On the $\partial$- and $\bar{\partial}$-Operators of a Generalized Complex Structure

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Abstract

In this note, we prove that the $\partial$- and $\bar{\partial}$-operators introduced by Gualtieri for a generalized complex structure coincide with the $\bar{\partial}_\ast$- and $\partial$-operators introduced by Alekseev-Xu for Evens-Lu-Weinstein modules of a Lie bialgebroid.

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1 Introduction

Generalized complex structures \cite{4,8,10} have been extensively studied recently due to their close connection with mirror symmetry. They include both symplectic and complex structures as extreme cases. Gualtieri defined the $\partial$- and $\bar{\partial}$-operators for any twisted generalized complex structure in the same way the $\partial$- and $\bar{\partial}$-operators are defined in complex geometry \cite{8,9}.

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A Lie bialgebroid, as introduced by Mackenzie \& Xu, is a pair of Lie algebroids $(A, A^*)$ satisfying some compatibility condition \cite{11,17}. They appear naturally in many places in Poisson geometry. In \cite{1}, two differential operators $d_e, \partial$ were introduced for Evens-Lu-Weinstein modules of a Lie bialgebroid as follows.

Consider a pair of (real or complex) Lie algebroid structures on a vector bundle $A$ and its dual $A^*$. Assume that the line bundle (real or complex) $\mathcal{L} = (\wedge^{\top} A^* \otimes \wedge^{\top} T^* M)^{\frac{1}{2}}$ exists, then $\mathcal{L}$ is a module over $A^*$, as discovered by Evens, Lu and Weinstein \cite{7}. The Lie algebroid structures of $A^*$ and $A$ induce two natural differential operators $\partial_e : \Gamma(\wedge^k A \otimes \mathcal{L}) \to \Gamma(\wedge^{k+1} A \otimes \mathcal{L})$ and $\partial : \Gamma(\wedge^k A \otimes \mathcal{L}) \to \Gamma(\wedge^{k-1} A \otimes \mathcal{L})$ (see Equations \cite{10} to \cite{16}).

Since a generalized complex structure $\mathcal{J}$ induces a (complex) Lie bialgebroid $(L, \mathcal{T})$, where $L$ and $\mathcal{T}$ are, respectively, the $(+i)$ and $(-i)$-eigenspaces of $\mathcal{J}$, it is tempting to investigate the relations between the operators $\partial, \partial_e, d_e, \partial$. In this note, we show that $\partial$ and $\partial_e$ essentially coincide with $d_e$ and $\partial_e$ respectively, under some natural isomorphisms.

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2 Courant algebroids and Lie bialgebroids

In this article, all vector bundles are complex vector bundles. Likewise, Lie algebroids are always complex Lie algebroids.

A (complex) Courant algebroid consists of a vector bundle $\pi : E \to M$, a nondegenerate pseudo-metric $\langle \cdot, \cdot \rangle$ on the fibers of $\pi$, a bundle map $\rho : E \to T_{\mathbb{C}}M$, called anchor, and a $\mathbb{C}$-bilinear operation $\circ$ on $\Gamma(E)$ called Dorfman bracket, which, for all $f \in C^\infty(M, \mathbb{C})$ and $z_1, z_2, z_3 \in \Gamma(E)$ satisfy the relations

\begin{align*}
  z_1 \circ (z_2 \circ z_3) &= (z_1 \circ z_2) \circ z_3 + z_2 \circ (z_1 \circ z_3); & (1) \\
  \rho(z_1 \circ z_2) &= [\rho(z_1), \rho(z_2)]; & (2) \\
  z_1 \circ f z_2 &= (\rho(x)f) z_2 + f(z_1 \circ z_2); & (3) \\
  z_1 \circ z_2 + z_2 \circ z_1 &= 2\mathcal{D}\langle z_1, z_2 \rangle; & (4) \\
  \mathcal{D}f \circ z_1 &= 0; & (5) \\
  \rho(z_1)\langle z_2, z_3 \rangle &= \langle z_1 \circ z_2, z_3 \rangle + \langle z_2, z_1 \circ z_3 \rangle; & (6)
\end{align*}

where $\mathcal{D} : C^\infty(M, \mathbb{C}) \to \Gamma(E)$ is the $\mathbb{C}$-linear map defined by $\langle \mathcal{D}f, z_1 \rangle = \frac{1}{2}\rho(z_1)f$.

The symmetric part of the Dorfman bracket is given by \cite{4}. The Courant bracket is defined as the skew-symmetric part $\llbracket z_1, z_2 \rrbracket = \frac{1}{2} (z_1 \circ z_2 - z_2 \circ z_1)$ of the Dorfman bracket. Thus we have the relation $z_1 \circ z_2 = \llbracket z_1, z_2 \rrbracket + \mathcal{D}\langle z_1, z_2 \rangle$.

The definition of a Courant algebroid can be rephrased using the Courant bracket instead of the Dorfman bracket \cite{19}.

A Dirac structure is a smooth subbundle $A \to M$ of the Courant algebroid $E$, which is maximal isotropic with respect to the pseudo-metric and whose space of sections is
closed under (necessarily both) brackets. Thus a Dirac structure inherits a canonical Lie algebroid structure \[15\].

Let \( A \to M \) be a vector bundle. Assume that \( A \) and its dual \( A^* \) both carry a Lie algebroid structure with anchor maps \( a : A \to T^*_CM \) and \( a_\#: A^* \to T^*_CM \), brackets on sections \( \Gamma(A) \otimes \Gamma(A) \to \Gamma(A) : X \otimes Y \mapsto [X, Y] \) and \( \Gamma(A^*) \otimes \Gamma(A^*) \to \Gamma(A^*) : \theta \otimes \xi \mapsto [\theta, \xi]_s \), and differentials \( d : \Gamma(A^* \wedge \cdot A^*) \to \Gamma(A^* \wedge \cdot 1_\cdot A^*) \) and \( d_\#: \Gamma(A^\wedge \cdot A) \to \Gamma(A^\wedge \cdot 1_\cdot A) \).

