The noncommutative replica procedure

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Abstract

The alternative to the replica procedure, which we call the noncommutative replica procedure, is discussed. The detailed comparison with the standard EA replica procedure is performed.

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1 Introduction

In the present note we discuss the noncommutative replica approach to statistical mechanics of disordered systems, proposed in [1]. This approach is an alternative to the standard Edwards–Anderson (or EA) replica approach introduced in [2], see for the review [3]. Using the operator mean field approach to noncommutative replica symmetry breaking, we are able to derive the ultrametric space of states, which describes the disordered system after the phase transition. In the considered simplest case the obtained space of states is a $p$–adic disk.

For the review of applications of ultrametricity in physics see [4]. For introduction to $p$–adic analysis see [5, 8]. $p$–Adic analysis and $p$–adic mathematical physics attract great interest, see [5, 11]. $p$–Adic analysis was applied to investigate the spontaneous breaking of the replica symmetry, cf. [12, 13].

We investigate disordered models with quenched disorder such as the Sherrington–Kirkpatrick (or SK) model. We discuss the general approach to quenched disorder and suggest, that the procedure to describe quenched disorder, which we call the general (not necessarily EA) replica procedure, is essentially non unique.

We propose the general definition of replica procedure as a morphism of noncommutative probability spaces, or the map of probability spaces, which (in the high temperature limit) will conserve the correlation functions.

In the standard replica approach quenched disorder is described by the EA replica procedure [2, 3]. We construct the family of examples of replica procedures, all of which are different from the EA replica procedure. We show, using the Wigner theorem, see [14, 15], that in the free case (in the high temperature limit) the disordered system will be described by the Fock state over the quantum Boltzmann algebra. For interacting system (for the finite temperature) the state will become non Fock.

The quantum Boltzmann algebra arises in the limit of large stochastic matrices [14], was used in the free probability [15] and describes the quantum system in the stochastic approximation [17–19].
To describe phase transitions in disordered systems we use the operator mean field approach and the free coherent states. The free coherent states were introduced and investigated in [21]–[23]. In [24] corresponding representation of the Cuntz algebra was constructed.

In the present paper we put the detailed comparison of the EA and noncommutative replica procedures. The structure of the paper is as follows.

In Section 2 we remind the EA replica procedure.

In Section 3 we discuss the noncommutative replica procedure of [1] and compare it with the EA replica procedure.

In Section 4 we discuss the relation between the high temperature limit of the noncommutative replica procedure and the Wigner theorem and propose a general definition of the replica procedure.

In Section 5 we discuss the noncommutative replica symmetry breaking.

In Section 6 we give the conclusion of our results.

2 The EA replica procedure

Consider the disordered model with the Hamiltonian $H[σ, J]$, where $σ$ enumerates the states of the system and $J = (J_{ij})$ is the random matrix of interactions with the matrix elements which are independent Gaussian random variables with the probability distribution

$$P[J] = \prod_{i \leq j} \exp\left( -\frac{J_{ij}^2}{2} \right) \quad (1)$$

For simplicity we neglect here the normalization of the random matrix in the exponent, taking into account that we may easily reproduce the normalization by proper rescaling of the correlators.

The averaging over the random interaction $J_{ij}$ is called in the literature the assumption of self-averaging: to calculate the expectation value of the observable first we have to take the average over the system degrees of freedom and then take the average over the Gaussian stochastic variables $J_{ij}$.

The typical disordered model is the Sherrington–Kirkpatrick (or the SK) model with the Hamiltonian

$$H[σ, J] = -\frac{1}{2} \sum_{i < j} J_{ij} σ^i σ^j \quad (2)$$

where the summation runs over the spins $σ_j$ in the $d$-dimensional lattice taking values $±1$. Here $σ = \{σ_i\}$ describes the orientations of all the spins.

The replica procedure is used to describe disordered models with quenched disorder. The disorder is called quenched when the Hamiltonian $H[σ, J]$ of the model depends on some typical realization of the random variable $J$ and from the beginning no averaging on $J$ is assumed.

In this case statistic sum of the disordered system is

$$Z[J] = \sum_{\{σ\}} \exp (-βH[σ, J])$$

where $β$ is the inverse temperature.

