MONOTONICITY AND SYMMETRY OF SINGULAR SOLUTIONS TO QUASILINEAR PROBLEMS

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ABSTRACT. We consider singular solutions to quasilinear elliptic equations under zero Dirichlet boundary condition. Under suitable assumptions on the nonlinearity we deduce symmetry and monotonicity properties of positive solutions via an improved moving plane procedure.

1. INTRODUCTION

We consider the problem

\[ (P_{\Gamma}) \begin{cases} 
-\Delta_p u = f(u) & \text{in } \Omega \setminus \Gamma \\
u > 0 & \text{in } \Omega \setminus \Gamma \\
u = 0 & \text{on } \partial \Omega,
\end{cases} \]

in a bounded smooth domain \( \Omega \subset \mathbb{R}^n \) and \( p > 1 \). The solution \( u \) has a possible singularity on the critical set \( \Gamma \) and in fact we shall only assume that \( u \) is of class \( C^{1,\alpha} \) far from the critical set. Therefore the equation is understood as in the following

**Definition 1.1.** We say that \( u \in C^{1,\alpha}(\overline{\Omega} \setminus \Gamma) \) is a solution to \( (P_{\Gamma}) \) if \( u = 0 \) on \( \partial \Omega \) and

\[ \int_{\Omega} |\nabla u|^{p-2}(\nabla u, \nabla \varphi) \, dx = \int_{\Omega} f(u) \varphi \, dx \quad \forall \varphi \in C_0^1(\Omega \setminus \Gamma). \]

The purpose of the paper is to investigate symmetry and monotonicity properties of the solutions when the domain is assumed to have symmetry properties. This issue is well understood in the semilinear case \( p = 2 \) when \( \Gamma = \emptyset \). The symmetry of the solutions in this case can be deduced by the celebrated moving plane method, see [1, 2, 8, 13]. In [14] the moving plane procedure has been adapted to the case when the singular set has zero capacity, in the semilinear setting \( p = 2 \).

The moving plane procedure has been developed for problems involving the \( p \)-Laplace operator, in the standard case \( \Gamma = \emptyset \), in [4] for \( 1 < p < 2 \) and in [6] for \( p > 2 \). In fact, in our proofs, we shall borrow many techniques and ideas from [4, 6] and from [14]. However the techniques cannot be applied straightforwardly manly for two reasons. First of all the technique in [14], that works the case \( p = 2 \), is strongly based on a homogeneity argument that fails for \( p \neq 2 \). Furthermore, since the gradient of the solution may blows up near the critical set, then the equation may exhibit both a degenerate and a singular nature at the same time. This causes in particular that it is no longer true that the case \( 1 < p < 2 \) allows to get stronger results in a easier way, as it is in the case \( \Gamma = \emptyset \). In fact, having in mind

\begin{itemize}
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\end{itemize}
this remark, we prefer to start the presentation of our results with the case $p > 2$. We have the following

**Theorem 1.2.** Let $p > 2$ and let $u \in C^{1,\alpha}(\Omega \setminus \Gamma)$ be a solution to $(P_\Gamma)$ and assume that $f$ is locally Lipschitz continuous with $f(s) > 0$ for $s > 0$, namely assume $(A_2^f)$. If $\Omega$ is convex and symmetric with respect to the $x_1$-direction, $\Gamma$ is closed with $\text{Cap}_p(\Gamma) = 0$, namely let us assume $(A_2^\Gamma)$, and

$$\Gamma \subset \{x \in \Omega : x_1 = 0\},$$

then it follows that $u$ is symmetric with respect to the hyperplane $\{x_1 = 0\}$ and increasing in the $x_1$-direction in $\Omega \cap \{x_1 < 0\}$.

Although the technique that we will develop to prove Theorem 1.2 works for any $p > 2$, the result is stated for $2 < p \leq n$ since there are no sets of zero $p$-capacity when $p > n$.

Surprisingly the case $1 < p < 2$ presents more difficulties related to the fact that, as already remarked, the operator may degenerate near the critical set even if $p < 2$. We will therefore need an accurate analysis on the behaviour of the gradient of the solution near $\Gamma$. We carry out such analysis exploiting the results of [11] (therefore we shall require a growth assumption on the nonlinearity) and a blow up argument. The result is the following:

**Theorem 1.3.** Let $1 < p < 2$ and let $u \in C^{1,\alpha}(\overline{\Omega} \setminus \Gamma)$ be a solution to $(P_\Gamma)$ and assume that $f$ is locally Lipschitz continuous with $f(s) > 0$ for $s > 0$ and has subcritical growth, namely let us assume $(A_1^f)$. Assume that $\Gamma$ is closed and that $\Gamma = \{0\}$ for $n = 2$, while $\Gamma \subseteq M$ for some compact $C^2$ submanifold $M$ of dimension $m \leq n - k$, with $k \geq \frac{n}{2}$ for $n > 2$, see $(A_1^\Gamma)$. Then, if $\Omega$ is convex and symmetric with respect to the $x_1$-direction and

$$\Gamma \subset \{x \in \Omega : x_1 = 0\},$$

it follows that $u$ is symmetric with respect to the hyperplane $\{x_1 = 0\}$ and increasing in the $x_1$-direction in $\Omega \cap \{x_1 < 0\}$.

The paper is organized as follows: we prove some technical results in Section 2 that we will exploit in Section 3 to prove Theorems 1.2 and 1.3.

## 2. Some Technical Results

**Notation.** Generic fixed and numerical constants will be denoted by $C$ (with subscript in some case) and they will be allowed to vary within a single line or formula. By $|A|$ we will denote the Lebesgue measure of a measurable set $A$.

For a real number $\lambda$ we set

$$\Omega_\lambda = \{x \in \Omega : x_1 < \lambda\}$$

(2.2)

$$x_\lambda = R_\lambda(x) = (2\lambda - x_1, x_2, \ldots, x_n)$$

(2.3)

which is the reflection through the hyperplane $T_\lambda := \{x \in \mathbb{R}^n : x_1 = \lambda\}$. Also let

