On the geodesic incompleteness of spacetimes containing marginally (outer) trapped surfaces

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Abstract
In a recent paper, Eichmair et al (2012 arXiv:1204.0278v1) have proved a Gannon–Lee-type singularity theorem based on the existence of marginally outer trapped surfaces (MOTS) on noncompact initial data sets for globally hyperbolic spacetimes. A natural question is whether the corresponding incomplete geodesics could still be complete in a possible non-globally hyperbolic extension of spacetime. In this paper, some variants of their result are given with weaker causality assumptions, thus suggesting that the answer is generically negative, at least if the putative extension has no closed timelike curves. We consider first marginally trapped surfaces (MTS) in chronological spacetimes, introducing the natural notion of a generic MTS, a notion also applicable to MOTS. In particular, a Hawking–Penrose-type singularity theorem is proven in chronological spacetimes with dimension $n \geq 3$ containing a generic MTS. Such surfaces naturally arise as cross-sections of quasi-local generalizations of black hole horizons, such as dynamical and trapping horizons, and we discuss some natural conditions which ensure the existence of MTS in initial data sets. Nevertheless, much of the more recent literature has focused on MOTS rather than MTS as quasi-local substitutes for the description of black holes, as they are arguably more natural and easier to handle in a number of situations. It is therefore pertinent to ask to what extent one can deduce the existence of singularities in the presence of MOTS alone. We address this issue and show that singularities indeed arise in the presence of generic MOTS, but under slightly stronger causal conditions than those in the case of MTS (specifically, for causally simple spacetimes). On the other hand, we show that with additional conditions on the MOTS itself, namely that it is either the boundary of a compact spatial region, or strictly stable in a suitable sense, a Penrose–Hawking-type singularity theorem can still be established for chronological spacetimes containing generic MOTS.

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1. Introduction

The existence of a closed trapped surface in spacetime is a well-known, fundamental criterion for gravitational collapse. This concept was introduced by Penrose [20] as an abstract indicator that the region containing it, say, describing some stellar core, had attained a ‘point of no return’ of matter concentration, and the formation of a black hole would presumably follow, at least if the collapse would not deviate too much from spherical symmetry. A mathematical justification for the appropriateness of this criterion stems from the fact that, under physically reasonable additional assumptions, the existence of such surfaces implies nonspacelike geodesic incompleteness of the spacetime manifold, which is the geometrical counterpart of gravitational collapse.

However, the existence of black holes cannot be directly deduced from either the existence of closed trapped surfaces or the existence of singularities, unless one assumes that some form of the Cosmic Censorship conjecture holds (see, e.g., [24, chapter 12] for a nice discussion). Part of the difficulty involved here lies in the global nature of the black hole and event horizon concepts, which makes it difficult to establish their existence from local considerations. This global character also makes the standard mathematical notion of black hole notoriously ill-suited to handle more realistic analyses of astrophysical black hole candidates, especially in their dynamical aspects, as well as in numerical studies. Accordingly, the theoretical focus in recent years has shifted to a systematic study of quasi-local notions of horizons. Among these are trapped, isolated or dynamical horizons (see, e.g., [4] for extensive discussions and references), whose chief defining feature is to be foliated by marginally trapped surfaces (MTS), understood as codimension-2 closed spacelike submanifolds with nonpositive expansion of the two normal families of null geodesics.

Another quasi-local notion especially suited for the study of dynamical black holes at the level of initial data sets is the apparent horizon, i.e. the boundary (in an underlying spatial slice of spacetime) of the region containing closed trapped surfaces. Apparent horizons have become increasingly important in recent times, not only due to the current emphasis on quasi-local notions, but also because they were found to have a number of mathematically interesting properties. Indeed, when smooth they are known to be marginally outer trapped surfaces (MOTS), codimension-2 closed spacelike submanifolds in which the outgoing family of normal geodesics is required to have zero convergence (see, e.g., [24, chapter 12] and [3]), but no restriction is made on the ingoing family. Now, both MOTS and MTS are natural initial data surrogates for event horizons, but MOTS are in some respect easier to study. For example, proving the existence of a MTS directly is more involved than that of a MOTS, because it entails control on the sign of certain quantities, which in general is a difficult problem. MOTS, on the other hand, have been extensively studied in the mathematical physics literature, and some existence results for them have been proven [1–3, 8, 9]. Apart from their appearance in relativity, they are particularly interesting to geometers because they have a number of properties similar to minimal surfaces and can be thought of as natural generalizations of the latter [12]. Moreover, they play an important role in the Schoen–Yau proof of the positive mass theorem [21, 22].

There is another, more fundamental reason to consider MOTS in connection with gravitational collapse. For while the existence of closed trapped surfaces is the mathematical criterion par excellence of the onset of gravitational collapse, a direct proof that they naturally appear for large enough concentrations of mass is not forthcoming, except of course in specific models. To the best of the author’s knowledge, the closest approach to such a general proof is...
the celebrated Schoen–Yau result in [23], which however proves the existence of MOTS rather than that of closed trapped surfaces. The question then naturally arises as to the conditions under which one can infer that the existence of MOTS implies gravitational collapse.

In a recent paper [10], Eichmair et al investigate the existence of MOTS in initial data sets, and they find that MOTS naturally form in asymptotically flat initial data sets with non-trivial topology; in addition, they prove that if the existence of such a MOTS $\Sigma$ contained in a noncompact Cauchy hypersurface in spacetime is granted, and if this MOTS is moreover ‘generic’ in the sense that future and past inextendible null geodesics normal to $\Sigma$ have nonzero tidal acceleration somewhere along them, then null geodesic incompleteness follows. (The null convergence condition is also assumed.) This result can therefore be understood as a Gannon–Lee-type [13, 14, 18, 7] singularity theorem. Clearly, some sort of genericity condition is needed to infer incompleteness. For instance, by performing simple isometric identifications in Minkowski spacetime, one can check that there are even globally hyperbolic spacetimes which are geodesically complete while having compact marginally (outer) trapped surfaces. (See also [2], where a singularity theorem in globally hyperbolic spacetimes containing MOTS is proven, with alternative generic conditions on the MOTS.) Now, since these results primarily address initial data sets, attention is restricted in [10, 2] to their globally hyperbolic Cauchy developments. This is usually enough for numerical relativity. However, from a more general perspective, one naturally wishes to know whether the geodesic incompleteness thus deduced would survive when further, non-globally hyperbolic extensions of spacetime are made. We emphasize that this is far from being a mere appendix to the globally hyperbolic setting, as there may well exist physically relevant, non-globally hyperbolic extensions of a maximal globally hyperbolic development of a given initial data set which is nonspacelike geodesically complete. Domains of dependence of partial Cauchy hypersurfaces in anti-de Sitter spacetime offer a simple illustration of this point. The specific proofs given in [10, 2] cannot be generalized to this setting without extra input.