This pair of Lie algebroids \((A, A^*)\) is a Lie bialgebroid (or Manin triple) \[11\]\[16\]\[17\] if \( d_\# \) is a derivation of the Gerstenhaber algebra \((\Gamma(A^\wedge \cdot A), \wedge, [\cdot, \cdot])\) or, equivalently, if \( d \) is a derivation of the Gerstenhaber algebra \((\Gamma(A^* \wedge \cdot A^*), \wedge, [\cdot, \cdot]_s)\). The link between Courant and Lie bialgebroids is given by the following

**Theorem 2.1** \[15\]. *There is a 1-1 correspondence between Lie bialgebroids and pairs of transversal Dirac structures in a Courant algebroid.*

More precisely, if the pair \((A, A^*)\) is a Lie bialgebroid, then the vector bundle \( A \oplus A^* \to M \) together with the pseudo-metric
\[
\langle X_1 + \xi_1, X_2 + \xi_2 \rangle = \frac{1}{2} (\xi_1(X_2) + \xi_2(X_1)),
\]
the anchor map \( \rho = a + a_\# \) (whose dual is given by \( Df = df + d_\# f \) for \( f \in C^\infty(M, \mathbb{C}) \)) and the Dorfman bracket
\[
(X_1 + \xi_1) \circ (X_2 + \xi_2) = \left( [X_1, X_2] + \mathcal{L}_x_1 X_2 - i_{\xi_2}(d_\# X_1) \right) + \left( [\xi_1, \xi_2]_s + \mathcal{L}_{\xi_1} \xi_2 - i_{\xi_2}(d\xi_1) \right)
\]
is a Courant algebroid of which \( A \) and \( A^* \) are transverse Dirac structures. It is called the double of the Lie bialgebroid \((A, A^*)\). Here \( X_1, X_2 \) denote arbitrary sections of \( A \) and \( \xi_1, \xi_2 \) arbitrary sections of \( A^* \).

An important example is that when \( A = T^*_CM \) is the tangent bundle of a manifold \( M \) and \( A^* = T^*_CM \) takes the trivial Lie algebroid structure. Then \( T^*_CM \oplus T^*_CM \) has the standard Courant algebroid structure. As observed by Severa and Weinstein in \[20\], the Dorfman bracket on \( T^*_CM \oplus T^*_CM \) can be twisted by a closed 3-form \( H \in Z^3(M) \):
\[
(x_1 + \eta_1) \circ_H (x_2 + \eta_2) = (x_1 + \eta_1) \circ (x_2 + \eta_2) + i_{x_2}^* \xi_1 H - i_{\xi_2}^* \xi_1 H.
\]

And \( \circ_H \) defines a Courant algebroid structure on \( T^*_CM \oplus T^*_CM \), using the same inner product and anchor. The corresponding Courant bracket is also twisted:
\[
[x_1 + \eta_1, x_2 + \eta_2]_H = [x_1 + \eta_1, x_2 + \eta_2] + i_{x_2}^* \xi_1 H - i_{\xi_2}^* \xi_1 H.
\]

### 3 Clifford modules and Dirac generating operators

Let \( V \) be a vector space of dimension \( r \) endowed with a nondegenerate symmetric bilinear form \( \langle \cdot, \cdot \rangle \). Its Clifford algebra \( \mathcal{C}(V) \) is defined as the quotient of the tensor
algebra \( \oplus_{k=0}^{r} V^\otimes r \) by the relations \( x \otimes y + y \otimes x = 2(x, y) \) \( (x, y) \in V \). It is naturally an associative \( \mathbb{Z}_2 \)-graded algebra. Up to isomorphisms, there exists a unique irreducible module \( S \) of \( \mathcal{C}(V) \), called spin representation \( \mathcal{O} \). The vectors of \( S \) are called spinors.

**Example 3.1.** Let \( W \) be a vector space of dimension \( r \). We can endow \( V = W \oplus W^* \) with the nondegenerate pairing defined in the same fashion as in Eq. (7). The representation of \( \mathcal{C}(V) \) on \( S = \oplus_{k=0}^{r} \Lambda^k W \) defined by \( u \cdot w = u \wedge w \) and \( \xi \cdot w = \iota_\xi w \), where \( u \in W, \xi \in W^* \) and \( w \in S \), is the spin representation. Note that \( S \) is \( \mathbb{Z}_2 \)- and thus also \( \mathbb{Z}_2 \)-graded.

Now let \( \pi : E \to M \) be a vector bundle endowed with a nondegenerate pseudo-metric \( \langle \cdot, \cdot \rangle \) on its fibers and let \( \mathcal{C}(E) \to M \) be the associated bundle of Clifford algebras. Assume there exists a smooth vector bundle \( S \to M \) whose fiber \( S_m \) over a point \( m \in M \) is the spin module of the Clifford algebra \( \mathcal{C}(E)_m \). Assume furthermore that \( S \) is \( \mathbb{Z}_2 \)-graded: \( S = S_0 \oplus S_1 \).

An operator \( O \) on \( \Gamma(S) \) is called even (or of degree 0) if \( O(S^i) \subset S^i \) and odd (or of degree 1) if \( O(S^i) \subset S^{i+1} \). Here \( i \in \mathbb{Z}_2 \).

**Example 3.2.** If the vector bundle \( E \) decomposes as the direct sum \( A \oplus A^* \) of two transverse Lagrangian subbundles as in Example 3.1, then \( S = \Lambda A \). The multiplication by a function \( f \in C^\infty(M, \mathbb{C}) \) is an even operator on \( \Gamma(S) \) while the Clifford action of a section \( e \in \Gamma(E) \) is an odd operator on \( \Gamma(S) \).

If \( O_1 \) and \( O_2 \) are operators of degree \( d_1 \) and \( d_2 \) respectively, then their commutator is the operator \( [O_1, O_2] = O_1 \circ O_2 - (-1)^{d_1d_2} O_2 \circ O_1 \).

**Definition 3.3.** A Dirac generating operator for \( (E, \langle \cdot, \cdot \rangle) \) is an odd operator \( D \) on \( \Gamma(S) \) satisfying the following properties:

(a) For all \( f \in C^\infty(M, \mathbb{C}) \), \( [D, f] \in \Gamma(E) \). This means that the operator \( [D, f] \) is the Clifford action of some section of \( E \).

(b) For all \( z_1, z_2 \in \Gamma(E) \), \( [[D, z_1], z_2] \in \Gamma(E) \).

(c) The square of \( D \) is the multiplication by some function on \( M \): that is \( D^2 \in C^\infty(M, \mathbb{C}) \).

Note that “deriving operators” of \([13]\) do not require the assumption \([13] \).

