The partition function of multicomponent log-gases

Christopher D Sinclair
Department of Mathematics, University of Oregon, Eugene, OR 97403, USA
E-mail: csinclai@uoregon.edu

Received 17 January 2012, in final form 9 March 2012
Published 4 April 2012
Online at stacks.iop.org/JPhysA/45/165002

Abstract
We give an expression for the partition function of a one-dimensional log-gas comprised of particles of (possibly) different integer charge at inverse temperature $\beta = 1$ (restricted to the line in the presence of a neutralizing field) in terms of the Berezin (Grassmann) integral of an associated non-homogeneous alternating tensor. This is the analogue of the de Bruijn integral identities (de Bruijn 1956 *J. Indian Math. Soc.* 19 133–51) (for $\beta = 1$ and $\beta = 4$) ensembles extended to multicomponent ensembles.

PACS numbers: 52.27.Cm, 05.20.Gg, 02.50.−r
Mathematics Subject Classification: 15B52, 82C22, 60G55

1. The ensembles

We imagine a finite number of charged particles interacting logarithmically on an infinite wire modeled by the real line. Different particles may have different charges (which we will assume are positive integers), but any two particles with the same charge, that is, of the same species, are assumed to be indistinguishable. A potential is placed on the wire to keep the particles from escaping to infinity. This system is placed in contact with a heat reservoir with inverse temperature $\beta$.

We will consider two ensembles.

(1) *The canonical ensemble*. We assume that the number of each species of particle is fixed.

(2) *The grand canonical ensemble*. We assume that the sum of the charges, that is, the total charge of the system, is fixed but the number of each species is variable.1

When all the particles have charge 1 (that is, there is only one species) and the inverse temperature is $\beta = 1$ the above ensembles are the same, and the statistics of the particles correspond to those of eigenvalues of the ensemble of orthogonally invariant random Hermitian matrices whose weight is determined by the corresponding potential. (This was originally

1 The standard notion of the grand canonical ensemble is that where the number of particles is not fixed. That is, in its traditional sense, the grand canonical ensemble is the direct sum over all possible values of the sum of the charges. What we refer to as the grand canonical ensemble might be better referred to as an isocharge or zero current grand canonical ensemble.
observed by Dyson\textsuperscript{2} [7].) When the particles all have charge 2 and the inverse temperature is $\beta = 1$, the particle statistics behave the same as the eigenvalues of matrices in the ensemble of symplectically invariant Hermitian matrices with weight function corresponding to the external potential. Classically the log-gas analogy of the symplectic ensembles consists of charge 1 particles at inverse temperature $\beta = 4$.

We remark that the quantity which controls the local behavior of the particles is not the inverse temperature $\beta$, but rather the quantity $q\sqrt{\beta}$ (this is obvious in the construction of the Boltzmann factor). That is, for instance, a system of interacting particles all with charge 1 at inverse temperature $\beta = 4$ will behave the same as a system of charge 2 particles at $\beta = 1$. Often, especially in the random matrix community, $\beta$ has been used as the parameter which distinguishes the behavior of the various ensembles (with $\beta = 1, 2$ and 4 corresponding to the ‘classical’ orthogonally invariant, unitarily invariant and symplectically invariant ensembles). Here, however, it is more useful (or at least more concrete) to fix $\beta = 1$ and view $q$ as the parameter which varies. In this context the classical orthogonal, unitary and symplectic ensembles correspond to systems of identical particles with charges $q = 1, \sqrt{2}$ and 2 respectively. This may seem a bit unnatural at first, especially the value $q = \sqrt{2}$, but in fact the formulas for the partition functions of the canonical and grand canonical ensembles considered here generalize the Pfaffian formulation for the partition function of the $\beta = 1, q = 1$ and 2 ensembles, and not the determinantal form of the $q = \sqrt{2}$ ensemble. We will recover some generality in the following by demanding only that $q\sqrt{\beta}$ be an integer, for the various allowed $q$ in our ensembles, however as this makes little difference in the interpretation of the results, it is useful to think of our systems consisting of particles with integer charge at inverse temperature $\beta = 1$.

In particular, when the potential is that of the classical harmonic oscillator, the charge 1 ensemble has the same statistics as the Gaussian orthogonal ensemble of matrices and the charge 2 ensemble has the same statistics as the Gaussian symplectic ensemble (GSE) (see, for instance, [27]).

Other (tridiagonal) matrix ensembles correspond to other possible values (notably non-integer values) of the charge [6], however the charge 1 and charge 2 ensembles are differentiated by the fact that their particle statistics determine a Pfaffian point process [26, 10, 8]. The simplest implication of this is that the partition function of these ensembles can be expressed as the Pfaffian of an antisymmetric matrix with a very special form—this is the fact that we will generalize in this paper. It should be noted that the charge $\sqrt{2}$ ensemble determines a determinantal point process [9] (and the partition function is the determinant of a Gram matrix). This ensemble is more naturally interpreted as an ensemble of charge 1 particles at inverse temperature $\beta = 2$, and there are some existing results regarding the partition function of multicomponent log-gases with integer charges at $\beta = 2$ with which we will compare our results. In particular, some recent analysis has been done on multicharge ensembles in the circular case when $\beta = 2$. In [19] the authors express the partition function of the multicharge ensemble on the unit circle at inverse temperature $\beta = 2$ in terms of series whose terms contain determinants of certain Toeplitz matrices. Their results use similar tools as those employed here.

Recent work by the author [37] shows that the partition function for $\beta = 1, q \in \mathbb{N}$ can be expressed as a hyperpfaffian of an algebraic object which generalizes the antisymmetric matrices whose Pfaffian gives the partition functions in the charge 1 and charge 2 cases. Specifically, the antisymmetric matrix that appears when $q = 1$ or 2 is replaced with an

\textsuperscript{2} Dyson’s original observation was for particles restricted to the unit circle and the ensemble of orthogonal matrices, but the analogy stands.
alternating \( q \)-tensor (or \( 2q \)-tensor in the case where \( q \) is odd); this generalization is natural since an antisymmetric matrix can be identified with an alternating 2-tensor. The hope is that there is a ‘hyperpfaffian’ point process in the background waiting to be discovered, but at this point this remains elusive.

Loosely speaking, the hyperpfaffian of an alternating \( L \)-tensor over a vector space with dimension \( LM \) is the projection of the \( M \)-fold wedge product of the \( L \)-tensor onto the determinantal line. The central result presented here is that the partition function of the grand canonical ensemble of the multicomponent ensemble can be expressed as the projection onto the determinantal line of the exponential of a non-homogeneous alternating tensor; this alternating tensor can be decomposed into a sum of homogeneous tensors, one for each species of particle, which correspond exactly to the alternating tensor which appears in the canonical ensemble consisting of particles of only that species. The projection onto the determinantal line of the exponentiated form is conveniently presented in terms of a Berezin integral.\(^3\) The use of Berezin integrals here is purely for convenience, though it is suggestive since the grand canonical ensemble allows for annihilation and creation of particles (so long as the total charge is conserved), the original motivation for Berezin’s construction.\(^2\)

The simplest example of a multicomponent grand canonical ensemble is that with two species with charge ratio 1:2. This ensemble, particularly in the presence of the harmonic oscillator potential, was introduced in [33]. A brief summary of the results and the relationship between this ensemble and the general multicomponent grand canonical ensemble will be discussed in section 4.1. Some discussion of the circular version of this ensemble can be found in [11, section 6.7].

1.1. The setup

Let \( J > 0 \) be an integer and suppose \( \mathbf{q} = (q_1, q_2, \ldots, q_J) \) is a vector of positive integers representing possible charge magnitudes so that each of the \( q_j \) is distinct. We imagine a system of particles consisting of \( M_1 \) indistinguishable particles of charge \( q_1 \), \( M_2 \) indistinguishable particles of charge \( q_2 \) and so on. We will refer to \( \mathbf{q} \) as the charge vector and \( \mathbf{M} = (M_1, M_2, \ldots, M_J) \) as the population vector of the system.

