A universal construction for moduli spaces of decorated vector bundles over curves

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Abstract
Let \( X \) be a smooth projective curve over the field of complex numbers, and fix a homogeneous representation \( \rho : \text{GL}(\mathcal{V}) \to \text{GL}(\mathcal{W}) \). Then, one can associate to every vector bundle \( E \) of rank \( r \) over \( X \) a vector bundle \( E^\rho \) with fibre \( V \). We would like to study triples \( (E, L, \varphi) \) where \( E \) is a vector bundle of rank \( r \) over \( X \), \( L \) is a line bundle over \( X \), and \( \varphi : E^\rho \to L \) is a non-trivial homomorphism. This set-up comprises well-known objects such as framed vector bundles, Higgs bundles, and conic bundles. In this paper, we will formulate a general (parameter dependent) semistability concept for such triples, which generalizes the classical Hilbert-Mumford criterion, and establish the existence of moduli spaces for the semistable objects. In the examples which have been studied so far, our semistability concept reproduces the known ones. Therefore, our results give in particular a unified construction for many moduli spaces considered in the literature.
Introduction

The present paper is devoted to the study of vector bundles with an additional structure from a unified point of view. We have picked the name “decorated vector bundles” suggested in [23].

Before we outline our paper, let us give some background. The first problem to treat is the problem of classifying vector bundles over an algebraic curve \(X\), assumed here to be smooth, projective and defined over \(\mathbb{C}\). From the point of view of projective geometry, this is important because it is closely related to classifying projective bundles over \(X\), so-called ruled manifolds. The basic invariants of a vector bundle \(E\) are its rank and its degree. They determine \(E\) as topological \(\mathbb{C}\)-vector bundle. The problem of classifying all vector bundles of fixed degree \(d\) and rank \(r\) is generally accessible only in a few cases:

The case \(r = 1\), i.e., the case of line bundles which is covered by the theory of Jacobian varieties.

The case \(X = \mathbb{P}^1\) where Grothendieck’s splitting theorem [18] provides the classification.

The case \(g(X) = 1\). In this case, the classification has been worked out by Atiyah [1].

As is clear from the theory of line bundles, over a curve of genus \(g \geq 1\), vector bundles of degree \(d\) and rank \(r\) cannot be parameterized by discrete data. Therefore, one seeks a variety parameterizing all vector bundles of given degree \(d\) and rank \(r\) characterized by a universal property like the Jacobian. Such a universal property was formulated by Mumford in his definition of a coarse moduli space [29]. However, one checks that the family of all vector bundles of degree \(d\) and rank \(r\) is not bounded which implies that a coarse moduli space cannot exist. For this reason, one has to restrict one’s attention to suitable bounded subfamilies of the family of all vector bundles of degree \(d\) and rank \(r\). Motivated by his general procedure to construct moduli spaces via his Geometric Invariant Theory [29], Mumford suggested that these classes should be the classes of stable and semistable vector bundles. His definition, given in [28], is the following: A vector bundle \(E\) is called (semi)stable, if for every non-trivial, proper subbundle \(F \subset E\)

\[
\mu(F) := \frac{\deg F}{\rk F} < \mu(E)
\]

Here, “( )” means that “<” is to be used for defining “semistable” and “<” for stable. Seshadri then succeeded to give a construction of the coarse moduli space of stable vector bundles, making use of Geometric Invariant Theory [42]. This moduli space is only a quasi-projective manifold. To
compactify it, one has also to look at semistable vector bundles. Seshadri formulated the notion of *S-equivalence* of semistable bundles which agrees with isomorphy for stable bundles but is coarser for properly semistable ones. The moduli space of S-equivalence classes exists by the same construction and is a normal projective variety compactifying the moduli space of stable bundles. Later Gieseker, Maruyama, and Simpson generalized the results to higher dimensions [14], [27], [43]. Their constructions also apply to curves and replace Seshadri’s (see [24]). Narasimhan and Seshadri related stable bundles to unitary representations of fundamental groups, a framework in which vector bundles had been formerly studied [31], [32].

The next step is to consider vector bundles with extra structures. Let us mention a few sources for this kind of problems:

**Classification of algebraic varieties.** We have already mentioned that the classification of vector bundles is related to the classification of projective bundles via the assignment $E \mapsto \mathbb{P}(E)$. Suppose, for example, that we want to study divisors in projective bundles. For this, let $E$ be a vector bundle, $\mathbb{P}(E)$ its associated projective bundle, $k$ a positive integer, and $M$ a line bundle on $X$. To give a divisor $D$ in the linear system $|\mathbb{P}(E)(k)|$ we have to give a section $\sigma: \mathcal{O}_{\mathbb{P}(E)}(-k) \rightarrow \mathcal{O}_{\mathbb{P}(E)}(k) \rightarrow \pi^* M$ which is the same as giving a non-zero homomorphism $\mathcal{O}_X \rightarrow S^k E \otimes M$, or $S^k E^* \rightarrow M$. Thus, we are led to classify triples $(E;M;\tau)$ where $E$ is a vector bundle over $X$, $M$ a line bundle, and $\tau: S^k E \rightarrow M$ a non-trivial homomorphism. In case the rank of $E$ is three and $k$ is two, this is the theory of *conic bundles*, recently studied by Gómez and Sols [15].

**Dimensional reduction.** Here, one looks at vector bundles $G$ on $X \subset \mathbb{P}^1$ which can be written as extensions

$$0 \rightarrow \pi_X F \rightarrow G \rightarrow \pi_X E \rightarrow \pi_{\mathbb{P}^1} \mathcal{O}_{\mathbb{P}^1}(2) \rightarrow 0$$

where $E$ and $F$ are vector bundles on $X$. These extensions are parameterized by $H^0(\mathcal{E} \rightarrow F) = \text{Hom}(\mathcal{E};F)$. The study of such vector bundles is thus related to the study of triples $(E;F;\varphi)$ where $E$ and $F$ are vector bundles on $X$ and $\varphi: E \rightarrow F$ is a non-zero homomorphism. These are the *holomorphic triples* of Bradlow and García-Prada [13] and [7]. They were also studied from the algebraic point of view by the author [39]. For the special case $E = \mathcal{O}_X$, we find the problem of vector bundles with a section, so-called *Bradlow pairs* [4]. An important application of Bradlow pairs was given by Thaddeus in his proof of the Verlinde formula [43].

**Representations of fundamental groups.** *Higgs bundles* are pairs $(E;\varphi)$, consisting of a vector bundle $E$ and a twisted endomorphism $\varphi: E \rightarrow E \otimes \omega_X$. Simpson used in [43] the higher dimensional analogues of these objects to study representations of fundamental groups of projective manifolds. This ties up nicely with the work of Narasimhan and Seshadri.

**Gauge theory.** Here, one starts with differentiable vector bundles together with an additional structure and considers certain differential equations associated to these data. The solutions of the equations then have — via a Kobayashi-Hitchin correspondence — interpretations as holomorphic decorated vector bundles over $X$, satisfying certain stability conditions. Again, the first case where this arose was the theory of Hermite-Einstein equations and stable vector bundles (see [26]) and was later studied in more complicated situations like the above examples. Recently, Banfield [2] and Mundet i Riera [30] investigated this in a broad context. We will come back to this again.

Now, for all of these problems and many more, there exist notions of semistability, depending on a rational parameter. The task of projective geometry is then to generalize the construction of Seshadri.
and the successors to obtain moduli spaces for the respective semistable and stable objects. These constructions, where existent, were done case by case and follow a certain pattern inspired by Gieseker’s, Maruyama’s, and Simpson’s constructions. One is therefore led to ask for a single unifying construction incorporating the known examples. This would complete the algebraic counterpart to the work of Banfield and Mundet i Riera.

We will consider this problem in the present article. Our framework is as follows: We fix a representation $\rho: \text{GL}(r) \to \text{GL}(V)$, such that the restriction to the centre $C = \text{GL}(r)$ is $z \mapsto z^\alpha \text{id}$ for some integer $\alpha$. Then, to any vector bundle $E$, we can associate a vector bundle $E_\rho$ of rank $\dim V$.

The objects we will treat are triples $(E; M; \tau)$ where $E$ is a vector bundle of rank $r$, $M$ is a line bundle, and $\tau: E_\rho \to M$ is a non-zero homomorphism. E.g., for $\rho: \text{GL}(3) \to \text{GL}(\mathbb{P}^2)$, we recover conic bundles. The list of problems we then have to solve is

1. Formulate an appropriate notion of semistability for the above objects!
2. Prove boundedness of the semistable triples $(E; M; \tau)$ where $\deg E$ and $\deg M$ are fixed!
3. Construct a parameter space $\mathcal{P}$ for the semistable objects together with an action of a general linear group $G$, such that the equivalence relation induced by this action is the natural equivalence relation on those triples!
4. Show that the categorical quotient $\mathcal{P} = \mathcal{P}/G$ exists!

The latter space will then be the moduli space. As one sees from this list, especially in view of the existing constructions, Geometric Invariant Theory will play a central rôle. Let us explain how one can find the semistability concept. First, assume that we are given a bounded family of triples $(E; M; \tau)$. Using the theory of quot-schemes it is by now not too hard a task to construct a parameter space $\mathcal{P}$ for the members of the family in such a way that we have a group action as required together with a family of linearizations — depending on a rational parameter — in line bundles over $\mathcal{P}$. Therefore, we have realized the input for the GIT process. The Hilbert-Mumford criterion now tells us how to find the semistable points. Thus, it is clear that our notion of semistability should mimic the Hilbert-Mumford criterion as closely as possible. Such an approach was also taken in gauge theory [2] and [30]. The structure of one parameter subgroups of the special linear group suggests that one parameter subgroups should be replaced by weighted filtrations of vector bundles. For weighted filtrations, one then defines the necessary numerical quantities resembling Mumford’s “$\mu$” and arrives at the desired semistability concept.

Our paper is organized as follows: In the first section, we collect the necessary background material from representation theory and GIT. Then, we come to the definition of semistability for the triples $(E; M; \tau)$ which depends on a positive rational parameter and describe the associated moduli functors. We state the main result, namely the existence of moduli spaces, and proceed to the proofs along the lines outlined before. The paper concludes with a long discussion of examples in order to show that the known problems in that context can be recovered from our results and that, in some cases, additional light is shed on them. The reader will notice that our general semistability concept is in the known cases more complicated than the existing ones and has to be simplified to recover the known ones. This is one of the key points of the paper: The notion of semistability should be simplified after doing the GIT construction and not before. This is why a unifying construction is feasible. However, we will present a general method to simplify the semistability concept in terms of the representation $\rho$. This method enables us to write down in every concrete situation the semistability concept in a more classical form. Applying this procedure, e.g., to framed bundles or conic bundles...
immediately reproduces the known semistability concepts. This provides us with a mechanism for finding the correct notion of semistability without guessing or referring to gauge theory.

Finally, we remark that we have confined ourselves to the case of curves in order to have a nice moduli functor associated to every representation of the general linear group. However, if one restricts to direct sums of tensor powers, the construction can also be performed over higher dimensional manifolds [16]. These higher dimensional versions have, in turn, important applications in the problem of compactifying moduli spaces of principal bundles with singular objects ([40], [17]). Finally, there is now also a version for product groups $GL(r_1) \times \cdots \times GL(r_s)$ over base manifolds of arbitrary dimension [41] the construction of which is based on the results of the present paper.

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Notations and conventions

All schemes will be defined over the field of complex numbers, $X$ will be a smooth projective curve of genus $g$. We denote by $\text{Sch}_C$ the category of separated schemes of finite type over $C$. A point will be a closed point unless otherwise mentioned.

For a vector bundle $E$ over a scheme $S$, we denote by $\mathbb{P}(E)$ the projective bundle of hyperplanes in the fibres of $E$.

Given a product $X \times Y$ of schemes, $\pi_X$ and $\pi_Y$ stand for the projections from $X \times Y$ onto the respective factors.

Let $V$ be a finite dimensional $C$-vector space and $\rho: G \to GL(V)$ a representation of the algebraic group $G$. This yields an action of $G$ on $\mathbb{P}(V)$ and a linearization $G \to \mathbb{P}(V) \to \mathbb{P}(V)$. We will denote this linearization again by $\rho$.

Let $E$ be a vector bundle of rank $r$. Then, the associated $GL(r)$-principal bundle is given as $\mathbb{P}(E) = \coprod_{x \in X} \text{Hom}(\mathcal{O}_x, E_x)$. If we are furthermore given an action $\Gamma: GL(r) \times \mathbb{P}(E) \to \mathbb{P}(E)$ of $GL(r)$ on a quasi-projective manifold $F$, we set $\mathbb{P}(E)^{GL(r)} := \mathbb{P}(E) / GL(r)$. Here, $GL(r)$ acts on $\mathbb{P}(E)$ on $F$ by $(x, y) \mapsto (x, \rho(y)x)$. If $F$ is a vector space and the action $\Gamma$ comes from a representation $\rho: GL(r) \to GL(F)$, we write $E_\rho$ for the vector bundle $\mathbb{P}(E)^{GL(r)}$.

For any $x \in \mathbb{R}$, we set $[x] := \max \{0, x\}$.
1 Preliminaries

1.1 Representations of the general linear group

First, let $\rho : \text{GL}(\mathcal{V}) \to \text{GL}(V)$ be an irreducible representation on the finite dimensional $\mathbb{C}$-vector space $V$.

**Theorem 1.1.** There are integers $a_1, \ldots, a_r$ with $a_i \geq 0$ for $i = 1; \ldots; r$, such that $\rho$ is a direct summand of the natural representation of $\text{GL}(r)$ on $S^{a_1} \mathbb{C}^r \otimes S^{a_r} \mathbb{C}^r$.

**Proof.** See [12], Proposition 15.47.

For any vector space $W$, the representations of $\text{GL}(W)$ on $S^i(W)$ and $V^iW$ are direct summands of the representation of $\text{GL}(W)$ on $W^i$. Setting $a := a_1 + \ldots + a_r$ and $b := a_n$, we see that $\rho$ is a direct summand of the representation $\rho_{a,b,c}$ of $\text{GL}(r)$ on $S^{a_1} \mathbb{C}^r \otimes S^{a_r} \mathbb{C}^r$.

**Corollary 1.2.** Let $\rho : \text{GL}(\mathcal{V}) \to \text{GL}(V)$ be a (not necessarily irreducible) representation of $\text{GL}(\mathcal{V})$ on the finite dimensional $\mathbb{C}$-vector space $V$, such that the centre $\mathbb{C}$ of $\text{GL}(\mathcal{V})$ acts by $z^\alpha$ for some $\alpha \in \mathbb{Z}$. Then, there exist $a, b, c \in \mathbb{Z}$, $c > 0$, such that $\rho$ is a direct summand of the natural representation $\rho_{a,b,c}$ of $\text{GL}(\mathcal{V})$ on $V^{a_1} \mathbb{C}^r \otimes V^{a_r} \mathbb{C}^r$.

**Proof.** We can decompose $\rho = \rho_1 \otimes \ldots \otimes \rho_c$ where the $\rho_i$'s are irreducible representations. By what we have said before, there are integers $a_i, b_i$, $i = 1; \ldots; c$, such that $\rho$ is a direct summand of $\rho_{a_i,b_i}$. Our assumption on the action of $\mathbb{C}$ implies that $a_1 + rb_1 = a_c + rb_c$. Let $b$ be a positive integer which is so large that $b_i + b > 0$ for $i = 1; \ldots; c$. Then, $\rho_{a,b,c}$ is the natural representation of $\text{GL}(\mathcal{V})$ on $V^{a_1} \mathbb{C}^r \otimes V^{a_r} \mathbb{C}^r$.

Now, the $\text{GL}(\mathcal{V})$-module $V^{a_1} \mathbb{C}^r \otimes V^{a_r} \mathbb{C}^r$ is a direct summand of $V^{a_1} \mathbb{C}^r \otimes V^{a_r} \mathbb{C}^r$ and we are done.

1.2 Basic concepts from GIT

We briefly summarize the main steps in Geometric Invariant Theory to fix the notation. References are [29] and [33].
1.2.1 The GIT-process

Let $G$ be a reductive algebraic group and $G \times F$ an action of $G$ on the projective scheme $F$. Let $L$ be an ample line bundle on $F$. A linearization of the given action in $L$ is a lifting of that action to an action $\rho: G \times L \rightarrow L$, such that for every $g \in G$ and $x \in F$ the induced map $L_x \rightarrow L_g \times L$ is a linear isomorphism. Taking tensor powers, $\rho$ provides us with linearizations of the action in any power $L_k$, $k > 0$, and actions of $G$ on $H^0(F;L^k)$ for any $k > 0$. A point $x_0 \in F$ is called semistable, if there exist an integer $k > 0$ and a $G$-invariant section $\sigma \in H^0(F;L^k)$ not vanishing in $x_0$. If, moreover, the action of $G$ on the set $\{x \in F \mid \sigma(x) \not= 0\}$ is closed and $\dim G \cdot \sigma = \dim G$, $x_0$ is called stable. The sets $F^{ss}$ of (semi)stable points are open $G$-invariant subsets of $F$. Finally, a point $x \in F$ is called polystable, if it is semistable and its $G$-orbit is closed in $F^{ss}$. Using this definition, the stable points are precisely the polystable points with finite stabilizer. The core of Mumford’s Geometric Invariant Theory is that the categorical quotients $F^{ss}=G$ and $F^{s}=G$ do exist and that $F^{ss}=G$ is a projective scheme whose closed points are in one to one correspondence to the orbits of polystable points, so that $F^{s}=G$ is in particular an orbit space.

A finite dimensional representation $\rho: G \rightarrow \text{GL}(V)$ provides an action of $G$ on $\mathbb{P}(V)$ and a linearization of this action in $\mathbb{P}(V)(1)$, called again $\rho$. A point $[v] \in \mathbb{P}(V)$ represented by $v \in V$ is then semistable if and only if the closure of the orbit of $v$ in $V$ does not contain 0. A point $x \in V$ is closed and the dimension of this orbit equals the dimension of $G$, and polystable if the orbit of $v$ in $V$ is closed.

1.2.2 Around the Hilbert-Mumford criterion

Let $F$ be a projective variety on which the reductive group $G$ acts. Suppose this action is linearized in the line bundle $L$. Call the linearization $\rho$. Then, given a one parameter subgroup $\lambda$ of $G$ and $x \in F$, we can form

$$x_\infty := \lim_{\varepsilon \rightarrow 1^- \infty} \lambda(\varepsilon) \cdot x.$$ 

The point $x_\infty$ is clearly a fix point for the $C$-action on $F$ induced by $\lambda$. Thus, $C$ acts on the fibre of $L$ over $x_\infty$, say, with weight $\gamma$. One defines

$$\mu_\rho(\lambda; x) := \gamma;$$ 

Theorem 1.3 (Hilbert-Mumford criterion [29]). A point $x \in F$ is (semi)stable, if and only if for every non-trivial one parameter subgroup $\lambda: C \rightarrow G$

$$\mu_\rho(\lambda; x) = 0;$$ 

Moreover, a point $x \in F$ is polystable if and only if it is semistable and, for every one parameter subgroup $\lambda$ of $G$ with $\mu_\rho(\lambda; x) = 0$, there is a $g \in G$ with $x_\infty = g \cdot x$.

As we have explained in the introduction, our concept of stability for decorated vector bundles is basically a Hilbert-Mumford criterion. To define the necessary numerical invariants, we need the following preparatory

Lemma 1.4. Let $S$ be a scheme and $\sigma: S \rightarrow F$ a morphism. Suppose the $G$-action on $F$ is linearized in the ample line bundle $L$. Then

$$\mu_\rho(\lambda; \sigma) := \max \mu_\rho(\lambda; \sigma(\xi)) \in \mathbb{P}(S) \text{ exists}.$$
Proof. We may assume that $L$ is a very ample line bundle. Set $V := H^0 (F; L)$. The linearization $\rho$ provides us with a representation $\rho: G \rightarrow \text{GL}(V)$ and a $G$-equivariant embedding $t: F \rightarrow \mathbb{P}(V)$. Since obviously $\mu_\rho(\lambda; x) = \mu_{\text{id}, \text{GL}, \rho}(\lambda; t(x))$ for all points $x \in F$ and all one parameter subgroups $\lambda$ of $G$, we can assume $F = \mathbb{P}(V)$. Now, there are a basis $v_1, \ldots, v_n$ of $V$ and integers $\gamma_1, \ldots, \gamma_n$ with

$$\lambda(x) \sum_{i=1}^n c_i v_i = \sum_{i=1}^n z_i^\gamma c_i v_i.$$ 

A point $[\ell] \in \mathbb{P}(V)$ can be thought of as the equivalence class of a linear form $l: V \rightarrow \mathbb{C}$.

Then,

$$\mu_\rho(\lambda; [\ell]) = \min \gamma \mid \ell(v_i) \neq 0 :$$

Therefore, $\mu_\rho(\lambda; \sigma(s)) \geq 0$ for all $s \geq 2$. In other words, if $S$ is irreducible, $\mu_\rho(\lambda; \sigma(s))$ is just the generic weight occurring for a point $\sigma(s), s \geq 2$.