$$a = \inf_{x \in \Omega} x_1.$$  

(2.4)
Finally we set
(2.5) \[ u_\lambda(x) = u(x_\lambda). \]
We recall also the definition of \( p \)-capacity of a compact set \( A \subset \mathbb{R}^n \). For \( 1 \leq p \leq n \) we define \( \text{Cap}_p(A) \) as
(2.6) \[ \text{Cap}_p(A) := \inf \left\{ \int_{\mathbb{R}^n} |\nabla \varphi|^p \, dx < +\infty : \varphi \in C_c^\infty(\mathbb{R}^n) \text{ and } \varphi \geq \chi_A \right\}, \]
where \( \chi_S \) denotes the characteristic function of a set \( S \). By the invariance under reflections of (2.6), it follows that
(2.7) \[ \text{Cap}_p(\Gamma) = \text{Cap}_p(\mathcal{R}_\lambda(\Gamma)). \]
Moreover it can be shown that, if \( \text{Cap}_p(\mathcal{R}_\lambda(\Gamma)) = 0 \), then we have that
(2.8) \[ \text{Cap}_p^D(\mathcal{R}_\lambda(\Gamma)) = 0, \]
where \( D \subset \mathbb{R}^n \) denotes a bounded subset and with \( \text{Cap}_p^D(A) \) \((A \subset D \text{ a compact set of } \mathbb{R}^n)\) we mean
\[ \text{Cap}_p^D(A) := \inf \left\{ \int_D |\nabla \varphi|^p \, dx < +\infty : \varphi \in C_c^\infty(D) \text{ and } \varphi \geq \chi_A \right\}. \]
Let \( \varepsilon > 0 \) small and let \( B_\varepsilon^\lambda \) be a \( \varepsilon \)-neighborhood of \( \mathcal{R}_\lambda(\Gamma) \). From (2.7) and (2.8) it follows that there exists \( \varphi_\varepsilon \in C_c^\infty(B_\varepsilon^\lambda) \) such that \( \varphi_\varepsilon \geq 1 \) on \( \chi_{\mathcal{R}_\lambda(\Gamma)} \) and
\[ \int_{B_\varepsilon^\lambda} |\nabla \varphi_\varepsilon|^p \, dx < \varepsilon. \]
To carry on our analysis we need to construct a function \( \psi_\varepsilon \in W^{1,p}(\Omega) \) such that \( \psi_\varepsilon = 1 \) in \( \Omega \setminus B_\varepsilon^\lambda \), \( \psi_\varepsilon = 0 \) in a \( \delta_\varepsilon \)-neighborhood \( B_{\delta_\varepsilon}^\lambda \) of \( \mathcal{R}_\lambda(\Gamma) \) (with \( \delta_\varepsilon < \varepsilon \)) and such that
(2.9) \[ \int_{R_\varepsilon^\lambda} |\nabla \psi_\varepsilon|^p \, dx \leq C\varepsilon, \]
for some positive constant \( C \) that does not depend on \( \varepsilon \). To construct such a test function we consider the real functions \( T : \mathbb{R} \to \mathbb{R}^{+} \) and \( g : \mathbb{R}^{+} \to \mathbb{R}^{+} \) defined by
(2.10) \[ T(s) := \max\{0; \min\{s; 1\}\}, s \in \mathbb{R} \text{ and } g(s) := \max\{0; -2s + 1\}, s \in \mathbb{R}^{+}. \]
Finally we set
(2.11) \[ \psi_\varepsilon(x) := g(T(\varphi_\varepsilon(x))). \]
By the definitions (2.10), it follows that \( \psi_\varepsilon \) satisfies (2.9).
To simplify the presentation we summarize the assumptions of the main results as follows:
\((A_1^f)\). For \( 1 < p < 2 \) we assume that \( f \) is locally Lipschitz continuous so that, for any \( 0 \leq t, s \leq M \), there exists a positive constant \( K_f = K_f(M) \) such that\[ |f(s) - f(t)| \leq K_f |s - t|. \]
Moreover \( f(s) > 0 \) for \( s > 0 \) and
\[ \lim_{t \to +\infty} \frac{f(t)}{t^q} = l \in (0, +\infty). \]
for some \( q \in \mathbb{R} \) such that \( p - 1 < q < p^* - 1 \), where \( p^* = np/(n - p) \).
(A²)

For $p \geq 2$ we only assume that $f$ is locally Lipschitz continuous so that, for $0 \leq t, s \leq M$ there exists a positive constant $K_f = K_f(M)$ such that

$$|f(s) - f(t)| \leq K_f|s - t|.$$ 

Furthermore $f(s) > 0$ for $s > 0$.

(A¹)

For $1 < p < 2$ and $n = 2$ we assume that $\Gamma = \{0\}$, while for $1 < p < 2$ and $n > 2$ we assume that $\Gamma \subseteq \mathcal{M}$ for some compact $C^2$ submanifold $\mathcal{M}$ of dimension $m \leq n - k$, with $k \geq \frac{n}{2}$.

(A²)

For $2 < p < n$ and $n \geq 2$, we assume that $\Gamma$ closed and such that

$$\text{Cap}_p(\Gamma) = 0.$$ 

Remark 2.1. We want just to remark that in the case $1 < p < 2$ and $N > 2$ if $\Gamma \subseteq \mathcal{M}$ for some compact $C^2$ submanifold $\mathcal{M}$ of dimension $m \leq n - k$ then $\text{Cap}_p(\Gamma) = 0$. In this case we consider $B_\varepsilon$ a tubular neighborhood of radius $\varepsilon$ of $\mathcal{M}$, i.e.

$$B_\varepsilon := \{x \in \Omega : \text{dist}(x, \mathcal{M}) < \varepsilon\},$$

with $\varepsilon > 0$ sufficiently small so that $\mathcal{M}$ has the unique nearest point property in the neighborhood of $\mathcal{M}$ of radius $\varepsilon$. We may and do also assume that Fermi coordinates are well defined in such neighborhood, see e.g. [10]. Therefore, using the definition (2.6) above, it can be shown that $\text{Cap}_p(\Gamma) = 0$.

Moreover (see for example [3]) in the following we further use the following inequalities: for all $\eta, \eta' \in \mathbb{R}^n$ with $|\eta| + |\eta'| > 0$ there exists positive constants $C_1, C_2, C_3, C_4$ depending on $p$ such that

$$|\eta|^{p-2}\eta - |\eta'|^{p-2}\eta'| \eta - \eta'| \geq C_1(|\eta| + |\eta'|)^{p-2}|\eta - \eta'|^2,$$

$$|\eta|^{p-2}\eta - |\eta'|^{p-2}\eta'| \leq C_2(|\eta| + |\eta'|)^{p-2}|\eta - \eta'|,$$

$$|\eta|^{p-2}\eta - |\eta'|^{p-2}\eta'| \eta - \eta'| \geq C_3|\eta - \eta'|^p \quad \text{if } p \geq 2,$$

$$|\eta|^{p-2}\eta - |\eta'|^{p-2}\eta'| \leq C_4|\eta - \eta'|^{p-1} \quad \text{if } 1 < p \leq 2.$$ 

(2.12)

In the following we will exploit the fact that $u_\lambda$ (in the sense of Definition 1.1) is a solution to

$$\int_{R_\lambda(\Omega)} |\nabla u_\lambda|^{p-2}(\nabla u_\lambda, \nabla \varphi)\, dx = \int_{R_\lambda(\Omega)} f(u_\lambda)\varphi\, dx \quad \forall \varphi \in C^1_c(R_\lambda(\Omega) \setminus R_\lambda(\Gamma)).$$

We set

$$w_\lambda(x) := (u - u_\lambda)(x), \quad x \in \overline{\Omega_\lambda} \setminus R_\lambda(\Gamma).$$

Lemma 2.2. Let $p > 1$ and let $u$ and $u_\lambda$ be solutions to (1.1) and (2.13) respectively and let $f : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function. Let us assume $\Gamma \subset \Omega$ closed and such that

$$\text{Cap}_p(\Gamma) = 0.$$ 

Let $a$ be defined as in (2.4) and $a < \lambda < 0$. 

