REMOVABILITY OF PRODUCT SETS FOR SOBOLEV FUNCTIONS IN THE PLANE

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Abstract. We study conditions on closed sets $C, F \subset \mathbb{R}$ making the product $C \times F$ removable or non-removable for $W^{1,p}$. The main results show that the Hausdorff-dimension of the smaller dimensional component $C$ determines a critical exponent above which the product is removable for some positive measure sets $F$, but below which the product is not removable for another collection of positive measure totally disconnected sets $F$. Moreover, if the set $C$ is Ahlfors-regular, the above removability holds for any totally disconnected $F$.

1. Introduction

In this paper we study the Sobolev-removability of closed subsets of the Euclidean plane. The Sobolev space $W^{1,p}(\Omega)$, for $1 \leq p \leq \infty$ and a domain $\Omega \subset \mathbb{R}^2$, consists of $f \in L^p(\Omega)$ for which the weak first order partial derivatives $\partial_i f$ are also in $L^p(\Omega)$. A subset $E \subset \mathbb{R}^2$ of Lebesgue measure zero is called removable for $W^{1,p}$, or simply $p$-removable, for $1 \leq p \leq \infty$, if $W^{1,p}(\mathbb{R}^2 \setminus E) = W^{1,p}(\mathbb{R}^2)$ as sets. Since $E$ has Lebesgue measure zero, $E$ is removable for $W^{1,p}$ if and only if every $u \in W^{1,p}(\mathbb{R}^2 \setminus E)$ has an $L^p$-representative that is absolutely continuous on almost every line-segment parallel to the coordinate axis.

Let us make some observations on $p$-removable sets. By Hölder’s inequality, $p$-removable sets are also $q$-removable for every $q > p$. In particular, each $p$-removable set is $\infty$-removable. Hence, the complement of a $p$-removable set is always quasi-convex meaning that any two points in the complement can be joined by a curve in the complement whose length is comparable to the distance between the points. Since the sets $E$ we consider have Lebesgue measure zero, the quasi-convexity of the complement implies that closed $p$-removable sets are actually metrically removable, see [11, Proposition 3.3].

Ahlfors and Beurling [1] studied removable sets for analytic functions with finite Dirichlet integral (see also the work of Carleson [4]). This class of sets coincides with planar 2-removable sets. Consequently, a lot of work was done on removable sets for quasiconformal maps that are globally homeomorphisms, see for instance [3] [7] [9] [10] [11] [13] [25] [26]. Let us point out that Sobolev-removability has also been considered for globally continuous functions, see for example [20] and references therein. The version of Sobolev-removability we consider here can be characterized via condenser capacities or extremal distances [11] [2] [8] [22] [24] [26] [28]. However, these conditions are not easy to check. Because of this, Koskela [17] and Wu [27] considered Sobolev-removability in terms of different kinds of porosities that are easier to verify. Removability of porous sets for weighted Sobolev spaces [6] and (weighted) Orlicz-Sobolev spaces [12] [13] has also been studied. Generalizations of the removability results in
Let $\text{Theorem 1.2}$. Theorem 1.2.

$x \leq \sum_{i} C$ considered the case $\{i\}$. Koskela proved in [17, Theorem 2.3] that thick (Theorem 1.2) or thin (Theorem 1.3) enough. However, we emphasize that in our results the positive measure set $C$ is the set $C \times F$ removable for $W^{1,p}$?

Examples of $p$-removable and non-removable product sets of the type considered in Problem 1.1 have appeared in [17, 27 Example 2], and [18, Lemma 4.4]. In [17] and [27] different porosity parameters of sets determined the $p$-removability. In our results the porosity type conditions have only a secondary role and the main parameter is the Hausdorff dimension of the set $C$. Some ideas of the proofs we present here were present in [18, Lemma 4.4], where also the $p$-removability results of Koskela [17]. He considered the case $C = \{0\}$ and observed in [17, Theorem 2.2] that $\{0\} \times F$ is not $p$-removable for $1 \leq p \leq 2$, when $\mathcal{H}^1(F) > 0$ and $F = [0,1] \setminus \bigcup_{i=1}^{\infty} I_i$ with $I_i$ pairwise disjoint open intervals with $\sum_{i=1}^{\infty} |I_i|^{2-p}$. The generalization of this result is done in Theorem 1.2 below. In the other direction, Koskela proved in [17, Theorem 2.3] that $\{0\} \times F$ is $p$-removable for $1 < p < 2$, if for almost every $x \in F$ there exist a sequence of numbers $r_i \searrow 0$ and a constant $c$ so that $B(x,r_i) \setminus F$ contains an interval of length $c_i^{1/(2-p)}$. We generalize this result in Theorem 1.3

