Sobolev-type embeddings for neural network approximation spaces

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This paper is dedicated to Ron DeVore on the occasion of his 80th birthday

Abstract

We consider neural network approximation spaces that classify functions according to the rate at which they can be approximated (with error measured in $L^p$) by ReLU neural networks with an increasing number of coefficients, subject to bounds on the magnitude of the coefficients and the number of hidden layers. We prove embedding theorems between these spaces for different values of $p$. Furthermore, we derive sharp embeddings of these approximation spaces into Hölder spaces. We find that, analogous to the case of classical function spaces (such as Sobolev spaces, or Besov spaces) it is possible to trade “smoothness” (i.e., approximation rate) for increased integrability.

Combined with our earlier results in [arXiv:2104.02746], our embedding theorems imply a somewhat surprising fact related to “learning” functions from a given neural network space based on point samples: if accuracy is measured with respect to the uniform norm, then an optimal “learning” algorithm for reconstructing functions that are well approximable by ReLU neural networks is simply given by piecewise constant interpolation on a tensor product grid.

1 Introduction

We consider approximation spaces associated to neural networks with ReLU activation function $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(x) = \max\{0, x\}$ as introduced in our recent work [12]: see also Section 2. Roughly speaking, these spaces $A^{\alpha,p}_{\ell,c}$ are defined as follows: Given a depth-growth function $\ell : \mathbb{N} \to \mathbb{N}_{\geq 2} \cup \{\infty\}$ and a coefficient growth function $c : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$, the approximation space $A^{\alpha,p}_{\ell,c}([0,1]^d)$ consist of all functions that can be approximated up to error $O(n^{-\alpha})$, with the error measured in the $L^p([0,1]^d)$ (quasi)-norm for some $p \in (0,\infty]$, by neural networks with at most $\ell(n)$ layers and $n$ coefficients (non-zero weights) of size at most $c(n)$, $n \in \mathbb{N}$. As shown in [12] these spaces can be equipped with a natural (quasi)-norm; see also Lemma 2.1 below. In most applications in machine learning, one is interested in “learning” such functions from a finite amount of samples. To render this problem well defined we would like the space $A^{\alpha,p}_{\ell,c}([0,1]^d)$ to embed into $C([0,1]^d)$. The question that we consider in this work is therefore to find conditions under which such an embedding holds true. More generally, we are interested in characterizing the existence of embeddings of the approximation spaces into the Hölder spaces $C^{\alpha,\beta}([0,1]^d)$ for $\beta \in (0,1)$. The existence of such embeddings is far from clear; for example, recent work [7, 10, 3, 5] shows that fractal and highly non-regular functions can be efficiently approximated by (increasingly deep) ReLU neural networks.

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The main contribution of this paper are sharp conditions for when the space $A_{\ell,c}^{\alpha,p}([0,1]^d)$ embeds into $C([0,1]^d)$ or $C^{0,\beta}([0,1]^d)$. These results in particular show that, similar to classical Sobolev spaces, one can trade smoothness for integrability in the context of neural network approximation, where “smoothness” of a function is understood as it being well approximable by neural networks.

Building on these insights we consider the (highly practically relevant) problem of finding optimal numerical algorithms for reconstructing arbitrary functions in $A_{\ell,c}^{\alpha,p}([0,1]^d)$ from finitely many point samples, where the reconstruction error is measured in the $L^\infty([0,1]^d)$ norm: As a corollary to our embedding theorems we establish the somewhat surprising fact that, irrespective of $\alpha$, an optimal algorithm is given by piecewise constant interpolation with respect to a uniform tensor-product grid in $[0,1]^d$.

1.1 Description of Results

In the following sections we sketch our main results. To simplify the presentation, let us assume that the coefficient growth function $c$ satisfies the asymptotic growth $c \asymp n^\theta \cdot (\ln(2n))^\kappa$ for certain $\theta \geq 0$ and $\kappa \in \mathbb{R}$ (our general results hold for any coefficient growth function). Let

$$\mathcal{L}^* := \sup_{n \in \mathbb{N}} \mathcal{L}(n) \in \mathbb{N}_2 \cup \{\infty\} \quad \text{and} \quad \gamma^* := \theta \cdot \mathcal{L}^* + [\mathcal{L}^*/2] \in [1,\infty].$$

1.1.1 Embedding Results

Our first main theorem provides sharp embedding results into $C([0,1]^d)$. A slight simplification of our main result reads as follows.

**Theorem 1.1.** Assume that $\alpha > \frac{d}{p} \cdot \gamma^*$. Then the embedding $A_{\ell,c}^{\alpha,p}([0,1]^d) \hookrightarrow C([0,1]^d)$ holds. Assume that $\alpha < \frac{d}{p} \cdot \gamma^*$. Then the embedding $A_{\ell,c}^{\alpha,p}([0,1]^d) \hookrightarrow C([0,1]^d)$ does not hold.

Theorem 1.1 is a special case of Theorems 3.3 and 3.6 below that will be proven in Section 3.

We would like to draw attention to the striking similarity between Theorem 1.1 and the classical Sobolev embedding theorem for Sobolev spaces $W^{\alpha,p}$ which states that $W^{\alpha,p}([0,1]^d) \hookrightarrow C([0,1]^d)$ holds if $\alpha > \frac{d}{p}$, but not if $\alpha < \frac{d}{p}$. Just as in the case of classical Sobolev spaces we show that, also in the context of neural network approximation, it is possible to trade “smoothness” (i.e., approximation order) for integrability.

We also prove sharp embedding results into the Hölder spaces $C^{0,\beta}([0,1]^d)$. A slight simplification of our main result reads as follows.

**Theorem 1.2.** Let $\beta \in (0,1)$.

- If $\alpha > \frac{\beta+\frac{d}{p}}{1-\beta} \cdot \gamma^*$, then the embedding $A_{\ell,c}^{\alpha,p}([0,1]^d) \hookrightarrow C^{0,\beta}([0,1]^d)$ holds.

- If $\alpha < \frac{\beta+\frac{d}{p}}{1-\beta} \cdot \gamma^*$, then the embedding $A_{\ell,c}^{\alpha,p}([0,1]^d) \hookrightarrow C^{0,\beta}([0,1]^d)$ does not hold.

Theorem 1.1 is a special case of Theorems 3.7 and 3.6 below that will be proven in Section 3.

Theorems 1.1 and 1.2 provide a complete characterization of the embedding properties of the neural network approximation spaces $A_{\ell,c}^{\alpha,p}([0,1]^d)$ away from the “critical” index $\alpha = \frac{d}{p} \cdot \gamma^*$ (embedding into $C([0,1]^d)$), respectively $\alpha = \frac{d+\frac{d}{p}}{1-\beta} \cdot \gamma^*$ (embedding into $C^{0,\beta}([0,1]^d)$). We leave the (probably subtle) question of what happens at the critical index open for future work.

1.1.2 Optimal Learning Algorithms

Our embedding theorems allow us to deduce an interesting and surprising result related to the algorithmic feasibility of a learning problem posed on neural network approximation spaces.

More precisely, we consider the problem of approximating a function in the unit ball

$$U_{\ell, c}^{\alpha,\infty}([0,1]^d) := \{ f \in A_{\ell,c}^{\alpha,\infty}([0,1]^d) : \| f \|_{A_{\ell,c}^{\alpha,\infty}([0,1]^d)} \leq 1 \}$$
of the neural network approximation space $A_{\ell,c}^{\alpha,\infty}([0,1]^d)$ under the constraint that the approximating algorithm is only allowed to access point samples of the functions $f \in U$. Formally, a (deterministic) algorithm $A : U_{\ell,c}^{\alpha,\infty}([0,1]^d) \to L^\infty([0,1]^d)$ using $m$ point samples is determined by a set of sample points $x = (x_1, \ldots, x_m) \in ([0,1]^d)^m$ and a map $Q : \mathbb{R}^m \to L^\infty([0,1]^d)$ such that

$$A(f) = Q(f(x_1), \ldots, f(x_m)) \quad \forall f \in U_{\ell,c}^{\alpha,\infty}([0,1]^d).$$

Note that $A$ need not be an “algorithm” in the usual sense, i.e., implementable on a Turing machine. The set of all such “algorithms” is denoted $\text{Alg}_m(U_{\ell,c}^{\alpha,\infty}([0,1]^d))$ and we define the optimal order for (deterministically) uniformly approximating functions in $U_{\ell,c}^{\alpha,\infty}([0,1]^d)$ using point samples as the best possible convergence rate with respect to the number of samples:

$$\beta^\text{det}_* (U_{\ell,c}^{\alpha,\infty}([0,1]^d)) := \sup \left\{ \lambda \geq 0 : \exists C > 0 \forall m \in \mathbb{N} : \inf_{A \in \text{Alg}_m(U_{\ell,c}^{\alpha,\infty}([0,1]^d))} \sup_{f \in U_{\ell,c}^{\alpha,\infty}([0,1]^d)} \| A(f) - f \|_{L^\infty([0,1]^d)} \leq C \cdot m^{-\lambda} \right\}.$$

