Proof of a conjecture on the infinite dimension limit of a unifying model for random matrix theory

Giovanni M. Cicuta\textsuperscript{1}, \textsuperscript{*} and Mario Pernici\textsuperscript{2}, \textsuperscript{†}

\textsuperscript{1} Dip. Fisica, Università di Parma, Parco Area delle Scienze 7A, 43100 Parma, Italy, \textsuperscript{2} Istituto Nazionale di Fisica Nucleare, Sezione di Milano, 16 Via Celoria, 20133 Milano, Italy

We study the large \(N\) limit of a sparse random block matrix ensemble. It depends on two parameters: the average connectivity \(Z\) and the size of the blocks \(d\), which is the dimension of an euclidean space. We prove the conjecture that, in the limit of large \(d\), with \(Zd\) fixed, the spectral distribution of the sparse random block matrix converges in the case of the Adjacency block matrix to the one of the effective medium approximation, in the case of the Laplacian block matrix to the Marchenko-Pastur distribution.

We extend previous analytical computations of the moments of the spectral density of the Adjacency block matrix and the Lagrangian block matrix, valid for all values of \(Z\) and \(d\).

\section{Introduction}

In the past 60 years the theory of random matrices had an impressive development in theoretical physics and in a variety of disciplines [1]. Further progress and usefulness of random matrices will be linked to the ability of a specific class of random matrices to encode the relevant properties of a specific problem.

Models with random matrices made of blocks seem capable to capture modular features of complex networks. Perhaps the most relevant model is the Stochastic Block Model, well known in the study of social and biological networks. The model describes a complex network with \(n\) nodes, partitioned into communities or blocks, often of equal size. If two nodes belong to different communities, there is an edge with probability which may depend on the chosen pair of communities. If two nodes belong to the same community there is an edge with different probability. The model is flexible enough to properly describe many nontrivial types of structures.

Some recent references [2] may introduce the reader to this vast subject. Random block matrices were often analyzed by the cavity method, familiar in statistical physics.

In [3] a sparse block random matrix has been presented, which is a straightforward picture of the Hessian of a system of points connected by springs. Its original motivation was the study of vibrational spectrum of glasses but in the present work we are not concerned with the physics insights. The reader is referred to [3], [4], [5], [6] and references quoted there for the usefulness of the present model in the study of a class of disordered systems.

In this model the dimension \(d\) of euclidean space is a parameter of the random matrix (it is the size of the blocks) and the qualitative features of the model as \(d\) varies from 1 to \(\infty\) agree with the expected behavior of disordered systems in spaces with different dimension.

By considering different regimes as the average connectivity \(Z\) and \(d\) are varied, several well known random matrix models are obtained. Then the sparse block random matrix here analyzed interpolates among the most famous random matrix models.

In particular, based on the computation of the first 5 non-trivial moments and numerical simulations, in [3] it has been conjectured that, for \(d \to \infty\) with \(Zd\) fixed, the spectral distribution of the Adjacency block matrix model tends to the one of the effective medium approximation, the spectral distribution of the Laplacian block matrix model tends to the Marchenko-Pastur distribution.

In this paper we prove this conjecture, using the following observation. While for \(d\) finite the computation of the moments reduces, for \(N \to \infty\), to contributions on tree paths, as in the \(d = 1\) case analyzed in [7], in the limit \(d \to \infty\), with \(Zd\) fixed, only the tree graphs in which the edges form a noncrossing partition [18] contribute.

We extend the analytical computation of the moments of the spectral distribution for the Adjacency and Laplacian \(d\)-dimensional block matrix models respectively through 26 and 15 order.

After summarizing in Sect.II the definition of the matrix ensemble and the limiting domains of the parameters, we present in Sect.III the algorithm that allows the exact automated evaluation of the several moments of the spectral distributions.

Sect.IV describes the proof of the convergence of this random matrix ensemble to two well known spectral dis-
tributions, in the limit of large dimension of the euclidean space.

In Appendix A we illustrate in detail the computation on the moments at the first three orders. In Appendix B we write down the moments of the spectral distribution for the Adjacency model through order 18, and those for the Laplacian model through order 10. In Appendix C we give some details on the distribution of the vertices in the Laplacian model.

II. THE SPARSE BLOCK RANDOM MATRIX AND THE LIMITING DOMAINS

We consider a real symmetric matrix \( M \) of dimension \( Nd \times Nd \) where each row or column has \( N \) random block entries, each being a \( d \times d \) matrix. The set \( \{ a_{ij} \} \), \( 1 \leq i < j \leq N \) is a set of \( N(N-1)/2 \) i.i.d. random variables, \( a_{ij} = \alpha_{ij} \), with the probability law:

\[
P(\alpha) = \left( \frac{Z}{N} \right) \delta(\alpha - 1) + \left( 1 - \frac{Z}{N} \right) \delta(\alpha)
\]

Every \( d \times d \) off-diagonal block is a rank one random matrix, \( X_{i,j} = X_{j,i} = (X_{i,j})^t = [\hat{a}_{ij}] \) where \( [\hat{a}_{ij}] \) is a \( d \)-dimensional random vector of unit length, chosen with uniform probability on the \( d \)-dimensional sphere.

In the formulation of the stiffness matrix \( W \), the unit vector \( [\hat{a}_{ij}] \) provides the direction between vertex \( i \) and vertex \( j \) (in a disordered solid or elastic network, between two atoms \( i \) and \( j \)). For more details on the Hessian matrix of disordered solids see Refs. [1], [5], [6].

We study two prototypes of such block random matrices called the Adjacency block matrix \( A \) and the Laplacian block matrix \( L \). In both the above matrices, the set of \( X_{i,j}, i < j \) is a set of \( N(N-1)/2 \) independent identically distributed random matrices and each \( X_{i,j} \) is a rank-one matrix and a projector.

\[
A = \begin{pmatrix}
0 & \alpha_{1,2}X_{1,2} & \alpha_{1,3}X_{1,3} & \cdots & \alpha_{1,N}X_{1,N} \\
\alpha_{2,1}X_{2,1} & 0 & \alpha_{2,3}X_{2,3} & \cdots & \alpha_{2,N}X_{2,N} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\alpha_{N,1}X_{N,1} & \alpha_{N,2}X_{N,2} & \alpha_{N,3}X_{N,3} & \cdots & 0
\end{pmatrix}
\]

\[
L = \begin{pmatrix}
\sum_{j \neq 1} \alpha_{1,j}X_{1,j} & -\alpha_{1,2}X_{1,2} & -\alpha_{1,3}X_{1,3} & \cdots & -\alpha_{1,N}X_{1,N} \\
-\alpha_{2,1}X_{2,1} & \sum_{j \neq 2} \alpha_{2,j}X_{2,j} & -\alpha_{2,3}X_{2,3} & \cdots & -\alpha_{2,N}X_{2,N} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
-\alpha_{N,1}X_{N,1} & -\alpha_{N,2}X_{N,2} & -\alpha_{N,3}X_{N,3} & \cdots & \sum_{j \neq N} \alpha_{N,j}X_{N,j}
\end{pmatrix}
\]

The present sparse random block matrix may be considered a block generalization of the Erdos-Renyi random graph, and indeed our model reduces to it for \( d = 1 \). It is useful to recall that, for \( d = 1 \), the moments of the spectral distributions of the adjacency matrix and Laplacian matrix were determined by recurrence relations at every order [7], [8].

