EXISTENCE OF NODAL SOLUTIONS FOR QUASILINEAR ELLIPTIC PROBLEMS IN $\mathbb{R}^N$

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Abstract. We prove the existence of one positive, one negative, and one sign-changing solution of a $p$-Laplacian equation on $\mathbb{R}^N$, with a $p$-superlinear subcritical term. Sign-changing solutions of quasilinear elliptic equations set on the whole of $\mathbb{R}^N$ have only been scarcely investigated in the literature. Our assumptions here are similar to those previously used by some authors in bounded domains, and our proof uses fairly elementary critical point theory, based on constraint minimization on the nodal Nehari set. The lack of compactness due to the unbounded domain is overcome by working in a suitable weighted Sobolev space.

1. Introduction

In this paper we study sign-changing solutions of the quasilinear elliptic equation

$$-\Delta_p u = \lambda A(x)|u|^{p-2}u + g(x,u) \text{ on } \mathbb{R}^N,$$

where $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ with $1 < p < N$, $N \geq 1$, and $\lambda$ is a real parameter. We suppose that the coefficient $A: \mathbb{R}^N \to \mathbb{R}$ satisfies:

$$A \text{ is measurable, with } A > 0 \text{ a.e., } A \in L^\infty(\mathbb{R}^N) \text{ and } A^{-\frac{1}{p-1}} \in L^1_{\text{loc}}(\mathbb{R}^N).$$

Note that the last condition in (A1) holds if, for instance, $\text{ess inf}_\Omega A > 0$ for any bounded open set $\Omega \subset \mathbb{R}^N$. An additional condition relating $A$ and $\lambda$, formulated in Section 1.1 below, will also play a crucial role in our analysis.

We suppose that the function $g$ satisfies the following assumptions, where $p^* = \frac{Np}{N-p}$:

$$\text{(g1) } g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \text{ is a Carathéodory function, } g(x,\cdot) \in C^1(\mathbb{R}) \text{ for a.e. } x \in \mathbb{R},$$

and there exist $q \in (p, p^*)$ and $B \in L^{p^*}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), B > 0$ a.e., such that

$$|g_s(x,s)| \leq B(x)|s|^{q-2}, \quad \text{for a.e. } x \in \mathbb{R}^N, s \in \mathbb{R}.$$

$$\text{(g2) There exist } \theta > p \text{ and } R > 0 \text{ such that}$$

$$g(x,s)s \geq \theta G(x,s) > 0, \quad \text{for a.e. } x \in \mathbb{R}^N, |s| \geq R,$$

where $G(x,s) = \int_0^s g(x,t) \, dt$.

$$\text{(g3) The mapping } s \mapsto \frac{g(x,s)}{|s|^{p-1}} \text{ is strictly increasing in } s \in \mathbb{R} \setminus \{0\}, \text{ for a.e. } x \in \mathbb{R}^N.$$

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The research work reported in this paper was initiated while A.D. was at the CeReMath, Université Toulouse 1 Capitole, and F.G. was at Heriot–Watt University. A.D. visited Heriot–Watt once and F.G. visited the CeReMath twice during this work. We are grateful to both institutions for their hospitality. We also thank the anonymous referee for helpful comments, Charles Stuart for insightful discussions, and Peter Takač for indicating useful references to us. F.G. acknowledges the support of the U.K. Engineering and Physical Sciences Research Council [EP/H030514/1], and of the ERC Advanced Grant “Nonlinear studies of water flows with vorticity”.
Assumptions $(g1)$ and $(g2)$ are often referred to by saying that $g$ is subcritical and $p$-superlinear, respectively. The hypothesis $B \in L^\infty(\mathbb{R}^N)$ will be convenient to establish compactness of a weighted Sobolev embedding which plays an important role in the paper. However, it can be relaxed to a local integrability assumption, namely $B \in L^s_{\text{loc}}(\mathbb{R}^N)$ for some large enough $s$ (see Remark 19 in Appendix A for a more a precise statement). Our assumptions on $B$ imply that $(1)$ is a compact perturbation (in a sense that will be made precise in Lemma 7) of the $p$-linear eigenvalue problem

$$-\Delta_p u = \lambda A(x)|u|^{p-2}u \quad \text{on } \mathbb{R}^N.$$  \hspace{1cm} (2)

The existence of eigenvalues and eigenfunctions for this problem has been discussed by Allegretto and Huang [1], even in the case of an indefinite weight $A$ (i.e. when $A$ does not have a constant sign), as long as $A$ is positive on a set of positive measure and satisfies an integrability condition at infinity. Bifurcation of solutions of $(1)$ from the principal eigenvalue of $(2)$ has been studied by Drábek and Huang [11], who obtained global continua of positive and negative solutions of $(1)$. Bifurcation from higher eigenvalues — which would provide sign-changing solutions — is a difficult problem. In fact, sign-changing solutions for quasilinear equations in the whole of $\mathbb{R}^N$ have only been scarcely investigated; see however [9], where a problem with cylindrical symmetry is considered, and [14], dealing with a $p$-asymptotically linear problem.

On the other hand, there is a fair amount of literature on solutions of quasilinear elliptic equations in bounded domains. In the radial case, the Dirichlet problem in a ball has been solved by Del Pino and Manásevich [10] by the bifurcation approach, yielding infinitely many nodal solutions. In the non-radial setting, an important contribution (probably the most general so far) is due to Bartsch et al. [3], who proved existence and multiplicity of nodal solutions for $p$-Laplacian Dirichlet problems in smooth bounded domains. Their results are obtained by critical point theory in Banach spaces, making clever use of a suitable pseudo-gradient flow. In fact in $\mathbb{R}^N$, using similar arguments, a sign-changing solution of an auxiliary $p$-superlinear problem (denoted $(P)_\mu$) is obtained in the course of the proof in [14]. However, the equation $(P)_\mu$ considered there has a different structure from ours, mainly due to the $p$-asymptotically linear nature of the original problem.

It is worth remarking that the hypotheses (H$_0$)–(H$_3$) used to prove the existence of a nodal solution in Bartsch et al. [3] are structurally similar to ours. However in the present work we use a more elementary method, based on constraint minimization on the `nodal Nehari set’. This approach originated in a series of works on the Dirichlet problem for semilinear equations, a review of which can be found in [4]. It has also been used more recently in the context of a prescribed mean-curvature problem in Bonheure et al. [6], from which some of our arguments are inspired. Note that the Nehari approach strongly relies on the monotonicity assumption $(g3)$, which is not needed in [3]. We will focus here on the case of a positive coefficient $A$ and, under the hypotheses (A1), $(g1)$–$(g3)$, we will give a sufficient condition relating $\lambda$ and $A$ for the existence of at least one positive, one negative, and one nodal solution of $(1)$. This condition — assumption $(A, \lambda)$ below — should be compared with hypothesis (H$_3$) in [3], involving the first eigenvalue of the $p$-linear problem. We will also deduce corresponding existence results for the problem

$$-\Delta_p u = \bar{A}(x)|u|^{p-2}u + g(x, u) \quad \text{on } \mathbb{R}^N,$$  \hspace{1cm} (3)

under appropriate conditions on the coefficient $\bar{A} \in L^\infty(\mathbb{R}^N)$.

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1The term ‘$(p-1)$-superhomogeneous’ is sometimes used instead of $p$-superlinear.

2Throughout the paper we will use the terms ‘nodal’ and ‘sign-changing’ interchangeably.
Our approach is based on a variational formulation of (1) in a weighted Sobolev space. More precisely, let us define the norm $\| \cdot \|_{W_A}$ on $C_0^\infty(\mathbb{R}^N)$ by

$$
\|u\|_{W_A} = \left( \int_{\mathbb{R}^N} |\nabla u|^p + A(x)|u|^p \, dx \right)^{1/p}.
$$

We will work in the space $W_A(\mathbb{R}^N) = C_0^\infty(\mathbb{R}^N)^{\perp_{W_A}}$. Some elementary properties of $W_A(\mathbb{R}^N)$ will be established in Section 2. In particular, $W_A(\mathbb{R}^N)$ is a separable and uniformly convex (hence reflexive) Banach space. Since $A \in L^\infty(\mathbb{R}^N)$, $W_A(\mathbb{R}^N) \supseteq W^{1,p}(\mathbb{R}^N)$, with equality if $A$ is bounded away from zero. We will often write $W_A$ for $W_A(\mathbb{R}^N)$, although some properties of $W_A(\Omega)$ will be given in Section 2 for more general open subsets $\Omega \subset \mathbb{R}^N$.

With $G$ defined as in (g2), we introduce the functional $S_\lambda : W_A \to \mathbb{R}$,

$$
S_\lambda(u) = \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p - \lambda A(x)|u|^p \, dx - \int_{\mathbb{R}^N} G(x,u) \, dx.
$$

Due to (g2) $S_\lambda$ is not coercive, so one cannot apply the direct method of the calculus of variations to find critical points of $S_\lambda$. We will see that constraint minimization on Nehari-type sets provides an efficient alternative to obtain critical points, and to discuss their nodal properties.

