D-branes and Matrix Theory in Curved Space

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We discuss the relation between supersymmetric gauge theory of branes and supergravity; as it was discovered in D-brane physics, and as it appears in Matrix theory, with emphasis on motion in curved backgrounds. We argue that gauged sigma model Lagrangians can be used as definitions of Matrix theory in curved space.

Lecture given at Strings ’97; June 20, 1997.

1. Introduction

In this lecture, we will discuss a class of quantum mechanical actions which we believe are one appropriate starting point for defining Matrix theory in curved backgrounds. This is in part based on the works [17] and [26], and on work in progress with A. Kato and H. Ooguri. But first, let us give some historical background.

A central lesson from the physics of D-branes [12,40,41] was a new and potentially deep relationship between supersymmetric gauge theory and supergravity. The prototype for this was the computation of the force between two D-branes [40,1]. At leading order in the string coupling, this is found by evaluating the world-sheet path integral on an annulus with one boundary on each D-brane. Although the static force between parallel branes vanishes, by considering velocity-dependent forces or by turning on fields on the branes, one finds non-zero interactions, which can be understood in field theory terms in two ways: either as the sum of all classical closed string exchanges between the branes, or using world-sheet duality, as the sum of one-loop amplitudes in the gauge theory of the open strings ending on the branes:

$$\sum_{\text{closed}} v^\alpha Q_1 Q_2 G(x, y; m_\alpha) =$$

with $v$ velocity, $Q_i$ appropriate charges and $G$ the Green function. The annulus amplitude is quite accessible to explicit computation, and was explored in [34,35] and many other works.

In general this is a relation between two descriptions within string theory, and requires the sum over the entire string spectrum for its validity, just like the modular invariance of closed string amplitudes. However, it was soon discovered that in certain amplitudes with residual supersymmetry (left unbroken by the velocities or field expectation values), the massive string states decouple, and the relation becomes a relation between interactions computed in two field theories: supergravity, and the gauge theory of the lightest open strings stretched between the two branes: maximally supersymmetric Yang-Mills theory in the case of parallel branes in flat space, and more generally an SYM with matter determined by the brane configuration.

The first example of this phenomenon (to our knowledge) was found in [24]. After obtaining $d = 4, N = 4$ SYM from parallel 3-branes, it was natural to ask what D-branes had to say about $N = 2$ SYM, and in that work pure $N = 2$ SYM was obtained by the simple expedient of wrapping 7-branes on K3. It was found that the one-loop prepotential in the gauge theory (which reduces to a sum over BPS states) was equal to the classi-
cal Green function for exchange of massless fields in the supergravity.

One interesting followup (for the issues raised in this talk) to this was \[2\], where it was shown that such gauge theory computations are finite, despite the lack of any explicit UV cutoff, thanks to cancellations between the loop divergences. This is ‘dual’ to the statement that, even in two dimensions, the Greens function (which determines the supergravity amplitude) will not have an IR divergence on a compact space. The result of \[24\] was also what led us to look for and find the analogous relation for the \(v^4/r^7\) interaction between D0-branes in \[23\], and thus the possibility of computing this interaction in gauge theory.

These observations found a natural place, along with other connections between supersymmetric quantum mechanics and supergravity (most notably, the description of the supermembrane developed in \[13\] and first cited in this context in \[13\]), and many other observations in D-brane physics, as part of the far-reaching Matrix theory conjecture of Banks, Fischler, Shenker and Susskind \[4\]. To put this in a nutshell, all of eleven-dimensional physics (or what is visible in the infinite momentum frame) is contained in maximally supersymmetric gauge theory, reduced to quantum mechanics and in the large \(N\) limit. The result of \[23\] then explains the leading long-distance supergravity interaction between D0-branes – it is produced as a one-loop effect in the quantum mechanics.

These observations also play an important role at weak string coupling – but there, they are a special case of a different relation \[24\]: gauge theory replaces gravity for D-branes at substringy distances and low velocities, but can in general give different predictions.

Let us compare the two limits in the context of D-branes in a background with spatial curvature. Both are potentially relevant to low-energy physics. After the comparison, we will concentrate on the large \(R_{11}\) (strong string coupling) limit, which after all is a new and fascinating regime which has become accessible to us by virtue of \[1\], but it is helpful to have the larger picture in mind.

1.1. Weak string coupling – \(l_s >> l_p\)

1. The D-brane world-volume action is defined by world-sheet computations in superstring theory, along the lines of \[13\], so can in principle be computed in any background.

2. At substringy distances, \(r \leq l_s\), gauge theory replaces gravity. The fact that gravity is produced by integrating out stretched strings, and the relation between their mass and separation \(m = T_s r\), implies that the UV limit for gravitational interactions is defined by the IR physics of the branes.

3. Even if the string coupling at infinity is weak, quantum effects on the branes can be enhanced by IR effects. In field theory language, the couplings can grow under renormalization. For D0-branes, loop effects are controlled by the dimensionless parameter \(g_s (\alpha')^{3/2} / r^3 \sim (l_{p11}/r)^3\).

4. A convenient way to study this type of ‘gravity’ is to introduce an auxiliary D-brane ‘probe’ on which open strings can end, and solve its gauge theory. We can interpret its moduli space, or more generally the configuration space visible at low energy, as the space-time geometry. This allows bringing all the techniques of supersymmetric gauge theory to bear, and thus can provide exact results for the metric and other fields.

5. For \(r > l_p\), these results need not agree with the predictions of ten or eleven-dimensional supergravity. In special cases, supersymmetry constrains the Lagrangian to force such agreement, but in general, there is a non-trivial interpolation between the long distance and short distance behavior, with a cross-over at the string scale.

