms of space.)

There is an alternative way to look at the problem of the distinguished (imaginary) time direction. Notice, that \( E^a_{\mu} \) enters all expressions in such a way, that there is no essential difference between the components with \( a, \mu = 1, 2, 3 \) and \( a \) (or \( \mu \)) equal to 4. We may extend the notion of the elastic deformations (and the corresponding displacement vectors \( u_\mu \)) from the 3-component vectors defined in 3-dimensional space to the 4-component vectors defined in 4-dimensional space-time. We may actually think that space-time is the four-dimensional superelastic crystal. This restores the complete symmetry between space and (imaginary) time components of all vectors and tensors. The dislocations will appear as the two-dimensional defects, such that for the closed contours that enclose them, the integrals of Eq. (8) are nontrivial. The disclinations also appear as the two-dimensional defects. The integrals over the contours that surround them given by Eq. (11) are nontrivial. \( \Sigma \) represents the generators of the \( SO(4) \) group (recall that we perform the Wick rotation to the Euclidean signature of space-time).

III. CONTINUUM MICROSCOPIC THEORY ON THE LANGUAGE OF WIGNER - WEYL FORMALISM

A. Wigner - Weyl formalism

In this section we consider the Wigner - Weyl formalism to be used for the description of the underlined microscopic theory. For more details of this formalism we refer to [32][35]. Let us start from the discussion of the

\[ ^3 \text{Notice that in graphene, for example, } \beta \sim 2. \]
model with the noninteracting fermions in $D + 1$ spatial - time dimensions. The corresponding partition function has the form

$$Z = \int D\hat{\Psi} D\Psi e^{\int d^{D+1}y(\hat{Q}\hat{\Psi})}$$

(14)

where operator $\hat{Q}$ contains an information about the microscopic physics including the periodic potential of the fermions in the crystal. It is useful to introduce the (Grassmann - valued) distribution function:

$$W(x, y) = \langle x|\hat{W}|y\rangle = \hat{\Psi}(x)\Psi(y)$$

we also denote the matrix elements of operator $\hat{Q}$ by

$$Q(x, y) = \langle x|\hat{Q}|y\rangle$$

As a result, the partition function may be rewritten as follows

$$Z = \int D\hat{\Psi} D\Psi e^{\text{Tr} \hat{W} \hat{Q}}$$

(15)

Here in the exponent the functional trace appears. The next step is the introduction of the Weyl symbol of operator $A$:

$$A_W(p, x) \equiv \int dy e^{-ipy} A(x + y/2, x - y/2)$$

(16)

We introduce the Weyl symbols of operators $\hat{Q}$ and $\hat{W}$, and come to

$$Z = \int D\hat{\Psi} D\Psi \exp\left( \int d^{D+1}x \int \frac{d^{D+1}p}{(2\pi)^{D+1}} \text{Tr} W_W[\Psi, \bar{\Psi}](p, x) * Q_W(p, x) \right)$$

(17)

We denote here by the star (Moyal) product the following expression

$$* = e^{\int (\bar{\Psi}_x \partial_x \Phi - \bar{\Phi}_x \partial_x \Psi)}$$

B. Derivative of the partition function as a topological invariant

Let us suppose that operator $\hat{Q}$ introduced above depends on a parameter $\eta$. We consider derivative of the logarithm of the partition function:

$$\frac{d}{d\eta} \log Z = \frac{1}{Z} \int D\hat{\Psi} D\Psi \left( \int d^{D+1}x \int \frac{d^{D+1}p}{(2\pi)^{D+1}} \text{Tr} W_W[\psi, \bar{\psi}](p, x) * Q_W(p, x) \right) \left( \int d^{D+1}x \int \frac{d^{D+1}p}{(2\pi)^{D+1}} \text{Tr} W_W[\bar{\psi}, \bar{\psi}](p, x) + \frac{d}{d\eta} Q_W(p, x) \right) \right)$$

$$= \int d^{D+1}x \int \frac{d^{D+1}p}{(2\pi)^{D+1}} \text{Tr} G_W(p, x) * \frac{d}{d\eta} Q_W(p, x)$$

(18)

As it was mentioned above, we made here rotation to space - time of Euclidean signature, i.e. we consider the theory in imaginary time. Here $Q_W$ depends on parameter $\eta$, and we introduced the Wigner transformed Green function $G_W$ as the Weyl symbol of $\hat{G}$, where

$$G(x, y) = \langle x|\hat{G}|y\rangle = \frac{1}{Z} \int D\hat{\Psi} D\Psi e^{\text{Tr} \hat{W} \hat{Q}}$$

As usual in field theory, in order to calculate this integral in practise we need to put the system into the large, but finite spatial volume. Correspondingly, the imaginary time extent (the inverse temperature) also should be large, but finite. Then, strictly speaking, the sums over the discrete momenta appear instead of the integral. Though, we are able to calculate the integrals instead of the sums because the overall volume is large. Correspondingly, the boundary conditions are to be imposed. The experience with the conventional quantum field theory prompts that for the large volume (and the large imaginary time extent) the type of the boundary condition is irrelevant for the physics in the bulk. For the definiteness we assume the periodical spacial boundary conditions, and the anti - periodic boundary conditions for the imaginary time direction.

