Morse index and linear stability of the Lagrangian circular orbit in a three-body-type problem via index theory

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Abstract

It is well known that the linear stability of the Lagrangian elliptic solutions in the classical planar three-body problem depends on a mass parameter $\beta$ and on the eccentricity $e$ of the orbit. We consider only the circular case ($e = 0$) but under the action of a broader family of singular potentials: $\alpha$-homogeneous potentials, for $\alpha \in (0, 2)$, and the logarithmic one. It turns out indeed that the Lagrangian circular orbit persists also in this more general setting.

We discover a region of linear stability expressed in terms of the homogeneity parameter $\alpha$ and the mass parameter $\beta$, then we compute the Morse index of this orbit and of its iterates and we find that the boundary of the stability region is the envelope of a family of curves on which the Morse indices of the iterates jump. In order to conduct our analysis we rely on a Maslov-type index theory devised and developed by Y. Long, X. Hu and S. Sun; a key role is played by an appropriate index theorem and by some precise computations of suitable Maslov-type indices.

Keywords: $n$-body problem, $\alpha$-homogeneous potential, logarithmic potential, Morse index, linear stability, Maslov index, Lagrangian solutions, relative equilibrium.

Introduction and main results

We consider a planar three-body-type problem governed by a singular potential function $U : X \subset \mathbb{R}^6 \to \mathbb{R}$, where $X := \{ q = (q_1, q_2, q_3) \in \mathbb{R}^6 | q_i \neq q_j \forall i \neq j \}$. We shall deal with homogeneous and logarithmic potentials of the form

$$
U_{\alpha}(q) := \sum_{i,j=1 \atop i < j}^{3} \frac{m_im_j}{|q_i - q_j|^\alpha}, \quad \alpha \in (0, 2); \quad U_{\log}(q) := \sum_{i,j=1 \atop i < j}^{3} m_im_j \log \frac{1}{|q_i - q_j|}.
$$

(0.1)

Newton’s equations for this problem (which as $U = U_{\alpha}$ is commonly known as the generalised 3-body problem) are

$$
m_i\ddot{q}_i = \frac{\partial U}{\partial q_i}
$$

(0.2)

and we seek solutions that satisfy periodic boundary conditions. By taking into account the conservation law of the centre of mass we see that the configuration space is 4-dimensional and is given by

$$
\hat{X} := \left\{ q \in \mathbb{R}^6 \ \bigg| \sum_{i=1}^{3} m_iq_i = 0, \ q_i \neq q_j \ \forall \ i \neq j \right\}.
$$

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Let \((q,v)\) be an element of the tangent bundle \(T\tilde{X}\), so that \(q \in \tilde{X}\) and \(v \in T_q\tilde{X}\). The **Lagrangian function** \(\mathcal{L} \in \mathcal{C}^\infty(T\tilde{X}, \mathbb{R})\) is given by

\[
\mathcal{L}(q, v) := \frac{1}{2} \sum_{i=1}^{3} m_i |v_i|^2 + U(q),
\]

(0.3)

Let \(W^{1,2}(\mathbb{R}/2\pi\mathbb{Z}, \tilde{X})\) be the Sobolev space of \(L^2\)-loops with weak \(L^2\)-derivatives and define on it the **Lagrangian action functional** \(\mathcal{A}\) as

\[
\mathcal{A}(\gamma) := \int_0^{2\pi} \mathcal{L}(\gamma(t), \dot{\gamma}(t)) \, dt,
\]

(0.4)

which is smooth on its domain, since it consists of collisionless loops. Its critical points in this space are the \(2\pi\)-periodic (classical) solutions of Equations (0.2).

The first solutions of the classical \((U = U_\alpha\) with \(\alpha = 1\)) planar three-body problem have been shown in 1772 by J.-L. Lagrange [Lag72]: for any choice of the three masses there exists a family of periodic motions during which the bodies are always arranged in an equilateral triangle that rotates around its barycentre, changing its size but not its shape; moreover, each particle describes a Keplerian conic. In the special case where the trajectory of each body around the centre of mass is a circle swept with some appropriate angular frequency, Lagrange’s triangular solution is an example of relative equilibrium, called Lagrange circular orbit. We observe that this kind of circular motion is maintained also in the case of the more general potentials defined in (0.1).

Given a periodic solution of (0.2), it is natural to investigate its stability properties in order to understand the dynamical behaviour of the orbits nearby. Linear stability of periodic orbits is a paradigm of a complex behaviour of a dynamical system: positive topological entropy, existence of transversal heteroclinic connections and KAM tori, presence of horseshoes. Our main concern is the linear stability of these circular Lagrangian solutions. It turns out that it depends on two parameters: the mass parameter

\[
\beta := 27 \frac{m_1m_2 + m_2m_3 + m_1m_3}{(m_1 + m_2 + m_3)^2} \in (0, 9]
\]

and the homogeneity parameter \(\alpha \in [0, 2]\). Note that we now include the value \(\alpha = 0\) because it will be shown that this corresponds to the logarithmic case. These two parameters define a family of Lagrangian circular solution, which we denote by \(\gamma_{\alpha, \beta}\).

In order to investigate the linear stability of this family we need to reformulate the Newtonian problem (0.2) in Hamiltonian language. A \(2\pi\)-periodic solution of this autonomous Hamiltonian system is spectrally stable if the spectrum of the monodromy matrix of the corresponding linearised system is contained in the unit circle of the complex plane; it is linearly stable if in addition such matrix is diagonalisable.

When analysing \(\gamma_{\alpha, \beta}\) we face a very degenerate situation because of the invariance of \(n\)-body-type problems under the symmetry group of Euclidean transformations and the presence of first integrals. It is possible, through a wise change of coordinates originally found by Meyer and Schmidt and here modified, to factorise the contributions of these constants of motion and split the phase space into a direct sum of invariant 4-dimensional symplectic subspaces: \(T^*X = E_1 \oplus E_2 \oplus E_3\) (see [Moe94, MS05, BJP]). It turns out that the degeneracy is confined in \(E_1\) and partly in \(E_2\), whereas \(E_3\) contains the essence of the dynamics. More precisely the subspace \(E_1\) corresponds to the four integrals of motion of the center of mass, whilst \(E_2\) includes the conservation of the angular momentum. Furthermore, the restriction of the Hamiltonian to the symplectic invariant subspace \(E_2\) of the phase space agrees with the Hamiltonian of a generalised Kepler problem (i.e. a Kepler problem with potential \(U\) of the form (0.1)). We note that the eigenvalues of the monodromy matrix restricted to \(E_2\) are \(1, 1, e^{\pm 2\pi i \sqrt{\alpha}}\); hence, for any \(\alpha \in (0, 2]\), the circular solutions of the Kepler-type problem (corresponding to the line \(\beta = 0\) in Figure 1) are spectrally stable and \(\alpha = 2\)
is the boundary of their stability region (which is also called in the literature \textit{elliptic region}). The portrait of the stability properties of $\gamma_{\alpha,\beta}$, which takes into account the essence of the dynamics, taking place on $E_3$, is depicted in Figure 1, where one can neatly distinguish three regions: that of spectral instability ($SI$), that of linear stability ($LS$) and the curve of spectral stability ($SS$) that separates them.

A very deep and intriguing question is the relation between the linear stability of a periodic solution or of a closed geodesic and the Morse index of its iterations [Bot56]: a famous result by H. Poincaré states that every closed minimising geodesic on a Riemannian surface is unstable. Motivated by this question we computed the Morse index of the Lagrangian circular orbit in the free loop space of $\dot{X}$. Very few results are known about this topic; a classical one is due to W. B. Gordon [Gor77], who proved that the minimisers of the Lagrangian action functional for the Kepler problem on the subspace of $W^{1,2}(\mathbb{R}/2\pi\mathbb{Z},\mathbb{R}^2 \setminus \{0\})$ of loops with winding number $\pm 1$ with respect to the origin are the ellipses. However M. Ramos and S. Terracini showed in [RT95] a sort of double variational characterisation of the set of all periodic solutions of the $\alpha$-homogeneous Kepler problem. This can give a heuristic explanation of the degeneracy occurring at $\alpha = 1$. In [Ven02] A. Venturelli proved that for $\alpha \in (1, 2)$ and winding numbers $\pm 1$ the minimisers are precisely the circular solutions, whilst for $\alpha \in (0, 1)$ the minima are attained by the ejection-collision solutions. He left, however, completely open the problem of computing the Morse index of the circular solutions in the case $\alpha \in (0, 1)$.

Our first main result concerns the computation of the Morse index of the circular solution of Kepler-type problems (we write $\gamma_{\alpha,0}$ for the Keplerian trajectory, in view of the formal correspondence with the case $\beta = 0$). As already observed, this means to compute the Morse index of the restriction of $\gamma_{\alpha,\beta}$ to the subspace $E_2$ (see Figure 2a). Note that this quantity does not depend on $\beta$; however, we represent its values in the plane $(\beta, \alpha)$ in order to relate them more clearly with the restriction of the system to $E_3$: the Morse index of the original problem is indeed given by the sum of the indices of the restrictions and it is easy to visualise this with the superposition of the graphs.

\textbf{Theorem.} The Morse index of the circular solution $(\gamma_{\alpha,0})$ of the generalised Kepler problem is

$$i_{\text{Morse}}(\gamma_{\alpha,0}) = \begin{cases} 0 & \text{if } \alpha \in [1, 2) \\ 2 & \text{if } \alpha \in [0, 1). \end{cases}$$

We can then go further by computing the Morse index of any $k$-th iteration $\gamma_{\alpha,0}^k$ of $\gamma_{\alpha,0}$ for $k \in \mathbb{N}$, $k > 1$: this is made possible by the $\omega$-index theory and the Bott-Long iteration formula. What we obtain is that $i_{\text{Morse}}(\gamma_{\alpha,0}^k)$ is a piecewise constant and non-increasing function of $\alpha$ for
For a fixed $\alpha$, the Morse index is a monotone decreasing function of $\beta$ that attains the minimum value in correspondence of equal masses. The dotted curve is the stability curve.

Figure 2: Values of the Morse index of the generalised Kepler problem (a) and of the Lagrangian circular solution (b).

every fixed $k > 1$. In particular, for any fixed $k$, there exists an interval $(2 - \frac{1}{k}, 2)$ on which $i_{\text{Morse}}(\gamma_{k, 0}) = 0$. On the other hand, for any fixed value of $\alpha$, the quantity $i_{\text{Morse}}(\gamma_{k, 0})$ diverges to $+\infty$ as $k \to +\infty$. Let us observe that $\alpha_k := 2 - \frac{1}{k}$ tends to the value 2 as $k$ diverges: this means that the jumps of the Morse index tend to the boundary of the stability region for the Kepler-type problem. See Figure 3 for some examples.

As for the Morse index of the family of circular Lagrangian solutions of the planar 3-body-type problem, an interesting result is due to Venturelli [Ven02, Theorem 3.1.7, page 25], who proved that for $\alpha = 1$ the minimisers of the Lagrangian action functional among the loops under a homological constraint are circular orbits. Moreover, he showed that for equal masses ($\beta = 9$) and $\alpha \in (1, 2)$ the periodic solution $\gamma_{\alpha, 9}$ is a strict local minimiser, whereas for $\alpha \in [0, 1)$ it is a saddle. The problem of determining the Morse index of the circular Lagrangian orbit for different masses and for any parameter $\alpha \in (0, 2)$ has been left unsolved until now.

**Theorem.** The Morse index of the Lagrangian circular solution $\gamma_{\alpha, \beta}$ is given by

$$i_{\text{Morse}}(\gamma_{\alpha, \beta}) = \begin{cases} 
0 & \text{if } \alpha \in [1, 2) \\
2 & \text{if } \beta \geq \frac{36(1 - \alpha)}{(\alpha + 2)^2} \text{ and } \alpha \in [0, 1) \\
4 & \text{if } 0 < \beta < \frac{36(1 - \alpha)}{(\alpha + 2)^2}.
\end{cases}$$

The result is depicted in Figure 2b.

In the particular case of $\alpha = 1$ we recover the results proved by X. Hu and S. Sun in [HS10, Formulas (55)–(56)]: they compute the Morse index of the Lagrangian elliptic orbits of the classical three-body problem taking as parameters the eccentricity of the orbit and $\beta$.

As for the generalised Kepler problem, we are able to determine, via $\omega$-index theory, any $i_{\text{Morse}}(\gamma_{k, \beta})$ for all $k \geq 2$. It is worth noting that $\alpha = 2$ is the limit of some values $\alpha_k$ that are the points where the Morse index of the $k$-th iteration of the circular Keplerian solution jumps. Moreover this limit value (which coincides with the lower bound of the strong force condition) is the boundary of the spectral stability region. The very same behaviour appears also in the restriction to the symplectic invariant subspace $E_3$, although the curves of the $(\beta, \alpha)$-plane over which the Morse index of all the $k$-iterations jumps are no longer straight lines. As we show at the end of Subsection 6.2, the boundary of the stability region is the enveloping curve of a
Figure 3: Values of the Morse index of the $k$-th iteration of the Kepler circular orbit $\gamma_{\alpha,\beta}^k$ for some values of $k$. The white upper band in each subfigure represents the value 0; going downwards and passing through the lower boundary of each band increases the Morse index by 2. As $k$ increases there is an accumulation of bands at the value $\alpha = 2$. 
Figure 4: Curves \( \{f_{k,l}\} \) on which the Maslov index of the \( k \)-th iteration of the problem restricted to \( E_3 \) jumps (the values of \( k \) taken into account are shown below each subfigure). The dotted line is the stability curve, which is approximated more and more accurately as \( k \) increases.
two-parameter family of curves representing the jumps in the Morse index of the iterations of the solution. It seems then quite plausible to conjecture that the points at which a transition of stability occurs could be locally approximated, in a suitable sense, by curves along which there is a change in the Morse index of all the iterates.

Let us now compare our result with some other important contributions on the subject. Being every relative equilibrium a zero-average loop solution, our theorem and [Ven02, Theorem 3.1.7, page 25] seem to be in striking contrast with the main theorem by A. Chenciner and N. Desolneux in [CD98], where they proved that $\gamma_{\alpha, \beta}$, for $\alpha \in (0, +\infty)$, are global minima of the action functional defined on the space of $W^{1,2}$-loops with zero average (and fixed centre of mass). However, although for $\alpha \in [0,1)$ we show that the Morse index is strictly positive, there is no contradiction because we do not restrict ourselves to the zero-average $W^{1,2}$-loop space. One might observe that the domain of the functional analysed by Chenciner and Desolneux includes collisions and ours does not, but this is not at all influential on the question: even taking into account those singularities the Morse index would not be affected, being it a local function and being relative equilibria always collisionless by definition. The main result in [CD98] has been recently generalised in [BT04], where V. Barutello and S. Terracini proved that for every $\alpha \in (0, +\infty)$ the absolute minimum among simple choreographies is attained on a relative equilibrium motion associated with the regular $n$-gon. We observe that in imposing the choreographic symmetry constraint the authors require as well that the masses be equal, so that the symmetry may act transitively on the bodies’ labels. This corresponds in our setting to fixing $\beta = 9$. Their result [BT04, Theorem 1] entails that the circular Lagrange solution is an absolute minimum of the action functional on the $W^{1,2}$-choreographies. However, by our theorem we have that for $\alpha \in [0,1)$ the Morse index is 2. We observe as above that this is not in contrast with our result since we are computing the Morse index in a strictly larger space.

The main tool we used to demonstrate these results is an index theory, namely a Morse index theorem that relates the Morse index of a critical point of the Lagrangian action functional and the Maslov index of the fundamental solution associated with the corresponding Hamiltonian system. The problem of computing the Morse index is then translated into the computation of the Maslov index. The key ingredient in order to switch from the Morse index to the Maslov index is the use of the Morse index theorem. In order to compute this symplectic invariant we avail ourselves of some canonical transformations that involve a symplectic change of coordinates. Such new coordinates provide two useful advantages: first, the linearised Hamiltonian system becomes autonomous; second, the reduced phase space is split into two symplectic 4-dimensional subspaces $E_2$ and $E_3$ which are invariant under the phase flow. As a consequence, the Maslov index is obtained as the sum of the Maslov indices of the restrictions of the fundamental solution to these subspaces. Although some formulas for the computation of the Maslov index exist for non-degenerate situations (involving for instance the Krein signature), we point out that $E_2$ gives rise to a really degenerate setting. We overcome all of these problems by using different notions of Maslov index available in the literature, all of which differ by the contribution at the endpoints and by their homotopy properties. In order to overcome the degeneracy on $E_2$ we used the axiomatic definition given by Cappell, Lee and Miller in their well-known paper [CLM94], while to manage the degeneracy represented by the boundary of the stability region on $E_3$ we mainly employ the Maslov index introduced by Long. In Section 2 we recall the puzzle of all these indices trying to point out their main properties as well as the intertwining relations between them. Due to the low dimension, in all of our computation a big role is played by the geometry of $Sp(2)$. To this end and for the sake of the reader we dedicate Appendix A to fix our notation and to recall some well-known facts scattered in the literature. As already observed, a key result is represented by the Morse index theorem stating the relation between the Morse index of the essentially positive Fredholm quadratic forms associated with the second variation and the Maslov index of the periodic solution. Appendix B is devoted to fixing and clarifying the functional-analytical setting.
1 Relative equilibria and a symplectic decomposition of the phase space

1.1 Relative equilibria and central configurations

1.2 A symplectic decomposition of the phase space for the linearised system

2 Maslov-type index theories

2.1 Maslov-type index theory for symplectic paths

2.2 The $\omega$-index theory and the iteration formula

2.3 Morse index of paths of Lagrangian subspaces and relation with other Maslov-type indices

2.4 Computation of the Maslov index via Krein signature and splitting numbers

3 Variational setting: an index theorem

4 Linear and spectral stability of the Lagrangian solution

5 Maslov index of the generalised Kepler problem

5.1 Computation of the Maslov index

5.2 Computation of the $\omega$-index on $E_2$

6 $\omega$-index associated with the restriction to $E_3$

6.1 Computation of the Maslov index

6.2 Computation of the $\omega$-index on $E_3$

7 The $\omega$-Morse index of the Lagrangian circular orbit

7.1 $\omega$-Morse index of the generalised Kepler problem

7.2 $\omega$-Morse index of the Lagrangian circular orbit

7.3 Relation between linear stability and Morse index

A The geometric structure of $Sp(2)$

B Morse index of Fredholm quadratic forms

References

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where $U : X \subset \mathbb{R}^6 \to \mathbb{R}$ is one of the two potential functions

$$U_\alpha(q) := \sum_{i,j=1}^{3} \frac{m_i m_j}{|q_i - q_j|^{\alpha}}, \quad \alpha \in (0, 2),$$  \hspace{1cm} (1.2a)$$

$$U_{\log}(q) := -\sum_{i,j=1}^{3} m_i m_j \log |q_i - q_j|$$  \hspace{1cm} (1.2b)$$

($\alpha = 1$ corresponds to the gravitational case) defined on the collision-free configuration space

$$X := \mathbb{R}^6 \setminus \{ q \in \mathbb{R}^6 \mid q_i = q_j \text{ for some } i \neq j \}.$$  

The symbol $| \cdot |$ indicates the Euclidean norm in $\mathbb{R}^2$, whilst $M \in \text{Mat}(6, \mathbb{R})$ is the diagonal mass matrix $\text{diag}(m_1 I_2, m_2 I_2, m_3 I_2)$ and $I_k$ is the $k \times k$ identity matrix.

In order to rewrite the second-order system (1.1) as a first-order Hamiltonian system we define the Hamiltonian function $H : \mathbb{T}^* \mathbb{R}^6 \to \mathbb{R}$ to be

$$H(p, q) := \frac{1}{2} \langle M^{-1} p^T, p^T \rangle - U(q),$$  \hspace{1cm} (1.3)$$

where $p := (p_1, p_2, p_3) \in \mathbb{R}^6$ is the row vector of the linear momenta conjugate to $q$. Hence System (1.1) becomes

$$\begin{cases} p^T = -\partial_q H = \nabla U(q) \\ q = \partial_p H = M^{-1} p^T. \end{cases}$$ (1.4)$$

Let us remark that summing up the equations of (1.1) we obtain that the centre of mass of the system moves uniformly along a straight line; therefore, without loss of generality, we can fix it at the origin and study the dynamics on the reduced (collision-free) configuration space

$$\hat{X} := \{ q \in X \mid \sum_{i=1}^{3} m_i q_i = 0 \}.$$  

the reduced phase space $\mathbb{T}^* \hat{X}$ is therefore 8-dimensional.

