Lecture hall $P$-partitions

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We introduce and study $s$-lecture hall $P$-partitions which is a generalization of $s$-lecture hall partitions to labeled (weighted) posets. We provide generating function identities for $s$-lecture hall $P$-partitions that generalize identities obtained by Savage and Schuster for $s$-lecture hall partitions, and by Stanley for $P$-partitions. We also prove that the corresponding $(P,s)$-Eulerian polynomials are real-rooted for certain pairs $(P,s)$, and speculate on unimodality properties of these polynomials.

1. Introduction

Let $s = (s_1, \ldots, s_n)$ be a sequence of positive integers. An $s$-lecture hall partition is an integer sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ satisfying $0 \leq \lambda_1/s_1 \leq \cdots \leq \lambda_n/s_n$. These are generalizations of lecture hall partitions, corresponding to the case when $s = (1, 2, \ldots, n)$, first studied by Bousquet-Mélo and Eriksson [3]. It has recently been made evident that $s$-lecture hall partitions serve as a rich model for various combinatorial structures with interesting generating functions, see [2, 3, 4, 13, 14, 19, 18, 20, 21] and the references therein.

In this paper we generalize the concept of $s$-lecture hall partitions to labeled posets. This constitutes a generalization of Stanley’s theory of $P$-partitions, see [24, Ch. 3.15]. In Section 3 we derive multivariate generating function identities for $s$-lecture hall $P$-partitions, and prove a reciprocity theorem (Theorem 3.9). When $P$ is a naturally labeled chain or an anti-chain, the generating function identities obtained produce results on $s$-lecture hall partitions and signed permutations, respectively (see Section 6). We also introduce and study a $(P,s)$-Eulerian polynomial. In Section 4 we prove that this polynomial is palindromic for sign-graded labeled posets with a specific choice of $s$. In Section 5 we prove that the $(P,s)$-Eulerian polynomial is real-rooted for certain choices of $(P,s)$, and we also speculate on unimodality properties satisfied by these polynomials.

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2. Lecture hall \( P \)-partitions

In this paper a \emph{labeled poset} is a partially ordered set on \( [p] := \{1, \ldots, p\} \) for some positive integer \( p \), i.e., \( P = ([p], \preceq) \), where \( \preceq \) denotes the partial order. We will use the symbol \( \preceq \) to denote the usual total order on the integers. If \( P \) is a labeled poset, then a \( P \)-partition\(^1\) is a map \( f : [p] \to \mathbb{R} \) such that

1. if \( x \prec y \), then \( f(x) \leq f(y) \), and
2. if \( x \prec y \) and \( x > y \), then \( f(x) < f(y) \).

The theory of \( P \)-partitions was developed by Stanley in his thesis and has since then been used frequently in several different combinatorial settings, see [24, 25].

Let \( O(P) = \{ f \in \mathbb{R}^p : f \) is a \( P \)-partition and \( 0 \leq f(x) \leq 1 \) for all \( x \in [p] \} \) be the \textit{order polytope} associated to \( P \). Note that if \( P \) is naturally labeled, i.e., \( x \prec y \) implies \( x < y \), then \( O(P) \) is a closed integral polytope. Otherwise \( O(P) \) is the intersection of a finite number of open or closed half-spaces. Recall that the \textit{Ehrhart polynomial} of an integral polytope \( \mathcal{P} \) in \( \mathbb{R}^p \) is defined for nonnegative integers \( n \) as

\[
i(\mathcal{P}, n) = |n\mathcal{P} \cap \mathbb{Z}^p|,
\]

where \( n\mathcal{P} = \{nx : x \in \mathcal{P}\} \), see [24, p. 497]. For order polytopes we have the following relationship due to Stanley:

\[
\sum_{n \geq 0} i(O(P), n)t^n = \frac{A_P(t)}{(1-t)^{p+1}},
\]

where \( A_P(t) \) is the \( P \)-Eulerian polynomial, which is the generating polynomial of the descent statistic over the set of all linear extensions of \( P \), see [24, Ch. 3.15].

The purpose of this paper is to initiate the study of a lecture hall generalization of \( P \)-partitions. Let \( P \) be a labeled poset and let \( s : [p] \to \mathbb{Z}_+ := \{1, 2, 3, \ldots\} \) be an arbitrary map. We define a \textit{lecture hall \( (P, s) \)-partition} to be a map \( f : [p] \to \mathbb{R} \) such that

\(^1\)What we call \( P \)-partitions are called reverse \( (P, \omega) \)-partitions in [24, 25]. However the theory of \( (P, \omega) \)-partitions and reverse \( (P, \omega) \)-partitions are clearly equivalent.
1. if $x < y$, then $f(x)/s(x) \leq f(y)/s(y)$, and
2. if $x < y$ and $x > y$, then $f(x)/s(x) < f(y)/s(y)$.

Let

$$O(P, s) = \{ f \in \mathbb{R}^P : f \text{ is a } (P, s)-\text{partition and}$$
$$0 \leq f(x)/s(x) \leq 1 \text{ for all } x \in [p] \}$$

be the lecture hall order polytope associated to $(P, s)$. We also let

$$C(P, s) = \{ f \in \mathbb{R}^P : f \text{ is a } (P, s)-\text{partition and}$$
$$0 \leq f(x)/s(x) \text{ for all } x \in [p] \}$$

be the lecture hall order cone associated to $(P, s)$. The $(P, s)$-Eulerian polynomial, $A_{(P,s)}(t)$, is defined by

$$\sum_{n \geq 0} i(O(P, s), n)t^n = \frac{A_{(P,s)}(t)}{(1 - t)^{p+1}}.$$

3. The main generating functions

In this section we derive formulas for the main generating functions associated to lecture hall $(P, s)$-partitions. The outline follows Stanley’s theory of $P$-partitions [24, Ch. 3.15]. We shall see in Section 6 that the special cases when $P$ is naturally labeled chain or an anti-chain automatically produce results on lecture hall polytopes and signed permutations, respectively.

Let $S_p$ denote the symmetric group on $[p]$. If $\pi = \pi_1 \pi_2 \cdots \pi_p \in S_p$ is a permutation written in one-line notation, we let $P_\pi$ denote the labeled chain $\pi_1 \prec \pi_2 \prec \cdots \prec \pi_p$. If $P = ([p], \preceq)$ is a labeled poset, let $L(P)$ denote the set

$$L(P) := \{ \pi \in S_p : \text{if } \pi_i \preceq \pi_j, \text{ then } i \leq j, \text{ for all } i, j \in [p] \},$$

of linear extensions (or the Jordan-Hölder set) of $P$. The following lemma is an immediate consequence of Stanley’s decomposition of $P$-partitions [24, Lemma 3.15.3].