Then
\[
\int_{\Omega_\lambda} \left( |\nabla u| + |\nabla u_\lambda| \right)^{p-2} \left| \nabla w_\lambda^+ \right|^2 \, dx \leq C(p, \lambda, \|u\|_{L^\infty(\Omega_\lambda)}).
\]

Proof. In all the proof, according to our assumptions, we assume that \(0 \leq t, s \leq M\), there exists a positive constant \(K_f = K_f(M)\) such that

\[|f(s) - f(t)| \leq K_f |s - t|.
\]

For \(\psi_\epsilon\) defined as in (2.11), we consider

\[\varphi_\epsilon := w_\lambda^+ \psi_\epsilon^p \chi_{\Omega_\lambda}.
\]

By standard arguments, since \(w_\lambda^+ \leq \|u\|_{L^\infty(\Omega_\lambda)}\) (recall that in particular \(u \in C(\overline{\Omega} \setminus \Gamma)\)) and by construction \(0 \leq \psi_\epsilon \leq 1\), we have that \(\varphi_\epsilon \in W^{1,p}_0(\Omega_\lambda)\). By a density argument we use \(\varphi_\epsilon\) as test function test function in (1.1) and (2.13). Subtracting we get

\[
\int_{\Omega_\lambda} \left( (|\nabla u|^{p-2} \nabla u - |\nabla u_\lambda|^{p-2} \nabla u_\lambda, \nabla w_\lambda^+) \psi_\epsilon^p \right) \, dx
\]

\[
+ p \int_{\Omega_\lambda} \left( (|\nabla u|^{p-2} \nabla u - |\nabla u_\lambda|^{p-2} \nabla u_\lambda, \nabla \psi_\epsilon) \psi_\epsilon^{p-1} w_\lambda^+ \right) \, dx
\]

\[
= \int_{\Omega_\lambda} (f(u) - f(u_\lambda)) w_\lambda^+ \psi_\epsilon^p \, dx.
\]

Now it is useful to split the set \(\Omega_\lambda\) as the union of two disjoint subsets \(A_\lambda\) and \(B_\lambda\) such that \(\Omega_\lambda = A_\lambda \cup B_\lambda\). In particular, for \(\hat{C} > 1\) that will be fixed large, we set

\[A_\lambda = \{x \in \Omega_\lambda : |\nabla u_\lambda(x)| < \hat{C} |\nabla u(x)|\} \quad \text{and} \quad B_\lambda = \{x \in \Omega_\lambda : |\nabla u_\lambda(x)| \geq \hat{C} |\nabla u(x)|\}.
\]

Then it follows that

- By the definition of \(A_\lambda\) it follows that there exists \(\hat{C}\) such that

\[
|\nabla u| + |\nabla u_\lambda| < \hat{C} |\nabla u|.
\]

- By the definition of the set \(B_\lambda\) and standard triangular inequalities, we can deduce the existence of a positive constant \(\hat{C}\) such that

\[
\frac{1}{\hat{C}} |\nabla u_\lambda| \leq |\nabla u_\lambda| - |\nabla u| \leq |\nabla w_\lambda| \leq |\nabla u_\lambda| + |\nabla u| \leq \hat{C} |\nabla u_\lambda|.
\]

We distinguish two cases:

Case 1: \(1 < p < 2\). From (2.15), using (2.12) and (\(A_f^1\)) we have

\[
C_1 \int_{\Omega_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} \left| \nabla w_\lambda^+ \right|^2 \psi_\epsilon^p \, dx \leq \int_{\Omega_\lambda} (|\nabla u|^{p-2} \nabla u - |\nabla u_\lambda|^{p-2} \nabla u_\lambda, \nabla w_\lambda^+) \psi_\epsilon^p \, dx
\]

\[
\leq p \int_{\Omega_\lambda} \left( |\nabla u|^{p-2} \nabla u - |\nabla u_\lambda|^{p-2} \nabla u_\lambda \right) \left| \nabla \psi_\epsilon \right| \psi_\epsilon^{p-1} w_\lambda^+ \, dx
\]

\[
+ \int_{\Omega_\lambda} \frac{f(u) - f(u_\lambda)}{u - u_\lambda} (w_\lambda^+)^2 \psi_\epsilon^p \, dx
\]

\[
\leq pC_4 \int_{\Omega_\lambda} \left| \nabla \psi_\epsilon \right| \psi_\epsilon^{p-1} w_\lambda^+ \, dx
\]

\[
+ K_f \int_{\Omega_\lambda} (w_\lambda^+)^2 \psi_\epsilon^p \, dx
\]

\[
\leq C \left( I_1 + I_2 \right) + C \int_{\Omega_\lambda} \psi_\epsilon^p \, dx,
\]
where

\[
I_1 := \int_{A_\lambda} |\nabla w_\lambda^+|^{p-1} |\nabla \psi_\varepsilon| \psi_\varepsilon^{-1} w_\lambda^+ dx,
\]

\[
I_2 := \int_{B_\lambda} |\nabla w_\lambda^+|^{p-1} |\nabla \psi_\varepsilon| \psi_\varepsilon^{-1} w_\lambda^+ dx,
\]

and \( C = C(p, \lambda, \|u\|_{L^\infty(\Omega_\lambda)}) \) is a positive constant.

**Step 1: Evaluation of \( I_1 \).** Using Young’s inequality and (2.16), we have

\[
I_1 = \int_{A_\lambda} |\nabla w_\lambda^+|^{p-1} |\nabla \psi_\varepsilon| \psi_\varepsilon^{-1} w_\lambda^+ dx \leq \left( \int_{A_\lambda} |\nabla w_\lambda^+|^{p} \psi_\varepsilon^{p} dx \right)^{\frac{p-1}{p}} \left( \int_{A_\lambda} |\nabla \psi_\varepsilon|^p (w_\lambda^+) p dx \right)^{\frac{1}{p}}
\]

\[
\leq \left( C \int_{A_\lambda} |\nabla u|^p \psi_\varepsilon^{p} dx \right)^{\frac{p-1}{p}} \left( \int_{A_\lambda} |\nabla \psi_\varepsilon|^p (w_\lambda^+) p dx \right)^{\frac{1}{p}}
\]

\[
\leq C \left( \int_{A_\lambda} |\nabla u|^p dx \right)^{\frac{p-1}{p}} \left( \int_{A_\lambda} |\nabla \psi_\varepsilon|^p dx \right)^{\frac{1}{p}},
\]

where \( C = C(p, \lambda, \|u\|_{L^\infty(\Omega_\lambda)}) \) is a positive constant.

**Step 2: Evaluation of \( I_2 \).** Using the weighted Young’s inequality and (2.17) we get

\[
I_2 = \int_{B_\lambda} |\nabla w_\lambda^+|^{p-1} |\nabla \psi_\varepsilon| \psi_\varepsilon^{-1} w_\lambda^+ dx \leq \delta \int_{B_\lambda} |\nabla w_\lambda^+|^{p} \psi_\varepsilon^{p} dx + \frac{1}{\delta} \int_{B_\lambda} |\nabla \psi_\varepsilon|^p (w_\lambda^+) p dx
\]

\[
\leq \delta \int_{B_\lambda} (|\nabla u| + |\nabla u|)^{p-2} (|\nabla u| + |\nabla u|)^2 \psi_\varepsilon^{p} dx + \frac{1}{\delta} \int_{B_\lambda} |\nabla \psi_\varepsilon|^p (w_\lambda^+) p dx
\]

\[
\leq \delta C \int_{B_\lambda} (|\nabla u| + |\nabla u|)^{p-2} |\nabla \psi_\varepsilon|^2 \psi_\varepsilon^{p} dx + \frac{1}{\delta} \int_{B_\lambda} |\nabla \psi_\varepsilon|^p (w_\lambda^+) p dx
\]

\[
\leq \delta C \int_{B_\lambda} (|\nabla u| + |\nabla u|)^{p-2} |\nabla \psi_\varepsilon|^2 \psi_\varepsilon^{p} dx + \frac{C}{\delta} \int_{B_\lambda} |\nabla \psi_\varepsilon|^p dx,
\]

