Strings in Kerr-Newmann Black Holes.

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Abstract

We study the evolution of strings in the equatorial plane of a Kerr-Newmann black hole. Writing the equations of motion and the constraints resulting from Hamilton’s principle, three classes of exact solutions are presented, for a closed string, encircling the black-hole. They all depend on two arbitrary, integration functions and two constants. A process for extracting energy is examined for the case of one the three families of solutions. This is the analog of the Penrose process for the case of a particle.

I. Introduction

In modern physics, string theory plays a prominent role in the effort to achieve a consistent quantization of gravity and to provide a unified description of all the fundamental interactions [1-2]. String theory has overcome several longstanding problems in high energy physics in a beautiful and elegant way and has given the most convincing perspective in the unification effort. So the formulation of the theory in a background that contains gravity has been recognized from the early days as necessary for extracting valuable information for its content. This is the strongest motivation for the study of strings in curved spacetimes [2-3].

In the context of Black-Hole physics, strings have been used as candidates for the resolution of the main paradoxes and unsolved problems associated with it, and effective string actions in background fields have been investigated [4-5].
The string equations in the Kerr-Newmann spacetime are highly non-linear and difficult to solve exactly. Various attempts have been made to reduce them using different kinds of ansatze [6-7], by considering the Nambu-Goto action or by studying generic null-string configurations [8-11]. The problem is simplified if strings are bound to move on the equatorial plane of the Kerr-Newmann black hole. One can consider then the main problems of black hole physics in the light of string theory. The aim of this paper is to provide a method of obtaining three families of solutions and to examine the process of energy extraction for one of these families.

This paper is organized as follows:

In section II, the general features of string theory in the Kerr-Newmann black-holes are given.

In section III, the equations of motion for strings in the equatorial plane of the Kerr-Newmann black-hole are given.

In section IV, three classes of solutions are presented and analyzed.

In section V, the process for extracting energy from the black-hole via strings is examined for a special case.

II. Strings in Black-Hole Spacetimes

The action for a bosonic string in a curved-spacetime background is given (for a D-dimensional spacetime) by [1,2]

\[
S = -\frac{T}{2} \int d\tau d\sigma \sqrt{-hh^{\alpha\beta}(\tau, \sigma)} G_{MN}(X) \partial_\alpha X^M \partial_\beta X^N
\]

where \(M, N = 0, 1, \ldots, (D-1)\) are spacetime indices, \(\alpha, \beta = 0, 1\) are worldsheet indices and \(T = (2\pi\alpha')^{-1}\) is the string tension.

Variation of the action with respect to the "fields" which are the string coordinates \(X^M(\tau, \sigma)\), gives the equations of motion and the constraints [2-3]

\[
\ddot{X}^M - (X^M)^{\prime\prime} + \Gamma^M_{AB} (\dot{X}^A \dot{X}^B - X'^A X'^B) = 0
\]

\[
G_{AB}(X)[\dot{X}^A \dot{X}^B + X'^A X'^B] = 0
\]

\[
G_{AB}(X) \dot{X}^A X'^B = 0
\]

where the dot stands for \(\partial_\tau\) and the prime stands for \(\partial_\sigma\). In the conformal gauge choice, \(h_{\alpha\beta}(\tau, \sigma) = \exp[\Sigma(\tau, \sigma)]\eta_{\alpha\beta}\) where \(\Sigma(\tau, \sigma)\) is an arbitrary function and \(\eta_{\alpha\beta} = diag(-1, +1)\) we have the Lagrangian density

\[
\mathcal{L} = G_{MN}(X)[-\dot{X}^M \dot{X}^N + X'^M X'^N]
\]

from which the equations of motion follow via the Euler equations

\[
\partial_\alpha \left( \frac{\partial \mathcal{L}}{\partial (\partial_\alpha X^M)} \right) - \frac{\partial \mathcal{L}}{\partial X^M} = 0
\]
This constitutes a two-dimensional field theory. The equations of motion and the constraints for null strings are given by [4]

\[ \ddot{X}^M + \Gamma_{AB}^M \dot{X}^A \dot{X}^B = 0 \quad (7) \]
\[ G_{AB}(X) \dot{X}^A \dot{X}^B = 0 \quad (8) \]
\[ G_{AB}(X) \dot{X}^A X'^B = 0 \quad (9) \]

For the case of the four-dimensional Kerr-Newmann black hole we have the string coordinates \((X^0, X^1, X^2, X^3)\), corresponding to the coordinates \((t, r, \theta, \phi)\), of the Boyer-Lindquist coordinate system.

