‘Mixed’ $\delta$-Jordan-Lie Superalgebra

I. Raptis*

Abstract

An algebra $A$ not encountered in either the usual algebraic varieties or super-varieties is introduced. $A$ is a $\mathbb{Z}_2$-graded and multiplicatively deformed version of the quaternions, with structure similar to that of a $\delta$-Jordan-Lie algebra as defined in [7], but it is shown to be neither that of a purely associative ($\delta = +1$) Lie superalgebra, nor that of a purely antiassociative ($\delta = -1$) Jordan-Lie superalgebra. Rather, it exhibits a novel kind of associativity, here called ‘ordered $\mathbb{Z}_2$-graded associativity’, that is somewhat ‘in between’ pure associativity and pure antiassociativity. In addition to graded associativity, the generators of $A$ obey graded commutation relations encountered in both the usual $\mathbb{Z}_2$-graded Lie algebras ($\delta = 1$) and in $\mathbb{Z}_2$-graded Jordan-Lie algebras ($\delta = -1$). They also satisfy new graded Jacobi identities that combine characteristics of the Jacobi relations obeyed by the generators of ungraded Lie, $\mathbb{Z}_2$-graded Lie and $\mathbb{Z}_2$-graded Jordan-Lie algebras. Mainly due to these three features, $A$ is called a ‘mixed’ $\delta$-Jordan-Lie superalgebra. The present paper defines $A$ and compares it with the $\delta$-Jordan-Lie algebra defined in [7].

*Theoretical Physics Group, Blackett Laboratory of Physics, Imperial College of Science, Technology and Medicine, Prince Consort Road, South Kensington, London SW7 2BZ, UK; e-mail: i.raptis@ic.ac.uk
1 Introduction

In theoretical physics, supersymmetry pertains to a symmetry between bosons and fermions. Supergroups, or $\mathbb{Z}_2$-graded Lie groups, are the mathematical structures modelling continuous supersymmetry transformations between bosons and fermions. As Lie algebras consist of generators of Lie groups—the infinitesimal Lie group elements tangent to the identity, so $\mathbb{Z}_2$-graded Lie algebras, otherwise known as Lie superalgebras, consist of generators of (or infinitesimal) supersymmetry transformations \[3\].

Like their ungraded Lie ancestors $L$, Lie superalgebras $\mathcal{L}$

- (i) Are complex vector spaces that are $\mathbb{Z}_2$-graded\[4\]

$$\mathcal{L} = \mathcal{L}^0 \oplus \mathcal{L}^1,$$

with grading function $\pi$ given by

$$\pi(x) := \begin{cases} 0, & \text{when } x \in \mathcal{L}^0, \\ 1, & \text{when } x \in \mathcal{L}^1. \end{cases}$$ (2)

- (ii) Are associative algebras with respect to a bilinear product $\cdot : \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}$ (simply write $x \cdot y \equiv xy = z \in \mathcal{L}$ for the associative product $\cdot$ of $x$ and $y$ in $\mathcal{L}$).

- (iii) Close under the so-called super-Lie bracket $\langle \cdot, \cdot \rangle : \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}$ represented by the non-associative, bilinear, $\mathbb{Z}_2$-graded (anti-commutator) Lie product $\{\cdot, \cdot\}$ defined as

$$\{x, y\} = \begin{cases} [x, y] = xy - yx \in \mathcal{L}^0, & \text{when } x, y \in \mathcal{L}^0, \\ \{x, y\} = xy + yx \in \mathcal{L}^0, & \text{when } x, y \in \mathcal{L}^1, \\ [x, y] = xy - yx \in \mathcal{L}^1, & \text{when } x \in \mathcal{L}^0 \text{ and } y \in \mathcal{L}^1. \end{cases}$$ (3)

- (iv) With respect to $\langle \cdot, \cdot \rangle$, they obey the so-called super-Jacobi identities\[4\].

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1It is tacitly assumed that both $\mathcal{L}^0$ and $\mathcal{L}^1$ in \[4\] are linear subspaces of $\mathcal{L}$ whose only common element is the zero vector 0. $\mathcal{L}^0$ is usually referred to as the even subspace of $\mathcal{L}$, while $\mathcal{L}^1$ as the odd subspace of $\mathcal{L}$.

2For more details about the properties (i)–(iv) of Lie superalgebras, the reader is referred to \[3\]. We will encounter them in a slightly different guise and in more detail when we define $\delta$-Jordan-Lie superalgebras in the next section.
In what follows we first recall the definition of an abstract $\delta$-Jordan-Lie ($\delta$-J-L) algebra $A$ given by Okubo and Kamiya in \[7\], which, as we shall see, includes as a particular case the Lie superalgebra defined in (i)-(iv) above (section 2), then we introduce the concrete algebra $A$ (section 3), and finally we compare the key defining properties of the two structures (section 4). Section 5 concludes the paper with some brief remarks about a possible physical application and interpretation of $A$.

### 2 $\delta$-Jordan-Lie Superalgebra

Let $A$ be a finite dimensional vector space over a field $K$ of characteristic not 2 which, for familiarity, one may wish to identify with $\mathbb{C}$. Also, let $A$ be $\mathbb{Z}_2$-graded

$$A = A^0 \oplus A^1,$$

with grader $\pi$ given by

$$\pi(x) := \begin{cases} 0, & \text{when } x \in A^0, \\ 1, & \text{when } x \in A^1, \end{cases}$$

as in (1) and (2) for $L$ above.

Next, we consider only homogeneous elements of $A$ (ie, either $x \in A^0$ or $x \in A^1$, but not $z = \alpha x + \beta y, \ x \in A^0, \ y \in A^1; \ \alpha, \beta \in \mathbb{C}$) and as in (1.3) of \[7\] we define

$$(-1)^{xy} := (-1)^{\pi(x)\pi(y)}.$$

Let also $xy$ be a bilinear product in $A$ satisfying

$$(xy)z = \delta x(yz), \ \delta = \pm 1,$$

with respect to which $A$ is said to be a $\delta$-associative algebra. In particular, for $\delta = +1$, $A$ is an associative algebra; while for $\delta = -1$, it is antiassociative.