As it is well known, any matrix elements of a power \((A^k)_{a,b}\) is the sum of all weighted walks of \( k \) steps from vertex \( a \) to vertex \( b \) on the \( N \)-vertex graph corresponding to the matrix, where the edge \((i,j)\) of the graph has the weight \( A_{i,j} \). We follow this traditional approach to evaluate moments of the matrices. In our model, \( d \geq 1 \), the walks to be considered in the \( N \to \infty \) limit are the same class of walks on trees of the simpler model \( d = 1 \), because of the probability law of the random variables \( \{ \alpha_{ij} \} \).

In order to deal with the diagonal terms occurring in the Laplacian matrix, \( L_{i,i} = \sum_{j \neq i} \alpha_{i,j}X_{i,j} \), Bauer and Golinelli, in the Appendix of [7], suggested to use the property \( \alpha_{i,j}X_{i,j} = \alpha_{i,j}X_{j,i} \). This extends the walk on trees on the graph to the case of the Laplacian matrix.

In the \( d = 1 \) case, the average of the product of \( l \) distinct weights of a path is \( Z^l \). The corresponding product of \( j \) weights, in our model is \( Z^l < tr (X_{1}X_{2} \ldots X_{j}) > \) where \( l \) is the number of distinct \( d \times d \) random blocks, \( 1 \leq l \leq j \). The value of the product depends on the order of the blocks, and this prevents recursion relations to evaluate the moments.

Every \( d \times d \) block \( X_k \) is the projector \( |\hat{a}_k \rangle \langle \hat{a}_k| \), and \( < tr (X_{1}X_{2} \ldots X_{j}) > = \langle \hat{a}_1 \cdot \hat{a}_2 \cdot \hat{a}_3 \ldots \hat{a}_j \cdot \hat{a}_1 \rangle > \) is the average of the product of \( j \) scalar products involving \( l \) distinct random unit vectors.

Fig.1 shows three limiting random matrix models reached by the present sparse block random matrix in different regimes of the parameters. In each case we compare our analytic evaluation of moments with the known moments of the limiting random matrix models.
Let us briefly recall the three limiting regimes.

A. \( d = 1 \), the random graph

For \( d = 1 \) the sparse block random matrix reproduces the random graph by Erdos and Renyi. The spectral moments of the Adjacency matrix and the Lagrangian matrix are known at arbitrary order and they provide a check of our analytic evaluations. More specifically, for \( d = 1 \) we set all \( c_j = 1 \) for all the moments of Eqs. (B1) , (B2). They are polynomials in \( Z \) and reproduce the moments of the random graph model. We used Table 1 and 2 in ref.[7] for the moments up to \( \mu_{20} \) and \( \nu_{10} \). Then we used the recursive relation (2.8) of Khorunzhy, Shcherbina, Vangerovsky [8] to evaluate higher moments \( \mu_{2n} \) and the recursive relation of Bauer and Golinelli to evaluate higher moments of \( \nu_{n} \). (We were unable to use the recursive relation (2.9) in [8].)

B. the dilute random graph, \( Z \sim N^{\alpha} \), \( 0 < \alpha < 1 \)

In this regime we let \( Z \) increase with \( N \). The average number of non-zero entries in each row of the random matrix is no longer finite. It is usually described as a transition from a sparse matrix to a dilute matrix. If \( d = 1 \) several papers [11, 12, 13], proved that the spectral density of the Adjacency matrix in the dilute regime is the Wigner semi-circle, and the spectral density of the Laplacian matrix is the free convolution of Wigner semi-circle with a Normal distribution.

In the physics literature it was called the Addition Theorem for two ensembles of random matrices [14, 15, 16]. The heuristic explanation of the above result is obvious: in the dilute regime, the Adjacency matrix becomes an ordinary real symmetric matrix then its spectral distribution is Wigner semi-circle. The diagonal terms of the Laplacian are strongly dependent on the off-diagonal terms. But in the \( N \to \infty \) and \( Z \to \infty \) limit they are sums of a large number of i.i.d. random variables then the diagonal part of the Laplacian is made of entries independent normal random variables. The Laplacian is sum of a Wigner matrix and a diagonal of Normal variables.

We are not aware of analogous results for \( d > 1 \), that is the case of random block matrices.

In the paper [3], it was indicated that the highest powers \( Z/d \) in each moment \( \mu_{2n} \) of the Adjacency matrix are the moments of the semi-circle distribution

\[
\rho(x) = \frac{\sqrt{4(Z/d) - x^2}}{2\pi(Z/d)}, \quad \mu_{2n} = \frac{(2n)!}{n!(n+1)!} \left( \frac{Z}{d} \right)^n
\]

It was also indicated that the two highest powers \( Z/d \) in each moment \( \nu_n \) of the Laplacian matrix are the moments of the shifted semi-circle distribution

\[
\rho(x) = \frac{\sqrt{8(Z/d) - (x - Z/d)^2}}{4\pi(Z/d)}
\]

This suggest that the above heuristic explanation may hold in the \( d > 1 \) case.

C. \( d = \infty \)

The third regime is the most interesting because it obtains rather unexpected relations. Let us consider the moments \( \mu_{2n} \) of the Adjacency matrix, let \( Z/d \) fixed and \( d \to \infty \). In [3] it has been verified that in this limit the
The first 5 non-trivial moments are equal to those of the effective medium (EM) approximation by Semerjian and Cugliandolo. It was indicated in how to obtain easily the moments of the effective medium approximation at arbitrary order, by Taylor expansion of the cubic equation of the resolvent.

We extend this check by computing the first 13 non-trivial orders (the first 9 are written down in Appendix B). Letting $Z/d$ fixed and $d \to \infty$, then all the $c_j$ defined in Eq. tend to zero, and $\mu_{2k}$ for $k \leq 13$ reduce to those in the EM approximation.

In analogous fashion, in it has been checked that the first 5 moments $\nu_n$ of the Laplacian matrix tend to those of a Marchenko-Pastur distribution. We extend this check to the first 15 moments (the first 10 are given in Appendix B). These simplified moments reproduce the moments of a Marchenko-Pastur distribution,

$$\rho_{M_P}(x) = \frac{\sqrt{(b-x)(x-a)}}{4\pi x}$$

$$a = \left(\sqrt{t} - \sqrt{2}\right)^2, \ b = \left(\sqrt{t} + \sqrt{2}\right)^2$$

$$t = \frac{Z}{d}$$

The moments are all evaluated in Appendix 3 of ref.

