THE PROBABILITY OF LONG CYCLES IN INTERCHANGE PROCESSES

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Abstract. We examine the number of cycles of length \( k \) in a permutation, as a function on the symmetric group. We write it explicitly as a combination of characters of irreducible representations. This allows to study formation of long cycles in the interchange process, including a precise formula for the probability that the permutation is one long cycle at a given time \( t \), and estimates for the cases of shorter cycles.

1. Introduction

A well known phenomenon in the theory of mixing times\(^1\) is that occasionally certain aspects of a system mix much faster than the system as a whole. Pemantle [19] constructed an example of a random walk on the symmetric group \( S_n \) which mixes in time \( n^{1+o(1)} \) while every \( k \) elements mix in \( \leq C(k)\sqrt{n} \) time. Schramm showed that for the interchange process on the complete graph — this is another random walk on \( S_n \); see below for details — the structure of the large cycles mix in time \( \approx n \), and it was known before [6] that the mixing time of this graph is \( \approx n \log n \). See [21] and also [7]. Schramm’s result is related to — in physics’ parlance, it is the mean-field case of — a conjecture of Bálint Tóth [22] that the cycle structure of the interchange process on the graph \( \mathbb{Z}^d \), \( d \geq 3 \), exhibits a phase-transition. In this paper we investigate the probability of long cycles, and obtain precise formulae for any graph, using the representation theory of \( S_n \). As an application, we analyse certain variations on Tóth’s conjecture.

Let us define the interchange process. Let \( G \) be a finite graph with vertex set \( \{1, \ldots, n\} \), and equip each edge \( \{i, j\} \) with an alarm clock that rings with exponential rate \( a_{i,j} \). Put a marble on every vertex of \( G \), all different, and whenever the clock of \( \{i, j\} \) rings, exchange the two marbles. Each marble therefore does a standard continuous-time random walk on the graph but the different walks are dependent. The positions of the marbles at time \( t \) is a permutation of their original positions, and viewed this way the process is a random walk on the symmetric group. Note that we have changed the timing from the previous paragraph. For example, if our graph is the complete graph and \( a_{i,j} = 1/n \) for all \( i \) and \( j \), then the process mixes in time \( \approx \log n \) and the large cycle structure mixes in time \( \approx 1 \). However, the added convenience of having each marble do the natural continuous time random walk outweighs the difference in notations from some of the literature.

The stronger results of this paper require representation theory to state, but let us start with two corollaries that can be stated elementarily. Let \( s_k(t) \) be the number of cycles of length \( k \) in our permutation at time \( t \). Let \( 0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1} \) be the eigenvalues of the continuous time Laplacian of the random walk on the graph \( G \). Then

\(^1\)We do not need the notion of mixing time in this paper, it is only used for comparison. The reader unfamiliar with it may peruse the survey [16] or the book [14].
Theorem 1. We have

$$\mathbb{P}(s_n(t) = 1) = \frac{1}{n} \prod_{i=1}^{n-1} (1 - e^{-\lambda_i t})$$

Let us demonstrate the utility of this formula on the graph $G = \{0, 1\}^d$ with weights equal to 1. There is nothing particular about this graph, but existing literature allows for easy comparison. For example, Wilson [24, §9] showed that the mixing time of the interchange process on $G$ is $\geq cd$ (see also [17, 18]). The eigenvalues of $G$ may be calculated explicitly: the eigenvectors are the Walsh functions, indexed by $y \in \{0, 1\}^d$ and given by $f_y(x) = (-1)^{\sum_{i=1}^{d} x_i y_i}$. We get that $2k$ is an eigenvalue with multiplicity $\binom{d}{k}$ for $k = 0, \ldots, d$. Inserting into the formula at times $1 \pm \epsilon \log d$ gives

$$\mathbb{P}\left(s_n\left(1 - \epsilon \log d\right) = 1\right) = 2^{-d} \prod_{k=1}^{d} \left(1 - e^{-(1-\epsilon)k \log d}\right) \binom{d}{k} \leq$$

and looking only at $k = K := \lfloor d^\epsilon / 2 \rfloor$,

$$\leq \exp\left(-d^{(\epsilon-1)K} \binom{d}{K}\right) \leq \exp\left(-d^{K}\right) \leq \exp(-\exp(cd^\epsilon))$$

where $(*)$ comes from

$$\binom{d}{K} = \binom{d}{K} \cdot \left(\frac{1 - 1/d}{1 - 1/K} \cdot \frac{1 - 2/d}{1 - 2/K} \cdot \ldots\right) \geq \binom{d}{K}.$$ 

On the other hand,

$$\mathbb{P}\left(s_n\left(1 + \epsilon \log d\right) = 1\right) = 2^{-d} \exp\left(\sum_{k=1}^{d} O(d^{-(1+\epsilon)k}) \binom{d}{k}\right) = 2^{-d} (1 + O(d^{-\epsilon})).$$

We see that the probability equilibrates at $\frac{1}{2} \log d$, before the mixing time of the whole chain. Further, the equilibration happens sharply — this is reminiscent of the cutoff phenomenon for mixing times. See [6], [15] or [14, §18] for the cutoff phenomenon.

We remark that taking $t \to 0$ in Theorem 1 one can get a new proof of Kirchoff’s matrix-tree theorem. We fill the details in the appendix.

