On Thermodynamic and Ultraviolet Stability of Yang-Mills

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We prove ultraviolet stability bounds for the pure Yang-Mills relativistic quantum theory in an imaginary-time, functional integral formulation. We consider the gauge groups \( G = U(N) \), \( SU(N) \) and let \( d(N) \) denote their Lie algebra dimensions. We start with a finite hypercubic lattice \( \Lambda \subset a\mathbb{Z}^d \), \( d = 2, 3, 4 \), \( a \in (0, 1] \), \( L \) sites on a side, and with free boundary conditions. The Wilson partition function \( Z_{\Lambda,a} \equiv Z_{\Lambda,a,g^2,d} \) is used, where the action is a sum over gauge-invariant plaquette actions with a pre-factor \( a^{d-4}/g^2 \), where \( g^2 \in (0, g_0^2] \), \( 0 < g_0 < \infty \), defines the gauge coupling. By a judicious choice of gauge fixing, which involves gauging away the bond variables belonging to a maximal tree in \( \Lambda \), and which does not alter the value of \( Z_{\Lambda,a} \), we retain only \( \Lambda_r \) bond variables, which is of order \( [(d - 1)L^d] \), for large \( L \). We prove that the normalized partition function \( Z_{\Lambda,a}^\ast = (a^{d-4}/g^2)^{d(N)}L^{d/2}Z_{\Lambda,a} \) satisfies the stability bounds \( c_a^{d(N)}L_r \leq Z_{\Lambda,a}^\ast \leq c_a^{d(N)}L_r \), with finite \( c_a, c_b \in \mathbb{R} \) independent of \( L \), the lattice spacing \( a \) and \( g^2 \). In other words, we have extracted the exact singular behavior of the finite lattice free-energy. For the normalized free-energy \( f_r = [d(N) L \Lambda_r]^{-1} \) in \( Z_{\Lambda,a}^\ast \), our stability bounds imply, at least in the sense of subsequences, that a finite thermodynamic limit \( \Lambda \not\to a\mathbb{Z}^d \) exists. Subsequently, the continuum \( a \not\to 0 \) limit also exists.

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I. INTRODUCTION AND RESULTS

To show the existence, the particle spectrum, the particle interaction and scattering in a quantum field theory (QFT) are amongst the most fundamental problems in physics \cite{1,2}. Unfortunately, in spite of much work and progress (see e.g. \cite{3,4}), we lack a physically relevant, mathematically well-defined QFT in spacetime dimension \( d = 4 \). For many reasons, Quantum Chromodynamics (QCD) is the best candidate for such a model.

In this context, the existence of nonabelian pure-gauge, Yang-Mills models was considered in a series of papers \cite{6,7} (and Refs. therein) using a lattice regularization for the continuum spacetime \cite{8,9} and employing intricate tools as e.g. multiscale methods based on the renormalization group, and small/large field decompositions. Within this framework, thermodynamic and ultraviolet stability bounds \cite{10} were proven for \( d = 3, 4 \). Recently, in \cite{11}, abelian gauge models with fermions were considered in \( d = 3 \). But, up to now, stability bounds have not been proved for gauge-matter models like QCD. Together with confinement, this is a very challenging problem.

Here, we provide a very simple proof of thermodynamic and ultraviolet stability of Yang-Mills in Euclidean dimension \( d = 2, 3, 4 \) and for abelian/nonabelian compact gauge Lie groups \( G \). We work in configuration space and our method is direct and does not employ sophisticated analysis. Instead, it exploits the pointwise positivity of the gauge-invariant Wilson plaquette action \cite{9,12}, gauge invariance, properties of the Haar measure on \( G \) \cite{13}, a relation with random matrices and, finally, the Weyl formula \cite{14} for the integration over \( G \) of class functions (functions with constant values in the conjugacy classes of \( G \)). Our treatment is rigorous, uses the gluon fields and differs from the character representation of \cite{13}.

Our analysis applies to other compact Lie groups, but we focus on \( G = U(N) \), \( SU(N) \), and denote by \( d(N) \) the dimension of their Lie algebras \([N^2] \text{ and } (N^2 - 1), \text{ respectively}] \). For simplicity, we will treat explicitly the case \( G = U(N) \). In the last section of the paper, we show how the proofs are modified for \( SU(N) \). Our starting point is the finite-lattice partition function. We let \( \Lambda \subset a\mathbb{Z}^d \), \( a \in (0, 1] \), be a finite hypercubic lattice with \( L \) sites on each side. A site is denoted by \( x = (x_0, x_1, \ldots, x_{d-1}) \); \( x_0 \) is the time coordinate. If \( \nu = 0, 1, \ldots, (d - 1) \) is a coordinate direction and \( e^\nu \) its unit vector, \( b_\nu(x) \) denotes a lattice nearest-neighbor bond starting at \( x \) and ending at \( x + ae^\nu \equiv x^\nu + \). To each \( b_\nu(x) \), we assign a bond variable which is a unitary matrix \( g_{b_\nu}(x) \). The partition function for our model is

