Quantum coin tossing and bit-string generation in the presence of noise

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We discuss the security implications of noise for quantum coin tossing protocols. We find that if quantum error correction can be used, so that noise levels can be made arbitrarily small, then reasonable security conditions for coin tossing can be framed so that results from the noiseless case will continue to hold. If, however, error correction is not available (as is the case with present day technology), and significant noise is present, then tossing a single coin becomes problematic. In this case, we are led to consider random n-bit string generation in the presence of noise, rather than single-shot coin tossing. We introduce precise security criteria for n-bit string generation and describe an explicit protocol that could be implemented with present day technology. In general, a cheater can exploit noise in order to bias coins to their advantage. We derive explicit upper bounds on the average bias achievable by a cheater for given noise levels.

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The cryptographic task of coin tossing was first introduced by Blum \textsuperscript{1}. Briefly, the idea is that two separated, mistrustful parties wish to generate a random bit and be sure that the other party cannot have biased the bit by more than a certain amount. Secure coin tossing is known to be impossible classically, unless either computational assumptions or special relativistic considerations \textsuperscript{2} are invoked. Upon using a quantum communication channel, however, it is possible to achieve levels of security that are impossible classically. Various results concerning the security of quantum coin tossing under different assumptions, and its relationship to other cryptographic tasks (in particular, bit commitment), have been obtained \textsuperscript{2,4,5,6,7,8,9,10,11,12,13,14,15}.

It would be highly desirable to implement quantum coin tossing experimentally. Long distance quantum communication is indeed possible with present technology \textsuperscript{16}, and it may be possible to adapt such techniques to coin tossing. In real life situations, however, state preparation, communication channels and measurements are all imperfect, while all the above results refer to the ideal situation where no noise is present. As the example of quantum key distribution illustrates, a large amount of theoretical work must be carried out before an idealized quantum communication protocol can be implemented experimentally with the effect of all experimental imperfections taken into account. The present work initiates this line of investigation in the case of quantum coin tossing.

In particular, we shall argue that if quantum error correction can be used, then noise can in principle be made arbitrarily small, and one can frame security definitions such that results holding in the noiseless case still apply. On the other hand, if techniques such as quantum error correction are not available (as is the case with present day technology), then there will be a significant level of noise that cannot be reduced. In this case, tossing a single coin is problematic, in the sense that it is not possible to frame sensible security conditions that can actually be satisfied. But tossing a string of coins such that the average bias is bounded is possible. We obtain detailed results concerning the relation between the amount of noise and this average bias.

We begin with some definitions that have been introduced for noiseless coin tossing (see, e.g., Ref. \textsuperscript{5}). A coin tossing protocol involves a sequence of rounds of communication, at the end of which either a bit $x$ is produced, whose value is agreed on by both parties, or one party or the other aborts, in which case we write symbolically $x = \infty$. We denote by $P_{\text{HA},\text{HB}}(x = c)$ the probability that $x = c$, assuming that Alice follows a strategy $S_A$ and Bob $S_B$. We denote Alice’s honest strategy (i.e., that defined by the protocol) as $H_A$ and Bob’s as $H_B$. A protocol is correct if, for $c = 0, 1$,

\begin{equation}
P_{H_A,H_B}(x = c) = 1/2.
\end{equation}

Security conditions are written as

\begin{align}
\forall S_A \quad & P_{S_A,H_B}(x = 0) \leq 1/2 + \epsilon_A \tag{2} \\
\forall S_A \quad & P_{S_A,H_B}(x = 1) \leq 1/2 + \epsilon_A \tag{3} \\
\forall S_B \quad & P_{H_A,S_B}(x = 0) \leq 1/2 + \epsilon_B \tag{4} \\
\forall S_B \quad & P_{H_A,S_B}(x = 1) \leq 1/2 + \epsilon_B. \tag{5}
\end{align}

(These conditions define the task as strong coin tossing. A weaker task can be defined by imposing only \textsuperscript{(2)} and \textsuperscript{(3)}. This is known as weak coin tossing. In this work we are concerned only with strong coin tossing. With this understood, we shall simply call it coin tossing.) A protocol is perfectly secure iff $\epsilon_A = \epsilon_B = 0$. A protocol is arbitrarily secure iff $\epsilon_A$ and $\epsilon_B$ can be made arbitrarily small as some parameter associated with the protocol increases. A protocol is partially secure iff
Perfectly secure coin tossing is shown to be impossible in Ref. 4. More recently, Kitaev has shown 12 that for any possible protocol, either $\epsilon_A$ or $\epsilon_B \geq 1/\sqrt{2} - 1/2$. This result implies that arbitrarily secure strong coin tossing is impossible. At present, the best protocol for strong coin tossing is due to Ambainis 3 and achieves $\epsilon_A = \epsilon_B = 1/4$ (close to Kitaev’s lower bound).