**Theorem 3.4.** Let \( D \) be a Dirac generating operator for a vector bundle \( \pi : E \to M \). Then there is a canonical Courant algebroid structure on \( E \). The anchor \( \rho : E \to T^*_C M \) is defined by \( \rho(z) f = 2\langle [D, f], z \rangle = [[D, f], z] \), while the Dorfman bracket reads \( z_1 \circ z_2 = [[D, z_1], z_2] \).

We follow the same setup as in \([1]\) and \([5]\).

Let \( (A, [\cdot, \cdot], a) \) and \( (A^*, [\cdot, \cdot], a_*) \) be a pair of Lie algebroids, where \( A \) is of rank \( r \) and the base manifold \( M \) is of dimension \( m \). The line bundle \( \Lambda^r A^* \otimes \Lambda^m T^*_C M \) is a module over the Lie algebroid \( A^* \) \([7]\): a section \( \alpha \in \Gamma(A^*) \) acts on \( \Gamma(\Lambda^r A^* \otimes \Lambda^m T^*_C M) \) by

\[
\alpha \cdot (\alpha_1 \wedge \cdots \wedge \alpha_r \otimes \mu) = \sum_{i=1}^{r} \left( \alpha_1 \wedge \cdots \wedge [\alpha, \alpha_i]_a \wedge \cdots \wedge \alpha_r \otimes \mu \right) + \alpha_1 \wedge \cdots \wedge \alpha_r \otimes \mathcal{L}_{a_*(\alpha)} \mu. \tag{10}
\]
If it exists, the square root \( \mathcal{L} = (\wedge^r A^* \otimes \wedge^m T^*_C M)^{1/2} \) of this line bundle is also a module over \( A^* \). One can thus define a differential operator

\[
\tilde{d}_* : \Gamma(\wedge^k A \otimes \mathcal{L}) \to \Gamma(\wedge^{k+1} A \otimes \mathcal{L}).
\] (11)

Similarly, \( (\wedge^r A \otimes \wedge^m T^*_C M)^{1/2} \) is — provided it exists — a module over \( A \). Hence we obtain a differential operator

\[
\Gamma(\wedge^k A^* \otimes (\wedge^r A \otimes \wedge^m T^*_C M)^{1/2}) \to \Gamma(\wedge^{k+1} A^* \otimes (\wedge^r A \otimes \wedge^m T^*_C M)^{1/2}).
\] (12)

But the isomorphisms of vector bundles

\[
\wedge^k A^* \cong \wedge^k A^* \otimes \wedge^{r-k} A^* \otimes \wedge^r A \cong \wedge^{r-k} A \otimes \wedge^r A^*
\] (13)

and

\[
\wedge^r A^* \otimes (\wedge^r A \otimes \wedge^m T^*_C M)^{1/2} \cong (\wedge^r A^* \otimes \wedge^m T^*_C M)^{1/2}
\] (14)

imply that

\[
\wedge^k A^* \otimes (\wedge^r A \otimes \wedge^m T^*_C M)^{1/2} \cong \wedge^{r-k} A \otimes \wedge^r A^* \otimes (\wedge^r A \otimes \wedge^m T^*_C M)^{1/2}
\]

\[
\cong \wedge^{r-k} A \otimes (\wedge^r A^* \otimes \wedge^m T^*_C M)^{1/2}. \] (15)

Therefore, one ends up with a differential operator

\[
\tilde{\partial} : \Gamma(\wedge^k A \otimes \mathcal{L}) \to \Gamma(\wedge^{k-1} A \otimes \mathcal{L}).
\] (16)

The following theorem is proved in [5].

**Theorem 3.5.** The pair of Lie algebroids \((A, A^*)\) is a Lie bialgebroid if, and only if, \( \hat{D}^2 \in C^\infty(M, \mathbb{C}) \), i.e., the square of the operator \( \hat{D} = \tilde{d}_* + \tilde{\partial} : \Gamma(\wedge A \otimes \mathcal{L}) \to \Gamma(\wedge^2 A \otimes \mathcal{L}) \) is the multiplication by some function \( \hat{f} \in C^\infty(M, \mathbb{C}) \). Moreover \( \hat{D}^2 = \hat{f} \), where \( \hat{D}_* = \hat{d} + \hat{\partial}_* \) is defined analogously to \( \hat{D} \) by exchanging the roles of \( A \) and \( A^* \).

## 4 Generalized complex geometry

In this section, we fix a real 2n-dimensional manifold \( M \) and denote the tangent and cotangent bundle of \( M \) by \( T \) and \( T^* \), respectively. Let \( T_C \) and \( T^*_C \) be, respectively, the complexification of \( T \) and \( T^* \). The first vital ingredient in \( T_C \oplus T^*_C \) is the natural pairing:

\[
\langle x_1 + \eta_1, x_2 + \eta_2 \rangle = \frac{1}{2}(\langle x_1|x_2 \rangle + \langle x_2|\eta_1 \rangle), \quad \forall x_i \in T_C, \eta_i \in T^*_C.
\] (17)

Thus we have the Clifford algebra \( \mathcal{C}(T_C \oplus T^*_C) \), which acts on the spinor bundle \( \mathcal{M} \triangleq \bigoplus_{i=0}^{2n} \wedge^i T^*_C \) via

\[
(x + \eta) \cdot \rho = i_x \rho + \eta \wedge \rho, \quad \forall \rho \in \mathcal{M}.
\]

Introduce a \( \mathbb{C} \)-linear map \( .^T : \mathcal{M} \to \mathcal{M} \) by

\[
(\eta_1 \wedge \cdots \wedge \eta_j)^T = \eta_j \wedge \cdots \wedge \eta_1.
\]
The Mukai pairing \( (\chi, \omega) : M \times M \rightarrow \wedge^{2n}(T^*_C) \) is defined by

\[
(\chi, \omega) = [\chi^T \wedge \omega]^{2n},
\]

where \([\cdot]^{2n}\) indicates the top degree component of the product. Explicitly, if \(\chi = \sum_{i=0}^{2n} \chi_i, \omega = \sum_{i=0}^{2n} \omega_i\), where \(\chi_i, \omega_i \in \wedge^i T^*_C\), then

\[
(\chi, \omega) = \sum_{i=0}^{2n} (-1)^{(i-1)/2} \chi_i \wedge \omega_{2n-i}.
\]

The following properties are standard [8]:

\[
(\chi, \omega) = (-1)^n (\omega, \chi),
\]

(18)

\[
(\phi \wedge \chi, \omega) + (\chi, \phi \wedge \omega) = 0,
\]

(19)

for all \(\chi, \omega \in M\), \(\phi \in \wedge^2 T^*_C\).

