A NEW CLASS OF IRREDUCIBLE MODULES OVER THE AFFINE-VIRASORO ALGEBRA OF TYPE $A_1$

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ABSTRACT. In this paper, we construct a class of non-weight modules over the affine-Virasoro algebra of type $A_1$ by taking tensor products of a finite number of irreducible modules $M(\lambda, \alpha, \beta, \gamma)$ with irreducible highest weight modules $V(\eta, \epsilon, \theta)$. We obtain the necessary and sufficient conditions for such tensor product modules to be irreducible, and determine the necessary and sufficient conditions for such two modules to be isomorphic. We also compare these modules with other known non-weight modules, showing that these irreducible modules are new.

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1. Introduction

Throughout the paper, we denote by $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{C}^*$, $\mathbb{Z}_+$, $\mathbb{N}$ the sets of complex numbers, integers, nonzero complex numbers, nonnegative integers and positive integers, respectively. All

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algebras (modules, vector spaces) are assumed to be over \( \mathbb{C} \). For a Lie algebra \( g \), we use \( U(g) \) to denote the universal enveloping algebra of \( g \). More generally, for a subset \( X \) of \( g \), we use \( U(X) \) to denote the universal enveloping algebra of the subalgebra of \( g \) generated by \( X \).

It is well known that representation theory of the Virasoro algebra and affine Kac-Moody Lie algebras plays an important role both in physics and in mathematics. The Virasoro algebra acts on any (except when the level is negative the dual coxeter number) highest weight module of the affine Lie algebra through the use of the famous Sugawara operators. The affine Lie algebras admit representations on the Fock space and hence admit representations of the Virasoro algebra. This close relationship strongly suggests that they should be considered simultaneously, i.e., as one algebraic structure, and hence has led to the definition of the so-called affine-Virasoro algebra. It is known that in literature this algebra is also named the conformal current algebra \([10, 17]\), the entire gauge algebra \([12]\). The physical context in which the affine-Virasoro algebra appears is a two-dimensional conformal field theory on the circle with an internal symmetry algebra. In particular, the even part of the \( N = 3 \) superconformal algebra \([9]\) is just the affine-Virasoro algebra of type \( A_1 \). The affine-Virasoro algebra of type \( A_1 \), denoted by \( \mathfrak{L} \), is defined as the Lie algebra with \( \mathbb{C} \)-basis \( \{ e_i, f_i, h_i, d_i, C \mid i \in \mathbb{Z} \} \) subject to the following Lie brackets:

\[
\begin{align*}
[e_i, f_j] &= h_{i+j} + i\delta_{i+j,0}C, \\
[h_i, e_j] &= 2e_{i+j}, \quad [h_i, f_j] = -2f_{i+j}, \\
[d_i, d_j] &= (j-i)d_{i+j} + \delta_{i+j,0} \frac{j^3 - i}{12} C, \\
[d_i, h_j] &= jh_{i+j}, \quad [h_i, h_j] = -2i\delta_{i+j,0}C, \\
[d_i, e_j] &= je_{i+j}, \quad [d_i, f_j] =jf_{i+j}, \\
[e_i, e_j] &= [f_i, f_j] = [C, \mathfrak{L}] = 0.
\end{align*}
\]

It is clear that \( \mathfrak{h} := \mathbb{C}d_0 + \mathbb{C}h_0 + \mathbb{C}C \) is the Cartan subalgebra of \( \mathfrak{L} \). Moreover, \( \mathfrak{L} \) admits a triangular decomposition:

\[
\mathfrak{L} = \mathfrak{L}_- \oplus \mathfrak{h} \oplus \mathfrak{L}_+,
\]

where

\[
\mathfrak{L}_- = \text{span}_\mathbb{C}\{e_{-i}, f_{-i}, h_{-i}, d_{-i}, f_0 \mid i \in \mathbb{N}\}
\]
and

\[ \mathfrak{L}_+ = \text{span}_C \{ e_i, f_i, h_i, d_i, e_0 \mid i \in \mathbb{N} \} . \]

Highest weight representations and integrable representations of the affine-Virasoro algebras have been extensively studied (cf. [2], [11], [13], [16]-[21], [30]). Quite recently, the authors gave the classification of irreducible quasi-finite modules over the affine-Virasoro algebras in [20].

In recent years, many authors constructed various irreducible non-Harish-Chandra modules and irreducible non-weight modules (cf. [1], [3, 4], [15], [23]-[29]). In particular, J. Nilsson [25] constructed a class of \( \mathfrak{sl}_{n+1} \)-modules that are free of rank one when restricted to the Cartan subalgebra. Since then, this kind of non-weight modules, which many authors call \( U(\mathfrak{h}) \)-free modules, have been extensively studied. Especially, the authors classified the \( U(\mathfrak{h}) \)-free modules of rank one for \( \mathfrak{L} \) in [7]. Moreover, the irreducibility and isomorphism classes of these modules were determined therein. However, the theory of representation over \( \mathfrak{L} \) is far more from being well-developed.

It is well known that an important way to construct new modules over an algebra is to consider the linear tensor product of known modules over the algebra (cf. [3, 6], [14], [27], [28], [31]). As a follow-up of our previous paper [8], the purpose of the present paper is to construct new irreducible non-weight \( \mathfrak{L} \)-modules by taking tensor products of a finite number of irreducible modules defined in [7] with irreducible highest weight modules.

This paper is organized as follows. In Section 2, we recall the definitions of various modules involved in this paper and some basic known results on them. In Section 3, we obtain a class of \( \mathfrak{L} \)-modules by taking tensor products of several modules of type \( M(\lambda, \alpha, \beta, \gamma) \) and the irreducible highest modules \( V(\eta, \epsilon, \theta) \). We also determine the necessary and sufficient conditions for such tensor product modules to be irreducible and study their submodule structures when they are reducible. Section 4 is devoted to determining the necessary and sufficient conditions for two such irreducible tensor modules to be isomorphic. In Section 5, we compare the irreducible tensor product modules constructed in Section 3 with other known non-weight irreducible modules, and conclude that these irreducible modules provide a class of new irreducible \( \mathfrak{L} \)-modules.
2. Preliminaries and related known results

We first recall the definitions of some $\mathfrak{L}$-modules considered in this paper. Denote by $\mathbb{C}[s, t]$ the polynomial algebra in variables $s$ and $t$ with coefficients in $\mathbb{C}$.

**Definition 2.1.** For $\lambda, \alpha \in \mathbb{C}^*, \beta, \gamma \in \mathbb{C}, i \in \mathbb{Z}$ and $g(s, t) \in \mathbb{C}[s, t]$, define the $\mathfrak{L}$-module action on $\mathbb{C}[s, t]$ as follows:

\[
\Omega(\lambda, \alpha, \beta, \gamma) : e_i(g(s, t)) = \lambda^i \alpha g(s - i, t - 2),
\]

\[
f_i(g(s, t)) = -\frac{\lambda^i}{\alpha} \left( \frac{t}{2} - \beta \right) \left( \frac{t}{2} + \beta + 1 \right) g(s - i, t + 2),
\]

\[
h_i(g(s, t)) = \lambda^i t g(s - i, t), \quad d_i(g(s, t)) = \lambda^i (s + i\gamma) g(s - i, t),
\]

\[
C(g(s, t)) = 0;
\]

\[
\Delta(\lambda, \alpha, \beta, \gamma) : e_i(g(s, t)) = -\frac{\lambda^i}{\alpha} \left( \frac{t}{2} + \beta \right) \left( \frac{t}{2} - \beta - 1 \right) g(s - i, t - 2),
\]

\[
f_i(g(s, t)) = \lambda^i \alpha g(s - i, t + 2), \quad h_i(g(s, t)) = \lambda^i t g(s - i, t),
\]

\[
d_i(g(s, t)) = \lambda^i (s + i\gamma) g(s - i, t),
\]

\[
C(g(s, t)) = 0;
\]

\[
\Theta(\lambda, \alpha, \beta, \gamma) : e_i(g(s, t)) = \lambda^i \alpha \left( \frac{t}{2} + \beta \right) g(s - i, t - 2),
\]

\[
f_i(g(s, t)) = -\frac{\lambda^i}{\alpha} \left( \frac{t}{2} - \beta \right) g(s - i, t + 2),
\]

\[
h_i(g(s, t)) = \lambda^i t g(s - i, t), \quad d_i(g(s, t)) = \lambda^i (s + i\gamma) g(s - i, t),
\]

\[
C(g(s, t)) = 0.
\]

These modules were introduced in [7] to characterize the $U(\mathfrak{h})$-free modules of rank one for $\mathfrak{L}$. It is worthwhile to point out that $\mathbb{C}[s, t]$ in each case has the same module structure over the subalgebra span\{\(h_i, d_i, C \mid i \in \mathbb{Z}\}). For later use, we need the following known result on conditions for irreducibility and a classification of isomorphism classes for the modules constructed above.