In order to discuss what happens when noise is present, we first describe a very simple protocol for coin tossing in the absence of noise (it is similar to the protocol for quantum gambling developed in Ref. 17). The protocol is for strong coin tossing and is partially secure. It is not as good as that of Ambainis, but is illustrative. We shall then consider how it is affected by noise. The protocol is as follows.

i) Alice generates a random bit $b \in \{0, 1\}$, and prepares a quantum state $|\phi_b\rangle$, where $0 < |\langle \phi_0 | \phi_1 \rangle|^2 < 1$. We write $|\langle \phi_0 | \phi_1 \rangle|^2 \equiv \cos^2 \theta$. Alice sends $|\phi_b\rangle$ to Bob.

ii) Bob generates a random bit $b'$ and sends it to Alice.

iii) Alice sends $b$ to Bob.

iv) Bob measures the quantum state in a basis containing $|\phi_b\rangle$ to check that Alice is not cheating. He aborts if he gets an outcome different from $|\phi_b\rangle$. In this case we write the outcome of the coin toss as $x = \infty$. Otherwise, the outcome of the coin toss is $x = b \oplus b'$.

It is easy to show that Alice’s best cheating strategy is to send a state $|\chi\rangle$ such that $|\langle \chi | \phi_0 \rangle|^2 = |\langle \chi | \phi_1 \rangle|^2 = \cos^2 \theta/2$, and then to declare $b$ as she requires. (In particular, it is never to her advantage to declare the opposite value of $b$, and this implies that strategies that involve her entangling two systems and sending one to Bob cannot help.) Bob’s best cheating strategy is to measure the quantum state as soon as he receives it, in order to determine as well as possible whether it is $|\phi_0\rangle$ or $|\phi_1\rangle$. We get that for $c = 0, 1,$

$$\forall S_A \Pr^{S_A,H_B}(x = c) \leq 1/2 + (1/2) \cos \theta$$

$$\forall S_B \Pr^{H_A,S_B}(x = c) \leq 1/2 + (1/2) \sin \theta. \quad (6)$$

Note that the protocol is correct.

How are this protocol, and the corresponding security considerations, affected if it is assumed that noise is present? We discuss mainly the case in which the quantum channel separating Alice and Bob is noisy, but in which all other devices are perfect. Other types of experimental imperfections that could be considered include, for example, noise in state preparation and measurement, or the limited efficiency of detectors. The arguments we use in the case of noisy channels can easily be adapted to these other situations. In general, the channel can be described by a linear, completely positive, trace-preserving map $\mathcal{E}$. Thus Bob will receive the noisy state $\rho_a = \mathcal{E}(|\psi_0\rangle)$. For long distance quantum communication the principal type of noise will be losses. Losses can in principle be included in the form of $\mathcal{E}$, but it may be convenient to consider them separately. An honest party must assume that the other, potentially dishonest, party can control everything outside the honest party’s laboratory. In particular, this means that a cheater can replace the noisy channel with a noiseless one, introducing noise only as and when he or she wants.

The presence of noise on the face of it affords several simple cheating strategies. We list a few.

i) If the channel is lossy, and the rules specify that if Bob’s detector does not fire then he simply has to accept whatever value of $b$ Alice declares, then Alice may cheat simply by not sending any quantum state at all. Bob will think that the state was lost in the channel. Alice will declare whatever value of $b$ she wants, winning with certainty.

ii) Suppose alternately that the rules specify that if Bob receives no state, he is allowed to declare “no fire,” and the protocol recommences. Then a cheating Bob may replace the noisy, lossy channel with a noiseless one, so that the states he must discriminate are pure. He then (using a perfect detector) performs a conclusive measurement on the quantum state, with outcomes “definitely $|\phi_0\rangle,” “definitely $|\phi_1\rangle,” and “inconclusive” 18. If the “inconclusive” result is obtained, then Bob declares “no fire,” and the protocol repeats. The protocol will keep repeating until a run occurs on which Bob finds out the value of $b$ with certainty. He can then obtain the outcome he wants with certainty by choosing $b'$ appropriately.

