Path integral description and direct interaction approximation for elastic plate turbulence

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Abstract

In this work, we apply the Martin-Siggia-Rose path integral formalism to the equations of a thin elastic plate. Using a diagrammatic technique, we obtain the direct interaction approximation (DIA) equations to describe the evolutions of the correlation function and the response function of the fields. Consistent with previous results, we show that DIA equations for elastic plates can be derived from a non-markovian stochastic process and that in the weakly nonlinear limit, the DIA equations lead to the kinetic equation of wave turbulence theory. We expect that this approach will allow a better understanding of the statistical properties of wave turbulence and that DIA equations can open new avenues for understanding the breakdown of weakly nonlinear turbulence for elastic plates.

1 Introduction

Turbulence represents one of the long-standing and unsolved problems which has challenged physicists for centuries. Although the turbulence phenomenon is rooted in the dynamics of fluid flows, turbulence-like behavior is rather ubiquitous in nature. A large class of systems ranging from quantum scales to astrophysical ones are characterized by out-of-equilibrium randomly fluctuating waves, a state referred to as wave turbulence (WT). These systems have many similarities with hydrodynamic turbulence. They display a Richardson-like cascade with an energy flux from large scales toward small scales, leading to a stationary out-of-equilibrium solution for the energy spectrum, called Kolmogorov-Zakharov. The significant advantage of random wave systems is that the dynamics can be considered typically as weakly nonlinear, in stark opposition to hydrodynamic turbulence. In consequence, it is possible to establish a closed kinetic equation for wave amplitudes evolution which capture the WT behavior. The wave turbulence theory has been extensively applied to quantum turbulence, surface water waves, Alfven waves, and even to describe the preheating stage of inflation in the early Universe.

The case of elastic plates has been considered experimentally and numerically in several works. It has been shown to be an excellent play-
ground for WT due to the simplicity of the experimental setup and of the dynamic equations for the surface elevation. Although important progress has been made, a full understanding of the probability density function evolution, as well as the nonlinear response of elastic plates, remains unclear. In addition, in the past decade, it has been reported that the weakly nonlinear regime breaks down at increasing forcing, and a new regime emerges. Under strongly nonlinear effects, elastic plates display an energy spectrum different from the WT theory prediction. Intermittency, which is responsible for breaking self-similarity, is also observed. Such a phenomenon, which is well known in hydrodynamic turbulence, goes beyond the standard description of the WT theory and remains largely unclear. In this work, using the Martin-Siggia-Rose formalism, we construct a path integral description for the stochastic dynamics of plates. This statistical description enables a full representation of the probability distribution function and a clear description of correlations and responses of the system under external perturbation. We expect that this formalism will provide new insights into the statistical properties of weakly nonlinear elastic plates and might elucidate some properties beyond the breakdown of the wave turbulence theory.

Different non-perturbative approximations have been developed to understand the properties of strong turbulence, ranging from the introduction of the so-called eddy viscosity constant to methods recycled from quantum field theory. Theories based on renormalization have led to several descriptions such as Wyld’s analysis, Local Energy Transfer Theory (LET), Direct Interaction Approximation (DIA) and others. Each of them leads to different results and descriptions, showing some degree of similarity, but in general, fail to give an accurate description of hydrodynamic turbulence. DIA presents several advantages compared to other renormalization methods. First, it lays on an exact correspondence with an underlying stochastic system which guarantees a well-defined evolution. Second, it correctly yields to the kinetic equation when applied to weakly nonlinear wave systems. Here, using the path integral formalism, we derive the DIA equations for elastic plates to extend previous results to cubic nonlinear systems which are not phase invariant. We obtain an extra non-trivial term in the DIA arising from one-loop correction diagrams, which could impact the phase mixing at increasing wave amplitude. This extra term does not lead to energy transfer along modes and in the weakly nonlinear regime, it is responsible for the frequency renormalization.

DIA equations have been largely applied to different nonlinear systems from the Duffing Oscillator to Langmuir Turbulence in plasmas or the nonlinear Schrödinger equation (NLS). For strongly nonlinear regimes, DIA is known to yield relatively unsatisfactory results. However, when a modified version of the DIA is imposed, the prediction compares well with numerical results. Thus, this seems to be the best starting point to study the breakdown of the WT theory, due to its simplicity and realizability.

2 Föppl-von Kármán equations for plates

Thin elastic plates are modeled by the Föppl-von Kármán (FvK) equations. These equations describe the time evolution of the vertical displacement $\zeta$ and the Airy stress function $\chi$ where

$$\rho \frac{\partial^2 \zeta}{\partial t^2} (x, t) = -\frac{E h^2}{12(1 - \sigma^2)} \Delta^2 \zeta(x, t) + \{\zeta, \chi\}(x, t), \quad (1)$$

$$\frac{1}{E} \Delta^2 \chi(x, t) = -\frac{1}{2} \{\zeta, \zeta\}(x, t) \quad (2)$$
and \( \{f, g\} := f_{xx}g_{yy} + f_{yy}g_{xx} - 2f_{xy}g_{xy} \). The physical parameters are given by the mass density \( \rho \), the thickness \( h \), the Young’s modulus \( E \) and the Poisson ratio \( \sigma \). The linear regime is characterized by bending waves, which display a relation dispersion \( \omega_k = \sqrt{\frac{Eh^3}{12(1-\sigma^2)\rho}} |k|^2 \). Using the canonical change of variables \( A_k(t) := \frac{1}{\sqrt{2}} \left( \sqrt{\omega_k \rho} \rho \zeta(t) + \frac{i}{\omega_k \rho} \rho \zeta(t) \right) \) for the Fourier transform of the displacement field \( \zeta \) and for the Fourier transforms of the momentum \( p = \rho \partial_t \zeta \), the FvK equations transform to the general form of cubic nonlinear wave systems \[17\]

\[
\left( \frac{\partial}{\partial t} + i \omega_k \right) A_k^\dagger(t) = \sum_{s_1 \ldots s_3} \int L_{kk_1k_2k_3}^{s_1s_2s_3} A_{k_1}^{s_1}(t) A_{k_2}^{s_2}(t) A_{k_3}^{s_3}(t) \times \delta(k_1 + k_2 + k_3 - k) d\mathbf{k}_{123}. \tag{3}
\]

We utilize Newell’s notation \[42\] : \( A_k^\dagger(t) := A_k(t) \), \( A_k^\ast(t) := A_{-k}(t) \) and \( d\mathbf{k}_{123} := d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \). It should be noted that for finite size plates, integrals in Fourier space must be replaced with sums, but the entire formalism developed in the following remains unaltered. For the elastic plate:

\[
L_{kk_1k_2k_3}^{s_1s_2s_3} = \frac{-is}{6\pi^2} X_{-k} X_{k_1} X_{k_2} X_{k_3} \times (T_{-k_1k_2k_3} + T_{-k_2k_1k_3} + T_{-k_3k_2k_1}) \tag{4}
\]

with

\[
T_{k_1k_2k_3k_4} = \frac{1}{16} \left( \frac{1}{|k_1 + k_2|^2} + \frac{1}{|k_3 + k_4|^2} \right) (k_1 \times k_2)^2 (k_3 \times k_4)^2.
\]

The canonical equations \[3\] can be encountered in several nonlinear wave systems such as gravity waves \[53, 54\] or capillary waves \[55, 43\]; however, beyond the weakly nonlinear regime, higher order corrections need to be included for those systems. Another important example is the Gross-Pitaevskii \[56, 57, 58, 59\] or NLS equations, which have been extensively studied including a description using DIA \[40, 60\]. However, it is known that the phase invariance of NLS implies an extra conserved quantity, leading to a complex statistical description even for initially weakly non linear amplitudes \[57\]. It is also interesting to observe that ideal incompressible fluids can be recast in a similar form making use of Clebsch pair \( v = -\frac{1}{\rho} \Delta^{-1} \nabla \times (\nabla \lambda \times \nabla \mu) \) with \( \rho \) the density. The dynamic equation in Fourier space thus takes the form \[3\] with \( \omega_k = 0 \), \( A_k = (\mu_k + i\lambda_k)/\sqrt{2} \) and the scattering matrix \( L_{kk_1k_2k_3}^{s_1s_2s_3} \) given in \[4\].