In a similar way one can define randomized (Monte Carlo) algorithms and consider the optimal order $\beta^\text{MC}_* (U_{\ell,c}^{\alpha,\infty}([0,1]^d))$ for approximating functions in $U_{\ell,c}^{\alpha,\infty}([0,1]^d)$ using randomized algorithms based on point samples; see [12, Section 2.4.2]. In [12, Theorem 1.1] we showed (under the assumption $\ell^* \geq 3$) that

$$\beta^\text{det}_* (U_{\ell,c}^{\alpha,\infty}([0,1]^d)) = \beta^\text{MC}_* (U_{\ell,c}^{\alpha,\infty}([0,1]^d)) = \begin{cases} \frac{2}{d} \cdot \frac{\alpha}{\gamma + \alpha} & \text{if } \ell^* < \infty \\ 0 & \text{else.} \end{cases} \quad (1.1)$$

We now describe a particularly simple algorithm that achieves this optimal rate. For $m \in \mathbb{N}_{\geq 2^t}$, let $n := [m^{1/d}] - 1$ and let

$$A_{m}^{\text{pw-const}}(f)(x) := \sum_{i_1=0}^{n} \cdots \sum_{i_d=0}^{n} f \left( \frac{i_1}{n}, \frac{i_2}{n}, \ldots, \frac{i_d}{n} \right) \mathbb{1}_{\left[ \frac{i_1}{n}, \ldots, \frac{i_d}{n} \right]^d} + [0, \frac{1}{n}]^d(x)$$

be the piecewise constant interpolation of $f$ on a tensor product grid with sidelenth $\frac{1}{n}$, where for $\tau \in [0,1]^d$, the function $\mathbb{1}_{\tau + [0,\frac{1}{n}]^d}$ denotes the indicator function of the set $\tau + [0, \frac{1}{n}]^d \subset \mathbb{R}^d$. Since $(n+1)^d \leq m$ it clearly holds that

$$A_{m}^{\text{pw-const}} \in \text{Alg}_m(U_{\ell,c}^{\alpha,\infty}([0,1]^d)).$$

We have the following theorem.

**Theorem 1.3.** Under the above assumptions on $\ell, c$, the piecewise constant interpolation operator $A_{m}^{\text{pw-const}}$ constitutes an optimal algorithm for reconstructing functions in $U_{\ell,c}^{\alpha,\infty}([0,1]^d)$ from point samples: we have

$$\beta^\text{det}_* (U_{\ell,c}^{\alpha,\infty}([0,1]^d)) = \beta^\text{MC}_* (U_{\ell,c}^{\alpha,\infty}([0,1]^d)) = \sup \left\{ \lambda \geq 0 : \exists C > 0 \forall m \in \mathbb{N} : \sup_{f \in U_{\ell,c}^{\alpha,\infty}([0,1]^d)} \| A_{m}^{\text{pw-const}}(f) - f \|_{L^\infty([0,1]^d)} \leq C \cdot m^{-\lambda} \right\}. \quad (1.2)$$

**Proof.** Equation (1.1) yields the best possible rate with respect to the number of samples. Without loss of generality we assume that $\ell^* < \infty$ (otherwise there is nothing to show as the optimal rate is zero). We now show that all rates below the optimal rate $\frac{2}{d} \cdot \frac{\alpha}{\gamma + \alpha}$ can be achieved by piecewise constant interpolation, which readily implies our theorem. To this end, let $\lambda \in (0, \frac{d}{2} \cdot \frac{\alpha}{\gamma + \alpha})$ be arbitrary and let $\beta := d \cdot \lambda \in (0, \frac{\alpha}{\gamma + \alpha}) \subset (0,1)$. It is easy to check that $\alpha > \frac{\beta}{\frac{2}{d} - \gamma^*}$. Hence,
Theorem 1.2 (applied with $p = \infty$) implies the embedding $A_{\ell,c}^{\alpha,\infty}([0,1]^d) \hookrightarrow C^{0,\beta}([0,1]^d)$, yielding the existence of a constant $C_1 > 0$ with

$$U_{\ell,c}^{\alpha,\infty}([0,1]^d) \subset C_1 \cdot U^\beta,$$

where $U^\beta$ denotes the unit ball in $C^{0,\beta}([0,1]^d)$. Therefore,

$$\sup_{f \in U_{\ell,c}^{\alpha,\infty}([0,1]^d)} \|A_{pw-\text{const}}(f) - f\|_{L^\infty([0,1]^d)} \leq \sup_{f \in C_1 \cdot U^\beta} \|A_{pw-\text{const}}(f) - f\|_{L^\infty([0,1]^d)} \leq C \cdot m^{-\lambda}, \quad (1.3)$$

where the last inequality follows from the elementary fact that piecewise constant interpolation on the uniform tensor-product grid $\{0, 1/n, \ldots, 1\}$ achieves $L^\infty$ error of $O(n^{-\beta}) \subset O(m^{-\beta/d}) = O(m^{-\lambda})$ for functions $f \in U^\beta$. Since $\lambda < \frac{d}{\gamma + \alpha}$ was arbitrary, Equation (1.3) proves Equation (1.2) and thus our theorem.

Theorem 1.3 proves that, provided that the reconstruction error is measured in the $L^\infty$ norm, the optimal “learning” algorithm for reconstructing functions that are well approximable by ReLU neural networks is simply given by piecewise constant interpolation on a tensor product grid, at least if no additional information about the target function is given.

We finally remark that we expect all our results to hold (in slightly modified form) if general bounded Lipschitz domains $\Omega \subset \mathbb{R}^d$ are considered in place of the unit cube $[0,1]^d$.

1.2 Related Work

The study of approximation properties of neural networks has a long history, see for example the surveys [2, 10, 11, 3] and the references therein. Function spaces related to neural networks have been studied in several works. For example, [1, 9, 4] study so-called Barron-type spaces that arise in a certain sense as approximation spaces of shallow neural networks. The paper [2] develops a calculus on spaces of functions that can be approximated by neural networks without curse of dimension. Closest to our work are the papers [11, 12] which study functional analytic properties of neural network approximation spaces similar to the ones studied in the present paper. The paper [11] also considers embedding results between such spaces with different constraints on their underlying network architecture. However, none of the mentioned works considered embedding theorems of Sobolev type comparable to those in the present paper. An important reason for this is that the spaces considered in [11] do not restrict the magnitude of the weights of the approximating networks, so that essentially $c \equiv \infty$ in our terminology. The resulting approximation spaces then contain functions of very low smoothness.

1.3 Structure of the Paper

The structure of this paper is as follows. In Section 2, we provide a definition of the neural network approximation spaces considered in this paper and establish some of their basic properties. In Section 3, we prove our embedding theorems.

2 Definition of neural network approximation spaces

In this section, we review the mathematical definition of neural networks, then formally introduce the neural network approximation spaces $A_{\ell,c}^{\alpha,\beta}([0,1]^d)$, and finally define the quantities $\ell^*$ and $\gamma^*(\ell,c)$ that will turn out to be decisive for characterizing whether the embeddings $A_{\ell,c}^{\alpha,\beta}([0,1]^d) \hookrightarrow C^{0,\beta}([0,1]^d)$ or $A_{\ell,c}^{\alpha,\beta}([0,1]^d) \hookrightarrow C([0,1]^d)$ hold.

2.1 The mathematical formalization of neural networks

In our analysis, it will be helpful to distinguish between a neural network $\Phi$ as a set of weights and the associated function $R_{\ell} \Phi$ computed by the network. Thus, we say that a neural network is a tuple $\Phi = ((A_1, b_1), \ldots, (A_L, b_L))$, with $A_\ell \in \mathbb{R}^{N_{\ell-1} \times N_{\ell-1}}$ and $b_\ell \in \mathbb{R}^{N_{\ell}}$. We then say that
\( a(\Phi) := (N_0, \ldots, N_L) \in \mathbb{N}^{L+1} \) is the architecture of \( \Phi \), \( L(\Phi) := L \) is the number of layers\(^4\) of \( \Phi \), and \( W(\Phi) := \sum_{j=1}^{L} (\|A_j\|_{\ell_0} + \|b_j\|_{\ell_0}) \) denotes the number of (non-zero) weights of \( \Phi \). The notation \( \|A\|_{\ell_0} \) used here denotes the number of non-zero entries of a matrix (or vector) \( A \). Finally, we write \( d_{\text{in}}(\Phi) := N_0 \) and \( d_{\text{out}}(\Phi) := N_L \) for the input and output dimension of \( \Phi \), and we set \( \|\Phi\|_{\ell_{\alpha,p}} := \max_{j=1,\ldots,L} [\|A_j\|_{\infty}, \|b_j\|_{\infty}] \), where \( \|A\|_{\infty} := \max_{i,j} \|A_{i,j}\| \).