**Proposition 2.2.** (cf. [7]) Keep notations as above, then the following statements hold.

1. $\Omega(\lambda, \alpha, \beta, \gamma)$ and $\Delta(\lambda, \alpha, \beta, \gamma)$ are irreducible for any $\lambda, \alpha \in \mathbb{C}^*$ and $\beta, \gamma \in \mathbb{C}$;

2. $\Theta(\lambda, \alpha, \beta, \gamma)$ is irreducible if and only if $2\beta \notin \mathbb{Z}_+$. 

Let \( \lambda_1, \lambda_2, \alpha_1, \alpha_2 \in \mathbb{C}^*, \beta_1, \beta_2, \gamma_1, \gamma_2 \in \mathbb{C} \). Then

\[
\Omega(\lambda_1, \alpha_1, \beta_1, \gamma_1), \Delta(\lambda_1, \alpha_1, \beta_1, \gamma_1), \Theta(\lambda_1, \alpha_1, \beta_1, \gamma_1)
\]

are pairwise non-isomorphic for all parameter choices. Moreover,

\[
\Omega(\lambda_1, \alpha_1, \beta_1, \gamma_1) \cong \Omega(\lambda_2, \alpha_2, \beta_2, \gamma_2) \iff (\lambda_1, \alpha_1, \beta_1, \gamma_1) = (\lambda_2, \alpha_2, \beta_2, \gamma_2)
\]

or \((\lambda_1, \alpha_1, \beta_1, \gamma_1) = (\lambda_2, \alpha_2, -\beta_2 - 1, \gamma_2)\);

\[
\Delta(\lambda_1, \alpha_1, \beta_1, \gamma_1) \cong \Delta(\lambda_2, \alpha_2, \beta_2, \gamma_2) \iff (\lambda_1, \alpha_1, \beta_1, \gamma_1) = (\lambda_2, \alpha_2, \beta_2, \gamma_2)
\]

or \((\lambda_1, \alpha_1, \beta_1, \gamma_1) = (\lambda_2, \alpha_2, -\beta_2 - 1, \gamma_2)\);

\[
\Theta(\lambda_1, \alpha_1, \beta_1, \gamma_1) \cong \Theta(\lambda_2, \alpha_2, \beta_2, \gamma_2) \iff (\lambda_1, \alpha_1, \beta_1, \gamma_1) = (\lambda_2, \alpha_2, \beta_2, \gamma_2)
\].

For any \( \eta, \epsilon, \theta \in \mathbb{C} \), let \( I(\eta, \epsilon, \theta) \) be the left ideal of \( U(\mathfrak{g}) \) generated by the following elements

\[
\{e_0, e_i, f_i, h_i, d_i \mid i \in \mathbb{N}\} \cup \{d_0 - \eta, h_0 - \epsilon, C - \theta\}.
\]

The Verma \( \mathfrak{g} \)-module with highest weight \((\eta, \epsilon, \theta)\) is defined as the quotient module

\[
\overline{V}(\eta, \epsilon, \theta) = U(\mathfrak{g})/I(\eta, \epsilon, \theta).
\]

By the PBW theorem, \( \overline{V}(\eta, \epsilon, \theta) \) has a basis consisting of all vectors of the form

\[
f_{-q} F_{-q} \ldots f_0 F_0 E_{-p} \ldots E_{-1} h_{-m} \ldots h_{-1} d_{-n} \ldots d_{-1} v_h,
\]

where \( v_h \) is the coset of 1 in \( \overline{V}(\eta, \epsilon, \theta) \), and

\[
D_{-1} \ldots, D_{-n}, H_{-1} \ldots, H_{-m}, E_{-1} \ldots, E_{-p}, F_0, \ldots, F_{-q} \in \mathbb{Z}_+.
\]

Then we have the irreducible highest weight module \( V(\eta, \epsilon, \theta) = \overline{V}(\eta, \epsilon, \theta)/J \), where \( J \) is the unique maximal proper submodule of \( \overline{V}(\eta, \epsilon, \theta) \). Readers can refer to [2, 17] for the structure of \( V(\eta, \epsilon, \theta) \).

In the rest of this paper, we will always assume \( \lambda, \alpha \in \mathbb{C}^*, \beta, \gamma, \eta, \epsilon, \theta \in \mathbb{C}, M(\lambda, \alpha, \beta, \gamma) = \Omega(\lambda, \alpha, \beta, \gamma), \Delta(\lambda, \alpha, \beta, \gamma) \) or \( \Theta(\lambda, \alpha, \beta, \gamma) \) \((2\beta \notin \mathbb{Z}_+)\) constructed in Definition [2.1] and \( V(\eta, \epsilon, \theta) \) is an irreducible highest weight \( \mathfrak{g} \)-module.
3. Irreducibility of Tensor Product Modules

In this section, we investigate the structures of the tensor products of the irreducible \( \mathfrak{L} \)-modules defined in the previous section, that is, the modules \( M(\lambda, \alpha, \beta, \gamma) \) and \( V(\eta, \epsilon, \theta) \). In particular, we determine their irreducibility.

Let \( s_1, s_2, \ldots, s_m, t_1, t_2, \ldots, t_m \) be commuting variables, where \( m \in \mathbb{N} \). Fix any complex numbers \( \lambda_k, \alpha_k \in \mathbb{C}^*, \beta_k, \gamma_k \in \mathbb{C} \), \( 1 \leq k \leq m \), we have the modules \( M(\lambda_k, \alpha_k, \beta_k, \gamma_k) \) defined as in Definition 2.1. Note also that \( 2\beta_k \notin \mathbb{Z}_+ \) when \( M = \Theta \). Then we have \( M(\lambda_k, \alpha_k, \beta_k, \gamma_k) = \mathbb{C}[s_k, t_k] \) for \( 1 \leq k \leq m \) as vector spaces. For an irreducible highest weight module \( V(\eta, \epsilon, \theta) \), we can form the tensor product

\[
T(\lambda, \alpha, \beta, \gamma, \eta, \epsilon, \theta)^M = (\otimes_{k=1}^{m} M(\lambda_k, \alpha_k, \beta_k, \gamma_k)) \otimes V(\eta, \epsilon, \theta),
\]

where \( \lambda = (\lambda_1, \ldots, \lambda_m), \alpha = (\alpha_1, \ldots, \alpha_m), \beta = (\beta_1, \ldots, \beta_m), \gamma = (\gamma_1, \ldots, \gamma_m) \). Denote simply \( T^M := T(\lambda, \alpha, \beta, \gamma, \eta, \epsilon, \theta)^M \). Take any nonzero element

\[
g(s_1, s_2, \ldots, s_m, t_1, t_2, \ldots, t_m) = \sum_{(p, q) \in E} s_1^{p_1} t_1^{q_1} \otimes \cdots \otimes s_m^{p_m} t_m^{q_m} \otimes v_{(p, q)} \in T^M,
\]

where \( (p, q) = (p_1, p_2, \ldots, p_m, q_1, q_2, \ldots, q_m) \), \( v_{(p, q)} \in V(\eta, \epsilon, \theta) \setminus \{0\} \) and \( E \) is a finite subset of \( \mathbb{Z}_+^{2m} \). For convenience, we may write \( s^p t^q = s_1^{p_1} t_1^{q_1} \otimes \cdots \otimes s_m^{p_m} t_m^{q_m} \) and the element \( g(s, t) \in T^M \) can be rewritten as

\[
g(s, t) = \sum_{(p, q) \in E} s^p t^q \otimes v_{(p, q)}.
\]

Denote \( P_k = \max \{p_k \mid (p, q) \in E\} \) for all \( 1 \leq k \leq m \). We simply write \((s^p t^q)(s'^p t'^q) = s^{p+p'} t^{q+q'} \) for short in the following.

For any positive integer \( l \), we can define a total order “\( \succ \)” on \( \mathbb{Z}^l \) by

\[
(a_1, \ldots, a_l) \succ (b_1, \ldots, b_l) \iff \text{there exists } k \in \mathbb{N} \text{ such that } a_k > b_k \text{ and } a_i = b_i, \forall 1 \leq i < k.
\]

Then we define the degree \( \deg(g(s, t)) \) of \( g(s, t) \) as the maximal \( (p, q) \in E \) with \( v_{(p, q)} \neq 0 \). Note that \( \deg(1 \otimes \cdots \otimes 1 \otimes v) = 0 = (0, 0, \ldots, 0) \) for \( v \in V(\eta, \epsilon, \theta) \setminus \{0\} \).

We need the following two crucial results which will be used repeatedly throughout the paper.

Lemma 3.1. (cf. [28]) Let \( \lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{C}^*, s_1, s_2, \ldots, s_m \in \mathbb{N} \) with \( s_1 + s_2 + \cdots + s_m = s \). Define a sequence of functions on \( \mathbb{Z} \) as follows: \( f_1(n) = \lambda_1^n, f_2(n) = n\lambda_1^n, \ldots, f_{s_1}(n) = n^{s_1-1}\lambda_1^n, f_{s_1+1}(n) = \lambda_2^n, \ldots, f_{s_1+s_2}(n) = n^{s_2-1}\lambda_2^n, \ldots, f_s(n) = n^{s_m-1}\lambda_m^n \). Let \( \Omega = (y_{pq}) \) be the
Let \( y_{pq} = f_q(p-1), \) \( q = 1, 2, \ldots, s, p = r + 1, r + 2, \ldots, r + s \) where \( r \in \mathbb{Z}_+ \).