If the averaging of the statistic sum is assumed from the beginning then the disorder is called annealed. The state for annealed disordered system is determined by the following annealed statistic sum

$$\langle\langle Z[J]\rangle\rangle = \int \sum_{\{σ\}} \exp (-βH[σ, J]) \exp\left( -\frac{1}{2} \sum_{i \leq j} J_{ij}^2 \right) \prod_{i \leq j} dJ_{ij} \quad (3)$$

In the following we will omit the $\langle\langle\cdot\rangle\rangle$ brackets.

Remind the EA replica procedure, see [2], [3].

The replica procedure [2] was developed in order to describe quenched disorder. This procedure consists in introduction of $n$ identical replicas of the original system. This means that we take the $n$–th
degree of the statistic sum
\[ Z^n[J] = \left( \sum_{\{\sigma\}} \exp(-\beta H[\sigma, J]) \right)^n \]

and perform the averaging with the random matrix \( \mathbf{M} \).

This implies the following expression for the quenched statistic sum
\[ Z_n = \int \sum_{\{\sigma^a\}} \exp \left( -\beta \sum_{a=1}^n H[\sigma^a, J] \right) \exp \left( -\frac{1}{2} \sum_{i \leq j} J_{ij}^2 \right)^n \prod dJ_{ij} \]  

where we sum over the replicas \( \sigma^a \) of the system.

Then one calculates thermodynamical characteristics of system, for example the free energy, and takes the limit \( n \to 0 \).

The averaging on \( J \) in models with quenched disorder is the physical assumption of selfaveraging: we assume that the behavior of the model depending on the typical realization of the disorder is the same as for the model with averaged statistic sum.

This procedure may be discussed as follows. Consider the self-averaged free energy
\[ F = -\beta^{-1} \langle \ln Z \rangle \]

Since
\[ \ln x = \lim_{n \to \infty} \frac{1}{n} \ln x^n = \lim_{n \to \infty} \frac{x^n - 1}{n} \]

the replica procedure implies the following expression for the free energy
\[ F_n = -\beta^{-1} \frac{1}{n} [Z_n - 1] = -\beta^{-1} \frac{1}{n} \left[ \int \sum_{\{\sigma^a\}} \exp \left( -\beta \sum_{a=1}^n H[\sigma^a, J] \right) \exp \left( -\frac{1}{2} \sum_{i \leq j} J_{ij}^2 \right)^n \prod dJ_{ij} - 1 \right] \]  

Then the quenched free energy will be equal
\[ F = \lim_{n \to \infty} F_n \]  

Let us note that (3), (4) in the EA replica approach is the definition of the free energy of the disordered system with quenched disorder. Similar definitions should be made for the correlators.

In the present paper we introduce the alternative replica procedure (which we call noncommutative), and discuss the generalizations of the EA replica procedure.

This suggests the following definition.

**Definition 1.** We call replica procedure the procedure which transforms the state describing the system with annealed disorder into the state describing the corresponding system with quenched disorder. This procedure does not necessarily coincide with the EA replica procedure.

The state of a disordered system with quenched disorder is not determined only by a Hamiltonian of the system. To describe a state with quenched disorder we also have to describe the replica procedure. In principle, this procedure may be non unique, and systems with the same Hamiltonian may have different quenched states.

This feature, which looks strange, may be justified by the remark, that to describe quenched states we always use the assumption of selfaveraging, where we average over the realizations of disorder. This means that in reality we fix not a Hamiltonian, but a class of Hamiltonians with similar structure of disorder. Different replica procedures in this approach may be related to different kinds of incomplete selfaveraging. Of course, different replica procedures should describe the states with similar behavior, at least in the high temperature limit. We will formulate the conditions which should be satisfied by replica procedure in Definition 8 below.

In the present paper we describe the replica procedure, called the noncommutative replica procedure (or NRP), which is different from the EA replica procedure.
3 The noncommutative replica procedure

Let us discuss the EA replica procedure using the well known interpretation of the replica procedure by nonequilibrium states. Let us assume that we have two temperatures: one temperature $\beta^{-1}$ for the system degrees of freedom (say for the spins $\sigma$ in the SK model), and the second temperature $\beta'^{-1}$ for the disorder (the matrix $J$ of spin interactions in the SK model). Then we have for the quenched statistic sum

$$\tilde{Z} = \int \exp\left(-\beta' F[J]\right) P[J]dJ = \int \exp\left(\frac{\beta'}{\beta} \ln Z[J]\right) P[J]dJ = \int Z^n[J] P[J]dJ \tag{7}$$

which, in the limit $n \to 0$, gives the EA replica statistic sum.