**Lemma 3.1.** If $P$ is a labeled poset and $s : [p] \to \mathbb{Z}_+$, then

$$C(P, s) = \bigsqcup_{\pi \in L(P)} C(P_\pi, s),$$

where $\bigsqcup$ denotes disjoint union.
Let $s : [p] \to \mathbb{Z}_+$. An $s$-colored permutation is a pair $\tau = (\pi, r)$ where $\pi \in \mathfrak{S}_p$, and $r : [p] \to \mathbb{N}$ satisfies $r(\pi_i) \in \{0, 1, \ldots, s(\pi_i) - 1\}$ for all $1 \leq i \leq p$. If $P = ([p], \preceq)$ is a labeled poset, let

$$\mathcal{L}(P, s) = \{ \tau : \tau = (\pi, r) \text{ where } \pi \in \mathcal{L}(P) \text{ and } \tau \text{ is an } s\text{-colored permutation} \}.$$ 

For $f : [p] \to \mathbb{N}$, let $q(f), r(f) : [p] \to \mathbb{N}$ be the unique functions satisfying

$$f(x) = q(f)(x) \cdot s(x) + r(f)(x), \quad \text{where } q(f)(x) \in \mathbb{N} \text{ and } 0 \leq r(f)(x) < s(x),$$

for all $x \in [p]$. Let further

$$F_{(P, s)}(x, y) = \sum_{f \in \mathbb{N}(P, s)} y^{r(f)} x^{q(f)},$$

where $x^r = x_1^{r(1)} x_2^{r(2)} \cdots x_p^{r(p)}$ and $\mathbb{N}(P, s) = C(P, s) \cap \mathbb{N}^p$. We say that $i \in [p - 1]$ is a descent of $\tau = (\pi, r)$ if

$$\begin{cases} 
\pi_i < \pi_{i+1} \text{ and } r(\pi_i)/s(\pi_i) > r(\pi_{i+1})/s(\pi_{i+1}), \text{ or,} \\
\pi_i > \pi_{i+1} \text{ and } r(\pi_i)/s(\pi_i) \geq r(\pi_{i+1})/s(\pi_{i+1}), 
\end{cases}$$

Let

$$D_1(\tau) = \{ i \in [p - 1] : i \text{ is a descent} \}.$$

**Theorem 3.2.** If $P$ is a labeled poset and $s : [p] \to \mathbb{Z}_+$, then

$$F_{(P, s)}(x, y) = \sum_{\tau = (\pi, r) \in \mathcal{L}(P, s)} \frac{\prod_{i \in D_1(\tau)} x_{\pi_{i+1}} \cdots x_{\pi_p}}{\prod_{i \in [p]} (1 - x_{\pi_i} \cdots x_{\pi_p})}. \tag{3.1}$$

**Proof.** By Lemma 3.1 we may assume that $P = P_\pi$ is a labeled chain. Let $f \in \mathbb{N}^p$, and write $f(t) = q(t)s(t) + r(t)$, where $0 \leq r(t) < s(t)$ and $q(t) \in \mathbb{N}$ for all $t \in [p]$. What conditions on $q$ and $r$ guarantee $f \in \mathbb{N}(P, s)$? Suppose $\pi_i < \pi_{i+1}$. Then we need

$$q(\pi_i) + \frac{r(\pi_i)}{s(\pi_i)} = \frac{f(\pi_i)}{s(\pi_i)} \leq \frac{f(\pi_{i+1})}{s(\pi_{i+1})} = q(\pi_{i+1}) + \frac{r(\pi_{i+1})}{s(\pi_{i+1})}. \tag{3.2}$$
If $r(\pi_i)/s(\pi_i) \leq r(\pi_{i+1})/s(\pi_{i+1})$, then (3.2) holds if and only if $q(\pi_i) \leq q(\pi_{i+1})$. If $r(\pi_i)/s(\pi_i) > r(\pi_{i+1})/s(\pi_{i+1})$, then (3.2) holds if and only if $q(\pi_i) < q(\pi_{i+1})$.

Suppose $\pi_i > \pi_{i+1}$. Then we need

\begin{equation}
q(\pi_i) + r(\pi_i)/s(\pi_i) = f(\pi_i)/s(\pi_i) < f(\pi_{i+1})/s(\pi_{i+1}) = q(\pi_{i+1}) + r(\pi_{i+1})/s(\pi_{i+1}).
\end{equation}

If $r(\pi_i)/s(\pi_i) < r(\pi_{i+1})/s(\pi_{i+1})$, then (3.3) holds if and only if $q(\pi_i) \leq q(\pi_{i+1})$. If $r(\pi_i)/s(\pi_i) \geq r(\pi_{i+1})/s(\pi_{i+1})$, then (3.3) holds if and only if $q(\pi_i) < q(\pi_{i+1})$.

Let $\tau = (\pi, r)$, where $r$ is fixed. Then $f = qs + r \in \mathbb{N}(P, s)$ with given (fixed) $r$ if and only if

\begin{equation}
0 \leq q(\pi_1) \leq q(\pi_2) \leq \cdots \leq q(\pi_p),
\end{equation}

where $q(\pi_i) < q(\pi_{i+1})$ if $i \in D_1(\tau)$. Hence $f = qs + r \in \mathbb{N}(P, s)$ if and only if for each $k \in [p]$:

$q(\pi_k) = \alpha_k + |\{i \in D_1(\tau) : i < k\}|$,

where $\alpha_k \in \mathbb{N}$ and $0 \leq \alpha_1 \leq \cdots \leq \alpha_p$. Hence

\[
\sum q \prod_{i=1}^{p} x^{q(\pi_i)} = \sum_{0 \leq \alpha_1 \leq \cdots \leq \alpha_p} x_{\pi_1}^{\alpha_1} \cdots x_{\pi_p}^{\alpha_p} \prod_{i \in D_1(\tau)} x_{\pi_{i+1}} \cdots x_{\pi_p} = \prod_{i \in D_1(\tau)} x_{\pi_{i+1}} \cdots x_{\pi_p} = \prod_{i \in [p]} (1 - x_{\pi_i} \cdots x_{\pi_p}),
\]

where the first sum is over all $q$ satisfying (3.4). The theorem follows. \(\square\)