These particles are restricted to lie on an infinite wire, identified with the real axis,\(^4\) and interact logarithmically, so that the energy contributed to the system by a pair particles with charges \( q \) and \( q' \) located at \( x \) and \( x' \) is given by \(-qq' \log |x - x'|\). (Infinite energy is allowed in the situation where \( x = x' \).) We suppose that the particles of charge \( q_j \) are identified with the location vector \( \mathbf{x}^j = (x_{1j}, x_{2j}, \ldots, x_{M_j}) \); the location vectors \( \mathbf{x}^3, \ldots, \mathbf{x}^J \) are similarly defined. We do not preclude \( M_j \) from equaling 0, and in this situation \( \mathbf{x}^j \) is taken to be the empty vector. The particles are placed in a neutralizing field with potential \( U \) so that the total potential energy of the system is given by

\[
E_M(\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^J) = \sum_{j=1}^{J} q_j \sum_{m=1}^{M_j} U(x_{mj}^j) - \sum_{j=1}^{J} q_j^2 \sum_{m<n} \log |x_{mj}^j - x_{mn}^j| - \sum_{j<k} q_j q_k \sum_{m=1}^{M_j} \sum_{n=1}^{M_k} \log |x_{mj}^j - x_{nk}^k|.
\]

\(^3\) Berezin integrals are simply Grassman integrals where the anticommuting variables of integration are expressed in terms of elements in the (alternating) exterior algebra of a fixed vector space.

\(^4\) With a minor modification, much of what is presented here can be shown mutatis mutandis for multicomponent log-gases confined to the unit circle. See [19] for the circular case for \( \beta = 2 \), and [20] for some physical application.
We assume that the system is in contact with a heat reservoir at inverse temperature $\beta$, but energy is allowed to flow between the reservoir and the system of particles. In this situation the Boltzmann factor, which gives the relative density of states, is given by

$$\Omega_M(x^1, x^2, \ldots, x^J) = e^{-\beta E(x^1, \ldots, x^J)}$$

$$= \left\{ \prod_{j=1}^{J} \prod_{m=1}^{M_j} e^{-\beta q_m(u_{mj})} \right\} \times \left\{ \prod_{j=1}^{J} \prod_{m<n} (x^j_m - x^j_n)^{\beta q_{mn}} \right\} \times \left\{ \prod_{j<k} \prod_{m=1}^{M_j} \prod_{n=1}^{M_n} |x^j_n - x^k_m|^{|\beta q_j|/|\beta q_k|} \right\}.$$

(1.1)

The probability (density) of finding the system in a state determined by the location vectors $x^1, x^2, \ldots, x^J$ is then given by

$$p_M(x^1, x^2, \ldots, x^J) = \frac{\Omega_M(x^1, x^2, \ldots, x^J)}{Z_M M_1! M_2! \cdots M_J!}.$$

where the partition function of the system is given by

$$Z_M = \frac{1}{M_1! M_2! \cdots M_J!} \int_{\mathbb{R}^{M_1}} \cdots \int_{\mathbb{R}^{M_J}} \Omega_M(x^1, x^2, \ldots, x^J) \, d\mu^{M_1}(x^1) \, d\mu^{M_2}(x^2) \cdots d\mu^{M_J}(x^J).$$

(1.2)

and $\mu^M$ is Lebesgue measure on $\mathbb{R}^M$. The factors of $M_1! M_2! \cdots M_J!$ appear since a generic state of the system $x^1, x^2, \ldots, x^J$ has this many different representatives. We will always assume that the external potential $U$ is such that $Z_M$ is finite.

For the grand canonical ensemble, we may view $M$ as a random vector and the probability (density) of finding the system with a prescribed population vector $M$ and state $x^1, x^2, \ldots, x^J$ is then given by $p_M(x^1, x^2, \ldots, x^J) \cdot \text{prob}(M)$. Classically, the probability of finding the system in a state with (allowed) population vector $M$ is taken to be

$$\text{prob}(M) = \frac{Z^M}{Z_N},$$

(1.3)

where

$$Z_N = \sum_{M_{q=1}^{M_j}} c_1^{M_1} c_2^{M_2} \cdots c_j^{M_j} Z_M$$

and $z = (z_1, \ldots, z_J)$ is a vector of positive real numbers called the fugacity vector.

It shall sometimes be convenient to view $z$ as a vector of indeterminants and $Z_N = Z_N(z)$ as a polynomial in these indeterminants. Our main result will be to show that, for certain values of $\beta$ and $q$, $Z_N(z)$ can be expressed as a Berezin integral with respect to the volume form in $\mathbb{R}^N$ of the exponential of an (explicitly given) alternating element (i.e. form) in the exterior algebra $\Lambda(\mathbb{R}^N)$. By construction, $Z_M$ is the coefficient of $z_1^{M_1} \cdots z_j^{M_j}$, and thus the integral formulation of $Z_N(z)$ is exactly the generating function we seek.

2. Wronskians, Berezin integrals and hyperpfaffians

Here we collect the machinery necessary to state our main results.

---

5 If $M_j = 0$ for some $j$ then we will use the convention that

$$\int_{\mathbb{R}^{M_j}} \Omega_M(x^1, x^2, \ldots, x^J) \, d\mu^{M_j}(x^J) = \Omega_M(x^1, x^2, \ldots, x^J),$$

alternatively, in this situation we may assume that the integral over $\mathbb{R}^{M_j}$ does not actually appear in our expression for $Z_M$. Likewise we will assume that sums and products over empty sets are respectively taken to be 0 and 1.
Given a non-negative integer $L$, let $L = \{1, 2, \ldots, L\}$, and, assuming $K \geq L$ is an integer, let
\[ t : L \not\to K \]
be a strictly increasing function,
\[ 0 < t(1) < t(2) < \ldots < t(L) \leq K. \]
We will use such functions to keep track of minors of matrices, elements in exterior algebras and Wronskians of families of polynomials. Such indexing functions will always be written as fraktur minuscules.

2.1. Wronskians

A complete family of monic polynomials is a sequence of polynomials $P = (p_1, p_2, \ldots)$ such that each $p_n$ is monic and $\deg p_n = n - 1$. We define the $L$-tuple $P_t = (p_{t(1)}, \ldots, p_{t(L)})$. And, given $0 \leq \ell < L$ we define the modified $\ell$th differentiation operator by
\[ D^\ell f(x) = f(x) \quad \text{and} \quad D^\ell f(x) = \frac{1}{\ell!} \frac{d^\ell f}{dx^\ell}. \]

The Wronskian of $P_t$ is then defined to be
\[ \text{Wr}(P_t; x) = \det[D^{\ell-1} p_{t(k)}(x)]_{k,\ell=1}^L. \]

The Wronskian is often defined without the $\ell!$ in the denominator of (2.1); this combinatorial factor will prove convenient in the following. The reader has likely seen Wronskians in elementary differential equations, where they are used to test for linear dependence of solutions.

2.2. The Berezin integral

If $e_1, \ldots, e_K$ is a basis for $\mathbb{R}^K$, then $\epsilon_{t} = e_{t(1)} \wedge \cdots \wedge e_{t(L)}$ is an element in $\Lambda^L(\mathbb{R}^K)$, and $\{\epsilon_t \mid t : L \not\to K\}$ is a basis for $\Lambda^L(\mathbb{R}^K)$. In particular, we will denote
\[ \epsilon_{vol} = e_1 \wedge e_2 \wedge \cdots \wedge e_K, \]
which generates the one-dimensional vector space $\Lambda^K(\mathbb{R}^K)$.

Given $0 < k \leq K$ we define the linear operator $\partial/\partial e_1 : \Lambda^L(\mathbb{R}^K) \to \Lambda^{L-1}(\mathbb{R}^K)$ by
\[ \frac{\partial}{\partial e_1} \epsilon_{t} = \begin{cases} (-1)^{\nu} e_{t(1)} \wedge \cdots \wedge e_{t(k-1)} \wedge e_{t(k+1)} \wedge \cdots \wedge e_{t(L)} & \text{if } k = t^{-1}(\nu); \\ 0 & \text{otherwise}. \end{cases} \]
That is, if $e_1$ appears in $\epsilon_t$ then $\partial \epsilon_t/\partial e_1$ is formed by shuffling $e_1$ to the front of $\epsilon_t$ (taking into account the alternation of signs) and then deleting it. Given $0 < k_1, \ldots, k_M \leq K$ we then define the Berezin integral as the linear operator on $\Lambda(\mathbb{R}^K) \to \Lambda^L(\mathbb{R}^K)$ specified by
\[ \int \epsilon_t dei_1 de_{i_1} \cdots de_{i_M} = \frac{\partial}{\partial e_{i_1}} \cdots \frac{\partial}{\partial e_{i_M}} \epsilon_{t}. \]

Berezin integrals were introduced in [2] as a Fermionic analogue to the Gaussian integrals which appear in Bosonic field theory.

We will mostly be interested in Berezin integrals of the form with respect to the $\epsilon_{vol}$. In this case, the Berezin integral is simply the projection operator $\Lambda(\mathbb{R}^K) \to \Lambda^K(\mathbb{R}^K) \cong \mathbb{R}$.