Another general, elementarily stated result is:

Theorem 2. We have, for any graph $G$ and any $1 \leq k \leq n$,

$$\left| \mathbb{E}(s_k(t)) - \frac{1}{k} \right| \leq \frac{3^n}{k} e^{-t\lambda_1}$$

The point about this result is its generality — it holds for any graph. In particular examples that we tried the estimate was worse than the known or conjectured mixing time. But for general graphs it seems to be the best known.

To proceed, let us recall a few basic facts about the representations of $S_n$. For a full treatment see the books [10, 13, 20]. A representation of $S_n$ is a group homomorphism $\tau : S_n \to \text{GL}_k(\mathbb{C})$ for some $k$, typically denoted by $\dim \tau$. Its character, denoted by $\chi$, is an element of $L^2(S_n)$ defined
by \( \chi_\tau(g) = \text{tr}(\tau(g)) \). Now, the irreducible representations of \( S_n \) are indexed by partitions of \( n \), namely, by sequences \( \lambda = [\lambda_1, \lambda_2, \ldots, \lambda_k] \) with \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0 \) and \( \sum_{i=1}^{k} \lambda_i = n \) (we denote this by \( n \vdash \lambda \)). A nice graphical representation of partitions is using Young diagrams, i.e. drawing each \( \lambda_i \) as a line of boxes from top to bottom, e.g.

\[
[5, 1] = \begin{array}{ccc}
\hline
\hline
\end{array} \\
[3, 2, 1] = \begin{array}{ccc}
\hline
\hline
\hline
\end{array} \\
[2, 1^3] = \begin{array}{ccc}
\hline
\hline
\hline
\end{array}.
\]

To each partition \( n \vdash \lambda \) (and hence, for each young diagram with \( n \) boxes) corresponds an irreducible representation, which we shall denote by \( U_\lambda \). For brevity, we denote the character of \( U_\lambda \) by \( \chi_\lambda \). Fix now some \( 1 \leq k \leq n \) and define

\[
\alpha_k(g) = \#\{\text{cycles of length } k \text{ in } g\}.
\]

Now, \( \alpha_k(g) \) depends only on the cycle structure of \( g \), i.e. is a class function, and hence it is a linear combinations of characters of irreducible representations. Our main result is the precise decomposition.

**Theorem 3.** For any \( n \) and \( k \),

\[
\alpha_k = \frac{1}{k} \sum_{\rho \vdash n} a_\rho \chi_\rho,
\]

where

\[
a_\rho = \begin{cases}
1 & \rho = [n] \\
(-1)^{i+1} & \rho = [k-i-1, n-k+1, 1^i] \text{ for some } i \in \{0, \ldots, 2k-n-2\} \\
(-1)^i & \rho = [n-k, k-i, 1^i] \text{ for some } i \in \{\max\{2k-n, 0\}, \ldots, k-1\} \\
0 & \text{otherwise}
\end{cases}
\]

Let us describe this verbally (ignoring the diagram \([n]\) which has a somewhat special role). If \( k > (n+1)/2 \), start with \([k-1, n-k+1]\), with a minus sign. Now drop boxes from the first row into the leftmost column until the first and second row are equal. Then drop in a single step two boxes, one from each of the first two rows to the leftmost column. Then start dropping boxes from the second row until you reached a hook-shaped diagram. The sign keeps changing in each step. If \( k \leq n/2 \) start with the diagram \([n-k, k]\) with a plus sign, and drop boxes from the second row to the leftmost column until reaching a hook-shaped diagram, again switching sign at each step. The case \( k = (n+1)/2 \) is similar except you start from \([n-k, k-1, 1]\) with a minus sign.

It is now clear what is special in the case \( k = n \). In this case only hook-shaped diagrams appear in the sum. For the hook-shaped diagrams there is an explicit formula for the relevant eigenvalues discovered by Bacher [3] (see also the appendix of [1]). Let us remark that for \( k < n \) the probability \( \mathbb{P}(s_k(t) = 1) \) is not a function of the eigenvalues of the graph. In other words, one may find two isospectral graphs for which these probabilities differ. We will explain both facts (i.e. the conclusion of Theorem 1 from Theorem 3 and the isospectral examples) in section 4 below. Tóth’s conjecture will be stated and discussed in section 5.

We remark that Theorem 3 strengthens results by Eriksen and Hultman [9, §5] who found the decomposition of \( \sum \alpha_k \), i.e. of the number of cycles of a permutations. The formulas of [9] are quite
short and reveal some patterns in the numbers $a_{\rho}$. For example, for every $\rho$ of the form $[a, b, 1^c]$, $a_{\rho} \neq 0$ for exactly two values of $k$, with opposite signs.

2. Notations and preliminaries

Let $A = \{a_{i,j}\}_{1 \leq i < j \leq n}$ be a collection of non-negative numbers which we consider as a weighted graph. The random walk on $S_n$ associated with the weighted graph $A$ is a process in continuous time starting from the identity permutation $1$ on $S_n$ and going from $g$ to $(ij)g$ with rate $a_{i,j}$. Formally, consider $L^2(S_n)$, both as a Hilbert space with the standard inner product, and as an $\mathbb{R}$-algebra, via the group ring structure. Define the Laplacian as the element of $L^2(S_n)$ given by

$$\Delta = \Delta_A = \sum_{i<j} a_{i,j} (1 - (ij))$$

where $1$ is the element of $L^2(S_n)$ equal to $1$ in the identity permutation, and $0$ everywhere else; and $(ij)$ is similarly a singleton at the transposition $(ij)$. The distribution of the location of our process at time $t$ is

$$e^{-t\Delta} = \sum_{k=0}^{\infty} \frac{(-t\Delta)^k}{k!}$$

In particular for $\alpha_k$ defined by (1),

$$\mathbb{E}(s_k(t)) = \sum_{g \in S_n} (e^{-t\Delta}) (g) \alpha_k(g) = n! \langle e^{-t\Delta}, \alpha_k \rangle$$

where here and below $\langle \cdot, \cdot \rangle$ stands for the standard inner product in $L^2(S_n)$, i.e. $\langle a, b \rangle = \frac{1}{n!} \sum_{g \in S_n} a(g)b(g)$.

