Percolation between \( k \) separated points in two dimensions

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We consider a percolation process in which \( k \) widely separated points simultaneously connect together \((k > 1)\), or a single point at the center of a system connects to the boundary \((k = 1)\), through adjacent connected points of a single cluster. These processes yield new thresholds \( p_{c,k} \) defined as the average value of \( p \) at which the desired connections first occur. These thresholds are not sharp as the distribution of values of \( p_{c,k} \) remains broad in the limit of \( L \to \infty \). We study \( p_{c,k} \) for bond percolation on the square lattice, and find that \( p_{c,k} \) are above the normal percolation threshold \( p_c = 1/2 \) and represent specific supercritical states. The \( p_{c,k} \) can be related to integrals over powers of the function \( P_{\infty}(p) = \text{the probability a point is connected to the infinite cluster; we find numerically from both direct simulations and from measurements of } P_{\infty}(p) \text{ on } L \times L \text{ systems that for } L \to \infty, P_{c_1} = 0.51761(3), P_{c_2} = 0.53220(3), P_{c_3} = 0.54458(3), \text{ and } P_{c_4} = 0.55530(3). \) The percolation thresholds \( p_{c,k} \) remain the same, even when the \( k \) points are randomly selected within the lattice. We show that the finite-size corrections scale as \( L^{-1/\nu} \) where \( \nu = \nu_2/4 + 1 \), with \( \beta = 5/36 \) and \( \nu = 4/3 \) being the ordinary percolation critical exponents, so that \( \nu_1 = 48/41, \nu_2 = 24/23, \nu_3 = 16/17, \nu_4 = 6/7 \), etc. We also study three-point correlations in the system, and show how for \( p > p_c \), the correlation ratio goes to 1 (no net correlation) as \( L \to \infty \), while at \( p_c \) it reaches the known value of 1.021….

PACS numbers:

I. INTRODUCTION

Percolation is the study of long-range connectiveness in systems such as graphs or lattices in which the sites or bonds are randomly occupied with probability \( p \). There is a well-defined threshold \( p_c \) at which the average size of a cluster first becomes infinite. The threshold can also be defined by considering finite systems (say an \( L \times L \) square), and studying the probability that a single cluster connects or spans two opposite sides. The average value of \( p \) at which spanning first occurs yields an estimate for \( p_c(L) \), and using finite-size scaling one can predict the value of \( p_c \) for \( L \to \infty \). For a square lattice with bond percolation, for example, one has \( p_c = 1/2 \) \[1\] \[2\].

Percolation has received a great deal of attention over the years; some recent papers include \[3\] \[19\].

For the ordinary percolation problem in \( d \) dimensions, the connectivity is usually considered between the pair of opposite \((d-1)\) dimensional hypersurfaces. Naturally, the question arises, what happens if the connectivity is considered between the \((d-2), (d-3), \ldots \) dimensional hypersurfaces? In this paper, we try with the simplest possible situation, that is the connectivity between the \((d-2)\) dimensional hypersurfaces in \( d = 2 \). More specifically, we study the percolation problem between the \( k \) widely separated points (dimension 0) on the two-dimensional square lattice, or between a single point and the boundary of the system.

The first threshold we consider is defined as the average value of \( p \) at which a point in the center of a square system first connects to any point on the boundary. This defines the threshold \( p_{c,1} \). The other thresholds are defined as the average value of \( p \) at which \( k \) points separated far apart in a periodic system all first connect; we call those thresholds \( p_{c,k} \). These thresholds are all greater than \( p_c \), indicating that we are in the supercritical regime of percolation where there is a percolating net throughout the system. Being in a supercritical state is expected since connecting a large cluster to a specific single point at the normal critical point \( p_c \) occurs with low probability (unlike connecting to a boundary, for example, which can occur through many paths and is much easier). Connecting to a boundary is a universal property that survives at the critical point when the lattice spacing goes to zero, while in that limit the probability of connecting to a single point goes to zero. When going to the supercritical regime, the probability of connecting to a point can be raised to a significant value, and this allows different points to connect together simultaneously with a sufficient probability to be observed.