6. At long distances \(r > l_s\), gravity replaces gauge theory – the interaction is better thought of as a sum over closed string states, which at low energies reduces to supergravity. Normally one thinks of the infinite sum over open string states as regulating the open string theory, but we could
phrase this relation in a different way: the UV limit of the gauge theories on the branes is defined by the IR behavior of supergravity.

Let us return to point 5, and the general statement that D-branes see both supergravity and gauge theory in different limits. This is how exact results for the annulus diagram generically behave. An interesting example can be found in the system of a D0-brane and a D6-brane. Although this breaks supersymmetry completely, it does so in a controllable way – the leading interaction is a repulsive potential, with two different limiting behaviors. The supergravity interpretation of the D6-brane is a KK monopole, around which the D0-brane sees a $1/r$ potential, while at short distances the potential is produced by integrating out fermionic stretched strings and has the generic quantum mechanical behavior $V \sim -r$. In general, a non-constant potential in the probe theory corresponds to a non-constant $g_{00}$ component of the probe metric, and we conclude that the D0-brane does not see the KK monopole metric at all scales.

This system is also a good example of the phenomenon (noted in a different context in [7]) of different probes seeing different metrics, as the D2-brane will see the KK monopole metric at all distances. In string theory, this is no contradiction as different probes can have different couplings to the massive closed string states, which from the world-volume point of view also contribute to the metric. Which probe sees “the” metric? In the example at hand, it is the D2-brane, which preserves enough supersymmetry to forbid such couplings, but in general there is no such argument and one would say only that the metric seen by the lightest objects is the most relevant one physically.

When does supersymmetry determine the metric? The essential distinction is between backgrounds breaking half the supersymmetry (e.g. ALE spaces or K3), and those breaking more. Eight real supersymmetries guarantee that the target space is hyperkähler (assuming the non-metric fields are zero) and thus that it satisfies the equation of motion. Four real supersymmetries are only enough to guarantee that it is Kähler, and do not imply specific equations of motion.

In fundamental string theory, the sigma model metric does not in general satisfy the low-energy supergravity equations of motion. These receive corrections

$$0 = \beta^{(g)}_{\mu\nu} = R_{\mu\nu} + \alpha' r^3 R^4 + \ldots$$

The sigma model metric is not directly observable and suffers from renormalization prescription ambiguities, but the D0-brane metric is observable. It is defined by a similar calculation, which at this writing has not been done, but has no reason not to also receive corrections. As Greene described in his lecture here, there is an alternate (D-brane on orbifold) technique for getting at this metric, and the results from this also suggest that it will not be Ricci flat.

1.2. Large $R_{11}$ as defined by Matrix theory

I will be brief, as this has been discussed by many speakers here. Furthermore, some of the points (2, 3 and 4) are the same in both contexts. However we have

1'. At present we can only conjecture that some action (or “base theory”) appropriate for each background exists (more on this in the next section). In general, this may not be the same as any weak coupling D-brane action.

An explicit example of this was given in [26]. At weak string coupling, the properties of D0-branes moving on the K3 manifold are explicitly calculable in the orbifold limit and fairly well understood in general, and using this it was shown that the annulus amplitude could not reproduce the supergravity interaction on K3 without including all excited open string states, and relying on world-sheet duality to relate this to closed string exchange.

What this means is not that Matrix theory cannot be defined on K3, only that keeping only the lightest states in the D0-brane action derived at weak string coupling is

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2It is interesting that by turning on gauge fields on the 6-brane, one can get the $1/r$ behavior at all scales [36].
not a correct definition. Even if the correct action involves the same degrees of freedom, terms not protected by supersymmetry can receive arbitrary corrections in the large $R_{11}$ limit. The non-renormalization arguments of [4] do not generalize to reduced supersymmetry.

5' For $r > l_{p11}$, gauge theory results should agree with predictions of supergravity in the IMF. In particular, on a background with small curvature $R_{p11} << 1$, we should have

1. The background must satisfy the equation of motion
   \[ R_{\mu\nu} = 0 \]  
   (4)

2. The leading interactions between gravitons are those of linearized gravity,
   \[ H_{eff} = v^4 G(x, y) \]  
   (5)
   \[ G(x, y) \sim d \to 0 d^{-1(x,y)} + \frac{R}{d^2} + \frac{R^2}{d^4} + \ldots \]  
   (6)
   where $d = d(x, y)$ is the distance (measured along the shortest geodesic) between the locations $x$ and $y$ of the branes. The short distance expansion for the Green function $G(x, y)$ can be derived using heat kernel techniques [14].

It may be necessary to take the large $N$ limit to get this agreement, as stated in the original proposal. To put this more physically, the elementary states are not just the D0-branes of the weak coupling limit, but bound states involving arbitrary numbers of D0-branes, and it could be that the structure of the bound states is important in the dynamics.

More recently, Susskind has proposed that the finite $N$ gauge theories also have an M theory interpretation, as the theory with a compact light-like dimension (sometimes called DLCQ) [45]. This leads to very strong predictions which have found quite non-trivial support, as described by the Beckers here.

6'. Gauge theory must reproduce gravity at all scales $r > l_p$ with no upper limit. Now the IR behavior of gravity corresponds to UV behavior in the gauge theory, because it is determined by effects from integrating out super-massive states. Therefore, it would seem, this UV behavior must be well-defined.

As we will review shortly, compactification of Matrix theory involves not just quantum mechanics but quantum field theory, and point (6') appears to predict that field theory has far more possible UV limits than we ever dreamed; indeed, one for each possible IR behavior in supergravity.

I would like to advocate a point of view closer to (6) above – eventually, we will come to regard the UV behavior of these theories as defined by the IR behavior of the corresponding gravity. The relation between gauge theory and gravity will be much more symmetric than in the Matrix theory work so far.