To define the function \( R_\theta \Phi \) computed by \( \Phi \), we need to specify an activation function. In this paper, we will only consider the so-called rectified linear unit (ReLU) \( g : \mathbb{R} \to \mathbb{R}, x \mapsto \max\{0, x\} \), which we understand to act componentwise on \( \mathbb{R}^n \), i.e., \( g((x_1, \ldots, x_n)) = (g(x_1), \ldots, g(x_n)) \). The function \( R_\theta \Phi : \mathbb{R}^{N_0} \to \mathbb{R}^{N_L} \) computed by the network \( \Phi \) (its realization) is then given by

\[
R_\theta \Phi := T_L \circ (g \circ T_{L-1}) \circ \cdots \circ (g \circ T_1) \quad \text{where} \quad T_k x = A_k x + b_k.
\]

### 2.2 Neural network approximation spaces

Approximation spaces \([8]\) classify functions according to how well they can be approximated by a family \( \Sigma = (\Sigma_n)_{n \in \mathbb{N}} \) of certain “simple functions” of increasing complexity \( n \), as \( n \to \infty \). Common examples consider the case where \( \Sigma_n \) is the set of polynomials of degree \( n \), or the set of all linear combinations of \( n \) waves. The notion of neural network approximation spaces was originally introduced in \([11]\), where \( \Sigma_n \) was taken to be a family of neural networks of increasing complexity. However, \([17]\) does not impose any restrictions on the size of the individual network weights, which plays an important role in practice and which is important in determining the regularity of the function implemented by the network. Since the function spaces introduced in \([11]\) ignore this quantity, the spaces defined there never embed into the Hölder-space \( C^{0,\beta} \) for any \( \beta > 0 \).

For this reason, and following our previous work \([12]\), we introduce a modified notion of neural network approximation spaces that also takes the size of the individual network weights into account. Precisely, given an input dimension \( d \in \mathbb{N} \) (which we will keep fixed throughout this paper) and non-decreasing functions \( \ell : \mathbb{N} \to \mathbb{N} \cup \{0\} \) and \( c : \mathbb{N} \to \mathbb{N} \cup \{\infty\} \) (called the depth-growth function and the coefficient growth function, respectively), we define

\[
\Sigma_n^{\ell,c} := \left\{ R_\theta \Phi : \Phi \text{ NN with } d_{\text{in}}(\Phi) = d, d_{\text{out}}(\Phi) = 1, W(\Phi) \leq n, L(\Phi) \leq \ell(n), \|\Phi\|_{\ell_{\infty}} \leq c(n) \right\}.
\]

Then, given a measurable subset \( \Omega \subset \mathbb{R}^d \) as well as \( p \in (0, \infty] \) and \( \alpha \in (0, \infty) \), for each measurable \( f : \Omega \to \mathbb{R} \), we define

\[
\Gamma_{\alpha,p}(f) := \max \left\{ \|f\|_{L^p(\Omega)}, \sup_{n \in \mathbb{N}} [n^\alpha \cdot d_p(f, \Sigma_n^{\ell,c})] \right\} \in [0, \infty],
\]

where \( d_p(f, \Sigma) := \inf_{g \in \Sigma} \|f - g\|_{L^p(\Omega)} \).

The remaining issue is that since the set \( \Sigma_n^{\ell,c} \) is in general neither closed under addition nor under multiplication with scalars, \( \Gamma_{\alpha,p} \) is not a (quasi)-norm. To resolve this issue, taking inspiration from the theory of Orlicz spaces (see e.g. \([14]\) Theorem 3 in Section 3.2), we define the neural network approximation space quasi-norm \( \|\cdot\|_{\ell_{\alpha,p}} \) as

\[
\|f\|_{\ell_{\alpha,p}} := \inf \{ \theta > 0 : \Gamma_{\alpha,p}(f/\theta) \leq 1 \} \in [0, \infty],
\]

giving rise to the approximation space

\[
A_{\ell,c}^{\alpha,p} := A_{\ell,c}^{\alpha,p}(\Omega) := \left\{ f \in L^p(\Omega) : \|f\|_{\ell_{\alpha,p}} < \infty \right\}.
\]

The following lemma summarizes the main elementary properties of these spaces.

**Lemma 2.1.** Let \( \varnothing \neq \Omega \subset \mathbb{R}^d \) be measurable, let \( p \in (0, \infty] \) and \( \alpha \in (0, \infty) \). Then, the approximation space \( A_{\ell,c}^{\alpha,p} := A_{\ell,c}^{\alpha,p}(\Omega) \) satisfies the following properties:

\(^4\)Note that the number of hidden layers is given by \( H = L - 1 \).
1. \( (A_{\ell, c}^{a,p}, \| \cdot \|_{A_{\ell, c}^{a,p}}) \) is a quasi-normed space. Precisely, given arbitrary measurable functions \( f, g : \Omega \to \mathbb{R} \), it holds that \( \| f + g \|_{A_{\ell, c}^{a,p}} \leq C \cdot (\| f \|_{A_{\ell, c}^{a,p}} + \| g \|_{A_{\ell, c}^{a,p}}) \) for \( C = C(\alpha, p) \).

2. We have \( \Gamma_{\alpha,p}(cf) \leq |c| \Gamma_{\alpha,p}(f) \) for \( c \in [-1, 1] \).

3. \( \Gamma_{\alpha,p}(f) \leq 1 \) if and only if \( \| f \|_{A_{\ell, c}^{a,p}} \leq 1 \).

4. \( \Gamma_{\alpha,p}(f) < \infty \) if and only if \( \| f \|_{A_{\ell, c}^{a,p}} < \infty \).

5. \( A_{\ell, c}^{a,p}(\Omega) \hookrightarrow L^p(\Omega) \).

Proof. For \( p \in [1, \infty] \) this is precisely \cite{[12]}. The case \( p \in (0, 1) \) can be proven in a completely analogous way and is left to the reader.

Remark 2.2. While we introduced the spaces \( A_{\ell, c}^{a,p}(\Omega) \) for general domains \( \Omega \), in what follows we will specialize to \( \Omega = [0, 1]^d \). We expect, however, that all our results will remain true (up to minor modifications) for general bounded Lipschitz domains.

2.3 Quantities characterizing the complexity of the network architecture

To conveniently summarize those aspects of the growth behavior of the functions \( \ell \) and \( c \) most relevant to us, we introduce three quantities that will turn out to be crucial for characterizing whether the embeddings \( A_{\ell, c}^{a,p}([0, 1]^d) \hookrightarrow C^{0, \beta}([0, 1]^d) \) or \( A_{\ell, c}^{a,p}([0, 1]^d) \hookrightarrow C([0, 1]^d) \) hold. First, we set

\[
\ell^* := \sup_{n \in \mathbb{N}} \ell(n) \in \mathbb{N}_{\geq 2} \cup \{ \infty \}. \tag{2.1}
\]

Furthermore, we define \( \gamma^*(\ell, c), \gamma^\diamond(\ell, c) \in (0, \infty) \) by

\[
\gamma^*(\ell, c) := \sup \left\{ \gamma \in [0, \infty) : \exists L \in \mathbb{N}_{\leq \ell^*} : \lim_{n \to \infty} \frac{(c(n))^L \cdot n^{[L/2]}}{n^{\gamma}} \in (0, \infty) \right\},
\]

\[
\gamma^\diamond(\ell, c) := \inf \left\{ \gamma \in [0, \infty) : \exists C > 0 \forall n \in \mathbb{N}, L \in \mathbb{N}_{\leq \ell^*} : (c(n))^L \cdot n^{[L/2]} \leq C \cdot n^{\gamma} \right\}, \tag{2.2}
\]

where we use the convention that \( \sup \emptyset = 0 \) and \( \inf \emptyset = \infty \). As we now show, the quantities \( \gamma^* \) and \( \gamma^\diamond \) actually coincide.

Lemma 2.3. We have \( \gamma^*(\ell, c) = \gamma^\diamond(\ell, c) \). Furthermore, if \( \ell^* = \infty \), then \( \gamma^*(\ell, c) = \gamma^\diamond(\ell, c) = \infty \).

Proof. For brevity, write \( \gamma^* := \gamma^*(\ell, c) \) and \( \gamma^\diamond := \gamma^\diamond(\ell, c) \).

Case 1 (\( \ell^* = \infty \)): In this case, given any \( \gamma > 0 \), one can choose an even \( L \in \mathbb{N} = \mathbb{N}_{\leq \ell^*} \) satisfying \( L > 2\gamma \). It is then easy to see \( \limsup_{n \to \infty} \frac{(c(n))^L \cdot n^{[L/2]}}{n^{\gamma}} = \limsup_{n \to \infty} 1 = 1 \in (0, \infty] \). By definition of \( \gamma^* \), this implies \( \gamma^* \geq \gamma \) for each \( \gamma > 0 \), and hence \( \gamma^* = \infty \).

