Count of eigenvalues in the generalized eigenvalue problem

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Abstract

We address the count of isolated and embedded eigenvalues in a generalized eigenvalue problem defined by two self-adjoint operators with a positive essential spectrum and a finite number of isolated eigenvalues. The generalized eigenvalue problem determines spectral stability of nonlinear waves in a Hamiltonian dynamical system. The theory is based on the Pontryagin’s Invariant Subspace theorem in an indefinite inner product space but it extends beyond the scope of earlier papers of Pontryagin, Krein, Grillakis, and others. Our main results are (i) the number of unstable and potentially unstable eigenvalues equals the number of negative eigenvalues of the self-adjoint operators, (ii) the total number of isolated eigenvalues of the generalized eigenvalue problem is bounded from above by the total number of isolated eigenvalues of the self-adjoint operators, and (iii) the quadratic form defined by the indefinite inner product is strictly positive on the subspace related to the absolutely continuous part of the spectrum of the generalized eigenvalue problem. Applications to solitons and vortices of the nonlinear Schrödinger equations and solitons of the Korteweg–De Vries equations are developed from the general theory.

Keywords: generalized eigenvalue problem, discrete and continuous spectrum, indefinite metric, invariant subspaces, isolated eigenvalues, Krein signature
1 Introduction

Stability of equilibrium points in a Hamiltonian system of finitely many interacting particles is defined by the eigenvalues of the generalized eigenvalue problem \([GK02]\),

\[
Au = \gamma Ku, \quad u \in \mathbb{R}^n, \tag{1.1}
\]

where \(A\) and \(K\) are symmetric matrices in \(\mathbb{R}^{n \times n}\) which define the quadratic forms for potential and kinetic energies, respectively. The eigenvalue \(\gamma\) corresponds to the normal frequency \(\lambda = i\omega\) of the normal mode of the linearized Hamiltonian system near the equilibrium point, such that \(\gamma = -\lambda^2 = \omega^2\). The linearized Hamiltonian system is said to have an unstable eigenvalue \(\gamma\) if \(\gamma < 0\) or \(\text{Im}(\gamma) \neq 0\). Otherwise, the system is weakly spectrally stable. Moreover, the equilibrium point is a minimizer of the Hamiltonian if all eigenvalues \(\gamma\) are positive and semi-simple and the quadratic forms for potential and kinetic energies evaluated at eigenvectors of (1.1) are strictly positive.

When the matrix \(K\) is positive definite, all eigenvalues \(\gamma\) are real and semi-simple (that is the geometric and algebraic multiplicities coincide). By the Sylvester’s Inertia Law theorem \([G61]\), the numbers of positive, zero and negative eigenvalues of the generalized eigenvalue problem (1.1) equal to the numbers of positive, zero and negative eigenvalues of the matrix \(A\). When \(K\) is not positive definite, a complete classification of eigenvalues \(\gamma\) in terms of real eigenvalues of \(A\) and \(K\) has been developed with the use of the Pontryagin’s Invariant Subspace theorem \([P44]\), which generalizes the Sylvester’ Inertia Law theorem.

We are concerned with spectral stability of spatially localized solutions in a Hamiltonian infinite-dimensional dynamical system. In many problems, a linearization of the nonlinear system at the spatially localized solution results in the generalized eigenvalue problem of the form (1.1) but \(A\) and \(K^{-1}\) are now self-adjoint operators on a complete infinite-dimensional metric space. There has been recently a rapidly growing sequence of publications on mathematical analysis of the spectral stability problem in the context of nonlinear Schrödinger equations and other nonlinear evolution equations \([CPV05, GKP04, KKS04, KP05, KS05, P05]\). Besides predictions of spectral stability or instability of spatially localized solutions in Hamiltonian dynamical systems, linearized Hamiltonian systems are important in analysis of orbital stability \([GSS87, GSS90, CP03]\), asymptotic stability \([P04, RSS05, C03]\), stable manifolds \([CCP05, S05]\), and blow-up of solutions in nonlinear equations \([P01]\).

It is the purpose of this article to develop analysis of the generalized eigenvalue problem in infinite dimensions by using the Pontryagin space decomposition \([P44]\). The theory of Pontryagin spaces was developed by M.D. Krein and his students (see books \([AI86, GK69, IKL82]\)) and partly used in the context of spectral stability of solitary waves by MacKay \([M87]\), Grillakis \([G90]\), and Buslaev & Perelman \([BP93]\) (see also a recent application in \([GKP04]\)). We shall give an elegant geometric proof of the Pontryagin’s Invariant Subspace theorem. An application of the theorem recovers the main results obtained in \([CPV05, KKS04, P05]\). Moreover, we obtain a new inequality on the number of positive eigenvalues of
the linearized Hamiltonian that extends the count of eigenvalues of the generalized eigenvalue problem.

The structure of the paper is as follows. Main formalism of the generalized eigenvalue problem is described in Section 2. The Pontryagin’s Invariant Subspace theorem is proved in Section 3. Main results on eigenvalues of the generalized eigenvalue problem are formulated and proved in Section 4. Section 5 contains applications of the main results to solitons and vortices of the nonlinear Schrödinger equations and solitons of the Korteweg–De Vries equations.

2 Formalism

Let \( L_+ \) and \( L_- \) be two self-adjoint operators defined on the Hilbert space \( \mathcal{X} \) with the inner product \( (\cdot, \cdot) \). Our main assumptions are listed below.

P1 The essential spectrum of \( L_{\pm} \) in \( \mathcal{X} \) includes the absolute continuous part \( \sigma_c(L_{\pm}) \) bounded from below by \( \omega_+ \geq 0 \) and \( \omega_- > 0 \) and finitely many embedded eigenvalues of finite multiplicities.

P2 The discrete spectrum of \( L_{\pm} \) in \( \mathcal{X} \) includes finitely many isolated eigenvalues of finite multiplicities with \( p(L_{\pm}) \) positive, \( z(L_{\pm}) \) zero, and \( n(L_{\pm}) \) negative eigenvalues.

We shall consider the linearized Hamiltonian problem defined by the self-adjoint operators \( L_{\pm} \) in \( \mathcal{X} \),

\[
L_+ u = -\lambda w, \quad L_- w = \lambda u,
\]

where \( \lambda \in \mathbb{C} \) and \((u, w) \in \mathcal{X} \times \mathcal{X}\). By assumption P1, the kernel of \( L_- \) is isolated from the essential spectrum. Let \( \mathcal{H} \) be the constrained Hilbert space,

\[
\mathcal{H} = \{ u \in \mathcal{X} : \ u \perp \ker(L_-) \},
\]

and let \( \mathcal{P} \) be the orthogonal projection from \( \mathcal{X} \) to \( \mathcal{H} \). The linearized Hamiltonian problem (2.1) for non-zero eigenvalues \( \lambda \neq 0 \) is rewritten as the generalized eigenvalue problem

\[
Au = \gamma Ku, \quad u \in \mathcal{H},
\]

where \( A = \mathcal{P} L_+ \mathcal{P}, \ K = \mathcal{P} L_-^{-1} \mathcal{P}, \) and \( \gamma = -\lambda^2 \). We note that \( K \) is a bounded invertible self-adjoint operator on \( \mathcal{H} \), while properties of \( A \) follow from those of \( L_+ \). Finitely many isolated eigenvalues of the operators \( A \) and \( K^{-1} \) in \( \mathcal{H} \) are distributed between negative, zero and positive eigenvalues away of \( \sigma_c(L_{\pm}) \). By the spectral theory of the self-adjoint operators, the Hilbert space \( \mathcal{H} \) can be equivalently decomposed into two orthogonal sums of subspaces which are invariant with respect to the operators \( K \) and \( A \):

\[
\mathcal{H} = \mathcal{H}_K^- \oplus \mathcal{H}_K^+ \oplus \mathcal{H}_K^{\sigma_c(K)},
\]

\[
\mathcal{H} = \mathcal{H}_A^- \oplus \mathcal{H}_A^0 \oplus \mathcal{H}_A^+ \oplus \mathcal{H}_A^{\sigma_c(A)},
\]
where notation \((-(+))\) stands for the negative (positive) isolated eigenvalues, 0 for the isolated kernel, and \(\sigma_e\) for the essential spectrum that includes the absolute continuous part and embedded eigenvalues. It is clear that \(\sigma_e(K)\) belongs to the interval \((0,\omega^{-1})\) and \(\sigma_e(A)\) belongs to the interval \([\omega_+,\infty)\). Since \(P\) is a projection defined by eigenspaces of \(L_-\) while \(K = P L_0^\perp P\), it is clear that \(\dim(H_K^-) = n(L_-)\) and \(\dim(H_K^+) = p(L_-)\).

**Proposition 2.1** Let \(\omega_+ > 0\). There exist \(n_0 \geq 0, z_0 \geq 0, \) and \(z_1 \geq 0,\) such that

\[
\dim(H_A^-) = n(L_+)-n_0, \quad \dim(H_A^0) = z(L_+)-z_0+z_1, \quad \dim(H_A^+) \leq p(L_+)+n_0+z_0-z_1. \tag{2.6}
\]

**Proof.** Let \(\ker(L_-) = \{v_1, v_2, \ldots, v_n\} \subset \mathcal{X}\) and define the matrix-valued function \(A(\mu)\):

\[
A_{ij}(\mu) = ((\mu - L_+)^{-1}v_i, v_j), \quad 1 \leq i, j \leq n
\]

for all \(\mu\) not in the spectrum of \(L_+\). When \(z(L_+) = 0\), the first two equalities (2.6) follow by the abstract Lemma 3.4 in [CPV05], where \(n_0\) is the number of nonnegative eigenvalues of \(A(0)\), \(z_0 = 0\), and \(z_1\) is the number of zero eigenvalues of \(A(0)\). When \(z(L_+) \neq 0\), the same proof is extended for \(A_0 = \lim_{\mu \to 0^-} A(\mu)\) where \(z_0\) is the number of eigenvectors in the kernel of \(L_+\) in \(\mathcal{X}\) which do not belong to the space \(\mathcal{H}\) (see proof of Theorem 2.9 in [CPV05]). The last inequality in (2.6) is obtained by extending the analysis of [CPV05] from \(\mu = 0\) to \(\mu = \omega_+ > 0\), where the upper bound is achieved if all eigenvalues of \(A_+ = \lim_{\mu \to \omega_+^-} A(\mu)\) are negative. \(\square\)

Since \(A\) has finitely many negative eigenvalues and \(K\) has no kernel in \(\mathcal{H}\), there exists a small number \(\delta > 0\) in the gap \(0 < \delta < |\sigma_-|\), where \(\sigma_-\) is the smallest (in absolute value) negative eigenvalue of \(K^{-1} A\). The operator \(A + \delta K\) is continuously invertible in \(\mathcal{H}\) and the generalized eigenvalue problem (2.3) is rewritten in the shifted form,

\[
(A + \delta K)u = (\gamma + \delta)K u, \quad u \in \mathcal{H}. \tag{2.7}
\]

By the spectral theory, an alternative decomposition of the Hilbert space \(\mathcal{H}\) exists for \(0 < \delta < |\sigma_-|\):

\[
\mathcal{H} = H_{A+\delta K}^- \oplus H_{A+\delta K}^+ \oplus H_{\sigma_e(A+\delta K)}, \tag{2.8}
\]

where \(\sigma_e(A + \delta K)\) belongs to the interval \([\omega_{A+\delta K}, \infty)\) and \(\omega_{A+\delta K}\) is the minimum of \(\sigma_e(A + \delta K)\). We will assume that \(\omega_{A+\delta K} > 0\) for \(\delta > 0\) even if \(\omega_+ = 0\).

**Proposition 2.2** Let the number of negative (positive) eigenvalues of \(A + \delta K\) bifurcating from the zero eigenvalues of \(A\) as \(\delta > 0\) be denoted as \(n_- (n_+)\). When \(\omega_+ > 0\), the splitting is complete so that

\[
\dim(H_{A+\delta K}^\pm) = \dim(H_A^\pm) + n_\pm, \quad \dim(H_A^0) = n_- + n_. \tag{2.9}
\]

When \(\omega_+ = 0\) and \(\dim(H_A^0) = 1\), the statement remains valid provided that the bifurcating eigenvalue of \(A + \delta K\) is smaller than \(\omega_{A+\delta K} > 0\).
Proof. For any \( \delta \in (0, |\sigma_1|) \), the operator \( A + \delta K \) has no zero eigenvalues in \( \mathcal{H} \). The equality (2.9) follows from the definition of \( n_- \) and \( n_+ \).

The Pontryagin’s Invariant Subspace theorem can be applied to the product of two bounded invertible self-adjoint operators \((A + \delta K)^{-1}\) and \( K \) in \( \mathcal{H} \).

3 The proof of the Pontryagin’s Invariant Subspace theorem

We shall develop an abstract theory of Pontryagin spaces with sign-indefinite metric, where the main result is the Pontryagin’s Invariant Subspace theorem.