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The main result of this paper is the following thermodynamic and ultraviolet stable, stability bound.

\[ Z_{\Lambda,a} \equiv Z_{\Lambda,a,g,d} = \int \exp \left[ -\frac{a^{d-4}}{g^2} \sum_p A_p \right] \, dg. \]  

Here, \( g^2 > 0 \) defines the pure-gauge coupling, the measure \( dg \) is a product of normalized Haar measures \( d\sigma(g_b) \) on \( G \), one for each bond \( b \), with no distinction of orientation. If \( p \in \Lambda \) is a plaquette in the \( \mu - \nu \) plane (a minimal square), \( \mu < \nu \), with vertices at the sites \( x, x'_+, x''_+, x'_- + \alpha e^\nu \) and \( x''_+ \), then the single plaquette \( p \) action is given by

\[ A_p = |1 - U_p|^2_{H-S}, \]  

where \( U_p = U_1 U_2 U_3 U_4, \) with \( U_1 = g_{x,x''}, U_2 = g_{x'_+,x''_+}, U_3 = g_{x'_+,x''_+ + \alpha e^\nu}, U_4 = g_{x'_-,x''_+}^{-1} \). Here, \( g^{-1} \) is the inverse element of \( g \) and, for a square matrix \( M \) with trace \( Tr M \) and adjoint \( M^1, [M^1_{H-S}] = [Tr(MM^1)]^{1/2} \) is the Hilbert-Schmidt norm. Note that each plaquette action \( A_p \) is pointwise positive. Formally, in \([2]\), it is shown \((a^{d-4}/g^2) \sum_p A_p\) in Eq. (11) approximates the continuum Yang-Mills action in the limit \( a \searrow 0 \). Also, our model verifies Osterwalder-Schrader positivity, for \( L \) even \( [8] \).

We now discuss the parametrization of the bond variables for the gauge group \( U(N) \). (Much of our discussion also applies to \( SU(N) \).) Fixing a lattice bond \( b \) and for the associated variable \( U_b \) near the identity, we can write \( U_b = e^{iX_b} \), where \( X_b = x_bC_\alpha \), with a sum over \( \alpha = 1,2, \ldots, d(N) = N^2 \). The self-adjoint \( N \times N \) matrices \( \theta_\alpha \) form a Lie algebra basis and are taken to obey the normalization \( Tr(\theta_\alpha\theta_\beta) = \delta_{\alpha\beta} \), with a Kronecker delta. We call the real parameters \( x_bC_\alpha \) the gluon fields.

In terms of this parametrization, the Haar measure is a product of a density and a \( d(N) \)-dimensional Lebesgue measure. Whenever \( N > 2 \), we know some global gluon parametrization, as e.g. the parametrizations for \( SU(N) \) in terms of Euler angles (see e.g Ref. \([10]\)). However, in general, it is not clear that we can characterize their domains such that the global parametrization is also an injection between a group element and a parameter value, leading to a good characterization of the Haar measure on the gauge group \( G \). With these limitations, it is difficult to bound integrals of functions on \( G \). More precisely, in our case e.g. the Euler angle parametrizations do not lend to use good quadratic approximations. (See the end of this section where this point is made more clear.) In the special case of \( SU(2), \) a global gluon parametrization, its domain and an explicit formula for the density of the measure is available.

We now consider another parametrization which is global for class functions on \( SU(N) \). An arbitrary element \( U \in U(N) \) (not necessarily near the identity) is unitarily equivalent to a diagonal matrix \( D = \text{diag}(e^{i\lambda_1}, \ldots, e^{i\lambda_N}) \), \( \lambda_j \in (-\pi, \pi) \). We refer to the \( \lambda_j \) as the angular eigenvalues. If we are integrating a class function \( f(U) = f(VUV^{-1}) \) for any \( V \in G = U(N) \), then \( f(U) \) is a function of the angular eigenvalues only. By the Weyl integration formula \([13, 14]\), the gauge group integral of \( f(U) \) is equal to an integral over the angular eigenvalues with a measure which is the product of \( N \)-dimensional Lebesgue measure and an explicit density function; the integration domain is \((-\pi, \pi)^N \).

The functions we encounter here are not class functions of each bond variable. However, the bounds we obtain are class functions so that the Weyl integration formula and the angular eigenvalue parametrization can and is used. This is an important tool and ingredient in our method.

Let us now discuss local gauge invariance, gauge fixing and the model degrees of freedom. Recall the property of gauge invariance. The group \( \otimes \sigma G \), with element \( \prod_i r_i \), acts on bond variables mapping \( g_{b_i(x)} \) to \( r_x g_{b_i(x)} r_x^{-1} e^{i\alpha \nu} \); \( A_p \) and the total action are invariant under this mapping. Due to local gauge invariance, there is an excess of bond variables. By a gauge fixing procedure (see Chap. 22 of \([3]\)), some bond variables can be gauged away by setting them to the identity in the action, i.e. \( g_b = 1 \), for a bond \( b \), and omit the bond integration variable (its integral gives 1). The value of the partition function \( Z_{\Lambda,a} \) is unchanged in this process, provided the associated bonds do not form closed loops. As each bond variable has \( d(N) \) gluon fields, we reduce the number of degrees of freedom by \( d(N)\Lambda_p \), where \( \Lambda_p \) is the number of gauged away bonds. We denote by \( \Lambda_r \) the number of retained bond variables and, after the gauge fixing process, we are left with \( d(N)\Lambda_r \) degrees of freedom.