**Theorem 1.2.** Let $2 \leq p < \infty$ and $s > \frac{p-2}{p-1}$. Then for any closed subset $C \subset \mathbb{R}$ with $\mathcal{H}^s(C) > 0$ and any set $F$ of the form $F = [0,1] \setminus \bigcup_{j=1}^{\infty} I_j$, where $I_j$ are open intervals satisfying

$$\sum_{j=1}^{\infty} |I_j|^{-(1-s)(p-1)} < \infty,$$

and $\mathcal{H}^1(F) > 0$, the set $C \times F$ is not $p$-removable.

We do not know what are the sets $C$ in Theorem 1.2 for which $C \times F$ is not $p$-removable for every closed set $F \subset \mathbb{R}$ of positive Lebesgue measure. On one hand, if $C$ is a singleton, the $p$-removability depends on $F$ by the results of Koskela [17], as we already discussed above.

the spirit of Ahlfors and Beurling have been done for weighted Sobolev spaces, see for example [3].

The sets $E$ whose $p$-removability we consider here are of the form $E = C \times F$ where $C,F \subset \mathbb{R}$ are closed. If $C$ or $F$ contains an interval of positive length, it is easy to see that the set $E$ is not $W^{1,p}$-removable for any $1 \leq p \leq \infty$. Therefore, we may assume that both $C$ and $F$ are totally disconnected. Now, on one hand, if both $C$ and $F$ have zero Lebesgue measure, the set $E$ is automatically $W^{1,p}$-removable since almost every line segment parallel to a coordinate axis has empty intersection with $E$. On the other hand, if $C$ and $F$ both have positive Lebesgue measure, the set $E$ has positive Lebesgue measure, hence cannot be removable. We have reduced our study to the following.

**Problem 1.1.** Let $1 \leq p \leq \infty$ and $C,F \subset \mathbb{R}$ be totally disconnected closed subsets with $C$ having zero Lebesgue measure and $F$ positive Lebesgue measure. Under what conditions on $C$ and $F$ is the set $C \times F$ removable for $W^{1,p}$?
On the other hand, in Section 4 we show that if $C$ is Ahlfors $s$-regular with $0 < s < 1$, then the $p$-removability is independent of $F$.

**Theorem 1.3.** Let $2 \leq p < \infty$ and $s < \frac{p-2}{p-1}$. Suppose that $C \subset \mathbb{R}$ is a closed set with $\mathcal{H}^s(C) < \infty$ and that $F \subset \mathbb{R}$ is a closed set for which at $\mathcal{H}^1$-almost every point $y \in F$ there exists $r_y > 0$ and $c_y > 0$ so that for any $0 < r < r_y$ we have

$$\mathcal{H}^1(B(y,r) \setminus F) \geq c_y r^{1-(1-s)(p-1)}. \quad (2)$$

Then the set $C \times F$ is $p$-removable.

Notice that $(1-s)(p-1) > 1$ in Theorem 1.3 with the choices of $p$ and $s$. Thus, there exist closed sets $F \subset \mathbb{R}$ of positive Lebesgue measure that satisfy (2) at every point $y \in F$. One might wonder why in Theorem 1.3 we require (2) for all small scales $r$ instead of a sequence of scales as in [17, Theorem A]. One reason for our stricter requirement is that in our proof we argue using a sequence of dyadic scales. Even if this could be avoided, the fact that we assume the Hausdorff measure of $C$ to be finite would force us to work on many scales at once. Replacing the Hausdorff measure assumption by box counting dimension assumption might yield the analogous result with a weaker requirement on $F$.

The proof of Theorem 1.2 is inspired by the proof of [18, Lemma 4.4] by the second named author together with Koskela and Zhang. In [18, Lemma 4.4], the non-removability was proven for a more regular set $C$, while the removability was done via a curve condition to which we return in Section 4. The proof of Theorem 1.2 is done in Section 2 while Theorem 1.3 is proven in Section 3. In the final Section 4 we study the relations between $p$-removability, curve conditions, and Ahlfors regularity and lower porosity of $C$. In particular, we show that for Ahlfors-regular $C$ the $p$-removability of $C \times F$ is independent of $F$. The non-removability of $C \times F$ for Ahlfors-regular $C$ might still depend on $F$.