Let $\mathbb{Z}_+(P, s) = C(P, s) \cap \mathbb{Z}_+^p$ and let

$$F_{(P, s)}^+(x, y) = \sum_{f \in \mathbb{Z}_+(P, s)} y^{r(f)} x^{q(f)}.$$

Let further

$$D_2(\tau) = \begin{cases} D_1(\tau), & \text{if } r(\pi_1) \neq 0, \\ D_1(\tau) \cup \{0\}, & \text{if } r(\pi_1) = 0. \end{cases}$$
Theorem 3.3. If $P$ is a labeled poset and $s : [p] \to \mathbb{Z}_+$, then

$$F^+_{(P,s)}(x,y) = \sum_{\tau = (\pi, r) \in \mathcal{L}(P,s)} y^r \prod_{i \in [p]} x_{\pi_i+1} \cdots x_{\pi_p} \prod_{i \in [p]} (1 - x_{\pi_i} \cdots x_{\pi_p}).$$

Proof. Consider $(P', s')$ where $P'$ is obtained from $P$ by adjoining a least element 0 labeled $p+1$, and $s' : [p+1] \to \mathbb{Z}_+$ is such that $s'$ restricted to $[p]$ agrees with $s$. Let also $s'(p+1) > \max\{s(t) : t \in [p]\}$. Then $f \in \mathbb{N}(P', s')$ if and only if $f|_{[p]} \in \mathbb{N}(P, s)$ and

$$0 \leq \frac{f(p+1)}{s'(p+1)} < \frac{f(x)}{s(x)}, \quad \text{for all } x \in [p].$$

Thus $F^+_{(P,s)}(x,y)$ is obtained from $F^+_{(P', s')}(x,y)$ when we restrict to all $f \in \mathbb{N}(P', s')$ with $f(p+1) = 1$, i.e., $q(p+1) = 0$ and $r(p+1) = 1$, and then shift the indices. Hence $i = 0$ is a descent in $((p+1)\pi_2 \cdots \pi_p, r)$ if and only if $r(\pi_1) = 0$, and the proof follows.

For $f : [p] \to \mathbb{Z}_+$, let $q'(f), r'(f) : [p] \to \mathbb{N}$ be the unique functions satisfying

$$f(x) = q'(f)(x) \cdot s(x) + r'(f)(x), \quad \text{where } q'(f)(x) \in \mathbb{N} \text{ and } 0 < r'(f)(x) \leq s(x),$$

for all $x \in [p]$. Let further

$$G_{(P,s)}(x,y) = \sum_{f \in \mathbb{Z}_+(P,s)} y^{r'(f)} x^{q'(f)}.$$

Let $D_3(\tau)$ be the set of all $i \in [p-1]$ for which

$$\pi_i < \pi_{i+1} \text{ and } (r(\pi_i) + 1)/s(\pi_i) > (r(\pi_{i+1}) + 1)/s(\pi_{i+1}), \text{ or,}$$

$$\pi_i > \pi_{i+1} \text{ and } (r(\pi_i) + 1)/s(\pi_i) \geq (r(\pi_{i+1}) + 1)/s(\pi_{i+1}).$$

Theorem 3.4. If $P$ is a labeled poset and $s : [p] \to \mathbb{Z}_+$, then

$$G_{(P,s)}(x,y) = \sum_{\tau = (\pi, r) \in \mathcal{L}(P,s)} y^{r+1} \prod_{i \in D_3(\tau)} x_{\pi_i+1} \cdots x_{\pi_p} \prod_{i \in [p]} (1 - x_{\pi_i} \cdots x_{\pi_p}).$$
where \( \mathbf{1} = (1, 1, \ldots, 1) \) is the all ones vector.

**Proof.** The proof is almost identical to that of Theorem 3.2, and is therefore omitted.

For \( n \in \mathbb{N} \), let

\[
N_{\leq n}(P, s) = \{ f \in \mathbb{N}(P, s) : f(x)/s(x) \leq n \text{ for all } x \in [p] \},
\]

and let

\[
F(P, s)(x, y; n) = \sum_{f \in N_{\leq n}(P, s)} y^r(f) x^q(f).
\]

The polynomials \( F^+(P, s)(x, y; n) \) and \( G(P, s)(x, y; n) \) are defined analogously over \( \{ f \in \mathbb{Z}_+(P, s) : f(x)/s(x) \leq n \text{ for all } x \in [p] \} \). Let also

\[
N_{< n}(P, s) = \{ f \in \mathbb{N}(P, s) : f(x)/s(x) < n \text{ for all } x \in [p] \},
\]

and

\[
F'(P, s)(x, y; n) = \sum_{f \in N_{< n}(P, s)} y^r(f) x^q(f).
\]

For \( \tau = (\pi, r) \in \mathcal{L}(P, s) \), define

\[
D(\tau) = \begin{cases} 
D_1(\tau), & \text{if } r(\pi_p) = 0, \\
D_1(\tau) \cup \{p\}, & \text{if } r(\pi_p) > 0,
\end{cases}
\]

and

\[
D_1(\tau) = \begin{cases} 
D_2(\tau), & \text{if } r(\pi_p) = 0, \\
D_2(\tau) \cup \{p\}, & \text{if } r(\pi_p) > 0.
\end{cases}
\]

**Proposition 3.5.** If \( P \) is a labeled poset and \( s : [p] \to \mathbb{Z}_+ \), then

\[
\sum_{n \geq 0} F(P, s)(x, y; n) t^n = \sum_{\tau = (\pi, r) \in \mathcal{L}(P, s)} y^r \prod_{i \in D(\tau)} x_{\pi_{i+1}} \cdots x_{\pi_{p}} \frac{t^{|D(\tau)|}}{1 - t}, \tag{3.5}
\]

\[
\sum_{n \geq 0} F'(P, s)(x, y; n) t^n = \sum_{\tau = (\pi, r) \in \mathcal{L}(P, s)} y^r \prod_{i \in D_1(\tau)} x_{\pi_{i+1}} \cdots x_{\pi_{p}} \frac{t^{|D_1(\tau)|+1}}{1 - t}, \tag{3.6}
\]
\[
\sum_{n \geq 0} F^+_{(P,s)}(x,y;n) t^n = \sum_{\tau = (\pi,r) \in \mathcal{L}(P,s)} y^\tau \prod_{i \in [p]} \frac{\prod_{i \in D_4(\tau)} x_{\pi_{i+1}} \cdots x_{\pi_p}}{(1 - x_{\pi_i} \cdots x_{\pi_p} t)^{t|D_4(\tau)|}} \frac{1}{1 - t},
\]

(3.7)

\[
\sum_{n \geq 0} G_{(P,s)}(x,y;n) t^n = \sum_{\tau = (\pi,r) \in \mathcal{L}(P,s)} y^{\tau+1} \prod_{i \in [p]} \frac{\prod_{i \in D_3(\tau)} x_{\pi_{i+1}} \cdots x_{\pi_p}}{(1 - x_{\pi_i} \cdots x_{\pi_p} t)^{t|D_3(\tau)|+1}} \frac{1}{1 - t}.
\]