Here, we choose the \textit{enhanced temporal} (axial) gauge. If we identify the sites of the \( \mu \)-th coordinate with \( 1,2, \ldots, L \), this gauge is defined by setting the following bond variables to 1. First, for any \( d = 2,3,4 \), we take \( g_{b_0(x)} = 1 \). For \( d = 2, \) take also \( g_{b_1(x^0=1,x^1)} = 1 \). For \( d = 3 \), set also \( g_{b_1(x^0=1,x^1,x^2)} = 1 \) and \( g_{b_2(x^0=1,x^1=1,x^2)} = 1 \). Similarly, for \( d = 4 \), take also \( g_{b_1(x^0=1,x^1,x^2,x^3)} = 1 \), \( g_{b_2(x^0=1,x^1=x^2=1,x^3)} = 1 \) and \( g_{b_3(x^0=1,x^1=1,x^2=1,x^3)} = 1 \). For \( d = 3 \), the gauged away bonds can be visualized as forming a scrub brush with bristles along the \( x^0 \) axis and the grip forming a comb. In \( d = 2,3,4 \), the gauged away bonds do not form loops, and there are \( \Lambda_r \equiv \Lambda_r(d) \) remaining variables. \( \Lambda_r \) has the values \( (L-1)^2 \), \( (2L+1)(L-1)^2 \), \( (3L^3-L^2-L-1)(L-1) \), for \( d = 2,3,4 \), respectively. Clearly, \( \Lambda_r \geq \infty \) as \( \Lambda \not\geq a\mathbb{Z}^d \). Note that, fixing the enhanced temporal gauge, the \( \Lambda_r \) retained gauge variables are associated with bonds in the hypercubic lattice \( \Lambda \) which form a maximal tree, so that by adding any other bond to it we form a closed loop. The main result of this paper is the following thermodynamic and ultraviolet stable, stability bound.
Theorem 1 Let $d = 2, 3, 4$, $a \in (0, 1]$ and $g^2 \in (0, g_0^2]$, with $0 < g_0 < \infty$. Then, the normalized partition function
\[ Z_{\Lambda,a}^n = (a^{d-4}/g^2)^{d(N)\Lambda_r}/2Z_{\Lambda,a} \text{ satisfies the stability bounds} \]
\[ \exp[c_\varepsilon d(N)\Lambda_r] \leq Z_{\Lambda,a}^n \leq \exp[c_a d(N)\Lambda_r], \]
with finite constants $c_\varepsilon, c_a \in \mathbb{R}$ independent of $\Lambda_r, a$ and $g^2$. In these stability bounds, the exact singular behavior of the finite lattice free-energy has been extracted and isolated. At least in the sense of subsequences, a thermodynamic limit $\Lambda \rightarrow a\mathbb{Z}^d$ of the normalized free energy $f_{\Lambda,a}^n = [d(N)\Lambda_r]^{-1} \ln Z_{\Lambda,a}^n$ exists. Subsequently, a continuum limit $a \rightarrow 0$ also exists. These limits are finite.

Remark 1 The existence of the above finite subsequential limits follow, first, by considering a sequence of hypercubic lattices, with a fixed spacing $a$ and with increasing $\Lambda$. Taking the sequence of the corresponding normalized free-energies, by the Bolzano-Weierstrass theorem (BWT), we show there is a convergent subsequence in the thermodynamic limit $\Lambda \rightarrow a\mathbb{Z}^d$. Subsequently, we take a sequence of lattices in $a\mathbb{Z}^d$ with decreasing spacings $a$, and look at their normalized free-energies. Applying the BWT to this sequence shows there is a convergent subsequence in the continuum limit $a \rightarrow 0$. Here, we do not consider the unicity of the limits, but only their finiteness. Model correlations and physically relevant properties like their decay rates will be analyzed in the future.

The rest of the paper is devoted to show our stability bounds, and Theorem 1 is proved. Here, some details will be omitted. A complete mathematical proof shall appear elsewhere. For the upper stability bound, a special role is played by a single-plaquette partition function with a single bond variable. For $G = U(N)$ and with $N(N) \equiv 1/[(2\pi)^N N!]$, it is given by (the integral is 1 if all $\lambda_k \equiv 0$)

\[ z(c) = \int d\sigma (g) = N(N) \int_{[-\pi,\pi]^N} e^{-c \sum_{k=1}^N 2(1 - \cos \lambda_k)} \rho(\lambda) d\lambda. \tag{3} \]