Note in particular that, if $\sigma \in S_K$ then
\[ \int e_{\sigma(1)} \wedge e_{\sigma(2)} \wedge \cdots \wedge e_{\sigma(K)} d\epsilon_{vol} = \text{sgn } \sigma. \]
2.3. Exponentials of forms and hyperpfaffians

Given \( \omega \in \Lambda(\mathbb{R}^K) \) we define \( \omega_m = \omega \wedge \cdots \wedge \omega \). Using this we define \( e^{\omega} = \sum_{m=0}^{\infty} \frac{\omega_m}{m!}. \)

If \( \omega = \omega_0 + \omega_1 + \cdots + \omega_k \) with \( \omega_k \in \Lambda^k(\mathbb{R}^K) \) then it is easily verified that \( e^{\omega} = e^{\omega_0} \wedge e^{\omega_1} \wedge \cdots \wedge e^{\omega_k}. \)

Moreover, \( e^{\omega_0} \) is a real number equal to its traditional definition, and if \( k > 0 \) then the sum defining \( e^{\omega_k} \) is a finite sum.

In the situation where \( k \) divides \( K \), that is \( K = km \), then we define the hyperpfaffian \( \text{PF}(\omega_k) \) to be the real number defined by \( \frac{\omega_k^m}{m!} = \text{PF}(\omega_k) \epsilon_{\text{vol}}. \)

Alternatively, \( \text{PF}(\omega_k) = \int e^{\omega_k} \epsilon_{\text{vol}}. \)

The hyperpfaffian is related to the Pfaffian of an antisymmetric \( K \times K \) matrix by associating the matrix to a 2-form in the obvious manner. We see therefore that the Berezin integral of exponentials of forms with respect to \( \epsilon_{\text{vol}} \) is a generalization of the notion of a hyperpfaffian to non-homogeneous forms. Hyperpfaffians themselves are generalizations of Pfaffians given by allowing 2-forms to be replaced with (homogeneous) \( k \)-forms.

3. Statement of results

Suppose \( b \) is a positive integer and \( \beta = b^2 \). Set \( K = bN \) and \( L_j = bq_j \) for \( j = 1, 2, \ldots, J \). For any complete family of monic polynomials \( \mathcal{P} \) we define \( \omega_1, \omega_2, \ldots, \omega_J \in \Lambda(\mathbb{R}^K) \) as follows.

1. If \( L_j \) is even,

\[
\omega_j = \sum_{t \in \mathbb{Z}^J} \left\{ \int_{\mathbb{R}^J} e^{-\beta q_j U(x)} \text{Wr}(\mathcal{P}; x) \, dx \right\} \epsilon_t;
\]

2. If \( L_j \) is odd,

\[
\omega_j = \sum_{t, u \in \mathbb{Z}^J} \left\{ \frac{1}{2} \int_{\mathbb{R}^J} \int_{\mathbb{R}^J} e^{-\beta q_j U(x)} e^{-\beta q_j U(y)} \text{Wr}(\mathcal{P}; x) \text{Wr}(\mathcal{P}; y) \text{sgn}(y - x) \, dx \, dy \right\} \epsilon_t \wedge \epsilon_u.
\]

Note that \( \omega_j \) is in \( \Lambda^{L_j}(\mathbb{R}^K) \) when \( L_j \) is even and is in \( \Lambda^{2L_j}(\mathbb{R}^K) \) when \( L_j \) is odd.

**Theorem 3.1.** Suppose \( \beta = b^2 \) and \( K = bN \) is even. Given a charge vector \( q \) let

\[
L_j = bq_j; \quad j = 1, 2, \ldots, J,
\]
and, for any complete family of monic polynomials, define the form \( \omega \in \Lambda(\mathbb{R}^N) \) by

\[
\omega(z) = \sum_{j=1}^{\ell} z_j \omega_j,
\]

where \( \omega_j \) is defined as in (3.1) or (3.2). If the \( L_j \) are positive integers, at most one of which is odd,\(^6\) then

\[
Z_N(z) = \int e^{\omega(z)} d\text{vol}.
\]

Remark. This is an algebraic identity which can be written more generally by replacing the integral over \( \mathbb{R} \) with integrals over other sets (for instance, the partition functions for multicharge circular ensembles can be likewise expressed in terms of Berezin integrals). The only analytic prerequisite is the finiteness of the \( Z_M \) which allows for the use of Fubini’s theorem.

4. Discussion

This theorem specializes to many known cases. For instance, when \( J = 1 \) we are in the situation of a one-component log-gas [5]. If \( J = 1 \) and \( L_1 = 1 \) or 2 theorem 3.1 specializes to de Bruijn’s Pfaffian identities for the partition functions of GOE, GSE (and their variations given by other weights). When \( J = 1 \) and \( L_1 \) is an even integer, theorem 3.1 reduces to previously known hyperpfaffian formulas given by Luque and Thibon [22]. These results were recently extended by the author to the case where \( J = 1 \) and \( L_1 \) is an odd integer [37]. As mentioned previously, there is no obstruction to replacing the line on which the particles are restricted with some other region. Thus, the Pfaffian form for the partition functions of the circular ensembles COE and CSE, where the particles (or eigenvalues) are restricted to lie on the unit circle in \( \mathbb{C} \) follow from the appropriate modification of theorem 3.1. Likewise, the partition function for circular ensembles where \( J = 1 \) and \( L_1 \) is an integer will have a hyperpfaffian formulation by the appropriate modification of theorem 3.1.

Some instances of two-component, i.e. \( J = 2 \), situations have been previously studied. Recent work of Rider, Xu and the author have exploited the fact that when \( L_1 = 1 \) and \( L_2 = 2 \), theorem 3.1 reduces the grand canonical partition function to a Pfaffian. When the restoring field corresponds to the harmonic oscillator potential, the Pfaffian structure of the partition function allows many exact asymptotic (\( N \to \infty \)) results to be computed [33]. Since this case is illustrative of the general situation, more specifics of the Pfaffian formulation for the partition function and the exact results will be recounted in section 4.1. A circular version of this ensembles is considered in [11, section 6.7].

In spite of the similarity between the Berezin integral formulation of the partition function of the grand canonical ensemble and those of well-studied one-component ensembles (and the two-component ensemble discussed in section 4.1) the general multicharge ensemble possesses obstacles which cannot, at least at the moment, be resolved using traditional methods. Specifically, for the classical ensembles, the partition function is given as the Pfaffian of a Gram matrix formed with respect to a skew-symmetric inner product of all pairs of polynomials in any family of monic polynomials. (This Gram matrix is exactly the skew-symmetric matrix associated to the 2-form for these ensembles introduced before theorem 3.1.) By choosing these polynomials to be skew-orthogonal with respect to this inner product, the correlation

\(^6\) If one of the \( L_j \) is odd, then we must have \( N \) even. A modification of this theorem is true when one of the \( L_j \) is odd and \( N \) is odd; the necessary modification can be extracted from [37, theorem 2.1].
functions (see section 4.2) can be expressed in terms of Pfaffians of matrix kernels using Dyson’s quaternion-determinant version [10] of the Gaudin–Mehta method [26], or the method of Tracy and Widom [38]. The matrix kernel is a fundamental invariant of the ensemble and allows for the computation of essentially all particle statistics of interest. In many cases, one is interested in the particle statistics as $N \to \infty$, and knowledge of the asymptotics of the skew-orthogonal polynomials allows for the limiting kernel in the appropriate scaling regime to be derived. The skew-orthogonal polynomials are often related to the orthogonal polynomials with the same potential [1]. Hence for many classical potentials, the necessary asymptotics for the analysis of the kernel can be derived from those of the related orthogonal polynomials [31, 30, 29].

Of fundamental importance is the fact that, when $L_1 = 1$ or 2, the number of polynomial coefficients at our disposal is exactly equal to the number of independent entries in the skew-symmetric Gram matrix. For instance, when $J = 1$ and $L_1 = 2$, the skew-orthogonal polynomials allow us to express $\omega_1$ as a linear combination of $e_1 \wedge e_2, e_1 \wedge e_3, \ldots, e_{N-1} \wedge e_N$. Contrast this with the one-component cases where $L_1 > 2$: here the partition function is given as the hyperpfaffian of an $L_1$ or $2L_1$-form (depending on if $L$ is even or odd) and, when $N$ is large, there are far more coefficients in this form than can be naively eliminated by tuning the coefficients of the underlying monic polynomials (when $L_1$ is even there are $(N^{L_1})$ coefficients in the $L_1$-form and only $(N^2)$ coefficients in the polynomials). One would naively hope that for other one-component ensembles we might be able to express $\omega_q$ as a linear combination of $e_1 \wedge \cdots \wedge e_{L_1}, e_{L_1+1} \wedge \cdots \wedge e_{2L_1}, \ldots$. Indeed, if this were possible, then the method of Tracy and Widom (and likely Gaudin–Mehta) could be extended to produce a hyperpfaffian formulation for the correlations. However, barring immense and magical relationships between the coefficients of the $\omega$ for some choice of monic polynomials (or perhaps more likely, between the Wronskians of subfamilies of polynomials), this seems unlikely.

