Existence theory for well-balanced Euler model

Shuyang Xiang, Yangyang Cao

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Abstract

We study the initial value problem for a kind of Euler equation with a source term. Our main result is the existence of a globally-in-time weak solution whose total variation is bounded on the domain of definition, allowing the existence of shock waves. Our proof relies on a well-balanced random choice method called Glimm method which preserves the fluid equilibria and we construct a sequence of approximate weak solutions which converges to the exact weak solution of the initial value problem, based on the construction of exact solutions of the generalized Riemann problem associated with initially piecewise steady state solutions.

1 Introduction

Our model of interest is a non-conservative Euler equation with a source term reading

\begin{align*}
\partial_t \rho + \partial_r (\rho v) + \frac{2}{r} \rho v &= 0, \\
\partial_t (\rho v) + \partial_r \left( \rho (v^2 + k^2) \right) + \frac{2}{r} \rho v^2 + \frac{1}{r^2} m \rho &= 0,
\end{align*}

(1.1)

defined for all \( r > 0 \) where the main unknowns are the density \( \rho > 0 \) and the velocity \( v \) of a fluid flow in consideration. The model (1.1) is indeed the "non-relativistic version" of the Euler equation on a Schwarzschild spacetime background studied by LeFloch and Xiang [8] where a well-posedness theory was given for the relativist model. Here, the parameters are given as the Schwarzschild black hole mass \( m \in (0, +\infty) \) and the constant sound speed \( k \in (0, +\infty) \). An interesting observation is that even if the Euler model (1.1) is non-relativistic in the sense that the velocity \( v \) is far from light speed, the mass of the black hole \( m \) is still reflected by the source term.

Our model has the form of a well-balanced hyperbolic system with the right-hand side source terms because of the geometry of the Schwarzschild space. Such well-balanced system was first investigated by Dafermos and Hsiao [1], Liu [11], for different applications. In our investigation, we closely follow LeFloch and Xiang [8], which treated the relativistic version of the Euler model by allowing the fluid speed comparable to the speed of light. However, in our non-relativistic case, we were able to get rid of the influence of the light speed and had some stronger results.

Our main contributions of the Euler model with a source terms (1.1) are listed as follows:

- A systematic study of the existence the steady state solutions.

\*MATHCCES, Department of Mathematics, RWTH Aachen University, Schinkelstrasse 2, D-52062 Aachen, Germany. E-mail : xiang@mathcces.rwth-aachen.de

\*Laboratoire Jacques-Louis Lions & Centre National de la Recherche Scientifique, Sorbonne Université, 4 Place Jussieu, 75252 Paris, France. E-mail : caoy@ljll.math.upmc.fr
• The global-in-time existence of the (triple) generalized Riemann problem, which is an initial problem of (1.1) with a given piecewise steady state. Moreover, we gave also an analytical formulation of the exact solution.

• The existence of the Euler model (1.1) with an arbitrary initial data with bounded total variation.

The organization of this paper is as follows. In Section 2, we give some basic properties of the homogenous Euler model without source term, including the hyperbolicity and the nonlinear properties which lead us to give the result of the standard Riemann problem whose wave interactions are analyzed as well.

We take into consideration the steady state solutions in Section 3, where we first study different families of smooth steady state solutions to the Euler model, serving as one of the main results of the present paper. The study coming after is the generalized Riemann problem of the Euler model with the initial data consisting of two steady state solutions separated by a discontinuity of jump. An exact solution is constructed in Section 4, with three steady states connected by two different families of generalized elementary waves and we have verified that the Rankie-Hugoniot jump condition and the Lax entropy condition are satisfied. We also give the evolution of the total variation of the solution of the Riemann problem.

Referring to Section 3, smooth steady states may not be extended on the whole space region \((0, +\infty)\). To give a complete construction of an initial value problem, it is necessary to consider a so-called triple Riemann problem, which is an initial problem with its initial data given as three steady state solutions separated by two given radius. Such problem was first studied by Lefloch and Xiang [9] for a Burgers model on the Schwarzschild spacetime. We provide a global-in-time solution of such problem for our model in Section 5.

In Section 6, we are then able to give an existence theory of our Euler model. Inspired by the classic Glimm method [2] and the application of such method in the case of fluid flows in a flat space [12, 13], we generalize the method based on the (triple) generalized Riemann problem, developed earlier in [3, 9] in a different geometric setup and provides us with the desired global-in-time result. For the fluids of the Euler model in consideration in the present paper, the geometry may leads to the growth of the total variation of the solution, but we prove that it is uniformly controlled on any compact interval of time and consequently, sequence is proved to converge to the exact global-in-time solution of the Euler model (1.1).

2 Homogenous system

2.1 Elementary waves

According to (1.1), we write the Euler system as

\[
\partial_t U + \partial_r F(U) = S(r, U),
\]

where

\[
U = \left( \begin{array}{c} \rho \\ \rho v \end{array} \right), \quad F(U) = \left( \begin{array}{c} \rho v \\ \rho (v^2 + k^2) \end{array} \right), \quad S(r, U) = \left( \begin{array}{c} -\frac{2}{r} \rho v \\ -\frac{2}{r} \rho v^2 - \frac{1}{r^2} m \rho \end{array} \right).
\]

We derive the pair of eigenvalues reading

\[
\lambda(\rho, v) = v - k, \quad \mu(\rho, v) = v + k.
\]

We give also the pair of corresponding Riemann invariants:

\[
w(\rho, v) = v + k \ln \rho, \quad z(\rho, v) = v - k \ln \rho.
\]

Following directly from (2.2), we have the following proposition:
Proposition 2.1. Let \( k > 0 \) be the sound speed and \( m > 0 \) the black hole mass, the non-conservative Euler model (1.1) is strictly hyperbolic and both characteristic fields are genuinely nonlinear.

Proposition 2.1 enables us to consider first the elementary waves of the homogenous Euler system:

\[
\partial_t U + \partial_x F(U) = 0,
\]

where we recall that \( U = (\rho, \rho v)^T \) and \( F(U) = (\rho v, \rho (v^2 + k^2))^T \) according to (2.1). Notice that \( (\rho, v) \to (\rho, \rho v) \) is a one-to-one map and we thus don’t distinguish \( U \) and \( (\rho, v) \) in the following for the sake of simplicity.

We consider first the rarefaction curves along which the corresponding Riemann invariants remain constant.

Lemma 2.2. Consider the homogenous Euler model given by (2.4). The 1-rarefaction curve issuing from constant \( U_L = (\rho_L, v_L) \) and the 2-rarefaction wave from the constant \( U_R = (\rho_R, v_R) \) are given by

\[
R_1^+(U_L) : \left\{ v - v_L = \ln \left( \frac{\rho}{\rho_L} \right)^{-k}, \; v < v_L \right\}, \quad R_2^-(U_R) : \left\{ v - v_R = \ln \left( \frac{\rho}{\rho_R} \right)^k, \; v < v_R \right\}.
\]

Proof. The 1-family Riemann invariant is a constant along the 1-rarefaction curve passing the point \( U_L \) and we have

\[
R_1^+(U_L) : w(\rho, v) = w(\rho_L, v_L), \; z(\rho, v) < z(\rho_L, v_L),
\]

which gives the form of the 1-rarefaction wave. Similarly, we have the 2-rarefaction wave. \( \square \)

We can also give the form of 1-shock and 2-shock associated with the constant states \( U_L \) and \( U_R \) respectively.

Lemma 2.3. The 1-shock wave and 2-shock wave of the Euler model without source term (2.4) associated with the constant states \( U_L \) and \( U_R \) respectively have the following forms:

\[
S_1^+(U_L) : \left\{ v - v_L = -k \sqrt{\frac{\rho}{\rho_L} - \frac{\rho_L}{\rho}}, \; v > v_L \right\},
\]

\[
S_2^-(U_R) : \left\{ v - v_R = k \sqrt{\frac{\rho}{\rho_R} - \frac{\rho_R}{\rho}}, \; v > v_R \right\}.
\]

And the 1-speed \( \sigma_1 \) and the 2-speed \( \sigma_2 \) are:

\[
\sigma_1((\rho_L, v_L), (\rho, v)) = v - k \sqrt{\frac{\rho_L}{\rho}}, \; \sigma_2((\rho, v), (\rho_R, v_R)) = v + k \sqrt{\frac{\rho_R}{\rho}}.
\]

Proof. The Rankine-Hugoniot jump condition gives

\[
\sigma[\rho] = [\rho v],
\]

\[
\sigma[\rho v] = [\rho (v^2 + k^2)],
\]

where \( \sigma \) denotes the speed of the discontinuity. Consider first the 1-shock which should satisfy the Lax entropy inequality in the sense that

\[
\lambda(\rho_L, v_L) > \sigma > \lambda(\rho, v),
\]

for the 1-shock wave. Eliminating the speed \( \sigma \), we obtain:

\[
v - v_L = -k \sqrt{\frac{\rho}{\rho_L} - \frac{\rho_L}{\rho}}, \; v > v_L.
\]

The form of the 2-shock wave follows from a similar calculation. The shock speeds can be obtained directly from (2.6), (2.8). \( \square \)
2.2 Standard Riemann problem

We now consider the solution of the standard Riemann problem of the homogeneous Euler system (2.4) associated with given initial data:

$$U_0(r) = \begin{cases} U_L, & 0 < r < r_0, \\ U_R, & r > r_0, \end{cases}$$  \hspace{1cm} (2.9)

where $r_0 > 0$ is a fixed radius and $U_L = (\rho_L, v_L)$, $U_R = (\rho_R, v_R)$ are constant states. To give the solution of the standard Riemann problem, we define now the 1-family-wave and the 2-family wave:

$$W_1^+(U_L) = S_1^+(U_L) \cup R_1^+(U_L), \quad W_2^-(U_R) = S_2^-(U_R) \cup R_2^-(U_R),$$  \hspace{1cm} (2.10)

where $S_1^+$, $S_2^-$ are 1 and 2-shocks while $R_1^+$, $R_2^-$ are 1 and 2-rarefaction waves. It is obvious that if $U_L \in W_2^-(U_R)$ or $U_R \in W_1^+(U_L)$, then the Riemann problem is solved by the left state $U_L$ and the right state $U_R$ connected by either a 1-family wave or a 2-family wave. Otherwise, more analysis are required.

