Geometrically non-linear forced vibrations of Euler–Bernoulli laminated composite beams

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Abstract. The objective of this research is to study geometrically non-linear free vibrations and forced vibrations subjected to harmonic excitations of laminated composite beams. On the basis of Euler Bernoulli's beam theory and Green-Lagrange's hypothesis of geometric non-linearity, the theoretical model has been established. Taking into account the harmonic response, the transverse displacement function of the non-linear beam determined by applying Hamilton's principle, the problem is reduced to a non-linear algebraic system solved by an approximate method. In order to verify this method, numerical examples for carbon/epoxy materials, under two boundary conditions, i.e. clamped-free and clamped-clamped, have been performed and the results are very consistent with those obtained in the literature. In addition, based on the approximate multimode method in the vicinity of the predominant mode, a non-linear forced response was performed for a wide range of vibration amplitudes. It should also be noted that the effects on the non-linear forced dynamic response of the fiber orientation and number of layers, the excitation frequency and the level of the applied harmonic force have been studied and illustrated by various examples.

1. Introduction

The dynamic analysis of structures is an important part of industrial and engineering applications: in large deformations, the vibratory behavior of the beam will not occur in direct proportion to the large amplitude. Make linear calculation impossible. It is with this non-linear analysis in mind that researchers are seeking to develop better and simpler analytical solutions with many advantages by using vibration control and modeling of these structures.

Studies of the literature have shown that under various hypotheses, a great deal of recent research has been carried out on the non-linear geometrical vibration of composite beams. Singh and Al [1] studied the large amplitude free vibration of an asymmetrical laminated beam based on von-Karman theory using one-dimensional finite elements based on classical lamination theory. Marur [2] studied the analysis of the non-linear vibration of beams. Boay and Wee [3] used the classical lamination beam theory to study the effect of coupling of laminated corner layers of composite beams on bending, buckling and vibration. Closed form solutions of embedded beams and simply supported beams are obtained, and the results are compared with the finite element method. In [4], the dynamic behavior of a fully clamped rectangular plate with a high amplitude vibration was analyzed. In another study, Harras and Benamar [5] studied the geometrically non-linear free vibration behavior of composite laminated plates. Moreover, the experimental measurements performed in this study show that they
are in good agreement with the theoretical analysis. In [6], the non-linear lateral vibration of Bernoulli-Euler beams was studied, and the existence of a limited number of masses at arbitrary positions and various types of springs held at both ends was investigated. A functionally graded carbon nanotube-reinforced composite (CNTRC) beam with surface-bonded piezoelectric layers was considered in [7], and a comparison between the different volume fractions of the CNT was performed, emphasizing the significant influence of the CNTRC piezoelectric beams on the vibratory behavior. More recently, the author of [8] studied the non-linear forced dynamic response for FGM beams containing multiple cracks. The analytical results for high amplitude vibration show that increasing the depth of the crack leads to a significant increase in the vibration amplitude of the beam.

In the present work, the objective is to apply the theoretical model developed in references [9-11] to analyze the nonlinear forced vibrations of laminated composite beams. The mathematical model is based on the Euler-Bernoulli beam theory and the Green-Lagrange non-linear hypothesis. Based on Hamilton's principle and spectral analysis, a general formula for the nonlinear vibration model of composite laminated beams is proposed. The nonlinear algebraic equations are solved using an approximate computational method called the second formula, which is used to facilitate the numerical analysis of nonlinear problems [12]. The nonlinear vibrations of the clamped-clamped (C-C) and clamped-free (C-F) beams in unidirectional orientation of the layers are studied, and the current solution is compared with the previous solution obtained by the Generalized Differential Quadrature Method (GDQM). In addition, uniformly distributed harmonic excitations and dynamic responses of different orientations of symmetry of the layers are obtained and compared to each other.

2. Theoretical formulations

2.1. Expression of non-linear problems
Consider the laminated composite beam shown in figure 1. The rectangular cross-section of the beam has a length L and a thickness H.

