Bulk–Boundary Correspondence for Sturmian Kohmoto-Like Models

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Abstract. We consider one-dimensional tight binding models on $\ell^2(\mathbb{Z})$ whose spatial structure is encoded by a Sturmian sequence $(\xi_n)_n \in \{a, b\}^\mathbb{Z}$. An example is the Kohmoto Hamiltonian, which is given by the discrete Laplacian plus an on-site potential $v_n$ taking value 0 or 1 according to whether $\xi_n$ is $a$ or $b$. The only non-trivial topological invariants of such a model are its gap labels. The bulk–boundary correspondence we establish here states that there is a correspondence between the gap label and a winding number associated with the edge states, which arises if the system is augmented and compressed onto half-space $\ell^2(\mathbb{N})$. This has been experimentally observed with polaritonic waveguides. A correct theoretical explanation requires, however, first a smoothing out of the atomic motion via phason flips. With such an interpretation at hand, the winding number corresponds to the mechanical work through a cycle which the atomic motion exhibits on the edge states.

1. Introduction

Inspired by the research on topological insulators [8,13,16,20,26,27,34,38], there is growing experimental effort to search for topological boundary resonances in many kinds of wave-supporting media, such as photonic [15,49,58] and phononic [22,43,46,47] crystals, as well as plasmonic [53] systems. A more recent and quite vigorous effort of the community is to go beyond the periodic table of topological insulators and superconductors [21,48,51] and to try to detect such topological boundary resonances in aperiodic systems, such as almost periodic [19,23,25,39,45], quasi-crystalline [1,9,11,35,37,54–57] and amorphous patterns [42]. This is now experimentally feasible because, with the new wave-supporting media mentioned above, one has nearly perfect control over the design of the system, something which is not at all the case for the electronic systems.
Quite generally, the bulk–boundary correspondence principle relates the topological invariants of the bulk to topological invariants of the boundary of a physical system. Since the boundary invariants are carried by boundary states, this relation explains the emergence of robust boundary spectrum in systems with non-trivial bulk–boundary correspondence. This has long been understood for the quantum Hall effect \[ 17,31,32,50 \] and, while the bulk–boundary principle is relatively well understood for the almost periodic cases \[ 45 \], it has never been rigorously formulated for the quasi-crystalline cases. Here we deal exclusively with the one-dimensional quasi-crystalline case, as, for instance, the Kohmoto model \[ 6,7,24 \], and, as we shall see, the bulk–boundary correspondence is quite subtle. One of the basic results about the topology of quasi-crystals is that the phason degree of freedom lives in a Cantor set \[ 14,36 \], which is a totally disconnected space. In contradistinction, for almost periodic systems, the phason lives on a circle. While that Cantor set differs from the circle only on a set of measure zero, this difference fundamentally changes the topology of both the bulk and boundary for the quasi-crystalline systems. Indeed, while the bulk gap labels remain unchanged, they no longer can be interpreted in terms of Chern numbers as done for almost periodic systems \[ 23,45 \]. Furthermore, the topological structure of the boundary states is quite different; in particular, any unitary operator constructed from the boundary states is stably homotopic to the identity. Hence, we are facing a situation where the bulk–boundary principle is trivial in the sense that all topological bulk invariants, provided by the bulk gap labels, are sent to the unique trivial boundary invariant. This is quite unfortunate because the one-dimensional quasi-crystalline systems are particularly interesting as their bulk energy spectrum is expected to be a Cantor set of zero measure \[ 6 \]; hence, the spectral gaps abound and, as we shall see, all existing gaps have non-trivial labels.

Recent experiments \[ 1,55 \] with polaritonic waveguides structured like Fibonacci chains have, nevertheless, reported interesting findings. In \[ 55 \], the resonance spectrum of the cavity was measured and its bulk gaps were labelled in agreement with the gap-labelling theorem \[ 4,5 \]. Furthermore, in \[ 1 \], by coupling the cavity with its mirror image, localized modes were created at the centre of the mirror symmetry and their frequencies were plotted against the phason. From these plots, a boundary winding number was read off and shown to coincide with the bulk gap labels. However, as we shall see, the set of boundary (or Dirichlet) eigenvalues for all different values of the phason has zero measure and, as a consequence, the boundary spectrum always displays gaps. As such, the boundary spectral patches plotted in \[ 1 \] need to be connected by imaginary lines in order to see the winding numbers. Such lines, unfortunately, are not justified by the experiments, and the winding numbers depend on the choice of these lines, which is not unique at all.

The purpose of our work is to reconcile these observations with the triviality of the boundary topology and to give a mathematically rigorous definition of the winding numbers observed in \[ 1 \], as well as to provide a rigorous explanation of the equality between the bulk gap labels and these winding numbers.
This is based on the idea of Denjoy [18] to augment the Cantor set to a circle. Specifically, we work with models which have a quasi-periodic potential that is encoded by a Sturmian sequence of parameter $\theta$, proving the following statement (Theorem 8.1): let $I$ be the integrated density of states below a gap $[E_0, E_1]$ of the bulk Hamiltonian of Kohmoto type. There is a unique integer $N$ such that $I - N \in [0, 1)$. Then $(N - I)/\theta$ is the winding number of the boundary eigenvalues in $[E_0, E_1]$ of the augmented system restricted to a half-line, as they move around the augmented Cantor set. We establish that this winding number is related to a physical response coefficient resulting from a change in the potential due to the phason motion, a characteristic feature of quasi-crystals. This is similar but seemingly not identical to the response coefficient associated with the pressure on the boundary [28, 33].

Our proofs are based on algebraic topological methods. They involve the $C^*$-algebras of the bulk, half-space and boundary physical observables for the Kohmoto-type models. The framework we use to prove the bulk–boundary correspondence is the $K$-theory of these and related $C^*$-algebras. While this might appear to be heavy machinery, we do not know at present of any other approach, as the underlying system is aperiodic. On the other hand, the $K$- and $C^*$-algebraic approach to aperiodic solid-state systems proposed by Jean Bellissard [4] is by now well known and our $K$-theory arguments are quite standard, following essentially the ideas of [30, 31].

Our paper is organized as follows. Section 2 collects results from symbolic dynamical systems which allow one to study continuity properties of spectra of Schrödinger operators and underlie the non-commutative geometry of such systems. In particular, the techniques developed in [2, 3] are adapted to the case with a boundary and continuity of the spectra for families of half-space Hamiltonians is established in Theorem 2.3 and Corollary 2.4. Section 3 introduces three dynamical systems naturally associated with Sturmian sequences, namely 1) the original Sturmian subshift $(\Xi, \mathbb{Z})$ whose space $\Xi$ is a Cantor set and whose sequences have finite local complexity, 2) the augmented subshift $(\tilde{\Xi}, \mathbb{Z})$ whose space $\tilde{\Xi}$ is topologically a circle and which contains the original subshift as its unique minimal component, and 3) an approximation system $(\Xi_{\epsilon}, \mathbb{Z})$ whose space $\Xi_{\epsilon}$ is also topologically a circle, but whose sequences have infinite local complexity. The main new results of Sect. 3 are about the convergence (in spectrum) of the Schrödinger operators associated with $(\Xi_{\epsilon}, \mathbb{Z})$ to those of the augmented system $(\tilde{\Xi}, \mathbb{Z})$ (Lemma 3.1) and, in particular, the continuity of the Dirichlet eigenvalues for the augmented system in the intercept parameter (as we call the phason degree of freedom) (see Corollary 4.1, which is based on the general Theorem 2.3). This continuity is essential for the definition of the winding number which we give in Sect. 4. Section 5 reports numerical experiments for both full- and half-space Kohmoto models, which illustrate the issues raised at the beginning of this section and also show how our proposed solution works. More precisely, direct evidence is provided that the $K$-theoretic labels for the bulk gaps correlate with the well-defined spectral flow of the boundary states of the augmented models. Section 6 introduces the $C^*$-algebras canonically associated with the dynamical systems introduced in
Sect. 3 and establishes various relations between them. Section 7 computes the $K$-theories of these algebras and the final proof of the bulk–boundary correspondence, and the physical interpretation of the boundary invariant is given in Sect. 8.

2. Preliminaries from Symbolic Dynamics

In this section we give a brief introduction to families of strongly pattern equivariant Schrödinger operators associated with symbolic dynamical systems. We recall a result from [2,3] on the continuity of spectra, which is adapted to patterns with a boundary in Theorem 2.3. This enables a statement about the continuity of the Dirichlet eigenvalues.

Consider a compact metric space $X$ together with a homeomorphism $\alpha : X \rightarrow X$. Iterating the homeomorphism defines an action of $\mathbb{Z}$ on $X$, i.e. a group homomorphism $\mathbb{Z} \ni n \mapsto \alpha^n \in \text{Homeo}(X)$. This is also referred to as a dynamical system and denoted $(X, \mathbb{Z})$. A subsystem $(Y, \mathbb{Z})$ of $(X, \mathbb{Z})$ is given by a closed subset $Y \subseteq X$ which is invariant under the action. The orbit of an arbitrary subset $Y \subset X$ is the subset \( \{ \alpha^n(y) | y \in Y, n \in \mathbb{Z} \} \subseteq X \), and we denote by $O(Y)$ the closure of the orbit of $Y$. The dynamical system is called minimal if the orbit of any point $x \in X$ is a dense subset of $X$, that is, $O(x) = X$ for any $x \in X$. Every dynamical system contains a minimal subsystem. Two minimal subsystems either are equal or do not share a common point, and for this reason, the minimal subsystems are also called minimal components.

Of importance below will be the universal dynamical system $(X_u, \mathbb{Z})$ associated with $(X, \mathbb{Z})$ [2,3], which is constructed as follows. Let $I(X)$ be the space of closed subspaces of $X$ which are invariant under the action. We equip $I(X)$ with the Hausdorff topology; that is, two closed subsets $X_1, X_2$ of $X$ are at most Hausdorff distance $\epsilon$ apart if within distance $\epsilon$ of each point of $X_1$ lies a point of $X_2$ and vice versa. Now the space of the universal dynamical system is

$$X_u = \{(Y, y) \in I(X) \times X | y \in Y\}$$

with subspace topology of the product topology on $I(X) \times X$. Clearly, $X_u$ is compact and metrizable. $\alpha(Y, y) = (Y, \alpha(y))$ defines the $\mathbb{Z}$-action on $X_u$.

Let $A \subset \mathbb{R}$ be a compact subset and $A^\mathbb{Z}$ be the space of two-sided infinite sequences with values in $A$ equipped with the product topology. This topology is metrizable, and we choose to work with the following metric:

$$d(x, y) = \inf \{ \epsilon > 0 | \forall |k| \leq \epsilon^{-1} : |x_k - y_k| \leq \epsilon \}$$  \hspace{1cm} (2.1)

$\mathbb{Z}$ acts on sequences by left shift, $\alpha(x)_n = x_{n+1}$, and this action is continuous and preserves $A^\mathbb{Z}$. $(A^\mathbb{Z}, \mathbb{Z})$ is a symbolic dynamical system.

We denote by $A^N$ the sequences of length $N$ with values in $A$. In analogy to symbolic dynamics, we call these sequences also the (allowed) words of length $N$. We put on $A^N$ the product topology.

**Definition 2.1.** A sliding block code of range $r \in \mathbb{N}$ over $A$ is a continuous function $b : A^{2r+1} \rightarrow \mathbb{C}$. Such a function extends to a function on $A^\mathbb{Z}$ by
reading from \( x \in A^\mathbb{Z} \) only the block of length \( 2r + 1 \) around 0, i.e. the word \( x_{-r}, \ldots, x_r \). A function \( f : \mathbb{Z} \to \mathbb{C} \) is called strongly pattern equivariant for \( x \in A^\mathbb{Z} \) if there exists a sliding block code \( b : A^N \to \mathbb{C} \) such that \( f(n) = (b \circ \alpha^n)(x) \).

Given a sliding block code \( b \) of range \( r \), we obtain, for every \( x \in A^\mathbb{Z} \), a strongly pattern equivariant function \( b_x : \mathbb{Z} \to \mathbb{C} \) by setting
\[
\tilde{b}_x(n) = b(\alpha^n(x)) = b(x_{n-r}, \ldots, x_{n+r}). \tag{2.2}
\]
A family of Schrödinger operators \( \{H_x\}_{x \in A^\mathbb{Z}} \) is called strongly pattern equivariant if there is a finite subset \( S \subset \mathbb{Z}^+ \) and for each \( k \in S \) a sliding block code \( b_k \) such that
\[
H_x = \sum_{k \in S} (\tilde{b}_k)_x T^k + h.c. \tag{2.3}
\]
where \( T \) is the left translation operator on \( \ell^2(\mathbb{Z}) \), \( T\psi(n) = \psi(n + 1) \), and \( (\tilde{b}_k)_x \) act as multiplication operators on the same Hilbert space. As usual, \( h.c. \) stands for the Hermitian conjugate. The largest value of \( S \) is the range of the operator. A simple example is given by the model
\[
H_x \psi(n) = \psi(n - 1) + \psi(n + 1) + v_x(n)\psi(n) \tag{2.4}
\]
where \( v = \tilde{b}_0 \) is a strongly pattern equivariant function defined by a sliding block code \( b_0 : A^1 \to \mathbb{R} \) of range 0. The original Kohmoto model \([6,24]\) is of the above form, with \( A = \{a, b\} \) and \( b_0(a) = 0, b_0(b) = 1 \).