The Wigner transformed Green function obeys the Groenewold equation:

$$Q_W * G_W = 1$$

The properties of the star product guarantee the commutativity

$$\int d^{D+1}x \int \frac{d^{D+1}p}{(2\pi)^{D+1}} \text{Tr} A_W(p, x) * B_W(p, x)$$

$$= \int d^{D+1}x \int \frac{d^{D+1}p}{(2\pi)^{D+1}} \text{Tr} B_W(p, x) * A_W(p, x)$$

$$= \text{Tr} \hat{A} \hat{B}$$

(19)

Here $\text{Tr}$ is the functional trace of an operator that obeys $\text{Tr} \hat{A} \hat{B} = \text{Tr} \hat{B} \hat{A}$. Therefore, since $\delta(Q_W * G_W) = 0$, the
variation of $\frac{d}{d\eta} \log Z$ is equal to zero:

$$\delta \frac{d}{d\eta} \log Z = \int d^{D+1} x \int \frac{d^{D+1} p}{(2\pi)^{D+1}}$$

$$\text{Tr} \left( G_W(p,x) * \delta Q_W(p,x) + \delta G_W(p,x) * \frac{d}{d\eta} Q_W(p,x) \right)$$

$$= \int d^{D+1} x \int \frac{d^{D+1} p}{(2\pi)^{D+1}} \text{Tr} \left( G_W(p,x) * \delta Q_W(p,x) - G_W(p,x) * \delta Q_W(p,x) * G_W(p,x) * \frac{d}{d\eta} Q_W(p,x) \right)$$

$$= \int d^{D+1} x \int \frac{d^{D+1} p}{(2\pi)^{D+1}} \text{Tr} G_W(p,x) * \delta Q_W(p,x) - \frac{d}{d\eta} G_W(p,x) * \delta Q_W(p,x)$$

$$= \int d^{D+1} x \int \frac{d^{D+1} p}{(2\pi)^{D+1}} \frac{d}{d\eta} \text{Tr} G_W(p,x) * \delta Q_W(p,x) \quad (20)$$

We come to an interesting conclusion that the derivative of (the logarithm of) the partition function is topological invariant provided that the last integral in Eq. (20) vanishes.

In the above proof it has been implied that the integrals are convergent. The absence of the infrared divergencies means that neither $G$ nor $Q$ have poles at finite values of momenta. As for the ultraviolet divergencies, we assume that the discussed theory does not contain them. In particular, we may imagine that this theory has the form of a lattice model (the imaginary time is discretized as well). The formalism described in Sect. II, which leads to the Groenewold equation, formally has been derived for the continuous theory. It remains valid for the lattice models as well if the external field (in our case this is the elastic deformation tensor) varies slowly, so that its variation at the distance of the order of the lattice spacing may be neglected (for the details see [32–34]). This condition is assumed in the theory of elasticity. However, we do not rely on the assumption that the considered theory is defined on the lattice. This is only one of the possible options. For our purposes it is enough to assume that the theory is free from the ultraviolet divergencies.

Thus Eq. (18) at zero temperature is topological invariant for the gapped systems that do not contain ultraviolet divergencies provided that

$$\int d^{D+1} x \int \frac{d^{D+1} p}{(2\pi)^{D+1}} \frac{d}{d\eta} \text{Tr} G_W(p,x) * \delta Q_W(p,x) = 0 \quad (21)$$

for the given class of variations $\delta Q_W(p,x)$. Notice, that the theories containing gapless fermions in the presence of interactions suffer from the infrared divergencies, we do not consider them here.

An example, when the derivative of partition function with respect to a parameter is topological invariant is given by the electric current density integrated over the whole system volume. In this case the role of parameter $\eta$ is played by constant external electromagnetic potential $A_k$ along the $k$ - th axis. Then in Eq. (21) we substitute $\frac{d}{d\eta}$ by $-\frac{d}{d\eta}$, and this condition is satisfied if we chose appropriate boundary conditions in momentum space. The topological invariance of the total electric current derived in this way represents a field theoretical generalization of the quantum - mechanical Bloch theorem [39].

C. Contribution of fermions to the stress - energy tensor

Here we will consider example of the topological invariant given by the traceless components of stress - energy tensor integrated over the whole volume. In the absence of nontrivial background metric this stress tensor is given by response of partition function to the variations of elastic deformations $\delta u_{ij}(x)$. In the presence of the background metric originated from distribution of disclinations and dislocations we consider the response of thermodynamical potential to the variation of effective metric given by

$$\delta g^{ij}(x) = -\delta u^k \partial_k g^{ij}(x) + g^{ik}(x) \partial_k \delta u^j(x) + g^{jk}(x) \partial_k \delta u^i(x)$$

and effective vielbein

$$\delta e_a^i(x) = -\delta u^k \partial_k e_a^i(x) + e_a^i(x) \partial_j \delta u^j(x)$$

In the considered theory the response to variation of $g^{ij}$ represents stress - energy tensor $T_{ij}(x)$. Response to the variation of $e_a^i$ gives tetrad components of the stress - energy tensor $\Theta^a_{\mu \nu}$. In our consideration it is more convenient to deal with the latter. Relations between those two tensors are given by

$$\Theta^a_{\mu} = \Theta^a_{\mu} e_a^i, \quad T_{ij} = \frac{1}{2} (\Theta^e_{\mu} g_{kj} + \Theta^e_{\nu} g_{ei})$$