This comparison was intended to emphasize the similarities as well as the differences between the two frameworks. Ultimately, if Matrix theory provides a complete formulation, it should be possible to derive all weak string coupling results from it, perhaps along the lines of [16]. In their picture, achieving the weak string coupling limit requires a non-trivial RG flow, which is consistent with point (1') made above.

2. Matrix theory in curved space

The original conjecture of [4] did not claim to define M theory in a general curved background. The curved backgrounds which it does treat are those which can be realized by inserting objects formed out of zero-branes as blocks in the large $N$ matrix, as described here by Dijkgraaf. This can clearly realize time-dependent backgrounds such as propagating gravitational waves. It can also realize some time-independent backgrounds, such as the longitudinal five-brane [6].

It is an interesting question to what extent the original conjecture covers the most general curved background. At present, there is no evidence that time-independent backgrounds with
non-trivial metric but zero three-form tensor, or with non-trivial topology can be described.

On the other hand, one might generalize the original conjecture and propose new “base theories” which serve as the definition of Matrix theory on curved backgrounds. Indeed, the general wisdom about field theory in the infinite momentum frame is that a change in the vacuum must be represented as a change in the light-cone Hamiltonian. An example of this was the proposal in \[8\] to represent the longitudinal five-brane by adding a hypermultiplet to the gauge theory.

The most straightforward way to put the theory in curved space is the following. A single D-brane in curved space will be described by a Nambu-Born-Infeld action, which in the \(\alpha' \to 0\) limit reduces to decoupled super-Maxwell and non-linear sigma model actions. The logical generalization of this to \(N\) D-branes is to promote the sigma model coordinates to matrices, and define Matrix theory using some supersymmetrized version of the action

\[
\int dt \quad \text{tr} \ g_{ij}(X)D_t X^i D_t X^j + \text{tr} \ g_{ij}(X)g_{kl}(X)[X^i, X^k][X^j, X^l].
\]

However, this expression is highly ambiguous, as we must now choose an ordering prescription for the matrices \(X^\mu\). Furthermore, since quantum mechanics has no obvious analog of the renormalizability constraint on quantum field theory, there are an infinite number of higher derivative terms we might add, and reduced supersymmetry gives only weak constraints on these.

What we will shortly propose, is that these ambiguities will be resolved by requiring that the IR gravitational physics be correctly reproduced, and show how this could work for the two points in (3’) above.

Before we do this, let us briefly mention some of the interesting new elements which appear when space has non-trivial topology. Non-trivial homology leads to new conserved charges and new BPS states, such as the membrane and five-brane wrapped around the homology cycle. An attractive feature of Matrix theory is that these are just as fundamental as the original D0-branes. The new conserved charges correspond to topological charges in the gauge theory \(\mathbb{C}\).

In \(\mathbb{F}\), a very simple class of modified Hamiltonians was proposed to describe toroidal compactification: compactifying \(p\) dimensions is accomplished by replacing D0-branes with D\(p\)-branes. The prescription is justified by constructing the torus as a quotient \(\mathbb{R}^p/\mathbb{Z}^p\), where \(\mathbb{Z}^p\) acts both on space-time and on the gauge indices: for each vector \(e_i\) in the \(\mathbb{Z}_n\) lattice we have

\[
X^i + e_i = U^i_j X^j U_j^i.
\]

Now, two new BPS states which arise are the KK states and wrapped strings, which correspond respectively to the electric and magnetic fluxes of the gauge theory. This leads to the beautiful result that T-duality, which only becomes a symmetry after compactifying three dimensions, is exactly S-duality of the underlying \(3+1\) gauge theory! \(\mathbb{G}\).

A similar orbifold prescription can be used to define orbifolds in the traditional string theory sense such as \(T^4/\mathbb{Z}_2\) or \(T^6/\mathbb{Z}_3\). Now an interesting part of this physics is localized to the fixed points, and one can study this in the context of the simpler orbifolds \(\mathbb{R}^4/\mathbb{Z}_2\) or \(\mathbb{R}^6/\mathbb{Z}_3\) \(\mathbb{H}\). As explained here by Greene, these are linear sigma models with Fayet-Iliopoulos terms, whose moduli spaces are smooth ALE spaces asymptotic to the original orbifold.

These models contain more degrees of freedom than the non-linear sigma model \(\mathbb{J}\), and there is a good physical reason for this. Orbifolds are singular limits, but the physics must remain non-singular in this limit. This is possible because the additional quantum mechanical degrees of freedom become massless in the limit.

All of these ALE spaces have non-trivial two-cycles and thus these theories also contain wrapped membranes. As argued in \(\mathbb{K}\), these are also distinct “topological” sectors of the quantum mechanics, realized by modifying the orbifold construction to use general representations of the point group (intuitively, leaving out some of the images of the D0-branes, which produces objects bound to the fixed point).

As described here by Seiberg, the gauge theory prescription encounters difficulties on compactifying more dimensions. This has led to a
fascinating series of works in which a series of six-dimensional string theories have been used to compactify Matrix theory on $T^4$ and $T^5$. A simple argument for the role of string theory on $T^5$ is that the M theory duality group $SO(5,5;\mathbb{Z})$ can be directly identified with the T-duality group of a string theory on $T^5$. \[15\]

Using these theories, Govindarajan \[29\] and Berkooz and Rozali \[9\] have proposed to define string theory compactified on the dual space, again topologically $K3 \times S^1$. As Berkooz described here, this reproduces the appropriate string dualities, notably to the heterotic string, and leads to a simple origin for the additional degrees of freedom in the existing Matrix constructions of the heterotic string.

