SOME SPECIAL COPRIME ACTIONS
AND THEIR CONSEQUENCES

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Abstract. Let a group $A$ act on the group $G$ coprimely. Suppose that the order of the fixed point subgroup $C_G(A)$ is not divisible by an arbitrary but fixed prime $p$. In the present paper we determine bounds for the $p$-length of the group $G$ in terms of the order of $A$, and investigate how some $A$-invariant $p$-subgroups are embedded in $G$ under various additional assumptions.

1. INTRODUCTION

All groups considered are finite. Let a group $A$ act on the group $G$. The nature of this action has some radical consequences on the structures of both $G$ and $A$ and also leads to some bounds for the invariants of both in terms of the other’s. So much research is devoted to studying coprime action, that is the case $(|G|, |A|) = 1$ due to the existence of well known nice relations between $G$ and $A$. The present paper is concerned with the consequences of some coprime actions with the additional condition common to all of them such that the order of the fixed point subgroup $C_G(A)$ is not divisible by an arbitrary but fixed prime $p$. In Section 2 we handle the case where $A$ acts with regular orbits, that is, for every proper subgroup $B$ of $A$ and every elementary abelian $B$-invariant section $S$ of $G$, there exists some $v \in S$ such that $C_B(v) = C_B(S)$; and bound the $p$-length of the group $G$. Namely, we prove

Theorem A. Let $A$ be a group acting coprimely and with regular orbits on the solvable group $G$. Suppose that $B$ is a subgroup of $A$ such that $\bigcap_{a \in A}[G, B]^a = 1$. If $p$ is a prime not dividing $|C_G(A)|$ then $\ell_p(G) \leq \ell(A : B)$.

Here, $\ell(A : B)$ denotes the length (by the number of inclusions) of the longest chain of subgroups of $A$ that starts with $B$ and ends with $A$. Simply we use $\ell(A)$ instead of $\ell(A : 1)$. The proof of Theorem A, one immediate consequence of which is presented below, involves a series of reductions similar to the techniques used in the proof of Theorem 2.1 of [17].

Corollary. Let $A$ be a group acting coprimely and with regular orbits on the solvable group $G$. For any prime $p$ not dividing $|C_G(A)|$ we have $\ell_p(G) \leq \ell(A)$.

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Recall that the coprime action of \( A \) guarantees the existence of \( A \)-invariant Sylow subgroups. In [10] Kızmaz studied the structure of a group \( G \) admitting a coprime automorphism \( \alpha \) such that \( G \) has a unique \( \alpha \)-invariant Sylow \( p \)-subgroup for a prime \( p \) where \( C_G(\alpha) \) is a \( p' \)-group and also asked about replacing \( \alpha \) by any subgroup \( A \) of \( \text{Aut}G \). The rest of the paper is concerned with this kind of extensions. More precisely, we consider the situation that \( G \) contains a unique \( A \)-invariant Sylow \( p \)-subgroup where \( C_G(A) \) is a \( p' \)-group. It should be noted that \( G \) contains a unique \( A \)-invariant Sylow \( p \)-subgroup \( P \) if and only if \( C_G(A) \) normalizes \( P \). A first partial answer in this direction is the second result of Section 2 which bounds the \( p \)-length of \( G \) by \( \ell(A) \) in the case where \( A \) is abelian and \( G \) is \( p \)-separable. Namely we have

**Theorem B.** Let \( A \) be an abelian group acting coprimely on the \( p \)-separable group \( G \). Suppose that \( G \) contains a unique \( A \)-invariant Sylow \( p \)-subgroup and that \( C_G(A) \) is a \( p' \)-group. Then \( \ell_p(G) \leq \ell(A) \).

Section 3 is devoted to the extensions of some results of [10] and firstly includes the following answer to its Question 4.1 when \( A \) is abelian.

**Theorem C.** Let \( A \) be an abelian group acting coprimely on the solvable group \( G \). Suppose that \( G \) contains a unique \( A \)-invariant Sylow \( p \)-subgroup \( P \) for an odd prime \( p \) where \( C_G(A) \) is a \( p' \)-group. Then \( P \leq F_{2\ell(A)}(G) \).

The next result extends Proposition 3.1 in [10] for arbitrary \( A \) under some additional assumptions.

**Theorem D.** Let \( A \) be a group acting coprimely on the group \( G \) and let \( P \) be an \( A \)-invariant Sylow \( p \)-subgroup of \( G \) for a prime \( p \) dividing the order of \( G \). Suppose that \( p \) does not divide \( |C_G(a)| \) and that \( C_G(a) \) normalizes \( P \) for all \( 1 \neq a \in A \) of prime order. If \( |GA| \) is odd then \( P \leq F_2(G) \). Furthermore, \( P \leq F(G) \) if \( A \) is of prime order and \( G \) is solvable.

It should be noted that the assumptions of Theorem D are indispensable because the group \( G = \text{PSL}(2, 2^4) \) admits an automorphism of order \( a \) of order 5 such that \( C_G(a) = \text{PSL}(2, 2) \) normalizes an \( \langle a \rangle \)-invariant 11-subgroup of \( G \) while \( F(G) = F_2(G) = 1 \).

Section 4 mainly includes Theorem E below which yields Theorem C in [10] as an immediate corollary.

**Theorem E.** Let \( A \) be a group acting coprimely on a group \( G \). Suppose that \( U \) is an \( A \)-invariant \( p \)-subgroup of \( G \) such that \( C_U(a) = 1 \) for each \( 1 \neq a \in A \) and that \( C_G(A) \) normalizes \( U \). Then \( U \leq O_p(G) \) if the following hold:

(i) \( G = \text{PSL}(2, 2^r) \) free for all \( 1 \neq r \) dividing \( |A| \) in case where \( p \mid 2^r + 1 \),
(ii) \( G = \text{Sz}(2^r) \) free for all \( 1 \neq r \) dividing \( |A| \) in case where \( p \mid 4^r + 1 \).

The notation and terminology are standard.