III. THE MOMENTS OF THE SPECTRAL DISTRIBUTIONS.

The generating function for the moments in the Laplacian matrix is

$$\sum_{n \geq 0} x^n \nu_n = \lim_{N \to \infty} \frac{1}{N^d} \sum_{n \geq 0} x^n < \text{Tr}L^n > =$$

$$= \lim_{N \to \infty} \frac{1}{N^d} \sum_{j=1}^{N} < \text{tr}T_j^{(L)}(x) >$$

where we define

$$T_j^{(L)}(x) = \sum_{n \geq 0} x^n (L^n)_{j,j}$$

and $\text{tr}$ is the trace on the $d \times d$ matrices. In the Adjacency matrix we have analogous equations,

$$\sum_{n \geq 0} x^n \mu_n = \lim_{N \to \infty} \frac{1}{N^d} \sum_{n \geq 0} x^n < \text{Tr}A^n > =$$

$$= \lim_{N \to \infty} \frac{1}{N^d} \sum_{j=1}^{N} < \text{tr}T_j^{(A)}(x) >$$

where

$$T_j^{(A)}(x) = \sum_{n \geq 0} x^n (A^n)_{j,j}$$

We also evaluate the spectral density of the $d \times d$ matrix $L_{1,1}$ in the $d \to \infty$ limit. In this case, replace $(L^n)_{j,j}$ with $(L_{j,j})^n$.

In the large $N$ limit, only walks on tree graphs contribute to the moments, in a way completely analogous to the Erdos-Renyi random graph $(\mathbb{Z}, \mathbb{S})$. This can be shown in the following way.

To compute the moments, separate the indices so they are all different. In the case of the Adjacency matrix

$$\sum_{j_0} (A^n)_{j_0,j_0} = \sum_{j_0,j_1,\ldots,j_{n-1}=1} A_{j_0,j_1} \cdots A_{j_{n-1},j_0}$$

separate the indices $j_1 = 1, \ldots, N$ in indices $j_0, \ldots, j_{n-1}$ all different from each other, $j_r \neq j_s$ for all $r, s$. One can do this separation starting from the left, following the algorithm "label and substitution algorithm" in : the first item is labelled 0 (index $j_0$), the next item is labelled 1 (index $j_1$), and so on. Then use the rule $A_{ij} = \alpha_{ij} X_{ij}$.

The case of the Laplacian matrix is similar, but with the rules

$$L_{i,j} = -\alpha_{ij} X_{i,j}; \quad L_{j,j} = \sum_j \alpha_{j,j} X_{j,j} X_{j,j}$$

so that one must again separate the index $J$ occurring in $L_{i,j}$ to be a previously occurring index $j_r$, or a new index, different from the previous ones.

One can associate a path to a product of $\alpha$'s thus obtained: to the term $-\alpha_{r,s} X_{r,s}$ coming from the off-diagonal term one associates the move $(r, s)$ on the graph; the term $\alpha_{r,s} X_{r,s} X_{r,s}$ coming from the diagonal term of $L$ can be interpreted as the move $(r, s, r)$. In the following we will use the notation $\alpha_{r,s} X_{r,s}^2$ for a diagonal term, instead of simplifying it to $\alpha_{r,s} X_{r,s}$, when we want to emphasize that it comes from a diagonal contribution.

In the case of the Adjacency matrix there is only the move $(r, s)$.

To each move associate the variable $x$. $(L^n)_{j,j}$ corresponds to all the paths from vertex $j$ to $j$ with $n$ moves, identified by the factor $x^n$. The number $E$ of distinct $\alpha$'s associated to a path is the length of the unoriented graph associated to the path, leading to a factor $(\frac{x}{d})^E$ after taking the averages on the $\alpha$'s. Averaging on the $X$ a term corresponding to a graph with $V$ vertices, the sum on the $V$ distinct indices corresponding to these vertices give a factor $N^V_{\mathbb{S}} \approx N^V$ for $N \to \infty$. Considering the extra $N^{-1}$ factor in the definition of the moment Eq. , the contribution of this term to the moment has a factor $N^V-E-1$ which for $N \to \infty$ does not vanish only for $V = E + 1$, that is only if the graph is a tree. Therefore the path is on a tree graph.

In our computer implementation of the computations of moments, after separating indices so they are all different, as described above, and retaining only the tree graphs, one remains with the averages in the $d$-dimensional space to be performed. In Appendix A we give in detail the computation of the first three moments in the Laplacian model; in the appendix the terms $\alpha_{ij} X_{ij}$
coming from the off diagonal terms of $L$, and the term $(\alpha_{ij} X_{ij})^2$ coming from the diagonal of $L$ are kept separated.

In Appendix A of ref. [3] it has been shown the average of a power of a scalar product with a random unit vector $\hat{y}$

\[ \langle (\hat{p} \cdot \hat{y})^{2m} \rangle = \frac{c_m}{d} \langle \hat{p} \cdot \hat{p} \rangle^m \]

where

\[ c_m = \langle (\hat{a} \cdot \hat{y})^{2m} \rangle = \frac{(2m-1)!! \Gamma \left( \frac{d}{2} \right)}{2^m \Gamma \left( m + \frac{d}{2} \right)} \],

\[ c_1 = 1, \quad c_2 = \frac{3}{(d+2)} \cdot c_3 = \frac{(d+2)(d+4)}{(d+2)} \cdots \]

Taking in this equation $\hat{p} = t_1 \hat{a}_1 + \cdots + t_r \hat{a}_r$ one gets

\[ \langle (\hat{a}_1 \cdot \hat{y})^{k_1} \cdots (\hat{a}_r \cdot \hat{y})^{k_r} \rangle > y = \frac{k_1! \cdots k_r! c_m}{(2m)!} \langle \hat{a}_1 \hat{a}_2 \cdots \hat{a}_r \rangle(p^2)^m, \]

\[ k_1 + \cdots + k_r = 2m \]

where $|M|f$ is the operation of extraction of the monomial $M$. With this formula these averages are easily implemented in a program using truncated products of polynomials.

For $j \neq i, k$ one has, using Eq. (9)

\[ \langle \hat{a}_i | X_j | \hat{a}_k \rangle > a_j = \frac{1}{d} \hat{a}_i \cdot \hat{a}_k \]

From this follows a simple property of the averages: if in a product of $X$’s one of them appears only once, averaging on it consists in replacing it with a factor $\frac{1}{d}$, unless it is the only $X$ present, in which case $\langle X \rangle = 1$.

This property has been used, together with idempotency, to reduce the number of terms contributing to the moments in the computer implementation.