Let us add to their evidence the comment \[26\] that requiring that the model be non-singular in the orbifold limit also appears to favor string theory over proposals using gauge theory such as \[19\]. Field theories are typically singular in the orbifold limit, and this appears to be the case for $4 + 1$ gauge theory on $T^4/\mathbb{Z}_2$. \[19\] On the other hand, it seems reasonable to hope that the good behavior of string theory on orbifolds will carry over to the six-dimensional string.

3. Gravity from gauge theory

As yet, none of the proposals mentioned in the previous section have passed the two tests that the moduli space should be a symmetric product of Ricci-flat metrics, and that the supergravity interaction should be correctly reproduced. Testing the supergravity interaction is not easy and so we should try to do this in the simplest context possible.

Given target space locality, the simplest models to test will be those with non-compact target space, for several reasons. On the practical side, these metrics and Green functions are much simpler. There is little hope to explicitly write the Green function on K3; even the metric is not known. Conceptually, in order to make the first test, we need to study backgrounds which are not solutions as well as those which are; the meaning of the “dual manifold” used in the compactification constructions is not at all clear in this case.

One can argue with the assumption of target space locality – indeed, Banks emphasized the non-locality of the theory in his talk here. We will discuss this point at length in section 6.

Even the proposed definition on $\mathbb{R}^7 \times \text{ALE}$ studied in \[27\] is more complicated than we want, because of the non-trivial topology. The simplest model to consider is clearly \[19\].

Thus we seek $U(N)$ gauged non-linear sigma models with a specified metric, and which reproduce the supergravity interaction as a one-loop effect. As we said, the first issue in using \[19\] is to resolve the matrix ordering ambiguities. Now for the problem at hand, only a small part of this ambiguity will be important, because we are only going to consider linearized fluctuations around the moduli space (which will again be diagonal matrices) to compute our one-loop amplitude. These will only see terms which are up to second order in commutators $[X^\mu, X^\nu]$.

How will we reproduce the supergravity interaction? To get the leading $v^4/d^7(x,y)$ term at short distances, we need a gauge theory in which the $U(N)$ gauge action is the same as in flat space, $\text{X}^i \rightarrow U \text{X}^i U$, but in which all states which had mass $m \propto r$ in flat space now have mass $m \propto d(x,y)$. More explicitly, whatever form of \[19\] we take, one loop amplitudes will only depend on the expansion of the Lagrangian to quadratic order in the off-diagonal matrix elements $W$ and $\theta$. The mass condition requires this to take the form

$$\mathcal{L}_{od} = K_B \left( D W \right)^2 - K_B d^2(x,y) W^2 + i K_F \theta D \theta + K_F \theta \Gamma_\mu \Gamma^\mu m_F \theta,$$

where $K_B$ and $K_F$ are arbitrary functions of the curved space positions, and $\Gamma_\mu m^\mu_F$ is a matrix with eigenvalues $\pm d(x,y)$.

If the velocity $v$ and polarizations are purely in the flat directions, we can remove $K_B$ and $K_F$ by rescaling the fields. Then, since the gauge coupling is universal, the one-loop gauge theory computation of \[23\] proceeds in exactly the same way, and enjoys the same supersymmetric cancellations, with the only difference being the replacement $m \rightarrow d(x,y)$.\[23\]
A similar computation can be done for velocity $v$ or polarizations in the curved dimensions, and getting this right requires additional conditions relating the boson and fermion kinetic terms, which remain to be formulated precisely.

The discussion so far allows us to formulate necessary conditions for our model to pass the two tests. Indeed, the condition on the mass of the stretched strings is intuitively obvious; the only surprise is that this is not automatic in the gauge theory description.

4. D-geometry

Let us state the problem in a self-contained way which we could give to a mathematician: Given a $d$-dimensional manifold with metric $\mathcal{M}$, find a $U(N)$ gauged non-linear sigma model satisfying the axioms below.

The low energy action will be determined by a configuration space $X_N$, a $dN^2$-dimensional manifold with metric; an action of $U(N)$ by isometries; and a potential $V$. The axioms are then

1. The classical moduli space,
   \[ \{X_N|V'=0\}/U(N), \]
   is the symmetric product $\mathcal{M}^N/S_N$.

2. The generic unbroken gauge symmetry is $U(1)^N$, while if two branes coincide the unbroken symmetry is $U(2) \times U(1)^{N-2}$, and so on.

3. Given two non-coincident branes at points $p_i \neq p_j$, all states charged under $U(1)_i \times U(1)_j$ have mass $m_{ij} = d(p_i,p_j)$.

4. The action is a single trace (in terms of matrix coordinates),
   \[ S = \text{tr}(\cdots). \]  
   (10)

The last axiom is familiar in the leading order of open string perturbation theory (it follows from the definition of Chan-Paton factors and the disk topology of the world-sheet). It is also appropriate for Matrix theory, both so that the action for block-diagonal matrices will be the sum of that for the individual blocks, and to get the correct relativistic dispersion relation for bound states. It is a non-trivial constraint, as was pointed out by Tseytlin \[ \text{(1)} \] in the context of the non-abelian Born-Infeld action.

We can give a physical proof that a solution to the problem exists, in the case that the background is a solution of the $\alpha' \to 0$ limit of string theory – just consider D-branes in this background. Although the stretched strings in this case have masses far above the string scale, the spacing to the first excited state stays finite and it will still be true that for sufficiently low energy processes (or length scales $L \gg r$) a field theory description is appropriate. The axioms can be proven in this context, as the masses of stretched strings are entirely classical. On the other hand, $\alpha'$ corrections could violate the axioms, in particular axiom 3.

Note that we did not state as an axiom the $U(N)$ gauge action $X^i \to U^\dagger X^i U$. We believe that this can be derived, in the following sense. When we write an explicit sigma model with matrix coordinates, we have implicitly chosen a coordinate system for the off-diagonal components. For any given coordinate system on $\mathcal{M}$, the conjecture is that there exists a choice of matrix coordinates for which the gauge action will take this form.