1.1. Main results. We now formulate the assumption relating $A$ and $\lambda$ that we shall use to prove our results. For $A$ satisfying (A1), let

$$
\lambda_A = \inf_{u \in W_A \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^p \, dx}{\int_{\mathbb{R}^N} A(x)|u|^p \, dx} \geq 0.
$$

Provided $A$ and $\lambda$ satisfy

(A, $\lambda$) $\lambda < \lambda_A,$

it will be shown in Section 2 that

$$
\| \cdot \|_\lambda = \left( \int_{\mathbb{R}^N} |\nabla u|^p - \lambda A(x)|u|^p \, dx \right)^{1/p}
$$

defines a (quasi)norm\(^3\) on $W_A$, which is equivalent to $\| \cdot \|_{W_A}$. This enables one to rewrite $S_\lambda : W_A \to \mathbb{R}$ as

$$
S_\lambda(u) = \frac{1}{p} \|u\|_\lambda^p - \int_{\mathbb{R}^N} G(x,u) \, dx,
$$

which is a convenient way to exhibit the main properties of the $p$-homogeneous part of $S_\lambda$ with respect to the variational procedure. Note that if $\lambda < 0$ then $(A, \lambda)$ is always satisfied, and $\| \cdot \|_\lambda$ is a norm, which is equivalent to $\| \cdot \|_{W_A}$ on $W_A$. The more interesting case of positive $\lambda$ requires $\lambda_A > 0$. As will be shown in Section 3, a sufficient condition for this to hold is

(A2) $A \in L^{N/p}(\mathbb{R}^N)$.

We shall see in Lemma 8 that $S_\lambda \in C^1(W_A, \mathbb{R})$ provided $A$ satisfies (A1). Our main results concern weak solutions of problems (1) and (3). A function $u \in W_A$ is called a solution of (1) if and only if $S_\lambda(u) = 0$, that is, if and only if

$$
\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \cdot \nabla \varphi \, dx - \lambda \int_{\mathbb{R}^N} A(x)|u|^{p-2}u \varphi \, dx - \int_{\mathbb{R}^N} g(x,u)u \, dx = 0 \quad \forall \varphi \in W_A,
$$

with a similar definition for solutions of (3). The existence of weak solutions will be proved by a variational approach in Section 3 under hypotheses (A1), (A, $\lambda$) and

\(^3\)see Lemma 8 and Remark 9.
(g1)–(g3). In addition to ensuring that \( \lambda_A > 0 \), assumption (A2) is also needed to obtain extra regularity of the solutions. Our main result is the following.

**Theorem 1.** Suppose that the hypotheses (A1), (A2), (A, \( \lambda \)) and (g1)–(g3) are satisfied. Then there exist three solutions \( u_1, u_2, u_3 \in C_{\text{loc}}^{1,\alpha}(\mathbb{R}^N) \) of (1), with \( u_1 > 0 \), \( u_2 < 0 \), and \( u_3^+ \neq 0 \). Furthermore, \( u_3 \) has exactly two nodal domains.

The definition of the class \( C_{\text{loc}}^{1,\alpha}(\mathbb{R}^N) \) is recalled in Section 4. As usual, we denote by \( u^\pm \) the positive and negative parts of \( u \). More precisely, we use here the convention \( u^\pm(x) := \pm \max\{\pm u(x), 0\} \), so that \( u = u^+ + u^- \). For a continuous function \( u : \mathbb{R}^N \rightarrow \mathbb{R} \), a *nodal domain* is a connected component of \( \mathbb{R}^N \setminus u^{-1}(\{0\}) \).

It is also possible to formulate a version of our results from which the parameter \( \lambda \) is absent, pertaining to problem (3).

**Corollary 2.** Suppose that (g1)–(g3) are satisfied. If either

(i) \( \bar{A} \) satisfies (A1), (A2) and \( \lambda_{\bar{A}} > 1 \),
(ii) or \(-A\) satisfies (A1) and (A2),
(iii) or \( A = 0 \) a.e.,

then (3) has three solutions in \( C_{\text{loc}}^{1,\alpha}(\mathbb{R}^N) \). Furthermore, the solutions have the same nodal structure as in Theorem 4, i.e. one is positive, one is negative, and one is sign-changing with two nodal domains.

**Proof.** In case (i) holds, the result follows from Theorem 4 by letting \( A = \bar{A} \) and \( \lambda = 1 < \lambda_{\bar{A}} \). Case (ii) follows by letting \( A = -\bar{A} \) and \( \lambda = -1 \), while for (iii) one can take any \( A \) satisfying (A1)–(A2), and \( \lambda = 0 \).

The rest of the paper is organized as follows. In Section 2 we first establish some properties of the spaces \( W_A(\Omega) \), in particular a compact embedding that plays a central role in the proof of our main result. The existence of three critical points is proved in Section 3 by minimization on Nehari-type sets. The regularity of solutions is discussed in Section 4, where the proof of Theorem 4 is then completed.

We shall use the letter \( C \) (possibly with an index) to denote various positive constants, the exact value of which is not relevant to the analysis.

## 2. Preliminaries

We start this section by some preliminary results about the functional setting, which will be useful to our existence theory. In particular we establish embedding properties of the space \( W_A(\Omega) \) which will play an important role in our analysis. For an arbitrary open set \( \Omega \subset \mathbb{R}^N \), let

\[
W_A(\Omega) = \{ u|_\Omega : u \in W_A(\mathbb{R}^N) \},
\]

where \( W_A(\mathbb{R}^N) \) has been defined in the introduction. Given a positive measurable function \( B : \Omega \rightarrow \mathbb{R}_+ \) and \( q \geq 1 \), we define the weighted Lebesgue space

\[
L^q_B(\Omega) = \{ u : \Omega \rightarrow \mathbb{R} : u \text{ is measurable and } \int_\Omega |u|^q B(x) \, dx < \infty \},
\]

endowed with its natural norm \( \|u\|_{L^q_B(\Omega)} = (\int_\Omega |u|^q B(x) \, dx)^{1/q} \). When there is no risk of confusion, we shall merely write \( W_A \) and \( L^q_B \) for \( W_A(\Omega) \) and \( L^q_B(\Omega) \). We may also use the shorthand notation \( \| \cdot \|_r \equiv \| \cdot \|_{L^r} \) for the usual Lebesgue norms.

Under appropriate assumptions, we will show that \( W_A(\Omega) \) is continuously and compactly embedded into \( L^q_B(\Omega) \). Our proof will rely on the classic theory of
Let us first state the following elementary properties.

**Proposition 3.** Let $A$ satisfy (A1) and $B : \Omega \to \mathbb{R}$ be measurable with $B > 0$ a.e.

(i) $W_A(\Omega)$ is a separable, uniformly convex — hence reflexive — Banach space satisfying $W^{1, p}(\Omega) \subseteq W_A(\Omega)$, with equality if $\text{ess inf}_{x \in \Omega} A > 0$.

(ii) For any $1 < q < \infty$, $L^q_B(\Omega)$ is a separable reflexive Banach space.

**Proof.** The fact that $W_A(\mathbb{R}^N) = W^{1, p}(\mathbb{R}^N)$ if $\text{ess inf}_{x \in \Omega} A > 0$ follows from the definition of $W_A(\mathbb{R}^N)$. The remaining statements can be found in [5] and [12]. □

**Proposition 4.** Suppose that $\Omega \subset \mathbb{R}^N$ is either $\mathbb{R}^N$ or a bounded open set with $C^1$ boundary. Let $q \in (p, p^*)$ and $B \in L^{\frac{q}{q - p^*}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then we have $W_A(\Omega) \subseteq L^q_B(\Omega)$ and there is a constant $C > 0$ such that

$$\|u\|_{L^q_B(\Omega)} \leq C\|u\|_{W_A(\Omega)}, \quad u \in W_A(\Omega).$$

Furthermore, the embedding is compact.