It is the purpose of this paper to investigate such singularity theorems with weakened causality condition, for both MTS and MOTS. In particular, after proving some auxiliary propositions of independent interest, we prove a Hawking–Penrose-type singularity theorem valid for all spacetime dimensions $n$ greater than 2 (see [17], and [16, theorem 2, p 266]—henceforth, we refer to this result as the Hawking–Penrose singularity theorem), in which one assumes, besides the chronological condition, the timelike convergence condition (which is just the strong energy condition for spacetimes satisfying the Einstein field equations), together with the (standard) generic condition on nonspacelike geodesics. Such a theorem holds in the presence of either a generic MTS or a generic MOTS (to be defined more precisely below), but in the latter case, we must assume in addition either that the MOTS bounds a compact spatial region (as will be the case, for instance, if the region of the spatial slice containing it is diffeomorphic to some $\mathbb{R}^n$), or that it is strictly stable in an appropriate sense. We also prove a version of the singularity theorem in [10] valid for causally simple spacetimes containing generic or strictly stable MOTS (without the bounding assumption). Causal simplicity is a condition only slightly weaker than global hyperbolicity, but it holds in important examples as Kerr–Newman (extended maximally except for the causality-violating region) or anti-de Sitter spacetimes.

In the case $n = 4$, the results in this paper together with those in [23] go a long way toward establishing on a model-independent, mathematically rigorous footing the basic heuristic expectation that large concentrations of mass do lead to gravitational collapse.

The rest of the paper is organized as follows. We first give some preliminary definitions in section 2 to set the conventions, general assumptions and notation, and we prove three general propositions which will be instrumental in establishing our main results. The proofs
of the main results, together with some simple consequences, are given in sections 3 and 5. In section 4, we discuss additional conditions on initial data sets which imply the existence of MTS thereon.

2. Preliminaries: basic definitions and auxiliary results

In what follows, we fix a spacetime, i.e. an n-dimensional \( n \geq 3 \), second-countable, connected, Hausdorff, smooth (i.e. \( C^\infty \)) Lorentz (signature \((-+,\ldots,+)\)) manifold \( M \) endowed with a smooth metric tensor \( g \), which is also time oriented. We assume that the reader is familiar with the basic definitions and results of global Lorentzian geometry and the causal theory of spacetimes, as found in the core references [16, 24, 19, 5], and in particular with the standard singularity theorems. We denote by \( d : M \times M \rightarrow [0, +\infty] \) the (lower semicontinuous) Lorentzian distance function on \((M,g)\), and by \( L_g(\gamma) \) the Lorentzian arc length of a nonspacelike curve segment \( \gamma : [a, b] \rightarrow M \). All submanifolds of \( M \) are embedded unless otherwise specified, and their topology is the induced topology. Finally, we follow the convention that nonspacelike vectors are always nonzero.

We start by recalling a few standard definitions and results (cf [5, chapter 8] for examples and further discussion). Our intention here is merely to establish additional notation and terminology which will be used later on.

**Definition 2.1.** A future-directed timelike (resp. null) geodesic ray in \((M, g)\) is a future-directed, future-inextendible timelike (resp. null) geodesic \( \gamma : [0, b) \rightarrow M \) for which \( d(\gamma(s), \gamma(t)) = L_g(\gamma|_{[s,t]}) \) for all \( s \leq t \). In either case (i.e. \( \gamma \) is timelike or null), \( \gamma \) is also called a future-directed nonspacelike geodesic ray.

A past-directed timelike (resp. null, nonspacelike) geodesic ray can be analogously defined. In what follows, unless explicitly stated otherwise, we shall always consider future-directed nonspacelike curves, and we henceforth drop explicit references to the causal orientation. According to the above definition, a nonspacelike geodesic ray is characterized by the fact that it maximizes the Lorentzian arc length between any two of its points. If \( \gamma \) is a null ray, the condition that \( d(\gamma(s), \gamma(t)) = L_g(\gamma|_{[s,t]}) \) for all \( s \leq t \) in the domain of \( \gamma \) with \( s \leq t \) is actually equivalent to the requirement that the image of \( \gamma \) be achronal in \((M,g)\), i.e. no two of its points can be connected by a timelike curve segment.

Recall that an achronal set \( A \subseteq M \) is future trapped if \( E^+(A) = J^+(A) \setminus I^+(A) \) is compact (a past-trapped set is defined time dually).

Let \( S \subseteq M \) be a smooth, connected, spacelike, partial Cauchy hypersurface. (Recall that a partial Cauchy hypersurface is by definition an acausal edgeless subset of a spacetime, which means in particular that it is a closed topological (i.e. \( C^0 \)) hypersurface [19].) Let \( \Sigma \subseteq M \) be a compact (without boundary), smooth spacelike submanifold of codimension 2 (loosely called surface in what follows), with \( \Sigma \subseteq S \). We assume that \( \Sigma \) is two-sided in \( S \). This means, in particular, that there are unique unit spacelike vector fields \( N_{\pm} \) on \( \Sigma \) normal to \( \Sigma \) in \( S \).

For simplicity, we shall assume throughout this paper that \( \Sigma \) is connected and separates \( S \), i.e. \( S \setminus \Sigma \) is not connected. (This assumption is not essential for the results in this paper, however, for one can always focus on one connected component and consider a covering of \( M \) in which this holds for that connected component of \( \Sigma \). Now, in dealing with geodesic incompleteness, one may as well work in covering manifolds.) Thus, \( S \setminus \Sigma \) is a disjoint union \( S^+_+ \cup S^- \) of open submanifolds of \( S \) having \( \Sigma \) as a common boundary. We shall loosely call \( S^+_+ \) (resp. \( S^- \)) the outside (resp. inside) of \( \Sigma \) in \( S \). (In most interesting examples, there is a natural choice for these.) The normal vector fields \( N_{\pm} \) on \( \Sigma \) are then chosen so that \( N_+ \) (resp. \( N_- \)) is outward-pointing (resp. inward-pointing), i.e. points into \( S^+_+ \) (resp. \( S^- \)).
Let $U$ be the unique timelike, future-directed, unit normal vector field on $\mathcal{S}$. Then, $K_\pm := U|_\Sigma + N_\pm$ are future-directed null vector fields on $\Sigma$ normal to $\Sigma$ in $M$. The expansion scalars of $\Sigma$ in $M$ are the smooth functions $\theta_\pm : \Sigma \to \mathbb{R}$ given by

$$\theta_\pm (p) = -(H_p, K_\pm (p))_p,$$

for each $p \in \Sigma$, where $H_p$ denotes the mean curvature vector of $\Sigma$ in $M$ at $p$ [19], and we denote $\langle \cdot, \cdot \rangle$ here and hereafter, if there is no risk of confusion.