Canonical equation \[3\] can be shown to derive from a Hamiltonian structure

\[
is \partial_t A_k(t) = \frac{\delta H}{\delta A_k^\ast(t)}
\]

where for the elastic plate

\[
H = \int d\mathbf{k} \omega_k A_k(t) A_k^\ast(t) + \frac{1}{4(2\pi)^2} \int d\mathbf{k}_{1234} X_{k_1} X_{k_2} X_{k_3} X_{k_4} T_{k_1k_2k_3k_4} \times \sum_{s_1 \ldots s_4} A_{k_1}^{s_1}(t) A_{k_2}^{s_2}(t) A_{k_3}^{s_3}(t) A_{k_4}^{s_4}(t) \delta(k_1 + k_2 + k_3 + k_4). \tag{5}
\]

To obtain an out-of-equilibrium turbulent state, we have to include forcing and damping terms into the dynamics

\[
is \partial_t A_k(t) = \eta_k - is \gamma_k A_k^\ast(t) + \frac{\delta H}{\delta A_k(t)}. \tag{6}
\]

where \( \gamma_k \) is the damping term and \( \eta_k \) the external force. We will assume \( \eta_k \) to be a complex random variable with a Gaussian distribution and delta correlated \(, i.e., \langle \eta_k^\ast(t)\eta_k^\ast(t') \rangle = 2F_q \delta(t - t') \delta_{s,k} \delta(k + q) \) with \( \langle \eta_k^\ast(t_1) \rangle^\ast = \eta_k^\ast(t_1) \)\(^1\). To study the statistical properties of this system, we will write in the next chapter the path integral description following the standard Martin-Siggia-Rose formalism \[31, 32, 33\].

\(^1\)The star superscript denotes the complex conjugate.
3 Path Integral formalism

The key idea in constructing the path integral functional is to realize that each solution \( A^s_k \) depends on the specific realization of the noise \( \eta^s_k \). Thus, we can obtain a probability distribution function (PDF) for the field \( A^s_k \) combining the PDF of the gaussian noise and the constraint of the function \( A^s_k \) to the stochastic differential equation (SDE). The average for an observable \( O = O(A^s_{k_1}(t), \ldots, A^s_{k_n}(t)) \) is

\[
\langle O \rangle = \int D\eta_k(t) P(\eta_k(t)) O(A^s_{k_1}, \eta^s_{k_1}, \ldots, A^s_{k_n}, \eta^s_{k_n}),
\]

(7)

where \( D\eta_k(t) := \lim_{N \to \infty} \prod_{k=k}^{s_n} \prod_{j=1}^{s_j} d\eta^j_k(t) \) (the subindex \( j \) is for the time discretization), \( P(\eta_k(t)) \) is a Gaussian PDF and \( A^s_{k_n}, \eta^s_{k_n} \) is the field under a given realization of the noise. Inserting a Dirac delta function with the SDE as an argument allows us to rewrite the average as integrating over the field \( A^s_k \). Averaging over the noise, by making use of the integral representation of the delta function leads to

\[
\langle O \rangle = \int D\tilde{A}^s_k(t) D\tilde{A}^s_k(t) O(A^s_{k_1}(t), \ldots) e^{-S[A^s_k(t), \tilde{A}^s_k(t)]}
\]

(8)

where \( \tilde{A}^s_k \) corresponds to the integration variable from the Dirac delta integral representation and the MSR action reads as

\[
S[A^s_k(t), \tilde{A}^s_k(t)] :=
\sum \int dtdt' \tilde{A}^s_k(t) (\partial_t A^s_k(t) + \gamma_k A^s_k(t) + i s \frac{\delta H}{\delta A^s_k(t)} - F_k \tilde{A}^s_k(t)).
\]

(9)

Therefore, we can define the generating functional in terms of the auxiliary fields \( J^s_k(t) \) and \( \tilde{J}^s_k(t) \) as

\[
Z[J, \tilde{J}] = \int D\tilde{A}^s_k D\tilde{A}^s_k e^{-S[A^s_k(t), \tilde{A}^s_k(t)] + \sum \int dt J^s_k(t) A^s_k(t) + \tilde{J}^s_k(t) \tilde{A}^s_k(t) dt},
\]

(10)

where \( Z[0,0] = 1 \). Correlation functions can then be easily obtained by differentiating over the field \( \tilde{J} \) and \( J \). In particular, the two point correlation function \( C^s_{p_1 p_2} (t_1, t_2) := \langle A^s_{p_1} (t_1) A^s_{p_2} (t_2) \rangle \) is given by

\[
C^s_{p_1 p_2} (t_1, t_2) = \left. \frac{\delta^2 Z[J, \tilde{J}]}{\delta J_{p_1}(t_1) \delta \tilde{J}_{p_2}(t_2)} \right|_{J=\tilde{J}=0}.
\]

(11)

The response function defined as \( R^s_{p_1 p_2} (t_1, t_2) := \langle \frac{\delta A^s_{p_1} (t_1)}{\delta \eta^s_{p_2} (t_2)} \rangle \) can also be obtained directly from the generating functional. This quantity simply measures (in average) the infinitesimal change of the field with respect to a source. It can be shown (Appendix [F]) that

\[
R^s_{p_1 p_2} (t_1, t_2) = \left. \frac{\delta^2 Z[J, \tilde{J}]}{\delta J_{p_1}(t_1) \delta \tilde{J}_{p_2}(t_2)} \right|_{J=\tilde{J}=0}.
\]

(12)

Performing the path integral for the generating functional ([10] is extremely complicated and only in the absence of the nonlinear term, called the free case, a general solution can be found. The moment generating functional for the free case \( Z_0[J, \tilde{J}] \) is obtained by keeping the linear term of the Hamiltonian \( H \to H_0 = \int dk \omega_k A^s_k(t) \). The free generating functional then reads as (see details in appendix [A])

\[
Z_0[J, \tilde{J}] = e^{-\frac{1}{2} \sum_{s=1}^2 \int dk^1 dk^2 \int dt_1 dt_2 \int dt_1' \int \tilde{t}_1'' \tilde{J}_{k_1}^s(t_1) A^s_{k_1} (t_1) R^{s_1 - s_2 s_1}_{k_1 k_2} (t_1, t_1') \tilde{J}_{k_2}^s (t_1')}
\]

(13)

where

\[
C^{s_1 s_2}_{k_1 k_2} (t, t') = \begin{pmatrix}
C^{s_1 s_2}_{0, k_1 k_2} (t, t') & R^{s_1 - s_2 s_1}_{0, k_1 k_2} (t, t') \\
R^{s_1 - s_2 s_1}_{0, k_2 - k_1} (t, t') & 0
\end{pmatrix}
\]

(14)

with \( \tilde{J}_{k_1}^s (t) := \begin{pmatrix} \tilde{J}_{k_1}^s (t) \\ \tilde{J}_{k_1}^s (t) \end{pmatrix} \) and the superscript \( T \) denotes transpose. The subindex \( 0 \) hereafter refers to functions associated with the linear Hamiltonian \( H_0 \). It is to be noted that we express this functional only in terms of the correlation and response function; this is an outstanding advantage of this construction and the reason will be clarified later. The explicit solutions for the correlation and response functions are known (Appendix F) that
for the free case. Considering the dynamic equation (6) with the linear Hamiltonian $H_0$, one can easily obtain closed equations for the two-point correlation and response function

$$(\partial_t + is\Omega_k^s)C_{0,kp_2}^{s_2}(t_1,t_2) = 2F_{p_2}R_{0,p_2,k}^{s_2}(t_2,t)$$

$$(\partial_t + is\Omega_k^s)R_{0,kp_2}^{s_2}(t_1,t_2) = \delta_{s_12}\delta(k-p_2)\delta(t-t_2)$$

where $\Omega_k^s := \omega_k - is\gamma_k$ and we made use of the identity $R_{0,kp_2}^{s_2}(t_2,t) = \frac{1}{2\pi^2}(-A_k(t)\eta_{-p_2}^{-i_2}(t_2))$ (see Appendix E for details). Finally the zero order correlation and response function are given by:

$$R_{0,1}^{l_1 l_2}(t_1,t_2) = \delta_{l_1 l_2}R_{0,0}^{l_1 l_2}(t_1,t_2)$$

$$C_{0,0}^{l_1 l_2}(t_1,t_2) = C_{0,0}^{l_1 l_2}(0,0)e^{il_1 l_2 t_1} + 2F_{p_2}\delta_{l_1 l_2}e^{il_1 l_2 t_2}\theta(t_2-t_1)$$

$$+ 2F_{p_2}^2\delta_{l_1 l_2}e^{il_1 l_2 t_2}\theta(t_2-t_1) + 2F_{p_2}^2\delta_{l_1 l_2}e^{il_1 l_2 t_2}\theta(t_2-t_1)$$

(16)

where $\theta$ is the step function.

4 DIA equations

Calculating the path integral of the full nonlinear generating functional is a complicated problem, and an analog expression to (13) for the exact $Z$ is not known. Thus, to study the nonlinear regime, some level of approximation needs to be used. The Direct interaction approximation (DIA) corresponds to a particular resummation scheme which leads to a closed set of coupled nonlinear integro-differential equations for the correlation and response functions. In contrast to the original derivation, the use of the Martin-Siggia-Rose formalism exposes naturally the role of the response function in the statistical dynamics, as shown in the equation for the free generating functional (13).