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$^3$In \[6\], $\sigma(x)$ is used instead of $\pi(x)$ to symbolize the grading function. See (1.2) in \[6\].

$^4$In theoretical physics, this forbidding of linear combinations between bosons and fermions is known as the Wick-Wightman-Wigner superselection rule \[11\]. The direct sum split between the even and the odd subspaces in (4) and (5) is supposed to depict precisely this constraint to free superpositions between quanta of integer and half-integer spin (ie, bosons and fermions, respectively). Mainly because of \[11\], we decided to symbolize the grading function in (3) and (5) by ‘$\pi$’ (for ‘intrinsic parity’) rather than by ‘$\sigma$’ as in \[6\]. In the literature, the set-theoretic (disjoint) union ‘$\cup$’ is sometimes used instead of ‘$\oplus$’ between the even and odd subspaces of a $\mathbb{Z}_2$-graded vector space \[3\]—it being understood that these two subspaces have only the trivial zero vector in common, as noted in footnote 1. ‘$\cup$’ too is supposed to represent the aforesaid spin-statistics superselection rule.
Consider also a second bilinear product \( \langle \cdot, \cdot \rangle : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A} \)

\[
\langle x, y \rangle := xy - \delta(-1)^{xy}yx,
\]

satisfying

\[
\pi(\langle x, y \rangle) = \pi(x) + \pi(y) \pmod{2},
\]

and

\[
\langle x, y \rangle = -\delta(-1)^{xy} \langle y, x \rangle,
\]

or equivalently

\[
(-1)^{xz} \langle x, y, z \rangle + (-1)^{yx} \langle y, z, x \rangle + (-1)^{zy} \langle z, x, y \rangle = 0,
\]

\[
(-1)^{xz} \langle x, y, z \rangle + (-1)^{yx} \langle y, z, x \rangle + (-1)^{zy} \langle z, x, y \rangle = 0.
\]

\(\mathcal{A}\), satisfying (4)–(11), is called a \(\delta\)-J-L algebra \([7]\). Also, one can easily verify that for \(\delta = 1\), \(\mathcal{A}\) is the associative \(\mathbb{Z}_2\)-graded Lie superalgebra \(\mathcal{L}\) defined in (i)–(iv) of section 1. The antiassociative (\(\delta = -1\)) case is coined Jordan-Lie superalgebra \([7]\)—here to be referred to as \(J-L\) algebra \(J\) for short. We may summarize all this as follows

\[
\mathcal{A} = \begin{cases} 
\mathcal{L}, & \text{for } \delta = +1, \\
J, & \text{for } \delta = -1.
\end{cases}
\]

For future use we quote, without proof, the following lemma and two corollaries from \([7]\):

- **Lemma:** In every antiassociative algebra \(A\), any product involving four or more elements of \(A\) is identically zero.\(^7\)

- **Corollary 1:** Antiassociative algebras have no idempotent elements and, as a result, no units (ie, identity elements).\(^8\)

- **Corollary 2:** Let \(J\) be a J-L algebra as defined above. Then \(J\) is nilpotent of length at most 3 (write: \(J_4 = 0\)).\(^9\)

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\(^5\)In particular, the expression (iii) in (iii) is encoded in (7)–(9) above, while the ‘graded Jacobi identities’ property (iv) of \(\mathcal{L}\) is expressed by (10) or (11).

\(^6\)Proofs can be read directly from \([7]\).

\(^7\)Lemma 1.1 in \([7]\).

\(^8\)Corollary 1.2 in \([7]\).

\(^9\)Corollary 1.1 in \([7]\).
3 Introducing \( \mathbb{A} \)

Let \( \mathbb{A} \) be a 4-dimensional vector space over \( \mathbb{R} \) spanned by \( g = \{a, b, c, d\} \)[10] and also be \( 2 \oplus 2 \)-dimensionally \( \mathbb{Z}_2 \)-graded thus

\[
\mathbb{A} = \mathbb{A}^0 \oplus \mathbb{A}^1 = \text{span}_\mathbb{R} \{a, b\} \oplus \text{span}_\mathbb{R} \{c, d\}.
\]

(12)

Let \( \circ : \mathbb{A} \otimes \mathbb{A} \rightarrow \mathbb{A} \) be a bilinear product in \( \mathbb{A} \) which, in terms of \( \mathbb{A} \)'s generators in \( g \), is encoded in the following multiplication table

\[
\begin{array}{cccc}
\circ & a & b & c & d \\
\hline
a & a & b & -d & -c \\
b & b & -a & -d & c \\
c & c & d & a & -b \\
d & d & -c & b & -a \\
\end{array}
\]

(13)

From table (13), one can easily extract the following information:

• \( \circ \) is not commutative. In particular, \( a \) commutes only with \( b \); while, \( b, c \) and \( d \) mutually anticommute. Moreover, \( a \) is a right-identity, but not a left one.

• \( \circ \) is not (anti)associative. For example, one can evaluate

\[
c = -ad = a(bc) \neq \begin{cases} (ab)c = bc = -d, & (\delta = +1); \\
-(ab)c = -bc = d, & (\delta = -1). \end{cases}
\]

• \( a \) and \( c \) are \( \sqrt{a} \), while \( b \) and \( d \) are \( \sqrt{-a} \).

