Abstract

Let \( \mathcal{M}_g(2) \) be the moduli space of curves of genus \( g \) with a level-2 structure. We prove that there is always a non hyperelliptic element in the intersection of four thetanull divisors in \( \mathcal{M}_6(2) \). We prove also that for all \( g \geq 3 \), each component of the hyperelliptic locus in \( \mathcal{M}_g(2) \) is a connected component of the intersection of \( g - 2 \) thetanull divisors.

1 Introduction

Let \( C \) be a curve of genus \( g \geq 3 \) over \( \mathbb{C} \) and \( \sigma : \mathbb{Z}^{2g} \longrightarrow H_1(C, \mathbb{Z}) \) a symplectic isomorphism. We associate to \( (C, \sigma) \) a period matrix \( Z \) in the Siegel upper half space \( H_g \) and the numbers

\[
\theta[k](Z) := \sum_{r \in \mathbb{Z}^g} \exp \left( \pi \sqrt{-1} \left[ t \left( r + \frac{k'}{2} \right) Z \left( r + \frac{k'}{2} \right) + 2 \left( r + \frac{k'}{2} \right) \frac{k''}{2} \right] \right).
\]

for each \( k = (k', k'') \) in \( (\mathbb{Z}/2\mathbb{Z})^{2g} \) such as \( k'k'' \) is even. The vanishing of \( \theta[k](Z) \) depends only of \( \sigma(\text{mod} \, 2) \), that is, on the class of \( (C, \sigma) \) in \( \mathcal{M}_g(2) \), the moduli space of curves of genus \( g \) on \( \mathbb{C} \) with a level-2 structure. Thus the zero locus of \( \theta[k] \) defines a divisor in \( \mathcal{M}_g(2) \) called thetanull divisor of characteristic \([k]\). In genus 3, each thetanull divisor is a component of the hyperelliptic locus in \( \mathcal{M}_3(2) \). In genus 4, we know since Riemann that each intersection of two thetanull divisors is an union of hyperelliptic components in \( \mathcal{M}_4(2) \). In genus 5, Accola has established in [A1], that with a condition on the three characteristics, the intersection of the corresponding divisors is an union of hyperelliptic components. We propose here to prove that this fails in genus 6:

**Theorem 1.1** Each sub-variety of \( \mathcal{M}_6(2) \) intersection of four thetanull divisors contains an element which is not hyperelliptic.
The first step of the proof is to classify the orbits of the action of $\text{Sp}_{2g}(\mathbb{Z}/2\mathbb{Z})$ on the set of quadruplets of thetanull divisors. Afterwards, we finish the proof by verifying that one element of each orbit (and then every quadruplet) defines a subvariety of $\mathcal{M}_{0}(2)$ which contains a bi-elliptic curve. In view of this result, it is unlikely that hyperelliptic components are intersection of $g - 2$ thetanull divisors in higher genus. Nevertheless, in the last part, we prove the following result:

**Theorem 1.2** Every component of the hyperelliptic locus in $\mathcal{M}_{g}(2)$ can be defined as a connected component of the intersection of $g - 2$ thetanull divisors.

### 2 Preliminaries

In what follows we denote by $C$ a curve of genus $g$ over $\mathbb{C}$. Let $A_{g,2}$ be the moduli space of $g$-dimensional principally polarized abelian varieties with a level-2 structure. It can be described as follows (see for instance [BL] chapter 4). Let $H_{g}$ be the Siegel generalized half-space:

$$H_{g} = \{ Z \in M_{g}(\mathbb{C}) \mid ^{t}Z = Z, \text{ Im}Z > 0 \}.$$ 

Then to each $Z$ in $H_{g}$ corresponds the complex torus $\mathbb{C}^{g}/(\mathbb{Z}^{g} + Z.\mathbb{Z}^{g})$ which comes with a natural principal polarization and a level-2 structure. Moreover we let

$$\Gamma_{g}(2) := \{ M \in \text{Sp}_{2g}(\mathbb{Z}) \mid M \equiv 1_{2g}(\text{mod } 2) \}.$$ 

this group acts on $H_{g}$ in the following way:

$$M.Z := (AZ + B)(CZ + D)^{-1}, \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_{g}(2).$$

