NEARBY SLOPES AND BOUNDEDNESS FOR \(\ell\)-ADIC SHEAVES IN POSITIVE CHARACTERISTIC

by

Jean-Baptiste Teyssier

Introduction

Let \(S\) be a strictly henselian trait of equal characteristic \(p > 0\). As usual, \(s\) denotes the closed point of \(S\), \(k\) its residue field, \(\eta = \text{Spec} \ K\) the generic point of \(S\), \(\overline{K}\) an algebraic closure of \(K\) and \(\overline{\eta} = \text{Spec} \ \overline{K}\). Let \(f : X \to S\) be a morphism of finite type, \(\ell \neq p\) a prime number, \(\mathcal{F}\) an object of the derived category \(D^b_{\text{c}}(X_s, \mathcal{O})\) of \(\ell\)-adic complexes with bounded and constructible cohomology.

Let \(\psi_f^! : D^b_c(X_\eta, \mathcal{O}_\eta) \to D^b_c(X_s, \mathcal{O}_s)\) be the moderate nearby cycle functor. We say that \(r \in \mathbb{R}_{\geq 0}\) is a nearby slope of \(\mathcal{F}\) associated to \(f\) if one can find \(\mathcal{N} \in \text{Sh}_c(\eta, \mathcal{O}_k)\) with slope \(r\) such that \(\psi_f^!(\mathcal{F} \otimes f^* \mathcal{N}) \neq 0\). We denote by \(\text{Sl}_{\text{nb}}^f(\mathcal{F})\) the set of nearby slopes of \(\mathcal{F}\) associated to \(f\).

The main result of [Tey15] is a boundedness theorem for the set of nearby slopes of a complex holonomic \(\mathcal{D}\)-module. The goal of the present (mostly programmatic) paper is to give some motivation for an analogue of this theorem for \(\ell\)-adic sheaves in positive characteristic.

For complex holonomic \(\mathcal{D}\)-modules, regularity is preserved by push-forward. On the other hand, for a morphism \(C' \to C\) between smooth curves over \(k\), a tame constructible sheaf on \(C'\) may acquire wild ramification by push-forward. If \(0 \in C\) is a closed point, the failure of \(C' \to C\) to preserve tameness above 0 is accounted for by means of the ramification filtration on the absolute Galois group of the function field of the strict henselization \(C_0^{\text{sh}}\) of \(C\) at 0. Moreover, the Swan conductor at 0 measures to which extent an \(\ell\)-adic constructible sheaf on \(C\) fails to be tame at 0.

In higher dimension, both these measures of wild ramification (for a morphism and for a sheaf) are missing in a form that would give a precise meaning to the following question raised in [Tey14]

**Question 1.** — Let \(g : V_1 \to V_2\) be a morphism between schemes of finite type over \(k\), and \(\mathcal{G} \in D^b_c(V_1, \mathcal{O}_k)\). Can one bound the wild ramification of \(Rg_* \mathcal{G}\) in terms of the wild ramification of \(\mathcal{G}\) and the wild ramification of \(g|_{\text{Supp} \mathcal{G}}\)?
Note that in an earlier formulation, "wild ramification of $g|_{\text{Supp} \mathcal{G}}" was replaced by "wild ramification of $g", which cannot hold due to the following example that we owe to Alexander Beilinson: take $f : A_1 \to S$, $P \in S[t]$ and $i_P : \{P = 0\} \to A_1$. Then $i_P \overline{Q}_\ell$ is tame but $f_*(i_P \overline{Q}_\ell)$ has arbitrary big wild ramification as $P$ runs through the set of Eisenstein polynomials.

If $f : X \to S$ is proper, proposition 2.2.1 shows that $\text{Sl}_{\text{nb}}^b(f)$ controls the slopes of $H^i(X_\eta, \mathcal{F})$ for every $i \geq 0$. It is thus tempting to take for "wild ramification of $G" the nearby slopes of $\mathcal{G}$.

So Question 1 leads to the question of bounding nearby slopes of constructible $\ell$-adic sheaves. Note that this question was raised imprudently in [Tey15]. It has a negative answer as stated in loc. it. since already the constant sheaf $\mathbb{Q}_\ell$ has arbitrary big nearby slopes. This is actually good news since for curves, these nearby slopes keep track of the aforementioned ramification filtration. Hence, one can use them in higher dimension to quantify the wild ramification of a morphism and in Question 1 take for "wild ramification of $g|_{\text{Supp} \mathcal{G}}" the nearby slopes of $\mathbb{Q}_\ell$ on $\text{Supp} \mathcal{G}$ associated with $g|_{\text{Supp} \mathcal{G}}$ (at least when $V_2$ is a curve).

To get a good boundedness statement, one has to correct the nearby slopes associated with a morphism by taking into account the maximal nearby slope of $\mathbb{Q}_\ell$ associated with the same morphism. That such a maximal slope exists in general is a consequence of the following

**Theorem 1.** — Let $f : X \to S$ be a morphism of finite type and $\mathcal{F} \in \mathcal{D}^b_c(X_\eta, \overline{Q}_\ell)$.