**Proof.** To prove (i), let us first consider the case where $\Omega = \mathbb{R}^N$. Notice that $W^{1, p}(\mathbb{R}^N)$ is a dense subspace of $W_A(\mathbb{R}^N)$. We shall thus start by proving (9) for $u \in W^{1, p}(\mathbb{R}^N)$, and then argue by density. For $u \in W^{1, p}(\mathbb{R}^N)$, it follows from Hölder’s inequality and the Sobolev embedding theorem that

$$\int_{\mathbb{R}^N} |u|^q B(x) \, dx \leq \|B\|_{L^{\frac{q}{q - p^*}}} \|u\|_{L^{p^*}_B}^q \leq \|B\|_{L^{\frac{q}{q - p^*}}} C \|\nabla u\|_{L^p}_B \leq C \|B\|_{L^{\frac{q}{q - p^*}}} \|u\|_{L^q_B},$$

and so $\|u\|_{L^q_B} \leq C\|u\|_{W_A}$ for all $u \in W^{1, p}(\mathbb{R}^N)$. Now for any $u \in W_A(\mathbb{R}^N)$, there is a sequence $(u_n) \subset W^{1, p}(\mathbb{R}^N)$ such that $u_n \to u$ in $W_A(\mathbb{R}^N)$. But for each $n$, there holds $\|u_n\|_{L^q_B} \leq C\|u_n\|_{W_A}$, so passing to the limit using Fatou’s lemma yields (9). The case where $\Omega$ is a bounded domain with smooth boundary follows from the case $\Omega = \mathbb{R}^N$ by adapting the extension theorem [3, Théorème IX.7] to the present context. The compactness of the embedding is proved in Appendix A. □

**Remark 5.** Observe that, by density, the classic Sobolev inequality, $\|u\|_{L^p_0} \leq C\|\nabla u\|_p$ for $u \in W^{1, p}(\Omega)$, extends to $u \in W_A(\Omega)$, so that

$$\|u\|_{L^{p^*}_B(\Omega)} \leq C\|\nabla u\|_{L^p_B(\Omega)} \leq C\|u\|_{W_A(\Omega)}, \quad u \in W_A(\Omega).$$

We are now in a position to prove that the functional $S_\lambda : W_A(\mathbb{R}^N) \to \mathbb{R}$ defined in (5) is of class $C^1$. We shall denote by $\langle \cdot, \cdot \rangle : W_A^* \times W_A \to \mathbb{R}$ the duality pairing between $W_A$ and its topological dual $W_A^*$.

**Lemma 6.** Let $A$ satisfy assumption (A1). Then we have $S_\lambda \in C^1(W_A(\mathbb{R}^N), \mathbb{R})$ and, for all $u, v \in W_A(\mathbb{R}^N)$,

$$\langle S_\lambda'(u), v \rangle = \int_{\mathbb{R}^N} \nabla u |\nabla v|^2 - \lambda A(x) |u|^{p-2} v u \, dx - \int_{\mathbb{R}^N} g(x, u)v \, dx.$$  

**Proof.** The proof of Lemma 6 follows from the continuity of the embedding in Proposition 4. To avoid disrupting the exposition with technicalities, we postpone it to Appendix B. □

**Lemma 7.** Under assumptions (g1), (A1) and (A2), the following statements hold.

(i) The functionals $W_A(\mathbb{R}^N) \to \mathbb{R}$, $u \mapsto \int_{\mathbb{R}^N} g(x, u)u \, dx$, $u \mapsto \int_{\mathbb{R}^N} G(x, u) \, dx$ are compact, in the sense that they map bounded sequences to relatively compact ones.

Note that an English translation of Brezis’ book is also available [8]. However, in the original (French) version [3], the compactness result we shall use is formulated in a way which is slightly better suited to our proof of Proposition 4 so we will rather refer to [3] throughout.
(ii) The functional \( W_A(\mathbb{R}^N) \rightarrow \mathbb{R}, u \mapsto \int_{\mathbb{R}^N} A(x)|u|^p \, dx \) is also compact.

**Proof.** (i) Consider a bounded sequence \((u_n) \subset W_A\). By Proposition 4 (iii) \((u_n)\) is bounded in \( L^q_B(\mathbb{R}^N) \), and there exists a subsequence (still denoted by \((u_n)\)) and an element \( u \in L^q_B(\mathbb{R}^N) \) such that \( u_n \rightarrow u \) in \( L^q_B(\mathbb{R}^N) \). It follows from \((g1)\) that

\[
|\Phi(u_n) - \Phi(u)| \leq \int_{\mathbb{R}^N} |g(x, u_n) - g(x, u)| \, dx
\]

\[
\leq \int_{\mathbb{R}^N} |g(x, u_n) - g(x, u)||u_n| \, dx + \int_{\mathbb{R}^N} |g(x, u)||u_n - u| \, dx
\]

\[
\leq \int_{\mathbb{R}^N} B||u_n||^{q-2}u_n - |u||^{q-2}u||u_n - u| \, dx + \int_{\mathbb{R}^N} B|u||^{q-1}|u_n - u| \, dx,
\]

where, by Hölder's inequality,

\[
\int_{\mathbb{R}^N} B||u_n||^{q-2}u_n - |u||^{q-2}u||u_n - u| \, dx \leq C\left( \int_{\mathbb{R}^N} B||u_n||^{q-2}u_n - |u||^{q-2}u\right) \|u_n - u\|_{L_B^q}^{q-1}.\]

Since \( u_n \rightarrow u \) in \( L^q_B(\mathbb{R}^N) \), we can suppose that \( u_n \rightarrow u \) pointwise a.e., and that \( B||u_n||^q \leq f \), uniformly in \( n \), for some \( f \in L^1(\mathbb{R}^N) \). It then follows by dominated convergence that the right-hand side of the above inequality goes to zero as \( n \rightarrow \infty \).

On the other hand, by Hölder’s inequality,

\[
\int_{\mathbb{R}^N} B|u||^{q-1}|u_n - u| \, dx \leq ||u||_{L_B^q}^{q-1}||u_n - u||_{L_B^q} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty,
\]

which concludes the proof. A similar argument shows that \( u \mapsto \int_{\mathbb{R}^N} G(x, u) \, dx \) is compact.

(ii) Consider again a bounded sequence \((u_n) \subset W_A\). By Proposition 3 (i) there exists a subsequence (still denoted by \((u_n)\)) and an element \( u \in W_A \) such that \( u_n \rightharpoonup u \) weakly in \( W_A \). By Hölder’s inequality we have, for any open set \( \Omega \subset \mathbb{R}^N \),

\[
\int_{\Omega} A(x)|u_n|^p - |u|^p \, dx \leq ||A||_{L^r(\Omega)}||u_n|^p - |u|^p||_{L^s(\Omega)},
\]

for some \( r > \frac{N}{p} \) and \( s < \frac{N}{p} \). Since \( u \mapsto |u|^p \) is continuous from \( L^{p^*}(\Omega) \) to \( L^s(\Omega) \) and, for \( \Omega \) bounded, the embedding \( W_A(\Omega) \subset L^{p^*}(\Omega) \) is compact, it follows that

\[
\int_{\Omega} A(x)|u_n|^p - |u|^p \, dx \rightarrow 0, \quad \Omega \text{ bounded}.
\]

On the other hand,

\[
\int_{\mathbb{R}^N \setminus \Omega} A(x)|u_n|^p - |u|^p \, dx \leq ||u_n|^p - |u|^p||_{L^{N/(p(N/p \wedge p))}(\mathbb{R}^N \setminus \Omega)} ||A||_{L^{N/(p(N/p \wedge p))}(\mathbb{R}^N \setminus \Omega)}
\]

\[
\leq \left( ||u_n||_{L^{N/p}(\mathbb{R}^N \setminus \Omega)} + ||u||_{L^{N/p}(\mathbb{R}^N \setminus \Omega)} \right) ||A||_{L^{N/(p(N/p \wedge p))}(\mathbb{R}^N \setminus \Omega)}
\]

\[
\leq \left( ||\nabla u_n||_{L^{p}(\Omega \setminus \Omega)} + ||\nabla u||_{L^{p}(\Omega \setminus \Omega)} \right) ||A||_{L^{N/(p(N/p \wedge p))}(\mathbb{R}^N \setminus \Omega)}
\]

\[
\leq C||A||_{L^{N/(p(N/p \wedge p))}(\mathbb{R}^N \setminus \Omega)},
\]

which can be made arbitrarily small by choosing \( |\Omega| \) large enough. \( \Box \)

We conclude this section by showing that, under hypothesis \((A, \lambda)\), the (quasi)norm \( \| \cdot \|_\lambda \) defined in (7) is equivalent to \( \| \cdot \|_{W_A} \).