Here, $\lambda = (\lambda_1, \ldots, \lambda_N)$, $dN\lambda = d\lambda_1 \ldots d\lambda_N$ and the density $\rho(\lambda)$ is, for $1 \leq j < k \leq N$,

\[ \rho(\lambda) = \prod_{j<k} |e^{i\lambda_j} - e^{i\lambda_k}|^2 = \prod_{j<k} [2 - \cos(\lambda_j - \lambda_k)]. \]

The second equality in Eq. (3) is an application of the Weyl integration formula for a class function. For a $N \times N$ unitary matrix with eigenvalues $e^{i\lambda_1}, e^{i\lambda_2}, \ldots, e^{i\lambda_N}, \lambda_k \in (-\pi, \pi]$, $\lambda_1, \ldots, \lambda_N$ are called the angular eigenvalues. Finally, $c \equiv c(a, g^2, d)$ is a constant depending on $[a^{d-4}/g^2]$ and will be specified later. Note that $A_p = |1 - U_p|_{H-S}$ is not a class function of each bond variable. Also, for $d = 2$, it is known (see \[15\]) the exact result $Z_{\Lambda,a} = z^{\Lambda_r}$, and $c = [1/(g^2 a^2)^2]$. In this case, $\Lambda_r$ is equal to the number of plaquettes.

For the lower bound, there is also a characteristic function in the integrand of $z(c)$ of Eq. (3). This function restricts each bond variable $g$ to be close to the identity.

The importance of Eq. (3) is that, by gauging away some of the bond variables, $Z_{\Lambda,a}$ is reduced to or bounded by a product of $z(c)$ (modified with a characteristic function, for the lower bound), and we can bound $Z_{\Lambda,a}$ by bounding $z(c)$. In turn, $z(c)$ is bounded by bounding the angular eigenvalue distribution. In this process, we remark that although the four-bond plaquette action $A_p$ is not a class function of each bond variable on $G$, the single bond partition function is.

Before proving the stability bounds, we give some intuition about how our methods work and also the relation between our gluon fields and the usual gauge potentials $A_p(x)$, where a gauge group element $g_{b_p}(x)$, associated with a lattice bond $b_p(x)$, is parametrized as $\exp[iga_{b_p}(x)]$. In the enhanced temporal gauge, we only have $\Lambda_r$ gauge bond variables. The group bond variable, in terms of gluon fields, is $e^{i\varepsilon x_4 b_4}$. Formally, in the quadratic approximation for the action the partition function is given by

\[ Z_{\Lambda,a} \simeq \int e^{-a^{d-4}/g^2 \sum_{p} x_p^2} d\bar{x}, \]

where, for $x_1, \ldots, x_4$ denoting the gluon fields or the gauge group parameters associated with the four consecutive sides of a plaquette $p$, we have set $x_p = x_1 + x_2 - x_3 - x_4$.

In terms of the scaled fields $y$ defined by $y_\ell = \frac{a^{(d-4)/2}}{g} x_\ell$, we have

\[ Z_{\Lambda,a} \simeq \left( \frac{a^{d-4}}{g^2} \right)^{-d(N)\Lambda_r/2} \int e^{-\sum_{p} y_p^2} d\tilde{y}. \]
Now, in terms of the gauge potentials \( A_\mu(x) = A_{\mu,a}(x)\theta_a \), we write the bond variables as \( g_\nu(x) = \exp(igaA_\mu(x)) \) and

\[
Z_{\Lambda,a} \simeq (ga)^d(N) \Lambda \int e^{-a^d-2 \sum_\nu A_\nu^2} d\tilde{A}.
\]

For a plaquette in the \( \mu-\nu \) plane, \( \mu < \nu \), with a corner at site \( x \), \( A_\nu^2 = (A_1 + A_2 - A_3 - A_4)^2 = a^2(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x))^2 \), where \( \partial_\mu A_\nu(x) \) is the finite difference derivative \( \partial_\mu f(x) = [f(x + a\epsilon^0) - f(x)]/a \). Hence, we obtain that

\[
Z_{\Lambda,a} \simeq (ga)^d(N) \Lambda \int e^{-a^d \sum_{\mu,\nu < \nu} (\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x))^2} d\tilde{A},
\]

so that the exponent is the Riemann sum approximation to the quadratic part of the classical (nonabelian) Yang-Mills action. The relation between the components of \( y \) fields and the \( A \) fields is \( y = a^{(d-2)/2}A \), which in terms of the dependence and for small lattice spacing \( a \), displays the good singularity structure factor.

Our lower and upper bounds on the lattice Yang-Mills action (for any field, in the upper bound case, and at least for small fields, in the lower bound case) are quadratic and validate the above formal argument, as shown below, in the proof of our stability bounds.

