DECOHERENCE IN QUANTUM
BROWNIAN MOTION

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ABSTRACT: We examine the dependence of decoherence on the spectral density of the environment as well as on the initial state of the system. We use two simple examples to illustrate some important effects.

Introduction

Decoherence plays a major role in the transition from quantum to classical and has attracted much attention in recent years (see Zurek (1991)). The analysis of this process may allow us to understand in detail the mechanism that prevents observations of some quantum systems in superpositions of macroscopically distinguishable states. In the light of new technologies it can also help us to devise experiments to probe the fuzzy boundary between the quantum and the classical world. The interaction with an external environment is the mechanism responsible for the suppression of quantum interference effects. Therefore, there are several questions that arise naturally: How dependent on the environment decoherence is? What are the time scales involved in this process? How are some preferred states of the system dynamically chosen? In this paper we will report on recent work where some of these questions are addressed.
As a first point, let us clarify what we mean here by decoherence. Within the Gell–Mann and Hartle version of the consistent histories formulation of quantum mechanics, based on earlier work by Griffiths and Omnès, (see contributions in this proceedings) the term decoherence is used in place of “consistency” which is the condition that, if satisfied, allows us to assign probabilities to members of sets of coarse grained histories of a closed system. The Decoherence Functional is the basic diagnostic tool used in this framework. On the other hand, in previous works originated in quantum measurement theory, a different notion of decoherence was used. Measurement devices are always open systems that interact with external environments. This interaction dynamically selects a preferred set of states of the apparatus, the so–called pointer basis. This is, in some sense, the set of the most stable states: if the apparatus is prepared in a pointer state, the interaction with the environment has a minimal effect and almost no predictive power is lost. On the contrary, if the initial state is a superposition of pointer states, the interaction with the environment induces correlations and the state of the system tends to evolve into a mixture of pointer states. This process was called decoherence and this is the sense in which we will use this word here. Within this context, there are several important issues that require further attention. The most important one seems to be the definition of an appropriate measure of stability that may be used to determine the pointer states (see Zurek’s contribution in this conference). We will study a model describing a particle interacting with an environment formed by a collection of harmonic oscillators. In this case the pointer states of the particle seem to be closely related to coherent states and decoherence is the
process that suppresses interference between coherent states (our diagnostic tool will be described later). The action of the model is the following:

\[ S[x, q] = \int_0^t ds \left[ \frac{1}{2} M (\dot{x}^2 - \Omega_0^2 x^2) + \sum_n \frac{1}{2} m_n (q_n^2 - \omega_n^2 q_n^2) - \sum_n C_n x q_n \right] \] (1)

We will assume that there are no initial correlations between the system and the environment (i.e. the initial density matrix factorizes) and that the initial state of the environment is in thermal equilibrium at temperature \( T \).

In Section 2 we will describe generic features of the evolution of a system interacting with a general environment. We will study in detail the case in which the initial state is a superposition of two coherent states. In Section 3 we will illustrate the fact that decoherence strongly depends on some properties of the environment. We will also illustrate in what sense position is a preferred observable in the model (where position eigenstates are not pointer states). In the Appendix we outline a simple derivation of the master equation for a general environment.

**General Properties of the Reduced Dynamics**

Due to the interaction with the environment, the evolution of the system is non–unitary since it is affected by a stochastic noise and a “dissipative” force (the word dissipation is used here in a rather vague sense). Noise and dissipation, are entirely determined by two properties of the environment: the spectral density \( I(\omega) \) and the initial temperature. The spectral density, defined as \( I(\omega) = \sum_n \delta(\omega - \omega_n)C_n^2 / 2m_n \omega_n \), characterizes the number density of oscillators in the environment and the strength of their coupling with the system. Therefore, in order to analyze how decoherence depends on the
environment we can study how this process changes when varying $I(\omega)$ and the temperature since these are the only two environmental properties “seen” by the system.

