SIMPLE POLYTOPES WITHOUT SMALL SEPARATORS*

BY

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For Gil Kalai on the occasion of his sixtieth birthday

ABSTRACT

We show that by cutting off the vertices and then the edges of neighborly cubical polytopes, one obtains simple 4-dimensional polytopes with \( n \) vertices such that all separators of the graph have size at least \( \Omega(n/\log^{3/2} n) \). This disproves a conjecture by Kalai from 1991/2004.

1. Introduction

The Lipton–Tarjan planar separator theorem from 1979 [9] states that for any constant \( c \) with \( 0 < c < 1/2 \), the vertex set of each planar graph on \( n \) vertices can be partitioned into three sets \( A, B, C \) with \( cn \leq |A| \leq |B| \leq (1-c)n \) and \( |C| = O(\sqrt{n}) \), such that \( C \) separates \( A \) from \( B \), that is, there is no edge between a vertex in \( A \) and a vertex in \( B \). Traditionally \( c = 1/3 \) is used. We call any partition of the vertex set of a graph into sets \( A, B, C \) with \( cn \leq |A| \leq |B| \leq (1-c)n \) without any edge from a vertex in \( A \) to a vertex in \( B \) a separator with separation constant \( c \) and we define the size of the separator to be the size of the set \( C \).

Miller, Thurston et al. [11] in 1990/1991 provided a geometric proof for the planar separator theorem, combining the fact that every 3-polytope has an edge

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tangent representation by the Koebe–Andreev–Thurston circle packing theorem with the center point theorem. Miller, Teng, Thurston and Vavasis [10] generalized the planar separator theorem to $d$ dimensions, that is, to the intersection graphs of suitable ball packings in $\mathbb{R}^d$. In view of this, Kalai noted that there is no separator theorem for general $d$-polytopes, due to the existence of the cyclic polytopes, whose graphs are complete for $d \geq 4$ and thus have no separators. However, he conjectured that the graphs of simple $d$-polytopes cannot be good expanders, that is, they all should have small separators. Specifically, in his 1991 paper on diameters and $f$-vector theory [7, Conj. 12.1] (repeated in the 1997 first edition of the Handbook of Discrete and Computational Geometry [8]) he postulated that for every $d \geq 3$ any simple $d$-polytope on $n$ vertices should have a separator of size

$$O\left(n^{1-\frac{3}{d+2}}\right),$$

which fails for $d = 3$, while for $d = 4$ it postulates separators of size $O(\sqrt{n})$. In the 2004 second edition of the Handbook he provided a corrected version of his conjecture:

**Kalai’s Conjecture** ([8, Conj. 20.2.12]): For any fixed $d \geq 2$, the graphs of the simple $d$-polytopes on $n$ vertices have separators of size

$$O\left(n^{1-\frac{1}{d+1}}\right).$$

This matches the bound provided by the planar separator theorem for general (not necessarily simple) polytopes of dimension $d = 3$, while for $d = 4$ this conjecture asked for the existence of separators of size $O(n^{2/3})$.

At that time Kalai also referred to [10] for the claim that there are triangulations of $S^3$ into $n$ tetrahedra whose dual graphs, which then have $n$ vertices, cannot be separated even by $O(n/ \log n)$ vertices. This claim, however, does not appear in the paper [10], but it refers to a construction of Thurston who had described to his coauthors an embedding of the cube-connected cycle graph in $\mathbb{R}^3$ as the dual graph of a configuration of tetrahedra. Details about this construction seem to be lost (Gary Miller, personal communication, April 2015).

In this note, we disprove the corrected 2004 version of Kalai’s conjecture, and come close to confirming Thurston’s claim. Our construction uses the existence of *neighborly cubical* 4-polytopes $NC_4(m)$, first established by Joswig and Ziegler [6]: For each $m \geq 4$ there is a 4-dimensional polytope $NC_4(m)$
whose graph is isomorphic to the graph $G_m$ of the $m$-cube and whose facets are combinatorial 3-cubes.

**Theorem 1:** “Cutting off the vertices, and then the original edges” from a neighborly cubical 4-polytope $NC_4(m)$ results in a simple 4-dimensional polytope $NC_4(m)''$ with $n := (6m - 12)2^m$ vertices ($m \geq 4$) whose graph has no separator of size less than

$$\Omega\left(\frac{n}{\log^{3/2} n}\right),$$

while separators of size

$$O\left(\frac{n}{\log n}\right)$$

exist.

