Ground state separability and criticality in interacting many-particle systems

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We analyze exact ground state (GS) separability in general $N$ particle systems with two-site couplings. General necessary and sufficient conditions for full separability, in the form of one and two-site eigenvalue equations, are first derived. The formalism is then applied to a class of interacting systems with $SU(n)$ symmetry where each constituent has access to $n$ local levels, and where the total number parity of each level is preserved. Explicit factorization conditions for parity-breaking GSs are obtained, which generalize those for $XYZ$ spin systems and correspond to a fundamental GS multilevel parity transition where the lowest $2^n-1$ energy levels cross. We also identify a multicritical factorization point with exceptional high degeneracy proportional to $N^{n-1}$, arising when the full occupation number of each level is preserved, in which any uniform product state is an exact GS. Critical entanglement properties (like full range separation independent pairwise entanglement) are shown to emerge in the immediate vicinity of factorization. Illustrative examples are provided.

I. INTRODUCTION

The ground state (GS) of strongly interacting spin systems, while normally entangled [1–3], can exhibit the remarkable phenomenon of factorization when a suitable magnetic field is applied [4–12]. This means that for such field, the spin system admits a completely separable exact GS, i.e. a product of single spin states, despite the presence of nonnegligible couplings between the spins and the finite value of the applied field. Moreover, such product state is not necessarily trivial, in the sense that it may break fundamental symmetries of the Hamiltonian. In this case factorization signals in finite systems a special critical point where two or more levels with definite symmetry cross and the GS becomes degenerate [9–11, 13–14], allowing for such symmetry breaking exact eigenstates. The exact GS then typically undergoes in this case a transition between states with distinct symmetry as the factorization point is traversed, leading to visible effects in system observables [9–10, 13–14]. Furthermore, critical entanglement properties emerge in the immediate vicinity [7–9, 13–14], stemming ultimately from the product nature of the closely lying eigenstate.

Most studies of GS factorization have so far been restricted to interacting spin systems (see also [15–18]), where factorization conditions remain analytically manageable due to the small number of parameters required to specify an individual spin state. The main aim of this work is to investigate exact GS factorization in more general interacting systems, i.e., beyond the standard $SU(2)$ spin scenario, where already the characterization of a single component state is more complex. With this goal, we first derive the necessary and sufficient conditions for factorization in the form of eigenvalue equations, either for effective pair Hamiltonians or for the mean field Hamiltonian and residual couplings.

We then apply the formalism to a general $N$-component interacting system with $SU(n)$ symmetry, in which each constituent has $n$ accessible local levels. The model reduces in the $n=2$ case to an $XYZ$ spin system in an applied transverse field, sharing with the latter the basic level number parity symmetry. For full range couplings it comprises schematic $SU(n)$ models employed in nuclear physics for describing collective excitations [19–21], while for first neighbor couplings and special choices of parameters it becomes $SU(n)$-type Heisenberg models [22]. The study of interacting many body systems with global $SU(n)$ symmetry has aroused great interest in recent years, becoming an active research topic that links the fields of condensed matter and atomic, molecular and optical physics [23–28]. Systems possessing high dimensional symmetry can unveil exotic many body physics and are suitable for describing a wide range of non-trivial phenomena. The paradigmatic $SU(n)$ Heisenberg model [22], first employed in solid state physics in connection with the integer quantum Hall effect [29–30], played also an important role in identifying unconventional magnetic states and phases [25–31, 38]. Interest on the subject has been stimulated by the unprecedented advances in quantum control techniques that offer the possibility of realizing strongly interacting many body systems with high symmetry in alkaline earth atomic gases in optical lattices [21, 25, 28]. These platforms have also received attention in relation with high precision atomic clocks [39] and quantum computation [40].

The factorization formalism is presented in section [11] and its application to the $SU(n)$ model is described in [13]. Explicit equations for the existence of uniform parity-breaking factorized GS are determined, and shown to correspond to a multilevel parity transition occurring for any size and range where the GS becomes $2^n-1$-fold degenerate (if $N \geq n-1$). A critical factorization point with exceptionally high degeneracy (which increases with size $N$) is also identified in systems with full level number symmetry, at which any uniform separable state is an exact GS. Entanglement properties in the vicinity of factorization together with signatures of factorization in small systems are as well discussed. Conclusions are drawn in Appendices A and B. Appendices discuss further details including the mean field approximation in the model, which admits an analytic solution in the uniform case for arbitrary $n$.
II. FORMALISM

A. General factorization conditions

We consider a system described by a Hilbert space $\mathcal{H} = \bigotimes_{p=1}^{N} \mathcal{H}_p$, such that it can be seen as a composite of $N$ subsystems with Hilbert spaces $\mathcal{H}_p$. In this scenario we assume a general Hamiltonian containing one-site terms $h_p$ plus two-site interactions $V_{pq}$:

$$H = \sum_p h_p + \frac{1}{2} \sum_{p \neq q} V_{pq},$$

$$h_p = \sum_{\mu} b^p_{\mu} \phi^p_{\mu}, \quad V_{pq} = \sum_{\mu,\nu} J^p_{\mu\nu} \phi^p_{\mu} \phi^q_{\nu},$$

where $\{\phi^p_{\mu}\}$ denotes a complete set of linearly independent operators over $\mathcal{H}_p$, and $J^p_{\mu\nu} = J^q_{\mu\nu}$ are the coupling strengths of the interaction between sites $p$ and $q$. In particular, any spin array with two-spin interactions in a general applied magnetic field fits into this form. We use the notation $o^p_{\mu} \equiv \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes o^p_{\mu} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$ when operators are applied to global states.

We are here interested in the conditions which ensure that a completely separable state $|\Psi\rangle = \bigotimes_p |\psi_p\rangle = |\psi_1, \ldots, \psi_N\rangle$, possibly breaking some fundamental symmetry of $H$, is an exact eigenstate of $H$:

$$H|\Psi\rangle = E|\Psi\rangle. \quad (4)$$

When applied to $|\Psi\rangle$, $H$ can just connect it with itself and with superpositions of one- and two-site “excitations”:

$$|\Phi_p\rangle = |\psi_1, \ldots, \phi_p, \ldots, \psi_N\rangle, \quad (5)$$

$$|\Phi_{pq}\rangle = |\psi_1, \ldots, \phi_p, \ldots, \phi_q, \ldots, \psi_N\rangle, \quad (6)$$

where $\langle \phi_p | \psi_q \rangle = 0$. Then Eq. (4) implies the necessary and sufficient conditions

$$\langle \Phi_p | H | \Psi \rangle = 0, \quad p = 1, \ldots, N, \quad (7)$$

$$\langle \Phi_{pq} | H | \Psi \rangle = 0, \quad 1 \leq p < q \leq N, \quad (8)$$

to be satisfied $\forall |\phi_p\rangle, |\phi_q\rangle$ orthogonal to $|\psi_p\rangle, |\psi_q\rangle$ respectively. Since

$$\langle \Phi_p | H | \Psi \rangle = \langle \phi_p | h_p | \psi_p \rangle, \quad \hat{h}_p = h_p + \sum_{q \neq p} v^{(q)}_p, \quad (9)$$

where $\hat{h}_p$ is the local MF Hamiltonian at site $p$ and

$$v^{(q)}_p = \langle \psi_q | V_{pq} | \psi_q \rangle = \sum_{\mu,\nu} J^p_{\mu\nu} \langle o^p_{\mu} \rangle o^q_{\nu}, \quad (10)$$

the average potential at $p$ due to the coupling with site $q$ ($\langle o^p_{\mu} \rangle = \langle \psi_q | o^p_{\mu} | \psi_q \rangle$), Eqs. (7) imply $\langle \phi_p | h_p | \psi_p \rangle = 0 \forall |\phi_p\rangle$ orthogonal to $|\psi_p\rangle$ and hence the eigenvalue equations

$$\hat{h}_p |\psi_p\rangle = \lambda_p |\psi_p\rangle, \quad p = 1, \ldots, N. \quad (11)$$

As expected, each local state $|\psi_p\rangle$ in $|\Psi\rangle$ should be an eigenstate of the local MF Hamiltonian $\hat{h}_p$ determined by the same $|\Psi\rangle$, implying self-consistency.

