Effective Field Theory of the Pion-Nucleon-Interaction*

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Abstract

In the first part of the talk, I discussed the nature of the effective theory that describes the properties of the pion-nucleon-interaction at low energies, using the static model as a starting point. In the second part, I then pointed out that the infrared singularities occurring in the πN scattering amplitude are stronger and of a more complex structure than those encountered in ππ scattering. A formulation of the effective theory that properly accounts for these was described and the results obtained thereby were illustrated with a few examples. In the following, I restrict myself to a discussion of some qualitative aspects of this work.

INTRODUCTION

The static model represents a forerunner of the effective theories of the pion-nucleon interaction used today. In this model, the kinetic energy of the nucleon is neglected: The nucleon is described as a fixed source that only carries spin and isospin degrees of freedom. For an excellent review of the model and its application to several processes of interest, I refer to the book of Henley and Thirring.[1]

The systematic formulation of the effective theory relies on an expansion of the effective Lagrangian in powers of derivatives and quark masses. Chiral symmetry implies that the leading term of this expansion is fully determined by the pion decay constant $F_\pi$ and by the nucleon matrix element of the axial charge, $g_A$. Disregarding vertices with three or more pions, the explicit expression for the leading term reads

$$L_{\text{eff}} = \frac{g_A}{2F_\pi} \bar{\psi} \gamma^\mu \gamma_5 \partial_\mu \pi \psi + \frac{1}{8F_\pi^2} \bar{\psi} \gamma^\mu i[\pi, \partial_\mu \pi] \psi + \ldots \quad (1)$$

The success of the static model derives from the fact that it properly accounts for the first term on the right hand side – in the nonrelativistic limit, where the momentum of the nucleons is neglected compared to the nucleon mass.

The static model is only a model. In order for the effective theory to correctly describe the properties of QCD at low energies, that framework must be extended, accounting for the second term in the above expression for the effective Lagrangian, for the vertices that contain three or more pion fields, for the contributions arising at higher orders of the derivative expansion, as well as for the chiral symmetry breaking terms generated by the quark masses $m_u$, $m_d$. This can be done in a systematic manner, using a nonrelativistic expansion for the nucleon kinematics. The resulting framework is called “Heavy Baryon Chiral Perturbation Theory” (HBCHPT). It represents an extension of the static model that correctly accounts for nucleon recoil, order by order in the nonrelativistic expansion (for reviews of this approach, see for instance ref.[2]).

As pointed out in ref.[3], the nonrelativistic expansion of the infrared singularities generated by pion exchange is a subtle matter. The HBCHPT representations of the scattering amplitude or of the scalar nucleon form factor, for example, diverge in the vicinity of the point $t = 4M_\pi^2$. The problem does not arise in the Lorentz invariant approach proposed earlier.[4] It originates in the fact that for some of the graphs, the loop integration cannot be interchanged with the nonrelativistic expansion.

The reformulation of the effective theory given in ref.[3] exploits the fact that the infrared singular part of the one loop integrals can unambiguously be separated from the remainder. To any finite order of the nonrelativistic expansion, the regular part represents a polynomial in the momenta. Moreover, the singular and regular pieces separately obey the Ward identities of chiral symmetry. This ensures that a suitable renormalization of

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the effective coupling constants removes the regular part altogether. The resulting representation for the various quantities of interest combines the virtues of the Heavy Baryon approach with those of the relativistic formulation of ref. [4]. The perturbation series can be ordered with the standard chiral power counting and manifest Lorentz invariance is preserved at every stage of the calculation.