![Figure 1. Laminated composite beam notation.](image)

According to the literature [13], non-linear geometric vibrations are divided into two main groups. On the one hand, the vibration modeled with a large displacement and a small deformation is tested under the Von-Karman hypothesis, on the other hand, it is the vibration affected by a finite deformation (rotation without displacement), which is defined as the Green-Lagrange hypothesis. Then [14] gives the first term of the Green-Lagrange deformation tensor up to the second-order term:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} - y \frac{\partial^2 v}{\partial x^2} - z \frac{\partial^2 w}{\partial x^2} + \frac{1}{2} \left( \frac{\partial u}{\partial x} - y \frac{\partial^2 v}{\partial x^2} - z \frac{\partial^2 w}{\partial x^2} \right)^2 \frac{1}{2} \left( \frac{\partial u}{\partial x} - z \frac{\partial v}{\partial x} \right)^2 + \frac{1}{2} \left( \frac{\partial w}{\partial x} - y \frac{\partial \theta_x}{\partial x} \right)^2$$

(1)

In the Green-Lagrange deformation tensor formula, the transverse displacement of the beam plane is considered, while the longitudinal displacement of this plane is ignored. References [13] and [14] have adopted this assumption and are often used in the literature. The first term of the Green-Lagrange deformation tensor ignoring the longitudinal displacement is obtained as follows:

$$\varepsilon_{xx} = -z \frac{\partial^2 w}{\partial x^2} + \frac{1}{2} \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2$$

(2)
The total elastic deformation energy \( V \) of Euler-Bernoulli beams is defined by:

\[
V = \frac{1}{2} \int \sigma_{xx} \varepsilon_{xx} \, dv
\]

The stress resultants for the composite beam have been defined as adopted with reference [15] by:

\[
N_x = \int \sigma_{xx} \, ds, \quad M_x = \int z \sigma_{xx} \, ds, \quad P_x = \int z^2 \sigma_{xx} \, ds
\]

Where \( N_x, M_x \) and \( P_x \) are respectively the internal axial force, the bending moment and the high-order resultants. They are calculated on the basis of the stiffness components of laminated composite materials, and their definitions are as used in references [16] and [17]:

\[
N_x = A_{11} \left( \frac{1}{2} \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \right) + D_{11} \left( \frac{1}{2} \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \right), \quad M_x = D_{11} \left( - \frac{\partial^2 w}{\partial x^2} \right) + F_{11} \left( \frac{1}{2} \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \right)
\]

Where \( A_{11}, D_{11} \) and \( F_{11} \) are respectively extension stiffness, bending stiffness and additional stiffness their expressions are defined as follows:

\[
A_{11} = \int_{-H/2}^{H/2} \frac{Q_{11}}{z} \, dz, \quad D_{11} = \int_{-H/2}^{H/2} \frac{Q_{11} z^2}{z} \, dz, \quad F_{11} = \int_{-H/2}^{H/2} \frac{Q_{11} z^4}{z} \, dz
\]

The transformed reduced stiffness constant \( \tilde{Q}_{11} \) is given by:

\[
\tilde{Q}_{11} = Q_{11} \cos^2 \theta + 2(Q_{12} + 2Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{22} \sin^4 \theta
\]

Where \( \theta \) is the orientation angle of the fiber with respect to the longitudinal direction (the x-axis). When the fiber is oriented clockwise, the angle is positive, although it is negative counterclockwise for orientation. The reduced stiffness constants \( Q_{11}, Q_{22}, Q_{12} \) and \( Q_{66} \) can be written as follows:

\[
Q_{11} = \frac{E_1}{1 - \nu_{12} \nu_{21}}, \quad Q_{22} = \frac{\nu_{12} E_2}{1 - \nu_{12} \nu_{21}}, \quad Q_{12} = \frac{E_2}{1 - \nu_{12} \nu_{21}}, \quad Q_{66} = G_{12}
\]

\[\text{Figure 2. Geometry of laminated composite beams.}\]

The integral of equation (6) must be applied to the \( k \)th layer of the beam and then added. Where \( \tilde{Q}_{11k} \) is the reduced stiffness after transformation, and \( z_k \) is the height of the \( k \)th layer, as shown in figure 2.

\[
A_{11} = \sum_{k=1}^{n} \tilde{Q}_{11k} (z_k - z_{k-1}), \quad D_{11} = \frac{1}{3} \sum_{k=1}^{n} \tilde{Q}_{11k} (z_k^3 - z_{k-1}^3), \quad F_{11} = \frac{1}{5} \sum_{k=1}^{n} \tilde{Q}_{11k} (z_k^5 - z_{k-1}^5)
\]

Then, the total elastic deformation energy of the Euler-Bernoulli composite beam becomes:

\[
V = A_{11} \int_0^L \left( \frac{\partial w}{\partial x} \right)^4 \, dx + F_{11} \int_0^L \left( \frac{\partial^2 w}{\partial x^2} \right)^4 \, dx + D_{11} \int_0^L \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \left( \frac{\partial w}{\partial x} \right)^2 \, dx + F_{11} \int_0^L \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \, dx
\]