• The even subspace of \( \mathbb{A} \) in (12), \( \mathbb{A}^0 := \text{span}_\mathbb{R} \{a, b\} \), is isomorphic to the complex numbers \( \mathbb{C} \) if we make the following correspondence between the unit vectors of \( \mathbb{A} \) and \( \mathbb{C} \)

\[
a \rightarrow 1, \ b \rightarrow i \ (i^2 = -1).
\]

\( \mathbb{A}^0 \) is the subalgebra of even elements of \( \mathbb{A} \).

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[10] The alphabetical symbolism of the four basis vectors (generators) in \( g \) will be explained subsequently.
• The product of an even and an odd generator is odd, while the product of two odd generators is even. Together with the second observation above, we may summarize this to the following

\[ \pi(xy) = \pi(x) + \pi(y) \pmod{2}. \]

• The inhomogeneous vector \( \mathbf{n}_1 = b + c \) and the odd vector \( \mathbf{n}_2 = c + d \) are nilpotent\(^{11}\).

Let us gain more insight into the non-associativity of \( \circ \) by making a formal correspondence between the ‘units’ of \( \mathbb{A} \) in \( g \) and the standard unit quaternions \( \mathbf{u} = \{1, i, j, k\} \) in \( \mathbb{H} \)

\[
\begin{align*}
  a &\rightarrow 1, & b &\rightarrow i, \\
  c &\rightarrow j, & d &\rightarrow k.
\end{align*}
\]

Then, one may wish to recall that the associative division algebra \( \mathbb{H}^{12} \) can be obtained from \( \mathbb{C} \) by adjoining \( j = \sqrt{-1} \) to the generators \( \{1, i\} \) of \( \mathbb{C} \) and by assuming that it commutes with 1: \( 1j = j1 = j \), but it anticommutes with \( i \): \( ij = -ji = k^{13} \). Also, by assuming associativity, one verifies that \( k \) too is a \( \sqrt{-1} \) that anticommutes with both \( i \) and \( j \)

\[
\begin{align*}
  k^2 &= (ij)(ij) = i(ji)j = -i^2j^2 = -1, \\
  ki &= (ij)i = i(ji) = -i(ij) = -ik,
\end{align*}
\]

thus one completes the following well-known multiplication table for the unit quaternions

\[
\begin{array}{cccc}
  \bullet & 1 & i & j & k \\
  1 & 1 & i & j & k \\
  i & i & -1 & k & -j \\
  j & j & -k & -1 & i \\
  k & k & j & -i & -1
\end{array}
\]

\(^{11}\)\( \mathbf{n}_1 \) violates the aforementioned Wick-Wightman-Wigner superselection rule \([11]\).

\(^{12}\)We may write \( \bullet \) for the associative product of quaternions (i.e., \( : \mathbb{H} \otimes \mathbb{H} \rightarrow \mathbb{H} \)), but omit it in actual products, that is to say, we simply write \( xy \) (\( x, y \in \mathbb{H} \)). We assumed the same thing for \( x \cdot y \) in \( \mathcal{L} \) and \( \mathcal{A} \), as well as for \( x \circ y \) in \( \mathbb{A} \) (for instance, see (ii) in section [\[6\])

\(^{13}\)In fact, one assumes that by transposing \( i \) with \( j \), \( i \) gets conjugated: \( ij = ji^* = -ji \leftrightarrow \{i, j\} = 0 \)
If we were to emulate the extension of $C$ to $H$ in the case of $A$, thus adjoin $c$ to $b$ in $A^0 \cong C$ and require according to (13) that they anticommute, as well as that $\circ$ is associative, we would get

$$d^2 = (cb)(cb) = cbcb = -c^2b^2 = -(a)(-a) = a$$

which disagrees with entry $(4,4)$ in table (13). Similarly for $c^2$. Clearly then, as also noted above, (the product $\circ$ in) $A$ is neither associative\(^{15}\) nor antiassociative\(^{16}\).

**Question:** How can we obtain agreement between products like the one in (16)—which arise rather naturally upon trying to extend $C$ to $A$ in the same manner that $C$ is extended to $H$—with the entries of the multiplication table (13)? Evidently, we need a new (anti)associativity-like law for $\circ$.

To this end we first define:

**Definition 1:** A product string $w$ of generators of $A$ in $g$ of length $l$ greater than or equal to 3 is said to be (n)ormally (o)rdered\(^{17}\) if it is of the following ‘right-to-left alphabetical order’ or ‘alphabetic-syntax’

$$\overleftarrow{w} := d^s c^r b^q a^p, \quad p, q, r, s \in \mathbb{N} \quad l(w) := p + q + r + s.$$  \hspace{1cm} (17)

Then we impose the following three rules or relations\(^{18}\) onto the total contraction of any word of length $l \geq 3$:

**Rule 0:** Before contracting totally a word $w$ of length $l \geq 3$ it should be brought into no-ed form in the following two steps:

\(^{14}\)Reader, try to calculate $c^2 = (bd)(bd)$ in a manner similar to (16) above.
\(^{15}\)\(\delta = 1\) in (16).
\(^{16}\)\(\delta = -1\) in (16).
\(^{17}\)In free algebra jargon, such product strings $w$ are called *words* and their factors, which are elements of $g$, are called *letters* (which, in turn, makes $g$ $A$’s 4-letter alphabet!). The number $l$ of letters in a word $w$ is its *length*, and we write $l(w)$. Formally speaking, a word $w$ of length $l$ is a member of $l$ factors $A \otimes A \otimes \cdots \otimes A$. The $4^2$ possible words of length 2 in $A$ are the ones depicted in table (13) above.
\(^{18}\)Write ‘no-ed’ and symbolize the word by $\overleftarrow{w}$.
\(^{19}\)Again, this is free algebra jargon.
\(^{20}\)By ‘total contraction’ of a word of length $l \geq 3$ we mean the reduction of the word to a single (signed) letter in $g$ after $l - 1$ pairwise contractions of its constituent letters according to (13). Again, formally speaking, the product $\circ : A \otimes A \rightarrow A$ in (13) represents the contraction of 2-words in $A$, so analogously, the total contraction of words of length $l$ may be cast as $c^{l-1} : A \otimes A \otimes \cdots \otimes A \rightarrow A$.  

