Twisted Classical Phase Space

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Abstract

We consider relativistic phase space constructed by the twist procedure from the translation sector of the standard, nondeformed Poincaré algebra. Using the concept of cross product algebra we derive two kinds of phase space with noncommuting configuration space. The generalized uncertainty relations are formulated.

1. In [1] the twisted classical Poincaré algebras were described using two-tensor procedure. Recently, some efforts are made to describe deformed, noncommutative phase spaces with noncommuting configuration space, via the duality between quantum algebras and quantum groups [2]. This construction of phase space leans on the concept of a cross product algebra (Heisenberg double).

It appears that we can start from the standard, nondeformed (classical) Poincaré algebra, making a twist and apply duality to twisted Poincaré algebra in order to obtain relativistic phase space with noncommuting positions.

This point of view is presented in this paper and we show that in the simplest case we obtain the configuration noncommuting space with the structure of two dimensional inhomogenous Euclidean or hiperbolic algebras.

At the beginning we summarize the basic notions for our considerations.

- **Twisted coproduct** Let \( \mathcal{A} = (A, \Delta, S, \varepsilon) \) be the Hopf algebra with comultiplication \( \Delta \), antipode \( S \) and counit \( \varepsilon \).

if there exists an invertible function \( F = \sum_i f_i \otimes f_i \in A \otimes A \) such that

\[
\Delta^F(a) = F \cdot \Delta(a) \cdot F^{-1}
\]  

(1.1)

determines new comultiplication, or equivalently:

\[
(\Delta^F \otimes 1)\Delta^F = (1 \otimes \Delta^F)\Delta^F \quad \text{coassociativity}
\]  

(1.2)

Comultiplication \( \Delta^F \) is called the **twisted coproduct**.

- Two Hopf algebras \( \mathcal{A} = (A, \Delta, S, \varepsilon) \) and \( \mathcal{A}^F = (A, \Delta^F, S^F, \varepsilon) \) are related by **two-tensor twisting** if there exists an invertible function

\[
F = \sum_i f_i \otimes f_i \in A \otimes A
\]  

(1.3a)

satisfying

\[
F_{23}(1 \otimes \Delta)F = F_{12}(\Delta \otimes 1)F \quad (\varepsilon \otimes 1)F = (1 \otimes \varepsilon)F = 1
\]  

(1.3b)
where 
\[ F_{12} = F \otimes 1 \quad F_{23} = 1 \otimes F \quad F_{13} = \sum_i f_i \otimes 1 \otimes f^i \] (1.3c)

- For the complex simple Lie algebras \( \hat{g} \) the twist function \( F \) is given by

\[ F = \exp f \quad f \in \hat{c} \otimes \hat{c} \] (1.4)

where \( \hat{c} \) is the commutative subalgebra of \( \hat{g} \) (Cartan or Borel subalgebra).

Let \( \mathcal{P} \) be an algebra, \( \mathcal{X} \) a vector space.
- A **left action** (representation) of \( \mathcal{P} \) on \( \mathcal{X} \) is a linear map

\[ \triangleright : \mathcal{P} \otimes \mathcal{X} \to \mathcal{X} : p \otimes x \to p \triangleright x \] (1.5a)

such that

\[ (p\tilde{p}) \triangleright x = p \triangleright (\tilde{p} \triangleright x) \quad 1 \triangleright x = x \] (1.5b)

- We say that \( \mathcal{X} \) is a **left \( \mathcal{P} \)-module**.

In the case where \( \mathcal{X} \) is an algebra and \( \mathcal{P} \) a bialgebra (or Hopf algebra)
- We say that \( \mathcal{X} \) is a **left \( \mathcal{P} \)-module algebra** if

\[ p \triangleright (x\tilde{x}) = (p(1) \triangleright x)(p(2) \triangleright \tilde{x}) \quad p \triangleright 1 = \epsilon(p)1 \] (1.6)

where \( \epsilon \) denotes counit and we use the Sweedler’s notation

\[ \Delta(p) = \sum p(1) \otimes p(2) \]

Let \( \mathcal{P} \) be a Hopf algebra and \( \mathcal{X} \) a left \( \mathcal{P} \)-module algebra.
- **Left cross product algebra** (smash product) \( \mathcal{X} \ltimes \mathcal{P} \) is a vector space \( \mathcal{X} \otimes \mathcal{P} \) with product \( \otimes \)

\[ (x \otimes p)(\tilde{x} \otimes \tilde{p}) = (x(p(1) \triangleright \tilde{x}) \otimes p(2) \tilde{p}) \quad \text{(left cross product)} \] (1.7)

with unit element \( 1 \otimes 1 \), where \( x, \tilde{x} \in \mathcal{X} \) and \( p, \tilde{p} \in \mathcal{P} \).

