GRAPHS OF GONALITY THREE

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Abstract. In 2013, Chan classified all metric hyperelliptic graphs, proving that divisorial gonality and geometric gonality are equivalent in the hyperelliptic case. We show that such a classification extends to combinatorial graphs of divisorial gonality three, under certain edge-connectivity assumptions. We also give a construction for graphs of divisorial gonality three, and provide conditions for determining when a graph is not of divisorial gonality three.

1. Introduction

Tropical geometry studies graphs as discrete analogues of algebraic curves. A motivating goal of this program is to prove theorems in algebraic geometry using combinatorial methods, as in [9]. In [2], Baker and Norine define a theory of divisors on combinatorial graphs similar to divisor theory on curves, proving a Riemann-Roch type theorem. This was extended by [14] and [17] to metric graphs, which have lengths associated to each edge. To model maps between curves, harmonic morphisms between simple graphs were introduced in [15], extended to multigraphs in [3], and finally to metric graphs in [7].

An important invariant of an algebraic curve is its gonality. This is the minimum degree of a divisor of rank 1, or equivalently, the minimum degree of a morphism from the curve to a line [12, Section 8C]. We can extend these definitions to combinatorial and metric graphs, using either divisor theory or morphisms from the graph to a tree. However, unlike in classical algebraic geometry, these two notions of gonality defined on graphs are in general inequivalent, as demonstrated in [10]. We thus define two different types of gonality: divisorial gonality and geometric gonality. (Whenever we refer to the gonality of a graph without specifying which type, we mean the divisorial gonality.)

Our two notions of gonality happen to agree when either is equal to 1: divisorial gonality is equal to 1 if and only if the graph is a tree, and the same is true of geometric gonality [3, Lemma 1.1 and Example 3.3]. This no longer holds when our graph has higher divisorial gonality; for example, the banana graph, which has two vertices and $n \geq 2$ edges connecting the two vertices, has divisorial gonality 2 and geometric gonality $n$ [10]. However, this turns out to be the only such example for graphs of divisorial gonality 2, as shown by the following result.

Theorem 1.1 (Theorem 1.3 in [7]). Let $\Gamma$ be a metric graph with no points of valence 1 and canonical loopless model $(G, \ell)$. Then the following are equivalent:

1. $G$ has (divisorial) gonality 2.
2. There exists an involution $i : G \to G$ such that $G/i$ is a tree.
3. There exists a non-degenerate harmonic morphism $\varphi : G \to T$ where $\deg(\varphi) = 2$ and $T$ is a tree, or $|V(G)| = 2$.

Note that the only (connected) graphs $G$ with $|V(G)| = 2$ are those belonging to the family of banana graphs. Hence, Theorem 1.1 implies that, for all other metric graphs, having divisorial gonality 2 and having geometric gonality 2 are equivalent.

Our main result in this paper is an analogue of Theorem 1.1 for graphs of divisorial gonality 3. Although Theorem 1.1 is stated for metric graphs, ours holds only for combinatorial graphs, without the data of lengths associated to the edges. Equivalently, due to Theorem 1.3 in [10], our result holds for metric graphs with all edge lengths equal to 1.

Theorem 3.2. If $G$ is a simple 3-edge-connected combinatorial graph, then the following are equivalent:

1. $G$ has (divisorial) gonality 3.
2. There exists a cyclic automorphism $\sigma : G \to G$ of order 3 that does not fix any edge of $G$ satisfying the property that $G/\sigma$ is a tree.
Theorem 3.2 to multigraphs. Finally, in Section 5, we present a construction for some, but not all, graphs and give a criterion for identifying graphs with gonality strictly greater than 3. In Section 4, we extend results on divisors and harmonic morphisms of graphs. In Section 3, we prove Theorem 3.2 for simple graphs.