7
• (a) When the right-identity letter $a$ is found in an extreme left or intermediate position in $w$, it should be contracted with the adjacent letter on its right according to (13).

• (b) The other three mutually anticommuting generators $b$, $c$ and $d$ in $g$ should be pairwise swapped within $w$ so that they are ultimately brought to the form $\pm d^s c^r b^q$.

A couple of comments are due here:

1) Above, (a) implies that the length of a word may change upon no-ing it. This is allowed to happen in $A$. For the algebraic structure of $A$ that we wish to explore here not all words assembled by free (arbitrary) $\circ$-concatenations of letters in $g$ are significant. Only no-ed words are structurally significant$^{22}$, and any given $w$ has a unique no-ed form $\overleftarrow{w}$ fixed according to (i) and (ii) above. Rule 0 prompts us to call $A$ ‘multiplicatively ordered’ and this alphabetico-syntactic ordering may be formally cast as follows

\[ d > c > b > a, \]  

since, once again, every no word is of the form $\overleftrightarrow{w} := d^s c^r b^q a p$ according to (17). The generators of $A$ are ordered thus. Since in its transition to its unique no-ed form a word may change length, the latter is not a significant structural trait of $A$, but the order (18) is.

2) Normal ordering respects superpositions of words in $A$. In other words, no-ing is a linear operation; symbolically

\[ \overleftarrow{w_1 + w_2} = \overleftarrow{w_1} + \overleftarrow{w_2}. \]

The other two rules that we impose on the total contraction of a no-ed word of length $l \geq 3$ are:

**Rule 1:** Every no-ed word of length $l$ greater than 2 contracts totally to a (signed) letter in $g$ by $l - 1$ sequential pair-contractions of letters in it according to (13) (from right to left) (ie, in the multiplicative order depicted in (18)). We may call this rule for $\circ$ ordered or directed associativity.

**Rule 2:** Moreover, ordered associativity is $\mathbb{Z}_2$-graded as follows

\[ \overleftarrow{w_1 + w_2} \]

\[ \overrightarrow{w_1 + w_2}. \]

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21 As it were, the ‘natural’ position of $a$ in a word is to the extreme right. This seems to suit $a$’s role as a right-identity in $A$ (13).

22 This will be amply justified in the sequel.

23 Write ‘frtl’.
where \( \vec{w} \) signifies the commencement of the pairwise sequential total contraction of the no-ed word \( w \) \( \text{frtl à la} \) rule 1; ‘e’ stands for \( (e) \)en and ‘o’ for \( (o) \)dd letters in \( \vec{w} \); and ‘\( e'' = (e' \)e \) from \( (13) \)’ end of the first row of \( (19) \) signifies the contraction and substitution of the product pair \( e' \)\( e \) by \( e'' \) according to \( (13) \). Thus, rule 2 essentially says that \textit{when an odd and an even letter contract within a no-ed word \( \vec{w} \), one must put a minus sign in front of \( \vec{w} \)}. In view of rules 0–2, we call \( o \) in \( \mathbb{A} \) a ‘\( \mathbb{Z}_2 \)-\textit{graded ordered associative product}’. The \( \mathbb{Z}_2 \)-\textit{graded ordered associativity} of \( \mathbb{A} \) is somewhat ‘in between’ the pure associativity of a Lie superalgebra \( L (\delta = 1) \) and the pure antiassociativity of a J-L algebra \( J (\delta = -1) \) as defined above.

Due to rules 0–2, \( \mathbb{A} \) may be called a \textit{multiplicatively ordered \( \mathbb{Z}_2 \)-\textit{graded associative algebra}}\(^{24}\).

Having rules 0–2 in hand, we are now in a position to show that words such as the one displayed in \( (16) \) contract consistently with the binary multiplication table \( (13) \), thus we provide an answer to the question following \( (16) \) above. So, we check that

\[
d^2 = (cb)(cb) = cbcb \xrightarrow{R0} -c^2b^2 \xrightarrow{R1} -c^2(b^2) \xrightarrow{[3]} cca \xrightarrow{R1} c(ca) \xrightarrow{R2} -c^2 \xrightarrow{[3]} -a,
\]

is in agreement with \( (13) \).\(^{25}\)

Now we can give the rest of the \( \mathbb{Z}_2 \)-\textit{graded Lie algebra-like structural properties} of \( \mathbb{A} \).

- First, there is a \textit{bilinear product} \( <.,.> \): \( \mathbb{A} \otimes \mathbb{A} \rightarrow \mathbb{A} \) represented by the non-\textit{associative, \( \mathbb{Z}_2 \)-\textit{graded (anti-commutator) Lie product}} \( [.,.] \) as follows

\[
[x, y] := \begin{cases} 
[ x, y ] = xy - yx \in \mathbb{A}^{0}, & \text{when } x, y \in \mathbb{A}^{0}, \\
\{ x, y \} = xy + yx \in \mathbb{A}^{1}, & \text{when } x, y \in \mathbb{A}^{1}, \\
\{ x, y \} = xy + yx \in \mathbb{A}^{1}, & \text{when } x \in \mathbb{A}^{0} \text{ and } y \in \mathbb{A}^{1},
\end{cases}
\]

\textit{similar to} \( (3) \), or equivalently that \( <.,.> \) satisfies

\(^{24}\)And from now on, \( (xy) \) in \( \mathbb{A} \) will indicate precisely this ‘contraction of \( xy \) and its substitution by the corresponding entry from \( (13) \)’ process.