We make the definitions

\[ E \equiv r^2 + \alpha^2 + Q^2 - 2Mr \quad (10) \]
\[ \Delta \equiv r^2 + \alpha^2 \cos^2 \theta \quad (11) \]
\[ \delta \equiv r^2 + \alpha^2 \quad (12) \]
\[ \epsilon \equiv r^2 - \alpha^2 \cos^2 \theta \quad (13) \]

where \(Q\) is the charge of the black hole, \(M\) is its mass and \(\alpha \equiv S/M\) is the angular momentum per unit mass.

The metric is given by

\[ ds^2 = -\frac{E}{\Delta} [dt - \alpha \sin^2 \theta d\phi]^2 + \frac{\sin^2 \theta}{\Delta} [\delta d\phi - \alpha dt]^2 + \frac{\Delta}{E} dr^2 + \Delta d\theta^2 \quad (14) \]

The static limit is given by [12,13]

\[ S_{\pm} : r = M \pm \sqrt{M^2 - Q^2 - \alpha^2 \cos^2 \theta} \quad (15) \]

The horizon is given by

\[ \Sigma_{\pm} : r = M \pm \sqrt{M^2 - Q^2 - \alpha^2} \quad (16) \]

where we assume that \(M^2 \geq Q^2 + \alpha^2\).

The open string boundary conditions demand

\[ (X^M)'(\tau, \sigma = 0) = (X^M)'(\tau, \sigma = \pi) = 0 \quad (17) \]

while for closed strings we must have

\[ X^M(\tau, \sigma = 0) = X^M(\tau, \sigma = 2\pi) \quad (18) \]
III. The Equations of Motion

The worldsheet light-cone variables are defined by $\chi^\pm = (\tau \pm \sigma)$ . The Jacobian of the transformation is given by

$$ J \equiv \frac{\partial (\chi^+, \chi^-)}{\partial (\tau, \sigma)} = -2, \quad d\chi^+ d\chi^- = -2d\tau d\sigma \quad (19) $$

We introduce the notation and the ansatz

$$ X^0(\tau, \sigma) = t(\tau, \sigma), \quad X^r(\tau, \sigma) = r(\tau, \sigma) \quad (20) $$

$$ X^\theta(\tau, \sigma) = (\pi/2), \quad X^\phi(\tau, \sigma) = \phi(\tau, \sigma) \quad (21) $$

Defining

$$ R(r) \equiv (1 - \frac{2M}{r} + \frac{Q^2}{r^2}) \quad (22) $$

the Lagrangian density is given by

$$ \bar{\mathcal{L}} = 4R(\partial_+ t)(\partial_- t) - 4\alpha(R - 1)[(\partial_- t)(\partial_+ \phi) + (\partial_+ t)(\partial_- \phi)] $$

$$ - 4[r^2 + 2\alpha^2 - \alpha^2 R](\partial_+ \phi)(\partial_- \phi) - \frac{4r^2}{\alpha^2 + Rr^2}(\partial_+ r)(\partial_- r) \quad (23) $$

The action is given by

$$ S = \int \mathcal{L} d\tau d\sigma = \int -\bar{\mathcal{L}}(\tau, \sigma) d\chi^+ d\chi^- \quad (24) $$

The equations of motion

$$ \frac{\delta \bar{\mathcal{L}}}{\delta t} = \frac{\delta \bar{\mathcal{L}}}{\delta r} = \frac{\delta \bar{\mathcal{L}}}{\delta \phi} = 0 \quad (25) $$

read

$$ \partial_+[R(t) - \alpha(R - 1)(\partial_+ \phi)] + \partial_- [R(t) - \alpha(R - 1)(\partial_+ \phi)] = 0 \quad (26) $$

$$ \partial_+[(r^2 + 2\alpha^2 - \alpha^2 R)(\partial_- \phi) + \alpha(R - 1)(\partial_- t)] + $$

$$ \partial_-[(r^2 + 2\alpha^2 - \alpha^2 R)(\partial_+ \phi) + \alpha(R - 1)(\partial_+ t)] = 0 \quad (27) $$