iii) In general, both $\rho_0$ and $\rho_1$ will have support on the entire Hilbert space. This means that a cheating Alice can send $|\phi_0\rangle$ to Bob, and then declare that $b = 0$ or $b = 1$ as she needs. Even in the event that Alice declares $b = 1$, there will be no measurement result of Bob’s that tells him for certain that Alice is cheating. In addition, if the channel is noisy and if a certain outcome of Bob’s leads him to abort the protocol, then there is a nonzero probability that a player aborts, even when both are honest.

It should be clear that most of these remarks will apply in some form or other to any conceivable coin tossing protocol. If correctness is redefined as

$$\Pr^{H_A,H_B}(x = c) = (1 - \delta)/2$$

$$\Pr^{H_A,H_B}(x = \infty) = \delta, \quad (7)$$

for $c = 0, 1$, where $\delta$ must tend to zero as some parameter associated with the protocol increases, and if similar modifications are made to the security conditions, then protocols can still be made (at least partially) secure, as long as error correction is available. In this work, however, we are interested in the case in which there is no technique for making noise arbitrarily small. Protocols for tossing a single coin will then either allow one of the parties to bias the coin completely, or will abort with unacceptably high probability even though both parties are...
honest. They are therefore of limited interest. For these reasons we consider instead a slightly different scenario, in which Alice and Bob want to generate a random \( n \)-bit string. 

The basic reason why random \( n \)-bit string generation is easier to implement in the presence of noise is that, since the quantum channel is used many times, one can test that the average noise level is as expected. This is to be contrasted with a single use of the channel, which only provides very partial information about the noise present. Furthermore the probability that the protocol aborts when both parties are honest decreases exponentially with \( n \). This is due to the fact that the channel is used many times, which makes large fluctuations about the expected noise highly improbable. On the other hand, because the noise is only constrained on average, a cheater can determine any bit of the string with certainty. This implies that one cannot extract from the bit string a single unbiased bit (for instance by taking the parity of the bit string). But the average bias of the bits can be bounded\(^\text{[2]}\).

The problem of generating a random \( n \)-bit string in the absence of noise has previously been considered by Kent\(^\text{[13]}\), who showed that it is not a straightforward extension of the problem of generating a single random bit. We now introduce some precise security criteria for \( n \)-bit string generation. The output of a protocol for \( n \)-bit string generation is \( \vec{x} \), where either \( \vec{x} \) is an \( n \)-bit string, or one or the other party aborts, in which case we write symbolically \( \vec{x} = \infty \).

One security condition that one could imagine for \( n \)-bit string generation is that each bit of the string has small bias. This could be expressed as

\[
\forall A \forall i\Pr[S_A,H_B(x_i = c)] \leq 1/2 + \epsilon_A, \tag{8}
\]

for \( c = 0, 1 \), along with a similar condition for Bob. Here, \( x_i \) is the \( i \)-th bit of \( \vec{x} \). As we have argued above, this type of security is not achievable in the presence of noise.

A weaker security condition is that on average the bias of the bits is small. We express this as

\[
\forall A \forall \vec{c} \frac{1}{n} \sum \Pr[S_A,H_B(x_i = c_i)] \leq 1/2 + \epsilon_A, \tag{9}
\]

along with a similar condition for Bob. We have that \( x_i, c_i \) are the \( i \)-th bits of \( \vec{x}, \vec{c} \), with \( \vec{c} \) an arbitrary \( n \)-bit string describing a possible result of the coin tosses.

Other security conditions are possible. Satisfaction of condition \( \text{[3]} \) or \( \text{[4]} \) is compatible with a cheater fixing things so that the outcome is either \( 00 \ldots 0 \) with probability \( 1/2 \) or \( 11 \ldots 1 \), with probability \( 1/2 \). It would be desirable to have a security condition expressing the fact that the entropy of the bit string is large, satisfaction of which would rule out such cheating. In the remainder of this article, however, we will restrict ourselves to the security condition expressed by Eq. \( \text{[9]} \). In general, \( \epsilon_A \) and \( \epsilon_B \) will depend on \( n \). For simplicity we shall only be interested in the values of these quantities in the limit of large \( n \).