The total stress - energy tensor is given by an integral of $e_a^i \Theta^a_{\mu}(x)$ over the whole volume averaged in time $\left< e_a^i \Theta^a_{\mu}(x) \right>$:

$$\left< \Theta^a_{\mu} \right> = T \int d^{D+1} x \left< e_a^i \Theta^a_{\mu}(x) | E(x) \right>$$

$$= 2T \int d^{D+1} z |E(z)| \int d^{D+1} x$$

$$\int \frac{d^{D+1} p}{(2\pi)^{D+1}} \text{Tr} G_W(p,x) * e_a^i(z) \frac{\delta}{\delta e_a^i(z)} Q_W(p,x)$$

Here $T$ is temperature, which is assumed to be small. If $Q_W$ and $G_W$ do not depend on (imaginary) time, then
$T \int d\omega^{D+1}$ becomes equal to unity. In Eq. (22) we ignore the possible contribution to stress - energy tensor that may appear due to the dependence of measure over fermionic fields on $|E(x)|$. This contribution will be discussed later.

Using technique developed in Appendix A it is shown in Appendix C that we are able to substitute here $e^\alpha_a(p_i)\frac{4}{(2\pi)^{D+1}}$ by $\frac{1}{2} \left( p_j \frac{\partial}{\partial p_i} - \delta_a^j \right) \delta(z-x)$ provided that $e^\alpha_a$ depend on coordinates only slightly. In Appendix B we show, in addition, that the covariant derivative of stress - energy tensor is a topological invariant.

The effective low energy theory for the model with a Fermi point gives rise to the following expression for $Q_W$:

$$Q_W(p, y) = \frac{|E(y)|}{2} e^\alpha_a(y) \{ \gamma^a, (p_\mu - A_\mu(y) - \omega_\mu(y)) \} - m|E(y)|$$

(23)

provided that the Gruneisen parameter is much smaller than unity. This situation may be realized in the ideal superplastic crystals, where the elastic deformations do not require energy. Here $\omega$ is the emergent spin connection resulted from the disclinations, while the elasticity vielbein leads to the presence of torsion. In this case the space components of the emergent stress - energy tensor appear to be identical to the stress tensor of elasticity. As it was mentioned above, we assume that the appearance of the Fermi point is broken by a small mass term. As a result the fermions become gapped, and the system already does not suffer from the infrared singularities (which appear for any gapless fermions in the presence of interactions). Thus we come to

$$Q_W(p, y) = \frac{|E(y)|}{2} e^\alpha_a(y) \{ \gamma^a, (p_\mu - A_\mu(y) - \omega_\mu(y)) \} - m|E(y)|$$

(24)

It was noticed above that Eq. (24) represents the low energy approximation for the theory with the Fermi point (broken weakly by a small mass term). For momenta far from the Fermi point the form of $Q$ is essentially different. The whole theory may be actually considered as the regularization of the relativistic theory with fermions and gravity.

Taking into account all mentioned above, we may try to consider the following expression as the definition of the stress - energy tensor (again, up to the possible contribution from the integration measure):

$$\langle \Theta^i_t \rangle = T \int d^{D+1}x |E(x)| \int d^{D+1}p \frac{p_i}{(2\pi)^{D+1}} \text{Tr} G_W(p, x) \ast \left( p_i \frac{\partial}{\partial p_j} - \delta^i_j \right) Q_W(p, x)$$

(25)

For the variation of stress - energy tensor we get

$$\langle \delta \Theta^i_t \rangle = T \int d^{D+1}x |E(x)| \int d^{D+1}p \frac{p_i}{(2\pi)^{D+1}} \text{Tr} G_W(p, x) \ast \left( \frac{\partial}{\partial p_j} - \delta^i_j \right) Q_W(p, x)$$

(26)

One can see, that under a suitable ultraviolet regularization the traceless components of stress - energy tensor $\Theta^i_j$ (those with $i \neq j$) are topological invariants provided that $|E(x)|$ remains constant during the given variations. This may always be achieved via a suitable reparametrization that accompanies the modification of the system.

A few comments are in order. First of all, we still did not take into account the possible dependence of integration measure on $|E|$. We suppose that there is no dependence on the other components of $e^\alpha_a$. Such a dependence may influence only the trace components of the stress - energy tensor. It will be taken into account in section following the requirement that the whole stress - energy tensor is a topological invariant. Next, the term in the brackets in Eq. (26) proportional to the product $\delta^i_j Q_W(p, x)$ does not appear if we use the naive expression for the stress - energy tensor (it is proportional to the averaged fermion lagrangian, it vanishes on the classical solutions). The whole expression Eq. (26) suffers from all possible types of divergencies - both ultraviolet and infrared. The infrared divergency is to be regularized through the finite overall volume (and finite temperature). The naive ultraviolet regularization (via the ultraviolet cutoff in the integral over momentum) will then give the ultraviolet divergent expression, which does not have a form of topological invariant. As it was mentioned above, we need the ultraviolet regularization that obeys the requirement that in the integrals over momentum $p$, its shift by a constant $a$ may be performed: $p \rightarrow p + a$. The superplastic crystal by itself represents an ultraviolet regularization that satisfies this requirement. It is worth mentioning, that metric is a quantity, which is well defined in continuum theory. Correspondingly the stress energy tensor being the response of the system to the variation of metric, is also well - defined in continuum theory only.