Henceforth, whenever we refer to a surface $\Sigma$ contained in a partial Cauchy hypersurface $\mathcal{S}$, all the above conventions and choices will be understood.

Physically, the expansion scalars measure the divergence of light rays emanating from $\Sigma$. If $\Sigma$ is a round sphere in a Euclidean slice of Minkowski spacetime, with the obvious choices of inside and outside, then we have $\theta_+ > 0$ and $\theta_- < 0$. One also expects this to be the case if $\Sigma$ is a ‘large’ sphere in an asymptotically flat spacetime. But in a region of strong gravity, one expects instead that we have both $\theta_\pm < 0$, in which case $\Sigma$ is a closed (future) trapped surface.

Recall that $\Sigma$ is a MOTS if $\theta_+ = 0$. We also adopt the following slightly less standard definition (see [24, p 310]).

**Definition 2.2.** A (future) MTS is a smooth, closed, codimension-2 spacelike manifold $\Sigma \subset M$ with trivial normal bundle in $M$ for which $\theta_\pm \leq 0$.

Note that a MTS may be a MOTS, but there is an additional requirement on the inner expansion scalar $\theta_-$. There are slightly different definitions of MTS in the literature. The condition given here for $\theta_-$ is weaker than that imposed for cross-sections of dynamical and trapping horizons, which require a strict inequality $\theta_- < 0$ (see [4]). It is well known (see, e.g., [24, p 310]) that either MTS or MOTS remain inside the black hole region, provided they are contained in a strongly asymptotically predictable spacetime in which the null convergence condition holds.

We shall consider the following additional notion.

**Definition 2.3.** A MTS (or MOTS) $\Sigma$ is generic (in $(M, g)$) if any future-directed, future-inextendible, or past-directed, past-inextendible null geodesic $\eta : [0, a) \to M$ $(0 < a \leq +\infty)$ starting at $\Sigma$ and normal to $\Sigma$ at $\eta(0)$ satisfies the generic condition, i.e. at some point $p$, and for some vector $v$ normal to $\eta'(p)$, $\langle v, R(v, \eta', \eta') \rangle \neq 0$.

The generic condition formulated above is a mild constraint which ensures (see, e.g., proposition 2.11 in [5]) that there is nonzero tidal acceleration somewhere along each null geodesic $\eta$ normal to $\Sigma$. This will occur, for example (see proposition 2.12 in [5]), if $\text{Ric}(\eta', \eta') \neq 0$ somewhere along $\eta$, which in turn, if the Einstein field equations hold for $(M, g)$, will likely happen whenever $\eta$ crosses (or is part of) some matter–energy cluster of positive density.

The condition in definition 2.3 should not be confused with the condition on the spacetime $(M, g)$ bearing the same name. Recall that the nonspacelike generic condition holds in $(M, g)$ when there are nonzero tidal accelerations $\langle v, R(v, \gamma')\gamma' \rangle$ along any past and future inextendible nonspacelike geodesic $\gamma : (a, b) \to M$ (see, e.g., [5, chapters 2, 12 and 14] for a thorough discussion of the generic condition). The latter condition neither implies nor is implied by the condition that a MTS or a MOTS be generic, which is a condition only on future or past inextendible null geodesics starting at $\Sigma$ and normal to it. However, the nonspacelike generic condition on $(M, g)$ will be adopted as an additional assumption in theorems 3.2 and 5.1, as well as in the latter theorem’s corollary 5.4.
Definition 2.4. Let $\eta : [0, a) \to M$ be a null geodesic ray starting at and normal to $\Sigma$. It is said to be outward-pointing (resp. inward-pointing) if $\eta'(0)$ is parallel to $K_+(\eta(0))$ (resp. $K_-(\eta(0))$). It is a $\Sigma$-ray, if the length of any segment starting at $\Sigma$ up to any point $p$ along the ray realizes the Lorentzian distance $d(\Sigma, p) = \sup_{q \in \Sigma} d(q, p)$ from $\Sigma$ to that point. (In that case, there exists no timelike curve from $\Sigma$ to the ray and no focal points to $\Sigma$ along the ray.)

Note that our definition of a $\Sigma$-ray is basically definition 14.4 in [5], but here we shall always assume that a $\Sigma$-ray is null. Its definition already implies that it is also normal to $\Sigma$.

The key ingredients in the proofs of the main results in the next sections are the next three propositions, which also have independent interest. Proposition 2.1 will be the basis for Hawking–Penrose-type singularity theorems for generic MTS. Proposition 2.4 will allow us to generalize the singularity theorem in [10] for MOTS in causally simple spacetimes. Recall that $(M, g)$ is causally simple if it is causal (i.e. has no closed nonspacelike curves) and $J^+(p)$ are closed sets for every point $p \in M$. It is not difficult to check that in that case, $J^+(K)$ are also closed for every $K \subset M$ compact. A detailed discussion of such spacetimes with some results and further references can be found, e.g., in [7, section 3]. Of course, every globally hyperbolic spacetime is causally simple. Finally, proposition 2.6 will be instrumental in adapting our main theorems to (certain types of) MOTS.

Proposition 2.1. Let $\Sigma \subset M$ be a surface contained in a partial Cauchy hypersurface $S$. Then, at least one of the following alternatives occurs:

(i) $\Sigma$ is a future-trapped set,
(ii) there exists an inward-pointing, future-directed $\Sigma$-ray starting at $\Sigma$,
(iii) there exists an outward-pointing, future-directed $\Sigma$-ray starting at $\Sigma$.

Proof. Fix a complete Riemannian metric $h$ on $M$ with the distance function $d_h$.

If $E^+(\Sigma) = \Sigma$, we are done, since $\Sigma$ is compact. Thus, suppose $E^+(\Sigma) \neq \Sigma$. Let $(q_n)$ be any sequence in $E^+(\Sigma) \setminus \Sigma$. For each $n \in \mathbb{N}$, there exists a future-directed, future-inextendible nonspacelike curve $\gamma_n : [0, +\infty) \to M$ parametrized by $h$-arc length, such that $\gamma_n(0) \in \Sigma$ and $\gamma_n(0, t_n) \subset E^+(\Sigma) \setminus \Sigma$ for some $t_n \in (0, +\infty)$.

