Abstract

Motivated by portfolio allocation and linear discriminant analysis, we consider estimating a functional $\mu \Sigma^{-1} \mu$ involving both the mean vector $\mu$ and covariance matrix $\Sigma$. We study the minimax estimation of the functional in the high-dimensional setting where $\Sigma^{-1} \mu$ is sparse. Akin to past works on functional estimation, we show that the optimal rate for estimating the functional undergoes a phase transition between regular parametric rate and some form of high-dimensional estimation rate. We further show that the optimal rate is attained by a carefully designed plug-in estimator based on de-biasing, while a family of naive plug-in estimators are proved to fall short. We further generalize the estimation problem and techniques that allow robust inputs of mean and covariance matrix estimators. Extensive numerical experiments lend further supports to our theoretical results.

Key words: Functional estimation, high dimension, $\ell_1$ regularization, minimax optimality, phase transition, sparsity, sub-gaussian distribution.

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1 Introduction

In multivariate statistics, the mean vector $\mu$ and covariance matrix $\Sigma$ play a critical role in a variety of statistical procedures such as linear discriminant analysis (LDA), multivariate analysis of variance and principle component analysis (PCA). See Anderson (2003) for a comprehensive mathematical treatment of classical multivariate analysis. Modern multivariate statistics confronts new statistical challenges due to arrival of big high-dimensional data (Johnstone and Titterington, 2009; Fan, Han, and Liu, 2014). For example, when the dimension $p$ is comparable to or much larger than the sample size $n$, sample covariance matrix is a poor estimate for $\Sigma$ (Bai, Silverstein, and Yin, 1988; Bai and Yin, 1993), and classical PCA becomes inconsistent (Paul, 2007; Johnstone and Lu, 2009; Wang and Fan, 2017) when the eigenvalues are not sufficiently spiked.

To the challenges arising from high dimensionality, many new theory and methods have been proposed. For example, various regularization techniques have been proposed to estimate large covariance matrix under different matrix structural assumptions such as sparsity (Karoui, 2008; Bickel and Levina, 2008b; Lam and Fan, 2009; Cai and Liu, 2011a), conditional sparsity (Fan, Fan, and Lv, 2008; Fan, Liao, and Mincheva, 2011, 2013), and smoothness (Furrer and Bengtsson, 2007; Bickel and Levina, 2008a; Cai, Zhang, and Zhou, 2010; Cai and Yuan, 2012). We refer to the two review papers (Fan, Liao, and Liu, 2016; Cai, Ren, and Zhou, 2016) and references therein for many other important related works. There have also been active researches on the inferential theory of the high-dimensional mean vector. A large body of work focuses on developing powerful one-sample or two-sample tests where conventional approaches like Hotelling’s $T^2$ test fail (Bai and Sarandasa, 1996; Srivastava and Du, 2008; Srivastava, 2009; Chen and Qin, 2010; Zhong, Chen, and Xu, 2013; Cai, Liu, and Xia, 2014; Wang, Peng, and Li, 2015).

Let $x_1, \ldots, x_n \in \mathbb{R}^p$ be independently and identically distributed (i.i.d.) random vectors with $E(x_i) = \mu$ and $\text{cov}(x_i) = \Sigma$. Motivated by the sparse portfolio allocations and sparse linear discriminants (see Section 2 for details), the primary goal of this paper is to estimate the functional $\mu^T \Sigma^{-1} \mu$ based on the observations $\{x_i\}_{i=1}^n$, under the assumption that $\Sigma^{-1} \mu$ is (approximately) sparse. Estimation of functionals has been studied in great generality in nonparametric statistics.
Minimax and adaptive theory for the estimation of linear functionals (Ibragimov and Khas’minskii 1984; Donoho and Liu 1991a, 1991b; Klemelä and Tsybakov 2001; Cai and Low 2004, 2005a; Butucea and Comte 2009), and quadratic functionals (Bickel and Ritov 1988; Donoho and Nussbaum 1990; Fan 1991; Efromovich and Low 1996; Cai and Low 2005b; Butucea 2007; Collier, Comminges, and Tsybakov 2017), have been originally established for Gaussian white noise model and then extended to the convolution model, among others. Along this line of works, an elbow phenomenon has been recurrently discovered: the optimal rate of convergence for some functionals exhibits a phase transition between the regular parametric rate and certain forms of nonparametric rate.

We will investigate the optimal rate of convergence for estimating the functional $\mu^T \Sigma^{-1} \mu$. Akin to the functional estimation in nonparametric statistics, we will reveal that the minimax estimation rate of $\mu^T \Sigma^{-1} \mu$, in the high-dimensional multivariate problem, undergoes a transition between parametric rate and some type of high-dimensional estimation rate. Moreover, we show that the optimal rate is achieved by a carefully designed plug-in estimator based on a de-biased $\ell_1$-regularized estimator of $\Sigma^{-1} \mu$. On the contrary, a family of naive plug-in estimators are proved to fall short. A similar phase transition phenomenon was uncovered on the estimation for quadratic functional of sparse covariance matrices $\Sigma$ (Fan, Rigollet, and Wang 2015). We also refer to Guo, Wang, Cai, and Li (2018) regarding applying de-biasing to obtain optimal estimators for some one-dimensional functionals in high-dimensional linear models.

The remainder of the paper is organized as follows. Section 2 introduces two motivating examples and the basic setup. Section 3 describes our estimator and studies in detail the minimax estimation property. Section 4 presents numerical performance of the proposed estimator on both synthetic and real datasets. To improve readability, all the proof is relegated to Section 5.

**Notation.** For $a \in \mathbb{R}^p$, denote $\|a\|_q = \left( \sum_{i=1}^p |a_i|^q \right)^{\frac{1}{q}}$ for $q \in (0, \infty)$, $\|a\|_0 = \sum_{i=1}^p |a_i|^0$, and $\|a\|_\infty = \max_{1 \leq i \leq p} |a_i|$. Given a symmetric matrix $A = (a_{ij}) \in \mathbb{R}^{p \times p}$, $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ represent its largest and smallest eigenvalues respectively, and $\|A\|_{\max} = \max_{i,j} |a_{ij}|$, $\delta_A = \max_{i} |a_{ii}|$. For $a, b \in \mathbb{R}$, $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$. Moreover, $f(n) \lesssim g(n)$ ($f(n) \gtrsim g(n)$) means there
exists some constant $C > 0$ such that $f(n) \leq Cg(n)$ ($f(n) \geq Cg(n)$) for all $n$; $f(n) \asymp g(n)$ if and only if $f(n) \lesssim g(n)$ and $f(n) \gtrsim g(n)$; $f(n) \gg g(n)$ is equivalent to $g(n) = o(f(n))$. We use $I_p$ to denote the $p \times p$ identity matrix, and $B_q(r) = \{u \in \mathbb{R}^p : \|u\|_q \leq r\}$ for the $\ell_q$ ball with radius $r$.

2 Preliminaries and examples

Suppose that $x_1, \ldots, x_n$ are independent copies of $x \in \mathbb{R}^p$ with $\mathbb{E}(x) = \mu$ and $\text{cov}(x) = \Sigma$. We consider the problem of estimating the functional $\mu^T \Sigma^{-1} \mu$ using the data $\{x_i\}_{i=1}^n$. Throughout the paper we assume, unless otherwise stated, that $x$ is sub-gaussian. That is, $x = \Sigma^{1/2}y + \mu$ and the zero-mean isotropic random vector $y$ satisfies

$$
P(|c^T y| \geq t) \leq 2 \exp(-t^2/\nu^2), \quad \text{for all } t \geq 0, \|c\|_2 = 1,
$$

with $\nu > 0$ being a constant. We study the estimation problem under minimax framework. The central goal is to characterize the minimax rate of the estimation error given by

$$
\inf_{\hat{\theta}} \sup_{(\mu, \Sigma) \in \mathcal{H}} \mathbb{E}|\hat{\theta} - \mu^T \Sigma^{-1} \mu|,
$$

where the infimum is taken over all measurable estimators, and $\mathcal{H}$ is some parameter space under consideration. We first derive a lower bound for the error to reveal the effect of high dimension.

**Proposition 1.** Consider $\mathcal{H} = \{(\mu, \Sigma) : \mu^T \Sigma^{-1} \mu \leq c\}$, where $c > 0$ is a fixed constant. If $p \geq n^2$, it holds that

$$
\inf_{\hat{\theta}} \sup_{(\mu, \Sigma) \in \mathcal{H}} \mathbb{E}|\hat{\theta} - \mu^T \Sigma^{-1} \mu| \geq \tilde{c},
$$

where $\tilde{c} > 0$ is a constant that depends on $c$.

Proposition 1 demonstrates that it is impossible to consistently estimate the functional $\mu^T \Sigma^{-1} \mu$, under the scaling $p \geq n^2$ which is not uncommon in high-dimensional problems. To overcome the
difficulty, we need a more structured parameter space. However, it is not clear what kind of simple constraints would make the problem solvable and practical. As a part of the contribution, we find the following parameter spaces with sparsity constraint

$$\mathcal{H}(s, \tau) = \left\{ (\mu, \Sigma) : \|\Sigma^{-1}\mu\|_0 \leq s, \; \mu^T\Sigma^{-1}\mu \leq \tau, \; c_L \leq \lambda_{\min}(\Sigma) \leq \delta \Sigma \leq c_U \right\},$$

or more generally the approximate sparsity constraint

$$\mathcal{H}_q(R, \tau) = \left\{ (\mu, \Sigma) : \|\Sigma^{-1}\mu\|_q^q \leq R, \; \mu^T\Sigma^{-1}\mu \leq \tau, \; c_L \leq \lambda_{\min}(\Sigma) \leq \delta \Sigma \leq c_U \right\}, \; q \in (0, 1],$$
suffice for our problem, where $\delta \Sigma = \max_{i \in [p]} \sigma_{ii}$ (recall the notation $\delta A = \max_i |a_{ii}|$), $0 < c_L < c_U$ are fixed constants; $s, R$ and $\tau$ can scale with $n$ and $p$. For notational simplicity, we have suppressed the dependence of $\mathcal{H}(s, \tau)$ and $\mathcal{H}_q(R, \tau)$ on $c_L, c_U$. The sparsity assumption on $\Sigma^{-1}\mu$ can be well justified in several multivariate statistics problems, of which two will be introduced shortly in Sections 2.1 and 2.2. Moreover, by setting $\Sigma = \sigma^2 I_p$ and assuming normality for $x$, our problem is reduced to quadratic functional estimation over sparsity classes in the Gaussian sequence model (Fan, 1991; Collier, Comminges, and Tsybakov, 2017). However, with the covariance matrix $\Sigma$ being unknown, results in Collier, Comminges, and Tsybakov (2017) can not be directly generalized. Delicate analyses are required to establish minimax optimality results, as will be shown in Section 3.

2.1 The mean-variance portfolio optimization

The mean-variance portfolio optimization method has been widely adopted by both institutional and retail investors ever since it was proposed by Markowitz (1952). Markowitz’s theory was highly influential and can be regarded as one of the foundations in modern finance. It can be expressed as an optimization problem with the solution determining proportion of each asset in a portfolio by maximizing the expected return under risk constraint, where the risk is measured by the variance of the portfolio. Specifically, let $x \in \mathbb{R}^p$ be the excess return of $p$ risky assets, with $\mathbb{E}(x) = \mu, \text{cov}(x) =$
Markowitz's portfolio optimization problem is

$$\max_w \mathbb{E}(w^T x) = w^T \mu \quad \text{subject to} \quad \text{var}(w^T x) = w^T \Sigma w \leq \sigma^2,$$

where $\sigma$ is the prescribed risk level. The optimal portfolio $w^*$ for the risky assets admits (the remaining invests in cash, including short positions) the explicit expression,

$$w^* = \frac{\sigma}{\sqrt{\mu^T \Sigma^{-1} \mu}} \Sigma^{-1} \mu.$$ (2.2)

The functional $\mu^T \Sigma^{-1} \mu$ is the square of the maximum Sharpe ratio, which measures the risk-adjusted performance of the optimal portfolio. We need both $\mu^T \Sigma^{-1} \mu$ and $\Sigma^{-1} \mu$ in order to construct the optimal portfolio allocation $w^*$ (See Section 4.2), which forms the focus of our studies.

Since the number of assets $p$ can be large compared to $n$, the number of observed return vectors, the estimation of mean-variance efficient portfolios is faced with great challenges in the high-dimensional regime (Kan and Zhou, 2007; Bai, Liu, and Wong, 2009; Karoui, 2010). One stream of research has been focused on the construction of sparse portfolios via regularizations (Brodie, Daubechies, De Mol, Giannone, and Loris, 2009; DeMiguel, Garlappi, Nogales, and Uppal, 2009; Fan, Zhang, and Yu, 2012). We refer to Ao, Li, and Zheng (2017) for a detailed list of references. Our sparsity assumption on the optimal portfolio weight, is well aligned with this line of works.

### 2.2 High-dimensional linear discriminant analysis

Linear discriminant analysis (LDA) is one of the most classical classification techniques in statistics and machine learning. Consider the binary classification problem where $x$ is a $p$-dimensional normal vector drawn with equal probability from one of the two distributions $N(\mu_1, \Sigma)$ and $N(\mu_2, \Sigma)$. It is well known that, Fisher’s linear discriminant rule: classify $x$ to class 1 if and only if $(\mu_1 - \mu_2)^T \Sigma^{-1} (x - \frac{\mu_1 + \mu_2}{2}) \geq 0$, achieves the optimal classification error given by

$$R_{opt} = \Phi(-\Delta/2), \quad \Delta = \sqrt{(\mu_1 - \mu_2)^T \Sigma^{-1} (\mu_1 - \mu_2)},$$
where $\Phi(\cdot)$ is the standard normal distribution function (Anderson, 2003). The functional $(\mu_1 - \mu_2)^T \Sigma^{-1} (\mu_1 - \mu_2)$ is the square of the signal-to-noise ratio $\Delta$, measuring the fundamental difficulty of the classification problem. The classical LDA procedure approximates Fisher’s rule by replacing the unknown parameters $\mu_1, \mu_2, \Sigma$ by their sample versions. Consistency results under the classical asymptotic framework when $p$ is fixed have been well established (Anderson, 2003). However, in the high-dimensional settings, the standard LDA can be no better than random guess (Bickel and Levina, 2004). Various high-dimensional LDA approaches have been proposed under the sparsity assumption on $\mu_1 - \mu_2$ or $\Sigma$ (Fan and Fan, 2008; Shao, Wang, Deng, and Wang, 2011; Mai, 2013). An alternative approach to sparse linear discriminant analysis imposes sparsity directly on $\Sigma^{-1}(\mu_1 - \mu_2)$, based on the key observation that Fisher’s rule depends on $\mu_1 - \mu_2$ and $\Sigma$ only through the product $\Sigma^{-1}(\mu_1 - \mu_2)$ (Cai and Liu, 2011b; Mai, Zou, and Yuan, 2012; Cai and Zhang, 2018). Such sparsity assumption is precisely what we have made in the paper. We should emphasize that different from the aforementioned works with the main focus on excess misclassification risk, our analysis centers on the estimation of $(\mu_1 - \mu_2) \Sigma^{-1}(\mu_1 - \mu_2)$, the quantity that characterizes the intrinsic difficulty of the classification problem.

It is important to observe that the functional estimation in the LDA problem, at first glance, looks different from our problem formulated at the beginning of Section 2 because it involves two sets of samples. However, it is possible to extend our results to the LDA setting by a simple adaptation. Towards that goal, let $\{x_i\}_{i=1}^{n_1}$ and $\{y_i\}_{i=1}^{n_2}$ be two sets of i.i.d. random samples from $N(\mu_1, \Sigma)$ and $N(\mu_2, \Sigma)$ respectively. Define the parameter space

$$
\mathcal{I}(s, \tau) = \left\{ (\mu_1, \mu_2, \Sigma) : \| \Sigma^{-1}(\mu_1 - \mu_2) \|_0 \leq s, (\mu_1 - \mu_2)^T \Sigma^{-1}(\mu_1 - \mu_2) \leq \tau, c_L \leq \lambda_{\min}(\Sigma) \leq \delta \Sigma \leq c_U \right\}.
$$

For the moment, we write $\hat{\theta}_{\{x_i\}_{i=1}^{n_1}, \{y_i\}_{i=1}^{n_2}}$ to clarify that the estimator is a function of two sets of samples. We are able to lower bound the minimax error in the two-sample problem by the error
from the one-sample problem,

\[
\inf \sup_{\theta \in \mathcal{I}(s, \tau)} \mathbb{E} \left| \tilde{\theta} \left( \{x_i\}_{i=1}^{n_1}, \{y_i\}_{i=1}^{n_2} \right) - (\mu_1 - \mu_2)^T \Sigma^{-1} (\mu_1 - \mu_2) \right|
\geq \inf \sup_{\theta \in \mathcal{H}(s, \tau)} \mathbb{E} \left| \tilde{\theta} \left( \{x_i\}_{i=1}^{n_1}, \{y_i\}_{i=1}^{n_2} \right) - \mu_1 \Sigma^{-1} \mu_1 \right|
\geq \inf \sup_{\theta \in \mathcal{H}(s, \tau)} \mathbb{E} \left| \tilde{\theta} \left( \{x_i\}_{i=1}^{n_1+n_2} \right) - \mu_1 \Sigma^{-1} \mu_1 \right|
\]

Here, the first inequality is obtained by setting \( \mu_2 = 0 \); the second inequality holds because each \( y_i \) (\( 1 \leq i \leq n_2 \)) can be replaced by \( x_{n_1+2i-1} - x_{n_1+2i} \sqrt{2} \), where \( \{x_{n_1+j}\}_{j=1}^{2n_2} \) are additional independent samples from \( N(\mu_1, \Sigma) \). As will be shown in Section 3.3, a matching upper bound can be derived when \( n_1 \asymp n_2 \). Similar arguments hold for the approximate sparsity class.

3 Minimax estimation of the functional

To fix ideas, we first present a detailed discussion for the exact sparsity class \( \mathcal{H}(s, \tau) \) in Sections 3.1, 3.2, and 3.3. Generalization of the main results to the approximate sparsity class \( \mathcal{H}_q(R, \tau) \) will be given in Section 3.4. For notational simplicity, we set \( \alpha = \Sigma^{-1} \mu \) and \( \theta = \mu^T \Sigma^{-1} \mu \), and denote the sample mean and sample covariance matrix by

\[
\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} x_i, \quad \hat{\Sigma} = \frac{1}{n} \sum_{i=1}^{n} (x_i - \hat{\mu})(x_i - \hat{\mu})^T,
\]

respectively.

3.1 Optimal estimation over exact sparsity classes

We first consider the estimation of \( \alpha \), which will pave our way to the estimation of the functional \( \theta \). Since \( \alpha = \arg\min_{\beta \in \mathbb{R}^p} \frac{1}{2} \beta^T \Sigma \beta - \beta^T \mu \) and \( \alpha \) is sparse, a natural estimator for the vector \( \alpha \) is the \( \ell_1 \)-regularized M-estimator:

\[
\tilde{\alpha} \in \arg\min_{\|\beta\|_2 \leq \gamma} \frac{1}{2} \beta^T \hat{\Sigma} \beta - \beta^T \hat{\mu} + \lambda \|\beta\|_1.
\] (3.1)
The constraint \( \|\beta\|_2 \leq \gamma \) is necessary to ensure the existence of \( \tilde{\alpha} \). Otherwise, when \( \hat{\Sigma} \) is degenerate, it may hold with positive probability that no finite solution exists in (3.1). Moreover, as will be seen in the proof of Theorem 1 without that constraint the probability of nonexistence of the solution vanishes asymptotically. Nevertheless, we should rule out the rare event for finite samples since the minimax error considered in this paper is measured in expectation.

Given the estimator \( \tilde{\alpha} \), we propose to estimate the functional \( \theta \) by

\[
\tilde{\theta} = 2\hat{\mu}^T\tilde{\alpha} - \hat{\alpha}^T\hat{\Sigma}\tilde{\alpha}.
\]

The above estimator is motivated by the de-biasing ideas for statistical inference in high-dimensional linear models (Zhang and Zhang, 2014; Javanmard and Montanari, 2014a,b; Van de Geer, Bühlmann, Ritov and Dezeure, 2014; Javanmard and Montanari, 2018). We now give a detailed explanation in our case. Since \( \theta = \mu^T\alpha \), we would like to construct a plug-in estimator \( \tilde{\theta} = (\mu^*)^T\alpha^* \), for some estimators \( \mu^* \) and \( \alpha^* \) of \( \mu \) and \( \alpha \) respectively. We set \( \mu^* = \hat{\mu} \) and choose a de-biased version of \( \tilde{\alpha} \) for \( \alpha^* \). In particular, it is known that \( \tilde{\alpha} \) defined in (3.1) satisfies the Karush-Kuhn-Tucker (KKT) conditions:

\[
\hat{\mu} - \hat{\Sigma}\tilde{\alpha} = \lambda\hat{g},
\]

where \( \hat{g} \) is a subgradient of \( \|\beta\|_1 \) at \( \beta = \tilde{\alpha} \). Multiplying both sides of (3.3) by \( \Sigma^{-1} \) and rearranging the terms yields

\[
\tilde{\alpha} = \alpha + \Sigma^{-1}(\hat{\mu} - \hat{\Sigma}\alpha) + (I_p - \Sigma^{-1}\hat{\Sigma})(\tilde{\alpha} - \alpha) - \lambda\Sigma^{-1}\hat{g},
\]

Observe that \( \mathbb{E}\Delta_1 \approx 0 \), and \( \Delta_2 \) is of small order. The major bias term of \( \tilde{\alpha} \) is \( -\lambda\Sigma^{-1}\hat{g} \). Therefore,

\*For simplicity, we assume \( \|\tilde{\alpha}\|_2 < \gamma \). In fact that holds with high probability as seen in the proof of Theorem 1.
the decomposition (3.4) suggests a bias-corrected estimator

\[ \alpha^* = \hat{\alpha} + \lambda \Sigma^{-1} \hat{g} = \hat{\alpha} + \Sigma^{-1}(\hat{\mu} - \hat{\Sigma} \hat{\alpha}), \]

where the second equality is due to (3.3). Consequently,

\[ \tilde{\theta} = (\mu^*)^T \alpha^* = \mu^T [\hat{\alpha} + \Sigma^{-1}(\hat{\mu} - \hat{\Sigma} \hat{\alpha})]. \quad (3.5) \]

Since \( \Sigma^{-1} \hat{\mu} \approx \Sigma^{-1} \mu = \alpha \), we replace \( \Sigma^{-1} \hat{\mu} \) in (3.5) by \( \hat{\alpha} \) to make \( \tilde{\theta} \) a legitimate estimator. This leads to the proposed estimator in (3.2).

The bias correction demonstrated in the above paragraph turns out to be crucial for the plug-in estimator \( \tilde{\theta} \) to achieve the optimal rate. Without the de-biasing step, vanilla plug-in estimators will fall short. These two statements are formally stated in this and next subsections, whose proofs are relegated to Section 5. As a by-product, the minimax estimation error of \( \tilde{\alpha} \) will be derived to shed more light on the functional estimation problem.