Consider a real, closed 3-form \(H \in Z^3(M)\) which induces a twisted differential operator

\[
d^H = d + H \wedge (\cdot).
\]

**Definition 4.1.** [3, 8] A twisted generalized complex structure with respect to \(H\) is determined by any of the following three equivalent objects:

(i) A real automorphism \(J\) of \(T \oplus T^*\), which squares to \(-1\), is orthogonal with respect to the natural pairing [7], and has vanishing Nijenhuis tensor, i.e., for all \(z_1, z_2 \in \Gamma(T \oplus T^*)\),

\[
N(z_1, z_2) \triangleq -[Jz_1, Jz_2]_H + J[[z_1, z_2]]_H + \frac{1}{3} [z_1, [z_1, z_2]]_H + [z_1, z_2]_H = 0.
\]

Here \([\cdot, \cdot]_H\) is the twisted Courant bracket defined in Eq. [9].

(ii) A twisted Dirac structure \(L \subset T_C \oplus T^*_C\) with respect to \(H\), which satisfies \(L \cap \overline{T} = \{0\}\).

(iii) A line subbundle \(N\) of \(\mathcal{M} = \wedge^\bullet(T^*_C)\) generated at each point by a form \(u\), such that

\[
L = \{ X \in T_C \oplus T^*_C \mid X \cdot u = 0 \}
\]

is maximally isotropic, \((u, \bar{u}) \neq 0\), and

\[
d^H u = e \cdot u,
\]

for some \(e \in \Gamma(T \oplus T^*)\).

The line bundle \(N\) in (iii) is called the pure spinor line bundle corresponding to \(L\).

**Remark 4.2.** Also see [1] for the relation between Dirac structures and Dirac generating operators.
As a generalization of the usual $\partial$- and $\bar{\partial}$-operators in complex geometry, Gualtieri introduced the $\partial$- and $\bar{\partial}$-operators for any twisted generalized complex structure $[8]$. We recall its construction briefly below.

Let $\mathbb{J}$ be a twisted generalized complex structure and $L \subset T_{\mathbb{C}} \oplus T^*_{\mathbb{C}}$ be the $+i$-eigenspace of $\mathbb{J}$. $L$ is a twisted Dirac structure and satisfies $L \cap \mathcal{T} = \{0\}$. We will regard $\mathcal{T} = L^*$ by defining the canonical pairing between $L$ and $\mathcal{T}$ by

$$\langle X, \theta \rangle = 2 \langle X, \theta \rangle, \quad \forall X \in L, \theta \in \mathcal{T}. \quad (20)$$

Set $N_0 = N$, $N_k = \wedge^k \mathcal{T} \cdot N$ ($k = 0, \ldots, 2n$). Then $\mathcal{T}_k = N_{2n-k}$ and especially, $N_{2n} = N$ is the pure spinor of $\mathcal{T}$, and we have a decomposition

$$\mathcal{M} = N_0 \oplus N_1 \oplus \cdots \oplus N_{2n}.$$ 

It is proved that one can decompose $[8]$

$$d^H = \partial + \bar{\partial}.$$ 

Here $\partial : \Gamma(N_*) \rightarrow \Gamma(N_{*-1})$ and $\partial : \Gamma(N_*) \rightarrow \Gamma(N_{*+1})$ (or $\Gamma(\mathcal{N}_*) \rightarrow \Gamma(\mathcal{N}_{*-1})$) are defined by:

$$\partial(n_k) \triangleq pr_{N_{k-1}}(d^H n_k), \quad \bar{\partial}(n_k) \triangleq pr_{N_{k+1}}(d^H n_k), \quad \forall n_k \in \Gamma(N_k).$$

5 The main theorem

Given a generalized complex structure $\mathbb{J}$ as above, it is clear that $(L, \mathcal{T})$ is a Lie bialgebroid (regarding $L^* = \mathcal{T}$ by Eq. (20)). We prove that the operators $\partial_*$ and $\bar{\partial}_*$ for this particular situation are essentially $\partial$ and $\bar{\partial}$ (Theorem 5.2).

We continue the notations in Section 4 and let $u$ be a nowhere vanishing local section of $N$. Assume that $V \in \Gamma(\wedge^{2n} L)$ satisfies $V \cdot \overline{u} = u$. Hence $\nabla \cdot u = \overline{u}$ and

$$\langle V, \nabla \rangle u = (-1)^n V \cdot \nabla \cdot u = (-1)^n u, \quad \langle V, \nabla \rangle = (-1)^n.$$

So the dual section of $V$ is given by $\Omega = (-1)^n \nabla \in \Gamma(\wedge^{2n} \mathcal{T}).$

**Proposition 5.1.** ([8] Prop. 2.22, see also [6] III.3.2) The line bundle $\mathcal{L} = (\wedge^{2n} \mathcal{T} \otimes \wedge^{2n} T^*_{\mathbb{C}})^{\frac{1}{2}}$ and $N_{2n} = N$ are canonically isomorphic. The isomorphism can be explicitly described by the following:

$$\begin{align*}
\mathcal{N} \otimes \overline{\mathcal{N}} &\rightarrow \mathcal{L} \\
\Omega \otimes (V \cdot \omega_1, \omega_2) &\rightarrow \mathcal{L} 
\end{align*} \quad (21)$$

The isomorphism (21) does not depend on the choice of $u$ and $V$.