The starting point is to consider perturbatively the non linear term of the generating functional (10) and expand the exponential into a power series. Every term in the series will corresponds to functional derivatives of $Z_0$ with respect to the auxiliary fields. Thus, we can write: $Z = Z_0[J,\tilde{J}] + Z_1[J,\tilde{J}] + Z_2[J,\tilde{J}] + \ldots$. Consequently, the correlation $C_{p_1p_2}^{l_1 l_2}(t_1,t_2)$ and response function $R_{p_1p_2}^{l_1 l_2}(t_1,t_2)$ can be written as expansions series making use of (11) and (12),

where the zero order term correspond precisely to the free correlation $C_{0,p_1p_2}^{l_1 l_2}(t_1,t_2)$ and the free response $R_{0,p_1p_2}^{l_1 l_2}(t_1,t_2)$, respectively (see details in Appendix B). To obtain an intuitive understanding of the procedure, we use a diagram technique.

So if we name every function (independent of the indices):

$$C \to \begin{array}{c} \overbrace{\begin{array}{c} t_2 \\ t_1 \end{array}} \end{array}$$

$$C_0 \to \begin{array}{c} \overbrace{\begin{array}{c} t_2 \\ t_1 \end{array}} \end{array}$$

$$R \to \begin{array}{c} \overbrace{\begin{array}{c} t_2 \\ t_1 \end{array}} \end{array}$$

$$R_0 \to \begin{array}{c} \overbrace{\begin{array}{c} t_2 \\ t_1 \end{array}} \end{array}$$

one could find the Feynman diagrams for the correlation and the response functions. However, for practical reasons, we define the differential operator $\hat{L}_{p_1,t_1}^{l_1} := \frac{\partial}{\partial t_1} + il_1 0_{p_1}^{l_1}$ and then expand the quantities $\hat{L}_{p_1,t_1}^{l_1}C_{p_1p_2}^{l_1 l_2}(t_1,t_2)$ and $\hat{L}_{p_1,t_1}^{l_1}R_{p_1p_2}^{l_1 l_2}(t_1,t_2)$ instead of the pure two-point correlation and response function.

Then the Feynman diagrams up to the second order take the form (showed in Appendix B):
Here, the single loops means that there is a 0-order correlation function evaluated at equal times. Also, the corresponding order of each diagram is associated with the number of vertices. Concurrently, this number indicates the number of coefficient
factors $L^{ss_{1} s_{2} s_{3}}_{kk_{1} k_{2} k_{3}}$ involved in wave vector integrals. Superscript sums and combinatory factors are ignored in the diagramatical representation.

To obtain the DIA equations, a resummation scheme is necessary. Such resummation process consists of considering an infinite subset of diagrams, to capture some effects beyond the weakly nonlinear regime. In practice, the resummation corresponds to replace the free correlations and free response function ($C_0$ or $R_0$) on the right side of equation (17) and (18) with the respective "exact" correlation and response function, i.e, $C_0 \rightarrow C$ and $R_0 \rightarrow R$ for each mode. Finally, we obtain the DIA equations, for the two point correlation

\begin{equation}
\tilde{L}_{p_{1}, t_{1}}^{l_{1}} t_{2} t_{3} t_{1} = \sum_{s} s L_{p_{1}, t_{1}}^{l_{1} s} C_{p_{2}}^{s_{1} s_{2}}(t_{1}, t_{2}) = \delta(t_{1}, t_{2}) + \int_{0}^{t_{1}} d\tau \sum_{x_{3}} s_{3}^{l_{1} s_{2}}(t_{1}, \tau) C_{p_{2}}^{s_{1} s_{2}}(\tau, t_{2}) + \int_{0}^{t_{2}} d\tau \sum_{x} s_{x}^{l_{1} t_{2}}(\tau, t_{1}) R_{p_{2}}^{x_{2} s_{2}}(t_{2}, \tau) \tag{19}
\end{equation}

and for the response function

\begin{equation}
\tilde{L}_{p_{1}, t_{1}}^{l_{1}} t_{2} t_{2} t_{1} t_{1} = \delta_{l_{1}, l_{2}} \delta(t_{1} - t_{2}) + \int_{0}^{t_{1}} d\tau \sum_{x_{3}} s_{3}^{l_{1} s_{2}}(t_{1}, \tau) C_{p_{2}}^{s_{1} s_{2}}(\tau, t_{2}) + \int_{0}^{t_{2}} d\tau \sum_{x} s_{x}^{l_{1} t_{2}}(\tau, t_{1}) R_{p_{2}}^{x_{2} s_{2}}(t_{2}, \tau) \tag{20}
\end{equation}

It is important to note that the diagrams in the final DIA equations corresponds to an infinite subset of the original expansion (17). One can show that all the diagrams up to the second order in (17) are considered in the DIA equations, but also many more (see for details Appendix C). For instance, one can easily notice that diagrams of order 1 and 2 of type A in (17) can be recovered from the type A diagram in the DIA equation (19). However, diagrams of order 2 type B and C in (17) are not obtained from the type A diagram in the DIA equation (19), but from diagrams type B and C in (19) (see Appendix C for details).

Considering statistical spatial homogeneity, the correlation and response function can be simplified as

\begin{equation}
C_{p_{1} p_{2}}^{l_{1} l_{2}}(t_{1}, t_{2}) = \delta(p_{1} + p_{2}) C_{p_{2}}^{l_{1} l_{2}}(t_{1}, t_{2}) 2\pi \quad \text{and} \quad R_{p_{1} p_{2}}^{l_{1} l_{2}}(t_{1}, t_{2}) = \delta(p_{1} + p_{2}) R_{p_{2}}^{l_{1} l_{2}}(t_{1}, t_{2}) \tag{21}
\end{equation}

Then the DIA equations read as

\begin{equation}
\tilde{L}_{p_{1}, t_{1}}^{l_{1}} t_{2} t_{3} t_{1} = \sum_{s} s L_{p_{2}}^{l_{1} s} C_{p_{2}}^{s_{1} s_{2}}(t_{1}, t_{2}) + 2F_{p_{2}} R_{p_{2}}^{l_{1} l_{2}}(t_{2}, t_{1}) + \int_{0}^{t_{1}} d\tau \sum_{x_{3}} s_{3}^{l_{1} t_{2}}(t_{1}, \tau) C_{p_{2}}^{s_{1} s_{2}}(\tau, t_{2}) + \int_{0}^{t_{2}} d\tau \sum_{x} s_{x}^{l_{1} t_{2}}(\tau, t_{1}) R_{p_{2}}^{x_{2} s_{2}}(t_{2}, \tau) \tag{21}
\end{equation}
\[ \mathcal{L}_{p_1 l_1}^{l_1 - l_2} R_{p_2 l_2}^{l_1 - l_2} (t_1, t_2) = \delta_{l_1, l_2} \delta(t_1 - t_2) + \sum_l f_{p_2}^{l} R_{p_2}^{l - l_2} (t_1, t_2) \]
\[ + \int_{t_2}^{t_1} d\tau \sum_{l_3} \sigma_{p_2}^{l_1, l_3} (t_1, \tau) R_{p_2}^{l_3 - l_2} (\tau, t_2) \]
\[ (22) \]

with
\[ f_{p}^{l} (t) := 3(2\pi) \sum_{s_1 s_2} \int d k L_{-p k k} C_{k}^{s_1 s_2} (t, t) \]
\[ (23) \]
\[ \sigma_{p}^{l_1, l_3} (t_1, t_2) := 18(2\pi)^2 \sum_{s_1 s_2 s_3} \int d k_{123} L_{p k k} C_{k}^{s_1 s_3} (t_1, t_1) C_{k}^{s_2 s_3} (t_1, t_1) \]
\[ \times R_{k_1}^{l_1 - l_2} (t_1, t_1) \delta(k_1 + k_2 + k_3 + p_2) \]
\[ (24) \]
\[ S_{p_2}^{l_1, l_3} (t_1, t_1) := 6(2\pi)^2 \sum_{s_1 s_2 s_3} \int d k_{123} L_{p k k} C_{k}^{s_1 s_3} (t_1, t_1) \]
\[ \times \sum_{s_1 s_2 s_3} L_{p k k} C_{k}^{s_1 s_3} (t_1, t_1) C_{k}^{s_2 s_3} (t_1, t_1) \]
\[ \times \delta(k_1 + k_2 + k_3 + p_2). \]
\[ (25) \]

Equations (21) and (22) correspond to the DIA equations for the elastic plate, similar to the DIA equations originally derived by Kraichnan in [38] for hydrodynamic turbulence. The main difference lies in the one-loop contribution proportional to \( f_{p}^{l} \), observed on the right hand side of equation (21) and (22). As one could notice, this nonlinear term alone is not capable for inducing an energy flux toward the small scales. If the amplitudes were zero above some wave number at some initial time, only the two-loop terms can activate those modes. However, the one-loop term could be very relevant for phase mixing and the decorrelation time scale in the system, which can be responsible for the breakdown of the weakly nonlinear regime. As one would anticipate, this term will be shown to be responsible for the frequency shift in the weakly nonlinear regime.