D. Lattice regularization

On the lattice we need an appropriate definition, which tends to a continuum definition in continuum limit. One may try to take as the definition of stress - energy tensor on rectangular lattice (up to the terms coming from the
In this expression the lattice spacing is equal to unity while momentum space is a torus \((0; 2\pi)^{D+1}\). We assume here that all fields vary slowly, so that their variation at the distance of lattice spacing may be neglected. Under these conditions \(\sin p_m\) commutes with \(*\). Under the same conditions we may substitute the sum over \(x\) by an integral. We obtain the following expression for the variation of \(\Theta_i^j\) (as above we assume that \(|E(x)|\) is not changed during this variation):

\[
(\delta \Theta_i^j) = T \int d^{D+1}x \int \frac{d^{D+1}p}{(2\pi)^{D+1}} \text{Tr} G_W(p, x) \ast \delta Q_W(p, x)
\]

One can see, that this expression is not a topological invariant even for \(i \neq j\). Therefore, we redefine the lattice expression for the stress - energy tensor as follows

\[
(\Theta_i^j) = T \sum_x |E(x)| \int \frac{d^{D+1}p}{(2\pi)^{D+1}} \text{Tr} G_W(p, x) \ast \delta Q_W(p, x)
\]

\[
\left(\sin p_i \frac{\partial}{\partial p_j} - \delta_i^j\right)Q_W(p, x)
\]

\[+ T \text{Tr}(\cos \hat{p}_i - 1) \delta_i^j \left(\text{Log} \hat{Q} - \text{Log} \hat{Q}_0\right)
\]

Here \(\hat{Q}_0\) is built with a certain particular fixed vielbein \(e_a^{(0)}\). By \(\text{Log} \hat{Q}\) we understand the functional Logarithm and by \(\text{Tr}\) we understand the functional trace. Their definitions and properties are discussed in Appendix D. To avoid the uncertainties in the definition we define both in the gauge \(|E(x)| = 1\). Their values for arbitrary \(|E(x)|\) are to be calculated for the configuration of the vielbein with \(|E(x)| = 1\) connected by reparametrization to the given one. If the fields vary slowly at the distance of the order of lattice spacing, and operation \(*\) commutes with \(\cos p\), then the variation of this expression is

\[
(\delta \Theta_i^j) = T \int d^{D+1}x |E| \int \frac{d^{D+1}p}{(2\pi)^{D+1}} \text{Tr} G_W(p, x) \ast \delta Q_W(p, x)
\]

\[
\left(\frac{\partial}{\partial p_j} \sin p_i - \delta_i^j\right)Q_W(p, x)
\]

\[+ T \text{Tr}(\cos p_i \delta_i^j \left(\text{Log} \hat{Q} - \text{Log} \hat{Q}_0\right)
\]

One can see, that being defined in this way the traceless components of stress energy tensor are topological invariants.

### E. Cosmological constant as the topological invariant

It has been noticed above that the traceless components \(\theta_i^j = \Theta_i^j + \frac{1}{D+1} \delta_i^j \Theta_k^k\) of stress - energy tensor of Eq. (27) (and of its lattice regularization of Eq. (29)) are robust to the smooth modifications of the system accompanied by suitable reparametrizations aimed at keeping \(|E(x)|\) unchanged. However, we did not take into account the contribution to the stress energy tensor of the integration measure over the fermionic fields. This integration measure is assumed to depend on the determinant \(|E(x)|\) and not on the other components of \(E_{\mu}^\nu\). Therefore, the corresponding contribution to the stress - energy tensor cannot modify the traceless components \(\theta_i^j\). We propose the hypothesis that the topological invariance is an essential property of the source of the gravitational excitations. And therefore, it is natural to require that not only the traceless components, but also the whole stress - energy tensor is a topological invariant. The term coming from the integration measure are supposed to be responsible for this. This hypothesis leads us to the redefinition of the whole stress - energy tensor in such a way, that its trace is the topological invariant as well. This gives the following expression

\[
(\Theta_i^j) = T \int d^{D+1}x |E(x)| \int \frac{d^{D+1}p}{(2\pi)^{D+1}} \text{Tr} G_W(p, x)\]

\[
\left(p_i \frac{\partial}{\partial p_j} - \delta_i^j\right)Q_W(p, x)
\]

\[+ T \text{Tr}(\cos p_i \delta_i^j \left(\text{Log} \hat{Q} - \text{Log} \hat{Q}_0\right)
\]

and its lattice regularized form

\[
(\Theta_i^j) = T \sum_x |E| \int \frac{d^{D+1}p}{(2\pi)^{D+1}} \text{Tr} G_W(p, x)
\]

\[
\left(p_i \frac{\partial}{\partial p_j} - \delta_i^j\right)Q_W(p, x)
\]

\[+ T \text{Tr}(\cos p_i \delta_i^j \left(\text{Log} \hat{Q} - \text{Log} \hat{Q}_0\right)
\]

By \(\text{Log} \hat{Q}\) and \(\text{Tr}\) we understand here again the functional logarithm and functional trace considered in Appendix D. With this new definition the fermionic stress - energy tensor integrated over all space is the topological invariant, provided that the fermions are massive while background metric depends on coordinates weakly.