We use $\Delta : \text{Div} (G) \to \text{Div} (G)$ with vertex set $V(G)$ and edge set $E(G)$ to be a finite, connected, loopless, combinatorial multigraph. Graphs with no multiedges are called simple. Given a vertex $v \in V(G)$ and an edge $e \in E(G)$, we use the notation $v \in e$ to indicate that $v$ is an endpoint of $e$. Let $E(u,v) := \{ e \in E(G) : u \in e, v \in e \}$ for $u,v \in V(G)$ and let $E(A,B) := \{ e \in E(G) : e \in E(a,b) \text{ for some } a \in A, b \in B \}$ for $A,B \subset V(G)$. The valence of a vertex $v \in V(G)$ is defined as $\text{val}(v) := \# \{ e \in E(G) : v \in e \}$. We define the genus of a graph $G$ as $g(G) := |E(G)| - |V(G)| + 1$.

A graph $G = (V,E)$ is $k$-edge-connected if, for any set $W$ of $k - 1$ edges, the subgraph $(V,E - W)$ is connected. A bridge of $G$ is an edge whose deletion strictly increases the number of connected components of $G$. A graph is bridgeless if it has no bridges, or equivalently if it is 2-edge-connected. A tree is a graph of genus 0, or equivalently a graph with no cycles.

2. Definitions and Notation

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2.1. Divisor Theory on Graphs. We now review the key concepts of divisor theory on graphs, as developed in [1]. A divisor $D$ on a graph $G$ is a $\mathbb{Z}$-linear combination of vertices. We will often explicitly write out divisors with the notation

$$D = \sum_{v \in V(G)} D(v) \cdot (v),$$

where $D(v)$ denotes the value of $D$ at $v$. The set of all divisors $\text{Div}(G)$ forms an abelian group under component-wise addition. The degree of a divisor $D$ is defined as the sum of its integer coefficients:

$$\text{deg}(D) := \sum_{v \in V(G)} D(v).$$

For a fixed $k \in \mathbb{Z}_{>0}$, let $\text{Div}^k(G)$ be the set of all divisors of degree $k$ on $G$. A divisor $D$ is effective if, for all $v \in V(G)$, $D(v) \geq 0$. Let $\text{Div}_+(G)$ be the set of all effective divisors on a graph $G$ and for $k \in \mathbb{Z}_{>0}$, let $\text{Div}_+^k(G)$ be the set of all effective divisors of degree $k$ on $G$. For a given effective divisor $D$, we define the support of $D$ as

$$\text{supp}(D) := \{ v \in V(G) : D(v) > 0 \}.$$

The Laplacian $\mathcal{L}(G)$ of a graph $G$ is the $|V| \times |V|$ matrix with entries

$$\mathcal{L}_{v,w} = \begin{cases} \text{val}(v) & \text{if } v = w \\ -|E(v,w)| & \text{if } v \neq w. \end{cases}$$

We use $\Delta : \text{Div}(G) \to \text{Div}(G)$ to denote the Laplace operator associated with the Laplacian matrix. A principal divisor is a divisor in the image of $\Delta$. We use $\text{Prin}(G)$ to denote the set of principal divisors on a graph $G$, i.e., $\text{Prin}(G) = \Delta(\text{Div}(G))$. Notice that $\text{Prin}(G)$ is a subgroup of $\text{Div}^0(G)$. We can therefore define the Jacobian $\text{Jac}(G)$ of a graph $G$ as the quotient group $\text{Div}^0(G)/\text{Prin}(G)$.

Now, define an equivalence relation $\sim$ on divisors such that $D \sim D'$ if and only if $D - E \in \text{Prin}(G)$. We say in this case that $D$ and $E$ are linearly equivalent and define the linear system associated with a divisor $D$ as

$$|D| := \{ E \in \text{Div}_+(G) : E \sim D \}.$$

For a divisor $D \in \text{Div}(G)$, we define the rank of $D$ as $r(D) := -1$ if $|D| = \emptyset$, and otherwise as

$$r(D) := \max \{ k \in \mathbb{Z} : |D - F| \neq \emptyset \text{ for all } F \in \text{Div}_+^k(G) \}.$$

The gonality of a graph $G$ is defined as

$$\text{gon}(G) := \min \{ \text{deg}(D) : D \in \text{Div}_+(G), r(D) \geq 1 \}.$$
Later, when we need to distinguish between two types of gonality, this will be referred to as \textit{divisorial gonality}.

