Fractional Generalizations of Gradient Mechanics

E. C. Aifantis

Aristotle University of Thessaloniki, Thessaloniki, 54124, Greece
Michigan Technological University, Houghton, MI 49931, USA
Togliatti State University, Togliatti 445020, Russia

mom@mom.gen.auth.gr, ORCID: 0000-0002-6846-5686, Tel.:+30-2310995921

Abstract

This short chapter provides a fractional generalization of gradient mechanics, an approach (originally advanced by the author in the mid 80s) that has gained world-wide attention in the last decades due to its capability of modeling pattern forming instabilities and size effects in materials, as well as eliminating undesired elastic singularities. It is based on the incorporation of higher-order gradients (in the form of Laplacians) in the classical constitutive equations multiplied by appropriate internal lengths accounting for the geometry/topology of underlying micro/nano structures. This review will focus on the fractional generalization of the gradient elasticity equations (GradEla) an extension of classical elasticity to incorporate the Laplacian of Hookean stress by replacing the standard Laplacian by its fractional counterpart. On introducing the resulting fractional constitutive equation into the classical static equilibrium equation for the stress, a fractional differential equation is obtained whose fundamental solutions are derived by using the Greens function procedure. As an example, Kelvins problem is analyzed within the aforementioned setting. Then, an extension to consider constitutive equations for a restrictive class of nonlinear elastic deformations and deformation theory of plasticity is pursued. Finally, the methodology is applied for extending the authors higher-order diffusion theory from the integer to the fractional case.
1 Introduction

This contribution concerns the fractional generalization of the authors gradient elasticity and higher-order diffusion. Both of these theories were introduced three decades ago to model deformation and transport problems in media with micro/nanostructures. A new Laplacian term was added in the standard constitutive equations of Hookean elasticity and Fickean diffusion to interpret experimental data that could not be modeled by classical theories. Among the new results were the elimination of undesirable elastic singularities in dislocation lines and crack tips, as well as new robust continuum models for grain boundary diffusion. All these problems were successfully and efficiently addressed by incorporating internal lengths in the standard constitutive equations of elasticity and diffusion, as scalar multipliers of newly introduced Laplacian terms of the persistent constitutive variables to account for weakly nonlocal effects. The resulting internal length gradient (ILG) framework and its applications to various areas of material mechanics are reviewed in a recent article by the author [1], where extensive bibliography can also be found. In the same article a brief account of fractional and fractal generalization of the ILG framework is given.

We expand on the aforementioned review by providing an updated summary of the fractional generalization of the ILG framework focusing on static elasticity with a few related remarks on plasticity and steady-state diffusion. In that connection, it is noted that the basic balance laws for the mass and momentum are assumed to retain their classical (integer) form. For stationary deformation and steady-state diffusion problems these laws lead to the standard balance equations

\[ \text{div}\sigma = 0 \quad \text{or} \quad \sigma_{ij,j} = 0, \]  
\[ (1.1) \]

for the stress tensor and

\[ \text{div}\mathbf{j} = 0 \quad \text{or} \quad j_{i,i} = 0, \]  
\[ (1.2) \]

for the diffusive flux vector.

The standard constitutive equations of the ILG framework for $\sigma$ and $\mathbf{j}$ are of the form
\[ \sigma = \lambda (\text{tr} \varepsilon) \mathbf{1} + 2\mu \varepsilon - l_\varepsilon^2 \nabla^2 \lambda (\text{tr} \varepsilon) \mathbf{1} + 2\mu \varepsilon; \]
\[ \sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} - l_\varepsilon^2 \nabla^2 \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}, \tag{1.3} \]

and

\[ j = -D \nabla (\rho - l_d^2 \nabla^2 \rho); \quad j_i = -D (\rho - l_d^2 \nabla^2 \rho)_i, \tag{1.4} \]

respectively. The classical elastic moduli \((\lambda, \mu)\) and diffusivity \((D)\) have their usual meaning, the quantities \(\varepsilon_{ij}\) and \(\rho\) denote strain and concentration respectively, while the newly introduced parameters \(l_\varepsilon\) and \(l_d\) are deformation–induced and diffusion-induced internal lengths (ILs) accounting for weakly nonlocal effects.

The fractional generalization of the above equations consists of replacing the standard (integer) Laplacian \(\Delta\) in Eqs. (1.3) and (1.4) with a fractional one of the Riesz form \((-R\Delta)^{\alpha/2}\) or the Caputo form \(C\Delta_W^\alpha\). Then, the corresponding fractional generalization of Eq. (1.3) reads

\[ \sigma_{ij} = (\lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}) - l_\varepsilon^2 (\alpha) (-R\Delta)^{\alpha/2} [\lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}], \tag{1.5} \]

where \((-R\Delta)^{\alpha/2}\) is the fractional generalization of the Laplacian in the Riesz form, and

\[ \sigma_{ij} = (\lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}) - l_\varepsilon^2 (\alpha) C\Delta_W^\alpha [\lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}], \tag{1.6} \]

where \(C\Delta_W^\alpha\) is the fractional Laplacian in the Caputo form \([3]\).

Equations (1.5) and (1.6) are fractional generalizations of the original GradEla model. Similar equations can be written down for the fractional generalization of Eq. (1.4). They read

\[ j = -D \nabla [\rho - l_d^2 (\alpha) \{(-R\Delta)^{\alpha/2} \rho\}]; \quad j_i = -D [\rho - l_d^2 (\alpha) \{(-R\Delta)^{\alpha/2} \rho\}]_i, \tag{1.7} \]

and

\[ j = -D \nabla [\rho - l_d^2 (\alpha) \{C\Delta_W^\alpha \rho\}]; \quad j_i = -D [\rho - l_d^2 (\alpha) \{C\Delta_W^\alpha \rho\}]_i, \tag{1.8} \]

respectively.
On introducing the aforementioned fractional gradient constitutive equations into the non-fractional balance laws given by Eqs. (1.1) and (1.2), the corresponding partial differential equations of fractional order are obtained which need to be solved with the aid of appropriate boundary conditions for finite domains. To dispense with the complication of higher-order fractional boundary conditions, we consider infinite domains and derive fundamental solutions for the respective problems by employing a fractional extension of the Greens function method. The above is illustrated in detail in the next section (Section 2) by considering the classical Thomson (Lord Kelvin) elasticity problem in the framework of fractional GradEla. In Section 3 we give a brief account on preliminaries of fractional gradient nonlinear elasticity or deformation theory of plasticity. We note that both of these sections are an update of the fractional considerations of [1] based on the detailed elaborations contained in the initial articles by Tarasov and the author [2],[3], as well as subsequent further discussions by Tarasov [4]–[8]. In Section 4 we return to the problem of fractional Laplacian and obtain fundamental solutions for a general fractional equation of Helmholtz type which turns out to govern both gradient elasticity and higher-order diffusion theory. Since the basics of fractional deformation have been outlined in Sections 2 and 3, we show in Section 4 how these basic results are directly applicable to fractional diffusion problems. In this connection, it is noted that solutions of fractional GradEla problems are reduced to solutions of an inhomogenous Helmholtz equation, which is also the governing equation of fractional generalization of electrostatics with Debye screening [9].

