New relaxed stability and stabilization conditions for T-S fuzzy systems with time-varying delays

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Abstract

This paper investigates the stability analysis and stabilization of T-S fuzzy systems with time-varying delays. First, a new augmented Lyapunov–Krasovskii functional is constructed, delay-dependent stability criteria in terms of linear matrix inequalities (LMIs) are obtained by combining them with the integral inequality technique and the reciprocally convex combination inequality. Based on the state space decomposition method, some piecewise membership functions are employed to approximate the membership functions. The piecewise membership functions can be locally represented in terms of the convex combinations of the supremum and infimum of some local basis functions. The boundary information of the membership functions is adequately taken into consideration in stability analysis, and then some relaxed membership-function-dependent stability results are obtained. Second, state feedback controllers for fuzzy systems with time-varying delays are presented under the imperfect premise-matching technique, whose membership functions and the number of fuzzy rules are allowed to be designed freely, consequently, the flexibility of controller design is improved. Finally, four numerical examples are given to demonstrate the effectiveness of the presented approaches.

1 | INTRODUCTION

The T-S fuzzy model offers a universal framework to represent complex nonlinear systems, dynamic nonlinear systems can be represented as weighted sum of several local linear sub-systems via nonlinear fuzzy weights, and each linear sub-system can effectively describe the local characteristics of the nonlinear systems [1]. With the help of the favourable semi-linear property, perfect analysis and synthesis methods of linear systems can be introduced into the control problem of nonlinear systems via the T-S fuzzy model, which provides a powerful method for the control of nonlinear systems [2]. On the other hand, the phenomenon of time delay is inevitable in various practical control systems, the existence of time delay often affects the control performance and may even lead to the instability of the systems [3, 4]. Consequently, the stability analysis and controller synthesis of time-delay systems are of great significance for practical engineering applications. In industrial production and practical control systems, time-delay systems are often highly nonlinear. Therefore, it is difficult to model and control the time-delay systems directly. The T-S fuzzy model can approximate nonlinear systems with arbitrary precision, consequently, the control problem of nonlinear time-delay systems based on the T-S fuzzy model has attracted more and more attention, and many plentiful results have been achieved [5–9].

Generally, constructing a suitable Lyapunov–Krasovskii functional is the main way to reduce the conservativeness of stability analysis for time-delay systems. In order to obtain more relaxed stability results, a fuzzy line-integral Lyapunov–Krasovskii functional was chosen in [10], and the less conservative conditions in terms of linear matrix inequalities (LMIs) were obtained. However, the case of constant time delays was studied in [10], rather than time-varying delays. On the basis of [10], an augmented Lyapunov–Krasovskii functional containing the fuzzy line-integral Lyapunov function was constructed in [11], then, more delay information was considered in stability analysis of T-S fuzzy systems with time-varying delays and more relaxed results were obtained. In order to further reduce the conservatism, a piecewise fuzzy Lyapunov–Krasovskii functional was employed in [12]. Nevertheless, the derivatives of
controllers based on the PDC strategy are required to share the parallel distributed compensation (PDC) technique. Fuzzy controllers of T-S fuzzy systems are designed based on a relaxed delay-dependent stability criterion. One of the motivations of this paper is to choose the more relaxed stability terms directly. Moreover, the stability criterion could be employed to evaluate the time-varying delays in the estimation results. A reciprocally convex combination lemma was presented in [25], which could be employed to evaluate the time-varying delays in the estimation terms directly. Moreover, the stability criterion could be obtained with fewer decision variables, and it will not increase the conservatism of stability analysis. Therefore, the reciprocally convex combination method has been widely used in the stability analysis of time-varying delay systems. By introducing relaxation matrix variables in [26], the reciprocally convex combination lemma in [25] was generalized, and the generalized reciprocally convex inequality was proved to be less conservative. Nevertheless, the introduction of relaxation matrix variables will increase the computational complexity of stability criteria. One of the motivations of this paper is to choose the more advanced reciprocal convex inequality and obtain some more relaxed delay-dependent stability criterion.

On the other hand, the results mentioned above and most of the controllers of T-S fuzzy systems are designed based on the parallel distributed compensation (PDC) technique. Fuzzy controllers based on the PDC strategy are required to share the same membership functions and the number of fuzzy rules with the fuzzy models, which facilitates the stability analysis of T-S fuzzy systems. However, the PDC strategy limits the flexibility of fuzzy controllers design and may complicate their construction [27]. A new PDC design approach for T-S fuzzy control systems with affine matched membership functions in the system and controller was presented in [45], and the conservative membership functions need to be bounded when the fuzzy Lyapunov–Krasovskii functional method is employed. Stability conditions of T-S fuzzy systems with time-varying delays were obtained by constructing novel Lyapunov–Krasovskii functions dependent on the delay-fractioning technique in [13, 14], where the conservatism could be reduced by increasing the segmentation step. However, while increasing the segmentation step, the delay-fractioning technique may introduce the larger decision variables and computational complexity. In [15, 16], novel augmented Lyapunov–Krasovskii functions were constructed, which had made full use of the time-delay information, and relaxed stability results of T-S fuzzy time-delay systems were gained. In order to further reduce the conservatism, delay-product-type augmented Lyapunov–Krasovskii functions were constructed to introduce more time-delay information into stability analysis in [17, 18]. Obviously, compared with the simple form of the Lyapunov–Krasovskii functional, the Lyapunov–Krasovskii functionals with augmented terms introduce several extra matrices, which provide more freedom for checking the feasibility of the LMs conditions in the criteria [19]. The conservative of stability analysis can be effectively reduced by choosing an appropriate augmented Lyapunov–Krasovskii functional. However, introducing too many augmented vectors will inevitably increase computational complexity, how to balance computational complexity and conservatism is worth exploring.

In addition, the utilization of powerful boundary techniques is another effective way to reduce the conservatism of stability and stabilization conditions for time-delay systems. Common methods include free-weighting matrix approach [20–22], the inequality technique [23] and convex combination technique [24]. The integral inequality technique is an effective approach to estimate the integral term of the derivative of Lyapunov–Krasovskii functional. In the analysis of time-varying delay systems via the integral inequality method, some other techniques often need to be combined to deal with the time-varying delay terms generated in the estimation results. A reciprocally convex combination lemma was presented in [25], which could be employed to evaluate the time-varying delays in the estimation terms directly. Moreover, the stability criterion could be obtained with fewer decision variables, and it will not increase the conservatism of stability analysis. Therefore, the reciprocally convex combination method has been widely used in the stability analysis of time-varying delay systems. By introducing relaxation matrix variables in [26], the reciprocally convex combination lemma in [25] was generalized, and the generalized reciprocally convex inequality was proved to be less conservative. Nevertheless, the introduction of relaxation matrix variables will increase the computational complexity of stability criteria. One of the motivations of this paper is to choose the more advanced reciprocal convex inequality and obtain some more relaxed delay-dependent stability criterion.

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On the other hand, the results mentioned above and most of the controllers of T-S fuzzy systems are designed based on the parallel distributed compensation (PDC) technique. Fuzzy controllers based on the PDC strategy are required to share the same membership functions and the number of fuzzy rules with
four numerical examples are given to demonstrate the effectiveness of the presented approaches.

The main contributions of this paper are as follows:

1. A new augmented Lyapunov–Krasovskii functional is constructed by introducing some double delay integral terms into the augmented vectors, some relaxed stability criteria are obtained by combining with integral inequality technique and the reciprocally convex combination inequality;
2. In order to obtain further relaxed stability criteria, some piecewise membership functions are constructed to approximate the membership functions. Therefore, some less conservative membership-function-dependent stability results are obtained;
3. Under the imperfect premise matching technique, a fuzzy state feedback control design method is to be presented for T-S fuzzy systems with time-varying delays, the fuzzy controller is not required to employ the same premise membership functions and the number of fuzzy rules as the fuzzy model, and then the flexibility of the controller design is improved.

This paper is organized as follows. The problem description and some useful lemmas are given in Section 2. The main derivation process is presented in Section 3. Four numerical examples are provided to demonstrate the effectiveness of the proposed approaches in Section 4. The last section concludes this paper.

Notations. Throughout this study, the superscripts $T$ and $−1$ denote the transpose and the inverse of a matrix, respectively. $I$ and $0$ denote the identity matrix and zero matrix with compatible dimensions, respectively. $\mathbb{R}^n$ and $\mathbb{R}^{m \times n}$ stand for n-dimensional Euclidean space and the set of all $n \times m$ real matrices, respectively. $P > 0 \geq 0$ means that $P$ is a positive definite (positive semidefinite) matrix. diag$\{\cdot\}$ means a block diagonal matrix, and $\delta_{\omega}(X) = X + X^T$. If not specially explained, matrices are supposed to be with suitable dimensions.

2 | PRELIMINARIES

2.1 | Fuzzy models

Consider a class of non-linear systems with time-varying delays, which can be described by the following T-S fuzzy model with $p$ plant rules:

Rule $i$: IF $\delta_1(x(t))$ is $F_i^1$ and $\delta_2(x(t))$ is $F_i^2$ and … and $\delta_\chi(x(t))$ is $F_i^\chi$, THEN

$$
\begin{align*}
\dot{x}(t) &= A_i x(t) + A_i \delta_i x(t - \tau_i(t)) + B_i u(t) \\
x(t) &= \phi(t), & t \in [-\tau_m, 0]
\end{align*}
$$

(1)

where $F_i^\alpha$ denote the premise variables with $\alpha = 1, 2, \ldots, \chi$, $i = 1, 2, \ldots, p$; $\delta_1(x(t)), \ldots, \delta_\chi(x(t))$ are the fuzzy membership functions. $\phi(t)$ is the vector-valued initial condition on $[-\tau_m, 0]$; $A_i$, $A_i \delta_i$, $B_i$ are system matrices with appropriate dimensions, and $\tau(t)$ is a time-varying delay satisfying

$$
\begin{align*}
0 \leq \tau(t) &\leq \tau_M \\
\mu_1 &\leq \tau(t) \leq \mu_2
\end{align*}
$$

(2)

where $\tau_M, \mu_1$ and $\mu_2$ are known constant scalars.

Employing the singleton fuzzifier, product inference, and centre-average defuzzifier, the global dynamics of the delayed fuzzy model can be inferred as follows:

$$
\dot{x}(t) = \sum_{i=1}^p m_i(x(t)) \left( A_i x(t) + A_i \delta_i x(t - \tau(t)) + B_i u(t) \right)
$$

(3)

where $m_i(x(t)) = \frac{\omega_i(x(t))}{\sum_{i=1}^\chi \omega_i(x(t))}$, $\omega_i(x(t)) = \prod_{\alpha=1}^\chi \mu_{F_i^\alpha} \delta_{\alpha}(x(t))$ with $\mu_{F_i^\alpha} \delta_{\alpha}(x(t))$ representing the grade of membership of $\delta_{\alpha}(x(t))$ in $F_i^\alpha$. Since $\omega_i(x(t)) \geq 0$, it holds that $\sum_{i=1}^p \omega_i(x(t)) > 0, \sum_{i=1}^p m_i(x(t)) = 1$.

2.2 | Fuzzy controllers

Based on imperfect premise matching technique, a fuzzy state feedback controller with $\epsilon$ rules is presented as follows:

$$
u(j) = K_j x(t), \quad j = 1, 2, \ldots, \epsilon
$$

(4)

Then, the overall controller is presented as:

$$
u(t) = \sum_{j=1}^\epsilon b_j(x(t)) K_j x(t)
$$

(5)

where $K_j \in \mathbb{R}^{m \times n}$ are control gains matrix to be determined.