$$ \partial_+\left[\frac{r^2}{\alpha^2 + Rr^2}(\partial_- r)\right] + \partial_-\left[\frac{r^2}{\alpha^2 + Rr^2}(\partial_+ r)\right] = $$

$$ \frac{\partial}{\partial r}\left[\frac{r^2}{\alpha^2 + r^2 R}(\partial_+ r)(\partial_- r)\right] + \frac{\partial}{\partial r}\left[r^2 + 2\alpha^2 - \alpha^2 R\right](\partial_+ \phi)(\partial_- \phi) + $$

$$ + \alpha\left(\frac{\partial R}{\partial r}\right)[(\partial_- t)(\partial_+ \phi) + (\partial_+ t)(\partial_- \phi)] - \left(\frac{\partial R}{\partial r}\right)(\partial_+ t)(\partial_- t) \quad (28) $$


The constraints become
\[-R(\partial_{\pm t})^2 + 2\alpha(R - 1)(\partial_{\pm t})(\partial_{\pm} \phi) + 2r \land 2\alpha^2 - \alpha^2 R)(\partial_{\pm} \phi)^2 + \frac{r^2}{\alpha^2 + r^2 R} (\partial_{\pm} r)^2 = 0\] (29)

It is straightforward to verify that these constitute the same set of equations to be satisfied with the set of Eqs (2)-(4) when one sets \(\theta = (\pi/2)\). The action functional, the equations of motion and the constraints are invariant under the residual gauge symmetry
\[\chi^\pm \Rightarrow \tilde{\chi}^\pm = f^\pm (\chi^\pm) \] (30)

where \(f^\pm\) are arbitrary functions of the respective arguments.

IV. Classes of solutions

i. We introduce the ansatz
\[(\partial_{\pm} \phi) = \frac{\alpha(1 - R)}{(r^2 + 2\alpha^2 - \alpha^2 R)} (\partial_{\pm} t) \] (31)

Equation of motion (27) is satisfied identically. Substituting into the constraints we obtain, along with the previous relation the set of equations
\[(\partial_{\pm} t) = \epsilon \left( r^2 + 2\alpha^2 - \alpha^2 R \right)^{1/2} \frac{r}{(\alpha^2 + r^2 R)} (\partial_{\pm} r) \] (32)
\[(\partial_{\pm} \phi) = \frac{\alpha(1 - R)}{(r^2 + 2\alpha^2 - \alpha^2 R)^{1/2} (\alpha^2 + r^2 R)} (\partial_{\pm} r) \] (33)

with \(\epsilon = \pm 1\) for expanding or collapsing solutions, as observed by an asymptotic observer. There exists no need for integrability conditions for \(t, \phi\), because, due to the form of eqs (32) and (33), they are satisfied identically. Substituting these relations into the two remaining equations of motion (26) and (28) we obtain the same equation to be satisfied by \(r = r(\chi^+, \chi^-)\),
\[\partial_+ \left[ \frac{r}{(r^2 + 2\alpha^2 - \alpha^2 R)^{1/2}} (\partial_- r) \right] + \partial_- \left[ \frac{r}{(r^2 + 2\alpha^2 - \alpha^2 R)^{1/2}} (\partial_+ r) \right] = 0 \] (34)

This can be written as a wave equation \(\partial_+ \partial_- f(r) = 0\) with
\[\frac{df(r)}{dr} = \frac{r}{(r^2 + 2\alpha^2 - \alpha^2 R)^{1/2}} > 0 \] (35)

The solution of the wave equation together with the periodicity requirement for closed strings gives
\[f(r) = f + g\alpha \left[ \chi^+ + \chi^- \right] + i\sqrt{\alpha} \sum_{n \neq 0} \frac{1}{n} \left[ f_n e^{-in\chi^-} + \tilde{f}_n e^{-in\chi^+} \right] \] (36)
with \( f, g, f_n, \tilde{f}_n \) constants, satisfying the reality conditions for the solution, \( f_{-n} = \tilde{f}_n^* \), and \( \tilde{f}_{-n} = \tilde{f}_n^* \).

