ON A CLASS OF SPECIAL LINEAR SYSTEMS OF $\mathbb{P}^3$

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Abstract. In this paper we deal with linear systems of $\mathbb{P}^3$ through fat points. We consider the behavior of these systems under a cubo-cubic Cremona transformation that allows us to produce a class of special systems which we conjecture to be the only ones.

Introduction

Let us take the projective space $\mathbb{P}^n$ and let us consider the linear system of hypersurfaces of degree $d$ having some points of fixed multiplicity. The virtual dimension of such systems is the dimension of the system of degree $d$ polynomials minus the conditions imposed by the multiple points and the expected dimension is the maximum between the virtual one and $-1$. The systems whose dimension is bigger than the expected one are called special systems.

There exists a conjecture due to Hirschowitz (see [8]), characterizing special linear systems on $\mathbb{P}^2$, which has been proved in some special cases (see [2, 3, 11, 10]). In this paper we describe a class of special linear systems on $\mathbb{P}^3$. The main tool will be the cubo-cubic Cremona transformation which allows us to transform a linear system into another one. The dimension of the two systems is the same, while the virtual one may be different. This is a new phenomenon which does not occur in $\mathbb{P}^2$. In Proposition 2.3 we give a formula expressing the difference between these virtual dimensions and in particular we prove that if a transformation decreases the degree of a system then it does not decrease its virtual dimension (Corollary 2.4). We will say that a system for which is no longer possible to decrease the degree using one of these transformations is in standard form. Starting from a special system in standard form it is possible to construct infinitely many special systems by applying a sequence of cubic Cremona transformations. In this paper we describe two types of special systems in standard form and we conjecture that they produce all the possible special cases.

The paper is organized as follows: in the first section we recall some definitions and notations. In Section 2 we give a description of a cubic Cremona transformation of $\mathbb{P}^3$ and its action on linear systems, while Section 3 deals with the resolution of the indeterminacy of this transformation. In Section 4 we state the conjecture and we give a procedure for evaluating the dimension of a system and in the next one we give some motivations for it. Section 6 deals with special homogeneous linear system according to the conjecture and finally Section 7 provides some examples related to the procedure.

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1. Preliminaries

We start by fixing some definitions and notations.

**Definition 1.1.** Given a sheaf $\mathcal{F} = \mathcal{O}_{\mathbb{P}^3}(d) \otimes \mathcal{I}_V$, where $\mathcal{I}_V$ is the ideal sheaf of a subscheme $V \subset \mathbb{P}^3$, we denote by $v(\mathcal{F})$ its *virtual dimension*, defined as

$$v(\mathcal{F}) = \chi(\mathcal{F}) - 1.$$ 

With $L = \mathcal{O}_{\mathbb{P}^3}(d, m_1, \ldots, m_r)$ we will denote the linear system associated to the sheaf $\mathcal{O}_{\mathbb{P}^3}(d) \otimes \mathcal{I}_Z$, where $Z = \sum m_i p_i$ is a zero-dimensional scheme of fat points. If no confusion arises, by abuse of notation we will use the same letter $L$ to denote also the sheaf. From the cohomology exact sequence associated to

$$0 \rightarrow L \rightarrow \mathcal{O}_{\mathbb{P}^3}(d) \rightarrow \mathcal{O}_Z \rightarrow 0,$$

we obtain that $h^i(L) = 0$ for $i = 2, 3$. Therefore we deduce that $v(L) = h^0(\mathcal{O}_{\mathbb{P}^3}(d)) - h^0(\mathcal{O}_Z) - 1$, which may be written also as

$$v(L) = \left(\frac{d + 3}{3}\right) - \sum_{i=1}^r \left(\frac{m_i + 2}{3}\right) - 1.$$ 

Let us denote by $e(L) = \max(v(L), -1)$ the *expected dimension* of $L$.

**Definition 1.2.** A non-empty linear system $L$ is *special* if its expected dimension is strictly smaller than the effective one or, which is the same, if $h^1(L) \neq 0$.

Let $(X, \pi)$ be the blow-up of $\mathbb{P}^3$ along $\{p_1, \ldots, p_r\}$; by abuse of notation we will denote by $L$ the linear system associated to $L = dH - \sum m_i E_i$, where $H$ is the pull-back of an hyperplane of $\mathbb{P}^3$ and $E_i = \pi^{-1}(p_i)$. Let $\langle h, e_1, \ldots, e_r \rangle$ and $\langle h, e_1, \ldots, e_r \rangle$ be two bases for the Chow groups $A^1(X)$ and $A^2(X)$ respectively, where $h$ is the pull-back of a line and $e_i$ is the class of a line in $E_i$. The intersection matrix, with respect to these two bases, is diagonal with the first element equal to 1 and the others equal to $-1$ (since $E_i e_i = -1$).

Given a curve $C \subset \mathbb{P}^3$, by abuse of notation we will denote by $LC$ the intersection product of their strict transforms in $X$. We will write $C \in \ell_3(\delta, \mu_1, \ldots, \mu_r)$ to denote a curve of degree $\delta$ with multiplicity $\mu_i$ at $p_i$. In this way, the intersection is given by the formula

$$LC = d\delta - \sum_{i=1}^r \mu_i m_i.$$ 

(1.1)

We recall the Riemann-Roch formula for a divisor $L$ on a smooth threefold $X$: 

$$\chi(L) = \frac{L(L - K_X)(2L - K_X) + e_2(X)L}{12} + \chi(\mathcal{O}_X).$$ 

If the linear system associated to $L$ can be written as $|L| = F + |M|$, where $F$ is a fixed divisor of $|L|$ and $|M|$ is the residual system, then the above formula implies:

$$v(L) = v(M) + v(F) + \frac{FM(L - K_X)}{2}.$$ 

(1.2)
2. Cubic Cremona transformations and linear systems

In this section we focus our attention on a class of cubic Cremona transformations of $\mathbb{P}^3$. Consider the system $\mathcal{L}_3(3, 2^4)$, by putting the four double points in the fundamental ones, the associated rational map is given by:

$$\text{Cr} : (x_0 : x_1 : x_2 : x_3) \rightarrow (x_0^{-1} : x_1^{-1} : x_2^{-1} : x_3^{-1}).$$

(2.1)

The birational map 2.1 induces an action on the Picard group of $X$ which can be described in the following way:

**Proposition 2.1.** The action of transformation 2.1 on $L = \mathcal{L}_3(d, m_1, \ldots, m_r)$ is given by:

$$\text{Cr} (L) = \mathcal{L}_3(d + k, m_1 + k, \ldots, m_4 + k, m_5, \ldots, m_r),$$

(2.2)

where $k = 2d - \sum_{i=1}^{4} m_i$.