It should be remarked that, strictly speaking, the skew-orthogonal polynomials are not necessary to derive the matrix kernel in the $L_1 = 1$ or $L_1 = 2$ case; any family of monic polynomials for which the Gram matrix of skew-symmetric inner products can be formally inverted will suffice (see [3, appendix A], [36] and [32]). However, to generalize this to the $L_1 > 2$ case, one needs to (among other things) determine the analogue of matrix inversion of the Gram matrix for the related 2-form, and how to extend this to $L_1$-forms. The one-component situation needs to be fully understood before tackling the general multicomponent ensembles (see however section 4.2 below).

In another direction, the density of states (1.1) for the canonical ensembles with fixed population vector specialize to certain trial wavefunctions which arise in the study of the anomalous quantum Hall effect. The original trial wavefunctions were introduced by Laughlin, and correspond to our one-component joint density of particles [21]. Recent work of Boussicault, Luque and Thibon gave an expansion for certain of these wavefunctions in terms of hyperdeterminants (related to the hyperpfaffians and Berezin integrals presented here) [4]. Shortly after the work of Laughlin, Halperin introduced trial wavefunctions which, at least in some instances, reduce to the density of states for the two-component ensembles studied here [18]. A Pfaffian formulation for certain of these trial wavefunctions (on the circle, not the line) is presented by Forrester and the author in [14]. More recently, trial wavefunctions corresponding in some instances to the density of states of our canonical ensembles with fixed population vectors have appeared in the study of the fractional quantum Hall effect in graphene sheets [17].
The partition functions presented here, viewed as linear combinations of multidimensional integrals, generalize Selberg-type integrals. The integral originally considered by Selberg [34],

\[ S_N(a, b, c) := \int_0^1 \cdots \int_0^1 \left\{ \prod_{n=1}^N e^{a_n - 1} (1 + t_n)^{b_n - 1} \right\} \left\{ \prod_{1 \leq m < n \leq N} |t_m - t_n|^{2c} \right\} \, dt_1 \cdots dt_n, \]

clearly corresponds to the partition function for a one component log-gas consisting of particles of charge 2 at \( \beta = 1 \) restricted to [0, 1] and in the presence of charged particles fixed at the endpoints of this interval. This integral, and its generalizations formed from other potentials have a storied mathematical history appearing in diverse areas of mathematics and physics. An interesting semi-historical account of how the Selberg integral went from relative obscurity to centrality in a variety of contexts is given by Forrester and Warnaar [15]. Recent work of Luque and Thibon [24, 23, 22] and Matsumoto [25] have explored various aspects of the relationship between Selberg type integrals, hyperdeterminants and hyperpfaffians.

4.1. The two-component ensemble with charge ratio 1:2

When we reduce to the case of two species with \( q_1 = 1 \) and \( q_2 = 2 \), \( \beta = 1 \) and \( N \) even,

\[ \omega_1 = \sum_{m,n=1}^N \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-U(x)} e^{-U(y)} p_m(x) p_n(y) \text{sgn}(y - x) \, dx \, dy \, e_m \wedge e_n, \]

and

\[ \omega_2 = \sum_{m,n=1}^N \int_{\mathbb{R}} e^{-2U(x)} \left[ p_m(x) p'_n(x) - p_n(x) p'_m(x) \right] \, dx \, e_m \wedge e_n. \]

Associating the 2-form \( \omega(z_1, z_2) = z_1 \omega_1 + z_2 \omega_2 \) to an antisymmetric matrix in the usual manner, we find the partition function for the grand canonical ensemble can be expressed as the Pfaffian of the sum of two anti-symmetric matrices:

\[ Z_N(z_1, z_2) = \text{Pf} (z_1 A_1 + z_2 A_2), \]

where

\[ A_1 = \left[ \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-U(x)} e^{-U(y)} p_m(x) p_n(y) \text{sgn}(y - x) \, dx \, dy \right]_{m,n=1}^N \]

and

\[ A_2 = \left[ \int_{\mathbb{R}} e^{-2U(x)} \left( p_m(x) p'_n(x) - p_n(x) p'_m(x) \right) \, dx \right]_{m,n=1}^N. \]

In the special case where \( U(z) = -z^2/2 \), the Pfaffian of \( A_1 \) produces the partition function for the Gaussian orthogonal ensemble, and the Pfaffian of \( A_2 \) produces the partition function of the GSE. That is, for this choice of potential, \( Z_N(z, 1 - z) \) is the partition function of a family of ensembles which interpolate between GOE and GSE.

This ensemble was originally inspired by Ginibre’s real ensemble [16], which is the matrix ensemble consisting \( N \times N \) antisymmetric matrices with independent real Gaussian entries. Like the two charge grand canonical ensemble considered here, Ginibre’s real ensemble has two species of particles (eigenvalues): real and complex conjugate pairs. The discovery of a Pfaffian formulation for the total partition function for Ginibre’s real ensemble [35] (corresponding to the grand canonical partition function presented here) was the inspiration for the study of the charge 1:2 ensemble, which eventually led to the current work. Note that the charge 1:2
ensemble in the harmonic oscillator potential can be thought of, psychologically at least, as the result of forcing the complex conjugate pairs of eigenvalues in Ginibre’s real ensemble to coalesce into a single charge 2 particle on the real line, with fugacities \( z_1 = z_2 = 1 \).

The analysis of particle statistics of both Ginibre’s real ensemble and the charge 1:2 ensemble have been aided by the discovery of the skew-orthogonal polynomials necessary to skew-diagonalize the matrices whose Pfaffian yields the corresponding partition functions (these were given in [13, 12] for Ginibre’s real ensemble and [33] for the 1:2 ensemble).

4.2. Correlation functions

Using a slight modification, the partition function gives a generating function for the correlation functions. For single species ensembles, the correlation functions are simply renormalized marginal densities. For multicomponent ensembles, however, the situation is more complicated (though the marginal probabilities are an important ingredient).

For fixed population vector \( \mathbf{M} \) and vector \( \mathbf{m} = (m_1, m_2, \ldots, m_J) \) with \( 0 \leq m_j \leq M \), we define

\[ \xi^j = (\xi_1^j, \ldots, \xi_{m_j}^j) \quad \text{and} \quad y^j = (y_1^j, \ldots, y_{M-m_j}^j), \]

and set

\[ \xi^j \cup y^j = (\xi_1^j, \ldots, \xi_{m_j}^j, y_1^j, \ldots, y_{M-m_j}^j). \]

The \( m \)th marginal probability density of \( p_{\mathbf{M}} \) is then given by

\[
p_{\mathbf{M},\mathbf{m}}(\xi^1, \xi^2, \ldots, \xi^J) = \int_{\mathbb{R}^{M_1}} \cdots \int_{\mathbb{R}^{M_J}} p_{\mathbf{M}}(\xi^1 \cup y^1, \ldots, \xi^J \cup y^J) \, d\mu_{M_1-m_1}(y^1) \cdots d\mu_{M_J-m_J}(y^J),
\]

and by symmetry, the probability (density) that our system is in a state \( (x^1, \ldots, x^J) \) which occupies the substate \((\xi_1, \ldots, \xi_J)\) (that is, viewed as sets, \( \xi^j \subseteq x^j \) for each \( j \)) is given by

\[
R_{\mathbf{M},\mathbf{m}}(\xi^1, \xi^2, \ldots, \xi^J) = \frac{M_1!}{(M_1-m_1)!} \cdots \frac{M_J!}{(M_J-m_J)!} p_{\mathbf{M},\mathbf{m}}(\xi^1, \xi^2, \ldots, \xi^J) = \frac{1}{Z_{\mathbf{M}}(M_1-m_1)! \cdots (M_J-m_J)!} \int_{\mathbb{R}^{M_1}} \cdots \int_{\mathbb{R}^{M_J}} \Omega_{\mathbf{M}}(\xi^1 \cup y^1, \ldots, \xi^J \cup y^J) \, d\mu_{M_1-m_1}(y^1) \cdots d\mu_{M_J-m_J}(y^J). \tag{4.1}
\]

This is the \( m \)th correlation function for the canonical ensemble with population vector \( \mathbf{M} \).