Consequently, the closed-loop delayed T-S fuzzy system can be described as:

$$
\dot{x}(t) = \sum_{i=1}^p \sum_{j=1}^\epsilon m_i(x(t)) b_j(x(t)) \left( \left( A_i + B K_j \right) x(t) + A_i \delta_i x(t - \tau(t)) \right)
$$

+ $A_i \delta_i x(t - \tau(t))$

(6)

with compact form

$$
\dot{x}(t) = (A(t) + B(t) K(t)) x(t) + A_i t x(t - \tau(t))
$$

where

$$
A(t) = \sum_{i=1}^p \sum_{j=1}^\epsilon m_i(x(t)) b_j(x(t)) A_i,

B(t) = \sum_{i=1}^p \sum_{j=1}^\epsilon m_i(x(t)) b_j(x(t)) B_i,
$$
Lemma 1 [30]. For a positive definite matrix $R > 0$, and a differentiable function $\phi(w) \in [a, b]$, the following inequality holds:

$$\int_a^b \dot{x}(s) R \dot{x}(s) ds \geq \frac{1}{b-a} \Omega_1^{T} R \Omega_1 + \frac{3}{b-a} \Omega_2^{T} R \Omega_2 + \frac{5}{b-a} \Omega_3^{T} R \Omega_3$$

where

$$\Omega_1 = x(b) - x(a),$$
$$\Omega_2 = x(b) + x(a) - \frac{2}{b-a} \int_a^b x(s) ds,$$
$$\Omega_3 = x(b) - x(a) + \frac{6}{b-a} \int_a^b x(s) ds.$$

Lemma 2 [31]. Let $R_1, R_2 \in \mathbb{R}^{n \times n}$ be real symmetric positive definite matrices and $w_1, w_2 \in \mathbb{R}^n$ and a scalar $\alpha \in (0, 1)$. Then for any $Y_1, Y_2 \in \mathbb{R}^{n \times n}$, the following inequality holds:

$$\frac{1}{\alpha} w_1^T R_1 w_1 + \frac{1}{1-\alpha} w_2^T R_2 w_2 \geq w_1^T \left[ R_1 + (1-\alpha) \left( R_1 - Y_1 R_1^{-1} Y_1^T \right) \right] w_1 + w_2^T \left[ R_2 + \alpha \left( R_2 - Y_2 R_2^{-1} Y_2^T \right) \right] w_2 + 2 \alpha Y_1^T [x Y_1 + (1-\alpha) Y_2] w_2$$

Lemma 3 [32]. For a given quadratic function $\ell(\cdot) = a_2 \xi^2 + a_1 \xi + a_0$, where $a_i \in \mathbb{R}$ ($i = 0, 1, 2$), if the following inequalities hold:

$$(i) \ell(0) < 0; \ (ii) \ell(b) < 0; \ (iii) \ b^2 a_2 + \ell(0) < 0$$

one has $\ell(\cdot) < 0$ for $\forall \xi \in [a, b]$.

Lemma 4 [33]. Consider a stochastic vector $\xi \in \mathbb{R}^n$, a symmetric matrix $\Gamma \in \mathbb{R}^{n \times n}$, and a matrix $B \in \mathbb{R}^{n \times n}$ with rank$(B) = r < n$. Let $B^{-1} \in \mathbb{R}^{n \times (n-r)}$ be the right orthogonal complement of $B$ by $BB^T = 0$ and $B^T B^{1, T} > 0$. Then the following statements are equivalent:

1. $E[\xi^T \Gamma \xi] < 0, \forall \xi \neq 0, E(B \xi) = 0$;
2. $B^T \Gamma B^T < 0$;
3. $\exists k \in \mathbb{R} \Gamma - kB^T B^T < 0$;
4. $\exists R \in \mathbb{R}^{n \times n} \Gamma + RB + B^T R^T < 0$.

3. **MAIN RESULTS**

3.1 **Membership-function-dependent stability analysis**

In this subsection, a novel augmented Lyapunov–Krasovskii functional is constructed, and a relaxed delay-dependent stability criterion is obtained by employing the auxiliary function integral inequality together with the extended reciprocally convex matrix inequality. The boundary information of membership functions is considered by introducing piecewise membership functions, and further relaxed membership-function-dependent stability results are presented.

In order to consider the information of membership functions, we introduced the following results in [34].

Consider the state space $\Gamma$ being divided into $k$ connected state subspaces $\Gamma_j$, $j = 1, 2, \ldots, k$, such that $\Gamma = \bigcup_{j=1}^k \Gamma_j$. Considering a state $X = [x_1 \cdots x_n]$ such that $X \in \Gamma_j$, define the minimum and maximum of $x_r$ as $\underline{x}_{r,j}$ and $\overline{x}_{r,j}$, respectively, $r = 1, 2, \ldots, n, j = 1, 2, \ldots, k$, such that $\underline{x}_{r,j} \leq x_r \leq \overline{x}_{r,j}$. Denoting the vertices of the subspaces $\Gamma_j$ as $x_{1,j} \cdots x_{n,j} = [x_{1,j} \cdots x_{n,j}]$,
the piecewise membership function is defined as:

\[
\bar{\omega}(x) = \sum_{j=1}^{k} \sum_{i=1}^{2} \cdots \sum_{i_1=1}^{n} \prod_{i=1}^{2} \nu_{r_{ij}}(x_i) \bar{\omega}(x_{i_1 \cdots i_2}) \quad (7)
\]

where \( \nu_{r_{ij}}(x_i) \) is a function to be determined, which exhibits the properties as follows:

1. \( 0 \leq \nu_{r_{ij}}(x_i) \leq 1, \sum_{i=1}^{2} \nu_{r_{ij}}(x_i) = 1 \) for all \( r, j, \) and \( x \in \Gamma_j \), otherwise, \( \nu_{r_{ij}}(x_i) = 0 \);

2. \( \sum_{i=1}^{k} \sum_{i_1=1}^{2} \cdots \sum_{i_1=1}^{n} \prod_{i=1}^{2} \nu_{r_{ij}}(x_i) \bar{\omega}(x_{i_1 \cdots i_2}) = 1 \).

An example is given below to better illustrate the definition of the piecewise membership function \( \bar{\omega}(x) \). Considering the state \( X = \{x_1, x_2\} \), the state space \( \Gamma \) is divided into 12 subspaces \( \Gamma_j, j = 1, 2, \ldots, 12 \). As shown in Figure 1, the black dots represent vertices, and each subspace \( \Gamma_j \) is represented by four vertices. The grades of membership corresponding to the four vertices of the \( l \)-th subspace are recorded as \( \bar{\omega}(x_{11}), \bar{\omega}(x_{12}), \bar{\omega}(x_{21}), \bar{\omega}(x_{22}) \), respectively. The functions \( \nu_{ij}(x_i), i = 1, 2, j = 1, 2 \) in (7) are predefined with \( \nu_{11}(x_1) + \nu_{21}(x_1) = 1 \) and \( \nu_{21}(x_2) + \nu_{22}(x_2) = 1 \) in \( l \)-th subspace, and we have \( \nu_{ij}(x_i) = 0 \) in other subspaces. Based on the definition of the piecewise membership function, the grades of membership in the \( l \)-th subspace can be expressed as \( \bar{\omega}(x) = \sum_{i=1}^{k} \sum_{i_1=1}^{2} \prod_{i=1}^{2} \nu_{r_{ij}}(x_i) \bar{\omega}(x_{i_1 i_2}) \). Furthermore, we can obtain the overall piecewise membership function \( \bar{\omega}(x) \) in (7).

Remark 2. In order to obtain further relaxed stability criteria, instead of introducing global membership function and premise variable knowledge among the overall state space, additional information is to be introduced into the stability analysis, this includes: 1) the regional bounds of piecewise membership functions; 2) the property knowledge of interpolation membership function of piecewise membership functions and regional bounds of premise variables corresponding to each substate region; and 3) the regional approximation error between piecewise membership functions and original membership functions. Furthermore, some less conservative membership-function-dependent stability results are obtained.

**Theorem 1.** For given scalars \( \tau_M > 0, \mu \in \{\mu_1, \mu_2\} \), T-S fuzzy system with time-varying delays (3) under \( u(t) = 0 \) is asymptotically stable, if there exist matrices \( 0 < P \in \mathbb{R}^{7 \times 7}, 0 < Q_1 \in \mathbb{R}^{6 \times 6}, 0 < Q_2 \in \mathbb{R}^{6 \times 6}, 0 < R \in \mathbb{R}^{6 \times 6}, 0 < F \in \mathbb{R}^{12 \times 12}, E = E^T \in \mathbb{R}^{12 \times 12}, \) and any matrices \( Y_1 \in \mathbb{R}^{3 \times 3}, Y_2 \in \mathbb{R}^{3 \times 3}, \) such that the following inequalities hold:

\[
\Sigma_{1i} (0, \mu) + E - F_i < 0, \quad \forall i = 1, 2, \ldots, p \quad (8)
\]

\[
\Sigma_{2i} (\tau_M, \mu) + E - F_i < 0, \quad \forall i = 1, 2, \ldots, p \quad (9)
\]

\[
\Sigma_{3i} (0, \mu) + E - F_i < 0, \quad \forall i = 1, 2, \ldots, p \quad (10)
\]

\[
\sum_{i=1}^{p} \left( \tilde{m}_i (x_{i_1 i_2 \cdots i_l}) \Sigma_{1i} (0, \mu) + \Delta^m_i (\Sigma_{1i} (0, \mu) + E) \right. \\
\left. + \left( \Delta^m_i - \Delta^m_i ight) F_i \right) < 0, \quad \forall i_1, i_2, \ldots, i_l \in \{1, 2\}, \quad l = 1, 2, \ldots, k
\]

\[
\sum_{i=1}^{p} \left( \tilde{m}_i (x_{i_1 i_2 \cdots i_l}) \Sigma_{2i} (\tau_M, \mu) + \Delta^m_i (\Sigma_{2i} (\tau_M, \mu) + E) \right. \\
\left. + \left( \Delta^m_i - \Delta^m_i ight) F_i \right) < 0, \quad \forall i_1, i_2, \ldots, i_l \in \{1, 2\}, \quad l = 1, 2, \ldots, k
\]

where the piecewise membership functions \( \tilde{m}_i(x) \) are defined in (26), \( k \) is the number of divided state subspaces for the piecewise membership functions, \( x_{i_1 i_2 \cdots i_l} \) represents the apexes of the \( l \)-th state subspace of \( \Gamma \). \( \Delta^m_i \) are predefined constant scalars satisfying \( \Delta^m \leq \Delta^m_i (\Delta^m_i) \leq \Delta^m \). Concurrently, some other matrices are defined as follows:

\[
\Sigma_{1i} (0, \hat{\tau} (t)) = \begin{bmatrix} Y_1 & \hat{A}_1 Y_1 \\ \hat{A}_2 Y_1 & -\hat{R}_1 \end{bmatrix},
\]

\[
\Sigma_{2i} (\tau_M, \hat{\tau} (t)) = \begin{bmatrix} Y_1 & \hat{A}_1 Y_1 \\ \hat{A}_2 Y_1 & -\hat{R}_1 \end{bmatrix}
\]
\[\Sigma_i (0, \tau (t)) = \begin{bmatrix} \tau^{-2}_M \Delta (\tau (t)) + \gamma_{i_1} (0, \tau (t)) & \Lambda_i^T Y_1 \\ Y_1^T \Lambda_i & -\mathbf{R} \end{bmatrix},\]

\[\lambda = \frac{\tau (t)}{\tau_M}, \quad \mathbf{R} = \text{diag} \{ 3\mathbf{R}, 5\mathbf{R} \}\]

\[\gamma_{i_1} (\tau (t), \dot{\tau} (t)) = \text{sym} \left\{ \Pi_1^T (\tau (t)) \Pi_2 (\dot{\tau} (t)) \right\} + \text{sym} \left\{ \Pi_3^T (\tau (t)) \Pi_4 (\dot{\tau} (t)) \right\} + \text{sym} \left\{ \Pi_5^T (\tau (t)) \Pi_6 (\dot{\tau} (t)) \right\} - (1 - \tau (t)) \Pi_7 \dot{\Pi}_8 (\dot{\tau} (t)) + (1 - \tau (t)) \Pi_9 \dot{\Pi}_10 + \tau^2_M U^T \mathbf{R} U_j - (2 - \lambda) \Lambda^T \dot{\Lambda} - (1 + \lambda) \Lambda^T \dot{\Lambda},\]