Here $n = \frac{\beta'}{\beta}$, $F[J]$ is the free energy for the disorder $J$ and $Z[J]$ is the corresponding statistic sum.

Then the limit $n \to 0$ corresponds to the regime when the temperature of the disorder is much higher than the temperature of the system. The physical meaning of this property is the independence of the disorder degrees of freedom $J$ from the system degrees of freedom $\sigma$.

Another possibility to introduce the statistic sum with two temperatures is to put

$$\tilde{Z} = \int Z[J] (P[J]dJ)^p$$

and take the limit $p \to \infty$ which corresponds to the limit $n \to 0$ in (7).

This discussion leads to the noncommutative replica procedure (or NRP), introduced in [1].

The EA replica procedure is the transformation

$$Z = \int \sum_{\{\sigma\}} \exp(-\beta H[\sigma, J]) \exp\left(-\frac{1}{2} \sum_{i \leq j}^N J_{ij}^2\right) \prod_{i \leq j} dJ_{ij} \mapsto$$

$$\mapsto Z_n = \int \left[ \sum_{\{\sigma\}} \exp(-\beta H[\sigma, J]) \right]^n \exp\left(-\frac{1}{2} \sum_{i \leq j}^N J_{ij}^2\right) \prod_{i \leq j}^N dJ_{ij} \tag{8}$$

together with taking the limit $n \to 0$.

The noncommutative replica procedure is the transformation

$$Z = \int \sum_{\{\sigma\}} \exp(-\beta H[\sigma, J]) \exp\left(-\frac{1}{2} \sum_{i \leq j}^N J_{ij}^2\right) \prod_{i \leq j} dJ_{ij} \mapsto$$

$$\mapsto Z^{(p)} = \int \sum_{\{\sigma\}} \exp(-\beta H[\sigma, J]) \left[ \exp\left(-\frac{1}{2} \sum_{i \leq j}^N J_{ij}^{(a)}\right) \prod_{i \leq j}^N dJ_{ij}^{(a)} \right]^p \tag{9}$$

together with taking the limit $p \to \infty$.

One can see that in this scheme we have the correspondence $n = p^{-1}$.

The difference between the above two transformations is that in (8) we replicate the system degrees of freedom, and in (9) we replicate the disorder.

Formula (9), of course, is just a rough scheme. In reality we have to introduce more complicated procedure.

First, instead of the $p$-th degree in $Z^{(p)}$ in (9)

$$\left[ \exp\left(-\frac{1}{2} \sum_{i \leq j}^N J_{ij}^{(a)}\right) \prod_{i \leq j}^N dJ_{ij}^{(a)} \right]^p$$

we have to consider the product

$$\prod_{a=0}^{p-1} \exp\left(-\frac{1}{2} \sum_{i \leq j}^N J_{ij}^{(a)}^2\right) \prod_{i \leq j}^N dJ_{ij}^{(a)} \tag{10}$$
where \( J_{ij}^{(a)} \) are independent Gaussian random variables distributed as in (1). Note that the product on \( \sigma \) is taken for both the exponent and the differentials \( dJ_{ij}^{(a)} \).

Second, in \( H[\sigma, J] \) in \( Z^{(p)} \) we have to make the following substitution for \( J \):

\[
\Delta : J_{ij} \mapsto \frac{1}{\sqrt{p}} \sum_{a=0}^{p-1} J_{ij}^{(a)}
\]

(11)

where \( J^{(a)} \), as above, are independent copies of the random matrix \( J \). The map \( \Delta \) is an example of coproduct used in the theory of quantum groups.

Note that the transformation (11), taken together with the distribution (10), is an embedding of probability spaces (it conserves all the correlation functions).