From now on we will identify $\mathcal{N}$ with $(\wedge^{2n} \mathcal{T} \otimes \wedge^{2n} T^*_{\mathbb{C}})^{\frac{1}{2}}$. As a consequence of the $\mathcal{T}$-module structure on $(\wedge^{2n} \mathcal{T} \otimes \wedge^{2n} T^*_{\mathbb{C}})^{\frac{1}{2}}$, we have two differential operators (see Eq. (11), (16)):

$$\begin{align*}
\partial_* : (\wedge^* L) \otimes \mathcal{N} &\rightarrow (\wedge^{*+1} L) \otimes \mathcal{N} \\
\bar{\partial} : (\wedge^* L) \otimes \mathcal{N} &\rightarrow (\wedge^{*-1} L) \otimes \mathcal{N}.
\end{align*}$$
It is also shown in \[8\] that \((\wedge^k \mathcal{L}) \otimes N \cong N_k\) and \((\wedge^k L) \otimes N \cong N_k\) respectively by the following two isomorphisms:

\[
\begin{align*}
I : (\wedge^k \mathcal{L}) \otimes N & \to N_k, \quad W \otimes p \to W \cdot p, \quad \forall W \in \wedge^k \mathcal{L}, p \in N, \\
\bar{I} : (\wedge^k L) \otimes N & \to N_k, \quad X \otimes \overline{p} \to X \cdot \overline{p}, \quad \forall X \in \wedge^k L, p \in N.
\end{align*}
\]

Our main theorem is

**Theorem 5.2.** The following two diagrams are commutative.

\[
\begin{align*}
(\wedge^k L) \otimes N & \xrightarrow{\bar{I}} (\wedge^{k+1} L) \otimes N \\
\downarrow I & \quad \downarrow I \\
N_k & \xrightarrow{\partial} N_{k+1},
\end{align*}
\]

\[
(\wedge^k L) \otimes N \xrightarrow{\partial} (\wedge^{k-1} L) \otimes N \\
\downarrow I \quad \downarrow I \\
N_k & \xrightarrow{\partial} N_{k-1}.
\]

The proof will be deferred to Section \[7\].

In \[9\], Gualtieri constructed an \(L\)-module structure on \(N\) and an \(\mathcal{L}\)-module structure on \(\overline{N}\), respectively by

\[
\begin{align*}
\nabla_X p & \triangleq X \cdot d^H p = X \cdot \bar{\partial} p, \quad (24) \\
\nabla_W \overline{p} & \triangleq W \cdot d^H \overline{p} = W \cdot \partial \overline{p}, \quad (25)
\end{align*}
\]

\(\forall p \in \Gamma(N), X \in \Gamma(L), W \in \Gamma(\mathcal{L}).\)

As a special situation of \(k = 0\) in Diagram (22), we have

**Proposition 5.3.** The above \(\mathcal{L}\)-module structure defined by Eq. (26) coincides with the \(\mathcal{L}\)-module structure defined by Eq. (10), under the isomorphism (21).

### 6 Modular cocycles of Lie algebroids

In this section we establish a list of important identities valid in any Lie bialgebroid \((A, A^*)\) and generalized complex structure, which are subsequently used in Section \[7\] to prove the statements of Section \[5\].

We continue the assumptions in Section \[8\] let \((A, [\cdot, \cdot], a)\) and \((A^*, [\cdot, \cdot]^*, a_*)\) be a pair of rank-\(r\) Lie algebroids over dimension-\(m\) base manifold \(M\).

Assume there exists a volume form \(s \in \Gamma(\wedge^m T^*_0 M)\) and a nowhere vanishing section \(\Omega \in \Gamma(\wedge^r A^*)\) so that \(\mathcal{L}\) is the trivial line bundle over \(M\). And let \(V \in \Gamma(\wedge^r A)\) be the section dual to \(\Omega\): \(\langle \Omega | V \rangle = 1\). These induce two bundle isomorphisms:

\[
\begin{align*}
\Omega^2 : \wedge^k A & \to \wedge^{r-k} A^* : X \mapsto \iota_X \Omega, \\
V^2 : \wedge^k A^* & \to \wedge^{r-k} A : \xi \mapsto \iota_\xi V,
\end{align*}
\]

(26) (27)
which are essentially inverse to each other:

\[
(V^\sharp \circ \Omega^\sharp)(X) = (-1)^{k(r-1)} X, \quad \forall X \in \wedge^k A; \quad (28)
\]

\[
(\Omega^\sharp \circ V^\sharp)(\varphi) = (-1)^{k(r-1)} \varphi, \quad \forall \varphi \in \wedge^k A^*.
\]

Consider the operator \( \partial \) dual to \( d \) with respect to \( V^\sharp \):

\[
\begin{array}{ccc}
\Gamma(\wedge^k A^*) & \xrightarrow{V^\sharp} & \Gamma(\wedge^{r-k} A) \\
\downarrow{(-1)^k d} & & \downarrow{\partial} \\
\Gamma(\wedge^{k+1} A^*) & \xrightarrow{V^\sharp} & \Gamma(\wedge^{r-k-1} A),
\end{array}
\]

or

\[
-V^\sharp d \alpha = (-1)^k \partial V^\sharp \alpha, \quad \forall \alpha \in \Gamma(\wedge^k A^*),
\]

which, due to (28) and (29), can be rewritten as

\[
\Omega^\sharp \partial \beta = (-1)^l d \Omega^\sharp \beta, \quad \forall \beta \in \Gamma(\wedge^l A).
\]

The operator \( \partial \) is a Batalin-Vilkovisky operator for the Lie algebroid \( A \) [12,14,18,22]. Similarly, we have the operator \( \partial^* \) dual to \( d^* \):

\[
\begin{array}{ccc}
\Gamma(\wedge^{r-k} A) & \xrightarrow{V^\sharp} & \Gamma(\wedge^k A^*) \\
\downarrow{(-1)^k d_*} & & \downarrow{\partial_*} \\
\Gamma(\wedge^{r-k+1} A) & \xrightarrow{V^\sharp} & \Gamma(\wedge^{k-1} A^*),
\end{array}
\]

or

\[
d_* V^\sharp \alpha = (-1)^k V^\sharp \partial_* \alpha, \quad \forall \alpha \in \Gamma(\wedge^k A^*).
\]

According to [7], there exists a unique \( X_0 \in \Gamma(A) \) such that

\[
\mathcal{L}_\theta (\Omega \otimes s) = (\mathcal{L}_\partial \Omega) \otimes s + \Omega \otimes (\mathcal{L}_{\alpha_\theta}(s)) = \langle X_0 | \theta \rangle \Omega \otimes s, \quad \forall \theta \in \Gamma(A^*). \quad (33)
\]

Similarly, there exists a unique \( \xi_0 \in \Gamma(A^*) \) such that

\[
\mathcal{L}_X (s \otimes V) = (\mathcal{L}_{\alpha_{(X)}} s) \otimes V + s \otimes (\mathcal{L}_X V) = \langle \xi_0 | X \rangle s \otimes V, \quad \forall X \in \Gamma(A).
\]

These sections \( X_0 \) and \( \xi_0 \) are called modular cocycles and their cohomology classes are called modular classes [7].