Since $\Sigma$ is compact, passing to a subsequence if necessary we can assume that $\gamma_n(0) \to p \in \Sigma$. By the limit curve lemma, we can assume that there exists a future-directed, future-inextendible nonspacelike curve $\gamma : [0, +\infty) \to M$ with $\gamma(0) = p$ and such that $\gamma_n \to \gamma$ uniformly in compact subsets.

Assume that $E^+(\Sigma)$ is not compact. Then, by the Hopf–Rinow theorem in $(M, h)$, it is either not closed, or not bounded in $d_h$. Suppose first that it is not bounded. In that case, the sequence $(q_n)$ could be chosen as diverging to infinity; then, the sequence $(t_n)$ is not bounded above, and we can assume that $t_n \to +\infty$.

Fix a number $t > 0$. Eventually, $t_n > t$, in which case $\gamma_n(t) \in J^+(\Sigma)$, and hence, $\gamma_n(t) \in E^+(\Sigma)$, and for otherwise, $\gamma_n(t) \in I^+(\Sigma)$, and therefore, $q_n \in I^+(\Sigma)$, a contradiction. Similarly, we cannot have $\gamma(t) \in I^+(\Sigma)$; therefore, $\gamma(t) \in E^+(\Sigma)$, and since $\gamma(t) \notin \Sigma$ from the acausality of $S$, we conclude that $\gamma \subset E^+(\Sigma)$. In particular, there cannot exist any timelike curve from $\Sigma$ to any point of $\gamma$; so the latter curve must be a reparametrization of a future-directed, null $\Sigma$-ray (normal to $\Sigma$) as desired (cf [19, theorem 51, p 298]). Since $\Sigma$ is two sided, this ray will be either inward-pointing or outward-pointing.

Now suppose that $E^+(\Sigma)$ is not closed. Then, we can take $(q_n)$, such that $q_n \to q$, with $q \notin E^+(\Sigma)$. If $(t_n)$ is not bounded above, then the argument proceeds as in the previous

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2 We say that a sequence $(p_n)_{n \in \mathbb{N}}$ in $M$ diverges to infinity if given any compact subset $C \subseteq M$, only finitely many elements of the sequence are contained in $C$ (see [5, chapter 8]).
situation. Thus, assume that \((t_0)\) is bounded above. In that case, up to passing to a subsequence, we can assume that \(t_n \not \to t_0 \in [0, +\infty)\), and hence, \(q = \gamma(t_0) \in J^+(\Sigma)\). If \(q \in I^+(\Sigma)\), which is open, then eventually so does \(q_n\), which is absurd. Thus, \(q \in E^+(\Sigma)\), again a contradiction. Therefore, \((t_0)\) is not bounded above.

The remaining possibility is that \(E^+(\Sigma)\) is compact, in which case \(\Sigma\) is future trapped. \(\square\)

**Remark 2.2.** Clearly, a time-dual statement of proposition 2.1 also holds.

**Remark 2.3.** The result in 2.1 cannot be improved without further assumptions. Indeed, the following simple example shows this. Consider the flat 2D Lorentzian cylinder \(S^1 \times \mathbb{R}\) obtained by isometrically identifying 2D Minkowski spacetime along the spatial direction. Fix any acausal circle \(C\) going around the cylinder with unit normal parallel to the ‘axis’ of the cylinder, and fix two antipodal points \(p\) and \(q\) on \(C\). Now, delete the whole causal past of \(q\) (including \(q\) itself). We denote the resulting manifold by \(N_0\) and its (flat) metric by \(g_0\). Choose any compact Riemannian manifold \((N, h)\). Let \(M = N_0 \times N\), \(g = g_0 \oplus h\), \(\Sigma = (C \setminus \{q\}) \times N\) and \(\Sigma = p \times N\). This spacetime is globally hyperbolic, and \(\Sigma\) is a partial Cauchy hypersurface (but not Cauchy). Now, \(\Sigma\) is a future-trapped set therein, but one can easily delete suitable points along the null generators of \(\partial I^+ (\Sigma)\) to show that it can have an inward-pointing \(\Sigma\)-ray but not an outward-pointing one or vice versa.

In [7], the notion of a piercing was also introduced, and we shall find it necessary to assume that one such exists in proposition 2.4. We recall that definition here. We say that a smooth future-directed timelike vector field in \(M\) pierces \(\Sigma\) (or pierces \(\Sigma\)) if every maximally extended integral curve of \(X\) intersects \(\Sigma\) exactly once. In physical terms, one may think of the integral curves of a piercing as worldlines of members of a family of observers who ‘witness’ the ‘whole universe at a certain instant of common time’, described by \(\Sigma\). This interpretation will hopefully convince the reader that it is a rather harmless assumption from a physical standpoint. Moreover, it is not difficult to check that a piercing does exist for suitable partial Cauchy hypersurfaces in basic solutions like Minkowski, Kerr–Newman and FRW spacetimes.

Of course, a piercing of \(\Sigma\) may not exist for general spacetimes. On the other hand, if \((M, g)\) is globally hyperbolic and \(\Sigma\) is a Cauchy hypersurface, then every smooth future-directed timelike vector field in \(M\) pierces \(\Sigma\). However, the existence of a piercing for \(\Sigma\) is strictly weaker than the requirement that \(\Sigma\) be Cauchy. For example, consider a four-dimensional anti-de Sitter spacetime, taken to be \(\mathbb{R}^4\) with the metric given by the line element (see, e.g., [16, p 131])

\[
\text{d}s^2 = -\cosh^2 r \text{d}t^2 + \text{d}r^2 + \sinh^2 r (\text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2),
\]

where the coordinate ranges are \(-\infty < t < \infty, r > 0, 0 < \theta < \pi\) and \(0 < \phi < 2\pi\). This spacetime is not globally hyperbolic, but each hypersurface \(t = \text{const.}\), although not Cauchy, is pierced by the vector field \(\frac{\text{d}}{\text{d}t}\).

**Proposition 2.4.** Let \(\Sigma \subset M\) be a surface contained in a partial Cauchy hypersurface \(\Sigma\). Suppose that the following holds:

(i) \((M, g)\) is causally simple,

(ii) \(\Sigma^+\) is noncompact and \(\Sigma\) admits a piercing \(X\).

Then, there exists an outward-pointing, future-directed \(\Sigma\)-ray starting at \(\Sigma\).
Proof. The idea for the first part of the proof presented here is due to Gannon [14] (see also theorem 7.1 in [2]), adapted to our more general setting.