Then $A_{g,2}$ is isomorphic to the quotient space $H_{g}/\Gamma_{g}(2)$. We denote by $\mathcal{M}_{g,2}$ the space which parametrizes the pairs $(C, \sigma)$ where $C$ is a curve of genus $g$ and $\sigma : (\mathbb{Z}/2\mathbb{Z})^{2g} \rightarrow H_{1}(C, \mathbb{Z}/2\mathbb{Z})$ a symplectic isomorphism. $\text{Sp}_{2g}(\mathbb{Z}/2\mathbb{Z})$ acts on $\mathcal{M}_{g,2}$ in a natural way. Let $k = (k', k'') \in (\mathbb{Z}/2\mathbb{Z})^{2g}$ such as $k'.k''$ is even (in what follows, we will say that $k$ is even). Let

$$\theta[k] : \quad H_{g} \rightarrow \mathbb{C}$$

be the function defined by

$$\theta[k](Z) = \sum_{r \in \mathbb{Z}^{g}} \exp \left( \pi \sqrt{-1} \left[ ^{t} \left( r + \frac{k'}{2} \right) Z \left( r + \frac{k'}{2} \right) + 2 \left( r + \frac{k'}{2} \right) \frac{k''}{2} \right] \right).$$
Let $M$ be in $\Gamma_g(2)$. As a consequence of the transformation formula (see for instance [I] page 176), $\theta[k](M,Z)$ is proportional to $\theta[k](Z)$. By this fact, the zero locus of $\theta[k]$ defines a divisor of $\mathcal{M}_g(2)$ denoted by $\theta[k]_0$ and called the **theta-null divisor** of characteristic $[k]$. Finally, the natural action of $\text{Sp}_{2g}(\mathbb{Z}/2\mathbb{Z})$ on $(\mathbb{Z}/2\mathbb{Z})^{2g}$ induces an action on these hypersurfaces: For each $k$ in $(\mathbb{Z}/2\mathbb{Z})^{2g}$ which is even, for each $M$ in $\text{Sp}_{2g}(\mathbb{Z}/2\mathbb{Z})$, we have

$$M.\theta[k]_0 = \theta[M.k]_0.$$

### 3 Theta characteristics, symplectic torsors

Each concept of this part can be found (for instance) in [S].

**Definition 3.1** Let $C$ be a curve of genus $g$ over $\mathbb{C}$ and $K$ its canonical divisor class. A **theta characteristic** $\Delta$ on $C$ is a degree $g-1$ divisor class such as

$$2\Delta \equiv K$$

Moreover $\Delta$ is called **even** (resp. **odd**) if $h^0(\Delta) \equiv 0 \pmod{2}$ (resp. $1 \pmod{2}$).

Let $S(C)$ be the set of theta characteristics on $C$ and let $S^+(C)$ be the set of even ones.

**Definition 3.2** Let $(J,.)$ be a symplectic pair (that is a $\mathbb{Z}/2\mathbb{Z}$-vector space endowed with a non degenerate, alternate, bilinear form), we say that a pair $(S,Q)$ is a symplectic torsor over $(J,.)$ if there is simply transitive action of $J$ on $S$ denoted $+$ and a mapping $Q : S \to \mathbb{Z}/2\mathbb{Z}$ having the property

$$Q(s) + Q(j_1 + s) + Q(j_2 + s) + Q(j_1 + j_2 + s) = j_1.j_2 \quad (\ast)$$

**Example 3.3** Let $W$ be a set of order $2g + 2$. We denote by $P_2(W)$ the set of partitions on $W$ into two subsets. For each subset $A$ of $W$, we denote by $\bar{A}$ the element $(A; \mathbb{C}_W.A)$ of $P_2(W)$. One has

