Wigner phase of photonic helicity states in the spacetime of the Earth

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We study relativistic effects on polarised photons that travel in a curved spacetime. As a concrete application, we specialise to photons propagating in the gravitational field of the Earth between a laboratory and satellites. We show that wave packets of light initially in quantum superpositions of momentum-helicity states acquire Wigner phases as a consequence of the rotation of polarisation due to the curved background spacetime of the Earth. Contrary to existing results, we find that this effect occurs for closed paths already in static spacetimes, such as Schwarzschild spacetime. We consider propagation of light pulses in a closed loop between different links at different heights, after which the received photons can be measured and compared to the emitted ones. Surprisingly, the Wigner phase that ensues is two orders of magnitude larger than any similar Wigner phase known. Our work applies also to closed paths of light in arbitrary curved spacetimes. The predictions of this work can potentially be measured with current technologies.

I. INTRODUCTION

The study of the effects of gravity and relativity on quantum features of physical systems has both foundational and practical significance. On the theoretical side, it can aid our search for a fundamental theory of Nature. On the applied side, it can promote the proposal of new experiments that can push the limits of known science. In particular, quantum systems for advanced space science are promising a revolution in our ability to communicate securely 1–4, to distribute quantum computing tasks 5, 6 and to test cutting-edge physics 7–8. Polarised photons are at the core of a large number of these recent quantum experiments and technologies involving satellites 3–6. However, the theoretical treatment of quantum polari- sation, namely helicity, remains unclear in curved spacetime. The inexorable development of increasingly sophisticated quantum technologies in space calls for a proper theoretical foundation of photonic helicity within the framework of general relativity. Conversely, space experiments using satellites can help confirm or invalidate the theory by testing its predictions. In the present article, we propose such an experiment.

A thorough treatment of light’s polarisation rotation in flat and curved spacetime exists. In the case of flat spacetime, we find studies of helicity states of photons and the structure of Wigner’s little group 9–11, the influence of the detector’s motion on entangled helicity states of light 12, 13 and a QFT formalism for the Einstein-Podolsky-Rosen (EPR) correlations of polarised photons 14. In curved spacetime 15–17, since the first inquiry on photon polarisation 15, many physical cases have been considered among which we find: rotation of polarisation induced by astrophysical 19 and cosmological gravitational waves 20, photons experiencing vector perturbations in the metric due to the Cosmic Microwave Background (CMB) 21 and quasars 22, gravitational lensing by massive objects passing through the light’s trajectory 23–24 with relativistic velocities 25–26, or with rotation 27–30. Furthermore, there have been studies of optical interferometry experiments in a near-Earth environment 31, namely with polarisation vectors of two light rays that are compared when they recombine after having travelled through two paths at different altitudes, and therefore under different gravitational potentials. In the end, regardless of all of the existing efforts, the proposed effects are small and leave little hope for direct measurement and corroboration.

In this work we look at the effect encoded in the rotation of the polarisation of a pulse of light after it has propagated through a curved background. We employ a general formalism developed in 32–35 and study the evolution of the helicity quantum states of photons sent from Earth to different satellites and finally reflected back to Earth. The helicity states are affected by propagation in a curved spacetime. Surprisingly, we find that already for a closed path in Schwarzschild spacetime the states acquire a non-trivial Wigner phase, contrary to previous studies 13–15. Even more interestingly, we are able to devise schemes where the total Wigner phase accrued by the state of the photon after its propagation around a closed loop is two orders of magnitude larger than the largest estimates available 25–28. Therefore, we believe that the predictions of this work provide new understanding of the effects of gravity on genuine quantum properties of physical systems, and suggest that detection in experiments is feasible with current technologies.

The work is organized as follows: in Section II we provide all necessary mathematical tools. In Section III we compute explicitly the Wigner rotation for the propagating light pulses. In Section IV, we present two specific operational schemes and we find the Wigner phase picked up by the photons’ helicity states along the paths. In Section V we estimate the amplitude of the acquired Wigner phases with realistic parameters.
II. MATHEMATICAL FORMALISM

In this section we introduce the mathematical tools necessary to our work. This includes background spacetime surrounding the Earth, the 4-velocities of static observers and their reference frames, the propagation of light rays and the polarisation 4-vectors, and finally the quantum states of the polarised photons.

Throughout this paper we use natural units $c = G = 1$. Vectors are defined in Schwarzschild coordinates in the usual notation from differential geometry, and the Einstein summation convention is implied for repeated indices: $A = A^\mu \partial_\mu = A^t \partial_t + A^r \partial_r + A^\theta \partial_\theta + A^\phi \partial_\phi$.

A. Schwarzschild spacetime and static observers

1. The Schwarzschild spacetime

The spacetime considered in this work is Schwarzschild spacetime, which is the spacetime background for the vacuum surrounding a static spherical mass distribution. This is a good approximation for the Earth where, for the purposes of this work, we can safely ignore its angular momentum and asphericity. More refined metrics, such as the Kerr [34] or the Hartle-Thorne [35] metric, can be employed to take into account these features. We leave more detailed and realistic calculations for further work.

The metric is a symmetric, bilinear form with matrix representation in the usual Schwarzschild coordinates $(t, r, \theta, \phi)$ that reads

$$ g = \text{diag} \left( -f(r), \frac{1}{f(r)}, r^2, r^2 \sin^2 \theta \right), \quad (1) $$

where $f(r) := 1 - \frac{2M}{r}$.

The surface of the Earth is placed at $r = R_E$. The extremal values of the polar angle, $\theta = 0$ and $\theta = \pi$, indicate the North and South poles respectively, while the equatorial plane is given by $\theta = \pi/2$. Notice that the metric is independent on the time and longitude coordinates $t$ and $\phi$. This feature provides the spacetime with two Killing vector fields, $\partial_t$ and $\partial_\phi$, respectively, and therefore two conserved quantities for the kinematics of test particles and light rays. In addition to these two, there are another two spacelike Killing vector fields that, together with $\partial_\phi$, generate the spacetime’s $SO(3)$ rotational symmetry. This aspect gives another constant of motion, which is related to the square of the total angular momentum.

The metric (1) is fully parametrised by the Earth’s mass $M$. With units restored, we see that the quotient $M/r$ reads $GM/rc^2 \ll 1$ for any radius $r \geq R_E$, since the Schwarzschild radius $r_S = 2M$ has the value $r_S \sim 9mm$ while $R_E \sim 6341km$. In this work we restrict to realistic scenarios where $r \geq R_E$ and can safely ignore the metric and equations of state for the Earth when $r < R_E$.