(2.20)

where \( C = C(p, \lambda, \|u\|_{L^\infty(\Omega_\lambda)}) \) is a positive constant. Finally, using (2.18), (2.19) and (2.20), we obtain

\[
\int_{\Omega_\lambda} (|\nabla u| + |\nabla u|)^{p-2} |\nabla w_\lambda^+|^2 \psi_\varepsilon dx
\]

\[
\leq \delta C \int_{\Omega_\lambda} (|\nabla u| + |\nabla u|)^{p-2} |\nabla \psi_\varepsilon|^2 \psi_\varepsilon dx + C \left( \int_{\Omega_\lambda} |\nabla u|^p dx \right)^{\frac{p-1}{p}} \left( \int_{\Omega_\lambda} |\nabla \psi_\varepsilon|^p dx \right)^{\frac{1}{p}}
\]

\[
+ \frac{C}{\delta} \int_{\Omega_\lambda} |\nabla \psi_\varepsilon|^p dx + C \int_{\Omega_\lambda} \psi_\varepsilon^p dx,
\]

for some positive constant \( C = C(p, \lambda, \|u\|_{L^\infty(\Omega_\lambda)}) \).
Case 2: \( p \geq 2 \). From (2.15), using (2.12) and (\( A_f^p \)) we have

\[
C_1 \int_{\Omega} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w^+_\lambda|^2 \psi^p \, dx \leq \int_{\Omega} (|\nabla u|^{p-2} \nabla u - |\nabla u_\lambda|^{p-2} \nabla u_\lambda, \nabla w^+_\lambda) \psi^p \, dx
\]

\[
= -p \int_{\Omega} (|\nabla u|^{p-2} \nabla u - |\nabla u_\lambda|^{p-2} \nabla u_\lambda, \nabla \psi_\epsilon) \psi_{\epsilon}^{p-1} w^+_\lambda \, dx + \int_{\Omega} (f(u) - f(u_\lambda)) w^+_\lambda \psi^p \, dx
\]

\[
\leq p \int_{\Omega} ||\nabla u||^{p-2} \nabla u - |\nabla u_\lambda|^{p-2} \nabla u_\lambda| |\nabla \psi_\epsilon| \psi_{\epsilon}^{p-1} w^+_\lambda \, dx + \int_{\Omega} \frac{f(u) - f(u_\lambda)}{u - u_\lambda} (w^+_\lambda)^2 \psi^p \, dx
\]

\[
\leq pC_2 \int_{\Omega} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w^+_\lambda| |\nabla \psi_\epsilon| \psi_{\epsilon}^{p-1} w^+_\lambda \, dx + Kf \int_{\Omega} (w^+_\lambda)^2 \psi^p \, dx
\]

\[
= pC_2 \int_{A_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w^+_\lambda| |\nabla \psi_\epsilon| \psi_{\epsilon}^{p-1} w^+_\lambda \, dx + Kf \int_{\Omega} (w^+_\lambda)^2 \psi^p \, dx
\]

\[
\leq C \left( I_1 + I_2 \right) + C \int_{\Omega} \psi^p \, dx,
\]

where

\[
I_1 := \int_{A_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w^+_\lambda| |\nabla \psi_\epsilon| \psi_{\epsilon}^{p-1} w^+_\lambda \, dx,
\]

\[
I_2 := \int_{B_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w^+_\lambda| |\nabla \psi_\epsilon| \psi_{\epsilon}^{p-1} w^+_\lambda \, dx,
\]

and \( C = C(p, \lambda, \|u\|_{L^\infty(\Omega)}) \) is a positive constant.

Step 1: Evaluation of \( I_1 \). Using the weighted Young’s inequality we have

\[
I_1 = \int_{A_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w^+_\lambda|^2 \psi^p \, dx
\]

\[
\leq \delta \int_{A_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w^+_\lambda|^2 \psi_{\epsilon}^p \, dx + \frac{1}{\delta} \int_{A_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla \psi_\epsilon|^2 \psi_{\epsilon}^{p-2} (w^+_\lambda)^2 \, dx.
\]

Using (2.16) and Hölder inequality, we obtain

\[
I_1 \leq \delta \int_{A_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w^+_\lambda|^2 \psi^p \, dx + \frac{C^{p-2}}{\delta} \int_{A_\lambda} |\nabla u|^{p-2} |\nabla \psi_\epsilon|^2 \psi_{\epsilon}^{p-2} (w^+_\lambda)^2 \, dx
\]

\[
\leq \delta \int_{A_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w^+_\lambda|^2 \psi^p \, dx + \frac{C}{\delta} \left( \int_{A_\lambda} |\nabla \psi_\epsilon|^p \, dx \right)^{\frac{p-2}{p}} \left( \int_{A_\lambda} |\nabla u|^{p} \, dx \right)^{\frac{2}{p}}
\]

\[
\leq \delta \int_{A_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w^+_\lambda|^2 \psi^p \, dx + \frac{C}{\delta} \left( \int_{\Omega} |\nabla u|^p \, dx \right)^{\frac{p-2}{p}} \left( \int_{\Omega} |\nabla \psi_\epsilon|^p \, dx \right)^{\frac{2}{p}},
\]

with \( C = C(p, \lambda, \|u\|_{L^\infty(\Omega)}) \) is a positive constant.
Step 2: Evaluation of $I_2$. By the weighted Young’s inequality
\[
I_2 := \int_{B_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w_\lambda^+| |\nabla \psi_\varepsilon|^p w_\lambda dx
\]
\[
\leq \delta \int_{B_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{\frac{p-2}{p-1}} |\nabla w_\lambda^+|^2 |\nabla \psi_\varepsilon|^p dx + \frac{1}{\delta} \int_{B_\lambda} |\nabla \psi_\varepsilon|^p (w_\lambda^+)^p dx
\]
\[
= \delta \int_{B_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{\frac{p-2}{p-1}} |\nabla w_\lambda^+|^2 |\nabla \psi_\varepsilon|^p dx + \frac{1}{\delta} \int_{B_\lambda} |\nabla \psi_\varepsilon|^p (w_\lambda^+)^p dx.
\]
Using (2.17) and noticing that
\[
\int_{\Omega} (\nabla u_\lambda + \nabla \psi_\varepsilon) \cdot (\nabla u_\lambda - \nabla \psi_\varepsilon) dx = 0
\]
we obtain the following estimate
\[
I_2 \leq \delta C^{\frac{(p-2)(p+1)}{p-1}} \int_{B_\lambda} |\nabla u_\lambda|^{p-2} |\nabla w_\lambda^+|^2 |\nabla \psi_\varepsilon|^p dx + \frac{1}{\delta} \int_{B_\lambda} |\nabla \psi_\varepsilon|^p (w_\lambda^+)^p dx
\]
\[
(2.24) \quad \leq \delta C \int_{B_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w_\lambda^+|^2 |\nabla \psi_\varepsilon|^p dx + \frac{C}{\delta} \int_{B_\lambda} |\nabla \psi_\varepsilon|^p dx
\]
\[
\leq \delta C \int_{\Omega} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w_\lambda^+|^2 |\nabla \psi_\varepsilon|^p dx + \frac{C}{\delta} \int_{\Omega} |\nabla \psi_\varepsilon|^p dx,
\]
with $C = C(p, \|u\|_{L^\infty(\Omega)})$. In the second line of (2.24) we exploited the fact that, since $p \geq 2$ then
\[
|\nabla u_\lambda|^{p-2} \leq (|\nabla u| + |\nabla u_\lambda|)^{p-2}.
\]
Collecting (2.22), (2.23) and (2.24) we deduce that
\[
\int_{\Omega} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w_\lambda^+|^2 |\nabla \psi_\varepsilon|^p dx
\]
\[
\leq \delta C \left( \int_{\Omega} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w_\lambda^+|^2 |\nabla \psi_\varepsilon|^p dx + \frac{C}{\delta} \left( \int_{\Omega} |\nabla u|^p dx \right)^{\frac{p-2}{p}} \left( \int_{\Omega} |\nabla \psi_\varepsilon|^p dx \right)^{\frac{2}{p}} \right)
\]
\[
+ \frac{C}{\delta} \int_{\Omega} |\nabla \psi_\varepsilon|^p dx + C \int_{\Omega} \psi_\varepsilon^p dx,
\]
for some positive constant $C = C(p, \lambda, \|u\|_{L^\infty(\Omega)})$.