Construct an augmented Lyapunov–Krasovskii functional candidate as:

\[V (t) = V_1 (t) + V_2 (t) + V_3 (t)\]

where

\[V_1 (t) = \eta^T (t) \Pi_1 \eta (t)\]

\[V_2 (t) = \int_{\tau-	au_M}^{t} \left[ \begin{array}{c} \dot{x} (s) \\ \dot{x} (s) \\ \dot{x} (s) \end{array} \right]^T \left[ \begin{array}{c} \mathbf{Q}_1 \\ \mathbf{Q}_1 \\ \mathbf{Q}_1 \end{array} \right] \left[ \begin{array}{c} x (s) \\ x (s) \\ x (s) \end{array} \right] ds + \int_{\tau-	au_M}^{t-	au_M} \left[ \begin{array}{c} \dot{x} (s) \\ \dot{x} (s) \\ \dot{x} (s) \end{array} \right]^T \left[ \begin{array}{c} \mathbf{Q}_2 \\ \mathbf{Q}_2 \\ \mathbf{Q}_2 \end{array} \right] \left[ \begin{array}{c} x (s) \\ x (s) \\ x (s) \end{array} \right] ds + \int_{\tau-	au_M}^{t-	au_M} \left[ \begin{array}{c} \dot{x} (s) \\ \dot{x} (s) \\ \dot{x} (s) \end{array} \right]^T \left[ \begin{array}{c} \mathbf{Q}_3 \\ \mathbf{Q}_3 \\ \mathbf{Q}_3 \end{array} \right] \left[ \begin{array}{c} x (s) \\ x (s) \\ x (s) \end{array} \right] ds\]

with

\[\eta (t) = \left[ \begin{array}{c} x (t-	au_M) x (t-	au (t)) \int_{t-	au (t)}^{t} x (s) ds \\ \int_{t-	au (t)}^{t} x (s) ds \\ \int_{t-	au (t)}^{t} x (s) ds \end{array} \right]^T\]

where \(\mathbf{P} \in \mathbb{R}^{7 \times 7}, \mathbf{Q}_1 \in \mathbb{R}^{6 \times 6}, \mathbf{Q}_2 \in \mathbb{R}^{6 \times 6}, \mathbf{R} \in \mathbb{R}^{6 \times 6}\) are positive definite matrices to be solved.
First, the derivatives of individual Lyapunov–Krasovskii functionals along the trajectory of (3) are computed as:

\[
\dot{V}_1(t) = \sum_{i=1}^{p} m_i(x(t)) \xi^T(t) \sum \left\{ \Pi_i^T (\tau(t)) \text{Sym} \left\{ \Pi_{i1}^T (\tau(t)) \right\} \xi(t) \right\}
\]

(15)

\[
\dot{V}_2(t) = \sum_{i=1}^{n} m_i(x(t)) \xi^T(t) \left[ \Pi_i^T \xi - \Pi_i^T (\tau(t)) Q_i \Pi_i (\tau(t)) \right] + \text{Sym} \left\{ \Pi_i^T (\tau(t)) Q_i \Pi_i (\tau(t)) \right\} \\
- (1 - \dot{\tau}(t)) \Pi_i^T (\tau(t)) Q_i \Pi_i (\tau(t)) \\
+ \sum_{i=1}^{n} \Pi_i^T \Pi_i \right\} \xi(t)
\]

(16)

\[
\dot{V}_3(t) = \tau_M \dot{x}^T(t) R \dot{x}(t) - \tau_M \int_{t-\tau_M}^{t-\tau_M} \dot{x}^T(t) R \dot{x}(t) ds \\
= \tau_M \sum_{i=1}^{p} m_i(x(t)) \dot{x}^T(t) U_i \dot{x}(t) - \tau_M \int_{t-\tau_M}^{t-\tau_M} \dot{x}^T(t) R \dot{x}(t) ds
\]

(17)

where

\[
\dot{x}(t) = \left[ x^T(t) x^T(t - \tau(t)) x^T(t - \tau_M) \right]^{1/2} \tau_M - \tau(t) \int_{t-\tau_M}^{t-\tau_M} x^T(t) ds \\
+ \frac{1}{(\tau_M - \tau(t))^2} \int_{t-\tau_M}^{t-\tau_M} x^T(t) (\tau(t)) d\tau ds \int_{t-\tau(t)}^{t} x^T(t) ds
\]

(18)

Then, by utilizing Lemma 1, the integral terms in (17) can be respectively estimated as:

\[
-\tau_M \int_{t-\tau(t)}^{t} \dot{x}^T(t) R \dot{x}(t) ds \\
\leq -\frac{\tau_M}{\tau(t)} \sum_{i=1}^{p} m_i(x(t)) \dot{x}^T(t) (e_1 - e_2)^T R (e_1 - e_2) \xi(t)
\]

(18)

\[
-\frac{3\tau_M}{\tau(t)} \sum_{i=1}^{p} m_i(x(t)) \dot{x}^T(t) (e_1 - e_2 - 2e_4)^T R (e_1 - e_2 - 2e_4) \xi(t)
\]

(18)

\[
-\frac{5\tau_M}{\tau(t)} \sum_{i=1}^{p} m_i(x(t)) \dot{x}^T(t) (e_1 - e_2 + 6e_4 - 12e_7)^T R (e_1 - e_2 + 6e_4 - 12e_7) \xi(t)
\]

(18)

Let \( \lambda = \frac{\tau(t)}{\tau_M} \), \( \Lambda_1 = [\Pi_{11}^T \Pi_{12}^T \Pi_{13}^T \Pi_{14}^T \Pi_{15}^T \Pi_{16}^T]^T \), \( \Lambda_2 = [\Pi_{14}^T \Pi_{15}^T \Pi_{16}^T]^T \), by applying Lemma 2 to deal with (18) and (19)
together, the following inequality can be obtained:

\[-\tau_M \int_{t-\tau(t)}^{t} x^T(s) R x(s) ds - \tau_M \int_{t-\tau(t)}^{t} x^T(s) R x(s) ds \leq -\frac{1}{\lambda} \sum_{i=1}^{p} m_i(x(t)) \left( \Lambda_i \xi(t) \right)^T \hat{R} \left( \Lambda_i \xi(t) \right) \]

\[-\frac{1}{1-\lambda} \sum_{i=1}^{p} m_i(x(t)) \left( \Lambda_i \xi(t) \right)^T \hat{R} \left( \Lambda_i \xi(t) \right) \]

\[\leq \sum_{i=1}^{p} m_i(x(t)) \xi^T(t) \left[ -\left( 2 - \lambda \right) \Lambda_i^T \hat{R} \Lambda_i - \left( 1 + \lambda \right) \Lambda_i^T \hat{R} \Lambda_i \right] \xi(t) \]

\[+ \left( 1 - \lambda \right) \Lambda_i^T Y_1 \hat{R}^{-1} Y_1^T \Lambda_i + \lambda \Lambda_i^T Y_2 \hat{R}^{-1} Y_2^T \Lambda_i \right] \xi(t) \]

\[\leq \sum_{i=1}^{p} m_i(x(t)) \xi^T(t) \left[ -\left( 2 - \lambda \right) \Lambda_i^T \hat{R} \Lambda_i - \left( 1 + \lambda \right) \Lambda_i^T \hat{R} \Lambda_i \right] \xi(t) \]

\[+ \left( 1 - \lambda \right) \Lambda_i^T Y_1 \hat{R}^{-1} Y_1^T \Lambda_i + \lambda \Lambda_i^T Y_2 \hat{R}^{-1} Y_2^T \Lambda_i \right] \xi(t) \]

(20)

Based on (15), (16), (17) and (20), the derivative of $V(t)$ can be calculated as follows:

\[\dot{V}(t) \leq \sum_{i=1}^{p} m_i(x(t)) \xi^T(t) \left[ \gamma_{1i}(b(t), \dot{b}(t)) + \gamma_{2i}(b(t)) \right] \xi(t) \]

where

\[\gamma_{1i}(\tau(t), \dot{\tau}(t)) = \text{Sym} \left\{ \Pi_i^T(\tau(t)) P \Pi_i(\dot{\tau}(t)) \right\} \]

\[+ \Pi_i^T Q_{i1} \Pi_i - \Pi_i^T Q_{i2} (\tau(t)) Q_{i2} (\tau(t)) \]

\[\text{Sym} \left\{ \Pi_i^T (\tau(t)) Q_{i1} \Pi_i (\dot{\tau}(t)) \right\} \]

\[+ \Pi_i^T (\tau(t)) Q_{i2} \Pi_i (\dot{\tau}(t)) \]

\[- \left( 1 - \dot{\tau}(t) \right) \Pi_i^T (\tau(t)) Q_{i1} \Pi_i (\dot{\tau}(t)) \]

\[+ \left( 1 - \dot{\tau}(t) \right) \Pi_i^T Q_{i1} \Pi_i + \tau_i^2 U_i^T \hat{R} U_i \]

\[- \left( 2 - \lambda \right) \Lambda_i^T \hat{R} \Lambda_i - \left( 1 + \lambda \right) \Lambda_i^T \hat{R} \Lambda_i \]

\[- \text{Sym} \left\{ \Lambda_i^T \left[ \lambda Y_1 + \left( 1 - \lambda \right) Y_2 \right] \Lambda_i \right\} \xi(t) \]

\[\gamma_{2i}(b(t)) = (1 - \lambda) \Lambda_i^T Y_1 \hat{R}^{-1} Y_1^T \Lambda_i + \lambda \Lambda_i^T Y_2 \hat{R}^{-1} Y_2^T \Lambda_i \]

Therefore, if $\gamma_{1i}(b(t), \dot{b}(t)) + \gamma_{2i}(b(t)) < 0$ holds, the T-S fuzzy system with time-varying delays (3) under $u(t) = 0$ is asymptotically stable. By Schur complement lemma, $\gamma_{1i}(b(t), \dot{b}(t)) + \gamma_{2i}(b(t)) < 0$ is equivalent to

\[\begin{bmatrix}
\gamma_{1i}(\tau(t), \dot{\tau}(t)) & \left( 1 - \lambda \right) \Lambda_i^T Y_1 + \lambda \Lambda_i^T Y_2 \\
Y_i^T \Lambda_i + Y_2 \Lambda_i & -\hat{R}
\end{bmatrix} < 0 \]  

