Note on a Differential-Geometrical Construction of Optimal Directions in Linearly-Constrained Systems

John Ellis\textsuperscript{a,b}, Jae Sik Lee\textsuperscript{c} and Apostolos Pilaftsis\textsuperscript{d}

\textsuperscript{a}Theory Division, CERN, CH-1211 Geneva 23, Switzerland
\textsuperscript{b}Theoretical Physics and Cosmology Group, Department of Physics, King’s College London, London WC2R 2LS, United Kingdom
\textsuperscript{c}Physics Division, National Center for Theoretical Sciences, Hsinchu, Taiwan 300
\textsuperscript{d}School of Physics and Astronomy, University of Manchester, Manchester M13 9PL, United Kingdom

ABSTRACT
This note presents an analytic construction of the optimal unit-norm direction $\hat{x} = x/||x||$ that maximizes or minimizes the objective linear expression, $B \cdot \hat{x}$, subject to a system of linear constraints of the form $[A] \cdot x = 0$, where $x$ is an unknown $n$-dimensional real vector to be determined up to an overall normalization constant, $0$ is an $m$-dimensional null vector, and the $n$-dimensional real vector $B$ and the $m \times n$-dimensional real matrix $[A]$ (with $0 \leq m < n$) are given. The analytic solution to this problem can be expressed in terms of a combination of double wedge and Hodge-star products of differential forms.

KEYWORDS: Linear programming; Linear algebra; Differential forms

AMS CLASSIFICATION 2010: 90C05, 15A69, 58A10.
1 Introduction

A basic problem in linear programming [1] is the optimization of the objective function, $B \cdot x$, which is linear in an $n$-dimensional real vector $x$, with non-negative components, that is subject to a system of linear constraints of the form $[A] \cdot x = A_0$, where $[A]$ is an $m \times n$-dimensional real matrix, with $0 \leq m < n$, and $A_0$ is an $m$-dimensional real vector with non-negative components.

This note presents an exact, analytic solution to a simple, but different class of linear programming problems, where $A_0$ is the $m$-dimensional null vector $0$ and the objective is $B \cdot \hat{x}$, where $\hat{x} = x/||x||$ is the optimal unit-norm direction to be determined. In the absence of any linear constraint ($m = 0$), the solution is trivially given by $\hat{x} = B/||B|| \equiv \hat{B}$.

For a large number $m$ of linear constraints, however, the solution becomes non-trivial and relies on a geometric construction expressed in terms of wedge and Hodge-dual products of differential forms [2-4].

As an example of this type of linear programming problem, this note first describes the simple case involving a single constraint $A \cdot x = 0$ on a 3-dimensional real vector $x$. The subspace of parameters satisfying such a linear constraint is characterized, and the analytic construction of a real vector $x$ that optimizes the linear expression, $B \cdot \hat{x}$, is then described. Subsequently, a generalization to multiple constraints in higher-dimensional spaces is described. We emphasize the formal analogies between the three- and higher-dimensional cases.

2 A Three-Dimensional Example

Let us first consider a simple three-dimensional (3D) example, where $x = (x_1, x_2, x_3)$. Using the standard notation of inner product multiplication between vectors, the aim is to find the unit-norm direction $\hat{x} = x/||x||$, along which the expression, $B \cdot \hat{x}$, is maximized or minimized, subject into the constraint: $A \cdot x = 0$, where the three-vectors $A$ and $B$ are given.

As represented schematically in Fig. 1, the solution to this problem is well known and is given by the ordinary vector triple product:

$$x(t) = t \ A \times (B \times A), \quad (2.1)$$

where $t \in \mathbb{R}$ is an arbitrary real parameter. Evidently, the optimal (maximal) direction is given by the unit-norm vector,

$$\hat{x} = \frac{A \times (B \times A)}{||A \times (B \times A)||}, \quad (2.2)$$
Figure 1: Three dimensional example that illustrates the geometric construction of the optimal direction for $B \cdot x$, subject to the constraint $A \cdot x = 0$. The optimal direction is given by the vector $x(t)$, which is the intersection of the indicated plane perpendicular to the vector $A$ with the plane defined by the vectors $B$ and $A$.

provided that $A \times B \neq 0$, i.e. the 3-vectors $A$ and $B$ are not parallel to each other. If $A$ and $B$ are parallel, the solution is not unique, given by $\hat{x} = A \times N / ||A \times N||$, where $N$ is an arbitrary 3-vector non-parallel to $A$ or $B$, i.e. $N \cdot A \neq 0$. In this case, the unit-norm vector $\hat{x}$ lies on a plane perpendicular to both $A$ and $B$, and $B \cdot \hat{x} = 0$.