To simplify the problem and get further constraints one can assume additional supersymmetry. In [1], four real supersymmetries ($N = 1, d = 4$) were assumed, so the target space must be a Kähler manifold. Thus the problem becomes, given a Kähler potential $K(Z^i, \bar{Z}^\dagger)$ on $\mathcal{M}$, find a Kähler potential $\text{tr} K_N(Z^i, \bar{Z}^\dagger)$ and superpotential satisfying the axioms.

We also started with the simplest possible case of one complex dimension, so the action in this case is a $U(N)$ sigma model with a single matrix chiral superfield. By dimensional reduction, a Lagrangian for D0-branes moving in $3+1$ flat and 2 curved real dimensions can be obtained.

To give the idea of the analysis, we show how the condition on the masses of off-diagonal gauge bosons is realized. Given the free gauge kinetic

\[ ^3 \text{An observation of Steve Shenker.} \]
term $\Re \int d^2 \theta W^2$ (in string language, this is constant dilaton), the mass term for the $ij$ gauge boson is

$$\frac{\partial^2 \text{tr} K}{\partial Z_{mn} \partial \bar{Z}_{nm}} [A, Z]\, [A, Z^\dagger]_{nm}.$$  \hspace{1cm} (11)

Expanding around diagonal matrices, one sees that the second derivative will be a function of the eigenvalues $z_i \equiv Z_{ii}$:

$$K''_N (z_m, \bar{z}_m, z_n, \bar{z}_n) = \frac{\partial^2 \text{tr} K}{\partial Z_{mn} \partial \bar{Z}_{nm}}$$  \hspace{1cm} (12)

determined by the ordering prescription; for example

$$K_N = ZZ\bar{Z}\bar{Z} \rightarrow K''_N = (z_m + z_n)(\bar{z}_m + \bar{z}_n)$$

$$K_N = ZZ\bar{Z}\bar{Z} \rightarrow K''_N = 2(z_m \bar{z}_m + z_n \bar{z}_n).$$

Computing the commutators in (11) we require

$$K''_N |z_m - z_n|^2 = d^2 (z_m, z_n)$$  \hspace{1cm} (13)

which we solve for $K''_N$ and hence for the terms in $K_N$ with up to two commutators.

At leading order in a normal coordinate expansion, the result is

$$K_N = \text{tr} |Z|^2 - \frac{R}{4} \text{Str} Z^2 \bar{Z}^2 + \ldots$$  \hspace{1cm} (14)

where $\text{Str}$ is the symmetrized trace, normalized as $\text{Str} X^k = \text{tr} X^k$.

The same ideas can be implemented in ten dimensions. In the framework of $\mathcal{N} = 1$, $d = 4$ gauge theory, we can describe three complex transverse dimensions on a general Kähler target space, and A. Kato, H. Ooguri and I are in the process of working out these actions [22].

Rather surprisingly, it appears that the axioms of D-geometry have no solution in this case unless the manifold $\mathcal{M}$ is Ricci flat! The condition that the masses of all strings (with any polarization) stretched between two D-branes have the same mass (“the isotropic mass condition”) is quite strong and cannot in general be accomplished with a holomorphic superpotential.

It seems likely that this result is another expression of the well-known fact that in compactifications of $d = 10$ supergravity to $d = 4$, the background will admit $\mathcal{N} = 2$ supersymmetry, allowing brane solutions with $\mathcal{N} = 1$, only if it is Ricci flat. Given the relation between these Lagrangians and supermembrane theory [13], perhaps it can be related to the standard arguments in this context. A test of this idea will be to check that, dropping the assumption of supersymmetry, the problem can be solved for general metrics.

This argument for Ricci flatness is known to be modified by corrections to the low energy supergravity Lagrangian (such as the $\alpha'^3$ term we mentioned in the superstring-derived Lagrangian) and thus it does not seem that this proves that Dp-brane metrics at weak string coupling must be Ricci flat. We do see that $\alpha'$ corrections to the metric must come with corrections to the isotropic mass condition.

It would be interesting to generalize this to target spaces of non-trivial topology. For example, the question discussed in [26] – what is the correct Lagrangian for D-branes in an ALE space, reproducing the supergravity interaction but remaining non-singular in the orbifold limit? – should be solvable along these lines.

One would also like to find physical arguments for the higher order commutator terms. An interesting example of this is [32].

5. Renormalization group in Matrix theory

The proposal we will now consider is that a gauged sigma model of the form we just described can be used as a definition of Matrix theory in curved but non-compact eleven dimensions. Although we have not completely specified the model, if it satisfies the axioms, it will reproduce the leading behavior of the supergravity interaction, at least for velocity $v$ in the flat directions.

What about the equation of motion? Clearly, we must consider the whole family of models for all $N$ (or else take the large $N$ limit) to see how the gauge theory could know about this. After all, the model with $N = 1$ is perfectly unitary and consistent with any target space metric, so it does not know about the supergravity equation.

[1] O. Aharony, S. Kachru and E. Silverstein (unpublished) have made this observation in the context of brane probe theories, along with the caveat in the next paragraph.
of motion.

The story is potentially more interesting for \( N \geq 2 \) as quantum corrections can always become large as branes coincide. In quantum mechanics, these are controlled by \( g_s/v^3 \).

