ON HIGHER-DIMENSIONAL DYNAMICS

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Abstract

Technical results are presented on motion in $N (> 4)D$ manifolds to clarify the physics of Kaluza-Klein theory, brane theory and string theory. The so-called canonical or warp metric in 5D effectively converts the manifold from a coordinate space to a momentum space, resulting in a new force (per unit mass) parallel to the 4D velocity. The form of this extra force is actually independent of the form of the metric, but for an unbound particle is tiny because it is set by the energy density of the vacuum or cosmological constant. It can be related to a small change in the rest mass of a particle, and can be evaluated in two convenient gauges relevant to gravitational and quantum systems. In the quantum gauge, the extra force leads to Heisenberg’s relation between increments in the position and momenta. If the 4D action is quantized then so is the higher-dimensional part, implying that particle mass is quantized, though only at a level of $10^{-65}$ gm or less which is unobservably small. It is noted that massive particles which move on timeline paths in 4D can move on null paths in 5D. This agrees with the view from inflationary quantum field theory, that particles acquire mass dynamically in 4D but are intrinsically massless. A general prescription for dynamics is outlined, wherein particles move on null paths in an $N (> 4)D$ manifold
which may be flat, but have masses set by an embedded 4D manifold which is curved.

1 Introduction

The motion of a test particle in a higher-dimensional manifold is a prime way to investigate extensions of 4D Einstein theory. In 5D Kaluza-Klein theory, older studies concentrated on the case where the 4D spacetime was independent of the extra coordinate [1-6]. This condition was relaxed in newer work on 5D induced-matter theory, where the 5D field equations for apparent vacuum are broken down into 4D ones with an energy-momentum tensor derived from the extra dimension [7-12]. Dynamical effects in 4D, when the metric is allowed to depend on one or more extra coordinates, have also become the subject of studies in string and membrane theory [13-16]. The main result is the appearance in 4D of extra forces [13,17,18]. These are expected to be small in gravitational problems, but could be significant in particle physics, where a unification of the interactions could be achieved via 10D superstrings or 11D supergravity [19,20]. In view of current interest in the subject, some results will be given aimed at clarifying higher-dimensional
2 Geodesics in ND and 4D

Consider an N-dimensional Riemannian manifold with a metric tensor $g_{AB}$ that depends on coordinates $x^C$, with a line element $dS^2 = g_{AB}dx^A dx^B$, through which a particle moves along a path described by an affine parameter $\lambda$. It contains a 4D submanifold with line element $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$. Here and below it is instructive to concentrate on the 5D case, when Latin indices run 0-4 and Greek indices run 0-3.

In general relativity, the particle moves along a geodesic which minimizes $s$ via $\delta \left[ \int ds \right] = 0$. For a particle with mass $m$, $\lambda = s$ is normally chosen, the geodesic is non-null, and the 4-momentum is defined by $p^\alpha \equiv mu^\alpha$ where the 4-velocity is $u^\alpha \equiv dx^\alpha/ds$. For a massless particle (photon), $\lambda$ is often unspecified because the geodesic is null and can be obtained directly from the metric. The 4-velocities are conventionally normalized for non-null and null paths via $u^\alpha u_\alpha = 1, 0$ respectively. However, $S$ contains $s$, and the former defines geodesics via $\delta \left[ \int dS \right] = 0$.

A question which arises in the literature is whether the particle should
follow a geodesic in ND or in 4D (in brane theory, this is connected with
whether the particle is constrained to move on the brane or can wander
through the bulk). The answer is that it is most natural to assume that the
motion is geodesic in $S$, provided that the extra terms which then appear
in the geodesic in $s$ are compatible with observation (see below). In this
regard, it should be recalled that even in 4D the acceleration of the particle
d$^2x^\alpha/ds^2$ is not a covariantly-defined vector. The appropriate quantity is
the absolute or covariant derivative, which defines the path via $Du^\alpha/ds \equiv
du^\alpha/ds + \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma = 0$, where $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols. It is the
latter which yield the forces on the particle, which are thereby recognized
as being inertial in origin, meaning that they arise from the motion of the
reference frame. In ND, the same philosophy should hold. We can use
$\delta \left[ \int dS \right] = 0$, and the absolute derivative or the Lagrange equations to obtain
the dynamics, but the latter will in general contain terms which arise from
the motion with respect to the larger reference frame.

The only comment which needs to be added to this concerns the case of
null geodesics with $dS = 0$. This has been considered by several workers
[1,11,21]. It should be recalled that 4D causality is defined by $ds^2 \geq 0$, and does not constrain $dS^2$ [5]. There is no impediment to assuming that
particles with \( m \neq 0 \) move along paths with \( dS^2 = 0 \), when their motions can be described consistently by choosing \( \lambda = s \).

There is, however, another issue which relates to geodesics and requires notice. Geodesics in general relativity really describe accelerations, not forces or changes in momentum. The distinction is often unnecessary, because the mass is constant. But even in Newtonian mechanics, a rocket changes mass as its fuel burns, feeling a force along the direction of its motion. And in inflationary quantum field theory, particles are intrinsically massless, gaining mass by a dynamical mechanism involving the Higgs field [22]. As noted occasionally in the literature [23], the correct dynamics in situations where the mass changes, follows not from \( \delta \int ds = 0 \) but from \( \delta \int mds = 0 \). That is, dynamics is a theory not of 4-velocities but of 4-momenta. In this regard, it should be noted that there is no contradiction between the normalization condition used in relativity for the 4-velocities \( u^\alpha u_\alpha = 1 \), and the condition used in particle physics for the 4-momenta \( p^\alpha p_\alpha = m^2 \). (The latter is often written \( E^2 - p^2 = m^2 \) and effectively uses the energy and 3-momentum to define the rest mass of a particle.) It is just that the conventional geodesic does not give any information about the origin or variability of mass, a problem which is of central importance in
The subjects of the preceding paragraphs, while perhaps familiar, underlie much of the recent work which has been done on higher-dimensional dynamics [1-23]. While it is not exclusive, we wish to concentrate in what follows on an approach which resolves most of the issues raised above [7-13]. Specifically, we wish to present new results on metrics of the so-called canonical or warp type in 5D (the extension to $N > 5$ is straightforward). This has line element

$$dS^2 = \frac{\ell^2}{L^2} g_{\alpha \beta} (x^\gamma, \ell) \, dx^\alpha dx^\beta - d\ell^2, \quad (2.1)$$

where $x^4 = \ell$ is the extra coordinate and $L$ is a constant length. Certain things are already known about this metric, and certain others may be deduced from the comments made above. It is convenient to list these here. (a) Mathematically (2.1) is general, insofar as the 5 available coordinate degrees of freedom have been used to set $g_{4\alpha} = 0, g_{44} = -1$. Physically, this removes the potentials of electromagnetic type and flattens the potential of scalar type. (b) The metric (2.1) has been extensively used in the field equations, which in terms of the Ricci tensor are $R_{AB} = 0$, and many solutions are known [11]. These include solutions for the 1-body problem [24] and cosmology [25] which have acceptable dynamics, and solutions with the opposite sign for $g_{44}$ which describe waves [26]. (c) When $\partial g_{\alpha \beta} / \partial \ell = 0$ in (2.1),