**Proof.** Since the transformation 2.1 is an isomorphism out from the base locus, we can reduce to the case $r = 4$. The set of monomials which generate $L$ (i.e. monomials of degree $d$ in $x_1, \ldots, x_4$ with multiplicity $m_i$ at $p_i$ for $i = 1 \ldots, 4$) can be represented by the set of their exponents

$$T := \{(a_1, \ldots, a_4) \in \mathbb{Z}_4 \mid 0 \leq a_i \leq d - m_i \text{ and } \sum_{i=1}^{4} a_i = d\}.$$  

(2.3)

In the same way $T'$ corresponds to $\mathcal{L}_3(d + k, m_1 + k, \ldots, m_4 + k)$. Observe that

$$\text{Cr} \left( \prod_{i=1}^{4} x_i^{a_i} \right) = \prod_{i=1}^{4} x_i^{d-a_i},$$

where the right term may be written as

$$(x_1^{m_1} \cdots x_4^{m_4}) \prod_{i=1}^{4} x_i^{d-a_i-m_i}.$$ 

This may be summarized by saying that the transformation 2.1 induces the map $f(a_1, \ldots, a_4) = (d - a_1 - m_1, \ldots, d - a_4 - m_4)$ from $\mathbb{Z}^4$ to $\mathbb{Z}^4$. We want to prove that $f$ is a bijection between $T$ and $T'$. First of all observe that if $(a_1, \ldots, a_4) \in T$ then $0 \leq d - a_i - m_i \leq (d + k) - (m_i + k)$ and $\sum_{i=1}^{4} (d - a_i - m_i) = d + k$, which implies that $f(T) \subseteq T'$. In the same way it is possible to prove that $f(T') \subseteq T$ and, since $f^2 = 1_{\mathbb{Z}^4}$, we get the thesis. \hfill $\square$

Observe that $\dim \text{Cr} (L) = \dim L$ but in general the virtual dimensions of the two systems may be different.

**Lemma 2.2.** Suppose that $2d - m_1 - m_2 - m_3 < 0$ then $\text{Bs} \mathcal{L}_3(d, m_1, m_2, m_3)$ contains the plane through the three points.

**Proof.** Consider $T$ as before (with $m_4 = 0$), then we have the inequality $a_1 + a_2 + a_3 \leq 3d - m_1 - m_2 - m_3 < d$. This implies that each monomial contains $x_4$ which gives the fixed plane. \hfill $\square$
Proposition 2.3. Let \( \mathcal{L} = \mathcal{L}_4(d, m_1, \ldots, m_r) \) be a linear system such that \( 2d \geq m_i + m_j + m_k \) for any choice of \( \{i, j, k\} \subset \{1, 2, 3, 4\} \) then

\[
v(\text{Cr} (\mathcal{L})) - v(\mathcal{L}) = \sum_{t_{ij} \geq 2} \left( \frac{1 + t_{ij}}{3} \right) - \sum_{t_{ij} \leq 1} \left( \frac{1 - t_{ij}}{3} \right),
\]

(2.4)

where \( t_{ij} = m_i + m_j - d \).

Proof. We can reduce to the case \( \mathcal{L} = \mathcal{L}_4(d, m_1, \ldots, m_4) \), since the contribution of the other points is 0 in equation (2.4). In this case, \( \dim \mathcal{L} = \#T \), where \( T \) is the set defined in 2.3. In order to evaluate this number, consider the sets

\[
\Delta := \{ (a_1, \ldots, a_4) \in \mathbb{Z}^4_{\geq 0} \mid \sum_{i=1}^{4} a_i = d \}, \\
\Delta_i := \{ (a_1, \ldots, a_4) \in \Delta \mid a_i \geq d - m_i + 1 \},
\]

and observe that \( T = \Delta \setminus \bigcup_{i=1}^{4} \Delta_i \). By the inclusion-exclusion principle,

\[
\#T = \#\Delta - \sum_{i} \#\Delta_i + \sum_{i < j} \#(\Delta_i \cap \Delta_j) - \sum_{i < j < k} \#(\Delta_i \cap \Delta_j \cap \Delta_k).
\]

The virtual dimension \( v(\mathcal{L}) \) is given by the first two terms on the right side of the equation above. A point \( (a_1, \ldots, a_4) \) belongs to the intersection of the first three \( \Delta_i \)'s if \( a_i > d - m_i \) for \( i = 1, 2, 3 \); summing up these inequalities one obtains

\[
d - a_4 > 3d - m_1 - m_2 - m_3
\]

which implies that \( 2d - m_1 - m_2 - m_3 < 0 \), a contradiction to our assumption. The same argument holds for any other triple of points, so this gives the following formula

\[
\dim \mathcal{L} = v(\mathcal{L}) + \sum_{i < j} \#(\Delta_i \cap \Delta_j).
\]

The set \( \Delta_1 \cap \Delta_2 \) is in one to one correspondence with the tetrahedron

\[
\Delta_{12} := \{ (b_1, \ldots, b_4) \in \mathbb{Z}^4_{\geq 0} \mid \sum_{i=1}^{4} b_i = m_1 + m_2 - d - 2 \}
\]

where \( b_i = a_i - d - 1 + m_i \) for \( i = 1, 2 \) and \( b_i = a_i \) for \( i = 3, 4 \). The same holds for the other intersections and, since \( m_i + m_j - d - 2 = t_{ij} - 2 \), we have \( \#(\Delta_i \cap \Delta_j) = \binom{t_{ij}}{3} \). The equality \( \dim \mathcal{L} = v(\text{Cr} (\mathcal{L})) \) implies that

\[
v(\text{Cr} (\mathcal{L})) - v(\mathcal{L}) = \sum_{t_{ij} \geq 2} \left( \frac{t_{ij} + 1}{3} \right) - \sum_{t_{ij} \leq 1} \left( \frac{t'_{ij} + 1}{3} \right),
\]

where \( t'_{ij} \) is defined in the same way of \( t_{ij} \) for the system \( \text{Cr} (\mathcal{L}) \). This implies that \( t'_{12} = (m_1 + k) + (m_2 + k) - (d + k) = t_{12} + k = -t_{34} \) and an analogous equality holds for each \( t'_{ij} \). \( \square \)

Corollary 2.4. Under the same assumptions of Proposition 2.3, if the degree of \( \text{Cr} (\mathcal{L}) \) is smaller than that of \( \mathcal{L} \), then \( v(\text{Cr} (\mathcal{L})) \geq v(\mathcal{L}) \).

Proof. The difference between the degree of \( \text{Cr} (\mathcal{L}) \) and that of \( \mathcal{L} \) is equal to \( k = 2d - \sum_{i=1}^{4} m_i \). From \( 2d < \sum_{i=1}^{4} m_i \) we deduce that, if \( t_{12} \geq 2 \) then \( d - m_4 - m_4 < m_1 + m_2 - d \) which is equivalent to \(-t_{34} < t_{12} \). The same holds for each \( t_{ij} \) such that \( t_{ij} \geq 2 \), hence the right side of equation (2.4) is non negative. \( \square \)
3. The isomorphisms of \( A^4 \) (\( Y \)) induced by the cubic transformation

The resolution of the indeterminacy of map 2.1 may be expressed by the following diagram:

\[
\begin{array}{ccc}
Y & \xrightarrow{p} & X \\
\pi \downarrow & & \pi' \downarrow \\
\mathbb{P}^3 & \xrightarrow{\pi'} & \mathbb{P}^3
\end{array}
\]

(3.1)