**Lemma 2.4.** On the $w-z$ plane where $w,z$ are the Riemann invariants of the Euler model given by (2.3), $S_1^+(U_L)$ defines a curve such that $0 \leq \frac{dw}{dz} < 1$, $S_2^-(U_R)$ defines a curve satisfying $0 \leq \frac{dw}{dz} < 1$ where $S_1^+, S_2^-$ are the 1 and 2-shocks given by (2.6).

**Proof.** Introduce functions $\Phi_{\pm}$:

$$\Phi_{\pm}(\gamma) := 1 + \gamma \left( 1 \pm \sqrt{1 + \frac{2}{\gamma}} \right).$$  \hspace{1cm} (2.11)

Taking $\gamma = \gamma(v,v_L) = \frac{(v-v_L)^2}{2\kappa^2}$ along the 1-shock, we have

$$w - w_L = v - v_L + k \ln \frac{\rho}{\rho_L} = -\sqrt{2\gamma k^2} + k \ln \Phi(\gamma),$$

$$z - z_L = v - v_L - k \ln \frac{\rho}{\rho_L} = -\sqrt{2\gamma k^2} - k \ln \Phi(\gamma).$$

The tangent of the shock wave curve $S_1^+(U_L)$ in the $w-z$ plane is given by

$$\frac{dw}{dz} = \frac{d(w - w_L)}{d(z - z_L)} = \frac{d(w - w_L)}{d\gamma} \frac{d\gamma}{d(z - z_L)}.$$ 

Hence, we have $0 \leq \frac{dw}{dz} < 1$. A similar calculation gives the result of the 2-shock. \hfill \Box

Together with Lemma 2.4 and the form of elementary waves given in Lemmas 2.5, 2.6 some direct observations are given in order, concerning the standard Riemann problem of the homogeneous Euler model (2.4):

- For different given states $U_L, U_L'$, the two 1-family wave curves $W_1^+(U_L) \cap W_1^+(U_L') = \emptyset$. Similarly, for $U_R \neq U_R'$, the 2-family wave curve $W_2^-(U_R)$ has no intersection point with $W_2^-(U_R')$.
- The two families of wave curves cover the whole upper half $\rho - v$ plane as a result of Lemma 2.4.
- For given constant states $U_L, U_R$, the waves $W_1^+(U_L)$ and $W_2^-(U_R)$ intersect once and only once at a point $U_M$.

We thus have the proposition:

**Proposition 2.5** (Solution of the standard Riemann problem). Given two constant states $U_L = (\rho_L, v_L)$ and $U_R = (\rho_R, v_R)$, the standard Riemann problem (2.4), (2.9) admits a unique entropic solution which only depends on $r^\pm r_0$. More precisely, the solution is realized by the left state $U_L$, the right state $U_R$ and a uniquely defined intermediate state $U_M$ where $U_L$ and $U_M$ are connected by a 1-wave while $U_M$ and $U_R$ are connected by a 2-wave.
2.3 Wave interactions

For the standard Riemann problem of the Euler model without source term (2.4) with left-hand side constant state $U_L$ and right-hand side constant state $U_R$, define the wave strength of the Riemann problem $S = S(U_L, U_R)$:

$$S(U_L, U_R) := |\ln \rho_L - \ln \rho_M| + |\ln \rho_R - \ln \rho_M|,$$

where $U_M$ is the unique intermediate state $U_M \in W_1^+(U_L) \cap W_2^+(U_R)$. We have the following lemma concerning $S$:

Lemma 2.6. Let $U_L, U_P, U_R$ be three given constant states. The wave strengths associated with the Riemann problem $(U_L, U_P)$, $(U_P, U_R)$ and $(U_L, U_R)$ satisfy the following inequality

$$S(U_L, U_R) \leq S(U_L, U_P) + S(U_P, U_R). \quad (2.12)$$

To prove Lemma 2.6, we first need the following calculation.

Lemma 2.7. Given an arbitrary state $U_0$, the 1 and 2-shock wave curves $S_1^+(U_0)$ and $S_2^+(U_0)$ are reflectional symmetric with respect to the straight line parallel to $w = z$ passing the point $U_0$ on the $w - z$ plane where $w, z$ are the Riemann invariants of the Euler model introduced by (2.3).

Proof. Denote by $(w_0, z_0)$ the point $U_0$ on the $w - z$ plane. For a given point $(w, z)$ along the 1-shock, we have

$$\Delta w_1 := w - w_0 = -\sqrt{2\gamma k^2 + k \ln \Phi_+(\gamma)}, \quad \Delta z_1 := z - z_0 = -\sqrt{2\gamma k^2 - k \ln \Phi_+(\gamma)},$$

while for a point along the 2-shock $(w, z)$:

$$\Delta w_2 := w - w_0 = -\sqrt{2\gamma k^2 + k \ln \Phi_-(\gamma)}, \quad \Delta z_2 := z - z_0 = -\sqrt{2\gamma k^2 - k \ln \Phi_-(\gamma)},$$

where the function $\Phi_\pm$ is defined by (2.11), which gives $\Phi_+(\gamma)\Phi_-(\gamma) = 1$. We have got the result by noticing that $\Delta w_1 = \Delta z_2, \quad \Delta z_1 = \Delta w_2$. $\Box$

We can thus continue the proof of Lemma 2.6.

Proof of Lemma 2.6. Again, we stay on $w - z$ plane. From Lemmas 2.4, 2.7, we can see that the shock waves $S_1^+, S_2^+$ passing the same point $U_0$ are symmetric with respect to the straight line parallel to $w = z$ passing the point $U_0$. According to the definition of the wave strength (2.12), which is actually measured along the line $w = z$, the symmetry of waves gives immediately the result. $\Box$

3 Fluid equilibria

3.1 Critical smooth steady state solutions

We now turn our attention to steady state solutions $\rho = \rho(r), v = v(r)$, which satisfies the ordinary differential system:

$$\frac{d}{dr}(r^2 \rho v) = 0, \quad (3.1)$$

$$\frac{d}{dr}(r^2 (v^2 + k^2)\rho) - 2k^2 \rho r + m \rho = 0,$$

with the initial condition $\rho_0 > 0, v_0$ posed at a given radius $r = r_0 > 0$,

$$\rho(r_0) = \rho_0 > 0, \quad v(r_0) = v_0. \quad (3.2)$$
We call to (3.1) the static Euler model. For a steady state solution \( \rho = \rho(r), v = v(r) \), it is straightforward to find a pair of algebraic relations:

\[
\begin{align*}
    r^2 \rho v &= r_0^2 \rho_0 v_0, \\
    \frac{1}{2} v^2 + k^2 \ln \rho - m \frac{1}{r} &= \frac{1}{2} v_0^2 + k^2 \ln \rho_0 - m \frac{1}{r_0},
\end{align*}
\]

from which we recover the equation for \( v \) by eliminating \( \rho \):

\[
\frac{1}{2} v^2 - k^2 \ln (r^2 \text{sgn}(v_0)v) - m \frac{1}{r} = \frac{1}{2} v_0^2 - k^2 \ln(r_0^2 |v_0|) - m \frac{1}{r_0}. \tag{3.3}
\]

Notice that once we get the value of \( v \), we can have the value \( \rho \) directly from the first equation of (3.1). Therefore, we focus on the analysis of the steady state velocity \( v \).

