Periodic discrete conformal maps.

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1 Introduction.

Recently there has been much interest in the theory of discrete surfaces in 3-space and its connection with the discretization of soliton equations (see e.g. [4, 11] and references therein). In this article we study a discrete geometry which is the simplest example for both theories. Following [1, 3] we will define a discrete conformal map (DCM) to be a map $z : \mathbb{Z}^2 \to \mathbb{P}^1$ with the property that the cross-ratio of each fundamental quadrilateral is the same. Specifically, for four points $a, b, c, d$ on $\mathbb{P}^1$ define their cross-ratio to be

$$[a : b : c : d] = \frac{(a - b)(c - d)}{(b - c)(d - a)}.$$

Then $z : \mathbb{Z}^2 \to \mathbb{P}^1$ is discrete conformal when

$$[z_{k,m+1} : z_{k,m} : z_{k+1,m} : z_{k+1,m+1}] = q \quad (1)$$

for some constant $q \neq 0, 1, \infty$ for all $k, m$.

![Figure 1: The points about $z_{k,m}$ with neighbours joined by edges.](image)

The motivation for this definition is that if $z : \mathbb{R}^2 \to \mathbb{C}^1$ is smooth then it is (weakly) conformal precisely when

$$\lim_{\epsilon \to 0} [z(x, y + \epsilon) : z(x, y) : z(x + \epsilon, y) : z(x + \epsilon, y + \epsilon)] = -1,$$

i.e. when $z^2_y/z^2_x = -1$. Moreover, Nijhoff & Capel [11] have shown that one can think of the equations (1) as being a discretization of the Schwarzian KdV (SKdV) equations, hence this geometry should be ‘integrable’ in some appropriate sense. From another perspective, Bobenko [1] has shown that
all the circle patterns of Schramm [3] correspond to DCM’s with cross-ratio −1. However, as we shall see, we achieve greater insight by allowing the cross-ratio to be any complex value.

Our main aim is to show that all periodic DCM’s (i.e. $z_{k+n,m} = z_{k,m}$ for some $n$ and all $k, m$) can be constructed using methods which are straight from integrable systems theory, viz, by relating each such map to a linear flow on the Jacobi variety of a compact Riemann surface $\Sigma$ (or more generally, an algebraic curve). Recall that this is the moduli space of degree zero holomorphic line bundles over the curve: it is a complex manifold with the structure of an abelian group and its dimension equals the genus of $\Sigma$. In our case the flow is discrete so by ‘linear’ we mean the flow is a map of $\mathbb{Z}^2$ into this Jacobian which is essentially a homomorphism (generically the map is a zigzag i.e. a homomorphism on a subgroup of index two, but each of these is just a deformation of a homomorphism). We show that every periodic DCM is determined, uniquely up to Möbius equivalence, by its spectral data, which consists of: a compact hyperelliptic Riemann surface $\Sigma$ (which may be singular) equipped with three marked points $O, S, Q$; a degree two rational function $\lambda$ on $\Sigma$ for which $\lambda(O) = 0, \lambda(S) = 1, \lambda(Q) = q$; and, a degree $g+1$ line bundle $L$ over $\Sigma$ satisfying a non-speciality condition, where $g$ is the genus of $\Sigma$.

The spectral data arises by considering the DCM as the ‘conformal flow’ of a periodic discrete curve i.e. each of the discrete periodic curves $\Gamma_{k,m} = \langle z_{k,m}, z_{k+1,m}, \ldots, z_{k+n−1,m} \rangle$ is considered to be the evolution of the initial curve $\Gamma_{0,0}$ according to the cross-ratio condition. Given $\Gamma_{0,0}$ and a cross-ratio $q$ one asks the question: what condition must a point $z \in \mathbb{P}^1$ satisfy for it to be a neighbour of $z_{0,0}$ in this flow? This is a question about the fixed points of a composite of Möbius transformations as we go around $\Gamma_{0,0}$. We call this composite the holonomy $H_{0,0}$ of the closed curve $\Gamma_{0,0}$. By treating the cross-ratio as a parameter (which we re-label $\lambda$) the holonomy becomes a rational function of $\lambda$ with values in $\mathbb{P}GL_2$. The fixed points of $H_{0,0}$ are the eigenlines of its matrix representation: these vary with $\lambda$. The characteristic polynomial of this matrix determines $\Sigma$ while $L$ is the dual of its bundle of eigenlines. As a result, the $\mathbb{P}^1$ in which the discrete map takes values gets identified with the projective space $\mathbb{P}\Gamma(\mathcal{L})^*$ of hyperplanes (i.e. dual lines) in $\Gamma(\mathcal{L})$, the space of globally holomorphic sections of $\mathcal{L}$.

When we do the same construction for $\Gamma_{k,m}$ we obtain another holonomy matrix, $H_{k,m}$, with its spectral curve and line bundle $\mathcal{L}_{k,m}$. But $H_{k,m}$ is conjugate to $H_{0,0}$ by a matrix of rational functions of $\lambda$, so the spectral curves
are isomorphic. Moreover, since the conjugacy maps eigenlines to eigenlines we obtain a rational section of the degree zero line bundle $\text{Hom}(\mathcal{L}_{k,m}; \mathcal{L}_{0,0}) \simeq \mathcal{L}_{0,0} \otimes \mathcal{L}_{k,m}^{-1}$. We can explicitly compute the divisor $D_{k,m}$ of poles and zeroes of this section. In the simplest case, where $\lambda = 0$ is a branch point, $D_{k,m}$ is the divisor $k(S - O) + m(Q - O)$ whence the conformal flow ‘linearises’ on the Jacobian of $\Sigma$.

However, the periodicity condition requires a little more: $n(S - O)$ must be the divisor of a rational function on the singularisation $\Sigma'$ of $\Sigma$ obtained by identifying the two points over $\lambda = \infty$. This suggests that we should think of the linearised flow as taking place on the generalised Jacobian $J'$ for this singular curve. This leads us to a fairly elegant formula for periodic DCM’s involving the $\theta$-function for $\Sigma$ pulled back to $J'$. This is analogous to the formula found in [5] for discrete surfaces of negative Gaussian curvature.

This much is contained in sections 2 and 3. Section 2 treats the holonomy matrix for a discrete curve and derives the spectral data. For simplicity we assume that $\Sigma$ is a non-singular curve (we show in the Appendix that this is the generic case). Section 3 applies this to the construction of periodic DCM’s and proves that the spectral data $(\Sigma, \lambda, \mathcal{L}, O, S, Q)$ characterizes the DCM uniquely in its Möbius equivalence class. We give the explicit formula for $z_{k,m}$ in terms of the $\theta$-function and show that these maps will have singularities whenever the flow passes through (a certain translate of) the $\theta$-divisor. These singularities manifest as the collapse of all four neighbours of a point $z_{k,m}$ onto that point: in this case the cross-ratio breaks down in the adjacent quadrilaterals. We also give a geometric interpretation for the $\theta$-function formula which supports the view that the Schwarzian KdV equations are (one) continuum limit of the equations (1). Geometrically this limit is very easy to describe. Let $\mathcal{A}' : \Sigma' \to J'$ be the Abel map for $\Sigma'$, then the SKdV limit arises as the secant $\tilde{O}S$ (on $\mathcal{A}'(\Sigma')$) tends to the tangent line at $O$ while $\tilde{O}Q$ tends to the third derivative $\partial^3 \mathcal{A}'/\partial \zeta^3$ at $O$ (where $\zeta$ is a local parameter about $O$). Finally, we compute some examples in the case where $\Sigma$ is a rational nodal curve. These we interpret as the soliton solutions for this theory: recall that the soliton solutions of the KdV equation have rational nodal spectral curves. Indeed, computer investigations show that the multi-soliton solutions behave like superposed 1-solitons (in this geometry 1-solitons can be distinguished by their rotational symmetries).

Section 4 addresses a different aspect of the integrable nature of this geometry: the existence of the Lax pair found by Nijhoff & Capel [11]. This,
together with the fact that DCM’s naturally occur in 1-parameter families indexed by $\lambda$, suggests that we can apply the loop group dressing action theory known for KdV [14, 16] and SKdV [10]. This is achieved without much difficulty and we show that the appropriate dressing action preserves the cross-ratio of the map. We concentrate on describing the corresponding ‘dressing orbit of the vacuum solution’. The term ‘vacuum solution’ comes from soliton theory and means the most elementary solution. In our case there is a vacuum solution for every cross-ratio: each is the tiling of the plane by a given parallelogram. We show that each dressing orbit of a vacuum solution is infinite dimensional and their union contains every periodic DCM which is not too great a perturbation of its continuum limit (by comparison, one knows from [14] that every solution of KdV arising from a spectral curve lies in the dressing orbit of the vacuum). The dressing construction also gives many DCM’s which are not periodic and we compute a simple example: the discrete cubic.

Finally, we prove a result about the connection between Darboux (Bäcklund) transforms of the KdV equation and DCM’s. In [2] Bobenko argues that whenever a smooth integrable geometry possesses a discrete infinite group of Bäcklund transforms this family can be taken to be an integrable discretization of the smooth geometry. We show that each DCM in the dressing orbit of the vacuum solution corresponds to a $\mathbb{Z}^2$ family of Darboux transforms of a solution of the KdV equation. In the case of finite type (i.e. those DCM’s possessing a spectral curve) these Darboux transforms are precisely the ones described in [6] as those preserving the spectral curve of the KdV solution.

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2 Discrete Curves in $\mathbb{P}^1$.

2.1 Preliminaries.

Let $\Gamma = (z_0, \ldots, z_{n-1})$ be distinct ordered points in $\mathbb{P}^1$ (and to avoid trivial-
λ-sphere $\mathbb{P}^1_\lambda$ (i.e. $\mathbb{P}^1$ with an affine coordinate $\lambda$) into the Möbius group,

$$\mathbb{P}^1_\lambda \to \text{PGL}_2, \quad \lambda \mapsto T^\lambda_k,$$

by requiring that, for all $z \in \mathbb{P}^1 \setminus \{z_k, z_{k+1}\}$, the cross-ratio condition

$$[z : z_k : z_{k+1} : T^\lambda_k(z)] = \lambda$$

is satisfied. A simple calculation shows that $T^\lambda_k$ is invertible except at $\lambda = 0, 1$ and that we can represent it in $\mathfrak{gl}_2$ by

$$T^\lambda_k = I - \lambda^{-1}A_k$$

where $A_k$ is the projection matrix with kernel $z_k$ and image $z_{k+1}$ (thinking of these as lines in $\mathbb{C}^2$).

We now introduce the holonomy of $\Gamma$ at the base point $z_0$ (see Figure 2):

$$H^\lambda_0 = T^\lambda_{n-1} \circ T^\lambda_{n-2} \circ \ldots \circ T^\lambda_0.$$  

We will use this notation for both the map into $\text{PGL}_2$ and the matrix representation corresponding to (3).

**Lemma 1** Let $d$ denote the degree of $H^\lambda_0$ (in $\lambda^{-1}$). Then $d \leq n/2$ for $n$ even and $d \leq (n+1)/2$ for $n$ odd.

**Proof.** Since $A_{k+1}A_k = 0$ the highest order term that can possibly appear in the expansion $H^\lambda_0 = I + h_1\lambda^{-1} + \ldots + h_d\lambda^{-d}$ is given by

$$h_d = \begin{cases} 
\lambda^{-n/2}(A_{n-2}B_1A_0 + A_{n-1}B_2A_1 + A_{n-1}B_3A_0) & \text{for } n \text{ even;} \\
\lambda^{-(n+1)/2}(A_{n-1}A_{n-3} \ldots A_0) & \text{for } n \text{ odd;}
\end{cases}$$

where the $B_j$ are some composites of the $A_k$. $\square$
2.2 Spectral Data.

The spectral data will be computed from the trace free part of the holonomy matrix. Set \( p(\lambda) = \text{Tr}(H^\lambda_0)/2 \); then one readily sees that \( p(\lambda) \) is a polynomial in \( \lambda^{-1} \) of degree \( d \) with \( p(\lambda) = 1 + p_1 \lambda^{-1} + \ldots + p_d \lambda^{-d} \).

**Remark.** For \( n \) odd we observe from the formula above that \( z_0 \) is both the kernel and image of \( h_d \), whence \( h_d^2 = 0 \) i.e. it is nilpotent. Therefore \( p_d = 0 \) for \( n \) odd.

Define
\[
M^\lambda_0 = \lambda(H^\lambda_0 - p(\lambda)I) \tag{5}
\]
This is a trace-free matrix polynomial in \( \lambda^{-1} \) of degree \( \leq d - 1 \). Set \( m(\lambda) = \det(M^\lambda_0) \): this is a polynomial in \( \lambda^{-1} \) with leading order term \( \lambda^{2-2d} \det(h_d - p_d I) \), so by the previous remark we see that \( \deg(m(\lambda)) \) is at most \( 2d - 2 \) for \( n \) even and \( 2d - 3 \) for \( n \) odd. By the lemma this means that for any \( n \) this degree is at most \( n - 2 \). From now on we will make a genericity assumption: we will assume that \( \deg(m(\lambda)) = n - 2 \), that \( m(\infty) \neq 0 \) and that \( m(\lambda) \) has distinct roots. Since the map \((z_0, \ldots, z_{n-1}) \mapsto \det(M^\lambda_0)\) is algebraic it is clear that the generic discrete curves occupy a Zariski open subset of \( \{(z_0, \ldots, z_{n-1}) : z_i \neq z_j\} \). We will show in the appendix that it is not empty, so such discrete curves exist and are indeed generic. With these assumptions we have \( d = n/2 \) for \( n \) even, \( d = (n + 1)/2 \) for \( n \) odd.

Define the spectral curve to be the isomorphism class \( \Sigma \) of the curve
\[
\Sigma_0 = \{(\lambda, [v]) \in \mathbb{P}^1 \times \mathbb{P}^1 : M^\lambda_0[v] = [v]\}.
\]
In this notation \([v]\) denotes the line through \( v \in \mathbb{C}^2 \).

**Proposition 1** This construction makes \( \Sigma \) a complete non-singular hyper-elliptic curve of genus \( g = d - 2 \) (equal to \( (n-4)/2 \) for \( n \) even, \( (n-3)/2 \) for \( n \) odd). This curve comes equipped with a rational function \( \lambda \) of degree two and a degree \( g + 1 \) map \( f : \Sigma \to \mathbb{P}^1 \). The function \( \lambda \) is unbranched at \( 1 \) and \( \infty \) but when \( n \) is odd it is branched at \( 0 \).

