An Interpretation of the Laminar-Turbulent Transition Startup against the Consideration of the Transverse Viscosity Factor

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Abstract. This paper proposes a rheological model of a fluid having the Newtonian model applicability limit and a potential for further “addition” of the transverse viscosity factor. The dynamic equations for a fluid that has such rheological model are discussed, the analysis of which demonstrates the possibility of “generating” the cross stream velocity components. The transition to the dimensionless notation introduces four dimensionless complexes of local characterization for the transition conditions in the neighborhood of the flow region point in question. Based on such dimensionless complexes and using the known experimental data, the empiric conditions of “generating” the cross stream velocity components and starting the laminar-turbulent transition are proposed.

1. Introduction
The key approaches to modeling the laminar-turbulent transition suggest superimposition of the pressure and velocity disturbance pulse background on an initially laminar flow that obeys the Navier-Stokes equations to further define the conditions under which such disturbance amplitudes or kinetic energies will rise indefinitely over time [1-5].

However, along with the conventional flow disturbance factors which result in an instability of such flow and, ultimately, in turbulence, other circumstances that impel the startup of transition and could be related to some specific “intrinsic” properties of fluids may as well be in place there. In the early 1970-s, Academician L I Sedov supposed the following in his discussion of the new physical models: “It is clear from practice that such rheological properties of fluids may manifest themselves in turbulent flows that are hidden and insignificant from the laminar liquid motion standpoint” [6]. Therefore, such hypothesis allows for possible consideration of the rheological factor impact on the startup of the laminar-turbulent transition as well. Quite naturally, such an assumption will demand respective adjustments of the rheological model of fluids in order to have not only their conventional properties taken into account, but also such properties that “initiate” the transition on certain conditions.

This paper is an attempt to study this sort of such rheological model and propose, on its basis, an interpretation of the startup of the laminar-turbulent transition.

Note that the preceding local characterizations of the laminar flow form transition into the turbulent form often involved various “local” dimensionless complexes similar to the Reynolds number which included, among other things, the “differential” properties of the velocity field. A fairly simple example of such complex in relation to the boundary layer flow was discussed in [7]. A little later, a
more complex parameter was introduced for the cylindrical channel flow in [8]. The result was developed in [9], which suggested another example of a dimensionless complex

\[
K = \frac{1}{2} \cdot \rho \cdot \left| \frac{\text{grad} (\vec{v} \cdot \vec{v})}{\vec{F} - \text{grad} (p)} \right|
\]

distributed across the flow region that characterizes the start of the transition locally. Here: \( \rho \) is density; \( \vec{v} \) is fluid velocity vector; \( \vec{F} \) is the body force density factor; and \( p \) is pressure.

Various other examples of such dimensionless complexes were suggested later at different times [10-14]. Despite that those complexes were different in form, their common trait was that they shaped up a basis for the following hypothesis. An excess of the maximum value of such complex, at a certain spatial point within the flow region area, of the respective threshold value triggers the transition of the laminar flow form to the turbulent flow form. Then the neighborhood of that point acts as a kind of “zone of initial instability” [7].

2. Rheological model.
Assume that there is some boundary to the deployment of the Newtonian incompressible fluid model beyond which the threshold transverse viscosity factor “joins in”.

It is well known, modeling of channel flows for fluids whose rheology involves the consideration of transverse viscosity, leads to solutions which show the existence of secondary flows [15]. These flows are characterized by cross stream velocity components in relation to the original flow lines (obtained by flow modeling without the consideration of transverse viscosity). In this respect, note that a similar phenomenon (in terms of “generation” of cross stream velocity components in relation to the original laminar flow lines) exists in the laminar-turbulent transition, too. Indeed, the “surge” of not previously existing cross stream components can be always seen on the initial stage of transition.

Given the relation between the transverse viscosity and secondary flows characterized by the cross stream velocity components, let us introduce the following rheological model with the key relationships represented in the Cartesian system

\[
\begin{align*}
\tau_{ij} &= -P \cdot \delta_{ij} + 2 \cdot \mu \cdot \varepsilon_{ij} + 4 \cdot \eta_c(I_2) \cdot \xi_{ij}, \\
I_2 &= \varepsilon_{11} \cdot \varepsilon_{22} + \varepsilon_{22} \cdot \varepsilon_{33} + \varepsilon_{33} \cdot \varepsilon_{11} - \varepsilon_{12}^2 - \varepsilon_{23}^2 - \varepsilon_{31}^2, \\
\varepsilon_{ij} &= \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \\
\xi_{ij} &= \sum_{k=1}^{3} \varepsilon_{ik} \cdot \varepsilon_{kj}, \quad i, j = 1, 2, 3, \\
\eta_c(I_2) &= \begin{cases} 
0, & |I_2| < I_{2\eta}, \\
\eta_q \cdot \left( |I_2| - I_{2\eta} \right)^q, & |I_2| \geq I_{2\eta},
\end{cases}
\end{align*}
\]

where \( \tau_{ij}, \varepsilon_{ij} \) are components of stress tensors and strain rates; \( P \) is pressure; \( \delta_{ij} \) is the Kronecker symbol; \( \mu = \text{const} \) is dynamic viscosity; \( u_i \) are velocity vector components; \( \eta_c(I_2) \) is transverse viscosity as a function of the second strain rate tensor invariant \( I_2 \); \( I_{2\eta} \) is some critical value of \( |I_2| \) above which the transverse viscosity factor “joins in” and, consequently, cross stream velocity components begin “generating” \( \eta_q \); \( q \) are the empiric constants of the model.

A similar rheological model, yet with a constant transverse viscosity value (\( q = 0 \)), was studied in [16]. A model with a linear law (\( q = 1 \)) of the dependence of the transverse viscosity on the second invariant was proposed in [17].

The analysis of (1) reveals the following. If, at some spatial point of the flow region, the condition \( |I_2| < I_{2\eta} \) is satisfied, the behavior of fluids satisfies the classical Newtonian model in the minor
neighborhood of such point. Otherwise, if \( \|I_{2}\| > I_{2}\eta \), the transverse viscosity factor “joins in” to create conditions for “generating” the cross stream velocity components. Such condition is suggested as a necessary, yet insufficient, condition for triggering the laminar-turbulent transition.

