KERNEL BASED DIRICHLET SEQUENCES

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Abstract. Let \( X = (X_1, X_2, \ldots) \) be a sequence of random variables with values in a standard space \((S, B)\). Suppose
\[
X_1 \sim \nu \text{ and } P(X_{n+1} \in \cdot \mid X_1, \ldots, X_n) = \frac{\theta \nu(\cdot) + \sum_{i=1}^{n} K(X_i)(\cdot)}{n + \theta} \quad \text{a.s.}
\]
where \( \theta > 0 \) is a constant, \( \nu \) a probability measure on \( B \), and \( K \) a random probability measure on \( B \). Then, \( X \) is exchangeable whenever \( K \) is a regular conditional distribution for \( \nu \) given any sub-\( \sigma \)-field of \( B \). Under this assumption, \( X \) enjoys all the main properties of classical Dirichlet sequences, including Sethuraman’s representation, conjugacy property, and convergence in total variation of predictive distributions. If \( \mu \) is the weak limit of the empirical measures, conditions for \( \mu \) to be a.s. discrete, or a.s. non-atomic, or \( \mu \ll \nu \) a.s., are provided. Two CLT’s are proved as well. The first deals with stable convergence while the second concerns total variation distance.

1. Introduction
Throughout, \( S \) is a Borel subset of a Polish space and \( B \) the Borel \( \sigma \)-field on \( S \). All random elements are defined on a common probability space, say \((\Omega, \mathcal{A}, P)\). Moreover,
\[
X = (X_1, X_2, \ldots)
\]
is a sequence of random variables with values in \((S, B)\) and
\[
\mathcal{F}_n = \sigma(X_1, \ldots, X_n).
\]
We say that \( X \) is a Dirichlet sequence, or a Polya sequence, if its predictive distributions are of the form
\[
P(X_{n+1} \in \cdot \mid \mathcal{F}_n) = \frac{\theta P(X_1 \in \cdot) + \sum_{i=1}^{n} \delta_{X_i}(\cdot)}{n + \theta}
\]
a.s.
for all \( n \geq 1 \) and some constant \( \theta > 0 \). The finite measure \( \theta P(X_1 \in \cdot) \) is called the parameter of \( X \). Here and in the sequel, for each \( x \in S \), we denote by \( \delta_x \) the unit mass at \( x \).
Let \( \mathcal{L}_0 \) be the class of Dirichlet sequences. As it can be guessed from the definition, each element of \( \mathcal{L}_0 \) is exchangeable. We recall that \( X \) is exchangeable if
\[
\pi(X_1, \ldots, X_n) \sim (X_1, \ldots, X_n) \quad \text{for all } n \geq 2 \text{ and all permutations } \pi \text{ of } S^n.
\]

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A permutation of $S^n$ is meant as a map $\pi : S^n \rightarrow S^n$ of the form

$$\pi(x_1, \ldots, x_n) = (x_{j_1}, \ldots, x_{j_n})$$

for all $(x_1, \ldots, x_n) \in S^n$, where $(j_1, \ldots, j_n)$ is a fixed permutation of $(1, \ldots, n)$. An i.i.d. sequence is obviously exchangeable while the converse is not true. However, the distribution of an exchangeable sequence (with values in a standard space) is a mixture of the distributions of i.i.d. sequences; see Subsection 1.2.

Since Ferguson, Blackwell and Mac Queen, $\mathcal{L}_0$ played a prevailing role in Bayesian statistics. It was for a long time the basic ingredient of Bayesian nonparametrics. And still today, the Bayesian nonparametrics machinery is greatly affected by $\mathcal{L}_0$ and its developments. In addition, $\mathcal{L}_0$ plays a role in various other settings, including population genetics and species sampling. The literature on $\mathcal{L}_0$ is huge and we do not try to summarize it. Without any claim of being exhaustive, we mention a few seminal papers and recent textbooks: [1], [8], [10], [12], [14], [17], [19], [22], [23], [24].

The object of this paper is a new class of exchangeable sequences, say $\mathcal{L}$, such that $\mathcal{L} \supset \mathcal{L}_0$. There are essentially two reasons for taking $\mathcal{L}$ into account. First, all main features of $\mathcal{L}_0$ are preserved by $\mathcal{L}$, including the Sethuraman’s representation, the conjugacy property and the simple form of predictive distributions. Thus, from the point of view of a Bayesian statistician, $\mathcal{L}$ can be handled as simply as $\mathcal{L}_0$. Second, $\mathcal{L}$ is more flexible than $\mathcal{L}_0$ and allows to model more real situations. For instance, if $X \in \mathcal{L}$, the weak limit of the empirical measures is not forced to be a.s. discrete, but it may be a.s. non-atomic or even a.s. absolutely continuous with respect to a reference measure.

1.1. Definition of $\mathcal{L}$. Obviously, the notion of Dirichlet sequence can be extended in various ways. In this paper, for $X$ to be an extended Dirichlet sequence, two conditions are essential. First, $X$ should be exchangeable. Second, the predictive distributions of $X$ should have a known (and possibly simple) structure. Indeed, to define a sequence $X$ via its predictive distributions has various merits. It is technically convenient (see the proof of Theorem 13) and makes the dynamics of $X$ explicit. Furthermore, having the predictive distributions in closed form makes straightforward the Bayesian predictive inference on $X$; see e.g. [7] and [15]. We also note that, as claimed in [16]: “There are very few models for exchangeable sequences $X$ with an explicit prediction rule”.

Let $\mathcal{P}$ be the collection of all probability measures on $\mathcal{B}$ and $\mathcal{C}$ the $\sigma$-field over $\mathcal{P}$ generated by the maps $p \mapsto p(A)$ for all $A \in \mathcal{B}$. A kernel on $(S, \mathcal{B})$ is a measurable map $K : (S, \mathcal{B}) \rightarrow (\mathcal{P}, \mathcal{C})$. Thus, $K(x) \in \mathcal{P}$ for each $x \in S$ and $x \mapsto K(x)(A)$ is a $\mathcal{B}$-measurable map for fixed $A \in \mathcal{B}$. Here, $K(x)(A)$ denotes the value attached to the event $A$ by the probability measure $K(x)$. (This notation is possibly heavy but suitable for this paper).

A quite natural extension of $\mathcal{L}_0$, among the possible ones, consists in replacing $\delta$ with any kernel $K$ in the predictive distributions of $X$. If $K$ is arbitrary, however, $X$ may fail to be exchangeable.
More precisely, fix $\nu \in \mathcal{P}$, a constant $\theta > 0$ and a kernel $K$ on $(S, \mathcal{B})$. By the Ionescu-Tulcea theorem, there is a sequence $X$ such that

1. $X_1 \sim \nu$ and $P(X_{n+1} \in \cdot \mid \mathcal{F}_n) = \frac{\theta \nu(\cdot) + \sum_{i=1}^{n} K(X_i)(\cdot)}{n + \theta}$ a.s.

for all $n \geq 1$. Generally, however, $X$ is not exchangeable. As an obvious example, take the trivial kernel $K(x) = \nu^*$ for all $x \in S$, where $\nu^* \in \mathcal{P}$ but $\nu^* \neq \nu$. Then, condition (1) implies that $X_2$ is not distributed as $X_1$.

Our starting point is that, for $X$ to be exchangeable, it suffices condition (1) and

2. $K$ is a regular conditional distribution (r.c.d.) for $\nu$ given $\mathcal{G}$

for some sub-$\sigma$-field $\mathcal{G} \subset \mathcal{B}$. We recall that $K$ is a r.c.d. for $\nu$ given $\mathcal{G}$ if $K(x) \in \mathcal{P}$ for each $x \in S$, the map $x \mapsto K(x)(A)$ is $\mathcal{G}$-measurable for each $A \in \mathcal{B}$, and

$$\nu(A \cap G) = \int_G K(x)(A) \nu(dx) \quad \text{for all } A \in \mathcal{B} \text{ and } G \in \mathcal{G}.$$  

Equivalently, $K$ is a r.c.d. for $\nu$ given $\mathcal{G}$ if $K(x) \in \mathcal{P}$ for each $x \in S$ and

$$K(\cdot)(A) = E_{\nu}(1_A \mid \mathcal{G}), \quad \nu\text{-a.s., for all } A \in \mathcal{B}.$$ 

Since $(S, \mathcal{B})$ is a standard space, for any sub-$\sigma$-field $\mathcal{G} \subset \mathcal{B}$, a r.c.d. for $\nu$ given $\mathcal{G}$ exists and is $\nu$-essentially unique. See e.g. [4] for more information on r.c.d.’s.

Condition (2) makes the next definition operational.

Say that $X$ is a kernel based Dirichlet sequence if it is exchangeable and satisfies condition (1) for some $\nu \in \mathcal{P}$, some constant $\theta > 0$ and some kernel $K$ on $(S, \mathcal{B})$. In particular, $X$ is a kernel based Dirichlet sequence if conditions (1)-(2) hold. In the sequel, $\mathcal{L}$ denotes the collection of all $X$ satisfying conditions (1)-(2).