\(^{25}\)From now on we will most often drop the adverb ‘\textit{multiplicatively}’ above and simply refer to \( \mathbb{A} \) as an ordered \( \mathbb{Z}_2 \)-\textit{graded associative algebra}.

\(^{26}\)We note that in \( (20) \), \( R0 \), for instance, refers to ‘Rule 0’ (similarly for \( R1 \) and \( R2 \)). Again, for ‘\textit{practice}’ the reader can also verify that \( c^2 = (bd)(bd) = \cdots = a \), in agreement with \( (13) \).
\[ \pi(<x, y>) = \pi(x) + \pi(y) \pmod{2} \] (22)

and

\[
<x, y> := xy - \delta(-1)^{xy}yx = -\delta(-1)^{xy} < y, x > = \\
\begin{cases} 
xy - (-1)^{xy}yx, & \text{when } x, y \in A^0 \text{ or } x, y \in A^1; \quad \delta = 1, \\
xy + (-1)^{xy}yx, & \text{when } x \in A^0 \text{ and } y \in A^1; \quad \delta = -1,
\end{cases}
\] (23)

similar to (7), (8) and (9).

\[ 
\bullet \text{ Second, the following eight possible super-Jacobi identities are satisfied. These are the analogues in } A \text{ of expressions (10) and (11) for the } \delta-\text{J-L algebra in } [7].
\]

\[ 
[d, c] + [c, a] + [a, d] + [a, d, c] = 0, \\
[d, c] + [c, b] + [b, d, c] = 0, \\
[a, b] + [b, d, a] + [d, a, b] = 0, \\
[a, b] + [b, c] + [c, a] + [c, a, b] = 0
\] (24)

and

\[ 
\{d, [c, a]\} + \{c, [a, d]\} + \{a, [d, c]\} = 0, \\
\{d, [c, b]\} + \{c, [b, d]\} + \{b, [d, c]\} = 0, \\
\{a, [b, d]\} + \{b, [d, a]\} + \{d, [a, b]\} = 0, \\
\{a, [b, c]\} + \{b, [c, a]\} + \{c, [a, b]\} = 0
\] (25)

are satisfied. These are the analogues in } A \text{ of expressions (11) and (12) for the } \delta-\text{J-L algebra } A \text{ in } [7].

In view of the novel and quite peculiar ordered } \mathbb{Z}_2 \text{-graded associative multiplication structure } \circ \text{ of } A \text{ (rules 0–2), we must specify to the reader who wishes to verify patiently that the graded Jacobi relations (24) and (25) hold how to actually contract them. To this end we define: }

**Definition 2:** The contraction of a super-Jacobi relation is said to be performed ‘(f)rom (i)nside (t)o (o)utside’ when the inner < .. >-brackets are opened and contracted first, and then the outer ones. Analogously, the contraction of a super-Jacobi relation is said to be ‘foti (ie, (f)rom (o)utside (t)o (i)nside)’ when the outer brackets are opened first, then the inner ones, and then the resulting superpositions of words of length 3 are totally contracted according to rules 0–2.

\[ ^{27} \text{We will comment further on (21) and (22)-(23) in the next section when we compare } A \text{ and the } \delta-\text{J-L algebra } A \text{ of } [7]. \]

\[ ^{28} \text{Write } \text{fito.} \]
Scholium: The conscientious reader can check, by using (13), that the super-Jacobi relations (24) and (25) are satisfied by the fito mode of contraction, but not by the foti one. For instance, also to give an analytical example of the two kinds of contraction, we evaluate the third expression in (25) by both fito and foti means

\[
\text{fito} : \{a, \{b, d\}\} + \{b, \{d, a\}\} + \{d, [a, b]\} = \{a, (bd) + (db)\} + \\
\{b, (da) + (ad)\} + \{d, (ab) - (ba)\} \quad \text{[13]} \quad \{b, d - c\} = \{b, d\} - \{b, c\} = 0
\]

\[
\text{foti} : \{a, \{b, d\}\} + \{b, \{d, a\}\} + \{d, [a, b]\} = a\{b, d\} + \{b, d\}a + b\{d, a\} + \\
\{d, a\}b + d[a, b] + [a, b]d = abd + adb + bda + dba + \\
bda + bad + dab + adb + dab - dba + adb - bad = 2(ab)d + 2(da)b + \\
2(ad)b + 2bda = 2(-db + da - cb - dba) = 2(c + d - d + c) = 2c \neq 0.
\]

This indicates that, by virtue of the ordered \(\mathbb{Z}_2\)-graded associative product structure of \(A\),

\[A\] is a Lie superalgebra-like structure with respect to the fito, but not the foti, mode of contraction of its graded Jacobi relations.

This is another peculiar feature of \(A\)—an immediate consequence of its ordered \(\mathbb{Z}_2\)-graded associative multiplication idiosyncrasy\(^{29}\).

4 Comparing \(A\) with \(A\)

We can now compare \(A\) with the abstract \(\delta\)-J-L algebra \(A\) defined in \([7]\). Below, we itemize this comparison:

- (i) As vector spaces, both \(A\) and \(A\) are finite dimensional and \(\mathbb{Z}_2\)-graded \([4), (12)\].
- (ii) With respect to multiplication, while \(A\) is \(\delta\)-associative (ie, associative \(L\) for \(\delta = 1\) or antiassociative \(J\) for \(\delta = -1\)), \(A\) is ordered \(\mathbb{Z}_2\)-graded associative—somewhat ‘in between’ pure associativity and pure antiassociativity \([13], 17, 18, 19]\).
- (iii) With respect to \(\mathbb{Z}_2\)-graded commutation relations \(<\ldots, \ldots>\), \(A\) combines characteristics of both Lie superalgebras \(L = A|_{\delta = 1}\) and J-L algebras \(J = A|_{\delta = -1}\). In particular, as \([5], 3, 23]\) depict:

\[\text{(a) } A\] is like \(L\) with respect to the ‘homogeneous’ \(<\ldots, \ldots>\)-relations obeyed by even and odd elements\(^{30}\).

\[\text{while:}\]

\(^{29}\)In the next two sections we will discuss in more detail these ‘multiplication oddities’ of \(A\).