For the proof of Theorem 3 we will need a second set of representations of $S_n$, this time reducible representations. For $n \vdash \rho$, let $T_{\rho} < S_n$ be the subgroup of all permutations fixing the sets $\{1, \ldots, \rho_1\}$, $\{\rho_1 + 1, \ldots, \rho_1 + \rho_2\}$, etc. As a group $T_{\rho} \cong S_{\rho_1} \times \cdots \times S_{\rho_r}$. Now, $S_n$ acts on the left cosets of $T_{\rho}$, i.e. $\{hT_{\rho}\}_{h \in S_n}$, and using these cosets as a basis we obtain a representation of $S_n$, which we will denote by $V_{\rho}$. Readers familiar with exclusion processes might find it convenient to think about $V_{\rho}$ as $\mathbb{R}^X$ where $X$ is the space of configurations of the exclusion process with $\rho_1$ particles of colour 1, $\rho_2$ particles of colour 2 etc. — considering $\Delta$ as an operator on $V_{\rho}$, it is easy to verify that one gets an identical process. We denote

$$\psi_{\rho} = \chi_{V_{\rho}}.$$

(3)

It is well known that the representations $V_{\rho}$ are generally reducible and their irreducible components, consist of all $U_{\sigma}$ for $\sigma \geq \rho$, where $\geq$ is the domination order [10, Corollary 4.39] — we say that $\sigma \geq \rho$ when you can reach $\rho$ from $\sigma$ by a series of “toppling” of a box of the Young diagram to a lower row which keep the structure of a Young diagram. Alternatively, $\sigma \geq \rho$ is equivalent to

$$\sum_{i=1}^{j} \sigma_i \geq \sum_{i=1}^{j} \rho_i \quad \forall j.$$
3. Character decomposition

In this section we prove Theorem 3. We go about it by describing a more general method for expressing a class function on $S_n$ as a linear combination of characters, and then applying it to our case.

Given a function $f : S_n \to \mathbb{R}$ that is a class function (i.e. satisfies $f(hgh^{-1}) = f(g)$ for all $g, h \in S_n$), it can be expressed as a linear combination of the characters of $S_n$ (see, e.g., [10, Proposition 2.30]). By the character orthogonality relations (ibid.), we have

$$f = \sum_{n \vdash \rho} \langle f, \chi_{\rho} \rangle \chi_{\rho}$$  \hspace{1cm} (4)

As it is often hard to calculate the inner products $\langle f, \chi_{\rho} \rangle$ directly, we start by calculating $\langle f, \psi_{\lambda} \rangle$, where $\psi_{\lambda} = \chi_{V_{\lambda}}$ and $V_{\lambda}$ are the “exclusion-like” reducible representations defined just before (3).

**Lemma 1.** We have $\langle f, \psi_{\lambda} \rangle = \frac{1}{\#(T_{\lambda})} \sum_{q \in T_{\lambda}} f(q)$.

**Proof.** We have

$$\langle f, \psi_{\lambda} \rangle = \frac{1}{n!} \sum_{g \in S_n} \psi_{\lambda}(g)f(g)$$

Recall from §2 that $V_{\lambda}$ is obtained from the action of $S_n$ on the cosets of a $T_{\lambda} < S_n$. By the definition of trace, $\psi_{\lambda}(g)$ equals the number of cosets of $T_{\lambda}$ fixed by $g$. A coset $hT_{\lambda}$ is fixed by $g$ iff $h^{-1}gh \in T_{\lambda}$. Hence,

$$\langle f, \psi_{\lambda} \rangle = \frac{1}{n!} \sum_{hT_{\lambda} \in S_n/T_{\lambda}} \sum_{g \in G} f(h^{-1}gh \in T_{\lambda})$$

Let us make a change of variables, $q = h^{-1}gh$. Since $f$ is a class function, we have

$$\langle f, \psi_{\lambda} \rangle = \frac{1}{n!\#(T_{\lambda})} \sum_{h \in G} \sum_{q \in T_{\lambda}} f(hqh^{-1}) = \frac{1}{n!\#(T_{\lambda})} \sum_{h \in G} \sum_{q \in T_{\lambda}} f(q) = \frac{1}{\#(T_{\lambda})} \sum_{q \in T_{\lambda}} f(q). \hspace{1cm} \square$$