We carried out computer simulations to find the values of \( p_{c,k} \) directly for \( k = 1, 2, 3 \) and 4. We also developed a theory to connect \( p_{c,k} \) to \( P_{\infty} \), the percolation function that gives the probability a given point belongs to the infinite cluster, or the largest cluster for a finite system. By directly simulating \( P_{\infty} \) for this system, we are able to verify numerically that the relation to \( p_{c,k} \) is valid. The analysis also shows that, unlike in the case of the usual percolation threshold, the distribution of \( p_{c,k} \) for individual systems is broad and does not become sharp as the system size goes to infinity. That is, there are...
large fluctuations in the states of these systems defined by these percolation criteria.

In Fig. 1 we show pictures of simulations of a 64 × 64 periodic system in which the first connection between the two anchor points occurred when 4415 bonds were placed down, or at \( p_{c2} = 4415/8192 \approx 0.53894 \), and the same system at the standard threshold \( p = 4096/8192 = 1/2 = p_c \), at which point no connection exists between the two anchor points for this system. The value of \( p_{c2} \) for this sample is close to the average value \( p_{c2} = 0.5312 \) found by averaging over many realizations. It can be seen that, at \( p_{c2} \), there is one overwhelming “infinite” cluster throughout the system, and finite clusters are very small. This behavior illustrates the idea behind our conjecture that in the supercritical region, the probability that \( k \) points are connected together is equal to \( [P(\infty)]_k \).

In Fig. 2 we show a very rare case where the connection between the anchor points occurred at a value substantially below \( p = 1/2 \); for large systems such cases appear with very low probability.

We also studied a ratio involving three-point correlations and two-point correlations, and show how that varies with the separation of the points compared with the size of the system. This ratio has been studied previously at the critical point only \([20, 21]\); here we study it for all \( p \).

In section II we develop our theory for \( \overline{p}_{ck} \), including the scaling of the estimates. In section III we describe out simulation methods, and in section IV we give the results of our simulations. In section V we consider the problem of the three-point correlation ratio. In section VI we discuss out results and give our conclusions.

II. THEORETICAL ANALYSIS

Here we develop a theory to predict \( \overline{p}_{ck} \) from \( P(\infty) \), and develop a scaling analysis that allows one to predict the convergence exponents for the \( \overline{p}_{ck} \).

A. Relation to \( P(\infty) \)

The first assumption is that we must be in the supercritical state, since only then will the \( k \) points be able to connect together via the infinite network. At \( p_c \), the infinite cluster is tenuous and fractal, and does not connect to given points with a significant probability, and below \( p_c \) the clusters are all small and it would be virtually impossible for points far apart to connect together.

Thus, for \( k \) widely separated points to be all connected together, we hypothesize that they are part of the infinite cluster in the supercritical state. The probability a single point belongs to the infinite cluster is denoted as \( P(\infty) \); for a finite system we can define \( P(\infty, p, L) = s_{\text{max}}/L^2 \) where \( s_{\text{max}} \) is the number of sites in the largest cluster in the system. Thus, we conjecture that the probability that \( k \) widely separated points are connected must be equal to \( [P(\infty)]_k \). The probability that they first connect when the occupation probability is \( p \) is then \( (d/dp)[P(\infty)]_k = k[P(\infty)]_{k-1}P(\infty, p) \), and the average value of \( p \) at which the \( k \) points first connect will be given by

\[
\overline{p}_{ck} = \langle p \rangle = \int_0^1 p (d/dp)[P(\infty)]_k dp
\]

Integrating by parts, we find

\[
\overline{p}_{ck} = 1 - \int_0^1 [P(\infty)]_k dp = \int_0^1 (1 - [P(\infty)]_k) dp
\]

For the problem of a single site connected to the boundary (corresponding to \( \overline{p}_{c1} \)), the above formulas also apply, taking \( k = 1 \). In this case, the largest cluster surely connects to the boundary, so we are asking for just the probability that a point connects to the largest cluster, which is given by \( P(\infty) \).

For \( k > 1 \) the value of \( \overline{p}_{ck} \) should be independent of the exact configuration of the \( k \) points, as long as their relative distances grow with \( L \), so that they become infinitely far apart as \( L \to \infty \) and greater than the correlation length \( \xi \), which is finite for any given \( p > p_c \). For finite systems, the specific configuration of the points will be relevant for the precise threshold.