Next, we show that \( \gamma^\diamond = \infty \) as well. To see this, assume towards a contradiction that \( \gamma^\diamond < \infty \). By definition of \( \gamma^\diamond \), this means that there exist \( \gamma \geq 0 \) and \( C > 0 \) satisfying

\[
C \cdot n^{\gamma} \geq (c(n))^L \cdot n^{[L/2]} \geq n^{[L/2]} \quad \forall n, L \in \mathbb{N}.
\]

But choosing an even \( L \in \mathbb{N} \) satisfying \( L > 2\gamma \), we have \( [L/2] = L/2 > \gamma \), showing that the preceding displayed inequality cannot hold. This is the desired contradiction.

Case 2 (\( \ell^* < \infty \)): We first show \( \gamma^* \leq \gamma^\diamond \). In case of \( \gamma^\diamond = \infty \), this is trivial, so that we can assume \( \gamma^\diamond < \infty \). Let \( \gamma_0 \in (0, \infty) \) such that there exists \( C > 0 \) satisfying \( (c(n))^L \cdot n^{[L/2]} \leq C \cdot n^{7\gamma_0} \) for all \( n \in \mathbb{N} \) and \( L \in \mathbb{N}_{\leq \ell^*} \). Then, given any \( \gamma \leq \gamma_0 \) and \( L \in \mathbb{N}_{\leq \ell^*} \), we have

\[
\limsup_{n \to \infty} \frac{(c(n))^L \cdot n^{[L/2]}}{n^{\gamma}} \leq \limsup_{n \to \infty} \frac{C \cdot n^{7\gamma_0}}{n^{\gamma}} = 0.
\]
This easily implies $\gamma^* \leq \gamma_0$. Since this holds for all $\gamma_0$ as above, we see by definition of $\gamma^\circ$ that $\gamma^* \leq \gamma^\circ$.

We finally show that $\gamma^\circ \leq \gamma^*$. In case of $\gamma^* = \infty$, this is clear; hence, we can assume that $\gamma^* < \infty$. Let $\gamma > \gamma^*$ be arbitrary. Then

$$0 \leq \liminf_{n \to \infty} \frac{(c(n))^\gamma \cdot n^{|\ell|/2}}{n^\gamma} \leq \limsup_{n \to \infty} \frac{(c(n))^\gamma \cdot n^{|\ell|/2}}{n^\gamma} = 0.$$  

Thus, since convergent sequences are bounded, we see that $(c(n))^\gamma \cdot n^{|\ell|/2} \leq C \cdot n^\gamma$ for all $n \in \mathbb{N}$ and a suitable $C > 0$. Given any $L \in \mathbb{N}$, we thus have $(c(n))^L \cdot n^{|\ell|/2} \leq (c(n))^\gamma \cdot n^{|\ell|/2} \leq C \cdot n^\gamma$. By definition of $\gamma^\circ$, this implies $\gamma^\circ \leq \gamma$. Since this holds for all $\gamma > \gamma^*$, we conclude $\gamma^\circ \leq \gamma^*$. \hfill \Box

3 Sobolev-type embeddings for neural network approximation spaces

In this section, we establish a characterization of the embeddings $A^{\alpha,p}_{\ell,c}([0,1]^d) \hookrightarrow C([0,1]^d)$ and $A^{\alpha,p}_{\ell,c}([0,1]^d) \hookrightarrow C^{0,\beta}([0,1]^d)$ solely in terms of $\gamma^*(\ell, c)$ and the quantities $d \in \mathbb{N}$, $\beta \in (0, 1)$, and $p \in (0, \infty]$. This is roughly similar to Sobolev embeddings, which allow one to “trade smoothness for increased integrability.”

3.1 Sufficient Conditions for Embedding into $C([0,1]^d)$

In this subsection we establish sufficient conditions for an embedding $A^{\alpha,p}_{\ell,c}([0,1]^d) \hookrightarrow C([0,1]^d)$ to hold. The following lemma—showing that functions with large (but finite) Lipschitz constant but small $L^p$ norm have a controlled $L^\infty$ norm—is an essential ingredient for the proof.

We first make precise the notion of Lipschitz constant that we use in what follows. Given $d \in \mathbb{N}$ and a function $f : [0,1]^d \to \mathbb{R}$ we define

$$\text{Lip}(f) := \sup_{x,y \in [0,1]^d, x \neq y} \frac{|f(x) - f(y)|}{\|x - y\|_{\ell^\infty}} \in [0, \infty].$$

Lemma 3.1. Let $p \in (0, \infty)$ and $d \in \mathbb{N}$. Then there exists a constant $C = C(d,p) > 0$ such that for each $0 < T \leq 1$ and each $f \in L^p([0,1]^d) \cap C^{0,1}([0,1]^d)$, we have

$$\|f\|_{L^\infty} \leq C \cdot (T^{-d/p} \|f\|_{L^p} + T \cdot \text{Lip}(f)).$$

Proof. For brevity, set $Q := [0,1]^d$. Let $f \in L^p([0,1]^d) \cap C^{0,1}([0,1]^d)$. Since $Q$ is compact and $f$ is continuous, there exists $x_0 \in Q$ satisfying $|f(x_0)| = \|f\|_{L^\infty}$. Set $P := Q \cap (\{x_0\} + [-T,T]^d)$. For $x \in P$, we have

$$\|f\|_{L^\infty} \leq |f(x_0)| \leq |f(x_0) - f(x)| + |f(x)| \leq \text{Lip}(f) \cdot \|x - x_0\|_{\ell^\infty} + |f(x)| \leq \text{Lip}(f) \cdot T + |f(x)|.$$

Now, denoting the Lebesgue measure by $\lambda$, taking the $L^p(P)$ norm of the preceding estimate, and recalling that $L^p$ is a (quasi)-Banach space, we obtain a constant $C_1 = C_1(p) > 0$ satisfying

$$\|f\|_{L^\infty} \cdot \|\lambda(P)\|^{1/p} = \|f\|_{L^\infty} \|\lambda(P)\|^{1/p} \leq \text{Lip}(f) \cdot T + \|f\|_{L^p(P)} \leq C_1 \cdot (\text{Lip}(f) \cdot T \cdot \|\lambda(P)\|^{1/p} + \|f\|_{L^p(P)}) \leq C_1 \cdot (\text{Lip}(f) \cdot T \cdot \|\lambda(P)\|^{1/p} + \|f\|_{L^p})).$$

Next, \cite{Lee} Lemma A.2 shows that $\lambda(P) \geq 2^{-d} T^d$, so that we finally see

$$\|f\|_{L^\infty} \leq C_1 \cdot (\text{Lip}(f) \cdot T + \|\lambda(P)\|^{-1/p} \|f\|_{L^p}) \leq C_1 \cdot \text{Lip}(f) \cdot T + 2^{d/p} T^{-d/p} \|f\|_{L^p},$$

which easily implies the claim. \hfill \Box
In order to derive a sufficient condition for the embedding $A^{α,p}_ε([0,1]^d) \hookrightarrow C([0,1]^d)$, we will use the following lemma that provides a bound for the Lipschitz constant of a function implemented by a neural network, in terms of the size (number of non-zero weights) of the network and of the magnitude of the individual network weights.

**Lemma 3.2. (Lemma 4.1)** Let $ε : \mathbb{N} \to \mathbb{N}_{>2} \cup \{∞\}$ and $c : \mathbb{N} \to [1, ∞)$. Let $n \in \mathbb{N}$ and assume that $L := ε(n)$ and $C := c(n)$ are finite. Then each $F ∈ Σ_n$ satisfies

$$Lip(F) ≤ d · C^L · n^{1/L^2}.$$  

Now, we can derive a sufficient condition for the embedding $A^{α,p}_ε([0,1]^d) \hookrightarrow C([0,1]^d)$. Below, in Section 3.3, we will see that this sufficient condition is essentially optimal.