2. Static observers in Schwarzschild spacetime

The observer that will prepare and measure the states in our proposed scheme is a static observer located on the surface of the (non-rotating) Earth. In Schwarzschild spacetime, there are static observers at any spacetime point with $r > 2M$, and they are defined by their 4-velocities

$$ v_E = \frac{1}{\sqrt{1 - \frac{2M}{r}}} \partial_t. \quad (2) $$

Our real observer (i.e., the experimental laboratory) will be located at a fixed radius $r = R_E$, and we will need to introduce a family of fiducial static observers all along the light’s trajectory in order to track the evolution of the initial observer’s reference frame (i.e., to parallel transport it). This is because the frame in which the polarisation is initially defined by the observer at $r = R_E$ changes as it is carried by the light propagating in the curved spacetime around the Earth. When the polarisation is measured at the end of the light’s propagation, the reference frame has changed in general, even if the light is measured at the same location and with the same direction as it was initially emitted.

B. Reference frames and tetrads

An observer defines its reference frame mathematically by a tetrad $e$, with tetrad elements $e_\mu^a$ [34]. The tetrad relates physical quantities from their expression in the global coordinates of the spacetime $(t, r, \theta, \phi)$ to the expression of these quantities in the local Cartesian frame of the observer. In other words, in its local frame the observer witnesses a flat, Minkowski spacetime with metric $\eta = \text{diag}(-1, 1, 1, 1)$. The domain of validity of this frame is therefore the spacetime in the vicinity of the observer’s world-line, where the effects of curvature can be neglected. In this local frame, the polarisation 4-vector $\psi$ of a light ray passing by will be given components

$$ \psi^a = \psi^\mu e_\mu^a = \psi^\mu e_\mu^r \eta^{ab} g_{\mu \nu}^{-1} \quad (a = 0, 1, 2, 3). $$

As discussed above, we need to keep track of the evolution of the static observer’s frame along the path followed by the light rays, in order to be able to properly compare the change in the polarisation of the field. Therefore, we will define a tetrad field $e(\lambda)$ along the light’s null geodesic parametrised by the affine parameter $\lambda$. The tetrad components are obtained explicitly as follows:

i) The timelike component $e_0^a$ is chosen to be the velocity of the observer, in our case $e_0^a := v_E$ defined in (2);
ii) The triad of spacelike components $e_1, e_2, e_3$ of the tetrad is obtained through the orthonormalisation relations $e^a_\nu e^\nu_b g_{\mu\nu} = \eta_{ab}$.

iii) Each of the three triad components are vectors that have four elements themselves, and these twelve elements are constrained by only nine orthonormalisation relations. Therefore, there is some gauge freedom in the choice of the triad, and thus of the tetrad. This mathematical gauge freedom represents the observer’s freedom of choice of the reference frame. In what follows, we will adapt our observer’s tetrad to the light’s null vector, which will simplify the polarisation rotation computations. To do so we will choose to set, say, the third component of the triad $e_3$ in a specific way that depends on the trajectory of the light ray. This procedure will be explicitly described in the next subsection. After having adapted the tetrad to the null vector of the emitted light ray, there are then only eight tetrad elements $e^a_\mu$ left to be determined and seven orthonormalisation equations. This implies that all of the remaining gauge freedom is encoded in one single tetrad element, and a choice of two signs due to the second order nature of the two remaining normalisation relations.

By extending this procedure to the whole null geodesic followed by the light ray, we obtain the tetrad field for our static observers.

C. The light’s null vector

In this work we neglect the deviation from the null geodesic due to the photon’s helicities, since such corrections are of order $\lambda/b \ll 1$, see [57], where $\lambda$ is the photon’s wavelength and $b$ is the shortest distance between the photon and the center of the Earth, i.e. $b = R_E$.

1. General null vector

A light ray defines the path followed by pulses of light and is defined formally by the null vector $k$ tangent to its null geodesic [38]. In the spacetime considered here, such a null vector has the general expression

$$k = E_p \left( \frac{1}{f(r)} \partial_t + \epsilon_t \sqrt{1 - f(r) \frac{\kappa^2}{r^2} - \frac{l_\phi^2 + \kappa}{r^2}} \partial_r + \epsilon_\phi \sqrt{\kappa - \frac{l_\phi^2 \cot^2 \theta}{r^2}} \partial_\theta + \frac{l_\phi}{r^2 \sin^2 \theta} \partial_\phi \right),$$

(3)

where $\epsilon_t = \pm 1$ depending on whether the light ray ascends or descends the gravitational field of the Earth, and $\epsilon_\phi = \pm 1$ when the photon’s polar angle $\theta$ increases or decreases along its path, respectively.

There are three constants of motion in [3]. One comes from the stationarity of the metric [1], while the other two are due to the spherical symmetry of the spacetime. The first constant is $E_p$, which gives the value of the energy of the photon as it would be measured by an inertial observer at spacelike infinity. We then have the rescaled azimuthal angular momentum and Carter constants $l_\phi = L_\phi/E_p$ and $\kappa = K/E_p^2$ that are independent on the energy of the photon encoded in $E_p$. The constants $L_\phi$ and $K \geq 0$ are the azimuthal angular momentum and a quantity related to the square of the total angular momentum [39] of the photon as seen by an inertial observer at spatial infinity, respectively. The constant of motion $E_p > 0$ can easily be related to the photon’s frequency $\Omega$ measured by an observer with velocity $v$ at any spacetime point through the relation $\hbar \Omega = -k \cdot v$. For example, our static observer (2) on Earth would measure

$$E_p = \hbar \Omega_E \sqrt{1 - \frac{2M}{R_E}},$$

(4)

where $\Omega_E$ is the photon’s frequency measured by the static observer on the surface of the Earth.

Unfortunately, it is more complicated to obtain such a simple analytic formula for the two other constants of motion $l_\phi$ and $\kappa$. We note that in the special case of radially propagating light rays, i.e. for constant polar angle $\theta$ and longitude angle $\phi$ along the null geodesic, or for an equatorial $\theta = \pi/2$ trajectory, we simply have $\kappa = 0$. We can anticipate that in the schemes that we will consider in Section [14] we will focus instead on light rays with constant longitude $\phi$, namely with $l_\phi = 0$. In the next paragraph, we briefly describe how this choice simplifies the quantities defined above.