**IV. THE LIMIT OF LARGE SPACE DIMENSION, $\frac{d}{\alpha}$ FIXED**

In this section it will be proved that, in the limit $d \to \infty$ with $t \equiv \frac{d}{\alpha}$ fixed, the resolvent of the sparse random block matrix is the resolvent of a Marchenko-Pastur random matrix, in the case of the Laplacian matrix, or it is the resolvent of the Effective Medium Approximation by Semerjian and Cugliandolo, in the case of the Adjacency matrix.

Our proof has the following steps: Proposition 1 shows that only tree graphs with edges forming a noncrossing partition are relevant to the evaluation of the spectral moments. Proposition 2 evaluates a sum of contributions. Proposition 3 shows that the average of certain generating functions of primitive paths may be expressed in terms of a single unknown function, both for the Laplacian matrix and the Adjacency matrix. Proposition 4 shows a factorization property which gives simple algebraic equations for the resolvents.

In the rest of this section neglect the powers of $N$, which always cancel for the tree graph contributions to the moments.

We say that a product of blocks $\prod X$ contains the pattern $abab$ if it contains $\ldots X_a \ldots X_b \ldots X_a \ldots X_b$, with $a \neq b$.

**Proposition 1.** All products of blocks, containing the pattern $abab$, correspond to non-leading contributions to the moments of the Adjacency matrix or the Laplacian matrix in the $d \to \infty$ limit, $t = \frac{d}{\alpha}$ fixed.

Any product of blocks $X$, where $m$ of them are distinct and it does not contain the pattern $abab$, contributes $t^m$ to the moments of the Adjacency matrix or the Laplacian matrix in the $d \to \infty$ limit.

The noncrossing partitions [13] are the products not containing the $abab$ pattern [19]. Therefore in this limit a tree graph contribution to the moment with $m$ distinct edges gives $t^m$ if its edges form a noncrossing partition, it is zero otherwise.

**Proof of the second part of Proposition 1.** Consider a term $\prod X$ without the pattern $abab$. If all its $X$’s appear only once, $\langle \text{tr} \prod X \rangle = dt^m$. Consider a closest pair of equal $X$’s, say $X_{a_1} \ldots X_{a_r}$. If there is no $X$ between them, it is trivially reduced, using idempotency. Let $X_b$ be inside the pair of $X_a$. There cannot be a second $X_b$ inside the pair of $X_a$, otherwise the latter would not be a closest pair; nor there can be a $X_b$ outside the $X_a$ pair, because there cannot be the pattern $abab$. Therefore $X_b$ appears only once, and can be replaced by $t$. The same is true for all the $X$’s between the pair of $X_a$, which then reduce to a single $X_a$. Continuing in the same way, all pairs can be eliminated, and one gets the contribution $d^{m-1} \langle \text{tr} \prod X \rangle = t^m$ to the moment.

**Proof of the first part of Proposition 1.**

Let us consider a product of blocks $I = \langle \text{tr} \prod X \rangle$ containing the pattern $abab$. Performing the averaging of the blocks occurring only once and using idempotency as far as possible, one gets $I = t^r < \text{tr} \prod X \rangle = t^m$. In the reduced term each block occurs at least twice. Let us say that in the reduced term there are $n$ blocks , $h$ of which are distinct, then $n \geq 2h$.

The average over the $h$ unit vectors $\hat{a}_i$ has the form

\[ < \text{tr} \prod X > = Z^h < \prod (\hat{a}_i \cdot \hat{a}_j) > \quad, \quad i \neq j \]

by the leading term of Eq. (9)

\[ < (\hat{a}_1 \hat{y})^{k_1} \cdots (\hat{a}_r \hat{y})^{k_r} > y \approx d^{-m} \sum c_{i,j} \prod_{1 \leq i < j \leq r} (\hat{a}_i \cdot \hat{a}_j)^{p_{ij}}, \]

\[ m = \frac{1}{2} \sum_{i=1}^r k_i \]
where \( c_{ij} = O(d^p) \) and the sum is over the non-negative integer powers \( p_{ij} = k_i \).

Averaging over a variable \( a \) gives terms of the form
\[
d^{-n_1} < \prod_{i=1}^{2n_1-2n_1+m_1} \hat{a}_i \hat{a}_j >,
\]
with \( r_1 \geq 0 \).
The effect of averaging on a variable is therefore to give a factor \( d^{-p} \), with \( p \geq \frac{1}{2} \), for an inner product. Therefore one gets that \( I \) gives a sum of terms of the form \( t^p \frac{Z^p}{\hat{a}^p} \), with \( p = \sum_{i=1}^{2h} p_i \geq h \), i.e. a vanishing contribution to \( I \) to the moment for \( d \to \infty \) with \( t = \frac{Z}{\hat{a}} \) fixed.

Let \( |v| > \) be an arbitrary \( d \)-dimensional vector, let \( \{ X_j \} \) be a set \( N \) distinct rank-one projectors, let \( < .. > \) indicate the average over all the random unit vectors associated to the projectors \( X_j \) inside the symbols \( < .. > \).

**Proposition 2.**
\[
\lim_{d \to \infty} \lim_{N \to \infty} < < v \left( \sum_{j=1}^{N} \alpha_j X_j \right) >^s |v| > = < < v | v > P_s(t)
\]
\[(11)\]
where the limits are done while keeping \( t = \frac{Z}{\hat{a}} \) fixed, \( P_s(t) \) is the Narayana polynomial of degree \( s \)
\[
P_s(t) = \sum_{j=1}^{s} N(s, j) t^j, \quad N(s, j) = \frac{1}{s} \binom{s}{j} \binom{s}{j-1}
\]
\[(12)\]

**Proof of Proposition 2.** By expanding the power of the sum, one obtains the sum of \( N^s \) terms, each one being the product of \( s \) non-commuting projectors. As indicated in Proposition 1, each averaged product has a non-vanishing contribution iff it does not contain \( abab \) sequences. It then contributes \( t^p \) where \( p \) is the number of distinct \( X_j \) in the product, \( 1 \leq p \leq s \).
The \( s \)-set of projectors \( \{ X_j \}, j = 1, \ldots, s \) is partitioned into \( p \) parts, such that in each part all projectors are equal. The number of noncrossing partitions [19] with \( p \) parts is \( N(s, p) \).

**Remark.** In Appendix C, it is recalled that Proposition 2 is consequence of an old theorem by Pastur. However the above combinatorial derivation is useful for the derivation of Proposition 4.

A primitive path on the graph, starting at vertex \( r \), returns to it only at the last step. Our proof will use the decomposition of a generic path into concatenated primitive paths and the generating functions corresponding to classes of primitive paths.

Let us define the \( d \times d \) matrix \( B^{(n)}_{r,s} \) to be the sum of the contributions of the primitive paths of \( n \) steps, such that the first edge is \( (r, s) \) and \( B_{r,s}(x) = \sum x^n B^{(n)}_{r,s} \) to be its generating function.