**Lemma 8.** Let \( A \) and \( \lambda \) satisfy the hypotheses \((A1)\) and \((A, \lambda)\). Then there exist constants \( c_1 = c_i(\lambda) > 0, \ i = 1, 2, \) such that

\[
c_1\|u\|_{W_A} \leq \|u\|_\lambda \leq c_2\|u\|_{W_A}, \quad u \in W_A(\mathbb{R}^N).
\]
that, when \((A, \lambda)\) is required to prove the first inequality in (12), which we do now. Let \(\varepsilon > 0\). By the definition of \(\lambda_A\) in (10) we have, for any \(u \in W_A\),
\[
\|u\|_A^2 = \varepsilon \int_{\mathbb{R}^N} |\nabla u|^p \, dx + (1 - \varepsilon) \int_{\mathbb{R}^N} |\nabla u|^p \, dx - \lambda \int_{\mathbb{R}^N} A(x)|u|^p \, dx
\]
\[
\geq \varepsilon \int_{\mathbb{R}^N} |\nabla u|^p \, dx + (1 - \varepsilon)\lambda_A \int_{\mathbb{R}^N} A(x)|u|^p \, dx - \lambda \int_{\mathbb{R}^N} A(x)|u|^p \, dx
\]
\[
= \varepsilon \int_{\mathbb{R}^N} |\nabla u|^p \, dx + [(1 - \varepsilon)(\lambda_A - \lambda) - \varepsilon \lambda] \int_{\mathbb{R}^N} A(x)|u|^p \, dx
\]
\[
\geq \min\{\varepsilon, [(1 - \varepsilon)(\lambda_A - \lambda) - \varepsilon \lambda]\}\|u\|_{W_A}^p.
\]
Since \((1 - \varepsilon)(\lambda_A - \lambda) - \varepsilon \lambda \to \lambda_A - \lambda > 0\) as \(\varepsilon \to 0\), we can choose \(\varepsilon > 0\) such that \(c_1 := (\min\{\varepsilon, [(1 - \varepsilon)(\lambda_A - \lambda) - \varepsilon \lambda]\})^{1/p}\) does the job. This concludes the proof. \(\square\)

Remark 9. It is worth noting that \(\|\cdot\|_A\) satisfies the usual properties of a norm except for the triangle inequality. However, it follows from (12) that
\[
\|u + v\|_A \leq c_2 c_1^{-1}(\|u\|_A + \|v\|_A).
\]
Hence \(\|\cdot\|_A\) is a norm or a quasinorm, depending on whether \(c_2 c_1^{-1}\) is smaller or larger than 1. In fact it can be seen that \(c_2 c_1^{-1} > 1\) if \(\lambda > 0\), so \(\|\cdot\|_A\) is only a quasinorm in this case.

3. Existence of weak solutions

We will prove the existence of at least one positive, one negative, and one sign-changing solution of (11) by constraint minimization of the functional \(S_\lambda\) defined in (5). Since it is easier to obtain solutions of a given sign, we will focus our attention on the existence of a sign-changing solution, and we will explain in the course of the proof how to modify it in order to get positive/negative solutions. The existence of a sign-changing solution is obtained by minimizing \(S_\lambda\) on the ‘nodal Nehari set’, which will be defined below.

We define the positive and negative parts \(u^\pm\) of a function \(u : \mathbb{R}^N \to \mathbb{R}\) by \(u^\pm(x) := \pm \max\{\pm u(x), 0\}\), \(x \in \mathbb{R}^N\), so that \(u = u^+ + u^-\), with \(\pm u^\pm \geq 0\). It follows from (10) that \(u^\pm \in W_A(\mathbb{R}^N)\) whenever \(u \in W_A(\mathbb{R}^N)\).

Theorem 10. Suppose that the hypotheses (A1), \((A, \lambda)\) and (g1)–(g3) are satisfied. Then there exist \(u_1, u_2, u_3 \in W_A(\mathbb{R}^N)\), with \(u_1 > 0\) a.e., \(u_2 < 0\) a.e., and \(u_3^\pm \neq 0\) a.e., such that \(S_\lambda'(u_i) = 0\), \(i = 1, 2, 3\).

Remark 11. When \(\lambda < 0\), the condition \((A, \lambda)\) of Theorem 10 is trivially satisfied. Then \(A\) needs only satisfy assumption (A1) and the conclusion of Theorem 10 holds. The choice \(A \equiv 1\) is allowed in this case, yielding solutions in \(W^{1,p}(\mathbb{R}^N)\). Observe that \(\lambda_A = 0\) for \(A \equiv 1\), reflecting the absence of Poincaré inequality on \(\mathbb{R}^N\). In order to apply Theorem 10 with \(\lambda > 0\), we need conditions on \(A\) such that \(\lambda_A > 0\).

Proposition 12. If \(A\) satisfies (A1) and (A2) then \(\lambda_A > 0\). Moreover, there exists \(u^* \in W_A(\mathbb{R}^N)\) such that \(\lambda_A = \frac{\int_{\mathbb{R}^N} |\nabla u^*|^p \, dx}{\int_{\mathbb{R}^N} A(x)|u^*|^p \, dx}\).

Proof. By Hölder’s inequality and Remark 4 there exists \(C > 0\) such that
\[
\int_{\mathbb{R}^N} A(x)|u|^p \, dx \leq C\|A\|_{N/p}\|\nabla u\|_{p'}^p, \quad u \in W_A(\mathbb{R}^N).
\]
Furthermore, variational arguments similar to the proof of [1, Theorem 1] show that, when (A1) and (A2) hold, the infimum in (6) is actually achieved. \(\square\)
Hence, letting \( \delta \) where, for

Clearly, Nehari set all sign-changing solutions of (1).

\[ g \quad \text{of hypotheses } (\text{the hypotheses of Theorem 10 hold throughout the rest of this section. Before we)} \]

Therefore, (i) follows immediately from (g1). To prove (ii), first note that, for \( s > R \),

\[ \frac{G'(x, s)}{G(x, s)} \geq \frac{\theta}{s} \implies G(x, s) \geq R^\theta G(x, R)s^\theta = R^\theta G(x, R)|s|^\theta, \]

whereas, for \( s < -R \),

\[ \frac{G'(x, s)}{G(x, s)} \leq \frac{\theta}{s} \implies \int_s^{-R} \frac{G'(x, t)}{G(x, t)} dt \leq \theta \int_s^{-R} \frac{dt}{t} \implies \ln \frac{G(x, -R)}{G(x, s)} \leq \ln \left( \frac{-s}{R} \right)^{-\theta} \]

\[ \implies G(x, s) \geq R^{-\theta} G(x, -R)(-s)^\theta = R^{-\theta} G(x, -R)|s|^\theta. \]

Therefore, \( G(x, s) \geq C(x)|s|^\theta \) for almost all \( x \in \mathbb{R}^N \) and for all \( |s| \geq R \), where \( C(x) := \min\{R^\theta G(x, R), R^{-\theta} G(x, -R)\} \). Then (g2) implies that

\[ |g(x, s)| \geq \theta C(x)|s|^{|\theta| - 1}, \quad \text{a.e. } x \in \mathbb{R}^N, \ |s| \geq R, \]

with \( C(x) := \min\{R^\theta G(x, R), R^{-\theta} G(x, -R)\} \), from which the limits in (ii) follow. Finally, (iii) follows from (g2) and (g3). \( \square \)

Let us now describe the variational setting we shall use to obtain critical points of the functional \( S_\lambda \) defined in (5). By Lemma 6 \( S_\lambda \in C^1(W_A, \mathbb{R}) \) and, recalling the definition of \( \| \cdot \|_\lambda \) in (7), for all \( \lambda < \lambda_A \) we define

\[ J_\lambda(u) = \langle S'_\lambda(u), u \rangle = \|u\|_{\lambda}^p - \int_{\mathbb{R}^N} g(x, u)u \, dx, \]

\[ N_\lambda = \{ u \in W_A \setminus \{0\} : J_\lambda(u) = 0 \}, \]

\[ M_\lambda = \{ u \in W_A : u^\pm \in N_\lambda \} \subset N_\lambda. \]

The sets \( N_\lambda \) and \( M_\lambda \) are respectively known as the Nehari manifold and the nodal Nehari set. Clearly, \( N_\lambda \) contains all non-trivial solutions of (1) while \( M_\lambda \) contains all sign-changing solutions of (1).

For \( u \in N_\lambda \), it follows from Lemma 13 (i), Proposition 4 and Lemma 8 that

\[ \|u\|_{\lambda}^p = \int_{\mathbb{R}^N} g(x, u)u \, dx \leq C \int_{\mathbb{R}^N} B(x)|u|^q \, dx \leq C\|u\|_{W_A}^q \leq C\|u\|_{\lambda}^q. \]

Hence, letting \( \delta_\lambda := C^{-1/(q-p)} \), we have

\[ \|u\|_{\lambda} \geq \delta_\lambda > 0, \quad u \in N_\lambda. \]  

(13)

Observing that

\[ J_\lambda(u) = 0 \iff \|u\|_{\lambda}^p = \int_{\mathbb{R}^N} g(x, u)u \, dx, \]
it follows from Lemma 13 (iii) that
\[ S_\lambda(u) = \int_{\mathbb{R}^N} \frac{1}{p} q(x, u)u - G(x, u) \, dx = \int_{\mathbb{R}^N} h(x, u) \, dx > 0, \quad u \in N_\lambda. \] (14)

Therefore
\[ m_\lambda := \inf_{M_\lambda} \sup_{N_\lambda} S_\lambda \geq 0. \] (15)

We will now show that the Nehari manifold is diffeomorphic to the unit sphere in \((W_A, \| \cdot \|_\lambda)\). Firstly, similar arguments to the proof of Lemma 6 show that \(J_\lambda \in C^1(W_A, \mathbb{R})\), and it follows from \((g3)\) that \(\langle J'_\lambda(u), u \rangle < 0\) for all \(u \in N_\lambda\).

