Electron motion in a 3-D undulator magnetic field

N Smolyakov, S Tomin, G Geloni
1 NRC «Kurchatov Institute», Moscow, Russia
2 European XFEL GmbH, Hamburg, Germany
# Email: smolyakovnv@mail.ru

Abstract. In this contribution we present an analysis of electron trajectories in the three dimensional magnetic field from a planar undulator. The differential equations of motion are analytically solved by means of a perturbation theory approach, and the expressions for the electrons trajectories are derived, without recourse to averaging over the undulator period. All small terms up to the third order inclusive are obtained. Analytically calculated trajectories are in excellent agreement with those computed numerically by the specially written code.

1. Introduction
The electron trajectory in an undulator is influenced by the focusing properties (both horizontal and vertical) of the magnetic field. In the European XFEL case, long segmented undulators (21 segments for the SASE3 beamline to 35 for SASE1 and SASE2) are planned to be installed, with quadrupole lenses between different segments [1]. These focusing properties should be taken into account in simulations of spontaneous radiation, which constitutes the background signal of the FEL.
As far as we know horizontal and vertical focal lengths of an undulator were first calculated in [2]. In a planar undulator with infinitely wide magnetic poles and hence without horizontal focusing, the vertical focusing was analyzed in [3, 4]. In [5 – 7] trajectories in focusing undulator magnetic field were calculated up to the lowest order in the initial positions and angles of the electrons. Some general relations dealing with undulator focal lengths were derived in [8 – 10]. Long-length-scale anharmonic betatron motion of electrons in very long undulators was studied in [11]. Notice that all these studies were carried out within the following limits: the focusing effects were calculated with averaging over the undulator period and only terms, which are linear in the electron initial positions and angles were taken into account. As a consequence in some instances the obtained results lack precision.
In this report we present a higher approximation to the solutions for the equation of motion in a 3-dimensional undulator field. We calculated all correction terms up to the third order inclusive. These expressions provide insight into nonlinear effects of electron beam dynamics as we obtained the terms, which are quadratic in electron initial parameters. In some instances these terms have effect on the electron trajectory. These expressions show that in fact the undulator field influence on the electron trajectory is complicated and often cannot be reduced to the well-known focusing phenomena along.

2. Trajectory equations in undulator field
We model the three-dimensional magnetic field \( \vec{B}(x, y, z) = -\hat{\nabla} V(x, y, z) \) of a planar undulator, Fig. 1, by means of the scalar potential \( V(x, y, z) = -\left( B_0/k_y \right) \cos(k_x x) \sinh(k_y y) \sin(k_z z) \). Here \( k_x = 1/a, k_z = 2\pi/\lambda_u, k_y = \sqrt{k_y^2 + k_z^2}, \lambda_u \) is the undulator period length. The linear parameter \( a \) gives the field non-uniformity along \( X \)-axis and is of the order of the width of the undulator poles.
In the vicinity of an undulator axis the magnetic field \( B_{x,y,z}(x,y,z) \) is equal to:

\[
B_x(x,y,z) \equiv -B_0 k_x^2 y \sin(k_z z),
\]

\[
B_y(x,y,z) \equiv B_0 \left[1 - 0.5(k_x x)^2 + 0.5(k_y y)^2\right] \sin(k_z z),
\]

\[
B_z(x,y,z) \equiv B_0 k_z y \cos(k_z z).
\]

We will use the exact trajectory equations in the fixed coordinate system \( \{x,y,z\} \):

\[
\begin{align*}
\dddot{x}' &= -q \sqrt{1 + (x')^2} \left(1 + (x')^2\right) B_y - y' B_z - x' y' B_x, \\
\dddot{y}' &= q \sqrt{1 + (y')^2} \left(1 + (y')^2\right) B_x - x' B_z - y' x' B_y,
\end{align*}
\]

where \( \beta \) and \( \gamma \) are the electron's reduced velocity and energy respectively, \( q = e/(mc^2 \beta \gamma) \) and an apostrophe indicates a derivative with respect to \( z \).

Substituting Eqs. (1) – (3) into Eqs. (4) and (5) and expanding them into a series about small values \( k_x x, k_y y, x' \) and \( y' \), we derive the following system of two nonlinear differential equations:

\[
\begin{align*}
\dddot{x}' &= p k_z \sin(\varphi) + p k_z \left\{0.5 \cdot \left[3(x')^2 - (k_x x)^2 + (y')^2 + (k_y y)^2\right] \sin(\varphi) - k_z y' \cos(\varphi)\right\}, \\
\dddot{y}' &= p k_z \left\{k_z^2 x y + x' y'\right\} \sin(\varphi) + k_z x' y' \cos(\varphi),
\end{align*}
\]

where \( K = (eB_0 \lambda_u)/(2 \pi mc^2) \) is the undulator deflection parameter, \( p = K/(\beta \gamma) \) and \( \varphi = k_z z \).