Any path starting and ending at vertex \( j_0 \) has a unique representation as concatenation of primitive paths, each one starting and ending at vertex \( j_0 \). This implies an equation, both for the Adjacency matrix and the Laplacian matrix
\[
T_{j_0}(x) = \sum_{n \geq 0} \left( \sum_j B_{j_0,j}(x) \right)^n
\]
\[(13)\]

In the case of the Adjacency matrix, the paths in \( B^{(A)}_{r,s}(x) \) start by definition with the edge \((r, s)\) and end with \((s, r)\); in between there are all possible tree paths with root \( s \), generated by \( T^{(A)}_{s}(x) \), which is isomorphic to \( T^{(A)}_{j}(x) \) in Eq.(6). Therefore for the Adjacency matrix the generating function of the tree primitive paths with first edge in \((r, s)\) is
\[
B^{(A)}_{r,s}(x) = x^2 \alpha_{r,s} X_{r,s} T^{(A)}_{s}(x) X_{s,r}
\]
\[(14)\]

In the Laplacian matrix, the primitive paths starting with \((r, s)\) can be either the single-edge path corresponding to \( X_{r,s} \), or can start with \( X_{r,s} \); in the latter case, it can continue with any tree path rooted in \( s \) and not going to \( r \) (including the trivial path), then it can either return to \( r \) with \( X_{s,r} \) ending the primitive path, or have an edge corresponding to \( X_{r,s}^2 \); in the latter case it can continue with any tree path rooted in \( s \) and not going to \( r \), and so on, so that
\[
B^{(L)}_{r,s}(x) = \alpha_{r,s} (x X_{r,s}^2 + x^2 X_{r,s} \hat{T}^{(r)}(x) X_{s,r})
\]
\[(15)\]

\[
\hat{T}^{(r)}(x) = T^{(L)}_{s}(x) \sum_{i \geq 0} (x X_{s,r} X_{r,s})^i
\]
\[(16)\]

Each of the \( T^{(L)}_{s}(x) \) is the generating function of primitive tree paths with root \( s \), formed by trees isomorphic to trees with root \( j \), generated by \( T^{(L)}_{j}(x) \) (notice that the trees rooted in \( s \) do not contain the vertex \( r \) otherwise \( B^{(L)}_{r,s} \) would not consist of primitive paths); each of these \( T^{(L)}_{s}(x) \) has different internal edges due to the \( abab \) exclusion rule: if there were an edge corresponding to \( X_{r} \) in common between two trees of two \( T^{(L)}_{s}(x) \), one would have the product \( X_{r,s} X_{r,s} \cdots X_{s,s} \cdots X_{r,s} \cdots \); with \( X_{s} = X_{r,s} = X_{s,r} \), we would get the forbidden \( abab \) pattern.

Each \( B_{r,s}(x) \) contains "internal" unit random vectors and one "external" unit random vector associated to \( X_{r,s} \). The averages can be separated in average on internal and external random vectors.

**Proposition 3.** In the limit \( d \to \infty \), the average of the generating function \( B^{(A)}_{r,s}(x) \) over the internal
variables, $< B_{r,s}^{(A)}(x) >_{I}$ is evaluated in terms of an unknown function $f_{A}(x)$. In a analogous way, the average $< B_{r,s}^{(L)}(x) >_{I}$ is evaluated in terms of the unknown function $f_{L}(x)$.

**Proof of Proposition 3**

For the Adjacency matrix, from Eq. (14), one has

$$B_{r,s}^{(A)}(x) = \alpha_{r,s}x^2|\hat{a}_{r,s}| < T_{s}^{(A)}(x)|\hat{a}_{r,s} > = \alpha_{r,s}X_{r,s}x^2F_{r,s}^{(A)}(x)$$  \hspace{1cm} (17)

where

$$F_{r,s}^{(A)}(x) = < \hat{a}_{r,s}|T_{s}^{(A)}(x)|\hat{a}_{r,s} >$$  \hspace{1cm} (18)

For each tree path, with $n$ steps and $k$ distinct edges, contributing to $T_{s}^{(A)}(x)$, by Proposition 1 the average over the internal edges gives $t^k$. After performing the internal averages, the external edge variable appears only in the form $< \hat{a}_{r,s}|\hat{a}_{r,s} > = 1$, so we can write

$$< F_{r,s}^{(A)}(x) >_{I} = f_{A}(x)$$  \hspace{1cm} (19)

and from Eq. (17)

$$< B_{r,s}^{(A)}(x) >_{I} = \alpha_{r,s}X_{r,s}x^2f_{A}(x)$$  \hspace{1cm} (20)

The analogous derivation, for the Laplacian matrix, obtains

$$< B_{r,s}^{(L)}(x) >_{I} = < \alpha_{r,s}X_{r,s}(x+y^2F_{r,s}^{(L)} \sum_{i=0}^{\infty} (xF_{r,s}^{(L)})^i) >_{I} = \alpha_{r,s}X_{r,s}x \frac{x}{1-xf_{L}(x)}$$  \hspace{1cm} (21)

where it has been used $< (F_{r,s}^{(L)})^i >_{I} = < (F_{r,s}^{(L)}) >_{I}^i = (f_{L}(x))^i$, which follows from the fact that the paths of two trees have different internal edges, due to the abab exclusion rule.

Eqs. (17) to (21) may be summarized:

$$\frac{1}{d} < T_{s}^{(M)}(x) >_{I} = f_{M}(x)$$

$\frac{1}{d} < B_{r,s}^{(M)}(x) >_{I} = \alpha_{r,s}X_{r,s}g^{(M)}(x)$

where $M = A$ or $L$ and $g^{(A)}(x) = x^2f_{A}(x)$, $g^{(L)}(x) = \frac{x}{1-xf_{L}(x)}$  \hspace{1cm} (22)

**Proposition 4.** In the limit $d \to \infty$, the expectation of $T_{j_{0}}(x)$, Eq. (13), has factorization properties which give simple algebraic equations for the resolvents of the Adjacency and Laplacian matrices, respectively those for the effective medium approximation and the Marchenko-Pastur distribution.