Here \( \pi \) is the blow-up of \( \mathbb{P}^3 \) along the points \( p_1, \ldots, p_4 \) and \( p \) is the blow-up of \( X \) along the strict transforms of the lines \( l_{ij} \) through \( p_i \) and \( p_j \). The Picard group of \( X \) is generated by \( \langle H, E_1, \ldots, E_4 \rangle \) where \( E_i \) is the exceptional divisor corresponding to \( p_i \). Let us denote by \( F_i = p^* H_i \), this means that \( F_i \) is the blow-up of a plane through three points, and let \( F_{ij} \) be the exceptional divisor corresponding to the blow-up of the line through \( E_i \) and \( E_j \). The divisor \( F_{ij} \) is a quadric and we will denote by \( F_{ij}^* \) its “vertical” ruling, i.e. the one given by the lines of the blow-up, and by \( F_{ij} \) the other one (the “horizontal”). The morphism \( p' \) contracts the quadrics \( F_{ij} \) along the horizontal rulings while the divisors contracted by \( \pi' \) are the four fundamental planes. A base for \( \text{Pic}(Y) \) is given by \( \langle H, F_1, \ldots, F_4, F_{12}, \ldots, F_{34} \rangle \), where, with abuse of notation, we write \( H \) instead of \( p^* H \). Let \( H_1 = H - F_2 - F_3 - F_4 - F_{23} - F_{24} - F_{34} \) be the strict transform of the plane of \( \mathbb{P}^3 \) through the points \( p_2, p_3, p_4 \) and the three lines of the triangle. In the same way we define \( H_i \) for \( i = 2, 3, 4 \). The involution \( \text{Cr} : \text{Pic}(Y) \longrightarrow \text{Pic}(Y) \) induced by 2.1 is given by

\[
\text{Cr} (F_i) = H_i, \quad \text{Cr} (F_{ij}) = F_{hk},
\]

(3.2)

where \( \{h, k\} = \{1, 2, 3, 4\} \setminus \{i, j\} \), which gives

\[
\text{Cr} (H) = 3H - \sum_i 2F_i - \sum_{i < j} F_{ij}.
\]

(3.3)

**Proposition 3.1.** Let \( Z \) be the 0-dimensional scheme of four points \( p_1, \ldots, p_4 \) with multiplicities \( m_1, \ldots, m_4 \) and \( W \) be the 1-dimensional scheme of the lines \( \langle p_i, p_j \rangle \) with multiplicities \( n_{ij} \). The map 2.1 transforms the sheaf \( \mathcal{F} = \mathcal{O}_{\mathbb{P}^3}(d) \otimes \mathcal{I}_Z \otimes \mathcal{I}_W \) into \( \mathcal{F}' = \mathcal{O}_{\mathbb{P}^3}(d + s) \otimes \mathcal{I}_Z' \otimes \mathcal{I}_{W'} \), where \( s = 2d - \sum n_{ij} m_i' = m_i + s \) and \( n_{ij}' = d - m_i - m_j + n_{hk} \).

**Proof.** Let us take the pull-back of \( \mathcal{F} \) on \( Y \):

\[
p^* \pi^* \mathcal{F} = dH - \sum_i m_i F_i - \sum_{i < j} n_{ij} F_{ij}.
\]

By 3.2 and 3.3 its image under the map \( \text{Cr} \) is

\[
\text{Cr} (p^* \pi^* \mathcal{F}) = (d + s)H - \sum_i (m_i + s) F_i - \sum_{i < j} (d - m_i - m_j + n_{hk}) F_{ij},
\]

which gives the thesis. \[\square\]
To each linear system $\mathcal{L} = \mathcal{L}_3(d, m_1, \ldots, m_r)$ we associate a 1-cycle defined as

$$\Gamma(\mathcal{L}) := \sum_{t_{ij} \geq 1} t_{ij}l_{ij},$$

where $t_{ij} = m_i + m_j - d$ and $l_{ij}$ is the line through $p_i$ and $p_j$. Observe that by definition $H^0(\mathcal{L} \otimes I_{\Gamma(\mathcal{L})}) = H^0(\mathcal{L})$, since each line $l_{ij} \in \Gamma(\mathcal{L})$ is contained into the base locus of $\mathcal{L}$ with multiplicity at least $t_{ij}$.

**Proposition 3.2.** With the preceding notations we have

$$\chi(\mathcal{L} \otimes I_{\Gamma(\mathcal{L})}) = \chi(\mathcal{L}) + \sum_{t_{ij} \geq 2} \left( \frac{t_{ij} + 1}{3} \right).$$

**Proof.** Consider the pull-back of $\mathcal{L} \otimes I_{\Gamma(\mathcal{L})}$ on $Y$. In what follows we will concentrate on one of the six lines and in order to simplify the notations we will omit the indexes. Let $l \in \mathbb{A}^2(X)$ be the strict transform of the line, then $N_{l \mid X} \cong \mathcal{O}_{p_1}(-1) \oplus \mathcal{O}_{p_1}(-1)$. From the evaluation of the tautological line bundle associated to the blow-up of $l$ (see [3]) one obtains that $F_{l \mid F} = -f^v - f^h$. The intersection $lF = -tF$. The exact sequence

$$0 \rightarrow p^*\mathcal{L} - (k + 1)F \rightarrow p^*\mathcal{L} - kF \rightarrow (p^*\mathcal{L} - kF)|_F \rightarrow 0,$$

and the preceding formulas imply that $h^0((p^*\mathcal{L} - kF)|_F) = h^0(kf^h + (k - t)f^v) = 0$ if and only if $k < t$. Therefore $tF \subseteq Bs(p^*\mathcal{L})$ and

$$\chi(p^*\mathcal{L} - tF) = \chi(p^*\mathcal{L}) - \sum_{k=0}^{t} \chi(kf^h + (k - t)f^v)).$$

By Riemann-Roch theorem on the quadric $F$ we have $\chi(kf^h + (k - t)f^v)) = (k + 1)(k + 1 - t)$. An easy calculation shows that the last sum of the above equation is equal to $-\left(\frac{t}{3} + 1\right)$, hence applying this procedure to each one of the $F_{l_{ij}}$ one obtains:

$$\chi(p^*\mathcal{L} - \sum_{t_{ij} \geq 2} t_{ij}F_{l_{ij}}) = \chi(p^*\mathcal{L}) + \sum_{t_{ij} \geq 2} \left( \frac{t_{ij} + 1}{3} \right).$$

$\square$

**Corollary 3.3.** Let $\mathcal{L} = \mathcal{L}_3(d, m_1, \ldots, m_r)$ be a linear system and $C_1, \ldots, C_n \in \mathbb{A}^2(X)$ be a set of irreducible rational curves such that $N_{C_i \mid X} \cong \mathcal{O}_{p_1}(-1) \oplus \mathcal{O}_{p_1}(-1)$ and $\mathcal{L}C_i = -t_i \leq -2$ for $i = 1, \ldots, n$. Then

$$\dim \mathcal{L} - v(\mathcal{L}) \geq \sum_{i=1}^{n} \left( \frac{t_i + 1}{3} \right) - h^2(\mathcal{L} \otimes I_{\Gamma}),$$

where $\Gamma := \sum t_iC_i$ is the 1-dimensional scheme of the multiple curves.