Luckily enough, the reduced dynamics has some very general features that are entirely independent of the spectral density and the temperature. One of the most striking and important ones is the fact that the reduced density matrix always satisfies a master equation that can be written as follows (we use $\hbar = 1$):

$$i \frac{\partial}{\partial t} \rho_r(x, x', t) = <x|[H_{\text{ren}}(t), \rho_r]|x' > - i \gamma(t)(x - x')(\partial_x - \partial_{x'}) \rho_r(x, x', t)$$

$$- i D(t)(x - x')^2 \rho_r(x, x', t) + f(t)(x - x')(\partial_x + \partial_{x'}) \rho_r(x, x', t)$$

This equation depends on the spectral density and the temperature only through the coefficients appearing in the right hand side: the physical frequency entering in $H_{\text{ren}}(t)$, the friction coefficient $\gamma(t)$ and the diffusion coefficients $D(t)$ and $f(t)$ are time dependent functions that vanish initially and depend on the environment in a fairly complicated way.

The validity of (2) for a general environment at arbitrary temperature has been recently demonstrated by Hu et al. (1992) and is an interesting discovery that generalizes previous results concerning the nature of the master equation (see Unruh and Zurek (1989), Caldeira and Leggett (1985), Haake and Reiboldt (1985)). The result is also surprising since a general environment generates a non–Markovian evolution for which one expects highly nonlocal integral kernels in the master equation. However, for this model the non–Markovian effects can be fully encoded in the time dependence of the coefficients. The interested reader can find a simple proof of equation
(2) (the simplest I could think off) in Appendix 1.

Equation (2) is a very useful tool to study generic properties of the evolution and can be exactly solved for some simple initial conditions. To study decoherence we will consider the following initial superposition of coherent states:

\[ \Psi(x, t = 0) = N \exp\left(-\frac{(x - L_0)^2}{2\delta^2} + iP_0x\right) + N \exp\left(-\frac{(x + L_0)^2}{2\delta^2} - iP_0x\right) \]  

where \( N \) is a constant. In this case it is possible to solve the master equation and show that the Wigner function constructed from the reduced density matrix is:

\[ W(x, p, t) = W_1(x, p, t) + W_2(x, p, t) + W_{int}(x, p, t) \]

where

\[ W_1(x, p, t) = N^2 \frac{\delta_2}{\delta_1} \exp\left(-\frac{(x \mp x_c)^2}{\delta_1^2} - \frac{(p \mp p_c - \beta(x \mp x_c))^2}{\delta_2^2}\right) \]

\[ W_{int}(x, p, t) = 2N^2 \frac{\delta_2}{\delta_1} \exp\left(-\frac{x^2}{\delta_1^2} - \frac{(p - \beta x)^2}{\delta_2^2}\right) \cos(\phi_p p + (\phi_x - \beta \phi_p) x) \exp(-A_{int}) \]

The functions \( x_c(t), P_c(t), \delta_1(t), \beta(t), \phi_x(t), \phi_p(t) \) and \( A_{int}(t) \) depend on the environment (and on the constants \( L_0, P_0 \) and \( \delta \) that appear in (3)) in a rather complicated way. For the sake of brevity, we will not discuss here the behavior of all these functions (see Paz et al (1992) for details) but concentrate on \( A_{int} \) which is the only one relevant for decoherence. Thus, to quantify the importance of interference at a given time we will use the peak to peak ratio between the interference and the direct terms in the Wigner function, a quantity closely related to \( A_{int} \):

\[ \exp(-A_{int}) = \frac{1}{2} \frac{W_{int}(x, p)|_{peak}}{\left(W_1(x, p)|_{peak}W_2(x, p)|_{peak}\right)^{1/2}} \]
As the two initial wave packets have a finite overlap, the above function satisfies $A_{int} \leq \delta^2 P_0^2 + L_0^2 / \delta^2$. The system decoheres when $A_{int}$ irreversibly grows to a value that is large with respect to unity (which can only occur if the initial peaks are well separated, i.e. $\delta^2 P_0^2 + L_0^2 / \delta^2 >> 1$).

To analyze how the evolution of $A_{int}$ is affected by the environment it is convenient to use the master equation (2) to show the following identity:

$$\dot{A}_{int} = D(t) \phi_p^2 - 2f(t)\phi_p(\phi_x - \beta \phi_p) \quad (7)$$

The first term carries the effect of normal diffusion and always produces decoherence since increases the value of $A_{int}$. On the contrary, the sign of the second term in (7) may vary in time depending upon the relation between $\phi_p$ and $\phi_x$. Equation (7) can be approximately solved if one neglects the anomalous diffusion and considers $D(t)$ as a constant, two conditions met by an ohmic environment in the high temperature regime (see next section).