\(^{30}\)That is, even elements obey antisymmetric commutation relations, while odd elements obey symmetric anticommutation relations. This is a concise algebraic statement of the celebrated spin-statistics connection \([8]\).
(b) $\mathcal{A}$ is like $\mathcal{J}$ with respect to the ‘inhomogeneous’ commutation relations between bosons and fermions\textsuperscript{31}.

moreover:

(c) The $\mathbb{Z}_2$-graded $\langle \ldots \rangle$-relations ‘close’ in $\mathcal{A}$ in exactly the same way that they close in $\mathcal{A}\{\delta\} \{8\}, \{22\}$.

• (iv) The generators of $\mathcal{A}$, unlike those in $\mathcal{A}$, obey ‘externally ungraded’ Jacobi relations\textsuperscript{33}. In this formal respect, $\mathcal{A}$ is like an ungraded Lie algebra $L$.

• (v) We return a bit to the comparison of the multiplication structure of the two algebras (ii), now also in connection with the Jacobi relations in (iv) above, and note that for the (anti)associative $\delta$-J-L superalgebras it is immaterial whether one evaluates their super-Jacobi relations $\{11\}$ and $\{11\}$ $fito$ or $foti$, because they are ‘multiplicatively unordered’ structures\textsuperscript{34}. On the other hand, as we saw in $\{29\}$ for example, exactly because of the ordered $\mathbb{Z}_2$-graded associative multiplication structure of $\mathcal{A}$, $fito$-contracted Jacobis are satisfied in $\mathcal{A}$, but $foti$ ones are not, therefore it crucially depends on the ordered multiplication structure $\circ$ whether $\mathcal{A}$ is a Lie-like algebra ($fito$) or not ($foti$). Such a dependence is absent from the multiplicative unordered $\mathcal{L}$ and $\mathcal{J}$ algebras\textsuperscript{35}.

• (vi) Also in connection with (v) above, we note in view of the lemma and the two corollaries concluding section 2 that:

$(\alpha)$ Because $\mathcal{A}$ is not purely antiassociative, words of length greater than or equal to 4 in it do not vanish identically as they do in $\mathcal{J}$ for instance\textsuperscript{36}.

$(\beta)$ Like the antiassociative $\mathcal{J}$, $\mathcal{A}$ has no idempotents and no two-sided identity. However, as we saw in the previous section, $\mathcal{A}$ has a right-identity, namely, $a\textsuperscript{37}$.

$(\gamma)$ As a corollary of (\alpha), and unlike $\mathcal{J}$, $\mathcal{A}$ is not nilpotent of length at most 4.

\textsuperscript{31}That is, the commutation relation between an even and an odd element of $\mathcal{A}$, like in $\mathcal{J}$, is symmetric (i.e., anticommutator).

\textsuperscript{32}That is, in both $\mathcal{A}$ and $\mathcal{A}$ the homogeneous $\langle \ldots \rangle$-relations close in their even subspaces, while the inhomogeneous ones in their odd subspaces.

\textsuperscript{33}That is, the three external factors $(−1)^{xz}$, $(−1)^{yx}$ and $(−1)^{zy}$ present in the Jacobi expressions $\{10\}$ and $\{11\}$ for $\mathcal{A}$ are simply missing in the corresponding ones, $\{24\}$ and $\{25\}$, for $\mathcal{A}$.

\textsuperscript{34}That is, it does not matter in what order one contracts pairs of generators in words of length greater than 2 in $\mathcal{A}$.

\textsuperscript{35}The ‘multiplicative unorderedness’ of both $\mathcal{A}\{\delta=1\} = \mathcal{L}$ and $\mathcal{A}\{\delta=-1\} = \mathcal{J}$ is encoded in the (anti)associativity relation $\{9\}$ imposed on their products, since on the one hand associativity simply means that the left-to-right contraction of a 3-letter word is the same as the right-to-left one, while on the other, antiasociativity means essentially the same thing under the proviso that one compensates with a minus sign for one order of contraction relative to the other. Both associativity and antiasociativity however, unlike the $\mathbb{Z}_2$-graded associativity in $\mathcal{A}$ $\{19\}$, do not depend on the grade of the letters involved in the binary contractions within words of length greater than or equal to 3.

\textsuperscript{36}See lemma in section 2.

\textsuperscript{37}See corollary 1 in section 2.
• (vii) Finally, in connection with (iii) and (iv) above, we note that our choice of the symmetric anticommutator relation (as in $\mathcal{J}$) instead of the antisymmetric commutation relation (as in $\mathcal{L}$) for the inhomogeneous $<.,.>$-relations in $\mathcal{A}$ can be justified as follows: had we assumed $[e,o]$ instead of $\{e,o\}$, the ito contraction of the first super-Jacobi expression in (24) would yield

$$\{[d,c],a\} + \{[c,a],d\} + \{[a,d],c\} = \{c + d, d\} + \{-c - d, c\} = \{d, d\} - \{c, c\} = -2a - 2a = -4a \neq 0,$$

hence the graded Jacobi identities would not have been obeyed by the generators of $\mathcal{A}$ and, as a result, the latter could not qualify as a Lie-like algebra.

## 5 Closing remarks about $\mathcal{A}$

Our concluding remarks about $\mathcal{A}$ concentrate on the following four issues:

• (1) We compare $\mathcal{A}$ against the other four possible Euclidean division rings, namely, the reals ($\mathbb{R}$), the complexes ($\mathbb{C}$), the quaternions ($\mathbb{H}$) and the octonions ($\mathbb{K}$).