GOLDBERGER-TREIMAN RELATION

As a first illustration of the method, I briefly discuss the relation between the pion-nucleon coupling constant and the axial charge of the nucleon, obtained on the basis of a calculation of the \( \pi N \) scattering amplitude to order \( q^4 \). A detailed account of this work is in preparation [5]. Throughout the following, I consider the isospin limit, \( m_u = m_d \), and replace the quark masses by the leading term in the expansion of \( M_\pi^2 \),

\[ M^2 \equiv (m_u + m_d)B. \]

The Goldberger-Treiman relation may be written in the form

\[ g_{\pi N} = \frac{g_A m_N}{F_\pi} \{1 + \Delta_{GT}\}. \] (2)

If the quark masses \( m_u, m_d \) are turned off, the strength of the \( \pi N \) interaction is fully determined by \( g_A \) and \( F_\pi \): \( \Delta_{GT} = 0 \). The effective theory allows us to analyze the correction that arises for nonzero quark masses. The quantities \( g_{\pi N}, g_A, m_N \) and \( F_\pi \) may be calculated in terms of the effective coupling constants. The result takes the form of an expansion in powers of \( M \), i.e. in powers of the quark masses.

Some of the graphs occurring within the effective theory develop infrared singularities when the pion mass is sent to zero. These manifest themselves through odd powers of \( M \) and through logarithms thereof. The expansion of the nucleon mass, for instance, is of the form

\[ m_N = m + k_1 M^2 + k_2 M^3 + k_3 M^4 \ln \frac{M^2}{m^2} + k_4 M^4 + O(M^5). \] (3)

The first term, \( m \), is the value of the nucleon mass in the chiral limit. The coefficients \( k_1, k_2, \ldots \) represent combinations of effective coupling constants that remain finite when the quark masses are turned off. In particular, the coefficient of the term proportional to \( M^3 \) is given by

\[ k_2 = -\frac{3g^2}{32\pi F^2}, \] (4)

where \( g \) and \( F \) represent the values of \( g_A \) and \( F_\pi \) in the chiral limit, respectively. The term is correctly described by the static model, where it arises from the self energy of the pion cloud that surrounds the nucleon. Numerically, this contribution lowers the nucleon mass by about 15 MeV.

Similar terms also occur in the chiral expansion of \( g_A \) – Kambor and Mojžiš [6] have worked out this quantity to order \( q^3 \). The expansion of \( F_\pi \) is known since a long time – it only contains even powers of \( M \), accompanied by logarithms. The coupling constant \( g_{\pi N} \) is obtained by evaluating the residue of the pole terms occurring in the scattering amplitude at \( s = m_N^2 \) and \( u = m_N^2 \). The representation of the amplitude to order \( q^4 \) yields an expression for \( g_{\pi N} \) in terms of the effective coupling constants and of the quark masses, valid up to and including order \( q^3 \).

Using these results, we may evaluate the chiral expansion of \( \Delta_{GT} \) to order \( q^3 \). Remarkably, the contributions of order \( M^2 \ln M^2 / m^2 \) as well as those of order \( M^3 \) cancel – the Goldberger-Treiman relation is free of infrared singularities, up to and including order \( M^3 \):

\[ \Delta_{GT} = k_{GT} M^2 + O(M^4). \] (5)
The coefficient $k_{GT}$ represents a combination of effective coupling constants – chiral symmetry does not determine its magnitude. The result shows that in the case of the Goldberger-Treiman relation, the breaking of chiral symmetry generated by the quark masses does not get enhanced by small energy denominators. Assuming that the scale of the symmetry breaking is the same as in the case of $F_K/F_\pi$, where it is set by the massive scalar states, $M_\pi \sim 1\,\text{GeV}$, we obtain the crude estimate $\Delta_{GT} \simeq M_\pi^2/M_N^2 \simeq 0.02$. The detailed analysis based on models and on the SU(3) breaking effects seen in the meson-baryon coupling constants confirms the expectation that $\Delta_{GT}$ must be very small – a discrepancy of order 4% or more would be very difficult to understand.

Since the days when the Goldberger-Treiman relation was discovered, the value of $g_A$ has increased considerably. Also, $F_\pi$ decreased a little, on account of radiative corrections. The main source of uncertainty is $g_{\pi NN}$. The comprehensive analysis of $\pi N$ scattering published by Höhler in 1983 led to $f^2 = g_{\pi NN}^2 M_\pi^2/(16\pi m_N^2) = 0.079$. With $g_A = 1.267$ and $F_\pi = 92.4\,\text{MeV}$, this value yields $\Delta_{GT} = 0.041$. As stressed by Pavan at this meeting, the data accumulated since then indicate that $f^2$ is somewhat smaller, numbers in the range from 0.076 to 0.077 looking more likely. This range corresponds to $0.021 < \Delta_{GT} < 0.028$.