1. An addition on $P_2(W)$: For each $\bar{A}$ and $\bar{B}$ in $P_2(W)$

$\bar{A} + \bar{B} := \overline{A \cup B - A \cap B}$

2. A map $p : P_2(W) \to \mathbb{Z}/2\mathbb{Z}$ defined by

$$p(\bar{A}) = |A| \pmod{2}.$$
3. A map \( e : P_2^+(W) \times P_2^+(W) \rightarrow \mathbb{Z}/2\mathbb{Z} \) defined by
\[
e(\bar{A}; \bar{B}) := |A \cup B| \pmod{2}.
\]
If \( g \) is even (resp odd) then we have a quadratic form defined on \( P_2^-(W) \) (resp on \( P_2^+(W) \)) by
\[
Q_-(\bar{A}) = \frac{|A| + 1}{2} \pmod{2}
\]
(resp \( Q_+(\bar{A}) = \frac{|A|}{2} \pmod{2} \)).

\((P_2^-(W), Q_-)\) (resp \((P_2^+(W), Q_+)\)) is a symplectic torsor over \((P_2^+(W), e)\) denoted by \( T(W) \).

**Example 3.4** For each symplectic pair \((J, .)\), the set of quadratic forms

\[\text{Quad}(J) := \{ q : J \rightarrow \mathbb{Z}/2\mathbb{Z} \mid \forall j_1 \in J, \forall j_2 \in J, q(j_1) + q(j_2) + q(j_1 + j_2) = j_1 \cdot j_2 \}\]endowed with the mapping \( \text{Arf} : \text{Quad}(J) \rightarrow \mathbb{Z}/2\mathbb{Z} \) (see for instance \([\text{Sc}]\)) is a symplectic torsor over \((J, .)\). Moreover we have the following property :
\[\forall q \in \text{Quad}(J), \forall j \in J, \text{Arf}(j + q) = \text{Arf}(q) + q(j)\]

**Proposition 3.5** Let \((J_2(C), .)\) be the \( \mathbb{Z}/2\mathbb{Z} \)-vector space of order two points of the Jacobian of \( C \) endowed with the intersection pairing. If we denote by \( Q \) the mapping
\[Q : S(C) \rightarrow \mathbb{Z}/2\mathbb{Z} \quad \Delta \mapsto h_0(\Delta) \pmod{2}\]
then \((S(C), Q)\) is a symplectic torsor over \((J_2(C), .)\).

**Remark 3.6** Let \( C \) be an hyperelliptic curve. If we denote by \( W(C) \) the set of its Weierstrass points, then we have a canonical isomorphism of symplectic torsors
\[\widetilde{\beta}_C : T(W(C)) \rightarrow (S(C), Q)\]compatible with the canonical symplectic isomorphism
\[\beta_C : (P_2^+(W(C)), e) \rightarrow (J_2(C), .)\].
Finally, we recall the result established by Mumford in [M]:

**Proposition 3.7** The map

\[
(S(C), Q) \rightarrow (\text{Quad}(J_2(C)), \text{Arf})
\]

\[
\Delta \rightarrow \left( q_\Delta : J_2(C) \rightarrow \mathbb{Z}/2\mathbb{Z} \\
j \mapsto Q(\Delta) + Q(j + \Delta) \right)
\]

is an isomorphism of symplectic torsors. In particular it preserves the parity.