Strongly pattern equivariant families of Schrödinger operators are covariant, \( T H_x T^{-1} = H_{\alpha(x)} \). Note that there is always a uniform (in \( x \)) upper bound for the norm of the \( H_x \). We can restrict such families to subsystems \( (X, \mathbb{Z}) \subset (A^\mathbb{Z}, \mathbb{Z}) \) where \( X \) is closed and shift invariant. We then denote by \( H_X \) the restriction of \( \{H_x\}_{x \in A^\mathbb{Z}} \) to points \( x \in X \) and \( \text{spec}(H_X) = \bigcup_{x \in X} \text{spec}(H_x) \). It is natural to view the latter as elements of \( \mathcal{K}(\mathbb{R}) \), the space of compact subsets of the real line equipped with the Hausdorff topology \([3]\).

The map \( X \ni x \mapsto H_x \) is strongly continuous. Indeed, as the distance between \( x \) and \( x' \) gets closer they coincide more and more on larger pieces and hence, when applied to a compactly supported vector near the origin, \( H_x \) and \( H_{x'} \) yield the same result. It follows that \( \text{spec}(H_x) \) contains \( \text{spec}(H'_{x'}) \) for all \( x' \in O(x) \) and hence is equal to \( \text{spec}(H_{O(x)}) \). In particular, if \( (X, \mathbb{Z}) \) is minimal then all \( H_x, x \in X \) have the same spectrum.

**Theorem 2.2** \([2,3]\). Let \( \{H_x\}_{x \in A^\mathbb{Z}} \) be a family of strongly pattern equivariant Schrödinger operators. The map \( \Sigma : I(A^\mathbb{Z}) \to \mathcal{K}(\mathbb{R}) \),
\[
\Sigma(X) = \text{spec}(H_X)
\]
is continuous.

**Proof.** For finite \( A \) this is Corollary 4.3.14 of \([2]\). For compact \( A \) the result follows with the same methods as every sliding block code defines a continuous function \( b \) on \( A^\mathbb{Z} \) by \( b(Y, y) = b(y) \). The result is then a consequence of Thm. 2 of \([3]\). \( \square \)
We now turn to the situation with a boundary, which is formalized as follows. We add a large real number \( z \) to the alphabet \( \mathcal{A} \), \( \hat{\mathcal{A}} = \mathcal{A} \cup \{ z \} \) and then construct the following shift space. If \( X \) is a closed invariant subspace of \( \mathcal{A}^\mathbb{Z} \) and \( r \in \mathbb{N} \), then we let \( X^+_r \) be the set of sequences \( x^+_r \in \hat{\mathcal{A}}^\mathbb{Z} \) of the following form: there is \( x \in X \) such that

\[
(x^+_r)_n = \begin{cases} 
x_n & \text{if } n \geq -r, \\
z & \text{otherwise.}
\end{cases}
\]

\( X^+_r \) is a closed subspace of \( \hat{\mathcal{A}}^\mathbb{Z} \), but it is not shift invariant. Note that \( O(x^+_r) \) contains the constant sequence \( x_n = z \) and the set \( M_0 \) containing only this sequence is a minimal component. If \( X \) is minimal then \( O(x^+_r) = \{ \alpha^n(x^+_r) | n \in \mathbb{Z} \} \cup X \cup M_0 \). This can be seen as follows: if \( y \in O(x^+_r) \) lies in a minimal component then it is either an accumulation point of the forward, or of the backward orbit. But all accumulation points of the forward orbits lie in \( X \), whereas the backward orbit has one single accumulation point, namely the constant sequence \( x_n = z \). Last but not least, let us remark that the map \( x \mapsto x^+_r \) from \( X \) to \( X^+_r \) is continuous and surjective.

The map \( p : \hat{\mathcal{A}}^1 \to \mathbb{C} \), \( p(\omega) = 1 \) if \( \omega \neq z \) and else 0 is a sliding block code. We use this to extend the family of strongly pattern equivariant Hamiltonians to the larger space \( \hat{\mathcal{A}}^\mathbb{Z} \),

\[
H_{x,s} = (p \circ \alpha^{-s})_x \left( \sum_{k \in S} (\tilde{b}_k)_x T^k \right) + (1 - (p \circ \alpha^{-s})_x)z + h.c., \quad s \in \mathbb{N}.
\]

Here the sliding block codes \( b_k \) are extended to \( \hat{\mathcal{A}} \) by 0 on words which contain \( z \).

**Theorem 2.3.** Let \( (X, \mathbb{Z}) \) be a minimal subsystem of \( (\mathcal{A}^\mathbb{Z}, \mathbb{Z}) \). The map \( X^+_r \ni x^+_r \mapsto \text{spec}(H_{x^+_r,s}) \in \mathcal{K}(\mathbb{R}) \) is continuous.

**Proof.** We show that \( O(y^+_r) \) and \( O(y'^+_r) \) are close if \( y^+_r \) and \( y'^+_r \) are close in \( X^+_r \). Let \( x \in O(y^+_r) \). We have \( d(x, O(y'^+_r)) = \inf \{ d(x, x') | x' \in O(y'^+_r) \} \). If \( x \) is in a minimal component of \( O(y^+_r) \) then \( d(x, O(y'^+_r)) = 0 \), because \( O(y^+_r) \) and \( O(y'^+_r) \) have the same minimal components. So let \( x \) not be in a minimal component, and hence \( x = \alpha^n(y^+_r) \) for some \( n \in \mathbb{Z} \).

Suppose that \( d(y^+_r, y'^+_r) < \epsilon \) and \( |(y'^+_r)_k - (y^+_r)_k| < \epsilon \) for all \( |k| < \epsilon^{-1} \). If \( 0 \leq n \leq \frac{1}{2}\epsilon^{-1} \) then \( d(\alpha^n(y^+_r), \alpha^n(y'^+_r)) < 2\epsilon \). If \( n \geq \frac{1}{2}\epsilon^{-1} \) then \( d(\alpha^n(y^+_r), \alpha^n(y'^+_r)) < 2\epsilon \) and since \( y \in X \subset O(y'^+_r) \) we see that, in both cases, \( d(\alpha^n(y^+_r), O(y'^+_r)) < 2\epsilon \). Finally, if \( n \) is negative then \( d(\alpha^n(y^+_r), \alpha^n(y'^+_r)) \leq d(y^+_r, y'^+_r) \). This shows that the Hausdorff distance between \( O(y^+_r) \) and \( O(y'^+_r) \) is bounded by twice the distance between \( y^+_r \) and \( y'^+_r \).

By Theorem 2.2 the map \( \Sigma : I(\hat{\mathcal{A}}^\mathbb{Z}) \to \mathcal{K}(\mathbb{R}) \) is continuous. We apply this to the subspace \( \{ O(x^+_r) | x^+_r \in X^+_r \} \subset I(\hat{\mathcal{A}}^\mathbb{Z}) \) to conclude with the above that \( x^+_r \mapsto \text{spec}(H_{O(x^+_r),s}) \) is continuous. We saw that \( \text{spec}(H_{O(x^+_r),s}) = \text{spec}(H_{x^+_r,s}) \).

\( \square \)
For \( x \in X \) we define \( \hat{H}_x \) to be the compression of \( H_x \) to the right half-space, i.e.
\[
\hat{H}_x : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}), \quad \hat{H}_x = \Pi^* H_x \Pi,
\]
with \( \Pi \) the usual partial isometry from \( \ell^2(\mathbb{N}) \) to \( \ell^2(\mathbb{Z}) \). In the case of strongly pattern equivariant Schrödinger operators \( \hat{H}_x \) is the restriction of \( \left( \sum_{k \in S} (b_k)_x T^k \right) + h.c. \) to \( \ell^2(\mathbb{N}) \).

**Corollary 2.4.** Let \((X, \mathbb{Z})\) be a minimal subsystem of \((A^\mathbb{Z}, \mathbb{Z})\) and \( \{H_x\}_{x \in X} \) a strongly pattern equivariant family of Schrödinger operators. Then the map \( X \ni x \mapsto \text{spec}(\hat{H}_x) \in \mathcal{K}(\mathbb{R}) \) is continuous.

**Proof.** Recall that the map \( x \mapsto x + r \) is continuous and take \( R \in \mathbb{N} \) greater than the maximum range of sliding blocks in \( H_x \). Then \( H_{x_R} = \hat{H}_x \oplus z(I - \Pi \Pi^*) \); hence, \( \text{spec}(H_{x_R}) = \text{spec}(\hat{H}_x) \cup \{z\} \). If we take \( z \) larger than a common bound on the norm of the \( \hat{H}_x \), then its contribution to the spectrum is well separated from \( \text{spec}(\hat{H}_x) \). The latter is then continuous in \( x \), which follows from the continuity of \( \text{spec}(H_{x_R}) \) w.r.t. \( x_R \). \( \square \)

### 3. Dynamical Systems and Models Associated with Sturmian Sequences

In this section Sturmian sequences and their associated dynamical systems are discussed in the framework of the cut-and-project formalism. Their associated spaces are totally disconnected and an augmentation is used to generate dynamical systems over the circle, whose minimal components coincide with the original dynamical systems. Smooth approximations of the augmentations are also introduced, and relations between these three classes of dynamical systems are established.

Sturmian sequences can be defined as the repetitive aperiodic sequences \( x \in \{a, b\}^\mathbb{Z} \) which have minimal word complexity function; namely, there are exactly \( n + 1 \) distinct words of length \( n \) in \( x \). In the seminal work by Morse and Hedlund [40, 41], various other characterizations of these sequences have been given and here we make substantial use of their geometric characterization by means of the cut-and-project formalism, illustrated in Fig. 1. Consider the integer lattice \( \mathbb{Z}^2 \) in \( \mathbb{R}^2 \), and parametrize the anti-diagonal \( D \) of the unit cell \([0, 1]^2 \subset \mathbb{R}^2 \) by \([0, 1] : [0, 1] \ni \varphi \mapsto i(\varphi) := \varphi e_2 + (1 - \varphi)e_1 \in D \). Let \( \theta \in [0, 1] \) be an irrational and \( L_\theta \) be the line in \( \mathbb{R}^2 \) which passes through the origin \( 0 \in \mathbb{R}^2 \) and the point on \( D \) parametrized by \( \theta \). Consider the orthogonal projection \( \pi^\perp \) onto the orthocomplement of \( L_\theta \), and let \( W \) be \( -\pi^\perp([0, 1]^2) \). These are the data of the cut-and-project scheme. To obtain a Sturmian sequence with intercept \( \phi \in [0, 1) \) consider the intersection of \( L_\phi := L_\theta + i(\phi - \theta) \) with \( \mathbb{Z}^2 + W \). This pattern of points of \( L_\phi \) can be ordered, as \( L_\phi \) is a line, and so is given by a sequence \( w(\phi) = (w_n(\phi))_{n \in \mathbb{Z}} \) of points. Each of these points can be uniquely specified by their \( \varphi_n \) coordinate on \( D \). Denote by \( NS \) the set of
Figure 1. The cut-and-project scheme. \(a\) A one-dimensional quasi-periodic point pattern (the dots 0, 1, 2, \ldots) is generated by projecting the points of \(\mathbb{Z}^2 \cap ([0,1]^2 + L_\phi)\) orthogonally onto the line \(L_\phi\). \(b\) When wrapped on \(\mathbb{R}^2/\mathbb{Z}^2\), the same point pattern is generated by the intersection of \(L_\phi\) (which now winds on the torus) with the transversal \(W\) shown as a blue line (dashed part excluded). The figure also illustrates the notation introduced and used in the text, in particular, the parameters \(l_0 = \sqrt{1+\theta^2}/1+\theta\) and \(\gamma = 1-\theta/\sqrt{2(1+\theta^2)}\) appearing in (3.1).

For \(\phi\) for which \(L_\phi\) does not intersect a boundary point of \(\mathbb{Z}^2 + W\). It turns out that, for all \(\phi \in NS\), the difference vectors \(w_n - w_{n-1}\) take only two different values \([40,41]\); we can distinguish them by their length which we denote by \(a\) and \(b\) with \(a < b\). Defining \(\xi_n\) to be the length of \(w_n - w_{n-1}\), we obtain in this case a sequence \(\xi = (\xi_n)_{n \in \{a, b\}}^\mathbb{Z}\) which has the desired minimal word complexity. To note its dependence on the intercept we also write \(\xi(\phi)\).

For \(\phi \in NS\) and with the notation from Fig. 1, we can give the formula:

\[
w_n(\phi) = n l_0 + \gamma (\varphi_n - \chi(\varphi_n - \theta)), \quad \varphi_n = \{ \phi + n \theta \}, \quad n \in \mathbb{Z}. \tag{3.1}
\]

Here \(\{ r \}\) denotes the fractional part of \(r \in \mathbb{R}\), \(\chi\) the indicator function on \(\mathbb{R}^+\). Note that the function \(r \mapsto \{ r \} - \chi(\{ r \} - \theta)\) is 0 on \(r = 0\) and on \(r = 1\) and has a discontinuity at \(r = \theta\). When \(\theta = \frac{3-\sqrt{5}}{2}\), \(b/a\) becomes the golden ratio and \(\xi(\phi)\) describes a Fibonacci sequence.

