DERIVATION OF A NONLINEAR SCHRÖDINGER EQUATION WITH A GENERAL POWER-TYPE NONLINEARITY

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ABSTRACT. In this paper we study the derivation of a certain type of NLS from many-body interactions of bosonic particles. We consider a model with a finite linear combination of $n$-body interactions, where $n \geq 2$ is an integer. We show that the $k$-particle marginal density of the BBGKY hierarchy converges when particle number goes to infinity, and the limit solves a corresponding infinite Gross-Pitaevskii hierarchy. We prove the uniqueness of factorized solution to the Gross-Pitaevskii hierarchy based on a priori space time estimates. The convergence is established by adapting the arguments originated or developed in [6], [12] and [2]. For the uniqueness part, we expand the procedure followed in [13] by introducing a different board game argument to handle the new contraction operator. This new board game argument helps us obtain a good estimate on the Duhamel terms. In [13], the relevant space time estimates are assumed to be true, while we give a prove for it.

1. Introduction

The nonlinear Schrödinger equation (NLS) is a macroscopic model for a quantum mechanical system, with different type of nonlinearities depending on the way we model the interaction potential (cubic, quintic, Hartree, etc.) in a quantum many body system. A derivation of the corresponding PDE that governs the system is a hot topic in mathematical physics that has been drawing much attention during the past decade. Some of the references in this direction include [18], [9], [5], [8], [6], [13], [12], [2], [11], [10], [1] etc. In particular, the sequence of crucial works by Elgart, Erdős, Schlein and Yau [9], [5], [8], [6] studied a model of Bose gas in $\mathbb{R}^3$ with pairwise interactions and rescaled potentials $V_p^{(p)}$ approaching a delta function. They proved that the $k$-particle density matrix for BBKGY hierarchy converges to that of the infinite hierarchy (GP hierarchy), which is actually governed by the solution of the cubic non-linear Schrödinger equation. In their work, uniqueness of solutions to the GP hierarchy is established via Feynman diagrams.

In this paper, we derive a nonlinear Schrödinger equation with a linear combination of power type nonlinearities. Our work is motivated by [12] and [2], in which the authors consider a quantum model with 2-body interactions [12] and 3-body interactions [2] respectively and obtain cubic and quintic NLS correspondingly that correctly describes the system. It is also worth mentioning that in [2], Chen and Pavlović predict that, if both 2-body and 3-body interactions are present in a quantum model, then that would lead (via Gross-Pitaevskii limit) to a NLS with a linear combination of cubic and quintic nonlinearities. In this paper, we will give a proof of that claim. Actually, we generalize the prediction from [2] and derive the NLS
with a finite linear combination of power nonlinearities. We also note that a particular example of such kind of NLS was studied by Tao-Visan-Zhang in [19], in which local and global wellposedness and related questions are explored.

1.1. BBGKY hierarchy. We consider a quantum mechanical system of \( N \) bosonic particles in \( \mathbb{R}^d \), with \( d \in \{1, 2\} \). Let \( p \) and \( p_0 \) be positive integers, fixed \( p_0 \), \( 1 \leq p \leq p_0 \). The time evolution of the \( N \) particle wave function \( \psi_N \in L^2_s(\mathbb{R}^{dN}) \) is governed by the Schrödinger equation

\[
(1.1) \quad i \partial_t \psi_{N,t} = H_N \psi_{N,t}
\]

with the Hamiltonian

\[
(1.2) \quad H_N := \sum_{i=1}^{N} (-\Delta_{x_i}) + \sum_{p=1}^{p_0} \frac{1}{N^p} \sum_{1 \leq i_1 < \cdots < i_{p+1} \leq N} N^{\beta(p)} V^{(p)}(N^\beta(x_{i_1} - x_{i_2}), \ldots, N^\beta(x_{i_{p+1}}))
\]

on Hilbert space \( L^2(\mathbb{R}^{dN}) \), which is the subspace of \( L^2(\mathbb{R}^{dN}) \) consisting of all functions satisfying

\[
\psi_N(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(N)}) = \psi_N(x_1, x_2, \ldots, x_N),
\]

for any permutation \( \sigma \in S_N \) and \( 0 < \beta < \frac{1}{2dp+2} \). Also we assume that for all \( 1 \leq p \leq p_0 \) the \( (p+1) \)-body interaction potential \( V^{(p)} \in W^{p,\infty}(\mathbb{R}^d) \) is a non-negative function with sufficient regularity and it is translation-invariant so that it can be written in the above form. For instance, when \( p = 2 \), we have that

\[
V^{(2)}(x_1 - x_2, x_2 - x_3, x_1 - x_3) = V^{(2)}(x_1 - x_2, -(x_1 - x_2) + (x_1 - x_3), x_1 - x_3) = V^{(2)}(x_1 - x_2, x_1 - x_3).
\]

The first part of the Hamiltonian represents the kinetic energy part, while the second is the sum of interaction potentials involving \( p+1 \) particles.

Note that (1.1) is linear, which together with the fact that \( H_N \) is a self-adjoint operator implies that global in time solutions can be written by means of the unitary group generated by \( H_N \) as

\[
(1.3) \quad \psi_{N,t} = e^{-iH_N t} \psi_{N}, \quad \forall t \in \mathbb{R}
\]

As in previous works on derivation of NLS from many body quantum dynamical systems [6, 12, 2], we define the corresponding \( N \)-particle density matrix as follows:

\[
(1.4) \quad \gamma_{N,t}(t, x_N; x_N') = \psi_{N,t}(x_N) \bar{\psi}_{N,t}(x_N')
\]

\( \bar{\psi}_{N,t} \) denotes the complex conjugate of \( \psi_{N,t} \). Then (1.3) implies that

\[
(1.5) \quad \dot{i} \partial_t \gamma_{N,t} = [H_N, \gamma_{N,t}],
\]

where the Heisenberg commutator is given as usual \([A, B] := AB - BA\). The \( L^2 \)-normalization of \( \psi_{N,t} \) implies that \( T^\gamma \gamma_{N,t} = 1 \). By taking partial trace of \( \gamma_{N,t} \) over the last \( N-k \) particles we define the \( k \)-particle marginal density:

\[
(1.6) \quad \gamma_{N,k}^{(k)}(t, x_k; x_k') = \int \gamma_{N,t}(t, x_k, x_{N-k}; x_k', x_{N-k}) d x_{N-k}
\]

where \( x_k = (x_1, \ldots, x_k) \), \( x_{N-k} = (x_{k+1}, \ldots, x_N) \), \( k = 1, \ldots, N \).
Let \( V^{(p)}(x_1, x_2, \cdots, x_p) := N^{d/p} V^{(p)}(N^{\beta} x_1, N^{\beta} x_2, \cdots, N^{\beta} x_p) \). We can verify that the marginal densities satisfy the so called BBGKY hierarchy

\[
(1.7) \quad i \partial_t \gamma^{(k)}_{N,t} = \sum_{i=1}^{k} \left[ -\Delta x_i + \gamma^{(k)}_{N,t} \right] + \sum_{p=1}^{\infty} \left\{ \frac{1}{N^p} \sum_{1 \leq i_1 < \cdots < i_{p+1} \leq k} \left[ V^{(p)}(x_{i_1} - x_{i_2}, \cdots, x_{i_1} - x_{i_{p+1}}), \gamma^{(k)}_{N,t} \right] + \frac{N - k}{N^p} \sum_{1 \leq i_1 < \cdots < i_{p+1} \leq k} Tr_{k+1} \left[ V^{(p)}(x_{i_1} - x_{i_2}, \cdots, x_{i_1} - x_{i_{p+1}}, x_{i_1} - x_{k+1}), \gamma^{(k+1)}_{N,t} \right] \right. \\
+ \frac{(N-k)(N-k-1)}{N^p} \sum_{1 \leq i_1 < \cdots < i_{p+1} \leq k} Tr_{k+1} Tr_{k+2} \left[ \sum_{1 \leq i_1 \leq k} Tr_{k+1} Tr_{k+2} \cdots Tr_{k+p} \right] \\
\left. \left[ V^{(p)}(x_{i_1} - x_{k+1}, x_{i_1} - x_{k+2}, \cdots, x_{i_1} - x_{k+p}), \gamma^{(k+p)}_{N,t} \right] \right\}.
\]

Here we use the convention that \( \gamma^{(k)}_{N,t} = 0 \), whenever \( k > N \). The symbol \( Tr_{k+1} \) denotes the partial trace over the \( m \)-th particle, i.e. the kernel of the \( k \)-particle operator \( Tr_{k+1} \left[ V^{(p)}(x_{i_1} - x_{i_2}, \cdots, x_{i_1} - x_{i_{p+1}}, x_{i_1} - x_{k+1}), \gamma^{(k+1)}_{N,t} \right] \) is given by

\[
(Tr_{k+1} V^{(p)}(x_{i_1} - x_{i_2}, \cdots, x_{i_1} - x_{i_{p+1}}, x_{i_1} - x_{k+1}), \gamma^{(k+1)}_{N,t})(x_k; x_k)
= \int V^{(p)}(x_{i_1} - x_{i_2}, \cdots, x_{i_1} - x_{i_{p+1}}, x_{i_1} - x_{k+1}) \gamma^{(k+1)}(x_k, x_k; x_k, x_{k+1}) dx_{k+1}
- \int V^{(p)}(\delta_{x_{i_1}'}, \cdots, \delta_{x_{i_1}'}, x_{i_1} - x_{i_{p+1}}, x_{i_1} - x_{k+1}) \gamma^{(k+1)}(x_k, x_{k+1}; x_k, x_{k+1}) dx_{k+1}
\]

Let us present a heuristic argument on what one expects when taking \( N \to \infty \). In particular, we note that all the terms in (1.7), except the first term on the RHS and the last term in the bracket, are expected to vanish for fixed \( k \) and sufficiently small \( \beta \), because \( \frac{1}{N^p} \to 0 \), \( \frac{\prod_{j=0}^{p-1}(N-k-j)}{N^p} \to 0 \), \( \forall 0 \leq j \leq p - 2 \). The last interaction term on the RHS is expected to survive thanks to \( \frac{\prod_{j=0}^{p-1}(N-k-j)}{N^p} \to 1 \). Indeed, one can make this heuristic precise and prove existence of a weak sequential limit of (1.7) in the same topology that was originally used in [6], and subsequently in [12, 2]. Details are presented in Section 2.

1.2. GP hierarchy. Following the convention in [1], we formally write down the limit of (1.7) as \( N \to \infty \), as follows:

\[
(1.9) \quad i \partial_t \gamma^{(k)}_{\infty,t} = \sum_{j=1}^{k} \left[ -\Delta x_j + \Delta x_j' \right] \gamma^{(k)}_{\infty,t} + \sum_{p=1}^{\infty} \sum_{j=1}^{k} b^{(p)}_{j} \sum_{p=1}^{\infty} B_{j; k+1, \ldots, k+p} \gamma^{(k+p)}_{\infty,t}
\]
for any \( k \geq 1 \). We call (1.14) cubic Gross-Pitaevskii (GP) hierarchy if \( p = 1 \); quintic GP hierarchy if \( p = 2 \) and septic GP hierarchy if \( p = 3 \), and so on. Here \( b_0^{(p)} \) is the \( L^1 \) norm of the non-negative potential: 
\[
b_0^{(p)} = \int_{\mathbb{R}^d} V^{(p)}(x_1, \cdots, x_p) dx_1 \cdots dx_p.
\]

The contraction operator is given via 
\[
B^+_{j;k+1,\ldots,k+p} = B^-_{j;k+1,\ldots,k+p} - B^\varepsilon_{j;k+1,\ldots,k+p}
\]
where 
\[
B^\varepsilon_{j;k+1,\ldots,k+p} = \frac{1}{\gamma_{x,t}}(t, x_k, x_k')
\]
\[
\gamma_{x,t}(t, x_k, x_k') = \int \delta(x_j - x_{k+1}) \delta(x_j - x_{k+p}) \delta(x_j - x_{k+p}') \cdot \gamma_{x,t}(t, x_1, \cdots, x_{k+p}; x_1', \cdots, x_{k+p}') dx_{k+1} \cdots dx_{k+p} dx_{k+p}'
\]
and 
\[
B^-_{j;k+1,\ldots,k+p} = \frac{1}{\gamma_{x,t}}(t, x_k, x_k')
\]
\[
\gamma_{x,t}(t, x_k, x_k') = \int \delta(x_j - x_{k+1}) \delta(x_j - x_{k+p}) \delta(x_j - x_{k+p}') \cdot \gamma_{x,t}(t, x_1, \cdots, x_{k+p}; x_1', \cdots, x_{k+p}') dx_{k+1} \cdots dx_{k+p} dx_{k+p}'
\]

We can check that 
\[
\gamma_{x,t}^{(k)} = \langle \phi_t \rangle \langle \phi_t \rangle^{\otimes k} = \prod_{j=1}^k \phi_t(x_j) \phi_t(x_j')
\]
is a solution to (1.10) if \( \phi_t \) is a solution to the nonlinear Schrödinger equation 
\[
i \hat{\Delta} \phi_t = -\nabla \cdot \int_{\mathbb{R}^d} \phi_t(x) \phi_t(x') dx + \sum_{p=1}^{p_0} b_0^{(p)} \phi_t^{2p} \phi_t.
\]

We hope to establish the uniqueness on solutions of the Gross-Pitaevskii hierarchy, and build the following convergence under appropriate topology:

\[
\gamma_{N,t}^{(k)} \rightarrow \gamma_{x,t}^{(k)}, \quad \text{as } \ N \rightarrow \infty, \quad \forall k \geq 1
\]
The uniqueness of solution to cubic GP hierarchy is proved in [6] by Erdős-Schlein-Yau in a suitable space. By use of sophisticated Feynman graph expansions, Fourier integrals associated to these graphs take most of the efforts in their analysis. Then later in [13], a new method has been developed by Klainerman and Machedon to deal with the uniqueness part in a different space of density matrices. This approach also uses the expansion introduced in [6], but the authors take advantage of the space-time estimate obtained from free evolving Schrödinger equations, and thus yielding a comparatively simpler analysis on the contributions of expansion terms. Subsequent works like [12] by Kirkpatrick, Schlein and Staffilani, and [2] by Chen and Pavlović proceeded along their lines when considering the Bose gas with pair and three-body interactions respectively, and the solutions obtained in both [12] and [2] satisfies the Klainerman-Machedon bounds.

In this paper, we prove the following result:
Theorem 1.1. Let $p_0 \geq 1$ be a fixed integer. Suppose that for all $1 \leq p \leq p_0$ the potential $V^{(p)} \in W^{p,\infty}(\mathbb{R}^d)$ and $V^{(p)} \geq 0$ is translation-invariant. Let $d \in \{1,2\}$ and $0 < \beta < \frac{1}{2d p_0 + 2}$. \{\psi_N\}_{N \geq 1}$ is a family of functions that satisfy

\begin{equation}
(1.16) \quad \sup_N \frac{1}{N} \langle \psi_N, H_N \psi_N \rangle < \infty
\end{equation}

and assume \{\psi_N\}_{N \geq 1} exhibits asymptotic factorization: \exists \phi \in L^2(\mathbb{R}^d)$ such that $Tr[\gamma_N^{(1)} | \phi \rangle \langle \phi |] \to 0$ as $N \to 0$. $\gamma_N^{(1)}$ is the 1-particle marginal density associated with $\psi_N$.

Then we have

\begin{equation}
(1.17) \quad \gamma_N^{(k)} - |\phi_t \rangle \langle \phi_t |^{\bigotimes k} \to 0 \quad \text{as} \quad N \to \infty
\end{equation}

Here $\gamma_N^{(k)}$ is the $k$-particle marginal density associated to $\psi_{N,t} = e^{-iH_{N,t}} \psi_N$, and $\phi_t$ solves the nonlinear Schrödinger equation: $i \frac{d}{dt} \phi_t = -\Delta \phi_t + \sum_{p=1}^{p_0} b_0^{(p)} |\phi_t|^p \phi_t$ with initial condition $\phi_0 = \phi$ and potential constant $b_0^{(p)} = \int_{\mathbb{R}^d} V^{(p)}(x) dx < \infty$.

The bulk of this paper is devoted to the proof of Theorem 1.1. The strategy we follow is to identify the limit of $\Gamma_{N,t} = \{\gamma_N^{(k)}\}_{k=1}^N$ as the unique solution to (1.9); or in other words, every limit (under suitable topology) of $\Gamma_{N,t}$ solves (1.9) uniquely, since (1.13) is a solution, then (1.17) follows by compactness.

The idea to prove uniqueness of the infinite hierarchy in (1.3) consists of the following three major steps. First, we express each solution $\gamma^{(k)}$ in terms of the future iterates $\gamma^{(k+p_0)}, ..., \gamma^{(k+kp_0)}$ using Duhamel formula (we choose all $p$ to be $p_0$ for a upper bound of the number of terms). Since for each $p_0$, the operator $B^{k+p_0} = \sum_{j=1}^k B_{j,k+1} \cdots k$ is a sum of $k$ operators, the iterated Duhamel formula involves up to $k(k+p_0) \cdots (k + p_0(n - 1)) \sim n!$ terms (see $J^k$ in (1.12)). Then in the second step, we use a combinatorial argument to group these iterated terms into equivalence classes that we can bound. Finally, we treat each equivalence class with the Strichartz type estimate (1.4).

Compared to [2], the main novelties are:

- in the proof of an a priori energy bound (Proposition 2.1) which had to be carefully done due to presence of many terms in the interacting potential;
- in the combinatorial argument, since in the case considered in this paper the matrices associated with iterated Duhamel terms reflect a combination of different interactions.

Organization of the paper. In section 2, we prove a priori energy bound for the BBGKY solutions and summarize main steps on establishing the convergence of $k$-particle marginals to the infinite hierarchy. In section 3, we obtain two space-time estimates for the limiting hierarchy. In section 4, a free evolving bound on the limiting hierarchy is presented, which is later used to prove the uniqueness of solutions in 2D case. Sections 5-7 are devoted to the proof of uniqueness of solutions to the limiting hierarchy. We prove 1D case in section 5. In section 6, we obtain the results from board game arguments (first introduced in [13]), which, combined with the bounds in section 3 and 4 lead to the uniqueness in 2D case. Finally, two technical lemmas are included in the appendix sections.
2. Convergence

2.1. A Priori Energy Bounds. From the energy estimates, following \[12, 2, 6, 5, 9\], we will be able to obtain the priori bounds below.

**Proposition 2.1.** Suppose \(0 < \beta < \frac{1}{2d p_0 + 2}\), then there exists a constant \(C\) (depends on \(p_0, V^{(p)}, d\)), such that for every \(k\), there exists \(N_0(k)\) such that

\[
\langle \psi, (H_N + N)^k \psi \rangle \geq C^k N^k \langle \psi, (1 - \Delta_{x_1}) \cdots (1 - \Delta_{x_k}) \psi \rangle
\]

for all \(N \geq N_0(k)\), \(\psi \in L^2_c(\mathbb{R}^d)\). The Hamiltonian \(H_N\) is defined as in \([12, 2]\).

**Proof.** We adapt the proof in \([12, 2]\) to the current case. It’s a two-step induction for all \(N \geq N_0(k)\), \(\psi \in L^2_c(\mathbb{R}^d)\). The Hamiltonian \(H_N\) is defined as in \([12, 2]\).