**Proof.** Observe that in the proof of Proposition 3.2 the only assumption needed on the curve $l$ is that its normal bundle in $X$ is $\mathcal{O}_{p_1}(-1) \oplus \mathcal{O}_{p_1}(-1)$. Since we are assuming the same for the $C_i$, we have that $\chi(\mathcal{L} \otimes I_{\Gamma}) = \chi(\mathcal{L}) + \sum (\frac{t_i + 1}{3})$. The equalities $h^0(\mathcal{L}) = h^0(\mathcal{L} \otimes I_{\Gamma})$ and $h^2(\mathcal{L}) = 0$ give us the thesis. $\square$
Remark 3.4. Let us take a system $\mathcal{L}$, chose four points and consider the associated sheaf $\mathcal{L} \otimes \mathcal{I}_{\Gamma(\mathcal{L})}$. Use Proposition 3.1 to transform this sheaf into $\mathcal{L}' \otimes \mathcal{I}_{\Gamma(\mathcal{L}')}$. The system $\mathcal{L}'$ is obtained from $\mathcal{L}$ by applying Proposition 2.1. Observe that $h^i(\mathcal{L} \otimes \mathcal{I}_{\Gamma(\mathcal{L})}) = h^i(\mathcal{L}' \otimes \mathcal{I}_{\Gamma(\mathcal{L}')})$ since the pull-back of the first system on $Y$ is just the pull-back of the second one obtained by a base change on $\text{Pic}(Y)$. In particular this implies that the virtual dimensions of the two systems are the same. This, together with Proposition 3.2 gives another proof of Proposition 2.3.

Let us consider now the action of transformation 2.1 on the curves of $Y$. A basis for $\mathcal{A}^3(Y)$ may be given by: $\langle h, f_1, \ldots, f_4, f_{12}, \ldots, f_{34} \rangle$, where $h$ is the pull-back of the class of a line in $\mathbb{P}^3$, $f_i$ is the pull-back of a line of $E_i$ and $f_{ij} = f_{ij}$ is the vertical ruling of $F_{ij}$.

**Proposition 3.5.** The intersection matrix $M : \mathcal{A}^1(Y) \times \mathcal{A}^2(Y) \rightarrow \mathbb{Z}$ with respect to the chosen bases of $\mathcal{A}^1(Y)$ is given by

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

**Proof.** By using the projection formula, since $(p \circ \pi)_* F_i = p_i, F_i$ has a non zero intersection only with $f_i$ and $F_i |_{F_i} = F_{ij} |_{F_i} = -f_i^2 = -1$ where the second equation is due to $F_i^2 = p^* E_i^2 = p^* N_{E_i|X} = p^*(-e_i) = -f_i$. In the same way, since $(p \circ \pi)_* F_{ij} = l_{ij}, F_{ij}$ has non vanishing intersection only with $f_{ij}$. Let $F$ and $f$ be $F_{ij}$ and $f_{ij}$ respectively, then $F f = F_{ij} f = (-f^v - f^h) f^v = -1$. Finally $H h = 1$ since it is the intersection of a plane with a line. \hfill $\square$

**Proposition 3.6.** With the same notation as before, we have that $H F_{ij} = f_{ij}$ and $F_{ij}^2 = -h + f_i + f_j - 2 f_{ij}$.

**Proof.** As before we will use $F_i f$ instead of $F_{ij}, f_{ij}$. The first equality is obvious, since $p_* H$ intersect $p_* F$ along a point and $f$ is just the vertical fiber of the quadric over that point. Observe that since $F^2$ has a non vanishing intersection only with $H, F_i, F_j$ and $F$, then by Proposition 3.4 we have $F^2 = a h + m_i f_i + m_j f_j + b f$. The coefficient $a$ is equal to $F^2 H = F f = -1$. In the same way $m_i = -F^2 F_i = -F f = 1$ and $b = -F^3 = (-f^v - f^h)^2 = -2$. \hfill $\square$

**Proposition 3.7.** The action of map 2.1 on $\mathcal{A}^2(Y)$ is given by

$$\text{Cr}(h) = 3h - \sum_i f_i, \quad \text{Cr}(f_i) = 2h - \sum_{j \neq i} f_j, \quad \text{Cr}(f_{ij}) = h + f_{ks} - f_k - f_s,$$

where $\{k, s\} = \{1, 2, 3, 4\} \setminus \{i, j\}$.
Proof. Since $h = H^2$, $f_1 = -F_1^2$ and $f_{12} = HF_{12}$, we have
\[
\begin{align*}
\text{Cr}(h) &= \text{Cr}(H)^2 \\
&= (3H - 2 \sum_i F_i - \sum_{i<j} F_{ij})^2 \\
&= 9h - 4 \sum_i f_i - \sum_{i<j} (h - f_i - f_j + 2f_{ij}) - 6 \sum_{i<j} f_{ij} + 8 \sum_{i<j} f_{ij} \\
&= 3h - \sum_i f_i
\end{align*}
\]
\[
\begin{align*}
\text{Cr}(f_1) &= -\text{Cr}(F_1)^2 \\
&= -(H - F_2 - F_3 - F_4 - F_{23} - F_{24} - F_{34})^2 \\
&= -(h - f_2 - f_3 - f_4 - 3h + 2(f_2 + f_3 + f_4) - 2(f_{23} + f_{24} + f_{34}) \\
&= 2h - f_2 - f_3 - f_4.
\end{align*}
\]
\[
\begin{align*}
\text{Cr}(f_{12}) &= \text{Cr}(H)\text{Cr}(F_{12}) \\
&= (3H - 2 \sum_i F_i - \sum_{i<j} F_{ij})F_{34} \\
&= 3f_{34} - 4f_{34} + h - f_3 - f_4 + 2f_{34} \\
&= h + f_{34} - f_3 - f_4.
\end{align*}
\]
\[
\square
\]
Now, with $\ell_3(\delta, \mu_1, \ldots, \mu_4, \beta_{12}, \ldots, \beta_{34})$ we will mean the system of curves of $\mathbb{P}^3$ of degree $\delta$ with multiplicities $\mu_i$ in the four points and intersecting the line $l_{ij}$ along $\beta_{ij}$ points.

**Proposition 3.8.** The image of the curve $\ell = \ell_3(\delta, \mu_1, \ldots, \mu_4, \beta_{12}, \ldots, \beta_{34})$ by transformation \(\mathbb{P}^3\) is $\text{Cr}(\ell) := \ell_3(\delta', \mu_1', \ldots, \mu_4', \beta_{12}', \ldots, \beta_{34}')$, where
\[
\delta' = 3\delta - \sum_i 2\mu_i - \sum_{i<j} \beta_{ij}, \quad \mu'_r = \delta - \sum_{j \neq r} \mu_j - \sum_{(i,j) \neq r} \beta_{ij}, \quad \beta'_{ij} = \beta_{ks}. \quad (3.4)
\]

**Proof.** The strict transform of a general element of $\ell$ may be written on $Y$ as $\delta h - \sum_i \mu_i f_i - \sum \beta_{ij} f_{ij}$. The image of this element by the map $\text{Cr}$ is
\[
\delta(3h - \sum_i f_i) - \sum_i \mu_i (2h - \sum_{j \neq i} f_j) - \sum_{i<j} \beta_{ij} (h + f_{ks} - f_k - f_s).
\]
This gives immediately the value of $\delta'$ and $\beta'_{ij}$. For $\mu'_r$, observe that the $f_r$ appear in the first sum with coefficient $-\delta$, in the second with coefficient $\sum_{j \neq r} \mu_j$ and in the third sum it appears each time that the index $r \in \{k, s\}$ or, which is the same, each time that $r \notin \{i, j\}$.
\[
\square
\]
If the system $\ell$ has no intersection with any one of the six lines, then we have the following:
Corollary 3.9. The action of transformation $\mathbf{2}$ on $\ell = \ell_3(\delta, \mu_1, \ldots, \mu_r)$ is given by:

$$Cr(\ell) := \ell(\delta + 2h, \mu_1 + h, \ldots, \mu_4 + h, \mu_5, \ldots, \mu_r),$$

(3.5)

where $h = \delta - \sum_{i=1}^{4} \mu_i$.