2. **Bounding the \( p \)-length**

Let \( G \) and \( A \) be groups where \( A \) acts on \( G \). The concept of an \( A \)-tower will be frequently used throughout the paper.
Definition 2.1. (Definition 1.1 and 1.2 of [16]) We say that a sequence $(S_i), i = 1, \ldots, t$ of $A$-invariant subgroups of $G$ is an $A$-tower of $G$ of height $t$ if the following are satisfied:

1. $S_i$ is a $p_i$-group, $p_i$ is a prime, for $i = 1, \ldots, t$;
2. $S_i$ normalizes $S_j$ for $i \leq j$;
3. Set $P_i = S_i$, $P_i = S_i/T_i$ where $T_i = C_{S_i}(P_{i+1})$, $i = 1, \ldots, t-1$ and we assume that $P_t$ is not trivial for $i = 1, \ldots, t$;
4. $p_i \neq p_{i+1}$, $i = 1, \ldots, t-1$.

An $A$-tower $(S_i), i = 1, \ldots, t$ of $G$ is said to be irreducible if the following are satisfied:

5. $\Phi(\Phi(P_i)) = 1$, $\Phi(P_i) \leq Z(P_i)$ and, if $p_i \neq 2$, then $P_i$ has exponent $p_i$ for $i = 1, \ldots, t$. Moreover $P_{t-1}$ centralizes $\Phi(P_t)$;
6. $P_t$ is elementary abelian;
7. There exists $H_i$, an elementary abelian $A$-invariant subgroup of $P_{t-1}$ such that $[H_i, P_i] = P_i$;
8. $(\prod_{i=1}^{t-1} S_i)A$ acts irreducibly on $P_t/\Phi(P_t)$.

Remark 2.2. When the action is coprime $G$ and $G$ is solvable, $G$ contains an $A$-invariant Sylow $p$-subgroup for every prime $p$ dividing $|G|$. This leads to the existence of $A$-towers so that the Fitting height of the group $G$ coincides with the maximum of the heights of all possible $A$-towers in $G$.

The essence of the proof of Theorem A lies in the following

Lemma 2.3. Let a group $G$ act on the solvable group $G$ coprimely and let $p$ be a prime dividing $|G|$. Then $G$ contains an $A$-tower having exactly $\ell_p(G)$-many $p$-terms.

Proof. We proceed by induction on $|G|$. Suppose that $O_{p'}(G) \neq 1$. Then the group $G = G/O_{p'}(G)$ contains an $A$-tower $S_1, \ldots, S_t$ of $G$ having exactly $\ell_p(G)$-many $p$-terms where $S_i$ is a subgroup of $G$ for each $i = 1, \ldots, t$. By Lemma 1.6 in [16] there is an $A$-tower $S_1, \ldots, S_t$ of $G$ which maps to $S_i$, $i = 1, \ldots, t$ having exactly $\ell_p(G)$-many $p$-terms. This contradiction shows that $O_{p'}(G) = 1$ and hence $F(G) = O_p(G)$.

Similarly an induction argument applied to the action of $G = G/O_{p,p'}(G)$ yields an $A$-tower $S_1, \ldots, S_t$ of $G$ having exactly $((\ell_p(G) - 1)$-many $p$-terms. Notice that $S_i$ is a subgroup of $O_p(G)$. By Lemma 1.6 in [16] again, we get an $A$-tower $S_1, \ldots, S_t$, of $G$ which maps to $S_i$, $i = 1, \ldots, t$. We see that $[S_1, \ldots, S_{t-1}, S_t] \neq 1$ where $S_t$ is a $p$-group contained in $O_{p',p'}(G)$. Since $S_t/S_i \cap O_p(G)$ acts faithfully on $O_{p',p'}(G)/O_p(G)$, there exists a $q$-group $Q \leq O_{p,p'}(G)$ for some prime $q \neq p$ such that $[S_1, \ldots, S_{t-1}, S_t, Q] \neq 1$. We may assume that $Q$ is $A$-invariant. It follows that the sequence

$S_1, \ldots, S_{t-1}, S_t, Q, F(G)$

forms an $A$-tower having exactly $\ell_p(G)$-many $p$-terms which is a contradiction completing the proof.

Proof of Theorem A. We choose a counterexample with minimum $|GA| + |A : B|$. If $A = B$ then $G = CG(A)$ whence $G = O_p(G)$ and the result holds. Hence we may assume that $\ell(A : B) \geq 1$. We may also assume that $O_{p'}(G) = 1$, that is $F(G) = O_p(G)$.
Lemma 2.2 guarantees the existence of an A-tower \( S_1, \ldots, S_t \) in \( G \) having exactly \( \ell_p(G) \)-many \( p \)-terms where \( S_1 \) and \( S_t \) are both \( p \)-groups with \( t \geq 3 \). We may assume that this tower is irreducible by Lemma 1.4 in [16]. An induction argument gives that \( G = \prod_{i=1}^{t-1} S_i \) with \( T_{t-1} = 1 \). Set \( H = \prod_{i=1}^{t-2} S_i \) and \( R = P_{t-1} = S_{t-1} \). From now on we shall proceed over a series of steps:

**Step 1.** For all \( C \leq A \) such that \( B \leq C \) and \( \ell(C : B) \geq 1 \) we have \( R = [R, C]^H \).

Assume the contrary, that there exists \( C \leq A \) such that \( B \leq C \) and \( \ell(C : B) \geq 1 \) so that \( R \neq [R, C]^H \). Set \( R_0 = [R, C]^H \Phi(R) \). Since \( R/\Phi(R) \) is irreducible as an \( HA \)-module, we have \( \cap_{a \in A} R_0^a = \Phi(R) \) and \( \cap_{a \in A} C_G(R/R_0)^a = C_G(R/\Phi(R)) \).

Set \( \mathcal{H} = H/C_H(R/\Phi(R)) \). As \( [\mathcal{S}_1, \ldots, \mathcal{S}_{t-2}] = \mathcal{S}_{t-2} \neq 1 \), the sequence \( \mathcal{S}_1, \ldots, \mathcal{S}_{t-2} \) is an \( A \)-tower of \( \mathcal{H} \) having exactly \( \ell_p(G) - 1 \)-many \( p \)-terms. This forces that

\[
\ell_p(\mathcal{H}) = \ell_p(G) - 1.
\]

Notice that \([R/R_0, C]\) = 1 and \( R_0 < \mathcal{H} \). Then, by the three subgroups lemma, \([H, C] \leq C_H(R/R_0)\) and hence

\[
\bigcap_{a \in A} [\mathcal{H}, C]^a = \bigcap_{a \in A} [H, C]^a \leq \bigcap_{a \in A} C_H(R/R_0)^a = 1.
\]

Now an induction argument applied to the action of \( A \) on \( \mathcal{H} \) implies

\[
\ell_p(G) - 1 = \ell_p(\mathcal{H}) \leq \ell(A : C) \leq \ell(A : B) - 1.
\]

This forces that \( \ell_p(G) \leq \ell(A : B) \), which is a contradiction establishing the claim.