4 **Proof of theorem 1.1**

4.1 **Orbits of the action of Sp\(_{12}(\mathbb{Z}/2\mathbb{Z})\) on the sets of four distincts even theta characteristics**

Let \(C\) be a curve of genus 6 over \(\mathbb{C}\). For each set \(\{\Delta_i\}_{i=1,4}\) of four distincts even theta characteristics on \(C\), we can associate the set \(\{\delta_i\}_{i=1,4}\) of theta characteristics on \(C\), defined as

\[
\delta_i \equiv 2K - \sum_{j \neq i} \Delta_j.
\]

**Proposition 4.1** Let \(C\) be a curve of genus 6 over \(\mathbb{C}\). The action of \(\text{Sp}_{12}(\mathbb{Z}/2\mathbb{Z})\) on the sets of four even theta characteristics has four orbits \(A_1, \ldots, A_4\), given respectively by the following conditions:

1. \(\sum_{i=1}^{4} \Delta_i \equiv 2K\).
2. \(\{\Delta_i\}_{i=1,4}\) is not in \(A_1\) and all the \(\delta_i\)’s are even.
3. Exactly two of the \(\delta_i\)’s are even.
4. All the \(\delta_i\)’s are odd.

By Proposition 3.7, we can study the action of \(\text{Sp}_{12}(\mathbb{Z}/2\mathbb{Z})\) on the sets of four distinct even quadratic forms on \(J_2(C)\):

**Lemma 4.2** Let \(C\) be a curve of genus 6 over \(\mathbb{C}\). Let \(\{q, q+a_1, q+a_2, q+a_3\}\) and \(\{q, q+a_1', q+a_2', q+a_3'\}\) be two sets of four distinct even quadratic forms on \(J_2(C)\) so that

\[
\dim < a_1, a_2, a_3 > = \dim < a_1', a_2', a_3' >.
\]
If it exists $\sigma \in \mathfrak{S}_3$ so that
\[ \forall k, l \in \{1, 2, 3\}, \; a_k.a_l = a'_{\sigma(k)}.a'_{\sigma(l)}, \]
then these two sets are in the same orbit of the action of $\text{Sp}_{12}(\mathbb{Z}/2\mathbb{Z})$ on the sets of four distinct even quadratic forms on $\mathcal{J}_2(C)$.

**Proof:** By taking into account the parity conditions on the quadratic forms, we have:
\[
q(a_1) = q(a'_1), \\
q(a_2) = q(a'_2), \\
q(a_3) = q(a'_3).
\]
Moreover, $\dim < a_1, a_2, a_3 > = \dim < a'_1, a'_2, a'_3 >$ and it exists $\sigma \in \mathfrak{S}_3$ so that
\[ \forall k, l \in \{1, 2, 3\}, \; a_k.a_k = a'_{\sigma(k)}.a'_{\sigma(l)}. \]
By Witt’s theorem (see for instance [Sc]) these conditions are equivalent to:
\[ \exists f \in \text{O}(q) \text{ so that } \forall k \in \{1, 2, 3\}, \; f(a_k) = a'_{\sigma(k)}. \]
Finally, this is equivalent to the existence of $f$ in $\text{O}(q)$ so that
\[
q_1 = q'_{\tau_1} \circ f^{-1} \\
q_2 = q'_{\tau_2} \circ f^{-1} \\
q_3 = q'_{\tau_3} \circ f^{-1}. \square
\]

**Proof of Proposition 4.1:** As the action preserves the parity, each orbit is contained in one of the $A_i$'s. By this fact, we have to show the transitivity of this action on these sets:

Let $\Delta_1, \Delta_2, \Delta_3$ and $\Delta_4$ be four even theta characteristics on $C$. Let $q_1, q_2, q_3$ and $q_4$ be the quadratic forms associated to these theta characteristics. For each $i$ in $\{1, 2, 3\}$, let $a_i$ be the element of $\mathcal{J}_2(C)$ so that
\[ q_i = q_4 + a_i. \]
If $a_1, a_2$ and $a_3$ are linearly dependent then $\Delta_4 + \Delta_1 = \Delta_2 + \Delta_3$.
If $a_1, a_2$ and $a_3$ are linearly independent then we have:
\[ \forall \{i, j\} \subseteq \{1, 2, 3\}, \]
\[ a_i.a_j = 0 \iff \Delta_4 + \Delta_i = \Delta_j \text{ even}. \]
Now let us suppose that \( \{ \Delta_i \}_{i=1\ldots4} \) is in the same orbit as \( \{ \Delta_i^* \}_{i=1\ldots4} \). By this fact, if \( \{ \Delta_i \}_{i=1\ldots4} \) and \( \{ \Delta_i^* \}_{i=1\ldots4} \) are both in \( A_1, A_2, A_4 \), then the transitivity is obvious by lemma 4.2.