The action of $Cr$ on the space of curves may be extended to a linear action on the space of polynomials $\mathbb{C}[\delta, \mu_1, \ldots, \mu_r]$. An easy calculation gives

Corollary 3.10. The following polynomials are invariant with respect to transformation $\mathbf{2}$:

$$2\delta - \sum_{i=1}^{s} \mu_i, \quad \delta^2 - \sum_{i=1}^{s} 2\mu_i^2 + 3.$$

4. Conjecture

In this section we state a conjecture which allows us to give a procedure for computing the dimension of a linear system $L = L_3(d, m_1, \ldots, m_r)$. First of all, since birational transformations do not change the effective dimension of $L$, we can perform Cremona transformations until we get a system in standard form. If at some step we get a system $L^k$ with a negative multiplicity $-\alpha_i$ at $p_i$, then the system contains $\alpha_i E_i$ in its base locus. We remark that if this is the case, then there exists also a fixed component contained $\alpha_i$ times in the base locus of the starting system $L$. This is the image of $E_i$ by the sequence of Cremona transformations sending $L^k$ back to $L$. This component can be removed without changing the dimension of $L$. In particular, we can remove $\alpha_i E_i$ from $L^k$ and keep performing Cremona if possible. We can then reduce to the study of the dimension of linear systems in standard form. In this direction let us state the following:

Conjecture 4.1. A linear system $L = L_3(d, m_1, \ldots, m_r)$ in standard form is special if and only if one of the following holds:

(i) there exists a quadric $Q = L_3(2, 1^9)$ such that $Q(L - Q)(L - K) < 0$;

(ii) there exists a line $\ell = \ell_3(1, 1^2)$ such that $\ell \leq -2$.

We remark that if condition (i) holds, then by equation 1.2 $v(L) < v(L - Q)$, while $\dim L \geq \dim(L - Q)$, which means that $L$ is special. In order to simplify the procedure we are going to prove that, under an extra assumption, condition (i) implies that the quadric $Q$ is contained in the base locus of $L$.

Lemma 4.2. Let us suppose that Harbourne-Hirschowitz Conjecture holds for linear systems on $\mathbb{P}^2$ with 10 fixed points. If $L = L_3(d, m_1, \ldots, m_r)$ is a system in standard form and such that $Q(L - Q)(L - K) < 0$, then $Q \subset Bs(L)$.

Proof. From the exact sequence

$$0 \rightarrow L - Q \rightarrow L \rightarrow L_{|Q} \rightarrow 0,$$

we get $v(L) = v(L - Q) + v(L_{|Q}) + 1$. If we compare with equation 1.2 we obtain that

$$v(L_{|Q}) = Q(L - Q)(L - K) \quad - 1 < 0.$$
The system $\mathcal{L}_Q = \mathcal{L}_Q((d, d), m_1, \ldots, m_9)$ is equivalent to the planar system $\mathcal{L}_2 = \mathcal{L}_2(2d - m_1, (d - m_1)^2, m_2, \ldots, m_9)$ (see [4]). Therefore in order to prove the thesis it is enough to prove that the system $\mathcal{L}_2$ is non-special or, since we assume that Harbourne-Hirschowitz conjecture holds for 10 multiple points, to prove that $\mathcal{L}_2$ is not $(-1)$-special. Let us compare the multiplicity $d - m_1$ with respect to the $m_i$’s. If $d - m_1 \geq m_3$, the system $\mathcal{L}_2$ is Cremona-stable (since $2d - m_1 \geq 2(d - m_1) + m_2$) and hence it is not special.

If $d - m_1 < m_4$ and $\mathcal{L}_2$ is not Cremona-stable, then $2d - m_1 < m_2 + m_3 + m_4$, which is not possible since we are assuming that $\mathcal{L}$ is in standard form.

Finally, if $m_4 \leq d - m_1 < m_3$ and $\mathcal{L}_2$ is not Cremona-stable, then $d \leq m_2 + m_3 - 1$.

If we write $d = m_2 + m_3 - t$, with $t \geq 1$, then $\mathcal{L}_2 = \mathcal{L}_2(2m_2 + 2m_3 - m_1 - 2t, m_2, m_3, (m_2 + m_3 - m_1 - t)^2, m_4, \ldots, m_9)$. Performing a Cremona transformation with the first three points we obtain a stable system (the degree equals the sum of the highest multiplicities).

Now we wonder: which system of quadrics may be contained into the base locus of a given linear system? The answer is given by the following:

**Lemma 4.3.** If $|\sum_{i=1}^n r_i Q_i|$ is contained in the base locus of a linear system, then the $Q_i$ must share 8 points, i.e. $Q_i \in \mathcal{L}_3(2, 1^8)$ for each $i$.

**Proof.** Consider any two of the given quadrics, say $Q_1, Q_2$, which share $s \leq 8$ points. Since $|Q_1 + Q_2| = \mathcal{L}_3(4s, 2^s, 1^{18-2s})$ has virtual dimension $16 - 2s$, the system moves unless $s = 8$. Now consider any three quadrics, say $Q_1, Q_2, Q_3$. If they have less than 8 common points, then $|Q_1 + Q_2 + Q_3| = \mathcal{L}_3(6, 3^7, 2^3)$ since, by the preceding discussion, each pair must share 8 points. This system has virtual dimension 1, which is not possible. If four or more quadrics share less than 8 points, then there exist three of them which do not share 8 points, and this is not possible as proved before.

On the other hand, let us prove that if $s = 8$, then the system $\mathcal{L} = |\sum_{i=1}^n r_i Q_i|$ cannot move. We can write $\mathcal{L} = \mathcal{L}_3(2r, r^8, r_1, \ldots, r_n)$, where $r = \sum_{i=1}^n r_i$. From the exact sequence of $Q_1$ we get

$$0 \longrightarrow \mathcal{L} - Q_1 \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}_{Q_1} \longrightarrow 0.$$ 

Let us consider the restricted system $\mathcal{L}_{Q_1} = \mathcal{L}_Q((2r, 2r), r^8, r_1)$. By [4] this is equivalent to the planar system $\mathcal{L}_2(3r, r^9, r_1)$, which is empty since $\mathcal{L}_2(3r, r^9)$ is the fixed cubic $\mathcal{L}_2(3, 1^9)$ counted $r$ times. Therefore also $\mathcal{L}_{Q_1} = \emptyset$ which implies that $Q_1$ is contained in the base locus of $\mathcal{L}$. This also implies that $r_1 Q_1$ is contained in the base locus of $\mathcal{L}$ and the same holds for $r_i Q_i$. Therefore $\mathcal{L} = \sum_{i=1}^n r_i Q_i$. □

In order to conclude the part of the procedure concerning quadrics we need the following:

**Lemma 4.4.** Let us suppose that Harbourne-Hirschowitz conjecture holds for linear systems on $\mathbb{P}^2$ with 10 points. If $\mathcal{L} = \mathcal{L}_3(d, m_1, \ldots, m_r)$ is in standard form and $Q(\mathcal{L} - Q)(\mathcal{L} - K) < 0$, where $Q$ is the quadric through the first 9 points, then $\mathcal{L} - Q$ is still in standard form.