5 DIA Properties

5.1 Underlying stochastic system

An important property of the DIA equations is that they correspond to an exact closure for an underlying stochastic system. This ensures the existence of a well-defined PDF and quantities like \( C_{k}^{l_1 - l_2} (t, t) \) are positive definite. Constructing backwards from equation (21) we get the stochastic system \[ (\partial_t + i \omega_p) A_{p}^{l_1} (t) - \int_{0}^{t} \sum_{l_3} \sigma_{p}^{l_1, l_3} (t_1, \tau) A_{p}^{l_3} (\tau) d\tau = \eta_{p}^{l_1} (t) \]
\[ (26) \]
where \( \eta_{p}^{l_1} (t) \) is the stochastic force which is statistical independent among modes and must satisfy \( \langle \eta_{p}^{l_1} (t) \eta_{p}^{l_2} (t') \rangle = S_{p}^{l_1, l_2} (t, t') 2\pi \delta(p + p_2) \). A simple way to accomplish this constraint is defining
\[ \eta_{p}^{l_1} (t) := \sqrt{6(2\pi)^3} \sum_{s_1 s_2 s_3} \int d k_{123} L_{p k k} C_{k}^{s_1 s_3} (t_1, t_1) \]
\[ \times \xi_{k_1}^{s_1} (t) \chi_{k_2}^{s_2} (t) \phi_{k_3}^{s_3} (t) \delta(k_1 + k_2 + k_3 + p) \]
\[ (27) \]
where \( \xi_{k_1}^{s_1} (t), \chi_{k_2}^{s_2} (t) \) and \( \phi_{k_3}^{s_3} (t) \) are independent complex stochastic variables which correlation \( \langle \xi_{k_1}^{s_1} (t_1) \xi_{k_1}^{s_1} (t_2) \rangle = \langle \chi_{k_2}^{s_2} (t_1) \chi_{k_2}^{s_2} (t_2) \rangle = \langle \phi_{k_3}^{s_3} (t_1) \phi_{k_3}^{s_3} (t_2) \rangle = \delta(k_1 + k_2) C_{k_3}^{s_1 s_2} (t_1, t_2) \). Therefore, the correlation and response functions of the SDE (20) under the noise (27) are precisely the ones given by the DIA equations.

5.2 Kinetic equation

It is important to note that the DIA contains all the diagrams related to the second order expansion and therefore, contains all the needed diagrams to derive the kinetic equation of wave turbulence. If we want to take the weakly nonlinear limit,
we have to rescale the order of the deformation, \( A_k \rightarrow \epsilon A_k \), the forcing \( F_p \rightarrow \epsilon^2 F_p \) and dissipation \( \gamma_p \rightarrow \epsilon^2 \gamma_p \). It is convenient to make a change of variables to the time difference: 
\((t_1, t_2) \rightarrow (\tau, t_2)\) with \( \tau := t_2 - t_1 \). A standard perturbation method leads at zero order to 
\[ C^{l_1 l_2}_{0, p_2}(\tau, t_2) = C^{l_1 l_2}_{0, p_2}(0, 0) e^{-i(\tau + l_2 \omega_{p_2} + \delta_{l_1} \omega_{p_2} \tau)} \]
\[ R^{l_1 l_2}_{0, p_2}(\tau) = \delta_{l_1, -l_2} e^{-i \delta_{l_1} \omega_{p_2} \tau} \]
which are the same as Eq.13 and Eq.16 but without injection and dissipation. At higher orders of the expansion, resonant terms emerge requiring a multiscale perturbation method. Keeping the time difference \( \tau \) as a fast time scale of order one and considering the long time behavior of \( t_2 \), one can rewrite \( C^{l_1 l_2}_{0, p_2}(\tau, t_2) = C^{l_1 l_2}_{0, p_2}(\tau, t_2, T_2, T_2) + \epsilon C^{l_1 l_2}_{p_2,1}(\tau, t_2, T_2, T_2) + \epsilon^2 C^{l_1 l_2}_{p_2,2}(\tau, t_2, T_2, T_2) + \ldots \) with \( T_2 := \epsilon t_2 \) and \( T_2 := \epsilon^2 t_2 \). The integration constant of the zero order perturbation can then depend on the slow time, thus \( C^{l_1 l_2}_{0, p_2}(0, 0, T_2, T_2) \). The slow time evolution of the correlation function \( C^{l_1 l_2}_{0, p_2}(0, 0, T_2, T_2) \) is then set by the secular condition necessary to ensure an asymptotic perturbation expansion. At the second order expansion, one obtains that the two point correlation function at \( \tau = 0 \) defined as \( n_{l_2 p_2}(t_2) := C^{l_2 l_2}_{0, p_2}(0, 0, 0, T_2) \) must satisfy the well known kinetic equation for elastic plates [17] (see Appendix D for details):
\[ \frac{dn_{l_2 p_2}}{dt} = 12l_2^4 \epsilon^4 \int dt_1 dt_2 dt_3 L_{p_2}^{l_2 s_1 s_2 s_3} \sum_{s_1 s_2 s_3} \frac{n_{s_1 k_1} n_{s_2 k_2} n_{s_3 k_3} n_{l_2 p_2}}{n_{l_2 p_2}} \]
\times \delta(t_2 - s_1 \omega_{k_1} - s_2 \omega_{k_2} - s_3 \omega_{k_3})
\times \delta(k_1 + k_2 + k_3 - p_2) \]

(28)

It should be noted that although the derivation of DIA equations does not require continuum variables in Fourier space, the derivation of the kinetic equation does (see Appendix D).

6 Discussion

The Martin-Siggia-Rose path integral formalism used in this work gives a natural and clear approach to derive DIA and study wave turbulence theory for the elastic plates and other systems. This simplicity arises from the explicit dependence of the path integral on the response function, which enables a straightforward derivation of DIA. Interestingly, a new term emerges in the DIA equations for elastic plates, which is not present in previously studied systems. It originated from one-loop corrections, but yet, its physical significance and its role in non-weak turbulent systems remain to be explored.

Since the weak turbulence theory is recovered from the DIA equations in the proper limit, we expect the present formalism will elucidate several open questions on Wave Turbulence. For instance, a description of the response function at the slow time scales of the kinetic equation or a better understanding of the full probability distribution function using the stochastic underlying equations in the weakly nonlinear regime. Since DIA equations are equally derived for finite size systems, questions regarding discreteness in wave turbulence could also be addressed. Finally and more interestingly, the DIA equations might shed light on the breakdown of wave turbulence and the critical balance regime [61].

Theoretically, the DIA equations enable us to compute the so-called transport power [38, 62]. This is the rate at which energy is transferred from modes \( k' \) below some particular \( k' \) \((k' < k)\) to modes above the same \( k\) \((k' > k)\). Therefore, we propose to study the evolution of this quantity for different amplitudes and consequently, attempt to categorize regimes according to the transport power.
Finally, it should be noted that for incompressible fluids using the Clebsch variable, similar DIA equations are obtained. Since no small parameter exists for such a system, the validity of DIA is questionable but it could be interesting to establish whether it leads to the same incorrect results obtained by Kraichnan for the DIA in fluids [38].

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References

[1] J. Saur, H. Politano, A. Pouquet, and W. Matthaeus, Astronomy & Astrophysics 386, 699 (2002).

[2] K. Wilson, E. Samson, Z. Newman, T. Neely, and B. Anderson, Annual Review of Cold Atoms and Molecules: Volume 1, 261 (2013).

[3] P. Denissenko, S. Lukaschuk, and S. Nazarenko, Physical review letters 99, 014501 (2007).

[4] V. E. Zakharov and V. Lvov, Radiofizika 18, 1470 (1975).

[5] V. E. Zakharov, V. S. L’vov, and G. Falkovich, Kolmogorov spectra of turbulence I: Wave turbulence (Springer Science & Business Media, 2012).

[6] R. Peierls, Annalen der Physik 395, 1055 (1929).

[7] D. Benney and P. G. Saffman, Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences 289, 301 (1966).

[8] D. Benney and A. C. Newell, Studies in Applied Mathematics 48, 29 (1969).

[9] K. Hasselmann, Journal of Fluid Mechanics 12, 481 (1962).

[10] W. Vinen and J. Niemela, Journal of low temperature physics 128, 167 (2002).

[11] E. Kozik and B. Svistunov, Physical review letters 94, 025301 (2005).

[12] S. Y. Annenkov and V. I. Shrita, Journal of Fluid Mechanics 449, 341 (2001).

[13] E. Falcon, C. Laroche, and S. Fauve, Physical review letters 98, 094503 (2007).

[14] R. Bedard, S. Lukaschuk, and S. Nazarenko, Jetp Letters 97, 459 (2013).