In this case it can be shown that, for an initial state with $P_0 = 0$, $A_{int}(t) \simeq 4L_0^2 D t / (1 + 4D \delta^2 t)$ and that the “decoherence rate” is $\Gamma_{dec} = 4L_0^2 D \simeq 8L_0^2 m \gamma_0 k_B T$ (see Paz et al. (1992)). However, this solution is no longer valid when one moves away from the ohmic environment in the high temperature regime or when considers more general initial states. In fact, the behavior of $A_{int}$ strongly depends on the initial conditions (that enter into (7) through the functions $\phi_x$ and $\phi_p$ whose initial data are $\phi_x = P_0$, $\phi_p = L_0$) and decoherence will be drastically different in the case $L_0 = 0$, $P_0 \neq 0$ where the two initial gaussian are spatially separated than when $P_0 = 0$ and $L_0 \neq 0$ (where the coherent states are separated in momentum).

Decoherence and the environment.
A wide and interesting class of environments is defined by a spectral density of the form $I(\omega) = \frac{2m\gamma_0}{\pi} \omega^n \Lambda^{-1} \exp(-\frac{\omega^2}{\Lambda^2})$ where $\Lambda$ is a high frequency cutoff and $n$ is an index that characterizes different environments. We will consider two examples: $n = 1$ which is the largely studied ohmic environment (Caldeira and Leggett (1985)) and $n = 3$ which is a supra–ohmic environment used to model the interaction between defects and phonons in metals, Grabert et al (1988), and also to mimic the interaction between a charge and its own electromagnetic field, Barone and Caldeira (1991).

Using these two environments we want to illustrate how strongly decoherence depends on the spectral density. It can be shown that the process is much more inefficient in the supraohmic than in the ohmic case because the final value of the diffusion coefficient $D(t)$ is much smaller in the former than in the latter environment (as $n = 3$ corresponds to a bath of oscillators with an infrared sector substantially weaker than $n = 1$, the dissipative and diffusive effects are expected to be weaker). The time dependence of the diffusion coefficient for these two environments has been described by Hu et al (1992) and has a rather generic feature: $D(t)$ vanishes initially and develops a very strong peak in a time scale of the order of the collision time $\tau_\Lambda = 1/\Lambda$. Its value after the initial peak is $D(\tau_\Lambda) \simeq m\gamma_0\Lambda$, approximately the same for all environments. After this initial cutoff dominated regime, $D(t)$ approaches (in a dynamical time scale) an asymptotic value that depends on the environment (in the high temperature regime, $D(t) \rightarrow 2m\gamma_0 k_B T$ for $n = 1$ while vanishes for $n = 3$). Thus, there is no generic long time behavior but a quite universal short time regime. One may thus wonder if this general initial behavior produces a rather universal
decoherence. We will argue here that this is not the case. The impact of the initial peak has been analyzed in detail (see Unruh and Zurek (1989)) and it was shown that in some cases may completely wipe out the interference effects. However, the physical significance of the decoherence produced by the initial peak is rather questionable since this jolt is certainly related to the initial conditions that do not contain correlations between the system and its environment. In fact, such correlations are likely to wash out the initial peak, Grabert et al (1988).

To discredit even more the role of the initial peak on decoherence we would like to point out that its effect can be made completely innocuous by appropriately choosing the initial conditions for the system. This is well illustrated by the supra–ohmic environment where the asymptotic value of the diffusion coefficient is too small to produce decoherence and all the effect, if any, should come from the initial peak. In Figure 1 we plotted $A_{int}$ for the ohmic and supraohmic environments. We considered an harmonic oscillator with renormalized frequency $\Omega_r$ and fixed $\gamma_0 = 0.3\Omega_r$, $\Lambda = 500\Omega_r$ and $k_B T = 25000\Omega_r$ (high temperature regime).