• (2) As a particular case of (1), we remark about the ordered $\mathbb{Z}_2$-graded associative $\mathcal{A}$ versus the multiplicatively unordered, because associative, quaternions $\mathbb{H}$, and we briefly comment on a possible representation of $\mathcal{A}$.

• (3) We abstract $\mathcal{A}$ to a general mixed $\delta$-Jordan-Lie superalgebra $\mathfrak{J}\mathcal{L}$.

• (4) Finally, we discuss a possible physical application and interpretation of $\mathcal{A}$ as originally anticipated in [9].

(1) To make the aforesaid comparison, we first recall how abstract algebraic structure gets lost in climbing the dimensional ladder from $\mathbb{R}$ to $\mathbb{K}$:

• Going from $\mathbb{R}$ of dimension $2^0 = 1$ to $\mathbb{C}$ of dimension $2^1 = 2$, one loses order $^{38}$.

• Going from $\mathbb{C}$ of dimension $2^1 = 2$ to $\mathbb{H}$ of dimension $2^2 = 4$, one loses commutativity.

• Going from $\mathbb{H}$ of dimension $2^2 = 4$ to $\mathbb{K}$ of dimension $2^3 = 8$, one loses associativity.

• And if one wished to extend the octonions to a ring-like structure of dimension $2^4 = 16$, which should be coined ‘decahexanions’ $\mathbb{D}$, there would be no more abstract algebraic structure to be lost $^{4,5}$.

We then note that $\mathcal{A}$ combines characteristics of all those four division rings $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{K}$, as follows:

$^{38}$Although, one gains algebraic completeness!
• (a) $A$ is a vector space over $\mathbb{R}$.

• (b) $A$’s even subalgebra $A^0$ is isomorphic to $\mathbb{C}$.

• (c) $A$ is a 4-dimensional vector space like $\mathbb{H}$, and its 3-subspace spanned by the mutually anticommuting $b, c$ and $d$ reminds one of the subspace of real quaternions (ie, $\mathbb{H}$ over $\mathbb{R}$) spanned by the three imaginary (ie, $\sqrt{-1}$) quaternion units $i, j$ and $k$. Also, by comparing the multiplication tables (13) and (15) for $A$ and $\mathbb{H}$ respectively, one immediately realizes that the former is a sort of deformation of the latter.

• (d) Like $\mathbb{K}$, $A$ is not associative.

• (e) Furthermore, the novel multiplicatively ordered $\mathbb{Z}_2$-graded associative structure of $A$ recalls a bit the linearly ordered $\mathbb{R}$. Could a structure like that be used to define somehow a $\mathbb{D}$-like ring, thus extend Hurwitz’s theorem in [4] to 16 dimensions?

(2) We stressed above the close similarities between $A$ and $\mathbb{H}$. Now we would like to gain some more insight into the novel non-associativity of $A$ by comparing it with the associative quaternions. As a bonus from such a comparison, we will also comment briefly on a possible representation of $A$.

So, we may recall from [8] the real 4-dimensional left ($L$) and right ($R$) matrix ‘self-representations’ of quaternions over $\mathbb{R}$

\[
\begin{align*}
\text{Left} : \quad ab &= c \rightarrow L(a)[b] = [c] \quad \text{and} \quad \text{Right} : \quad bc &= d \rightarrow R(c)[b] = [d],
\end{align*}
\]

\[39\] With the important difference that $c$ in $A$ is a ‘real’ unit (ie, $b = \sqrt{a} \neq \sqrt{-a}$).

\[40\] With most notable ‘deformation features’ of the generators of $A$ relative to those of $\mathbb{H}$ being $c$’s squaring to $a$ unlike $j$’s squaring to $-1$ mentioned in the last footnote, and $a$’s role only as a right-identity unlike 1’s role in $\mathbb{H}$ as being both a right and a left-identity. In fact, from the diagonals of their respective multiplication tables (13) and (15), one could say that the unit quaternions in $\mathbb{H}$ naturally support a metric of Lorentzian signature $\text{diag}(1, -1, -1, -1)$ (absolute trace 2) $\mathbb{I}$ $\mathbb{I}$, while the units of $A$ in $\mathbb{g}$ support a metric of Kleinian signature $\text{diag}(1, -1, 1, -1)$ (traceless). Otherwise, see correspondence (14) in section 3.

\[41\] For example, since the extension from $\mathbb{R}$ to $\mathbb{C}$, to $\mathbb{H}$, and finally, to $\mathbb{K}$, involves a complexification-like process (ie, one adjoins an imaginary unit to the existing generators and demands algebraic closure—thus, in effect, one doubles the dimension), further extension of $\mathbb{K}$ to $\mathbb{D}$ could involve an ‘algebraic degree of freedom’ coming from $\mathbb{Z}_2$-grading (ie, somehow assume that the usual octonion units are even and that the other eight needed to comprise the decahexanions are odd, so that some kind of $\mathbb{Z}_2$-graded associative multiplication à la $A$, not covered by Hurwitz [4], could be evoked; for instance, one could assume that the $j$ ($j^2 = -1$) adjoined to the eight ‘even’ octonion units $\{1, i_1, \cdots, i_7\}$ is odd which, by the displayed expression at the top of page 6, would make the other seven resulting decahexanion units $\{i_1j, \cdots, i_7j\}$ odd as well). However, at this stage this is purely ‘heuristic speculation’.

\[42\] The epithet ‘self’ refers to the representation of $\mathbb{H}$ (by real matrices) induced by the quaternions’ own algebraic product.
where \( \mathbf{b} \) is a column vector in \( \mathbb{R}^{4} \), while both \( L(a) \) and \( R(c) \) are \( 4 \times 4 \) real matrices. The crucial point is that, because \( \mathbb{H} \) is associative,

\[
(ab)c = a(bc) \Rightarrow R(c)L(a)[\mathbf{b}] = L(a)R(c)[\mathbf{b}] \iff [L(\mathbb{H}), R(\mathbb{H})] = 0,
\]

and similarly, for a purely antiassociative algebra like \( \mathcal{J} \) before, it follows that

\[
\{L(\mathcal{J}), R(\mathcal{J})\} = 0.
\]

We may summarize (28) and (29) to the following:

The left and right self-representations of an associative algebra commute, while those of an antiassociative algebra anticommute.