If $X \in \mathcal{L}$ and $\mathcal{G} = \mathcal{B}$, then $K = \delta$ and $X \in \mathcal{L}_0$. At the opposite extreme, if $\mathcal{G} = \{\emptyset, S\}$, then $K(x) = \nu$ for $\nu$-almost all $x \in S$ and $X$ is i.i.d. Various other examples come soon to the fore. The following are from [7] (even if, when writing [7], we didn’t know yet that $X$ is exchangeable).

**Example 1.** Let $\mathcal{G} = \sigma(\mathcal{H})$, where $\mathcal{H} \subset \mathcal{B}$ is a countable partition of $S$ such that $\nu(H) > 0$ for all $H \in \mathcal{H}$. A r.c.d. for $\nu$ given $\mathcal{G}$ is

$$K(x) = \sum_{H \in \mathcal{H}} 1_H(x) \nu(\cdot \mid H) = \nu[\cdot \mid H(x)]$$

where $H(x)$ denotes the only $H \in \mathcal{H}$ such that $x \in H$. Therefore, $X \in \mathcal{L}$ whenever

$$X_1 \sim \nu \quad \text{and} \quad P(X_{n+1} \in \cdot \mid \mathcal{F}_{n}) = \frac{\theta \nu(\cdot) + \sum_{i=1}^{n} \nu[\cdot \mid H(X_i)]}{n + \theta} \quad \text{a.s.}$$

Note that

$$P(X_{n+1} \in \cdot \mid \mathcal{F}_n) \ll \nu(\cdot) \quad \text{a.s.}$$

This fact highlights a stricking difference between $\mathcal{L}$ and $\mathcal{L}_0$. In this example, if $\nu$ is non-atomic, the probability distributions of $X$ and $Y$ are singular for any $Y \in \mathcal{L}_0$. 
Example 2. Let $S = \mathbb{R}^2$ and $G = \sigma(f)$ where $f(u, v) = u$ for all $(u, v) \in \mathbb{R}^2$. Let $\mathcal{B}_0$ be the Borel $\sigma$-field on $\mathbb{R}$ and $\mathcal{N}(u, 1)$ the Gaussian law on $\mathcal{B}_0$ with mean $u$ and variance 1. Fix a probability measure $r$ on $\mathcal{B}_0$ and define

$$\nu(A \times B) = \int_A \mathcal{N}(u, 1)(B) r(du) \quad \text{for all } A, B \in \mathcal{B}_0$$

where $\mathcal{N}(u, 1)(B)$ denotes the value attached to $B$ by $\mathcal{N}(u, 1)$. Then, a r.c.d. for $\nu$ given $G$ is

$$K(u, v) = \delta_u \times \mathcal{N}(u, 1) \quad \text{for all } (u, v) \in \mathbb{R}^2.$$ 

Hence, letting $X_i = (U_i, V_i)$, one obtains $X \in \mathcal{L}$ provided $(U_1, V_1) \sim \nu$ and

$$P(U_{n+1} \in A, V_{n+1} \in B \mid \mathcal{F}_n) = \theta \nu(A \times B) + \sum_{i=1}^{n} \delta_{X_i} \mathcal{N}(U_i, 1)(B) a.s.$$ 

Example 3. Let $f : S \to S$ be a measurable map. If $\nu$ is $f$-invariant, that is $\nu = \nu \circ f^{-1}$, it may be reasonable to take $G = \{ A \in \mathcal{B} : f^{-1}(A) = A \}$. As a trivial example, if $S = \mathbb{R}$, $f(x) = -x$ and $\nu$ is symmetric, then

$$K(x) = \frac{\delta_x + \delta_{-x}}{2}$$

is a r.c.d. for $\nu$ given $G$. Hence, $X \in \mathcal{L}$ whenever $X_1 \sim \nu$ and

$$P(X_{n+1} \in \cdot \mid \mathcal{F}_n) = \frac{2 \theta \nu + \sum_{i=1}^{n} (\delta_{X_i} + \delta_{-X_i})}{2(n + \theta)} a.s.$$ 

This example is related to [7], [9] and [18]. We will take it up again in forthcoming Example 17.

1.2. Sethuraman’s representation and conjugacy for $\mathcal{L}_0$. Before going on, a few basic properties of $\mathcal{L}_0$ are to be recalled.

A random probability measure on $(S, \mathcal{B})$ is a measurable map $\mu : (\Omega, \mathcal{A}) \to (\mathcal{P}, \mathcal{C})$. Let $X$ be exchangeable. Since $(S, \mathcal{B})$ is a standard space, there is a random probability measure $\mu$ on $(S, \mathcal{B})$ such that

$$\mu(A) \overset{\text{a.s.}}{=} \lim_n \frac{1}{n} \sum_{i=1}^{n} 1_A(X_i) \overset{\text{a.s.}}{=} \lim_n P(X_{n+1} \in A \mid \mathcal{F}_n)$$

for each fixed $A \in \mathcal{B}$. Moreover, $X$ is i.i.d. conditionally on $\mu$, in the sense that

$$P(X \in B \mid \mu) = \mu^\infty(B) \quad \text{a.s. for all } B \in \mathcal{B}^\infty$$

where $\mu^\infty = \mu \times \mu \times \ldots$; see e.g. [6, p. 2090].

Suppose now that $X \in \mathcal{L}_0$ and define

$$\mathcal{D}(C) = P(\mu \in C) \quad \text{for all } C \in \mathcal{C}.$$ 

Such a $\mathcal{D}$ is a probability measure on $\mathcal{C}$, called the Dirichlet prior, and admits the following representation. Define a random probability measure $\mu^*$ on $(S, \mathcal{B})$ as

$$\mu^* = \sum_j V_j \delta Z_j,$$
where \((Z_j)\) and \((V_j)\) are independent sequences, \((Z_j)\) is i.i.d. with \(Z_1 \sim \nu\), and \((V_j)\) has the stick-breaking distribution with parameter \(\theta\); see Section 2. Then,

\[ D(C) = P(\mu^* \in C) \quad \text{for all } C \in \mathcal{C}. \]

Thus, \(D\) can be also regarded as the probability distribution of \(\mu^*\). This fact, proved by Sethuraman [24], is fundamental in applications; see e.g. [11].

Finally, we recall the conjugacy property of \(L_0\). Write \(D(\lambda)\) (instead of \(D\)) if \(X \in L_0\) has parameter \(\lambda\). In this notation, if \(X\) has parameter \(\theta \nu\), then

\[ P(\mu \in C \mid \mathcal{F}_n) = D\left(\theta \nu + \sum_{i=1}^n \delta_{X_i}\right)(C) \quad \text{a.s. for all } C \in \mathcal{C} \text{ and } n \geq 1. \]

Roughly speaking, the posterior distribution of \(\mu\) given \((X_1, \ldots, X_n)\) is still of the Dirichlet type but the parameter turns into \(\theta \nu + \sum_{i=1}^n \delta_{X_i}\). Once again, this fact plays a basic role in applications.

### 1.3. Our contribution.

As claimed above, this paper aims to introduce and investigate the class \(L\).

Our first result is that conditions (1)-(2) suffice for exchangeability of \(X\). Thus, each \(X \in L\) is a kernel based Dirichlet sequence, as defined in Subsection 1.1.

The next step is to develop some theory for \(L\). The obvious hope is that, at least to a certain extent, such a theory is parallel to that of \(L_0\). This is exactly the case. Essentially all main results concerning \(L_0\) extend nicely to \(L\). To illustrate, we assume \(X \in L\) and we mention a few facts.

- Up to replacing \(\delta\) with \(K\), the Sethuraman’s representation remains exactly the same. Precisely, \(P(\mu \in C) = P(\mu^* \in C)\) for all \(C \in \mathcal{C}\), where

  \[ \mu^* = \sum_j V_j K(Z_j) \]

  and \((V_j)\) and \((Z_j)\) are as in Subsection 1.2.

- The predictive distributions converge in total variation, that is

  \[ \sup_{A \in \mathcal{B}} \left| P(X_{n+1} \in A \mid \mathcal{F}_n) - \mu(A) \right| \overset{\text{a.s.}}{\to} 0 \quad \text{as } n \to \infty. \]

- If \(X \in L_0\), it is well known that \(\mu\) is a.s. discrete. This result extends to \(L\) as follows. Denote by \(D_1, D_2, D_3\) the collections of elements of \(\mathcal{P}\) which are, respectively, discrete, non-atomic, or absolutely continuous with respect to \(\nu\). Then, for each \(1 \leq j \leq 3\),

  \[ P(\mu \in D_j) = 1 \quad \iff \quad K(x) \in D_j \text{ for } \nu\text{-almost all } x \in S. \]

Since \(\delta_x \in D_1\) for all \(x \in S\), the classical result is recovered. But now, with a suitable \(K\), one obtains \(P(\mu \in D_2) = 1\) or \(P(\mu \in D_3) = 1\). This fact may be useful in applications.