### 1.1 Relative equilibria and central configurations

Among all the non-colliding solutions of Newton’s Equations (1.1), maybe the simplest are represented by a special class of periodic solutions called relative equilibria: they are special motions which are at rest in a uniformly rotating frame. In the following and throughout all this paper, the matrix

$$J_{2n} := \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

will denote the complex structure in $\mathbb{R}^{2n}$, but it will always be written simply as $J$, its dimension being clear from the context. The symplectic form on $\mathbb{R}^{2n}$ is then represented through the scalar product $\langle \cdot, \cdot \rangle$. Let $e^{\omega J t}$ be the matrix representing the rotation in the plane with angular velocity $\omega$. With the symplectic change of coordinates

$$\begin{cases} y^T := e^{\omega K t} p^T \\ x := e^{\omega K t} q, \end{cases}$$

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where $K$ is the $6 \times 6$ block-diagonal matrix $\text{diag}(J, J, J)$, we rewrite Hamilton’s Equations (1.4) in a frame uniformly rotating about the origin with a period $2\pi/\omega$:

$$\begin{cases}
\dot{y}^T = -\partial_x \mathcal{H} = \omega Ky^T + \nabla U(x) \\
\dot{x} = \partial_y \mathcal{H} = M^{-1}y^T + \omega Kx,
\end{cases} \quad (1.5)$$

where $\mathcal{H}$ is the new Hamiltonian function given by

$$\mathcal{H}(y, x) := \frac{1}{2} \langle M^{-1}y^T, y^T \rangle - U(x) - \omega \langle Ky^T, x \rangle. \quad (1.6)$$

From the physical point of view, the terms involving $K$ come from the Coriolis force. A relative equilibrium $(\bar{y}, \bar{x}^T)$ is then an equilibrium point for System (1.5) and must satisfy the conditions

$$\begin{cases}
M^{-1}\nabla U(\bar{x}) + \omega^2 \bar{x} = 0 \\
\bar{y}^T = -\omega MK\bar{x}.
\end{cases} \quad (1.7)$$

Note that the first equation just involves the configuration $\bar{x}$ and it is the well known central configuration equation (for further details see [BJP]). Using Euler’s Theorem for homogeneous functions one can compute

$$\omega^2 = \begin{cases} 
\lambda_\alpha := \frac{\alpha U_\alpha(\bar{x})}{\mathcal{I}(\bar{x})} & \text{if } U = U_\alpha \\
\lambda_{\log} := \frac{1}{\mathcal{I}(\bar{x})} \sum_{i,j=1}^n m_i m_j & \text{if } U = U_{\log},
\end{cases} \quad (1.8)$$

where

$$\mathcal{I}(\bar{x}) := \langle M\bar{x}, \bar{x} \rangle = \sum_{i=1}^3 m_i |\bar{x}_i|^2,$$

is the (double of) the moment of inertia (and a norm in $\mathbb{R}^6$). Hence if we let three bodies, distributed in a planar central configuration, rotate with an angular velocity $\omega$ equal to $\sqrt{\lambda_\alpha}$ or to $\sqrt{\lambda_{\log}}$ we get a relative equilibrium, which becomes an equilibrium in a uniformly rotating coordinate system.

Remark 1.1. We observe that $\bar{x}$ is a central configuration if and only if it is a constrained critical point of $U_\alpha$ on a level surface of $\mathcal{I}$; furthermore if $\bar{x}$ is a central configuration then $c\bar{x}$ and $O\bar{x}$ are, for any $c \in \mathbb{R} \setminus \{0\}$ and any $6 \times 6$ block-diagonal matrix $O$ with entries given by a $2 \times 2$ fixed matrix in $\text{SO}(2)$. Because of these facts, it is standard practice to take the quotient of the configuration space $\hat{X}$ with respect to homotheties and rotations about the origin, which gives the so-called shape sphere $S$. It is well known by the studies of Lagrange and Euler ([BJP]) that on $S$ (for any choice of the masses) there are exactly five central configurations: three of them are collinear (the three bodies lie on the same line), while in the other two the bodies are arranged at the vertices of a regular triangle.

1.2 A symplectic decomposition of the phase space for the linearised system

Consider the Hamiltonian System (1.4) in $\mathbb{R}^{12}$

$$\dot{\zeta}(t) = J\nabla \mathcal{H}(\zeta(t)), \quad (1.9)$$

where $\zeta := (p, q^T)^T$ and $\mathcal{H}$ is the Hamiltonian of the 3-body problem defined in (1.3). We linearise it around a relative equilibrium $\bar{\zeta}$ and write

$$\dot{\zeta}(t) = JD^2 \mathcal{H}(\bar{\zeta}) \zeta(t). \quad (1.10)$$
The presence of the first integrals of motion and the invariance of the problem under some isometries gives rise to three symplectic invariant subspaces of the phase space: $E_1$, carrying the information about the translational invariance, $E_2$, generated by the conservation of the angular momentum and by the invariance by dilations, and $E_3$, defined as the symplectic orthogonal complement of the first two.

Indeed, a basis for the position and momentum of the centre of mass is given by the four vectors in $\mathbb{R}^{12}

\begin{align*}
G_1 &:= \begin{pmatrix} Mv \\ 0 \end{pmatrix}, & G_2 &:= \begin{pmatrix} KMv \\ 0 \end{pmatrix}, & g_1 &:= \begin{pmatrix} 0 \\ v \end{pmatrix}, & g_2 &:= \begin{pmatrix} 0 \\ Kv \end{pmatrix}
\end{align*}

with $v := (1, 0, 1, 0, 1, 0)^T \in \mathbb{R}^6$. If we let $E_1$ be the space spanned by these vectors, it turns out that it is invariant and also symplectic. Note that the symplectic complement of $E_1$ is the space where the barycentre of the system is fixed at the origin and the total linear momentum is zero.

The scaling and rotational symmetries generate another linear symplectic invariant subspace $E_2$, a basis of which is given by the four vectors in $\mathbb{R}^{12}

\begin{align*}
Z_1 &:= \begin{pmatrix} M\bar{q} \\ 0 \end{pmatrix}, & Z_2 &:= \begin{pmatrix} KM\bar{q} \\ 0 \end{pmatrix}, & z_1 &:= \begin{pmatrix} 0 \\ \bar{q} \end{pmatrix}, & z_2 &:= \begin{pmatrix} 0 \\ K\bar{q} \end{pmatrix}.
\end{align*}

The coordinates on third subspace $E_3$ will be denoted by $(W, w^T)^T$; note that this also is 4-dimensional.

We now derive a useful expression of the matrix of the linearised system by adapting the proof of Meyer and Schmidt in [MS05, Lemma 3.1, pages 271–273] to the case of the $\alpha$-homogeneous potential, but restricting ourselves to the circular case, i.e. with zero eccentricity. In order to simplify the computations we set, without loss of generality,

$$m_1 + m_2 + m_3 = 1;$$

furthermore we introduce the key parameter

$$\beta := \frac{27(m_1m_2 + m_1m_3 + m_2m_3)}{(m_1 + m_2 + m_3)^2} = 27(m_1m_2 + m_1m_3 + m_2m_3) \in (0, 9].$$

**Proposition 1.2 ($\alpha$-homogeneous case).** There exists a system of symplectic coordinates $\xi := (\bar{Z}, \bar{W}, \bar{z}^T, \bar{w}^T)^T \in \mathbb{R}^8$ and a rescaled time $\tau$ such that the linearised System (1.10) restricted to $E_2 \oplus E_3 = T^* \bar{X}$ has the form

$$\frac{d\xi}{d\tau} = \Lambda \xi, \quad (1.11)$$

where

$$A := \begin{pmatrix}
0 & 1 & 0 & 0 & \alpha + 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & \frac{1}{2}(\alpha + \frac{\alpha + 2}{3} \sqrt{9 - \beta}) & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & \frac{1}{2}(\alpha - \frac{\alpha + 2}{3} \sqrt{9 - \beta}) & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 & 0
\end{pmatrix}. \quad (1.12)$$

**Proof.** The Hamiltonian of the system in the fixed reference frame is

$$\mathcal{H}(p, q) := \frac{1}{2}(M^{-1}p^T, p^T) - U_\alpha(q),$$

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We make the following symplectic change of coordinates:

\[ p^T = C^{-T} \begin{pmatrix} G^T \\ Z^T \\ W^T \end{pmatrix}, \quad q = C \begin{pmatrix} g \\ z \\ w \end{pmatrix}, \]  

(1.13)

where \( C \) is given by (cf. [MS05, pages 268–269])

\[
C := \begin{pmatrix}
1 & 0 & \frac{9(m_2+m_3)}{2\sqrt{\beta}} & \frac{3\sqrt{3}(m_2-m_3)}{2\sqrt{\beta}} & 0 & -\frac{3\sqrt{3}(m_2m_3)}{\sqrt{\beta}\sqrt{m_2m_3}} \\
0 & 1 & -\frac{3\sqrt{3}(m_2-m_3)}{2\sqrt{\beta}} & \frac{9(m_2+m_3)}{2\sqrt{\beta}} & \frac{3\sqrt{3}(m_2m_3)}{2\sqrt{\beta}\sqrt{m_2m_3}} & 0 \\
1 & 0 & -\frac{9m_3}{2\sqrt{\beta}} & \frac{3\sqrt{3}(m_1+2m_3)}{2\sqrt{\beta}} & -\frac{3\sqrt{3}(m_1m_3)}{2\sqrt{\beta}\sqrt{m_2m_3}} & \frac{9m_1m_3}{2\sqrt{\beta}\sqrt{m_2m_3}} \\
0 & 1 & \frac{9m_3}{2\sqrt{\beta}} & \frac{3\sqrt{3}(m_1+2m_3)}{2\sqrt{\beta}} & -\frac{3\sqrt{3}(m_1m_3)}{2\sqrt{\beta}\sqrt{m_2m_3}} & \frac{9m_1m_3}{2\sqrt{\beta}\sqrt{m_2m_3}} \\
1 & 0 & -\frac{9m_1}{2\sqrt{\beta}} & \frac{3\sqrt{3}(m_1+2m_3)}{2\sqrt{\beta}} & -\frac{3\sqrt{3}(m_1m_3)}{2\sqrt{\beta}\sqrt{m_2m_3}} & \frac{9m_1m_3}{2\sqrt{\beta}\sqrt{m_2m_3}} \\
0 & 1 & \frac{9m_1}{2\sqrt{\beta}} & \frac{3\sqrt{3}(m_1+2m_3)}{2\sqrt{\beta}} & -\frac{3\sqrt{3}(m_1m_3)}{2\sqrt{\beta}\sqrt{m_2m_3}} & \frac{9m_1m_3}{2\sqrt{\beta}\sqrt{m_2m_3}} 
\end{pmatrix}
\]

It is a straightforward computation to verify that \( C \) is invertible and it satisfies the relations

\[ C^TMC = I, \quad C^{-1}JC = J. \]

After fixing the centre of mass at the origin (i.e. setting \( g = G^T = 0 \), thus restricting the system to \( E_2 \oplus E_3 \)), the Hamiltonian of the system becomes

\[ \mathcal{H}(Z, W, z, w) = \frac{1}{2}(Z_1^2 + Z_2^2 + W_1^2 + W_2^2) - U_\alpha(z, w). \]

Consider now the rotation in the plane

\[ R(t) := \begin{pmatrix}
\cos(\lambda_\alpha t) & -\sin(\lambda_\alpha t) \\
\sin(\lambda_\alpha t) & \cos(\lambda_\alpha t)
\end{pmatrix}, \]

where \( \lambda_\alpha \) is the Lagrange multiplier (1.8) of the central configuration, corresponding to the square of the angular velocity of each body. Accordingly, we move to a uniformly rotating reference frame in the following way:

\[
\begin{cases}
Z^T = R(t)\tilde{Z}^T \\
W^T = R(t)\tilde{W}^T \\
z = R(t)\tilde{z} \\
w = R(t)\tilde{w}.
\end{cases}
\]

(1.14)

Since we are moving to a new set of canonical coordinates (see for instance [GPS80, Chapter 9]) via the time-depending generating function

\[ F(Z, W, \tilde{Z}, \tilde{W}, t) := -ZR(t)\tilde{z} - WR(t)\tilde{w}, \]

the new Hamiltonian function (still denoted by \( \mathcal{H} \)) must contain the extra term \( \frac{dF}{dt} \):

\[ \mathcal{H}(\tilde{Z}, \tilde{W}, \tilde{z}, \tilde{w}) = \frac{1}{2}((\tilde{Z}_1^2 + \tilde{Z}_2^2 + \tilde{W}_1^2 + \tilde{W}_2^2) - U_\alpha(\tilde{z}, \tilde{w}) + \lambda_\alpha(\tilde{Z}_1\tilde{z}_2 - \tilde{Z}_2\tilde{z}_1 + \tilde{W}_1\tilde{w}_2 - \tilde{W}_2\tilde{w}_1). \]
Then we operate the following symplectic scaling with multiplier $\lambda^{-\alpha_{ff}}$

\[
\begin{cases}
\tilde{Z} = \lambda^{\alpha_{ff}+1}\hat{Z} \\
\tilde{W} = \lambda^{\alpha_{ff}+2} \hat{W} \\
\tilde{z} = \lambda^{\alpha_{ff}+1} \hat{z} \\
\tilde{w} = \lambda^{\alpha_{ff}} \hat{w}
\end{cases}
\]  \tag{1.15}

going thus

\[
\mathcal{H}(\tilde{Z}, \tilde{W}, \tilde{z}, \tilde{w}) = \frac{\lambda}{2} (\tilde{Z}_1^2 + \tilde{Z}_2^2 + \tilde{W}_1^2 + \tilde{W}_2^2) - U_\alpha(\tilde{z}, \tilde{w})
\]

+ $\lambda_\alpha (\tilde{Z}_1\tilde{z}_2 - \tilde{Z}_2\tilde{z}_1 + \tilde{W}_1\tilde{w}_2 - \tilde{W}_2\tilde{w}_1)$.

The next step consists in a time scaling: define $\tau := \lambda t$ and rewrite System (1.9) as

\[
\frac{d\zeta(t)}{d\tau} = J \nabla \mathcal{H}(\zeta(t)),
\]

or equivalently as

\[
\zeta'(\tau)\lambda_\alpha = J \nabla \mathcal{H}(\zeta(\tau)), \tag{1.16}
\]

where the prime $'$ denotes the derivative with respect to $\tau$. Hence a division of both sides of (1.16) by $\lambda_\alpha$ yields the equivalent system

\[
\zeta'(\tau) = J \nabla \mathcal{H}(\zeta(\tau)),
\]

where

\[
\mathcal{H}(\tilde{Z}, \tilde{W}, \tilde{z}, \tilde{w}) = \frac{1}{2} (\tilde{Z}_1^2 + \tilde{Z}_2^2 + \tilde{W}_1^2 + \tilde{W}_2^2) - \frac{1}{\lambda_\alpha} U_\alpha(\tilde{z}, \tilde{w})
\]

+ $\tilde{Z}_1\tilde{z}_1 - \tilde{Z}_2\tilde{z}_2 + \tilde{W}_1\tilde{w}_2 - \tilde{W}_2\tilde{w}_1$

= $\frac{1}{\lambda_\alpha} \mathcal{H}(\tilde{Z}, \tilde{W}, \tilde{z}, \tilde{w})$.

Finally, in order to shift the equilibrium point into the origin, we operate a translation and set

\[
\begin{cases}
Z_1 := \tilde{Z}_1 \\
Z_2 := \tilde{Z}_2 - 1 \\
W_1 := \tilde{W}_1 \\
W_2 := \tilde{W}_2 \\
z_1 := \tilde{z}_1 - 1 \\
z_2 := \tilde{z}_2 \\
w_1 := \tilde{w}_1 \\
w_2 := \tilde{w}_2
\end{cases}
\]  \tag{1.17}

whence

\[
\mathcal{H}(Z, W, z, w) = \frac{1}{2} [Z_1^2 + (Z_2 + 1)^2 + W_1^2 + W_2^2] - \frac{1}{\lambda_\alpha} U_\alpha(z, w)
\]

+ $Z_1z_2 - (Z_2 + 1)(z_1 + 1) + W_1w_2 - W_2w_1$.

The matrix of the linearised system is (J times) the Hessian of this Hamiltonian, evaluated at the origin. In order to write it down we need the Hessian of the potential $U_\alpha$ expressed in the
coordinates \((\bar{z}, \bar{w})\), but since the computations are quite long and tedious we shall omit them and indicate only the way in which we obtained the result. We have that
\[
U_\alpha(\bar{z}, \bar{w}) = \frac{m_1 m_2}{d_{12}^3} + \frac{m_1 m_3}{d_{13}^3} + \frac{m_2 m_3}{d_{23}^3},
\]
where
\[
d_{12} := \frac{3 \sqrt{3}}{\sqrt{\beta}} \left[ (\bar{z}_1 + 1)^2 + \bar{z}_2^2 + \frac{m_3(m_1^2 + m_1 m_2 + m_2^2)}{m_1 m_2} (\bar{w}_1^2 + \bar{w}_2^2) \right.
\]
\[
+ \sqrt{\frac{3 m_2 m_3}{m_1}} \left( \bar{w}_2 \bar{w}_1 - (\bar{z}_1 + 1) \bar{w}_2 \right) - (2 m_1 + m_2) \sqrt{\frac{m_3}{m_1 m_2}} \left( (\bar{z}_1 + 1) \bar{w}_1 + \bar{z}_2 \bar{w}_2 \right) \bigg]^2,
\]
\[
d_{13} := \frac{3 \sqrt{3}}{\sqrt{\beta}} \left[ (\bar{z}_1 + 1)^2 + \bar{z}_2^2 + \frac{m_2 (m_1^2 + m_1 m_3 + m_3^2)}{m_1 m_3} (\bar{w}_1^2 + \bar{w}_2^2) \right.
\]
\[
+ \sqrt{\frac{3 m_2 m_3}{m_1}} \left( \bar{w}_2 \bar{w}_1 - (\bar{z}_1 + 1) \bar{w}_2 \right) + (2 m_1 + m_3) \sqrt{\frac{m_2}{m_1 m_3}} \left( (\bar{z}_1 + 1) \bar{w}_1 + \bar{z}_2 \bar{w}_2 \right) \bigg]^2,
\]
\[
d_{23} := \frac{3 \sqrt{3}}{\sqrt{\beta}} \left[ (\bar{z}_1 + 1)^2 + \bar{z}_2^2 + \frac{m_1 (m_2^2 + m_2 m_3 + m_3^2)}{m_2 m_3} (\bar{w}_1^2 + \bar{w}_2^2) \right.
\]
\[
- (m_2 + m_3) \sqrt{\frac{3 m_1}{m_2 m_3}} \left( \bar{w}_1 - (\bar{z}_1 + 1) \bar{w}_2 \right) + (m_2 - m_3) \sqrt{\frac{m_1}{m_2 m_3}} \left( (\bar{z}_1 + 1) \bar{w}_1 + \bar{z}_2 \bar{w}_2 \right) \bigg]^2.
\]

Now, calculating the Hessian of \(\frac{1}{\lambda_\alpha} U_\alpha\) and evaluating it at the origin yields
\[
\frac{1}{\lambda_\alpha} D^2 U_\alpha(0, 0) = \begin{pmatrix}
\alpha + 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & a & b \\
0 & 0 & b & c
\end{pmatrix},
\]
with \(a := \frac{1}{4} \left[ 4(\alpha + 1) m_1 + (\alpha - 2)(m_2 + m_3) \right], \quad b := \frac{1}{4} \left[ \sqrt{3}(\alpha + 2)(m_2 - m_3) \right] \) and \(c := \frac{1}{4} \left[ -4 m_1 + (3 \alpha + 2)(m_2 + m_3) \right]\). The matrix of the linearised system is thus
\[
\begin{pmatrix}
0 & 1 & 0 & 0 & \alpha + 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & a & b & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & b & c \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 & 0
\end{pmatrix}.
\]
(1.18)

Extracting from it the submatrix representing the dynamics on \(E_3\) (i.e. the one acting on the \(W\)’s and \(w\)’s only):
\[
\begin{pmatrix}
0 & 1 & \frac{1}{4} \left[ 4(\alpha + 1) m_1 + (\alpha - 2)(m_2 + m_3) \right] \\
-1 & 0 & \frac{1}{4} \left[ \sqrt{3}(\alpha + 2)(m_2 - m_3) \right] \\
1 & 0 & 0 \\
0 & 1 & -1
\end{pmatrix},
\]
we apply a rotation to both positions \(w\) and momenta \(W\) and obtain
\[
\begin{pmatrix}
0 & 1 & \frac{1}{2} \left( \alpha + \frac{\alpha + 2}{3} \sqrt{9 - \beta} \right) & 0 \\
-1 & 0 & 0 & \frac{1}{2} \left( \alpha - \frac{\alpha + 2}{3} \sqrt{9 - \beta} \right) \\
1 & 0 & 0 & \frac{1}{4} \\
0 & 1 & -1 & 0
\end{pmatrix},
\]
(1.19)
so that the final matrix depends only on \( \alpha \) and \( \beta \). Now substitute (1.19) back into (1.18) to get (1.12).