- **commutation relations in a cross product algebra**. The obvious isomorphism \( \mathcal{X} \sim \mathcal{X} \otimes 1, \mathcal{P} \sim 1 \otimes \mathcal{P} \) gives us the following cross relations between the algebras \( \mathcal{X} \) and \( \mathcal{P} \)

\[ [x, p] = x \circ p - p \circ x \quad \text{where} \quad x \circ p = x \otimes p \quad p \circ x = (p(1) \triangleright x) \otimes p(2) \] (1.8)

**2.** We consider following two choices of the commuting subalgebra \( \hat{c} \) giving noncommuting configuration space [3].

i) Let \( \hat{c} = (M_3 = M_{12}, P_3, P_0) \), then the twist function is given by \( (r, s = 3, 0) \):

\[ F = e^{M_3 \otimes A_1} e^{P_3 \otimes B_r} = e^{A_2 \otimes M_3} e^{C_r \otimes P_r} \] (2.1)

where:

\[ A_1 = \alpha_+ M_3 + (\delta^+_r + \delta^+_c) P_r \quad B_r = (\delta^+_r - \delta^+_c) M_3 + \rho^+_r P_s \]

\[ A_2 = \alpha_+ M_3 + (\delta^+_r - \delta^+_c) P_r \quad C_r = \rho^+_r P_s + (\delta^+_r + \delta^+_c) M_3 \] (2.2)

Using the formula (1.1) we obtain the twisted coproduct in the form

\[ \Delta^F(M_\pm) = M_\pm \otimes e^{\pm A_1} + e^{\pm A_2} \otimes M_\pm \pm P_\pm \otimes B_3 e^{\pm A_1} \pm e^{\pm A_2} C_3 \otimes P_\pm \]

\[ \Delta^F(M_3) = M_3 \otimes 1 + 1 \otimes M_3 \] (2.3a)
\[ \Delta^F(N_\pm) = N_\pm \otimes e^{\pm A_1} + e^{\pm A_2} \otimes N_\pm - iP_\pm \otimes B_0 e^{\pm A_1} + C_0 e^{\pm A_2} \otimes P_\pm \]
\[ \Delta^F(N_3) = N_3 \otimes 1 + 1 \otimes N_3 - iP_3 \otimes B_0 + C_0 \otimes P_3 + P_0 \otimes B_3 + C_3 \otimes P_0 \quad (2.3b) \]
\[ \Delta^F(P_1) = P_1 \otimes \cosh(A_1) + \cosh(A_2) \otimes P_1 + iP_2 \otimes \sinh(A_1) + i \sinh(A_2) \otimes P_2 \]
\[ \Delta^F(P_2) = P_2 \otimes \cosh(A_1) + \cosh(A_2) \otimes P_2 - iP_1 \otimes \sinh(A_1) - i \sinh(A_2) \otimes P_1 \]
\[ \Delta^F(P_3) = P_3 \otimes 1 + 1 \otimes P_3 \]
\[ \Delta^F(P_0) = P_0 \otimes 1 + 1 \otimes P_0 \quad (2.3c) \]

ii) For the second choice of the commuting subalgebra \( \hat{c} = (P_1, P_2, N_3 = M_{30}) \) the twist function takes the form \((a,b=1,2)\):
\[ F = e^{N_3 \otimes A_1} e^{P_2 \otimes B_2} = e^{A_2 \otimes N_3} e^{C_2 \otimes P_2} \quad (2.4) \]

where:
\[ A_1 = (\xi^a_+ - \xi^a_-)P_a + \gamma_+ N_3 \quad B_a = (\rho^{ab}_+ P_b + (\xi^a_+ + \xi^a_-)N_3 \]
\[ A_2 = (\xi^a_+ - \xi^a_-)P_a + \gamma_+ N_3 \quad C_a = (\rho^{ab}_+ P_b + (\xi^a_+ + \xi^a_-)N_3 \]

