A RIEMANN-HILBERT PROBLEM FOR AN ENERGY DEPENDENT SCHröDINGER OPERATOR

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Abstract. We consider an inverse scattering problem for Schrödinger operators with energy dependent potentials. The inverse problem is formulated as a Riemann-Hilbert problem on a Riemann surface. A vanishing lemma is proved for two distinct symmetry classes. As an application we prove global existence theorems for the two distinct systems of partial differential equations $u_t + (u^2/2 + w)_x = 0$, $w_t + u_{xxx} + (uw)_x = 0$ for suitably restricted, complementary classes of initial data.

Dedicated to Hugh Turrittin on his 90th birthday

1. Introduction

In this paper the scattering theory of the energy dependent Schrödinger operator

$$\left(D^2 + k^2\right)\psi = (ikp(x) + q(x))\psi, \quad D = \frac{d}{dx}, \quad (1.1)$$

is investigated and used to prove global existence theorems for the “isospectral flows”:

$$u_t + (w + u^2/2)_x = 0, \quad w_t + u_{xxx} + (uw)_x = 0; \quad p = \frac{u}{2}, \quad q = \frac{u^2}{16} - \frac{w}{4} \quad (1.2)$$

and

$$u_t + (w + u^2/2)_x = 0, \quad w_t - u_{xxx} + (wu)_x = 0; \quad p = \frac{iu}{2}, \quad q = \frac{w}{4} - \frac{u^2}{16} \quad (1.3)$$

The scattering problem for such energy dependent Schrödinger operators was first considered by Jaulent [8], Jaulent and Jean [9], Kaup [10], and more recently by the present authors [14]. In [8], [9] and [14] the problem was considered for purely imaginary $p$ and real $q$ tending to zero at infinity. The inverse problem was solved by a modification of the Gel’fand-Levitan-Marchenko method.

Kaup studied the scattering problem under the assumption of real $p$, and $q$ tending to non-zero limits at infinity, in connection with a long wave approximation.
of Boussinesq type (equations (1.2)). He obtained a coupled pair of equations of GLM type, but did not investigate their solvability.

The GLM equations are of Fredholm type, and so their solvability is a consequence of uniqueness. Such uniqueness theorems are valid, (though usually not proved in the literature) in the case of the standard Schrödinger equation, or the isospectral problem obtained by Zakharov and Shabat in their study of the nonlinear Schrödinger equation [15]. But they fail in general for $2 \times 2$ AKNS systems without further symmetry assumptions on the potential, for example, that it is symmetric or skew symmetric, Hermitian or skew-Hermitian.

We investigate the solvability of the inverse scattering problem for (1.1) in the two cases $p$ real or imaginary, corresponding to the two flows (1.2) and (1.3). Uniqueness theorems are proved in these two cases under complementary, restricted conditions on the scattering data. Global existence theorems for the flows (1.2), (1.3) can then be proved by showing that the constraints on the scattering data are invariant under the flows.

This, however, entails restrictions on the initial data. Since global existence theorems can be proved for the two flows only for restricted initial data, the physical relevance of these two models is unclear. Our interest in the matter stems rather from the novel features of the associated inverse scattering problem, and its formulation as a Riemann-Hilbert problem on a Riemann surface.

Riemann-Hilbert problems arise in integrable systems as a more general formulation of inverse scattering problems. In second order cases, such as the Schrödinger and Zakharov-Shabat scattering problems, the Gel’fand-Levitan-Marchenko theory and Riemann-Hilbert method are equivalent, being essentially Fourier transforms of one another. But there is no GLM theory for $n \times n$ first order systems or $n^{th}$ order ordinary differential operators ($n > 2$); and the inverse scattering problem must be formulated as a Riemann-Hilbert problem [1, 2]. Moreover, even for second order problems the method of Riemann-Hilbert problems, coupled with the method of steepest descent, can be used to obtain precise asymptotic behavior of the solutions [5].

One can go from (1.2) to (1.3) by the transformation

$$u(x,t) \rightarrow iu(x,it), \quad w(x,t) \rightarrow -w(x,it);$$

but this transformation is complex, so the two equations may be considered to be distinct real forms of one another.

Sachs [13] found a pair of tau functions for (1.2) and used them to construct rational solutions similar to the Calogero-Moser solutions of the KdV equation. Matveev and Yavor [11] used Riemann surface theory to construct finite gap solutions, and, in the limiting case, multi-soliton solutions.
Isospectral flows for energy dependent operators arise in a wide variety of applications. Recently, for example, Camassa and Holm [4] found a special shallow water wave equation generated by an energy dependent iso-spectral operator related to the Dym hierarchy. For a formal discussion of isospectral flows generated by operators with energy dependent potentials, see Fordy [7].

The Gel’fand-Dikki hierarchy of isospectral flows of $n^{th}$ order ordinary differential operators can be formally extended to operators $L$ in which the coefficient of $D^j$ is a polynomial of degree $j-1$ in the spectral parameter. These considerations suggest, at least on a formal level, the existence of a large class of flows, of which the Gel’fand-Dikki flows are a special case. We conjecture that all these flows are local. The corresponding inverse problems lead to a broader class of Riemann-Hilbert problems than those considered in [3].

2. The forward problem

Throughout this paper we assume that $q \to 1$ as $x \to \pm \infty$ and rewrite (1.1) as

$$(D^2 + E^2 - (ikp + q))\psi = 0, \quad E^2 = k^2 + 1, \quad (2.1)$$

where now $p$ and $q$ belong to Schwartz class $S$. Although the restriction to potentials in Schwartz class could be weakened considerably in the theory of the forward and inverse scattering problem, the class is a convenient one for discussing completely integrable systems with an infinite number of conservation laws, and so we shall make this assumption throughout.

The scattering problem may be formulated as a Riemann-Hilbert problem on the Riemann surface $E^2 = k^2 + 1$. Following Kaup, we introduce the uniformizing parameter $z$:

$$E = \frac{1}{2} \left( z + \frac{1}{z} \right), \quad k = \frac{1}{2} \left( z - \frac{1}{z} \right). \quad (2.2)$$

The transformation $z \mapsto E$ is a double covering of the $E$ plane by the $z$ plane. Define four regions $\mathcal{U}_j$, $j = 1, 2, 3, 4$, where $\mathcal{U}_1 = \{ z : \Im z > 0 \cap |z| > 1 \}$, $\mathcal{U}_2$ is its image under complex conjugation, $\mathcal{U}_3$ the upper half of the unit disk, and $\mathcal{U}_4$ the lower half of the unit disk. Let

$$\mathcal{U}_+ = \mathcal{U}_1 \cup \mathcal{U}_4 = \{ z : \Re E > 0 \}, \quad \mathcal{U}_- = \mathcal{U}_2 \cup \mathcal{U}_3 = \{ z : \Re E < 0 \}.$$ 

Finally, let $\Sigma$ denote the union of the real line and the unit circle in the $z$ plane, less the origin.

We construct two sets of wavefunctions of (2.1), $\psi$ and $\phi$ defined by the following asymptotic behavior:

$$\psi_\pm(x, z) = m_\pm^{\text{out}} e^{\pm iEx}, \quad z \in \mathcal{U}_\pm \quad \phi_\pm(x, z) = m_\pm^{\text{in}} e^{\mp iEx}, \quad z \in \mathcal{U}_\pm$$
where \( m_{\pm}^{\text{out}} \) and \( m_{\pm}^{\text{in}} \) are analytic on \( \mathcal{U}_{\pm} \) respectively, and

\[
\lim_{x \to -\infty} m_{\pm}^{\text{in}} = \lim_{x \to \infty} m_{\pm}^{\text{out}} = 1.
\]

Equation (2.1) can be converted to a Volterra integral equation, for example,

\[
\psi_{\pm}(x, z) = e^{\pm iEx} - \int_{x}^{\infty} \frac{\sin E(x - y)}{E} (ikp(y) + q(y))\psi_{\pm}(y, z) \, dy, \quad z \in \mathcal{U}_{\pm}.
\]

This leads to the following integral equations for \( m_{\pm}^{\text{out}} \):

\[
m_{\pm}^{\text{out}}(x, z) = 1 \mp \int_{x}^{\infty} \frac{1 - e^{\pm 2iE(x-y)}}{2iE} (ikp(y) + q(y))m_{\pm}^{\text{out}}(y, z) \, dy, \quad z \in \mathcal{U}_{\pm}.
\] (2.3a)

The wave functions \( \phi_{\pm} \) and \( m_{\pm}^{\text{in}} \) satisfy similar Volterra integral equations. For example,

\[
m_{\pm}^{\text{in}}(x, z) = 1 \mp \int_{-\infty}^{x} \frac{1 - e^{\pm 2iE(x-y)}}{2iE} (ikp(y) + q(y))m_{\pm}^{\text{in}}(y, z) \, dy, \quad z \in \mathcal{U}_{\pm}. \] (2.3b)

The integral equations for \( m_{\pm}^{\text{in}}, m_{\pm}^{\text{out}} \) may be solved by successive approximations for all \( z \in \mathcal{U}_{\pm} \setminus \{0, \infty\} \). (See also the form of the wave functions given in the Appendix.)

Throughout this paper we shall understand by \( \psi \) the sectionally analytic function

\[
\psi(x, z) = \begin{cases} 
\psi_{+}(x, z) & z \in \mathcal{U}_{+}; \\
\psi_{-}(x, z) & z \in \mathcal{U}_{-}.
\end{cases}
\]

The other wave functions are denoted similarly.