II. UPPER STABILITY BOUND

To prove the upper bound, we use the pointwise positivity of \( A_p \) to write a lower bound on \( A = \sum_p A_p \). We have

\[
e^{-A} \leq e^{-A_h} - A^4,
\]

where \( A_h \) is the sum over the actions of horizontal plaquettes in the plane \( x^0 = 1 \), only. Hence,

\[
Z_{\Lambda,a} \leq \int \exp[-c(A_h + A_p)] \, d\tilde{g} = z^{\Lambda_h}.
\]

The bound of Eq. (4) saturates for \( d = 2 \) and gives the exact result. For \( d = 3 \), the right-hand-side is \([z(c(a, g, d = 3))]^{(2L+1)(L-1)^2} \). To understand this result (it is similar for \( d = 2, 4 \)), we observe that the integral in Eq. (4) is performed applying the following procedure. First, we treat the integration over the \( 2L(L - 1) \) horizontal bond variables in the \( x^0 = L \) plane. Each bond variable appears in only one retained vertical plaquette. Using the right/left invariance of the Haar measure \([13, 14] \), we can perform a matrix change of variable. The integral over each of the \( x^0 = L \) plane variables factorizes. For each vertical plaquette, the remaining three integrals are done and give one. Altogether, we extract a factor \( z^{2L(L-1)} \). Similarly, we treat the other vertical plaquettes and integrate over the horizontal bond variables of the \( x^0 = (L - 1), \ldots, 2 \) planes. We get a factor of \( z^{2L(L-1)^2} \). With this, we are left with the integral over the \( x^0 = 1 \) plane, horizontal bond variables, which can be carried out in various ways. For instance, by integrating over the \( (L - 1) \) variables in the column between the planes \( x^1 = 1 \) and \( x^1 = 2 \), starting at \( x^1 = 1 \), we extract a factor of \( z^{L-1} \). Repeating this procedure over the remaining \( (L - 2) \) columns we get \( z^{L(L-1)^2} \). Using this and Eq. (4) we obtain the upper bound in Theorem 1. Note that, in the above procedure, we have not used local gauge invariance or gauge fixing. The same result holds using the enhanced temporal gauge defined above.

With Eq. (4) in mind, the upper bound on \( Z_{\Lambda,a} \) is proved by giving an upper bound on \( z(c) \) of Eq. (3). We need a lower bound on the actions and an upper bound on the density of the measure. The bounds we obtain are global, i.e. they hold for the whole gauge group \( G \). For this, we use the lower bound \( 17 \). \( 2(1 - \cos \theta) \geq 4\theta^2/\pi^2 \), \( |\theta| < \pi \), on \( A_p \) and the upper bound \( 2(1 - \cos \theta) \leq \theta^2 \), \( \theta \in \mathbb{R} \), on the density \( \rho(\lambda) \), to get \( |e^{i\lambda j} - e^{i\lambda k}|^2 \leq (\lambda_j - \lambda_k)^2 \). Doing this, for \( d = 3 \), with \( c(a, g^2, d = 3) = 1/(ag^2) \) and setting \( y_j = \frac{2\sqrt{2}}{\pi\sqrt{ag}} \lambda_j \), we obtain

\[
z(c(a, g^2, 3)) \leq N(N) \int_{[-\pi, \pi]^N} \exp \left[ -\frac{4}{\pi^2 ag^2} \sum_{j=1}^N \lambda_j^2 \right] \prod_{1 \leq j < k \leq N} (\lambda_j - \lambda_k)^2 \, d^N \lambda \]

\[
\leq [\pi \sqrt{ag}/(2\sqrt{2})]^N I(2\sqrt{2}/\sqrt{ag}),
\]

where we defined

\[
I(u) \equiv N(N) \int_{[-u, u]^N} \frac{e^{-y^2/2}}{\prod_{1 \leq j < k \leq N} (y_j - y_k)^2} \, d^N y.
\]

Note that we have extracted, in the above bound on \( z(c(a, g^2, 3)) \), the factor \((\sqrt{ag})^N\), which gives a singularity in the free energy in the continuum limit \( a \searrow 0 \). Note also that the function \( I(u) \) is continuous, monotone increasing,
verifies $I(0) = 0$ and is bounded from above by $I(\infty) = N(N)(2\pi)^{N/2} \prod_{j=1,...,N} j! = (2\pi)^{-N/2} \prod_{j=1,...,(N-1)} j!$. Up to a constant factor, the integrand of $I(u)$ is the probability distribution for the eigenvalues of a self-adjoint matrix in the Gaussian Unitary Ensemble (GUE) (see [18, 19]), and arises naturally in the context of our problem. For $d = 2$, the thermodynamic limit of $f_{a,n}^{\Lambda}$ exists and, by dominated convergence, the continuum limit also exists and is

$$f'' = -\ln \sqrt{2} - \frac{1}{2N} \ln(2\pi) + \frac{1}{N^2} \sum_{1 \leq j \leq (N-1)} \ln(j!).$$

III. LOWER STABILITY BOUND

For the lower bound on $Z_{\Lambda,a}$, we fix the enhanced temporal gauge so that we have $\Lambda_r$ retained bond variables. Next, we reduce the integration domain in $Z_{\Lambda,a}$ so that each retained variable is close to the identity. With this reduction, we obtain an upper bound on each plaquette action which is quadratic and local in the gluon fields of each plaquette variable. The lower bound for $Z_{\Lambda,a}$ factorizes over the retained bond variables. Each factor is a single-bond partition function $\tilde{\varpi}$, with a quadratic action and small field restrictions. The single-bond quadratic action is a class function on $G$ and the small field restriction is also. Hence, by the Weyl integration formula, the group integral reduces to an integration over angular eigenvalues. A lower bound for the density $\rho(\lambda)$ is used. Last, changing the variables in the retained variables gives the lower stability bound of Theorem 1. With the enhanced temporal gauge, we remark we have the same exponent $\Lambda_r$ of $z$ that occurs in Eq. (1).