In the logarithmic case there is a wholly similar result.

**Proposition 1.3** (Logarithmic case). There exist a system \( \xi := (\hat{Z}, \hat{W}, \hat{z}, \hat{w})^T \in \mathbb{R}^8 \) of symplectic coordinates and a rescaled variable \( \tau \) such that the linearised System (1.10) restricted to \( E_2 \oplus E_3 = T^* \hat{X} \) has the form

\[
\frac{d\xi}{d\tau} = \Lambda \xi,
\]

where

\[
\Lambda := \begin{pmatrix}
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 & 0
\end{pmatrix}.
\]

**Proof.** We proceed exactly as in Proposition 1.2 with some slight modifications. After the symplectic change of coordinates (1.13), we have of course to replace \( \lambda \alpha \) with \( \lambda_{\log} \log \). Hence we apply the rotation in the plane

\[
R(t) := \begin{pmatrix}
\cos \lambda_{\log} t & -\sin \lambda_{\log} t \\
\sin \lambda_{\log} t & \cos \lambda_{\log} t
\end{pmatrix},
\]

in the same way as in (1.14), getting

\[
\mathcal{H}(\hat{Z}, \hat{W}, \hat{z}, \hat{w}) = \frac{1}{2} (\hat{Z}_1^2 + \hat{Z}_2^2 + \hat{W}_1^2 + \hat{W}_2^2) - U_{\log}(\hat{z}, \hat{w})
\]

\[
+ \lambda_{\log} (\hat{Z}_1 \hat{z}_2 - \hat{Z}_2 \hat{z}_1 + \hat{W}_1 \hat{w}_2 - \hat{W}_2 \hat{w}_1).
\]

Transformation (1.15) is now the following:

\[
\begin{align*}
\hat{Z} &= \lambda_{\log}^{1/2} \hat{Z} \\
\hat{W} &= \lambda_{\log}^{1/2} \hat{W} \\
\hat{z} &= \lambda_{\log}^{-1/2} \hat{z} \\
\hat{w} &= \lambda_{\log}^{-1/2} \hat{w}
\end{align*}
\]

and gives

\[
\mathcal{H}(\hat{Z}, \hat{W}, \hat{z}, \hat{w}) = \frac{\lambda_{\log}}{2} (\hat{Z}_1^2 + \hat{Z}_2^2 + \hat{W}_1^2 + \hat{W}_2^2) - U_{\log}(\lambda_{\log}^{-1/2} \hat{z}, \lambda_{\log}^{-1/2} \hat{w})
\]

\[
+ \lambda_{\log} (\hat{Z}_1 \hat{z}_2 - \hat{Z}_2 \hat{z}_1 + \hat{W}_1 \hat{w}_2 - \hat{W}_2 \hat{w}_1).
\]

Then we rescale time by setting \( \tau := \lambda_{\log} t \) and obtain

\[
\hat{\mathcal{H}}(\hat{Z}, \hat{W}, \hat{z}, \hat{w}) = \frac{1}{2} (\hat{Z}_1^2 + \hat{Z}_2^2 + \hat{W}_1^2 + \hat{W}_2^2) - \frac{1}{\lambda_{\log}} U_{\log}(\lambda_{\log}^{-1/2} \hat{z}, \lambda_{\log}^{-1/2} \hat{w})
\]

\[
+ \hat{Z}_1 \hat{z}_2 - \hat{Z}_2 \hat{z}_1 + \hat{W}_1 \hat{w}_2 - \hat{W}_2 \hat{w}_1
\]

\[
= \frac{1}{\lambda_{\log}} \mathcal{H}(\hat{Z}, \hat{W}, \hat{z}, \hat{w}).
\]
Translation (1.17) sets the equilibrium point at the origin and we have
\[
\mathcal{H}(\bar{Z}, \bar{W}, \bar{z}, \bar{w}) = \frac{1}{2} [\bar{Z}_1^2 + (\bar{Z}_2 + 1)^2 + \bar{W}_1^2 + \bar{W}_2^2] - \frac{1}{\lambda_{\log}} U_{\log}(\lambda_{\log}^{-1/2} \bar{z}, \lambda_{\log}^{-1/2} \bar{w}) \\
+ \bar{Z}_1 \bar{z}_2 - (\bar{Z}_2 + 1)(\bar{z}_1 + 1) + \bar{W}_1 \bar{w}_2 - \bar{W}_2 \bar{w}_1.
\]
The Hessian of \(\frac{1}{\lambda_{\log}} U_{\log}\) evaluated at the origin is
\[
\frac{1}{\lambda_{\log}} D^2 U_{\log}(0, 0) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & \frac{1}{2}(2m_1 - m_2 - m_3) & \sqrt{\frac{3}{2}}(m_2 - m_3) \\
0 & 0 & \frac{\sqrt{3}}{2}(m_2 - m_3) & -\frac{1}{2}(2m_1 - m_2 - m_3)
\end{pmatrix};
\]
an orthogonal transformation applied on the subspace \(E_3\) to both positions \(\bar{w}\) and momenta \(\bar{W}\) diagonalises the lower right corner of \(\lambda_{\log}^{-1} D^2 U_{\log}(0, 0)\), making it dependent only on \(\beta\):
\[
\frac{1}{\lambda_{\log}} D^2 U_{\log}(0, 0) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & \frac{1}{3}(9 - \beta) & 0 \\
0 & 0 & 0 & -\frac{1}{3}(9 - \beta)
\end{pmatrix}.
\]
By computing the Hessian of \(\mathcal{H}\) and multiplying on the left by \(J\), we find the matrix \(\Lambda\) of the statement.

Remark 1.4. We observe that (1.20) can be obtained from (1.12) simply by setting \(\alpha = 0\). Therefore in the analysis that will follow we shall consider the logarithmic case as a subcase of the \(\alpha\)-homogeneous one. Note that this is a remark \textit{a posteriori}, since we could not deduce it directly from the relation \(U_{\alpha}(q) - \frac{1}{\alpha} U_{\log}(q) \sim U_{\log}(q)\) as \(\alpha \to 0^+\), which is only asymptotic.

2 Maslov-type index theories

The aim of this section is to briefly describe some Maslov-type index theories for paths of symplectic matrices as well as for paths of Lagrangian subspaces. In Subsection 2.1 we recall a geometric definition of the Maslov index for symplectic paths exploiting the intersection number of a curve and a singular cycle (an algebraic variety of codimension 1 in the symplectic group). Then, in Subsection 2.2, we recollect the basic definitions of the \(\omega\)-index theory, essentially developed by Long and his school, and exhibit the relation with the geometric Maslov-type index. Our main sources for these two subsections are [CZ84, LZ90, LZ00] and references therein. Subsection 2.3 is devoted to a brief presentation of other Maslov-type index theories defined through a suitable intersection theory in the Lagrangian Grassmannian manifold by means of the crossing forms. We also show the relationship with the Maslov-type index theories previously introduced in the symplectic context. Our basic references for all this are [RS93, CLM94, Por08, GPP04, Lon02, HS10, HS11, Ara67, PPT04, MPP05, Por10].

\(^1\)Here and in the following, with a slight abuse of notation, we denote by \(\lambda_{\log}^{-1/2} \bar{z}\) the vector \((\lambda_{\log}^{-1/2} z_1 + 1, \lambda_{\log}^{-1/2} z_2)^T\).
2.1 Maslov-type index theory for symplectic paths

Following Long and Zhu in [LZ00], we define for all \( n \in \mathbb{N} \setminus \{0\} \) the complex and real symplectic groups

\[
\text{Sp}(2n, \mathbb{C}) := \{ M \in \text{GL}(2n, \mathbb{C}) \mid M^t JM = J \} \\
\text{Sp}(2n) := \{ M \in \text{GL}(2n, \mathbb{R}) \mid M^T J M = J \}
\]

and for \( 0 \leq k \leq 2n \) we set

\[
\text{Sp}_k(2n, \mathbb{C}) := \{ M \in \text{Sp}(2n, \mathbb{C}) \mid \dim_{\mathbb{C}} \ker_{\mathbb{C}}(M - I) = k \} \\
\text{Sp}_k(2n) := \{ M \in \text{Sp}(2n, \mathbb{R}) \mid \dim \ker(M - I) = k \}.
\]

It is clear that one has the following stratifications:

\[
\text{Sp}(2n, \mathbb{C}) = \bigcup_{k=0}^{2n} \text{Sp}_k(2n, \mathbb{C}), \quad \text{Sp}(2n) = \bigcup_{k=0}^{2n} \text{Sp}_k(2n).
\]

For the sake of the reader we recall the following well-known result, which gives the properties of the stratification.

**Proposition 2.1.** The subsets \( \text{Sp}_k(2n, \mathbb{C}) \) and \( \text{Sp}_k(2n) \) are, respectively speaking, smooth submanifolds of \( \text{Sp}(2n, \mathbb{C}) \) and \( \text{Sp}(2n) \), with codimension \( k^2 \) and \( \frac{1}{2}k(k+1) \). Moreover, \( \text{Sp}_1(2n, \mathbb{C}) \) and \( \text{Sp}_1(2n) \) are co-oriented, the transverse orientation being given by the vector field \( \frac{d}{dt}(Me^{jt})|_{t=0} \).

We have in addition that

\[
\overline{\text{Sp}_k(2n, \mathbb{C})} = \bigcup_{t \geq k} \text{Sp}_t(2n, \mathbb{C}) \quad \text{and} \quad \overline{\text{Sp}_k(2n)} = \bigcup_{t \geq k} \text{Sp}_t(2n).
\]

By Proposition 2.1 the intersection points of the curve

\[
\gamma(t) := Me^{jt}, \quad M \in \text{Sp}_1(2n, \mathbb{C})
\]

with the cycle \( \text{Sp}_1(2n, \mathbb{C}) \) form a discrete subset of \( \gamma(\mathbb{R}) \). We recall that a matrix in \( \text{Sp}(2n, \mathbb{C}) \) is called non-degenerate if it does not admit 1 as an eigenvalue. A straightforward computation allows us to see that for a continuous path \( \gamma : [a, b] \to \text{Sp}(2n, \mathbb{C}) \) there exists \( \delta > 0 \) such that for any \( \varepsilon \in (-\delta, \delta) \setminus \{0\} \) (the perturbed) path \( s \mapsto \gamma(s)e^{-\varepsilon J} \) is non-degenerate, meaning that it has non-degenerate endpoints.

**Definition 2.2.** Let \( \gamma : [a, b] \to \text{Sp}(2n, \mathbb{C}) \). We define its geometric Maslov-type index to be the intersection number of \( s \mapsto \gamma(s)e^{-\varepsilon J} \) with \( \text{Sp}_1(2n, \mathbb{C}) \) for all \( \varepsilon \in (0, \delta) \) (where \( \delta \) is such that the perturbed path is non-degenerate).

\[
i_{\text{geo}}(\gamma) := \left[ \gamma e^{-\varepsilon J} : \text{Sp}_1(2n, \mathbb{C}) \right], \tag{2.1}
\]

where the right-hand side of (2.1) is the usual homotopy intersection number.

For any \( \omega \in U := \{ z \in \mathbb{C} \mid |z| = 1 \} \) and \( T > 0 \) it is convenient to define the set

\[
\mathcal{P}_T(2n) := \{ \gamma \in C^0([0, T]; \text{Sp}(2n, \mathbb{C})) \mid \gamma(0) = I_{2n} \}
\]

and its subset

\[
\mathcal{P}_T^\omega(2n) := \{ \gamma \in \mathcal{P}_T(2n) \mid \gamma(T) \in \omega \text{Sp}_1(2n, \mathbb{C}) \}.
\]

Consider now two square matrices \( M_1 \) and \( M_2 \) of sizes \( 2m_1 \times 2m_1 \) and \( 2m_2 \times 2m_2 \) respectively (with \( m_1, m_2 \in \mathbb{N} \setminus \{0\} \)) such that they can both be written in the form

\[
M_k := \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}, \quad k = 1, 2,
\]

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Definition 2.3

The diamond product of $M_1$ and $M_2$ is defined (see [Lon02, page 17]) as the following $2(m_1 + m_2) \times 2(m_1 + m_2)$ matrix:

$$M_1 \diamond M_2 := \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix},$$

(2.2)

The $k$-fold diamond product of $M$ with itself is denoted by $M^{\otimes k}$. The symplectic sum of two paths $\gamma_j \in \mathcal{P}_T(2n_j)$, with $j = 1, 2$ and $n_1, n_2 \in \mathbb{N} \setminus \{0\}$, is defined in a natural way:

$$(\gamma_1 \oplus \gamma_2)(t) := \gamma_1(t) \oplus \gamma_2(t), \quad \forall t \in [0, T].$$

We list the basic properties of the geometric Maslov-type index that we need in the paper.

(i) (Path additivity) Let $\gamma : [a, b] \to \text{Sp}(2n, \mathbb{C})$ and $c \in [a, b]$. Then

$$i_{\text{geo}}(\gamma) = i_{\text{geo}}(\gamma|_{[a,c)}) + i_{\text{geo}}(\gamma|_{[c,b]}).$$

(ii) (Composition) Let $\gamma_1 : [a, b] \to \text{Sp}(2k, \mathbb{C})$ and $\gamma_2 : [a, b] \to \text{Sp}(2l, \mathbb{C})$ be two symplectic paths. Then we have

$$i_{\text{geo}}(\gamma_1 \circ \gamma_2) = i_{\text{geo}}(\gamma_1) + i_{\text{geo}}(\gamma_2).$$

(iii) (Homotopy invariance) For any two paths $\gamma_1$ and $\gamma_2$, if $\gamma_1$ is homotopic to $\gamma_2$ (written $\gamma_1 \sim \gamma_2$) in $\text{Sp}(2n, \mathbb{C})$ with either fixed or always non-degenerate endpoints, there holds

$$i_{\text{geo}}(\gamma_1) = i_{\text{geo}}(\gamma_2).$$

(iv) (Normalisation) If $n = 1$ then

$$i_{\text{geo}}(e^{it}I, t \in [0, a]) = \begin{cases} 1 & \text{if } a \in (0, 2\pi) \\ 0 & \text{if } a = 2\pi. \end{cases}$$

(v) (Affine scale invariance) For all $k > 0$ and $\gamma \in \mathcal{P}_{kT}(2n)$ we have

$$i_{\text{geo}}(\gamma(kt), t \in [0, \tau]) = i_{\text{geo}}(\gamma(t), t \in [0, k\tau]).$$

2.2 The $\omega$-index theory and the iteration formula

For any two continuous paths $\gamma, \delta : [0, T] \to \text{Sp}(2n, \mathbb{C})$ such that $\gamma(T) = \delta(0)$, we define their concatenation $\gamma \ast \delta : [0, T] \to \text{Sp}(2n, \mathbb{C})$ as

$$(\gamma \ast \delta)(t) := \begin{cases} \gamma(2t) & \text{if } 0 \leq t \leq \frac{T}{2} \\ \delta(2t - T) & \text{if } \frac{T}{2} \leq t \leq T. \end{cases}$$

For any $n \in \mathbb{N} \setminus \{0\}$ we also define a special continuous symplectic path $\xi_n : [0, T] \to \text{Sp}(2n)$ as follows:

$$\xi_n(t) := \begin{pmatrix} 2 - \frac{t}{T} & 0 \\ 0 & 2 - \frac{t}{T} \end{pmatrix}^n \begin{pmatrix} 0 & 0 \\ 0 & 2 - \frac{t}{T} \end{pmatrix}, \quad \forall t \in [0, T].$$

(2.3)

**Definition 2.3** ([Lon99, HS09]). Let $\omega \in \mathbb{U}$. If $\gamma \in \mathcal{P}_T(2n)$, we define

$$\nu_\omega(\gamma) := \dim_{\mathbb{C}} \ker_{\mathbb{C}} (\gamma(T) - \omega I_{2n}).$$
If \( \gamma \in \mathcal{P}_{T,\omega}(2n) \) the \( \omega \)-index is defined as
\[
i_\omega(\gamma) := \begin{bmatrix} \omega^* \xi_n : \text{Sp}_1(2n, \mathbb{C}) \end{bmatrix}.
\] (2.4)

If \( \gamma \in \mathcal{P}_{T}(2n) \setminus \mathcal{P}_{T,\omega}(2n) \), we let \( \mathcal{F}(\gamma) \) be the set of all open neighbourhoods \( U \) of \( \gamma \) in \( \mathcal{P}_{T}(2n) \), and define
\[
i_\omega(\gamma) := \sup_{U \in \mathcal{F}(\gamma)} \inf \{ i_\omega(\delta) \mid \delta \in U \cap \mathcal{P}_{T,\omega}(2n) \}.
\]

Finally the \( \omega \)-geometric Maslov index is defined as
\[
i_{\text{geo,}\omega}(\gamma) := \begin{bmatrix} \omega^* \xi ne^{-J} : \text{Sp}_1(2n, \mathbb{C}) \end{bmatrix},
\] (2.5)

The right-hand side of (2.4) and (2.5) is the usual homotopy intersection number, the orientation of \( \omega^* \xi_n \) is its positive time direction under homotopies with fixed end-points and \( \varepsilon \) is a positive real number sufficiently small.

We list the basic properties of the \( \omega \)-index that we need in the sequel.

(i) (Lower semicontinuity) For all \( \gamma : [a,b] \to \mathcal{P}_{T}(2n) \) and \( c \in [a,b] \) we have
\[
i_\omega(\gamma) = \inf \{ i_\omega(\beta) \mid \beta \in \mathcal{P}_{T}(2n) \text{ is sufficiently } C^0 \text{-close to } \gamma \}.
\]

(ii) (\( \omega \)-additivity) Let \( \gamma_1 : [a,b] \to \text{Sp}(2k, \mathbb{C}) \) and \( \gamma_2 : [a,b] \to \text{Sp}(2l, \mathbb{C}) \) be two symplectic paths. Then we have
\[
i_\omega(\gamma_1 \circ \gamma_2) = i_\omega(\gamma_1) + i_\omega(\gamma_2).
\]

(iii) (Homotopy invariance) For any two paths \( \gamma_1 \) and \( \gamma_2 \), if \( \gamma_1 \sim \gamma_2 \) in \( \text{Sp}(2n, \mathbb{C}) \) with either fixed or always non-degenerate endpoints, there holds
\[
i_\omega(\gamma_1) = i_\omega(\gamma_2).
\]

(iv) (Affine scale invariance) For all \( k > 0 \) and \( \gamma \in \mathcal{P}_{kT}(2n) \), we have
\[
i_\omega(\gamma(kt), t \in [0,T]) = i_\omega(\gamma(t), t \in [0,kT])
\]

The proofs of these properties are consequences of [LZ00, Lemma 2.2 (3), Corollary 2.1, Theorem 2.1] and of the index theory contained in [Lon99].

Let \( \gamma \in \mathcal{P}_{T}(2n) \) and \( m \in \mathbb{N} \setminus \{0\} \). The \( m \)-th iteration of \( \gamma \) is \( \gamma^m : [0, mT] \to \text{Sp}(2n) \) defined as
\[
\gamma^m(t) := \gamma(t - jT)(\gamma(T))^{j}, \quad \text{for } jT \leq t \leq (j+1)T, \quad j = 0, \ldots, m-1.
\]

The next Bott-type iteration formula is crucial in order to study the geometric multiplicity of periodic orbits and plays a big role in the question of linear stability.

**Lemma 2.4** (Bott-Long iteration formula, [Lon02, Theorem 9.2.1]). For any \( z \in U, \gamma \in \mathcal{P}_{T}(2n) \) and \( m \in \mathbb{N} \setminus \{0\} \) the following formula holds:
\[
i_z(\gamma^m) = \sum_{\omega = 1}^m i_\omega(\gamma).
\] (2.6)

In particular one has \( i_1(\gamma^2) = i_1(\gamma) + i_{-1}(\gamma) \).
2.3 Morse index of paths of Lagrangian subspaces and relation with other Maslov-type indices

Let \((\mathbb{C}^{2n}, \cdot, \cdot)\) be the complex symplectic space whose complex symplectic structure can be represented through the Hermitian product \((\cdot, \cdot)\) as

\[
\{v, w\} := (Jv, w), \quad \forall v, w \in \mathbb{C}^{2n}.
\]

We denote by \(\text{Lag}(\mathbb{C}^{2n})\) the space of all Lagrangian subspaces in \(\mathbb{C}^{2n}\).