To get the \( m \)th correlation function for the grand canonical ensemble we need to sum over the related correlation function for the canonical ensemble over all allowable population vectors \( \mathbf{M} \) with \( m_j \leq M_j \) for each \( j \) (a situation we will abbreviate by \( \mathbf{m} \leq \mathbf{M} \)), taking into account the probability of being in a state with prescribed population vector. That is, the probability (density) of the (grand canonical) system is in a state \( (x^1, \ldots, x^J) \) which occupies the substate \((\xi_1, \ldots, \xi_J)\) is given by

\[
\text{prob}(\xi_1, \ldots, \xi_J) \subseteq (x^1, \ldots, x^J) = \sum_{\mathbf{M} \geq \mathbf{m}} \text{prob}(\mathbf{M}) \cdot R_{\mathbf{M},\mathbf{m}}(\xi^1, \xi^2, \ldots, \xi^J).
\]

Denoting this density by \( R_{\mathbf{M},\mathbf{m}} \), (1.3) and (4.1) yield

\[
R_{\mathbf{M},\mathbf{m}}(\xi_1, \ldots, \xi_J) = \frac{1}{Z_{\mathbf{N}}(\mathbf{z})} \sum_{\mathbf{M}, \mathbf{m} \geq \mathbf{M}} \frac{z_1^{M_1} \cdots z_J^{M_J}}{(M_1-m_1)! \cdots (M_J-m_J)!}
\times \int_{\mathbb{R}^{M_1}} \cdots \int_{\mathbb{R}^{M_J}} \Omega_{\mathbf{M}}(\xi^1 \cup y^1, \ldots, \xi^J \cup y^J) \, d\mu_{M_1-m_1}(y^1) \cdots d\mu_{M_J-m_J}(y^J).
\]
Note that, by ignoring the prefactor \( Z_0(z) \), and up to an easily recoverable constant, the coefficient of \( c_1^{M_1} \cdots c_N^{M_N} \) in \( R_{M,N} \) is the \( m \)th correlation function for the corresponding canonical ensemble.

We can in turn give a generating function for the correlation functions for the grand canonical ensemble as follows: Let \( e^j = (e_1^j, e_2^j, \ldots, e_N^j) \) and \( \zeta^j = (\zeta_1^j, \zeta_2^j, \ldots, \zeta_N^j) \) and define the measures \( \nu_j \) and \( \eta_j \) by

\[
\frac{d\nu_j}{d\mu_j}(x) = e^{-\beta q,U(x)} \quad \text{and} \quad \eta_j(x) = e^{-\beta q,U(x)} \sum_{n=1}^N c_j^n \delta(x - \zeta_j^n),
\]

where \( \delta(x) \) is the probability measure with unit mass at \( x = 0 \).

It is convenient at this point to index the forms \( \omega_j \) from (3.1) and (3.2) by \( v_j \) so that, for instance when \( L_j \) is even,

\[
\omega_j^{v_j} = \sum_{v_{j+1},v_j} \left\{ \int_R \text{Wr}(\beta_k) \, dv_j \right\} \epsilon_j.
\]

Quantities which are dependent on \( v = (v_1, \ldots, v_j) \) will be denoted by, for instance, \( Z_N^{v,M}, Z_N^{v} \) and \( \omega^{v}(z) \). Theorem 3.1 is purely algebraic, and thus, we have, for instance that

\[
Z_N^{v}(z) = \int \exp(\omega^{v}(z)) \, dv_{\text{vol}}.
\]

We can generalize these quantities by replacing \( v \) with other vectors of measures. The following theorem gives particular relevance to \( Z_N^{v+\nu}(z, e^1, \ldots, e^j) \), where the notation indicates the additional dependence on the \( e^j \).

**Claim 4.1.** The \( m \)th correlation function of the grand canonical ensemble is the coefficient of

\[
\prod_{j=1}^m \prod_{I=1}^j c_j^I \quad \text{in} \quad \frac{Z_N^{v+\nu}(z, e^1, \ldots, e^j)}{Z_N(z)}.
\]

That is, if \( \zeta^j = (\zeta_1^j, \ldots, \zeta_N^j) \), and we define

\[
\frac{\partial^m}{\partial \epsilon^j} = \frac{\partial}{\partial \epsilon_1^j} \cdots \frac{\partial}{\partial \epsilon_m^j},
\]

then,

\[
R_{N,M}(\xi^1, \ldots, \xi^j) = \frac{1}{Z_N(z)} \left[ \frac{\partial^m}{\partial \epsilon^j} \cdots \frac{\partial^m}{\partial \epsilon^1} Z_N^{v+\nu}(z, e^1, \ldots, e^j) \right]_{\epsilon^1=\cdots=\epsilon^j=0}.
\]

Moreover, the \( m \)th correlation function of the canonical ensemble with population vector \( M \) is given by

\[
R_{M,N}(\xi^1, \ldots, \xi^j) = \frac{1}{Z_M M! \cdots M!} \left[ \frac{\partial^m}{\partial \epsilon^j} \cdots \frac{\partial^m}{\partial \epsilon^1} Z_M^{v+\nu}(z, e^1, \ldots, e^j) \right]_{z=e^1=\cdots=e^j=0}.
\]

The proof of this claim is standard (it is the multicomponent version of the ‘functional differentiation’ method), and follows *mutatis mutandis* that for Ginibre’s real ensemble [3, proposition 6].

To write the correlation functions explicitly in terms of a Berezin integral (taking all of the \( L_j \) to be even for convenience), we note that

\[
\omega_j^{v+\nu} = \sum_{v \in \mathbb{R}^L} \left\{ \int_R \text{Wr}(\beta_k) \, dv_j + \eta_j \right\} \epsilon_t = \omega_j^{v} + \omega_j^{\nu},
\]
and
\[ \omega^y + \eta = \sum_{j=1}^J \omega_{y_j}^{\nu_j} + \eta_j = \omega^y + \omega^\eta, \]

Hence, \( \exp(\omega^y + \eta) = \exp(\omega^y) \land \exp(\omega^\eta). \)

This is useful, since the first term in the right-hand side is independent of the \( c_j. \) The following maneuvers are elementary
\[
\frac{\partial^{m_1}}{\partial c_1} \exp(\omega^y) \land \exp(\omega^\eta) \bigg|_{c_1=0} = \exp(\omega^y) \land \exp(\omega^\eta) \land \left\{ \bigwedge_{\ell=1}^{m_1} e^{-\beta q_1(U(\zeta^\ell_1))} \sum_{t^\ell_{\ell_1} \neq K} \text{Wr}(P_1; \zeta^\ell_1)\epsilon_t \right\},
\]

note that since all forms are even, we do not have to specify their order. It follows that
\[
\frac{\partial^{m_1}}{\partial c_1} \exp(\omega^y) \land \exp(\omega^\eta) \bigg|_{c_1=0} = \exp(\omega^y) \land \exp(\omega^\eta) \land \left\{ \bigwedge_{\ell=1}^{m_1} e^{-\beta q_1(U(\zeta^\ell_1))} \sum_{t^\ell_{\ell_1} \neq K} \text{Wr}(P_1; \zeta^\ell_1)\epsilon_t \right\},
\]

and that
\[
\frac{\partial^{m_1} \cdots \partial^{m_J}}{\partial c^1 \cdots \partial c^J} \exp(\omega^y) \land \exp(\omega^\eta) \bigg|_{c^1=\cdots=c^J=0} = \exp(\omega^y) \land \exp(\omega^\eta) \land \left\{ \bigwedge_{j=1}^J \bigwedge_{\ell=1}^{m_j} e^{-\beta q_j(U(\zeta^\ell_j))} \sum_{t^\ell_{\ell_1} \neq K} \text{Wr}(P_1; \zeta^\ell_j)\epsilon_t \right\}.
\]

We therefore have the following corollary to claim 4.1.