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In order to illustrate the techniques here, we check one more case before the running of induction. Write \(S_i = (1 - \Delta_{x_i})^{\frac{1}{2}}\) and the interactions in two groups \(h_1\) and \(h_2\), such that \(H_N + N = h_1 + h_2\):

\[
h_1 = \sum_{j=n+1}^{N} S_j^2
\]

\[
h_2 = \sum_{j=1}^{n} S_j^2 + \sum_{p=1}^{p_0} \sum_{i_1 < i_2 < \cdots < i_{p+1}} N^{-p} V^{(p)}_N(x_{i_1} - x_{i_2}, \cdots, x_{i_1} - x_{i_{p+1}})
\]

For \(k = 2\), let \(h_1 = \sum_{j=1}^{N} S_j^2\) and \(h_2 = \sum_{p=1}^{p_0} \sum_{i_1 < i_2 < \cdots < i_{p+1}} N^{-p} V^{(p)}_N(x_{i_1} - x_{i_2}, \cdots, x_{i_1} - x_{i_{p+1}})\), then since \(h_2^2 \geq 0\),

\[
\langle \psi, (H_N + N)^2 \psi \rangle = \langle \psi, h_1^2 \psi \rangle + \langle \psi, h_1 h_2 \psi \rangle + \langle \psi, h_2 h_1 \psi \rangle + \langle \psi, h_2^2 \psi \rangle
\]

\[\geq \langle \psi, h_1^2 \psi \rangle + \langle \psi, h_1 h_2 \psi \rangle + \langle \psi, h_2 h_1 \psi \rangle + \langle \psi, h_2^2 \psi \rangle\]

\[= N(N-1)\langle \psi, S_1^2 S_2^2 \psi \rangle + N\langle \psi, S_1^4 \psi \rangle\] (leading terms)

\[+ \sum_{p=1}^{p_0} \sum_{i_1 < i_2 < \cdots < i_{p+1}} N^{-p} \langle \psi, S_1^2 V^{(p)}_N(x_{i_1} - x_{i_2}, \cdots, x_{i_1} - x_{i_{p+1}}) \psi \rangle + c.c\] (error terms)

where \(c.c\) denotes “complex conjugate”. We keep the “leading terms” in RHS of \([2, 2]\) and look for a lower bound of the terms in the last line (“error terms”). As in \([2]\), let \(\hat{S}_j = (\hat{S}_j)_{4j} := i \nabla_{x_j}\), then \(S_j^2 = 1 + \hat{S}_j^2 = 1 - \Delta_{x_j}\). For sufficiently large \(N\), by the permutation symmetry of \(\psi\):

\[
N \sum N^{-p} \langle \psi, S_1^2 V^{(p)}_N(x_{i_1} - x_{i_2}, \cdots, x_{i_1} - x_{i_{p+1}}) \psi \rangle + c.c
\]

\[= N^{1-p} (N-1) \cdots (N-p-1) \langle \psi, S_1^2 V^{(p)}_N(x_{i_1} - x_{i_2}, \cdots, x_{i_1} - x_{i_{p+1}}) \psi \rangle + c.c\]

\[+ N^{1-p} (N-1) \cdots (N-p) \langle \psi, S_1^2 V^{(p)}_N(x_{i_1} - x_{i_2}, \cdots, x_{i_1} - x_{i_{p+1}}) \psi \rangle + c.c\]

\[\geq C^2 N^{2} \langle \psi, S_1^2 V^{(p)}_N(x_{i_1} - x_{i_2}, \cdots, x_{i_1} - x_{i_{p+1}}) \psi \rangle + c.c\]

\[+ C N \langle \psi, (1 + \hat{S}_1^2) V^{(p)}_N(x_{i_1} - x_{i_2}, \cdots, x_{i_1} - x_{i_{p+1}}) \psi \rangle + c.c\]
Then it follows that
\[ \|\nabla V_N(p)\|_{L^\infty(\mathbb{R}^d)}^2 \geq 2CN\rho \langle \psi, S_1^2 \rangle - \frac{CN}{\rho} \|\nabla V_N(p)\|_{L^2(\mathbb{R}^d)}^2 \langle \psi, S_1^2 \rangle \]

The last term above is the error term we want to control. Again by permutation symmetry of \( \psi \), we can further break down the interactions of the last term in (2.7) for big enough \( N \):

\[ \langle \psi, (H_N + N)^{n+2} \rangle \]

\[ \geq C^n N^n \langle \psi, S_1^4 \cdots S_n^4 (H_N + N) \rangle \]

Then it follows that

\[ \langle \psi, (H_N + N) S_1^2 \cdots S_n^2 (H_N + N) \rangle \]

\[ = \langle \psi, S_1^2 \cdots S_n^2 h_1 \rangle + \langle \psi, S_1^2 \cdots S_n^2 h_2 \rangle \]

Note that \( h_2 S_1^2 \cdots S_n^2 h_2 \geq 0 \). Combine (2.5) and (2.6) and use the permutation symmetry of \( \psi \) to get:

\[ \langle \psi, (H_N + N)^{n+2} \rangle \]

\[ \geq C^n N^n \langle \psi, h_1 S_1^2 \cdots S_n^2 h_1 \rangle + \langle \psi, h_1 S_1^2 \cdots S_n^2 h_2 \rangle + \langle \psi, h_2 S_1^2 \cdots S_n^2 h_1 \rangle \]

\[ \geq C^n N^n (N - n)(N - n - 1) \langle \psi, S_1^2 \cdots S_{n+2}^2 \rangle + C^n N^n (N - n)n \langle \psi, S_1^2 \cdots S_{n+1}^2 \rangle \]

\[ + \sum_{p=1}^{n_0} \sum_{i_1 < \cdots < i_p} \left( \langle \psi, S_1^2 \cdots S_{n+1}^2 V_N^{(p)}(x_{i_1} - x_{i_2}, \ldots, x_{i_1} - x_{i_1+p}) \rangle + c.c \right) \]

The last term above is the error term we want to control. Again by permutation symmetry of \( \psi \), we can further break down the interactions of the last term in (2.7) for big enough \( N \):

\[ \langle \psi, (H_N + N)^{n+2} \rangle \]

\[ \geq C^n N^n +2 \langle \psi, S_1^2 \cdots S_{n+2}^2 \rangle + C^{n+1} N^{n+1} \langle \psi, S_1^2 \cdots S_{n+1}^2 \rangle \]
and the "rest" in group $h$

\begin{equation}
(2.11)
\end{equation}

For (2.10), the sum over

\begin{equation}
(2.14)
\end{equation}

(2.12)

We split terms as follows: \((2.9)-(2.11)\): we put the “first” \(n\) particles in group \(h_2\) and the “rest” in group \(h_1\). Then the term \((2.9)\) comes exclusively from group \(h_1\) interactions; and term \((2.11)\) is contributed purely by group \(h_2\) interactions; \((2.10)\) are mixture of inter-group and inner-group \((h_2)\) interactions. We will handle each of these terms individually.

Our goal is to show that \((2.9)-(2.11)\) are dominated by \((2.8)\). Since \(p_0\) is a finite number and \(N\) can be arbitrarily large, thus it suffices to show the goal for a single \(p\) with \(1 \leq p \leq p_0\).

First of all, term \((2.9)\) is non-negative and thus can be dropped for purpose of a lower bound. To see this, note \(V_N^{(p)} \geq 0\) and commutes with all derivatives \(S_1, S_2, \ldots, S_{n+1}\), we have

\begin{align}
\langle \psi, S_1^2 \cdots S_{n+1}^2 V_N^{(p)}(x_{n+2} - x_{n+3}, \ldots, x_{n+2} - x_{n+2+p}) | \psi \rangle \\
= \int d\mathbf{x}_N V_N^{(p)}(x_{n+2} - x_{n+3}, \ldots, x_{n+2} - x_{n+2+p}) \left( |(S_1 \cdots S_{n+1})| \right)^{2} \geq 0
\end{align}

For \((2.10)\), the sum over \(j\) consists of \(p\) terms (if \(1 + p > n + 1\), \((2.10)\) is a sum of \(n\) terms, and \((2.11)\) vanishes). Consider the first term which corresponding to \(j = 2\):

\begin{align}
\langle \psi, S_1^2 \cdots S_{n+1}^2 V_N^{(p)}(x_1 - x_{n+2}, x_1 - x_{n+3}, \ldots, x_1 - x_{n+1+p}) | \psi \rangle \\
\geq \langle \psi, S_{n+1} \cdots S_2 V_N^{(p)}(x_1 - x_{n+2}, x_1 - x_{n+3}, \ldots, x_1 - x_{n+1+p}) S_2 \cdots S_{n+1} | \psi \rangle \\
- |\langle \psi, S_{n+1} \cdots S_2 \tilde{S}_1 (\nabla_{x_1} V_N^{(p)}(x_1 - x_{n+2}, x_1 - x_{n+3}, \ldots, x_1 - x_{n+1+p}) S_2 \cdots S_{n+1} | \psi \rangle| \\
(2.12) \\
\geq -|\langle \psi, S_{n+1} \cdots S_2 \tilde{S}_1 (\nabla_{x_1} V_N^{(p)}(x_1 - x_{n+2}, x_1 - x_{n+3}, \ldots, x_1 - x_{n+1+p}) | \psi \rangle| \\
(2.13) \\
\geq -\rho \langle \psi, S_{n+1}^2 \cdots S_2^2 | \psi \rangle \\
- \frac{1}{\rho} \langle \psi, S_{n+1} \cdots S_2 | \nabla_{x_1} V_N^{(p)}(x_1 - x_{n+2}, x_1 - x_{n+3}, \ldots, x_1 - x_{n+1+p}) |^2 S_2 \cdots S_{n+1} | \psi \rangle \\
(2.14) \\
\geq -\rho \langle \psi, S_{n+1}^2 \cdots S_2^2 | \psi \rangle - \frac{1}{\rho} \left[ \nabla_{x_1} V_N^{(p)} \right]_{L^2(\mathbb{R}^{d}\psi)}^2 \langle \psi, S_{n+1}^2 \cdots S_2^2 | \psi \rangle \\
= -\rho \langle \psi, S_{n+1}^2 \cdots S_2^2 | \psi \rangle - \frac{C N^{(2p+2)\beta}}{\rho} \langle \psi, S_{n+1}^2 \cdots S_2^2 | \psi \rangle
\end{align}
which are dominated by the leading terms in (2.8) when $\beta < \frac{1}{2p+2}$ (which is fine since $p$ is at most $p_0$). The constant $C$ depends on $\|\nabla x_j V^{(p)}\|_{L^\infty(\mathbb{R}^{d+p})}$. Here we use the positivity of $V^{(p)}$ to obtain (2.12). Note $S_j^2 = S_j^2 - 1 < S_j^2$, $\rho > 0$ in (2.13) can be chosen arbitrarily, and in (2.14) we have applied (2.22) with $l = 2$.

For the term corresponding to $j = 3$ in (2.11),

\[
\langle \psi, S_1^2 \cdots S_{n+1}^2 V_N^{(p)}(x_1 - x_2, x_1 - x_{n+2}, \ldots, x_1 - x_{n+p})\rangle
\]
(2.15)
\[
\geq \langle \psi, S_{n+1} \cdots S_3 V_N^{(p)}(x_1 - x_2, x_1 - x_{n+2}, \ldots, x_1 - x_{n+p})S_3 \cdots S_{n+1}\psi \rangle
\]
(2.16)
\[
+ \langle \psi, S_{n+1} \cdots S_3(S_1^2 + \hat{S}_2^2) V_N^{(p)}(x_1 - x_2, x_1 - x_{n+2}, \ldots, x_1 - x_{n+p})S_3 \cdots S_{n+1}\psi \rangle
\]
(2.17)
\[
+ \langle \psi, S_{n+1} \cdots S_3 \hat{S}_2 \hat{S}_1(S_1 \hat{S}_2 V_N^{(p)}(x_1 - x_2, x_1 - x_{n+2}, \ldots, x_1 - x_{n+p}))S_3 \cdots S_{n+1}\psi \rangle.
\]

We know (2.15) is positive and thus can be discarded for a lower bound. (2.15) can be treated as in the case $j = 2$. Note that

\[ [\hat{S}_1 \hat{S}_2, V_N^{(p)}] = [\hat{S}_1, V_N^{(p)}]\hat{S}_2 + \hat{S}_1[V_N^{(p)}] \]

Hence

\[
(2.17) \geq -\langle \psi, S_{n+1} \cdots S_3 \hat{S}_2 \hat{S}_1[S_1 V_N^{(p)}(x_1 - x_2, x_1 - x_{n+2}, \ldots, x_1 - x_{n+p})]\hat{S}_2 S_3 \cdots S_{n+1}\psi \rangle
\]
\[
- \langle \psi, S_{n+1} \cdots S_3 \hat{S}_2 \hat{S}_1[S_1 V_N^{(p)}(x_1 - x_2, x_1 - x_{n+2}, \ldots, x_1 - x_{n+p})]S_3 \cdots S_{n+1}\psi \rangle
\]
\[
\geq -\rho_1 \langle \psi, S_{n+1}^2 \cdots S_3^2 S_1^4 \psi \rangle - \frac{1}{\rho_1} \|\nabla V_N^{(p)}\|_{L^\infty(\mathbb{R}^{d+p})}^2 \langle \psi, S_{n+1}^2 \psi \rangle
\]
\[
- \rho_2 \langle \psi, S_{n+1}^2 \cdots S_3^2 S_1^4 \psi \rangle - \frac{1}{\rho_2} \|V_N^{(p)}\|_{W^{1,\infty}(\mathbb{R}^{d+p})}^2 \langle \psi, S_{n+1}^2 \psi \rangle
\]

We shall prove the estimate for general terms in (2.11) by running a one-step induction in $j$. Note that the $j$-th term $T_j$ in (2.11), with $2 \leq j \leq 1 + p$, has the coefficient of order $O(N^{n-j+3})$. Assume we have the desired bound for $j$ from 2 through $j_0$, then is

\[
T_2 \geq -(CN)^{n+1+(2pd+2)}\beta \langle \psi, S_1^2 \cdots S_{n+2}^2 \psi \rangle,
\]
\[
T_3 \geq -(CN)^{n+(2pd+2)}\beta \langle \psi, S_1^2 \cdots S_{n+2}^2 \psi \rangle,
\]
\[
\cdots
\]
\[
T_{j_0} \geq -(CN)^{n-j_0+3+\delta_{j_0}(\beta)} \langle \psi, S_1^2 \cdots S_{n+2}^2 \psi \rangle.
\]

Function $\delta_j(\beta)$ ($2 \leq j \leq j_0$) takes values in interval $(0, 1)$, this small piece of power on $N$ is contributed by appropriate norm of $V_N^{(p)}$. By the cases we have already checked, we know that $j_0 \geq 3$. Rewrite the main part of $T_{j_0+1}$ as the following

\[
\langle \psi, S_1^2 \cdots S_{n+1}^2 V_N^{(p)}(x_1 - x_2, \ldots, x_1 - x_{j_0}, x_1 - x_{n+2}, \cdots, x_1 - x_{n+3+p-j_0-1})\rangle
\]
\[
= \langle \psi, (1 + \hat{S}_1^2) \cdots (1 + \hat{S}_{j_0}^2) S_{j_0+1}^2 \cdots S_{n+1}^2 V_N^{(p)} \psi \rangle
\]
\[
= \langle \psi, S_{n+1} \cdots S_{j_0+1} V_N^{(p)} S_{j_0+1} \cdots S_{n+1}\psi \rangle
\]
Both terms appear in the previous induction, but with one order higher derivative by the leading terms in (2.8) except the last term. By the definition of where a hat denotes a missing term. Thanks to the induction assumption we may conclude that the lower bounds of all the terms in the RHS of (2.18) are controlled by the leading terms in (2.8) except the last term. By the definition of $\hat{S}_j$, we can prove the following decomposition:

$$[\hat{S}_1 \cdots \hat{S}_{j-1}, V_N^{(p)}] = [\hat{S}_1, V_N^{(p)}] \hat{S}_2 \cdots \hat{S}_{j-1} + \hat{S}_1 [\hat{S}_2, V_N^{(p)}] \hat{S}_3 \cdots \hat{S}_{j-1} + \cdots + \hat{S}_1 \cdots \hat{S}_{j-2} [\hat{S}_{j-1}, V_N^{(p)}]$$

Therefore

$$\langle \psi, \hat{S}_1^{2} \hat{S}_2^{2} \cdots \hat{S}_{j-1}^{2} V_N^{(p)} \rangle$$

$$= \langle \psi, S_{n+1}^{2} \cdots \hat{S}_{j-1} [\hat{S}_2, V_N^{(p)}] S_{j-2} \cdots \hat{S}_{j-1} \hat{S}_1 \hat{S}_2 \cdots \hat{S}_{j-1} + \hat{S}_1 [\hat{S}_2, V_N^{(p)}] S_{j-2} \cdots \hat{S}_{j-1} + \cdots + \hat{S}_1 \cdots \hat{S}_{j-2} [\hat{S}_{j-1}, V_N^{(p)}] S_{j-2} \cdots \hat{S}_{j-1} \hat{S}_1 \hat{S}_2 \cdots \hat{S}_{j-1} \rangle$$

Again, by induction assumption all terms in the RHS of the above are bounded as we need except the one in the last line. However, we can reduce it into previous case since (for $j \geq 4$):

$$\langle \psi, \hat{S}_1 \cdots \hat{S}_{j-1}, V_N^{(p)} \rangle = \hat{S}_1 \cdots \hat{S}_{j-3} [\hat{S}_{j-1}, (\hat{S}_{j-2} V_N^{(p)})] + \hat{S}_1 \cdots \hat{S}_{j-3} [\hat{S}_{j-1}, V_N^{(p)}] \hat{S}_{j-2}$$

Then

$$\langle \psi, S_{n+1}^{2} \cdots \hat{S}_{j-2} \hat{S}_{j-1} [\hat{S}_{j-1}, V_N^{(p)}] S_{j-2} \cdots \hat{S}_{j-1} \rangle$$

Both terms appear in the previous induction, but with one order higher derivative on $V_N^{(p)}$. Since $\|V_N^{(p)}\|_{W_{j-1}^{\infty} \cap \mathbb{R}^{2p}} \sim N^{2(pd\beta+\delta_j)} \|V_N^{(p)}\|_{W_{j-1}^{\infty} \cap \mathbb{R}^{2p}}$, we may set $\delta_j = 2pd\beta + 2(j_0 - 1)\beta < 1$ (with $j_0 \geq 3$ since (2.19) requires $j \geq 4$). In general, the $j$-th term $T_j$ in (2.19) has the following bound:

$$T_j \geq -N^{n-j+3} N^{2(pd\beta+(j-2)\beta)} \langle \psi, S_1^{2} \cdots S_{j+2}^{2} \psi \rangle$$

for $V_N^{(p)} \in W_{j-2, \infty}^{\infty}$, $3 \leq j \leq 1 + p$
And $T_2 \geq -N^{n+1+(2dp+2)\beta} \langle \psi, S_1^2 \cdots S_{n+2}^2 \psi \rangle$. Admissible value for $\beta$ will not send the total power of $N$ to be greater than or equal to $n + 2$. Thus for each $p \leq p_0$, $\beta$ can take values in $(0, \frac{1}{2dp+2})$, which is actually determined by the base case $j = 2$.