Now, by Young’s rule [10, Corollary 4.39], the characters $\psi_{\lambda}$ and the characters $\chi_{\lambda}$ are related by the linear equations

$$\psi_{\lambda} = \sum_{\mu \vdash \lambda} K_{\mu\lambda} \chi_{\mu}$$

Where the numbers $K_{\mu\lambda}$, called the Kostka numbers, are defined as follows: Let $\lambda = [\lambda_1, \ldots, \lambda_r]$, then $K_{\mu\lambda}$ is the number of ways the Young diagram $\mu$ can be filled with $\lambda_1$ 1’s, $\lambda_2$ 2’s, etc., such that each row is nondecreasing, and each column is strictly increasing. The numbers $K_{\mu\lambda}$ satisfy $K_{\mu\lambda} = 0$ whenever $\mu < \lambda$ (with respect to the lexicographic order), and $K_{\mu\mu} = 1$. (See [10], appendix A). Hence,

$$\langle f, \psi_{\lambda} \rangle = \sum_{\mu \vdash \lambda} K_{\mu\lambda} \langle f, \chi_{\mu} \rangle$$

In other words the numbers $\langle f, \chi_{\mu} \rangle$ satisfy a system of linear equations, whose coefficient matrix $(K_{\mu\lambda})$ is triangular with 1’s on the diagonal, hence invertible.
The resulting system of equations has a more elegant form when expressed in terms of symmetric polynomials.

Fix an integer \( m \geq n \) (whose value is not important), and consider the ring of symmetric polynomials in \( m \) variables \( x_1, \ldots, x_m \) over \( \mathbb{C} \). Consider the following homogeneous symmetric polynomials of degree \( n \) (see [10], i.e., for more details):

- For \( n \vdash \lambda = [\lambda_1, \ldots, \lambda_r] \), \( M_\lambda = \sum_\alpha x^\alpha \), where \( \alpha = (\alpha_1, \ldots, \alpha_n) \) goes over all the possible permutations of \((\lambda_1, \ldots, \lambda_r, 0, \ldots, 0)\).
- The Schur polynomials \( S_\mu = \sum_\lambda K_{\mu\lambda} M_\lambda \)
- The full homogeneous polynomial \( H_n \), defined as the sum of all monomials of degree \( n \). It is easy to see that for all \( n \vdash \lambda \), \( K_{[n]\lambda} = 1 \). Hence, \( H_n = \sum_{n-\lambda} M_\lambda = \sum_{n-\lambda} K_{[n]\lambda} M_\lambda = S_\lambda \).

Recall also the Frobenius characteristic map \( \chi \), defined on the class functions of \( S_n \), which sends an irreducible character \( \chi_\mu \) to its corresponding Schur polynomial \( S_\mu \), and is extended by linearity. Clearly, decomposing a class function into irreducible characters, \( f = \sum_\mu a_\mu \chi_\mu \) is equivalent to decomposing its image \( \chi(f) \) into Schur polynomials, \( \chi(f) = \sum_\mu a_\mu S_\mu \). By (3), we have

\[
\chi(f) = \sum_\mu \langle f, \chi_\mu \rangle S_\mu = \sum_\mu \langle f, \chi_\mu \rangle \sum_\lambda K_{\mu\lambda} M_\lambda = \sum_\lambda \langle f, \psi_\lambda \rangle M_\lambda
\]

We conclude:

**Lemma 2.** Let \( f \) be a class function on \( S_n \). Then

\[
\chi(f) = \sum_\lambda \left( \frac{1}{\#(T_\lambda)} \sum_{g \in T_\lambda} f(g) \right) M_\lambda.
\]

We now apply this to the class functions \( \alpha_k \) (Recall the definition of \( \alpha_k \), (1)).

**Lemma 3.** We have for all \( 1 \leq k \leq n \), \( \chi(\alpha_k) = \frac{1}{k} (\sum_{i=1}^m x_i^k) H_{n-k}(x_1, \ldots, x_m) \).

**Proof.** Let us define a function \( \beta_k \) on the set of partitions of \( n \) by

\[
\beta_k([\lambda_1, \ldots, \lambda_r]) = \# \{ i : \lambda_i \geq k \}.
\]

By lemma 1,

\[
\langle \alpha_k, \psi_\lambda \rangle = \frac{1}{\#(T_\lambda)} \sum_{q \in T_\lambda} \alpha_k(q).
\]

The sum \( \sum_{q \in T_\lambda} \alpha_k(q) \) can be evaluated by summing over all possible \( k \)-cycles \( c \in T_\lambda \), the number of elements of \( T_\lambda \) such that \( c \) is one of their cycles. For any \( i \) such that \( \lambda_i \geq k \), there are \( \binom{\lambda_i}{k} \cdot (k-1)! \) choices for a cycle \( c \) in the \( S_{\lambda_i} \)-factor of \( T_\lambda \), and \( \lambda_1! \lambda_2! \cdots (\lambda_i - k)! \cdots \lambda_r! \) choices for an element \( g \in T_\lambda \) with \( c \) as a cycle. Hence each such \( i \) contributes to the sum

\[
\binom{\lambda_i}{k} \cdot (k-1)! \cdot \lambda_1! \lambda_2! \cdots (\lambda_i - k)! \cdots \lambda_r! = \frac{\#(T_\lambda)}{k}
\]

Obviously, if \( \lambda_i < k \) then there are no \( k \)-cycles in the \( S_{\lambda_i} \)-factor, and the contribution is 0. Hence,

\[
\langle \alpha_k, \psi_\lambda \rangle = \frac{1}{\#(T_\lambda)} \sum_{i : \lambda_i \geq k} \frac{\#(T_\lambda)}{k} = \frac{1}{k} \beta_k(\lambda).
\]
By lemma 2,
\[ \text{ch}(\alpha_k) = \frac{1}{k} \sum_{\lambda} \beta_k(\lambda) M_\lambda. \]

A moment’s reflection shows that \( \sum_{\lambda} \beta_k(\lambda) M_\lambda = (\sum_{i=1}^m x_i^k) H_{n-k} \). Indeed, each monomial \( x_1^{a_1} \cdots x_m^{a_m} \) of degree \( n \) appears on the left-hand side with coefficient \( \# \{ i : \alpha_i \geq k \} \) (by the definition of \( \beta \)), and the same is on the right-hand side. This finishes the lemma. \( \Box \)

Our goal is to express \( \text{ch}(\alpha_k) \) as a linear combination of Schur polynomials. Let us start with the case of \( k = n \).