The kinetic energy of the beam is equal to [18]:

\[
T = \frac{\Delta P}{2} \int_0^L \left( \frac{\partial w}{\partial t} \right)^2 \, dx
\]
The transverse displacement function is developed into a series of basic spatial functions, while the time function is considered harmonic [18]:

\[ w = a_i w_i \sin \omega t \]  

The expression of the transverse displacement function defined above can be used for potential and kinetic energy:

\[ V = \frac{1}{2} a_i a_j k_{ij} \sin^2 \omega t + \frac{1}{2} a_i a_j a_k a_l b_{ijkl} \sin^4 \omega t + \frac{1}{2} a_i a_j a_k V_{ijkl} \sin^4 \omega t \]  

\[ T = \frac{1}{2} \omega^2 a_i a_j m_{ij} \cos^2 \omega t \]  

\( k_{ij} \) is the classical linear stiffness tensor:

\[ k_{ij} = D_1 \int_0^L \left( \frac{\partial^2 w_i}{\partial x^2} \right) \left( \frac{\partial^2 w_j}{\partial x^2} \right) dx \]  

\( b_{ijkl} \) and \( V_{ijkl} \) represent the non-linear stiffness tensor:

\[ b_{ijkl} = \frac{A_1}{4} \int_0^L \left( \frac{\partial w_i}{\partial x} \right) \left( \frac{\partial w_j}{\partial x} \right) \left( \frac{\partial w_k}{\partial x} \right) \left( \frac{\partial w_l}{\partial x} \right) dx + \frac{F_{11}}{4} \int_0^L \left( \frac{\partial^2 w_i}{\partial x^2} \right) \left( \frac{\partial^2 w_j}{\partial x^2} \right) \left( \frac{\partial^2 w_k}{\partial x^2} \right) \left( \frac{\partial^2 w_l}{\partial x^2} \right) dx , \]

\[ V_{ijkl} = \frac{D_1}{2} \int_0^L \left( \frac{\partial^2 w_i}{\partial x^2} \right) \left( \frac{\partial^2 w_j}{\partial x^2} \right) \left( \frac{\partial^2 w_k}{\partial x^2} \right) \left( \frac{\partial^2 w_l}{\partial x^2} \right) dx \]  

And \( m_{ij} \) the mass tensor:

\[ m_{ij} = A \rho \int_0^L w_i w_j dx \]  

#### 2.2. Governing equation

The guiding equation of the non-linear forced vibration is considered for a composite beam excited by the force \( F(x,t) \) over the range \( S \) (where \( S \) is the length of the beam or part of the beam). The physical force \( F(x,t) \) excites the mode of the structure through a set of generalized forces \( F_i(t) \). These forces depend on the expression of \( F \), the forces distributed over the length of the excitation and the mode considered. The generalized force \( F_i(t) \) is given by [19]:

\[ F_i(t) = \int_S F(x,t) w_i(x,t) dx \]  

In our case, the dynamic behavior is examined to study the distributed harmonic force \( F^d(x,t) \) defined by the following formula:

\[ F^d(x,t) = F^d \sin \omega t \]  

The corresponding generalized force is given by:

\[ F^d_i(t) = F^d \sin \omega t \int_S w_i(x) dx = f_i^d \sin \omega t \]  

According to Hamilton's principle, the dynamic behavior of the structure is expressed as follows:

\[ \delta \int_0^{2\pi / \omega} (V - T + W) dt = 0 \]  

The integral function is calculated in the range \([0, 2\pi / \omega]\) by replacing the expression of energy and force.

\[ \phi = \int_0^{2\pi / \omega} \left( \frac{1}{2} a_i a_j k_{ij} \sin^2 \omega t + \frac{1}{2} a_i a_j a_k a_l b_{ijkl} \sin^4 \omega t + \frac{1}{2} a_i a_j a_k V_{ijkl} \sin^4 \omega t \ight) dt \]  

\[ - \frac{1}{2} \omega^2 a_i a_j m_{ij} \cos^2 \omega t + f_i^d \sin \omega t \]
The function $\phi$ is defined as a function of the constant to be determined $a_i$, $i=1,...,n$ and their partial derivatives. Taking into account the symmetry of the mass matrix, the linear stiffness and the non-linear stiffness, the non-linear algebraic equation becomes:

$$a_k a_{i} + \frac{3}{2} a_k a_{i} b_{ijk} + \frac{3}{4} a_j a_{i} V_{ij} + \frac{3}{4} a_j a_{i} V_{ijk} - \alpha^2 a_{i} m_r = f_r^d, \quad r=1,...,n$$  \hspace{1cm} (23)