3. Dynamic equations

Let us consider the following problem. Assume that the respective velocity and pressure regions, which exactly satisfy the Navier-Stokes equations and continuity condition, have been formed in a set liquid flow region having the rheological model (1) by a certain moment of time conventionally set as the initial moment of time \( t = 0 \). With such initial conditions, let us, however, admit that the following condition is satisfied at a certain point and its minor neighborhood at the initial moment of time

\[
\|I_{2}\| > I_{2}\eta .
\]

(2)

Condition (2) means that the transverse viscosity factor manifests itself here right after the initial moment of time.

Let us introduce the Cartesian system and place its origin at that point in question. One of the axes, e.g. \( Ox_{i} \), runs tangentially to the current line at this flow region point. Now we have the equations that describe the evolution of velocity component and pressure distribution in the neighborhood of that point over time

\[
u_{i} = u_{i}(x_{1}, x_{2}, x_{3}, t), \quad i = 1, 2, 3, \quad P = P(x_{1}, x_{2}, x_{3}, t).
\]

(3)

For characterizing the velocity and pressure fields (3), let us introduce the two following vectors

\[
\bar{E} = \text{grad} \left[ P + \frac{\rho \cdot u^{2}}{2} \right], \quad \bar{D} = \text{grad} \left[ 2 \cdot \mu \cdot \sqrt{\|I_{2}\|} \right], \quad u = \sqrt{u_{1}^{2} + u_{2}^{2} + u_{3}^{2}}.
\]

In this case, vector \( \bar{E} \) characterizes the maximum increase direction and “speed” (along the space axes) of the full mechanical liquid flow region energy at the flow region point in question. Vector \( \bar{D} \), in its turn, is again a characteristic of the maximum increase direction and speed, but it relates to viscous dissipation at the flow region point in question.

Let us introduce, along with scale velocity \( u_{s} \), the following parameters also taken as scale values for measuring length, time, pressure and the second invariant of the strain rate tensor, respectively

\[
L_{s} = \frac{\mu \cdot E_{s}}{\rho \cdot u_{s} \cdot D_{s}}, \quad t_{s} = \frac{L_{s}}{u_{s}}, \quad P_{s} = \frac{\rho^{3} \cdot u_{s}^{3}}{\mu \cdot E_{s}}, \quad I_{s} = \left( \frac{E_{s}}{\rho \cdot u_{s}} \right)^{2},
\]

(4)

where \( E_{s}, D_{s} \) are vector \( \bar{E}, \bar{D} \) modules, respectively.

Here and hereafter, subscript \( s \) shows that the respective functions have been calculated at the spatial point in questions (zero point) for \( t = 0 \).

Given the scale values (4) introduced above, the incompressible fluid equations having rheological model (1) for special case \( q = 1 \) with (2) having the dimensionless form, acquire the form of

\[
\frac{\partial u'_{i}}{\partial t'} = -K_{2} \cdot \frac{\partial P'}{\partial x_{i}} + \sum_{j=1}^{3} \left\{ \frac{2}{K_{1}} \frac{\partial e'_{ij}}{\partial x_{j}} - u'_{j} \cdot \frac{\partial u'_{i}}{\partial x_{j}} - 4 \cdot K_{4} \cdot \frac{\partial}{\partial x_{j}} \left[ \left( I_{2}' + K_{2g} \right) \sum_{k=1}^{3} e'_{ik} \cdot e'_{kj} \right] \right\},
\]

(5)

\[
I_{2}' = K_{5} \left( e'_{11} \cdot e'_{22} + e'_{12} \cdot e'_{23} + e'_{13} \cdot e'_{21} - e'_{11}^{2} - e'_{22}^{2} - e'_{33}^{2} \right), \quad e'_{ij} = \frac{1}{2} \left( \frac{\partial u'_{i}}{\partial x_{j}} + \frac{\partial u'_{j}}{\partial x_{i}} \right), \quad i, j = 1, 2, 3,
\]

3
\[ K_j = \frac{E_s}{D_s}, \quad [I_2'] = \left| \frac{I_2}{I_{2s}} \right| = K_2 > K_{2g}, \quad K_{2g} = \frac{I_2 \eta}{I_{2s}}, \quad K_3 = \frac{\rho^2 \cdot u_s^3}{\mu \cdot E_s}, \quad K_4 = \frac{\eta_1 \cdot D_s^2}{\rho \cdot \mu^2}, \quad K_5 = \frac{K_3}{K_4}, \] (6)

\[ t' = \frac{t}{t_s}, \quad P' = \frac{P}{P_s}, \quad x' = \frac{x_i}{L_s}, \quad u_{ij}' = \frac{u_i}{u_j}, \]

where \( K_j, \ K_{2g}, \ K_3, \ K_4 \) are the individual dimensionless complexes built of invariant values and not connected directly with the specific linear dimensions of the flow region.

Note that, in terms of its meaning and “location” in equations (5), the dimensionless array \( K_j \) is similar to the Reynolds number. Given that, the inequality in (6) results from condition (2) of “generating” the cross stream velocity components via the transition to the dimensionless form.

4. Modeling the development of the initial stage of transition

Let us consider an approach to building the first approximation of the problem about the initial (with \( t = 0 \)) velocity distribution development in the minor neighborhood of a certain spatial point where condition (2) is satisfied.

Assume that the initial velocity and pressure distributions

\[ t = 0, \quad u_i = U_i(0, i)(x_1, x_2, x_3), \quad i = 1, 2, 3, \quad P = P_i(0)(x_1, x_2, x_3), \] (7)

exactly satisfy the Navier-Stokes equations and flow continuity condition.

In (7) and further on in this section, the strokes over the dimensionless values have been omitted.