**Theorem 1.** Set \( \lambda = t \nu \sqrt{s \log p \over n}, \gamma = 2 \sqrt{s \log p \over c_L \nu} \) in (3.1). If \( s \log p \over n < \tilde{c} \), then for all \( t > 1 \lor 2 \),

\[
\sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \mathbb{E} \| \tilde{\alpha} - \alpha \|^2_2 \leq c_1 \cdot \left( \frac{t^2 (1 + \tau) s \log p}{n} + \tau p^{-(c_2 t - 1) \wedge c} \right),
\]

\[
\sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \mathbb{E} |\tilde{\theta} - \theta| \leq c_3 \cdot \left( \frac{t^2 (1 + \tau) s \log p}{n} + (\tau + \sqrt{\tau}) \cdot (n^{-1/2} + p^{-(c_2 t - 2) \wedge (c - 1)}) \right).
\]

Here, \( c_1, c_2, c_3 > 0 \) are constants possibly depending on \( \nu, c_L, c_U \); \( c \) is an arbitrary positive constant; \( \tilde{c} \in (0, 1) \) is a constant dependent on \( c, v, c_L, c_U \) and \( \tilde{c} \to 0 \) as \( c \to \infty \).

The two terms \( \tau p^{-(c_2 t - 1) \wedge c} \) and \( p^{-(c_2 t - 2) \wedge (c - 1)} \) appearing in the above bounds might not be optimally derived. However, they are both negligible by choosing a sufficiently large constant. In typical applications \( \tau \) is constant. Therefore, the results of Theorem 1 are simplified to

\[
\sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \mathbb{E} \| \tilde{\alpha} - \alpha \|^2_2 = O \left( \frac{s \log p}{n} \right), \quad \sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \mathbb{E} |\tilde{\theta} - \theta| = O \left( \frac{s \log p}{n} + \frac{1}{\sqrt{n}} \right). \quad (3.6)
\]
See Corollary 1 for more general results. Before we discuss in details the upper bounds, we present complementary lower bounds in the next theorem, which show that the rates in (3.6) are optimal.

**Theorem 2.** There exist positive constants \( \{c_i\}_{i=1}^6 \) possibly depending on \( c_L, c_U, \nu \) such that,

(a) if \( \frac{\log p}{n} < c_1, \frac{p}{s} > c_2 \), then

\[
\inf_{\hat{\alpha}} \sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \mathbb{E} \|\hat{\alpha} - \alpha\|_2^2 \geq c_3 \cdot \left[ \tau \land \left( \frac{(1 + \tau)s \log(p/s)}{n} \right) \right],
\]

(b) if \( \frac{\log p}{n} < 1 \), then

\[
\inf_{\hat{\theta}} \sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \mathbb{E} |\hat{\theta} - \theta| \geq c_4 \cdot \left[ \tau \land \frac{\tau + \sqrt{\tau}}{\sqrt{n}} \right] + c_5 \cdot \left[ \tau \land \left( \frac{(1 + \tau)s \log p}{n} \right) \right] c_0 \exp(-e^{2s^2 p/\nu c_0 - 1}),
\]

where \( c_0 \) can be any constant in \([0, 1]\).

According to Theorems 1 and 2, we conclude several important points as follows.

(1) **Estimation of \( \alpha \).** Consider the scaling \( \frac{\log p}{n} = o(1) \). Suppose \( p^{-\delta} \gtrsim \frac{\log p}{n} \) for some \( \delta > 0 \). It is clear that we may choose the constants \( t \) and \( c \) in Theorem 1 large enough so that

\[
\sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \mathbb{E} \|\hat{\alpha} - \alpha\|_2^2 \lesssim \frac{(1 + \tau)s \log p}{n} \quad (3.7)
\]

On the other hand, if \( s \lesssim p^{1-\tilde{\delta}} \) for some \( \tilde{\delta} > 0 \), Theorem 2 implies

\[
\inf_{\hat{\alpha}} \sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \mathbb{E} \|\hat{\alpha} - \alpha\|_2^2 \gtrsim \tau \land \left( \frac{(1 + \tau)s \log p}{n} \right). \quad (3.8)
\]

Hence, as long as \( \tau \gtrsim \frac{\log p}{n} \), the estimator \( \hat{\alpha} \) is rate-optimal. Moreover, when \( \tau \lesssim \frac{\log p}{n} \), the trivial estimator \( 0 \) is optimal, because its maximum error

\[
\sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \mathbb{E} \|0 - \alpha\|_2^2 \leq \sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} c_L^{-1} \alpha^T \Sigma \alpha = c_L^{-1} \sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \mu^T \Sigma^{-1} \mu \lesssim \tau,
\]

matches the lower bound in (3.8).
(2) *Estimation of $\theta$. Consider the same scaling $\frac{s\log p}{n} = o(1)$, and $p^{-\delta} \lesssim \frac{s\log p}{n}$ for some $\delta > 0$. It is straightforward to confirm that choosing large enough $t$ and $c$ in Theorem 1 yields

$$
(\tau + \sqrt{\tau})p^{-(c_{2t-2}^\tau)^{2}} \lesssim \frac{(1 + \tau)s\log p}{n},
$$

thus

$$
\sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \mathbb{E}|\tilde{\theta} - \theta| \lesssim \frac{(1 + \tau)s\log p}{n} + \frac{\tau + \sqrt{\tau}}{\sqrt{n}}. \quad (3.9)
$$

On the other hand, since $c_0 \in [0, 1]$ can be any constant in Theorem 2(b), the second term in the lower bound there is not negligible only when $s^2 \lesssim p^{1-\tilde{\delta}}$ for some $\tilde{\delta} > 0$. In such a case, we can choose sufficiently small $c_0 > 0$ so that $e^{2s^2p^\delta c_0^{-1}} = 1 + o(1)$ and obtain

$$
\inf_{\hat{\theta}} \sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \mathbb{E}|\hat{\theta} - \theta| \gtrsim \left[ \tau \wedge \frac{(1 + \tau)s\log p}{n} \right] + \left[ \tau \wedge \frac{\tau + \sqrt{\tau}}{\sqrt{n}} \right]. \quad (3.10)
$$

It can be directly verified that the above lower bound will match the upper bound in (3.9) when $\tau \gtrsim \frac{s\log p}{n}$. Hence $\tilde{\theta}$ is rate-optimal in the regime $\tau \gtrsim \frac{s\log p}{n}$. Furthermore, in the other regime $\tau \lesssim \frac{s\log p}{n}$, the lower bound in (3.10) is simplified to be of order $\tau$. So the trivial estimator 0 attains the optimal rate since its error

$$
\sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \mathbb{E}|0 - \theta| \leq \tau,
$$

matches the lower bound.

We summarize the preceding discussions in the corollary below.

**Corollary 1.** Consider the scaling $\frac{s\log p}{n} = o(1)$, $p^{-\delta} \lesssim \frac{s\log p}{n}$ for some $\delta > 0$. Set $\lambda \asymp \sqrt{\frac{(1 + \tau)\log p}{n}}$, $\gamma \asymp \sqrt{\tau}$ in (3.1).
(a) Suppose \( s \lesssim p^{1-\delta} \) for some \( \delta > 0 \), then

\[
\inf_{\hat{\alpha}} \sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \mathbb{E} \| \hat{\alpha} - \alpha \|^2_2 \times \tau \wedge \left( \frac{(1+\tau)s \log p}{n} \right) \times \begin{cases} 
\frac{(1+\tau)s \log p}{n} & \text{if } \tau \gtrsim \frac{s \log p}{n} \\
\tau & \text{if } \tau \lesssim \frac{s \log p}{n} 
\end{cases}
\]

The estimator \( \hat{\alpha} \) is minimax rate-optimal in the regime \( \tau \gtrsim \frac{s \log p}{n} \), and the trivial estimator \( 0 \) attains the optimal rate when \( \tau \lesssim \frac{s \log p}{n} \).

(b) Suppose \( s^2 \lesssim p^{1-\delta} \) for some \( \delta > 0 \), then

\[
\inf_{\hat{\theta}} \sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \mathbb{E} |\hat{\theta} - \theta| \asymp \left[ \tau \wedge \frac{\tau + \sqrt{\tau}}{\sqrt{n}} \right] + \left[ \tau \wedge \frac{(1+\tau)s \log p}{n} \right] \times \begin{cases} 
\frac{\tau + \sqrt{\tau}}{\sqrt{n}} + \frac{(1+\tau)s \log p}{n} & \text{if } \tau \gtrsim \frac{s \log p}{n} \\
\tau & \text{if } \tau \lesssim \frac{s \log p}{n} 
\end{cases}
\]

(3.11)

The estimator \( \hat{\theta} \) is minimax rate-optimal in the regime \( \tau \gtrsim \frac{s \log p}{n} \), and the trivial estimator \( 0 \) obtains the optimal rate when \( \tau \lesssim \frac{s \log p}{n} \).

There are a few remarks we should make about the results in Corollary 1.

**Remark 1.** The scaling \( \frac{s \log p}{n} = o(1) \) considered in Corollary 1 is standard for high-dimensional sparse models. The condition \( p^{-\delta} \lesssim \frac{s \log p}{n} \) is very mild. For instance, it holds when \( p \geq n^\epsilon \) with \( \epsilon > 0 \) being any positive constant.

**Remark 2.** Regarding the estimation for \( \alpha \), since \( \| \alpha \|^2_2 = O(\tau) \) in the parameter space \( \mathcal{H}(s, \tau) \), we may consider \( \tau \) as the signal strength. Under the additional assumption on the sparsity \( s \lesssim p^{1-\delta} \), the minimax estimation rate for \( \alpha \) is \( \tau \wedge \frac{(1+\tau)s \log p}{n} \). It is interesting to observe that the optimal rate depends on the signal strength in a non-linear fashion. Moreover, in the regime \( \tau \lesssim \frac{s \log p}{n} \) where the signal is sufficiently weak, a trivial estimator \( 0 \) achieves the optimal rate. Such a result can be well explained by the bias-variance tradeoff: for estimating very weak signals, the variance of an estimator plays the dominant role in the resulting error. Once \( \tau \) is above the threshold \( \frac{s \log p}{n} \), our estimator \( \hat{\alpha} \) becomes rate-optimal. Note that our analysis is focused on the absolute error.
The study of the relative error is an interesting problem and left for a future research.

**Remark 3.** Estimation of the functional $\alpha = \Sigma^{-1}\mu$ has been studied in various contexts such as portfolio selection (Ao, Li, and Zheng 2017), time series (Chen, Xu, and Wu 2016), and linear discriminant analysis (Cai and Zhang 2018). Different Lasso (Tibshirani 1996) or Dantzig-selector (Candes and Tao 2007) type estimators have been proposed and analyzed. However, our result characterizes the optimality of the proposed estimator $\hat{\alpha}$ over a wide range of the signal strength $\tau$, which is not available in the existing works.

**Remark 4.** For the estimation of the functional $\theta = \alpha^T\Sigma^{-1}\alpha$, given the fact that $\theta \leq \tau$ in $\mathcal{H}(s, \tau)$, we can regard $\tau$ as the signal strength when estimating $\theta$. Under the sparsity condition $s^2 \lesssim p^{1-\delta}$, the minimax rate takes the form $\left[ \tau \wedge \frac{\tau + \sqrt{\tau}}{\sqrt{n}} \right] + \left[ \tau \wedge \frac{(1+\tau)s\log p}{n} \right]$. The signal strength $\tau$ appears in the rate in a rather complicated way. Consider the common case $\tau \asymp 1$, the rate is reduced to $\frac{1}{\sqrt{n}} \vee \frac{s\log p}{n}$. It is clear that the rate of convergence undergoes a transition between the parametric rate $\frac{1}{\sqrt{n}}$ and the high-dimensional rate $\frac{s\log p}{n}$. We refer to Fan, Rigollet, and Wang (2015); Guo, Wang, Cai, and Li (2018) and reference therein for similar phenomenon in other high-dimensional problems. As in the estimation of $\alpha$, the trivial estimator $0$ attains the optimal rate when the signal is weak $\tau \lesssim \frac{s\log p}{n}$, while our proposed estimator $\hat{\theta}$ is rate-optimal in the other regime $\tau \gtrsim \frac{s\log p}{n}$.

**Remark 5.** When the covariance matrix $\Sigma$ is known and equal to $I_p$, the minimax estimation of the functional $\theta$ has been thoroughly studied in Collier, Comminges, and Tsybakov (2017). The authors derived non-asymptotic minimax rates for $\theta$ without any assumption on the sparsity $s$. The obtained rate shares some similarity with the rate we have derived in the present setting. Hence, our analysis might be considered as an extension of Collier, Comminges, and Tsybakov (2017) to unknown $\Sigma$.

### 3.2 Sub-optimality of a class of plug-in estimators

This section demonstrates the sub-optimality of the naive plug-in estimator. Consider more generally

$$
\hat{\theta}_c = c\hat{\mu}^T\hat{\alpha} + (1-c)\hat{\alpha}^T\hat{\Sigma}\hat{\alpha}
$$

(3.12)
for a given constant \( c \). The naive plug-in estimator mentioned in Section 3.1 corresponds to \( c = 1 \) and our proposed debias-based estimator \( \tilde{\theta} \) in (3.2) corresponds to \( c = 2 \). The question is then whether the optimality of \( \tilde{\theta} \) carries over to the other plug-in estimators \( \tilde{\theta}_c \) with \( c \neq 2 \). The answer turns out to be negative. We give a formal statement in the next proposition.

**Proposition 2.** Consider the scaling \( \frac{s \log p}{n} = o(1), p^{-\delta} \lesssim \frac{s \log p}{n} \) for some \( \delta > 0 \). Choose \( \lambda \propto \sqrt{(1+\tau) \log p n}, \gamma \propto \sqrt{\tau} \) in (3.1). If \( \tau \gtrsim \frac{s \log p}{n} \), then for any given constant \( c \neq 2 \),

\[
\sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} E |\tilde{\theta}_c - \theta| \gtrsim \sqrt{\frac{\tau(1+\tau)s \log p}{n}}.
\]

When \( \tau \gg \frac{s \log p}{n} \), it is straightforward to verify that

\[
\sqrt{\frac{\tau(1+\tau)s \log p}{n}} \gg \frac{(1+\tau)s \log p}{n} + \frac{\tau + \sqrt{\tau}}{\sqrt{n}}.
\]

According to Corollary 1, the lower bound in the above inequality is precisely the minimax rate when \( \tau \gtrsim \frac{s \log p}{n} \). Therefore, Proposition 2 shows that in the regime \( \tau \gg \frac{s \log p}{n} \), the plug-in estimator \( \tilde{\theta}_c \) can not achieve the optimal rate for any constant \( c \neq 2 \).

The sub-optimality of \( \tilde{\theta}_c \) \((c \neq 2)\) is essentially due to the bias of \( \tilde{\alpha} \) induced by the \( \ell_1 \) regularization. Specifically, since \( \tilde{\alpha} \) is the global solution of the convex optimization problem (3.1), the first-order optimality condition shows that

\[
(\tilde{\Sigma} \tilde{\alpha} - \tilde{\mu} + \lambda \tilde{g}, \beta - \tilde{\alpha}) \geq 0,
\]

where \( \tilde{g} \) is a subgradient of \( \|\beta\|_1 \) at \( \beta = \tilde{\alpha} \), and \( \beta \) can be any vector with \( \|\beta\|_2 \leq \gamma \). Setting \( \beta = \frac{1}{2} \tilde{\alpha} \) in (3.13) gives

\[
\tilde{\mu}^T \tilde{\alpha} - \tilde{\alpha}^T \tilde{\Sigma} \tilde{\alpha} \geq \lambda \|\alpha\|_1,
\]

thus yielding

\[
|\tilde{\theta}_c - \tilde{\theta}| = |c - 2| \cdot |\tilde{\mu}^T \tilde{\alpha} - \tilde{\alpha}^T \tilde{\Sigma} \tilde{\alpha}| \geq |c - 2| \cdot \lambda \|\tilde{\alpha}\|_1.
\]
The regularization term $\lambda \|\tilde{\alpha}\|_1$ is so large that the gap between $\tilde{\theta}_c (c \neq 2)$ and the optimal estimator $\tilde{\theta}$ exceeds the optimal rate. We refer to the proof in Section 5.4 for detailed calculations.

The preceding arguments suggest a remedy to address the sub-optimality of $\tilde{\theta}_c$ for $c \neq 2$. If the bias arising from $\ell_1$ regularization can be attenuated, the resulting plug-in estimator might be possibly improved. Indeed, once the “biased” estimator $\tilde{\alpha}$ in (3.12) is replaced by the less-biased $\ell_0$-estimator

$$\hat{\alpha} \in \text{argmin} \|\beta\|_{0,s} \|\beta\|_2 \leq \gamma \frac{1}{2} \beta^T \tilde{\Sigma} \beta - \beta^T \tilde{\mu},$$

the optimal estimation rate will be ultimately obtained. We show it formally in Theorem 3. Define the new class of plug-in estimators as

$$\tilde{\theta}_c = c \hat{\mu}^T \hat{\alpha} + (1 - c) \hat{\alpha}^T \tilde{\Sigma} \hat{\alpha}, \quad \forall c \in \mathbb{R}.$$ 

**Theorem 3.** Consider the scaling $\frac{s \log p}{n} = o(1), p^{-\delta} \lesssim \frac{s \log p}{n}$ for some $\delta > 0$. Set $\gamma \asymp \sqrt{\tau}$ in (3.14). In the regime $\tau \gtrsim \frac{s \log p}{n},$

$$\sup_{(\mu, \Sigma) \in H(s, \tau)} \mathbb{E} \|\hat{\alpha} - \alpha\|_2^2 \lesssim \frac{(1 + \tau) s \log p}{n},$$

$$\sup_{(\mu, \Sigma) \in H(s, \tau)} \mathbb{E} |\tilde{\theta}_c - \theta| \lesssim \frac{\tau + \sqrt{\tau}}{\sqrt{n}} + \frac{(1 + \tau) s \log p}{n}, \quad \forall c \in \mathbb{R}.$$ 

We emphasize a few points about Proposition 2 and Theorem 3.

**Remark 6.** The upper bound for $\hat{\alpha}$ in Theorem 3 matches the minimax rate in Corollary 1 when $\tau \gtrsim \frac{s \log p}{n}$. As expected, the $\ell_0$-estimator $\hat{\alpha}$ attains the optimal rate as the $\ell_1$-regularized M-estimator $\tilde{\alpha}$ does in the same regime. However, despite $\hat{\alpha}$ is rate-optimal for $\alpha$, it does not guarantee that the projection of $\hat{\alpha}$ along every direction is optimal for estimating the same projection of $\alpha$. This is the main reason why the plug-in estimator $\tilde{\theta}_c (c \neq 2)$ using $\hat{\alpha}$ turns out to be sub-optimal.

**Remark 7.** Combined with Corollary 1, Theorem 3 reveals that the new plug-in estimator $\tilde{\theta}_c$ is rate-optimal in the same regime as the de-biased estimator $\tilde{\theta}$ given in (3.2), and this optimality holds for every choice of $c \in \mathbb{R}$. On the other hand, we should emphasize that $\tilde{\theta}_c$ is computationally
infeasible, while \( \tilde{\theta} \) can be computed in polynomial time. Hence, our proposed estimator \( \tilde{\theta} \) achieves the best of both worlds in terms of computational feasibility and statistical efficiency.

### 3.3 A general result for the de-biased estimator

In this section, we discuss some generalization of our method to estimate the functional that takes the form

\[
\vartheta = \xi^T \Upsilon^{-1} \xi,
\]

where \( \xi \in \mathbb{R}^p, \Upsilon \in \mathbb{R}^{p \times p} \) are unknown parameters. When \( \xi = \mu, \Upsilon = \Sigma \), \( \vartheta \) equals to the functional \( \theta \) that we have studied in the previous two sections. It is direct to observe that the de-biased estimator \( \tilde{\theta} \) for \( \theta \) proposed in (3.2) is simply a function of the sample mean \( \hat{\mu} \) and sample covariance matrix \( \hat{\Sigma} \). More broadly, we may estimate \( \vartheta \) in the same way by replacing \( \hat{\mu} \) and \( \hat{\Sigma} \) with some other good estimators for \( \xi \) and \( \Upsilon \) respectively. Thus motivated, our generalization is to take two estimators \( \hat{\xi} \in \mathbb{R}^p \) and \( \hat{\Upsilon} \succeq 0 \) as inputs, and construct the estimator \( \hat{\vartheta} \) for \( \vartheta \) in the following way,

\begin{align}
\hat{\varpi} &\in \arg\min_{\beta} \| \beta \|_2 \leq \gamma_1 \beta^T \hat{\Upsilon} \beta - \beta^T \hat{\xi} + \lambda \| \beta \|_1, \tag{3.15} \\
\hat{\vartheta} &= 2 \hat{\xi}^T \hat{\varpi} - \hat{\varpi}^T \hat{\Upsilon} \hat{\varpi}. \tag{3.16}
\end{align}

This framework includes estimating \( \theta = \mu^T \Sigma^{-1} \mu \) using different inputs of estimators for \( \mu \) and \( \Sigma \) such as robustified estimators \footnote{Fan, Wang, Zhu (2016) Fan, Wang, Zhong and Zhu (2018) Ke, Minster, Ren, Sun, and Zhou (2019)}. It also includes the two-sample problem as to be elaborated below.

The estimation risk of \( \hat{\vartheta} \) will critically depend on the approximation accuracy of the inputs \( \hat{\xi} \) and \( \hat{\Upsilon} \). We give a general upper bound for \( \hat{\vartheta} \) in the next proposition. As a by-product, we include an upper bound for \( \hat{\varpi} \) as an estimator of \( \varpi = \Upsilon^{-1} \xi \). Towards that goal, define

\[
K(s) = \left\{ u \in \mathbb{R}^p : \| u_s^- \|_1 \leq 3 \| u_S \|_1, |S| \leq s, S \subseteq [p] \right\},
\]

\[
A(\lambda) = \left\{ \| \hat{\Upsilon} \varpi - \hat{\xi} \|_\infty \leq \lambda/2 \right\}.
\]
\[ B(s, \kappa) = \left\{ \max_{u \in K(s) \cap B_2(1)} \left| \sqrt{u^T \bar{\Sigma} u} - \sqrt{u^T \hat{\Sigma} u} \right| \leq \kappa \right\}. \]

We drop the sub-gaussian assumption on \( x \).

**Proposition 3.** Set \( \gamma \asymp \sqrt{\tau} \) in \((3.15)\). If there exist some constants \( c_1, c_2 > 0 \) such that

\[
\mathbb{P}(\|\hat{\xi} - \xi\|_\infty > t) \leq c_1 p e^{-c_2 mt^2}, \quad \forall t > 0, \tag{3.17}
\]

then the followings hold

\[
\sup_{(\xi, \Upsilon) \in H(s, \tau)} \mathbb{E}\|\hat{\sigma} - \sigma\|^2 \lesssim \lambda^2 s + \tau [\mathbb{P}(A^c(\lambda)) + \mathbb{P}(B^c(s, \sqrt{c_1L}/2))],
\]

\[
\sup_{(\xi, \Upsilon) \in H(s, \tau)} \mathbb{E}|\hat{\theta} - \theta| \lesssim \lambda s(\lambda + \sqrt{\log p/m}) + (\sqrt{\tau} + \tau)p \sqrt{\mathbb{P}(A^c(\lambda)) + \mathbb{P}(B^c(s, \sqrt{c_1L}/2))}
\]

\[ + \sup_{(\xi, \Upsilon) \in H(s, \tau)} \mathbb{E}|\sigma^T (\hat{\Upsilon} - \Upsilon) \sigma| + \sup_{(\xi, \Upsilon) \in H(s, \tau)} \sqrt{\mathbb{E}|\sigma^T (\hat{\xi} - \xi)|^2}. \]

Proposition 3 has a few implications we should discuss.