Then

\[
det(\mathfrak{M}) = \prod_{j=1}^{m} (s_j - 1)! \lambda_j^{s_j(s_j + 2r - 1)/2} \prod_{1 \leq i < j \leq m} (\lambda_j - \lambda_i)^{s_is_j},
\]

where \( s_j!! = s_j! \times (s_j - 1)! \times \cdots \times 2! \times 1! \) with \( 0!! = 1 \).

**Proposition 3.2.** Suppose that \( \lambda_1, \ldots, \lambda_m \) are distinct. Let \( W \) be a subspace of \( \mathbf{T}^M \) which is stable under the action of \( h_i, d_i \) for any \( i \) sufficiently large. Then for any \( g(s, t) = \sum_{(p,q) \in E} s^{pt} \otimes v_{(p,q)} \in W, \) we have

\[
(3.3) \quad \sum_{(p,q) \in E, p_k = p_k} s^{p - p_k \omega_k} t^{q + \omega_k} \otimes v_{(p,q)} \in W,
\]

\[
(3.4) \quad t_k g(s, t) = \sum_{(p,q) \in E} s^{pt} t^{q + \omega_k} \otimes v_{(p,q)} \in W,
\]

\[
(3.5) \quad s_k g(s, t) = \sum_{(p,q) \in E} s^{pt} t^{q} \otimes v_{(p,q)} \in W,
\]

where \( \omega_k = (\delta_{k,1}, \delta_{k,2}, \ldots, \delta_{k,m}) \) and \( k = 1, \ldots, m \).

**Proof.** For sufficiently large integer \( i \), we have

\[
(3.6) \quad h_i(c) = \sum_{(p,q) \in E} \sum_{k=1}^{m} s_1^{p_1} t_1^{q_1} \otimes \cdots \otimes h_i(s_k^{p_k} t_k^{q_k}) \otimes \cdots \otimes s_m^{p_m} t_m^{q_m} \otimes v_{(p,q)}
\]

\[
= \sum_{(p,q) \in E} \sum_{k=1}^{m} s_1^{p_1} t_1^{q_1} \otimes \cdots \otimes (s_k^{p_k} t_k^{q_k} + 1) \otimes \cdots \otimes s_m^{p_m} t_m^{q_m} \otimes v_{(p,q)}
\]

\[
= \sum_{(p,q) \in E} \sum_{k=1}^{m} \sum_{x=0}^{p_k} (-1)^x i^x \lambda_k s_1^{p_1} t_1^{q_1} \otimes \cdots \otimes \left( \frac{p_k}{x} \right) s_k^{p_k-x} t_k^{q_k+1} \otimes \cdots \otimes s_m^{p_m} t_m^{q_m} \otimes v_{(p,q)}
\]

and

\[
d_i(c) = \sum_{(p,q) \in E} \sum_{k=1}^{m} s_1^{p_1} t_1^{q_1} \otimes \cdots \otimes d_i(s_k^{p_k} t_k^{q_k}) \otimes \cdots \otimes s_m^{p_m} t_m^{q_m} \otimes v_{(p,q)}
\]

\[
= \sum_{(p,q) \in E} \sum_{k=1}^{m} s_1^{p_1} t_1^{q_1} \otimes \cdots \otimes s_k^{p_k} t_k^{q_k} \otimes \cdots \otimes s_m^{p_m} t_m^{q_m} \otimes v_{(p,q)}
\]

\[
= \sum_{(p,q) \in E} \sum_{k=1}^{m} \sum_{y=0}^{p_k} (-1)^y y^i \lambda_k s_1^{p_1} t_1^{q_1} \otimes \cdots \otimes \left( \frac{p_k}{y} \right) s_k^{p_k-y+1} t_k^{q_k+1}
\]
where we make the convention that \( (0, 0) = 1 \) and \( (p_k/v, y) = 0 \) whenever \( y > p_k \) or \( y < 0 \). Thanks to Lemma 3.1 we know that the coefficients of \( i^x \lambda_k^t \) and \( i^y \lambda_k^t \) in (3.6) and (3.7) belong to \( W \) for any \( 0 \leq x \leq P_k, 0 \leq y \leq P_k + 1 \) and \( 1 \leq k \leq m \), respectively. That is,

\[
(3.7) \quad \otimes \cdots \otimes s_m^p q_m v (p,q) \in W,
\]

For any \( 1 \leq k \leq m \), taking \( x = P_k, 0 \) and \( y = 0 \) in the above two elements, respectively, we get (3.8)-(3.10). This completes the proof.

The following result implies that any element of the form \( 1 \otimes \cdots \otimes 1 \otimes v \) generates the whole tensor product module \( T^M \) for any \( 0 \neq v \in V(\eta, \epsilon, \theta) \).

**Proposition 3.3.** Suppose that \( \lambda_1, \ldots, \lambda_m \) are distinct. Then \( 1 \otimes \cdots \otimes 1 \otimes v \) generates the module \( T^M \) for any \( 0 \neq v \in V(\eta, \epsilon, \theta) \).

**Proof.** Fix any nonzero \( v \in V(\eta, \epsilon, \theta) \). Let \( W \) be the submodule of \( T^M \) generated by \( 1 \otimes \cdots \otimes 1 \otimes v \). On one hand, it follows from (3.4) that \( v^q \otimes v \in W \) for all \( q \in \mathbb{Z}_+^m \) by induction on \( q \). On the other hand, we further deduce from (3.3) that \( s_p v^q \otimes v \in W \) for all \( (p,q) \in \mathbb{Z}_+^{2m} \) by induction on \( p \). That is \( C[s,t] \otimes v \subseteq W \). Let \( V' := \{ u \in V(\eta, \epsilon, \theta) \mid C[s,t] \otimes u \subseteq W \} \). The previous argument implies that \( V' \neq \{0\} \). Note that \( V' \) is an \( \mathfrak{L} \)-submodule of \( V(\eta, \epsilon, \theta) \). It follows that \( W = T^M \) by the irreducibility of \( V(\eta, \epsilon, \theta) \), as desired.

We are now in a position to determine a sufficient condition for the tensor product module \( T^M \) to be irreducible.

**Theorem 3.4.** Let \( m \in \mathbb{N} \), \( \lambda_k, \alpha_k \in \mathbb{C}^*, \beta_k, \gamma_k, \eta, \epsilon, \theta \in \mathbb{C} \) for \( k = 1, 2, \ldots, m \) with the \( \lambda_k \) pairwise distinct. Then the tensor product module \( T^M \) is irreducible provided that \( 2\beta_k \notin \mathbb{Z}_+ \) for any \( 1 \leq k \leq m \) when \( M = \Theta \).

**Proof.** Let \( W \) be a nonzero submodule of \( T^M \) and take a nonzero element \( g(s,t) \in W \) with minimal degree. We claim \( \deg (g(s,t)) = 0 \) and hence \( g(s,t) = 1 \otimes \cdots \otimes 1 \otimes v \) for some nonzero \( v \in V(\eta, \epsilon, \theta) \). Therefore, by Proposition 3.3 we have \( W = T^M \) and \( T^M \) is irreducible.
Assume conversely that \( \deg (g(s, t)) > 0 \). Write \( g(s, t) \) in the form of (3.2), i.e., \( g(s, t) = \sum_{(p, q) \in E} s^p t^q \otimes v_{(p, q)} \). We claim that \( p = 0 \) for any \((p, q) \in E \). If this is not true, let \( k_0 := \min \{k \mid p_k > 0, (p, q) \in E \text{ for some } q\} \). It follows from (3.3) that

\[
0 \neq \sum_{(p, q) \in E, p_k = p_{k_0}} s^{p-p_{k_0}} t^{q+q_{k_0}} \otimes v_{(p, q)} \in W,
\]

which has lower degree than \( g(s, t) \). This contradicts the choice of \( g(s, t) \). Thus, the claim follows, i.e., \( p = 0 \) for any \((p, q) \in E \). Then there exists a minimal \( j_0 \) with \( 1 \leq j_0 \leq m \) such that \( q_{j_0} > 0 \). Set \( Q_{j_0} = \max \{q_{j_0} \mid q_{j_0} \neq 0, (0, q) \in E\} \). We divide the following discussion into three cases.