**Corollary 4.2.** If, for each \( 1 \leq j \leq J, L_j \) is even, \( \zeta^\ell = (\zeta^\ell_1, \ldots, \zeta^\ell_{m_j}) \in \mathbb{R}^{m_j}, \) then
\[
R_{N,M}(\zeta^1, \ldots, \zeta^J) = \frac{1}{Z_N(z)} \int \exp(\omega^y) \land \left\{ \bigwedge_{j=1}^J \bigwedge_{\ell=1}^{m_j} e^{-\beta q_j(U(\zeta^\ell_j))} \sum_{t^\ell_{\ell_1} \neq K} \text{Wr}(P_1; \zeta^\ell_j)\epsilon_t \right\} \text{dvol}.
\]

Note that we do not have to justify the exchange of the derivatives and the ‘integral’ in claim 4.1, since the Berezin integral is not an integral in the traditional sense. That is, claim 4.1 is an algebraic, not an analytic, identity. Notice also, that the quantity in braces is an \((m \cdot L)\)-form, and therefore only the projection of \( \exp \omega^y \) onto the space of \((K - m \cdot L)\)-forms will make a contribution to the \( m \)th correlation function. Finally, we note that a similar formula for the partial correlation function \( R_{M,N} \) is available via functional differentiation with respect to the \( z \) variables.

5. The proof of theorem 3.1

5.1. The confluent vandermonde determinant

Suppose \( 0 < L < K, x \in \mathbb{R} \) and \( P = (p_1, p_2, \ldots) \) is any complete family of monic polynomials. We define the \( K \times L \) matrix
\[
V^L(x) = \left[ D^\ell p^\ell_n(x) \right]_{\ell=1}^{K_L},
\]
and given an admissible population vector \( \mathbf{M} \) and \( \mathbf{x}^1, \ldots, \mathbf{x}^t \) with \( \mathbf{x}^t \in \mathbb{R}^{M_j} \), we define the \( K \times K \) confluent Vandermonde matrix by

\[
\mathbf{V}^\mathbf{M} (\mathbf{x}^1, \ldots, \mathbf{x}^t) = \begin{bmatrix}
\mathbf{V}^{L_1} (\mathbf{x}_1^1) & \cdots & \mathbf{V}^{L_1} (\mathbf{x}_1^{M_1}) \\
& \ddots & \\
\mathbf{V}^{L_t} (\mathbf{x}_t^1) & \cdots & \mathbf{V}^{L_t} (\mathbf{x}_t^{M_t})
\end{bmatrix}.
\]

(Recall that \( L_j = \sqrt{\beta q_j} \).) In this case, the confluent Vandermonde determinant identity [28] has that

\[
\det \mathbf{V}^\mathbf{M} (\mathbf{x}^1, \ldots, \mathbf{x}^t) = \prod_{j=1}^t \prod_{m<n} \left( x_m^j - x_n^j \right) \times \prod_{j<k,m=1}^{M_j} \left( x_m^j - x_m^k \right).
\]

(5.1)

When all of the \( L_j \) are even, it follows from (1.3) that

\[
\Omega^\mathbf{M} (\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^t) = \prod_{j=1}^t \prod_{m=1}^{M_j} \sum_{\mathbf{v}^j \in J^m} e^{-\beta q_j (\mathbf{v}^j)} \det \mathbf{V}^\mathbf{M} (\mathbf{x}^1, \ldots, \mathbf{x}^t). \tag{5.2}
\]

We will deal with the situation where one of the \( L_j \) is odd in section 5.5.

5.2. The Laplace expansion of the determinant

Each \( t : L \not\rightarrow K \) specifies a unique \( t' : K - L \not\rightarrow N \) whose range is disjoint from \( t \). Given a \( K \times K \) matrix \( \mathbf{V} = [\mathbf{v}_{m,n}] \) and \( t, u : L \not\rightarrow K \) then we may create a \( L \times L \) minor of \( \mathbf{V} \) by selecting the rows and columns from the ranges of \( t \) and \( u \). That is, we write

\[
\mathbf{V}_{t,u} = \left[ \mathbf{v}_{\ell(t),\ell(u)} \right]_{\ell,\ell=1}^L \text{.}
\]

Notice that the complementary minor to \( \mathbf{V}_{t,u} \) is given by \( \mathbf{V}_{t',u'} \).

We define \( \text{sgn} \ t \) by

\[
\text{sgn} \ t = \int \epsilon_t \wedge \epsilon_{t'} \, d\epsilon_{\text{vol}}.
\]

More generally let

\[
\mathbf{t}^j_m : L_j \to K_j \quad \text{where} \quad j = 1, 2, \ldots J \quad \text{and} \quad m = 1, 2, \ldots, M_j,
\]

and set

\[
\mathbf{t}^j = \left( \mathbf{t}_1^j, \ldots, \mathbf{t}_{M_j}^j \right)
\]

We will use \( \mathbf{t}^j \) to select minors of \( \mathbf{V}^\mathbf{M} (\mathbf{x}^1, \ldots, \mathbf{x}^t) \) each of which depends only on a single location variable. We denote the set of all such \( \mathbf{t}^j \) by \( \mathcal{J}_m \).

We define \( \text{sgn} \ \mathbf{t}^j \) by

\[
\text{sgn} \ \mathbf{t}^j = \int \epsilon_{t_1^j} \wedge \cdots \wedge \epsilon_{t_{M_j}^j} \wedge \epsilon_{t_{M_j}^j} \wedge \cdots \wedge \epsilon_{t_{M_j}^j}^j \, d\epsilon_{\text{vol}}. \tag{5.3}
\]

Clearly, \( \text{sgn} \ \mathbf{t}^j = 0 \) unless the ranges of the various \( t_m^j \) are mutually disjoint, and otherwise \( \text{sgn} \ t \) is the signature of the permutation defined by concatenating the ranges of the various \( t_m^j \) in the appropriate order.
We will reserve the symbol \( \vec{t} \) for the vector whose coordinate functions are given by

\[
i_{m}(\ell) = \ell + (m - 1)L_{j} + M_{1}L_{j} + \cdots + M_{j-1}L_{j-1}.
\]

That is, for instance, if \( \mathbf{L} = (2, 3) \) and \( \mathbf{M} = (2, 2) \) then the ranges of \( i_{1}^{'}, i_{2}^{'}, i_{1}^{\prime} \) and \( i_{2}^{\prime} \) are given respectively by \( [1, 2], [3, 4], [5, 6, 7] \) and \( [8, 9, 10] \). Clearly \( \text{sgn} \ \vec{t} = 1 \).

This notation is convenient to represent the Laplace expansion of the determinant (which we will write in the form most useful for our ultimate goal).

\[
\det \mathbf{V} = \sum_{\vec{t} \in \mathcal{S}_{M}} \text{sgn} \ \vec{t} \prod_{j=1}^{M_{j}} \det \mathbf{V}_{i_{j}^{'}, i_{j}^{\prime}}.
\]

Applying (5.4) to \( \mathbf{V}^{\mathbf{M}}(x^{1}, \ldots, x^{J}) \) we find

\[
\det \mathbf{V}^{\mathbf{M}}(x^{1}, \ldots, x^{J}) = \sum_{\vec{t} \in \mathcal{S}_{M}} \text{sgn} \ \vec{t} \prod_{j=1}^{M_{j}} \det \mathbf{V}^{\mathbf{M}}_{t_{j}^{1}, t_{j}^{\prime}}(x_{m}^{j}),
\]

where the notation reflects the fact that \( \mathbf{V}^{\mathbf{M}}_{t_{j}^{1}, t_{j}^{\prime}}(x_{m}^{j}) \) in independent of all location variables except \( x_{m}^{j} \). From the definition of \( \mathbf{V}^{\mathbf{M}} \) we see that \( \det \mathbf{V}^{\mathbf{M}}_{t_{j}^{1}, t_{j}^{\prime}}(x_{m}^{j}) = \text{Wr}(\mathcal{P}_{t_{j}^{1}, t_{j}^{\prime}}; x_{m}^{j}) \), and therefore

\[
\det \mathbf{V}^{\mathbf{M}}(x^{1}, \ldots, x^{J}) = \sum_{\vec{t} \in \mathcal{S}_{M}} \text{sgn} \ \vec{t} \prod_{j=1}^{M_{j}} \text{Wr}(\mathcal{P}_{t_{j}^{1}, t_{j}^{\prime}}; x_{m}^{j}).
\]

5.3. Fubini’s theorem

From (1.2), (5.2) and (5.5) we have that

\[
Z_{\mathbf{M}} = \frac{1}{M_{1}!M_{2}! \cdots M_{J}!} \sum_{\vec{t} \in \mathcal{S}_{M}} \text{sgn} \ \vec{t} \int_{\mathbb{R}^{M_{1}}} \int_{\mathbb{R}^{M_{2}}} \cdots \int_{\mathbb{R}^{M_{J}}}
\times \prod_{j=1}^{J} \prod_{m=1}^{M_{j}} e^{-\beta q_{j}/U(x_{m}^{j})} \text{Wr}(\mathcal{P}_{t_{j}^{1}, t_{j}^{\prime}}; x_{m}^{j}) \ d\mu^{M_{1}}(x^{1}) d\mu^{M_{2}}(x^{2}) \cdots d\mu^{M_{J}}(x^{J}).
\]