2. Null vector with constant longitude $\phi$

In order to simplify the computations involved and in particular to provide simple analytical formulas for the Wigner phases without loss of generality, we will assume that the light rays can be sent with a constant azimuthal angular momentum $[39]$ of the photon as seen by an inertial observer at spatial infinity. We then have the energy of the photon as it would be measured by an inertial observer at spacelike infinity. We then have the constant of motion $E_p > 0$ can easily be related to the photon’s frequency $\Omega$ measured by an observer with velocity $v$ at any spacetime point through the relation $\hbar \Omega = -k \cdot v$. For $l_\phi = 0$, the null vector $k$ takes the simplified expression

$$k = E_p \left( \frac{1}{f(r)} \partial_t + \epsilon_t \sqrt{1 - f(r) \frac{\kappa}{r^2}} \partial_r + \frac{\epsilon_\theta}{r} \sqrt{\frac{\kappa}{r^2 \sin^2 \theta}} \partial_\theta \right).$$

(5)

Since the light rays are confined to $\phi = \text{const}$. planes, the quantity $\kappa$ now plays the role of the square of the polar angular momentum (i.e., it is related to the $\theta$ polar coordinate). Formally, the constant $\epsilon_\theta \sqrt{\kappa}$ is here equivalent to the $l_\phi$ constant of motion of a null vector [3] confined to the equatorial plane $\theta = \pi/2$. To obtain $\kappa$ in this case amounts to finding the single angular-momentum-like constant on a geodesic whose plane is already known. Note that due to the spacetime’s spherical symmetry, we could have worked equivalently in any other plane containing the center of the Earth. We chose to work in a
we need to make the simple choice of our observer’s tetrad with the light’s propagation direction. In mathematical terms, that means that we have $k^1 = k^2 = 0$ and $k^3 = -\kappa \hat{0}$ in the adapted frame. With this choice for the third component of the triad, the polarisation components in the observer’s adapted frame $\psi^1$ and $\psi^2$ are gauge invariant under transformations (7).

2. Polarisation rotation in curved spacetime

In the adapted frames that we have just defined, the polarisation rotation equations take a simple form [32]:

$$\frac{d\psi^A}{d\lambda} + u^\mu w^B \Gamma^A_{\mu B} \psi^B = 0.$$  

(9)

Capital letters $A, B \in \{1, 2\}$ denote the indices of the polarisation’s gauge free components. We have defined the rescaled null vector $u = k/k^0$ and we recall that $\lambda$ is the affine parameter along the light’s null geodesic. In these two coupled first order differential equations, we need to compute the Wigner rotation $w^\mu w^\nu A_{\mu \nu}$ in order to obtain the rotation of the polarisation. This term is expressed through to the spin-1 connection as

$$w^\mu A_{\mu \nu} = e^A_{\nu} \partial_\nu e^A_{\mu} + \Gamma^\sigma_{\mu \rho} e^A_{\sigma} e^\rho_{\mu}.$$  

(10)

In order to compute the Wigner rotation we will thus need to use the expression of the light’s null vector [1], the adapted tetrad field $e(\lambda)$ of static observers along the light’s null geodesic, and the Christoffel symbols defined by the metric components and their derivatives as $\Gamma^\sigma_{\mu \rho} = g^{\sigma \nu}(\partial_\mu g_{\nu \rho} + \partial_\rho g_{\nu \mu} - \partial_\nu g_{\rho \mu})$. The explicit expressions for the Christoffel symbols in Schwarzschild spacetime are known [10]. We can simplify (9) by noticing that $w_{AB}$ is an antisymmetric tensor, and we obtain

$$\frac{d\psi^1}{d\lambda} + \tilde{\Psi} \psi^2 = 0,$$  

(11)

$$\frac{d\psi^2}{d\lambda} - \tilde{\Psi} \psi^1 = 0,$$  

(12)

with the Wigner rotation $\tilde{\Psi} = u^\mu w^1_{\mu 2} = -u^\mu w^2_{\mu 1}$. Notice that the system of coupled equations above can be decoupled by taking derivatives with respect to the affine parameter $\lambda$ and then feeding back the first order equations in to the second order ones. We do not follow this path, but it can be of interest perhaps when employing numerical simulations.

In the next subsection, we will describe how the Wigner rotation of the classical polarisation vector described by the coupled equations (11) and (12) yields a phase change in the quantum state of the photons that propagate through curved spacetime.

D. Adapted frames and polarisation rotation in curved spacetime

So far, we have defined mathematically the observers’ frames and the null vector of light. We now proceed to implement the formalism required to describe the light’s polarisation and its rotation in a curved spacetime background.

1. Polarisation vectors and adapted frames

We have defined the light ray through the null vector [3], and we will now define the light’s polarisation vector in the standard way [32]. The polarisation vector $\psi$ is a spacelike vector orthogonal to the light’s null vector $k$, namely $\psi^\mu k_\mu = \psi^a k_a = 0$. Because $k$ is null, i.e., $k^a k_a = 0$, there exist gauge transformations of the polarisation vector that leave the orthogonality relation invariant. These read

$$\psi \rightarrow \psi + C k,$$  

(7)

where $C$ is an arbitrary real constant. The light’s polarisation vector is therefore not uniquely defined by its orthogonality with the light’s null vector. This particular gauge freedom is unwelcome as it prevents a unique definition of the polarisation vector that is going to be initial prepared and later on measured in the laboratory. Here we discuss how to eliminate this freedom. We start by noting that we can construct the observer’s tetrad in such a way that two of the the polarisation components measured in that particular frame will be invariant under gauge transformations like (7). This construction amounts to adapting the frame to the null vector of the light ray considered. To adapt our tetrad to the light ray, we need to make the simple choice [32]

$$e^3 = \frac{k}{k^0} - e^0.$$  

(8)

Physically, this choice amounts to aligning the third component $e^3$ of our observer’s tetrad with the light’s propagation direction.
E. Photonic wave packets and Wigner phases

A pulse of light, whether composed by a single photon or many, can be modelled by a wave packet, i.e., a continuous superposition of momentum states weighted by an appropriate “shape” function. For each of these momentum states, the corresponding helicity state is a superposition of positive and negative helicity eigenstates giving the momentum-helicity eigenstate \( |p, s\rangle \). If an observer prepares a photon with an equal proportion of positive and negative helicity eigenstates for each momentum state, the initial pure quantum state \( |\gamma\rangle \) can be written as

\[
|\gamma\rangle = \frac{1}{\sqrt{2}} \sum_{s=\pm 1} \int dp \, F(p) \, |p, s\rangle ,
\]

where \( p = (k^0, k^1, k^2, k^3) \) is the photon’s null vector as seen by the observer in its chosen reference frame, \( s \) is the helicity which can only take values \( \pm 1 \) for photons and the function \( F(p) = F(\vec{p}) = \delta(\vec{p})^2 - (k^0)^2 \delta(\vec{k}) \) is the distribution of whole 4-momenta, where \( \delta \) the step function and \( F \) is the distribution function of the 3-momenta. The distribution \( F(\vec{p}) \) is normalised through \( \int d\vec{p} \, |F(\vec{p})|^2 = 1 \). This implies that the corresponding annihilation operators \( a_{\vec{p}, s} \) of the photons have the usual commutation relations \( [a_{\vec{p}_{1}, s_1}, a_{\vec{p}_{2}, s_2}^\dagger] = \delta(\vec{p}_{1} - \vec{p}_{2}) \delta_{s_1 s_2} \).