Definition 3.1 Let \( \mathcal{H} \) be a Hilbert space equipped with the inner product \((\cdot, \cdot)\) and the sesquilinear form \([\cdot, \cdot]_1\). The Hilbert space \( \mathcal{H} \) is called the Pontryagin space (denoted as \( \Pi_\kappa \)) if it can be decomposed into the sum, which is orthogonal with respect to \([\cdot, \cdot]_1\),

\[ \mathcal{H} \cong \Pi_\kappa = \Pi_+ \oplus \Pi_- , \quad (3.1) \]

where \( \Pi_+ \) is a Hilbert space with the inner product \((\cdot, \cdot) = [\cdot, \cdot]_1\), \( \Pi_- \) is a Hilbert space with the inner product \((\cdot, \cdot) = -[\cdot, \cdot]_1\), and \( \kappa = \dim(\Pi_-) < \infty \).

Remark 3.2 We shall write components of an element \( x \) in the Pontryagin space \( \Pi_\kappa \) as a vector \( x = \{x_-, x_+\} \). The orthogonal sum (3.1) implies that any non-zero element \( x \neq 0 \) is represented by two terms,

\[ \forall x \in \Pi_\kappa : \ x = x_+ + x_- , \quad (3.2) \]

such that

\[ [x_+, x_-] = 0 , \ [x_+, x_+] > 0 , \ [x_-, x_-] < 0 , \quad (3.3) \]

and \( \Pi_+ \cap \Pi_- = \emptyset \).

Definition 3.3 We say that \( \Pi \) is a non-positive subspace of \( \Pi_\kappa \) if \([x, x] \leq 0 \) \( \forall x \in \Pi \). We say that the non-positive subspace \( \Pi \) has the maximal dimension \( \kappa \) if any subspace of \( \Pi_\kappa \) of dimension higher than \( \kappa \) is not a non-positive subspace of \( \Pi_\kappa \). Similarly, \( \Pi \) is a non-negative (neutral) subspace of \( \Pi_\kappa \) if \([x, x] \geq 0 \) \( ([x, x] = 0) \forall x \in \Pi \). The sign of \([x, x]\) on the element \( x \) of a subspace \( \Pi \) is called the Krein signature of the subspace \( \Pi \).

Theorem 1 (Pontryagin) Let \( T \) be a self-adjoint bounded operator in \( \Pi_\kappa \), such that \([T, \cdot] = [\cdot, T\cdot]\). There exists a \( T \)-invariant non-positive subspace of \( \Pi_\kappa \) of the maximal dimension \( \kappa \).

\(^1\)We say that a complex-valued form \([u, v]\) on the product space \( \mathcal{H} \times \mathcal{H} \) is a sesquilinear form if it is linear in \( u \) for each fixed \( v \) and linear with complex conjugate in \( v \) for each fixed \( u \).
**Remark 3.4** There are historically two completely different approaches to the proof of this theorem. A proof based on theory of analytic functions was given by Pontryagin [P44], while a proof based on angular operators was given by Krein [GK69] and later developed by students of M.G. Krein [A186, IKL82]. Theorem 1 was rediscovered by Grillakis [G90] with the use of topology. We will describe a geometric proof of Theorem 1 based on the Fixed Point theorem. The proof uses the Cayley transformation of a self-adjoint operator in $\Pi_\kappa$ to a unitary operator in $\Pi_\kappa$ (Lemma 3.5) and the Krein's representation of the maximal non-positive subspace of $\Pi_\kappa$ in terms of a graph of the contraction map (Lemma 3.7). While many statements of our analysis are available in the literature, details of the proofs are missing. Our presentation gives full details of the proof of Theorem 1 (see [GKP04] for a similar treatment in the case of compact operators).

**Lemma 3.5** Let $T$ be a linear operator in $\Pi_\kappa$ and $z \in \mathbb{C}$, $\text{Im}(z) > 0$ be a regular point of the operator $T$, such that $z \in \rho(T)$. Let $U$ be the Cayley transform of $T$ defined by $U = (T - \bar{z})(T - z)^{-1}$. The operators $T$ and $U$ have the same invariant subspaces in $\Pi_\kappa$.

**Proof.** Let $\Pi$ be a finite-dimensional invariant subspace of the operator $T$ in $\Pi_\kappa$. It follows from $z \in \rho(T)$ that $(T - z)\Pi = \Pi$ then $(T - z)^{-1}\Pi = \Pi$ and $(T - \bar{z})(T - z)^{-1}\Pi \subseteq \Pi$, i.e. $U\Pi \subseteq \Pi$. Conversely, let $\Pi$ be an invariant subspace of the operator $U$. It follows from $U - I = (z - \bar{z})(T - z)^{-1}$ that $1 \in \rho(U)$ therefore $\Pi = (U - I)\Pi = (T - z)^{-1}\Pi$. From there, $\Pi \subseteq \text{dom}(T)$ and $(T - z)\Pi = \Pi$ so $T\Pi \subseteq \Pi$. ■

**Corollary 3.6** If $T$ is a self-adjoint operator in $\Pi_\kappa$, then $U$ is a unitary operator in $\Pi_\kappa$.

**Proof.** We shall prove that $[Ug, Ug] = [g, g]$, where $g \in \text{dom}(U)$, by the explicit computation:

$$
[Ug, Ug] = [(T - z)f, (T - z)f] = [Tf, Tf] - z[f, Tf] - \bar{z}[Tf, f] + |z|^2[f, f],
$$

$$
[g, g] = [(T - z)f, (T - z)f] = [Tf, Tf] - \bar{z}[f, Tf] - z[Tf, f] + |z|^2[f, f],
$$

where we have introduced $f \in \text{dom}(T)$ such that $f = (T - z)^{-1}g$. ■

**Lemma 3.7** A linear subspace $\Pi \subseteq \Pi_\kappa$ is a $\kappa$-dimensional non-positive subspace of $\Pi_\kappa$ if and only if it is a graph of the contraction map $\mathcal{K} : \Pi_- \rightarrow \Pi_+$, such that $\Pi = \{x_-, \mathcal{K}x_+\}$ and $\|\mathcal{K}x_-\| \leq \|x_-\|$.

**Proof.** Let $\Pi = \{x_-, x_+\}$ be a $\kappa$-dimensional non-positive subspace of $\Pi_\kappa$. We will show that there exist a contraction map $\mathcal{K} : \Pi_- \rightarrow \Pi_+$ such that $\Pi$ is a graph of $\mathcal{K}$. Indeed, the subspace $\Pi$ is a graph of a linear operator $\mathcal{K}$ if and only if it follows from $\{0, x_+\} \in \Pi$ that $x_+ = 0$. Since $\Pi$ is non-positive with respect to $[\cdot, \cdot]$, then $[x, x] = \|x_+\|^2 - \|x_-\|^2 \leq 0$, where $\|\cdot\|$ is a norm in $\mathcal{H}$. As a result, $0 \leq \|x_+\| \leq \|x_-\|$ and if $x_- = 0$ then $x_+ = 0$. Moreover, for any $x_- \in \Pi_-$, it is true that $\|\mathcal{K}x_-\| \leq \|x_-\|$ such that $\mathcal{K}$ is a...
contraction map. Conversely, let $\mathcal{K}$ be a contraction map $\mathcal{K} : \Pi_- \mapsto \Pi_+$. The graph of $\mathcal{K}$ belongs to the non-positive subspace of $\Pi_\kappa$ as

$$ [x,x] = \|x_+\|^2 - \|x_-\|^2 = \|\mathcal{K}x_-\|^2 - \|x_-\|^2 \leq 0. $$

Let $\Pi = \{x_-, \mathcal{K}x_-\}$. Since $\dim(\Pi_-) = \kappa$, then $\dim(\Pi) = \kappa. \quad \Box$

**Proof of Theorem 1** Let $z \in \mathbb{C}$ and $\text{Im}(z) > 0$. Then, $z$ is a regular point of the self-adjoint operator $T$ in $\Pi_\kappa$. Let $U = (T - \bar{z})(T - z)^{-1}$ be the Cayley transform of $T$. By Corollary 3.6, $U$ is a unitary operator in $\Pi_\kappa$. By Lemma 3.5 $T$ and $U$ have the same invariant subspaces in $\Pi_\kappa$. Therefore, the existence of the maximal non-positive invariant subspace for the self-adjoint operator $T$ can be proved from the existence of such a subspace for the unitary operator $U$. Let $x = \{x_-, x_+\}$ and

$$ U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} $$

be the matrix representation of the operator $U$ with respect to the decomposition (3.1). Let $\Pi$ denote a $\kappa$-dimensional non-positive subspace in $\Pi_\kappa$. Since $U$ has an empty kernel in $\Pi_\kappa$ and $U$ is unitary in $\Pi_\kappa$ such that $[Ux_-, Ux_-] = [x_-, x_-] \leq 0$, then $\tilde{\Pi} = U\Pi$ is also a $\kappa$-dimensional non-positive subspace of $\Pi_\kappa$. By Lemma 3.7, there exist two contraction mappings $\mathcal{K}$ and $\tilde{\mathcal{K}}$ for subspaces $\Pi$ and $\tilde{\Pi}$, respectively. Therefore, the assignment $\tilde{\Pi} = U\Pi$ is equivalent to the system,

$$ \begin{pmatrix} \tilde{x}_- \\ \tilde{\mathcal{K}}\tilde{x}_- \end{pmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{pmatrix} x_- \\ \mathcal{K}x_- \end{pmatrix} = \begin{pmatrix} (U_{11} + U_{12}\mathcal{K})x_- \\ (U_{21} + U_{22}\mathcal{K})x_- \end{pmatrix}, $$

such that $U_{21} + U_{22}\mathcal{K} = \tilde{\mathcal{K}}(U_{11} + U_{12}\mathcal{K})$.

We shall prove that the operator $(U_{11} + U_{12}\mathcal{K})$ is invertible. By contradiction, we assume that there exists $x_- \neq 0$ such that $\tilde{x}_- = (U_{11} + U_{12}\mathcal{K})x_- = 0$. Since $\tilde{x}_- = 0$ implies that $\tilde{x}_+ = \tilde{\mathcal{K}}\tilde{x}_- = 0$, we obtain that $[x_-, \mathcal{K}x_-]^T$ is an eigenvector in the kernel of $U$. However, $U$ has an empty kernel in $\Pi_\kappa$ such that $x_- = 0$. Let $F(\mathcal{K})$ be an operator-valued function in the form,

$$ F(\mathcal{K}) = (U_{21} + U_{22}\mathcal{K})(U_{11} + U_{12}\mathcal{K})^{-1}, $$

such that $\tilde{\mathcal{K}} = F(\mathcal{K})$. By Lemma 3.7, the operator $F(\mathcal{K})$ maps the operator unit ball $\|\mathcal{K}\| \leq 1$ to itself. Since $U$ is a continuous operator and $U_{12}$ is a finite-dimensional operator, then $U_{12}$ is a compact operator. Hence the operator ball $\|\mathcal{K}\| \leq 1$ is a weakly compact set and the function $F(\mathcal{K})$ is continuous with respect to weak topology. By the Schauder’s Fixed-Point Principle, there exists a fixed point $\mathcal{K}_0$ such that

\footnote{Extending arguments of Lemma 3.7 one can prove that the subspace $\Pi$ is strictly negative with respect to $[\cdot, \cdot]$ if and only if it is a graph of the strictly contraction map $\mathcal{K} : \Pi_- \mapsto \Pi_+$, such that $\Pi = \{x_-, \mathcal{K}x_-\}$ and $\|\mathcal{K}x_-\| < \|x_-\|$.}
$F(K_0) = K_0$ and $\|K_0\| \leq 1$. By Lemma 3.7, the graph of $K_0$ defines the $\kappa$-dimensional non-positive subspace $\Pi$, which is invariant with respect to $U$.

It remains to prove that the $\kappa$-dimensional non-positive subspace $\Pi$ has the maximal dimension that a non-positive subspace of $\Pi_\kappa$ can have. By contradiction, we assume that there exists a $(\kappa + 1)$-dimensional non-positive subspace $\tilde{\Pi}$. Let $\{e_1, e_2, ..., e_\kappa\}$ be a basis in $\Pi_\kappa$ in the canonical decomposition (3.2). We fix two elements $y_1, y_2 \in \tilde{\Pi}$ with the same projections to $\{e_1, e_2, ..., e_\kappa\}$, such that

$$y_1 = \alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_\kappa e_\kappa + y_{1p},$$

$$y_2 = \alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_\kappa e_\kappa + y_{2p},$$

where $y_{1p}, y_{2p} \in \Pi_\kappa$. It is clear that $y_1 - y_2 = y_{1p} - y_{2p} \in \Pi_\kappa$ such that $[y_{1p} - y_{2p}, y_{1p} - y_{2p}] > 0$.

On the other hand, $y_1 - y_2 \in \tilde{\Pi}$, such that $[y_1 - y_2, y_1 - y_2] \leq 0$. Hence $y_{1p} = y_{2p}$ and then $y_1 = y_2$.

We proved that any vector in $\tilde{\Pi}$ is uniquely determined by the $\kappa$-dimensional basis $\{e_1, e_2, ..., e_\kappa\}$ and the non-positive subspace $\tilde{\Pi}$ is hence $\kappa$-dimensional.