(22)

If conditions $\tau(t) \in [0, \tau_M]$ and $\dot{\tau}(t) \in [\mu_1, \mu_2]$ hold, by Lemma 3, (22) is equivalent to

\[\Sigma_{1i}(0, \hat{\tau}(t)) = \begin{bmatrix}
\gamma_{1i}(0, \hat{\tau}(t)) & \Lambda_i^T Y_1 \\
Y_i^T \Lambda_i & -\hat{R}
\end{bmatrix} < 0 \]

(23)

\[\Sigma_{2i}(\tau_M, \hat{\tau}(t)) = \begin{bmatrix}
\gamma_{1i}(\tau_M, \hat{\tau}(t)) & \Lambda_i^T Y_2 \\
Y_i^T \Lambda_i & -\hat{R}
\end{bmatrix} < 0 \]

(24)

\[\Sigma_{3i}(0, \hat{\tau}(t)) = \begin{bmatrix}
-\tau_M^2 \Delta(\hat{\tau}(t)) + \gamma_{1i}(0, \hat{\tau}(t)) & \Lambda_i^T Y_1 \\
Y_i^T \Lambda_i & -\hat{R}
\end{bmatrix} < 0 \]

(25)

To reduce the conservativeness of stability analysis of T-S fuzzy system (3) with time-varying delays, piecewise membership functions $\hat{m}_i(\cdot)$ are constructed based on (7) as:

\[\hat{m}_i(\cdot) = \sum_{i=1}^{k} \sum_{\eta_1=1}^{2} \sum_{\eta_2=1}^{2} \prod_{r=1}^{p} \nu_{ri}(\xi(t)) \hat{m}_i(\eta_1, \eta_2) \]

(26)

In the following parts, the piecewise membership functions $\hat{m}_i(\cdot)$ are used to approximate the membership functions $m_i(\cdot)$ of the fuzzy model, and some less conservative membership-function-dependent stability criterion are obtained.

Let $\Delta m_i(\cdot) = m_i(\cdot) - \hat{m}_i(\cdot)$, the maximum and minimum of $\Delta m_i(\cdot)$ are denoted as $\Delta \hat{m}_i$ and $\Delta \hat{m}_i$, respectively, such that $\Delta \hat{m}_i \leq \Delta m_i(\cdot) \leq \Delta \hat{m}_i$. From the properties of the piecewise membership functions and the definition of $\Delta m_i(\cdot)$, we obtain

\[\sum_{i=1}^{p} \Delta m_i(\cdot) = 0.\]

Introducing a slack matrix $E = E^T \in \mathbb{R}^{12 \times 12}$, and the following equation is obtained:

\[\sum_{i=1}^{p} \Delta m_i(\cdot) E = 0 \]

(27)

Let $n = 1, 2, 3$, combining (27) with (23)–(25), we arrive at:

\[\dot{V}(t) \leq \sum_{i=1}^{p} m_i(x(t)) \xi^T(t) \Sigma_n \xi(t) \]

\[= \sum_{i=1}^{p} \hat{m}_i(\cdot) \xi^T(t) \Sigma_n \xi(t) \]

\[+ \sum_{i=1}^{p} \Delta m_i(\cdot) \xi^T(t) (\Sigma_n + E) \xi(t) \]

\[= \sum_{i=1}^{p} \hat{m}_i(\cdot) \xi^T(t) \Sigma_n \xi(t) \]

\[+ \sum_{i=1}^{p} (\Delta m_i(\cdot) \xi^T(t) (\Sigma_n + E) \xi(t)) \]

\[= \sum_{i=1}^{p} \hat{m}_i(\cdot) \xi^T(t) \Sigma_n \xi(t) \]

(28)
\[ + \sum_{j=1}^{p} \left( \Delta m_j(x(t)) - \Delta m_j \right) \dot{\xi}^T(t) \left( \Sigma_m + E \right) \xi(t) \]
\[ + \sum_{j=1}^{p} \Delta m_j \dot{\xi}^T(t) \left( \Sigma_m + E \right) \xi(t) \]

(28)

Furthermore, slack matrices \( 0 < F_i \in \mathbb{R}^{12 \times 12n} \), \( i = 1, 2, \ldots, p \) are introduced into (28), and then we can obtain the following equation:

\[ V'(t) \leq \sum_{j=1}^{k} \sum_{\eta=1}^{2} \sum_{\nu_{ri,j} \equiv 1} \sum_{m=1}^{n} \nu_{ri,j}(x_{ri}) \xi^T(t) \]
\[ + \left( \Delta m_j(x(t)) - \Delta m_j \right) \xi^T(t) \left( \Sigma_m + E \right) \xi(t) \]
\[ + \sum_{j=1}^{p} \Delta m_j \xi^T(t) \left( \Sigma_m + E \right) \xi(t) \]

(29)

Expanding \( \hat{m}_j(x) \) as (26) in (29), we obtain:

\[ V'(t) \leq \sum_{j=1}^{k} \sum_{\eta=1}^{2} \sum_{\nu_{ri,j} \equiv 1} \sum_{m=1}^{n} \nu_{ri,j}(x_{ri}) \xi^T(t) \]
\[ + \left( \Delta m_j(x(t)) - \Delta m_j \right) \xi^T(t) \left( \Sigma_m + E \right) \xi(t) \]
\[ + \sum_{j=1}^{p} \Delta m_j \xi^T(t) \left( \Sigma_m + E \right) \xi(t) \]

(30)

Consequently, if LMIs in (8)-(13) are feasible, and it implies that \( V'(t) < 0 \), which in turn guarantees the asymptotic stability of the T-S fuzzy system (3) under \( u(t) = 0 \). Then, the proof is completed.

Remark 3. Considering the issue of balancing conservatism and computational complexity, we do not select the delay-fractioning technique in [13] and [14] to construct Lyapunov–Krasovskii functionals. It is well known that the augmented Lyapunov–Krasovskii functionals can effectively reduce the conservatism of stability analysis. In Theorem 1, a novel augmented Lyapunov–Krasovskii functional is constructed, which has made full use of the information of time delays. Double integral terms of delays \( \frac{1}{\tau(t)} \int_{\tau(t)}^{\tau(t)} f(t) x(t) dt \) and \( \frac{1}{\tau(t)} \int_{\tau(t)}^{\tau(t)} f(t) x(t) dt \) are introduced into the augmented vector of the functional \( V_1(t) \), and the further relaxed stability criteria are obtained by considering more time-delay information in the double integral terms.

Remark 4. In order to obtain further relaxed stability results based on Lyapunov–Krasovskii functionals method, an augmented Lyapunov–Krasovskii functional was constructed in [35], which contains single, double, triple and quadruple-integral terms. Meanwhile, multiple integral inequalities were introduced to estimate the integral terms of the derivative of Lyapunov–Krasovskii functional. However, simply introducing multiple integral terms to construct complex Lyapunov–Krasovskii functionals may increase computational complexity, and it is not conducive to controller synthesis. In this paper, a novel augmented Lyapunov–Krasovskii functional with the double integral terms is constructed, and we employ the auxiliary function-based integral inequalities to estimate the integral terms. The simulation results show that the stability results obtained in this paper are less conservative. This processing approach well balances the computational complexity and conservatism, and also facilitates the design of the fuzzy controllers.

In order to further illustrate the benefits of the membership-function-dependent method, Corollary 1 is given based on the membership-function-independent method. In this case, the information of membership functions is not considered in stability analysis, and the following corollary can be obtained.

Corollary 1. For given scalars \( \tau_M > 0, \mu \in \{\mu_1, \mu_2\} \), T-S fuzzy system with time-varying delays (3) under \( u(t) = 0 \) is asymptotically stable, if there exist matrices \( 0 < P \in \mathbb{R}^{7 \times 7n} \), \( 0 < Q_1 \in \mathbb{R}^{6 \times 6n} \), \( 0 < Q_2 \in \mathbb{R}^{6 \times 6n} \), \( 0 < R \in \mathbb{R}^{3 \times 3n} \), \( Y_1 \in \mathbb{R}^{3 \times 3n} \), \( Y_2 \in \mathbb{R}^{3 \times 3n} \), such that the following inequalities hold:

\[ \Sigma_{1i}(0, \tau(t)) = \begin{bmatrix} Y_{1i} \tau(t) & \Lambda_1^T Y_1 \\ Y_{1i} \Lambda_1 & -R \end{bmatrix} < 0 \]

(31)

\[ \Sigma_{2i}(\tau_M, \tau(t)) = \begin{bmatrix} Y_{1i} \tau_M \tau(t) & \Lambda_2^T Y_2 \\ Y_{1i} \Lambda_2 & -R \end{bmatrix} < 0 \]

(32)

\[ \Sigma_{3i}(0, \tau(t)) = \begin{bmatrix} \tau_M \Delta(t) \tau(t) & Y_{1i} \tau(t) \\ Y_{1i} \Lambda_1 & -R \end{bmatrix} < 0 \]

(33)

where

\[ Y_{1i} \tau(t), \tau(t) = Sym\left\{ \Pi_{1i}^T (\tau(t)) P \Pi_{1i} (\tau(t)) \right\} \]

\[ + \Pi_{1i}^T Q_1 \Pi_{1i} - \Pi_{1i}^T (\tau(t)) Q_2 \Pi_{1i} (\tau(t)) \]

\[ + \Pi_{1i}^T (\tau(t)) Q_2 \Pi_{1i} (\tau(t)) + \Pi_{1i}^T (\tau(t)) Q_3 \Pi_{1i} (\tau(t)) \]

\[ - (1 - \tau(t)) \Pi_{1i}^T (\tau(t)) Q_1 \Pi_{1i} (\tau(t)) \]

\[ + (1 - \tau(t)) \Pi_{1i}^T (\tau(t)) Q_2 \Pi_{1i} (\tau(t)) \]

\[ + (1 - \tau(t)) \Pi_{1i}^T Q_2 \Pi_{1i} + \tau_M \Delta(t) \]

\[ - (2 - \lambda) \Lambda_1^T R \Lambda_1 - (1 + \lambda) \Lambda_2^T \]

\[ - Sym\left\{ \Lambda_1^T [\lambda Y_1 + (1 - \lambda) Y_2] \Lambda_2 \right\} \]
Controller design

In this subsection, the stabilization problem for the T-S fuzzy closed-loop system (6) with time-varying delays is considered. On the basis of Theorem 1, novel state feedback controllers are presented under the imperfect premise matching design scheme. In order to reduce the conservatism of design conditions, some piecewise membership functions are employed to approximate the membership functions, and then some relaxed membership-function-dependent stabilization results are obtained by considering the boundary information of the membership functions.