**Proof of Proposition 4.** From Eqs. (22, 13)

$$d f(x) = < T_{j_{0}}(x) >_{E} = \sum_{n \geq 0} < \left( \sum_{j} B_{j_{0},j}(x) \right)^n >_{I}$$

$$= \sum_{n \geq 0} \left( g(x) \sum_{j} \alpha_{j_{0},j}X_{j_{0},j} \right)^n >_{E}$$  \hspace{1cm} (23)

Using Eq. (11)

$$f(x) = 1 + \sum_{n \geq 1} P(t) \left( g(x) \right)^n$$

from which,  \hspace{1cm} (24)

so that one obtains

$$g(x) \left( f^2(x) - f(x)(1-t) \right) = f(x) - 1$$  \hspace{1cm} (25)

By use of Eq. (22) one obtains the algebraic equations for the Adjacency matrix and for the Laplacian matrix

$$x^2f_{A}(x)^3 - x^2f_{A}(x)^2(1-t) - f_{A}(x) + 1 = 0$$

$$2xf_{L}(x)^2 + f_{L}(x)(xt - 1 - 2x) + 1 = 0$$  \hspace{1cm} (26)

The average resolvent is $r_{M}(z) = \lim_{d \to \infty} \lim_{N \to \infty} \frac{1}{dN} < Tr(zI - M)^{-1} >$, that is $r_{M}(z) = f_{M}(x)$ with $z = 1/x$. Then the first Eq. (26) is a cubic equation for the resolvent of the Adjacency matrix

$$r_{A}^3(z) + \frac{t-1}{z}r_{A}^2(z) - r_{A}(z) + \frac{1}{z} = 0$$

which is the effective medium approximation by Semerjian and Cugliandolo.  \hspace{1cm} (27)

The second Eq. (26) is a quadratic equation for the resolvent of the Laplacian matrix

$$2zr_{L}^2(z) + (t - 2 - z)r_{L}(z) + 1 = 0$$

It corresponds to the Marchenko-Pastur spectral distribution in Eq. (3).
V. CONCLUSIONS

We argued that the sparse block random matrix considered in this paper deserves to be studied. It seems unlikely that an analytic representation for spectral (limiting) distributions for the Adjacency matrix and for the Laplacian matrix will be found in terms of elementary functions. Indeed an analytic expression for the spectral density of the Erdos-Renyi ensemble is not known. In paper [3] the first five moments were analytically evaluated and some unexpected relations with other matrix models were conjectured. Here, many more moments were analytically evaluated by computer exact enumeration. They agree with the conjectured limiting matrix models in the proper limits of the two parameters, the average connectivity $Z$ and the block dimension $d$. It seems likely that the availability of several analytically exact spectral moments will be useful for approximate evaluations of the spectral functions.

In Sect.IV we proved that in the limit of large physical dimension, $d \to \infty$, with $Z$ fixed, the sparse block model for the Adjacency model and the Laplacian model converge respectively to the Semerjian-Cugliandolo and the Marchenko-Pastur distributions, as conjectured in (1). In paper [3] the first five moments were analytically evaluated by computer exact enumeration. They agree with the conjectured limiting matrix models in the proper limits of the two parameters, the average connectivity $Z$ and the block dimension $d$. It seems likely that the availability of several analytically exact spectral moments will be useful for approximate evaluations of the spectral functions.

VI. ACKNOWLEDGMENTS

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Appendix A: Derivation of the first three moments in the Laplacian matrix

We derive the first three moments of the limiting Laplacian matrix. At the first three orders there are only noncrossing partitions, so by Proposition 1 the contribution to a moment of a product of $X$ blocks, $m$ of which are distinct, is $t^m$. Furthermore the expansion in Eqs. (13, 16) holds; we verify them here for the laplacian matrix through order $x^3$, by comparing Eq. (5) with the primitive path decomposition Eq. (13), which at the first three orders reads

\begin{equation}
T_{j_0}^{(L)}(x) = 1 + x L_{j_0,j_0} + x^2 (L^2)_{j_0,j_0} + x^3 (L^3)_{j_0,j_0} + O(x^4)
\end{equation}

where here and in the following the sums are over $j_r \neq j_s$ for $r \neq s$.

We write for short $(\alpha X)_{i,j} = \alpha_{i,j} X_{i,j}$, and $(\alpha X)^2_{i,j} = \alpha_{i,j} X_{i,j} X_{j,i}$ for a diagonal term. From Eq. (5)

\begin{equation}
T_{j_0}^{(L)}(x) = 1 + x L_{j_0,j_0} + x^2 (L^2)_{j_0,j_0} + x^3 (L^3)_{j_0,j_0} + O(x^4)
\end{equation}

(A2)

Denote by $[x^k]$ the term $x^k$ of the series in $x$ of $f$.

One has

\begin{equation}
[x] T_{j_0}^{(L)} = \sum_{j_1} (\alpha X)^2_{j_0,j_1}
\end{equation}

(A3)

\begin{equation}
\nu_1 = \lim_{N \to \infty} \frac{1}{Nd} \sum_{j_0} < tr([x] T_{j_0}^{(L)}) > = \lim_{N \to \infty} \frac{1}{Nd} N(N-1) < tr(\alpha X)_1 > = \frac{Z}{d} = t
\end{equation}

(A4)

One can compute order by order $(L^n)_{j_0,j}$ using the following formula: let $P_S$ be a product of $(\alpha X)$’s with a set $S$ of indices, all different. In $P_S \sum_{j_1} L_{k,j_1} L_{j_1,j}$ separate $J_1$ in one of the indices in $S - \{ k \} \ or \ k$ or a new index $J'$; one of the resulting terms contains $L_{k,k} = \sum_{j_2} (\alpha X)^2_{k,j_2}$; separating $J_2$ in the indices present on in a new index, we get

\begin{equation}
P_S \sum_{j_1} L_{k,j_1} L_{j_1,j} = \sum_{h \in S - \{ k \}} P_S \left( - (\alpha X)_{k,h} L_{h,j} + (\alpha X)_{k,h}^2 L_{k,j} \right) + \sum_{J'} P_S \left( - (\alpha X)_{k,J'} L_{J',j} + (\alpha X)_{k,J'}^2 L_{k,j} \right)
\end{equation}

(A5)
where \( j' \) is a new index, not in \( S + \{k\} \), and where a term of the sum on \( h \) contributes only if no loop in the associated graph is formed. For \( P_S = 1 \) and \( k = j_0 \) one gets

\[
(L^2)_{j_0,J} = \sum_{j_1} \left( (\alpha X)_{j_0,j_1}^2 L_{j_0,J} - (\alpha X)_{j_0,j_1} L_{j_1,J} \right) \tag{A6}
\]

Taking in this equation \( J = j_0 \) and expanding \( L_{j_0,j_0} \) one gets

\[
[x^2]T^{(L)}_{j_0} = \sum_{j_1} \left( (\alpha X)_{j_0,j_1}^2 (\alpha X)_{j_0,j_0} + (\alpha X)_{j_0,j_1} (\alpha X)_{j_1,j_0} + \sum_{j_2} (\alpha X)_{j_0,j_1} (\alpha X)_{j_2,j_0} \right) \tag{A7}
\]

from which

\[
\nu_2 = \lim_{N \to \infty} \frac{1}{N^d} \sum_{j_0} < \text{tr}([x^2]T^{(L)}_{j_0})> = 2t + t^2 \tag{A8}
\]

From Eqs. (A3, A7, A1) one gets

\[
B_{j_0,j_1} = x(\alpha X)_{j_0,j_1}^2 + x^2(\alpha X)_{j_0,j_1}(\alpha X)_{j_1,j_0} + O(x^3) \tag{A9}
\]

which agrees Eqs. (15, 16) with \( \hat{T}_{j_0}^{(j_1)} = 1 + O(x) \).