1. **Upper bound for two-sample problems.** Recall the high-dimensional LDA problem discussed in Section 2.2. Since two sets of samples are present, the interested functional is \( \hat{\theta} = (\mu_1 - \mu_2)^T \Sigma^{-1} (\mu_1 - \mu_2) \). We have shown in Section 2.2 that the minimax error for \( \hat{\theta} \) can be lower bounded by the minimax error for \( \theta \) in the one-sample problem. We now use Proposition 3 to derive a matching upper bound. Consider the estimator \( \hat{\theta} \) in \((3.16)\) by setting

\[
\hat{\xi} = \hat{\mu}_1 - \hat{\mu}_2, \quad \hat{\mu}_1 = \frac{1}{n_1} \sum_{i=1}^{n_1} x_i, \quad \hat{\mu}_2 = \frac{1}{n_2} \sum_{i=1}^{n_2} y_i,
\]

\[
\hat{\Upsilon} = \frac{1}{n_1 + n_2} \left[ \sum_{i=1}^{n_1} (x_i - \hat{\mu}_1)(x_i - \hat{\mu}_1)^T + \sum_{i=1}^{n_2} (y_i - \hat{\mu}_2)(y_i - \hat{\mu}_2)^T \right].
\]

Suppose the two samples \( \{x_i\}_{i=1}^{n_1} \) and \( \{y_i\}_{i=1}^{n_2} \) follow sub-gaussian distributions with \( n_1 \asymp n_2 \). Choose \( \lambda \asymp \sqrt{\frac{(1+\tau)\log p}{n_1 + n_2}} \) in \((3.15)\). Under the scaling \( p^{-\delta} \lesssim \frac{s \log p}{n_1 + n_2} \) for some \( \delta > 0 \), a minor modification of the arguments in the proof of Theorem 1 enables us to simplify the upper
bound in Proposition 3 to obtain

\[ \sup_{(\xi, \mathbf{Y}) \in \mathcal{H}(s, \tau)} \mathbb{E}|\hat{\vartheta} - \vartheta| \lesssim \frac{\tau + \sqrt{\tau}}{n_1 + n_2} + \frac{(1 + \tau)s \log p}{n_1 + n_2}. \]

In light of the discussion preceding Corollary 1, we thus can conclude that the minimax rate in the one-sample problem continues to hold in the two-sample case.

(2) **Robust estimation of the functionals.** The main results regarding the functional \( \theta = \mathbf{\mu}^T \Sigma^{-1} \mathbf{\mu} \) in Sections 3.1 and 3.2 are established for sub-gaussian distributions. When the data possesses heavier tails, it might be necessary to substitute the sample mean and sample covariance matrix used in \( \tilde{\alpha} \) and \( \tilde{\theta} \) by some robustified versions, in order to achieve a better bias-variance tradeoff. Motivated by recent advances in nonasymptotic deviation analyses of tail-robust estimators for the mean vector and covariance matrix (see Fan, Wang, Zhu (2016), Ke, Minster, Ren, Sun, and Zhou (2019) and references therein), to estimate \( \alpha \) and \( \theta \) under heavier-tailed distributions, we consider the estimators \( \hat{\varpi} \) and \( \hat{\vartheta} \) in (3.15) and (3.16), with element-wise truncated mean and covariance matrix estimators defined as follows:

\[ \hat{\xi} = (\hat{\xi}_1, \ldots, \hat{\xi}_p), \quad \hat{\xi}_j = \frac{1}{n} \sum_{i=1}^{n} \varphi_{\tau_j}(x_{ij}), \quad j = 1, \ldots, p. \]

\[ \hat{\Upsilon} = (\hat{\Upsilon}_{k\ell})_{1 \leq k, \ell \leq p}, \quad \hat{\Upsilon}_{k\ell} = \frac{1}{N} \sum_{i=1}^{N} \varphi_{\tau_{k\ell}}(y_{ik}y_{i\ell}/2), \quad 1 \leq k, \ell \leq p. \]

It holds that

\[ \frac{1}{N} \sum_{i=1}^{N} \varphi_{\tau_{k\ell}}(y_{ik}y_{i\ell}/2) \leq c_1 \]

under appropriate choice of the thresholds \( \{\tau_{k\ell}\}_{1 \leq k, \ell \leq p} \). According to Theorem 3.1 in Ke, Minster, Ren, Sun, and Zhou (2019), if \( \max_{k\ell} \mathbb{E}(y_{ik}^2y_{i\ell}^2) \leq c_1 \), then for any \( 0 < \delta < 1 \),

\[ \mathbb{P}\left( \|\hat{\Upsilon} - \Upsilon\|_{\max} \geq c_2 \sqrt{\frac{\log p + \log \delta^{-1}}{n}} \right) \leq \delta. \]

The deviation analyses in Ke, Minster, Ren, Sun, and Zhou (2019) lead to a concentration
result for $\hat{\xi}$ as well. Specifically, if $\max_j \mathbb{E}(x_{1j}^2) \leq c_3$, then for any $t > 0$, setting $\tau_j = 2\mathbb{E}(x_{1j}^2)/t$ gives that

$$\mathbb{P}(\|\hat{\xi} - \xi\|_{\infty} > t) \leq c_4 p e^{-ctn^2},$$

(3.19)

Note that the condition (3.17) in Proposition 3 is stronger than (3.19), because the truncated estimator $\hat{\xi}$ in (3.19) depends on $t$. We can further use the confidence interval method in Devroye, Lerasle, Lugosi, and Oliveira (2016) to turn the $t$-dependent estimators into an estimator $\hat{\xi}$ invariant of $t$ such that

$$\mathbb{P}(\|\hat{\xi} - \xi\|_{\infty} > t) \leq c_6 p e^{-ctn^2}, \quad \forall t \geq 0.$$  

(3.20)

In the above results, all the $c_i$'s are positive constants. Therefore, under bounded fourth moments conditions, we can apply Proposition 3 to derive upper bounds for $\hat{\omega}$ and $\hat{\vartheta}$. In particular, set $\lambda \asymp \sqrt{(1+\tau)s\log p}/n$ in (3.15), $\log \delta^{-1} \asymp \log p$ in (3.18), and assume the scaling $s^2\log p/n = o(1)$ and $p^{-\delta} \lesssim s^2\log p/n$ for some $\delta > 0$. It is straightforward to simplify the upper bounds in Proposition 3 to obtain

$$\sup_{(\xi, \Upsilon) \in H(s, \tau)} \mathbb{E}\|\hat{\omega} - \varpi\|^2 \lesssim \frac{(1 + \tau)s^2\log p}{n},$$

(3.21)

$$\sup_{(\xi, \Upsilon) \in H(s, \tau)} \mathbb{E}\left|\hat{\theta} - \theta\right| \lesssim \frac{(1 + \tau)s^2\log p}{n} + \sup_{(\xi, \Upsilon) \in H(s, \tau)} \mathbb{E}|\varpi^T(\hat{\Upsilon} - \Upsilon)\varpi|$$

$$+ \sup_{(\xi, \Upsilon) \in H(s, \tau)} \sqrt{n\mathbb{E}|\varpi^T(\hat{\xi} - \xi)|^2}. $$

(3.22)

Compared to the one in the sub-gaussian case, the bound for $\hat{\omega}$ in (3.21) has an additional multiplicative factor $s$. A similar rate was derived in the problem of high-dimensional Huber regression with heavy-tailed designs (Sun, Zhou, and Fan, 2018). Whether such a rate is minimax optimal for heavy-tailed distributions seems unknown. Regarding the estimator $\hat{\theta}$, the upper bound in (3.22) has one term identical to the rate for $\hat{\omega}$, as in the sub-gaussian scenario. The other two terms
are more subtle. When \( \hat{\xi} = \hat{\mu}, \hat{\Upsilon} = \hat{\Sigma} \), it is direct to compute the sum of these two terms to be of the order \( \frac{\sqrt{\tau + \tau}}{\sqrt{n}} \). Hence both terms contribute to the parametric rate part in the minimax rate of \( \theta \) (cf. Corollary 1). However, in the present heavier-tailed setting, the estimators \( \hat{\xi} \) and \( \hat{\Upsilon} \) are constructed by truncation operations to trade bias for robustness. It becomes difficult to characterize the accurate dependence of the two terms on \( n \) and \( \tau \). More fundamentally, what is the minimax rate for estimating the functional \( \theta \) under heavy-tailed distributions? Does the rate undergo a transition between parametric rate and high-dimensional rate as we revealed in the sub-gaussian situation? We leave a thorough minimax analysis of the functional estimation under heavy-tailed distributions for a future research.

3.4 Generalization to approximate sparsity classes

In Section 3.1, we have derived the minimax rate for the functional \( \theta = \mu^T \Sigma^{-1} \mu \) when \( \alpha = \Sigma^{-1} \mu \) is sparse. We investigate the performance of \( \tilde{\theta} \) over the approximate sparsity classes

\[
\mathcal{H}_q(R, \tau) = \left\{ (\mu, \Sigma) : \|\alpha\|_q^q \leq R, \theta \leq \tau, c_L \leq \lambda_{\min}(\Sigma) \leq \delta \Sigma \leq c_U \right\}, \quad q \in (0, 1].
\]

Observe that in \( \mathcal{H}_q(R, \tau) \), in addition to the \( \ell_q \) ball constraint \( \|\alpha\|_q^q \leq R \), the vector \( \alpha \) has to satisfy the quadratic inequality \( \theta = \alpha^T \Sigma \alpha \leq \tau \). The equality may not hold simultaneously in the preceding two constraints. In fact, if we define

\[
\tilde{\tau} = (c_U R^{\frac{2}{q}}) \wedge \tau, \quad \tilde{R} = (p^{1-\frac{q}{2}} c_L \tau^\frac{q}{2}) \wedge R,
\]

Lemma 5 from Section 5.6 shows that \( \mathcal{H}_q(R, \tau) = \mathcal{H}_q(\tilde{R}, \tilde{\tau}) \). Therefore, the effective scaling parameters under \( \mathcal{H}_q(R, \tau) \) are \( (\tilde{R}, \tilde{\tau}) \) rather than \( (R, \tau) \). We should expect \( (\tilde{R}, \tilde{\tau}) \) to play the role in the minimax results. We derive the upper and lower bounds for \( \tilde{\alpha} \) and \( \tilde{\theta} \) in the next two theorems.

**Theorem 4.** Consider the scaling \( \tilde{R}(1 + \tilde{\tau})^{-\frac{q}{2}} (\frac{\log p}{n})^{1-\frac{q}{2}} = o(1) \), and \( p^{-\delta} \lesssim \tilde{R}(\frac{\log p}{n})^{1-\frac{q}{2}} (\tilde{\tau}^{-\frac{q}{2}} \vee \tilde{\tau}^{-1}) \)
for some $\delta > 0$. Set $\lambda \asymp \sqrt{\frac{(1+\tau)\log p}{n}}, \gamma \asymp \sqrt{\tau}$ in \eqref{3.1}. Then, it holds the followings:

$$
\sup_{(\mu, \Sigma) \in H_q(R, \tau)} E\|\hat{\alpha} - \alpha\|_2^2 \lesssim (1 + \hat{\tau})^{1 - \frac{q}{2}} \hat{R} \left( \frac{\log p}{n} \right)^{1 - \frac{q}{2}},
$$
$$
\sup_{(\mu, \Sigma) \in H_q(R, \tau)} E|\hat{\theta} - \theta| \lesssim (1 + \hat{\tau})^{\frac{q}{2}} \hat{R} \left( \frac{\log p}{n} \right)^{1 - \frac{q}{2}} + \hat{\tau} + \sqrt{\tau}.
$$

**Theorem 5.** Consider the scaling $\tilde{R}(1 + \tilde{\tau})^{-\frac{q}{2}} \left( \frac{\log p}{n} \right)^{1 - \frac{q}{2}} = o(1)$.

(a) If $1 \lesssim \tilde{R}(1 + \tilde{\tau})^{-\frac{q}{2}} \left( \frac{\log p}{n} \right)^{-\frac{q}{2}} \lesssim p^{1-\delta}$ for some $\delta > 0$, then

$$
\inf_{\hat{\alpha}} \sup_{(\mu, \Sigma) \in H_q(R, \tau)} E\|\hat{\alpha} - \alpha\|_2^2 \gtrsim \hat{\tau} \wedge \left[ (1 + \hat{\tau})^{1 - \frac{q}{2}} \hat{R} \left( \frac{\log p}{n} \right)^{1 - \frac{q}{2}} \right].
$$

(b) If $1 \lesssim \tilde{R}^2(1 + \tilde{\tau})^{-q} \left( \frac{\log p}{n} \right)^{-q} \lesssim p^{1-\delta}$ for some $\delta > 0$, then

$$
\inf_{\hat{\theta}} \sup_{(\mu, \Sigma) \in H_q(R, \tau)} E|\hat{\theta} - \theta| \gtrsim \hat{\tau} \wedge \left[ (1 + \hat{\tau})^{1 - \frac{q}{2}} \hat{R} \left( \frac{\log p}{n} \right)^{1 - \frac{q}{2}} \right] + \left[ \hat{\tau} \wedge \frac{\hat{\tau} + \sqrt{\tau}}{\sqrt{n}} \right].
$$

Theorems 4 and 5 can be seen as a generalization of Theorems 1 and 2, respectively. The quantity $\tilde{R}(1 + \tilde{\tau})^{-\frac{q}{2}} \left( \frac{\log p}{n} \right)^{-\frac{q}{2}}$ plays the same role as the sparsity level $s$ in the exactly sparse case. Setting $q = 0$ in the two theorems, we fully recover the upper bounds \eqref{3.7} and \eqref{3.9} and the lower bounds \eqref{3.8} and \eqref{3.10}, for the exact sparsity classes under the same scaling conditions. Moreover, observe that

$$
\sup_{(\mu, \Sigma) \in H_q(R, \tau)} E\|0 - \alpha\|_2^2 = \sup_{(\mu, \Sigma) \in H_q(R, \hat{\tau})} E\|0 - \alpha\|_2^2 \lesssim \hat{\tau}
$$

$$
\sup_{(\mu, \Sigma) \in H_q(R, \tau)} E|0 - \theta| = \sup_{(\mu, \Sigma) \in H_q(R, \hat{\tau})} E|0 - \theta| \lesssim \hat{\tau}.
$$

The above, combined with Theorems 4 and 5 gives us the generalization of Corollary 1.

**Corollary 2.** Consider the scaling $\tilde{R}(1 + \tilde{\tau})^{-\frac{q}{2}} \left( \frac{\log p}{n} \right)^{1 - \frac{q}{2}} = o(1)$, and $p^{-\delta} \lesssim \tilde{R} \left( \frac{\log p}{n} \right)^{1 - \frac{q}{2}} (\tilde{\tau}^{-\frac{q}{2}} \vee \tilde{\tau}^{-1})$ for some $\delta > 0$. Set $\lambda \asymp \sqrt{\frac{(1+\tau)\log p}{n}}, \gamma \asymp \sqrt{\tau}$ in \eqref{3.1}.
(a) Suppose \(1 \preceq \tilde{R}(1 + \tau)^{-\frac{q}{2}}(\log \frac{p}{n})^{-\frac{q}{2}} \preceq p^{1-\delta}\) for some \(\delta > 0\), then

\[
\inf_{\alpha} \sup_{(\mu, \Sigma) \in \mathcal{H}_0(R, \tau)} \mathbb{E} \|\hat{\alpha} - \alpha\|^2 \asymp \frac{1}{1 + \tau} \wedge \left( (1 + \tau)^{-\frac{q}{2}} \tilde{R} \left( \frac{\log \frac{p}{n}}{n} \right)^{1-\frac{q}{2}} \right).
\]

The estimator \(\hat{\alpha}\) is minimax rate-optimal in the regime \(\tilde{\tau} \gtrsim \tilde{R}(1 + \tau)^{-\frac{q}{2}}(\log \frac{p}{n})^{-\frac{q}{2}}\), and the trivial estimator 0 attains the optimal rate when \(\tilde{\tau} \lesssim \tilde{R}(1 + \tau)^{-\frac{q}{2}}(\log \frac{p}{n})^{-\frac{q}{2}}\).

(b) Suppose \(1 \preceq \tilde{R}^2(1 + \tau)^{-q}(\log \frac{p}{n})^{-q} \preceq p^{1-\delta}\) for some \(\delta > 0\), then

\[
\inf_{\theta} \sup_{(\mu, \Sigma) \in \mathcal{H}_0(R, \tau)} \mathbb{E} |\hat{\theta} - \theta| \asymp \frac{1}{1 + \tau} \wedge \left( (1 + \tau)^{-\frac{q}{2}} \tilde{R} \left( \frac{\log \frac{p}{n}}{n} \right)^{1-\frac{q}{2}} \right) + \left[ \tilde{\tau} \wedge \tilde{\tau} + \sqrt{\tilde{\tau}} \right] \sqrt{\frac{\log \frac{p}{n}}{n}} \mathbb{E} \|\hat{\theta} - \theta\| \asymp \frac{1}{1 + \tau} \wedge \left( (1 + \tau)^{-\frac{q}{2}} \tilde{R} \left( \frac{\log \frac{p}{n}}{n} \right)^{1-\frac{q}{2}} \right) + \left[ \tilde{\tau} \wedge \tilde{\tau} + \sqrt{\tilde{\tau}} \right] \sqrt{\frac{\log \frac{p}{n}}{n}} \mathbb{E} \|\hat{\theta} - \theta\|.
\]

The estimator \(\hat{\theta}\) is minimax rate-optimal in the regime \(\tilde{\tau} \gtrsim \tilde{R}(1 + \tau)^{-\frac{q}{2}}(\log \frac{p}{n})^{-\frac{q}{2}}\), and the trivial estimator 0 obtains the optimal rate when \(\tilde{\tau} \lesssim \tilde{R}(1 + \tau)^{-\frac{q}{2}}(\log \frac{p}{n})^{-\frac{q}{2}}\).

### 4 Numerical experiments

#### 4.1 Simulation

In this section, we perform simulation studies to validate our theoretical results. In particular, we compute the empirical convergence rates over some instances and compare them with the theoretical forms we have derived in Section 3. We further evaluate and compare our proposed estimators with some alternative methods.

To empirically verify the convergence rate, we set \(\mu = \xi \cdot (1_s^T, 0^T)^T \in \mathbb{R}^p\), where \(1_s\) is a \(s\)-dimensional vector with all entries equal to 1 and \(0\) is a vector with all entries equal to 0, and \(\Sigma = \eta \cdot I_p\). Under these parameters, we have \(\alpha = \xi \eta^{-1} \cdot (1_s^T, 0^T)^T\) and \(\theta = s \xi^2 \eta^{-1}\). We generate data from normal distribution with the sample size \(n = \lfloor 2^k \rfloor\) for \(k = 10.5, 11, 11.5, \ldots, 15\). Now we take \(p = \lfloor 0.5 \cdot n^{0.5} \rfloor + 8, \eta = 2\) and consider the following three settings:

1. \(s = 2, \xi = 1\)
2. \(s = \lfloor n^{0.24} \rfloor, \xi = \lfloor n^{0.24} \rfloor^{-0.5}\)
(3) \( s = \lfloor n^{0.24} \rfloor, \xi = 3 \cdot n^{-0.45} \)

For each setting we repeat the experiment for 200 times, and for each specific \( n \) we use formulas (3.1) and (3.2) to obtain estimators \( \hat{\alpha}^{(1)}, \hat{\theta}^{(1)} \) in setting (1), \( \hat{\alpha}^{(2)}, \hat{\theta}^{(2)} \) in setting (2) and \( \hat{\alpha}^{(3)}, \hat{\theta}^{(3)} \) in setting (3). The tuning parameters are picked optimally to minimize the estimation error of \( \alpha \).

Figure 1: Averaged error v.s. sample size on logarithmic scale for the three settings (solid curves) of Section 4.1 along with their corresponding theoretical rates of convergence (dashed lines) for setting 1 (green), 2 (red) and 3 (blue).

Figure 1 depicts the averaged error (logarithmic scale for estimating \( \alpha \)) and \( \theta \) versus sample size in the logarithmic scale. To verify our theoretical results, we also show the theoretical rate of convergence in the figure. Specifically, for setting (1), it’s easy to check that the form of theoretical convergence rate for estimating \( \hat{\alpha} \) on logarithmic scale is \( C + \log_2 \log_2 n \), where \( C \) is a constant. Since the plots are on log-scale, we include the following function in the left plot of Figure 1:

\[
f_\alpha(x) = C_1 + \log_2 x - x
\]

with an appropriately calibrated constant \( C_1 \). In the same way, we add the other two functions for setting (2) and setting (3):

\[
g_\alpha(x) = C_2 + \log_2 x - 0.76 \cdot x,
\]
\[ h_\alpha(x) = C_3 + \log_2 x - 0.76 \cdot x. \]

For the estimation of \( \theta \), similarly, we attach the three functions below to the right plot of Figure 1:

\[ f_\theta(x) = C_4 - 0.5 \cdot x, \]
\[ g_\theta(x) = C_5 - 0.5 \cdot x, \]
\[ h_\theta(x) = C_6 + \log_2 x - 0.76 \cdot x. \]

From Figure 1, we can see that all the empirical convergence rates (solid curves) are well matched with the theoretical ones (dashed curves).

We next compare the performance of our method with some alternative estimators. The benchmark would be the plug-in estimators:

\[ \hat{\alpha}_P = \Sigma^{-1} \hat{\mu}, \quad \hat{\theta}_P = \frac{\hat{\mu}^T \Sigma^{-1} \hat{\mu} - a}{b}, \]

where \( a = \frac{p}{n-p}, b = \frac{n}{n-p} \). The estimator \( \hat{\theta}_P \) is a bias-corrected plug-in estimator that was proposed in Karoui (2010) for normal distributions under the scaling \( p < n \). We also consider the following Dantzig-type estimator for \( \alpha \) (Chen, Xu, and Wu, 2016):

\[ \hat{\alpha}_D = \text{argmin}_{\alpha \in \mathbb{R}^p: \|\Sigma \alpha - \hat{\mu}\|_\infty \leq \lambda} \|\alpha\|_1. \]

We then construct a family of plug-in estimators for \( \theta \):

\[ \hat{\theta}_{D,c} = c \hat{\mu}^T \hat{\alpha}_D + (1 - c) \hat{\alpha}_D^T \hat{\Sigma} \hat{\alpha}_D. \]

We perform the comparison under two different scenarios.

In the first scenario, we follow the same model setup from the part that verifies the convergence rate, with parameters \( p = \lfloor 0.5 \cdot n \rfloor, s = 5, \xi = 2, \eta = 1 \) where \( n = 60, 80, \cdots, 200 \). For a fair
comparison, all the estimators are obtained under optimal tuning. Each experiment is repeated
300 times and the results are shown in Figure 2. Regarding the estimation for $\alpha$, we see that
both $\hat{\alpha}_D$ and $\tilde{\alpha}$ have much better performance compared with $\hat{\alpha}_P$. This is expected because
the estimator $\hat{\alpha}_P$ does not exploit the sparsity structure in the data. For the estimation of the
functional $\theta$, we also observe that $\hat{\theta}_{D,2}$ and $\tilde{\theta}_2$ outperform $\hat{\theta}_P$ by a large margin. Moreover, it is clear
that $\tilde{\theta}_2$ has a better convergence rate compared with the naive plug-in estimators $\tilde{\theta}_0$ and $\tilde{\theta}_1$. This is
consistent with our theoretical conclusions in Section 3. As expected, the estimators $\hat{\alpha}_D$ and $\tilde{\alpha}$ have
similar performance, and estimators $\hat{\theta}_{D,c}$ and $\tilde{\theta}_c$ for $c = 0, 1, 2$ have very similar performance too.
Such a phenomenon that Lasso and Dantzig type estimators exhibit similar behavior has strong
theoretical support in high-dimensional sparse regression (Bickel, Ritov, and Tsybakov, 2009). We
leave the theoretical analysis for estimating functional using Danzig estimator as a future research.