Now, let’s assume that the photon with state \( |13\rangle \) has been prepared by the source at \( R_e \) and \( p \) thus represents the photon’s momentum as seen by the observer located at the emission event, parametrised by \( \gamma_e \) on the null geodesic. We are interested in knowing the expression for the quantum state of the photon that will be observed by the receiver at affine parameter \( \gamma_r \). We will give here a simple procedure to obtain this final state and all details can be found in [33].

Let the photons take a certain path in spacetime, that is the union of propagation along geodesics connected by, for example, mirror reflections on satellites. Let \( U \) be the operator that encodes the propagation of the photons from the beginning to the end of a specific segment of the path, denoted by affine parameters \( \lambda_1 \) and \( \lambda_2 \) respectively, with here \( \lambda_1 = \gamma_e \) and in general \( \lambda_e \leq \lambda_1 < \lambda_2 \leq \gamma_e \). Each momentum-helicity eigenstate \( |p, s\rangle \) that constitutes the photon acquires a different Wigner phase factor because of its dependence on both the photon’s propagation direction \( \vec{n} = \vec{p}^\prime/k^0 \) and helicity \( s \), see [32]. Therefore we have

\[
\hat{U} \, |p, s\rangle = e^{i\gamma(p)} \, |p^\prime, s\rangle ,
\]

where the prime denotes evaluation at the endpoint of the null geodesic considered. Since every momentum-helicity eigenstate picks up a different phase factor, the quantum state acquires a measurable, not-global phase. In the evolution equation (14) we find the Wigner phase \( \gamma(p) \) that is obtained by integration of the Wigner rotation along the photon’s null geodesic, namely

\[
\Psi = \int_{\lambda_1}^{\lambda_2} \hat{\Psi} \, d\lambda .
\]  

We apply the propagator \( \hat{U} \) to the initial state \( |13\rangle \) and we use equation (15) to obtain the transformed state \( |\gamma\rangle \) with expression

\[
|\gamma\rangle = \frac{1}{\sqrt{2}} \sum_{s=\pm 1} \int dp \, e^{i\gamma(p)} F(p) \, |p^\prime(p), s\rangle .
\]  

We note that the photons have followed a single null geodesic between the points labelled by the affine parameters \( \lambda_1 \) and \( \lambda_2 \). This means that the null vectors \( p^\prime \) and \( p \) are constructed through the same constants of motion, although they are obtained by evaluation at different spacetime coordinates. This implies that \( p^\prime \) is a function of the initial momentum \( p \).

We also note that our initial state \( |13\rangle \) prepared at \( \lambda_1 \) can be written as \( |\gamma_p\rangle \otimes |\gamma_s\rangle \), where \( |\gamma_p\rangle = \int dp \, F(p) \, |p\rangle \) and \( |\gamma_s\rangle = (|+1\rangle + |−1\rangle)/\sqrt{2} \). Therefore, it is a separable state. This is not true anymore for the received state \( |\gamma\rangle \) observed at \( \lambda_2 \), since momentum and helicity states now appear entangled.

We are able to provide a reference state where no Wigner phase is acquired. To do this we use wavepackets with of the Bell-type \( (+p−p) \pm |−p+p\rangle)/\sqrt{2} \) instead of the simple superposition of \( |\gamma_s\rangle \) states. Indeed, in such a Bell-type state each helicity eigenstate in the tensor products gives opposite phase contributions. Therefore, it is easy to check that the total state remains invariant under the rotation of the polarisation [11, 41].

On the contrary, one can increase the effect by using GHZ-like states of helicity \( (|+p\rangle \otimes m |−p\rangle \pm |−p\rangle \otimes m)/\sqrt{2} \). In this case, the \( |+p\rangle \otimes m \) component acquires \( m \) times the individual Wigner phase \( \Psi \), while the \( |−p\rangle \otimes m \) component picks up a total phase of \( −m\Psi \). The total relative phase acquired by these states is therefore \( 2m\Psi \). In other words, there is an increase of the effect by a factor \( 2m \). For the same purpose, one can also use a product of \( m \) qubit states, namely \( (|+\rangle + |−\rangle) \otimes m \), which is easier to prepare in the lab.

We have defined all the mathematical formalisms required in the rest of the work. We proceed to the next section where we compute explicitly the Wigner phase [15] between two static observers in the background spacetime of the Earth.

III. POLARISATION ROTATION AND WIGNER PHASE IN SCHWARZSCHILD SPACETIME

In this section, we proceed and compute the Wigner rotation of the polarisation vector of a light ray sent and received by two static observers at different heights in a gravitational field. We then compute the associated
Wigner phase acquired by the quantum state of the propagating photons.

A. Polarisation rotation for static observers

Our goal requires us to construct the tetrad field for the chosen observers. The tetrad field associated to the family of static observers along the light ray’s null geodesic has \( e_0(\lambda) = v_E(\rho(\lambda)) \) as its zeroth component, where \( \rho(\lambda) \) stands for the radial coordinate of the photon at affine parameter \( \lambda \). We then follow the adaptation procedure introduced above [5], and we use the expression [3] for a general null vector, with \( k^0 = k \cdot e_0 \eta^{00} = -k \cdot v_E = E_p/\sqrt{f(r)} \). We obtain the expression of the third triad component \( e_3 \) which reads

\[
e_3 = \sqrt{f(r)} \left( \epsilon_r \sqrt{1 - f(r)} \frac{\epsilon_\theta}{r^2} \frac{l_\phi + \kappa}{E(r)} + \frac{\epsilon_\theta}{r} \sqrt{\frac{\kappa - l_\phi^2 \cot^2 \theta}{r^2}} \frac{\partial \phi}{\partial \theta} + \frac{l_\phi}{r^2 \sin^2 \theta} \partial \phi \right). \tag{17}
\]

The remaining first and second triad components \( e_1 \) and \( e_2 \) are then obtained using the orthonormalisation relations of the tetrad, as explained in Section III B. We give their cumbersome expressions in appendix [11, A14]. It should be clear from the form of these expressions that the computation of the Wigner rotation \( \Psi \) for the general null vector [3] and for the general adapted frame would be very involved.