Therefore, by the submersion theorem, \(N_\lambda\) is a \(C^1\) manifold of codimension 1 in \(W_A\), such that the tangent space \(T_u N_\lambda\) is transversal to \(\mathbb{R}_+ u\), for all \(u \in N_\lambda\).

**Lemma 14.** For any fixed \(u \in W_A \setminus \{0\}\), there exists a unique \(t = t_\lambda(u) > 0\) such that \(t_\lambda(u) u \in N_\lambda\). Furthermore the map \(u \mapsto t_\lambda(u) u\) is a \(C^1\) diffeomorphism from \(\{u \in W_A : \| u \|_\lambda = 1\}\) onto \(N_\lambda\), with inverse \(u \mapsto u/\| u \|_\lambda\). Moreover, for any \(u \in W_A \setminus \{0\}\), we have
\[ t_\lambda(u) < 1 \text{ if } J_\lambda(u) < 0 \quad \text{and} \quad t_\lambda(u) > 1 \text{ if } J_\lambda(u) > 0. \] (16)

Finally, \(S_\lambda(t_\lambda(u)u)\) is increasing for \(t \in (0, t_\lambda(u))\) and decreasing for \(t \in (t_\lambda(u), \infty)\), with
\[ S_\lambda(t_\lambda(u)u) = \max_{t > 0} S_\lambda(tu), \quad \text{for all } u \in W_A \setminus \{0\}. \] (17)

**Proof.** Define a \(C^1\) function \(\varphi : (0, \infty) \to \mathbb{R}\) by
\[ \varphi_u(t) = \frac{1}{tp^p} J'_\lambda(tu) = \| u \|_\lambda^p - \int_{\mathbb{R}^N} g(x, tu) \cdot tu \, dx. \]

It follows from Lemma 13 (i) and (ii) that
\[ \lim_{t \to 0^+} \varphi_u(t) = \| u \|_\lambda^p > 0, \quad \lim_{t \to +\infty} \varphi_u(t) = -\infty. \]

Furthermore, by \((g3)\), \(t \mapsto \varphi_u(t)\) is strictly decreasing on \((0, \infty)\), from which the existence and uniqueness of \(t_\lambda(u)\) follow. The diffeomorphism statement is a consequence of the implicit function theorem and the transversality of \(T_u N_\lambda\) and \(\mathbb{R}_+ u\). It follows easily from the properties of \(\varphi_u\), while the behaviour of \(S_\lambda(tu), \ t > 0\), follows from the calculation
\[ \frac{d}{dt} S_\lambda(tu) = \langle S'_\lambda(tu), u \rangle \]
\[ = \int_{\mathbb{R}^N} t^{p-1} (|\nabla u|^p - \lambda A(x)|u|^p) \, dx - \int_{\mathbb{R}^N} g(x, tu) tu \, dx \]
\[ = t^{-1} \left( \| tu \|_\lambda^p - \int_{\mathbb{R}^N} g(x, tu) tu \, dx \right) = t^{-1} J_\lambda(tu). \]

The lemma is proved. \(\square\)

**Proposition 15.** The infimum \(m_\lambda\) defined in (15) is achieved.

**Proof.** In the course of this proof, we will take the liberty of passing to subsequences when necessary, without mentioning it explicitly. The proof proceeds in two steps.

1. **Boundedness of a minimizing sequence.** Consider \((u_n) \subset M_\lambda\) such that \(S_\lambda(u_n) \to m_\lambda\), and suppose by contradiction that \(\| u_n \|_\lambda \to \infty\) as \(n \to \infty\). Now let
\[ v_n = p[m_\lambda + 1]^{1/p} \frac{u_n}{\| u_n \|_\lambda}. \]

That \(M_\lambda \neq \emptyset\) is easily seen from step 2 in the proof of Proposition 15.
Since the sequence \((v_n)\) is bounded, we can suppose that there exists \(v \in W_A\) such that \(v_n \rightharpoonup v\) weakly in \(W_A\). By Lemma 17 we have

\[ S(v_n) \to m_\lambda + 1 - \int_{\mathbb{R}^N} G(x, v) \, dx. \]

On the other hand, since \(u_n \in N_\lambda\), it follows from Lemma 17 that \(S(v_n) \leq S(u_n)\). We shall thus reach a contradiction by showing that \(v \equiv 0\). If it is not the case, there exists a set \(\Omega \subset \mathbb{R}^N\) with positive measure, and a number \(\delta > 0\), such that \(\text{ess inf}_\Omega |v| \geq \delta\). Invoking Proposition 4 and Egorov’s theorem, we can suppose that for some large enough \(n_0 \in \mathbb{N}\).

\[ \text{ess inf}_\Omega |v_n| \geq \frac{\delta}{2} > 0, \quad n \geq n_0, \]

for some large enough \(n_0 \in \mathbb{N}\). Since \(G(x,0) \equiv 0\) and the supports of \(v^+_n\) and \(v^-_n\) are disjoint, it follows from Lemma 13 (iii) that

\[
\int_{\mathbb{R}^N} G(x, u_n) \, dx \geq \int_{\Omega} G(x, \frac{\|u_n\|}{p(m_\lambda + 1)^{1/p}} v_n) \, dx
\]

\[
= \int_{\Omega} G(x, \frac{\|u_n\|}{p(m_\lambda + 1)^{1/p}} v^+_n) + G(x, \frac{\|u_n\|}{p(m_\lambda + 1)^{1/p}} v^-_n) \, dx
\]

\[
\geq \int_{\Omega} G(x, \frac{\|u_n\|}{p(m_\lambda + 1)^{1/p}} \frac{\delta}{2} + G(x, \frac{\|u_n\|}{p(m_\lambda + 1)^{1/p}} (-\frac{\delta}{2})) \, dx, \quad n \geq n_0.
\]

Then Lemma 18 (ii) yields

\[ \int_{\mathbb{R}^N} G(x, u_n) \, dx \to \infty \quad \text{as} \ n \to \infty. \]

However, on the other hand,

\[ \int_{\mathbb{R}^N} G(x, u_n) \, dx = \frac{1}{p} \frac{\|u_n\|_p^p - S_\lambda(u_n)}{\|u_n\|_\lambda^p} \to \frac{1}{p} \quad \text{as} \ n \to \infty, \]

which gives the desired contradiction. Therefore, any minimizing sequence \((u_n)\) is indeed bounded.

2. Existence of a minimizer. Let \((u_n) \subset M_\lambda\) such that \(S_\lambda(u_n) \to m_\lambda\). Since \((u_n)\) is bounded in \(W_A\), there exists \(u \in W_A\) such that \(u_n \rightharpoonup u\) and \(u^\pm_n \rightharpoonup u^\pm\) weakly in \(W_A\) as \(n \to \infty\). It immediately follows from the weak lower semicontinuity of \(u \mapsto \|\nabla u\|_p^p\), and from Lemma 7 that

\[ S_\lambda(u) \leq \liminf_{n \to \infty} S_\lambda(u_n) = m_\lambda. \]

Hence we need only prove that \(u^\pm \in N_\lambda\). We first observe that \(u^\pm \neq 0\). Indeed, by Lemma 7 and (13),

\[ \int_{\mathbb{R}^N} g(x, u^\pm) u^\pm \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^N} g(x, u_n^\pm) u_n^\pm \, dx = \lim_{n \to \infty} \|u_n^\pm\|_\lambda^p \geq \delta_\lambda^p > 0. \quad (18) \]

Invoking again the weak lower semicontinuity of \(u \mapsto \|\nabla u\|_p^p\) and Lemma 7 it follows from (18) that

\[ J_\lambda(u^\pm) = \|u^\pm\|_\lambda^p - \lim_{n \to \infty} \|u_n^\pm\|_\lambda^p \leq 0. \]

Suppose by contradiction that

\[ \|u^\pm\|_\lambda^p < \liminf_{n \to \infty} \|u_n^\pm\|_\lambda^p. \]
Then \( t^+ := t_\lambda(u^+) < 1 \) and \( t^- := t_\lambda(u^-) \leq 1 \), \( t^+u^+ + t^-u^- \in M_\lambda \subset N_\lambda \), and so
\[
S_\lambda(t^+u^+ + t^-u^-) = \int_{\mathbb{R}^N} h(x, t^+u^+ + t^-u^-) \, dx
\]
by (14). Since \( h(x, 0) \equiv 0 \) and the supports of \( u^+ \) and \( u^- \) are disjoint, it follows from Lemma 13 (iii) that
\[
S_\lambda(t^+u^+ + t^-u^-) = \int_{\mathbb{R}^N} h(x, t^+u^+ + t^-u^-) \, dx
\]
\[
= \int_{\mathbb{R}^N} h(x, t^+u^+) \, dx + \int_{\mathbb{R}^N} h(x, t^-u^-) \, dx
\]
\[
< \int_{\mathbb{R}^N} h(x, u^+) \, dx + \int_{\mathbb{R}^N} h(x, u^-) \, dx = \int_{\mathbb{R}^N} h(x, u^+ + u^-) \, dx
\]
\[
= \int_{\mathbb{R}^N} h(x, u) \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^N} h(x, u_n) \, dx = \lim_{n \to \infty} S_\lambda(u_n) = m_\lambda.
\]
This contradiction concludes the proof. \( \square \)

**Remark 16.** If \( \lambda < 0 \) then \( \| \cdot \|_\lambda \) is a norm, \( (W_\lambda, \| \cdot \|_\lambda) \) is uniformly convex, and the proof shows that \( u_n \to u \) in \( W_\lambda \) (up to a subsequence).