1.2.3 Semistability for actions coming from direct sums of representations

Let $G$ be a reductive algebraic group and $V_1, \ldots, V_s$ finite dimensional vector spaces. Suppose we are given representations $\rho_i: G \rightarrow \text{GL}(V_i)$, $i = 1, \ldots, s$. The direct sum $\rho := \oplus_{i=1}^s \rho_i$ provides us with a linear action of $G$ on $\mathbb{P}(V) := \mathbb{P}(V_1) \oplus \cdots \oplus \mathbb{P}(V_s)$. Furthermore, for any $t < s$, $t_1; \ldots; t_s \in \mathbb{P}(V)$ and $t_1 < \cdots < t_s$, the $\rho_i$'s yield an action $\sigma_k$ of $G$ on $\mathbb{P} := \mathbb{P}(V_1) \oplus \cdots \oplus \mathbb{P}(V_s)$ and, for any sequence of positive integers $k_1; \ldots; k_s$, a linearization of $\sigma_k$ in the very ample line bundle $O(\ell_1; \ldots; k_s)$. The computation of the semistable points in $\mathbb{P}(V)$ can be reduced to the computation of the semistable points in the $\mathbb{P}(V_i)$'s by means of the following

**Theorem 1.6.** Let $w^0 := (w_1; w_2; \ldots; w_k)$ be a point in the space $\mathbb{P}(V_1, V_2) \oplus \cdots \oplus \mathbb{P}(V_s, V_t)$. Then $w^0$ is semistable (polystable) w.r.t. the given linearization in the line bundle $O(\ell_1; \ldots; k_s)$, if and only if either $(w_1; w_2; \ldots; w_k)$ is semistable (polystable) in $\mathbb{P}(V_1, V_2) \oplus \cdots \oplus \mathbb{P}(V_s, V_t)$ w.r.t. the linearization in $O(\ell_1; \ldots; k_s)$ for either $i = 1$ (and $w_2 = 0$) or $i = 2$ (and $w_1 = 0$), or there are positive natural numbers $n$, $k_1$, and $k_2$, such that $k_1 + k_2 = nk$ and the point $(w_1; w_2; \ldots; w_k)$ is semistable (polystable) in $\mathbb{P}(V_1, V_2) \oplus \cdots \oplus \mathbb{P}(V_s, V_t)$ w.r.t. the linearization in $O(\ell_1; \ldots; k_s)$ for either $i = 1$ (and $w_2 = 0$) or $i = 2$ (and $w_1 = 0$), or there are positive natural numbers $n$, $k_1$, and $k_2$, such that $k_1 + k_2 = nk$ and the point $(w_1; w_2; \ldots; w_k)$ is semistable (polystable) in $\mathbb{P}(V_1, V_2) \oplus \cdots \oplus \mathbb{P}(V_s, V_t)$ w.r.t. the linearization in $O(\ell_1; \ldots; k_s)$.

**Remark 1.7.** As one easily checks, for stable points only the “if”-direction remains true.

**Proof.** This theorem can be proved with the methods developed in [35] for $s = 2$. A more elementary approach is contained in the note [38].

□
1.3 One parameter subgroups of $SL(\mathfrak{g})$

Let $GL(\mathfrak{g}) = F \acts V$ be an action of the general linear group on the projective manifold $F$. For our definition of semistability, only the induced action of $SL(\mathfrak{g}) = F \acts V$ will matter. Since the Hilbert-Mumford criterion will play a central rôle throughout our considerations, we will have to describe the one parameter subgroups of $SL(\mathfrak{g})$.

Given a one parameter subgroup $\lambda : \mathbb{C} \rightarrow SL(\mathfrak{g})$, we can find a basis $w = (\omega_1; \ldots; \omega_r)$ of $\mathbb{C}^r$ and a weight vector $\gamma = (\gamma_1; \ldots; \gamma_r)$ with integral entries, such that

$$\gamma \text{ and } \sum_{i=1}^r \gamma_i = 0,$$

$$\lambda(\xi) \sum_{i=1}^r c_i\omega_i = \sum_{i=1}^r z^i c_i\omega_i.$$

Conversely, a basis $w$ of $\mathbb{C}^r$ and a weight vector $\gamma$ with the above properties define a one parameter subgroup $\lambda(w; \gamma)$ of $SL(\mathfrak{g})$.

To conclude, we remark that, for any vector $\gamma = (\gamma_1; \ldots; \gamma_r)$ of integers with $\gamma_1 \cdots \gamma_r$ and $\sum \gamma_i = 0$, there is a decomposition

$$\gamma = \sum_{i=1}^r \frac{1}{r} \gamma_i \gamma_i^\circ,$$

with

$$\gamma_i^\circ := \frac{i}{r} \sum_{j=1}^r \gamma_j \left( \frac{\gamma_j}{\gamma_i} \right) \quad i = 1; \ldots; r.$$

1.4 Estimates for the weights of some special representations

In the following, $\rho_{a; \mathfrak{g}_{a; \mathfrak{g}}}$ will stand for the induced representation of $GL(\mathfrak{g})$ on the vector space $V_{a; \mathfrak{g}_{a; \mathfrak{g}}} := \mathbb{C}^r$, where $a; b; c \in \mathbb{Z}$. Then, $\mathbb{P}(V_{a; \mathfrak{g}_{a; \mathfrak{g}}}) = \mathbb{P}(V_{a; \mathfrak{g}_{a; \mathfrak{g}}})$ and $V_{a; \mathfrak{g}_{a; \mathfrak{g}}} = V_{a; \mathfrak{g}_{a; \mathfrak{g}}}$ as $SL(\mathfrak{g})$-modules.

Let $w = (\omega_1; \ldots; \omega_r)$ be a basis for $\mathbb{C}^r$ and $\gamma = \sum_{i=1}^r \alpha_i \gamma^\circ_i$, $\alpha_i \in \mathbb{Q}$, an integral weight vector. Let $L'$ be the set of all $\alpha$-tuples $l = (l_1; \ldots; l_n)$ with $j \geq 1$; $\ldots; r$, $j = 1; \ldots; a. For \frac{1}{2} L'$ and $k \geq 1; \ldots; c \in \mathbb{Z}$, we define $w_{\mathfrak{g}_{a; \mathfrak{g}}} := w_{l_1} \cdots w_{l_n}$ and $w^k := \{0; \ldots; 0; w_{l_1} \cdots w_{l_n} \cdots 0\}$, $w_{l_1} \cdots w_{l_n}$ occupying the $k$-th entry. The elements $w_{l_1} \cdots w_{l_n}$ and $k \geq 1; \ldots; c \in \mathbb{Z}$ form a basis for $V_{a; \mathfrak{g}_{a; \mathfrak{g}}}$. We let $w_{\mathfrak{g}_{a; \mathfrak{g}}} := \mathbb{C}$ and $k \geq 1; \ldots; c \in \mathbb{Z}$ be the dual basis of $V_{a; \mathfrak{g}_{a; \mathfrak{g}}}$. Now, let $l \in \mathbb{P}(V_{a; \mathfrak{g}_{a; \mathfrak{g}}})$ where $l = \sum_{i} w_{l_i}$ Then, there exist $k_0$ and $l_0$ with $a_{l_0}^k \neq 0$ and

$$\mu_{\rho_{a; \mathfrak{g}_{a; \mathfrak{g}}}} \lambda \left( w_l; \gamma \right) \left( \frac{\gamma}{\gamma_i} \right) = \mu_{\rho_{a; \mathfrak{g}_{a; \mathfrak{g}}}} \lambda \left( w_l; \gamma \right) \left( \frac{\gamma_i}{\gamma_i} \right) = \mu_{\rho_{a; \mathfrak{g}_{a; \mathfrak{g}}}} \lambda \left( w_l; \gamma \right) \left( \frac{\gamma_i}{\gamma_i} \right)$$

and for any other $k$ and $l$ with $a_{l_0}^k \neq 0$

$$\mu_{\rho_{a; \mathfrak{g}_{a; \mathfrak{g}}}} \lambda \left( w_l; \gamma \right) \left( \frac{\gamma}{\gamma_i} \right) = \mu_{\rho_{a; \mathfrak{g}_{a; \mathfrak{g}}}} \lambda \left( w_l; \gamma \right) \left( \frac{\gamma_i}{\gamma_i} \right)$$

We also find that for $i \geq 1; \ldots; a$

$$\mu_{\rho_{a; \mathfrak{g}_{a; \mathfrak{g}}}} \lambda \left( w_l; \gamma \right) \left( \frac{\gamma_i}{\gamma_i} \right) = \nu_i a_i; \nu = \# j \quad j = 1; \ldots; a$$

One concludes
Lemma 1.8. i) For every basis \( w = (w_1; : : : ; w_r) \) of \( \mathbb{C}^r \), every integral weight vector \( \gamma = \sum_{i=1}^r \alpha_i \gamma^i \), \( \alpha_i \in \mathbb{Q} \setminus 0 \), and every point \( [l] \in \mathbb{P} \left( \mathcal{M}_{a,b} \right) \)

\[
\sum_{i=1}^r \alpha_i \ a \ (\rho \ 1) = \mu_{\rho, a} \lambda \ (w; \gamma); [l] \quad \sum_{i=1}^r \alpha_i \ a \ (\rho \ 1);
\]

ii) For every basis \( w = (w_1; : : : ; w_r) \) of \( \mathbb{C}^r \), every two integral weight vectors \( \gamma_i = \sum_{i=1}^r \alpha_i \gamma^i \), \( \alpha_i \in \mathbb{Q} \setminus 0 \), \( \gamma_2 = \sum_{i=1}^r \beta_i \gamma^i \), \( \beta_i \in \mathbb{Q} \setminus 0 \), and every point \( [l] \in \mathbb{P} \left( \mathcal{M}_{a,b} \right) \)

\[
\mu_{\rho, a} \lambda \ (w; \gamma_i + \gamma_j); [l] \quad \mu_{\rho, a} \lambda \ (w; \gamma_i); [l] \quad \sum_{i=1}^r \beta_i \ a \ (\rho \ 1);
\]

2 Decorated Vector Bundles

2.1 The moduli functors

In this section, we will introduce the vector bundle problems we would like to treat. The main topic will be the definition of the semistability concept. Having done this, we describe the relevant moduli functors to be studied throughout the rest of this chapter.

2.1.1 Semistable objects

The input data for our construction are:

- a positive integer \( r \),
- an action of the general linear group \( GL(\rho) \) on the projective manifold \( F \), such that the centre \( C \) of \( GL(\rho) \) acts trivially.

The objects we want to classify are pairs \( \langle E; \sigma \rangle \) where

\( E \) is a vector bundle of rank \( r \), and

\( \sigma : X \rightarrow F \) is a section.

Here, \( \mathbb{P} \langle E \rangle \) is the principal \( GL(\rho) \)-bundle associated with \( E \). Uninspired as we are, we call \( \langle E; \sigma \rangle \) an \( F \)-pair. Two \( F \)-pairs \( \langle E^1; \sigma^1 \rangle \) and \( \langle E^2; \sigma^2 \rangle \) are called equivalent, if there exists an isomorphism \( \psi : E^1 \rightarrow E^2 \) such that \( \sigma^1 \psi = \sigma^2 \psi \) being the induced isomorphism.

It will be our task to formulate a suitable semistability concept for these objects and to perform a construction of the moduli spaces. Let \( E \) be a vector bundle over \( X \). A weighted filtration of \( E \) is a pair \( \langle E; \alpha \rangle \) consisting of a filtration \( E : 0 \rightarrow E_1 \rightarrow \cdots \rightarrow E_{s+1} E \) of \( E \) by non-trivial proper subbundles and a vector \( \alpha = (\alpha_1; : : : ; \alpha_s) \) of positive rational numbers. Given such a weighted filtration, we set

\[
M E ; \alpha := \sum_{j=1}^s \alpha_j \deg (E) \text{rk} E_j \deg (E_j) \text{rk} E \quad :
\]

Suppose we are also given a linearization \( \rho \) of the \( GL(\rho) \)-action on \( F \) in an ample line bundle \( L \). Let \( \langle E; \sigma \rangle \) be as above and \( \langle E; \alpha \rangle \) be a weighted filtration of \( E \). We define \( \mu_{\rho} \langle E; \alpha; \sigma \rangle \) as follows:
Let $\underline{w} = (w_1; \ldots; w_r)$ be an arbitrary basis of $W := \mathbb{C}^r$. For every $i \geq 1; \ldots; r$, we set $W_{\underline{w}}^i := hw_1; \ldots; w_i$. Define $i_j := \text{rk} E_j$, $j = 1; \ldots; s$. This provides a flag

$$W : 0 \to W_{\underline{w}}^{i_1} \to \ldots \to W_{\underline{w}}^{i_s} \to W$$

and thus a parabolic subgroup $P := \text{SL}(\mathfrak{g})$, namely the stabilizer of the flag $W$. Finally, set $\gamma := \sum_{j=1}^s \alpha_j \gamma^{i_j}$. Next, let $U$ be an open subset of $X$ over which there is an isomorphism $\psi: E_U \to W \to O_U$ with $\psi(E_U) = W \to O_U$. Then, $\psi$ gives us an isomorphism $\mathcal{F}(E_U) ! U$ $F$ and $\sigma$ a morphism $\sigma : U \to F$ $! U$ $U$ $F$ $! F$. If $\gamma$ is a vector of integers, we set $\mu_\rho(E; \sigma: \gamma) := \mu_\rho(\underline{w}; \gamma; \sigma)$ as in Lemma[3]. Otherwise, we choose $k > 0$ such that $k \gamma$ is a vector of integers and set $\mu(E; \sigma; \gamma) := (1+\gamma k)\mu_\rho(\underline{w}; \gamma k; \sigma)$. Since for an integral weight vector $\gamma$ and a positive integer $k$, one has $\mu_\rho(\underline{w}; \gamma k; \sigma) = k^\mu_\rho(\underline{w}; \gamma; \sigma)$, this is well-defined. Note that the weight vector $\gamma$ is canonically defined by $(E; \sigma; \gamma)$, but that we have to verify that definition does not depend on the basis $\underline{w}$ and the trivialization $\psi$. First, let $\underline{w}^0 = (w_1^0; \ldots; w_r^0)$ be a different basis. Let $g_2 \text{GL}(\mathfrak{g})$ be the element which maps $w_i$ to $w_i^0$, $i = 1; \ldots; r$, and set $\psi_0 := (g \cdot \text{id}_{O_U}) \psi$. This defines the morphism $\sigma^0 : U \to F$. Then, $\lambda(\underline{w}; \gamma) = g \lambda(\underline{w}; \gamma)$, for $k$ and $\sigma_0(\chi) = g \sigma(\chi)$ for every $x \in U$. Since $\mu_\rho(\underleftarrow{\lambda(\underline{w}; \gamma)}; \sigma_0(x)) = \mu_\rho(\underline{g \lambda(\underline{w}; \gamma)}; \sigma_0(x)) = \mu_\rho(\underline{w}; \gamma \sigma_0(x)) = \mu_\rho(\underline{w}; \gamma \sigma_0(x)) = \mu_\rho(\underline{g \lambda(\underline{w}; \gamma)}; \sigma_0(x))$, we may fix the basis $\underline{w}$. Any other trivialization $\psi$ defined w.r.t. $\underline{w}$ differs from $\psi$ by a map $U \to P$. Now, for every $g_2 \text{P}$ and every point $x \in U$, $\mu_\rho(\lambda; \sigma(x)) = \mu_\rho(\underline{g \lambda}; \sigma(x)) = \mu_\rho(\underline{g \lambda}; \sigma(x))$. The last equality results from [29], Prop. 2.7, p. 57. This shows our assertion. To conclude, Remark[5] shows that the definition is also independent of the choice of the open subset $U$.

Fix also a number $\delta \geq 0$. With these conventions, we call $(E; \sigma; \gamma)$ $\delta$-stable, if for every weighted filtration $(E; \sigma; \gamma)$ of $E$

$$M \leq \sigma ; \gamma + \delta \mu(E; \sigma) \quad (\chi) = 0 :$$

Next, we remark that we should naturally fix the degree of $E$. Then, the topological fibre space $\pi : \mathbb{F}_d \times X$ underlying $\mathbb{F}(E)$ will be independent of $E$, so that it makes sense to fix the homology class $[\sigma(X)] \in H_2(\mathbb{F}_d; \mathbb{Z})$. Given $d \geq 2 \mathbb{Z}$, $r \geq 2 \mathbb{Z}$, and $h \geq 2 \mathbb{H}_2(\mathbb{F}_d; \mathbb{Z})$, we say that $(E; \sigma)$ is of type $(\delta, h, \rho)$, if $E$ is a vector bundle of degree $d$ and rank $r$, and $\sigma(X) = h$. Before we define the moduli functor, we enlarge our scope.

For a given linearization of the $GL(\mathfrak{g})$-action on $\mathbb{F}$ in the line bundle $L$, we can choose a positive integer $k$ such that $L^k$ is very ample. Therefore, we obtain a $GL(\mathfrak{g})$-equivariant embedding $\mathbb{F} ! \mathbb{P}(\mathbb{V})$, $\mathbb{V} := H^0(\mathbb{F}; L^k)$. Note that $\mathbb{C}$ acts trivially on $\mathbb{P}(\mathbb{V})$. Therefore, we formulate the following classification problem: The input now consists of

- a positive integer $r$, a finite dimensional vector space $V$, and
- a representation $\rho : GL(\mathfrak{g})$ $! GL(\mathbb{V})$ whose restriction to the centre $\mathbb{C}$ is of the form $z \gamma \sigma$ $\lambda^\alpha \cdot \text{id}$ for some integer $\alpha$,
- and the objects we want to classify are pairs $(E; \sigma)$ where $E$ is a vector bundle of rank $r$, and
- $\sigma : X$ $! \mathbb{P}(\mathbb{V})$ is a section. Here, $E_\rho$ is the vector bundle of rank $\text{dim} V$ associated to $E$ via the representation $\rho$. 

Rem.
The equivalence relation is the same as before. Now, giving a section \( \sigma:X \rightarrow \mathbb{P}(E_\rho) \) is the same as giving a line bundle \( M \) on \( X \) and a surjection \( \tau:E_\rho \rightarrow M \). Remember that \((M;\tau)\) and \((M';\tau')\) give the same section if and only if there exists an isomorphism \( M \cong M' \) which carries \( \tau \) into \( \tau' \). Moreover, fixing the homology class \( [\sigma(X)] \) amounts to the same as fixing the degree of \( M \). Since the condition that \( \tau \) be surjective will be an open condition in a suitable parameter space, we formulate the following classification problem: The input data are

- a tuple \((d;r;m)\) called the type, where \( d, r, \) and \( m \) are integers, \( r > 0 \),
- a representation \( \rho:GL(r) \rightarrow GL(V) \),

and the objects to classify are triples \((E;M;\tau)\) where

- \( E \) is a vector bundle of rank \( r \) and degree \( d \),
- \( M \) is a line bundle of degree \( m \), and
- \( \tau:E_\rho \rightarrow M \) is a non-zero homomorphism.

Then, \((E;M;\tau)\) is called a \( \rho \)-pair of type \((d;r;m)\), and \((E^1;M^1;\tau^1)\) and \((E^2;M^2;\tau^2)\) are said to be equivalent, if there exist isomorphisms \( \psi:E^1 \rightarrow E^2 \) and \( \chi:M^1 \rightarrow M^2 \) with \( \tau^1 = \chi \circ \tau \circ \psi \), where \( \psi: E^1_\rho \rightarrow E^2_\rho \) is the induced isomorphism. Let \((E;M;\tau)\) be a \( \rho \)-pair of type \((d;r;m)\). A weak automorphism of \((E;M;\tau)\) is the class \([\psi] \in \mathbb{P}(\text{End}(E))\) of an automorphism \( \psi:E \rightarrow E \) with \( \tau = \psi \cdot \tau \). We call \((E;M;\tau)\) simple, if there are only finitely many weak automorphisms.

Remark 2.1. i) A representation \( \rho:GL(r) \rightarrow GL(V) \) of the general linear group with \( \rho(\xi) = \z^\alpha \text{id} \) is called homogeneous of degree \( \alpha \). Every representation of \( GL(r) \) obviously splits into a direct sum of homogeneous representations. Some cases of inhomogeneous representations \( \rho \) can be treated within our framework. Indeed, if \( \rho \) is a representation, such that its homogeneous components \( \rho_1;\cdots;\rho_n \) have positive degrees \( \alpha_1;\cdots;\alpha_n \), let \( \kappa \) be a common multiple of the \( \alpha_i \). Then, we pass to the homogeneous representation

\[
\rho^0 := \bigoplus_{\alpha \in \kappa \alpha_{\nu} = \kappa} S^{\nu_1} \rho_{\nu_1} \quad \rho_{\nu_1} \colon
\]

The solution of the moduli problem associated with \( \rho^0 \) can be used to solve the moduli problem associated with \( \rho \). This trick was used in [35] and will be recalled in the section on examples.

ii) The identification of \( \tau \) and \( \lambda \cdot \tau \), or equivalently, considering sections in \( \mathbb{P}(\xi) \) rather than in \( E_\rho \) seems a little artificial. First of all, this identification is mandatory to get projective moduli spaces. Second, for homogeneous representations of degree \( \alpha \notin 0 \), this is naturally forced upon us. Third, if we are given a homogeneous representation \( \rho \) of degree zero and are interested in the moduli problem without the identification of \( \tau \) and \( \lambda \cdot \tau \), we may pass to the representation \( \rho^0 \) obtained from \( \rho \) by adding the trivial one dimensional representation. Then, one gets from the solution of the moduli problem associated with \( \rho^0 \) a compactification of the moduli problem associated with \( \rho \). This will be explained within the context of Hitchin pairs in the examples.

In order to define a functor, we first fix a Poincaré line bundle \( L \) on \( \text{Jac}^m X \). For every scheme \( S \) and every morphism \( \kappa:S \rightarrow \text{Jac}^m X \), we define \( L(\kappa) := (\kappa \cdot \text{id}_X) \cdot L \). Now, let \( S \) be a scheme of finite type over \( \mathbb{C} \). Then, a family of \( \rho \)-pairs of type \((d;r;m)\) parameterized by \( S \) is a tuple \((E_S;\kappa;S;\tau_S)\) with

\( E_S \) a vector bundle of rank \( r \) having degree \( d \) on \( \text{Ext} X \), for all \( s \in S \),
\( \kappa : S \xrightarrow{\sigma} \text{Jac}^m \) a morphism,

\( \mathbb{N}_S \) a line bundle on \( S \),

\( \tau : E_{S;\rho} \xrightarrow{\mathbb{L}} L [\kappa_S] \quad \pi_S N_S \) a homomorphism whose restriction to \( \mathbb{F}_S \mathcal{G} \quad X \) is non-zero for every closed point \( s \in S \).