**Step 2.** Recall that \( P_{t-2} = S_{t-2}/T_{t-2} \). Set \( Q = S_{t-2} \) and \( K = \prod_{i=1}^{t-3} S_i \). Then for all \( D \leq A \) such that \( B \leq D \) and \( \ell(D : B) \geq 2 \) we have \( Q = [Q, D]^K \Phi \) where \( \Phi = \Phi(Q)T_{t-2} \).

Assume the contrary, that there exists \( D \leq A \) such that \( B \leq D \) and \( \ell(D : B) \geq 2 \) so that \( Q \neq [Q, D]^K \Phi \). Let \( Q_0 = [Q, D]^K \Phi \). Since \( Q/\Phi \) is an irreducible \( KA \)-module, \( \cap_{a \in A} Q_0^a = \Phi \) and \( \cap_{a \in A} C_G(Q/Q_0)^a = C_G(Q/\Phi) \).

Set \( \mathcal{K} = K/C_K(Q/\Phi) \). As \( [\mathcal{S}_1, \ldots, \mathcal{S}_{t-3}] = \mathcal{S}_{t-3} \neq 1 \), the sequence \( \mathcal{S}_1, \ldots, \mathcal{S}_{t-3} \) forms an \( A \)-tower of \( \mathcal{K} \). It follows that

\[
\ell_p(G) - 2 \leq \ell_p(\mathcal{K}) \leq \ell_p(G) - 1.
\]

Then we also have \( \ell_p(K/C_K(Q/Q_0)) \geq \ell_p(G) - 2 \). Since \( D \) acts trivially on \( Q/Q_0 \), \([K, D] \leq C_K(Q/Q_0)\). Therefore we have

\[
\bigcap_{a \in A} [\mathcal{K}, D]^a = \bigcap_{a \in A} [K, D]^a \leq \bigcap_{a \in A} C_K(Q/Q_0)^a = 1.
\]

By induction applied to the action of \( A \) on \( \mathcal{K} \) we get

\[
\ell_p(G) - 2 \leq \ell_p(\mathcal{K}) \leq \ell(A : D) \leq \ell(A : B) - 2
\]

which forces that \( \ell_p(G) \leq \ell(A : B) \). This contradiction establishes the claim.
Step 3. Final contradiction.

Since the $A$-tower $S_1, \ldots, S_t$ is irreducible, the groups $P$ and $Q/C_Q(P)$ are special. Furthermore they are of exponent $p_{t-1}$ (resp. $p_{t-2}$) if $p_{t-1}$ and $p_{t-2}$ are odd. We are now ready to apply [17, Theorem 1.1], to the action of $S_{t-1}S_{t-2}A$ on the Frattini factor group of $S_t$ and get $C_{S_t}(A) \neq 1$. This contradiction completes the proof. □

Proof of Theorem 3.4 Let $GA$ be a minimal counterexample to the theorem. We may assume that $O_{p'}(G) = 1$. Let $O_i(G)$, $i = 1, \ldots, \ell(G)$ be defined for $\pi = \{p\}$ as in [12] where $\ell(G)$ is the least positive integer such that $G = O_{\ell(G)}(G)$. If $P \leq O_{\ell(G)-1}(G)$, an induction argument applied to the action of $A$ on the group $O_{\ell(G)-1}(G)$ implies that $\ell_p(G) \leq \ell(A)$, which is not the case. Therefore we may assume that $G = O_{\ell(G)-1}(G)P$. By [12, Lemma 4.3] there exists a sequence $A_1, \ldots, A_{\ell(G)}$ of $A$-invariant sections of $G$ satisfying the conditions (1.10.a) – (1.10.f) of [12]. Furthermore, as a consequence of [12, Lemma 4.3 (a)], the following are satisfied:

(a) $A_i$ is a $p$-group (or a $p'$-group), respectively $A_i+1$ is a $p'$-group (or a $p$-group), and $A_{\ell(G)} \leq G$. In our case we see that $A_1$ and $A_{\ell(G)}$ are both $p$-groups. In particular $C_{A_1}(A) = 1$

(b) $\ell_p(G)$ is equal to the number of $p$-groups among the sections $A_i$, for $i = 1, \ldots, \ell(G)$.

(c) $[A_i, A_{i-1}] = A_i$, for $i = 2, \ldots, \ell(G)$.

More precisely, the sequence $A_1, \ldots, A_{\ell(G)}$ is an $A$-tower. Since $A$ acts fixed point freely on $A_1$ there is a nonidentity element $a \in A$ of prime order such that $[A_1, a] \neq 1$. It follows by Theorem 3.1 in [16] that there is a sequence of $A$-invariant subgroups $C_2, \ldots, C_{\ell(G)}$ each of which is centralized by $a$ so that it forms an $A$-tower. This forces that the $C_{O_{\ell(G)}(G)}(a)$ has $p$-length $\ell_p(G) - 1$. We then apply induction to the action of $A/\langle a \rangle$ on $C_{O_{\ell(G)}(G)}(a)$ and get $\ell_p(G) - 1 \leq \ell - 1$. This contradiction completes the proof.

3. Embedding of the unique $A$-invariant Sylow $p$-subgroup

Proof of Theorem 3.4 We proceed by induction on $[GA]$. Let $k$ be the largest such that $p$ divides the order of $F_k(G)/F_{k-1}(G)$. Assume that $k > 2\ell(A)$. Then there is an $A$-tower $S_1, S_2, \ldots, S_{2\ell(A)+1}$ of $G$ where $S_1$ is a $p$-group. We may assume that this tower is irreducible. Set $V = P_2/\Phi(P_2)$. We have $C_V(A) = 1$ as $[C_{P_2}(A), P_1] = 1$. If the group $P_1A$ is Frobenius we would have $C_V(A) \neq 1$, which is a contradiction. Thus there exists $1 \neq a \in A$ such that $C_{P_2}(a) \neq 1$. By [16, Theorem 3.1], we see that

$$[C_{S_1}(a), \ldots, C_{S_{j-1}}(a), C_{S_j+1}(a), \ldots, C_{S_{2\ell(A)+1}}(a)] \neq 1.$$  

Indeed we have one of the two cases:
Consider the semidirect product $V \rtimes H$.