The solution of equation system (5) is suggested in the form of

\[ u_i(x_1, x_2, x_3, t) = U_i(0, i)(x_1, x_2, x_3) + u_i(t, i)(x_1, x_2, x_3, t), \quad i = 1, 2, 3, \] (8)

\[ P(x_1, x_2, x_3, t) = P_i(0)(x_1, x_2, x_3) + p_i(t, i)(x_1, x_2, x_3, t), \]

where \( u_i(t, i) \); \( p_i(t, i) \) are the non-stationary components of the pressure and velocity fields which “originate” in the minor neighborhood of the point in question on the initial stage of flow development from original condition (7) and satisfy the following conditions

\[ u_i(t, i)(x_1, x_2, x_3, 0) = 0, \quad i = 1, 2, 3, \quad p_i(t, i)(x_1, x_2, x_3, 0) = 0. \] (9)

By linearizing equations (5) with reference to (8) and conventional, for such procedures, assumptions, we arrive at the following equation system [17]

\[
\frac{\partial u_i(t, j)}{\partial t} = -K_3 \cdot \frac{\partial p(t, j)}{\partial x_i} + \sum_{j=1}^{3} \left\{ \frac{2}{K_j} \cdot \frac{\partial \varepsilon_i(t, j)}{\partial x_j} \right\} - u_i(t, j) \cdot \frac{\partial U_i(0, j)}{\partial x_j} - U_i(0, j) \cdot \frac{\partial u_i(t, j)}{\partial x_j}

- 4 \cdot K_4 \cdot \frac{\partial}{\partial x_j} \sum_{k=1}^{3} \left\{ \left( I_2^{(i, j)} + I_2^{(0)} + K_{2g} \right) \cdot \varepsilon_i^{(0)} \cdot \varepsilon_k^{(0)} 

+ \left( I_2^{(0)} + K_{2g} \right) \cdot \left( \varepsilon_i^{(0)} \cdot \varepsilon_k^{(0)} + \varepsilon_i^{(t)} \cdot \varepsilon_k^{(0)} \right) \right\}, \quad i = 1, 2, 3. \] (10)

In developing (10), it was assumed that

\[ \varepsilon_{ij} = \varepsilon_{ij}^{(0)} + \varepsilon_{ij}^{(t)}, \quad I_2 = I_2^{(0)} + I_2^{(t)}, \]
\[ e_{ij}^{(1)} = \frac{1}{2} \left( \frac{\partial u^{(i,j)}}{\partial x_j} + \frac{\partial u^{(i,j)}}{\partial x_i} \right), \quad e_{ij}^{(0)} = \frac{1}{2} \left( \frac{\partial U^{(0,j)}}{\partial x_j} + \frac{\partial U^{(0,j)}}{\partial x_i} \right), \quad i,j = 1,2,3. \]

\[ I_2^{(0)} = K_2^2 \left( \epsilon_{11}^{(0)} - \epsilon_{22}^{(0)} + \epsilon_{12}^{(0)} \right) + \epsilon_{33}^{(0)} - \frac{1}{2} (\epsilon_{12}^{(0)})^2 - \frac{1}{2} (\epsilon_{11}^{(0)})^2, \]

\[ I_2^{(1)} = K_2^2 \left( \epsilon_{11}^{(1)} + \epsilon_{22}^{(1)} + \epsilon_{12}^{(1)} \right) - 2 \epsilon_{12}^{(1)}, - \epsilon_{23}^{(1)} - 2 \epsilon_{31}^{(1)}, - \epsilon_{32}^{(1)}. \]

Let us represent the initial velocity component distributions and non-stationary components for the pressure and velocity components in the minor neighborhood of the point in question (zero point) through space coordinate power expansion

\[ U^{(0,1)}(x_1,x_2,x_3) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} U_{kmn}^{(0,1)}(t) x_1^k x_2^m x_3^n, \]

\[ p^{(1)}(x_1,x_2,x_3,t) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} p_{kmn}^{(1)}(t) x_1^k x_2^m x_3^n, \]

\[ u^{(i)}(x_1,x_2,x_3,t) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} u_{kmn}^{(i)}(t) x_1^k x_2^m x_3^n, \quad i = 1,2,3. \quad (11) \]

Here, \( U_{kmn}^{(0,1)} \) are the known expansion coordinates; \( u_{kmn}^{(i)}(t), \ p_{kmn}^{(1)}(t) \) are the expansion factors representing the yet unknown time functions which, with reference to (9), satisfy the following initial conditions

\[ u_{kmn}^{(i,0)}(0) = 0, \quad p_{kmn}^{(1)}(0) = 0, \quad i = 1,2,3, \quad k,m,n = 0,1,2,.... \quad (12) \]

First of all, let us estimate the tendency for the consequential velocity component evolution over time at the spatial point in question at the initial moment of time. Let us plug (11) into (10). Assuming that \( x_1 = 0 \); \( x_2 = 0 \); \( x_3 = 0 \), in the resulting equations and taking into account initial conditions (12), we deduce that

\[ \frac{du_{kmn}^{(i)}(t)}{dt} \bigg|_{t=0} = W^{(i,1)}, -4K_d \sum_{j=1}^{3} \left( I_2^{(0,0)}(x_1,x_2,x_3) e_{ij}^{(0,0)} + \epsilon_{i,j}^{(0,1)} x_j + \epsilon_{i,j}^{(0,2)} x_2 + \epsilon_{i,j}^{(0,3)} x_3, \right) \]

\[ I_2^{(0,0)}(x_1,x_2,x_3) \approx \epsilon_{ij}^{(0,0)} \epsilon_{ij}^{(0,0)} + \epsilon_{ij}^{(0,1)} x_j + \epsilon_{ij}^{(0,2)} x_2 + \epsilon_{ij}^{(0,3)} x_3, \]

\[ I_2^{(0,1)}(x_1,x_2,x_3) \approx \epsilon_{ij}^{(0,1)} \epsilon_{ij}^{(0,1)} + \epsilon_{ij}^{(0,0)} x_j + \epsilon_{ij}^{(0,2)} x_2 + \epsilon_{ij}^{(0,3)} x_3, \]

\[ I_2^{(0,2)}(x_1,x_2,x_3) \approx \epsilon_{ij}^{(0,2)} \epsilon_{ij}^{(0,2)} + \epsilon_{ij}^{(0,0)} x_j + \epsilon_{ij}^{(0,1)} x_2 + \epsilon_{ij}^{(0,3)} x_3. \]

All coefficients in the latter relationships are expressed in the known manner for \( t = 0 \) via the set initial velocity distribution parameters \( U_{kmn}^{(0,1)} \).