**Case 1.** \( M = \Omega \).

In this case, a straightforward computation yields that

\[
e_i(g(s, t)) = \sum_{(0, q) \in E} \sum_{k=1}^{m} t_1^q \otimes \cdots \otimes e_i(t_k^q) \otimes \cdots \otimes t_m^q \otimes v_{(0, q)}
\]

\[
(3.9)
\]

where \( i \) is sufficiently large. Applying Lemma 3.1 to the above element, we see that the coefficient of \( \lambda_{j_0}^i \) lies in \( W \), which together with the fact that \( \alpha_{j_0} \neq 0 \) implies

\[
(t_{j_0} - 2)^{q_{j_0}} t^{q - q_{j_0} \omega_{j_0}} \otimes v_{(0, q)} \in W.
\]

Subtracting the above element from \( g(s, t) \), we have

\[
\sum_{(0, q) \in E} (t_{j_0}^q - (t_{j_0} - 2)^{q_{j_0}}) t^{q - q_{j_0} \omega_{j_0}} \otimes v_{(0, q)}
\]

\[
= \sum_{(0, q) \in E, q_{j_0} = Q_{j_0}} (t_{j_0}^{Q_{j_0}} - (t_{j_0} - 2)^{Q_{j_0}}) t^{q - Q_{j_0} \omega_{j_0}} \otimes v_{(0, q)} + \text{lower terms w. r. t. } t_{j_0},
\]

which is a nonzero element in \( W \) and has lower degree than \( g(s, t) \). This is a contradiction with the choice of \( g(s, t) \). So \( \deg (g(s, t)) = 0 \).

**Case 2.** \( M = \Delta \).

Using similar arguments as for Case 1 but replacing \( e_i \) with \( f_i \), we see that \( \deg (g(s, t)) = 0 \).

**Case 3.** \( M = \Theta \).
In this case, for sufficiently large $i$, we have

\[ e_i(g(s, t)) = \sum_{(0,q) \in E} \sum_{k=1}^{m} t_k^{q_1} \otimes \cdots \otimes e_i(t_k^{q_k}) \otimes \cdots \otimes t_m^{q_m} \otimes v(0,q) \]

\[ = \sum_{(0,q) \in E} \sum_{k=1}^{m} \lambda_k t_k^{q_1} \otimes \cdots \otimes (\frac{t_k}{2} + \beta_k) (t_k - 2)^{q_k} \otimes \cdots \otimes t_m^{q_m} \otimes v(0,q), \]

\[ f_i(g(s, t)) = \sum_{(0,q) \in E} \sum_{k=1}^{m} t_k^{q_1} \otimes \cdots \otimes f_i(t_k^{q_k}) \otimes \cdots \otimes t_m^{q_m} \otimes v(0,q) \]

\[ = - \sum_{(0,q) \in E} \sum_{k=1}^{m} \lambda_k t_k^{q_1} \otimes \cdots \otimes (\frac{t_k}{2} - \beta_k) (t_k + 2)^{q_k} \otimes \cdots \otimes t_m^{q_m} \otimes v(0,q) \]

and

\[ h_i(g(s, t)) = \sum_{(0,q) \in E} \sum_{k=1}^{m} t_k^{q_1} \otimes \cdots \otimes h_i(t_k^{q_k}) \otimes \cdots \otimes t_m^{q_m} \otimes v(0,q) \]

\[ = \sum_{(0,q) \in E} \sum_{k=1}^{m} \lambda_k t_k^{q_1} \otimes \cdots \otimes t_k^{q_k+1} \otimes \cdots \otimes t_m^{q_m} \otimes v(0,q). \]

Applying Lemma 3.1 to the above three elements, respectively, we know that the coefficients of $\lambda_j^{q_j}$ belong to $W$, i.e.,

\[ 0 \neq \alpha_{j_0} g_{1,j_0} := \sum_{(0,q) \in E} \alpha_{j_0} \left( \frac{t_{j_0}}{2} + \beta_{j_0} \right) (t_{j_0} - 2)^{q_{j_0}} t^{q-q_{j_0}w_{j_0}} \otimes v(0,q) \in W, \]

\[ 0 \neq -\frac{1}{\alpha_{j_0}} g_{2,j_0} := -\sum_{(0,q) \in E} \frac{1}{\alpha_{j_0}} \left( \frac{t_{j_0}}{2} - \beta_{j_0} \right) (t_{j_0} + 2)^{q_{j_0}} t^{q-q_{j_0}w_{j_0}} \otimes v(0,q) \in W, \]

\[ 0 \neq g_{3,j_0} := \sum_{(0,q) \in E} t_{j_0}^{q_{j_0}+1} t^{q-q_{j_0}w_{j_0}} \otimes v(0,q) \in W. \]

Considering a linear combination of $g_{1,j_0}, g_{2,j_0}, g_{3,j_0}$, we have

\[ g_{1,j_0} + g_{2,j_0} - g_{3,j_0} \]

\[ = \sum_{(0,q) \in E} \left( (\frac{t_{j_0}}{2} + \beta_{j_0}) (t_{j_0} - 2)^{q_{j_0}} + (\frac{t_{j_0}}{2} - \beta_{j_0}) (t_{j_0} + 2)^{q_{j_0}} - t_{j_0}^{q_{j_0}+1} \right) t^{q-q_{j_0}w_{j_0}} \otimes v(0,q) \]

\[ = \sum_{(0,q) \in E, q_0 = Q_{j_0}} 2Q_{j_0} (Q_{j_0} - 1 - 2\beta_{j_0}) t_{j_0}^{Q_{j_0}-1} t^{q-Q_{j_0}w_{j_0}} \otimes v(0,q) \]

+ lower terms w. r. t. $t_{j_0}$,
which is a nonzero element in $W$ and has lower degree than $g(s, t)$. Hence, $\deg(g(s, t)) = 0$. We complete the proof. □

Now we consider the case when $\lambda_1, \ldots, \lambda_m$ are not distinct. Actually, $T^M$ is reducible in this case and it suffices to show the reducibility of $M(\lambda, \alpha_1, \beta_1, \gamma_1) \otimes M(\lambda, \alpha_2, \beta_2, \gamma_2)$. We still use the notations as before, only taking $m = 2$ and $V(\eta, \epsilon, \theta)$ as the 1-dimensional trivial module. For convenience, we identify $M(\lambda, \alpha_1, \beta_1, \gamma_1) \otimes M(\lambda, \alpha_2, \beta_2, \gamma_2) = \mathbb{C}[s_1, s_2, t_1, t_2]$. For any $l \in \mathbb{Z}_+$, denote

$$W_l = \text{span} \{ s_1^r(s_1 + s_2)^p \mathbb{C}[t_1, t_2] \mid r, p \in \mathbb{Z}_+, r \leq l \}.$$ 

Clearly, $W_l \subset W_{l+1}$. And, we have the following

**Theorem 3.5.** Keep notations as above, then each $W_l$ is a proper submodule of $M(\lambda, \alpha_1, \beta_1, \gamma_1) \otimes M(\lambda, \alpha_2, \beta_2, \gamma_2)$.

**Proof.** We only show the assertion for the case $M = \Omega$, the other two cases can be treated similarly. Fix any $l \in \mathbb{Z}_+$. For any $s_1^r(s_1 + s_2)^pg_1(t_1)g_2(t_2) \in W_l$, where $r, p \in \mathbb{Z}_+, r \leq l, g_1(t_1) \in \mathbb{C}[t_1], g_2(t_2) \in \mathbb{C}[t_2]$, we can compute

$$\begin{align*}
\lambda^{-i}h_i \left( s_1^r(s_1 + s_2)^pg_1(t_1)g_2(t_2) \right) \\
= \lambda^{-i}h_i \left( \sum_{j=0}^{p} \binom{p}{j} s_1^{r+j}s_2^{p-j}g_1(t_1)g_2(t_2) \right) \\
= \lambda^{-i} \sum_{j=0}^{p} \binom{p}{j} \left( d_i(s_1^{r+j}g_1(t_1))s_2^{p-j}g_2(t_2) + s_1^{r+j}g_1(t_1)d_i(s_2^{p-j}g_2(t_2)) \right) \\
= (s_1 - i)^r(s_1 + i\gamma_1)g_1(t_1)g_2(t_2) \sum_{j=0}^{p} \binom{p}{j} (s_1 - i)^j s_2^{p-j} \\
+s_1^r(s_2 + i\gamma_2)g_1(t_1)g_2(t_2) \sum_{j=0}^{p} \binom{p}{j} (s_2 - i)^{p-j} s_1^j \\
= (s_1 - i)^r(s_1 + i\gamma_1)g_1(t_1)g_2(t_2)(s_1 + s_2 - i)^p + s_1^r(s_2 + i\gamma_2)g_1(t_1)g_2(t_2)(s_1 + s_2 - i)^p \\
= (s_1((s_1 - i)^r - s_1^r) + i\gamma_1(s_1 - i)^r)g_1(t_1)g_2(t_2)(s_1 + s_2 - i)^p \\
+s_1^r(s_1 + s_2 + i\gamma_2)g_1(t_1)g_2(t_2)(s_1 + s_2 - i)^p,
\end{align*}$$