From this we obtain

\[
\frac{r\dot{r}}{(r^2 + 2\alpha^2 - \alpha^2 R)^{1/2}} = 2g\alpha' + \sqrt{\alpha} \sum_{n \neq 0} [f_n e^{-in\chi} + \tilde{f}_n e^{-in\chi}] \tag{37}
\]

The two dimensional worldsheet is in fact a null hypersurface. This is easily seen, if we substitute into the relation for the induced metric

\[
h_{\alpha\beta} = G_{AB}(X)\partial_\alpha X^A \partial_\beta X^B \tag{38}
\]

We can state equivalently that the invariant string size [2], vanishes.

\[
ds^2 = G_{AB}(X)\partial_+ X^A \partial_- X^B (dr^2 - d\sigma^2) = 0 \tag{39}
\]

The quantity in the square root is non-negative. Indeed for \( (R_- \leq r \leq R_+) \), we have \( R(r) \leq 0 \) so this is evident while for \( (R_+ \leq r) \) we have for \( M^2 \geq (\alpha^2 + Q^2) \)

\[
(r^2 + 2\alpha^2 - \alpha^2 R) = r^2 + 2\alpha^2 - \frac{\alpha^2}{r^2} [r^2 - 2Mr + Q^2] = \\
= \frac{1}{r^2} [(r^2 + \alpha^2)^2 - \alpha^2(r^2 - 2Mr + \alpha^2 + Q^2)] \geq \\
\geq \frac{1}{r^2} [(r^2 + \alpha^2)^2 - \alpha^2(r - M)^2] \geq 0 \iff \\
\iff (r^2 + \alpha^2) \geq \alpha |r - M| \iff \\
\iff (r^2 + \alpha^2) \geq \alpha(r - M) \quad (r \geq M) \tag{40}
\]

which holds for \( (r \geq R_+) \). So for \( R_- \leq r < +\infty \) from eq (25) we have that the function \( f(r) \) is strictly increasing in this interval and therefore invertible. Denoting the inverse by \( (\phi) \) we have \( r = r(\chi^\pm) = \phi[\Phi_+(\chi^+) + \Phi_-(\chi^-)] \), with \( (\Phi_\pm) \) arbitrary functions of the indicated arguments. This family of solutions depends on these two arbitrary functions and the two arbitrary constants coming from the integration of eqs (22), (23).

ii. We perform the ansatz

\[
(\partial_\pm t) = \frac{\alpha(1-R)}{(-R)}(\partial_\pm \phi) \tag{41}
\]

Equation (26) is satisfied identically. Substituting into the constraints, we have non-trivial solutions, provided that

\[
R(r) \leq 0 \iff R_- \leq r \leq R_+ \\
R_\pm = M \pm \sqrt{M^2 - Q^2} \tag{42}
\]
being the Outer and Inner static limits. Also we have
\[
\rho(r) \equiv \alpha^2 + r^2 R(r) = \alpha^2 + r^2 - 2M r + Q^2 \leq 0 \iff \\
\rho_- \leq r \leq \rho_+ \quad \rho_\pm = M \pm \sqrt{M^2 - Q^2 - \alpha^2}
\]
which are the Inner and Outer Horizons.

From these we obtain the useful approximation relations
\[
\rho(r) \simeq \begin{cases} 
0 & r \simeq \rho_+ \\
\alpha^2 & r \simeq R_+ \\
r^2 & r \gg R_+
\end{cases}
\]
(44)

From the first we have that \( r^2 \simeq (\alpha^2/(-R)) \) so we obtain
\[
[r^2 + 2\alpha^2 - \alpha^2 R] \simeq \begin{cases} 
\frac{(r^2 + \alpha^2)^2}{(r^2 + 2\alpha^2)} & r \simeq \rho_+ \\
\frac{(r^2 + 2\alpha^2)^2}{(r^2 + \alpha^2)} & r \simeq R_+ \\
(r^2 + \alpha^2) & r \gg R_+
\end{cases}
\]
(45)

We have therefore the solutions
A) \((\rho_+ \leq r \leq R_+)\). Ergosphere

\[
(\partial_{\pm} \phi) = \epsilon \frac{r \sqrt{-R}}{\rho(r)} (\partial_{\pm} r)
\]
(46)

\[
(\partial_{\pm} t) = \epsilon \frac{\alpha(1 - R)}{\sqrt{-R}} \frac{r}{\rho(r)} (\partial_{\pm} r)
\]
(47)

with \( \epsilon = \pm 1 \) for expanding or collapsing solutions.