The Sturmian sequences obtained in the above geometric way, for fixed \(\theta\) but varying intercept \(\phi\) (but such that \(L_\phi\) does not intersect the boundary of \(\mathbb{Z}^2 + W\)), form a shift-invariant non-closed subset of \(\{a, b\}^\mathbb{Z}\). From a topological point of view, it is most interesting to study what has to be added to this system to get a complete dynamical system. There are several options.

(i) Identifying the end points of \([0,1]\) to obtain a circle, \(NS\) becomes a dense set of \(S^1\) and we refer to its points as the non-singular values for \(\phi\). One easily sees that its complement of the so-called singular values is \(\theta \mathbb{Z}\) modulo 1. One way to obtain a completion of the dynamical system \((NS, \mathbb{Z})\) is to complete \(NS\) in the Euclidean topology to obtain all of \(S^1\). The Sturmian sequence with intercept \(\phi\) shifted by one unit (to the left) corresponds to the sequence with intercept \(\phi + \theta\). The completed system
is therefore the rotation action by $\theta$ on the circle $S^1$. It is minimal as we assume $\theta$ to be irrational.

(ii) From the point of view of symbolic dynamics it is natural to complete the shift-invariant subset of sequences $\xi(\phi)$ with intercept $\phi \in NS$ in the topology of $\{a, b\}^\mathbb{Z}$. The result is, by definition, the Sturmian subshift with parameter $\theta$. We will denote it by $(\Xi, \mathbb{Z})$. It is a minimal system. But its space $\Xi$ is totally disconnected, as $\{a, b\}^\mathbb{Z}$ is a Cantor set. We can describe $\Xi$ in a geometric way by means of limits. Indeed, let $\phi$ be a singular value. Then we can approach it from the left or from the right by sequences of non-singular values. While on $S^1$ the limits are the same, $\xi(\phi^+) := \lim_{\phi \to \phi^+} \xi(\phi)$ will differ from $\xi(\phi^-) := \lim_{\phi \to \phi^-} \xi(\phi)$ by exactly one word of length 2: the first can be obtained from the second by flipping exactly one pair $ab$ to $ba$. We call such Sturmian sequences also singular; they are characterized by the fact that they contain a flipping pair, that is a pair $(ba$ or $ab$) which can be flipped so that the result remains a Sturmian sequence.

Geometrically, we can think of $\Xi$ as the cut up circle: whenever $\phi$ is singular we replace it by its two half-sided limits $\phi^+$ and $\phi^-$. This can be done topologically by means of inverse limits: consider a chain of finite subsets $S_k \subset S_{k+1}$ such that $\bigcup_{k \in \mathbb{N}} S_k$ consists of all singular points. At step $k$ take out the points of $S_k \setminus S_{k-1}$ and complete the individual components by adding boundary points. In the limit we have disconnected the circle along $\theta \mathbb{Z}$.

(iii) If, instead of replacing a singular point $\phi$ by its two half-sided limits $\phi^+$ and $\phi^-$, we replace it by a closed interval $[0, 1]$ in the above construction, we obtain the augmented system $(\tilde{\Xi}, \mathbb{Z})$. It can be parametrized as follows: it contains $NS$ and for each singular $\phi$ we parametrize the added in interval by $(\phi, t)$, $t \in [0, 1]$. $\tilde{\Xi}$ contains $\Xi$ as a closed subspace where, for each singular $\phi$, $\phi^-= (\phi, 0)$ and $\phi^+ = (\phi, 1)$.

Let us give this a symbolic interpretation. If we would construct a sequence of points $(w_n)$ as above, but for a singular choice of the intercept, then there would be exactly one position $n$ at which $w_n - w_{n-1}$ has a third length $c = b-a$, and both $w_{n+1} - w_n$ and $w_{n-1} - w_{n-2}$ would have length $a$. Therefore, the sequence $(w_n - w_{n-1})_n$ would not be a Sturmian sequence. However, if we take out either $w_n$ or $w_{n-1}$ then the length of the difference vectors form a Sturmian sequence. (The above-described limiting procedure can be interpreted in making this choice.) The augmented Sturmian sequence can now be symbolically understood by saying that instead of taking out one of the two points one replaces the two by a convex combination $tw_n + (1-t)w_{n-1}$, $t \in [0, 1]$. $(\phi, t)$ therefore corresponds to the sequence $\xi(\phi, t) \in [a, b]^\mathbb{Z}$ where $\xi_n(\phi, t) = w_n(\phi, t) - w_{n-1}(\phi, t)$ with

$$w_n(\phi, t) = \begin{cases} w_n(\phi) & \text{if } \phi \neq \{(1-n)\theta\} \\ n\ell_0 - t\gamma & \text{if } \phi = \{(1-n)\theta\} \end{cases}$$

(3.2)
and \( w_n(\phi) \) is as in (3.1). In particular, if \( \phi \) is singular the flipping pair \( ba \) has been replaced by the word \( b_\ell a_\ell \) where \( b_\ell = ta + (1 - t)b \) and \( a_\ell = (1 - t)a + tb \). The shift map acts as \( (\phi, t) \mapsto (\{ \phi + \theta \}, t) \). \((\Xi, \mathbb{Z})\) is the unique minimal component of \((\tilde{\Xi}, \mathbb{Z})\). The augmented version is crucial for a correct understanding of the bulk–boundary correspondence.

(iv) For numerical approximation and comparison with other results from the literature we also consider a smoothed out version of the Sturmian subshift. To describe this it is convenient to consider the cut-and-project scheme on the torus \( \mathbb{R}^2/\mathbb{Z}^2 \), as in Fig. 1b. There, the line \( L_\phi \) wraps densely around this torus and \( W \) appears as a one-dimensional line segment of constant irrational slope. The constant slope guarantees that the resulting sequence \( (w_n - w_{n-1})_n \) has only finitely many values, but inhibits (because of the irrationality of the slope) that the segment is a closed continuous curve on the torus. On the other hand, if \( W \) is deformed into a closed continuous curve, which is transversal to \( L_\phi \) for all \( \phi \), then the map \( \phi \mapsto \xi(\phi) \) from above becomes continuous for all \( \phi \in S^1 \). As a consequence, the symbolic dynamical system corresponding to such a deformed transversal is topologically conjugate to the rotation by \( \theta \) on \( S^1 \). One should be aware, however, that \( \xi(\phi) \) takes now a dense set of values in the interval \([a, b] \); hence, the system has infinite local complexity [29].

We consider here a deformation \( W_\epsilon \) of \( W \) which is obtained if one replaces the indicator function \( \chi \) in (3.1) by \( \chi_\epsilon(x) = \frac{1}{2}(1 + \tanh(\frac{x}{\epsilon})) \). Indeed, if we plot the segment \( W \) as a function of \( \phi \) with an \( y \)-axis which is along \( L_\theta \) (thus a bit tilted) then it has a discontinuity at \( \phi = \theta \). We smooth this discontinuity out with the help of the above approximation to the step function (see Fig. 2). We denote the corresponding system by \((\Xi_\epsilon, \mathbb{Z})\) and the sequence with intercept \( \phi \) by \( \xi_\epsilon(\phi) \). It is given thus by \( \xi_n(\phi) = w_n^\epsilon(\phi) - w_{n-1}^\epsilon(\phi) \) where

\[
    w_n^\epsilon(\phi) = n l_0 + \gamma(\varphi_n - \chi_\epsilon(\varphi_n - \theta)), \quad \varphi_n = \{ \phi + n\theta \}, \quad n \in \mathbb{Z}. \tag{3.3}
\]

For \( \theta \) irrational, \((\Xi_\epsilon, \mathbb{Z})\) is a minimal subsystem of \(([a, b]^\mathbb{Z}, \mathbb{Z})\) which is topologically conjugate to the first of the above systems, the rotation by \( \theta \) on \( S^1 \).

Above we defined three symbolic dynamical systems \( \Xi, \tilde{\Xi} \) and \( \Xi_\epsilon \), which will play a role in what follows. They are all subsystems of \([a, b]^\mathbb{Z}\). A strongly pattern equivariant family of Hamiltonians \( H_x \) for \( x \in [a, b]^\mathbb{Z} \) defines, therefore, by restriction to the three spaces \( \Xi, \tilde{\Xi}, \) and \( \Xi_\epsilon \), families for all these subsystems. We simplify the notation by writing \( H_{\phi, t} \) for \( H_{\xi(\phi, t)} \). (Here \( \phi \) is a singular value.)

How are these different systems related? We have already commented on various relations between them, but now we want to compare more closely the smoothed out version with the augmented Sturmian subshift. Before doing that, let us point out that, from a dynamical point of view and therefore also, as we will see, from an operator algebraic point of view, the smoothed out
system \((\Xi_\epsilon, \mathbb{Z})\) does not converge to the Sturmian subshift \((\Xi, \mathbb{Z})\) when \(\epsilon\) tends to 0.

**Lemma 3.1.** With the notation of Sect. 2 and for \(\theta\) irrational, we have

\[
\lim_{\epsilon \to 0} d_H(\Xi_\epsilon, \tilde{\Xi}) = 0.
\]

**Proof.** Let \(\delta > 0\) and \(D_\delta(\phi) = \{(\phi+n\theta)| |n| \leq \delta^{-1}+1\}\). Let \(2r_\delta := \inf\{|x-y|: x \neq y \in D_\delta(\phi), \phi \in [0,1]\}\). Let \(\epsilon_\delta = \sup\{\epsilon > 0| \forall|\epsilon| \geq r_\delta : |\chi(x) - \chi_\epsilon(x)| < \delta\}\).

We will show that, for all \(\epsilon \leq \epsilon_\delta\), within distance \(2\delta\) of any point in \(\Xi_\epsilon\) there is a point in \(\tilde{\Xi}\) and vice versa.

By construction \(D_\delta(\phi) \cap (\theta-r_\delta, \theta+r_\delta)\) contains at most one point. If this point is above \(\theta\) (or there is no point at all) then we can find \(\eta\) with \(0 \leq \eta \leq r_\delta\) such that \(D_\delta(\phi+\eta) \cap (\theta-r_\delta, \theta+r_\delta) = \emptyset\), and if the point is below \(\theta\) then the same statement is true but for some \(0 \geq \eta \geq -r_\delta\). It follows that, for \(\phi \neq \{k\theta\}\), \(|k| \leq \delta^{-1}+1\), we have that for all \(|n| \leq \delta^{-1}+1\), \(w_n(\phi) = w_n(\phi + \eta)\) and, if \(\epsilon \leq \epsilon_\delta\) then also \(|w_n(\phi + \eta) - w_n(\phi + \eta)| \leq \delta\). By definition of the metric (2.1) we see that if \(\epsilon \leq \epsilon_\delta\) then for any non-singular value of \(\phi\) there is \(\eta\) such that \(d(\xi(\phi), \xi(\phi+\eta)) < \delta\).

Now let \(\phi = \{k\theta\}\) be singular and \(t \in [0,1]\). We need to show that there is \(\xi^\epsilon \in \Xi_\epsilon\) such that \(d(\xi(\phi,t), \xi^\epsilon) < 2\delta\). This essentially boils down to consider the distance between \(w_{1-k}(\phi,t)\) and the point \(w^\epsilon_{1-k}\) of the point pattern determining \(\xi^\epsilon\). The situation is depicted in Fig. 3, with \(k = 1\). We give the precise arguments:

Consider first the case \(t \geq 1 - \delta\). Then \(|w_{1-k}(\phi,t) - w_{1-k}(\phi,1)| < \delta\) and hence \(d(\xi(\phi,t), \xi(\phi,0)) < \delta\). Since \(\xi(\varphi,0)\) is an accumulation point of the non-singular sequences we conclude from the above that there is \(\eta\) such that \(d(\xi(\varphi,0), \xi(\varphi+\eta)) \leq \delta\); hence, \(d(\xi(\varphi,t), \xi^\epsilon(\varphi+\eta)) < 2\delta\).

If \(t \leq \delta\) we use \(d(\xi(\phi,t), \xi(\varphi,0)) < \delta\) and argue similarly.

Finally, let \(\delta < t < 1 - \delta\). Then there exists \(|\eta| < r_\delta\) with \(\chi_\epsilon(\eta) = t\). It follows that \(w_{1-k}^\epsilon(\phi + \eta) = w_{1-k}(\phi,t)\) and for all \(|n| \leq \delta^{-1}, n \neq 1-k\), \(|w_n^\epsilon(\phi + \eta) - w_n(\phi,t)| < \delta\). Hence, \(d(\xi(\phi,t), \xi^\epsilon(\phi + \eta)) < \delta\). Thus, we have shown that within distance \(2\delta\) of any point \(\xi \in \tilde{\Xi}\) one finds a point of \(\Xi_\epsilon\).
We now consider a point $\xi^\epsilon(\phi) \in \Xi_{\epsilon}$, $\epsilon \leq \epsilon_\delta$. If $D_\delta(\phi) \cap (\theta - r_\delta, \theta + r_\delta) = \emptyset$ then for all $|n| \leq \delta^{-1} + 1$ we have $|w_n^\epsilon(\phi) - w_n(\phi)| < \delta$ and thus $d(\xi(\phi, t), \xi^\epsilon(\phi)) < \delta$. Otherwise, $D_\delta(\phi) \cap (\theta - r_\delta, \theta + r_\delta)$ contains a single point, $\{\phi + k\theta\}$, and $\eta := \{\phi + k\theta\} - \theta$ satisfies $|\eta| < r_\delta$. Let $t = \chi_\epsilon(\eta)$. Then $d(\xi(\phi - \eta, t), \xi^\epsilon(\phi)) < \delta$. Thus, within distance $\delta$ of any point of $\Xi_{\epsilon}$ one finds a point of $\Xi$.