**Proof.** By the genericity assumption \( \Sigma \) is modelled by the non-singular completion of the affine curve with equation \( \det(\mu I - M^\lambda_0) = \mu^2 + m(\lambda) = 0 \). This is clearly a hyperelliptic curve with hyperelliptic cover \( \lambda : \Sigma \to \mathbb{P}^1 \).

Since we have assumed \( m(\infty) \neq 0 \) there is no branch point at \( \infty \). Since \( \deg(m(\lambda)) = n - 2 \) this cover is branched at \( \lambda = 0 \) (i.e. \( \lambda^{-1} = \infty \)) when \( n \) is odd and we read off the genus from \( \deg m(\lambda) = 2g + 2 \) for \( n \) even and
deg $m(\lambda) = 2g + 1$ for $n$ odd, giving $g = d - 2$. To show $\lambda$ is unbranched at 1 it suffices to observe that $z_1, z_{n-1}$ are distinct eigenlines of $M_0^\lambda$. To see this, simply note that $H_0^1 = (I - A_{n-1}) \circ \ldots \circ (I - A_0)$ and $z_1 = \ker(I - A_0)$ while $z_{n-1} = \text{im}(I - A_{n-1})$.

In $\Sigma \times \mathbb{C}^2$ we have the kernel line bundle of $\mu I - M_0^\lambda$, which is clearly holomorphic. Its projectivisation is a holomorphic map $\Sigma \to \Sigma \times \mathbb{P}^1$ and we obtain $f : \Sigma \to \mathbb{P}^1$ by composing this with projection on the second factor. Clearly $\Sigma_0$ is the image of $\lambda \times f : \Sigma \to \mathbb{P}^1 \times \mathbb{P}^1$. We have to show that this is an embedding. Certainly it is injective, since the eigenlines of $M_0^\lambda$ are distinct away from branch points of $\lambda$. Moreover it is an embedding, for when $d\lambda = 0$ we are at ramification points, which lie over the roots of $m(\lambda)$ (and the point over $\lambda = 0$ when $n$ is odd). By the genericity assumption at these points $M_0^\lambda$ is transverse to the determinant conic, whence its eigenlines have distinct tangents i.e. $df \neq 0$. Finally, since the image curve has genus $g$ in the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$ it must be of type $(2, g + 1)$ (see e.g. [8]) i.e. the degree of $f$ is $g + 1$.

Let us define two triples $(\Sigma, \lambda, f)$ and $(\Sigma', \lambda', f')$, of data of the type in the previous proposition, to be isomorphic if there is an isomorphism $\Sigma \simeq \Sigma'$ which identifies $\lambda$ with $\lambda'$ and equates $f$ with a Möbius transform of $f'$. Then we have the following lemma.

**Lemma 2** Any Möbius transformation of $\Gamma$ leaves the isomorphism class of $(\Sigma, \lambda, f)$ fixed.

**Proof.** Let $\Gamma' = (g_0z_0, \ldots, g_{n-1}z_{n-1})$ for some Möbius transformation $g$. Clearly all the maps/matrices $T^\lambda, H^\lambda, M^\lambda$ are conjugated by $g$ and so we have

$$\Sigma'_0 = \{(\lambda, [v]) \in \mathbb{P}^1 \times \mathbb{P}^1 : gM_0^\lambda g^{-1}[v] = [v]\}$$

whence the map $(\lambda, [v]) \mapsto (\lambda, g[v])$ is an isomorphism between $\Sigma_0$ and $\Sigma'_0$ which identifies $\lambda$ with $\lambda'$ and equates $f$ with a Möbius transform of $f'$. □

Henceforth we will use $(\Sigma, \lambda, f)$ to denote this isomorphism class. The spectral data for the based curve $\Gamma$ is the quintuple $(\Sigma, \lambda, f, O, S)$ where $O, S$ are points on $\Sigma$ such that $(\lambda, f)(O) = (0, z_0)$ and $(\lambda, f)(S) = (1, z_1)$. That such points exist follows from:

**Lemma 3** The points $(0, z_0)$ and $(1, z_1)$ both lie on $\Sigma_0$.

**Proof.** We have already shown that $(1, z_1)$ lies on $\Sigma_0$ in the proof of the previous proposition. Now, given the calculations in the proof of lemma □
we want to show that $z_0$ is an eigenline of $h_d - \text{Tr}(h_d)/2$. When $n$ is odd, $h_d$ is nilpotent with kernel $z_0$, so $(0, z_0)$ is the ramification point over $\lambda = 0$. When $n$ is even we know $h_d$ is a sum of three matrices: the first has kernel $z_0$ and image $z_{n-2}$, the second has kernel $z_1$ and image $z_0$ while the third has kernel $z_0$ and image $z_{n-1}$. It follows that $h_d$ maps the line $z_0$ to itself i.e. it is an eigenline. □

It is much more useful to take, in place of the Möbius class of $f$, the line bundle $L = f^*\mathcal{O}_{\mathbb{P}}(1)$ i.e. the pullback of the hyperplane line bundle over $\mathbb{P}^1$. Indeed, by definition $L$ is the dual to the bundle of eigenlines over $\Sigma$ and therefore contains exactly the information we require. It clearly has degree $g + 1$ since $f$ does. As a result of the next lemma we may recover $f$ up to isomorphism from $L$ as the map $\Sigma \rightarrow P\Gamma(L)^*$ in which $P$ is mapped to the hyperplane $\Gamma(L(-P))$ of all sections vanishing at $P$.

**Lemma 4** $\Gamma(L(-P - \bar{P})) = 0$ for each point $P \in \Sigma$ (where $\bar{P}$ denotes the hyperelliptic involute of $P$). Thus $\dim \Gamma(L) = 2$ and the map $\Sigma \rightarrow P\Gamma(L)^*$; $P \mapsto \Gamma(L(-P))$ separates points in hyperelliptic involution.

**Proof.** Let $V = \mathbb{P}^1 \times \mathbb{C}^2$, we will show that this is isomorphic to the direct image $\lambda_* L$. It follows that $\Gamma(L) = \Gamma(V)$ and a global section of $L$ vanishes at $P + \bar{P}$ precisely when the corresponding section of $V$ vanishes at $\lambda(P)$. Since all global sections of $V$ are constant this will prove the lemma. Observe that the sheaf of local sections of the dual, $V^* \simeq V$, is an $\mathcal{O}_\Sigma$-module: for any local section $\sigma$ of $V^*$ and locally regular function $r(\lambda, \mu)$ on $\Sigma$ we define $r(\lambda, \mu)\sigma = \sigma \circ r(\lambda, \mu)$. Now let $E \subset \Sigma \times \mathbb{C}^2$ denote the eigenline bundle whose dual is $L$, then the natural pairing gives rise to an injective $\mathcal{O}_\Sigma$-module homomorphism of $V^*$ into $\text{Hom}(E, \mathcal{O}_\Sigma)$. Therefore as an $\mathcal{O}_\Sigma$-module $V^* \simeq L(-D)$ for some positive divisor $D$ of degree $d$. But as an $\mathcal{O}_P$-module $V^* \simeq \lambda_* L(-D)$ and $V^*$ has Euler characteristic $\chi(V^*) = 2$, which gives $\chi(L(-D)) = 2$. But in fact $D$ must be the trivial divisor, since by Riemann-Roch $\chi(L(-D)) = (g + 1 - d) + 1 - g$ so $d = 0$, whence $V^* \simeq \lambda_* L$. □

### 2.3 Change of base point.

Given the spectral data we wish ultimately to recover the discrete curve $\Gamma$. We have seen that the point $O$ corresponds to the base point $z_0$ of $\Gamma$ via the line bundle $L$. Here we will show that the change of base point corresponds to moving only the line bundle $L$, not the other spectral data. We examine what happens when the based curve $\Gamma_0 = \Gamma$ is subjected to a cyclic permutation to
give $\Gamma_k = (z_k, z_{k+1}, \ldots, z_{k-1})$. For $\Gamma_k$ we have the corresponding holonomy $H_k^\lambda$ with base point $z_k$ and its trace free part $M_k^\lambda$. Clearly we have the relationship

$$M_{k+1}^\lambda = T_k \circ M_k^\lambda \circ T_k^{-1}. \tag{6}$$

Let the spectral data for $\Gamma_k$ be $(\Sigma_k, \lambda_k, \mathcal{L}_k, O_k, S_k)$. In particular, $f_k(O_k) = z_k$ and $f_k(S_k) = z_{k+1}$. Recall that when $n$ is odd $O_k$ is a ramification point of $\lambda$ and therefore a fixed point of the hyperelliptic involution $P \mapsto \tilde{P}$.

**Proposition 2** For each $k$ we have an isomorphism $(\Sigma_k, \lambda_k) \simeq (\Sigma, \lambda)$ such that $S_k$ is mapped to $S$ but $O_k$ maps to $O$ for $k$ even and $\tilde{O}$ for $k$ odd. Further

$$\mathcal{L}_{k+1} \otimes \mathcal{L}_k^{-1} \simeq O_{\Sigma}(\tilde{O}_k - S).$$

**Proof.** We will construct the isomorphisms $(\Sigma_k, \lambda_k) \simeq (\Sigma_{k+1}, \lambda_{k+1})$ and then deduce the result from these. Fix $k$ and consider the map

$$\begin{array}{c c c}
\Sigma_k & \rightarrow & \Sigma_{k+1} \\
(\lambda, [v]) & \mapsto & (\lambda, [T_kv])
\end{array} \tag{7}$$

which we deduce from (6). Since $T_k$ is invertible except at $\lambda = 0, 1$ this map is certainly biholomorphic off the points over $\lambda = 0, 1$ and equates $\lambda_k$ with $\lambda_{k+1}$. Now we consider $T_k^\lambda$ about $\lambda = 0, 1$, where it has at most simple zeroes or poles.

Set $\eta = \lambda^{-1} - 1$: this is a local coordinate about both points $S_k, \tilde{S}_k$ over $\lambda = 1$. Let $v_\eta = v_0 + \eta v_1 + \ldots$ be the expansion for a locally holomorphic family of eigenvectors for $M_k^\lambda$ about $\eta = 0$. Then from $T_k^\lambda = (I - A_k) - \eta A_k$ we see that

$$T_k^\lambda v_\eta = (I - A_k)v_0 + \eta[(I - A_k)v_1 - A_kv_0] + O(\eta^2).$$

If $[v_0] = z_{k+1} = \text{im} A_k$ this has a simple zero, otherwise it has no zero. Since we may rescale $T_k$ without changing the map (7) we see that replacing $T_k^\lambda$ by $\eta^{-1}T_k^\lambda$ exhibits (7) as a biholomorphic map about $S_k = (1, z_{k+1})$. Further, to see that $S_k$ is mapped to $S_{k+1} = (1, z_{k+2})$ it is enough to see that $\tilde{S}_k$ is not mapped to it. But $\text{im}(I - A_k) = z_k$ so from the expression above $\tilde{S}_k$ is mapped to $(1, z_k)$.

To perform a similar calculation about $\lambda = 0$ we have to consider the two cases: $\lambda = 0$ is a branch or is not a branch. In the latter case $\lambda$ is a
local parameter about each point $O_k, 	ilde{O}_k$. Any locally holomorphic family of eigenvectors for $M_k^\lambda$ has expansion $v_\lambda = v_0 + \lambda v_1 + \ldots$ about $\lambda = 0$, whence

$$T_k^\lambda v_\lambda = (I - \lambda^{-1}A_k)v_\lambda = -\lambda^{-1}A_kv_0 + (v_0 - A_kv_1) + O(\lambda).$$

This has a simple pole unless $[v_0] = z_k = \ker A_k$ i.e. a simple pole only at $\tilde{O}_k$. By replacing $T_k$ with $\lambda T_k$ about $\tilde{O}_k$ we see that (7) is biholomorphic here also. Moreover, since $\text{im} A_k = z_{k+1}$ we see that (7) maps $\tilde{O}_k$ to $O_{k+1}$ and therefore $O_k$ maps to $\tilde{O}_{k+1}$. When $\lambda = 0$ is a branch we choose $\zeta = \sqrt{\lambda}$ to be a local parameter. A locally holomorphic family $v_\zeta = v_0 + \zeta v_1 + \ldots$ of eigenvectors now yields

$$T_k^\zeta^2 v_\zeta = -\zeta^{-2}A_kv_0 - \zeta^{-1}A_kv_1 + O(1).$$

But $[v_0] = z_k$ since there is only one point over $\lambda = 0$ hence $T_k$ has a simple pole at $O_k$. Again, the image of $O_k$ under (7) is $O_{k+1}$ since $z_{k+1} = \text{im} A_k$.

Finally, since $T_k$ maps eigenlines to eigenlines it represents a rational section of $L_k \otimes L_{k+1}^{-1}$ since $L_k$ is the dual of the eigenline bundle of $M_k$. By the discussion above this section has divisor $S_k - \tilde{O}_k$ so $L_k \otimes L_{k+1}^{-1} \simeq O_\Sigma(S_k - \tilde{O}_k)$. But (7) maps $(O_k, \tilde{O}_k)$ to $(\tilde{O}_{k+1}, O_{k+1})$ so we find that $O_k$ is $O$ for $k$ even and $\tilde{O}$ for $k$ odd. This completes the proof. □

Let us define a periodic map $L : \mathbb{Z} \to \text{Jac}(\Sigma)$ into the Jacobi variety (i.e. the group of isomorphism classes of line bundles of degree zero over $\Sigma$) by setting $L_0 = O_\Sigma$ and $L_{k+1} \otimes L_k^{-1} = L_{k+1} \otimes L_k^{-1}$ (with $L_{k+n} = L_k$).

The previous proposition shows that when $n$ is odd this is a homomorphism, whereas when $n$ is even we call it a zigzag since it is only a homomorphism on $2\mathbb{Z}$. In fact in either case

$$O_\Sigma \simeq L_{2n} \simeq O_\Sigma(O - S + \tilde{O} - S)^n \simeq O_\Sigma(\tilde{S} - S)^n$$

(8) using the fact that $S + \tilde{S} \sim O + \tilde{O}$ (linear equivalence). Therefore the divisor $\tilde{S} - S$ is a torsion divisor (in which case $S$ is called a division point on $\Sigma$). Indeed $\tilde{S} - S$ satisfies a slightly stronger condition.

**Lemma 5** The divisor $n(\tilde{S} - S)$ is the divisor of a rational function on $\Sigma$ which takes the same value over the two points $P_\infty, \tilde{P}_\infty$ over $\infty$.