2.2. \textbf{Baker-Norine Chip-Firing.} The definition of gonality we have provided has an equivalent statement in terms of chip-firing games on graphs. Given a graph \(G\) and an initial placement of chips on the vertices of \(G\), a \textit{chip-firing move} from a vertex \(v \in V(G)\) is made by subtracting \(\text{val}(v)\) chips from \(v\) and adding \(|E(v, v')|\) chips to all adjacent vertices \(v' \in V(G)\).

The \textit{Baker-Norine chip-firing game} is played with the following rules.

1. The player places \(k\) chips on the vertices \(V(G)\) of a graph \(G\).
2. The opponent chooses a vertex \(v \in V(G)\) from which to subtract a chip. They can subtract a chip from a vertex without any chips, placing that vertex “in debt”.
3. The player attempts to reach a configuration of chips where no vertex is in debt via a sequence of chip-firing moves.

Notice that these “chip configurations” correspond to divisors on graphs. By standard results as in [2], chip-firing moves correspond to subtracting principal divisors; the divisors present before and after chip-firing are equivalent; and the gonality of a graph is equivalent to the minimum number of chips required to guarantee a winning strategy in the Baker-Norine chip-firing game. Hence, we define a \textit{winning divisor} \(D\) to be a divisor satisfying \(r(D) \geq 1\). Since chip-firing is a commutative operation, we can chip-fire from an entire subset \(A \subset V(G)\) at once by sending a chip along each edge outgoing from the subset. Let \(\mathbb{I}_A\) denote the indicator function on \(A\). Then, given a divisor \(D\), the resulting divisor after chip-firing from the subset \(A\) is \(D - \Delta \mathbb{I}_A\). We define the \textit{outdegree} of \(A\) from a vertex \(v \in A\) to be the number of edges leaving \(A\) from \(v\) so

\[
\text{outdeg}_v(A) := |E\{v\}, V(G) - A|.
\]

Hence, a chip-firing move from a subset \(A \subset V(G)\) sends \(\text{outdeg}_v(A)\) chips from each vertex \(v \in A\) into \(V(G) - A\). The total outdegree of \(A\) is defined as \(\text{outdeg}_A(A) := \sum_{v \in A} \text{outdeg}_v(A)\). The following result is proven in [20].

**Lemma 2.1.** Given an effective divisor \(D\) and an equivalent effective divisor \(D'\), there exists a finite sequence of subset-firing moves which transforms \(D\) into \(D'\) without ever inducing debt in any vertex of the graph.

This means that, if we have a divisor \(D\) with \(r(D) \geq 1\), we can move at least one chip onto every vertex of our graph (in turn) without ever putting any of the vertices of the graph into debt. For a given divisor \(D\), we say \(D\) is \(v\)-\textit{reduced} with respect to some vertex \(v \in V(G)\) if

1. for each \(v' \in V(G) - \{v\}, D(v') \geq 0\), and
2. for any nonempty subset \(A \subset V(G) - \{v\}\), there exists \(v' \in A\) such that \(\text{outdeg}_{v'}(A) < D(v')\).

This means that every vertex (except possibly \(v\)) is out of debt, and that there exists no way to fire from any subset of \(V(G) - \{v\}\) without inducing debt. The following two results are proven in [2].

**Lemma 2.2.** Given a divisor \(D \in \text{Div}(G)\) and a vertex \(v \in V(G)\), there exists a unique \(v\)-reduced divisor \(D'\) such that \(D' \sim D\).