The described procedure implies for (9) the following

\[
Z^{(p)} = \int \sum_{\{\sigma\}} \exp \left( -\beta H[\sigma, \Delta J] \right) \prod_{a=0}^{p-1} \exp \left( -\frac{1}{2} \sum_{i \leq j} J_{ij}^{(a)^2} \right) \prod_{i \leq j} dJ_{ij}^{(a)}
\]

(12)

This is the noncommutative replica expression for the quenched statistic sum. We arrive to the following definition.

Definition 2. The noncommutative replica expression for correlation functions of quenched disordered system is given by the correlations of the order parameter \( \Delta J \) given by (11), defined by the statistic sum (12), after the limit \( p \to \infty \).

The noncommutative replica procedure and the EA replica procedure are different examples of replica procedure. In the next section we discuss properties of the noncommutative replica procedure, give a general definition of replica procedure and construct a family of examples.

4 Large random matrices and the Wigner theorem

Discuss the noncommutative replica procedure and compare it with the described below generalization of the Wigner theorem.

The order parameter for the SK model in the NRP approach is given by the matrix \( J = (J_{ij}) \), with matrix elements corresponding to correlation functions

\[ (\sigma_i \sigma_j) \]

After the noncommutative replica procedure this order parameter is equal to \( \Delta J \).

The quenched NRP state with the statistic sum (12) defines the state on the algebra generated by \( p \) matrices with independent components. In the high temperature regime we have \( \beta \to 0 \), and one can neglect the \( \beta H \) term in (12). In this case the NRP state reduces to the state described by a variant of the Wigner theorem.

The following result was presented in [20], [15] and was used in the \( N \to \infty \) limit of the matrix model, see [16], [20].

Consider the space \( P(N, \mathbb{R}) \) of polynomials of symmetric \( N \times N \) matrices over real numbers. Introduce \( p \) independent copies of the space \( P(N, \mathbb{R}) \) and the tensor degree \( P(N, \mathbb{R}) \otimes^p \). Introduce the state on \( P(N, \mathbb{R}) \otimes^p \) in the following way

\[
\langle f \left( J^{(0)}, \ldots, J^{(p-1)} \right) \rangle_N =
\]

\[
= \frac{1}{Z_N} \int f \left( \frac{J^{(0)}}{N}, \ldots, \frac{J^{(p-1)}}{N} \right) e^{-\sum_{k=0}^{p-1} \text{tr} \, J^{(k)}_k \prod_{k=0}^{p-1} \prod_{i \leq j} dJ^{(k)}_{ij}};
\]

(13)

\[ Z_N = \langle 1 \rangle_N. \]
Theorem 3. The limits \( \lim_{N \to \infty} \langle f \left( J^{(0)}, \ldots, J^{(p-1)} \right) \rangle_N \) exist for each polynomial \( f \) and are equal to

\[
\lim_{N \to \infty} \langle f \left( J^{(0)}, \ldots, J^{(p-1)} \right) \rangle_N = (\Omega, f(Q_0, \ldots, Q_{p-1})\Omega);
\]

where

\[ Q_a = A_a + A^\dagger_a; \]

Here \( A^\dagger_a \) and \( A_a \) are the quantum Boltzmann creators and annihilators and \( \Omega \) is the vacuum in the free Fock space:

\[ A_a\Omega = 0. \]

The quantum Boltzmann algebra is generated by the Boltzmannian creation and annihilation operators \( A_a, A^\dagger_a, a = 0, \ldots, p-1 \) with the relations

\[ A_a A_b^\dagger = \delta_{ab} \quad (14) \]

The Fock representation is constructed in the free, or quantum Boltzmann, Fock space, generated from the vacuum \( \Omega, A_a\Omega = 0 \), by the action of creators \( A^\dagger_a \).

We see, that the thermodynamic \( N \to \infty \) limit of random matrices with the distribution \( \mathcal{H} \), by the Wigner theorem \([14],[15]\), gives rise to the quantum Boltzmann algebra in the Fock representation.

The order parameter of the noncommutative replica approach, which is equal to the \( \Delta \) of large random matrix, takes the form of the following operator from the Fock representation of the quantum Boltzmann algebra

\[
Q = \lim_{N \to \infty} \frac{1}{N} \Delta J = \frac{1}{\sqrt{p}} \sum_{a=0}^{p-1} Q_a = \frac{1}{\sqrt{p}} \sum_{a=0}^{p-1} (A_a + A^\dagger_a)
\]

where \( A_a \) and \( A^\dagger_a \) are quantum Boltzmann annihilator and creator correspondingly.