**Lemma 4.** \( \text{ch}(\alpha_n) = \frac{1}{n} \sum_{i=0}^{n-1} (-1)^i S_{[n-i,1^i]} \).

**Proof.** By lemma 3, \( \text{ch}(\alpha_n) = \frac{1}{n} \sum_i x_i^n = \frac{1}{n} M_{[n]} \). On the other hand, for all \( 0 \leq i \leq n - 1 \) we have
\[ S_{[n-i,1^i]} = \sum_{\lambda} K_{[n-i,1^i] \lambda} M_\lambda. \]

Let \( \lambda \) have \( r \) rows. By definition of the Kostka numbers, we have \( K_{[n-i,1^i] \lambda} = (r - 1)^i \), and \( K_{[n-i,1^i] \lambda} = 0 \) for \( i \geq r \), since the top left box of \( [n-i,1^i] \) has to be numbered 1, and the whole configuration is determined by the choice of distinct \( i \) numbers out of 2, \ldots, \( r \) to be placed in the leftmost column in ascending order. Denoting by \( r(\lambda) \) the number of rows in \( \lambda \), we get
\[ \sum_{i=0}^{n-1} (-1)^i S_{[n-i,1^i]} = \sum_{\lambda} M_\lambda \sum_{i=0}^{n-1} (-1)^i \binom{r(\lambda) - 1}{i} \]

By the binomial identity, the inner sum is 0 unless \( r(\lambda) = 1 \), i.e. \( \lambda = [n] \), in which case the inner sum is 1. We get \( \sum_{i=0}^{n-1} (-1)^i S_{[n-i,1^i]} = M_{[n]} \), as desired. \( \Box \)

**Remark.** Lemma 4 can be proved more directly by using the Murnaghan-Nakayama rule [20, Theorem 4.10.2] to express \( \alpha_n \) as a linear combination of characters: for any \( n \vdash \lambda \), the scalar product \( \langle \chi_\lambda, \alpha_n \rangle \) is, up to a constant, the value of \( \chi_\lambda \) at one specific permutation, namely a cycle of length \( n \). The Murnaghan-Nakayama rule, when applied to such a cycle, takes a simple form.

We immediately conclude:

**Corollary 1.** We have
\[ \alpha_n = \frac{1}{n} \sum_{i=0}^{n-1} (-1)^i \chi_{[n-i,1^i]} \]

which is Theorem 3 for \( k = n \).

Let us now treat the general case, using the case we already proved. By lemma 4, applied to \( k \),
\[ \frac{1}{k} \sum_{i=1}^m x_i^k = \frac{1}{k} \sum_{i=0}^{k-1} (-1)^i S_{[k-i,1^i]} \]
Hence, by lemma 3,

\[ \text{ch}(\alpha_k) = \frac{1}{k} \left( \sum_{i=0}^{k-1} (-1)^i S_{[k-i,1^i]} \right) H_{n-k}. \]

We now apply Pieri’s formula (see [10]), according to which, \( S_{[k-i,1^i]} H_{n-k} \) is the sum of all polynomials of the form \( S_{\lambda'} \), where \( \lambda' \) is obtained by adding \( n-k \) boxes to \([k-i,1^i]\), without adding two boxes in the same column. Since we have a hook-shaped diagram, our possibilities are rather limited: we may add a box at the leftmost column or not, and the rest of the boxes go in the first two rows. Denote therefore

\[ S_{[k-i,1^i]} H_{n-k} = A_i + B_i \]

where \( A_i \) is the sum when one does not add a square at the leftmost column, and \( B_i \) is when one does. Denote also \( x(i,j) = S_{[n-i-j,1+j,1^{i-1}]} \) (the contribution coming from adding \( j \) boxes to the second row of \([k-i,1^i]\), and the remaining \( n-k-j \) boxes to the first row). Then

\[
\begin{align*}
A_0 &= S_{[n]} & A_i &= \sum_{j=0}^{\min(n-k,k-i-1)} x(i,j) \\
B_i &= \sum_{j=0}^{\min(n-k-1,k-i-1)} x(i+1,j).
\end{align*}
\]

We now sum over \( i \) and get,

\[ \left( \sum_i x_i^k \right) H_{n-k} = \sum_{i=0}^{k-1} (-1)^i (A_i + B_i) \]