**Theorem 3.3.** Let $d ∈ \mathbb{N}$ and $p ∈ (0, ∞)$, and suppose that $γ^∗(ε, c) < ∞$. If $α > \frac{d}{p} · γ^∗(ε, c)$, then $A^{α,p}_ε([0,1]^d) \hookrightarrow C([0,1]^d)$. Furthermore, for each $γ ∈ (γ^∗(ε, c), \frac{d}{p} α)$, we have the embedding

$$A^{α,p}_ε([0,1]^d) \hookrightarrow A^{α(1-\frac{d}{p}γ),∞}_ε([0,1]^d).$$

**Proof.** As shown in Lemma 2.3, the assumption $γ^∗(ε, c) < ∞$ ensures that $L := ε^∗ < ∞$. Furthermore, since $γ^∗(ε, c) = γ^∗(ε, c) < \frac{d}{p} α$ we see by definition of $γ^∗$ (see Equation (2.2)) that there exist $γ < \frac{d}{p} α$ (arbitrarily close to $γ^∗(ε, c)$) and $C = C(γ, ε, c) > 0$ satisfying $(c(n))^L · n^{1/L^2} ≤ C · n^γ$ for all $n ∈ \mathbb{N}$. Let $f ∈ A^{α,p}_ε([0,1]^d)$ with $∥f∥_{A^{α,p}_ε} ≤ 1$. By Lemma 2.1 this implies $Γ_{α,p}(f) ≤ 1$, so that for each $m ∈ \mathbb{N}$, we can choose $F_m ∈ Σ_2^m$ satisfying $∥f − F_m∥_{L^p} ≤ 2 · 2^{−αm}$. Since furthermore $∥f∥_{L^p} ≤ 1$, this remains true for $m = 0$ if we set $F_0 := 0$. Note that there exists a constant $C_1 = C_1(p) > 0$ satisfying

$$∥F_{m+1} − F_m∥_{L^p} ≤ C_1 · (∥F_{m+1} − f∥_{L^p} + ∥f − F_m∥_{L^p}) ≤ C_1 · (2 · 2^{−α(m+1)} + 2 · 2^{−αm}) ≤ 4C_1 · 2^{−αm}.$$  

Furthermore, note as a consequence of Lemma 3.2 and because of $ε(n) ≤ L$ for all $n ∈ \mathbb{N}$ that

$$Lip(F_{m+1} − F_m) ≤ Lip(F_{m+1}) + Lip(F_m) ≤ d · (c(2m+1))L · (2m+1)^{L/2} + |c(2m)|L · (2m)^{L/2}) ≤ d · ((2^{m+1})^γ + (2^m)^γ) ≤ C_2 · 2^{γm}$$

for a suitable constant $C_2 = C_2(d, γ, ε, c) > 0$. Now, set $θ := \frac{d}{p+α}(γ + α) > 0$. Then, for each $m ∈ \mathbb{N}$, we can apply Lemma 3.1 with $T = T_m := 2^{−αθm} ∈ [0, 1]$ to deduce for a suitable constant $C_3 = C_3(d, p) > 0$ that

$$∥F_{m+1} − F_m∥_{L^∞} ≤ C_3 · (T_m^{−d/p} · ∥F_{m+1} − F_m∥_{L^p} + T_m · Lip(F_{m+1} − F_m)) ≤ C_3 · (4C_1 · 2^{θm} · 2^{−αm} + C_2 · 2^{−θm} · 2^{γm}) ≤ C_4 · 2^{−μm},$$

for a suitable constant $C_4 = C_4(d, p, γ, ε, c) > 0$ and $μ := \frac{p}{d+α}(γ + α) > 0$. Therefore, we see for $M ≥ m ≥ m_0$ that

$$∥F_M − F_m∥_{L^∞} ≤ ∑_{ℓ=m}^{M−1} ∥F_{ℓ+1} − F_ℓ∥_{L^∞} ≤ C_4 · ∑_{ℓ=m_0}^∞ 2^{−μℓ} → 0,$$  

showing that $(F_m)_{m ∈ \mathbb{N}} ⊂ C([0,1]^d)$ is a Cauchy sequence. Since $F_m → f$ in $L^p$, this implies (after possibly redefining $f$ on a null-set) that $f ∈ C([0,1]^d)$ and $F_m → f$ uniformly. Finally, we see

$$∥f∥_{L^∞} = \lim_{m→∞} ∥F_m∥_{L^∞} ≤ \lim_{m→∞} ∑_{ℓ=0}^{m−1} ∥F_{ℓ+1} − F_ℓ∥_{L^∞} ≤ C_4 · ∑_{ℓ=0}^∞ 2^{−μℓ} ≤ C_5 = C_5(d, p, γ, α, ε, c).$$

Since this holds for all $f ∈ A^{α,p}_ε([0,1]^d)$ with $∥f∥_{A^{α,p}_ε} ≤ 1$, we have shown $A^{α,p}_ε([0,1]^d) \hookrightarrow C([0,1]^d)$. 

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We complete the proof by showing \( A^{\alpha, p}_{\ell, c}([0, 1]^d) \hookrightarrow A^{\mu, \infty}_{\ell, c}([0, 1]^d) \). By definition of \( \mu \) and since \( \gamma < \frac{\mu}{p} \) can be chosen arbitrarily close to \( \gamma \) (\( \ell, c \)), this implies the second claim of the theorem. To see \( A^{\alpha, p}_{\ell, c}([0, 1]^d) \hookrightarrow A^{\mu, \infty}_{\ell, c}([0, 1]^d) \), note thanks to Equation (3.1) and because of \( F_m \to f \) uniformly that

\[
\|f - F_m\|_{L^\infty} = \lim_{M \to \infty} \|F_M - F_m\|_{L^\infty} \leq C_4 \sum_{\ell = m}^{\infty} 2^{-\mu \ell} \leq \frac{C_4}{1 - 2^{-\mu}} 2^{-\mu m} =: C_6 \cdot 2^{-\mu m},
\]

for a suitable constant \( C_6 = C_6(d, p, \gamma, \alpha, \ell, c) \).

Now, given any \( t \in \mathbb{N}_{\geq 2} \), choose \( m \in \mathbb{N} \) with \( 2^m \leq t \leq 2^{m+1} \). Then, if \( F_m \in \Sigma_{2^m} \subseteq \Sigma_t \), we then see \( t^\mu \|d\|_{\ell, c} \leq t^\mu \|f - F_m\|_{L^\infty} \leq 2^{(m+1)C_6 \cdot 2^{-\mu m}} = 2^m C_6 \). For \( t = 1 \), we have \( t^\mu \|d\|_{\ell, c} \leq \|f\|_{L^\infty} \leq C_5 \); see Equation (3.2). Overall, setting \( C_7 := \max\{1, 2^\mu C_6, C_5\} \) and recalling Lemma 3.1 we see \( \Gamma_{2^\mu} (C_7^{-1} f) \leq 1 \) and hence \( \|f\|_{A^{\mu, \infty}_{\ell, c}} \leq 1 \), meaning \( \|f\|_{A^{\mu, \infty}_{\ell, c}} \leq C_7 \).

All in all, this shows that indeed \( A^{\alpha, p}_{\ell, c}([0, 1]^d) \hookrightarrow A^{\mu, \infty}_{\ell, c}([0, 1]^d) \).

### 3.2 Sufficient Conditions for Embedding into \( C^{0, \beta}([0, 1]^d) \)

Our next goal is to prove a sufficient condition for the embedding \( A^{\alpha, p}_{\ell, c}([0, 1]^d) \hookrightarrow C^{0, \beta}([0, 1]^d) \). We first introduce some notation. For \( d \in \mathbb{N} \), \( \beta \in (0, 1] \) and \( f : [0, 1]^d \to \mathbb{R} \) we define

\[
\text{Lip}_\beta(f) := \sup_{x, y \in [0, 1]^d, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{1 - \beta}} \in [0, \infty].
\]

Then, the Hölder space \( C^{0, \beta}([0, 1]^d) \) consists of all (necessarily continuous) functions \( f : [0, 1]^d \to \mathbb{R} \) satisfying

\[
\|f\|_{C^{0, \beta}} := \|f\|_{L^\infty} + \text{Lip}_\beta(f) < \infty.
\]

It is well-known that \( C^{0, \beta}([0, 1]^d) \) is a Banach space with this norm.

The proof of our sufficient embedding condition relies on the following analogue of Lemma 3.1

**Lemma 3.4.** Let \( p \in (0, \infty], \beta \in (0, 1) \), and \( d \in \mathbb{N} \). Then there exists a constant \( C = C(d, p, \beta) > 0 \) such that for each \( 0 < T \leq 1 \) and each \( f \in L^p([0, 1]^d) \cap C^{0,1}([0, 1]^d) \), we have

\[
\text{Lip}_\beta(f) \leq C \cdot (T^{-\frac{\beta}{p}} \|f\|_{L^p} + T^{1 - \beta} \text{Lip}(f)).
\]

**Proof.** For \( x, y \in [0, 1]^d \), we have \(|f(x) - f(y)| \leq (2 \|f\|_{L^\infty})^{1 - \beta} \) and hence

\[
|f(x) - f(y)| = |f(x) - f(y)|^{1 - \beta} \cdot |f(x) - f(y)|^{1 - \beta} \leq 2^{1 - \beta} \|f\|_{L^\infty}^{1 - \beta} \cdot (\text{Lip}(f)) \cdot |x - y|^{1 - \beta},
\]

and this easily implies \( \text{Lip}_\beta(f) \leq 2^{1 - \beta} \cdot \|f\|_{L^\infty}^{1 - \beta} \cdot (\text{Lip}(f))^{\beta} \).