The set $\text{Sl}_{\text{nb}}^b(\mathcal{F})$ is finite.

The proof of this theorem follows an argument due to Deligne [Del77, Th. finitude 3.7]. For a $\mathcal{D}$-module version, let us refer to [Del07]. Thus, $\text{Max} \text{Sl}_{\text{nb}}^b(\mathcal{F})$ makes sense if $\text{Sl}_{\text{nb}}^b(\mathcal{F})$ is not empty. Otherwise, we set $\text{Max} \text{Sl}_{\text{nb}}^b(\mathcal{F}) = +\infty$. Proposition 2.3.4 suggests and gives a positive answer to the following question for smooth curves

**Question 2.** — Let $V/k$ be a scheme of finite type and $\mathcal{F} \in \mathcal{D}^b_c(V, \overline{Q}_\ell)$. Is it true that the following set

$$\{r/(1 + \text{Max} \text{Sl}_{\text{nb}}^b(\overline{Q}_\ell)), \text{ for } r \in \text{Sl}_{\text{nb}}^b(\mathcal{F}) \text{ and } f \in \mathcal{O}_V\}$$

is bounded?

Let us explain what $\text{Sl}_{\text{nb}}^b(\mathcal{F})$ means in this global setting. A function $f \in \Gamma(U, \mathcal{O}_V)$ reads as $f : U \to A_1$. If $S$ is the strict henselianization of $A_1$ at a geometric point over the origin, we set $\text{Sl}_{\text{nb}}^b(\mathcal{F}) := \text{Sl}_{\text{nb}}^b(\mathcal{F}_U)$ where the subscripts are synonyms of pull-back.

For smooth curves, the main point of the proof of boundedness is the concavity of Herbrand $\varphi$ functions. In case $f$ has generalized semi-stable reduction (see [1.3]), the above weighted slopes are the usual nearby slopes. This is the following

1. see [2.1.2 (6)] for a precise statement.
Theorem 2. — Suppose that $f : X \to S$ has generalized semi-stable reduction. Then we have $\text{Sl}^{\text{nb}}_f(\overline{Q_L}) = \{0\}$.

We owe the proof of this theorem to Joseph Ayoub. For the vanishing of $\mathcal{H}^0\psi_f^t$, we also give an earlier argument based on the geometric connectivity of the connected components of the moderate Milnor fibers in case of generalized semi-stable reduction.

As a possible application of a boundedness theorem in the arithmetic setting, let us remark that for every compactification $j : V \to \overline{V}$, one could define a separated decreasing $\mathbb{R}_{\geq 0}$-filtration on $\pi_1(V)$ by looking for each $r \in \mathbb{R}_{\geq 0}$ at the category of $\ell$-adic local systems $L$ on $V$ such that the weighted slopes $[0,1]$ of $j_*L$ are $\leq r$.

Let us also remark that on a smooth curve $C$, the tameness of $\mathcal{F} \in \text{Sh}_u(C, \overline{Q}_L)$ at $0 \in C$ is characterized by $\text{Sl}_f^{\text{nb}}(\mathcal{F}) \subset [0, \text{Max} \text{Sl}_f^{\text{nb}}(\overline{Q}_L)]$ for every $f \in \mathcal{O}_C$ vanishing only at 0. This suggests a notion of tame complex in any dimension that may be of interest.

I thank Joseph Ayoub for his willingness to know about nearby slopes and for generously explaining me a proof of Theorem 2 during a stay in Zurich in May 2015. I also thank Kay Rülling for a useful discussion. This work has been achieved with the support of Freie Universität/Hebrew University of Jerusalem joint post-doctoral program. I thank Hélène Esnault and Yakov Varshavsky for their support.

1. Notations

1.1. — For a general reference on wild ramification in dimension 1, let us mention [Ser68]. Let $\eta$ be the point of $S$ corresponding to the tamely ramified closure $K_i$ of $K$ in $\overline{K}$ and $P_K := \text{Gal}(\overline{K}/K_i)$ the wild ramification group of $K$. We denote by $(G_K^r)_{r \in \mathbb{R}_{\geq 0}}$ the upper-numbering ramification filtration on $G_K$ and define

$$G_K^{r+} := \bigcup_{r' > r} G_K^{r'}$$

If $L/K$ is a finite extension, we denote by $S_L$ the normalization of $S$ in $L$ and $\nu_L$ the valuation on $L$ associated with the maximal ideal of $S_L$.

If moreover $L/K$ is separable, we denote by $q : G_K \to G_K/G_L$ the quotient morphism and define a decreasing separated $\mathbb{R}_{\geq 0}$-filtration on the set $G_K/G_L$ by $(G_K/G_L)^r := q(G_K^r)$. We also define $(G_K/G_L)^{r+} := q(G_K^{r+})$.

In case $L/K$ is Galois, this filtration is the upper numbering ramification filtration on $\text{Gal}(L/K)$. If $L/K$ is non separable trivial, the jumps of $L/K$ are the $r \in \mathbb{R}_{\geq 0}$ such that $(G_K/G_L)^{r+} \subsetneq (G_K/G_L)^r$. If $L/K$ is trivial, we say by convention that 0 is the only jump of $\text{Gal}(L/K)$.