2. **Proof of Theorem 1.2**

In this section we prove Theorem 1.2. The proof is similar to the proof of [18, Lemma 4.4], where a standard Cantor staircase function was extended by hand from horizontal lines passing through $F$ to the whole set $\mathbb{R}^2 \setminus (C \times F)$. This was possible because of the regularity of the Cantor set $C$ that was used. In the proof of Theorem 1.2 we give a more general construction of a suitable Cantor staircase function via Frostman’s Lemma, and an extension of the staircase function via averages.

Up to taking a subset of $C$, we can assume that $0 < \mathcal{H}^s(C) < \infty$ and that $C$ is compact (say, $C \subset [0,1]$). For $R > 1$, we will construct a function $u \in W^{1,p}(B(0,R)^2 \setminus (C \times F))$ which is not absolutely continuous on any segment $\{y\} \times (-R,R)$ for $y \in F$. It follows that $u$ cannot be in $W^{1,p}(B(0,R)^2)$.

By Frostman’s Lemma (see for instance [19, Theorem 8.8]), there exists a Borel probability measure $\mu$ concentrated on $C$ satisfying

$$\mu(B(x,r)) \leq c_F r^s \quad (3)$$

for some constant $c_F > 0$.

We define on $[0,1]$ the non-decreasing function $f(x) = \mu([0,x])$ and we extend it to $B(0,R)$ by letting $f = 0$ on $(-R,0)$ and $f = 1$ on $(1,R)$. Observe that $f$ is not absolutely continuous, since it is constant $\mathcal{H}^1$-a.e. on $[0,1]$, but $f(1) - f(0) = 1$. 


We extend the function $f$ to $\mathbb{R} \times [0, +\infty)$ by letting
\[
v(x, y) = \frac{1}{2y} \int_{x-y}^{x+y} f(t) \, dt.
\] (4)

**Lemma 2.1.** The extension $v$ defined in (4) is differentiable on $B(0, R) \times (0, +\infty)$, and
\[
\nabla v(x, y) = \frac{1}{2y} (f(x + y) - f(x - y), f(x + y) + f(x - y) - 2v(x, y)).
\]

In particular,
\[
|\nabla v(x, y)| \leq \frac{f(x + y) - f(x - y)}{\sqrt{2y}} = \frac{\mu(B(x, y))}{\sqrt{2y}}.
\]

**Proof.** By the Leibniz integral rule we have
\[
\frac{dv}{dx}(x, y) = \frac{1}{2y} \frac{d}{dx} \int_{x-y}^{x+y} f(t) \, dt = \frac{f(x + y) - f(x - y)}{2y}
\]
and
\[
\frac{dv}{dy}(x, y) = -\frac{1}{2y^2} \int_{x-y}^{x+y} f(t) \, dt + \frac{1}{2y} \frac{d}{dy} \int_{x-y}^{x+y} f(t) \, dt
\]
\[
= \frac{1}{2y} (-2v(x, y) + f(x + y) + f(x - y)).
\]

The final estimate comes from the fact that $f$ is non-decreasing, which implies
\[
v(x, y) = \frac{1}{2y} \int_{x-y}^{x+y} f(t) \, dt \geq f(x - y).
\]

\[\square\]

It will be useful to estimate the integral of $|\nabla v|$ on a rectangle $(-R, R) \times (0, r)$. For $0 < y < r$, let $\{B(x, r_i)\}_{i}$ be a finite cover of $C$ with disjoint balls of radii $r_i < y$. Then we have
\[
\int_{-R}^{R} \mu(B(x, y)) \, dx = \int_{-R}^{R} \sum_{i} \mu(B(x, y) \cap B(x, r_i)) \, dx \leq \sum_{i} \int_{x_i-2y}^{x_i+2y} \mu(B(x, y) \cap B(x, r_i)) \, dx
\]
\[
\leq 4y \sum_{i} \mu(B(x, r_i)) = 4y,
\]
where we used that $\mu$ is a probability measure on $C$. Combining this with **Lemma 2.1** and (4) yields
\[
\int_{0}^{r} \int_{-R}^{R} |\nabla v|^p \, dx \, dy \leq 2^{-\frac{p}{2}} \int_{0}^{r} \int_{-R}^{R} \frac{\mu(B(x, y))}{y^p} \, dx \, dy
\]
\[
\leq 2^{-\frac{p}{2}} \sum_{i} y^{s(p-1)} \int_{-R}^{R} \mu(B(x, y)) \, dx \, dy
\]
\[
\leq 2^{2-p} 2^{-\frac{p}{2}} \sum_{i} y^{s(1-p)} \, dy.
\]