the 15 field equations $R_{AB} = 0$ contain as a subset the 10 field equations of general relativity, which in terms of the Einstein tensor are $G_{\alpha\beta} = 3g_{\alpha\beta}/L^2$. The scale $L$ is thereby identified, in cosmology in terms of the cosmological constant via $\Lambda = 3/L^2$ and in other situations as the characteristic size of the 4-space [10]. (d) This kind of local embedding of a 4D Riemann space in a 5D Ricci-flat space can be applied to any N, and is guaranteed by Campbell’s theorem [27-30]. (e) The factorization in (2.1) says in effect that the 4D part of the 5D interval is $(\ell/L)\, ds$, which defines a momentum space rather than a coordinate space if $\ell$ is related to $m$. This resolves the issue of forces versus accelerations noted above. (f) Partial confirmation of this comes from a study of the 5D geodesic and a comparison of the constants of the motion in 5D and 4D [11,32,33]. In the Minkowski limit, the energy of a particle moving with velocity $v$ is $E = \ell (1 - v^2)^{-1/2}$ in 5D, which agrees with the expression in 4D if $\ell = m$. (g) The 5 components of the geodesic equation for (2.1) split naturally into 4 spacetime components and an extra component. For $\partial g_{\alpha\beta}/\partial \ell \neq 0$, the former contain terms parallel to the 4-velocity $u^\alpha$ as noted above for the case of a rocket [10,33]. For $\partial g_{\alpha\beta}/\partial \ell = 0$, the motion is not only geodesic in 5D but geodesic in 4D.
3 The Nature of the Canonical Metric

The metric (2.1) is not mathematically unique but is physically rich, which prompts a deeper examination of its nature.

Consider a 4D space with $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$ embedded in a 5D space with line element

$$dS^2 = ds^2 - d\ell^2 \quad .$$

(3.1)

Then the transformation

$$s \rightarrow \ell \sinh (s/L)$$

$$\ell \rightarrow \ell \cosh (s/L)$$

(3.2)

causes (2) to become

$$dS^2 = \frac{\ell^2}{L^2} ds^2 - d\ell^2 \quad .$$

(3.3)

This is of the canonical form, which is therefore recognized to be a spherical form of a 2-plane (the opposite sign for $g_{44}$ may be obtained by replacing the hyperbolic functions by their trigonometric counterparts). Physically, there is an analogy with the angular momentum (per unit mass) of a particle $rv$ moving with velocity $v$ at distance $r$ from the centre of a circle. Recalling
from above that $\ell$ plays the role of $m$ in the canonical metric, we see that $\mu^\alpha$ is the product of the velocity in 4D with the distance in the orthogonal fifth direction. In other words, $p^\alpha$ is a true 5D moment.

The above suggests that physically significant 4D structure may even be present in a flat 5D manifold. The latter in spherical polars has line element

$$dS^2 = dt^2 - dr^2 - r^2 d\Omega^2 - d\ell^2,$$

(3.4)

where $d\Omega^2 \equiv (d\theta^2 + \sin^2 \theta d\phi^2)$. Introducing a dimensionless parameter $\alpha$, consider the transformation

$$t \rightarrow \left(\frac{\alpha}{2}\right) t^{1/\alpha} \ell^{1/(1-\alpha)} \left(1 + \frac{r^2}{\alpha^2}\right) - \frac{\alpha}{2(1-2\alpha)} \left[t^{-1} \ell^{\alpha/(1-\alpha)}\right]^{(1-2\alpha)/\alpha}$$

$$r \rightarrow rt^{1/\alpha} \ell^{1/(1-\alpha)}$$

$$\ell \rightarrow \left(\frac{\alpha}{2}\right) t^{1/\alpha} \ell^{1/(1-\alpha)} \left(1 - \frac{r^2}{\alpha^2}\right) + \frac{\alpha}{2(1-2\alpha)} \left[t^{-1} \ell^{\alpha/(1-\alpha)}\right]^{(1-2\alpha)/\alpha}$$

(3.5)

with $\theta \rightarrow \theta$, $\phi \rightarrow \phi$. Then some algebra shows that (3.4) becomes

$$dS^2 = \ell^2 dt^2 - t^{2/\alpha} \ell^{2/(1-\alpha)} (dr^2 + r^2 d\Omega^2) - \alpha^2 (1 - \alpha)^{-2} t^2 d\ell^2.$$

(3.6)
This is the metric of a class of cosmological models first found as solutions of the field equations $R_{AB} = 0$ by Ponce de Leon [34]. They are separable in $\ell$, $t$ and reduce to the 4D Friedmann-Robertson-Walker models with flat 3D sections on the hypersurfaces $\ell = \text{constant}$. (The dust or Einstein-de Sitter solution has $\alpha = \frac{3}{2}$ while the radiation solution has $\alpha = 2$.) For this reason they are often regarded as the standard 5D cosmologies, but (3.5) shows that they are actually canonical forms of 5D Minkowski space.

The same cannot be said of the standard 1-body solution of $R_{AB} = 0$ [24]. This has the line element

$$dS^2 = \frac{\Lambda \ell^2}{3} \left\{ \left[ 1 - 2\frac{M}{r} - \frac{\Lambda r^2}{3} \right] dt^2 - \left[ 1 - 2\frac{M}{r} - \frac{\Lambda r^2}{3} \right]^{-1} dr^2 \right. - \left. r^2 d\Omega^2 \right\} - d\ell^2 , \quad (3.7)$$

where $M$ is a constant usually identified with the mass at the centre of the 3-geometry. Metric (3.7) is pure canonical in the sense that it has the form (2.1) with $\Lambda = 3/L^2$ and $\partial g_{\alpha\beta}/\partial \ell = 0$. However, it is not 5D flat like (3.6), as may be verified by computer. This agrees with the well-known fact that the 4D Schwarzschild-de Sitter solution (given by the part inside the big brackets in the last expression) can only be embedded in a flat space.
with dimension $N \geq 6$. However, since any 4D solution can be embedded in $N \geq 10$, the use of the canonical form obviously has relevance to superstring theory.

The field equations $R_{AB} = 0$ mentioned above clearly contain physical information which is relevant to why the canonical metric (2.1) is so effective. These 15 relations can in general be broken down into 1 wave equation, 4 conservation equations, and 10 Einstein equations $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ with an effective or induced energy-momentum tensor [35]. However, only the last quantity contains $u^\alpha$, and via the 4D covariant derivative $T^\alpha_{\beta;\beta} = 0$ is usually interpreted as describing the dynamics of a fluid consisting of particles. We will return to the field equations below, but let us here take a look at particle dynamics.