3. Solutions of the equations of motion

The averaging of the short-length oscillating equations (6) and (7) gives the following solutions in first (linear) approximation in the electron initial coordinates \( x_0, x'_0, y_0 \) and \( y'_0 \) [5 – 7]:

\[
\begin{align*}
x_1(z) &= x_0 \cosh(\omega_x \varphi) + \left(\theta(\omega_x k_z)\right) \sinh(\omega_x \varphi) - \left(p/k_z\right) \sin(\varphi), \\
y_1(z) &= y_0 \cos(\omega_y \varphi) + \left(y'_0/(\omega_y k_z)\right) \sin(\omega_y \varphi),
\end{align*}
\]

where \( \theta = x'_0 + p, \omega_x = (p k_x)/\sqrt{2 k_z} \) and \( \omega_y = (p k_y)/\sqrt{2 k_z} \) are dimensionless periods (in units of \( \lambda_u \)) of the undulator betatron oscillations in horizontal and vertical directions respectively.

Below we will consider relatively short undulators, with the number of periods \( N \) so that:

\[
2\pi N \omega_{x,y} \ll 1.
\]

In the case of European XFEL for an electron beam energy of 17.5 GeV, \( \gamma \equiv 35000, K \equiv 4, k_{x,y} \sim k_z, \omega_{x,y} \sim p \equiv 10^{-4}, N = 124 \), conditions (10) are clearly fulfilled.
Differentiating Eqs. (8) and (9) with respect to \( z \), we get for relatively short undulators:

\[
x'(z) = x_0 \omega_z^2 \kappa z \varphi + \theta(1 + 0.5 \omega_z^2 \varphi^2) - p \cos(\varphi)
\]

\[
y'(z) = -y_0 \omega_z^2 \kappa z \varphi + y'_0(1 - 0.5 \omega_z^2 \varphi^2)
\]

Inserting \( z = N \lambda_u, \varphi = 2 \pi N, \theta = 0 \) in Eq. (11) and using the relation \([8, 9]: \frac{1}{f_x} = \frac{dx'(z = N \lambda_u)}{dz_0} \)

we get: \( 1/f_x = -2 \pi N \kappa_z \omega^2 \) [2]. The expression \( 1/f_y = 2 \pi N \kappa_y \omega^2 \) can be derived similarly.

Numerical simulations show (see below) that sometimes Eqs. (8), (9), (11) and (12) are not sufficiently precise and closer analytical solution of the equations of motion (6) and (7) are required.

The right-hand sides of Eqs. (6) and (7) are proportional to the small parameter \( p \). The functions \( x(z), y(z), x'(z) \) and \( y'(z) \) are in turn proportional to the small parameters \( x_0, \theta, y_0, y'_0 \) and \( p \). This makes possible solving Eqs. (6) and (7) by the methods of perturbation theory. The functions \( x(z) \) and \( y(z) \) can be written as \( x(z) = x_1(z) + \Delta x(z) \) and \( y(z) = y_1(z) + \Delta y(z) \) with new unknown small functions \( \Delta x(z), \Delta y(z) \). We can substitute \( x(z) = x_1(z) + \Delta x(z) \) and \( y(z) = y_1(z) + \Delta y(z) \) into the left-hand sides of Eqs. (6), (7) and let \( x(z) = x_1(z), y(z) = y_1(z) \) into the right-hand sides of Eqs. (6) and (7). Using explicit forms for \( x_1(z) \) and \( y_1(z) \) given by Eqs. (8) and (9), we derive differential equations for \( \Delta x(z) \) and \( \Delta y(z) \) in the form: \( (\Delta x)' = g_{x,y}(z) \) and \( (\Delta y)' = g_{x,y}(z) \) with the well-defined functions \( g_{x,y}(z) \), which are of the third order in small parameters \( x_0, \theta, y_0, y'_0 \) and \( p \). Hence, it follows that the functions \( \Delta x(z) \) and \( \Delta y(z) \) are also of the third order in the small parameters, while \( x_1(z) \) and \( y_1(z) \) are of the first order only. As a result, the functions \( \Delta x(z) \) and \( \Delta y(z) \) can be found by direct integration of \( g_{x,y}(z) \). After some algebraic manipulations we get:

\[
x'(z) = \theta - p \cos(\varphi) + X_0 \omega_z^2 \varphi + 0.5 \theta \omega_z^2 \varphi^2 + \frac{p}{2 A^2}(X_0 + \theta \varphi - 0.25 p \sin(\varphi))^2 \cos(\varphi) - \frac{p}{2 A^2} X_0^2
\]

\[
-\frac{p}{A^2}(X_0 + \theta \varphi)(\theta \cos(\varphi)) \sin(\varphi) - \frac{\omega_z^2}{p} Y_0^2 - \frac{\omega_z^2}{p}(Y_0 + y'_0 \varphi)^2 \cos(\varphi) \cos(\varphi) + \frac{p}{A^2} y'_0(Y_0 + y'_0 \varphi) \sin(\varphi) \sin(\varphi)
\]