To prove this, we only use the existence of neighborly cubical 4-polytopes, and the fact that the $f$-vector of any such polytope is of the form

$$f(NC_4(m)) = (f_0, f_1, f_2, f_3) = 2^{m-2}(4, 2m, 3m - 6, m - 2).$$

Thus, in particular, we do not need to specify the combinatorial types of the neighborly cubical polytopes. Indeed, it was established by Sanyal and Ziegler [12] that there are many different combinatorial types, and any sequence of such polytopes can be used for our proof of Theorem 1. It may still be that for some specific neighborly cubical 4-polytopes, such as the family for which a complete combinatorial description was given in [6, Thm. 18], all separators in the resulting simple polytopes have size at least $\Omega(n/\log n)$. This would strongly confirm Thurston’s claim. On the other hand, all simple 4-polytopes that are constructed according to Theorem 1 have separators of size $O(n/\log n)$. However, we do not know whether such separators exist for arbitrary simple 4-polytopes on $n$ vertices.

This paper is structured as follows. In Section 2 we describe the construction of $NC_4(m)''$, compute its $f$-vector, establish that it is simple, and give a “coarse” description of the graph $G''_m := G(NC_4(m)'')$. In Section 3 we show that the graph $G''_m$ has no small separators. This is derived from elementary and well-known expansion properties of the cube graph $G_m$. In Section 4 we establish the existence of separators of size $O(n/\log n)$ in the graphs $G''_m$. Finally, in Section 5 we extend all this to simple $d$-dimensional polytopes for $d \geq 4$. 
2. Doubly truncated neighborly cubical polytopes

A neighborly cubical \(d\)-polytope \(NC_d(m)\) is a \(d\)-dimensional convex polytope whose \(k\)-skeleton for \(2k + 2 \leq d\) is isomorphic to that of the \(m\)-cube. It is required to be cubical, which means that all of its faces are combinatorial cubes. The existence of such polytopes was established by Joswig and Ziegler [6], solving a problem of Kalai.

For 4-dimensional polytopes, the complete flag vector (that is, the extended \(f\)-vector of Bayer and Billera [1]) is determined by the \(f\)-vector together with the number \(f_{03}\) of vertexfacet incidences. Thus we will use \((f_0, f_1, f_2, f_3; f_{03})\) to represent the flag vector of a 4-polytope.

Let \(m \geq 4\). We start our constructions with a neighborly cubical 4-polytope \(NC_4(m)\) with the graph (1-skeleton) of the \(m\)-cube, so \(f_0 = 2^m\) and \(f_1 = m2^{m-1}\). The rest of the flag vector is now obtained from the Euler equation together with the fact that \(NC_4(m)\) is cubical: Each facet has 6 2-faces and 8 vertices, which yields \(6f_3 = 2f_2\) and \(8f_3 = f_{03}\). Thus we obtain

\[
\text{flag}(NC_4(m)) := (f_0, f_1, f_2, f_3; f_{03}) = (2^m, m2^{m-1}, 3(m-2)2^{m-2}, (m-2)2^{m-2}; 8(m-2)2^{m-2}) = (4, 2m, 3m - 6, m - 2; 8m - 16) \cdot 2^{m-2}.
\]

We generate the polytope \(NC_4(m)'\) from \(NC_4(m)\) by cutting off all of its vertices. The resulting polytope thus has the following facets:

- \((m - 2)2^{m-2}\) facets that are 3-cubes whose vertices have been cut off, with \(f\)-vector \((24, 36, 14)\). These are the vertex-truncated facets of \(NC_4(m)\).
- \(2^m\) facets that are simplicial 3-polytopes, each with \(f\)-vector \((m, 3m - 6, 2m - 4)\).

The latter facets are the vertex figures of \(NC_4(m)\), which are simplicial since the facets of \(NC_4(m)\), which are 3-cubes, are simple. The resulting 4-polytope has the following flag vector:

\[
\text{flag}(NC_4(m)') = (4m, 14m - 24, 11m - 22, m + 2; 28m - 48) \cdot 2^{m-2}.
\]

We now generate \(NC_4(m)''\) from \(NC_4(m)\) by cutting off the edges which come from edges in the original polytope \(NC_4(m)\) (but have been shortened in the transition to \(NC_4(m)\)'). The resulting polytope has three types of facets:
• $(m - 2)2^{m-2}$ facets that are 3-cubes whose vertices and subsequently the original cube edges have been cut off, with $f$-vector $(48, 72, 26)$. These correspond to the facets of $\mathrm{NC}_4(m)$.