We will use \(\text{Red}_v(D)\) to denote this unique \(v\)-reduced divisor.

**Lemma 2.3.** For a divisor \(D \in \text{Div}(G), r(D) \geq 1\) if and only if \(\text{Red}_v(D)(v) \geq 1\) for each \(v \in V(G)\).

Thus, we can determine if a divisor is winning divisor by checking that, for each \(v \in V(G)\), the associated \(v\)-reduced divisor satisfies \(v \notin \text{supp}(\text{Red}_v(D))\). Furthermore, given a divisor \(D\) and a vertex \(v\) for which \(D\) is effective away from \(v\), Algorithm [4] developed by Dhar in [11], computes \(\text{Red}_v(D)\).

We refer the reader to [4] for a proof that Algorithm [4] terminates and that the resulting divisor is indeed \(\text{Red}_v(D)\). As a corollary of Lemma [2.3] we have the following result.

**Corollary 2.4.** For an effective divisor \(D \in \text{Div}_+(G)\), if there exists some \(v \in V(G)\) such that \(v \notin \text{supp}(D)\) and for which beginning Dhar’s burning algorithm at \(v\) results in the entire graph burning, then \(r(D) < 1\).
If Riemann-Roch for Graphs.

2.3. Riemann-Roch for Graphs. For a graph $G$, we define the canonical divisor as

$$K := \sum_{v \in V(G)} (\text{val}(v) - 2)(v).$$

The canonical divisor has degree $2g(G) - 2$. In [2], Baker and Norine proved the following Riemann-Roch theorem for graphs, analogous to the classical Riemann-Roch theorem on algebraic curves.

**Theorem 2.5** (Riemann-Roch for graphs). If $G$ is a graph with $D \in \text{Div}(G)$,

$$r(D) - r(K - D) = \text{deg}(D) + 1 - g(G).$$

Notice that this implies that $r(K) = g(G) - 1$. As a consequence, we can prove the following result.

**Proposition 2.6.** If $G$ is a graph with genus $g(G) \leq 2$, then $\text{gon}(G) \leq 2$.

**Proof.** If $g(G) = 0$, then $G$ must be a tree, giving $\text{gon}(G) = 1$. If $g(G) = 1$ and $D \in \text{Div}(G)$ satisfies $\text{deg}(D) = 2$, then by Riemann-Roch for graphs, we see that

$$r(D) = \text{deg}(D) + 1 - g + r(K - D) = 2 + r(K - D) \geq 1.$$

Finally, if $g(G) = 2$, then the canonical divisor $K$ has $\text{deg}(K) = 2$ and $r(D) = 1$, providing an upper bound on the gonality of $G$. □

2.4. Harmonic Morphisms of Graphs. The notion of gonality we have described in the previous sections is often referred to as divisorial gonality in the literature. We can also define a notion of gonality called geometric gonality in terms of maps between graphs.

If $G$ and $G'$ are combinatorial graphs, a morphism $\varphi : G \to G'$ is a map sending $V(G) \to V(G')$ and $E(G) \to E(G') \cup V(G')$, satisfying:

1. if $e = uv \in E(G)$ and $\varphi(u) = \varphi(v)$, then $\varphi(e) = \varphi(u) = \varphi(v)$
2. if $e = uv \in E(G)$ and $\varphi(u) \neq \varphi(v)$, then $\varphi(e) = \varphi(u)\varphi(v)$.

This definition comes from [3]. Morphisms defined on combinatorial graphs are sometimes indexed, as in [10]. In this paper, we will only consider non-indexed morphisms. For a vertex $v \in V(G)$, we define the multiplicity of $\varphi$ at $v$ with respect to an edge $e' \ni \varphi(v)$ as

$$m_\varphi(v) := |\{e \in E(G) : v \in e, \varphi(e) = e'\}|,$$

for some choice of $e' \in E(G')$ adjacent to $\varphi(v)$. A morphism is harmonic if the value of $m_\varphi(v)$ does not depend on the choice of $e' \in E(G')$. A harmonic morphism is non-degenerate if $m_\varphi(v) > 0$ for all $v \in V(G)$.