Let \(l: [a, b] \to \text{Lag}(\mathbb{C}^{2n})\) be a \(C^1\)-curve of Lagrangian subspaces and let \(L_0 \in \text{Lag}(\mathbb{C}^{2n})\). Fix \(t \in [a, b]\) and let \(W\) be a fixed Lagrangian complement of \(l(t)\). If \(s\) belongs to a suitable small neighbourhood of \(t \in [a, b]\) for every \(v \in l(t)\) we can find a unique vector \(w(s) \in W\) in such a way that \(v + w(s) \in l(s)\).

**Definition 2.5.** The crossing form \(\Gamma(l, L_0, t)\) at \(t^*\) is the quadratic form \(\Gamma(l, L_0, t^*): l(t) \cap L_0 \to \mathbb{R}\) defined by

\[
\Gamma(l, L_0, t^*)[\xi] := \frac{d}{ds}\{v, w(s)\} |_{s=t^*}.
\]

(7.2)

The number \(t^*\) is said to be a crossing instant for \(l\) with respect to \(L_0\) if \(l(t^*) \cap L_0 \neq \{0\}\) and it is called regular if the crossing form is non-degenerate.

Let us remark that regular crossings are isolated and hence on a compact interval they are finitely many. Following [LZ00, Definition 3.1, Theorem 3.1] we give the next definition.

**Definition 2.6.** If \(l\) has only regular crossings with respect to \(L_0\), the Maslov index of \(l\) with respect to \(L_0\) is defined as

\[
i_{\text{CLM}}(L_0, l, [a, b]) := m^+ (\Gamma(l, L_0, a)) + \sum_{t^* \in (a, b)} \text{sgn} \Gamma(l, L_0, t^*) - m^- (\Gamma(l, L_0, b))
\]

where the summation runs over all crossings \(t^* \in (a, b)\), the symbols \(m^+, m^-\) denote the dimension of the positive and negative spectral subspaces respectively and \(\text{sgn} := m^+ - m^-\) is the signature.

Let \(V := \mathbb{C}^{2n} \oplus \mathbb{C}^{2n}\), and \((\cdot, \cdot)\) be the standard Hermitian product of \(V\). We define

\[
\{v, w\}_J := (Jv, w), \quad \forall v, w \in V
\]

where

\[
J := \begin{pmatrix} -J & 0 \\ 0 & J \end{pmatrix}.
\]

By a direct calculation it follows that if \(M \in \text{Sp}(2n, \mathbb{C})\) then the complex subspace

\[
\text{Gr}(M) := \left\{ \begin{pmatrix} x \\ Mx \end{pmatrix} \mid x \in \mathbb{C}^{2n} \right\}
\]

is a Lagrangian subspace of the (complex) symplectic space \((V, \cdot, \cdot)_J\).

Given a path of symplectic matrices \(\gamma: [a, b] \to \text{Sp}(2n, \mathbb{C})\) then the graph of the path \(\gamma\), \(\text{Gr}(\gamma)\), is defined as the path of graphs: \(\text{Gr}(\gamma)(t) := \text{Gr}(\gamma(t)), t \in [a, b]\), and it is indeed a path of Lagrangian subspaces of \((V, \cdot, \cdot)_J\). The next result gives the relationship between the geometric index of a path of symplectic matrices and the Maslov index of the corresponding path of Lagrangian subspaces with respect to the diagonal \(\Delta := \text{Gr}(I_{2n})\).

**Proposition 2.7.** For all path \(\gamma: [a, b] \to \text{Sp}(2n, \mathbb{C})\) we have

\[
i_{\text{geo}}(\gamma) = i_{\text{CLM}}(\Delta, \text{Gr}(\gamma), [a, b]),
\]

where the crossing forms involved in the right-hand side are calculated using the symplectic structure \((\cdot, \cdot)_J\) in \(\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}\).
Lemma 2.8. For any path $\gamma \in \mathcal{P}_T(2n)$ we have the following equalities:

1. $i_1(\gamma) + n = i_{\text{CLM}}(\Delta, \text{Gr}(\gamma), [0, T])$;
2. $i_\omega(\gamma) = i_{\text{CLM}}(\text{Gr}(\omega_{2n}), \text{Gr}(\gamma), [0, T])$ for all $\omega \in U \setminus \{1\}$.

Remark 2.9. We observe that the integer $i_1$ is sometimes denoted by $i_{\text{CZ}}$ and it is called the Conley-Zehnder index. For further details we refer to [LZ00] and references therein.

We now show some examples of computation of $i_1(\gamma)$ passing through $i_{\text{CLM}}$ of some paths of matrices in $\text{Sp}(2) \subset \text{Sp}(2, \mathbb{C})$. Let $\gamma : [a, b] \to \text{Sp}(2)$ be the path

$$
\gamma(t) := \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix},
$$

with $a, b, c, d \in \mathcal{C}^1([a, b], \mathbb{R})$, let $l$ be the induced path of Lagrangian subspaces in $\mathbb{R}^4$ defined by $l(t) := \text{Gr}(\gamma(t))$. In order to compute the crossing form (2.7) we consider the Lagrangian subspace complementary to $\Delta$:

$$
W := \{0\} \times \mathbb{R} \times \mathbb{R} \times \{0\}.
$$

Thus the Lagrangian splitting $\mathbb{R}^4 = \Delta \oplus W$ holds and for any $v := (x_0, y_0, x_0, y_0) \in \Delta$ let us choose $w(t) := (0, \eta(t), \xi(t), 0) \in W$ in order that $v + w(t) \in l(t)$. This means that $\eta(t)$ and $\xi(t)$ solve the equations

$$
\begin{align*}
x_0 + \xi(t) &= a(t)x_0 + b(t)(y_0 + \eta(t)), & y_0 &= c(t)x_0 + d(t)(y_0 + \eta(t)).
\end{align*}
$$

(2.9)

Since in a crossing instant $t^*$ we have $\xi(t^*) = \eta(t^*) = 0$, differentiating the above identities gives

$$
\begin{align*}
\xi'(t^*) &= a'(t^*)x_0 + b'(t^*)y_0 - \frac{b(t^*)}{d(t^*)} [c'(t^*)x_0 + d'(t^*)y_0],
\eta'(t^*) &= -\frac{1}{d(t^*)} [c'(t^*)x_0 + d'(t^*)y_0].
\end{align*}
$$

(2.10a)

(2.10b)

By a direct computation we obtain

$$
\{v, w(t)\}_2 = \langle 3v, w(t) \rangle = -\left\langle J \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \begin{pmatrix} \eta(t) \\ 0 \end{pmatrix} \right\rangle + \left\langle J \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \begin{pmatrix} \xi(t) \\ 0 \end{pmatrix} \right\rangle = -x_0\eta(t) - y_0\xi(t).
$$

Hence the crossing form at the crossing instant $t = t^*$ is given by

$$
\Gamma(\{l, \Delta, t^*\}(v) = \frac{d}{dt} \{v, w(t)\}_2 \big|_{t=t^*} = -x_0\eta'(t^*) - y_0\xi'(t^*).
$$

(2.11)

Example 2.10. Let us consider the path $R_\alpha : [0, 2\pi] \to \text{Sp}(2)$ with

$$
R_\alpha(t) = \begin{pmatrix} \cos(\sqrt{2 - \alpha} t) & -\sqrt{2 - \alpha} \sin(\sqrt{2 - \alpha} t) \\ \frac{1}{\sqrt{2 - \alpha}} \sin(\sqrt{2 - \alpha} t) & \cos(\sqrt{2 - \alpha} t) \end{pmatrix},
$$

$\alpha \in (0, 2)$,

that means $a = 0$, $b = 2\pi$, and

$$
\begin{align*}
a(t) &= d(t) = \cos(\sqrt{2 - \alpha} t), & b(t) &= -\sqrt{2 - \alpha} \sin(\sqrt{2 - \alpha} t), & c(t) &= \frac{1}{\sqrt{2 - \alpha}} \sin(\sqrt{2 - \alpha} t).
\end{align*}
$$

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For any value of the parameter $\alpha$, $t^* = 0$ is a crossing instant and $a(0) = 1$, $a'(0) = 0$, $b(0) = 0$, $b'(0) = -(2 - \alpha)$, $c(0) = 0$, $c'(0) = 1$. Using Equations (2.10) we get
\begin{equation}
\Gamma(l_\alpha, \Delta, 0)[v] = -x_0 v'_\alpha(0) - y_0 c'_\alpha(0) = x_0^2 + (2 - \alpha)y_0^2
\end{equation}
where $l_\alpha$ is the path of Lagrangian subspaces associated to $R_\alpha$. Since $\Gamma(l_\alpha, \Delta, 0)$ is a positive definite quadratic form, its signature is 2. Thus, according to Formula (2.8), the contribution to $i_{\text{CLM}}$ at the starting point of the path is 2.

In order to find out all the crossing instants, we observe that they are in one-to-one correspondence with the zeros of the function $\det(R_\alpha(t) - I_2)$, and hence with the solutions in $[0, 2\pi]$ of the equation
\begin{equation}
\cos(\sqrt{2 - \alpha} t) = 1,
\end{equation}
that we write as $t_\alpha^0 := 2k\pi / \sqrt{2 - \alpha}$, with $k \in \mathbb{Z}$. It is readily seen that

- if $\alpha \in (1, 2)$ then the only solution of (2.13) is $t_\alpha^0 = 0$, hence there are no other contributions to the computation of $i_{\text{CLM}}$.
- if $\alpha = 1$ then we have two solutions: $t_1^0 = 0$ and $t_1^1 = 2\pi$. We need to add $m^-(\Gamma(l_\alpha, \Delta, 2\pi))$ to the contribution of the initial instant, but this quantity is actually 0.
- if $\alpha \in (0, 1)$ then (2.13) admits also the non-zero solution $\alpha^0 = \frac{2\pi}{\sqrt{2 - \alpha}}$. The contribution of this crossing is $\text{sgn}\Gamma(l_\alpha, \Delta, t_\alpha^0) = 2$.

Summing up all these computations we obtain
\begin{equation}
i_{\text{CLM}}(R_\alpha) = \begin{cases}
2 & \text{if } \alpha \in [1, 2) \\
4 & \text{if } \alpha \in (0, 1).
\end{cases}
\end{equation}

**Example 2.11.** We now consider the path $N_\alpha : [0, 2\pi] \to \text{Sp}(2)$ with
\[ N_\alpha(t) = \begin{pmatrix} 1 & 0 \\ f_\alpha(t) & 1 \end{pmatrix} \]
where the function
\[ f_\alpha(t) := \frac{1}{36\pi^2} \left( \frac{4\sin(\sqrt{2 - \alpha} t)}{(2 - \alpha)^{3/2}} - \frac{2 + \alpha}{2 - \alpha} t \right) \]
is drawn in Figure 5a for $\alpha = 1$ (the other cases for different $\alpha$’s are all similar).

We first observe that we are in a very degenerate situation, in the sense that $N_\alpha(t) \subset \text{Sp}_1(2)$. Furthermore, the function $f_\alpha$ admits two zeros in the interval $[0, 2\pi]$, $t_1 = 0$ and $t_2^* \in (0, 2\pi)$. Thus the path is not contained in a fixed stratum of the Maslov cycle.

However, by taking into account the very definition of the Maslov index in the degenerate case given in Definition 2.3, we need to compute the contributions of the crossing of the graph of the perturbed matrix
\[ N_{\varepsilon\alpha}(t) := N_\alpha(t) e^{-\varepsilon J} \quad \forall t \in [0, 2\pi] \]
and for $\varepsilon > 0$ sufficiently small. By a direct computation we get:
\begin{equation}
N_{\varepsilon\alpha}(t) := \begin{pmatrix} \cos \varepsilon & \sin \varepsilon \\ -\sin \varepsilon + f_\alpha(t) \cos \varepsilon & \cos \varepsilon + f_\alpha(t) \sin \varepsilon \end{pmatrix}
\end{equation}
The crossing instants are the zeros of the equation:
\begin{equation}
2 - 2\cos \varepsilon - f_\alpha(t) \sin \varepsilon = 0.
\end{equation}
\[\text{We observe that this value coincides with the apsidal angle for the } \alpha\text{-homogeneous potential.}\]
The function \( f_1(t) \) in the interval \([0, 2\pi]\).

(b) The function \( 2 - 2\cos\varepsilon - f_1(t)\sin\varepsilon \) in the interval \([0, 2\pi]\).

Figure 5: The function \( f_\alpha \) (a) and its deformation (b) for \( \alpha = 1 \).

The path \( N_\alpha \) (in red) and its deformation \( N_{\varepsilon,\alpha} \) (in blue). The first path starts at the identity, then goes downwards right, then comes back to the identity and finally bends downward left. The second follows the same trajectory, just rotated clockwise by an angle \( \varepsilon \). See Appendix A for more details about the coordinates and the underlying curves.

The function whose zeros we are searching is depicted in Figure 5b.

It is easy to see that for \( \varepsilon \) sufficiently small and for any \( \alpha \in (0, 2) \) this equation admits two distinct solutions \( t_1^\alpha \) and \( t_2^\alpha \) in \((0, 2\pi)\).

Denoting by \( t^\alpha \) a generic solution (crossing) we easily compute

\[
\eta_\alpha'(t^\alpha) = -f_\alpha'(t^\alpha)x_0 \\
\xi_\alpha'(t^\alpha) = 0,
\]

whence

\[
\Gamma(N_{\varepsilon,\alpha}, \Delta, t^\alpha) = f_\alpha'(t^\alpha)x_0^2.
\]

Summing up the two contributions, from the monotonicity of \( f_\alpha \) we immediately obtain that \( \iota_{\text{CLM}}(N_\alpha, \Delta, [0, 2\pi]) = 0 \). The path \( N_\alpha \) and its deformation \( N_{\varepsilon,\alpha} \) are represented in Figure 6.

2.4 Computation of the Maslov index via Krein signature and splitting numbers

In the case of autonomous Hamiltonian systems and under the assumption of non-degeneracy it is possible, at least theoretically, to compute the Maslov index (see for instance \[Abb01\] and references therein). Let \( M \in \text{Sp}(2n, \mathbb{R}) \) act on \( \mathbb{C}^{2n} \) in the usual way:

\[
M(\xi + i\eta) := M\xi + iM\eta, \quad \forall \xi, \eta \in \mathbb{R}^{2n}
\]
and consider the Hermitian form $g$ on $\mathbb{C}^{2n}$ defined as
\[ g(v, w) := (iJv, w). \]

**Definition 2.12.** Let $\lambda \in \mathbb{U}$ be an eigenvalue of a complex symplectic matrix. The *Krein signature* of $\lambda$ is the signature of the restriction of the Hermitian form $g$ to the generalised eigenspace associated with $\lambda$. If $g$ is positive definite on this subspace then $\lambda$ is said to be *Krein-positive*.

The next result will be useful in the following.

**Proposition 2.13** ([Abb01, Theorem 1.5.1]). Let $B$ be a real symmetric matrix. Let $i\theta_1, \ldots, i\theta_k$ be the Krein-positive purely imaginary eigenvalues of $JB$, counted with their algebraic multiplicity. Then the linear autonomous Hamiltonian system
\[ \zeta'(t) = JB\zeta(t) \]
is non-degenerate at time $T$ if and only if $\theta_j T \notin 2\pi\mathbb{Z}$, for any $j = 1, \ldots, k$. If $\psi$ denotes the fundamental solution, we get:
\[ i_1(\psi) = -\sum_{j=1}^{k} \left[ \left[ T\theta_j / \pi \right] \right] \]
provided that it is non-degenerate at time $T$. The function $[\cdot]$ is defined as follows:
\[ [\theta] := \begin{cases} \theta & \text{if } \theta \in \mathbb{Z} \\ \text{the closest odd integer} & \text{if } \theta \in \mathbb{R} \setminus \mathbb{Z}. \end{cases} \]

Now, following [Lon02], we recall the definition of the so-called *splitting numbers* as well as their basic properties, which will be crucial later. For this we refer to [Lon02, Chapter 6, pages 190–199].

**Definition 2.14.** For any $M \in \text{Sp}(2n)$ and every $\omega \in \mathbb{U}$, the splitting numbers $S^+_{M}(\omega)$ of $M$ at $\omega$ are defined by
\[ S^+_{M}(\omega) := \lim_{\varepsilon \to 0^+} i\omega \exp(\pm i\varepsilon\gamma) - i\omega(\gamma), \quad (2.17) \]
where $\gamma \in \mathcal{P}_T(2n)$ is such that $\gamma(T) = M$.

In the next proposition we recall the basic properties of the splitting numbers. For their computation we introduce the *normal forms*
\[ R(\theta) := \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad \theta \in (0, 2\pi) \setminus \{0\}, \quad N(\lambda, a) := \begin{pmatrix} \lambda & a \\ 0 & \lambda \end{pmatrix}, \quad \lambda \in \mathbb{R}^*, \; a \in \mathbb{R}. \]

**Proposition 2.15** ([Lon02, Chapter 6]). For $M, M_0, M_1 \in \text{Sp}(2n)$ and all $\omega \in \mathbb{U}$, $\theta \in (0, \pi)$, the following properties hold:

1. The splitting numbers $S^+_{M}(\omega)$ are well defined, i.e. they are independent of the choice of the path $\gamma \in \mathcal{P}_T(2n)$ satisfying $\gamma(\tau) = M$ in Definition (2.17).
2. The splitting numbers $S^+_{M}(\omega)$ are constant in the set $\Omega^0(M)$, that is the path-connected component $M$ of the set
\[ \Omega(M) := \{ N \in \text{Sp}(2n) \mid \sigma(N) \cap \mathbb{U} = \sigma(M) \cap \mathbb{U} \text{ and } \nu_{\lambda}(N) = \nu_{\lambda}(M) \; \forall \lambda \in \sigma(M) \cap \mathbb{U} \}. \]
3. $S^+_{M}(\omega) = 0$ if $\omega \notin \sigma(M)$.
4. $S^+_{M}(\bar{\omega}) = S^+_{M}(\omega)$.
5. $0 \leq S^+_{M}(\omega) \leq \dim \ker(M - \omega I)$.
6. $S^+_{M}(\omega) + S^-_{M}(\omega) \leq \dim \ker(M - \omega I)^{2n}$ if $\omega \in \sigma(M)$.
7. \( S_{M_0 \circ M_1}^+(\omega) = S_{M_0}^+(\omega) + S_{M_1}^+(\omega) \).

8. \( i_\omega(\gamma) - i_1(\gamma) = S_{M_1}^+(1) + \sum_{\omega_0} (S_{M}^+(\omega_0) - S_{M}^-(\omega_0)) - S_{M}^-(\omega), \) where \( \Im(\omega) > 0 \) and \( \omega_0 \in \sigma(M) \) lies in the interior of the arc of the upper unit semicircle connecting 1 and \( \omega \).