The convergence is understood in the sense of correlators, see \([14],[15],[20]\).

In the Fock representation the expectation of this order parameter is zero:

\[ \langle \Omega, Q\Omega \rangle = 0 \]

This state will describe disordered system in the high temperature regime, when we can neglect the interaction of the system degrees of freedom. We see that we can reformulate Theorem 3 as follows:

Proposition 4. The free theory (the high temperature regime) of disordered systems is described by the Fock state over the quantum Boltzmann algebra. For finite temperature the state will become non Fock.

Discuss now the noncommutative replica procedure. Consider the linear coproduct map, which was used in the NRP:

\[
\Delta : P(N, R) \longrightarrow P(N, R)^{\otimes p};
\]

\[
\Delta : J \mapsto \frac{1}{\sqrt{p}} \left( J \otimes 1 \otimes 1 \otimes \ldots \otimes 1 + 1 \otimes J \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes 1 \otimes \ldots \otimes 1 \otimes J \right);
\]

or equivalently

\[
\Delta : J \mapsto \frac{1}{\sqrt{p}} \sum_{a=0}^{p-1} J_a;
\]

where \( J_a \) belongs to the \( a \)-th component of the tensor product. This map coincides with the map used in the central limit theorem. For example, for the gaussian random variables \( J \) the map \( \mathcal{H} \) is an embedding of probability spaces (all the correlation functions are invariant). For large random matrices,
in the thermodynamic (large $N$) limit, the central limit theorem becomes the free central limit theorem, see [15]. For instance, the Wigner state will be invariant under (16). We formulate the following:

**Theorem 5.** The map (16) with the state (13), where we put

$$f(J^{(0)}, \ldots, J^{(p-1)}) = f(\Delta J);$$

is an embedding of algebraic probability spaces. In the limit $N \to \infty$ this embedding becomes the $\ast$-homomorphism of the quantum Boltzmann algebra with one degree of freedom maps into the quantum Boltzmann algebra with $p$ degrees of freedom defined according to the formula

$$A \mapsto \frac{1}{\sqrt{p}} \sum_{a=0}^{p-1} A_a, \quad A^\dagger \mapsto \frac{1}{\sqrt{p}} \sum_{a=0}^{p-1} A^\dagger_a; \quad (17)$$

and the Fock state $\langle \Omega, x \Omega \rangle$ will map onto the Fock state $\langle \Omega, x \Omega \rangle_p$ in the quantum Boltzmann Fock space with $p$ degrees of freedom

$$\langle \Omega, x \Omega \rangle \mapsto \langle \Omega, x \Omega \rangle_p. \quad (18)$$

For instance the map (17), (18) conserves all the correlators.

Let us remind the following well known definition.

**Definition 6.** Noncommutative (or quantum) probability space is a pair $(A, \phi)$, where $A$ is an algebra over the field of complex numbers with unit and involution, and $\phi$ is a positive normed state on the algebra $A$:

$$\phi(a^* a) \geq 0, a \in A, \quad \phi(1) = 1$$

Category of noncommutative probability spaces is the category, where the objects are noncommutative probability spaces, and the morphisms $\ast$-homomorphisms of algebras, conserving correlation functions.

This means that if $f$ is a morphism of noncommutative probability spaces

$$f : (A, \phi) \to (B, \psi)$$

and $P \in A$ is a (noncommutative) polynomial $P(a_1, \ldots, a_k)$ in $A$, then the properties of a homomorphism

$$f(P(a_1, \ldots, a_k)) = P(f(a_1), \ldots, f(a_k))$$

involution

$$f(a^*) = f(a)$$

and conservation of correlators

$$\psi(f(a)) = \phi(a)$$

are satisfied.

We call the morphism $f$ an embedding, if it is an embedding as a map of algebras.

The Wigner theorem and related results (theorems 3, 5, proposition 4) describe the free case (or the high temperature limit) of disordered systems. The NRP in this case takes the form (16) (or (17) in the $N \to \infty$ limit), where for the states the NRP takes the form (18) (in the $N \to \infty$ limit).