We now define the function $u$ as $u(x, y) = v(x, \text{dist}(y, F))$. Observe that $u(x, y) = f(x)$ for every $y \in F$, so $u$ is not absolutely continuous on every segment $\{y\} \times (-R, R), \forall y \in F$. 


Since by hypothesis \((s - 1)(p - 1) > -1\), making use of (5), for each interval \(I_j\) in the complement of \(F\) we have
\[
\int_{I_j} \int_{-R}^R |\nabla u|^p \, dx \, dy = 2 \int_0^{|I_j|/2} \int_{-R}^R |\nabla u|^p \, dx \, dy \leq c(p, s) c_F |I_j|^{1-(1-s)(p-1)},
\]
where \(c(p, s) = 2^{1+s-2s(p-1)}\). By summing over \(j\) and using (1) we obtain that \(u \in W^{1,p}(B(0, R)^2 \setminus (C \times F))\), as wanted.

3. Proof of Theorem 1.3

Let \(u \in W^{1,p}(\mathbb{R}^2 \setminus E)\). We aim at showing that \(u \in W^{1,p}(\mathbb{R}^2)\), which holds exactly when \(u\) has an \(L^p\)-representative that is ACL in \(\mathbb{R}^2\). Without changing the notation, let \(u\) be the ACL (in \(\mathbb{R}^2 \setminus E\)) representative of \(u\). Since \(H^1(C) = 0\), \(u\) is absolutely continuous on almost every vertical line-segment in \(\mathbb{R}^2\). Hence, we only need to verify that \(u\) is absolutely continuous on almost every horizontal line-segment.

Let us write \(\alpha = (1-s)(p-1)\). Let \(y \in F\) be such that there exist \(c_y > 0\) and \(r_y > 0\) so that for any \(0 < r < r_y\) we have
\[
H^1(B(y, r) \setminus F) \geq c_y r^\alpha. \tag{6}
\]
By assumption, such constants exist for almost every \(y\). Let us abbreviate \(f(x) = u(x, y)\). It remains to show that \(f\) is absolutely continuous.

Let \(0 < \delta < r_y\). Recalling that \(H^s(C) < \infty\), we take a collection of open intervals \(\{J_i\}_{i=1}^n\) such that \(C \subset \bigcup_{i=1}^n J_i\), \(|J_i| < \delta\) for all \(i\) and
\[
\sum_{i=1}^n |J_i|^s \leq 2\mathcal{H}^s(C). \tag{7}
\]
Without loss of generality, we may assume that no point in \(\mathbb{R}\) is contained in more than two different intervals \(J_i\). Define for every \(i\) the open square
\[
Q_i = J_i \times B \left( y, \frac{1}{2} |J_i| \right).
\]

**Lemma 3.1.** For every \(i\) we have the inequality
\[
\left| \inf_{x \in J_i} f(x) - \sup_{x \in J_i} f(x) \right| \leq c(s, p, y) |J_i|^\frac{\alpha}{p} \|\nabla u\|_{L_p(Q_i)}, \tag{8}
\]
where \(\frac{1}{p} + \frac{1}{q} = 1\) and \(c(s, p, y) > 0\) is a constant depending only on \(s, p, \) and \(y\).

Assuming for the moment **Lemma 3.1** we conclude the proof as follows. By Hölder’s inequality, (7), and (8) we obtain
\[
\sum_{i=1}^n \left| \inf_{x \in J_i} f(x) - \sup_{x \in J_i} f(x) \right| \leq \left( \sum_{i=1}^n |J_i|^s \right)^\frac{1}{s} \left( \sum_{i=1}^n |J_i|^\frac{s}{p} \left| \inf_{x \in J_i} f(x) - \sup_{x \in J_i} f(x) \right|^p \right)^\frac{1}{p} \\
\leq (2\mathcal{H}^s(C))^{\frac{1}{p}} \left( \sum_{i=1}^n c(s, p, y) \|\nabla u\|_{L_p(Q_i)}^p \right)^\frac{1}{p} \\
\leq c(s, p, y) (2\mathcal{H}^s(C))^{\frac{1}{p}} \|\nabla u\|_{L_p(\mathbb{R} \times [y-\delta, y+\delta])} \to 0
\]
as $\delta \to 0$. Since $f$ is absolutely continuous outside $C$, the above shows that $f$ is absolutely continuous on the whole $\mathbb{R}$.