Expand \( (L^2)_{j_0,J} = \sum_{j_1} (L^2)_{j_0,j_1} L_{j_1,J} \) using Eqs. (A5, A6)

\[
(L^2)_{j_0,J} = \sum_{j_1} \left( (\alpha X)_{j_0,j_1}(\alpha X)_{j_1,j_0} + (\alpha X)_{j_0,j_1}^2 + \sum_{j_2} (\alpha X)_{j_0,j_2}^2 \right) L_{j_0,J} - \left( (\alpha X)_{j_0,j_1}(\alpha X)_{j_0,j_1} + (\alpha X)_{j_0,j_1}^2 + \sum_{j_2} (\alpha X)_{j_1,j_2}^2 \right) L_{j_1,J} + \sum_{j_2} (- (\alpha X)_{j_0,j_1}(\alpha X)_{j_0,j_2} + (\alpha X)_{j_0,j_1}(\alpha X)_{j_1,j_2}) L_{j_2,J} \tag{A10}
\]

Then expand this expression for \( J = j_0 \) to get \( [x^3]T^{(L)}_{j_0} \); the last term in Eq. (A10) becomes \(- \sum_{j_0,j_1,j_2,j_3} (\alpha X)_{j_0,j_1}(\alpha X)_{j_1,j_2}(\alpha X)_{j_2,j_0} \), which gives vanishing contribution to \( \nu_3 \), since the corresponding graph is a loop.

We separate the contributions to \( [x^3]T^{(L)}_{j_0} \) of the various \( \prod B \) terms in Eq. (A1) and match them with the expression for \( [x^3]T^{(L)}_{j_0} \) obtained above:

\[
[x^3] \sum_{j_0} B_{j_0,j_1} = \sum_{j_0} (\alpha X)_{j_0,j_1}((\alpha X)_{j_1,j_0}^2 + \sum_{j_2} (\alpha X)_{j_1,j_2}^2)(\alpha X)_{j_1,j_0} \tag{A11}
\]

\[
[x^3] \sum_{j_0} (B_{j_0,j_1})^2 = \sum_{j_0} (\alpha X)_{j_0,j_1}(\alpha X)_{j_1,j_0}(\alpha X)_{j_0,j_1}^2 + (\alpha X)_{j_0,j_1}(\alpha X)_{j_0,j_1}(\alpha X)_{j_1,j_0} \tag{A12}
\]

\[
[x^3] \sum_{j_0} B_{j_0,j_1} B_{j_0,j_2} = \sum_{j_0} (\alpha X)_{j_0,j_1}(\alpha X)_{j_1,j_0}(\alpha X)_{j_0,j_2}^2 + (\alpha X)_{j_0,j_1}(\alpha X)_{j_0,j_2}(\alpha X)_{j_2,j_0} \tag{A13}
\]

\[
[x^3] \sum_{j_0} (B_{j_0,j_1})^3 = \sum_{j_0} ((\alpha X)_{j_0,j_1})^3 + \sum_{j_1,j_2,j_3} ((\alpha X)_{j_0,j_1})^2(\alpha X)_{j_0,j_2}^2 + (\alpha X)_{j_0,j_1}(\alpha X)_{j_0,j_2}(\alpha X)_{j_0,j_3} \tag{A14}
\]

so that \( \nu_3 = 4t + 6t^2 + t^3 \).

From Eqs. (A11, A11) we get

\[
\hat{T}_{j_0}^{(j_1)}(x) = 1 + x \sum_{j_2} (\alpha X)_{j_1,j_2}^2 + x(\alpha X)_{j_1,j_0}^2 + O(x^2) \tag{A15}
\]

which satisfies Eq. (16).

Eqs. (A12, A13, A14) follow trivially from Eq. (A9).
Appendix B: The moments in the Adjacency and Laplacian matrices, for generic $d$.

We report here the analytic evaluation of several moments, helped by computer symbolic enumeration. In the case of the Adjacency matrix, we evaluated the moments up to $\mu_{26} = \lim_{N \to \infty} \frac{1}{N^d} < \text{Tr} A^{26} >$.

In the case of the Laplacian matrix, we evaluated the moments up to $\nu_{15} = \lim_{N \to \infty} \frac{1}{N^d} < \text{Tr} L^{15} >$.

We report here the moments only up to $\mu_{18}$ and $\nu_{10}$. The moments are displayed in terms of the two variables $t = Z/d$ and $d$, but the latter variable only appears in the coefficients $c_m$. This representation is useful to read the spectral moments for fixed $t = Z/d$ and the extreme values of $d$: $d = 1$ and $d = \infty$. Indeed every $c_m = 1$ for $d = 1$ and every $c_m = 0$ for $d = \infty$.

Let us note that the form taken by the averages depends on the order in which the variables are integrated; the results are given for a particular choice of order in the averages, so they could be written in other equivalent forms.

\begin{align}
\mu_k &= \lim_{N \to \infty} \frac{1}{N^d} < \text{Tr} A^k > , \quad \mu_0 = 1 , \quad \mu_{2k+1} = 0 \\
\mu_2 &= t , \quad \mu_4 = t + 2t^2 , \quad \mu_6 = t + 6t^2 + 5t^3 \\
\mu_8 &= t + t^2 (12 + 2c_2) + 28t^3 + 14t^4 \\
\mu_{10} &= t + t^2 (20 + 10c_2) + t^3 (90 + 20c_2) + 120t^4 + 42t^5 \\
\mu_{12} &= t + t^2 (30 + 30c_2 + 2c_3) + t^3 \left(220 + \frac{5}{3}c_2(88 + 5c_2)\right) + \\
&\quad + t^4 \left(550 + 132c_2 + 495t^5 + 132t^6\right) \\
\mu_{14} &= t + t^2 \left(42 + 70c_2 + 14c_3\right) + t^3 \left(455 + \frac{c_2}{3} (1820 + 301c_2) + 28c_3\right) + \\
&\quad + t^4 \left(1820 + c_2^2 \left(3934 + 350c_2\right)\right) + t^5 \left(3003 + 728c_2 + 2002t^6 + 429t^7\right) \\
\mu_{16} &= t + t^2 \left(56 + 140c_2 + 56c_3 + 2c_4\right) + \\
&\quad + t^3 \left(840 + 160c_2^2 \left(35 + c_3\right) + 256c_3 + 612(c_2)^2\right) + \\
&\quad + t^4 \left(4900 + \frac{c_2}{3} (10597 + 74(c_2)^2) + 1808(c_2)^2 + 240c_3\right) + \\
&\quad + t^5 \left(12740 + 9280c_2 + 1000(c_2)^2\right) + t^6 \left(15288 + 3640c_2 + 800t^7 + 1430t^8\right) \\
\mu_{18} &= t + t^2 \left(72 + 252c_2 + 168c_3 + 18c_4\right) + \\
&\quad + t^3 \left(1428 + 4760c_2 + \frac{c_3}{5} (103c_3 + 3342c_2) + 2596(c_2)^2 + 1296c_3 + 36c_4\right) + \\
&\quad + t^4 \left(11424 + 27462c_2 + \frac{c_2}{3} (10372 + 815c_2) + 960c_2c_3 + 2754c_3\right) + \\
&\quad + t^5 \left(42840 + 62484c_2 + 1632c_3 + 19185(c_2)^2 + 888(c_2)^3\right) + \\
&\quad + t^6 \left(79968 + 57256c_2 + 6800(c_2)^2\right) + t^7 \left(74256 + 17136c_2\right) + \\
&\quad + 31824t^8 + 4862t^9
\end{align}
\[ \nu_k = \lim_{N \to \infty} \frac{1}{N d} \mathbb{E} \left[ \text{Tr} L^k \right], \quad \nu_0 = 1 \]