Let $V$ be an elementary abelian 5-group and $C_i$.

Both are impossible by hypothesis and the first claim follows.

Proof. Let $V, P, B, G$ be two successive terms of an $A$-tower such that $S_{i-1}$ is a $p$-group. We may assume that this tower is irreducible. Then $C_{P_2}(a) \leq \Phi(P_2)$ for all $1 \neq a \in A$ as $[C_{P_2}(a), P_2] = 1$. It follows that $P_2 = [P_2, a]$ for all $1 \neq a \in A$, that is, the group $P_2 A$ is Frobenius-like with kernel $P_2$.

By [10, Corollary C], applied to the action of $P_2 A$ on $P_3$ we observe that $C_{P_3}(A) \neq 1$ which forces that $P_3 \neq P$.

If the group $P_3 A$ were Frobenius then we would have $C_{P_2}(A) \leq \Phi(P_2)$ which is not the case. Thus there exists $1 \neq b \in A$ such that $C_{P_2}(b) \neq 1$. We may assume that $b$ is of prime order. It follows that $[C_{P_2}(b), P_2, P_3] \neq 1$ as $P_2$ acts faithfully on $P_3$. Since $p$ is odd, by [10, Theorem 3.1], we get either $[C_{P_2}(b), C_{P_2}(b)] \neq 1$ or $[C_{P_2}(b), C_{P_2}(b)] \neq 1$.

Both are impossible by hypothesis and the first claim follows.

Finally assume that $A$ is of prime order and $G$ is solvable. In this case let $S_i$ and $S_{i-1}$ be two successive terms of an $A$-tower such that $S_{i-1}$ is a $p$-group. Notice that $C_{S_{i-1}}(A)$ and $C_{P_i, \Phi(P_i)}(A)$ are both trivial. This forces by Thompson’s celebrated theorem that $S_{i-1}$ centralizes $P_i$ which is a contradiction. Hence the proof is complete.

Example 3.1. Let $H = P \times A$ where $P \triangleleft H$ is cyclic of order 7 and $A$ is cyclic of order 9. Suppose that $B = \Omega_i(A) \triangleleft H$ and $H/B$ is a Frobenius group of order 21. Then $Soc(H) = P \times B$ and is cyclic of order 21. By Theorem 10.3 on page 173 in [1] there exists an elementary abelian 5-group $V$ which is a faithful and irreducible $H$-module.

Consider the semidirect product $V H$ and let $G$ be the subgroup $V P$. Then $A$ acts coprimely on $G$, $[V, P] = V$, and $[V, B] = V$ whence the group $VA$ is Frobenius. Now $C_G(A) = 1$ and $C_G(a) = P$ for any $1 \neq a \in A$ of prime order. This example shows that $P$ is not necessarily contained in $F(G)$ under the hypothesis of Theorem 10.

4. EMBEDDING OF SOME $A$-INvariant $p$-SUBGROUPS WITHIN THE GROUP

Although this section is devoted to a proof of Theorem E we want first to emphasize a special case of this result due to the simplicity of its proof.

Theorem 4.1. Let $A$ be a group acting on the $p$-separable group $G$ coprimely and let $U$ be an $A$-invariant $p$-subgroup of $G$ such that $C_U(a) = 1$ for each $1 \neq a \in A$. If $C_G(A) \triangleleft G$, then $U \leq O_p(G)$.

Proof. Let $G$ be a minimal counterexample to the theorem. We can easily observe that by an induction argument applied to the action of $A$ on $G/O_p(G)$ we get $O_p(G) = 1$. 
Another induction argument applied to the action of $A$ on $O_{p'}(G)U$ yields that $G = O_{p'}(G)U$. By hypothesis, the group $UA$ is Frobenius with kernel $U$. Let $Q$ be a $UA$-invariant Sylow $q$-subgroup of $O_{p'}(G)$ on which $U$ is nontrivial. Set $V$ be the Frattini factor group of $Q$. W.l.o.g. we may assume that $V$ is irreducible as a $UA$-module. It is well known that $C_V(A) \neq 1$. On the other hand $[C_Q(A), U] = 1$ and so $C_Q(A) \leq \Phi(Q)$, that is $C_V(A) = 1$, which is a contradiction. \qed

We now prove some lemmas which will be used in the proof of Theorem E.

**Lemma 4.2.** Let $G = PGL(2, p^r)$ for some positive integer $r$ and let $P$ be a Sylow $p$-subgroup of $G$. Then $C_G(x) \leq P$ for any nonidentity $x \in P$.

**Proof.** Let $\Gamma = GL(2, p^r)$ and $F$ be a field of order $p^r$. Let $A = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ for some $t \in F^*$ and pick $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ such that $AB = BA$. It follows easily that $a = d$ and $c = 0$, that is,

$$B = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}^{1} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix} \in Z(\Gamma)Q$$

where $Q$ is the Sylow $p$-subgroup of $\Gamma$ which consists of upper triangular matrices. Thus, we obtain $C_\Gamma(A) \leq Z(\Gamma)Q$. Write $\overline{\Gamma} = \Gamma/Z(\Gamma) = G$. Since $Z(\Gamma)$ is a $p'$-subgroup of $\Gamma$, we get $C_G(\overline{A}) = C_{\overline{\Gamma}}(A) = C_{\overline{\Gamma}}(A) \leq \overline{Q}$ by [9] Lemma 7.7. Note that $\overline{Q}$ is a Sylow $p$-subgroup of $G$, and so by taking an appropriate conjugate we obtain that $C_G(x) \leq P$ for any $1 \neq x \in P$. \qed

**Lemma 4.3.** Let $G \cong PSL(2, 3^r)$ and $H \cong PSL(2, 3)$ be a subgroup of $G$. Suppose that $U$ is a subgroup of $G$ which is normalized by $H$ and which has trivial intersection with $H$. Then $U = 1$.