[15] C. Ng and A. Bhattacharjee, Physics of Plasmas 4, 605 (1997).

[16] R. Micha and I. I. Tkachew, Physical review letters 90, 121301 (2003).

[17] G. Düring, C. Josserand, and S. Rica, Physical review letters 97, 025503 (2006).

[18] A. Boudaoud, P. Patricio, Y. Couder, and M. Ben Amar, Nature 407, 718 (2000).

[19] J. Chopin, D. Vella, and A. Boudaoud, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences 464, 2887 (2008).

[20] N. Mordant, Physical review letters 100, 234505 (2008).

[21] N. Mordant, The European Physical Journal B 76, 537 (2010).

[22] P. Cobelli, P. Petitjeans, A. Maurel, V. Pagneux, and N. Mordant, Physical review letters 103, 204301 (2009).

[23] G. Düring, C. Josserand, and S. Rica, Physica D: Nonlinear Phenomena 347, 42 (2017).
[24] G. Düring and G. Krstulovic, Physical Review E 97, 020201 (2018).
[25] R. Hassaini, N. Mordant, B. Miquel, G. Krstulovic, and G. Düring, Physical Review E 99, 033002 (2019).
[26] N. Yokoyama and M. Takaoka, Physical Review Letters 110, 105501 (2013).
[27] N. Yokoyama and M. Takaoka, Physical Review E 89, 012909 (2014).
[28] B. Miquel, A. Alexakis, C. Josserand, and N. Mordant, Physical review letters 111, 054302 (2013).
[29] G. Düring, C. Josserand, G. Krstulovic, and S. Rica, Physical Review Fluids 4, 064804 (2019).
[30] U. Frisch and A. N. Kolmogorov, Turbulence: the legacy of AN Kolmogorov (Cambridge university press, 1995).
[31] P. C. Martin, E. Siggia, and H. Rose, Physical Review A 8, 423 (1973).
[32] S. Gauthier, M.-E. Brachet, and J.-D. Fournier, Journal of Physics A: Mathematical and General 14, 2969 (1981).
[33] T. Castellani and A. Cavagna, Journal of Statistical Mechanics: Theory and Experiment 2005, P05012 (2005).
[34] J. Boussinesq, Essai sur la théorie des eaux courantes (Impr. nationale, 1877).
[35] H. Wyld Jr, Annals of Physics 14, 143 (1961).
[36] S. F. Edwards and W. McComb, Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences 325, 313 (1971).
[37] W. McComb, Journal of Physics A: Mathematical, Nuclear and General 7, 632 (1974).
[38] R. H. Kraichnan, Journal of Fluid Mechanics 5, 497 (1959).
[39] W. D. McComb, Oxford (1990).
[40] W. McComb and V. Shanmugasundaram, Journal of Fluid Mechanics 143, 95 (1984).
[41] J. A. Krommes, Physics Reports 360, 1 (2002).
[42] A. C. Newell, S. Nazarenko, and L. Biven, Physica D: Nonlinear Phenomena 152, 520 (2001).
[43] G. Düring and C. Falcón, Physical review letters 103, 174503 (2009).
[44] S. Galtier and S. V. Nazarenko, Physical review letters 119, 221101 (2017).
[45] J. Morton and S. Corrsin, Journal of Statistical Physics 2, 153 (1970).
[46] D. DuBois and H. Rose, Physical Review A 24, 1476 (1981).
[47] G.-Z. Sun, D. R. Nicholson, and H. A. Rose, The Physics of fluids 28, 2395 (1985).
[48] G.-Z. Sun, D. R. Nicholson, and H. A. Rose, The Physics of fluids 29, 1011 (1986).
[49] S. A. Orszag and G. Patterson Jr, Physical Review Letters 28, 76 (1972).
[50] J. S. Frederiksen and A. G. Davies, Geophysical & Astrophysical Fluid Dynamics 98, 203 (2004).
[51] J. C. Bowman, J. A. Krommes, and M. Ottaviani, Physics of Fluids B: Plasma Physics 5, 3558 (1993).
[52] L. D. Landau, E. M. Lifshitz, A. M. Kosevich, and L. P. Pitaevskii, Theory of elasticity: volume 7, Vol. 7 (Elsevier, 1986).
[53] A. I. Dyachenko, A. O. Korotkevich, and V. E. Zakharov, Physical review letters 92, 134501 (2004).

[54] V. E. Zakharov and N. Filonenko, in Doklady Akademii Nauk, Vol. 170 (Russian Academy of Sciences, 1966) pp. 1292–1295.

[55] V. E. Zakharov and N. Filonenko, Journal of applied mechanics and technical physics 8, 37 (1967).

[56] S. Nazarenko and M. Onorato, Physica D: Nonlinear Phenomena 219, 1 (2006).

[57] S. Dyachenko, A. C. Newell, A. Pushkarev, and V. E. Zakharov, Physica D: Nonlinear Phenomena 57, 96 (1992).

[58] S. Nazarenko and M. Onorato, Journal of Low Temperature Physics 146, 31 (2007).

[59] V. E. Zakharov, S. Musher, and A. Rubenchik, Physics reports 129, 285 (1985).

[60] P. J. Hansen and D. R. Nicholson, The Physics of Fluids 24, 615 (1981).

[61] A. C. Newell and V. E. Zakharov, Physics Letters A 372, 4230 (2008).

[62] R. H. Kraichnan, The Physics of Fluids 7, 1030 (1964).
A Generating Functional

To obtain the solution for the generating functional at order $0$, we start with the linear version of the path integral. As we would like to have a quadratic form in the argument of the exponential, we write $S_0$ (the linear version of MSR action $S$) as:

$$ S_0 = \frac{1}{2} \sum_{s_1 s_2} \int dk_1 \int dk_2 \int \phi^{s_1 T} \phi^{s_2} dt dt' G^{-1,s_1 s_2}_{k_1 k_2}(t, t') \phi^{s_2}(t') dt dt' $$

where $G^{-1,s_1 s_2}_{k_1 k_2}(t, t') := \begin{pmatrix} G^{-1,s_1 s_2}_{k_1 k_2,1} & G^{-1,s_1 s_2}_{k_1 k_2,2} \\ G^{-1,s_1 s_2}_{k_1 k_2,3} & G^{-1,s_1 s_2}_{k_1 k_2,4} \end{pmatrix}$.

Then, by comparison we find out that:

$$ G^{-1,s_1 s_2}_{k_1 k_2,1}(t, t') = 0 $$
$$ G^{-1,s_1 s_2}_{k_1 k_2,2}(t, t') = \delta(t - t') \delta_{s_1 s_2} \delta(k_1 - k_2)(-\partial_{\epsilon} + is_2 \Omega_{-k_2}) $$
$$ G^{-1,s_1 s_2}_{k_1 k_2,3}(t, t') = \delta(t - t') \delta_{s_1 s_2} \delta(k_1 - k_2)(\partial_{\epsilon} + is_2 \Omega_{k_2}) $$
$$ G^{-1,s_1 s_2}_{k_1 k_2,4}(t, t') = -2F_{k_2} \delta(k_1 + k_2) \delta_{s_1 s_2} \delta(t - t') $$

As already mentioned, here we add the auxiliar fields compressed in $J_k^s(t)$. So we get:

$$ Z_0[J, \tilde{J}] = \int \mathcal{D}[A_k^s(t)] \mathcal{D}[\tilde{A}_k^s(t)] e^{-\frac{1}{2} \sum_{ij \tau} \int dk_1 \int dk_2 \int dt \int dt' \phi^{s_1 T} \phi^{s_2} G^{-1,s_1 s_2}_{k_1 k_2}(t, t') \phi^{s_2}(t') dt dt' } $$

As usual, we now follow the common procedure to integrate quadratic forms. That is, we first make a translation of variable $y_k^s(t) := \phi_k^s(t) - \phi_k^s(t)$ where \[ \frac{\delta S_0^s}{\delta \phi_k^s} \bigg|_{\phi_k^s = \phi^s} = 0. \] From this last condition we get:

$$ \sum_s \int dk \int_0^T dt' G^{-1,s_1 s_2}_{k_1 k_2}(t, t') \phi_k^{s_2}(t') = J_k^{s_1}(t) $$

Thus, to obtain $\phi_k^{s_2}(t')$, we need to invert this equation and that means we need to have the inverse of $G^{-1,s_1 s_2}_{k_1 k_2}(t, t')$:

$$ \sum_s \int dk_3 \int d\tau G^{-1,s_1 s_2}_{k_1 k_3}(\tau, t)G^{s_2 s_3}_{k_3 k_2}(\tau, t') = \delta_{s_1 s_2} \delta(k_1 - k_2) \delta(t - t') $$