Two families \((E^1_S; \kappa^1_S; N^1_S; \tau^1_S)\) and \((E^2_S; \kappa^2_S; N^2_S; \tau^2_S)\) are called equivalent, if \( \kappa^1_S = \kappa^2_S = : \kappa_S \) and there exist isomorphisms \( \psi_S : E^1_S \xrightarrow{1} E^2_S \) and \( \chi_S : N^1_S \xrightarrow{1} N^2_S \) with

\[
\tau^1_S = (d_{L; \kappa_S}) \cdot \pi_S X_S \cdot \frac{1}{8} \cdot \frac{2}{8} \cdot \psi_S : \quad \tau^2_S
\]

To define the semistability concept for \( \rho \)-pairs, observe that for given \( (E; M; \tau) \), the homomorphism \( \tau : E \xrightarrow{1} M \) will be generically surjective, therefore we get a rational section \( \sigma^0 \mathcal{X} \xrightarrow{1} \mathbb{P}(E_{\rho}) \) which can, of course, be prolonged to a section \( \sigma : X \xrightarrow{1} \mathbb{P}(E_{\rho}) \), so that we can define for every weighted filtration \( (E ; \mathcal{G}) \) of \( E \)

\[
\mu_{\rho} E ; \mathcal{G} ; \tau := \mu_{\rho} E ; \mathcal{G} ; \sigma
\]

We will occasionally use the following short hand notation: If \( E^0 \) is a non-zero, proper subbundle of \( E \), we set

\[
\mu_{\rho} (E^0; \tau) := \mu_{\rho} (0) E^0 E ; (1); \tau
\]

Now, for fixed \( \delta \geq \mathbb{Q}_>0 \), call a \( \rho \)-pair \( (E; M; \tau) \) \( \delta \)-(semi)stable, if for every weighted filtration \( (E ; \mathcal{G}) \)

\[
M E ; \mathcal{G} + \delta \mu E ; \mathcal{G} ; \tau (1) \text{ } 0
\]

**Remark 2.2.** For the \( F \)-pairs, one can formulate the semistability concept in a more intrinsic way. For this, one just has to choose a linearization \( \rho \) of the given action in an ample \( \mathcal{Q} \)-line bundle. Then, \( \mu_{\rho}(E ; \mathcal{G}; \Phi) \) can still be defined, and an \( F \)-pair \( (E; \Phi) \) will be called \( \rho \)-(semi)stable, if

\[
M E ; \mathcal{G} + \mu_{\rho} E ; \mathcal{G} ; \Phi (1) \text{ } 0
\]

In gauge theory, one would say that the notion of semistability depends only on the metric chosen on the fibre \( F \). If \( \rho \) is a linearization in an ample line bundle \( L \) and \( \delta \geq \mathbb{Q} \), we can pass to the induced linearization “\( \rho \) \( \delta \) ” in the \( \mathcal{Q} \)-line bundle \( \delta L \) to recover \( \delta \rho \)-semistability. For the moduli problems associated with a representation \( \rho \), the formulation with the parameter \( \delta \) seems more appropriate and practical and, since we treat \( F \)-pairs only as special cases of \( \rho \)-pairs, we have decided for the given definition of \( \delta \rho \)-semistability.

We define the functors

\[
\mathcal{M}(\rho)_{\delta \rho} : \text{Sch}_S \xrightarrow{\delta} \text{Set} \quad \rho \text{-pairs } \in \text{type } (d; r; m) \text{ parameterized by } S
\]

**Remark 2.3.** The definition of the moduli functor involves the choice of the Poincaré sheaf \( L \). Nevertheless, the above moduli functor is independent of that choice. Indeed, choosing another Poincaré line bundle \( L^0 \) on \( \text{Jac}^m \) \( X \), there is a line bundle \( N \text{Jac}^m \) on \( \text{Jac}^m \) with \( L = L^0 \cdot \pi_{\text{Jac}^m} \text{Jac}^m \). Therefore, assigning to a family \( (E_S; \kappa_S; N_S; \tau_S) \) defined via \( L \) the family \( (E_S; \kappa_S; N_S; \kappa_S \cdot N \text{Jac}^m \cdot \tau_S) \) defined via \( L^0 \) identifies the functor which is defined w.r.t. \( L \) with the one defined w.r.t. \( L^0 \).
We also define the open subfunctors \( \mathbb{M}(\varphi)^{s} \) of equivalence classes of families \( (E_s ; \mathcal{K}_s ; \mathcal{N}_s ; \tau_s) \) where \( \tau_s \) is surjective for all \( s \in S \).

Next, let \( (E ; M ; \tau) \) be a \( \rho \)-pair where \( \tau \) is surjective, and let \( \mathbb{D}^{d} \) be the oriented topological projective bundle underlying \( \mathbb{P}(\varphi) \). This is independent of \( \varphi \), and as explained before, the degree of \( M \) determines the cohomology class \( h_m := [\varphi \times \mathbb{K}^{1}] \subset H_2(\mathbb{D}^{d}; \mathbb{Z}) \) where \( \sigma \) is the section associated with \( \tau \). Set \( h := h_m \setminus \mathbb{D}^{d} \subset H_2(\mathbb{D}^{d}; \mathbb{Z}) \). We can now define \( \mathbb{M}(\varphi)^{s} \) as the closed subfunctor of \( \mathbb{M}(\varphi)^{s} \) of equivalence classes of families \( (E_s ; \mathcal{K}_s ; \mathcal{N}_s ; \tau_s) \) for which the section \( S \setminus X \) ! \( \mathbb{P}(\varphi) \) factorizes over \( \mathbb{P}(\varphi) \).

2.1.2 Polystable pairs

Fix a basis \( \omega = (\omega_1 ; \cdots ; \omega_r) \) of \( \mathbb{C}^r \). Let \( (E ; M ; \tau) \) be a \( \delta \)-polystable \( \rho \)-pair of type \( (d ; \rho ; m) \). We call \( (E ; M ; \tau) \) \( \delta \)-polystable, if for every weighted filtration \( (E ; \mathcal{G}) \), \( E : 0 = : E_0 : E_1 : \cdots : E_{s+1} := E \), with

\[
M \ E ; \mathcal{G} + \delta \ \mu \ E ; \mathcal{G} ; \tau = 0
\]

the following holds true

\[
E = \bigoplus_{j=0}^{s+1} E_j \quad \text{and} \quad \rho(\varphi) \mathcal{G} ; \tau = 0.
\]

Here, one uses the fact that giving an isomorphism \( E \) ! \( L \bigoplus_{j=0}^{s+1} E_j \) is the same as giving a cocycle for \( E \) in the group

\[
Z \lambda : \omega ; \gamma := \bigoplus_{j=0}^{s+1} \lambda_{j} \omega_j \gamma_j(\xi) = \lambda_{i} \omega_i \gamma_i(\xi) \quad \text{for } g \in \mathbb{C}.
\]

It follows that \( E \rho = E \tau^{\gamma} \) where \( \gamma_1 \ldots \gamma_r \) are the weights of \( \omega ; \gamma \) on \( V \) and \( E \gamma^{\omega} \) is the “eigenbundle” for the weight \( \gamma_i \). As before, \( W ; 0 \leftarrow W_{\omega}^{k \mathcal{G}} \leftarrow W_{\omega}^{k \mathcal{G}(E_j)} \) and \( \gamma := \sum_{j=1}^{s} \gamma_j \). The stated condition is again independent of the involved choices.

Remark 2.4. i) If \( (E ; M ; \tau) \) is \( \delta \)-stable, the stated condition is void, so that \( (E ; M ; \tau) \) is also \( \delta \)-polystable.

ii) It will follow from our GIT construction that \( (E ; M ; \tau) \) is \( \delta \)-stable, and if only if it is \( \delta \)-polystable and has only finitely many weak automorphisms.

iii) For the description of S-equivalence in the case of \( \rho = \rho_{a \neq b} \) for some \( a \neq b \), \( a \neq c \), \( c \neq 0 \), the reader may consult [16].

2.2 The main result

Theorem 2.5. i) There exist a projective scheme \( \mathbb{M}(\varphi)^{s} \) and an open subscheme \( \mathbb{M}(\varphi)^{s} \) \( \mathbb{M}(\varphi)^{s} \) together with natural transformations

\[
\varphi ; \mathbb{M}(\varphi)^{s} \quad \text{and} \quad h : \mathbb{M}(\varphi)^{s} \mathbb{M}(\varphi)^{s} \quad \text{with the following properties:}
\]

1. For every scheme \( N \) and every natural transformation \( \varphi : \mathbb{M}(\varphi)^{s} \mathbb{M}(\varphi)^{s} \), there exists a unique morphism \( \varphi : \mathbb{M}(\varphi)^{s} \mathbb{M}(\varphi)^{s} \) with \( \varphi^0 = h(\varphi) \).
2. \( M(\rho)^{d-r-m} \) is a coarse moduli space for the functor \( M(\rho)^{d-r-m} \).

3. \( \delta^{ss} (\text{Spec } C) \) induces a bijection between the set of equivalence classes of \( \delta \)-polystable \( \rho \)-pairs of type \( (d; r; m) \) and the set of closed points of \( M(\rho)^{d-r-m} \).

ii) There exist a locally closed subscheme \( M(\rho)^{d-r-h}_{d-r-h} \) of \( M(\rho)^{d-r-m} \) and a natural transformation

\[
\partial_F: M(\rho)^{d-r-h}_{d-r-h} \rightarrow h \cdot \rho^{d-r-h}_{d-r-h}
\]

which turns \( M(\rho)^{d-r-h}_{d-r-h} \) into the coarse moduli space for \( M(\rho)^{d-r-h}_{d-r-h} \).

2.3 The proof of the main result

Given any homogeneous representation \( \rho: \text{GL}(\tau) \rightarrow \text{GL}(V) \), we have seen in Section 2.2.1 that we can find integers \( a; b; c > 0 \), such that \( \rho \) is a direct summand of the representation \( \rho_{a,b,c} \). Write \( \rho_{a,b,c} = \rho \cdot T \). For every vector bundle \( E \) of rank \( r \), we find \( E_{a,b,c} = E \cdot E_{a,b,c} \). Every \( \rho \)-pair \( (E; M; \tau) \) can therefore also be viewed as a \( \rho_{a,b,c} \)-pair. Since \( \mu_{\rho}(E; \tau; \tau) = \mu_{\rho_{a,b,c}}(E; \tau; \tau) \) for every weighted filtration \( (E; \tau) \), the triple \( (E; M; \tau) \) is \( \delta \)-stable as \( \rho \)-pair, if and only if it is \( \delta \)-polystable as \( \rho_{a,b,c} \)-pair. More precisely, we can recover \( M(\rho)^{d-r-m}_{d-r-m} \) as closed subfunctor of \( M(\rho_{a,b,c})^{d-r-m}_{d-r-m} \). Indeed, for every scheme of finite type over \( C \),

\[
\left[ \begin{array}{c} \rho \\ \tau \end{array} \right]_{d-r-m} \left[ \begin{array}{c} (S) \\ \left[ \begin{array}{c} E_{\rho_{a,b,c}}^{r,0,1} \\ \pi_0 \end{array} \right]_{d-r-m} \end{array} \right] = \left[ \begin{array}{c} \rho \\ \tau \end{array} \right]_{d-r-m} \left[ \begin{array}{c} (S) \\ \left[ \begin{array}{c} E_{\rho_{a,b,c}}^{r,0,1} \\ \pi_0 \end{array} \right]_{d-r-m} \end{array} \right]
\]

Therefore, we will assume from now on that \( \rho = \rho_{a,b,c} \) for some \( a; b; c \).

2.3.1 Boundedness

**Theorem 2.6.** There is a non-negative constant \( C_1 \), depending only on \( r, a, \) and \( \delta \), such that for every \( \delta \)-semistable \( \rho_{a,b,c} \)-pair \( (E; M; \tau) \) of type \( (d; r; m) \) and every non-trivial proper subbundle \( E_0 \) of \( E \)

\[
\mu(E_0) < \frac{d}{r} + C_1
\]

**Proof.** Let \( 0 \leq E_0 \leq E \) be any subbundle. By Lemma 2.3.1i), \( \mu_{\rho_{a,b,c}}(E_0; \tau) \leq \frac{d}{r} + C_1 \), so that \( \delta \)-semistability gives

\[
\begin{align*}
d \text{rk} E_0 & \geq \deg E_0 + \delta \geq a \cdot \tau + 1 \\
d \text{rk} E_0 & \geq \deg E_0 + \delta \geq \mu_{\rho_{a,b,c}} E_0; \tau + 0;
\end{align*}
\]

i.e.,

\[
\mu(E_0) \geq \frac{d}{r} + \delta + \frac{a}{r} + 1)
\]

so that the theorem holds for \( C_1 := \delta + \frac{a}{r} + 1 = r \).
2.3.2 Construction of the parameter space

Recall that, for a scheme $S$ of finite type over $\mathbb{C}$, a family of $\rho_{\phi\psi}$-pairs parameterized by $S$ is a quadruple $(E_S; \mathcal{K}_S; \mathcal{N}_S; \mathcal{T}_S)$ where $E_S$ is a vector bundle of rank $r$ on $S$, $\mathcal{K}_S$ is a vector bundle of rank $r$ on $S$ such that for every $n \geq 0$ and every $\delta$-semistable $\rho_{\phi\psi}$-pair $(\mathcal{E}; M; \tau)$ of type $(d; r; n)$

\[ H^0(\mathcal{E}(n)) = 0 \text{ and } E(n) \text{ is globally generated,} \]

\[ H^0(\det(\mathcal{E}(n))) = 0 \text{ and } \det(\mathcal{E}(n)) \text{ is globally generated,} \]

\[ H^0(\det(\mathcal{E}))^b M \mathcal{O}_X(na) = 0 \text{ and } \det(\mathcal{E})^b M \mathcal{O}_X(na) \text{ is globally generated.} \]

Choose some $n \geq 0$ and set $p := d + rn + r(1 + g)$. Let $U$ be a complex vector space of dimension $p$. We define $O^0$ as the quasi-projective scheme parameterizing equivalence classes of quotients $q: U \to \mathcal{O}_X(n)$ such that $E$ is a vector bundle of rank $r$ and degree $d$ on $X$ and $H^0(q(\mathcal{E}))$ is an isomorphism. Then there exists a universal quotient

\[ q^0: U \to \mathcal{O}_X(n) ! E^0 \]

on $O^0$ $X$. Let

\[ q^0 \circ \text{Jac}^m: U \to \mathcal{O}_X(n) ! E^0 \circ \text{Jac}^m \]

be the pullback of $q^0$ to $O^0$ $\text{Jac}^m$ $X$. Set $U_{ax} := U^a c$. By our assumption, the sheaf

\[ \text{Hom}_U U_{ax} \mathcal{O}_X(n) ! E_{ax} \circ \text{Jac}^m \text{ det}(E^0) \circ \text{Jac}^m \text{ L} \mathcal{K} \text{ Jac}^m \text{ } \mathcal{O}_X(na) \]

is locally free, call it $H$, and set $H := \mathbb{P}(H \backslash -)$. We let

\[ q_H: U \to \mathcal{O}_X(n) ! E_H \]

be the pullback of $q^0 \circ \text{Jac}^m$ to $H$ $X$. Now, on $H$ $X$, there is the tautological homomorphism

\[ s_H: U_{ax} \to \mathcal{O}_H \text{ det}(E_H) \text{ L} \mathcal{K} \mathcal{O}_X(na) \]

Here, $\kappa_H ! H \to O^0 \text{ Jac}^m \to \text{ Jac}^m$ is the natural morphism. Let $T$ be the closed subscheme defined by the condition that $s_H \mathcal{O}_X(na)$ vanish on $X$.

\[ \ker U_{ax} \mathcal{O}_X(na) ! E_{ax} \circ \text{Jac}^m \]

Let

\[ q_T: U \to \mathcal{O}_X(n) ! E_T \]

be the restriction of $q_H$ to $T$ $X$. By definition, there is a universal homomorphism

\[ \tau_T: E_T \circ \text{Jac}^m \to \mathcal{O}_T \mathcal{K} \mathcal{O}_X(na) \]

Here, $N$ and $\kappa_T$ are the restrictions of $O_H$ to $T$. Note, that the parameter space $T$ is equipped with a universal family $(E_T; \mathcal{K}_T; \mathcal{N}_T; \mathcal{T}_T)$.

Remark 2.7. Let $S$ be a scheme of finite type over $\mathbb{C}$. Call a tuple $(q_S; \mathcal{K}_S; \mathcal{N}_S; \mathcal{T}_S)$ where
\( q: U \to \pi_X \circ X (n) \) \( E_s \) is a family of quotients, such that its restriction to \( \mathcal{E}_s \) \( X \) lies in \( Q^0 \) for every \( s \in S \).

\( \mathfrak{g}: S \to \text{Jac}^m \) is a morphism, \( N_s \) is a line bundle on \( S \), and

\( f: E_s \to S, \quad \det (E_s) = L \quad [k_S] \quad \pi_S N_s \) is a homomorphism which is non trivial on all fibres \( \mathcal{E}_s \) \( X, s \in S \),

a quotient family of \( \rho_{a,b,x} \)-pairs of type \( (d; rm) \) parameterized by \( S \). We say that the families \( (q_1^1, \kappa_1^1; N_1^1; \tau_1^1) \) and \( (q_2^2, \kappa_2^2; N_2^2; \tau_2^2) \) are equivalent, if \( \kappa_1^1 = \kappa_2^2 = \kappa_S \) and there are isomorphisms \( \psi_S: E_1^1 \to E_2^2 \) and \( \chi_S: N_1^1 \to N_2^2 \) with \( q_2^2 = \psi_S q_1^1 \) and

\[
\tau_1^1 = \text{id}_{\mathcal{L}_1} \quad \pi_S (\chi_S) \quad \frac{q_2^2}{\psi_S} \quad \psi_S a^c.
\]

It can be easily inferred from the construction of \( T \) and the base change theorem that \( T \) represents the functor which assigns to a scheme \( S \) of finite type over \( \mathbb{C} \) the set of equivalence classes of quotient families of \( \rho_{a,b,x} \)-pairs of type \( (d; rm) \) parameterized by \( S \).

**Proposition 2.8 (Local universal property).** Let \( S \) be a scheme of finite type over \( \mathbb{C} \), and \( \mathcal{E}_S; \kappa_S; N_S; \tau_S \) a family of \( \delta \)-semistable \( \rho_{a,b,x} \)-pairs parameterized by \( S \). Then, there exist an open covering \( S_i, i \geq 1 \) of \( S \), and morphisms \( \beta_i: S_i \to S \), \( i \geq 1 \), such that the restriction of the family \( \mathcal{E}_S; \kappa_S; N_S; \tau_S \) to \( S_i \) \( X \) is equivalent to the pullback of \( \mathcal{E}_T, \kappa_T, N_T; \tau_T \) via \( \beta_i \text{id}_X \) for all \( i \geq 1 \).

**Proof.** By our assumptions, the sheaf \( \pi_S (\mathcal{E}_S, \pi_X \circ X (n)) \) is locally free of rank \( p \). Therefore, we can choose a covering \( S_i, i \geq 2 \) of \( S \), such that it is free over \( S_i \) for all \( i \geq 2 \). For each \( i \), we can choose a trivialization

\[
U \to S_i \quad \pi_S E_S \pi_X \circ X (n)_{S_i} ;
\]

so that we obtain a surjection

\( q_{S_i}: U \to \pi_X \circ X (n) \) \( E_{S_i} \) \( X \)

on \( S_i \) \( X \). Therefore, \( q_{S_i; \kappa S_i; N S_i; \tau S_i; \pi X \circ X (n)} \) is a quotient family of \( \rho_{a,b,x} \)-pairs of type \( (d; rm) \) parameterized by \( S_i \), and we can conclude by Remark 2.7.

\[ \square \]

### 2.3.3 The group action

Let \( m: U \to \mathcal{SL} \mathcal{U} \) \( \mathcal{SL} \mathcal{U} \) be the universal automorphism over \( \mathcal{SL} \mathcal{U} \). Let \( \mathcal{E}_S; \kappa_S; N_S; \tau_S \) be the pullback of the universal family on \( T \) \( X \) to \( \mathcal{SL} \mathcal{U} \) \( T \) \( X \). Define

\[
q_{\mathcal{SL} \mathcal{U}}(T): U \to \pi_X \circ X (n) \quad \pi_{\mathcal{SL} \mathcal{U}} \quad \text{id}_{\pi \circ X (n)} \quad U \to \pi_X \circ X (n) \to \mathcal{E}_{\mathcal{SL} \mathcal{U}}(T) ;
\]

Thus, \( q_{\mathcal{SL} \mathcal{U}}(T): \kappa_{\mathcal{SL} \mathcal{U}} \kappa_{\mathcal{SL} \mathcal{U}} N_{\mathcal{SL} \mathcal{U}} \tau_{\mathcal{SL} \mathcal{U}} \tau_{\mathcal{SL} \mathcal{U}} \) is a quotient family of \( \rho_{a,b,x} \)-pairs parameterized by \( \mathcal{SL} \mathcal{U} \) \( T \), and hence, by 2.7 defines a morphism

\[
\Gamma: \mathcal{SL} \mathcal{U} \quad T \to T ;
\]

It is not hard to see that \( \Gamma \) is indeed a group action. Note that this action descends to a \( \text{PGL} \mathcal{U} \)-action!
Remark 2.9. By construction, the universal family \((E_T; \kappa_T; N_T; \tau_T)\) comes with a linearization, i.e., with an isomorphism

\[
\Gamma \overset{\text{id}_X}{\longrightarrow} E_T; \kappa_T; N_T; \tau_T \overset{\pi_T}{\longrightarrow} E_T; \kappa_T; N_T; \tau_T.
\]

Therefore, elements of the PGL\((U)\)-stabilizer of a point \(t \in T\) correspond to weak automorphisms of the \(\rho_{a,b,c}\)-pair \((E_t; M_t; \tau_t) := (E_T; \kappa_T; N_T; \tau_T) \big|_{x \in X}\). In particular, the SL\((U)\)-stabilizer of \(t\) is finite if and only if \((E_t; M_t; \tau_t)\) has only finitely many weak automorphisms.