Fubini’s theorem implies then that

\[
Z_{\mathbf{M}} = \frac{1}{M_{1}!M_{2}! \cdots M_{J}!} \sum_{\vec{t} \in \mathcal{S}_{M}} \text{sgn} \ \vec{t} \prod_{j=1}^{J} \prod_{m=1}^{M_{j}} \int_{\mathbb{R}} e^{-\beta q_{j}/U(x)} \text{Wr}(\mathcal{P}_{t_{j}^{1}, t_{j}^{\prime}}; x) \ dx.
\]

Thus,

\[
Z_{\mathbf{N}}(x) = \sum_{\mathbf{M} \in \mathcal{S}} \frac{M_{1}!M_{2}! \cdots M_{J}!}{M_{1}!M_{2}! \cdots M_{J}!} \sum_{\vec{t} \in \mathcal{S}_{M}} \text{sgn} \ \vec{t} \prod_{j=1}^{J} \prod_{m=1}^{M_{j}} \int_{\mathbb{R}} e^{-\beta q_{j}/U(x)} \text{Wr}(\mathcal{P}_{t_{j}^{1}, t_{j}^{\prime}}; x) \ dx
\]

\[
= \sum_{\mathbf{M} \in \mathcal{S}} \sum_{\vec{t} \in \mathcal{S}_{M}} \text{sgn} \ \vec{t} \prod_{j=1}^{J} \prod_{m=1}^{M_{j}} \int_{\mathbb{R}} e^{-\beta q_{j}/U(x)} \text{Wr}(\mathcal{P}_{t_{j}^{1}, t_{j}^{\prime}}; x) \ dx.
\]
5.4. Enter the Berezin integral

Using the definition of $\text{sgn} \, \epsilon$ (5.3) we find

$$Z_N(\mathbf{x}) = \sum_{\mathbf{M} = N} \sum_{\mathbf{q} = \mathbf{M}} \left\{ \int_{\mathbb{R}^M} \frac{1}{M_j!} \prod_{m=1}^{M_j} e^{-\beta q_j U(x)} \mathbf{Wr}(\mathcal{P}_{\psi_j}; x) \, d\mathbf{e}_{\mathbf{vol}} \right\}$$

$$\times \prod_{j=1}^{J} \frac{1}{M_j!} \prod_{m=1}^{M_j} z_j \int_{\mathbb{R}} e^{-\beta q_j U(x)} \mathbf{Wr}(\mathcal{P}_{\psi_j}; x) \, d\mathbf{e}_{\mathbf{vol}}.$$

Exploiting the linearity of the Berezin integral,

$$Z_N(\mathbf{x}) = \int \left[ \sum_{\mathbf{M} = N} \sum_{\mathbf{q} = \mathbf{M}} \prod_{m=1}^{M_j} e^{-\beta q_j U(x)} \mathbf{Wr}(\mathcal{P}_{\psi_j}; x) \, d\mathbf{e}_{\mathbf{vol}} \right].$$

where the wedge products are taken in the standard order.

Next we may expand the sum over $\mathbf{f} \in \mathcal{F}_M$ as

$$\sum_{\mathbf{f} \in \mathcal{F}_M} (\cdots) \Rightarrow \sum_{t_1} \cdots \sum_{t_{M_j}} (\cdots),$$

so that

$$Z_N(\mathbf{x}) = \int \left[ \sum_{\mathbf{M} = N} \sum_{\mathbf{q} = \mathbf{M}} \prod_{m=1}^{M_j} e^{-\beta q_j U(x)} \mathbf{Wr}(\mathcal{P}_{\psi_j}; x) \, d\mathbf{e}_{\mathbf{vol}} \right].$$

We observe that

$$\sum_{t_1, \ldots, t_{M_j}} \left\{ \int_{\mathbb{R}} e^{-\beta q_j U(x)} \mathbf{Wr}(\mathcal{P}_{\psi_j}; x) \, d\mathbf{e}_{\mathbf{vol}} \right\} \epsilon_i = (z_j, \mathbf{q}_j)^{\wedge M_j},$$

and hence

$$Z_N(\mathbf{x}) = \int \left[ \sum_{\mathbf{M} = N} \sum_{j=1}^{J} \frac{(z_j, \mathbf{q}_j)^{\wedge M_j}}{M_j!} \right] \, d\mathbf{e}_{\mathbf{vol}}.$$

Now, we can remove the restriction $\mathbf{M} \cdot \mathbf{q} = N$ from the sum in this expression, since the Berezin integral will be zero for any $\mathbf{M}$ not satisfying this condition. (If $\mathbf{M}$ does not satisfy this condition the form in the integrand will not be in $\Lambda^k(\mathbb{R}^k)$ and hence its projection onto $\Lambda^k(\mathbb{R}^k) \equiv \mathbb{R}$ will be 0.)

Thus,

$$Z_N(\mathbf{x}) = \int \left[ \sum_{j=1}^{J} \sum_{M_j=0}^{\infty} \sum_{M_j=0}^{\infty} \sum_{M_j=0}^{\infty} \frac{(z_j, \mathbf{q}_j)^{\wedge M_j}}{M_j!} \right] \, d\mathbf{e}_{\mathbf{vol}}$$

$$= \int \left[ \sum_{j=1}^{J} \sum_{M_j=1}^{\infty} \sum_{M_j=1}^{\infty} \sum_{M_j=1}^{\infty} \frac{(z_j, \mathbf{q}_j)^{\wedge M_j}}{M_j!} \right] \, d\mathbf{e}_{\mathbf{vol}}$$

$$= \int \left[ \sum_{j=1}^{J} \sum_{M_j=1}^{\infty} \frac{(z_j, \mathbf{q}_j)^{\wedge M_j}}{M_j!} \right] \, d\mathbf{e}_{\mathbf{vol}}.$$
\[
\int e^{i \phi_1} \wedge e^{i \phi_2} \wedge \cdots \wedge e^{i \phi_J} \, d\text{vol} = \int e^{i \phi(x)} \, d\text{vol},
\]
as desired.

5.5. When one of the \(L_j\) is odd

In the case where exactly one of the \(L_j\) is odd, we will reorder the \(q_j\) so that \(L_1\) is odd and \(L_2, \ldots, L_J\) are even. In this situation, (1.1) and (5.1) imply that

\[
\Omega_M(x^1, x^2, \ldots, x^J) = \left\{ \prod_{j=1}^J \prod_{m=1}^{M_j} e^{-\beta q_j U(x^j_m)} \right\} \left\{ \prod_{1 \leq m < n \leq M_1} \text{sgn}(x^1_m - x^1_n) \right\} \det V^M(x^1, x^2, \ldots, x^J),
\]

where the additional factors of the form \(\text{sgn}(x^1_m - x^1_n)\) exist in order to make the expression non-negative for all choices of \(x^1, x^2, \ldots, x^J\). Defining the \(M_1 \times M_1\) antisymmetric matrix

\[
T(x^1) = \left[ \text{sgn}(x^1_m - x^1_n) \right]_{m,n=1}^{M_1},
\]

when \(K\) is even, so is \(M_1\), and in this situation

\[
\text{Pr} T(x^1) = \prod_{1 \leq m < n \leq M_1} \text{sgn}(x^1_m - x^1_n).
\]

Thus,

\[
\Omega_M(x^1, x^2, \ldots, x^J) = \left\{ \prod_{j=1}^J \prod_{m=1}^{M_j} e^{-\beta q_j U(x^j_m)} \right\} \text{Pr} T(x^1) \det V^M(x^1, x^2, \ldots, x^J),
\]

Following the analysis of the case where all \(L_j\) are even, we find the analogue of (5.6) in the current situation is

\[
Z_M = \frac{1}{M_1! M_2! \cdots M_J!} \sum_{\bar{t} \in \mathcal{S}_M} \text{sgn} \left\{ \prod_{j=1}^J \prod_{m=1}^{M_j} \int_{\mathbb{R}} e^{-\beta q_j U(x^j)} \text{Wr}(P_{t^j_1 m}; x) \, dx \right\} 
\times \int_{\mathbb{R}^{M_1}} \text{Pr} T(x) \left\{ \prod_{m=1}^{M_1} e^{-\beta q_1 U(x^1_m)} \text{Wr}(P_{t^1_1 m}; x) \right\} \, d\mu^{M_1}(x).
\]