\section{Bounds on eigenvalues of the generalized eigenvalue problem}

We shall apply Theorem 1 to the product of two bounded invertible self-adjoint operators $B = (A+\delta K)^{-1}$ and $K$, where $\delta \in (0, |\sigma_-|)$ and $\sigma_-\kappa$ is the smallest negative eigenvalue of $K^{-1}A$. Properties of self-adjoint operators $A$ and $K^{-1}$ in $\mathcal{H}$ follow from properties of self-adjoint operators $L_{\pm}$ in $\mathcal{X}$, which are summarized in the main assumptions P1–P2. With a slight abuse of notations, we shall denote eigenvalues of the operator $T = BK$ by $\lambda$, which is expressed in terms of the eigenvalue $\gamma$ of the shifted generalized eigenvalue problem (2.7) by $\lambda = (\gamma + \delta)^{-1}$. We note that $\lambda$ here does not correspond to $\lambda$ used in the linearized Hamiltonian problem (2.1).

\begin{lemma}
Let $\mathcal{H}$ be a Hilbert space with the inner product $(\dot{}, \dot{})$ and $B, K : \mathcal{H} \to \mathcal{H}$ be bounded invertible self-adjoint operators in $\mathcal{H}$. Define the sesquilinear form $[\dot{}, \dot{}] = (K\dot{}, \dot{})$ and extend $\mathcal{H}$ to the Pontryagin space $\Pi_\kappa$, where $\kappa$ is the finite number of negative eigenvalues of $K$ counted with their multiplicities. The operator $T = BK$ is self-adjoint in $\Pi_\kappa$ and there exists a $\kappa$-dimensional non-positive subspace of $\Pi_\kappa$ which is invariant with respect to $T$.
\end{lemma}

\begin{proof}
It follows from the orthogonal sum decomposition (2.4) that the quadratic form $(K\dot{}, \dot{})$ is strictly negative on the $\kappa$-dimensional subspace $\mathcal{H}_K^+$ and strictly positive on the infinite-dimensional subspace $\mathcal{H}_K^- \oplus \mathcal{H}_K^{\sigma_\kappa(K)}$. By continuity and Gram–Schmidt orthogonalization, the Hilbert space $\mathcal{H}$ is extended to the Pontryagin space $\Pi_\kappa$ with respect to the sesquilinear form $[\dot{}, \dot{}] = (K\dot{}, \dot{})$. The bounded operator $T = BK$ is self-adjoint in $\Pi_\kappa$, since $B$ and $K$ are self-adjoint in $\mathcal{H}$ and

$$[T\dot{}, \dot{}] = (KBK\dot{}, \dot{}) = (K\dot{}, BK\dot{}) = [\dot{}, T\dot{}].$$

\end{proof}
Existence of the \(\kappa\)-dimensional non-positive \(T\)-invariant subspace of \(\Pi_\kappa\) follows from Theorem 1.

**Remark 4.2** The decomposition (3.1) of the Pontryagin space \(\Pi_\kappa\) is canonical in the sense that \(\Pi_+ \cap \Pi_- = \emptyset\). We will now consider various sign-definite subspaces of \(\Pi_\kappa\) which are invariant with respect to the self-adjoint operator \(T = BK\) in \(\Pi_\kappa\). In general, these invariant sign-definite subspaces do not provide a canonical decomposition of \(\Pi_\kappa\). Let us denote the invariant subspace of \(T\) associated with complex eigenvalues in the upper (lower) half-plane as \(\mathcal{H}_{c+} (\mathcal{H}_{c-})\) and the non-positive (non-negative) invariant subspace of \(T\) associated with real eigenvalues as \(\mathcal{H}_{n} (\mathcal{H}_{p})\). The invariant subspace \(\mathcal{H}_n\), prescribed by Lemma 4.1, may include both isolated and embedded eigenvalues of \(T\) in \(\Pi_\kappa\). We will show that this subspace does not include the residual and absolutely continuous parts of the spectrum of \(T\) in \(\Pi_\kappa\).

### 4.1 Residual and absolutely continuous spectra of \(T\) in \(\Pi_\kappa\)

**Definition 4.3** We say that \(\lambda\) is a point of the residual spectrum of \(T\) in \(\Pi_\kappa\) if \(\ker(T - \lambda I) = \emptyset\) but \(\range(T - \lambda I) \neq \Pi_\kappa\) and \(\lambda\) is a point of the continuous spectrum of \(T\) in \(\Pi_\kappa\) if \(\ker(T - \lambda I) = \emptyset\) but \(\range(T - \lambda I) \neq \range(T - \lambda I) = \Pi_\kappa\).

**Lemma 4.4** No residual part of the spectrum of \(T\) in \(\Pi_\kappa\) exists.

**Proof.** By a contradiction, assume that \(\lambda\) belongs to the residual part of the spectrum of \(T\) in \(\Pi_\kappa\) such that \(\ker(T - \lambda I) = \emptyset\) but \(\range(T - \lambda I)\) is not dense in \(\Pi_\kappa\). Let \(g \in \Pi_\kappa\) be orthogonal to \(\range(T - \lambda I)\), such that

\[
\forall f \in \Pi_\kappa : \quad 0 = [(T - \lambda I)f, g] = [f, (T - \bar{\lambda}I)g].
\]

Therefore, \((T - \bar{\lambda}I)g = 0\), that is \(\bar{\lambda}\) is an eigenvalue of \(T\). By symmetry of eigenvalues, \(\lambda\) is also an eigenvalue of \(T\) and hence it can not be in the residual part of the spectrum of \(T\).

**Remark 4.5** It is assumed in [G90] that the residual part of spectrum is empty and that the kernels of operators \(A\) and \(K\) are empty. The first assumption is now proved in Lemma 4.4 and the second assumption is removed with the use of the shifted generalized eigenvalue problem (2.7).

**Lemma 4.6** The absolutely continuous part of the spectrum of \(T\) in \(\Pi_\kappa\) is real.

**Proof.** Let \(P^+\) and \(P^-\) be orthogonal projectors to \(\Pi^+\) and \(\Pi^-\) respectively, such that \(I = P^+ + P^-\). Since \(\Pi^\pm\) are defined by \((\cdot, K^\cdot)\), the self-adjoint operator \(K\) admits the polar decomposition \(K = J|K|\), where \(J = P^+ - P^-\) and \(|K|\) is a positive operator. The operator \(T = BK\) is similar to the operator

\[
|K|^{1/2}BJ|K|^{1/2} = |K|^{1/2}B|K|^{1/2}(J + 2P^-) = |K|^{1/2}B|K|^{1/2} + 2|K|^{1/2}BJ|K|^{1/2}P^-.
\]
Since $P^-$ is a projection to a finite-dimension subspace, the operator $|K|^{1/2}BJ|K|^{1/2}$ is the finite-dimensional perturbation of the self-adjoint operator $|K|^{1/2}B|K|^{1/2}$. By perturbation theory [K76], the absolutely continuous part of the spectrum of the self-adjoint operator $|K|^{1/2}B|K|^{1/2}$ is the same as that of $|K|^{1/2}BJ|K|^{1/2}$. By similarity transformation, it is the same as that of $T$.

**Lemma 4.7 (Cauchy-Schwartz)** Let $\Pi$ be either non-positive or non-negative subspace of $\Pi_\kappa$. Then,

$$\forall f, g \in \Pi : \quad ||f, g||^2 \leq [f, f][g, g].$$

**Proof.** The proof resembles that of the standard Cauchy–Schwartz inequality. Let $\Pi$ be a non-positive subspace of $\Pi_\kappa$. Then, for any $f, g \in \Pi$ and any $\alpha, \beta \in \mathbb{C}$, we have $[\alpha f + \beta g, \alpha f + \beta g] \leq 0$ and

$$[\alpha f + \beta g, \alpha f + \beta g] = [f, f]|\alpha|^2 + [f, g]|\alpha\beta + [g, f]|\alpha\beta + [g, g]|\beta|^2.$$  

(4.2)

If $[f, g] = 0$, then (4.1) is satisfied as $[f, f] \leq 0$ and $[g, g] \leq 0$. If $[f, g] \neq 0$, let $\alpha \in \mathbb{R}$ and

$$\beta = \frac{[f, g]}{||f, g||},$$

such that

$$[f, f]|\alpha|^2 + 2\alpha||f, g|| + |g, g| \leq 0.$$  

The last condition is satisfied if the discriminant of the quadratic equation is non-positive such that $4||f, g||^2 - 4[f, f][g, g] \leq 0$, that is (4.1). Let $\Pi$ be a non-negative subspace of $\Pi_\kappa$. Then, for any $f, g \in \Pi$ and any $\alpha, \beta \in \mathbb{C}$, we have $[\alpha f + \beta g, \alpha f + \beta g] \geq 0$ and the same arguments result in the same inequality (4.1).

**Corollary 4.8** Let $\Pi$ be either non-positive or non-negative subspace of $\Pi_\kappa$. Let $f \in \Pi$ such that $[f, f] = 0$. Then $[f, g] = 0 \forall g \in \Pi$.

**Proof.** The proof follows from (4.1) since $0 \leq ||f, g||^2 \leq 0$.

**Lemma 4.9** Let $\Pi$ be an invariant subspace of $\Pi_\kappa$ with respect to operator $T$ and $\Pi^\perp$ be the orthogonal compliment of $\Pi$ in $\Pi_\kappa$ with respect to $[\cdot, \cdot]$. Then, $\Pi^\perp$ is also invariant with respect to $T$.

**Proof.** For all $f \in \text{Dom}(T) \cap \Pi$, we have $Tf \in \Pi$. Let $g \in \text{Dom}(T) \cap \Pi^\perp$. Then $[g, Tf] = [Tg, f] = 0$.

**Theorem 2** Let $\Pi_c$ be a subspace related to the absolute continuous spectrum of $T$ in $\Pi_\kappa$. Then $[f, f] > 0$ $\forall f \in \Pi_c$. 

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Proof. By contradiction, assume that there exists \( f_0 \in \Pi_\kappa \) such that \([f_0, f_0] < 0\). Since \( \Pi_\kappa \) is a subspace for the absolutely continuous spectrum, there exists a continuous family of functions \( f_\alpha \in \Pi_\kappa \) such that \([f_\alpha, f_\alpha] < 0\) and hence \( f_\alpha \in \Pi_{-\kappa} \) in the decomposition (3.1). However, this contradicts to the fact that \( \dim(\Pi_{-\kappa}) = \kappa < \infty \). Therefore, \( \Pi_\kappa \) is a non-negative subspace of \( \Pi_\kappa \). Assume that there exists an element \( f_0 \in \Pi_\kappa \) such that \([f_0, f_0] = 0\). By Corollary 4.8, \( f_0 \in \Pi_\kappa^+ \). By Lemma 4.9, \( \Pi_\kappa^+ \) is an invariant subspace of \( \Pi_\kappa \). The intersection of invariant subspaces is invariant, such that \( \Pi_\kappa \cap \Pi_\kappa^+ \) is a neutral invariant subspace of \( \Pi_\kappa \). By Lemma 4.1, \( \dim(\Pi_\kappa \cap \Pi_\kappa^+) \leq \kappa \). Therefore, \( f_0 \) is an element of a finite-dimensional invariant subspace of \( \Pi_\kappa \), which is a contradiction to the fact that \( f_0 \in \Pi_\kappa \).

4.2 Isolated and embedded eigenvalues of \( T \) in \( \Pi_\kappa \)

Definition 4.10 We say that \( \lambda \) is an eigenvalue of \( T \) in \( \Pi_\kappa \) if \( \text{Ker}(T - \lambda I) \neq \emptyset \). The eigenvalue of \( T \) is said to be semi-simple if it is not simple and the algebraic and geometric multiplicities coincide. Otherwise, the non-simple eigenvalue is said to be multiple. Let \( \lambda_0 \) be an eigenvalue of \( T \) with algebraic multiplicity \( n \) and geometric multiplicity one. The canonical basis for the corresponding \( n \)-dimensional eigenspace of \( T \) is defined by the Jordan chain of eigenvectors,

\[
f_j \in \Pi_\kappa : \quad Tf_j = \lambda_0 f_j + f_{j-1}, \quad j = 1, \ldots, n,
\]

where \( f_0 = 0 \).

Lemma 4.11 Let \( \mathcal{H}_\lambda \) and \( \mathcal{H}_\mu \) be eigenspaces of the eigenvalues \( \lambda \) and \( \mu \) of the operator \( T \) in \( \Pi_\kappa \) and \( \lambda \neq \mu \). Then \( \mathcal{H}_\lambda \) is orthogonal to \( \mathcal{H}_\mu \) with respect to \([\cdot, \cdot]\).