Let us discuss now the properties of the NRP in the interacting case. In this case the NRP for observables has the same form, while for the states the NRP state will be determined by the NRP statistic sum (12).

If in (12) one can interchange the summation over $\sigma$ and the integration over $J_{ij}$ we get for the expectation of $(\Delta J)^k$:

$$\langle (\Delta J)^k \rangle = \frac{1}{Z^{(p)}} \sum_{\{\sigma\}} \int (\Delta J)^k \exp (-\beta H[\sigma, \Delta J]) \prod_{a=0}^{p-1} \exp \left( -\frac{1}{2} \sum_{i \leq j} N \cdot J_{ij}^{(a)^2} \right) \prod_{i \leq j} dJ_{ij}^{(a)}$$
Then, expanding \( \exp(-\beta H[\sigma, \Delta J]) \) into series over the degrees of \( \beta \) and interchanging again (if possible) the summation with integration, we get that the NRP (11) will conserve all the expectations of the degrees of the random matrix \( J \):

\[
\langle (\Delta J)^k \rangle = \langle J^k \rangle
\]

where at the LHS of this formula the state is defined by the NRP statistic sum (12), and at the RHS the state is defined by the annealed statistic sum (3).

The analyticity of the statistic sum with respect to \( \beta \) and the possibility to interchange summations with integration in the formula above are related to the existence of phase transitions in the system: if there is no phase transitions, then the statistic sum is analytic with respect to \( \beta \). In this case the arguments above are correct and all the correlation functions are invariant under the NRP. If we have phase transitions then the NRP will modify the state.

Let us formulate now the following proposition.

**Proposition 7.** For the case when there is no phase transition the NRP defined by (11), (12) defines the state equivalent to the annealed state. After phase transition the state defined by the NRP will be not equivalent to the annealed state.

Let us note that the EA replica procedure has similar properties: it conserves the correlators in the high temperature limit.

Summing up, we see that the Wigner theorem allows to compute the thermodynamic limit of the SK model in the high temperature regime. We introduce the noncommutative replica transformation (16) and show, that in the high temperature regime it realizes an equivalent description of the disordered model (by Theorems 3 and 5). The Proposition 7 shows, that this will be also true for finite temperatures above the temperature of phase transition.

One can easily construct a family of generalizations of transformation (16) which are also the embeddings of probability spaces.

For instance, one can consider the map

\[
\Delta' : J \mapsto \frac{1}{\sqrt{p}} \sum_{a=0}^{p-1} c_a J_a;
\]

where \( c_a \) are complex valued coefficients, which should satisfy the condition

\[
\sum_{a=0}^{p-1} |c_a|^2 = p
\]

which guarantee that the map (19) with the Wigner state used in Theorem 5 is an embedding of probability spaces).

To make this map the replica procedure for the disordered system, one has to extend it on the states, by defining the quenched statistic sum

\[
Z^{(p)} = \int \sum_{\{\sigma\}} \exp (-\beta H[\sigma, \Delta' J]) \prod_{a=0}^{p-1} \exp \left( -\frac{1}{2} \sum_{i\leq j} J^{(a)}_{ij} \right) \prod_{i\leq j} d J^{(a)}_{ij}
\]

Different replica procedures of the form (19), (20) should correspond to physically different disordered systems, which have different behavior after phase transition.

Let us discuss the obtained results from the point of view of noncommutative probability theory. In the theory of disordered systems we the following probability space. The algebra is the \(*\)-algebra \( J \) generated by large symmetric random matrix, and the state \( \langle \cdot \rangle \) (depending on the inverse temperature \( \beta \)) is the annealed Gibbs state generated by the annealed statistic sum (3). Therefore the annealed case is described by a commutative probability space. We will see that the quenched state should be described by noncommutative probability space.

The discussed in the present paper replica procedures are the maps which, in the high temperature regime, conserve the correlation functions. This suggests to formulate the following definition.
Definition 8. We call the replica procedure the family $f_\beta$ of maps of the probability space $(\mathcal{J}, \langle \cdot \rangle)$, parameterized by the inverse temperature $\beta$, into the other probability space $(\mathcal{A}, \phi)$ (which also depends on $\beta$), which in the high temperature regime $\beta \to 0$ becomes a morphism in the category of quantum probability spaces.