Finally, the term (2.11) is actually a special case in (2.10) corresponding to $j = 2 + p$, thus can be handled as above (the highest regularity of the potential is used here). This completes the proof. 

\[ \Box \]

**Lemma 2.2.** For $d \geq 1$, $m \geq 1$ and $\psi \in L^2_s(\mathbb{R}^{md})$, we have

\[ \langle \psi, V(x_1, \cdots, x_m) \psi \rangle \leq \| V \|_{L^\infty_{x_1, \cdots, x_m}} \langle \psi, (1 - \Delta_{x_1}) \cdots (1 - \Delta_{x_m}) \psi \rangle \]

for any $r > 2$ if $d \leq \frac{2}{m}$, and for $r \geq md$ if $d > \frac{2}{m}$. Moreover for any $1 \leq l \leq m$, we have

\[ \langle \psi, V(x_1, \cdots, x_m) \psi \rangle \leq \| V \|_{L^\infty_{x_1, \cdots, x_m}} \langle \psi, \prod_{j=1}^l (1 - \Delta_{x_j}) \psi \rangle_{L^2_{x_1, \cdots, x_m}} \]

**Proof.** By Hölder inequality with $\frac{1}{q} + \frac{1}{r} = 1$ and Sobolev embedding we have

\[ \langle \psi, V(x_1, \cdots, x_m) \psi \rangle \leq \| V \|_{L^\infty_{x_1, \cdots, x_m}} \| \psi \|_{L^2_{x_1, \cdots, x_m}} \| \psi \|_{L^2_{x_1, \cdots, x_m}} \]

\[ \leq \| V \|_{L^\infty_{x_1, \cdots, x_m}} \| \psi \|_{L^2_{x_1, \cdots, x_m}} \| \psi \|_{H^1_{x_1, \cdots, x_m}} \]

\[ \leq \| V \|_{L^\infty_{x_1, \cdots, x_m}} \| \psi \|_{H^1_{x_1, \cdots, x_m}} \]

\[ = \| V \|_{L^\infty_{x_1, \cdots, x_m}} \| (1 + |\xi_1|^2 + |\xi_2|^2 + \cdots + |\xi_m|^2)^{\frac{1}{2}} \hat{\psi} \|_{L^2} \]

\[ \leq \| V \|_{L^\infty_{x_1, \cdots, x_m}} \| (1 + |\xi_1|^{\frac{2}{q}}(1 + |\xi_2|^{\frac{2}{q}})^{\frac{1}{2}} \cdots (1 + |\xi_m|^{\frac{2}{q}})^{\frac{1}{2}} \hat{\psi} \|_{L^2_{x_1, \cdots, x_m}} \]

The Sobolev embedding requires that $q$ is finite and satisfying $2 \leq q \leq \frac{2md}{md-2}$, which is equivalent to $2 \leq q \leq \frac{2md}{md-2}$ when $d > \frac{2}{m}$ and $2 \leq q < \infty$ when $d \leq \frac{2}{m}$. From the Hölder conjugate relations $\frac{1}{r} = \frac{1}{2} - \frac{1}{q}$, we know the constrains on $r$ must be $r > 2$ if $d \leq \frac{2}{m}$ and $r \geq md$ if $d > \frac{2}{m}$.

To prove (2.22), choose $q = 2, r = \infty$ in the above proof, then replace $L^2$ norm by $H^1$ norm in the first $l$ variables to obtain:

\[ \langle \psi, V(x_1, \cdots, x_m) \psi \rangle \leq \| V \|_{L^\infty_{x_1, \cdots, x_m}} \| \psi \|_{H^1_{x_1, \cdots, x_m}} \]

\[ \leq \| V \|_{L^\infty_{x_1, \cdots, x_m}} \| \psi \|_{L^2_{x_1, \cdots, x_m}} \| \psi \|_{L^2_{x_1, \cdots, x_m}} \]

\[ = \| V \|_{L^\infty_{x_1, \cdots, x_m}} \| (1 + |\xi_1|^2 + |\xi_2|^2 + \cdots + |\xi_m|^2)^{\frac{1}{2}} \hat{\psi} \|_{L^2_{x_1, \cdots, x_m}} \]

\[ \leq \| V \|_{L^\infty_{x_1, \cdots, x_m}} \| (1 + |\xi_1|^2)^{\frac{1}{q}}(1 + |\xi_2|^2)^{\frac{1}{q}} \cdots (1 + |\xi_m|^2)^{\frac{1}{q}} \hat{\psi} \|_{L^2_{x_1, \cdots, x_m}} \]

Here the Fourier transform and its inverse transform of $\psi$ are taken only on the first $l$ variables with $1 \leq l \leq m$. 

\[ \Box \]

After regularization of the initial data, we have
Corollary 2.3 (A priori bound). Let $\chi$ be a bump function with support on $[0,1]$ and $\kappa > 0$. Define

\begin{equation}
\gamma^*_N := \frac{\chi(\frac{\kappa}{N}H_N)\psi_N}{\|\chi(\frac{\kappa}{N}H_N)\psi_N\|}
\end{equation}

Let $\gamma^*_N, \tilde{\gamma}^*_N, \gamma^{(k)}_{N,t}$ be the corresponding $k$-marginal density. Then there exists a constant $\tilde{C} > 0$ depending on $\kappa, p_0, V^{(p)}$ for all $1 \leq p \leq p_0$ but independent of $k, t$, and there exists an integer $N_0(k)$ for every $k \geq 1$, such that for all $N > N_0(k)$, we have

\begin{equation}
\text{Tr}(1 - \Delta_x) \cdots (1 - \Delta_x) \gamma^{(k)}_{N,t} \leq \tilde{C}^k
\end{equation}

Proof. The proof is simple when we have Proposition 2.1, since we have

$\text{Tr}(1 - \Delta_x) \cdots (1 - \Delta_x) \gamma^{(k)}_{N,t} = \langle \gamma^{(k)}_{N,t}, S^2 \cdots S^2 \gamma^{(k)}_{N,t} \rangle$

\begin{equation}
\leq \frac{1}{C^k N^k} \langle \gamma^{(k)}_{N,t}, (H_N + N)^k \gamma^{(k)}_{N,t} \rangle
\end{equation}

\begin{equation}
= \frac{1}{C^k N^k} \langle \gamma^{(k)}_{N,t}, H_N^k \gamma^{(k)}_{N,t} \rangle + \frac{2}{C^k} \| \gamma^{(k)}_{N,t} \|^2
\end{equation}

In the first inequality we use Proposition 2.1 and in the last inequality we use the fact that $\langle \gamma^{(k)}_{N,t}, H_N^k \gamma^{(k)}_{N,t} \rangle \leq C^k N^k$ with the constant $C$ depending on $\kappa$ (see Proposition 5.1 in [6]). \hfill \Box

2.2. Compactness and Convergence. The compactness of the $k$-particle marginal density sequence and the convergence to the infinite hierarchy are established in [6, 12, 12], since the arguments are essentially the same, we outline the main steps here for completeness.

We introduce the following Banach spaces of density matrices. Denote by $\mathcal{K}_k = \mathcal{K}(L^2(\mathbb{R}^{dk}))$ the space of compact operators on $L^2(\mathbb{R}^{dk})$, equipped with the operator norm topology. And let $\mathcal{L}^1_k = L^1(L^2(\mathbb{R}^{dk}))$ denote the space of trace operators on $L^2(\mathbb{R}^{dk})$ equipped with the trace class norm. Then we know (see Theorem VI.26 in [10] for details)

\begin{equation}
\mathcal{L}^1_k = K^*_k
\end{equation}

The closed unit ball in $\mathcal{L}^1_k$ is weak* compact by Banach-Alaoglu theorem, and thus is metrizable in the weak* topology. Since $\mathcal{K}_k$ is separable, there exists a sequence $\{J_i^{(k)}\}_{i \geq 1} \in \mathcal{K}_k$, with $\|J_i^{(k)}\| \leq 1$, dense in the unit ball of $\mathcal{K}_k$. Then

\begin{equation}
\eta_k(\gamma^{(k)}, \tilde{\gamma}^{(k)}) := \sum_{i=1}^{\infty} 2^{-i} |\text{Tr} J_i^{(k)} (\gamma^{(k)} - \tilde{\gamma}^{(k)})|
\end{equation}

is a metric on $\mathcal{L}^1_k$, and the induced topology by $\eta_k$ is equivalent to the weak* topology on any weak* compact subset of $\mathcal{L}^1_k$ (Theorem 3.16 in [17]). Therefore a uniformly bounded sequence $\gamma^{(k)}_N \in \mathcal{L}^1_k$ converges to $\gamma^{(k)} \in \mathcal{L}^1_k$ with respect to the
Proposition 2.4. Let \( \tilde{\gamma} = (\gamma^{(k)}_N)_{k \geq 1} \) be the sequence of marginal densities \( \tilde{\gamma} = \{\tilde{\gamma}^{(k)}_{N,t}\}_{k \geq 1} = \bigoplus_{k \geq 1} C([0,T], \mathcal{L}^1_k) \) compact with respect to the product topology \( \tau_{\text{prod}} \). Then the sequence of marginal densities \( \tilde{\gamma} = \{\tilde{\gamma}^{(k)}_{N,t}\}_{k \geq 1} \) is compact with respect to the product topology \( \tau_{\text{prod}} \) generated by the metric \( \hat{\eta} \). If \( \Gamma_{x,t} = (\gamma^{(k)}_{x,t})_{k \geq 1} \) is an arbitrary subsequential limit point, then its component \( \gamma^{(k)}_{x,t} \) is non-negative and symmetric under permutations, and

\[
\text{Tr} \gamma^{(k)}_{x,t} \leq 1
\]

for every \( k \geq 1 \).

**Scheme of the proof.** The proof is completely analogous to those in [12], [2], [6]. First of all, by a Cantor diagonal argument, it is sufficient to prove the compactness of \( \tilde{\gamma}^{(k)}_{N,t} \) for some fixed \( k \). Thanks to Arzelà-Ascoli theorem, this can be done by showing the equicontinuity of \( \tilde{\gamma}^{(k)}_{N,t} \) with respect to the metric \( \hat{\eta} \). Then it should be enough to show that for every observable \( J^{(k)} \) from a dense subset of \( \mathcal{K}_k \) and for every \( \epsilon > 0 \), there exists \( \delta = \delta(J^{(k)}, \epsilon) \) such that

\[
\sup_{N \geq 1} |\text{Tr} J^{(k)}(\tilde{\gamma}^{(k)}_{N,t} - \tilde{\gamma}^{(k)}_{N,s})| < \epsilon
\]

for all \( t, s \in [0,T] \) with \( |t - s| \leq \delta \).

In order to prove (2.29), use (17) to rewrite \( \tilde{\gamma}^{(k)}_{N,t} - \tilde{\gamma}^{(k)}_{N,s} \) in integral form and bound \( |\text{Tr} J^{(k)}(\tilde{\gamma}^{(k)}_{N,t} - \tilde{\gamma}^{(k)}_{N,s})| \), which consists of \( p + 2 \) terms, by the following:

\[
\sup_{N \geq 1} |\text{Tr} J^{(k)}(\tilde{\gamma}^{(k)}_{N,t} - \tilde{\gamma}^{(k)}_{N,s})| \leq C \|J^{(k)}\| |t - s|.
\]

For this purpose, [2], [12] introduced an operator norm:

\[
\|J^{(k)}\| := \sup_{p^*_k} \int d\mathbf{p}_k \left| \prod_{j=1}^k \langle p_j \rangle \langle p_j' \rangle (|\tilde{J}^{(k)}(p_k; \mathbf{p}_k)| + |\tilde{J}^{(k)}(p'_k; \mathbf{p}_k)|) \right|
\]

where \( \tilde{J}^{(k)}(p_k; p'_k) \) denotes the kernel of the compact operator \( J^{(k)} \) in momentum space. Then use the fact that the set of all \( J^{(k)} \in \mathcal{K}_k \) with finite norm is dense in \( \mathcal{K}_k \) to reach the conclusion. \( \square \)

From the above proposition, we know that the sequence \( \tilde{\gamma}_{N,t} = (\tilde{\gamma}^{(k)}_{N,t})_{k \geq 1} \) admits at least one limit point in \( \bigoplus_{k \geq 1} C([0,T], \mathcal{L}^1_k) \) with respect to the product topology \( \tau_{\text{prod}} \).

**Theorem 2.1.** Let \( \tilde{\psi}_N \) be defined as in [22], \( \tilde{\psi}_N, \tilde{\psi}_N = e^{-itH_N \tilde{\psi}_N} \) and \( \tilde{\gamma}^{(k)}_{N,t} \) be the corresponding \( k \)-marginal density. Suppose that \( \Gamma_{x,t} = (\gamma^{(k)}_{x,t})_{k \geq 1} \) is a limit point of
\( \hat{\Gamma}_{N,t} = \{ \hat{\Gamma}^{(k)}_{N,t} \}_{k=1}^N \) in \( \bigoplus_{k \in \mathbb{N}} C([0,T], L^p_k) \) with respect to the product topology \( \tau_{\text{prod}} \). Then \( \Gamma_{N,t} \) is a solution to the infinite hierarchy

\[ \gamma^{(k)}_{N,t} = U^{(k)}(t) \gamma^{(k)}_{N,0} - i \sum_{p=1}^{p_0} b_p \sum_{j=1}^{k} \int_0^t ds U^{(k)}(t-s) B_{j,k+1,...,k+p} \gamma^{(k+p)}_{N,s} \]

with initial data \( \gamma^{(k)}_{N,0} = \langle \phi \rangle \langle \phi \rangle^{(k)} \). \( U^{(k)}(t) \) is the free evolution operator defined by \( U^{(k)}(t) \gamma^{(k)} = e^{it(\Delta_{x_k} - \Delta^{(k)}_{x_k})} \gamma^{(k)} \).

**Proof.** We adapt the proof in [2, 12]. Let \( k \geq 1 \) be fixed. Up to a subsequence, we can assume that for every \( J^{(k)} \in \mathcal{K}_k \)

\[ \sup_{t \in [0,T]} \text{Tr} J^{(k)}(\gamma^{(k)}_{N,t} - \hat{\gamma}^{(k)}_{N,t}) \to 0, \quad \text{as} \quad N \to \infty \]

It is enough to test (2.32) for observables in a dense subset of \( \mathcal{K}_k \). So we choose an arbitrary \( J^{(k)} \in \mathcal{K}_k \) with \( \| J^{(k)} \| < \infty \). We need to prove

\[ \text{Tr} J^{(k)}(\gamma^{(k)}_{N,0} - \langle \phi \rangle \langle \phi \rangle^{(k)}) \to 0, \quad \text{as} \quad N \to \infty \]

Here \( \gamma^{(k)}_{N,0} = \hat{\gamma}^{(k)}_{N} \). We provide the proof of (2.36) in Appendix A.

For (2.35), we use the notation \( J^{(k)}_t := J^{(k)} U^{(k)}(t) \), and go back to the BBGKY hierarchy (1.7) in the integral form as

\[ \text{Tr} J^{(k)}_t(\hat{\gamma}^{(k)}_{N,t}) = \]

\[ \sum_{p=1}^{p_0} i \frac{(N-k)}{N^p} \sum_{1 \leq i_1 < \cdots < i_{p+1} \leq k} \int_0^t ds \text{Tr} J^{(k)}_{t-s} \left[ V^{(p)}_N(x_{i_1} - x_{i_2} - \cdots - x_{i_{p+1}}), \hat{\gamma}^{(k)}_{N,s} \right] \]

\[ - \sum_{p=1}^{p_0} i \frac{(N-k)(N-k-1)}{N^p} \sum_{1 \leq i_1 < \cdots < i_{p+1} \leq k} \int_0^t ds \text{Tr} J^{(k)}_{t-s} \left[ V^{(p)}_N(x_{i_1} - x_{i_2} - \cdots - x_{i_{p+1}}, x_{i_{p+1}} - x_{k+1}, x_{k+2} - x_{k+2}), \hat{\gamma}^{(k+2)}_{N,s} \right] \]

\[ - \cdots \]

\[ - \sum_{p=1}^{p_0} i \frac{(N-k)(N-k-1) \cdots (N-k-p+1)}{N^p} \sum_{1 \leq i_1 \leq k} \int_0^t ds \text{Tr} J^{(k)}_{t-s} \left[ V^{(p)}_N(x_{i_1} - x_{k+1}, x_{i_1} - x_{k+2} - \cdots - x_{i_{p+1}}, \hat{\gamma}^{(k+p)}_{N,s} \right] \]

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Let us look at the behavior of the above terms when $N \to \infty$. It is obvious that by (2.33), (2.37), converges to the LHS of (2.35); and (2.38) converges to the first term on the RHS of (2.35). We also observed that all the terms between (2.33) and (2.42) vanish as $N \to \infty$. Therefore, our goal is to show (2.42) converges to the last term on the RHS of (2.35). It suffices to prove that for fixed $T, k, J^{(k)}$ and $p$, 

\[(2.43)\]

\[
\sup_{t \in [s, T]} |\text{Tr}J^{(k)}_{t-s}(V_N(x_j - x_{k+1}, \ldots, x_j - x_{k+p})\gamma_{N,s}^{(k+p)} - b_0^{(k)}(x_j - x_{k+1}) \cdots \delta(x_j - x_{k+p})\gamma_{N,s}^{(k+p)})| \rightarrow 0, \quad \text{as} \quad N \to \infty.
\]

To bound (2.43), we choose a non-negative probability measure $h$, i.e $h \geq 0$ and $\int h = 1$. Define $\tilde{h}_\epsilon(x) = \frac{1}{\epsilon h(\frac{x}{\epsilon})}, \epsilon > 0$. Then

\[(2.44)\]

\[|\text{Tr}J^{(k)}_{t-s}(V_N(x_j - x_{k+1}, \ldots, x_j - x_{k+p})\gamma_{N,s}^{(k+p)} - b_0^{(k)}(x_j - x_{k+1}) \cdots \delta(x_j - x_{k+p})\gamma_{N,s}^{(k+p)})| \leq |	ext{Tr}J^{(k)}_{t-s}(V_N(x_j - x_{k+1}, \ldots, x_j - x_{k+p}) - b_0^{(k)}(x_j - x_{k+1}) \cdots \delta(x_j - x_{k+p})\gamma_{N,s}^{(k+p)})| \]

\[(2.45)\]

\[+ b_0^{(k)}|\text{Tr}J^{(k)}_{t-s}(\delta(x_j - x_{k+1}) \cdots \delta(x_j - x_{k+p}) - h_\epsilon(x_j - x_{k+1}) \cdots h_\epsilon(x_j - x_{k+p})\gamma_{N,s}^{(k+p)})| \]

\[(2.46)\]

\[+ b_0^{(k)}|\text{Tr}J^{(k)}_{t-s}(h_\epsilon(x_j - x_{k+1}) \cdots h_\epsilon(x_j - x_{k+p})(\gamma_{N,s}^{(k+p)} - \gamma_{N,s}^{(k+p)})| \]

\[(2.47)\]

\[+ b_0^{(k)}|\text{Tr}J^{(k)}_{t-s}(h_\epsilon(x_j - x_{k+1}) \cdots h_\epsilon(x_j - x_{k+p}) - \delta(x_j - x_{k+1}) \cdots \delta(x_j - x_{k+p})\gamma_{N,s}^{(k+p)})| \]

We conclude that (note $\int \frac{V_N(x)}{b_0^{(k)}} = 1$):

- term (2.44) converges to 0 as $N \to \infty$ by Lemma 3.1 and Corollary 2.3.
- term (2.45) converges to 0 uniformly in $N$ as $\epsilon \to 0$ by Lemma 3.1 and Corollary 2.3.
- term (2.46) converges to 0 as $N \to \infty$, for every fixed $\epsilon$. (see (6.8) of [12]).
- term (2.47) converges to 0 as $\epsilon \to 0$ by Lemma 3.1 and 3.2.