Corollary 3.2. Let $H_x$ be a strongly pattern equivariant Hamiltonian associated with $([a, b]^2, \mathbb{Z})$. Let $\xi_n \in \Xi_{1/n}$. Then

$$\lim_{n \to +\infty} \text{spec}(H_{\xi_n}) = \text{spec}(H_{\Xi}) = \bigcup_{t \in [0, 1]} \text{spec}(H(\phi, t))$$

for any singular $\phi$.

Proof. $(\Xi_{\epsilon}, \mathbb{Z})$ is minimal, and thus all $H_x$, $x \in \Xi_{\epsilon}$ have the same spectrum and hence $\text{spec}(H_x) = \text{spec}(H_{\Xi})$. The statement follows therefore from Lemma 3.1 and Theorem 2.2. The last equality follows from the fact that $\text{spec}(H(\phi, t))$ is independent of the choice of the singular value $\phi$, as different choices lead to unitarily equivalent operators.

4. The Winding Number of Dirichlet Eigenvalues

Here we discuss the continuity of Dirichlet eigenvalues when varying a half-space Hamiltonian inside the family defined by one of the three dynamical systems introduced above. Our aim is to define winding numbers of Dirichlet eigenvalues. We will see that this cannot work if the underlying space is $\Xi$, as in this case the union of all Dirichlet eigenvalues is a set of zero Lebesgue measure. Nevertheless, the Dirichlet eigenvalues of the augmented systems
display continuous flows of eigenvalues and their winding numbers can be rigorously defined.

It is a generic feature of operators $\hat{H}_x$ obtained by compression of strongly pattern equivariant operators $H_x$ on $\ell^2(\mathbb{Z})$ onto the half-space $\ell^2(\mathbb{N})$ that their spectrum is richer than that of $H_x$. Indeed, $H_x$ and $\hat{H}_x \oplus \tilde{\Pi}^* H_x \tilde{\Pi}$, with $\tilde{\Pi}$ the partial isometry from $\ell^2(\mathbb{Z}\setminus \mathbb{N})$ to $\ell^2(\mathbb{Z})$, differ just by a finite rank perturbation. In this case it is known (see, for example, [52]) that $\text{spec}(\hat{H}_x \oplus \tilde{\Pi}^* H_x \tilde{\Pi}) = \text{spec}(\hat{H}_x) \cup \text{spec}(\tilde{\Pi}^* H_x \tilde{\Pi})$ contains $\text{spec}(H_x)$ (which is essential spectrum) plus additional discrete spectrum. As such, if $\Delta$ is a gap in the spectrum of $H_x$, then $\hat{H}_x$ may contain eigenvalues in that gap, that is, isolated spectral values $\mu$ such that

$$\hat{H}_x \psi = \mu \psi$$

for $\psi \in \ell^2(\mathbb{N})$. The number of eigenvalues in each gap $\Delta$, counted with their degeneracy, cannot exceed the rank of the perturbation [52]. A square integrable solution $\psi$ to that equation will be referred to as a boundary state, as it is localized near the boundary. Since such a solution can be interpreted as satisfying Dirichlet boundary conditions at $n = 0$ we call $\mu$ a Dirichlet eigenvalue. Due to the mirror symmetry, the localized states measured at the mirror symmetric points in the experiments of [1] can be theoretically described as boundary states of $\hat{H}_x$. Of course, $\mu = \mu(x)$ depends on $x$ and we are interested in its behaviour under variation of $x$.

**Corollary 4.1.** Let $X$ be any of the three $\Xi$, $\Xi_\epsilon$ or $\tilde{\Xi}$. The map $X \ni x \mapsto \text{spec}(\hat{H}_x)$ is continuous.

**Proof.** When $X$ is $\Xi$ or $\Xi_\epsilon$, this is a restatement of Corollary 2.4, as both systems are minimal. In the third case, $\tilde{\Xi}$ is not minimal, but $\Xi$ is its minimal component. As such, we only need to investigate the remaining sequences of $\tilde{\Xi}$, parametrized by a fixed singular $\phi \in \theta \mathbb{Z}$ and $t \in [0,1]$. As any $\tilde{H}_{(\phi,t)}(\phi,0)$ is a finite rank perturbation of $\tilde{H}_{(\phi,0)}$ and $\tilde{H}_{(\phi,1)}$ continuity in $t$ at fixed $\phi$ follows from analytic perturbation theory. The continuity extends therefore from $\Xi$ to $\tilde{\Xi}$. \hfill $\square$

For $\Xi$, which is a Cantor set, the continuity proven above does not exclude a sort of singular behaviour of the boundary spectrum. Indeed, let $\Delta$ be a spectral gap of $H_\Xi$, i.e. a connected component of the resolvent set of $H_\Xi$. Consider a closed interval $\Delta' \subset \Delta$. Pick $\xi$ arbitrary from $\Xi$. Then $\text{spec}(\hat{H}_\xi) \cap \Delta'$ consists of at most finitely many isolated points; let $K$ be an upper bound to their number. Let $\mu(\xi)$ be such a point, if it exists at all, and let $\psi$ be its normalized eigenvector. Since the distance of $\Delta'$ to the spectrum of $H_\Xi$ is strictly positive, a direct application of Thomas–Combes theory shows that $\psi(n) \leq c e^{-\beta n}$ with real constants $c$ and $\beta > 0$ which do not depend on $\xi$ (as long as $\mu(\xi) \in \Delta'$).

Consider now the first $N$ singular points $\phi_n = n\theta$, $n = 1, \ldots, N$. They divide the circle $S^1$ into $N$ connected open sets and consequently $\Xi$ into $N$ clopen sets $I_j$. Any two sequences from the same $I_j$ agree on their first $N$
entries. The $\xi$ chosen above necessarily falls into one of these, say $I_k$. Thus, for any $\xi' \in I_k$ such that $\mu(\xi') \in \Delta'$ we have $(\hat{H}_{\xi'} - \mu(\xi))\psi(n) = 0$ provided $n < N - r$ where $r$ is the sum of the range of $\hat{H}_\xi$ and the largest range of the sliding block codes. It follows that
\[
\| (\hat{H}_{\xi'} - \mu(\xi))\psi \| \leq 2\|H_\xi\| \left( \sum_{n \geq N-r} \|\psi(n)\|^2 \right)^{1/2} \leq c' e^{-\beta N}
\]
for some constant $c'$ which again does not depend on $\xi$. This implies that the distance between $\mu(\xi)$ and the spectrum of $\hat{H}_{\xi'}$ is bounded from above by $c' e^{-\beta N}$. We conclude that $\text{spec}(\hat{H}_{I_k}) \cap \Delta'$ is contained in (at most) $K$ intervals of widths less than $c' e^{-\beta N}$. As such, $\text{spec}(\hat{H}_\Xi) \cap \Delta' = \bigcup_{k=1}^N \text{spec}(\hat{H}_{I_k}) \cap \Delta'$ is contained in at most $KN$ intervals of widths at most $c' e^{-\beta N}$. Hence, the Lebesgue measure of $\text{spec}(\hat{H}_\Xi) \cap \Delta'$ cannot exceed $c' K N e^{-\beta N}$. But this upper bound goes rapidly to zero as $N \to \infty$. We have just proved:

**Proposition 4.2.** The spectrum $\text{spec}(\hat{H}_\Xi) \cap \Delta$ has Lebesgue measure 0. In particular, it contains gaps.

Because of the last result, we cannot define a spectral flow through the gap as $x$ varies in $\Xi$, and neither a winding number. Consider, however, the situation in which $X$ is homeomorphic to the circle $S^1$ and there is one Dirichlet eigenvalue $\mu(x)$ in the gap for some $x \in X$. The continuity of $x \mapsto \text{spec}(\hat{H}_x)$ implies that Dirichlet eigenvalues depend continuously on $x$. Therefore, the eigenvalue $\mu(x)$ cannot disappear in the interior of the gap under variation of $x$ but only be absorbed by the gap-boundaries or have a trajectory that closes into itself without touching the gap-boundaries. Identifying the gap-boundaries, the gap becomes a circle and the function $x \mapsto \mu(x)$ defines a winding number (with the convention that, if there is no eigenvalue at $x$ inside the gap $\Delta$, then $\mu(x)$ coincides with the gap-boundary). Now if there are several Dirichlet eigenvalues they define several continuous functions $x \mapsto \mu_i(x)$ and the winding number of the gap $\Delta$ is the sum of the winding numbers defined by the individual curves in $\Delta$. Strictly speaking, if there is level crossing then we have to make choices in the definition of these functions, but the sum of their winding numbers can be easily seen to be independent of these choices. Stated differently, the winding number of gap $\Delta$ depends only on the spectral flow of the boundary states in $\Delta$, that is, the restriction of $x \mapsto \text{spec}(\hat{H}_x)$ to the gap.

### 5. Numerical Experiments

In this section we present the results of the numerical calculations of various spectra. We will restrict to one specific Hamiltonian of the type (2.4):
\[
H_\xi = T + T^* + 2b_\xi, \quad \xi \in [a, b]^Z,
\]
where the sliding block code is $b : [a, b] \to \mathbb{R}$, $b(t) = (t - a)/(b - a)$. The above Hamiltonian will also be subjected to Dirichlet boundary conditions.
We follow the notation from Sect. 2; hence, \( H_{\Xi} \) denotes the family \( \{ H_\xi \}_{\xi \in \Xi} \), etc. Standard numerical approximants are employed, and the targeted accuracy was such that the numerical errors will not be detectable by the eye in the plots reported here. The value of \( \theta \) has been fixed to \( \frac{3-\sqrt{5}}{2} \); hence, we simulate Fibonacci sequences.

### 5.1. Bulk Spectra

In Fig. 4 we show the bulk spectra for the models associated with different spaces. The bulk spectra are computed with periodic boundary conditions on a finite approximant. The figure confirms and exemplifies the statement of Corollary 3.2. Indeed, it shows clearly that \( \text{spec}(H_{\Xi}) \) converges to \( \text{spec}(\tilde{H}_{\Xi}) \) in the limit \( \epsilon \to 0 \), and not to \( \text{spec}(H_{\Xi}) \). We recall that, whereas \( \text{spec}(H_{\Xi}) = \text{spec}(H_\xi) \) for any point \( \xi \in \Xi \), and \( \text{spec}(H_{\Xi}) = \text{spec}(H_\xi) \) for any point \( \xi \in \Xi \), we have \( \text{spec}(\tilde{H}_{\Xi}) = \bigcup_{t \in [0,1]} \text{spec}(H_{\phi,t}) \) for any singular choice of \( \phi \). While the numerical data show that the prominent spectral gaps in \( \text{spec}(H_{\Xi}) \) remain open when augmenting to \( \tilde{\Xi} \), this is not the case for all of the spectral gaps. We will show, however, below, that the exponential map of the six-term exact sequence associated with the left short exact sequence of Diagram 6.3 is trivial, which is a sign that the closing of these gaps is not topological. (The \( K \)-group elements defined by the spectral gaps in \( \text{spec}(H_{\Xi}) \) lift to \( K \)-group elements in the \( K_0 \)-group for the augmented model.)

### 5.2. Boundary Spectra and Dirichlet Eigenvalues

The set

\[
\text{spec}_b(X) = \bigcup_{x \in X} \text{spec}(\hat{H}_x) \setminus \text{spec}(H_X)
\]

may be referred to as the boundary spectrum of the family \( H_X \), as all its values correspond to eigenstates localized at the boundary. We call the boundary spectrum topological if it covers a bulk gap completely, regardless of the imposed boundary conditions. This property, while interesting in itself, is a necessary condition for a topologically non-trivial spectral flow at the boundary.