This definition means, that for high temperatures any quenched states, generated by the replica procedure, should be equivalent to the annealed state. For low temperature this equivalence will broken. For instance, different replica procedures may generate different quenched states. This may be discussed as the consequence of different self-averaging of the quenched disorder.

The noncommutative replica procedure, discussed in the present paper, maps the annealed probability space $(\mathcal{J}, \langle \cdot \rangle)$ into the quenched, or replica probability space $(\mathcal{J}_\infty, \langle \cdot \rangle)$, generated by infinite number of random matrices $J^a$, $a = 0, \ldots$ with independent matrix elements and the state generated by the replica statistic sum [12]. In the limit of large random matrices this probability space becomes the quantum Boltzmann algebra with infinite number of degrees of freedom, where the state is the $N \to \infty$ limit of the replica state [12].

Since two different replica procedures $f_\beta$ and $g_\beta$ (morphisms of probability spaces) may have non-commuting images, we see that the general approach in the replica theory should be described in the framework of noncommutative probability theory.

5 The noncommutative RSB: the operator mean field theory

To describe phase transitions in disordered systems in the EA replica approach the procedure of the replica symmetry breaking, or RSB, is used [3]. In this procedure the properties of the system are described by the Parisi replica matrix, which in $n \to 0$ limit becomes the operator in infinite dimensional space. $p$–Adic parameterization for the Parisi matrix was obtained by Avetisov, Bikulov, Kozyrev [12], and by Parisi, Sourlas [13].

In the present section we discuss an approach to describe phase transitions in disordered systems, called the noncommutative replica symmetry breaking, introduced in [1]. This approach is a kind of a mean field approach, where the mean field (the order parameter) is an operator. The operator mean field approach was used, for instance, in the Bogoliubov approach in theory of superconductivity and superfluidity [25].

In this approach the phase transition was described by the quantum mean field condition

$$ A\Psi = \lambda \Psi $$

where $A$ is the Bose annihilation operator. This condition means that the Bose condensate is in the coherent state.

In the theory of superconductivity and superfluidity the operator mean field theory [25] describes the changing of the kinematics of the system, which results in the effects of superconductivity and superfluidity.

For quenched disordered systems the statistics is described by the replica transformed large random matrices. By the Wigner theorem, large random matrix has the quantum Boltzmann statistics, and after the noncommutative replica transformation the order parameter becomes

$$ Q = \frac{1}{\sqrt{p}} \sum_{a=0}^{p-1} (A_a + A_a^\dagger), \quad A_a A_b^\dagger = \delta_{ab} $$

The condition of the noncommutative mean field, similar to used in [25], takes the form

$$ \frac{1}{\sqrt{p}} \sum_{a=0}^{p-1} A_a \Psi = \lambda \Psi \quad (21) $$

We will see, that condition [21], due to the quantum Boltzmann statistics of the order parameter for disordered systems, will lead to nontrivial effects already on the level of statics.

Let us describe now the noncommutative replica symmetry breaking procedure. It includes the following constructions.
The first step  To describe the phase transition, one has to consider the representation in which the order parameter has non-zero expectation. To do this, we apply the noncommutative mean field approach and consider \[1\] the free coherent states, constructed in \[21\]–\[23\]. The free coherent state \(\Psi\) is the eigenvector of the free annihilation

\[ A\Psi = \lambda \Psi, \quad A = \frac{1}{\sqrt{p}} \sum_{a=0}^{p-1} A_a, \quad \lambda \in \mathbb{R} \]  \hspace{1cm} (22)

Note that here \(\Psi\) is a function of \(\lambda\).

One easily observes that the expectation of the order parameter \(Q\) between the free coherent states is non-zero:

\[ \langle Q \rangle \neq 0 \]

In the works \[21\]–\[23\] it was shown that, in the renormalized scalar product,

\[ (\Psi, \Phi) = \lim_{\lambda \to 1^{-}} \langle \Psi, \Phi \rangle \]

the space of the free coherent states is isomorphic to the space of (complex valued) generalized functions on \(p\)-adic disk \(\mathbb{Z}_p\).