Thus by taking first the limit $N \to \infty$, and then $\epsilon \to 0$, we obtain (2.43). \(\square\)

So far, with the uniqueness theorems in Section 5, we know that for each fixed $\kappa > 0$ and $k \geq 1$, $\hat{h}_\epsilon(\gamma_{N,t}^{(k)})|\phi_t\rangle\langle \phi_t|^{(k)} \rightarrow 0$ as $N \to \infty$. Or in other words,

\[(2.48)\]

\[
\gamma_{N,t}^{(k)} \rightarrow |\phi_t\rangle\langle \phi_t|^{(k)}
\]

in the weak* topology of $L_k^1$. It remains to prove that $\gamma_{N,t}^{(k)}$, the k-particle marginal density associated with the original wave functions $\psi_N$, converges to $|\phi_t\rangle \langle \phi_t|^{(k)}$ as $N \to \infty$. For any given $\epsilon > 0$, and compact operator $J^{(k)} \in K_k$, we can find a small enough $\kappa$ such that (see (4.19))

\[(2.49)\]

\[|\text{Tr}J^{(k)}(\gamma_{N,t}^{(k)} - \tilde{\gamma}_{N,t}^{(k)})| \leq \| J^{(k)} \| \| \psi_N - \tilde{\psi}_N \| < C \kappa^2 \leq \epsilon \]

uniformly in $N$. With this fixed $\kappa$, by (2.48), we can pick large enough $N$ to have

\[(2.50)\]

\[|\text{Tr}J^{(k)}(\gamma_{N,t}^{(k)} - |\phi_t\rangle\langle \phi_t|^{(k)})| \leq \epsilon \]

This shows that for any given $\epsilon > 0$ and $J^{(k)} \in K_k$, $\exists N_0 > 0$ such that

\[(2.51)\]

\[|\text{Tr}J^{(k)}(\gamma_{N,t}^{(k)} - |\phi_t\rangle\langle \phi_t|^{(k)})| \leq \epsilon \]
whenever \( N > N_0 \). So for each \( t \in [0, T] \) and every \( k \), \( \gamma^{(k)}_{N,t} \rightarrow |\phi_t\rangle \langle \phi_t|^{\otimes k} \) in the weak* topology of \( L^1_1 \). Since the limiting hierarchy is an orthogonal projection, the convergence in weak* topology is equivalent to the trace norm convergence. This concludes Theorem 1.1.

3. A Priori Energy Bounds on the Limiting Hierarchy

This section is a preparation for proving uniqueness theorems in section 5 using the approach introduced in [13]. In order to apply [13] we have to establish some energy bounds on the limiting hierarchy. The results are stated in theorems. From now on, we denote \( S^{(k,\alpha)} \) as:

\[
S^{(k,\alpha)} = \prod_{j=1}^{k} (1 - \Delta x_j)_{\hat{\gamma}} (1 - \Delta x'_j)_{\hat{\gamma}}
\]

**Theorem 3.1 (A priori energy bound).** Suppose that \( d \in \{1, 2\} \), \( 0 < \beta < \frac{1}{2d(p_0+2)} \), \( p \) satisfies \( 1 \leq p \leq p_0 \). If \( \Gamma_{N,t} = (\gamma^{(k)}_{N,t})_{k=1} \) is a limit point of the sequence \( \Gamma_{N,t} = (\gamma^{(k)}_{N,t})_{k=1} \) with respect to the product topology \( \tau_{prod} \), then for every \( \alpha < 1 \) if \( d = 2 \), and every \( \alpha \leq 1 \) if \( d = 1 \), there exists \( C_\alpha > 0 \) (also has \( \kappa, p_0, V(p), d \) dependence) such that

\[
(3.1) \quad \left\| S^{(k,\alpha)} B_{j,k+1,k+p}^{(k+p)} \gamma^{(k+p)}_{\Gamma_{N,t}} \right\|_{L^2(\mathbb{R}^{dk} \times \mathbb{R}^{dk})} \leq C^{k+p}_\alpha
\]

for all \( k \geq 1 \) and all \( t \in [0, T] \).

**Proof.** Since the inequality in Corollary 2.3 is uniformly true for all large \( N \), we can extract an estimate on limit points \( (\gamma^{(k)}_{\Gamma_{N,t}})_{k=1} \) by taking \( N \rightarrow \infty \):

\[
(3.2) \quad Tr(1 - \Delta x_1) \cdots (1 - \Delta x_k)\gamma^{(k)}_{\Gamma_{N,t}} \leq C^k
\]

It is enough to prove that

\[
(3.3) \quad \left\| S^{(k,\alpha)} B_{j,k+1,k+p}^{(k+p)} \gamma^{(k+p)}_{\Gamma_{N,t}} \right\|_{L^2(\mathbb{R}^{dk} \times \mathbb{R}^{dk})} \leq Tr(1 - \Delta x_1) \cdots (1 - \Delta x_{k+p})\gamma^{(k+p)}_{\Gamma_{N,t}}
\]

Further, it should be enough to show the case that \( k = 1 \) and \( j = 1 \), since the proof of the argument for other values of \( k, j \) is extremely similar. Also, by the definition of the contraction operator \( B_{j,k+1,k+p} \), we only need to deal with \( B_{j,k+1,k+p} \) (same way works for \( B_{j,k+1,k+p} \)). Switching to the Fourier space we have \( q_i \) and \( q_i' \) are Fourier conjugate variables of \( x_i \) and \( x_i' \) respectively:

\[
(\langle B_{1,2,\ldots,k+1,p}^{(1+p)} \rangle^{1+p} x_{i+1+p}) \rightarrow (q_i; q_i')
\]

\[
= \int dx_1 dx_2 e^{-ix_1 \cdot q_i} e^{ix_2 \cdot q_i'} \int dx_3 dx_4 \cdots dx_{1+p} dx_{1+p}'
\]

\[
\times \delta(x_1 - x_2) \delta(x_1 - x_3) \cdots \delta(x_1 - x_{1+p})\delta(x_1 - x_{1+p}')
\]

\[
\times (\gamma^{(1+p)}_{\Gamma_{N,t}}(x_1, \ldots, x_{1+p}; x_1', \ldots, x'_{1+p})
\]

\[
= \int dq_1 dq_2 \cdots dq_{1+p} dq'_{1+p} \int dx_1 dx_2 \cdots dx_{1+p} dx_{1+p}'
\]

\[
\times e^{-ix_1 \cdot q_i} e^{ix_2 \cdot q_i'} e^{-ix_3 \cdot q_{1+p}} e^{-ix_4 \cdot q'_{1+p}} \cdots e^{-ix_{1+p} \cdot q_{1+p}} e^{-ix_{1+p}' \cdot q'_{1+p}}
\]

\[
\times (\gamma^{(1+p)}_{\Gamma_{N,t}}(x_1, \ldots, x_{1+p}; x_1', \ldots, x'_{1+p})
\]

\[
\int (\gamma^{(1+p)}_{\Gamma_{N,t}}(x_1, \ldots, x_{1+p}; x_1', \ldots, x'_{1+p})
\]

\[
16
\]
\[ = \int dq_2 dq_2' \cdots dq_{1+p} dq'_{1+p} \int dx_1 dx_1' \cdots dx_{1+p} dx'_{1+p} \]
\[ \times e^{-i x_1 (q_1 - q_2 + q_2' - \cdots - q_{1+p} + q'_{1+p})} e^{-i x_1' (q_1' - q_2' + q_2'' - \cdots - q'_{1+p} + q''_{1+p})} \]
\[ \times \gamma_{\tan_t}^{(1+p)} (x_1, \cdots, x_{1+p}; x_1', \cdots, x'_{1+p}) \]
\[ = \int dq_2 dq_2' \cdots dq_{1+p} dq'_{1+p} \]
\[ \times \gamma_{\tan_t}^{(1+p)} (q_1 - q_2 + q_2' - \cdots - q_{1+p} + q'_{1+p}, q_2, \cdots, q_{1+p}; q_1', q_2', \cdots, q'_{1+p}) \]

Thus
\[ (S^{(1,\alpha)} B^+_{1:2, \cdots, 1+p} \gamma_{\tan_t}^{(1+p)})^\alpha (q_1; q'_1) \]
\[ = \langle q_1 \rangle^\alpha \langle q'_1 \rangle^\alpha \int dq_2 dq_2' \cdots dq_{1+p} dq'_{1+p} \]
\[ \times \gamma_{\tan_t}^{(1+p)} (q_1 - q_2 + q_2' - \cdots - q_{1+p} + q'_{1+p}, q_2, \cdots, q_{1+p}; q_1', q_2', \cdots, q'_{1+p}) \]

which implies
\[ || S^{(1,\alpha)} B^+_{1:2, \cdots, 1+p} \gamma_{\tan_t}^{(1+p)} ||_{L^2(\mathbb{R}^d \times \mathbb{R}^d)}^2 \]
\[ = \int dq_1 dq_1' dq_2 dq_2' \cdots dq_{1+p} dq_1'_{1+p} dq_1''_{1+p} dq''_{1+p} \langle q_1 \rangle^{2\alpha} \langle q'_1 \rangle^{2\alpha} \]
\[ \times \gamma_{\tan_t}^{(1+p)} (q_1 - q_2 + q_2' - \cdots - q_{1+p} + q'_{1+p}, q_2, \cdots, q_{1+p}; q_1', q_2', \cdots, q'_{1+p}) \]
\[ \times \gamma_{\tan_t}^{(1+p)} (q_1 - q_2 + q_2' - \cdots - q_{1+p} + q'_{1+p}, q_2, \cdots, q_{1+p}; q_1', q_2', \cdots, q'_{1+p}) \]

Note that \( \gamma^{(k+p)} \) is non-negative as an operator with trace less than or equal to 1 (see Proposition 2.3). We have the following decomposition
\[ \gamma_{\tan_t}^{(1+p)} (q_1, q_2, \cdots, q_{1+p}; q_1', q_2', \cdots, q'_{1+p}) = \sum_j \lambda_j \psi_j (q_1, q_2, \cdots, q_{1+p}) \bar{\psi}_j (q_1', q_2', \cdots, q'_{1+p}) \]
with \( \{ \psi_j \} \) an orthonormal system, \( \lambda_j \geq 0, \forall j \) and \( \sum_j \lambda_j \leq 1 \). Applying this decomposition in (3.6) yields:
\[ || S^{(1,\alpha)} B^+_{1:2, \cdots, 1+p} \gamma_{\tan_t}^{(1+p)} ||_{L^2(\mathbb{R}^d \times \mathbb{R}^d)}^2 \]
\[ = \sum_{i,j} \lambda_i \lambda_j \int dq_1 dq_1' dq_2 dq_2' \cdots dq_{1+p} dq_1'_{1+p} dq_1''_{1+p} \langle q_1 \rangle^{2\alpha} \langle q'_1 \rangle^{2\alpha} \]
\[ \times \psi_i (q_1 - q_2 + q_2' - \cdots - q_{1+p} + q'_{1+p}, q_2, \cdots, q_{1+p}) \bar{\psi}_j (q_1', q_2', \cdots, q'_{1+p}) \]
\[ \times \psi_j (q_1 - q_2 + q_2' - \cdots - q_{1+p} + q'_{1+p}, q_2, \cdots, q_{1+p}) \bar{\psi}_i (q_1', q_2', \cdots, q'_{1+p}) \]

It is obviously true that
\[ \langle q_1 \rangle^\alpha \leq C \left( \langle q_1 - q_2 + q_2' - \cdots - q_{1+p} + q'_{1+p} \rangle^\alpha + \langle q'_2 \rangle^\alpha + \cdots + \langle q'_{1+p} \rangle^\alpha \right) \]
and
\[ \langle q_1 \rangle^\alpha \leq C \left( \langle q_1 - q_2 + q_2' - \cdots - q_{1+p} + q'_{1+p} \rangle^\alpha + \langle q'_2 \rangle^\alpha + \cdots + \langle q'_{1+p} \rangle^\alpha \right) \]

multiplying them together we have the following estimate:
\[ \langle q_1 \rangle^{2\alpha} \leq C \left( \langle q_1 - q_2 + q_2' - \cdots - q_{1+p} + q'_{1+p} \rangle^\alpha + \langle q'_2 \rangle^\alpha + \cdots + \langle q'_{1+p} \rangle^\alpha \right) \]
\[
\times \left( \langle q_1 - \tilde{q}_2 + \tilde{q}_2' - \cdots - \tilde{q}_{1+p} + \tilde{q}_{1+p}' \rangle^{\alpha} + \langle \tilde{q}_2 \rangle^{\alpha} + \langle \tilde{q}_2' \rangle^{\alpha} + \cdots + \langle \tilde{q}_{1+p} \rangle^{\alpha} + \langle \tilde{q}_{1+p}' \rangle^{\alpha} \right).
\]

After substituting the above bound in (3.7), we will obtain \((2p + 1)^2\) contributed terms. However, it is enough to illustrate how to control just one of them, since the remaining cases are essentially the same. For instance, the first contribution comes from the replacement of the factor \(\langle q_1 \rangle^{2\alpha}\) on the RHS of (3.7) by \(\langle q_1 - \tilde{q}_2 + \tilde{q}_2' - \cdots - \tilde{q}_{1+p} + \tilde{q}_{1+p}' \rangle^{\alpha} \langle q_1 - \tilde{q}_2 + \tilde{q}_2' - \cdots - \tilde{q}_{1+p} + \tilde{q}_{1+p}' \rangle^{\alpha}\). Using Schwartz inequality we find

\[
(3.9)
\]
\[
\int dq_1 dq_1' d\tilde{q}_2 d\tilde{q}_2' d\tilde{q}_3 d\tilde{q}_3' \cdots d\tilde{q}_{1+p} d\tilde{q}_{1+p}' \int d\tilde{q}_1 d\tilde{q}_1' d\tilde{q}_2 d\tilde{q}_2' d\tilde{q}_3 d\tilde{q}_3' \cdots d\tilde{q}_{1+p} d\tilde{q}_{1+p}' (q_1')^{2\alpha} \]
\[
\times \langle q_1 - \tilde{q}_2 + \tilde{q}_2' - \cdots - \tilde{q}_{1+p} + \tilde{q}_{1+p}' \rangle^{\alpha} \langle q_1 - \tilde{q}_2 + \tilde{q}_2' - \cdots - \tilde{q}_{1+p} + \tilde{q}_{1+p}' \rangle^{\alpha} \psi_i(q_1 - \tilde{q}_2, \tilde{q}_2', \cdots, \tilde{q}_{1+p}) \psi_i(q_1', \tilde{q}_2', \cdots, \tilde{q}_{1+p}) \]
\[
\times \psi_j(q_1 - \tilde{q}_2, \tilde{q}_2', \cdots, \tilde{q}_{1+p}) \psi_j(q_1', \tilde{q}_2', \cdots, \tilde{q}_{1+p}) \]
\[
\leq A + B
\]

where
\[
A = \int dq_1 dq_1' d\tilde{q}_2 d\tilde{q}_2' d\tilde{q}_3 d\tilde{q}_3' \cdots d\tilde{q}_{1+p} d\tilde{q}_{1+p}' (q_1')^{2\alpha} \]
\[
\times \langle q_1 - \tilde{q}_2 + \tilde{q}_2' - \cdots - \tilde{q}_{1+p} + \tilde{q}_{1+p}' \rangle^{\alpha} \langle q_1 - \tilde{q}_2 + \tilde{q}_2' - \cdots - \tilde{q}_{1+p} + \tilde{q}_{1+p}' \rangle^{\alpha} \psi_i(q_1 - \tilde{q}_2, \tilde{q}_2', \cdots, \tilde{q}_{1+p}) \psi_i(q_1', \tilde{q}_2', \cdots, \tilde{q}_{1+p}) \]
\[
\times \psi_j(q_1 - \tilde{q}_2, \tilde{q}_2', \cdots, \tilde{q}_{1+p}) \psi_j(q_1', \tilde{q}_2', \cdots, \tilde{q}_{1+p}) \]

and
\[
B = \int dq_1 dq_1' d\tilde{q}_2 d\tilde{q}_2' d\tilde{q}_3 d\tilde{q}_3' \cdots d\tilde{q}_{1+p} d\tilde{q}_{1+p}' (q_1')^{2\alpha} \]
\[
\times \langle q_1 - \tilde{q}_2 + \tilde{q}_2' - \cdots - \tilde{q}_{1+p} + \tilde{q}_{1+p}' \rangle^{\alpha} \langle q_1 - \tilde{q}_2 + \tilde{q}_2' - \cdots - \tilde{q}_{1+p} + \tilde{q}_{1+p}' \rangle^{\alpha} \psi_i(q_1 - \tilde{q}_2, \tilde{q}_2', \cdots, \tilde{q}_{1+p}) \psi_i(q_1', \tilde{q}_2', \cdots, \tilde{q}_{1+p}) \]
\[
\times \psi_j(q_1 - \tilde{q}_2, \tilde{q}_2', \cdots, \tilde{q}_{1+p}) \psi_j(q_1', \tilde{q}_2', \cdots, \tilde{q}_{1+p}) \]

Now let us focus on \(A\) below, because \(B\) can be handled similarly. Performing integration on \(\tilde{q}_2, \tilde{q}_3, \ldots, \tilde{q}_{1+p}\) we obtain:

\[
(3.10)
\]
\[
A \leq C \int dq_1 dq_1' d\tilde{q}_2 d\tilde{q}_2' d\tilde{q}_3 d\tilde{q}_3' \cdots d\tilde{q}_{1+p} d\tilde{q}_{1+p}' (q_1')^{2\alpha} \]
\[
\times \langle q_1 - \tilde{q}_2 + \tilde{q}_2' - \cdots - \tilde{q}_{1+p} + \tilde{q}_{1+p}' \rangle^{\alpha} \langle q_1 - \tilde{q}_2 + \tilde{q}_2' - \cdots - \tilde{q}_{1+p} + \tilde{q}_{1+p}' \rangle^{\alpha} \psi_i(q_1 - \tilde{q}_2, \tilde{q}_2', \cdots, \tilde{q}_{1+p}) \psi_i(q_1', \tilde{q}_2', \cdots, \tilde{q}_{1+p}) \]
\[
\times \psi_j(q_1 - \tilde{q}_2, \tilde{q}_2', \cdots, \tilde{q}_{1+p}) \psi_j(q_1', \tilde{q}_2', \cdots, \tilde{q}_{1+p}) \]

where we used the bound

\[
(3.11)
\]
\[
\int_{\mathbb{R}^d} dy \frac{dy}{(W - y)^{2 - 2\alpha}} \leq C (W)^{2 - 2\alpha}
\]

for all \(\alpha < 1\) when \(d = 2\) and for \(\alpha \leq 1\) when \(d = 1\).