2 Gradient Elasticity (GradEla): Revisiting Kelvins Problem

To shed light on the implications of fractional GradEla, a specific 3D configuration with spherical symmetry is considered below. The model of Eq. (1.3) is employed due to the fact that definite results are available for the fractional Laplacian of Riesz type. The corresponding most general fractional GradEla governing equation is of the form [2]

\[ c_\alpha ((-\Delta)^{\alpha/2}u)(\mathbf{r}) + c_\beta ((-\Delta)^{\beta/2}u)(\mathbf{r}) = f(\mathbf{r}) \quad (\alpha > \beta), \quad (2.1) \]

where \( \mathbf{r} \in \mathbb{R}^3 \) and \( r = |\mathbf{r}| \) are dimensionless, \( (-\Delta)^{\alpha/2} \) is the Riesz fractional...
Laplacian of order $\alpha$, with the same for the symbols characterized by $\beta$, and the (fractional) gradient coefficients $(c_\alpha, c_\beta)$ are material constants related to the elastic moduli and the internal length, respectively. The rest of the symbols have their usual meaning: $u$ denotes displacement and $f(r)$ body force. For $\alpha > 0$ and suitable functions $u(r)$, the Riesz fractional derivative can be defined in terms of the Fourier transform $\mathcal{F}$ by

$$((-\Delta)^{\alpha/2}u)(r) = \mathcal{F}^{-1}(|k|\alpha(\mathcal{F}u)(k)),$$

where $k$ denotes the wave vector. If $\alpha = 4$ and $\beta = 2$, we have the well-known GradEla equation

$$c_2\Delta u(r) - c_4\Delta^2 u(r) + f(r) = 0,$$

Equation (2.3) is a fractional partial differential equation with a solution of the form

$$u(r) = \int_{\mathbb{R}^3} G_{\alpha,\beta}(r - r') f(r') \, d^3r',$$

with the Green-type function $G_{\alpha,\beta}$ given by

$$G_{\alpha,\beta}(r) = \int_{\mathbb{R}^3} \frac{1}{c_\alpha |k|\alpha + c_\beta |k|\beta} e^{ik\cdot r} \, d^3k = \frac{1}{(2\pi)^{3/2} |r|} \int_0^\infty \frac{\lambda^{3/2} J_{1/2}(\lambda|r|)}{c_\alpha \lambda\alpha + c_\beta \lambda\beta} \, d\lambda,$$

where $J_{1/2}(z) = \sqrt{2/\pi z} \sin(z)$ is the Bessel function of the first kind and the dot denotes inner product.

To proceed further, we consider Thomson’s problem of an applied point load $f_0$, i.e.

$$f(r) = f_0 \delta(r) = f_0 \delta(x)\delta(y)\delta(z).$$

Then, the displacement field $u(r)$ has a simple form given by the particular solution

$$u(r) = f_0 G_{\alpha,\beta}(r),$$

with the Green’s function given by Eq. (2.5), i.e.
\[ u(\mathbf{r}) = \frac{f_0}{2\pi^2|\mathbf{r}|} \int_0^\infty \frac{\lambda \sin(\lambda|\mathbf{r}|)}{c_\alpha \lambda^\alpha + c_\beta \lambda^\beta} d\lambda, \quad (\alpha > \beta). \quad (2.8) \]

It turns out that the asymptotic form of the solution given by Eq. (2.8) for \( 0 < \beta < 2 \), and \( \alpha \neq 2 \), reads

\[ u(\mathbf{r}) \approx \frac{f_0 \Gamma(2 - \beta) \sin(\pi\beta/2)}{2\pi^2 c_\beta} \frac{1}{|\mathbf{r}|^{3 - \beta}}, \quad (|\mathbf{r}| \to \infty). \quad (2.9) \]

This asymptotic behavior for \( |\mathbf{r}| \to \infty \) does not depend on the parameter \( \alpha \), and (as will be seen below) the corresponding asymptotic behavior for \( |\mathbf{r}| \to 0 \) does not depend on the parameter \( \beta \), where \( \alpha > \beta \). It follows that the displacement field at large distances from the point of load application is determined only by the term \((-\Delta)^{\beta/2}\), where \( \beta < \alpha \). This can be interpreted as a fractional non-local “deformation” counterpart of the classical elasticity result based on Hooke’s law. We can also note the existence of a maximum for the quantity \( u(\mathbf{r}) \cdot |\mathbf{r}| \) in the case \( 0 < \beta < \alpha < 2 \). Indeed, these observations become clear by considering in detail the following two special cases that emerge.

**A)** Sub-GradEla model: \( \alpha = 2; 0 < \beta < 2 \). In this case Eq. (2.1) becomes

\[ c_2 \Delta u(\mathbf{r}) - c_\beta ((-\Delta)^{\beta/2} u)(\mathbf{r}) + f(\mathbf{r}) = 0, \quad (0 < \beta < 2). \quad (2.10) \]

The order of the fractional Laplacian \((-\Delta)^{\beta/2}\) is less than the order of the first term related to the usual Hooke’s law. For example, one can consider the square of the Laplacian, i.e. \( \beta = 1 \). In general, the parameter \( \beta \) defines the order of the power-law non-locality. The particular solution of Eq. (2.10) in the present case, reads

\[ u(\mathbf{r}) = \frac{f_0}{2\pi^2|\mathbf{r}|} \int_0^\infty \frac{\lambda \sin(\lambda|\mathbf{r}|)}{c_2 \lambda^2 + c_\beta \lambda^\beta} d\lambda, \quad (0 < \beta < 2). \quad (2.11) \]

The following asymptotic behavior for Eq. (2.11) can be derived in the form

\[ u(\mathbf{r}) \approx \frac{f_0 \Gamma(2 - \beta) \sin(\pi\beta/2)}{2\pi^2 c_\beta} \frac{1}{|\mathbf{r}|^{3 - \beta}} + \sum_{k=1}^{\infty} \frac{C_k(\beta)}{|\mathbf{r}|^{(2 - \beta)(k+1)+1}} \quad (|\mathbf{r}| \to \infty), \quad (2.12) \]
where

\[
C_0(\beta) = \frac{f_0 \Gamma(2 - \beta) \sin(\pi \beta/2)}{2\pi^2 c_\beta},
\]

\[
C_k(\beta) = -\frac{f_0c_\beta^k}{2\pi^2 c_\beta^{k+1}} \int_0^\infty z^{(2-\beta)(k+1)-1} \sin(z) \, dz.
\] (2.13)

As a result, the displacement field generated by the force that is applied at a point in the fractional gradient elastic continuum described by the fractional Laplacian \((-\Delta)^{\beta/2}\) with \(0 < \beta < 2\) is given by

\[
u(r) \approx C_0(\beta) |r|^{3-\beta} (0 < \beta < 2),
\] (2.14)

for large distances \(|r| \to \infty\).