$$\lambda^{-i}h_i \left( s_1^r(s_1 + s_2)^pg_1(t_1)g_2(t_2) \right)$$
\[
\begin{align*}
&= \lambda^{-i} \sum_{j=0}^{p} \binom{p}{j} \left( h_i(s_1^{r+j} g_1(t_1)) s_2^{p-j} g_2(t_2) + s_1^{r+j} g_1(t_1) h_i(s_2^{p-j} g_2(t_2)) \right) \\
&= (s_1 - i)^r t_1 g_1(t_1) g_2(t_2) \sum_{j=0}^{p} \binom{p}{j} (s_1 - i)^j s_2^{p-j} \\
&\quad + s_1^{r} g_1(t_1) t_2 g_2(t_2) \sum_{j=0}^{p} \binom{p}{j} (s_2 - i)^{p-j} s_1^{j} \\
&= (s_1 - i)^r t_1 g_1(t_1) g_2(t_2)(s_1 + s_2 - i)^p + s_1^{r} g_1(t_1) t_2 g_2(t_2)(s_1 + s_2 - i)^p,
\end{align*}
\]

and

\[
\begin{align*}
&= \lambda^{-i} e_i(s_1^r(s_1 + s_2)^p g_1(t_1) g_2(t_2)) \\
&= \lambda^{-i} \sum_{j=0}^{p} \binom{p}{j} \left( e_i \cdot (s_1^{r+j} g_1(t_1)) s_2^{p-j} g_2(t_2) + s_1^{r+j} g_1(t_1) e_i(s_2^{p-j} g_2(t_2)) \right) \\
&= \alpha_1 (s_1 - i)^r g_1(t_1 - 2) g_2(t_2) \sum_{j=0}^{p} \binom{p}{j} (s_1 - i)^j s_2^{p-j} \\
&\quad + \alpha_2 s_1^{r} g_1(t_1) g_2(t_2 - 2) \sum_{j=0}^{p} \binom{p}{j} (s_2 - i)^{p-j} s_1^{j} \\
&= \alpha_1 (s_1 - i)^r g_1(t_1 - 2) g_2(t_2)(s_1 + s_2 - i)^p + \alpha_2 s_1^{r} g_1(t_1) g_2(t_2 - 2)(s_1 + s_2 - i)^p
\end{align*}
\]
Lemma 4.1. For any nonzero \( \lambda \) irreducible, we suppose that

\[
T \text{ tensor product modules tinct.}
\]

for all \( i \) \( \beta \) distinct and 2 \( (3.2) \), there exists a minimal positive integer \( I \) and

\[
The Corollary 3.6. The \( \Sigma \)-module \( T^M \) is irreducible if and only if \( \lambda_1, \ldots, \lambda_m \) are pairwise distinct.

4. ISOMORPHISM CLASSES OF THE TENSOR PRODUCT MODULES

In this section, we will determine the necessary and sufficient conditions for two irreducible tensor product modules \( T^M \) to be isomorphic. By Corollary 3.6 to ensure that \( T^M \) is irreducible, we suppose that \( \lambda_k, \alpha_k \in \mathbb{C}^*, \beta_k, \gamma_k \in \mathbb{C} \) for \( k = 1, 2, \ldots, m \) with the \( \lambda_k \) pairwise distinct and \( 2\beta_k \notin \mathbb{Z}_+ \) when \( M = \Theta \) throughout this section.

For any \( g := g(s, t) \in T^M \), we define

\[
R_g = \begin{cases} 
\lim_{l \to \infty} \text{rank} \{ g, h_i(g), e_i(g) \mid i \geq l \}, & \text{if } g \in T^\Omega, \\
\lim_{l \to \infty} \text{rank} \{ g, h_i(g), f_i(g) \mid i \geq l \}, & \text{if } g \in T^\Delta, \\
\lim_{l \to \infty} \text{rank} \{ g, h_i(g), e_i(g), f_i(g) \mid i \geq l \}, & \text{if } g \in T^\Theta,
\end{cases}
\]

and

\[
R_{T^M} = \inf \{ R_g \mid 0 \neq g \in T^M \},
\]

where \( \text{rank}(Y) = \dim \text{span}(Y) \) for any subset \( Y \) in a vector space. If we write \( g \) in the form \( \Theta \), there exists a minimal positive integer \( I(g) \) such that \( h_i v_{(p, q)} = e_i v_{(p, q)} = f_i v_{(p, q)} = 0 \) for all \( i \geq I(g) \) and \( (p, q) \in E \).

We have the following result describing the property of the invariants \( R_g \) and \( R_{T^M} \).

Lemma 4.1. For any nonzero \( g \in T^M \), the following statements hold.

(1) For all \( l \geq I(g) \),

\[
R_g = \begin{cases} 
\text{rank} \{ g, h_i(g), e_i(g) \mid i \geq l \}, & \text{if } g \in T^\Omega, \\
\text{rank} \{ g, h_i(g), f_i(g) \mid i \geq l \}, & \text{if } g \in T^\Delta, \\
\text{rank} \{ g, h_i(g), e_i(g), f_i(g) \mid i \geq l \}, & \text{if } g \in T^\Theta.
\end{cases}
\]
Proposition 3.2 (3.3), (3.4) with (4.1), we see that

Consequently, \( R_{\mathbf{T}^m} = m + 1 \).

Proof. (1) we only tackle the case \( g \in \mathbf{T}^\Omega \), since a similar argument can be applied to the other two cases. Denote \( R_{g,l} = \text{rank} \{ g, h_i(g), e_i(g) \mid i \geq l \} \) for any \( l \in \mathbb{N} \), then it suffices to show that \( R_{g,l} = R_{g,I(g)} \) for all \( l \geq I(g) \). For any \( i \geq l \geq I(g) \), an explicit calculation yields that

\[
e_i(g) = \sum_{(p,q) \in E} \sum_{k=1}^m s_1^{p_1 t^{q_1}} \cdots \cdots e_i(s_k^{p_k t^{q_k}}) \cdots \cdots s_m^{t^{q_m}} v_{(p,q)}
\]

\[
= \sum_{(p,q) \in E} \sum_{k=1}^m s_1^{p_1 t^{q_1}} \cdots \cdots \lambda_i^k \alpha_k(s_i - i)^{p_k} (t_k - 2)^{q_k} \cdots \cdots s_m^{t^{q_m}} v_{(p,q)}
\]

\[
= \sum_{(p,q) \in E} \sum_{k=1}^m \sum_{z=0}^I \sum_{z=0}^{z} (-1)^z \lambda_i^k \alpha_k s_1^{p_1 t^{q_1}} \cdots \cdots \left( \frac{p_k}{z} \right) s_k^{p_k z} (t_k - 2)^{q_k} \cdots \cdots s_m^{t^{q_m}} v_{(p,q)}.
\]

Considering the coefficient of \( i^z \lambda_i^k \) for any \( 0 \leq z \leq P_k \) and \( 1 \leq k \leq m \) in the above element and noting that \( \alpha_k \neq 0 \), we get

\[
(4.1) \quad c_{z,k} := \sum_{(p,q) \in E} s_1^{p_1 t^{q_1}} \cdots \cdots \left( \frac{p_k}{z} \right) s_k^{p_k z} (t_k - 2)^{q_k} \cdots \cdots s_m^{t^{q_m}} v_{(p,q)} \in W,
\]

which is independent of \( l \) provided that \( l \geq I(g) \). This along with (3.8) gives

\[
\text{span} \{ g, h_i(g), e_i(g) \mid i \geq l \} = \text{span} \{ g, a_x, c_{z,k}, c_{k,z} \mid 0 \leq x, z \leq P_k, 1 \leq k \leq m \}.
\]

Consequently, \( R_{g,l} = R_{g,I(g)} \) for all \( l \geq I(g) \), proving (1).

(2) We first consider the case \( g \in \mathbf{T}^\Omega \), and the proof for \( g \in \mathbf{T}^\Delta \) is similar. Combining Proposition 3.2 (3.3), (3.4) with (4.1), we see that

\[
ap_{p_k} = \sum_{(p,q) \in E, p_k = P_k} s^{p - P_k \omega_k} t^{q + \omega_k} v_{(p,q)} \in W;
\]

\[
a_0 = t_k g(s, t) = \sum_{(p,q) \in E} s^{p} t^{q + \omega_k} v_{(p,q)} \in W;
\]

\[
c_0 = \sum_{(p,q) \in E} (t_k - 2)^{q} s^{p} t^{q - \omega_k} v_{(p,q)} \in W.
\]
If \( \deg (g) = 0 \), that is, \( g = 1 \otimes \cdots \otimes 1 \otimes v \) for some \( 0 \neq v \in V(\eta, \epsilon, \theta) \), then we have 
\[ R_g = \text{rank} \{ g, a_{0,k}, c_{0,k} \mid 1 \leq k \leq m \} = m + 1. \]
Now suppose \( \deg (g) = (p', q') \succ 0 \). It is clear that \( \deg (a_{0,k}) = (p', q' + \omega_k) \) for \( 1 \leq k \leq m \). If there exists \( 1 \leq k \leq m \) such that \( p'_k > 0 \), let \( k_1 \) be the minimal \( k \) with \( p'_k > 0 \). Then we have \( \deg (a_{p_{k_1},k_1}) \prec (p', q') \). Hence the space spanned by \( g, a_{0,k}, a_{p_{k_1},k_1}, 1 \leq k \leq m \) has dimension \( m + 2 \), which means \( R_g \geq m + 2 \). If \( p_k = 0 \) for all \( 1 \leq k \leq m \), then there exists \( 1 \leq k \leq m \) such that \( q'_k > 0 \). Let \( k_2 \) be the minimal \( k \) with \( q'_k > 0 \). One can observe that \( 0 \leq \deg (g - c_{0,k_2}) \prec (p', q') \). Thus the space spanned by \( g, a_{0,k}, c_{0,k_2}, 1 \leq k \leq m \) has dimension \( m + 2 \), so that \( R_g \geq m + 2 \). Thus, (2) holds for \( M = \Theta \), proving (2).

