Terminal Angle Constraint Guidance Law Based on SDRE Method

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Abstract. In this paper, a new guidance law based on SDRE method is designed to solve the problem of terminal angle constraint of a missile by using nonlinear kinematics model and dynamic model of a missile. The constraint of collision angle can be realized by the guidance law under the condition of satisfying the requirement of miss distance of terminal guidance. In addition, the guidance instructions in the form of feedback and the analytic solution of the guidance law are obtained in the two-dimensional plane. Compared with previous terminal guidance methods, the advantage of the method is that it does not depend on the estimation of the time to go. The effectiveness of the proposed guidance law can be verified by digital simulation.

1. Introduction

The main role of guidance system in application is to generate appropriate guidance commands to make the terminal miss distance zero for most guided weapons. However, in many cases, miss distance is not the only tactical and technical indicators. When the targets of attack show great military value, such as command centers, fighter planes in the air, ballistic missile transport vehicles, and refueling aircraft in the air, miss distance constraint and collision angle constraint have to be satisfied simultaneously to ensure a better damaging effect. Therefore, in application of the terminal guidance stage, it still requires the missile to strike the target in a specific direction and attitude based on satisfaction of the minimum of miss distance in order to enhance the damaging effect of the direct strike. Therefore, the guidance problem with collision constraints is emerging in response to this problem. Besides, at the same time domestic and foreign scholars have carried out a lot of research and achieved fruitful results in the past decades.

Kim first proposed the concept of guidance with collision angle constraint in 1973 [1]. After that, many researchers have designed a series of improved proportional guidance methods to meet the terminal angle constraint on the application basis of the proportional guidance method [2-6]. Meanwhile, the synovial control method was gradually applied to solve the terminal guidance problem with collision angle constraint [7-9]. However, these methods often need accurate time to go of missile in their application while the performance of the guidance system will receive a greatly decline that is constrained by the current imprecise measurement system and the measurement error of time to go. In [10], an angle of collision constraint guidance law based on SDRE was designed. However, the
guidance law needs to accurately estimate time to go. Once the information was estimated inaccurately, the guidance performance of the system would be greatly affected.

In this paper, a new guidance law is proposed based on the terminal constraint angle guidance law. In the design, the guidance law does not need to estimate time to go which can greatly improve the possibilities for engineering applications. The main contents of this paper are as follows: firstly, a guidance law is designed for the missile two-dimensional nonlinear particle kinematics model. Especially, the problem for the dependence of time to go have been solved effectively and the terminal accuracy can be guaranteed largely by introducing the state-dependent weight matrix. Then, the two-dimensional guidance law is improved and the collision angle range of the guidance law is greatly enlarged. Finally, the effectiveness and stability of the proposed guidance law have been verified by numerical simulation.

2. Missile mathematical model

The main research tasks of missile guidance system are six related equations which are composed of space position $[x, y, z]^T$ and velocity, trajectory inclination angle and trajectory deflection angle $[V, \theta, \psi]$. Combining the kinematics and dynamics equations of the missile's motion of mass center and the kinematics and dynamics equations of the missile's rotation around the missile's mass center, the following three-dimensional particle model of the missile can be obtained:

$$
\dot{x} = V \cos \theta \cos \psi, \quad \dot{y} = V \sin \theta, \quad \dot{z} = -V \cos \theta \sin \psi, \quad \dot{V} = -\frac{D}{m} - g \sin \theta, \quad \dot{\theta} = \frac{L \cos \gamma_c}{mV} - \frac{g \cos \theta}{V}, \quad \dot{\psi} = -\frac{L \sin \gamma_c}{m \cos \theta}
$$

(1)

Where $m$ is the mass of the missile, $g$ is the acceleration of gravity, $P$ is the thrust of engine, $\gamma_c$ is the tilt angle of the velocity, the total aerodynamic force acting on the missile body is divided into drag $D$, lift $L$ and lateral force $F$, along the velocity coordinate system. From the above formula, $\frac{L \cos \gamma_c}{m}$ and $\frac{L \sin \gamma_c}{m}$ are denoted as $a_y$ and $a_z$, respectively. Meanwhile, $a_y$ and $a_z$ are denoted as normal acceleration and lateral acceleration, respectively. $a_y$ and $a_z$ are perpendicular to the velocity vector of the missile. Therefore, the velocity of the missile cannot be changed while the direction of the velocity can be changed. Then Eq. (1) can be rewritten as:

$$
\dot{x} = V \cos \theta \cos \psi, \quad \dot{y} = V \sin \theta, \quad \dot{z} = -V \cos \theta \sin \psi, \quad \dot{V} = -\frac{D}{m} - g \sin \theta, \quad \dot{\theta} = \frac{a_y - g \cos \theta}{V}, \quad \dot{\psi} = -\frac{a_z}{V \cos \theta}
$$