For $\delta$ small, from (2.21) and (2.25), using (2.9) and the fact that for $\lambda < 0$ the solution $u \in W^{1,p}(\Omega_\lambda)$, letting $\varepsilon \to 0$ by Fatou’s Lemma we obtain
\[
\int_{\Omega_\lambda} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w_\lambda^+|^2 dx \leq C(p, \lambda, \|u\|_{L^\infty(\Omega_\lambda)}),
\]
concluding the proof.

3. Proof of Theorem 1.2 and Theorem 1.3

We recall the fact that $u_\lambda$ (in the sense of Definition 1.1) is a solution to
\[
(3.26) \quad \int_{R_\lambda(\Omega)} |\nabla u_\lambda|^{p-2}(\nabla u_\lambda, \nabla \varphi) dx = \int_{R_\lambda(\Omega)} f(u_\lambda) \varphi dx \quad \forall \varphi \in C_c^1(R_\lambda(\Omega) \setminus R_\lambda(\Gamma)).
\]
We set
\[ w_\lambda(x) := (u - u_\lambda)(x), \quad x \in \Omega \setminus (\Gamma \cup R_\lambda(\Gamma)). \]
Since in the following we will exploit weighted Sobolev inequalities, it is convenient to set weight
\[ (3.27) \quad \hat{\varrho} := |\nabla u|^{p-2} \quad \frac{1}{\hat{\varrho}} := |\nabla u|^{2-p}. \]
We have the following

**Lemma 3.1.** Let \( 1 < p < 2 \). Under the same assumption of Theorem 1.2, define
\[ \Omega_\lambda^+ := \Omega_\lambda \cap \text{supp}(w_\lambda^+). \]
Then
\[ (3.28) \quad |\nabla u|^{2-p} \in L^t(R_\lambda(\Omega_\lambda^+)), \]
for some \( t > \frac{n}{2} \).

**Proof.** By definition of \( \Omega_\lambda^+ \) we have
\[ \|u\|_{L^\infty(R_\lambda(\Omega_\lambda^+))} = \|u_\lambda\|_{L^\infty(\Omega_\lambda^+)} \leq \|u\|_{L^\infty(\Omega_\lambda)} \leq C(\lambda, \|u\|_{L^\infty(\Omega_\lambda)}). \]
Taking \( x_0 \in R_\lambda(\Omega_\lambda^+) \setminus \Gamma \), we set:
\[ (3.29) \quad g(x) := u(dx + x_0) \quad \text{in} \quad B_{\frac{d}{2}}(0), \]
where \( d := \text{dist}(x_0, \Gamma) \). Since \( u \) is a solution (in the sense of Definition 1.1) to \((P_\Gamma)\), we deduce that for any \( \varphi \in C^1_c(B_{1/2}(0)) \)
\[ (3.30) \quad \int_{B_{\frac{d}{2}}(0)} |\nabla g|^{p-2}(\nabla g, \nabla \varphi) \, dx \]
\[ = \int_{B_{\frac{d}{2}}(0)} |\nabla u(dx + x_0)|^{p-2}(\nabla u(dx + x_0), \nabla \varphi) \, dx \]
\[ = \int_{B_{\frac{d}{2}}(x_0)} |\nabla u(x)|^{p-2}(\nabla u(x), \nabla \left(\varphi\left(\frac{x-x_0}{d}\right)\right)) \, dx = \int_{B_{\frac{d}{2}}(x_0)} f(u(x))\varphi\left(\frac{x-x_0}{d}\right) \, dx \]
\[ = \int_{B_{\frac{d}{2}}(0)} f(u(dx + x_0))\varphi(x) \, dx = \int_{B_{\frac{d}{2}}(0)} c(x)(g(x))^{p-1}\varphi(x) \, dx, \]
with
\[ (3.31) \quad c(x) := \frac{d^p f(u(dx + x_0))}{u^{p-1}(dx + x_0)}. \]
From (3.30) we deduce that in distributional sense
\[ -\Delta_p g = c(x)g^{p-1} \quad \text{in} \quad B_{\frac{d}{2}}(0). \]
Moreover \( u \) as well (in distributional sense) is a positive solution to \(-\Delta u = f(u)\) in \( B_d(x_0) \). Therefore using [11, Theorem 3.1] we have
\[ (3.32) \quad 0 < u(x) \leq C(1 + d^{-\frac{p}{n+1-p}}), \]
where \( C = C(f, n, p) > 0 \). By (3.31), using (A_1^1) we have
\[
(3.33) \quad c(x) = Cd^p(1 + u^{q+1-p}),
\]
with \( C = C(l, p, K_f) \) is a positive constant. Finally, collecting (3.32) and (3.33) we deduce
\[
c(x) \leq Cd^p(1 + d^{-p}) \leq C,
\]
with \( C = C(f, l, n, p, q, K_f, \Omega) \). Hence \( c(x) \in L^\infty(B_{1/2}(0)) \). By [12, Theorem 7.2.1], recalling (3.29), for every \( x \in B_{1/8}(0) \) it follows
\[
(3.34) \quad g(x) \leq \sup_{x \in B_{\frac{1}{4}}(0)} g(x)
\]
\[
\leq C_H \inf_{x \in B_{\frac{1}{4}}(0)} g(x) \leq C_H g(0) \leq C
\]
where \( C = C(f, l, n, p, q, K_f, \Omega) \) is a positive constant. Hence \( g(x) \) is bounded in \( B_{1/8}(0) \) and as consequence, see e.g. [7, 15]
\[
E_1 := \int_{\mathcal{M}} d\sigma \int_0^\varepsilon \frac{1}{r(2-p)t-(k-1)} \, dr < +\infty,
\]
if \( t < k/(2 - p) \), recalling that \( 1 < p < 2 \). Hence, since \( k \geq n/2 \), inequality (3.37) holds for some
\[
t \in \left( \frac{n}{2}, \frac{k}{2 - p} \right),
\]
being $2k > n(2 - p)$ under our assumption. \hfill \Box

Let us now set

\[ Z_\lambda := \{ x \in \Omega_\lambda \setminus R_\lambda(\Gamma) \mid \nabla u(x) = \nabla u_\lambda(x) = 0 \}. \]

We have the following

**Lemma 3.2.** Let $u$ be a solution to \((1.1)\) with $f : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function such that $f(s) > 0$ for $s > 0$. Let $a < \lambda < 0$. If $C_\lambda \subset \Omega_\lambda \setminus (R_\lambda(\Gamma) \cup Z_\lambda)$ is a connected component of $\Omega_\lambda \setminus (R_\lambda(\Gamma) \cup Z_\lambda)$ with $u = u_\lambda$ in $C_\lambda$, then $C_\lambda = \emptyset$.