The asymptotic behaviors of the wave functions \( m_{\pm}^{\text{out}}, m_{\pm}^{\text{in}} \) as \( z \to 0, \infty \) are easily determined. We note that

\[
\frac{k}{E} \to \begin{cases} 
1 & z \to \infty; \\
-1 & z \to 0.
\end{cases}
\]

To simplify the discussion, we assume, as in [14], that

\[
\int_{-\infty}^{\infty} p \, dy = 0.
\]

This assumption is not essential and could be dropped, though with a consequent increase in the computational details. Letting

\[
\lambda = e^{P/2}, \quad P = \int_{x}^{\infty} p(y) \, dy,
\]
we easily deduce from the integral equations (2.3a,b), that
\[
\lim_{z \to \infty} m^{\text{out}}_{\pm}(x, z) = \lambda^{\pm 1}, \quad \lim_{z \to 0} m^{\text{out}}_{\pm}(x, z) = \lambda^{\pm 1}, \quad z \in \mathcal{U}_{\pm};
\]
\[
\lim_{z \to \infty} m^{\text{in}}_{\pm}(x, z) = \lambda^{\pm 1}, \quad \lim_{z \to 0} m^{\text{in}}_{\pm}(x, z) = \lambda^{\mp 1}, \quad z \in \mathcal{U}_{\pm}.
\]
(2.4)

Note that $\lambda$ is real when $p$ is real and has modulus 1 when $p$ is imaginary.

If $p$ and $q$ are in the Schwartz class, the reduced wave functions $m^{\text{out}}$ have asymptotic expansions in powers of $z$. Substituting $\psi_+ = m^{\text{out}}_+ e^{iEx}$ into (2.1) we get the following equation for $m = m^{\text{out}}_+$:
\[
m'' + 2iEm' - (ikp + q)m = 0,
\]
We seek a formal asymptotic expansion of $m$ in inverse powers of $z$; that is, we set
\[
m(x, z) \sim \sum_{j=0}^{\infty} m_j(x) z^{-j}
\]
and substitute this asymptotic expansion into the differential equation for $m$. At orders 0 and 1 we obtain
\[
im_0' - \frac{i}{2}pm_0 = 0, \quad m''_0 + im_1' - \frac{ipm_1}{2} - qm_0 = 0.
\]

From these two equations we obtain
\[
m_0 = e^{-p/2} = \lambda^{-1}, \quad q = \frac{p^2}{4} + \frac{d}{dx}\left(\frac{p}{2} + im_1 e^{p/2}\right)
\]
(2.5)

As in the theory of the standard Schrödinger operator, the wave functions satisfy the usual relations on $\Sigma$:
\[
\phi_+(x, \xi) = a(\xi)\psi_-(x, \xi) + b(\xi)\psi_+(x, \xi)
\]
\[
\phi_-(x, \xi) = c(\xi)\psi_-(x, \xi) + d(\xi)\psi_+(x, \xi)
\]
(2.6)

where $\phi_+(x, \xi)$ denotes the limiting value of $\phi_+(x, z)$ as $z \to \xi$ from $\mathcal{U}_+$, etc. The coefficients in (2.6) relate the incoming and scattered waves. Thus, for example, from (2.6) it follows that for $\xi \in \Sigma$
\[
\phi_+(x, \xi) \sim \begin{cases} e^{-iEx} & x \to -\infty; \\
 a(\xi)e^{-iEx} + b(\xi)e^{iEx} & x \to \infty.
\end{cases}
\]

The matrix of coefficients
\[
S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]
is sometimes called the scattering matrix ([6]). The coefficients of $S$ may be computed in terms of Wronskians of the wave functions, just as for the standard Schrödinger equation. Since the Wronskian of solutions of (2.1) is independent of $x$, we have $W(\phi_-, \phi_+) = W(e^{iEx}, e^{-iEx}) = 2iE$. It then follows from (2.6) that,

$$
a(\xi) = \frac{W(\phi_+, \psi_+)}{2iE}, \quad b(\xi) = \frac{W(\phi_+, \psi_-)}{-2iE},
$$

$$
c(\xi) = \frac{W(\phi_-, \psi_+)}{2iE}, \quad d(\xi) = \frac{W(\phi_-, \psi_-)}{-2iE}.
$$

Moreover,

$$W(\phi_+, \phi_-) = -2iE = W(\alpha\psi_+ + b\psi_-, c\psi_- + d\psi_+) = W(\psi_-, \psi_+)(ad - bc) = -2iE \det S;$$

hence $\det S = 1$.

At a zero of $a(z)$ at $z^+_j \in \mathcal{U}_+$ the wave functions $\phi_+$ and $\psi_+$ are linearly dependent, and we can write

$$\phi_\pm(x, z^+_j) = c^\pm_j \psi_\pm(x, z^+_j),$$

where $c^+_j$ are the associated coupling coefficients. Since $\phi_+$ decays exponentially as $x \to -\infty$ and $\psi_+$ decays exponentially as $x \to \infty$, this common function belongs to $L_2$ hence constitutes an eigenfunction of (2.1), called the bound state. Similar considerations hold on $\mathcal{U}_-$. Thus, the bound states of the problem are given by zeroes of $a(z)$ in $\mathcal{U}_+$ and $d(z)$ in $\mathcal{U}_-$. We denote these zeroes by $z^\pm_j$ respectively, and the associated coupling coefficients by $c^\pm_j$.

We define generalized reflection coefficients

$$r_+(\xi) = \frac{b(\xi)}{a(\xi)}, \quad r_-(\xi) = \frac{c(\xi)}{d(\xi)}, \quad \xi \in \Sigma.
$$

When the zeroes of $a$ and $d$ are simple, the scattering data for the operator (2.1) is the set

$$\{r_\pm(\xi), \ \xi \in \Sigma; \ z^\pm_j; \ c^\pm_j \}$$

Under the mapping $z \mapsto -1/z$, $E \mapsto -E$ and $k \mapsto k$ so the differential equation (2.1) is invariant. We denote this mapping by $s$, and write $s\psi(x, z) = \psi(x, -1/z)$.

**Lemma 2.1.** Under the mapping $s$ the wave functions transform as $s\psi_+ = \psi_-$, $s m^+_{out} = m^-_{out}$, i.e. $\psi_+(x, -1/z) = \psi_-(x, z)$, etc. Moreover, for $z \in \mathcal{U}_+$, $a(z) = d(-1/z)$; while $b(\xi) = c(-1/\xi)$ and $r_+(\xi) = r_-(-1/\xi)$ for $\xi \in \Sigma$.

**Proof.** Under the mapping $z \mapsto -1/z$ the Volterra integral equations in (2.3a) for $m^\pm_{out}$ are interchanged. Since the solutions are uniquely determined, they
are the same. The result extends immediately to $\psi_{\pm}$ and $\phi_{\pm}$. The relations on the generalized reflection coefficients then follow from their expressions in terms of the Wronskians. □

When $q$ is real and $p$ is real or imaginary, the wave functions possess additional symmetries. Since $k = \sqrt{E^2 - 1}$, $k$ is real on $E^2 > 1$ and imaginary on the slit $-1 < E < 1$. Hence (2.1) is invariant under Schwarz reflection across the slit if $p$ is real, and under Schwarz reflection across $E^2 > 1$ if $p$ is imaginary. The slit lifts to the unit circle in the $z$-plane, while the rays $E^2 > 1$ lift to the real line in the $z$-plane. These observations lead to the following result:

**Lemma 2.2.** When $p$ is real, (2.1) is invariant under Schwarz reflection across the unit circle and across the imaginary axis in the $z$-plane; and the wave functions $\phi$ and $\psi$ satisfy the symmetry conditions $\phi_+(x, z) = \phi_-(x, 1/\bar{z}) = \phi_+(x, -\bar{z})$.

When $p$ is imaginary, (2.1) is invariant under Schwarz reflection across the real axis $\Im z = 0$, as well as under the reflection $z \rightarrow -\bar{z}^{-1}$; the wave functions possess the corresponding symmetries: $\phi_+(x, z) = \phi_-(x, \bar{z}) = \phi_+(x, -1/\bar{z})$.

**Proof.** Take the case of real $p$, and consider the wave function $\phi$. An easy calculation shows that, for $z \in U_+$, both $\phi_-(x, 1/\bar{z})$, and $\phi_+(x, z)$ are asymptotic to $e^{-iEx}$ as $x \rightarrow -\infty$ and satisfy the same differential equation. Since solutions of (2.1) are uniquely determined by their asymptotics as $x \rightarrow -\infty$, they in fact coincide. The other cases are handled in the same way. □

**Theorem 2.3.** When $p$ is real or imaginary the scattering data possess additional symmetries:

For $p$ real, $a(\xi) = a(-\xi) = d(1/\xi)$ and $r_+(\xi) = r_-(\xi) = r_-(1/\xi)$. Moreover, if $z_j$ is a bound state, then so are $-\bar{z}_j$, $z_j^{-1}$ and $-z_j^{-1}$. The four associated coupling coefficients are respectively $c_j$, $\overline{c}_j$, $\overline{c}_j$, and $c_j$.

When $p$ is imaginary, $a(\xi) = a(-1/\bar{\xi}) = d(\bar{\xi})$ and $r_+(\xi) = r_+(1/\bar{\xi}) = r_-(\xi)$. The bound states also appear in fours: $z_j$, $\bar{z}_j$, $-z_j^{-1}$, $-\bar{z}_j^{-1}$, with corresponding coupling coefficients $c_j$, $\overline{c}_j$, $c_j$, $\overline{c}_j$.