We can notice that in the ohmic environment decoherence is very fast. For the initial condition I ($L_0 = 3\delta, P_0 = 0$) it takes place in a time of the order of $\tau_\Lambda$ while for condition II ($L_0 = 0, P_0 = 3/\delta$) it requires a time that is also much smaller than $\Omega_r^{-1}$. On the other hand, in the supraohmic environment of Figure 1.b decoherence goes as in the ohmic case for condition I while no net decoherence is achieved for condition II. In this case the initial growth of $A_{int}$ is followed by a plateau and a decreasing regime during which coherence is recovered! The reason for the drastic difference
between the fate of conditions I and II in the supraohmic environment is clear: decoherence can only be produced by the initial peak but the interaction between the system and the environment is initially effective only if the two coherent states are spatially separated. The non monotonic behavior of $A_{int}$ seen in curve (II) of Figure 1.b is due to the anomalous diffusion that cannot produce any net decoherence since the sign of the second term in the r.h.s. of (7) changes with time.

The above example not only illustrates the strong dependence of decoherence on the spectral density but also clarifies in what sense position is an observable that is preferred by the interaction. In fact, in general, coherent states that are spatially separated decohere much faster than those separated only in momentum (see Paz et al (1992) for more details).

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**Appendix: Derivation of the Master Equation**

I outline here a simple derivation of the master equation (2) based on the properties of the evolution operator of the reduced density matrix. This propagator, which we denote as $J(t, t_0)$ and is defined so as to satisfy $\rho_{\text{red}}(t) = J(t, t_0)\rho_{\text{red}}(t_0)$, has a path integral representation of the following form:

$$J(x, x', t \mid x_0, x'_0, t_0) = \int_{x_0}^{x} D\tilde{x} \int_{x'_0}^{x'} D\tilde{x}' \exp \frac{i}{\hbar} \left\{ S[\tilde{x}] - S[\tilde{x}'] \right\} F[\tilde{x}, \tilde{x}'] \quad (8)$$

where $F(x, x')$ is the Feynman–Vernon influence functional that arises due
to the integration of the environment variables. For the model we are considering, this functional is well known and can be written as (see Grabert et al (1988)):

\[
i \ln(F[x, y]) = \int_0^t ds \int_0^s ds'(x - y)(s)[\eta(s - s')(x + y)(s') - i\nu(s - s')(x - y)(s')]
\]

where \(\nu(s)\) and \(\eta(s)\) are the noise and dissipation kernels defined in terms of the spectral density:

\[
\eta(s) = -\int_0^\infty d\omega I(\omega) \sin(\omega s), \quad \nu(s) = \int_0^\infty d\omega I(\omega) \coth(\omega / 2k_B T) \cos(\omega s)
\]

As the integrand of (A.1) is gaussian, the integral can be exactly computed and the result is (written in terms of the variables \(\xi = x - x', \ X = x + x'\)):

\[
J(X, \xi; X_0, \xi_0, t_0) = \frac{b_3}{2\pi} \exp(ib_1 X \xi + ib_2 X_0 \xi - ib_3 X \xi_0 - ib_4 X_0 \xi_0) \times \exp(-a_{11} \xi^2 - a_{12} \xi \xi_0 - a_{22} \xi_0^2)
\]

where the functions \(b_k(t)\) and \(a_{ij}(t)\) depend on the environment and can be constructed in terms of solutions to the equation:

\[
\ddot{u}(s) + \Omega_0^2 u(s) + 2 \int_0^s ds' \eta(s - s') u(s') = 0
\]

Thus, if \(u_1\) and \(u_2\) are two solutions of (A.3) that satisfy the boundary conditions \(u_1(0) = u_2(t) = 1\) and \(u_1(t) = u_2(0) = 0\) we can write:

\[
2 \ b_1(t) = \dot{u}_2(t), \quad 2 \ b_2(t) = \dot{u}_2(0), \quad 2 \ b_3(t) = \dot{u}_1(t), \quad 2 \ b_4(t) = \dot{u}_1(0)
\]

\[
a_{ij}(t) = \frac{1}{1 + \delta_{ij}} \int_0^t \int_0^t ds \ ds' u_i(s) u_j(s') \nu(s - s')
\]