![Figure 2: Comparison of different optimally tuned estimators for $\alpha$ and $\theta$ in the first scenario.](image)

In addition to the comparison of optimally tuned estimators, we also perform the comparison
of the aforementioned estimators under 5-fold cross-validation (cv). The implementation of cv for
$\hat{\alpha}_D$ can be found in Chen, Xu, and Wu (2016). We now describe cv for our estimator $\tilde{\alpha}$. Suppose
we split the data into $m$ folds. For each $j = 1, \ldots, m$, we construct the sample estimates $\hat{\mu}_j$ and $\hat{\Sigma}_j$
from the data in $j$-th fold, and obtain estimator $\tilde{\alpha}_{-j}$ from the rest of the data. We then compute
the cv error as
\[
\bar{l} = \frac{1}{m} \sum_{j=1}^{m} \left( \frac{1}{2} \hat{\alpha}_j^T \hat{\Sigma}_j \hat{\alpha}_j - \hat{\mu}_j^T \hat{\alpha}_j \right).
\]

We choose the parameters that minimize \( \bar{l} \) and obtain the corresponding estimator \( \hat{\alpha} \). The comparison is shown in Figure 3. As is clear from the figure, similar comparison results to the ones under optimal tuning are found in the case of cross-validation.

![Figure 3: Comparison of different cross-validation tuned estimators for \( \alpha \) and \( \theta \) in the first scenario.](image)

In the second scenario, as a preliminary investigation for our empirical study, we use financial data to calibrate for a low signal-to-noise ratio regime. Specifically, we randomly select \( p = 100 \) stocks from S&P500 to compute the sample mean \( \hat{\mu} \) and the sample covariance matrix \( \hat{\Sigma} \), using daily data in 2017 and 2018. We hard threshold the vector \( \hat{\Sigma}^{-1} \hat{\mu} \) to keep the top 10 entries with largest absolute values to obtain a sparse vector as the choice for \( \alpha \). We then use \( \hat{\Sigma} \alpha \) and \( \hat{\Sigma} \) as the values for the mean and covariance matrix parameters to generate multivariate Gaussian data.

The average signal-to-noise ratio, defined as \( \sum_{j=1}^{p} |\mu_j| / (\text{tr}(\Sigma))^{1/2} \), is 0.096, which is similar to the one from real data, despite the thresholding. Let \( n = 260, 280, \cdots, 500 \). Due to the low signal-to-noise ratio, we repeat the experiment for 600 times. The comparisons under optimal tuning and cross-validation are shown in Figures 4 and 5 respectively. Our estimators \( \hat{\alpha} \) and \( \hat{\theta}_2 \) perform much better than benchmark estimators and naive plug-in estimators. The functional estimator \( \tilde{\theta}_2 \) also
outperforms Dantzig-type estimators. Also $\tilde{\theta}_0, \tilde{\theta}_1$ have worse performance compared with $\tilde{\theta}_2$.

Figure 4: Comparison of different optimally tuned estimators for $\alpha$ and $\theta$ in the second scenario.

Figure 5: Comparison of different cross-validation tuned estimators for $\alpha$ and $\theta$ in the second scenario.

4.2 Empirical Study

In this section, we use the daily data for the constituents in the S&P500 from 2012 to 2018 to construct portfolios and test the portfolio performance in a 5-year horizon starting from 2014 to 2018. Since returns of stocks are highly correlated, we use a single factor, the market portfolio, to
adjust their dependence and construct the portfolios based on the factor-adjusted returns (residuals after regressed on the returns of S&P 500). A 2-year training window is employed: we use the first 7 quarters of data to estimate $\hat{\mu}$ and $\hat{\Sigma}$ and tune the parameters in estimating $\alpha$ and $\theta$ to yield the highest Sharpe ratio in the following quarter, which serves as a validation window. Note that we obtain and hedge the market beta in the training window and validation window combined. The testing window is 1 month where we will hedge the beta previously obtained and test the portfolio trained with data in past 2-year, and then we roll the window forward after the testing period, i.e. we re-balance our portfolio monthly. For comparison, we provide benchmarks the market and equal-weighted portfolios, and also the minimum-variance portfolio with gross-exposure constraint (Fan, Zhang, and Yu 2012). In addition, we impose the constraint on short-selling such that a short position for each individual stock cannot exceed 25% of the principal and the total short position cannot exceed 50%. Whenever this constraint is violated, we would rescale all the positions on risky assets to satisfy this constraint (the rest invested in cash).

![Figure 6: Portfolio performance.](image-url)
Figure 6 depicts the cumulative excess returns of the aforementioned strategies for constructing portfolios. Obviously, the sparse portfolio constructed by our estimators $\tilde{\alpha}, \tilde{\theta}_2$ (see (2.2)) outperforms other portfolios, in terms of the annualized return and Shape ratio. Moreover, during two correction periods in these 5 years, the sparse portfolio has smaller pullback compared with the market, and the same phenomenon can be observed for the minimum variance portfolio with gross exposure constraint, which is expected to performance in terms of stability and maximum drawdown.

|                | Value-Weighted | Equal-Weighted | Gross Exposure | Sparse Portfolio |
|----------------|----------------|----------------|----------------|------------------|
| Annual Return  | 8.15%          | 0.36%          | 4.93%          | 12.94%           |
| Volatility     | 13.53%         | 13.73%         | 7.76%          | 12.26%           |
| Sharpe Ratio   | 0.60           | 0.03           | 0.64           | 1.06             |
| Maximum Draw-down | -22.92%     | -30.29%        | -13.58%        | -12.75%          |
| Alpha          | 0.00%          | -7.64%         | 1.87%          | 13.28%           |
| Beta           | 1.00           | 0.98           | 0.38           | -0.04            |

Table 1: Portfolio characteristics

Table 1 provides some details for each portfolio. All numbers except maximum draw-down and beta are annualized. We can see that the sparse portfolio has low correlation with the market and its alpha over the market is more than 13%, and at the same time it shares low maximum draw-down as the minimum variance portfolio with gross exposure constraint. Understandably, its Sharpe ratio is the highest.

5 Proof of the main results

The section contains the proof of all the main results. The organization is as follows:

1. Section 5.1 collects a few lemmas that will be useful in the later proof.

2. Section 5.2 proves Theorem 1 and Proposition 3

3. Section 5.3 proves Theorem 2 and Proposition 1
4. Section 5.4 proves Proposition 2.

5. Section 5.5 proves Theorem 3.

6. Section 5.6 proves Theorem 4.

7. Section 5.7 proves Theorem 5.

8. Section 5.8 puts together some reference materials.

We introduce more notations below.

**Notation.** For an integer $k \geq 1$, let $[k] = \{1, 2, \ldots, k\}$. For a set $S \subseteq [k]$, its cardinality is $|S|$; $u_S \in \mathbb{R}^{|S|}$ is the subvector of $u \in \mathbb{R}^k$ indexed by $S$, and $A_{SS} \in \mathbb{R}^{|S| \times |S|}$ is the submatrix of $A \in \mathbb{R}^{k \times k}$ whose rows and columns are both indexed by $S$. For a matrix $A \in \mathbb{R}^{k \times k}$, $\|A\|_F$ and $\|A\|_2$ represent its Frobenius norm and spectral norm, respectively. We use $\{e_i\}_{i=1}^n$ to denote the standard basis in $\mathbb{R}^n$, and $\text{diag}(v_1, v_2, \ldots, v_k)$ to represent a $k \times k$ diagonal matrix with diagonal elements $v_1, \ldots, v_k$. Further define

$$K(s) = \left\{ u \in \mathbb{R}^p : \|u_{S^c}\|_1 \leq 3\|u_S\|_1, |S| \leq s, S \subseteq [p]\right\},$$

$$A(\kappa) = \left\{ \|\hat{\Sigma} - \Sigma\|_\infty \leq \kappa/4, \|\hat{\mu} - \mu\|_\infty \leq \kappa/4 \right\},$$

$$B(s, \kappa) = \left\{ \max_{u \in K(s) \cap B_2(1)} \left| \sqrt{u^T \Sigma u} - \sqrt{u^T \hat{\Sigma} u} \right| \leq \kappa \right\},$$

$$C(s, \kappa) = \left\{ \max_{u \in B_0(2s) \cap B_2(1)} \left| \sqrt{u^T \Sigma u} - \sqrt{u^T \hat{\Sigma} u} \right| \leq \kappa \right\}.$$

### 5.1 Technical lemmas

**Lemma 1.** The followings hold:

(i) For any given $\lambda > 0$,

$$\mathbb{P}(A(\lambda)) \geq 1 - 8p \exp \left( - \frac{c_1 n \lambda^2}{\nu^2} \cdot \min \left( \frac{1}{\lambda(\delta \Sigma + 1)\sqrt{\theta}}, \frac{1}{\nu^2 \theta + 1}\delta \Sigma \right) \right).$$

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(ii) For any given $t \geq 0$,

$$
E\left( \|\hat{\mu} - \mu\|_\infty \cdot 1_{\|\hat{\mu} - \mu\|_\infty > t} \right) \leq 2p \cdot (t^2 + (c_2n)^{-1}\delta \Sigma \nu^2) \cdot e^{-\frac{c_2nt^2}{\nu^2\delta \Sigma}}.
$$

Here $c_1, c_2 > 0$ are absolute constants.

**Proof.** Throughout the proof, $C_i$ ($i = 1, 2, \ldots$) are positive absolute constants.

Part (i): We first bound $\|\hat{\mu} - \mu\|_\infty$. Since for each $1 \leq j \leq p$, $e_j^T(x_i - \mu)$ is zero-mean and $\|e_j^T(x_i - \mu)\|_\infty \leq \sqrt{e_j^T \Sigma e_j} \cdot \|y_i\|_\psi_2 \leq \sqrt{\delta \Sigma} \cdot \nu$,

Hoeffding’s inequality (cf. Theorem A) enables us to obtain $\forall t \geq 0$,

$$
\mathbb{P}(\|\hat{\mu} - \mu\|_\infty > t) \leq \sum_{j=1}^p \mathbb{P}\left( \left| \frac{1}{n} \sum_{i=1}^n e_j^T(x_i - \mu) \right| > t \right) \leq 2p \exp\left( -\frac{C_1nt^2}{\nu^2\delta \Sigma} \right).
$$

Regarding the bound for $\|(\hat{\Sigma} - \Sigma)\alpha\|_\infty$, because of the following identity

$$
(\hat{\Sigma} - \Sigma)\alpha = \frac{1}{n} \sum_{i=1}^n \left[ (x_i - \mu)(x_i - \mu)^T \alpha - \Sigma \alpha \right] - (\hat{\mu} - \mu)(\hat{\mu} - \mu)^T \alpha,
$$

we bound the two terms on the above right-hand side, respectively. For each $1 \leq j \leq p$,

$$
\|e_j^T(x_i - \mu)(x_i - \mu)^T \alpha - e_j^T \Sigma \alpha\|_\psi_1 \leq C_2 \cdot \|e_j^T(x_i - \mu)(x_i - \mu)^T \alpha\|_\psi_1
$$

$$
\leq C_2 \cdot \|e_j^T(x_i - \mu)\|_\psi_2 \cdot \|(x_i - \mu)^T \alpha\|_\psi_2 \leq C_2 \cdot \sqrt{\delta \Sigma} \nu \cdot \sqrt{\theta \nu},
$$

where in the first two inequalities we have used basic properties of sub-exponential norm (cf. Lemma 2.7.7 and Exercise 2.7.10 in Vershynin [2018]), and the last inequality holds because $x_i - \mu = \Sigma^{1/2} y_i$ with $\|y_i\|_\psi_2 = \nu$. We can then apply Bernstein’s inequality (cf. Theorem A) to obtain

$$
\mathbb{P} \left( \frac{1}{n} \sum_{i=1}^n \left[ (x_i - \mu)(x_i - \mu)^T \alpha - \Sigma \alpha \right] \|_\infty > \frac{\lambda}{8} \right)
$$
\begin{equation}
\leq \sum_{j=1}^{p} \mathbb{P}\left(\left|\frac{1}{n} \sum_{i=1}^{n} [e_j^T (x_i - \mu)(x_i - \mu)^T \alpha - e_j^T \Sigma \alpha] \right| > \frac{\lambda}{8}\right)
\leq 2p \exp \left( - \frac{C_3 n \lambda^2}{\nu^2} \cdot \min \left( \frac{1}{\nu^2 \delta \Theta}, \frac{1}{\lambda \sqrt{\delta \Theta}} \right) \right). \tag{5.3}
\end{equation}

Moreover, since for each $1 \leq i \leq n$, $\alpha^T (x_i - \mu)$ is zero-mean and $\|\alpha^T (x_i - \mu)\|_2 \leq \sqrt{\theta \nu}$, Hoeffding's inequality (cf. Theorem A) gives us

\begin{equation}
\mathbb{P}(|\alpha^T (\hat{\mu} - \mu)| > t) = \mathbb{P}\left(\left|\frac{1}{n} \sum_{i=1}^{n} \alpha^T (x_i - \mu) \right| > t\right) \leq 2 \exp \left( - \frac{C_4 n t^2}{\theta \nu^2} \right), \quad \forall t > 0. \tag{5.4}
\end{equation}

Based on the results from (5.1) and (5.4), it holds that

\begin{align*}
\mathbb{P}\left(\|\hat{\Sigma} - \Sigma\|_\infty > \frac{\lambda}{4}\right) &\leq \mathbb{P}\left(\|\hat{\mu} - \mu\|_\infty > \frac{\lambda^{1/2}}{4 \theta^{1/4}}\right) + \mathbb{P}\left(\|\hat{\mu} - \mu\|_\infty > \frac{\lambda^{1/2} \theta^{1/4}}{2}\right) \\
&\leq 2p \exp \left( - \frac{C_1 n \lambda}{16 \nu^2 \delta \Theta \sqrt{\theta}} \right) + 2 \exp \left( - \frac{C_4 n \lambda}{4 \sqrt{\theta} \nu^2} \right) \\
&\leq 4p \exp \left( - \frac{C_5 n \lambda}{\nu^2 \sqrt{\theta} (\delta \Theta + 1)} \right). \tag{5.5}
\end{align*}

Combining the results (5.2), (5.3) and (5.5) we obtain

\begin{equation}
\mathbb{P}\left(\|\hat{\Sigma} - \Sigma\|_\infty > \frac{\lambda}{4}\right) \leq 6p \exp \left( - \frac{C_6 n \lambda^2}{\nu^2} \cdot \min \left( \frac{1}{\nu^2 \delta \Theta}, \frac{1}{\lambda \sqrt{\delta \Theta} (\delta \Theta + 1)} \right) \right). \tag{5.6}
\end{equation}

Therefore, the proof is completed by using (5.1) and (5.6) in the following bound,

\begin{equation}
\mathbb{P}(A(\lambda)) \geq 1 - \mathbb{P}(\|\hat{\Sigma} - \Sigma\|_\infty > \lambda/4) - \mathbb{P}(\|\hat{\mu} - \mu\|_\infty > \lambda/4).
\end{equation}

Part (ii): Using the integral identity $\mathbb{E}(z) = \int_0^\infty \mathbb{P}(z > s)ds$ for any non-negative random $z \in \mathbb{R}_+$, we have

\begin{equation}
\mathbb{E}\left(\|\hat{\mu} - \mu\|_\infty^2 \cdot 1_{\|\hat{\mu} - \mu\|_\infty > t}\right) = \int_0^\infty \mathbb{P}(\|\hat{\mu} - \mu\|_\infty^2 \cdot 1_{\|\hat{\mu} - \mu\|_\infty > t} > s)ds
\end{equation}
\[
= t^2 \mathbb{P}(\|\hat{\mu} - \mu\|_\infty > t) + \int_t^\infty s \mathbb{P}(\|\hat{\mu} - \mu\|^2 > s) ds
\]
\[
= t^2 \mathbb{P}(\|\hat{\mu} - \mu\|_\infty > t) + 2 \int_t^\infty s \mathbb{P}(\|\hat{\mu} - \mu\|_\infty > s) ds
\]
\[
\leq 2t^2 p e^{-\frac{t^2}{2\delta^2 \Sigma}} + 4p \int_t^\infty s e^{-\frac{t^2}{2\delta^2 \Sigma}} ds = 2p \cdot (t^2 + (nC_1)^{-1} \delta^2 \nu^2) \cdot e^{-\frac{t^2}{2\delta^2 \Sigma}},
\]
where the inequality above is due to (5.1).

Lemma 2. For any given constant \( c > 0 \), with probability at least \( 1 - 4p^{-c} \), it holds that
\[
\left| \sqrt{u^T \Sigma u} - \sqrt{u^T \hat{\Sigma} u} \right| \leq c_1 \nu^2 \max(\delta^{1/2}, 1) \sqrt{\log_p n} \|u\|_1 + c_2 \nu^2 \delta^2 \frac{\log_p n}{n} \|u\|_2^2, \quad \forall u \in \mathbb{R}^p,
\]
where \( c_1, c_2 > 0 \) are constants only depending on \( c \), and \( c_1, c_2 \to \infty \), as \( c \to \infty \).

Proof. Throughout the proof, we use \( C_i \) (\( i = 1, 2, \ldots \)) to denote positive absolute constants. Since
\( \hat{\Sigma} = \frac{1}{n} \sum_{i=1}^n (x_i - \mu)(x_i - \mu)^T - (\hat{\mu} - \mu)(\hat{\mu} - \mu)^T \), we have
\[
\left| \sqrt{u^T \hat{\Sigma} u} - \sqrt{u^T \Sigma u} \right| \leq \left| \left( n^{-1} \sum_{i=1}^n |u^T(x_i - \mu)|^2 \right)^{1/2} - \sqrt{u^T \Sigma u} \right| + \left| \sqrt{u^T \Sigma u} - \left( n^{-1} \sum_{i=1}^n |u^T(x_i - \mu)|^2 \right)^{1/2} \right|
\]
\[
\leq \left| \left( n^{-1} \sum_{i=1}^n |u^T(x_i - \mu)|^2 \right)^{1/2} - \sqrt{u^T \Sigma u} \right| + \frac{|u^T(\hat{\mu} - \mu)|^2}{\left( n^{-1} \sum_{i=1}^n |u^T(x_i - \mu)|^2 \right)^{1/2}}, \quad (5.7)
\]
We bound the two terms on the right-hand side of the last inequality, respectively. Regarding the second term, from (5.1) we have that \( \forall a \geq 0 \),
\[
|u^T(\hat{\mu} - \mu)|^2 \leq \|\hat{\mu} - \mu\|^2_\infty \cdot \|u\|^2 \leq a \nu^2 \delta^2 \frac{\log_p n}{n} \|u\|^2_1
\]
holds with probability at least \( 1 - 2p^{1-C_1 a} \). Once a bound for the first term is derived, using (5.8) we can obtain the bound for the second term. The main part of the proof uses the matrix deviation inequality (cf. Theorem C) together with a slicing argument (Van Handel, 2016) to bound the first term.
Towards that goal, let $A$ be a $n \times p$ matrix whose $i$th row is $(x_i - \mu)^T \Sigma^{-1/2}$, for $i = 1, 2, \ldots, n$. It is straightforward to verify that
\[
\left| \left( n^{-1} \sum_{i=1}^{n} u^T (x_i - \mu) \right)^{1/2} - \sqrt{u^T \Sigma u} \right| = n^{-1/2} \|A \Sigma^{1/2} u\|_2 - \|\Sigma^{1/2} u\|_2 \quad \text{(5.9)}
\]
Define
\[
\mathcal{T}(r) = \left\{ u \in \mathbb{R}^p : \|\Sigma^{1/2} u\|_\infty = 1, \|\Sigma^{-1/2} u\|_1 \leq r \right\}, \quad \text{for } r > 0
\]
with which it is clear that
\[
\sup_{\|\Sigma u\|_\infty = 1, \|u\|_1 \leq r} \left| n^{-1/2} \|A \Sigma^{1/2} u\|_2 - \|\Sigma^{1/2} u\|_2 \right| = \sup_{u \in \mathcal{T}(r)} \left| n^{-1/2} \|A u\|_2 - \|u\|_2 \right|. \quad \text{(5.10)}
\]
The matrix deviation inequality (cf. Theorem C) enables us to conclude that $\forall b \geq 0$,
\[
P\left( \sup_{u \in \mathcal{T}(r)} \left| n^{-1/2} \|A u\|_2 - \|u\|_2 \right| \leq C_2 n^{-1/2} \nu^2 \left( w(\mathcal{T}(r)) + b \cdot \text{rad}(\mathcal{T}(r)) \right) \right) \geq 1 - 2e^{-b^2}. \quad \text{(5.11)}
\]
Here,
\[
\text{rad}(\mathcal{T}(r)) = \sup_{u \in \mathcal{T}(r)} \|u\|_2 \leq \sup_{u \in \mathcal{T}(r)} \sqrt{\|\Sigma^{1/2} u\|_\infty \cdot \|\Sigma^{-1/2} u\|_1} \leq \sqrt{r},
\]
and
\[
w(\mathcal{T}(r)) = E \sup_{u \in \mathcal{T}(r)} g^T u \leq E \sup_{u \in \mathcal{T}(r)} \|\Sigma^{1/2} g\|_\infty \cdot \|\Sigma^{-1/2} u\|_1 \leq rE\|\Sigma^{1/2} g\|_\infty \leq C_3 \delta^{1/2}_\Sigma \sqrt{r \log p},
\]
where the last inequality follows from Theorem B. Therefore, setting $b = \max(\delta^{1/2}_\Sigma, 1) \tilde{b} C_3 \sqrt{r \log p}$ in (5.11) combined with (5.10) yields that $\forall r > 0, \tilde{b} \geq 0$,
\[
P\left( \sup_{\|\Sigma u\|_\infty = 1, \|u\|_1 \leq r} \left| n^{-1/2} \|A \Sigma^{1/2} u\|_2 - \|\Sigma^{1/2} u\|_2 \right| \geq C_4 (1+\tilde{b}) \nu^2 r \max(\delta^{1/2}_\Sigma, 1) \sqrt{\frac{\log p}{n}} \right) \leq 2p^{-C_5 \tilde{b}^2 r \max(\delta\Sigma, 1)}. \quad \text{(5.12)}
\]
We now utilize a slicing method to derive an upper bound on $|n^{-1/2} \|A \Sigma^{1/2} u\|_2 - \|\Sigma^{1/2} u\|_2|$. 35
Towards that end, denote $r_k = 2^{k}$, $k = k_0, k_0 + 1, k_0 + 2\ldots$, where $k_0 = \lfloor \log_2 \delta_{\Sigma}^{-1} \rfloor$. We then have

$$
P \left( \sup_{\|\Sigma u\|_\infty = 1} \left| n^{-1/2} \|A \Sigma^{1/2} u\|_2 - \|\Sigma^{1/2} u\|_2 \right| > t \right)
$$

$$
= P \left( \sup_{k \geq k_0} \sup_{r_k \leq \|u\|_1 \leq r_{k+1}} \left( \left| n^{-1/2} \|A \Sigma^{1/2} u\|_2 - \|\Sigma^{1/2} u\|_2 \right| > t \right) \right)
$$

$$
\leq \sum_{k \geq k_0} P \left( \sup_{r_k \leq \|u\|_1 \leq r_{k+1}} \left| n^{-1/2} \|A \Sigma^{1/2} u\|_2 - \|\Sigma^{1/2} u\|_2 \right| > t \right),
$$

where the first equality holds since $\|\Sigma u\|_\infty \leq \delta_{\Sigma} \cdot \|u\|_1$. Choosing $t = 2C_4(1 + \bar{b}) \nu^2 \max(\delta_{\Sigma}^{1/2}, 1) \sqrt{\frac{\log p}{n}}$ and using the result (5.12), we can continue from (5.13) to obtain that $\forall \bar{b} \geq 0$

$$
P \left( \sup_{\|\Sigma u\|_\infty = 1} \left| n^{-1/2} \|A \Sigma^{1/2} u\|_2 - \|\Sigma^{1/2} u\|_2 \right| > 2C_4(1 + \bar{b}) \nu^2 \max(\delta_{\Sigma}^{1/2}, 1) \sqrt{\frac{\log p}{n}} \right)
$$

$$
\leq 2 \sum_{k \geq k_0} \left( p^{-2C_5 \bar{b}^2 \max(\delta_{\Sigma}, 1)} \right)^{2^{k}} \leq 2 \sum_{m=1}^{\infty} \left( p^{-C_5 \bar{b}^2 \max(\delta_{\Sigma}, 1) \delta_{\Sigma}^{-1}} \right)^{m} \leq \frac{2}{p^{C_5 \bar{b}^2 - 1}}.
$$

This is equivalent to saying that with probability at least $1 - 2(p^{C_5 \bar{b}^2} - 1)^{-1}$,

$$
\left| n^{-1/2} \|A \Sigma^{1/2} u\|_2 - \|\Sigma^{1/2} u\|_2 \right| \leq 2C_4(1 + \bar{b}) \nu^2 \max(\delta_{\Sigma}^{1/2}, 1) \sqrt{\frac{\log p}{n}} \|u\|_1, \ \forall u \in \mathbb{R}^p.
$$

The above result together with (5.9) shows that with probability at least $1 - 2(p^{C_5 \bar{b}^2} - 1)^{-1}$,

$$
\left| \left( n^{-1} \sum_{i=1}^{n} |u^T(x_i - \mu)|^2 \right)^{1/2} - \sqrt{u^T \Sigma u} \right| \leq 2C_4(1 + \bar{b}) \nu^2 \max(\delta_{\Sigma}^{1/2}, 1) \sqrt{\frac{\log p}{n}} \|u\|_1, \ \forall u \in \mathbb{R}^p. \ (5.14)
$$

Combining the results (5.7), (5.8) and (5.14) gives us that with probability at least $1 - 2(p^{C_5 \bar{b}^2} - 1)^{-1} - 2p^{1-C_1 a}$,

$$
\left| \sqrt{u^T \Sigma u} - \sqrt{u^T \Sigma u} \right| \leq 2C_4(1 + \bar{b}) \nu^2 \max(\delta_{\Sigma}^{1/2}, 1) \sqrt{\frac{\log p}{n}} \|u\|_1 +
$$
\[
\frac{a^2 \delta \Sigma^{-\frac{\log p}{n}} \|u\|_1^2}{(\sqrt{u^T \Sigma u} - 2C_4(1 + \bar{b})\nu^2 \max(\delta_{\Sigma}^{-\frac{1}{2}}, 1)\sqrt{\frac{\log p}{n}}\|u\|_1^2)}.
\]

Finally since the above inequality holds for all \(a \geq 0\) and \(\bar{b} \geq 0\), thus for any given constant \(c > 0\), setting \(a = \frac{1 + c}{C_1}, \bar{b} = \sqrt{\frac{\log(p^2 + 1)}{C_5 \log p}}\) establishes the desired result.