We apply this consideration to the fermions of the Standard Model. Then the vacuum average \(\langle \Theta_i^j \rangle_{\text{vacuum}}\) appears to be the topological invariant if the background metric does not depend on coordinates. Phenomenological gravitational action, which appears at low energies as the effective description of the disclinations and the dislocations, is assumed to have the standard form

\[
S_g = -m_p^2 \int R \sqrt{-g} d^4 x - 2\lambda m_p^2 \int \sqrt{-g} d^4 x
\]
It contains the cosmological constant $\lambda$. Variation of the corresponding term in the action with respect to $\eta_{\mu}$ gives the following "vacuum" stress-energy tensor:

$$\Theta^\mu_{\nu,\text{vacuum}} = -\lambda m_\nu^2 \delta^\mu_\nu$$

One can see, that $\lambda$ is the topological invariant. Smooth modification of the system does not change value of the cosmological constant. Assuming that the smooth modification connects our world with the world with vanishing vacuum energy, we come to the conclusion that the cosmological constant $\lambda$ vanishes. Its small deviation from zero results, therefore, from the contributions that were not taken into account in our consideration.

### IV. CONCLUSIONS

In the present paper we discuss the scenario, in which the Standard Model of elementary particles and the quantum gravity appear in the effective low energy description of a condensed matter system. The latter system describes the ideal superplastic crystal, in which smooth elastic deformations do not require energy. As a result, the emergent theory is invariant under the general coordinate transformations of space. In this theory the disclinations result in curvature, while the dislocations result in torsion. The action, which describes the dynamics of dislocations and disclinations, may have at low energies the form of the Einstein action of general relativity.

Alternatively, the superplastic crystals may be considered as the ultraviolet regularization of quantum gravity coupled to fermions. The advantage of this regularization is that the integration measure over fermions does not depend on the gauge fields and on the gravitational fields. Instead, the dependence of operator $Q_W(x, p)$ on momenta is responsible for the quantum anomalies. Therefore, the variational derivative of the integration measure does not contribute the total stress energy tensor.

We show that in the discussed theory the fermionic stress energy tensor defined properly is a topological invariant, provided that the fermions are massive and the background metric depends on coordinates weakly. With our definition of the stress-energy tensor the cosmological constant becomes topological invariant. Finally, if our World appears to be a smooth modification of the system with vanishing vacuum energy, we come to vanishing value of the cosmological constant. The extremely small observable value of the cosmological constant may follow from the deviations of the system from this state related to dependence of metric on both time and space coordinates. One of such contributions may come from the scale anomaly, i.e. the contribution due to curvature to the trace of the stress-energy tensor (for the discussion of such contributions see [10] and references therein).

The extra terms proportional to the functional trace of the logarithm of $Q$ added to Eq. [31] and Eq. [33] are responsible for the topological invariance of the trace components of the stress-energy tensor. The appearance of those terms reflects difficulties with the definition of measure over the fermionic fields in the presence of nontrivial background metric. In fact, measure over the fermionic fields in the superplastic crystal is trivial for the trivial metric only. It is then defined as the integration over the Grassmann variables attached to the lattice sites. The presence of nontrivial metric means that there are dislocations and disclinations. In the presence of these defects it is not sufficiently clear ab initio how to define the integration measure. Thus, our above definition assumes, that the measure is defined in such a way, that its variation with respect to variation of metric gives the mentioned above extra terms in Eqs. [31] and [33], so that not only the traceless components of $\Theta$, but also its trace components are the topological invariants.

In the proposed scheme the bosonic fields of the Standard Model were not taken into account (as well as in [3]). There should be the reason, why this contribution vanishes or appears to be small. In this sense the solution of the cosmological constant problem proposed here is incomplete, and requires the further elaboration. However, this remains out of the scope of the present paper.

The author is grateful for the discussions to G.E. Volovik, who actually proposed him to look for the topological nature of the cosmological constant using an analogy with the superplastic crystals.

### Appendix A. Variation of Weyl symbol of an operator with respect to reparametrizations

Let us consider variation of Wigner transformed matrix elements of an operator $A$ caused by the following transformation:

$$A(y_1, y_2) \rightarrow A(y_1 - \delta u(y_1), y_2 - \delta u(y_2))$$

---

4 We only mention one of the possible ways to understand the reason, why the gauge field contribution to the vacuum stress energy tensor vanishes. One may accept the scheme, in which the gauge fields appear dynamically. Namely, the variation of the position of the Fermi point describes the emergent gauge field [35]. The fluctuations of the Fermi point occur due to the interactions between the fermions. These interactions may be caused by the emergent gravity. In principle, all gauge fields of the Standard Model (or of its ultraviolet completion) may appear in this way. Those fields appear as the composite excitations existing at the relatively low energies (the energies much smaller than the Plank mass). At the same time, at high energies, we deal with the system discussed in the present paper.
with small elastic deformation displacement $\delta u^k(x)$:

$$A_W(p, x) + \delta A_W(p, x)$$

$$= \int dy_1 dy_2 e^{-i(y_1 - y_2)p} A(y_1 - \delta u(y_1), y_2 - \delta u(y_2))$$

$$\delta(x - (y_1 + y_2)/2)$$

(34)