In order to simplify the computations, let us now consider light rays constrained to a plane with constant longitude \( \phi \), with null vector [5]. This is a reasonable assumption in a spherically symmetric spacetime where the light’s trajectories are confined to propagate on a plane. We also make a simplifying choice for the reference frame: let us give the value \( B = \frac{1}{r} \sqrt{1 - \frac{2\kappa}{E}} \) to the free component of the tetrad in (A1). This choice implies that the expression of the Wigner rotation vanishes in the flat spacetime limit \( M \to 0 \). Finally, to uniquely fix the reference frame, we set \( \eta_1 = \eta_2 = +1 \). The triad associated to this choice of reference frame is displayed in appendix [A2]. We now proceed to calculate the Wigner rotation using (10), (A6), (A7) and (5). Expanding up to lowest order in the dimensionless perturbative parameter \( \epsilon = \sqrt{M/R_E} \ll 1 \), we find

\[
\Psi = -\frac{\epsilon_r 3 r^2 - \kappa}{r^2 - \kappa} \sqrt{\frac{\kappa}{2r^2} \frac{R_E}{r}} + O(\epsilon^3). \tag{18}
\]

For a light ray trajectory with a vanishing Carter constant \( \kappa \), there is no Wigner rotation as measured in the chosen frame. Such trajectories correspond to radial light rays, where both the polar angle \( \theta \) and the longitude angle \( \phi \) remain constant along the null geodesic. Notice that the Wigner rotation [13] is independent on the energy of the photons which is encoded in \( E_p \). This feature is already known in the literature [9] and it remains true for the Wigner phase that will be computed in the next subsection. Note that one can obtain a reference channel where no Wigner rotation occurs by making the choice \( B = 0 \) or \( B = \pm \frac{1}{r} \) instead of the choice we made above. This could be interesting for experimental purposes: comparing a channel with the Wigner rotation [13] and a channel without any Wigner rotation simply by changing the reference frame enables to get rid of all unwanted noise in the signal that we want to measure.

B. Wigner phase between static observers

We can now proceed and compute the Wigner phase \( \Psi \) acquired by the momentum-helicity quantum states of photons travelling through Schwarzschild spacetime, from \( r(\lambda_1) = R_1 \) to \( r(\lambda_2) = R_2 \). We employ (15) and (18), and obtain to lowest order

\[
\Psi = \int_{R_1}^{R_2} \sqrt{\frac{E}{E'}} \frac{\tilde{\Psi}}{u'} dr = \epsilon \sqrt{\frac{R_E \kappa}{2}} \int_{R_1}^{R_2} - \frac{3 r^2 - \kappa}{(r^3 - \kappa r)^2} dr + O(\epsilon^3)
\]

\[
= \sqrt{2} \left( \sqrt{\frac{R_E}{R_2}} \sqrt{\frac{\kappa}{R_2^2 - \kappa}} - \sqrt{\frac{R_E}{R_1}} \sqrt{\frac{\kappa}{R_1^2 - \kappa}} \right) \epsilon + O(\epsilon^3). \tag{19}
\]

We emphasize that, for the static observers and for the frame choice we have made, there would be no Wigner phase for a light ray with a purely radial trajectory, i.e., \( \Psi = 0 \) for \( l_\phi = \kappa = 0 \). Note that the amplitude of the Wigner phase can become arbitrarily large if \( \kappa \) approaches its limit value \( R^2_{\min} \), where \( R_{\min} = \min\{R_1, R_2\} \). This limit value is however never reached as it corresponds to a null vector with all the energy of the photon in its angular momentum, with thus no radial momentum, which is impossible in the schemes considered here.

In the next section we apply our result (19) to schemes where quantum optical signals are exchanged between laboratories on Earth using the aid of reflecting mirrors placed on satellites.

IV. POLARISED PHOTONIC SIGNALS IN EARTH-SATELLITE SCHEMES

The schemes described in this section consist of a station placed on Earth that emits polarised photonic pulses of light to a satellite, which in turn reflects these signals to another satellite or to another laboratory on Earth. The signal is then communicated back to the emitter’s laboratory in order for the trajectory of the pulse to form a closed path. We are interested in those schemes where light travels closed paths since it is possible to compare polarisation states in a “natural” way only when the associated light rays have parallel null vectors [32]. This
is a consequence of the fact that the polarisation vector is always defined by its orthogonality to the light’s null vector, i.e., \( k \cdot \psi = 0 \). Therefore, to talk about polarisation without specifying the corresponding null vector (or momentum) is meaningless. One can thus only compare polarisations associated to parallel null vectors.

In curved spacetime this comparison is not easy to achieve in general, unless in stationary spacetime backgrounds (like in the present study), where the light ray is bound to back to the same space location it was emitted from, and reflected to acquire the same initial propagation direction. To align the received light ray with the emitted one, it suffices to have a mirror at the reception location, oriented in such a way that the light ray would take the same propagation direction that it had when it was emitted. Placing the detection device very close to the emitting source after this last reflection would thus enable to measure photons which null vectors are parallel to the emitted ones.

**A. Earth-satellite(s)-Earth schemes**

We consider here setups where the light rays follow trajectories with a constant longitude, for which we have obtained the explicit expression of the Wigner phase \( \Psi \). This requires that the laboratories on Earth and the satellite’s reflections events are all placed at the same longitude \( \phi_0 \). The satellites themselves, however, are not constrained to the \( \phi = \text{const.} \) plane and can have an arbitrary motion as long as they are located at longitude \( \phi_0 \) when the reflection event occurs.

The first scheme consists of two neighbouring laboratories placed on Earth at the same longitude, but at different polar angles, see Figure 1. The optical signals are emitted from one station and propagate to a satellite, they are reflected by the satellite to the second laboratory (in the same fashion as in recent satellite experiments [42, 43]), which in turn sends them back to the first laboratory. In this scheme we will consider different satellite orbits, ranging between very low Earth orbits (VLEO) to geostationary ones (GEO). Note that if we were to simply reflect the light rays back from the satellite to the first laboratory there would be no Wigner rotation.

Another scheme we will consider consists of a station on Earth emitting light pulses towards a satellite, which then reflects them towards a second satellite, in turn reflecting the signals back to the emitter’s laboratory, see Figure 2. Alternatively, the downlink from the second satellite to Earth could also reach a second station that would then send the photons back to the emitter’s laboratory, as shown in Figure 2. Since the light ray’s radial coordinate remains approximately constant along the null geodesic in an extra small Earth-to-Earth segment, there is no Wigner phase contribution \( \Psi \) that arises from it, and the two variants for this scheme are equivalent. In practice, we consider this second configuration, where one satellite is located at a low Earth orbit (LEO) and the other at a GEO one.

**B. Wigner phase**

We have described the physical implementation of the two schemes of interest. We proceed and give the explicit expression for the total Wigner phase accrued by the photons along the closed paths designed above.

1. **Wigner phase: one satellite**

For the first scheme, we use (19) for each different geodesic segment of the closed trajectory, and find the full expression for the Wigner phase \( \Psi_1 \), which reads

\[
\Psi_1 = \left[ \sqrt{\frac{R_E}{R_s}} \Delta K_s(\kappa, \kappa') + \Delta K_E(\kappa', \kappa) \right] \epsilon + O(\epsilon^3), \tag{20}
\]

\[2\] Note that in practice, due to the Earth’s rotation, the signals reflected by the satellite would need to be received at a slightly different longitudinal spacetime coordinates in order to be at the same longitude on Earth’s surface. However, we ignore this additional effect here since we assume the Earth to be static since we are using the Schwarzschild metric to model the spacetime around the Earth. We leave it to further work to assess the practical implications of these additional effects.
emitter’s one, since then the receiver’s laboratory was at the same location as the
at the same altitude, see (19). This is also the case if
the light’s propagation vanishes since its endpoints are
phase $\Psi$ repeat the procedure above and obtain the total Wigner

different polar angles.

the emitter and the receiver at different locations, i.e., at
the reason why, in the first scenario, we need to have

where we have denoted the radius of the satellite
by $R_s$ and we have defined $K_X(y) := \sqrt{\frac{2y}{R_s^2 - y}}$ and
$\Delta K_X(y, z) := K_X(y) - K_X(z)$.