If $A$ and $B$ are not parallel to each other and the norm of the vector $x(t)$ is fixed to be $||x(t)|| = R$ at $t = t_\ast > 0$, then the largest value of $B \cdot x$ at radius $R$ is determined by

$$B \cdot x(t_\ast) = t_\ast ||A \times B||^2 = t_\ast \left[ ||A||^2 ||B||^2 - (A \cdot B)^2 \right].$$

Likewise, the minimum of $B \cdot x$ at radius $R$ occurs at $t = -t_\ast$.

### 3 A Higher-Dimensional Analytic Construction

The above geometric construction for the 3D example can be generalized to higher dimensions, where $B$ is an $n$-dimensional real vector and $[A]$ is an $m \times n$-dimensional real matrix (with $0 \leq m < n$) that imposes a number $m$ of real constraints on the $n$-dimensional real vector $x$ through the condition $[A] \cdot x = 0$, with $0$ being the null vector in $m$ dimensions.

Our analytic construction proceeds along the lines of the 3D example. In detail, in analogy to the 3D vector $A$, we define the $m$-form $A^{(m)}$ through the $m$-fold exterior or
wedge product:

\[(A^{(m)})_{\alpha_1\alpha_2...\alpha_m} = (A_1 \wedge A_2 \wedge \ldots \wedge A_m)_{\alpha_1\alpha_2...\alpha_m} = A[1\alpha_1 A_2\alpha_2 \ldots A_m\alpha_m], \quad (3.4)\]

where the Greek indices \(\alpha_1,2,\ldots,m = 1,2,\ldots,n\) label the columns of the \(m \times n\)-dimensional matrix \([A] = (A_1, A_2, \ldots, A_m)^T\). The square brackets on the RHS of (3.4) indicate that the \(m\)-rank tensor \((A^{(m)})_{\alpha_1\alpha_2...\alpha_m} \equiv A_{\alpha_1\alpha_2...\alpha_m}\) is fully antisymmetrized in all Greek indices, entailing that \(A^{(m)}\) is an \(m\)-form.

Correspondingly, the analogue of the direction \(A \times B\), which determines the normal to the plane defined by \(A\) and \(B\) in the 3D example of Section 2, is the \((n - m - 1)\)-form:

\[
(C^{(n-m-1)})_{\gamma_1\gamma_2...\gamma_{(n-m-1)}} = \varepsilon_{\alpha_1\alpha_2...\alpha_m\beta_1\beta_2...\beta_{n-m-1}} A_{\alpha_1\alpha_2...\alpha_m} B_{\beta_1\beta_2...\beta_{n-m-1}}, \quad (3.5)
\]

where summation over repeated indices is implied and \(\varepsilon_{\alpha_1\alpha_2...\alpha_m\beta_1\beta_2...\beta_{n-m-1}}\) is the usual fully antisymmetric Levi–Civita tensor generalized to \(n\) dimensions. In the language of differential forms, \((C^{(n-m-1)})_{\gamma_1\gamma_2...\gamma_{(n-m-1)}} \equiv C_{\gamma_1\gamma_2...\gamma_{(n-m-1)}}\) is, up to an irrelevant overall factor, the Hodge-dual product between the 1-form \((B^{(1)})_{\beta} \equiv B_{\beta}\), representing the components of the \(n\)-dimensional real vector \(B\), and the \(m\)-form \(A^{(m)}\), i.e.,

\[
C^{(n-m-1)} = \star (B^{(1)} \wedge A^{(m)}), \quad (3.6)
\]

where \(\star\) denotes the standard Hodge-star operation applied within the entire \(n\)-dimensional vector space.