It is now convenient to rewrite $H$ as

$$H = \sum_p \hat{h}_p + \frac{1}{2} \sum_{p \neq q} \hat{V}_{pq}, \quad (12)$$

where $\hat{V}_{pq} = V_{pq} - v^{(q)}_p - v^{(p)}_q$ is a residual coupling satisfying $\langle \Phi_p | V_{pq} | \Psi \rangle = \langle \Phi_q | \hat{V}_{pq} | \Psi \rangle = 0$. Then

$$\langle \Phi_{pq} | H | \Psi \rangle = \langle \phi_p, \phi_q | \hat{V}_{pq} | \psi_p, \psi_q \rangle, \quad (13)$$

and Eqs. (8) together with previous property imply that $|\Psi\rangle$ should be an eigenstate of all $\hat{V}_{pq}$:

$$\hat{V}_{pq} |\psi_p, \psi_q\rangle = \lambda_{pq} |\psi_p, \psi_q\rangle, \quad 1 \leq p < q \leq N, \quad (14)$$

with $\lambda_{pq} = -\langle \hat{V}_{pq} \rangle$. As $\lambda_p = \langle h_p \rangle + \frac{1}{2} \sum_{q \neq p} \lambda_{pq} = \langle H \rangle$. Therefore, we can state the following theorem:

The product state $|\Psi\rangle$ is an exact eigenstate of the Hamiltonian $\hat{H}$ iff $|\Psi\rangle$ is a simultaneous eigenstate of all one-site MF Hamiltonians $\hat{h}_p$ and all residual couplings $\hat{V}_{pq}$.

Once Eqs. (11) and (14) are fulfilled, additional single site terms having $|\psi_p\rangle$ as GS ($\Delta h_p |\psi_p\rangle = \Delta \lambda_p |\psi_p\rangle$) can be added to $H$ without affecting the product eigenstate. They can be used to remove the eventual degeneracy of $|\Psi\rangle$ and bring down its energy ($E \rightarrow E + \sum_p \Delta \lambda_p$), making it a nondegenerate GS for sufficiently large $\Delta \lambda_p < 0 \forall p$.

B. Pair equations and the uniform case

Eqs. (11) and (14) imply that $H$ can be written as a sum of pair Hamiltonians $H_{pq} = h_{pq} (p \neq q)$ having the pair product state $|\psi_p, \psi_q\rangle$ as eigenstate:

$$H = \frac{1}{2} \sum_{p \neq q} H_{pq}, \quad (15)$$

$$H_{pq} |\psi_p, \psi_q\rangle = E_{pq} |\psi_p, \psi_q\rangle, \quad 1 \leq p < q \leq N. \quad (16)$$

For instance, we can set $H_{pq} = r_{pq} (\hat{h}_p + \hat{h}_q) + \hat{V}_{pq}$, with $r_{pq} = r_{qp}$ numbers satisfying $\sum_q r_{pq} = 1 \forall p$ (and $r_{pp} = 0$) in which case $E_{pq} = r_{pq} (\lambda_p + \lambda_q) + \lambda_{pq}$. The converse is trivially true: Eqs. (15)–(16) imply Eq. (4) for the state (3), with

$$E = \frac{1}{2} \sum_{p \neq q} E_{pq}. \quad (17)$$

Moreover, if $|\psi_p, \psi_q\rangle$ is a GS of $H_{pq} \forall p \neq q$, $|\Psi\rangle$ will clearly be a GS of $H$, since it will minimize each average $\langle H_{pq} \rangle$ in (15), and hence the full average $\langle H \rangle$.

The pair Hamiltonians will have the general form

$$H_{pq} = h_{pq}^{(a)} + h_{pq}^{(r)} + V_{pq}, \quad (18)$$
with $\sum_{q \neq p} h_p^{(q)} = h_p$. Then, when multiplied by $\langle \psi_q |$, Eq. (16) leads to $(h_p^{(q)} + v_p^{(q)}) | \psi_p \rangle = \lambda_p^{(q)} | \psi_p \rangle$, with $\lambda_p^{(q)} = E_{pq} - h_p^{(q)}$, implying Eq. (11) when summed over $q$ with $\lambda_p = \sum_q \lambda_p^{(q)}$ and also Eq. (14) (with $\lambda_{pq} = E_{pq} - \lambda_p^{(q)} - \lambda_p^{(q)}$). Eqs. (15–16) and (11–14) are then equivalent.

By expanding the local states $| \psi_p \rangle$ in an orthogonal basis, $| \psi_p \rangle = \sum_i f_i^{\dagger} | i_p \rangle$ with $f_i^{\dagger} = \langle i_p | \psi_p \rangle$, $\sum_i | f_i^{\dagger} |^2 = 1$, Eq. (16) becomes, explicitly,

$$\sum_{i,j} [\delta_{kl} (i_p h_p^{(q)} | j_p) + \delta_{ij} (k_p h_q^{(p)} | l_q) + (i_p k_p | V_{pq} | j_q l_q)] f_j^{\dagger} f_k^{\dagger} = E_{pq} f_i^{\dagger} f_k^{\dagger},$$

(19)
to be fulfilled $\forall i, k$. For $\dim H_{pq} = n_{pq} \geq 2$ and general couplings, Eq. (10) imposes $m = n_{pq} - 1$ complex equations to be satisfied by product states $| \psi_p, \psi_q \rangle$ having $l = n_p + n_q - 2 < m$ complex parameters $f_j^{\dagger}$, $f_k^{\dagger}$, hence entailing restrictions on the feasible coupling strengths $J_{pq}^{\mu}$ and “fields” $b_{pq}^{\mu}$. Factorization will then take place at special “points” or “curves” in parameter space. In particular, if $H_{pq}$ is real in the previous pair product basis, one could always satisfy (19) by adjusting the diagonal elements $(i_p k_p | V_{pq} | i_q l_q)$.

A simple realization of Eqs. (15–16) is the case of a uniform system where all local Hilbert spaces $H_p$ and operators $\sigma_{ij}^\mu$ are identical, while couplings between sites are all proportional (or zero) such that $J_{pq}^{\mu} = r_{pq} J_{\mu\nu}$ and

$$V_{pq} = r_{pq} V, \quad V = \sum_{\mu,\nu} J_{\mu\nu} \sigma_{\mu \otimes \nu},$$

$$h_p^{(q)} = r_{pq} h, \quad h = \sum_{k} h_{k}^\mu \sigma_{k},$$

(20)
(21)
in (13), with $V$ and $h$ independent of $p$ and $q$ and $(J_{\mu\nu} = J_{\nu\mu})$. Here $r_{pq}$ determines the relative strength of the coupling between $p$ and $q$, and hence the range of the interaction. Eqs. (20)–(21) imply $h_p = r_{pq} h$, with $r_p = \sum_{q \neq p} r_{pq}$, and $H_p = r_{pq} h (h \otimes 1 + 1 \otimes h + V)$, such that all $H_{pq}$ become proportional.

Then a uniform product eigenstate with $| \psi_p \rangle = | \psi \rangle$ $\forall p$ may be feasible for special couplings, as all pair equations (16) reduce in this case to the single equation

$$(h \otimes 1 + 1 \otimes h + V) | \psi, \psi \rangle = E_2 | \psi, \psi \rangle,$$

(22)
after setting $E_{pq} = r_{pq} E_2$. The total energy (17) becomes

$$E = \frac{1}{2} E_2 \sum_p r_p.$$  

(23)

Here $E_2$ represents a common pair energy while $r_p$ a sort of coordination number for site $p$. In uniform cyclic systems $r_p$ is constant $\forall p$ and $E = r_p N_2 E_2$, while in open systems $r_p$ is typically smaller at the borders due to the smaller number of coupled neighbors, entailing edge corrections in $h_p = r_p h$. We will normalize the factors $r_p$ such that $r_p = 1$ for inner “bulk” sites (e.g. $r_p = \frac{1}{2} \delta_{p,N-1}$ for first neighbor couplings in a linear chain, $r_p = \frac{1}{N-1}$ for fully and equally connected systems).