Note that because \((M, g)\) is causally simple, \(E^+(\Sigma) = \partial I^+(\Sigma)\). Given \(p \in E^+(\Sigma) \setminus \Sigma\), let \(\eta : [0, 1] \to M\) be a future-directed null curve with \(\eta(1) = p\) and \(\eta(0) \in \Sigma\). \(\eta\) must of course be a segment of a null generator of \(\partial I^+(\Sigma)\), and it must be normal to \(\Sigma\) at \(\eta(0)\). Therefore, \(\eta'(0)\) is either parallel to \(K_+(\eta(0))\), or to \(K_-(\eta(0))\). We denote by \(\mathcal{H}_+\) the set of points of \(E^+(\Sigma) \setminus \Sigma\) for which the former case occurs, and by \(\mathcal{H}_-\) the set of points in \(E^+(\Sigma) \setminus \Sigma\) for which the latter case occurs. Thus, \(E^+(\Sigma) \setminus \Sigma = \mathcal{H}_+ \cup \mathcal{H}_-\).

We claim that \(\mathcal{H}_+ \cap \mathcal{H}_- = \emptyset\). Suppose not, and let \(p \in \mathcal{H}_+ \cap \mathcal{H}_-\). Let \(\eta_{\pm} : [0, 1] \to M\) be future-directed null curves with \(\eta_{\pm}(1) = p\) and \(\eta_{\pm}(0) \in \Sigma\), such that \(\eta_{+}\) (resp. \(\eta_{-}\)) is outward-pointing (resp. inward-pointing). Let \(\eta : [0, 1] \to M\) be given by

\[
\eta(t) = \begin{cases} 
\eta_-(2t), & \text{if } 0 \leq t < 1/2 \\
\eta_+(2t - 1), & \text{if } 1/2 \leq t \leq 1.
\end{cases}
\]

Now, using the piercing \(X\), we can define by a standard argument (see, e.g., [19, proposition 31, chapter 14]) a continuous open map \(\rho_X : M \to S\) leaving \(S\) pointwise fixed (i.e. a retraction) by ‘sliding’ points of \(M\) along the maximal integral curves of \(X\) into \(S\). Consider the continuous curve \(\eta_X : [0, 1] \to S\) given by \(\eta_X(t) = \rho_X(\eta(t))\).

Clearly, there exists a number \(0 < \epsilon < 1\) for which \(\eta_X(0, \epsilon) \subset S_-\) and \(\eta_X(1- \epsilon, 1) \subset S_+\). Therefore, since \(\Sigma\) separates \(S\), there exists \(t_0 \in [\epsilon, 1 - \epsilon]\) for which \(\eta_X(t_0) \in \Sigma\). But then \(\eta_X(t_0) \in \Sigma \cap I^-(p) \neq \emptyset\), i.e. \(I^+(\Sigma) \cap \partial I^+(\Sigma) \neq \emptyset\), contradicting the fact that \(I^+(\Sigma)\) is open. Thus, the claim is established.

From standard results on the structure of achronal boundaries, \(\mathcal{H}_\pm\) are \(C^0\) achronal connected hypersurfaces in \(M\).

Write \(T := \mathcal{H}_+ \cup \Sigma\). We now claim that the restriction \(\rho_X : \mathcal{H}_+ \to S\) maps \(\mathcal{H}_+\) into a connected open subset of \(S_+\) (and indeed it is a homeomorphism onto its image by invariance of domain). Otherwise, if \(\rho_X(p) \in S_-\cup \Sigma\) for some \(p \in \mathcal{H}_+\), then we could pick a future-directed timelike curve from \(\rho_X(p)\) to \(p\), compose it with an outward-pointing null curve from \(p\) to \(\Sigma\) and project it through \(\rho_X\) onto \(S\), and a repetition of the above argument for \(\eta\) would yield a contradiction.

We also claim that \(\rho_X(T) = S_+\). Indeed, clearly \(\rho_X(T) \subset S_+\), and \(\Sigma \subset T\), so \(S_+ = \rho_X(\Sigma) \subset \rho_X(T)\). Thus, \(S_+ \setminus \rho_X(T) \subset S_+\). Suppose that the claim is false. Then, \(S_+ \setminus \rho_X(T) \neq \emptyset\), and we must have \(\partial_S \rho_X(T) \cap S_+ \neq \emptyset\). Hence, pick \(p \in \partial_S \rho_X(T) \cap S_+\). Now, \(\rho_X(T)\) is clearly closed, so \(p = \rho_X(q)\) for some \(q \in T\). If \(q \in \Sigma\), then \(\rho_X(q) = q = p\) in \(S_+\), a contradiction. Therefore, we can assume that \(q \in \mathcal{H}_+\). Since the latter set is a topological hypersurface, and the restriction of \(\rho_X\) to it is a homeomorphism onto its image, we can choose a neighborhood \(V_0\) of \(q\) in \(M\) with \(V_0 \cap S = \emptyset, V_0 \cap T\) open in \(\mathcal{H}_+\) and \(p \in \rho_X(V_0 \cap T) \subset S_+\). Since \(\rho_X(V_0 \cap T)\) is open in \(S\) because \(\rho_X\) is an open map, we conclude that \(p\) is in the \(S\)-interior of \(\rho_X(T)\), in contradiction with the fact that \(p\) must be in the \(S\)-boundary of \(\rho_X(T)\). Therefore, \(\rho_X(T) = S_+\) as claimed.

But now our assumption about \(S_+\) establishes that \(T\) is not compact.

The remaining part of the proof is basically a repetition of the argument in proposition 2.1, working with \(T\) instead of \(E^+(\Sigma)\). We thereby obtain a future-directed, null \(\Sigma\)-ray starting at \(\Sigma\) and contained in \(T\). In particular, it is then outward-pointing as desired.

Remark 2.5. The conclusion of proposition 2.4 is false without the piercing assumption. Indeed, in the globally hyperbolic spacetime constructed in remark 2.3, there are no future-directed null \(\Sigma\)-rays emanating from \(\Sigma\). This does not contradict our conclusions, since in this example, clearly there are no piercings of \(S\).
As mentioned in remark 2.3, one cannot expect much more than the result in proposition 2.1 in general. But there is an additional assumption on $\Sigma$ which can be quite natural in examples. This is that $\Sigma$ bounds (in $S$), i.e. $\overline{S}_{-}$ is compact. This will be a key assumption in proposition 2.6.

**Proposition 2.6.** Let $\Sigma \subset M$ be a surface contained in a partial Cauchy hypersurface $S$. Suppose that $\Sigma$ bounds. Then, either $\Sigma$ is a future-trapped set, or else there exists an outward-pointing $\Sigma$-ray starting at $\Sigma$.

**Proof.** Suppose that $\Sigma$ is not a future-trapped set. We must show that it does not admit an inward-pointing $\Sigma$-ray, and so by proposition 2.1, there exists an outward-pointing $\Sigma$-ray starting at $\Sigma$.