One concrete proposal for the physical consequences of this is that since we are seeking models which have a good large \( N \) limit in the sense of \[10\], we should try to formulate a large \( N \) renormalization group, whose basic operation is to integrate out a row and column of the matrix. Such an approach was first used by Brezin and Zinn-Justin \[10\] for the original matrix models of random surfaces, where it led to good qualitative results for the critical behavior. Further motivation for this idea is the relation to string theory. \[10\] If we compactify another dimension, the resulting 1+1 theory renormalization, and fixed points will satisfy the equations of motion. \[10\]

We thus start with an action for \( N \) D0-branes of the type described in the previous section, and decompose each matrix into an \((N-1) \times (N-1)\) matrix \( X \), an \( N-1 \) component vector \( v \) and the position of the \( N \)'th brane \( x_N \):

\[
\begin{pmatrix} X & v \\ v^\dagger & x_N \end{pmatrix}
\]  
(15)

We then integrate out the \( N \)'th D0-brane in two steps. First, we integrate out the vector \( v \). As long as \( x_i \neq x_j \forall i \neq j \), this is a completely well-defined problem.

Let us consider the one loop renormalization of the metric, which should give a good description for small curvature \( R_{\mu\nu}^2/\ell_{p,11}^4 \ll 1 \). We would like to start with a general target space metric and from the previous discussion, this will require using ten-dimensional actions with no supersymmetry assumed. To illustrate the idea, let us grant that such actions exist with a kinetic term of the same form as \[14\] – in normal coordinates,

\[
\text{tr} \left( D_t X^i \right)^2 - \frac{1}{3} R_{ijkl}(0) \text{Str} \left( D X^i X^j D X^k X^l + \ldots \right)
\]  
(16)

In flat ten-dimensional space, the action has maximal supersymmetry, and zero metric renormalization. Therefore the leading renormalization of the metric will come from a one loop diagram with one insertion of the leading supersymmetry breaking operator, the curvature operator in \[10\]. (From \[1\], there is also an \( RX^6 \) term in the potential, which does not contribute at one loop.)

This leads to the same one-loop diagram as in the usual two-dimensional sigma model renormalization group, but with a different propagator:

\[
\delta L = g^2 \sum_{\mu} D_{\mu} X^i_n D_{\mu} X^i_n R_{ijkl}(0) g^{ij}(0) \quad (17)
\]

\[
\times \int \frac{dk}{k^2} \left( \frac{\delta k}{|x_n - x_N|^2} \right) + \ldots
\]

The result is no longer UV divergent, and the IR divergence is controlled for \( x_N \neq x_n \).

The second step would be to integrate over \( x_N \). Now the IR divergence at \( x_N = x_n \) is not physical, because the bound state wave function is not singular. Completing the definition of the RG requires cutting off the IR divergence, and arguing that physical quantities are independent of this cutoff. We do not know enough about the bound states to make this precise at present, but it is quite plausible that for small curvature this works the same way as in flat space, leading to a universal result, on dimensional grounds

\[
\delta L = \frac{g^2}{\ell_{p,11}} R_{ij}(0) \text{tr} D_t X^j D_t X^j + \ldots
\]  
(18)

Whether we should be able to make sense of flow towards a fixed point in this framework is not yet clear (it did not have a clear interpretation in light-cone string theory, either). What we can say is that the fixed points will be Ricci flat manifolds, at this order.

6. The final formulation and locality

Although we have argued that gauge theory can reproduce the equation of motion and the leading behavior of the supergravity interaction, this does not complete our two tests – we need to reproduce the exact Green function. The leading correction in \[3\] vanishes on a Ricci flat manifold, but the \( R^2/d^4 \) term does not.

In \[20\] it was shown that this is not possible with a simple truncation of the weakly coupled open string theory, or indeed with any model whose expansion to quadratic order takes the form \[3\]. It is necessary to add additional
terms to the Lagrangian, perhaps higher derivative terms. However, we cannot just add the long-range interactions explicitly to the Lagrangian, as they are singular as $r \to 0$.

Suppose this were not possible – what would we conclude? Perhaps the simplest way out would be to assert that supergravity is reproduced only in the large $N$ limit, as in the original conjecture, and that getting the subleading interactions right requires detailed understanding of the bound states.

The other out is to assert that these problems cannot be studied in non-compact backgrounds. Rather, one must embed the background in a compact background, and even in the limit that its volume goes to infinity, we need to keep the additional degrees of freedom. This cannot be true in the gauge theory definitions of Matrix theory, where these states have energies going to infinity in the limit, but perhaps if sufficiently exotic base theories are used to define Matrix theory this out will need to be reconsidered.

An interesting related idea is that compactified theories will be more constrained than the uncompactified theory, since the base theory is higher dimensional. Certainly the number of relevant and marginal perturbations around the Gaussian limit decreases as we go up in dimension. This leads to the idea that field theories in higher dimensions with sensible UV limits are few and far between, perhaps only coming in a few series, like Lie algebras.

Let us consider the consequences of the modest assumption, that there exist finitely many such series of field theories with well-defined UV limits. Perhaps the simplest is that not all solutions of the supergravity equations of motion can actually be realized as backgrounds of the theory! Let us consider a non-compact space; we would be saying that only the spaces which are subspaces of the spaces on our finite list can actually be realized as backgrounds. There may be some room to extend the list by allowing “objects” in the background; as we mentioned earlier it is not clear that one can make pure deformations of the metric in this way.

This is a highly nonlocal constraint and as such extremely interesting. Indeed, such nonlocal effects might shed a new light on problems such as vacuum selection and the vanishing of the cosmological constant, which have resisted real understanding in the context of local physics.

But do we really believe it? A less radical interpretation is just that any solution of supergravity is an allowed background, and the finite list of theories comes from a finite list of compact manifolds which admit such solutions. On the other hand, there could be infinitely many theories with non-compact moduli spaces.

Still, since locality is not at all manifest in Matrix theory, we should not dismiss such ideas out of hand. However, I will argue that so far, we have no good reason to believe in non-locality in Matrix theory for low energy processes in a time-independent background. Let us examine the arguments one might make.