We can make a very useful approximation for calculating \( \overline{p}_{ck} \) from \( P(\infty) \) for finite systems by simply assuming \( P(\infty) = 0 \) for \( p < p_c \), which is true for an infinite system. Then the integrand in the second form of equation (2) is exactly 1 in the interval 0 < \( p < p_c \), and we can write as an alternative to (2)

\[
\overline{p}_{ck} = p_c + \int_{p_c}^1 (1 - [P(\infty)]_k) dp
\]

Equations (2) and (3) are identical when \( L \to \infty \), but it will turn out that (3) gives a much better estimate of \( \overline{p}_{ck} \) for finite \( L \).

B. Scaling of the estimates

If we assume that the mapping of our problem to \( [P(\infty)]_k \) is correct for finite systems characterized by \( P(\infty, L) \), we can then estimate the scaling behavior of the estimates from finite-size scaling theory. That theory states that for \( L \to \infty \) and \( p - p_c \to 0 \) with \( (p - p_c)L^{1/\nu} \) constant,

\[
P(\infty, p, L) \sim a L^{-\beta/\nu} F(b(p - p_c)L^{1/\nu})
\]

where \( a \) and \( b \) are system-dependent constants (“metric factors”) while \( \beta, \nu \) and \( F(z) \) are universal quantities, having the same values and behavior for all systems of a given dimensionality, and also a given system shape for the case of \( F(z) \). For \( d = 2 \), one has \( \beta = 5/36 \) and \( \nu = 4/3 \) \([1]\).

We will apply this to the estimate for \( \overline{p}_{ck} \) given by Eq. (2). First we consider the interval \( p = (0, p_c) \). In this
FIG. 1: Two illustrations of the system are presented for a lattice of size 64 × 64 with periodic boundary conditions, and with k = 2 anchor points (marked by filled blue circles) separated by a distance of 32 lattice units. Bonds of the largest occupied cluster are shown in red, and all other occupied bonds are shown in black. (a) A system where the number of bonds is exactly 4096 or \( p = 4096/8192 = 1/2 = p_c \), without connection between the two anchor points. (b) The same system where the number of occupied bonds is increased to 4415 bonds or \( p_s = 4415/8192 \approx 0.53894 \), at which point connection between the anchors first occurred. This is a typical example where the threshold is near the average value of \( p_c \).

interval, we assume that the finite-size effects are essentially those given by the scaling function \( F(z) \), because when \( p < p_c \), \( P_\infty(p, \infty) = 0 \). That is, we assume the non-scaling corrections are unimportant for large \( L \) for \( p < p_c \).

Putting [4] into the integral in Eq. [2] over the interval

\[
p = (0, p_c), \quad \text{we find}
\]

\[
\int_0^{p_c} [P_\infty(p)]^k dp = a^k \int_0^{p_c} L^{-k\beta/\nu} [F(b(p - p_c)L^{1/\nu})]^k dp
\]

and a change of variables yields

\[
\int_0^{p_c} [P_\infty(p)]^k dp = a^k b^{-1} L^{-(k\beta+1)/\nu} \int_{-bp_cL^{1/\nu}}^0 [F(z)]^k dz
\]
where $z = b(p - p_c)L^{1/\nu}$. In the second integral in (6) we extended the lower limit to $-\infty$, valid for large $L$ because the integrand decays exponentially for negative $z$.

Therefore, this contribution to the integral in (2) should scale as $L^{-1/\nu_k}$ with

$$1/\nu_k = (k\beta + 1)/\nu = \frac{36 + 5k}{48}$$

so that $1/\nu_1 = 41/48 = 0.854166$ and $1/\nu_2 = -23/24 = 0.958333$ etc.

For $p > p_c$, it is not clear how to attack the finite-size corrections to $P_\infty$ whose behavior we do not know, but it seems reasonable to assume that the finite-size corrections for $p > p_c$ scale the same as those we found for $p < p_c$, so we conjecture that the exponents $\nu_k$ above should characterize the full finite-size corrections to $\overline{p}_{ck}$. We will verify this conjecture numerically.

Thus, we conclude that the scaling part of $k[P_\infty(p)]^{k-1}P'_\infty(p)$ goes to zero as $L^{-(k\beta+1)/\nu}$ as $L \to \infty$, while the remainder yields $p_{ck} - p_c$, when put in the integral in Eq. (2).