**B) Super-GradEla model:** \(\alpha > 2\) and \(\beta = 2\). In this case, Eq. (2.1) becomes

\[
c_2 \Delta u(r) - c_\alpha ((-\Delta)^{\alpha/2} u)(r) + f(r) = 0, \quad (\alpha > 2).
\] (2.15)

The order of the fractional Laplacian \((-\Delta)^{\alpha/2}\) is greater than the order of the first term related to the usual Hooke’s law. The parameter \(\alpha > 2\) defines the order of the power-law non-locality of the elastic continuum. If \(\alpha = 4\), Eq. (2.15) reduces to Eq. (2.3). The case can be viewed as corresponding as closely as possible \((\alpha \approx 4)\) to the usual gradient elasticity model of Eq. (2.3). The asymptotic behavior of the displacement field \(u(r)\) for \(r \to 0\) in the case of super-gradient elasticity is given by

\[
u(r) \approx \frac{f_0 \Gamma((3 - \alpha)/2)}{2^{\alpha} \pi^2 \sqrt{\pi c_\alpha \Gamma(\alpha/2)}} \frac{1}{|r|^{3-\alpha}}, \quad (2 < \alpha < 3),
\]

\[
u(r) \approx \frac{f_0}{2\pi^{3/2}} \frac{c_\alpha^{1-3/\alpha}}{c_\alpha^{3/\alpha}} \frac{1}{3\pi/\alpha} \sin(3\pi/\alpha), \quad (\alpha > 3).
\] (2.16)

Note that the above asymptotic behavior does not depend on the parameter \(\beta\), and that the corresponding relation of Eq. (2.16) does not depend on \(c_\beta\). The displacement field \(u(r)\) for short distances away from the point of load application is determined only by the term with \((-\Delta)^{\alpha/2}, (\alpha > \beta)\), i.e. the fractional counterpart of the usual extra non-Hookean term of gradient elasticity. More details for the above results can be found in [2]–[3]. (Should we add more?)
3 Fractional Gradient Plasticity

In this section we provide an introductory account of fractional deformation theory of plasticity (as opposed ot the flow theory of plasticity) by simply elaborating on a specific constitutive equation which could also be viewed as a very special form of nonlinear elasticity.

The corresponding (nonlinear) fractional gradient constitutive equation involves scalar measures of the stress and strain tensors; i.e. their second invariants, as these quantities enter in both theories of nonlinear elasticity and plasticity. In plasticity theory, in particular, we employ the second invariants of the deviatoric stress and plastic strain tensors. The effective (equivalent) stress $\sigma$ is defined by the equation

$$\sigma = \sqrt{(1/2)\sigma'_{ij}\sigma'_{ij}}; \quad \sigma'_{ij} = \sigma_{ij} - \frac{1}{3}\sigma'_{kk}\delta_{ij},$$

(3.1)

where $\sigma_{ij}$ is the stress tensor. The effective (equivalent) plastic strain is defined as

$$\varepsilon = \int dt \sqrt{2\dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij}},$$

(3.2)

where $\dot{\varepsilon}_{ij}$ is the plastic strain rate tensor, which is assumed to be traceless in order to satisfy plastic incompressibility.

Motivated by the above, we propose the following form of nonlinear fractional differential equation for the scalar quantities $\sigma$ and $\varepsilon$ which can be used as a basis for future tensorial formulation of nonlinear elasticity and plasticity theories

$$\sigma(r) = E \varepsilon(r) + c(\alpha) ((-\Delta)^{\alpha/2}\varepsilon)(r) + \eta K(\varepsilon(r)) + \eta K(\varepsilon(r)), \quad (\alpha > 0),$$

(3.3)

where $K(\varepsilon(r))$ is a nonlinear function, which describes the usual (homogeneous) material's response (linear hardening plasticity); $c(\alpha)$ is an internal parameter, that measures the nonlocal character of deformation mechanisms; $E$ is a shear-like elastic modulus; $\eta$ is a small parameter of nonlinearity; and $(-\Delta)^{\alpha/2}$ is the fractional Laplacian in the Riesz form. As a simple example of the nonlinear function, we can consider

$$K(\varepsilon) = \varepsilon^\beta(r), \quad (\beta > 0).$$

(3.4)
Equation (3.3), where $K(\varepsilon(r))$ is defined by Eq. (3.4), is the fractional Ginzburg–Landau equation. For various choices of the parameters $(E, \eta, \beta)$ characterizing the homogenous material response, different models of nonlinear elastic and plastic behavior may result.

Let us derive a particular solution of Eq. (3.3) with $K(\varepsilon(r)) = 0$. To solve the linear fractional differential equation

$$\sigma(r) = E \varepsilon(r) + c(\alpha) \left[(-\Delta)^{\alpha/2}\varepsilon(r)\right], \quad (3.5)$$

we apply the Fourier method and the fractional Green function method. Using Theorem 5.22 of Kilbas et al. ([10] pages 342 and 343, see also [11]) for the case $E \neq 0$ and $\alpha > (n - 1)/2$, we see that Eq. (3.5) is solvable, and its particular solution is given by the equation

$$\varepsilon(r) = G_{n,\alpha} \ast \sigma = \int_{\mathbb{R}^n} G_{n,\alpha}(r - r') \sigma(r') \, d^n r', \quad (3.6)$$

where the symbol $\ast$ denotes convolution, and $G_{n,\alpha}(r)$ is defined by

$$G_{n,\alpha}(r) = \frac{|r|^{(2-n)/2}}{(2\pi)^{n/2}} \int_0^\infty \frac{\lambda^{n/2} J_{(n-2)/2}(\lambda|r|)}{c(\alpha) \lambda^{\alpha} + E} \, d\lambda, \quad (3.7)$$

where $n = 1, 2, 3$, $\alpha > (n - 1)/2$, and $J_{(n-2)/2}$ is the Bessel function of the first kind.

Let us consider a deformation of unbounded fractional nonlocal continuum, where the stress is applied to an infinitesimally small region in this continuum. In this case, we can assume that the strain $\varepsilon(r)$ is induced by a point stress $\sigma(r)$ at the origin of coordinates, i.e.

$$\sigma(r) = \sigma_0 \delta(r), \quad (3.8)$$

i.e. the particular solution is proportional to the Green’s function. As a result, the stress field is

$$\varepsilon(r) = \frac{1}{2\pi^2 |r|} \int_0^\infty \frac{\lambda \sin(\lambda|r|)}{E + c(\alpha) \lambda^{\alpha}} \, d\lambda. \quad (3.9)$$
3.1 Perturbation of Linearized Fractional Deformations by Nonlinear Hardening

Suppose that \( \varepsilon(r) = \varepsilon_0(r) \) is the solution of Eq. (3.3) with \( \eta = 0 \), i.e. is the solution of the linear equation

\[
\sigma(x) = E \varepsilon_0(r) + c(\alpha) ((-\Delta)^{\alpha/2} \varepsilon_0)(r) \quad (3.10)
\]

The solution of this fractional differential equation may be written in the form

\[
\varepsilon(r) = \varepsilon_0(r) + \eta \varepsilon_1(r) + \ldots \quad (3.11)
\]

This means that we consider perturbations to the strain field \( \varepsilon_0(r) \) of the fractional gradient deformation state, which are caused by weak nonlinear hardening effects. The first order approximation with respect to \( \eta \) gives the equation

\[
E \varepsilon_1(r) + c(\alpha) ((-\Delta)^{\alpha/2} \varepsilon_1)(r) + K(\varepsilon_0(r)) = 0, \quad (3.12)
\]

which is equivalent to the linear equation

\[
\sigma_{\text{eff}}(r) = E \varepsilon_1(r) + c(\alpha) ((-\Delta)^{\alpha/2} \varepsilon_1)(r), \quad (3.13)
\]

where the effective stress \( \sigma_{\text{eff}}(x) \) is defined by the equation

\[
\sigma_{\text{eff}}(r) = -K(\varepsilon_0(r)). \quad (3.14)
\]