Another way of saying this is to say that $\tilde{S} - S$ is a torsion divisor on the singular curve $\Sigma'$ obtained from $\Sigma$ by identifying the two points at infinity to obtain an ordinary double point.
Proof. By the proof of the previous proposition 2 \( S - (O + \tilde{O}) \) is the divisor of the rational section of \( L_{2j} \otimes L_{-1}^{j+2} \) represented by \( T_{2j+1} \circ T_{2j} \), therefore \( S - \tilde{S} \) is the divisor of \( (1 - \lambda^{-1})^{-1}T_{2j+1} \circ T_{2j} \). Observe that

\[
(H_0^\lambda)^2 = \prod_{j=n-1}^{0} T_{2j+1} \circ T_{2j}
\]

(with the indices counted modulo \( n \)), so \( (1 - \lambda^{-1})^{-n} (H_0^\lambda)^2 \) is a rational section of \( L_0 \otimes L_0^{-1} \) with divisor \( n(S - \tilde{S}) \). But \( H_0^\lambda \) is itself a section of \( L_0 \otimes L_0^{-1} \simeq \mathcal{O}_\Sigma \) and from [3] we see that any section \( v \) of \( L_0^{-1} \) satisfies

\[
H_0^\lambda v = (p + \lambda^{-1} \mu) v,
\]

where we recall that \( p(\lambda) = \text{Tr}(H_0^\lambda)/2 \). Therefore \( H_0^\lambda \) represents \( p + \lambda^{-1} \mu \), which takes the value 1 at any point where \( \lambda^{-1} = 0 \) since \( p(\infty) = 1 \). Therefore \( n(S - \tilde{S}) \) is the divisor for \( (1 - \lambda^{-1})^{-n} (p + \lambda^{-1} \mu)^2 \), which also takes the value 1 wherever \( \lambda^{-1} = 0 \).

\( \blacksquare \)

2.4 Recovery of the discrete curve from its spectral data.

The spectral data \( (\Sigma, \lambda, L, O, S) \) determines each \( L_k \) by proposition [2] if we take \( L_0 = L \). This is enough to give each map \( f_k : \Sigma \to \mathbf{P}^1 \) up to a Möbius transform: we take it from the natural map \( \Sigma \to \mathbf{P}\Gamma(L_k)^* \) which assigns to each point \( P \) the hyperplane \( \Gamma(L_k(-P)) \). By lemma [3] we know \( f_k(O_k) \) gives \( z_k \) upon an appropriate identification of \( \mathbf{P}\Gamma(L_k) \) with \( \mathbf{P}^1 \). So to recover the curve \( \Gamma \) we need only understand how this identification is fixed. Indeed it is clear that since \( \Gamma \) is only to be determined up to Möbius transformation what we really want to see is how each \( \mathbf{P}\Gamma(L_k) \) is identified with, say, \( \mathbf{P}\Gamma(L_0) \). This is achieved by first identifying each space \( \Gamma(L_k) \) with the sum of the two fibres of \( L_k \) over \( \lambda = \infty \). We then interpret \( T_{k}^{\infty} = I \) as identifying these fibres for \( L_k \) with the fibres for \( L_{k+1} \).

More precisely, for each \( k \) let \( \mathcal{E}_k \subset \Sigma \times \mathbf{C}^2 \) denote the eigenline bundle with dual \( L_k \). It follows that any linear form \( e \in (\mathbf{C}^2)^* \), being a global section of \( \Sigma \times (\mathbf{C}^2)^* \), induces a global section \( \sigma_k \) of \( L_k \). Now let \( \tau_k \) denote the section of \( \text{Hom}(\mathcal{E}_k, \mathcal{E}_{k+1}) \simeq L_k \otimes L_{k+1}^{-1} \) corresponding to the map \( T_{k}^\lambda \). Then \( e \circ T_{k}^\lambda \) represents the rational section \( \sigma_{k+1} \otimes \tau_k \) of \( L_k \). Since \( T_{k}^{\infty} = I \) we have the identity

\[
(\sigma_{k+1} \otimes \tau_k)|P = \sigma_k|P \quad \text{for} \quad \lambda(P) = \infty.
\]
Here $\sigma|P$ denotes the section $\sigma$ restricted to $P$. This uniquely determines $\sigma_{k+1}$ given $\sigma_k$ since no global section vanishes at both points over $\lambda = \infty$ (by lemma 4). Thus we have maps

$$t_k : \Gamma(\mathcal{L}_{k+1}) \to \Gamma(\mathcal{L}_k) \quad \text{where} \quad t_k(\sigma)|\infty = (\sigma \otimes \tau_k)|\infty \quad (9)$$

and $\sigma|\infty$ denotes $(\sigma|P_\infty, \sigma|\tilde{P}_\infty)$. This uses the identification

$$\Gamma(\mathcal{L}_k) \to \mathcal{L}_k|P_\infty \oplus \mathcal{L}_k|\tilde{P}_\infty$$

which restricts sections to the two fibres over infinity.

For simplicity let $V_k$ denote the sum of fibres on the right. Notice that to each point $P$ on $\Sigma$ we have a line in $V_k$, by evaluating the section vanishing at $P$ at the two fibres over infinity. The lines for $P_\infty$ and $\tilde{P}_\infty$ are independent and we choose a third point $P$ to fix the identification of $PV_0$ with $\mathbb{P}^1$ by sending these three lines to 0, 1 and 1 respectively. Combining this with the map $t_k$ from (9) gives the identification of $PV_k$ with $\mathbb{P}^1$ for every $k$ (notice this only depends on the divisor of $\tau$ and not its scale). Since $\mathcal{L}$ has no sections which vanish at both $O, \tilde{O}$ any globally holomorphic section of $\mathcal{L}_k(\cdot - O_k)$ has divisor $D_k + O_k$ where $D_k$ is a positive and non-special divisor of degree $g$.

**Lemma 6** Given a discrete curve $\Gamma$ with spectral data as above, let $\psi_k$ be the (unique up to scaling) non-zero rational function on $\Sigma$ with divisor $D_k + E_k - D_0$ where $E_k = \sum_{j=0}^{k-1}(S - O_j)$. Then we recover $\Gamma$, up to a M"obius transform, as the image of the map $z : \mathbb{Z} \to \mathbb{P}^1$ given by $z_k = \psi_k(P_\infty)/\psi_k(\tilde{P}_\infty)$.

**Proof.** Let $\sigma_k$ generate $\Gamma(\mathcal{L}_k(\cdot - O_k))$, then $\sigma_k$ has divisor $D_k + O_k$. According to (9) it determines a line in $V_0$ by evaluating the section $s_k = \sigma_k \otimes \tau_k \otimes \ldots \otimes \tau_0$ at $P_\infty$ and $\tilde{P}_\infty$. The resulting line $[s_k|P_\infty, s_k|\tilde{P}_\infty]$ is then mapped to the line in $\mathbb{P}^1$ with homogeneous coordinates $[(s_k/\sigma)|P_\infty, (s_k/\sigma)|\tilde{P}_\infty]$ where $\sigma$ is any section generating the line $\Gamma(\mathcal{L}_0(\cdot))$ corresponding to our third point $P$, according to the identification of $PV_0$ with $\mathbb{P}^1$ fixed above. Since $\psi_k = s_k/s_0 = (s_k/\sigma)/(s_0/\sigma)$ this rational function determines a M"obius equivalent discrete curve. This function has divisor

$$D_k + O_k + \sum_{j=0}^{k-1}(S - \tilde{O}_j) - D_0 - O_0 = D_k + \sum_{j=0}^{k-1}(S - O_j) - D_0,$$

since $\tilde{O}_j = O_{j+1}$. Notice that we will not have $\psi_k(P_\infty) = 0 = \psi_k(\tilde{P}_\infty)$ since we have assumed that $\mathcal{L}(-P_\infty - \tilde{P}_\infty)$ has no global sections. We will postpone the explicit computation of the $z_k$ until we have introduced discrete conformal maps.
3 Discrete Conformal Maps.

A discrete conformal map is a map $z : \mathbb{Z}^2 \to \mathbb{P}^1$ with the property that

\[
[z_{k,m+1} : z_{k,m} : z_{k+1,m} : z_{k+1,m+1}] = q
\]  

(10)

for some constant $q \neq 0, 1, \infty$ for all $k, m$. We will be principally concerned with discrete conformal maps with one period i.e. we will assume there is an $n$ such that $z_{k+n,m} = z_{k,m}$ for all $k, m \in \mathbb{Z}^2$. In that case we can also think of the map as describing the conformal flow of the discrete curve $\Gamma_{0,0} = (z_{0,0}, \ldots, z_{n-1,0})$.

To each discrete curve $\Gamma_{k,m}$ in this flow let us assign its spectral data $(\Sigma_{k,m}, \lambda_{k,m}, \mathcal{L}_{k,m}, O_{k,m}, S_{k,m})$.

Lemma 7 The point $Q_{k,m} = (q, z_{k,m+1})$ lies on $\Sigma_{k,m}$.

Proof. By (10) we see that $T^q_{k,m}(z_{k,m+1}) = z_{k+1,m+1}$ for all $k, m$. It follows that $H^q_{k,m}(z_{k,m+1}) = z_{k,m+1}$.

As earlier, we use $(\Sigma, \lambda)$ to denote the isomorphism class of $(\Sigma_{0,0}, \lambda_{0,0})$. We define $Q \in \Sigma$ to be the point corresponding to $Q_{0,0}$ on $\Sigma_{0,0}$.

Proposition 3 For each $k, m$ there is an isomorphism $(\Sigma_{k,m}, \lambda_{k,m}) \simeq (\Sigma, \lambda)$ such that $S_{k,m}$ is mapped to $S$, $Q_{k,m}$ is mapped to $Q$ but $O_{k,m}$ is mapped to $O$ for $k + m$ even and $\tilde{O}$ for $k + m$ odd. Further:

\[
\mathcal{L}_{k+1,m} \otimes \mathcal{L}_{k,m}^{-1} \simeq \mathcal{O}_\Sigma(\tilde{O}_{k,m} - S);
\]

\[
\mathcal{L}_{k,m+1} \otimes \mathcal{L}_{k,m}^{-1} \simeq \mathcal{O}_\Sigma(\tilde{O}_{k,m} - Q).
\]  

(11)

The proof of this is identical to the proof of proposition 2 given the next lemma, which tells us how the holonomy changes under the conformal flow. Let us introduce the map $\hat{T}_{k,m} : \mathbb{P}^1 \to PGL_2$ characterised by

\[
[z : z_{k,m} : z_{k,m+1} : \hat{T}_{k,m}^\lambda(z)] = \lambda.
\]

By earlier remarks this has matrix representation

\[
\hat{T}_{k,m}^\lambda = I - \lambda^{-1} \hat{A}_{k,m}
\]

where $\hat{A}_{k,m}$ is the projection matrix with kernel $z_{k,m}$ and image $z_{k,m+1}$. The following lemma tells us how the holonomy evolves as we change the base point (cf. [9] for a similar result about discrete isothermic nets).
Lemma 8 The trace free part $M_{k,m}^\lambda$ of the holonomy for $\Gamma_{k,m}$ evolves according to

$$
M_{k+1,m}^\lambda = T_{k,m}^\lambda \circ M_{k,m}^\lambda \circ (T_{k,m}^\lambda)^{-1};
$$

$$
M_{k,m+1}^\lambda = \hat{T}_{k,m}^{\lambda/q} \circ M_{k,m}^\lambda \circ (\hat{T}_{k,m}^{\lambda/q})^{-1}.
$$

(12)

Proof. The first identity we know from earlier. To prove the second identity it suffices to show that

$$
\hat{T}_{k+1,m}^{\lambda/q} \circ T_{k,m}^\lambda = T_{k,m+1}^\lambda \circ \hat{T}_{k,m}^{\lambda/q},
$$

when (11) holds. If we expand the matrix representations for these maps as functions of $\lambda^{-1}$ we see that this is equivalent to showing that:

(a) $\hat{A}_{k+1,m} A_{k,m} = A_{k,m+1} \hat{A}_{k,m}$, and

(b) $A_{k,m} + q \hat{A}_{k+1,m} = A_{k,m+1} + q \hat{A}_{k,m}$.

In (a) it is clear that on both sides the image of the first matrix is the kernel of the second, hence both sides are identically zero. For (b) we can compute the matrices explicitly. But this can be made easier by first mapping $(z_{k,m+1}, z_{k,m}, z_{k+1,m}, z_{k+1,m+1})$ to $(\infty, 1, 0, q)$ by Möbius transform. If we lift $z \in \mathbb{P}^1$ to $(z, 1)^t$ (or $(1, 0)^t$ when $z = \infty$), then elementary calculations show that:

$$
A_{k,m} = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}; \quad q \hat{A}_{k+1,m} = \begin{pmatrix} q & 0 \\ 1 & 0 \end{pmatrix};
$$

$$
A_{k,m+1} = \begin{pmatrix} 0 & q \\ 0 & 1 \end{pmatrix}; \quad q \hat{A}_{k,m} = \begin{pmatrix} q & -q \\ 0 & 0 \end{pmatrix}.
$$

The identity required follows immediately. □

By arguments identical to those in the proof of proposition 2 we see that $\hat{T}_{k,m}^{\lambda/q}$ represents a rational section of $L_{k,m+1}^{-1} \otimes L_{k,m}$ with divisor $Q_{k,m} - \hat{O}_{k,m}$ and combining this with proposition 3 we deduce proposition 4. Naturally this means the complete spectral data for a periodic discrete conformal map is the sextuple $(\Sigma, \lambda, L, O, S, Q)$. We will see later that any sextuple $(\Sigma, \lambda, L, O, S, Q)$ possessing the properties of proposition 1 is spectral data for a discrete conformal map.

3.1 Explicit formula for the discrete conformal map.

Given a discrete conformal map with generic spectral data we can write down an explicit formula for it (up to Möbius equivalence) in terms of the Riemann
theta function of $\Sigma$. For this we need the analogue of lemma 6 proved earlier. Recall we identify each $\Gamma(L_{k,m})$ with $\Gamma(L_{0,0})$ in the following way. To $T_{k,m}^\lambda$ and $\hat{T}_{k,m}^\lambda$ we have corresponding sections $\tau_{k,m}$ of $L_{k,m} \otimes L_{k+1,m}^{-1}$ and $\hat{\tau}_{k,m}$ of $L_{k,m} \otimes L_{k,m+1}^{-1}$. Since every section $\sigma \in \Gamma(L_{k,m})$ is determined entirely by its restriction $\sigma|\infty = (\sigma|P_\infty, \sigma|\bar{P}_\infty)$ we may define bijective linear maps

\[ t_{k,m} : \Gamma(L_{k+1,m}) \to \Gamma(L_{k,m}) \quad \text{where} \quad t_{k,m}(\sigma)|\infty = (\sigma \otimes \tau_{k,m})|\infty; \]

\[ \hat{t}_{k,m} : \Gamma(L_{k+1,m}) \to \Gamma(L_{k,m}) \quad \text{where} \quad \hat{t}_{k,m}(\sigma)|\infty = (\sigma \otimes \hat{\tau}_{k,m})|\infty. \] (13)

To write this in terms of global sections we need to introduce the following divisors. First, let $D_{k,m}$ be the unique positive divisor in the linear system of $L_{k,m}(O_{k,m})$. Now define the divisors $E_{k,m}$ by taking $E_{0,0}$ to be trivial and making

\[ E_{k+1,m} - E_{k,m} = S - O_{k,m}, \]

\[ E_{k,m+1} - E_{k,m} = Q - O_{k,m}. \]

**Lemma 9** Up to Möbius equivalence the discrete conformal map with spectral data $(\Sigma, \lambda, L, O, S, Q)$ is given by the map $z : \mathbb{Z}^2 \to \mathbb{P}^1$ for which $z_{k,m} = \psi_{k,m}(P_\infty)/\psi_{k,m}(\bar{P}_\infty)$, where $\psi_{k,m}$ is the (unique up to scaling) rational function on $\Sigma$ with divisor

\[ (\psi_{k,m}) = D_{k,m} + E_{k,m} - D_{0,0}. \] (14)

The proof is the same as for lemma 6.