**Proof.** By hypothesis $2d \geq \sum_{i=1}^4 m_i$; moreover the degree of $\mathcal{L} - Q$ is $d - 2$ while the first 9 multiplicities are $m_i - 1$. Therefore $\mathcal{L} - Q$ is in standard form unless
$m_{10} > m_4 - 1$. But this can happen only if $m_i = m$ for $i = 4, \ldots, 10$. In this case $\mathcal{L}$ would contain all the quadrics through $p_1, p_2, p_3$ and 6 of the points $p_4, \ldots, p_{10}$, but by Lemma 4.3 this is not possible. \hfill $\Box$

Therefore, assuming that Harbourne-Hirschowitz conjecture holds up to 10 points, we can proceed computing $Q(\mathcal{L} - Q)(\mathcal{L} - K)$. If it is negative we can remove the quadric $Q$ and consider the system $\mathcal{L} - Q$, which is still in standard form. We reorder the multiplicities if necessary and we keep removing the quadric as far as $Q(\mathcal{L} - Q)(\mathcal{L} - K) < 0$.

Let us consider part (ii) of Conjecture 4.1. We are going to prove that, in fact, such a system $\mathcal{L}$ is special.

**Proposition 4.5.** Let $\mathcal{L} = \mathcal{L}_3(d, m_1, \ldots, m_r)$ be a non-empty linear system and let $l$ be the line through $p_1, p_2$. If $\mathcal{L}l = - t \leq -1$ then $h^2(\mathcal{L} \otimes \mathcal{I}_l) = 0$.

**Proof.** Let $Z = Z' + Z''$, where $Z' = m_1 p_1 + m_2 p_2$ and $Z'' = m_3 p_3 + \ldots + m_r p_r$ and let $\mathcal{L}' = O_{\mathbb{P}^2}(d) \otimes I_Z'$. The tensor product of the defining sequence of $Z''$ with $\mathcal{L}' \otimes \mathcal{I}_l$ gives

$$0 \longrightarrow \mathcal{L} \otimes \mathcal{I}_l \longrightarrow \mathcal{L}' \otimes \mathcal{I}_l \longrightarrow O_{Z''} \longrightarrow 0.$$ 

Since $h^i(O_{Z''}) = 0$ for $i \geq 1$, it is enough to prove that $h^2(\mathcal{L}' \otimes \mathcal{I}_l) = 0$.

Let us take a plane $V \in \mathcal{L}(1, 1^2)$ and denote by $W$ the corresponding element of $\mathcal{L}(1, 1^2) \otimes \mathcal{I}_l$. From the defining sequence of $W$ we obtain

$$0 \longrightarrow (\mathcal{L}' - V) \otimes \mathcal{I}_{(t-1)}l \longrightarrow \mathcal{L}' \otimes \mathcal{I}_l \longrightarrow (\mathcal{L}' \otimes \mathcal{I}_l)|_W \longrightarrow 0.$$ 

Observe that $h^2((\mathcal{L}' \otimes \mathcal{I}_l)|_W) = h^2(O_{\mathbb{P}^2}(d-t) \otimes I_{Z''}) = 0$, since $t = m_1 + m_2 - d \leq d$ (otherwise $m_1 > d$ and the system would be empty). This means that

$$h^2((\mathcal{L}' - V) \otimes \mathcal{I}_{(t-1)}l) \geq h^2(\mathcal{L}' \otimes \mathcal{I}_l),$$

so we can proceed by induction on $t$ until we obtain the system $\mathcal{L}_3(d - t, m_1 - t, m_2 - t)$ whose $h^2$ vanishes. \hfill $\Box$

An easy consequence of Propositions 4.3 and 4.5 is the following:

**Corollary 4.6.** Let $\mathcal{L}$ and $l$ be as before. If $\mathcal{L}l = - t \leq -2$, then $\mathcal{L}$ is special and $\dim \mathcal{L} - v(\mathcal{L}) \geq \binom{t+1}{3}$.

The preceding discussion allows us to give a procedure to calculate $\dim \mathcal{L}$.

**Table 1.**

| While $\mathcal{L}$ is not in standard form |
|---------------------------------------------|
| put $\mathcal{L} := \mathcal{C}_r(\mathcal{L})$. |
| If $m_i = - \alpha_i < 0$ then put $\mathcal{L} := \mathcal{L} - \alpha_i E_i$. |
| While $Q(\mathcal{L} - Q)(\mathcal{L} - K) < 0$ |
| put $\mathcal{L} := \mathcal{L} - Q$. |
| Put $t_{ij} := m_i + m_j - d$. Return $\dim \mathcal{L} = v(\mathcal{L}) + \sum_{t_{ij} \geq 2} \left( \frac{t_{ij} + 1}{3} \right)$. |
Observe that in the last step of the procedure we assume that \( h^2(\mathcal{L} \otimes \mathcal{I}_\ell) = 0 \), where \( \Gamma = \sum t_{ij} l_{ij} \) is the 1-dimensional scheme of the lines \( l_{ij} \), such that \( \mathcal{L} l_{ij} = -t_{ij} < 0 \).

5. Motivations

In [4] Conjecture 4.1 has been proved for systems \( \mathcal{L}_3(d, m_1, \ldots, m_r) \) with \( r \leq 8 \). With the help of a procedure written with Singular [5], we verified the conjecture for systems \( \mathcal{L}_3(d, m^r) \) with \( m \leq 7 \) and \( r \leq 20 \).

A consequence of the conjecture is that the only curves \( \ell \) which can give speciality are the images of the line \( \ell_3(1, 1^2) \) by a finite set of Cremona transformations. In one direction, observe that if there exists a curve \( \ell \) which is the image of \( \ell_3(1, 1^2) \) by a finite set of Cremona transformations and such that \( \mathcal{L} \ell = -t \leq -2 \), then \( \mathcal{L} \) is special.