To discuss the restriction on the domain, we write the $N \times N$ matrix $U_b$, associated with the bond $b$, as $U_b = e^{iX_b}$. Imposing the condition $A_b \equiv |U_b - 1|_{H-S}^2 < 1$, we have a well-defined self-adjoint $X_b = -i \ln[1 + (U_b - 1)] = -i \sum_{j \geq 1} (-1)^{j+1} (U_b - 1)^j/j$ where, for $H = U(N)$, $X_b = \sum_{\alpha=1,...,d(N)} x\lambda^b_{\alpha} \theta_{\alpha}$, and $\theta_{\alpha}$ are the corresponding $d(N) = N^2$ Lie algebra generators, verifying $\text{Tr}(\theta_{\alpha} \theta_{\beta}) = \delta_{\alpha\beta}$, and the $x\lambda^b_{\alpha}$ are the gauge or gluon fields. Using the eigenvalues $\lambda_j$ of $X_b$, $U_b$ is unitarily equivalent to $\text{diag}(e^{i\lambda_1}, \ldots, e^{i\lambda_N})$, and $A_b \equiv |1 - U_b|_{H-S}^2 = 2\text{Tr}(1 - \cos X_b)$. Also,

$$|X_b|_{H-S}^2 = \text{Tr}[\ln[1 + (1 - U_b)^{-1}]] = \ln[1 + (1 - U_b)^{-1}].$$

Note that both, $A_b$ and $|X_b|_{H-S}^2$, are class functions. Furthermore, introducing the Euclidean norm $|x^b|$ in $\mathbb{R}^N$ and $|\lambda^b|$ in $\mathbb{R}^N$, we have the important identity

$$|X_b|_{H-S}^2 = \sum_{\alpha=1}^{d(N)} |x\lambda^b_{\alpha}|^2 = |x^b|^2 = \sum_{k=1}^{N} (\lambda_k^b)^2 = |\lambda^b|^2.$$

Continuing, for the abelian case $H = U(1)$, $|x^b| < \pi$, we have that

$$A_p = 2[1 - \cos(x^1 + x^2 - x^3 - x^4)] \leq 4[(x^1)^2 + \ldots + (x^4)^2].$$

This global bound may be not true for a nonabelian $G$, without a restriction on the fields. To obtain a quadratic bound in this case, we take the gluon fields to be small. In the following, we still take $H = U(N)$. For the single plaquette, with subsequent bonds $b_1, b_2, b_3$ and $b_4$, with action $A_p = |U_1U_2U_3U_4 - 1|_{H-S}^2$, and $U_j = \exp(i \sum_{\alpha=1}^{d(N)} x\lambda^b_{\alpha} \theta_{\alpha})$. Below, we prove:

**Lemma 1** Let $U_p = U_1U_2U_3U_4^1$, $U_j = e^{Cj}$, $1 \leq j \leq 4$ and $L_j = i \sum_{\alpha=1,...,d(N)} x\lambda^b_{\alpha} \theta_{\alpha}$. Then, if $|L_j| < N^{-1/2}$, we have that

$$A_p \leq 4 \left(1 + N^2 \sum_{j=1}^{4} |L_j|^2 + N^4 \sum_{j=1}^{4} \sum_{j'=1}^{4} |L_j|^2 \right) \sum_{j=1}^{4} |L_j|^2 \leq C^2 \sum_{k=1}^{4} |x^b|^2,$$

where $C = 2(1 + 2N^{3/2})$ and $|x^b| = |L_j|_{H-S}$.

Now, we obtain a lower bound of $Z_{\Lambda,a}$ in terms of the modified single-bond partition function $\tilde{\varpi}$. By Taylor expanding $U_b(\alpha) \equiv e^\alpha L_\alpha$, $(U_b - 1) = \int_0^1 L_\alpha U_b(\alpha) d\alpha$ and $|U_b - 1|_{H-S} \leq |L_b|_{H-S} |U_b|_{H-S}$. The logarithms in Eq. (6) are defined if, for the retained bonds, we impose $|x^b| < (1/\sqrt{N})$, so that $|U_b - 1|_{H-S} < 1$. Lemma 1 applies and, in $A = \sum_p A_p$, we replace $A_p$ by the quadratic bound. Since the integrand is now a class function of each retained bond variable, we replace the integration variables by the angular eigenvalues with $|\lambda_k^b| < (1/N) = \gamma$. Each bond $b$ appears...
at most in \([2(d-1)]\) terms of \(A\). Hence, paying with a factor \([2(d-1)]\), \(\sum_{\kappa}\) is replaced by the sum over retained bonds. With this, \(Z_{\Lambda,\alpha}\) factorizes over the retained bonds to give (compare with Eq. 43),

\[
Z_{\Lambda,\alpha} \geq \left[ \mathcal{N}(N) \int_{|\lambda_k|<\gamma} e^{-2c(d-1)c^2\sum_{k=1}^{N} \lambda_k^2\rho(\lambda)dN_\lambda} \lambda^\Lambda \right]^{\Lambda},
\]