- The conjugacy property of \(L_0\) is still available. For each \(n \geq 1\), let

  \[ V^{(n)} = (V_j^{(n)} : j \geq 1) \quad \text{and} \quad Z^{(n)} = (Z_j^{(n)} : j \geq 1) \]

be two sequences such that...
(i) \( V^{(n)} \) and \( Z^{(n)} \) are conditionally independent given \( F_n \);
(ii) \( V^{(n)} \) has the stick-breaking distribution, with parameter \( n + \theta \), conditionally on \( F_n \);
(iii) \( Z^{(n)} \) is i.i.d., conditionally on \( F_n \), with

\[
P(Z_1^{(n)} \in \cdot \mid F_n) = P(X_{n+1} \in \cdot \mid F_n) = \frac{\theta \nu(\cdot) + \sum_{i=1}^{n} K(X_i)(\cdot)}{n + \theta} \quad \text{a.s.}
\]

Then,

\[
P(\mu \in \cdot \mid F_n) = P(\mu_n^* \in \cdot \mid F_n)
\]

where

\[
\mu_n^* = \sum_j V_j^{(n)} K(Z_j^{(n)}).
\]

Again, if \( K = \delta \), this result reduces to the classical one.

- A stable CLT holds true. Let \( S = \mathbb{R}^p \) and \( \int \|x\|^2 \nu(dx) < \infty \), where \( \|\cdot\| \) is the Euclidean norm. Suppose that \( K \) has mean 0, in the sense that

\[
\int y_i K(x)(dy) = 0 \quad \text{for all} \ x \in \mathbb{R}^p \ \text{and} \ i = 1, \ldots, p
\]

where \( y_i \) denotes the \( i \)-th coordinate of a point \( y \in \mathbb{R}^p \). Then, \( n^{-1/2} \sum_{i=1}^{n} X_i \) converges stably (in particular, in distribution) to the Gaussian kernel \( \mathcal{N}_p(0, \Sigma) \), where \( \Sigma \) is the (random) covariance matrix

\[
\Sigma = \left( \int y_i y_j \mu(dy) : 1 \leq i, j \leq p \right).
\]

Moreover, under some additional conditions, \( n^{-1/2} \sum_{i=1}^{n} X_i \) converges in total variation as well.

This is a brief summary of our main results. Before closing the introduction, however, two remarks are in order.

First, to prove such results, we often exploit the fact that

\[(3) \quad (K(X_n) : n \geq 1) \quad \text{is a classical Dirichlet sequence with values in} \ (\mathcal{P}, \mathcal{C}). \]

Condition (3) is not surprising. We give a simple proof of it, based on predictive distributions, but condition (3) could be also obtained via some known results on \( \mathcal{L}_0 \).

Second, the above results are potentially useful in Bayesian nonparametrics. Define in fact

\[
\Pi(C) = P(\mu \in C) = P(\mu_n^* \in C) \quad \text{for all} \ C \in \mathcal{C}.
\]

Such a \( \Pi \) is a new prior to be used in Bayesian nonparametrics. In real problems, working with \( \Pi \) is as simple as working with the classical Dirichlet prior \( \mathcal{D} \). In both cases, the posterior can be easily evaluated. Unlike \( \mathcal{D} \), however, \( \Pi \) can be chosen such that \( \Pi(C) = 1 \) for some meaningful sets \( C \) of probability measures. For instance, \( C = D_j \) with \( D_j \) defined as above for \( j = 1, 2, 3 \). Or else, \( C \) the set of invariant probability measures under a countable class of measurable transformations; see forthcoming Example 17. Finally, just because of its definition, \( \mathcal{L} \) is particularly suitable in Bayesian predictive inference. And predicting future observations is one of the main tasks of Bayesian nonparametrics.
2. Preliminaries

For all \( \lambda \in \mathcal{P} \) and bounded measurable \( f : S \rightarrow \mathbb{R} \), the notation \( \lambda(f) \) stands for \( \lambda(f) = \int f \, d\lambda \). Moreover, \( \mathcal{N}_p(0, \Sigma) \) denotes the \( p \)-dimensional Gaussian law (on the Borel \( \sigma \)-field of \( \mathbb{R}^p \)) with mean 0 and covariance matrix \( \Sigma \).

Let \( \theta > 0 \) be a constant, \( (W_n) \) an i.i.d. sequence with \( W_1 \sim \text{beta}(1, \theta) \) and \( T_1 = W_1, \ T_n = W_n \prod_{i=1}^{n-1} (1 - W_i) \) for \( n > 1 \).

A sequence \( (V_n) \) of real random variables has the stick-breaking distribution with parameter \( \theta \) if \( (V_n) \sim (T_n) \). Note that \( V_n > 0 \) for all \( n \) and \( \sum_n V_n = 1 \) a.s.

Stable convergence is a strong form of convergence in distribution. Let \( N \) be a random probability measure on \((S, \mathcal{B})\). Then, \( X_n \) converges to \( N \) stably if
\[
E[N(f) \mid H] = \lim_n E[f(X_n) \mid H]
\]
for all bounded continuous \( f : S \rightarrow \mathbb{R} \) and all \( H \in \mathcal{A} \) with \( P(H) > 0 \). In particular, \( X_n \) converges in distribution to the probability measure \( A \mapsto E[N(A)] \).

We next report an useful characterization of exchangeability due to [13]; see also [5] and [7]. Let \( \sigma_0 = \{\emptyset, \Omega\} \) be the trivial \( \sigma \)-field and
\[
\sigma_n(x) = P[X_{n+1} \in \cdot \mid (X_1, \ldots, X_n) = x] \quad \text{for all } x \in S^n.
\]

Theorem 4. ([13, Theorem 3.1]). The sequence \( X \) is exchangeable if and only if
\[
P[(X_{n+1}, X_{n+2}) \in \cdot \mid \mathcal{F}_n] = P[(X_{n+2}, X_{n+1}) \in \cdot \mid \mathcal{F}_n] \quad \text{a.s.}
\]
for all \( n \geq 0 \) and
\[
\sigma_n(x) = \sigma_n(\pi(x))
\]
for all \( n \geq 2, \) all permutations \( \pi \) of \( S^n, \) and almost all \( x \in S^n. \) (Here, “almost all” is with respect to the marginal distribution of \((X_1, \ldots, X_n))\).

We conclude this section with two technical lemmas. Let \( \sigma(K) = \{\{x \in S : K(x) \in C\} : C \in \mathcal{C}\} \) be the \( \sigma \)-field over \( S \) generated by the kernel \( K \).

**Lemma 5.** (Lemma 10 of [4]). Under condition (2), there is a set \( F \in \sigma(K) \) such that \( \nu(F) = 1 \) and
\[
K(x)(B) = \delta_x(B) \quad \text{for all } B \in \sigma(K) \text{ and } x \in F.
\]

**Proof.** This is basically [4, Lem. 10] but we give a proof to make the paper self-contained. The atoms of the \( \sigma \)-field \( \sigma(K) \) are sets of the form
\[
B(x) = \{y \in S : K(y) = K(x)\} \quad \text{for all } x \in S.
\]
Hence, each \( B \in \sigma(K) \) can be written as
\[
B = \bigcup_{x \in B} B(x).
\]
Moreover, by [4, Lem. 10], there is a set $F \in \sigma(K)$ such that $\nu(F) = 1$ and $$K(x)(B(x)) = 1 \quad \text{for all } x \in F.$$ Having noted these facts, fix $x \in F$ and $B \in \sigma(K)$. If $x \in B$, then $$K(x)(B) \geq K(x)(B(x)) = 1.$$ If $x \notin B$, since $B^c \in \sigma(K)$, then $K(x)(B) = 1 - K(x)(B^c) = 0$. Hence, $K(x)(B) = \delta_x(B)$. \hfill \Box

Lemma 6. Under condition (2), there is a set $F \in \sigma(K)$ such that $\nu(F) = 1$ and $$\int_A K(y)(B)\, K(x)(dy) = K(x)(A) K(x)(B) \quad \text{for all } x \in F \text{ and } A, B \in \mathcal{B}.$$ Moreover, $$\int_A K(y)(B)\, \nu(dy) = \int_B K(y)(A)\, \nu(dy) \quad \text{for all } A, B \in \mathcal{B}.$$ Proof. Let $F$ be as in Lemma 5. Fix $x \in F$ and $A, B \in \mathcal{B}$. Define $$G = \{ y \in S : K(y)(B) = K(x)(B) \}$$ and note that $x \in G$ and $G \in \sigma(K)$. Since $x \in G$, then $\delta_x(G) = 1$. Since $G \in \sigma(K)$ and $x \in F$, Lemma 5 implies $$K(x)(G) = \delta_x(G) = 1.$$ Therefore, $$\int_A K(y)(B)\, K(x)(dy) = K(x)(B) \int_A K(x)(dy) = K(x)(A) K(x)(B).$$ Finally, $$\int_A K(y)(B)\, \nu(dy) = \int_A E_{\nu}(1_B \mid \mathcal{G})\, d\nu = \int_B E_{\nu}(1_A \mid \mathcal{G})\, d\nu = \int_B K(y)(A)\, \nu(dy). \hfill \Box

3. Results

Recall that $\mathcal{L}$ is the class of sequences satisfying conditions (1)-(2) for some $\nu \in \mathcal{P}$ and some constant $\theta > 0$. In this section, $X \in \mathcal{L}$ and $\mu$ is a random probability measure on $(S, \mathcal{B})$ such that $$\mu(A) \overset{a.s.}{=} \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} 1_A(X_i) \overset{a.s.}{=} \lim_{n \to \infty} P(X_{n+1} \in A \mid \mathcal{F}_n) \quad \text{for all } A \in \mathcal{B}.$$ Existence of $\mu$ depends on $X$ is exchangeable and $(S, \mathcal{B})$ is a standard space; see Subsection 1.2.