In fact, performing back Cremona, we get a system \( \mathcal{L}' \) such that \( \mathcal{L}' \ell_3(1, 1^2) = -t \), since the intersection product is invariant under Cremona. Therefore, by Proposition 4.5, since cohomology groups are invariant under birational transformations, \( h^2(\mathcal{L} \otimes \mathcal{I}_\ell) = 0 \), and \( \mathcal{L} \) is special. On the other hand, for other classes of curves one can have the following problem:

**Example 5.1.** Let \( \ell = \ell_3(4, 1^8) \) be the system of quartics through 8 fixed points, and suppose that \( \mathcal{L} \ell = -t \leq -2 \). The intersection product is \( 4d - \sum m_i \leq -2 \), and hence it must be \( 2d - \sum m_i \leq -1 \), which implies that \( \mathcal{L} \) is not in standard form. Let us perform a Cremona transformation involving the first 4 points. Since the intersection product is invariant under this transformation, \( \mathcal{L}' \ell' = \mathcal{L} \ell = -t \), where \( \mathcal{L}' = \text{Cr} (\mathcal{L}) \) and \( \ell' = \text{Cr} (\ell) \). Moreover the degree \( d' \) of \( \mathcal{L}' \) is strictly smaller than \( d \) and, by Proposition 4.5, \( \ell' = \ell \). Arguing as before we can say that \( \mathcal{L}' \) is not in standard form and we can perform another transformation. Iterating this procedure we can transform \( \mathcal{L} \) until we get a system having negative degree, and hence we conclude that it must be \( \mathcal{L} = \emptyset \). Therefore the quartic can not give speciality. The fact that \( \ell_3(4, 1^8) \) can not be obtained from \( \ell_3(1, 1^2) \) by a finite set of Cremona transformations is an easy consequence of Corollary 4.10 since the invariant \( \delta^2 - \sum 2 \mu_i^2 + 3 \) is 3 for the quartic and 0 for the line.

Another consequence of Conjecture 4.1 is that if \( \mathcal{L} \) is in standard form then we can not find any curve \( \ell \neq \ell_3(1, 1^2) \) which is the image of a line and such that \( \mathcal{L} \ell \leq -2 \).

In order to give a motivation for this we prove the following:

**Lemma 5.2.** Let us suppose that \( \mathcal{L} = \mathcal{L}_3(d, m_1, \ldots, m_r) \) and \( \ell \) is obtained from \( \ell_3(1, 1^2) \) by a finite set of Cremona transformations such that at each step the degree of the curve increases. Then we can write

\[
\mathcal{L} \ell = \beta_1 (2d - \sum_{i=1}^4 m_i^{(1)}) + \cdots + \beta_6 (2d - \sum_{i=1}^4 m_i^{(s)}) + (d - m_h - m_k),
\]

where \( \beta_j \geq 1 \), \( m_i^{(j)} \) are chosen between \( m_1, \ldots, m_r \) and \( h, k \geq 5 \).

**Proof.** We argue by induction on the number of Cremona transformations necessary to obtain \( \ell \) from the line \( \ell_3(1, 1^2) \). First of all, after one transformation the image of the line is the rational normal cubic \( \ell_3(3, 1^6) \), having intersection product with \( \mathcal{L} \) equal to \( 3d - \sum m_i = (2d - \sum m_i) + (d - m_5 - m_6) \). Now we assume that the formula is true for \( \ell = \ell_3(\delta, \mu_1, \ldots, \mu_s) \) and we prove it for the curve
\( \ell' = \text{Cr}(\ell) = \ell_3(\delta', \mu'_1, \ldots, \mu'_r) \) obtained from \( \ell \) performing one more Cremona transformation increasing the degree. We can suppose that the transformation is based on the first 4 points. By formula (3.5), \( \delta' = \delta + 2h \) and \( \mu'_i = \mu_i + h \) for \( i = 1, \ldots, 4 \), where \( h = \delta - \sum_{i=1}^{4} \mu_i > 0 \), and \( \mu'_i = \mu_i \) for \( i \geq 5 \). Therefore \( \mathcal{L}\ell' - \mathcal{L}\ell = h(2d - \sum_{i=1}^{4} m_i) \), which gives the thesis.

\[ \Box \]

Corollary 5.3. If \( \mathcal{L} \) is not empty and in standard form and \( \ell \) is obtained from \( \ell_3(1,1^2) \) by a finite set of Cremona transformations such that at each step the degree of the curve increases, then \( \mathcal{L}\ell \geq 0 \).

\textit{Proof.} Since \( \mathcal{L} \) is in standard form, \( 2d \geq \sum m_i^{(j)} \). Moreover \( d - m_h - m_k \geq 0 \) since otherwise \( 2d < m_1 + m_2 + m_h + m_k \). Hence all the terms on the right side of equation 5.1 are non-negative. \( \Box \)

Therefore, if we assume that the following conjecture holds, we have that a system in standard form can not have negative intersection product with the image of a line.

Conjecture 5.4. Let \( \ell = \ell_3(\delta, \mu_1, \ldots, \mu_r) \) be a curve that can be obtained from the line \( \ell_3(1,1^2) \) by a finite set of Cremona transformations. Then \( \ell \) can be obtained by a finite set of Cremona transformations such that at each step the degree of the curve increases.

6. Homogeneous Linear Systems

In this section we study special homogeneous systems \( \mathcal{L} = \mathcal{L}_3(d, m^r) \).

Proposition 6.1. The system \( \mathcal{L} \) is empty for \( d \leq 2m - 1 \) and \( r \geq 8 \).

\textit{Proof.} It is enough to show that the system \( \mathcal{L}_3(2m - 1, m^8) \) is empty. We first prove by induction that performing Cremona on the first four points and the last four alternatively, at the \( i \)-th step we get the system \( \mathcal{L}' = \mathcal{L}_3(2m - 2i^2 - 1, (m - i^2 + i)^4, (m - i^2 - i)^4) \). The basis of induction holds since applying Cremona to the starting system we get \( \mathcal{L}' = \mathcal{L}_3(2m - 3, (m - 2)^4, m^4) \). If we now perform Cremona to \( \mathcal{L}' \) taking the first four points we have that \( k = 2(2m - 2i^2 - 1) - 4(m - i^2 + i) = -4i - 2 \) and hence the transform system is \( \text{Cr}(\mathcal{L}') = \mathcal{L}_3(2m - 2(i + 1)^2 - 1, (m - (i + 1)^2 - (i + 1))^4, (m - i^2 - i)^4) \). The multiplicity \( m - i^2 - i \) can be written as \( m - (i + 1)^2 + (i + 1) \) and hence, reordering the points, we get that \( \text{Cr}(\mathcal{L}') = \mathcal{L}' + 1 \), which gives the inductive step.

We keep performing these transformations until \( m - i^2 - i \leq 0 \). At this step we have that \( 2m - 2i^2 - 1 < m - i^2 + i \) and hence the system \( \mathcal{L}' \) is empty since it has some multiplicity bigger than the degree. \( \Box \)

By assuming Conjecture 4.1 and Harbourne-Hirschowitz conjecture for linear systems on \( \mathbb{P}^2 \) with 10 points, we can prove the following:

Proposition 6.2. If \( d \geq 2m \) the system \( \mathcal{L} = \mathcal{L}_3(d, m^r) \) is special if and only if \( r = 9 \) and \( 2m \leq d < [-1 + \sqrt{2m^2 + 2m}] \).
Proof. Our assumption on the degree implies that $\mathcal{L}$ is in standard form. According to Conjecture 4.1, the system $\mathcal{L}$ is special if and only if there exists either a quadric $Q \in \mathcal{L}(2,1^9)$ such that $Q(\mathcal{L} - Q)(\mathcal{L} - K) < 0$, or a line $\ell = \ell_3(1,1^2)$ such that $\ell \mathcal{L} \leq -2$. But under our hypothesis on the degree, a line through 2 fixed points has non-negative intersection product with $\mathcal{L}$, and hence $\mathcal{L}$ is special if and only if $Q(\mathcal{L} - Q)(\mathcal{L} - K) < 0$. But we can write $Q(\mathcal{L} - Q)(\mathcal{L} - K) = 2(d - 2)(d + 4) - 9(m - 1)(m + 2)$, which is negative if and only if $2m \leq d < [-1 + \frac{3}{2}\sqrt{2m^2 + 2m}]$. We end the proof showing that if $\mathcal{L}$ is special then the number $\mathcal{r}$ of fixed points can not be bigger than 9. If we suppose by contradiction that $\mathcal{r} \geq 10$, then $Q_i(\mathcal{L} - Q_i)(\mathcal{L} - K) < 0$, where $Q_i$ is the quadric through all the first 10 points but $p_i$. By Lemma 4.2 $\mathcal{L}$ must contain the system of quadrics $\sum_{i=1}^{10} Q_i$, and this is not possible because of Lemma 4.3.