Theorem 2. For given scalars \( \tau_M > 0, \mu \in \{\mu_1, \mu_2\} \) and tuning parameter \( \sigma \), T-S fuzzy closed-loop system (6) with time-varying delays is asymptotically stable, if there exist matrices \( 0 < \bar{P} \in \mathbb{R}^{13\times13}, 0 < \bar{Q}_1 \in \mathbb{R}^{6\times6}, 0 < \bar{Q}_2 \in \mathbb{R}^{6\times6}, 0 < \bar{K} \in \mathbb{R}^{13\times13}, \bar{E} = \bar{E}^T \in \mathbb{R}^{13\times13} \), and any matrices \( \bar{Y}_1 \in \mathbb{R}^{13\times3n}, \bar{X} \in \mathbb{R}^{3n}, G_j \in \mathbb{R}^{3\times3n}, \) such that the following inequalities hold:

\[
\bar{X} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}, \quad i = 1, 2, \ldots, 9
\]

3.2 Controller design

\[
\Pi_{18} = \begin{bmatrix} 0 & 0 & 0 & 0 & e_i^T \end{bmatrix}^T, \\
\Pi_{19} = \begin{bmatrix} 0 & 0 & 0 & 0 & e_i^T \end{bmatrix}^T, \\
\Pi_{20} = \begin{bmatrix} 0 & 0 & 0 & 0 & e_i^T \end{bmatrix}^T, \\
\Pi_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 & e_i^T \end{bmatrix}^T, \\
\Pi_{22} = \begin{bmatrix} 0 & 0 & 0 & 0 & e_i^T \end{bmatrix}^T
\]

\[
U_i = A_i e_i + A_{ii} e_{ii} \\
e_i = \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix}^T, i = 1, 2, \ldots, 9
\]
The corresponding state feedback control gain matrices are given by $K_j = G_j X^{-1}, j = 1, 2, ..., c$.

Where the piecewise membership functions $\tilde{\omega}_{ij}(x)$ are defined in (49), $k$ is the number of divided state subspaces for the piecewise membership functions, $\xi_{ij}(t)$ represents the apexes of the $i$-th state subspace of $\Gamma$. $\Delta \tilde{\omega}_{ij}, \Delta \omega_{ij}$ are predefined constant scalars satisfying $\Delta \omega_{ij} \leq \Delta \tilde{\omega}_{ij}(x) \leq \Delta \omega_{ij}$. Concurrently, some other matrices are defined as follows:

$$\Sigma_{1ij}(0, \tau(t)) = \begin{bmatrix} \tilde{\gamma}_{1ij}(0, \tau(t)) & \tilde{A}_{1}^T \tilde{Y}_{1} \\ \tilde{Y}_{1} \tilde{A}_{1} & -\tilde{R} \end{bmatrix}$$

$$\Sigma_{2ij}(\tau(t), \tau(t)) = \begin{bmatrix} \tilde{\gamma}_{1ij}(\tau(t), \tau(t)) & \tilde{A}_{2}^T \tilde{Y}_{2} \\ \tilde{Y}_{2} \tilde{A}_{2} & -\tilde{R} \end{bmatrix}$$

$$\Sigma_{3ij}(0, \tau(t)) = \begin{bmatrix} -\tau_{M}^2 \Delta(\tau(t)) + \tilde{\gamma}_{1ij}(0, \tau(t)) & \tilde{A}_{1}^T \tilde{Y}_{1} \\ \tilde{Y}_{1} \tilde{A}_{1} & -\tilde{R} \end{bmatrix}$$

$$\lambda = \frac{\tau(t)}{\tau_{M}}$$

$$\tilde{R} = \text{diag} \{ R, 3R, 5R \}$$

$$\tilde{\varphi}_{1ij}(\tau(t), \tau(t)) = \text{sym} \left\{ \tilde{n}_{7}^T(\tau(t)) \tilde{P} \tilde{n}_{2}(\tau(t)) \right\}$$

$$+ \tilde{n}_{4}^T \tilde{P} \tilde{n}_{3} - \tilde{n}_{4}^T(\tau(t)) \tilde{P} \tilde{n}_{4}(\tau(t))$$

$$+ \text{sym} \left\{ \tilde{n}_{5}^T(\tau(t)) \tilde{P} \tilde{n}_{5}(\tau(t)) + \tilde{n}_{7}^T(\tau(t)) \tilde{P} \tilde{n}_{7}(\tau(t)) \right\}$$

$$- (1 - \tau(t)) \tilde{n}_{9}^T(\tau(t)) \tilde{P} \tilde{n}_{9}(\tau(t))$$

$$+ (1 - \tau(t)) \tilde{n}_{10}^T \tilde{P} \tilde{n}_{10} + \tau_{M}^2 \tilde{e}_{10}^T \tilde{R} \tilde{e}_{10}$$

$$- (2 - \lambda) \tilde{A}_{1}^T \tilde{R} \tilde{A}_{1} - (1 + \lambda) \tilde{A}_{2}^T \tilde{R} \tilde{A}_{2}$$

$$- \text{sym} \left\{ \tilde{A}_{1}^T \tilde{R} \tilde{A}_{1} + (1 - \lambda) \tilde{A}_{2}^T \tilde{R} \tilde{A}_{2} \right\}$$

$$+ \text{sym} \left\{ (\tilde{\tau}_{10}^T + \sigma \tilde{e}_{10}) (A \tilde{X} \tilde{e}_{1} + B \tilde{G} \tilde{e}_{1} + A_{d} \tilde{X} \tilde{e}_{2} - \tilde{X} \tilde{e}_{10}) \right\}$$

$$\tilde{\Pi}_{1}(\tau(t)) = \begin{bmatrix} \tilde{e}_{1}^T \tilde{e}_{1}^T \tilde{e}_{2}^T \tilde{e}_{2}^T \tau(t) \tilde{e}_{6}^T(\tau_{M} - \tau(t)) \tilde{e}_{6}^T \tau(t) \tilde{e}_{8}^T \\ (\tau_{M} - \tau(t)) \tilde{e}_{8}^T \end{bmatrix}^T$$

$$\tilde{\Pi}_{2}(\tau(t)) = \begin{bmatrix} \tilde{e}_{10}^T \tilde{e}_{10}^T (1 - \tau(t)) \tilde{e}_{5}^T \tilde{e}_{5}^T (1 - \tau(t)) \tilde{e}_{5}^T \\ (1 - \tau(t)) \tilde{e}_{5}^T - \tilde{e}_{5}^T - (1 - \tau(t)) \tilde{e}_{5}^T - \tau(t) \tilde{e}_{5}^T \\ (1 - \tau(t)) \tilde{e}_{5}^T - \tilde{e}_{5}^T + \tau(t) \tilde{e}_{5}^T \end{bmatrix}^T$$

$$\tilde{e}_{i} = \begin{bmatrix} 0 & 0 & 0 & 0 & \tilde{e}_{i}^T \end{bmatrix}^T, \quad i = 1, 2, ..., 10$$
Proof. For the convenience of presentation, the following matrices are defined as:

\[
\tilde{\xi}(\tau) = \begin{bmatrix} x^T(t) x^T(t - \tau(t)) x^T(t - \tau_M) \frac{1}{\tau_M - \tau(t)} \\
\frac{1}{\tau^2(\tau)} \int_{t_0}^{t-\tau_M} x^T(u) du \frac{1}{\tau(t)} \int_{t-\tau_M}^{t} x^T(s) ds \int_{t_0}^{t} x^T(s) ds \end{bmatrix}^T 
\]

(40)

\[
\Psi(t) = \begin{bmatrix} A(t) + B(t) K(t) & A_L(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \]

(41)

On the basis of Theorem 1, it follows from (21) and (40) that

\[
\hat{\dot{V}}(t) \leq \sum_{i=1}^{p} \sum_{j=1}^{q_i} m_j(x(t)) b_j(x(t)) \tilde{\xi}^T(t) 
\times \left[ \hat{\gamma}_1(b(t), b(t)) + \hat{\gamma}_2(\dot{b}(t)) \right] \tilde{\xi}(t)
\]

(42)

where

\[
\hat{\gamma}_1(t, \dot{t}(t)) = \text{Sym} \left\{ \tilde{\Pi}_1^T(t) \tilde{P} \tilde{\Pi}_2(t) \right\} 
+ \tilde{\Pi}_1^T \tilde{Q}_1 \tilde{\Pi}_3 - \tilde{\Pi}_4^T(t) \tilde{Q}_2 \tilde{\Pi}_4(t) 
+ \text{Sym} \left\{ \tilde{\Pi}_5^T(t) \tilde{Q}_1 \tilde{\Pi}_6(t) \right\} 
+ \tilde{\Pi}_7^T(t) \tilde{Q}_1 \tilde{\Pi}_6(t) 
- (1 - \dot{t}(t)) \tilde{\Pi}_1^T(t) \tilde{Q}_1 \tilde{\Pi}_9(t) 
+ (1 - \dot{t}(t)) \tilde{\Pi}_1^T \tilde{Q}_2 \tilde{\Pi}_9(t) 
+ \text{Sym} \left\{ \tilde{\Pi}_7^T(t) \tilde{Q}_1 \tilde{\Pi}_9(t) \right\} 
- (2 - \lambda) \tilde{\Lambda}_1^T \tilde{R} \tilde{A}_1 - (1 + \lambda) \tilde{\Lambda}_2^T \tilde{R} \tilde{A}_2 
- \text{Sym} \left\{ \tilde{\Lambda}_1^T \left[ \lambda \tilde{Y}_1 + (1 - \lambda) \tilde{Y}_2 \right] \tilde{A}_2 \right\}
\]

(43)

From (40) and (41), we obtain \(\Psi(t)\tilde{\xi}(t) = 0\). By applying the statements of (1) and (4) of Lemma 4, the closed-loop T-S fuzzy system (6) is asymptotically stable if there exists \(L \in \mathbb{R}^{n_{X} \times 10_t\tau} \) such that

\[
\hat{\gamma}_1(b(t), \dot{b}(t)) + \hat{\gamma}_2(b(t)) + L \psi(t) + \psi^T(t) L^T < 0
\]

(44)

Let

\[
L = \begin{bmatrix} X^{-T} & 0 & \ldots & 0 \end{bmatrix} \sigma X^{-T}, \quad \sigma = \text{diag}(\text{XXXXXXX}), \quad X \in \mathbb{R}^{n_{X} \times n_{X}} \text{ is any invertible matrix, and } \sigma \text{ is a tuning parameter.} \]

By pre-and post-multiplying both sides of (43) with \(\varpi^T\) and \(\varpi\), respectively, we can obtain

\[
\varpi^T \left( \hat{\gamma}_1(b(t), \dot{b}(t)) + \hat{\gamma}_2(b(t)) + L \psi(t) + \psi^T(t) L^T \right) 
\varpi = \hat{\gamma}_{ij}(t, \dot{t}(t)) + \hat{\gamma}_2(b(t)) < 0
\]

(44)

where

\[
\hat{\gamma}_{ij}(t, \dot{t}(t)) = \text{Sym} \left\{ \tilde{\Pi}_1^T(t) \tilde{P} \tilde{\Pi}_2(t) \right\} 
+ \tilde{\Pi}_1^T \tilde{Q}_1 \tilde{\Pi}_3 - \tilde{\Pi}_4^T(t) \tilde{Q}_2 \tilde{\Pi}_4(t) 
+ \text{Sym} \left\{ \tilde{\Pi}_5^T(t) \tilde{Q}_1 \tilde{\Pi}_6(t) \right\} 
+ \tilde{\Pi}_7^T(t) \tilde{Q}_1 \tilde{\Pi}_6(t) 
- (1 - \dot{t}(t)) \tilde{\Pi}_1^T(t) \tilde{Q}_1 \tilde{\Pi}_9(t) 
+ (1 - \dot{t}(t)) \tilde{\Pi}_1^T \tilde{Q}_2 \tilde{\Pi}_9(t) 
+ \text{Sym} \left\{ \tilde{\Pi}_7^T(t) \tilde{Q}_1 \tilde{\Pi}_9(t) \right\} 
- (2 - \lambda) \tilde{\Lambda}_1^T \tilde{R} \tilde{A}_1 - (1 + \lambda) \tilde{\Lambda}_2^T \tilde{R} \tilde{A}_2 
- \text{Sym} \left\{ \tilde{\Lambda}_1^T \left[ \lambda \tilde{Y}_1 + (1 - \lambda) \tilde{Y}_2 \right] \tilde{A}_2 \right\}
\]