Proof. Let \( n \) and \( m \) be dimensions of \( \mathcal{H}_\lambda \) and \( \mathcal{H}_\mu \), respectively, such that \( n \geq 1 \) and \( m \geq 1 \). By Definition 4.10, it is clear that

\[
f \in \mathcal{H}_\lambda \iff (T - \lambda I)^n f = 0, \quad (4.4)
g \in \mathcal{H}_\mu \iff (T - \mu I)^m g = 0. \quad (4.5)
\]

We should prove that \([f, g] = 0\) by induction for \( n + m \geq 2 \). If \( n + m = 2 \) (\( n = m = 1 \)), then it follows from the system (4.4)–(4.5) that

\[
(\lambda - \mu)[f, g] = 0, \quad f \in \mathcal{H}_\lambda, \quad g \in \mathcal{H}_\mu,
\]

such that \([f, g] = 0\) for \( \lambda \neq \mu \). Let us assume that subspaces \( \mathcal{H}_\lambda \) and \( \mathcal{H}_\mu \) are orthogonal for \( 2 \leq n+m \leq k \) and prove that extended subspaces \( \mathcal{H}_\lambda \) and \( \mathcal{H}_\mu \) remain orthogonal for \( \tilde{n} = n + 1 \), \( \tilde{m} = m \) and \( \tilde{n} = n \), \( \tilde{m} = m + 1 \). In either case, we define \( \tilde{f} = (T - \lambda I)f \) and \( \tilde{g} = (T - \mu I)g \), such that

\[
f \in \mathcal{H}_\lambda \iff (T - \lambda I)^{\tilde{n}} f = (T - \lambda I)^n \tilde{f} = 0, \quad (4.4)\tilde{f}
g \in \mathcal{H}_\mu \iff (T - \mu I)^{\tilde{m}} g = (T - \mu I)^m \tilde{g} = 0. \quad (4.5)\tilde{g}
\]
By the inductive assumption, we have \( \hat{f}, g = 0 \) and \( f, \tilde{g} = 0 \) in either case, such that
\[
(T - \lambda I)f, g = 0 \quad [f, (T - \mu I)g] = 0. \tag{4.6}
\]
By using the system (4.4)–(4.5) and the relations (4.6), we obtain that
\[
(\lambda - \mu)[f, g] = 0, \quad f \in \tilde{\mathcal{H}}_{\lambda}, \quad g \in \tilde{\mathcal{H}}_{\mu},
\]
from which the statement follows by the induction method.

**Lemma 4.12** Let \( \mathcal{H}_{\lambda_0} \) be eigenspace of a multiple real isolated eigenvalue \( \lambda_0 \) of \( T \) in \( \Pi_\kappa \) and \( \{f_1, f_2, \ldots, f_n\} \) be the Jordan chain of eigenvectors. Let \( \mathcal{H}_0 = \text{span}\{f_1, f_2, \ldots, f_k\} \subset \mathcal{H}_{\lambda_0} \), where \( k = \text{round}(n/2) \), and \( \mathcal{H}_0 = \text{span}\{f_1, f_2, \ldots, f_k, f_{k+1}\} \subset \mathcal{H}_{\lambda_0} \).

- If \( n \) is even \( (n = 2k) \), the neutral subspace \( \mathcal{H}_0 \) is the maximal sign-definite subspace of \( \mathcal{H}_{\lambda_0} \).
- If \( n \) is odd \( (n = 2k+1) \), the subspace \( \mathcal{H}_0 \) is the maximal non-negative subspace of \( \Pi_\kappa \) if \( [f_1, f_n] > 0 \) and the maximal non-positive subspace of \( \Pi_\kappa \) if \( [f_1, f_n] < 0 \), while the neutral subspace \( \mathcal{H}_0 \) is the maximal non-positive subspace of \( \Pi_\kappa \) if \( [f_1, f_n] > 0 \) and the maximal non-negative subspace of \( \Pi_\kappa \) if \( [f_1, f_n] < 0 \).

**Proof.** Without loss of generality we will consider the case \( \lambda_0 = 0 \) (if \( \lambda_0 \neq 0 \) the same argument is applied to the shifted self-adjoint operator \( T' = T - \lambda_0 I \)). We will show that \( [f, f] = 0 \forall f \in \mathcal{H}_0 \). By a decomposition over the basis in \( \mathcal{H}_0 \), we obtain
\[
[f, f] = \sum_{i=1}^{k} \sum_{j=1}^{k} \alpha_i \tilde{\alpha}_j \, [f_i, f_j]. \tag{4.7}
\]
We use that
\[
[f_i, f_j] = [Tf_{i+1}, Tf_{j+1}] = \cdots = \begin{bmatrix} T^k f_{i+k}, T^k f_{j+k} \end{bmatrix} = \begin{bmatrix} T^{2k} f_{i+k}, f_{j+k} \end{bmatrix},
\]
for any \( 1 \leq i, j \leq k \). In case of even \( n = 2k \), we have \( [f_i, f_j] = [T^n f_{i+k}, f_{j+k}] = 0 \) for all \( 1 \leq i, j \leq k \). In case of odd \( n = 2k+1 \), we have \( [f_i, f_j] = [T^{n+1} f_{i+k+1}, f_{j+k+1}] = 0 \) for all \( 1 \leq i, j \leq k \). Therefore, \( \mathcal{H}_0 \) is a neutral subspace of \( \mathcal{H}_{\lambda_0} \). To show that it is actually the maximal neutral subspace of \( \mathcal{H}_{\lambda_0} \), let \( \mathcal{H}'_0 = \text{span}\{f_1, f_2, \ldots, f_k, f_{k_0}\} \), where \( k + 1 \leq k_0 \leq n \). Since \( f_{n+1} \) does not exist in the Jordan chain (4.3) (otherwise, the algebraic multiplicity is \( n+1 \)) and \( \lambda_0 \) is an isolated eigenvalue, then \( [f_1, f_n] \neq 0 \) by the Fredholm theory. It follows from the Jordan chain (4.3) that
\[
[f_1, f_n] = [T^{n-1} f_m, f_n] = [f_m, T^{m-1} f_n] = [f_m, f_{n-m+1}] \neq 0. \tag{4.8}
\]
When \( n = 2k \), we have \( 1 \leq n - k_0 + 1 \leq k \), such that \( [f_{k_0}, f_{n-k_0+1}] \neq 0 \) and the subspace \( \mathcal{H}'_0 \) is sign-indefinite in the decomposition (4.7). When \( n = 2k+1 \), we have \( 1 \leq n - k_0 + 1 \leq k \), such that \( k_0 \geq k + 2 \) and
n − k_{0} + 1 = k + 1 for k_{0} = k + 1. In either case, \([f_{k_{0}}, f_{n−k_{0}+1}] \neq 0\) and the subspace \(\mathcal{H}'_{0}\) is sign-indefinite in the decomposition \((4.7)\) unless \(k_{0} = k + 1\). In the latter case, we have \([f_{k+1}, f_{k+1}] = [f_{1}, f_{n}] \neq 0\) and \([f_{j}, f_{k+1}] = [T^{2k} f_{j+k}, f_{n}] = 0\) for \(1 \leq j \leq k\). As a result, the subspace \(\tilde{\mathcal{H}}_{0} \equiv \mathcal{H}'_{0}\) is non-negative for \([f_{1}, f_{n}] > 0\) and non-positive for \([f_{1}, f_{n}] < 0\).

**Corollary 4.13** Let \(\mathcal{H}_{\lambda_{0}}\) be eigenspace of a semi-simple real isolated eigenvalue \(\lambda_{0}\) of \(T\) in \(\Pi_{\kappa}\). Then, the eigenspace \(\mathcal{H}_{\lambda_{0}}\) is either strictly positive or strictly negative subspace of \(\Pi_{\kappa}\) with respect to \([\cdot, \cdot]\).

**Proof.** The proof follows by contradiction. Let \(\tilde{f}\) be a particular linear combination of eigenvectors in \(\mathcal{H}_{\lambda_{0}}\), such that \([f, \tilde{f}] = 0\). By the Fredholm theory, there exists a solution of the Jordan chain \(\lambda_{0}\), where \(f_{1} \equiv \tilde{f}\). Then, the eigenvalue \(\lambda_{0}\) could not be semi-simple.

**Remark 4.14** If \(\lambda_{0}\) is a real (multiple or semi-simple) embedded eigenvalue of \(T\), the Jordan chain can be truncated at \(f_{n}\) even if \([f_{1}, f_{n}] = 0\). In the latter case, the neutral subspaces \(\mathcal{H}_{0}\) for \(n = 2k\) and \(\tilde{\mathcal{H}}_{0}\) for \(n = 2k + 1\) in Lemma 4.12 do not have to be maximal non-positive or non-negative subspaces, while the subspace \(\mathcal{H}_{\lambda_{0}}\) in Corollary 4.13 could be sign-indefinite. The construction of a maximal non-positive subspace for (multiple or semi-simple) embedded eigenvalues depend on the computations of the projection matrix \([f_{i}, f_{j}]\) in the eigenspace of eigenvectors \(\mathcal{H}_{\lambda} = \text{span}\{f_{1}, \ldots, f_{n}\}\). We shall simplify this unnecessary complication by an assumption that all embedded eigenvalues are simple, such that the corresponding one-dimensional eigenspace \(\mathcal{H}_{\lambda_{0}}\) is either positive or negative or neutral with respect to \([\cdot, \cdot]\).

**Remark 4.15** By Lemma 4.12 and Corollary 4.13 if \(\lambda_{0}\) is a real (multiple or semi-simple) isolated eigenvalue, then the sum of dimensions of the maximal non-positive and non-negative subspaces of \(\mathcal{H}_{\lambda_{0}}\) equals the dimension of \(\mathcal{H}_{\lambda_{0}}\) (although the intersection of the two subspaces can be non-empty). By Remark 4.14 the dimension of \(\mathcal{H}_{\lambda_{0}}\) for a real embedded eigenvalue can however be smaller than the sum of dimensions of the maximal non-positive and non-negative subspaces. Therefore, the presence of embedded eigenvalues introduces a complication in the count of eigenvalues of the generalized eigenvalue problem \((2.3)\). This complication was neglected in the implicit count of embedded eigenvalues in [KKS04].

**Lemma 4.16** Let \(\lambda_{0} \in \mathbb{C}, \text{Im}(\lambda_{0}) > 0\) be an eigenvalue of \(T\) in \(\Pi_{\kappa}\). \(\mathcal{H}_{\lambda_{0}}\) be the corresponding eigenspace, and \(\tilde{\mathcal{H}}_{\lambda_{0}} = \{\mathcal{H}_{\lambda_{0}}, \mathcal{H}_{\lambda_{0}}\} \subset \Pi_{\kappa}\). Then, the neutral subspace \(\mathcal{H}_{\lambda_{0}}\) is the maximal sign-definite subspace of \(\tilde{\mathcal{H}}_{\lambda_{0}}\), such that \([f, f] = 0\ \forall f \in \mathcal{H}_{\lambda_{0}}\).

**Proof.** By Lemma 4.11 with \(\lambda = \mu = \lambda_{0}\), the eigenspace \(\mathcal{H}_{\lambda_{0}}\) is orthogonal to itself with respect to \([\cdot, \cdot]\), such that \(\mathcal{H}_{\lambda_{0}}\) is a neutral subspace of \(\tilde{\mathcal{H}}_{\lambda_{0}}\). It remains to prove that \(\mathcal{H}_{\lambda_{0}}\) is the maximal sign-definite subspace in \(\tilde{\mathcal{H}}_{\lambda}\). Let \(\mathcal{H}_{\lambda_{0}} = \text{span}\{f_{1}, f_{2}, \ldots, f_{n}\}\), where \(\{f_{1}, f_{2}, \ldots, f_{n}\}\) is the Jordan chain of

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3The Fredholm theory gives a necessary but not a sufficient condition for existence of the solution \(f_{n+1}\) in the Jordan chain \((4.3)\) if the eigenvalue \(\lambda_{0}\) is embedded into the continuous spectrum.
eigenvectors (4.3). Consider a subspace $\tilde{H}_0 = \text{span}\{f_1, f_2, ..., f_n, \bar{f}_j\}$, where $1 \leq j \leq n$ and construct a linear combination of $f_{n+1-j}$ and $\bar{f}_j$:

$$[f_{n+1-j} + \alpha \bar{f}_j, f_{n+1-j} + \alpha \bar{f}_j] = 2\text{Re} \left( \alpha [\bar{f}_j, f_{n+1-j}] \right), \quad \alpha \in \mathbb{C}. $$

By the Fredholm theory, $[f_n, \bar{f}_1] \neq 0$ and by virtue of (4.8), $[\bar{f}_j, f_{n+1-j}] \neq 0$. As a result, the linear combination $f_{n+1-j} + \alpha \bar{f}_j$ is sign-indefinite with respect to $[\cdot, \cdot]$.

**Lemma 4.17** Let $\mathcal{H}_0$ be eigenspace of a multiple zero eigenvalue of $K^{-1}A$ in $\mathcal{H}$ and $\{f_1, ..., f_n\}$ be the Jordan chain of eigenvectors, such that $f_1 \in \text{Ker}(A)$. Let $0 < \delta < |\sigma_{-1}|$, where $\sigma_{-1}$ is the smallest negative eigenvalue of $K^{-1}A$. If $\omega_+ > 0$ then $[f_1, f_n] = (Kf_1, f_n) \neq 0$, and

- If $n$ is odd, the subspace $\mathcal{H}_0$ corresponds to a positive eigenvalue of the operator $(A + \delta K)$ if $[f_1, f_n] > 0$ and to a negative eigenvalue if $[f_1, f_n] < 0$.

- If $n$ is even, the subspace $\mathcal{H}_0$ corresponds to a positive eigenvalue of the operator $(A + \delta K)$ if $[f_1, f_n] < 0$ and to a negative eigenvalue if $[f_1, f_n] > 0$.