Let us first consider the case \( p < \infty \). In this case, we combine the preceding bound with the estimate of \( \|f\|_{L^\infty} \) provided by Lemma 3.1 to obtain a constant \( C_1 = C_1(d, p) > 0 \) satisfying

\[
\text{Lip}_\beta(f) \leq 2^{1 - \beta} C_1^{1 - \beta} \cdot (T^{-\frac{\beta}{p}} \|f\|_{L^p} + T \cdot \text{Lip}(f))^{1 - \beta} \cdot (\text{Lip}(f))^{\beta} \leq 2^{1 - \beta} C_1^{1 - \beta} \cdot (T^{-\frac{\beta}{p}} \|f\|_{L^p} + T \cdot \text{Lip}(f))^{\beta} + T^{1 - \beta} \cdot \text{Lip}(f)).
\]

The second step used the elementary estimate \((a + b)^\theta \leq a^\theta + b^\theta \) for \( \theta \in (0, 1) \) and \( a, b \geq 0 \).

The preceding estimate shows that it suffices to prove that

\[
T^{-\beta} \|f\|_{L^p} \cdot (\text{Lip}(f))^{\beta} \leq C_2 \cdot (T^{-\frac{\beta}{p}} \|f\|_{L^p} + T \cdot \text{Lip}(f))
\]

for a suitable constant \( C_2 = C_2(d, p, \beta) > 0 \). To prove the latter estimate, note for \( a, b \geq 0 \) and \( \theta \in (0, 1) \) that

\[
a^\theta \cdot b^{1 - \theta} = \exp(\theta \ln a + (1 - \theta) \ln b) \leq \theta \exp(\ln a) + (1 - \theta) \exp(\ln b) = \theta a + (1 - \theta) b, \quad (3.3)
\]
thanks to the convexity of $\exp$. The preceding estimate clearly remains valid if $a = 0$ or $b = 0$. Applying this with $\theta = \beta$, $a = T^{1-\beta} \Lip(f)$, and $b = T^{-\frac{d}{p}}\|f\|_{L^p}$, we thus obtain the needed estimate

$$T^{-(1-\beta)\frac{d}{p}} \cdot \|f\|_{L^p}^{-\beta} \cdot (\Lip(f))^\beta = (T^{1-\beta} \Lip(f))^\beta \cdot (T^{-\frac{d}{p}} \|f\|_{L^p})^{1-\beta} \leq \beta T^{1-\beta} \Lip(f) + (1-\beta) T^{-\frac{d}{p}} \|f\|_{L^p} \leq T^{1-\beta} \Lip(f) + T^{-\frac{d}{p}} \|f\|_{L^p}.$$ 

This completes the proof in the case $p < \infty$.

In case of $p = \infty$, we combine the estimate from the beginning of the proof with Equation (3.3) to obtain

$$\Lip_\beta(f) \leq 2^{1-\beta} \cdot \|f\|_{L^\infty}^{1-\beta} \cdot (\Lip(f))^\beta = 2^{1-\beta} \cdot (T^{-\beta} \|f\|_{L^\infty})^{1-\beta} \cdot (T^{1-\beta} \Lip(f))^\beta \leq 2^{1-\beta} \cdot \left((1-\beta)T^{-\beta}\|f\|_{L^\infty} + \beta T^{1-\beta} \Lip(f)\right) \leq 2^{1-\beta} \cdot (T^{1-\beta} \Lip(f)),$$

where the last step used that $p = \infty$. \hfill \Box

We now prove our sufficient criterion for the embedding $A^{\alpha,p}_{\ell,c}([0,1]^d) \hookrightarrow C^{0,\beta}([0,1]^d)$.

**Theorem 3.5.** Let $d \in \mathbb{N}$, $\beta \in (0,1)$, and $p \in (0,\infty]$, and suppose that $\gamma^*(\ell, c) < \infty$. If

$$\alpha > \frac{\beta + \frac{d}{p}}{1 - \beta} \cdot \gamma^*(\ell, c),$$

then $A^{\alpha,p}_{\ell,c}([0,1]^d) \hookrightarrow C^{0,\beta}([0,1]^d)$.

**Proof.** The assumption on $\alpha$ implies in particular that $\alpha > \frac{d}{p} \gamma^*(\ell, c)$. Hence, Theorem 3.3 shows in case of $p < \infty$ that $A^{\alpha,p}_{\ell,c}([0,1]^d) \hookrightarrow C([0,1]^d)$, and this embedding trivially holds if $p = \infty$. Thus, it remains to show for $f \in A^{\alpha,p}_{\ell,c}([0,1]^d)$ with $\|f\|_{A^{\alpha,p}_{\ell,c}} \leq 1$ that $\Lip_\beta(f) \lesssim 1$.

Let $\nu := (\beta + \frac{d}{p})/(1 - \beta)$. Recall from Lemma 2.3 and the assumptions of the present theorem that $\gamma^*(\ell, c) = \gamma^*(\ell, c) < \alpha/\nu < \infty$ and hence that $L := \ell^* \in \mathbb{N}$. By definition of $\gamma^*(\ell, c)$, we thus see that there exist $\gamma > 0$ and $C = C(\gamma, \ell, c) > 0$ satisfying $\gamma < \alpha/\nu$ and $(e(n))^{\frac{1}{2}} \cdot n^{1/2} \leq C \cdot n^\gamma$ for all $n \in \mathbb{N}$.

Let $f \in A^{\alpha,p}_{\ell,c}([0,1]^d)$ with $\|f\|_{A^{\alpha,p}_{\ell,c}} \leq 1$. By Lemma 2.3 this implies $\Gamma_{\alpha,p}(f) \leq 1$, meaning that for each $m \in \mathbb{N}$ we can choose $F_m \in \Sigma^{\alpha}_{\ell,c}$ satisfying $\|f - F_m\|_{L^p} \leq 2^{-\alpha \mu}$. Since furthermore $\|f\|_{L^p} \leq 1$, this remains true for $m = 0$ if we set $F_0 := 0$. As in the proof of Theorem 3.3 we then obtain constants $C_1 = C_1(p) > 0$ and $C_2 = C_2(d, \gamma, \ell, c) > 0$ satisfying

$$\|F_{m+1} - F_m\|_{L^p} \leq 4C_1 \cdot 2^{-\alpha \mu} \quad \text{and} \quad \Lip(F_{m+1} - F_m) \leq C_2 \cdot 2^\gamma m \quad \forall m \in \mathbb{N}.$$ 

Now, set $\theta := (\alpha + \gamma)/(1 + \frac{d}{p})$. A direct computation shows that

$$\theta \cdot \left(\frac{d}{p} + \beta\right) - \alpha = \gamma - \theta \cdot (1 - \beta) = \frac{1 - \beta}{1 + \frac{d}{p}} \cdot (\nu \gamma - \alpha) =: \mu < 0.$$ 

Thus, applying Lemma 3.4 with $T = 2^{-\beta m}$, we obtain a constant $C_3 = C_3(d, p, \beta) > 0$ satisfying

$$\Lip_\beta(F_{m+1} - F_m) \leq C_3 \cdot (T^{-\beta} \|F_{m+1} - F_m\|_{L^p} + T^{1-\beta} \Lip(F_{m+1} - F_m)) \leq C_3 \cdot (4C_1 \cdot 2^m(\theta + \beta) - \alpha) + C_2 \cdot 2^m(\gamma - (1 - \beta))) \leq C_3 \cdot (4C_1 + C_2) \cdot 2^m =: C_4 \cdot 2^m.$$ 

Next, note for $g \in C^{0,\beta}([0,1]^d)$ and $x, y \in [0,1]^d$ that

$$|g(x)| \leq |g(x) - g(y)| + |g(y)| \leq \Lip_\beta(g) \cdot |x - y|_\infty + |g(y)| \leq \Lip_\beta(g) + |g(y)|.$$
Taking the $L^p$-norm (with respect to $y$) of this estimate shows that
\[ |g(x)| = \|g(x)\|_{L^p([0,1]^d,dy)} \leq \|\text{Lip}_\beta(g) + |g(y)|\|_{L^p([0,1]^d,dy)} \leq C_5 \cdot (\text{Lip}_\beta(g) + \|g\|_{L^p}) \quad \forall x \in [0,1]^d, \]
for a suitable constant $C_5 = C_5(p) > 0$. Setting $\sigma := \max\{\mu, -\alpha\} < 0$ and applying the preceding estimate to $g = F_{m+1} - F_m$, we see for a suitable constant $C_6 = C_6(d, p, \beta) > 0$ that $\|F_{m+1} - F_m\|_{\sup} \leq C_6 \cdot 2^{\sigma m}$ for all $m \in \mathbb{N}_0$. Hence, the series $F := \sum_{m=0}^{\infty} (F_{m+1} - F_m)$ converges uniformly, so that we get $F \in C^{\alpha,\beta}([0,1]^d)$ with
\[ \text{Lip}_\beta(F) \leq \sum_{m=0}^{\infty} \text{Lip}_\beta(F_{m+1} - F_m) \leq C_4 \cdot \sum_{m=0}^{\infty} 2^{\mu m} = \frac{C_4}{1 - 2^\mu} =: C_7 < \infty. \]

Since $\sum_{m=0}^{N} (F_{m+1} - F_m) = F_{N+1} \to f$ with convergence in $L^p([0,1]^d)$ as $N \to \infty$, we see that $F = f$ almost everywhere, and hence (after changing $f$ on a null-set) $f \in C^{0,\beta}([0,1]^d)$ with $\text{Lip}_\beta(f) \leq C_7$. As seen above, this completes the proof. \hfill \square

### 3.3 Necessary Conditions

In this section we complement our sufficient conditions by corresponding necessary conditions. Our main result reads as follows.