A simple computation yields the following formulas, which are given in [11].

**Proposition 6.1.** With the above notations, the differential operators defined by Eq. (11) and (16) are given respectively:

\[
\bar{d}_* (X \otimes l) = (d_* X + \frac{1}{2} X_0 \wedge X) \otimes l
\]

and

\[
\bar{\partial}(X \otimes l) = (-\partial X + \frac{1}{2} \iota_{\xi_0} X) \otimes l,
\]

for all \( X \in \Gamma(\wedge A) \) and \( l \in \Gamma(\mathcal{L}) \).
Hence the operator $\tilde{D}$ in Theorem 3.5 reads

$$\tilde{D} = \tilde{d} + \partial = d^* - \nu + \frac{1}{2}(X_0 \wedge \cdot + i\xi_0).$$

This construction of Dirac generating operators using modular cocycles appeared in [115].

Now we consider a twisted generalized complex structure $J$ on a $2n$-dimensional manifold $M$ and let $L$ and $\overline{L}$ be respectively the $+i$ and $-i$-eigenspace of $J$. Again we assume that $u$ is a nowhere vanishing local section of $N$, the pure spinor bundle of $L$.

We need the following basic fact:

**Lemma 6.2.** [9] There exists some $e = x + \eta \in \Gamma(L)$ such that

$$d^H u = \partial u = du + H \wedge u = e \cdot u = i_x u + \eta \wedge u,$$

$$d^H \overline{u} = \overline{\partial u} = d\overline{u} + H \wedge \overline{u} = \overline{e} \cdot \overline{u} = i_{\overline{x}} \overline{u} + \overline{\eta} \wedge \overline{u}.$$  \hspace{1cm} (37)

The main result in this section is the following.

**Proposition 6.3.** Let $V \in \Gamma(\wedge^{2n} L)$ such that $V \cdot \overline{u} = u$. Then the modular cocycle of $L$ with respect to the top form $V$ and the volume form $s = (u,\overline{u})$ is $2e$, where $e \in \Gamma(L)$ is given by Lemma 6.2.

Similarly, the modular cocycle of $\overline{L}$ with respect to $\Omega = (-1)^n V \in \Gamma(\wedge^{2n} \overline{L})$ and $s$ is $2\overline{e}$.

Before the proof, we need a couple of lemmas. The first one can be easily verified.

**Lemma 6.4.** For all $W \in \wedge^j L$, $X \in \wedge^i L$, and $i \leq j \leq 2n$, one has

$$X \cdot W \cdot u = (-1)^{(i-1)j/2}(i_X W) \cdot u.$$  \hspace{1cm} (39)

Since $L$ is a Lie algebroid and $L^* = \overline{L}$, we have the differential

$$d_L: \Gamma(\wedge^* L) \rightarrow \Gamma(\wedge^{*+1} L).$$

Moreover, we have the following equality:

$$\tilde{\partial}(W \cdot u) = (d_L W) \cdot u + (-1)^k W \cdot \tilde{\partial} u = (d_L W) \cdot u + (-1)^k (W \wedge e) \cdot u = (d_L W + e \wedge W) \cdot u, \quad \forall W \in \Gamma(\wedge^k \overline{L}),$$

which encodes the $L$-module structure on $N$ defined by Eq. (24).

Similarly, one has

$$\partial(X \cdot \overline{u}) = (d_{\overline{T}} X) \cdot \overline{u} + (-1)^i X \cdot \partial \overline{u} = (d_{\overline{T}} X + \overline{T} \wedge X) \cdot \overline{u}, \quad \forall X \in \Gamma(\wedge^i L).$$

\hspace{1cm} (40)
Lemma 6.5. For any $X = a + \zeta \in \Gamma(L)$, we have
\begin{equation}
(L_X \nabla) \cdot u = \partial(X \cdot \nabla) - \langle e|X\rangle \nabla,  \tag{42}
\end{equation}

\begin{equation}
L_a u = \langle e|X\rangle u - (d\zeta + \iota_a H) \wedge u,  \tag{43}
\end{equation}

\begin{equation}
L_a \nabla = \partial(X \cdot \nabla) + (d_L X) \cdot \nabla - (d\zeta + \iota_a H) \wedge \nabla.  \tag{44}
\end{equation}

Proof. A basic fact is that
\begin{equation}
0 = X \cdot u = \iota_a u + \zeta \wedge u,  \tag{45}
\end{equation}
for any $X = a + \zeta \in \Gamma(L)$. Hence
\[
\begin{align*}
\partial(X \cdot \nabla) &= \partial((\iota_X \nabla) \cdot u) \\
&= (dl_L X \nabla) \cdot u - (\iota_X \nabla \cdot e) \cdot u \quad \text{(by \ref{40})} \\
&= (dl_L X \nabla + \iota_X dl_L \nabla) \cdot u + (\langle e|X\rangle \nabla) \cdot u \\
&= (L_X \nabla) \cdot u + \langle e|X\rangle \nabla.
\end{align*}
\]
This proves Eq. \ref{42}. For Eq. \ref{43}, we have
\[
\begin{align*}
L_a u &= \iota_a du + dt_a u \\
&= \iota_a (\iota_X u + \eta \wedge u - H \wedge u) - d(\zeta \wedge u) \quad \text{(by \ref{37} and \ref{45})} \\
&= -\iota_a \iota_X u + (a|\eta)u - \eta \wedge \iota_a u - \iota_a H \wedge u + H \wedge \iota_a u - d\zeta \wedge u + \zeta \wedge du \\
&= \iota_a (\zeta \wedge u) + (a|\eta)u - \zeta \wedge u - (H \wedge u) \wedge (\zeta \wedge u) \\
&= (\langle x|\zeta \rangle + \langle a|\eta \rangle) u - \iota_a H \wedge u - d\zeta \wedge u.
\end{align*}
\]
To prove Eq. \ref{44}, we observe that, on one hand
\[
\begin{align*}
d^H(X \cdot \nabla) &= \partial(X \cdot \nabla) + \partial(X \cdot \nabla) \\
&= \partial(X \cdot \nabla) + (d_L X) \cdot \nabla - (X \wedge \nabla) \cdot \nabla. \quad \text{(by \ref{41}).}
\end{align*}
\]
On the other hand, we have
\[
\begin{align*}
d^H(X \cdot \nabla) &= d(\iota_a \nabla + \zeta \wedge \nabla) + H \wedge (X \cdot \nabla) \\
&= dt_a \nabla + d\zeta \wedge \nabla - \zeta \wedge d\nabla + H \wedge (X \cdot \nabla) \\
&= (dt_a \nabla + \iota_a d\nabla) + d\zeta \wedge \nabla - \iota_a (\zeta \wedge d\nabla + H \wedge (X \cdot \nabla) \\
&= L_a \nabla + d\zeta \wedge \nabla - \zeta \wedge (\nabla \cdot \nabla - H \wedge (X \cdot \nabla) \\
&= L_a \nabla + d\zeta \wedge \nabla - (X \wedge H \wedge \nabla) + H \wedge (X \cdot \nabla) \quad \text{(by \ref{35})} \\
&= L_a \nabla + d\zeta \wedge \nabla - (X \wedge H \wedge \nabla) + (\iota_a + \zeta \wedge (H \wedge \nabla) + H \wedge (\iota_a \nabla + \zeta \wedge \nabla) \\
&= L_a \nabla + d\zeta \wedge \nabla - (X \wedge \nabla) \cdot \nabla + \iota_a H \wedge \nabla.
\end{align*}
\]
This proves the last equation. \qed