Proposition 2.10. Let \(S\) be a scheme of finite type over \(\mathbb{C}\) and \(\beta_1, \beta_2 : S \rightarrow T\) two morphisms, such that the pullbacks of \((E_T; \kappa_T; N_T; \tau_T)\) via \(\beta_1, \beta_2 : \text{id}_S\) are equivalent. Then, there exist an étale covering \(\eta : T \rightarrow S\) and a morphism \(\Xi : T \rightarrow \text{SL}(U)\), such that the morphism \(\beta_2 \circ \eta : T \rightarrow T\) equals the morphism

\[
T \overset{\Xi}{\longrightarrow} \eta \circ \text{SL}(U) \overset{\Gamma}{\longrightarrow} T.
\]

Proof. The two morphisms \(\beta_1, \beta_2\) provide us with quotient families \((q_S^1; \kappa_S^1; \eta_S^1; \tau_S^1)\) and \((q_S^2; \kappa_S^2; \eta_S^2; \tau_S^2)\) of \(\rho_{a,b,c}\)-pairs parameterized by \(S\). By hypothesis, \(\kappa_S^1 = \kappa_S^2 =: \kappa_S\), and we have isomorphisms \(\psi_S : E_S^1 \overset{\sim}{\longrightarrow} E_S^2\) and \(\chi_S^1 : N_S^1 \overset{\sim}{\longrightarrow} N_S^2\) with

\[
\tau_S^1 = (\text{id}_X \circ \kappa_S) \overset{\pi_S \chi_S}{\longrightarrow} \pi_S N_S^1 \overset{\psi_S}{\longrightarrow} \pi_S N_S^2 \overset{\text{id}_X \circ \kappa_S}{\longrightarrow} \tau_S^2.
\]

In particular, there is an isomorphism

\[
U \circ_S \overset{\pi_S}{\longrightarrow} E_S \overset{\pi_X \circ \chi_S}{\longrightarrow} E_S \overset{\psi_S}{\longrightarrow} E_S \overset{\text{id}_X \circ \chi_S}{\longrightarrow} U \circ_S.
\]

This yields a morphism \(\Xi_S : S \rightarrow \text{GL}(U)\) and \(\Delta_S := \text{det} \Xi_S \circ \Delta_S \rightarrow S \rightarrow \mathbb{C}\). Let \(T := S \times \mathbb{C}\) be the fibre product taken w.r.t. \(\Delta_S\) and \(\mathbb{C}\) as \(z \mapsto z^p\). The morphism \(\eta : T \rightarrow S\) is then a \(p\)-sheeted étale covering coming with the projection map \(\text{pr}_1 : T \rightarrow S\). In the following, we set \(\Xi^e := (\Xi \circ \text{pr}_1) \circ \eta\). One has \(\Xi^p = \Delta_S \circ \eta\). By construction, the morphism

\[
T \overset{\Xi}{\longrightarrow} \eta \circ \text{SL}(U) \overset{\Gamma}{\longrightarrow} T
\]

factorizes over a morphism \(\Xi : T \rightarrow \text{SL}(U)\). The quotient family defined by the morphism

\[
T \overset{\Xi}{\longrightarrow} \eta \circ \text{SL}(U) \overset{\Gamma}{\longrightarrow} T
\]

is just \((q_S^1; \kappa_S; \eta \circ \text{id}_X \circ \tau_S^1)\) with

\[
q_S^1 : U \overset{\pi_X \circ \chi_S}{\longrightarrow} E_S \overset{\text{id}_X \circ \chi_S}{\longrightarrow} U \overset{\pi_X \circ \chi_S}{\longrightarrow} E_S \overset{\text{id}_X \circ \chi_S}{\longrightarrow} U.
\]

The assertion of the proposition is that this family is equivalent to the quotient family \((\eta \circ \text{id}_X \circ \tau_S^2)\). But this is easily seen, using

\[
\psi_T := \Xi \circ (\eta \circ \text{id}_X \circ \psi_S)(\eta \circ \text{id}_X \circ \psi_S) \circ (\eta \circ \text{id}_X \circ \psi_S) \circ (\eta \circ \text{id}_X \circ \psi_S) \circ (\eta \circ \text{id}_X \circ \psi_S)
\]

and \(\Xi^e_T := \Xi^e \circ (\eta \circ \chi_S)(\eta \circ \chi_S)(\eta \circ \chi_S)\).
2.3.4 The Gieseker space and map

Choose a Poincaré sheaf $\mathcal{P}$ on $\text{Jac}^d X$. By our assumptions on $n$, the sheaf

$$G_1 := \text{Hom}_X \big( \mathcal{P}, \pi_{\text{Jac}^d X} \big)$$

is locally free. We set $G_1 := \mathcal{P}(G^-)$. By replacing $\mathcal{P}$ with $\mathcal{P}$ (sufficiently ample), we may assume that $\mathcal{O}_{G_1}(l)$ is very ample. Let $\mathcal{E}_T$ be the morphism associated with $\mathcal{E}_T$, and let $A_T$ be a line bundle on $T$ with $\pi^* E_T = (\mathcal{O} \xrightarrow{\pmb{id}} \mathcal{P}) \pi_T A_T$. Then

$$\big( \pi \xrightarrow{\text{id} \pi_X(V)} \big) : \mathcal{P} \xrightarrow{\mathcal{P}} \pi_X(V \mathcal{P}) \xrightarrow{\pi_T} \mathcal{P}$$

defines a morphism $t_1: T \mapsto G_1$ with $t_1 \mathcal{O}_{G_1}(l) = A_T$.

Set $J^{d,m} := \text{Jac}^d \text{Jac}^m$. The sheaf

$$G_2 := \text{Hom}_X \big( U_{\mathcal{P}}, \pi_{\text{Jac}^d X}(\mathcal{P}) \big)$$

on $J^{d,m}$ is also locally free. Set $G_2 := \mathcal{P}(G^-)$. Making use of Remark 2.3, it is clear that we can assume $\mathcal{O}_{G_2}(l)$ to be very ample. The homomorphism

$$U_{\mathcal{P}} \xrightarrow{\text{id} \pi_X(V)} \big( \mathcal{P} \xrightarrow{\pi_X(V \mathcal{P})} \mathcal{P}_T \big)$$

provides a morphism $t_2: T \mapsto G_2$ with $t_2 \mathcal{O}_{G_2}(l) = A_T \otimes \mathcal{N}_T$. Altogether, setting $G := G_1 \setminus G_2$ and $t := t_1 \setminus t_2$, we have an injective and $\text{SL}(U)$-equivariant morphism

$$t: T \mapsto G$$

Linearize the $\text{SL}(U)$-action on $G$ in $\mathcal{O}_G(\mathcal{P}; 1)$ with

$$\mathcal{E} := \frac{p \otimes a}{r \otimes \delta}$$

and denote by $G^{e \otimes p}$ the sets of points in $G$ which are $\text{SL}(U)$-(semi/poly)stable w.r.t. the given linearization.

**Theorem 2.11.** For $n$ large enough, the following two properties hold true:

i) The preimages $t_1^{-1}(G^{e \otimes p})$ consist exactly of those points $t_2 \in T$ for which $(\mathcal{E}_T; M; \tau_T)$ (notation as in Rem. 2.9) is a $\delta$-(semi/poly)stable $\rho_{d,\tilde{\mathcal{P}}}$-pair of type $(d, p; m)$. 

ii) The restricted morphism $t_3^*: G^{e \otimes p} : t_1^{-1}(G^{e \otimes p})$ is $\text{proper}$.

The proof of this theorem will be given in a later section.

2.3.5 Proof of Theorem 2.5

Set $T^\delta \otimes \mathcal{P} := t_1^{-1}(G^{e \otimes p})$. Then $\text{Theorem 2.11}$ now shows that the categorical quotients

$$M \otimes \mathcal{P}_{d,\tilde{\mathcal{P}}; \otimes \mathcal{P}; \otimes m} := T^\delta \otimes \mathcal{P} = \text{SL}(U)$$
exist and that $\mathcal{M} (\varphi_{a \varphi_x})^{\delta \cdot s}_{d-r-m}$ is an orbit space. Proposition 2.8 and Proposition 2.10 tell us that we have a natural transformation of the functor $\mathcal{M} (\varphi_{a \varphi_x})_{d-r-m}$ into the functor of points of $\mathcal{M} (\varphi_{a \varphi_x})^{\delta \cdot s}_{d-r-m}$. The asserted minimality property of $\mathcal{M} (\varphi_{a \varphi_x})^{\delta \cdot s}_{d-r-m}$ and $\mathcal{M} (\varphi_{a \varphi_x})_{d-r-m}$'s being a coarse moduli space follow immediately from the universal property of the categorical quotient. Finally, the assertion about the closed points is a consequence of the “polystable” part of 2.11 Therefore, Theorem 2.5 is settled for representations of the form $\rho_{a \varphi_x}$.

For an arbitrary representation $\rho$, we may find $a \varphi c$ and a decomposition $\rho_{a \varphi_x} = \rho \bar{\rho}$. Define $\mathcal{T}(\rho)$ as the closed subscheme of $\mathcal{T}$ where the homomorphism

$$\mathfrak{e}_x: E_{\mathcal{S} \varphi_{a \varphi_x}} = E_{\mathcal{S} \rho \bar{\rho}}^{a \bar{c} \rho \bar{b} \mathcal{L}}$$

vanishes on $E_{\mathcal{S} \bar{\rho}}$. Set $\mathcal{T}(\rho)^{\delta \cdot s} := \mathcal{T}(\rho) \setminus \mathcal{T}^{\delta \cdot s}$. It follows that the categorical quotients

$$\mathcal{M}(\rho)^{\delta \cdot s}_{d-r-m} := \mathcal{T}(\rho)^{\delta \cdot s} \setminus \mathcal{T}_{\text{surj}}.$$ 

also exist. By our characterization of $\mathcal{M}(\rho)^{\delta \cdot s}_{d-r-m}$ as a closed subfunctor of the functor $\mathcal{M}(\varphi_{a \varphi_x})^{\delta \cdot s}_{d-r-m}$, the theorem follows likewise for $\rho$.

Next, we let $\mathcal{T}_{\text{surj}}$ be the open subscheme of $\mathcal{T}$ consisting of those points $t$ for which $\mathfrak{e}_x: E_S \not\subseteq X$ is surjective and set $\mathcal{T}(\rho)^{\delta \cdot s}_{\text{surj}} := \mathcal{T}(\rho)^{\delta \cdot s} \setminus \mathcal{T}_{\text{surj}}$. Thus, there is a section

$$\sigma_{\mathcal{T}(\rho)^{\delta \cdot s}_{\text{surj}}} : \mathcal{T}(\rho)^{\delta \cdot s}_{\text{surj}} \to \mathcal{P}(E_S \rho):$$

Moreover, $\mathcal{P}(E_S \mathcal{S} \rho)^{\mathcal{GL}(\rho)} F$ is a closed subscheme of $\mathcal{P}(E_S \rho)$. Now, we define $\mathcal{T}(\mathcal{F} \rho)^{\delta \cdot s}$ as the closed subscheme of those points $t \in \mathcal{T}(\rho)^{\delta \cdot s}_{\text{surj}}$ for which the restricted morphism $\sigma_{\mathcal{T}(\rho)^{\delta \cdot s}_{\text{surj}}} : \mathcal{T}(\rho)^{\delta \cdot s}_{\text{surj}} \to \mathcal{P}(E_S \rho)$ factorizes over $\mathcal{P}(E_S \mathcal{S} \rho)^{\mathcal{GL}(\rho)} F$. Since the action of $\mathcal{S}(\rho)$ on $\mathcal{T}(\rho)^{\delta \cdot s}$ is closed, the categorical quotient $\mathcal{T}(\rho)^{\delta \cdot s}_{\text{surj}} = \mathcal{S}(\rho)$ exists as an open subscheme of the moduli space $\mathcal{M}(\rho)^{\delta \cdot s}_{d-r-m}$, whence

$$\mathcal{M}(\mathcal{F} \rho)^{\delta \cdot s}_{d-r-h} := \mathcal{T}(\mathcal{F} \rho)^{\delta \cdot s} \mathcal{S}(\rho)$$

exists as a closed subscheme of $\mathcal{T}(\rho)^{\delta \cdot s}_{\text{surj}} = \mathcal{S}(\rho)$ and hence as a locally closed subscheme of $\mathcal{M}(\rho)^{\delta \cdot s}_{d-r-m}$ as asserted.

### 2.3.6 Proof of Theorem 2.13

### 2.3.7 Notation and Preliminaries

The remarks about one parameter subgroups of $\mathcal{S}(\rho)$ in Section 1.3 naturally apply to one parameter subgroups of $\mathcal{S}(\rho)$. We set

$$\gamma^{\lambda}_{\rho_i} := \frac{\lambda}{\rho_i} \rho_i^{\lambda \rho_i} \rho_i^{\lambda \rho_i}; \ i = 1; \cdots; p; \ 1$$

Given a basis $u = \mathfrak{u}_1; \cdots; \mathfrak{u}_p$ of $U$ and a weight vector $\mathfrak{v} = \sum_{i=1}^p \mathfrak{v}_i \gamma^{\lambda}_{\rho_i}$, we denote the corresponding one parameter subgroup of $\mathcal{S}(\rho)$ by $\mathfrak{u} (\mathfrak{u} \rho)$, we hope that these conventions will not give rise to too much confusion. Having fixed a basis $u = \mathfrak{u}_1; \cdots; \mathfrak{u}_p$ of $U$ and an index $l \geq 1; \cdots; \rho \mathfrak{g}$, we set $U^{\rho_i}_{\mathfrak{u}} := \mathfrak{u} \mathfrak{u}_1; \cdots; \mathfrak{u}_l$.
Let $\rho_{G_1}$ be the natural linearization of the $\mathcal{G}$-action on $G_1$ in $\mathcal{O}_{G_1}(1)$. Then, we write $\mu_{G_1}$ instead of $\rho_{G_1}$, and in the same way, $\mu_{G_2}$ is to be read. Finally, $\mu_{G_0}^{E_0} = \mu_{G_1}^{E_0} + \mu_{G_2}^{E_0}$, i.e., $\mu_{G_0}^{E_0} = \mu_{G_1}^{E_0}$, where $\rho_{G_1}$ stands for the linearization of the $\mathcal{G}$-action on $G$ in $\mathcal{O}(\mathcal{E}; 1)$, $\mathcal{E} \cong \mathcal{O}_{\mathcal{E}; 1}$.

Let $q: U \to \mathcal{O}_X^{(n)}$ be a generically surjective homomorphism and $E$ a vector bundle of degree $d$ and rank $r$. Set $Z := H^0(\mathcal{E}(\mathcal{E}^{(n)}))$. Then $h := q(\mathcal{E}(\mathcal{E}^{(n)}))$ is non trivial, and we can look at $[h] \in \mathcal{P}(\mathcal{Hom}(\mathcal{E}; \mathcal{E}Z)$. On this space, there is a natural $\mathcal{G}$-action. Then, it is well-known (e.g., [21]) that for any basis $\mu = \mu_1 \cdots \mu_p$ and any two weight vectors $\underline{\gamma} = \underline{\gamma}_1 \cdots \underline{\gamma}_p$ with $\gamma_i^j$ and $\sum_i \gamma_i^j = 0$, $i = 1, 2$,

$$
\mu \cdot \lambda(\underline{\gamma}^1); \mu_1 \cdots \mu_p \cdot \lambda(\underline{\gamma}^2); \mu_1 \cdots \mu_p = \mu \cdot \lambda(\underline{\gamma}^1; \mu_1 \cdots \mu_p)$$

and for every $l \geq 1$,

$$
\mu \cdot \lambda(\underline{\gamma}^1); \mu_1 \cdots \mu_p \cdot \lambda(\underline{\gamma}^2); \mu_1 \cdots \mu_p = \mu \cdot \lambda(\underline{\gamma}^1; \mu_1 \cdots \mu_p) + \mu \cdot \lambda(\underline{\gamma}^2; \mu_1 \cdots \mu_p)
$$

Here, $E_i$ stands for the subbundle generated by $q_i \mathcal{U}_\mathcal{E}^{(n)} \in \mathcal{O}_X^{(n)}$.

### 2.3.8 Sectional semi stability

**Theorem 2.12.** Fix the tuple $(\mathfrak{d}; r; a; \mathcal{H})$ and $a \neq 0$ as before. Then, there exists an $n_1$, such that for every $n > n_1$ and every $d$-(semi)stable $\rho_{a \mathfrak{d} \mathcal{H}}$-pair $(\mathcal{E}; \mathcal{M}; \mathcal{F})$, the following holds true: For every weighted filtration $(\mathcal{E}; \underline{\mathfrak{a}})$, $\mathcal{E} = 0 \cdots \mathcal{E}_1 \cdots \mathcal{E}_s \cdot \mathcal{E}$, of $E$

$$
\sum_{j=1}^s \alpha_i \chi(\mathcal{E}^i) \operatorname{rk} E_i \cdot h^0(\mathcal{E}^i) \cdot \operatorname{rk} E_i + \delta \cdot \mu(\mathcal{E}; \underline{\mathfrak{a}}; \mathcal{F}) (\mathcal{E}; \mathcal{M}; \mathcal{F}) = 0.
$$

**Proof.** First, suppose that we are given a weighted filtration $(\mathcal{E}; \underline{\mathfrak{a}})$, $\mathcal{E} = 0 \cdots \mathcal{E}_1 \cdots \mathcal{E}_s \cdot \mathcal{E}$, such that $\mathcal{E}_i(\mathcal{E})$ is globally generated and $H^1(\mathcal{E}_i(\mathcal{E})) = 0$ for $i = 1, \ldots, s$. Then, for $i = 1, \ldots, s$,

$$
\chi(\mathcal{E}_i) \operatorname{rk} E_i \cdot h^0(\mathcal{E}_i) \cdot \operatorname{rk} E_i = d \cdot (r \cdot (n+1) \cdot g) \cdot \operatorname{deg} (\mathcal{E}_i) + \operatorname{rk} E_i \cdot (n+1) \cdot g) 
$$

so that the claimed condition follows from $(\mathcal{E}; \mathcal{M}; \mathcal{F})$ being $\delta$-(semi)stable.

Next, recall that we have found a universal positive constant $C_1$ depending only on $r$, $a$, and $\delta$, such that for every $d$, every semistable $\rho_{a \mathfrak{d} \mathcal{H}}$-pair $(\mathcal{E}; \mathcal{M}; \mathcal{F})$, and every non-trivial subbundle $E^0$ of $E$

$$
\mu(\mathcal{E}; \mathcal{M}; \mathcal{F}) \leq \frac{d}{r} + C_1.
$$

If we fix another positive constant $C_2$, then the set of isomorphism classes of vector bundles $E^0$ such that $\mu(\mathcal{E}; \mathcal{M}; \mathcal{F}) \leq \frac{d}{r} + C_2$, $\mu_{\mathcal{O}}(\mathcal{E}; \mathcal{M}; \mathcal{F}) \leq \frac{d}{r} + C_2 + 1$, and $\operatorname{rk} E^0 \leq r$ is bounded. From this, we infer that there is a natural number $n(C_2)$, such that for every $n > n(C_2)$, every semistable $\rho_{a \mathfrak{d} \mathcal{H}}$-pair $(\mathcal{E}; \mathcal{M}; \mathcal{F})$ of type $(\mathfrak{d}; r; a; \mathcal{H})$, and every proper subbundle $E^0$ of $E$

$$
either \mu(\mathcal{E}; \mathcal{M}; \mathcal{F}) < \frac{d}{r} + C_2 \text{ or } E^0(\mathcal{E}; \mathcal{M}; \mathcal{F})$$

or $E^0(\mathcal{E}; \mathcal{M}; \mathcal{F})$ is globally generated and $H^1(\mathcal{E}; \mathcal{M}; \mathcal{F}) = 0$. 

Moreover, the Le Potier-Simpson estimate (cf. [24], Lemma 7.1.2 and proof of 7.1.1, p. 106) gives in
the first case
\[ h^0 E^0 (\eta_j) = \text{rk} E^0 \left( \frac{1}{m} + C_1 + n + 1 \right) + \frac{1}{\text{rk} E^0} \frac{h^d}{r} C_2 + n + 1 ; \]
\[ \text{i.e., for large } n \]
\[ h^0 E^0 (\eta_j) = \text{rk} E^0 \left( \frac{d}{r} + n + 1 + \psi 2 \right) C_1 \frac{C_2}{r} ; \]
and thus
\[ \chi E (\eta_j) \text{rk} E^0 h^0 E^0 (\eta_j) r = K (g, r; C_1, C_2) \]
\[ := r (\psi 1) g (r + 1) \psi 2 C_1 C_2 ; \]
Our contention is now that for \( C_2 \) with
\[ K (g, r; C_1, C_2) > \delta a (\psi 1) \]
and \( n_1 := n (C_2) \), the theorem holds true.

So, assume that we are given a weighted filtration \( (E, \mu) \) with \( E : 0 \leftarrow E_1 \leftarrow \ldots \leftarrow E_s \) and \( \alpha = (\alpha_1, \ldots, \alpha_s) \). Let \( j_1, \ldots, j_t \) be the indices such that \( \mu (E_{j_i}) = \delta = r \leftarrow C_2 \), for \( i = 1, \ldots, t \), so that \( E_{j_i} (\eta_i) \) is globally generated and \( H^1 (E_{j_i} (\eta_i)) = 0, i = 1, \ldots, t \). We let \( \bar{i}, \ldots, \bar{s} \) be the indices in \( \{1, \ldots, s\} \cap \{j_1, \ldots, j_t\} \) in increasing order. We introduce the weighted filtrations \( (E_1, \mu_1) \) and \( (E_2, \mu_2) \) with \( E_1 : 0 \leftarrow E_{j_1} \leftarrow \ldots \leftarrow E_{j_t} \leftarrow E, \alpha_1 := (\alpha_{j_1}, \ldots, \alpha_{j_t}) \) and \( E_2 : 0 \leftarrow E_1 \leftarrow \ldots \leftarrow E, \alpha_2 := (\alpha_{j_1}, \ldots, \alpha_{j_t}) \).

Lemma 7.8(ii) yields
\[ \mu_\rho E \not\subset \alpha; \tau \quad \mu_\rho E_1 \not\subset \alpha_1; \tau \quad \sum_{i=1}^{t} \alpha_i \not\subset \delta a (\psi 1) ; \]
whence
\[ \sum_{j=1}^{t} \alpha_{j} \chi (E (\eta_j)) \text{rk} E_{j} h^0 (E_{j} (\eta_j)) \text{rk} E + \delta \mu E \not\subset \alpha; \tau \]
\[ \sum_{i=1}^{t} \alpha_{j_i} \chi (E (\eta_j)) \text{rk} E_{j_i} h^0 (E_{j_i} (\eta_j)) \text{rk} E + \delta \mu E_1 \not\subset \alpha_1; \tau \]
\[ + \sum_{i=1}^{t} \alpha_{i} \chi (E (\eta_j)) \text{rk} E_{i} h^0 (E_{i} (\eta_j)) \text{rk} E + \delta \mu E_2 \not\subset \alpha_2; \tau \]
\[ \sum_{j=1}^{t} \alpha_{j} \chi (E (\eta_j)) \text{rk} E_{j} h^0 (E_{j} (\eta_j)) \text{rk} E + \delta \mu E_1 \not\subset \alpha_1; \tau \]
\[ + \sum_{i=1}^{t} \alpha_{i} K (g, r; C_1, C_2) \sum_{i=1}^{t} \alpha_i \not\subset \delta a (\psi 1) ; \]
Since this last expression is positive by assumption, we are done. \( \square \)
The implication $t \not\preceq (G^e \not\prec s)$ \quad (E_i;\mathcal{M}_i; \tau) \not\prec \delta-(\text{semi})\text{stable}

To begin with, we fix a constant $K$ with the property that
\[
  rK > \max_d d(\psi, r) + \delta a \quad (\rho \ 1) \ \forall \ \rho = 1; \ldots; r: \quad k.
\]

Now, let $t = [\psi; U \odot X (\ n) \not\prec E_i;\mathcal{M}_i; \tau]$ be a point with $t(\psi) \not\prec G^e \not\prec s$.