To visualize the spectra of \( \hat{H}_\xi \), \( \xi \in X \) where \( X \) is \( \Xi \) or \( \Xi_{\epsilon} \), we first note that the (uncountable) union over all space \( \bigcup_{\xi \in X} \text{spec}(\hat{H}_\xi) \) yields the same as the countable union along a single orbit \( \bigcup_{n \in \mathbb{Z}} \text{spec}(\hat{H}_{\alpha^n(\xi)}) \) where \( \xi \) can be any choice. This follows from the fact that the systems \( (X, \mathbb{Z}) \), \( X = \Xi \) or \( \Xi_{\epsilon} \), are forward minimal. (Every forward orbit is dense.) Now we approximate \( \text{spec}_b(\hat{H}_\xi) \) by \( \text{spec}_b(H^{(D)}_{\xi_L}) \), where \( \text{spec}_b(H^{(D)}_{\xi_L}) \) is \( \text{spec}(H^{(D)}_{\xi_L}) \) restricted to the bulk gaps, \( \xi_L = \{ \xi_0, \ldots, \xi_{L-1} \} \) is the restriction to the finite length \( L \) part of \( \xi \), and we impose Dirichlet conditions at the two boundaries. Of course, \( \text{spec}_b(H^{(D)}_{\xi_L}) \) includes the Dirichlet eigenvalues of both boundaries, that is the eigenstates localized at the left and at the right boundaries. But due to the inversion symmetry of Sturmian sequences, namely that \( \xi_n \mapsto \xi_{-n} \) preserves the spaces \( \Xi \) and \( \Xi_{\epsilon} \), the spectrum coming from the left Dirichlet eigenvalues...

Figure 4. Bulk spectra of $H_X$ for the three choices for $X$.

The first three panels show $\text{spec}(H_{\epsilon X})$ with $\epsilon = 0.1$ (a), $\epsilon = 0.01$ (b), $\epsilon = 0.001$ (c). Panel d shows $\text{spec}(H_{\tilde{X}})$ and panel e how $\text{spec}(H_{\phi,t})$ depends on $t \in [0,1]$. Finally, panel f shows $\text{spec}(H_{\Xi})$. The computations were performed with periodic boundary conditions on a finite system of size 6765, which came from the rational approximation: $\theta = \frac{3 - \sqrt{5}}{2} = \frac{2584}{6765} - 9.77 \times 10^{-9}$ and that coming from the right Dirichlet eigenvalues have to coincide and thus $\bigcup_{n=0}^{L-1} \text{spec}_b(H_{\alpha^n(\xi L)})$ approximates $\bigcup_{\xi \in X} \text{spec}_b(\hat{H}_{\xi})$, where the index $k$ in the shift action $\alpha(\xi_k) = \xi_{k+1}$ is to be taken modulo $L$.

Figure 5 shows that the boundary spectrum $\text{spec}_b(H_{\epsilon X})$ ($\epsilon = 0.1$) is topological. Indeed, as higher $n$ is considered, i.e. more spectra are overlapped, the union of the Dirichlet eigenvalues is seen to fill out more and more the bulk gaps. With the present resolution, some of the bulk gaps already appear completely filled out (Fig. 5d).

Figure 6 shows that the situation is very different for the Kohmoto model. We have verified that, even when we take the maximal value for $n$ that is allowed by the finite size of the system, the bulk gaps remain mostly empty of boundary spectrum, in agreement with Proposition 4.2. This already indicates that the Kohmoto model does not support topological boundary spectrum.

5.3. Dependence of the Dirichlet Eigenvalues on the Intercept

Whereas in Figs. 5 and 6 we have superimposed the spectra of $H_x^{(D)}$ for different $x \in X$, in Fig. 7 we show their dependence on the intercept $\phi \in [0,1]$. We refer to the red lines in the panels as spectral flow lines, along the parameter $\phi$; they may also be understood as band functions for the boundary spectrum.
Figure 5. Boundary spectrum $\text{spec}_b(\hat{H}_\Xi), \epsilon = 0.1$ (shown in red). For reference, the spectrum of the periodic model from Fig. 4b is overlapped and shown in black. The panels show $\bigcup_n \text{spec}(H^{(D)}_{\alpha^n(\xi_L)})$ for one $\xi \in \Xi_\epsilon$ ($L = 6765$ as in Fig. 4), with $a$ $n = 0$, $b$ $n = 0, \ldots, 9$, $c$ $n = 0, \ldots, 99$ and $d$ $n = 0, \ldots, 999$ (colour figure online)

Figure 6. The boundary spectrum of the Kohmoto model with Dirichlet boundary condition (shown in red). For reference, the spectrum of the periodic model from Fig. 4f is overlapped and shown in black. The panels show $\bigcup_n \text{spec}(H^{(D)}_{\alpha^n(\xi_L)})$ with, as in Fig. 4, $L = 6765$ and, for $a$ $n = 0$, $b$ $n = 0, \ldots, 9$, $c$ $n = 0, \ldots, 99$ and $d$ $n = 0, \ldots, 999$ (colour figure online)

with $\phi$ playing the role of quasi-momentum. It should be kept in mind that the curves show the left and right Dirichlet eigenvalues, together. However, by modifying the potential at one end of the chain and observing which lines have been affected, we can easily decide which values correspond to eigenstates localized at the left and which at the right boundary. Indeed, our observation is that half of the lines remain virtually unaffected, and so we can be sure that
Figure 7. Dirichlet eigenvalues displayed as functions of the intercept $\phi$ (plotted along the $y$-axis). Panels a and d replot the data from Figs. 5d and 6d, respectively, b is same as panel a but for $\epsilon = 0.01$, while c is a plot of $\cup_{t\in[0,1]}\text{spec}(\hat{H}_{\phi,t})$. The labels of the most prominent gaps (see next figure) are included for the reader’s convenience. The marks in panel a indicate which bands are localized at the left edge (colour figure online).

the boundary spectra at the two edges are entirely decoupled. This being said, we can focus on the spectral flow of the states localized at the left boundary, which have been marked in panel (a) of the figure (and hence can be also identified in the rest of the panels).

We now analyse Fig. 7 more closely. As alluded to already in the previous section, the smooth models shown in panels (a) and (b) display spectral
flow lines visible through the red lines connecting the lower and upper gap-boundaries. If we identify the gap-boundaries as in Sect. 4, then the flow lines wind once or several times around the gap, which is now a circle, and we can easily read off their winding numbers. The numerical data confirm that the spectral flow of the boundary states of the smooth models converges in the $\epsilon \to 0$ limit to that of the augmented model, shown in panel (c). For this case too, we can define without any difficulty a winding number. But note that in panel (c) the lines are piecewise either horizontal or vertical. The horizontal lines are, of course, not functions of $\phi$, but they arise as we take the union over $t$. The non-horizontal pieces of the red lines in panel (c) look vertical to the eye, and thus, the winding number depends only on the horizontal ($t$-dependent) part of the lines. We exploit this to give an interpretation of the winding number in Sect. 8.1.

We can clearly see in panel (d) that the spectral flow lines are discontinuous as a function of $\phi$ for the Kohmoto model and therefore there is no proper way of defining winding numbers. This picture ought to be compared with Figure 3 (a),(c) of [1], except that in [1] $\phi$ is plotted along the $x$-axis. The vertical pieces of the curves of [1] Fig. 3 (c) remain unexplained, and there is no way to guess them, as is clear, for instance, if one considers the gap with label 1 of our panel (d). How should one conclude from panel (d) that the winding number of the edge states in that gap should be $-1$?

For the panels (a), (b) and (c), where the winding numbers are well defined, the winding numbers corresponding to the prominent gaps can be seen to coincide with the so-called gap labels provided by the $K$-theory and listed above panel (d).

### 5.4. Integrated Density of States at Gap Energies

Figure 8 shows a numerical representation of the integrated density of states (IDS) for the Kohmoto model. It has been computed as

$$\text{IDS}(E) = \frac{\text{number of eigenvalues of } H_{\xi L} \text{ smaller than } E}{L}, \quad (5.2)$$

where $L$ is the finite size of the periodic approximant used in the simulations. For Kohmoto model, it is known that the IDS, when evaluated inside the spectral gaps, can take only very specific values, namely

$$\text{IDS} \subseteq \{n + m\theta, \, n, m \in \mathbb{Z}\} \cap [0, 1]. \quad (5.3)$$

This is a famous result of Bellissard [4,5]. It follows from $K$-theory. The labels $(n, m)$ are bulk topological invariants which are supplied by the $K$-theory of the algebra of bulk observables. Figure 8 also reports the label $m$ of the prominent gaps, as computed from the numerical IDS. One central issue in our work is how to relate this bulk topological invariant to the topological edge spectrum. Note that all gap labels $m$ in Fig. 8 are different from zero, which is a direct consequence of (5.3). Indeed, the only gap labels $(n, m)$ with $m = 0$ for which $\text{IDS}$ fits inside the interval $[0, 1]$ are $n = 0$ or 1, but these cases correspond to the empty or fully populated spectrum, respectively.
Figure 8. The bulk spectrum \( \text{spec}(H_\Xi) \) (shaded regions), replotted from Fig. 4, and its integrated density of states (red curve). Several prominent spectral gaps can be observed, together with their labels (colour figure online).

6. C*-Algebras

Our proof of the bulk–boundary correspondence uses the C*-algebraic description of covariant families of Schrödinger operators. This section describes the standard C*-algebras associated with topological dynamical systems and the canonical representations that generate the bulk and half-space families of covariant Hamiltonians [31]. Particular attention is given to the relations between the C*-algebras associated with Sturmian system and its augmentation, which are summarized in Diagram (6.3). The exact sequence of C*-algebras which will supply the bulk–boundary correspondence is listed in (6.2).

6.1. Preliminaries

Two C*-algebras associated with a topological dynamical system given by a homeomorphism \( \alpha \) on a compact space \( X \) will be of importance. The first one is the crossed product algebra \( C(X) \times_\alpha \mathbb{Z} \). It is the universal C*-closure of the *-algebra \( C(X)_\alpha \mathbb{Z} \) given by finite sums of finite products of elements of \( C(X) \) and a unitary \( u \) such that \( uf u^* = f \circ \alpha \). Each \( x \in X \) induces a representation \( \pi_x \) of \( C(X)_\alpha \mathbb{Z} \) on \( \ell^2(\mathbb{Z}) \),

\[
\pi_x(f)\psi(n) = f(\alpha^n(x))\psi(n), \quad \pi_x(u) = T
\]

where \( f \in C(X) \) and \( T \) is the left translation operator, \( T\psi(n) = \psi(n + 1) \). The universal C* norm on \( C(X)_\alpha \mathbb{Z} \) is given by \( \|H\| := \sup_{x \in X} \|\pi_x(H)\| \).

In the context of symbolic dynamics where \( X \) is a symbolic subshift, a sliding block code \( b \), extended to \( C(X) \), defines an element of \( C(X)_\alpha \mathbb{Z} \) and \( \pi_x(b) = \tilde{b}_x \) as defined in (2.2). Thus, the covariant family (2.3) coincides with
the family \((\pi_x(H))_{x \in X}\), where \(H = \sum_{k \in S}(b_k u^k + u^{-k}b_k)\). The covariant family is thus described by a single element in the \(C^*\)-algebra and, since the family of representations \(\pi_x\), \(x \in X\) is faithful, this yields a one-to-one correspondence. A single representation \(\pi_x\) is faithful if the orbit of \(x\) is dense, and then we may identify \(C(X) \rtimes_\alpha \mathbb{Z}\) also with the norm closed subalgebra of \(\mathcal{B}(\ell^2(\mathbb{Z}))\) of that representation.

A second algebra associated with the dynamical system is the Toeplitz extension algebra \(\mathcal{T}(C(X), \alpha)\). It is the universal \(C^*\)-closure of the *-algebra \(C(X)_\alpha \mathbb{N}\) given by finite sums of finite products of elements of \(C(X)\) and a proper coisometry \(v\) such that \(vfv^* = f \circ \alpha\). \(v\) being an coisometry means that \(vv^* = 1\) and \(v^*v = 1 - \hat{e}\), where \(\hat{e}\) is a nonzero projection which commutes with the elements of \(C(X)\). Each \(x \in X\) induces a representation \(\hat{\pi}_x\) of \(C(X)_\alpha \mathbb{N}\) on \(\ell^2(\mathbb{N})\),

\[
\hat{\pi}_x(f)\psi(n) = f(\alpha^n(x))\psi(n), \quad \hat{\pi}_x(v) = \hat{T}
\]

where \(\hat{T}\) is the left translation operator on \(\ell^2(\mathbb{N})\), i.e. \(\hat{T}\psi(n) = \psi(n + 1)\). Again, the family of representations \(\hat{\pi}_x\), \(x \in X\) is faithful. Note that \(\hat{T}\) is the compression of \(T\) to \(\ell^2(\mathbb{N})\) and hence, in the context of symbolic subshifts, our family of half-space operators (7.2) is faithfully represented by the element \(\hat{H} = \sum_{k \in S}(b_k v^k + v^*k b_k) \in \mathcal{T}(C(X), \alpha)\).

The map \(\hat{e} \mapsto 0\) induces a unital surjective *-algebra morphism \(C(X)_\alpha \mathbb{N} \to C(X)_\alpha \mathbb{Z}\) which extends to a surjective *-algebra morphism \(\pi : \mathcal{T}(C(X), \alpha) \to C(X) \rtimes_\alpha \mathbb{Z}\) whose kernel can be seen to be isomorphic to \(C(X) \otimes \mathcal{K}\). The associated short exact sequence (SES)

\[
0 \to C(X) \otimes \mathcal{K} \overset{i}{\to} \mathcal{T}(C(X), \alpha) \overset{\pi}{\to} C(X) \rtimes_\alpha \mathbb{Z} \to 0 \tag{6.1}
\]

is called the Toeplitz extension.