(3.8)

**Proof.** For (3.5) consider \((P',s')\) where \(P'\) is obtained from \(P\) by adjoining a greatest element \(\hat{1}\) labeled \(p+1\), and \(s' : [p+1] \to \mathbb{Z}_+\) restricted to \([p]\) agrees with \(s\), while \(s'(p+1) = 1\). If we set \(x_{p+1} = t\), then

\[
\sum_{n \geq 0} F_{(P,s)}(x,y;n) t^n = F_{(P',s')},
\]

and

\[
\mathcal{L}(P',s') = \{(\pi_1 \cdots \pi_p(p+1), r') : (\pi_1 \cdots \pi_p, r'|P) \in \mathcal{L}(P,s) \text{ and } r'(p+1) = 0\}.
\]

The identity (3.5) follows by noting that \(i = p\) is a descent of \((\pi_1 \cdots \pi_p(p+1), r')\) if and only if \(r(\pi_p)/s(\pi_p) > r'(p+1)/s'(p+1) = 0\).

The other identities follows similarly. For example (3.6) follows by considering \((P',s')\) where \(P'\) is obtained from \(P\) by adjoining a greatest element \(\hat{1}\) labeled 0 (and then relabel so that \(P'\) has ground set \([p+1]\)). For (3.8) consider again \((P',s')\), where \(P'\) is obtained from \(P\) by adjoining a greatest element \(\hat{1}\) labeled \(p+1\), and \(s'\) is defined as for the case of (3.5). Note that since \(r'(p+1) = 1\) we have \(q'(p+1) = n - 1\) if \(f(p+1) = n\). This explains the shift by one in the exponent on the right hand side of (3.8), i.e., \(|D_3(\tau)| + 1\).

If \(q\) is a variable, let \([0]_q := 0\) and \([n]_q := 1 + q + q^2 + \cdots + q^{n-1}\) for \(n \geq 1\). For the special case of (3.5) when \(P\) is an anti-chain we acquire the following corollary, which is a generalization of [1, Theorem 5.23].
Corollary 3.6. If $P$ is an anti-chain and $s : [p] \to \mathbb{Z}_+$, then

$$\sum_{n \geq 0} \prod_{i=1}^{p} (x_i^n + [n]x_i[s(i)]y_i) t^n = \sum_{\tau = (\pi, r) \in \mathcal{L}(P, s)} y^r \prod_{i \in [p]} \frac{x_{\pi+1} \cdots x_{\pi_p}}{(1 - x_{\pi_i} \cdots x_{\pi_p} t)^{1-t}}$$

Proof. Let $P$ be an anti-chain and let $s : [p] \to \mathbb{Z}_+$. Consider $f \in \mathbb{N} \leq n(P, s)$. Since $P$ is an anti-chain, $f(i)$ and $f(j)$ are independent for all $1 \leq i < j \leq p$, and the only restriction is $0 \leq f(i) \leq ns(i)$ for all $1 \leq i \leq p$. We write $f(i) = s(i)q(i) + r(i)$, where $0 \leq r(i) < s(i)$. Then $f \in \mathbb{N} \leq n(P, s)$ if and only if either $q(i) = n$ and $r(i) = 0$, or $0 \leq q(i) \leq n - 1$ and $0 \leq r(i) \leq s(i) - 1$. Hence

$$\sum_{f \in \mathbb{N} \leq n(P, s)} y^{q(f)} x^{q(f)} = \prod_{i=1}^{p} (x_i^0[s(i)]y_i + \cdots + x_i^{n-1}[s(i)]y_i + x_i^n)$$

$$= \prod_{i=1}^{p} (x_i^n + [n]x_i[s(i)]y_i).$$

The corollary now follows from (3.5). \qed

Note that the special case of (3.5) when $P$ is a naturally labeled chain gives an analogue (by an appropriate change of variables) to one of the main results in [20], see Theorem 5 therein. From (3.5) we also get an interpretation of the Eulerian polynomial $A_{(P, s)}(t)$. For $\tau \in \mathcal{L}(P, s)$, let $\text{des}_s(\tau) = |D(\tau)|$.

Corollary 3.7. If $P$ is a labeled poset and $s : [p] \to \mathbb{Z}_+$, then

$$A_{(P, s)}(t) = \sum_{\tau \in \mathcal{L}(P, s)} t^{\text{des}_s(\tau)}.$$ 

The next corollary follows from Proposition 3.5 by setting the $x$- and $y$-variables to 1.

Corollary 3.8. If $P$ is a labeled poset and $s : [p] \to \mathbb{Z}_+$, then

$$\sum_{\tau \in \mathcal{L}(P, s)} t^{|D_4(\tau)|} = \sum_{\tau \in \mathcal{L}(P, s)} t^{|D_4(\tau)|+1},$$
and if \( s(x) = 1 \) for all minimal elements \( x \) in \( P \), then

\[
A_{(P,s)}(t) = \sum_{\tau \in \mathcal{L}(P,s)} t^{|D(\tau)|} = \sum_{\tau \in \mathcal{L}(P,s)} t^{|D_3(\tau)|}.
\]

Let \( P = ([p], \preceq) \) be a labeled poset. For \( i \in [p] \), let \( i^* = p + 1 - i \), and let \((P^*, s^*)\) be defined by \( P^* = ([p], \preceq^*) \) with

\[
i \preceq j \text{ in } P \quad \text{if and only if} \quad i^* \preceq^* j^* \text{ in } P^*, \quad \text{for all } i, j \in [p],
\]

and \( s^*(i^*) = s(i) \) for all \( i \in [p] \). The poset \( P^* \) is called the dual of \( P \).

**Theorem 3.9** (Reciprocity theorem). If \( P \) is a labeled poset and \( s : [p] \to \mathbb{Z}_+ \), then

\[
G_{(P^*, s^*)}(x^*, y^*) = (-1)^p \frac{y_1^{s(1)} \cdots y_p^{s(p)}}{x_1 \cdots x_p} F_{(P,s)}(x^{-1}, y^{-1}),
\]

where \( x^* = (x_p, x_{p-1}, \ldots, x_1) \) and \( x^{-1} = (x_1^{-1}, \ldots, x_p^{-1}) \).