We define correctness by

\[
\forall \vec{c} \Pr[H_A,H_B(\vec{x} = \vec{c})] = (1 - \delta)/2^n \quad \Pr[H_A,H_B(\vec{x} = \infty)] = \delta, \tag{10}
\]

where \( \delta \) must tend to zero as \( n \) becomes large.

We now introduce a protocol for random \( n \)-bit string generation that is adapted from the simple coin tossing protocol above. Then we will consider the security of this protocol in the presence of noise.

\textbf{1) For } \( i = 1 \text{ to } n \):

\begin{itemize}
\item[i)] Alice generates a random bit \( b_i \in \{0, 1\} \), and prepares a quantum state \( |\phi_{b_i}\rangle \), where \( 0 < |\langle \phi_{0}|\phi_{1}\rangle|^2 < 1 \). Alice sends \( |\phi_{b_i}\rangle \) to Bob.
\item[ii)] Bob generates a random bit \( b'_i \) and sends it to Alice.
\item[iii)] Alice sends \( b_i \) to Bob.
\item[iv)] Bob measures the quantum state in a random basis. If his detector fails, this is considered as a null outcome.
\end{itemize}

\textbf{Next } \( i \).

\textbf{2) Bob} uses his measurement statistics to estimate \( \rho_0 \), the average state he received when Alice declared \( b_i = 0 \), and \( \rho_1 \), the average state he received when Alice declared \( b_i = 1 \). If either of the fidelities, \( \langle \phi_{0}|\rho_0|\phi_{0}\rangle \) and \( \langle \phi_{1}|\rho_1|\phi_{1}\rangle \), is less than \( 1 - \gamma \) (where \( 0 < \gamma \leq 1 \) is decided in advance), then Bob aborts the protocol. Otherwise the output of the protocol is an \( n \)-bit string with \( x_i = b_i \oplus b'_i \).

This protocol is essentially the simple protocol above repeated \( n \) times, with two modifications. First, Bob measures each time in a random basis - he is performing a sort of state estimation in order that he can bound any potential cheating by Alice. Second, an honest Bob does not have the option of aborting until the end of the protocol when he has collected all his statistics. It is easy to see that if the fidelity of the whole process of state preparation, transmission and measurement is \( F > 1 - \gamma \), then the protocol is correct.

We investigate available cheating strategies for this protocol, assuming that the only noise is noise in the quantum channel, described by \( \mathcal{E} \), and that either there are no losses or they are included in the form of \( \mathcal{E} \). We can consider two cases. In the first, a cheater’s actions on different runs (that is, different values of \( i \) in the protocol above) are uncorrelated. Thus, we simply need to consider one strategy, perhaps involving random choices, that is repeated for each run. In the second, a cheater’s actions on different runs may be correlated, and may even involve entanglement across the different runs\(^{[22]}\).

In this paper, we will restrict ourselvess to uncorrelated cheating. Elsewhere, we show that this protocol in fact satisfies an entropic security condition, even when correlated or entangled attacks are considered\(^{[16]}\).

A cheating Bob is easiest to deal with. His best strategy is to replace the noisy channel with a noiseless one,
thus ensuring that the states he receives are \(|\phi_0\rangle\) and \(|\phi_1\rangle\). He can measure each state as soon as he receives it, in order to determine as well as possible the identity of the state. He can then choose \(b_i'\) appropriately. As mentioned before, this gives \(\epsilon_B = 1/2 \sin \theta\), where \(|\langle \phi_0 | \phi_1 \rangle|^2 = \cos^2 \theta\).

The most general strategy for a cheating Alice is to prepare a pure state \(|\psi\rangle_{AB}\), and send the \(B\) subsystem to Bob via a noiseless channel. (In general, of course, Alice may prepare an overall mixed state, perhaps resulting from a probabilistic mixture of pure states. We lose no generality, however, by supposing that Alice prepares a pure state, as Alice can always introduce an extra ancilla such that \(|\psi\rangle_{AB}\) is a purification of the mixed state.) We denote the reduced density matrix for Bob’s subsystem by \(\rho_B\). Alice then waits for the bit \(b_i'\). The value of \(b_i'\) and the outcome of the coin toss that she wants to determine together the value of \(b_i\) that Alice wants to declare. If Alice wants to declare \(b_i = 0\), then she performs a two-outcome positive operator-valued (POV) measurement \(M_0\) on the \(A\) subsystem. Denoting the outcomes \(M_{00}\) and \(M_{01}\), Alice declares \(b_i = 0\) (thus winning) if she obtains \(M_{00}\) and \(b_i = 1\) (thus losing) if she obtains \(M_{01}\). If Alice wants to declare \(b_i = 1\), on the other hand, then she performs a POV measurement \(M_1\). She declares \(b_i = 0\) (losing) if she obtains \(M_{10}\) and \(b_i = 1\) (winning) if she obtains \(M_{11}\).