Finally,

$$1 - |r(\xi)|^2 = 1/|a(\xi)|^2 \quad \left\{ \begin{array}{ll} |\xi| = 1 & p \text{ real}; \\
\Im \xi = 0 & p \text{ imaginary}. \end{array} \right.$$  

**Proof.** The lemma follows from Lemma 2.2 and the computation of the coefficients of $S$ in terms of Wronskians. From $\det S = ad - bc = 1$ we get $1 - r_+ r_- = 1/ad$; and the last statement above follows from the relationships between $a$ and $d$, $r_+$ and $r_-$ on $\Sigma$ when $p$ is real or imaginary. □

The following result will be useful in our discussion of the vanishing lemma, in §3.
Lemma 2.4. Let \( p \) be real and consider the bound states lying on \( i\mathbb{R} \setminus \{0\} \). Then
\[
\frac{1}{2} \int_{-\infty}^{\infty} (2 \cosh \alpha_j - p) \phi_j^2 \, dx = \begin{cases} 
\omega_j a'(i\omega_j) c_j & 1 < \omega_j < \infty; \\
-i\omega_j d'(i\omega_j) c_j & 0 < \omega_j < 1,
\end{cases}
\] (2.9a)
and
\[
\frac{1}{2} \int_{-\infty}^{\infty} (2 \cosh \alpha_j + p) \phi_j^2 \, dx = \begin{cases} 
-i\omega_j a'(i\omega_j) c_j & -1 < \omega_j < 0; \\
i\omega_j d'(i\omega_j) c_j & -\infty < \omega_j < -1,
\end{cases}
\] (2.9b)
where \( |\omega_j| = e^{\alpha_j} \) in both cases, \( c_j \) is the coupling coefficient associated with the bound state, and \( \phi_j = \phi_{\pm}(x, i\omega_j) \) according as \( \omega_j \in \mathcal{U}_{\pm} \).

Proof. Differentiating (2.1) with respect to \( z \) we have
\[
(D^2 + E^2 - V) \phi' = (V' - 2EE') \phi \quad (D^2 + E^2 - V) \phi = 0,
\]
where primes denote differentiation with respect to \( z \), and \( V = ikp + q \). Multiplying the first of these equations by \( \phi \), the second by \( \phi' \) and subtracting, we obtain
\[
DW(\phi, \phi') = (V' - 2EE') \phi^2, \quad W(\phi, \phi') = \phi D\phi' - \phi' D\phi.
\]
Now \( \phi_j = \phi_+(x, i\omega) \) tends to zero exponentially as \( x \to \pm\infty \), while \( \phi'_j \) tends to zero exponentially as \( x \to -\infty \). Integrating the above expression over the real line we obtain
\[
\lim_{x \to \infty} W(\phi_j, \phi'_j) = \frac{i}{\omega_j} \int_{-\infty}^{\infty} [p(y) \sinh \alpha_j - 2 \sinh \alpha_j \cosh \alpha_j] \phi_j^2 \, dy
\]
\[
= \frac{i \sinh \alpha_j}{\omega_j} \int_{-\infty}^{\infty} [p(y) - 2 \cosh \alpha_j] \phi_j^2 \, dy,
\]
where \( \omega_j = e^{\alpha_j} \) for the case \( \omega_j > 0 \). From (2.7), \( W(\phi_+, \psi_+) = 2iEa(z) \) for \( z \in \mathcal{U}_+ \); differentiating this identity with respect to \( z \), and evaluating at a bound state \( z = i\omega_j \) we obtain
\[
W(\phi'_j, \psi_j) + W(\phi_j, \psi'_j) = 2iEa'(i\omega_j) = -2 \sinh \alpha_j a'(i\omega_j),
\]
with \( \phi'_j = \phi'_+(x, i\omega_j) \), etc.

We now use the relation \( \phi_j = c_j^+ \psi_j \), in the above result, and let \( x \to \infty \). The left side becomes
\[
\frac{1}{c_j^+} W(\phi'_j, \phi_j) + c_j^+ W(\psi_j, \psi'_j).
\]
The second term tends to zero, since \( \psi_j \) and \( \psi'_j \) tend to zero exponentially as \( x \to \infty \). The identity (2.9a) now follows. The identity (2.9b) is obtained in the same manner. \( \square \)

As an immediate consequence of Lemma 2.4 we have:
Theorem 2.5. Let $p$ be real and $|p| < 2$. Then all the bound states $z_j = i\omega_j \in i\mathbb{R} \setminus \{0\}$ are simple, and $C_j > 0$, where

$$C_j = \begin{cases} -\frac{A_j}{\omega_j} & 1 < \omega_j < \infty; \\ A_j & -1 < \omega_j < 0, \end{cases} \quad A_j = -\frac{c_j}{a'(z_j)} \quad (2.10)$$

We now formulate the forward problem as a Riemann-Hilbert problem. Define the row vector $m$ by

$$m(x, z) = \begin{cases} (\psi(x, z), \frac{\phi(x, z)}{\sigma(\xi)} e^{-iEx}\sigma_3) & z \in \mathcal{U}_+; \\ (\frac{\phi(x, z)}{\sigma(\xi)}, \psi(x, z)) e^{iEx}\sigma_3) & z \in \mathcal{U}_-, \end{cases}$$

$$= \begin{cases} (m^+_{\text{out}}(x, z), m^+_{\text{in}}(x, z)) & z \in \mathcal{U}_+; \\ (m^-_{\text{out}}(x, z), m^-_{\text{in}}(x, z)) & z \in \mathcal{U}_-. \end{cases} \quad (2.11)$$

Here, as usual, $\sigma_3$ denotes the Pauli spin matrix

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

A simple computation shows that (2.6) are equivalent to the jump relations

$$m^+(x, \xi) \begin{pmatrix} 1 & -r_+ e^{2iEx} \\ 0 & 1 \end{pmatrix} = m^-(x, \xi) \begin{pmatrix} 1 & 0 \\ -r_- e^{-2iEx} & 1 \end{pmatrix}.$$ 

We write this as

$$m^+(x, \xi) = m^-(x, \xi) v(x, \xi), \quad \xi \in \Sigma \quad (2.12)$$

where

$$v(x, \xi) = \begin{pmatrix} 1 & r_+ (\xi) e^{2iEx} \\ -r_- (\xi) e^{-2iEx} & 1 - r_+ r_- \end{pmatrix}. \quad (2.13)$$

The row vector $m$ has prescribed asymptotic behavior as $x \to \infty$ and as $z$ tends to 0 or $\infty$, namely:

$$m(x, z) \to (1, 1), \quad \text{as } x \to \infty \quad (2.14)$$

and by (2.4)

$$m(x, z) \to \begin{cases} (\lambda, \lambda^{-1}) & z \to 0; \\ (\lambda^{-1}, \lambda) & z \to \infty. \end{cases} \quad (2.15)$$

Moreover,

$$\psi^+(x, i) = m^+_{\text{out}}(x, i) = \psi^-(x, i) = m^+_{\text{in}}(x, i). \quad (2.16)$$

This identity follows by observing that $\psi^\pm(x, i)$ satisfy the same Volterra integral equation, viz.

$$\psi^\pm(x, i) = 1 - \int_x^\infty (x - y)(q - p)\psi^\pm(y, i) \, dy.$$ 

We also have $\psi^-(x, -i) = \psi^+(x, -i)$.

In the case of a reflectionless potential, $r_+ (\xi) \equiv 0$, and $v(x, \xi) = I$ everywhere on $\Sigma$. 

Theorem 2.6. The inequality $|r(\xi)|^2 < 1$ holds on the entire real line in the case $p$ imaginary, and everywhere on the unit circle except at $\pm i$ in the case $p$ real. When $p$ is real we have generically, (that is, except in the case of reflectionless potentials) $r_+(\pm i) = r_-(\pm i) = -1$ and

$$v(x, \pm i) = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$  

In all cases, $r_+(0) = 0$, and $v(x, 0) = I$.

Proof. By Theorem 2.3 we have $|a|^2 - |b|^2 = 1$ on either the real line or the unit circle, according as $p$ is imaginary or real; hence $a$ cannot vanish on the corresponding portion of $\Sigma$. Writing this identity as $1 - |r(\xi)|^2 = |a(\xi)|^{-2}$, we see that $|r(\xi)| < 1$ whenever

$$\frac{1}{a(\xi)} = \frac{2iE}{W(\phi_+, \psi_+)} \neq 0.$$  

We see that this holds whenever $E \neq 0$, hence everywhere except at $\xi = \pm i$.

Writing the first equation in (2.6) in the form

$$\phi_+(x, \xi) a(\xi) = \psi_-(x, i) + r_+(\xi) \psi_+ (x, \xi),$$

and noting that $1/a(i) = 0$ (note that, since we have already observed that $a(i) \neq 0$ for real $p$, $W(\phi_+(x, i), \psi_+ (x, i)) \neq 0$) we see that $r_+(i) = 0$. A similar argument shows that $r_-(\pm i) = 0$. The expression for $v(x, \pm i)$ above follows from the fact that $E(\pm i) = 0$.

We write $r_+(\xi) = -W(\phi_+, \psi_-) / W(\phi_+, \psi_+).$ Then

$$W(\phi_+, \psi_+) = W(e^{-iEx} m_+^\text{in}, e^{iEx} m_+^\text{out}) = -2iEm_+^\text{out} m_+^\text{in} + W(m_+^\text{in}, m_+^\text{out}).$$

As $\xi \to 0$, $m_+^\text{out} \to \lambda$, and $m_+^\text{in} \to \lambda^{-1}$, so that

$$\lim_{\xi \to 0} W(m_+^\text{in}, m_+^\text{out}) = W(\lambda^{-1}, \lambda) = -\frac{2}{\lambda} \frac{d}{dx} \log \lambda = p(x).$$

A similar calculation shows that

$$\lim_{\xi \to 0} W(\phi_+, \psi_-) = \lim_{\xi \to 0} e^{-2iEx} W(m_+^\text{in}, m_+^\text{out}) = 0$$

since $m_+^\text{in}$ and $m_+^\text{out}$ both tend to $\lambda^{-1}$ as $\xi \to 0$. Noting that $E(\xi) \to \infty$ as $\xi \to 0$, we see that $r_+(\xi) \to 0$ as $\xi \to 0$. A similar argument shows that $r_-(0) = 0$. $\square$

Since $1/a(i) = 1/d(i) = 0$ we have, in the generic case,

$$m_+(x, i) = (m_+^\text{out}(x, i), 0), \quad m_-(x, i) = (0, m_+^\text{out}(x, i));$$

(2.17)
Lemma 2.7. The row vector \( m \) possesses the symmetries
\[
sm = m(x, -1/z) = m(x, z)R, \quad R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};
\]
and \( v(x, \xi) = Rv^{-1}(x, -1/\xi)R \). When \( q \) is real, \( m(x, z) = m(x, 1/z)R \) if \( p \) is real; and \( m(x, z) = m(x, \bar{z})R \) when \( p \) is imaginary.