Let \(q_1 = q_1 - \tilde{q}_2 + \tilde{q}_2' - \cdots - \tilde{q}_{1+p} + \tilde{q}_{1+p}'\) in (3.10). Since \(\alpha \leq 1\), we can replace
\[ \langle q' \rangle^{2\alpha} \text{ with } \langle q' \rangle^2 \text{ for an upper bound:} \]

(3.12) \[
A \leq C \int d\tilde{q}_1 d\tilde{q}_2 d\tilde{q}_3 \cdots d\tilde{q}_1 + p d\tilde{q}_1 + p \]
\[
\times \left( \langle \tilde{q}_1 \rangle^{2} \langle \tilde{q}_2 \rangle^{2} \cdots \langle \tilde{q}_1 + p \rangle^{2} \langle \tilde{q}_2 \rangle^{2} \cdots \langle \tilde{q}_1 + p \rangle^{2} \right) \times \left( \psi_{\tilde{j}}(\tilde{q}_1, \tilde{q}_2, \cdots, \tilde{q}_1 + p) \right)^2 \]
\[
\times \left( \psi_{\tilde{j}}(\tilde{q}_1, \tilde{q}_2, \cdots, \tilde{q}_1 + p) \right)^2
\]
\[
\leq C' \int d\tilde{q}_1 d\tilde{q}_2 d\tilde{q}_3 \cdots d\tilde{q}_1 + p \langle \tilde{q}_1 \rangle^{2} \langle \tilde{q}_2 \rangle^{2} \cdots \langle \tilde{q}_1 + p \rangle^{2} \left| \psi_{\tilde{j}}(\tilde{q}_1, \tilde{q}_2, \cdots, \tilde{q}_1 + p) \right|^2
\]
\[
\times \left( \psi_{\tilde{j}}(\tilde{q}_1, \tilde{q}_2, \cdots, \tilde{q}_1 + p) \right)^2
\]
\[
\leq C' \left( \int d\tilde{q}_1 d\tilde{q}_2 d\tilde{q}_3 \cdots d\tilde{q}_1 + p \langle \tilde{q}_1 \rangle^{2} \langle \tilde{q}_2 \rangle^{2} \cdots \langle \tilde{q}_1 + p \rangle^{2} \left| \psi_{\tilde{j}}(\tilde{q}_1, \tilde{q}_2, \cdots, \tilde{q}_1 + p) \right|^2
\]
\[
\times \left( \psi_{\tilde{j}}(\tilde{q}_1, \tilde{q}_2, \cdots, \tilde{q}_1 + p) \right)^2
\]
\[
= C' \left( Tr(1 - \Delta_{x_1}) \cdots (1 - \Delta_{x_k}) \gamma_{\tilde{j}, t}^{(k+p)} \right)^2
\]
Therefore (3.13) follows.

\[ \square \]

**Theorem 3.2.** Suppose that \( d \geq 1 \). If \( \Gamma_{x,t} = \{ \gamma_{\tilde{j}, t}^{(k)} \}_{k \geq 1} \) is a limit point of the sequence \( \tilde{\Gamma}_{N,t} = \{ \tilde{\gamma}_{\tilde{j}, t}^{(k)} \}_{k = 1}^N \) with respect to the product topology \( \tau_{\text{prod}} \), then, for every \( \alpha > \frac{d}{2} \) there exists a constant \( C_\alpha \) (also depends on \( p, d \)) such that the estimate

(3.14) \[
\left\| S^{(k, \alpha)} B_{j_1, \cdots, j_{k+p}} \right\|_{L^2(\mathbb{R}^d)} \leq C_\alpha \left\| S^{(k+p, \alpha)} \gamma_{\tilde{j}, t}^{(k+p)} \right\|_{L^2(\mathbb{R}^{d(k+p)} x \mathbb{R}^{d(k+p)})}
\]
holds.

**Proof.** We will work on the Fourier side of spatial coordinates. Let \( (u_k, u'_k) \), \( q := (q_1, q_2, \cdots, q_p) \) and \( q' := (q'_1, q'_2, \cdots, q'_p) \) be the Fourier conjugate variables corresponding to \( (x_k, x'_k) \), \( (x_{k+1}, x_{k+2}, \cdots, x_{k+p}) \) and \( (x'_{k+1}, x'_{k+2}, \cdots, x'_{k+p}) \) respectively.
Assume \( j = 1 \) in \( B_{j_1, \cdots, j_{k+p}} \) without loss of generality, and we replace the contraction operator by its positive part \( B^+_{j_1, \cdots, j_{k+p}} \) here, since the negative part is
Because of our simplifications at the beginning (specification of \(j\), the negative part of \(\gamma\)) similar. By Plancherel’s theorem

\[
(3.15) \quad \left\| S^{(k,\alpha)} B_{1:k+1,\ldots,k+p} \gamma_{x,t}^{(k+p)} \right\|^2_{L^2(\mathbb{R}^{dk} \times \mathbb{R}^k)}
\]

\[
= \int du_k du'_k \prod_{j=1}^k \langle u_j \rangle^{2\alpha} \langle u'_j \rangle^{2\alpha}
\]

\[
\times \left( \int dq dq' \gamma^{(k+p)}(u_1 + q_1 + \cdots + q_p - q'_1 - \cdots - q'_p, u_2, \cdots, u_k, q; u'_1, q') \right)^2
\]

Cauchy-Schwartz inequality gives us an upper bound

\[
(3.16) \quad \leq \int du_k du'_k F_\alpha(u_k, u'_k) \prod_{j=2}^k \langle u_j \rangle^{2\alpha} \prod_{j=1}^k \langle u'_j \rangle^{2\alpha} \int dq dq'
\]

\[
\times \left( \langle u_1 + q_1 + \cdots + q_p - q'_1 - \cdots - q'_p \rangle^{2\alpha} \langle \gamma^{(k+p)}(u_1 + q_1 + \cdots + q_p) \rangle^{2\alpha} \ldots \langle \gamma^{(k+p)}(u_1) \rangle^{2\alpha} \cdots \langle \gamma^{(k+p)}(u_2) \rangle^{2\alpha} \cdots \langle \gamma^{(k+p)}(u_{k-1}) \rangle^{2\alpha} \cdots \langle \gamma^{(k+p)}(u_k) \rangle^{2\alpha}\right)^2
\]

\[
\leq \sup_{u_k, u'_k} F_\alpha(u_k, u'_k) \times \left\| S^{(k+p,\alpha)} \gamma_{x,t}^{(k+p)} \right\|^2_{L^2(\mathbb{R}^{d(k+p)} \times \mathbb{R}^d)}
\]

Where

\[
(3.17) \quad F_\alpha(u_k, u'_k) := \int \frac{dq dq'}{\langle u_1 + q_1 + \cdots + q_p - q'_1 - \cdots - q'_p \rangle^{2\alpha} \langle \gamma^{(k+p)}(u_1 + q_1 + \cdots + q_p) \rangle^{2\alpha} \ldots \langle \gamma^{(k+p)}(u_1) \rangle^{2\alpha} \cdots \langle \gamma^{(k+p)}(u_2) \rangle^{2\alpha} \cdots \langle \gamma^{(k+p)}(u_{k-1}) \rangle^{2\alpha} \cdots \langle \gamma^{(k+p)}(u_k) \rangle^{2\alpha}}
\]

Because of our simplifications at the beginning (specification of \(j\) and neglect of the negative part of \(B_{j:k+1,\ldots,k+p}\)), function \(F_\alpha(u_k, u'_k)\) only depends on \(u_1\) here. From

\[
\langle u_1 \rangle^{2\alpha} \leq C \langle u_1 + q_1 + \cdots + q_p - q'_1 - \cdots - q'_p \rangle^{2\alpha} + \langle \gamma^{(k+p)}(u_1 + q_1 + \cdots + q_p) \rangle^{2\alpha} + \cdots + \langle \gamma^{(k+p)}(u_1) \rangle^{2\alpha} \cdots + \langle \gamma^{(k+p)}(u_2) \rangle^{2\alpha} \cdots + \langle \gamma^{(k+p)}(u_{k-1}) \rangle^{2\alpha} \cdots + \langle \gamma^{(k+p)}(u_k) \rangle^{2\alpha}
\]

we shift some of the momentum variables to obtain

\[
(3.18) \quad \sup_{u_k, u'_k} F_\alpha(u_k, u'_k) \leq C \int \frac{dq dq'}{\langle q_1 \rangle^{2\alpha} \cdots \langle q_p \rangle^{2\alpha} \langle q'_1 \rangle^{2\alpha} \cdots \langle q'_p \rangle^{2\alpha}}
\]

The RHS of (3.18) is always finite when \(\alpha > \frac{d}{2}\). This proves the theorem. \(\square\)

The above estimate (3.14) requires that \(\alpha > \frac{d}{2}\). Recall the conditions on \(\alpha\) (\(\alpha < 1\) if \(d > 2\), \(\alpha \leq 1\) if \(d = 1\)) in Theorem 3.1. If we want to use both theorems, only \(d = 1\) gives us a nonempty intersection of the two conditions, so we cannot afford this when \(d > 1\). However we need a bound like (3.14) for iterative computations in the proof of uniqueness of the limiting hierarchy. We build such a bound in next section.

4. \textbf{Bounds on the Free Evolution of Infinite Hierarchy}

In this section, we consider the case when the interactions among particles are neglected (\(U_0^{(p)} = 0\)). We will prove a Strichartz estimate that can be used when dealing with recursive Duhamel expansion terms. The approach we followed in this paper is exhibited in [13] [12] [2]. From now on, we will use \(\gamma^{(k)}(t, x_{k+p}, x'_{k+p})\) to replace \(\gamma^{(k)}_{x,t}(t, x_{k+p}, x'_{k+p})\) for convenience.
Applying the Cauchy-Schwarz inequality, the above integral is further bounded by:

\[ (4.6) \]

Then, there exists a constant \( C = C_\alpha \) (also depends on \( p_0 \)) but independent of \( j, k \) such that

\[ (4.4) \]

where \( H^\alpha \) denotes the space of density matrices with finite Hilbert-Schmidt type Sobolev norms:

\[ (4.3) \]

Then, there exists a constant \( C = C_\alpha \) (also depends on \( p_0 \)) but independent of \( j, k \) such that

\[ (4.4) \]

holds.

**Proof.** Following [2], since the two norms are both \( L^2 \) norms, by Plancherel’s theorem, it suffices to prove the estimate (4.3) for the Fourier transform of functions \( u \).

We may also assume that \( j = 1 \) in \( B_{j_1,j_2,\ldots,j_k} \) without loss of generality. Then

\[ (4.6) \]

Applying the Cauchy-Schwarz inequality, the above integral is further bounded by:

\[ (4.9) \]
Such a function \( h \) (which only depends on \( \alpha \)) is a symmetric function, so we have the following:

\[
I(\alpha) := \frac{\delta(\cdots)\langle u_1 \rangle^{2\alpha}}{\langle u_1 + q_1 + \cdots + q_p - q_1' - \cdots - q_p' \rangle^{2\alpha}\langle q_1 \rangle^{2\alpha}\cdots\langle q_p \rangle^{2\alpha}\langle q_1' \rangle^{2\alpha}\cdots\langle q_p' \rangle^{2\alpha}}.
\]

If we can show that the supremum of \( I(\alpha) \) over \( \tau, u_k, u_k' \) is bounded by a constant (which only depends on \( \alpha \)) then we are done. Now, observe that

\[
\langle u_1 \rangle^{2\alpha} \leq C(\langle u_1 + q_1 + \cdots + q_p - q_1' - \cdots - q_p' \rangle^{2\alpha} + \langle q_1 \rangle^{2\alpha} + \cdots + \langle q_p \rangle^{2\alpha} + \langle q_1' \rangle^{2\alpha} + \cdots + \langle q_p' \rangle^{2\alpha})
\]

So we have the following:

\[
I(\alpha) \leq \sum_{l=1}^{2p+1} J_l,
\]

where \( J_l \) is obtained by using (4.8) and canceling the corresponding term in the denominator of (4.7). For example,

\[
J_1 \leq C \int dq_1 dq_1' \frac{\delta(\cdots)}{\langle q_1 \rangle^{2\alpha}\cdots\langle q_p \rangle^{2\alpha}\langle q_1' \rangle^{2\alpha}\cdots\langle q_p' \rangle^{2\alpha}}
\]

and each \( J_l \) for \( l = 2, 3, \ldots, 2p+1 \) can be brought into a similar form by appropriately translating one of the momenta \( q_j, q_j' \). Following [13, 12, 2], we observe the argument of the \( \delta \) distribution equals to

\[
\text{Arg}[\delta] = \tau + (u_1 + q_1 + \cdots + q_p - q_1' - \cdots - q_p')^2 + \sum_{j=2}^k u_j^2 + |q|^2
\]

\[
- |u_k|^2 - |q|^2 + (q_p')^2 - 2(u_1 + q_1 + \cdots + q_p - q_1' - \cdots - q_p') \cdot q_p
\]

Then we integrate out the \( \delta \) distribution using the component of \( q_j' \) parallel to \( u_1 + q_1 + \cdots + q_p - q_1' - \cdots - q_p' \), which yields

\[
J_1 \leq C_\alpha C \int [u_1 + q_1 + \cdots + q_p - q_1' - \cdots - q_p']^2\langle q_1 \rangle^{2\alpha}\cdots\langle q_p \rangle^{2\alpha}\langle q_1' \rangle^{2\alpha}\cdots\langle q_p' \rangle^{2\alpha}
\]

Where

\[
C_\alpha := \int_R \frac{d\zeta}{\langle \zeta \rangle^{2\alpha}}
\]

Obviously, \( C_\alpha \) is finite when \( \alpha > \frac{1}{2} \) (Note \( \alpha > 1 - \frac{1}{2(2p_0 - 1)} \geq 1 - \frac{1}{2(2p - 1)} \geq \frac{1}{2} \)). Following [2], in order to bound \( J_1 \), we introduce a non-negative spherically symmetric function \( h \) with rapid decay away from the unit ball in \( \mathbb{R}^2 \), such that \( h(x) \geq 0 \) decays fast outside the unit ball in \( \mathbb{R}^2 \), and

\[
\frac{1}{\langle q \rangle^{2\alpha}} < \left( h \ast \frac{1}{| \cdot |^{2\alpha}} \right)(q)
\]

Such a function \( h \) does exists. For example we can take \( h(y) = c_1 e^{-c_2 y^2} \) with appropriate \( c_1, c_2 \). Here we need \( \alpha < 1 \) for \( h \ast \frac{1}{| \cdot |^{2\alpha}} \) to stay in \( L^\infty(\mathbb{R}^2) \). Then (take
Now assume the initial condition \( \gamma \) with \( \Delta p \).

The infinite hierarchy (1.9) can be rewritten in integral form as

\[
\text{in the theorems below. Recall that we use } \gamma \text{ previous sections. Before doing that we need to introduce some notation appeared (4.1)}.
\]

We are getting close to prove the conclusions on uniqueness with the results in previous sections. Before doing that we need to introduce some notation appeared in the theorems below. Recall that we use \( \gamma(k, t, \cdot) \) to replace \( \gamma_{x, t}(\cdot) \) when there is no confusion. The infinite hierarchy (1.9) can be rewritten in integral form as

\[
\gamma^{(k)}(t, \cdot) = U^{(k)}(t) \gamma^{(k)}(0, \cdot) - i \sum_{p=1}^{p_0} \sum_{j=1}^{k} ds U^{(k)}(t - s) B_{j; k+1, \ldots, k+p} \gamma^{(k+p)}(s, \cdot)
\]

Here \( b_0^{(p)} = \int_{\mathbb{R}^d} V^{(p)}(x) dx \). Recall the free evolution operator \( U^{(k)}(t) \) given by

\[
U^{(k)}(t) \gamma^{(k)} = e^{it\Delta^{(k)}} \gamma^{(k)}
\]

with \( \Delta^{(k)} = \Delta_{x_1} - \Delta_{x_2} \).

Now assume the initial condition \( \gamma^{(k)}(0, \cdot) = 0 \). For fixed positive integer \( k \), thanks to Duhamel formula, we can write \( \gamma^{(k)} \) in terms of the future iterates \( \gamma^{(k+p_1)}, \gamma^{(k+p_1+p_2)}, \ldots, \gamma^{(k+p_1+\cdots+p_n)} \), where \( p_1, p_2, \ldots, p_n \) are integers chosen from set

\[
S_{p_0} := \{1, 2, 3, \ldots, p_0\}.
\]

Also let \( Q_j \) be half of the running sum over \( p_1, p_2, p_3, \ldots \):

\[
Q_j := p_1 + p_2 + \cdots + p_j \leq p_0 j, \quad j = 1, 2, \ldots
\]

5. Uniqueness of Solutions

We are getting close to prove the conclusions on uniqueness with the results in previous sections. Before doing that we need to introduce some notation appeared in the theorems below. Recall that we use \( \gamma^{(k)}(t, \cdot) \) to replace \( \gamma_{x, t}(\cdot) \) when there is no confusion. The infinite hierarchy (1.9) can be rewritten in integral form as

\[
\gamma^{(k)}(t, \cdot) = U^{(k)}(t) \gamma^{(k)}(0, \cdot) - i \sum_{p=1}^{p_0} \sum_{j=1}^{k} ds U^{(k)}(t - s) B_{j; k+1, \ldots, k+p} \gamma^{(k+p)}(s, \cdot)
\]

Here \( b_0^{(p)} = \int_{\mathbb{R}^d} V^{(p)}(x) dx \). Recall the free evolution operator \( U^{(k)}(t) \) given by

\[
U^{(k)}(t) \gamma^{(k)} = e^{it\Delta^{(k)}} \gamma^{(k)}
\]

with \( \Delta^{(k)} = \Delta_{x_1} - \Delta_{x_2} \).

Now assume the initial condition \( \gamma^{(k)}(0, \cdot) = 0 \). For fixed positive integer \( k \), thanks to Duhamel formula, we can write \( \gamma^{(k)} \) in terms of the future iterates \( \gamma^{(k+p_1)}, \gamma^{(k+p_1+p_2)}, \ldots, \gamma^{(k+p_1+\cdots+p_n)} \), where \( p_1, p_2, \ldots, p_n \) are integers chosen from set

\[
S_{p_0} := \{1, 2, 3, \ldots, p_0\}.
\]

Also let \( Q_j \) be half of the running sum over \( p_1, p_2, p_3, \ldots \):

\[
Q_j := p_1 + p_2 + \cdots + p_j \leq p_0 j, \quad j = 1, 2, \ldots
\]
Conventionally let $Q_0 = 0$. Then we have

(5.2) \[
\gamma^{(k)}(t, \cdot) = \sum_{p_1 \in S_{p_0}} b_{p_1} \int_0^{t_k} e^{i(t_k - t_{k+Q_1}^1) \Delta^1_{+}} B_{k+Q_1}^{k} (\gamma^{(k+Q_1)}(t_{k+Q_1})) dt_{k+Q_1},
\]

\[
= \sum_{p_1, p_2 \in S_{p_0}} b_{p_1} b_{p_2} \int_0^{t_k} e^{i(t_k - t_{k+Q_1}) \Delta^1_{+}} B_{k+Q_1}^{k} \left( \int_0^{t_{k+Q_1}} e^{i(t_{k+Q_1} - t_{k+Q_2}) \Delta^1_{+}} B_{k+Q_2}^{k} (\gamma^{(k+Q_2)}(t_{k+Q_2})) dt_{k+Q_2} \right) dt_{k+Q_1},
\]

\[= \cdots \]

\[
= \sum_{p_1, \ldots, p_n \in S_{p_0}} \left( \prod_{j=1}^{n} b_{p_j} \right) \int_0^{t_k} \cdots \int_0^{t_{k+Q_{n-1}}} J^{k}(t_{k+Q_n}) dt_{k+Q_{n-1}} \cdots dt_{k+Q_n},
\]

where

\[
\mathcal{L}_{k+Q_n} = (t_k, t_{k+Q_1}, \ldots, t_{k+Q_n})
\]

\[
B_{k+Q_n}^{k+Q_n-1} := \sum_{j=1}^{k+Q_n-1} B_{j; k+Q_n-1+1, k+Q_n-1+2, \ldots, k+Q_n}
\]

\[J^{k}(t_{k+Q_n}) :=
\]

\[
e^{i(t_k - t_{k+Q_1}) \Delta^1_{+}} B_{k+Q_1}^{k} \cdots e^{i(t_{k+Q_{n-1}} - t_{k+Q_n}) \Delta^1_{+}} B_{k+Q_n}^{k+Q_n-1} (\gamma^{(k+Q_n)}(t_{k+Q_n}))
\]

Here are the main uniqueness theorems for $d = 1, 2$:

**Theorem 5.1.** Assume that $d = 1$, $t \in [0, T]$ and $\frac{1}{2} < \alpha \leq 1$. The maximal potential constant $b_0 = \max \{b_0^{(1)}, b_0^{(2)}, \ldots, b_0^{(p_0)}\} \in (0, \infty)$. Then we have

(5.3) \[
\left\| S^{(k, \alpha)} \gamma^{(k)}(t, \cdot) \right\|_{L^2(\mathbb{R}^k \times \mathbb{R}^k)} \leq C^k (C_0 T)^n
\]

for arbitrary $n$ and constants $C, C_0$ that are depending on $b_0$, $p_0$, $\kappa$ and $\alpha$, but are independent of $k$ and $T$.