1.2. — For $M \in D^b_c(\eta, \overline{Q}_L)$, we denote by $\text{Sl}(M) \subset \mathbb{R}_{\geq 0}$ the set of slopes of $M$ as defined in [Kat88, Ch 1]. We view $M$ in an equivalent way as a continuous representation of $G_K$. 
1.3. — Let $f : X \to S$ be a morphism of finite type and $F \in D^b_c(X_0, \overline{\mathbb{Q}}_\ell)$. Consider the following diagram with cartesian squares

$$
\begin{array}{ccc}
X_s & \xrightarrow{i} & X \\
\downarrow f & & \downarrow \\
S & \xrightarrow{\eta} & \overline{\mathbb{Q}}_\ell
\end{array}
$$

Following [DK73] XIII, we define the nearby cycles of $F$ as

$$\psi_F := i^* Rj_{\ast} F$$

By [Del77] Th. finitude 3.2, the complex $\psi_F$ is an object of $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ endowed with a continuous $G_K$-action. Define $X_t := X \times_S \eta_t$ and $j_t : X_t \to X$ the projection. Following [Gro72] I.2, we define the moderate nearby cycles of $F$ as

$$\psi^M_F := j_{\ast}^* Rj_t^* F$$

It is a complex in $D^b_c(X_t, \overline{\mathbb{Q}}_\ell)$ endowed with a continuous $G/P_K$-action. Since $P_K$ is a pro-$p$ group, we have a canonical identification

$$\psi^M_F \cong (\psi_F)^{P_K}$$

Note that by proper base change [AGV73] XII, $\psi^M$ and $\psi_F$ are compatible with proper push-forward.

1.4. — By a generalized semi-stable reduction morphism, we mean a morphism $f : X \to S$ of finite type such that etale locally on $X$, $f$ has the form

$$S[x_1, \ldots, x_n]/(\pi - x_1^{a_1} \cdots x_n^{a_m}) \to S$$

where $\pi$ is a uniformizer of $S$ and where the $a_i \in \mathbb{N}^\ast$ are prime to $p$.

1.5. — If $X$ is a scheme, $x \in X$ and if $\overline{x}$ is a geometric point of $X$ lying over $X$, we denote by $X^\mathrm{sh}_x$ the strict henselization of $X$ at $x$.

2. Nearby slopes in dimension one

2.1. — We show here that nearby slopes associated with the identity morphism are the usual slopes as in [Kat88] Ch 1.

**Lemma 2.1.1.** — For every $M \in \text{Sh}_c(\eta, \overline{\mathbb{Q}}_\ell)$, we have

$$\text{Sl}^\text{nb}_{\text{id}}(M) = \text{Sl}(M)$$

**Proof.** — We first remark that $\psi^M_{\text{id}}$ is just the "invariant under $P$" functor. Suppose that $r \in \text{Sl}(M)$. Then $M$ has a non zero quotient $N$ purely of slope $r$. The dual $N^\vee$ has pure slope $r$. Since $N$ is non zero, the canonical map

$$N \otimes N^\vee \to \overline{\mathbb{Q}}_\ell$$
is surjective. Since taking $P$-invariants is exact, we obtain that the maps in

$$(M \otimes N)^{P} \longrightarrow (N \otimes N)^{P} \longrightarrow \mathfrak{T}$$

are surjective. Hence $(M \otimes N)^{P} \neq 0$, so $r \in \text{Sl}_{\text{nd}}^{\text{nb}}(M)$.

If $r$ is not a slope of $M$, then for any $N$ of slope $r$, the slopes of $M \otimes N$ are non zero. This is equivalent to $(M \otimes N)^{P} = 0$. \hfill \Box

We deduce the following

**Lemma 2.1.2.** — Let $f : X \longrightarrow S$ be a finite morphism with $X$ local and $\mathcal{F} \in \text{Sh}_{c}(X_{\eta}, \mathbb{Q}_{\ell})$.

1. $\text{Sl}_{\text{nb}}^{\text{nb}}(\mathcal{F}) = \text{Sl}(f_{*} \mathcal{F})$.
2. Suppose that $X$ is regular connected and let $L/K$ be the extension of function fields induced by $f$. Suppose that $L/K$ is separable. Then $\text{Max} \text{Sl}_{f}^{\text{nb}}(\mathbb{Q}_{\ell})$ is the highest jump in the ramification filtration on $G_{K}/G_{L}$.
3. Suppose further in (2) that $L/K$ is Galois and set $G := \text{Gal}(L/K)$. Then $\text{Sl}_{f}^{\text{nb}}(\mathbb{Q}_{\ell})$ is the union of $\{0\}$ with the set of jumps in the ramification filtration on $G$.