To explicitly compute the $z_{k,m}$, we need to fix a basis $\{a_j, b_j\}_{j=1}^g$ for the first homology of $\Sigma$ with the standard intersection pairing. We use the a-cycles to fix a dual basis $\{\omega_j\}_{j=1}^g$ of holomorphic one forms and thereby equate $Jac(\Sigma)$ with $\mathbb{C}^g/\Lambda$ where $\Lambda$ is the lattice representing the first homology via integration of the vector $(\omega_1, \ldots, \omega_g)$ over each homology class. Given a base point $B$ on $\Sigma$ we have the Abel map

\[ \mathcal{A} : \Sigma \to \mathbb{C}^g/\Lambda, \quad P \mapsto \int_B^P (\omega_1, \ldots, \omega_g) \mod \Lambda, \]

and, more generally, its extension to divisors by addition i.e. $\mathcal{A}(P + Q) = \mathcal{A}(P) + \mathcal{A}(Q)$. Further, let $\omega_{g+1}$ be the unique meromorphic differential satisfying: (1) $\omega_{g+1}$ is holomorphic except at $P_\infty$ and $\bar{P}_\infty$ where it has simple poles of residue $1/2\pi i$ and $-1/2\pi i$ respectively; (2) its integral around any a-cycle is zero.
Given all this, we define maps $\alpha_j : \mathbb{Z}^2 \to \mathbb{C}$ for $j = 1, \ldots, g + 1$ by setting $\alpha_j(0, 0) = 0$ and

\[
\alpha_j(k + 1, m) - \alpha_j(k, m) = \int_{O_k, m}^S \omega_j;
\]
\[
\alpha_j(k, m + 1) - \alpha_j(k, m) = \int_{O_k, m}^Q \omega_j.
\]

In the formula to follow we will write $\alpha'_{k,m} : \mathbb{Z}^2 \to \mathbb{C}^{g+1}$ for the map whose $j$-th component is $\alpha_j(k, m)$, for $j = 1, \ldots, g + 1$ and $\alpha_{k,m} : \mathbb{Z}^2 \to \mathbb{C}^g$ for its projection onto the first $g$ components. We think of the former as lying over the generalised Jacobian $J'$ of the curve $\Sigma'$ obtained by identifying $P_\infty$ with $\tilde{P}_\infty$ to obtain a node. The group $J'$ may be analytically realised as $\mathbb{C}^{g+1}/\Lambda'$ where $\Lambda'$ represents the first homology of $\Sigma - \{P_\infty, \tilde{P}_\infty\}$, the open variety of smooth points on $\Sigma'$, via integration of the augmented vector $(\omega_1, \ldots, \omega_{g+1})$ (see e.g. [15] p101). This point of view is useful for computing the periodicity conditions.

**Theorem 1** The formula

\[
z_{k,m} = \exp[2\pi i \alpha_{g+1}(k, m)] \frac{\theta(A(P_\infty) + \alpha_{k,m} - A(D_{0,0}) - \kappa)}{\theta(A(\tilde{P}_\infty) + \alpha_{k,m} - A(D_{0,0}) - \kappa)},
\]

where $\kappa$ is the vector of Riemann constants, recovers the discrete conformal map with spectral data $(\Sigma, \lambda, L, O, S, Q)$ up to Möbius transform. This map is periodic (in $k$) with period $n$ precisely when $\alpha'_{k+n,m} \equiv \alpha'_{k,m} \mod \Lambda'$ for some (and hence all) $k, m$.

**Proof.** We begin by computing a function $\psi_{k,m}$ with divisor (14). For any $A \in \Sigma$ let $\eta^A_{k,m}$ be the unique meromorphic differential on $\Sigma$ with zero $a$-periods and simple poles only at $A$ and $O_{k,m}$, where it has residues $1/2\pi i$ and $-1/2\pi i$ respectively. On the universal cover of $\Sigma$ we may define functions $\beta_{k,m}$ by setting $\beta_{0,0} \equiv 0$ and

\[
\beta_{k+1,m}(P) - \beta_{k,m}(P) = \int_{B}^P \eta^A_{k,m};
\]
\[
\beta_{k,m+1}(P) - \beta_{k,m}(P) = \int_{B}^P \eta^Q_{k,m}.
\]

Then define

\[
\psi_{k,m} = \exp(2\pi i \beta_{k,m}(P)) \frac{\theta(A(P) + \alpha_{k,m} - A(D_{0,0}) - \kappa)}{\theta(A(P) - A(D_{0,0}) - \kappa)}
\]

(16)
where every integral from $B$ to $P$ is along the same path. By a standard reciprocity formula for differentials (see e.g. [4])

$$
\oint_{b_j} \eta_{k,m}^A = \int_{O_{k,m}}^A \omega_j.
$$

Using this it is a simple exercise to check that $\psi_{k,m}$ is well-defined on $\Sigma$. By construction, the denominator vanishes precisely on $D_{0,0}$ and the exponential term contributes the divisor $E_{k,m}$. Moreover, by definition $\alpha_{k,m} = A(E_{k,m})$ and $A(D_{k,m}) = A(D_{0,0}) - A(E_{k,m})$ so the numerator vanishes precisely on $D_{k,m}$. Hence $\psi_{k,m}$ has divisor (14). By lemma 9 the discrete conformal map is, up to Möbius transform, given by $\psi_{k,m}(P_\infty)/\psi_{k,m}(\tilde{P}_\infty)$. However, by another reciprocity formula

$$
\int_{O_{k,m}}^A \omega_{g+1} = \int_{\tilde{P}_\infty}^{P_\infty} \eta_{k,m}^A
$$

hence $\beta_{k,m}(P_\infty) - \beta_{k,m}(\tilde{P}_\infty) = \alpha_{g+1}(k,m)$. Hence, up to multiplication by a constant independent of $k,m$, we obtain (15).

For the periodicity, we will give a proof which works even when $\Sigma$ is singular, since we will compute examples with such spectral curves shortly. First recall (from e.g. [14]) that the curve $\Sigma'$ has its own Abel map:

$$
A' : \Sigma - \{P_\infty, \tilde{P}_\infty\} \to J', \quad P \mapsto \int_B^P (\omega_1, \ldots, \omega_{g+1}) \mod \Lambda'.
$$

With this notation we have

$$
\alpha'_{k+n,m} - \alpha'_{k,m} = A'(E_{k+n,m} - E_{k,m}) = \begin{cases} 
A'(nS - \frac{n}{2}O - \frac{n}{2}\tilde{O}), & n \text{ even;} \\
A'(nS - nO), & n \text{ odd.}
\end{cases}
$$

In particular this is independent of $k,m$. Also recall that Abel’s theorem holds for $\Sigma'$ i.e. $A'(D) \equiv 0$ if and only if $D$ is the divisor of a rational function on $\Sigma'$ (equally, $D$ is the divisor of a rational function on $\Sigma$ taking the same value at $P_\infty, \tilde{P}_\infty$). Now the rational function $\psi_{k+n,m}/\psi_{k,m}$ has divisor $D_{k+n,m} - D_{k,m} + E_{k+n,m} - E_{k,m}$ so if $A'(E_{k+n,m} - E_{k,m}) \equiv 0$ then $D_{k+n,m} - D_{k,m}$ is itself the divisor of a rational function on $\Sigma$. But each $D_{k,m}$ is positive of degree $g$ and non-special, therefore $D_{k+n,m} = D_{k,m}$. Thus $\psi_{k+n,m}/\psi_{k,m}$ must take the same value at $P_\infty, \tilde{P}_\infty$, whence $z_{k+n,m} = z_{k,m}$.
On the other hand, we have seen earlier that when the map is periodic the function $p + \lambda^{-1} \mu$ (representing the eigenvalues of the holonomy matrix) has divisor $E_{k+n,m} - E_{k,m}$. Since $H_0^\infty = I$ this function takes the same values at $P_\infty, \tilde{P}_\infty$. □

**Singularities and the $\theta$-divisor.** The formula (15) need not give a discrete conformal map for all $k,m$. Indeed we expect there to be singularities when both translates of the $\theta$-function are zero: this will occur on some codimension two subvariety of $\mathbb{C}^g$. However, and perhaps less obviously, the map will also have singularities whenever $L_{k,m}(-P_\infty - \tilde{P}_\infty)$ fails to be non-special i.e. on some translate of the $\theta$-divisor. It is interesting to see what happens in this circumstance: typically the map fails to be a discrete immersion i.e. adjacent points fail to be distinct (see Figure 3).

![Figure 3: The collapsing of points at a singularity (with close up on the right). Points with the same $m$ are joined by bolder lines.](image)

Let us see why this occurs. Suppose that $L(-P_\infty - \tilde{P}_\infty)$ is special. Since all pairs $P + \tilde{P}$ are linearly equivalent this means the divisor class for $L(-O - \tilde{O})$ contains a positive divisor (of degree $g-1$), $E$, say, and we take $D = E + O$. Now let us define

$$f(P) = \exp(2\pi i \int_P^O \omega_{g+1}) \frac{\theta(A(P) - A(D - P_\infty) - \kappa)}{\theta(A(P) - A(D - \tilde{P}_\infty) - \kappa)}$$

with the base point for the Abel map at $O$. In that case, unless $D - P_\infty, D - \tilde{P}_\infty$ are both special (which is the codimension two condition mentioned
above), this is a well-defined rational function on Σ and a careful comparison with (13) shows that

\begin{align*}
z_{0,0} = f(O), \quad z_{1,0} = f(S), \quad z_{-1,0} = f(\tilde{S}), \quad z_{0,1} = f(Q), \quad z_{0,-1} = f(\tilde{Q}).
\end{align*}

The divisor for \( f \) is \( P_\infty - \tilde{P}_\infty + C - C' \) where \( C, C' \) are the unique positive divisors of degree \( g \) for which \( C - O \sim D - P_\infty \) and \( C' - O \sim D - \tilde{P}_\infty \). But \( D = E + O, \quad O + \tilde{O} \sim P_\infty + \tilde{P}_\infty \) and \( C, C' \) are unique, hence \( C = E + P_\infty \) and \( C' = E + P_\infty \). Therefore the poles and zeroes of \( f \) cancel i.e. \( f \) is constant, so the five points above are identical. In that case the cross-ratio condition breaks down locally.

**Geometric interpretation.** One of the reasons for choosing to write (13) in this form is to exhibit an elegant geometric interpretation which supports the claim (made in [11]) that the discrete conformal map equations are a discretization of the Schwarzian KdV (SKdV) equations:

\begin{align*}
z_t = S(z) z_x, \quad S(z) = z_{xxx} z_x^{-1} - \frac{3}{2} (z_{xxx} z_x^{-1})^2. \tag{17}
\end{align*}

Here \( S(z) \) is the Schwarzian derivative: it is well-known that \( u(x, t) = 2 S(g) \) satisfies the KdV equation. The ‘finite gap’ solutions of (17) are related to the formula (13) in the following way.

Let us assume that in the spectral data the point \( O \) is ramified (i.e. \( O = \tilde{O} \)). It is not hard to see that the projection \( p : \mathbf{C}^{g+1} \to \mathbf{C}^g \) onto the first \( g \) coordinates projects the lattice \( \Lambda' \) onto \( \Lambda \) and therefore it induces a surjective homomorphism \( p : J' \to \text{Jac}(\Sigma) \) whose kernel is \( \Lambda'/\Lambda \simeq \mathbf{C}/\mathbf{Z} \simeq \mathbf{C}^* \). Further, one can compute by means of multipliers that the pullback of the \( \theta \)-line bundle over \( \text{Jac}(\Sigma) \) to \( J' \) has its space of globally holomorphic sections spanned by the infinite collection

\begin{align*}
\{ \theta_k(Z) = \exp(2\pi i k Z_{g+1}) \theta(p(Z) + k A(P_\infty - \tilde{P}_\infty)) : k \in \mathbf{Z}, \quad Z \in \mathbf{C}^{g+1} \}.
\end{align*}

We see, therefore, that if we let \( U_P \) denote \( \mathcal{A}'(P - O) \) then up to a scaling the formula (13) can be re-expressed as

\begin{align*}
z_{k,m} = \frac{\theta_1(k U_S + m U_Q + \tau)}{\theta_0(k U_S + m U_Q + \tau)}
\end{align*}

for some constant \( \tau \in \mathbf{C}^{g+1} \). Geometrically \( U_S \) and \( U_Q \) are the secants \( \tilde{O} \tilde{S} \) and \( O \tilde{Q} \) on \( \mathcal{A}'(\Sigma_0) \) respectively.
On the other hand, if \( U_1, U_3 \in \mathbf{C}^{g+1} \) represent, respectively, the tangent to \( A'(\Sigma_0) \) at \( O \) and its third derivative there (i.e. \( U_1 = (\partial A'/\partial \zeta)(0) \) and \( U_3 = (\partial^3 A'/\partial \zeta^3)(0) \) for the local parameter \( \zeta = \sqrt{\lambda} \)), then

$$z(x, t) = \frac{\theta_1(xU_1 + tU_3 + \tau)}{\theta_0(xU_1 + tU_3 + \tau)}$$

satisfies the SKdV equation (17) (cf. the formula given at the end of [10]). Hence when viewed in the Jacobi variety (that is to say, after the equations have been linearised) the discretization is nothing other than the perturbation of a tangent into a secant on the spectral curve.

### 3.2 Examples.

Here we will present three examples, all based on taking \( \Sigma \) to be a rational nodal curve. The theory works equally well in this case and it is a good deal easier to calculate since the \( \theta \)-functions are simply polynomials in exponentials. Also, the periodicity condition is easy to satisfy and we can obtain discrete maps with any period.