Equation (3.12) gives a particular solution in the form

\[
\varepsilon(r) = \varepsilon_0(r) + \varepsilon_1(r) = G_{n,\alpha} * \sigma + \eta G_{n,\alpha} * \sigma_{\text{eff}}, \quad (3.15)
\]

where the convolution operation and \( G_{n,\alpha} \) are defined by Eqs. (3.6), (3.7). Upon substitution of Eq. (3.14) into Eq. (3.15), we obtain

\[
\varepsilon(r) = G_{n,\alpha} * \sigma - \eta G_{n,\alpha} * K(G_{n,\alpha} * \sigma). \quad (3.16)
\]

For a “point stress” of the form given by Eq. (3.8), Eq. (3.15) can be written in the form

\[
\varepsilon(r) = \sigma_0 G_{n,\alpha}(r) - \eta \left( G_{n,\alpha} * K(\sigma_0 G_{n,\alpha}) \right)(r). \quad (3.17)
\]
which, for $K(\cdot)$ given by Eq. (3.4), results to

$$\varepsilon(r) = \sigma_0 G_{n,\alpha}(r) - \eta \sigma_0^3 \left( G_{n,\alpha} * (G_{n,\alpha})^\beta \right)(r).$$

(3.18)

### 3.2 Perturbation of Plasticity by Fractional Gradient Nonlocality

Let us now consider an equilibrium state by setting $\varepsilon_0 = const.$ (i.e. $(-\Delta)^{\alpha/2} \varepsilon_0 = 0$) and $\sigma(r) = \sigma = const.$ in Eq. (3.3), i.e.

$$E \varepsilon_0 + \eta K(\varepsilon_0) = \sigma.$$  

(3.19)

This, for the case, where the function $K$ is defined by Eq. (3.4) with $\beta = 3$, becomes

$$E \varepsilon_0 + \eta \varepsilon_0^3 = \sigma.$$  

(3.20)

For $\sigma \neq 0$, there is no solution $\varepsilon_0 = 0$. For $E > 0$ and the weak scalar stress field $\sigma \ll \sigma_c$ with respect to the critical value $\sigma_c = \sqrt{E/\eta}$, there exists only one solution

$$\varepsilon_0 \approx \sigma/E.$$  

(3.21)

For negative stiffness materials ($E < 0$) and $\sigma = 0$, we have three solutions

$$\varepsilon_0 \approx \pm \sqrt{|E|/\eta}, \quad \varepsilon_0 = 0.$$  

(3.22)

For $\sigma < (2\sqrt{3}/9)\sigma_c$, also exist three solutions. For $\sigma \gg \sigma_c$, we can neglect the first term ($E \approx 0$),

$$\eta \varepsilon_0^3 \approx \sigma,$$  

(3.23)

and obtain

$$\varepsilon_0 \approx (\sigma/\eta)^{1/3}.$$  

(3.24)

In general, the equilibrium values $\varepsilon_0$ are solutions of the nonlinear algebraic relation given by Eq. (3.19).

Let us consider a deviation $\varepsilon_1(r)$ of the field from the equilibrium value $\varepsilon_0(r)$. For this purpose we will seek a solution in the form
\[ \varepsilon(\mathbf{r}) = \varepsilon_0 + \varepsilon_1(\mathbf{r}) \] (3.25)

In general, the stress field is not constant, i.e. \( \sigma(\mathbf{x}) \neq \sigma \). In a first approximation, we obtain the equation

\[ \sigma(\mathbf{r}) = c(\alpha) \left((-\Delta)^{\alpha/2} \varepsilon_1\right)(\mathbf{r}) + \left(E + \eta \varepsilon'_0(\varepsilon_0)\right)\varepsilon_1(\mathbf{r}), \] (3.26)

where \( K'_\varepsilon = \partial K(\varepsilon)/\partial \varepsilon \). Equation (3.26) is equivalent to the linear fractional differential equation

\[ \sigma(\mathbf{x}) = E_{eff} \varepsilon_1(\mathbf{r}) + c(\alpha) \left((-\Delta)^{\alpha/2} \varepsilon_1\right)(\mathbf{r}), \] (3.27)

with the effective modulus \( E_{eff} \) defined by

\[ E_{eff} = E + \eta \varepsilon'_0(\varepsilon_0). \] (3.28)

For the case \( K(\varepsilon) = \varepsilon^\beta \), we have

\[ E_{eff} = E + \beta \eta \varepsilon_0^{\beta-1}. \] (3.29)

A particular solution of Eq. (3.26) can be written in the form of Eq. (3.6), where we use \( E_{eff} \) instead of \( E \). For the “point stress” (see Eqs. (3.8)–(3.9)), Eq. (3.16) gives

\[ \varepsilon_1(\mathbf{r}) = \frac{1}{2\pi^2} \frac{\sigma_0}{|\mathbf{r}|} \int_0^\infty \frac{E + E_{eff} + 2c(\alpha) \lambda^\alpha}{(c(\alpha) \lambda^\alpha + E)(c(\alpha) \lambda^\alpha + E_{eff})} \sin(\lambda|\mathbf{r}|) \, d\lambda. \] (3.30)

For the case \( \alpha = 2 \), the field \( \varepsilon_1(\mathbf{r}) \) is given by the equation

\[ \varepsilon_1(\mathbf{r}) = \frac{\sigma_0}{4\pi c(\alpha)|\mathbf{r}|} e^{-|\mathbf{r}|/r_c}, \] (3.31)

where \( r_c \) is defined by

\[ r_c^2 = \frac{c(\alpha)}{E + \eta K'_s(\varepsilon_0)}. \] (3.32)

It should be noted that on analogous situation exists in classical theory of electric fields. In the electrodynamics the field \( \varepsilon_1(\mathbf{r}) \) describes the Coulomb potential with the Debye screening. For a fractional differential field equation \( (\alpha \neq 2) \), we have a power-law type of screening that is described in [9]. The
electrostatic potential for media with power-law spatial dispersion differs from the Coulomb potential by the factor $C_{\alpha,0}(|r|) = \frac{2}{\pi} \int_0^{\infty} \frac{\lambda \sin(\lambda |r|)}{E_{eff} + c(\alpha) \lambda^\alpha} d\lambda$.  

Note that the Debye potential differs from the Coulomb potential by the exponential factor $C_D(|x|) = \exp(-|r|/r_D)$.