9. \( \left( S_{N_1(1,a)}^+(1), S_{N_1(1,a)}^-(1) \right) = \begin{cases} (1,1) & \text{if } a \in \{0,1\} \\ (0,0) & \text{if } a = -1. \end{cases} \)

10. \( \left( S_{N_1(-1,a)}^+(1), S_{N_1(-1,a)}^-(1) \right) = \begin{cases} (1,1) & \text{if } a \in \{-1,0\} \\ (0,0) & \text{if } a = 1. \end{cases} \)

11. \( \left( S_{R(\theta)}^+(e^{i\theta}), S_{R(\theta)}^-(e^{i\theta}) \right) = (0,1). \)

12. \( \left( S_{R(2\pi - \theta)}^+(e^{i\theta}), S_{R(2\pi - \theta)}^-(e^{i\theta}) \right) = (1,0). \)

3 Variational setting: an index theorem

We recall here some basic facts about the Lagrangian and Hamiltonian dynamics (for further details see for instance [Fat08, AF07, APS08]). The elements of the tangent bundle \( TR^n \cong \mathbb{R}^n \times \mathbb{R}^n \) are denoted by \((q,v)\) where \( q \in U \) and \( v \in T_qU \). Let \( \mathcal{L} \in \mathcal{C}^\infty(TR^n; \mathbb{R}) \) be a regular Lagrangian, meaning that \( \mathcal{L} \) is assumed to satisfy

(L1) \( \partial_{v} \mathcal{L}(q,v) > 0 \) for all \((q,v)\) \( \in TR^n \);

(L2) There is a constant \( l_1 > 0 \) such that

\[
\|\partial_{v} \mathcal{L}(q,v)\| \leq l_1, \quad \|\partial_{q_0} \mathcal{L}(q,v)\| \leq l_1(1 + |v|), \quad \|\partial_{q_0} \mathcal{L}(q,v)\| \leq l_1(1 + |v|^2).
\]

As a direct consequence of the Inverse Function Theorem, under Condition (L1) the Legendre transformation

\[
\mathcal{L}_{\mathcal{F}} : TR^n \to T^*R^n, \quad (q,v) \mapsto (D_v \mathcal{L}(q,v), q),
\]

is a smooth local diffeomorphism. The Fenchel transformation of \( \mathcal{L} \) is the autonomous Hamiltonian on \( T^*R^n \)

\[
\mathcal{H}(p,q) := \max_{v \in T_qR^n} (p[v] - \mathcal{L}(q,v)) = p[v(p,q)] - \mathcal{L}(q,v,p(q)),
\]

where \((q,v(p,q)) = \mathcal{L}_{\mathcal{F}}^{-1}(p,q)\). Under the above assumptions on \( \mathcal{L} \), the function \( \mathcal{H} \) is smooth on \( T^*R^n \). The associated autonomous Hamiltonian vector field \( X_{\mathcal{H}} \) on \( T^*R^n \), defined by

\[
\langle JX_{\mathcal{H}}(p,q), \xi \rangle = -D_{\mathcal{H}}(p,q)[\xi], \quad \forall (p,q) \in T^*R^n, \forall \xi \in T_{(p,q)}T^*R^n,
\]

is then smooth, so it defines an autonomous smooth local flow on \( T^*R^n \). The corresponding flow on \( TR^n \) obtained by conjugating the Hamiltonian flow \( \varphi_{\mathcal{H}} \) by the Legendre transform \( \mathcal{L}_{\mathcal{F}} \) is denoted by

\[
\varphi_{\mathcal{L}} : TR^n \to TR^n
\]

and its orbits have the form \( t \mapsto (\gamma(t), \gamma'(t)) \), where \( \gamma \) solves the Euler-Lagrange equation

\[
\frac{d}{dt} \partial_v \mathcal{L}(\gamma(t), \gamma'(t)) = \partial_q \mathcal{L}(\gamma(t), \gamma'(t)).
\]

Let us consider the Lagrangian action functional \( A : W^{1,2}(\mathbb{R}/2\pi\mathbb{Z}, \widehat{X}) \to \mathbb{R} \) defined by

\[
A(\gamma) := \int_0^{2\pi} \mathcal{L}(\gamma(t), \gamma'(t)) dt.
\]
We recall that if $\mathcal{L}$ satisfies (L2) then $A$ is of class $C^2$ (cf. [AF07, Proposition 4.1]). Moreover if the first variation of $A$ vanishes at $\gamma \in W^{1,2}(\mathbb{R}/2\pi\mathbb{Z}, \hat{X})$ for every $\xi \in W^{1,2}(\mathbb{R}/2\pi\mathbb{Z}, \hat{X})$, then $\gamma$ is a (classical) solution of class $C^2$ of the Euler-Lagrange equation (3.1) such that $\gamma(2\pi) = \gamma(0)$. Given a classical solution $\gamma$ of (3.1) the second variation of $A$ is given by

$$d^2A(\gamma)[\xi, \eta] = \int_0^{2\pi} \left[ (P(t)\xi' + Q(t)\xi)\eta' + Q^T(t)\xi'\eta + R(t)\xi\eta \right] dt,$$

(3.2)

where

$$P(t) := D_{vv}\mathcal{L}(\gamma(t), \gamma'(t)), \quad Q(t) := D_{vq}\mathcal{L}(\gamma(t), \gamma'(t)), \quad R(t) := D_{qq}\mathcal{L}(\gamma(t), \gamma'(t)).$$

Linearising the Euler-Lagrange equations (3.1) around a critical point $\gamma$ we obtain the Sturm system

$$- (P(t)\gamma'(t) + Q(t)\gamma(t))' + Q^T(t)\gamma'(t) + R(t)\gamma(t) = 0$$

(3.3)

Let now $\zeta(t) := (D_v\mathcal{L}(\gamma(t), \gamma'(t), \gamma(t)))$ be the solution of the Hamiltonian system associated with (3.3), whose fundamental solution $\phi$ satisfies

$$\begin{align*}
\phi'(t) &= JB(t)\phi(t) \\
\phi(0) &= I_{2n},
\end{align*}$$

(3.4)

with

$$B(t) := \begin{pmatrix}
P^{-1}(t) & -P^{-1}(t)Q(t) \\
-Q^T(t)P^{-1}(t) & Q^T(t)P^{-1}(t)Q(t) - R(t)
\end{pmatrix}.$$ 

For any $\omega \in U$ let $h(\gamma)$ be the quadratic form on $D(\omega) := \{ \xi \in W^{1,2}([0, 2\pi], \mathbb{C}^n) \mid \xi(2\pi) = \omega\xi(0) \}$ induced by $d^2A(\gamma)$. Then it is possible to show (see [MPP05] for further details) that $h(\gamma)$ is an essentially positive Fredholm quadratic form in the sense specified in Appendix B.

**Definition 3.1.** Let $\gamma \in D(\omega)$ be a critical point of $A$. We define the $\omega$-Morse index of $\gamma$, denoted by $i^\omega_{\text{Morse}}(\gamma)$, as the dimension of the largest subspace of $D(\omega)$ such that the quadratic form $h(\gamma)$ is negative definite.

We observe that the $\omega$-Morse index is the number of negative eigendirections counted according to their multiplicities on which $h(\gamma)$ is negative definite. We also define

$$n_\omega(\gamma) := \dim \ker h(\gamma).$$

The following Morse-type index theorem relates the Morse index of a solution with the $\omega$-index introduced in Subsection 2.2.

**Lemma 3.2** (Morse Index Theorem, [Lou02, page 172]). Let $\gamma$ be a critical point of the Lagrangian action functional (hence a classical solution of the Euler-Lagrange equation (3.1)) and let $\phi$ be the fundamental solution of the linearised system around $\gamma$ (that is, $\phi$ satisfies (3.4)). Then

$$i^\omega_{\text{Morse}}(\gamma) = i_\omega(\phi), \quad n_\omega(\gamma) = n_\omega(\gamma), \quad \forall \omega \in U.$$

We close this section by recalling two important results about the minimising properties of the circular periodic solutions of the $\alpha$-homogeneous Kepler problem and the circular Lagrangian solution of the three-body problem under $\alpha$-homogeneous potential. The first one is due to Gordon (cf. [Gor77]) for the case $\alpha = 1$ and was generalised to different homogeneity degrees by Venturelli in [Ven02, Proposition 2.2.3].

**Lemma 3.3.** In the $\alpha$-homogeneous Kepler problem with $\alpha \in [1, 2]$, circular solutions are local minimisers of the Lagrangian action functional in the space of loops with winding number $\pm 1$ around the origin.
As regards the circular Lagrangian solution for the \( \alpha \)-homogeneous 3-body problem (without any restriction on the choice of the masses), from [Ven02, Theorem 3.1.17] we infer the following result.

**Lemma 3.4.** In the \( \alpha \)-homogeneous 3-body problem the circular Lagrange relative equilibrium is a local minimum of the Lagrangian action functional when \( \alpha \in [1, 2) \) (it is actually a strict minimiser if \( \alpha \neq 1 \)). It is a non-degenerate saddle point when \( \alpha \in (0, 1) \).

### 4 Linear and spectral stability of the Lagrangian solution

Recall that in Section 1.2 we established that there exists a system of symplectic coordinates such that the linearised system restricted to \( E_2 \oplus E_3 \) is represented in the standard basis of \( \mathbb{R}^3 \) by the matrix \( \Lambda \) defined in (1.12). Note now that \( \Lambda \) can be expressed as the diamond product \( \Lambda_2 \odot \Lambda_3 \) of two matrices \( \Lambda_2 \) and \( \Lambda_3 \) defined by

\[
\Lambda_2 := \begin{pmatrix}
0 & 1 & \alpha + 1 & 0 \\
-1 & 0 & 0 & -1 \\
1 & 0 & 0 & 1 \\
0 & 1 & -1 & 0
\end{pmatrix}, \quad \Lambda_3 := \begin{pmatrix}
0 & 1 & \frac{1}{2} \left( \alpha + \frac{\alpha^2 + 2}{3} \sqrt{9 - \beta} \right) & 0 \\
-1 & 0 & 0 & \frac{1}{2} \left( \alpha - \frac{\alpha^2 + 2}{3} \sqrt{9 - \beta} \right) \\
1 & 0 & 0 & 1 \\
0 & 1 & -1 & 0
\end{pmatrix}, \tag{4.1}
\]

the range of \( \alpha \) now being \([0, 2)\) (cf. Remark 1.4). The former encodes the dynamics on the symplectic invariant subspace \( E_2 \), whereas the latter governs the motion on \( E_3 \).

System (1.11) thus decouples into two linear autonomous Hamiltonian subsystems on \( E_2 \) and \( E_3 \) respectively, and it follows that its fundamental solution \( \Phi \in \mathcal{P}_{2\pi}(8) \) can be written as the diamond product of the fundamental solutions \( \phi_2 \in \mathcal{P}_{2\pi}(4) \) and \( \phi_3 \in \mathcal{P}_{2\pi}(4) \) of these subsystems.

**Remark 4.1.** By the discussion given in Section 1 (see also [MS05, page 271] for the gravitational case) the Hamiltonian system on the invariant subspace \( E_2 \) is equivalent to the generalised \( \alpha \)-homogeneous and logarithmic Kepler problem. It is worthwhile noting that the matrix \( \Lambda_3 \) coincides with \( \Lambda_2 \) when \( \beta = 0 \); in this case then the essential part of the fundamental solution of the Lagrangian circular orbit coincides with the fundamental solution of the Kepler orbit.

**Definition 4.2.** The linear autonomous Hamiltonian system (1.11) is spectrally stable if the spectrum \( \sigma(\Lambda) \) of \( \Lambda \) is contained in the imaginary axis \( i\mathbb{R} \); we call it linearly stable if in addition the matrix \( \Lambda \) is diagonalisable. We say that System (1.11) is degenerate if \( \ker \Lambda \neq \{0\} \).

Note that the spectrum \( \sigma(\Lambda) \) of \( \Lambda \) is the union \( \sigma(\Lambda_2) \cup \sigma(\Lambda_3) \) of the spectra of \( \Lambda_2 \) and \( \Lambda_3 \) respectively. The eigenvalues of \( \Lambda_2 \) are

\[0, \ 0, \ \pm i\sqrt{2 - \alpha};\]

hence the system is always degenerate for every \( n \geq 3 \). It is then natural, following Moeckel in [Moe94], to adopt the following terminology.

**Remark 4.3.** We observe that when \( \alpha = 2 \) the spectrum of \( \Lambda_2 \) reduces to \( \{0\} \), while for \( \alpha > 2 \) such matrix admits also two non-zero real eigenvalues. More precisely, when \( \alpha = 2 \) the two non-zero purely imaginary eigenvalues of \( \Lambda_2 \) collapse into the origin (this corresponds to a Krein collision in 1 for the eigenvalues of the monodromy matrix) and split into a pair of non-zero real eigenvalues when \( \alpha > 2 \). The value \( \alpha = 2 \) is then the threshold of linear stability on \( E_2 \).

**Definition 4.4.** A relative equilibrium is non-degenerate if the remaining 4 eigenvalues (relative to \( \Lambda_3 \)) are different from 0; we say that it is spectrally stable if these eigenvalues are purely imaginary and linearly stable if, in addition to this condition of spectral stability, \( \Lambda_3 \) is diagonalisable.

\[\text{Technically speaking we ruled out the possibility that the parameter } \beta \text{ could be equal to 0 for two reasons. The first is that at some point of the derivation of the matrix of the linearised system we divided by } \beta \text{ (cf. Section 1); the second is due to the fact that if } \beta = 0 \text{ then two masses would vanish and therefore there would be no dynamics at all. However we consider the limit } \beta \to 0 \text{ and the extension by continuity.}\]
The eigenvalues of the Hamiltonian matrix $\Lambda_3$ are
\[
\lambda^\pm_1 := \pm \frac{1}{6} i \sqrt{36 - 18 \alpha + 6 \sqrt{9(\alpha - 2)^2 - \beta(\alpha + 2)^2}}
\]
\[
\lambda^\pm_2 := \pm \frac{1}{6} i \sqrt{36 - 18 \alpha - 6 \sqrt{9(\alpha - 2)^2 - \beta(\alpha + 2)^2}}
\]
and their direct study leads to a picture of the zones of stability and instability in the parameter space (see Figure 1 on page 3).

**Proposition 4.5.** The rectangle $(0, 9] \times [0, 2)$ is divided into three regions, depending on the stability of the relative equilibrium determined by the parameters $\alpha$ and $\beta$:

1. **Region of linear stability**
   \[
   LS := \left\{ (\beta, \alpha) \in (0, 9] \times [0, 2) \mid \beta < 9 \left( \frac{\alpha - 2}{\alpha + 2} \right)^2 \right\};
   \]

2. **Curve of spectral (but not linear) stability**
   \[
   SS := \left\{ (\beta, \alpha) \in (0, 9] \times [0, 2) \mid \beta = 9 \left( \frac{\alpha - 2}{\alpha + 2} \right)^2 \right\};
   \]

3. **Region of spectral instability**
   \[
   SI := \left\{ (\beta, \alpha) \in (0, 9] \times [0, 2) \mid \beta > 9 \left( \frac{\alpha - 2}{\alpha + 2} \right)^2 \right\}.
   \]

**Proof.** A direct computation shows that the eigenvalues of $\Lambda_3$ are purely imaginary in $LS \cup SS$; however on the stability curve $SS$ they collide and form two pairs of purely imaginary eigenvalues which give rise to two Jordan blocks, so that diagonalisability is lost. In the region $SI$ their real part is different from 0.

**Remark 4.6.** Let us observe that as $\beta$ is arbitrarily small (which corresponds to the presence of a dominant mass) and $\alpha$ is bounded away from 2 we lie in the region of linear stability. Such a result agrees with Moeckel’s conjecture on the dominant mass, according to which relative equilibria with a dominant mass are linearly stable.

## 5 Maslov index of the generalised Kepler problem

The aim of this section is to compute the $\omega$-index of the restriction of the Hamiltonian system (1.11) to the invariant subspace $E_2$ of the phase space. As already observed, the Hamiltonian function on this subspace coincides with the Hamiltonian of the generalised (i.e. $\alpha$-homogeneous and logarithmic) Kepler problem.

### 5.1 Computation of the Maslov index

Consider the linear autonomous Hamiltonian initial value problem
\[
\begin{cases}
\dot{\phi}_2(\tau) = \Lambda_2 \phi_2(\tau) \\
\phi_2(0) = I_4.
\end{cases}
\]

(5.1)

Here $\phi_2$ is the restriction to $E_2$ of the fundamental solution $\Phi$ of the Lagrangian circular orbit.
Proposition 5.1. The Maslov index of the fundamental solution $\phi_2$ of System (5.1) is

$$i_1(\phi_2) = \begin{cases} 0 & \text{if } \alpha \in [1, 2) \\ 2 & \text{if } \alpha \in [0, 1). \end{cases}$$

Proof. Here is the fundamental solution $\phi_2(\tau) := \exp(\tau A_2)$, with $\tau \in [0, 2\pi]$:

$$
\phi_2(\tau) = \begin{pmatrix} 2 - \alpha \cos(\sqrt{2 - \alpha} \, \tau) & 2 \alpha \sin(\sqrt{2 - \alpha} \, \tau) & 4 \alpha \sin(\sqrt{2 - \alpha} \, \tau) & 0 \\
2 \alpha \sin(\sqrt{2 - \alpha} \, \tau) & 2 - \cos(\sqrt{2 - \alpha} \, \tau) & 2 \alpha \sin(\sqrt{2 - \alpha} \, \tau) & 0 \\
2 \alpha \sin(\sqrt{2 - \alpha} \, \tau) & 2 \alpha \cos(\sqrt{2 - \alpha} \, \tau) & 2 - \alpha \cos(\sqrt{2 - \alpha} \, \tau) & 0 \\
2 \alpha \sin(\sqrt{2 - \alpha} \, \tau) & 2 \alpha \cos(\sqrt{2 - \alpha} \, \tau) & 2 \alpha \sin(\sqrt{2 - \alpha} \, \tau) & 0 \end{pmatrix} = P^{-1} \phi_2(\tau) P,
$$

where $P := \begin{pmatrix} 1 & 0 & 0 & 6\pi \\
0 & -1 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -6\pi \end{pmatrix}$

we see that $\phi_2(\tau)$ is symplectically equivalent to $\tilde{\phi}_2(\tau) := P^{-1} \phi_2(\tau) P$, which is given by

$$
\tilde{\phi}_2(\tau) := \begin{pmatrix} \cos(\sqrt{2 - \alpha} \, \tau) & -\frac{2 \sin(\sqrt{2 - \alpha} \, \tau)}{6\pi \sqrt{2 - \alpha}} & -\sqrt{2 - \alpha} \sin(\sqrt{2 - \alpha} \, \tau) & 0 \\
0 & 1 & 0 & 0 \\
\frac{\sin(\sqrt{2 - \alpha} \, \tau)}{\sqrt{2 - \alpha}} & \frac{2 \cos(\sqrt{2 - \alpha} \, \tau) - 2}{6\pi (2 - \alpha)} & \cos(\sqrt{2 - \alpha} \, \tau) & 0 \\
\frac{2 - 2 \cos(\sqrt{2 - \alpha} \, \tau)}{6\pi (2 - \alpha)} & \frac{2 \sin(\sqrt{2 - \alpha} \, \tau) - 2 \sqrt{2 - \alpha}}{6\pi \sqrt{2 - \alpha}} & \frac{2 \sin(\sqrt{2 - \alpha} \, \tau)}{6\pi \sqrt{2 - \alpha}} & 1 \end{pmatrix}.
$$

It follows, by the naturality property, that $i_1(\phi_2) = i_1(\tilde{\phi}_2)$. Take now the homotopy $F : [0, 1] \times [0, 2\pi] \to \text{Sp}(4)$ defined by

$$
F(s, \tau) := \begin{pmatrix} \cos(\sqrt{2 - \alpha} \, \tau) & -s \frac{2 \sin(\sqrt{2 - \alpha} \, \tau)}{6\pi \sqrt{2 - \alpha}} & -\sqrt{2 - \alpha} \sin(\sqrt{2 - \alpha} \, \tau) & 0 \\
0 & 1 & 0 & 0 \\
\frac{\sin(\sqrt{2 - \alpha} \, \tau)}{\sqrt{2 - \alpha}} & s \frac{2 \cos(\sqrt{2 - \alpha} \, \tau) - 2}{6\pi (2 - \alpha)} & \cos(\sqrt{2 - \alpha} \, \tau) & 0 \\
s \frac{2 - 2 \cos(\sqrt{2 - \alpha} \, \tau)}{6\pi (2 - \alpha)} & \frac{2 \sin(\sqrt{2 - \alpha} \, \tau) - 2 \sqrt{2 - \alpha}}{6\pi \sqrt{2 - \alpha}} & s \frac{2 \sin(\sqrt{2 - \alpha} \, \tau)}{6\pi \sqrt{2 - \alpha}} & 1 \end{pmatrix}.
$$

It is admissible because we have that $F(1, \tau) = \tilde{\phi}_2(s, \tau) \in \text{Sp}(4)$ and $F(s, 0) = I_4$ for all $s \in [0, 1]$ and all $\tau \in [0, 2\pi]$. Moreover, $F(1, \tau) = \phi_2(\tau)$ and

$$
\tilde{\phi}_2(0, \tau) = \begin{pmatrix} \cos(\sqrt{2 - \alpha} \, \tau) & 0 & -\sqrt{2 - \alpha} \sin(\sqrt{2 - \alpha} \, \tau) & 0 \\
0 & 1 & 0 & 0 \\
\frac{\sin(\sqrt{2 - \alpha} \, \tau)}{\sqrt{2 - \alpha}} & 0 & \cos(\sqrt{2 - \alpha} \, \tau) & 0 \\
0 & \frac{1}{\sqrt{2 - \alpha}} & \frac{1}{36\pi^2} & 1 \end{pmatrix} \circ \begin{pmatrix} \cos(\sqrt{2 - \alpha} \, \tau) & 0 & -\sqrt{2 - \alpha} \sin(\sqrt{2 - \alpha} \, \tau) & 0 \\
0 & 1 & 0 & 0 \\
\frac{\sin(\sqrt{2 - \alpha} \, \tau)}{\sqrt{2 - \alpha}} & 0 & \cos(\sqrt{2 - \alpha} \, \tau) & 0 \\
0 & \frac{1}{\sqrt{2 - \alpha}} & \frac{1}{36\pi^2} & 1 \end{pmatrix} = R_\alpha(\tau) \circ N_\alpha(\tau).
$$
Therefore, being the Maslov index a homotopic invariant, we have

\[ i_1(\phi_2) = i_1(R_\alpha) + i_1(N_\alpha). \]  

From Example 2.10, Example 2.11 and Lemma 2.8 we find

\[ i_1(R_\alpha) = \begin{cases} 1 & \text{if } \alpha \in (1, 2) \\ 3 & \text{if } \alpha \in [0, 1), \end{cases} \quad i_1(N_\alpha) = -1 \quad \forall \alpha \in [0, 2), \]  

and the thesis follows.