In order to carry out a general parametric study, we use a non-dimensional formulation by setting up:

$$x^* = \frac{x}{L}, \quad w_i(x) = H w_i^*(x^*)$$  \hspace{1cm} (24)

Where $m_{ij}^*$, $k_{ij}^*$, $V_{ijkl}^*$ and $b_{ijkl}^*$ are the general non-dimensional matrices which are defined by:

$$m_{ij}^* = \int_0^1 w_i^* w_j^* dx^*, \quad k_{ij}^* = \int_0^1 \left( \frac{\partial^2 w_i^*}{\partial x^2} \right) \left( \frac{\partial^2 w_j^*}{\partial x^2} \right) dx^*, \quad b_{ijkl}^* = a_k^* \int_0^L \left( \frac{\partial w_i^*}{\partial x^*} \right) \left( \frac{\partial w_j^*}{\partial x^*} \right) \left( \frac{\partial w_k^*}{\partial x^*} \right) \left( \frac{\partial w_l^*}{\partial x^*} \right) dx^*, \quad V_{ijkl}^* = a_s \int_0^L \left( \frac{\partial^2 w_i^*}{\partial x^2} \right) \left( \frac{\partial^2 w_j^*}{\partial x^2} \right) \left( \frac{\partial^2 w_k^*}{\partial x^2} \right) \left( \frac{\partial^2 w_l^*}{\partial x^2} \right) dx^*$$  \hspace{1cm} (25)

$$\alpha_1 = \frac{A_{11} H^2}{D_{11} L^2}, \quad \alpha_2 = \frac{F_{1} H^2}{D_{11} L^2}, \quad \alpha_3 = \frac{H^2}{L^2}$$  \hspace{1cm} (26)

And $f_r^d$ the dimensionless generalized force is defined as:

$$f_r^d = F_d^d \frac{L^2}{D_{11} H} \int_{x^*} w_i^*(x^*) dx^*$$  \hspace{1cm} (27)

Taking these notations into consideration, we obtain the following non-linear algebraic equation:

$$a_k a_{i} + \frac{3}{2} a_k a_{i} b_{ijk} + \frac{3}{4} a_j a_{i} V_{ij} + \frac{3}{4} a_j a_{i} V_{ijk} - \alpha^2 a_{i} m_r = f_r^d, \quad r=1,...,n$$  \hspace{1cm} (28)

Equation (28) gives a set of non-linear algebraic equations, leading to the contribution coefficient $a_i$. Use the second formulation defined in [12] to calculate the numerical solution related to the contribution coefficient. This formulation is an approximation, it ignores the second-order term. This leads to:

$$a_k a_{i} a_{i} b_{ijk}^* = a_k^* b_{i1} + a_i^* e_{i1}^* $$  \hspace{1cm} (29)

After replacing, reorganizing and exploiting the symmetry of the matrix, equation (28) can be written as follows

$$\left( K_R^* - \alpha^2 M_R^* \right) A_R + \frac{3}{2} \alpha^* \{ A_R \} = \left\{ f_r^d - \frac{3}{2} a_k^* b_{i1} + \frac{3}{4} a_i^* V_{i1} + \frac{3}{4} a_i^* V_{i1} \right\}, \quad i=2,...,n$$  \hspace{1cm} (30)

Among them, $[M_R^*]$ and $[K_R^*]$ represent respectively the mass matrix and the linear stiffness matrix associated with the first non-linear mode. Where $[\alpha^*]$ is the matrix defined by

$$a_i^* b_{i1} + \frac{1}{2} a_i^* V_{i1} + a_i^* V_{i1}$$  \hspace{1cm} (31)

which depends on $a_i$. $\{ A_R \}^T = \{ e_2, e_3, ..., e_i \}$ is a vector of contribution coefficients, which can be easily determined by solving the approximate linear system (31).