B) \((\rho_- \leq r \leq \rho_+)\). In the Horizon.

\[
(\partial_{\pm} \phi) = \epsilon \frac{r \sqrt{-R}}{(-\rho(r))} (\partial_{\pm} r)
\]
(48)

\[
(\partial_{\pm} t) = \epsilon \frac{\alpha(1 - R)}{\sqrt{-R}} \frac{r}{(-\rho(r))} (\partial_{\pm} r)
\]
(49)

with \( \epsilon = \pm 1 \) for expanding or collapsing solutions.

In both cases substitution into the equations (27) and (28) results after straightforward calculations, to the same equation to be satisfied by \( r = r(\chi^+, \chi^-) \), which is,
\[
\partial_+ \left[ \frac{r}{\sqrt{-R}} (\partial_{-} r) \right] + \partial_- \left[ \frac{r}{\sqrt{-R}} (\partial_{+} r) \right] = 0
\]
(50)

This can be written as a wave equation
\[
\partial_+ \partial_- f(r) = 0 \quad R_- \leq r \leq R_+
\]
(51)
where
\[ f(r) = -[2R_+ + \frac{1}{2}(r - R_-) + \frac{3}{4}(R_+ - R_-)]\sqrt{(R_+ - r)(r - R_-)} + \]
\[ + \frac{3}{4}(R_+ - R_-)^2 + 2R_-(R_+ - R_-) + 2(R_-)^2\arctan\left[\frac{(r - R_-)}{(R_+ - r)}\right] \]
(52)

Again the function \( f(r) \) is invertible and the class depends on two arbitrary integration functions and two constants. This class of solutions also gives a null hypersurface as in the first case from eqs (38) and (39).

iii. We introduce the ansatz
\[
(\partial_- \phi) = \frac{1}{(\alpha^2 + r^2 R)} (\partial_- \Phi) \]
(53)
\[
(\partial_+ \phi) = -\frac{1}{(\alpha^2 + r^2 R)} (\partial_+ \Phi) \]
(54)
\[
(\partial_- t) = \frac{(r^2 + \alpha^2)}{\alpha(\alpha^2 + r^2 R)} (\partial_- \Phi) \]
(55)
\[
(\partial_+ t) = -\frac{(r^2 + \alpha^2)}{\alpha(\alpha^2 + r^2 R)} (\partial_+ \Phi) \]
(56)
\[
\alpha(\partial_- r) = (\partial_- \Phi) \]
(57)
\[
\alpha(\partial_+ r) = -(\partial_+ \Phi) \]
(58)

The equations of motion (26)-(28) and the constraints (29) are satisfied identically, provided that \( \Phi = \Phi(\chi^+, \chi^-) \) is a solution of the wave equation \( \partial_+ \partial_- \Phi = 0 \). Again the integrability conditions for the three functions are satisfied by the combined use of the wave equation for \( \Phi = \Phi(\chi^+, \chi^-) \) and eqs (57)-(58). All the three classes of solutions represent tensionfull strings. However, they also satisfy the null-tensionless string eqs (7)-(9) provided that we insert the small dimensionless parameter \( c^2 = 2\lambda T \) of the perturbation expansion into the terms that contain derivatives with respect to the \( (\sigma) \) and let \( c \to 0 \) [4]. For example we will have \( (\partial_r^2 - c^2 \partial_{\sigma}^2)\Phi(\chi^+) = 0 \). Thus, although they have null worldsheet manifold, they satisfy the null string equations of motion and constraints, only when their small tension limit is taken explicitly.

V. The Extraction of Energy

In the Boyer-Lindquist coordinate system \((t, r, \theta, \phi)\), the Killing vector fields are given by \( \xi^\mu = \delta^\mu_\nu \) and \( \xi^\mu_{(\phi)} = \delta^\mu_\phi \).

The energy-density along the string is given by [5],[13]
\[
U \equiv -p \cdot \xi_{(t)} = -G_{\alpha\beta}p^\alpha \xi^\beta_{(t)} = -p_t
\]
(59)
where \( p_{\alpha} = \left( \frac{\partial L}{\partial \dot{X}^{\alpha}} \right) \).

