Spectral response of Cantor multilayers made of materials with negative refractive index

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ABSTRACT—Whereas Cantor multilayers made of an isotropic dielectric–magnetic material with positive refractive index will show power–law characteristics, low–order Cantor multilayers made of materials with negative refractive index will not exhibit the power–law nature. A reason for this anomalous behavior is presented.

Key words: Cantor multilayers; filters, fractals, left–handed materials, negative index of refraction, negative phase velocity

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1 Introduction

This letter addresses the incorporation of isotropic materials with negative refractive index \( \frac{1}{3}, \frac{2}{3} \) in fractal filters inspired by Cantor dusts \( \frac{1}{3}, \frac{2}{3} \).

The emergence of Cantor dusts, bars and cakes during the late 19th century has been described at some length by Mandelbrot \( \frac{1}{3}, \frac{2}{3} \). Briefly, the simplest Cantor dust is formed by dividing the closed interval \([0, 1]\) into 3 pieces and removing the center open piece \((1/3, 2/3)\), repeating the trifurcation–and–removal process on the remaining intervals \([0, 1/3]\) and \([2/3, 1]\), and continuing in that fashion \textit{ad infinitum}. The fractal

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(similarity) dimension of the resulting dust is \( \log 2 / \log 3 \sim 0.6309 \). Similar structures in \( p \)-dimensional space, \( (p = 1, 2, \ldots) \), can be constructed \( \text{via} \) spatial convolution \([3]\). In particular, the constructs called Cantor bars appear to have captured the imagination of optical–filter researchers, as recounted recently by Lehman \([3]\), because of their putatively self–similar response properties in the frequency domain \([3]\).

The materials of choice for optical Cantor filters are isotropic dielectric with relative permittivity \( \epsilon_r \). Although \( \epsilon_r \) is a complex–valued function of frequency, the usual practice in optics is to ignore dissipation by setting \( \text{Im} [\epsilon_r] = 0 \). In the area of fractal optics, with emphasis still on understanding basic interactions in nonperiodic multilayers, dispersion is also ignored \([3]–[8]\). The structural self–similarity of the Cantor bars is then expected to result in the self–similarity of the spectral reflectance/transmittance responses of optical Cantor filters to normally incident light \([3]\). Truly, physically realizable Cantor filters are not actually fractal but pre–fractal instead \([9]\) — so that the spectral self–similarity can only be approximate \([10]\).

On examining the available literature, two questions arise. First, will the situation change for Cantor filters made of isotropic dielectric–magnetic materials (with relative permeability denoted by \( \mu_r > 1 \))? Second, will the situation change if both \( \epsilon_r < 0 \) and \( \mu_r < 0 \)?

The second question arose because of the supposed verification of the existence of negative refractive index (NRI) by Shelby et al. \([1]\) last year. Experiments performed on certain composite materials with oriented microstructure suggested that these materials are endowed with negligible dissipation as well as NRI in some appreciably wide frequency band in the centimeter–wave regime. Also called left–handed materials by some researchers (despite possessing no handedness), in NRI materials the phase velocity is pointed opposite to the direction of energy flow (and attenuation) \([2], [11]\). Although the extant experimental results are not perfect \([12], [13]\), the essential conclusion of the existence of NRIs appears undeniable. As NRIs can potentially lead to exciting new technologies \([14]\), theoretical consideration is warranted.
In this letter, we answer the two questions posed earlier in a unified way. Section 2 is devoted to the theory of reflection and transmission of normally incident plane waves by Cantor multilayers. Numerical results are presented and discussed in Section 3.

2 Theory

A Cantor multilayer is constructed sequentially as follows: Take a layer of thickness \( \ell_0 \) made of a certain material with \( \epsilon_r \) and \( \mu_r \) as its constitutive parameters. Call this layer a multilayer of order \( N = 0 \). Next, cascade two multilayers of order \( N = 0 \) inserting a space of thickness \( \ell_0/f \), \( f \geq 1 \), in between. Call this a multilayer of order \( N = 1 \). Its total thickness \( \ell_1 = (2 + 1/f)\ell_0 \). Continue in this manner. Thus, a multilayer of order \( N + 1 \) is formed by inserting a space of thickness \( \ell_N/f \) between two multilayers of order \( N \). The thickness of a multilayer of order \( N + 1 \) is then \( \ell_{N+1} = (2 + 1/f)\ell_N = (2 + 1/f)^{N+1} \ell_0 \).

The fractal dimension of the multilayer is given by

\[
D = \frac{\log 2}{\log(2 + 1/f)} ,
\]

which concept is applicable strictly in the limit \( N \to \infty \).