**Proof.** Let $G$ be a minimal counter example to the lemma. Note that the order of $G$ is $q(q - 1)(q + 1)/2$ where $q = 3^r$. Let now $M$ be a maximal subgroup of $G$ that contains the subgroup $HU$. The possible structure of $M$ is given in [11] Corollary 2.2[(a)-(h)]. Since $\pi(H) = \{2, 3\} \subseteq \pi(M)$ and $q = 3^r$ for $r \geq 2$, the group $M$ can be only one of the groups described in (e), (d), (c) or (h) of [11] Corollary 2.2. We shall complete the proof by obtaining a separate contradiction for each case below.

Suppose that [11] Corollary 2.2 (c) holds. Then $M$ is of order $q(q - 1)/2$ as $q$ is odd, and so the normalizer of a Sylow 3-subgroup. It follows that $M$ is 3-closed and hence $H \cong A_4$ is also 3-closed, which is not the case.

Suppose that [11] Corollary 2.2 (d) holds, that is, $M \cong PSL(2, 3^r)$ for some $r_0 < r$. It follows that $U = 1$ by the minimality of $G$, which is a contradiction.

Suppose that [11] Corollary 2.2 (e) holds. Then $M \cong PGL(2, 3^r)$ for some $r_0 < r$. Set $q_0 = 3^r$ and let $M_0$ be the subgroup of $M$ isomorphic to $PSL(2, 3^r)$. Note that the order of $M$ is $q_0(q_0^2 - 1)$, and so $|M : M_0| = 2$. Clearly $H \leq M_0$ and $H$ normalizes $U \cap M_0$ as $M_0 < M$. By the minimality of $G$, we see that $U \cap M_0 = 1$, and so $|U| = 2$ as $|U| > 1$. It follows then that $[H, U] = 1$ as $H$ normalizes $U$. Let now $h \in H$ be of order
3. Then $U \leq C_M(h) \leq P$ where $P$ is a Sylow 3-subgroup of $M$ by Lemma 4.2 which leads to a contradiction.

Finally suppose that \cite{11} Corollary 2.2 ([h]) holds. Then $M \cong A_5$ and so it is apparent that the subgroup $H \cong A_4$ does not normalize any nontrivial such $U$.

\begin{lemma}

Let $G = L(q^f)$ be a simple group of Lie type over the field $F$ of order $q$ where $q$ is a power of a prime $u$, and let $a$ be the automorphism of $G$ induced by the field automorphism $x \mapsto x^q$. Suppose that $\O''(G(a)) = L(q)$ is not isomorphic to one of $\PSL(2,2)$, $\PSL(2,3)$ and $\Sz(2)$. Then there is no nontrivial subgroup $K$ of $G$ which is normalized by $CG(a)$ and which has trivial intersection with $CG(a)$.

\begin{proof}

Let $K$ be a subgroup of $G$ which is normalized by $CG(a)$ and which has trivial intersection with $CG(a)$. By \cite{3} Theorem 1, there exists a positive integer $r$ dividing $f$ such that 

$$L(q^r) \cong \O''(G(a^r)) \leq CG(a)K \leq CG(a)^r$$

where $CG(a^r)$ is generated by $\text{Inn}(CG(a^r))$ and some diagonal automorphisms of the simple group $\O''(CG(a^r))$. Note that $r < f$ because otherwise $G = CG(a)K$ which implies by the simplicity of $G$ that $K = 1$. If $r = 1$ then $CG(a)K \leq \Aut(CG(a))$ and so $K$ normalizes $CG(a)$, that is, $K = 1$ as desired. Then by induction applied to the action of $\langle a \rangle / \langle a^r \rangle$ on $CG(a^r)$ with $CG(a^r) \cap K$ we obtain $CG(a^r) \cap K = 1$. It follows that $CG(a^r) = CG(a)$ and so $[CG(a^r), K] \leq CG(a^r) \cap K = 1$. Due to the faithful action of $K$ on $CG(a)$ we get $K = 1$.

\end{proof}

\end{lemma}

\begin{lemma}

Let $A$ be a nontrivial automorphism group of a nonabelian simple group $G$ and $(|A|, |G|) = 1$. Then $G$ is a simple group of Lie type and $A$ is cyclic.

\begin{proof}

It is clear that $A$ is isomorphic to a subgroup of $\Out(G) = \Aut(G)/\Inn(G)$. According to the classification of finite simple groups \cite{1} Theorem 0.1.1 and the well-known information on outer automorphism groups (see \cite{8} Theorem 5.2.1 and Tables 5.3a–5.3z), one of the following statements holds:

\begin{itemize}
  \item $G$ is isomorphic to an alternating group of degree $\geq 5$ and $|\Out(G)| \in \{2, 4\}$;
  \item $G$ is isomorphic to one of 26 sporadic groups and $|\Out(G)| \in \{1, 2\}$;
  \item $G$ is isomorphic to a simple group of Lie type.
\end{itemize}

Since $|G|$ is even (by the Feit–Thompson theorem \cite{5}, for example), $G$ must be isomorphic to a simple group of Lie type. Then $\Out(G)$ is solvable. More exactly, according to \cite{8} Theorem 2.5.12 $\Out(G)$ has a normal subgroup $\Outdiag(G)$ such that $\pi(\Outdiag(G)) \subseteq \pi(G)$. Moreover, $\overline{G} = \Out(G)/\Outdiag(G)$ has a normal cyclic subgroup $\Phi$ (isomorphic to the automorphism group of the ground field) such that $\overline{G}/\Phi \in \{1, 2\}$ or $G \cong D_4(q)$ and $\overline{G}/\Phi \cong S_3$. In all cases $\pi(\overline{G}/\Phi) \subseteq \pi(G)$. This implies that $A$ is isomorphic to a subgroup of $\Phi$. In particular, $A$ is cyclic.