**Proof.** Let $C := C_\lambda \cup R_\lambda(C_\lambda)$. Arguing by contradiction we assume $C \neq \emptyset$. Now for $\varepsilon > 0$, we define $h_\varepsilon(t) : \mathbb{R}_0^+ \to \mathbb{R}$ as

\[ h_\varepsilon(t) = \begin{cases} \frac{G_\varepsilon(t)}{t} & \text{if } t > 0 \\ 0 & \text{if } t = 0, \end{cases} \]

where $G_\varepsilon(t) = (2t - 2\varepsilon)t \chi_{[\varepsilon, 2\varepsilon]}(t) + t \chi_{[2\varepsilon, \infty)}(t)$ for $t \geq 0$. Moreover we consider the cut-off function $\psi_\varepsilon$ on the set $\Gamma \cup R_\lambda(\Gamma)$ defined in a similar way as in \((2.11)\). Hence we define the test function

\[ \varphi_\varepsilon := h_\varepsilon(|\nabla u|)\psi_\varepsilon^2 \chi_C. \]

We point out that the supp $\varphi_\varepsilon \subset C$ and therefore we can use it as test function in \((1.1)\). We obtain

\[ 0 < \int_C f(u)h_\varepsilon(|\nabla u|)\psi_\varepsilon^2 \, dx = \int_C |\nabla u|^{p-2}(\nabla u, \nabla |\nabla u|)h_\varepsilon'(|\nabla u|)\psi_\varepsilon^2 \, dx + 2 \int_C |\nabla u|^{p-2}(\nabla u, \nabla \psi_\varepsilon)h_\varepsilon(|\nabla u|)\psi_\varepsilon \, dx. \]

Using Schwartz inequality, observing that $h_\varepsilon(t) \leq 1$ and $h_\varepsilon'(t) \leq 2/\varepsilon$,

we obtain

\[ 0 < \int_C f(u) \frac{G_\varepsilon(|\nabla u|)}{|\nabla u|} \psi_\varepsilon^2 \, dx \]

\[ \leq 2 \int_{C \cap \{ \varepsilon < |\nabla u| < 2\varepsilon \}} |\nabla u|^{p-2} \| D^2 u \| |\nabla \psi_\varepsilon| \frac{|\nabla u|}{\varepsilon} \, dx + 2 \int_C |\nabla u|^{p-1} |\nabla \psi_\varepsilon| \psi_\varepsilon \, dx \]

\[ \leq 4 \int_{C \cap \{ \varepsilon < |\nabla u| < 2\varepsilon \}} |\nabla u|^{p-2} \| D^2 u \| |\nabla \psi_\varepsilon| \psi_\varepsilon \, dx + 2 \int_C |\nabla u|^{p-1} |\nabla \psi_\varepsilon| \psi_\varepsilon \, dx \]

\[ \leq 4 \int_C |\nabla u|^{p-2} \| D^2 u \| \psi_\varepsilon \chi_{A_\varepsilon} \, dx + 2 \left( \int_C |\nabla u|^{p} \, dx \right)^{\frac{p-1}{p}} \left( \int_C |\nabla \psi_\varepsilon|^{p} \, dx \right)^{\frac{1}{p}}, \]

where $A_\varepsilon := C \cap \{ \varepsilon < |\nabla u| < 2\varepsilon \}$. Now we note that by the definition of the region $C$ and because $u = u_\lambda$ in $C_\lambda$, then the solution $u$ is bounded and $C^{1,\alpha}$ by classical regularity results. Moreover

\[ |\nabla u|^{p-2} \| D^2 u \| \psi_\varepsilon \chi_{A_\varepsilon} \leq |\nabla u|^{p-2} \| D^2 u \| \]

and $|\nabla u|^{p-2} \| D^2 u \| \in L^1(C)$ by \([6]\) (see also \([9]\) Lemma 5 for details). It is important to note that the regularity of the solution in $R_\lambda(C_\lambda)$ is induced by symmetry by the regularity
in $C_\lambda$. Noticing that $|\nabla u|^{p-2}D^2u\psi^2\chi_\lambda \to 0$ as $\varepsilon$ goes to 0, then letting $\varepsilon \to 0$ in (3.38), by Dominated Convergence Theorem and (2.9) it follows

$$0 < \int_C f(u) \, dx \leq 0,$$

and this gives a contradiction. Hence $C = \emptyset$.

Proof of Theorem 1.3. Since the singular set $\Gamma$ is contained in the hyperplane $\{x_1 = 0\}$, then the moving plane procedure can be started in the standard way (see e.g. [4, 5, 6]) and, for $a < \lambda < a + \sigma$ with $\sigma > 0$ small, we have that $w_\lambda \leq 0$ in $\Omega_\lambda$ (see (2.14)) by the weak comparison principle in small domains. Note that the crucial point here is that $w_\lambda$ has a singularity at $\Gamma$ and at $R_\lambda(\Gamma)$. For $\lambda$ close to $a$ the singularity does not play a role. To proceed further we define

$$\Lambda_0 = \{a < \lambda < 0 : u \leq u_t \, \text{in} \, \Omega_t \setminus R_t(\Gamma) \, \text{for all} \, t \in (a, \lambda]\}$$

and $\lambda_0 = \sup \Lambda_0$, since we proved above that $\Lambda$ is not empty. To prove our result we have to show that $\lambda_0 = 0$. To do this we assume that $\lambda_0 < 0$ and we reach a contradiction by proving that $u \leq u_{\lambda_0 + \tau}$ in $\Omega_{\lambda_0 + \tau} \setminus R_{\lambda_0 + \tau}(\Gamma)$ for any $0 < \tau < \tilde{\tau}$ for some small $\tilde{\tau} > 0$. We remark that $|Z_{\lambda_0}| = 0$, see [8]. Let us take $A_{\lambda_0} \subset \Omega_{\lambda_0}$ be an open set such that $Z_{\lambda_0} \cap \Omega_{\lambda_0} \subset A_{\lambda_0} \subset \Omega$. Such set exists by Hopf’s Lemma. Moreover note that, since $|Z_{\lambda_0}| = 0$, we can take $A_{\lambda_0}$ of arbitrarily small measure. By continuity we know that $u \leq u_{\lambda_0}$ in $\Omega_{\lambda_0} \setminus R_{\lambda_0}(\Gamma)$. We can exploit the strong comparison principle, see e.g. [12, Theorem 2.5.2] to get that, in any connected component of $\Omega_{\lambda_0} \setminus Z_{\lambda_0}$, we have

$$u < u_{\lambda_0} \quad \text{or} \quad u \equiv u_{\lambda_0}.$$

The case $u \equiv u_{\lambda_0}$ in some connected component $C_{\lambda_0}$ of $\Omega_{\lambda_0} \setminus Z_{\lambda_0}$ is not possible, since by symmetry, it would imply the existence of a local symmetry phenomenon and consequently that $\Omega \setminus Z_{\lambda_0}$ would be not connected, in spite of what we proved in Lemma 3.2. Hence we deduce that $u < u_{\lambda_0}$ in $\Omega_{\lambda_0} \setminus R_{\lambda_0}(\Gamma)$. Therefore, given a compact set $K \subset \Omega_{\lambda_0} \setminus (R_{\lambda_0}(\Gamma) \cup A_{\lambda_0})$, by uniform continuity we can ensure that $u < u_{\lambda_0 + \tau}$ in $K$ for any $0 < \tau < \tilde{\tau}$ for some small $\tilde{\tau} > 0$.