**Proof.** Let $\mu(\delta)$ be an eigenvalue of the operator $A + \delta K$ related to the subspace $\mathcal{H}_0$. By standard perturbation theory for isolated eigenvalues [K76], each eigenvalue $\mu_j(\delta)$ is a continuous function of $\delta$ and

$$\lim_{\delta \rightarrow 0^+} \frac{\mu(\delta)}{\delta^n} = (-1)^{n+1} \frac{(Kf_1, f_n)}{(f_1, f_1)}. \quad (4.9)$$

If $\omega_+ > 0$, the zero eigenvalue of $A$ is isolated from the essential spectrum of $K^{-1}A$, such that $[f_1, f_n] = (Kf_1, f_n) \neq 0$ by the Fredholm theory. The statement of the lemma follows from the limiting relation (4.9). Since no eigenvalues of $K^{-1}A$ exists in $(-\sigma_{-1}, 0)$, the eigenvalue $\mu(\delta)$ remains sign-definite for $0 < \delta < |\sigma_{-1}|$.

**Remark 4.18** The statement holds for the case $\omega_+ = 0$ provided that $n = 1$, $[f_1, f_1] \neq 0$, and $\mu(\delta) < \omega_{A+\delta K}$, where $\omega_{A+\delta K} > 0$ is defined below the decomposition (2.8).

**Remark 4.19** We are concerned here in the bounds on the numbers of eigenvalues in the generalized eigenvalue problem (2.3) in terms of the numbers of isolated eigenvalues of self-adjoint operators $A$ and $K^{-1}$. Let $N_p^-$ ($N_n^-$) be the number of negative eigenvalues of the bounded or unbounded operator $K^{-1}A$ with the account of their multiplicities whose eigenvectors are associated to the maximal non-negative (non-positive) subspace of $\Pi_\kappa$ with respect to $[\cdot, \cdot] = (K\cdot, \cdot)$. Similarly, let $N_p^0$ ($N_n^0$) be the number of zero eigenvalues of $K^{-1}A$ with the account of their multiplicities and $N_p^+$ ($N_n^+$) be the number of positive eigenvalues of $K^{-1}A$ with the account of their multiplicities, such that the corresponding eigenvectors are associated to the maximal non-negative (non-positive) subspace of $\Pi_\kappa$. In the case of real isolated
eigenvalues, the sum of dimensions of the maximal non-positive and non-negative subspaces equals the dimension of the subspace $\mathcal{H}_{\lambda_0}$ by Remark 4.15. The splitting of the dimension of $\mathcal{H}_{\lambda_0}$ between $N_p(\lambda_0)$ and $N_n(\lambda_0)$ is obvious for each semi-simple isolated eigenvalue $\lambda_0$ in Corollary 4.13. In the case of a multiple real isolated eigenvalue $\lambda_0$ of algebraic multiplicity $n$, the same splitting is prescribed in Lemma 4.12.

(i) If $n = 2k$, $k \in \mathbb{N}$, then $N_p(\lambda_0) = N_n(\lambda_0) = k$.

(ii) If $n = 2k + 1$, $k \in \mathbb{N}$ and $[f_1, f_n] > 0$, then $N_p(\lambda_0) = k + 1$ and $N_n(\lambda_0) = k$.

(iii) If $n = 2k + 1$, $k \in \mathbb{N}$ and $[f_1, f_n] < 0$, then $N_p(\lambda_0) = k$ and $N_n(\lambda_0) = k + 1$.

In the case of a simple real embedded eigenvalue $\lambda_0$, the numbers $N_p(\lambda_0)$ and $N_n(\lambda_0)$ for a one-dimensional subspace $\mathcal{H}_{\lambda_0}$ are prescribed in Remark 4.14 as follows:

(i) If $[f_1, f_1] > 0$, then $N_p(\lambda_0) = 1$, $N_n(\lambda_0) = 0$.

(ii) If $[f_1, f_1] < 0$, then $N_p(\lambda_0) = 0$, $N_n(\lambda_0) = 1$.

(iii) If $[f_1, f_1] = 0$, then $N_p(\lambda_0) = N_n(\lambda_0) = 1$.

In the case (iii), the sum $N_p(\lambda_0) + N_n(\lambda_0)$ exceeds the dimension of $\mathcal{H}_{\lambda_0}$. Let $N_{c^+}$ ($N_{c^-}$) be the number of complex eigenvalues in the upper half plane $\gamma \in \mathbb{C}$, $\text{Im}(\gamma) > 0$ ($\text{Im}(\gamma) < 0$). The maximal sign-definite subspace of $\Pi_\kappa$ associated to complex eigenvalues is prescribed by Lemma 4.16.

**Theorem 3** Let assumptions P1–P2 be satisfied. Eigenvalues of the generalized eigenvalue problem (2.3) satisfy the pair of equalities:

$$N_p^{-} + N_0^{-} + N_0^{+} + N_{c^+} = \dim(\mathcal{H}_{A+\delta K}^{-}) \quad (4.10)$$

$$N_n^{-} + N_0^{-} + N_0^{+} + N_{c^+} = \dim(\mathcal{H}_{K}^{-}) \quad (4.11)$$

**Proof.** We use the shifted eigenvalue problem (2.7) with sufficiently small $\delta > 0$ and consider the bounded operator $T = (A + \delta K)^{-1} K$. By Lemma 4.1, the operator $T$ is self-adjoint with respect to $[\cdot, \cdot] = (K \cdot, \cdot)$ and it has a non-positive invariant subspace of dimension $\kappa = \dim(\mathcal{H}_{K}^{-})$. Counting all eigenvalues of the shifted eigenvalue problem (2.7) in Remark 4.19, we establish the equality (4.11). The other equality (4.10) follows from a count for the bounded operator $\tilde{T} = K(A + \delta K)^{-1}$ which is self-adjoint with respect to $[\cdot, \cdot] = ((A + \delta K)^{-1} \cdot, \cdot)$. The self-adjoint operator $(A + \delta K)^{-1}$ defines the indefinite metric in the Pontryagin space $\tilde{\Pi}_\kappa$, where $\kappa = \dim(\mathcal{H}_{A+\delta K}^{-})$. For any semi-simple eigenvalue $\gamma_0$ of the shifted eigenvalue problem (2.7), we have

$$\forall f, g \in \mathcal{H}_{\gamma_0}, \quad ((A + \delta K)f, g) = (\gamma_0 + \delta)(Kf, g).$$
If $\gamma_0 \geq 0$ or $\text{Im}(\gamma_0) \neq 0$, the maximal non-positive eigenspace of $\tilde{T}$ in $\tilde{\Pi}_\kappa$ associated with $\gamma_0$ coincides with the maximal non-positive eigenspace of $T$ in $\Pi_\kappa$. If $\gamma_0 < 0$, the maximal non-positive eigenspace of $\tilde{T}$ in $\tilde{\Pi}_\kappa$ coincides with the maximal non-negative eigenspace of $T$ in $\Pi_\kappa$. The same statement can be proved for the case of multiple isolated eigenvalues $\gamma_0$ when the eigenspace is defined by the Jordan block of eigenvectors,

$$A f_j = \gamma_0 K f_j + f_{j-1}, \quad j = 1, \ldots, n,$$

where $f_0 = 0$. The dimension of the maximal non-positive eigenspace of $\tilde{T}$ in $\tilde{\Pi}_\kappa$ is then $N_p^+ + N_0^0 + N_n^+ + N_c^+$. 

**Corollary 4.20** Let $N_{\text{neg}} = \dim(\mathcal{H}_{A+\delta K}^-) + \dim(\mathcal{H}_K^-)$ be the total negative index of the shifted generalized eigenvalue problem (2.7). Let $N_{\text{unst}} = N_p^- + N_n^- + 2N_c^+$ be the total number of unstable eigenvalues that include $N^- = N_p^- + N_n^-$ negative eigenvalues $\gamma < 0$ and $N_c = N_c^+ + N_c^-$ complex eigenvalues with $\text{Im}(\gamma) \neq 0$. Under the assumptions of Theorem 3, it is true that

$$\Delta N = N_{\text{neg}} - N_{\text{unst}} = 2N_n^+ + 2N_n^0 \geq 0. \quad (4.12)$$

**Proof.** The equality (4.12) follows by the sum of (4.10) and (4.11). 

**Theorem 4** Let assumptions P1–P2 be satisfied and $\omega_+ > 0$. Let $N_A = \dim(\mathcal{H}_A^- \oplus \mathcal{H}_A^0 \oplus \mathcal{H}_A^+)$ be the total number of isolated eigenvalues of $A$. Let $N_K = \dim(\mathcal{H}_K^- \oplus \mathcal{H}_K^+)$ be the total number of isolated eigenvalues of $K$. Assume that no embedded eigenvalues of the generalized eigenvalue problem (2.3) exist. Isolated eigenvalues of the generalized eigenvalue problem (2.3) satisfy the inequality:

$$N_p^- + N_0^0 + N_p^+ + N_c^+ \leq N_A + N_K. \quad (4.13)$$

**Proof.** We prove this theorem by contradiction. Let $\Pi$ be a subspace in $\Pi_\kappa$ spanned by eigenvectors of the generalized eigenvalue problem (2.3) which belong to $N_p^-$ negative eigenvalues $\gamma < 0$, $N_0^0$ zero eigenvalues $\gamma = 0$, $N_p^+$ positive isolated eigenvalues $0 < \gamma < \omega_+ \omega_-$, and $N_c^+$ complex eigenvalues $\text{Im}(\gamma) > 0$. Let us assume that $N_p^- + N_p^0 + N_p^+ + N_c^+ > N_A + N_K$. By Gram–Schmidt orthogonalization, there exist a vector $h$ in $\Pi$ such that $(h, f) = 0$ and $(h, g) = 0$, where $f \in \mathcal{H}_A^- \oplus \mathcal{H}_A^0 \oplus \mathcal{H}_A^+$ and $g \in \mathcal{H}_K^- \oplus \mathcal{H}_K^+$, such that $h \in \mathcal{H}_A^{\sigma_+}(A) \cap \mathcal{H}_K^{\sigma_+}(K)$. As a result,

$$(Ah, h) \geq \omega_+(h, h), \quad (Kh, h) \leq \omega_-^{-1}(h, h),$$

and

$$(Ah, h) \geq \omega_+ \omega_-(Kh, h).$$
On the other hand, since \( h \in \Pi \), we represent \( h \) by a linear combination of the eigenvectors in the corresponding subspaces of \( \Pi \), such that

\[
(Ah, h) = \sum_{i,j} \alpha_i \bar{\alpha}_j (Ah_i, h_j)
\]

\[
= \sum_{\gamma_i = \gamma_j < 0} \alpha_i \bar{\alpha}_j (Ah_i, h_j) + \sum_{\gamma_i = \gamma_j = 0} \alpha_i \bar{\alpha}_j (Ah_i, h_j) + \sum_{0 < \gamma_i = \gamma_j < \omega_+ \omega_-} \alpha_i \bar{\alpha}_j (Ah_i, h_j),
\]

where we have used Lemma 4.11 and Corollary 4.16. By Lemma 4.12 the non-zero values in \((Ah, h_j)\) occur only for eigenvalues with odd algebraic multiplicity \( n = 2k + 1 \) for eigenvectors \((Af_{k+1}, f_{k+1})\), where \( f_{k+1} \) is the generalized eigenvector in the basis for \( \mathcal{H}_0 \). Since all these cases are similar to the case of simple eigenvalues, we obtain

\[
(Ah, h) = \sum_{\gamma_j < 0} |\alpha_j|^2 (Ah_j, h_j) + \sum_{\gamma_j = 0} |\alpha_j|^2 (Ah_j, h_j) + \sum_{0 < \gamma_j < \omega_+ \omega_-} |\alpha_j|^2 (Ah_j, h_j)
\]

\[
= \sum_{\gamma_j < 0} \gamma_j |\alpha_j|^2 (Kh_j, h_j) + \sum_{0 < \gamma_j < \omega_+ \omega_-} \gamma_j |\alpha_j|^2 (Kh_j, h_j)
\]

\[
< \omega_+ \omega_- \sum_{0 < \gamma_j < \omega_+ \omega_-} |\alpha_j|^2 (Kh_j, h_j),
\]

where we have used the fact that \((Kh_j, h_j) \geq 0 \) in \( \Pi \). On the other hand,

\[
(Kh, h) = \sum_{i,j} \alpha_i \bar{\alpha}_j (Kh_i, h_j)
\]

\[
= \sum_{\gamma_j < 0} |\alpha_j|^2 (Kh_j, h_j) + \sum_{\gamma_j = 0} |\alpha_j|^2 (Kh_j, h_j) + \sum_{0 < \gamma_j < \omega_+ \omega_-} |\alpha_j|^2 (Kh_j, h_j)
\]

\[
\geq \sum_{0 < \gamma_j < \omega_+ \omega_-} |\alpha_j|^2 (Kh_j, h_j).
\]

Therefore, \((Ah, h) < \omega_+ \omega_- (Kh, h)\), which is a contradiction. \( \blacksquare \)

**Corollary 4.21** Let \( N_{\text{total}} = N_A + N_K \) be the total number of isolated eigenvalues of operators \( A \) and \( K \). Let \( N_{\text{isol}} = N_{A^-} + N_{K^-} + N_{A^0} + N_{K^0} + N_{A^+} + N_{K^+} + N_{c^+} + N_{c^-} \) be the total number of isolated eigenvalues of the generalized eigenvalue problem \((2.3)\). Under the assumptions of Theorem 4, it is true that

\[
N_{\text{isol}} \leq N_{\text{total}} + \dim(\mathcal{H}^-_K).
\]  

(4.14)

**Proof.** The inequality (4.14) follows by the sum of (4.11) and (4.12). \( \blacksquare \)

**Remark 4.22** If a simple embedded eigenvalue with the corresponding eigenvector \( f_1 \) is included into consideration according to Remark 4.19, then the left-hand-side of the inequality (4.13) is increased by the number of simple embedded eigenvalues with \([f_1, f_1] \leq 0\). For each embedded eigenvalue, the right-hand-side in the inequality (4.14) is reduced by two if \([f_1, f_1] \leq 0\) and it remains the same if \([f_1, f_1] > 0\).
5 Applications

We shall describe three applications of the general analysis which are related to recent studies of stability of solitons and vortices in the nonlinear Schrödinger equation and solitons in the Korteweg–de Vries equation.