Let a Cantor multilayer of order \( N \) occupy the space \( 0 \leq z \leq \ell_N \). Suppose a plane wave is normally incident on this multilayer from the vacuous half-space \( z \leq 0 \), with \( \lambda_0 \) denoting its wavelength. Therefore, a reflected plane wave also exists in the same half-space. Furthermore, a transmitted plane wave is engendered in the vacuous half-space \( z \geq \ell_N \). The corresponding electric field phasors are given by

\[
\vec{E}(z) = u_x \begin{cases} 
\exp(ik_0z) + \rho_N \exp(-ik_0z), & z \leq 0 \\
\tau_N \exp[ik_0(z - \ell_N)], & z \geq \ell_N 
\end{cases}
\]

where \( k_0 = 2\pi/\lambda_0 \) is the wavenumber in vacuum; \( \rho_N \) and \( \tau_N \) are the reflection and the transmission coefficients, respectively, both complex-valued; and \((u_x, u_y, u_z)\) is the triad of cartesian unit vectors. An \( \exp(-i\omega t) \) time-dependence is implicit, where \( \omega = k_0/(\epsilon_0\mu_0)^{1/2} \) is the angular frequency, while \( \epsilon_0 \) and \( \mu_0 \) are the permittivity and the permeability of vacuum, respectively.
The coefficients $\rho_N$ and $\tau_N$ can be easily determined using a $2\times2$ matrix algebra \[15\]. After defining the two matrixes

$$A = \begin{bmatrix} 0 & \mu_0 \\ \epsilon_0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & \mu_0\mu_r \\ \epsilon_0\epsilon_r & 0 \end{bmatrix}, \quad (3)$$

the matrices $M_p$, $0 \leq p \leq N$, are iteratively computed as

$$M_{p+1} = M_p \cdot e^{i\omega(\ell_p/f)} A \cdot M_p, \quad 0 \leq p \leq N - 1, \quad (4)$$

beginning with

$$M_0 = e^{i\omega\ell_0} B. \quad (5)$$

The boundary value problem for the electromagnetic fields then involves the solution of the equation

$$\tau_N \begin{bmatrix} 1 \\ \eta_0^{-1} \end{bmatrix} = M_N \cdot \begin{bmatrix} (1 + \rho_N) \\ \eta_0^{-1}(1 - \rho_N) \end{bmatrix}, \quad (6)$$

where $\eta_0 = (\mu_0/\epsilon_0)^{1/2}$ is the intrinsic impedance of vacuum. The principle of conservation of energy entails that $|\rho_N|^2 + |\tau_N|^2 \leq 1$, with the equality coming in when the multilayer is made of a non–dissipative material.

3 Numerical results and discussion

Following normal practice, we implemented the foregoing equations to compute $\rho_N$ and $\tau_N$ for non–dissipative and non–dispersive materials. We varied the quantity $\zeta = k_0\ell_0$ for various values of $N$, while keeping $\epsilon_r$ and $\mu_r$ fixed.

The spectrums of $|\rho_N|^2$ and $|\tau_N|^2$ turned to be identical to the ones reported in the literature \[3\], \[4\] for optical Cantor filters (i.e., with $\epsilon_r > 1$ and $\mu_r = 1$). Those for $\{\epsilon_r > 1, \mu_r > 1\}$ and $\{\epsilon_r < 0, \mu_r < 0\}$ turned to be qualitatively similar, and therefore do not need reproduction here.

As $\zeta$ increases from zero, the fundamental layer thickness $\ell_0$ becomes an increasingly significant fraction of the wavelength $\lambda_0 = 2\pi/k_0$, and eventually surpasses $\lambda_0$. In
other words, layers are electrically thin for small $\zeta$, and an increase in $\zeta$ amounts to magnification. Therefore we evaluated the value $\tilde{\zeta}_N$ of $\zeta$ at which the first minimum of $|\tau_N|$ occurs as $\zeta$ increases from zero, thereby reckoning $\tilde{\zeta}_N$ as a reasonable parameter containing structural information on the chosen multilayers. If indeed the structural self–similarity of Cantor multilayers would result in their spectral self–similarity, we expect the relationship

$$\tilde{\zeta}_N = 2^{-N/D} \tilde{\zeta}_0$$

(7)

to emerge from our numerical investigations.

Figures 1 and 2 contain plots of $\log \tilde{\zeta}_N$ versus $N$ for Cantor multilayers made with positive refractive index (PRI) materials ($\{\epsilon_r = 3, \mu_r = 1.02\}$ or $\{\epsilon_r = 4, \mu_r = 1.02\}$), and for Cantor multilayers made with their NRI analogs ($\{\epsilon_r = -3, \mu_r = -1.02\}$ or $\{\epsilon_r = -4, \mu_r = -1.02\}$). The factor $f = 1$ for Figure 1, and $f = 2$ for Figure 2.

Two conclusions can be immediately drawn from these two figures as follows:

A. The relationship $\tilde{\zeta}_N = 2^{-N/D_{PRI}} \tilde{\zeta}_0$ satisfied by Cantor multilayers with PRI materials is a power law with $D_{PRI} > D$, and could be fractalesque [16, 17].