We assume that at \((\tau = 0)\), the string is at spatial infinity and as the class of solution implies, every point of it moves along a null curve. So since the radius of the string decreases, we must have

\[
\begin{align*}
  r[\tau = 0, \sigma] &= +\infty \quad (60) \\
  \dot{r}[\tau = 0, \sigma] &\equiv p_0(\sigma) \leq 0 \quad (61) \\
  U[\tau = 0, \sigma] &= |p_0(\sigma)| \quad (62)
\end{align*}
\]

Equation (62) is the special-relativistic formula \( E^2 - p^2 = 0 \) for massless particles, or photons. Computing this, using the first class of solutions we get for the energy of the string, \( \dot{r}(\tau, \sigma) \leq 0 \),

\[
U(\tau, \sigma) = \left[-\frac{2r\dot{r}}{(r^2 + 2\alpha^2 - \alpha^2 R)^{1/2}}\right]
\]

However the canonical momenta used above have been defined with the ambiguity of the above Lagrangian density. This is because (see [12] p.553) if \( L, \tilde{L} \) are two Lagrangian densities connected by

\[
\tilde{L} = L + \frac{\partial}{\partial \tau} \lambda(\phi, \phi, \alpha, \tau, \sigma)
\]

where we denote all the “fields” collectively by \( (\phi) \), then the dynamics that stem from eq (24) is unaltered. This is valid provided that in the variation, in addition to \( (\delta \phi)_{\text{boundary}} = 0 \) one requires that the solutions of our initial Lagrangian \( L \) also satisfy \( (\delta \phi_{\alpha})_{\text{boundary}} = 0 \). But from eq (37) this occurs for this subclass when we set \( f_n = \tilde{f}_n = 0 \) because then the variation of \( \ddot{r} \) and consequently from eqs (32)-(33) of all the other field derivatives are proportional to the variation of \( r \) and therefore vanish at the boundary, \( (\delta \phi_{\alpha})_{\text{boundary}} = 0 \). We shall retain the constants \( f_n, \tilde{f}_n \) for the sake of generality only and set them equal to zero at the end. This gauge freedom allows one to write the energy properly, by choosing the function \( (\lambda) \), as

\[
E(\tau) \equiv \int_0^{2\pi} d\sigma U(\tau, \sigma) =
\int_0^{2\pi} d\sigma \left[ \frac{2r\dot{r}[1]}{(r^2 + 2\alpha^2)^{1/2}} - \frac{1}{(r^2 + 2\alpha^2 - \alpha^2 R)^{1/2}} - p_0(\sigma) \right]
\]

Using eqs (59), (63), (64) and (65), the function \( (\lambda) \) has to satisfy

\[
\frac{\partial^2 \lambda}{\partial \tau \partial \dot{t}} = p_0(\sigma) - \frac{2r\dot{r}}{(r^2 + 2\alpha^2)^{1/2}}
\]

which is directly integrable for \( (\lambda) \) giving \( \lambda = p_0(\sigma)\tau - 2(r^2 + 2\alpha^2)^{1/2}\dot{t} \).

The criterion that enforces this choice is twofold. First, the first term in eq (65) is introduced to correctly reproduce the asymptotic form of the energy. Indeed as
$r = r(\tau, \sigma) \to +\infty$ the energy of the string must irrespectively of the functional form of $\dot{r}(\tau, \sigma) \leq 0$, assume the required value (see eq (72) below). This is satisfied by this choice since we now have $\bar{U}(\tau = 0, \sigma) = |p_0(\sigma)|$.

Secondly one sees that orbits of negative energy exist in a region somewhat larger than the ergosphere $r(\tau, \sigma) \leq R_+$, because the factor $R(r)$ in the second term becomes negative. This is in conformity with what one expects in analogy with the particle case.

So we examine the case that the string decays into two parts and one of them asymptotically $(t \to +\infty)$, as observed by the asymptotic observer, reaches a stable, negative-at infinity energy state [13]. Indeed this happens because, for $r \simeq \rho_+$, from eq (32) we have that

$$\left. \frac{dr}{dt} \right|_{r \simeq \rho_+} \simeq \frac{\alpha^2 + \rho_+^2 R(\rho_+)}{\rho_+(\rho_+^2 + \alpha^2)^{1/2}} = 0$$

(67)

using eq (43).