One argument is the lack of any manifest locality in the underlying quantum mechanics. In particular, the energies of the off-diagonal modes (stretched strings) are functions of two D-brane positions, leading to apparently instantaneous interactions between the D-branes.

However, on reflection, we remember that these interactions are supposed to be one component of the gravitational interaction, which we know is local. This type of apparent non-locality is familiar in gauge theories – for example, in Coulomb gauge, although there are explicit non-local interactions in the Hamiltonian, all non-causal effects are cancelled by other interactions, leading to a causal theory. The new element here is that the interactions which we would try to “un-gauge fix” to get a manifestly local formulation are produced as quantum effects, making it unclear how to realize this locality – at present.

A better argument is the existence of fundamental extended objects in the theory. Does the

5 One might think that Seiberg’s zero string coupling limit of five-brane theories could be used to construct such an infinite set of theories, by putting the five-branes at points in a general Ricci-flat manifold $M_4$ and repeating the construction. This is not true, essentially because they are bound (in the language of the defining gauge theory is on the Higgs branch) and cannot separate from each other to explore the $M_4$ metric.

6 This section is expanded from the original talk; I found a useful foil.
need to introduce new fundamental objects at each stage of compactification mean that the theory was non-local? Yes, in the sense that we don’t yet know how to define the compact theory given the local definition of theory. But, I would claim, no, not in the usual sense of the word, that one point can influence another without something propagating in between.

To define locality in a theory of extended objects, we must allow for the possibility that the extended objects themselves have internal degrees of freedom which can be localized. Once we accept this, it is not obvious why having more than one topological class of fundamental object is essentially more difficult.

A final motivation is the derived nature of space-time in Matrix theory – it arises as the moduli space of a supersymmetric gauge theory, just as for D-branes at short distances. Since space-time is not fundamental, we can even imagine situations with no space-time interpretation – as Seiberg pointed out in his talk here, this may be the general situation when the base theory is strongly coupled. How can we say that such a theory is local in space-time?

Obviously we don’t know how to say it, but that does not mean that we will never know how to say it – this is an issue which will take time to understand. Again, this issue already arose in perturbative string theory; non-linear sigma models with highly curved target spaces ($\alpha' R >> 1$) are strongly coupled and can be equivalent to conformal field theory constructions with no obvious space-time description; nevertheless geometric descriptions have been found in many cases, as Greene discussed here.

It is hard to prove a no go theorem; in this case the suggestion that there are compactifications with no definition of locality. However, superstring duality provides many examples of models with multiple space-time interpretations. How could the natural ideas of locality in the various large volume limits all be valid?

The simplest proposal one could make for a definition of locality in this situation is a principle of “simultaneous locality,” which asserts the following:

For every large volume limit of the space-time there exists a definition of locality, which agrees with the conventional one at large distances but can be extended to cover the entire parameter space of vacua. All of these definitions will be exactly valid in all regimes.

What makes this idea not obviously wrong is that a given space-time has at most a single large volume interpretation. All the other definitions of locality will degenerate and lead to no constraints at low energies. However, since they are supposed to be exactly valid, they will lead to constraints at high energies.

While on this subject, I cannot resist mentioning a striking property: although nothing in the proposal seems to require it, many (and perhaps all) of the “base theories” used to define Matrix theory are local. This locality does not seem to be observable in the large $N$ limit, but it is certainly visible at finite $N$, and it will be interesting to interpret this in Susskind’s DLCQ proposal.

7. Conclusions

We know how to study the behavior of D-branes in curved space at weak string coupling. At sub-stringy distances these questions can be reformulated in terms of the world-volume gauge theory, which is the context which has been most studied, but this is no more the general prescription than supergravity was. A framework valid at all distances has been proposed, but there are many questions which remain to be answered in this framework, even very basic ones such as what metric is seen by D-branes on various curved spaces. We have numerous pieces of evidence that this metric does not always satisfy the low energy equations of motion of supergravity.

There are even interesting questions about the behavior of D-branes in curved space in the $\alpha' \to 0$ limit, where supergravity is a good descrip-
tion. The first question is to write world-volume theories which reproduce the known physics of enhanced gauge symmetry. It turns out that very general consistency conditions provide much stronger constraints on these actions than one might have guessed, determining the leading non-abelian terms uniquely in the simplest case.

Matrix theory in curved space is not yet understood. There is no general proposal for how to do it, and problems have arisen in the detailed comparison of many conjectures with supergravity results. On the other hand, some spectacular successes in flat space motivate continued efforts to try.

The most straightforward approach is to adapt the non-linear sigma model approach which we know in string theory. We proposed here that actions for D-branes in a curved background in the $\alpha' \to 0$ limit are a valid starting point which include all necessary degrees of freedom in the case of non-compact space with trivial topology, and we found that such actions can reproduce the leading behavior of the gravitational interaction as a one-loop quantum effect.

These actions are only a starting point as there is no reason that the precise weak coupling string theory action should work in the large $R_{11}$ limit, but the correct actions will be determined (perhaps uniquely) by checking that they precisely reproduce supergravity predictions. The ability to do this at finite $N$ should provide another test of Susskind’s DLCQ conjecture.

This might serve as a guiding principle for defining the higher dimensional base theories corresponding to compactifying further flat directions as well. For example, we could consider Matrix theory on $M^4 \times T^3 \times \mathbb{R}^4$ to get non-linear sigma models in $3+1$ dimensions. Although these sigma models appear highly non-renormalizable, the fact that these spaces are sensible solutions of supergravity suggests that they exist as sensible field theories all the way up to energies $E \sim L \to \infty$ in the non-compact limit. Perhaps we will eventually regard their seemingly ill-defined UV behavior as defined by IR predictions in supergravity.