1. **\( \Sigma \) is the Riemann sphere.** Take \( \Sigma \) to be the Riemann sphere equipped with an affine coordinate \( \zeta \). In this case the \( \theta \)-function is a constant which we may as well take to equal 1. We take the hyperelliptic involution to be \( \zeta \mapsto -\zeta \) and prescribe the spectral data by \( \zeta(O) = \epsilon, \zeta(S) = a, \zeta(Q) = b \) and \( \zeta(P_{\infty}) = y \) where these are all distinct and different from 0, \( \infty \). It follows that

$$\lambda = \frac{(a^2 - y^2)(\zeta^2 - \epsilon^2)}{(\zeta^2 - y^2)(a^2 - \epsilon^2)}.$$ 

Since \( g = 0 \) in this case we only need to compute \( \omega_1 \) and its integrals. It is easy to see that \( \omega_1 = \omega_y \) where

$$\omega_y = \frac{1}{2\pi i} \left( \frac{1}{\zeta - y} - \frac{1}{\zeta + y} \right) d\zeta$$

and therefore, following the procedure above, we see that \( z_{k,m} = h_{k,m}(y) \) where

$$h_{k,m}(y) = \exp[2\pi i \alpha_1(k, m)] = \begin{cases} 
    (\frac{a-y}{a+y})^k (\frac{b-y}{b+y})^m (\frac{\epsilon+y}{\epsilon-y}) \quad & k + m \text{ odd}, \\
    (\frac{a-y}{a+y})^k (\frac{b-y}{b+y})^m \quad & k + m \text{ even}. 
\end{cases}$$

(19)
This map has cross-ratio $\lambda(Q)$ and period $n$ whenever $(a + y)/(a - y)$ is an $n$-th root of unity ($n$ must be even if $\epsilon \neq \infty$). Observe that this agrees with the periodicity condition given in theorem [1], for in this example $\Sigma'$ is a one node curve and $J' = \mathbb{C}/\mathbb{Z} \langle f \omega_1 \rangle$, where the integral is around any cycle separating $y$ from $-y$. It is not hard to see that parameter values can be chosen to give any cross-ratio with any period $\geq 4$.

This is the general case of the zigzag: when $\epsilon = \infty$ we have a homomorphism $\mathbb{Z}^2 \to \mathbb{C}^*$. It is clear that this is the discrete exponential function $k, m \mapsto \exp(kA + mB)$ for constants $A, B$.

![Figure 4: The discrete exponential (left) and a zigzag version (right).](image)

2. **$\Sigma$ is a rational curve with one node.** Here we take $\Sigma$ to be $\mathbb{P}^1$ with the points $\pm x \neq 0, \infty$ identified together to obtain a node. In this case the $\theta$-function is given by $\theta(Z) = \exp(2\pi iZ) - 1$. Here the space of regular one forms corresponds to the meromorphic forms on $\mathbb{P}^1$ with simple poles at $x, -x$ only (see [13] p68), hence the space is one dimensional and the (arithmetic) genus is $g = 1$. We keep the same notation as in the previous example and make the same choice for $\lambda$ (so $a, b, x, y, \epsilon$ are all distinct). One easily checks that the appropriate choices for $\omega_1, \omega_2$ in this case are to take $\omega_1 = \omega_x$ and $\omega_2 = \omega_y$ using (13). By computing the integrals and applying the formula (13) we see that we can write

$$z_{k,m} = h_{k,m}(y) \left( \frac{y-x}{y+x} \right) h_{k,m}(x) - e^{2\pi i c} \left( \frac{y+x}{y-x} \right) h_{k,m}(x) - e^{2\pi i c}$$
where $h_{k,m}(y)$ is given by (19) and the constant $c$ represents the terms $A(D_{0,0}) + \kappa$ in (14). In this case $\Sigma'$ is a two node curve with Jacobian
\[ J' = \frac{C}{Z}\langle \int \omega_1, \int \omega_2 \rangle \]
where the integrals are around $x$ and $y$ respectively. According to theorem the map is periodic when both $(a-y)/(a+y)$ and $(a-x)/(a+x)$ are $n$-th roots of unity (distinct, so that $x \neq \pm y$). In this case the periodicity problem can be solved for any period $\geq 5$ for any value of the cross-ratio $\lambda(Q)$.

![Figure 5: Two discrete 1-solitons: showing 2-fold and 3-fold symmetries.](image)

Each of these maps behaves like a discrete 1-soliton in the sense that it has asymptotics in $m$ like the discrete exponential (see Figure 5). Let $B = (b-y)/(b+y)$, then for $|B| < 1$ we have
\[
z_{k,m}/h_{k,m}(y) \to \begin{cases} 1 & \text{as } m \to \infty; \\ (y-x)^2/(y+x)^2 & \text{as } m \to -\infty. \end{cases}
\]
For $|B| > 1$ the limits are interchanged.

3. $\Sigma$ is a rational curve with two nodes. Take $\Sigma$ to be $\mathbb{P}^1$ with $\pm x_1$, $\pm x_2$ identified in pairs. The $\theta$-function here is given by
\[
\theta(Z) = F(e^{2\pi i Z_1}, e^{2\pi i Z_2}), \quad F(X,Y) = \det \begin{pmatrix} X-1 & (Y+1)x_1 \\ Y-1 & (X+1)x_2 \end{pmatrix}.
\]
Figure 6: A discrete 2-soliton: superposition of 2-fold and 3-fold symmetries.

The arithmetic genus is $g = 2$ and $(\omega_1, \omega_2, \omega_3) = (\omega_{x_1}, \omega_{x_2}, \omega_y)$ using $\omega$. Again, with $\lambda$ chosen as above the appropriate computation yields

$$z_{k,m} = h_{k,m}(y) \frac{F((\frac{y-x_1}{y+x_1}) h_{k,m}(x_1) e^{-2\pi i c_1}, (\frac{y-x_2}{y+x_2}) h_{k,m}(x_2) e^{-2\pi i c_2})}{F((\frac{y-x_1}{y+x_1}) h_{k,m}(x_1) e^{-2\pi i c_1}, (\frac{y+x_2}{y-x_2}) h_{k,m}(x_2) e^{-2\pi i c_2})}$$

where $c_1, c_2$ are parameters corresponding to the initial point $A(D_{0,0}) + \kappa$. The periodicity conditions can be solved for any period $\geq 7$ and any cross-ratio $\lambda(Q)$. Each of these maps behaves like a 2-soliton in the sense that: (a) as $m \to \pm \infty$ it behaves like the discrete exponential, and; (b) by suitable choice of $c_1, c_2$ we observe that it behaves like two interacting 1-solitons (see Figure 6).
3.3 Spectral data which produces discrete conformal maps.

The aim of this section is to prove that any choice of spectral data satisfying the conditions described above yields a periodic discrete conformal map.

**Theorem 2** Let \( \Sigma \) be a compact hyperelliptic Riemann surface of genus \( g \) with: a degree two function \( \lambda \) unbranched at \( \lambda = 1, \infty \); a degree \( g+1 \) line bundle \( \mathcal{L} \) for which \( \mathcal{L}(-P_{\infty} - \tilde{P}_{\infty}) \) is non-special; and non-singular points \( O, S, Q \) with \( \lambda(O) = 0, \lambda(S) = 1 \) and \( \lambda(Q) \neq 0, 1, \infty \). Then the data \( (\Sigma, \lambda, \mathcal{L}, O, S, Q) \) is the spectral data for a discrete conformal map \( z : \mathbb{Z} \rightarrow \mathbb{P}^1 \) of cross-ratio \( \lambda(Q) \). This map is periodic with period \( n \) if and only if either: a) \( n \) is even and \( \frac{n}{2}(2S - O - \tilde{O}) \) is the divisor of a rational function on \( \Sigma \) taking the same value at each point over \( \lambda = \infty \); or, b) \( n \) is odd and \( n(S - O) \) is such a divisor.

**Remarks.** (i) Strictly speaking it may be that \( z_{k,m} \) has some singularities. These will occur only when \( \mathcal{L}_{k,m}(-P_{\infty} - \tilde{P}_{\infty}) \) (where \( \mathcal{L}_{k,m} \) is given by (11)) is special i.e. has a non-zero global section. (ii) As we have seen in the examples above, the condition that \( \Sigma \) be smooth can be weakened to allow irreducible singular algebraic curves, provided the points \( O, \tilde{O}, S, Q \) are all smooth. The proof we give works in this generality, in particular, we will not rely on the \( \theta \)-function formula given earlier.

We can immediately deduce from this theorem a simple fact about smooth families of periodic discrete conformal maps.

**Corollary 1** Set \( \Sigma^o = \Sigma - \lambda^{-1}(\{0, 1, \infty\}) \). Up to Möbius equivalence, each discrete conformal map lies in a \( g+1 \)-parameter family \( z(Q, \mathcal{L}) \) parameterised by \( \Sigma^o \times \text{Jac}(\Sigma) \).

Notice that if we swap \( Q \) with its involute \( \tilde{Q} \) we get \( z_{k,m}(\tilde{Q}, \mathcal{L}) = z_{k,-m}(Q, \mathcal{L}) \), since \( \mathcal{O}_\Sigma(\tilde{Q} - O) \cong \mathcal{O}_\Sigma(O - Q) \).

Now let us turn to proving the theorem. We saw in the previous sections that the point \( z_{k,m} \) is the image of the hyperplane \( \Gamma(\mathcal{L}_{k,m}(-O_{k,m})) \subset \Gamma(\mathcal{L}_{k,m}) \), where \( O_{k,m} = O \) for \( k + m \) even and \( \tilde{O} \) otherwise, under an identification

\[
\zeta_{k,m} : \mathbb{P}\Gamma(\mathcal{L}_{k,m})^* \rightarrow \mathbb{P}^1
\]  

of \( \mathbb{P}^1 \) with the space \( \mathbb{P}\Gamma(\mathcal{L}_{k,m})^* \) of hyperplanes in the two dimensional space \( \Gamma(\mathcal{L}_{k,m}) \). This identification is defined as follows. For each \( k, m \) there is, up
to scaling, a unique section $\tau_{k,m}$ of $\mathcal{L}_{k,m} \otimes \mathcal{L}_{k+1,m}^{-1}$ with divisor $S - \hat{O}_{k,m}$. Using (13) this fixes an isomorphism $t_{k,m} : \Gamma(\mathcal{L}_{k+1,m}) \to \Gamma(\mathcal{L}_{k,m})$ which identifies sections by their behaviour over $\lambda = \infty$. Similarly using (13) we fix an isomorphism $\hat{t}_{k,m} : \Gamma(\mathcal{L}_{k,m+1}) \to \Gamma(\mathcal{L}_{k,m})$ by choosing a section $\hat{\tau}_{k,m}$ of $\mathcal{L}_{k,m} \otimes \mathcal{L}_{k,m+1}^{-1}$ with divisor $Q - \hat{O}_{k,m}$. Notice that $\hat{t}_{k,m} \circ t_{k,m+1}$ and $t_{k,m} \circ \hat{t}_{k+1,m}$ differ only by a scaling, since $\hat{\tau}_{k,m} \otimes \tau_{k,m+1}$ and $\tau_{k,m} \otimes \hat{\tau}_{k+1,m}$ differ only by a scaling (they are sections of the same line bundle and have the same divisor). Therefore we obtain, by composition, a well-defined isomorphism $\text{P}(\Gamma(\mathcal{L}_{k,m})) \simeq \text{P}(\Gamma(\mathcal{L}))$. Since the only freedom in the choice of $\tau_{k,m}$ and $\hat{\tau}_{k,m}$ is the scale, which is irrelevant when we pass to the projective spaces, this construction depends only on the spectral data. Finally, by fixing an identification of $\text{P}(\Gamma(\mathcal{L}))$ with $\text{P}^1$ we obtain the maps $\zeta_{k,m}$. Since the last step is not determined by the spectral data the discrete map is only determined up to Möbius equivalence.

At this point we can see why, under the conditions of the theorem, the map $z$ should be periodic. It suffices to show that $\zeta_{k,n,m} = \zeta_{k,m}$. Consider the map $r_{k,m} = t_{k,n+1,m} \circ \ldots \circ t_{k,m}$: by (13) it is characterised by

$$r_{k,m} : \Gamma(\mathcal{L}_{k,m}) \to \Gamma(\mathcal{L}_{k,m}) \quad \text{where} \quad r_{k,m}(\sigma)|_{\infty} = (\sigma \otimes h_{k,m})|_{\infty},$$

for $h_{k,m} = \tau_{k,n-1,m} \otimes \ldots \otimes \tau_{k,m}$. Notice that $h_{k,m}$ is a rational section of $\mathcal{L}_{k,m} \otimes \mathcal{L}_{k+n,m}^{-1}$ with divisor $\frac{n}{2}(2S - O - \hat{O})$ for $n$ even and $n(S - O)$ for $n$ odd. Therefore, under the conditions of the theorem, $h_{k,m}$ is a rational function on $\Sigma$ with $h_{k,m}(P_\infty) = h_{k,m}(\hat{P}_\infty)$. It follows that $r_{k,m}$ is a scalar multiple of the identity whence $\zeta_{k,n,m} = \zeta_{k,m}$.

Now we must show that this recipe produces discrete conformal maps. First we observe the following convenient result.

**Lemma 10** The map $\zeta_{k,m}$ sends $\Gamma(\mathcal{L}_{k,m}(-S))$ and $\Gamma(\mathcal{L}_{k,m}(-Q))$ to $z_{k+1,m}$ and $z_{k,m+1}$ respectively.

**Proof.** It suffices to show that the maps $t_{k,m}$ and $\hat{t}_{k,m}$ in (13) send, respectively, $\Gamma(\mathcal{L}_{k+1,m}(-O_{k+1,m}))$ to $\Gamma(\mathcal{L}_{k,m}(-S))$ and $\Gamma(\mathcal{L}_{k,m+1}(-O_{k,m+1}))$ to $\Gamma(\mathcal{L}_{k,m}(-Q))$. We will prove the former: the latter use the same proof with $S$ replaced by $Q$. Let $\sigma$ generate the line $\Gamma(\mathcal{L}_{k+1,m}(-O_{k+1,m}))$, then $\sigma$ is a holomorphic section with a zero at $O_{k+1,m} = \hat{O}_{k,m}$ and this is where $\tau_{k,m}$ has a simple pole. Hence $\sigma \otimes \tau_{k,m}$ is a holomorphic section of $\mathcal{L}_{k,m}$ with a zero at $S$ i.e. it generates $\Gamma(\mathcal{L}_{k,m}(-S))$. \(\square\)

To prove the theorem we will show that the action of tensoring sections with $\tau_{k,m}$ is represented by a matrix of the form (13) and deduce from this that
the cross-ratios are constant. Since this matrix really acts on the eigenline bundle $E_{k,m}$ dual to $L_{k,m}$ we must take some care to describe the relationship between rational sections of $E_{k,m}$ and global sections of $L_{k,m}$.