We can conclude that if the system $\mathcal{L}$ has more than 9 fixed points (or exactly 8 points) then it is not special. If it has 9 fixed points, it is special if and only if $2m \leq d < [-1 + \frac{3}{2}\sqrt{2m^2 + 2m}]$. If $\mathcal{r} \leq 7$ and $d \geq 2m$, the system can not be special. Finally, if $\mathcal{r} \leq 7$ and $d \leq 2m - 1$, we have to follow the procedure of table 1.

7. Examples

We end the paper by presenting many examples of special systems and computing their dimension following the procedure of table 1. We remark that for each example the dimension we found in this way agrees with the dimension computed with the help of a Singular program.

Notation 7.1. We will use the symbols $(i)$, $(ii)$ and $(iii)$ to signify that we are applying the first, the second and the third step of the procedure (in $(i)$ we are applying a Cremona transformation, in $(ii)$ we are removing a plane and in $(iii)$ we are removing a quadric).

Example 7.2. Consider the system $\mathcal{L} := \mathcal{L}_3(7,4^6)$ with $v(\mathcal{L}) = -1$. Following the procedure of table 1 we can apply four Cremona transformations:

$$\mathcal{L}_3(7,4^6) \overset{(i)}{\rightarrow} \mathcal{L}_3(5,4^2,2^4) \overset{(i)}{\rightarrow} \mathcal{L}_3(3,2^4) \overset{(i)}{\rightarrow} \mathcal{L}_3(1),$$

so that $\dim \mathcal{L} = 3$. Observe that since $\mathcal{L}_3(7,4^6)\ell_3(3,1^6) = -3$ the starting system contains the rational normal curve. After the first transformation $\ell_3(3,1^6) \overset{\cong}{\rightarrow} \ell_3(1,1^2)$ and, in fact, $\mathcal{L}_3(5,4^2,2^4)\ell_3(1,1^2) = -3$.

Example 7.3. Consider the system $\mathcal{L} := \mathcal{L}_3(12,7^6)$ with $v(\mathcal{L}) = -50$. As before we can apply the following Cremona transformations:

$$\mathcal{L}_3(12,7^6) \overset{(i)}{\rightarrow} \mathcal{L}_3(8,7^2,3^4) \overset{(i)}{\rightarrow} \mathcal{L}_3(4,3^4,1^2) \overset{(i)}{\rightarrow} \mathcal{L}_3(4,3^4) \overset{\cong}{\rightarrow} \mathcal{L}_3(0,1^4) \overset{(ii)}{\rightarrow} \mathcal{L}_3(0),$$

so that $\dim \mathcal{L} = 0$. From the procedure we may deduce that $\mathcal{L}$ is given by the union of six surfaces of type $\mathcal{L}_3(2,2,1^5)$ (each surface can be obtained from a plane $\mathcal{L}_3(1,1^3)$ applying one Cremona transformation).

In these two first examples, performing some Cremona transformations and removing fixed planes we get a system in standard form which is not special. Let us give two examples where the system in standard form is still special.
Example 7.4. Consider the system \( \mathcal{L} := \mathcal{L}_3(10, 6^5) \) with \( v(\mathcal{L}) = 5 \). Apply the Cremona transformation:

\[
\mathcal{L}_3(10, 6^5) \overset{(i)}{\dashrightarrow} \mathcal{L}_3(6, 6, 2^4).
\]

The last system is in standard form and has virtual dimension 11 while its dimension is 15, as expected by our conjecture. In fact each of the four lines \( l_{ij}, j = 2, \ldots, 5 \) has intersection product \(-2\) with the system.

Example 7.5. Consider the system in standard form \( \mathcal{L} := \mathcal{L}_3(16, 11, 7^8) \) with \( v(\mathcal{L}) = 10 \). Since \( Q(\mathcal{L} - Q)(\mathcal{L} - K) = -2 \) following the procedure, we may remove the quadric from the base locus of \( \mathcal{L} \):

\[
\mathcal{L}_3(16, 11, 7^8) \overset{(iii)}{\dashrightarrow} \mathcal{L}_3(14, 10, 6^8).
\]

The virtual dimension of the last system is 11 and its dimension is 19 (each of the 8 lines \( l_{ij}, j = 2, \ldots, 9 \) has intersection product \(-2\) with the system).

We remark that in the two examples above, when the linear system is in standard and it does not contain any quadric, then its speciality is given exactly by the sum of the binomials \( \binom{t_{ij}+1}{3} \), where \(-t_{ij}\) is the intersection product of the line \( l_{ij} \) with the system. In the following example we show that if a linear system \( \mathcal{L} \) is not in standard form, then its speciality can be strictly smaller than the sum of these binomials, or equivalently \( h^2(\mathcal{L} \otimes \mathcal{I}_\Gamma) \neq 0 \) (where \( \Gamma \) is the 1-dimensional scheme of the multiple lines).

Example 7.6. The system \( \mathcal{L} = \mathcal{L}_3(3, 3^3) \) with \( v(\mathcal{L}) = -11 \) is not in standard form, hence

\[
\mathcal{L}_3(3, 3^3) \overset{(i)}{\dashrightarrow} \mathcal{L}_3(0, -3) \overset{(ii)}{\dashrightarrow} \mathcal{L}_3(0).
\]

This implies that \( \dim \mathcal{L} = 0 \), since it is three times the plane \( \mathcal{L}_3(1, 1^3) \). For each line \( l_{ij} \) through two of the three points we have \( \mathcal{L}l_{ij} = -3 \). Let \( \Gamma = 3(l_{12} + l_{13} + l_{23}) \). By Proposition 3.3 \( \dim \mathcal{L} - v(\mathcal{L}) \geq 12 - h^2(\mathcal{L} \otimes \mathcal{I}_\Gamma) \) which implies that \( h^2(\mathcal{L} \otimes \mathcal{I}_\Gamma) \geq 1 \).

In the next example we give a way to construct a class of systems whose speciality is due to multiple quadrics in the fixed locus.

Example 7.7. Let \( r_1, \ldots, r_n \) be positive integers and let \( r = \sum r_i \). As we proved in Lemma 4.3, the system \( \mathcal{L} = \mathcal{L}_3(2r, r^8, r_1, \ldots, r_n) \) has dimension 0, while its virtual dimension is \( \sum (r_i - \binom{r_i+2}{3}) \leq 0 \). Moreover this sum is 0 if and only if each \( r_i = 1 \), otherwise the system is special.

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