Vortex formation in weakly-interacting inhomogeneous Bose condensates

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We solve the time-dependent Gross-Pitaevskii in 2-D to simulate the flow of an object through a dilute Bose condensate trapped in a harmonic well. We demonstrate vortex formation and analyze the process in terms of the accumulation of phase-slip and the evolution of the fluid velocity.

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The origin of drag in quantum liquids is central to the understanding of superfluidity \[\text{[1]}.\] For a weakly-interacting fluid, Bogoliubov showed that macroscopic occupation of the ground state leads to a linear dispersion curve, and hence superfluidity for motion slower than a critical velocity. However, for liquid helium, the transition to normal flow is observed at a much lower velocity than can be explained by the dispersion curve. This led Feynman \[\text{[2]}\] to suggest that the onset of dissipation may arise due to vortex shedding, but experimental verification of this idea has been impeded by the two-fluid nature of liquid helium, which complicates quantitative comparison with theory.

The recent experimental discovery of Bose condensation in dilute alkali vapours \[\text{[3–5]}\] presents a near-perfect system for the study of quantum fluids: the condensates are almost pure, and sufficiently dilute that the interactions can be accurately parameterized in terms of a scattering length. As a result, a relatively simple non-linear Schrödinger equation (NLSE), known as the Gross-Pitaevskii equation \[\text{[6]},\] gives a precise description of condensate dynamics. Experiments have confirmed that the NLSE is remarkably accurate in the limit of low temperature \[\text{[7–9]}\].

Recent experiments have also demonstrated that far-off resonant laser light may be used to split \[\text{[10]}\] or excite the condensate \[\text{[11]}\]. The question arises whether such light forces may be used to study the phenomenon of vortex formation, and thereby shed some light on the issue of superfluidity. The appearance of vortices in solutions of the 2-D NLSE has been considered by Frisch et al. \[\text{[12]}\]. However, they consider homogeneous fluid flow past an impenetrable obstacle, which is inappropriate for atomic condensates. Here, we solve the time-dependent 2-D NLSE to simulate the flow of an ‘object’ through a dilute inhomogeneous condensate. The ‘object’ is the potential barrier produced by far-off resonant blue-detuned laser beam \[\text{[1]}\]. The condensate is formed in a harmonic trap with the obstacle at the center. Subsequently, the obstacle is withdrawn at a constant speed, \(v\) (from the Galilean invariance of the NLSE \[\text{[1]}\], this is equivalent to flow past a stationary object). Vortices are formed when the condensate, whose motion is limited by the speed of sound, cannot adjust sufficiently quickly to the density wave induced by the object. We show how the time scale for vortex formation can be extracted from the phase-slip of the wavefunction.

The evolution of the condensate is described by the NLSE \[\text{[13]}:\]

\[
i\partial_t \psi = \left(-\nabla^2 + V + C|\psi|^2\right) \psi ,
\]

where \(C = 8\pi Na\) is the nonlinear coefficient, \(N\) is the number of atoms in the condensate, and \(a\) is the s-wave scattering length. The potential term, \(V = \frac{1}{2}(x^2 + y^2) + \alpha \exp(-\beta x^2 - \beta(y - vt)^2)\), describes a symmetric harmonic trap with a light-induced ‘Gaussian’ potential barrier, moving with velocity, \(v\). We have considered a variety of obstacle parameters but present detailed results for \(C = 500\), \(\alpha = 30\), and \(\beta = 3\).

The eigenvalue problem, \(\psi = \phi(x, y)e^{-\mu t}\), (where \(\mu\) is the chemical potential), and \(v = 0\), is easily solved using finite difference methods. With \(\alpha = 0\), we find \(\mu = 9.003\). The depletion of density in the trap center due to the presence of the obstacle, (with \(\alpha = 30\), \(\beta = 3\), \(v = 0\)), raises the chemical potential to \(\mu = 9.208\). A change in the object potential creates a local disturbance which propagates through the fluid. The time evolution of \(\psi\) was determined by the split-operator technique \[\text{[14,15]}\]. Extinction of the object, excites sound waves which propagate with speed, \(c = \sqrt{2C|\psi|^2}\), \(\sim 4.3\) at the peak density). Translation of the object displaces the condensate center-of-mass \[\text{[15]}\], and leads to vortex-pair creation as illustrated in Fig. \[\text{[1]}\]. The vortex minima are highlighted by the
surface plot shown in Fig. 2. The half-width of the vortex core is comparable to the healing length, \( \lambda \sim C^{-1/4} \). We have simulated the free expansion of the condensate and observe that the vortices also expand, permitting experimental detection using optical imaging.