For the remaining case \( g \in T^\Theta \), by a similar argument in the preceding paragraph, we see that if \( \deg (g) = 0 \), then \( R_g = m + 1 \). If \( \deg (g) = (p', q') \succ 0 \) and there exists \( 1 \leq k \leq m \) such that \( p'_k > 0 \), let \( k_3 \) be the minimal \( k \) with \( p'_k > 0 \). Then dim span \{ \( g, a_{0,k}, a_{p_{k_3},k_3} \mid 1 \leq k \leq m \} = m + 2 \). So, \( R_g \geq m + 2 \). If \( p_k = 0 \) for all \( 1 \leq k \leq m \) and there exists \( 1 \leq k \leq m \) such that \( q_k > 0 \), let \( k_4 \) be the minimal one. It follows from the computation in Theorem 3.4 Case 3 that \( 0 \leq \deg (g_{1,k_4} + g_{2,k_4} - g_{3,k_4}) \prec (p', q') \), forcing \( R_g \geq m + 2 \). Hence, (2) holds for \( M = \Theta \), proving (2).

(3) is an obvious consequence of (2).

We complete the proof. \( \square \)

Now we are ready to prove our isomorphism theorem. Let \( T^M \) be the tensor module defined before. Now we take another tensor module,

\[ T^M := T'(\lambda', \alpha', \beta', \gamma', \eta', \epsilon', \theta')^M = (\otimes_{j=1}^{m'} M(\lambda'_j, \alpha'_j, \beta'_j, \gamma'_j)) \otimes V(\eta', \epsilon', \theta'), \]

where \( \lambda' = (\lambda'_1, \ldots, \lambda'_{m'}) \), \( \alpha' = (\alpha'_1, \ldots, \alpha'_{m'}) \), \( \beta' = (\beta'_1, \ldots, \beta'_{m'}) \), \( \gamma' = (\gamma'_1, \ldots, \gamma'_{m'}) \) and \( V(\eta', \epsilon', \theta') \) is an irreducible highest weight module. We identify \( M(\lambda'_j, \alpha'_j, \beta'_j, \gamma'_j) = \mathbb{C}[s'_j, t'_j] \) for \( 1 \leq j \leq m' \). To ensure that \( T^M \) and \( T^M \) are irreducible, we suppose that \( \lambda_1, \ldots, \lambda_m \) are pairwise distinct as well as \( \lambda'_{1,\ldots,\lambda'_{m'}} \) are pairwise distinct, \( \alpha_k, \alpha'_j \in \mathbb{C}^* \) for \( 1 \leq k \leq m \) and \( 1 \leq j \leq m' \) and \( 2\beta_k, 2\beta'_j \not\in \mathbb{Z}_+ \) when \( M = \Theta \).

**Theorem 4.2.** \( T^M \cong T^M \) as \( \mathcal{L} \)–modules if and only if \( m = m' \), \( V(\eta, \epsilon, \theta) \cong V(\eta', \epsilon', \theta') \)
and \( M(\lambda_k, \alpha_k, \beta_k, \gamma_k) \cong M(\lambda'_k, \alpha'_k, \beta'_k, \gamma'_k) \) for \( 1 \leq k \leq m \) after renumbering the indices \( (\lambda'_k, \alpha'_k, \beta'_k, \gamma'_k) \) if necessary.
Proof. The sufficiency is obvious and it suffices to show the necessity. Let $\phi$ be an $\mathcal{L}$-module isomorphism from $T^M$ to $T^M$. From Lemma 4.1 (3), we see that $m + 1 = R_{T^M} = R_{T^M} = m' + 1$, i.e., $m = m'$.

Take $g = 1 \otimes \cdots \otimes 1 \otimes v$ for some $0 \neq v \in V(\eta, \epsilon, \theta)$. From Lemma 4.1 (2), we have $R_{\phi(g)} = R_{g} = m + 1$, yielding

$$\phi(g) = 1 \otimes \cdots \otimes 1 \otimes v' \quad \text{for some} \; 0 \neq v' \in V(\eta', \epsilon', \theta').$$

Since $v'$ is uniquely determined by $v$, we may denote $\tau(v) = v'$. It is obvious that $\tau$ is a linear bijection from $V(\eta, \epsilon, \theta)$ to $V(\eta', \epsilon', \theta')$. We only tackle the case $M = \Omega$ in the following, since the other two cases can be treated similarly.

Now for any $i \in \mathbb{Z}$ with $i \geq \max \{I(g), I(\phi(g))\}$, by (4.2), we have

$$0 = \phi(d_i(1 \otimes \cdots \otimes 1 \otimes v)) - d_i(1 \otimes \cdots \otimes 1 \otimes v) = \sum_{k=1}^{m} \lambda_k^i \phi(1 \otimes \cdots \otimes s_k \otimes \cdots \otimes 1 \otimes v) + \sum_{k=1}^{m} i\lambda_k^i \gamma_k(1 \otimes \cdots \otimes 1 \otimes \tau(v))$$

$$- \sum_{k=1}^{m} (\lambda_k^i)^t(1 \otimes \cdots \otimes s_k \otimes \cdots \otimes 1 \otimes \tau(v)) - \sum_{k=1}^{m} i(\lambda_k^i)^t \gamma_k^t(1 \otimes \cdots \otimes 1 \otimes \tau(v)).$$

(4.3)

From Lemma 3.1, we know that for any $1 \leq k \leq m$, if $\lambda_k \neq \lambda_j$ for any $1 \leq j \leq m$, then the coefficient of $\lambda_k$ in (4.3) should be zero, i.e., $\phi(1 \otimes \cdots \otimes s_k \otimes \cdots \otimes 1 \otimes v) = 0$, which is absurd. So, we may assume that $\lambda_k = X_k$ for all $1 \leq k \leq m$ by reordering the indices $1 \leq k \leq m$. Now the coefficients of $\lambda_k$ and $i\lambda_k^i, 1 \leq k \leq m$ in (4.3) become $\phi(1 \otimes \cdots \otimes s_k \otimes \cdots \otimes 1 \otimes v) - 1 \otimes \cdots \otimes s_k \otimes \cdots \otimes 1 \otimes \tau(v)$ and $(\gamma_k - \gamma_k^t)1 \otimes 1 \otimes \cdots \otimes 1 \otimes \tau(v)$, respectively, forcing

$$\phi(1 \otimes \cdots \otimes s_k \otimes \cdots \otimes 1 \otimes v) = 1 \otimes \cdots \otimes s_k' \otimes \cdots \otimes 1 \otimes \tau(v)$$

and $\gamma_k = \gamma_k^t$ for $1 \leq k \leq m$. Now for any $i \geq \max \{I(g), I(\phi(g))\}$, the equations

$$\phi(e_i(1 \otimes \cdots \otimes 1 \otimes v)) - e_i(\phi(1 \otimes \cdots \otimes 1 \otimes v)) = 0,$$

$$\phi(h_i(1 \otimes \cdots \otimes 1 \otimes v)) - h_i(\phi(1 \otimes \cdots \otimes 1 \otimes v)) = 0,$$

$$\phi(f_i(1 \otimes \cdots \otimes 1 \otimes v)) - f_i(\phi(1 \otimes \cdots \otimes 1 \otimes v)) = 0$$

are respectively equivalent to

$$\sum_{k=1}^{m} \lambda_k^i (\alpha_k - \alpha_k^i)(1 \otimes 1 \otimes \cdots \otimes 1 \otimes \tau(v)) = 0,$$
\[ \sum_{k=1}^{m} \lambda^i_k \phi(1 \otimes \cdots \otimes t_k \otimes \cdots \otimes 1 \otimes v) - 1 \otimes \cdots \otimes t'_k \otimes \cdots \otimes 1 \otimes \tau(v) \] 

\[ - \sum_{k=1}^{m} \frac{\lambda^i_k}{\alpha_k} \phi(1 \otimes \cdots \otimes \left( \frac{t^2_k}{4} + \frac{t_k}{2} - \beta_k(\beta_k + 1) \right) \otimes \cdots \otimes 1 \otimes v) \]

\[ + \sum_{k=1}^{m} \frac{\lambda^i_k}{\alpha'_k} \left( 1 \otimes \cdots \otimes \left( \frac{t^2_k}{4} + \frac{t'_k}{2} - \beta'_k(\beta'_k + 1) \right) \otimes \cdots \otimes 1 \otimes \tau(v) \right) = 0. \]