In the vicinity of the $r_{th}$ mode, the Eq. (31) can be written as follows
\[
\left( \left[ K^*_{Rr} \right] - \omega^2 \left[ M^*_{Rr} \right] \right) \left\{ A_{Rr} \right\} + \frac{3}{2} \left[ \alpha_r \right] \left\{ A_{Rr} \right\} = \left\{ f_r^d - \frac{3}{2} a_1^2 V_{i rf}^* - \frac{3}{4} a_1^2 V_{i rf}^* - \frac{3}{4} a_1^2 V_{i rf}^* \right\}, \quad r = 1,...,n \quad (32)
\]

With
\[
\left[ \alpha_r \right] = \left[ a_r^2 b_{ijr}^* + \frac{1}{2} a_1^2 V_{ijr}^* + \frac{1}{2} a_1^2 V_{ijr}^* \right]
\quad (33)
\]

Equation (32) presents a basic function to obtain the non-linear mode shapes and non-linear frequencies of a beam with various boundary conditions.

3. Presentation and discussion of numerical results

This section presents the results obtained from the free and forced vibration of the Euler-Bernoulli laminated composite beams under the hypothesis of finite deformation. In addition, the purpose of this section is to evaluate the accuracy of the results obtained from this analysis and the current approximation used with respect to the results produced by the GDQM defined in the literature.

In table 1, three parameters \( A_{11} \), \( D_{11} \), and \( F_{11} \) related to the non-linear stiffness matrix of the composite beam are given. In order to study the effect of non-linear vibrations on symmetrically laminated composite beams, we consider a carbon-epoxy laminated beams whose material properties are as follows:

\[ E_1 = 181 \text{ GPa} , \quad E_2 = 10.3 \text{ GPa} , \quad G_{12} = 7.17 \text{ GPa} , \quad \nu_{12} = 0.28 \text{ GPa} , \quad \rho = 1600 \text{ kg/m}^3 \]

Table 1. Coupling coefficients for different symmetrical orientations of laminated composite beams.

|          | Unidirectional | \((0/90)_S\) | \((+/\mp45)_S\) | \((0/\pm45/\pm90)_S\)
|----------|----------------|--------------|-----------------|------------------|
| \( A_{11}(N/m) \) | 363.622278.10^9 | 192.1572977.10^9 | 104.696950872.10^9 | 163.603848777.10^9 |
| \( D_{11}(N/m) \) | 121.207426.10^9 | 106.9186776.10^9 | 34.898983624.10^9 | 94.59748994523.10^9 |
| \( F_{11}(N/m^3) \) | 72.724455610^9 | 70.58114334.10^9 | 20.9393901744.10^9 | 65.92205382993.10^9 |

3.1. Comparison with previous results

Figures 3 and 4 show the normal shape of the non-linear mode of the unidirectional composite beam. It is clear from these figures that for different amplitudes, the effect of geometrical non-linearity can be observed.

![Figure 3](image)

Figure 3. The first normalized non-linear mode of a C-C unidirectional composite beam with different vibration amplitude values.

The backbone curves figures 5, 6 and 7 of the first non-linear mode of the unidirectional composite beam in a wide range of amplitudes are shown. On these figures, the comparison between the results of this method and the results obtained in the reference [13] is presented. In addition, three ratios
(thickness by length) are considered, ranging from thick beams \((H / L = 0.2)\) to thin beams \((H / L = 0.02)\). It should be noted that the dimensionless vibration amplitude of the beam \(W_{\text{max}} / H\) amounts to a value corresponding to 2, the result is very close to the result calculated by the GDQM method, and the error caused by the current solution is lower than that obtained in the reference.

![Graph](image1)

Figure 4. The first normalized non-linear mode of a C-F unidirectional composite beam with different vibration amplitude values.

![Graph](image2)

Figure 5. Comparison of backbone curves obtained by present results \((-\)) and that published in reference [13] \((\square)\), read from the graph. \(H / L = 0.02\).

![Graph](image3)

Figure 6. Comparison of backbone curves obtained by present results \((-\)) and that published in reference [13] \((\square)\), read from the graph. \(H / L = 0.1\).

In another verification study, table 2 shows a comparison of the dimensionless frequency values of the two boundary conditions C-C and C-F using Von-Karman theory and finite stress theory. The numerical results correspond to a dimensionless vibration amplitude of 0.4, and for laminates with several configurations. Various laminates show that, compared to isotropic beams, the effect of the order of lamination can significantly affect the vibration frequency of the composite beam.
Figure 7. Comparison of backbone curves obtained by present results (-) and that published in reference [13] (○, read from the graph). $H/L = 0.2$.