### C. Formulation for fermion and boson systems

Previous equations admit a second quantized formulation for systems of fermions or bosons. For $N$ of such particles at $N$ distinct (orthogonal) sites labelled by $p$, having each $n_p = \dim H_p$ accessible local states labelled by $i$, we can define the corresponding creation and annihilation operators $c_{p_1}^{\dagger}, c_{p_1}$ satisfying

$$[c_{p_1}, c_{q_2}^{\dagger}] = \delta_{pq} \delta_{ij}, \quad [c_{p_1}^{\dagger}, c_{q_2}] = [c_{p_1}, c_{q_2}] = 0,$$

(24)
for fermions $(\pm)$ or bosons $(-)$ $| [a, b] \pm = ab \pm ba \rangle$. Setting $\sigma_{pq}^\mu = g_{pq}^{\mu} = | i_p / j_p \rangle$ and replacing it with $c_{p_1}^{\dagger} c_{p_1}$, we can express the equivalent of Hamiltonian (1) as

$$H = \sum_{p,i,j} b_{ij}^{p} c_{p_1}^{\dagger} c_{p_2} + \frac{1}{2} \sum_{p \neq q, i,j,k,l} J_{ijkl}^{pq} c_{p_1}^{\dagger} c_{p_2} c_{q_1} c_{q_2},$$

(25)
with $b_{ij}^{p} = \delta_{ij}^{p} h_{ij}^{p}, J_{ijkl}^{pq} = J_{kl}^{pq}$ and $J_{ijkl}^{pq} = J_{ijkl}^{pq}$ for $H$ hermitian. It preserves the total occupancy at each site:

$$[H, N_p] = 0, \quad N_p = \sum_{i} c_{p_1}^{\dagger} c_{p_1},$$

(26)
(where $[a, b] = [a, b] \ldots$). We will consider the single occupancy sector $N_p = 1 \forall p$, where the formulation in the previous form (1) is equivalent. The commutators

$$[c_{p_1}^{\dagger} c_{p_2}, c_{q_1}^{\dagger} c_{q_2}] = \delta_{pq} (\delta_{jk} c_{p_1} c_{p_2} - \delta_{kl} c_{p_1} c_{p_2}),$$

(27)
are the same for fermions and bosons and are identical to those satisfied by $g_{pq}^{\mu} = | i_p / j_p \rangle$ $([g_{pq}^{\mu}, g_{qj}^{\mu}] = \delta_{pq} (\delta_{jk} g_{pq}^{\mu} - \delta_{kl} g_{pq}^{\mu}))$, defining an $U(n_p)$ algebra at each site.

The product state (3) corresponds in the fermionic or bosonic scenario to an independent particle state

$$| \Psi \rangle = \prod_p a_{p_1}^{\dagger} | 0 \rangle, \quad a_{p_2}^{\dagger} = \sum_i U_{ji}^{p} a_{p_1}^{\dagger},$$

(28)
where $U_{ji}^{p}$ are the elements of a unitary matrix $U^p$ such that the same relations (24) are fulfilled by the new operators $a_{p_1}^{\dagger}, a_{p_1}$. Then the one and two-site excitations (7–8) can be written as

$$| \Phi_p \rangle = a_{p_1}^{\dagger} a_{p_1} | \Psi \rangle, \quad | \Phi_{pq} \rangle = a_{p_1}^{\dagger} a_{q_1} a_{q_1} a_{p_1} | \Psi \rangle$$

(29)
for $| \phi_p \rangle = a_{p_1}^{\dagger} | 0 \rangle, | \phi_q \rangle = a_{q_1}^{\dagger} | 0 \rangle$ and $i,j \geq 2$. Thus, we can employ expression (19) with $f_{ij}^{p} = U_{ij}^{p}$ and

$$\langle i_p k_q | V_{pq} | j_q l_q \rangle = J_{ijkl}^{pq}$$

(30)

### III. APPLICATION TO SU(n) MODELS

We will now consider the problem of factorization in a general $n$-level model with two-site interactions. It can be formulated as a system of $N$ particles at $N$ distinct sites.
p, having each access to n local levels with unperturbed
energies $\epsilon_i^p$. The Hamiltonian reads

$$H = \sum_{i,p} c_i^p c_i^\dagger - \frac{1}{2} \sum_{p \neq q} r_{pq} \sum_{i,j} (U_{ij} c_i^p c_j^q + c_j^q c_i^p) + V_{ij} c_i^p c_j^q c_j^q c_p$$

where $U_{ij} = U_{ji}$, $V_{ij} = V_{ji}$ and $W_{ij} = W_{ji}$ are real
coupling strengths and $r_{pq} = r_{qp}$ determines the coupling
range. The $V_{ij}$ terms promote two particles at sites $p,q$
from level $j$ to $i$, while the $W_{ij}$ terms interchange the
occupancies of these levels at these sites (Fig. 1).

As discussed in Appendix A, for full range couplings
($r_{pq} = \frac{1}{N-1} \forall p \neq q$) and $W_{ij} = U_{ij} = 0$, the present
model becomes the fully connected $SU(n)$ fermionic nu-
clear model employed in [19] [20] [31] [32], which is an $n$-level
generalization of the well-known Lipkin model
[33] [34], recovered from [19] for $n = 2$. Some $SU(n)$
spin models and magnets [35] [36] [37] [38] also correspond to
special cases of [31], with the $SU(n)$ Heisenberg coupling
[22] [26] [39] [40] recovered for $V_{ij} = U_{ij} = 0 (i \neq j)$
and $W_{ij} = U_{ii} = I$. In its distinguishable formulation, (31)
is an $n$ level extension of the $XYZ$ spin $1/2$ Hamiltonian
with an applied magnetic field along the $z$ axis, recovered
from [19] for $n = 2$. Besides, for $n = 2s+1$ Eq. (31)
can be formulated as a system of spins $s$ with couplings
depending on powers of the spin operators.

Since particles are moved in pairs between levels, the
Hamiltonian (31) has, for any value of the coupling
strengths and range, the number parity symmetries

$$[H, P_i] = 0, \quad i = 1, \ldots, n,$$

$$P_i = \exp[-i\pi N_i], \quad N_i = \sum_{p} c_i^p c_i^\dagger,$$

where $P_i$ is the parity of the total occupation $N_i$ of level
$i$. Since $\prod_{i=1}^{n} P_i = e^{-i\pi N}$ is fixed, just $n-1$ parities
are independent. The exact eigenstates of $H$ will then have
definite parities when non-degenerate, and can be
characterized by their $n-1$ values $\sigma_i = \pm 1$ for $i = 2, \ldots, n$.

In the MF approximation, which in the uniform at-
ttractive case can be determined analytically (see App.
B) the GS of (31) will typically exhibit a series of transitions as they increase from 0, from the unperturbed
phase with all particles in the lowest $i = 1$ level, to a
final full parity-breaking phase where all $n$ levels are oc-
cupied, with intermediate steps where just $m < n$ levels
are nonempty. These transitions become smoothed out in
the actual entangled exact GS for finite $N$, which may in-
stead exhibit number parity transitions (secs. III B–IV).

The parity-breaking MF GS becomes however exact at
the factorization point, as discussed below.

### A. Uniform factorized GS

We now determine the conditions for which the Hamiltonian (31) possesses a uniform factorized GS

$$|\Psi\rangle = \prod_{p} a_{p1} c_{p1} \bigg| 0 \bigg> , \quad a_{p1} = \sum_{i} f_{i} c_{pi}^\dagger ,$$

with $f_i$, $p$-independent and $\sum_{i} |f_i|^2 = 1$. We set $\epsilon_i^p = r_{p} \epsilon_i$ with $r_p = \sum_{q \neq p} r_{pq}$ according to (21), such that
this factorization is determined by the single Eq. (22).