Introduce the function \( G = G(r, v) \):

\[
G(r, v) := \frac{1}{2} v^2 - k^2 \ln(r^2 \text{sgn}(v_0)v) - m \frac{1}{r}, \tag{3.4}
\]

and we see if \( v = v(r) \) is a solution of (3.1) with the condition \( v(r_0) = v_0 \), then \( G(r, v(r)) \equiv G(r_0, v_0) \) always holds. Differentiating \( G \) with respect to \( v \) and \( r \), we obtain

\[
\partial_v G = v - \frac{k^2}{v}, \quad \partial_r G = \frac{1}{r^2} (m - 2k^2 r). \tag{3.5}
\]

We can immediately deduce the first-order derivative of the steady state velocity \( v = v(r) \):

\[
\frac{dv}{dr} = \frac{v}{r^2} \frac{2k^2 r - m}{v^2 - k^2}. \tag{3.6}
\]

It is obvious to see that \( \partial_v G = 0 \) if and only if \( v = \pm k \) while \( \partial_r G = 0 \) if and only if \( r = \frac{m}{2k^2} \) from (3.5). This observation motivates us to find the steady state curves passing the points \((\frac{m}{2k^2}, \pm k)\) on the \( r - v \) plane \((0, +\infty) \times (-\infty, +\infty)\). We call the solution \( v = v(r) \) on the subset of \( r - v \) plane \((0, +\infty) \times (-\infty, +\infty)\) the critical steady state solution of the static Euler model (3.1) if and only if satisfies \( S(r, v(r)) \equiv 0 \) where \( S = S(r, v) \) is given by

\[
S(r, v) := \frac{1}{2} v^2 - k^2 \ln (r^2 |v|) - \frac{m}{r} + \frac{3}{2} k^2 + k^2 \ln \frac{m^2}{4k^2}.
\tag{3.7}
\]

It is direct to check that \( S(\frac{m}{2k^2}, \pm k) = 0 \). We now have the following lemma concerning the critical steady state curve.

**Proposition 3.1.** The static Euler model (3.1) admits four smooth critical steady state curves on the subset of \( r - v \) plane \((0, +\infty) \times (-\infty, +\infty)\) denoted by \( v^{P,\flat}_s, v^{P,\sharp}_s, v^{N,\flat}_s, v^{N,\sharp}_s \). Moreover, we have the following properties:

- The sign of each solution does not change on the space domain \((0, +\infty)\).
- On the interval \((0, \frac{m}{2k^2})\), we have
  \[
  v^{N,\flat}_s < -k < v^{N,\sharp}_s < 0 < v^{P,\flat}_s < k < v^{P,\sharp}_s,
  \]
  while on the interval \((\frac{m}{2k^2}, +\infty)\), we have
  \[
  v^{N,\flat}_s < -k < v^{N,\sharp}_s < 0 < v^{P,\flat}_s < k < v^{P,\sharp}_s.
  \]
- The solutions \( v^{P,\flat}_s, v^{N,\flat}_s \) intersect once at \((\frac{m}{2k^2}, -k)\) while \( v^{P,\sharp}_s, v^{N,\sharp}_s \) intersect once at \((\frac{m}{2k^2}, k)\).
The derivatives of each solution at \( \frac{m}{2k^2}, \pm k \) are give by

\[
\frac{dv^{\beta}_{+}}{dr} \left( \frac{m}{2k^2} \right) = \frac{dv^{\beta}_{-}}{dr} \left( \frac{m}{2k^2} \right) = \frac{2k^3}{m}, \quad \frac{dv^{\beta}_{+}}{dr} \left( \frac{m}{2k^2} \right) = \frac{dv^{\beta}_{-}}{dr} \left( \frac{m}{2k^2} \right) = \frac{2k^3}{m} \tag{3.8}
\]

**Proof.** We would like to show that for every fixed radius \( r > 0 \) and \( r \neq \frac{m}{2k^2} \), there exist four different values \( v \) satisfying (3.7). Observing \( S(r, v) = S(r, -v) \), we first consider the case where \( v > 0 \). According to (3.5), for every fixed \( r > 0 \), \( S(r, \cdot) \) reaches its minimum at \( v = k \) and the value is given as

\[
S^k(r) := 2k^2 - \frac{k^2}{r} - m^2 \frac{4k^3}{r^4}.
\]

Since \( \partial_r S^k \geq \frac{1}{2} \), \( m - 2k^2 r \), we have \( S^k(r) < S^k(\frac{m}{2k^2}) = 0 \). Moreover, we have \( \lim_{r \to 0} S(r, v) = +\infty \) and \( \lim_{r \to +\infty} S(r, v) = +\infty \). Therefore, for every fixed \( r \neq \frac{m}{2k^2} \), \( S(r, v) \) admits two different positive roots \( v_1 \leq k \leq v_2 \) on \((0, +\infty)\) where the equality holds only once at the point \( r = \frac{m}{2k^2} \). The symmetry of \( S(r, \cdot) \) with respect to \( v = 0 \) gives two other negative roots \( v_3 \leq -k \leq v_4 \).

Since \( S_r \neq 0 \) when \( v \neq \pm k \), there exist four smooth different solutions on the interval \((0, \frac{m}{2k^2})\) and \((\frac{m}{2k^2}, +\infty)\) respectively. To extend the steady solution on the whole domain \((0, +\infty)\), we have to treat the very points \( \frac{m}{2k^2}, \pm k \). Indeed, we have, by the L'Hôpital's rule, \( \frac{dv}{dr} \left( \frac{m}{2k^2} \right) = \frac{k}{(m/2k^2)^2} k^3 / \left( \frac{k}{(m/2k^2)^2} k^3 \right) \), which gives

\[
\frac{dv}{dr} \left( \frac{m}{2k^2} \right) = \pm \frac{2k^3}{m}, \tag{3.9}
\]

whose sign depends on the choice of the branch of curves. According to (3.9), we are able to to keep the solution smooth on the whole domain \((0, +\infty)\) by keeping the sign of the derivative of \( v \) at \( r = \frac{m}{2k^2} \). We thus define the four different solutions on \((0, +\infty)\): 

\[
v^+_{\pm}(r) = \begin{cases} v_1(r) & r \in (0, \frac{m}{2k^2}), \\ v_2(r) & r \in (\frac{m}{2k^2}, +\infty), \end{cases} \quad v^-_{\pm}(r) = \begin{cases} v_2(r) & r \in (0, \frac{m}{2k^2}), \\ v_1(r) & r \in (\frac{m}{2k^2}, +\infty), \end{cases} \tag{3.10}
\]

The derivative of the velocities in (3.8) follows directly from (3.9) and (3.10). \( \square \)

### 3.2 Families of steady state solutions

The former construction gives that the relation \( S(r, v) \equiv 0 \) admits four different solutions on the whole domain \((0, +\infty)\). We would like now to give all families of solutions according to the sign of \( S(r, v) \) defined in (3.7). We now study general cases of the steady state solutions.

We then have the following lemma.

**Lemma 3.2.** Let \( S = S(r, v) \) be the function defined by (3.9), then:

- If \( S = \text{const.} > 0 \), then there exists four solutions \( v = v(r) \) satisfying the algebraic equation (3.3) on the whole space interval out of the black hole \((0, +\infty)\).
- If \( S = \text{const.} < 0 \), then there exist two radius \( 0 < r_S < \frac{m}{2k^2} < r_S \) such that there exist four solutions \( v = v(r) \) satisfying the algebraic equation (3.3) on the interval \((0, r_S)\) and four solutions satisfying (3.1) on the interval \((r_S, +\infty)\).

**Proof.** We now focus on the case where \( S = \text{const.} > 0 \). Again, \( S(r, v) = S(r, -v) \) allows us to consider the case where \( v > 0 \). Now we notice that \( G(r, v) = G(\frac{m}{2k^2}, k) = S(r, v) \) where \( G \) is defined by (3.4). By
the formula of (3.5), for all the fixed $r \in (0, +\infty)$, the equation $G(r, v) - G(\frac{m}{2k^2}, k) = \text{const.} > 0$ admits two positive roots $v^{P,\rho}_S > k > v^{P,\rho}_S$ if and only if $G(r, k) < G(\frac{m}{2k^2}, k)$. Moreover, (3.5) gives the fact that $G(r, k)$ reaches its maximum at the point $r = \frac{m}{2k^2}$ and we thus have $G(r, k) < G(\frac{m}{2k^2}, k)$. We have another two negative roots $v^{N,\rho}_S < -k < v^{N,\rho}_S$ following from the same analysis.

Now if $S = \text{const.} < 0$, there exist two points $0 < r_S < \frac{m}{2k^2} < r_S$ such that $S(r_S, k) = S(\bar{r}_S, k) = 0$ and $S(r, k) < 0$ for all $r \in (r_S, \bar{r}_S)$. We have four roots satisfying (3.3) among which two are defined only on $(0, r_S)$ while two on $(\bar{r}_S, +\infty)$ respectively.

We can now give the existence result of the steady state solution of the Euler model (1.1).