Our starting point is the following.
Theorem 7. Under condition (1), $X$ is exchangeable if and only if

\[
\int_A K(y)(B) \nu(dy) = \int_B K(y)(A) \nu(dy)
\]

and

\[
\int_A K(y)(B) K(x)(dy) = \int_B K(y)(A) K(x)(dy)
\]

for all $A, B \in B$ and $\nu$-almost all $x \in S$. In particular, $X$ is exchangeable whenever $X \in \mathcal{L}$ (because of Lemma 6).

**Proof.** For all $A, B \in B$, condition (1) implies

\[
P(X_1 \in A, X_2 \in B) = E\left\{ 1_A(X_1) P(X_2 \in B \mid \mathcal{F}_1) \right\}
\]

\[
= E\left\{ 1_A(X_1) \frac{\theta \nu(B) + K(X_1)(B)}{1 + \theta} \right\}
\]

\[
= \frac{\theta}{1 + \theta} \nu(B) \nu(A) + \frac{1}{1 + \theta} \int_A K(y)(B) \nu(dy).
\]

Therefore,

\[
\text{condition (4)} \iff (X_1, X_2) \sim (X_2, X_1).
\]

Similarly, under (1), one obtains

\[
P(X_2 \in A, X_3 \in B \mid \mathcal{F}_1) = E\left\{ 1_A(X_2) P(X_3 \in B \mid \mathcal{F}_2) \mid \mathcal{F}_1 \right\}
\]

\[
= E\left\{ 1_A(X_2) \frac{\theta \nu(B) + K(X_1)(B) + K(X_2)(B)}{2 + \theta} \mid \mathcal{F}_1 \right\}
\]

\[
= \frac{1 + \theta}{2 + \theta} P(X_2 \in B \mid \mathcal{F}_1) P(X_2 \in A \mid \mathcal{F}_1) + \frac{1}{2 + \theta} E\left\{ 1_A(X_2) K(X_2)(B) \mid \mathcal{F}_1 \right\} \quad \text{a.s.}
\]

and

\[
E\left\{ 1_A(X_2) K(X_2)(B) \mid \mathcal{F}_1 \right\} = \frac{\theta}{1 + \theta} \int_A K(y)(B) \nu(dy) + \frac{1}{1 + \theta} \int_A K(y)(B) K(X_1)(dy) \quad \text{a.s.}
\]

Next, if $X$ is exchangeable, condition (4) follows from $(X_1, X_2) \sim (X_2, X_1)$. Moreover, $P(X_2 \in A, X_3 \in B \mid \mathcal{F}_1) = P(X_2 \in B, X_3 \in A \mid \mathcal{F}_1)$ a.s. implies

\[
E\left\{ 1_A(X_2) K(X_2)(B) \mid \mathcal{F}_1 \right\} = E\left\{ 1_B(X_2) K(X_2)(A) \mid \mathcal{F}_1 \right\} \quad \text{a.s.}
\]

Therefore, (5) follows from (4) and the above condition.

Conversely, assume conditions (4)-(5). Define

\[
\sigma_n(x) = \frac{\theta \nu + \sum_{i=1}^n K(x_i)}{n + \theta} \quad \text{for all } n \geq 1 \text{ and } x = (x_1, \ldots, x_n) \in S^n.
\]

By (1), $P(X_{n+1} \in \cdot \mid \mathcal{F}_n) = \sigma_n(X_1, \ldots, X_n)$ a.s. Moreover, $\sigma_n(x) = \sigma_n(\pi(x))$ for all $n \geq 2$, all permutations $\pi$ of $S^n$ and all $x \in S^n$. Hence, by Theorem 4, it suffices to show that

\[
P[(X_{n+1}, X_{n+2}) \in \cdot \mid \mathcal{F}_n] = P[(X_{n+2}, X_{n+1}) \in \cdot \mid \mathcal{F}_n] \quad \text{a.s. for all } n \geq 0.
\]
For \( n = 0 \), the above condition is equivalent to (4) (recall that \( \mathcal{F}_0 \) is the trivial \( \sigma \)-field). Therefore, it is enough to show that

\[
\int_A \sigma_{n+1}(x, y)(B) \sigma_n(x)(dy) = \int_B \sigma_{n+1}(x, y)(A) \sigma_n(x)(dy)
\]

for all \( n \geq 1 \), all \( A, B \in \mathcal{B} \) and almost all \( x \in S^n \) (where “almost all” refers to the marginal distribution of \((X_1, \ldots, X_n)\)).

Fix \( m \geq 1 \) and \( A \in \mathcal{B} \). If \( X_i \sim \nu \) for \( i = 1, \ldots, m \), then

\[
E\{K(X_i)(A)\} = \int K(y)(A) \nu(dy) = \nu(A)
\]

for \( i = 1, \ldots, m \), where the second equality is by (4) (applied with \( B = S \)). Hence,

\[
P(X_{m+1} \in A) = E\{P(X_{m+1} \in A \mid \mathcal{F}_m)\} = \frac{\theta \nu(A)}{m + \theta} + \frac{\sum_{i=1}^m E\{K(X_i)(A)\}}{m + \theta} = \nu(A).
\]

By induction, it follows that \( X_i \sim \nu \) for all \( i \geq 1 \).

Finally, fix \( n \geq 1 \) and \( A, B \in \mathcal{B} \). By (5), there is a set \( M \in \mathcal{B} \) such that \( \nu(M) = 1 \) and

\[
\int_A K(y)(B) K(x)(dy) = \int_B K(y)(A) K(x)(dy)
\]

for all \( x \in M \).

Thanks to this fact and condition (4), if \( x = (x_1, \ldots, x_n) \in M^n \), one obtains

\[
\int_A K(y)(B) \sigma_n(x)(dy) = \frac{\theta \int_A K(y)(B) \nu(dy) + \sum_{i=1}^n \int_A K(y)(B) K(x_i)(dy)}{n + \theta}
\]

\[
= \frac{\theta \int_B K(y)(A) \nu(dy) + \sum_{i=1}^n \int_B K(y)(A) K(x_i)(dy)}{n + \theta} = \int_B K(y)(A) \sigma_n(x)(dy).
\]

It follows that

\[
\int_A \sigma_{n+1}(x, y)(B) \sigma_n(x)(dy) = \int_A \frac{\theta \nu(B) + \sum_{i=1}^n K(x_i)(B) + K(y)(B)}{n + 1 + \theta} \sigma_n(x)(dy)
\]

\[
= \frac{n + \theta}{n + 1 + \theta} \sigma_n(x)(B) \sigma_n(x)(A) + \frac{\int_A K(y)(B) \sigma_n(x)(dy)}{n + 1 + \theta}
\]

\[
= \frac{n + \theta}{n + 1 + \theta} \sigma_n(x)(B) \sigma_n(x)(A) + \frac{\int_B K(y)(A) \sigma_n(x)(dy)}{n + 1 + \theta}
\]

\[
= \int_B \sigma_{n+1}(x, y)(A) \sigma_n(x)(dy).
\]

Therefore, equation (6) holds for each \( x \in M^n \). To conclude the proof, it suffices to note that, since \( \nu(M) = 1 \) and \( X_i \sim \nu \) for all \( i \),

\[
P((X_1, \ldots, X_n) \in M^n) = 1.
\]

\[\square\]

In view of Theorem 7, \( X \) is a kernel based Dirichlet sequence, as defined in Subsection 1.1, if and only if conditions (1) and (4)-(5) hold. Since (2) ⇒ (4)-(5) (because of Lemma 6), a sufficient condition for \( X \) to be a kernel based Dirichlet sequence is that \( X \in \mathcal{L} \). We do not know whether (4)-(5) ⇒ (2). In the sequel, however, we always assume \( X \in \mathcal{L} \), namely, we always assume conditions (1)-(2).
The next step is to develop some theory for $L$. To this end, the following result is useful.