From Lemma 3.1 we have for any \(1 \leq k \leq m\),

\[ (4.5) \quad \alpha_k = \alpha'_k, \]

\[ (4.6) \quad \phi(1 \otimes \cdots \otimes t_k \otimes \cdots \otimes 1 \otimes v) = 1 \otimes \cdots \otimes t'_k \otimes \cdots \otimes 1 \otimes \tau(v), \]

\[ - \frac{1}{\alpha_k} \phi(1 \otimes \cdots \otimes \left( \frac{t^2_k}{4} + \frac{t_k}{2} - \beta_k(\beta_k + 1) \right) \otimes \cdots \otimes 1 \otimes v) \]

\[ = - \frac{1}{\alpha_k} \left( 1 \otimes \cdots \otimes \left( \frac{t^2_k}{4} + \frac{t'_k}{2} - \beta'_k(\beta'_k + 1) \right) \otimes \cdots \otimes 1 \otimes \tau(v) \right). \]

For any \(1 \leq k \leq m\) and \(i \geq \max \{I(g), I(\phi(g))\}\), from (1.6) and

\[ \phi(h_i(1 \otimes \cdots \otimes t_k \otimes \cdots \otimes 1 \otimes v)) - h_i \phi(1 \otimes \cdots \otimes t_k \otimes \cdots \otimes 1 \otimes v) = 0, \]

we have

\[ \sum_{j \in \{1, \ldots, m\} \setminus \{k\}} \lambda^i_j \phi(1 \otimes \cdots \otimes t_j \otimes \cdots \otimes t_k \otimes \cdots \otimes 1 \otimes v) \]

\[ + \lambda^i_k \phi(1 \otimes \cdots \otimes t'_k \otimes \cdots \otimes 1 \otimes v) \]

\[ - \sum_{j \in \{1, \ldots, m\} \setminus \{k\}} \lambda^i_j \left( 1 \otimes \cdots \otimes t'_j \otimes \cdots \otimes t'_k \otimes \cdots \otimes 1 \otimes \tau(v) \right) \]

\[ + \lambda^i_k \left( 1 \otimes \cdots \otimes t'^2_k \otimes \cdots \otimes 1 \otimes \tau(v) \right) = 0, \]

Also, it follows from Lemma 3.1 that the coefficient of \(\lambda^i_k\) should be zero, that is,

\[ (4.8) \quad \phi(1 \otimes \cdots \otimes t'^2_k \otimes \cdots \otimes 1 \otimes v) = 1 \otimes \cdots \otimes t'^2_k \otimes \cdots \otimes 1 \otimes \tau(v). \]

Applying (4.5), (4.6), (4.8) to (4.7) gives \(\beta_k = \beta'_k\) or \(\beta_k = -\beta'_k - 1\) for any \(1 \leq k \leq m\).

Hence, \(\Omega(\lambda_k, \alpha_k, \beta_k, \gamma_k) \cong \Omega(\lambda'_k, \alpha'_k, \beta'_k, \gamma'_k)\) by Proposition 2.2 (2). Combining (4.2), (4.4), (4.6) with (4.8), we obtain

\[ \phi(X_i(1 \otimes \cdots \otimes 1) \otimes v) = X_i(1 \otimes \cdots \otimes 1) \otimes \tau(v), \quad \text{where } X_i \in \{d_i, h_i, e_i, f_i \mid \forall i \in \mathbb{Z}\}. \]

This together with

\[ \phi(X_i(1 \otimes \cdots \otimes 1 \otimes v)) = X_i(\phi(1 \otimes \cdots \otimes 1 \otimes v)), \quad \text{where } X_i \in \{d_i, h_i, e_i, f_i \mid \forall i \in \mathbb{Z}\} \]

gives

\[ \phi(1 \otimes \cdots \otimes 1 \otimes X_i(v)) = 1 \otimes \cdots \otimes 1 \otimes X_i(\tau(v)), \quad \text{where } X_i \in \{d_i, h_i, e_i, f_i \mid \forall i \in \mathbb{Z}\}. \]
Therefore,
\[ \tau(X_i(v)) = X_i(\tau(v)), \quad \text{where } X_i \in \{d_i, h_i, e_i, f_i \mid \forall i \in \mathbb{Z}\}, v \in V(\eta, \epsilon, \theta). \]

From
\[ \phi(C(1 \otimes \cdots \otimes 1 \otimes v)) = C(\phi(1 \otimes \cdots \otimes 1 \otimes v)), \quad \forall v \in V(\eta, \epsilon, \theta), \]
we see that \( \tau(C(v)) = C(\tau(v)) \). Thus, \( \tau \) is a nonzero \( \mathfrak{L} \)-module homomorphism. Since \( V(\eta, \epsilon, \theta) \) and \( V(\eta', \epsilon', \theta') \) are simple \( \mathfrak{L} \)-modules, \( \tau \) is an \( \mathfrak{L} \)-module isomorphism. We complete the proof. \( \square \)

5. Comparison of Tensor Product Modules with Known Non-weight Modules

In this section, we compare the tensor product modules constructed in the present paper with all other known non-weight \( \mathfrak{L} \)-modules, i.e., \( U(\mathfrak{h}) \)-free modules of rank one and Whittaker modules (cf. [7]). We fix an irreducible tensor module \( T^M \) as defined in (3.1), where \( \lambda_k, \alpha_k \in \mathbb{C}^* \), \( \beta_k, \gamma_k \in \mathbb{C} \) for \( k = 1, 2, \ldots, m \) with the \( \lambda_k \) pairwise distinct and \( 2\beta_k \notin \mathbb{Z}_+ \) when \( M = \Theta \).

Let \( \underline{\mu} = (\mu_1, \ldots, \mu_4) \in \mathbb{C}^4 \). Assume that \( J_{\underline{\mu}} \) is the left ideal of \( U(\mathfrak{L}_+) \) generated by \( \{d_1 - \mu_1, d_2 - \mu_2, e_0 - \mu_3, f_1 - \mu_4, d_j, e_k, f_l, h_m \mid j \geq 3, k \geq 1, l \geq 2, m \geq 1\} \). Denote \( N_{\underline{\mu}} := U(\mathfrak{L}_+)/J. \) Then \( \text{Ind}(N_{\underline{\mu}}) := U(\mathfrak{L}) \otimes_{U(\mathfrak{L}_+)} N_{\underline{\mu}} \) is a universal Whittaker module, and any Whittaker module is a quotient of some universal Whittaker module.

For any \( r \in \mathbb{Z}_+, l, j \in \mathbb{Z} \), as in [22], we denote
\[ \omega^{(r)}_{l,j} = \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} d_{l-j-i} d_{j+i} \in U(\mathfrak{L}). \]

Lemma 5.1. Keep notations as above. Then the following statements hold.

1. For sufficiently large \( i \in \mathbb{N} \), the action of \( d_i \) on \( T^M \) is not locally finite.
2. If \( V(\eta, \epsilon, \theta) \) is not the 1-dimensional trivial module, then for any \( r \in \mathbb{Z}_+ \), there exist \( l, j \in \mathbb{Z} \) and \( v \in V(\eta, \epsilon, \theta) \) such that \( \omega^{(r)}_{l,j} (1 \otimes \cdots \otimes 1 \otimes v) \neq 0. \)
3. Assume that \( m \geq 2 \) and \( V(\eta, \epsilon, \theta) \) is the 1-dimensional trivial module. Then for any \( r > 2 \), there exist \( l, j \in \mathbb{Z} \) such that \( \omega^{(r)}_{l,j} (1 \otimes \cdots \otimes 1) \neq 0. \)

Proof. (1) For any \( g(s, t) \in T^M \) and sufficiently large \( i \in \mathbb{N} \), it follows from Proposition 3.2 that \( \deg(d_i(g(s, t))) > \deg(g(s, t)) \), so we have \( g(s, t), d_i(g(s, t)), d_i^2(g(s, t)), \ldots \) are linearly independent. Hence, (1) follows.
(2) Take $v$ to be the highest weight vector of $V(\eta, \epsilon, \theta)$. It is important to observe that the vectors $v, d_{-2}v, d_{-3}v, \ldots, d_{-r-2}v$ are linearly independent in $V(\eta, \epsilon, \theta)$, since they belong to different weight subspaces of $V(\eta, \epsilon, \theta)$. Take $l = r + 1$ and $j = -r - 2$. We compute

$$\omega(l,j)(1 \otimes \cdots \otimes 1 \otimes v) = \sum_{i=0}^{r} \left( \begin{array}{c} r \\ i \end{array} \right) (-1)^{r-i} d_{l-j-i} d_{j+i}(1 \otimes \cdots \otimes 1 \otimes v)$$

$$= \sum_{i=0}^{r} \left( \begin{array}{c} r \\ i \end{array} \right) (-1)^{r-i} (d_{l-j-i}(1 \otimes \cdots \otimes 1) \otimes d_{j+i}(v)$$

$$+ d_{l-j-i}(d_{j+i}(1 \otimes \cdots \otimes 1)) \otimes v),$$

which is nonzero.