**Theorem 3.3 (Families of steady state solutions).** Consider the family of steady state solutions of the Euler model (3.1). Then, for any given radius $r_0 > 0$, the density $\rho_0 > 0$ and the velocity $v_0$, we have: there exists a unique smooth steady state solution $\rho = \rho(r), v = v(r)$ satisfying (3.1) together with the initial condition $\rho_0 = \rho(r_0), v(r_0) = v_0$ such that the velocity satisfies $\text{sgn}(v) = \text{sgn}(v_0)$ and $\text{sgn}(|v| - k) = \text{sgn}(|v_0| - k)$ on the corresponding domains of definition. Furthermore, we have different families of solutions:

- If $G(r_0, v_0) > -\frac{3}{2}k^2 - k^2 \ln \frac{m^2}{4k^2}$ in which the parameter $G = G(r, v)$ was introduced in (3.4), then the steady state solution is defined on the whole space interval $(0, +\infty)$.
- If $G(r_0, v_0) = -\frac{3}{2}k^2 - k^2 \ln \frac{m^2}{4k^2}$, then we have the critical steady state solution on the whole interval $(0, +\infty)$ whose formula is given by (3.10).
- If $G(r_0, v_0) < -\frac{3}{2}k^2 - k^2 \ln \frac{m^2}{4k^2}$, then the solution is defined on $(0, r_S)$ if $r_0 < \frac{m}{2k^2}$ or $(\bar{r}_S, +\infty)$ if $r_0 > \frac{m}{2k^2}$ where $r_S, \bar{r}_S$ satisfies $G(r_S, k) = G(\bar{r}_S, k) = G(r_0, v_0)$.

![Figure 3.1: Plot of steady state solutions.](image)

### 3.3 Steady shock

We now consider the steady shock which is also a solution of the static Euler equation (3.1) but contains one discontinuity satisfying also the entropy condition. We give the following lemma.

**Lemma 3.4 (Jump conditions for steady state solutions).** A steady state discontinuity of the Euler model (1.1) associated with left/right-hand limits $(\rho_L, v_L)$ and $(\rho_R, v_R)$ must satisfy

$$\frac{\rho_R}{\rho_L} = \frac{v_L^2}{k^2}, \quad v_L v_R = k^2, \quad v_L \in (-k, 0) \cup (k, +\infty).$$
Proof. From the steady Rankine-Hugoniot relations
\[ [ \rho v] = 0, \quad [\rho (k^2 + v^2)] = 0, \]
where the bracket \([\cdot]\) denoted the value of the jump and we deduce that
\[ \rho_R v_R = \rho_L v_L, \quad \rho_R (v_R^2 + k^2) = \rho_L (v_L^2 + k^2), \]
which gives the relation of the left-hand side and the right-hand side limit of the jump. Then the Lax entropy condition requires that \(\lambda(\rho_L, v_L) > 0 > \lambda(\rho_R, v_R), \mu(\rho_L, v_L) > 0 > \mu(\rho_R, v_R)\) for 1 and 2-waves.

Lemma 3.4 permits us to construct a steady shock wave of the Euler model (1.1) with a zero speed, that is, a function composed of a pair of steady state solutions \((\rho_L, v_L) = (\rho_L, v_L)(r), (\rho_R, v_R) = (\rho_R, v_R)(r)\) separated by a discontinuity at a fixed point \(r_0\) with the relation
\[
\begin{align*}
v_R(r_0) &= \frac{k^2}{v_L(r_0)}, & \rho_R(r_0) &= \frac{v_L(r_0)^2}{k^2} - \rho_L(r_0),
\end{align*}
\]
with
\[
v_L(r_0) \in v_L \in (-k, 0) \cup (k, +\infty).
\]

4 The generalized Riemann problem

4.1 The rarefaction regions

The generalized Riemann problem of the Euler model is a Cauchy problem of (1.1) with given initial data given as
\[
U_0(r) = \begin{cases} U_L(r) & \leq r < r_0, \\ U_R(r) & r_0 < r < \bar{r}, \end{cases}
\]
for a fixed radius \(r_0 > 0\) and two steady state solutions \(U_L = (\rho_L, v_L)\) and \(U_R = (\rho_R, v_R)\) such that the static Euler equation (3.1) holds.

For simplicity, we write \((\rho_L, v_L)(r_0) = (\rho_L^0, v_L^0) = U_L^0\) and \((\rho_R, v_R)(r_0) = (\rho_R^0, v_R^0) = U_R^0\). To solve the generalized Riemann problem, we need first to fix the point \(r = r_0\) and solve the standard Riemann problem (2.4) with initial data
\[
U_0(r) = \begin{cases} U_L^0 & \leq r < r_0, \\ U_R^0 & r_0 < r < \bar{r}. \end{cases}
\]
The standard Riemann problem at a fixed radius is solved by three constant states \(U_L^0 = (\rho_L^0, v_L^0), U_M^0 = (\rho_M^0, v_M^0)\) and \(U_R^0 = (\rho_R^0, v_R^0)\) connected to each other with 1-wave and 2-wave respectively where the intermediate constant state is given by
\[
U_M^0 \in W_1^{-}(U_L^0) \bigcap W_2^{+}(U_R^0).
\]

Coming back to the Euler equation with source term (1.1), we would like to construct a solution of the generalized Riemann problem (1.1), (4.1) with three steady state solutions connected by generalized elementary curves. We give the intermediate steady state solution denoted by \((\rho_M, v_M) = (\rho_M, v_M)(r)\) by the static Euler equation (3.1) with initial data \((\rho_M^0, v_M^0)\) at the point \(r = r_0\), that is
\[
(\rho_M, v_M)(r_0) = (\rho_M^0, v_M^0). \quad (4.3)
\]

To work on different types of elementary waves, we consider the following differential equations:
The curves to the sonic point on Lemma 4.1. Let
\[
  t \to \lambda (\rho_M(r_M^+, v_M(r_M^+))), \quad v_L < v_M,
\]
\[
  t \to \sigma_1 \left( (\rho_L(r_L^+, v_L(r_L^+)), (\rho_M(r_M^+, v_M(r_M^+))) \right), \quad v_L > v_M,
\]
\[
  t \to \lambda (\rho_L(r_L^-, v_L(r_L^-))), \quad v_L < v_M, \quad (4.4)
\]
\[
  t \to \sigma_1 \left( (\rho_L(r_L^-, v_L(r_L^-)), (\rho_M(r_M^-, v_M(r_M^-))) \right), \quad v_L > v_M,
\]
where \( \sigma_1, \sigma_2 \) are speeds of 1 and 2-shocks respectively and \( \lambda, \mu \) are eigenvalues given by (2.2).

**Lemma 4.1.** Let \((\rho_1, v_1) = (\rho_L, v_L)(r), (\rho_M, v_M) = (\rho_R, v_R)(r)\) be two steady state solutions given by (3.1). The curves \( r_L^{M\pm}, r_R^{M\pm} \) are uniquely defined by (4.4), (4.5) for all \( t > 0 \) respectively, with bounded derivatives.

**Proof.** We first consider the 1-wave. If \((\rho_L^0, v_L^0)\) and \((\rho_M^0, v_M^0)\) are connected by a 1-rarefaction, then we have
\[
  \frac{dr_L^{M+}}{dt} = \lambda (\rho_M(r_M^{M+}, v_M(r_M^{M+}))), \quad \frac{dr_L^{M-}}{dt} = \lambda (\rho_L(r_L^{M-}, v_L(r_L^{M-}))),
\]
Following from the existence theory of ordinary differential equations, there exists a time \( T > 0 \) such that the curves are well-defined on \( 0 < t < T \). To prove that these curves are indeed defined globally in time, we have to show that steady state solutions can not be sonic along the wave curves, referring to Theorem 3.3. We take into account two cases:

- When \( r_0 < \frac{m}{2k} \), \( v_L = v_L(r) \) cannot be sonic for all \( r < r_0 \). Then we only have to consider the case where \( r_L^{M-}(t) > r_0 \), which gives \( \frac{dr_L^{M-}(t)}{dt} > 0 \), providing \( v_L \geq k \). If there exists a finite time \( t_1 \) such that \( v_L(r_L^{M-}(t_1)) = k \), then \( \frac{dr_L^{M-}(t)}{dt}|_{t=t_1} = v_L(r_L^{M-}(t_1)) - k \), which provides a contradiction.
- When \( r_0 \geq \frac{m}{2k} \), the following equality holds \( r_L^* \leq r < r_0 \) where \( r_L^* \) is the sonic point of \((\rho_L, v_L)\), then we have at once the result.

Now if \((\rho_L^0, v_L^0)\) and \((\rho_M^0, v_M^0)\) is connected by a 1-shock, the result will hold if \((\rho_L, v_L)\) will not reach to the sonic point on \((r, r_L^{M-}(t))\) for \( 0 < t < T \). We consider the two cases as follows.

- When \( r_0 < \frac{m}{2k} \), we only have to consider the case where \( \sigma_1 > 0 \). The entropy condition gives \( \lambda(\rho_L, v_L) > \sigma_1 > \lambda(\rho_M, v_M) \), leading to \( v_L > k \). Then we have the result.
- When \( r_0 \geq \frac{m}{2k} \), we have \( r_L^* \leq r < r_0 \) and the result holds.

A similar calculation gives all the curves listed in the lemma. 