Our next goal is to find the alternating sum \( \sum_{i=0}^{k-1} (-1)^i (A_i + B_i) \). There are further cancellations here because \( B_i \) and \( A_{i+1} \) are quite similar — \( B_i \) corresponds to adding a box to the first column of \([k-i,1^i]\) while \( A_{i+1} \) corresponds to not adding a box to the first column of \([k-i-1,1^{i+1}]\). Hence most of the terms cancel out. We get

\[
\begin{align*}
A_{i+1} &= \left\{ \begin{array}{l}
\sum_{j=0}^{n-k} x(i+1,j) \quad 0 \leq i \leq 2k-n-2 \\
\sum_{j=0}^{k-i-2} x(i+1,j) \quad 2k-n-1 \leq i \leq k-2
\end{array} \right. \\
B_i &= \left\{ \begin{array}{l}
\sum_{j=0}^{n-k-1} x(i+1,j) \quad 0 \leq i \leq 2k-n-1 \\
\sum_{j=0}^{k-i-1} x(i+1,j) \quad 2k-n \leq i \leq k-1
\end{array} \right.
\]

Hence (putting \( A_k = 0 \)),

\[
\begin{align*}
B_i - A_{i+1} &= \left\{ \begin{array}{l}
\sum_{j=0}^{n-k-1} x(i+1,j) - \sum_{j=0}^{n-k} x(i+1,j) = -x(i+1,n-k) \quad 0 \leq i \leq 2k-n-2 \\
\sum_{j=0}^{n-k-1} x(i+1,j) - \sum_{j=0}^{n-k-1} x(i+1,j) = 0 \quad i = 2k-n-1 \\
\sum_{j=0}^{k-i-1} x(i+1,j) - \sum_{j=0}^{k-i-2} x(i+1,j) = x(i+1,k-i-1) \quad 2k-n \leq i \leq k-1
\end{array} \right.
\]
and

\[\sum_{i=0}^{k-1} (-1)^i (A_i + B_i) = A_0 + \sum_{i=0}^{k-1} (-1)^i (B_i - A_{i+1}) =\]

\[= S[n] - \sum_{i=0}^{2k-n-2} (-1)^i x(i + 1, n - k) + \sum_{i=2k-n}^{k-1} (-1)^i x(i + 1, k - i - 1) =\]

\[= S[n] - \sum_{i=0}^{2k-n-2} (-1)^i S[k-i-1, n-k+1, 1] + \sum_{i=2k-n}^{k-1} (-1)^i S[n-k, k-i, 1] = \sum_{\rho} a_{\rho} S_{\rho}\]

where the numbers \(a_{\rho}\) were defined in the statement of Theorem 3. Hence, by lemma 3,

\[k \cdot \text{ch}(\alpha_k) = \left(\sum_i x_i^k\right) H_{n-k} = \sum_{\rho} a_{\rho} S_{\rho}.\]

This ends the proof of Theorem 3. \(\square\)

4. THE PROBABILITY OF LONG CYCLES

Let \(\rho\) be a partition of \(n\), and let \(U_{\rho} : S_n \to \text{GL}(\mathbb{C}^{\dim U_{\rho}})\) be the corresponding irreducible representation. Let \(D = \sum d_g g\) be any element of the group ring. Then \(U_{\rho}(D)\) is the element of \(\text{GL}(\mathbb{C}^{\dim U_{\rho}})\) given by

\[\sum_{g} d_g U_{\rho}(g).\]

(it might be useful to think about \(U_{\rho}(D)\) as a non-commutative Fourier transform of \(D\), with the fact that \(U_{\rho}(D_1 D_2) = U_{\rho}(D_1) U_{\rho}(D_2)\) being the non-commutative analog of \(\hat{f} \ast g = \hat{f} \hat{g}\)). In the case that \(D = \Delta_A\) we will denote the eigenvalues of this matrix by \(0 \leq \lambda_1(A, \rho) \leq \ldots \leq \lambda_{\dim(\rho)}(A, \rho)\) (it is well-known that \(U_{\rho}(\Delta_A)\) is positive semidefinite and in particular diagonalizable, see e.g. [1]).

**Lemma 5.** For any \(n\) and \(k\) we have

\[E(s_k(t)) = \frac{1}{k} \sum_{n \vdash \rho} a_{\rho} \sum_{j=1}^{\dim U_{\rho}} e^{-t\lambda_j(A, \rho)}\]

where \(a_{\rho}\) are as in Theorem 3.

**Proof.** As discussed in §2,

\[E(s_k(t)) = n!(\alpha_k, e^{-t\Delta_A}) = \frac{1}{k} \sum_{\rho} a_{\rho} n!(e^{-t\Delta_A}, \chi_{\rho})\]

By definition, \(\chi_{\rho}\) attaches to each \(g \in S_n\) the trace of \(g\) acting on the representation \(U_{\rho}\). By the linearity of the trace,

\[n!(e^{-t\Delta_A}, \chi_{\rho}) = \text{tr} \left( U_{\rho} (e^{-t\Delta_A}) \right)\]
where \( U_\rho(\cdot) \) is the action of a representation on an element of the group ring as above. Further, for every representation \( U \) and any element \( D \) of the group ring,

\[
U(e^D) = e^{U(D)}
\]

where the exponentiation on the left-hand side is in the group ring while on the right-hand side we have exponentiation of matrices. Since \( U_\rho(-t\Delta) \) is diagonalizable,

\[
\text{tr} \left( U_\rho \left( e^{-t\Delta} \right) \right) = \sum_j e^{-t\lambda_j(A,\rho)}.
\]

The proof now follows from Theorem 3. \( \square \)