This subject has been studied using both the geodesic equation [7,8] and the Lagrange equations [10,12]. These approaches are compatible of course, but the latter is the more instructive for investigating the special status of (2.1). Thus consider a Lagrangian which generalizes (2.1) and has the form

$$L = \left\{ \frac{\ell^2}{L^2} g_{\alpha\beta}(x^\gamma, \ell) \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} - \Phi^2(x^\gamma, \ell) \left( \frac{d\ell}{d\lambda} \right)^2 \right\}^{1/2}. \quad (3.8)$$

This is dimensionless, and contains a scalar field $g_{44} = -\Phi^2$ which is the
classical analog of the Higgs potential that is responsible for particle masses in quantum field theory [11,22]. The Lagrangian (3.8) defines an action
\[ I = \int L (x^A, \dot{x}^A) \, d\lambda, \]
where \( \lambda \) is an affine parameter along the path of the particle and \( \dot{x}^A \equiv dx^A/d\lambda \) is its 5-velocity. The action is an extremum if
\[
\frac{d}{d\lambda} \left( \frac{\partial L}{\partial \dot{x}^A} \right) - \frac{\partial L}{\partial x^A} = 0.
\] (3.9)
The spacetime and extra components of this can be worked out explicitly once \( \lambda \) is chosen. A natural choice might appear to be \( \lambda = S \) [8], in which case (3.9) is equivalent to
\[
\frac{dU^A}{dS} + \Gamma^A_{BC} U^B U^C = 0.
\] (3.10)
the 5D geodesic equation in \( U^A \equiv dx^A/dS \) with appropriately defined Christoffel symbols (see Section 2). However, this is undefined for 5D null geodesics, and if another affine parameter \( \lambda \) is used instead then (3.10) acquires an extra term on the rhs equivalent to \( U^A L^{-1} dL/d\lambda \) [36]. Also, the object of the exercise is to understand 4D dynamics in \( u^\alpha \equiv dx^\alpha/ds \) rather than 5D dynamics in \( U^A \equiv dx^A/dS \). For these reasons, we choose \( \lambda = s \) [7,10,12]. We also choose \( u_\alpha u^\alpha = 1 \) for a massive particle following a timelike 4D path. Then the substitution of (3.8) into (3.9) results in two rather complicated expressions for the \( \alpha, 4 \) components of the motion, of which the second is the
more enlightening:

\[
\ell \left( \frac{\Phi^2 \dot{\ell}}{\ell^2} \right) - \frac{L^2 \Phi \dot{\ell}^3}{\ell^2} - \frac{1}{L \dot{\Phi}} \left( 1 - \frac{L^2 \Phi^2 \dot{\ell}^2}{\ell^2} \right) \left( 1 + \frac{\ell u^\alpha u^\beta}{2} \frac{\partial g_{\alpha\beta}}{\partial \ell} - \frac{L^2 \Phi \dot{\ell}^2}{\ell} \right) = 0 .
\]

(3.11)

Remarkably, this is satisfied with no constraint on the last parenthesis by 
\((L \dot{\Phi} / \ell)^2 = 1,\) which by (3.8) implies \(\mathcal{L} = 0.\) That is, the particle is 
travelling along a timelike path in 4D but a null path in 5D.

We end this section by stating explicitly the equations of motion 
which follow from (3.9) or (3.10). The spacetime components can be written 
as a part which is geodesic in \(s\) and an extra part:

\[
\frac{du^\mu}{ds} + \Gamma^\mu_{\beta\gamma} u^\beta u^\gamma = F^\mu 
\]

(3.12)

\[
P^\mu \equiv \left( -g^{\mu\alpha} + \frac{u^\mu u^\alpha}{2} \right) u^\beta \frac{\partial g_{\alpha\beta}}{\partial \ell} \frac{d\ell}{ds} .
\]

(3.13)

Here \(F^\mu\) is a force per unit (inertial) mass, or acceleration. It can be written 
as a sum of components normal and parallel to \(u^\mu,\) so \(F^\mu = N^\mu + P^\mu\) where

\[
N^\mu = \left( -g^{\mu\alpha} + \frac{u^\mu u^\alpha}{2} \right) u^\beta \frac{\partial g_{\alpha\beta}}{\partial \ell} \frac{d\ell}{ds}
\]

(3.14)
\[ P^\mu = -\frac{u^\mu}{2} \left( u^\alpha u^\beta \frac{\partial g_{\alpha\beta}}{\partial \ell} \right) \frac{d\ell}{ds} . \] (3.15)

The normal component obeys \( N^\mu u_\mu = 0 \) (by construction), which is the behaviour typical of Einstein gravity and Maxwell electromagnetism [10,11]. The parallel component obeys

\[ P^\mu u_\mu = -\frac{u^\alpha u^\beta}{2} \frac{\partial g_{\alpha\beta}}{\partial \ell} \frac{d\ell}{ds} \equiv \beta . \] (3.16)

Here the 4-velocities are still normalized via \( u^\alpha u_\alpha = 1 \) (see above and the next section). But the scalar quantity \( \beta \) is finite if \( \partial g_{\alpha\beta}/\partial \ell \neq 0 \) in the canonical metric (2.1), and there is motion in the extra dimension as measured with the particle’s proper 4D time \( s \). The quantity \( \beta \) is a kind of power per unit (inertial) mass. It has no analog in standard 4D field theory. The magnitude of \( \beta \) depends on \( d\ell/ds \), which is given by the extra component of the equation of motion:

\[ \frac{d^2 \ell}{ds^2} - \frac{2}{\ell} \left( \frac{d\ell}{ds} \right)^2 + \frac{\ell}{L^2} = -\frac{1}{2} \left[ \frac{\ell^2}{L^2} - \left( \frac{d\ell}{ds} \right)^2 \right] u^\alpha u^\beta \frac{\partial g_{\alpha\beta}}{\partial \ell} . \] (3.17)

This implies that there is no intrinsic state of rest for the particle in the extra dimension. (Formally, the last equation is satisfied with \( \ell = \ell_0, u^{123} = 0, u^0 = 1 \) and \( g_{\alpha\beta} = L^2/\ell_0^2 \), but then \( d\ell = 0 \) and the metric reverts to
This is basically because the 4D proper time has been used as a parameter for the 5D motion. Thus in general, \( d\ell/ds \neq 0 \) in (3.17), \( \partial g_{\alpha\beta}/\partial \ell \neq 0 \) in (2.1) and \( \beta \neq 0 \) in (3.16). The existence of finite scalar quantities like \( \beta \) is expected to be typical of the dynamics of any \( N(>4) \)-dimensional theory.

### 4 Extra Forces in \( N(>4)D \) Theory

In section 2, it was noted that the word “force” has to be treated with some caution, because theories like general relativity describe accelerations while theories of particle physics describe momenta, and the two concepts can only be consistently joined via a suitable definition of mass. The canonical metric (2.1) opens a route to this, as well as providing a number of other interesting results. In Section 3, it was seen that the effectiveness of the canonical metric can be traced partly to the fact that it is an algebraically convenient way to parametrize a 5D manifold, but mainly to the fact that it is a natural basis for 5D dynamics. In the present section we wish to go beyond the canonical metric and note some general results on forces.