It follows directly from the definition that \( r_L^{M-}(t) \leq r_L^{M+}(t) \leq r_M^{M+}(t) \leq r_M^{M+}(t) \), which permits us to define five disjoint regions below for all fixed \( t > 0 \): \((r_L^{M-}(t), r_M^{M+}(t))\), \((r_M^{M+}(t), r_M^{M+}(t))\), \((r_M^{M+}(t), r_M^{M-}(t))\), \((r_M^{M-}(t), r_M^{M+}(t))\), \((r_M^{M+}(t), r_M^{M+}(t))\) and we denote by \((r_L^{M-}(t), r_M^{M+}(t))\) and \((r_M^{M+}(t), r_M^{M+}(t))\) the 1-rarefaction region and the 2-rarefaction region.

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4.2 Exact solution to Riemann problem

We now give the solution \( U = (\rho, v) = (\rho, v)(t, r) \) for the generalized Riemann problem. Write

\[
U(r) := \begin{cases} 
U_L(r) & r < r_L^{M-}(t), \\
U_1(t, r) & r_L^{M-}(t) < r < r_L^{M+}(t), \\
U_M(r) & r_L^{M+}(t) < r < r_M^{R-}(t), \\
U_2(t, r) & r_M^{R-}(t) < r < r_M^{R+}(t), \\
U_R(r) & r_M^{R+}(t) < r < \bar{r}, 
\end{cases}
\tag{4.6}
\]

where \( r_L^{M\pm}, r_M^{R\pm} \) are boundaries of the rarefaction regions defined by (4.4), (4.5). Here, \( U_L = (\rho_L, v_L) \), \( U_M = (\rho_M, v_M) \), \( U_R = (\rho_R, v_R) \) are three steady state solutions and \( U_1 \) and \( U_2 \) are generalized rarefaction waves to be given by the integro-differential problem following from Liu [11]. Indeed, we give the function \( \tilde{U}_j(t, \theta_j) = (\tilde{\rho}_j, \tilde{\sigma}_j)(t, \theta_j), j = 1, 2 \) and the new variable \( \tilde{r} = \tilde{r}(t, \theta_j) \). To seek for the form of \( \tilde{U}_j \) and \( \tilde{r} \), we consider the following problem:

\[
\begin{align*}
\partial_t \tilde{r} & \tilde{r}_t \tilde{U}_j + \left( \partial_{tU} F(\tilde{U}_j) - \lambda(\tilde{U}_j) \right) \partial_{\theta_j} \tilde{U}_j = S(\tilde{U}_j) \partial_{\theta_j} \tilde{r}, \\
\partial_t \tilde{r} & = \lambda(\tilde{U}_j(t, \theta_j)),
\end{align*}
\tag{4.7}
\]

with boundary and initial conditions reading

\[
\begin{align*}
\tilde{U}_j(t, \theta_j^0) & = U_k^0(\tilde{r}(t, \theta_j^0)), & \tilde{U}_j(0, \theta_j) & = h_1(\theta_j), \\
\partial_t \tilde{r}(t, \theta_j^0) & = \lambda(U_k^0(\tilde{r})), & \tilde{r}(0, \theta_j) & = r_0,
\end{align*}
\tag{4.8}
\]

where we give \( \theta_j^0 = \lambda(U_k^0), j = 1, 2, k = L, R \) and the function \( h_j \) defined by

\[
\xi = \lambda_j(h_j(\xi)) = \frac{r - r_0}{t},
\tag{4.9}
\]

where \( \lambda_1 = \lambda, \lambda_2 = \mu \) are the eigenvalues of the 1 and 2 families.

**Lemma 4.2.** The integro-differential problem (4.7), (4.8) admits a unique \( \tilde{U}_j \) smooth for all fixed time \( t > 0 \).

**Proof.** To prove the lemma, we use a standard fixed point argument. Without loss of generality, we consider the 1-rarefaction wave. Denote by \( l_1, l_2 \) two linearly independent vectors corresponding to \( \lambda, \mu \) respectively. Multiplying (4.7) by \( l_2 \), we have

\[
\begin{align*}
DV_2 & = \frac{\partial_{\theta_1} \tilde{r}}{\mu - \lambda} l_2 \cdot S + Dl_2 \cdot V_1, \\
\partial_t V_1 & = l_2 \cdot S + \partial_1 l_2 \cdot V_1,
\end{align*}
\]

where we have defined \( V_1 = l_1 \cdot \tilde{U}_1, V_2 = l_2 \cdot \tilde{U}_1, \) and the operator reads \( D = \frac{\partial_{\theta_1} \tilde{r}}{\mu - \lambda} l_2 + \partial_2 \) whose integral curves starting from \( (\tau, \lambda(U_0)) \) is denoted by \( \xi \). We thus have

\[
\begin{align*}
V_2(t, \theta_1) & = V_2(\tau, \lambda(U_0)) + \int_{\xi} \left( \frac{\partial_{\theta_1} \tilde{r}}{\mu - \lambda} l_2 \cdot S + Dl_2 \cdot V_1 \right) d\theta_1, \\
V_1(t, \theta_1) & = V_1(0, \xi) + \int_0^t \left( l_2 \cdot S + \partial_1 l_2 \cdot V_1 \right) d\theta_1.
\end{align*}
\tag{4.10}
\]

Now let \( \mathcal{F} \) be the operator of the right-hand side of (4.10) and we study the iteration method \( \tilde{U}_1^{(l)} = \mathcal{F}^{(l)} \tilde{U}_1^0, l \geq 1 \) where \( \tilde{U}_1^0 \) is an arbitrary smooth function satisfying the initial-boundary condition \( \tilde{U}_1^0(t, \theta_1^0) = \).
where we have used the continuous dependence property in the max norm of \( \bar{U}_1(0, \theta_j) = \bar{U}_1(0, \theta_j) \). It is direct to check that for sufficiently small \( t_1 \), \( F \) is contractive in the max norm of \( \bar{U}_1 \). By iterating the operator \( F \), we prove that there exists a unique solution \( \bar{U}_1 \) for all \( 0 < t \leq \Delta t_1 \). Then we repeat the process by taking \( \bar{U}_1(t_1, \cdot) \) as initial condition and there exists a time \( \Delta t_2 \) such that \( \bar{U}_1 \) is defined by all \( \Delta t_1 < t < \Delta t_1 + \Delta t_2 \) and it is directly to see that \( \Delta t_1 \leq \Delta t_2 \) by the definition of the operator \( F \).

According to the construction above, we conclude the following theorem.

**Theorem 4.3** (The solution of the generalized Riemann problem). Consider the generalized Riemann problem for the Euler model (1.1), (4.1). There exists a weak solution to the generalized problem on \( t > 0 \) whose exact form is given by (4.6), satisfying the Rankie-Hugoniot jump condition and the Lax entropy condition.

### 4.3 Evolution of total variation

It is obvious that the total variation of \( \ln \rho \) of the solution of the standard Riemann problem (2.4), (2.9) stays as a constant when time passes. However, it is a different story for the generalized Riemann problem (1.1), (4.1). We have the following lemma.

**Lemma 4.4.** Let \( U = (\rho, v) = (\rho, v)(t, r) \) be the solution of the generalized Riemann problem of the Euler model (1.1) whose initial data \( U_0 = (\rho_0, v_0) = (\rho_0, v_0)(r) \) has the form (4.1). Then we have

\[
TV_{[t\bar{t}]}(\ln \rho(t, \cdot)) < TV_{[t\bar{t}]}(\ln \rho(0+, \cdot))(1 + O(t)),
\]

for all \( t > 0 \).

**Proof.** Let \( U_M = U_M(r) \) be the intermediate steady state solution associated with the left state \( U_L \) and the right state \( U_R \) given in the initial data. According to (4.4), we have

\[
U_L(r_M^{M^-(t)}) - U_M(r_M^{M^-(t)}) = U_L(r_0) - U_M(r_0) + |U_L(r_0) - U_M(r_0)|O(r_M^{M^-(t)} - r_0) = U_L(r_0) - U_M(r_0) + |U_L(r_0) - U_M(r_0)|O(t).
\]

Moreover, according to the construction of the generalized Riemann problem, we give

\[
TV_{[t\bar{t}]}(\ln \rho(t+, \cdot)) - TV_{[t\bar{t}]}(\ln \rho(0+, \cdot)) \leq (\ln \rho_L(r_0) - \rho_M(r_0) + \ln \rho_L(r_0) - \rho_M(r_0))O(t) = TV_{[t\bar{t}]}(\ln \rho(0+, \cdot))O(t),
\]

where we have used the continuous dependence property \( |U_L(r_0) - U_M(r_0)| = O(1)|\ln \rho_L(r_0) - \rho_M(r_0)| \).

This ends the proof of the lemma.