Let \( q_1^*, q_2^*, q_3^* \) and \( q_4^* \) be four even quadratic forms on \( J_2(C) \) corresponding to \( \{ \Delta_i^* \}_{i=1\ldots4} \), an other set of even theta characteristics on \( C \). The transitivity of the action (on even quadratic forms) implies that there exists \( M \) in \( \text{Sp}_{12}(\mathbb{Z}/2\mathbb{Z}) \), so that \( M.q_1^* = q_4 \). If we put

\[
q_1' := M.q_1^* \\
q_2' := M.q_2^* \\
q_3' := M.q_3^*
\]

then \( \{ q_1', q_2', q_3', q_4' \} \) is in the same orbit as \( \{ q_1^*, q_2^*, q_3^*, q_4^* \} \).

By this fact, if \( \{ \Delta_i \}_{i=1\ldots4} \) and \( \{ \Delta_i^* \}_{i=1\ldots4} \) are both in \( A_1, A_2, A_4 \), then the transitivity is obvious by lemma 4.2.

Now let us suppose that \( \{ \Delta_i \}_{i=1\ldots4} \) and \( \{ \Delta_i^* \}_{i=1\ldots4} \) are both in \( A_3 \). For each \( i \) in \( \{ 1, 2, 3 \} \), let \( a_i' \) be the element of \( J_2(C) \) so that

\[
q_i = q_4 + a_i'.
\]

If Card (\( \{ i, j \} \subset \{ 1, 2, 3 \} \mid a_i, a_j \} = \text{Card} (\{ i, j \} \subset \{ 1, 2, 3 \} \mid a_i', a_j' \})

then we can conclude again with lemma 4.2. If not, we can suppose for instance, that

\[
\begin{align*}
a_1.a_2 &= 1 & a_1', a_2' &= 1 \\
a_1.a_3 &= 0 & a_1', a_3 &= 1 \\
a_2.a_3 &= 0 & a_2', a_3 &= 0.
\end{align*}
\]

By lemma 4.2 there exist \( a_2'' \) and \( a_3'' \) in \( J_2(C) \) so that

\[
\begin{align*}
a_2.a_2'' &= 1 \\
a_2.a_3'' &= 1 \\
a_2.a_3 &= 0.
\end{align*}
\]

and \( \{ q_4 + a_2, q_4 + a_3'', q_4 + a_3'', q_4 \} \) is in the same orbit as \( \{ q_4 + a_1', q_4 + a_2', q_4 + a_3', q_4 \} \). If we put

\[
q := q_4 + a_2,
\]
then
\[ \{q_1, q_2, q_3, q_4\} = \{q + (a_1 + a_2), q, q + (a_3 + a_2), q + a_2\} \]
\[ \{q_4 + a_2, q_4 + a_2', q_4 + a_3', q_4\} = \{q, q + (a_2 + a_2'), q + (a_2 + a_3'), q + a_2\}. \]

We can verify that
\[ (a_1 + a_2).(a_2 + a_3) = 1 \quad \text{and} \quad a_2.(a_2 + a_2') = 1 \]
\[ (a_1 + a_2).a_2 = 1 \quad \text{and} \quad a_2(a_2 + a_3) = 1 \]
\[ (a_2 + a_3).a_2 = 0 \quad \text{and} \quad (a_2 + a_2')(a_2 + a_3') = 0. \]
and conclude with the lemma \[\square\]

### 4.2 Theta characteristics on bi-elliptic curves

Let \( C \) be a bi-elliptic curve of genus 6. By definition this means that there exists an elliptic curve \( X \) and a degree 2 morphism:
\[ \pi : C \longrightarrow X \]
The Riemann-Hurwitz theorem implies that \( \pi \) has 10 ramification points denoted \( R_1, \ldots, R_{10} \).