It is straightforward to see that any \( N(>4)D \) theory will yield extra
accelerations as viewed in 4D, which modulo an appropriate definition of mass will be interpreted as extra forces [13,17,18]. Given an ND line element $dS^2 = g_{AB} dx^A dx^B$, the N-velocities $u^A \equiv dx^A / dS$ are normalized for a non-null path via $U^A U_A = 1$, and the path is extremized in terms of an ND covariant derivative via $U^B U^A :_B = 0$. This when contracted gives $U_A F^A = 0$, where the $F^A$ are forces per unit (inertial) mass. However, this implies $U_\alpha F^\alpha = -U_{(N-\alpha)} F^{(N-\alpha)} \neq 0$ when viewed from 4D.

To relate this to what was done in Section 3, consider one extra coordinate and the normalization condition

$$g_{\alpha\beta} (x^\gamma, \ell) \ u^\alpha u^\beta = 1 \quad . \quad (4.1)$$

Differentiating this wrt an affine parameter $\lambda$ gives

$$g_{\alpha\beta,\gamma} \ u^\gamma u^\alpha u^\beta + \frac{\partial g_{\alpha\beta}}{\partial \ell} \frac{d\ell}{d\lambda} \ u^\alpha u^\beta + 2 g_{\alpha\mu} \frac{du^\mu}{d\lambda} \ u^\alpha = 0 \quad , \quad (4.2)$$

where $u^\alpha \equiv dx^\alpha / d\lambda$ and $g_{\alpha\beta,\gamma} \equiv \partial g_{\alpha\beta} / \partial \ell$. Introducing the Christoffel symbols and noting symmetries under the exchange of $\alpha$ and $\beta$, the first term on the lhs of (4.2) can be rewritten as

$$(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha}) \ u^\gamma u^\alpha u^\beta = 2 g_{\alpha\mu} \Gamma^\mu_{\beta\gamma} u^\gamma u^\alpha u^\beta \quad . \quad (4.3)$$
Then (4.2) reads
\[ 2g_{\alpha\mu}u^\alpha \left( \frac{du^\mu}{d\lambda} + \Gamma^\mu_{\beta\gamma} u^\beta u^\gamma \right) + \frac{\partial g_{\alpha\beta}}{\partial \ell} \frac{d\ell}{d\lambda} u^\alpha u^\beta = 0 \] . \tag{4.4}

With \( F^\mu \equiv (du^\mu/d\lambda + \Gamma^\mu_{\beta\gamma} u^\beta u^\gamma) \) as in (3.12) this says
\[ u_\mu F^\mu = -\frac{u^\alpha u^\beta}{2} \frac{\partial g_{\alpha\beta}}{\partial \ell} \frac{d\ell}{d\lambda} \] . \tag{4.5}

There is clearly a force per unit (inertial) mass parallel to \( u^\mu \) given by
\[ P^\mu = -\frac{u^\mu}{2} \left( u^\alpha u^\beta \frac{\partial g_{\alpha\beta}}{\partial \ell} \right) \frac{d\ell}{d\lambda} \] . \tag{4.6}

When \( \lambda = s \) this is identical to (3.15) above, which was derived starting from the canonical metric (2.1). But here, we started from the normalization condition (4.1) for a massive particle following a 4D timelike path. This means that the existence of (4.6) does not depend on the form of the metric. It is a consequence of defining and normalizing 4-velocities in the conventional way when spacetime is part of a bigger manifold.

Given the manner in which 4D dynamics is conventionally set up, it is difficult to conceive of any way in which a force parallel to the 4-velocity could be interpreted other than by relating it to the mass of the particle which feels it. For a particle moving under the influence of standard gravity and the extra force, the equation of motion in spacetime is
\[
\frac{du^\mu}{ds} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta = P^\mu,
\]
(4.7)

with \(P^\mu\) given by (4.6) with \(\lambda = s\). Below, we will consider some significant applications of (4.7). Here, as an illustration, we can take a canonical metric with \(g_{\alpha\beta} = (\ell^2/L^2)\eta_{\alpha\beta}\) where \(\eta_{\alpha\beta}\) = diagonal \((1,-1,-1,-1)\). Then \(\partial g_{\alpha\beta}/\partial \ell = (2/\ell)g_{\alpha\beta}\) and \(u^\alpha u^\beta \partial g_{\alpha\beta}/\partial \ell = 2/\ell\) in (4.6). The equation of motion (4.7) reads

\[
\frac{du^\mu}{ds} = -\frac{u^\mu}{\ell} \frac{d\ell}{ds},
\]
(4.8)

which yields \(\ell u^\mu = \ell_0\) where \(\ell_0\) is a constant of the (4D) motion. The last is clearly the momentum \(mu^\mu\), confirming that in canonical coordinates the extra coordinate plays the role of particle mass. [See Section 2 and refs. 8, 10, 11 and 12. Equation (4.8) above is the analog of what is sometimes called the rocket equation in Newtonian mechanics, which just says that \(d(mv)/dt = 0\) or \(dv/dt = -(v/m)dm/dt\).] However, while \(p^\mu = mu^\mu\) is a constant of the 4D motion, it should be noted that \(m = m(s)\) in general.

This cannot be fixed in the conventional approach to 4D dynamics, except by appeal to some external condition. But in 5D dynamics it can,
notably by the extra component of the geodesic (3.17). This in general requires a solution of the field equations, but as noted in Section 3 a natural parametrization of 5D geodesics is via $dS = 0$. Then for the canonical metric (2.1) we have

$$dS^2 = 0 = \frac{\ell^2}{L^2} ds^2 - d\ell^2,$$

(4.9)

which yields

$$\ell = \ell_0 e^{\pm s/L}.$$

(4.10)

The rate of variation of $\ell$ depends on the characteristic dimension of the 4-space $L$. As noted in Section 2, for pure-canonical metrics with $\partial g_{\alpha\beta}/\partial \ell = 0$, $L = (3/\Lambda)^{1/2}$ where $\Lambda$ is the cosmological constant [10,11]. Thus from (4.10), with the identification $\ell = m$, the rate of variation of the rest mass is given by

$$\frac{1}{m} \left| \frac{dm}{ds} \right| \approx \frac{1}{m} \left| \frac{dm}{dt} \right| = \left( \frac{\Lambda}{3} \right)^{1/2}.$$

(4.11)

The value of $\Lambda$ is severely constrained by astrophysical data [37-39]. These indicate $|\dot{m}|/m \lesssim 2 \times 10^{-18}\text{sec}^{-1}$ by (4.11), which is observationally acceptable. There are also other constraints on extra forces like (4.6), but these
are relatively weak [40]. However, it should be noted that $\Lambda$ measures the energy density of the vacuum in general relativity [11], and this could be larger on small scales [22], so in principle mass variation and extra forces could be measured.