It remains to prove Lemma 3.1.

Proof of Lemma 3.1. Fix $i \in \{1, \ldots, n\}$ and let $I \subset J \subset J_i$ be intervals such that $|J| = 2|I|$. By (3), the open set $K = B(y, |J|) \setminus F$ satisfies $\mathcal{H}^1(K) \geq c_y |J|^\alpha$. Define a collection $\gamma_i : [0, 1] \to Q_i$, $t \in [0, 1]$ of curves so that each $\gamma_i$ is the concatenation of three line-segments $\gamma_1^i$, $\gamma_2^i$, and $\gamma_3^i$ that are defined as follows. Write $J = [a, b]$, $I = [c, d]$ and set $x_1(t) = a + t(b - a)$ and $x_2(t) = c + t(d - c)$. Define

$$y(t) = \inf \{ \tilde{y} \in [y - |J|, y + |J|] : \mathcal{H}^1(K \cap (-\infty, \tilde{y})] \geq t\mathcal{H}^1(K) \}.$$ 

Now, $\gamma_1^i$ is taken to be the line-segment from $(x_1(t), y)$ to $(y, y(t))$, $\gamma_2^i$ the line-segment from $(x_1(t), y(t))$ to $(x_2(t), y(t))$, and $\gamma_3^i$ the line-segment from $(x_2(t), y(t))$ to $(x_2(t), y)$. Notice that the image of $\gamma_i$ does not intersect $E$ for $\mathcal{H}^1$-almost every $t \in [0, 1]$.

By integrating over the curves $\gamma_i$ we obtain

$$\left| \frac{1}{|I|} \int_J f(x) \, dx - \frac{1}{|J|} \int_J f(x) \, dx \right| = \left| \int_0^1 u(\gamma_i(1)) - u(\gamma_i(0)) \, dt \right|$$

$$\leq \int_0^1 |u(\gamma_i(1)) - u(\gamma_i(0))| \, dt$$

$$\leq \int_0^1 \int_{\gamma_i} |\nabla u(z)| \, ds(z) \, dt$$

$$= \sum_{k=1}^3 \int_0^1 \int_{\gamma_i^k} |\nabla u(z)| \, ds(z) \, dt$$

First we treat the integrals along the vertical lines $\gamma_1^i$, $\gamma_3^i$. By Hölder’s inequality we have

$$\int_0^1 \int_{\gamma_1^i} |\nabla u(z)| \, ds(z) \, dt \leq \int_0^1 \int_{y-\frac{1}{2}|J_i|}^{y+\frac{1}{2}|J_i|} |\nabla u(x_1(t), z)| \, dz \, dt$$

$$= \int_{y-\frac{1}{2}|J_i|}^{y+\frac{1}{2}|J_i|} \int_{\gamma_1^i} |\nabla u(x, z)| \, dz \, dx$$

$$\leq (|I| \cdot |J|)^{\frac{1}{p}} \|\nabla u\|_{L^P(Q_i)}$$

$$= c(p)\delta^{\frac{2-q}{q}} |J|^{\frac{1}{q}} \|\nabla u\|_{L^P(Q_i)}.$$ 

A similar computation shows that $\int_0^1 \int_{\gamma_3^i} |\nabla u(z)| \, ds(z) \, dt \leq c(p)\delta^{\frac{2-q}{q}} |J|^{\frac{1}{q}} \|\nabla u\|_{L^P(Q_i)}$.