I conclude that, within the current experimental uncertainties to be attached to the pion-nucleon coupling constant, the Goldberger-Treiman relation does hold to the expected accuracy. Note that at the level of 1 or 2%, isospin breaking cannot be ignored. In particular, radiative corrections need to be analyzed carefully. Also, the coupling constant relevant for the neutral pion picks up a significant contribution from $\pi^0 - \eta$ interference. A precise determination of the pion-nucleon coupling constant is essential to arrive at reliable results for small quantities such as the $\sigma$-term. The various discussions at this meeting show that the issue is under close scrutiny by several groups and I am confident that the uncertainties will soon be reduced.

**LOW ENERGY THEOREM FOR D**

As a second example, I consider the low energy theorem for the value of the scattering amplitude $D^+(s,t,u)$ at the Cheng-Dashen point: $s = u = m_N^2$, $t = 2M_\pi^2$. The theorem relates this amplitude to the scalar form factor of the nucleon,

$$\langle N' | m_u \bar{u} u + m_d \bar{d} d | N \rangle = \sigma(t) \bar{u}' u.$$  \hspace{1cm} (6)

The relation may be written in the form

$$F_\pi^2 D^+(m_N^2, 2M_\pi^2, m_N^2) = \sigma(2M_\pi^2) + \Delta_{CD}.$$  \hspace{1cm} (7)

The theorem states that the term $\Delta_{CD}$ vanishes up to and including contributions of order $M^2$. The explicit expression obtained for $F_\pi^2 D^+(m_N^2, 2M_\pi^2, m_N^2)$ when evaluating the scattering amplitude to order $q^4$ again contains infrared singularities proportional to $M^3$ and $M^4 \ln M^2/m^2$. Precisely the same singularities, however, also show up in the scalar form factor at $t = 2M_\pi^2$, so that the result for $\Delta_{CD}$ is free of such singularities.

$$\Delta_{CD} = k_{CD} M^4 + O(M^5).$$  \hspace{1cm} (8)

Crude estimates like those used in the case of the Goldberger-Treiman relation indicate that the term $\Delta_{CD}$ must be very small, of order 1 MeV.

The value of the scalar form factor at $t = 0$ is referred to as the $\sigma$-term,

$$\sigma = \sigma(0).$$

This quantity is of particular interest, because it represents the response of the nucleon mass to a change in the quark masses:

$$\sigma = m_u \frac{\partial m_N}{\partial m_u} + m_d \frac{\partial m_N}{\partial m_d}.$$  \hspace{1cm} (9)

*The cancellation of the terms of order $M^3$ was pointed out in ref. and the absence of logarithmic contributions of order $M^4$ was shown in ref.*
The difference $\sigma(2M^2) - \sigma(0)$ is well understood: The results found on the basis of a dispersive analysis \cite{11} are confirmed by the evaluation of the scalar form factor within the effective theory \cite{12,3}. The net result is that an accurate determination of the scattering amplitude at the Cheng-Dashen point amounts to an accurate determination of the $\sigma$-term.

Unfortunately, the experimental situation concerning the magnitude of $D^+$ at the Cheng-Dashen point leaves much to be desired (for a recent review, see ref. \cite{13}). The low energy theorem makes it evident that we are dealing with a small quantity here – the object vanishes in the chiral limit. The inconsistencies among the various data sets available at low energies need to be clarified to arrive at a reliable value for $g_{\pi N}$. Only then will it become possible to accurately measure small quantities such as the $\sigma$-term.