The row vector \( m \) has poles at the zeroes of \( a \) and \( d \). At a simple pole of \( m \), there is a triangular matrix \( v_j(z) \) such that
\[
\rho_j(x, z) = me^{iE_j\sigma_3}v_j(z)e^{-iE_j\sigma_3},
\]
is regular at \( z = z_j \) [1]. In fact, from the bound state relation \( \phi_{\pm}(z, z_j^\pm) = c_j^\pm \psi_{\pm}(z, z_j^\pm) \) we find this to be true if we take
\[
v_j = \begin{pmatrix} 1 & A_j(z - z_j^\pm)^{-1} \\ 0 & 1 \end{pmatrix}, \quad z_j^\pm \in \mathcal{U}_+ \quad A_j = -\frac{c_j^\pm}{a'(z_j^\pm)}; \tag{2.18a}
\]
\[
v_j = \begin{pmatrix} 1 & 0 \\ D_j(z - z_j^-)^{-1} & 1 \end{pmatrix}, \quad z_j^- \in \mathcal{U}_- \quad D_j = -\frac{c_j^-}{a'(z_j^-)} \tag{2.18b}
\]
\[\]
From the symmetry \( a(z) = d(-1/z) \), it follows that if \( m \) has a pole at \( z_j \in \mathcal{U}_+ \), with coupling coefficient \( c_j \), then it also has one at \( -1/z_j \in \mathcal{U}_- \), with the same coupling coefficient. In fact, from \( \phi_+(x, z_j) = c_j\psi_+(x, z_j) \) and the \( s \)-symmetry we find \( \phi_-(x, -1/z_j) = c_j\psi_-(x, -1/z_j) \). From the identity \( a'(z) = -z^{-2}d'(-1/z) \), we find
\[
\left. D_j \right|_{-1/z_j} = -\frac{c_j}{z_j a'(z_j)} = A_j \frac{z_j}{z_j}, \quad (2.19)
\]
\[\]
From (2.18) we deduce that \( m_1 \) is regular in \( \mathcal{U}_+ \) while \( m_2 \) is regular in \( \mathcal{U}_- \), and that
\[
\text{Res } m(x, z) \big|_{z=z_j} = \begin{cases} \(0, -A_jm_1(x, z_j)e^{2ixE_j}\), & z_j \in \mathcal{U}_+; \\
\(-D_jm_2(x, z_j)e^{-2ixE_j}, 0\), & z_j \in \mathcal{U}_-, \end{cases} \tag{2.20}
\]
where \( 2E_j = (z_j + 1/z_j) \).

It is easily seen that both components of the row vector \( me^{iE_j\sigma_3} \) satisfy the differential equation (2.1). Moreover, \( \rho_j \) satisfies (2.14), (2.15). Now choose a small circle \( \mathcal{C}_j \) around \( z_j \), containing no other bound states and not intersecting \( \Sigma \), and define a new vector \( \bar{m} \) by
\[
\bar{m}(x, z) = \begin{cases} \rho_j(x, z) & \text{inside } \mathcal{C}_j; \\
m(x, z) & \text{outside } \mathcal{C}_j. \end{cases}
\]
Then the jump of \( \bar{m} \) across the contour \( \mathcal{C}_j \) is
\[
\bar{m}_+ = \bar{m}_- v_j(x, z), \quad v_j(x, z) = e^{iE_j\sigma_3}v_j(z)e^{-iE_j\sigma_3}, \tag{2.21}
\]
where the limits \( m_{\pm} \) are taken relative to a counterclockwise orientation of \( \mathcal{C}_j \) and \( E_j = E(z_j) \). If there are only a finite number of simple poles, we can modify the original contour by adding a small circle around each pole, and redefining \( m \) accordingly, so that the original Riemann-Hilbert problem is replaced by a modified one on the extended contour.
Theorem 2.8. Let \( p, q \in S \), and let \( m \) be the reduced vector valued wave function defined in (2.11). Then:

1. \( m \) satisfies the Riemann-Hilbert problem (2.12, 2.21) with \( v \) and \( v_j \) given in (2.13), (2.18);
2. \( m \) satisfies the asymptotic conditions (2.14), (2.15), and the conditions (2.16) at \( z = i \) together with corresponding conditions at \( z = -i \);
3. in the generic case, \( m \) satisfies the constraints (2.17), plus similar constraints at \( \xi = -i \);
4. \( m \) satisfies the symmetries of Lemma 2.7.

3. Two Vanishing Lemmas

A general approach to the solution of Riemann-Hilbert problems such as (2.8) is given in [3]. The inverse problem is reduced to the inversion of a linear operator of the form \( I + T_1 + T_2 \) where \( T_1 \) is of small norm and \( T_2 \) is compact. Once this reduction has been obtained, the Fredholm alternative applies, and the proof of existence and uniqueness is reduced to proving a uniqueness theorem for the Riemann-Hilbert problem. Such a uniqueness theorem in the context of Riemann-Hilbert problems is sometimes called a “Vanishing Lemma” [3]. We prove the vanishing lemma for our problem in this section. The remaining details of the inverse problem are the same as, but somewhat simpler than, those given in Part II of [3]. The inverse problem treated in that book, the Riemann-Hilbert problem for an \( n \)th order ordinary differential operator, is complicated by the intersection of rays at the origin.

The proof of the vanishing lemma is based on the symmetries on the reflection coefficients given in Theorem 2.3. For the purposes of the vanishing lemma, \( x \) appears only as a parameter, so we suppress it for the discussion. We first prove a vanishing lemma for the Riemann-Hilbert problem that arises in the case when \( p \) is imaginary and there are no bound states.

Theorem 3.1. Let \( m \) be a piecewise analytic function of \( z \) in \( \Im z \neq 0 \), with boundary values \( m_\pm \in L_2(\mathbb{R}) \); and let \( m \) satisfy the Riemann-Hilbert problem

\[
m_+ (\xi) = m_- (\xi) v(\xi), \quad \xi \in \mathbb{R}; \quad m = O(z^{-1}) \quad \text{as} \quad z \to \infty,
\]

where

\[
v(\xi) = \begin{pmatrix} \frac{1}{1-r(\xi)} & r(\xi) \\ -r(\xi) & 1 - |r(\xi)|^2 \end{pmatrix}.
\]

Assume \(|r| < 1\) a.e. Then \( m \) vanishes identically.

Proof. We define

\[
F = \begin{cases} \frac{m(z) - m'_\xi(\xi)}{2} & \Im z > 0; \\ -\frac{m(z) - m'_\xi(\xi)}{2} & \Im z < 0, \end{cases}
\]
where

\[ m^\dagger = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}. \]

Thus, \( m(z) \cdot m^\dagger(\bar{z}) = m_1(z)m_1(\bar{z}) + m_2(z)m_2(\bar{z}). \)

By the Schwarz reflection principle, \( F \) is sectionally analytic in \( \Im z \neq 0 \). Its boundary values on the real line are

\[ F_+(\xi) = \frac{m_+(\xi) \cdot m^\dagger_-(\xi)}{2} = \frac{m_-(\xi)v(\xi)m^\dagger_-(\xi)}{2}, \]

and

\[ F_-(\xi) = -\frac{m_-^{\dagger}(\xi) \cdot m^\dagger_+(\xi)}{2} = -\frac{m_-^{\dagger}(\xi)v^\ast(\xi)m^\dagger_+(\xi)}{2}. \]

Therefore, the jump of \( F \) across the real line is

\[ [F] = F_+ - F_- = m_-\left(\frac{v + v^\ast}{2}\right)m^\dagger_- = |m^\dagger_-(\xi)|^2 + (1 - |r(\xi)|^2)|m^\dagger_-(\xi)|^2. \]

Since \( F \) has no poles and is \( O(z^{-2}) \) as \( z \to \infty \), we can integrate it around a circle of radius \( R_\alpha \) centered at the origin, and let \( R_\alpha \to \infty \) to obtain

\[ \int_{-\infty}^{\infty} [F] \, d\xi = \int_{-\infty}^{\infty} |m^\dagger_-(\xi)|^2 + (1 - |r(\xi)|^2)|m^\dagger_-(\xi)|^2 \, d\xi = 0. \]

Since \( |r| < 1 \) a.e. the boundary values \( m_-(\xi) \) and \( m_+(\xi) = m_-(\xi)v(\xi) \) vanish a.e. on the real line. Since \( m \to 0 \) as \( z \to \infty \) and has no other singularities, \( m \) must vanish identically. \( \square \)

We now prove a vanishing lemma for the case corresponding to real \( p \). The operator (2.1) is self-adjoint only for \( z \) on the imaginary axis or the unit circle, so in this case we restrict the scattering data to lie on the union of the imaginary axis and the unit circle. Note also that the data and wave functions are invariant under Schwarz reflection in the unit circle and in the imaginary axis.