**Theorem 5.2.** Assume that $d = 2$ and $t \in [0, T]$, $1 - \frac{1}{2(2p_0 - 1)} < \alpha \leq 1$. The maximal potential constant $b_0 = \max \{b_0^{(1)}, b_0^{(2)}, \ldots, b_0^{(p_0)}\} \in (0, \infty)$. Then we have

(5.4) \[
\left\| S^{(k, \alpha)} \gamma^{(k)}(t, \cdot) \right\|_{L^2(\mathbb{R}^{2k} \times \mathbb{R}^{2k})} \leq C^k (C_0 \sqrt{T})^n
\]

for arbitrary $n$ and constants $C, C_0$ that are depending on $b_0$, $p_0$, $\kappa$ and $\alpha$, but are independent of $k$ and $T$.

Based on the above theorems, if we are given sufficiently small $T$, then for all $t \in [0, T]$:

(5.5) \[
\left\| S^{(k, \alpha)} \gamma^{(k)}(t, \cdot) \right\|_{L^2(\mathbb{R}^{dk} \times \mathbb{R}^{dk})} \rightarrow 0 \text{ as } n \rightarrow 0.
\]

Which implies that $\gamma^{(k)}(t, \cdot) = 0$. Since $k$ is arbitrary, therefore solutions to the infinite hierarchy (1.5) with zero initial conditions are unique in the above norm.
Proof of Theorem 5.7  

The idea of the proof is an iterative applications of spatial bound 3.14 and Cauchy-Schwarz inequality and at last followed by the use of the bound 3.11. Noticed that $\alpha$ is a constant in $(\frac{1}{2}, 1]$ and $e^{i(t_k - t_{k+1})} \Delta_{\pm}^{(k)}$ is a unitary operator and commutes with the operator $S^{(k, \alpha)}$, thus we have

\[
\begin{align*}
\left\| S^{(k, \alpha)} \int_0^{t_k} \cdots \int_0^{t_k+Q_n-1} J^k (L_{k+Q_n}) dt_{k+Q_1} \cdots dt_{k+Q_n} \right\|_{L^2(\mathbb{R}^k \times \mathbb{R}^k)} \\
\leq \int_0^{t_k} \cdots \int_0^{t_k+Q_n-1} \left\| S^{(k, \alpha)} e^{i(t_k - t_{k+1})} \Delta_{\pm}^{(k)} B_{k+Q_1} \cdots B_{k+Q_n} \right\|_{L^2(\mathbb{R}^k \times \mathbb{R}^k)} dt_{k+Q_1} \cdots dt_{k+Q_n} \\
\leq k C_{\alpha} \int_0^{t_k} \cdots \int_0^{t_k+Q_n-1} \left\| S^{(k, \alpha)} e^{i(t_k - t_{k+1})} \Delta_{\pm}^{(k)} B_{k+Q_1} \cdots B_{k+Q_n} \right\|_{L^2(\mathbb{R}^k \times \mathbb{R}^k)} dt_{k+Q_1} \cdots dt_{k+Q_n} \\
\leq \cdots
\end{align*}
\]

(5.6)

\[
\begin{align*}
\leq \sum_{j=0}^{n-2} \left( (k + Q_j) C_{\alpha} \right) \int_0^{t_k} \cdots \int_0^{t_k+Q_n-1} \left\| S^{(k + Q_j - 1, \alpha)} e^{i(t_k + Q_j - 1 - t_{k+Q_n})} \Delta_{\pm}^{(k + Q_j - 1)} \right\|_{L^2(\mathbb{R}^k \times \mathbb{R}^k)} dt_{k+Q_1} \cdots dt_{k+Q_n} \\
\leq k C_{\alpha} \int_0^{t_k} \cdots \int_0^{t_k+Q_n-1} \left\| S^{(k + Q_n - 1, \alpha)} e^{i(t_k + Q_n - 1 - t_{k+Q_n})} \Delta_{\pm}^{(k + Q_n - 1)} \right\|_{L^2(\mathbb{R}^k \times \mathbb{R}^k)} dt_{k+Q_1} \cdots dt_{k+Q_n} \\
\leq C_{\alpha}^{-1} \prod_{j=0}^{n-1} (k + p_{Q_j}) \int_0^{t_k} \cdots \int_0^{t_k+Q_n-1} C^{k+p_{Q_n}} dt_{k+Q_1} \cdots dt_{k+Q_n} \\
\leq p_0^n C_{\alpha}^{-1} \left( \frac{k}{p_0} + 1 \right) \cdots \left( \frac{k}{p_0} + n - 1 \right) C^{k+p_{Q_n}} n! \\
= C^{k+p_{Q_n}} p_0^n C_{\alpha}^{-1} \left( \frac{1}{p_0} + n - 1 \right) n! \\
\leq C^k (p_0 C_{\alpha}^n t_k)^n C_{\alpha}^{-1} \frac{k}{p_0^n} n! + n - 1
\end{align*}
\]

Thus

\[
\begin{align*}
\left\| S^{(k, \alpha)} \gamma (t_{k+1}) \right\|_{L^2(\mathbb{R}^k \times \mathbb{R}^k)} \\
\leq \sum_{p_1, \cdots, p_n \in S_{p_0}} \left( \prod_{j=1}^n b_{p_j} \right) \left\| S^{(k, \alpha)} \int_0^{t_k} \cdots \int_0^{t_k+Q_n-1} J^k (L_{k+Q_n}) dt_{k+Q_1} \cdots dt_{k+Q_n} \right\|_{L^2(\mathbb{R}^k \times \mathbb{R}^k)} \\
\leq p_0 (b_0) n C^k (p_0 C_{\alpha}^n t_k) C_{\alpha}^{-1} \frac{k}{p_0^n} n! + n - 1
\end{align*}
\]
\[ \leq C^h(C_0 T)^n \]

where (5.6) is based on (3.14) and we keep using (3.14) to obtain (5.7). Since \[ e^{i(t_{k+Q_n} - t_{k+Q_{n-1}}) \Delta_{k+Q_{n-1}}} \] is unitary and commutes with \( S_{k+Q_{n-1,n}} \), then after applying Theorem 3.1 we have (5.8). \( [x] \) is the ceiling function. In the last line, choose appropriate \( C \) and \( C_0 \) to finish the proof. \( \square \)

For the proof of Theorem 5.2, we run the following combinatorial argument which is inspired by [13].

### 6. Combinatorial Arguments

#### 6.1. Graphical Representations

The key point in the proof of Theorem 5.2 is to handle the iterative terms from Duhamel formula. Throughout this section, we will prove some lemmas to help us group these terms and also derive some bounds on certain equivalence classes. The technique we worked here is analogous to [13] and [2], but in a much more generalized setting.

For the reader’s convenience, recall some notations we have defined before:

\[ \forall 1 \leq j \leq n, \quad p_j \in S_{p_0} = \{1, 2, 3, \ldots, p_0\}. \]

Also, \( B_{k+Q_n} = \sum_{j=1}^{k+Q_n} B_{j; k+Q_n} \), we can rewrite \( J^k(u_{k+Q_n}) \) as the following:

\[
J^k(u_{k+Q_n}) = \sum_{\mu \in M} J^k(u_{k+Q_n}; \mu)
\]

where

\[
J^k(u_{k+Q_n}; \mu) = e^{i(t_k - t_{k+Q_n}) \Delta_{k+Q_n}} B_{\mu(k+Q_n)} e^{i(t_{k+Q_n} - t_{k+Q_2}) \Delta_{k+Q_1}} \ldots \]

\[ \times e^{i(t_{k+Q_n} - t_{k+Q_{n-1}}) \Delta_{k+Q_{n-1}}} B_{\mu(k+Q_{n-1}+1), k+Q_{n-1}+1, \ldots, k+Q_n} \]

and \( \mu \) is a map from \( \{k + 1, k + 2, \ldots, k + Q_{n-1} + 1\} \) to \( \{1, 2, \ldots, k + Q_{n-1}\} \) such that \( \mu(2) = 1 \) and \( \mu(j) < j \) for all \( j \). \( M \) is the set of all these mappings.

By the definition of \( \mu \), we can represent it by highlighting exactly one nonzero entry in each column of a \( (k + Q_{n-1}) \times n \) matrix like:

\[
\begin{pmatrix}
B_{1;k+1, \ldots, k+Q_1} & B_{1;k+Q_1+1, \ldots, k+Q_2} & \cdots & B_{1;k+Q_{n-1}, \ldots, k+Q_n} \\
B_{2;k+1, \ldots, k+Q_1} & B_{2;k+Q_1+1, \ldots, k+Q_2} & \cdots & B_{2;k+Q_{n-1}, \ldots, k+Q_n} \\
\vdots & \vdots & \ddots & \vdots \\
B_{k;k+1, \ldots, k+Q_1} & B_{k;k+Q_1+1, \ldots, k+Q_2} & \cdots & B_{k;k+Q_{n-1}, \ldots, k+Q_n} \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
\end{pmatrix}
\]

Henceforth we can rewrite (5.2) as

\[
\gamma^k(t_k, \cdot) = \int_0^{t_k} \ldots \int_0^{t_k + Q_{n-1}} \sum_{\mu \in M} J^k(u_{k+Q_n}; \mu) dt_{k+Q_1} \ldots dt_{k+Q_n}
\]
So the basic term of the above sum is the following integral

\[
I(\mu, \sigma) = \int_{t_{q(k+Q_1)} \geq t_{q(k+Q_2)} \geq \cdots \geq t_{q(k+Q_n)}} J^k(t_{k+Q_n}; \mu)dt_{k+Q_1} \cdots dt_{k+Q_n}
\]

where \( \sigma \) is a permutation of \( k + Q_1, k + Q_2, \ldots, k + Q_n \). We will associate the integral \( I(\mu, \sigma) \) to the following \((k + Q_{n-1} + 1) \times n\) matrix. Matrix (6.5) is also helpful to visualize \( B_{\mu(k+Q_{j-2}+1),k+Q_{j-1}+1} \) for \( j = 1, 2, \ldots, n \) and \( \sigma \):

\[
\begin{pmatrix}
  t_{\sigma^{-1}(k+Q_1)} & t_{\sigma^{-1}(k+Q_2)} & \cdots & t_{\sigma^{-1}(k+Q_n)} \\
  B_{1:k+1, \ldots, k+Q_1} & B_{2:k+1, \ldots, k+Q_1} & \cdots & B_{k:k+1, \ldots, k+Q_1} \\
  0 & 0 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & 0
\end{pmatrix}
\]

We label the columns of matrix (6.5) by 1 through \( n \) while rows 0 through \( k + Q_{n-1} \).

6.2. Acceptable Moves. It is an important step to introduce the so-called “acceptable move” on the set of matrices like (6.5). In particular, if \( \mu(k + Q_j + 1) < \mu(k + Q_{j-1} + 1) \), we are allowed to do the following changes at the same time:

- exchange the highlights in columns \( j \) and \( j + 1 \)
- exchange the highlights in rows \( k + Q_{j-1} + 1 \) and \( k + Q_j + 1 \)
- exchange the highlights in rows \( k + Q_{j-1} + 2 \) and \( k + Q_j + 2 \)
- \ldots
- exchange the highlights in rows \( k + Q_{j-1} + r_0 \) and \( k + Q_j + r_0 \)
- exchange \( t_{\sigma^{-1}(k+Q_1)} \) and \( t_{\sigma^{-1}(k+Q_{j+1})} \)

with \( r_0 := \min(p_j, p_{j+1}) \).

For instance, if \( k = 1, n = 4, p_1 = 2, p_2 = 1, p_3 = 3, p_4 = 2 \), then we go from

\[
\begin{pmatrix}
  t_{\sigma^{-1}(1+Q_1)} & t_{\sigma^{-1}(1+Q_2)} & t_{\sigma^{-1}(1+Q_3)} & t_{\sigma^{-1}(1+Q_4)} \\
  B_{1:2,3} & B_{1:4} & B_{1:5,6,7} & B_{1:8,9} \\
  0 & B_{2:4} & B_{2:5,6,7} & B_{2:8,9} \\
  0 & 0 & B_{3:4} & B_{3:5,6,7} \\
  0 & 0 & 0 & B_{4:5,6,7} \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{pmatrix}
\]

to

\[
\begin{pmatrix}
  t_{\sigma^{-1}(1+Q_1)} & t_{\sigma^{-1}(1+Q_2)} & t_{\sigma^{-1}(1+Q_3)} & t_{\sigma^{-1}(1+Q_4)} \\
  B_{1:2,3} & B_{1:4} & B_{1:5,6,7} & B_{1:8,9} \\
  0 & B_{2:4} & B_{2:5,6,7} & B_{2:8,9} \\
  0 & B_{3:5,6,7} & B_{3:5,6,7} & B_{3:8,9} \\
  0 & 0 & B_{4:5,6,7} & B_{4:8,9} \\
  0 & 0 & 0 & B_{5:8,9} \\
  0 & 0 & 0 & B_{6:8,9} \\
  0 & 0 & 0 & B_{7:8,9}
\end{pmatrix}
\]

The reason for taking such moves is explained by the following lemma.
Lemma 6.1. Let \((\mu, \sigma)\) be transformed into \((\mu', \sigma')\) by an acceptable move. Then, for the corresponding integrals \(I(\mu, \sigma) = I(\mu', \sigma')\)

Proof. This is a relatively straightforward proof but somewhat tedious, as in [13] and [2]. We modify the proof of Lemma 7.1 in [2] so that it can be used here. Since there is only one acceptable move between the two integrals, most part of their expressions share the same terms. Let us fix \(j \geq 3\), select two integers \(i, l\) such that \(i < l < j < j + 1\) and compare \(I(\mu, \sigma)\) and \(I(\mu', \sigma')\)

\[
(6.6) \quad I(\mu, \sigma) = \int_{t_k \geq \cdots t_{\sigma(k+Q_j)} \geq t_{\sigma(k+Q_j+1)} \cdots \geq t_{\sigma(k+Q_n)}} J^k(t_{k+Q_n}; \mu)dt_{k+Q_1} \cdots dt_{k+Q_n}
\]

\[
= \int_{t_k \geq \cdots t_{\sigma(k+Q_j)} \geq t_{\sigma(k+Q_j+1)} \cdots \geq t_{\sigma(k+Q_n)}} \cdots e^{i(t_{k+Q_j-1} - t_{k+Q_j})} \Delta_{\pm}^{(k+Q_j)}
\]

\[
\times B_i; k+Q_j, l, \cdots, k+Q_{j+1} e^{i(t_{k+Q_j} - t_{k+Q_j+1})} \Delta_{\pm}^{(k+Q_j)} B_{i; k+Q_j, l, \cdots, k+Q_{j+1}}
\]

\[
\times e^{i(t_{k+Q_j+1} - t_{k+Q_j+2})} \Delta_{\pm}^{(k+Q_j+1)} (\cdots)dt_{k+Q_1} \cdots dt_{k+Q_n}
\]

and

\[
(6.7) \quad I(\mu', \sigma') = \int_{t_k \geq \cdots t_{\sigma'(k+Q_j)} \geq t_{\sigma'(k+Q_j+1)} \cdots \geq t_{\sigma'(k+Q_n)}} J^k(t_{k+Q_n}; \mu')dt_{k+Q_1} \cdots dt_{k+Q_n}
\]

\[
= \int_{t_k \geq \cdots t_{\sigma'(k+Q_j)} \geq t_{\sigma'(k+Q_j+1)} \cdots \geq t_{\sigma'(k+Q_n)}} \cdots e^{i(t_{k+Q_j-1} - t_{k+Q_j})} \Delta_{\pm}^{(k+Q_j)}
\]

\[
\times B_i; k+Q_j, l, \cdots, k+Q_{j+1} e^{i(t_{k+Q_j} - t_{k+Q_j+1})} \Delta_{\pm}^{(k+Q_j)} B_{i; k+Q_j, l, \cdots, k+Q_{j+1}}
\]

\[
\times e^{i(t_{k+Q_j+1} - t_{k+Q_j+2})} \Delta_{\pm}^{(k+Q_j+1)} (\cdots)dt_{k+Q_1} \cdots dt_{k+Q_n}
\]

The \(\cdots\) in \((6.6)\) and \((6.7)\) coincide.

For \(1 \leq r \leq r_0 = \min\{p_j, p_{j+1}\}\), \(s \geq Q_j\) and index \(m\): \(j + 1 \leq m \leq n\), any \(B_{k+Q_j, r; s+1, \cdots, s+p_m}\) (when it is highlighted) in \((\cdots)\) of \((6.6)\) will become \(B_{k+Q_j, r; s+1, \cdots, s+p_m}\) in \((\cdots)'\) of \((6.7)\) and any \(B_{k+Q_j, r; s+1, \cdots, s+p_m}\) (when it is highlighted) in \((\cdots)\) of \((6.7)\) become \(B_{k+Q_j, r; s+1, \cdots, s+p_m}\) in \((\cdots)'\) of \((6.6)\).

All the changes are illustrated in the table below:

| \(\cdots\) | \(\cdots)'\) |
|----------------|----------------|
| \(B_{k+Q_j-1+1}; s+1, \ldots, s+p_m\) | \(B_{k+Q_j+1}; s+1, \ldots, s+p_m\) |
| \(B_{k+Q_j-1+2}; s+1, \ldots, s+p_m\) | \(B_{k+Q_j+2}; s+1, \ldots, s+p_m\) |
| \(\vdots\) | \(\vdots\) |
| \(B_{k+Q_j-1+r_0}; s+1, \ldots, s+p_m\) | \(B_{k+Q_j+r_0}; s+1, \ldots, s+p_m\) |

In order to prove \(I(\mu, \sigma) = I(\mu', \sigma')\) we introduce \(P\) and \(\tilde{P}\) which are defined as:

\[
(6.8) \quad P = B_{i; k+Q_j-1+1, \cdots, k+Q_{j+1}} e^{i(t_{k+Q_j} - t_{k+Q_{j+1}})} \Delta_{\pm}^{(k+Q_j)} B_{i; k+Q_j, l, \cdots, k+Q_{j+1}}
\]

\[
(6.9) \quad \tilde{P} = B_{i; k+Q_j+1, \cdots, k+Q_{j+1}} e^{-i(t_{k+Q_j} - t_{k+Q_{j+1}})} \Delta_{\pm}^{(k+Q_j)} B_{i; k+Q_j, l, \cdots, k+Q_{j+1}}
\]
where

\[ \tilde{\Delta}_{\pm}^{(k+Q_j)} = \Delta_{\pm}^{(k+Q_j)} - \Delta_{\pm, x_k+Q_j} - \Delta_{\pm, x_k+Q_{j-1}} - \cdots - \Delta_{\pm, x_k+Q_{j-1}+1} + \Delta_{\pm, x_k+Q_{j+1}} + \Delta_{\pm, x_k+Q_{j+2}} + \cdots + \Delta_{\pm, x_k+Q_j+1} \]

We’ve used this notion above: \( \Delta_{\pm, x_j} = \Delta_{x_j} - \Delta_{x_j'} \).