Suppose first that $(M, g)$ is globally hyperbolic and that $S$ is a Cauchy hypersurface. In that case, using the notation in the proof of proposition 2.4, and by completely analogous arguments using a piercing by any future-directed timelike vector field, we can show that $\mathcal{H}_{-} \cup \Sigma$ must be compact, as it is homeomorphic to $\overline{S}_{-}$. But any inward-pointing null $\Sigma$-ray would have to be contained in $\mathcal{H}_{-} \cup \Sigma$, which cannot occur since the spacetime is in particular strongly causal and the latter set is compact. Thus, such a ray cannot exist.

Now, let us turn to the general case. Since $S$ is a partial Cauchy hypersurface, its Cauchy development (domain of dependence) $D(S)$ is open, and viewed as a spacetime in its own right, it is globally hyperbolic with $S$ as a Cauchy hypersurface (see, e.g., [19, chapter 14] for proofs of these facts). Suppose there exists an inward-pointing $\Sigma$-ray $\eta : [0, a) \to M$ starting at $\Sigma$. Then, it is contained in $\partial I^{+}(\Sigma)$. However, it is easy to check, due to the fact that $D(S)$ is open and causally convex (in the sense that nonspacelike curves cannot leave and reenter it), that $\partial_{D(S)}I^{+}(\Sigma, D(S)) = \partial I^{+}(\Sigma) \cap D(S)$. Thus, the portion of $\eta$ contained in $D(S)$ would be an inward-pointing $\Sigma$-ray in $(D(S), g|_{D(S)})$, contradicting the conclusion in the previous paragraph. This last contradiction completes the proof. $\square$

### 3. Singularity theorems in the presence of a MTS

Throughout this section, unless otherwise specified, we shall consider only MTS. The following technical lemma proven by Guimaraes [15] for solutions of Riccati-type ODE’s will be useful.

**Lemma 3.1.** Let $u : [0, +\infty) \to \mathbb{R}$ be a continuous function. Then, for any $c > 0$,

$$
\lim\sup_{t \to +\infty} \left( u(s) + c \int_{0}^{s} (u(t))^{2} \, dt \right) \geq 0,
$$

with equality iff $u \equiv 0$.

We are now ready to state and prove our first main theorem.

**Theorem 3.2.** Let $(M, g)$ be a spacetime of dimension $n \geq 3$ satisfying the following requirements.

1. $(M, g)$ contains no closed timelike curves.
2. $(M, g)$ obeys the timelike convergence condition, i.e. its Ricci tensor satisfies $\text{Ric}(v, v) \geq 0$, for all timelike vectors $v$.
3. $(M, g)$ contains a generic MTS $\Sigma$ contained in a partial Cauchy hypersurface.
4. The nonspacelike generic condition holds in $(M, g)$.

Then, $(M, g)$ is nonspacelike geodesically incomplete.
Proof. Suppose first that $\Sigma$ is not a future-trapped set. Then, by proposition 2.1, there exists a future-directed, affinely parametrized (null) $\Sigma$-ray $\eta: [0, a) \rightarrow M$ normal to $\Sigma$. Because we are assuming that $\Sigma$ is a MTS and not only a MOTS, $\eta$ can be either outward- or inward-pointing, but since the argument is the same in both cases, we may take it to be outward-pointing for definiteness. Suppose first that $\eta$ is future-complete, so that we put $a = +\infty$. Since $\eta$ is a $\Sigma$-ray, a standard analysis (see [5, chapter 12]) shows that there exists a globally defined Lagrange tensor field $A(t)$ on $\eta$ such that $b(t) = A(t)A^{-1}(t)$ obeys the Riccati-type equation
\begin{equation}
\dot{b}^2 + b^2 + \mathcal{R} = 0,
\end{equation}
where $\mathcal{R}(t) : N(\eta(t)) \rightarrow N(\eta(t))$ takes the values of $\mathcal{R}(v, \eta')\eta'$ on normal vectors $v \perp \eta(t)$ (taken modulo $\eta'$), $N(\eta(t))$ being its span, and so that $b_\theta(t) = \text{tr}(b(t))$ satisfies $\theta_\theta(0) = \theta_\eta(\eta(0)) \leq 0$. Now, tracing equation (3.1), we obtain the Raychaudhuri equation
\begin{equation}
\theta_\theta + \frac{1}{n-2}\theta_\theta^2 = -\text{Ric} (\eta', \eta') - \sigma^2,
\end{equation}
where the shear scalar $\sigma$ is the trace of the square of the trace-free part of $b$. If we define $f : [0, +\infty) \rightarrow \mathbb{R}$ by
\begin{equation}
f(t) = -\text{Ric} (\eta'(t), \eta'(t)) - \sigma^2(t),
\end{equation}
then the assumption on the Ricci tensor implies that $f \leq 0$, so that integrating equation (3.2) from 0 up to an arbitrary $s$, we obtain
\begin{equation}
\lim_{s \rightarrow +\infty} \left( \theta(b(s)) + \frac{1}{n-2} \int_0^s (\theta(b))^2 \, dt \right) = \theta(b(0)) + \lim_{s \rightarrow +\infty} \int_0^s f(t) \, dt \leq 0,
\end{equation}
and using lemma 3.1 (for $c = 1/(n-2)$) we conclude that $\theta_{\theta}(0) \equiv 0$ and, again by equation (3.2), that $f \equiv 0$. Note, moreover, that we must have $\theta_\eta(\eta(0)) = 0$. But then $b \equiv 0$ along $\eta$ and hence $\mathcal{R} \equiv 0$ by equation (3.1), contradicting the generic condition on normal null geodesics arising from $\Sigma$.

Therefore, either $\eta$ is future incomplete and we are done, or $\Sigma$ is a future-trapped set. But in the latter case, the (standard) Hawking–Penrose singularity theorem applies and yields the conclusion, so the proof is complete.

Remark 3.3. Note that the proof we give works as well if one assumes only averaged non-negativity of the Ricci tensor on nonspacelike inextendible geodesics [6].

In [10, section 3], the authors call an initial data singularity theorem any result which proves the existence of closed trapped surfaces in initial data sets from suitable conditions on the geometry and/or energy–momentum density. Of course, what is really relevant for singularity theorems are trapped sets, closed trapped surfaces being just one class of such sets when some additional conditions hold (strong causality plus the null convergence condition suffice in this case). However, they are particularly important for, say, numerical relativity, because the signs of the expansion scalars can be ascertained in a compact region of the initial data underlying manifold and can be expressed solely in terms of the initial data. MTS or MOTS also are likewise quasi-locally detectable. In particular, the proof of theorem 3.2 actually means that any proof of the existence of a MTS in initial data sets can also be viewed as an initial data singularity theorem in this sense.