and the following formulae for coproduct can be easy obtain:
\[ \Delta^F(M_{\pm}) = M_\pm \otimes \cos(A_1) + \cos(A_2) \otimes M_\pm \pm \{N_\pm \otimes \sin(A_1) + \sin(A_2) \otimes N_\pm \}
\]
\[ \mp \{P_3 \otimes (B_1 \pm iB_2) \cos(A_1) + (C_1 \pm iC_2) \cos(A_2) \otimes P_3 \}
\]
\[ \mp i\{P_0 \otimes (B_1 \pm iB_2) \sin(A_1) + (C_1 \pm iC_2) \sin(A_2) \otimes P_0 \}
\]
\[ \Delta^F(M_3) = M_3 \otimes 1 + 1 \otimes M_3 
\]
\[ -i\{P_2 \otimes B_1 + C_1 \otimes P_2\} + i\{P_1 \otimes B_2 + C_2 \otimes P_1\} \quad (2.6a) \]
\[ \Delta^F(N_{\pm}) = N_{\pm} \otimes \cos(A_1) + \cos(A_2) \otimes N_{\pm} \mp \{M_\pm \otimes \sin(A_1) + \sin(A_2) \otimes M_\pm \}
\]
\[ -i\{P_0 \otimes (B_1 \pm iB_2) \cos(A_1) + (C_1 \pm iC_2) \cos(A_2) \otimes P_0\}
\]
\[ +P_3 \otimes (B_1 \pm iB_2) \sin(A_1) + (C_1 \pm iC_2) \sin(A_2) \otimes P_3 \]
\[ \Delta^F(N_3) = N_3 \otimes 1 + 1 \otimes N_3 \quad (2.6b) \]
\[ \Delta^F(P_{\pm}) = P_{\pm} \otimes 1 + 1 \otimes P_{\pm} \]
\[ \Delta^F(P_3) = P_3 \otimes \cos(A_1) + \cos(A_2) \otimes P_3 + iP_0 \otimes \sin(A_1) + i \sin(A_2) \otimes P_0 \]
\[ \Delta^F(P_0) = P_0 \otimes \cos(A_1) + \cos(A_2) \otimes P_0 + iP_3 \otimes \sin(A_1) + i \sin(A_2) \otimes P_3 \quad (2.6c) \]

where:
\[ B_1 \pm iB_2 = (\rho^{1b}_+ \pm i\rho^{2b}_+) P_b + (\xi^1_+ \pm i\xi^1_-)(\xi^2_+ \pm i\xi^2_-))N_3 \quad (2.7a) \]
\[ C_1 \pm iC_2 = (\rho^{1b}_+ \pm i\rho^{2b}_+) P_b + (\xi^1_+ \pm i\xi^1_-)(\xi^2_+ \pm i\xi^2_-))N_3 \quad (2.7b) \]
in both cases we assume the following hermiticity condition for coproduct
\[ (\Delta^F)^+ = \Delta^F \Rightarrow A_1^+ = -A_1, A_2^+ = -A_2 \quad (2.8) \]
which leads to the hermitian configuration space.

3. Let us consider the first choice of the commuting subalgebra (i). The hermiticity condition \((8)\) give us
\[ A_1 = i(\alpha_+ M_3 + (\delta^+_r + \delta^-_r)P_r \quad A_2 = i(\alpha_+ M_3 + (\delta^+_r - \delta^-_r)P_r \quad \alpha_+, \delta^+_r, \delta^-_r \in \mathbb{R} \quad (3.1) \]
In the simple case \( \alpha_+ \equiv 0, \delta_+^\alpha \equiv 0 \) the coproduct for the momentum takes the form
\[
\Delta F(P_r) = P_r \otimes I + I \otimes P_r
\]
\[
\Delta F(P_1) = P_1 \otimes \cos(\hat{A}) + \cos(\hat{A}) \otimes P_1 + \cos(\hat{A}) \otimes P_1 + \sin(\hat{A}) \otimes P_2
\]
\[
\Delta F(P_2) = P_2 \otimes \cos(\hat{A}) + \cos(\hat{A}) - P_1 \otimes \sin(\hat{A}) + \sin(\hat{A}) \otimes P_1
\]
(3.2)

\[
\hat{A} = \hat{A}_1 = -\hat{A}_2 = \delta^\alpha_+ P_r
\]

We obtain the twisted classical phase space using (3.2), (1.8) and duality relations
\[
<x_\mu, P_\nu> = -i\hbar g_{\mu\nu} \quad g_{\mu\nu} = (-1, 1, 1, 1)
\]

and we get the nonvanishing commutator relations
- noncommuting configuration space
\[
[x_0, x_3] = 0 \quad [x_0, x_1] = 2i\hbar\delta^0_2 x_2 \quad [x_3, x_1] = -2i\hbar\delta^3_2 x_2
\]
\[
[x_0, x_2] = -2i\hbar\delta^0_2 x_1 \quad [x_3, x_2] = 2i\hbar\delta^3_2 x_1
\]
(3.3a)

- position-momentum cross relations
\[
[p_1, x_0] = -i\hbar\delta^0_2 p_2 \quad [p_2, x_0] = i\hbar\delta^0_2 p_1
\]
\[
[p_1, x_3] = i\hbar\delta^3_2 p_2 \quad [p_2, x_3] = -i\hbar\delta^3_2 p_1
\]
\[
[p_1, x_1] = -i\hbar \cos(\hat{A}) \quad [p_2, x_1] = i\hbar \sin(\hat{A})
\]
\[
[p_1, x_2] = -i\hbar \sin(\hat{A}) \quad [p_2, x_2] = -i\hbar \cos(\hat{A})
\]
(3.3b)

and commuting momentum space.