**Proof of Theorem 1.** \( s_n(t) \) can take only the values 0 and 1. Hence, using lemma 5 for \( k = n \), we get

\[
\mathbb{P}(s_n(t) = 1) = \mathbb{E}(s_n(t)) = \frac{1}{n} \sum_{i=0}^{n-1} (-1)^i \sum_j e^{-t\lambda_j(A, [n-i,1^i])}
\]

Since \([n-i,1^i]\) is a hook-shaped diagram, the eigenvalues \( \lambda_j(A, [n-i,1^i]) \) are simply all the sums of \( i \)-tuples of the eigenvalues \( \lambda_1(A), \ldots, \lambda_{n-1}(A) \). (See [3] and also the appendix of [1]). Hence,

\[
\mathbb{P}(s_n(t) = 1) = \frac{1}{n} \left( 1 + \sum_{i=1}^{n-1} (-1)^i \sum_{1 \leq j_1 < j_2 < \ldots < j_i \leq n-1} e^{-t(\lambda_{j_1} + \ldots + \lambda_{j_i})} \right) = \frac{1}{n} \prod_{i=1}^{n-1} (1 - e^{-\lambda_it}). \quad \square
\]

**Proof of Theorem 2.** The partitions that appear in lemma 5 are of the form \([a, b, 1^c]\), where \( a+b+c = n, \ a \geq b > 0, \) and \( c \geq 0 \). For such a partition a simple calculation with the hook formula [10, §4.12] gives

\[
dim U_\lambda = \frac{b(a-b+1)}{(b+c)(a+c+1)} \frac{n!}{a!b!c!} \leq \binom{n}{a,b,c}
\]

Hence, the total number of summands in lemma 5 is bounded by \( \sum_{a+b+c=n} \binom{n}{a,b,c} = 3^n \). Also, by the celebrated Caputo-Liggett-Richthammer theorem [5], we have for all \( j \),

\[
\lambda_j(A, [a, b, 1^c]) \geq \lambda_1(A)
\]

The result now follows from lemma 5. \( \square \)

**Remark.** For a non-hook-shaped partition \( \rho \), the eigenvalues \( \lambda(A, \rho) \) are, in general, not a function of the eigenvalues of the graph \( A \). Such examples exist for \( n \) as low as 4. In other words, one can find two isospectral (weighted) graphs \( A_1, A_2 \) with 4 vertices for which \( \lambda(A_i, [2,2]) \) differ. By lemma 5, these two isospectral graphs also have different values for \( \mathbb{P}(s_3(t) = 1) \) for general \( t \). Such examples can be found by constructing \( A_2 \) as a conjugation of \( A_1 \) (for a generic \( A_1 \)) by an orthogonal perturbation of the identity which preserves the vector \( (1, \ldots, 1) \).
Let us start by describing Tőth’s work on the quantum Heisenberg ferromagnet [22]. Building on earlier work by Conlon and Solovej, he found what physicists term a graphical representation of the model, i.e. a rigorous translation to an (interacting) random walk question. Most relevant for us is Tőth’s formula for the spontaneous magnetization \( m(\beta) \) of the quantum Heisenberg ferromagnet at inverse temperature \( \beta \). Let \( c_\beta(0) \) be the size of the cycle of 0 at time \( \beta \) for the interchange process on \([-r, r]^3\). Then \[22, (5.2)\]

\[
m(\beta) = \frac{1}{2} \lim_{n \to \infty} \lim_{r \to \infty} \frac{\mathbb{E}\left(1\{c_\beta(0) > n\}2\sum_{k \geq 1} s_k(\beta)\right)}{\mathbb{E}\left(2\sum_{k \geq 1} s_k(\beta)\right)}
\]

(recall that \( s_k(\beta) \) is the number of cycles of length \( k \) at time \( \beta \), so their sum is just the total number of cycles, again for the interchange process on \([-r, r]^3\)). Notice the somewhat counterintuitive fact that the inverse temperature becomes the time in this representation. With this formula (which some readers might feel more convenient to simply take as the definition of \( m(\beta) \)), Tőth’s conjecture is

**Conjecture 1.** \( m(\beta) \) admits a phase transition, i.e. there exists some \( \beta_c \) such that \( m(\beta) = 0 \) for \( \beta < \beta_c \) and \( m(\beta) > 0 \) for \( \beta > \beta_c \).

It is natural to try first to remove the weights and investigate only \( \mathbb{P}(c_\beta(0) > n) \) (Tőth himself hints that this might be an interesting toy model). One then gets the following:

**Conjecture 2.** The function

\[
\lim_{n \to \infty} \lim_{r \to \infty} \mathbb{P}(c_\beta(0) > n)
\]

Undergoes a phase transition in \( \beta \): it is zero for \( \beta < \beta_c \) (not necessarily the same \( \beta_c \) as in the previous conjecture) and positive for \( \beta > \beta_c \).

For both conjectures, it is not difficult to show that for \( \beta \) sufficiently small the corresponding limits are zero. What is wide open, for both conjectures, is that for \( \beta \) sufficiently large, the limits are non-zero. In other words, the big open problem at this point is not sharpness or uniqueness of the phase transition, but the actual existence of the high \( \beta \) phase (the so-called ordered phase).