• $2^m$ simple 3-polytopes with $f$-vector $(6m - 12, 9m - 18, 3m - 4)$, which arise by cutting off the vertices of a simplicial 3-polytope with $m$ vertices, and

• $m2^{m-1}$ prisms over polygons that may range between triangles and $(m - 1)$-gons. They arise from the hyperplane cutting off an edge of $\mathrm{NC}_4(m)$. If this edge was contained in $k$ facets of $\mathrm{NC}_4(m)$, then its vertices in $\mathrm{NC}_4(m)'$ will have degree $k + 1$, and cutting it off results in a prism with a $k$-gon base.

Again from the available information one can easily work out the flag vector,

$$\text{flag}(\mathrm{NC}_4(m)'') = (24m - 48, 48m - 96, 27m - 46, 3m + 2; 28m - 48) \cdot 2^{m-2}.$$ 

In particular, we can see from the $f$-vector that $f_1 = 2f_0$, so $\mathrm{NC}_4(m)''$ is simple. Indeed, from any 4-polytope one gets a simple polytope by first cutting off the vertices, then the original edges. This may also be visualized in the dual picture: Any 4-polytope may be made simplicial by first stacking onto the facets, and then onto the ridges of the original polytope. After the first step, the facets are pyramids over the original ridges. The second step corresponds to subdivisions of the pyramids in a point in the base, which subdivides them into tetrahedra.

More generally, for $d$-polytopes we observe the following.

**Proposition 2** (see Ewald and Shephard [3]): For $d \geq 2$ and $0 < k < d$ let $P$ be a $d$-polytope. Denote by $P^{(k)}$ the result of truncating the vertices, edges, etc. up to the $(k-1)$-faces of $P$, in this order. Then the polytopes $P^{(d-2)}$ and $P^{(d-1)}$ are simple.

Indeed, in the dual picture stacking onto facets etc. down to edges, which yields $P^{(d-1)}$, corresponds to the barycentric subdivision of the boundary complex of the polytope. Subdividing the edges is unnecessary for our purpose, since these are already simplices.

### 3. No small separators

Let $G_m$ be the graph of the $m$-cube, whose vertex set we identify with $\{0, 1\}^m$. It has $2^m$ vertices and $m2^{m-1}$ edges. For any subset $S \subseteq V$ of the vertex set,
its **neighborhood** is defined as

\[
N(S) := \{ v \in V \setminus S : \{u, v\} \text{ is an edge for some } u \in S \}.
\]

Harper solved the “discrete isoperimetric problem” in the \(m\)-cube in the sixties [4]: For fixed \(m\) and given cardinality \(|S|\), the cardinality of its neighborhood \(|N(S)|\) is minimized by taking \(S' = \{v \in V, \sum_i v_i \leq d\}\) for the maximal possible \(d\) and adding in a suitable set of \(|S| - |S'|\) vertices with coordinate sum \(d + 1\). See Bollobás [2] or Harper [5] for expositions. Thus optimal separators in the cube graph \(G_m\) are obtained by taking level sets, of size \(\binom{m}{k}\). Here the usual asymptotics for binomial coefficients (as given by the central limit theorem, see e.g. [13, Sect. 6.4]) tell us that all the separators in the cube graph \(G_m\) have cardinality at least \(\Omega(2^m/\sqrt{m})\), where the implied constant depends on the separation constant \(c\).

The graph \(G'_m\) of the polytope \(NC_4(m)\) is obtained from the cube graph \(G_m\) by replacing each node by a maximal planar graph on \(m\) vertices and \(3m - 6\) edges. (Note that this description does not specify the graph \(G'_m\) completely.) In the transition from \(G'_m\) to the graph \(G''_m\) of the polytope \(NC_4(m)''\), the \(2^m\) planar graphs grow into cubic (3-regular) graphs on \(2(3m - 6) = 6m - 12\) vertices each, which we call the **clusters** of \(G''_m\). Here \(G''_m\) is a 4-regular graph on \(n := (6m - 12)2^m\) vertices.