The Wigner’s phase contribution for the last segment of
the light’s propagation vanishes since its endpoints are
at the same altitude, see (19). This is also the case if
the receiver’s laboratory was at the same location as the
emitter’s one, since then $\kappa' = \kappa$ and $\Psi_1 = 0$. This is
the reason why, in the first scenario, we need to have
the emitter and the receiver at different locations, i.e., at
different polar angles.

2. Wigner phase: two satellites

We now turn our attention to the second scheme. We
repeat the procedure above and obtain the total Wigner
phase $\Psi_2$, which reads

$$\Psi_2 = \left[ \frac{R_E}{R_{s_2}} \Delta K_{s_2}(\kappa', \kappa'') + \frac{R_E}{R_{s_1}} \Delta K_{s_1}(\kappa, \kappa') + \Delta K_E(\kappa'', \kappa) \right] \epsilon + O(\epsilon^3).$$

(21)

The heights of the first and the second satellite used to
to reflect the light ray are denoted by $R_{s_1}$ and $R_{s_2}$
respectively.

3. Wigner phase: an explicit example

We can obtain more physical insights by employing a
perturbative expansion up to first order in $\kappa/r^2$ in order
to solve (6) analytically. In fact, we always have $\kappa < R_{\min}$,
with $R_{\min} = \min\{R_1, R_2\}$, which can be seen from
the expression of the radial component of the null vector
in (3). Assuming $\kappa \ll R_{\min}^2$ amounts to consider almost
radial light rays. In this constrained case, we obtain:

$$\kappa \approx \frac{R_1^2 R_2^2}{(R_1 - R_2)^2} \Delta \theta^2.$$  

(22)

This approximate expression is valid provided that
$|\Delta \theta| \ll \frac{|R_1 - R_2|}{R_{\max}} =: \Delta \theta_c$, with $R_{\max} = \max\{R_1, R_2\}$. The
critical angular difference $\Delta \theta_c$ is unfortunately typically
small in most realistic cases, which is why in general one
has to solve (6) numerically instead of using the simple
expression given in (22). However, one can use (22)
in scenarios such as when a light ray is exchanged between a
laboratory close to the equator and a geostationary satel-
lite. We can then expand the expression of the phase (20)
for $\kappa, \kappa' \ll R_E^2$, in the same fashion as for the approxi-
mate expression of the rescaled Carter constant (22) in
the special case of small angular deviations. We then have

$$\Psi_1 \approx \sqrt{2(|\Delta \theta'|-|\Delta \theta|)} \frac{1 - \left(\frac{R_E}{R_c}\right)^2}{1 - \frac{R_c}{R_s}} \epsilon + O(\epsilon^3).$$

(23)

From this simple explicit formula, we see that when
$|\Delta \theta'| = |\Delta \theta|$ there is no Wigner phase for this scheme.
This feature is due to the symmetry of the spacetime,
and it is simply a more explicit way of stating that (20)
vanishes for $\kappa = \kappa'$. The expression (23) is however not
a good approximation in most of the realistic cases. In
these setups one must use the full expression (20). One of
the few cases where (23) can be used to predict Wigner
phases with good accuracy is if the both laboratories are
close to the equator, and exchange signals with a geostas-
tionary satellite.

4. Wigner phase: existence in Schwarzschild

We have shown that both total phases (20) and (21)
do not vanish in general in Schwarzschild spacetime, and
this can be seen even more explicitly for the first phase
(20) in its perturbative expeansion (23). This is one of
our results.

This result contradicts the statement found in the
literature that the total Wigner phased acquired along a
closed path in Schwarzschild spacetime would al-
ways vanish regardless of the chosen gauge convention
[15, 16, 31]. In Section V we will give the numerical val-
umes for the parameters appearing in (20) and (21) in or-
der to estimate the amplitude of the total Wigner phases
obtained here for the two schemes.
C. Changes due to reflections on mirrors

Before proceeding to our main result, we review the particular effects induced by the reflections of the pulses by the moving satellites.

In the schemes described in IV A, the light rays are reflected by the satellite(s), and then by the mirrors of a static observer on Earth prior to their measurement since we need to align the momenta of the received photons with those of the emitted ones. This last reflection doesn’t change the energy of the received photons as seen by the static observer that will perform the measurement, because the mirror is also static and located at the same distance $r_E$. Its only effect is to change the directional parameters $\epsilon_{\tau}$, $\epsilon_{\theta}$ and $\kappa$ of the null vector [5] back to the initial values.

However, the orbiting satellites do impart some of their energy to the photons during the reflection events, which changes the value of the constant $E_p$ for the new geodesic segments that follow these reflection events. The total state of the photon is affected by these events, and in particular the momentum states are. However, the Wigner phases are energy independent, therefore the helicity-part of the quantum states is not affected.

Finally, the reflections of light rays by the mirrors change the ray’s direction, and therefore the polarisation vector as well. This will in principle induce extra Wigner phases, although we can safely ignore these effects in this work. This is a consequence of the fact that the relativistic corrections on the polarisation vector due to reflections on the mirrors are of one order higher in the perturbative expansion than the polarisation rotation induced by the propagation in the curved spacetime [31].

V. TOTAL WIGNER PHASE MEASURED

In this section we use experimental values for the parameters that appear in the expression of the Wigner phase [19], in order to evaluate the phase for realistic implementations of the different configurations proposed above IV A.

A. Physical parameters

The Wigner phases [20] and [21] encode physical parameters of the Earth and of the trajectory of the light pulses. Here we list some values to be used for the evaluation of the Wigner phase. The Earth’s (mean) radius is $R_E = 6371$ km; the Earth’s mass in natural units takes a value of $M = 4.43$ mm. GEO orbits are located at $R_s(\text{GEO}) = R_E + 35784$ km, LEO orbits at $R_s(\text{LEO}) = R_E + 2000$ km (the highest orbit to be considered as LEO) and VLEO orbits at $R_s(\text{VLEO}) = R_E + 255$ km, which is the radius at which the Gravity field and steady-state Ocean Circulation Explorer (GOCE) satellite orbited the Earth.