Note that to do this we implicitly assume, with no loss of generality, that $R_{\lambda_0}(\Gamma)$ remains bounded away from $K$. Arguing in a similar fashion as in Lemma 2.2 we consider

$$\varphi_\varepsilon := w^{+}_{\lambda_0 + \tau} \psi^p \chi_{\Omega_{\lambda_0} + \tau}.$$

By density arguments as above, we plug $\varphi_\varepsilon$ as test function in (1.1) and (3.20) so that, subtracting, we get

$$\int_{\Omega_{\lambda_0 + \tau} \setminus K} (|\nabla u|^{p-2} \nabla u - |\nabla u_{\lambda_0 + \tau}|^{p-2} \nabla u_{\lambda_0 + \tau}, \nabla \psi^p) \, dx$$

$$+ p \int_{\Omega_{\lambda_0 + \tau} \setminus K} (|\nabla u|^{p-2} \nabla u - |\nabla u_{\lambda_0 + \tau}|^{p-2} \nabla u_{\lambda_0 + \tau}, \nabla \psi^p) \psi^{-1} \psi^{p-1} \psi^p \, dx$$

$$= \int_{\Omega_{\lambda_0 + \tau} \setminus K} (f(u) - f(u_{\lambda_0})) w^{+}_{\lambda_0 + \tau} \psi^p \, dx.$$

(3.40)
Now we split the set $\Omega_{\lambda_0+\tau} \setminus K$ as the union of two disjoint subsets $A_{\lambda_0+\tau}$ and $B_{\lambda_0+\tau}$ such that $\Omega_{\lambda_0+\tau} \setminus K = A_{\lambda_0+\tau} \cup B_{\lambda_0+\tau}$. In particular, for $\bar{C} > 1$, we set

$$A_{\lambda_0+\tau} = \{ x \in \Omega_{\lambda_0+\tau} \setminus K : |\nabla u_{\lambda_0+\tau}(x)| < \bar{C}|\nabla u(x)| \}$$

and

$$B_{\lambda_0+\tau} = \{ x \in \Omega_{\lambda_0+\tau} \setminus K : |\nabla u_{\lambda_0+\tau}(x)| \geq \bar{C}|\nabla u(x)| \}.$$

From (3.45), using (2.12) and (A_1^\delta), repeating verbatim arguments in (2.18), (2.19) and in (2.20) we have

$$\int_{\Omega_{\lambda_0+\tau} \setminus K} (|\nabla u| + |\nabla u_{\lambda_0+\tau}|)^{p-2} |\nabla w_{\lambda_0+\tau}^+|^2 \psi^p dx$$

$$\leq \delta C \int_{\Omega_{\lambda_0+\tau} \setminus K} (|\nabla u| + |\nabla u_{\lambda_0+\tau}|)^{p-2} |\nabla w_{\lambda_0+\tau}^+|^2 \psi^p dx + C \left( \int_{\Omega_{\lambda_0+\tau} \setminus K} |\nabla u|^p dx \right)^{\frac{p-1}{p}} \left( \int_{\Omega_{\lambda_0+\tau} \setminus K} |\nabla \psi|^p dx \right)^{\frac{1}{p}}$$

$$+ \frac{C}{\delta} \int_{\Omega_{\lambda_0+\tau} \setminus K} |\nabla \psi|^p dx + K_f \int_{\Omega_{\lambda_0+\tau} \setminus K} (w_{\lambda_0+\tau}^+)^2 \psi^p dx,$$

for some positive constant $C = C(p, \lambda, \|u\|_{L^\infty(\Omega_{\lambda_0+\tau})})$. Taking $\delta > 0$ sufficiently small and using (A_1^\delta), as we did above passing to the limit for $\varepsilon \to 0$ we obtain

$$\int_{\Omega_{\lambda_0+\tau} \setminus K} (|\nabla u| + |\nabla u_{\lambda_0+\tau}|)^{p-2} |\nabla w_{\lambda_0+\tau}^+|^2 \psi^p dx \leq CK_f \int_{\Omega_{\lambda_0+\tau} \setminus K} (w_{\lambda_0+\tau}^+)^2 dx.$$

Now we set

$$\varrho := 1 + \frac{|\nabla u|^2 + |\nabla u_{\lambda_0}|^2}{\varphi^2}$$

in order to exploit the weighted Sobolev inequality from [16]. The results of [16] apply if $\varrho \in L^1(\Omega_{\lambda_0+\tau} \setminus K)$ and

$$\frac{1}{\varrho} \in L^t(\Omega_{\lambda_0+\tau} \setminus K),$$

for some $t > n/2$. In particular, $H^1_{0,\varrho}(\Omega')$ (see [6, 16]) coincides with the closure of $C_c^\infty(\Omega')$ with respect to the norm

$$\|w\|_{\varrho} := \|\nabla w\|_{L^2(\Omega', \varrho)} := \left( \int_{\Omega'} \varrho|\nabla w|^2 dx \right)^{\frac{1}{2}}$$

and it holds that

$$\|w\|_{L^{2^*_{\varrho}}(\Omega')} \leq C_S \|\nabla w\|_{L^2(\Omega', \varrho)}$$

for any $w \in H^1_{0,\varrho}(\Omega')$,

where

$$\frac{1}{2^*_{\varrho}} := \frac{1}{2} \left( 1 + \frac{1}{t} \right) - \frac{1}{n}.$$ 

Note that

$$(3.42) \quad (1 + |\nabla u|^2 + |\nabla u_{\lambda_0+\tau}|^2)^{\frac{2^*_{\varrho}}{2}} \leq K_1 + K_2 |\nabla u_{\lambda_0+\tau}|^{2-p},$$
in $\Omega^{+}_{\lambda_{0}+\tau} := \Omega_{\lambda_{0}+\tau} \cap \text{supp}(w^{+}_{\lambda_{0}+\tau})$, where $K_{1}$ and $K_{2}$ are positive constants depending only on $p$ and on $\|u\|_{C^{1}(\Omega_{\lambda_{0}+\tau})}$. By Lemma 3.1 and (3.42), we deduce that
\[
\frac{1}{\rho} := \left(1 + |\nabla u|^{2} + |\nabla u_{\lambda_{0}+\tau}|^{2}\right)^{\frac{2p}{p-2}} \in L^{1}(\Omega_{\lambda_{0}+\tau}),
\]
for some $t > n/2$ and this allows us to use the above mentioned results of [16]. We shall use the fact that
\[
(\nabla u + |\nabla u_{\lambda_{0}+\tau}|)^{2-p} \leq 2^{\frac{2p}{p-2}} \left(1 + |\nabla u|^{2} + |\nabla u_{\lambda_{0}+\tau}|^{2}\right)^{\frac{2-p}{p-2}} \leq 2^{\frac{2p}{p-2}} \left(1 + |\nabla u|^{2} + |\nabla u_{\lambda_{0}+\tau}|^{2}\right)^{\frac{2-p}{2}}.
\]
In particular, by (3.44), H"{o}lder inequality and weighted Sobolev inequality, in (3.41), we obtain
\[
\int_{\Omega_{\lambda_{0}+\tau}\setminus K} \rho |\nabla w^{+}_{\lambda_{0}+\tau}|^{2} \, dx \leq 2^{\frac{2p}{p-2}} \int_{\Omega_{\lambda_{0}+\tau}\setminus K} (|\nabla u| + |\nabla u_{\lambda_{0}+\tau}|)^{p-2} |\nabla w^{+}_{\lambda_{0}+\tau}|^{2} \, dx
\leq 2^{\frac{2p}{p-2}} CK_{f} \int_{\Omega_{\lambda_{0}+\tau}\setminus K} (w^{+}_{\lambda_{0}+\tau})^{2} \, dx
\leq 2^{\frac{2p}{p-2}} CK_{f} |\Omega_{\lambda_{0}+\tau}\setminus K|^{\frac{1}{2p}} \left(\int_{\Omega_{\lambda_{0}+\tau}\setminus K} (w^{+}_{\lambda_{0}+\tau})^{2} \, dx\right)^{\frac{2}{p-2}}
\leq 2^{\frac{2p}{p-2}} CK_{f} C_{p} (|\Omega_{\lambda_{0}+\tau}\setminus K|) \int_{\Omega_{\lambda_{0}+\tau}\setminus K} \rho |\nabla w^{+}_{\lambda_{0}+\tau}|^{2} \, dx,
\]
where $C_{p}(\cdot)$ tends to zero if the measure of the domain tends to zero. For $\bar{\tau}$ small and $K_{f}$ large, we may assume that
\[
2^{\frac{2p}{p-2}} CK_{f} C_{p} (|\Omega_{\lambda_{0}+\tau}\setminus K|) < \frac{1}{2}
\]
so that by (3.44), we deduce that
\[
\int_{\Omega_{\lambda_{0}+\tau}} \rho |\nabla w^{+}_{\lambda_{0}+\tau}|^{2} \, dx = \int_{\Omega_{\lambda_{0}+\tau}\setminus K} \rho |\nabla w^{+}_{\lambda_{0}+\tau}|^{2} \, dx \leq 0,
\]
proving that $u \leq u_{\lambda_{0}+\tau}$ in $\Omega_{\lambda_{0}+\tau}\setminus R_{\lambda_{0}+\tau}(\Gamma)$ for any $0 < \tau < \bar{\tau}$ for some small $\bar{\tau} > 0$. Such a contradiction shows that
\[
\lambda_{0} = 0.
\]
Since the moving plane procedure can be performed in the same way but in the opposite direction, then this proves the desired symmetry result. The fact that the solution is increasing in the $x_{1}$-direction in $\{x_{1} < 0\}$ is implicit in the moving plane procedure.