**Proof.** For \( \tau = (\pi, r) \in \mathcal{L}(P,s) \), let \( \tau^* = (\pi^1 \pi^2_1 \cdots \pi^p_1, r^*) \) where \( r^*(i^*) = s(i) - 1 - r(i) \) for all \( i \in [p] \). Clearly the map \( \tau \mapsto \tau^* \) is a bijection between \( \mathcal{L}(P,s) \) and \( \mathcal{L}(P^*, s^*) \). Moreover if \( i \in [p-1] \), then \( i \in D_3(\tau) \) if and only if

\[
\begin{cases}
\pi_i < \pi_{i+1} \text{ and } (r(\pi_i) + 1)/s(\pi_i) > (r(\pi_{i+1}) + 1)/s(\pi_{i+1}), \text{ or }, \\
\pi_i > \pi_{i+1} \text{ and } (r(\pi_i) + 1)/s(\pi_i) \geq (r(\pi_{i+1}) + 1)/s(\pi_{i+1}),
\end{cases}
\]

if and only if

\[
\begin{cases}
\pi_i^* > \pi_{i+1}^* \text{ and } r^*(\pi_i^*)/s^*(\pi_i^*) < r^*(\pi_{i+1}^*)/s^*(\pi_{i+1}^*), \text{ or },
\pi_i^* < \pi_{i+1}^* \text{ and } r^*(\pi_i^*)/s^*(\pi_i^*) \leq r^*(\pi_{i+1}^*)/s^*(\pi_{i+1}^*)
\end{cases}
\]

if and only if \( i \in [p-1] \setminus D_1(\tau^*) \). Thus

\[
D_3(\tau) = [p-1] \setminus D_1(\tau^*) \quad \text{and} \quad D_1(\tau) = [p-1] \setminus D_3(\tau^*),
\]

for all \( \tau \in \mathcal{L}(P,s) \). Now

\[
F_{(P,s)}(x, y) = \sum_{\tau \in \mathcal{L}(P,s)} y^r \prod_{i \in D_1(\tau)} x_{\pi_i+1} \cdots x_{\pi_p} \prod_{i \in [p]} (1 - x_{\pi_i} \cdots x_{\pi_p}).
\]
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\[
\sum_{\tau \in \mathcal{L}(P,s)} y^s_{\tau} \frac{\prod_{i \in [p]} x_{\pi_i+1} \cdots x_{\pi_p}}{\prod_{i \in [p]} (1 - x_{\pi_i} \cdots x_{\pi_p})}
\]

\[
= \sum_{\tau \in \mathcal{L}(P,s)} y^s_{\tau} \prod_{i \in [p]} x_{\pi_i+1} \cdots x_{\pi_p} \prod_{i \in [p]} (1 - x_{\pi_i} \cdots x_{\pi_p})
\]

\[
= (-1)^p y^s_{1} \cdots y^s_{p} \sum_{\tau \in \mathcal{L}(P,s)} (y^s_{\tau})^{-r^* + 1} \prod_{i \in [p]} x_{\pi_i+1} \cdots x_{\pi_p} \prod_{i \in [p]} (1 - x_{\pi_i} \cdots x_{\pi_p})
\]

from which the theorem follows. \(\square\)

**Remark 3.1.** Theorem 3.9 generalizes the reciprocity theorem in [4] which follows as the special case when \(P\) is a naturally labeled chain.

### 4. Sign-ranked posets

Let \(P = \{1 \prec 2 \prec \cdots \prec p\}\) be a naturally labeled chain, and let \(s(i) = i\) for all \(i \in [p]\). Savage and Schuster [20, Lemma 1] proved that \(A_{(P,s)}(t)\) is equal to the Eulerian polynomial

\[
A_p(t) = \sum_{\pi \in S_p} t^{\text{des}(\pi)},
\]

where \(\text{des}(\pi) = |\{i \in [p] : \pi_i > \pi_{i+1}\}|\). Recall that a polynomial \(g(t)\) is *palindromic* if \(t^N g(1/t) = g(t)\) for some integer \(N\). It is well known that \(A_p(t)\) is palindromic (in fact \(t^{p-1} A_p(1/t) = A_p(t)\)). The same is known to be true for the \(P\)-Eulerian polynomial of any naturally labeled graded poset, see [24, Corollary 3.15.18], and more generally for \(P\)-Eulerian polynomials of so called sign-graded labeled posets [10, Corollary 2.4]. We shall here generalize these results to \((P,s)\)-Eulerian polynomials.

Recall that a pair of elements \((x, y)\) taken from a labeled poset \(P\) is a *covering relation* if \(x \prec y\) and \(x \prec z \prec y\) for no \(z \in P\). Let \(\mathcal{E}(P)\) denote the set of covering relations of \(P\). If \(P\) is a labeled poset define a
function \( \epsilon : \mathcal{E}(P) \to \{-1, 1\} \) by

\[
\epsilon(x, y) = \begin{cases} 
1, & \text{if } x < y, \text{ and} \\
-1, & \text{if } x > y.
\end{cases}
\]

Sign-graded (labeled) posets, introduced in [10], generalize graded naturally labeled posets. A labeled poset \( P \) is sign-graded of rank \( r \), if

\[
\sum_{i=1}^{k} \epsilon(x_{i-1}, x_{i}) = r
\]

for each maximal chain \( x_0 < x_1 < \cdots < x_k \) in \( P \). A sign-graded poset is equipped with a well-defined rank-function, \( \rho : P \to \mathbb{Z} \), defined by

\[
\rho(x) = \sum_{i=1}^{k} \epsilon(x_{i-1}, x_{i}),
\]

where \( x_0 < x_1 < \cdots < x_k = x \) is any unrefinable chain, \( x_0 \) is a minimal element and \( x_k = x \). Hence a naturally labeled poset is sign-graded if and only if it is graded. A labeled poset \( P \) is sign-ranked if for each maximal element \( x \in P \), the subposet \( \{ y \in P : y \preceq x \} \) is sign-graded. Note that each sign-ranked poset has a well-defined rank function \( \rho : P \to \mathbb{Z} \). Thus a naturally labeled poset is sign-ranked if and only if it is ranked.

**Theorem 4.1.** Let \( P \) be a sign-ranked labeled poset and suppose its rank function attains non-negative values only. Let \( s(x) = \rho(x) + 1 \) for each \( x \in [p] \), and define \( u : \mathbb{N}(P, s) \to \mathbb{Z}^P \) by \( u(f)(x^*) = f(x) + \rho(x) \). Then \( u : \mathbb{N}_{\leq n}(P, s) \to \mathbb{N}_{< n+1}(P^*, s^*) \) is a bijection for each \( n \in \mathbb{N} \).

**Proof.** We first prove \( u : \mathbb{N}(P, s) \to \mathbb{N}(P^*, s^*) \). Note that \( f \) is a \((P, s)\)-partition if and only if

1. if \((x, y) \in \mathcal{E}(P)\), then \( f(x)/s(x) \leq f(y)/s(y) \), and
2. if \((x, y) \in \mathcal{E}(P)\) and \( \epsilon(x, y) = -1 \), then \( f(x)/s(x) < f(y)/s(y) \).