5.2 Proof of Theorem 1 and Proposition 3

Proof. Given that Theorem 1 is a specialized result of Proposition 3, we will only present the proof for Theorem 1. The proof of Proposition 3 follows directly by replacing \(\tilde{\alpha}, \tilde{\theta}, \hat{\mu}, \hat{\Sigma}\) with \(\hat{\nu}, \hat{\vartheta}, \hat{\xi}, \hat{\Upsilon}\) respectively. Throughout the proof, \(C_1, C_2, \ldots\) are used to denote positive constants that possibly depend on \(\nu, c_L, c_U\). Some of them may depend on additional quantities, and clarification will be made in such cases.

Lemma 3 combined with the fact that \(\|\alpha\|_2 \leq \gamma, \|\tilde{\alpha}\|_2 \leq \gamma\) shows that

\[
E\|\tilde{\alpha} - \alpha\|_2^2 = E\|\tilde{\alpha} - \alpha\|_2^2 1_{A(\lambda) \cap B(s, \kappa)} + E\|\tilde{\alpha} - \alpha\|_2^2 1_{A^c(\lambda) \cup B^c(s, \kappa)} \\
\leq \frac{9t^2\nu^2(1 + \tau)s \log p}{n(\lambda_{\min}^{1/2}(\Sigma) - \kappa)^4} + \frac{16\tau}{c_L} (p(A^c(\lambda)) + p(B^c(s, \kappa))).
(5.15)
\]

According to Lemma 1 Part (i), it is straightforward to obtain the following bound,

\[
P(A^c(\lambda)) \leq 8p(e^{-C_1 t^2 \log p} + e^{-C_2 t \sqrt{n \log p}}) \leq 8p^{1-C_3 t},
(5.16)
\]

where we have used the condition \(\frac{s \log p}{n} < 1\) and \(t > 1\). Moreover, since \(\|u\|_2 \leq 1, \|u\|_1 \leq 4\sqrt{s}\) for \(u \in \mathcal{K}(s) \cap B_2(1)\), Lemma 2 implies that for \(\forall c > 0\), as long as \(\frac{s \log p}{n} \leq \frac{cL}{4c_4^{\nu^2 \kappa c_L}}\),

\[
P(B^c(s, \kappa)) \leq 4p^{-c},
(5.17)
\]

with \(\kappa = C_4 \sqrt{\frac{s \log p}{n}}\). Here, \(C_4 > 0\) is a constant depending on \(c\) (in addition to \(\nu, c_L, c_U\), and \(C_4 \to \infty\), as \(c \to \infty\). Observe that the constant \(\tilde{c}\) in Theorem 1 can be chosen as \(\min(\frac{cL}{4c_4^{\nu^2 \kappa c_L}}, \frac{1}{2}, \frac{cL}{4c_4^2})\), thus \(\lambda_{\min}^{1/2}(\Sigma) - \kappa \geq \frac{\sqrt{c_L}}{2c}\). Therefore, putting together the results (5.15), (5.16) and (5.17) establishes the
desired bound for $\|\hat{\alpha} - \alpha\|_2^2$.

We now bound $\mathbb{E} |\tilde{\theta} - \theta|$. Set the same value for $\kappa$ as in the preceding proof. According to Lemma 4 we obtain

$$
\mathbb{E} |(\tilde{\theta} - \theta)1_{A(\lambda) \cap B(s,\kappa)}| \leq \frac{C_{3}t^{2}(1 + \tau)s\log p}{n} + 2\mathbb{E} |\alpha^T(\hat{\mu} - \mu)| + \mathbb{E} |\alpha^T(\hat{\Sigma} - \Sigma)\alpha|,
$$

(5.18)

Moreover, observe that

$$
\mathbb{E} |\alpha^T(\hat{\mu} - \mu)| \leq \sqrt{\text{var}(\alpha^T(\hat{\mu} - \mu))} = \sqrt{\alpha^T\Sigma\alpha n} \leq \sqrt{\tau n},
$$

(5.19)

and

$$
\mathbb{E} |\alpha^T(\hat{\Sigma} - \Sigma)\alpha| \leq \mathbb{E} \left| \frac{1}{n} \sum_{i=1}^{n} (\alpha^T(x_i - \mu))^2 - \alpha^T\Sigma\alpha \right| + \mathbb{E} |\alpha^T(\hat{\mu} - \mu)|^2
$$

$$
\leq \sqrt{\mathbb{E} |\alpha^T(x_1 - \mu)|^4} + \frac{\tau}{n} \leq C_{6}^{\tau \nu^2} + \frac{\tau}{n},
$$

(5.20)

where the last inequality is due to $\|\alpha^T(x_1 - \mu)\|_{\psi^2} \leq \sqrt{\tau \nu}$ and the moments inequality for sub-gaussian variables (cf. Proposition 2.5.2 in Vershynin (2018)).

The rest of the proof is to bound $\mathbb{E} |(\tilde{\theta} - \theta)1_{A(\lambda) \cup B(s,\kappa)}|$. Towards that goal, we first bound $\mathbb{E} |\hat{\theta}|^2$. By the definition of $\hat{\alpha}$,

$$
\frac{1}{2} \alpha^T \hat{\Sigma} \hat{\alpha} - \hat{\alpha}^T \hat{\mu} + \lambda \|\hat{\alpha}\|_1 \leq 0,
$$

yielding $\hat{\theta} \geq 2\lambda \|\hat{\alpha}\|_1 \geq 0$. Hence,

$$
\mathbb{E} |\hat{\theta}|^2 \leq 4\mathbb{E} |\hat{\alpha}^T \hat{\mu}|^2 \leq 12\mathbb{E} |\mu^T \hat{\alpha}|^2 + 12\mathbb{E} |(\hat{\mu} - \mu)^T \alpha|^2 + 12\mathbb{E} |(\hat{\mu} - \mu)^T (\hat{\alpha} - \alpha)|^2
$$

$$
\leq \frac{48pcU^2}{c_L} \frac{\tau^2}{n} + \frac{12\tau}{n} + 12\mathbb{E} (\|\hat{\mu} - \mu\|_{\infty}^2 : \|\hat{\alpha} - \alpha\|_1^2),
$$

(5.21)

(b)

Here, in (a) we have used that $|\mu^T \hat{\alpha}|^2 \leq \|\mu\|^2 \cdot \|\hat{\alpha}\|^2 \leq \gamma^2 \|\mu\|^2 \|\hat{\alpha}\|^2 \leq 4pcU^2 c_L^{-1} \tau^2$; (b) is due to
\[ \|\tilde{\alpha} - \alpha\|^2 \leq p \|\tilde{\alpha} - \alpha\|^2 \leq 4p\gamma^2. \]
Furthermore, we use Lemma 3 and the upper bound for \( \mathbb{E}\|\tilde{\alpha} - \alpha\|^2 \) to obtain

\[ \mathbb{E}\|\tilde{\alpha} - \alpha\|^2 = \mathbb{E}(\|\tilde{\alpha} - \alpha\|^2 1_{A(\lambda)}) + \mathbb{E}(\|\tilde{\alpha} - \alpha\|^2 1_{A^c(\lambda)}) \leq 16s\mathbb{E}\|\tilde{\alpha} - \alpha\|^2 + 16p\gamma^2 \]

(5.22)

According to Lemma 1 Part (ii), it is possible to set \( b = C_8 \frac{\log p}{n} \) in (5.21) to have

\[ \mathbb{E}(\|\hat{\mu} - \mu\|^2 1_{\|\hat{\mu} - \mu\| > b}) \leq \frac{C_9}{np}. \]

(5.23)

Based on the results (5.21), (5.22) and (5.23), we can derive an upper bound for \( \mathbb{E}|\tilde{\theta}|^2 \),

\[ \mathbb{E}|\tilde{\theta}|^2 \leq C_{10} \left( p\tau + \frac{1}{n} \log \frac{p}{n} + \frac{t^2(1 + \tau)^2 s^2}{n^2} \right). \]

As a result, we are able to conclude

\[ \mathbb{E}|(\tilde{\theta} - \theta) 1_{A^c(\lambda) \cup B^c(s, \kappa)}| \leq \sqrt{2(\mathbb{E}|\tilde{\theta}|^2 + \mathbb{E}||\theta||^2)} \cdot \sqrt{\mathbb{P}(A^c(\lambda)) + \mathbb{P}(B^c(s, \kappa))} \]

\[ \leq C_{11} \cdot \left( \sqrt{\frac{1}{n}} + \sqrt{\frac{t^2(1 + \tau)s \log p}{n}} + \sqrt{\tau p^{-\frac{(C_3t - 2)\lambda c}{2}}}, \right) \]

\[ \leq C_{11} \cdot \left( \frac{1}{n} + \frac{t^2(1 + \tau)s \log p}{n} + (\tau + \sqrt{\tau}) p^{-\frac{(C_3t - 2)\lambda (e - 1)}} \right), \]

where in the last step we have used the condition \( C_3t - 2 > 0 \). The above result combined with (5.18), (5.19), and (5.20) yields the desired bound for \( \mathbb{E}|\tilde{\theta} - \theta| \). □

**Lemma 3.** Denote \( s = \|\alpha\|_0 \), and set \( \gamma \) in (3.1) such that \( \|\alpha\|_2 \leq \gamma \). It holds that

(i) On the event \( A(\lambda) \), \( \tilde{\alpha} - \alpha \in K(s) \).
(ii) On the event \( A(\lambda) \cap B(s, \kappa) \) with \( \kappa < \lambda_{\text{min}}^{1/2}(\Sigma) \),

\[
\|\tilde{\alpha} - \alpha\|_2 \leq \frac{3\lambda\sqrt{s}}{(\lambda_{\text{min}}^{1/2}(\Sigma) - \kappa)^2}.
\] (5.24)

The above results might be obtained by the general analysis framework for high-dimensional M-estimator developed in Negahban, Ravikumar, Wainwright, and Yu (2012). For completeness, we give a proof tailored for our problem.

Proof. Let the support of \( \alpha \) be indexed by the set \( S \subseteq [p] \). Since \( \mu = \Sigma \alpha \), it is clear that on \( A(\lambda) \),

\[
\|\hat{\Sigma} \alpha - \hat{\mu}\|_{\infty} \leq \|\hat{\Sigma} - \Sigma\|_1 \alpha \| + \|\hat{\mu} - \mu\|_{\infty} \leq \frac{\lambda}{2}.
\] (5.25)

Then by the definition of \( \tilde{\alpha} \) in (3.1), we have on the event \( A(\lambda) \):

\[
0 \geq \frac{1}{2} \alpha^T \hat{\Sigma} \tilde{\alpha} - \tilde{\alpha}^T \hat{\mu} - \frac{1}{2} \alpha^T \hat{\Sigma} \alpha + \alpha^T \hat{\mu} + \lambda(\|\tilde{\alpha}\|_1 - \|\alpha\|_1)
\]

\[
= \frac{1}{2}(\hat{\alpha} - \alpha)^T \hat{\Sigma}(\hat{\alpha} - \alpha) + (\hat{\Sigma} - \tilde{\alpha})^T (\hat{\alpha} - \alpha) + \lambda(\|\tilde{\alpha}\|_1 - \|\alpha\|_1)
\]

\[
\geq \frac{1}{2}(\hat{\alpha} - \alpha)^T \hat{\Sigma}(\hat{\alpha} - \alpha) - \|\hat{\Sigma} \alpha - \hat{\mu}\|_{\infty} \cdot \|\tilde{\alpha} - \alpha\|_1 + \lambda(\|\tilde{\alpha}\|_1 - \|\alpha\|_1).
\]

\[
\overset{(a)}{\geq} \frac{1}{2}(\hat{\alpha} - \alpha)^T \hat{\Sigma}(\hat{\alpha} - \alpha) + \frac{\lambda}{2}(-\|\tilde{\alpha} - \alpha\|_1 + 2\|\tilde{\alpha}\|_1 - 2\|\alpha\|_1)
\]

\[
= \frac{1}{2}(\hat{\alpha} - \alpha)^T \hat{\Sigma}(\hat{\alpha} - \alpha) + \frac{\lambda}{2}[\|\hat{\alpha}_{S^c}\|_1 - \|\tilde{\alpha}_{S^c} - \alpha_{S^c}\|_1 + 2(\|\tilde{\alpha}_{S^c}\|_1 - \|\alpha_{S^c}\|_1)]
\]

\[
\overset{(b)}{\geq} \frac{1}{2}(\hat{\alpha} - \alpha)^T \hat{\Sigma}(\hat{\alpha} - \alpha) + \frac{\lambda}{2}(\|\hat{\alpha}_{S^c}\|_1 - 3\|\tilde{\alpha}_{S^c} - \alpha_{S^c}\|_1)
\]

\[
\geq \frac{\lambda}{2}(\|\tilde{\alpha}_{S^c}\|_1 - 3\|\tilde{\alpha}_{S^c} - \alpha_{S^c}\|_1),
\]

where \((a)\) holds by (5.25), and \((b)\) is due to \( \|\alpha_{S^c}\|_1 \leq \|\hat{\alpha}_{S^c} - \alpha_{S^c}\|_1 + \|\tilde{\alpha}_{S^c}\|_1 \). We thus obtain \( \|\tilde{\alpha}_{S^c}\|_1 \leq 3\|\tilde{\alpha}_{S^c} - \alpha_{S^c}\|_1 \), i.e., \( \hat{\alpha} - \alpha \in K(s) \).

To prove the second part, using the fact that \( \hat{\alpha} - \alpha \in K(s) \) on \( A(\lambda) \), we can conclude that on
the event $\mathcal{A}(\lambda) \cap \mathcal{B}(s, \kappa)$,

$$\sqrt{(\hat{\alpha} - \alpha)^T \hat{\Sigma}(\hat{\alpha} - \alpha)} \geq \|\hat{\alpha} - \alpha\|_2 \cdot \left(\min_{u \in B_2(1)} \sqrt{u^T \Sigma u} - \max_{u \in K(s) \cap B_2(1)} |\sqrt{u^T \Sigma u} - \sqrt{u^T \hat{\Sigma} u}|\right)$$

$$\geq \|\hat{\alpha} - \alpha\|_2 \cdot (\lambda_{\min}^{1/2}(\Sigma) - \kappa) \quad (5.27)$$

Therefore, on the event $\mathcal{A}(\lambda) \cap \mathcal{B}(s, \kappa)$, we can continue from (5.26) to obtain that when $\kappa < \lambda_{\min}^{1/2}(\Sigma)$,

$$0 \geq \frac{1}{2}(\lambda_{\min}^{1/2}(\Sigma) - \kappa)^2 \|\hat{\alpha} - \alpha\|_2^2 + \frac{\lambda}{2}(\|\hat{\alpha}_S\|_1 - 3\|\hat{\alpha}_S - \alpha_S\|_1)$$

$$\geq \frac{1}{2}(\lambda_{\min}^{1/2}(\Sigma) - \kappa)^2 \|\hat{\alpha} - \alpha\|_2^2 - \frac{3\lambda\sqrt{s}}{2} \|\hat{\alpha} - \alpha\|_2$$

$$= \frac{\|\hat{\alpha} - \alpha\|_2^2}{2} \left[(\lambda_{\min}^{1/2}(\Sigma) - \kappa)^2 \|\hat{\alpha} - \alpha\|_2^2 - 3\lambda\sqrt{s}\right].$$

where (c) is simply by Cauchy-Schwarz inequality $\|\hat{\alpha}_S - \alpha_S\|_1 \leq \sqrt{|S|} \cdot \|\hat{\alpha}_S - \alpha_S\|_2 \leq \sqrt{s} \|\hat{\alpha} - \alpha\|_2$.

The upper bound on $\|\hat{\alpha} - \alpha\|_2$ follows. □

**Lemma 4.** Denote $s = \|\alpha\|_0$, and set $\gamma$ in (3.1) such that $\|\alpha\|_2 \leq \gamma$. On the event $\mathcal{A}(\lambda) \cap \mathcal{B}(s, \kappa)$ with $\kappa < \lambda_{\min}^{1/2}(\Sigma)$, it holds that

$$|\hat{\theta} - \theta| \leq \frac{24s\lambda^2(5\lambda_{\min}(\Sigma) + 8\kappa^2)}{(\lambda_{\min}^{1/2}(\Sigma) - \kappa)^4} + 2|\alpha^T(\hat{\mu} - \mu)| + |\alpha^T(\hat{\Sigma} - \Sigma)\alpha|.$$

**Proof.** Let the support of $\alpha$ be indexed by the set $S \subseteq [p]$. Define the function $f : \mathbb{R}^p \to \mathbb{R}$ as $f(x) = 2\mu^T x - x^T \Sigma x$. It is straightforward to verify that $f(\alpha) = \theta$, and $x = \alpha$ is the global maximizer of $f(x)$. Therefore,

$$\theta = f(\alpha) \geq f(\hat{\alpha}) = 2\mu^T \hat{\alpha} - \hat{\alpha}^T \Sigma \hat{\alpha}, \quad (5.28)$$

from which we can proceed to derive an upper bound on $\hat{\theta} - \theta$,

$$\hat{\theta} - \theta = (2\hat{\mu}^T \hat{\alpha} - \hat{\alpha}^T \hat{\Sigma} \hat{\alpha}) - f(\alpha) \leq (2\hat{\mu}^T \hat{\alpha} - \hat{\alpha}^T \hat{\Sigma} \hat{\alpha}) - f(\hat{\alpha})$$

$$= (2\hat{\mu}^T \hat{\alpha} - \hat{\alpha}^T \hat{\Sigma} \hat{\alpha}) - (2\mu^T \hat{\alpha} - \hat{\alpha}^T \Sigma \hat{\alpha})$$
\[ \begin{array}{l}
\leq -2\alpha^T (\Sigma - \hat{\Sigma})(\alpha - \hat{\alpha}) + (\alpha - \hat{\alpha})^T \Sigma (\alpha - \hat{\alpha}) - \alpha^T (\hat{\Sigma} - \Sigma) \alpha \\
\quad - 2(\hat{\alpha} - \hat{\alpha})^T (\mu - \hat{\mu}) - 2\alpha^T (\mu - \hat{\mu}) \\
\leq 2\| (\hat{\Sigma} - \Sigma) \alpha \|_\infty \cdot \| \alpha - \hat{\alpha} \|_1 + (\alpha - \hat{\alpha})^T \Sigma (\alpha - \hat{\alpha}) + |\alpha^T (\hat{\Sigma} - \Sigma) \alpha | \\
\quad + 2\| \hat{\mu} - \mu \|_\infty \cdot \| \alpha - \hat{\alpha} \|_1 + 2|\alpha^T (\hat{\mu} - \mu) |.
\end{array} \] (5.29)

Moreover, (5.26) implies that on \( A(\lambda) \),
\[ (\alpha - \hat{\alpha})^T \hat{\Sigma} (\alpha - \hat{\alpha}) \leq 3\lambda \| \hat{\alpha}_S - \alpha_S \|_1 \leq 3\lambda \sqrt{s} \| \alpha - \hat{\alpha} \|_2, \]
which further yields that on \( A(\lambda) \cap B(s, \kappa) \),
\[ (\alpha - \hat{\alpha})^T \Sigma (\alpha - \hat{\alpha}) \leq 2(\alpha - \hat{\alpha})^T \hat{\Sigma} (\alpha - \hat{\alpha}) + 2\| \alpha - \hat{\alpha} \|_2 \cdot \max_{u \in K(\hat{\alpha}) \cap B_2(1)} |\sqrt{u^T \Sigma u} - \sqrt{u^T \hat{\Sigma} u}|^2 \\
\leq 6\lambda \sqrt{s} \| \alpha - \hat{\alpha} \|_2 + 2\kappa^2 \| \alpha - \hat{\alpha} \|_2^2 \] (5.30)

Since \( \| \hat{\alpha}_S \|_1 \leq 3\| \hat{\alpha}_S - \alpha_S \|_1 \) on event \( A(\lambda) \) by Lemma [3], it holds that
\[ \| \hat{\alpha} - \alpha \|_1 \leq 4\| \hat{\alpha}_S - \alpha_S \|_1 \leq 4\sqrt{s} \| \alpha - \hat{\alpha} \|_2. \] (5.31)

Hence on the event \( A(\lambda) \cap B(s, \kappa) \), putting together the results (5.24), (5.29), (5.30) and (5.31) gives us that
\[ \hat{\theta} - \theta \leq 2\kappa^2 \| \alpha - \hat{\alpha} \|_2^2 + 10\sqrt{s} \lambda \| \alpha - \hat{\alpha} \|_2 + |\alpha^T (\hat{\Sigma} - \Sigma) \alpha | + 2|\alpha^T (\hat{\mu} - \mu) |. \]
\[ \leq \frac{6s\lambda^2 (5\lambda_{\min}(\Sigma) + 8\kappa^2)}{(\lambda_{\min}(\Sigma) - \kappa)^4} + |\alpha^T (\hat{\Sigma} - \Sigma) \alpha | + 2|\alpha^T (\hat{\mu} - \mu) |. \] (5.32)

We now turn to lower bounding \( \hat{\theta} - \theta \). On the event \( A(\lambda) \),
\[ \hat{\theta} - \theta \overset{(a)}{=} (2\hat{\mu}^T \hat{\alpha} - \hat{\alpha}^T \hat{\Sigma} \hat{\alpha}) - (2\mu^T \alpha - \alpha^T \Sigma \alpha) \]
\[ \overset{(a)}{\geq} (2\hat{\mu}^T \alpha - \alpha^T \hat{\Sigma} \alpha - 2\lambda \| \alpha \|_1 + 2\lambda \| \hat{\alpha} \|_1) - (2\mu^T \alpha - \alpha^T \Sigma \alpha) \]

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\[
\begin{align*}
= -\alpha^T(\widehat{\Sigma} - \Sigma)\alpha - 2\alpha^T(\mu - \widehat{\mu}) + 2\lambda(\|\alpha\|_1 - \|\alpha\|_1) \\
\geq -|\alpha^T(\widehat{\Sigma} - \Sigma)\alpha| - 2|\alpha^T(\mu - \widehat{\mu})| - 2\lambda\|\alpha\|_1 \\
\geq (b) -8\lambda\sqrt{s}\|\alpha - \alpha\|_2 - |\alpha^T(\widehat{\Sigma} - \Sigma)\alpha| - 2|\alpha^T(\widehat{\mu} - \mu)| \\
\geq (c) -24s\lambda^2 \left(\frac{1}{\lambda_{\min}^{1/2}(\Sigma) - \kappa}\right)^2 - |\alpha^T(\widehat{\Sigma} - \Sigma)\alpha| - 2|\alpha^T(\widehat{\mu} - \mu)|.
\end{align*}
\]

(5.33)

(5.34)

Here, (a) holds by the definition of $\tilde{\alpha}$; (b) is by (5.31); and (c) is due to (5.24). Combining the upper bound (5.32) and lower bound (5.34) for $\theta - \theta$ completes the proof.