**Proof.** — Point (1) comes from [2.1.1] and the compatibility of $\psi_{f}$ with proper push-forward.

From point (1) and $f_{*} \mathbb{Q}_{\ell} \cong \mathbb{Q}_{\ell}[G_{K}/G_{L}]$, we deduce

$$\text{Sl}_{f}^{\text{nb}}(\mathbb{Q}_{\ell}) = \text{Sl}(\mathbb{Q}_{\ell}[G_{K}/G_{L}])$$

If $L/K$ is trivial, (2) is true by our definition of jumps in that case. If $L/K$ is non trivial, $r_{\text{max}} = \text{Max} \text{Sl}(\mathbb{Q}_{\ell}[G_{K}/G_{L}])$ is characterized by the property that $G_{K}^{\text{tr}}$ acts non trivially on $\mathbb{Q}_{\ell}[G_{K}/G_{L}]$ and $G_{K}^{\text{ram}+}$ acts trivially. On the other hand, the highest jump $r_{0}$ in the ramification filtration on $G_{K}/G_{L}$ is such that $q(G_{K}^{\text{tr}}) \neq \langle G_{L} \rangle$ and $q(G_{K}^{\text{ram}+}) = \langle G_{L} \rangle$, that is $G_{K}^{\text{tr}} \subseteq G_{L}$ and $G_{K}^{\text{ram}+} \subseteq G_{L}$. The condition $G_{K}^{\text{tr}} \subseteq G_{L}$ ensures that $G_{K}^{\text{tr}}$ acts non trivially on $\mathbb{Q}_{\ell}[G_{K}/G_{L}]$.

$$h \cdot (gG_{L}) = hgG_{L} = gg^{-1}hgG_{L} = gG_{L}$$

where the last equality comes from the fact that since $G_{K}^{\text{tr}}$ is a normal subgroup in $G_{K}$, we have $g^{-1}hg \in G_{K}^{\text{tr}} \subset G_{L}$. So (2) is proved.

Let $S$ be the union of (2) with the set of jumps in the ramification filtration of $G$. To prove (3), we have to prove $\text{Sl}(\mathbb{Q}_{\ell}[G]) = S$. If $r \in \mathbb{R}_{\geq 0}$ does not belong to $S$, we can find an open interval $J$ containing $r$ such that $G^{r} = G^{r'}$ for every $r' \in J$. In particular, the image of $G_{K}$ by $G_{K} \longrightarrow G_{L}(\mathbb{Q}_{\ell}[G])$ does not depend on $r'$ for every $r' \in J$. So $r$ is not a slope of $\mathbb{Q}_{\ell}[G]$.

Reciprocally, $\mathbb{Q}_{\ell}[G]$ contains a copy of the trivial representation, so $0 \in \text{Sl}(\mathbb{Q}_{\ell}[G])$. Let $r \in S \setminus \{0\}$. The projection morphism $G \longrightarrow G/G^{r+}$ induces a surjection of $G_{K}$-representations

$$\mathbb{Q}_{\ell}[G] \longrightarrow \mathbb{Q}_{\ell}[G/G^{r+}] \longrightarrow 0$$
So $\text{Sl}(\mathbb{Q}_p[G/G^r]) \subset \text{Sl}(\mathbb{Q}_p[G])$. Note that $G^r$ acts trivially on $\mathbb{Q}_p[G/G^r]$. By definition $G^r \subset G^r$, so $G^r$ acts non trivially on $\mathbb{Q}_d[G/G^r]$. So $r = \max \text{Sl}(\mathbb{Q}_d[G/G^r])$ and point (3) is proved.

### 2.2. — Let us draw a consequence of 2.1.1

We suppose that $f : X \to S$ is proper. Let $\mathcal{F} \in D^b_c(X_\eta, \mathcal{O}_\ell)$. The $G_K$-module associated to $R^k f_* \mathcal{F} \in D^b_c(\eta, \mathcal{O}_L)$ is $H^k(X_\eta, \mathcal{F})$.

From 2.1.1 we deduce

$$\text{Sl}(H^k(X_\eta, \mathcal{F})) = \text{Sl}^{\text{sh}}_{i_0}(R^k f_* \mathcal{F})$$

$$\subset \text{Sl}^{\text{sh}}_{i_0}(Rf_* \mathcal{F})$$

where the inclusion comes from the fact that taking $P_{k}$-invariants is exact. For every $N \in \text{Sh}_{c}(\eta, \mathcal{O}_L)$, the projection formula and the compatibility of $\psi_f$ with proper push-forward gives

$$\psi_f^i(Rf_* \mathcal{F} \otimes N) = \psi_f^i(Rf_*(\mathcal{F} \otimes f^* N))$$

$$\simeq Rf_* \psi_f^i(\mathcal{F} \otimes f^* N)$$

Hence we have proved the following

**Proposition 2.2.1.** — Let $f : X \to S$ be a proper morphism, and let $\mathcal{F} \in D^b_c(X_\eta, \mathcal{O}_L)$. For every $i \geq 0$, we have

$$\text{Sl}(H^i(X_\eta, \mathcal{F})) \subset \text{Sl}^{\text{sh}}_{i_0}(\mathcal{F})$$

### 2.3. Boundedness. — We first need to see that the upper-numbering filtration is unchanged by purely inseparable base change. This is the following

**Lemma 2.3.1.** — Let $K'/K$ be a purely inseparable extension of degree $p^n$. Let $L/K$ be finite Galois extension, $L' := K' \otimes_K L$ the associated Galois extension of $K'$. Then, the isomorphism

$$(2.3.2) \quad \text{Gal}(L/K) \xrightarrow{\sim} \text{Gal}(L'/K')$$

$$(2.3.3) \quad g \mapsto \text{id} \otimes g$$

is compatible with the upper-numbering filtration.