**DEPENDENCE OF THE $\sigma$-TERM ON THE QUARK MASSES**

In the following, I do not discuss the magnitude of $\sigma$ as such, but instead consider the dependence of this quantity on the quark masses, which is quite remarkable. In this discussion, the precise value of $\sigma$ is not of crucial importance. For definiteness, I use the value $\sigma = 45\text{ MeV}$ \cite{11}. The chiral expansion of the $\sigma$-term is readily obtained by applying the Feynman-Hellman theorem \cite{8} to the formula \cite{3}, with the result

$$
\sigma = k_1 M^2 + \frac{3}{2} k_2 M^3 + k_3 M^4 \left\{ 2 \ln \frac{M^2}{m^2} + 1 \right\} + 2k_4 M^4 + O(M^5),
$$

(10)

As discussed above, the term proportional to $M^3$ arises from an infrared singularity in the self energy of the pion cloud. It lowers the magnitude of the $\sigma$-term by $3/2 \times 15\text{ MeV} \simeq 23\text{ MeV}$. The coefficient $k_3$ can also be expressed in terms of measurable quantities \cite{3}. Numerically the contribution from this term amounts to $-7\text{ MeV}$, thus amplifying the effect seen at $O(M^3)$. Chiral symmetry does not determine all of the effective coupling constants entering the regular contribution $k_4 M^4$, which is of the same type as the correction $\Delta_{\text{CD}} = k_{\text{CD}} M^4$ to the low energy theorem \cite{6}. As discussed above, corrections of this type are expected to be very small – I simply drop the term $k_4 M^4$. The value of $k_1$ is then fixed by the input $\sigma = 45\text{ MeV}$ for the total, so that we can now discuss the manner in which $\sigma$ changes when the quark masses are varied.

At leading order, $\sigma$ is given by $k_1 M^2$. In figure 1 this contribution is shown as a dashed straight line. The full curve includes the contributions generated by the infrared singularities, $k_2 M^3$ and $k_3 M^4 \{2 \ln M^2/m^2 + 1\}$. The figure shows that the expansion of the $\sigma$-term in powers of the quark masses contains large contributions from infrared singularities. These must show up in evaluations of the $\sigma$-term on a lattice: The ratio
Figure 2. Square of the pion mass as a function of the quark masses. The dash-dotted line indicates the physical value of $M_\pi^2$.

$\sigma/(m_u + m_d)$ must change significantly if the quark masses are varied from the chiral limit to their physical values. Note that in this discussion, the mass of the strange quark is kept fixed at its physical value – the curvature seen in the figure exclusively arises from the perturbations generated by the two lightest quark masses.

It is instructive to compare this result with the dependence of $M_\pi^2$ on the quark masses. In that case, the expansion only contains even powers of $M$:

$$M_\pi^2 = M^2 + \frac{M^4}{32\pi^2 F^2} \ln \frac{M^2}{\Lambda^2} + O(M^6).$$

The quantity $\Lambda_3$ stands for the renormalization group invariant scale of the effective coupling constant $l_3$. The SU(3) estimate for this coupling constant given in ref. [14] reads $l_3 \equiv -\ln M_\pi^2/\Lambda_3^2 = 2.9 \pm 2.4$. The error bar is so large that the estimate barely determines the order of magnitude of the scale $\Lambda_3$. Figure 2 shows, however, that this uncertainty does not significantly affect the dependence of $M_\pi^2$ on the quark masses, because the logarithmic contribution is tiny: the range of $\bar{l}_3$ just quoted corresponds to the shaded region shown in the figure.

The logarithmic term occurring in the chiral expansion of $M_\pi^2$ gets enhanced by about a factor of two if we consider the pion $\sigma$-term,

$$\sigma = \langle \pi | m_u \bar{u} u + m_d \bar{d} d | \pi \rangle = m_u \frac{\partial M_\pi^2}{\partial m_u} + m_d \frac{\partial M_\pi^2}{\partial m_d}.$$  

I do not show the corresponding curve, because it is also nearly a straight line.

The main point here is that the infrared singularities encountered in the self energy of the nucleon are much stronger than those occurring in the self energy of the pion. A detailed account of the work on the low energy structure of the $\pi N$ scattering amplitude done in collaboration with Thomas Becher is in preparation.

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