In Table I we give the numerical values of the quantities appearing in the expressions of the Wigner phases. We compute the values of the rescaled Carter constant $\kappa$ by integrating numerically the constraint equation [3]. We use the parameters given above together with the polar angles $\theta_{E1} = 42^\circ$ and $\theta_{E2} = 43^\circ$ for laboratories on Earth, $\theta_s(\text{VLEO}) = 30^\circ$ and $\theta_s(\text{LEO}) = 15^\circ$ for the reflecting VLEO and LEO satellites respectively, and the GEO satellite lies in the equatorial plane $\theta_s(\text{GEO}) = 90^\circ$.

| Quantity | Quantity | Position | Light ray | Value |
|----------|----------|----------|-----------|-------|
| $M/R_E$ | $\frac{GM}{R_Ec^2}$ | Earth | / | $6.95 \times 10^{-10}$ |
| $M/R_s$ | $\frac{GM}{R_s c^2}$ | GEO | / | $1.05 \times 10^{-10}$ |
| | | LEO | / | $5.29 \times 10^{-10}$ |
| | | VLEO | / | $6.69 \times 10^{-10}$ |

| $\kappa$ | $\kappa$ | / |
|-----------|-----------|--------|
| E1 $\rightarrow$ GEO | $2.73 \times 10^{13}$ m$^2$ |
| GEO $\rightarrow$ E2 | $2.66 \times 10^{13}$ m$^2$ |
| E1 $\rightarrow$ LEO | $3.75 \times 10^{13}$ m$^2$ |
| LEO $\rightarrow$ E2 | $3.80 \times 10^{13}$ m$^2$ |
| E1 $\rightarrow$ VLEO | $4.03 \times 10^{13}$ m$^2$ |
| VLEO $\rightarrow$ E2 | $4.46 \times 10^{13}$ m$^2$ |
| GEO $\rightarrow$ LEO | $6.98 \times 10^{13}$ m$^2$ |
| LEO $\rightarrow$ GEO | $6.98 \times 10^{13}$ m$^2$ |

TABLE I: Physical parameters used to compute the magnitude of the Wigner phase in this work.

B. Wigner phases

We now have all the numerical values required to evaluate the Wigner phases [20] and [21]. We give their amplitudes for the different schemes and satellites’ configurations in Table II below.

| Scheme | Light’s trajectory | $\Psi$ (rad) |
|--------|--------------------|--------------|
| 1 | E1 $\rightarrow$ VLEO $\rightarrow$ E2 | $1.97 \times 10^{-4}$ |
| | E1 $\rightarrow$ LEO $\rightarrow$ E2 | $1.32 \times 10^{-5}$ |
| | E1 $\rightarrow$ GEO $\rightarrow$ E2 | $-2.09 \times 10^{-6}$ |
| 2 | E1 $\rightarrow$ LEO $\rightarrow$ GEO $\rightarrow$ E2 | $-5.56 \times 10^{-4}$ |
| | E1 $\rightarrow$ GEO $\rightarrow$ LEO $\rightarrow$ E2 | $5.67 \times 10^{-4}$ |

TABLE II: Wigner phase values for different satellite configurations in both schemes.

Unsurprisingly, the magnitude of the effect is larger in the second scheme, where the light rays propagate along larger distances (covering longer areas) across the curved spacetime surrounding the Earth. However, we can notice that in the first scheme higher values are obtained for lower orbits. This is mostly due to the Wigner phase’s dependence on the inverse of the satellite’s radius, but also because of its dependence on the light’s angular momen-
tum, which generally has larger values when the light rays are sent to such low orbits: the null geodesic segments are shorter in such cases, and this leads to a larger angular momentum for a given angular difference between endpoints of the segment.

Finally, we present our main result. As explained in [11], we can facilitate the detection of such phases by using more refined helicity states like GHZ states. However, the values we have obtained for the Wigner phases are much larger than those found in the literature for other scenarios that study the rotation of the polarisation of light [19, 25, 26, 28]. Furthermore, this result also seems to invalidate the statement that there would be no Wigner phase for a closed trajectory in Schwarzschild spacetime [15, 16, 31], even when it is due to judiciously positioned mirrors and not because of initial conditions [10].

We believe that this result opens a window on the possibility of measuring, and perhaps exploiting, Wigner phases induced on quantum states of light due to the curved spacetime.

VI. CONCLUSION

We have considered pulses of polarised light that travel through the curved spacetime around the Earth. We have specialised to closed paths, where light is emitted and reflected by different links to reach again the emitter, and we found that the helicity quantum states of the photons pick up a non-zero Wigner phase. We have studied two setups that can be implemented with current satellite technologies and we have found that amplitudes of the Wigner phases are, to our knowledge, larger than any other phase obtained so far in the context of the rotation of the polarisation of light by at least two orders of magnitude. Our result also questions the statement found in the literature that there is no Wigner phase accrued by photons propagating along a closed path in Schwarzschild spacetime.

We presented two schemes involving reflecting satellites and located at near-Earth distances. We find that the amplitude of the effect is higher when considering lower orbits for a scheme with a single satellite. This is mainly a consequence of the dependence of the Wigner phase in the inverse of the radius of the satellite. With multiple reflecting satellites, the configurations with the longest paths with the greatest angular differences produce the largest effects. These effects can be enhanced by using GHZ-like helicity states, in which case the Wigner phase is effectively multiplied by the number of input probes.

The schemes we propose in this article are achievable with current technology. In particular, single photon experiments are already performed with and without dedicated satellites [5, 6, 44]. The measurement of the phase we predict can potentially help to deepen our understanding of the influence of general relativistic effects on quantum states of polarised photons, and therefore of the interplay between quantum physics and relativity.

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Appendix A: Tetrad

1. General tetrad

We give here the general first and second triad components for a static observer [2], adapted to an arbitrary null vector [3]. Using the expressions of the zeroth $e^i_0 = v_E$ and third [17] tetrad components, together with the orthonormalisation relations, we have $e^i_1 = e^i_2 = 0$ and we find
\[ e_1^r = -\eta_1 \epsilon_r (r - 2M) \sin \theta \]
\[ e_1^\theta = -\eta_1 \eta_2 \frac{\sqrt{Q - r^5 B^2 \sin^2 \theta}}{r^2 \sin \theta}, \]
\[ e_1^\phi = \eta_1 \frac{r^4 B \sin^2 \theta \sqrt{r^3 - (r - 2M)(\kappa + l_\phi^2)}}{r^2 \sin \theta Q}, \]
\[ e_2^r = -\epsilon_r (r - 2M) \frac{r^4 B \sin^2 \theta \sqrt{r^3 - (r - 2M)(\kappa + l_\phi^2)}}{r^2 \sin \theta Q}, \]
\[ e_2^\theta = -\epsilon_\phi (r - 2M) \frac{2M \kappa \sqrt{1 - \frac{2M}{r}}}{r^2} \partial_r - \frac{1}{r} \sqrt{2M \kappa \sqrt{1 - \frac{2M}{r}}} \partial_\theta + \frac{\csc \theta}{r} \frac{\sqrt{1 - \frac{2M}{r}}}{\sqrt{1 - \frac{2M}{r}}} \partial_\phi, \]
\[ e_2^\phi = -\epsilon_\phi (r - 2M) \frac{\kappa \sqrt{1 - \frac{2M}{r}}}{r^2} \partial_r + \frac{1}{r} \sqrt{1 - \frac{2M}{r}} \partial_\theta + \frac{\csc \theta}{r} \frac{\sqrt{2M \kappa \sqrt{1 - \frac{2M}{r}}}}{\sqrt{1 - \frac{2M}{r}}} \partial_\phi, \]
\[ e_3 = \sqrt{1 - \frac{2M}{r}} \left( \epsilon_r \sqrt{1 - \left( \frac{2M}{r} \right)^2} \partial_r + \frac{\epsilon_\phi}{\sqrt{1 - \left( \frac{2M}{r} \right)^2}} \partial_\phi \right). \]