**Proof.** — Note that for every $g \in \text{Gal}(L/K)$, $\text{id} \otimes g \in \text{Gal}(L'/K')$ is determined by the property that its restriction to $L$ is $g$.

Let $\pi$ be a uniformizer of $S$ and $\pi_L$ a uniformizer of $S_L$. We have $K \simeq k((\pi))$ and $L \simeq k((\pi_L))$. Since $L$ is perfect and since $K'/K$ and $L'/L$ are purely inseparable of degree $p^n$, we have $K' = k((\pi_1/p^n))$ and $L' = k((\pi_L/p^n))$. So $\pi_L^{1/p^n}$ is a uniformizer of $S_L$. For every $\sigma \in \text{Gal}(L'/K')$ we have

$$(\sigma(\pi_L^{1/p^n}) - \pi_L^{1/p^n})p^n = \sigma_L(\pi_L) - \pi_L$$
Nearby slopes and boundedness for $\ell$-adic sheaves in char. > 0

\[ v_{L'}(\pi_L^{1/p^n} - \pi_L^{1/p^n}) = \frac{1}{p^n} v_{L'}(\sigma(L(\pi_L) - \pi_L) = v_L(\sigma(L(\pi_L) - \pi_L) \]

So Lemma 2.3.2 commutes with the lower-numbering filtration. Hence, Lemma 2.3.1 is proved.

Boundedness in case of smooth curves over $k$ is a consequence of the following

Proposition 2.3.4. — Let $S_0$ be an henselian trait over $k$, let $\eta_0 = \text{Spec} K_0$ be the generic point of $S_0$ and $M \in \text{Sh}_c(\eta_0, \mathbb{Q}_{\ell})$. There exists a constant $C_M \geq 0$ depending only on $M$ such that for every finite morphism $f : S_0 \to S$, we have

\[ \text{Sl}_{f}^{nh}(M) \subset [0, \max(C_M, \max \text{Sl}_{f}^{nh}(\mathbb{Q}_{\ell}))] \]

In particular, the quantity

\[ \max \text{Sl}_{f}^{nh}(M) / (1 + \max \text{Sl}_{f}^{nh}(\mathbb{Q}_{\ell})) \]

is bounded uniformly in $f$.

Proof. — By Lemma 2.1.2 (1), we have to bound $\text{Sl}(f_* M)$ in terms of $\max \text{Sl}(f_* \mathbb{Q}_{\ell})$. Using \cite{Kat88} I.1.10, we can replace $\mathbb{Q}_{\ell}$ by $F_{\lambda}$, where $\lambda = \ell^n$. Hence, $G_{K_0}$ acts on $M$ via a finite quotient $H \subset \text{GL}_{F_{\lambda}}(M)$. Let $L/K_0$ be the corresponding finite Galois extension and $f_M : S_L \to S_0$ the induced morphism. We have $H = \text{Gal}(L/K_0)$. Let us denote by $r_M$ the highest jump in the ramification filtration of $H$. Using Herbrand functions \cite{Ser68} IV 3, we will prove that the constant $C_M := \psi_{L/K_0}(r_M)$ does the job.

Using Lemma 2.3.1 we are left to treat the case where $K_0/K$ is separable. The adjunction morphism

\[ M \to f_M * f_M^* M \]

is injective. Since $f_M^* M \cong F_{\lambda}^* M$, we obtain by applying $f_*$ an injection

\[ f_! M \to F_{\lambda} [\text{Gal}(L/K)]^* M \]

So we are left to bound the slopes of $F_{\lambda} [\text{Gal}(L/K)]$ viewed as a $G_K$-representation, that is by Lemma 2.1.2 (2) the highest jump in the upper-numbering ramification filtration of $\text{Gal}(L/K)$. By Lemma 2.1.2 (2), $r_0 := \max \text{Sl}_{f}^{nh}(\mathbb{Q}_{\ell})$ is the highest jump in the ramification filtration of $\text{Gal}(L/K)/H$. Choose $r > \max(r_0, \varphi_{L/K} \psi_{L/K_0}(r_M))$. We have

\[ \text{Gal}(L/K)^r = H \cap \text{Gal}(L/K)^r = H \cap \text{Gal}(L/K)^r \psi_{L/K}(r) = H \psi_{L/K}(r) = H^r \psi_{L/K}(r) = \{1\} \]
The first equality comes from $r > r_0$. The third equality comes from the compatibility of the lower-numbering ramification filtration with subgroups. The last equality comes from the fact that $r > \varphi_{L/K} \psi_{L/K_0}(r_M)$ is equivalent to $\varphi_{L/K_0} \psi_{L/K}(r) > r_M$. Hence, 
\[ \text{Sh}^b_f(M) \subset [0, \text{Max}(r_0, \varphi_{L/K} \psi_{L/K_0}(r_M))] \]
Since $\varphi_{L/K} : [-1, +\infty] \to \mathbb{R}$ is concave, satisfies $\varphi_{L/K}(0) = 0$ and is equal to the identity on $[-1, 0]$, we have 
\[ \varphi_{L/K} \psi_{L/K_0}(r_M) \leq \psi_{L/K_0}(r_M) \]
and we obtain (2.3.5) by setting $C_M := \psi_{L/K_0}(r_M)$. \(\square\)