To do this in practice, though, requires solutions of the field equations. These in turn require the specification of a system of coordinates or gauge. In this context, it should be noted that the extra force $P^{\mu}$ of (4.6) for the 5D case is a 4-vector. As such, it is covariant under the usual group of 4D coordinate transformations $x^{\mu} \rightarrow x^{\mu}(x^{\nu})$, but will in general change under the group of 5D coordinate transformations $x^{A} \rightarrow x^{A}(x^{B})$. This is inevitable given that the field equations $R_{AB} = 0$ are covariant in 5D, but may represent a problem as regards the interpretation of observations made in 4D. This problem will be greater in the (algebraically straightforward) extension of the results of this section to $N(> 5)D$. There is a considerable relevant literature on gauges [11,19,20,41,42]. Fortunately, if attention is restricted to dynamics there are only two natural gauges, to which we now turn our attention.
5 The Einstein and Planck Gauges

It has been seen that the pure canonical gauge, namely (2.1) with a factor $\ell^2$ attached to an $\ell$-independent spacetime, is remarkably successful as a basis for conventional 4D dynamics. This is because the use of $\ell = m$ effectively converts the 4D part of the 5D manifold from a coordinate space to a momentum space. However, that success concerns the classical concept of momentum as the product of mass and velocity. In modern quantum field theory the mass of a particle is not defined \textit{a priori} [22], and even in old quantum theory the momentum is described by a de Broglie wave and derived from a wave function. A superior formulation of dynamics ought to address both the classical and quantum nature of momentum. In this section, we will assume that the differences in description are due to differences in gauge choices for a higher-dimensional metric, and narrow the choices for the gauges using field equations.

The latter are still the subject of discussion in ND field theory, but in 5D there is a consensus that they are given by the Ricci tensor as

$$R_{AB} = 0 \quad (A, B = 0, 123, 4).$$

\text{(5.1)}
Let us consider these for a generalized form of the pure canonical metric (2.1) with line element

\[ dS^2 = \left( \frac{L}{\ell} \right)^{2a} \bar{g}_{\alpha\beta}(x^\gamma) dx^\alpha dx^\beta - \left( \frac{L}{\ell} \right)^{4b} d\ell^2 . \]  

(5.2)

Here \( a, b \) are constants which it is desired to constrain using (5.1). These 15, 5D relations can be decomposed into 4D ones under only the 4 coordinate conditions \( g_{\alpha4} = 0 \) [35]. For (5.2) it is convenient to take the components in the order \( AB = 44, 4\alpha, \alpha\beta \). The result is:

\[ a^2 - 2ab + a = 0 \]  

(5.3)

\[ V^\beta_{\alpha;\beta} = 0 \]

\[ V^\beta_{\alpha;\beta} \equiv \frac{1}{2 |g_{44}|^{1/2}} \left( g^{\beta\sigma} \frac{\partial g_{\sigma\alpha}}{\partial \ell} - \delta^\beta_\alpha g^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial \ell} \right) \]

\[ = \frac{3a\delta^\beta_\alpha}{\ell} \left( \frac{\ell}{L} \right)^{2b} \]  

(5.4)

\[ G_{\alpha\beta} = \frac{(2a^2 + 2ab - a)}{\ell^2} \left( \frac{L}{\ell} \right)^{2a-4b} \bar{g}_{\alpha\beta} \]  

(5.5)
Here a semicolon denotes the usual 4D covariant derivative, $g_{\alpha\beta} = (L/\ell)^2 \bar{g}_{\alpha\beta}$, $g_{44} = -(L/\ell)^4 b$ and indices have been raised and lowered using $g_{\alpha\beta}$ in order to get the 4D Einstein tensor. This in mixed form is $G_{\alpha}^{\beta} = (2a^2 + 2ab - a)\ell^{-2}(L/\ell)^4 b^{\delta}_{\alpha}$, and as usual $G_{\alpha}^{\beta} \equiv R_{\alpha}^{\beta} - R\delta_{\alpha}^{\delta}/2$ in terms of the 4D Ricci tensor and Ricci scalar. The last may be found by direct calculation, and is

$$R = \frac{-12a^2}{\ell^2} \left( \frac{L}{\ell} \right)^{4b}.$$  

(5.6)

This determines the curvature of the 4D part of the manifold, which by (5.5) has the form of a vacuum space with an effective cosmological constant $\Lambda = (2a^2 + 2ab - a)\ell^{-2}(L/\ell)^2 - 4b$. This is zero for $a = b = 0$, in which case (5.2) describes general relativity embedded in a flat and physically innocuous extra dimension. For $a = -1, b = 0$ we have $\Lambda = 3/L^2$, and (5.2) is the pure canonical metric already discussed. Interestingly, the same value of $\Lambda$ results for $a = +1, b = +1$ which implies that (5.5) would give the same 4D physics even though the metric (5.2) does not have the canonical form. We will return to this below. Here we note that the 4D physics is contained in the 10 Einstein equations (5.5), while the 4 conservation equations (5.4) are satisfied, and the 1 scalar relation (5.3) provides a constraint between the constants $a, b$. In this regard it should be noted that the relation $G = -R$ (which follows from the definition of the Einstein tensor), when combined
with (5.6) reproduces (5.3), which is the only meaningful constraint.

The comments of the preceding paragraph imply that (5.2) as constrained by (5.3) contains interesting physics in (5.5). For example, the fact that the 4D cosmological constant depends in general on 5D parameters opens the way to a resolution of the conflict in its size as inferred from cosmology and particle physics [43-45]. However, it is apparent that for dynamics there are two natural gauges, namely those with \( a = -1, b = 0 \) and \( a = +1, b = +1 \). For these (5.2) has the forms

\[
\begin{align*}
    dS^2 &= \frac{\ell^2}{L^2} \bar{g}_{\alpha\beta}(x^\gamma) dx^\alpha dx^\beta - d\ell^2 \\
    dS^2 &= \frac{L^2}{\ell^2} \bar{g}_{\alpha\beta}(x^\gamma) dx^\alpha dx^\beta - \frac{L^4}{\ell^4} d\ell^2.
\end{align*}
\]

Mathematically, these are equivalent since (5.7) \( \rightarrow \) (5.8) under the simple coordinate transformation

\[
\ell \rightarrow L^2 / \ell.
\]

Physically, this corresponds to changing the way the rest mass of a particle is described. We saw above that for the pure canonical metric the dynamics
implies the identification $\ell = m$. Let us restore physical units for the speed of light $c$, the gravitational constant $G$ and Planck's constant $h$. Then (5.7) corresponds to the shift from gravitational “units” to quantum “units”, where the extra coordinate in (5.7), (5.8) is given respectively by

$$\ell = \frac{Gm}{c^2}, \quad \ell = \frac{h}{mc}.$$  \hspace{1cm} (5.10)

There is nothing really fundamental about the presence of the dimensional constants here. Dimensional analysis is an elementary group-theoretic technique based on the Pi theorem [46-48]. The purpose of $c, G$ and $h$ in (5.10) is merely to transpose the dimensions of mass to length so that it can be geometrized. And since the dimensions of these quantities are $LT^{-1}$, $M^{-1}L^3T^{-2}$ and $ML^2T^{-1}$ and are not degenerate [47], they can all be set to unity as is the common practice. This said, it is convenient to rename the pure canonical metric (5.7) the Einstein gauge and its other form (5.8) the Planck gauge.