\[\square\]

**Proof of Theorem 1.2.** Arguing verbatim as in the previous case up to (3.39), we consider
\[
\varphi_{\varepsilon} := w^{+}_{\lambda_{0}+\tau} \psi_{\varepsilon}^{p} \chi_{\Omega_{\lambda_{0}+\tau}}
\]
and by a density arguments, we plug it as test function in (1.1) and (2.13). Subtracting, we get

\[
\int_{\Omega_{\lambda_0+\tau}\setminus K} (|\nabla u|^{p-2}\nabla u - |\nabla u_{\lambda_0+\tau}|^{p-2}\nabla u_{\lambda_0+\tau}, \nabla w^+_{\lambda_0+\tau}) \psi^p \, dx
\]

\[+ p \int_{\Omega_{\lambda_0+\tau}\setminus K} (|\nabla u|^{p-2}\nabla u - |\nabla u_{\lambda_0+\tau}|^{p-2}\nabla u_{\lambda_0+\tau}, \nabla \psi) \psi^{p-1} w^+_{\lambda_0+\tau} \, dx \]

\[= \int_{\Omega_{\lambda_0+\tau}\setminus K} (f(u) - f(u_{\lambda})) w^+_{\lambda_0+\tau} \psi^p \, dx.
\]

Using the split

\[A_{\lambda_0+\tau} = \{ x \in \Omega_{\lambda_0+\tau} \setminus K : |\nabla u_{\lambda_0+\tau}(x)| < \bar{C}|\nabla u(x)| \},\]

\[B_{\lambda_0+\tau} = \{ x \in \Omega_{\lambda_0+\tau} \setminus K : |\nabla u_{\lambda_0+\tau}(x)| \geq \bar{C}|\nabla u(x)| \},\]

from (3.45), using (2.12), (A1) and arguing as in Lemma 2.2, we obtain

\[
\int_{\Omega_{\lambda_0+\tau}\setminus K} (|\nabla u| + |\nabla u_{\lambda_0+\tau}|)^{p-2} |\nabla w^+_{\lambda_0+\tau}|^2 \psi^p \, dx
\]

\[\leq \delta C \int_{\Omega_{\lambda_0+\tau}\setminus K} (|\nabla u| + |\nabla u_{\lambda_0+\tau}|)^{p-2} |\nabla w^+_{\lambda_0+\tau}|^2 \psi^p \, dx
\]

\[+ \frac{C}{\delta} \left( \int_{\Omega_{\lambda_0+\tau}\setminus K} |\nabla u|^p \, dx \right)^{\frac{p-2}{p}} \left( \int_{\Omega_{\lambda_0+\tau}\setminus K} |\nabla \psi| \, dx \right)^{\frac{2}{p}}
\]

\[+ Kf \int_{\Omega_{\lambda_0+\tau}\setminus K} (w^+_{\lambda_0+\tau})^2 \psi^p \, dx,
\]

for some positive constant \(C = C(p, \lambda, ||u||_{L^\infty(\Omega_{\lambda_0+\tau})})\). As we did above passing to the limit for \(\varepsilon \to 0\), by Fatou’s Lemma we obtain

\[
\int_{\Omega_{\lambda_0+\tau}\setminus K} (|\nabla u| + |\nabla u_{\lambda_0+\tau}|)^{p-2} |\nabla w^+_{\lambda_0+\tau}|^2 \, dx \leq CKf \int_{\Omega_{\lambda_0+\tau}\setminus K} (w^+_{\lambda_0+\tau})^2 \, dx.
\]

In this case we have \(|\nabla u|^{p-2} \leq (|\nabla u| + |\nabla u_{\lambda_0+\tau}|)^{p-2}\) since \(p > 2\). Then we set \(\varrho := |\nabla u|^{p-2}\) and we see that \(\varrho\) is bounded in \(\Omega_{\lambda_0+\tau}\), hence \(\varrho \in L^1(\Omega_{\lambda_0+\tau})\). By applying the weighted Poincaré inequality to (3.46), see [6, Theorem 1.2], we deduce that

\[
\int_{\Omega_{\lambda_0+\tau}\setminus K} \varrho |\nabla w^+_{\lambda_0+\tau}|^2 \, dx \leq CKf \int_{\Omega_{\lambda_0+\tau}\setminus K} (w^+_{\lambda_0+\tau})^2 \, dx
\]

\[\leq CKf \int_{\Omega_{\lambda_0+\tau}\setminus K} (w^+_{\lambda_0+\tau})^2 \, dx
\]

\[\leq CKf C_p(|\Omega_{\lambda_0+\tau} \setminus K|) \int_{\Omega_{\lambda_0+\tau}\setminus K} \varrho |\nabla w^+_{\lambda_0+\tau}|^2 \, dx
\]

where \(C_p(\cdot)\) tends to zero if the measure of the domain tends to zero. For \(\bar{\tau}\) small and \(K\) large, we may assume that

\[CKf C_p(|\Omega_{\lambda_0+\tau} \setminus K|) < \frac{1}{2}\]
so that by (3.44), we deduce that
\[
\int_{\Omega_{\lambda_0+\tau}} \rho |\nabla w_{\lambda_0+\tau}^+|^2 \, dx = \int_{\Omega_{\lambda_0+\tau}} \rho |\nabla w_{\lambda_0+\tau}^-|^2 \, dx \leq 0,
\]
proving that \( u \leq u_{\lambda_0+\tau} \) in \( \Omega_{\lambda_0+\tau} \setminus R_{\lambda_0+\tau}(\Gamma) \) for any \( 0 < \tau < \bar{\tau} \) for some small \( \bar{\tau} > 0 \). Such a contradiction shows that
\[
\lambda_0 = 0.
\]
Since the moving plane procedure can be performed in the same way but in the opposite direction, then this proves the desired symmetry result. The fact that the solution is increasing in the \( x_1 \)-direction in \( \{x_1 < 0\} \) is implicit in the moving plane procedure.

\[\square\]

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