### 5 Triple Riemann problem

#### 5.1 Preliminary

Considering the fact that a steady state solution of the steady Euler model (5.1) may not be defined globally as is the result of Theorem 3.5, and we are obliged to introduce the **triple Riemann problem** in order to complete the Glimm method in the coming section, that is, a Cauchy problem associated with initial data composed of three steady state solutions:

\[
U_0(r) = \begin{cases} 
U_{b}(r) & r < r_a, \\
U_{b}(r) & r_a < r < r_b, \\
U_{c}(r) & r_b < r < \bar{r},
\end{cases}
\]

(5.1)
for fixed radius $0 < r < r_1 < r_2 < \tilde{r}$ and steady states $U_a = (\rho_a, v_a)$, $U_\beta = (\rho_\beta, v_\beta)$, $U_\gamma = (\rho_\gamma, v_\gamma)$. We denote by $U_a(r_s) = U^s_a = (\rho^s_a, v^s_a)$, $U_\beta(r_s) = U^s_\beta = (\rho^s_\beta, v^s_\beta)$, $U_\gamma(r_b) = U^b_\gamma = (\rho^b_\gamma, v^b_\gamma)$.

We first give the main conclusion of this section:

Theorem 5.1. Consider a given initial data composed of three steady state solutions $U_a, U_\beta, U_\gamma$. Then for all $t > 0$, the triple Riemann problem of the Euler model (1.1), (5.1) admits a weak solution $U = (\rho, v) = (\rho, v)(t, r)$ such that for all $t > 0$, we have:

$$TV[\rho(t, \cdot)] < TV[\rho(0+, \cdot)] (1 + O(\bar{r} - \tilde{r})).$$

We define the left-hand problem as a generalized Riemann problem with initial data

$$U_0(r) = \begin{cases} U_a(r) & r < r_1, \\ U_\beta(r) & r > r_1, \end{cases}$$

and the right-hand problem as a generalized Riemann problem with initial data

$$U_0(r) = \begin{cases} U_\beta(r) & r < r_2, \\ U_\gamma(r) & r > r_2, \end{cases}$$

Since the Euler model (1.1) is strictly hyperbolic following from Proposition 2.1, for a small enough time $t > 0$, both the left-hand and the right-hand problem admit a solution denoted by $U_L = U_L(t, r)$ and $U_R = U_R(t, r)$ respectively and the wave curves of the solutions do not interact. We denote by $r^L_{M \pm}, r^L_{M \pm}$ the rarefaction regions boundaries of the left-hand side problem and $r^R_{M \pm}, r^R_{M \pm}$ of the right-hand side problem (4.4), (4.5). We then define the moment of the first interaction denoted by $T_f$:

$$T_f := \sup \{ t > 0 | r^L_{M \pm}(t) \leq r^R_{M \pm}(t) \}.$$  (5.3)

Clearly, if $T_f = +\infty$, the triple Riemann problem (1.1), (5.1) exists a solution reading

$$U^f(t, r) = \begin{cases} U_a(t, r) & r < r_2, \\ U_R(t, r) & r_2 < r < \tilde{r}. \end{cases}$$  (5.4)

5.2 Possible interactions

If the moment of the first interaction $T_f < +\infty$, then the waves of the left and the right-hand Riemann problem did have interactions. Possible interactions are given in order:

- 2-shock of the left-hand problem and 1-shock of the right-hand problem,
- 2-shock of the left-hand problem and 1-rarefaction of the right-hand problem,
- 2-rarefaction of the left-hand problem and 1-shock of the right-hand problem,

which are denoted by Problems $P - ss, P - sr, P - rs$ respectively. For later use, we denote by $U^L_{M \pm}, U^R_{M \pm}$ the intermediate states of the left and right-hand problems respectively. We consider different kinds of interactions separately.

Lemma 5.2. If $T_f < +\infty$ where $T_f$ is defined by (5.3) and we have the 2-shock of the left-hand problem and the 1-shock of the right-hand problem of the Euler model (1.1), then there exists a time $T_{ss}$ such that Problem $P - ss$ admits a solution on $0 < t < T_{ss}$.
Proof. We only have to consider the solution after \( t > T_f \). We denote by \( U_{ss}^M = U_{ss}^M(t, r) \) the solution of the generalized problem with initial states \( U_{M}^{\alpha, \beta}, U_{M}^{\beta, \gamma} \) separated by \( r = r_{L}^{M+}(T_f) = r_{MR}^{R-}(T_f) \) at \( t = T_f \). Then for \( T_f < t < T_{ss} \), we give

\[
U_{ss}^M(t, r) = \begin{cases} 
U_{L}(t, r) & r < r_{L}^{M+}(t), \\
U_{R}^M(t, r) & r_{L}^{M+}(t) < r < r_{MR}^{R-}(t), \\
U_{R}^M(t, r) & r_{MR}^{R-}(t) < r < r, 
\end{cases}
\]

(5.5)

where

\[
T_{ss} = \min \left( \sup \{ t > T_f | r_{L}^{M-}(t) > r_{L}^{M+}(t) \}, \sup \{ t > T_f | r_{MR}^{R-}(t) > r_{MR}^{R+}(t) \} \right),
\]

(5.6)

where \( r_{L}^{M-} \) are boundaries of the rarefaction regions of the state \( U_{ss}^M \) given by (4.4), (4.5). Thus Problem P-ss admits a solution for all \( t < T_{ss} \).

We now consider Problem P — rs.

Lemma 5.3. Let \( T_f \) be the first moment of interaction and we suppose \( T_f < +\infty \) and the Euler model \([7] \) has 2-rarefaction of the left-hand problem and the 1-shock of the right-hand problem. Then there exists a time \( T_{rs} \) such that we have a solution of Problem P — rs for all \( 0 < t < T_{rs} \).

Proof. Again, we only have to construct a solution after \( t > T_f \). Let us first write \( \tilde{U}_{2}^{\alpha, \beta} = \tilde{U}_{2}^{\alpha, \beta}(t, r) \) the 2-rarefaction wave of the left-hand problem which evolves in the region \( (r_{L}^{M-}(t), r_{L}^{M+}(t)) \). Then we give

\[
U_{0}^{rs}(t, r) = \begin{cases} 
U_{L}(t, r) & r < r_{L}^{M+}(t), \\
U_{M}^{rs}(t, r) & r_{L}^{M+}(t) < r < r_{MR}^{R-}(t), \\
U_{R}(t, r) & r_{MR}^{R-}(t) < r < r, 
\end{cases}
\]

(5.7)

where the function \( U_{M}^{rs}(t, r) \) is given by

\[
U_{M}^{rs,0}(t, r) = \begin{cases} 
\tilde{U}_{2}^{\alpha, \beta}(t, r) & r_{L}^{M+}(t) < r < \tilde{r}_{MR}^{R-}(t), \\
\tilde{U}_{2}^{\alpha, \beta}(t, r) & \tilde{r}_{MR}^{R+}(t) < r < \tilde{r}_{MR}^{R-}(t), \\
\tilde{U}_{2}^{\alpha, \beta}(t, r) & r_{MR}^{R+}(t) < r < \tilde{r}_{MR}^{R-}(t), \\
U_{M}^{rs,0}(r) & r_{MR}^{R+}(t) < r < \tilde{r}_{MR}^{R+}(t), \\
U_{\gamma}(r) & \tilde{r}_{MR}^{R+}(t) < r < \tilde{r}_{MR}^{R+}(t). 
\end{cases}
\]

(5.8)

Here, \( U_{MM}^{rs} = U_{MM}^{rs}(r) \) is a steady state with

\[
U_{MM}^{rs}(r_{L}^{M}(T_f)) \in W_{1}^{-}((\tilde{U}_{2}^{\alpha, \beta}(T_f, r_{L}^{M}(T_f))) \cap W_{2}^{-}(U_{\gamma}(r_{MR}^{R}(T_f)))
\]

and we recall that \( W_{1}^{-} \) and \( W_{2}^{-} \) are elementary waves given by (2.10). The wave curves \( \tilde{r}_{MR}^{R+}, \tilde{r}_{MR}^{R-} \) satisfy (4.4), (4.5) with three states \( \tilde{U}_{2}^{\alpha, \beta}, U_{MM}^{rs}, U_{\gamma} \). The functions \( \tilde{U}_{12}^{rs}(t, r) \) are given by (4.7), (4.8), (4.9). Denote by

\[
T_{rs}^{0} = \sup \{ t > T_f | \tilde{r}_{MR}^{R+}(t) < r_{L}^{M+}(t) \}
\]

(5.9)

and we see immediately that (5.7) provides an exact solution for Problem P — rs for all \( 0 < t < T_{rs}^{0} \). Now for \( t > T_{rs}^{0} \), we give

\[
U_{M}^{rs}(t, r) = \begin{cases} 
U_{L}(t, r) & r < r_{L}^{M+}(t), \\
U_{M}^{rs,1}(t, r) & r_{L}^{M+}(t) < r < \tilde{r}_{MR}^{R-}(t), \\
U_{M}^{rs,0}(r) & \tilde{r}_{MR}^{R+}(t) < r < \tilde{r}_{MR}^{R-}(t), \\
U_{\gamma}(r) & \tilde{r}_{MR}^{R+}(t) < r < \tilde{r}_{MR}^{R+}(t), 
\end{cases}
\]

(5.10)
with \( U_{M}^{rs,0} \) given by (5.8) and \( U_{M}^{rs,1} \) the solution of the Riemann problem generated by initial data \( U_{M}^{\alpha,\beta} \). \( U_{MM}^{rs,0} \) at the radius \( r = r_{M}^{-}\left(T_{r}^{-}\right) = r_{M}^{-}\left(T_{r}^{0}\right) \) from the very moment \( t = T_{r}^{-} \). Now we denote by

\[
T_{rs} = \min \left( \sup \{ t > T_{rs}^{-} | \frac{U_{M}^{-}}{L_{rs}}(t) > \frac{U_{M}^{-}}{L_{L}}(t) \}, \sup \{ t > T_{rs}^{-} | \frac{U_{M}^{-}^{R}}{M_{rs}^{-}}(t) > \frac{R_{M}^{-}}{M_{rs}^{-}}(t) \} \right),
\]

(5.11)

where \( \frac{U_{M}^{-}}{L_{rs}}(t) \) is the lower bound of the 1-wave of the solution \( U_{M}^{rs,1} = U_{M}^{rs,1}(t, r) \). Together with (5.4), (5.7), (5.10), we have a solution of Problem \( P - rs \) for all \( 0 < t < T_{rs} \).