The dynamics in these gauges can be studied for (5.7) by using (3.12)-(3.17), and for both (5.7) and (5.8) by using (4.6), (4.7). As an illustration, let us revisit the short calculation which led to (4.8) but now in both gauges. That is, we take $g_{\alpha\beta} = (L/\ell)^2 \eta_{\alpha\beta}$ so $\partial g_{\alpha\beta}/\partial \ell = -(2a/\ell) g_{\alpha\beta}$
and $u^\alpha u^\beta \partial g_{\alpha\beta}/\partial \ell = -2a/\ell$, which causes the 4D equation of motion (4.7) with the parallel force (4.6) to read

$$
\frac{du^\mu}{ds} = P^\mu = - \frac{u^\mu}{2} \left( u^\alpha u^\beta \frac{\partial g_{\alpha\beta}}{\partial \ell} \right) \frac{d\ell}{ds} = \frac{au^\mu}{\ell} \frac{d\ell}{ds} .
$$

(5.11)

This yields $u^\mu = (\ell/\ell_0)^a$ where $\ell_0$ is a constant of the 4D motion. Alternatively, the last equation can be written as

$$
d \left( \ell^{-a} u^\mu \right) = 0 .
$$

(5.12)

This says that in the Einstein gauge ($a = -1$) $\ell u^\mu$ is conserved, while in the Planck gauge ($a = +1$) $u^\mu/\ell$ is conserved. That is, by (5.10), the conserved quantities are the classical momentum $(G/c^2)mu^\mu$ and the quantum momentum or inverse de Broglie wavelength $(c/h)mu^\mu$. These are as expected, but as before in both quantities $m = m(s)$ in general. If as in (4.9) we take a null 5D geodesic, (5.2) gives

$$
dS^2 = 0 = \left( \frac{L}{\ell} \right)^{2a} ds^2 - \left( \frac{L}{\ell} \right)^{4b} d\ell^2 .
$$

(5.13)

This yields for both gauges

$$
\ell = \ell_0 e^{\pm s/L} = \ell_0 \exp \left[ \pm (\Lambda/3)^{1/2} s \right] ,
$$

(5.14)
which is the same as (4.10) and has similar implications for mass variation.

This process is intrinsic to 5D dynamics and warrants a closer examination, because it opens the way to understanding the Heisenberg uncertainty relation. The latter is not a part of classical 4D dynamics, and neither is the parallel 4D acceleration \( P^\mu \) derived above. This was calculated from the canonical metric (2.1) in (3.15) and has an associated scalar power per unit (inertial) mass (3.16), but was also calculated from the normalization condition (4.1) in (4.6) where it was found to be of general form. This agrees with the Hamiltonian approach to Kaluza-Klein theory, where a (4+1) split can always be performed in order to recover Einstein theory \([49-51]\). In the Einstein and Planck gauges, \( P^\mu \) has the form (5.11), whose associated scalar is \( P^\mu u_\mu = (a/\ell) d\ell/ds \) where \( a = \pm 1 \). This quantity has no analog in conventional classical dynamics, whose forces as was noted before obey \( F^\mu u_\mu = 0 \). However, an observer could interpret \( P^\mu \) as causing an anomalous change in momentum \( dp^\mu \), such that by (5.11)

\[
\frac{1}{m} \frac{dp^\mu}{ds} = P^\mu = \frac{au^\mu}{\ell} \frac{d\ell}{ds} .
\]  

(5.15)

This implies the associated scalar quantity
\[ dx_\mu d\bar{p}^\mu = \frac{am}{\ell} \, d\ell \, ds \]  \hspace{1cm} (5.16)

For both the Einstein gauge and the Planck gauge, this reads

\[ dx_\mu d\bar{p}^\mu = -dm \, ds \]  \hspace{1cm} (5.17)

That is, there is a Heinsenberg-type relation in 4D which depends on the mass change associated with 5D. The relation (5.17) is general, insofar as no use has been made of (5.14) for \( m = m(s) \) which follows from (5.13) for a null 5D geodesic. The generality of (5.17) may also be appreciated by recalling from above that the momentum is really conserved along a 5D path, so \( d(mu^\mu) = 0 \) or \( mdu^\mu + u^\mu dm = 0 \), where the second term represents an anomalous change in momentum \( d\bar{p}^\mu = -u^\mu dm \), so there is a scalar \( dx_\mu d\bar{p}^\mu = -dx_\mu (dx^\mu/ds) dm = -dmds \).

Let us now ask how a 5D null path with \( dS^2 = 0 \) relates to the motion of a particle of mass \( m \) moving along a 4D timelike path with \( ds^2 > 0 \). We work in the Planck gauge with conventional units. Then (5.14) gives \( dm = \pm L^{-1} mds \), and (5.17) may be written

\[ dx_\mu d\bar{p}^\mu = \frac{(mcds)^2}{h} \left( \frac{\ell}{L} \right) \]  \hspace{1cm} (5.18)
The first term in parentheses here is the 4D action, which as usual is defined and quantized for integers $n$ via

$$ I \equiv \int mc ds, \quad dI = nh \quad . \quad (5.19) $$

The second term in parentheses in (5.18) is the ratio of the Compton wavelength of the particle $l = h/mc$ and the characteristic dimension of the 4-space it inhabits $L = (3/\Lambda)^{1/2}$. Now the 5D null condition (5.13) in the Planck gauge ($a = 1$, $b = 1$) shows that if the 4D action is quantized via (5.19) then so must be the extra part of the 5D line element. We therefore put $L/\ell = n$, which says that the Compton wavelength is an excitation of the fundamental mode. The result is that (5.18) reads

$$ dx_\mu d\vec{p}^\mu = nh \quad , \quad (5.20) $$

which is Heisenberg’s relation. The above approach can clearly be generalized to other box sizes. But if free particles have masses set by the energy density of the vacuum [22], constraints on the cosmological constant [37-39] show that mass is quantized in units of

$$ m = \frac{h}{cL} = \frac{h}{c} \left( \frac{\Lambda}{3} \right)^{1/2} \lesssim 10^{-65} \text{ gm} \quad . \quad (5.21) $$
This is too small to detect with current methods, and miniscule compared to the so-called Planck mass \((hc/G)^{1/2} \approx 10^{-5}gm\), which is seen to be an artefact produced by a mixture of the Einstein and Planck gauges discussed in this section.