4 Fractional Helmholtz Equation

On introducing the fractional GradEla constitutive relation given by Eq. (1.5) into the equilibrium relation given by Eq. (1.1), we obtain

$$[1 + l_\varepsilon^\alpha (-\Delta)^{\alpha/2}] [\lambda \nabla tr\varepsilon + 2\mu \text{div}\varepsilon] = 0,$$

where the notation $l_\varepsilon^\alpha(\alpha) \equiv l_\alpha^\alpha$, and $(-R \Delta)^{\alpha/2} \equiv (-\Delta)^{\alpha/2}$ was used for simplicity. Noting the fact that the operators $\nabla$ and $(-\Delta)^{\alpha/2}$ commute and that the second bracket in Eq. (4.1) is also zero by replacing $\varepsilon$ with $\varepsilon_0$, where $\varepsilon_0$ denotes the solution of the corresponding equation for classical elasticity, (i.e. $\lambda \nabla tr\varepsilon + 2\mu \text{div}\varepsilon = 0$), we can easily deduce that the solution of Eq. (4.1) satisfies the reduced fractional partial differential equation

$$[1 + l_\varepsilon^\alpha (-\Delta)^{\alpha/2}] \varepsilon = \varepsilon_0,$$

which for the case $\alpha = 2$ reduces to the inhomogeneous Helmholtz equation derived for the non-fractional GradEla (Ru-Aifantis theorem [12]) and was used successfully to derive non-singular solutions for dislocations and cracks [13]–[17]. It turns out that compatible displacements $u$ ($\varepsilon_{ij} = (1/2)[u_{i,j} + u_{j,i}]$) also obey Eq. (4.2) and the same holds for corresponding fields in electrostatics with Debye screening [9], as well as for steady-state higher-order diffusion problems [18], [19].

It is thus critical to derive fundamental solutions for Eq. (4.2); i.e. for the equation

$$[1 + l_\varepsilon^\alpha (-\Delta)^{\alpha/2}] G_\alpha(r) = \delta(r),$$

where $G_\alpha(r)$ denotes the fundamental solution, $\delta(r)$ denotes the delta function and $r$ is the radial coordinate in a 3D space.
To compute the fundamental solution of Eq. (4.3) with the natural boundary condition $G_\alpha(r) \to 0$ as $r \to \infty$, we employ the method of Fourier transforms. Using the properties of the Fourier transform of the Riesz fractional Laplacian for every “well-behaved” scalar function $f(r)$

$$\mathcal{F}( (-\Delta)^{\alpha/2} f(r) )(k) = |k|^{\alpha} \mathcal{F}(f(r))(k),$$

and the well-known transform of the delta function $\mathcal{F}(\delta(r))(k) = 1$, we obtain the following algebraic equation for the fundamental solution

$$[1 + l^\alpha \varepsilon |k|^\alpha] G_\alpha(k) = 1,$$

which gives

$$G_\alpha(k) = \frac{1}{1 + l^\alpha \varepsilon |k|^\alpha}. \quad (4.6)$$

Consequently, the fundamental solution of Eq. (4.3) in the physical space is obtained through inversion of Eq. (4.6)

$$G_\alpha(r) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{1}{1 + l^\alpha \varepsilon |k|^\alpha} e^{i k \cdot r} d^3 k. \quad (4.7)$$

To simplify Eq. (4.7), we perform a change of variables $k \to l^{-1} \varepsilon k$, which results a factor of $l^{-3}$ and a change in scale $r \to r/l$. Therefore, for simplicity, we omit those factors, and restore them at the end result.

The integral given by Eq. (4.7) is defined in a 3-dimensional Euclidean space and can be analytically computed in spherical coordinates by applying a well-known relationship (see, for example Lemma 25.1 of Samko et al [11])

$$\frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} f(|k|) e^{i k \cdot r} d^3 k = \frac{1}{(2\pi)^{3/2} |r|^{3/2}} \int_{0}^{\infty} k^{3/2} f(k) J_{1/2}(k |r|) dk. \quad (4.8)$$

In Eq. (4.8) $k$ denotes the magnitude of the wave vector and $J_{1/2} = \sqrt{2/(\pi z)} \sin(z)$ denotes the Bessel function of order 1/2. Introduction of Eq. (4.8) in Eq. (4.7) by omitting the scaling factors, results to

$$G_\alpha(r) = \frac{1}{(2\pi)^{3/2} |r|^{3/2}} \int_{0}^{\infty} \frac{k^{3/2}}{1 + k^\alpha} J_{1/2}(k |r|) dk,$$

$$= \frac{1}{2\pi^2 |r|} \int_{0}^{\infty} \frac{k}{1 + k^\alpha} \sin(k |r|) dk. \quad (4.9)$$
The integral in Eq. (4.9) can be computed using the convolution property of the Mellin transform, defined in [21] by the relationship
\[ \mathcal{M}(f(x))(s) = \int_{0}^{\infty} f(x) x^{s-1} \, dx. \] (4.10)
Its inverse is given by
\[ f(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \mathcal{M}(f(x))(s) x^{-s} \, ds, \] (4.11)
where the path of integration is a vertical strip separating the poles of \( \mathcal{M}(f(x))(s) \), defined in \( \gamma_1 < \text{Re}(s) < \gamma_2 \). For more details about the Mellin transform, we refer the reader to [20]. Here we only use the basic results
\[ \mathcal{M}(\frac{1}{1+x^s})(s) = \frac{1}{\alpha} \Gamma\left(\frac{s}{\alpha}\right) \Gamma(1-\frac{s}{\alpha}), \]
\[ \mathcal{M}(x^{3/2} J_{1/2}(x))(s) = 2^{1/2+s} \Gamma\left(\frac{1}{2} + s\right) \Gamma\left(\frac{1}{2} - \frac{s}{2}\right). \] (4.12)
where we made use of the Mellin transform of the Bessel function (see also Section 6.8 of [21])
\[ \mathcal{M}(J_{\sigma}(2\sqrt{u}))(s) = \frac{\Gamma\left(\frac{\sigma}{2} + s\right)}{\Gamma\left(\frac{\sigma}{2} + 1 - s\right)}. \] (4.13)
Consequently, Eq. (4.9) can be evaluated using the above results and performing the inverse Mellin transform by computing the Mellin-Barnes integral
\[ G_\alpha(r) = \frac{1}{2\alpha \pi^{3/2} |r|^2} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma\left(\frac{1}{\alpha} - \frac{s}{\alpha}\right) \Gamma(1-\frac{1}{\alpha} + \frac{s}{\alpha}) \Gamma\left(1+\frac{s}{\alpha}\right)}{\Gamma\left(\frac{1}{2} - \frac{s}{2}\right)} \left(\frac{|r|}{2}\right)^{-s} \, ds. \] (4.14)
The Mellin-Barnes integral representation of Eq. (4.14) can be expressed in terms of the corresponding Fox-H function of fractional analysis (see, for example, [22]–[26])
\[ G_\alpha(r) = \frac{1}{2\alpha \pi^{3/2} |r|^2} H_{1,3}^{2,1} \frac{|r|}{2} \left[ \begin{array}{c} (1 - \frac{1}{\alpha} + \frac{1}{\alpha}) \\ (1 - \frac{1}{\alpha}, \frac{1}{\alpha}, \frac{1}{2}, \frac{1}{2}) \end{array} \right]. \] (4.15)
The integral (4.14) has poles at the points $s = 1 - \alpha(\nu - 1)$ and $s = 1 - (3 + 2\nu)$, $\nu \in \mathbb{N}$. To evaluate it we apply first the residue theorem to the poles of $f \Gamma(1 - \frac{1-s}{\alpha})$, since they correspond to the singularity near the origin $r \approx 0.$