We define the degree of a harmonic morphism to be

$$\text{deg}(\varphi) := |\{e \in E(G) : \varphi(e) = e'\}| = |\varphi^{-1}(e')|.$$
Figure 2.1. A non-degenerate harmonic morphism of degree 3 from $G \to T$

for some choice of $e' \in E(G')$. The degree of a harmonic morphism is well-defined and independent of the choice of $e'$ [3, Lemma 2.4]. Figure 2.1 depicts an example of a non-degenerate harmonic morphism. Notice that for each edge $e \in T = \varphi(G)$, we have $|\varphi^{-1}(e)| = 3$.

We define the geometric gonality of a graph $G$ to be

$$ggon(G) := \min\{\deg(\varphi) : \varphi : G \to T \text{ is a non-degenerate harmonic morphism onto a tree} \}.$$  

We remark that there are multiple inequivalent notions of geometric gonality defined in the literature. In particular, some authors consider refinements of the original graph [8], while other authors only consider graph morphisms that are also homomorphisms [19, Section 1.3]. The results in our paper hold specifically for our definition of geometric gonality above.

2.5. **Bounds on Gonality.** For a graph $G$, let $\eta(G)$ denote the edge-connectivity of the graph. That is, $\eta(G)$ is the maximum integer $k$ such that $G$ is $k$-edge-connected. The following result is stated in [10] and proven here for the reader’s convenience.

**Lemma 2.7.** For a graph $G$, $gon(G) \geq \min\{|V(G)|, \eta(G)\}$.

**Proof.** Suppose that $D \in \text{Div}^+(G)$ is a divisor with $\deg(D) < \min\{|V(G)|, \eta(G)\}$. This means that $D$ does not contain all of the vertices of $G$ in its support, nor can we fire from any subset of $\text{supp}(D)$ because any such subset $A \subseteq \text{supp}(D)$ will have $\text{outdeg}_A(A) > \sum_{v \in A} D(v)$. Hence, $D$ is not a winning divisor. \qed

The treewidth $\text{tw}(G)$ of a graph $G$ is defined to be the minimum width amongst all possible tree decompositions of $G$. The following result is proven in [21].

**Lemma 2.8.** For a graph $G$, $\text{gon}(G) \geq \text{tw}(G)$.

It is shown in [6] that, for a simple graph $G$, $\text{tw}(G) \geq \min\{\text{val}(v) : v \in V(G)\}$. Hence, we have the following result.

**Lemma 2.9.** For a simple graph $G$, $\text{gon}(G) \geq \min\{\text{val}(v) : v \in V(G)\}$.

We also have the following “trivial” upper bound on gonality.

**Lemma 2.10.** For a graph $G$, $\text{gon}(G) \leq |V(G)|$.

This upper bound is typically only attained when the edge-connectivity of the graph is high relative to the number of vertices. In fact, if $G$ has a vertex not incident to any multiple edges, then $\text{gon}(G) \leq |V(G)| - 1$, since placing one chip on every vertex except one forms a winning divisor.

3. **Simple Graphs of Gonality Three**

We begin our study of graphs of gonality 3 with the following result, which holds for all combinatorial graphs (simple or otherwise).

**Proposition 3.1.** If $G$ is a combinatorial graph with $\text{gon}(G) = 3$, then

1. $g(G) \geq 3$, and
2. $G$ is at most 3-edge-connected, or $|V(G)| = 3$. 

Proof. Note that (1) comes as a corollary of Proposition 2.6. For (2), assume that $|V(G)| \geq 4$ and $\eta(G) \geq 4$. Then, by Lemma 2.7 we have $\text{gon}(G) \geq 4$.