**Theorem 3.2.** Let \( m(z) \) be a piecewise meromorphic function of \( z \) in \( |z| \neq 1 \) satisfying the Riemann-Hilbert problem \( m_+(\xi) = m_-(\xi)v(\xi) \) on \( |\xi| = 1 \), where \( v \) has the form (3.1) on the unit circle. Assume that \( |r(\xi)| < 1 \) a.e. on the unit circle, and that

1. the poles of \( m \) are simple and lie on the imaginary axis;
2. the residue matrices at \( z_j = i\omega_j \) and \( z_{-j} = i/\omega_j \) are given by (2.18a) and (2.18b) respectively, where \( A_j \) and \( D_j \) satisfy (2.19) and \( A_j/z_j < 0 \);
3. both components of \( m \) are \( O(z^{-1}) \) as \( z \to \infty \);
Then $m$ vanishes identically.

Proof. We proceed in three steps.

Step 1. We first prove the theorem in the case there are no bound states. Define the function

$$F(z) = \begin{cases} 
  -\frac{m(z) \cdot m^\dagger(1/\bar{z})}{2z} & |z| > 1; \\
  \frac{m(z) \cdot m^\dagger(1/\bar{z})}{2z} & |z| < 1.
\end{cases}$$

By the Schwarz reflection principle, $F$ is sectionally analytic in $|z| \neq 1$; and also

$$F(z) = \begin{cases} 
  O(z^{-2}) & z \to \infty; \\
  O(1) & z \to 0.
\end{cases} \quad (3.2)$$

We orient the unit circle in the counterclockwise direction, and denote the limiting values of $F$ from the interior and exterior of the unit circle by $F_+, F_-$ respectively. On $|\xi| = 1$,

$$F_+(\xi) = \frac{m_-(\xi) \cdot m^\dagger_+(\xi)}{2\xi} = \frac{m_-(\xi) v^*(\xi) m^\dagger_-(\xi)}{2\xi};$$

and

$$F_-(\xi) = -\frac{m_+(\xi) \cdot m^\dagger_-(\xi)}{2\xi} = -\frac{m_-(\xi) v(\xi) m^\dagger_-(\xi)}{2\xi}.$$  \hspace{1cm} (Note that $\xi = 1/\bar{\xi}$ on the unit circle.) The jump of $F$ across the unit circle is therefore

$$[F] = m_- \left( \frac{v + v^*}{2\xi} \right) m^\dagger_-
= |m^\dagger_-(\xi)|^2 + (1 - |r(\xi)|^2)|m^\dagger_-(\xi)|^2, \quad |\xi| = 1. \quad (3.3)$$

Let $\Gamma^\pm_{R_0}$ be the contour consisting of the unit circle traversed in the clockwise direction and the circle $|z| = R_0 > 1$ in the counterclockwise direction; and let $\Gamma_{R_0}$ consist of the unit circle traversed in the counterclockwise direction together with the circle $|z| = 1/R_0$ traversed in the clockwise direction. We let $\Gamma_{R_0} = \Gamma^+_{R_0} \cup \Gamma^-_{R_0}$.

For all $R_0 > 1$ we get, by Cauchy’s theorem,

$$\frac{1}{2\pi i} \int_{\Gamma_R} F(z) \, dz = 0.$$  \hspace{1cm} since $F$ has no poles. Letting $R_0 \to \infty$ we obtain, in virtue of (3.2),

$$\frac{1}{2\pi} \int_{|\xi|=1} |m^\dagger_-(\xi)|^2 + (1 - |r(\xi)|^2)|m^\dagger_-(\xi)|^2 \frac{d\xi}{i\xi} = 0.$$  \hspace{1cm} Now $d\xi/i\xi$ is a positive measure on the unit circle oriented in the counterclockwise direction, and $|r_+(\xi)| < 1$ a.e. The boundary values $m_-$ and $m_+ = m_- v$ must
therefore vanish a.e. on the unit circle. Since \( m \) has no singularities in the finite plane and vanishes at \( z = \infty \), \( m \) vanishes identically. This completes step 1.

**Step 2.** We next prove the theorem when \( m \) has simple poles at \( i\omega_j \) and \( i/\omega_j \), where \( \omega_j > 1 \) and \( 1 \leq j \leq N \) and there are no jumps across the unit circle. We define

\[
F(z) = \begin{cases} 
-\frac{m(z)m^*(-z)}{2z} & |z| > 1; \\
\frac{\overline{m(z)m^*(-z)}}{2z} & |z| < 1.
\end{cases}
\]

We now calculate the residue of the first term at a pole \( z_j = i\omega_j \), \( \omega_j > 1 \). Since \( \omega_j > 1 \), \( m_1 \) is regular at \( z_j = i\omega_j \), and, by (2.20),

\[
\text{Res } m_2(z) \mid_{z=i\omega_j} = -A_j m_1(i\omega_j).
\]

Now observe that if the function \( f(z) \) has a simple pole at \( z_0 \) with residue \( f_0 \), then the function \( f(-\bar{z}) \) has a pole at \( -\bar{z}_0 \) with residue \(-f_0 \). Therefore

\[
\text{Res } m_2(-\bar{z}) \mid_{z=i\omega_j} = A_j m_1(i\omega_j),
\]

and

\[
\text{Res } \frac{m_1(z)m_2(-\bar{z})}{2z} \mid_{z=i\omega_j} = \frac{A_j}{2i\omega_j} |m_1(i\omega_j)|^2 = -\frac{A_j}{2z_j} |m_1(i\omega_j)|^2
\]

(Note that, since \( A_j/z_j < 0 \), \( A_j \) is imaginary.)

The residue of the second term is calculated in a similar manner, and we find that

\[
\text{Res } F \mid_{z=i\omega_j} = \frac{A_j}{z_j} |m_1(i\omega_j)|^2.
\]

A similar calculation yields

\[
\text{Res } F \mid_{z=i/\omega_j} = \frac{D_j}{-1/z_j} |m_1(i\omega_j)|^2 = \frac{A_j}{z_j} |m_1(i\omega_j)|^2.
\]

Here we have used (2.19).

We again integrate \( F \) around the contour \( \Gamma_{R_a} \) and let \( R_a \to \infty \). By the Cauchy residue theorem,

\[
\lim_{R_a \to \infty} \frac{1}{2\pi i} \int_{\Gamma_{R_a}} F(z) \, dz = 2 \sum_j C_j |m_1(i\omega_j)|^2 = 0.
\]

where \( C_j = -A_j/z_j \).

Since \( C_j \geq 0 \) by assumption 2, each term in the above sum must vanish. Since \( m \) has no singularities in the plane and tends to 0 at infinity, \( m \) must vanish identically. This completes step 2.

**Step 3.** When \( m \) has both poles and jumps across the unit circle, we construct a gauge transformation that removes the poles and reduces the problem to the case
where there are only jumps. We construct a matrix valued function of $z$ with the following properties:

i) $M$ is meromorphic in the extended complex plane with simple poles at $z_j$ and $M \to A$ as $z \to \infty$, where $A$ is an invertible matrix.

ii) $Mv_j$ is regular at $z_j$.

iii) $M(\xi)$ is unitary on the unit circle.

Given such an $M$ we put $m = wM$, where $m$ is the null solution of the original problem. Since $mv_j$ and $Mv_j$ are regular at $z_j$, it follows that $w$ is regular at $z_j$. Moreover, $w = O(z^{-1})$ at infinity. The jumps of $w$ on the unit circle are given by

$$w_+ = w_-MvM^{-1} = w_-MvM^*$$

since $M$ is unitary on the unit circle. We now apply the argument of step 1 to the row vector $w$, which has no poles in the complex plane, with $v$ replaced by $MvM^*$.

Then

$$M^*\frac{v + v^*}{2}M$$

is also positive definite, and we may conclude that $w_\pm$ vanish on $|\xi| = 1$.

It remains to construct the matrix $M$. Each of the rows of $M$ separately satisfy the same vector Riemann-Hilbert problem as the row vector $m$, namely each row vector of $Mv_j$ is regular at $z_j$. So we need two row vector solutions of this problem, with distinct limits at infinity. We start by taking these limits to be given by $(1,0)$ and $(0,1)$ respectively, so that initially $M(\infty) = I$. These are finite dimensional, algebraic problems, so that uniqueness implies existence. But we have already proved uniqueness of this problem in Step 2. So there exists a unique $M$ satisfying i) and ii) with $M(\infty) = I$.

We next prove that

$$M(z)M^*\left(\frac{1}{\bar{z}}\right) = M(0). \tag{3.4}$$

By the symmetry $v_j^{-1}(z) = v^*(1/\bar{z})$, we have, in the neighborhood of $z = z_j$,

$$M(z)M^*(1/\bar{z}) = M(z)v_j(z)v_j^*(1/\bar{z})M^*(1/\bar{z}).$$

By assumption the poles of $M$ are such that the row vectors of $Mv_j$ are regular at $z = z_j$. Now

$$v_j^{-1}\left(\frac{1}{\bar{z}}\right) = \begin{pmatrix} 1 & 0 \\ -\frac{1}{z + 1/z_j} & 1 \end{pmatrix};$$

and this matrix has the same residue at $z = -1/z_j$ as the matrix $v_j^-(z)$ in (2.18b). So $v_j^*(1/\bar{z})M^*(1/\bar{z})$ is also regular in a neighborhood of $z = z_j$. Since $M(z)M^*(1/\bar{z})$
has no singularities in the finite $z$-plane and is bounded at infinity, it is a constant matrix, by Louiville’s theorem. Letting $z \to 0$ we obtain (3.4).