(44)

Here, if LMI in (44) holds, the closed-loop T-S fuzzy system (6) is asymptotically stable. By Schur complement lemma, (44) is equivalent to

\[
\begin{bmatrix} \hat{\gamma}_{ij}(t, \dot{t}(t)) & (1 - \lambda) \tilde{\Lambda}_1^T \tilde{Y}_1 + \lambda \tilde{\Lambda}_2^T \tilde{Y}_2 \n \tilde{\Lambda}_1 \tilde{A}_2 \tilde{A}_2 \n - \tilde{R} \n \tilde{Y}_1 \tilde{A}_2 \tilde{A}_2 \end{bmatrix} < 0
\]

(45)

Based on lemma 3, for \(\tau(t) \in [0, \tau_M] \), \(\dot{t}(t) \in \{\mu_1, \mu_2\} \), (45) is equivalent to

\[
\Sigma_{ij}(0, \dot{t}(t)) = \begin{bmatrix} \hat{\gamma}_{ij}(0, \dot{t}(t)) & \tilde{\Lambda}_1^T \tilde{Y}_1 \n \tilde{Y}_1 \tilde{A}_1 \n - \tilde{R} \end{bmatrix}
\]

(46)
To reduce the conservatism of stabilization conditions of the closed-loop T-S fuzzy system (6), piecewise membership functions \( \tilde{\omega}_{ij}(x) \) are constructed based on (7) as:

\[
\hat{\omega}_{ij}(x) = \sum_{i=1}^{2} \sum_{\eta=1}^{2} \prod_{r=1}^{\eta} \nu_{\eta,i}(x_r) \hat{\omega}_{ij}\left(x_{\eta_1},...,x_{\eta_\eta}\right)
\]

In the following parts, the piecewise membership functions \( \hat{\omega}_{ij}(x) \) are used to approximate the product of membership functions \( \omega_{ij}(x) = m_i(x)d_j(x) \), and some less conservative membership-function-dependent stabilization conditions are obtained.

Let \( \Delta \omega_{ij}(x) = \omega_{ij}(x) - \hat{\omega}_{ij}(x) \), the maximum and minimum of \( \Delta \omega_{ij}(x) \) are denoted as \( \Delta \omega_{ij} \) and \( \Delta \omega_{ij}^- \), respectively, such that \( \Delta \omega_{ij} \leq \Delta \omega_{ij}(x) \leq \Delta \omega_{ij}^- \). Introducing slack matrices \( 0 < F_{ij} \in \mathbb{R}^{13 \times 13} \), \( E = E^T \in \mathbb{R}^{12 \times 12} \), it can be found that

\[
\hat{V}(t) = \sum_{j=1}^{p} \sum_{i=1}^{c} \hat{\omega}_{ij}(x(t)) \overline{\xi}(t) \Sigma_{ij} \overline{\xi}(t) + \sum_{j=1}^{p} \sum_{i=1}^{c} \Delta \omega_{ij} \overline{\xi}(t) (\Sigma_{ij} + \overline{E}) \overline{\xi}(t) + \sum_{j=1}^{p} \sum_{i=1}^{c} \left( \Delta \omega_{ij} \overline{\xi}(t) \right) \left( \Sigma_{ij} + \overline{E} - F_{ij} \right) \overline{\xi}(t)
\]

where \( n = 1, 2, 3 \).

Expanding the piecewise membership functions \( \hat{\omega}_{ij}(x(t)) \) as (49) in (50), we obtain:

\[
\hat{V}(t) \leq \sum_{j=1}^{p} \sum_{i=1}^{c} \sum_{\eta=1}^{2} \sum_{\eta_1=1}^{2} \sum_{\eta_2=1}^{2} \prod_{r=1}^{\eta} \nu_{\eta,i}(x_r) \overline{\xi}_{\eta_1}(t) \Sigma_{ij} \overline{\xi}_{\eta_2}(t) + \sum_{j=1}^{p} \sum_{i=1}^{c} \Delta \omega_{ij} \overline{\xi}_{\eta_1}(t) (\Sigma_{ij} + \overline{E}) \overline{\xi}_{\eta_2}(t) + \sum_{j=1}^{p} \sum_{i=1}^{c} \left( \Delta \omega_{ij} \overline{\xi}_{\eta_1}(t) \right) \left( \Sigma_{ij} + \overline{E} - F_{ij} \right) \overline{\xi}_{\eta_2}(t)
\]

4 | SIMULATION EXAMPLES

In this subsection, four numerical examples are presented to demonstrate the advantages and effectiveness of the proposed stability criteria and stabilization approach. For comparison, the first two numerical examples effectively demonstrate that the membership-function-dependent stability results presented in this paper is less conservative, and the last two simulation examples show the effectiveness of the fuzzy state feedback controllers presented under the imperfect premise matching technique. In Tables 2 and 3, “–” means results are not provided.

### Table 1
| Methods | \( \mu = 0 \) | \( \mu = 0.1 \) | \( \mu = 0.5 \) | Number of variables |
|---------|---------------|---------------|---------------|------------------|
| Theorem 1 (\( k = 20 \)) | 14.0290 | 8.7800 | | 295.2 \^2 + 28 \^2 |

Therefore, if LMI in (34)–(39) are feasible, and it implies that \( \hat{V}(t) < 0 \), which in turn guarantees the asymptotic stability of the closed-loop T-S fuzzy system (6). This completes the proof of Theorem 2.
Example 1. Consider the following non-linear system with time-varying delay given in [36]:

\[
\begin{align*}
\dot{x}_1(t) &= 0.5 \left( 1 - \sin^2 (\theta(t)) \right) x_2(t) \\
- x_1(t - \tau(t)) - (1 + \sin^2 (\theta(t))) x_1(t) \\
\dot{x}_2(t) &= \text{sgn} \left( \left( \theta(t) - \frac{\pi}{2} \right) \left( 0.9 \cos^2 (\theta(t)) - 1 \right) \right) \\
x_1(t - \tau(t)) - x_2(t - \tau(t)) - (0.9 + 0.1 \cos^2 (\theta(t))) x_2(t)
\end{align*}
\]

(52)

which can be described by the following two-rule fuzzy model

**Rule 1:** If \( \theta(t) \) is \( \pm \frac{\pi}{2} \), THEN \( \dot{x}(t) = A_1 x(t) + A_2 x(t - \tau(t)) \)

**Rule 2:** If \( \theta(t) = 0 \), THEN \( \dot{x}(t) = A_2 x(t) + A_2 x(t - \tau(t)) \)

where

\[
A_1 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}
\]

and the membership functions are defined by

\[
m_1(x_1(t)) = 1 - \frac{1}{1 + \exp \left( -\frac{x_1}{12} \right)},
\]

\[
m_2(x_1(t)) = 1 - m_1(x_1(t))
\]

As the membership functions \( m_i(x_1(t)) \), \( i = 1, 2 \) depend on the system state \( x_1(t) \), it is logical to construct the piecewise membership functions \( \hat{m}_i(x) \) in the form of (26) depending on \( x_1 \) as \( \hat{m}_i(x_1) = \sum_{i=1}^{k} \sum_{l=1}^{2} \nu_{1l_i}(x_1) \hat{m}_i(x_1) \). First, the state space \( x_1(t) \in [a, b] \) is divided into \( k \) connected state subspaces, and the \( l \)-th subspace is described by \( \left( x_1 - \frac{a + \frac{a + b}{k} (l - \frac{k}{2})}{k} \right) \leq x_1 \leq \left( x_1 - \frac{a + \frac{a + b}{k} (l - \frac{k}{2})}{k} \right) \), \( l = 1, 2, \ldots, k \), where \( a \) and \( b \) are positive real numbers. In this example, we select \( \nu_{11}(x_1) = 1 - \frac{x_1 - x_1}{x_1 - x_1} \), \( \nu_{12}(x_1) = 1 - \nu_{11}(x_1) \), where \( x_1 = \left( x_1 - \frac{a + \frac{a + b}{k} (l - \frac{k}{2})}{k} \right) \), \( x_2 = \left( x_2 - \frac{a + \frac{a + b}{k} (l - \frac{k}{2})}{k} \right) \). In addition, \( a = b = 10, k = 20 \) are employed, respectively. Therefore, based on the definition of \( \Delta m_i(x) = m_i(x) - \hat{m}_i(x) \), we can obtain the minimum and maximum of \( \Delta m_i(x) \) as shown in Table 1.

The example is introduced for conservatism comparison, and an open-loop delayed fuzzy system is explored to demonstrate the improvements of the developed methods. This system is also considered in [1, 11, 17, 36–42]. The objective is to calculate the maximum allowable delay \( \tau_M \) which guarantees the asymptotic stability of the system, and the conservatism is evaluated by the calculated \( \tau_M \). Under the same conditions, larger \( \tau_M \) usually means wider stability regions.

From Table 2, the maximum allowable delay \( \tau_M \) obtained from Theorem 1 with \( k = 20 \) is larger than the results given in [1, 11, 17, 36–42] at different \( \mu \) values, this shows that the membership-function-dependent stability criteria given in Theorem 1 are less conservative than those proposed in the aforementioned literatures.

Furthermore, as shown in Table 2, the maximum allowable delay \( \tau_M \) obtained by Theorem 1 is much larger than that by Corollary 1, which demonstrates that the membership-function-dependent analysis method can effectively reduce conservatism. Compared with the membership-function-independent approach, the membership-function-dependent method can obtain further relaxed results.

Example 2. Consider the T-S fuzzy time-delay system with \( u(t) = 0 \) given in [11], which is in the form of (3) with two plant rules

**Rule 1:** If \( x_1(t) \) is \( F_1^1 \), THEN \( \dot{x}(t) = A_{11} x(t) + A_{12} x(t - \tau(t)) \)

**Rule 2:** If \( x_1(t) \) is \( F_1^2 \), THEN \( \dot{x}(t) = A_{21} x(t) + A_{22} x(t - \tau(t)) \)

(53)

where

\[
A_1 = \begin{bmatrix} -2.1 & 0.1 \\ -0.2 & -0.9 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} -1.1 & 0.1 \\ -0.8 & -0.9 \end{bmatrix}
\]

\[
A_2 = \begin{bmatrix} -1.9 & 0 \\ -0.2 & -1.1 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} -0.9 & 0 \\ -1.1 & -1.2 \end{bmatrix}
\]

and the membership functions are defined by

\[
m_1(x_1(t)) = 1 - \frac{1}{1 + \exp \left( -\frac{x_1}{12} \right)},
\]

\[
m_2(x_1(t)) = 1 - m_1(x_1(t))
\]

According to Example 1, we select the number of divided state subspaces for the piecewise membership functions \( k = 20 \), moreover, the minimum and maximum of \( \Delta m_i(x) \) are obtained as shown in Table 1. In this example, the maximum allowable delays \( \tau_M \) are obtained, which guarantee the asymptotic stability of the T-S fuzzy system with time-varying delays (53) by different literatures. As shown in Table 3, we choose \( \mu = 0, 0.1 \) and 0.5, respectively, and the maximum allowable delays \( \tau_M \) is obtained via Theorem 1 at 5.6897, 4.9091 and 3.6397, respectively, which are larger than the results given in [10, 11, 43, 44, 46]. This indicates that the membership-function-dependent stability criteria given in Theorem 1 are less conservative than those proposed in the aforementioned literature.