Hence it suffices to consider covering relations when proving that \( u : \mathbb{N}(P, s) \to \mathbb{N}(P^*, s^*) \).

Let \( f \in \mathbb{N}(P, s) \). Suppose \( y \) covers \( x \) and \( \epsilon(x, y) = 1 \). Then \( f(x)/s(x) \leq f(y)/s(y) \) and \( s(x) < s(y) \), and thus

\[
\frac{u(f)(x^*)}{s^*(x^*)} = \frac{f(x) + s(x) - 1}{s(x)} \leq \frac{f(y)}{s(y)} + 1 - \frac{1}{s(x)} < \frac{f(y)}{s(y)} + 1 - \frac{1}{s(y)} = \frac{u(f)(y^*)}{s^*(y^*)},
\]
Thus $\eta$ and $u$ as desired. Note that $Hence (s(y) + 1)f(y) - s(y)f(x)$ is a positive integer, so that
\[
\frac{f(y)}{s(y)} - \frac{f(x)}{s(y) + 1} \geq \frac{1}{s(y)(s(y) + 1)},
\]
as desired. Note that $u(f)$ is nonnegative since it is increasing and $u(f)(x^*) = f(x)$ when $x^*$ is a minimal element in $P^*$. Hence $u(f) \in \mathbb{N}(P^*, s^*)$.

Let $\eta : \mathbb{N}(P^*, s^*) \rightarrow \mathbb{Z}^P$ be defined by $\eta(g)(x) = g(x^*) - \rho(x) = g(x^*) + \rho^*(x^*)$, where $\rho^*$ is the rank function of $P^*$. Clearly $\eta : \mathbb{N}(P^*, s^*) \rightarrow \mathbb{N}(P, s)$ by the exact same arguments as above. Thus $u^{-1} = \eta$ and $u : \mathbb{N}(P, s) \rightarrow \mathbb{N}(P^*, s^*)$ is a bijection.

Now $u(f)(x^*)/s^*(x^*) = f(x)/s(x) + (s(x) - 1)/s(x) < n + 1$ if $f \in \mathbb{N}_{\leq n}(P, s)$ and $x \in P$, so that $u : \mathbb{N}_{\leq n}(P, s) \rightarrow \mathbb{N}_{\leq n+1}(P^*, s^*)$ for each $n \in \mathbb{N}$.

On the other hand if $g \in \mathbb{N}_{\leq n+1}(P^*, s^*)$, then $g(x^*) = q(x^*)(\rho(x) + 1) + r(x^*)$ where $0 \leq q(x^*) \leq n$ and $0 \leq r(x^*) \leq \rho(x)$. Hence
\[
\frac{\eta(g)(x)}{s(x)} = \frac{g(x^*)}{\rho(x) + 1} - \frac{\rho(x)}{\rho(x) + 1} \leq n + \frac{r(x^*)}{\rho(x) + 1} - \frac{\rho(x)}{\rho(x) + 1} \leq n.
\]
Thus $\eta : \mathbb{N}_{\leq n+1}(P^*, s^*) \rightarrow \mathbb{N}_{\leq n}(P, s)$ which proves the theorem.

**Theorem 4.2.** If $P$ is a sign-ranked labeled poset with nonnegative rank function $\rho$ and $s = \rho + 1$, then
\[
A_{(P,s)}(t) = t^{\rho-1}A_{(P,s)}(t^{-1})
\]
and
\[
(-1)^{P}i(O(P, s), -t) = i(O(P, s), t - 2).
\]
Proof. By (3.5), (3.6) and Theorem 4.1

\[ A(P, s)(t) = \sum_{\tau \in \mathcal{L}(P, s)} t^{D(\tau)} = \sum_{\tau^* \in \mathcal{L}(P^*, s*)} t^{D_1(\tau^*)}. \]

The first part of the theorem now follows from (3.9) and (3.10). The second part follows from e.g., [24, Lemma 3.15.11]. \qed

5. Real-rootedness and unimodality

The Neggers-Stanley conjecture asserted that for each labeled poset \( P \), the Eulerian polynomial \( A(P, t) \) is real-rooted. Although the conjecture is refuted in its full generality [9, 26], it is known to hold for certain classes of posets [6, 27]. Moreover, when \( P \) is sign-graded, then the coefficients of \( A(P, t) \) form a unimodal sequence [10, 16]. It is natural to ask for which pairs \((P, s)\)

(a) is \( A(P, s)(t) \) real-rooted?

(b) do the coefficients of \( A(P, s)(t) \) form a unimodal sequence?

We first address (a). Suppose \( P = ([p], \preceq_P) \), \( Q = ([q], \preceq_Q) \) and \( R = ([p + q], \preceq_R) \) are labeled posets such that \([p + q]\) is the disjoint union of the two sets \([u_1 < u_2 < \cdots < u_p]\) and \([v_1 < v_2 < \cdots < v_q]\), and \( x \preceq_R y \) if and only if either

- \( x = u_i \) and \( y = u_j \) for some \( i, j \in [p] \) with \( i \preceq_P j \), or
- \( x = v_i \) and \( y = v_j \) for some \( i, j \in [q] \) with \( i \preceq_Q j \).

We say that \( R \) is a disjoint union of \( P \) and \( Q \) and write \( R = P \sqcup Q \). Moreover if \( s_P : [p] \to \mathbb{Z}_+ \) and \( s_Q : [q] \to \mathbb{Z}_+ \), then we define \( s_{P \sqcup Q} : [p + q] \to \mathbb{Z}_+ \) as the unique function satisfying \( s_{P \sqcup Q}(u_i) = s_P(i) \) and \( s_{P \sqcup Q}(v_j) = s_Q(j) \).

Proposition 5.1. If the polynomials \( A(P, s_P)(t) \) and \( A(Q, s_Q)(t) \) are real-rooted, then so is the polynomial \( A(P \sqcup Q, s_P \sqcup s_Q)(t) \).