### 6.2. An Augmented Extension for Sturmian Systems

In the context of Sturmian systems we are interested in the case where \(X\) is \(\Xi\) or \(\check{\Xi}\), and the action is induced by the left shift. The inclusion of \(\Xi\) into \(\check{\Xi}\) induces a surjection \(C(\check{\Xi}) \to C(\Xi)\) which commutes with the \(\mathbb{Z}\)-action and thus gives rise to a surjection \(q : C(\check{\Xi}) \rtimes_\alpha \mathbb{Z} \to C(\Xi) \rtimes_\alpha \mathbb{Z}\). For the bulk–boundary correspondence, which we discuss below, we compose the two surjective maps, \(\mathcal{T}(C(\check{\Xi}), \alpha) \to C(\check{\Xi}) \rtimes_\alpha \mathbb{Z} \to C(\Xi) \rtimes_\alpha \mathbb{Z}\) to obtain the \textit{augmented} exact sequence

\[
0 \to J \hookrightarrow \mathcal{T}(C(\check{\Xi}), \alpha) \overset{\check{\pi}}{\to} C(\Xi) \rtimes_\alpha \mathbb{Z} \to 0 \tag{6.2}
\]

where \(J\) is by definition the kernel of \(\check{\pi} := q \circ \pi\). To better understand \(J\) consider the following diagram of short exact sequences in which all squares commute
The first horizontal SES is the Toeplitz extension (6.1) for $X = \tilde{\Xi}$. The right vertical SES comes from the inclusion $\Xi \subset \tilde{\Xi}$ which induces the surjection $q : C(\tilde{\Xi}) \rtimes_\alpha \mathbb{Z} \to C(\Xi) \rtimes_\alpha \mathbb{Z}$. With $S := \tilde{\Xi} \setminus \Xi$ the ideal in that sequence is just $C_0(S) \rtimes_\alpha \mathbb{Z}$.

The second horizontal SES defines $J$ as the kernel $J = \ker q \circ \pi$. A diagram chase shows that the left vertical sequence is short exact: let $x \in T(\tilde{\Xi}, \alpha)$, then $x \in J$ iff $q(\pi(x)) = 0$ iff $\pi(x) \in C_0(S) \rtimes_\alpha \mathbb{Z}$. Let $E$ be the preimage under $\pi$ of $C_0(S) \rtimes_\alpha \mathbb{Z}$, then $\pi(x) \in C_0(S) \rtimes_\alpha \mathbb{Z}$ iff $\exists y \in E : \pi(y) = \pi(x)$ iff $x \in E + I$ where $I = i(C(\tilde{\Xi}) \otimes \mathcal{K})$. Hence, $J = E + I$. It follows that $J/I = E/(E \cap I) = E/\ker \pi \cong C_0(S) \rtimes_\alpha \mathbb{Z}$.

7. $K$-Theory

We describe in Sect. 5 that the $K$-theory of the bulk $C^*$-algebra supplies labels for the spectral gaps of the covariant Hamiltonians. We have also observed (so far numerically) that for the covariant operators associated with $\Xi$, a non-trivial gap label leads to topological boundary spectrum, and further numerical evidence suggests that this is also true for the augmented system $\tilde{\Xi}$. However, for the original Sturmian system this was not at all the case. In this section we use $K$-theory to make all these statements precise. In particular, we show how the exponential map from $K$-theory connects the bulk gap labels of the Sturmian system with the $K_1$-labels of the boundary spectrum of the augmented system.

7.1. Preliminaries

We recall that any SES of $C^*$-algebras $0 \to I \xrightarrow{i} E \xrightarrow{q} A \to 0$ gives rise to a six-term exact sequence in $K$-theory

\[
\begin{array}{ccc}
K_0(I) & \xrightarrow{i_*} & K_0(E) \\
\uparrow \text{ind} & & \downarrow \exp \\
K_1(A) & \xrightarrow{q_*} & K_1(E) \\
& \xrightarrow{i_*} & K_1(I)
\end{array}
\]

This sequence yields a calculational tool to determine $K$-groups, but we will see that in particular the exponential map has significant physical interpretation.
Applied to the Toeplitz extension (6.1) the six-term exact sequence has the particular form (called Pimsner–Voiculescu exact sequence) [44]

$$
\begin{align*}
K_0(C(X)) & \xrightarrow{\text{id} - \alpha} K_0(C(X)) \xrightarrow{\iota} K_0(C(X) \rtimes_\alpha \mathbb{Z}) \\
& \uparrow \exp \\
K_1(C(X) \rtimes_\alpha \mathbb{Z}) & \xrightarrow{\iota} K_1(C(X)) \xleftarrow{\text{id} - \alpha} K_1(C(X))
\end{align*}
$$

It splits into two SESs

$$
0 \to C_\alpha, K_i(C(X)) \xrightarrow{\iota} K_i(C(X) \rtimes_\alpha \mathbb{Z}) \to I_\alpha, K_{i-1}(C(X)) \to 0
$$

for \( i = 0, 1 \). Here \( C_\alpha \) is the coinvariant and \( I_\alpha \) the invariant functor. More precisely, for a \( \mathbb{Z} \)-module \( M \) with an isomorphism \( \beta \),

\[
I_\beta M = \ker(\text{id} - \beta), \quad C_\beta M = \text{coker}(\text{id} - \beta)
\]

If \( X = \Xi \) above, then \( K_1(C(X)) \) is trivial. Therefore, there cannot be any topological bulk–boundary correspondence. The boundary modes of the Sturmian system cannot support a non-trivial topological invariant. That is why our attention will be on the augmented system, more precisely, on the connecting maps in \( K \)-theory stemming from (6.2).

7.2. Application to Sturmian Systems

Our interest here is in exponential map associated with (6.2):

\[
\exp_J : K_0(C(\Xi) \rtimes_\alpha \mathbb{Z}) \to K_1(J).
\]

(Since there are more boundary maps around, we add for clarity the subscript \( J \).) For that, we need first to determine \( K_*(J) \) which can be done using the left vertical SES of (6.3). We start therefore to investigate the \( K \)-theory of the ideal of that SES, that is, \( K_*(C_0(\mathcal{S}) \rtimes_\alpha \mathbb{Z}) \). \( \mathcal{S} \) is parametrized by \((\phi, t)\) where \( \phi \) is a singular value for the intercept and \( t \in (0, 1) \). Since the singular values correspond to the orbit \( \theta \mathbb{Z} \) of the rotation action on the circle, we may identify \( \mathcal{S} = \mathbb{Z} \times (0, 1) \), and the \( \mathbb{Z} \) action on \( \mathcal{S} \) is then simply given by the action of \( \mathbb{Z} \) on \( \mathbb{Z} \) in the left variable. It follows that \( C_0(\mathcal{S}) \rtimes_\alpha \mathbb{Z} \cong C_0(0, 1) \otimes \mathcal{K} \) and thus

\[
K_0(C_0(\mathcal{S}) \rtimes_\alpha \mathbb{Z}) = 0, \quad K_1(C_0(\mathcal{S}) \rtimes_\alpha \mathbb{Z}) \cong \mathbb{Z}.
\]

Moreover, the generator of \( K_1(C_0(\mathcal{S}) \rtimes_\alpha \mathbb{Z}) \) is given by the class of the function \( f(n, t) = \delta_{n0}(e^{2\pi it} - 1) + 1 \) in the unitization of \( C_0(\mathbb{Z} \times (0, 1)) \). Furthermore, \( q(f - 1) = 0 \). Diagram (6.3) shows therefore that \( f - 1 \) lifts under \( \pi \) to an element of \( J \). It follows that the class of the function \( f \) lies in the kernel of the exponential map of the six-term exact sequence for the left vertical SES of (6.3). In other words, the exponential map of the six-term exact sequence for the left vertical SES of (6.3) is trivial. This implies, first, that the six-term exact sequence reduces to

\[
K_0(J) \cong K_0(C(\Xi)) \cong \mathbb{Z}
\]

and

\[
0 \to K_1(C(\Xi)) \xrightarrow{\iota} K_1(J) \xrightarrow{\pi} \mathbb{Z} \to 0,
\]
and second, that any element of $K_0(C(\Xi) \rtimes_\alpha \mathbb{Z})$ lifts to an element of $K_0(C(\tilde{\Xi}) \rtimes_\alpha \mathbb{Z})$ so that the image of $\exp_J$ lies in the kernel of $\pi_*$. While the above sequence splits the isomorphism is not canonical. It depends on a choice of section $s : \mathbb{Z} \to K_i(J)$, that is, preimage under $\pi_*$ of the generator of $\mathbb{Z}$. Given such a section, $\sigma : K_i(J) \to K_i(C(\tilde{\Xi}))$, $\sigma(x) = i_*(x - s \circ \pi_*(x))$ is a left inverse for $i_*$ and

$$K_i(J) \xrightarrow{(\sigma, \pi_*)} K_i(\tilde{\Xi}) \oplus \mathbb{Z}$$

an isomorphism. The six-term exact sequence of SES (6.2) can therefore be written

$$K_0(C(\tilde{\Xi})) \xrightarrow{i_*} K_0(\mathcal{T}(\tilde{\Xi}, \alpha)) \xrightarrow{q_* \circ \pi_*} K_0(C(\Xi) \rtimes_\alpha \mathbb{Z}) \xrightarrow{\downarrow (\sigma, \pi_*) \circ \exp_J}$$

$$K_1(C(\Xi) \rtimes_\alpha \mathbb{Z}) \xrightarrow{q_* \circ \pi_*} K_1(\mathcal{T}(\tilde{\Xi}, \alpha)) \xrightarrow{\tilde{\psi}} K_1(C(\tilde{\Xi})) \oplus \mathbb{Z}$$

where $\tilde{\psi}([u] + n) = i_*([u]) + s(n)$. Under the isomorphism $K_1(\mathcal{T}(\tilde{\Xi}, \alpha)) \cong K_i(C(\tilde{\Xi}))$ from [44] this becomes

$$K_0(C(\tilde{\Xi})) \xrightarrow{1 - \alpha_*} K_0(C(\tilde{\Xi})) \xrightarrow{q_* \circ \pi_*} K_0(C(\Xi) \rtimes_\alpha \mathbb{Z}) \xrightarrow{\downarrow (\sigma, \pi_*) \circ \exp_J}$$

$$K_1(C(\Xi) \rtimes_\alpha \mathbb{Z}) \xrightarrow{q_* \circ \pi_*} K_1(C(\tilde{\Xi})) \xrightarrow{\psi} K_1(C(\tilde{\Xi})) \oplus \mathbb{Z}$$

where $\psi([u] + n) = (1 - \alpha_*)([u]) + s(n)$. Since $\tilde{\Xi}$ is homeomorphic to the circle $\alpha_*$ is the identity on $K_i(C(\tilde{\Xi}))$, $i = 1, 2$. It follows that $\ker \psi = K_1(C(\tilde{\Xi}))$. In particular, $\pi_* \circ \exp_J = 0$. From this we conclude that $\sigma \circ \exp_J$ is independent of the chosen lift $s$ and $\im \sigma \circ \exp_J = K_1(C(\tilde{\Xi}))$. The above six-term exact sequence cuts therefore down to

$$0 \to K_0(C(\tilde{\Xi})) \xrightarrow{q_* \circ \pi_*} K_0(C(\Xi) \rtimes_\alpha \mathbb{Z}) \xrightarrow{\sigma \circ \exp_J} K_1(C(\tilde{\Xi})) \to 0$$

(7.1)

The image of $q_* \circ i_*$ is generated by the class of the identity. The exponential map can be described as follows: if a projection $p$ represents an element $[p] \in K_0(C(\Xi) \rtimes_\alpha \mathbb{Z})$, we lift $p$ to a self-adjoint element $a \in \mathcal{T}(\tilde{\Xi}, \alpha)$. Then $\exp_J([p]) = [e^{2\pi ia}] \in K_1(J)$. As $\pi_* \circ \exp_J = 0$ there is even a representative $a$ such that $e^{2\pi ia} - 1 \in C(\tilde{\Xi}) \otimes \mathbb{K}$. Thus, $\sigma \circ \exp_J([p]) = [e^{2\pi ia}] \in K_1(C(\tilde{\Xi}))$.

7.3. The Exponential Map for Covariant Families of Operators

We apply the exponential map to the $K_0$-class defined by the Fermi projection $P_F$ of a strongly pattern equivariant family of Hamiltonians $H_\Xi$ on $l^2(\mathbb{Z})$,

$$H_\xi = \left( \sum_{k \in S} (\tilde{b}_k)_\xi T^k \right) + h.c., \quad \xi \in \Xi$$

where $b_k$ are (finitely many) sliding block codes for $\Xi$. These represent an element in $H \in C(\Xi) \rtimes_\alpha \mathbb{Z}$. We suppose that the Fermi energy lies in a gap $\Delta = (E_0, E_1)$ of their (common) spectrum. Then also $P_F$ corresponds to an element in the crossed product, and we wish to express its image under the
exponential map in terms of a strongly pattern equivariant family of operators. We first note that

\[
f(E) = \begin{cases} 
1 & \text{if } E \leq E_0 \\
\frac{E_1 - E}{E_1 - E_0} & \text{if } E \in (E_0, E_1) \\
0 & \text{if } E \geq E_1
\end{cases}
\]

is a continuous function such that the Fermi projection of \( H_\xi \) is given by

\[ P_F(H_\xi) = f(H_\xi). \]

We can extend the sliding block codes \( b_k \) to sliding block codes on \( \tilde{\Xi} \) by linear interpolation: a word in a sequence of \( \tilde{\Xi} \) may contain one single, or a pair of consecutive letters whose lengths differ from \( a \) and \( b \). These depend then on the parameter \( t \in [0,1] \), and we can just extend \( b_k \) on such a word by taking the convex combination of the extreme values. In this way, we define the family \( H_{\tilde{\Xi}} \).