To evaluate the integrals along $\gamma_2^i$, observe that the map $t \mapsto y(t)$ is piecewise affine (on a countable union of open intervals) with $y'(t) = \frac{1}{\mathcal{H}^1(K)}$ a.e. on $(0, 1)$. Thus we have

$$\int_0^1 \int_{\gamma_2^i} |\nabla u(z)| \, ds(z) \, dt \leq \int_{J \times K} \mathcal{H}^1(K)^{-1} |\nabla u(w, z)| \, dw \, dz$$

$$\leq |J|^{\frac{1}{q}} \mathcal{H}^1(K)^{-\frac{1}{q}} \|\nabla u\|_{L^P(J \times K)}$$

$$\leq 2^\frac{1}{p} c_y^{-\frac{1}{p}} |J|^{\frac{1}{q}} \|\nabla u\|_{L^P(Q_i)} = 2^\frac{1}{p} c_y^{-\frac{1}{p}} |J|^{\frac{1}{q}} \|\nabla u\|_{L^P(Q_i)},$$

where we used $\mathcal{H}^1(K) \geq 2^{-\alpha} c_y |J|^\alpha$ and the definition of $\alpha$. 


By putting all together we get
\[ \left| \frac{1}{|I|} \int_I f(x) \, dx - \frac{1}{|J|} \int_J f(x) \, dx \right| \leq c(s, p, y) |J|^{\frac{q}{q'}} \| \nabla u \|_{L^p(Q_i)}. \] (9)

Now, let \( z_1, z_2 \in J_i \) be such that
\[ \left| \inf_{x \in J_i} f(x) - \sup_{x \in J_i} f(x) \right| \leq 2 |f(z_1) - f(z_2)|. \]

Let \( \{I_k\}_{k=1}^\infty \) be subintervals of \( J_i \) so that \( I_1 = J_i \), \( z_1 \in I_k \) for every \( k \in \mathbb{N} \), and \( |I_k| = 2 |I_{k+1}| \) for every \( k \in \mathbb{N} \). Now, by (9), we have
\[ \left| f(z_1) - \frac{1}{|J|} \int_J f(x) \, dx \right| \leq \sum_{k=1}^\infty \frac{1}{|I_k|} \left| \int_{I_k} f(x) \, dx - \frac{1}{|I_{k+1}|} \int_{I_{k+1}} f(x) \, dx \right| \]
\[ \leq \sum_{k=1}^\infty c(s, p, y) |I_k|^{\frac{q}{q'}} \| \nabla u \|_{L^p(Q_i)} \]
\[ \leq c(s, p, y) |J|^{\frac{q}{q'}} \| \nabla u \|_{L^p(Q_i)}. \]

Together with an analogous estimate for \( z_2 \), we obtain
\[ \frac{1}{2} \left| \inf_{x \in J_i} f(x) - \sup_{x \in J_i} f(x) \right| \leq |f(z_1) - f(z_2)| \]
\[ \leq \left| f(z_1) - \frac{1}{|J|} \int_J f(x) \, dx \right| + \left| f(z_2) - \frac{1}{|J|} \int_J f(x) \, dx \right| \]
\[ \leq 2c(s, p, y) |J|^{\frac{q}{q'}} \| \nabla u \|_{L^p(Q_i)}. \]
\[ \square \]

4. Curve-condition, porosity and Ahlfors-regular sets

In this section we study the case where the set \( E = C \times F \) consists of a set \( F \) of positive measure and a zero measure set \( C \) with more regularity. The most regular case is when \( C \) is (Ahlfors) \( s \)-regular, that is, if there exists a constant \( c_R > 0 \) so that
\[ \frac{1}{c_R} r^s \leq \mathcal{H}^s(B(x, r) \cap C) \leq c_R r^s \]
for every \( x \in C \) and \( 0 < r < \text{diam}(C) \). The set \( C \) in [18, Lemma 4.4] was not exactly \( s \)-regular, but almost. A small perturbation to \( s \)-regularity was required there to have the nonremovability at the critical exponent.

In [18, Lemma 4.4], the \( p \)-removability of \( C \times F \) was proven via the following sufficient condition from [16, 23]. Suppose \( E \subset \mathbb{R}^2 \) is closed set of measure zero and \( 2 \leq p < \infty \). If there exists a constant \( c_T > 0 \) such that for every \( z_1, z_2 \in \mathbb{R}^2 \setminus E \) there exists a curve \( \gamma \subset \mathbb{R}^2 \setminus E \) connecting \( z_1 \) to \( z_2 \) and satisfying
\[ \int_{\gamma} \text{dist}(z, E)^{\frac{1}{1-p}} \, ds(z) \leq c_T |z_1 - z_2|^\frac{2}{p-1}, \] (10)
then \( E \) is \( p \)-removable. If the above holds, we say that \( E \subset \mathbb{R}^2 \) satisfies the curve condition [10].