A similar analysis gives the result of Problem \( P - rs \).

**Lemma 5.4.** If the first moment of interaction \( T_{f} < +\infty \) and the Euler model (1.1) admits 1-rarefaction of the left-hand problem and the 2-shock of the right-hand problem. That we have a solution of Problem \( P - sr \) for all \( 0 < t < T_{sr} \) for a given moment \( T_{sr} \).

We now consider interactions after these moments \( T_{ss}, T_{rs}, T_{sr} \). Indeed, following from the constructions in Lemmas 5.2, 5.3, 5.4 it is clear that possible interactions after these moments are also interplays of shock waves and rarefaction waves as is listed at the beginning of this section. Thus, for any fixed moment \( t > 0 \), we have the solution of the triple Riemann problem. The estimation of the total variation given by (5.2) follows directly from Lemmas 2.6, 4.4. We thus obtain the main conclusion of this section, that is, Theorem 5.1.

6 The initial value problem

6.1 The Glimm method

We construct an approximate solution of the Euler model (1.1) with initial data

\[
U(t, r) = U_{0}(r) = (\rho_{0}, v_{0})(r), \quad r > 0,
\]

(6.1)

by using a random choice method based or equivalently, the Glimm method on the generalized problem. Let \( \Delta r \) and \( \Delta t \) denote the mesh lengths in space and in time respectively, and let \( (r_{j}, t_{n}) \) denotes the mesh point of the grid, where \( r_{j} = j \Delta r, t_{n} = 0 + n \Delta t \). We assume the so-called *CFL condition*:

\[
\frac{\Delta r}{\Delta t} > \max(\{\lambda, |\mu|\}),
\]

(6.2)

insuring that elementary waves other than those in the triple Riemann problem do not interact within one time interval.

To construct the approximate solution \( U_{\Delta r} = U_{\Delta r}(t, r) \), we would first like to approximate the initial data by a piecewise steady state solution of the Euler model given by (3.1). However, note that some steady state solutions cannot be defined globally on \( r > 0 \), we need more constructions. Recall first that there exists four critical steady state solutions which pass the point \( (\frac{m}{2\pi}, \pm k) \) denoted by \( U_{s}^{P,\alpha}, U_{s}^{P,\beta}, U_{s}^{N,\alpha}, U_{s}^{N,\beta} \) according to (3.10). Another important remark is given in Theorem 3.3 that is, for given \( r_{0}, U_{0} \), there exists always a steady solution \( U = U(r) \) with \( U(r_{0}) = U_{0} \) defined on \((0, r_{0})\) if \( r_{0} < \frac{m}{2\pi} \) or \((r_{0}, +\infty)\) if \( r_{0} > \frac{m}{2\pi} \). Now we denote by \( U_{\Delta r,0}^{i+1} = U_{\Delta r,0}^{i+1}(r) = (\rho_{\Delta r,0}^{i+1}, v_{\Delta r,0}^{i+1})(r) \) the steady state solution of the Euler model satisfying (3.1) such that \( U_{\Delta r,0}^{i+1}(r_{j+1}) = U_{0}(r_{j+1}) \) and we define:

\[
r_{j+1}^{i} := \sup \{ r > 0 | v_{\Delta r,0}^{i+1}(r) \neq \pm k \} \chi_{\{r_{j+1} < \frac{m_{\Delta r}}{2\pi}\}}(r) + \inf \{ r > 0 | v_{\Delta r,0}^{i+1}(r) \neq \pm k \} \chi_{\{r_{j+1} > \frac{m_{\Delta r}}{2\pi}\}}(r).
\]

(6.3)
Note that if $r_{j+1}^s \neq 0$ or $r_{j+1}^s \neq +\infty$, $r_{j+1}^s$ is the sonic point of the steady state $U_{\Delta r, 0}^{j+1}$. We now denote by $U_{0,s}^{j+1} = (\rho_{0,s}^{j+1}, v_{0,s}^{j+1})$ the unique critical steady state solution satisfying

$$
\text{sgn}(v_{0,s}^{j+1}) = \text{sgn}(v_{\Delta r, 0}^{j+1}), \quad \text{sgn}(|v_{0,s}^{j+1}| - k) = \text{sgn}(|v_{\Delta r, 0}^{j+1}| - k).
$$

On the interval $(r_j, r_{j+2})$, we have the following possible constructions.

- If $U_{\Delta r, 0}^{j+1}$ is well-defined on $(r_j, r_{j+2})$, we approximate the initial data $U_0$ by $U_{\Delta r, 0}^{j+1}$ on the interval.

- If $U_{\Delta r, 0}^{j+1}$ vanishes at $r_{j+1}^s$ and $r_{j+1} < \frac{m}{2\Delta r}$, then we approximate the initial data on $(r_{j+1}^s, r_{j+2})$ by
  - $U_{\Delta r, 0}^{j+1}$ if $r_{j+1}^s \notin (r_{j+1}^s, r_{j+2})$;
  - $U_{0,s}^{j+1}$ if $r_{j+1}^s \in (r_{j+1}^s, r_{j+2})$ for $U_{0,s}^{j+1}$ given by (6.4). Note that this case happens at most once if $r_{j+1} < \frac{m}{2\Delta r} < r_{j+3}$ and $r_{j+3} > r_{j+2}$.

- If $U_{\Delta r, 0}^{j+1}$ vanishes at $r_{j+1}^s$ and $r_{j+1} > \frac{m}{2\Delta r}$, then we approximate the initial data on $(r_j, r_{j+1}^s)$ by
  - $U_{\Delta r, 0}^{j+1}$ if $r_{j-1}^s \notin (r_j, r_{j+1}^s)$;
  - $U_{0,s}^{j+1}$ if $r_{j-1}^s \in (r_j, r_{j+1}^s)$. Also, this case happens at most one time if $r_{j-1} < \frac{m}{2\Delta r} < r_{j+1}$ and $r_{j-1}^s < r_j$.

Following the ideas above, we can now approximate the initial data on $(r_j, r_{j+2})$ for $j$ even:

$$
U_{\Delta r, 0}(r) = \begin{cases} 
U_{\Delta r, 0}^{j+1} \frac{2\Delta r}{m} (r) & r_j < r < M(r_j, r_{j+1}^s), \\
U_{\Delta r, 0}^{j+1} (r) & M(r_j, r_{j+1}^s) < r < M(r_{j+1}^s, r_{j+2}), \\
U_{0,s}^{j+1} \frac{2\Delta r}{m} (r) & M(r_{j+1}^s, r_{j+2}) < r < r_{j+2},
\end{cases}
$$

(6.5)

where we give the operator $M$ by

$$
M(x, y) = \begin{cases} 
\min(x, y) & r < \frac{m}{2\Delta r}, \\
\max(x, y) & r > \frac{m}{2\Delta r},
\end{cases}
$$

(6.6)

and

$$
U_{\Delta r, 0}^{j+1} \frac{2\Delta r}{m} (r) = \begin{cases} 
U_{\Delta r, 0}^{j+1} \frac{2\Delta r}{m} (r) & r_{j+1}^s < r < r_{j+1}^s - \frac{m}{2\Delta r}, \\
U_{\Delta r, 0}^{j+1} (r) & r_{j+1}^s - \frac{m}{2\Delta r} \leq r_{j+1}^s - \frac{m}{2\Delta r} \notin (r_j, r_{j+1}^s) \cup (r_{j+1}^s, r_{j+2}), \\
\text{else,}
\end{cases}
$$

with the sonic point $r_{j+1}^s$ given by (6.3) and the critical steady state solution $U_{0,s}^{j+1}$ satisfying (6.4). Assume now that the approximate solution has been defined for $t_{n-1} \leq t < t_n$. To complete the definition of $U_{\Delta r}$, it suffices to define the solution on $t_n \leq t < t_{n+1}$. Let $\theta_n$ be a given equidistributed sequence on the interval $(-1, 1)$ and introduce the point related to the randomly choose values:

$$
r_{n, j+1} := (\theta_n + j)\Delta r, \quad j > 0.
$$

(6.7)

Following the idea before, we denote by $U_{\Delta r, n}^{j+1} = U_{\Delta r, n}^{j+1} (r)$ the steady state solutions passing the point $(r_{n,j+1}, U_{\Delta r, (nt-r, r_{n,j+1})})$ and the sonic point

$$
r_{n,j+1}^s := \sup \{ r > 0 | v_{\Delta r, n}^{j+1} (r) \neq \pm k \} \chi_{t_{n,j+1} < \frac{m}{2\Delta r}} (r) + \inf \{ r > 0 | v_{\Delta r, n}^{j+1} (r) \neq \pm k \} \chi_{t_{n,j+1} > \frac{m}{2\Delta r}} (r),
$$

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For any given initial density \( \rho \)

Theorem 6.1

The Glimm scheme provides as an approximate solution which indeed converges to an exact weak solution.