**Lemma 11** Let $L$ be a line bundle of degree $g + 1$ over the genus $g$ hyperelliptic curve $\Sigma$ with $\Gamma(L(\omega - P - \tilde{P})) = 0$ (for some $P \in \Sigma$) and let $E$ denote the dual line bundle. Then $\Gamma(L)$ is canonically dual to $\Gamma(E(R))$ where $R$ is the ramification divisor of $\lambda$. This duality identifies the hyperplane $\Gamma(L(-P))$ with the line $\Gamma(E(-P))$.

**Proof.** Let $F$ be the field of rational functions on $X$ and let $K$ denote the subfield of rational functions of $\lambda$. Clearly $F$ is a two dimensional $K$-space and we have a $K$-linear map $\text{Tr} : F \to K$ which gives to each element $f \in F$ the trace of the matrix in $\text{gl}_2(K)$ representing the multiplication map $a \mapsto fa$ on $F$ (since the trace is invariant this is independent of the $K$-basis chosen for $F$). It is easy to check that: (a) $\text{Tr}(f)$ is globally holomorphic precisely when its divisor of poles is no worse than the ramification divisor $R$ of $\lambda$; (b) for any such function $f$, $\text{Tr}(f) = 0$ if its divisor of zeroes includes $P + \tilde{P}$ (for some $P \in \Sigma$). We have a non-degenerate $K$-bilinear form on $F(E) \times F(L)$ by $(v, e) \mapsto \text{Tr}(e(v))$ which, by the properties (a) and (b), pairs $\Gamma(L)$ non-degenerately with $\Gamma(E(R))$. Further, if $e \in \Gamma(L(-P))$ then $\text{Tr}(e(v)) = 0$ if and only if $v \in \Gamma(E(R - P))$.\qed

Since $E(R)$ also has the property $\Gamma(E(R - P - \tilde{P})) = 0$ it follows that $F(E)) = K \otimes C \Gamma(E(R))$. As a result we may draw the following commuting diagram with which we define the map $T_{k,m}$:

$$
\begin{array}{ccc}
F(E_{k,m}) & \overset{\otimes T_{k,m}}{\longrightarrow} & F(E_{k+1,m}) \\
\downarrow & & \downarrow \\
K \otimes \Gamma(L_{k,m})^* & \overset{T_{k,m}}{\longrightarrow} & K \otimes \Gamma(L_{k+1,m})^*
\end{array}
$$

Now when we identify $\Gamma(L_{k,m})^*$ with $C^2$ using any lift of $\zeta_{k,m}$ we obtain a $K$-valued matrix we will define to be $T_{k,m}^\lambda$. The theorem will follow from the next proposition.

**Proposition 4** The matrix $T_{k,m}^\lambda$ obtained from (21) is of the form $I - \lambda^{-1}A_{k,m}$ where: (i) $\ker(A_{k,m}) = z_{k,m}$; (ii) $\text{im}(A_{k,m}) = z_{k+1,m} = \ker(I - A_{k,m})$; and (iii) $I - q^{-1}A_{k,m}$ maps $z_{k,m+1}$ to $z_{k+1,m+1}$ for $q = \lambda(Q)$. It follows that $[z_{k,m+1} : z_{k,m} : z_{k+1,m+1}] = q$ and therefore the map $z$ is discrete conformal.
Proof. To begin, observe that \( T_{k,m}^{\infty} \) represents the map \( \sigma \to \sigma \otimes \tau_{k,m} \) over \( \lambda = \infty \). By (3) this is the identity. Further, since \( \tau_{k,m} \) has degree one it follows that there is a constant matrix \( A_{k,m} \) such that \( T_{k,m}^{\lambda} = I - \lambda^{-1} A_{k,m} \).

To prove (i), (ii) and (iii) we consider three different values of \( \lambda \).

First, \( A_{k,m} = (\lambda T_{k,m}^{\lambda})|_{\lambda=0} \) represents \( \lambda \tau_{k,m} \) over \( \lambda = 0 \). But \( \lambda \tau_{k,m} \) has a simple zero at \( O_{k,m} \) and none at \( \tilde{O}_{k,m} \), hence \( (\sigma \otimes \lambda \tau_{k,m})|_{\lambda=0} = 0 \) if and only if \( \sigma \) vanishes at \( \tilde{O}_{k,m} \). By lemma [1] \( z_{k,m} \) corresponds to \( \Gamma(\mathcal{E}_{k,m}(R - \tilde{O}_{k,m})) \) and \( z_{k+1,m} \) corresponds to \( \Gamma(\mathcal{E}_{k+1,m}(R - O_{k,m})) \) (since \( \tilde{O}_{k+1,m} = O_{k,m} \)) so \( \ker(A_{k,m}) = z_{k,m} \) while \( \text{im}(A_{k,m}) = z_{k+1,m} \).

Secondly, consider \( I - A_{k,m} \), which represents \( \tau_{k,m} \) over \( \lambda = 1 \). But \( \tau_{k,m} \) has a zero at \( S \) so \( (\sigma \otimes \tau_{k,m})|_{\lambda=1} = 0 \) if and only if \( \sigma \) has a zero at \( S \). Recall that \( z_{k+1,m} \) corresponds to \( \Gamma(\mathcal{E}_{k,m}(R - \tilde{S})) \) by lemmas [3] and [4] so \( z_{k+1,m} = \ker(I - A_{k,m}) \).

Thirdly, \( I - q^{-1} A_{k,m} \) represents \( \tau_{k,m} \) over \( \lambda = q \). Here \( \tau_{k,m} \) has neither zeroes nor poles so \( \sigma \otimes \tau_{k,m} \) has a zero at \( \tilde{Q} \) if and only if \( \sigma \) does. Since \( z_{k,m+1} \) corresponds to \( \Gamma(\mathcal{E}_{k,m}(R - \tilde{Q})) \) and \( z_{k+1,m+1} \) corresponds to \( \Gamma(\mathcal{E}_{k+1,m}(R - \tilde{Q})) \) (by lemmas [3] and [4]) we see that \( I - q^{-1} A_{k,m} \) maps \( z_{k,m+1} \) to \( z_{k+1,m+1} \).

Finally, it is an elementary computation (which we will leave to the reader) to establish that for \( A_{k,m} \) to have these properties we are obliged to have \( [z_{k,m+1}: z_{k,m}: z_{k+1,m}: z_{k+1,m+1}] = q. \square \)

4 The Lax pair and a loop group action.

In [11] Nijhoff and Capel wrote down a Lax pair for the discrete conformal map equations. Using this we will show that all discrete conformal maps:

(i) come in 1-parameter families obtained by deforming the cross-ratio (we have already seen this for periodic maps in corollary [1]);

(ii) admit non-trivial transformations by elements of an infinite dimensional Lie group (in fact, a loop group). To begin, let us define \( \mathcal{Z}_q = \{ z : \mathbb{Z}^2 \to \mathbb{C} \} \) discrete conformal with cross-ratio \( q \) and \( z_{0,0} = 0 \). We will say maps of this type are based at 0. Since we are working with maps into the plane we may define

\[
\begin{align*}
  u_{k,m} &= z_{k+1,m} - z_{k,m}, \\
  v_{k,m} &= z_{k,m+1} - z_{k,m}.
\end{align*}
\]

Now fix a pair \( \alpha, \beta \in \mathbb{C} \) for which \( \beta^2 / \alpha^2 = q \) and define

\[
\begin{align*}
  U_{k,m}(\lambda) &= \begin{pmatrix} 1 & \alpha^2 u_{k,m}^{-1} \lambda^{-1} \\ u_{k,m} & 1 \end{pmatrix}, \\
  V_{k,m}(\lambda) &= \begin{pmatrix} 1 & \beta^2 v_{k,m}^{-1} \lambda^{-1} \\ v_{k,m} & 1 \end{pmatrix}.
\end{align*}
\]

(22) (23)
One readily computes that the pair \( U, V \) is a discrete Lax pair i.e.

\[
U_{k,m}V_{k+1,m} = V_{k,m}U_{k,m+1}.
\]  

(24)

We consider these as maps \( Z^2 \to \mathcal{G} \) where \( \mathcal{G} \) is the infinite dimensional Lie group of holomorphic maps from \( P^1_\lambda \backslash \{0, \alpha^2, \beta^2\} \) to \( GL_2 \). By the Lax equations (24) there exists a unique map \( \Phi : Z^2 \to \mathcal{G} \) satisfying:

\[
\Phi_{k+1,m} = \Phi_{k,m}U_{k,m} ; \quad \Phi_{k,m+1} = \Phi_{k,m}V_{k,m} ; \quad \Phi_{0,0} = I.
\]  

(25)

We will call such maps extended frames. Each \( z \in Z_q \) has a unique extended frame \( \Phi \) and \( z \) is recovered from the first column of \( \Phi(\infty) \). To see this simply observe that \( \Phi \) is constructed inductively from (25) and is uniquely specified by the initial condition \( \Phi_{0,0} = I \). One readily checks that

\[
\begin{pmatrix}
1 & 0 \\
z_{k,m} & 1
\end{pmatrix}
\]

satisfies (25) for \( \lambda = \infty \) so, by uniqueness, this must be \( \Phi_{k,m}(\infty) \).

**Example: the vacuum solution.** The simplest example of a discrete conformal map is the tiling of the plane by a parallelogram of cross-ratio \( q \) i.e. 

\[
z_{k,m} = k\alpha + m\beta.
\]

We will adopt a common terminology from soliton theory and call this solution the ‘vacuum solution’: it corresponds to having \( u_{k,m} = \alpha, v_{k,m} = \beta \) for all \( k, m \). In this case it is easy to check that for this map the extended frame is

\[
\Phi_{k,m} = (I + \alpha \Lambda)^k(I + \beta \Lambda)^m , \quad \text{where} \quad \Lambda = \begin{pmatrix} 0 & \lambda^{-1} \\ 1 & 0 \end{pmatrix}.
\]  

(26)

Proposition 5 The map \( z(\lambda) \) is discrete conformal with cross-ratio \( \beta^2(1 - \lambda^{-1} \alpha^2)/\alpha^2(1 - \lambda^{-1} \beta^2) \). If \( z, \tilde{z} \in Z_q \) are Möbius equivalent maps then so are \( z(\lambda), \tilde{z}(\lambda) \) for each \( \lambda \).
Proof. The cross-ratio of the map $z(\lambda)$ may be written using the lift $F(\lambda)$ as
\[
\det(F_{k,m+1}(\lambda) \ F_{k,m}(\lambda)) \det(F_{k+1,m}(\lambda) \ F_{k+1,m+1}(\lambda))
\]
\[
\det(F_{k,m}(\lambda) \ F_{k+1,m}(\lambda)) \det(F_{k+1,m+1}(\lambda) \ F_{k,m+1}(\lambda))
\]
where if $a, b \in \mathbb{C}^2$ then $(a \ b)$ denotes the matrix with these columns. But $F_{k+1,m} = \Phi_{k,m}U_{k,m}e_1$ and $F_{k,m+1} = \Phi_{k,m}V_{k,m}e_1$. Inserting these expressions and using (23), (24) this reduces to
\[
\begin{align*}
\det \left( \begin{array}{cc} 1 & 1 \\ v_{k,m} & 0 \end{array} \right) & \det \left( \begin{array}{cc} 1 & 1 \\ 0 & v_{k+1,m} \end{array} \right) \det U_{k,m} \\
\det \left( \begin{array}{cc} 1 & 1 \\ u_{k,m} & 0 \end{array} \right) & \det \left( \begin{array}{cc} 1 & 1 \\ 0 & u_{k,m+1} \end{array} \right) \det V_{k,m}
\end{align*}
\]
\[
\frac{\beta^2(1 - \lambda^{-1}\alpha^2)}{\alpha^2(1 - \lambda^{-1}\beta^2)},
\]
Now suppose $z, \hat{z}$ are Möbius equivalent. Since they are both based at 0 it suffices to consider two types of Möbius transforms: 1) $\hat{z} = cz$; 2) $\hat{z} = z/(1 - cz)$ for any non-zero constant $c$. We wish to show that in each case there exists $a : \mathbb{Z}^2 \to B_0$ for which
\[
\hat{U}_{k,m} = a_{k,m}^{-1}U_{k,m}a_{k+1,m}, \quad \hat{V}_{k,m} = a_{k,m}^{-1}V_{k,m}a_{k+1,m}.
\]
For then the extended frames $\Phi, \hat{\Phi}$ satisfy $\hat{\Phi}_{k,m} = a_{0,0}^{-1}\Phi_{k,m}a_{k,m}$, whence $\hat{z}(\lambda) = a_{0,0}^{-1} \circ z(\lambda)$ (Möbius action). We leave it to the reader to verify that for the two cases above we can take:
\[
1) \ a_{k,m} = \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}, \quad 2) \ a_{k,m} = \begin{pmatrix} z_{k,m} - c^{-1} & -1 \\ 0 & 1/(z_{k,m} - c^{-1}) \end{pmatrix}.
\]
Remark. Notice that each $z(\lambda)$ will be based at 0, although we cannot guarantee that it maps into $\mathbb{C} \subset \mathbb{P}^1$. The parameterisation of this family $z(\lambda)$ depends upon the choice of $\alpha^2, \beta^2$ but the family itself clearly does not, since a rescaling of $\lambda$ will allow us to move through all pairs $\alpha, \beta$ such that $\beta^2/\alpha^2 = q$.

Corollary 2 Any Lax pair of the form (23) produces a 1-parameter family of discrete conformal maps.

This follows immediately from the first part of the previous proof.
4.1 The dressing action.