One can also ask whether taking the large $N$ limit independently determines the action or leads to new consistency conditions. We described one framework in which one can study this issue, a large $N$ renormalization group analogous to the string world-sheet renormalization group, and found evidence that its fixed points would have Ricci flat metrics.

The approach we are following might be regarded as an “effective theory” approach and leaves open the question of what fundamental principles determine these actions, but the study of these effective theories should provide valuable information about what these fundamental principles might be.

It is a pleasure to acknowledge fruitful collaborations on these topics with M. Berkooz, D. Finnell, B. Greene, D. Kabat, A. Kato, M. Li, G. Moore, D. R. Morrison, H. Ooguri, J. Polchinski, P. Pouliot, S. H. Shenker, and A. Strominger; and valuable discussions with P. Aspinwall, C. Bachas, T. Banks, B. de Wit, D.-E. Diaconescu, M. Green, S. Kachru, J. Maldacena, J. Schwarz, N. Seiberg, E. Silverstein, L. Susskind, P. K. Townsend and E. Witten.

This research was supported in part by DOE grant DE-FG02-96ER40959.

REFERENCES

1. C. Bachas, Phys. Lett. B374 (1996) 37-42; [hep-th/9511043].
2. C. Bachas and C. Fabre; Nucl.Phys. B476 (1996) 418; [hep-th/9605028].
3. T. Banks, “The State of Matrix Theory,” this proceedings; [hep-th/9706168].
4. T. Banks, W. Fischler, S. H. Shenker and L. Susskind, Phys. Rev. D55 (1997) 5112-5128; [hep-th/9610043].
5. T. Banks and N. Seiberg; [hep-th/9702187].
6. T. Banks, N. Seiberg and S. H. Shenker, Nucl. Phys. B490 (1997) 91-106; [hep-th/9612157].
7. T. Banks, N. Seiberg and E. Silverstein, Phys.Lett. B401 (1997) 30-37; [hep-th/9703092].
8. M. Berkooz and M. R. Douglas, Phys. Lett. B395 (1997) 196-202; [hep-th/9610230].
9. M. Berkooz and M. Rozali, [hep-th/9705175].
10. E. Brezin and J. Zinn-Justin, Phys. Lett. B288 (1992) 54; hep-th/9206033.
11. C.G. Callan, E. Martinec, M. Perry and D. Friedan, Nucl. Phys. B262 (1985) 593.
12. J. Dai, R. G. Leigh and J. Polchinski, Mod. Phys. Lett. A4 (1989) 2073.
13. B. de Wit, J. Hoppe and H. Nicolai, Nucl. Phys. B305 [FS23] (1988) 545.
14. B. de Witt, Phys. Rep. 19C (1975) 295.
15. R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. B486 (1997) 77, 89; hep-th/9603126 and 9604055.
16. R. Dijkgraaf, E. Verlinde and H. Verlinde, hep-th/9703030.
17. M. R. Douglas, “D-branes in curved space,” hep-th/9703035.
18. M. R. Douglas, “Enhanced Gauge Symmetry in M(atrix) Theory,” hep-th/9612120.
19. M. R. Douglas and D. Finnell, unpublished.
20. M. R. Douglas, and B. Greene, hep-th/9707214.
21. M. R. Douglas, B. Greene and D. R. Morrison, hep-th/9704151.
22. M. R. Douglas, A. Kato and H. Ooguri, to appear.
23. M. R. Douglas, D. Kabat, P. Pouliot and S. H. Shenker, Nucl. Phys. B485 (1997) 85-127; hep-th/9608024.
24. M. R. Douglas and M. Li, “D-Brane Realization of N=2 Super Yang-Mills Theory in Four Dimensions,” hep-th/9604041.
25. M. R. Douglas and G. Moore, hep-th/9603167.
26. M. R. Douglas, H. Ooguri and S. H. Shenker, Phys. Lett. B402 (1997) 36-42, hep-th/9702203.
27. W. Fischler and R. Rajaraman, hep-th/9704123.
28. O. Ganor, S. Ramgoolam and W. Taylor, Nucl. Phys. B492 (1997) 191-204; hep-th/9611202.
29. S. Govindarajan, hep-th/9705113.
30. M. T. Grisaru, A. E. M. Van de Ven and D. Zanon, Nucl. Phys. B277 (1986) 388 and 409.
31. D. J. Gross and E. Witten, Nucl. Phys. B277 (1986) 1.
32. A. Hashimoto and W. Taylor, hep-th/9703217.
33. R. G. Leigh, Mod. Phys. Lett. A4 (1989) 2767.
34. M. Li, Nucl. Phys. B460 (1996) 351; hep-th/9510161.
35. G. Lifschytz, Phys. Lett. B388 (1996) 720; hep-th/9604156.
36. G. Lifschytz, hep-th/9612223.
37. C. Lovelace, Phys. Lett. 155B (1984) 75.
38. D. Lowe, J. Polchinski, L. Susskind, L. Thorlacius and J. Uglum, Phys. Rev. D52 (1995) 6997, hep-th/9506138.
39. E. Martinec, hep-th/9311129.
40. J. Polchinski, Phys. Rev. Lett. 75 (1997) 4724; hep-th/9510017.
41. J. Polchinski, “TASI Lectures on D-Branes,” hep-th/9611050.
42. J. Polchinski, Phys. Rev. D55 (1997) 6423-6428, hep-th/9606163.
43. N. Seiberg, hep-th/9705221.
44. N. Seiberg and S. Sethi, hep-th/9708083.
45. L. Susskind, hep-th/9704080.
46. L. Susskind, hep-th/9611164.
47. W. Taylor, Phys. Lett. B394 (1997) 283-287; hep-th/9611042.
48. P. K. Townsend, Phys. Lett. B373 (1996) 68; hep-th/9512062.
49. A. Tseytlin, hep-th/9701125.