This gives

$$\delta A_W(p, x) = \int dy_1 dy_2 \frac{dq}{(2\pi)^4} e^{-i(y_1 - y_2)p} \delta(x - (y_1 + y_2)/2)$$

$$\left(-\delta u^k(y_1) \partial_{\phi^k} - \delta u^k(y_2) \partial_{\phi^k}\right)$$

$$A(y_1, y_2)$$

$$= \int dy_1 dy_2 \frac{dq}{(2\pi)^4} e^{-i(y_1 - y_2)p} \delta(x - (y_1 + y_2)/2)$$

$$\left(-\delta u^k(y_1)(iq_k + z_k/2) - \delta u^k(y_2)(-iq_k + z_k/2)\right)$$

$$e^{i(y_1 - y_2)q} A(q, x)$$

$$= \int dy_1 dy_2 \frac{dq}{(2\pi)^4} e^{i(y_1 - y_2)(q - p)} \delta(x - (y_1 + y_2)/2)$$

$$\left(-\delta u^k(y_1)(iq_k + z_k/2) - \delta u^k(y_2)(-iq_k + z_k/2)\right)$$

$$A_W(q, x)$$

(35)

Therefore,

$$\frac{\delta A_W(p, x)}{\delta u^k(z)} = \int \frac{dq}{(2\pi)^4} e^{i(z - y_2)(q - p)} \delta(x - (z + y_2)/2)$$

$$(-iq_k - z_k/2) A_W(q, x)$$

$$+ \int \frac{dq}{(2\pi)^4} e^{i(y_1 - z)(q - p)} \delta(x - (z + y_1)/2)$$

$$(iq_k - z_k/2) A_W(q, x)$$

$$= 2 \int \frac{dq}{(2\pi)^4} e^{i(z - x)(q - p)}$$

$$(-iq_k - z_k/2) A_W(q, x)$$

$$+ 2 \int \frac{dq}{(2\pi)^4} e^{i(z - x)(q - p)}$$

$$(iq_k - z_k/2) A_W(q, x)$$

(36)

We assume that boundary conditions in momentum space are organized in such a way that we may shift the integration variable $q \rightarrow q - p = k$. This may always be achieved via the appropriate ultraviolet regularization. As a result we are able to represent the above written variational derivative as

$$\frac{\delta A_W(p, x)}{\delta u^k(z)} = -2 \int \frac{dk}{(2\pi)^4} e^{2i(z - x)k}$$

$$\left(i(k_k + p_k) + \partial_{x^k}/2\right) A_W(k + p, x)$$

$$- 2 \int \frac{dk}{(2\pi)^4} e^{2i(x - z)k}$$

$$\left(-i(k_k + p_k) + \partial_{x^k}/2\right) A_W(k + p, x)$$

(37)

Appendix B. Covariant derivative of stress - energy tensor in terms of Wigner transformed Green function

We know that with respect to reparametrizations $y \rightarrow x(y) = y - \delta u$ the action $S = \int \frac{dq}{(2\pi)^4} e^{i(y_1 - y_2)(q - p)} \delta(x - (y_1 + y_2)/2)$
\[ S = \int d^{D+1}y_1 d^{D+1}y_2 \bar{\Psi}(y_1) \langle y_1 | \hat{Q} | y_2 \rangle \Psi(y_2) \]

is invariant:

\[ \langle x(y_1) | \hat{Q} | x(y_2) \rangle \]

\[ \langle x(y_1) | \hat{Q} | x(y_2) \rangle \frac{\partial x}{\partial y_1} \]

\[ \langle x(y_1) | \hat{Q} | x(y_2) \rangle \frac{\partial x}{\partial y_2} \]

\[ \langle x(y_1) | \hat{Q} | x(y_2) \rangle \]

Here \( e^i \) is the corresponding transformation of vielbein. Thus the reparametrizations inside \( Q \)

\[ \langle y_1 | \hat{Q} | y_2 \rangle \quad \text{(38)} \]

compensate transformation \( \Psi(y) \quad \text{to} \quad \Psi(y - \delta u) \)

correspond to the transformation of vielbein. Recall that spinors are subject to \( SU(2) \) transformations that are not related to the reparametrizations of coordinates. Spin connection is transformed as vector with respect to the transformations of coordinates. However, in Riemannian geometry the spin connection is expressed through the vielbein, and therefore, its transformation is caused by the appropriate transformation of the latter. Therefore, indeed the transformation of Eq. (38) is reduced to the transformation of \( e^i_a \). This transformation gives:

\[ Q_W \to (1 - \partial_i \delta u^i) \star (Q_W + \delta Q_W) \star (1 - \partial_i \delta u^i) \quad \text{(40)} \]

where \( \delta Q_W \) has been calculated in Appendix A. Let us denote the resulting overall variation (that takes into account both the transformation of matrix elements of \( \hat{Q} \)

\[ \text{and the extra factors } [1 - \frac{\partial \delta u}{\partial y}] \text{ by } \delta_1. \]