We first claim that there can be no subbundle $E^0 \subset E_i$ with $\deg(E^0) = d + K$. Let $E^0$ be such a subbundle. Then, for every natural number $n$,
\[
  h^0(E^0(n)) = d + K + \text{rk}E^0(n + 1) g);
\]

Let $\mathcal{E}$ be the subbundle of $E_i$ which is generated by $\text{Im} \psi \subset H^0(E^0(\psi)) \odot X (\ n) \not\prec E_i)$. Thus, $H^0(\mathcal{E}^0(\psi)) = H^0(E^0(\psi))$ and $\mathcal{E}$ is generically generated by its global sections. Now, choose a basis $u_1; \ldots; u_i$ for $H^0(E^0(\psi))$, complete it to a basis $\mathcal{U} := \psi_1; \ldots; \psi_p)$ of $U$, and set $\mathcal{V} := \lambda(\psi_1; \ldots; \psi_p)$. Then, we have seen that
\[
  \mu_{G_1} \lambda; t_1 \psi) = p \ \text{rk}E \ h^0(E^0(\psi)) r
\]
\[
  \mu_{G_2} \lambda; t_2 \psi) = a \ (p \ i);
\]

Therefore,
\[
  \mu_{G}^e \lambda; t \psi) = \varepsilon \ \mu_{G_1} \lambda; t_1 \psi) + \mu_{G_2} \lambda; t_2 \psi)
\]
\[
  = p^2 \text{rk}E^0 \ h^0(E^0(\psi)) r + a \ (p \ i)
\]

Next, we multiply the last expression by the positive number $r\delta = p$ in order to obtain
\[
p \text{rk}E^0 \ h^0(E^0(\psi)) r + a \ r \ \text{rk}E^0
\]
\[
d + r(n + 1) g \ \text{rk}E^0 \ r + K + \text{rk}E^0(n + 1) g + a \ r
\]
\[
= d \ \text{rk}E^0 \ r + a \ r + 1 \ \ rK < 0;
\]

by our choice of $K$. This obviously contradicts the assumption $t(\psi) \not\prec G^e \not\prec s$. We can also assume that $d + K > 0$. Set $C_3 := (\rho \ 1)d=r + K$. Then our arguments show that $t(\psi) \not\prec G^e \not\prec s$ implies
\[
  \mu_{\max}(E_i) = \frac{d}{r} + C_3;
\]

independently of the number $n$ with which we performed the construction of $G_i$. An argument similar to the one used in the proof of Theorem 2.12 shows that a $\rho_{a,b,e}$-pair $(E;\mathcal{M}; \tau)$ is $\delta-(\text{semi})\text{stable}, if and only if for every weighted filtration $(E;\mathcal{M}; \tau)$, such that
\[
  \mu(E_j) = \frac{d}{r} \ \delta \ a \ \frac{\varphi}{r} 1) = \frac{d}{r} \ C_1; \quad j = 1; \ldots; s;\]
one has
\[ M E \otimes \alpha + \delta \mu \beta E \otimes \tau \] ( ) 0:

Therefore, we choose \( n \) so large that for every vector bundle \( E^0 \) with \( d=r+C_3 \mu_{\max}(E^0) \mu(E^0) \) \( d=r \) \( C_1 \) and \( 1 - \text{rk} E^0 r \) \( 1 \), one has that \( E^0(\eta) \) is globally generated and \( H^1(E^0(\eta)) \) vanishes.

Now, let \( (E \otimes \mathbb{R}) \) be a weighted filtration with
\[ \mu(E_j) \frac{d}{r} C_1 j = 1 ; \cdots ; s : \]

Fix a basis \( \mathbf{w} = (w_1; \cdots ; w_r) \) of \( W := \mathbb{C}^r \), and let \( W : 0 W^2 \) \( W^0 \) \( W^2 \) \( W \) be the associated flag, \( i_j := \text{rk} E_j j = 1 ; \cdots ; s \). Let \( \mathbf{u} = (u_1 ; \cdots ; u_p) \) be a basis of \( U \) such that there are indices \( l_1 ; \cdots ; l_s \) with \( U^1 = H^0(E_j(\eta)) j = 1 ; \cdots ; s \). Define
\[ \Phi := \sum_{j=1}^s \alpha_j \gamma^{1/j} : \]

We also set, for \( j = 1 ; \cdots ; s + 1 l_{s+1} := p l_0 := 0 i_{s+1} := r i_0 := 0 \),
\[ \text{gr}_j(U \otimes \mathbb{R}) := U^1 \gamma^{1/j} = U^0(\eta) ; \text{and} \]
\[ \text{gr}_j(W \otimes \mathbb{R}) := W^1 \gamma^{1/j} = W^0(\eta) : \]

The fixed bases \( \mathbf{w} \) for \( W \) and \( \mathbf{u} \) for \( U \) provide us with isomorphisms
\[ U = \text{gr}_j(U \otimes \mathbb{R}) ; \text{and} \]
\[ W = \text{gr}_j(W \otimes \mathbb{R}) : \]

Let \( J^a := 1 ; \cdots ; s \). For every index \( \perp 2 J^a \), we set
\[ W^{2 \perp} := \text{gr}_i(W \otimes \mathbb{R}) : \]

Analogously, we define \( U^{2 \perp} \). Moreover, for \( k 2 1 ; \cdots ; c \), and \( \perp 2 J^a \), we let \( W^{2 \perp} \) be the subspace of \( W^{2 \perp} := W^a c \) which is \( W^{2 \perp} \) living in the \( k \)-th copy of \( W^a \) in \( W^{2 \perp} \), and similarly we define \( U^{2 \perp} \). The spaces \( W^{2 \perp} \) and \( U^{2 \perp} \), \( k 2 1 ; \cdots ; c \) and \( \perp 2 J^a \), are eigenspaces for the actions of the one parameter subgroups \( \lambda(U_j\gamma^{1/j}) \) and \( \lambda(U_j\gamma^{1/j}) \), respectively, \( j = 1 ; \cdots ; s \). Define
\[ \nu_j(\mathbf{u}) := \# i \perp j = (1 ; \cdots ; u_p) ; i = 1 ; \cdots ; a : \]

Then \( \lambda(U_j\gamma^{1/j}) \) acts on \( W^{2 \perp} \) with weight \( \nu_j(\mathbf{u}) \) \( r a ; \lambda(U_j\gamma^{1/j}) \) acts on \( U^{2 \perp} \) with weight \( \nu_j(\mathbf{u}) \) \( p a j : \]

Let \( Z_t := H^0(\text{det}(E_t) \bigwedge M_r \bigwedge \chi(\eta)) \). Then \( Z_2 \) \( \mathbb{P}(\text{Hom}(U^{2 \perp} ; Z_t) \bigwedge) \) can be represented by a homomorphism
\[ L_t : U^{2 \perp} ! \bigwedge Z_t : \]

One readily verifies
\[ \mu_{\xi_1} \lambda(U_j \gamma^{1/j}) ; \mu_{\xi_1} = \min \sum_{j=1}^n \alpha_j \nu_j(\mathbf{u}) p a j : \]

(3)
Next, we observe that we can choose a small open subset $X_0 \subseteq X$ over which $E_i$ and $M_i$ are trivial and there is an isomorphism $\psi: E_i|_{X_0} = W \circ X_0$ with $\psi(E_i|_{X_0}) = W \circ X_0$. This trivialization and the $\rho_{a+b}$-pair $(E_i; M_i; \tau_i)$ provide us with

$$l_i: W_{a+b} \circ X_0 \to W \circ X_0.$$

We observe that for every $k \geq 1; \ldots; \leq c \leq q$ and every $1 \leq J^a$

$$W_{k, J^a}^k \circ X_0 \leq \ker l_i \quad \text{and} \quad U_{k, J^a}^k \leq \ker L_i,$$

and that

$$\mu_{\rho_{a+b}}(E; \alpha_i; \tau_i) = \min \sum_{j=1}^s \alpha_j \nu_j(\underline{u})$$

for $\rho = a + b$ and $\rho = a - b$.

Now, let $k_0 \geq 1; \ldots; \leq c \leq q$ and $k_0 \geq 2; \ldots; \leq J^a$ be such that the minimum in (3) is achieved by $\sum_{j=1}^s \alpha_j(\nu_j, \underline{u})$, $p = a$ and $U_{k_0, J^a}^k \leq \ker L_i$. We obtain

$$0 \quad (\quad \mu_{\rho_{a+b}}(E; \alpha_i; \tau_i))$$

and

$$0 \quad (\quad \mu_{\rho_{a+b}}(E; \alpha_i; \tau_i))$$

for $\rho = a + b$ and $\rho = a - b$.

We multiply this inequality by $r \delta = p$ and find

$$0 \quad (\quad \sum_{j=1}^s \alpha_j \nu_j(\underline{u}), \rho \circ l_i) \quad \rho \circ l_i$$

Since $h^1(E_j(\nu)) = 0$, $j = 1; \ldots; s$, we have $\rho \circ l_i \circ h^0(E_j(\nu)) = d \circ l_i \circ h^0(E_j(\nu)) = 0$. Moreover, $\rho \circ l_i = \nu_j$, by definition, and $\mu_{\rho_{a+b}}(E; \alpha_i; \tau_i) \sum_{j=1}^s \alpha_j(\nu_j(\underline{u})) \rho \circ l_i$, by (4) and (5), whence we finally see

$$M \circ E; \alpha_i + \delta \circ \mu_{\rho_{a+b}}(E; \alpha_i; \tau_i) \quad (\quad 0,$$

as required.

The implication $(E_i; M_i; \tau_i) \leq (\text{semistable}) \iff E \geq 0)$ (by (4) and (5))

By the Hilbert-Mumford criterion, we have to show that for every basis $\nu = (\nu_1; \ldots; \nu_p)$ of $U$ and every weight vector $\underline{\nu} = (\nu_1; \ldots; \nu_p)$ with $\nu = \sum_{j=1}^p \gamma_j = 0$

$$\mu_{\rho_{a+b}}(E; \alpha_i; \tau_i) = \epsilon \mu_{G_1}(\nu) \circ \mu_{G_2}(\nu) \circ l_i(\nu) + \mu_{G_2}(\nu) \circ l_i(\nu) \circ \mu_{G_2}(\nu) \circ l_i(\nu) \quad (\quad 0,$$

as required.
So, let \( \mu = (\mu_1; \ldots; \mu_p) \) be an arbitrary basis for \( U \) and \( \overline{\phi} = \sum_{i=1}^{\rho} \beta_i \overline{\gamma_i}^\phi \) a weight vector. Let \( l_1; \ldots; l_v \) be the indices with \( \beta_{i_0} \neq 0, h = 1; \ldots; v \). For each \( h \in \{1; \ldots; v\} \), let \( E_{i_0} \) be the subbundle of \( E_l \) generated by \( \text{Im} \{ U_{l_0}^{(h)} \} \text{ of X( n)} \). Note that for \( h^0 \), we have \( E_{i_0} = E_{l_0} \) if and only if \( U_{l_0}^{(h_0)} \text{ H}^0 (E_{l_0} (n)) \). We let \( E : 0 = E_0 \leq E_1 \leq \cdots \leq E_{s+1} := E \) be the filtration by the distinct vector bundles occurring among the \( E_{i_0} \)’s.

Recall that we know (1)

\[
\mu_{G_1} \lambda (\overline{\phi} (\overline{\gamma_1}^\phi); \iota_1 (\psi)) = \sum_{h=1}^{v} \beta_{i_0} p \text{rk} E_{i_0} \ i_h r \ \sum_{h=1}^{v} \beta_{i_0} p \text{rk} E_{i_0} \ h^0 (E_{i_0} (n)) r :
\]

Set, for \( j = 1; \ldots; s \),

\[
\alpha_j := \sum_{h; E_{i_0} = E_j} \beta_{i_0} ;
\]

so that we see

\[
\mu_{G_1} \lambda (\overline{\phi} (\overline{\gamma_1}^\phi); \iota_1 (\psi)) = \sum_{j=1}^{s} \alpha_j p \text{rk} E_j \ h^0 (E_j (n)) r :
\]

Next, we define for \( j = 0; \ldots; s \)

\[
h(j) := \max \ h = 1; \ldots; v \ \overline{\phi} (\overline{\gamma_1}^\phi) \ h^0 (E_j (n)) ;
\]

With these conventions, \( h = h(j) + 1 \) is the minimal index, such that \( U_{l_0}^{(h_0)} \text{ O X}(n) \) generically generates \( E_{j+1} \), \( j = 0; \ldots; s \). We now set

\[
\text{gr}_j (U_{l_\mu}) := U_{l_0}^{(h(j)+1)} = U_{l_0}^{(h_j+1)} ; \quad j = 1; \ldots; s + 1 ;
\]

The space \( \text{gr}_j (U_{l_\mu}) \) can be identified with a subspace of \( U \), via \( \text{gr}_j (U_{l_\mu}) = \text{rk} h_j + 1 ; \ldots; l_{h(j)+1} \mu, j = 1; \ldots; s \).

For any index tuple \( \underline{l} = (l_1; \ldots; l_a) \in J^a := \{1; \ldots; c \}^a \), we define

\[
\mathbb{E}^{k}_{l_\mu} := \text{gr}_1 (U_{l_\mu}) \quad \text{gr}_h (U_{l_\mu}) ;
\]

Again, for \( \underline{l} \in J^a \) and \( k \leq 1; \ldots; c \), \( \mathbb{E}^{k}_{l_\mu} \) will be \( \mathbb{E}^{k}_{l_\mu} \) viewed as a subspace of the \( k \)-th summand of \( U_{l_\mu} \).

The effect of our definition of the \( h(j) \)’s is that the spaces \( \mathbb{E}^{k}_{l_\mu} \) \( \text{O}_a \) \( J^a \) and \( k \leq 1; \ldots; c \) are eigenspaces for all the one parameter subgroups \( \lambda (\overline{\phi} (\overline{\gamma_1}^\phi)) \), \( h = 1; \ldots; v \), with respect to the weight \( v_j (\underline{l}) p \ \text{al}_h, v_j (\underline{l}) \) as in (2).

Now, let \( \underline{w} = (w_1; \ldots; w_p) \) be a basis for \( W \) and \( W : 0 = W_{l_0}^{(h)} \subset W \), \( i_j := \text{rk} E_j, j = 1; \ldots; s \), the corresponding flag. Then, the spaces \( W_{l_0}^{(j)} \) \( J^a \) and \( k \leq 1; \ldots; c \) are defined as before. We can find a small open set \( X_0 \) \( X \), such that

\[ M \text{ and } E_l \text{ are trivial over } X_0, \]

there is an isomorphism \( \psi : E_{l_0} \text{ O}_X \) with \( \psi (E_{l_0}) = W \text{ O}_X \),

\[ E_{l_0} = \sum_{j=1}^{s+1} (E_j (n)) \]

the homomorphism \( \sum_{j=1}^{s+1} (E_j (n)) \text{ O}_X \) \( \text{O}_X (n) \) \( \text{O}_X (n) \) is surjective.
As before, let \( Z_{2} := H^{0}(\det(\mathcal{E}_{j}) \circ M_{l} \circ X(\eta a)), \) so that \( t_{2} \psi \) \( 2 \mathbb{P}(\text{Hom}(U_{aX}; Z_{2})) \) induces a homomorphism
\[
\mathcal{E}_{j} : \mathbb{P}^{k} \rightarrow Z_{2}.
\]
Letting
\[
l_{j} : W_{aX} \circ X_{0} \rightarrow W^{b} \circ X_{0}
\]
be the resulting homomorphism, we find that for every \( k \geq 1 ; \cdots ; c \) and every \( l_{j} \geq J^{a}
\[
W_{aX}^{k} \circ X_{0} 6 \ker l_{j} \quad , \quad \mathbb{P}^{k} \circ X_{0} \quad \ker \mathcal{E}_{j} :
\]
By Theorem 2.12, we have
\[
\sum_{j=1}^{s} \alpha_{j} \cdot \text{prk} \cdot E_{j} \cdot h^{0}(\mathcal{E}_{j}(\eta)) \cdot r + \delta \cdot \mu_{\varphi_{0}} E ; (\alpha_{1} ; \cdots ; \alpha_{s}) ; \tau_{i} \quad ( \quad 0 ;
\]
Now, we choose \( k_{0} \geq 1 ; \cdots ; c \) and \( l_{j} \geq J^{a} \) with \( W_{aX}^{k_{0}} \circ X_{0} 6 \ker l_{j} \) and \( \mu_{\varphi_{0}} E ; (\alpha_{1} ; \cdots ; \alpha_{s}) ; \tau_{i} \) \( = \sum_{j=1}^{s} \alpha_{j} \cdot (\nu_{j}(\eta)) \cdot r \cdot \text{prk} \cdot E_{j} \). Plugging this into (8) and multiplying by \( p = (r \delta) \) yields
\[
0 \quad ( \quad \sum_{j=1}^{s} \alpha_{j} \cdot \text{prk} \cdot E_{j} \cdot h^{0}(\mathcal{E}_{j}(\eta)) \cdot r + \sum_{j=1}^{s} \alpha_{j} \cdot \nu_{j}(\eta) \cdot p \quad a \quad 0(\mathcal{E}_{j}(\eta)) ;
\]
By our definition of the \( \alpha_{j} \), and (7), we know
\[
\mu_{G_{2}} (\varphi_{2} ; t_{2} \psi) \quad \sum_{h=1}^{v} \beta_{h} \cdot \nu_{j}(\eta) \cdot p \quad a \cdot h
\]
\[
\sum_{j=1}^{s} \alpha_{j} \cdot \nu_{j}(\eta) \cdot p \quad a \cdot h^{0}(\mathcal{E}_{j}(\eta)) ;
\]
Here, we have set \( j(\eta) \) to be the element \( j \geq 1 ; \cdots ; c \) with \( E_{h_{j}} = E_{j} \). This together with (6) finally shows \( \mu_{G_{2}} (\varphi_{2} ; t_{2} \psi) \) ( \( \psi \)).

The identification of the polystable points
By the Hilbert-Mumford criterion, a point \( t_{2} \psi \) is polystable if and only if it is semistable and, for every one parameter subgroup \( \lambda \) of \( SL(U) \) with \( \mu_{G_{2}} (\lambda ; t_{2} \psi) \) \( = 0 \), \( \lim_{t \rightarrow 0} \cdot \lambda \cdot (\varphi_{2} ; t_{2} \psi) \) lies in the orbit of \( t_{2} \psi \). Now, let \( u = \eta_{1} ; \cdots ; \eta_{p} \) be a basis for \( U \) and \( \varphi = \sum_{j=1}^{s} \beta_{j}^{(j)} \cdot (\eta_{1}) \) be a weight vector with \( \beta_{j} \neq 0 \) and \( l_{j} \geq 1 ; \cdots ; c \) such that \( \mu_{G_{2}} (\lambda ; (\varphi_{2} ; t_{2} \psi)) = 0 \). Then, our previous considerations show that the following must be satisfied
\[
U_{L}^{(j)} = H^{0}(\mathcal{E}_{j}(\eta)), \quad j = 1 ; \cdots ; s,
\]
\[
\mathcal{E}_{j}(\eta) \quad \text{is generated by global sections and} \quad H^{1}(\mathcal{E}_{j}(\eta)) \quad = 0.
\]
Set $E_j := E_j$, $i_j := \text{rk} E_j$, $\alpha_j := \beta_j$, $j = 1; \cdots; s$, and choose a basis $w_1; \cdots; w_r$ for $E$. As before, we associate to these data a flag $W$. Consider the weighted filtration $(\mathcal{E}, \mathcal{F})$ with $E : 0 \to E_1 \to E_2 \to \cdots \to E_s$ and $\mathcal{E} = (\mathcal{E}_1; \cdots; \mathcal{E}_s)$, so that the condition $\mu_G^s (\lambda \mathcal{U} \mathcal{F} \mathcal{C}_j ; \mathcal{F}_j) = 0$ becomes equivalent to

$$ M E : \mathcal{E} + \delta \quad \mu_{i}^s E : \mathcal{F}_j \mathcal{C}_j = 0; $$

Let $t_m := \lim_{\gamma \to \lambda (\xi)} t$ and $(\mathcal{E}_i (\mathcal{M}_i ; \mathcal{T}_i))$ be the corresponding $\rho_{i}^s$ pair. Then, clearly $M_{n}^s = M_{t}$, and it is well known that $E_{n} = \sum_{j=1}^{s} E_{j} t_{j}$. Let $U_{ax} := U^{s}$ be the decomposition of $U_{ax}$ into eigenspaces w.r.t. the $C$-action coming from $\lambda \mathcal{U} \mathcal{F} \mathcal{C}_j$ and $\mathcal{E}_{s} = \mu_{G} (\lambda \mathcal{U} \mathcal{F} \mathcal{C}_j ; t_{j} \mathcal{F}_j )$. If $L_{1} : U_{ax} \to Z_{i}$ and $L_{n} : U_{ax} \to Z_{i}$ are the homomorphisms representing $t$ and $t_{n}$, respectively, then $L_{m}$ is just the restriction of $L_{1}$ to $U_{ax}$ extended by zero to the other weight spaces. As we have seen before, the condition that $L_{m}$ be supported only on $U_{ax}$ is equivalent to the fact that over each open subset $X_{0}$ over which $\mathcal{T}_i$ is surjective and we have a trivialization $\psi : E_{n} \to W \otimes X_{0}$ with $\psi (\mathcal{E}_{s}) = W \otimes X_{0}$, the induced morphism $X_{0} \to \mathbb{P} (W_{ax})$, where $W_{ax}$ is the eigenspace for the weight $g := \mu_{i}^s E (\mathcal{F}_j \mathcal{C}_j , \mathcal{T}_i )$. Thus, we have shown that $(E_{i} ; M_{t} ; \mathcal{T}_i)$ being $\delta$-polystable implies that $t \mathcal{F}_j$ is a polystable point. The converse is similar.