This shows that the thermodynamic limit \(N \to \infty\), together with the procedure of noncommutative replica symmetry breaking \[\langle Q \rangle \neq 0\] in the noncommutative mean field approach, implies the derivation of the ultrametric (and even \(p\)-adic) space of states, which was postulated in the EA replica approach in the procedure of the replica symmetry breaking. All the procedure of construction of the ultrametric space in the noncommutative replica approach is encoded into the simple algebraic condition \[22\]. No conditions of the kind \(n \to 0\) or \(p \to \infty\) are relevant to ultrametricity.

In \[1\], \[23\] the condition \[22\] was discussed as the equation of the noncommutative line

\[ A = 1, \quad A = \frac{1}{\sqrt{p}} \sum_{a=0}^{p-1} A_a \]  \hspace{1cm} (23)

in the noncommutative plane with the coordinates \(A_a\). The results of \[21\]–\[23\] in this language may be reformulated as the equivalence of the noncommutative line \[23\] and \(p\)-adic disk.

This, together with the arising of the noncommutative quantum Boltzmann algebra in the Wigner theorem, explains the name noncommutative replica approach.

The second step  We showed already that the degrees of freedom of the system in the \(N \to \infty\) limit will be described by the quantum Boltzmann algebra \[14\], which before the phase transition acts in the Fock representation, and after the phase transition will act in the non Fock representation, related to the free coherent states. In \[1\], \[24\] the following expressions for this representation were obtained: the operators from the quantum Boltzmann algebra

\[ A_a A_b^\dagger = \delta_{ab}. \]

in this representation satisfy the Cuntz relation

\[ \sum_{a=0}^{p-1} A_a^\dagger A_a = 1. \]

and are realized in the space \(L^2(\mathbb{Z}_p)\) of quadratically integrable functions on \(p\)-adic disk, where act as follows:

\[ A_a^\dagger \xi(x) = \sqrt{\theta_1(x - a)} \xi([\frac{1}{p}x]); \]

\[ A_a \xi(x) = \frac{1}{\sqrt{p}} \xi(a + px). \]  \hspace{1cm} (24)

Here

\[ [x] = x - x(\text{mod } 1) \]

for \(x \in \mathbb{Q}_p\) is the integer part of \(x\). \(\theta_1(x - a)\) is the characteristic function of the \(p\)-adic disk with the center in \(a\) and the radius \(p^{-1}\).

Physically this representation will describe the disordered system in the limit of zero temperature.
6 Discussion of the NRP

Let us make the conclusion of our discussion. In the present note we described the noncommutative replica procedure and compared it with the EA replica procedure.

We conjecture, that the transition from annealed to quenched disorder can be described by the noncommutative replica procedure (11), (12).

For the high temperatures (above the temperature of phase transition) the NRP is an embedding of probability spaces, that means that (11) conserves the correlation functions:

$$\langle f(J) \rangle = \langle f(\Delta J) \rangle$$

Here $f(J)$ lies in the algebra generated by the large random matrix $J$, the state at the LHS is given by (1), and the state at the RHS is given by (10).

We will say that in (26) the observable $J$ at the LHS lies in the annealed algebra of observables, and $\Delta J$ at the RHS lies in the quenched algebra (and correspondingly for the states).

The noncommutative replica procedure implies that in the limit of infinite temperature, by the Wigner theorem, the state of a disordered system will be equal to the vacuum state in the quantum Boltzmann Fock space.

We propose the definition of the replica procedure as the map between noncommutative probability spaces which becomes a morphism for high temperatures.

We described the noncommutative replica symmetry breaking condition in the operator mean field approach $\langle Q \rangle \neq 0$, $Q = A + A^\dagger$, $A = 1/p \sum_{a=0}^{p-1} A_a$, as the equation of quantum line $A \Psi = \lambda \Psi$. The application of the theorem about the isomorphism between the quantum line and $p$–adic disk allows to derive the ultrametric space of states which is postulated in the Parisi replica symmetry breaking approach [3].

The correlation functions of the disordered model after the noncommutative replica symmetry breaking will be calculated using the $p$–adic representation of the Cuntz algebra (24)–(25) (for zero temperature).

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