III. SIMULATION METHODS

A. Simulation method to find $\overline{p}_{ck}$

We carried out computer simulations of these processes on systems of size $L \times L$ for bond percolation, with periodic boundary conditions. For the case $k = 1$, we consider $L$ odd and add bonds until the center point connects to the boundary for $p = p_{c1}$. Repeating this process many times, we average the values of $p_{c1}$ to find $\overline{p}_{c1}$. For $k = 2$, 3 and 4, we consider periodic $L \times L$ systems with $L = 2^n$, $n = 5, 6, \ldots, 12$. For $k = 2$ we consider the connectivity between a point at the origin $(0,0)$ and a point at $(0, L/2)$. For $k = 3$, the connectivity between the three points $(0,0)$, $(L/2,0)$, and $(0, L/2)$, and for $k = 4$, the connectivity between the four points $(0,0)$, $(L/2,0)$, $(0, L/2)$, and $(L/2, L/2)$ are considered. The average value of $p$ at the first connection gives $\overline{p}_{ck}$.

It is clear from the Eq. (3) that the values of the thresholds $\overline{p}_{ck}$ should depend only on the value of $k$, and not on the actual distribution of the $k$ points. We have numerically verified this issue for $k = 2$ by randomly distributing these two points on the lattice for every configuration. Our simulation results show that the values of $\overline{p}_{ck}$ remain unchanged.

We also studied the average $\bar{p}$ at which the origin connects to point $x = 1$, $x = 2$, $\ldots$, $x = L/2$ and $y = 0$. 

FIG. 4: Plots of $2P_\infty(p)P'_\infty(p)$ for $L = 128$, 256 and 512. These are the analogous curves as given in Fig. 3 calculated directly from $P_\infty(p)$ rather than by direct simulation.

FIG. 5: Scaling plot of $L^{(2\beta-1)/\nu}2P_\infty(p)P'_\infty(p)$ vs. $(p-p_c)L^{1/\nu}$ for $L = 128$, 256 and 512. The curves collapse well to a universal curve, except for the tail for large $(p-p_c)L^{1/\nu}$ which represents the non-scaling part of this quantity.
We discovered that \( p_{c2}(x) \) does not noticeably depend upon \( L \) as along as \( x \ll L \), indicating that the size of the system is irrelevant for shorter-range connections.

**B. Simulation method to find \( P_\infty \)**

To test the conjecture relating \( \overline{p}_{ck} \) to \( P_\infty \), we carried out simulations of \( P_\infty(p) \) using the method of Newman and Ziff (NZ) \[22, 23\], which involves adding bonds one at a time to the system and using the union-find procedure to merge clusters and keep track of the cluster distribution. This method allows one to effectively measure a quantity \( Q(p) \) (such as \( P_\infty(p) \)) for all values of \( p \) in a single simulation. In this method, one first determines the “microcanonical” \( Q_n \) (here \( P_{\infty,n} \)) when exactly \( n \) bonds have been placed down, and then determines the “canonical” \( Q(p) \) (here \( P_\infty(p) \)) by going through a convolution with the binomial distribution \( B(N, n, p) = \binom{N}{n}p^n(1-p)^{N-n} \):

\[
Q(p) = \sum_{n=0}^{N} \binom{N}{n}p^n(1-p)^{N-n}Q_n
\]  

(8)

where \( N \) is the total number of bonds in the system, in this case \( 2L^2 \). For large systems, the differences between the microcanonical \( Q_n \), with \( n = pN \) and \( Q(p) \) are small, except for regions of high curvature or second derivative, but the convolution serves a further purpose of smoothing out the data, and connecting it with a continuous curve, rather than the discrete values \( p = 1/N, 2/N, \ldots \). To integrate \( P_\infty(p) \) (as required for \( \overline{p}_{ck} \) according Eqs. \[3\] and \[6\]), one can just as well sum the microcanonical values, because of the identity \[24\]

\[
\int_{0}^{1} Q(p)dp = \sum_{n=0}^{N} \binom{N}{n}Q_n \int_{0}^{1} p^n(1-p)^{N-n}dp
\]

\[= \frac{1}{N+1} \sum_{n=0}^{N} Q_n \]  

(9)

Likewise it follow that

\[
\int_{0}^{1} pQ(p)dp = \frac{1}{(N+1)(N+2)} \sum_{n=0}^{N} (n+1)Q_n
\]  

(10)

To integrate \( [Q(p)]^k \) \( [P_\infty(p)]^k \) for \( k > 1 \) with respect to \( p \), it is most straightforward to first carry out the convolution to find \( P_\infty(p) \), and then numerically integrate the \( [P_\infty(p)]^k \) at equally spaced values of \( p \).