The properness of the Gieseker map

In this section, we will prove that the Gieseker morphism $t$ is proper, using the (discrete) valuative criterion.

Thus, let $(\mathcal{C} ; 0)$ be the spectrum of a DVR $R$ with quotient field $K$. Suppose we are given a morphism $h : C ! G^{s s}$ which lifts over $\text{Spec} K$ to $T$. This means that we are given a quotient family $(\psi_{G} ; U \pi_{X} \otimes X (n) ! E_{C} ; \kappa_{C} ; t_{C})$ of $\rho_{i}^s$-pairs parameterized by $\text{Spec} K$ (we left out $\mathbb{N}$, because it is trivial). This can be extended to a certain family $(\psi_{C} ; U \pi_{X} \otimes X (n) ! E_{C} ; \kappa_{C} ; t_{C})$, consisting of

- a surjection $\psi$ onto the flat family $E_{C}$, where $E_{C} \otimes_{\mathcal{O}_{X}} X$ may have torsion
- the continuation $\pi_{X}$ of $\kappa_{X}$ into $0$
- a homomorphism $\pi : E_{C} \to \mathcal{F}$ such that $\det (\mathcal{F}) \Im L [\kappa_{C}]$ whose restriction to $\mathcal{O}_{X}$ $\otimes X$ is non trivial and whose restriction to $\text{Spec} K$ $X$ differs from $t_{C}$ by an element in $K$.

The resulting datum $L : U_{ax} \to \pi \pi_{X} \otimes X (n) ! E_{C} ; \kappa_{C} \pi_{X} \otimes X (n)$ defines a morphism $C ! G_{2}$ which coincides with the second component $h_{2}$ of $h$.

Set $E_{C} := E_{C}$. This is a reflexive sheaf on the smooth surface $C \otimes X$, whence it is locally free and thus flat over $C$. Therefore, we have a family

$$ q_{C} : U \pi_{X} \otimes X (n) ! E_{C} $$

where the kernel of the homomorphism $\pi : U \pi_{X} \otimes X (n)$ of $E_{C} \otimes_{\mathcal{O}_{X}} X$ is isomorphic to the torsion $T$ of $E_{C} \otimes_{\mathcal{O}_{X}} X$. One gets a homomorphism $\mathcal{O} \otimes X (n) ! \pi_{C} \det (\mathcal{E}_{C}) \pi_{X} \otimes X (n)$ which defines a morphism $C ! G_{1}$ which coincides with the first component $h_{1}$ of $h$.

Set $E_{0} := E_{C} \otimes_{\mathcal{O}_{X}} X$. Our claim is that $H^{0} (q_{C} \otimes_{\mathcal{O}_{X}} X (n) ! \pi_{X} \otimes X (n)) : U ! H^{0} (E_{0} (n))$ must be injective. This implies, in particular, that $E_{C} \otimes_{\mathcal{O}_{X}} X$ is torsion free and, hence, $E_{C} = E_{C}$ and $q_{C} = q_{C}$. If $H := \ker (H^{0} (q_{C} \otimes_{\mathcal{O}_{X}} X (n) ! \pi_{X} \otimes X (n)))$ is non trivial, we choose a basis $u_{1}; \cdots; u_{r}$ for $H$ and complete it to a basis $u = (u_{1}; \cdots; u_{r})$ of $U$. Set $\mathcal{T} = \mathcal{H} u_{1}; \cdots; u_{r}$. We first note

$$ \mu_{G_{1}} (\lambda \mathcal{U} \mathcal{F} \mathcal{C}_{j} ; \mathcal{F}_{j}) = \det \mu_{G_{1}} (\lambda \mathcal{U} \mathcal{F} \mathcal{C}_{j} ; \mathcal{F}_{j}) = j r: $$

where $\mu_{G_{1}} (\lambda \mathcal{U} \mathcal{F} \mathcal{C}_{j} ; \mathcal{F}_{j})$ denotes the homomorphism representing $t_{j}$ for $C$ and $r$ the rank of $E_{C}$.
The spaces $H_l := H^l / \overline{H}^{(l)}$, $l = 1, \ldots, a$, are the eigenspaces of $U^a$ for the $C$-action coming from $\lambda, \mu; \gamma_p^{(j)}).$ Let $H^k_i$ be $H_i$ embedded into the $k$-th component of $U_{l,x}, k = 1, \ldots, c, l = 1, \ldots, a.$ For every $k \geq 1; \ldots, c$ and every $l \geq 1; \ldots, a$, $H^k_i \circ X (n)$ generates a torsion subsheaf of $E_{C \cdot \overline{Q}_n}$, so that $H^k_i \ker L.$ This implies

$$
\mu_{G^2, \lambda, \nu; \gamma_p^{(j)} ; h_2 (0)} = \mu_{G^2, \lambda, \nu; \gamma_p^{(j)} ; h_2 (0)} = a j;
$$

and thus

$$
\mu_{G^2, \lambda, \nu; \gamma_p^{(j)} ; h_2 (0)} = ejr a j < 0;
$$
in contradiction to the assumption $h (0) \geq 0 \in \mathcal{G}^{ss}$.

We identify $U$ with its image in $H^0 (E_0 (n)).$ Let $K$ be a positive constant such that $rK > \max \varepsilon \delta d \wedge (r + 1)$ and $1 \geq r \geq 1$. We assert that for every non-trivial and proper quotient bundle $Q$ of $E_0$ we must have $\deg Q < K (r \geq 1)$ and $g$. For this, let $Q$ be the minimal destabilizing quotient bundle. Set $E^0 := \ker (E \rightarrow Q)$. It suffices to show that $\deg Q < K (r \geq 1)$ and $g$ implies $\dim (H^0 (E^0 (n)) \setminus U)$ $d + r + k E_0 (n + 1) g$, because then a previously given argument applies. Note that we have an exact sequence

$$
0 \rightarrow H^0 (E^0 (n)) \rightarrow U \rightarrow U \rightarrow H^0 (Q (n));
$$

Assume first that $h^0 (Q (n)) = 0$. Thus, $\dim (H^0 (E^0 (n)) \setminus U) = p = d + r k E_0 (n + 1) g + \deg Q + r k E_0 (n + 1) g + n + 1 g$, since we can assume $n + 1 g > K$, this is impossible.

Therefore, the Le Potier-Simpson estimate gives $h^0 (Q (n)) \deg Q + r k E_0 (n + 1)$ and thus

$$
dim H^0 (E^0 (n)) \setminus U = p \rightarrow h^0 (Q (n))
$$

$$
d + r (n + 1) g \deg Q + r k E_0 (n + 1) g
$$

$$
d \deg Q + g r k Q + (r \rightarrow r k Q + (r \rightarrow r k E_0 (n + 1) g)
$$

This gives the claim. We see

$$
\mu_{\min} (E_0) = \frac{K \rightarrow (n + 1) g}{rk Q} + K \rightarrow (n + 1) g;
$$

This bound does not depend on $n$. Since the family of isomorphism classes of vector bundles $G$ of degree $d$ and rank $r$ with $\mu_{\min} (G) \rightarrow K \rightarrow (n + 1) g$ is bounded, we can choose $n$ so large that $H^1 (G (n)) = 0$ for every such vector bundle. In particular, $H^1 (E_0 (n)) = 0$, i.e., $U = H^0 (E_0 (n))$. This means that the family $\{Q_{C, \overline{Q}_C} : r \in \mathbb{C} \}$ we started with is a quotient family of $\rho_{\mu, \overline{P}_C}$-pairs parameterized by $C$ and thus defines a morphism from $C$ to $\mathbb{T}$ which lifts $h$. By Theorem 2.11, this morphism factorizes through $\mathbb{T}^\delta \rightarrow \mathbb{G}^{ss}$, and we are done.

3 Examples

This section is devoted to the treatise of the known examples within our general context. First, we discuss two important methods of simplifying the stability concept. Second, we will consider some easy specializations of the moduli functors. Then, we briefly discuss the variation of the stability parameter and prove an “asymptotic irreducibility” result. Afterwards, we turn to the examples. In the examples, we will show how many of the known stability concepts and constructions of the moduli spaces over curves can be obtained via our construction. In two cases we will see that our results give
a little more than previous constructions. We have also added the stability concept for conic bundles of rank 4. The main aim of the examples is to illustrate that the complexity of the stability concept only results from the complexity of the input representation \( \rho: \text{GL}(\mathcal{V}) \to \text{GL}(\mathcal{V}) \) and to illustrate how the understanding of \( \rho \) can be used to simplify the stability concept.

### 3.1 Simplifications of the stability concept

In this part, we will formulate several ways of restating the concept of \( \delta \)-semistability in different, easier ways which will be used in the study of examples to recover the known notions of semistability. The first one uses a well-known additivity property to reduce the stability conditions to conditions on subbundles. The second one generalizes this to a method working for all representations. This provides the mechanism alluded to in the introduction. The third one is a method to express the concept of \( \delta \)-semistability for \( \rho \)-pairs associated with a direct sum \( \rho = \rho_1 \oplus \cdots \oplus \rho_n \) representations in a certain sense in terms of the semistability concepts corresponding to the summands \( \rho_i \). Further methods of simplifying the semistability concept will be discussed in the examples.

**A certain additivity property**

Let \( \rho: \text{GL}(\mathcal{V}) \to \text{GL}(\mathcal{V}) \) be a representation such that the following property holds true: For any basis \( \mathcal{w} = (w_1; \cdots; w_r) \) of \( \mathbb{C}^r \), any two weight vectors \( \gamma_1 \) and \( \gamma_2 \), and any point \( \ell \) of \( \mathbb{P}(\mathcal{V}) \)

\[
\mu_{\rho}(\lambda; \mathcal{w}; \gamma_1 + \gamma_2; \ell) = \mu_{\rho}(\lambda; \mathcal{w}; \gamma_1; \ell) + \mu_{\rho}(\lambda; \mathcal{w}; \gamma_2; \ell) : \quad (9)
\]

Now, let \( (\mathcal{E}; M; \tau) \) be a \( \rho \)-pair and \( \delta \) a positive rational number. For every weighted filtration \( \mathcal{E} : \mathcal{G} \), \( \mathcal{E} : 0 \rightarrow E_1 \rightarrow \cdots \rightarrow E_s \rightarrow \mathcal{E} \), the definition of \( \mu_{\rho}(\mathcal{E}; \mathcal{G}; \tau) \) and \( \mathcal{G} \) imply

\[
M \ E : \mathcal{G} + \delta \mu_{\rho} \ E : \mathcal{G}; \tau
\]

\[
= \sum_{j=1}^{s} \alpha_j \ d \text{rk} \ E_j \ r \text{deg} \ E_j + \delta \mu_{\rho} \ E_j; \tau : \quad (10)
\]

We see that the semistability condition becomes a condition on subbundles of \( \mathcal{E} \): The \( \rho \)-pair \( (\mathcal{E}; M; \tau) \) is \( \delta \)-(semi)stable, if and only if for every non trivial proper subbundle \( E^0 \) of \( \mathcal{E} \) one has

\[
\mu(\mathcal{E}^0) \ ( \ ) \mu(\mathcal{E}) + \frac{\mu_{\rho}(\mathcal{E}^0; \tau)}{\text{rk} \ \mathcal{E}} : \quad (10)
\]

**The general procedure**

Let \( \rho: \text{GL}(\mathcal{V}) \to \text{GL}(\mathcal{V}) \) be a representation on \( \mathcal{V} \) and \( \rho^0 \text{SL}(\mathcal{V}) \to \text{GL}(\mathcal{V}) \) its restriction to \( \text{SL}(\mathcal{V}) \). We fix a basis \( \mathcal{w} = (w_1; \cdots; w_r) \) of \( \mathbb{C}^r \). This basis determines a maximal torus \( T \) of \( \text{SL}(\mathcal{V}) \). First, we observe that the Hilbert-Mumford criterion can be restated in the following form: A point \( \ell \) of \( \mathbb{P}(\mathcal{V}) \) is \( \rho^0 \)-semistable, if and only if for every element \( g \) of \( \text{SL}(\mathcal{V}) \) and every weight vector \( \gamma = (\gamma_1; \cdots; \gamma_r) \) with \( \gamma_1 \)

\[
\mu_{\rho}(\lambda; \mathcal{w}; \gamma; \ell) \quad (\ ) \quad 0 : \quad (11)
\]

The representation \( \rho_{\gamma}: T \to \text{GL}(\mathcal{V}) \) yields a decomposition

\[
\mathcal{V} = \bigoplus_{\chi \in \mathcal{X}(\mathcal{G})} V_{\chi}
\]
with
\[ V_{\chi} := V^2 \mathcal{P} \psi(v) = \chi \psi(v) \quad \forall t \geq T. \]

The set \( \text{ST}(\rho) := \varepsilon \chi \odot X(T) \mathcal{P} \rho \in \mathcal{P} \) is the set of states of \( \rho \). We look at the rational polyhedral cone
\[ C := \{ \gamma_1, \ldots, \gamma_r \} : \sum_r \gamma_r = 0 = \mathbb{R}_0 \mathbb{Z}^+ + \mathbb{R}_0 \mathbb{Z}^1; \]

For every subset \( A \in \text{ST}(\rho) \), we obtain a decomposition
\[ C = \frac{C^X_A}{\chi \neq A} \quad \text{with} \quad C^X_A := \sum C_A \gamma; \varphi \text{ with } \gamma \text{ critical for } \rho \text{ of } \mathcal{P} \text{ if the set } \text{ST}(\rho) \text{ is critical.} \]

We observe that for a point \( \sum \mathcal{P} \rho \) and a weight vector \( \gamma \in C \) one has
\[ \mu_\rho \lambda(\gamma) \in C \quad \text{with} \quad \gamma = \min \lambda(\gamma) \text{ for } \gamma \in C. \]

This means that Equation (3) remains valid, if there exists a character \( \chi \in \text{ST}(\rho) \), such that \( C_{\chi} \) contains both \( \gamma_1 \) and \( \gamma_2 \). We infer

**Corollary 3.1.** A point \( \sum \mathcal{P} \rho \) is \( \rho \)-stable if and only if it satisfies the following two conditions:

1. **For every element** \( g \in \text{SL}(\rho) \) **and every** \( i \in \{1, \ldots, r\} \)
\[ \mu_\rho \lambda(\gamma) \in C \quad \text{with} \quad \gamma \in C \text{ for } \chi \in \text{ST}(\rho). \]

2. **For every** \( g \in \text{SL}(\rho) \) **which is critical for** \( \rho \) **and every weight vector** \( \gamma \in K_{\text{ST}(\rho)} \in \mathcal{P} \rho \)
\[ \mu_\rho \lambda(\gamma) \in C \quad \text{with} \quad \gamma \in C \text{ for } \chi \in \text{ST}(\rho). \]

In particular, it suffices to test (3) for the weight vectors belonging to the finite set
\[ K_{\rho} := \{ \lambda \in \text{ST}(\rho) \}. \]

**Remark 3.2.** A similar procedure works for all semisimple groups \( G \). Indeed, one fixes a pair \( (B; T) \) consisting of a Borel subgroup of \( G \) and a maximal torus \( T \). With analogous arguments, one obtains decompositions of the Weyl chamber \( W(B; T) \). See [3] for a precise discussion.
Let’s now turn to the \(\rho\)-pairs. Let \(W\) be the complete flag \(0 \subset h_{\mathbf{1}} \subset \cdots \subset h_{\mathbf{r}} \subset C'\). For a \(\rho\)-pair \((E; \mathcal{M}; \mathcal{V})\) and a filtration \(0 < E_1 < \cdots < E_r \subset E\) with \(\text{rk} E_i = i, i = 1; \cdots; r\), we define \(\text{ST}(E)\) as follows: Choose an open subset \(U\) and a trivialization \(\psi: E_U \to \mathcal{O}_U\). Then, for each \(\chi \in \text{ST}(\mathcal{V})\), there is a rational map

\[ U \to \mathbb{P}(E_{\rho U}) = \mathbb{P}(\mathcal{V}) \]}

An element \(\chi \in \text{ST}(\mathcal{V})\) now belongs to \(\text{ST}(E)\), if and only if this rational map is defined on a non-empty subset of \(U\). As before, one verifies that \(\text{ST}(E)\) is well-defined. The filtration \(E\) is called critical for \(\mathcal{V}\), if \(\text{ST}(E)\) is critical. Corollary \[\text{1.6}\] now shows

**Theorem 3.3.** i) The \(\rho\)-pair \((E; \mathcal{M}; \mathcal{V})\) is \(\delta\)-(semi)stable, if and only if it meets the following two requirements:

1. For every proper non-trivial subbundle \(E^0\) of \(E\)

\[ \mu(E^0) \leq \mu(E) + \frac{\mu(E^0; \mathcal{V})}{\text{rk} E} := \delta \]

2. For every filtration \(E\) which is critical for \(\mathcal{V}\) and every element \(\sum_{j=1}^s \alpha_j \gamma_{\mathcal{V}}(j) \geq \sum_{j=1}^s \gamma_{\mathcal{V}}^0 \geq 0\) \(\alpha_j > 0, i_j := \text{rk} E_{i_j}, j = 1; \cdots; s,\)

\[ M = 0 = E_1 \quad \text{EE}; (\alpha_1; \cdots; \alpha_s) + \]

\[ \delta = \mu = 0 = E_1 \quad \text{EE}; (\alpha_1; \cdots; \alpha_s); \mathcal{V} \quad 0 = \]

Direct sum of representations

Let \(\rho: \text{GL}(\psi) \to \text{GL}(\mathcal{V})\) be representations of the general linear group and assume there is an integer \(\alpha\) with \(\rho(z) = z^\alpha \text{id}\) for all \(z \in \mathbb{C}, i = 1; \cdots; \mathcal{V}\). Define \(\rho := \rho_1 \cdots \rho_r\). Note that for every rank \(r\) vector bundle \(E\) one has \(E_\rho = E_{\rho_1} \cdots E_{\rho_r}\).

The following result is a counterpart to Theorem \[\text{1.6}\] in the first part.

**Proposition 3.4.** Let \((E; \mathcal{M}; \tau)\) be a \(\rho\)-pair of type \(\mu; \tau; \rho; m\) and \(\delta \geq 0\). Then the following conditions are equivalent:

1. \((E; \mathcal{M}; \tau)\) is \(\delta\)-semistable (\(\delta\)-polystable).

2. There exist pairwise distinct indices \(t_1; \cdots; t_s \neq 1; \cdots; t; s\) \(t, \text{ such that} \)

\[ j \geq t_1; \cdots; t_s \quad \tau_{i_{\rho_j}}: E_{i_{\rho_j}} \quad M = \text{non-zero}, \]

and positive rational numbers \(\sigma_1; \cdots; \sigma_s\) with \(\sum_{j=1}^s \sigma_j = 1\) such that for every weighted filtration \((E; \mathcal{G})\)

\[ M = E; \mathcal{G} + \delta = \sum_{j=1}^s \sigma_j \mu_{\rho_{i_j}}: E; \mathcal{G}; \tau_{i_{\rho_{i_j}}} = 0 : \]

(And if equality holds)

\[ E = \sum_{j=1}^s \rho_{i_j} E_{j} = E_{j} \]
the \( \rho_j \)-pair \( (E; M; \tau \ph_{\rho_j}) \) is equivalent to the \( \rho_{ij} \)-pair \( (E; M; \tau \ph_{\rho_{ij}}) \), \( j = 1; \cdots; s \) (Compare with definition of polystability above).

Here, \( \ph \) is a basis for \( C' \), \( W : 0 \rightarrow W_{\ph E_1} \rightarrow W_{\ph E_2} \rightarrow \cdots \rightarrow W_{\ph E_d} \rightarrow W \), and \( \gamma := \sum_{j=1}^{s} \alpha_j \phi_{\gamma j} \).

3. There exist pairwise distinct indices \( t_1; \cdots; t_s \) and positive rational numbers \( \sigma_1; \cdots; \sigma_s \) with \( \sum_{j=1}^{s} \sigma_j = 1 \) such that for every positive integer \( v \) with \( v \sigma_j \geq 2, j = 1; \cdots; s \), the associated \( (\varphi_{t_1; \sigma_1}; \cdots; \varphi_{t_s; \sigma_s}) \)-pair

\[
E; M \rightarrow \bigoplus_{i=1}^{s} \varphi_{\rho_{t_i}} V_{\rho_{t_i}} \bigotimes_{E_{\rho_{t_i}}} \ph \bigotimes_{E_{\rho_{t_i}}} \ph \bigotimes_{E_{\rho_{t_i}}} \ph \bigotimes_{E_{\rho_{t_i}}} \ph
\]

of type \( \langle d; r; v \rangle \) is \( (\delta = v) \)-semistable \((\delta = v)\)-polystable.

Proof. To see the equivalence between 2. and 3., observe that \( \varphi (\sigma_1; \cdots; \sigma_s) \) provides an equivariant embedding of \( \mathbb{P}(V_{t_i}) \rightarrow \mathbb{P}(\mathbb{S}^{\sigma_1} V_{t_i}) \). Via the canonical surjection \( V_{t_i} \twoheadrightarrow \mathbb{S}^{\sigma_1} V_{t_i} \), the latter space becomes embedded into \( \mathbb{P}(V_{t_i}) \). For every point \( x = (x_1; \cdots; x_s) \in \mathbb{P}(V_{t_i}) \), and every one parameter subgroup \( \lambda: \mathbb{C} \rightarrow \text{GL}(r) \)

\[
\sum_{j=1}^{s} \sigma_j \mu_{\rho_{t_j}} \lambda \cdot x_j = \frac{1}{v} \mu_{\rho_{t_i}} \lambda \cdot (x);
\]

the claimed equivalence is easily seen.