(2)

3. Design of SDRE guidance law

3.1. Design of two-dimensional guidance law

Assume that the target is not fixed, $(x, y, z)$ and $(x_f, y_f, z_f)$ are the coordinates of the missile and the target, respectively. $(\theta, \psi)$ are the trajectory inclination angle and trajectory deflection angle of the missile, respectively. $(\theta_f, \psi_f)$ are the terminal value of the trajectory inclination angle and the trajectory deflection angle of the missile, respectively. Therefore, when the suitable $a_y$ and $a_z$ are found, the guidance problem with collision angle constraint can be expressed by the following equations:

$$
\lim_{t \to 0} (x, y, z) = (x_f, y_f, z_f), \quad \lim_{t \to 0} (\theta, \psi) = (\theta_f, \psi_f)
$$

(3)

In consideration of the unpredictability of the time to go $t_{go}$ of the missile, then the Eq. (3) is converted into the following equation:

$$
\lim_{y \to 0} (x, z) = (x_f, z_f), \quad \lim_{y \to 0} (\theta, \psi) = (\theta_f, \psi_f)
$$

(4)

The current guidance law is designed with the Eq. (4) as the mission target. When the 3-dimensional guidance problem is simply transformed into two vertical 2-dimensional guidance
problems, the guidance law design can be conducted on the condition of 2-dimensional plane. Considering the case of a longitudinal plane in two dimensions, there is $z = y_c = 0$, then the Eq. (2) can be reduced to the following form:

$$
\dot{x} = V \cos \theta, \dot{y} = V \sin \theta, \dot{V} = -\frac{D}{m} - g \sin \theta, \dot{\theta} = \frac{a_y - g \cos \theta}{V}
$$

The guided target equation can be simplified accordingly as follows:

$$
\lim (x) = x_f, \lim (\theta) = \theta_f
$$

Define a new variable $Y = y_0 - y$, $y_0$ is the initial height of the missile, $Y$ is the height of the missile and “"""" represents the differential operation on the height variable $Y$. The equation is as follows:

$$
x' = \frac{dx}{dY} = \frac{dx}{dt} \frac{dt}{dY} = \frac{dx}{dt} \frac{dt}{dy}
$$

Substituting Eq. (5) into the upper Eq. (7) gives:

$$
x' = -\cot \theta
$$

According to Eq.(8), $\theta$ can be determined by $x'$, which means $\theta$ can obtain a corresponding certain value when $x'$ is determined. Therefore, the terminal value of desired collision angle $\theta$ can be controlled by changing $x'$. Now two new variables $\sigma_1$ and $\sigma_2$ are defined and the corresponding expressions are as follows:

$$
\sigma_1 = x - x_f + \cot \theta_f (Y - Y_f), \sigma_2 = \sigma_1' + \cot \theta_f = -\cot \theta + \cot \theta_f
$$

when the collision occurred, $Y - Y_f$ converges to zero. Eq. (6) will be satisfied if $\sigma_1$ and $\sigma_2$ converge to zero at this time in combination with Eq. (9). Let $\sigma_2$ take the derivative over $Y$ and then the equation is obtained as follows:

$$
\sigma_2' = (-\cot \theta + \cot \theta_f)' = \frac{1}{\sin^2 \theta} \theta'
$$

Let $X = [\sigma_1, \sigma_2]^T$ be a state variable, $\theta'$ be a control variable. Combining (9) and (10), the following state space equation can be obtained:

$$
X' = AX + B\theta'
$$

Where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ \frac{1}{\sin^2 \theta} \end{bmatrix}$. Obviously, $A$ is a linear time-invariant matrix and $B$ is a nonlinear time-varying matrix. And by formula (9):

$$
\theta = \text{arccot(cot } \theta_f - \sigma_2 \text{)}
$$

Obviously, the control matrix $B$ is state-dependent. Considering the problem of the infinite time nonlinear regulator, the value function $J$ can be designed as follows:

$$
J = \frac{1}{2} \int_0^\infty (X'^T Q X + R \theta'^2) dY
$$

Where $Y_0 = 0$ is the initial value of $Y$, $Q$ and $R$ denote the state weight matrix and the control weight matrix, respectively. Let their expressions are as follows:

$$
Q = \begin{bmatrix} q_1^2 & 0 \\ 0 & q_2^2 \end{bmatrix}, R = 1
$$

Now check whether the system can satisfy the using conditions of the control method $SDRE$:

1. The state weight $Q$ is positive semidefinite and the control weight $R$ is positive definite.
2. $f(X) = AX = [\sigma_2, 0]^T$ and $f'(0) = 0$.
3. From $\|B A\| = -\frac{1}{\sin^2 \theta} \neq 0$, we can know that the system is computer controllable for $X$.  