Formula (13) produces the following result. If condition (2) is satisfied at the point in question (zero point) at the initial moment of time, time derivatives \( W^{(i,1)} \) for the cross stream velocity
components, that were not here before, may not be equal to zero \( W(t,i) \neq 0 \). This means that the cross stream velocity components can “originate” at this point right after the initial moment of time. Such “generation” of the cross stream velocity components will result in changing the velocity field in the minor neighborhood of the spatial point in question that was formed around the initial moment of time. A similar result was achieved in [16] for the plane problem statement with an example of a rheological model similar to (1), having a constant transverse viscosity value \( q = 0 \).

Let us further restrict ourselves just to a concept of the simplest approximation to a solution of the problem stated above. Let us plug (11) into (10) and, after the respective transformations, equate the multiplier coefficients in the form of \( X_1^k \cdot X_2^m \cdot X_3^n \) to the same exponent combination \( k,m,n \) for the respective coordinates in the left and right sides of the equation. In such case, each of the three equations (10) may become a basis for an infinite ordinary differential equation system in relation to expansion coefficients \( u_{k,m,n}^{(t,i)}(t), \ f_{k,m,n}^{(t,i)}(t) \). As the first approximation, let us restrict ourselves only to the first equation in such infinite system built of the coefficients before complex \( X_1^k \cdot X_2^m \cdot X_3^n \) with zero exponents \( k = 0; m = 0; n = 0 \). As a result, we arrive at the following system of three ordinary differential first-order equations which may be structurally represented as

\[
\begin{align*}
\frac{du_{000}^{(t,i)}}{dt} &= -K_3 \cdot p_{000}^{(t,i)} + W(t,i) + \sum_{k=0}^{2} \sum_{m=0}^{2} \sum_{n=0}^{2} \left( \sum_{j=1}^{3} \alpha_{k,m,n}^{(j,i)} \cdot u_{k,m,n}^{(j,i)} \right), \\
\frac{du_{001}^{(t,i)}}{dt} &= -K_3 \cdot p_{001}^{(t,i)} + W(t,2) + \sum_{k=0}^{2} \sum_{m=0}^{2} \sum_{n=0}^{2} \left( \sum_{j=1}^{3} \alpha_{k,m,n}^{(j,i)} \cdot u_{k,m,n}^{(j,i)} \right), \\
\frac{du_{002}^{(t,i)}}{dt} &= -K_3 \cdot p_{002}^{(t,i)} + W(t,3) + \sum_{k=0}^{2} \sum_{m=0}^{2} \sum_{n=0}^{2} \left( \sum_{j=1}^{3} \alpha_{k,m,n}^{(j,i)} \cdot u_{k,m,n}^{(j,i)} \right).
\end{align*}
\]

(14)

The set of coefficients \( \alpha_{k,m,n}^{(j,i)}; i,j = 1,2,3; k,m,n = 0,1,2; k+m+n \leq 2 \) is assumed as assigned in this case and may be represented in the known manner through velocity component initial distribution (11) coefficients \( U_k^{(0,i)} \) or dimensionless complexes \( K_1, K_2 \& K_4 \).

Equation system (14) is not locked as, for the approximation in question, it should contain 34 unknown time functions: \( p_{000}^{(t,i)}, p_{100}^{(t,i)}, p_{001}^{(t,i)}, p_{010}^{(t,i)}, p_{002}^{(t,i)}, u_{000}^{(t,i)}, u_{001}^{(t,i)}, u_{010}^{(t,i)}, u_{002}^{(t,i)}, u_{011}^{(t,i)}, u_{011}^{(t,i)}, u_{110}^{(t,i)}, u_{111}^{(t,i)} \), \( j = 1,2,3 \).

First of all, in order to lock the three equation system (14), the continuity condition should be used which provides for four additional equations. The other 27 equations should result from meeting the respective “boundary” conditions which must be placed at eight spatial nodal points \( M_{\beta}(x_1, x_2, x_3) \), \( \beta = 1,2,...,8 \); three velocity components, and in three nodal points \( M_{\gamma}(x_1, x_2, x_3) \), \( \gamma = 1,2,3 \), for pressure. Assuming that such nodal points \( M_{\beta} \) may act as vertexes of a “hexagon”, let us proceed to building the respective finite element. Note that such elements with six, generally speaking, “curvy” faces and eight vertexes should “cover” the entire flow region in the general case.
As a result, the system of three differential equations (14) and 31 linear algebraic equation for each such finite elements may, after all respective transformations, be simplified to a simply translated equation, such as
\[
\frac{d^2 u^{(1,2)}_{000}}{dt^2} + a_1 \frac{du^{(1,2)}_{000}}{dt} + a_0 u^{(1,2)}_{000} = H(t),
\]
(15)

Coefficients \(a_0\), \(a_1\) of equation (15) and its constituent function \(H(t)\) are determined by the totality of initial process parameters with reference to the relationships mentioned above (including the “boundary” conditions on the finite element vertexes).

Any further evolution scenarios for the cross stream velocity components are determined by a specific type of solving equation (15) with reference to initial conditions (12), (13), and may be drastically different in meaning. Given certain additional conditions (overlaid on the value and signs of coefficients \(a_0\), \(a_1\) and, consequently, dimensionless complexes \(K_1\), \(K_2\), \(K_3\), \(K_4\), the “generated” velocity components may decay. There may also be a situation where the originating cross stream velocity components, which generate the secondary flow, will “convert” the existing laminar flow pattern into another still laminar pattern. Finally, in the event of infinite growth, such components may become a basis for instigating the initial stage of the laminar transition to turbulent flow. In the latter case, it may concern a continuous stage cascade where the cross stream velocity components generate that are “new” in relation to the recently (at the preceding moments of time) existing “old” cross stream velocity components.