Proof. Clearly

\[ i((P \sqcup Q, s_P \sqcup s_Q), t) = i(O(P, s_P), t) \cdot i(O(Q, s_Q), t), \]

so the proposition follows from [28, Theorem 0.1]. \qed

It was proved in [22] that if \( P = \{1 < 2 < \cdots < p\} \) and \( s : [p] \to \mathbb{Z}_+ \) is arbitrary, then \( A(P, s)(t) \) is real-rooted. In Theorem 5.2 below we generalize this result to ordinal sums of anti-chains. If \( P = (X, \preceq_P) \) and \( Q = (Y, \preceq_Q) \) are posets on disjoint ground sets, then the ordinal sum, \( P \oplus Q = (X \cup Y, \preceq) \), is the poset with relations
Let $f$ and $g$ be two real-rooted polynomials in $\mathbb{R}[t]$ with positive leading coefficients. Let further $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_m$ be the zeros of $f$ and $g$, respectively. If
\[
\ldots \leq \alpha_2 \leq \beta_2 \leq \alpha_1 \leq \beta_1
\]
we say that $f$ is an interleaver of $g$ and we write $f \ll g$. We also let $f \ll 0$ and $0 \ll f$. We call a sequence $F_n = (f_i)_{i=1}^n$ of real-rooted polynomials interlacing if $f_i \ll f_j$ for all $1 \leq i < j \leq n$. We denote by $\mathcal{F}_n$ the family of all interlacing sequences $(f_i)_{i=1}^n$ of polynomials and we let $\mathcal{F}^{\pm}_n$ be the family of $(f_i)_{i=1}^n \in \mathcal{F}_n$ such that $f_i$ has nonnegative coefficients for all $1 \leq i \leq n$.

To avoid unnecessary technicalities we here redefine a labeled poset to be a poset $P = (S, \preceq)$, where $S$ is any set of positive integers. Thus $\mathcal{L}(P)$ is now the set of rearrangements of $S$ that are also linear extensions of $P$.

Equip $X(P,s) := \{(k,x) : x \in P \text{ and } 0 \leq k < s(x)\}$ with a total order defined by $(k,x) < (\ell,y)$ if $k/s(x) < \ell/s(y)$, or $k/s(x) = \ell/s(y)$ and $x < y$. For $\gamma \in X(P,s)$, let
\[
A^\gamma_{(P,s)}(t) = \sum_{\tau = (\pi,r) \in \mathcal{L}(P,s) \atop (r(\pi_1),\pi_1) = \gamma} t^{\des(\tau)}.
\]

**Theorem 5.2.** Suppose $P = A_{p_1} \oplus \cdots \oplus A_{p_m}$ is an ordinal sum of anti-chains, and let $s : P \to \mathbb{Z}_+$ be a function which is constant on $A_{p_i}$ for $1 \leq i \leq m$. Then $\{A^\gamma_{(P,s)}(t)\}_{\gamma \in X}$, where $X = X(P,s)$, is an interlacing sequence of polynomials.

In particular $A^\gamma_{(P,s)}(t)$ and $A^\gamma_{(P,s)}(t)$ are real-rooted for all $\gamma \in X$.

**Proof.** The proof is by induction over $m$. Suppose $m = 1$, $p_1 = n$, $A_n$ is the anti-chain on $[n]$, and $s(A_n) = \{s\}$. We prove the case $m = 1$ by induction over $n$. If $n = 1$ we get the sequence $1, t, t, \ldots, t$ which is interlacing. Otherwise if $\gamma = (k, \pi_1)$, then
\[
A^\gamma_{(A_n,s)}(t) = \sum_{\kappa < \gamma} tA^\kappa_{(A_{n-1},s')}^\gamma(t) + \sum_{\kappa \geq \gamma} A^\kappa_{(A_{n-1},s')}^\gamma(t),
\]
where $s'$ is $s$ restricted to $A_{n-1}$. This recursion preserves the interlacing property, see [22, Theorem 2.3] and [11], which proves the case $m = 1$ by induction.
Suppose $m > 1$. The proof for $m$ is again by induction over $p_1 = n$. If $p_1 = 1$, then

$$A^\gamma_{(P,s)}(t) = \sum_{\kappa < \gamma} tA^\kappa_{(P',s')}(t) + \sum_{\kappa > \gamma} A^\kappa_{(P',s')}(t),$$

where $P' = A_2 \oplus \cdots \oplus A_m$, and where $s'$ is the restrictions to $P'$. Hence the case $p_1 = 1$ follows by induction (over $m$) since this recursion preserves the interlacing property, see [22, Theorem 2.3].

The case $m > 1$ and $p_1 > 1$ follows by induction over $p_1$ just as for the case $m = 1$, $n > 1$.

Hence $\{A^\gamma_{(P,s)}(t)\}_\gamma$ is an interlacing sequence, and thus

$$A_{(P,s)}(t) = \sum_\gamma A^\gamma_{(P,s)}(t),$$

is real-rooted by e.g., [22, Theorem 2.3].

Next we address (b). A palindromic polynomial $g(t) = a_0 + a_1 t + \cdots + a_n t^n$ may be written uniquely as

$$g(t) = \sum_{k=0}^{\lfloor d/2 \rfloor} \gamma_k(g) t^k (1 + t)^{d-2k},$$

where $\{\gamma_k(g)\}_{k=0}^{\lfloor d/2 \rfloor}$ are real numbers. If $\gamma_k(g) \geq 0$ for all $k$, then we say that $g(t)$ is $\gamma$-positive, see [11]. Note that if $g(t)$ is $\gamma$-positive, then $\{a_i\}_{i=0}^{n}$ is a unimodal sequence, i.e., there is an index $m$ such that $a_0 \leq \cdots \leq a_m \geq a_{m+1} \geq \cdots \geq a_n$.

**Conjecture 5.3.** Suppose $P$ is a sign-ranked labeled poset with nonnegative rank function $\rho$ and $s = \rho + 1$, then $A_{(P,s)}(t)$ is $\gamma$-positive.

**Remark 5.1.** Let $P$ be a sign-ranked labeled poset with a rank function $\rho$ with values only in $\{0, 1\}$, and let $s = \rho + 1$. Following the proof of [10, Theorem 4.2], with the use of Theorem 5.2, it follows that Conjecture 5.3 holds for $(P,s)$. We omit the technical details in recalling the proof here.

If $P$ is a naturally labeled ranked poset and $s = \rho + 1$, then $O(P,s)$ is a closed integral polytope and $A_{(P,s)}(t)$ is the so called $h^*$-polynomial of $O(P,s)$. If the following conjecture is true, then the coefficients of $A_{(P,s)}(t)$ form a unimodal sequence by a powerful theorem of Bruns and Römer [8, Theorem 1].
Conjecture 5.4. Suppose $P$ is a naturally labeled ranked poset, and let $s = \rho + 1$. Then $O(P,s)$ (or some related polytope with the same Ehrhart polynomial) has a regular and unimodular triangulation.

Remark 5.2. Evidence for Conjectures 5.3 and 5.4 is provided by [23] where it is proved that the coefficients of $A_{\tau}(P,s)(t)$ form unimodal sequence whenever $P$ is a naturally labeled ranked poset with a least element, and $s = \rho + 1$.