We see that the twisting of nondeformed, standard Poincaré algebra provides the noncommuting configuration space (noncommuting position operators) for suitable choice of commuting subalgebra \( \hat{c} \). The simplest structure of noncommutative configuration space \( \{x_0, x_1, x_2\} \) is realized by two dimensional inhomogenous Euclidean algebra \( \text{iso}(2) \) if we assume \( \delta^2_\alpha = 0, \alpha = \delta^6_\alpha \).

Then we get the commutation relations in the form
\[
[x_0, x_1] = 2i\hbar \alpha x_2 \quad [x_0, p_1] = i\hbar \alpha p_2
\]
\[
[x_0, x_2] = 2i\hbar \alpha x_1 \quad [x_0, p_2] = -i\hbar \alpha p_1
\]
(3.4a)

\[
[p_1, x_1] = -i\hbar \cos(\alpha p_0) \quad [p_1, x_2] = -i\hbar \sin(\alpha p_0)
\]
\[
[p_2, x_1] = i\hbar \sin(\alpha p_0) \quad [p_2, x_2] = -i\hbar \cos(\alpha p_0)
\]
(3.4b)

other commutators vanish.

The phase space obtained in this way has a natural realization by the standard quantum-mechanical momentum and position operators. In fact, let \( \hat{x}_\mu \) and \( \hat{p}_\nu \) satisfy the Heisenberg commutation relations \( [\hat{x}_\mu, \hat{p}_\nu] = i\hbar g_{\mu\nu} \quad g_{\mu\nu} = (-1, 1, 1, 1) \), then we define
\[
x_0 = \hat{x}_0 + \alpha(\hat{x}_1\hat{p}_2 - \hat{p}_1\hat{x}_2) = \hat{x}_0 + \alpha M_3
\]
(3.5a)
\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos(\alpha p_0) & -\sin(\alpha p_0) \\ \sin(\alpha p_0) & \cos(\alpha p_0) \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix}
\]
(3.5b)
Therefore, the noncommuting configuration space one can obtain as \( \hat{p}_0 \) - depending rotation in two dimensional commuting space \( (\hat{x}_1, \hat{x}_2) \).

Introducing the dispersion of observable \( a \) in quantum mechanical sense by

\[
\Delta(a) = \sqrt{< a^2 > - < a >^2} \quad \Delta(a)\Delta(b) \geq \frac{1}{2} < c > | \quad (3.6)
\]

where \( c = [a, b] \), we obtain the generalized Heisenberg uncertainty relations in the form

\[
\begin{align*}
\Delta(x_0)\Delta(x_1) &\geq \hbar|a_0| < x_2 > | \quad \Delta(x_0)\Delta(p_1) \geq \frac{1}{2}\hbar|a_0| < p_2 > | \\
\Delta(x_0)\Delta(x_2) &\geq \hbar|a_0| < x_1 > | \quad \Delta(x_0)\Delta(p_2) \geq \frac{1}{2}\hbar|a_0| < p_1 > | \\
\Delta(p_1)\Delta(x_1) &\geq \frac{1}{2}\hbar|\cos(\alpha p_0) | \quad \Delta(p_1)\Delta(x_2) \geq \frac{1}{2}\hbar|\sin(\alpha p_0) | \\
\Delta(p_2)\Delta(x_1) &\geq \frac{1}{2}\hbar|\sin(\alpha p_0) | \quad \Delta(p_2)\Delta(x_2) \geq \frac{1}{2}\hbar|\cos(\alpha p_0) | \quad (3.7a)
\end{align*}
\]

Let us notice, that the standard Heisenberg relations we obtain in the limit \( \alpha \to 0 \) or for the quantized energy \( \alpha p_0 = n\pi \quad n = 0, \pm 1, \pm 2, ... \).