We get:

\[ \delta_2 Q_W = \delta Q_W - \partial_i \delta u^i \star Q_W - Q_W \star \partial_i \delta u^i \quad \text{and} \]

\[ \delta_2 Q_W = -i \left[ \delta u^k(x) \star \left[ p_k Q_W(p, x) \right] \right. \]

\[ - \left[ p_k Q_W(p, x) \star \delta u^k(x) \right) \]

\[ - \frac{1}{2} \left( \delta u^k(x) \star \partial_{x^k} Q_W(p, x) \right) \]

\[ + \partial_{x^k} Q_W(p, x) \star \delta u^k(x) \]

\[ - \partial_i \delta u^i \star Q_W - Q_W \star \partial_i \delta u^i \quad \text{(41)} \]

First we demonstrate the power of the proposed technique and calculate response of the thermodynamical potential to constant \( \delta u^k \). Such a \( \delta u^k \) causes variation of metric equal to \( \delta g^{ij} = -\delta u^k \partial_k g^{ij} \). Response of log \( Z \) to the variation \( \delta u^k \) gives the covariant derivative of stress-energy tensor. According to Eq. (41) the corresponding response of \( Q_W(p, x) \) to \( \delta u^i \) is const is:

\[ \delta^{(0)} Q_W(p, x) = -\delta u^i \partial_{x^i} Q_W(p, x) \quad \text{(42)} \]

Then:

\[ \langle D_i T_{ij} \rangle = T \int d^{D+1}x \sqrt{g(x)} \quad \text{(43)} \]

\[ = -T \int d^{D+1}x \frac{d^{D+1}p}{(2\pi)^{D+1}} \text{Tr} G_W(p, x) \star \frac{\partial}{\partial \delta u^i} Q_W(p, x) \]

\[ = T \int d^{D+1}x \frac{d^{D+1}p}{(2\pi)^{D+1}} \text{Tr} G_W(p, x) \star \partial_{x^i} Q_W(p, x) \quad \text{(44)} \]

According to Eq. (41) variation of the covariant derivative of total stress-energy tensor is given by:

\[ \delta \langle D_i T_{ij} \rangle = T \int d^{D+1}x \delta \langle D_i T_{ij} \rangle \sqrt{g(x)} \quad \text{(45)} \]

\[ = -T \int d^{D+1}x \frac{d^{D+1}p}{(2\pi)^{D+1}} \text{Tr} G_W(p, x) \star \delta Q_W(p, x) \]

\[ = T \int d^{D+1}x \frac{d^{D+1}p}{(2\pi)^{D+1}} \partial_{x^i} \text{Tr} G_W(p, x) \star \delta Q_W(p, x) \quad \text{(46)} \]

The last integral vanishes if we take appropriate boundary conditions in coordinate space. Therefore, the covariant derivative of stress-energy tensor integrated over the whole volume is robust to the smooth modification of the system. This means that the global Einstein anomaly is a topological invariant. (Recall that at \( D = 3 \) the field theory in conventional regularization predicts no Einstein (Weyl) anomaly.)

**Appendix C. Stress-energy tensor in terms of Wigner - Weyl calculus**

Equipped with Eq. (41) we derive here response of partition function to the variation of an induced vielbein. Recall, that the latter variation has the form:

\[ \delta e^i_a(x) = -\delta u^k \partial_{x^k} e^i_a(x) + e^k_a(x) \partial_k \delta u^i(x) \]

in the presence of background vielbein \( e^i_a \). First we calculate the response of \( \delta_2 Q_W(p, x) \) to \( \partial_{x^i} \delta u^i \). This calculation does not give an immediate expression for the response to the vielbein variation. But it allows to calculate it indirectly. In order to obtain this response we use derivative expansion in Eq. (41). The term proportional to the first order in derivatives gives:

\[ \delta^{(1)}_2 Q_W(p, x) = \partial_{p_j} \left( p_k Q_W \right) \partial_j \delta u^k(x) - 2Q_W \partial_j \delta u^j(x) \]

\[ = \partial_{p_j} \left( p_k Q_W \right) E_j^a e^a_k \partial_j \delta u^k(x) - 2Q_W E_j^a \partial_{p_j} e^a_k \delta u^j(x) \]

\[ = \left[ \partial_{p_j} \left( p_k Q_W(p, x) \right) E_j^a \right] \left( \delta u^k_a(x) + \delta u^m_a(x) \partial_m e^k_a(x) \right) \]

\[ - 2Q_W(p, x) E_j^a \delta \left( \epsilon^i_a(x) + \delta u^m(x) \partial_m e^i_a(x) \right) \quad \text{(47)} \]
Let us suppose, that vielbein is given by a slight variation of the constant one $e_a^i(x) = h_a^i(x) + e_a^{(0) i}$, where $e_a^{(0) i} = \text{const}(x)$. We have for $\delta^{(0)} Q W(p, x)$:

$$\delta_2^{(0)} Q W(p, x) = -\delta u^k \partial_x^k Q W(p, x)$$

$$\approx -\delta u^k \int dz \partial_x^k h_a^i(z) \frac{\delta}{\delta e_a^i(z)} Q W(p, x)$$