It is then seen that for $k = i$, Eq. (19) leads here to

$$\sum_{j} [(2\epsilon_i - U_{ii}) \delta_{ij} - V_{ij}] f_j^2 = E_2 f_i^2 ,$$

for $i = 1, \ldots, n$, which is a standard eigenvalue equation
for the vector $f_i^2$ of elements $f_i^2$ (i.e., for the “squared
wave function”) and matrix $M_{ij} = (2\epsilon_i - U_{ii}) \delta_{ij} - V_{ij}$:

$$M f^2 = E_2 f^2 .$$

It represents the $n \times n$ $ii$-$jj$ block in (19).

On the other hand, for $k = j \neq i$, Eq. (19) leads here to
the $2 \times 2$ $ij$-$jj$ block

$$\begin{pmatrix} \epsilon_i + \epsilon_j - U_{ij} - W_{ij} & -W_{ij} \\ -W_{ij} & \epsilon_i + \epsilon_j - U_{ij} \end{pmatrix} \begin{pmatrix} f_i f_j \\ f_j f_i \end{pmatrix} = E_2 \begin{pmatrix} f_i f_j \\ f_j f_i \end{pmatrix} .$$

Eq. (36) entails, for $f_i f_j \neq 0$, the constraint

$$U_{ij} + W_{ij} = \epsilon_i + \epsilon_j - E_2 .$$

Hence, given an arbitrary single site spectrum $\epsilon_i$ and
couplings $V_{ij}, U_{ii}$, the factorized eigenstate and pair
energy $E_2$ are first determined from the eigenvalue equation
(35b). The couplings $W_{ij}$ or $U_{ij}$ for which such state be-
comes an exact eigenstate are then obtained from (37). These
conditions are independent of coupling range $r_{pq}$
and system size $N$, implying that this factorization will
emerge for any $N \geq 2$ and range $r_{pq}$ if (37) is satisfied.
The total energy is determined by $E_2$ through Eq. (23).

For GS factorization, the lowest eigenvalue $E_2$ of (35)
should be chosen. In this case, as the eigenvalues of the matrix
in (36) are $\epsilon_i + \epsilon_j - U_{ij} \pm W_{ij}$, i.e. $E_2$ and $E_2 + 2W_{ij}$
when (37) is fulfilled, the uniform factorized state will be
a GS of the full pair Hamiltonian (and hence of the full $H$) for any signs of the $V_{ij}$'s if

$$W_{ij} \geq 0 \quad \forall \ i \neq j,$$  

(38)
i.e. $E_2 \leq \epsilon_i + \epsilon_j - U_{ij} \forall i \neq j$. Since the lowest eigenvalue of (35) satisfies $E_2 \leq \text{Min}_{2i \neq U_{ii}} \leq 2\epsilon_i - U_{ii} \forall i$, a sufficient condition for the validity of (38) at fixed $U_{ij}$ is

$$U_{ij} \leq (U_{ii} + U_{jj})/2,$$  

(39)

$\forall i \neq j$. In particular, (38) will be always satisfied for the lowest eigenvalue of (35) associated to the lowest eigenvalue $E$ which indicates the number of distinct parity levels exactly crossing at this point.

On the other hand, for small systems with $N < n - 1$, the number $D$ of linearly independent states obtained with such sign changes in the $f_i$'s, and hence the degeneracy at factorization, is smaller. We obtain in general

$$D = \left\{ \begin{array}{ll} \frac{2^{n-1}}{\sum_{k=0}^{n-1} (n-1)^k}, & N \geq n - 1 \\ \frac{2}{N}, & N < n - 1 \end{array} \right..$$  

(44)
such that signs are to be changed in just $k \leq N$ levels. For a single pair $(N = 2)$, $D = \binom{n}{2} + 1$.

We have so far assumed that the matrix $M$ in (35) has a non-degenerate GS, with a full rank eigenvector $f^2$. If $f_i = 0$ for some $i$, then factorization (and the ensuing degeneracy) becomes equivalent to that for $n \rightarrow n - 1$. And if the GS of $M$ is itself degenerate, the coefficients $f^2_i$ will no longer be unique (after normalization). The GS of $H$ will then exhibit additional degeneracy, since a continuous set of factorized GS's becomes feasible. We will consider below a special extreme case.

C. The $W$-case: Number symmetry and exceptional degeneracy at factorization

We now consider the special case where $V_{ij} = 0 \forall i \neq j$ in (33). For $n = 2$ it corresponds to the $XXZ$ model, which conserves the total $S_z$ and hence has eigenstates with definite magnetization. Accordingly, for $V_{ij} = 0$ Eq. (34) exhibits an additional symmetry: not only parity but also the total occupation of each level $i$ is conserved:

$$[H, N_i] = 0, \quad i = 1, \ldots, n$$  

(45)
since the $U$ and $W$ couplings preserve all $N_i$'s. This higher symmetry entails, first, a trivial factorization: the $n$ states with all particles in just one level $i$,

$$|\Psi_i\rangle = \prod_{p} c^\dagger_{pi} |0\rangle, \quad i = 1, \ldots, n,$$  

(46)
are clearly exact eigenstates: $H|\Psi_i\rangle = E|\Psi_i\rangle$ with $E = \sum_p r_p \epsilon_p - \frac{1}{2} U_{ii}$. For $n = 2$ they become the fully aligned spin states with maximum magnetization $|M|$.

But in addition, non-trivial symmetry-breaking uniform factorized eigenstates of the form (34) may also arise: Eqs. (35)–(37) remain valid, but Eq. (35) becomes
Thus, \( f_i \) remains here completely arbitrary. For vanishing \( V_{ij} \) any uniform factorized state is an exact eigenstate with the same energy when \((47) - (45)\) are fulfilled, as the matrix \( M \) becomes proportional to the identity. And if \( W_{ij} \geq 0 \forall i \neq j \), i.e. if Eq. (50) holds \( \forall i \neq j \), they will be GS’s by the same previous arguments. Such continuous set of factorized exact GS’s reflects their breaking of all number symmetries \((48)\) when \( 0 < f_i < 1 \forall i \), as they lead to non-zero fluctuations \( \langle N_i^2 \rangle - \langle N_i \rangle^2 = N f_i (1 - f_i) > 0 \).

Moreover, since they contain terms with all possible values \( 0 \leq N_i \leq N \), all number projected states with definite values \( N_i = n_i \) \( \forall i \) derived from \( \Psi \),

\[
|\Psi_{n_1 \ldots n_n} \rangle \propto P_{n_1} \ldots P_{n_n} |\Psi \rangle,
\]

(49)
satisfying \( N_i |\Psi_{n_1 \ldots n_n} \rangle = n_i |\Psi_{n_1 \ldots n_n} \rangle \) with \( \sum_{i=1}^{N} n_i = N \), will also be exact eigenstates with the same energy due to \((45)\). Here \( P_{n_i} = \frac{1}{2 \pi} \int_{0}^{2 \pi} e^{-i \phi (N_i - n_i)} d\phi \) are number projectors \((|P_{n_i}, H\rangle = 0 \forall i\)\).

Remarkably, when normalized these projected states become independent of the arbitrary coefficients \( f_i \) determining the product state \( |\Psi \rangle \), since each term in their expansion \((41)\) will have exactly \( n_i \) particles in level \( i \) and hence all coefficients become identical: \( f_{i1} \ldots f_{i_N} = \prod_{i=1}^{n} (f_i)^{n_i} = C_{n_1 \ldots n_n} \). Therefore,

\[
|\Psi_{n_1 \ldots n_n} \rangle = |n_1 \ldots n_n \rangle,
\]

(50)
where \( |n_1 \ldots n_n \rangle \) is the fully symmetric state having \( N_i = n_i \) particles in each level \( i \). The total degeneracy at factorization is then given by the number of distinct projected states \((50)\), which is just the number of ways of distributing \( N \) undistinguishable particles on \( n \) levels:

\[
D = \frac{N + n - 1}{n - 1},
\]

(51)
with \( D \approx \frac{N^{n-1}}{(n-1)!} \) for \( N \gg n \). Then factorization arises at an exceptional critical point where the \( D \) lowest levels with distinct values of the \( N_i \)’s cross and become degenerate. The ensuing degeneracy grows with system size, in contrast with previous \( N \)-independent parity degeneracy.