This condition yields to equations for the components and we know functions that satisfy them: The correlation function and the response function, both at order $0$. From the linear version of the dynamic equation for the fields, we know that:

$$ (\partial_\epsilon + is_1 \Omega_{k_1}) C^{s_1 s_2}_{0,k_1 k_2}(t, t') = 2F_{k_1} R^{s_2 s_3}_{0,k_1 k_2}(t, t') $$

Then we conclude that a possible solution is:

$$ G^{s_1 s_2}_{k_1 k_2}(t, t') = \begin{pmatrix} C^{s_1 s_2}_{0,k_1 k_2}(t, t') & R^{s_2 s_3}_{0,k_1 k_2}(t, t') \\ R^{s_2 s_3}_{0,k_1 k_2}(t, t') & 0 \end{pmatrix} $$

Therefore, we can invert \[ \mathcal{D}[\tilde{A}_k^s(t)] \] and make the change of variable. Finally, to diagonalize the matrix, we make a unitary rotation which yields a gaussian integral. Because of the unitary rotation and the linear translation, the jacobian of the variable change is 1. Finally, after integration, we get:

$$ Z_0[J, \tilde{J}] = e^{-\frac{1}{2} \sum_{ij \tau} \int dk_1 \int dt \phi_{k_1}^{s_1 T}(t) \phi_{k_1}^{s_1}(t) } $$

(37)
B Obtaining DIA

Since we already know the generating functional for the linear case, we start from the definition of the exact generating functional and separate \( S = S_0 + S_{N.L} \):

\[
Z_0 = \int \mathcal{D}A_k(t)\mathcal{D}\tilde{A}_k(t)e^{-S_0[A_k^*(t),\tilde{A}_k^*(t)]-S_{N,L}[A_k(t),\tilde{A}_k(t)]} \tag{38}
\]

We then convert it into a power series and add the auxiliary fields:

\[
Z[J, \tilde{J}] = \int \mathcal{D}[A_k(t)]\mathcal{D}[\tilde{A}_k(t)] \sum_{n=0}^{\infty} \frac{1}{n!} (S_{N.L})^n \times \exp \left( -S_0 + \sum_s \int dk \int J_k(t)A_k^*(t) + J_k^*(t)\tilde{A}_k(t)dt \right) \tag{39}
\]

If we expand \((S_{N,L})^n\), we observe that each term can be obtained if we perform derivatives with respect to the auxiliary fields and then set them to 0:

\[
Z[J, \tilde{J}] = \int \mathcal{D}[A_k(t)]\mathcal{D}[\tilde{A}_k(t)] \left( 1 + \int dk \int dt \sum_{s_{1,2,3}} \int I_{kk_kk_2k_3} \times \tilde{A}_k^*(t)A_k(t)A_{k_2}^*(t)A_{k_3}(t)\delta(K_{k_2k_3})d^4k + O(2) \right) \\
\times \exp \left( -S_0 + \sum_s \int dk \int J_k(t)A_k^*(t) + J_k^*(t)\tilde{A}_k(t)dt \right) = \int \mathcal{D}[A_k(t)]\mathcal{D}[\tilde{A}_k(t)] \left( 1 + \int dk \int dt \sum_{s_{1,2,3}} \int I_{kk_kk_2k_3} \times \frac{\delta^4}{\delta J_k^3(t)\delta J_{k_2}^3(t)\delta J_{k_3}^3(t)} \delta(K_{k_2k_3})d^4k + O(2) \right) \\
\times \exp \left( -S_0 + \sum_s \int dk \int J_k(t)A_k^*(t) + J_k^*(t)\tilde{A}_k(t)dt \right)
\]

where \(\delta(K_{k_2k_3}) := \delta(k_1 + k_2 + k_3 - k)\). Since the integrals are over the \(A\) and \(\tilde{A}\) fields, the series pass out the integrals and \(Z\) can be viewed as a series of terms, each of them involving derivatives of free case \(Z_0\) with respect to the auxiliary fields. Finally, to obtain the DIA, we go up to second order of the expansion and compute 10 derivatives (8 for the nonlinear expression of \(Z_2\) and 2 to obtain either the correlation function or the response function). Nevertheless, instead of actually computing the 10 derivatives, we can make use of the diagram technique as follows. We need second order terms, which means that we will have 2 internal vertices corresponding to the 2 factors \(I_{kk_kk_2k_3}\) and 2 external legs associated with the 2 times where the correlation/response function is actually evaluated. In this case, one of the external legs will have the time \(t_2\) and the other \(t_1\). Also, the 2 internal vertices correspond to 2 time variables (say \(t\) and \(t'\)) which, obviously, are integrated. Next, we observe the symmetries that arise from the derivatives with respect to the \(x, q, s\) and \(k\), because they are all dummy variables. Since \(Z_0\) is expressed in terms of the 0-order statistical functions and these functions come from differentiate twice with respect to the auxiliary fields as seen in \(\{\text{for the } c^{(1)}_{l_1l_2} (t_1, t_2) \text{ the derivatives are } \frac{\delta J_{l_2}^2(t_2)}{\delta J_{l_1}^2(t_1)}\}\), while for \( R_{l_1l_2}^{(1)} (t_1, t_2) \) the derivatives are \(\frac{\delta J_{l_2}^2(t_2)}{\delta J_{l_1}^2(t_1)}\), we can view the problem of derivatives as a combinatorial problem where we have to count every possible outcome that can be made if we take the derivatives by pairs. In this way, the only possible outcomes for the correlation function come from:

a) \[
\left( \frac{\delta}{\delta J_{12}^2(t_1)} \right) \left( \frac{\delta}{\delta J_{12}^2(t')} \right) \left( \frac{\delta}{\delta J_{12}^2(t_1)} \right) \left( \frac{\delta}{\delta J_{12}^2(t')} \right) \left( \frac{\delta}{\delta J_{12}^2(t_1)} \right) \left( \frac{\delta}{\delta J_{12}^2(t')} \right)
\]

and

\[
\left( \frac{\delta}{\delta J_{12}^2(t_1)} \right) \left( \frac{\delta}{\delta J_{12}^2(t')} \right) \left( \frac{\delta}{\delta J_{12}^2(t_1)} \right) \left( \frac{\delta}{\delta J_{12}^2(t')} \right) \left( \frac{\delta}{\delta J_{12}^2(t_1)} \right) \left( \frac{\delta}{\delta J_{12}^2(t')} \right)
\]

which will represent:

\[
\begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2}
\end{array}
\]

and because of symmetry, there will be 18 diagrams of the same type.

b) \[
\left( \frac{\delta}{\delta J_{12}^2(t_1)} \right) \left( \frac{\delta}{\delta J_{12}^2(t')} \right) \left( \frac{\delta}{\delta J_{12}^2(t_1)} \right) \left( \frac{\delta}{\delta J_{12}^2(t')} \right) \left( \frac{\delta}{\delta J_{12}^2(t_1)} \right) \left( \frac{\delta}{\delta J_{12}^2(t')} \right)
\]

and

\[
\left( \frac{\delta}{\delta J_{12}^2(t_1)} \right) \left( \frac{\delta}{\delta J_{12}^2(t')} \right) \left( \frac{\delta}{\delta J_{12}^2(t_1)} \right) \left( \frac{\delta}{\delta J_{12}^2(t')} \right) \left( \frac{\delta}{\delta J_{12}^2(t_1)} \right) \left( \frac{\delta}{\delta J_{12}^2(t')} \right)
\]

which will represent:

\[
\begin{array}{c}
\text{Diagram 3} \\
\text{Diagram 4}
\end{array}
\]

and because of symmetry, there will be 18 diagrams of the same type.
C Resummation process and expansion for order 1

To explain and clarify the resummation process, let us use the first order as an example. That is, we will only consider the diagrams with only one vertex at the most: the first and second one at the right hand side of the expansion of the correlation function. Then we make the mentioned replacements for the exact functions; so, the final equation corresponds to

\[ \hat{L}^l_{\mathbf{p},t_1} t_2 \mathbf{t}_2 \mathbf{t}_1 t_1 = t_2 \mathbf{t}_2 t_1 + 3 \times t_2 \mathbf{t}_2 t_1 \]

and

\[ \hat{L}^l_{\mathbf{p},t_1} t_2 \mathbf{t}_2 t_1 t_1 = \delta_{t_1,t_2} t_1 - t_2 + 3 \times t_2 \]

Thus, if we use the Green’s function of the operator \( \hat{L}^l_{\mathbf{p},t_1} \) which is nothing but adding a solid line as a right external leg, we can expand the recursive relation for the function and eventually replace this expression in the loop and leg of the second term in the right hand side of equation. Explic-

\[ \text{Here we show the combinatory factor "}3\times\text{" as it is not explicitly written in the equation at first order.} \]
D Kinetic Equation and frequency correction