\textbf{4} The second choice (ii) of the commuting subalgebra \( \hat{c} \) one can consider analogously to the case (i). Using the hermicity condition (2.8) and the relations (2.5) and (2.6c), assuming for simplicity \( \gamma = 0, \xi^0 = 0 \) we obtain \( (a, b = 1, 2) \)

- \textbf{noncommuting configuration space}

\[
\begin{align*}
[x_0, x_a] &= -2i\hbar\xi^a x_3 \\
[x_3, x_a] &= -2i\hbar\xi^a x_0
\end{align*} \quad (4.1a)
\]

- \textbf{position-momentum cross relations}

\[
\begin{align*}
[p_0, x_0] &= i\hbar \cosh(\xi^a p_a) & [p_3, p_0] &= -i\hbar \sinh(\xi^a p_a) \\
[p_0, x_1] &= -i\hbar\xi^1 p_3 & [p_3, x_1] &= -i\hbar\xi^1 p_0 \\
[p_0, x_2] &= -i\hbar\xi^2 p_3 & [p_3, x_2] &= -i\hbar\xi^2 p_0 \\
[p_0, x_3] &= i\hbar \sinh(\xi^a p_a) & [p_3, x_3] &= -i\hbar \cosh(\xi^a p_a)
\end{align*} \quad (4.1b)
\]

and commuting momentum space.

Similar to the Euclidean case (i) the simplest structure of noncommutative configuration space \( \{x_0, x_1, x_2\} \) is realized by two dimensional inhomogenous hiperbolic algebra \( iso(1, 1) \) assuming \( \xi^2 \equiv 0, \beta = \xi^1 \). Now, we obtain the commutation relations in the form

\[
\begin{align*}
[x_0, x_1] &= -2i\hbar\beta x_3 \\
[x_3, x_1] &= -2i\hbar\beta x_0
\end{align*} \quad (4.2a)
\]

\[
\begin{align*}
[p_0, x_1] &= -i\hbar\beta p_3 & [p_3, x_1] &= -i\hbar\beta p_0 \\
[p_0, x_3] &= i\hbar \sinh(\beta p_1) & [p_3, x_0] &= -i\hbar \sinh(\beta p_1) \\
[p_0, x_0] &= i\hbar \cosh(\beta p_1) & [p_3, x_3] &= -i\hbar \cosh(\beta p_1)
\end{align*} \quad (4.2b)
\]

This kind of phase space one can also realized using quantum-mechanical momentum and position operators (see (3.5))
\[ x_1 = \hat{x}_1 + \beta(\hat{x}_3\hat{p}_0 - \hat{p}_3\hat{x}_0) = \hat{x}_1 + \beta N_3 \]  
\[ \begin{pmatrix} x_0 \\ x_3 \end{pmatrix} = \begin{pmatrix} \cosh(\beta\hat{p}_1) & \sinh(\beta\hat{p}_1) \\ \sinh(\beta\hat{p}_1) & \cosh(\beta\hat{p}_1) \end{pmatrix} \begin{pmatrix} \hat{x}_0 \\ \hat{x}_3 \end{pmatrix} \]

therefore, noncommuting configuration space one can obtain as \( \hat{p}_1 \) - depending hiperbolic rotation in two dimensional commuting space \( (\hat{x}_0, \hat{x}_3) \).

The choice (ii) of twisting provides the generalized Heisenberg uncertainty relations as follow

\[ \Delta(x_0)\Delta(x_1) \geq \hbar\beta < x_3 > | \]
\[ \Delta(x_3)\Delta(x_1) \geq \hbar\beta < x_0 > | \]  
\[ \Delta(p_0)\Delta(x_1) \geq \frac{1}{2}\hbar\beta < p_3 > | \quad \Delta(p_3)\Delta(x_1) \geq \frac{1}{2}\hbar\beta < p_0 > | \]
\[ \Delta(p_0)\Delta(x_3) \geq \frac{1}{2}\hbar < \sinh(\beta p_1) > | \quad \Delta(p_3)\Delta(x_0) \geq \frac{1}{2}\hbar < \sinh(\beta p_1) > | \]
\[ \Delta(p_0)\Delta(x_0) \geq \frac{1}{2}\hbar < \cosh(\beta p_1) > | \quad \Delta(p_3)\Delta(x_3) \geq \frac{1}{2}\hbar < \cosh(\beta p_1) > | \]

We see that two generalizations of the Heisenberg uncertainty relations Eqs.(3.7), (4.6) have different behaviour for high momentum limit.

In the case (i) they are bounded by \( \frac{\hbar}{2} \) because of sine and cosine functions. However in the hiperbolic case (ii) they are strongly divergent for \( p_1 \to \infty \).

References

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[2] Majid S.: Foundations of Quantum Group Theory, Cambridge 1995.