### 5.2 Computation of the \( \omega \)-index on \( E_2 \)

Next we compute the \( \omega \)-index \( i_\omega(\phi_2) \) for all \( \omega \in U \setminus \{1\} \). To this end we have to compute first the splitting numbers of the monodromy matrix

\[ M_2 := \phi_2(2\pi) \sim R_\alpha(2\pi) \circ N_1(1, 1) \quad \text{for every } \alpha \in [0, 2). \]

We note that \( R_\alpha(\tau) \) is not a normal form for every \( \tau \in [0, 2\pi] \); however, it is homotopic to the rotation \( R(\sqrt{2} - \alpha \tau) \) via the map \( G : [0, 1] \times [0, 2\pi] \to \Omega^0(R_\alpha) \) defined by

\[ G(s, \tau) := \begin{pmatrix} \cos(\sqrt{2} - \alpha \tau) & -\sqrt{2} - \alpha \sin(\sqrt{2} - \alpha \tau) \\ (1 - s + s\sqrt{2 - \alpha}) \sin(\sqrt{2 - \alpha} \tau) & \cos(\sqrt{2 - \alpha} \tau) \end{pmatrix}. \]

Accordingly, for all \( \alpha \in [0, 2) \)

\[ M_2 \sim R(\theta_\alpha) \circ N_1(1, 1), \]  

where, modulo \( 2\pi \),

\[ \theta_\alpha := 2\pi \sqrt{2 - \alpha} \in \begin{cases} \{0\} & \text{if } \alpha = 1 \\ (0, \pi) & \text{if } \alpha \in [0, 1) \cup (\frac{7}{4}, 2) \\ \{\pi\} & \text{if } \alpha = \frac{7}{4} \\ (\pi, 2\pi) & \text{if } \alpha \in (1, \frac{7}{4}) \end{cases} \]  

**Proposition 5.2.** The \( \omega \)-index \( i_\omega(\phi_2) \) of the fundamental solution \( \phi_2 \) is given by:

- (i) \( \alpha \in [\frac{7}{4}, 2) \):
  \[ i_\omega(\phi_2) = \begin{cases} 1 & \text{if } 0 < \theta < \theta_\alpha \\ 0 & \text{if } \theta_\alpha \leq \theta \leq \pi \end{cases} \]

- (ii) \( \alpha \in (1, \frac{7}{4}) \):
  \[ i_\omega(\phi_2) = \begin{cases} 1 & \text{if } 0 < \theta \leq -\theta_\alpha \\ 2 & \text{if } -\theta_\alpha < \theta \leq \pi \end{cases} \]

- (iii) \( \alpha = 1 \):
  \[ i_\omega(\phi_2) = 2 \quad \text{for all } \theta \in (0, \pi] \]

- (iv) \( \alpha \in [0, 1) \):
  \[ i_\omega(\phi_2) = \begin{cases} 3 & \text{if } 0 < \theta < \theta_\alpha \\ 2 & \text{if } -\theta_\alpha \leq \theta \leq \pi \end{cases} \]

where \( \omega = e^{i\theta} \neq 1 \).
Proof. Item 8 of Proposition 2.15 gives

$$i_\omega(\phi_2) = i_1(\phi_2) + S_{M_2}^+(1) + \sum_{\omega_0} \left(S_{M_2}^+(\omega_0) - S_{M_2}^-\left(\omega_0\right)\right) - S_{M_2}^-\left(\omega\right),$$

(5.5)

where \(\omega \in U \setminus \{1\}\) is such that \(\Im(\omega) > 0\) and \(\omega_0 \in \sigma(M_2)\) lies in the interior of the arc of the upper unit semicircle connecting 1 and \(\omega\) (see Figure 7). Note that the assumption \(\Im(\omega) > 0\) does not imply any loss of generality: by virtue of Item 4 of Proposition 2.15 we have indeed that

$$i_\sigma(\phi_2) = i_\omega(\phi_2).$$

From (5.3) we find that for every \(\omega \in U\) with \(\Im(\omega) \geq 0\)

$$S_{M_2}^+(\omega) = \begin{cases} S_{R(\theta_0)}^+(\omega) + S_{N_1(1,1)}^+(\omega) & \text{if } \alpha \in [0,2) \setminus \{1,\frac{7}{4}\}, \\ S_{f_2}^+(\omega) + S_{N_1(1,1)}^+(\omega) & \text{if } \alpha = \frac{7}{4}, \\ S_{f_2}^+(\omega) + S_{N_1(1,1)}^+(\omega) & \text{if } \alpha = 1. \end{cases}$$

Thanks to the results collected in Proposition 2.15 we know that if \(\omega \notin \sigma(M_2) = \{1, e^{i\theta_0}, e^{-i\theta_0}\}\) then \(S_{M_2}^+(\omega) = 0\); moreover the splitting numbers involved are the following:

$$\begin{align*}
(S_{N_1(1,1)}^+(1), S_{N_1(1,1)}^-(1)) &= (1,1), \\
(S_{R(\theta_0)}^+(e^{i\theta_0}), S_{R(\theta_0)}^-(e^{i\theta_0})) &= (0,1), \quad \forall \alpha \in [0,2) \setminus \{1,\frac{7}{4}\} \\
(S_{f_2}^+(1), S_{f_2}^-(1)) &= (1,1), \\
(S_{f_2}^+(1), S_{f_2}^-(1)) &= (1,1). \quad (5.6a), (5.6b), (5.6c), (5.6d)
\end{align*}$$

Writing \(\omega := e^{i\theta}\), we are now able to compute the \(\omega\)-index depending on \(\alpha\) and on the position of \(\omega\) with respect to the eigenvalues \(e^{\pm i\theta_0}\) (modulo \(2\pi\)). Using Formula (5.5), we distinguish the following cases:

(i) \(\alpha \in \left(\frac{\pi}{4}, \pi\right)\):

$$i_\omega(\phi_2) = \begin{cases} i_1(\phi_2) + S_{M_2}^+(1) & \text{if } \theta \in (0, \theta_\alpha), \\ i_1(\phi_2) + S_{M_2}^+(1) - S_{M_2}^-\left(e^{i\theta_\alpha}\right) & \text{if } \theta = \theta_\alpha, \\ i_1(\phi_2) + S_{M_2}^+(1) + S_{M_2}^+(e^{i\theta_\alpha}) - S_{M_2}^-\left(e^{i\theta_\alpha}\right) & \text{if } \theta \in (\theta_\alpha, \pi], \end{cases}$$

leading to

$$i_\omega(\phi_2) = \begin{cases} 1 & \text{if } \theta \in (0, \theta_\alpha), \\ 0 & \text{if } \theta \in [\theta_\alpha, \pi]. \end{cases}$$
(ii) $\alpha = \frac{7}{4}$:

$$i_\omega(\phi_2) = \begin{cases} i_1(\phi_2) + S_{M_2}^+(1) & \text{if } \theta \in (0, \pi) \\
i_1(\phi_2) + S_{M_2}^+(1) - S_{M_2}^-(1) & \text{if } \theta = \pi, \end{cases}$$

giving

$$i_\omega(\phi_2) = \begin{cases} 1 & \text{if } \theta \in (0, \pi) \\
0 & \text{if } \theta = \pi. \end{cases}$$

(iii) $\alpha \in (1, \frac{7}{4})$:

$$i_\omega(\phi_2) = \begin{cases} i_1(\phi_2) + S_{M_2}^+(1) & \text{if } \theta \in (0, -\theta_\alpha) \\
i_1(\phi_2) + S_{M_2}^+(1) - S_{M_2}^-(e^{-i\theta_\alpha}) & \text{if } \theta = -\theta_\alpha \\
i_1(\phi_2) + S_{M_2}^+(1) + S_{M_2}^-(e^{i\theta_\alpha}) - S_{M_2}^-(e^{-i\theta_\alpha}) & \text{if } \theta \in (-\theta_\alpha, \pi], \end{cases}$$

yielding

$$i_\omega(\phi_2) = \begin{cases} 1 & \text{if } \theta \in (0, -\theta_\alpha] \\
2 & \text{if } \theta \in (-\theta_\alpha, \pi]. \end{cases}$$

(iv) $\alpha = 1$:

$$i_\omega(\phi_2) = i_1(\phi_2) + S_{M_2}^+(1) = 2 \quad \text{for all } \theta \in (0, \pi].$$

(v) $\alpha \in [0, 1)$:

$$i_\omega(\phi_2) = \begin{cases} i_1(\phi_2) + S_{M_2}^+(1) & \text{if } \theta \in (0, \theta_\alpha) \\
i_1(\phi_2) + S_{M_2}^+(1) - S_{M_2}^-(e^{i\theta_\alpha}) & \text{if } \theta = \theta_\alpha \\
i_1(\phi_2) + S_{M_2}^+(1) + S_{M_2}^-(e^{i\theta_\alpha}) - S_{M_2}^-(e^{i\theta_\alpha}) & \text{if } \theta \in (\theta_\alpha, \pi], \end{cases}$$

obtaining

$$i_\omega(\phi_2) = \begin{cases} 3 & \text{if } \theta \in (0, \theta_\alpha) \\
2 & \text{if } \theta \in [\theta_\alpha, \pi]. \end{cases} \quad \square$$

The following result is a direct consequence of Lemma 2.4 and generalises [HS10, Proposition 3.6] to the $\alpha$-homogeneous case.

**Proposition 5.3.** Let $\phi_2$ be the fundamental solution of System (5.1) and $k \in \mathbb{N} \setminus \{0\}$. Then the Maslov index of the $k$-th iteration $\phi_k^\omega$ of $\phi_2$ is given by $i_1(\phi_k^\omega) = \sum_{\omega=1}^k i_\omega(\phi_2)$ and is equal to:

(i) $\alpha \in (\frac{7}{4}, 2)$:

$$i_1(\phi_k^\omega) = 2(n_{k,\alpha}^- - 1),$$

where $n_{k,\alpha}^-$ is the number of $k$-th roots of unity in the arc $[1, e^{i\theta_\alpha})$;

(ii) $\alpha \in (1, \frac{7}{4})$:

$$i_1(\phi_k^\omega) = \begin{cases} 2(n_{k,\alpha}^- - 1) + 4(n_{k,\alpha}^+ - 1) + 2 & \text{if } k \text{ is even} \\
2(n_{k,\alpha}^- - 1) + 4n_{k,\alpha}^+ & \text{if } k \text{ is odd} \end{cases}$$

where $n_{k,\alpha}^-$ is the number of $k$-th roots of unity in the arc $[1, e^{-i\theta_\alpha}]$ and $n_{k,\alpha}^+$ is the number of $k$-th roots of unity in the arc $(e^{-i\theta_\alpha}, -1]$;

(iii) $\alpha = 1$:

$$i_1(\phi_k^\omega) = 2(k - 1)$$

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(iv) $\alpha \in [0, 1)$:

$$i_1(\phi_2^k) = \begin{cases} 6(n_{k,\alpha}^- - 1) + 4(n_{k,\alpha}^+ - 1) + 4 & \text{if } k \text{ is even} \\ 6(n_{k,\alpha}^- - 1) + 4n_{k,\alpha}^+ + 2 & \text{if } k \text{ is odd}, \end{cases}$$

where $n_{k,\alpha}^-$ is the number of $k$-th roots of unity in the arc $[1, e^{\pm i \theta_\alpha})$ and $n_{k,\alpha}^+$ is the number of $k$-th roots of unity in the arc $[e^{\pm i \theta_\alpha}, -1]$.

We observe that, for fixed $k$, the index $i_1(\phi_2^k)$ is constant on horizontal bands of the rectangle $(0, 9) \times [0, 2)$, since it is independent of $\beta$ (see Figure 3 on page 5). From the previous proposition it is evident that the index is monotonically non-increasing as $\alpha$ increases for every $k \in \mathbb{N} \setminus \{0\}$.

Since the computation of the Maslov index of the iterate is based on the Bott-Long formula, it is clear that the only contributions to this value are given by those $\omega$-indices for which $\omega$ is a root of unity. This means that one has a jump in the index of the $k$-th iterate only when the angle $\theta_\alpha$ (defined in (5.4)) is a rational multiple of $2\pi$, i.e. $\theta_\alpha = \frac{2l\pi}{k}$ for some $l \in \mathbb{N} \setminus \{0\}$. Now, since $\theta_\alpha \in [0, 2\sqrt{2}\pi]$ it follows that $l$ actually ranges in the set $\{1, \ldots, [\sqrt{2}k]\}$.

In particular the Maslov index vanishes when $0 < \theta_\alpha < \frac{2\pi}{k}$, that is when $\alpha > 2 - \frac{1}{k}$. As $k$ increases, the horizontal lines corresponding to the jumps of $i_1(\phi_2^k)$, which are characterised by the double sequence $(\alpha_{k,l})$ with $\alpha_{k,l} : = 2 - \frac{l^2}{k^2}$, accumulate at the stability threshold $\alpha = 2$ as $k \to +\infty$ (see Remark 4.3).

Let us now fix $\alpha \in [0, 2)$. The number of $k$-th roots of unity in the arc $[1, e^{\pm i \theta_\alpha})$ increases with $k$ and diverges to $+\infty$ as $k \to +\infty$, hence $i_1(\phi_2^k) \to +\infty$ as $k \to +\infty$.

## 6 $\omega$-index associated with the restriction to $E_3$

In this section we perform the computation of the $\omega$-index of the restriction $\phi_3$ to $E_3$ of the fundamental solution $\Phi$ of the Lagrangian circular orbit. This will be achieved, as before, by means of the splitting numbers.

### 6.1 Computation of the Maslov index

The restriction $\phi_3$ to $E_3$ of the fundamental solution $\Phi$ of the Lagrangian circular orbit satisfies the linear autonomous Hamiltonian initial value problem

$$\begin{cases} \dot{\phi}_3(\tau) = \Lambda_3 \phi_3(\tau) \\ \phi_3(0) = I_4. \end{cases}$$

(6.1)

By taking into account Proposition 4.5, we immediately get the following result.

**Proposition 6.1.** The Maslov index $i_1(\phi_3)$ is zero for all $(\beta, \alpha) \in SI$.

**Proof.** The eigenvalues that contribute to the Maslov index are only the ones contained in $U$. If $9(\alpha - 2)^2 - \beta(\alpha + 2)^2 < 0$ (i.e. in the region $SI$) the spectrum is contained in $\mathbb{C} \setminus (U \cup R)$ and the result follows. \(\square\)

The monodromy matrix $M_3 := \phi_3(2\pi) := \exp(2\pi \Lambda_3)$ is non-degenerate in the whole region $LS$ of linear stability, except on the curve of equation

$$\beta = \frac{36(1-\alpha)}{(\alpha + 2)^2},$$

(6.2)

where two of the four eigenvalues are equal to 1. On the stability curve $SS$ of equation

$$\beta = 9 \left(\frac{\alpha - 2}{\alpha + 2}\right)^2,$$
in instead, $M_3$ is non-degenerate but not diagonalisable. We can compute its Maslov index in the non-degenerate subzone of $LS$ by using again the formula of Proposition 2.13: the Krein-positive eigenvalues of $\Lambda_3$ are

$$\lambda_1^- = -\frac{1}{6}i\sqrt{36 - 18\alpha + 6\sqrt{9(\alpha - 2)^2 - \beta(\alpha + 2)^2}}$$

and

$$\lambda_2^+ = \frac{1}{6}i\sqrt{36 - 18\alpha - 6\sqrt{9(\alpha - 2)^2 - \beta(\alpha + 2)^2}}$$

for all $(\beta, \alpha) \in LS$, so that

$$i_1(\phi_3) = \begin{cases} 0 & \text{if } \frac{36(1 - \alpha)}{(\alpha + 2)^2} < \beta < \frac{9(\alpha - 2)^2}{(\alpha + 2)^2} \\ 2 & \text{if } 0 < \beta < \frac{36(1 - \alpha)}{(\alpha + 2)^2}. \end{cases}$$

However, since the Maslov index is a lower semicontinuous function, we conclude that $i_1(\phi_3) = 0$ also on the curve (6.2) and on the stability curve:

$$i_1(\phi_3) = \begin{cases} 0 & \text{if } \beta \geq \frac{36(1 - \alpha)}{(\alpha + 2)^2} \\ 2 & \text{if } 0 < \beta < \frac{36(1 - \alpha)}{(\alpha + 2)^2}. \end{cases}$$

The result is depicted in Figure 8.

### 6.2 Computation of the $\omega$-index on $E_3$

The monodromy matrix $M_3 := \exp(2\pi \Lambda_3)$ is similar to the diagonal matrix

$$\text{diag}(e^{2\pi \lambda_1^-}, e^{2\pi \lambda_2^+}, e^{2\pi \lambda_1^+}, e^{2\pi \lambda_2^-})$$

and can consequently be expressed as

$$M_3 = R(\theta^{(1)}_{\alpha, \beta}) \circ R(\theta^{(2)}_{\alpha, \beta}),$$

with $\theta^{(1)}_{\alpha, \beta} := \Im(2\pi \lambda_1^+)$ and $\theta^{(2)}_{\alpha, \beta} := \Im(2\pi \lambda_2^-)$. 

Figure 8: Values of $i_1(\phi_3)$. The dotted curve is the stability curve.
Remark 6.2. Note that these two angles correspond to the Krein-negative eigenvalues; the reason is the following. When $\beta \to 0$ the dynamics of the problem reduces to that of a generalised Kepler problem, i.e. to the restriction to $E_2$ previously analysed. The values of the $\omega$-index must then agree with the ones found in the previous study when approaching the segment $\{0\} \times [0,2)$ as $\beta$ tends to 0, and this forces the choice of the two eigenvalues.

Observe that in the region $LS$ these angles take the following values (modulo $2\pi$):

$$
\theta^{(1)}_{\alpha,\beta} \in \begin{cases} 
\{0\} & \text{if } \beta = \frac{36(1-\alpha)}{(\alpha+2)^2} \\
(0, \pi) & \text{if } \beta < \frac{36(1-\alpha)}{(\alpha+2)^2} \text{ or } \left( \beta > \frac{9(7-4\alpha)}{4(\alpha+2)^2} \text{ and } \alpha > \frac{3}{2} \right) \\
\{\pi\} & \text{if } \beta = \frac{9(7-4\alpha)}{4(\alpha+2)^2} \text{ and } \alpha > \frac{3}{2} \\
(\pi, 2\pi) & \text{if } \beta > \frac{36(1-\alpha)}{(\alpha+2)^2} \text{ and } \left( \beta < \frac{9(7-4\alpha)}{4(\alpha+2)^2} \text{ or } \alpha < \frac{3}{2} \right)
\end{cases}
\quad (6.3)
$$

$$
\theta^{(2)}_{\alpha,\beta} \in \begin{cases} 
\{0, \pi\} & \text{if } \beta > \frac{9(7-4\alpha)}{4(\alpha+2)^2} \text{ and } \alpha < \frac{3}{2} \\
\{\pi\} & \text{if } \beta = \frac{9(7-4\alpha)}{4(\alpha+2)^2} \text{ and } \alpha < \frac{3}{2} \\
(\pi, 2\pi) & \text{if } \beta < \frac{9(7-4\alpha)}{4(\alpha+2)^2} \text{ or } \alpha > \frac{3}{2}
\end{cases}
\quad (6.4)
$$

Figure 9a and Figure 9b show the involved regions, and they are superposed in Figure 10a. In order to compute the splitting numbers and eventually find the $\omega$-index we have to determine not only the absolute position of $\theta^{(1)}_{\alpha,\beta}$ and $\theta^{(2)}_{\alpha,\beta}$ on $U$ (which is the one given above), but also how their relative position changes as the parameters $\alpha$ and $\beta$ vary. This is represented in Figure 10b.