It follows that the self-representations of \( \mathcal{A} \), which is neither purely associative nor purely associative (but somewhat in between the two), will neither commute nor anticommute with each other. As a matter of fact, since \( \mathcal{A} \) is multiplicatively ordered \( \text{frtl} \), only its left self-representation would be relevant (if it actually existed).

(3) The abstraction of \( \mathcal{A} \) to a general mixed \( \delta\)-J-L algebra \( \mathfrak{J} \mathfrak{L} \) is straightforward:

A finite dimensional \( \mathbb{Z}_2 \)-graded vector space \( \mathfrak{J} \mathfrak{L} \) over a field \( K \) of characteristic not 2, together with an ordered \( \mathbb{Z}_2 \)-graded associative multiplication between its elements and a \( \mathbb{Z}_2 \)-graded Lie-like bracket \( \langle \cdot, \cdot \rangle \) satisfying (21)–(25), is called an abstract mixed \( \delta\)-Jordan-Lie superalgebra.

(4) We conclude the present paper by allowing ourselves some latitude so as to discuss briefly a possible physical application and concomitant interpretation of \( \mathcal{A} \).

\( \mathcal{A} \) was originally conceived in [9], but not in the rather sophisticated \( \delta\)-J-L guise presented above. The basic intuition in [9] was to give a simple ‘generative grammar’-like theoretical scenario for the creation of spacetime from a finite number of quanta (generators) which were supposed to inhabit the quantum spacetime substratum commonly known as the vacuum [2]. Thus, it was envisaged that a spacetime-like structure could arise from the algebraic combinations of a finite number of quanta, as it were, a combinatorial-algebraic process modelling the aufbau of spacetime from quantum spacetime numbers filling the vacuum [46]. Furthermore,

\[\text{That is, in the expansion of the real quaternion } \mathbf{b} \text{ in the standard unit quaternion basis } \mathbf{u}: \]

\[\mathbf{b} = b_0 \mathbf{1} + b_j \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k}, \text{ the entries of the 4-vector } [\mathbf{b}] \text{ are the real numbers } b_\mu.\]

\[\text{It is easy to check that the maps } L \text{ and } R \text{ are homomorphisms of } \mathbb{H} (\text{ie, representations of } \mathbb{H}).\]

\[\text{This author has not been able to construct yet a matrix representation of } \mathcal{A} \text{ based on its ordered } \mathbb{Z}_2 \text{-graded associative product. Of course, like with all the usual Lie algebraic varieties and supervarieties, we could alternatively look directly into a possible representation of the non-associative (under the Lie bracket } \langle \cdot, \cdot \rangle \text{ now!) } \mathcal{A} \text{ by a (possibly graded) Lie algebra } \text{End}(V) \text{ of endomorphisms of a suitable vector space } V. \text{ However, this alternative has not been seriously explored yet.}\]

\[\text{Thus, } \mathcal{A} \text{ could be coined ‘quantum spacetime arithmetic’ and the imagined process of building spacetime from such abstract numbers is akin, at least in spirit, to how relativistic spacetime was assembled from abstract digits and a suitable code or ‘algorithm’ for them in [1].}\]
by the very alphabetic character of \( A \) and its alphabetically ordered algebraic structure, this syntactic *lexicographic* process representing the building of spacetime was envisaged to encode the germs of the primordial ‘quantum arrow of time’ in the sense that a primitive ‘temporal directedness’ is already built into the algebraic structure of those quantum spacetime numbers—a basic order or ‘taxis’ inherent in the very rules for the algebraic combinations of the generators of \( A \), as we saw before. In view of the intimate structural similarities between \( A \) and the quaternion division algebra \( \mathbb{H} \) mentioned above, and since the latter are so closely tied to the structure of relativistic spacetime and the best unification between quantum mechanics and relativity that has been achieved so far, namely, the Dirac equation \[6\], we can imagine that \( A \) could be somehow used in the future to represent algebraically a ‘time-directed’ sort of Minkowski spacetime and a time-asymmetric version of the Dirac equation that would appear to be supported rather naturally by the former. However, the quest in this direction is far from its completion.

We would like to close the present paper in the spirit of the last paragraph with a suitable quote from the end of \[5\]:

“In the beginning was the word.
The word became self-referential/periodic.
In the sorting of its lexicographic orders,
The word became topology, geometry and
The dynamics of forms;
Thus were chaos and order
Brought forth together
From the void.”
(from CODA)

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\[47\] For example, in \[4\] Minkowski vectors are represented by hermitian biquaternions, Lorentz transformations by unimodular complex quaternions (essentially, the biquaternion analogues of the elements of \( SL(2, \mathbb{C}) \)—the double covering of the Lorentz group), the 3-generators \( \sigma_i \) of the Pauli spin Lie algebra \( su(2) \) are just the three mutually anticommuting ‘imaginary’ quaternions multiplied by the complex number \( i \) in front (\( i^2 = -1 \)), and, most importantly, the Dirac equation can be derived very simply and entirely algebraically from \( \mathbb{H} \) over \( \mathbb{C} \) (i.e., from biquaternions). Also, as noted in footnote 40, the Lorentzian signature (and even the dimensionality!) of Minkowski spacetime is effectively encoded in (the diagonal of) the multiplication table \( \mathbb{H} \) of the unit quaternions in \( \mathbb{H} \).
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