For low object speeds, no vortices are observed implying that the process requires motion faster than a critical speed. For an impenetrable cylinder of diameter \( d \), the critical speed is related to the energy required to create a vortex-pair, \( \epsilon = (C/2a) \ln(d/\lambda) \). For a finite potential barrier, we find that the vortices emerge from a point, i.e., initially \( d \sim \lambda \), so the pair creation energy, \( \epsilon \sim 0 \), and the critical speed depends principally on the speed of sound. However, due to the condensate inhomogeneity, the speed of sound and hence the critical speed are spatially dependent. For this reason, we choose to study the process of vortex formation in terms of the fluid velocity and the phase of the wavefunction.

Fig. 3 shows a quiver plot of the fluid velocity, \( \mathbf{v}_s = (\psi^* \nabla \psi - \psi^* \nabla \psi^*)/i|\psi|^2 \), in the vicinity of the object, for \( t = 3.0 \) and \( v = 2.0 \) (as in Figs. 1 and 2). One sees that vortices are formed in pairs with opposing vorticity, and that the velocity is inversely proportional to distance from the vortex line, as expected. Two pairs are discernable in Fig. 3: one at \( y = 3.5 \), which corresponds to the density zeros in Fig. 2, and a second at \( y = 5.9 \) (not visible in Fig. 2 because it has yet to separate from the object). The second pair appears due to the rapid accumulation of phase-slip at the edge of the condensate, as discussed below.

By calculating the wavefunction phase, \( S(x, y, t) \), one may verify that the phase changes by \( 2\pi \) on circulation of the vortex line. In addition, the evolution of the phase determines the time-scale for vortex formation. At \( t = 0.0 \), the phase is uniform, then the motion induces a dephasing or ‘phase-slip’ centred on the object. The phase-slip between two neighbouring points along the \( y \)-axis is defined as: \( \Delta S(y, t) = \arg \{ \psi(0, y + \Delta y, t) \} - \arg \{ \psi(0, y, t) \} \), with \( -\pi < \arg \{ z \} \leq \pi \). The maximum phase-slip, \( \Delta S_{\text{max}} \), occurs at a \( y \)-coordinate close to the center of the object.
Fig. 4 shows a plot $\Delta S_{\text{max}}$ for different object speeds. For $v = 2.0$ and 3.0, the phase-slip accumulates gradually, reaching a value of $\pi$, and then changing sign at a critical time, $t_c$. The sign change indicates a reversal of the flow, and may be taken to define the moment of vortex creation. The critical time is proportional to the speed of sound: vortices are produced faster for smaller $c$, as the condensate requires longer to adjust to the perturbation (compare curves for $C = 200$ and $C = 500$ in Fig. 4). For $t > t_c$, the phase-slip builds again until another vortex-pair is formed. For $v = 1.3$, (upper plot in Fig. 4) the phase-slip initially saturates and then increases towards the edge of the condensate, where the density and hence the local speed of sound are lower. This explains the sudden formation of a vortex-pair as the object leaves the condensate, (e.g. the second pair for $v = 2.0$ appearing in Figs. 3 and 4). For $v = 3.0$, a third pair is created at $t = 2.5$: the object is still surrounded by condensate at this time because of a motion induced ‘stretching’.

To summarize, we have solved the time-dependent NLSE in 2-D to simulate the flow of an object through a dilute Bose condensate trapped in a harmonic well. The process of vortex formation has been analyzed in terms of the accumulation of phase-slip and the evolution of the velocity field. We find that the vortices emerge from a point close to the center of the object, and that the vortex shedding frequency is higher when the speed of sound is lower.
A complete description of dilute atomic vapours should include the effects of the non-condensate density, finite temperature, and dissipation. To what extent the NLSE provides an accurate description of vortex formation requires quantitative comparison between simulation and experiment. Such a comparison will provide the focus for future work.

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[12] T. Frisch, Y. Pomeau, and S. Rica, Phys. Rev. Lett. **69**, 1644 (1992).
[13] We use scaled harmonic oscillator units (h.o.u.) throughout, i.e., for a symmetric harmonic trap with angular frequency \( \omega \) and particles of mass \( m \), the units of length, time and energy are \( (\hbar/2m\omega)^{1/2} \), \( \omega^{-1} \) and \( \hbar \omega \), respectively.
[14] Most of the calculations were performed using a square box of side 10 h.o.u. divided into a grid of 256×256 points. A larger grid (20 h.o.u., 512×512) was used to check for edge-effects. More details of the numerical methods employed are discussed elsewhere: B. Jackson, J. F. McCann, and C. S. Adams, to be published.
[15] The center-of-mass motion provides a useful diagnostic: the initial acceleration is \( v \) divided by the integration time-step; after a few steps the solution is only sensitive to \( v \); and when the object leaves the condensate, the center-of-mass oscillates at the trap frequency.