The components \(x_\alpha\) of the optimal \(n\)-dimensional real vector \(x\) are given by the Hodge-dual product of the \(m\)-form \((A^{(m)})_{\alpha_1\alpha_2...\alpha_m} \equiv A_{\alpha_1\alpha_2...\alpha_m}\) and the \((n - m - 1)\)-form \((C^{(n-m-1)})_{\gamma_1\gamma_2...\gamma_{(n-m-1)}} \equiv C_{\gamma_1\gamma_2...\gamma_{(n-m-1)}}\), which is the 1-form:

\[
x_\alpha(t) = t' \varepsilon_{\alpha \alpha_1\alpha_2...\alpha_m\gamma_1\gamma_2...\gamma_{(n-m-1)}} A_{\alpha_1\alpha_2...\alpha_m} C_{\gamma_1\gamma_2...\gamma_{(n-m-1)}}
\]

\[
= t \varepsilon_{\alpha \alpha_1\alpha_2...\alpha_m\gamma_1\gamma_2...\gamma_{(n-m-1)}} \varepsilon_{\beta_1\beta_2...\beta_{m}\gamma_1\gamma_2...\gamma_{(n-m-1)}} 
\times A_{1\alpha_1} A_{2\alpha_2} \ldots A_{m\alpha_m} B_{\beta_1\beta_2} A_{1\beta_1} A_{2\beta_2} \ldots A_{m\beta_m}, \quad (3.7)
\]

where \(t\) and \(t'\) are arbitrary real parameters. Equation (3.7) represents the central result of this note, and can be equivalently cast into the more compact form:

\[
x(t) = t' \star (A^{(m)} \wedge C^{(n-m-1)}) = t \star [A^{(m)} \wedge \star (B^{(1)} \wedge A^{(m)})]. \quad (3.8)
\]

We emphasize the complete analogy between the general solution given by the RHS of (3.8) and the 3D result of (2.1).

By construction, the \(n\)-dimensional 1-form vector \(x_\alpha(t) \equiv x_\alpha(t)\) satisfies the linear system of constraints: \([A] \cdot x = 0\). Within a fixed given radius \(||x(t_*)|| = R\) at \(t = t_* > 0\), the maximum value of \(B \cdot x(t_*)\) is given by

\[
B \cdot x(t_*) = t_* \varepsilon_{\alpha \alpha_1\alpha_2...\alpha_m\gamma_1\gamma_2...\gamma_{(n-m-1)}} \varepsilon_{\beta_1\beta_2...\beta_{m}\gamma_1\gamma_2...\gamma_{(n-m-1)}} 
\times B_{\alpha} A_{1\alpha_1} A_{2\alpha_2} \ldots A_{m\alpha_m} B_{\beta_1} A_{1\beta_1} A_{2\beta_2} \ldots A_{m\beta_m}. \quad (3.9)
\]
The minimum value is obtained correspondingly as \( t = -t_\ast \). Obviously, a non-zero value for \( \mathbf{B} \cdot \mathbf{x}(t_\ast) \) is obtained, iff the \( n \)-dimensional vectors \( A_{1\alpha}, A_{2\alpha}, \ldots, A_{m\alpha} \) and \( B_\beta \) are all linearly independent of each other.

**Proof.** We now present a simple proof of the analytic construction given in Eq. (3.9). To this end, we consider the non-trivial case, where the \( n \)-dimensional vectors \( A_{1\alpha}, A_{2\alpha}, \ldots, A_{m\alpha} \) and \( B_\beta \) span a non-degenerate \((m+1)\)-dimensional subspace. Consequently, a linear transformation represented by the \( m \times m \) matrix \([\mathbf{R}]\) can be performed on the left of the \( m \times n \)-dimensional matrix \([\mathbf{A}]\), i.e. \([\mathbf{A}'] = [\mathbf{R}] \cdot [\mathbf{A}]\), such that \([\mathbf{A}'] \cdot \mathbf{x}(t_\ast) = \mathbf{0}\) and \( A'_{k\alpha} A'_{l\alpha} = \text{const.} \delta_{kl} \), with \( k, l = 1, 2, \ldots, m \). Specifically, this linear transformation exemplifies the usual Gram–Schmidt approach [5] to orthogonalizing a set of \( m \) linearly-independent vectors which span the same \( m \)-dimensional subspace as the original vectors. In addition, the complete \( n \)-dimensional space can be rotated by an orthogonal transformation, i.e., \( \mathbf{x}(t_\ast) \rightarrow \mathbf{x}'(t_\ast) = [\mathbf{O}] \cdot \mathbf{x}(t_\ast)\), without affecting the norm of all \( n \)-dimensional vectors, such that \( A''_{k\alpha} = a_k \delta_{ka} \). Here, \( a_k \) are positive constants that give the norms of the orthogonalized vectors \( \mathbf{A}'_k \) which are equal to those of the orthogonally-rotated ones \( \mathbf{A}''_k \). In this linear basis, only the components of \( B_\beta \) lying in the complementary dimensions \( m+1, m+2, \ldots, n \) give a non-zero contribution in Eq. (3.9). If we denote this orthogonally projected vector with \( B_\beta^\perp \), the linear expression \( \mathbf{B} \cdot \mathbf{x}(t_\ast) \) becomes