What advantage does this strategy give Alice? Suppose that Bob’s (normalized) reduced density matrices, conditioned on Alice getting the outcomes \(M_{00}\), \(M_{01}\), \(M_{10}\) and \(M_{11}\), are \(\sigma, \bar{\sigma}, \tau\) and \(\bar{\tau}\) respectively. Then we can write

\[
\begin{align*}
\rho_B &= q \sigma + (1 - q) \bar{\sigma}, \\
\rho_B &= q' \tau + (1 - q') \bar{\tau},
\end{align*}
\]

where \(q\) is the probability of Alice getting outcome \(M_{00}\), given that she performs measurement \(M_0\), and \(q'\) is the probability of her getting outcome \(M_{11}\), given that she performs measurement \(M_1\). It can be shown via a symmetry argument that we do not lose generality if we suppose that

\[
q = q'.
\]

We can also write

\[
\begin{align*}
\rho_B &= \frac{1}{2} (\rho_0 + \rho_1), \\
\rho_0 &= q \sigma + (1 - q) \bar{\tau}, \\
\rho_1 &= q' \tau + (1 - q') \bar{\sigma}.
\end{align*}
\]

The probability of Alice getting the outcome she wants is given by \(q\), so we have that \(\epsilon_A = q - 1/2\). The problem is now to maximize \(q\) subject to the constraints of Eqs. (11)-(16) (and of course the constraints that \(0 \leq q \leq 1\) and that \(\sigma, \bar{\sigma}, \tau, \bar{\tau}\) are valid normalized density operators). Note that if we find a solution for valid \(\sigma, \bar{\sigma}, \tau, \bar{\tau}\), then the Hughston-Jozsa-Wootters (HJW) theorem ensures that there does indeed exist a strategy of Alice’s that corresponds to this solution. In other words, there is a state \(|\psi\rangle_{AB}\), and measurements \(M_0\) and \(M_1\), that give rise to \(\sigma, \bar{\sigma}, \tau, \bar{\tau}\) when we condition on Alice’s outcomes.

We have obtained an upper bound on Alice’s cheating capacity that applies for arbitrary quantum states and noise. We write the fidelity between a general state \(\rho\) and a pure state \(|\psi\rangle\) as \(F(\rho, |\psi\rangle) = \langle \psi | \rho | \psi \rangle\). We write the trace distance between two general states \(\rho\) and \(\rho'\) as \(D(\rho, \rho') = 1/2 \| \rho - \rho' \|_1\), where \(|A| = \text{Tr} \sqrt{A^\dagger A}\). All the results concerning these quantities used below can be found in Ref. [21] (although note that the fidelity is defined slightly differently).

**Theorem 1** For all uncorrelated strategies of Alice, we have that for large \(n\),

\[
\epsilon_A \leq \frac{\sqrt{2} \pi}{\sin^2 \theta}.
\]

To prove this bound, note that if Bob is not to abort we must have \(F(\rho_0, |\phi_0\rangle) \geq 1 - \gamma\), and that this, along with Eq. (15), gives

\[
q \langle \phi_0 | \sigma | \phi_0 \rangle + (1 - q) \langle \phi_0 | \bar{\tau} | \phi_0 \rangle \geq 1 - \gamma.
\]

This in turn implies

\[
\langle \phi_0 | \sigma | \phi_0 \rangle \geq 1 - \gamma/q.
\]

Using the fact that \(D(\sigma, |\phi_0\rangle) \leq \sqrt{1 - F(\rho, \rho')}\) for arbitrary states \(\rho\) and \(\rho'\), this gives us

\[
D(\sigma, |\phi_0\rangle) \leq \sqrt{\gamma/q}.
\]