5.1 Solitons of the scalar nonlinear Schrödinger equation

Consider a scalar nonlinear Schrödinger (NLS) equation,

\[ i\psi_t = -\Delta \psi + F(|\psi|^2)\psi, \quad \Delta = \partial^2_{x_1x_1} + \ldots + \partial^2_{x_dx_d}, \]

(5.1)

where \((x, t) \in \mathbb{R}^d \times \mathbb{R}\) and \(\psi \in \mathbb{C}\). For a suitable nonlinear function \(F(|\psi|^2)\), where \(F \in C^\infty_0\) and \(F(0) = 0\), the NLS equation (5.1) possesses a solitary wave solution \(\psi = \phi(x)e^{i\omega t}\), where \(\omega > 0\), \(\phi : \mathbb{R}^d \to \mathbb{R}\), and \(\phi(x)\) is exponentially decaying \(C^\infty\) function. See [M93] for existence and uniqueness of ground state solutions to the NLS equation (5.1). Linearization of the NLS equation (5.1) with the anzats,

\[
\psi = \left(\phi(x) + [u(x) + iw(x)]e^{\lambda t} + [\bar{u}(x) + i\bar{w}(x)]e^{\bar{\lambda}t}\right)e^{i\omega t},
\]

(5.2)

where \(\lambda \in \mathbb{C}\) and \((u, w) \in \mathbb{C}^2\), results in the linearized Hamiltonian problem (2.1), where \(L^{\pm}\) are Schrödinger operators,

\[
L^{+} = -\Delta + \omega + F(\phi^2) + 2\phi^2 F'(\phi^2),
\]

(5.3)

\[
L^{-} = -\Delta + \omega + F(\phi^2).
\]

(5.4)

We note that \(L^{\pm}\) are unbounded operators and \(\sigma_c(L^{\pm}) = [\omega, \infty)\) with \(\omega_+ = \omega_- = \omega > 0\). The kernel of \(L^-\) includes at least one eigenvector \(\phi(x)\) and the kernel of \(L^+\) includes at least \(d\) eigenvectors \(\partial_{x_j}\phi(x)\), \(j = 1, \ldots, d\). The Hilbert space \(\mathcal{X}\) is defined as \(\mathcal{X} = H^1(\mathbb{R}^d, \mathbb{C})\) and the main assumptions P1-P2 are satisfied due to exponential decay of the potential functions \(F(\phi^2)\) and \(\phi^2 F'(\phi^2)\). Theorems 3 and 4 give precise count of eigenvalues of the stability problem \(L_- L^+ u = -\lambda^2 u\), provided that the numbers \(\dim(\mathcal{H}^-_{\delta K}), \dim(\mathcal{H}^-_{\delta A + \delta K}), N_K\) and \(N_A\) can be computed from the count of isolated eigenvalues of \(L^{\pm}\). We shall illustrate these computations with two examples.

Example 1. Let \(\phi(x)\) be the ground state solution such that \(\phi(x) > 0\) on \(x \in \mathbb{R}^d\). By spectral theory, \(\text{Ker}(L^-) = \{\phi(x)\}\) is one-dimensional and the subspace \(\mathcal{H}^-_{\delta K}\) in (2.4) is empty, such that the corresponding Pontryagin space is \(\Pi_0\) with \(\kappa = 0\).

- By the equality (4.11), it follows immediately that \(N^-_n = N^0_n = N^+_n = N^c_+ = 0\) such that the spectrum of the problem (2.3) is real-valued and all eigenvalues are semi-simple.
• Since eigenvectors of \( \text{Ker}(A) \) are in the positive subspace of \( K \), the number of zero eigenvalues of the problem (2.3) is given by \( N^0_p = \dim(\mathcal{H}_A^0) \). If \( \frac{d}{dx} \| \phi \|^2_{L^2} = 0 \), then \( L_+ \partial_x \phi(x) = -\phi(x), \partial_x \phi(x) \in \mathcal{H}, P \phi = 0 \), and the eigenvector \( \partial_x \phi(x) \in \text{Ker}(PL_+P) \), such that \( z_1 = 1 \). If \( z(L_+) = d \), then \( z_0 = 0 \) since \( \partial_x \phi(x) \in \mathcal{H} \) for \( j = 1, \ldots, d \). In particular, \( L_- x \phi(x) = -2 \partial_x \phi(x) \) and \((K \partial_x \phi, \partial_x \phi) = \frac{1}{4} \| \phi \|^2_{L^2} > 0 \).

• By the equality (4.10), the number of negative eigenvalues of the problem (2.3) is given by \( N^-_p = \dim(\mathcal{H}_{A+\delta K}) \). By Lemma 4.17 with \( n = 1 \) and \( [f_1, f_1] > 0 \), all zero eigenvalues of \( A \) become positive eigenvalues of \( A + \delta K \) for \( \delta > 0 \). By Propositions 2.1 and 2.2, we have \( \dim(\mathcal{H}_{A+\delta K}) = \dim(\mathcal{H}_A) = n(L_+) - n_0 \). By Theorem 3.1 in [GSS90], \( n_0 = 1 \) if \( \frac{d}{dx} \| \phi \|^2_{L^2} > 0 \) and \( n_0 = 0 \) otherwise.

• By the inequality (4.13), the number of positive eigenvalues of the problem (2.3) are bounded from above by \( N^+_p \leq \dim(\mathcal{H}_A^+) + \dim(\mathcal{H}_K^+) \). By Proposition 2.1, it is then \( N^+_p \leq p(L_+) + p(L_-) + n_0 + z_0 - z_1 \).

Remark 5.1 Under the assumptions that \( n(L_+) = 1, z(L_+) = d, p(L_+) = p(L_-) = 0 \) and \( \frac{d}{dx} \| \phi \|^2_{L^2} < 0 \), it follows from the above properties that \( N^-_p = 1, N^0_p = d, \) and \( N^+_p = 0 \). The assumptions \( n(L_+) = 1, z(L_+) = d \) and \( \frac{d}{dx} \| \phi \|^2_{L^2} < 0 \) are verified for the super-critical NLS equation with the power nonlinearity \( F = |\psi|^p \) [S05]. Under the further assumptions \( p(L_+) = p(L_-) = 0 \), the statement \( N^-_p = 0 \) is proved directly in Proposition 2.1.2 [P01] and Proposition 9.2 [KS05] for \( d = 1 \) and in Lemma 1.8 [S05] for \( d = 3 \). The same statement follows here by the count of eigenvalues from Theorem 4.

Remark 5.2 Systems of coupled NLS equations generalize the scalar NLS equation (5.1). Stability of vector solitons in the coupled NLS equations results in the same linearized Hamiltonian system (2.1) with the matrix Schrödinger operators \( L_\pm \). General results for non-ground state solutions are obtained in [KKS04, P05] for \( d = 1 \) and in [CPV05] for \( d = 3 \). Multiple and embedded eigenvalues were either excluded from analysis by an assumption [P05, CPV05] or were treated implicitly [KKS04]. The present manuscript generalizes these results for an abstract case with a precise count of multiple and embedded eigenvalues.

Example 2. Let the scalar NLS equation (5.1) with \( F = |\psi|^2 \) be discretized with \( \Delta \equiv \epsilon \Delta_{\text{disc}} \), where \( \Delta_{\text{disc}} \) is the second-order discrete Laplacian and \( \epsilon \) is a small parameter. We note that \( \Delta_{\text{disc}} \) is a bounded operator and \( \sigma_{\epsilon}(-\Delta_{\text{disc}}) \in [0, 4d] \). By the Lyapunov–Schmidt reduction method, the solution \( \psi = \phi e^{i\omega t} \) with \( \omega > 0 \) bifurcates from the limit \( \epsilon = 0 \) with \( N \) non-zero lattice nodes to \( \epsilon \neq 0 \), such that \( \frac{d}{d\epsilon} \| \phi \|^2_{L^2} > 0 \), \( \ker(L_+) = \emptyset \), and \( \ker(L_-) = \{ \phi \} \) is one-dimensional for \( 0 < |\epsilon| < \epsilon_0 \) with \( \epsilon_0 > 0 \) (see [PKF06a] for \( d = 1 \) and [PKF06b] for \( d = 2 \)). By the equalities (4.10) and (4.11), we have

\[
N^-_p + N^+_n + N_{c^+} = n(L_+) - 1,
\]
\[ N_n^- + N_n^+ + N_{c^+} = n(L_-), \]

where \( n(L_+) = N \) and \( n(L_-) \leq N - 1 \) in the domain \( 0 < |e| < \epsilon_0 \). When small positive eigenvalues of \( L_- \) are simple for \( \epsilon \neq 0 \), it is true that \( N_n^+ = n(L_-) \), \( N_n^- = N_{c^+} = 0 \), and \( N_p^- = N - 1 - n(L_-) \) (see Corollary 3.5 in [PKF06a]). It is clear that the Lyapunov–Schmidt reduction method gives a more precise information on numbers \( N_n^+, N_p^- \), and \( N_{c^+} \), compared to the general equalities above. Similarly, by the inequality (4.13) and the counts above for \( 0 < |e| < \epsilon_0 \), we have

\[ N_p^+ \leq 2n(L_-) + \dim(H^+_A) + \dim(H^+_K). \]

When \( 0 < |e| < \epsilon_0 \), the numbers \( N_p^+ \) and \( \dim(H^+_A) \) give precisely the numbers of edge bifurcations from the essential spectrum of \( K^{-1}A \) and \( A \) respectively, while the number \( \dim(H^+_K) \) exceeds the number of edge bifurcations from the essential spectrum of \( K^{-1} \) by \( N - 1 - n(L_-) \). When the solution \( \phi \) is a ground state, we have \( N = 1 \) and \( n(L_-) = 0 \), such that the edge bifurcations in the spectrum of \( K^{-1}A \) may only occur if there are edge bifurcations in the spectrum of \( A \) and \( K^{-1} \) for \( \epsilon \neq 0 \). When \( N > 1 \), the relation between edge bifurcations in the self-adjoint and non-self-adjoint problems become less direct.

**Remark 5.3** The Lyapunov–Schmidt reduction method was also used for continuous coupled NLS equations with and without external potentials. See [KK04, KK06, PY05] for various results on the count of unstable eigenvalues in parameter continuations of the NLS equations.

### 5.2 Vortices of the scalar nonlinear Schrödinger equation

Consider the scalar two-dimensional NLS equation (5.1) with \( d = 2 \) in polar coordinates \((r, \theta)\):

\[
i \psi_t = -\Delta \psi + F(|\psi|^2) \psi, \quad \Delta = \partial_{rr} + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_{\theta \theta},
\]

(5.5)

where \( r > 0 \) and \( \theta \in [0, 2\pi] \). Assume that the NLS equation (5.5) possesses a charge-\( m \) vortex solution \( \psi = \phi(r)e^{im\theta+i\omega t} \), where \( \omega > 0, m \in \mathbb{N}, \phi : \mathbb{R}_+ \to \mathbb{R}, \) and \( \phi(r) \) is exponentially decaying \( C^\infty \) function for \( r > 0 \) with \( \phi(0) = 0 \). See [PW02] for existence results of charge-\( m \) vortices in the cubic-quintic NLS equation with \( F = -|\psi|^2 + |\psi|^4 \). Linearization of the NLS equation (5.5) with the anzats,

\[
\psi = \left( \phi(r)e^{im\theta} + \varphi_+(r, \theta)e^{\lambda t} + \varphi_-(r, \theta)e^{\bar{\lambda} t} \right) e^{i\omega t},
\]

(5.6)

where \( \lambda \in \mathbb{C} \) and \((\varphi_+, \varphi_-) \in \mathbb{C}^2\), results in the stability problem,

\[
\sigma_3 H \varphi = i\lambda \varphi,
\]

(5.7)

---

4Corollary 3.5 in [PKF06a] is valid only when small positive eigenvalues of \( L_- \) are simple. It is shown in [PKF06b] that the case of multiple small positive eigenvalues of \( L_- \) leads to splitting of real eigenvalues \( N_p^- \) of the generalized eigenvalue problem (5.5) to complex eigenvalues \( N_{c^+} \) beyond the leading-order Lyapunov–Schmidt reduction. The case of multiple small negative eigenvalues of \( L_- \) does not lead to this complication since the semi-simple purely imaginary eigenvalues \( N_n^+ \) do not split to complex eigenvalues \( N_{c^+} \).
where \( \varphi = (\varphi_+, \varphi_-)^T, \sigma_3 = \text{diag}(1, -1), \) and
\[
H = \begin{pmatrix}
-\Delta + \omega + F(\phi^2) + \phi^2 F'(\phi^2) & \phi^2 F'(\phi^2)e^{2im\theta} \\
\phi^2 F'(\phi^2)e^{-2im\theta} & -\Delta + \omega + F(\phi^2) + \phi^2 F'(\phi^2)
\end{pmatrix}.
\]