\end{proof}

\end{lemma}

\begin{lemma}

Let $G = \Sz(q)$, where $q = 2^r$ for some odd $r$. Then every solvable subgroup $H$ of $G$ such that $5 \in \pi(H)$ is contained in a subgroup $M$ of $G$ with the following properties.

\begin{proof}

\end{proof}

\end{lemma}
• $|M| = 4(q - \varepsilon\sqrt{2q} + 1)$, where $\varepsilon \in \{+, -\}$ is uniquely defined by the relation $5 | (q - \varepsilon\sqrt{2q} + 1)$;
• $M$ is the Frobenius group with cyclic kernel of order $q - \varepsilon\sqrt{2q} + 1$ and cyclic complement of order 4.

In particular, if $U$ is a nontrivial $p$-subgroup of $G$ such that 5 divides $|N_G(U)|$ then $U$ is cyclic and $|U|$ divides $q - \varepsilon\sqrt{2q} + 1$.

Proof. According to [15, Theorem 9], every proper subgroup of $G$ is conjugate to a subgroup of one of the following subgroups:
• Frobenius group of order $q^2(q - 1)$;
• dihedral group $B_0$ of order $2(q - 1)$;
• the normalizer $B_1$ of a cyclic group $A_1$ of order $q - \sqrt{2q} + 1$, $|B_1| = 4|A_1|$;
• the normalizer $B_2$ of a cyclic group $A_2$ of order $q + \sqrt{2q} + 1$, $|B_2| = 4|A_2|$;
• $Sz(2^r)$ where $r_0$ divides $r$.

It is easy to see that 5 divides only the orders of $Sz(2^r)$ and exactly one of $B_1$ and $B_2$. We choose $i \in \{1, 2\}$ such that 5 divides $|B_i|$. It follows by induction from this remark that $H$ contains a normal cyclic 2-complement $T$ and $T$ contains a cyclic subgroup of order 5. Since $A_i$ contains a Sylow 5-subgroup of $G$, we may assume that $A_i \cap T \neq 1$.

By [15, Proposition 16], the centralizer of every nontrivial element of $A_i$ coincides with $A_i$. Therefore, $T \leq C_G(A_i \cap T) \leq A_i$ and $H$ normalizes $C_G(T) = A_i$. This means that $H \leq B_i$. Moreover, $B_i/A_i = B_i/C_{B_i}(Z)$ is isomorphic to a subgroup of the automorphism group of the subgroup $Z \leq A_i$ of order 5. Consequently, $B_i/A_i$ is cyclic. Since $C_{B_i}(a) = A_i$ for every nontrivial $a \in A_i$, $B_i$ is a Frobenius group with the cyclic kernel $A_i$ and a cyclic complement of order 4. \hfill \Box

We need an extension of [10, Lemma 3.3] in the proof of Theorem E.

Lemma 4.7. Let $G$ be a nonabelian simple group and let $\alpha$ be a coprime automorphism of $G$ of order $r$. Let $U$ be a nontrivial $\alpha$-invariant $p$-subgroup of $G$ such that $C_U(\alpha) = 1$. If $C_G(\alpha)$ normalizes $U$, then one of the following holds:

a) $G = PSL(2, 2^r)$ and $r \geq 5$. Moreover, $p \geq 5$ and $p$ is a divisor of $2^r + 1$.

b) $G = Sz(2^r)$ and $r \geq 7$. Moreover, $p \geq 5$ and $p$ is a divisor of $2^r \pm \sqrt{2^r + 1} + 1$ where the sign $\pm$ is chosen such that 5 divides $2^r \pm \sqrt{2^r + 1} + 1$.

Proof. We see that $G$ is a simple group of Lie type by Lemma 4.5. It follows that $G = PSL(2, 2^r)$ or $Sz(2^r)$ by Lemmas 4.3 and 4.4. Set $C = C_G(\alpha)$.

Let $G = PSL(2, 2^r)$ and set $q = 2^r$. Then we see that $C = PSL(2, 2) \cong S_3$. Suppose first that $p$ is odd. A Sylow $p$-subgroup $P$ of $G$ is cyclic by [13, Theorem 8.6.9]. If $p = 3$, then $U \cap C$ contains an element of order 3, which is impossible by the hypothesis. Thus $p \geq 5$. We also have $r \geq 5$ as $r$ is coprime to $|C| = 6$. We see that $C$ is contained in a maximal subgroup $D$, which is a dihedral group of order $2(q + 1)$ (see [11, Corollary 2.2 (f) and (i)]). Since $r$ is odd, 3 is coprime to $q - 1 = 2^r - 1$, and so $|D| = 2(q + 1) = 2^r + 1$.

Let $T$ be the subgroup of $C$ of order 3. Clearly, $T$ is normalized by $D$, and so $D = N_G(T)$ as $G$ is simple and $D$ is a maximal subgroup of $G$. Now we claim that
p | q + 1. Since U is cyclic, Aut(U) is abelian. It follows that C/C_G(U) is abelian. We get that T \leq C_G(U) as T = C_g, and so U \leq C_G(T) \leq N_G(T) = D. As p is odd and |D| = 2(q + 1), we have that p divides q + 1. Consequently, we observe that if such an α-invariant p-subgroup U of G exists, it must be contained in D = N_G(T). On the other hand, D = N_G(T) is α-invariant and π(D) ≠ \{2, 3\} as r ≥ 5. Pick an α-invariant Sylow p-subgroup P of D for p ≥ 5. Clearly, P is normalized by C and C_P(α) = 1, which completes the proof for this case.

Assume now that p = 2 and take a Sylow p-subgroup P of G such that U \leq P. In this case, P is elementary abelian of order 2^r and |N_G(P)| = 2^r(2^r - 1) (see [2, Table 1]). Since r is odd, we have (|N_G(P)|, 3) = 1. It follows from Lemma 4.2 that C_G(U) = P which means that P is a normal subgroup of N_G(U). Therefore, N_G(U) \leq N_G(P) and (|N_G(U)|, 3) = 1, which contradicts the fact that N_G(U) ≥ C \cong S_3. Thus, the case p = 2 is impossible.

Next let G = Sz(q) where q = 2^r and r is odd. Note that C \cong Sz(2) which is a Frobenius group of order 20. Denote T = O_5(C). Then T \leq C \leq N_G(U). In particular, |N_G(U)| is divisible by 5. By Lemma 4.6 U must be cyclic and |U| must divide q - ε√2q + 1, where ε = ± and 5 divides q - ε√2q + 1.