We are now in a position to complete the

**Proof of Theorem 10.** Proposition 15 yields an element \( u_3 \in W_\lambda \) that minimizes \( S_\lambda \) on the nodal Nehari set \( M_\lambda \). To conclude the proof of Theorem 10 we will now show that \( S_\lambda'(u_3) = 0 \). The existence of the critical points \( u_1 \) and \( u_2 \) (with \( u_1 > 0 \) a.e. and \( u_2 < 0 \) a.e.) follows similarly, by minimizing \( S_\lambda \) over \( N_{\lambda}^{\pm} \) instead of \( M_\lambda \), where
\[
N_{\lambda}^{\pm} = \{ u \in N_\lambda : u^\pm = 0 \}.
\]
Since \( M_\lambda \) is not a submanifold of \( W_\lambda \), we cannot use the Lagrange multiplier theorem to infer that the minimizer \( u_3 \) is indeed a critical point of \( S_\lambda \). To overcome this difficulty, we appeal to a theorem of Miranda 14, which was first established in 15. For the reader’s convenience, we recall here the two-dimensional version of this result. An elegant proof can be found in 18.

**Lemma 17.** Let \( L > 0 \), \( R = (-L, L)^2 \subset \mathbb{R}^2 \) and consider a continuous function \( F = (F_1, F_2) : \overline{R} \to \mathbb{R}^2 \) which satisfies \( F(t, s) \neq 0 \) for all \( (s, t) \in \partial R \), and the following conditions on the boundary \( \partial R \):
\[
F_1(-L, t) \geq 0, \quad F_1(L, t) \leq 0, \quad F_2(s, -L) \geq 0, \quad F_2(s, L) \leq 0.
\]
In other words, the vector field \( F \), evaluated on the boundary \( \partial R \), always points towards the interior of \( R \). Then there exists \( (s_0, t_0) \in R \) such that \( F(s_0, t_0) = 0 \).

We apply this lemma in the following way. Suppose by contradiction that \( S_\lambda'(u_3) \neq 0 \). Then there is \( \varphi \in W_\lambda \) such that \( \langle S_\lambda'(u_3), \varphi \rangle = -2 \) and so, by continuity of \( S_\lambda' \), there is an \( \varepsilon > 0 \) such that
\[
\langle S_\lambda'(tu_3^+ + su_3^- + r\varphi), \varphi \rangle < -1
\]
for all \( r \in (0, \varepsilon] \) and all \( (s, t) \in \overline{R} \), where \( R = (1 - \varepsilon, 1 + \varepsilon)^2 \subset \mathbb{R}^2 \). Now consider a continuous function \( \eta : \overline{R} \to [0, \varepsilon] \) such that \( \eta(1, 1) = \varepsilon, \eta(\partial R) = 0 \), and \( \eta \neq 0 \) on \( R \). We define \( F : \overline{R} \to \mathbb{R}^2 \) by
\[
F(s, t) = \left( J_\lambda((tu_3^+ + su_3^- + \eta(t, s)\varphi)^-), J_\lambda((tu_3^+ + su_3^- + \eta(t, s)\varphi)^+) \right),
\]
6which is essentially a version of Brouwer’s fixed point theorem
First of all, it is clear that $F$ is continuous. Next, for $s = 1 - \varepsilon$ and $t \in [1 - \varepsilon, 1 + \varepsilon]$, using the function $\varphi_u(t)$ introduced in the proof of Lemma [14] we have

$$\frac{F_i(1 - \varepsilon, t)}{(1 - \varepsilon)^p} = \frac{J_\lambda((1 - \varepsilon)w_3)}{(1 - \varepsilon)^p} = \varphi_{u_3}^*(1 - \varepsilon) > \varphi_{u_3}^*(1) = 0$$

since $u_3 \in N_\lambda$. Similar arguments show that $F_i(1 + \varepsilon, t) < 0$ and $F_2(s, 1 \pm \varepsilon) > 0$. Hence, the hypotheses of Lemma [17] are satisfied and there exists $(s_0, t_0) \in R$ such that $F(s_0, t_0) = 0$. Remarking that $t_0u_3^+ + s_0u_3^- + \eta(t_0, s_0)\varphi \neq 0$ by [19], it follows that $t_0u_3^+ + s_0u_3^- + \eta(t_0, s_0)\varphi \in M_\lambda$. We will reach a contradiction by showing that $S_\lambda(t_0u_3^+ + s_0u_3^- + \eta(t_0, s_0)\varphi) < m_\lambda$. By [19] we have

$$S_\lambda(t_0u_3^+ + s_0u_3^- + \eta(t_0, s_0)) = S_\lambda(t_0u_3^+ + s_0u_3^-)$$

$$+ \int_0^{\eta(t_0, s_0)} \langle S_\lambda'(tu_3^+ + su_3^- + r\varphi), \varphi \rangle \, dr$$

$$< S_\lambda(t_0u_3^+ + s_0u_3^-) - \eta(t_0, s_0).$$

If $(s_0, t_0) = (1, 1)$ then $S_\lambda(t_0u_3^+ + s_0u_3^- + \eta(t_0, s_0)\varphi) < m_\lambda - \varepsilon$ and we are done. So suppose that $(s_0, t_0) \neq (1, 1)$. Since $u_3^2 \in N_\lambda$ it follows from [17] that

$$S_\lambda(t_0u_3^+ + s_0u_3^- + \eta(t_0, s_0)\varphi) < S_\lambda(t_0u_3^+ + s_0u_3^-) - \eta(t_0, s_0)$$

$$= S_\lambda(t_0u_3^+ + s_0u_3^-) - \eta(t_0, s_0)$$

$$\leq S_\lambda(u_3^+ + s_\lambda(u_3^-) - \eta(t_0, s_0)$$

$$= S_\lambda(u_3^-) - \eta(t_0, s_0) < m_\lambda,$$

yielding the desired contradiction. This completes the proof of Theorem [10].

4. Regularity and conclusion of the proof

We will now prove a regularity result for the solutions given by Theorem [10]. We say that $u \in C^{1,\alpha}_{loc}(\mathbb{R}^N)$ if for any compact $K \subset \mathbb{R}^N$ there exists $\alpha = \alpha(K)$ such that $u \in C^{1,\alpha}(K)$. Once $C^{1,\alpha}$ regularity is proved, we will prove the remaining statements of Theorem [11] about the nodal properties of the solutions.

Proposition 18. Suppose that the hypotheses (A1), (A2) and (g1) hold. Then any weak solution $u$ of [11] satisfies $u \in C^{1,\alpha}_{loc}(\mathbb{R}^N)$.

Proof. Assumption (A1) implies that $u \in W^{1,p}_{loc}(\mathbb{R}^N) \subset L^p_{loc}(\mathbb{R}^N)$. It then follows from [17] that $u \in C^{1,\alpha}_{loc}(\mathbb{R}^N)$, provided we know a priori that $u \in L^\infty_{loc}(\mathbb{R}^N)$. For a fixed compact $K \subset \mathbb{R}^N$, we briefly explain how the results of [13] imply that $u \in L^\infty(K)$. Firstly, the integrability conditions on $A$ and $B$ given in hypotheses (A1), (A2) and (g1) precisely ensure that the assumptions (7.1) and (7.2) in [12, Chap. 4, Sec. 7] (together with conditions 1–3) there) hold. Thus the idea is to apply Theorem 7.1 of [13, Chap. 4, Sec. 7]. In the present context, the parameters $m$ and $q$ appearing there are given by $m = p$ and $q = p^*$. We need only explain here why the conclusion of this theorem still holds without the hypothesis that

$$\text{ess sup}_{\partial K} |u| < \infty. \quad (20)$$

This assumption is used in the proof of [13, Theorem 7.1, Chap. 4] in the two following instances. First, to derive the estimate (7.3), the test function $\eta(x) = \max\{u(x) - k, 0\}$ is used, where $k$ is a positive parameter such that $k \geq \text{ess sup}_{\partial K} |u|$. This restriction is due to the definition of a weak solution in [13, Chap. 4], requiring that $\eta \in W_0^{1,p}(K)$. Our definition of a weak solution (see [11]) allows us to merely consider $\eta \in W^{1,p}(K)$, and we can derive the estimate (7.3) in the same way as in [13]. Finally, assumption (20) is used to conclude the proof of Theorem 7.1 by...
invoking Theorem 5.1 of [13, Chap. 2]. It turns out that Theorem 5.2 of [13, Chap. 2] does the job as well, and does not require [20]. This completes the proof. □