An analysis of the above problem solution scheme regarding the development of the initial flow stage in the minor neighborhood of a certain spatial point where condition (2) is satisfied shows that obvious analytical conditions for the startup of the laminar-turbulent transition are quite difficult to build even for the simplest approximation cited above due to cumbersomeness of the aforementioned relationships. In this respect, it is suggested that the anticipated consequences of the approach discussed be represented in the form of empiric conditions based on the known experimental data.

5. Empiric “generation” conditions
The class of studies associated with the “generation” of cross stream velocity components should also include quite spectacular experiments related to the study of secondary flows (in the form of interspersing, toroidal, vortex structures) in a gap between the two coaxial cylinders [18-26]. Such experimental data may be admitted as a basis for an empiric estimate of the startup conditions for the “generation” of cross stream velocity components.

Let us restrict ourselves to a well-known and fairly simple special case of such flows where the internal cylinder with the radius \(R_1\) rotates with a constant angular speed \(\omega_1\), and the external cylinder whose radius is \(R_2\) remains motionless.

In a gap between the cylinders and against a fairly low angular speed values for the internal cylinder rotation, the distribution of velocity and second invariant of the strain rate tensor is determined by the following relationships
\[
\dot{u}(r) = \frac{\omega_1 \cdot R_1^2}{r} \left( \frac{R_2^2 - r^2}{R_2^2 - R_1^2} \right), \quad I_2(r) = -\varepsilon_{\varphi \varphi}^2 = -\frac{\omega_1^2 \cdot R_1^4 \cdot R_2^4}{(R_2^2 - R_1^2)^2 \cdot r^4},
\]
(16)
where \(r\) is the radial coordinate measured from the cylinders’ axis.

Let us interpret the startup of the toroidal structures formation from the standpoint of the initial “generation” of cross stream velocity components due to “joining in” the transverse viscosity factor in the respective flow region areas. Let us also assume that condition \(|I_2| = I_2 \eta\) satisfies the startup of this process satisfies, with reference to the fulfillment of (2).
Taking into consideration (16), we arrive at the expressions that determine the distribution of primary dimensionless complexes (6) in the form of

\[
K_1'(r') = \frac{Re \cdot (R_2^2 - r'^2) \cdot r'^2}{2 \cdot R_2^2 \cdot (R_2^2 - 1) \cdot (R_2^2 - 1)}, \quad K_2'(r') = \frac{1}{4} \left( \frac{R_2^2}{r'^2} \right)^4, \\
K_3'(r') = \frac{Re \cdot (R_2^2 - r'^2)^2}{2 \cdot r'^2 \cdot (R_2^2 - 1) \cdot (R_2^2 - 1)}, \quad Re = \frac{\rho \cdot \omega_1 \cdot R_j \cdot (R_2 - R_1)}{\mu}, \quad r' = \frac{r}{R_j}, \quad R_2' = \frac{R_2}{R_j}. \quad (17)
\]

It follows from (16) that the second invariant module is a monotonic decreasing radial coordinate function. That is why its maximum value can be achieved on the internal cylinder surface. This means that, as the angular speed increases and against its critical value of \( \omega_1 = \omega_{1\eta} \), threshold level \([I_2'] = I_{2\eta}\) may be primarily achieved on the surface of the second rotating cylinder. In this respect, we will further associate the critical value of \( \omega_{1\eta} \) with such angular speed that instigates the origination of toroidal structures determined by the “generation” of the cross stream velocity components. Then primary dimensionless arrays (17) in such situation will be defined as follows

\[
K_{1g} = K_1(1) = \frac{Re_g}{2 \cdot R_2^2 \cdot (R_2^2 - 1)}, \quad K_{2g} = K_2(1) = \frac{R_2^4}{4}, \quad K_{3g} = K_3(1) = \frac{Re_g \cdot (R_2^2 + 1)}{2}, \quad (18)
\]

\[
Re_g = \frac{\rho \cdot \omega_{1\eta} \cdot R_j \cdot (R_2 - R_1)}{\mu}.
\]

These critical values (18) of dimensionless complexes calculated on the basis of the known experimental data taken from various authors are shown in the first 17 lines of Table 1. The results from paper [26] have been assumed by reference in [27], and the results of [19] – by reference in monograph [1].

These experimental data may be amplified with the results related to the Couette flow in a plane channel with the following velocity distribution \( u(y) = V_w \cdot y / h \). Here \( V_w \) is the velocity of the channel’s mobile wall; \( h \) is the channel width; \( y \) is the coordinate measured from the motionless channel wall. Quite often this flow pattern is treated as a specific flow case in a gap between the coaxial cylinders in the extreme case \( R_2' \rightarrow 1 \). At this point, we can add that paper [28] sets forth experimental evidence of vortex structures associated with the originating cross stream velocity components for the plane Couette flows, and offers a current line pattern for such vortexes.

For such flow pattern, complexes (6) may be represented in the form of

\[
K_1 \rightarrow \infty, \quad K_2 = \frac{1}{4}, \quad K_3 = Re \left( \frac{y}{h} \right)^2, \quad Re = \frac{\rho \cdot h \cdot V_w}{\mu}. \quad (19)
\]

Using known experimental data [29-33] related to the laminar-turbulent transition startup and assuming a priori that this process is concomitant with the “generation” of the cross stream velocity components in the vicinity of the movable channel wall (for \( y = h \)), we arrive at the critical values of dimensionless arrays (19) shown in the last four lines of Table 1.
Table 1. Relationships between the primary dimensionless complexes on the “generation” stage obtained by processing the known experimental data.