Since any uniform factorized state is an exact GS at the factorizing point, the GS subspace is here clearly invariant under arbitrary \( U(n) \) unitary transformations

\[
U = \exp \left[ -i \sum_{i,j} T_{ij} \sum_{p} c_{p_i} c_{p_j} \right],
\]

(52)
where \( T \) is an arbitrary hermitian matrix, as \( U \) transforms any product state \((34)\) into another uniform product state and these states span the GS subspace:

\[
|\Psi \rangle \rightarrow U|\Psi \rangle \implies f \rightarrow \exp[-i T] f.
\]

(53)
It corresponds to \( U = e^{-i T} \otimes \ldots \otimes e^{-i T} \) in the distinguishable formulation.

The question which now arises is whether the full \( H \) also becomes \( SU(n) \) invariant when the factorizing conditions \((47) - (48)\) are fulfilled. For \( n = 2 \) this is indeed the case: as shown in App. \( A \) they lead to a Heisenberg Hamiltonian \( H \propto -\sum_{p<q} r_{pq} s_p \cdot s_q \) plus constant terms, where \( s_p \) is the (dimensionless) spin operator at site \( p \). This \( H \) is obviously invariant under arbitrary global rotations \( e^{-i \phi_0} \sum_{p} s_p \), with \( k \) an arbitrary unit vector, and admits any aligned product state \(|k, \ldots, k\rangle\), with \( \langle k|s_p|k\rangle = \frac{1}{2} k \), as exact GS for arbitrary \( k \).

However, for \( n \geq 3 \) only the GS subspace remains invariant in general, i.e., \([H, U] \neq 0\), with \([H, U] \) having just \( D \) zero eigenvalues, corresponding to the GS subspace. Therefore, the \( SU(n) \) Heisenberg Hamiltonian \((22)\)

\[
H = -J \sum_{p<q} r_{pq} \sum_{i,j} c_{pi} \cdot c_{qj} c_{qi} c_{pj}
\]

(54)
is just a particular case of present factorizing Hamiltonian, corresponding to \( \epsilon_i = 0 \forall i \) and hence \( W_{ij} = U_{ii} = J = -E_2 \forall i \neq j \), according to Eqs. \((47) - (48)\).

D. Definite parity eigenstates and entanglement at the border of factorization

We now examine the GS in the immediate vicinity of factorization. We consider first the \( V \neq 0 \) case. Since away from factorization the exact GS is normally non-degenerate for finite \( N \), it will have definite parities \( P_i \). The same holds for the other levels which meet at the factorization point. Therefore, their side-limits at factorization will be given by the parity projected states

\[
|\Psi_{\sigma_2 \ldots \sigma_n} \rangle \propto (1 + \sigma_2 P_2) \ldots (1 + \sigma_n P_n) |\Psi \rangle,
\]

(55)
where \( \sigma_i = \pm 1 \), satisfying \( P_i |\Psi_{\sigma_2 \ldots \sigma_n} \rangle = \sigma_i |\Psi_{\sigma_2 \ldots \sigma_n} \rangle \). This projection just selects from the expansion \((41)\) those terms with the specified level parities. The GS will then exhibit a parity transition as the factorization point is crossed \([9] [12] [37] \) (when some Hamiltonian parameter is varied), having distinct parities \( \sigma_i \) at each side.

These projected states are entangled, i.e., they are no longer product states. They exhibit critical entanglement properties since the product state \(|\Psi\rangle\) from which they are derived is uniform and has lost all information about the range \( r_{pq} \) of the coupling and the distance between sites. Accordingly, the side-limits at factorization of GS entanglement entropies will be range-independent. Moreover, pairwise entanglement, though small (in compliance with monogamy), will be independent of the separation \( |p - q| \) between sites.

These properties can be seen, for instance, in the reduced state of site \( p, \rho_p = \text{Tr}_{p' \neq p} |\Psi_0\rangle \langle \Psi_0|\) of elements

\[
(\rho_p)_{ij} = \langle c_{p_i}^\dagger c_{p_j} \rangle,
\]

(56)
and eigenvalues $\lambda_{pi}$. Its entropy

$$S_p = -\text{Tr} \rho_p \log_2 \rho_p = -\sum_{i=1}^{n} \lambda_{pi} \log_2 \lambda_{pi}$$

(57)

is a measure of the (mode) entanglement between this site and remaining sites. In the fermion case it is also a measure of fermionic entanglement [44,48], in the sense of indicating the deviation of the state from an independent fermion state (SD), since it is the $p$-block of the one-body density matrix $\rho^{(1)}$:

$$\rho^{(1)}_{p;ij} = \langle c_{ij}^\dagger c_{pi} | = \delta_{pj} \langle c_{ij}^\dagger c_{pi} \rangle,$$

(58)

whose blocked structure is due to the fixed fermion number $N_p$ at each site. Its entropy $S(\rho^{(1)}) = \sum_p S_p$ is a quantity which vanishes iff $|\Psi_0\rangle$ is a SD, i.e. $(\rho^{(1)})^2 = \rho^{(1)}$ [45] and is just $NS_p$ in the uniform case. In the factorized state $|\Psi\rangle$, $\langle c_{ij}^\dagger c_{pi} | = f_i^p f_j^p$, implying obviously $\rho^{(1)}_p = \rho_p$, i.e., $\lambda_{pi} = 6\delta_{ij}$, as directly seen in the MF basis $\langle c_{ij}^\dagger a_{pi} | = \delta_{ij} \delta_{1i}$, and hence $S_p = 0$.

In contrast, in states $|\Psi_0\rangle$ with definite parity all off-diagonal elements in the standard basis are cancelled by parity conservation $\langle \rho_p, e^{i\pi c_{ij} c_{pi}} \rangle = 0 \forall i$, implying

$$\langle c_{ij}^\dagger c_{pi} | = \delta_{ij} \langle c_{ij}^\dagger c_{pi} \rangle.$$

(59)

Hence the eigenvalues of $\rho_p$ are just the average occupations $\lambda_{pi} = \langle c_{ij}^\dagger c_{pi} |$ and $S_p > 0$ whenever $\langle c_{ij}^\dagger c_{pi} | \in (0,1)$.

In the projected states [55], these occupations depend on the parities $\sigma_1, \ldots, \sigma_n$. For instance, for $n = 3$ in the uniform case, we obtain

$$\langle \Psi_{\sigma_1 \sigma_2 \sigma_3} | c_{ij}^\dagger c_{pi} | \Psi_{\sigma_2 \sigma_3} \rangle = | f_i |^{2+\sum_j (-) \sigma_j (1-2|f_j|^2)^{N-1} / (1+\sum_j \sigma_j (1-2|f_j|^2)^{N-1}}$$

(60)

where $j = 1, \ldots, n$, and $\sigma_1 \sigma_2 \sigma_3 = (-1)^N$. Hence, for large $N$, $\lambda_{pi} \approx | f_i |^2$ plus corrections of order $1 - 2|f_j|^2 = 1$ which depend on the parities $\sigma_i$.

For finite $N$ these corrections are, nonetheless, appreciable and their parity dependence originates the splitting of the degeneracy in the immediate vicinity of factorization (App. C). Moreover, the occupations [60] determine the exact side-limits of the GS entanglement entropies $S_p$ at factorization, which will then exhibit a discontinuity at this point due to the change in the GS parities $\sigma_i$ and will be range independent.

Finally, pairwise entanglement can be measured through the negativity [60,52] $N_{pq}$ of the reduced pair state $\rho_{pq} = \text{Tr}_{p\neq q} |\Psi_0\rangle \langle \Psi_0 |$ of a pair $p \neq q$, given by

$$N_{pq} = \frac{1}{2} \left| \text{Tr} [\rho_{pq}^\tau] - 1 \right|$$

(61)

where $T_p$ denotes the partial transpose. The side-limits at factorization of the exact GS negativities $N_{pq}$ will be again determined by the projected states [55], and will then be independent of $p,q$ (and the coupling range) in the uniform case, undergoing a discontinuity due to the transition in the GS parities.