From eq (37) we have, using the above notation conventions, that substitution of eq (61) gives

$$p_0(\sigma) = 2g\alpha' + \sqrt{\alpha'} \sum_{n \neq 0} [f_n e^{in\sigma} + \tilde{f}_n e^{-in\sigma}]$$

(68)

and from these

$$g = \frac{1}{4\pi\alpha'} \int_0^{2\pi} d\sigma p_0(\sigma)$$

(69)

$$f_n = \frac{1}{2\pi \sqrt{\alpha'}} \int_0^{2\pi} d\sigma p_0(\sigma) e^{-in\sigma}$$

(70)

$$\tilde{f}_n = \frac{1}{2\pi \sqrt{\alpha'}} \int_0^{2\pi} d\sigma p_0(\sigma) e^{+in\sigma}$$

(71)

The energy of the string at infinity is given

$$E_i = E(\tau = 0) = -\int_0^{2\pi} d\sigma p_0(\sigma)$$

(72)

and the energy at later times when it approaches a stable negative-at infinity energy state by

$$E(\tau) = \int_0^{2\pi} d\sigma \left[ 2 \left( \frac{(r^2 + 2\alpha^2 - \alpha^2 R)^{1/2}}{(r^2 + 2\alpha^2)^{1/2}} - 1 \right) \cdot [2g\alpha' + \sqrt{\alpha'} \sum_{n \neq 0} (f_n e^{-in\chi} + \tilde{f}_n e^{-in\chi})] - p_0(\sigma) \right]$$

(73)

where use of eq (37) has been made. We now introduce the previous result that for this orbit we have with high approximation $r \simeq \rho_+$ and set $f_n = \tilde{f}_n = 0$.
Factoring out the term with the \((r)\) dependance and performing the integral, we obtain for the energy in a stable negative-energy state, with the help of the relation \(r^2 \simeq (\alpha^2/(-R))\)

\[
\frac{E(\text{stable})}{E(\tau = 0)} = 1 - 2 \left[ \frac{(\rho_+^2 + \alpha^2)}{\rho_+ \sqrt{\rho_+^2 + 2\alpha^2}} - 1 \right] < 1
\]

(74)

This means that the energy of the string as measured asymptotically has been reduced from its initial value due to the trapping in the negative energy state. Now if the string decays at two parts with a proportionality factor \(\bar{\lambda}\) \((0 < \bar{\lambda} < 1)\) where the \((\bar{\lambda})\) part is trapped in the negative energy state, while the \((1 - \bar{\lambda})\) part escapes at infinity, then the energy that one obtains with respect to the energy that was thrown in the black-hole, is given by the efficiency ratio,

\[
\epsilon \equiv \frac{E_f}{E_i} = 1 + 2\bar{\lambda} \left[ \frac{(\rho_+^2 + \alpha^2)}{\rho_+ \sqrt{\rho_+^2 + 2\alpha^2}} - 1 \right] > 1
\]

(75)

This occurs as follows. The second term on the r.h.s. of eq (74) is negative and so, for the \((\bar{\lambda})\) part of the string it represents the excess energy that is obtained asymptotically. One can verify in eq (75) that the quantity in the brackets is positive. Expanding the square root one sees that the remaining term is of the order of \((\alpha^4)\).

When no decay occurs \((\bar{\lambda} = 0)\) we have, as expected, equality of the emitted and received, energies. Also when the black-hole is non-rotating, \((\alpha = 0)\), we have the same result, that is one obtains no energy gain.

VII. Discussion

Three classes of solutions for null strings, that reside on the equatorial plane of a Kerr-Newmann black hole were presented. All of them depend on two arbitrary integration functions and two integration constants, coming from wavelike differential equations.

From the first of these classes, an efficiency coefficient for the energy extraction was calculated, for a closed string that encircles the black hole and decays into two parts, one of which is trapped at a stable, negative-at infinity, energy state [13-14]. It would be interesting to generalize this argument to the case where the string is not bound to move on the equatorial plane. Also an important case is to consider bosonic strings in their classical as well their quantum description. Of the same importance is to examine the case where one has initially a stationary axisymmetric metric, try to obtain cosmic string solutions of solitonic nature and examine their properties [15]. This however is extremely difficult due to the absence of exact solutions in the general case. Also it would be very interesting to consider more general extended objects such as null bosonic p-branes and examine similar energy-extraction processes for these spacetime backgrounds [16]. Work along these directions is in progress and will reported promptly.
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