| Item | References | $R_1'$ | $Re$ | $K_1g$ | $K_2g$ | $K_3g$ |
|------|------------|--------|------|--------|--------|--------|
| 1    | [18]       | 1.345  | 94.1 | 75.37  | 0.818  | 110    |
| 2    | [22]       | 1.250  | 94.4 | 120.8  | 0.610  | 106    |
| 3    | [19]       | 1.190  | 94.6 | 175.3  | 0.502  | 104    |
| 4    | [21]       | 1.177  | 106  | 216.1  | 0.480  | 116    |
| 5    | [20]       | 1.144  | 114  | 302.4  | 0.428  | 122    |
| 6    | [21]       | 1.143  | 117  | 313.0  | 0.426  | 125    |
| 7    | [23]       | 1.142  | 128  | 345.3  | 0.425  | 137    |
| 8    | [18]       | 1.137  | 122  | 344.8  | 0.417  | 130    |
| 9    | [24]       | 1.133  | 120  | 352.8  | 0.410  | 128    |
| 10   | [21]       | 1.111  | 130  | 475.1  | 0.381  | 138    |
| 11   | [21]       | 1.081  | 150  | 793.2  | 0.341  | 156    |
| 12   | [18]       | 1.062  | 170  | 1217   | 0.318  | 175    |
| 13   | [21]       | 1.053  | 183  | 1571   | 0.307  | 188    |
| 14   | [22]       | 1.043  | 217  | 2335   | 0.295  | 221    |
| 15   | [21]       | 1.039  | 212  | 2521   | 0.291  | 216    |
| 16   | [25]       | 1.018  | 316  | 8711   | 0.268  | 319    |
| 17   | [26]       | 1.010  | 350  | 16890  | 0.260  | 352    |
| 18   | [29]       | 1.000  | 280  | $\infty$ | 0.250  | 280    |
| 19   | [33]       | 1.000  | 325  | $\infty$ | 0.250  | 325    |
| 20   | [31]       | 1.000  | 360  | $\infty$ | 0.250  | 360    |
| 21   | [30, 32]   | 1.000  | 370  | $\infty$ | 0.250  | 370    |

The data from Table 1 demonstrate that complex $K_{2g}$ introduced earlier is not a constant value. With this in mind, Figure 1 shows the experimental points of dependence of $K_{2g}$ on $K_{3g}$.

Analyzing the behavior of the data produced, it may be assumed that the approximating function is a hyperbolic curve having the form of

$$K_{2g} = F_g(K_{3g}) = k_0 + \frac{k_1}{K_{3g} - k_2}, \quad k_0 = 0.221, \ k_1 = 8.572, \ k_2 = 84.015.$$ (20)

Processing of the experimental data did not include the results of the first, third and seventh lines of Table 1 since they demonstrate an evident disruption of monotony in varying the values of complex $K_{3g}$ against the variations of complex values $K_{2g}$. Note that the data excluded from processing largely reflect the general dependence (20) trend detected. Thus the correlating points are shown in figure 1.

The above results allow for the following interpretation. If condition (6) is satisfied for any flow point for complex $K_2$ representable, with regard to (20), in the form of

$$K_2 > k_0 + \frac{k_1}{K_3 - k_2},$$ (21)

that means that the “generation” of cross stream velocity components starts in the neighborhood of such point.
Figure 1. The nature of dependence of $K_{2 \phi}$ on $K_{3 \phi}$ built on the basis of known experimental data [18-26, 29-33]; 1 is the approximating function (20); graph. 2, 3 is the horizontal asymptote $K_{2 \phi} = k_0$ and vertical asymptote $K_{3 \phi} = k_2$, respectively.

Quite naturally, general case “generation” condition (21) for the cross stream velocity components may be satisfied not for a separate spatial point, but in a certain flow region area.

6. Empiric condition of the transition startup

The empiric relationship (21) of the cross stream velocity component generation may be seen as a necessary yet insufficient startup condition for the laminar-turbulent transition. The sufficient startup condition may be sought empirically with reference to the known experimental data in the form of restricted dimensionless complex $K_f$ (a Reynolds number analog) in the following form

$$K_{f_{\text{max}}} > K_{f_{c}} = F_c(K_{3c})$$

(22)

because this is exactly the complex that characterizes the relationship between the full energy density increase factor for the spatial point in question and viscous dissipation factor. In condition (22), $K_{f_{c}}$ is the critical threshold for complex $K_f$, and $K_{f_{\text{max}}}$ is the maximum value of dimensionless array $K_f$ in that flow region area where the necessary transition startup (21) is satisfied, i.e. the condition of “generating” the cross stream velocity components. Complex $K_{3c}$ should be defined in the same spatial point as where $K_f = K_{f_{\text{max}}}$ is achieved. In that case, it is assumed that the spatial point at which conditions (21) and (22) are satisfied at the same time starts to act as an “initiator” of the laminar-turbulent transition.

With the exemplary transition startup for the rotational Couette flow, such approach may be interpreted, with reference to (17), as follows.

Due to the monotonic decrease of function $K_2(r')$, the flow region area where necessary transition condition (21) may be satisfied is essentially a ring with an internal radius $r' = 1$. Formally, function $K_f(r')$, with reference to (17), achieves the maximum type extremum at the point $r' = r'_{\text{max}} = R_2' / \sqrt{2}$. However, the condition $r'_{\text{max}} < 1$ is satisfied for values $R_2'$ set forth in Table 1. In such situation, the maximum value of function $K_f(r')$ is achieved on the internal cylinder surface, i.e. $K_{f_{\text{max}}} = K_f(1)$. Using the known experimental data for the critical Reynolds number $Re_c$ in relation
to the transition startup [19, 20, 23, 34-38], the respective critical values of dimensionless complexes $K_{Ic}$ and $K_{3c}$ set forth in Table 2 have been calculated. The experimental data from [38] were assumed by reference in [39].

**Table 2.** Relationships between the primary dimensionless arrays on the transition-to-turbulence stage obtained by processing the known experimental data.

| Item | References | $Re_c$ | $K_{3c}$ | $K_{Ic}$ |
|------|------------|--------|----------|----------|
| 1    | [36]       | 3357   | 3551     | 11620    |
| 2    | [37]       | 3076   | 3296     | 8232     |
| 3    | [20]       | 2877   | 3057     | 9094     |
| 4    | [34]       | 2879   | 3024     | 11760    |
| 5    | [38]       | 2687   | 2821     | 11100    |
| 6    | [23, 35]   | 2608   | 2791     | 7168     |
| 7    | [37]       | 2603   | 2789     | 6967     |
| 8    | [19]       | 1892   | 2072     | 3516     |
| 9    | [40-43]    | 2653   | 1021     | 1021     |
| 10   | [44-46]    | 3344   | 967      | 965      |
| 11   | [47, 48]   | 1743   | 380      | 168      |

Similarly, the calculations have been performed for some other flow patterns. The most well-known are the transition-to-turbulence results for the Poiseuille flow in the cylindrical channel with the radius $R$. It can be proven that, in such case, the maximum value of complex $K_I$ is achieved at a point with the radial coordinate of $r = R/\sqrt{3}$. Given the scatter of the known experimental data with various authors, both classical transition results [40] and recent year results [41-43] have been averaged. The resulting values are shown in line 9, Table 2. The average channel flow velocity has been conventionally assumed as the characteristic velocity for building the Reynolds number used in the calculation.