Recall that the compression of the \( H_\xi \) to \( \ell^2(\mathbb{N}) \),

\[ \hat{H}_\xi = \Pi \left( \sum_{k \in S} (\tilde{b}_k)_{\xi} T^k \right) \Pi + \text{h.c.}, \quad \tilde{\xi} \in \tilde{\Xi} \quad (7.2) \]

is a strongly pattern equivariant family of Hamiltonians representing an element \( \hat{H} \) of \( \mathcal{F}(C(\tilde{\Xi}), \alpha) \). Furthermore, \( \hat{H} \) is a lift of \( H \) under the map \( q \circ \pi \). We can therefore take \( a = f(\hat{H}) \) and thus

\[
\sigma \circ \exp_J([P_F(H_\Xi)]) = \left[ e^{2\pi i f(\hat{H}_\Xi)} \right] = \left[ e^{-it_\Delta \hat{H}_\Xi} P_\Delta(\hat{H}_\Xi) + P_\Delta(\hat{H}_\Xi)^\perp \right] \quad (7.3)
\]

where \( t_\Delta = \frac{2\pi}{E_1 - E_0} \) and we have used that \( e^{it_\Delta \hat{E}_1} \) is homotopic to 1. The boundary map applied to the \( K_0 \)-class of the Fermi projection is thus the \( K_1 \)-class of the unitary of time evolution defined by the family of half-space operators \( \hat{H}_\Xi \) by the characteristic time \( t_\Delta \), restricted to the states in the gap.

8. Bulk–Boundary Correspondence

Pairings of \( K \)-theory and cyclic cohomology furnish numerical invariants which often connect with physical measurements. We now formulate such pairings which, together with Eq. (7.3), lead to the bulk–boundary correspondence. This correspondence relates the gap labels to a numerical boundary topological invariant, a winding number, whose physical interpretation we develop.

8.1. The Winding Number Cocycle

Recall that \( \tilde{\Xi} \) is homeomorphic to the circle. It follows that \( K_1(C(\tilde{\Xi})) \) is generated by the class of the function \( S^1 \ni z \mapsto z \in \mathbb{C} \) once we have made an identification of \( \tilde{\Xi} \) with \( S^1 \). In particular, by taking the winding number of a unitary function representing the class of an element of \( K_1(C(\tilde{\Xi})) \), one obtains a group isomorphism \( \mathcal{W} : K_1(C(\tilde{\Xi})) \to \mathbb{Z} \).
Quite generally, a group homomorphism like $W$ can be understood as a pairing of $K_1(C(\hat{\Xi}))$ with a cyclic cocycle $[10]$. In our case this cocycle $\eta$ is defined on the dense subalgebra $A \subset A = C(\hat{\Xi})$ of continuous functions on $\hat{\Xi}$ which are locally constant on the subspace $\Xi$, continuously differentiable on $S = \hat{\Xi} \setminus \Xi \cong \theta \mathbb{Z} \times (0, 1)$ and such that their derivative is integrable over $S$. Here the integral over $S$ is given by the Lebesgue integral on the components: $\int_S f(\theta, t) \, d\lambda = \sum_{n \in \mathbb{Z}} \int_0^1 f(n\theta, t) \, dt$. The winding number cocycle $\eta$ is now defined on the algebra $M_m(A)$ of matrices with entries in $A$ as

$$\eta(f_1, f_2) = \frac{1}{2\pi i} \int_S \text{Tr}_m f_1 \dot{f}_2 \, d\lambda$$

where $\dot{f}_2$ is the derivative of $f_2$ w.r.t. $t$ and $\text{Tr}_m$ the matrix trace on $m \times m$-matrices. In particular, $\eta(f^{-1}, f)$ is the winding number of an invertible function $f \in A$ and

$$W(y) = \eta(f^{-1}, f) \quad (8.1)$$

where $f$ is a representative in $M_m(A)$ for $y$.

We apply this to the unitary class given in (7.3). The only spectrum of $\hat{H}_\Xi$ in the gap $\Delta$ is given by the Dirichlet eigenvalues $\mu_i$, $i = 1, \ldots, m$. Hence, for singular $\phi$,

$$\text{Tr}_m(e^{it\hat{H}_\phi} \partial_t e^{-it\hat{H}_\phi} P_\Delta(\hat{H}_\Xi)) = -it\Delta \sum_{i=1}^m \partial_t \mu_i(\phi, t).$$

By Prop. 4.2 the set of values $\{\mu(x)|x \in \Xi\}$ has Lebesgue measure 0. The variation of $\mu(x)$ along $\Xi$ has thus zero weight and does not contribute to the winding number. It follows that

$$W(\sigma \circ \exp_J([P_F(H_\Xi)])) = \frac{1}{|\Delta|} \sum_{n \in \mathbb{Z}} \int_0^1 \sum_{i=1}^m \partial_t \mu_i(n\theta, t) \, dt$$

This formula has a physical interpretation. Indeed, $\int_0^1 \partial_t \mu_i(n\theta, t) \, dt$ is the work performed on the $i$th boundary state in the gap $\Delta$ by the motion of moving an atom to perform the flip $ab$ to $ba$ at the position which is encoded by the singular value $\phi = n\theta$. $|\Delta|W$ is therefore the total work performed by all such (phason) flips. Note that the winding number can be inferred graphically from a plot like Fig. 7 by reading off the spectral flow.

It is not easy to compute analytically the winding number from the above explicit formula. We therefore make use of a very general result which allows to reformulate the winding number as a Chern number.

Dual to the boundary map $\delta : K_i(A \rtimes_\alpha \mathbb{Z}) \to K_{i-1}(A)$ of the associated Pimsner–Voiculescu six-term exact sequence is a map $\#_\alpha : HC^i(A) \to HC^{i+1}(A \rtimes_\alpha \mathbb{Z})$ on cyclic cohomology $[12, 31]$. In particular, if $\eta$ is a 1-cocycle then this duality corresponds to the formula

$$\eta(f^{-1}, f) = \#_\alpha \eta(p, p, p) \quad (8.2)$$

provided $f$ is a representative for the $K_1$-class $\exp([p])$ defined by the projection $p \in A \rtimes_\alpha \mathbb{Z}$.
Note that $\mathcal{A}$, the integral and the derivative are $\alpha$ invariant. The general theory therefore yields that $\#_\alpha \eta$ is the 2-cocycle on $\mathcal{A}_\alpha \mathbb{Z}$ given by [31]

$$\#_\alpha \eta(a_1, a_2, a_3) = -i \text{Tr}(a_1 [a_2, a'_3])$$

where the second derivation is given by $(fu^n)' = \inf u^n$ and the trace $\text{Tr}$ by

$$\text{Tr}(fu^n) = \delta_{n0} \int_S f d\lambda.$$ 

Above, $u$ is as in (6.1).

8.2. Gap-Labelling

The winding number defined in the last section will be related to the gap labels of the bulk spectrum. We recall therefore the construction of these labels which is due to [4,5].

The dynamical system $(\Xi, \alpha)$ is uniquely ergodic. The unique ergodic probability measure $m$ defines a normalized trace $\text{tr} : C(\Xi) \rtimes_\alpha \mathbb{Z} \to \mathbb{C}$ via

$$\text{tr} \left( \sum_n f_n u^n \right) = \int_\Xi f_0(\xi) dm(\xi).$$

Since traces are cyclic $\text{tr}^*([p]) = \text{tr}(p)$ is well defined on classes of projections and so defines a tracial state $\text{tr}^* : K_0(C(\Xi) \rtimes_\alpha \mathbb{Z}) \to \mathbb{R}$. The image of this tracial state is called the gap-labelling group. This name comes from the application to Schrödinger operators. Indeed, any spectral projection $P_{\leq E}$ of the operator onto its states of energy below $E$ is an element of the $C^*$-algebra, provided $E$ lies in the gap of the spectrum. The class of the spectral projection then defines an element in the $K_0$-group, and $\text{tr}^*([P_{\leq E}])$ equals the integrated density of states up to energy $E$ [5]. Its numerical representation was already supplied in Sect. 5.4.

For Sturmian systems the gap-labelling group is known to be $\mathbb{Z} + \theta \mathbb{Z}$ and generated by its values on two projections, namely the identity $1 \in C(\Xi) \subset C(\Xi) \rtimes_\alpha \mathbb{Z}$ and a projection $\chi_{[\phi_1^+, \phi_2^-]} \in C(\Xi) \subset C(\Xi) \rtimes_\alpha \mathbb{Z}$ [5]. More precisely, if we identify $\Xi$ with the cut up circle, then the indicator functions on subsets $[\phi_1^+, \phi_2^-]$ are continuous and hence projections, provided $\phi_1$ and $\phi_2$ are singular values. The ergodic probability measure corresponds to the Lebesgue measure on the circle (normalized to 1) and hence $\text{tr}(\chi_{[\phi_1^+, \phi_2^-]}) = \phi_2 - \phi_1$.

8.3. The Correspondence

We are now ready to prove our main result. Recall that $\sigma$ is a left inverse for the map induced on $K$-theory by the inclusion $C(\tilde{\Xi}) \otimes \mathcal{K} \hookrightarrow J$ and that the composition $\sigma \circ \exp_J : K_0(C(\Xi) \rtimes_\alpha \mathbb{Z}) \to K_1(C(\tilde{\Xi}))$ does not depend on its choice. $W$ denotes the winding number (8.1) of the last section. The following theorem applied to the Fermi projection $P_F(H_\Xi)$ of $H_\Xi$ yields the statement of the introduction, relating the value of the integrated density of states of $H_\Xi$ if the Fermi energy lies in a gap to the winding number of the boundary eigenvalues of $\hat{H}_\Xi$ in that gap.
Theorem 8.1. Let $p$ be a projection in $C(\Xi) \rtimes_\alpha \mathbb{Z}$. There are $N, n \in \mathbb{Z}$ such that

$$\text{tr}_*(\{p\}) = N + n\theta.$$ 

Moreover, $N$ is the unique integer such that $\text{tr}_*(\{p\}) \in [0, 1]$ and

$$n = -\mathcal{W}([\sigma \circ \exp_J([p])]).$$ 

Proof. We verify the result on the generators of $K_0(C(\Xi) \rtimes_\alpha \mathbb{Z})$. We infer from (7.1) that there are two generators: the generator coming from $K_0(C(\tilde{\Xi}))$ and the one from $K_1(C(\tilde{\Xi}))$. The first is given by the class $[1]$ of the identity. It satisfies $\text{tr}_*(\{1\}) = 1$ and $\exp_J([1]) = 0$ and hence verifies the theorem.

Consider the projection $\chi_{[0^+, \theta^[-]}$ whose trace is, as we have seen, $\theta$. We need to determine $\mathcal{W}([\sigma \circ \exp_J([\chi_{[0^+, \theta^[-]}])].$ For that we first show that $\chi_{[0^+, \theta^[-]}$ lifts to a projection $P_0 \in C(\Xi) \rtimes_\alpha \mathbb{Z}$. Let $f$ be a real function on $\tilde{\Xi}$ which, in the open interval parametrized by $0 \times (0, 1)$, increases smoothly from 0 to 1, in the open interval parametrized by $\theta \times (0, 1)$, decreases smoothly from 1 to 0, and is otherwise constant. In particular, $f$ restricts to $\chi_{[0^+, \theta^[-]}$ on $\Xi$. We then set

$$P_0 = gu + f + u^*g$$

where $g = \sqrt{f(1 - f)}\chi_{(0,1)}.$ (So the support of $g$ is where $f$ decreases to 0.) Clearly, $q(g) = 0$ and $q(f) = \chi_{[0^+, \theta^[-]}$ so $P_0$ is a lift of $\chi_{[0^+, \theta^[-]}$. Moreover, $P_0$ is a projection and therefore $\sigma \circ \exp_J([\chi_{[\theta^+, \theta^-, \theta^+]}) = \exp([P_0]) \in K_1(C(\tilde{\Xi}))$ where $\exp$ is the exponential map for the Toeplitz extension of $(\tilde{\Xi}, \alpha)$. According to Lemma 8.2, proven below, $\exp([P_0])$ has winding number $-1$. It follows that $\exp([P_0])$ is a generator of $K_1(C(\tilde{\Xi}))$ and hence $\chi_{[0^+, \theta^[-]}$ the other generator of $K_0(C(\Xi) \rtimes_\alpha \mathbb{Z})$. We have thus verified the statement of the theorem on the two generators.

The reader may have noticed that the projection $P_0$ constructed in the proof of the last theorem is reminiscent of the Rieffel projection, the difference being that Rieffel’s projection is an element of $C(S^1) \rtimes_\alpha \mathbb{Z}$ where $\alpha$ is the rotation by $\theta$, a minimal action, whereas the action on $C(\tilde{\Xi})$ is non-minimal.