If r = 3 then q - √2q + 1 = 5. Consequently, ε = + and p = |U| = 5. But in this case UT is contained in a Frobenius subgroup with a cyclic kernel and a cyclic complement of order 4 by Lemma 4.6. This means that U = T is contained in C, a contradiction. Hence, r > 3.

Since 5 divides |G| and |α| = r, we have r > 5. Now the desired statement follows from Lemma 4.6.

Proof of Theorem E. Let G be a minimal counterexample to the theorem and choose U of minimal possible order. Then U > 1. It can be easily observed by an induction argument applied to the action of A on G/O_p(G) that O_p(G) = 1. Let N be a minimal normal A-invariant subgroup of G. We shall separate the proof into two cases:

Case 1. Assume that N = G. Then G is characteristically simple, that is, G = G_1 × ... × G_n where G_i are isomorphic nonabelian simple groups and A acts transitively on \{G_i : i = 1, ..., n\}. Let now B = N_A(G_1) and let X = G_2 × G_3 × ... × G_n. Note that X is a B-invariant normal subgroup of G. Assume that X > 1 and set \overline{G} = G/X. Let A = ∪_i=1^n B_{a_i} be a coset decomposition of A with respect to B where a_1 = 1. We observe that C_G(A) = \{ \prod_{i=1}^n g^{a_i} : g \in C_{G_1}(B) \}, and hence

$$C_G(A) = C_{G_1}(B) = C_{\overline{G}_1}(B) = C_{\overline{G}}(B).$$

Then \overline{C}(B) normalizes \overline{U}. Since \overline{U}B is a Frobenius group, an induction argument applied to the action of B on \overline{G} yields that \overline{U} \leq O_p(\overline{G}) = 1, that is, \overline{U} ≤ X. It follows that U = 1 as A acts transitively on \{G_i : i = 1, ..., n\}. By this contradiction, we get X = 1, that is, G is simple. We observe by Lemma 4.3 that A is cyclic. Then, appealing to Lemma 4.7 we obtain the final contradiction in Case 1.
**Case 2.** Assume that $N < G$. By induction applied to the action of $A$ on $N$, it holds that $U \cap N \leq O_p(N) \leq O_p(G) = 1$. Write $\overline{G} = G/N$. Then by induction applied to the action of $A$ on $\overline{G}$, we get $1 < \overline{U} \leq O_p(\overline{G})$. Let $H/N = O_p(\overline{G})$. Assume $H < G$. Clearly, $U \subseteq H$, and so $U \leq O_p(H) \leq O_p(G)$ by induction applied to $H$. This contradiction shows that $H = G$. Since $\overline{G}$ is a $p$-group, $NU$ is subnormal in $G$. Thus, if $NU < G$, then we get $U \leq O_p(G)$, which is not the case. Thus, $G = NU$. We get $\Phi(U) = O_p(G) = 1$ by the minimality of $U$, and so $U$ is an elementary abelian $p$-group.

Now $N$ is characteristically simple, that is, $N = N_1 \times \cdots \times N_k$ where $N_i$, $i = 1, \ldots, k$, are simple. Notice that $N$ is nonabelian because otherwise $G$ is $p$-separable and the result follows by Theorem 4.1.

Let $\Omega$ denote the set of $N_i$, $i = 1, \ldots, k$. Then $UA$ acts transitively on $\Omega$. Let $\Omega_1$ be the $U$-orbit on $\Omega$ containing $N_1$, and set $A_1 = Stab_A(\Omega_1)$.

Suppose first that $A_1 = 1$. Clearly, we have $Stab_A(N_1) \leq A_1 = 1$. Consider the group $X = \prod_{a \in A} N_1^a$. Then $C_X(A) = \{\prod_{a \in A} n^a : n \in N_1\}$. Since $U$ centralizes $C_X(A)$, $X$ is $UA$-invariant and hence $X = N$ by the minimality of $N$. That is, $k = |A|$ and so there is a $U$-orbit of length 1 because otherwise we would have $p$ divides $|A|$. Suppose that $U$ normalizes $N_1$. Then $U$ normalizes $N_i$ for each $i$. This forces that $|N_i, U| = 1$ for each $i$ as $[C_N(A), U] = 1$, and so $[N, U] = 1$. This contradiction shows that $A_1 \neq 1$.

Let now $S = Stab_{UA_1}(N_1)$ and $U_1 = U \cap S$. Then $|U : U_1| = |\Omega_1| = |UA_1 : S|$. Notice next that $(|S : U_1|, |U_1|) = 1$ as $(|U|, |A_1|) = 1$. Let $S_1$ be a complement of $U_1$ in $S$. Then we have $|U : U_1| = |U : A_1|/|U_1 : S_1|$ which implies that $|A_1| = |S_1|$. Therefore we may assume that $S = U_1A_1$, that is, $N_1$ is $A_1$-invariant.

Let $x \in U$ and $1 \neq a \in A_1$ such that $(N_1^a)^x = N_1^x$ holds. Then $[a, x^{-1}] \in U_1$ and so $U_1x = U_1x^a = (U_1x)^a$ implying the existence of an element $g \in U_1x \cap C_U(a)$. Hence $x \in U_1$. It follows that $Stab_{A_1}(N_1^x) = 1$ for every $x \in U \setminus U_1$. More precisely we have shown that $A_1$ is a nontrivial subgroup of $A$ stabilizing exactly one element, namely $N_1$, and all the remaining orbits of $A_1$ are of length $|A_1|$.

The group $A$ acts transitively on $\{\Omega_i : i = 1, 2, \ldots, s\}$, the collection of $U$-orbits on $\Omega$. Let now $M_i = \prod_{s \in \Omega_i} M$ for $i = 1, 2, \ldots, s$. Suppose that $s > 1$. Then $A_1 = Stab_{A_1}(\Omega_1)$ is a proper subgroup of $A$. Let $A = \bigcup_{i=1}^{m} A_1g_i$ be the coset decomposition of $A$ with respect to $A_1$. Notice that $C_N(A) = \prod_{i=1}^{m} n^{g_i} : n \in C_{M_i}(A_i)\}$. Since $[C_N(A), U] = 1$, we have $[C_{M_i}(A_1), U] = 1$. Applying induction to the action of $A_1$ on $M_iU$ we obtain $U = O_p(M_iU)$, that is $[M_i, U] = 1$. Then $[M_i, U] = 1$ for each $i$, which is impossible. Thus $A_1 = A$ and $\Omega = \Omega_1$, that is, $U$ acts transitively on $\Omega$.