Now, for every $\omega \in U$ we have that

$$
S^\pm_{M_5}(\omega) = S^\pm_{R(\theta^{(1)}_{\alpha,\beta})}(\omega) + S^\pm_{R(\theta^{(2)}_{\alpha,\beta})}(\omega)
$$

and $S^\pm_{M_5}(\omega) = 0$ if $\omega \notin \sigma(M_3) \equiv \{e^{\pm i\theta^{(1)}_{\alpha,\beta}}, e^{\pm i\theta^{(2)}_{\alpha,\beta}}\}$. In order to compute the $\omega$-index we use the formula

$$
i_\omega(\phi_3) - i_1(\phi_3) = S^+_M(1) + \sum_{\omega_0} (S^+_M(\omega_0) - S^-_M(\omega_0)) - S^-_M(\omega),
$$
(a) Light shade: $\theta^{(1)}_{\alpha,\beta} \in (0, \pi)$ and $\theta^{(2)}_{\alpha,\beta} \in (\pi, 2\pi)$;
Medium shade: $\theta^{(1)}_{\alpha,\beta}, \theta^{(2)}_{\alpha,\beta} \in (\pi, 2\pi)$; Heavy shade: $\theta^{(1)}_{\alpha,\beta} \in (\pi, 2\pi)$ and $\theta^{(2)}_{\alpha,\beta} \in (0, \pi)$; Dark shade: $\theta^{(1)}_{\alpha,\beta}, \theta^{(2)}_{\alpha,\beta} \in (0, \pi)$.

(b) If $\tilde{\theta}^{(1)}_{\alpha,\beta}$ and $\tilde{\theta}^{(2)}_{\alpha,\beta}$ are the representatives of $\pm \theta^{(1)}_{\alpha,\beta}$ and $\pm \theta^{(2)}_{\alpha,\beta}$ in the upper unit semicircle, then the colours have to be interpreted in the following way.

Light shade: $\tilde{\theta}^{(1)}_{\alpha,\beta} < \tilde{\theta}^{(2)}_{\alpha,\beta}$;
Solid line: $\tilde{\theta}^{(1)}_{\alpha,\beta} = \tilde{\theta}^{(2)}_{\alpha,\beta}$;
Dark shade: $\tilde{\theta}^{(1)}_{\alpha,\beta} > \tilde{\theta}^{(2)}_{\alpha,\beta}$.

Figure 10: Values of $\theta^{(1)}_{\alpha,\beta}$ and $\theta^{(2)}_{\alpha,\beta}$ (a) and their relative position (b) modulo $2\pi$.

where $\omega \in \mathbb{U} \setminus \{1\}$ is such that $\Im(\omega) \geq 0$ and $\omega_0 \in \sigma(M_3)$ lies in the interior of the arc of the upper unit semicircle connecting 1 and $\omega$ (see Figure 7). The splitting numbers involved are the following:

$$(S_{M_3}^+(1), S_{M_3}^-(1)) = \begin{cases} (1,1) & \text{if } \beta = \frac{36(1-\alpha)}{(\alpha+2)^2} \\ (0,0) & \text{otherwise} \end{cases}$$

$$(S_{M_3}^+(-1), S_{M_3}^-(-1)) = \begin{cases} (1,1) & \text{if } \beta = \frac{9(7-4\alpha)}{4(\alpha+2)^2} \text{ and } \alpha \neq \frac{3}{2} \\ (0,0) & \text{otherwise} \end{cases}$$

$$(S_{M_3}^+(e^{i\theta^{(1)}_{\alpha,\beta}}), S_{M_3}^-(e^{i\theta^{(1)}_{\alpha,\beta}})) = \begin{cases} (0,1) & \text{if all } \theta^{(1)}_{\alpha,\beta} \notin \{0, \pi, \pm \theta^{(2)}_{\alpha,\beta}\} \\ (0,2) & \text{if } \theta^{(1)}_{\alpha,\beta} = \theta^{(2)}_{\alpha,\beta} \\ (1,1) & \text{if } \theta^{(1)}_{\alpha,\beta} = -\theta^{(2)}_{\alpha,\beta} \end{cases}$$

$$(S_{M_3}^+(e^{i\theta^{(2)}_{\alpha,\beta}}), S_{M_3}^-(e^{i\theta^{(2)}_{\alpha,\beta}})) = \begin{cases} (0,1) & \text{if all } \theta^{(2)}_{\alpha,\beta} \notin \{0, \pi, \pm \theta^{(1)}_{\alpha,\beta}\} \\ (0,2) & \text{if } \theta^{(2)}_{\alpha,\beta} = \theta^{(1)}_{\alpha,\beta} \\ (1,1) & \text{if } \theta^{(2)}_{\alpha,\beta} = -\theta^{(1)}_{\alpha,\beta} \end{cases}$$

The $\omega$-index depends therefore on the values of $\alpha$ and $\beta$. Writing $\omega := e^{i\theta}$, we have

i) $\beta > \frac{9(7-4\alpha)}{4(\alpha+2)^2}$ and $\alpha > \frac{3}{2}$,

$$i_{\omega}(\phi_3) = \begin{cases} 0 & \text{if } 0 < \theta \leq -\theta^{(2)}_{\alpha,\beta} \\ 1 & \text{if } -\theta^{(2)}_{\alpha,\beta} < \theta < \theta^{(1)}_{\alpha,\beta} \\ 0 & \text{if } \theta^{(1)}_{\alpha,\beta} \leq \theta \leq \pi \end{cases}$$
\[
\begin{align*}
\text{ii) } \beta &= \frac{9(7 - 4\alpha)}{4(\alpha + 2)^2} \text{ and } \alpha > \frac{3}{2}; \\
& \quad i_\omega(\phi_3) = \\
& \quad \begin{cases} 
0 & \text{if } 0 < \theta \leq -\theta_{\alpha,\beta}^{(2)} \\
1 & \text{if } -\theta_{\alpha,\beta}^{(2)} < \theta < \pi \\
0 & \text{if } \theta = \pi
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\text{iii) } \beta &< \frac{9(7 - 4\alpha)}{4(\alpha + 2)^2} \text{ and } \beta < \frac{9(\alpha - 1)^2}{(\alpha + 2)^2} \text{ and } \alpha > 1; \\
& \quad i_\omega(\phi_3) = \\
& \quad \begin{cases} 
0 & \text{if } 0 < \theta \leq -\theta_{\alpha,\beta}^{(2)} \\
1 & \text{if } -\theta_{\alpha,\beta}^{(2)} < \theta < -\theta_{\alpha,\beta}^{(1)} \\
2 & \text{if } -\theta_{\alpha,\beta}^{(1)} < \theta \leq \pi
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\text{iv) } \beta &= \frac{(\alpha - 1)^2}{(\alpha + 2)^2} \text{ and } \alpha > 1; \\
& \quad i_\omega(\phi_3) = \\
& \quad \begin{cases} 
0 & \text{if } 0 < \theta \leq -\theta_{\alpha,\beta}^{(2)} = \theta_{\alpha,\beta}^{(1)} \\
2 & \text{if } \theta_{\alpha,\beta}^{(2)} = \theta_{\alpha,\beta}^{(1)} < \theta \leq \pi
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\text{v) } \beta &< \frac{9(7 - 4\alpha)}{4(\alpha + 2)^2} \text{ and } \beta > \frac{9(\alpha - 1)^2}{(\alpha + 2)^2} \text{ and } \beta > \frac{36(1 - \alpha)}{4(\alpha + 2)^2}; \\
& \quad i_\omega(\phi_3) = \\
& \quad \begin{cases} 
0 & \text{if } 0 < \theta < -\theta_{\alpha,\beta}^{(1)} \\
1 & \text{if } -\theta_{\alpha,\beta}^{(1)} < \theta < -\theta_{\alpha,\beta}^{(2)} \\
2 & \text{if } -\theta_{\alpha,\beta}^{(2)} < \theta \leq \pi
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\text{vi) } \beta &= \frac{9(7 - 4\alpha)}{4(\alpha + 2)^2} \text{ and } \beta > \frac{36(1 - \alpha)}{(\alpha + 2)^2} \text{ and } \alpha < \frac{3}{2}; \\
& \quad i_\omega(\phi_3) = \\
& \quad \begin{cases} 
0 & \text{if } 0 < \theta \leq -\theta_{\alpha,\beta}^{(1)} \\
1 & \text{if } -\theta_{\alpha,\beta}^{(1)} < \theta < \pi \\
0 & \text{if } \theta = \pi
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\text{vii) } \beta &> \frac{9(7 - 4\alpha)}{4(\alpha + 2)^2} \text{ and } \beta > \frac{36(1 - \alpha)}{(\alpha + 2)^2}; \\
& \quad i_\omega(\phi_3) = \\
& \quad \begin{cases} 
0 & \text{if } 0 < \theta \leq -\theta_{\alpha,\beta}^{(1)} \\
1 & \text{if } \theta_{\alpha,\beta}^{(1)} < \theta < \theta_{\alpha,\beta}^{(2)} \\
0 & \text{if } \theta_{\alpha,\beta}^{(2)} < \theta \leq \pi
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\text{viii) } \beta &= \frac{36(1 - \alpha)}{(\alpha + 2)^2} \text{ and } \beta < \frac{9(7 - 4\alpha)}{4(\alpha + 2)^2}; \\
& \quad i_\omega(\phi_3) = \\
& \quad \begin{cases} 
1 & \text{if } 0 < \theta \leq -\theta_{\alpha,\beta}^{(2)} \\
2 & \text{if } -\theta_{\alpha,\beta}^{(2)} < \theta \leq \pi
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\text{ix) } \beta &= \frac{36(1 - \alpha)}{(\alpha + 2)^2} \text{ and } \beta = \frac{9(7 - 4\alpha)}{4(\alpha + 2)^2}; \\
& \quad i_\omega(\phi_3) = \begin{cases} 1 & \text{if } 0 < \theta < \pi \\
0 & \text{if } \theta = \pi
\end{cases}
\end{align*}
\]
\[ \beta = \frac{36(1-\alpha)}{(\alpha+2)^2} \text{ and } \beta > \frac{9(7-4\alpha)}{4(\alpha+2)^2}. \]

\[ i_\omega(\phi_3) = \begin{cases} 1 & \text{if } 0 < \theta < \theta_{(1)}^{(1)}_{\alpha,\beta} \\ 0 & \text{if } \theta_{(1)}^{(1)}_{\alpha,\beta} \leq \theta \leq \pi \end{cases} \]

x) \ \beta < \frac{(\alpha - 1)^2}{(\alpha + 2)^2} \text{ and } \alpha < 1:

\[ i_\omega(\phi_3) = \begin{cases} 2 & \text{if } 0 < \theta \leq -\theta_{(2)}^{(2)}_{\alpha,\beta} \\ 1 & \text{if } -\theta_{(2)}^{(2)}_{\alpha,\beta} < \theta < \theta_{(1)}^{(1)}_{\alpha,\beta} \\ 2 & \text{if } -\theta_{(1)}^{(1)}_{\alpha,\beta} \leq \theta \leq \pi \end{cases} \]

xi) \ \beta < \frac{9(\alpha - 1)^2}{(\alpha + 2)^2} \text{ and } \alpha < 1:

\[ i_\omega(\phi_3) = \begin{cases} 2 & \text{if } 0 < \theta \leq -\theta_{(2)}^{(2)}_{\alpha,\beta} \\ 1 & \text{if } -\theta_{(2)}^{(2)}_{\alpha,\beta} < \theta < \theta_{(1)}^{(1)}_{\alpha,\beta} \\ 2 & \text{if } -\theta_{(1)}^{(1)}_{\alpha,\beta} \leq \theta \leq \pi \end{cases} \]

xii) \ \beta = 9\frac{(\alpha - 1)^2}{(\alpha + 2)^2} \text{ and } \alpha < 1:

\[ i_\omega(\phi_3) = \begin{cases} 2 & \text{if } 0 < \theta \leq -\theta_{(2)}^{(2)}_{\alpha,\beta} \\ 1 & \text{if } -\theta_{(2)}^{(2)}_{\alpha,\beta} < \theta < \theta_{(1)}^{(1)}_{\alpha,\beta} \\ 2 & \text{if } -\theta_{(1)}^{(1)}_{\alpha,\beta} \leq \theta \leq \pi \end{cases} \]

xiii) \ \beta > \frac{9(\alpha - 1)^2}{(\alpha + 2)^2} \text{ and } \beta < \frac{36(1-\alpha)}{(\alpha+2)^2} \text{ and } \beta < \frac{9(7-4\alpha)}{4(\alpha+2)^2}:

\[ i_\omega(\phi_3) = \begin{cases} 2 & \text{if } 0 < \theta < \theta_{(1)}^{(1)}_{\alpha,\beta} \\ 1 & \text{if } \theta_{(1)}^{(1)}_{\alpha,\beta} \leq \theta \leq -\theta_{(2)}^{(2)}_{\alpha,\beta} \\ 2 & \text{if } -\theta_{(2)}^{(2)}_{\alpha,\beta} < \theta \leq \pi \end{cases} \]

xiv) \ \beta < \frac{36(1-\alpha)}{(\alpha+2)^2} \text{ and } \beta = \frac{9(7-4\alpha)}{4(\alpha+2)^2}:

\[ i_\omega(\phi_3) = \begin{cases} 2 & \text{if } 0 < \theta < \theta_{(1)}^{(1)}_{\alpha,\beta} \\ 1 & \text{if } \theta_{(1)}^{(1)}_{\alpha,\beta} \leq \theta \leq \pi \end{cases} \]

xv) \ \beta < \frac{36(1-\alpha)}{(\alpha+2)^2} \text{ and } \beta > \frac{9(7-4\alpha)}{4(\alpha+2)^2}:

\[ i_\omega(\phi_3) = \begin{cases} 2 & \text{if } 0 < \theta < \theta_{(1)}^{(1)}_{\alpha,\beta} \\ 1 & \text{if } \theta_{(1)}^{(1)}_{\alpha,\beta} \leq \theta < \theta_{(2)}^{(2)}_{\alpha,\beta} \\ 0 & \text{if } \theta_{(2)}^{(2)}_{\alpha,\beta} \leq \theta \leq \pi \end{cases} \]

As we did analogously for \( E_2 \), we now turn our attention to the computation of the Maslov index \( i_1(\phi_k^3) \) of the iterates of \( \phi_3 \). Once again we have that the Maslov index jumps in correspondence of those \( \omega \) that are roots of unity, due to the structure of Bott-Long formula. Hence, in the region \( LS \), there are jumps of the index of the \( k \)-th iterate if and only if

\[ \theta_{(i)}^{(i)}_{\alpha,\beta} = \frac{2l\pi}{k}, \quad (6.5) \]

for some \( i = 1, 2 \) and \( l \in \mathbb{N} \setminus \{0\} \) (here \( \theta_{(i)}^{(i)}_{\alpha,\beta} \) are the angles defined in (6.3) and (6.4)). In actual fact \( \theta_{(2)}^{(2)}_{\alpha,\beta} \) ranges in \((0, 2\pi)\), whereas \( \theta_{(1)}^{(1)}_{\alpha,\beta} \) varies in \((0, 2\sqrt{2}\pi)\): this implies that \( l \) takes values in the finite set \( \{1, \ldots, [\sqrt{2}k]\} \).
Condition (6.5) defines a family of curves \( \{f_{k,l}\} \) in the plane \((\beta, \alpha)\), parameterised by \(k\) and \(l\), that are defined by the equations
\[
\beta = -\frac{36}{(\alpha + 2)^2} \frac{l^2}{k^2} \left( \frac{l^2}{k^2} + \alpha - 2 \right).
\]
Each of these curves is convex and for \(l \in \{1, \ldots, k\}\) they are tangent at exactly one point to \(SS\), namely
\[
\left( \frac{9l^4}{(2k^2 - l^2)^2}, 2 \left( 1 - \frac{l^2}{k^2} \right) \right),
\]
and it turns out that the stability curve is actually the envelope of the one-parameter family \(\{f_t\}_{t \in (0, 1]}\) consisting of curves of equations
\[
\beta = -\frac{36}{(\alpha + 2)^2} t^2 (t^2 + \alpha - 2),
\]
into which the collection \(\{f_{k,l}\}\) is contained. We observe that at every point in \(LS\) the Maslov index \(i_1(\phi^2)\) increases with \(k\) and that, for each fixed \(k \in \mathbb{N} \setminus \{0\}\), it decreases along half-lines from the origin. The index is also monotonically increasing when one crosses any of the curves \(f_{k,l}\) (going towards the origin). Note that the intersections of these curves with the line \(\beta = 0\) yield exactly the values of the sequence \(\alpha_{k,l}\) introduced in \(E_2\) that tends to \(\alpha = 2\) as \(k \to +\infty\).

In Figure 11 we present, as an example, a complete computation of \(i_1(\phi^2)\), whereas Figure 4 on page 6 shows some of the curves \(f_{k,l}\) for some values of \(k\).

### 7 The \(\omega\)-Morse index of the Lagrangian circular orbit

Let \(\mathcal{L} \in \mathcal{C}^\infty(T\hat{X}, \mathbb{R})\) and \(\mathbb{A} : W^{1,2}(\mathbb{R}/2\pi\mathbb{Z}, \hat{X}) \to \mathbb{R}\) be the Lagrangian function and the Lagrangian action functional respectively, as given in (0.3) and (0.4). Since the Euler-Lagrange equation for \(\mathbb{A}\), which is smooth on collisionless loops, coincides with the Newton’s equations given in (0.2), for each pair \((\beta, \alpha) \in (0, 9) \times [0, 2]\) the Lagrangian circular solution \(\gamma_{\alpha,\beta}\) of Newton’s equation can be found (up to a standard bootstrap argument) as a critical point of \(\mathbb{A}\).

From Equation (3.2) we see that the second variation at the critical point \(\gamma_{\alpha,\beta}\) is
\[
d^2 \mathbb{A}(\gamma_{\alpha,\beta})[\xi, \eta] = \int_0^{2\pi} \langle M\dot{\xi}', \eta' \rangle + \langle D^2 U(\gamma_{\alpha,\beta}(t)) \xi, \eta \rangle dt.
\]
(7.1)

Using the Sobolev Embedding Theorem it follows that the second variation is a (bounded) essentially positive Fredholm quadratic form, being a weakly compact perturbation of an invertible
quadratic form (cf. for instance [MPP05, Section 2, Proposition 3.1] and references therein). This in particular ensures that the $\omega$-Morse index $i_{\text{Morse}}^{\omega}$ is finite.

By taking into account the Morse index theorem (Lemma 3.2), in order to compute the $i_{\text{Morse}}^{\omega}(\gamma_{\beta,\alpha})$ it is enough to compute the $\omega$-index $i_{\omega}(\psi)$, where $\psi : [0, 2\pi] \rightarrow \text{Sp}(8)$ is the fundamental solution of the first-order Hamiltonian system obtained from the associated Sturm system through the Legendre transformation, i.e. $\psi$ satisfies

$$
\begin{cases}
\psi'(t) = JB_{\alpha,\beta}(t)\psi(t) \\
\psi(0) = I_{2n}
\end{cases}
$$

(7.2)

where

$$
B_{\alpha,\beta}(t) := \begin{pmatrix} M & 0 \\
0 & -D^2U(\gamma_{\alpha,\beta}(t)) \end{pmatrix}.
$$

Taking into account [MS05, Theorem 2.1] there exists a linear symplectomorphism between $T^*\tilde{X}$ and $E_2 \oplus E_3$. By the symplectic invariance of $i_{\text{CLM}}$ (cf. [CLM94, Property V, page 128]) and hence of $i_{\omega}$ (as a direct consequence of Lemma 2.8), it follows that

$$
i_{\omega}(\psi) = i_{\omega}(\Phi),
$$

where $\Phi$ was defined in Section 4. Since $\Phi = \phi_2 \circ \phi_3$, by using the symplectic additivity property of $i_{\omega}$ and considering the previous discussion it follows that

$$
i_{\text{Morse}}^{\omega}(\gamma_{\alpha,\beta}) = i_{\omega}(\phi_2) + i_{\omega}(\phi_3).
$$

**Remark 7.1.** We assume that $H$ is a Hilbert space and there exist $H_1, \ldots, H_n$ such that $H = \bigoplus_{k=1}^n H_k$. Let $A$ be a self-adjoint essentially positive bounded Fredholm operator such that $A(H_k) \subseteq H_k$ for $i = 1, \ldots, n$. Setting $A_k := A|H_k$ we have

$$
i_{\text{Morse}}^{\omega}(A) = \sum_{k=1}^n i_{\text{Morse}}^{\omega}(A_k).
$$

It is worth noting that in correspondence of the 4-dimensional subspaces $E_2$ and $E_3$ there exist two 2-dimensional subspaces $\tilde{X}_2$ and $\tilde{X}_3$ of $\tilde{X}$ such that $E_2 = T^*\tilde{X}_2$ and $E_3 = T^*\tilde{X}_3$. Hence

$$W^{1,2}(\mathbb{R}/2\pi\mathbb{Z}, \tilde{X}) = W^{1,2}(\mathbb{R}/2\pi\mathbb{Z}, \tilde{X}_2) \times W^{1,2}(\mathbb{R}/2\pi\mathbb{Z}, \tilde{X}_3).$$

In the next two subsections we shall compute the Lagrangian functions on the aforementioned subspaces $\tilde{X}_2$ and $\tilde{X}_3$ as well as the differential operators on such subspaces.