Since the kinetic equation corresponds to the slow evolution of the same-time correlation function, an appropriate change of variables must be made to capture the required dynamics. First of all, \((t_1, t_2) \rightarrow (\tau, t_2)\) where \(\tau := t_2 - t_1\) but then as we allow different time scales \((\tau, t_2) \rightarrow (\tau, t_2, T_2)\) with \(T_2 := \epsilon t_2\) and \(T_1 := \epsilon^2 t_2\). This change yields to a respective expansion for the correlation function

\[
C^{l_1l_2}_{p_2}(t_2, T_2) = C^{l_1l_2}_{0,p_2}(\tau, t_2, T_2) + \epsilon C^{l_1l_2}_{1,p_2}(\tau, t_2, T_2) + \epsilon^2 C^{l_1l_2}_{2,p_2}(\tau, t_2, T_2) + \ldots.
\]

It is to be observed that in order to justify the expansion of the correlation function for every time, it has to remain bounded for times at any order. In this way, we will find a secular equation to ensure that the constant terms will vanish and therefore, don’t grow linearly with \(t_2\). The usual procedure starts with the solution for the zero order functions without forcing and dissipation:

\[
\frac{\partial}{\partial \tau} C^{l_1l_2}_{0,p_2}(\tau, t_2, T_2) + i l_1 \omega_{p_2} C^{l_1l_2}_{0,p_2}(\tau, t_2, T_2) = 0 \quad (48)
\]

\[
\left( \frac{\partial}{\partial \tau} + \frac{\partial}{\partial t_2} + i l_2 \omega_{p_2} \right) C^{l_1l_2}_{0,p_2}(\tau, t_2, T_2) = 0 \quad (49)
\]

Then,

\[
C^{l_1l_2}_{0,p_2}(\tau, t_2, T_2) = C^{l_1l_2}_{0,p_2}(0, 0, T_2) e^{-i(\tau + t_2)\omega_{p_2} t_2 + il_1 \omega_{p_2} \tau} \quad (50)
\]

where all diagrams up to order 2 are written. It should be clearly noted that the diagrams that are multiplied from the right with solid line are the ones corresponding to the expansion of the response function. In this way, we observe that a subset of diagrams, already included in the original expansion, appear in this last expansion and it is only because of the resummation for the first order term. Despite the appearing subset of diagrams, we can also observe that there are diagrams that cannot be made up to this order. In consequence, the resummation replacement for \(l_1 l_2\) will only take place at those diagrams that cannot be recovered from the order 1.

4 These are also the ones involving an extra time variable in comparison with the diagrams of the same order.
D.1 Order $\epsilon$: Frequency shift and nonexistent energy transfer

The equations are:

\[
\left( -\frac{\partial}{\partial \tau} + il_1 \omega_{p_2} \right) C^{l_1 l_2}_{1, p_2}(\tau, t_2, T_2) = 6\pi^2 \sum_{s_1 s_2 s_3} \int \tau_{l_1 s_1 s_2 s_3}^{l_2 s_2 s_3} \frac{C^{s_1 s_2}_{o, k_2}(0, t_2 - \tau, T_2)}{\partial \tau} \partial C^{l_1 l_2}_{0, p_2}(\tau, t_2, T_2)
\]

(51)

If we add the equations we obtain:

\[
\left( -\frac{\partial}{\partial t_2} + il_1 + il_2 \omega_{p_2} \right) C^{l_1 l_2}_{1, p_2}(\tau, t_2, T_2) = -\frac{\partial C^{l_1 l_2}_{0, p_2}}{\partial t_2} + 6\pi^2 \sum_{s_1 s_2 s_3} \int \tau_{l_2 s_1 s_2 s_3}^{l_1 s_1 s_2 s_3} \frac{C^{s_1 s_2}_{o, k_2}(0, t_2)}{\partial \tau} \partial C^{l_1 l_2}_{0, p_2}(\tau, t_2, T_2) + 6\pi^2 \sum_{s_1 s_2 s_3} \int \tau_{l_1 l_2 s_1 s_2 s_3}^{l_2 s_2 s_3} \frac{C^{s_1 s_2}_{o, k_2}(0, t_2, T_2)}{\partial \tau} \partial C^{l_1 l_2}_{0, p_2}(\tau, t_2, T_2)
\]

(52)

(53)

Therefore, the nonlinear terms cancel each other, yielding to the secular condition:

\[
\frac{\partial C^{l_1 l_1}_{0, p_2}}{\partial t_2}(0, 0, T_2) = 0
\]

(55)

which shows that there is no energy cascade at this order. For the case $l_2 = l_1$ the procedure is similar; therefore we only present the secular condition:

\[
\frac{\partial C^{l_1 l_1}_{0, p_2}}{\partial t_2}(0, 0, T_2) = 6\pi \sum_{s_1} \int \tau_{l_1 s_1}^{l_1 s_1} \frac{C^{s_1 s_1}_{0, k_2}(0, 0, T_2) C^{l_1 l_1}_{0, p_2}(0, 0, T_2)}{\partial t_2} + 6\pi \sum_{s_1} \int \tau_{l_1 s_1}^{l_1 s_1} \frac{C^{s_1 s_1}_{0, k_2}(0, 0, T_2) C^{l_1 l_1}_{0, p_2}(0, 0, T_2)}{\partial t_2} + 6\pi \sum_{s_1} \int \tau_{l_1 s_1}^{l_1 s_1} \frac{C^{s_1 s_1}_{0, k_2}(0, 0, T_2) C^{l_1 l_1}_{0, p_2}(0, 0, T_2)}{\partial t_2}
\]

(57)

which leads to a frequency correction of order $\epsilon$:

\[
C^{l_1 l_1}_{0, p_2}(0, 0, T) = C^{l_1 l_1}_{0, p_2}(0, 0, 0) e^{i\omega_{p_2} T}
\]

(58)

with

\[
\omega_{p_2} := -6\pi \sum_{s_1} \int \tau_{l_1 s_1}^{l_1 s_1} \frac{C^{s_1 s_1}_{0, k_2}(0, 0, 0) d^2 k_1}{\partial t_2}
\]

(59)

For the case $l_2 = -l_1$, the only constant terms with respect to $t_2$ are: \(\frac{\partial C^{l_1 l_1}_{0, p_2}}{\partial t_2}(0, 0, T_2)\), the first nonlinear term for the case $s_1 = -s_2, s_3 = -l_1$ and finally, the second nonlinear term for the case $s_1 = -s_2, s_3 = l_1$. Then, because of properties of the scattering coefficient

\[
\tau_{l_1 l_1 s_1 s_2 s_3}^{l_1 l_2 s_1 s_2 s_3} = -\tau_{l_1 l_2 s_1 s_2 s_3}^{l_1 l_1 s_1 s_2 s_3}
\]

(54)
D.2 Order $\epsilon^2$: Kinetic equation

The second order equations read as:

$$
\left[ -\frac{\partial}{\partial \tau} + i\omega_{p2} \right] C_{2,p2}^{t_2}(\tau, t_2, T_2) = 6\pi \sum_{x_1 x_2 x_3} L_{p2-k_2-k_2-p2}^{t_1 x_1 x_2 x_3} \int_{t_2}^{t_2-\tau} \frac{dt}{\tau} \sum_{x_1 x_2 x_3} L_{k_1-x_1-k_2-p2}^{t_1 x_1 x_2 x_3} x_1 x_2 x_3 \nu \int_{t_2}^{t_2-\tau} \frac{dt}{\tau} \sum_{x_1 x_2 x_3} L_{p1-k_1-k_2-k_2-p2}^{t_1 x_1 x_2 x_3} x_1 x_2 x_3 \nu \left( C_{2,p2}^{t_1}(t_2-\tau, t_2, T_2) \right)
$$

and

$$
\left[ -\frac{\partial}{\partial \tau} + i\omega_{p2} \right] C_{2,p2}^{t_2}(\tau, t_2, T_2) = 6\pi \sum_{x_1 x_2 x_3} L_{p2-k_2-k_2-p2}^{t_1 x_1 x_2 x_3} \int_{t_2}^{t_2-\tau} \frac{dt}{\tau} \sum_{x_1 x_2 x_3} L_{k_1-x_1-k_2-p2}^{t_1 x_1 x_2 x_3} x_1 x_2 x_3 \nu \int_{t_2}^{t_2-\tau} \frac{dt}{\tau} \sum_{x_1 x_2 x_3} L_{p1-k_1-k_2-k_2-p2}^{t_1 x_1 x_2 x_3} x_1 x_2 x_3 \nu \left( C_{2,p2}^{t_1}(t_2-\tau, t_2, T_2) \right)
$$

slow scales and therefore we take

$$
\lim_{t_2 \to \infty} \frac{1}{t_2} \int dt \left\{ \frac{\partial C_{2,p2}^{t_1-t_2}}{\partial \tau_2} + \frac{\partial C_{0,p2}^{t_1-t_2}}{\partial \tau_1} \right\} = \lim_{t_2 \to \infty} \frac{1}{t_2} \int dt \left\{ 6\pi \sum_{x_1 x_2 x_3} L_{p2-k_2-k_2-p2}^{t_1 x_1 x_2 x_3} \int_{t_2}^{t_2-\tau} \frac{dt}{\tau} \sum_{x_1 x_2 x_3} L_{k_1-x_1-k_2-p2}^{t_1 x_1 x_2 x_3} \nu \int_{t_2}^{t_2-\tau} \frac{dt}{\tau} \sum_{x_1 x_2 x_3} L_{p1-k_1-k_2-k_2-p2}^{t_1 x_1 x_2 x_3} x_1 x_2 x_3 \nu \left( C_{2,p2}^{t_1}(t_2-\tau, t_2, T_2) \right) \right\}
$$

