A SMALL GENERATING SET FOR THE TWIST SUBGROUP OF THE MAPPING CLASS GROUP OF A NON-ORIENTABLE SURFACE BY DEHN TWISTS

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Abstract. We give a small generating set for the twist subgroup of the mapping class group of a non-orientable surface by Dehn twists. The difference between the number of the generators and a lower bound of numbers of generators for the twist subgroup by Dehn twists is one. The lower boundary is obtained from an argument of Hirose [5].

1. Introduction

Let $\Sigma_{g,n}$ be a compact connected oriented surface of genus $g \geq 0$ with $n \geq 0$ boundary components and we put $\Sigma_g := \Sigma_{g,0}$. The mapping class group $\mathcal{M}(\Sigma_{g,n})$ of $\Sigma_{g,n}$ is the group of isotopy classes of orientation preserving self-diffeomorphisms on $\Sigma_{g,n}$ fixing the boundary pointwise. Dehn [2] proved that $\mathcal{M}(\Sigma_g)$ is generated by $2g(g - 1)$ Dehn twists. The generating set includes Dehn twists along separating simple closed curves. Mumford [10] showed that $\mathcal{M}(\Sigma_g)$ is generated by Dehn twists along non-separating simple closed curves, and Lickorish [8] gave a finite generating set for $\mathcal{M}(\Sigma_g)$ by $3g - 1$ Dehn twists along non-separating simple closed curves. By an argument in Proof of Theorem 4.13 in [4], $\mathcal{M}(\Sigma_{g,1})$ is also generated by $3g - 1$ Dehn twists along non-separating simple closed curves. After that, Humphries [6] proved that $\mathcal{M}(\Sigma_{g,n})$ is generated by a subset of Lickorish’s generating set whose cardinality is $2g + 1$ for $g \geq 2$ and $n \in \{0, 1\}$, and he also proved that the generating set is minimal in generating sets for $\mathcal{M}(\Sigma_{g,n})$ by Dehn twists.

Let $N_{g,n}$ be a compact connected non-orientable surface of genus $g \geq 1$ with $n \geq 0$ boundary components. The surface $N_g := N_{g,0}$ is a connected sum of $g$ real projective planes. The mapping class group $\mathcal{M}(N_{g,n})$ of $N_{g,n}$ is the group of isotopy classes of self-diffeomorphisms on $N_{g,n}$ fixing the boundary pointwise. For $n \in \{0, 1\}$, $\mathcal{M}(N_{1,n})$ is the trivial group (see [3, Theorem 3.4]). For $g \geq 2$, Lickorish proved that $\mathcal{M}(N_g)$ is not generated by Dehn twists in [7], and $\mathcal{M}(N_{g,n})$ is generated by Dehn twists and a “Y-homeomorphism” in [7, 9]. The Y-homeomorphism is introduced by Lickorish in [7]. Lickorish [7] also showed that $\mathcal{M}(N_2)$ is generated by a Dehn twist and a Y-homeomorphism. In generally, Chillingworth [11] gave a finite generating set for $\mathcal{M}(N_g)$ which consists of $\frac{3g - 5}{2}$ (resp. $\frac{3g - 6}{2}$) Dehn twists and a Y-homeomorphism for odd (resp. even) $g$. After that, Szepietowski [13] proved that $\mathcal{M}(N_g)$ is generated by a subset of Chillingworth’s generating set which consists of $g$ Dehn twists and a Y-homeomorphism, and Hirose [5] showed that the generating set is minimal in generating sets for $\mathcal{M}(N_g)$ by Dehn twists and Y-homeomorphisms. By Stukow’s finite presentation for $\mathcal{M}(N_{g,1})$ in [12] and an argument in [5] (see...
Remark 3.3, \( \mathcal{M}(N_{g,1}) \) also has a minimal generating set by Dehn twists and \( Y \)-homeomorphisms which consists of \( g \) Dehn twists and a \( Y \)-homeomorphism.