If $|V(G)| < 3$, we know that $G$ is either a single point or the path $P_2$ (both of which have gonality 1), or that $G$ is a banana graph on two vertices, which has gonality 2 (see Figure 3.1(a)). Hence, if $|V(G)| \leq 3$ and $\text{gon}(G) = 3$, then $|V(G)| = 3$. Notice that, as in Figure 3.1(b), we can have a 4-edge-connected graph $G$ with $\text{gon}(G) = 3$ and $|V(G)| = 3$. \hfill \Box

We now state the main result of this section.

Theorem 3.2. If $G$ is a simple, 3-edge-connected combinatorial graph, then the following are equivalent:

1. $G$ has gonality 3.
2. There exists a cyclic automorphism $\sigma : G \to G$ of order 3 that does not fix any edge of $G$, such that $G/\sigma$ is a tree.
3. There exists a non-degenerate harmonic morphism $\varphi : G \to T$, where $\text{deg}(\varphi) = 3$ and $T$ is a tree.

While statements (1) and (3) in Theorem 3.2 are nearly identical to those given in Theorem 1.1, statement (2) requires the extra condition that the automorphism $\sigma$ does not fix any edge of $G$. In our proof of Theorem 3.2, we will show that this condition is required so that the implication from statement (2) to statement (3) holds.

To prove this theorem, we begin with a few lemmas. In each of these lemmas, we assume $G$ is a simple, 3-edge-connected graph with $\text{gon}(G) = 3$, and that $D \in \text{Div}_+(G)$ with $r(D) = 1$ and $\text{deg}(D) = 3$.

Lemma 3.3. For any vertex $v \in V(G)$, there is a unique divisor $D' \in \text{Div}_+(G)$ such that $D \sim D'$ and $v \in \text{supp}(D')$.

Proof. Recall that we are assuming $r(D) = 1$. It follows that for any vertex $v_1 \in V(G)$, there exist (not necessarily distinct) vertices $v_2, v_3 \in V(G)$ such that $D \sim (v_1) + (v_2) + (v_3)$. Thus for any $v \in V(G)$, there exists at least one divisor $D' \in \text{Div}_+(G)$ such that $D \sim D'$ and $v \in \text{supp}(D')$.

For uniqueness of $D'$, consider the Abel-Jacobi map $S^{(k)} : \text{Div}_+^{k}(G) \to \text{Jac}(G)$ with basepoint $v_0$, defined as follows:

$$S^{(k)}((v_1) + \cdots + (v_k)) = [(v_1) - (v_0)] + \cdots + [(v_k) - (v_0)],$$

where $[(v)]$ denotes the equivalence class associated to the divisor $(v)$ under the usual equivalence relation on divisors. Then, by Theorem 1.8 from [2], $S^{(k)}$ is injective if and only if $G$ is $(k+1)$-edge-connected. Suppose now that $D \sim (v_1) + (v_2) + (v_3)$ and that there exist two other vertices $v'_2, v'_3$ satisfying $D \sim (v_1) + (v'_2) + (v'_3)$. Then, we see that

$$(v_2) + (v_3) - 2(v_1) \sim D - 3(v_1) \sim (v'_2) + (v'_3) - 2(v_1).$$

Since $G$ is 3-edge-connected, up to relabelling we have $v_2 = v'_2$ and $v_3 = v'_3$. Thus $D'$ is unique. \hfill \Box

Lemma 3.4. If $D \sim (v_1) + (v_2) + (v_3)$, then either $v_1 = v_2 = v_3$ or $v_1 v_2$ and $v_1 v_3$ are all distinct.