Now suppose that \( U_{\Delta_r} \) is constructed for all \( t < t_n \). The construction of the approximate solution on the time interval \( t_n \leq t < t_{n+1} \) is similar to the approximation of the initial data:

- **The steady state solution step.** On the level \( t = t_n \), on the interval \( (r_j, r_{j+2}) \) with \( n + j \) even

\[
U_{\Delta_r,n}(r) = \begin{cases} 
U_{\Delta_r,n}^{j+1-2\text{sgn}(\rho_{n,j+1} - \frac{m}{2})} (r) & r_j \leq r < M(r_j, r_{n,j+1}^s), \\
U_{\Delta_r,n}^{j+1} (r) & M(r_j, r_{n,j+1}^s) < r < M(r_{n,j+1}^s, r_j + 2), \\
U_{\Delta_r,n}^{j+1-2\text{sgn}(\rho_{n,j+1} - \frac{m}{2})} (r) & M(r_{n,j+1}^s, r_j + 2) < r < r_{j+2},
\end{cases}
\]

where \( M(\cdot, \cdot) \) is the operator given by (6.6) and

\[
U_{\Delta_r,n}^{j+1-2\text{sgn}(\rho_{n,j+1} - \frac{m}{2})} (r) = \begin{cases} 
U_{\Delta_r,n}^{j+1-2\text{sgn}(\rho_{n,j+1} - \frac{m}{2})} (r) & r_j < r < r_{j+2}, \\
U_{\Delta_r,n}^{j+1} (r) & r_j \leq r < r_{j+2},
\end{cases}
\]

with \( U_{n,s}^{j+1} \) given by (6.8). It is direct to observe that if a steady state solution reaches its sonic point in a cell, then the nearest discontinuity is replaced by this sonic point, then this construction guarantees that there exists at most one point of discontinuity in \( (r_{j-1}, r_{j+1}) \), \( n + j \) even.

- **The generalized Riemann problem step.** Denote by \( r_j^d \) the point of discontinuity in \( r_j-1 < r < r_j+1 \) and we then define the approximate solution \( U_{\Delta_r} \) on the rectangle \( \{ t_n < t < t_{n+1}, r_{j-1} < r < r_{j+1} \} \), \( n + j \) even:

\[
U_{\Delta r}(t, r) = \begin{cases} 
U_{R}^{j-1} (t, r), & r_d - r_{j-2}^d = 2\Delta r \text{ and } r_{j+2}^d < r^d = 2\Delta r, \\
U_{R}^{j-3} (t, r), & r_d^j - r_{j-2}^d < 2\Delta r, \\
U_{R}^{j-1} (t, r), & r_d^j - r_{j+2}^{d-1} < 2\Delta r,
\end{cases}
\]

where \( U_{R}^{j-1} \) is the solution of the generalized Riemann problem at the time level \( t = t_n \) on \( (r_{j-1}, r_{j+1}) \) with two steady states separated by a discontinuity at \( r^d_j \) and \( U_{R}^{j-3} \) the solution of the triple Riemann problem at the time level \( t = t_n \) on the interval \( (r_{j-3}, r_{j+1}) \) with the three steady states separated by discontinuities at \( r_{j-2}^d, r_j^d \).

This completes the construction of the approximate solution \( U_{\Delta r} = U_{\Delta r}(t, r) \) on \( [0, +\infty) \times (0, +\infty) \) by the Glimm scheme.

### 6.2 Existence of Cauchy problem

The Glimm scheme provides as an approximate solution which indeed converges to an exact weak solution.

**Theorem 6.1** (Global existence theory). Consider the Euler model with source term describing fluid flows. For any given initial density \( \rho_0 = \rho_0(r) > 0 \) and velocity \( v_0 \) such that

\[
TV(\ln \rho_0) + TV(v_0) < +\infty,
\]

and any given time interval (possibly infinite) \( (0, T) \subset (0, +\infty) \), there exists a weak solution \( \rho = \rho(t, r), v = v(t, r) \) defined on \( (0, T) \) such that the initial condition holds in the sense that \( \rho(0, \cdot) = \rho_0, v(0, \cdot) = v_0 \) and for any fixed moment \( T' \in (0, T) \)

\[
\sup_{t \in [0, T']} \left( TV(\ln \rho(t, \cdot)) + TV(v) \right) < +\infty.
\]
To prove Theorem 6.1, we first need an estimation of the total variation. See the following lemma.

**Lemma 6.2.** Let $U_{\Delta t} = (\rho_{\Delta t}, v_{\Delta t})$ be the approximate solution of the Euler model (1.1) constructed by the Glimm method, then for any two neighboring time interval $t_n, t_{n+1}$, we have a constant $C > 0$ such that

$$TV\left(\ln \rho_{\Delta t}(t_{n+1}+, \cdot)\right) - TV\left(\ln \rho_{\Delta t}(t_n+, \cdot)\right) \leq C \Delta t.$$  

From Lemma 6.2, we have, for any given $0 < t < +\infty$,

$$TV\left(\ln \rho_{\Delta t}(t, \cdot)\right) \leq TV\left(\ln \rho_{\Delta t}(0, \cdot)\right)e^{C_1 t},$$  

(6.11)

where $C_1$ is a constant.

**Proof.** On the time level $t = t_{n+1}$, we consider the interval $(r_{n+1,j-1}, r_{n+1,j+1})$ with $n + j$ even. According to (6.7), $r_{n+1,j+1}$ is the point determined by a chosen random value. Following from the construction of the Glimm method, $(r_{n+1,j-1}, r_{n+1,j+1})$ only contains one point of discontinuity which we write as $r_{j,n}^d$. According to Lemma 4.4, we have

$$TV\left(\ln \rho(t_{n+1}+, \cdot)\right) = \sum_j \left| \ln \rho(t_{n+1}+, r_{j,n+1}^d) - \ln \rho(t_{n+1}+, r_{j,n}^d) \right| \left(1 + C(\Delta t)\right).$$

Now we notice that there are portions of three possible waves generated by either the generalized Riemann problem or the triple Riemann problem lying in the interval $(r_{n+1,j-1}, r_{n+1,j+1})$. We write these waves as $\omega_{j,m,i}$ from left to right, staring from points of discontinuity (reading $r_{j,n}^d, r_{m,n}^d, r_{i,n}^d$, respectively) in $(r_{j-2,n}, r_{j,n}), (r_{j,n}, r_{j+2,n}), (r_{j+2,n}, r_{j+4,n})$ at the time level $t = t_n$ respectively.

We observe that the wave $\omega_1$ is either a zero strength wave or a second Riemann problem lying in the interval $(r_{n+1,j-1}, r_{n+1,j+1})$. We write these waves as $\omega_{j,m,i}$ from left to right, staring from points of discontinuity (reading $r_{j,n}^d, r_{m,n}^d, r_{i,n}^d$, respectively) in $(r_{j-2,n}, r_{j,n}), (r_{j,n}, r_{j+2,n}), (r_{j+2,n}, r_{j+4,n})$ at the time level $t = t_n$ respectively.

Now since the uniform BV bound on a given time interval $(0, T)$ (established below) is known, Helly’s theorem gives immediately the fact that there exists a subsequence of $\Delta t \to 0$ such that we have a limit function $U = U(t, r)$ and $U_{\Delta t}(t, r) \to U(t, r)$ pointwise a.e. in $L^1_{\text{loc}}$ at each fixed time $t$. Moreover, the limit function $U = U(t, r)$ is a weak solution of the Euler model (1.1), (6.1). This ends the proof of Theorem 6.1.

**7 Conclusion**

In the article, we considered a kind of Euler equation with a particular source term depending on the sound speed and the body mass. We first presented the hyperbolicity and the nonlinear genuinity of the equation. We gave then an analysis of the steady state solutions of this model and give a classification of these steady states with respect to the behaviour of the sonic points. We then considered the generalized Riemann problem whose initial data are two constant steady state solutions and proved their existence by giving an analytical formula of the solution. We also proved the existence for a so-called triple Riemann problem with three different steady state solutions. We were then able to use the Glimm method to construct a sequence of the solutions to the initial value problem of the Euler equation and prove it existence with a control of the total variation.
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