Similarly, we can derive \(D(\bar{\tau}, |\phi_0\rangle) \leq \sqrt{\gamma/(1 - q)}\), \(D(\bar{\sigma}, |\phi_1\rangle) \leq \sqrt{\gamma/(1 - q)}\), and \(D(\tau, |\phi_1\rangle) \leq \sqrt{\gamma}q\). We now recall, from Eqs. (11)-(13), that

\[
\rho_B = q \sigma + (1 - q) \bar{\sigma} = q \tau + (1 - q) \bar{\tau}.
\]

Combining this with the above, and using the fact that \(D(\rho, \rho') = \max_P |\text{Tr}(P \rho) - \text{Tr}(P \rho')|\), where the maximum is over all projection operators, we get that

\[
q'(\text{Tr}(P |\phi_0\rangle \langle \phi_0|) - \sqrt{\gamma/q}) + (1 - q)(\text{Tr}(P |\phi_1\rangle \langle \phi_1|) - \sqrt{\gamma/(1 - q)}) \leq q'\text{Tr}(P |\phi_1\rangle \langle \phi_1|) + \sqrt{\gamma/q} + (1 - q)\text{Tr}(P |\phi_0\rangle \langle \phi_0|) + \sqrt{\gamma/(1 - q)},
\]

for any projection operator \(P\). Setting \(P = |\phi_0\rangle \langle \phi_0|\) then gives Theorem 1.

We have also analyzed in detail the simple case in which the quantum states are qubit states and the channel is a depolarizing channel, acting as \(\rho \rightarrow E(\rho) \equiv f \rho + (1 - f) I/2\). In this case, Alice’s optimal cheating strategy can be found explicitly:
Theorem 2: For the qubit depolarizing channel, if Alice adopts her optimal uncorrelated cheating strategy, then for large $n$

$$\epsilon_A = \frac{1}{2} (1 - f \sin \theta) \quad \text{if } f \leq f^*, $$

$$\epsilon_A = \frac{1}{2} \sqrt{f^2 (1 - f^2) \cos^2 \theta} \quad \text{if } f > f^*, $$

where $f^* \equiv (\sqrt{1 + 3 \cos^2 \theta} - \sin \theta)/2 \cos^2 \theta$.

The proof of Theorem 2 is given in the appendix.

We can compare this result with the upper bound above. If we set $f = 1 - 2\gamma$, then for fixed $\theta$, we find that $\epsilon_A \rightarrow \sqrt{\gamma} \cot \theta$, as $\gamma \rightarrow 0$. This shows that the $\gamma$ dependence of Eq. 16 is close to optimal.

In conclusion, we have shown that the attainable security in quantum coin tossing is qualitatively affected by the presence of noise. Indeed in the presence of significant noise, tossing a single coin does not give acceptable security. Rather, in this case one should consider protocols for generation of strings of random bits. As we explain above, generating a string of random bits is a weaker protocol than tossing a single coin. However in situations where one needs to toss coins many times in succession (for instance if one wants to play repeatedly with a quantum casino), then bit-string generation can be useful. The importance of bit-string generation is that even in situations where tossing a single coin is impossible, it will be possible to generate a string of bits such that the average bias of the bits is bounded. We have illustrated this by a simple protocol for which we prove bounds on the average bias in the case where uncorrelated cheating strategies are used.

Our work is motivated by the present status of quantum communication. Indeed with present day optical technology, quantum communication can be performed over short distances (e.g., laboratory length scales) with minimal noise and absorption. In this case, Theorem 1 above indicates that quantum $n$-bit string generation, using our protocol, should be practically possible with good security. Over longer distances (kilometers and above), losses in particular are significant, and our results would need to be generalized.

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Appendix: Proof of Theorem 2

To prove Theorem 2, note that without loss of generality we can write

$$|\phi_0\rangle\langle\phi_0| = 1/2 (I + \alpha \sigma_z - \beta \sigma_x),$$

$$|\phi_1\rangle\langle\phi_1| = 1/2 (I + \alpha \sigma_z + \beta \sigma_x),$$

where $\sigma_x$ and $\sigma_z$ are Pauli sigma matrices, $\alpha^2 + \beta^2 = 1$, and we have that $\alpha = \cos \theta$, $\beta = \sin \theta$. Recall that the channel acts as