5.3 Proof of Theorem 2 and Proposition 1

Proposition 1 is a simple by-product of Theorem 2. In the following, we present the proof for Theorem 2. The proof of Proposition 1 will be appropriately mentioned at some point in the proof.

To obtain the lower bounds, it is sufficient to consider Gaussian distributions and covariance matrices with bounded maximum eigenvalue $\lambda_{\max}(\Sigma) \leq c_U$. In this section, a given pair $(\mu, \Sigma)$ that belongs to $\mathcal{H}(s, \tau)$ represents a Gaussian distribution with mean $\mu$ and covariance matrix $\Sigma$. Let $\mathbb{P}_n(\mu, \Sigma)$ be the joint distribution of $n$ i.i.d samples from $N(\mu, \Sigma)$. We use $C_1, C_2, \ldots$ to denote positive constants possibly depending on $c_L, c_U$.

5.3.1 Lower bound for $\mathbb{E}|\widehat{\theta} - \theta|$ 

We first derive the lower bound $\tau \wedge \frac{\tau + \sqrt{\tau}}{\sqrt{n}}$. This can be obtained by reducing the estimation problem to a problem of testing between two distributions $N(\mu, \Sigma)$ and $N(\tilde{\mu}, \tilde{\Sigma})$. Specifically, according to Theorem 2.2 (iii) in Tsybakov [2009], a lower bound

\[
\inf_{\hat{\theta}} \sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \mathbb{E}|\widehat{\theta} - \theta| \geq \frac{c}{4} e^{-\eta}
\]

holds if the followings are true:

1. $(\mu, \Sigma), (\tilde{\mu}, \tilde{\Sigma}) \in \mathcal{H}(s, \tau)$. 

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(2) $D_{KL}(\mathbb{P}_n(\mu, \Sigma) \| \mathbb{P}_n(\hat{\mu}, \hat{\Sigma})) \leq \eta$, where $D_{KL}(\cdot \| \cdot)$ is the KL-divergence.

(3) $|\mu^T \Sigma^{-1} \mu - \hat{\mu}^T \hat{\Sigma}^{-1} \hat{\mu}| \geq 2\zeta$.

Depending on the scaling of $\tau$, we construct different testing problems.

(i) $\tau \geq 1$. We consider $\mu = \hat{\mu} = (\sqrt{\tau}c_L, 0, \ldots, 0)^T$, $\Sigma = \text{diag}(c_L, c_L, \ldots, c_L)$, $\hat{\Sigma} = \text{diag}(c_L + \frac{c_U - c_L}{\sqrt{n}}, c_L, \ldots, c_L)$. We verify (1)-(3) one by one. Part (1) is obvious. For Part (3), it is straightforward to see

$$|\mu^T \Sigma^{-1} \mu - \hat{\mu}^T \hat{\Sigma}^{-1} \hat{\mu}| = \frac{\tau(c_U - c_L)}{(\sqrt{n} - 1)c_L + c_U} \geq \frac{\tau(c_U - c_L)}{\sqrt{n}(c_U + c_L)}.$$ 

Regarding Part (2), we have

$$D_{KL}(\mathbb{P}_n(\mu, \Sigma) \| \mathbb{P}_n(\hat{\mu}, \hat{\Sigma})) = n^2 \left[ \log |\hat{\Sigma}| - \log |\Sigma| - p + \text{tr}(\Sigma \hat{\Sigma}^{-1}) \right]$$

$$= n^2 \left( \log \left( 1 + \frac{c_U/c_L - 1}{\sqrt{n}} \right) - \frac{c_U/c_L - 1}{\sqrt{n} + c_U/c_L - 1} \right)$$

$$\leq n^2 \left( \frac{c_U/c_L - 1}{\sqrt{n}} + \frac{(c_U/c_L - 1)^2}{2n} - \frac{c_U/c_L - 1}{\sqrt{n} + c_U/c_L - 1} \right) \leq \frac{(c_U - c_L)^2}{c_L^2},$$

where we have used the fact that $\log(1 + z) \leq z + \frac{z^2}{2}$ for $z \geq 0$. Above all, we obtain the lower bound

$$\inf_{\hat{\theta}} \sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \mathbb{E}|\hat{\theta} - \theta| \geq C_1 \frac{\tau}{\sqrt{n}}.$$ 

(ii) $\frac{1}{n} \leq \tau \leq 1$. We set up the parameters: $\mu = (\sqrt{\tau}c_L, 0, \ldots, 0)^T$, $\hat{\mu} = (\sqrt{\tau}c_L + \frac{c_U - c_L}{2n}, 0, \ldots, 0)^T$, $\Sigma = \hat{\Sigma} = \text{diag}(c_U, \ldots, c_U)$. Since $\frac{1}{n} \leq \tau$,

$$\mu^T \Sigma^{-1} \mu \leq c_U^{-1}(\tau c_L + \frac{c_U - c_L}{n}) \leq \tau.$$

Then Part (1) follows directly. For Part (2),

$$D_{KL}(\mathbb{P}_n(\mu, \Sigma) \| \mathbb{P}_n(\hat{\mu}, \hat{\Sigma})) = n \left\| \mu - \hat{\mu} \right\|_2^2 = \frac{c_U - c_L}{4c_U}.$$ 

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Finally, \( |\mu^T \Sigma^{-1} \mu - \hat{\mu}^T \hat{\Sigma}^{-1} \hat{\mu}| \geq \sqrt{\frac{c_L (2c_L - c_L)}{c_L^2 n}} \). We thus have the bound

\[
\inf_{\hat{\theta}} \sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \mathbb{E} |\hat{\theta} - \theta| \geq C_2 \sqrt{\frac{\tau}{n}}.
\]

(iii) \( \tau \leq \frac{1}{n} \). We consider the parameters \( \mu = (\sqrt{\tau c_L} 0, \ldots, 0)^T, \hat{\mu} = (\sqrt{\tau c_L} 0, \ldots, 0)^T, \Sigma = \hat{\Sigma} = \text{diag}(c_L, \ldots, c_L) \). Part (1) clearly holds. Regarding Part (2), since \( \tau \leq \frac{1}{n} \)

\[
D_{KL}(\mathbb{P}^n_\mu, \Sigma) || \mathbb{P}^n_{\hat{\mu}, \hat{\Sigma}}) = n \frac{\|\mu - \hat{\mu}\|_2^2}{2c_L} = \frac{n \tau}{8} \leq 1.
\]

For Part (3), we have \( |\mu^T \Sigma^{-1} \mu - \hat{\mu}^T \hat{\Sigma}^{-1} \hat{\mu}| = \frac{3}{4} \). Hence,

\[
\inf_{\hat{\theta}} \sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \mathbb{E} |\hat{\theta} - \theta| \geq C_3 \tau.
\]

We proceed to obtain the other lower bound \( \left[ \tau \wedge \left( \frac{(s+1) \log p}{n} \right) c_0 \exp(-e^{2s^2 p^2 c_0^{-1}}) \right] \). We apply the method of two fuzzy hypotheses, a generalization of the two-point testing technique. The method has been used to obtain minimax lower bound for functional estimation problems (Fan, Rigollet, and Wang, 2015). Denote \( S = \{S \subseteq [p - 1] : |S| = s - 1\} \). We reduce the estimation problem to a testing between \( \mathbb{P}^n_\mu, \Sigma \) and \( \frac{1}{|S|} \sum_{S \in S} \mathbb{P}^n_{\mu, \Sigma} \), where the parameters \( (\mu^0, \Sigma^0), (\mu^S, \Sigma^S) \) will be constructed adaptively depending on the scaling of \( \tau \). According to Theorem 2.15 in Tsybakov (2009), if we are able to show the following:

1. \( (\mu^0, \Sigma^0), (\mu^S, \Sigma^S) \in \mathcal{H}(s, \tau), \forall S \in S \),
2. \( \chi^2 \left( \frac{1}{|S|} \sum_{S \in S} \mathbb{P}^n_{\mu^S, \Sigma^S}, \mathbb{P}^n_\mu, \Sigma^0 \right) \leq \eta \), where \( \chi^2(\cdot, \cdot) \) is the \( \chi^2 \)-divergence,
3. \( |(\mu^0)^T (\Sigma^0)^{-1} \mu^0 - (\mu^S)^T (\Sigma^S)^{-1} \mu^S| \geq 2 \zeta, \forall S \in S \),

then

\[
\inf_{\hat{\theta}} \sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \mathbb{E} |\hat{\theta} - \theta| \geq \frac{\zeta}{4} e^{-\eta}.
\]

We perform the analyses for three different cases, respectively.
(i) \( \tau \geq 1 \). Let \( 1 \in \mathbb{R}^{p-1} \) be the vector with all the entries equal to 1. Set \( \mu^0 = \mu^S = (\sqrt{\frac{(c_L + c_U)^2}{4}}, 0, \ldots, 0)^T, \Sigma^0 = \frac{c_L + c_U}{2} \mathbf{1}_p \), and for \( S \in \mathcal{S} \)

\[
\Sigma^S = \frac{c_L + c_U}{2} \begin{bmatrix} 1 & c_1^T \frac{1}{s} \\ c_1 \mathbf{1}_{p-1} \end{bmatrix}, 
\]

\[ c = \frac{(c_U - c_L) \rho}{(c_U + c_L) \sqrt{2(s-1)}}, \quad 0 \leq \rho \leq 1, \]

where \( \rho \) is a constant that we will specify later. For Part (1), first note that \((\mu^0, \Sigma^0) \in H(s, \tau)\) is trivially true. Moreover, by blockwise matrix inversion we obtain

\[
(\Sigma^S)^{-1} = \frac{2}{c_L + c_U} \mathbf{1}_p + \frac{2}{c_L + c_U} \begin{bmatrix} \frac{c^2(s-1)}{1-c^2(s-1)} & \frac{-c}{1-c^2(s-1)} \frac{1}{s} \\ \frac{-c}{1-c^2(s-1)} & \frac{c^2(s-1)}{1-c^2(s-1)} \end{bmatrix}.
\]

Hence,

\[
\|(\Sigma^S)^{-1} \mu^S\|_0 \leq s, \quad (\mu^S)^T(\Sigma^S)^{-1} \mu^S = \frac{\tau}{2 - \frac{(c_U - c_L)^2 \rho^2}{(c_L + c_U)^2}} \leq \tau.
\]

To prove \((\mu^S, \Sigma^S) \in H(s, \tau)\), it remains to show \(c_L \leq \lambda_{\min}(\Sigma^S) \leq \lambda_{\max}(\Sigma^S) \leq c_U\). This holds because by Weyl’s inequality,

\[
\max(|\lambda_{\max}(\Sigma^S) - \frac{c_L + c_U}{2}|, |\lambda_{\min}(\Sigma^S) - \frac{c_L + c_U}{2}|) \
\leq \|\Sigma^S - \Sigma^0\|_F = \frac{(c_U - c_L) \rho}{2} \leq \frac{c_U - c_L}{2}.
\]

Regarding Part (3), a straightforward calculation delivers

\[
|(\mu^0)^T(\Sigma^0)^{-1} \mu^0 - (\mu^S)^T(\Sigma^S)^{-1} \mu^S| \geq \frac{(c_U - c_L)^2 \rho^2 \tau}{4(c_L + c_U)^2}.
\]

Finally, we bound the \( \chi^2 \)-divergence in Part (2),

\[
\chi^2\left( \frac{1}{|\mathcal{S}|} \sum_{S \in \mathcal{S}} \mathbb{P}^{\mu^S, \Sigma^S} / \mathbb{P}^{\mu^0, \Sigma^0} \right) = \int \left( \frac{1}{|\mathcal{S}|} \sum_{S \in \mathcal{S}} \prod_{i=1}^n \mathbb{P}^{\mu^S, \Sigma^S}(x_i) \right)^2 d\mathbf{x}_1 \cdots d\mathbf{x}_n - 1
\]
The key step is to bound the \( \chi^2 \)-divergence in Part (2):

\[
\chi^2 \left( \frac{1}{|S|^2} \sum_{S \in \mathcal{S}} \mathbb{P}(\mu_0, \Sigma^0), \mathbb{P}(\mu^S, \Sigma^S) \right)
= \frac{1}{|S|^2} \sum_{S \in \mathcal{S}} \sum_{\tilde{S} \in \mathcal{S}} \left(2\pi\right)^{-p/2} \left| \frac{2}{c_L + c_U}(\Sigma^S + \Sigma^{\tilde{S}}) - \frac{4}{(c_L + c_U)^2} \Sigma^S \Sigma^{\tilde{S}} \right|^{-n/2} - 1
= \frac{1}{|S|^2} \sum_{S \in \mathcal{S}} \sum_{\tilde{S} \in \mathcal{S}} |1 - c^2 \mathbf{1}_S^T \frac{T}{S}| - n - 1 \leq \left[ (e^{2\rho^2} - 1) \frac{s}{p} + 1 \right]^{s} \leq e^{2\rho^2 s^{2\rho^2 - 1}},
\]

where in the first inequality we have used the result of Lemma A.1 in Fan, Rigollet, and Wang (2015), and the second inequality is due to the fact that \( \log(1 + x) \leq x, \forall x \geq 0 \). Therefore, by choosing \( \rho = \rho_0 \sqrt{\frac{\log p}{n}} \), we can conclude the lower bound,

\[
\inf_{\hat{\theta}} \sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \mathbb{E}[\hat{\theta} - \theta] \geq C_1 \rho_0^2 \frac{s \log p}{n} \exp(-e^{2\rho^2 \rho_0^2 - 1}), \forall \rho_0 \leq \sqrt{\frac{n}{s \log p}}.
\]

(ii) \( \frac{s \log p}{n} \leq \tau \leq 1 \). We set the parameters: \( \mu^0 = 0, \mu^S = (\gamma_0 \sqrt{\frac{c_L \log p}{n}} \mathbf{1}_S, 0)^T \) for \( \gamma_0 \in (0, 1) \), and \( \Sigma^0 = \Sigma^S = c_L \mathbf{I}_p \). Clearly, \( (\mu^0, \Sigma^0) \in \mathcal{H}(s, \tau) \). Also since \( \frac{s \log p}{n} \leq \tau \) and \( \gamma_0 \leq 1 \),

\[
(\mu^S)^T (\Sigma^S)^{-1} \mu^S = \frac{\gamma_0^2 (s - 1) \log p}{n} \leq \tau,
\]

thus \( (\mu^S, \Sigma^S) \in \mathcal{H}(s, \tau), \forall S \in \mathcal{S} \). For Part (3), we obtain

\[
| (\mu^0)^T (\Sigma^0)^{-1} \mu^0 - (\mu^S)^T (\Sigma^S)^{-1} \mu^S | = \frac{\gamma_0^2 (s - 1) \log p}{n}, \forall S \in \mathcal{S}.
\]

The key step is to bound the \( \chi^2 \)-divergence in Part (2):
randomly chosen from \( S \); the first inequality is again due to the result in Lemma A.1 from Fan, Rigollet, and Wang (2015); and the last inequality holds because \( \log(1 + x) \leq x, \forall x \geq 0. \)

As a result, we obtain the lower bound,

\[
\inf_{\hat{\theta}} \sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \mathbb{E}|\hat{\theta} - \theta| \geq C_3 \gamma_0^2 s \log p \frac{p}{n} \exp(-e^{2s^2 p \gamma_0^2 - 1}), \quad \forall \gamma_0 \in (0, 1].
\]

**Remark:** Proposition 1 can be proved similarly as in Case (ii), by setting \( \mu^0 = 0, \mu^S = (\gamma_0 \sqrt{s} 1_S, 0)^T, \Sigma^0 = \Sigma^S = c_L I_p, \) where \( s = \frac{p}{n} \) and \( \gamma_0 > 0 \) is some fixed constant. We thus do not repeat the details.

(iii) \( \tau \leq \frac{s \log p}{n} \). We construct the same parameters as we did in Case (ii) except setting \( \mu^S = (\gamma_0 \sqrt{s} 1_S, 0)^T \). It is direct to confirm that Part (1) holds, and in Part (3),

\[
|\left((\mu^0)^T (\Sigma^0)^{-1} \mu^0 - (\mu^S)^T (\Sigma^S)^{-1} \mu^S\right)| = \frac{\gamma_0^2 (s - 1) \tau}{s}.
\]

Regarding Part (2), since \( \tau \leq \frac{s \log p}{n} \), the bound for the \( \chi^2 \)-divergence derived in Case (ii) continues to hold here. Hence, we have

\[
\inf_{\hat{\theta}} \sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \mathbb{E}|\hat{\theta} - \theta| \geq C_4 \gamma_0^2 \tau \exp(-e^{2s^2 p \gamma_0^2 - 1}), \quad \forall \gamma_0 \in (0, 1].
\]

### 5.3.2 Lower bound for \( \mathbb{E}\|\hat{\alpha} - \alpha\|_2^2 \)

Since

\[
\inf_{\hat{\alpha}} \sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \mathbb{E}\|\hat{\alpha} - \alpha\|_2^2 \geq \left( \inf_{\hat{\alpha}} \sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \mathbb{E}\|\hat{\alpha} - \alpha\|_2 \right)^2,
\]

the rest of the proof will be focused on obtaining

\[
\inf_{\hat{\alpha}} \sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \mathbb{E}\|\hat{\alpha} - \alpha\|_2 \geq C_1 \left[ \sqrt{\tau} \wedge \sqrt{(1 + \tau)s \log(p/s)} \right].
\]
We use Fano’s inequality (Cover and Thomas, 2012) to lower bound $E\|\hat{\alpha} - \alpha\|_2$. More specifically, according to Corollary 2.6 in Tsybakov (2009), if the followings hold,

1. $(\mu^j, \Sigma^j) \in \mathcal{H}(s, \tau), j = 0, 1, \ldots, M$, for $M \geq 2$,
2. \(1 \over M+1 \sum_{j=1}^{M} D_{KL}(P_n(\mu^j, \Sigma^j) \parallel P_n(\mu^0, \Sigma^0)) \leq \eta \log M \) for some $\eta \in (0, 1),$
3. $\| (\Sigma^i)^{-1} \mu^i - (\Sigma^j)^{-1} \mu^j \|_2 \geq 2 \zeta, 0 \leq i < j \leq M$,

then

$$\inf_{\hat{\alpha}} \sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} E\|\hat{\alpha} - \alpha\|_2 \geq \zeta \cdot \left(\log(M + 1) - \log 2 \frac{\log M}{\log M} - \eta\right).$$

As in Section 5.3.1, we will construct different parameters $\{(\mu^j, \Sigma^j)\}_{j=0}^{M}$ based on the scaling of $\tau$, and verify Parts (1)-(3) to derive the lower bounds.

(i) $\tau \geq 1$. Let $\Lambda = \{1, 2, \ldots, p - \frac{s}{2}\}$, $\mathcal{Q}$ be the set of all subsets of $\Lambda$ with cardinality $\frac{s}{2}$, and $1 \in \mathbb{R}^{p-s/2}$ be the vector with all the elements equal to 1. For each $0 \leq j \leq M$, we consider

$$\mu^j = \bar{\mu} = (\sqrt{c_L / s}, \sqrt{c_L / s}, \ldots, \sqrt{c_L / s}, 0, \ldots, 0)^T, \quad \Sigma^j = c_L I_p + \frac{(c_0 - c_L) \mathbf{a}^j (\mathbf{a}^j)^T}{\|\mathbf{a}^j\|_2^2},$$

where $c_0 \in [c_L, c_U]$, $\mathbf{a}^j = \bar{\mu} + \sqrt{\frac{c_L}{s}}(0, 1_{Q^j}), Q^j \in \mathcal{Q}$. The specific choice of $c_0, Q^j, M$ will be made clear later. We first verify Part (1). Observe that $\lambda_{\min}(\Sigma^j) = c_L, \lambda_{\max}(\Sigma^j) = c_0 \leq c_U$.