Expand \( \varphi(r, \theta) \) in the Fourier series
\[
\varphi = \sum_{n \in \mathbb{Z}} \varphi(n)(r)e^{in\theta}
\]
and reduce the problem to a sequence of spectral problems for ODEs:
\[
\sigma_3 H_n \varphi_n = i\lambda \varphi_n, \quad n \in \mathbb{Z},
\]
where \( \sigma_3 \) is given by (5.3)–(5.4) with
\[
H_n = \begin{pmatrix}
-\partial_{rr} - \frac{1}{r} \partial_r + \frac{(n+m)^2}{r^2} + \omega + F(\phi^2) + \phi^2 F'(\phi^2) & \phi^2 F'(\phi^2) \\
\phi^2 F'(\phi^2) & -\partial_{rr} - \frac{1}{r} \partial_r + \frac{(n-m)^2}{r^2} + \omega + F(\phi^2) + \phi^2 F'(\phi^2)
\end{pmatrix}.
\]

When \( n = 0 \), the stability problem (5.8) transforms to the linearized Hamiltonian system (2.1), where \( L_\pm \) is given by (5.3)–(5.4) with \( \Delta = \partial^2_{rr} + \frac{1}{r} \partial_r - \frac{m^2}{r^2} \) and \( (u, w) \) are given by \( u = \varphi_+(m) + \varphi_-(m) \) and \( w = -i(\varphi_+(m) - \varphi_-(m)) \). When \( n \in \mathbb{N} \), the stability problem (5.8) transforms to the linearized Hamiltonian system (2.1) with \( L_+ = H_n \) and \( L_- = \sigma_3 H_n \sigma_3 \), where
\[
L_+ = L_- + 2\phi^2 F'(\phi^2) \sigma_1, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]
and \( (u, w) \) are given by \( u = \varphi_n \) and \( w = -i\sigma_3 \varphi_n \). When \( -n \in \mathbb{N} \), the spectrum of the stability problem (5.8) can be obtained from that for \( n \in \mathbb{N} \) by the correspondence \( H_{-n} = \sigma_1 H_n \sigma_1 \).

Let us introduce the weighted inner product for functions \( r \geq 0 \):
\[
(f, g) = \int_0^\infty f(r)g(r)rdr.
\]
In all cases \( n = 0, n \in \mathbb{N} \) and \( -n \in \mathbb{N}, \) \( L_\pm \) are unbounded self-adjoint differential operators and \( \sigma_0(L_\pm) = [\omega, \infty), \) such that \( \omega_+ = \omega_- = \omega > 0. \) The kernel of \( H_n \) includes at least one eigenvector for \( n = \pm 1 \)
\[
\phi_{\pm 1} = \phi'(r)1 \pm \frac{m}{r} \phi(r)\sigma_3 1, \quad 1 = (1, 1)^T,
\]
and at least one eigenvector for \( n = 0 \): \( \phi_0 = \phi(r)\sigma_3 1. \) The Hilbert space \( \mathcal{X} \) associated with the weighted inner product is defined as \( \mathcal{X} = H^1(\mathbb{R}_+, \mathbb{C}) \) for \( n = 0 \) and \( \mathcal{X} = H^1(\mathbb{R}_+, \mathbb{C}^2) \) for \( \pm n \in \mathbb{N}. \) In all cases, the main assumptions P1-P2 are satisfied due to exponential decay of the potential functions \( F(\phi^2) \) and \( \phi^2 F'(\phi^2). \)

The case \( n = 0 \) is the same as for solitons (see Section 5.1). We shall hence consider adjustments in the count of eigenvalues in the case \( \pm n \in \mathbb{N}, \) when the stability problem (5.8) is rewritten in the form,
\[
\begin{cases}
\sigma_3 H_n \varphi_n = i\lambda \varphi_n & n \in \mathbb{N}, \\
\sigma_3 H_{-n} \varphi_{-n} = i\lambda \varphi_{-n}
\end{cases}
\]
Let $L_+$ be a diagonal composition of $H_n$ and $H_{-n}$ and $L_-$ be a diagonal composition of $\sigma_3 H_n \sigma_3$ and $\sigma_3 H_{-n} \sigma_3$.

**Lemma 5.4** Let $\lambda$ be an eigenvalue of the stability problem (5.9) with the eigenvector $(\varphi_n, 0)$. Then there exists another eigenvalue $-\lambda$ with the linearly independent eigenvector $(0, \sigma_1 \varphi_n)$. If $\text{Re}(\lambda) > 0$, there exist two more eigenvalues $\tilde{\lambda}, -\tilde{\lambda}$ with the linearly independent eigenvectors $(0, \sigma_1 \tilde{\varphi}_n), (\varphi_n, 0)$.

**Proof.** We note that $\sigma_1 \sigma_3 = -\sigma_3 \sigma_1$ and $\sigma_1^2 = \sigma_3^2 = \sigma_0$, where $\sigma_0 = \text{diag}(1, 1)$. Therefore, each eigenvalue $\lambda$ of $H_n$ with the eigenvector $\varphi_n$ generates eigenvalue $-\lambda$ of $H_{-n}$ with the eigenvector $\varphi_{-n} = \sigma_1 \varphi_n$. When $\text{Re}(\lambda) \neq 0$, each eigenvalue $\lambda$ of $H_n$ generates also eigenvalue $-\tilde{\lambda}$ of $H_n$ with the eigenvector $\tilde{\varphi}_n$ and eigenvalue $\tilde{\lambda}$ of $H_{-n}$ with the eigenvector $\varphi_{-n} = \sigma_1 \tilde{\varphi}_n$.

**Theorem 5** Let $N_{\text{real}}$ be the number of real eigenvalues in the stability problem (5.9) with $\text{Re}(\lambda) > 0$, $N_{\text{comp}}$ be the number of complex eigenvalues with $\text{Re}(\lambda) > 0$ and $\text{Im}(\lambda) > 0$, $N_{\text{imag}}^-$ be the number of purely imaginary eigenvalues with $\text{Im}(\lambda) > 0$ and $(\varphi_n, H_n \varphi_n) \leq 0$, and $N_{\text{zero}}^-$ is the algebraic multiplicity of the zero eigenvalue of $\sigma_3 H_n \varphi_n = i\lambda \varphi_n$ counted in Remark 4.19. Assume that all embedded eigenvalues are simple. Then,

$$\frac{1}{2} N_{\text{real}} + N_{\text{comp}} = n(H_n) - N_{\text{zero}}^- - N_{\text{imag}}^-,$$

where $N_{\text{real}}$ is even. Moreover, $n_0 = n_-$, where $n_0$ and $n_-$ are defined by Propositions 2.7 and 2.2.

**Proof.** By Lemma 5.4, a pair of real eigenvalues of $\sigma_3 H_n \varphi_n = i\lambda \varphi_n$ corresponds to two linearly independent eigenvectors $\varphi_n$ and $\tilde{\varphi}_n$. Because $(H_n \varphi_n, \varphi_n)$ is real-valued and hence zero for $\lambda \in \mathbb{R}$, we have

$$(H_n(\varphi_n \pm \tilde{\varphi}_n), (\varphi_n \pm \tilde{\varphi}_n)) = \pm 2 \text{Re}(H_n \varphi_n, \tilde{\varphi}_n).$$

By counting multiplicities of the real negative and complex eigenvalues of the problem (2.3) associated to the stability problem (5.9), we have $N_{\text{comp}} = N_{\text{imag}}^-$, and $N_{\text{real}} = 2 N_{\text{comp}}$. By Lemma 5.4, a pair of purely imaginary and zero eigenvalues of the stability problem (5.9) corresponds to two linearly independent eigenvectors $(\varphi_n, 0)$ and $(0, \varphi_{-n})$, where $\varphi_{-n} = \sigma_1 \varphi_n$ and $(H_n \varphi_{-n}, \varphi_{-n}) = (H_n \varphi_n, \varphi_n)$. By counting multiplicities of the real positive and zero eigenvalues of the problem (2.3) associated to the stability problem (5.9), we have $N_{\text{real}}^0 = 2 N_{\text{zero}}^-$ and $N_{\text{real}}^+ = 2 N_{\text{imag}}^-$. Since the spectra of $H_n$, $\sigma_1 H_n \sigma_1$, and $\sigma_3 H_n \sigma_3$ coincide, we have $n(L_+) = n(L_-) = 2 n(H_n)$. As a result, the equality (5.10) follows by the equality (4.11) of Theorem 3. By Lemma 5.4, the multiplicity of $N_{\text{real}}$ is even in the stability problem (5.9). The other equality (4.10) of Theorem 3 recovers the same answer provided that $\text{dim}(H^-_{A+\delta K}) = \text{dim}(H^-_{A}) + n_- = n(L_+) - n_0 + n_- = n(L_-)$, such that $n_0 = n_-$. ■

**Example 3.** Let $\phi(r)$ be the fundamental charge-$m$ vortex solution such that $\phi(r) > 0$ for $r > 0$ and $\phi(0) = 0$. By spectral theory, $\text{Ker}(H_0)$ is one-dimensional with the eigenvector $\phi_0$. The analysis of $n = 0$
is similar to Example 1. In the case \( n \in \mathbb{N} \), we shall assume that \( \text{Ker}(H_1) = \{ \phi_1 \} \) and \( \text{Ker}(H_n) = \emptyset \) for \( n \geq 2 \).

- Since \((\sigma_3 \phi_1, \phi_1) = 0\) and \( \text{Ker}(\sigma_3 H_1 \sigma_3) = \{ \sigma_3 \phi_1 \} \), then \( \phi_1 \in \mathcal{H} \), such that \( z_0 = 0 \).
- By direct computation, \((\sigma_3 H_1 \sigma_3)^{-1} \phi_1 = -\frac{1}{2} r \phi(r) \mathbf{1}\) and
  \[
  (\sigma_3 H_1 \sigma_3)^{-1} \phi_1, \phi_1) = \int_0^\infty r \phi^2(r) dr > 0.
  \]
  By Lemma 4.17 we have \( N_n^0 = n_+ = 0 \) for \( n = 1 \), as well as for \( n \geq 2 \). By Proposition 2.1, we have \( A(0) < 0 \) such that \( n_0 = z_1 = 0 \) for all \( n \in \mathbb{N} \).
- By Theorem 5 we have
  \[
  N_{\text{real}} + 2 N_{\text{comp}} = 2 n(H_n) - 2 N_{\text{imag}}. \tag{5.11}
  \]
  If all purely imaginary eigenvalues are semi-simple and isolated, \( N_{\text{imag}}^– \) gives the total number of eigenvalues in the stability problem (5.9) with \( \text{Re}(\lambda) = 0, \text{Im}(\lambda) > 0 \), and negative Krein signature \((H_n \varphi_n, \varphi_n) < 0\).

Remark 5.5 Stability of vortices was considered numerically in [PW02], where Lemma 5.4 was also obtained. The closure relation (5.11) was also discussed in [KKS04] in a more general context. Detailed comparison of numerical results and the closure relation (5.11) can be found in [K05]. Vortices in the discretized scalar NLS equation were considered with the Lyapunov–Schmidt reduction method in [PKF06b]. Although the reduced eigenvalue problems were found in a much more complicated form compared to the reduced eigenvalue problem for solitons, the relation (5.11) was confirmed in all particular vortex configurations considered in [PKF06b].

Remark 5.6 Since \( N_{\text{real}} \) is even, splitting of simple complex eigenvalues \( N_{\text{comp}} \) into double real eigenvalues \( N_{\text{real}} \) is prohibited by the closure relation (5.11). Simple complex eigenvalues may either coalesce into a double real eigenvalue that persists or reappear again as simple complex eigenvalues.