Let us now fix $q$ and choose $\alpha, \beta$ so that $|\alpha|, |\beta| < 1$ (such a pair can clearly be found for each $q$). Then we may view $U, V$ (and hence $\Phi$) as taking values in the connected loop group $LGL_2 = \{g \in C^\infty(S^1, GL_2) : \det(g)$ has winding number zero\}. In fact $U, V$ and $\Phi$ all take values in the subgroup $N = \{g \in LGL_2 : g$ extends holomorphically into the disc $|\lambda^{-1}| < 1$ with $g(\infty)$ lower unipotent $\}$ (we will say a matrix $A$ is lower unipotent if $I - A$ is strictly lower triangular). The dressing action uses the fact that almost every $g \in LGL_2$ can be factorised into a product $g = g_N g_B$ where $g_N \in N$ and $g_B$ belongs to the subgroup $B = \{g \in LGL_2 : g$ extends holomorphically into $|\lambda| < 1$ with $g(0) \in B_0\}$. One knows (from e.g. [12]) that the space $N.B$ of all products is open dense in $LGL_2$. Given an extended frame $\Phi$ and $g \in B$ we might expect $g\Phi$ to map into $N.B$, in which case we could define $g \circ \Phi = (g\Phi)_N$ and therefore we have

$$g\Phi_{k,m} = (g \circ \Phi)_{k,m} \Psi_{k,m}, \quad \text{for some } \Psi : \mathbb{Z}^2 \to B.$$  \hspace{1cm} (27)

**Lemma 12** For any $g \in B$ for which it exists, the map $g \circ \Phi : \mathbb{Z}^2 \to N$ is again the extended frame for a discrete conformal map $g \circ z$ of cross-ratio $\beta^2 / \alpha^2$.

**Proof.** It suffices to show that the Lax pair for $g \circ \Phi$ is again of the form (23), since it is clear that $g \circ \Phi_{0,0} = g_N = I$. Observe that $g \circ \Phi = g\Phi\Psi^{-1}$ so that we can define

$$\hat{U}_{k,m} = (g \circ \Phi)^{-1}_{k,m}(g \circ \Phi)_{k+1,m} = \Psi_{k,m} U_{k,m} \Psi_{k+1,m}^{-1},$$
$$\hat{V}_{k,m} = (g \circ \Phi)^{-1}_{k,m}(g \circ \Phi)_{k,m+1} = \Psi_{k,m} V_{k,m} \Psi_{k,m+1}^{-1}.$$  

Since $\Psi_{k,m} \in B$ we can expand it in Fourier series as

$$\Psi_{k,m} = \begin{pmatrix} a_{k,m} & b_{k,m} \\ 0 & c_{k,m} \end{pmatrix} + O(\lambda).$$

Consequently we compute

$$\Psi_{k,m} U_{k,m} \Psi_{k+1,m}^{-1} = \alpha^2 \lambda^{-1} \begin{pmatrix} 0 & e_{k,m} \\ 0 & 0 \end{pmatrix} + O(1),$$

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for some expression $e_{k,m}$. But $g \circ \Phi$ takes values in $N$ so the only other terms in this expression are constant in $\lambda$ and lower unipotent i.e.

$$\hat{U}_{k,m} = \begin{pmatrix} 1 & \alpha^2 \lambda^{-1} e_{k,m} \\ \hat{u}_{k,m} & 1 \end{pmatrix},$$

for some $\hat{u}_{k,m}$. Now we claim that $\det(\Psi_{k,m}) = \det(g)$ for all $k, m$, whence $\det(\hat{U}_{k,m}) = \det(U_{k,m}) = 1 - \alpha^2 \lambda^{-1}$ so that $e_{k,m} = \hat{u}_{k,m}^{-1}$ and we are done. To see the claim, notice that

$$\det(\Phi) \det(g \circ \Phi)^{-1} = \det(g)^{-1} \det(\Psi).$$

The terms on the left extend holomorphically into $|\lambda| < 1$ while on the right they extend holomorphically into $|\lambda^{-1}| < 1$. Therefore both sides are constant and by evaluating at $k, m = 0$ we see this constant is 1. A similar argument for $\hat{V}$ finishes the proof. \(\square\)

Formally, at least, one can check that $\Phi \mapsto g \circ \Phi$ is a (left) group action and therefore we obtain an action of $B$ on $Z_q$. By analogy with the theory of the KdV equation we call this the dressing action (see e.g. [10]). The obstruction to making this more rigorous is that we cannot guarantee that $g \circ \Phi$ exists off $(0,0) \in Z^2$ (compare with the KdV theory where one has at least local solutions guaranteed).

**Remark:** the effect of the scaling $\alpha, \beta \mapsto k\alpha, k\beta$. For fixed $q$ we are only permitted to change $\alpha, \beta$ by the scaling $k\alpha, k\beta$, where we require $|k| < \min\{|\alpha|^{-1}, |\beta|^{-1}\}$. If $U(\lambda), V(\lambda)$ is the Lax pair for $z, \alpha, \beta$ then the Lax pair for $z, k\alpha, k\beta$ is $U(k^{-2}\lambda), V(k^{-2}\lambda)$. It is clear that our choice of working with loops on the unit circle limits us if the scaling is not unimodular. But this limitation is unnecessary: we could equally well work with loops on any circle and apply the dressing theory. It is not hard to see that by rescaling the unit circle the dressed map $g \circ z$ obtained using $\alpha, \beta$ can be obtained using $k\alpha, k\beta$ and $g(k^{-2}\lambda)$.

For the next result, we think of $B_0$ as the subgroup of constant loops in $B$.

**Lemma 13** The dressing action of the subgroup $B_0 \subset B$ corresponds precisely to the action of $B_0$ as the full group of base point preserving Möbius transformations.

**Proof.** Let $z \in Z_q$ and let $\Phi$ be its extended frame, so $F = \Phi(\infty)e_1$ lifts $z$ into $C^2$. Take $g \in B_0$, then $g \Phi$ extends holomorphically into $|\lambda^{-1}| < 1$. If
$g \Phi_{k,m}$ factorises then $g \circ z$ (i.e. the action of $g$ by Möbius transforms) has lift

$$gF = (g\Phi)_N(\infty)(g\Phi)_B(\infty)e_1 = k(g \circ \Phi)(\infty)e_1,$$

for some $k \in \mathbb{C}$, since $(g\Phi)_B(\infty)$ takes value in $B_0$ which is the stabilizer of the line generated by $e_1$. Now we observe that $g \Phi_{k,m}$ factorises precisely when $(g\Phi_{k,m})(\infty)e_1 \neq e_2$, i.e. when $z_{k,m}$ belongs to the plane $\mathbb{C}$. □

Now observe that $B_0$ is a normal subgroup of $B$ and $B/B_0 \simeq LG_+ = \{g \in B : g(0) = I\}$. Since the quotient space $\mathcal{Z}_q/B_0$ can be identified with the space of all Möbius equivalence classes of discrete conformal maps we have an induced action of $LG_+$ on $\mathcal{Z}_q/B_0$. This is what we shall focus on from now on.

### 4.2 Dressing orbit of the vacuum solution.

Our main interest in the dressing action is to examine the dressing orbit of the vacuum solution, which we will denote by $z^{(0)}$. Throughout this section $\Phi$ will be the extended frame (26) for that solution. First we will describe this dressing orbit as a quotient space. It can be shown (by a tedious but straightforward argument using Fourier expansions) that $g \circ \Phi = \Phi$ if and only if $g \in \Gamma_+ = \{g \in B : g \Lambda = \Lambda g\}$. Therefore the dressing orbit of $\Phi$ is identifiable with $B/\Gamma_+$, so the dressing orbit of the vacuum solution in $\mathcal{Z}_q/B_0$ is identifiable with the double coset space $B_0 \setminus B/\Gamma_+$. Since $B_0$ is a normal subgroup the group $LG_+$ acts transitively on the left so we may identify $B_0 \setminus B/\Gamma_+$ with the quotient space $LG_+/\Gamma_+$. Here the action of $\Gamma_+$ on $LG_+$ is given by $\gamma \cdot g = \gamma(0)^{-1}g\gamma$.

The principal result in this section is that this dressing orbit contains all those periodic discrete conformal maps which are "small perturbations" of their corresponding SKdV solution in the following sense.

**Theorem 3** Let $z : \mathbb{Z}^2 \to \mathbb{P}^1$ be a periodic discrete conformal map (of cross-ratio $q$) with spectral data $(\Sigma, \lambda, O, S, Q)$ for which: (a) $\lambda = 0$ is a branch point (i.e. $O = \bar{O}$); (b) the disc $|\lambda| \leq \max\{1, |q|\}$ contains no other branch points. Then $z$ is in the dressing orbit of the vacuum solution.

To prove this we will use the Grassmannian picture of the dressing orbit of the KdV vacuum solution developed by Segal and Wilson [14] (see also [12]). Let $H = L^2(S^1, \mathbb{C})$ and recall that $GL_2$ acts on this space by 'interleaving Fourier series' i.e. we use the isometric isomorphism between $H$ and
$L^2(S^1, \mathbb{C}^2)$ given by $f(\zeta) \mapsto (f_0(\lambda), f_1(\lambda))^t$ where $f_0(\zeta^2) + \zeta^{-1}f_1(\zeta^2) = f(\zeta)$. The usual action of $LGL_2$ on $\mathbb{C}^2$-valued functions then passes to an action on $H$ which we will denote by $g \cdot f$. In particular, the action of the commutative subgroup $\Gamma = \{g \in LGL_2 : g\Lambda = \Lambda g\}$ on $H$ corresponds to the multiplication action of $C^\omega(S^1, \mathbb{C}*)$. This follows from the observation that $\Lambda \cdot f(\zeta) = \zeta^{-1}f(\zeta)$. Therefore we have

$$\Phi_{k,m} \cdot f(\zeta) = (1 + \alpha \zeta^{-1})^k(1 + \beta \zeta^{-1})^m f(\zeta)$$

(28)

for the vacuum extended frame.

Next, recall that $H$ has an orthogonal decomposition $H_- \oplus H_+$ where $H_-$ (respectively, $H_+$) is the subspace of functions whose Fourier series possess only non-positive (respectively, positive) powers of $\zeta$. [Readers familiar with [12, 14] should note that we have applied the involution $\lambda \mapsto -\lambda$ to the picture described there.] We will define the Grassmannian to be $Gr = \{g \cdot H_- : g \in LGL_2\}$. We recall that the orbit of $H_-$ under the subgroup $LG_+$ is open dense (the ’big cell’) and that it is characterised as being the set of all $W \in Gr$ for which the orthogonal projection $pr_- : W \to H_-$ is invertible. Our assumption about the existence of $g \circ \Phi$ implies that $\Phi_{k,m}^{-1} \cdot W$ belongs to the big cell for all $k, m$, where $W = g^{-1} \cdot H_-$. We will now associate to each $W \in Gr$ the discrete analogue of the Baker function used in [14]. First we observe that we may refine our current factorisation $N.B$ into $N.T.N$ where $T$ denotes the subgroup of constant diagonal loops and $\tilde{N} = \{g \in B : g(0) \text{ is upper unipotent}\}$. Thus any $g \in N.B$ can be factorised into $g_N g_T g_{\tilde{N}}$. Let us now fix a $g \in LG_+$ and define

$$\psi = \Phi \Psi_{-1}^{-1} \cdot 1 = g^{-1}(g \circ \Phi)\Psi_T \cdot 1,$$

using (27). Since both subgroups $N$ and $T$ preserve $H_-$ we see that $\psi : \mathbb{Z}^2 \to W = g^{-1} \cdot H_-.$

**Lemma 14** Each $\psi_{k,m} \in W$ given in this way is uniquely determined by the property that

$$pr_-((1 + \alpha \zeta^{-1})^{-k}(1 + \beta \zeta^{-1})^{-m}\psi_{k,m}) = 1.$$  

**Proof.** Since we are assuming $\Phi^{-1} \cdot W$ is always in the big cell it suffices to show that $pr_-((\Phi_{-1})_{k,m} \cdot \psi_{k,m}) = 1$, given (28). But $\Phi^{-1} \cdot \psi = \Psi_{-1}^{-1} \cdot 1$. It is easy to check that for any $n \in \tilde{N}$, $n \cdot 1 = 1 + O(\zeta). \Box$

We will call this function the discrete Baker function for $W$. Now we want to reconstruct the dressed map $g \circ z^{(0)}$ from $\psi$. First we observe that
\( \zeta^{-2}W \subset W \) and the quotient \( W/\zeta^{-2}W \) is two dimensional (since \( g \cdot \zeta^{-2}f = \zeta^{-2}g \cdot f \) and \( H_-/\zeta^{-2}H_- \) is two dimensional).

**Lemma 15** Choose any linear identification of \( W/\zeta^{-2}W \) with \( \mathbb{C}^2 \) and let \( [\psi] : \mathbb{Z}^2 \to \mathbb{P}^1 \) be the map given by \( (k,m) \mapsto \psi_{k,m} + \zeta^{-2}W \). Then \([\psi]\) and \( g \circ z^{(0)} \) are Möbius equivalent (so in particular this gives a discrete conformal map).

**Proof.** Observe that \( 1 + \zeta^{-2}H_- \) and \( \zeta^{-1} + \zeta^{-2}H_- \) span \( H_-/\zeta^{-2}H_- \). Therefore by setting \( e_1 = g^{-1} \cdot 1 \) and \( e_2 = g^{-1} \cdot \zeta^{-1} \) we obtain a basis \( e_1 + \zeta^{-2}W, e_2 + \zeta^{-2}W \) for \( W/\zeta^{-2}W \). We can write \( g \circ \Phi \) and \( \Psi_T \) in the form

\[
g \circ \Phi = \left( \begin{array}{cc} a & 0 \\ b & c \end{array} \right) + O(\lambda^{-1}), \quad \Psi_T = \left( \begin{array}{cc} s & 0 \\ 0 & * \end{array} \right)
\]

so that the map \( g \circ z^{(0)} \) has homogeneous coordinates \( [a,b] \) and \( \psi_T \cdot 1 = s \).

Now observe that \( (g \circ \Phi) \Psi_T \cdot 1 = sa + \zeta^{-1}sb + O(\zeta^{-2}) \) and therefore

\[
s(ae_1 + be_2) \equiv g^{-1}(g \circ \Phi)\Psi_T \cdot 1 \mod \zeta^{-2}W \\
\equiv \psi \mod \zeta^{-2}W.
\]

It follows that in this basis the map \([\psi]\) has homogeneous coordinates \([a,b]\). \( \square \)

Now we are in a position to prove theorem 3. Given a periodic discrete conformal map \( z \) with spectral data satisfying the conditions of the theorem we will construct a \( W \in \text{Gr} \) for which the map \( \psi : \mathbb{Z}^2 \to W \) recovers \( z \) as \([\psi]\). By the previous lemma this will prove the theorem.