Another lemma needed is

Lemma 6.6. \[3\] The Mukai pairing vanishes in $N_1 \times N_k$, unless $i + k = 2n$, in which case it is nondegenerate.
Proof of Proposition 6.3. For an $X = a + \zeta \in \Gamma(L)$, we assume that $\bar{\partial}(X \cdot \overline{w}) = f \overline{w}$ for some function $f \in C^\infty(M, \mathbb{C})$. Then Eq. (12) implies that $\mathcal{L}_X \overline{V} = (f - \langle e | X \rangle) \overline{V}$ and hence

$$\mathcal{L}_X V = (\langle e | X \rangle - f) V.$$  \hfill (46)

We also have, according to Eq. (43) and (44)

$$\mathcal{L}_\rho L (X) s = L_a(u, \overline{w}) = (\langle e | X \rangle, u, \overline{w}) + (u, \mathcal{L}_a u) \overline{w} = (\langle e | X \rangle + f, u, \overline{w}) = (\langle e | X \rangle + f) V \otimes \overline{s}.$$  \hfill (47)

In the last step, we have applied Eq. (19) and Lemma 6.6. In turn, we get

$$\mathcal{L}_X V \otimes s + V \otimes \mathcal{L}_\rho L (X) s = 2 \langle e | X \rangle V \otimes s.$$  \hfill (48)

This proves the first claim. By symmetry, we also have

$$\mathcal{L}_W V \otimes \overline{s} + \overline{V} \otimes \mathcal{L}_\sigma (W) \overline{s} = 2 \langle \overline{e} | W \rangle \overline{V} \otimes \overline{s},$$

for all $W \in \Gamma(\overline{L})$. By Eq. (18), we know

$$\overline{s} = (w, u) = (-1)^n (u, \overline{w}) = (-1)^n s.$$  \hfill (49)

Thus there holds

$$\mathcal{L}_W \Omega \otimes s + \Omega \otimes \mathcal{L}_\sigma (W) s = 2 \langle \overline{e} | W \rangle \Omega \otimes s,$$

which implies that $2\overline{e}$ is the modular cocycle of $\overline{L}$ with respect to $\Omega$ and $s$. \hfill \Box

7 Proof of the main theorem

We first finish the proof of Proposition 5.3.

Proof. $\overline{N}$ has an induced $\overline{L}$-module structure arising from the $\overline{L}$-module $\mathcal{L}$ (see Eq. (10)). According to the second statement of Proposition 6.3 we know that this module structure is determined by the following equation:

$$\mathcal{L}_W \overline{w} = \langle \overline{e} | W \rangle \overline{w}, \quad \forall W \in \Gamma(\overline{L}).$$

This just coincides with the standard $\overline{L}$-module structure defined by Eq. (25) because

$$W \cdot \partial \overline{w} = W \cdot \overline{w} \cdot \overline{w} = \langle \overline{e} | W \rangle \overline{w},$$

by Lemma 6.4. \hfill \Box

Now we are ready to prove the main theorem in this paper.

Proof of Theorem 5.2. By Proposition 6.3 and Eq. (35), (36), we conclude that

$$\partial_s (X \otimes \overline{w}) = (d_{\overline{L}} X + \overline{w} \wedge X) \otimes \overline{w},$$

$$\partial (X \otimes \overline{w}) = (-\partial X + e \cdot X) \otimes \overline{w},$$

12
for all $X \in \Gamma(\Lambda^k L)$.

Compare with the expression of $\partial$ in Eq. (11), we immediately know that Diagram (22) is commutative. To prove the commutativity of Diagram (23), it suffices to prove:

$$\bar{\partial}(X \cdot \bar{u}) = (\partial X + \iota_e X) \cdot \bar{u}, \quad \forall X \in \Gamma(\Lambda^i L).$$  \hfill (47)

In fact, by Eq. (40),

$$\text{LHS of Eq. (47)} = \bar{\partial}(X \cdot V \cdot u) = (-1)^{(i-1)/2} \bar{\partial}((\iota_X V) \cdot u) = (-1)^{(i-1)/2} (d_{L \iota_X V} + e \wedge \iota_X V) \cdot u.$$  

We also have

$$(\iota_e X) \cdot \bar{u} = e \cdot X \cdot \bar{V} \cdot u = (-1)^{(i-1)/2} e \cdot (\iota_X \bar{V}) \cdot u = (-1)^{(i-1)/2} (e \wedge \iota_X \bar{V}) \cdot u.$$  

And by Eq. (28), (32),

$$\partial X = (-1)^{(i-1)(2n-1)} V^\sharp \Omega^\sharp \partial X$$

$$= (-1)^{(i-1)(2n-1)+i} V^\sharp d_{L \iota_X} \Omega X = -V^\sharp d_{L \iota_X} \Omega X.$$  

Hence

$$-(\partial X) \cdot \bar{u} = (V^\sharp d_{L \iota_X} \Omega X) \cdot \bar{u}$$

$$= (-1)^{(2n-i+1)(2n-i)/2} (d_{L \iota_X} \Omega \cdot V \cdot \bar{u} = (-1)^{(i-1)/2} (d_{L \iota_X} \bar{V}) \cdot u.$$  

This proves Eq. (47), and the proof of Theorem 5.2 is thus completed. $\square$

8 Some corollaries

The first obvious result is that, by the isomorphisms

$$\Lambda^k L \otimes \mathcal{L} \cong (\Lambda^k L) \cdot \bar{N} = \bar{N}_k = N_{2n-k},$$

the Dirac generating operator constructed by Theorem 3.5 for $E = L \oplus \bar{L}$ is exactly

$$\tilde{D} = \tilde{d} + \tilde{d} = \partial + \bar{\partial} = d^H,$$

and especially $\tilde{f} = \tilde{D}^2 = 0$.