We will show that

\[ e^{i(t_k+Q_j-t_k+Q_{j+1})} \Delta_{\pm}^{(k+Q_{j-1})} P e^{i(t_k+Q_j-t_k+Q_{j+2})} \Delta_{\pm}^{(k+Q_{j+1})} = e^{i(t_k+Q_j-t_k+Q_{j+1})} \Delta_{\pm}^{(k+Q_{j-1})} P e^{i(t_k+Q_j-t_k+Q_{j+2})} \Delta_{\pm}^{(k+Q_{j+1})} \]

Indeed in (6.8) we can write \( \Delta_{\pm}^{(k+Q_j)} = \Delta_{\pm, x_i} + (\Delta_{\pm}^{(k+Q_j)} - \Delta_{\pm, x_i}) \). Therefore,

\[ e^{i(t_k+Q_j-t_k+Q_{j+1})} \Delta_{\pm}^{(k+Q_{j-1})} = e^{i(t_k+Q_j-t_k+Q_{j+1})} \Delta_{\pm, x_i} + e^{i(t_k+Q_j-t_k+Q_{j+1})} (\Delta_{\pm}^{(k+Q_j)} - \Delta_{\pm, x_i}) \]

Observe that the first term on the RHS of the above equation can be commuted to the left of \( B_{t; k+Q_j-1+1, \ldots, k+Q_j} \) and the second one to the right of \( B_{t; k+Q_j+1, \ldots, k+Q_{j+1}} \), thus after two commutations

\[ P \equiv e^{i(t_k+Q_j-t_k+Q_{j+1})} B_{t; k+Q_j-1+1, \ldots, k+Q_j} B_{t; k+Q_j+1, \ldots, k+Q_{j+1}} \]

(6.11)

and the LHS of (6.10) becomes

\[ e^{i(t_k+Q_j-t_k+Q_{j+1})} \Delta_{\pm}^{(k+Q_{j-1})} P e^{i(t_k+Q_j-t_k+Q_{j+2})} \Delta_{\pm}^{(k+Q_{j+1})} \]

(6.12)

\[ = e^{i(t_k+Q_j-t_k+Q_{j+1})} \Delta_{\pm}^{(k+Q_{j-1})} e^{i(t_k+Q_j-t_k+Q_{j+1})} \Delta_{\pm, x_i} B_{t; k+Q_j-1+1, \ldots, k+Q_j} \]

\[ \times B_{t; k+Q_j+1, \ldots, k+Q_{j+1}} e^{i(t_k+Q_j-t_k+Q_{j+1})} (\Delta_{\pm}^{(k+Q_j)} - \Delta_{\pm, x_i}) e^{i(t_k+Q_j-t_k+Q_{j+1})} \Delta_{\pm}^{(k+Q_{j+1})} \]

\[ = e^{i(t_k+Q_j-t_k+Q_{j+1})} \Delta_{\pm}^{(k+Q_{j-1})} e^{i(t_k+Q_j-t_k+Q_{j+1})} \Delta_{\pm, x_i} B_{t; k+Q_j+1, \ldots, k+Q_{j+1}} \]

\[ \times B_{t; k+Q_j-1, \ldots, k+Q_j} e^{i(t_k+Q_j-t_k+Q_{j+1})} (\Delta_{\pm, x_i} + \Delta_{\pm, x_k+Q_{j+1}} + \cdots + \Delta_{\pm, x_k+Q_j+1}) \]

\[ \times e^{i(t_k+Q_j-t_k+Q_{j+2})} \Delta_{\pm, x_1} + \Delta_{\pm, x_k+Q_{j+1}} + \cdots + \Delta_{\pm, x_k+Q_j+1} \]

where a hat denotes a missing term.

Similarly, we can rewrite \( \tilde{\Delta}_{\pm}^{(k+Q_j)} \) as

\[ \tilde{\Delta}_{\pm}^{(k+Q_j)} = \Delta_{\pm}^{(k+Q_j)} - \Delta_{\pm, x_k+Q_j} - \cdots - \Delta_{\pm, x_k+Q_{j-1}+1} + \Delta_{\pm, x_k+Q_{j+1}} + \Delta_{\pm, x_k+Q_{j+2}} + \cdots + \Delta_{\pm, x_k+Q_j+1} \]

\[ = \Delta_{\pm}^{(k+Q_{j-1})} + \Delta_{\pm, x_k+Q_{j+1}} + \cdots + \Delta_{\pm, x_k+Q_{j+1}} \]

\[ = (\Delta_{\pm}^{(k+Q_{j-1})} - \Delta_{\pm, x_i}) + (\Delta_{\pm, x_i} + \Delta_{\pm, x_k+Q_{j+1}} + \cdots + \Delta_{\pm, x_k+Q_{j+1}}) \]
Hence the factor \( e^{-i(t_k+Q_j-t_{k+Q_{j+1}})\Delta_{\pm}^{(k+Q_j)}} \) appearing in the definition of \( \tilde{P} \) can be rewritten as
\[
e^{-i(t_k+Q_j-t_{k+Q_{j+1}})\Delta_{\pm}^{(k+Q_j)}} = e^{-i(t_k+Q_j-t_{k+Q_{j+1}})(\Delta_{\pm}^{(k+Q_{j-1})}-\Delta_{\pm,x_i})} \times e^{-i(t_k+Q_j-t_{k+Q_{j+1}})(\Delta_{\pm,x_i}+\Delta_{\pm,x_k+Q_{j+1}}+\cdots)}
\]
and consequently,
\[
(6.13) \quad \tilde{P} = e^{-i(t_k+Q_j-t_{k+Q_{j+1}})(\Delta_{\pm}^{(k+Q_{j-1})}-\Delta_{\pm,x_i})} B_{t_k+Q_{j+1}, t_k+Q_{j+1}+1, \ldots, t_k+Q_j} \times e^{-i(t_k+Q_j-t_{k+Q_{j+1}})(\Delta_{\pm,x_i}+\Delta_{\pm,x_k+Q_{j+1}}+\cdots)}
\]

The RHS of (6.10) equals to (6.14)
\[
e^{i(t_k+Q_{j-1}-t_k+Q_{j+1})\Delta_{\pm}^{(k+Q_j)}} \tilde{P} e^{i(t_k+Q_j-t_{k+Q_{j+2}})\Delta_{\pm}^{(k+Q_{j+1})}} = e^{i(t_k+Q_{j-1}+t_{k+Q_{j+1}})\Delta_{\pm}^{(k+Q_j-1)}} e^{-i(t_k+Q_j-t_{k+Q_{j+1}})(\Delta_{\pm}^{(k+Q_j-1)}-\Delta_{\pm,x_i})} \times B_{t_k+Q_{j+1}, t_k+Q_{j+1}+1, \ldots, t_k+Q_j} \times e^{-i(t_k+Q_j-t_{k+Q_{j+1}})(\Delta_{\pm,x_i}+\Delta_{\pm,x_k+Q_{j+1}}+\cdots)} e^{i(t_k+Q_j-t_{k+Q_{j+2}})\Delta_{\pm}^{(k+Q_{j+1})}}
\]
which is the same as (6.12), so (6.10) is proved.

Note \( r_0 = \min\{p_j, p_{j+1}\} \). By the symmetry property of \( \gamma^{(k+Q_n)} \), we can perform the following exchanges without changing its value
- exchange \((x_{k+Q_{j-1}+1}, x'_{k+Q_{j-1}+1})\) with \((x_{k+Q_{j}+1}, x'_{k+Q_{j}+1})\)
- exchange \((x_{k+Q_{j-1}+2}, x'_{k+Q_{j-1}+2})\) with \((x_{k+Q_{j}+2}, x'_{k+Q_{j}+2})\)
- \( \ldots \)
- exchange \((x_{k+Q_{j-1}+r_0}, x'_{k+Q_{j-1}+r_0})\) with \((x_{k+Q_{j}+r_0}, x'_{k+Q_{j}+r_0})\)

After performing these exchanges only in the arguments of \( \gamma^{(k+Q_n)} \) we can rewrite (6.10) based on (6.10) as follows:
\[
(6.15) \quad I(\mu, \sigma) = \int_{t_k \geq \cdots \geq t_{\sigma(k+Q_j)} \geq t_{\sigma(k+Q_{j+1})} \cdots \geq t_{\sigma(k+Q_n)} \geq 0} \cdots \times e^{i(t_k+Q_{j-1}-t_k+Q_j)\Delta_{\pm}^{(k+Q_j-1)}} e^{i(t_k+Q_{j+1}-t_k+Q_{j+2})\Delta_{\pm}^{(k+Q_{j+1})}} (\cdots) dt_{k+Q_1} \cdots dt_{k+Q_n}
\]
In (6.15) we perform the change of variables that

\[ P e^{i((t_k+Q_j)-(t_k+Q_j+1)) \Delta \frac{(k+Q_j)}{2}} \Delta \frac{(k+Q_j+1)}{2} (\ldots) dt_{k+Q_j} \ldots dt_{k+Q_n} \]

\[ \int_{(k+Q_j)} \int_{(k+Q_j+1)} \ldots \int_{(k+Q_n)} e^{i((t_k+Q_j)-(t_k+Q_j+1)) \Delta \frac{(k+Q_j)}{2}} \Delta \frac{(k+Q_j+1)}{2} (\ldots) dt_{k+Q_j} \ldots dt_{k+Q_n} \]

\[ \times \delta_{i,k+Q_j+1,\ldots,k+Q_j+1} e^{-i((t_k+Q_j)-(t_k+Q_j+1)) \Delta \frac{(k+Q_j)}{2}} \delta_{i,k+Q_j+1,\ldots,k+Q_j} \]

\[ \times e^{i((t_k+Q_j)-(t_k+Q_j+1)) \Delta \frac{(k+Q_j+1)}{2}} (\ldots) dt_{k+Q_j} \ldots dt_{k+Q_n} \]

in which \( \delta_{j,s+1,\ldots,s+p_m} \) denotes the abbreviated kernel of the operator \( B_{j,s+1,\ldots,s+p_m} \):

(6.16)

\[ \delta_{j,s+1,\ldots,s+p_m} = \delta(x_j - x_{s+1}) \delta(x_j - x'_{s+1}) \delta(x_j - x_{s+p_m}) \delta(x_j - x'_{s+p_m}) \]

\[ - \delta(x'_j - x_{s+1}) \delta(x'_j - x'_{s+1}) \delta(x'_j - x_{s+p_m}) \delta(x'_j - x'_{s+p_m}) \]

In (6.16) we perform the change of variables that

- exchange \((t_k+Q_j+1, x_k+Q_j+1, x'_{k+Q_j-1}+1)\) with \((t_k+Q_j+1, x_k+Q_j+1, x'_{k+Q_j-1}+1)\)
- exchange \((t_k+Q_j+2, x_k+Q_j+2, x'_{k+Q_j-1}+2)\) with \((t_k+Q_j+2, x_k+Q_j+2, x'_{k+Q_j-1}+2)\)
- \ldots
- exchange \((t_k+Q_j-1+r_0, x_k+Q_j-1+r_0, x'_{k+Q_j-1}+r_0)\) with \((t_k+Q_j-1+r_0, x_k+Q_j-1+r_0, x'_{k+Q_j-1}+r_0)\)

in the whole integral. Under the same change of variables \( \Delta \frac{(k+Q_j)}{2} \) becomes

\[ \Delta \frac{(k+Q_j)}{2} = \Delta \frac{(k+Q_j)}{2} - \Delta \frac{(k+Q_j)}{2} - \cdots - \Delta \frac{(k+Q_j)}{2} + \Delta \frac{(k+Q_j)}{2} + \cdots + \Delta \frac{(k+Q_j)}{2} + \Delta \frac{(k+Q_j)}{2} \]

Note that \( \Delta \frac{(k+Q_j)}{2} \) stay unchanged under this change of variable. Therefore, we obtain:

(6.17)

\[ I(\mu, \sigma) = \int_{(k+Q_j)} \int_{(k+Q_j+1)} \ldots \int_{(k+Q_n)} e^{i((t_k+Q_j)-(t_k+Q_j+1)) \Delta \frac{(k+Q_j)}{2}} \Delta \frac{(k+Q_j+1)}{2} (\ldots) dt_{k+Q_j} \ldots dt_{k+Q_n} \]

where \( \sigma' = (k + Q_j, k + Q_j+1) \circ \sigma \). \((k + Q_j, k + Q_j+1)\) denotes the permutation which reverses \( k + Q_j \) and \( k + Q_j+1 \).

Next, let us consider the subset \( \{ \mu_{s} \} \subset M \) of “special upper echelon” matrices in which each highlighted element of a higher row is to the left of each highlighted
element of a lower row. A simple example of a “special upper echelon” matrix is given below (with $k = 1, n = 4, p_1 = 2, p_2 = 1, p_3 = 3, p_4 = 2$)

$$
\begin{bmatrix}
B_{1:2,3} & B_{1:4} & B_{1:5,6,7} & B_{1:8,9} \\
0 & B_{2:4} & B_{2:5,6,7} & B_{2:8,9} \\
0 & B_{3:4} & B_{3:5,6,7} & B_{3:8,9} \\
0 & 0 & B_{4:5,6,7} & B_{4:8,9} \\
0 & 0 & 0 & B_{5:8,9} \\
0 & 0 & 0 & B_{6:8,9} \\
0 & 0 & 0 & B_{7:8,9}
\end{bmatrix}
$$

Lemma 6.2. For each element of $M$ there is a finite number of acceptable moves which brings the matrix to upper echelon form.

Proof. We start from the first row and take acceptable moves to bring all highlighted entries in the first row in consecutive order. Since our goal is the upper echelon form, the updated highlighted entries will occupy $B_{1:k+1-1}+1,\ldots,k+Q_2$. Then if there are any highlighted entries on the second row, bring them to positions $B_{2:k+Q_1+1-1}+1,\ldots,k+Q_2$ through $B_{2:k+Q_2+1-1}+1,\ldots,k+Q_2$. Here $j_1 < j_2$. Noticed that this will not effect the highlighted positions of the first row. If there is no highlighted entire on the second row, just leave it and move to the third row. Keep repeating these steps and we will end up with a special upper echelon matrix after finitely many steps. □

Lemma 6.3. Let $C_{k,n}$ be the number of $(k + Q_{n-1}) \times n$ special upper echelon matrices of the type discussed above. Then $C_{k,n} \leq 2^{k+(p_0+1)(n-1)}$.

Proof. The proof consists of two steps. First of all, we dis-assemble the matrix by “lifting” all highlighted entries to the first row and put them in the same subsets if they were originally from the same row. In this way, the first row is partitioned into many subsets. Let $P_n$ denote the number of all possible partitions, then

$$P_n = \sum_{i=0}^{n-1} \binom{n-1}{i} = 2^{n-1}$$

The idea is to put $n-1$ pads in the space among the $n$ elements to separate them. Since we can separate them into different numbers (from 1 to $n$) of subsets, we can choose to use 0 pads, 1 pads, …, up to $n-1$ pads. Hence (6.18) follows.

The second step is to re-assemble the upper echelon matrix by “lowering” the first subset to the first used row, the second subset to the second used row, etc. Note here, we do not require that only the upper triangle matrix is used, which may result in more matrices. This does not matter since we are looking for an upper bound of the number of such matrices. Suppose an arbitrary partition of $n$ has $i$ subsets. Then there will be exactly $(k+Q_{n-1})$ ways to lower them in an order preserving way to the $k + Q_{n-1}$ available rows. Thus

$$C_{k,n} \leq P_n \sum_{i=0}^{n} \binom{k+Q_{n-1}}{i} \leq 2^{k+Q_{n-1}+n-1} \leq 2^{k+(p_0+1)(n-1)}$$

as desired (since $Q_{n-1} = p_1 + p_2 + \cdots + p_{n-1} \leq (n-1)p_0$). □
6.3. Equivalence Classes.

**Corollary 6.4.** Let $\mu_s$ be a special upper echelon matrix. We write $\mu \sim \mu_s$ if $\mu$ can be transformed to $\mu_s$ in finitely many acceptable moves. Then there exists a subset $D$ of $[0, t_k]^{n}$ such that

(6.19) \[ \sum_{\mu \sim \mu_s} \int_0^{t_k} \cdots \int_0^{t_k+Q_n-1} J^k(\mu; \mu) dt_{k+Q_1} \cdots dt_{k+Q_n} = \int_D J^k(\mu; \mu) dt_{k+Q_1} \cdots dt_{k+Q_n} \]

**Proof.** Consider the following integral

\[ I(\mu, id) = \int_0^{t_k} \cdots \int_0^{t_k+Q_n-1} J^k(\mu; \mu) dt_{k+Q_1} \cdots dt_{k+Q_n} \]

and perform finitely many acceptable moves on the matrix associated to $I(\mu, id)$ until it is transformed to the special upper echelon matrix associated with $I(\mu_s, \sigma)$. By Lemma 6.3

\[ I(\mu, id) = I(\mu_s, \sigma). \]

Assume that $(\mu_1, id)$ and $(\mu_2, id)$ with $\mu_1 \neq \mu_2$ yield the same echelon form $\mu_s$, then the corresponding permutations $\sigma_1$ and $\sigma_2$ must be different. Therefore, $D$ can be chosen to be the union of all \{ $t_k \geq t_{\sigma(k+Q_1)} \geq t_{\sigma(k+Q_2)} \geq \cdots \geq t_{\sigma(k+Q_n-1)}$ \} for all permutations $\sigma$ which occur in a given equivalence class of some $\mu_s$. \( \square \)

7. PROOF OF THEOREM 5.2

Now we are ready to prove Theorem 5.2.