After a change of variables $s \to s + 1 - \alpha$, (4.14) becomes

\[
G_\alpha(r) = \frac{1}{\alpha 2^\alpha \frac{\pi}{2} |r|^{3-\alpha}} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(1 - \frac{s}{\alpha}) \Gamma(\frac{s}{\alpha}) \Gamma\left(\frac{3}{2} - \frac{\alpha}{2} + \frac{s}{2}\right)}{\Gamma\left(\frac{\alpha}{2} - \frac{s}{2}\right)} \left(\frac{|r|}{2}\right)^{-s} ds.
\]

(4.16)

Next, we perform another change of variables $s \to \alpha s$ in Eq. (4.16), which results to

\[
G_\alpha(r) = \frac{1}{2^\alpha \frac{\pi}{2} |r|^{3-\alpha}} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(1 - s) \Gamma(s) \Gamma\left(\frac{3}{2} - \frac{\alpha}{2}(1-s)\right)}{\Gamma\left(\frac{\alpha}{2}(1-s)\right)} \left(\frac{|r|}{2}\right)^{-\alpha s} ds,
\]

(4.17)

which provides an alternative representation in terms of the Fox-H function, i.e.

\[
G_\alpha(r) = \frac{1}{2^\alpha \frac{\pi}{2} |r|^{3-\alpha}} H_{1,3}^{2,1} \left[ \left(\frac{|r|}{2}\right)^\alpha \begin{array}{c}(0,1) \\ (0,1), \left(\frac{3}{2} - \frac{\alpha}{2}, \frac{\alpha}{2}\right), \left(1 - \frac{\alpha}{2}, \frac{\alpha}{2}\right) \end{array} \right].
\]

(4.18)

The contour integral in Eq. (4.18) can be evaluated using the method of residues from complex analysis, by closing the contour encircling all poles at $s = -\nu$ and then applying the Cauchy residue theorem

\[
G_\alpha(r) = \frac{1}{2^\alpha \frac{\pi}{2} |r|^{3-\alpha}} \sum_{\nu=0}^{\infty} \lim_{s \to -\nu} \left\{ (s + \nu) \Gamma(s) \Gamma(1 - s) \Gamma\left(\frac{3}{2} - \frac{\alpha}{2}(1-s)\right) \Gamma\left(\frac{\alpha}{2}(1-s)\right) \left(\frac{|r|}{2}\right)^{-\alpha s} \right\}.
\]

(4.19)

Eq. (4.19) can be evaluated using the relation

\[
\lim_{s \to -\nu} (s + \nu) \Gamma(s) = \lim_{s \to -\nu} \frac{\Gamma(s + \nu + 1)}{s(s+1) \ldots (s + \nu - 1)} = \frac{(-1)^\nu}{\nu!}.
\]

(4.20)

This gives
\[ G_\alpha(r) = \frac{1}{2^\alpha \pi^{3/2} |r|^{3-\alpha}} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \Gamma(1+\nu) \Gamma \left( \frac{3}{2} - \frac{\alpha}{2}(1+\nu) \right)}{\nu! \Gamma \left( \frac{3}{2}(1+\nu) \right)} \left( \frac{|r|}{2} \right)^{\alpha \nu}, \]

which can be simplified by noting that \( \Gamma(1+\nu) = \nu! \), for \( \nu \in \mathbb{N} \). The final result is

\[ G_\alpha(r) = \frac{1}{2^\alpha \pi^{3/2} |r|^{3-\alpha}} \sum_{\nu=0}^{\infty} \frac{\Gamma \left( \frac{3}{2} - \frac{\alpha}{2}(1+\nu) \right)}{\Gamma \left( \frac{3}{2}(1+\nu) \right)} (-1)^\nu \left( \frac{|r|}{2} \right)^{\alpha \nu}. \] (4.22)

An asymptotic expression near the origin is obtained from the dominating term of Eq. (4.22) for \( r \to 0 \), i.e.

\[ G_\alpha(r) \approx \frac{\Gamma \left( \frac{3}{2} - \frac{\alpha}{2} \right)}{2^\alpha \pi^{3/2} \Gamma \left( \frac{\alpha}{2} \right)} \frac{1}{|r|^{3-\alpha}}, \quad (r \to 0). \] (4.23)

This asymptotic form cancels the singularity of the fundamental solution of corresponding classical theories. To see this, one can compute the contributions from the poles of \( \Gamma(1+s/2) \) in Eq. (4.14), which correspond to non-singular asymptotic behavior near the origin, using the same techniques. The result is

\[ G_\alpha(r) = \frac{1}{2^\alpha \pi^{3/2} |r|^{3-\alpha}} \sum_{\nu=0}^{\infty} \frac{\Gamma \left( \frac{3}{2} - \frac{\alpha}{2}(1+\nu) \right)}{\Gamma \left( \frac{3}{2}(1+\nu) \right)} (-1)^\nu \left( \frac{|r|}{2} \right)^{\alpha \nu} + \frac{2}{\alpha(4\pi)^{3/2}} \sum_{\nu=0}^{\infty} \frac{\Gamma \left( \frac{3+2\nu}{\alpha} \right) \Gamma \left( 1 - \frac{1}{\alpha}(3+2\nu) \right)}{\Gamma \left( \frac{3}{2} + \nu \right) \nu!} \left( \frac{|r|}{2} \right)^{2\nu}. \] (4.24)

In the special case \( \alpha \to 2 \), Eq. (4.24) reduces to the Green’s function of the classical Helmholtz equation, i.e.

\[ G_\alpha(r) = \frac{1}{4\pi |r|} e^{-|r|} \] (4.25)

It is easily checked that Eqs. (4.23), (4.24) give the same results as Eq. (64) of [9], since it solves the same mathematical equation (i.e. the fractional inhomogeneous Helmholtz equation (their Eq. (59)) for a different physical problem – the problem of a point charge.
5 Fractional Higher-Order Diffusion

On introducing the fractional diffusion constitutive relation given by Eq. (1.7) into the classical (non-fractional) mass balance law

\[
\frac{\partial \rho}{\partial t} + \text{div} \mathbf{j} = 0 \quad \text{or} \quad \rho,_{t} + j,_{i} = 0,
\]

we obtain the fractional high-order diffusion equation

\[
\frac{\partial \rho}{\partial t} = \mathcal{D} \Delta \rho + \mathcal{D} l_{d}^{\alpha} \nabla \cdot \{(-\Delta)^{\alpha/2} \nabla \rho\},
\]

along with the auxiliary conditions \(\rho(\mathbf{r}, 0) = \delta(\mathbf{r})\), \(\rho(\mathbf{r}, t) \to 0\) as \(|\mathbf{r}| \to \infty\) and \(\delta(\mathbf{r})\) denoting, as usual, the delta function. [The notation \(l_{d}^{\alpha}(\alpha) \equiv l_{d}^{\alpha}\), and \((-\RDelta)^{\alpha/2} \equiv (-\Delta)^{\alpha/2}\) was used for simplicity]