**Theorem 8.** If $X \in L$, the sequence $(K(X_n) : n \geq 1)$ is a Dirichlet sequence with values in $(\mathcal{P}, \mathcal{C})$ and parameter the image measure $\theta \nu \circ K^{-1}$.

**Proof.** By Lemma 5, there is a set $F \in \sigma(K)$ such that
\[ \nu(F) = 1 \quad \text{and} \quad K(x)(B) = \delta_x(B) \quad \text{for all } B \in \sigma(K) \text{ and } x \in F. \]

Since $P(X_n \in F) = \nu(F) = 1$ for all $n$, it follows that
\[ P(X_{n+1} \in B \mid \mathcal{F}_n) = \frac{\theta \nu(B) + \sum_{i=1}^{n} \delta_{X_i}(B)}{n + \theta} \quad \text{for all } B \in \sigma(K) \text{ a.s.} \]

Having noted this fact, define
\[ K_n = \sigma[K(X_1), \ldots, K(X_n)]. \]

Since $K_n \subset \mathcal{F}_n$ and $P(X_{n+1} \in \cdot \mid \mathcal{F}_n)$ is $K_n$-measurable,
\[ P(X_{n+1} \in \cdot \mid K_n) = P(X_{n+1} \in \cdot \mid \mathcal{F}_n) \quad \text{a.s.} \]

Finally, fix $C \in \mathcal{C}$ and define $B = \{ K \in C \}$. Since $B \in \sigma(K)$, one obtains
\[ P\left[K(X_{n+1}) \in C \mid K_n\right] = P\left(X_{n+1} \in B \mid K_n\right) = P\left(X_{n+1} \in B \mid \mathcal{F}_n\right) \]
\[ = \frac{\theta \nu(B) + \sum_{i=1}^{n} \delta_{X_i}(B)}{n + \theta} \quad \text{a.s.} \]

\[ \square \]

We next turn to a Sethuraman-like representation for $L$. Let $\mu^*$ be the random probability measure on $(S, \mathcal{B})$ defined as
\[ \mu^* = \sum_j V_j K(Z_j), \]

where $(Z_j)$ and $(V_j)$ are independent sequences, $(Z_j)$ is i.i.d. with $Z_1 \sim \nu$, and $(V_j)$ has the stick-breaking distribution with parameter $\theta$; see Section 2.

**Theorem 9.** If $X \in L$, then
\[ P(\mu \in C) = P(\mu^* \in C) \quad \text{for all } C \in \mathcal{C}. \]

**Proof.** Let $\mu_0$ and $\mu_0^*$ be the restrictions of $\mu$ and $\mu^*$ on $\sigma(K)$. Then, $\mu_0 \sim \mu_0^*$ by [24] and since $(K(X_n) : n \geq 1)$ is a classical Dirichlet sequence. Hence,
\[ \left(\mu(g_1), \ldots, \mu(g_k)\right) \sim \left(\mu^*(g_1), \ldots, \mu^*(g_k)\right) \]

whenever $g_1, \ldots, g_k : S \to \mathbb{R}$ are bounded and $\sigma(K)$-measurable. In addition, for fixed $A \in \mathcal{B}$, one obtains
\[ \int K(x)(A) \mu(dx) = \lim_{n} \frac{\sum_{i=1}^{n} K(X_i)(A)}{n} = \lim_{n} P(X_{n+1} \in A \mid \mathcal{F}_n) = \mu(A) \quad \text{a.s.} \]

Similarly, Lemma 6 (applied with $B = S$) implies
\[ \int K(x)(A) K(Z_j)(dx) = K(Z_j)(A) \quad \text{a.s. for all } j \geq 1. \]
Thus,
\[
\int K(x)(A) \mu^*(dx) = \sum_j V_j \int K(x)(A) K(Z_j)(dx)
\]
\[
= \sum_j V_j K(Z_j)(A) = \mu^*(A) \quad \text{a.s.}
\]
Having noted these facts, fix \(k \geq 1\), \(A_1, \ldots, A_k \in \mathcal{B}\), and define \(g_i(x) = K(x)(A_i)\) for all \(x \in S\) and \(i = 1, \ldots, k\). Then,
\[
\left(\mu(A_1), \ldots, \mu(A_k)\right) \overset{a.s.}{=} \left(\mu(g_1), \ldots, \mu(g_k)\right) \sim \left(\mu^*(g_1), \ldots, \mu^*(g_k)\right) \overset{a.s.}{=} \left(\mu^*(A_1), \ldots, \mu^*(A_k)\right).
\]
This concludes the proof. \(\square\)

Theorem 9 plays for \(\mathcal{L}\) the same role played by [24] for \(\mathcal{L}_0\). Among other things, it provides a simple way to approximate the probability distribution of \(\mu\) and to obtain its posterior distribution; see forthcoming Theorem 13 and its proof. For a further implication, define
\[
D_1 = \{p \in \mathcal{P} : p \text{ discrete}\}, \quad D_2 = \{p \in \mathcal{P} : p \text{ non-atomic}\}, \quad D_3 = \{p \in \mathcal{P} : p \ll \nu\}.
\]
Then, Theorem 9 implies the following result.

**Theorem 10.** If \(j \in \{1, 2, 3\}\) and \(X \in \mathcal{L}\), then \(P(\mu \in D_j) \in \{0, 1\}\) and
\[
P(\mu \in D_j) = 1 \iff K(x) \in D_j \text{ for } \nu\text{-almost all } x \in S.
\]
In addition,
\[
\sup_{A \in \mathcal{B}} |P(X_{n+1} \in A | \mathcal{F}_n) - \mu(A)| \overset{a.s.}{\rightarrow} 0 \quad \text{as } n \rightarrow \infty.
\]

**Proof.** Fix \(j \in \{1, 2, 3\}\) and define \(a_j = \nu\{x : K(x) \in D_j\}\). If \(a_j = 1\), Theorem 9 yields
\[
P(\mu \in D_j) = P(\mu^* \in D_j) = P(K(Z_i) \in D_j \text{ for all } i \geq 1) = 1.
\]
Similarly, if \(a_j < 1\),
\[
P(\mu \in D_j) \leq P(K(Z_i) \in D_j \text{ for } 1 \leq i \leq n) = a_j^n \rightarrow 0 \quad \text{as } n \rightarrow \infty.
\]
It remains to prove (7). Define the random probability measure
\[
\lambda_n = \frac{1}{n} \sum_{i=1}^n K(X_i).
\]
To prove (7), it is enough to show that \(\lim_n \sup_{A \in \mathcal{B}} |\lambda_n(A) - \mu(A)| \overset{a.s.}{\rightarrow} 0\) and this limit relation is actually true if \(X \in \mathcal{L}_0\); see e.g. [22, Prop. 11]. Hence, since \((K(X_n) : n \geq 1)\) is a classical Dirichlet sequence, one obtains
\[
\sup_{A \in \sigma(K)} |\lambda_n(A) - \mu(A)| \overset{a.s.}{\rightarrow} 0.
\]
Now, we argue as in the proof of Theorem 9. Precisely, for each \(A \in \mathcal{B}\), Lemma 6 (applied with \(B = S\)) yields
\[
\int K(x)(A) \lambda_n(dx) = \frac{1}{n} \sum_{i=1}^n \int K(x)(A) K(X_i)(dx) = \frac{1}{n} \sum_{i=1}^n K(X_i)(A) = \lambda_n(A) \quad \text{a.s.}
\]
Similarly, \( \int K(x)(A) \mu(dx) = \mu(A) \) a.s. Therefore, after fixing a countable field \( \mathcal{B}_0 \) such that \( \mathcal{B} = \sigma(\mathcal{B}_0) \), one finally obtains
\[
\sup_{A \in \mathcal{B}} |\lambda_n(A) - \mu(A)| = \sup_{A \in \mathcal{B}_0} |\lambda_n(A) - \mu(A)| \xrightarrow{\text{a.s.}} \sup_{A \in \sigma(K)} |\lambda_n(A) - \mu(A)| \xrightarrow{\text{a.s.}} 0.
\]

\[\square\]

It is worth noting that, for an arbitrary exchangeable sequence \( X \), convergence in total variation of \( P(X_{n+1} \in \cdot \mid F_n) \) is not guaranteed; see e.g. [6].

A further consequence of Theorem 9 is a stable CLT (stable convergence is briefly recalled in Section 2). For each \( y \in \mathbb{R}^p \), let \( y_i \) denote the \( i \)-th coordinate of \( y \).