B. The data for Cantor multilayers with NRI materials indicates two different regimes, one for small $N$ and the other for large $N$, the second regime characterized by a power law.

The foregoing conclusions suggest that the effect of NRI materials on electromagnetic fields must be substantively different from that of PRI materials, for the anomalous first regime to arise for Cantor multilayers with NRI materials. Furthermore, in the present context, the difference must be evident definitely for order $N = 0$.

Hence, we analyzed the planewave response of a single layer to obtain

$$\rho_0 = \frac{(\eta_r^2 - 1) \sin \beta}{(\eta_r^2 + 1) \sin \beta + 2i\eta_r \cos \beta},$$

(8)

and

$$\tau_0 = \frac{2i\eta_r}{(\eta_r^2 + 1) \sin \beta + 2i\eta_r \cos \beta}.$$  

(9)
Here, the relative impedance $\eta_r = \pm \sqrt{\mu_r/\varepsilon_r}$ must be positive real, while the sign of $\beta = k_0\ell_0\sqrt{\mu_r\varepsilon_r}$ has to be positive/negative for PRI/NRI materials [2, 18]. Denoting the phase of a complex number $\xi$ by $\angle\xi$, we conclude from the foregoing equations that

$$\{\varepsilon_r \rightarrow -\varepsilon_r, \mu_r \rightarrow -\mu_r\} \Rightarrow \{\rho_0\rightarrow |\rho_0|, |\tau_0| \rightarrow |\tau_0|, -\angle\rho_0 \rightarrow -\angle\rho_0, -\angle\tau_0 \rightarrow -\angle\tau_0\}. \quad (10)$$

In light of the relationship (10), let us compare a PRI layer and a NRI layer — labeled $a$ and $b$, respectively — such that $\mu_{ra} = -\mu_{rb} > 0$ and $\varepsilon_{ra} = -\varepsilon_{rb} > 0$, while the wavenumber $k_0$ is fixed. If the thicknesses of the two layers are such that the sum $\beta_a + |\beta_b|$ is an integral multiple of $2\pi$, then (8) and (9) yield $\rho_{0a} = \rho_{0b}$ and $\tau_{0a} = \tau_{0b}$. Thus, a PRI layer of a certain thickness is equivalent to a NRI layer of different thickness, in terms of the complex–valued reflection and transmission coefficients at a fixed wavelength. But the thickness of the equivalent NRI layer is wavelength–dependent — which implies that a PRI Cantor multilayer is equivalent at different wavelengths to different NRI Cantor multilayers. Not surprisingly therefore, the spectral characteristics of a PRI and a NRI Cantor multilayers with the same $\ell_0$ are not isomorphic.

The difference is very noticeable for small $N$ in Figures 1 and 2. As $N$ increases, the value of $\tilde{\zeta}_N$ decreases for both PRI and NRI multilayers — in other words, the fundamental layer of thickness $\ell_0$ becomes electrically thinner at the first transmittance minimum and, therefore, a weaker reflector as well as a stronger transmitter. Structural characteristics then dominate over the consequences of (10), because $|\rho_0| \simeq 0$ and $|\tau_0| \simeq 1$. As the difference between PRI and NRI multilayers lessens with increasing $N$, the latter also begin to evince power–law characteristics.

The crossover between the anomalous and the power–law regimes for NRI multilayers takes place at a higher value of $N$ as $f$ increases. This general trend is indicated by Figures 1 and 2 as well as calculations for other values of $f$.

To conclude, we have shown that the planewave reflection and transmission spectra of a Cantor multilayer made of an isotropic dielectric–magnetic material with positive refractive index shows power–law characteristics which indicate spectral self–similarity. However, if the same multilayer were to be made of a material with negative
refractive index, then the power–law nature is not going to be evident when the interaction between the material layers and the interleaving vacuous spaces is substantial (the small–$N$ regime). The existence of this anomalous regime can be attributed to the reflection/transmission phase reversal of a NRI layer in relation to its PRI analog. If that interaction is insubstantial (the large–$N$ regime), the structural features would dominate the constitutive features, and the power–law characteristics would be evident also for the NRI Cantor multilayer.

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Figure 1: Calculated values of $\log \tilde{\zeta}_N$ for Cantor multilayers of orders $N$ when $f = 1$. Dotted lines join points for PRI Cantor multilayers, dashed-dotted lines for NRI Cantor multilayers, and solid lines for $\tilde{\zeta}_N = 2^{-N/D} \tilde{\zeta}_0$. (a) $\epsilon_r = \pm 3$ and $\mu_r = \pm 1.02$; (b) $\epsilon_r = \pm 4$ and $\mu_r = \pm 1.02$. 
Figure 2: Same as Figure 1, but for $f = 2$. 