**Theorem 3.6.** Let $p \in (0, \infty]$, $\beta \in (0, 1]$, and $d \in \mathbb{N}$. Then:

- If $p < \infty$ and $\gamma^*(\ell, c) > \alpha/(d/p)$, then the embedding $A_{\ell,c}^{\alpha,p}([0,1]^d) \hookrightarrow C([0,1]^d)$ does not hold.

- If $\gamma^*(\ell, c) > \frac{1-\beta}{\beta+\alpha}$, then the embedding $A_{\ell,c}^{\alpha,p}([0,1]^d) \hookrightarrow C^{0,\beta}([0,1]^d)$ does not hold.

**Remark.** A combination of the preceding result with Theorem 3.3 provides a complete characterization of the embedding $A_{\ell,c}^{\alpha,p}([0,1]^d) \hookrightarrow C([0,1]^d)$, except in the case where $\gamma^*(\ell, c) = \alpha/(d/p)$.

This remaining case is more subtle and seems to depend on the precise growth of the function $c$.

Likewise, combining the preceding result with Theorem 3.5 provides a complete characterization of the embedding $A_{\ell,c}^{\alpha,p}([0,1]^d) \hookrightarrow C^{0,\beta}([0,1]^d)$, except in the “critical” case where $\gamma^*(\ell, c) = \frac{1-\beta}{\beta+\alpha}$.

**Proof.** The proof is divided into several steps, where the first two steps are concerned with constructing and analyzing a certain auxiliary function $\zeta_M$ which is then used to construct counterexamples to the considered embeddings.

**Step 1:** Given $M' \geq 0$, define
\[ \zeta_{M'} : \quad \mathbb{R}^d \to \mathbb{R}, \quad x \mapsto p \left(1 - M' \sum_{i=1}^{d} x_i\right). \]

In this step, we show for any $n \in \mathbb{N}$, $0 \leq C \leq c(n)$, $L \in \mathbb{N}_{\geq 2}$ with $L \leq \ell(n)$ and any $M' \geq 1$ and $0 \leq M \leq C^{L,n}[L/2]$ that
\[ \frac{M}{M'} \cdot \zeta_{M'} \in \Sigma_{(d+L)n}. \quad (3.4) \]

To prove this, set $\mu := C^L n[L/2]$ and $D := C \cdot (1) \in \mathbb{R}^{1 \times 1}$, as well as
\[ b_1 := \frac{C}{M'} \cdot (1, \ldots, 1)^T \in \mathbb{R}^n, \quad A := C \cdot (1, \ldots, 1) \in \mathbb{R}^{1 \times n}, \quad B := C \cdot (1, \ldots, 1)^T \in \mathbb{R}^{n \times 1}, \]
and finally
\[ A_1 := -C \cdot \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \in \mathbb{R}^{n \times d}. \]
Note because of \( M' \geq 1 \) that \( \|b_1\|_\infty, \|A\|_\infty, \|B\|_\infty, \|D\|_\infty, \|A_1\|_\infty \leq C \leq c(n) \leq c((L + d)n) \). Now, we distinguish two cases.

**Case 1 (\( L \geq 2 \) is even):** In this case, define
\[
\Phi := \{(A_1, b_1), (A, 0), (B, 0), (A, 0), \ldots, (B, 0), (A, 0)\}.
\]

Note that \( W(\Phi) \leq dn + n + n + 2\frac{L}{2}(n + n) \leq (d + L)n \) and \( L(\Phi) = L \leq \ell(n) \leq L((d + L)n) \), showing that \( R_\ell \Phi \in \Sigma^{\ell, c}_{(d+L)n} \). Because of \( A(g(Bx)) = C \sum_{j=1}^n g(Cx) = C^2 n \cdot g(x) \) for \( x \in \mathbb{R} \) and because of
\[
A \cdot g(A_1x + b_1) = C \sum_{j=1}^n g(Cx) = C^2 n \cdot g(1 - M' \sum_{i=1}^d x_i) = \frac{C^2 n}{M'} \cdot \zeta_M(x)
\]
for \( x \in \mathbb{R}^d \), it follows that \( R_\ell \Phi = C^{L-2} \cdot n^{L-2} \cdot \frac{C^2 n}{M'} \cdot \zeta_M \). Since \( \sum_{(d+L)n}^{\ell, c} \subset \Sigma^{\ell, c}_{(d+L)n} \) for all \( c \in [-1, 1] \) and because of \( 0 \leq M \leq \mu \), this proves Equation (33) in this case.

**Case 2 (\( L \geq 3 \) is odd):** In this case, define
\[
\Phi := \{(A_1, b_1), (A, 0), (B, 0), (A, 0), \ldots, (B, 0), (A, 0), (D, 0)\}.
\]

Then \( W(\Phi) \leq dn + n + n + \frac{L}{2}(n + n) + 1 \leq (d + L)n \) and \( L(\Phi) = L \leq \ell(n) \leq L((d + L)n) \), so that \( R_\ell \Phi \in \Sigma^{\ell, c}_{(d+L)n} \). Using essentially the same arguments as in the previous case, we see \( R_\ell \Phi = \frac{C^2 n}{M'} \cdot \zeta_M \), which then implies that Equation (33) holds.

**Step 2:** We show \( \|\zeta_M\|_{L_{\ell}^p([0,1]^d)} \leq (M^{'})^{-d/p} \). To see this, note for \( x \in [0,1]^d \) that \( \sum_{i=1}^d x_i \geq 0 \) and hence \( 1 - M' \sum_{i=1}^d x_i \leq 1 \), showing that \( 0 \leq \zeta_M(x) = g(1 - M' \sum_{i=1}^d x_i) \leq g(1) = 1 \). Next, note that if \( 0 \neq \zeta_M(x) = g(1 - M' \sum_{i=1}^d x_i) \), then \( 1 - M' \sum_{i=1}^d x_i > 0 \), which is only possible if \( x_j \sum_{i=1}^d x_i \leq \frac{1}{M'} \) for all \( j \in \{1, \ldots, d\} \), that is, if \( x \in \frac{1}{M'}[0,1]^d \). Therefore, we see as claimed that
\[
\|\zeta_M\|_{L_{\ell}^p([0,1]^d)} \leq \left( \lambda \left( \frac{1}{M'} \right) \right)^{1/p} = (M^{'})^{-d/p}.
\]

**Step 3:** In this step, we prove the first part of the theorem. By the assumptions of that part, we have \( p < \infty \) and \( \gamma^*(\ell, c) > \alpha/(d/p) \). By definition of \( \gamma^* \), this implies that there exists \( \gamma > \alpha/(d/p) \) and some \( L \in \mathbb{N} \) satisfying
\[
c_0 := \limsup_{n \to \infty} \left[ (c(n))^L \cdot n^{L/2} \right]^{-\gamma} \in (0, \infty).
\]

Since \( \ell^* \geq 2 \), we can clearly assume that \( L \geq 2 \). Choose a subsequence \( (n_k)_k \in \mathbb{N} \) such that \( c_0 = \lim_{n \to \infty} \left[ (c(n_k))^L \cdot n_k^{L/2} \right]^{-\gamma} \). Then, setting \( c := \min\{1, c_0/2\} \), there exists \( k_0 \in \mathbb{N} \) satisfying \( (c(n_k))^L \cdot n_k^{L/2} \geq c \cdot n_k^L > 0 \) and \( L \leq \ell(n_k) \) for all \( k \in \mathbb{N} \).

Choose \( k \in (0,1] \) satisfying \( k \leq c \) and \( k \cdot (d + L)^\alpha \leq 1 \). Furthermore, define
\[
\theta := \frac{1}{2} (\gamma + 2p) \in \left( \frac{2p}{\alpha}, \gamma \right) \text{ and } \beta := \theta + \min\{\delta_1, \delta_2\}, \text{ where } \delta_1 := \frac{d}{p} - \alpha > 0 \text{ and } \delta_2 := \gamma - \theta > 0.
\]

and choose \( k_1 \in \mathbb{N} \) satisfying \( n_k^\beta \leq \kappa \cdot n_k^\beta \) for all \( k \geq k_1 \).

We now claim for all \( k \geq \max\{k_0, k_1\} \) that if we set \( M := \kappa \cdot n_k^\beta \) and \( M' := n_k^\beta \), then
\[
\| M \cdot \zeta_M \|_{L_{\ell}^p([0,1]^d)} \leq 1. \]

This will prove the claim, since \( \| M \cdot \zeta_M \|_{C([0,1]^d)} \geq \frac{M}{M'} \to \infty \) as \( k \to \infty \); here, we used that \( \zeta_M(0) = 1 \) and that \( \beta > \theta \).