We know that $\delta u^k$ may enter the expression for $\delta Q W$ only through the variation $\delta e_a^i$. Combining the two leading terms in derivative expansion we get up to the terms linear in $h$ and its first derivatives the following equation:

$$\delta_2 Q W(p, x) = \left[ \partial_p \left( p \delta Q W(p, x) \right) \right] E_j^a$$

$$\left( \delta e_a^i(x) + \delta u^m(x) \partial_m e_a^i(x) \right)$$

$$-2Q W(p, x)E_j^a \delta e_a^i(x) + \delta u^m(x) \partial_m e_a^i(x)$$

$$-\delta u^k \int dz \partial_x^k h_a^i(z) \frac{\delta}{\delta e_a^i(z)} Q W(p, x)$$

Solution of this equation gives

$$\frac{\delta Q W(p, x)}{\delta e_a^i(z)} = \left[ E_k^a p_k \partial_p - E_k^a \right] Q W(p, x) \delta(x - z)$$

The total stress-energy tensor receives the form

$$\langle e_a^i(x) \Theta_a^i(x) \rangle = T \int d^{D+1} x d^{D+1} z |E(z)|$$

$$\int \frac{d^{D+1} p}{(2\pi)^{D+1}} \text{Tr} G W(p, x) \star e_a^i \frac{\delta}{\delta e_a^i(z)} Q W(p, x)$$

$$= T \int d^{D+1} x |E| \int \frac{d^{D+1} p}{(2\pi)^{D+1}} \text{Tr} G W(p, x)$$

$$\star \left[ p_i \partial_p - \delta_a^i \right] Q W(p, x)$$

This expression being understood naively is divergent both in ultraviolet and in infrared. Infrared regularization goes through the consideration of the system in large but finite volume. The superplastic crystal by itself represents an ultraviolet regularization. It is worth mentioning, that metric is a quantity, which is well defined in continuum theory. Correspondingly the stress energy tensor being the response of the system to the variation of metric, is also well-defined in continuum theory only.

**Appendix D. Functional logarithm**

In the main text the functional logarithm is used. It enters expression for the partition function $Z \equiv \text{Det} \hat{Q} = \exp \text{Tr} \log \hat{Q}$. One can also define this logarithm using a suitably regularized representation:

$$\text{Tr} O \left( \log \hat{Q} - \log \hat{Q}_0 \right) = -\lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} ds \int d^{D+1} x$$

$$\langle x | e^{i s \hat{Q} - e^{i s \hat{Q}_0}} | x \rangle$$

Here $O$ is a certain operator. Convergence of the last expression is guaranteed if all eigenvalues of $\hat{Q}$ are real, and we add to each of them the small imaginary part $i \delta$, which gives the mentioned above regularization. Thus it is assumed, that after the Wick rotation $\hat{Q}$ is Hermitian. Under such a regularization the integral over $s$ becomes convergent at $s \to \infty$.

One can check that Eq. (49) is manifestly not invariant under the reparametrizations. One can see this expanding the exponent as a series in powers of $\hat{Q}$. An example of $\hat{Q}$ given by Eq. (24) shows how the invariance under the reparametrizations is broken due to the product of determinants $|E(x)|$. In order to have the expression invariant under the reparametrizations we, therefore, have to fix the gauge $|E(x)| = 1$. Then, Eq. (49) gives the proper definition of the functional logarithm in this gauge. It has the same value for the configurations related to the given one by reparametrizations, where already $|E(x)| \neq 1$.

Thus dealing with Eq. (49) we assume the above mentioned gauge fixing. If $O$ commutes with $\hat{Q}$ and $\delta \hat{Q}$, one can easily calculate (in the gauge $|E(x)| = 1$)

$$\delta \text{Tr} \hat{O} \log \hat{Q} = -i \int d^{D+1} x |E| \int_{\epsilon}^{\infty} ds \text{Tr} O e^{i s \hat{Q} - e^{i s \hat{Q}_0}} |x \rangle$$

$$\int d^{D+1} x e^{i s \hat{Q} - e^{i s \hat{Q}_0}} |x \rangle$$

$$= -i \int_{\epsilon}^{\infty} ds \int d^{D+1} x e^{i s \hat{Q} - e^{i s \hat{Q}_0}} |\hat{O} \hat{Q}^{-1} \delta \hat{Q}| |x \rangle$$

$$= \int d^{D+1} x \langle x | \hat{O} \hat{Q}^{-1} \delta \hat{Q} |x \rangle$$

$$= \int d^{D+1} x \frac{d^{D+1} p}{(2\pi)^{D+1}} O W \star G W \star \delta Q W$$

For $O_W = O(p)$ that is the function of $p$ only the last expression may be easily generalized to its covariant form adding the extra factor $|E(x)|$:

$$\delta \text{Tr} \hat{O} \log \hat{Q} = \int d^{D+1} x |E(x)| \frac{d^{D+1} p}{(2\pi)^{D+1}} O(p)$$

$$G_W \star \delta Q_W$$

\[ (51) \]

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