Taking $z = i$ in (3.5), we see that $M(0) = M(i)M^*(i)$, and so is a positive definite matrix. Let $A$ be the positive definite square root of $M(0)^{-1}$. Replacing $M(z)$ by $AM(z)$ we obtain the required gauge transformation satisfying i)-iii) above. □

4. The Inverse Problem

In the formulation of the forward problem, the asymptotics of the row vector $m$ at $z = 0, \infty$ are given in terms of $\lambda(x)$ by (2.15). In the inverse problem, $\lambda$ is determined by requiring that the solution of the Riemann-Hilbert problem satisfy (2.15). In addition, (2.16) must be satisfied, and this leads to an additional constraint on $\lambda$. In the generic case, the constraint (2.17) must also be satisfied, leading to yet another constraint on $\lambda$. One must show that these various determinations of $\lambda$ are consistent.

Consider the matrix Riemann-Hilbert problem

\begin{equation}
H_+(x, \xi) = H_-(x, \xi)v(x, \xi), \quad \xi \in \Sigma \quad Hv_j \text{ regular at } z = z_j
\end{equation}

\begin{equation}
H \to I \text{ as } z \to \infty.
\end{equation}

Each row of $H$ satisfies the Riemann-Hilbert problem of the previous section. Hence the vanishing lemmas of the previous section imply that this problem has a unique solution $H$ when $v$ satisfies the appropriate symmetries. Since $\det v = \det v_j = 1$, the scalar function $\det H$ has no jumps and no singularities, hence is identically 1.

Put

\begin{equation}
m = (1, 1)\Lambda H, \quad \Lambda = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}.
\end{equation}

Then $m$, being a linear combination of the rows of $H$, satisfies the Riemann-Hilbert problem, and $m \sim (\lambda^{-1}, \lambda)$ as $z \to \infty$. We determine $\lambda(x)$ by requiring that (2.15) be satisfied, i.e. $m \sim (\lambda, \lambda^{-1})$, $z \to 0$. (By Theorem 2.6 we may assume that $v(x, 0) = I$, hence that $H_+(x, 0) = H_-(x, 0)$; hence we may write unambiguously $H(x, 0)$ for $H_\pm(x, 0)$. This leads to:

\begin{equation}
\lambda = \lambda^{-1}H_{11}(x, 0) + \lambda H_{21}(x, 0), \quad \lambda^{-1} = \lambda^{-1}H_{12}(x, 0) + \lambda H_{22}(x, 0);
\end{equation}

hence to two determinations for $\lambda$:

\begin{equation}
\lambda^2 = \frac{H_{11}(x, 0)}{1 - H_{21}(x, 0)}, \quad \lambda^2 = \frac{1 - H_{12}(x, 0)}{H_{22}(x, 0)}.
\end{equation}

Moreover, (2.16) implies, $\lambda^{-1}H_{11}^+(x, i) + \lambda H_{21}^+(x, i) = \lambda^{-1}H_{12}^+(x, i) + \lambda H_{22}^+(x, i)$, so

\begin{equation}
\lambda^2 = \frac{H_{11}^+(x, i) - H_{12}^+(x, i)}{H_{22}^+(x, i) - H_{21}^+(x, i)}.
\end{equation}
In the case of reflectionless potentials, or when \( v = I \) on the unit circle, we may take \( H_+(x, i) = H_-(x, i) \) in (4.2). In addition, in the generic case, the equations in (2.17) imply

\[
\lambda^{-1}H_{12}^+(x, i) + \lambda H_{22}^+(x, i) = 0, \quad \lambda^{-1}H_{11}^-(x, i) + \lambda H_{21}^-(x, i) = 0;
\]

hence

\[
\lambda^2 = \frac{H_{12}^+(x, i)}{H_{22}^+(x, i)} = \frac{-H_{11}^-(x, i)}{H_{21}^-(x, i)}.
\]

(4.3)

A similar determination of \( \lambda \) is given at \(-i\).

We must show the equivalence of these various determinations of \( \lambda \). We first prove

**Lemma 4.1.** Assume the Riemann-Hilbert problem above has a unique solution. Then

\[
RH(x, 0)R = H^{-1}(x, 0),
\]

(4.4)

\[
H_+(x, i) = H(x, 0)RH_-(x, i)R.
\]

(4.5)

**Proof.** Let \( \bar{H}(x, z) = RH(x, -1/z)R \). By the \( s \)-symmetry of the data, \( \bar{H} \) and \( H \) satisfy the same Riemann-Hilbert problem. Since the solution of that problem is uniquely determined by its behavior at \( \infty \), we must have

\[
\bar{H}(x, z) = RH(x, -1/z)R = MH(x, z),
\]

(4.6)

where \( M = M(x) \) is a matrix independent of \( z \), and \( \det M(x) = 1 \). Letting \( z \to 0, \infty \) we find \( RH(x, \infty)R = I = MH(x, 0) \) and \( RH(x, 0)R = M \), hence (4.4). Writing (4.6) as \( H(x, z) = H(x, 0)RH(x, -1/z)R \) and letting \( z \to i \) from \( U_+ \), we obtain (4.5). \( \Box \)

We are now ready to show the equivalence of (4.1), (4.2), and, in the generic case, (4.3). The two expressions in (4.1) are equivalent provided

\[
H_{12}(x, 0) + H_{21}(x, 0) = 1 - \det H(x, 0).
\]

From the fact that \( \det H = 1 \), we see that we must show that \( H_{12}(x, 0) + H_{21}(x, 0) = 0 \); but this follows from the identity (4.4).

The equivalence of (4.1) and (4.2) reduces to the identity

\[
H_{11}(x, 0)(H_{22}^-(x, i) - H_{22}^+(x, i)) = H_{12}^+(x, i) - H_{12}^-(x, i) - H_{21}(x, 0)(H_{11}^+(x, i) - H_{21}^-(x, i)).
\]

(4.7)

In the case of reflectionless potentials, \( v(x, \xi) = I \), and \( H_+(x, i) = H_-(x, i) \). In (4.7), replace \( H_-(x, i) \) by \( H_+(x, i) \), \( H_{21}(x, 0) \) by \(-H_{12}(x, 0)\); one then obtains the
same expression as that obtained by operating on the column vector \((1, -1)\dagger\) with both sides of the matrix equation (4.5).

In the generic case we write (4.5) in the form \(H(x, 0)RH_+(x, i) = H_+(x, i)Rv(i)\) and use the expression for \(v(i)\) given in Lemma 2.6. We obtain four equations, one of which is

\[
H_{11}(x, 0)H_{22}^+(x, i) + H_{12}(x, 0)H_{12}^+(x, i) = -H_{12}^+(x, i).
\]

This may be used to show the equivalence of (4.1), (4.2). The proofs of the equivalence of (4.3) with (4.1) in the generic case proceeds along similar lines, and is left to the reader.

**Theorem 4.2.** Let \(m(x, z) = (1, 1)\Lambda H(x, z)\). Then \(m(x, -1/z) = m(x, z)R\).

**Proof.** Since \(\lambda\) has been chosen so that \(m(x, 0) = (\lambda, \lambda^{-1})\) we have \((\lambda, \lambda^{-1}) = (1, 1)\Lambda H(x, 0)\); so

\[
m(x, z) = (1, 1)\Lambda H(x, 0)H^{-1}(x, 0)H(x, z) = (\lambda, \lambda^{-1})H(x, z)
\]

\[
= (\lambda, \lambda^{-1})RH(x, -1/z)R
\]

\[
= (\lambda^{-1}, \lambda)H(x, -1/z)R = (1, 1)\Lambda H(x, -1/z)R
\]

\[
= m(x, -1/z)R. \quad \Box
\]

**Theorem 4.3.** Let the scattering data \(v\) satisfy the symmetry \(\overline{v(\xi)} = Rv^{-1}(\xi)R\). Then \(|\lambda| = 1\).

**Proof.** Define \(\tilde{H}(x, z) = \overline{H(x, z)}\). A short calculation shows that

\[
\tilde{H}_+(x, \xi) = \lim_{z \to \xi^+} \overline{H(x, z)} = \overline{H_-(x, \xi)}
\]

\[
= \overline{H_+(x, \xi)v^{-1}(\xi)} = \overline{H_+(x, \xi)Rv(\xi)R}
\]

\[
= \overline{\tilde{H}_-(x, \xi)Rv(\xi)R}.
\]

Since the solution of the Riemann-Hilbert problem is uniquely determined by its limit as \(z \to \infty\) and since both \(H\) and \(\tilde{H}\) tend to \(I\) as \(z \to \infty\) we have \(\tilde{H}(x, z) = RH(x, z)R\). Letting \(z \to 0^+\), and recalling that \(H_+(x, 0) = H_-(x, 0)\), we obtain \(H(x, 0) = RH(x, 0)R\), hence \(H_{11}(x, 0) = \overline{H_{22}(x, 0)}\) and \(H_{12}(x, 0) = \overline{H_{12}(x, 0)}\). The two expressions for \(\lambda^2\) in (4.1) then show that \(\lambda^2 = \overline{\lambda^{-2}}\). \(\Box\)

For real \(p\), we find that \(H_{12}(x, z) = \overline{H_{12}(x, -\bar{z})}\), hence \(H_{12}(x, i) = \overline{H_{12}(x, i)}\). From (4.2) we see that \(\lambda^2\) is real. We still need to show that this quantity is positive.

Finally, as in [3] one may prove that \(H \to I\) as \(x \to \infty\), so that (2.14) is satisfied.
5. The Initial Value Problems

We want to solve the initial value problems for (1.2) and (1.3) by the inverse scattering method. We begin by determining the associated evolution of the scattering data.

The evolution of the scattering data is obtained by standard asymptotic arguments. From [14] the flows (1.2), (1.3) are obtained from the Lax pair

\[ L = D^2 - u, \quad u = ikp + q; \quad P = -i\frac{ep_x}{4} + \epsilon\left(\frac{ip}{2} + k\right)D \]

We take \( \epsilon = 2i \) for the flow (1.2) and \( \epsilon = 2 \) for the flow (1.3) [14]. The flows are obtained from the Lax equations

\[ \dot{L} = [P, L] \]

when the commutator is evaluated on the submanifold of wave functions of \( L \); that is, on functions satisfying \( L\psi + k^2\psi = 0 \).