Similarly, the results of experimental data processing [44-46] for the startup transition with the Poiseuille flow in a plane channel, width $2\cdot h$ have been obtained and produced in line 10, Table 2.

In line 11, Table 2, also averaged, the experimental data [47, 48] processing results are produced for the laminar-turbulent transition with the Poiseuille flow in a gap between the two coaxial cylinders. Here, the double gap width $2\cdot(R_2 - R_f)$ between the external and internal cylinders has been used as the characteristic linear dimension for the Reynolds number, and the average velocity for the annular cross-section has been taken as the characteristic velocity.

After processing these known experimental results, the data from Table 2 are represented as an assembly of points in Figure 2.
Figure. 2. The nature of dependence of \( K_{1,c} \) on \( K_{3,c} \) built on the basis of known experimental data [19, 20, 23, 34-48]; 1 is the approximating function (23) graph.

For such point assembly, the approximating function has been obtained in the form of the following power law

\[
K_{1,c} = F_c(K_{3,c}) = q_0 \cdot (K_{3,c})^{q_1}, \quad q_0 = 9.382 \cdot 10^{-4}, \quad q_1 = 2.008. \tag{23}
\]

Thus, if condition (21) related to the “generation” of cross stream velocity components is satisfied at a certain flow region area, and, simultaneously, condition (22) is satisfied with reference to (23) at a certain point of that area where the maximum value of dimensionless complex \( K_j \) is obtained, we suggest that such situation be interpreted as a startup of the laminar-turbulent transition.

Note that the laminar-to-turbulent transition startup hypothesis used in this section and based on the excess of a critical threshold level by the maximum value of a dimensionless complex (similar to the Reynolds number) was assumed as a startup condition in a series of well-known works [7-14]. Quite naturally, those works discussed the dimensionless complexes that were different from \( K_j \) that was introduced in accordance with relationship (6).

7. Conclusion.
The proposed rheological model, which, within a certain variation range of the second invariant of the strain rate tensor, perfectly coincides with the classical Newtonian fluid model, and takes into account the transverse viscosity factor beyond such range, allowed us to arrive at the respective notation for the fluid dynamics equations of such kind. In translating the dimensionless forms of such equations, all scale values were determined using the invariant local characteristics of the velocity and pressure field. As a result, this introduced four dimensionless complexes that are not directly associated with macrocharacteristics of the set flow region. The analysis of the fluid dynamics equations demonstrates that a process of “generating” the cross stream velocity components starts in the neighborhood of a
spatial point where “joining in” of the transverse viscosity factor is satisfied. In the event that the additional condition of infinite increase for such “generated” cross stream velocity components is satisfied, we suggest that the latter circumstance be interpreted as a startup of the laminar-turbulent transition. Based on the analysis of the known experimental data related, among other things, to the secondary flow origination due to the generation of cross stream velocity components, certain (necessary and sufficient) empiric conditions overlaid on the local dimensionless complexes are proposed for the startup of laminar-turbulent transition.