### 7.1 $\omega$-Morse index of the generalised Kepler problem

Define the Lagrangian function on $W^{1,2}(\mathbb{R}/2\pi\mathbb{Z}, \tilde{X}_2)$ as

$$L_2(x, \dot{x}) := \frac{1}{2} \|\dot{x}\|^2 + \langle Jx, \dot{x} \rangle + \frac{1}{2} \langle S_2x, x \rangle,$$

(7.3)

where $S_2 := \begin{pmatrix} \alpha & 0 \\
0 & \beta \end{pmatrix}$. By a straightforward calculation it follows that the origin in the configuration space is a solution of the corresponding Euler-Lagrange equation

$$-\ddot{x} - 2J\dot{x} + S_2x = 0;
$$

(7.4)

associated with $L_2$. Let $B_2: W^{1,2}(\mathbb{R}/2\pi\mathbb{Z}, \tilde{X}_2) \times W^{1,2}(\mathbb{R}/2\pi\mathbb{Z}, \tilde{X}_2) \rightarrow \mathbb{R}$ be defined as follows:

$$B_2(x, y) := \int_0^{2\pi} [\langle \dot{x}, \dot{y} \rangle + \langle Jy, x \rangle + \langle J\dot{x}, y \rangle + \langle S_2x, y \rangle] \, dt.$$
Once again it follows from the Sobolev Embedding Theorem that $B_2$ is a (bounded) essentially positive Fredholm quadratic form, being a weakly compact perturbation of an invertible quadratic form. This in particular ensures that the Morse index $i_{\omega \text{Morse}}$ is finite.

By taking into account the Legendre transformation, the corresponding autonomous Hamiltonian function is

$$H_2(v) := \frac{1}{2} \langle B_2 v, v \rangle, \quad \forall v \in \mathbb{R}^4,$$

where

$$B_2 := \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & -(\alpha + 1) & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}. \quad (7.5)$$

Clearly the origin in the phase space is the corresponding solution of the linear autonomous Hamiltonian initial value problem

$$\begin{cases} \phi'_2(\tau) = \Lambda_2 \phi_2(\tau) \\ \phi_2(0) = I_4 \end{cases} \quad (7.6)$$

where $\Lambda_2 = JB_2$ agrees with the one given in formula (4.1).

**Theorem 7.2.** For all $\omega \in U$, the $\omega$-Morse index of the circular solution $\gamma_{0,0}$ of the generalised Kepler problem coincides with $i_{\omega}(\phi_2)$, which has been computed in Propositions 5.1 and 5.2.

**Proof.** First of all we observe that as a direct consequence of the results proved in Section 1 the subspace $E_2$ is invariant under the phase flow of the Hamiltonian (1.3). Moreover on this subspace the aforementioned Hamiltonian reduces to the Hamiltonian of the generalised Kepler problem. Now, by the above construction System (7.6) is the Legendre transformation of the Euler-Lagrange system (7.4). The thesis is then a direct consequence of Lemma 3.2. \hfill \square

**Remark 7.3.** It is worthwhile noting that this result perfectly agrees with [HS10, Proposition 3.6] and [Ven02, Proposition 2.2.3]. Moreover we point out that in the last quoted reference the author only states that for $\alpha \in (0, 1)$ the circular solutions are not local minimisers, without any further information on the Morse index. The logarithmic case has not been treated thus far from this point of view.

### 7.2 $\omega$-Morse index of the Lagrangian circular orbit

We proceed exactly as in the previous subsection, by introducing the Lagrangian

$$\mathcal{L}_3(x, \dot{x}) := \frac{1}{2} \| \dot{x} \|^2 + \langle Jx, \dot{x} \rangle + \frac{1}{2} \langle S_3 x, x \rangle$$

on the Sobolev space $W^{1,2}(\mathbb{R}/2\pi \mathbb{Z}, \hat{X}_3)$, with

$$S_3 := \begin{pmatrix} \frac{1}{\alpha} [6 + 3\alpha + (\alpha + 2)\sqrt{9 - \beta}] & 0 \\ 0 & \frac{1}{\alpha} [6 + 3\alpha - (\alpha + 2)\sqrt{9 - \beta}] \end{pmatrix}.$$  

Defining a symmetric bilinear form $B_3$ in a completely analogous way as above, we obtain the Hamiltonian system

$$\begin{cases} \phi'_3(\tau) = \Lambda_3 \phi_3(\tau) \\ \phi_3(0) = I_4, \end{cases} \quad (7.7)$$

where $\Lambda_3 = JB_3$, being

$$B_3 := \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & -(\alpha + 2)\sqrt{9 - \beta} & 0 \\ 1 & 0 & 0 & -\frac{1}{2} (\alpha - \frac{2 + 2}{3} \sqrt{9 - \beta}) \end{pmatrix}. \quad (7.8)$$
Theorem 7.4. For all \( \omega \in U \) the \( \omega \)-Morse index of the Lagrangian circular solution \( \gamma_{\alpha, \beta} \) is given by \( i(\Phi) = i(\phi_2) + i(\phi_3) \). In particular for \( \omega = 1 \) we have

\[
i_{\text{Morse}}(\gamma_{\alpha, \beta}) = \begin{cases} 
0 & \text{if } \alpha \in [1, 2) \\
2 & \text{if } \beta \geq \frac{36(1 - \alpha)}{(\alpha + 2)^2} \text{ and } \alpha \in [0, 1) \\
4 & \text{if } 0 < \beta < \frac{36(1 - \alpha)}{(\alpha + 2)^2}.
\end{cases}
\]

Proof. Arguing as in the proof of Theorem 7.2, it is enough to apply Theorem 3.2, use the calculations performed in Subsections 6.1 and 5.1 and the additivity of the Maslov index \( i_1 \). \( \square \)

7.3 Relation between linear stability and Morse index

We have shown how both in \( E_2 \) and in \( E_3 \) there is a sequence of curves (possibly straight lines) that “converge”, in a suitable sense, to the boundary of the region of linear stability. By virtue of the Index Theorem also the Morse index of the iterates jumps when crossing each of those curves.

Since the angles \( \theta^{(1)}_{\alpha, \beta} \) and \( \theta^{(2)}_{\alpha, \beta} \) introduced in Subsection 6.2 cover the whole of \( U \) as \( \alpha \) and \( \beta \) vary, it may happen that for some values of these parameters one of them is a rational multiple of \( 2\pi \) (so that its exponential is a root of unity). When this occurs then the corresponding curve in the plane \( (\beta, \alpha) \) is tangent to the stability curve at the point whose coordinates are given by \((6.6)\). Instead, in the case when the aforementioned angles do not give rise to roots of unity, one obtains tangency to the stability curve at some point only after taking the limit as \( k \to +\infty \). The reason of this fact is simply due to the density of roots of unity in \( U \).

A The geometric structure of \( \text{Sp}(2) \)

Every real invertible matrix \( A \) can be decomposed in polar form

\[
A = PO,
\]
where \( P := (AA^T)^{1/2} \) is symmetric and positive definite and \( O := P^{-1}A \) is orthogonal. If \( A \in \text{Sp}(2) \) then \( \det P = 1 \) and therefore \( P \in \text{Sp}(2) \) as well. This entails that \( O \in \text{Sp}(2) \); in fact, being orthogonal, it belongs to \( \text{SO}(2) \cong U \), i.e. it is a proper rotation:

\[
O = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.
\]

Let \( u : \text{Sp}(2) \to U \) be the map which associates every \( 2 \times 2 \) real symplectic matrix with the angle of rotation of its orthogonal part:

\[
u(A) = u(PO) := e^{i\theta}.
\]

Now, the eigenvalues of \( P \) are all real, positive and reciprocal of each other. Therefore we have that \( \text{tr} \, P \geq 2 \) and we may introduce a coordinate \( \xi \) ranging in \( [0, +\infty) \) by setting \( \text{tr} \, P = 2 \cosh \xi \). Hence we can write

\[
P = \begin{pmatrix} \cosh \xi + a & b \\ b & \cosh \xi - a \end{pmatrix}
\]

for some \( a, b \in \mathbb{R} \) such that \( \cosh^2 \xi - a^2 - b^2 = 1 \). Thus \( b^2 = \sinh^2 \xi - a^2 \), which is meaningful if and only if \( |a| \leq |\sinh \xi| \). Hence we are allowed to set \( a := \sinh \xi \cos \eta \) for some \( \eta \in \mathbb{R} \), so that \( b = \sinh \xi \sin \eta \) and \( P \) becomes

\[
P = \begin{pmatrix} \cosh \xi + \sinh \xi \cos \eta & \sinh \xi \sin \eta \\ \sinh \xi \sin \eta & \cosh \xi - \sinh \xi \cos \eta \end{pmatrix}.
\]
Figure 12: The singular surface $\text{Sp}(2)_0^{1,0}$. The representation is in Cartesian coordinates $(x, y, z) = (r \cos \theta, r \sin \theta, z)$.

Figure 13: Intersection of $\text{Sp}(2)_1^{0,0}$ with the plane $z = 0$. The representation is in Cartesian coordinates $(x, y) = (r \cos \theta, r \sin \theta)$.

Setting now $r \equiv \cosh \xi + \sinh \xi \cos \eta$ and $z \equiv \sinh \xi \sin \eta$ yields

$$P = \begin{pmatrix} r & z \\ z & \frac{1+z^2}{r} \end{pmatrix}$$

and then every symplectic matrix $M$ of size 2 can be written as the product

$$M = \begin{pmatrix} r & z \\ z & \frac{1+z^2}{r} \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix},$$

where $(r, \theta, z) \in (0, +\infty) \times [0, 2\pi) \times \mathbb{R}$. Viewing $(r, \theta, z)$ as cylindrical coordinates in $\mathbb{R}^3 \setminus \{z\text{-axis}\}$ we obtain a representation of $\text{Sp}(2)$ in $\mathbb{R}^3$; more precisely, we obtain a smooth global diffeomorphism $\psi : \text{Sp}(2) \to \mathbb{R}^3 \setminus \{z\text{-axis}\}$. We shall henceforth identify elements in $\text{Sp}(2)$ with their image under $\psi$.

The eigenvalues of a symplectic matrix $M$ written as in (A.1) are

$$\lambda_{\pm} := \frac{1}{2r} \left( 1 + r^2 + z^2 \right) \cos \theta \pm \sqrt{(1 + r^2 + z^2)^2 \cos^2 \theta - 4r^2}.$$ 

For $\omega = e^{i\varphi} \in U$ we get

$$D_\omega(M) := (-1)^{n-1} \omega^{-n} \det(M - \omega I)|_{n=1} = e^{-i\varphi} \det(M - e^{i\varphi} I)$$

$$= 2 \cos \varphi - \left( \frac{1 + z^2}{r} \right) \cos \theta.$$
and define
\[ \text{Sp}(2)^{\pm}_{\omega} := \{ (r, \theta, z) \in (0, +\infty) \times [0, 2\pi) \times \mathbb{R} \mid \pm (1 + r^2 + z^2) \cos \theta > 2 r \cos \varphi \} , \]
\[ \text{Sp}(2)_{\omega}^{0} := \{ (r, \theta, z) \in (0, +\infty) \times [0, 2\pi) \times \mathbb{R} \mid \pm (1 + r^2 + z^2) \cos \theta = 2 r \cos \varphi \} . \]

The set \text{Sp}(2)^{\pm}_{\omega} \cup \text{Sp}(2)_{\omega}^{0} is named the \emph{\omega-regular part} of \text{Sp}(2), while \text{Sp}(2)_{\omega}^{0} is its \emph{\omega-singular part}; the former corresponds to the subset of \( 2 \times 2 \) symplectic matrices which do not have \( \omega \) as an eigenvalue, whereas those matrices admitting \( \omega \) in their spectrum belong to the latter.

We are particularly interested in \( \text{Sp}(2)_{\omega}^{0} \), the singular part of \( \text{Sp}(2) \) associated with the eigenvalue \( 1 \), a representation of which is depicted in Figure 12. The “pinched” point is the identity matrix, and it is the only element satisfying \( \dim \ker(M - I) = 2 \). If we denote by
\[ \text{Sp}(2)_{\omega}^{0, \pm} := \{ (r, \theta, z) \in \text{Sp}(2)_{\omega}^{0} \mid \pm \sin \theta > 0 \} , \]
we see that \( \text{Sp}(2)_{\omega}^{0} \setminus \{ I \} = \text{Sp}(2)_{\omega}^{0, +} \cup \text{Sp}(2)_{\omega}^{0, -} \), and each subset is a path-connected component of \( \mathbb{R}^2 \setminus \{ 0 \} \).

The \emph{stratum homotopy property} of the Maslov index states that the Maslov index of a path does not change if to that path is applied a homotopy that maintains each endpoint in its original stratum. Thanks to this property we can simplify the visualisation of paths involving \( \text{Sp}(2)_{\omega}^{0} \) by considering only their deformation (in the sense just described) onto the intersection of the surface with the plane \( z = 0 \) (which is the curve represented in Figure 13).

## B Morse index of Fredholm quadratic forms

In this section we recall the definition of Morse index of Fredholm quadratic forms acting on a (real) separable Hilbert space and for a certain class of unbounded self-adjoint Fredholm operators.

Let \( (H, \langle \cdot, \cdot \rangle) \) be a real separable Hilbert space. As usual we denote by \( \mathcal{L}(H) \) the Banach space of all bounded linear operators on \( H \) and by \( \mathcal{F}(H) \subset \mathcal{L}(H) \) the subspace consisting of all (bounded) Fredholm operators. An operator in \( \mathcal{L}(H) \) defined on all of \( H \) is self-adjoint if and only if it is symmetric. We denote by \( \mathcal{F}^s(H) \) the subspace of all (bounded) self-adjoint Fredholm operators.

For \( T \in \mathcal{F}^s(H) \), if 0 belongs to the spectrum \( \sigma(T) \), then (being \( T \) Fredholm) 0 is an isolated point of \( \sigma(T) \) and therefore it follows from the Spectral Decomposition Theorem that there is an orthogonal decomposition of \( H \),
\[ H = E_-(T) \oplus \ker T \oplus E_+(T), \]
that reduces the operator \( T \) and has the property that
\[ \sigma(T) \cap (-\infty, 0) = \sigma(T|_{E_-(T)}) \quad \text{and} \quad \sigma(T) \cap (0, +\infty) = \sigma(T|_{E_+(T)}). \]
If \( \dim E_-(T) < +\infty \), then \( T \) is called \emph{essentially positive} and if it is also an isomorphism its Morse index \( i_{\text{Morse}}(T) \) is defined as
\[ i_{\text{Morse}}(T) := \dim E_-(T). \]

Let us consider a bounded quadratic form \( q: H \to \mathbb{R} \) and we let \( b = b_q: H \times H \to \mathbb{R} \) be the bounded symmetric bilinear form such that
\[ q(u) = b(u, u), \quad \forall u \in H. \]
By the Riesz Representation Theorem there exists a bounded self-adjoint operator \( A_q : H \to H \) such that \( b_q(u, v) = \langle A_q u, v \rangle \), \( u, v \in H \).

**Definition B.1.** We call \( q: H \to \mathbb{R} \) a \emph{Fredholm quadratic form} if \( A_q \) is Fredholm; i.e. \( \ker A_q \) is finite-dimensional and \( \text{Ran} A_q \) is closed.
Recall that the space $Q(H)$ of bounded quadratic forms is a Banach space with respect to the norm

$$||q|| := \sup_{||u||=1} |q(u)|.$$  

The subset $Q_F(H)$ of all Fredholm quadratic forms is an open subset of $Q(H)$ which is stable under perturbations by weakly continuous quadratic forms. A quadratic form $q \in Q_F(H)$ is called 

\begin{definition}{non-degenerate}\end{definition} 

if the corresponding Riesz representation $A_q$ is invertible. 

\begin{remark}{B.2}\end{remark} 

It is worth noting that if the representation of a quadratic form on $H$ is either invertible, Fredholm or compact then so is its representation with respect to any other Hilbert product on the (real) vector space $H$. 

\begin{proposition}{B.3}\end{proposition} 

A quadratic form on the Hilbert space $H$ is weakly continuous if and only if one (and hence any by Remark B.2) of its representations is a compact (self-adjoint) operator in $\mathcal{L}(H)$. 

\begin{proof}\end{proof} 

Recall that $K$ is compact if and only if it maps weakly convergent sequences to strongly convergent sequences. 

We prove ($\Leftarrow$). Suppose that $K$ is compact and let $(u_n)$ be a sequence in $H$ such that $u_n \overset{w}{\to} u_0$. Then $(Ku_n)$ strongly converges to $Ku_0$. Thus we get

$$\lim_{n \to +\infty} q(u_n) = \lim_{n \to +\infty} \langle Ku_n, u_n \rangle = \langle Ku_0, u_0 \rangle = q(u_0),$$

so the quadratic form is weakly sequentially continuous (and hence weakly continuous because $H$ is first-countable). 

Now suppose that $q$ is weakly sequentially continuous. By the polarisation identity applied to the bilinear form $(u, v) \mapsto \langle Ku, v \rangle$ with $v = Ku$ we get

$$\langle Ku, Ku \rangle = \frac{1}{4} \left[ \langle K(u + Ku), u + Ku \rangle - \langle K(u - Ku), u - Ku \rangle \right] \text{ for all } u \in H. \quad (B.1)$$

Let us assume that $(u_n) \subset H$ weakly converges to $u_0$. Since $K \in \mathcal{L}(H)$ then $Ku_n \overset{w}{\to} Ku_0$. Thus $(u_n \pm Ku_n)$ weakly converges to $u_0 \pm Ku_0$. Therefore by the weak sequential continuity of $q$ and by the identity (B.1) applied to $u = u_n$ and $u = u_0$ we get

$$\lim_{n \to +\infty} \|Ku_n\|^2 = \|Ku_0\|^2.$$ 

Since $(Ku_n)$ converges to $Ku_0$ weakly and in norm, it follows that it converges pointwise to $Ku_0$ (strongly) in $H$. Thus $K$ is compact and this conclude the proof. 

\begin{definition}{B.4}\end{definition} 

A Fredholm quadratic form $q : H \to \mathbb{R}$ is said \textit{essentially positive} if it is the perturbation of a positive definite Fredholm quadratic form by a weakly continuous quadratic form.

By this discussion it follows that

\begin{proposition}{B.5}\end{proposition} 

A Fredholm quadratic form $q$ is essentially positive if and only if it is represented by an essentially positive self-adjoint Fredholm operator $A_q$. 

\begin{proof}\end{proof} 

By the Riesz representation theorem there exists a bounded self-adjoint Fredholm operator $A_q : H \to H$ such that $b_q(u, v) = \langle A_q u, v \rangle$ for all $u, v \in H$. Now since a bounded self-adjoint Fredholm operator is essentially positive if and only if it is a self-adjoint compact perturbation of a self-adjoint positive definite (and hence Fredholm, being invertible) operator, the conclusion follows by applying Proposition B.3.
**Definition B.6.** The *Morse index of an essentially positive Fredholm quadratic form* \( q : H \to \mathbb{R} \) is the Morse index of the (self-adjoint) bounded Fredholm operator \( A_q : H \to H \) uniquely determined by the Riesz Representation Theorem, i.e.

\[ b_q(u, v) = \langle A_q u, v \rangle \text{ for all } u, v \in H \]

where \( b_q \) is the bounded symmetric form induced by \( q \) through the polarisation identity.

**Remark B.7.** It is worth noting that it is possible to show that the *Morse index* of an essentially positive Fredholm quadratic form depends only on the quadratic form and not on the Hilbert structure on \( H \).

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