3. Proof of Theorem \[1\]

3.1. Preliminary. — Let us consider the affine line $\mathbb{A}^1_S \to S$ over $S$. Let $s'$ be the generic point of $\mathbb{A}^1_S$ and $S'$ the strict henselianization of $\mathbb{A}^1_S$ at $s'$. We denote by $\overline{S}$ the normalization of $S$ in $\overline{\mathbb{F}}$, by $\kappa$ the function field of the strict henselianization of $\mathbb{A}^1_S$ at $s'$, and by $\overline{\kappa}$ an algebraic closure of $\kappa$. We have $\kappa \simeq K' \otimes_K \overline{\kappa}$ and

(3.1.1) 
$G_K \simeq \text{Gal}(\kappa/K')$ 

Let $L/K$ be a finite Galois extension of $K$ in $\overline{K}$. Set $L' := K' \otimes_K L$. At finite level, (3.1.1) reads

(3.1.2) $\text{Gal}(L/K) \to \text{Gal}(L'/K')$ 

(3.1.3) $g \to \text{id} \otimes g$

Since a uniformizer in $S_L$ is also a uniformizer in $S'_L$, we deduce that (3.1.2) is compatible with the lower-numbering ramification filtration on $\text{Gal}(L/K)$ and $\text{Gal}(L'/K')$. Hence, (3.1.2) is compatible with the upper-numbering ramification filtration on $\text{Gal}(L/K)$ and $\text{Gal}(L'/K')$. We deduce that through (3.1.1), the canonical surjection $G_{K'} \to G_K$ is compatible with the upper-numbering ramification filtration.

3.2. The proof. — We can suppose that $\mathcal{F}$ is concentrated in degree 0. In case $\dim X = 0$, there is nothing to prove. We first reduce the proof of Theorem 1 to the case where $\dim X = 1$ by arguing by induction on $\dim X$.

Since the problem is local on $X$, we can suppose that $X$ is affine. We thus have a digram

(3.2.1) 
\[
\begin{array}{ccc}
X & \longrightarrow & \mathbb{A}^0_S \\
\downarrow & & \downarrow \\
S & \longrightarrow & \mathbb{P}^0_S
\end{array}
\]

Let $\overline{X}$ be the closure of $X$ in $\mathbb{P}^0_S$ and let $j : X \to \overline{X}$ be the associated open immersion. Replacing $(X, \mathcal{F})$ by $(\overline{X}, j_*\mathcal{F})$, we can suppose $X/S$ projective. Then Theorem 1 is a consequence of the following assertions

(A) There exists a finite set $E_A \subset \mathbb{R}_{\geq 0}$ such that for every $N \in \text{Sh}_c(\eta, \overline{\mathbb{Q}}_L)$ with slope not in $E_A$, the support of $\psi_j^*(\mathcal{F} \otimes f^*N)$ is punctual.
(B) There exists a finite set $E_B \subseteq \mathbb{R}_{\geq 0}$ such that for every $N \in \text{Sh}_c(\eta, \mathbb{M}_\ell)$ with slope not in $E_B$, we have

$$R\Gamma(X, \psi^*_f(\mathcal{F} \otimes f^*N)) \approx 0$$

Let us prove (A). This is a local statement on $X$, so we can suppose $X$ to be a closed subset in $\mathbb{A}^n_S$ and consider the factorisations

$$\begin{array}{ccc}
X \ar@/_/[r]_{p_i} & \mathbb{A}^1_S \ar@{-->}[l]_f \\
\downarrow & \downarrow \\
S & \mathbb{A}^1_S \ar@/^/[u]_h
\end{array}$$

where $p_i$ is the projection on the $i$-th factor of $\mathbb{A}^n_S$. Using the notations in 3.1, let $X'/S'$ making the upper square of the following diagram cartesian. Let us set $\mathcal{F}' := \lambda^*\mathcal{F}$ and $N' := h^*N$. From [Del77, Th. finitude 3.4], we have

$$\lambda^*\psi^*_f(\mathcal{F} \otimes f^*N) \approx \psi_{h^*}(\mathcal{F}' \otimes p_1^*N') \approx \psi_{p_i}(\mathcal{F}' \otimes p_i^*N')^{G_\kappa}$$

where $G_\kappa$ is a pro-$p$ group sitting in an exact sequence

$$1 \longrightarrow G_\kappa \longrightarrow G_{K'} \longrightarrow G_K \longrightarrow 1$$

In particular, $G_\kappa$ is a subgroup of the wild-ramification group $P_{K'}$ of $G_{K'}$. So applying the $P_{K'}$-invariants on (3.2.2) yields

$$\lambda^*\psi^*_f(\mathcal{F} \otimes f^*N) \approx \psi_{p_i}(\mathcal{F}' \otimes p_i^*N')$$

If $N$ has pure slope $r$, we know from 3.1 that $N'$ has pure slope $r$ as a sheaf on $\eta'_i$. Applying the recursion hypothesis gives a finite set $E_i \subseteq \mathbb{R}_{\geq 0}$ such that the right-hand side of (3.2.3) is 0 for $N$ of slope not in $E_i$. The union of the $E_i$ for $1 \leq i \leq n$ is the set $E_A$ sought for in (A).