The twist subgroup \( \mathcal{T}(N_{g,n}) \) of \( \mathcal{M}(N_{g,n}) \) is the subgroup of \( \mathcal{M}(N_{g,n}) \) which is generated by all Dehn twists. \( \mathcal{T}(N_{g,n}) \) is an index 2 subgroup of \( \mathcal{M}(N_{g,n}) \) (see [9] and [11, Corollary 6.4]). In particular, \( \mathcal{T}(N_{g,n}) \) is finitely generated. Chillingworth [1] showed that \( \mathcal{T}(N_{g}) \) is generated by a Dehn twist for \( g = 2 \), two Dehn twists for \( g = 3 \), \( 3g - 1 \) Dehn twists for the other odd \( g \) and \( 3g - 2 \) Dehn twists for the other even \( g \). By an argument as in [6], we can reduce the number of Chillingworth's generators to \( g + 2 \) for odd \( g > 3 \) and \( g + 3 \) for even \( g > 3 \). For \( n \in \{0, 1\} \), Stukow [13] gave a finite presentation for \( \mathcal{T}(N_{g,n}) \) whose generators are \( g + 2 \) Dehn twists essentially by relations of the presentation (see Proof of Theorem 3.1).

In this paper we proved that \( \mathcal{T}(N_{g,n}) \) is generated by \( g + 1 \) Dehn twists for \( g \geq 4 \) (Theorem 3.1). The generating set is a proper subset of the generating set of Stukow’s finite presentation in [13]. By applying Hirose’s argument in [5], the difference between the number of the generators in Theorem 3.1 and a lower bound of numbers of generators for \( \mathcal{T}(N_{g,n}) \) by Dehn twists is one (see Remark 3.3). The author does not know whether the generating set for \( \mathcal{T}(N_{g,n}) \) in Theorem 3.1 is minimal in generating sets for \( \mathcal{T}(N_{g,n}) \) by Dehn twists or not.

2. Preliminaries

For a two-sided simple closed curve \( c \) on \( N_{g,n} \), we take an orientation of the regular neighborhood of \( c \) in \( N_{g,n} \). Then we denote by \( t_c \) the right-handed Dehn twist along \( c \) with respect to the orientation. In particular, for a given explicit two-sided simple closed curve, an arrow on a side of the simple closed curve indicates the direction of the Dehn twist (see Figure 1).

![Figure 1. The right-handed Dehn twist \( t_c \) along a two-sided simple closed curve \( c \) on \( N_{g,n} \).](image)

Let \( e_i : D \hookrightarrow \Sigma_0 \) for \( i = 1, 2, \ldots, g + 1 \) be smooth embeddings of the unit disk \( D \) to a 2-sphere \( \Sigma_0 \) such that \( D_i := e_i(D) \) and \( D_j \) are disjoint for distinct \( 1 \leq i, j \leq g + 1 \). Then we take a model of \( N_g \) (resp. \( N_{g,1} \)) as the surface obtained from \( \Sigma_0 - \text{int}(D_1 \sqcup \cdots \sqcup D_g) \) (resp. \( \Sigma_0 - \text{int}(D_1 \sqcup \cdots \sqcup D_{g+1}) \)) by identifying antipodal points of the boundary components of \( D_1, \ldots, D_g \) and we describe the identification of \( \partial D_i \) by the x-mark as in Figure 2.

For \( n \in \{0, 1\} \), we denote by \( \alpha_1, \ldots, \alpha_{g-1} \) and \( \beta \) two-sided simple closed curves on \( N_{g,n} \) as in Figure 2 and denote by \( \gamma, \varepsilon, \zeta \) and \( \psi \) two-sided simple closed curves on \( N_{g,n} \) as in Figure 3 respectively. Then we define \( a_i := t_{\alpha_i} \) (\( i = 1, \ldots, g - 1 \)), \( b := t_{\beta}, c := t_\varepsilon, f := t_\zeta, y^2 := t_\psi \) and \( e := t_\gamma \).

3. Main result

The main theorem in this paper is as follows.
For $g \geq 4$ and $n \in \{0,1\}$, $\mathcal{T}(N_{g,n})$ is generated by $a_1, \ldots, a_{g-1}, b$ and $e$. In particular, $\mathcal{T}(N_{g,n})$ is generated by $g+1$ Dehn twists along non-separating simple closed curves.

**Proof.** Assume $g \geq 4$ and $n \in \{0,1\}$. Stukow's presentation for $\mathcal{T}(N_{g,n})$ in [13] has the following generating set:

- $X := \{a_1, \ldots, a_{g-1}, b, c, f, y^2, e\}$ for odd $g$ and $n = 0$ or $g = 4$ and $n = 1$,
- $X := X \cup \{b_0, b_1, \ldots, b_{g-2}, b_{g-4}, \ldots, b_{2g-2}\}$ for even $g \geq 6$ and $n = 1$,
- $X \cup \{\rho\}$ for odd $g$ and $n = 0$,
- $X \cup \{\bar{\rho}\}$ for $g = 4$ and $n = 0$,
- $X' \cup \{\bar{\rho}\}$ for even $g \geq 6$ and $n = 0$.