For the equivalence between 1. and 3., we have to go into the GIT construction of the moduli space of \( \delta \)-semistable \( \rho \)-pairs. We choose \( a, b, c \), such that \( \rho \) is a direct summand of \( \rho_{a,b,c} \). Therefore, \( \rho_i \) is also a direct summand of \( \rho_{a,b,c} \), \( i = 1; \cdots; t \), so that we can assume \( \rho_i = \rho_{a,b,c} \) for \( i = 1; \cdots; t \). For a tuple \( \langle t_1; \cdots; t_s \rangle \), positive rational numbers \( \sigma_1; \cdots; \sigma_s \), and \( v \in \mathbb{N} \) as in the statement, we thus find

\[
\rho_{t_1; \sigma_1} \cdots \rho_{t_s; \sigma_s} = \rho_{v \sigma_1; \cdots; v \sigma_s}.
\]

for some \( c^0 > 0 \). Recall that in our GIT construction of the moduli space of \( \delta \)-semistable \( \rho_{a,b,c} \) pairs of type \( \langle d; r; v \rangle \), we had to fix some natural number \( n \) which was large enough. Being large enough but not depending on \( a, b, c \) and \( K \) which in turn depended only on \( d, r, a, \) and \( \delta \). One now checks that \( d, r, a, \) and \( \delta = v \) yield exactly the same constants, so that the construction will work also — for all \( v \) and all \( c^0 \) — for \( \rho_{v \sigma_1; \cdots; v \sigma_s} \)-semistable \( \rho_{v \sigma_1; \cdots; v \sigma_s} \)-pairs of type \( \langle d; r; v \rangle \). Fix such an \( n \). We can now argue as follows. Set \( p := d + r + 1 \), and let \( U \) be a complex vector space of dimension \( p \). Given a \( \delta \)-semistable \( \rho_{a,b,c} \)-pair \( (E; M; \tau) \) of type \( \langle d; r; v \rangle \), we can write \( E \) as a quotient \( q: U \rightarrow \mathbb{O}(\n) \) is an isomorphism. Set \( Z := \text{Hom}(U; H^0(\text{det} E \otimes \mathbb{O}(\n))) \) and \( W := \text{Hom}(U_{a,b,c}; H^0(\text{det} E \otimes \mathbb{O}(\n))) \). Then \( q: U \rightarrow \mathbb{O}(\n) \) defines a Gieseker point (\( x: \mathbb{V}_{t_1; \cdots; t_s} \)) which is semistable for the linearization of the SL(\( U \))-action in \( \mathbb{O}(\n) \) with \( e = (p; a, \delta) \). By Theorem 16 we find indices \( t_1; \cdots; t_s \) and positive rational numbers \( \sigma_1; \cdots; \sigma_s \) with \( \sum_{j=1}^{s} \sigma_j = 1 \), such that \( w_j \in 0 \), \( j = 1; \cdots; s \), and the point \( (x; \mathbb{V}_{t_1; \cdots; t_s}) \) is semistable w.r.t. the linearization of the SL(\( U \))-action in \( \mathbb{O}(\n) \). As before, there is an embedding \( t: \mathbb{P}(Z) \rightarrow \mathbb{P}(W) \) such that the pullback of \( \mathbb{O}(\mathbb{V}_{1}) \) is \( \mathbb{O}(\mathbb{V}_{1; \sigma_1; \cdots; \sigma_s}) \). The point \( y := t(\mathbb{V}_{t_1; \cdots; t_s}) \) is thus semistable w.r.t. the linearization in \( \mathbb{O}(\mathbb{V}_{1; \sigma_1; \cdots; \sigma_s}) \). Now, the second component of \( y \)
is defined by the homomorphism \( U_{\text{val}} = U_{\alpha} \neq 1 \neq H^0(\det E \otimes M \otimes \chi(\eta a)) \neq 0 \) obtained from \( \alpha \) and the components \( \tau_{\mathbb{F}_p} \), \( j = 1; \ldots; n \). Composing this homomorphism with the natural map \( H^0(\det E \otimes M \otimes \chi(\eta a)) \neq 0 \neq H^0(\det E \otimes M \otimes \chi(\eta a)) \neq 0 \), we find a point \( y^0 \in \mathbb{P}(\mathcal{Z}^{-}) \mathbb{P}(E) \neq 0 \). Here, we will discuss several properties of the moduli spaces which we have constructed.

3.2 Some features of the moduli spaces

Here, we will discuss several properties of the moduli spaces which we have constructed.

**Trivial specializations**

Let \( \rho: \text{GL}(\mathcal{V}) \to \text{GL}(\mathcal{W}) \) be a representation. Very often, one fixes the determinant of the vector bundles under consideration. So, let \( L_0 \) be a line bundle of degree \( d \). If we want to consider only \( \rho \)-pairs \( (\mathcal{E}; \mathcal{M}; \tau) \) of type \( (d; r; m) \) with \( \det E = L_0 \), we say that the type of \( (\mathcal{E}; \mathcal{M}; \tau) \) is \( (0_0; r; m) \). We then obtain a closed subfunctor \( \mathcal{M}(\rho)_{\delta \in \mathcal{W}} \) of \( \mathcal{M}(\rho)_{\delta \in \mathcal{V}} \). Note that our construction shows that we have a morphism \( \mathcal{M}(\rho)_{\delta \in \mathcal{W}} \to \mathcal{M}(\rho)_{\delta \in \mathcal{V}} \) that the concept of \( \delta \)-semistability is constant within each interval \( (\delta_{v}; \delta_{v+1} \in \mathbb{Q}_{>0}) \), \( \delta_{v} \) and, for given \( \tau \), \( \delta \)-semistability for \( \delta \geq 2 \), \( \delta \in (\delta_{v}; \delta_{v+1} \in \mathbb{Q}_{>0}) \) implies \( \delta_{v} \) and \( \delta_{v+1} \)-semistability and both \( \delta_{v} \) and \( \delta_{v+1} \)-semistability imply \( \delta \)-stability. In particular, there are maps \( \mathcal{M}(\rho)_{\delta \in \mathcal{W}} \to \mathcal{M}(\rho)_{\delta \in \mathcal{V}} \) ("chain of flips", [45]).

1. There is an increasing sequence \( (\delta_{v}; \delta_{v+1} \in \mathbb{Q}_{>0}) \), \( \delta_{v} = 0; 1; 2; \ldots \); which is discrete in \( \mathbb{R} \), such that the concept of \( \delta \)-semistability is constant within each interval \( (\delta_{v}; \delta_{v+1}) \), \( \delta_{v} = 0; 1; 2; \ldots \); and, for given \( \tau \), \( \delta \)-semistability for \( \delta \geq 2 \), \( \delta \in (\delta_{v}; \delta_{v+1}) \) implies \( \delta_{v} \) and \( \delta_{v+1} \)-semistability and both \( \delta_{v} \) and \( \delta_{v+1} \)-semistability imply \( \delta \)-stability. In particular, there are maps \( \mathcal{M}(\rho)_{\delta \in \mathcal{W}} \to \mathcal{M}(\rho)_{\delta \in \mathcal{V}} \) ("chain of flips", [45]).

2. For \( \delta \geq 2 \), \( (\delta; \delta_{0}) \) and \( (\mathcal{E}; \mathcal{M}; \tau) \) a \( \delta \)-semistable \( \rho \)-pair, the vector bundle \( E \) must be semistable, and there is a morphism \( \mathcal{M}(\rho)_{\delta \in \mathcal{W}} \to \mathcal{M}(\rho)_{\delta \in \mathcal{V}} \) to the moduli space of semistable bundles of degree \( d \) and rank \( r \). Conversely, if \( E \) is a stable bundle, then \( (\mathcal{E}; \mathcal{M}; \tau) \) will be \( \delta \)-stable.
3. In the studied examples, there are only finitely many critical values, i.e., there is a $\delta_\text{cr}$, such that the concept of $\delta$-semistability is constant in $(\delta_\text{cr}, \infty)$. We refer to [5], [3], [5], [37] and the examples for explicit discussions of this phenomenon. It would be interesting to know whether this is true in general or not, i.e., to check it for $\rho_{ab}$. We note that in two of the examples, namely the example of oriented framed modules and the example of Hitchin pairs, only a parameter independent stability concept has been treated so far. Our discussions will therefore complete the picture in view of the above observations.

**Asymptotic irreducibility**

Fix the representation $\rho$, the integers $d$ and $r$ as well as the stability parameter $\delta > 0$. Suppose that $\rho$ is a direct summand of the representation $\rho_{ab}$. Since the estimate in Theorem 2.6 does not depend on the integer $m$, we conclude that the set $S$ of isomorphism classes of vector bundles $E$, such that there exist an $m \geq 2\mathbb{Z}$ and a $\delta$-semistable $\rho$-pair $(E; M; \tau)$ of type $(d; r; m)$ is still bounded. The same goes for the set $S_\rho$ of vector bundles of the form $E_\rho$ with $[E] \geq S$. Thus, there is a constant $m_0$, such that for every $m > m_0$ and every $\delta$-semistable $\rho$-pair $(E; M; \tau)$ of type $(d; r; m)$, one has

$$\text{Ext}^1(E_\rho; M) = H^1(E_\rho; M) = 0.$$ 

Our construction and standard arguments [24], 8.5, now show that the natural parameter space for $\delta$-semistable $\rho$-pairs of type $(d; r; m)$ is a projective bundle over the product of a smooth, irreducible, and quasi-projective quotient scheme and the Jacobian of degree $m$ line bundles. In particular, it is smooth and irreducible. We infer

**Theorem 3.5.** Given the data $\rho$, $d$, $r$, and $\delta$ as above, there exists a constant $m_0$ such that the moduli space $M_\rho^{SS}_{d; r; m}$ is a normal and irreducible quasi-projective variety for every $m > m_0$.

**Remark 3.6.** Given $m_0$, $m$ with $m = l > 0$ and a point $p_0 \in X$, the assignment $(E; M; \tau) \mapsto (E; M; \tau)$ induces a closed embedding $M_\rho^{SS}_{d; r; m} \subseteq M_\rho^{SS}_{d; r; m}$.

### 3.3 Extension pairs

Fix positive integers $0 < s < r$, and let $F$ be the Grassmannian of $s$-dimensional quotients of $\mathbb{C}^r$. An $F$-pair is thus a pair $(E; g; q)$, where $E$ is a vector bundle of rank $r$ and $q$ is a homomorphism onto a vector bundle $Q$ of rank $s$. Setting $K := \ker q$, we obtain a pair $(E; K)$ with $E$ as before and $K$ a subbundle of rank $r - s$. These objects were introduced by Bradlow and García-Prada [6] as holomorphic extensions and called (smooth) extension pairs in [9]. In that work, $q$ is not required to be surjective.

We embed $F$ via the Plücker embedding into $\mathbb{P}(\mathbb{V}^s \mathbb{C}^r)$, i.e., we consider the representation $\rho; GL(r) \rightarrow GL(\mathbb{V}^s \mathbb{C}^r)$. To describe the notion of $(\delta, \rho)$-semistability, we observe that for points $[\psi] \in F$, $\mathbb{P}(\mathbb{V}^s \mathbb{C}^r)$, bases $w$ of $\mathbb{C}^r$, and weight vectors $\gamma$, $\gamma$, Equation (9) holds true. Furthermore, for a point $[\psi] \in \mathbb{P}(\mathbb{V}^s \mathbb{C}^r)$, a basis $w = (w_1; \ldots; w_r)$ of $\mathbb{C}^r$, and $i \geq 1; \ldots; r$:

$$\mu_\rho (w; \gamma^{(i)}); [\psi] = i \dim \ker v \\ \dim h w_1; \ldots; w_i \setminus \ker v :$$

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Therefore, according to (10), an $F$-pair $(\mathcal{E};\rho;E)$ is $(\delta;\rho)$-(semi)stable, if and only if for every non trivial proper subbundle $E^0$ of $E$ one has
\[
\mu(\mathcal{E}) + \frac{\delta \cdot \text{rk}(E^0 \setminus \ker q)}{\text{rk}E^0} \geq \mu(E) + \frac{\delta \cdot \text{rk}\ker q}{\text{rk}E};
\]
This is the same notion (9) provides for the extension pair $(E;\ker q)$.

### 3.4 Framed Modules

The case of framed modules is one of the most thoroughly studied examples of a decorated vector bundle problem (see, e.g., [4, 13, 43, 25, 21, 22]).

First, we fix a positive integer $r$, an integer $d$, and a line bundle $M_0$ on $X$ and look at the $\rho$-pairs of type $(d;r;M_0)$ associated with the representation $\rho: \text{GL}(r)$ of $\text{GL}(\mathcal{C}';\mathcal{C}')$, i.e., at pairs $(\mathcal{E};\varphi)$ consisting of a vector bundle $E$ of degree $d$ and rank $r$ and a homomorphism $\varphi:E \to M_0$. For the representation $\rho$, the Additivity Property (9) is clearly satisfied, and given a non-trivial proper subbundle $E^0$ of $E$ one has $\mu_\rho(\mathcal{E};\varphi) = \frac{\text{rk}E^0}{\text{rk}E}$ or $\frac{\text{rk}E^0}{\text{rk}E}$ if $\mathcal{E}$ is $(\text{semi})$-stable.

Given $\delta \geq 0$, Equation (10) thus shows that $(\mathcal{E};\varphi)$ is $(\delta$-(semi)stable, if for every non-trivial proper subbundle $E^0$ of $E$
\[
\mu(\mathcal{E}) + \frac{\delta}{\text{rk}E^0} \geq \mu(E) + \frac{\delta}{\text{rk}E};
\]
Finally, one has the following result on the stability parameter $\delta$:

**Lemma 3.7.** Fix integers $d;r$, $r > 0$, and a line bundle $M_0$. The set of isomorphism classes of vector bundles $E$ for which there exist a parameter $\delta \geq 0$ and a $(\text{semi})$-stable $\rho$-pair of type $(d;r;M_0)$ of the form $(\mathcal{E};\varphi)$ is bounded.

This is proved as Prop. 2.2.2. in [35]. From this boundedness result, it follows easily that the set of isomorphism classes of vector bundles of the form $\ker \varphi$, $(\mathcal{E};\varphi)$ a $\rho$-pair of type $(d;r;M_0)$ for which there exists a $\delta \geq 0$ w.r.t. which it becomes semistable is bounded, too. We infer

**Corollary 3.8.** There exists a positive rational number $\delta_\infty$ such that for every $\delta \geq \delta_\infty$ and every $\rho$-pair $(\mathcal{E};\varphi)$ of type $(d;r;M_0)$, the following conditions are equivalent

1. $(\mathcal{E};\varphi)$ is $(\delta$-(semi)stable.
2. $\varphi$ is injective.

Now, fix a vector bundle $E_0$ on $X$. Recall that a framed module of type $(d;r;E_0)$ is a pair $(\mathcal{E};\psi)$ consisting of a vector bundle $E$ of degree $d$ and rank $r$ and a non-zero homomorphism $\psi:E \to E_0$. Fix a sufficiently ample line bundle $M_0$ on $X$ and an embedding $i:E_0 \to M_0$ for some $s$. Therefore, any framed module $(\mathcal{E};\psi)$ of type $(d;r;E_0)$ gives rise to the $\rho$-pair $(\mathcal{E};\varphi := i \circ \psi)$ of type $(d;r;M_0)$, and the $\rho$-pair $(\mathcal{E};\varphi)$ is $(\delta$-(semi)stable, if and only if $(\mathcal{E};\psi)$ is a $(\delta$-(semi)stable framed module in the sense of [21]. Finally, a family of framed modules of type $(d;r;E_0)$ parameterized by $S$ is a triple $(\mathcal{E}_s;\psi_s;\pi_s)$ consisting of a rank $r$ vector bundle $E_s$ on $S \times X$, a line bundle $\mathbb{N}_S$ on $S$, and a homomorphism $\psi_s:E_s \to \pi_s E_0 \otimes \mathbb{N}_S$ which is non trivial on every fibre $\pi_s X$, $s \in S$. Associate to such a family $(\mathcal{E}_s;\psi_s;\pi_s)$ the family $(\mathcal{E}_s;\kappa_s;\pi_s)\psi_s$ of $\rho$-pairs of type $(d;r;M_0)$ where $\kappa_s(s) = $
\([M_0]\) for all \(s \geq S\) and \(\varphi_S = \psi_X (t) \cdot \text{id}_{N_S} \cdot \psi\). This exhibits the functor associated to a scheme \(S\) the set of equivalence classes of families of \(\delta\)-(semi)stable framed modules of type \((\mathcal{L}_d; \mathcal{M}_0)\) as the subfunctor of \(\mathcal{M}(\mathcal{L}^\delta_{d-r-M_0})\) of those families \((L_S; \kappa_S; N_S; \varphi_S)\) where \(\kappa_S\) is the constant morphism \(s \mapsto \mathcal{M}_0\) and the composite \(E_S \rightarrow \mathcal{L}_S \rightarrow \mathcal{N}_S \rightarrow \mathcal{M}_0\) vanishes. Since all these conditions are closed conditions, the moduli spaces of \(\delta\)-(semi)stable framed modules on curves \([45], [21]\) become closed subschemes of our moduli spaces \(\mathcal{M}(\mathcal{L}^\delta_{d-r-M_0})\).

**Remark 3.9.** We have used a slightly different, more general notion of family than \([21]\). This choice only destroys the property of being a fine moduli space and does not affect the construction of the moduli space of framed modules.

### 3.5 Oriented framed modules

We begin with the representations \(\rho_1: \text{GL}(\mathfrak{g}) \rightarrow \text{GL}(\text{Hom}(C'; C'))\) and \(\rho_2: \text{GL}(\mathfrak{g}) \rightarrow \text{GL}(C')\), and set \(\rho := \rho_1 \cdot \rho_2\). Fix line bundles \(L_0\) and \(M_0\). Then, a \(\rho\)-pair of type \((\mathcal{L}_d; \mathcal{M}_0)\) is a triple \((E; \rho; \sigma)\), consisting of a vector bundle \(E\) of rank \(r\) with \(\det E = L_0\), a homomorphism \(\varphi: \mathcal{S}^r E \rightarrow M_0\), and a homomorphism \(\sigma: \det E \rightarrow M_0\). Next, assume we are given a line bundle \(N_0\) with \(N_0^r = M_0\) and \(t \in \mathbb{C}\) such that \(s = \# \{ \gamma_1; \cdots; \gamma_i \} \leq t \leq \# \{ \gamma_1; \cdots; \gamma_i; \cdots; \gamma_i \leq t \} = r\), i.e., \(S' N_0^r = M_0\). Then, to any triple \((E; \psi; \sigma)\) where \(E\) is a vector bundle of rank \(r\) with \(\det E = L_0\) and \(\psi: E \rightarrow N_0^r\), we can associate the \(\rho\)-pair \((E; \mathcal{S}^r \psi; \sigma)\) of type \((\mathcal{L}_d; \mathcal{M}_0)\).

Observe that for any weighted filtration \((\mathcal{E}; \mathcal{G})\) one has

\[
\mu_{\rho_2} E : \mathcal{G}; \sigma = 0 \quad \text{and} \quad \mu_{\rho_1} E : \mathcal{G}; \mathcal{S}^r \psi = r \quad \mu_{\varphi} E : \mathcal{G}; \psi
\]

where \(\rho_1^\mathfrak{g}: \text{GL}(\mathfrak{g}) \rightarrow \text{GL}(\text{Hom}(C'; C'))\). Therefore, Proposition \([52]\) and the discussion of framed modules show

**Lemma 3.10.** Let \((E; \psi; \sigma)\) be a triple where \(E\) is a vector bundle of rank \(r\) with \(\det E = L_0\) and \(\psi: E \rightarrow N_0^r\) and \(\sigma: \det E \rightarrow N_0^r\) are homomorphisms, and \(\delta > Q_0\). Then, the following conditions are equivalent:

1. The associated \(\rho\)-pair \((E; \mathcal{S}^r \psi; \sigma)\) of type \((\mathcal{L}_d; \mathcal{M}_0)\) is \(\delta\)-(semi)stable.

2. One of the following three conditions is verified:
   
   i. \(E\) is a semistable vector bundle.

   ii. The homomorphisms \(\psi\) and \(\sigma\) are non-zero and there exists a positive rational number \(\delta_0\) such that \((E; \psi)\) is a \(\delta_0\)(semi)stable \(\rho_1^\mathfrak{g}\)-pair of type \((\mathcal{L}_d; \mathcal{M}_0)\).

   iii. The homomorphism \(\sigma\) vanishes and \((E; \psi)\) is an \((\psi; \delta)\)-(semi)stable \(\mathbb{P}\)-pair of type \((\mathcal{L}_d; \mathcal{M}_0)\).

We omit the “polystable version” of this Lemma. In particular, for \(r \delta > \delta_\infty\) (cf. Corollary \([38]\)), one finds

**Corollary 3.11.** Let \((E; \psi; \sigma)\) be a triple where \(E\) is a vector bundle of rank \(r\) with \(\det E = L_0\) and \(\psi: E \rightarrow N_0^r\) and \(\sigma: \det E \rightarrow N_0^r\) are homomorphisms, and \(\delta > \delta_\infty\). Then, the following conditions are equivalent:

1. The associated \(\rho\)-pair \((E; \mathcal{S}^r \psi; \sigma)\) of type \((\mathcal{L}_d; \mathcal{M}_0)\) is \(\delta\)-(semi)stable.

2. One of the following three conditions is verified:
i. $E$ is a semistable vector bundle.

ii. The homomorphisms $\psi$ and $\sigma$ are non-zero and there exists a positive rational number $\delta^0$, such that $(E;\psi)$ is a $\delta^0$-semistable $\rho_1^0$-pair of type $(\mathcal{L}_0;\mathcal{L};\mathcal{N}_0)$.

iii. The homomorphism $\sigma$ vanishes and $\psi$ is injective.

Now, we turn to the moduli problem we would like to treat. For this, we fix a line bundle $L_0$ and a vector bundle $E_0$. Then, an oriented framed module of type $(\mathcal{L}_0;\mathcal{L};E_0)$ is a triple $(E;\varepsilon;\psi)$ where $E$ is a vector bundle of rank $r$ with $\det E = L_0$ and $\varepsilon: \det E \to L_0$ and $\psi: E \to E_0$ are homomorphisms, not both zero. The corresponding moduli problem was treated in [35]. Over curves, we can recover it from our theory in the following way: If $N_0$ is sufficiently ample, there are embeddings $t_1: L_0 \to N_0^r$ and $t_2: E_0 \to N_0^r$. Thus, setting $M_0 := N_0^r$, we can define $\sigma := t_1: \varepsilon: \det E \to M_0$ and $\varphi := \sigma^r \in \operatorname{End}(\mathcal{L}_0;\mathcal{L};\mathcal{N}_0)$. By Corollary 3.11 for $\delta = \delta_0 = r$, the $\rho$-pair $(E;\varphi;\sigma)$ is $\delta$-semistable if and only if $(E;\varepsilon;\psi)$ is a semistable oriented framed module in the sense of [35].