\[ \nu_1 = t, \quad \nu_2 = 2t + t^2, \quad \nu_3 = 4t + 6t^2 + t^3 \]

\[ \nu_4 = 8t + t^2(24 + c_2) + 12t^3 + t^4 \]

\[ \nu_5 = 16t + t^2(80 + 10c_2) + t^3(80 + 5c_2) + 20t^4 + t^5 \]

\[ \nu_6 = 32t + t^2(240 + 60c_2 + c_3) + t^3 \left(400 + 72c_2 + 4c_2 \frac{1 + 2c_2}{3} \right) + \]

\[ + t^4(200 + 15c_2) + 30t^5 + t^6 \]

\[ \nu_7 = 64t + t^2(672 + 280c_2 + 14c_3) + \]

\[ + t^3 \left(1680 + 588c_2 + 14(c_2)^2 + 7c_3 + 56 \frac{c_2}{3}(1 + 2c_2) \right) + \]

\[ + t^4 \left(1400 + 294c_2 + 28 \frac{c_2}{3}(1 + 2c_2) \right) + t^5(420 + 35c_2) + 42t^6 + t^7 \]

\[ \nu_8 = 128t + t^2(1792 + 1120c_2 + 112c_3 + c_4) + \]

\[ + t^3 \left(6272 + 128c_3 + 50 \frac{c_2^2}{3}(c_3 + 200) + 544(c_2)^2 \right) + \]

\[ + t^4 \left(7840 + 28c_3 + \frac{c_2}{3}(9925 + 1412c_2) + 12(c_2)^3 \right) + \]

\[ + t^5 \left(3920 + 112 \frac{c_2}{3}(25 + 2c_2) \right) + t^6(784 + 70c_2) + 56t^7 + t^8 \]

\[ \nu_9 = 256t + t^2(4608 + 4032c_2 + 672c_3 + 18c_4) + \]

\[ + t^3 \left(21504 + 32c_2(595 + 131c_2) + 1296c_3 + 4 \frac{c_2}{5}(11c_3 + 504c_2) + 9c_4 \right) + \]

\[ + t^4 \left(37632 + 4\left(\frac{c_2}{3}\right)^2(4727 + 286c_2) + 168c_2c_3 + 25950c_2 + 648c_3 \right) + \]

\[ + t^5 \left(28224 + 108(c_2)^3 + 2388(c_2)^2 + 12975c_2 + 84c_3 \right) + \]

\[ + t^6 \left(9408 + 2380c_2 + 224(c_2)^2 \right) + t^7(1344 + 126c_2) + 72t^8 + t^9 \]

\[ \nu_{10} = 512t + t^2(11520 + 13440c_2 + 3360c_3 + 180c_4 + c_5) + \]

\[ + t^3 \left(69120 + 26240c_2^2 + 4 \frac{c_2^2}{3}(25c_4 + 3584c_3) + 85120c_2 + 248c_3^2 + 9600c_3 + 200c_4 \right) + \]

\[ + t^4 \left(161280 + 20\left(\frac{c_2}{3}\right)^2(26897 + 2792c_2 + 137c_3) + 4447c_2c_3 + 164300c_2 + \right. \]

\[ + 88(c_3)^2 + 8100c_3 + 45c_4) + \]

\[ + t^5 \left(169344 + 2\frac{c_2^2}{27}(1705527 + 56807(c_2)^2 + 1036(c_2)^3) + 39438c_2^2 + 840c_2c_3 + \right. \]

\[ + 2400c_3) + \]

\[ + t^6 \left(84672 + 540c_2^3 + 8860c_2^2 + 41075c_2 + 210c_3 \right) + t^7 \left(560c_2^2 + 5320c_2 + 20160 \right) + \]

\[ + t^8 (2160 + 210c_2) + 90t^9 + t^{10} \]

(B2)

---

**Appendix C: Spectral density of \( L_{1,1} \).**

\( N \) is the Poisson distribution of parameter \( Z \)

Every diagonal block of the Laplacian matrix is the sum of \((N - 1)\) identically distributed random matrices.

\[ L_{1,1} = \sum_{j=2}^{N} \alpha_{1,j} X_{1,j} \]

In the \( d = 1 \) case, each block \( X_{1,j} \) is replaced by one and the probability law of a diagonal entry of \( L \) for large

\[ P \left( L_{1,1} = \sum_{j=2}^{N} \alpha_{1,j} = k \right) = \frac{Z^k}{k!} e^{-Z} \]
The moments \( m_s \) of Poisson distribution are

\[
m_s = \lim_{N \to \infty} \left< \left( \sum_{j=2}^{N} \alpha_{1,j} \right)^s \right> = \sum_{k=0}^{\infty} k^s \frac{Z^k}{k!} e^{-Z} = \sum_{i=1}^{s} Z^i S(s, i)
\]

where \( S(s, i) \) are the Stirling numbers of second kind, that is the number of partitions of a \( s \)-set \( \{X\} = \{X_1, X_2, \ldots, X_s\} \) into \( i \) parts.

For generic dimension, \( 1 < d < \infty \), we computed the moments through \( m_{15} \). We report here the first five.

\[
m_s = \frac{1}{d} \lim_{N \to \infty} \left< \text{tr} \left( \sum_{j=2}^{N} \alpha_{1,j} X_{1,j} \right)^s \right>, \quad m_1 = t = Z/d
\]

\[
m_2 = t + t^2
\]

\[
m_3 = t + 3t^2 + t^3
\]

\[
m_4 = t + t^2 (6 + c_2) + 6t^3 + t^4
\]

\[
m_5 = t + t^2 (10 + 5c_2) + 3t^3 (20 + 5c_2) + 10t^4 + t^5
\]

\[\text{(C1)}\]

Narayana polynomials

\[
P_s(t) = \sum_{j=1}^{s} N(s, j) t^j
\]

In the Section IV Proposition 2, this result is derived by a combinatorial argument; the difference with respect to the \( d = 1 \) case is that the restriction to the terms without the \( abab \) pattern reduce the number of partitions of \( n \) elements with \( k \) blocks from \( S(n, k) \) to \( N(n, k) \).

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