Recall that, \(c \equiv c(a,g^2,d)\). For the lower bound on \(Z_{\Lambda,\alpha}\), we use a lower bound on the eigenvalue density \(\rho(\lambda)\). Namely, we use \(|e^{i\lambda_j} - e^{i\lambda_k}|^2 \geq 4(\lambda_j - \lambda_k)^2/\pi^2, |\lambda_k| \leq \pi/2\). After the change of variables \(y_k = [4c(d-1)]^{1/2}C\lambda_k\), we get

\[
Z_{\Lambda,\alpha} \geq \left[ \Theta(g^2/\alpha d^{-4})d(N)/\sqrt{I(2(d-1)/2 C\gamma(\alpha d^{-4}/g^2))} \right]^{\Lambda'},
\]

with \(\Theta \equiv \mathcal{N}(N)/(4/\pi^2)^{N(N-1)/2[(d-1)/2 C\gamma]} - d(N)\) which displays the factor \((g^2/\alpha d^{-4})d(N)\lambda^\Lambda/\sqrt{I(\alpha d^{-4}/g^2)}\). We emphasize this is the same factor which occurs above Eq. 43 for the upper bound and, by the monotonicity of \(I(u)\), the integral is bounded below for \(a = 1\) and \(g^2 = g_0^2 < \infty\).

We now write \(|\cdot|\) for \(|\cdot|_{H-S}\) and prove Lemma 1. For \(\delta \in [0,1]\), let \(U_p(\delta) = U_1(\delta)U_2(\delta)U_3(\delta)U_4(\delta)\), where \(U_j(\delta) = e^{i\delta c_j}\). Clearly, \(U_j = U_j(\delta = 1)\). With \(L = \sum_{j=1,...,4} L_j\) and applying a Taylor expansion, we have

\[
U_p = 1 + L + \int_0^1 \left[ dU_p(\delta_1)/d\delta_1^2 \right] d\delta_1 d\delta_2 = 1 + L + R,
\]

Since \((dU_j/d\delta) = U_j L_j = L_j U_j\), suppressing the \(\delta\)’s, we have \([d^2U_p(\delta)/d\delta^2] = L_1 L_2 U_p + L_1 U_1 L_2 U_4 U_3 + \ldots + U_p L_4 L_4\).

Thus, since \(|U_p| = N^{1/2}\) and \(2ab \leq a^2 + b^2, a, b \in \mathbb{R}\), we obtain \(|R| \leq (N/2) \sum_{j,k} |L_j||L_k| \leq 2N^2 \sum_j |L_j|^2 \equiv 2N^2 Q\).

With the small field condition,

\[
A_p = |U_p - 1|^2 = |L + R|^2 \leq (|L| + |R|)^2 = |L|^2 + 2|L||R| + |R|^2.
\]

But \(|L|^2 \leq (\sum_j |L_j|^2)^2 \leq 4 \sum_j |L_j|^2\), by the Cauchy-Schwarz inequality. So,

\[
A_p \leq 4 \left( 1 + N^2 \sum_j |L_j| + N^4 Q \right) \leq C^2 \sum_{k=1}^{d} |x^k|^2.
\]

IV. EXTENSION OF RESULTS TO SU(N)

Our analysis and results extend from the gauge group \(G = U(N)\) to \(G = SU(N)\). To do this, besides noticing there are now \(d(N) = (N^2 - 1)\) self-adjoint and traceless Lie algebra generators, we make the following changes:

a) Using a Dirac delta, in the Weyl integration formula, insert \([2\pi \delta(\xi)]\) in the integrand, with \(\xi \equiv \lambda_1 + \ldots + \lambda_N\);

b) For the lower bound of \(\sum_{k=1,...,N}[2(1 - \cos \lambda_k)], \xi = 0\), replace \(N\) by \((N - 1)\) and use \(\sum_{k=1,...,N-1}[2(1 - \cos \lambda_k)] \geq \sum_{k=1,...,N-1}[2\lambda_k^2/\pi^2]; |\lambda_k| < \pi\). For the upper bound of \(\sum_{k=1,...,N-1}[2(1 - \cos \lambda_k)] + 2[1 - \cos(\xi - \lambda_N)], \) use \([N(\lambda_1^2 + \ldots + \lambda_{N-1}^2)];\)

c) For the upper bound on the density, we have \(\rho(\lambda) \leq \tilde{\rho}(\lambda), \) where, for \(j, k = 1, \ldots, (N - 1), \tilde{\rho}(\lambda) \equiv \prod_{j<k}(\lambda_j - \lambda_k)^2 \prod_j(\lambda_j + \xi - \lambda_N^2).\) For the lower bound, restrict the set \(\{\lambda_k\}\) so that we can use \((1 - \cos u) \geq 2u^2/\pi, |u| < \pi\). Taking \(|\lambda_k| \leq \pi/N\), we have \(\rho(\lambda) \geq (2/\pi^2)^{N(N-1)/2}\tilde{\rho}(\lambda)\).