Remark 5. Comparing with the existing results, the number of utilized decision variables of Theorem 1 is larger than the reported ones listed in Tables 2 and 3. The main reason for obtaining such a larger number is that Theorems 1
is obtained based on membership-function-dependent method, the slack matrices
\( 0 < F_i \in \mathbb{R}^{12 \times 12} \), \( i = 1, 2, \ldots, p \), \( E = E^T \in \mathbb{R}^{12 \times 12} \) are introduced to derive the further relaxed membership-function-dependent stability results, this greatly increases the computational complexity of Theorem 1. In fact, the number of utilized decision variables of Corollary 1 is smaller than the ones of [11, 17, 38] and Theorem 1, because no slack matrix is employed in Corollary 1. However, as shown in Table 2 and 3, the proposed criteria provide larger delay bounds than the previous results by sacrificing a larger number of decision variables. It is important to balance conservativeness and computational complexity in the membership-function-dependent method. This issue should be investigated further.

**Example 3.** In order to prove the effectiveness of the imperfect-premise-matching-based fuzzy state feedback controllers presented in Theorem 2, considering the following modified truck-trailer model with time-varying delays given in [37]:

\[
\begin{align*}
\dot{x}_1(t) &= -\frac{\tilde{v}^2}{L_0} x_1(t) - (1 - a) \frac{\tilde{v}^2}{L_0} x_1(t - \tau(t)) + \frac{\tilde{v}^2}{L_0} u(t) \\
\dot{x}_2(t) &= a \frac{\tilde{v}^2}{L_0} x_1(t) + (1 - a) \frac{\tilde{v}^2}{L_0} x_1(t - \tau(t)) \\
\dot{x}_3(t) &= \frac{\tilde{v}^2}{L_0} \sin \left( x_3(t) + a \frac{\tilde{v}^2}{2L} x_1(t) + (1 - a) \frac{\tilde{v}^2}{2L} x_1(t - \tau(t)) \right)
\end{align*}
\]

where \( x_1(t) \) is the angular difference between the truck and trailer, \( x_2(t) \) is the angle of the trailer, and \( x_3(t) \) is the vertical position of rear end of the trailer; \( l \) and \( L \) are the lengths of truck and trailer, and \( v \) is the constant speed of backing up. The model parameters are given as \( l = 2.8, L = 5.5, \tau = 1.0, a = 0.7, \tilde{v} = 2.0, t_0 = 0.5 \).

Let \( \tilde{\theta}(t) = x_2(t) + a \frac{\tilde{v}^2}{2L} x_1(t) + (1 - a) \frac{\tilde{v}^2}{2L} x_1(t - \tau(t)) \), \( x(t) = [x_1(t), x_2(t), x_3(t)]^T \), and the nonlinear truck-trailer system can be approximated by the T-S fuzzy model (3) with two plant rules:

**Rule 1:** IF \( \tilde{\theta}(t) \) is 0 rad, THEN \( \dot{x}(t) = A_1 x(t) + A_{d1} (x(t) - \tau(t)) + B_1 u(t) \)

**Rule 2:** IF \( \tilde{\theta}(t) \) is \( -\pi \) rad or \( \pi \) rad, THEN \( \dot{x}(t) = A_1 x(t) + A_{d2} (x(t) - \tau(t)) + B_2 u(t) \) where

\[
\begin{align*} 
A_1 &= \begin{bmatrix} -a \frac{\tilde{v}^2}{L_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & A_{d1} &= \begin{bmatrix} -(1 - a) \frac{\tilde{v}^2}{L_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
A_2 &= \begin{bmatrix} \frac{\tilde{v}^2}{L_0} \\ 0 \\ 0 \end{bmatrix}, & A_{d2} &= \begin{bmatrix} (1 - a) \frac{\tilde{v}^2}{L_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
B_1 &= \begin{bmatrix} \frac{\tilde{v}^2}{L_0} \\ 0 \\ 0 \end{bmatrix}, & B_2 &= \begin{bmatrix} \frac{\tilde{v}^2}{L_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{align*}
\]

Based on the imperfect premise matching method, the two-rule fuzzy controllers are presented:

**Rule 1:** IF \( \tilde{\theta}(t) \) is 0 rad, THEN \( u(t) = K_1 x(t) \)

**Rule 2:** IF \( \tilde{\theta}(t) \) is \( -\pi \) rad or \( \pi \) rad, THEN \( u(t) = K_2 x(t) \)

In this example, the following simple membership functions are selected for the imperfect-premise-matching-based fuzzy controllers:

\[
m_1(x) = 1 - \frac{1}{1 + \exp \left(-3 \left(x - \frac{1}{2} \pi \right)\right)} \\
m_2(x) = 1 - m_1(x)
\]

Utilizing the calculation method in Example 1, the minimum and maximum of \( \Delta \omega_{ij}(x_k) \) with \( k = 20 \) are shown in Table 4, which are obtained by the six different controllers:

**Theorem 2 (Example 3)**

| \( i \) | \( \Delta \omega_{ij}(x_1) \) | \( \Delta \omega_{ij}(x_2) \) |
|---|---|---|
| 1 | \( -4.3837 \times 10^{-2} \) | \( 6.0457 \times 10^{-2} \) |
| 2 | \( -5.6831 \times 10^{-2} \) | \( 8.4357 \times 10^{-2} \) |

**Table 4** The minimum and maximum of \( \Delta \omega_{ij}(x_k) \) with \( k = 20 \)

\[
A_{d2} = \begin{bmatrix} -(1 - a) \frac{\tilde{v}^2}{L_0} & 0 & 0 \\ (1 - a) \frac{\tilde{v}^2}{L_0} & 0 & 0 \\ (1 - a) \frac{d \tilde{v}^2}{2L_0} & \frac{\tilde{v}^2}{L_0} & 0 \end{bmatrix}, \\
B_2 = \begin{bmatrix} \frac{\tilde{v}^2}{L_0} \\ 0 \\ 0 \end{bmatrix}
\]

The truck-trailer model is also studied in [29, 37, 38, 44, 46]. The objective is to calculate the maximum allowable delay \( \tau_M \) which guarantees the asymptotic stability of the closed-loop system, and the conservatism of stabilization conditions is evaluated by the calculated \( \tau_M \). The maximum allowable delays \( \tau_M \) are shown in Table 5, which are obtained by the six different approaches.

As shown in Table 5, the maximum allowable delays \( \tau_M \) calculated in [29, 37, 38, 44, 46] are 4465, 3.4815, 246.0938, 8.1951 and 262.4908, respectively, which guarantee the asymptotic stability of the closed-loop system. However, the maximum allowable delay \( \tau_M \) calculated by Theorem 2 is 66,691, which is far bigger than the above ones. Consequently, the...
TABLE 5 The maximum allowable delay \( \tau_M \) (Example 3)

| Methods | \( \tau_M \) |
|---------|-------------|
| \{1\}[37] \{1\}[44] (Corollary 1) | 3.4815 |
| \{1\}[44] | 8.1951 |
| \{1\}[38] \{1\}[38] \{1\}[38] (\( \delta = 1.4 \)) | 246.0938 |
| \{1\}[46] \{1\}[46] (\( \delta = 1.8 \)) | 262.4908 |
| \{1\}[29] \{1\}[29] (\( g = 20 \)) | 4465 |
| Theorem 2 (\( k = 20 \)) | 66691 |

FIGURE 2 State responses of the closed-loop system for Example 3

proposed membership-function-dependent stabilization conditions can obtain further relaxed results.

Especially, assuming \( \tau = 12, \mu = \mu_2 = -\mu_1 = 0.2 \) and \( \sigma = 0.05 \), the imperfect-premise-matching-based fuzzy state feedback controllers are obtained based on Theorem 2:

\[
K_1 = G_1X^{-1} = \begin{bmatrix} 9.0444 & -18.2054 & 2.1799 \end{bmatrix},
K_2 = G_2X^{-1} = \begin{bmatrix} 14.6220 & -30.2457 & 3.6886 \end{bmatrix}
\]

For simulation, we choose the initial condition \( \phi(0) = [0.5\pi, 0.75\pi, -5]^T \) and \( \tau(t) = 6 + 6\sin(t) \). The state responses and control signals of the closed-loop system are shown in Figures 2 and 3, respectively. Figure 2 shows that all the state trajectories go to zero as time increases. Therefore, the closed-loop system is asymptotically stable under the above controllers, and the effectiveness of the imperfect-premise-matching-based controller design approach in Theorem 2 is illustrated.

Remark 6. It can be seen from Example 3 that the approaches proposed in this paper can obtain more relaxed results. Moreover, the fuzzy controllers presented in [29, 37, 38, 44, 46] should employ the same membership functions with the fuzzy models, which is not required in Theorem 2. In Example 3, we select some simpler membership functions to replace the complex ones in the fuzzy models, and the complexity and the implementation cost of the controllers are reduced.

Example 4. In order to further illustrate the advantages and effectiveness of the imperfect-premise-matching-based fuzzy state feedback controllers presented in Theorem 2, considering the following continuous stirred tank reactor (CSTR) system given in [47, 48], the T-S fuzzy models with three rules for CSTR with time-varying delays can be represented as follows:

Rule 1: IF \( x_2(t) \) is 0.8862 (temperature is low), THEN \( \dot{x}_3(t) = A_1 \dot{x}_3(t) + A_{1d} \dot{x}(t - \tau(t)) + B_1 u(t) \)

Rule 2: IF \( x_2(t) \) is 2.7520 (temperature is middle), THEN \( \dot{x}_3(t) = A_2 \dot{x}_3(t) + A_{2d} \dot{x}(t - \tau(t)) + B_2 u(t) \)

Rule 3: IF \( x_2(t) \) is 4.7052 (temperature is high), THEN \( \dot{x}_3(t) = A_3 \dot{x}_3(t) + A_{3d} \dot{x}(t - \tau(t)) + B_3 u(t) \) where \( \dot{x}(t) = x(t) - x_d, \delta x(t - \tau(t)) = x(t - \tau(t)) - x_d, \delta u(t) = u(t) - u_d \), and \( (x_d, u_d) \) is an expected operating point, and

\[
A_1 = \begin{bmatrix} -1.4274 & 0.0757 \\ 1.4189 & -0.9442 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2.0508 & 0.3958 \\ -6.4066 & 1.6168 \end{bmatrix},
A_3 = \begin{bmatrix} -4.5279 & 0.3167 \\ -26.2228 & 0.9837 \end{bmatrix},
A_{1d} = A_{2d} = A_{3d} = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.25 \end{bmatrix},
B_1 = B_2 = B_3 = \begin{bmatrix} 0 \\ 0.3 \end{bmatrix}
\]