Derivatives of \( Q(p) \) can also be found directly from the \( Q_n \) \[24\]:

\[
Q'(p) = \sum_{n=0}^{N} \binom{N}{n} \frac{d}{dp} p^n(1-p)^{N-n}
\]

\[= \frac{1}{p(1-p)} \sum_{n=0}^{N} (n-Np) \binom{N}{n} (p^n(1-p)^{N-n})Q_n \]

(11)

and likewise

\[
Q''(p) = \frac{(n^2Q_n) - (nQ_n)p + 1(nQ_n) + N(N-1)p^2(Q_n)}{p^2(1-p)^2}
\]

\[= \frac{(n-Np)Q_n - 2Np(1-p)(Q_n)}{p^2(1-p)^2} \]

where the averages are over the binomial distribution \( B(N, n, p) \). Note that in \[24\], there is a typo in Eq. (32) for \( Q''(p) \), in which the last term should have the factor \((N-n-1)\) rather than \((N-n+1)\).

![FIG. 6: A plot of (d/dp)[P_\infty(p)]^k vs. p for k = 1 (red), k = 2 (orange), k = 3 (green), and k = 4 (blue) for a system with L = 512, calculated from the results of the numerical simulations of P_\infty(p) using the NZ method, including using Eqs. (11) to find P_\infty(p). The estimates of \overline{p}_{ck} are the means of these distributions according to Eq. (11), and it can be seen that the distribution spreads to the right as \( k \) increases, yielding larger values of \overline{p}_{ck}. To find the accurate values of \overline{p}_{ck}, one has to consider systems of different \( L \) and take the limit that \( L \to \infty \), although there is not much change for systems larger than \( L = 512 \). Note that the distribution is broad and the large fluctuations in the individual values of \overline{p}_{ck} persist as \( L \to \infty \).](attachment:image.png)

To find \( P_\infty(p) \) we simulated 10^7 samples each for \( L = 64, 128, 256 \) and 512 on \( L \times L \) periodic systems, saving the 2L^2 microcanonical values of \( s_{\text{max}} \) in a file. For the largest system \( L = 512 \), the simulations took several days on a (laptop) computer. Then we used a separate program to read the files and calculate \( P_\infty(p) = s_{\text{max}}/L^2 \) for 10^4 points \( p = 0, 0.0001, \ldots, 1.0000 \) using the convolution \[6\]. We also calculated \( P'_\infty(p) \) and \( P''_\infty(p) \) using the formulas of Eqs. \[11\] and \[12\]. We used the recursive method described in Ref. \[23\] to calculate the binomial distribution for each \( p \). To find the integrals of \( [P_\infty(p)]^k \), we carried out numerical integration of the 10^4 points using the trapezoidal rule (namely counting the two endpoints with weight 1/2 and all other points with weight 1). We compared some of the integrals using 10^3 and 10^5 points and did not find significant difference in the results, and used 10^4 values of \( p \) in our calculations.
IV. RESULTS

Fig. 6 shows plots of the predicted distribution of the probabilities of first connection, \((d/\partial p)[P_\infty(p)]^k\), for \(k = 1, 2, 3,\) and \(4\), based upon measurements \(P_\infty(p)\), for a system of \(L = 512\). As can be seen, the distribution is broad, meaning that the thresholds we find \(p_c, k\) have large fluctuations from system to system and persist as \(L \to \infty\).

In Fig. 7 we plot estimates for \(p_{c1}\) found from direct simulations of a point at the center to the boundary of an \((L + 1) \times (L + 1)\) square system (magenta triangles), by integrating \(P_\infty(p)\) on \(L \times L\) periodic systems using Eq. (2) for \(k = 1\) (black squares), and by integrating \(P_\infty(p)\) using Eq. (3) (blue circles). The estimates are all plotted vs. \(L^{-41/48}\) according to the prediction of Eq. (7). The equations of the linear fits through the points are \(p_{c1} = a + bL^{-41/48}\) with \(a = 0.55520, b = -0.33805\) (black), \((middle) a = 0.55532, b = -0.27962\) (magenta), \(a = 0.55530, b = -0.06323\) (blue).