In Section 6 we defined a pair of operators $d_s$ and $\partial$ on $\Gamma(\Lambda A)$, and similarly $d$ and $\partial_s$ on $\Gamma(\Lambda A^*)$, for any Lie bialgebroid $(A, A^*)$. Let $D = d_s + \partial$, $D_s = d + \partial_s$. Their squares yield the pair of Laplacian operators

$$\Delta = D^2 = d_s \partial + \partial d_s : \Gamma(\Lambda^k A) \to \Gamma(\Lambda^k A),$$  \hfill (48)

$$\Delta_s = D_s^2 = d \partial_s + \partial_s d : \Gamma(\Lambda^k A^*) \to \Gamma(\Lambda^k A^*).$$  \hfill (49)

As an immediate corollary of the following result

**Theorem 8.1.** [5] Theorem 3.4] If $(A, A^*)$ is a Lie bialgebroid, then

$$\Delta = \frac{1}{2}(\mathcal{L}_{X_0} + \mathcal{L}_{\xi_0}) : \Gamma(\Lambda A) \to \Gamma(\Lambda A),$$

$$\Delta_s = \frac{1}{2}(\mathcal{L}_{X_0} + \mathcal{L}_{\xi_0}) : \Gamma(\Lambda A^*) \to \Gamma(\Lambda A^*).$$
we have

**Corollary 8.2.** Let \((L, \overline{L})\) be the Lie bialgebroid coming from a twisted generalized complex structure \(\mathcal{J}\). The Laplacian operators \(\Delta\) and \(\Delta_*\) defined by Eq. (48) and (49) are given respectively by

\[
\Delta f = \Delta_* f = \frac{1}{2} pr_T(e + \overline{e})(f), \quad \forall f \in C^\infty(M, \mathbb{C}),
\]

\[
\Delta X = (\overline{e} + \frac{1}{2} e) \circ_H X - \frac{i}{2} J(\overline{e} \circ_H X), \quad \forall X \in \Gamma(L),
\]  

(50)

\[
\Delta_* W = (e + \frac{1}{2} \overline{e}) \circ_H W + \frac{i}{2} J(\overline{e} \circ_H W), \quad \forall W \in \Gamma(L).
\]  

(51)

**Proof.** For \(e \in \Gamma(T\mathcal{L}), \overline{e} \in \Gamma(L)\), the Lie derivations \(\mathcal{L}_e\) and \(\mathcal{L}_{\overline{e}}\) on \(\Gamma(L)\) are given respectively by:

\[
\mathcal{L}_e X = pr_L(e \circ_H X),
\]

\[
\mathcal{L}_{\overline{e}} X = \overline{e} \circ_H X, \quad \forall X \in \Gamma(L).
\]

The projections of \(T\mathcal{C} \oplus T\mathcal{C}^*\) to \(L\) and \(\overline{L}\) are given respectively by:

\[
pr_L(z) = \frac{1}{2}(z - i\mathbb{J}z), \quad pr_{\overline{L}}(z) = \frac{1}{2}(z + i\mathbb{J}z), \quad \forall z \in T\mathcal{C} \oplus T\mathcal{C}^*.
\]

Thus Eq. (50) and (51) follow directly from Theorem 8.1. \(\square\)

It is well known [17] that, if \(a\) and \(a_*\) denote the anchor maps of a Lie bialgebroid \((A, A^*)\), the bundle map

\[
\pi^* = a \circ (a_*)^*: T\mathcal{C}^*M \to T\mathcal{C} M
\]  

(52)

defines a (complex) Poisson structure on \(M\).

In particular, for the Lie bialgebroid \((L, \overline{L})\) coming from the twisted generalized complex structure \(\mathcal{J}, -i\pi\) is a real Poisson structure. In fact, up to a factor of 2, it is given by [2][9]

\[
P(\xi, \eta) = \langle \mathcal{J}\xi, \eta \rangle.
\]  

(53)

Let us briefly recall the definition of the modular vector field of a Poisson manifold \((M, \pi)\) [21]. Let \(\omega \in \Omega^{\text{top}}(M)\) be a volume form, the modular vector field with respect to \(\omega\) is the derivation \(X_\omega\) of the algebra of functions \(C^\infty(M)\) characterized by

\[
\mathcal{L}_{\pi^*(d\omega)} \omega = X_\omega(f) \omega.
\]  

(54)

For the Poisson structure induced from a Lie bialgebroid, the relation between modular cocycles and modular vector fields is established in the following:

**Lemma 8.3.** [5 Corollary 3.8] Let \(M\) be an orientable manifold with volume form \(s \in \Omega^{\text{top}}(M)\) and let \((A, A^*)\) be a real Lie bialgebroid over \(M\) with associated Poisson bivector \(\pi\) defined by Eq. (52). Then the modular vector field of the Poisson manifold \((M, \pi)\) with respect to \(s\) is

\[
X_s = \frac{1}{T}(a_*(\xi_0) - a(X_0)),
\]  

(55)

where \(\xi_0\) and \(X_0\) are modular cocycles defined by Eq. (33) and (34) (choosing arbitrary \(V\) and \(\Omega\)).
Hence as an immediate consequence, we obtain the following result of Gualtieri [9, Proposition 3.27].

**Corollary 8.4.** The modular vector field of the Poisson structure $\mathcal{P}$ defined in Eq. (53) is given by

$$\frac{i}{2} \operatorname{pr}_2(e - \tau),$$

with respect to the volume form $s = (u, \overline{u})$.

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