When the Lax equations are satisfied,

\[ (\frac{\partial}{\partial t} - P)(D^2 - u + E^2)\phi = (D^2 - u + E^2)(\frac{\partial}{\partial t} - P)\phi = 0 \]

so

\[ (\frac{\partial}{\partial t} - P)\phi \]

satisfies (2.1) whenever \( \phi \) does.

Since \( p \to 0 \) to as \( x \to \pm\infty \),

\[ \frac{\partial}{\partial t} - P \sim \frac{\partial}{\partial t} - \epsilon k D, \quad \text{as} \quad x \to \pm\infty. \]

so

\[ \left(\frac{\partial}{\partial t} - P\right)\phi \sim -\epsilon k D e^{-iEx} \sim i\epsilon k E e^{-iEx}, \quad \text{as} \quad x \to -\infty. \]

Since the wave functions are uniquely determined by their asymptotic behavior as \( x \to -\infty \),

\[ (\frac{\partial}{\partial t} - P)\phi = i\epsilon k E \phi. \]

On the other hand, \( \phi_+ = a\psi_- + b\psi_+ \), so on \( \Sigma \),

\[ (\frac{\partial}{\partial t} - P)\phi_+ = (\frac{\partial}{\partial t} - P)(a\psi_- + b\psi_+) \]

\[ \sim (\dot{a} + ikEa)e^{-iEx} + (\dot{b} - ikEb)e^{iEx} \quad x \to \infty. \]

Therefore,

\[ (\frac{\partial}{\partial t} - P)\phi_+ = (\dot{a} + ikEa)\psi_- + (\dot{b} - ikEb)\psi_+ = i\epsilon k E(a\psi_- + b\psi_+) \]
and
\[ \dot{a} = 0, \quad \dot{b} = 2i\epsilon kEb. \]

Similarly \( \dot{c} = -2i\epsilon kEc. \) The evolution of the coupling coefficients for the bound states, defined by,
\[ \phi(x,t,z_j^\pm) = c_j^\pm(t)\psi(x,t,z_j^\pm) \]
is derived by similar arguments. We have shown

**Theorem 5.1.** Under the flows (1.2) or (1.3) the evolution of the scattering data is given by
\[ r_\pm(x,t,\xi) = r_\pm(\xi)e^{\pm2iE(x+\epsilon kt)}; \quad c_j^\pm(t) = c_j^\pm(0)e^{2i\epsilon k_j^\pm E_j^\pm t}, \quad (5.1) \]
where \( k_j^\pm \) and \( E_j^\pm \) are the values of \( k \) and \( E \) at \( z_j^\pm \).

Global existence theorems for each of the flows (1.2), (1.3) are proved by showing that the scattering problem can be inverted at all later times, hence, by showing that the hypotheses on the scattering data in the two vanishing lemmas are invariant under the evolution of the scattering data (5.1). Now, the support of \( r_\pm \) and the location of the bound states is invariant under the evolution of the scattering data, and \( E \) is real on \( \Sigma \). Therefore, the only remaining constraints to be verified are that \( \epsilon k(\xi) \) is real on the support of \( r_\pm(\xi) \) and that \( \epsilon k_j^\pm E_j^\pm \) is real on the bound states \( z_j^\pm \) in the case of real \( p \).

We restrict the support of \( r_\pm \) to lie on the unit circle in the case \( p \) real, and on the real line for the case \( p \) imaginary. Since \( k \) is real on the real line, and imaginary on the unit circle, and since \( \epsilon = 2i \) in the Boussinesq case and \( \epsilon = 2 \) for imaginary \( p \), \( \epsilon k(\xi) \) is then always real on the support of \( r_\pm \).

The condition that \( k_j^\pm E_j^\pm \) be real on the bound states is met for \( z_j = i\omega_j \) since in that case, \( E_j = i\sinh\omega_j \), and \( k_j = i\cosh\omega_j \).

The following theorem now follows as a consequence of the invariance of the scattering data and the solvability of the inverse scattering problem.

**Theorem 5.2.** The initial value problem (1.2) is solvable for \( -\infty < t < \infty \) when, at some finite time, the support of the reflection coefficients \( r_\pm(\xi) \) is contained in the unit circle, and the bound states are restricted to the imaginary axis and satisfy (2.10).

The initial value problem (1.3) is globally solvable if at some finite time the support of the reflection coefficients is contained in the real line, and there are no bound states.
Global solutions of (1.3) do not exist when there are bound states for the case $p$ imaginary. This is illustrated by the example in §6 where we construct a soliton for (1.3). In order to prove the vanishing lemma, we need a positivity condition, as that of the expression on the right side of (3.5). In the case of real $p$ we used the fact that $m_1^+(x, i\omega)$ was real in (3.4). In order to get real wave functions we need the coefficients in (1.1) to be real. This means that $E$ is real or imaginary and $ikp$ is real, hence that $k$ is real.

6. CONSTRUCTION OF SOLITONS

We construct simple solitary waves by solving the Riemann-Hilbert problem in the case of reflectionless potentials, that is, when $v$ is the identity matrix and there are only discrete scattering data in the problem. When $p$ is real the bound state eigenvalues are symmetric with respect to the imaginary axis and the unit circle, by Lemma 2.4. In the simplest case we may consider a pair of eigenvalues $i\omega$ and $i/\omega$, with $\omega > 1$.

Our problem is

$$
\phi_+(x, z) = a(z)\psi_-(x, z), \quad \phi_+(x, i\omega) = c\psi_+(x, i\omega) \tag{6.1}
$$

where $c$ is the coupling constant and

$$
a(z) = \frac{z - i\omega}{z - i/\omega}, \quad \psi_-(x, z) = \overline{\psi_+(x, \bar{z}^{-1})}. \tag{6.2}
$$

We take $\psi_+ = m_+^{\text{out}} e^{izE}$ and assume $m_+^{\text{out}}$ is a meromorphic function in the extended $z$-plane with a single pole at $i/\omega$:

$$
m_+^{\text{out}} = \lambda^{-1} + \frac{w(x)}{z - i/\omega}.
$$

From (2.4),

$$
\lim_{z \to 0} m_+^{\text{out}}(x, z) = \lambda
$$

hence

$$
\lambda - \lambda^{-1} = i\omega w. \tag{6.3}
$$
Invariance of the wave function under Schwarz reflection in the imaginary axis, viz.
\[ \psi_+(x, -z) = \psi_+(x, z), \]
leads to the constraint
\[ \lambda^{-1} - \bar{\omega}(x) \frac{\omega}{z - i/\omega} = \lambda^{-1} + \frac{w(x)}{z - i/\omega} \]
for all \( z \). Hence we must have
\[ \bar{\lambda} = \lambda, \quad \bar{w} = -w. \]

Setting
\[ W = \frac{w}{2i\beta}, \quad \lambda = e^{P/2}, \quad \omega = e^{i\alpha}, \quad \beta = -iE(i\omega) = \sinh \alpha, \]
we obtain from equations (6.1-6.3) the relations
\[ -\omega^2 W = e[\lambda^{-1} + W]e^{-2\beta x}, \quad W = -\omega \beta \sinh \frac{P}{2}. \]
Solving for \( P \), we obtain
\[ e^P = 1 + \frac{2c}{\beta\omega^3 \exp(2\beta x) + c\omega \beta} \quad (6.4) \]
Then, from (2.5),
\[ q = \frac{p^2}{4} + \frac{d}{dx} \left( \frac{p}{2} - 2\beta We^{P/2} \right) \]
\[ = \frac{p^2}{4} + \frac{d}{dx} \left[ \frac{p}{2} + \frac{2\beta c}{\omega^2 \exp(2\beta x) + c} \right] \quad (6.5) \]

By Theorem 5.1 the time evolution of the coupling constant \( c \) is
\[ c(t) = c_0 e^{2t \sinh 2\alpha}. \]
The eigenvalues for imaginary \( p \) appear in fours, with the symmetries given in Theorem 2.3. We construct a soliton corresponding to four bound states with purely imaginary \( z_0 = i\omega \), hence
\[ i\omega, \quad i/\omega, \quad -i/\omega, \quad -i\omega. \]
Then
\[ a(z) = \frac{z - i\omega \frac{z + i/\omega}{z + i/\omega} z - i/\omega}. \]
and
\[ \phi_+(x, z) = a(z)\psi_-(x, z), \quad \psi_-(x, z) = \overline{\psi_+(x, \overline{z})}. \]  \hspace{1cm} (6.6)

From Theorem 2.3
\[ \phi_+(x, i\omega) = c\psi_+(x, i\omega), \quad \phi_+(x, -i/\omega) = \overline{c}\psi_+(x, -i/\omega) \]  \hspace{1cm} (6.7)
where \( c \) is the coupling coefficient.

By Theorem 5.1, the time evolution of \( c \) is given by
\[ c(t) = c(0)e^{2i\epsilon k E t} = c(0)e^{-2\omega(\sinh 2\alpha)t}. \]

Now \( \psi_+(x, z) = m^\text{out}_+(x, z)e^{ixE} \). We look for a wave function of the form
\[ m^\text{out}_+(x, z) = \lambda^{-1} + \frac{w_1(x)}{z + i\omega} + \frac{w_2(x)}{z - i/\omega}. \]

The symmetry
\[ m^\text{out}_+(x, z) = m^\text{out}_+(x, -\overline{z}^{-1}) \]
implies that
\[ m^\text{out}_+(x, z) = \overline{\lambda^{-1} - \frac{z\overline{w}_1}{1 + i\omega z} - \frac{z\overline{w}_2}{1 - iz/\omega}}. \]
The requirement that \( m^\text{out}_+(x, 0) = \lambda(x) \) implies that \( \lambda = \overline{\lambda}^{-1} \), hence that \( |\lambda| = 1 \).