FIG. 7: Values of \(p_{c1}(L)\) found from simulations of connections of a point at the center to the boundary of an \((L + 1) \times (L + 1)\) square system (magenta triangles), by integrating \(P_\infty(p)\) on \(L \times L\) periodic systems using Eq. (2) for \(k = 1\) (black squares), and by integrating \(P_\infty(p)\) using Eq. (3) (blue circles). The equations of the linear fits through the points are \(p_{c1} = a + bL^{-41/48}\) with \(a = 0.55520, b = -0.33805\) (black), \((middle) a = 0.55532, b = -0.27962\) (magenta), \(a = 0.55530, b = -0.06323\) (blue).

FIG. 8: Values of \(p_{c2}(L)\) found from direct simulations on an \(L \times L\) periodic system with the two points at \((0,0)\) and \((0,L/2)\) (magenta triangles), and the predictions from Eqs. (2) (black squares) and (3) (blue circles) based upon \(P_\infty(p)\) on an \(L \times L\) periodic system, all plotted as a function of \(L^{-46/48} = L^{-23/24}\) as predicted by Eq. (7). Here \(L = 32, 64, 128, 256,\) and \(512\) for the upper two sets of data, and also \(L = 1024\) for the lower set.

V. CORRELATIONS

We also considered a related question for two- and three-point correlations. Studying this problem sheds light on the correlations that occur in the system in the critical vs. the post-critical regime where the connectivity between the anchor points mainly occurs.

In [20, 21] the following ratio was considered:

\[
R = \frac{P(r_1, r_2, r_3)}{\sqrt{P(r_1, r_2)P(r_1, r_3)P(r_2, r_3)}}
\]

where \(r_1, r_2\) and \(r_3\) are three points in the system, \(P(r_i, r_j)\) is the probability that points \(r_i\) and \(r_j\) connect, and \(P(r_1, r_2, r_3)\) is the probability that all three points connect.

This ratio has previously been studied, to our knowledge, only at \(p = p_c\), where the value of \(R\) approaches...
Based upon P\textsuperscript{∞}(p) using Eqs. (2) and (3), both based upon P\textsuperscript{∞}(p) on an L × L periodic system. All data are plotted as a function of L\textsuperscript{-56/48} = L\textsuperscript{-7/6} as predicted by Eq. (7). Lines show linear fits through the data. It can be seen that estimates based upon Eq. (3) exhibit the fastest convergence with system size.

![Graph](image1)

**TABLE I**: Our best estimates for the extrapolated values of the percolation thresholds \( p_{c,k} \) from direct measurements (second column) and from \( P_{\infty} \) via Eqs. (2) and (3) for different values of \( k \).

| \( k \) | \( p_{c,k} \) measured | Eq. (2) | Eq. (3) |
|---|---|---|---|
| 1 | 0.51749(5) | 0.51761(3) | 0.51759(5) |
| 2 | 0.53212(5) | 0.53220(3) | 0.53226(3) |
| 3 | 0.54450(5) | 0.54458(3) | 0.54460(3) |
| 4 | 0.55520(5) | 0.55530(3) | 0.55531(3) |

Here we consider the three points in a right triangle, \((0, 0), (0, L/n), \) and \((L/n, 0)\), in an \( L \times L \) periodic system, for \( n = 2, 4 \) and 8. As \( n \) increases for large \( L \) (that is, as the separation of the three points is small compared to the size of the system), \( R \) approaches the value \( C_1 \). Using the NZ method, we were able to calculate \( R \) as a function of \( p \) after executing a microcanonical simulation where we found the \( P \)'s as a function of the number of bonds added. We then carried out the convolution to the canonical \((p\text{-dependent}) \) functions for all \( P \) separately, and calculated \( R \) according to Eq. (13). The results are shown in Fig. 11.

As can be seen, at \( p = p_c \), \( R(p_c) \) approaches \( C_1 \) as \( n \) increases (in which case the points get closer together compared to the size of the system). In the limit that \( L \to \infty \), \( R \) becomes a discontinuous function, with \( R = 0 \) for \( p < p_c \), \( R = C_1 \) for \( p = p_c \), and \( R = 1 \) for \( p > p_c \). Notice in Fig. 11 that there is a maximum for \( \tilde{R} \) in finite systems at \( z = (p - p_c)L^{1/\nu} \approx -0.5 \), meaning for some values of \( p < p_c \), \( R \) is greater than the value at \( p_c \). However, it is not clear what the behavior is as \( n \to \infty \); it is possible that the peak for negative \( z \) disappears and the peak occurs only at \( z = 0 \) or \( p = p_c \). The scaling limit here can use further investigation.