By adapting the proof in [18], we get a \( p \)-removability result that is independent of the structure of \( F \).
Theorem 4.1. Let $C \subset \mathbb{R}$ be a closed $s$-regular set with $0 < s < 1$, and $F \subset \mathbb{R}$ totally disconnected closed set. Then $C \times F$ is $p$-removable for every $p > \frac{2s}{1-s}$.

A slightly more general result for $p$-removability via the curve condition (10) than the one stated in Theorem 4.1 is in terms of porosity. Recall that a set $C \subset \mathbb{R}$ is called uniformly lower $\alpha$-porous, if for every $x \in C$ and $r > 0$ there exists $y \in B(x,r)$ so that $B(y,\alpha r) \cap C = \emptyset$.

Theorem 4.2. Let $C \subset \mathbb{R}$ be a closed uniformly lower $\alpha$-porous set and $F \subset \mathbb{R}$ totally disconnected closed set. Then $C \times F$ is $p$-removable for every $p > \hat{p}$, where $\hat{p} > 2$ depends only on the parameter $\alpha$.

Proof of Theorems 4.1 and 4.2. Both of the theorems are proven by verifying the condition (10). Towards verifying this condition, let $z_1, z_2 \in \mathbb{R}^2 \setminus E$. Write these points in coordinates as $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$. Let us abbreviate $r = |z_1 - z_2|$. Since $F$ is totally disconnected and $E$ is closed, we may assume that $y_1, y_2 \notin F$.

Notice that an $s$-regular set is uniformly lower porous. Thus, in both cases by porosity of $C$ there exists a point $x \in B(x_1, r)$ so that $B(x,\alpha r) \cap C = \emptyset$. We now connect $z_1$ to $z_2$ by concatenating three line-segments $\gamma_1$, $\gamma_2$, and $\gamma_3$. The curve $\gamma_1$ connects $(x_1, y_1)$ to $(x, y_1)$, $\gamma_2$ connects $(x, y_1)$ to $(x, y_2)$, and $\gamma_3$ connects $(x, y_2)$ to $(x_2, y_2)$. The choice of $x$ now gives

$$\int_{\gamma_2} \text{dist}(z, E) \frac{1}{1-p} \, ds(z) \leq \int_{\gamma_2} (\alpha r)^{1-p} \, ds(z) = (\alpha r)^{1-p} |y_1 - y_2| \leq \alpha^{1-p} r^{p-2}$$

for the vertical part $\gamma_2$.

For the horizontal parts $\gamma_1$ and $\gamma_3$ we first show that the following condition holds for $s$-regular sets $C$ and for uniformly lower $\alpha$-porous sets $C$ with some $0 < s < 1$: there exists a constant $c_s < \infty$ such that for all $0 < \delta \leq 1$ and every $-\infty < a < b < \infty$, the set $(a, b) \setminus C$ contains at most $c_s \delta^{-s}$ connected components of length more than $\delta |b - a|$.

Let us first show this for an $s$-regular set $C$. Suppose that $\{I_i\}_{i=1}^n$ are the connected components of $(a, b) \setminus C$ of length more than $\delta |b - a|$. For each $i$ let $v_i$ be the left-most point of $I_i$. The sets $(B(v_i, \delta |b - a|) \cap C) \subset [a - |b - a|, b + |b - a|]$ are pairwise disjoint. Thus, by $s$-regularity (notice that the left-most $v_i$ might not be in $C$)

$$\frac{n-1}{c_R} (\delta |b - a|)^{s} \leq \mathcal{H}^s([a - |b - a|, b + |b - a|] \cap C) < c_R (2 |b - a|)^s,$$

which gives the claim for $s$-regular sets $C$.

Let us now suppose that $C$ is uniformly lower $\alpha$-porous, fix $\delta$ and denote by $\{I_i\}_{i=1}^n$ the intervals of $(a, b) \setminus C$ of length at least $\delta |b - a|$, and by $\{I_i\}_{i=1}^\infty$ the remaining intervals of $(a, b) \setminus C$. Consider the set $C' = C + B(0, \delta |b - a|)$. By a result of A. Salli [21, Theorem 3.5], we have