**Proof of Theorem 5.2** Fix $t_k$. Recall the expansion of $\gamma^{(k)}$:

(7.1) \[ \gamma^{(k)}(t_k, \cdot) = \sum_{\mu \in M} \int_0^{t_k} \cdots \int_0^{t_k+Q_n-1} J^k(\mu; \mu) dt_{k+Q_1} \cdots dt_{k+Q_n} \]

and $J^k$:

\[ J^k(\mu; \mu) = e^{i(t_k-t_{k+Q_1})} B_{\mu(k+1); k+1, \cdots, k+Q_1} e^{i(t_{k+Q_1}-t_{k+Q_2})} B_{\mu(k+Q_1+1); k+Q_1+1, \cdots, k+Q_n} \gamma^{(k+Q_n)}(t_{k+Q_n}) \]

Thanks to Corollary 6.3 and Lemma 6.3 we can write $\gamma^{(k)}(t_k, \cdot)$ as a sum of at most $2^{k+(Q_1+1)(n-1)}$ terms of the form

(7.2) \[ \int_D J^k(\mu; \mu_s) dt_{k+Q_1} \cdots dt_{k+Q_n}. \]

Let $I_k^n = [0, t_k] \times [0, t_k] \times \cdots \times [0, t_k]$ and $D_{t_k+Q_1} = \{(t_{k+Q_1}, \cdots, t_{k+Q_n}) | (t_{k+Q_1}, t_{k+Q_2}, \cdots, t_{k+Q_n}) \in D \}$, then

\[ \left\| S^{(k, n)} \right\|_{L^2(\mathbb{R}^{2k} \times \mathbb{R}^{2k})} \]

\[ \left\| S^{(k, n)} \right\|_{L^2(\mathbb{R}^{2k} \times \mathbb{R}^{2k})} \]

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\[ \times B_{\mu_t}(k+Q_1+1);k+Q_1+1,\ldots,k+Q_2 \cdots dt_{k+Q_1} \cdots dt_{k+Q_n} \|_{L^2(\mathbb{R}^{2k} \times \mathbb{R}^{2k})} \]

\[ = \left\| \int_0^{t_k} e^{i(t_k-t_{k+Q_1})} \Delta_{k+Q_1} \left( \int_{D_{t_k+Q_1}} S^{(k,\alpha)} B_{\mu_t}(k+1);k+1,\ldots,k+Q_1 e^{i(t_k+Q_1-t_{k+Q_2})} \Delta_{k+Q_2} \right) dt_{k+Q_1+1},\ldots,k+Q_2 \cdots dt_{k+Q_2} \cdots dt_{k+Q_n} \right\|_{L^2(\mathbb{R}^{2k} \times \mathbb{R}^{2k})} dt_{k+Q_1} \]

\[ \leq C \alpha t_k \frac{2^k}{t_k} \int_0^{t_k} \left| \int_{D_{t_k+Q_1+1}} S^{(k,\alpha)} B_{\mu_t}(k+1);k+1,\ldots,k+Q_1 e^{i(t_k+Q_1-t_{k+Q_2})} \Delta_{k+Q_2} \right| dt_{k+Q_2+1},\ldots,k+Q_2+1 \cdots dt_{k+Q_2+1} \cdots dt_{k+Q_n} \]

we choose appropriate \( C \) and \( C_0 \) to obtain the last line. As we have already seen in the proof of Theorem 4.31, the sum over \( p_1, p_2, \ldots, p_n \) and product of potential

\[ \text{the implicit scalar is proportional to } 2^{k+(p_0+1)(n-1)} \text{ which can be absorbed in } C^{k} (C_0 \sqrt{T})^n. \]
constants $b_0^{(p_1)}, b_0^{(p_2)}, \ldots, b_0^{(p_n)}$ will contribute extra factor $p_0(b_0)^n$, which can be absorbed in constant $C$ and $C_0$. This completes the proof. \hfill $\Box$

**Remark 7.1.** The main ingredients in the above proof are the free evolution bound (4.4) and a priori energy bound (3.1). The a priori energy bound usually requires $\alpha \leq 1$ (may see [3.13]). While in (4.4), we will need $\alpha > \frac{d}{2} - \frac{1}{2(2p-1)}$ which is at least 1 when $d \geq 3$ (see (4.14)). Therefore only the cases $d = 1, 2$ give a nonempty intersection for the survival of $\alpha$. Which implies that, under this setting, the method we used here to prove the uniqueness fails for the higher dimensional cases, unless we have better constrains on $\alpha$. Klainerman and Machedon obtained a better estimate (on a different space) here which allows them to prove the uniqueness for the case $d = 3$, $p = 1$. Actually, we are answering the same questions on the convergence of BBGKY hierarchy to p-GP hierarchy as in [12] (for $d = 2$, $p = 1$) and [2] (for $d \leq 2$, $p = 2$), for any positive integer $p$. The case when $d = 3$, $p = 1$ is covered by [3] recently with a new approach.

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**Appendix A. Approximation of the initial wave function**

Recall the proof of the a priori bound in Corollary 2.3, we need the expectation of $H_N^k$ to be of the order $N^k$ at time 0. The main idea to obtain this is to approximate the initial wave function with cutoffs. We will prove Lemma A.1 in this section, with which (2.36) is immediate.

**Lemma A.1.** Suppose $\psi_N \in L^2(\mathbb{R}^{dN})$ with $\|\psi_N\| = 1$ is a family of $N$-particle wave functions with the associated marginal densities $\gamma_N^{(k)}$, $k = 1, 2, \cdots$. Let $\chi$ be a bump function such that $0 \leq \chi \leq 1$, $\chi(s) = 1$ for $s \in [0, 1]$ and $\chi(s) = 0$ for $s \geq 2$. $\kappa > 0$ is a parameter. Define

$$\bar{\psi}_N := \frac{\chi(\frac{\kappa}{N}H_N)\psi_N}{\|\chi(\frac{\kappa}{N}H_N)\psi_N\|}$$

We denote by $\bar{\gamma}_N^{(k)}$ the corresponding $k$-marginal density associated with $\bar{\psi}_N$. We also assume that

$$\langle \psi_N, H_N\psi_N \rangle \leq CN$$

and

$$\gamma_N^{(1)} \to |\phi\rangle\langle\phi| \quad \text{as} \quad N \to \infty$$

with $\phi \in H^1(\mathbb{R}^d)$. Then, for $\kappa > 0$ small enough and for every $k \leq 1$ we have

$$\lim_{N \to \infty} \text{Tr}|\bar{\gamma}_N^{(k)} - |\phi\rangle\langle\phi|^{\otimes k}| = 0$$

**Proof.** The proof is similar in spirit to the proof for the two-body interactions case, which can be found in [6, 8, 7]. Sketch of the key steps are listed below. We just need to show

$$\text{Tr}|\bar{\gamma}_N^{(1)} - |\phi\rangle\langle\phi| | \to 0, \quad \text{as} \quad N \to \infty$$

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The proof of (A.6) is divided into five steps.

**Step 1.** By (A.3), we know that there exists a sequence $\xi_{N}^{(N-1)} \in L^{2}(\mathbb{R}^{d(N-1)})$, \( \|\xi_{N}^{(N-1)}\| = 1 \) satisfying

\[
\psi_{N} - \phi \otimes \xi_{N}^{(N-1)} \rightarrow 0, \quad \text{as} \quad N \rightarrow 0
\]

This was proved by Alessandro Michelangeli in [13]. The proof in current case is identical to the proof presented in [7].

**Step 2.** There exists $\phi_{\ast} \in H^{2}(\mathbb{R}^{d})$ with $\|\phi_{\ast}\| = 1$ such that

\[
\|\phi - \phi_{\ast}\| \leq \frac{\epsilon}{32\|J^{(1)}\|}.
\]

**Step 3.** Let $\Xi = \chi(\frac{\kappa}{N} H_{N})$. Then by (A.2):

\[
\|1 - \Xi\|_{\psi_{N}}^{2} = \langle \psi_{N}, (1 - \Xi)^{2} \psi_{N}\rangle \leq \langle \psi_{N}, 1(\kappa H_{N} \geq N)\psi_{N}\rangle \leq \frac{\kappa}{N} \langle \psi_{N}, H_{N}\psi_{N}\rangle \leq C\kappa
\]

is uniformly in $N$. Since $\|\psi_{N}\| = 1$, by triangle inequality we know

\[
\|\psi_{N} - \tilde{\psi}_{N}\| = \|\frac{\psi_{N}}{\|\psi_{N}\|} - \frac{\Xi \psi_{N}}{\|\Xi \psi_{N}\|}\| \leq \frac{2}{\|\psi_{N}\|} \|\psi_{N} - \Xi \psi_{N}\| = 2 \|1 - \Xi\|_{\psi_{N}} \leq C\kappa\|\psi_{N}\|.
\]

The above inequality is needed in (2.40). One can find $\kappa > 0$ small enough such that $\|\Xi \psi_{N}\| \geq \frac{1}{2}$. Use triangle inequality and note that $\|\Xi\| \leq 1$. We have

\[
\|\Xi \psi_{N} - \Xi(\phi_{\ast} \otimes \xi_{N}^{(N-1)})\| = \|\Xi \psi_{N} - \Xi(\phi_{\ast} \otimes \xi_{N}^{(N-1)})\| + \|\Xi(\phi_{\ast} \otimes \xi_{N}^{(N-1)}) - \Xi(\phi_{\ast} \otimes \xi_{N}^{(N-1)})\| \leq \frac{1}{\|\Xi \psi_{N}\|} \|\Xi \psi_{N} - \Xi(\phi_{\ast} \otimes \xi_{N}^{(N-1)})\| + \frac{1}{\|\Xi \psi_{N}\|} \|\Xi \psi_{N} - \Xi(\phi_{\ast} \otimes \xi_{N}^{(N-1)})\| \leq \frac{2}{\|\Xi \psi_{N}\|} \|\Xi \psi_{N} - \Xi(\phi_{\ast} \otimes \xi_{N}^{(N-1)})\| \leq 4\|\psi_{N} - \phi_{\ast} \otimes \xi_{N}^{(N-1)}\| \leq 4\|\psi_{N} - \phi \otimes \xi_{N}^{(N-1)}\| + 4\|\phi \otimes \xi_{N}^{(N-1)} - \phi_{\ast} \otimes \xi_{N}^{(N-1)}\| \leq 4\|\psi_{N} - \phi \otimes \xi_{N}^{(N-1)}\| + 4\|\phi - \phi_{\ast}\| \leq \frac{\epsilon}{6\|J^{(1)}\|}
\]

for large $N$. Here in the last inequality we used (A.7) and (A.8).
Step 4. As in [7] and [2], we define a similar Hamiltonian after taking into account of the \((p + 1)\)-particle interactions studied in this paper.

\[(A.12) \quad \tilde{H}_N := \sum_{i=2}^{N} (-\Delta x_i) + \frac{1}{N^p} \sum_{2 \leq i_1 \cdots < i_{p+1} \leq N} V^{(p)}_N(x_{i_1} - x_{i_2}, \ldots, x_{i_1} - x_{i_{p+1}})\]

Instead of acting on all variables, the new Hamiltonian only acts on the last \(N - 1\) variables. Let \(\hat{\Xi} = \chi(H_N)\). Then by (A.11), we will have

\[(A.13) \quad \| \hat{\Xi} \psi_N \| - \| \hat{\Xi}(\phi_\# \otimes \xi_N^{(N-1)}) \| \leq \frac{\epsilon}{3\|J(1)\|}\]

We refer the proof of (A.13) to [8].

Step 5. For (A.6), we define

\[(A.14) \quad \gamma^{(1)}_N(x_1; x_1') := \int \tilde{\psi}_N(x_1; x_{N-1}) \psi_N(x_1', x_{N-1}) dx_{N-1}\]

Note that \(\tilde{\psi}_N\) is not symmetric in all variables, but it is symmetric in the last \(N - 1\) variables. Clearly, \(\gamma^{(1)}_N\) is a density matrix and

\[\gamma^{(1)}_N = |\phi_\#\rangle \langle \phi_\#|\]

Thus, using \(\|\tilde{\psi}_N - \tilde{\psi}_N\| \leq \frac{\epsilon}{3\|J(1)\|}\), which is equivalent to (A.13) and \(\|\phi - \phi_\#\| \leq \frac{\epsilon}{3\|J(1)\|}\) from (A.6), we obtain

\[(A.16) \quad |\text{Tr} J(1)(\gamma^{(1)}_N - |\phi\rangle \langle \phi|)| \leq |\text{Tr} J(1)(\gamma^{(1)}_N - \gamma^{(1)}_N)| + |\text{Tr} J(1)(|\phi_\#\rangle \langle \phi_\#| - |\phi\rangle \langle \phi|)|\]

\[\leq 2\|J(1)\|\|\tilde{\psi}_N - \tilde{\psi}_N\| + 2\|J(1)\|\|\phi_\# - \phi\|\]

\[\leq \epsilon\]

for sufficiently large \(N\) with arbitrary \(\epsilon\) and small enough \(\kappa\). Hence (A.6) follows.

\[\square\]

Appendix B. A Poincaré type inequality

Lemma B.1. Let \(h\) be a non-negative probability measure on \(\mathbb{R}^d\) satisfying \(\int_{\mathbb{R}^d} (1 + x^2)^{\frac{1}{2}} h(x) dx < \infty\). Then for \(h_\epsilon(x) = \frac{1}{\epsilon^d} h\left(\frac{x}{\epsilon}\right)\), \(\epsilon > 0\), and every \(0 \leq \kappa < 1\), there exists a \(C > 0\) such that

\[(B.1) \quad |\text{Tr} J^{(k)}(h_\epsilon(x_j - x_{k+1}) \cdots h_\epsilon(x_j - x_{k+p}) - \delta(x_j - x_{k+1}) \cdots \delta(x_j - x_{k+p})) \gamma^{(k+p)}|\]

\[\leq C \kappa \|J^{(k)}\| \text{Tr} |S_j S_{k+1} \cdots S_{k+p} \gamma^{(k+p)} S_{k+p} \cdots S_{k+1} S_j|\]

for all non-negative \(\gamma^{(k+p)} \in \mathcal{L}^{k+p}_{k+p}\).
Proof. Following [12, 2], we prove the case \( k = 1 \). For case of \( k > 1 \), the proof is analogous. Since \( 1 \leq j \leq k \), so \( j = 1 \) in current case. By the non-negativity of \( \gamma^{(1+p)} \), we can decompose it as \( \gamma^{(1+p)} = \sum \lambda_i \langle \psi_i \rangle \langle \psi_i \rangle \), with \( \psi_i \in L^2(\mathbb{R}^{1+p}) \) and \( \lambda_i \geq 0 \). Then

\[
(B.2)
\]

\[
Tr J^{(1)} \left( h_c(x_1 - x_2) \cdots h_c(x_1 - x_{1+p}) - \delta(x_1 - x_2) \cdots \delta(x_1 - x_{1+p}) \right) \gamma^{(1+p)}
= \sum_i \lambda_i \langle \psi_i \rangle \left( h_c(x_1 - x_2) \cdots h_c(x_1 - x_{1+p}) - \delta(x_1 - x_2) \cdots \delta(x_1 - x_{1+p}) \right) \psi_i
= \sum_i \lambda_i \langle \psi_i \rangle \left( h_c(x_1 - x_2) \cdots h_c(x_1 - x_{1+p}) - \delta(x_1 - x_2) \cdots \delta(x_1 - x_{1+p}) \psi_i \right)
\]

where \( \Psi_i = (J^{(1)} \otimes 1) \psi_i \). Next we switch to Fourier side to obtain

\[
(B.3)
\]

\[
\langle \Psi_i, (h_c(x_1 - x_2) \cdots h_c(x_1 - x_{1+p}) - \delta(x_1 - x_2) \cdots \delta(x_1 - x_{1+p})) \psi_i \rangle
= \int dq_1 \cdots dq_{1+p} dq'_1 \cdots dq'_{1+p} \hat{\Psi}_i(q_1, \ldots, q_{1+p}) \hat{\psi}_i(q'_1, \ldots, q'_{1+p})
\times \int dx_2 \cdots dx_{1+p} h(x_2) \cdots h(x_{1+p}) (e^{ix_2(q_2 - q'_2)} \cdots e^{ix_{1+p}(q_{1+p} - q'_{1+p})} - 1)
\times \delta(q_1 + \cdots + q_{1+p} - q'_1 - \cdots - q'_{1+p})
\]

Since for \( x \in \mathbb{R} \), \( |e^{ix} - 1| = 2|\sin \frac{x}{2}| \leq C|x|^\kappa \) is always true with arbitrary \( 0 < \kappa < 1 \) and constant \( C > 0 \) independent of \( \kappa \), we have the following

\[
|e^{ix_2(q_2 - q'_2)} \cdots e^{ix_{1+p}(q_{1+p} - q'_{1+p})} - 1| \leq C \kappa \left( \sum_{i=2}^{1+p} |x_i(q_i - q'_i)| \right)^\kappa
\]

\[
(B.4)
\]

\[
\leq C \kappa \sum_{i=2}^{1+p} |x_i(q_i - q'_i)|^\kappa
\leq C \kappa \sum_{i=2}^{1+p} |x_i|^\kappa (|q_i|^\kappa + |q'_i|^\kappa)
\]

The last inequality follows from \( (a + b)^\kappa \leq a^\kappa + b^\kappa \) for \( \kappa \in (0, 1) \) and \( a, b \) both nonnegative. And the second to the last inequality follows in a similar way, but with an implicit constant depending on \( p \). Thus

\[
(B.5)
\]

\[
|\langle \Psi_i, (h_c(x_1 - x_2) \cdots h_c(x_1 - x_{1+p}) - \delta(x_1 - x_2) \cdots \delta(x_1 - x_{1+p})) \psi_i \rangle|
\leq C \kappa \left( \sum_{i=2}^{1+p} |x_i(q_i - q'_i)| \right)^\kappa
\times \left( \prod_{i=2}^{1+p} \int |x_i|^\kappa h(x_i) dx_i \right) \left( \sum_{i=2}^{1+p} |q_i|^\kappa + |q'_i|^\kappa \right) \delta(q_1 + \cdots + q_{1+p} - q'_1 - \cdots - q'_{1+p})
\]

Clearly the \( p \) copies of integrations involving \( h \) are finite by assumption. And the summation term \( \sum_{i=2}^{1+p} (|q_i|^\kappa + |q'_i|^\kappa) \) contains a total of \( p \) terms. We will show how to control one of them, say \( |q_2|^\kappa \). The final upper bound on this part will be the same (up to a constant \( p \)).

\[
\int dq_1 \cdots dq_{1+p} dq'_1 \cdots dq'_{1+p} \delta(q_1 + \cdots + q_{1+p} - q'_1 - \cdots - q'_{1+p})
\times |\hat{\psi}_i(q_1, \ldots, q_{1+p})||\hat{\psi}_i(q'_1, \ldots, q'_{1+p})| |q_2|^\kappa
\]
by taking
\[ \rho \sup_q \int dq_1 \cdots dq_p \left| \frac{\langle q_1^2 \rangle^2 \cdots \langle q_p^2 \rangle^2}{\langle q_1 \rangle^2 (Q - q_1 - \cdots - q_p)^2} \right|^2 \]
\[ + \frac{1}{\rho} \sup_q \int dq_1 dq_2 \cdots dq_{1+p} \left| \frac{\langle q_1^2 \rangle^2 \cdots \langle q_p^2 \rangle^2 \langle q_1 \rangle^2 (Q - q_1 - \cdots - q_p)^2 \langle q_1 \rangle^2 (Q - q_1 - \cdots - q_p)^2 \langle q_1 \rangle^2 (Q - q_1 - \cdots - q_p)^2}{\langle q_1 \rangle^2 (Q - q_1 - \cdots - q_p)^2 (Q - q_1 - \cdots - q_p)^2 (Q - q_1 - \cdots - q_p)^2} \right|^2 \]
for arbitrary \( \rho > 0 \). We can apply \((3.11)\) to the last two integrations \((3.6)\) and \((3.7)\) for all \( \kappa \in (0, 1) \) to have
\[ |\text{Tr} J^{(1)}(h_\kappa(x_1 - x_2) \cdots h_\kappa(x_1 - x_{1+p}) - \delta(x_1 - x_2) \cdots \delta(x_1 - x_{1+p}))| \]
\[ \leq C e^\kappa \left( \rho \text{Tr} S_1^2 S_2^2 \cdots S_{1+p}^2 J^{(1)}(1)^{1+p} + \frac{1}{\rho} \text{Tr} S_1^2 S_2^2 \cdots S_{1+p}^2 J^{(1)}(1)^{1+p} \right) \]
\[ \leq C e^\kappa \left( \rho \right) S_1^{-1} J^{(1)} S_1 \| S_1 J^{(1)} S_1^{-1} \| + \frac{1}{\rho} \text{Tr} S_1^2 S_2^2 \cdots S_{1+p}^2 J^{(1)}(1)^{1+p} \]
\[ \leq C e^\kappa \| J^{(1)} \| \text{Tr} S_1^2 S_2^2 \cdots S_{1+p}^2 J^{(1)}(1)^{1+p} \]
by taking \( \rho = \| J^{(1)} \|^{-1} \) in the last inequality. \( \square \)

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