\[
\mathbf{B} \cdot \mathbf{x}(t_\ast) = t_\ast \prod_{k=1}^{m} a_k^2 \sum_{\alpha=m+1}^{n} (B'^\perp_{\alpha})^2.
\]  

(3.10)

This represents indeed the largest value for the above linear expression. Note that the optimal direction \( \mathbf{x}(t) \) is parallel to the reduced vector: \( \mathbf{B}^\perp \).

### 4 Conclusions

The problem studied in this note deals with a specific class of problems within the wider context of linear programming. It has been shown that the optimal unit-norm direction \( \hat{\mathbf{x}} \) of the objective \( \mathbf{B} \cdot \hat{\mathbf{x}} \), subject to a system of linear constraints of the form \([\mathbf{A}] \cdot \mathbf{x} = \mathbf{0}\), is given by (3.7), where \([\mathbf{A}]\) is an \( m \times n \)-dimensional real matrix. In particular, the optimal solution can be expressed in terms of wedge and Hodge-dual products of differential forms, as stated in (3.8). The latter provides an explicit geometric connection of the 3D result in (2.1) with the general \( n \)-dimensional solution given in (3.8).

A straightforward extension of the problem studied in the present note would be to consider a complexification of the constraints, i.e., \([\mathbf{A}]\) is an \( m \times n \) complex matrix, in which case \( \mathbf{x} \) is a complex \( n \)-dimensional vector. Such an extension makes sense if the objective is a real number, e.g., the real or imaginary part of the expression \( \mathbf{B} \cdot \hat{\mathbf{x}} \). This problem can
be solved along the lines of the presented method, where $x$ is treated as $2n$-dimensional real vector and turning the complex $m \times n$ matrix $[A]$ into a $2m \times 2n$ real matrix.

The present geometric solution may suggest the existence of a more profound connection between differential forms and linear programming, beyond the limitations of the specific problem under study. It may therefore be interesting to pursue this direction of mathematical research in linear programming with greater vigour in future. Likewise, it would be interesting to explore the potential applications of this geometric solution to a plethora of problems related to physics, computer science, engineering, biology, actuarial science and economics.

Acknowledgements

We thank Luis Alvarez-Gaumé and Nikolaos Papadopoulos for useful comments and suggestions, as well as Frank Deppisch for a critical reading of the note.
References

[1] James K. Strayer, “Linear Programming and Its Applications,” Springer-Verlag, New York, 1989.

[2] David Hestenes and Garret Sobczyk, “Clifford Algebra to Geometric Calculus: a Unified Language for Mathematics and Physics,” Reidel, Dordrecht, 1984.

[3] Samuel I. Goldberg, “Curvature and Homology,” Academic Press, New York, 1982.

[4] Sean M. Carroll, “Spacetime and Geometry: An Introduction to General Relativity,” Addison-Wesley Publishing Company, Reading, 2003.

[5] Serge Long, “Linear Algebra,” Second Edition, Addison-Wesley Publishing Company, Reading, 1970.

[6] For six- and seven-dimensional particle-physics examples that motivated the present note, please see: J. Ellis, J. S. Lee and A. Pilaftsis, arXiv:1006.3087 [hep-ph].