Theorem 3.4. Let $(M, g)$ be a spacetime of dimension $n \geq 3$ in which the null convergence condition holds (i.e. its Ricci tensor satisfies $\text{Ric}(v, v) \geq 0$, for all null vectors $v$), and which contains a generic MTS $\Sigma$ in a partial Cauchy hypersurface. Then, either $(M, g)$ is null geodesically incomplete, or $\Sigma$ is a future-trapped set.
Now, by theorem 3.4 and the standard Penrose theorem [20], the analogue for MTS of the singularity theorem in [10] then follows.

**Corollary 3.5.** Let \((M, g)\) be a globally hyperbolic spacetime of dimension \(n \geq 3\) in which the null convergence condition holds, and which contains a generic MTS \(\Sigma\) in a noncompact Cauchy hypersurface \(\mathcal{S}\). Then, \((M, g)\) is null geodesically incomplete.

**Proof.** Suppose not. Then, by theorem 3.4, \(\mathcal{S}\) is future-trapped. A standard argument (see, e.g., [19, theorem 61, p 437]) establishes that \(\mathcal{S}\) and \(E^+(\Sigma)\) are homeomorphic, a contradiction. \(\square\)

### 4. Digression on the existence of a MTS in initial data sets

An important part of the interest in MOTS as quasi-local versions of black hole horizons springs from the fact that their existence and properties can be studied at the initial data set level. Our goal in this section is to make a few general remarks on the existence of MTS in such initial data sets.

Fix, for the remaining of this section, an \((n\text{-dimensional})\) initial data set, by which we mean a triple \(\left(\mathcal{N}, h, K\right)\), where \(\mathcal{N}, h\) is a smooth \(n\)-dimensional Riemannian manifold and \(K\) is a smooth, symmetric \((0, 2)\) tensor field over \(\mathcal{N}\). We can then define a real-valued function \(\rho\) and a 1-form \(J\) on \(\mathcal{N}\) by

\[
\rho := \frac{1}{2} (R_N - |K|^2 + (\text{tr}_N K)^2),
\]

\[
J := \text{div}_N (K - (\text{tr}_N K) h),
\]

where \(R_N\) denotes the scalar curvature of \(\mathcal{N}\).

Of course, the definitions above are purely geometric, but they acquire physical meaning when applied in the initial-value formulation of general relativity. In this setting, \(\mathcal{N}\) is to be thought of as an embedded spacelike hypersurface in an \((n+1)\)-dimensional spacetime \((M, g)\), with \(h\) being the induced metric and \(K\) being the second fundamental form, taken with respect to the unique future-directed, unit timelike normal vector field \(U\) over \(\mathcal{N} \hookrightarrow M\). Moreover, \((M, g)\) is assumed to satisfy the Einstein field equation

\[
\text{Ric}_M - \frac{1}{2} R_M g + \Lambda g = T,
\]

for a suitable \((0, 2)\) symmetric energy–momentum tensor \(T\), and a (possibly vanishing) cosmological constant \(\Lambda\). One can check that the Gauss–Codazzi equations for the embedding \(\mathcal{N} \hookrightarrow M\) imply that the initial data set \((\mathcal{N}, h, K)\) automatically satisfies equations (4.1) and (4.2) with the identifications

\[
\rho \equiv T(U, U) + \Lambda,
\]

\[
J \equiv - T(U, \cdot).
\]

In this context, one may consider how to express the expansion scalars \(\theta_{\pm}\) of a surface \(\Sigma^{n-1} \subset \mathcal{N}\), viewed as a codimension-2 submanifold of \(M\), in terms of initial data quantities. With the conventions of the previous section, a straightforward computation gives

\[
\theta_{\pm} = \text{tr}_{\Sigma} K \pm H_{\Sigma},
\]

where \(H_{\Sigma}\) denotes the mean curvature of \(\Sigma\) as a (codimension 1) submanifold of \(\mathcal{N}\) with respect to the outward-pointing normal \(N_{\Sigma}\), and given any orthonormal frame \(\{E_1, \ldots, E_{n-2}\}\) on \(\Sigma\),

\[
\text{tr}_{\Sigma} K \equiv \sum_{i=1}^{n-2} K(E_i, E_i).
\]
Using equation (4.6), we conclude that a MOTS is a MTS iff it is mean-convex, i.e. iff $H_{\Sigma} \geq 0$. In [10], it is discussed how a non-trivial topology of $N$ can give rise to the existence of MOTS, but to the best of the author’s knowledge, there is no result in the literature ensuring the existence of a mean-convex MOTS solely under geometrical conditions of this sort.

But given some physically motivated extrinsic conditions on the initial data, one can say a little more. The condition that $\Sigma$ be a MTS, $\theta_\pm \leq 0$, implies, from equation (4.6), that
\[ \text{tr}_\Sigma K \leq H_{\Sigma} \leq -\text{tr}_\Sigma K, \]
so in particular we must have $\text{tr}_\Sigma K \leq 0$. Now, adapting a definition given by Galloway in [11], we can introduce the following notion.

**Definition 4.1.** $N$ is non-expanding in all directions at a point $p \in N$ if $K$ is negative semi-definite on the tangent space $T_pN$. A set $A \subseteq N$ is non-expanding if $N$ is non-expanding in all directions at every $p \in A$.

The intuitive meaning of this definition becomes clear if $N$ describes (a portion of) the Universe at a given ‘instant’: one would expect that $N$ is non-expanding in all directions at all points of a region of $A \subseteq N$ undergoing gravitational collapse. In fact, with our sign conventions, a constant $t$ surface in a Robertson–Walker spacetime $(I \times S, -dt^2 \oplus f^2(t) \, d\sigma^2)$ is non-expanding iff $f'(t) \leq 0$ for that surface. However, for time-symmetric initial data, i.e. when $K \equiv 0$, $N$ will be non-expanding in all directions at all points. Then, $N$ would be non-expanding even if there is no gravitational collapse. Moreover, since this is an extrinsic condition on the initial data, even in spacetimes describing gravitational collapse, this condition might not hold for any initial data set thereof.

Therefore, as long as the additional condition of non-expansion holds for appropriate subsets in the initial data underlying manifold, one can benefit from extant results on the existence of MOTS and/or minimal surfaces in initial data sets (see section 4 of [10] and references therein for some of these). In fact, we have the following.

**Proposition 4.1.** Suppose that $\Sigma$ is either a MOTS or a minimal surface in $(N, h)$, contained in a non-expanding set $A \subseteq N$. Then, $\Sigma$ is a MTS.