5.3 Solitons of the fifth-order Korteweg–De Vries equation

Consider a general fifth-order KdV equation,
\[
v_t = a_1 v_x - a_2 v_{xxx} + a_3 v_{xxxx} + 3 b_1 v v_x - b_2 (v v_{xxx} + 2 v_x v_{xx}) + 6 b_3 v^2 v_x, \tag{5.12}
\]
where \((a_1, a_2, a_3)\) and \((b_1, b_2, b_3)\) are real-valued coefficients for linear and nonlinear terms, respectively. Without loss of generality, we assume that \( a_3 > 0 \) and
\[
c_{\text{wave}}(k) = a_1 + a_2 k^2 + a_3 k^4 \geq 0, \quad \forall k \in \mathbb{R}. \tag{5.13}
\]
For suitable values of parameters, there exists a traveling wave solution \( v(x, t) = \phi(x - ct) \), where \( c > 0 \) and \( \phi : \mathbb{R} \rightarrow \mathbb{R} \), such that the function \( \phi(x) \) is even and exponentially decaying as \( |x| \rightarrow \infty \). Existence of traveling waves was established in \([Z87, HS88, AT92]\) for \( b_2 = b_3 = 0 \), in \([CG97]\) for \( b_3 = 0 \), in \([K97]\) for \( b_1 = -b_2 = b_3 = 1 \), and in \([L99]\) for \( b_3 = 0 \) or \( b_1 = b_2 = 0 \). Linearization of the fifth-order KdV equation \([5.12]\) with the anzats

\[
v(x, t) = \phi(x - ct) + w(x - ct)e^{\lambda t}
\]

results in the stability problem

\[
\partial_x L_- w = \lambda w,
\]

(5.14)

where \( L_- \) is an unbounded fourth-order operator,

\[
L_- = a_3 \frac{d^4}{dx^4} - a_2 \frac{d^2}{dx^2} + a_1 + c + 3b_1 \phi(x) - b_2 \frac{d}{dx} \phi(x) \frac{d}{dx} - b_2 \phi''(x) + 6b_3 \phi^2(x).
\]

(5.15)

With the account of the condition \([5.13]\), the continuous spectrum of \( L_- \) is located for \( \sigma_c(L_-) \in [c, \infty) \), such that \( \omega_- = c > 0 \). The kernel of \( L_- \) includes at least one eigenvector \( \phi'(x) \). Since the image of \( L_- \) is in \( L^2(\mathbb{R}) \), the eigenfunction \( w(x) \in L^1(\mathbb{R}) \) for \( \lambda \neq 0 \) satisfies the constraint:

\[
(1, w) = \int_{\mathbb{R}} w(x)dx = 0.
\]

(5.16)

Let \( w = u'(x) \), where \( u(x) \rightarrow 0 \) as \( |x| \rightarrow \infty \) and define \( L_+ = -\partial_x L_- \partial_x \). The continuous spectrum of \( L_+ \) is located for \( \sigma_c(L_+) \in [0, \infty) \), such that \( \omega_+ = 0 \). The kernel of \( L_+ \) includes at least one eigenvector \( \phi(x) \).

Let the Hilbert space \( \mathcal{X} \) be defined as \( \mathcal{X} = H^3(\mathbb{R}, \mathbb{C}) \). The main assumptions P1-P2 for \( L_- \) and \( L_+ \) are satisfied due to exponential decay of the potential function \( \phi(x) \). The stability problem \([5.14]\) is equivalent to the linearized Hamiltonian system \([2.1]\). The operator \( L_- \) defines the constrained subspace \( \mathcal{H} \) and hence the Pontryagin space \( \Pi_\kappa \) with \( \kappa = \kappa(L_-) \). The operator \( L_+ \) has the embedded kernel to the endpoint of the essential spectrum of \( L_+ \). This introduces a technical complication in computations of the inverse of \( L_+ \) \([KP05]\), which we avoid here with the use of the shifted generalized eigenvalue problem \([2.7]\) with \( \delta > 0 \). It is easy to estimate that

\[
\omega_{A+\delta K} = \inf_{k \in \mathbb{R}} \left[ k^2(c + c_{\text{wave}}(k)) + \frac{\delta}{c + c_{\text{wave}}(k)} \right] \geq \frac{\delta}{c} > 0,
\]

where \( \omega_{A+\delta K} \) is defined below the decomposition \([2.8]\). Theorem \([3]\) can be applied after appropriate adjustments in the count of isolated and embedded eigenvalues in the stability problem \([5.14]\). Since \( \omega_+ = 0 \), Theorem \([4]\) is not applicable to the fifth-order KdV equation \([5.12]\). Because the continuous spectrum of \( \partial_x L_- \) is on \( \lambda \in i\mathbb{R} \), all real and complex eigenvalues are isolated and all purely imaginary eigenvalues including the zero eigenvalue are embedded.
**Lemma 5.7** Let $\lambda_j$ be a real eigenvalue of the stability problem (5.14) with the real-valued eigenvector $w_j(x)$, such that $\text{Re}(\lambda_j) > 0$ and $\text{Im}(\lambda_j) = 0$. Then there exists another eigenvalue $-\lambda_j$ in the problem (5.14) with the linearly independent eigenvector $w_j(-x)$. The linear combinations $w_j^{\pm}(x) = w_j(x) \pm w_j(-x)$ are orthogonal with respect to the operator $L_-$,

$$\left( L_- w_j^\pm, w_j^\pm \right) = \pm 2 \rho_j, \quad \left( L_- w_j^\mp, w_j^\mp \right) = 0,$$

where $\rho_j = (L_- w_j(-x), w_j(x))$.

**Proof.** Since $\phi(-x) = \phi(x)$, the self-adjoint operator $L_-$ is invariant with respect to the transformation $x \mapsto -x$. The functions $w_j(x)$ and $w_j(-x)$ are linearly independent since $w_j(x)$ has both symmetric and anti-symmetric parts provided that $\lambda_j \neq 0$. Under the same constraint,

$$(L_- w_j(\pm x), w_j(\pm x)) = \pm \lambda_j^{-1} (L_- w_j(\pm x), \partial_x L_- w_j(\pm x)) = 0,$$

and the orthogonality relations (5.17) hold by direct computations.

**Corollary 5.8** Let $\lambda_j$ be a complex eigenvalue of the stability problem (5.14) with the complex-valued eigenvector $w_j(x)$, such that $\text{Re}(\lambda_j) > 0$ and $\text{Im}(\lambda_j) > 0$. Then there exist eigenvalues $\bar{\lambda}_j$, $-\lambda_j$, and $-\bar{\lambda}_j$ in the problem (5.14) with the linearly independent eigenvectors $\bar{w}_j(x)$, $w_j(-x)$, and $\bar{w}_j(-x)$, respectively.

**Lemma 5.9** Let $\lambda_j$ be an embedded eigenvalue of the stability problem (5.14) with the complex-valued eigenvector $w_j(x)$, such that $\text{Re}(\lambda_j) = 0$ and $\text{Im}(\lambda_j) > 0$. Then there exists another eigenvalue $-\lambda_j = \bar{\lambda}_j$ in the problem (5.14) with the linearly independent eigenvector $w_j(-x) = \bar{w}_j(x)$. The linear combinations $w_j^{\pm}(x) = w_j(x) \pm \bar{w}_j(x)$ are orthogonal with respect to the operator $L_-$,

$$\left( L_- w_j^\pm, w_j^\pm \right) = 2 \rho_j, \quad \left( L_- w_j^\mp, w_j^\mp \right) = 0,$$

where $\rho_j = \text{Re} (L_- w_j(x), w_j(x))$.

**Proof.** Since operator $L_-$ is real-valued, the eigenvector $w_j(x)$ of the problem (5.14) with $\text{Im}(\lambda_j) > 0$ has both real and imaginary parts, which are linearly independent. Under the constraint $\lambda_j \neq 0$,

$$(L_- w_j, \bar{w}_j) = \lambda_j^{-1} (L_- w_j, \partial_x L_- \bar{w}_j) = 0,$$

and the orthogonality equations (5.18) follow by direct computations.

**Theorem 6** Let $N_{\text{real}}$ be the number of real eigenvalues of the stability problem (5.14) with $\text{Re}(\lambda) > 0$, $N_{\text{comp}}$ be the number of complex eigenvalues with $\text{Re}(\lambda) > 0$ and $\text{Im}(\lambda) > 0$, and $N_{\text{imag}}$ be the number
of imaginary eigenvalues with $\text{Im}(\lambda) > 0$ and $\rho_j \leq 0$ in (5.18). Assume that all embedded (zero and imaginary) eigenvalues of the stability problem (5.14) are simple, such that $\text{Ker}(L_+) = \{ \phi \} \in \mathcal{H}$. Then,

$$N_{\text{real}} + 2N_{\text{comp}} + 2N_{\text{imag}}^- = n(L_-) - N_{\text{zero}}, \quad (5.19)$$

where $N_{\text{zero}} = 1$ if $(L_-^{-1}\phi, \phi) \leq 0$ and $N_{\text{zero}} = 0$ otherwise.

**Proof.** Each isolated and embedded eigenvalue $\gamma_j = -\lambda_j^2$ of the generalized eigenvalue problem (2.3) is at least double with two linearly independent eigenvectors $u_j^\pm(x)$, such that $w_j^\pm = \partial_x u_j^\pm$. By Lemma 5.7 and Corollary 5.8, the dimension of non-positive invariant subspace of $L$ is invariant with respect to the transformation $x \mapsto -x$. For gradient dynamical systems, all negative eigenvalues of the stability problem (5.14) are simple, such that $N_n^- = N_p^- = N_{\text{real}}$ and $N_{c^+} = 2N_{\text{comp}}$. By convention in Remark 4.14 all embedded (zero and purely imaginary) eigenvalues are assumed to be simple. By Lemma 5.9 and the relation for eigenvectors of the stability problem (5.14),

$$\text{dim}(\mathcal{H}_L) = \{(L_+ u, u) = (L_- u', u') = (L_- w, w), \quad (5.20)\}
$$

we have $N_n^+ = 2N_{\text{imag}}^-$ and $N_n^0 = N_{\text{zero}}^0$. The count (5.19) follows by equality (4.11) of Theorem 3. \qed

**Remark 5.10** Theorem 6 can be generalized to any KdV-type evolution equation, when the linearization operator $L_-$ is invariant with respect to the transformation $x \mapsto -x$. When $N_{\text{imag}}^- = N_n^0 = 0$, the relation (5.19) extends the Morse index theory from gradient dynamical systems to the KdV-type Hamiltonian systems. For gradient dynamical systems, all negative eigenvalues of $L_-$ are related to real unstable eigenvalues of the stability problem. For the KdV-type Hamiltonian system, negative eigenvalues of $L_-$ may generate both real and complex unstable eigenvalues in the problem (5.14).

**Example 4.** Let $\phi(x)$ be a single-pulse solution on $x \in \mathbb{R}$ (which does not have to be positive). Assume that the operator $L_-$ has a one-dimensional kernel in $\mathcal{X}$ with the eigenvector $\phi'(x)$. Then the kernel of $L_+$ is one-dimensional in $\mathcal{X}$ with the eigenvector $\phi(x)$, such that $z(L_+) = 1$.

- Since $(\phi, \phi') = 0$, then $\text{Ker}(L_+) \in \mathcal{H}$ and $z_0 = 0$. Since $L_- \partial_x \phi(x) = -\phi(x)$, we have $z_1 = 1$ if $\frac{d}{dx} \| \phi \|^2_{L^2} = 0$. Since the latter case violates the assumption that the embedded kernel is simple in $\mathcal{H}$, we shall consider the case $\frac{d}{dx} \| \phi \|^2_{L^2} \neq 0$, such that $z_1 = 0$.

- Since $(K\phi, \phi) = -(\partial_x \phi, \phi) = -\frac{1}{2} \frac{d}{dx} \| \phi \|^2_{L^2}$, we have $N_{\text{zero}}^- = 1$ if $\frac{d}{dx} \| \phi \|^2_{L^2} > 0$ and $N_{\text{zero}}^- = 0$ if $\frac{d}{dx} \| \phi \|^2_{L^2} < 0$. By Lemma 4.17 and Remark 4.18 dim($\mathcal{H}_{A+\delta K}$) = dim($\mathcal{H}_A$) + $N_{\text{zero}}^-$. By the relation (4.10) of Theorem 4 with $N_p^- = N_{\text{real}}$, we have

$$N_{\text{real}} + 2N_{\text{comp}} + 2N_{\text{imag}}^- = \text{dim}(\mathcal{H}_{A+\delta K}) - N_{\text{zero}}^- = \text{dim}(\mathcal{H}_A). \quad (5.21)$$
• Recall that \( \dim(\mathcal{H}_A^+) = n(L_+) - n_0 \) in Proposition 2.1 (extended to the case \( \omega_+ = 0 \)). By the Sylvester Inertia Law Theorem and the relation \( (5.20) \), we have \( n(L_+) = n(L_-) \) in \( \mathcal{X} \). Since \( A(0) = -(L_+^{-1}\phi', \phi') = -(L_-^{-1}\phi, \phi) \), we have \( n_0 = N_{\text{zero}}^+ \) in \( \mathcal{H} \). As a result, the relation \( (5.21) \) recovers the same equality \( (5.19) \).

**Remark 5.11** General stability-instability results for the traveling waves of the KdV-type equations were obtained in [BSS87, SS90] when \( n(L_-) = 1 \). In this case, \( N_{\text{comp}} = N_{\text{imag}}^- = 0 \) and \( N_{\text{real}}^- = 1 - N_{\text{zero}}^- \), such that stability follows from \( \frac{d}{dc} \| \phi \|_{L^2}^2 > 0 \) and instability follows from \( \frac{d}{dc} \| \phi \|_{L^2}^2 < 0 \). By a different method, Lyapunov stability of positive traveling waves \( \phi(x) \) was considered in [W87]. Specific studies of stability for the fifth-order KdV equation \( (5.12) \) were reported in [S92, DK99] with the energy-momentum methods. Extension of the stability-instability theorems of [BSS87, W87] with no assumption on a simple negative eigenvalue of \( L_- \) was developed in [L99, P03] with a variational method. The variational theory is limited however to the case of homogeneous nonlinearities, e.g. \( b_3 = 0 \) or \( b_1 = b_2 = 0 \). Our treatment of stability in the fifth-order KdV equation \( (5.12) \) is completely new and it exploits a similarity between stability problems of KdV and NLS solitons. The pioneer application of the new theory to stability of N-solitons in the KdV hierarchy is reported in [KP05]. Further progress on the same topic will appear soon in [BCD06, S06].

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