3
(4) The control matrix \( B = \begin{bmatrix} 0 & \frac{1}{\sin^2 \theta} \end{bmatrix} \neq 0 \).

In conclusion, the optimal control law can be rewritten as:
\[
\theta' = -R^{-1}B^T P X = -B^T P X
\]  
(15)

Where \( P = \begin{bmatrix} p_{11} & p_{12} \\
 p_{12} & p_{22} \end{bmatrix} \) satisfies the following equation:
\[
A^T P + P A - P B R^{-1} B P + Q = 0
\]  
(16)

Substituting the expression of \( A \), \( P \), \( B \), \( R \) and \( Q \) into Eq. (16), respectively, the following equation is obtained:
\[
\begin{bmatrix} 0 & 0 \\
 1 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\
 p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\
 p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\
 1 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\
 p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} q_1^2 & 0 \\
 0 & q_2^2 \end{bmatrix} = 0
\]  
(17)

To solve the above equation, the following results can be expressed:
\[
p_{11} = q_1 \sqrt{2q_1 \sin^2 \theta + q_2^2}, \quad p_{12} = q_1 \sin^2 \theta, \quad p_{22} = \sin^2 \theta \sqrt{2q_1 \sin^2 \theta + q_2^2}.
\]  
(18)

Let \( q_1 = q_2^2 \). Substituting Eq. (18) into Eq. (15) gives the control instruction as follows:
\[
\theta' = -(q_1, \sigma_1 + \sigma_2 \sqrt{q_1 (1 + 2 \sin^2 \theta)})
\]  
(19)

\( q_1 \) is designed as a state-dependent matrix, whose expression is selected as follows:
\[
q_1 = \frac{N^4}{1 + 2 \sin^2 \theta}
\]  
(20)

Where \( N \) is an arbitrary constant. It can be seen from the above formula that the value of \( q_1 \) changes with the change of trajectory inclination angle, and the applied weight also changes correspondingly to ensure the guidance accuracy.

Combining Eqs. (19) and (20), the expression of the control input quantity \( \theta' \) can be obtained:
\[
\theta' = -(N^4 \sigma_1 + \sigma_2 N^2)
\]  
(21)

Substituting Eq. (9) into Eq. (21) and taking into account \( Y - Y_f = y_0 - y - (y_0 - y_f) = y_f - y \), the final form of \( \theta' \) can be given as:
\[
\theta' = -(N^4 \frac{x-x_f}{1+2 \sin^2 \theta} + N^4 \frac{y-y_f}{1+2 \sin^2 \theta} \cot \theta_f + N^2 \cot \theta_f - \cot \theta)
\]  
(22)

By substituting Eq. (20) into Eq. (22), the final simplified form of \( \theta' \) is obtained:
\[
\theta' = -(q_1 (x-x_f) + q_2 (y-y_f) \cot \theta_f + N^2 \cot \theta_f - \cot \theta)
\]  
(23)

Then, the actual guidance command \( a_y \) is derived as follows:
\[
\theta' = \frac{d \theta}{d Y} = -\frac{d \theta}{dt} \frac{dt}{dy} = -\frac{a_y - g \cos \theta}{V^2 \sin \theta}
\]  
(24)

The above formula gives:
\[
a_y = g \cos \theta - V^2 \theta' \sin \theta
\]  
(25)

Combining Eqs. (23) and (25) produces the final expression of the guidance instruction \( a_y \) which is expressed as:
\[
a_y = g \cos \theta + V^2 \sin \theta q_1 (x-x_f) + (y-y_f) \cot \theta_f + V^2 N^2 \sin \theta \cot \theta_f - V^2 N^2 \cos \theta
\]  
(26)

The proposed guidance law can make the state variables \( \sigma_1 \) and \( \sigma_2 \) converge to zero at the height information \( Y = Y_f \), so that Eq. (6) can be satisfied.
3.2. Perfection of two-dimensional guidance law

According to Eq. (26), the guidance law cannot be realized when the trajectory inclination angle is expected to be $\theta = 0^\circ$ or $\theta = -180^\circ$. It is because the control singularity would be occurred by Eq. (26) when the desired trajectory inclination angle is $0^\circ$ or $-180^\circ$. In this case, the missile-target reference frame is moved to a new reference coordinate system $O_x - y$. The new reference coordinate system is obtained based on the original reference coordinate system $O - x$ and then rotate a certain angle $\omega_n$ around the target point. Obviously, $\omega_n$ is positive in the case of $\theta = 0^\circ$, and $\omega_n$ is negative in the case of $\theta = -180^\circ$.