where \( B = e_2^\theta \) is the gauge component of the tetrad.

We have defined \( Q := \left( r^3 - (r - 2M)(\kappa + l_\phi^2) \right) \sin^2 \theta + l_\phi^2 (r - 2M) \) for compactness. The gauge signs \( \eta_1 = \pm 1 \) and \( \eta_2 = \pm 1 \), and the tetrad component \( e_2^\phi \) is arbitrary. The gauge component \( B \) is of dimension inverse of length. These three parameters encode all the gauge from the freedom of choice of our adapted frame.

\section{Chosen tetrad}

In III A we focused on \( l_\phi = 0 \) light rays and we fixed the gauge parameters \( B = \frac{1}{2} \sqrt{1 - \frac{2M}{r}} \) and \( \eta_1 = \eta_2 = +1 \) in order to simplify the expression of the Wigner rotation. Our triad then simplifies to:

\[ e_1 = \epsilon_\phi \kappa \frac{\sqrt{2M \kappa \sqrt{1 - \frac{2M}{r}}}}{r^2} \partial_r - \frac{1}{r} \sqrt{2M \kappa \sqrt{1 - \frac{2M}{r}}} \partial_\theta + \frac{\csc \theta}{r} \frac{\sqrt{1 - \frac{2M}{r}}}{\sqrt{1 - \frac{2M}{r}}} \partial_\phi, \]

\[ e_2 = -\epsilon_\phi \kappa \frac{\sqrt{1 - \frac{2M}{r}}}{r} \partial_r + \frac{1}{r} \sqrt{1 - \frac{\kappa}{r^2}} \partial_\theta + \frac{\csc \theta}{r} \frac{\sqrt{2M \kappa \sqrt{1 - \frac{2M}{r}}}}{\sqrt{1 - \frac{2M}{r}}} \partial_\phi, \]

\[ e_3 = \sqrt{1 - \frac{2M}{r}} \left( \epsilon_r \sqrt{1 - \left( \frac{2M}{r} \right)^2} \partial_r + \frac{\epsilon_\phi}{\sqrt{1 - \left( \frac{2M}{r} \right)^2}} \partial_\phi \right). \]
[16] A. Brodutch and D. R. Terno, Phys. Rev. D 84, 121501 (2011).
[17] S. Kopeikin and B. Mashhoon, Phys. Rev. D 65, 064025 (2002).
[18] G. V. Skrotskii, Soviet Physics Doklady 2, 226 (1957).
[19] A. R. Prasanna and S. Mohanty, EPL (Europhysics Letters) 60, 651 (2002).
[20] V. Faraoni, New Astronomy 13, 178 (2008).
[21] L. Dai, Phys. Rev. Lett. 112, 041303 (2014).
[22] J. A. Morales and D. Sáez, Phys. Rev. D 75, 043011 (2007).
[23] M. Sereno, Phys. Rev. D 69, 087501 (2004).
[24] S. Kopeikin and B. Mashhoon, Phys. Rev. D 65, 064025 (2002).
[25] M. Lyutikov, Phys. Rev. D 95, 124003 (2017).
[26] U.-L. Pen, X. Wang, and I.-S. Yang, Phys. Rev. D 95, 044034 (2017).
[27] M. Nouri-Zonoz, Phys. Rev. D 69, 024013 (1999).
[28] M. L. Ruggiero and A. Tartaglia, Monthly Notices of the Royal Astronomical Society 374, 847 (2007).
[29] M. Sereno, Monthly Notices of the Royal Astronomical Society 356, 381 (2005).
[30] H. Ishihara, M. Takahashi, and A. Tomimatsu, Phys. Rev. D 38, 472 (1988).
[31] A. Brodutch, A. Gilchrist, T. Guff, A. R. H. Smith, and D. R. Terno, Phys. Rev. D 91, 064041 (2015).
[32] M. C. Palmer, M. Takahashi, and H. F. Westman, Annals of physics 327, 1078 (2012).
[33] M. C. Palmer, Relativistic quantum information theory and quantum reference frames, Ph.D. thesis, University of Sydney, School of Physics (2013).
[34] D. L. Wiltshire, M. Visser, and S. M. Scott, The Kerr spacetime: Rotating black holes in general relativity (Cambridge University Press, 2009).
[35] D. Bini, K. Boshkayev, R. Ruffini, and I. Siutsou, Il Nuovo Cimento 36 C (2013), 10.1393/ncc/i2013-11483-8.
[36] F. de Felice and C. Clarke, Relativity on Curved Manifolds (Cambridge University Press, Cambridge, 1992).
[37] P. Gosselin, A. Bérard, and H. Mohrbach, Phys. Rev. D 75, 084035 (2007).
[38] F. de Felice and D. Bini, Classical measurements in curved space-times (Cambridge University Press, Cambridge, 2010).
[39] F. de Felice and G. Preti, Classical and Quantum Gravity 16, 2929 (1999).
[40] T. Müller and F. Grave, “Catalogue of spacetimes,” arXiv:0904.4184 (2010).
[41] S. D. Bartlett and D. R. Terno, Phys. Rev. A 71, 012302 (2005).
[42] G. Vallone, D. Bacco, D. Dequal, S. Gaiarin, V. Luceri, G. Bianco, and P. Villoresi, Phys. Rev. Lett. 115, 040502 (2015).
[43] G. Vallone, D. Dequal, M. Tomasin, F. Vedovato, M. Schiavon, V. Luceri, G. Bianco, and P. Villoresi, Phys. Rev. Lett. 116, 253601 (2016).
[44] D. Dequal, G. Vallone, D. Bacco, S. Gaiarin, V. Luceri, G. Bianco, and P. Villoresi, Phys. Rev. A 93, 010301 (2016).