Table 2. Comparison of non-dimensional frequency with different orientations of a laminated composite beam.

| Boundary conditions | Layup orientations | Von Karman | Finite strain |
|---------------------|--------------------|------------|---------------|
|                     | GDQM [20]          | Present work | % Error   | GDQM [20] | Present work | % Error   |
| C-C                 | Unidirectional    | 23.32261   | 23.3224326   | 0.00735   | 23.32339   | 23.30312   | 0.0868   |
|                     | $(0/90)_s$        | 22.94903   | 22.9459796   | 0.01329   | 22.94983   | 22.9339746 | 0.0455   |
|                     | $(+45/-45)_s$     | 23.32264   | 23.3224326   | 0.00088   | 23.32342   | 23.3033392 | 0.0860   |
|                     | $(0/±45/±90)_s$   | 22.98793   | 22.9226075   | 0.28415   | 22.98873   | 22.9111294 | 0.3375   |
| C-F                 | Unidirectional    | 3.94434    | _             | _          | 3.94436    | 3.9904312   | 1.1680   |
|                     | $(0/90)_s$        | 3.78877    | _             | _          | 3.78879    | 3.8071641   | 0.4849   |
|                     | $(+45/-45)_s$     | 3.94435    | _             | _          | 3.94437    | 3.9904532   | 1.1685   |
|                     | $(0/±45/±90)_s$   | 3.80569    | _             | _          | 3.80571    | 3.7957080   | 0.2628   |

3.2. Numerical illustrations and discussion
In this sub-section, the resonance of laminated composite beams is examined for different symmetrical layer orientations, excitation levels, thickness/length ratios and for two boundary conditions: C-C and C-F. All amplitude-frequency response curves illustrate the hardening behavior. The results shown in figures 8 and 9 correspond to harmonic forces uniformly distributed along the length of the unidirectional laminated composite beam, and correspond to three excitation levels of $F^d = 50, 500$ and 1000. Obviously, the applied distributed force has different effects on the two boundary conditions of the beam. It can be seen from these figures that for the three levels of excitation, the amplitude peaks increase with the excitation. In addition, for higher excitation values, the frequency range of the solution is wider. Figures 10 and 11 show the effect of layer orientation on the amplitude-frequency response of the composite beam under the excitation of uniformly distributed harmonics with $F^d = 500$. The change in the layer structure not only increases the peak amplitude, but also enlarges the resonance area. For both boundary conditions, beams with a $(+45/-45)_s$-directional and unidirectional orientation have the lowest peak amplitude. Furthermore, it is evident that $(+45/-45)_s$-orientation has virtually no effect compared to unidirectional orientation, and this similar
phenomenon is also observed in table 2 and in the literature. The peak amplitude of the \((0/\pm45/\pm90)_S\) beam is lower than that of the other two beams \((+45/-45)_S\) and unidirectional), and the highest peak amplitude belongs to the \((0/90)_S\) beam.

**Figure 8.** Comparison of forced vibration resonance curves of unidirectional laminated composite beams with C-C boundary conditions. The resonance curve is subjected to uniformly distributed harmonic forces and is obtained under three levels of excitation.

**Figure 9.** Comparison of forced vibration resonance curves of unidirectional laminated composite beams with C-F boundary conditions. The resonance curve is subjected to uniformly distributed harmonic forces and is obtained under three levels of excitation.

**Figure 10.** Comparison of the resonance curves of the forced vibration of the laminated composite beam when the boundary condition is C-C. The curve is obtained from different directions of the layer.
4. Conclusions
In this study, the geometrically non-linear vibration of a laminated composite beam under transverse harmonic excitation was studied. Applying Hamilton's principle, the system of non-linear algebraic equations is established and solved by an approximate method. The amplitude-frequency response curve and the dimensionless non-linear frequency are compared with the results obtained by the GDQM method to verify this research. Under both C-C and C-F boundary conditions, the influence of symmetrical layer orientation and uniformly distributed harmonic excitation on the resonance response curve is illustrated. The following conclusions can be drawn:

- In all forced vibration resonance curves, laminated composite beams show non-linear behaviour such as hardening, jumping and bifurcation.
- In the resonance response curve, the increase in harmonic excitation applied to the unidirectional laminated composite beam causes the curve to gradually increase as the frequency increases. The curve with the largest aperture belongs to the beam with the C-F boundary condition.
- For the harmonic excitation resonance response $F^{d} = 500$, the $(0/90)$, beam has the highest peak amplitude, while the $(+45/-45)$, beam has the lowest amplitude and the lowest hardening performance. When the beam has boundary conditions C-F, this configuration will also widen the resonance curve.

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