\[
-0.5 p^2 \theta(3 - 0.5 / A^2) \sin^2(\varphi) + 0.5 p^2 \theta^2(3 + 2 / A^2)(1 - \cos(\varphi)) + 0.5 p(y'_0)^2(1 - 2 / A^2) (1 - \cos(\varphi))
\]

\[
+ 0.125 p^3 (1 + 13 / (4A^2))(1 - \cos(3 \varphi)) + 0.125 p^3 (3 - 47 / (16A^2))(1 - \cos(\varphi))
\]

\[
y'(z) = y'_0 - Y_0 \omega_z^2 \varphi - 0.5 y'_0 \omega_z^2 \varphi^2 + \sin^2(\varphi) + \frac{p}{A^2} X_0 Y_0 - \frac{p}{A^2}(Y_0 + y'_0 \varphi)(X_0 + \theta \varphi - p \sin(\varphi)) \cos(\varphi)
\]

\[
+ p(1 + 1 / A^2)(Y_0 + y'_0 \varphi)(\theta - 0.5 p \cos(\varphi)) \sin(\varphi) + y'_0(p / A^2)(X_0 + \theta \varphi) \sin(\varphi - 2 \theta(1 - \cos(\varphi)))
\]

where \( X_0 = k_z x_0 \) and \( Y_0 = k_z y_0 \) are the electron dimensionless initial coordinates: \( |X_0|, |Y_0| << 1 \)

\[ A = k_a \pi / \lambda_u \]

\( A \) is of the order of unit since \( a \) is about the magnetic poles width. Integrating Eqs. (13) and (14) over \( z \), we can derive the expressions for transversal coordinates \( x(z) \) and \( y(z) \). Eqs. (13), (14) include all terms with the total order not greater than 3 in the small values \( X_0, Y_0, \theta, y'_0 \) and \( p \). The terms \( X_0 \omega_z^2 \varphi \) in Eq. (13) and \( Y_0 \omega_z^2 \varphi \) in Eq. (14) are responsible for the focusing properties itself. Recall that focusing means that there is a difference in the velocities of two outgoing electrons which move parallel to each other at the undulator entrance. These two terms are quadratic in
the small parameter $p$, are linear in the small dimensionless coordinates $X_0, Y_0$ and are also proportional to $\varphi$, with a maximum value $2\pi N$, that is about 780 at $N = 124$. At the same time one can find other terms in Eqs. (13) and (14), which are proportional to $\varphi^2$ with the maximum value $(2\pi N)^2$, for example the terms $\left(p/2A^2\right)\varphi^2$ in Eq. (13) and $\left(p/A^2\right)\varphi^0$ in Eq. (14). If $X_0\alpha_x^2 \leq 2\pi N\theta^2 p/(2A^2), Y_0\alpha_y^2 \leq 2\pi N\theta^0 p/A^2$ or $\theta^2 \geq (x_0 K)/(\lambda_0 N)\gamma, \varphi^0 \geq (y_0 K A^2)/(2\lambda_0 N)\gamma$, these terms exceed the correspondent focusing terms and should be taken properly into consideration. Moreover, these terms include the fast oscillating factors $\sin(\varphi)$ and $\cos(\varphi)$, which additionally contribute to the radiation phase during the photon generation.

A computer code was written, which relies on the Runge-Kutta algorithm. The analytical and numerical methods could then be compared, showing an excellent agreement. Computer simulations, which may be thought of as a figure of merit, are done with the following European XFEL parameters: electron beam reduced energy $\gamma \equiv 35000$, undulator period length $\lambda_0 = 40$ mm, its deflection parameter $K = 4, a = 40$ mm, number of periods $N = 124$. Let us consider an electron with the initial conditions: $x_0 = 0.1$ mm, $\theta = 0$, $y_0 = 0.1$ mm, $y'_0 = 0$. The maximum of the absolute discrepancy between numerically computed $x'(z)$ and new expression given by Eq. (13) is equal to $6 \cdot 10^{-11}$ with the relative accuracy about $6 \cdot 10^{-7}$ since $x'(z) \sim p$. For the case of $y'(z)$ the correspondent discrepancy between Eq. (14) and numerically computed trajectory is equal to $1.2 \cdot 10^{-11}$. At the same time the maximum discrepancy between numerically computed $x'(z)$ and $x'_i(z)$ given by Eq. (11), is equal to $2.8 \cdot 10^{-8}$. This value is 14 times greater than the term $x_0\alpha_x^2 k_\varphi \equiv 2 \cdot 10^{-9}$ in Eq. (11) which is standardly interpreted as the term which is responsible for the horizontal focusing effect. This is an indication that the correction terms derived here are not unconditionally negligible. In other words, in some cases Eqs. (8), (9), (11) and (12) are not accurate enough and more sophisticated and accurate Eqs. (13) and (14) should be employed for analysis of electron trajectories in undulator magnetic field.

**Acknowledgment**

This work was supported by Ministry of Education and Science of the Russian Federation program of "Physics with Accelerators and Reactors in West Europe (except CERN)". We would like to thank V. Nosik and M. Rychev for supporting this direction of research.

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