And,

\[
Z_N(x) = \sum_{\vec{M} \in \mathcal{S}_N} \frac{z_1^{M_1} \cdots z_J^{M_J}}{M_1! M_2! \cdots M_J!} \sum_{\bar{t} \in \mathcal{S}_M} \text{sgn} \left\{ \prod_{j=2}^J \prod_{m=1}^{M_j} \int_{\mathbb{R}} e^{-\beta q_j U(x^j)} \text{Wr}(P_{t^j_1 m}; x) \, dx \right\} 
\times \int_{\mathbb{R}^{M_1}} \text{Pr} T(x) \left\{ \prod_{m=1}^{M_1} e^{-\beta q_1 U(x^1_m)} \text{Wr}(P_{t^1_1 m}; x) \right\} \, d\mu^{M_1}(x).
\]

Using the same maneuvers as before, we can write

\[
Z_M = \int \left[ \sum_{\mathcal{M} \in \mathcal{S}_N} \left( \frac{(z_i t^j m)_{t^j_1 \cdots t^j_m}}{M_j !} \right) \sum_{t_1 \cdots t^j_j} \right] \left( \int_{\mathbb{R}^{M_1}} \text{Pr} T(x) \left\{ \prod_{m=1}^{M_1} e^{-\beta q_1 U(x^1_m)} \text{Wr}(P_{t^1_1 m}; x_m) \right\} \, d\mu^{M_1}(x) \right) e_1 \wedge e_2 \wedge \cdots \wedge e_{M_1} \, d\text{vol}.
\]
It is shown in [37, section 4.2] that
\[
\sum_{t_1, \ldots, t_M} \left( \int_{\mathbb{R}^M} \mathcal{P} T(x) \prod_{m=1}^{M_1} e^{-\beta q_1 U(x_m)} \text{Wr}(\mathcal{P}_{\mathcal{L}_1}; x_m) \right) \prod_{m=1}^{M_1} e^{-\beta q_1 U(x_m)} \prod_{m=1}^{M_1} \epsilon_{t_m}^1 \wedge \epsilon_{t_m}^2 \wedge \cdots \wedge \epsilon_{t_m}^{M_1} \left( z_1 \omega_1 \right)^{M_1}. \]

(The left hand side of this expression is the partition function of a system of \(M_1\) particles each of charge \(q_1\) when \(\beta\) is an odd square; showing that partition functions of such systems are hyperpfaffian was one of the goals of [37].)

We therefore have that
\[
Z_N(z) = \int \left[ \sum_M \left(\frac{z_1 \omega_1}{M_1!}\right)^{M_1} \right] d\epsilon_{\text{vol}} = \int e^{\omega(z)} d\epsilon_{\text{vol}},
\]
as desired.

Acknowledgment

This research was supported in part by the National Science Foundation (DMS-0801243).

References

[1] Adler M, Forrester P J, Nagao T and Moerbeke P van 2000 Classical skew orthogonal polynomials and random matrices J. Stat. Phys. 99 141–70
[2] Berezin F A 1966 The Method of Second Quantization (Pure and Applied Physics vol 24) (New York: Academic) (Engl. transl. by N Mugibayashi and A Jeffrey)
[3] Borodin A and Sinclair C D 2009 The Ginibre ensemble of real random matrices and its scaling limits Commun. Math. Phys. 291 177–224
[4] Boussaïcult A, Luque J-G and Tollu C 2009 Hyperdeterminantal computation for the Laughlin wavefunction J. Phys. A: Math. Theor. 42 145301
[5] de Bruijn N G 1956 On some multiple integrals involving determinants J. Indian Math. Soc. 19 133–51
[6] Dumitriu I and Edelman A 2002 Matrix models for beta ensembles J. Math. Phys. 43 5830–47
[7] Dyson F J 1962 Statistical theory of the energy levels of complex systems: I J. Math. Phys. 3 140–56
[8] Dyson F J 1962 Statistical theory of the energy levels of complex systems: II J. Math. Phys. 3 157–65
[9] Dyson F J 1962 Statistical theory of the energy levels of complex systems: III J. Math. Phys. 3 166–75
[10] Dyson F J 1970 Correlations between eigenvalues of a random matrix Commun. Math. Phys. 19 235–50
[11] Forrester P 2010 Log-Gases and Random Matrices (London Mathematical Society Monographs) (Princeton, NJ: Princeton University Press)
[12] Forrester P J and Nagao T 2007 Eigenvalue statistics of the real Ginibre ensemble Phys. Rev. Lett. 99 050603
[13] Forrester P J and Nagao T 2008 Skew orthogonal polynomials and the partly symmetric real Ginibre ensemble J. Phys. A: Math. Theor. 41 375003
[14] Forrester P J and Sinclair C D 2011 A generalized plasma and interpolation between classical random matrix ensembles J. Stat. Phys. 143 326–45
[15] Forrester P J and Warnaar S O 2008 The importance of the Selberg integral Bull. Am. Math. Soc. 45 489–534
[16] Ginibre J 1965 Statistical ensembles of complex, quaternion and real matrices J. Math. Phys. 6 440–9
[17] Goergig M O and Regnault N 2007 Analysis of a SU(4) generalization of Halperin’s wavefunction as an approach towards a SU(4) fractional quantum hall effect in graphene sheets Phys. Rev. B 75 241405
[18] Halperin B I 1983 Theory of the quantized hall conductance Helv. Phys. Acta 56 75
[19] Jokela N, Järvinen M and Keski-Vakkuri E 2008 The partition function of a multi-component Coulomb gas on a circle J. Phys. A: Math. Theor. 41 145003
[20] Jokela N, Järvinen M, Keski-Vakkuri E and Majumder J 2008 Disk partition function and oscillating rolling tachyons J. Phys. A: Math. Theor. 41 015402
[21] Laughlin R B 1983 Anomalous quantum hall effect: an incompressible quantum fluid with fractionally charged excitations Phys. Rev. Lett. 50 1395–8
[22] Luque J-G and Thibon J-Y 2002 Pfaffian and Hafnian identities in shuffle algebras Adv. Appl. Math. 29 620–46
[23] Luque J-G and Thibon J-Y 2003 Hankel hyperdeterminants and Selberg integrals J. Phys. A: Math. Gen. 36 5267–92
[24] Luque J-G and Thibon J-Y 2004 Hyperdeterminantal calculations of Selberg’s and Aomoto’s integrals Mol. Phys. 102 1351–9
[25] Masumoto S 2008 Hyperdeterminantal expressions for jack functions of rectangular shapes J. Algebra 320 612–32
[26] Mehta M L and Gaudin M 1960 On the density of eigenvalues of a random matrix Nucl. Phys. 18 420–7
[27] Mehta M L 2004 Random Matrices 3rd edn (Pure and Applied Mathematics vol 142) (Amsterdam: Elsevier)
[28] Meray C 1899 Sur un determinant dont celui de Vandermonde n’est qu’un particulier Rev. Math. Spéc. 9 217–9
[29] Nagao T and Wadati M 1992 Correlation functions of random matrix ensembles related to classical orthogonal polynomials: III J. Phys. Soc. Japan 61 1910–8
[30] Nagao T and Wadati M 1992 Correlation functions of random matrix ensembles related to classical orthogonal polynomials: II J. Phys. Soc. Japan 61 78–88
[31] Nagao T and Wadati M 1992 Correlation functions of random matrix ensembles related to classical orthogonal polynomials: I J. Phys. Soc. Japan 60 3298–322
[32] Rains E M 2000 Correlation functions for symmetrized increasing subsequences arXiv:math/0006097
[33] Rider B, Sinclair C D and Xu Y 2011 A solvable mixed charge ensemble on the line: global results Probab. Theory Relat. Fields at press (doi:10.1007/s00440-011-0394-z)
[34] Selberg A 1944 Bemerkninger om et multipelt integral Nor. Mat. Tidsskr. 26 71–8
[35] Sinclair C D 2007 Averages over Ginibre’s ensemble of random real matrices Int. Math. Res. Not. 2007 1–15
[36] Sinclair C D 2009 Correlation functions for β = 1 ensembles of matrices of odd size J. Stat. Phys. 136 17–33
[37] Sinclair C D 2010 Ensemble averages when β is a square integer submitted arXiv:1008.4362
[38] Tracy C A and Widom H 1998 Correlation functions, cluster functions and spacing distributions for random matrices J. Stat. Phys. 92 809–35