To solve Eq. (5.2) we employ the method of Fourier transform and exploit the properties of the Riesz fractional Laplacian, along with the symmetry of the problem. This gives

\[
\frac{\partial \rho(\mathbf{k}, t)}{\partial t} = -D |\mathbf{k}|^{2} \rho(\mathbf{k}, t) - D l_{d}^{\alpha} |\mathbf{k}|^{\alpha} |\mathbf{k}|^{2} \rho(\mathbf{k}, t),
\]

where \(\mathbf{k}\) denotes the wave vector. Equation (5.3) is a first order ordinary differential equation with respect to time with the initial condition \(\rho(\mathbf{k}, 0) = \mathcal{F}(\delta(\mathbf{r})) = 1\). Its solution is

\[
\rho(\mathbf{k}, t) = \exp(-D t |\mathbf{k}|^{2}) \exp(-D_{\alpha} t |\mathbf{k}|^{\alpha+2}),
\]

where we defined \(D_{\alpha} \equiv D l_{d}^{\alpha}\). The solution of Eq. (5.4) in configuration space is obtained by inversion of the Fourier transform

\[
\rho(\mathbf{r}, t) = \frac{1}{(2\pi)^{3}} \int_{-\infty}^{\infty} \exp(-D t |\mathbf{k}|^{2}) \exp(-D_{\alpha} t |\mathbf{k}|^{\alpha+2}) \exp(i\mathbf{k} \cdot \mathbf{r}) d^{3}k.
\]

Equation (5.5) is the inverse Fourier transform of the product of two independent terms and can be expressed as the convolution of the corresponding solutions in the physical space using the following well-known property of the Fourier transform
\[ F((f * g)(r, t))(k) = F(f(r, t))(k) F(g(r, t))(k), \]  
\[ (f * g)(r, t) = \int_{-\infty}^{\infty} f(r - r', t) g(r', t) \, d^3r'. \]  

Using Eq. (5.6), we recognize Eq. (5.5) as the convolution

\[ \rho(r, t) = (G_2 * G_{\alpha+2})(r, t), \]  

where we defined the set of functions \( G_{\alpha} \) as

\[ G_{\alpha}(r, t) = \mathcal{F}^{-1}\{\exp(-D_{\alpha} t |k|^\alpha)\} \]
\[ = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \exp(-D_{\alpha} t |k|^\alpha) \exp(i k \cdot r) \, d^3k. \]  

Equation (5.9) is the fundamental solution (i.e. the Greens function) for the fractional diffusion equation

\[ \frac{\partial G_{\alpha}(r, t)}{\partial t} = -D_{\alpha} (-\Delta)^{\alpha/2} G_{\alpha}(r, t), \]  

The corresponding fundamental solution of Eq. (5.2) is then deduced from Eq. (5.9) through the convolution property of Eq. (5.8)

Applying Eq. (5.8) into the fundamental solution \( G_{\alpha}(r) \) of Eq. (5.9), we obtain

\[ G_{\alpha}(r, t) = \frac{1}{(2\pi)^{3/2} \sqrt{|r|}} \int_0^{\infty} k^{3/2} \exp(-D_{\alpha} t k^\alpha) J_{3/2}(k|r|) \, dk \]
\[ = \frac{1}{2\pi^2 |r|} \int_0^{\infty} k \exp(-D_{\alpha} t k^\alpha) \sin(k|r|) \, dk. \]  

The integral in Eq. (5.11) can be computed using the convolution property of the Mellin transform, as in the previous section. The final result is the following series expansion expression [27]

\[ G_{\alpha}(r, t) = \frac{2}{\alpha(4\pi)^{3/2} (D_{\alpha} t)^{3/\alpha}} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \Gamma\left(\frac{3}{\alpha} + \frac{2\nu}{\alpha}\right)}{\nu! \Gamma\left(\frac{3}{2} + \nu\right)} \left(\frac{|r|^2}{4(D_{\alpha} t)^{2/\alpha}}\right)^\nu. \]
Equation (5.12) can be represented in terms of the Wrights function \( \Psi_1 \) as

\[
G_\alpha(r, t) = \frac{2}{\alpha(4\pi)^{3/2}(D_\alpha t)^{3/\alpha}} \Psi_1 \left[ \left( \frac{3}{2}, \frac{2}{\alpha} \right) ; -\frac{|r|^2}{4(D_\alpha t)^{\frac{2}{\alpha}}} \right].
\]  

The generalized Wrights function is defined by the following series [10, 11]

\[
p \Psi_q(z) = p \Psi_q \left[ (a_1, A_1) \ldots (a_p, A_p) ; (b_1, B_1) \ldots (b_q, B_q) ; z \right] = \sum_{\nu=0}^{\infty} \prod_{j=1}^{p} \Gamma(a_j + A_j \nu) \prod_{j=1}^{q} \Gamma(b_j + B_j \nu) \frac{z^\nu}{\nu!}. \]  

It is easy to check that when \( \alpha = 2 \), the series expansion reduces to the Greens function of the ordinary diffusion equation in 3-dimensional space. This is readily seen by letting \( \alpha \to 2 \) in Eqs. (5.12) and (5.13), resulting to the expression

\[
G_2(r, t) = \frac{1}{(4\pi D t)^{3/2}} \Psi_1 \left[ \left( \frac{3}{2}, 1 \right) ; -\frac{|r|^2}{4D t} \right] = \frac{1}{(4\pi D t)^{3/2}} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu!} \left( \frac{|r|^2}{4D t} \right)^\nu = \frac{1}{(4\pi D t)^{3/2}} \exp(-\frac{|r|^2}{4D t}).
\]  

Consequently, the fundamental solution of the second-order fractional diffusion equation (5.2), denoted as \( G(r, t) \), is obtained through convolution of Eq. (5.6), with \( G_\alpha(r, t) \) given by Eq. (5.13) and (5.15) for \( \alpha = 2 \), i.e.

\[
G(r, t) = \int_{-\infty}^{\infty} G_{\alpha+2}(r - r', t)G_2(r', t) d^3r'.
\]  

We can extend Eq. (5.1) to include distributed sources (e.g. chemical reaction or trapping) with density/concentration rate \( q(r, t) \). In this particular case, the classical mass balance law becomes

\[
\frac{\partial \rho}{\partial t} + \text{div} j = q,
\]  

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and the corresponding inhomogeneous fractional diffusion equation reads

\[
\frac{\partial \rho}{\partial t} = D \Delta \rho + D l_d^\alpha \nabla \cdot \{(\Delta)^{\alpha/2} \nabla \rho\} + q. \tag{5.18}
\]

Using the Fourier transform method, we can obtain the fundamental solution of Eq. (5.18) as follows

\[
\rho(\mathbf{r}, t) = \int_0^t \int_{-\infty}^{\infty} G(\mathbf{r} - \mathbf{r}', t - \tau) q(\mathbf{r}', \tau) \, d^3 \mathbf{r}' \, d\tau, \tag{5.19}
\]

where \(G(\mathbf{r}, t)\) is given by Eq. (5.16). For the special case of a unit point source \(q(\mathbf{r}, t) = \delta(\mathbf{r})\delta(t)\), it is readily seen that Eq. (5.19) reduces to the fundamental solution \(G(\mathbf{r}, t)\).