**Lemma 4.3 ([A2])** A general bi-elliptic curve has forty even, effective theta characteristics of the form
\[ R_i + \pi^*(D_i) \]
where \( R_i \) is one of the ramification points of \( \pi \) and \( D_i \) is a degree two divisor on \( X \).

**Remark 4.4**
- \( \pi^*(\mathcal{J}(X)) \) has 3 non zero points of order 2 denoted by \( F_1, F_2 \) and \( F_3 \). If for each \( i \) in \( \{1, \ldots, 10\} \), we choose a divisor \( D_i \) so that \( R_i + \pi^*(D_i) \) is an even theta characteristic on \( C \), then the three other even, effective theta characteristics with fixed point \( R_i \) will be
\[ R_i + \pi^*(D_i + F_j) \quad j = 1 \ldots 3 \]

**Lemma 4.5** Let \( \Delta_1 = R_1 + \pi^*(D_1) \), \( \Delta_2 = R_2 + \pi^*(D_2) \) and \( \Delta_3 = R_3 + \pi^*(D_3) \) be three even effective theta characteristics on a general bi-elliptic curve \( C \). One has
\[ \Delta_1 + \Delta_2 - \Delta_3 \text{ even } \iff \begin{cases} \text{two at least of} \\ \text{the } R_i \text{ are the same} \end{cases} \]
Proof: First of all, let us notice that for each \( j, k, l \) so that 
\[ \{j, k, l\} = \{1, 2, 3\}, \]
\[ \Delta_1 + \Delta_2 - \Delta_3 = R_j + R_k - R_l + \pi^* D_j + \pi^* D_k - \pi^* D_l \]
then \( D_j + D_k - D_l \equiv E_l \) where \( E_l \) is a degree two, effective divisor on \( X \); thus \((\Leftarrow)\) is evident.

Now, if we call \( P_l \), the point on \( X \) so that \( \pi^* P_l = 2 R_l \), as on an elliptic curve every linear system of degree \( d \) is of dimension \( d \), there exists \( Q_l \), a point on \( X \) so that 
\[ R_j + R_k - R_l + \pi^* P_l + \pi^* Q_l \equiv R_j + R_k - R_l + \pi^* D_j + \pi^* D_k - \pi^* D_l \]
But then, by lemma 4.5, for a general bi-elliptic curve, if we are considering, for instance, that \( R_1 \) is the fixed point of the even theta characteristic \( \Delta_1 + \Delta_2 - \Delta_3 \), then \( R_2 + R_3 + \pi^* Q_l \) must be the pullback of an effective degree two divisor. This is possible if and only if \( R_2 = R_3 \).

□

With this lemma, one checks easily the next result which ends our proof.

**Proposition 4.6** Let \( C \) be a smooth curve of genus 6 with a degree two morphism 
\[ \pi: C \rightarrow X \]
onto an elliptic curve \( X \). Let \( R_1, \ldots, R_{10} \) be the ramification points of this morphism and let \( D_1, \ldots, D_{10} \) be some degree two effective divisors on \( X \) so that 
\[ \{R_i + \pi^* D_i \mid i \in \{1, \ldots, 10\}\} \]
is a set of the even, effective theta characteristic on \( C \). The following sets are in the four orbits of the action of \( \text{Sp}_{12}(\mathbb{Z}/2\mathbb{Z}) \) on the sets of four distinct even theta characteristics on \( C \) :
\[
\begin{align*}
\{R_1 + \pi^* D_1, R_2 + \pi^* (D_2 + F_1), R_1 + \pi^* (D_1 + F_1), R_2 + \pi^* D_2\} & \in A_1 \\
\{R_1 + \pi^* D_1, R_2 + \pi^* (D_2 + F_2), R_1 + \pi^* (D_1 + F_1), R_2 + \pi^* D_2\} & \in A_2 \\
\{R_1 + \pi^* D_1, R_2 + \pi^* (D_2), R_3 + \pi^* (D_3), R_1 + \pi^* (D_2 + F_1)\} & \in A_3 \\
\{R_1 + \pi^* D_1, R_2 + \pi^* D_2, R_3 + \pi^* D_3, R_4 + \pi^* D_4\} & \in A_4.
\end{align*}
\]
5 Proof of theorem 1.2