Conjecture 2 was investigated when \([-r, r]^3\) is replaced by the complete graph, the so-called mean-field case. The mean-field case was solved first by Berestycki & Durrett [8] (who arrived at this problem from a different angle) and then by Schramm [21], who gave much more information on the structure of the large cycles. In the mean-field case, \( \beta_c \) is explicitly known. An analog of conjecture 2 for infinite graphs was investigated for trees [2, 11, 12]. Notably, for trees of sufficiently high degree, [12] shows that there is a phase transition without calculating the value of \( \beta_c \).

We consider our Theorem 3 as a stepping stone for a representation-theoretic attack on both conjectures. For conjecture 2, it reduces the problem to a calculation or estimate of the eigenvalues of only some representations. In the mean-field case, these eigenvalues are explicitly known [6] which leads to a simple analysis of this problem, see [4]. For conjecture 1, this requires an extra ingredient even in the mean-field case: the interaction between the function \( c_\beta(0) \) and the function \( 2\sum s_k(\beta) \). We hope to tackle this problem in the future.
To gain some more insight on the non-mean-field case in conjecture 2, let us examine the case $k = n$, i.e. apply Theorem 1 to the graph $[-r, r]^3$. For this graph the eigenfunctions and eigenvalues are explicitly known. Every vector $\xi \in \{0, \ldots, 2r - 1\}^3$ the function $f(v) = \exp(2\pi i \langle \xi, v \rangle / (2r - 1))$ is an eigenvector with the eigenvalue being $\sum (1 - \cos(2\pi \xi_j / (2r - 1)))$. Plugging these values into Theorem 1 with a little calculation shows, for example,

$$\min \left\{ t : \Pr(s_n(t) = 1) \geq \frac{1}{2n} \right\} \approx r^2 \approx n^{2/3}.$$ 

In other words, the probability starts approaching the limit value $\frac{1}{n}$ only when $t$ is of the order of $n^{2/3}$ (for general dimensions, i.e. the graphs $[-r, r]^d$, the value would be $n^{2/d}$).

Thus we see that, unlike what one would expect from a naive extrapolation of conjecture 2, the probability of a cycle of length $n$ does not equilibrate at constant time but after much longer time. The culprit for this slow equilibration lies in the representation $[n - 1, 1]$ appearing in the sum when $k = n$. It is therefore reassuring to notice that this representation appears only when $k = n$. Again, at this point our estimates for the eigenvalues $\lambda_j([-r, r]^3, \rho)$ are too weak to give good information on Tóth’s conjecture. See [23] for more information on these eigenvalues.

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Appendix. Kirchoff’s matrix-tree theorem

Here we give a new proof of the following old theorem, essentially due to Kirchoff.

Theorem. Let $G$ be any weighted graph, and denote by $w_e$ the weight of the edge $e$. For a spanning tree $T$, denote $w(T) = \prod_{e \in T} w_e$ where the product is over all edges $e$ of $T$. Finally denote by $0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1}$ the eigenvalues of the continuous time Laplacian $\Delta_G$. Then

$$\sum_T w(T) = \frac{1}{n} \prod_{i=1}^{n-1} \lambda_i$$

(it is quite common to replace the product on the right-hand side by the absolute value of a cofactor of the Laplacian $\Delta_G$, which gives an equivalent formulation, but for the approach here this formulation is the more natural one).

Proof. Apply Theorem 1 with $t \to 0$. On the right-hand side one gets

$$\frac{1}{n} \prod_{i=1}^{n-1} (1 - e^{-\lambda_i t}) = \frac{1}{n} \prod_{i=1}^{n-1} (\lambda_i t + O(t^2)) = \frac{1}{n} t^{n-1} \prod_{i=1}^{n-1} \lambda_i + O(t^n).$$

(5)

To estimate the left-hand side we use the coagulation-fragmentation view of the interchange process, see e.g. [21]. By this we mean the observation that when one applies the transposition $(i, j)$, if $i$ and $j$ belong to different cycles in the permutations, then the application of $(i, j)$ causes the cycles to merge; while if $i$ and $j$ belong to the same cycle, then this causes the cycle to split. In particular, if one draws an auxiliary graph $A_t$ with an edge between every $i$ and $j$ for which the transposition
\((i, j)\) was applied by time \(t\), then the cycles of the permutation at time \(t\) are subsets of the connected components of \(A_t\).

Now, since we are interested in the case that \(s_n(t) = 1\), i.e. in the case that the permutation is one big cycle, then this can happen only when \(A_t\) is connected. But a connected graph with \(n\) vertices must have at least \(n - 1\) edges, and if it has \(n - 1\) edges precisely then it is a spanning tree. Further, if for some spanning tree \(T\) the edges of \(T\) are exactly those that have rung by time \(t\), and each one rang exactly once, then the permutation is one big cycle, because a fragmentation event never happened (these require closed paths in the graph \(A_t\)) and \(n - 1\) coagulations lead to one big cycle. Hence we get for the left-hand side,

\[
\mathbb{P}(s_n(t) = 1) = \sum_T \mathbb{P}(\text{the edges of } T \text{ are exactly those that rang by time } t) + O(t^n).
\]

Now, for an edge \(e\) the probability that it rang exactly once by time \(t\) is \(w_e t + O(t^2)\). Further, all these events (for various \(e\)) are independent. So we can continue to write

\[
\mathbb{P}(s_n(t) = 1) = \sum_T \prod_{e \in T} (w_e t + O(t^2)) + O(t^n) = t^{n-1} \sum_T \prod_{e \in T} w_e + O(t^n).
\]

Comparing to (5) we get Kirchoff’s theorem. \(\square\)

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