**Proof.** If $\Sigma$ is a minimal surface, this follows immediately from the definition of a non-expanding set, and from equations (4.6) with $H_{\Sigma} \equiv 0$. If $\Sigma$ is a MOTS, then equations (4.6) imply that $H_{\Sigma} = -\text{tr}_\Sigma K$, and therefore,
\[ \theta_\pm = 2\, \text{tr}_\Sigma K \leq 0. \]

5. Singularity theorems in the presence of a MOTS

We are now ready to extend our main results to the MOTS setting. The first one is a Hawking–Penrose-type theorem [17, 5] in the presence of a generic MOTS that bounds. Actually, the proof also yields a singularity theorem for outer trapped surfaces (which bound), and we include this case as well. Using the notation in the previous sections, we say that $\Sigma$ is outer trapped if $\theta_+ < 0$.

**Theorem 5.1.** Let $(M, g)$ be a spacetime of dimension $n \geq 3$ satisfying the following requirements:

1. $(M, g)$ contains no closed timelike curve.
(2) \((M, g)\) obeys the timelike convergence condition, i.e. its Ricci tensor satisfies \(\text{Ric}(v, v) \geq 0\), for all timelike vectors \(v\).

(3) \((M, g)\) contains, in a partial Cauchy hypersurface, a surface \(\Sigma\) which bounds, and which is either a generic MOTS or else it is outer trapped.

(4) The nonspacelike generic condition holds in \((M, g)\).

Then \((M, g)\) is nonspacelike geodesically incomplete.

**Proof.** Suppose first that \(\Sigma\) is not a future-trapped set. Then, by proposition 2.6, there exists a future-directed, affinely parametrized, outward-pointing, \(\Sigma\)-ray \(\eta : [0, a) \to M\) normal to \(\Sigma\). The remaining of the proof is now identical to the proof of theorem 3.2 applied to this ray. \(\square\)

**Remark 5.2.** As mentioned in the introduction, in the spacetime dimension \(n = 4\), a celebrated result by Schoen and Yau establishes that compact MOTS appear naturally for large concentrations of matter [23]. In their setting, \(\mathcal{S}\) is a three-dimensional and asymptotically flat manifold bearing initial data. In the particular case when \(\mathcal{S}\) is diffeomorphic to \(\mathbb{R}^3\), any closed embedded MOTS will bound\(^3\).

Again, any proof of the existence of a bounding MOTS in initial data sets can also be viewed as an initial data singularity theorem in this sense.

**Theorem 5.3.** Let \((M, g)\) be a spacetime of dimension \(n \geq 3\) in which the null convergence condition holds (i.e. \(\text{Ric}(v, v) \geq 0\) for all null vectors \(v \in T M\)). Let \(\Sigma \subset M\) be a surface contained in a partial Cauchy hypersurface, which bounds therein, and which is either a generic MOTS or else it is outer trapped. Then, either \((M, g)\) is null geodesically incomplete, or \(\Sigma\) is a future-trapped set.

There is another special setting which is also of interest, that of **strictly stable** MOTS. It is well known that MOTS have a notion of stability [1–3] similar to that of minimal surfaces. Let us briefly recall the setting and refer the reader to [1–3, 12] and references therein for further details and proofs of the facts mentioned. Consider a normal variation, i.e. a variation \(t \to \Sigma_t\) in \(\mathcal{S}\) with \(\Sigma_0 = \Sigma\) and variation vector field \(V = \phi N + \), with \(\phi \in C^\infty(\Sigma)\). Then, one can show that

\[
\frac{\partial \theta_v}{\partial t}|_{t=0} = L\phi,
\]

where \(L : C^\infty(\Sigma) \to C^\infty(\Sigma)\) is an elliptic linear operator, the **stability operator**, whose specific form need not concern us here. It suffices to say that \(L\) has an eigenvalue with smallest real part \(\lambda_0\) which is real and admits a strictly positive smooth eigenfunction \(\phi_0\). Now, the MOTS \(\Sigma\) is said to be **stable** (resp. **strictly stable**) if \(\lambda_0 \geq 0\) (resp. \(\lambda_0 > 0\)). (An important situation in which a MOTS is stable [12] is when it is outermost, i.e. when there are neither outer trapped surfaces nor MOTS outside of \(\Sigma\) which are homologous to \(\Sigma\).)

Therefore, if \(\Sigma\) is strictly stable, we can use \(\phi_0\) in our variation, and in that case equation (5.1) implies that for some \(\epsilon > 0\), \(\Sigma_{-\epsilon}\) is an **outer trapped** closed surface. It can be chosen to bound if \(\Sigma\) bounds. Taking these facts into consideration yields the following.

**Corollary 5.4.** Let \((M, g)\) be a spacetime of dimension \(n \geq 3\) satisfying the following requirements.

(1) \((M, g)\) contains no closed timelike curve.

\(^3\) I thank Gregory Galloway for suggesting this example, together with [23].
(2) \((M, g)\) obeys the timelike convergence condition, i.e. its Ricci tensor satisfies \(\text{Ric}(v, v) \geq 0\), for all timelike vectors \(v\).

(3) \((M, g)\) contains a strictly stable MOTS \(\Sigma\), contained in a partial Cauchy hypersurface, and which bounds therein.

(4) The nonspacelike generic condition holds in \((M, g)\).

Then, \((M, g)\) is nonspacelike geodesically incomplete.

If we do not assume that the MOTS bounds, then we can still obtain a singularity theorem provided we strengthen the causal assumption on \((M, g)\) by using proposition 2.4. In this case, the nonspacelike generic condition on spacetime is not needed, and one can assume only the null convergence condition.

**Theorem 5.5.** Let \(\Sigma \subset M\) be a surface contained in a partial Cauchy hypersurface \(S\). Suppose that the following conditions hold.

(i) \((M, g)\) is causally simple and obeys the null convergence condition.

(ii) \(\overline{\Sigma}_{+}\) is noncompact and \(S\) admits a piercing \(X\).

(iii) \(\Sigma\) is a MOTS which is either generic or else it is a strictly stable.

Then, \((M, g)\) is null geodesically incomplete.

**Proof.** In the case that \(\Sigma\) is strictly stable, we can assume it is outer trapped by the observation before corollary 5.4. By proposition 2.4, conditions (i) and (ii) imply that there exists a future-directed, affinely parametrized, outward-pointing, \(\Sigma\)-ray \(\eta : [0, a) \to M\) normal to \(\Sigma\). A repetition of the proof of theorem 3.2 (without the trapped case, which does not apply here) now yields the conclusion. \(\square\)

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