Moreover, applying Woodbury matrix identity gives

$$(\Sigma^j)^{-1} = \frac{1}{c_L} I_p - \frac{(c_0 - c_L) \mathbf{a}^j (\mathbf{a}^j)^T}{c_L c_0 \|\mathbf{a}^j\|_2^2}.$$ 

Hence, $(\mu^j)^T (\Sigma^j)^{-1} \mu^j \leq \frac{\|\mu\|_2^2}{c_L} < \tau$, and $\| (\Sigma^j)^{-1} \mu^j \|_0 \leq s$. For Part (3), $\forall 0 \leq i < j \leq M$,

$$\| (\Sigma^i)^{-1} \mu^i - (\Sigma^j)^{-1} \mu^j \|_2 = \frac{c_0 - c_L}{2c_0 c_L} \| \mathbf{a}^i - \mathbf{a}^j \|_2 = \frac{(c_0 - c_L) \sqrt{\tau}}{2c_0 \sqrt{c_L} s} \sqrt{s - 2 |Q^j \cap Q^i|}.$$ (5.35)
Regarding Part (2), we can do the following calculations,

$$
\frac{1}{M+1} \sum_{j=1}^{M} D_{KL}(P^{n}(\mu^j, \Sigma^j)||P^{n}(\mu^{0}, \Sigma^{0})) = \frac{1}{M+1} \sum_{j=1}^{M} \frac{n}{2} \left( \log |\Sigma^0| - \log |\Sigma^j| - p + \text{tr}(\Sigma^j(\Sigma^0)^{-1}) \right)
$$

$$
= \frac{1}{M+1} \sum_{j=1}^{M} \frac{n(c_0 - c_L)^2}{2c_0c_L} \leq \frac{n(c_0 - c_L)^2}{2c_0c_L}.
$$

(5.36)

Based on (5.35) and (5.36), we now choose the value for \( c_0, M \) and \( Q^j (j = 0, \ldots, M) \). To obtain tighter lower bounds, it is desirable to make \( M \) as large as possible while keeping the norms in (5.35) “not small” (equivalently \(|Q^j \cap Q^i|, i \neq j “not large”\)). Towards that goal, we utilize an existing combinatorics result (cf. Lemma 4 in Birgé and Massart (2001)), which implies that if \( p \geq 2s \) then there exists a subset \( C \) of \( Q \), such that \(|Q^i \cap Q^j| \leq s^4\) for all \( Q^i \neq Q^j \in C \), and

$$
\log |C| \geq \frac{s}{4} \log \left( \frac{e(p - s/2)}{4s} \right) \geq \frac{s}{4} \log \left( \frac{3ep}{16s} \right).
$$

Thus we choose \( \{Q^j\}_{j=0}^{M} = C \), and \( M = |C| - 1 \). Accordingly, from (5.36) we can set \( c_0 = c_L(1 + \sqrt{\frac{\log p}{n} + \frac{2 \log s}{n}}) \) to satisfy Part (2) with \( \eta = \frac{1}{2} \), given the condition that \( p/s > C_2 \). It is also clear that \( c_0 \in [c_L, c_U] \) as long as \( \frac{s \log p/s}{n} \leq C_3 \). Finally, using the property \(|Q^i \cap Q^j| \leq s^4\) for \( i \neq j \), Equation (5.35) implies that Part (3) holds for \( \zeta = C_4 \sqrt{\frac{\tau s \log(p/s)}{n}} \). Above all, we reach the lower bound,

$$
\inf_{\hat{\alpha}} \sup_{(\mu, \Sigma) \in H(s, \tau)} \mathbb{E}||\hat{\alpha} - \alpha||_2 \geq C_5 \sqrt{\frac{\tau s \log(p/s)}{n}}.
$$

(ii) \( \frac{s \log(p/s)}{n} \leq \tau \leq 1 \). With a bit abuse of notations, let \( Q \) be the set of all subsets of \{1, 2, \ldots, p\} with cardinality \( s \). We set the parameters:

$$
\mu^j = \sqrt{\frac{\rho c_L \log(p/s)}{n}} 1_{Q^j}, \quad \rho \in (0, 1), \quad Q^j \in Q, \quad \Sigma^j = c_L I_p, \quad j = 0, 1, \ldots, M.
$$

We will specify \( M \) and \( \{Q^j\}_{j=0}^{M} \) shortly. Since \( \frac{s \log(p/s)}{n} \leq \tau \), it is straightforward to confirm
that Part (1) holds. In Parts (2) and (3), we obtain respectively,

\[
\frac{1}{M + 1} \sum_{j=1}^{M} D_{KL}(P^n_{\mu_j, \Sigma_j} || P^n_{\mu_0, \Sigma_0}) = \frac{\rho \log(p/s)}{M + 1} \sum_{j=1}^{M} (s - |Q^j \cap Q^0|) \leq \rho s \log(p/s),
\]

and

\[
\| (\Sigma^j)^{-1} \mu^j - (\Sigma^0)^{-1} \mu^0 \|_2 = \sqrt{\frac{2 \rho \log(p/s)}{n c_L}} \sqrt{s - |Q^j \cap Q^0|}.
\]

We then use the same arguments as in Case (i) to choose \( M \) and \( \{Q^j\}_{j=0}^{M} \). Under the condition that \( p/s > C_2 \), we will be able to choose sufficiently small \( \rho \) such that \( \rho s \log(p/s) \leq \frac{1}{2} \log M \). As a result, we will achieve the lower bound,

\[
\inf_{\hat{\alpha}} \sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \mathbb{E} \| \hat{\alpha} - \alpha \|_2 \geq C_6 \sqrt{s \log(p/s) / n}.
\]

(iii) \( \tau \leq \frac{s \log(p/s)}{n} \). We consider

\[
\mu^j = \sqrt{\frac{p c_L \tau}{s}} 1_{Q^j}, \quad \rho \in (0, 1), \quad Q^j \in \mathcal{Q}, \quad \Sigma^j = c_L I_p, \quad j = 0, 1, \ldots, M.
\]

Here \( \mathcal{Q} \) is the same as in Case (ii). Clearly Part (1) holds. For Parts (2) and (3), with the same choice of \( \rho, M \) and \( \{Q^j\}_{j=0}^{M} \) as in Case (ii), it is not hard to verify

\[
\frac{1}{M + 1} \sum_{j=1}^{M} D_{KL}(P^n_{\mu^j, \Sigma^j} || P^n_{\mu^0, \Sigma_0}) = \frac{\rho n \tau}{s(M + 1)} \sum_{j=1}^{M} (s - |Q^j \cap Q^0|) \leq \frac{1}{2} \log M,
\]

\[
\| (\Sigma^j)^{-1} \mu^j - (\Sigma^0)^{-1} \mu^0 \|_2 = \sqrt{\frac{2 \rho \tau}{c_L s}} \sqrt{s - |Q^j \cap Q^0|} \geq C_7 \sqrt{\tau},
\]

where we have used the fact \( \tau \leq \frac{s \log(p/s)}{n} \) in the first inequality. Therefore, we obtain the lower bound,

\[
\inf_{\hat{\alpha}} \sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \mathbb{E} \| \hat{\alpha} - \alpha \|_2 \geq C_8 \sqrt{\tau}.
\]
5.4 Proof of Proposition 2

Proof. By the definition of $\tilde{\alpha}$ in (3.1), it is clear that the quadratic function $\frac{1}{2} \tilde{\alpha}^T \tilde{\Sigma} \tilde{\alpha}w^2 + (\lambda \parallel \tilde{\alpha} \parallel_1 - \tilde{\alpha}^T \tilde{\mu})w$ achieves minimum over $[0, 1]$ at $w = 1$, which implies

$$\tilde{\mu}^T \tilde{\alpha} - \tilde{\alpha}^T \tilde{\Sigma} \tilde{\alpha} \geq \lambda \parallel \tilde{\alpha} \parallel_1 \geq \lambda (\parallel \alpha \parallel_1 - \parallel \tilde{\alpha} - \alpha \parallel_1).$$

The above inequality results in the following bounds,

$$|\tilde{\theta}_c - \theta| = |\tilde{\theta}_c - \tilde{\theta} + \tilde{\theta} - \theta| \geq |c - 2| \cdot |\tilde{\mu}^T \tilde{\alpha} - \tilde{\alpha}^T \tilde{\Sigma} \tilde{\alpha}| - |\tilde{\theta} - \theta| \geq \lambda |c - 2| \cdot \parallel \alpha \parallel_1 - \lambda |c - 2| \cdot \parallel \tilde{\alpha} - \alpha \parallel_1 - |\tilde{\theta} - \theta|.$$

Hence we can obtain

$$\sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \mathbb{E} |\tilde{\theta}_c - \theta| \geq \lambda |c - 2| \cdot \sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \parallel \alpha \parallel_1 - \lambda |c - 2| \cdot \parallel \tilde{\alpha} - \alpha \parallel_1 - |\tilde{\theta} - \theta| \geq \sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \mathbb{E} \parallel \tilde{\alpha} - \alpha \parallel_1^2$$

$$\sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \mathbb{E} |\tilde{\theta}_c - \theta| \geq \lambda |c - 2| \cdot \sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \parallel \alpha \parallel_1 - \lambda |c - 2| \cdot \parallel \tilde{\alpha} - \alpha \parallel_1 - |\tilde{\theta} - \theta| \geq \sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \mathbb{E} \parallel \tilde{\alpha} - \alpha \parallel_1^2$$

Let $\mu^* = (\sqrt{s c_U/s, \ldots, \sqrt{s c_U}/s, 0, \ldots, 0})$, $\Sigma^* = \text{diag}(c_U, c_U, \ldots, c_U)$. It is straightforward to confirm that $(\mu^*, \Sigma^*) \in \mathcal{H}(s, \tau)$. So

$$\sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \parallel \alpha \parallel_1 \geq \parallel (\Sigma^*)^{-1} \mu^* \parallel_1 = \sqrt{\frac{s \tau}{c_U}},$$

from which it holds that $J_1 \geq \sqrt{\frac{\tau(1 + \tau)s \log p}{n}}$. Regarding $J_2$, under the scaling $\frac{s \log p}{n} = o(1), p^{-\delta} \lesssim \frac{s \log p}{n}$, the upper bound we have derived for $\mathbb{E} \parallel \tilde{\alpha} - \alpha \parallel_1^2$ in (5.22) leads to $J_2 \lesssim \frac{(1 + \tau)s \log p}{n}$. For $J_3$, we already know from (3.9),

$$J_3 \lesssim \frac{(1 + \tau)s \log p}{n} + \frac{\tau + \sqrt{\tau}}{\sqrt{n}}.$$
Given the condition $\tau \gtrsim \frac{s \log p}{n}$, it is not hard to verify that $J_1$ is the dominant term among $J_i, i = 1, 2, 3$. This completes the proof. 

5.5 Proof of Theorem 3

Proof. Throughout the proof, we use $C_1, C_2, \ldots$ to denote positive constants that possibly depend on $\nu, c_L, c_U$. Some of them may depend on additional quantities, and clarification will be made when necessary.

Recall the definition of $\hat{\alpha}$ in (3.14). The basic inequality holds that

$$\frac{1}{2} \hat{\alpha} \hat{\Sigma} \hat{\alpha} - \hat{\alpha}^T \hat{\mu} \leq \frac{1}{2} \alpha \hat{\Sigma} \alpha - \alpha^T \hat{\mu},$$

which is equivalent to

$$\frac{1}{2} (\alpha - \hat{\alpha})^T \hat{\Sigma} (\alpha - \hat{\alpha}) \leq \alpha^T (\hat{\Sigma} - \Sigma)(\alpha - \hat{\alpha}) + (\mu - \hat{\mu})^T (\alpha - \hat{\alpha}).$$

Since $\|\hat{\alpha} - \alpha\|_0 \leq 2s$, on the event $A(\lambda) \cap C(s, \kappa)$, using a similar inequality to (5.27) we can continue from the above inequality to obtain,

$$\frac{1}{2} (\lambda^{1/2}_{\min}(\Sigma) - \kappa)^2 \|\hat{\alpha} - \alpha\|_2^2 \leq \frac{\sqrt{s} \lambda}{\sqrt{2}} \|\hat{\alpha} - \alpha\|_2,$$

leading to for $0 < \kappa < \lambda^{1/2}_{\min}(\Sigma)$,

$$\|\hat{\alpha} - \alpha\|_2^2 \leq \frac{2s \lambda^2}{(\lambda^{1/2}_{\min}(\Sigma) - \kappa)^4}.$$ 

The rest of the derivation of the upper bound for $E\|\hat{\alpha} - \alpha\|_2^2$ is similar to the one for $E\|\hat{\alpha} - \alpha\|_2^2$ in Theorem 1, we thus do not repeat the arguments.

Regarding the bound for $E|\hat{\theta}_c - \theta|$, we first consider the case $c = 2$. With a minor modification of the proof of Lemma 4, it can be obtained that on the event $A(\lambda) \cap C(s, \kappa)$, the same upper bound (up to constants) as the one in Lemma 4 holds for $|\hat{\theta}_2 - \theta|$. Accordingly, the bound for $E|\hat{\theta}_2 - \theta|$
can be derived in the same way as for $\mathbb{E}|\tilde{\theta} - \theta|$ in Theorem 1. We will not repeat the details here.

When $c \neq 2$, we use

$$
\sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \mathbb{E} |\tilde{\theta}_c - \theta| \leq \sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \mathbb{E} |\tilde{\theta}_2 - \theta| + |c - 2| \sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \mathbb{E} |\hat{\alpha}^T \hat{\mu} - \hat{\alpha}^T \hat{\Sigma} \hat{\alpha}|. \tag{5.37}
$$

We have already shown for the case $c = 2$ that

$$
\sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \mathbb{E} |\tilde{\theta}_2 - \theta| \lesssim \frac{(1 + \tau) s \log p}{n} + \frac{\tau + \sqrt{\tau}}{\sqrt{n}}. \tag{5.38}
$$

The remainder of the proof is to bound $\mathbb{E} |\hat{\alpha}^T \hat{\mu} - \hat{\alpha}^T \hat{\Sigma} \hat{\alpha}|$. Define the event

$$
\mathcal{D} = \left\{ \sup_{|S| \leq s} \| \tilde{\Sigma}_{SS}^{-1} \tilde{\mu}_S \|_2 \leq \gamma \right\}.
$$

Based on the definition of $\hat{\alpha}$ in (3.14), it is straightforward to verify that on the event $\mathcal{D}$, $\hat{\alpha}$ will take the following form,

$$
\hat{\alpha}_\hat{S} = \tilde{\Sigma}_{\hat{S} \hat{S}}^{-1} \tilde{\mu}_{\hat{S}},
$$

where $\hat{S}$ is the support of $\hat{\alpha}$, and $|\hat{S}| \leq s$. As a result, on the event $\mathcal{D}$ it holds that

$$
\hat{\alpha}^T \hat{\mu} - \hat{\alpha}^T \hat{\Sigma} \hat{\alpha} = 0,
$$

thus we can bound $\mathbb{E} |\hat{\alpha}^T \hat{\mu} - \hat{\alpha}^T \hat{\Sigma} \hat{\alpha}|$ in the following way,

$$
\mathbb{E} |\hat{\alpha}^T \hat{\mu} - \hat{\alpha}^T \hat{\Sigma} \hat{\alpha}| = \mathbb{E} |\hat{\alpha}^T \hat{\mu} - \hat{\alpha}^T \hat{\Sigma} \hat{\alpha}| 1_{\mathcal{D}^c} \leq 3 \mathbb{E} (\hat{\alpha}^T \hat{\mu} 1_{\mathcal{D}^c}) \\
= 3 \mathbb{E} (\hat{\alpha}^T \mu 1_{\mathcal{D}^c}) + 3 \mathbb{E} (\hat{\alpha}^T (\hat{\mu} - \mu) 1_{\mathcal{D}^c}), \tag{5.39}
$$

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where the inequality is due to the fact that \( \frac{1}{2} \alpha^T \hat{\Sigma} \hat{\alpha} - \hat{\alpha}^T \hat{\mu} \leq 0 \). Moreover, we have

\[
|E(\hat{\alpha}^T \mu 1_{D^c})| \leq \|\mu\|_2 \cdot \|\hat{\alpha}\|_2 \cdot \mathbb{P}(D^c) \leq \gamma \sqrt{p c_U \tau} \mathbb{P}(D^c),
\]

(5.40)

and

\[
|E(\hat{\alpha}^T (\hat{\mu} - \mu) 1_{D^c})| \leq E(\|\hat{\alpha}\|_1 \cdot \|\hat{\mu} - \mu\|_{\infty} 1_{D^c}) \leq \sqrt{s} \gamma E(\|\hat{\mu} - \mu\|_{\infty} 1_{D^c})
\]

\[
\leq \sqrt{s} \gamma \sqrt{E\|\hat{\mu} - \mu\|_2^2 \cdot \mathbb{P}(D^c)} \leq C_1 \sqrt{s} \gamma \sqrt{E\|\hat{\mu} - \mu\|_2^2 \cdot \mathbb{P}(D^c)}.
\]

(5.41)

Here, to obtain the last inequality we have used (5.1) to compute

\[
E\|\hat{\mu} - \mu\|_2^2 = \int_0^{C_2 \log p \over n} \mathbb{P}(\|\hat{\mu} - \mu\|_2^2 > t) dt + \int_{C_2 \log p \over n}^\infty \mathbb{P}(\|\hat{\mu} - \mu\|_2^2 > t) dt
\]

\[
\leq \frac{C_2 \log p \over n} + 2p \int_{C_2 \log p \over n}^\infty \exp(-C_3 nt) dt \leq \frac{C_4 \log p \over n}.
\]

We now bound \( \mathbb{P}(D^c) \). We first have

\[
\sup_{|S| \leq s} \|\hat{\Sigma}^{-1}_S \hat{\mu}_S\|_2^2 \leq \left[ \inf_{|S| \leq s} \lambda_{\min}(\hat{\Sigma} SS) \right]^{-1} \cdot \left[ \inf_{|S| \leq s} \lambda_{\min}(\Sigma^{-1/2}_S \hat{\Sigma} SS \Sigma^{-1/2}_S) \right]^{-1} \sup_{|S| \leq s} \|\Sigma^{-1/2}_S \hat{\mu}_S\|_2^2.
\]

We bound the three terms on the above right-hand side, respectively. According to Lemma 2, it holds with probability at least \( 1 - 4p^{-c} \) that

\[
\inf_{|S| \leq s} \lambda_{\min}(\hat{\Sigma} SS) = \inf_{\|u\|_2 = 1, \|u\|_0 \leq s} u^T \hat{\Sigma} u \geq \left( \inf_{\|u\|_2 = 1} \sqrt{u^T \Sigma u} - \sup_{u \in B_0(s) \cap B_2(1)} \left| \sqrt{u^T \Sigma u} - \sqrt{u^T \Sigma u} \right| \right)^2
\]

\[
\geq \left( \sqrt{c_L} - C_5 \sqrt{s \log p \over n} \right)^2 - \frac{C_6 \log p \over n} {\left( \sqrt{c_L} - C_5 \sqrt{s \log p \over n} \right)}^2,
\]

and with similar arguments that

\[
\inf_{|S| \leq s} \lambda_{\min}(\Sigma^{-1/2}_S \hat{\Sigma} SS \Sigma^{-1/2}_S) \geq \left( 1 - C_7 \sqrt{s \log p \over n} - \frac{C_8 \log p \over n} {1 - C_7 \sqrt{s \log p \over n}} \right)^2,
\]

\[
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\]
where for $j = 5, 6, 7, 8$, $C_j > 0$ depend on $c_U, \nu, c$ and $C_j \to \infty$ as $c \to \infty$. Regarding the third term,

$$
\sup_{|S| \leq s} \| \Sigma^{-1/2}_{SS} \hat{\mu}_S \|_2^2 \leq 2 \left( \sup_{|S| \leq s} \| \Sigma^{-1/2}_{SS} \mu_S \|_2^2 + \sup_{|S| \leq s} \| \Sigma^{-1/2}_{SS} (\hat{\mu}_S - \mu_S) \|_2^2 \right)
$$

$$
\leq 2 (\tau + c_L^{-1} s \| \hat{\mu} - \mu \|_\infty^2).
$$

We further upper bound the above using (5.1) to yield

$$
\mathbb{P}(\| \hat{\mu} - \mu \|_\infty^2 \leq \tau t/s) \geq 1 - 2p \exp(-C_8 n \tau t/s), \quad \forall t > 0.
$$

Above all, as long as $s \log p/n$ is sufficiently small, we will be able to set $\gamma = C_9 \sqrt{(1 + t) \tau}$ such that

$$
\mathbb{P}(D^c) = \mathbb{P} \left( \sup_{|S| \leq s} \| \hat{\Sigma}_{SS}^{-1/2} \hat{\mu}_S \|_2^2 > \gamma^2 \right) \leq 4p^{-c} + 2p \exp(-C_8 n \tau t/s). \quad (5.42)
$$

Under the conditions that $s \log p/n = o(1), p^{-\delta} \lesssim \frac{s \log p}{n}, \tau \gtrsim \frac{s \log p}{n}$, we can choose the constants $c, t$ in (5.42) large enough so that

$$
\mathbb{P}(D^c) \lesssim \frac{p^{-1/2} \log p}{n}.
$$

The above combined with (5.39), (5.40) and (5.41) shows that

$$
\sup_{(\mu, \Sigma) \in \mathcal{H}(s, \tau)} \mathbb{E} | \alpha^T \hat{\mu} - \hat{\alpha}^T \hat{\Sigma} \hat{\alpha} | \lesssim \frac{(\tau + \sqrt{\tau}) s \log p}{n}. \quad (5.43)
$$

Putting together the results (5.37), (5.38), and (5.43) establishes the upper bound for $\mathbb{E} | \hat{\theta}_c - \theta |$. \qed

5.6 Proof of Theorem 4

The proof is a direct generalization of the one in the exact sparsity setting. We simply highlight the major different steps, and refer to the proof of Theorem 1 for details. With approximate sparsity,
we define a “pseudo-support” set of $\alpha$ as
\[
S = \left\{ i \in [p] : |\alpha_i| > (1 + \sqrt{\tau})\sqrt{\frac{\log p}{n}} \right\},
\]
and let $|S| = s$. The following two lemmas are useful throughout the proof.

**Lemma 5.** For any $(\mu, \Sigma) \in \mathcal{H}_q(R, \tau)$, it holds that
\[
\mu^T \Sigma^{-1} \mu \leq \tilde{\tau}, \quad \sum_{i=1}^{p} |\alpha_i|^q \leq \tilde{R}.
\]

**Proof.** Recall $\alpha = \Sigma^{-1} \mu$. Given $(\mu, \Sigma) \in \mathcal{H}_q(R, \tau)$, it satisfies that
\[
\mu^T \Sigma^{-1} \mu \leq \tau, \quad \sum_{i=1}^{p} |\alpha_i|^q \leq R.
\]
Hence,
\[
\mu^T \Sigma^{-1} \mu = \alpha^T \Sigma \alpha \leq \delta \|\alpha\|_2^q \leq c_U \left( \sum_{i=1}^{n} |\alpha_i|^q \right)^{\frac{q}{2}} \leq c_U \tilde{R}^q,
\]
yielding that $\mu^T \Sigma^{-1} \mu \leq (c_U \tilde{R}^q) \wedge \tau = \tilde{\tau}$. Moreover, by Hölder’s inequality
\[
\sum_{i=1}^{p} |\alpha_i|^q \leq \left( \sum_{i=1}^{p} |\alpha_i|^2 \right)^{\frac{q}{2}} \cdot \left( \sum_{i=1}^{p} 1 \right)^{1-\frac{q}{2}} \leq p^{1-\frac{q}{2}} c_L^{-\frac{q}{2}} (\alpha^T \Sigma \alpha)^{\frac{q}{2}} \leq p^{1-\frac{q}{2}} c_L^{-\frac{q}{2}} \tau^{\frac{q}{2}}.
\]
Thus $\sum_{i=1}^{p} |\alpha_i|^q \leq (p^{1-\frac{q}{2}} c_L^{-\frac{q}{2}} \tau^{\frac{q}{2}}) \wedge R = \tilde{R}$. \qed

**Lemma 6.** For any $\alpha$ with $\|\alpha\|_q^q \leq \tilde{R}$, it holds that
\[
s \leq \tilde{R} \left( 1 + \sqrt{\tau} \right)^{-q} \left( \frac{\log p}{n} \right)^{-\frac{q}{2}}, \quad \|\alpha_{S^c}\|_1 \leq \tilde{R} \left( 1 + \sqrt{\tau} \right)^{-q} \left( \frac{\log p}{n} \right)^{-\frac{q}{2}}.
\]

\[\text{†With a bit abuse of notations, we have adopted some notations from the exact sparsity case with slightly different meanings.}\]
Proof. For the bound on \(s\), observe that
\[
\hat{R} \geq \sum_{i=1}^{p} |\alpha_i|^q \mathbb{1}_{|\alpha_i| > (1 + \sqrt{\tau}) \sqrt{\frac{\log p}{n}}} \geq s(1 + \sqrt{\tau})^q \left( \frac{\log p}{n} \right)^\frac{q}{2}.
\]
The bound for \(\|\alpha_{S^c}\|_1\) is due to
\[
\hat{R} \geq \sum_{i=1}^{p} |\alpha_i| \cdot |\alpha_i|^{q-1} \mathbb{1}_{|\alpha_i| \leq (1 + \sqrt{\tau}) \sqrt{\frac{\log p}{n}}} \geq (1 + \sqrt{\tau})^{q-1} \left( \frac{\log p}{n} \right)^\frac{q-1}{2} \|\alpha_{S^c}\|_1.
\]

We are in the position to derive the bounds for \(\mathbb{E}\|\hat{\alpha} - \alpha\|_2^2\) and \(\mathbb{E}|\hat{\theta} - \theta|\). Throughout the proof, \(C_1, C_2, \ldots\) are used to denote positive constants that possibly depend on \(\nu, c_L, c_U\). Some of them may depend on additional quantities, and clarification will be made when necessary.