Similar considerations hold for the $V = 0$ case. The level number projected states [49–50] represent the exact side-limits at factorization of the $D$ crossing states. Except for the states [46] with just one level occupied, all remaining states are entangled and lead again to critical entanglement properties (independence of coupling range and separation) due to their fully symmetric nature. In particular, they lead again to single site reduced states $\rho_p$ diagonal in the standard basis,

$$\langle n_1 \ldots n_n | c_{ij}^\dagger c_{pi} | n_1 \ldots n_n \rangle = \delta_{ij} n_i / N,$$

(62)

implying $\lambda_{pi} = n_i / N$ and hence a single-site entanglement entropy $S(\rho_p) > 0$ whenever $1 \leq n_i \leq N - 1$.

**E. Factorization signatures in small systems**

We discuss here typical illustrative results in small $n$-level systems. We consider first the $n = 3$ case with uniform single site spectrum $\epsilon_1^p = -1/2\epsilon$, $\epsilon_2^p = 0$, $\epsilon_3^p = 1/2\epsilon$ and couplings $V_{ij} = v$, with $U_{ij} = 0$. We set $W_{ij} = \alpha(\epsilon_i + \epsilon_j - E_c^{(2)})$ for $i \neq j$, with $\alpha = v/v_c$ and $E_c^{(2)}$ the pair energy obtained from (35) at $v = v_c$, such that GS factorization is reached at $\alpha = 1$ (Eq. (37)). We have set $v_c = \frac{2}{\epsilon}$, for which $E_c^{(2)} \approx -1.26\epsilon$.

We first depict in Fig. 2 the spectrum of $H$ for a single pair $(N = 2, r_{12} = 1)$ and for a cyclic four-particle chain with first-neighbor couplings $(N = 4, r_{pq} = \frac{1}{4} \delta_{q,p\pm 1})$, as a function of $v/v_c$. In both cases there is a GS band
of $2^n-1 = 4$ states which cross exactly at the factorization point $v = v_c$, where a GS number parity transition takes place: The GS changes from the $(\sigma_1, \sigma_2) = (+, +)$ state for $v < v_c$, to the $(\sigma_1, \sigma_2) = (-, -)$ state for $v > v_c$. These states form the border of the GS band, the remaining crossing levels $(\sigma_1, \sigma_2) = (\pm, \mp)$ lying in between.

Further results for a ring of $N = 4$ particles are shown in Fig. 3 It is verified that the first three exact excitation energies, together with the difference $E_{HF} - E_0$ and the mean field (HF) GS energy, exactly vanish at $v = v_c$. (top left), confirming factorization. The exact values represent the eigenvalues of the single site reduced density matrix and exhibit a discontinuity at $v = v_c$. Bottom: The one-site entanglement entropy (right), which shows a stepwise increase at factorization, and the exact negativities between first $(N_1 = N_{p,p+1})$ and second $(N_2)$ neighbors (left), measuring pairwise entanglement. Both reach the same side-limits at factorization, exhibiting there a stepwise decrease.

HF results (dotted lines) reproduce qualitatively the general trend but miss the jump at factorization: Though exact at this point, the HF GS corresponds to a superposition of the crossing definite parity exact eigenstates. It exhibits instead transitions at $v/v_c \approx 0.44$ and 0.65, where the second and third level respectively start to be populated (App. B). Thus, factorization lies within the full parity-breaking MF phase.

Entanglement properties are depicted in the lower panels. The exact single site entanglement entropy (bottom right) increases monotonously as $\alpha$ increases, and displays a stepwise increase precisely at the factorizing point, due to the transition in the average level occupations. The negativities $N_1$ and $N_2$ (bottom left), measuring the pairwise entanglement between first and second neighbors, exhibit instead a stepwise decrease at factorization, indicating multipartite entanglement effects of the parity projected states. They are also verified to approach the same side-limits at factorization, confirming the independence from separation in its immediate vicinity, as predicted by the projected states (55).

In Fig. 4 we show the same quantities for a ring of $N = 6$ particles with the same parameters, to view the trend for larger systems. Their behavior remains similar, with factorization located at the same point, where the four lowest levels with distinct parities cross (top left). However, the GS now exhibits in the range considered two further parity transitions, at $v_{c2} \approx 1.52v_c$ and $v_{c3} \approx 1.74v_c$, not related to factorization, where just two levels cross and the GS parity changes from $(\sigma_2, \sigma_3) = (+, +)$ for $v < v_{c2}$ to $(-, -)$ for $v_{c2} < v < v_{c3}$, $(+, -)$ for $v_{c3} < v < v_{c3}$ and back to $(+, +)$ for $v > v_{c3}$.

These transitions lead to further steps in the single site occupation numbers and entropy (right panels), though the larger step occurs again at the factorizing transition. All three pair negativities $N_i$ are verified to reach the same side-limits at the factorizing point, a characteristic signature of uniform factorization, exhibiting there a stepwise decrease. These patterns are not repeated at the other GS parity transitions, where $N_1$ increases but $N_3$ decreases, vanishing for $v > v_{c3}$. Full range pairwise entanglement is thus centered at the factorizing point, where it becomes independent of separation.

In Fig. 5 we show the eigenvalues $p_i$ (entanglement spectrum) of the two-site density matrix $\rho_{pq}$ (left panel),
which determine the entanglement of the pair with the rest of the chain (just 4 of them are nonnegligible). They also exhibit steps at the parity transitions, with the larger step again at the factorizing point. The ensuing mutual information

$$I_{pq} = S(p_p) + S(p_q) - S(p_{pq})$$

(63)

where $S(p_p) = S_{p}$ is the single site entropy, is shown on the right panel for the first three neighbors. It is a measure of the total correlation between sites. It is seen that all three values merge at the side-limits of the factorizing point, confirming again that in its vicinity correlations become independent of separation. Since it does not satisfy monogamy, its behavior is, however, different from that of the negativity, steadily increasing up to $v = v_c$. 

Finally Fig. 5 shows the spectrum of $H$ in the special $W$ case ($V_{ij} = 0$) of sec. IIIC for a ring of $N = 4$ particles with $n = 3$ (top) and $4$ (bottom) levels at each site. We have used a uniform spectrum $\epsilon_1 = -\epsilon$, $\epsilon_2 = 0$, $\epsilon_3 = 0.8\epsilon$ (and $\epsilon_4 = 2.2\epsilon$ for $n = 4$), and set $U_{ij} = \delta_{ij}(2\epsilon_i - E^{(2)})$, $W_{ij} = \alpha(\epsilon_i + \epsilon_j - E^{(2)})$ for $i \neq j$ and $\alpha = w/w_c$, with $w_c = \epsilon$, such that factorization occurs at $w = w_c$.

It is verified that all $\begin{pmatrix} N \times 1 \end{pmatrix}$ levels (15 for $n = 3$ and 35 for $n = 4$) forming the GS band cross at the factorization point $w = w_c$, where any uniform product state (real or complex) is confirmed to be an exact GS. The crossing states are symmetric states with definite occupations in all $n$ levels, the GS changing at $w_c$ from $|\Psi_1\rangle$ (Eq. 46, all particles in the first level) to $|\Psi_n\rangle$.

**IV. CONCLUSIONS**

We have analyzed the problem of GS factorization beyond the standard interacting spin system scenario ($SU(2)$ symmetry). We have first derived general necessary and sufficient factorization conditions for Hamiltonians with two-site couplings, showing that they can be recast as pair eigenvalue equations. These conditions were then applied to interacting $SU(n)$ systems, where each constituent has access to $n$ local levels. For the $UVW$ class of Hamiltonians [31] they can be worked out explicitly, leading in the uniform case to the eigenvalue equation (35) for the squared local wave function and the constraint (47) on the coupling strengths, valid for any number $n$ of levels. They are independent of size $N$ and coupling range, and generalize those for XYZ spin systems, recovered for $n = 2$. The ensuing product state is shown to be a GS when conditions (58) are fulfilled, which are directly satisfied for vanishing $U_{ij}$.

The full rank factorized GS breaks all level number parities, preserved by the Hamiltonian, therefore having a $2^{n-1}$ degeneracy (for $N \geq n - 1$). Factorization then arises at a special point where all $2^{n-1}$ definite parity levels of the GS band cross and become degenerate, signaling a fundamental GS level parity transition emerging for any size $N$ and range.