(3) In this case, we identify $T^M$ with $\mathbb{C}[s, t]$ as a vector space. We compute

$$\omega(l,j)(1) = \sum_{i=0}^{r} \left( \begin{array}{c} r \\ i \end{array} \right) (-1)^{r-i} d_{l-j-i} d_{j+i}(1)$$

$$= \sum_{i=0}^{r} \left( \begin{array}{c} r \\ i \end{array} \right) (-1)^{r-i} d_{l-j-i} (\sum_{k=1}^{m} \lambda_{k}^{j+i} (s_k + (j+i)\gamma_k))$$

$$= \sum_{i=0}^{r} \left( \begin{array}{c} r \\ i \end{array} \right) (-1)^{r-i} \sum_{k=1}^{m} \sum_{k' \in \{1, \ldots, m\} \setminus \{k\}} \lambda_{k'}^{l-j-i} \lambda_{k}^{j+i} (s_k + (l-j-i)\gamma_{k'}) (s_k + (j+i)\gamma_k)$$

$$+ \sum_{i=0}^{r} \left( \begin{array}{c} r \\ i \end{array} \right) (-1)^{r-i} \sum_{k=1}^{m} \lambda_{k}^{l} (s_k + (l-j-i)\gamma_k) (s_k + (j+i)\gamma_k - l + j + i).$$

The following identity

$$\sum_{i=0}^{r} \left( \begin{array}{c} r \\ i \end{array} \right) (-1)^{r-i} i^j = 0, \quad \forall j, r \in \mathbb{Z}_+ \text{ with } j < r$$

forces that the second summand of $\omega(l,j)(1)$ vanishes provided $r > 2$. Since $r > 2$, the coefficient of $s_1 s_2$ in $\omega(l,j)(1)$ is

$$\sum_{i=0}^{r} \left( \begin{array}{c} r \\ i \end{array} \right) (-1)^{r-i}(\lambda_1^{l-j-i}\lambda_2^{j+i} + \lambda_2^{l-j-i}\lambda_1^{j+i}) = (\lambda_1^{l-j-r}\lambda_2^{j} + (-1)^r\lambda_2^{l-j-r}\lambda_1^{j})(\lambda_2 - \lambda_1)^r,$$

which is nonzero provided that $(\lambda_1/\lambda_2)^{l-2j-r} + (-1)^r \neq 0$. Thus (3) follows. \qed

Finally, we have the following result which asserts that the tensor product modules constructed in the present paper are different from the other known non-weight modules.
Proposition 5.2. The tensor product module $T^M$ is a new non-weight $L$-module.

Proof. For the case $m = 1$, the assertion has been proved in [8, Proposition 5.2]. Next we assume that $m \geq 2$ in the following. We need to show that the tensor product module $T^M$ is neither isomorphic to a Whittaker module nor isomorphic to a $U(h)$-free modules of rank one in [7]. For that, let $W$ be a Whittaker module, then $W$ is isomorphic to a quotient of $N_{\mu}$ for some $\mu = (\mu_1, \ldots, \mu_4) \in \mathbb{C}^4$. Noticing that the action of $d_i$ on $W$ is locally finite for sufficiently large $i \in \mathbb{N}$. It follows from Lemma 5.1 (1) that $T^M \not\cong W$. Combining [8, Lemma 5.1(2)] with Lemma 5.1 (2) and (3), we see that $T^M$ is not isomorphic to the irreducible non-weight modules defined in [7]. We complete the proof. \qed

References

[1] P. Batra, V. Mazorchuk, Blocks and modules for Whittaker pairs, J. Pure Appl. Algebra 215, 1552–1568 (2011).
[2] Y. Billig, A category of modules for the full toroidal Lie algebra, Int. Math. Res. Notices, 46pp (2006).
[3] Y. Cai, H. Tan, K. Zhao, Module structure on $U(h)$ for Kac-Moody algebras (in Chinese), Sci. Sin. Math. 47(11), 1491–1514 (2017).
[4] Y. Cai, K. Zhao, Module structure on $U(h)$ for basic Lie superalgebras, Toyama Math. J. 37, 55–72 (2015).
[5] H. Chen, X. Guo, Tensor product weight modules over the Virasoro algebra, J. Lond. Math. Soc. 88, 829–844 (2013).
[6] H. Chen, J. Han, Y. Su, X. Yue, Two classes of non-weight modules over the twisted Heisenberg-Virasoro algebra, Manuscripta Math. 160(1-2), 265–284 (2019).
[7] Q. Chen, J. Han, Non-weight modules over the affine-Virasoro algebra of type $A_1$, J. Math. Phys. 60, 071707 (2019).
[8] Q. Chen, Y. Yao, Irreducible tensor product modules over the affine-Virasoro algebra of type $A_1$, Linear Multilinear A., in press, doi: 10.1080/03081087.2021.1998309 (2021).
[9] S. Cheng, N. Lam, Finite conformal modules over the $N = 2, 3, 4$ superconformal algebras, J. Math. Phys. 42(2), 906–933 (2001).
[10] S. Cheng, V. Kac, Conformal modules, Asian J. Math. 1, 181–193 (1997).
[11] S. Eswara Rao, C. Jiang, Classification of irreducible integrable representations for the full toroidal Lie algebras, J. Pure Appl. Algebra 200, 71–85 (2005).
[12] P. Etingof, I. Frenkel, A. Kirillov, Lectures on Representation Theory and Knizhnik-Zamolodchikov Equations, Mathematical Surveys and Monographs 58, American Mathematical Society, Providence, 1998.
[13] Y. Gao, N. Hu, D. Liu, Representations of the affine-Virasoro algebra of type $A_1$, J. Geom. Phys. 106, 102–107 (2016).
[14] X. Guo, X. Liu, J. Wang, New irreducible tensor product modules for the Virasoro algebra, *Asian J. Math.* **24**(2), 191–206 (2020).

[15] J. Han, Q. Chen, Y. Su, Modules over the algebras $\text{Vir}(a,b)$, *Linear Algebra Appl.* **515**, 11–23 (2017).

[16] C. Jiang, H. You, Irreducible representations for the affine-Virasoro Lie algebras of type $B_l$, *Chinese Ann. Math. Ser. B* **25**(3), 359–368 (2004).

[17] V. Kac, Highest weight representations of conformal current algebras, *Symposium on Topological and Geometric Methods in Field Theory, Espoo, Finland, World Scientific*, 3–16 (1986).

[18] V. Kac: Infinite-Dimensional Lie Algebras, 3rd ed., Cambridge University Press, Cambridge (1990).

[19] G. Kuroki, Fock space representations of an affine Lie algebras and integral representations in the Wess-Zumino-Witten models, *Comm. Math. Phys.* **142**(3), 511–542 (1991).

[20] D. Liu, Y. Pei, L. Xia, Classification of quasi-finite irreducible modules over affine-Virasoro algebras, *J. Lie Theory* **31**(2), 575–582 (2021).

[21] X. Liu, M. Qian, Bosonic Fock representations of the affine-Virasoro algebra, *J. Phys. A* **27**(5), 131–136 (1994).

[22] G. Liu, R. Lü, K. Zhao, A class of simple weight Virasoro modules, *J. Algebra* **424**, 506–521 (2015).

[23] V. Mazorchuk, E. Weisner, Simple Virasoro modules induced from codimension one subalgebras of the positive part, *Proc. Amer. Math. Soc.* **142**(11), 3695–3703 (2012).

[24] V. Mazorchuk, K. Zhao, Simple Virasoro modules which are locally finite over a positive part, *Selecta Math. (N.S.)* **20**(3), 839–854 (2014).

[25] J. Nilsson, Simple $\mathfrak{sl}_{n+1}$-module structures on $U(\mathfrak{h})$, *J. Algebra* **424**, 294–329 (2015).

[26] J. Nilsson, $U(\mathfrak{h})$-free modules and coherent families, *J. Pure Appl. Algebra* **220**, 1475–1488 (2016).

[27] H. Tan, K. Zhao, Irreducible Virasoro modules from tensor products, *Ark. Mat.* **54**, 181–200 (2016).

[28] H. Tan, K. Zhao, Irreducible Virasoro modules from tensor products (II), *J. Algebra* **394**, 357–373 (2013).

[29] H. Tan, K. Zhao, $\mathfrak{w}_n^+$ and $\mathfrak{w}_n$-module structures on $U(\mathfrak{h}_n)$, *J. Algebra* **424**, 257–375 (2015).

[30] L. Xia, N. Hu, Irreducible representations for Virasoro-toroidal Lie algebras, *J. Pure Appl. Algebra* **194**, 213–237 (2004).

[31] H. Zhang, A class of representations over the Virasoro algebra, *J. Algebra* **190**, 1–10 (1997).