6. Applications

In this section we derive some applications of the generating function identities obtained in Section 3. If $\alpha = (\alpha_1, \ldots, \alpha_p)$ is a sequence, let $|\alpha| = \alpha_1 + \cdots + \alpha_p$. For $\tau = (\pi, r) \in L(P,s)$, let

- $\text{comaj}(\tau) = \sum_{i \in D(\tau)} p - i$, and
- $lhp(\tau) = |r| + \sum_{i \in D(\tau)} s(\pi_i + 1) + \cdots + s(\pi_p)$

Theorem 6.1. If $P$ is a labeled poset and $s : [p] \to \mathbb{Z}_+$, then

$$
\sum_{n \geq 0} \left( \sum_{f \in \mathbb{N} \leq n(P,s)} q^{|r(f)|} u^{|q(f)|} \right) t^n = \frac{\sum_{\tau \in L(P,s)} q^{|r|} u^{\text{comaj}(\tau)} t^{\text{des}(\tau)}}{\prod_{i=0}^{p} (1 - u^i t)}.
$$

Proof. Set $x_i = u$ and $y_i = q$ for all $1 \leq i \leq p$ in (3.5). Then

$$
\sum_{\tau \in L(P,s)} y^{|r|} \prod_{i \in D(\tau)} x_{\pi_{i+1}} \cdots x_{\pi_p} \frac{q^{|D(\tau)|}}{1 - t} = \sum_{\tau \in L(P,s)} q^{|r|} u^{\text{comaj}(\tau)} t^{\text{des}(\tau)} \prod_{i \in [p]} (1 - tu^{p+1-i})(1 - t) \prod_{i \in [p]} (1 - tu^i)(1 - t).
$$

The theorem follows. \qed
Theorem 6.2. If \( P \) is a labeled poset and \( s : [p] \rightarrow \mathbb{Z}_+ \), then

\[
\sum_{n \geq 0} \left( \sum_{f \in \mathcal{N}_{\leq n}(P, s)} q^{|f|} \right) t^n = \sum_{\tau \in \mathcal{L}(P, s)} \frac{q^{\text{hlp}(\tau)} t^{\text{des}(\tau)}}{\prod_{i \in [p]} \left( 1 - tq^{\sum_{j=1}^p s(i_j)} \right)} (1 - t).
\]

Proof. Set \( x_i = q^{s(i)} \) and \( y_i = q \) for all \( 1 \leq i \leq p \) in (3.5). \( \square \)

Corollary 6.3. If \( P \) is an anti-chain and \( s : [p] \rightarrow \mathbb{Z}_+ \), then

\[
\sum_{n \geq 0} \left( \sum_{f \in \mathcal{N}_{\leq n}(P, s)} q^{|f|} \right) t^n = \sum_{\tau \in \mathcal{L}(P, s)} \frac{q^{\text{r}\cdot\text{comaj}(\tau)} t^{\text{des}(\tau)}}{\prod_{i=0}^p (1 - u^{i}t)}.
\]

Proof. The corollary follows from Theorem 6.1 and Corollary 3.6. \( \square \)

The wreath product of \( \mathfrak{S}_p \) with a cyclic group of order \( k \) has elements

\[ \mathbb{Z}_k \wr \mathfrak{S}_p = \{(\pi, r) : \pi \in \mathfrak{S}_p \text{ and } r : [p] \rightarrow \mathbb{Z}_k \}. \]

The elements of \( \mathbb{Z}_k \wr \mathfrak{S}_p \) are often thought of as \( r \)-colored permutations. We may identify \( \mathbb{Z}_k \wr \mathfrak{S}_p \) with \( \mathcal{L}(P, s) \) where \( P \) is an anti-chain on \( [p] \) and \( s(i) = k \) for all \( k \in [p] \). For \( \tau = (\pi, r) \in \mathbb{Z}_k \wr \mathfrak{S}_p \) define

\[ \text{fmaj}(\tau) = |r| + k \cdot \text{comaj}(\tau). \]

Note that \( \text{hlp}(\tau) \) agrees with \( \text{fmaj}(\tau) \) when \( s = (k, k, \ldots, k) \).

Below we derive a Carlitz formula for \( \mathbb{Z}_k \wr \mathfrak{S}_p \) first proved by Chow and Mansour in [12].

Corollary 6.4. For positive integers \( p \) and \( k \),

\[
\sum_{n \geq 0} [kn + 1] q^{|f|} t^n = \sum_{\tau \in \mathcal{L}(P, s)} \frac{t^{\text{des}(\tau)} q^{\text{ fmaj}(\tau)}}{\prod_{i=0}^p \left( 1 - t q^{k_i} \right)}.
\]


Proof. Let \( s = (k, k, \ldots, k) \) and set \( u = q^k \) in (6.3). Then
\[
\prod_{i=1}^{p} (u^n + [n]_q [s(i)]_q) = (q^{nk} + [n]_q [k]_q)^p
\]
\[
= (q^{nk} + \frac{q^{kn} - 1}{q^k - 1} q^k - 1)^p
\]
\[
= [nk + 1]_q^p.
\]
The right hand side follows since \( s(i) = k \) for all \( 1 \leq i \leq p \), and thus we sum over all \( \tau \in \mathbb{Z}_k \wr S_p \).

Theorem 6.1. The definition of \( \text{fmaj} \) above differs from the definition of the flag major index \( \text{fmaj}_r \) in [12]. By the change in variables \( q \to q^{-1} \) and \( t \to tq^p \) and by noting that \( [kn + 1]_q^p t^n \) is invariant under this change of variables we find that the two flag major indices have the same distribution.

Corollary 6.5. For positive integers \( p \) and \( k \),
\[
\sum_{n \geq 0} \prod_{i=1}^{p} (1 + n[k]_q) t^n = \sum_{\tau \in \mathbb{Z}_k \wr S_p} q_1^{\tau_1} q_2^{\tau_2} \cdots q_p^{\tau_p} t^{\text{des}_s(\tau)}(1 - t)^{p+1}.
\]

Proof. Let \( s = (k, k, \ldots, k) \) and set \( x_i = 1 \) for all \( 1 \leq i \leq p \) in the equation displayed in Corollary 3.6.

Remark 6.2. Note that when \( q_i \geq 0 \) for all \( 1 \leq i \leq p \), the polynomial
\[
n \mapsto \prod_{i=1}^{p} (1 + n[k]_q)
\]
has all its zeros in the interval \([-1, 0]\). By an application of [28, Theorem 0.1] it follows that the polynomial
\[
\sum_{\tau \in \mathbb{Z}_k \wr S_p} q_1^{\tau_1} q_2^{\tau_2} \cdots q_p^{\tau_p} t^{\text{des}_s(\tau)}
\]
is real-rooted in \( t \). This generalizes [7, Theorem 6.4], where the case \( k = 2 \) was obtained.
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