**Upper bound for \(\mathbb{E}\|\hat{\alpha} - \alpha\|_2^2\)**. Due to the approximate sparsity, we have a generalization of \(5.26\): on the event \(A(\lambda)\)
\[
0 \geq \frac{1}{2} (\hat{\alpha} - \alpha)^T \hat{\Sigma} (\hat{\alpha} - \alpha) + \frac{\lambda}{2} (\|\hat{\alpha}_{S^c} - \alpha_{S^c}\|_1 - 3\|\hat{\alpha}_S - \alpha_S\|_1 - 4\|\alpha_{S^c}\|_1), \quad (5.44)
\]
implying
\[
\|\hat{\alpha}_{S^c} - \alpha_{S^c}\|_1 \leq 3\|\hat{\alpha}_S - \alpha_S\|_1 + 4\|\alpha_{S^c}\|_1.
\]

Thus we obtain
\[
\|\hat{\alpha} - \alpha\|_1 \leq 4\|\hat{\alpha}_S - \alpha_S\|_1 + 4\|\alpha_{S^c}\|_1 \leq 4\sqrt{s}\|\hat{\alpha} - \alpha\|_2 + 4\|\alpha_{S^c}\|_1. \quad (5.45)
\]

Combining this result with Lemma 2, we can proceed from \(5.44\) to conclude, with probability at least \(1 - 4p^{-c} - \mathbb{P}(A^c(\lambda))\),
\[
0 \geq \frac{1}{2} \left( \sqrt{c_L} \|\hat{\alpha} - \alpha\|_2 - C_1 \sqrt{\frac{\log p}{n}} \|\hat{\alpha} - \alpha\|_1 - \frac{C_2 \log p}{n} \|\hat{\alpha} - \alpha\|_1^2 \left( \sqrt{c_L} \|\hat{\alpha} - \alpha\|_2 - C_1 \sqrt{\frac{\log p}{n}} \|\hat{\alpha} - \alpha\|_1 \right) \right)^2 + \]

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\[-\frac{\lambda}{2}(3\sqrt{s}\|\tilde{\alpha} - \alpha\|_2 + 4\|\alpha_{S^c}\|_1), \quad (5.46)\]

where the positive constants \(C_1, C_2 \to \infty\) as \(c \to \infty\). We now show that the above inequality leads to the result

\[\|\tilde{\alpha} - \alpha\|_2^2 \lesssim (1 + \tilde{\tau})^{1-\frac{q}{2}} \tilde{R}\left(\frac{\log p}{n}\right)^{1-\frac{q}{2}}. \quad (5.47)\]

Towards that goal, under the scaling \(\tilde{R}(1 + \tilde{\tau})^{-\frac{q}{2}}(\log p/n)^{1-\frac{q}{2}} = o(1)\), we can assume

\[\|\tilde{\alpha} - \alpha\|_2 \gg (1 + \sqrt{\tilde{\tau}})^{1-q/2} \tilde{R}\left(\frac{\log p}{n}\right)^{1-\frac{q}{2}}.\]

Otherwise \((5.47)\) trivially holds. Using this assumption together with Lemma 6 and \((5.45)\), it is straightforward to verify that

\[\sqrt{\frac{\log p}{n}} \|\tilde{\alpha} - \alpha\| \ll \|\tilde{\alpha} - \alpha\|_2.\]

Hence, \((5.46)\) can be simplified to

\[\|\tilde{\alpha} - \alpha\|_2^2 - (1 + \tilde{\tau})^{1-\frac{q}{2}} \tilde{R}\left(\frac{\log p}{n}\right)^{1-\frac{q}{2}} \cdot \|\tilde{\alpha} - \alpha\|_2 - (1 + \tilde{\tau})^{1-\frac{q}{2}} \tilde{R}\left(\frac{\log p}{n}\right)^{1-\frac{q}{2}} \lesssim 0, \quad (5.48)\]

where we have used Lemma 6. The bound \((5.47)\) follows directly from \((5.48)\). We thus have shown

\[\mathbb{P}\left(\|\tilde{\alpha} - \alpha\|_2^2 \lesssim (1 + \tilde{\tau})^{1-\frac{q}{2}} \tilde{R}\left(\frac{\log p}{n}\right)^{1-\frac{q}{2}} \right) \geq 1 - 4p^{-c} - \mathbb{P}(A^c(\lambda)). \quad (5.49)\]

Finally, by Lemma 5 it holds that \(\|\tilde{\alpha}\|_2 \lesssim \tilde{\tau}, \|\alpha\|_2 \lesssim \tilde{\tau}\). We obtain

\[\mathbb{E}\|\tilde{\alpha} - \alpha\|_2^2 \lesssim (1 + \tilde{\tau})^{1-\frac{q}{2}} \tilde{R}\left(\frac{\log p}{n}\right)^{1-\frac{q}{2}} + \tilde{\tau}(p^{-c} + \mathbb{P}(A^c(\lambda)).\]

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The proof is completed by the fact that under the condition \( p^{-\delta} \lesssim \tilde{R} \left( \frac{\log p}{n} \right)^{1-\frac{q}{2}} (\tilde{\tau}^{-\frac{q}{2}} \lor \tilde{\tau}^{-1}) \),

\[
\tilde{\tau} (p^{-c} + \mathbb{P}(A^c(\lambda))) \lesssim (1 + \tilde{\tau})^{1-\frac{q}{2}} \tilde{R} \left( \frac{\log p}{n} \right)^{1-\frac{q}{2}},
\]

where \( \mathbb{P}(A^c(\lambda)) \lesssim p^{-c} \) holds by Lemma 1 Part (i).

**Upper bound for \( \mathbb{E}|\tilde{\theta} - \theta| \).** The key step is to obtain an analog of Lemma 4. Observe that the inequality (5.29) continues to hold here. Hence on the event \( A(\lambda) \) we have

\[
\tilde{\theta} - \theta \leq \lambda \|\hat{\alpha} - \alpha\|_1 + (\alpha - \hat{\alpha})^T \Sigma (\alpha - \hat{\alpha}) + |\alpha^T (\hat{\Sigma} - \Sigma)\alpha| + 2|\alpha^T (\hat{\mu} - \mu)|
\]

From the proof of Lemma 2, we can directly obtain that with probability at least \( 1 - 4p^{-c} \),

\[
\sqrt{u^T \Sigma u} \leq \sqrt{u^T \hat{\Sigma} u} + |u^T (\hat{\mu} - \mu)|^2 + C_3 \left( \frac{\log p}{n} \right) \|u\|_1 \leq \sqrt{u^T \hat{\Sigma} u} + C_4 \left( \frac{\log p}{n} \right) \|u\|_1, \quad \forall u \in \mathbb{R}^p.
\]

This result combined with (5.44) enables us to bound \((\alpha - \hat{\alpha})^T \Sigma (\alpha - \hat{\alpha})\),

\[
(\alpha - \hat{\alpha})^T \Sigma (\alpha - \hat{\alpha}) \lesssim \lambda (\sqrt{\bar{s}} \|\hat{\alpha} - \alpha\|_2 + \|\alpha_{S^c}\|_1) + \frac{\log p}{n} \|\alpha - \hat{\alpha}\|_1^2.
\]

We thus have that with probability at least \( 1 - 4p^{-c} - \mathbb{P}(A^c(\lambda)) \),

\[
\tilde{\theta} - \theta \lesssim \frac{\log p}{n} \|\hat{\alpha} - \alpha\|_1^2 + \lambda (\sqrt{\bar{s}} \|\hat{\alpha} - \alpha\|_2 + \|\alpha_{S^c}\|_1) + \lambda \|\hat{\alpha} - \alpha\|_1 + |\alpha^T (\hat{\Sigma} - \Sigma)\alpha| + 2|\alpha^T (\hat{\mu} - \mu)|.
\]

Based on the results (5.43), (5.49) and Lemma 6, it is direct to verify that

\[
\mathcal{J}_1 = O\left((1+\tilde{\tau})^{1-q} R^2 \left( \frac{\log p}{n} \right)^{2-q}\right), \quad \mathcal{J}_2 = O\left((1+\tilde{\tau})^{1-\frac{q}{2}} R\left( \frac{\log p}{n} \right)^{1-\frac{q}{2}}\right), \quad \mathcal{J}_3 = O\left((1+\tilde{\tau})^{1-\frac{q}{2}} R\left( \frac{\log p}{n} \right)^{1-\frac{q}{2}}\right),
\]

and under the scaling \( R(1+\tilde{\tau})^{-\frac{q}{2}} \left( \frac{\log p}{n} \right)^{1-\frac{q}{2}} = o(1) \), \( \mathcal{J}_3 \) is the dominant term. Hence, it holds with
probability at least \( 1 - 4p^{-c} - \mathbb{P}(\mathcal{A}^c(\lambda)) \) that

\[
\hat{\theta} - \theta \lesssim (1 + \hat{\tau})^{1 - \frac{q}{2}} \tilde{R}(\frac{\log p}{n})^{1 - \frac{q}{2}} + |\alpha^T(\hat{\Sigma} - \Sigma)\alpha| + |\alpha^T(\hat{\mu} - \mu)|.
\]

Regarding the lower bound, (5.33) remains valid. Given the order of \( J_3 \) we have derived, it holds that with probability at least \( 1 - 4p^{-c} - \mathbb{P}(\mathcal{A}^c(\lambda)) \),

\[
\hat{\theta} - \theta \gtrsim -(1 + \hat{\tau})^{1 - \frac{q}{2}} \tilde{R}(\frac{\log p}{n})^{1 - \frac{q}{2}} - |\alpha^T(\hat{\Sigma} - \Sigma)\alpha| - |\alpha^T(\hat{\mu} - \mu)|.
\]

Equipped with Lemma 5 and under the scaling condition \( p^{-\delta} \lesssim \tilde{R}(\frac{\log p}{n})^{1 - \frac{q}{2}} (\hat{\tau}^{1 - \frac{q}{2}} \vee \hat{\tau}^{-1}) \), the remainder of the proof follows the same line of arguments in the proof of Theorem 1. We thus do not repeat the details.

5.7 Proof of Theorem 5

The proof generalizes the one of Theorem 2. Hence we do not detail out every step, and will refer to the proof of Theorem 2 on many occasions. We use \( C_1, C_2, \ldots \) to denote positive constants possibly depending on \( c_L, c_U \).

5.7.1 Lower bound for \( \mathbb{E}|\hat{\theta} - \theta| \)

The derivation for the lower bound \( \hat{\tau} \wedge \frac{\hat{\tau} + \sqrt{\hat{\tau}}}{\sqrt{n}} \) is almost the same as for the lower bound \( \tau \wedge \frac{\tau + \sqrt{\tau}}{\sqrt{n}} \) in Theorem 2. The only modification is to replace \( \tau \) by \( \hat{\tau} \). Then all the arguments there continue to hold (up to constants). The next step is to obtain the other lower bound \( \hat{\tau} \wedge \left[(1 + \hat{\tau})^{1 - \frac{q}{2}} \tilde{R}(\frac{\log p}{n})^{1 - \frac{q}{2}}\right] \). We follow closely the arguments that were used to derive the lower bound \( [\tau \wedge \frac{(\tau + 1)s\log p}{n}] c_0 \exp(-e^{2s^2 p^c c_0^{-1}}) \) in Theorem 2. For simplicity, we merely point out the differences in the following.

(i) \( \hat{\tau} \geq 1 \). Replace \( s \) by \( \tilde{R}(\frac{\log p}{n})^{-\frac{q}{2}} \hat{\tau}^{-\frac{q}{2}} \), and \( \tau \) by \( \hat{\tau} \). Under the scaling \( \tilde{R}(\frac{\log p}{n})^{-\frac{q}{2}} \hat{\tau}^{-\frac{q}{2}} = o(1) \), all the arguments remain valid (up to constants), except that the verification of \( \|\Sigma^0 \|^{-\frac{q}{2}} \leq \frac{61}{61} \)
\( R, \| (\Sigma^S)^{-1} \mu^S \|_q \leq R \) is required. This can be easily verified as

\[
\| (\Sigma^0)^{-1} \mu^0 \|_q \lesssim \tilde{\tau}^{\frac{q}{2}} \leq R, \quad \| (\Sigma^S)^{-1} \mu^S \|_q \lesssim \tilde{\tau}^{\frac{q}{2}} + C_2 \tilde{\tau}^{\frac{q}{2}} s \lesssim \tilde{\tau}^{\frac{q}{2}} + \tilde{R} \lesssim R.
\]

Thus we obtain the lower bound,

\[
\forall \rho_0 \leq (\tilde{R} (\log \frac{p}{n})^{1-\frac{q}{2}} \tilde{\tau}^{-\frac{q}{2}})^{-\frac{1}{2}}
\]

\[
\inf_{\hat{\theta}} \sup_{(\mu, \Sigma) \in \mathcal{H}_q(R, \tau)} \mathbb{E} |\hat{\theta} - \theta| \geq C_1 \rho_0^{\frac{2}{1-\frac{q}{2}}} \tilde{R} \left( \frac{\log p}{n} \right)^{1-\frac{q}{2}} \exp \left( -c_2 \tilde{R}^2 \left( \frac{\log p}{n} \right)^{-q} - C_2 \rho_0^{\frac{2}{1-\frac{q}{2}}} - 1 \right).
\]

Given the condition \( \tilde{R}^2 \tilde{\tau}^{-q} (\log \frac{p}{n})^{-q} \lesssim p^{1-\delta} \) for some \( \delta > 0 \), we can choose \( \rho_0 \approx 1 \) so that

\[
2\tilde{R} \left( \frac{\log p}{n} \right)^{-q} \tilde{\tau}^{-q} p^{C_2 \rho_0^{\frac{2}{1-\frac{q}{2}}}} \leq 1,
\]

leading to the lower bound

\[
\inf_{\hat{\theta}} \sup_{(\mu, \Sigma) \in \mathcal{H}_q(R, \tau)} \mathbb{E} |\hat{\theta} - \theta| \gtrsim \tilde{\tau}^{1-\frac{q}{2}} \tilde{R} \left( \frac{\log p}{n} \right)^{1-\frac{q}{2}}.
\]

(ii) \( \tilde{R} (\log \frac{p}{n})^{1-\frac{q}{2}} \leq \tilde{\tau} \leq 1 \). Replace \( s \) by \( \tilde{R} (\log \frac{p}{n})^{-\frac{q}{2}} \), and \( \tau \) by \( \tilde{\tau} \). With the scaling \( \tilde{R} (\log \frac{p}{n})^{1-\frac{q}{2}} = o(1) \), all continue to hold, up to constants. The result \( \| (\Sigma^0)^{-1} \mu^0 \|_q \leq R, \| (\Sigma^S)^{-1} \mu^S \|_q \leq R \)

is also straightforward to confirm. Using the condition \( \tilde{R}^2 (\log \frac{p}{n})^{-q} \lesssim p^{1-\delta} \) for some \( \delta > 0 \), we can choose \( \gamma_0 \approx 1 \) to obtain the bound,

\[
\inf_{\hat{\theta}} \sup_{(\mu, \Sigma) \in \mathcal{H}_q(R, \tau)} \mathbb{E} |\hat{\theta} - \theta| \gtrsim \tilde{R} \left( \frac{\log p}{n} \right)^{1-\frac{q}{2}}.
\]

(iii) \( \tilde{\tau} \leq \tilde{R} (\log \frac{p}{n})^{1-\frac{q}{2}} \). The same modification as in Case (ii). We will have

\[
\inf_{\hat{\theta}} \sup_{(\mu, \Sigma) \in \mathcal{H}_q(R, \tau)} \mathbb{E} |\hat{\theta} - \theta| \gtrsim \tilde{\tau}.
\]
5.7.2 Lower bound for $E\|\hat{\alpha} - \alpha\|^2$

Again, we adapt the derivation of the lower bound for $E\|\hat{\alpha} - \alpha\|^2$ in Theorem 2 (cf. Section 5.3.2). We summarize the modifications below.

(i) $\hat{\tau} \geq 1$. Replace $s$ by $\hat{R}\tau^{-\frac{q}{2}}(\log \frac{p}{n})^{-\frac{q}{2}}$, $\tau$ by $\hat{\tau}$, and set $\mu^j = \mu = (\sqrt{c_L\tau}, 0, 0, \ldots, 0)^T$. Under the condition $1 \lesssim \hat{R}(1 + \hat{\tau})^{-\frac{q}{2}}(\log \frac{p}{n})^{-\frac{q}{2}} \lesssim p^{1-\delta}$ for some $\delta$, the support size $\hat{R}\tau^{-\frac{q}{2}}(\log \frac{p}{n})^{-\frac{q}{2}}$ is a legitimate number between 1 and $p$. Moreover, it is straightforward to confirm that under the scaling $\hat{R}(1 + \hat{\tau})^{-\frac{q}{2}}(\log \frac{p}{n})^{-\frac{q}{2}} \lesssim p^{1-\delta}$, $\hat{R}(1 + \hat{\tau})^{-\frac{q}{2}}(\log \frac{p}{n})^{-\frac{q}{2}} = o(1)$, all the arguments in Case (i) of Section 5.3.2 remain valid (up to constants). Additionally, we need verify $\|\Sigma_j^{-1}\mu_j\|_q \leq R$. This is true because

$$\|\Sigma_j^{-1}\mu_j\|_q \lesssim (\sqrt{\tau})^q + (c_0 - c_L)^q \tau^q \left(\hat{R}(\log \frac{p}{n})^{-\frac{q}{2}}\right)^{1-\frac{q}{2}} \lesssim R,$$

where $c_0 - c_L \asymp \sqrt{\hat{R}(\log \frac{p}{n})^{1-\frac{q}{2}}}$ according to Section 5.3.2. Hence we obtain

$$\inf_{\hat{\alpha}} \sup_{(\mu, \Sigma) \in \mathcal{H}_q(R, \tau)} E\|\hat{\alpha} - \alpha\|^2 \gtrsim \hat{\tau}^{1-\frac{q}{2}} \left(\frac{\log p}{n}\right)^{1-\frac{q}{2}} \hat{R}.$$

(ii) $\hat{R}(\log \frac{p}{n})^{1-\frac{q}{2}} \leq \hat{\tau} \leq 1$. Replace $s$ by $\hat{R}(\log \frac{p}{n})^{-\frac{q}{2}}$, and $\tau$ by $\hat{\tau}$. Then under the scaling $\hat{R}(\log \frac{p}{n})^{-\frac{q}{2}} \lesssim p^{1-\delta}$, $\hat{R}(\log \frac{p}{n})^{1-\frac{q}{2}} = o(1)$, the arguments in Section 5.3.2 continue to hold, and $\|\Sigma_j^{-1}\mu_j\|_q \leq R$ can be verified easily (up to constants). Therefore, we can conclude

$$\inf_{\hat{\alpha}} \sup_{(\mu, \Sigma) \in \mathcal{H}_q(R, \tau)} E\|\hat{\alpha} - \alpha\|^2 \gtrsim \hat{R}(\log \frac{p}{n})^{1-\frac{q}{2}}.$$

(iii) $\hat{\tau} \leq \hat{R}(\log \frac{p}{n})^{1-\frac{q}{2}}$. Do the same change as in Case (ii) and we are able to obtain

$$\inf_{\hat{\alpha}} \sup_{(\mu, \Sigma) \in \mathcal{H}_q(R, \tau)} E\|\hat{\alpha} - \alpha\|^2 \gtrsim \hat{\tau}.$$
5.8 Reference Material

**Theorem A.** (General Hoeffding’s inequality). Let \( x_1, \ldots, x_n \in \mathbb{R} \) be independent, zero-mean, sub-gaussian random variables. Then for every \( t \geq 0 \), we have

\[
P\left( \left| \sum_{i=1}^{n} x_i \right| \geq t \right) \leq 2 \exp \left( -\frac{ct^2}{\sum_{i=1}^{n} \|x_i\|_2^2} \right),
\]

where \( c > 0 \) is an absolute constant, and \( \| \cdot \|_\psi \) is the sub-gaussian norm defined as \( \|x\|_\psi = \inf\{t > 0 : E e^{x^2/t^2} \leq 2\} \).

(Bernstein’s inequality). Let \( x_1, \ldots, x_n \in \mathbb{R} \) be independent, zero-mean, sub-exponential random variables. Then for every \( t \geq 0 \), we have

\[
P\left( \left| \sum_{i=1}^{n} x_i \right| \geq t \right) \leq 2 \exp \left[ -c \min \left( \frac{t^2}{\sum_{i=1}^{n} \|x_i\|_\psi^2}, \frac{t}{\max_{1 \leq i \leq p} \|x_i\|_\psi} \right) \right],
\]

where \( c > 0 \) is an absolute constant, and \( \| \cdot \|_\psi \) is the sub-exponential norm defined as \( \|x\|_\psi = \inf\{t > 0 : E |x|/t \leq 2\} \).

The above two results are Theorem 2.6.2 and Theorem 2.8.1, respectively in Vershynin (2018). Please refer there to see the details.

**Theorem B.** Let \( x_1, \ldots, x_p \in \mathbb{R} \) be sub-gaussian random variables, which are not necessarily independent. Then there exists an absolute constant \( c > 0 \) such that for all \( p > 1 \),

\[
E \max_{1 \leq i \leq p} |x_i| \leq c \sqrt{\log p} \max_{1 \leq i \leq p} \|x_i\|_\psi_2.
\]

The above result can be found in Lemma 2.4 of Boucheron, Lugosi, and Massart (2013).

**Theorem C.** (Matrix deviation inequality). Let \( A \) be an \( m \times n \) matrix whose rows \( \{A_i\}_{i=1}^{n} \) are independent, isotropic and sub-gaussian random vectors in \( \mathbb{R}^n \). Then for any subset \( T \subseteq \mathbb{R}^n \), we
have for any $u \geq 0$, the event
\[
\sup_{x \in T} \left| \|Ax\|_2 - \sqrt{m}\|x\|_2 \right| \leq C K^2 (w(T) + u \cdot \text{rad}(T))
\]
holds with probability at least $1 - 2e^{-u^2}$. Here $C > 0$ is an absolute constant, $K = \max_i \|A_i\|_{\psi^2}$, and $w(T), \text{rad}(T)$ are defined as:
\[
w(T) = E \sup_{x \in T} g^T x, \ g \sim N(0, I_p); \ \text{rad}(T) = \sup_{x \in T} \|x\|_2.
\]

Theorem [C] appears as Exercise 9.1.8 in [Vershynin (2018)].

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