We have also examined the special $V = 0$ case, where the Hamiltonian preserves the total occupation of each level. Here the factorization conditions allowed us to identify an exceptional critical point, again emerging for any size and range, where all levels with definite occupations $N_{ij}$ forming the GS band coalesce and become degenerate. This leads to a GS degeneracy which increases with system size ($D \propto N^{n-1}$). At this point all uniform product states, including those breaking all occupation number symmetries, are exact degenerate GSs, implying a full $SU(n)$ invariant GS subspace, in a Hamiltonian which for $n \geq 3$ is not necessarily $SU(n)$ invariant.

Finally, we have analyzed the entanglement properties in the immediate vicinity of factorization. Pairwise en-
tanglement (as detected by the negativity) reaches thus full range and becomes independent of separation, thus constituting a quantum critical point for the small finite sample. Moreover, in such systems the parity transition occurring at the factorizing point entails finite discontinuities in most quantities (single site entanglement, negativity, level occupations, mutual information, etc.), whose magnitude can be analytically determined through projection of the factorized GS.

In summary, in addition of providing nontrivial analytic exact GSs in strongly coupled systems which are not exactly solvable (which could be used as benchmarks for approximate numerical techniques), symmetry-breaking factorization enables one to identify quantum critical points in small samples with exceptional GS degeneracy and entanglement properties. Amidst increasing quantum control capabilities, present results open the way to explore factorization in $\text{SU}(n)$ many-body physics and complex systems beyond the usual $\text{SU}(2)$ spin scenario.

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**Appendix A: Special cases**

We consider here special cases of the Hamiltonian \(31\). The fully connected fermionic nuclear $U(n)$ model used in \(33\) corresponds to \(r_{pq} = \frac{1}{N-1} \quad \forall \, p \neq q \) and $U_{ij} = 0$:

\[
H = \sum_{i=1}^{n} c_{pj}G_{i}^{p} - \frac{1}{2(N-1)} \sum_{i,j} V_{ij}G_{ij}^{2} + W_{ij}(G_{ij}G_{ji} - G_{ii})
\]

where $G_{ij} = \sum_{p=1}^{\Omega} c_{pj}^{p} c_{pj}$ are collective operators satisfying the same $U(n)$ algebra as the operators $g_{ij} = c_{pj}^{p} c_{pj}$:

\[
[G_{ij}, G_{kl}] = \delta_{jk}G_{il} - \delta_{il}G_{jk},
\]

for both fermions and bosons. It is a simplified schematic model of collective excitations. For $n = 2$ and $\epsilon_{2} = -\epsilon_{1} = \epsilon/2$ it becomes the well known Lipkin Hamiltonian

\[
H = \epsilon S_{z} - \frac{1}{2(N-1)}[V(S_{+}^{2} + S_{-}^{2}) + W(S_{+}S_{-} + S_{-}S_{+} - N)]
\]

where $S_{z} = \frac{1}{2}(G_{21} - G_{11})$, $S_{+} = G_{21}$, $S_{-} = G_{12}$, are collective spin operators satisfying the standard $SU(2)$ spin algebra \(20, 44\) \([S_{+}, S_{-}] = \pm S_{z}, \quad [S_{+}, S_{+}] = 2S_{z}\) and $V = V_{12}, \quad W = W_{12}$. These fully connected models have been frequently used to test several approximations, as the exact GS for any size $N$ can be obtained by diagonalization of $H$ in the irreducible representations of $U(n)$ (in particular in the fully symmetric representation in the case of attractive couplings). For $n = 2$ level number parity conservation reduces to the $S_{z}$-parity symmetry $[H, P_{z}] = 0$, where $P_{z} = e^{-i\pi S_{z}} = P_{\pi} e^{-i\pi N}$.

On the other hand, in the distinguishable formulation, the Hamiltonian \(31\) corresponds, for $g_{ij}^{p} = |i/p\rangle\langle j/p|$, to

\[
H = \sum_{i,p} e_{i}^{p} g_{ij}^{p} - \sum_{p<q,i,j} r_{pq}(U_{pq}g_{ij}^{p}g_{ij}^{q} + V_{ij}g_{ij}^{p}g_{ij}^{q} + W_{ij}g_{ij}^{q}g_{ij}^{p}).
\]

For $n = 2$, $e_{i}^{p} = -e_{i}^{p} = b_{i}/2$, $V_{ij} = (J_{x} - J_{y})/2$, $W_{ij} = (J_{x} + J_{y})/2$ and $U_{ij} = U_{22} = -U_{12} = J_{z}/2$, with $p = 1, \ldots, N$, it becomes the Hamiltonian of $N$ spins $1/2$ interacting through anisotropic $XYZ$ couplings of general range in a nonuniform field $b^{p}$:

\[
H = \sum_{p} b_{p} s_{pz} - \sum_{p<q} r_{pq}(J_{x} s_{p} s_{q} + J_{y} s_{p} s_{q} + J_{z} s_{p} s_{q})
\]

where $s_{pz} = g_{z}^{2} - g_{ij}^{1}$, $s_{px} = g_{x}^{2} - g_{ij}^{1}$, $s_{py} = g_{y}^{2} - g_{ij}^{1}$, are again spin operators satisfying the $SU(2)$ algebra.

Besides, in the $n$-level case the operators $g_{ij}^{p}$ can always be expressed in terms of powers of spin-$s$ operators with $2s + 1 = n$. For instance, for $n = 3$ all $g_{ij}^{p}$ can be written in terms of spin-1 operators $s_{pz}$ and $s_{p} = s_{pz} \pm is_{py}$ as

\[
g_{ij}^{1} = \frac{1}{2}(s_{pz} \pm s_{py}), \quad g_{ij}^{2} = \frac{1}{2} s_{x}^{2} - s_{z}^{2}, \quad g_{ij}^{3} = \frac{1}{2} s_{y}^{2} - s_{z}^{2} + s_{p}^{2} - 2s_{pz}, \quad g_{ij}^{4} = \frac{1}{2} s_{x}^{2} - s_{y}^{2} + s_{p}^{2} - 2s_{pz},
\]

with $g_{ij}^{3} = 1/2 s_{p}^{2}$, $g_{ij}^{4} = (g_{ij}^{1})^{4}$ and $s_{x}^{2} = s_{x}^{2} + s_{y}^{2} + s_{z}^{2} = 2s_{p}^{2}$. Thus, single site operators become in general quadratic in the local spin components $s_{p}$. We now verify that for $n = 2$, the factorization conditions \(35\) become those for the $XYZ$ Hamiltonian in a uniform field $b^{p} = b$. Eq. \(35a\) leads for $n = 2$ to

\[
E_{2} = -J_{z}/2 - \sqrt{b^{2} + V_{12}^{2}}
\]

for the lowest pair energy, with \(37\) implying $W_{12} = -E_{2} - U_{12}$. We then obtain

\[
|b| = \sqrt{(W_{12} - J_{z})^{2} - V_{12}^{2}} = \sqrt{(J_{y} - J_{z})(J_{y} - J_{z})}
\]

which is the known expression for the factorizing field $b$ at given couplings $J_{x}, J_{y}, J_{z}$ (valid for $J_{y} < J_{x} < J_{z}$, corresponding to $W_{12} > 0, V_{12} > 0$). Setting now $f = (\cos s_{p}^{2}, \sin s_{p}^{2})$ for the local eigenvector, Eq. \(35a\) leads to

\[
\cos \theta = \frac{b - J_{z}/2 - E_{2} - V_{12}}{b - J_{z}/2 + E_{2} + V_{12}} = \frac{J_{x} - J_{z}}{J_{x} - J_{z}},
\]

which coincides with the known expression for the spin orientation angle $\theta$ of the uniform product GS \(39\).