$$\rho \rightarrow E(\rho) \equiv f \rho + (1 - f) I/2.$$  

We then begin by writing

$$\rho_0 = 1/2 (I + \alpha f \sigma_z - \beta f \sigma_x),$$

$$\rho_1 = 1/2 (I + \alpha f \sigma_z + \beta f \sigma_x)$$

and

$$q\sigma = 1/2 (q I + s_x \sigma_x + s_y \sigma_y + s_z \sigma_z),$$

$$(1 - q)\bar{s} = 1/2 ((1 - q) I + \bar{s}_x \sigma_x + \bar{s}_y \sigma_y + \bar{s}_z \sigma_z),$$

$$q\tau = 1/2 (q I + t_x \sigma_x + t_y \sigma_y + t_z \sigma_z),$$

$$(1 - q)\bar{t} = 1/2 ((1 - q) I + \bar{t}_x \sigma_x + \bar{t}_y \sigma_y + \bar{t}_z \sigma_z),$$

where $-1 \leq s_x, s_y, s_z \leq 1$, and so on. Conditions 11-16 imply

$$s_z + \bar{s}_z = f\alpha, \quad s_x + \bar{s}_x = 0,$$

$$t_z + \bar{t}_z = f\alpha, \quad t_x + \bar{t}_x = 0,$$

$$s_z + \bar{t}_z = f\alpha, \quad s_x + \bar{t}_x = -f\beta,$$

$$\bar{s}_z + t_z = f\alpha, \quad \bar{s}_x + t_x = f\beta,$$

while symmetry considerations imply

$$s_y = \bar{s}_y = t_y = \bar{t}_y = 0.$$  

From the positivity of the matrices $\sigma, \bar{\sigma}, \tau, \bar{\tau}$, we then have that

$$s_x^2 + s_z^2 \leq q^2,$$

$$(f\alpha - s_z)^2 + s_y^2 \leq (1 - q)^2,$$

$$s_x^2 + (f\beta + s_z)^2 \leq q^2,$$

$$(f\alpha - s_z)^2 + (f\beta + s_z)^2 \leq (1 - q)^2.$$  

Our aim is to maximize $q$ with respect to $s_z$ and $s_x$, subject to the various constraints. The HJW theorem will ensure that there do exist measurements $M_0$ and $M_1$, such that conditions 11, 12, and 16 are satisfied.

By inspection we see that the maximum value of $q$ can be obtained when $s_z = -f\beta/2$. This leaves

$$s_x^2 + 1/4 (f^2 \beta^2) \leq q^2,$$

$$(f\alpha - s_x)^2 + 1/4 (f^2 \beta^2) \leq (1 - q)^2.$$  

If we consider the equation derived from each of these inequalities, we see that each represents a hyperbola in the $q_{s_z}$ plane. Geometrical considerations tell us that there are two cases to be considered. In the first case, we need only find a turning point of the second hyperbola, and it is guaranteed to lie above the first hyperbola, so
that the first inequality will be satisfied. This occurs if 
\[ f^2 \alpha^2 \leq 1 - f|\beta| \quad (i.e., \ f < f^*) \]
In this case, it is easy to see that we maximize \( q \) by setting \( s_z = f \alpha \), giving

\[
q = 1 - (1/2)(f|\beta|), \\
\epsilon_A = (1/2)(1 - \sin \theta).
\]  

(28)

In the second case, the relevant turning point of the second hyperbola lies below the first, implying that this is not a solution that satisfies both inequalities. This occurs if \( f^2 \alpha^2 > 1 - f|\beta| \) (i.e., \( f > f^* \)). In this case, we find the maximum \( q \) by finding the intersection of the two hyperbolae, i.e., by considering both inequalities as equalities. A short calculation then gives

\[
q = \frac{1}{2} + \frac{1}{2} \sqrt{\frac{f^2 \alpha^2(1 - f^2)}{1 - f^2 \alpha^2}}, \\
\epsilon_A = \frac{1}{2} \sqrt{\frac{f^2(1 - f^2) \cos^2 \theta}{1 - f^2 \cos^2 \theta}},
\]  

(29)

which is achieved when

\[
s_z = \frac{1}{2} f \alpha + \frac{q}{f \alpha} - \frac{1}{2f \alpha}.
\]  

(30)

Throughout, we have ignored solutions corresponding to unphysical values of the variables. The explicit forms for Alice’s measurements \( M_0 \) and \( M_1 \) can be calculated from the values for the ss and the ts, although we have not done this here.

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