At the point \( p_{c,k} \approx 0.5445 \) where three points first connect, it can be seen that \( R \) approaches 1, since that would correspond to \( (p - p_c)L^{1/\nu} \) going to infinity as \( L \) goes to infinity. This result reiterates that at the places where multiple points connect, there are no correlations among connections between different pairs of points.

**VI. DISCUSSION**

We have shown that exploring the average value of the probability \( p \) of bond occupation at which a certain number of separated points first connect leads to a new set of average thresholds. The distribution of the values of \( p \) is broad, so that this threshold is not sharp as in the usual case of thresholds in percolation. For example, the median rather than the mean of the distribution would give a different values. We have shown that the values can be related to \( P_{\infty}(p) \), and confirm this relation by simulation. From this theory it is apparent that while the percolation thresholds \( p_{c,k} \) indeed depends on the number \( k \) of points, their values are robust with respect to the

![Graph](image2)

FIG. 10: Values of \( p_{c,k}(L) \) found from direct simulations on an \( L \times L \) periodic system with the four points at the corners of a square of length \( L/2 \) (magenta triangles), and the predictions based upon \( P_{\infty}(p) \) using Eqs. (2) (black squares) and (3) (blue circles), both based upon \( P_{\infty}(p) \) on an \( L \times L \) periodic system. All data are plotted as a function of \( L^{-56/48} = L^{-7/6} \) as predicted by Eq. (7). Lines show linear fits through the data. It can be seen that estimates based upon Eq. (3) exhibit the fastest convergence with system size.

FIG. 11: Scaling plot of \( R \) vs. \((p - p_c)L^{1/\nu} \) where \( R \) is given in Eq. (13), for three points at \((0, 0), (0, L/n), \) and \((L/n, 0)\), for \( n = 2 \) (the two curves with the highest peaks), \( n = 4 \), (two curves with the medium peak), and \( n = 8 \) (curve with the lowest peak), for \( L = 128 \) (blue curves) and 256 (violet curves). There is a small \( L \)-dependence on the results, but the main variation is due to \( n \). At \( p = p_c = 1/2 \), the value of \( R \) approaches the theoretical value \( C_1 = 1.022 \) \cite{20, 21} as \( n \) gets large, in which case the three points are close together compared to the size of the system.

the value \( C_1 = 1.02201 \) when the three points are far separated and the system size is infinite. This value of \( C_1 \) was first observed numerically in \cite{20} and then derived analytically from conformal field theory in \cite{21}. The fact that this ratio is unequal to 1 implies a three-point correlation between the three points in the system. If we make the assumption that \( P(r_1, r_2) = P_{\infty}(p)^2 \) and \( P(r_1, r_2, r_3) = P_{\infty}(p)^3 \), which we expect to be the case for \( p > p_c \), then we would have \( R = 1 \). At \( p_c \), where the infinite cluster does not span throughout the system, one would not expect this to be valid and indeed \( R \neq 1 \), although quite close to it.
actual spatial distribution of the $k$ points. For example, the $k$ points may either be symmetrically placed on the lattice or, they could be randomly distributed (for $L \to \infty$).

This work suggests further research in a variety of areas. It might be interesting to study these thresholds in higher dimensions, where the relations to $P_{\infty}(p)$ in Eqs. [2] and [3], and the scaling in [7] (but with $\nu$ and $\beta$ being the three-dimensional result) should still hold, for connections to points as we considered here. Furthermore, connections between higher-dimensional objects (lines, surfaces, ...) can also be considered. One question to consider is whether the thresholds continue to have broad distributions as found here, and how those thresholds scale with $L$.

With respect to the correlations $R$, one can consider a point in the center of a cylinder (that is a square with periodic b.c. in one direction), and find the probability of connecting the center to one boundary or to both boundaries of the cylinder. At $p_c$, the corresponding $R$ should go to the value $C_0 = 27/2 - 3/4 - 3/2 - 3/2 = 1.0299268\ldots$ [20] while the behavior away from $p_c$ has not been studied before. Likewise, similar correlations in higher dimensions have not been studied. Many aspects of correlations in percolation are yet to be explored.

VII. ACKNOWLEDGMENT

The authors would like to thank Deepak Dhar for a careful reading and constructive comments on the paper.

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