In the $V = 0$ case of sec. III C conditions \(47\)–\(48\) imply, for $n = 2$, $b_{p} = 0$ and $W_{12} = J_{x} = J_{y} = J_{z} = J$, such that $H$ becomes a Heisenberg Hamiltonian:

\[
H = -\sum_{p<q} r_{pq}(J_{s} s_{p} s_{q} + C),
\]
with $J = E_2 + U_{12}$ and $C = (E_2 - U_{12})/4$ a constant term (for $N_p = 1 \forall p$). Both $E_2$ and $U_{12}$ are free parameters. It is then verified that for $J > 0$, any uniform product state, i.e. any state with all spins aligned in a fixed arbitrary direction $\theta, \phi$ ($f = (\cos \frac{\theta}{2}, e^{i\phi} \sin \frac{\theta}{2})$) is an exact GS with pair energy $E_2$ (as $s_p \cdot s_q \langle \psi, \psi \rangle = \frac{1}{4} |\psi, \psi \rangle \forall |\psi\rangle$).

**Appendix B: Mean field approximation**

We show here that the mean field (MF) approximation for the Hamiltonian \([31]\) (which corresponds to the Hartree-Fock (HF) scheme in the fermionic case) can be solved analytically in the uniform attractive case, for any values of $n$, $N$ and the coupling range $r_{pq}$.

We look for the product state $|\Psi\rangle$ (or equivalently, the independent particle state \([31]\)) which minimizes $\langle H \rangle = \langle \Psi | H | \Psi \rangle$ with $\varepsilon^p = r_p^c \varepsilon_i$ and non-geometric couplings $U_{ij}$, $V_{ij}$, $W_{ij}$. As $\langle c_{pi} c_{qi} \rangle = \delta_{pq} \langle f_i^p \rangle$ and $\langle c_{pi} c_{qi} c_{qi} c_{pk} \rangle = \langle f_i^p \rangle \langle f_i^q \rangle \langle f_i^p \rangle$, it is easily seen that in case (B) cannot be minimized, such uniform energy coefficients $f_i^p$ vanishes. For decreasing coupling strengths, this is to be repeated until the trivial solution $f_i = \delta_{1i}$ (valid for sufficiently small $J_{ij}$) is reached.

Therefore, as $J_{ij}$ increases from 0, a series of $n - 1$ MF transitions normally arise, associated with the onset of occupation of the $i_{th}$ level. For instance, for $U_{ii} = 0$ and $J_{ij} = J(1 - \delta_{ij})$, $J > 0$, Eq. \([B3]\) leads to

$$f_i^2 = 1/n - \tilde{\epsilon}_i/J, \quad i = 1, \ldots, n,$$

**Appendix C: Splitting of energy levels at the border of factorization**

Let us assume that $H = H_f + \delta H$, where $H_f = H_0 + V_{int}$ is the Hamiltonian having the factorized GS and

$$\delta H_0 = \sum_i \delta \epsilon_i \sum_p c^\dagger_p c_p$$

a small perturbation of the single particle term. For instance, a perturbation $\delta V_{int} = \gamma N_{11}$ leads to $\delta H = \gamma H_f - \gamma H_0$, implying $\delta \epsilon_i = -\gamma \epsilon_i$ plus a constant energy shift $\delta E = \gamma E_f$. At first order in $\delta \epsilon_i$, the remaining correction on the definite parity energy levels is

$$\delta E_{\sigma_1 \ldots \sigma_n} = \sum_i \delta \epsilon_i \langle N_i \rangle_{\sigma_1 \ldots \sigma_n}$$

where $N_i = \sum_p c^\dagger_p c_p$ and the average is taken on the parity projected states \([55]\) derived from the factorized GS. In the $n = 3$ level case, $\langle N_i \rangle_{\sigma_2 \sigma_3}/N$ is given in Eq. \([60]\). We then obtain, setting $u_j = 1 - 2|f_j|^2$,

$$\frac{\delta E_{\sigma_2 \sigma_3}}{N} = \sum_i \delta \epsilon_i |f_i|^2 \left[ 1 + \sum_j \sigma_{ij} (-1)^{\delta_{ij}} u_j^{-1} \right]$$

$$\approx \sum_i \delta \epsilon_i |f_i|^2 \left[ 1 + \sum_j \sigma_{ij} (-1)^{\delta_{ij}} + 2|f_i|^2 - 1 |u_j^{-1} \right]$$

where $\sigma_{12} \sigma_{23} = (-1)^N$ and last expression holds for sufficiently large $N$. For $\delta \epsilon_1 = -\delta \epsilon_2 = \delta \epsilon_3 = 0$, this leads to $\delta E_{++} < \delta E_{-+} < \delta E_{--} < \delta E_{+--}$ for $\delta \epsilon > 0$. This is the case of Fig. 2, where $\delta \epsilon = (1 - \frac{\alpha}{2})e > 0$ ($< 0$) on the left (right) side of the factorization point $v = \nu_c$. In the $V = 0$ case, $\langle N_i \rangle = n_i$ is just the occupation of level $i$ in the projected states \([49], [50]\).
[36] J. Dufour, P. Nataf, and F. Mila, “Variational Monte Carlo investigation of $SU(N)$ Heisenberg chains,” Phys. Rev. B 91, 174427 (2015).

[37] P. Nataf and F. Mila, “Density matrix renormalization group simulations of $SU(N)$ Heisenberg chains using standard young tableaux: Fundamental representation and comparison with a finite-size Bethe ansatz,” Phys. Rev. B 97, 134420 (2018).

[38] Y. Yao, C. T. Hsieh, and M. Oshikawa, “Anomaly matching and symmetry-protected critical phases in $su(n)$ spin systems in 1 + 1 dimensions,” Phys. Rev. Lett. 123, 180201 (2019).

[39] B. J. Bloom et al, “An optical lattice clock with accuracy and stability at the 10-18 level.” Nature 506, 71 (2014).

[40] A. J. Daley, M. M. Boyd, J. Ye, and P. Zoller, “Quantum computing with alkaline-earth-metal atoms.” Phys. Rev. Lett. 101, 170504 (2008).

[41] R. Rossignoli and A. Plastino, “Truncation, statistical inference, and single-particle description,” Phys. Rev. C 36, 1595 (1987).

[42] N. Canosa, A. López, A. Plastino, and R. Rossignoli, “Systematic procedure for going beyond the time-dependent Hartree-Fock approximation,” Phys. Rev. C 37, 320 (1988).

[43] H. J. Lipkin, N. Meshkov, and A. J. Glick, “Validity of many-body approximation methods for a solvable model: (i). exact solutions and perturbation theory,” Nucl. Phys. A 62, 188–198 (1965).

[44] M. Di Tullio, R. Rossignoli, M. Cerezo, and N. Gigena, “Fermionic entanglement in the Lipkin model,” Phys. Rev. A 100, 062104 (2019).

[45] M. E. Beverland, G. Alagic, M. J. Martin, A. P. Koller, A. M. Rey, and A. V. Gorshkov, “Realizing exactly solvable $SU(N)$ magnets with thermal atoms,” Phys. Rev. A 93, 051601(R) (2016).

[46] C. Romen and A. M. Läuchli, “Structure of spin correlations in high-temperature $SU(n)$ quantum magnets,” Phys. Rev. Research 2, 043009 (2020).

[47] R. Rossignoli, N. Canosa, and J. M. Matera, “Factorization and entanglement in general XYZ spin arrays in nonuniform transverse fields,” Phys. Rev. A 80, 062325 (2009).

[48] N. Gigena and R. Rossignoli, “Entanglement in fermion systems,” Phys. Rev. A 92, 042326 (2015).

[49] M. Di Tullio, N. Gigena, and R. Rossignoli, “Fermionic entanglement in superconducting systems,” Phys. Rev. A 97, 062109 (2018).

[50] G. Vidal and R. F. Werner, “Computable measure of entanglement,” Phys. Rev. A 65, 032314 (2002).

[51] K. Zyczkowski, P. Horodecki, A. Sanpera, and M. Lewenstein, “Volume of the set of separable states,” Phys. Rev. A 58, 883 (1998).

[52] M. B. Plenio, “Logarithmic negativity: A full entanglement monotone that is not convex,” Phys. Rev. Lett. 95, 090503 (2005).

[53] R. Rossignoli and P. Ring, “Maximum overlap, critical phenomena and the coherence of generating functions,” Nucl. Phys. A 444, 35 (1984).