We can now finish the

Proof of Theorem 1. It only remains to show that \( u_1 > 0, u_2 < 0, \) and \( u_3 \) has exactly two nodal domains, as defined in Section 1.1. The positivity of \( u_1 \) and the negativity of \( u_2 \) follow from the strong maximum principle, see e.g. Theorem 5 in [13]. Regarding \( u_3 \), we already know from the previous results that it has two nodal domains. Suppose by contradiction that \( u_3 \) has (at least) three distinct nodal domains \( \Omega_i \subset \mathbb{R}^N, \ i = 1, 2, 3, \) and define

\[
v_i(x) = \begin{cases} u_3(x) & \text{if } x \in \Omega_i, \\ 0 & \text{otherwise.} \end{cases}
\]

Clearly \( v_i \in W_A, \ i = 1, 2, 3, \) and without loss of generality we can suppose that \( v_1 > 0 \) and \( v_2 < 0. \) Since \( S^r_{\lambda}(u_3) = 0, \) it follows that \( v_i \in N_\lambda, \ i = 1, 2, 3, \) \( v_1 + v_2 \in M_\lambda, \) and so \( [14] \) implies that

\[ m_\lambda \leq S_{\lambda}(v_1 + v_2) < S_{\lambda}(u_3) = m_\lambda. \]

This contradiction concludes the proof. □

Appendix A. Compactness

Since we were not able to find the exact result we need in the literature, we now give a proof of the compactness of the embedding \( W_A(\Omega) \subset L^p_B(\Omega) \) in Proposition[4]. We shall make extensive use of the classic H"older and interpolation inequalities, which hold in the weighted spaces \( L^p_B(\Omega), \) as in the usual case \( B = 1. \)

Proof of Proposition 2 (continued). We start by assuming that \( \Omega \subset \mathbb{R}^N \) is an open bounded domain with \( C^1 \) boundary. We will first explain how the proof of the classic Rellich-Kondrachov theorem can be adapted to the present case. We follow the proof of Brezis [7, Théorème IX.16], which is based on a criterion of strong compactness in \( L^p \) spaces [7, Corollaire IV.26]. Note that [7, Corollaire IV.26] is a consequence of the famous Riesz-Fréchet-Kolmogorov theorem [7, Théorème IX.25]. It is easy to see that the proof of both [7, Théorème IX.25] and [7, Corollaire IV.26] remain virtually unchanged in the case of a weighted \( L^p \) space such as \( L^p_B(\Omega). \) Therefore, we can merely follow the proof of [7, Théorème IX.16] in the case \( p < N. \)

Letting \( \mathcal{F} \) be the unit ball in \( W_A(\Omega), \) this amounts to verifying assumptions (IV.23) and (IV.24) of [7, Corollaire IV.26]. Following Brezis, for an open set \( \omega \subset \Omega \) such that \( \overline{\omega} \) is compact and \( \overline{\omega} \subset \Omega, \) we write \( \omega \subset \subset \Omega. \) For a given \( h \in \mathbb{R}^N, \) we also define the translate \( \tau_h u \) of a function \( u \) by \( \tau_h u(x) = u(x + h), \ x \in \mathbb{R}^N. \) Assumptions (IV.23) and (IV.24) of [7, Corollaire IV.26] now read

\[
\forall \varepsilon > 0 \quad \exists \omega \subset \subset \Omega \quad \forall h \in \mathbb{R}^N \text{ s.t. } |h| < \delta, \quad \forall u \in \mathcal{F}.
\]

and

\[
\forall \varepsilon > 0 \quad \exists \omega \subset \subset \Omega \quad \text{s.t. } \|u\|_{L^p_B(\omega)} < \varepsilon, \quad \forall u \in \mathcal{F}.
\]

To prove (21), consider \( \alpha \in (0, 1) \) such that \( \frac{1}{q} = \frac{\alpha}{p} + \frac{1-\alpha}{p} \). By interpolation,

\[
\|\tau_h u - u\|_{L^p_B(\omega)} \leq \|\tau_h u - u\|_{L^p_B(\omega)}^{\alpha} \|\tau_h u - u\|_{L^p_B(\omega)}^{1-\alpha}.
\]

Now following the proof of [7, Proposition IX.3], it is easily seen that

\[
\|\tau_h u - u\|_{L^p_B(\omega)} \leq \|B\|_{\infty} \|\nabla u\|_{L^1(\Omega)} |h|.
\]
Hence, recalling that $\Omega$ is bounded and using \((10)\), we have

$$
\|u_n - u\|_{L_h^p(\Omega)} \leq \|B\|_\infty \|\nabla u\|_{L^1(\Omega)} |h|^\alpha (2\|u\|_{L_h^\alpha(\Omega)})^{1 - \alpha}
\leq 2^{1 - \alpha} \|B\|_\infty \|\nabla u\|_{L^1(\Omega)} |h|^\alpha
\leq C\|\nabla u\|_{L^p(\Omega)} \|\nabla u\|_{L_h^\alpha(\Omega)} |h|^\alpha
\leq C|h|^\alpha \text{ (since } \|\nabla u\|_{L_h^\alpha(\Omega)} \leq 1 \text{ for } u \in \mathcal{F}),
$$

which proves \((21)\). On the other hand, for all $u \in \mathcal{F}$, it follows from Hölder’s inequality and \((10)\) that

$$
\|u\|_{L_h^p(\Omega) \setminus \omega} \leq C|\Omega \setminus \omega|^{\frac{1}{q} - \frac{1}{p}},
$$

which proves \((22)\) and concludes the proof that the embedding is compact when $\Omega$ is a smooth bounded domain.

We now consider the case $\Omega = \mathbb{R}^N$. Let $(u_n) \subset W_A(\mathbb{R}^N)$ be a bounded sequence. We will show that $(u_n)$ is relatively compact in $L_h^p(\mathbb{R}^N)$. Firstly, since $W_A(\mathbb{R}^N)$ is reflexive, we can suppose that $u_n \rightharpoonup u$ weakly in $W_A(\mathbb{R}^N)$, for some $u \in W_A(\mathbb{R}^N)$. Also, denoting by $B(0, R)$ the ball of radius $R$ centred at $x = 0$ in $\mathbb{R}^N$, $u_n|_{B(0, R)} \rightharpoonup u|_{B(0, R)}$ weakly in $W_A(B(0, R))$, and we already know that (up to a subsequence) $u_n|_{B(0, R)} \to u|_{B(0, R)}$ in $L_h^p(B(0, R))$. Furthermore, by Hölder’s inequality and \((10)\),

$$
\int_{|x| \geq R} |u_n - u|^q B(x) \, dx \leq C \int_{|x| \geq R} \left( |u_n|^q + |u|^q \right) B(x) \, dx
\leq C \left( \int_{|x| \geq R} \left( |u_n|^q + |u|^q \right)^{p^* q / q} \, dx \right)^{q / p^*}
\leq C \left( \|u_n\|_{L_h^p}^{q} + \|u\|_{L_h^p}^{q} \right) \|B\|_{L_h^{p^* q} \setminus \mathbb{R}}(\{ |x| \geq R \})
\leq C \left( \|\nabla u_n\|_{L_h^p}^{q} + \|\nabla u\|_{L_h^p}^{q} \right) \|B\|_{L_h^{p^* q} \setminus \mathbb{R}}(\{ |x| \geq R \})
\leq C \|B\|_{L_h^{p^* q} \setminus \mathbb{R}}(\{ |x| \geq R \}).
$$

Since $B \in L_h^{p^* q} (\mathbb{R}^N)$, the right-hand side of \((25)\) can be made arbitrarily small by choosing $R > 0$ large enough, uniformly in $n$, which concludes the proof. \(\square\)

**Remark 19.** Let us now explain how the hypothesis $B \in L_\infty(\mathbb{R}^N)$ in \((g1)\) can be relaxed to a local integrability condition in the above proof, and hence throughout the whole paper. Choosing $t \in (q, p^*)$ and $\alpha \in (0, 1]$ such that $\frac{1}{q} = \frac{1}{t} + \frac{1 - \alpha}{\alpha}$, we start by replacing the interpolation inequality \((23)\) by

$$
\|u_n - u\|_{L_h^p(\Omega)} \leq \|\nabla u_n - u\|_{L_h^\alpha(\Omega)} \|\nabla u\|_{L_h^{1 - \alpha}(\Omega)}.
$$

Then, instead of \((21)\), the proof of \([3\text{ Proposition IX.3}]\) can be modified to show that

$$
\|u_n - u\|_{L_h^p(\Omega)} \leq \|B\|_{L_h^{p^* q} \setminus \mathbb{R}}(\{ |x| \geq R \}) \|\nabla u\|_{L_h^p(\Omega)} |h|.
$$
we see that Proposition 4 holds provided \( \Omega \) is a domain with smooth boundary. The case \( \Omega = \mathbb{R}^N \) will again take advantage of the Hölder inequality in the weighted Lebesgue spaces \( L_p^q(\mathbb{R}^N) \).