First of all, we need to recall a result given by Teixidor I Bigas in \[T\] :

Let \( S_1 \) be the scheme which parametrizes the pairs \((C, \Delta)\), where \( C \) is curve of genus \( g \) on \( C \) and \( \Delta \) is a theta characteristic of projective dimension 1 on \( C \). For each \( t = (C, \Delta) \in S_1 \), there is an injective map:

\[
f : T_{S_1(t)} \longrightarrow H^1(T_C)
\]

As by Serre duality \( H^1(T_C) \) is dual to \( H^0(2K) \), we have:

**Lemma 5.1 (Teixidor I Bigas)** If \( F \) is the fixed part of the linear system \(|\Delta|\) and \( R \) the ramification divisor of the corresponding morphism \( \varphi_\Delta : C \longrightarrow \mathbb{P}^1 \) then \( \text{Im}(f) = (\omega)^\perp \), where \( \omega \) is an element of \( H^0(2K) \) with divisor \( R + 2F \).

By using the particular expression of the theta characteristics on an hyperelliptic curve (see [A,C,G,H] p 288), this lemma has the following consequence: Let \( C \) be an hyperelliptic curve of genus \( g \geq 3 \), \( W := \{p_1, \ldots, p_{2g+2}\} \) the set of its Weierstrass points, \( H \) its hyperelliptic divisor, then for each even theta characteristic \( \Delta \) on \( C \) of dimension 1,

\[
\Delta = H + p_{i_1} + \cdots + p_{i_{g-3}}.
\]

\( \text{Im}(f) \) is the orthogonal of \( \omega \) so that

\[
\text{div}(\omega) = p_1 + \cdots + p_{2g+2} + 2(p_{i_1} + \cdots + p_{i_{g-3}})
\]

**Proposition 5.2** Let \((C, \sigma)\) be an hyperelliptic element of \( \mathcal{M}_g(2) \ (g \geq 3) \). Let \( H \) be the hyperelliptic divisor on \( C \). We choose \( p_1, \ldots, p_{g-2}, g-2 \) Weierstrass points on \( C \) and let \( E \) be the divisor \( p_1 + \cdots + p_{g-2} \). Then the \( g-2 \) thetanull divisors associated to the theta characteristics \( \{H+E-p_1, \ldots, H+E-p_{g-2}\} \) intersect transversally at \((C, \sigma)\).

**Proof:** By the last remark, we have to prove that the linear subsystem of \( H^0(2K) \) generated by

\[
\{F_k = p_1 + \cdots + p_{2g+2} + 2(E-p_k) \mid k = 1, \ldots, g-2\}
\]

has rank \( g-2 \). Let us prove that for each \( k \) in \( \{2, \ldots, g-2\} \), \( F_k \) is not in the linear system generated by \( F_1, \ldots, F_{k-1} \). Let \( C \) be the Riemann surface

\[
y^2 = \prod_{i=1}^{2g+2} (x - x_i) , \quad x : C \longrightarrow \mathbb{P}^1
\]
so that for each $i$ in $\{1, \ldots, 2g+2\}$, $x(p_i) = x_i$. We have then to verify that there does not exist $(\lambda_1 : \cdots : \lambda_{k-1})$ in $\mathbb{P}^{k-2}$ so that $2(E - p_k)$ is the divisor $f \ast 0$ where

$$f = \lambda_1(x-x_2) \cdots (x-x_{g-2}) + \cdots + \lambda_{k-1}(x-x_1) \cdots (x-x_{k-2})(x-x_k) \cdots (x-x_{g-2})$$