Given \((\Sigma, \lambda, \mathcal{L}, O, S, Q)\) we follow [14] and define \( W \) in the following manner. By assumption there exists \( r > \max\{1, |q|\} \) for which the disc \( |\lambda| \leq r \) contains no branch points. We set \( \zeta = \sqrt{\lambda/r} \) so that the unit \( \zeta \)-circle may be identified with the boundary of a disc \( \Delta \) on \( \Sigma \) about \( O \) which contains no ramification points other than \( O \) i.e. this is a coordinate disc with coordinate \( \zeta \) and contains the points \( O, S, Q \). We define \( W \) to be the \( L^2 \)-closure of the space \( W^0 \) of all \( f \in H \) which extend meromorphically into \( \Sigma - \Delta \). We define \( W \) to be the \( L^2 \)-closure of the space \( W^0 \) of all \( f \in H \) which extend meromorphically into \( \Sigma - \Delta \) where they have divisor of poles no worse than \( D \), where this is the unique positive divisor of degree \( g \) in the class \( \mathcal{L}(-O) \). One knows that \( W \in \text{Gr} \) (even the small Grassmannian we have chosen, since the whole construction is analytic). Now, with the definitions from previous sections, we define \( \psi_{k,m} \) to be (the boundary of) a meromorphic function with divisor \( (14) \). Since
the points $O, S, Q$ are not in $\Sigma - \Delta$ this function belongs to $W$ for all $k, m$. Moreover, if we set $\alpha = -\zeta(S)$ and $\beta = -\zeta(Q)$ we see that

$$(1 + \alpha \zeta^{-1})^{-k}(1 + \beta \zeta^{-1})^{-m}\psi_{k,m}$$

extends holomorphically into $\Delta$ and $\psi_{k,m}$ will be uniquely determined by requiring this expression to equal 1 when evaluated at $\zeta = 0$. By lemma 14 this must be the discrete Baker function for $W$ and by lemma 9 the map $(k, m) \rightarrow \psi_{k,m}(\hat{P}_\infty)/\psi_{k,m}(\hat{P}_\infty)$ recovers $z : \mathbb{Z}^2 \rightarrow \mathbb{P}^1$ up to Möbius equivalence. But the map $W^0 \rightarrow \mathbb{P}^1$ given by $f \mapsto [f(\hat{P}_\infty), f(\hat{P}_\infty)]$ induces an isomorphism $\mathbb{P}(W^0/\zeta^{-2}W^0) \rightarrow \mathbb{P}^1$. Since $W^0$ is dense in $W$ the spaces $W^0/\zeta^{-2}W^0$ and $W/\zeta^{-2}W$ are equal. Therefore, by the previous lemma the maps $[\psi]$ and $z$ are equivalent, hence $z$ is in the dressing orbit of the vacuum solution.

The discrete cubic. Lemma 15 allows us to compute examples which are difficult to obtain otherwise. For example, take $W = C\langle \zeta \rangle + \zeta^{-1}H_-$, where $C\langle \zeta \rangle$ is the vector space generated by $\zeta$. One knows that $W \in Gr$ (see [14] §7): indeed $W$ lies in the Grassmanian for rational loops in $GL_2$. It is elementary to show that in this case

$$\psi_{k,m} = (1 + \alpha \zeta^{-1})^k(1 + \beta \zeta^{-1})^m(1 - \frac{\zeta}{k\alpha + m\beta}).$$

Since $W/\zeta^{-2}W \simeq C\langle \zeta, \zeta^{-2} \rangle$ we may take $\zeta + \zeta^{-2}W$ and $\zeta^{-2} + \zeta^{-2}W$ as a basis. A straightforward computation shows that

$$\psi_{k,m} \equiv \frac{1}{k\alpha + m\beta} \zeta - \frac{z_{k,m}}{6(k\alpha + m\beta)} \zeta^{-2} \text{ mod } \zeta^{-2}W,$$

where

$$z_{k,m} = (k + 1)k(k - 1)\alpha^3 + 3k^2m\alpha^2\beta + 3km^2\alpha\beta^2 + (m + 1)m(m - 1)\beta^3.$$ 

It follows that this is a discrete conformal map with cross-ratio $\beta^2/\alpha^2$. Indeed this is the discrete analogue of the cubic $z(x) = x^3$. In the smooth (binomial) limit $(1 + \alpha \zeta^{-1})^k(1 + \beta \zeta^{-1})^m \rightarrow \exp(x\zeta^{-1})$ we obtain the smooth Baker function $\psi(x) = \exp(x\zeta^{-1})(1 - \zeta/x)$ and

$$\psi(x) \equiv -\frac{1}{x} \zeta + \frac{2}{3}x^2 \zeta^{-2} \text{ mod } \zeta^{-2}W.$$ 

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See [14] §7 for a description of how this corresponds to taking the spectral curve \( \mu^2 = \lambda^{-3} \) and for the similar spaces corresponding to the spectral curves with equations of the form \( \mu^2 = \lambda^{-2g-1} \). All of these should produce (discrete) rational functions but it is difficult to describe a priori which rational functions arise in this way.

Figure 7: The discrete cubic.

4.3 Discrete conformal maps as Darboux transformations of the KdV hierarchy.

In this section we will prove rigorously a rather surprising relationship between discrete conformal maps and Darboux (Bäcklund) transforms of solutions to the KdV equation. We will work inside the dressing orbit of the vacuum solution of KdV. Recall from [14] that to every \( W \in Gr \) there exists a solution \( u_W(x, t) \) of the KdV equation, given by

\[
(\partial^2 + u_W)\psi_W = \zeta^{-2}\psi_W, \quad \partial = \partial/\partial x,
\]

where \( \psi_W : \mathbb{C}^2 \to W \) is the (smooth) Baker function for \( W \) i.e. the unique function with the property that \( \text{pr}_- (\exp(-x\zeta^{-1} - t\zeta^{-3})\psi_W) = 1 \) for almost all \( x, t \in \mathbb{C} \). If \( \Phi_{k,m} \) denotes the vacuum frame (26) then \( W_{k,m} = \Phi_{k,m}^{-1} \cdot W \) defines a map \( \mathbf{Z}^2 \to Gr \). We will show that \( \{u_{W_{k,m}}\} \) is a \( \mathbf{Z}^2 \)-family of Darboux transforms of \( u_W \).

To begin, let us recall, from e.g. [1] the Darboux transform for the KdV hierarchy. Given any solution \( u(x, t) \) of the KdV equation (or any other
equation in the hierarchy) we can produce other solutions using the following procedure. Set $L = \partial^2 + u$. For any pair $(\psi, c)$, consisting of a function $\psi(x, t)$ and a complex number $c$ for which $L\psi = c\psi$, define $v = \psi_x\psi^{-1}$. One readily checks that $u = -v_x - v^2 + c$ and therefore

$$L - c = (\partial + v)(\partial - v).$$

The Darboux transform for the pair $(\psi, c)$ maps $u$ to $\tilde{u}$ where

$$\tilde{L} - c = (\partial - v)(\partial + v), \quad \text{i.e. } \tilde{u} = v_x - v^2 + c.$$

One can think of this as a formal conjugation of $L - c$ into $\tilde{L} - c$ in the algebra of pseudo-differential operators and deduce that $\tilde{u}$ too satisfies the KdV equation (see [6]).

**Proposition 6** Whenever $0 < |\alpha| < 1$ the map $W \mapsto V = (1 + \alpha\zeta^{-1})W$ on $Gr$ induces the Darboux transform on $u_W$ determined by taking $\psi = \psi_W(x, t; -\alpha)$ and $c = \alpha^{-2}$.

**Proof.** First, since $W$ is a linear space we can write $V = (\zeta^{-1} + \alpha^{-1})W$. For convenience, set $\gamma(x, t) = \exp(x\zeta^{-1} + t\zeta^{-3})$, then

$$\psi_W = \gamma(1 + a_W \zeta + O(\zeta^2)),$$

for some function $a_W$ of $x, t$. Comparing the Fourier series for $\partial\psi_W, \psi_W$ and $(\zeta^{-1} + \alpha^{-1})\psi_V$, all of which take values in $W$, we see that if we set $v = a_W - a_V - \alpha^{-1}$ then

$$\gamma^{-1}[(\partial - v)\psi_W - (\zeta^{-1} + \alpha^{-1})\psi_V] : \{(x, t)\} \to H_+, \quad \text{i.e. it has only positive powers of } \zeta \text{ in its Fourier expansion. But } \gamma^{-1}W \cap H_+ = \{0\} \text{ for almost all } x, t, \text{ hence this expression must be identically zero. Therefore}$$

$$(\partial - v)\psi_W = (\zeta^{-1} + \alpha^{-1})\psi_V.$$

Further, $(\zeta^{-1} - \alpha^{-1})V = (\zeta^{-2} - \alpha^{-2})W = W$ so that a similar calculation gives

$$(\partial + v)\psi_V = (\zeta^{-1} - \alpha^{-1})\psi_W.$$

Therefore

$$(\partial + v)(\partial - v)\psi_W = (\zeta^{-2} - \alpha^{-2})\psi_W,$$

$$(\partial - v)(\partial + v)\psi_V = (\zeta^{-2} - \alpha^{-2})\psi_V,$$
so in particular $u_W = -v_x - v^2 + \alpha^{-2}$ and $u_V = v_x - v^2 + \alpha^{-2}$. Hence if we take $\psi = \psi_W(x, t; -\alpha)$ and $c = \alpha^{-2}$ then $(\partial^2 + u_W)\psi = c\psi$ and $(\partial - v)\psi = 0$. Thus $u_V$ is the Darboux transform of $u_W$ for the pair $(\psi, c)$.

**Remark.** In fact if one examines [6] p5, Theorem (ii), one sees that a Darboux transform of a certain type preserves the KdV spectral curve and shifts the line bundle by $\mathcal{L} \mapsto \mathcal{L}(Q - O)$ for some $Q \in \Sigma$ (recall that KdV solutions of finite type have spectral data $(\Sigma, \lambda, \mathcal{L}, O)$ with all the properties of the spectral data above). From this we see that this relationship between discrete conformal maps and Darboux transforms is true for all periodic maps, not just those in the dressing orbit of the vacuum.
Appendix: Non-singular spectral curves are generic.

Here we will prove our earlier claim that non-singular spectral curves exist (and are therefore generic) for periodic discrete curves with \( n \) points for any \( n \geq 4 \). We will use the notation of section 1 throughout.

Let \( X_\infty \subset \mathbb{P}^1 \times \ldots \times \mathbb{P}^1 \) be the space of periodic discrete curves of period \( n \). It is clearly an irreducible affine open subvariety. Let \( Y_\infty \) be the space of data \( (\Sigma, O, S, P_\infty, [y]) \) where: \( \Sigma \) is a complete irreducible algebraic curve of arithmetic genus \( g \) (equal to \( (n - 4)/2 \) for \( n \) even and \( (n - 3)/2 \) for \( n \) odd) admitting a rational function \( \lambda \) of degree 2; \( O, S, P_\infty \) are smooth points on \( \Sigma \) with \( \lambda \)-values 0, 1, \( \infty \) respectively, at which \( \lambda \) is unramified (unless \( n \) is odd in which case \( O \) is a ramification point); \( y \) is a rational function on \( \Sigma \) with divisor of poles \( nS \) and \([y]\) denotes its image in the complete linear system \( P_{\Gamma}(O_\Sigma(nS)) \). Since \( n > 2g + 2 \) this linear system has dimension \( n - g \). Notice that the map \( (\Sigma, O, S, P_\infty, [y]) \mapsto (\Sigma, O, S, P_\infty) \) displays \( Y_\infty \) as a \( \mathbb{P}^{n-g} \)-bundle over the subvariety of \( \mathbb{P}^2_{2g+2} \) corresponding to the possible configurations of branch divisors. Inside \( Y_\infty \) we consider two subvarieties: \( Y^s_\infty \), wherein \( \Sigma \) is singular; \( Y^r_\infty \), wherein \( \Sigma \) is rational with nodes only. In particular, \( Y^s_\infty \) is a hypersurface in \( Y \) while \( Y^r_\infty \subset Y^s_\infty \) clearly has codimension \( g \) in \( Y_\infty \).

To each \( \Gamma \in X_\infty \) we assign the data \( (\Sigma, O, S, P_\infty, [y]) \) where \( \Sigma, O, S, P_\infty \) are given by the characteristic polynomial of \( M_0^\lambda \) and \( y = \det(H_0^\lambda) \). This \( y \) has divisor
\[
D_n = \begin{cases} 
  n(O - S) & \text{for } n \text{ odd}, \\
  \frac{n}{2}(O + \hat{O} - 2S) & \text{for } n \text{ even},
\end{cases}
\]
and satisfies \( y(P_\infty) = y(\hat{P}_\infty) \). Thus we have an algebraic map \( F : X_\infty \rightarrow Y_\infty \) with image
\[
V = \{ (\Sigma, O, S, P_\infty, [y]) \in Y_\infty : (y) = D_n, y(P_\infty) = y(\hat{P}_\infty) \}.
\]

Since \( V \) is irreducible, either \( V \subset Y^s_\infty \) or there exists a generic curve. Locally \( Y^s_\infty \) is given by a single equation, say \( Z = 0 \), in \( Y \). We will show that the codimension of \( V \cap Y^r_\infty \) in \( V \) is \( g \) hence \( Z \) cannot vanish identically on \( V \) since \( Y^r_\infty \) has only codimension \( g - 1 \) in \( Y^s_\infty \).

First let us describe \( V \cap Y^r_\infty \). Each irreducible rational curve \( \Sigma \) with \( g \) nodes has normalisation \( \mathbb{P}^1 \). We choose a rational parameter \( t \) on \( \mathbb{P}^1 \) such that the hyperelliptic involution is \( t \mapsto 1/t \) and \( t(S) = \infty \). The preimage of the singularities under the normalisation will be \( g \) pairs of the form \( a_j, 1/a_j \) \((j = 1, \ldots, g)\) and all such curves \( \Sigma \) arise this way. For simplicity set \( b = t(P_\infty) \).
and $c = t(O)$. Then the parameters $a_j, b, c$ determine $(\Sigma, O, S, P_\infty)$. If $y$ has divisor $D_n$ then, up to scaling,

$$y = \begin{cases} (t - c)^n & \text{for } n \text{ odd}, \\ (t - c)^m(t - c^{-1})^m & \text{for } n = 2m, \end{cases}$$

and $a_1, \ldots, a_g, b$ must all be roots of $y(t) = y(1/t)$. Certainly this many distinct roots exist, so $V \cap Y_n^r$ is non-empty and has dimension 1 for $n$ even, since the only free parameter is $c$, while for $n$ odd it has dimension zero, since $O = \tilde{O}$ forces $c = \pm 1$. Now let us compute the dimension of $V$. The fibre of the map $F : X_n \to Y_n$ over data involving any nodal rational curve is $PSL_2 \times Jac(\Sigma)$ since we have ignored Möbius invariance and the line bundle $\mathcal{L}$. Therefore $V$ has dimension $n - (g + 3)$ which equals $g + 1$ for $n$ even and $g$ for $n$ odd. Therefore $V \cap Y_n^r$ has codimension $g$ in $V$ (for every $n$) whence the result follows.

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