6 Waves in \(N(>4)D\) Theory

The preceding section showed that Heisenberg’s relation follows as a consequence of the extra force which results when a causal 4D manifold is extended to a null 5D one. It is therefore natural to ask if other aspects of particles, including their wave nature, can be understood as manifestations of an \(N(>4)\)-dimensional space. That quantum field theory and general relativity are in principle compatible has been shown by work on the Hartle-Hawking and Vilenkin wave functions [52-54]. And exact solutions of Einstein’s equations are known which can describe non-gravitational waves [26, 42, 55-60]. There is of course the potential problem that the metric may be complex [61-66], but this can be avoided if the view is taken that only the physically-relevant quantities calculated from it need to be real [26, 42, 60]. In this section we will therefore proceed to see if it is possible to
set up a consistent framework for 4D wave mechanics in $N(> 4)D$ theory, concentrating as before on the 5D case.

The 4D Klein-Gordon equation for a relativistic particle with zero spin and finite mass should be derivable, based on what has been shown above, from the 5D equation for a null geodesic. However, since the Klein-Gordon equation is a second-order relation in a complex wave function, we take the line element to have signature $(+ - - - +)$. Then

$$dS^2 = 0 = \left(\frac{L}{\ell}\right)^{2a} ds^2 + \left(\frac{L}{\ell}\right)^{4b} d\ell^2 ,$$

where $g_{\alpha\beta} = (L/\ell)^{2a} \bar{g}_{\alpha\beta} dx^\alpha dx^\beta$ and $a = (-1, +1)$ with $b = (0, +1)$ for the Einstein and Planck gauges as before. The last relation is satisfied by

$$\ell = \ell_0 e^{\pm is/L} ,$$

which is the complex analog of (4.10). The mass (squared) involves $\ell\ell^* = \ell_0^2$ and is constant and real. Without loss of generality we can take the upper sign in (6.2) and define a dimensionless wave function

$$\psi = e^{is/L} .$$
This satisfies a hierarchy of wave equations

\[
\frac{d^n \psi}{ds^n} = \left( \frac{i}{L} \right)^n \psi, \quad (6.4)
\]

where we are interested primarily in the cases \( n = 1, 2 \). [The \( n \) in (6.4) should not be confused with that in (5.19).] The first-order equation of (6.4) implies \( i\psi/L = dx/ds = (\partial \psi/\partial x^\alpha)(dx^\alpha/ds) = (\partial \psi/\partial x^\alpha)u^\alpha \) or \( 1 = (L/i\psi)(\partial \psi/\partial x^\alpha)u^\alpha \). But \( 1 = u^\alpha u_\alpha \), so

\[
u_\alpha = \frac{L}{i\psi} \frac{\partial \psi}{\partial x^\alpha}. \quad (6.5)
\]

This for the fundamental mode of the Planck gauge with \( L = h/mc \) just says \( p_\alpha = (h/ic\psi)\partial \psi/\partial x^\alpha \), which is the usual prescription for obtaining the momenta from the wave function. The second-order equation of (6.4) can be treated similarly, and with (6.5) yields

\[
\frac{u^\alpha u^\beta}{\psi} \frac{\partial^2 \psi}{\partial x^\alpha \partial x^\beta} + \frac{1}{L^2} + \frac{iu_\alpha}{L} \frac{du^\alpha}{ds} = 0. \quad (6.6)
\]

The imaginary part of this is \( u_\alpha du^\alpha/ds = 0 \) or the usual orthogonality relation. The parallel acceleration (4.6) or (5.11) which follows from the metric in form (5.2) does not appear, which agrees with the fact that the effective mass is a constant for the metric in form (6.1). In this regard, it is instructive to consider the geodesic equation \( dU^\mu/d\lambda + \Gamma^\mu_{\beta \gamma} U^\beta U^\gamma = 0 \) with
$U^\alpha \equiv dx^\alpha/d\lambda$, $d\lambda = e^{\mp is/L} ds$ and $\Gamma^\mu_{\beta\gamma}$ constructed from $g_{\alpha\beta} = (L/\ell)^{\pm 1} \bar{g}_{\alpha\beta}$, as implied by (6.1). This geodesic, it may be verified, splits naturally into real and imaginary parts:

$$\frac{du^\mu}{ds} + \bar{\Gamma}^\mu_{\beta\gamma} u^\beta u^\gamma = 0 \quad (6.7)$$

$$u^\mu = \bar{g}^{\mu\alpha} \left( g_{\alpha\gamma} \frac{\partial s}{\partial x^\beta} + g_{\alpha\beta} \frac{\partial s}{\partial x^\gamma} - \bar{g}_{\gamma\beta} \frac{\partial s}{\partial x^\alpha} \right) u^\gamma . \quad (6.8)$$

Here $u^\mu \equiv dx^\mu/ds$ and $\bar{\Gamma}^\mu_{\beta\gamma}$ is constructed from $\bar{g}_{\alpha\beta}$. We see from (6.7) that the motion is geodesic in the embedded 4-space. And (6.8) is identically satisfied if $s = \int u_\alpha dx^\alpha$ so $\partial s/\partial x^\alpha = u_\alpha$ as usual for the 4-interval. In other words, the 4D dynamics is standard. Returning now to (6.6), its real part may be rewritten by noting that $u^\alpha u^\beta \partial^2 \psi/\partial x^\alpha \partial x^\beta = \bar{g}^{\alpha\beta} \partial^2 \psi/\partial x^\alpha \partial x^\beta$, which can be shown using (6.5), $u_\alpha = \partial s/\partial x^\alpha$ and $u^\alpha u_\alpha = 1$. Then for the fundamental mode of the Planck gauge, the real part of (6.6) reads

$$\bar{g}^{\alpha\beta} \frac{\partial^2 \psi}{\partial x^\alpha \partial x^\beta} + \frac{m^2 c^2 \psi}{L^2} = 0 \quad (6.9)$$

This is the standard 4D Klein-Gordon equation.