**Theorem 11.** Let \( S = \mathbb{R}^p \) and \( X \in \mathcal{L} \). Suppose \( \int \|x\|^2 \nu(dx) < \infty \), where \( \|\cdot\| \) is the Euclidean norm, and
\[
\int y_i K(x)(dy) = 0 \quad \text{for all} \ x \in \mathbb{R}^p \ \text{and} \ i = 1, \ldots, p.
\]

Then,
\[
\frac{\sum_{i=1}^n X_i}{\sqrt{n}} \xrightarrow{\text{stably}} \mathcal{N}_p(0, \Sigma) \quad \text{as} \ n \to \infty,
\]

where \( \Sigma \) is the random covariance matrix
\[
\Sigma = \left( \int y_i y_j \mu(dy) : 1 \leq i, j \leq p \right).
\]

**Proof.** By standard arguments, it suffices to show that
\[
\frac{\sum_{i=1}^n b^i X_i}{\sqrt{n}} \xrightarrow{\text{stably}} \mathcal{N}_1(0, b^\prime \Sigma b) \quad \text{for each} \ b \in \mathbb{R}^p,
\]

where points of \( \mathbb{R}^p \) are regarded as column vectors and \( b^\prime \) denotes the transpose of \( b \). Define
\[
\sigma^2_b = E[(b^\prime X_1)^2 \mid \mu] - E(b^\prime X_1 \mid \mu)^2.
\]

For fixed \( b \in \mathbb{R}^p \), one obtains
\[
n^{-1/2} \sum_{i=1}^n \left\{ b^i X_i - E(b^\prime X_1 \mid \mu) \right\} \xrightarrow{\text{stably}} \mathcal{N}_1(0, \sigma^2_b);\]

see e.g. [3, Th. 3.1] and the subsequent remark. Furthermore,

\[
E(b^\prime X_1 \mid \mu) = \int (b^\prime y) \mu(dy) = \sum_{i=1}^p b_i \int y_i \mu(dy) \quad \text{a.s. and}
\]

\[
E[(b^\prime X_1)^2 \mid \mu] = \int (b^\prime y)^2 \mu(dy) = \sum_{i=1}^p \sum_{j=1}^p b_i b_j \int y_i y_j \mu(dy) = b^\prime \Sigma b \quad \text{a.s.}
\]
Hence, it suffices to show that $\int y_i \mu(dy) \overset{a.s.}{\to} 0$ for all $i$, and this follows from Theorem 9. In fact, $\int y_i \mu(dy) \sim \int y_i \mu^*(dy)$ and

$$\int y_i \mu^*(dy) = \sum_j V_j \int y_i K(Z_j)(dy) = 0.$$ 

This concludes the proof. \hfill \qed

Theorem 11 applies to Examples 3 and 16. In fact, in Example 3, one has $p = 1$ and $K(x) = (\delta_x + \delta_{-x})/2$. Hence, $\int y K(x)(dy) = 0$ for all $x \in \mathbb{R}$. Example 16 is discussed below. Here, we give conditions for convergence in total variation of $n^{-1/2} \sum_{i=1}^n X_i$.

**Theorem 12.** In addition to the conditions of Theorem 11, suppose that $K(x)$ is not singular, with respect to Lebesgue measure, for $\nu$-almost all $x \in \mathbb{R}^p$. Define

$$Y_n = n^{-1/2} \sum_{i=1}^n X_i \quad \text{and} \quad \lambda(A) = E\left\{ N_p(0, \Sigma^*)(A) \right\}$$

for all $A \in \mathcal{B}$, where $\Sigma^* = \left( \int y_i y_j \mu^*(dy) : 1 \leq i, j \leq p \right)$.

Then,

$$\lim_{n \to \infty} \sup_{A \in \mathcal{B}} \left| P(Y_n \in A) - \lambda(A) \right| = 0.$$ 

**Proof.** Let $D$ be the collection of elements of $\mathcal{P}$ which are not singular with respect to Lebesgue measure. By Theorem 9, $P(\mu \in D) = P(\mu^* \in D) = 1$. Hence, conditionally on $\mu$, the sequence $X$ is i.i.d. and the common distribution $\mu$ belongs to $D$ a.s. Arguing as in Theorem 11, one also obtains $\int y_i \mu(dy) = 0$ and $\int \|y\|^2 \mu(dy) < \infty$ a.s. for all $i$. Thus, conditionally on $\mu$, $Y_n$ converges to $N_p(0, \Sigma)$ in total variation (see e.g. [2]) that is

$$\sup_{A \in \mathcal{B}} \left| P(Y_n \in A \mid \mu) - N_p(0, \Sigma)(A) \right| \overset{a.s.}{\to} 0.$$ 

Finally, $\Sigma \sim \Sigma^*$ implies $\lambda(\cdot) = E\left\{ N_p(0, \Sigma)(\cdot) \right\}$. Hence,

$$\sup_{A \in \mathcal{B}} \left| P(Y_n \in A \mid \mu) - \lambda(A) \right| = \sup_{A \in \mathcal{B}} \left| P(Y_n \in A) - E\left\{ N_p(0, \Sigma)(A) \right\} \right|$$

$$\leq E \left\{ \sup_{A \in \mathcal{B}} \left| P(Y_n \in A \mid \mu) - N_p(0, \Sigma)(A) \right| \right\} \to 0 \quad \text{as } n \to \infty. \hfill \Box$$

Our last result deals with the posterior distribution of $\mu$. We aim to find the conditional distribution of $\mu$ given $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. To this end, for each $n \geq 1$, we denote by $V^{(n)} = (V_j^{(n)} : j \geq 1)$ and $Z^{(n)} = (Z_j^{(n)} : j \geq 1)$ two sequences such that:

(i) $V^{(n)}$ and $Z^{(n)}$ are conditionally independent given $\mathcal{F}_n$;

(ii) $V^{(n)}$ has the stick-breaking distribution with parameter $n + \theta$ conditionally on $\mathcal{F}_n$. 


(iii) $Z^{(n)}$ is i.i.d. conditionally on $\mathcal{F}_n$ with
\[ P(Z_1^{(n)} \in \cdot \mid \mathcal{F}_n) = P(X_{n+1} \in \cdot \mid \mathcal{F}_n) = \frac{\theta \nu(\cdot) + \sum_{i=1}^{n} K(X_i)(\cdot)}{n + \theta} \quad \text{a.s.} \]

Moreover, we let
\[ \mu^*_n = \sum_j V_j^{(n)} K(Z_j^{(n)}). \]

**Theorem 13.** If $X \in \mathcal{L}$, then
\[ P(\mu \in C \mid \mathcal{F}_n) = P(\mu^*_n \in C \mid \mathcal{F}_n) \quad \text{a.s. for all } C \in \mathcal{C} \text{ and } n \geq 1. \]

We recall that, if $X \in \mathcal{L}_0$ and $X$ has parameter $\theta \nu$ (i.e., if $K = \delta$) then
\[ P(\mu \in C \mid \mathcal{F}_n) = D(\theta \nu + \sum_{i=1}^{n} \delta X_i)(C) = P(\mu^*_n \in C \mid \mathcal{F}_n) \quad \text{a.s.} \]

Hence, Theorem 13 extends to $\mathcal{L}$ the conjugacy property of $\mathcal{L}_0$. Such a property is clearly useful as regards Bayesian statistical inference. On one hand, the Bayesian analysis of $X \in \mathcal{L}$ is as simple as that of $X \in \mathcal{L}_0$. On the other hand, $\mathcal{L}$ is able to model much more situations than $\mathcal{L}_0$. As an obvious example, for $X \in \mathcal{L}$, it may be that $P(X_i = X_j) = 0$ if $i \neq j$. See e.g. Example 1 and Theorem 10.

Theorem 13 can be proved in various ways. We report here the simplest and most direct proof. Such a proof relies on Theorem 9 and the definition of $\mathcal{L}$ in terms of predictive distributions.

**Proof of Theorem 13.** Throughout this proof, if $X$ satisfies conditions (1)-(2), we say that $X \in \mathcal{L}$ and $X$ has parameter $(\theta \nu, K)$.

Fix $n \geq 1$ and define the sequence
\[ X^{(n)} = (X_i^{(n)} : i \geq 1) = (X_{n+i} : i \geq 1). \]

Define also the random measure
\[ J_n = \theta \nu + \sum_{i=1}^{n} K(X_i). \]

It suffices to show that, conditionally on $\mathcal{F}_n$, one obtains
\[ X^{(n)} \in \mathcal{L} \quad \text{and} \quad X^{(n)} \text{ has parameter } (J_n, K) \text{ a.s.} \tag{8} \]

In fact, under (8), Theorem 9 implies that $\mu \sim \mu^*_n$ conditionally on $\mathcal{F}_n$, namely
\[ P(\mu \in \cdot \mid \mathcal{F}_n) = P(\mu^*_n \in \cdot \mid \mathcal{F}_n) \quad \text{a.s.} \]

In turn, condition (8) follows directly from the definition. Define in fact
\[ P^{\mathcal{F}_n}(X^{(n)} \in B) = P(X^{(n)} \in B \mid \mathcal{F}_n) \quad \text{for all } B \in \mathcal{B}^\infty \text{ a.s.} \]

Then,
\[ P^{\mathcal{F}_n}(X_1^{(n)} \in \cdot) = P(X_{n+1} \in \cdot \mid \mathcal{F}_n) = \frac{\theta \nu(\cdot) + \sum_{i=1}^{n} K(X_i)(\cdot)}{n + \theta} = \frac{J_n}{n + \theta} \quad \text{a.s.} \]
and
\[ P^{F_n}\left(X_{m+1}^{(n)} \in \cdot \mid X_1^{(n)}, \ldots, X_m^{(n)}\right) = P\left(X_{n+m+1} \in \cdot \mid F_{n+m}\right) = \frac{\theta \nu + \sum_{i=1}^{n+m} K(X_i)}{n + m + \theta} = \frac{J_n + \sum_{i=1}^{m} K(X_i^{(n)})}{(n + \theta) + m} \quad \text{a.s. for all } m \geq 1. \]
This concludes the proof. \(\square\)

4. Open problems and examples

This section is split into two parts. First, we discuss some hints for future research and then we give three further examples.