It follows that $N_1$ is $A$-invariant. Let $Y = \prod_{n \in \Omega} N_2^a$. Since $[C_Y(A), U] = 1$, we see that $Y$ is $UA$-invariant which is impossible by the minimality of $N$. Therefore we may assume that $N$ is simple. Moreover, $C_G(N) \cap N = Z(N) = 1$. Consequently, $C_G(N)$ is isomorphic to a subgroup of $G/N \cong U$. Therefore, $C_G(N)$ is a normal $p$-subgroup of $G$ and $O_p(G) = 1$ implies $C_G(N) = 1$. It follows from the three subgroups lemma and the equalities

$$[C_N(N), G] = 1 \text{ and } [N, G, C_A(N)] = [N, C_A(N)] = 1$$
that
\[ [G, C_A(N), N] = 1, \quad [G, C_A(N)] \leq C_G(N) = 1 \quad \text{and} \quad C_A(N) = C_A(G) = 1. \]

This means that \( G \) and \( A \) are isomorphically embedded in \( Aut(N) \). Moreover, the kernel \( C_GA(N) \) of the natural homomorphism \( GA \rightarrow Aut(N) \) is also trivial because \( |G| \) and \( |A| \) are coprime. Thus we may consider \( GA \) as a subgroup of \( Aut(N) \).

Note that \( N \) must be isomorphic to a group of Lie type by Lemma 4.5 as it admits a coprime automorphism. We need now some information about the automorphism groups of the simple groups of Lie type given in [5, Theorem 2.5.12]. There are three subgroups \( Inndiag(N) \), \( \Phi \), and \( \Gamma \) in \( Aut(N) \) such that every two of them have the trivial intersection and
\[ Aut(N) = Inndiag(N)\Phi \Gamma. \]

Here \( \Phi \) is the field automorphism group of \( N \), \( \Gamma \) is the graph automorphism group, and \( Inndiag(N) \) is the inner-diagonal automorphism group of \( N \). The subgroup \( Inndiag(N) \) is normal in \( Aut(N) \) and contains \( Inn(N) \) by [8, Theorem 2.5.12]. We have that
\[ \pi(\Gamma) \cup \pi(Outdiag(N)) \subseteq \pi(N), \]
where \( Outdiag(N) = Inndiag(N)/Inn(N) \). Moreover \( [\Phi \Gamma, \Phi] = 1 \).

It follows from the Schur-Zassenhaus theorem that \( A \) is conjugate in \( Aut(N) \) to a subgroup of \( \Phi \) and we may assume that \( A \leq \Phi \). Moreover, as \( UA \) is a Frobenius group, we have
\[ U = [U, A] \leq [Aut(N), \Phi] \leq [\Phi \Gamma, \Phi] Inndiag(N) = Inndiag(N). \]

Furthermore, \( U \cap N = 1 \) implies that \( U \) is isomorphic to a subgroup of \( Outdiag(N) \). In particular, \( d = |Outdiag(N)| > 1 \). This means that \( N \) is not a Suzuki group and \( 2, 3 \in \pi(N) \) by [5] and [6, Chapter II, Corollary 7.3].

Assume that \( N \) is not isomorphic to
\[ PSL^+(n, q) = PSL(n, q) \cong A_{n-1}(q) \quad \text{and} \quad PSL^-(n, q) = PSU(n, q) \cong 2A_{n-1}(q). \]

Then \( d \leq 4 \) by [8, Theorem 2.5.12], and \( |U| \leq 4 \). In this case, \( A \) is a \( \{2, 3\} \)-group since \( A \leq Aut(U) \), which contradicts the fact that \( (|A|, |G|) = 1 \).

Thus, we may assume that \( N = PGL^\varepsilon(n, q) \), where \( \varepsilon \in \{+, -\} \). It follows from [8, Theorem 2.5.12], that \( Outdiag(N) \) is cyclic of order \( d = (n, q-\varepsilon 1) \) (in fact, \( Inndiag(N) \cong PGL^\varepsilon(n, q) \) in this case). This means that the elementary abelian \( p \)-group \( U \) is cyclic, \( |U| = p \), and \( p \leq d \leq n \). Now, take \( r \in \pi(A) \). Then
\[ r \leq |A| \leq |Aut(U)| = p - 1 < n. \]

Moreover, \( (|A|, |G|) = 1 \) implies \( (r, 2q) = 1 \) and \( r \) divides \( q^r-1 = q^r-1-(\varepsilon 1)^r-1 \). But this means that \( r \) divides
\[ |N| = \frac{1}{d}q^n(q-1)^{n-1} \prod_{i=1}^{n}(q^i - (\varepsilon 1)^i), \]
which contradicts the fact that \( |A| \) and \( |G| \) are coprime. This completes the proof. \( \Box \)
Suppose that $A$ acts on $G$. Let $S_p(G, A)$ denote the set of all $A$-invariant Sylow $p$-subgroups of $G$, and $(O, A)_p(G)$ denote the intersection of all $P \in S_p(G, A)$.

**Corollary 4.8.** Let $A$ act coprimely on $G$ and let $p$ be a prime which is coprime to $|C_G(a)|$ for all $1 \neq a \in A$. Then $(O, A)_p(G) = O_p(G)$ if the following hold:

(i) $G$ is $PSL(2, 2^r)$ free for all $1 \neq r$ dividing $|A|$, in case where $p | 2^r + 1$,

(ii) $G$ is $Sz(2^r)$ free for all $1 \neq r$ dividing $|A|$ in case where $p | 4^r + 1$.

**Proof.** We clearly have $O_p(G) \leq (O, A)_p(G)$. On the other hand, it is easy to see that $(O, A)_p(G)$ is normalized by $C_G(A)$. Since $p$ is coprime to $|C_G(a)|$ for all $1 \neq a \in A$, we have $(O, A)_p(G) \leq O_p(G)$ by Theorem E as claimed. ∎

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