The terms of order 1 will not contribute for reasons already clarified. For the second order terms, we can see that only particular cases will vanish. To see that, we have to use the Riemann-Lebesgue Lemma, because we are dealing with terms of the type:

$$
\lim_{t \to \infty} \int_{-\infty}^{\infty} f(x) \int_{0}^{t} e^{ixr} \int_{0}^{t} e^{-ixr'} dr' dr dx
$$

Here, we can get rid easily of derivatives with respect to $\tau$ and focus on $t_2$ for the special case where $l_2 = -l_1$. We can find the secular terms by integration and then divide everything by $t_2$. Finally, we are interested in the limit of dependence of the
Thus, we find that the secular terms will correspond to the ones where the correlation functions have opposite signs in the upper index, that is: for example for the first term of second order $x_3 = l_1, x_1 = s_3$ and $x_2 = s_2$. On the left side of the equation we have that the only one which is resonant is $\frac{\partial c^{l_{1}-l_{2}}}{\partial k_{p_{2}}}$. 

Finally, the secular equation becomes:

$$
\frac{\partial c^{l_{1}-l_{2}}}{\partial T_{2}} = -12\pi(2\pi)^{2} \sum_{s_{1} s_{2} s_{3}} \int dk_{123} T_{123} \left| \left[ L_{p_{2}q_{1}q_{2}q_{1}}^{l_{1}l_{2}} \right] \right|^{2} \\
\times c_{p_{2}l_{1}}^{-l_{2}}(T_{2}) c_{k_{3}s_{1}s_{3}}^{-s_{2}s_{3}}(T_{2}) c_{k_{2}l_{2}}^{-s_{2}s_{3}}(T_{2}) \\
\times \frac{\partial}{\partial k_{l_{1}}} \delta(K_{l_{1}}) \delta(l_{1} \omega_{p_{2}} + s_{1} \omega_{l_{1}} + s_{2} \omega_{k_{2}} + s_{3} \omega_{k_{3}}) \\
+ 12\pi(2\pi)^{2} \sum_{s_{1} s_{2} s_{3}} \int dk_{123} \left| L_{p_{2}q_{1}q_{2}q_{1}}^{l_{1}l_{2}} \right|^{2} \\
\times c_{k_{1}l_{1}}^{-s_{1}s_{1}}(T_{2}) c_{k_{2}l_{2}}^{-s_{2}s_{2}}(T_{2}) c_{k_{3}l_{3}}^{-s_{3}s_{3}}(T_{2}) \\
\delta(K_{l_{1}}) \delta(l_{1} \omega_{p_{2}} + s_{1} \omega_{l_{1}} + s_{2} \omega_{k_{2}} + s_{3} \omega_{k_{3}})
$$

(62)

which is the kinetic equation.

**E** \hspace{1em} DIA exact equations from stochastic system

We first begin with the DIA equation for the correlation function. From the nonlinear dynamic equation \[26\] we multiply both sides by $A_{p_{2}}^{l_{2}}(t_{2})$ and ensemble average to get:

$$
\mathcal{L}_{p_{1}} \mathcal{C}_{pp_{2}}^{l_{2}}(t_{2}) = \sum_{s} \int_{0}^{t} \sigma_{p_{1}}^{l_{1}l_{2}}(t_{1}, \tau) \mathcal{C}_{pp_{2}}^{l_{2}}(\tau, t_{2}) d\tau \\
= \langle \eta_{p_{1}}^{l_{1}}(t) A_{p_{2}}^{l_{2}}(t_{2}) \rangle 
$$

(63)

Then, to handle the right hand side of the equation, we return to the definition of the (non-averaged) response function \[38\]:

$$
A_{p_{2}}^{l_{2}}(t_{2}) = \sum_{s} \int_{0}^{t_{2}} d\tau \int_{-\infty}^{\infty} d\omega \mathcal{R}_{p_{2}-k}(t_{2}, \tau) \eta_{k}^{l_{2}}(\tau)
$$

(64)

so we obtain for the right hand side:

$$
\langle \eta_{p_{1}}^{l_{1}}(t) A_{p_{2}}^{l_{2}}(t_{2}) \rangle = \int_{0}^{t_{2}} d\tau \sum_{s} \int d\omega \mathcal{R}_{p_{2}-k}(t_{2}, \tau) \langle \eta_{p_{1}}^{l_{1}}(t) \eta_{k}^{l_{2}}(\tau) \rangle
$$

(65)

where $\langle \mathcal{R} \rangle = R$. Considering spatial homogeneity and using the imposed condition for the noise of the stochastic system, we obtain the DIA equation for the correlation function. The DIA equation for the response function can be obtained trivially by taking the functional derivative with respect to the noise and then averaging.

**F** \hspace{1em} Some useful identities about the response function

The first identity to show is $\langle \eta_{p_{1}}^{l_{1}}(t_{1}) A_{p_{2}}^{l_{2}}(t_{2}) \rangle$.

$$
\langle \eta_{p_{1}}^{l_{1}}(t_{1}) \rangle \frac{\delta A_{p_{2}}^{l_{2}}}{\delta \eta_{p_{2}}^{l_{2}}(t_{2})} = \int D\eta P(\eta) \int D[A, A'] \frac{\delta A_{p_{1}}^{l_{1}}(t_{1})}{\delta \eta_{p_{2}}^{l_{2}}(t_{2})} \sum_{s} \int dt \int dt' \hat{A}_{k}^{l_{1}}(t) \langle A_{k}^{l_{2}}(t) + is \omega_{k} A_{k}^{l_{2}}(t) - \eta_{k}^{l_{2}}(t) + NL \rangle
$$

$$
= \int D\eta P(\eta) \int D[A, A'] A_{p_{1}}^{l_{1}}(t_{1}) \left( -\delta_{p_{1}} \sum_{s} \int dt \int dt' \hat{A}_{k}^{l_{1}}(t) \langle A_{k}^{l_{2}}(t) + is \omega_{k} A_{k}^{l_{2}}(t) - \eta_{k}^{l_{2}}(t) + NL \rangle \\
\right) \\
= \int D\eta P(\eta) \int D[A, A'] A_{p_{1}}^{l_{1}}(t_{1}) A_{p_{2}}^{l_{2}}(t_{2}) \sum_{s} \int dt \int dt' \hat{A}_{k}^{l_{1}}(t) \langle A_{k}^{l_{2}}(t) + is \omega_{k} A_{k}^{l_{2}}(t) - \eta_{k}^{l_{2}}(t) + NL \rangle
$$

(66)
Next, we show that the response function also corresponds to the correlation of the field with the noise:

\[ \langle A_k'(t') \eta_k(t) \rangle = \int D\eta(t) \int D[A, \tilde{A}] A_k''(t') \eta_k(t) e^{\int -\tilde{A}(t)(\tilde{A}(t) + is\Omega A(t) - \eta(t) + NL)} \]

\[ = \int D\eta(t) \int D[A, \tilde{A}] A_k''(t') \eta_k(t) e^{\int [\tilde{A}(t)(\tilde{A}(t) + is\Omega A(t) - \eta(t) + NL)]} \bigg|_{b=0} \]

\[ = \int D\eta(t) \int D[A, \tilde{A}] A_k''(t') \frac{\delta}{\delta b_k(t)} e^{\int -\tilde{A}(t)(\tilde{A}(t) + is\Omega A(t) + NL)dt} \int D\eta(t) e^{\int (s\tilde{A}(t) + b(t))\eta(t)dt} \bigg|_{b=0} \]

\[ = \int D[A, \tilde{A}] A_k''(t') \left( \int \tilde{A}_{k1}^* (t_1) \delta_{-s,s_1} \delta(k_1 + k) \delta(t_1 - t) \right) e^{\int -\tilde{A}(t)(\tilde{A}(t) + is\Omega A(t) + NL)dt} \int \tilde{A}(t_1) \delta(t_1 - t_2) A(t_2) \bigg|_{b=0} \]

\[ = \left( \int R_{k' - k_1}^* (t', t_1) 2F_k \delta_{-s,a} \delta(k + k_1) \delta(t_1 - t) \right) \]

\[ = 2F_k R_{k' - k}^* (t', t) \]