$$\mathcal{H}^1(C') \leq c(\alpha) |b - a| \delta^{1-s},$$

where $s = \frac{\log 2}{\log (\frac{2}{\alpha})} \in (0,1)$ and $c(\alpha)$ is a positive constant depending on $\alpha$. Observe that $\bigcup I_i \subset C'$ and, for every interval $I_i$, $|I_i \setminus C'| \leq |I_i| - \frac{\delta}{2} |b - a|$. Thus, using (11), we have

$$|b - a| = \sum_{i=1}^n |I_i| + \sum_{i=1}^\infty |I_i| = \sum_{i=1}^n |I_i \setminus C'| + \mathcal{H}^1(C') \leq |b - a| - \frac{\delta}{2} |b - a| + c(\alpha) |b - a| \delta^{1-s},$$

yielding $n \leq 2c(\alpha)\delta^{-s}$.
Let us then estimate the integral along $\gamma_1$. Without loss of generality we may assume that $x_1 < x$. Denote by $\{J_i\}_i$ the collection of open intervals constituting the connected components of $(x_1, x) \backslash C$. Let $k_0 \in \mathbb{Z}$ be so that $2^{-k_0} < |x - x_1| \leq 2^{-k_0 + 1}$. Then

$$\int_{\gamma_1} \operatorname{dist}(z, E)^{\frac{1}{1-p}} \, ds(z) \leq \sum_i 2 \int_0^{|J_i|} t^{\frac{1}{1-p}} \, dt = 2^{\frac{p-1}{p-2}} \sum_i |J_i|^{\frac{p-2}{p-1}}$$

$$\leq c(p) \sum_{k=k_0}^{\infty} \# \left\{ i : 2^{-k-1} < |J_i| \leq 2^{-k} \right\} 2^{-k \frac{p-2}{p-1}}$$

$$\leq c(p) \sum_{k=k_0}^{\infty} c_d 2^{(k-k_0)s} 2^{-k \frac{p-2}{p-1}}$$

$$\leq c(p) \sum_{k=k_0}^{\infty} c_d 2^{(k-k_0)} (s^{\frac{p-2}{p-1}}) |x - x_1|^{\frac{p-2}{p-1}}$$

$$\leq c(p, s) |x - x_1|^{\frac{p-2}{p-1}} \leq c(p, s) |z_1 - z_2|^{\frac{p-2}{p-1}}$$

as long as $s < \frac{p-2}{p-1}$.

The integral along $\gamma_3$ is handled analogously. \qed

We end this section by showing that the $p$-removability results that are proven via the curve condition (10) give removability only for porous sets.

**Proposition 4.3.** Suppose that $E = C \times F \subset \mathbb{R}^2$ is a compact set satisfying the curve condition (10) and that $F \subset \mathbb{R}$ is a totally disconnected set with positive Lebesgue measure. Then $C$ is uniformly lower $\alpha$-porous for some $\alpha > 0$.

**Proof.** Let $c_T > 0$ be the constant in (10). Let $y \in F$ be a Lebesgue density-point of $F$ and $\varepsilon := \sqrt{2c_T^{1-p}}$. Then there exists $r_0 > 0$ such that for all $0 < r < r_0$ we have

$$\mathcal{H}^1(B(y, r) \setminus F) < \varepsilon r. \quad (12)$$

Let $x \in C$ and $0 < r < r_0$. Define $z_1 = (x - r/2, y)$ and $z_2 = (x + r/2, y)$, and select points $z_1 \in B(z_1, r/4) \setminus E$ and $z_2 \in B(z_2, r/4) \setminus E$. Let $\gamma \subset \mathbb{R}^2 \setminus E$ be a curve connecting $z_1$ to $z_2$ and satisfying (10). Define $A := \overline{B(x, 7r/8)} \times \overline{B(y, r/2)}$ and $d := \max \{ \operatorname{dist}(z, E) : z \in \gamma \cap A \}$. Now, by (10)

$$\frac{1}{1-p} \int_{\gamma \cap A} \operatorname{dist}(z, E)^{\frac{1}{1-p}} \, ds(z) \leq c_T |z_1 - z_2|^{\frac{p-2}{p-1}} \leq c_T (2r)^{\frac{p-2}{p-1}}.$$

Thus,

$$d \geq 2c_T^{1-p} r = \sqrt{2} \varepsilon r,$$

which together with (12) gives the $\varepsilon$-porosity of $C$ at $x$ at the scale $r$. From the compactness of $C$ it then follows that $C$ is uniformly lower $\alpha$-porous for some $\alpha > 0$. \qed

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