The membership functions for this fuzzy system are as the same as those defined in [47, 48]

\[
m_1(x_2) = \begin{cases} 1 & \text{if } x_2 \leq 0.8862 \\ 1 - \frac{x_2 - 0.8862}{2.7520 - 0.8862} & \text{if } 0.8862 < x_2 < 2.7520 \\ 0 & \text{if } x_2 \geq 2.7520 \end{cases}
\]

FIGURE 3 Control signals of the closed-loop system for Example 3
Remark 8. From Examples 3 and 4, it can be proved that the imperfect premise matching method is the extension of the traditional PDC technique and has the ability to deal with more general T-S fuzzy time-delay systems.
5 CONCLUSIONS

In this paper, the membership-function-dependent stability and stabilization for T-S fuzzy systems with time-varying delays are investigated. In Theorem 1, a new augmented Lyapunov–Krasovskii functional with more time-delay information is constructed, and some relaxed stability criteria are obtained by combining with integral inequality technique and the reciprocally convex combination inequality. In order to obtain further relaxed stability criteria, some piecewise membership functions are constructed to approximate the membership functions, the boundary information of membership functions is taken into consideration adequately, meanwhile, some slack matrices are employed, and then some more relaxed membership-function-dependent stability results are obtained. The membership-function-independent stability conditions are presented in Corollary 1, which is to demonstrate the effectiveness of the augmented Lyapunov–Krasovskii functional constructed in this paper for reducing conservatism, and further illustrate the advantages of the membership-function-independent method. The imperfect-premise-matching-based fuzzy state feedback controllers are presented in Theorem 2. By constructing a smaller number of fuzzy rules and choosing some simpler membership functions to replace the complex ones in the fuzzy models, the flexibility of the controller design is increased, and the complexity and the implementation cost of the controllers are reduced. Finally, four numerical examples are given to demonstrate the effectiveness of the presented approaches.

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REFERENCES

1. Lian, Z., et al.: A new filter design method for a class of fuzzy systems with time delays. IEEE Trans. Syst. Man Cybern. Syst. 1-11, (2020) https://doi.org/10.1109/TSMC.2019.2961143.
2. Yan, S., et al.: A distributed delay method for event-triggered control of T-S fuzzy networked systems with transmission delay. IEEE Trans. Fuzzy Syst. 27(10), 1963–1973 (2019)
3. Chen, J., et al.: Stability analysis of systems with time-varying delay: a quadratic-partitioning method. IET Control Theory Appl. 13(18), 3184–3189 (2019)
4. Chen, J., et al.: Improved stability criteria for delayed neural networks using a quadratic function negative-definiteness approach. IEEE Trans. Neural Netw. Learn. Syst. 1–7, (2020) https://doi.org/10.1109/TNNLS.2020.3042307.
5. Wang, Z.P., et al.: Sampled-data fuzzy control with space-varying gains for nonlinear time-delay parabolic PDE systems. Fuzzy Sets Syst. 392, 170–194 (2020)
6. Li, S., et al.: Adaptive fuzzy control of switched nonlinear time-varying delay systems with prescribed performance and unmodeled dynamics. Fuzzy Sets Syst. 371, 40–60 (2019)
7. Li, Z.C., et al.: Stability and stabilization with additive freedom for delayed Takagi–Sugeno fuzzy systems by intermediary-polynomial-based functions. IEEE Trans. Fuzzy Syst. 28(4), 692–705 (2019)
8. Zhang, R.M., et al.: Fuzzy adaptive event-triggered sampled-data control for stabilization of T-S fuzzy memristive neural networks with reaction-diffusion terms. IEEE Trans. Fuzzy Syst. 1-1, 29(3), 34–45, (2020) https://doi.org/10.1109/TFUZZ.2020.2985334.
9. Zhang, R.M., et al.: Fuzzy sampled-data control for synchronization of T-S fuzzy reaction-diffusion neural networks with additive time-varying delays. IEEE Trans. Cybern. 51, 2384-2397, (2020) https://doi.org/10.1109/TCYB.2020.2996619.
10. Zhang, Z.Y., et al.: New stability and stabilization conditions for T-S fuzzy systems with time delay. Fuzzy Sets Syst. 263, 82–91 (2015)
11. Kwon, O.M., et al.: Stability and stabilization of T-S fuzzy systems with time-varying delays via augmented Lyapunov-Krasovskii functionals. Inf. Sci. 372, 1–15 (2016)
12. Sun, S.X., et al.: Delay-dependent H∞ guaranteed cost control for uncertain switched T-S fuzzy systems with multiple interval time-varying delays. IEEE Trans. Fuzzy Syst. 29, 1065-1080, (2020) https://doi.org/10.1109/TFUZZ.2020.2968877.
13. Jia, F.J., et al.: Adaptive sliding-mode-based control for stochastic nonlinear systems subject to probabilistic interval delay: a delay-fractioning method. J. Franklin Inst. 357(2) 1002–1025 (2020)
14. Zhang, P.P., et al.: H∞ sliding mode control for Markovian jump systems with randomly occurring uncertainties and repeated scalar nonlinearities via delay-fractioning method. ISA Trans. 101, 10–22 (2020)
15. Liu, K., et al.: Bessel–Laguerre inequality and its application to systems with infinite distributed delays. Automatica 109, 108562 (2019)
16. Zeng, H.B., et al.: New results on stability analysis of systems with time-varying delays using a generalized free-matrix-based inequality. J. Franklin Inst. 356(13), 7312–7321 (2019)
17. Lian, Z., et al.: Robust H∞ control for T-S fuzzy systems with state and input time-varying delays via delay-product-type functional method. IEEE Trans. Fuzzy Syst. 27(10), 1917–1930 (2019)
18. Lee, T.H., Park, J.H.: Improved stability conditions of time-varying delay systems based on new Lyapunov functionals. J. Franklin Inst. 355(3), 1176–1191 (2018)
19. Zhang, C.K., et al.: Notes on stability of time-delay systems: bounding inequalities and augmented Lyapunov-Krasovskii functionals. IEEE Trans. Autom. Control. 62(10), 5331–5336 (2017)
20. Du, Z.B., et al.: Interval type-2 fuzzy sampled-data control of time-delay systems. Inf. Sci. 487, 193–207 (2019)
21. Shen, J., Lam, J.: On the algebraic Riccati inequality arising in cone-preserving time-delay systems. Automatica 113, 108820 (2020)
22. Islam, S.I., et al.: Robust fault detection of T-S fuzzy systems with time-delay using fuzzy functional observer. Fuzzy Sets Syst. 392, 1–23 (2020)
23. Sheng, Y., et al.: Stability and stabilization of Takagi–Sugeno fuzzy systems with hybrid time-varying delays. IEEE Trans. Fuzzy Syst. 27(10), 2067–2078 (2019)
24. Zhang, C.K., et al.: A relaxed quadratic function negative-determination lemma and its application to time-delay systems. Automatica 113, 108764 (2020)
25. Park, P., et al.: Reciprocally convex approach to stability of systems with time-varying delays. Automatica 47(1), 235–238 (2011)
26. Scaret, A., Gouaisbaut, F.: Delay-dependent reciprocally convex combination lemma, Rapport LAAS n16006, (2016)
27. Lam, H.K., Leung, F.H.E.: Stability Analysis of Fuzzy-Model-Based Control Systems, Springer, Berlin Heidelberg (2011)
28. Zhao, T., et al.: Finite-time control for interval-type-2 fuzzy time-delay systems with norm-bounded uncertainties and limited communication capacity. Inf. Sci. 483, 153–173 (2019)
29. Zhou, K., et al.: Membership-function-dependent stability and stabilization conditions for T-S fuzzy time-delay systems. IEEE J. Res. 65(3), 351–364 (2019)
30. Park, P., et al.: Auxiliary function-based integral inequalities for quadratic functions and their applications to time-delay systems. J. Franklin Inst. 352(4), 1378–1396 (2015)
31. Zhang, X.M., et al.: An improved reciprocally convex inequality and an augmented Lyapunov–Krasovskii functional for stability of linear systems with time-varying delay. Automatica. 84, 221–226 (2017)
32. Kim, J.H.: Further improvement of Jensen inequality and application to stability of time-delayed systems. Automatica 64, 121–125 (2016)
33. Chen, Y., Zheng, W.X.: Exponential H∞ filtering for stochastic Markovian jump systems with time delays. Int. J. Robust Nonlinear Control 24(4), 625–643 (2014)
34. Xie, W.B., et al.: New approaches to observer design and stability analysis for T–S fuzzy system with multiplicative noise. J. Franklin Inst. 354(2), 887–901 (2017)
35. Qian, W., et al.: Robust stability criteria for uncertain systems with interval time-varying delay based on multi-integral functional approach. J. Franklin Inst. 355(2), 849–861 (2018)
36. Feng, Z.G., Zheng, W.X.: Improved stability condition for Takagi–Sugeno fuzzy systems with time-varying delay. IEEE Trans. Cybern. 47(3), 661–670 (2017)
37. Datta, R., et al.: Improved delay-range-dependent stability condition for T–S fuzzy systems with variable delays using new extended affine Wirtinger inequality. Int. J. Fuzzy Syst. 22(3), 985–998 (2020)
38. Lian, Z., et al.: Stability and stabilization of T–S fuzzy systems with time-varying delays via delay-product-type functional method. IEEE Trans. Cybern. 50(6), 2580–2589 (2020)
39. Wang, L.K., Liu, J.J.: Local stability analysis for continuous-time Takagi–Sugeno fuzzy systems with time delay. Neurocomputing 273, 152–158 (2018)
40. Wang, L.K., Lam, H.K.: A new approach to stability and stabilization analysis for continuous-time Takagi–Sugeno fuzzy systems with time delay. IEEE Trans. Fuzzy Syst. 26(4), 2460–2465 (2018)
41. Yang, J., et al.: Further improved stability criteria for uncertain T–S fuzzy systems with time-varying delay by (m, N)-delay-partitioning approach. ISA Trans. 59, 20–28 (2015)
42. Chen, H., et al.: Stability criteria for T–S fuzzy systems with interval time-varying delays and nonlinear perturbations based on geometric progression delay partitioning method. ISA Trans. 63, 69–77 (2016)
43. Lian, Z., et al.: Stability analysis for T–S fuzzy systems with time-varying delay via free-matrix-based integral inequality. Int. J. Control Autom. Syst. 14(1), 21–28 (2016)
44. Lian, Z., et al.: Further robust stability analysis for uncertain Takagi–Sugeno fuzzy systems with time-varying delay via relaxed integral inequality. Inf. Sci. 409–410, 139–150 (2017)
45. Lee, S.: Novel stabilization criteria for T–S fuzzy systems with affine matched membership functions. IEEE Trans. Fuzzy Syst. 27(3), 540–548 (2019)
46. Lian, Z., et al.: Stability and stabilization for delayed fuzzy systems via reciprocally convex matrix inequality. Fuzzy Sets Syst. 402, 124–141 (2021)
47. Lien, C.H.: Stabilization for uncertain Takagi–Sugeno fuzzy systems with time-varying delays and bounded uncertainties. Chaos Solitons Fractals 32(2), 645–652 (2007)
48. Teng, L., et al.: Fuzzy model predictive control of discrete-time systems with time-varying delay and disturbances. IEEE Trans. Fuzzy Syst. 26(3), 1192–1206 (2018)