To prove (B), we observe that the compatibility of $\psi^*_f$ with proper morphisms and the projection formula give

$$R\Gamma(X, \psi^*_f(\mathcal{F} \otimes f^*N)) \approx \psi^*_{id}(Rf_*\mathcal{F} \otimes N)$$

By 2.1.1, the set $E_B := \text{Sl}(Rf_*\mathcal{F})$ has the required properties.

We are thus left to prove Theorem 1 in the case where $\dim X = 1$. At the cost of localizing, we can suppose that $X$ is local and maps surjectively on $S$. Let $x$ be the closed point of $X$. Note that $k(x)/k(s)$ is of finite type but may not be finite. Choosing a transcendence basis of $k(x)/k(s)$ yields a factorization $X \longrightarrow S' \longrightarrow S$
satisfying $\text{trdeg}_{k(s)} k(x) = \text{trdeg}_{k(s)} k(x) - 1$.

So we can further suppose that $k(x)/k(s)$ is finite. Since $k(s)$ is algebraically closed, we have $k(x) = k(s)$. If $\hat{S}$ denotes the completion of $S$ at $s$, we deduce that $X \times_S \hat{S}$ is finite over $\hat{S}$. By faithfully flat descent [Gro71, VIII 5.7], we obtain that $X/S$ is finite. We conclude the proof of Theorem 1 with 2.1.2 (1).

4. Proof of Theorem 2

4.1. — That $0 \in \text{Spec} \mathbb{Q}$ is easy by looking at the smooth locus of $f$. We are left to prove that for every $N \in \text{Spec} \mathbb{Q}$ with slope $> 0$, the following holds

\begin{equation}
(4.1.1)
\psi_f^* N = 0
\end{equation}

Since the problem is local on $X$ for the étale topology, we can suppose that $X = S[x_1, \ldots, x_n]/(\pi - x_1^{a_1} \cdots x_n^{a_n})$ and we have to prove (4.1.1) at the origin $0 \in X$. Let $a$ be the lowest common multiple of the $a_i$ and define $b_i = a/a_i$. Note that $a$ and the $b_i$ are prime to $p$. Hence the morphism $h$ defined as

\begin{align*}
Y := S[t_1, \ldots, t_n]/(\pi - t_1^{b_1} \cdots t_n^{b_n}) & \longrightarrow X \\
(t_1, \ldots, t_n) & \longmapsto (t_1^{b_1}, \ldots, t_n^{b_n}, t_{m+1}, \ldots, t_n)
\end{align*}

is finite surjective and finite étale above $\eta$ with Galois group $G$. Set $g = fh$. Then

\begin{equation}
(\mathcal{H}^i \psi_f^* N)_{\overline{\eta}} \cong (\mathcal{H}^i \psi_g^* N)_{\overline{\eta}}
\end{equation}

for every $i \geq 0$, so we can suppose $a_1 = \cdots = a_m = a$. Since $a$ is prime to $p$, the map of absolute Galois groups induced by $S[\pi^{1/a}] \longrightarrow S$ induces an identification at the level of the ramification groups. By compatibility of nearby cycles with change of trait [Del77, Th. finitude 3.7], we can suppose $a = 1$.

Let us now reduce the proof of Theorem 2 to the case where $m = 1$. We argue by induction on $m$. The case $m = 1$ follows from the compatibility of nearby cycles with smooth morphisms. We thus suppose that Theorem 2 is true for $m < n$ with all $a_i$ equal to 1 and prove it for $m + 1$ with all $a_i$ equal to 1. Let $h : \tilde{X} \longrightarrow X$ be the blow-up of $X$ along $x_m = x_{m+1} = 0$. Define $g := fh$ and denote by $E$ the exceptional divisor of $\tilde{X}$. Since $h$ induces an isomorphism on the generic fibers, and since $\psi_f$ is compatible with proper push-forward, we have

\begin{equation}
(4.1.2)
R\text{h}_* \psi_g^* N \cong \psi_f^* N = 0
\end{equation}

By proper base change, (4.1.2) gives

\begin{equation}
(4.1.3)
R\Gamma(h^{-1}(0), (\psi_g^* N)_{|h^{-1}(0)}) = 0
\end{equation}

The scheme $\tilde{X}$ is covered by a chart $U$ affine over $S$ given by

\begin{equation}
S[(u_i)_{1 \leq i \leq n}]/(\pi - u_1 \cdots u_m)
\end{equation}

with $E \cap U$ given by $u_m = 0$, and a chart $U'$ affine over $S$ given by

\begin{equation}
S[(u_i)_{1 \leq i \leq n}]/(\pi - u_1 \cdots u_{m+1})
\end{equation}
with $E \cap U'$ given by $u_{m+1} = 0$. By recursion hypothesis, $(\psi^*_g g^* N)_{|U' \cap \{0\}}$ is a skyscraper sheaf supported at the origin 0 of $U'$. Hence, \[1.1.3\] gives

$$(\psi^*_g g^* N)_{|\mathcal{P}} \simeq 0$$

This finishes the induction, and thus the proof of Theorem 2.