References
[1] Schlichting H 1955 Boundary-layer Theory (New York: McGraw-Hill) p 535
[2] Joseph D D 1976 Stability of Fluid Motions, Vol. I, II (Berlin-Heidelberg-New York: Springer-Verlag) Vol. I: p 282, Vol. II: p 274
[3] Goldschtick M A and Stern V N 1977 Hydrodynamic Stability and Turbulence (Novosibirsk: Nauka Publishers) p 367
[4] Zhitulev V N and Tumin A N 1987 Origination of Turbulence. (Novosibirsk: Nauka Publishers) p 283
[5] Boiko A V, Grek G R, Dovgal A V and Kozlov V V 1999 Origination of Turbulence in Wall-adjacent Flows (Novosibirsk: Nauka Publishers) p 328
[6] Sedov L I 1980 Invention of Physical Models In: Sedov L I Thoughts on Science and Scientists (Moscow: Nauka Publishers) pp 119–123
[7] Rose H 1946 Elementary Mechanics of Fluids (New York: Dover Publications) p 376
[8] Ryan N W and Johnson M 1959 Transition from laminar to turbulent flow in pipes AIChE J. 5 No 4 pp 433–435
[9] Hanks R W 1963 The Laminar-Turbulent Transition for flow in Pipes, Concentric Annuli, and Parallel Plates AIChE J. 9 (1) pp 45–48
[10] Artyushkov L S 1974 Transition from Laminar to Turbulent Flow for Pure Viscous Power Law Non-Newtonian Fluids Proc. of the Leningrad shipbuilding Institute (Leningrad: Leningrad shipbuilding Institute Press) 89 pp 19–24
[11] Artyushkov L S 1979 Dynamics of the Non-Newtonian Fluid (Leningrad: Leningrad shipbuilding Institute Press) p 228
[12] Kolodezhnov V N. 2005 On a Dimensionless Complex for Modeling Viscous Non-Newtonian Fluid Flow The Modern Issues of Mechanics and Applied Mathematics. In two parts. Part 1 (Voronezh: Voronezh State University Press) pp 166–168
[13] Dou H S 2006 Mechanism of flow instability and transition to turbulence Int. J. Non-Linear Mechanics 41(4) pp 512–517
[14] Tao J J, Chen S Y and Su W D 2013 Local Reynolds number and thresholds of transition in shear flows Science China Physics, Mechanics & Astronomy 56 (2) pp 263–269
[15] Litvinov V G 1982 Motion of Non-linear Viscous Fluid (Moscow: Nauka Publishers) p 376
[16] Kolodezhnov V N 2012 On a Possible Model of the Initial Stage of the Laminar-turbulent Transition Proc. of Voronezh State Technical University (Voronezh: Voronezh State Technical University Press) 8 (5) pp 25–30
[17] Kolodezhnov V N 2016 Modeling the initial stage of the flow of non-Newtonian fluid, taking into account the threshold threshold “connection” of the transverse viscosity factor Actual problems of applied mathematics, Informatics and mechanics (Voronezh: Voronezh State University Press) pp 338–340
[18] Taylor G I 1923 Stability of a Viscous Liquid contained between Two Rotating Cylinders Phil. Trans. Royal Society, A 223 pp 289–343
[19] Schultz-Grunow F and Hein H 1956 Beitrag zur Couettestromung Z. Flugwiss No 4 pp 28–30
[20] Coles D 1965 Transition in circular Couette flow J. of Fluid Mechanics 21 (03) pp 385–425
[21] Donnelly R J and Schwarz K W 1965 Experiments on the stability of Viscous Flow Between Rotating Cylinders. VI Finite-Amplitude Experiments Proc. of the Royal Society A 283 pp 531–556
[22] Snyder H A 1968 Stability of rotating Couette flow. II. Comparison with numerical results The Physics of Fluids 11 (8) pp 1599–1605
[23] Gollab J P and Swinney H L 1975 Onset of turbulence in a rotating fluid Phisical Review Letters 36 (14) pp 927–930
[24] Andereck C D, Liu S S and Swinney H L 1986 Flow regimes in circular Couette system with independently rotating cylinders J. of Fluid Mechanics 164 pp 155–183
[25] Prigent A and Dauchot O 2002 “Barder pole turbulence” in large aspect ratio Taylor-Couette flow Phisical Review Letters 89 (1) p 014501
[26] Hinko K A 2003 Transition in the Small Gap Limit of Taylor – Couette Flow The Ohio State University Physics Summer Institute, REU Summer 2003; Advisor: Dr. C.D. Andereck, Department of Physics, The Ohio State University
[27] Dou H S, Khoo B C and Yeo K S 2008 Instability of Taylor – Couette Flow between Concentric Rotating Cylinders Inter. J. of Termal Science 47 (11) pp 1422–35
[28] Bottin S, Dauchot O, Daviaud F and Manneville P. 1998 Experimental evidence of streamwise vortices as finite amplitude solutions in transitional plane Couette flow Physics of Fluids 10 (10) pp 2597–607
[29] Aydin M and Leutheusser H J 1979 Novel experimental facility for the study of plane Couette flow Review of Scientific Instruments 50 (11) pp 1362–66
[30] Daviaud F, Hegseth J and Berge P 1992 Subcritical Transition to Turbulence in Plane Flow Physical Review Letters 69 (17) pp 2511–14
[31] Tillmark N and Alfredsson P H 1992 Experiments on transition in plane Couette flow J. Fluid Mech. 235 pp 89–102
[32] Malerud S, Maloy K J and Goldburg W I 1995 Measurements of turbulent velocity fluctuations in a planar Couette cell Physics of Fluids 7 (8) pp 1949–55
[33] Dauchot O and, Daviaud F 1995 Finite amplitude perturbation and growth mechanism in plane Couette flow Physics of Fluids 7 (2) pp 335–343
[34] Barcilon A, Brindley J, Leesen M and Modds F R 1979 Marginal instability in Taylor-Couette flows at very high Taylor number J. of Fluid Mechanics 94(3) pp 453–463
[35] Fenstermacher P R, Swinney H L and Gollab J P 1979 Dinamical instabilities and the transition to chaotic Taylor vortex flow J. of Fluid Mechanics 94 (1) pp 103–128
[36] Koschmieder E L 1979 Turbulent Taylor vortex flow J. of Fluid Mechanics 93 (3) pp 515–527
[37] Welden R W and Donnelly R J 1979 Reemergent order of chaotic circular Couette flow Phisical Review Letters 42 (5) p 301–304
[38] Bouabdallah A and Cognet G 1980 Laminar-turbulent transition in Taylor-Couette flow Laminar-Turbulent Transition, (IUTAM Conference) pp 368–377
[39] Di Prima R S and Sweeney H L. 1984 Instabilities and Flow Transition between Concentric Rotating Cylinders In: Hydrodynamic Instabilities and Transition to Turbulence (Moscow: Mir Publishers) pp 169–217
[40] Nikuradse J 1933 Stromungsgesetze in rauhen Rohren VDI. Forschungsheft No 361 pp 1–22
[41] Novopashin S and Muriel A 2002 The critical Reynolds number universal J. of Experimental and Theoretical Physics 122 Issue 2 (8) pp 306–309
[42] Swanson C J, Julian B, Ihas G G and Donnelly R J 2002 Pipe flow measurements over a wide range of Reynolds numbers using liquid helium and various gases J. of Fluid Mechanics 461 pp 51–60
[43] Pavlevie A A, Reshmin A I, Teplovodsky S H and Fedoseev S G 2003 On the lower Critical Reynolds Number for Flow in Circular Pipe Fluid Dynamics 38 (4) pp 545–551
[44] Karnitz M A, Potter M C and Smith M C 1974 An experimental investigation of transition of a plane Poiseuille flow Trans. ASME. J. 96 (4) pp 384–388
[45] Carlson D R., Widnall S E and Peeters M F 1982 A flow-visualization study of transition in plane Poiseuille flow J. of Fluid Mechanics 121 pp 487–505
[46] Lemoult G, Gumowski K, Aider J and Wesfreid J 2014 Turbulent spots in channel: An
experimental study *The European Physical Journal E* 37 (25) p 11

[47] Walker J E, Whan G A and Rothfus RR 1957 Fluid Friction in Noncircular Ducts. *AIChE. J.* 3 (4) pp 484–489

[48] Hanks R W and Bonner W F 1971 Transition flow phenomena in concentric annuli *Industrial & Engineering Chemistry Fundamentals* 10 (1) pp 105–113