Moreover, we have

\[
\|u\|_{L_p^q(\Omega)} = \int_{\Omega} |u|^q B(x) \, dx \leq \|B\|_{L_p^{p^*}(\Omega)} \|u\|_{L_p^{p^*}(\Omega)} \leq C \|\Delta u\|_{L_p^{p^*}(\Omega)},
\]

and so (22) follows from the assumption that \( B \in L_p^{p^*}(\mathbb{R}^N) \). This completes the proof of compactness of the embedding \( W_A(\Omega) \subset L_p^q(\Omega) \) when \( \Omega \) is a bounded domain with smooth boundary. The case \( \Omega = \mathbb{R}^N \) then follows as before. Hence, we see that Proposition 4 holds provided \( B \in L_p^{p^*}(\mathbb{R}^N) \) and \( B \) satisfies (22) for some \( t \in (q, p^*) \).

**Appendix B. Differentiability**

This appendix is devoted to the proof of Lemma 6. Before we proceed with the proof, let us first remark that, thanks to Proposition 4, \( u \in W_A(\mathbb{R}^N) \implies u \in L_p^q(\mathbb{R}^N) \) and \( u \in L_p^q(\mathbb{R}^N) \),

where \( B \) and \( q \) have been introduced in hypothesis (g1). As in Appendix A, we will again take advantage of the Hölder inequality in the weighted Lebesgue spaces \( L_p^q(\mathbb{R}^N) \) and \( L_p^q(\mathbb{R}^N) \).

**Proof of Lemma 6.** For given \( u, v \in W_A(\mathbb{R}^N) \) we start by computing the Gâteaux derivative \( D_S(u) \) of \( S \) in the direction \( v \), and we show that it is equal to the right side of (11). That is, we compute \( \lim_{t \to 0} \frac{1}{t} [S(\lambda + tu) - S(\lambda)(u)] \). It follows from Hölder’s inequality that \( |\nabla u|^{p-2}\nabla u \cdot \nabla v \in L^1(\mathbb{R}^N) \). The derivation of the first term then follows in a standard manner, using the mean-value theorem and the dominated convergence theorem. Let us now consider the other two terms in more details. For \( t \neq 0 \), and \( x \in \mathbb{R}^N \), it follows from the mean-value theorem that there exists \( s = s(t, x) \in [0, 1] \) such that

\[
|u + tv|^p - |u|^p = p|u + tv|^{p-2}(u + tv)tv,
\]

and so

\[
\frac{1}{t} A(x)(|u + tv|^p - |u|^p) \to A(x)|u|^{p-2}uv \quad \text{as } t \to 0, \text{ for a.e. } x \in \mathbb{R}^N.
\]

Moreover,

\[
\frac{1}{t} A(x)(|u + tv|^p - |u|^p) \leq A(x)|u + tv|^{p-1}v \leq C A(x)(|u|^{p-1}|v| + |v|^p),
\]

where \( A|v|^p \in L^1(\mathbb{R}^N) \) since \( v \in W_A \), and \( A(x)|u|^{p-1}|v| \in L^1(\mathbb{R}^N) \) by Hölder’s inequality in \( L_p^q(\mathbb{R}^N) \). Hence, by dominated convergence,

\[
\frac{1}{t} \int_{\mathbb{R}^N} A(x)(|u + tv|^p - |u|^p) \, dx \to \int_{\mathbb{R}^N} A(x)|u|^{p-2}uv \, dx \quad \text{as } t \to 0.
\]

\[\text{Note that } \frac{p}{p-1} > \frac{p^*}{p^* - t} \iff t < \frac{N}{N-p}.\]
To deal with the last term, we apply again the mean-value theorem, which yields a number \( s = s(t, x) \in [0, 1] \) such that
\[
\frac{1}{t}(G(x, u + tv) - G(x, u)) = \frac{1}{t}g(x, u + stv)tv \to g(x, u)v \quad \text{as } t \to 0, \text{ for a.e. } x \in \mathbb{R}^N.
\]
Also, by (g1),
\[
|g(x, u + stv)v| \leq B(x)|u + stv|^{q-1}|v| \leq CB(x)(|u|^{q-1}|v| + |v|^q) \in L^1(\mathbb{R}^N),
\]
thanks to Proposition \([4]\) with \( r = q \). It then follows by dominated convergence that
\[
\frac{1}{t} \int_{\mathbb{R}^N} (G(x, u + tv) - G(x, u)) \, dx \to \int_{\mathbb{R}^N} g(x, u)v \, dx \quad \text{as } t \to 0.
\]
We have thus proved that the Gâteaux derivative \( DS_\lambda(u)v \) exists and is equal to the right-hand side of (11).

To complete the proof, we will now show that \( DS_\lambda(u) \in W_A^* \) for all \( u \in W_A \), and that the mapping \( u \mapsto DS_\lambda(u) \) is continuous. Hölder’s inequality yields
\[
\left| \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx \right| \leq \int_{\mathbb{R}^N} |\nabla u|^{p-1}|\nabla v| \, dx \leq ||\nabla u|^{p-1}||_p \|\nabla v\|_p = ||u||_W^{|p-1}|v|_W^p,
\]
\[
\left| \int_{\mathbb{R}^N} A(x)|u|^{p-2}uv \, dx \right| \leq \int_{\mathbb{R}^N} |u|^{p-1}|v|A(x) \, dx \leq ||u|^{p-1}||_L^p \|v\|_L^p = ||u||_W^{|p-1}|v|_W^p.
\]
and
\[
\left| \int_{\mathbb{R}^N} g(x, u)v \, dx \right| \leq \int_{\mathbb{R}^N} |g(x, u)| |v| \, dx \leq \int_{\mathbb{R}^N} |u|^{q-1}|v|B(x) \, dx \leq ||u|^{q-1}||_L^b \|v\|_L^b = ||u||_W^{|q-1}|v|_W^q \leq C||u||_W^{q-1}|v|_W^q,
\]
where the last inequality follows from Proposition \([4]\). These estimates show that \( DS_\lambda(u) \in W_A^* \) for all \( u \in W_A \).

To prove that \( u \mapsto DS_\lambda(u) \) is continuous, consider \( (u_n) \subset W_A \) such that \( u_n \to u \) in \( W_A \). We will show that
\[
||DS_\lambda(u_n) - DS_\lambda(u)|| = \sup_{v \in W_A \setminus \{0\}} \frac{|(DS_\lambda(u_n) - DS_\lambda(u), v)|}{\|v\|_W^q} \to 0 \quad \text{as } n \to \infty. \quad (30)
\]
We have
\[
|(DS_\lambda(u_n) - DS_\lambda(u), v)| \leq \int_{\mathbb{R}^N} ||\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u|| |\nabla v| \, dx + |\lambda| \int_{\mathbb{R}^N} A(x)||u_n|^{p-2}u_n - |u|^{p-2}u|| |v| \, dx + \int_{\mathbb{R}^N} |g(x, u_n) - g(x, u)||v| \, dx.
\]
Using Hölder’s inequality in the same fashion as above, we get
\[
\int_{\mathbb{R}^N} ||\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u|| |\nabla v| \, dx \leq ||\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u||_L^p \|\nabla v\|_W^p, \quad (31)
\]
\[
\int_{\mathbb{R}^N} A(x)||u_n|^{p-2}u_n - |u|^{p-2}u|| |v| \, dx \leq ||u_n|^{p-2}u_n - |u|^{p-2}u||_L^p \|v\|_L^p \quad (32)
\]
and

\[
\int_{\mathbb{R}^N} |g(x,u_n) - g(x,u)|v| \, dx \leq C\|u_n|^{q-2}u_n - |u|^{q-2}u\|_{L^q}^q \|v\|_{W^1_A}.
\]  

(33)

We now observe that, since \(u_n \to u\) in \(W_A\), we have (up to a subsequence) \(u_n \to u\) and \(\nabla u_n \to \nabla u\) pointwise a.e., \(A|u_n|^p \leq f\) and \(B|u_n|^q \leq g\) (by Proposition 4) for some functions \(f, g \in L^1(\mathbb{R}^N)\), uniformly in \(n\). The limit in (30) then follows from estimates (31)-(33) by dominated convergence, up to a subsequence. Since the previous argument can be applied to any subsequence of \((u_n)\), this completes the proof.

\[\square\]

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