The fractional diffusion equation admits steady-state solutions, under the presence of external sources/sinks with density/rate \(q(\mathbf{r})\). The governing equation for this time independent configuration is

\[
D \Delta \rho + D l_d^\alpha \nabla \cdot \{(\Delta)^{\alpha/2} \nabla \rho\} + q = 0. \tag{5.20}
\]

Equation (5.20) can be generalized to a higher-order steady-state fractional diffusion equation of the form

\[
D_\alpha((\Delta)^{\alpha/2} \rho)(\mathbf{r}) + D_\beta((\Delta)^{\beta/2} \rho)(\mathbf{r}) = q(\mathbf{r}), \quad (\alpha > \beta), \tag{5.21}
\]

where \((\alpha, \beta)\) denote arbitrary positive fractional order and \((D_\alpha, D_\beta)\) are corresponding fractional diffusion coefficients. Equation (5.21) can be derived by considering a fractional extension of the conservation law given by Eq. (5.17), along with the constitutive relation given by Eq. (1.7) and/or a further fractional extension for its classical gradient \((\nabla)\) part.

Equation (5.21) is a fractional partial differential equation, whose solution reads

\[
\rho(\mathbf{r}) = \int_{\mathbb{R}^3} G_{\alpha,\beta}(|\mathbf{r} - \mathbf{r}'|) q(\mathbf{r}') \, d^3 \mathbf{r}', \tag{5.22}
\]

with the Green-type function \(G_{\alpha,\beta}(\mathbf{r})\) given by

\[
G_{\alpha,\beta}(\mathbf{r}) = \int_{\mathbb{R}^3} \frac{1}{D_\alpha |\mathbf{k}|^{\alpha} + D_\beta |\mathbf{k}|^{\beta}} e^{i\mathbf{k} \cdot \mathbf{r}} \, d^3 \mathbf{k} = \frac{1}{(2\pi)^{3/2} \sqrt{|\mathbf{r}|}} \int_0^{\infty} \frac{\chi^{3/2} J_{1/2}(\chi |\mathbf{r}|)}{D_\alpha \chi^{\alpha} + D_\beta \chi^{\beta}} \, d\chi. \tag{5.23}
\]
Let us now consider the particular problem of a unit point source located at the origin of the form

\[ q(r) = q_0 \delta(r) = q_0 \delta(x)\delta(y)\delta(z). \]  

(5.24)

Upon substitution of Eq. (5.24) into Eq. (5.22), we obtain the particular solution

\[ \rho(r) = q_0 G_{\alpha,\beta}(r), \]  

(5.25)

with the Green function \( G_{\alpha,\beta}(r) \) given by Eq. (5.23). By using then the particular expression for the Bessel function of the first kind, we obtain

\[ \rho(r) = \frac{q_0}{2\pi^2} \int_0^\infty \frac{\lambda \sin(\lambda |r|)}{c_\alpha \lambda^\alpha + c_\beta \lambda^\beta} d\lambda, \quad (\alpha > \beta). \]  

(5.26)

Two distinct modes of diffusion arise, depending on the particular form of the fractional parameters \((\alpha, \beta)\), which are discussed in detail below.

A) Sub-GradDiffusion model: \( \alpha = 2; 0 < \beta < 2 \). In this case Eq. (5.20) becomes

\[ D\Delta \rho(r) - D_\beta((-\Delta)^{\beta/2}\rho)(r) + q(r) = 0, \quad (0 < \beta < 2). \]  

(5.27)

The order of the fractional Laplacian \((-\Delta)^{\beta/2}\) is less than the order of the first term related to the usual Fick’s law. The parameter \(\beta\) defines the order of the power-law non-locality. The particular solution of Eq. (5.27) reads

\[ \rho(r) = \frac{q_0}{2\pi^2 |r|} \int_0^\infty \frac{\lambda \sin(\lambda |r|)}{D\lambda^2 + D_\beta \lambda^\beta} d\lambda, \quad (0 < \beta < 2). \]  

(5.28)

The following asymptotic behavior for Eq. (5.28) can be derived in the form

\[ \rho(r) = \frac{q_0}{2\pi^2 |r|} \int_0^\infty \frac{\lambda \sin(\lambda |r|)}{D\lambda^2 + D_\beta \lambda^\beta} \approx \frac{C_0(\beta)}{|r|^{\beta-3}} + \sum_{k=1}^{\infty} \frac{C_k(\beta)}{|r|^{(2-\beta)(k+1)}} \quad (|r| \to \infty), \]

(5.29)

where
\[ C_0(\beta) = \frac{q_0 \Gamma(2 - \beta) \sin(\pi\beta/2)}{2\pi^2 D_\beta}, \]

\[ C_k(\beta) = -\frac{q_0 D^k}{2\pi^2 D_{\beta}^{k+1}} \int_0^\infty z^{(2-\beta)(k+1)-1} \sin(z) \, dz. \quad (5.30) \]

As a result, the density of the diffusive species generated by the source that is concentrated at a single point in space, for large distances from the source, is given asymptotically by the expression

\[ \rho(r) \approx \frac{C_0(\beta)}{|r|^{3-\beta}} \quad (0 < \beta < 2), \quad (5.31) \]

for large distances \(|r| \to \infty\).

**B) Super-GradDiffusion:** \( \alpha > 2 \) and \( \beta = 2 \). In this case, Eq. (5.20) becomes

\[ D\Delta \rho(r) - D_\alpha((\Delta)^{\alpha/2}\rho)(r) + q(r) = 0, \quad (\alpha > 2). \quad (5.32) \]

The order of the fractional Laplacian \((\Delta)^{\alpha/2}\) is greater than the order of the first term related to the usual Fick’s law. The asymptotic density \(\rho(r)\) for \( r \to 0 \) in this case is given by

\[ \rho(r) \approx \frac{q_0 \Gamma((3 - \alpha)/2)}{2^{\alpha/2} \sqrt{\pi} D_\alpha \Gamma(\alpha/2)} \frac{1}{|r|^{3-\alpha}}, \quad (2 < \alpha < 3), \]

\[ \rho(r) \approx \frac{q_0}{2\pi\alpha D^{1-3/\alpha} D_\alpha^{3/\alpha} \sin(3\pi/\alpha)} \quad (\alpha > 3). \quad (5.33) \]

Note that the above asymptotic behavior does not depend on the parameter \( \beta \), and that the corresponding relation of Eq. (5.33) does not depend on \( D_\beta \). The density \(\rho(r)\) for short distances away from the point of source application is determined only by the term with \((\Delta)^{\alpha/2}, (\alpha > \beta)\).

Finally, and especially for the case of more complicated boundary value problems, we mention that the steady-state diffusion of Eq. (5.20) can be factored as

\[ D\nabla \cdot \nabla \left\{ 1 + l_d^\alpha (\Delta)^{\alpha/2} \right\} \rho + q = 0. \quad (5.34) \]

By defining the “classical” operator \( L^0 \equiv D\nabla \cdot \nabla \), and similarly its “fractional gradient” counterpart \( L^\alpha \equiv 1 + l_d^\alpha (\Delta) \), we can prove that \( L^\alpha \) satisfies
the classical steady-state Fickean diffusion equation. This is a direct consequence of the fact that the operators $L^0$ and $L^\alpha$ commute. Therefore, we arrive at the following “operator-split” scheme

$$(1 + t^\alpha_d(-\Delta)^{\alpha/2}) \rho = \rho_0; \quad D \nabla \cdot \nabla \rho_0 + q = 0. \quad (5.35)$$

Equation (5.35) is the fractional counterpart of the Ru-Aifantis theorem [12], for the steady-state fractional higher–order diffusion equation.

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