**Remark 3.12.** The corresponding stability concept can be recovered via Proposition 3.4 and the characterisation “stable=polystable+simple” (Remark [24][ii]).

We conclude by observing that applying Lemma 3.10 yields new semistability concepts for oriented framed modules.

### 3.6 Hitchin Pairs

The theory of Hitchin pairs or Higgs bundles is also a famous example of a decorated vector bundle problem ([20], [43], [11], [34], [47], [19], [36]).

To begin with, we fix integers $d$ and $r > 0$, a line bundle $M_0$, and the representation $\rho: \mathbb{G} \to \mathbb{G}$. In this case, a $\rho$-pair of type $(\mathcal{L};\mathcal{L};M_0)$ is a triple $(E;\varphi;\sigma)$ consisting of a vector bundle $E$ of degree $d$ and rank $r$, a twisted endomorphism $\varphi: E \to E$ of $M_0$, and a section $\sigma: \mathcal{O}_X \to M_0$.

**Lemma 3.13.** There is a positive rational number $\delta_0$, such that for all $\delta < \delta_0$ and all $\rho$-pairs $(E;\varphi;\sigma)$ of type $(\mathcal{L};\mathcal{L};M_0)$ the following conditions are equivalent:

1. $(E;\varphi;\sigma)$ is a $\delta$-(semi)stable $\rho$-pair

2. for every non trivial subbundle $E^0$ of $E$ with $\varphi(E^0) \to M_0$

   \[ \mu(E^0) \to \mu(E); \]

   and either $\sigma \neq 0$ or $\varphi$ is not nilpotent, i.e., $(\varphi \mid_{M_0^{r+1}}) \neq 0$.

**Proof.** First, assume 1. Let $f: \mathbb{C}^r \to \mathbb{C}^r$ be a homomorphism. Call a sub vector space $V \subseteq \mathbb{C}^r$ $f$-superinvariant, if $V \ker f$ and $f(V) = V$.

**Lemma 3.14.** Let $[f;\varepsilon] \in \mathbb{P}((\mathbb{C}^r;\mathbb{C}^r) \to \mathbb{C})$. Given a basis $w = (w_1; \ldots; w_r)$ of $W$ and $i \neq 1; \ldots; r$ let set $W_0^{(i)} := h w_1; \ldots; h w_i$. Then

i. $\mu_p \lambda([w_0;\gamma^{(i)}];[f;\varepsilon]) = r$, if $W_0^{(i)}$ is not $f$-invariant.

ii. $\mu_p \lambda([w_0;\gamma^{(i)}];[f;\varepsilon]) = r$, if $W_0^{(i)}$ is $f$-superinvariant and $\varepsilon = 0$.

iii. $\mu_p \lambda([w_0;\gamma^{(i)}];[f;\varepsilon]) = 0$ in all the other cases.
Now, let $(E; \varphi; \sigma)$ be a $\rho$-pair of type $(\mathcal{L}; \tau; \mathcal{M}_0)$. For any subbundle $E^0$ of $E$ with $\varphi (E^0) = E^0 \mathcal{M}_0$, we find $\mu_\rho (E^0; (\varphi; \sigma)) = 0$.

**Corollary 3.15.** Let $\delta > 0$ and $(E; \varphi; \sigma)$ a $\delta$-(semi)stable $\rho$-pair of type $(\mathcal{L}; \tau; \mathcal{M}_0)$. Then $\mu (E^0) (\mu (E))$ for every non-trivial proper subbundle $E^0$ of $E$ with $\varphi (E^0) = E^0 \mathcal{M}_0$.

This condition implies that for every $\delta > 0$, every $\delta$-semistable $\rho$-pair $(E; \varphi; \sigma)$ of type $(\mathcal{L}; \tau; \mathcal{M}_0)$, and every subbundle $E^0$ of $E$

$$\mu (E^0) \max \mu (E) \mu (E) + \frac{(r^2 - 1)}{r} \deg \mathcal{M}_0$$

See, e.g., [34]. Therefore, the set of isomorphy classes of bundles $E$, such that there exist a positive rational number $\delta$ and a $\delta$-semistable $\rho$-pair of type $(\mathcal{L}; \tau; \mathcal{M}_0)$ of form $(E; \varphi; \sigma)$, is bounded.

Now, the only thing we still have to show is that for every sufficiently large positive rational number $\delta$ and every $\delta$-semistable $\rho$-pair $(E; \varphi; \sigma)$ of type $(\mathcal{L}; \tau; \mathcal{M}_0)$, such that $\sigma = 0$, the homomorphism $\varphi$ can't be nilpotent. First, let $(E; \varphi; \sigma)$ be a $\rho$-pair of type $(\mathcal{L}; \tau; \mathcal{M}_0)$, such that there exists a positive rational number $\delta$ w.r.t. which $(E; \varphi; \sigma)$ is semistable and such that $\varphi$ is nilpotent. Then, there is a filtration

$$0 = : E_0 \rightarrow \cdots \rightarrow E_{s} := E$$

with $E_j \mathcal{M}_0 = \varphi (E_{j+1})$, $j = 0; \ldots; s$. It is clear by the boundedness result that the $E_j$'s occurring in this way live in bounded families, so that we can find a positive constant $C$ with

$$dr E_j \deg E_j r < C; \quad j = 1; \ldots; s$$

for all such filtrations. One checks $\mu_\rho (E; (1; \ldots; 1); (\varphi; \sigma)) = r$, so that the semistability assumption yields

$$0 \leq M E ; (1; \ldots; 1) + \delta \mu_\rho E ; (1; \ldots; 1); (\varphi; \sigma)$$

This is impossible if $\delta < C$.

To see the converse, let $(E; \varphi; \sigma)$ be a $\rho$-pair satisfying 2. Let $m_0 := \max \{\deg \mathcal{M}_0 r \geq 1\} r = r > 0$. Then, as before, $\mu (E^0) \mu (E^0) + m_0$ for every non-trivial proper subbundle $E^0$ of $E$, i.e., $dr E^0 m_0 r E^0 m_0$ ($r \leq 1$). First, consider a weighted filtration $(E; \alpha)$ such that $\varphi (E_j) E_j \mathcal{M}_0$, $j = 1; \ldots; s$. Then, the condition that $\varphi$ be not nilpotent if $\sigma = 0$ implies $\mu_\rho (E; \alpha; (\varphi; \sigma)) = 0$, so that $M (E; \alpha; (\varphi; \sigma)) = 0$ follows from 2. Second, suppose that we are given a weighted filtration $(E; \alpha)$ such that, say, $E_j ; \ldots; E_j$ are not invariant under $\varphi$, i.e., $\varphi (E_j) \mathcal{M}_0 i = 1; \ldots; t$, and $t > 0$. Let $\alpha := \max \{\alpha_j; \ldots; \alpha_j, \sigma\}$. One readily verifies $\mu_\rho (E; \alpha; (\varphi; \sigma)) = r = r > 0$. We thus find

$$M E ; (\alpha) + \delta \mu_\rho E ; (\alpha) \leq \sum_{i=1}^{t} \alpha_j d r E_j \deg E_j + r \alpha \delta$$

so that $M E ; (\alpha) + \delta \mu_\rho E ; (\alpha)$ will be positive if we choose $\delta > r \geq 1$.

□
Example 3.16. For small values of $\delta$, the concept of $\delta$-(semi)stability seems to become rather difficult. However, in the rank two case we have: A $\rho$-pair $(E; \varphi; \sigma)$ of type $(d; 2; M_0)$ is $\delta$-(semi)stable if for every line subbundle $E^0$ of $E$ one has

1. $\deg E^0(\_d)_d = 2 + \delta$,
2. $\deg E^0(\_d)_d = 2$ if $E^0$ is invariant under $\varphi$,
3. $\deg E^0(\_d)_d = 2$ if $E^0 = \ker \varphi, \varphi(E)$ $E^0 M_0$, and $\sigma = 0$.

Fix a line bundle $L$ on $X$. We remind the reader \[36\] that a Hitchin pair of type $(\mathcal{U}; \mathcal{V})$ is a triple $(E; \psi; \epsilon)$ where $E$ is a vector bundle of degree $d$ and rank $r$, $\psi: E \to L$ is a twisted endomorphism, and $\epsilon$ is a complex number. Two Hitchin pairs $(E_1; \psi_1; \epsilon_1)$ and $(E_2; \psi_2; \epsilon_2)$ are called equivalent, if there exist an isomorphism $h: E_1 \to E_2$ and a non zero complex number $\lambda$ with $\lambda \psi_1 = (h \cdot \text{id}_L)^1 \psi_2$ and $\lambda \epsilon_1 = \epsilon_2$. We fix a point $x_0$ and choose $n$ large enough, so that $M_0 := L(x_0)$ has a non trivial global section. Fix such a global section $\sigma_0; O_X = M_0$ and an embedding $:L \to M_0$. To every Hitchin pair $(E; \psi; \epsilon)$ of type $(\mathcal{U}; \mathcal{V})$, we can assign the $\rho$-pair $(E; \varphi; \sigma)$ with $\varphi := (\delta) \psi$ and $\sigma = \epsilon \sigma$. Note that this assignment is compatible with the equivalence relations. By Lemma \[5.15\] for $\delta$, the $\rho$-pair $(E; \varphi; \sigma)$ is $\delta$-(semi)stable if and only if $(E; \psi; \epsilon)$ is a (semi)stable Hitchin pair in the sense of \[36\]. Again, the above assignment carries over to families, so that the general construction also yields a construction of the moduli space of semistable Hitchin pairs on curves, constructed in \[36\] and \[19\]. This space is a compactification of the “classical” Hitchin space \[20\], \[11\], \[34\].

As we have seen, the semistability concept for Hitchin pairs is parameter dependent in nature, though it might be difficult to describe for low values of $\delta$. To illustrate that we get new semistable objects for small values of $\delta$, let us look at an

Example 3.17. i) Let $x_0 \to X$ be a point, and set $O(1) := O_X(1)$. Define $E := O(1)$, and $\psi: E \to O(1) = O(1) \subset O(2)$ as the homomorphism whose restriction to $O$ is zero and, moreover, the induced homomorphisms $O(1) \to O(1)$ and $O(1) \to O(2)$ are the identity and zero, respectively. First, consider the Hitchin pair $(E; \psi; 1)$, then, the third condition in \[3.16\] is void and the second condition is satisfied. Indeed, a $\psi$-invariant subbundle $E^0$ of $E$ of rank one cannot be contained in $O(1)$, whence $\deg E^0 = 0 < 1$ is $E^0$ has degree at most one, and $E^0 := O(1)$ is a subbundle of degree exactly one. The first condition then reads $1(1) = 2 + \delta$. In other words, $(E; \psi; 1)$ is $\delta$-stable for $\delta > 1$, properly $(1; 2)$-semistable, and not semistable for $\delta < 1$. Finally, we claim that $(E; \varphi; 0)$ is properly $(1; 2)$-semistable (although $\psi$ is nilpotent). For this, we only have to check the condition for $E^0 := O$, i.e., $0 < 1$, and this is clearly satisfied.

ii) To see the role of $\delta$ in the whole theory, let us look at Hitchin pairs of type $(1; 2; \omega_X)$. Let $\delta_0$ be as in Lemma \[5.13\]. For $\delta < \delta_0$, denote by $H \omega_X$ the moduli space of stable (in the usual sense) Hitchin pairs of type $(1; 2; \omega_X)$. Let $\delta = \delta_0: \delta < \delta_0$, the moduli space of $\delta$-stable Hitchin pairs of type $(1; 2; \omega_X)$ equals $P(O_N \cong T_N)$, the compactified cotangent bundle of $N$, the moduli space of stable rank two bundles of degree one. Furthermore, let $M_{\delta_X}$ be the moduli space of $\delta$-stable Hitchin pairs of type $(1; 2; \omega_X)$ where $\delta \leq (\delta_1; \delta_2)$, $i = 0; \cdots; m$, and $M^{1}_{\delta_X}$ the moduli space of $\delta$-semistable Hitchin pairs of type $(1; 2; \omega_X)$, $i = 0; \cdots; m$. Between these spaces, we have morphisms

\[
\begin{array}{cccc}
P(O_N \cong T_N) & \cong & M^{0}_{\omega_X} & \cong & M^{1}_{\omega_X} \\
& & & \cong & H \omega_X \\
N & \cong & M^{0}_{\omega_X} & \cong & M^{1}_{\omega_X} \\
\end{array}
\]
As in [45], this is the factorization of the birational correspondence $\mathbb{P}(O_M \otimes T_M) \to \mathbb{P}(\mathcal{E})$ into flips and is thus related to the factorization into blow ups and downs (cf. [19]).

**Remark 3.18 (A. Teleman).** It might seem odd that we also obtain new semistability concepts for the classical Higgs bundles $(\mathcal{E};\phi)$ where the semistability concept is known to be parameter independent. In gauge theory, the reason is that, for studying Higgs bundles, one fixes a flat metric of infinite volume on the fibre $F = \text{End} \left( C' \right)$ whereas we use a metric of bounded volume induced by the embedding $\text{End} \left( C' \right) \to \mathbb{P}(\text{End} \left( C' \right) \otimes \mathcal{O})$ which yields a different moment map. If we let the parameter $\delta$ tend to infinity, we approximate the flat metric and therefore recover the parameter independent semistability concept.

The related moduli problems of framed and oriented framed Hitchin pairs discussed in [44] and [57] can also be dealt with in our context. We leave this to the interested reader.

### 3.7 Conic bundles

Consider the representation $\rho: \text{GL}(\mathfrak{g}) \to \text{GL}(S^2C')$ and fix a line bundle $M_0$ on $X$. A $\rho$-pair of type $(\mathfrak{g};\phi;M_0)$ is thus a pair $(\mathcal{E};\phi)$ consisting of a vector bundle $E$ of rank $r$ and degree $d$ and a non-zero homomorphism $\rho:S^2E \to M_0$. For $r = 3$, these objects have been studied in [15]. We apply Theorem 3.3 to analyze the notion of semistability, using slightly different notation.

To simplify the stability concept, we have to understand the weights occurring for the action of $\text{SL}(\mathfrak{g}) \to \mathbb{P}(S^2C')$. For this, let $\mathcal{F} \subset \mathbb{P}(S^2C')$ be a point represented by the linear form $\mathcal{F}$.

**Remark 3.18** Given subbundles $F_1$ and $F_2$, we write $F_1 \sim F_2$ for the subbundle of $S^2E$ generated by local sections of the form $f_1 \otimes f_2$ where $f_i$ is a local section of $F_i$, $i = 1, 2$. For any non-trivial proper subbundle $E'$ of $E$, one sets

$$\varepsilon \left( E^2 \right) = 2,$$  if $\tau_{E'}^E \not\equiv 0$,

$$\varepsilon \left( E^2 \right) = 1,$$  if $\tau_{E'}^E \equiv 0$ and $\tau_{E'}^E \not\equiv 0$,

$$\varepsilon \left( E^2 \right) = 0,$$  if $\tau_{E'}^E \equiv 0$.  


One checks
\[ \mu_\rho \; E_0; \tau \; = \; c_\tau \; (E_0) \text{rk} \; E \; 2 \text{rk} \; E_0; \] (13)
Finally, call a filtration \( E : 0 \; E_1 \; E_2 \; E \) with \( \text{rk} \; E_i = i, \; i = 1; 2 \), critical, if \( \tau_{E_1} \; \rho = 0 \), and \( \tau_{E_1} \; E \) and \( \tau_{E_2} \; \rho \) are both non-zero. Then
\[ \mu_\rho \; E \; (1; 1) ; \tau \; = \; 0; \]
Putting everything together, we find

**Lemma 3.19.** A \( \rho \)-pair \( (E ; \tau) \) of type \( \psi_1 ; 3; M_0 \) is \( \delta \)-(semi)stable if and only if it satisfies the following two conditions

1. For every non-zero proper subbundle \( E_0 \) one has
   \[ \mu \; (E_0) \; \delta \frac{c_\tau \; (E_0)}{\text{rk} \; E_0} \; ( ) \; \mu \; (E) \; \delta \frac{2}{3}; \]
2. For every critical filtration \( 0 \; E_1 \; E_2 \; E \)
   \[ \text{deg} \; E_1 + \text{deg} \; E_2 \; ( ) \; \text{deg} \; E; \]

This is the stability condition formulated by Gómez and Sols [15]. Next, we look at the case \( r = 4 \).

Set \( \nu \; (w;l) := \min \; i_1 + i_2 \; \nu \; w_{i_1/i_2} \; 0; \; \psi_1 \; i_2) \; 2 \; l \); Suppose we are given a linear form \( l:S^2 \mathbb{C}^4 \to \mathbb{C} \). Then, for a basis \( w = (w_1; \ldots; w_4) \), we have \#(w;l) = 1 except for the following cases

1. \( \nu \; (w;l) = 4, \; l \; (w)_{12} \; 0 \) and \( l \; (w)_{13} \; 0 \),
2. \( \nu \; (w;l) = 4, \; l \; (w)_{12} \; 0, \; l \; (w)_{13} = 0, \) and \( l \; (w)_{14} \; 0 \),
3. \( \nu \; (w;l) = 5, \; l \; (w)_{14} \; 0, \) and \( l \; (w)_{23} \; 0 \),
4. \( \nu \; (w;l) = 5, \; l \; (w)_{14} \; 0, \; l \; (w)_{23} = 0, \) and \( l \; (w)_{33} \; 0 \),
5. \( \nu \; (w;l) = 6, \; l \; (w)_{24} \; 0, \) and \( l \; (w)_{33} \; 0 \).

Straightforward computations show

**Lemma 3.20.** i) In case 1., \( C_{13} \; (w;l) \) is generated by \( \gamma^{(1)} , \gamma^{(3)}, \) and \( \gamma^{(1)} + \gamma^{(3)} \), and \( C_{23} \; (w;l) \) by \( \gamma^{(2)} \), \( \gamma^{(3)} \), and \( \gamma^{(1)} + \gamma^{(3)} \).
   ii) In case 2., \( C_{14} \; (w;l) \) is generated by \( \gamma^{(3)}, \gamma^{(1)} + \gamma^{(3)}, \) and \( \gamma^{(2)} + \gamma^{(3)} \), and \( C_{23} \; (w;l) \) by \( \gamma^{(1)} + \gamma^{(3)} \), \( \gamma^{(2)} + \gamma^{(3)} \), and \( \gamma^{(1)} + \gamma^{(3)} \).
   iii) In case 3., \( C_{14} \; (w;l) \) is generated by \( \gamma^{(3)}, \gamma^{(2)} \), and \( \gamma^{(1)} + \gamma^{(3)} \), and \( C_{23} \; (w;l) \) by \( \gamma^{(1)} + \gamma^{(3)} \), \( \gamma^{(2)} + \gamma^{(3)} \), and \( \gamma^{(1)} + \gamma^{(3)} \).
   iv) In case 2., \( C_{14} \; (w;l) \) is generated by \( \gamma^{(2)} \), \( \gamma^{(3)} \), \( \gamma^{(1)} + \gamma^{(3)} \), and \( \gamma^{(3)} \), and \( C_{33} \; (w;l) \) by \( \gamma^{(1)} \), \( \gamma^{(3)} \), and \( \gamma^{(1)} + \gamma^{(3)} \).
   v) In case 5., \( C_{24} \; (w;l) \) is generated by \( \gamma^{(1)}, \gamma^{(2)} \), and \( \gamma^{(2)} + \gamma^{(3)} \), and \( C_{33} \; (w;l) \) by \( \gamma^{(1)} \), \( \gamma^{(3)} \), and \( \gamma^{(2)} + \gamma^{(3)} \).
Now, let \((E; \tau)\) be a \(\rho\)-pair of type \((d; 4; M_0)\). For any non-zero, proper subbundle \(E^0\) of \(E\), we define \(c_\tau(E^0)\) as before. One checks that (13) remains valid. Call a filtration \(0 \to E_1 \to E_2 \to E_3 \to E\) with \(\text{rk} E_i = i\) critical of type (I), (II), (III), (IV), (V), if

(I) \(\tau_{E_1} E = 0\), and \(\tau_{E_1} E\) and \(\tau_{E_2} E\) are both non-zero;

(II) \(\tau_{E_1} E = 0\), and \(\tau_{E_1} E\) and \(\tau_{E_2} E\) are both non-zero;

(III) \(\tau_{E_1} E = 0\), \(\tau_{E_2} E = 0\), and both \(\tau_{E_1} E\) and \(\tau_{E_2} E\) are non-zero;

(IV) \(\tau_{E_2} E = 0\), and both \(\tau_{E_1} E\) and \(\tau_{E_3} E\) are non-zero;

(V) \(\tau_{E_3} E = 0\), \(\tau_{E_2} E = 0\), and both \(\tau_{E_1} E\) and \(\tau_{E_3} E\) are non-zero.

respectively. In these cases, one has

\[
\mu(H) 0 \quad E_1 \quad E_2 \quad E; \quad (1; 1); \quad \tau = 2 \quad \text{for type (I), (IV)}
\]

\[
\mu(H) 0 \quad E_1 \quad E_3 \quad E; \quad (1; 1); \quad \tau = 0 \quad \text{for type (II), (III), (IV)}
\]

\[
\mu(H) 0 \quad E_2 \quad E_3 \quad E; \quad (1; 1); \quad \tau = 2 \quad \text{for type (II), (V)}
\]

Gathering all information, we find

**Lemma 3.21.** The \(\rho\)-pair \((E; \tau)\) of type \((d; 4; M_0)\) is \(\delta\)-(semi)stable if and only if it satisfies the following two conditions

1. For every non-zero proper subbundle \(E^0\) one has

\[
\mu(E^0) \quad \frac{c_\tau(E^0)}{\text{rk} E^0} \quad \mu(E) \quad \frac{\delta}{2}:
\]

2. For every critical filtration \(0 \to E_1 \to E_2 \to E_3 \to E\)

\[
4 \deg E_1 + 4 \deg E_2 \quad (\text{if it is of type (I), (IV)}
\]

\[
\deg E_1 + \deg E_3 \quad \deg E_2 \quad \text{if it is of type (II), (III), (IV)}
\]

\[
4 \deg E_2 + 4 \deg E_3 \quad (\text{if it is of type (II), (V)}:
\]

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