$b_0, b_1, \ldots, b_{g-2}, \bar{b}_{g-2}, \bar{b}_{g-4}, \ldots, \bar{b}_{2g-2}$, $\rho$ and $\bar{\rho}$ are products of elements in $X$ by the relations (A7), (A8), (A7a)-(A8b), (C1a) and (C4) in Theorem 2.1, Theorem 2.2, Theorem 3.1 and Theorem 3.2 of [13]. Thus $\mathcal{T}(N_{g,n})$ is generated by $X$. By the relation (B21) in Theorem 3.1 of [13], $y^2$ is a product of elements in $X - \{y^2\}$, and by the relation (B61) in Theorem 3.1 of [13], $c$ is a product of $a_1, \ldots, a_{g-1}, b, e$ and $f$.

Finally, we can check $a_3^{-1}a_2^{-1}ba_1^{-1}a_2^{-1}a_3^{-1}(\varepsilon) = \zeta$ and the orientation of the regular neighborhood of $a_3^{-1}a_2^{-1}ba_1^{-1}a_2^{-1}a_3^{-1}(\varepsilon)$ is different from one of $\varepsilon$ as in Figure 4. Hence we have $f = (a_3^{-1}a_2^{-1}ba_1^{-1}a_2^{-1}a_3^{-1})\varepsilon^{-1}(a_3^{-1}a_2^{-1}ba_1^{-1}a_2^{-1}a_3^{-1})^{-1}$. Therefore $\mathcal{T}(N_{g,n})$ is generated by $a_1, \ldots, a_{g-1}, b$ and $c$, and we have completed the proof of Theorem 3.1.  

**Remark 3.2.** The regular neighborhood $N$ of the union of $\alpha_1, \ldots, \alpha_{g-1}$ is an orientable subsurface of $N_{g,n}$ and $\{a_1, \ldots, a_{g-1}, b\}$ is the minimal generating set for $\mathcal{M}(N)$ by Dehn twists which is given by Humphries [6]. Remark that $N_{g,n} - \text{int}N$ is not a disjoint union of disks, and an element of the subgroup of $\mathcal{T}(N_{g,n})$ which is generated by $a_1, \ldots, a_{g-1}$ and $b$ is represented by a diffeomorphism of $N_{g,n}$ whose restriction to $N_{g,n} - \text{int}N$ is the identity map. However, $e$ does not fix
Figure 4. Proving that $a_3^{-1}a_2^{-1}ba_1^{-1}a_2^{-1}a_3^{-1}(\varepsilon) = \zeta$.

$N_{g,n} = \text{int}N$ up to ambient isotopies of $N_{g,n}$. Hence $T(N_{g,n})$ is not generated by $a_1, \ldots, a_{g-1}$ and $b$. Define $X_0 := \{\alpha_1, \ldots, \alpha_{g-1}, b, \varepsilon\}$. For $x_0 \in \{\alpha_4, \ldots, \alpha_{g-1}, \varepsilon\}$, the complement $N_{g,n} - \bigcup_{x \in X_0 \setminus \{x_0\}} x$ has a non-disk component. Thus $T(N_{g,n})$ is not also generated by $X_0 - \{x_0\}$ for $x_0 \in \{\alpha_4, \ldots, \alpha_{g-1}, \varepsilon\}$.

**Remark 3.3.** We can apply Hirose’s argument in [5] to $M(N_{g,1})$ and $T(N_{g,n})$ for $g \geq 4$ and $n \in \{0, 1\}$. However, we should note that he take $\phi_j \in M(N_{g,n})$ such that $\phi_j(c_1) = \gamma_j$ in the proof of Lemma 6 in [5]. To apply Hirose’s argument in [5] to $T(N_{g,n})$, we must take such $\phi_j$ as an element of $T(N_{g,n})$. By using Lemma 7.2 in [11], we can take $\phi_j$ as an element of $T(N_{g,n})$. Therefore the minimum number of generators for $T(N_{g,n})$ by Dehn twists is at least $g$ for $g \geq 4$ and $n \in \{0, 1\}$, and the difference between the number of the generators for $T(N_{g,n})$ in Theorem 3.1 and the lower bound of numbers of generators for $T(N_{g,n})$ by Dehn twists is one.

Finally we raise the following problem.

**Problem 3.4.** Which of $g$ and $g + 1$ is the minimum number of generators for $T(N_{g,n})$ by Dehn twists when $g \geq 4$ and $n \in \{0, 1\}$?

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