Lemma 8.2. With the notation used in the proof of the last theorem $\exp[P_0]$ has winding number $-1$.

Proof. By (8.2) the winding number of $\exp[P_0]$ is given by $-i\text{Tr}(P_0[P_0, P_0'])$ where $P_0 = \dot{g}u + \dot{f} + u^*\dot{g}$ and $P_0' = igu - iu^*g.$ Then

$$-iP_0[P_0, P_0'] = gu[\dot{f}, u^*g] + f([u^*\dot{g}, gu] - [\dot{g}u, u^*g]) + u^*g[\dot{f}, gu] + R$$

where $R$ is a term which is proportional to nonzero powers of $u$ and hence vanishes under $\text{Tr}$. Since the integral is invariant under the action we can replace $\alpha^{-1}(g^2)(\alpha^{-1}(\dot{f}) - \dot{f})$ by $\frac{g^2}{2}(f - \alpha(f))$ and $f(\alpha^{-1}(g^2) - \dot{g}^2)$ by $(\alpha(f) - f)\dot{g}^2 = -(\alpha(f) - \dot{f})g^2 + T$ where $T$ is a total derivative. Thus,

$$-i\text{Tr}(P_0[P_0, P_0']) = \int_S 3(\dot{f} - \alpha(\dot{f}))g^2d\lambda = 6\int_0^1 \dot{f}(1,t)g^2(1,t)dt.$$
as \( \alpha(\dot{f}) = -\dot{f} \) on the support of \( g \). Finally, 
\[
\int_0^1 \dot{f}(1,t)g^2(1,t)dt = \int_0^1 (-1)(t-t^2)dt = -\frac{1}{6}
\]
and thus \(-i\text{Tr}(P_\theta[\dot{P}_\theta, P'_\theta]) = -1 \). \( \square \)

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**References**

[1] Baboux, F., Levy, E., Lemaitre, A., Gómez, C., Galopin, E., Gratiet, L.L., Sagnes, I., Amo, A., Bloch, J., Akkermans, E.: Measuring topological invariants from generalized edge states in polaritonic quasicrystals. Phys. Rev. B 95, 161114(R) (2017)

[2] Beckus, S.: Spectral approximation of aperiodic Schrödinger operators. PhD Thesis, Friedrich-Schiller-University, Jena (2016)

[3] Beckus, S., Bellissard, J., de Nittis, G.: Spectral continuity for aperiodic quantum systems I. General theory. J. Func. Anal. 275, 2917–2977 (2018)

[4] Bellissard, J.: K-theory of \( C^*\)-algebras in solid state physics. In: Dorlas, T.C., Hugenholtz, N.M., Winnink, M. (eds.) Statistical Mechanics and Field Theory: Mathematical Aspects, pp. 99–156. Springer, Berlin (1986)

[5] Bellissard, J.: Gap labeling theorems for Schrödinger operators. In: Waldschmidt, M., Moussa, P., Luck, J.-M., Itzykson, C. (eds.) From Number Theory to Physics. Springer, Berlin (1992)

[6] Bellissard, J., Iochum, B., Scoppola, E., Testard, D.: Spectral properties of one-dimensional quasi-crystals. Commun. Math. Phys. 125(3), 527–543 (1989)

[7] Bellissard, J., Iochum, B., Testard, D.: Continuity properties of the electronic spectrum of 1D quasicrystals. Commun. Math. Phys. 141, 353–380 (1991)

[8] Bernevig, B.A., Hughes, T.L., Zhang, S.-C.: Quantum spin Hall effect and topological phase transition in HgTe quantum wells. Science 314, 1757 (2006)

[9] Bandres, M.A., Rechtsman, M.C., Segev, M.: Topological photonic quasicrystals: fractal topological spectrum and protected transport. Phys. Rev. X 6, 011016 (2016)

[10] Connes, A.: Non Commutative Geometry. Acad. Press, San Diego (1994)

[11] Dareau, A., Levy, E., Aguilera, M.B., Bouguanne, R., Akkermans, E., Gerbier, F., Beugnon, J.: Direct measurement of Chern numbers in the diffraction pattern of a Fibonacci chain. arXiv:1607.00901v1 (2016)

[12] Elliott, G., Natsume, T., Nest, R.: Cyclic cohomology for one-parameter smooth crossed products. Acta Math. 160, 285–305 (1988)

[13] Fu, L., Kane, C.L.: Topological insulators in three dimensions. Phys. Rev. B 76, 045302 (2007)

[14] Forrest, A.H., Hunton, J.R., Kellendonk, J.: Cohomology of canonical projection tilings. Commun. Math. Phys. 226, 289–322 (2002)

[15] Hafezi, M., Mittal, S., Fan, J., Migdall, A., Taylor, J.M.: Imaging topological edge states in silicon photonics. Nat. Photonics 7, 1001–1005 (2013)

[16] Haldane, F.D.M.: Model for a quantum Hall-effect without Landau levels: condensed-matter realization of the parity anomaly. Phys. Rev. Lett. 61, 2015 (1988)
[17] Hatsugai, Y.: Chern number and edge states in the integer quantum Hall effect. Phys. Rev. Lett. 71, 3697–3700 (1993)
[18] Herman, M.R.: Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations. Pub. Math. IHES 49, 5–234 (1979)
[19] Hu, W., Pillay, J.C., Wu, K., Pasek, M., Shum, P.P., Chong, Y.D.: Measurement of a topological edge invariant in a microwave network. Phys. Rev. X 5, 011012 (2015)
[20] Hsieh, D., Qian, D., Wray, L., Xia, Y., Hor, Y.S., Cava, R.J., Hasan, M.Z.: A topological Dirac insulator in a quantum spin Hall phase. Nature 452, 970 (2008)
[21] Kitaev, A.: Periodic table for topological insulators and superconductors (Advances in Theoretical Physics: Landau Memorial Conference). In: AIP Conference Proceedings, vol. 1134, pp. 22–30 (2009)
[22] Kane, C.L., Lubensky, T.: Topological boundary modes in isostatic lattices. Nat. Phys. 10, 39–45 (2013)
[23] Kraus, Y.E., Lahini, Y., Ringel, Z., Verbin, M., Zilberberg, O.: Topological states and adiabatic pumping in quasicrystals. Phys. Rev. Lett. 109, 106402 (2012)
[24] Kohmoto, M., Oono, Y.: Cantor spectrum for an almost periodic Schrödinger operator and a dynamical map. Phys. Lett. 102A, 145–148 (1984)
[25] Kraus, Y.E., Ringel, Z., Zilberberg, O.: Four-dimensional quantum Hall effect in a two-dimensional quasicrystal. Phys. Rev. Lett. 111, 226401 (2013)
[26] Kane, C.L., Mele, E.J.: Quantum spin Hall effect in graphene. Phys. Rev. Lett. 95, 226801 (2005)
[27] Kane, C.L., Mele, E.J.: Z(2) topological order and the quantum spin Hall effect. Phys. Rev. Lett. 95, 146802 (2005)
[28] Kellendonk, J.: Gap labelling and the pressure on the boundary. Commun. Math. Phys. 258, 751–768 (2005)
[29] Kellendonk, J., Lenz, D.: Equicontinuous Delone dynamical systems. Can. J. Math. 65, 149–170 (2013)
[30] Kellendonk, J., Richard, S.: Topological boundary maps in physics. In: Perspectives in Operator Algebras and Mathematical Physics, Theta Ser. Adv. Math. vol. 8, pp. 105–121. Theta, Bucharest (2008)
[31] Kellendonk, J., Richter, T., Schulz-Baldes, H.: Edge current channels and Chern numbers in the integer quantum Hall effect. Rev. Math. Phys. 14, 87–119 (2002)
[32] Kellendonk, J., Schulz-Baldes, H.: Boundary maps for C*-crossed products with $\mathbb{R}$ with an application to the quantum Hall effect. Commun. Math. Phys. 249, 611–637 (2004)
[33] Kellendonk, J., Zois, I.: Rotation numbers, boundary forces and gap labelling. J. Phys. A Math. Gen. 38(18), 3937 (2005)
[34] Koenig, M., Wiedmann, S., Bruene, C., Roth, A., Buhmann, H., Molenkamp, L.W., Qi, X.-L., Zhang, S.-C.: Quantum spin hall insulator state in HgTe quantum wells. Science 318, 766 (2007)
[35] Kraus, Y.E., Zilberberg, O.: Topological equivalence between the Fibonacci quasicrystal and the Harper model. Phys. Rev. Lett. 109, 116404 (2012)
[36] Le, T.T.Q.: Local rules for quasiperiodic tilings. In: Moody, R.V. (ed.) The Mathematics of Long Range Aperiodic Order, pp. 331–366. Kluwer, Dordrecht (1997)
[37] Levy, E., Barak, A., Fisher, A., Akkermans, E.: Topological properties of Fibonacci quasicrystals: a scattering analysis of Chern numbers. arXiv:1509.04028v3 (2015)

[38] Moore, J.E., Balents, L.: Topological invariants of time-reversal-invariant band structures. Phys. Rev. B 75, 121306 (2007)

[39] Madsen, K.A., Bergholtz, E.J., Brouwer, P.W.: Topological equivalence of crystal and quasicrystal band structures. Phys. Rev. B 88, 125118 (2013)

[40] Morse, M., Hedlund, G.A.: Symbolic dynamics. Am. J. Math. 60, 815–866 (1938)

[41] Morse, M., Hedlund, G.A.: Symbolic dynamics II. Sturmian trajectories. Am. J. Math. 62, 1–42 (1940)

[42] Mitchell, N.P., Nash, L.M., Hexner, D., Turner, A., Irvine, W.T.M.: Amorphous gyroscopic topological metamaterials. arXiv:1612.09267v1 (2016)

[43] Nash, L.M., Kleckner, D., Read, A., Vitelli, V., Turner, A.M., Irvine, W.T.M.: Topological mechanics of gyroscopic metamaterials. Proc. Nat. Acad. Sci. 112, 14495–14500 (2015)

[44] Pimsner, M., Voiculescu, D.: Exact sequences for K-groups of certain cross-products of C* algebras. J. Oper. Theory 4, 93–118 (1980)

[45] Prodan, E.: Virtual topological insulators with real quantized physics. Phys. Rev. B 91, 245104 (2015)

[46] Prodan, E., Prodan, C.: Topological phonon modes and their role in dynamic instability of microtubules. Phys. Rev. Lett. 103, 248101 (2009)

[47] Paulose, J., Chen, B.-G., Vitelli, V.: Topological modes bound to dislocations in mechanical metamaterials. Nat. Phys. 11, 153–156 (2015)

[48] Ryu, S., Schnyder, A.P., Furusaki, A., Ludwig, A.W.W.: Topological insulators and superconductors: tenfold way and dimensional hierarchy. New J. Phys. 12, 065010 (2010)

[49] Rechtsman, M.C., Zeuner, J.M., Plotnik, Y., Lumer, Y., Podolsky, D., Dreisow, F., Nolte, S., Segev, M., Szameit, A.: Photonic floquet topological insulators. Nature 496, 196–200 (2013)

[50] Schulz-Baldes, H., Kellendonk, J., Richter, T.: Simultaneous quantization of edge and bulk Hall conductivity. J. Phys. A Math. Gen. 33, L27–L32 (2000)

[51] Schnyder, A.P., Ryu, S., Furusaki, A., Ludwig, A.W.W.: Classification of topological insulators and superconductors in three spatial dimensions. Phys. Rev. B 78, 195125 (2008)

[52] Simon, B.: Spectral averaging and the Krein spectral shift. Proc. AMS 126, 1409–1413 (1998)

[53] Song, J.C.W., Rudner, M.S.: Chiral plasmons without magnetic field. Proc. Nat. Acad. Sci. (PNAS) 113, 4658–4663 (2016)

[54] Tran, D.-T., Dauphin, A., Goldman, N., Gaspard, P.: Topological Hofstadter insulators in a two-dimensional quasicrystal. Phys. Rev. B 91, 085125 (2015)

[55] Tanese, D., Gurevich, E., Baboux, F., Jacqmin, T., Lemaitre, A., Galopin, E., Sagnes, I., Amo, A., Bloch, J., Akkermans, E.: Fractal energy spectrum of a polariton gas in a Fibonacci quasiperiodic potential. Phys. Rev. Lett. 112, 146404 (2014)
Verbin, M., Zilberberg, O., Lahini, Y., Kraus, Y.E., Silberberg, Y.: Observation of topological phase transitions in photonic quasicrystals. Phys. Rev. Lett. 110, 076403 (2013)

Verbin, M., Zilberberg, O., Lahini, Y., Kraus, Y.E., Silberberg, Y.: Topological pumping over a photonic Fibonacci quasicrystal. Phys. Rev. B 91, 064201 (2015)

Wang, Z., Chong, Y., Joannopoulos, J.D., Soljacic, M.: Observation of unidirectional backscattering-immune topological electromagnetic states. Nature 461, 772–775 (2009)

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Communicated by Jean Bellissard.
Received: December 21, 2017.
Accepted: February 27, 2019.