Let us suppose it is false. As $2(E - p_k)$ is $2p_1 + \cdots + 2p_{k-1} + 2p_{k+1} + \cdots + 2p_{g-2}$ this implies

$$f(x_1) = \lambda_1(x_1 - x_2)(x_1 - x_2) \cdots (x_1 - x_{g-2}) = 0 \Rightarrow \lambda_1 = 0$$
then

$$f(x_2) = 0 \Rightarrow \lambda_2 = 0$$
then

$$\vdots$$

$$f(x_{k-1}) = \lambda_{k-1}(x_{k-1} - x_1) \cdots (x_{k-1} - x_{k-2})(x_{k-1} - x_k) \cdots (x_{k-1} - x_{g-2}) = 0$$

$$\Rightarrow \lambda_{k-1} = 0$$

Finally we would have $\lambda_1 = \cdots = \lambda_{k-1} = 0$ which it is absurd. □

Each connected component $Y$ of $\mathcal{H}_g(2)$ is uniquely determined by a symplectic isomorphism

$$c_Y : (\mathbb{Z}/2\mathbb{Z})^{2g} \longrightarrow P_2^+\left(\{1, \ldots, 2g + 2\}\right)$$
and an isomorphism between symplectic torsors

$$\widetilde{c}_Y : (\mathbb{Z}/2\mathbb{Z})^{2g} \longrightarrow T(\{1, \ldots, 2g + 2\})$$
in the following way:

Let $U$ be the open subvariety of $\mathbb{C}^{2g+2}$ consisting of points with distinct coordinates. To each point $\xi = (\xi_1, \ldots, \xi_{2g+2})$ in $U$, we associate the hyperelliptic curve $C(\xi)$ with Weierstrass points $\xi_1, \ldots, \xi_{2g+2}$. We denote by $W(\xi)$ this set. For each $\xi$ in $U$, the bijection $[1, \ldots, 2g+2] \longrightarrow [\xi_1, \ldots, \xi_{2g+2}]$ induces a symplectic isomorphism

$$\alpha_\xi : P_2^+\left(\{1, \ldots, 2g + 2\}\right) \longrightarrow P_2^+(W(\xi))$$
and an isomorphism between symplectic torsors:

$$\widetilde{\alpha}_\xi : T(\{1, \ldots, 2g + 2\}) \longrightarrow T(W(\xi)).$$
$Y$ is the subspace of $\mathcal{M}_g(2)$ defined as

$$\{(C(\xi), \sigma_\xi) \in \mathcal{H}_{g,2} \mid \xi \in U\},$$

so that for each $\xi$ in $U$, the following diagrams

$$\begin{array}{c}
P_2^+(\{1, \ldots, 2g + 2\}) \xrightarrow{\alpha_\xi} P_2^+(C(\xi)) \xrightarrow{\beta_{C(\xi)}} J_2(C(\xi)) \\
(\mathbb{Z}/2\mathbb{Z})^{2g} \xrightarrow{\sigma_\xi}
\end{array}$$

$$\begin{array}{c}
T(\{1, \ldots, 2g + 2\}) \xrightarrow{\tilde{\alpha}_\xi} T(W(\xi)) \xrightarrow{\tilde{\beta}_{C(\xi)}} S(C(\xi)) \\
(\mathbb{Z}/2\mathbb{Z})^{2g} \xrightarrow{\sigma_\xi}
\end{array}$$

are commutative. Now let $\xi_0$ be in $U$. Let us choose $g - 2$ thetanull divisors which correspond on $C(\xi_0)$, to $g - 2$ theta characteristics in the configuration of Proposition 5.2. By the last diagram, one sees that for each $\xi$ in $U$, these $g - 2$ thetanull divisors correspond to $g - 2$ theta characteristics on $C(\xi)$ in the same configuration as on $C(\xi_0)$. By this fact, $Y$ is a component of the intersection of these $g - 2$ thetanull divisors. □

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