4.2. — Let us give a geometric-flavoured proof of

$$\mathcal{H}^0(\psi^*_f f^* N) \simeq 0$$

in case $X = S[x_1, \ldots, x_n]/(\pi - x_1^{a_1} \cdots x_m^{a_m})$. By constructibility [Del77] Th. finitude 3.2], it is enough to work at the level of germs at a geometric point $\mathfrak{p}$ lying over a closed point $x \in X$.

Hence, we have to prove $\mathcal{H}^0(C, f^* N) \simeq 0$ for every connected component $C$ of $X^\nu_{x,\eta}$. For such $C$, denote by $\rho_C : \pi_1(C) \twoheadrightarrow \pi_1(\eta) = P_K$ the induced map. Then $\mathcal{H}^0(C, f^* N) \simeq N^{\im \rho_C}$. Since by definition $N^{\im \rho_C} = 0$, it is enough to prove that $\rho_C$ is surjective. From V 6.9 and IX 3.4 of [Gro71], we are left to prove that $C$ is geometrically connected. To do this, we can always replace $X^\nu_{x,\eta}$ by its formalization $\hat{X}_x = \text{Spec } R[[\pi - x_1^{a_1} \cdots x_m^{a_m}]]$.

By hypothesis, $d := \text{gcd}(a_1, \ldots, a_m)$ is prime to $p$, so $\pi$ has a $d$-root in $K_1$. Hence $\hat{X}_x,\eta$ is a direct union of $d$ copies of

$$\text{Spec } K_1 \otimes_R R[\pi]/(\pi^{1/d} - x_1^{a_1} \cdots x_m^{a_m})$$

where $a_i = da'_i$. So we have to prove the following

**Lemma 4.2.1.** Let $a_1, \ldots, a_m, d \in \mathbb{N}^*$ with $\text{gcd}(a_1, \ldots, a_m) = 1$. Then

$$(4.2.2) \quad \text{Spec } K_1 \otimes_R R[\pi]/(\pi^{1/d} - x_1^{a_1} \cdots x_m^{a_m})$$

is connected.

**Proof.** One easily reduces to the case $d = 1$. If $R'$ is the normalization of $R$ in a Galois extension of $K$ in $\overline{K}$, it is enough to prove that $\text{Spec } R'[\pi]/(\pi - x_1^{a_1} \cdots x_m^{a_m})$ is irreducible. If $\pi'$ is a uniformizer of $R'$, we have $R' \simeq k[\pi']$, we write $\pi = P(\pi')$ where $P \in k[X]$ and then we are left to prove that $f_{a,p} := P(\pi') - x_1^{a_1} \cdots x_m^{a_m}$ is irreducible in $k[x_1, \ldots, x_n, \pi']$. This follows from $\text{gcd}(a_1, \ldots, a_m) = 1$ via Lypkovski’s indecomposability criterion [Lip88] 2.10] for the Newton polyhedron associated to $f_{a,p}$.

References

[AGV73] M. Artin, A. Grothendieck, and J.-L. Verdier, *Théorie des Topos et Cohomologie Étale des Schémas*, Lecture Notes in Mathematics, vol. 305, Springer-Verlag, 1973.

[Del77] P. Deligne, *Cohomologie étale*, vol. 569, Springer-Verlag, 1977.

[Del07] ________, *Lettre à Malgrange. 20 décembre 1983*, Singularités irrégulières (Société Mathématique de France, ed.), Documents Mathématiques, vol. 5, 2007.
[DK73] P. Deligne and N. Katz, *Groupes de Monodromie en Géométrie Algébrique. SGA 7 II*, Lecture Notes in Mathematics, vol. 340, Springer-Verlag, 1973.

[Gro71] A. Grothendieck, *Revêtements étalés et groupe fondamental*, Lecture Notes in Mathematics, vol. 263, Springer-Verlag, 1971.

[Gro72] , *Groupes de Monodromie en Géométrie Algébrique. SGA 7 I*, Lecture Notes in Mathematics, vol. 288, Springer-Verlag, 1972.

[Kat88] N. Katz, *Gauss Sums, Kloosterman Sums, and Monodromy Groups*, The Annals of Mathematics Studies, vol. 116, 1988.

[Lip88] A. Lipkovski, *Newton Polyhedra and Irreductibility*, Math. Zeitschrift 199 (1988).

[Ser68] J.-P. Serre, *Corps Locaux*, Hermann, 1968.

[Tey14] J.-B. Teyssier, *Mail to H. Esnault*, March 2014.

[Tey15] , *A boundedness theorem for nearby slopes of holonomic D-modules. Preprint*, 2015.