Now let, in the notation given in the introduction, \((A, B, C)\) denote a separator of the graph \(G''_m\), with separation constant \(c\). We want to show that \(C\) has size at least \(\Omega(n/\log^{3/2} n)\). For this we will now describe how to get from \((A, B, C)\) a separator for the cube graph \(G_m\), for any smaller separation constant \(c' < c\). Label the vertices of \(G_m\) by \(\alpha\) or \(\beta\) if the corresponding cluster in \(G''_m\) has vertices only in \(A\) or only in \(B\), respectively. The remaining vertices will be labeled by \(\gamma\). There cannot be any neighboring vertices in \(G_m\) labeled by \(\alpha\) and \(\beta\), since this would imply neighboring \(A\) and \(B\) clusters in \(G''_m\). The set of vertices of \(G_m\) labeled \(\alpha\) has size at most \((1 - c)2^m\), and the same is true for the set of vertices labeled \(\beta\). Thus, for any fixed \(c' < c\), unless the set of vertices labeled \(\gamma\) has linear size (in which case also \(C\) has linear size and we are done), both the sets of vertices with labels \(\alpha\) and \(\beta\) have size at least \(c'2^m\), and thus we have constructed a separator for \(G_m\). By the isoperimetric inequality for the \(m\)-cube graph, there must be \(\Omega(2^m/\sqrt{m})\) vertices labeled by \(\gamma\) and hence at least as many vertices in the separator for \(G''_m\).
Thus all separators for the graph $G''_m$ of $NC_4(m)''$ have size at least

$$\Omega\left(\frac{2^m}{\sqrt{m}}\right) = \Omega\left(\frac{n}{\log^{3/2}n}\right).$$

4. Small separators

Here we argue that for any neighborly cubical 4-polytope $NC_4(m)$, the derived simple 4-polytope $NC_4(m)''$ on $n = (6m - 12)2^m$ vertices has a separator of size

$$O(2^m) = O\left(\frac{n}{\log n}\right).$$

Indeed, with respect to the identification of the vertex set of $G_m$ with $\{0, 1\}^m$, choose a random coordinate ("edge direction"), and divide the vertices of $G_m$ into two sets by whether the corresponding vertex label is 0 or 1. This corresponds to cutting the $m$-cube into two $(m - 1)$-cubes, with $2^m - 1$ edges between them.

This cutting also divides the vertex set of $G''_m$ into two equal halves, containing $n/2 = (3m - 6)2^m$ vertices each. Each of the $m2^m - 1$ edges between two vertices of $G_m$ resp. between the maximal planar graphs in $G'_m$ gives rise to a number of edges (at least 3, at most $m - 1$) between the corresponding two clusters in $G''_m$. While the cube graph $G_m$ has $m2^m - 1$ edges, the modified graph $G''_m$ has $(6m - 12)2^m - 1$ edges between clusters. Thus in the transition from $G_m$ to $G''_m$, the cube graph edges are replaced by less than 6 edges on average. For a random coordinate direction, the expected number of edges between the two equal halves of $G''_m$ is less than $6 \cdot 2^{m-1} = 3 \cdot 2^m$. Thus by choosing a suitable coordinate, and removing one end vertex of each edge of $G''_m$ in the corresponding direction, we obtain a separator of size less than $3 \cdot 2^m$.

5. More generally

We can extend the result of Theorem 1 to dimensions $d > 4$ by taking the product of $NC_4(m)''$ and the standard $(d - 4)$-cube. For a fixed dimension $d$ this gives a sequence of polytopes with $2^{d-4}$ times as many vertices. We can find a separator in this graph by taking a product of a separator in $NC_4(m)''$ and the standard $(d - 4)$-cube, so these polytopes are at least as easy to separate as $NC_4(m)''$. On the other hand, the graph of this polytope again has a cube-like
structure, with $2^m$ clusters that are products of a cubic planar graph on $6m - 12$ vertices with the fixed graph $C_{d-4}$. Again we need to remove at least

$$\Omega \left( \frac{2^m}{\sqrt{m}} \right) = \Omega \left( \frac{n}{\log^{3/2} n} \right).$$

vertices to separate it.

**Corollary 3:** For each $d \geq 4$ there is a sequence of simple $d$-polytopes whose graphs (on $n$ vertices) have no separators that are smaller than

$$\Omega \left( \frac{n}{\log^{3/2} n} \right),$$

but which have separators of size

$$O \left( \frac{n}{\log n} \right).$$

Alternatively, one could try to start with neighborly cubical $d$-polytopes $NC_d(m)$ and to simplify them by Proposition 2. The resulting simple $d$-polytopes have graphs that are again similar to those of $m$-cubes, where however the clusters have a size of the order of $\Theta(m^{|d/2| - 1})$ for fixed $d$, and thus we get $n = \Theta(m^{|d/2| - 1} 2^m)$ vertices in total, and thus separators of size

$$O(2^m) = O \left( \frac{n}{\log^{|d/2| - 1} n} \right).$$

So the product construction sketched above is better for $d > 5$.

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