The distance between the missile and the target is expressed by $r$, and line-of-sight angle is indicated by $\lambda$. All variables defined in the new reference coordinate have the subscript $n$. The following equation holds:

$$x - x_f = -r \cos \lambda, y - y_f = -r \sin \lambda$$

Where $r$ and $\lambda$ can be solved by the following formula:

$$r = \sqrt{(x - x_f)^2 + (y - y_f)^2}, \lambda = \begin{cases} -\arcsin\left(\frac{y - y_f}{r}\right), & -90^\circ < \lambda \leq 0^\circ \\ \arcsin\left(\frac{y - y_f}{r}\right) - \pi, & -180^\circ \leq \lambda \leq -90^\circ \end{cases}$$

Substituting Eq. (27) into Eq. (26) gives:

$$a_x = g \cos \theta + V^2 \sin \theta \sin \lambda \cos \theta_f + V^2 N^2 \sin \theta \cos \theta_f - V^2 N^2 \cos \theta$$

In the new reference coordinate, the trajectory inclination $\theta$ and the line-of-sight angle $\lambda$ can be described as follows:

$$\theta_n = \theta - \omega_n, \lambda_n = \lambda - \omega_n$$

Combining Eqs. (2) and (30) produces the following final guidance law:

$$a_x = g \cos(\theta - \omega_n) + V^2 \sin(\theta - \omega_n) \sin(\lambda - \omega_n) \lambda \cos(\theta_f - \omega_n) + V^2 N^2 \sin(\theta - \omega_n) \cos(\theta_f - \omega_n) - V^2 N^2 \cos(\theta - \omega_n)$$

It can be seen from the above analysis that the control singularity problem caused by the collision angle $\theta = 0^\circ$ or $\theta = -180^\circ$ can be avoided by the guidance law shown in Eq. (31).

4. Simulation verification

Now the effectiveness of the proposed SDRE guidance law is verified by numerical simulation. The aerodynamic data is derived from a real missile. Set the initial position of the missile at $(x_0, y_0) = (10km, 10km)$. The initial velocity of the missile is chosen to be $V_0 = 300m/s$. The initial position of the target is $(x_f, y_f) = (95km, 35km)$ and the initial velocity of the target is $V_0 = 100m/s$.

After repeated verification, the choice of the guidance parameter $N$ should be more reasonable in the range of $N \in (5, 7)$ to ensure the precision of the terminal of the guidance law. In the simulation, we choose the parameters $N = 6$. The first group of simulations can verify that the missile collides with the target under the constraint of different trajectory inclination angles which are expected to be $\theta = -20^\circ, -40^\circ, -80^\circ$ when the initial trajectory inclination angle $\theta_f$ is chosen to be $\theta_f = 20^\circ$. The simulation results are shown in Figure 1, 2 and 3. In these simulation figures, the red curve represents the moving track of the target, the blue curve represents the missile parameters in the condition of $\theta = -20^\circ$, the yellow curve represents the missile parameters of $\theta = -40^\circ$, and the green curve represents the missile parameters of $\theta = -80^\circ$.

In these figures, it can be seen that the missile can accurately strike the target with desired terminal constraint trajectory inclination angle, and the guidance law can achieve a wide range of collision angles. The results verify the effectiveness of the proposed guidance law. Combined with Figure 2 and
Figure 3, when the missile is in the case of a large terminal constraint angle, large overload may produce during the collision.

In addition, this group of simulation verifies that the missile with different initial trajectory inclination angles collides with the target under the constraint of the same terminal trajectory inclination angle. The initial trajectory inclination angles of the missile are $\theta_0 = 20^\circ, 40^\circ, 80^\circ$.

The terminal trajectory inclination angle is chosen to be $\theta_f = -40^\circ$. The simulation results are shown in Figures 4, 5 and 6. In these figures, the red curve represents the moving track of the target, the blue curve represents the missile parameters of $\theta_0 = 20^\circ$, the yellow curve represents the missile parameters of $\theta_0 = 40^\circ$, and the green curve represents the missile parameters of $\theta_0 = 80^\circ$.

It can be seen from the results that although the initial launch angle is different, the missiles can strike the target from the same terminal impact angle of the constraint, which verifies the robustness of the guidance law to the initial launch angle error of the missile. Combining with Figure 5 and 6, it can be seen that the amplitude of the guidance instruction increases accordingly with the increase of initial launch angle of the missile.

Through the above two groups of simulations, the effectiveness of the guidance law based on different initial and terminal constraint angles is successfully verified.

5. Conclusion
The guidance law based on the terminal angle constraint is studied in this paper. The guidance law can not only satisfy the miss distance, but also meet the terminal angle constraint. Compared with previous terminal guidance methods, the advantage of the method is that it does not depend on the estimation of the time to go, so the guidance error caused by the inaccuracy of the measurement of the time to go can be avoided. Furthermore, the effectiveness and stability of the designed guidance law are verified by theoretical analysis and numerical simulation.
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