We then have
\[ \lambda - \lambda^{-1} = \left( \frac{w_1}{z + i\omega} + \frac{z\overline{w}_2}{1 - iz/\omega} \right) + \left( \frac{w_2}{z - i/\omega} + \frac{z\overline{w}_1}{1 + i\omega z} \right). \]  \hspace{1cm} (6.8)

Since the right side is independent of \( z \),
\[ \overline{w}_1 + \omega^2 w_2 = 0; \]
and
\[ m^\text{out}_+(x, z) = \lambda^{-1} + \frac{w_1(x)}{z + i\omega} + \frac{\overline{w}_1(x)}{i\omega(1 + i\omega z)}. \]

Equations (6.6) and (6.7) now lead to an equation for \( w = w_1 \):
\[ \bar{w} = \kappa \left( \lambda^{-1} + \frac{w}{2i\omega} - \frac{\bar{w}}{2i\omega^2 \sinh \alpha} \right) e^{-2x \sinh \alpha} \]  \hspace{1cm} (6.9)
where \( e^\alpha = \omega, \quad \kappa = 2i\omega c \tanh \alpha. \)

Taking complex conjugates of (6.9) we obtain the following system of equations for \( w \) and \( \bar{w} \):
\[ \begin{pmatrix} a & b \\ b & \bar{a} \end{pmatrix} \begin{pmatrix} w \\ \bar{w} \end{pmatrix} = \begin{pmatrix} \sigma \\ \bar{\sigma} \end{pmatrix}, \]
where
\[ a = 1 + \frac{\bar{c}}{\omega} \operatorname{sech} \alpha e^{-2x \sinh \alpha}, \quad b = -\bar{c} \tanh \alpha e^{-2x \sinh \alpha}, \]
and
\[ \sigma = -2i\omega \bar{c} \lambda \tanh \alpha e^{-2x \sinh \alpha}. \]

The solution of this 2 × 2 system is
\[ w = \frac{\bar{a}\sigma - b\sigma}{\Delta}, \quad (6.10) \]
where \( \Delta = |a|^2 - |b|^2 \). To determine whether a global solution exists, we must evaluate \( \Delta \) with \( c(t) \) given as above.

Taking \( z = 0 \) in (5.8) we find
\[ \Im \lambda = \frac{\lambda - \bar{\lambda}}{2i} = -\frac{\Re w(x)}{\omega}. \quad (6.11) \]

Equations (6.10), (6.11) are coupled equations for \( \lambda \) and \( w \).

We note that the solitons for imaginary \( p \) always have poles on the real axis. These occur at the zeros of \( \Delta \), hence, whenever \( |a| = |b| \). To show that \( \Delta \) always vanishes, first set \( C = c_0 e^{-2x \sinh \alpha} \), so that \( 0 < |C| < \infty \). As \( t \) varies, \( |b| = |C \tanh \alpha| \) is constant, while \( |a| \) ranges over the interval \((1 - |C|e^{-\alpha} \operatorname{sech} \alpha, 1 + |C|e^{-\alpha} \operatorname{sech} \alpha)\). Therefore \( \Delta \) vanishes whenever (we may assume \( \alpha > 0 \))
\[ 1 - |C|e^{-\alpha} \operatorname{sech} \alpha < |C| \tanh \alpha < 1 + |C|e^{-\alpha} \operatorname{sech} \alpha; \]
this inequality simplifies to
\[ 0 < (e^{2\alpha} + 1)(1 - 1/|C|) < 4, \]
which is always satisfied for a range of \( |C| \) in \((0, \infty)\). Hence the soliton always has poles for real \( x \) and \( t \).

7. Conclusions

We have proved the existence of global solutions of the initial value problems of the flows (1.2) and (1.3) under complementary conditions on the scattering data. Two questions are often raised about these results. Firstly, can the conditions on the initial data be expressed in terms of the initial values \( u(x, 0) \) and \( w(x, 0) \)? Secondly, are the conditions sharp or merely sufficient conditions for global existence?

As to the first question, it is typical in the theory of the scattering transform that the picture is often simpler on the scattering side than on the potential, or spatial side. This is perfectly analogous to the features of the Fourier transform. For example, under the Fourier transform, convolution, a non-local operation, is
mapped into pointwise multiplication, a local operation. In the case of completely integrable systems, action-angle variables are expressed very conveniently on the scattering side. [NMPZ]. Thus, without saying that simple conditions for global existence cannot be placed directly on $u$ and $w$, we should also not be surprised that the solvability conditions are expressed so simply in terms of the scattering data.

The second question is in fact one we wish to address in future work, but we conjecture that the conditions are in fact sharp. One already sees in the example of the soliton in §6 for the case of imaginary $p$: it has poles for all time. We conjecture further that if one takes as initial data reflection coefficients $r_{\pm}(\xi)$ with support on the wrong section of $\Sigma$, then singularities will develop in finite time. For example, we conjecture that if for real $p$ one chooses for $r_{\pm}(\xi)$ smooth functions with compact support on the real line, singularities will develop in finite time. It would be interesting to use the Riemann-Hilbert formulation to investigate the nature of the blow up of the solution. Presumably it would develop a pole, since that is the nature of the singularities of the solution of the inverse scattering problem.

The situation is in some respects similar to that in the AKNS system, in which some kind of symmetry on the potential is needed in order to guarantee global solvability. For example, the nonlinear Schrödinger equation is obtained for Hermitian or skew-Hermitian potentials, the modified KdV equation is obtained for real symmetric or skew symmetric potentials. The symmetry of the potentials is related to the positivity of the quadratic conservation laws. Without the symmetry, these conservation laws are not positive definite; and global solutions of the unsymmetric AKNS systems do not in general exist (cf. the discussion in [2]).

Appendix

**Proposition A1.** Let $p, q \in L^1(\mathbb{R})$. Then for $z \in \mathcal{U}_1$ the wave function $\psi_+$ can be written in the form

$$\psi_+(x, z) = \left(A_0(x, E) + \frac{1}{z}A_1(x, E)\right) e^{iEx}, \quad z \in \mathcal{U}_1,$$

(A1)

where $A_0, A_1$ are analytic in $\Im E > 0$, continuous onto the real $E$-axis, and

$$A_0 \sim \lambda^{-1}, \quad A_1 \sim 0, \quad E \to \infty, \Im E > 0.$$

The form of the wave functions in the other domains can be obtained from the $s$ symmetry.

**Proof.** Using the identities

$$z = 2E - \frac{1}{z}, \quad \frac{1}{z^2} = \frac{2E}{z} - 1, \quad k = E - \frac{1}{z},$$
and the \textit{ansatz} (A1), the integral equation (2.3a) can be written as the coupled system of Volterra integral equations
\begin{align*}
A_0(x, z) &= 1 - \int_x^\infty \frac{1 - e^{-2iE(x-y)}}{2iE} \left( (iEp(y) + q(y))A_0(y, z) + ipA_1(y, z) \right) dy, \\
A_1(x, z) &= -\int_x^\infty \frac{1 - e^{-2iE(x-y)}}{2iE} \left( (q(y) - 2iEp(y))A_1(y, z) - ipA_0(y, z) \right) dy.
\end{align*}

These may be solved by successive approximations for $\Im E > 0$ in the standard way, and the analyticity of $A_0$ and $A_1$ established. Letting $E \to \infty$ we obtain the limiting equations for $A_0$ and $A_1$,
\begin{align*}
A_0(x) &= 1 - \frac{1}{2} \int_x^\infty p(y)A_0(y) dy, \\
A_1(x) &= \int_x^\infty p(y)A_1(y) dy,
\end{align*}
from which the result follows. \hfill $\square$

\textbf{Lemma A2.} The wave function $\psi_+(x, z)$ has the representation
\begin{equation}
\psi_+(x, z) = \lambda \left( e^{iEx} + \int_x^\infty (K_0(x,y) + \frac{1}{z}K_1(x,y))e^{iEy} dy \right).
\end{equation}

Similar representations hold for the other wave functions $\psi_-, \phi_\pm$. Here, $K_0$ and $K_1$ are smooth kernel functions, with properties similar to the kernels in the Gel’fand-Levitan-Marchenko integral equations in the scattering theory of the Schrödinger equation.

\textit{Remark:} Kaup, ([10], (3.7)) assumes the representation (A2) in his derivation of the GLM equations.

\textbf{Proof.} These Fourier representations are proved by arguments similar to those used in the standard GLM theory. Since $A_0$ and $A_1$ are analytic in the upper half $E$-plane, and since $A_0 - \lambda^{-1} \to 0$ as $E \to \infty$, they can be represented, for each $x$, as Fourier transforms
\begin{align*}
A_0 &= \lambda^{-1} + \int_{-\infty}^0 H_0(x, s)e^{-iEs} ds, \\
A_1 &= \int_{-\infty}^0 H_1(x, s)e^{-iEs} ds,
\end{align*}
Writing
\begin{equation*}
e^{iEx}A_0 = \lambda e^{iEx} + \int_{-\infty}^0 H_0(x, s)e^{iE(x-s)} ds,
\end{equation*}
putting $y = x - s$, and changing variables in the integral, we obtain
\begin{equation*}
A_0 e^{iEx} = \lambda \left( e^{iEx} + \int_x^\infty K_0(x,y)e^{iEy} dy \right),
\end{equation*}
where $K_0(x,y) = \lambda^{-1}H_0(x,x-y)$. The other term is dealt with in the same way. \hfill $\square$
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