- **An enlargement of \( \mathcal{L} \).** The class \( \mathcal{L} \) could be made larger. In this case, however, some of the basic properties of \( \mathcal{L}_0 \) would be lost. As an example, suppose that

\[ X_1 \sim \nu \quad \text{and} \quad P\left(X_{n+1} \in \cdot \mid F_n\right) = c_n \nu(\cdot) + \frac{1}{n} \frac{\theta}{(n + \theta) + m} \sum_{i=1}^{n} K(X_i)(\cdot) \quad \text{a.s.,} \]

where the kernel \( K \) satisfies condition (2) and \( c_n \in [0, 1] \) is a constant. To make \( X \) closer to \( \mathcal{L}_0 \), suppose also that \( \lim_n c_n = 0 \). Then, \( X \) is exchangeable and \( X \in \mathcal{L} \) provided \( c_n = \theta/(n + \theta) \). Furthermore, various properties of \( \mathcal{L}_0 \) are preserved, including \( \mu \sim \sum_j V_j K(Z_j) \) where \( (V_j) \) and \( (Z_j) \) are independent sequences and \( (Z_j) \) is i.i.d. with \( Z_1 \sim \nu \). Unlike Theorem 9, however, the probability distribution of \( (V_j) \) is unknown (to us). Similarly, we do not know whether some form of Theorem 13 is still valid.

- **A characterization of \( \mathcal{L} \).** Denoting by \( \mathcal{L}^* \) the class of kernel based Dirichlet sequences, it is tempting to conjecture that \( \mathcal{L} = \mathcal{L}^* \). Since \( \mathcal{L} \subset \mathcal{L}^* \), the question is whether there is an exchangeable sequence satisfying condition (1) but not condition (2). Lemma 6 and Theorem 7 may be useful to address this issue.

- **Self-similarity.** Suppose \( X \in \mathcal{L}_0 \) and take \( A \in \mathcal{B} \) such that \( 0 < \nu(A) < 1 \). Then, the distribution of the random probability measure \( \mu(\cdot \mid A) \) is still of the Dirichlet type with \( \nu \) and \( \theta \) replaced by \( \nu(\cdot \mid A) \) and \( \theta \nu(A) \), respectively. In addition, \( \mu(A), \mu(\cdot \mid A) \) and \( \mu(\cdot \mid A^c) \) are independent random elements; see [14, p. 61]. A question is whether this property of \( \mathcal{L}_0 \), called self-similarity, is still true for \( \mathcal{L} \). Suppose \( X \in \mathcal{L} \) and \( K \) is a r.c.d. for \( \nu \) given the sub-\( \sigma \)-field \( \mathcal{G} \subset \mathcal{B} \). If \( A \in \mathcal{G} \), then \( K(X_i)(A) = 1_A(X_i) \) a.s. for all \( i \). Based on this fact, \( \mu(\cdot \mid A) \) can be shown to have the same distribution as \( \mu \) with \( \nu \) and \( \theta \) replaced by \( \nu(\cdot \mid A) \) and \( \theta \nu(A) \). Hence, \( \mathcal{L} \) satisfies some form of self-similarity when \( A \in \mathcal{G} \). However, we do not know whether \( \mu(A), \mu(\cdot \mid A) \) and \( \mu(\cdot \mid A^c) \) are independent. Similarly, we do not know what happens if \( A \notin \mathcal{G} \).

- **Topological support.** The topological support of a Borel probability \( \lambda \) on a separable metric space, denoted \( \mathcal{S}(\lambda) \), is the smallest closed set \( A \) satisfying \( \lambda(A) = 1 \). Let \( \mathcal{P} \) be equipped with the topology of weak convergence, i.e., the weakest topology on \( \mathcal{P} \) which makes continuous the maps
\( p \mapsto \int f \, dp \) for all bounded continuous functions \( f : S \to \mathbb{R} \). Moreover, let \( \Pi(C) = P(\mu \in C), \quad C \in \mathcal{C} \), be the prior corresponding to \( \mu \). It is well known that
\[
\mathcal{S}(\Pi) = \{ p \in \mathcal{P} : S(p) \subset S(\nu) \}
\]
whenever \( X \in \mathcal{L}_0 \); see \([12]\) and \([21]\). As a consequence, \( \mathcal{S}(\Pi) = \mathcal{P} \) if \( \mathcal{S}(\nu) = S \). A (natural) question is whether, under some conditions on \( K \), this basic property of \( \mathcal{L}_0 \) is preserved by \( \mathcal{L} \). The next result provides a partial answer.

**Proposition 14.** If \( X \in \mathcal{L} \) and \( \mathcal{S}(\Pi) = \mathcal{P} \), then
\[
\nu\{ x \in S : K(x)(A) \leq u \} < 1
\]
for all \( u < 1 \) and all non-empty open sets \( A \subset S \).

**Proof.** First note that \( \mathcal{S}(\Pi) = \mathcal{P} \) if and only if \( \Pi(U) > 0 \) for each non-empty open set \( U \subset \mathcal{P} \). Having noted this fact, suppose \( \nu\{ x \in S : K(x)(A) \leq u \} = 1 \), for some \( u < 1 \) and some non-empty open set \( A \subset S \), and define
\[
U = \{ p \in \mathcal{P} : p(A) > u \}.
\]
Then, \( U \) is open and non-empty. Moreover, if \( V_j, Z_j \) and \( \mu^* \) are as in Section 3, one obtains \( K(Z_j)(A) \leq u \) for all \( j \) a.s. and
\[
\mu^*(A) = \sum_j V_j K(Z_j)(A) \leq u \sum_j V_j = u \quad \text{a.s.}
\]
By Theorem 9, it follows that
\[
\Pi(U) = P(\mu(A) > u) = P(\mu^*(A) > u) = 0.
\]
Hence, \( \mathcal{S}(\Pi) \) is a proper subset of \( \mathcal{P} \). \( \square \)

Possibly, some version of condition (9) suffices for \( \mathcal{S}(\Pi) = \mathcal{P} \). However, condition (9) alone suggests that \( \mathcal{S}(\Pi) \) is usually a proper subset of \( \mathcal{P} \). In Example 3, for instance, condition (9) fails (just take \( A = (0, \infty) \) and note that \( K(x)(A) \leq 1/2 \) for all \( x \)). Finally, we mention here a property of \( \mathcal{L}_0 \) which is preserved by \( \mathcal{L} \). If \( X \in \mathcal{L} \) and \( \mathcal{S}(\nu) = S \), the prior \( \Pi \) a.s. selects probability measures with full support, i.e.
\[
\Pi\{ p \in \mathcal{P} : S(p) = S \} = 1.
\]
We next turn to examples.