The preceding paragraph started with the complex metric (6.1) and ended with the relativistic wave equation (6.9). Both are statements about
dynamics, and neither uses the field equations. Solutions of the latter of relevant type were mentioned above [26, 42,55-60]. It would be inappropriately long to discuss these here; but to show that there is a match between the dynamics and the field equations, let us take the metric (6.1) in the Planck gauge \((a = 1, b = 1)\) and consider the 5D field equations \(R_{AB} = 0\) \((A, B = 0, 123, 4)\). The latter may be shown to be satisfied, either tardily using algebra [35] or quickly using a computer package [67], by

\[
dS^2 = \left(\frac{L}{\ell}\right)^2 \left(dt^2 - e^{i(\omega t + k_x x)}dx^2 - e^{i(\omega t + k_y y)}dy^2 - e^{i(\omega t + k_z z)}dz^2\right) + \left(\frac{L}{\ell}\right)^4 d\ell.
\]

(6.10)

Here \(k_{xyz}\) are arbitrary wave-numbers along the Cartesian axes and \(\omega\) is a frequency fixed by the field equations as \(\omega = \pm 2/L\). This solution may be regarded as the canonical one in 5D, since not only does the Ricci tensor \(R_{AB}\) vanish but the Riemann tensor \(R_{ABCD}\) does also. That is, (6.10) describes a wave propagating in the 4D part of a flat 5D manifold. However, the 4D part of (6.10) is curved. In the induced-matter picture [11], it is curved by a cosmological constant \(\Lambda = -3\ell^2/L^4\). If the latter is modelled as in general relativity by a classical pressure and density, the wave in 4D is supported by a medium with the equation of state \(8\pi p = -8\pi \rho = -\Lambda = 3\ell^2/L^4\) typical of
the classical vacuum. Another way of appreciating what is involved here is by considering the extra coordinate $\ell$, or equivalently the inertial rest mass of the particle $m$. By (6.2), $\ell$ oscillates in and out of the 4D spacetime hypersurface defined by $s$. The average value of $\ell$ is zero, agreeing with the fact that $R_{ABCD}$ describes a flat 5D manifold; but the average value of its square is finite, agreeing with the fact that the 4D manifold is curved. By (6.10) directly, or by (5.6) modulo a sign due to the change in signature from (5.2) to (6.1), the 4D Ricci scalar is

$$R = \frac{12\ell^2}{L^4} = \frac{12m^2c^2}{h^2} \quad .$$

(6.11)

Here we have taken the fundamental mode in the Planck gauge ($\ell = L = h/mc$). The last relation just says that the scalar curvature of the 4D space is set by the Compton wavelength or mass of the particle which inhabits it. This agrees with Mach’s principle [11], and with the idea from inflationary quantum field theory that particles are intrinsically massless [22]. There is no contradiction, as long as the view is taken that particles with finite rest mass are 4D objects in a 5D vacuum.

The generalization of the above is expected to be straightforward, both for spin–1/2 particles described by the Dirac equation in 5D [68,69]
and spin-0 particles described by scalar wave equations in ND [11,70]. In the latter context, it is clear that the defining equation for classical dynamics in terms of the N-velocities should be $U^A U_A = 0$. The corresponding quantum wave function can be derived from the metric as fixed by solutions of $R_{AB} = 0$ ($A, B = 0, 123, ...N$). Probabilities should be defined from the metric using the element of proper volume $(|g|)^{1/2} dx^N$, so $|g|$ will be a non-trivial factor. [This is already evident in 5D from metrics like (6.10), where the 4D wavefunction may be augmented by $g_{44}$ to yield a 5D one whose extra component will be related to the spectrum of particle masses.] It is expected to report on these issues in the future.

7 Conclusion

In Section 2, we noted that the canonical 5D metric (2.1) justifies its name by providing a basis for $N(>4)D$ geodesics and leading to many useful results. The utility of the canonical (or warp) metric can be understood as a result of embeddings, which on 4D hypersurfaces reduces to physically acceptable solutions for cosmology (3.6) and the 1-body problem (3.7). However, as shown elsewhere in Section 3, its efficacy is mainly due to the fact
that it converts a coordinate space to a momentum space, with the extra coordinate playing the role of rest mass even for null 5D geodesics. In general, the velocity in the extra dimension results in a new inertial force (per unit mass) given by (3.15). This and its associated power (per unit mass) given by (3.16) have no analog in conventional 4D dynamics. The same is true of any non-trivial ND manifold. In Section 4, it was seen that the normalization condition (4.1) results in an extra acceleration parallel to the 4-velocity which has the form (4.6) independent of the coordinate system. This can most logically be handled by connecting it to the change in the (inertial) rest mass of the particle which feels it, which means that the 4D motion is technically non-geodesic in the 4-velocity (4.7), even though it agrees with the conventional law for the conservation of the 4-momentum as applied (say) to the motion of a rocket (4.8). However, the rate of variation of mass in the Minkowski limit is set by the cosmological constant as in (4.11) and is tiny, so for apparently free particles there is no conflict with observed dynamics. In Section 5, the field equations (5.1) were used to constrain the form of a generalized canonical metric (5.2), leading to the recognition of two natural choices for the extra coordinate in (5.7) and (5.8). These are related by an elementary coordinate transformation (5.9), whose physical meaning is
however significant: classical physics uses the gravitational constant of Newton or Einstein to geometrize the mass, while quantum physics uses Planck’s constant to geometrize the mass, as in (5.10). That is, classical and quantum dynamics in 4D are descriptions of 5D dynamics in what can be termed the Einstein and Planck gauges. In both gauges, the extra or parallel force (per unit mass) of (5.15) leads to a relation between the increments in the coordinates and momenta (5.17) which is reminiscent of Heisenberg’s relation. Further study confirms that the extra force - which is inertial is the Einstein sense of coming from the motion of the (extra part of the) coordinate frame - results in Heisenberg’s uncertainty relation (5.20). A corollary of this is that the inertial rest mass of a particle is quantized, though the unit is set by the cosmological constant and is less than $10^{-65}$ gm by (5.21) and so too small to be observed using current methods. These results prompted the brief study in Section 6 of waves in $N(>4)D$. In 5D, what is in effect a Wick rotation of the extra part of the null metric (6.1) was found to lead to a wave function (6.3) which satisfies a hierarchy of wave equations (6.4). The first of these (6.5) is a restatement of the standard prescription for the derivation of 4-momenta from a wave function. The second (6.6) splits naturally into two parts, of which one is a restatement of the geodesic equation of classical
theory (6.7), while the other is equivalent to the Klein-Gordon equation of relativistic quantum theory (6.9). It should be noted that the latter was derived without the use of operators, and of course contains the Schrödinger equation in the non-relativistic limit. More importantly, it should be noted that solutions of the 5D field equations exist which have complex metrics but which result (because of the structure of the field equations) in measurable quantities which are real. An example is (6.10), which is the canonical solution for a wave in a 5D manifold. It describes an oscillation of the extra coordinate in and out of the hypersurface called spacetime. The mass of the particle associated with the wave also oscillates, with a mean value which is zero, in accordance with inflationary quantum field theory and the flatness of the 5D manifold. However, the square of the mass is finite and in fact set by (6.11), which says that the square of the inverse Compton wavelength of the particle is proportional to the Ricci scalar of the embedded and curved 4D manifold. This last property is manifestly Machian, and is expected to be generic to $N(>4)D$ manifolds which contain submanifolds that prescribe the 4D “boxes” which particles inhabit. The general prescription for dynamics in N-dimensional spaces would appear to be a null product of N-vectors (specifying the coordinate velocities in the classical case or the wave
numbers in the quantum case). This means that particles are in causal contact in ND even though they appear to be out of contact in 4D, so there are obvious implications for the Aharanov-Bohm effect and the double-slit experiment. These phenomena, and other consequences for membranes and strings especially, will surely repay further investigation.

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