V. CONCLUDING REMARKS

We consider pure Yang-Mills model in dimension \(d = 2, 3, 4\), defined with the Wilson plaquette action and with a compact Lie group \(G\), on a hypercubic finite lattice \(\Lambda \subset a\mathbb{Z}^d, a \in (0,1]\), with \(L\) sites on a side. We take \(G = U(N), SU(N)\) and \(d(N)\) is the dimension of the corresponding Lie algebras. The lattice provides an ultraviolet, short-distance regularization.

In a series of papers (see Refs. 7 and Refs. therein), ultraviolet stable stability bounds were proved for the \(d = 3, 4\) pure-gauge cases, using intricate rigorous renormalization group methods and field decompositions.
In our treatment, the bond variables are parametrized using \( d(N) \) gluon fields. By local gauge invariance, a gauge fixing procedure allows us to gauge away some bond variables by setting them equal to the identity group element in the Wilson plaquette action. The corresponding gauge group integral gives one. This procedure can be carried out without changing the value of the finite lattice model partition function \( Z_{\Lambda,a} \). We use an enhanced temporal (axial) gauge where the temporal bond variables are set to one; some additional bond variables on the lattice boundary are also set to one. The set of gauged away bonds (variables) are associated with a maximal tree with bonds of \( \Lambda \). A closed loop is formed if any other bond is added and gives a different partition function. The number of retained bond variables is denoted by \( \Lambda_r \), and is of order \( (d-1)L^d \), for large \( L \).

By extracting a factor \( (a^{d-4}/g^2)^{-d(N)/\Lambda_r} \) from \( Z_{\Lambda,a} \), so that \( Z_{\Lambda,a} = (a^{d-4}/g^2)^{-d(N)/\Lambda_r} Z_{\Lambda,a}^\mu \), we show that the normalized partition function \( Z_{\Lambda,a}^\mu \) obeys the ultraviolet stable, stability bounds

\[
\exp \left[ c_{\ell}(N)\Lambda_r \right] \leq Z_{\Lambda,a}^\mu \leq \exp \left[ c_u d(N)\Lambda_r \right],
\]

with \( c_{\ell}, c_u \in \mathbb{R} \) independent of \( L, a \) and the gauge coupling \( g^2 \in (0, g_0^2), 0 < g_0 < \infty \). In other words, we extracted the exact singular behavior of the finite lattice free-energy \( f_{\Lambda,a} = \ln Z_{\Lambda,a}^\mu / (d(N)\Lambda_r) \) and the normalized free-energy \( f_{\Lambda,a}^\mu = \ln Z_{\Lambda,a}^\mu / [d(N)\Lambda_r] \) has both a thermodynamic limit \( (\Lambda \rightarrow \infty) \), at least in the subsequential sense, and subsequently a continuum limit \((a \downarrow 0)\).

The bounds on \( Z_{\Lambda,a}^\mu \) reduce to the \( d(N)\Lambda_r \) power of a single plaquette, single bond partition function \( z \) (for the lower bound, with a small gluon field restriction). The integrand of \( z \) is a class function on \( G \), and we may apply the Weyl integration formula. The bounds reduce to bounds on the angular eigenvalue distribution of unitary matrices and arise from quadratic bounds on the action, with a restriction on the size of the fields for the lower bound. This appears to indicate that the apparently high nonlinearities of gauge models are actually not that bad. Our bounds saturate for \( d = 2 \).

Extending the stability bounds to partition functions with a uniform source coupled with a sequence of well-known techniques (e.g., multiple reflection \([3]\)), may lead to the thermodynamic and continuum limits of correlations. More analysis is needed for correlation decays.

Our treatment extends to other compact groups \( G \), and when matter fields are present. Indeed, neglecting the pure-gauge action and using a priori locally scaled matter fields (not canonical scaling!) the coupling of matter and gauge fields was treated in \([20, 22]\). Stability bounds were proven for a Bose-gauge model; only upper bounds in the Fermi case. The bounds do not depend on \( a \) and a normalized free-energy exists in the thermodynamic and continuum limits. We expect to combine our pure-gauge and matter-gauge results to show the existence of QCD.

Stability bounds give the existence of the model but do not give information on the energy-momentum spectrum, local clustering properties and particles \([3]\). For lattice QCD, with fixed \( a \) and in the strong coupling regime (with \( g^{-2} \gg 0 \) much smaller than a small hopping parameter), we have results validating the Gell’Mann-Ne’eman eightfold way, the exponential decay of the Yukawa interaction and the existence of some two-hadron bound states (see e.g. \([23]\) and references therein). It would be nice to rigorously derive general properties of nuclear physics from first principles, i.e. from fundamental quarks and gluons, and QCD dynamics.

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