The derivation of the master equation can be done as follows by simply using equations (A.2) and (A.4): Let us take the time derivative of (A.2)
and write

\[
\dot{J}(t, t_0) = \left[ \frac{\dot{b}_3}{b_3} + i b_1 X \xi + i b_2 X_0 \xi - i \dot{b}_3 X \xi_0 - i \dot{b}_4 X_0 \xi_0 - \dot{a}_{11} \xi^2 - \dot{a}_{12} \xi \xi_0 - \dot{a}_{22} \xi_0^2 \right] J(t, t_0)
\]

(12)

If we multiply (A.5) by an arbitrary initial density matrix and integrate over the initial coordinates \(\xi_0\) and \(X_0\), we will obtain an equation whose left hand side is \(\dot{\rho}_r(x, x', t)\). In the right hand side we will find terms proportional to \(\rho(x, x', t)\) that look like some of the ones appearing in the right hand side of equation (2). The only potentially problematic terms are the ones that in (A.5) are proportional to the initial coordinates \(\xi_0\) and \(X_0\). However, their contribution can be easily shown to be local by realizing that the propagator \(J(t, t_0)\) satisfies:

\[
\xi_0 J(X, \xi, t; X_0, \xi_0, t_0) = (\frac{b_1}{b_3} \xi + \frac{i}{b_3} \partial_X) J(X, \xi, t; X_0, \xi_0, t_0)
\]

\[
X_0 J(X, \xi, t; X_0, \xi_0, t_0) = (-X \frac{b_1}{b_2} - \frac{i}{b_2} \partial_\xi - \frac{2a_{11}}{b_2 b_3} + \frac{a_{12} b_1}{b_2 b_3} \partial_X) J(X, \xi, t; X_0, \xi_0, t_0)
\]

Using these equations, the right hand side of (A.5) can be written in terms of the final coordinates \(X\) and \(\xi\) and the derivatives with respect to them, which implies that the master equation is local. To show that this equation is given by (2), we just have to demonstrate that the coefficients associated to terms like \(\partial_X^2\) or \(X \partial_X\) cancel and this can be done by exploiting some general properties of the coefficients \(b_k\) and \(a_{ij}\) that follow directly from their definition in (A.4). In fact, using relations such as \(\dot{a}_{22} = -\dot{b}_4 a_{12}/b_2\) (whose proof we omit), it is possible to show that the coefficients of equation (2) are given by

\[
\frac{b_1}{b_3} \xi + \frac{i}{b_3} \partial_X
\]
(2) are:

\[ \Omega_{ren}^2(t) = 2\left(\frac{\dot{b_2}b_1}{b_2} - \dot{b}_1\right); \quad \gamma(t) = -(b_1 + \frac{\dot{b_2}}{2b_2}) \]

\[ D(t) = \dot{a}_{11} - 4a_{11}b_1 + \dot{a}_{12}\frac{b_1}{b_3} - \frac{\dot{b}_2}{b_2}(2a_{11} + a_{12}\frac{b_1}{b_3}) \]

\[ 2f(t) = \frac{\dot{a}_{12}}{b_3} - \frac{\dot{b}_2a_{12}}{b_2b_3} - 4a_{11} \]

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Figure Captions

Figure 1: The degree of decoherence $A_{int}$ is plotted as a function of time (which is measured in units of $\Omega^{-1}$) for the ohmic (a) and supra-ohmic (b) environments in the high temperature regime. Curve (I) corresponds to an initial condition where the initial coherent states are spatially separated ($L_0 = 3\delta, P_0 = 0$) while for curve (II) the initial separation is in momentum ($L_0 = 0, P_0 = 3/\delta$).
Questions

Unruh: Your master equation is local in time. For an arbitrary spectral density I would strongly expect that the equations are strongly non local in time. Why aren’t yours?

Paz: There are two observations one can intuitively make for the model described by equation (1). On the one hand we expect that a general environment will produce non–Markovian effects and that the master equation will be non local in time. On the other hand, the (reduced) evolution operator must be gaussian since the problem is linear. The crucial observation is that if one admits a gaussian evolution operator, the master equation is always local provided the matrix mixing “old” and “new” coordinates in the propagator can be inverted. Taking this into account, it is surprising to me that the existence of a local master equation has not been noticed until so recently.

Morikawa: Why do you get a local coefficient $\gamma(t)$?

Paz: I refer to the answer I gave to Prof. Unruh’s question.