**Example 15.** (Example 1 continued). Let \( \mathcal{H} \subset \mathcal{B} \) be a countable partition of \( S \) such that \( \nu(H) > 0 \) for all \( H \in \mathcal{H} \). Then, \( K(x) = \nu[\cdot \mid H(x)] \) is a r.c.d. for \( \nu \) given \( \sigma(\mathcal{H}) \), where \( H(x) \) is the only \( H \in \mathcal{H} \) such that \( x \in H \). Therefore, \( X \in \mathcal{L} \) provided \( X_1 \sim \nu \) and
\[
P(X_{n+1} \in \cdot \mid \mathcal{F}_n) = \frac{\theta \nu(\cdot) + \sum_{i=1}^n \nu[\cdot \mid H(X_i)]}{n + \theta} \quad \text{a.s.}
\]
In this example, for each \( A \in \mathcal{B} \), one obtains
\[
\mu(A) = \lim_n P(X_{n+1} \in A \mid \mathcal{F}_n) = \sum_{H \in \mathcal{H}} \mu(H) \nu(A \mid H) \quad \text{a.s.}
\]
where $\mu(H) \overset{a.s.}{=} \lim_n (1/n) \sum_{i=1}^{n} 1_H(X_i)$. To grasp further information about $\mu$, define

$$b(H) = \sum_j V_j 1_H(Z_j), \quad H \in \mathcal{H},$$

where $(V_j)$ and $(Z_j)$ are independent, $(Z_j)$ is i.i.d. with $Z_1 \sim \nu$, and $(V_j)$ has the stick breaking distribution with parameter $\theta$. Then, Theorem 9 yields

$$\mu \sim \mu^* = \sum_{H \in \mathcal{H}} b(H) \nu(\cdot | H).$$

Therefore,

$$(\mu(H) : H \in \mathcal{H}) \sim (\mu^*(H) : H \in \mathcal{H}) = (b(H) : H \in \mathcal{H}).$$

To evaluate the posterior distribution of $\mu$, fix $n \geq 1$ and take two sequences $V^{(n)} = (V_j^{(n)} : j \geq 1)$ and $Z^{(n)} = (Z_j^{(n)} : j \geq 1)$ satisfying conditions (i)-(ii)-(iii). Recall that, by (iii), $Z^{(n)}$ is i.i.d. conditionally on $\mathcal{F}_n$ with

$$P(Z_1^{(n)} \in \cdot | \mathcal{F}_n) = P(X_{n+1} \in \cdot | \mathcal{F}_n) \quad \text{a.s.}$$

Define

$$b_n(H) = \sum_j V_j^{(n)} 1_H(Z_j^{(n)}) \quad \text{and} \quad \mu_n^* = \sum_{H \in \mathcal{H}} b_n(H) \nu(\cdot | H).$$

Then, Theorem 13 implies $\mu \sim \mu_n^*$ conditionally on $\mathcal{F}_n$.

**Example 16.** Let $\|\cdot\|$ be the Euclidean norm on $S = \mathbb{R}^p$. For $t \geq 0$, let $\mathcal{U}_t \in \mathcal{P}$ be uniform on the spherical surface $\{x : \|x\| = t\}$ (with $\mathcal{U}_0 = \delta_0$) and

$$\nu(A) = \int_0^\infty \mathcal{U}_t(A) e^{-t} dt \quad \text{for all } A \in \mathcal{B}.$$  

Then, $K(x) = \mathcal{U}_{\|x\|}$ is a r.c.d. for $\nu$ given $\sigma(\|\cdot\|)$. Hence, $X \in \mathcal{L}$ whenever $X_1 \sim \nu$ and

$$P(X_{n+1} \in \cdot | \mathcal{F}_n) = \frac{\theta \nu(\cdot) + \sum_{i=1}^{n} \mathcal{U}_{\|X_i\|}(\cdot)}{n + \theta} \quad \text{a.s.}$$

Theorem 11 applies to this example. To see this, first note that

$$\int \|x\|^2 \nu(dx) = \int_0^\infty \int \|x\|^2 \mathcal{U}_t(dx) e^{-t} dt = \int_0^\infty t^2 e^{-t} dt < \infty.$$  

Moreover, since $\mathcal{U}_t$ is invariant under rotations,

$$\int y_i \mathcal{U}_t(dy) = \int y_i y_j \mathcal{U}_t(dy) = 0 \quad \text{and} \quad \int y_i^2 \mathcal{U}_t(dy) = t^2 / p$$

for all $t$, all $i$ and all $j \neq i$. (Recall that $y_i$ denotes the $i$-th coordinate of a point $y \in \mathbb{R}^p$). Because of (10),

$$\int y_i K(x)(dy) = \int y_i \mathcal{U}_{\|x\|}(dy) = 0 \quad \text{for all } x \in \mathbb{R}^p \text{ and } i = 1, \ldots, p.$$  

Therefore, Theorem 11 yields

$$\frac{\sum_{i=1}^{n} X_i}{\sqrt{n}} \overset{\text{stably}}{\longrightarrow} \mathcal{N}_p(0, \Sigma).$$
where $\Sigma$ is the random covariance matrix with entries

$$
\sigma_{ij} = \int y_i y_j \mu(dy) = \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} \int y_i y_j U_{r\| \|}(dy)
$$

a.s.

It is even possible be more precise about $\Sigma$. In fact, using (10) again, one obtains $\sigma_{ij} = 0$ for $i \neq j$ and

$$
\sigma_{ii} = \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} \int y_i^2 U_{r\| \|}(dy) = \frac{1}{p} \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} \|X_r\|^2 = \frac{1}{p} \int \|x\|^2 \mu(dx)
$$

a.s.

Hence, if $I$ denotes the $p \times p$ identity matrix,

$$
\Sigma = \sigma_{11} I \quad \text{where} \quad \sigma_{11} = (1/p) \int \|x\|^2 \mu(dx).
$$

Two last remarks are in order. First, in the notation of Theorem 9,

$$
\int \|x\|^2 \mu(dx) \sim \int \|x\|^2 \mu^*(dx) = \sum_j V_j \|Z_j\|^2.
$$

Second, exploiting stable convergence and $\sigma_{11} > 0$ a.s., one also obtains

$$
\sqrt{p} \frac{\sum_{i=1}^{n} X_i}{\sqrt{n-1} \sum_{i=1}^{n} \|X_i\|^2} \xrightarrow{\text{stably}} \mathcal{N}(0, I).
$$

**Example 17.** Let $F$ be a countable class of measurable maps $f : S \to S$ and

$$
\mathcal{I} = \{ \lambda \in \mathcal{P} : \lambda = \lambda \circ f^{-1} \text{ for each } f \in F \}
$$

the set of $F$-invariant probability measures. Let

$$
\mathcal{G} = \{ A \in \mathcal{B} : f^{-1}(A) = A \text{ for all } f \in F \}
$$

be the sub-$\sigma$-field of $F$-invariant measurable sets. In this example, we assume that $\nu \in \mathcal{I}$ and conditions (1)-(2) hold with $\mathcal{G}$ as above.

Under these conditions, it is not hard to see that $K(x) \in \mathcal{I}$ for $\nu$-almost all $x \in S$; see e.g. [20]. Hence, $P(X_{n+1} \in A \mid F_n) \in \mathcal{I}$ a.s. which in turn implies

$$
\mu(f^{-1}A) \overset{a.s.}{=} \lim_n P(f(X_{n+1}) \in A \mid F_n) \overset{a.s.}{=} \lim_n P(X_{n+1} \in A \mid F_n) \overset{a.s.}{=} \mu(A)
$$

for fixed $A \in \mathcal{B}$ and $f \in F$. Since $F$ is countable and $\mathcal{B}$ countably generated, one finally obtains

$$
P(\mu \in \mathcal{I}) = 1.
$$

This fact is meaningful from the Bayesian point of view. It means that the prior corresponding to $\mu$ (namely, $\Pi(C) = P(\mu \in C)$ for all $C \in \mathcal{C}$) selects $F$-invariant laws a.s. Such priors are actually useful in some practical problems; see e.g. [9] and [18].

Example 3 is a special case of the previous choice of $\mathcal{G}$. Another example, borrowed from [7, Ex. 12], is $S = \mathbb{R}^d$ and $F$ the class of all permutations of $\mathbb{R}^d$. In this case, $\mathcal{I}$ is the set of exchangeable probabilities on the Borel sets of $\mathbb{R}^d$. Moreover, if $\nu$ is exchangeable, $K$ can be written as

$$
K(x) = \frac{\sum_{\pi \in \mathcal{E}} \delta_{\pi(x)}}{d!} \quad \text{for all } x \in \mathbb{R}^d.
$$
A last remark is in order.

**Claim:** If \( A_1, \ldots, A_k \) is a partition of \( S \) such that \( A_i \in \mathcal{G} \) for all \( i \), then the \( k \)-dimensional vector \( (\mu(A_1), \ldots, \mu(A_k)) \) has Dirichlet distribution with parameters \( \theta \nu(A_1), \ldots, \theta \nu(A_k) \).

To prove the Claim, because of Theorem 9, it suffices to show that \( (\mu^*(A_1), \ldots, \mu^*(A_k)) \) has the desired distribution. In addition, \( K(x)(A_i) = 1_{A_i}(x) = \delta_x(A_i) \), for \( \nu \)-almost all \( x \in S \), since \( A_i \in \mathcal{G} \) and \( K \) is a r.c.d. for \( \nu \) given \( \mathcal{G} \). Therefore,

\[
\mu^*(A_i) = \sum_j V_j K(Z_j)(A_i) \overset{a.s.}{=} \sum_j V_j \delta_{Z_j}(A_i) \quad \text{for all } i,
\]

and this implies that \( (\mu^*(A_1), \ldots, \mu^*(A_k)) \) has Dirichlet distribution with parameters \( \theta \nu(A_1), \ldots, \theta \nu(A_k) \).

In view of the Claim, \( \mu \) is a Dirichlet invariant process in the sense of Definition 2 of [9]. Thus, arguing as above, a large class of such processes can be easily obtained. Note also that, unlike [9], \( F \) is not necessarily a group.

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