To see that \( \| M \cdot \zeta_M \|_{L_{\ell}^p([0,1]^d)} \leq 1, \) note that \( \beta \leq \theta + \delta_2 = \gamma \) and hence
\[
M = \kappa \cdot n_k^\beta \leq c \cdot n_k^\beta \leq (c(n_k))^L \cdot n_k^{L/2},
\]

so that \( M \leq C L n_k^{|f|} \) for a suitable \( 0 \leq C \leq c(n_k) \). Thus, Equation (3.3) shows \( t^\alpha \zeta_{M'} \in \Sigma_{(d+L)n_k}^{\ell^e} \).

Now, for \( t \in \mathbb{N} \) there are two cases: If \( t \geq (d+L) \cdot n_k \), then \( t^\alpha \zeta_{M'}(M', \Sigma_{t}^{\ell^e}) \leq 0 \leq 1 \). If otherwise \( t < (d+L) \cdot n_k \), then the estimate from Step 2 combined with our choice of \( M, M' \) and \( \kappa \) shows that

\[
\begin{align*}
t^\alpha \zeta_{M'}(M', \Sigma_{t}^{\ell^e}) &\leq t^\alpha \frac{M}{M'} \| \zeta_{M'} \|_{L^p([0,1]^d)} \leq \kappa \cdot (d+L)^\alpha \cdot n_k^{\alpha - \theta - \theta _{\frac{\beta}{p} + \beta} + \delta_1} \leq n_k^{\alpha - \theta - \theta _{\frac{\beta}{p} + \beta} + \delta_1} = 1.
\end{align*}
\]

Finally, we also have \( \frac{M}{M'} \zeta_{M'} \|_{L^p([0,1]^d)} \leq \frac{M}{M'} \cdot (M')^{-d/p} \leq \kappa \cdot n_k^{\beta - \theta - \theta _{\frac{\beta}{p} + \beta}} \leq n_k^{\theta + \delta_1 - \theta - \theta _{\frac{\beta}{p} + \beta}} = n_k^{-\alpha} \leq 1 \).

Overall, this shows \( \Gamma_{\alpha, p}(\frac{M}{M'} \zeta_{M'} \|_{A_{\alpha, p}^{\ell^e}}) \leq 1 \), so that Lemma 2.1 shows \( \| \frac{M}{M'} \zeta_{M'} \|_{A_{\alpha, p}^{\ell^e}} \leq 1 \) as well. As seen above, this completes the proof of the first part of the theorem.

**Step 4:** In this step, we prove the second part of the theorem. By the assumptions of that part, we have \( \gamma^*(\ell, c) > \alpha \frac{1 - \beta}{\beta + \frac{\beta}{p}} = \frac{\delta}{\bar{p}} \) for \( \nu := \beta + \frac{\beta}{p} \). By definition of \( \gamma^* \), we can thus choose \( \gamma > \frac{\delta}{\bar{p}} \) and \( L \in \mathbb{N} \leq \ell^e \) satisfying

\[
c_0 := \limsup \sup_{n \to \infty} \left( (c(n_k))^L \cdot n_k^{L/2} - \gamma \right) \in (0, \infty).
\]

Since \( \ell^e \geq 2 \), we can clearly assume that \( L \geq 2 \). Choose a subsequence \( (n_k)_{k \in \mathbb{N}} \) such that

\[
c_0 = \lim_{k \to \infty} (c(n_k))^L \cdot n_k^{L/2} - \gamma.
\]

Then, setting \( c := \min \{1, c_0/2\} \), there exists \( k_0 \in \mathbb{N} \) satisfying \( (c(n_k))^L \cdot n_k^{L/2} \geq c \cdot n_k^2 \geq 0 \) and \( L \leq \ell(n_k) \) for all \( k \in \mathbb{N} \geq k_0 \).

Since \( \gamma > \frac{\delta}{\bar{p}} = \frac{1 - \beta}{\beta + \frac{\beta}{p}} \), we have \( \frac{\delta}{\bar{p}} < \frac{\gamma}{\alpha} \). Hence, we can choose \( \tau \in \left( \frac{\delta}{\bar{p}} \cdot \frac{\gamma}{\alpha} \right) \subset (0, \infty) \).

Then \( \tau \cdot (1 - \beta) < \gamma \) and furthermore

\[
\tau \cdot \beta > \alpha \quad \Longrightarrow \quad \tau \cdot (1 - \beta - 1 - \frac{\beta}{p}) < -\alpha \quad \Longrightarrow \quad \tau \cdot (1 - \beta) < \tau \cdot (1 + \frac{\beta}{p}) - \alpha.
\]

Hence, we can choose \( \theta \in (\tau \cdot (1 - \beta), \min \{\gamma, \tau \cdot (1 + \frac{\beta}{p}) - \alpha\}) \subset (0, \infty) \).

Now, choose \( k \in (0, 1] \) such that \( k \leq c \) and \( k \cdot (L + d)^\beta \leq 1 \). Fix \( k \in \mathbb{N} \geq k_0 \) for the moment, and set \( M := k \cdot n_k^\beta \) and \( M' := n_k^\beta \geq 1 \). Since \( (c(n_k))^L \cdot n_k^{L/2} \geq c \cdot n_k^2 \geq k \cdot n_k^\beta \), we can choose \( 0 \leq C \leq c(n_k) \) satisfying \( C \cdot n_k^{L/2} \geq k \cdot n_k^\beta \geq k \cdot n_k^\beta = M \).

Then, Step 1 shows that \( f := M \cdot \zeta_{M'} \in \Sigma_{(L+d)n_k}^{\ell^e} \).

Now, for \( t \in \mathbb{N} \) there are two cases: If \( t \geq (L+d) \cdot n_k \), then \( t^\alpha \cdot \zeta_{M'}(\Sigma_{t}^{\ell^e}) \leq 0 \leq 1 \). If otherwise \( t < (L+d) \cdot n_k \), then the estimate from Step 2 combined with the bound \( \theta \leq \tau \cdot (1 + \frac{\beta}{p}) - \alpha \) shows

\[
\begin{align*}
t^\alpha \cdot \zeta_{M'}(\Sigma_{t}^{\ell^e}) &\leq (L+d)^\alpha \cdot n_k^\alpha \cdot \| f \|_{L^p} \\
&\leq (L+d)^\alpha \cdot n_k^\alpha \cdot \frac{M}{M'} \cdot (M')^{-d/p} \\
&\leq (L+d)^\alpha \cdot n_k^\alpha \cdot \kappa \cdot n_k^\beta \\
&\leq n_k^{\alpha + \gamma \cdot (1 + \frac{\beta}{p})} \leq n_k^{-\alpha - \theta - \theta _{\frac{\beta}{p} + \beta} + \delta_1} = 1.
\end{align*}
\]

Furthermore, Step 2 also shows that

\[
\| f \|_{L^p} = \frac{M}{M'} \cdot \zeta_{M'} \|_{L^p} \leq k \cdot n_k^{\theta - \tau} \cdot (M')^{-d/p} \leq n_k^{\theta - \tau - \theta _{\frac{\beta}{p} + \beta}} \leq n_k^{\theta - \tau - \theta _{\frac{\beta}{p} + \beta}} = n_k^{-\alpha} \leq 1.
\]

Overall, we see that \( \Gamma_{\alpha, p}(f) \leq 1 \), so that Lemma 2.1 shows \( \| f \|_{A_{\alpha, p}^{\ell^e}} \leq 1 \).

We now show that \( \| f \|_{C^0, \beta} \to \infty \) as \( k \to \infty \); since \( \| f \|_{A_{\alpha, p}^{\ell^e}} \leq 1 \), this will imply that the embedding \( A_{\alpha, p}^{\ell^e}([0,1]^d) \hookrightarrow C^0, \beta([0,1]^d) \) does not hold. To see that \( \| f \|_{C^0, \beta} \to \infty \) as \( k \to \infty \), note that \( f(0) = 0 \) and \( f(\frac{1}{M}, 0, \ldots, 0) = 0 \). Hence,

\[
\| f \|_{C^0, \beta} \geq L \| f \|_{L^p, \beta}(f) \geq \frac{\| f(0) - f(\frac{1}{M}, 0, \ldots, 0) \|_{C^0, \beta}^2}{\| 0 - (\frac{1}{M}, 0, \ldots, 0) \|_{C^0, \beta}^2} \geq (M')^{-\beta} \frac{M}{M'} \cdot n_k^{\alpha} \cdot \kappa \cdot n_k^\beta = n_k^{\alpha - \theta - \theta _{\frac{\beta}{p} + \beta}} \cdot \kappa \cdot n_k^{\theta - \tau - (1 - \beta)} \cdot \kappa \cdot n_k^\beta \to \infty, \quad k \to \infty.
\]

since \( \theta > \tau \cdot (1 - \beta) \).
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