Proof. Suppose for the sake of contradiction that there exists a divisor $D' \in \text{Div}_+(G)$ such that $D' \sim D$ and $D' = 2(v_1) + (v_2)$ where $v_1 \neq v_2$. Note that because $G$ is 3-edge-connected, we are required to fire the entirety of $\text{supp}(D')$ in each firing move to avoid inducing debt. We claim that if we have a pair of vertices $(v_1, v_2)$ on a simple, 3-edge-connected graph $G$, there exists some connected component $C \subset G - \{v_1, v_2\}$ containing at least two edges incident to each vertex: if this were not the case, our graph would be at most 2-edge-connected. If we pick a vertex $v_0 \in C$ and begin Dhar’s burning algorithm at $v_0$, we can see that...
Choose a vertex once.\footnote{\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.2.png}
\caption{Three possible edge cases for Lemma 3.5}
\end{figure}}

In Figure 3.2 are thus the only possible edge configurations between equivalence classes. Furthermore, note that there are at most three edges and that the number of edges outgoing from each vertex is \(|\supp(D')| = 1\) or \(|\supp(D')| = 3\).

For a vertex \(v \in V(G)\), let \(D_v\) denote the unique divisor satisfying both \(D_v \sim D\) and \(v \in \supp(D_v)\). Define a new equivalence relation \(\sim_D\) on \(V(G)\) with \(v_1 \sim_D v_2\) if and only if \(v_1 \in \supp(D_{v_2})\) and \(v_2 \in \supp(D_{v_1})\). The equivalence classes associated with this relation are

\[ [v]_D := \{v' \in V(G) : v' \in \supp(D_v)\}. \]

We will use this relation to define a permutation \(\sigma\) of the vertices of \(G\), which we will then prove to be a cyclic automorphism of order 3. First, we prove the following claim.

**Claim.** If \(e = uv \in E(G)\) such that \([u]_D \neq [v]_D\), then there exist exactly three edges between the vertices in \([u]_D\) and the vertices in \([v]_D\). Furthermore, for any given vertex \(u \in [u]_D\),

\[ |E(u, [v]_D)| = D_u(u). \]

**Proof.** Suppose we have an edge \(e = uv \in E(G)\) and we begin with the divisor \(D_v\) satisfying \(\supp(D_v) = [u]_D\). By Lemmas 2.1 and 2.2 there exists a unique \(v\)-reduced divisor equivalent to \(D_v\) which can be reached by a finite sequence of chip-firing moves. Furthermore, by Lemma 3.21 of [20], we never need to fire from the vertex \(v\) itself during the reduction process. Since there is an edge from \(u\) to \(v\), this means that, after our first chip-firing move, we must have moved at least one chip onto \(v\). However, by the uniqueness of the divisor \(D_v\) with \(v \in \supp(D_v)\), we must have moved all three chips onto \([v]_D\). This implies that there exists at least three edges from \([u]_D\) to \([v]_D\) because only one chip can be sent along any given edge. On the other hand, because we were able to successfully fire our three chips from \([u]_D\) onto \([v]_D\) without inducing debt, this also implies that there are at most three edges and that the number of edges outgoing from each vertex \(u \in [u]_D\) is equivalent to \(D_u(u)\). This establishes our claim. □

Note that, by Lemma 3.4, we have \(|[v]_D| = 1\) or \(|[v]_D| = 3\) for each vertex \(v \in V(G)\). The cases depicted in Figure 3.2 are thus the only possible edge configurations between equivalence classes. Furthermore, note that the third case in Figure 3.2 is impossible because we cannot send all three chips along a single edge at once.

**Claim.** If \(e = uv \in E(G)\) such that \(|\supp(D_u)| = 3\). Such a vertex must exist because, otherwise, since our graph is connected and not a single vertex, we would have the third case in Figure 3.2 which we have already shown to be impossible. For the unique vertices \(v_1, v_2, v_3 \in [v]_D\), let \(\sigma(v_1) = v_2\), \(\sigma(v_2) = v_3\), and \(\sigma(v_3) = v_1\). For each edge from a vertex in \([v_1]_D\) to a vertex in another equivalence class \([v_1]_D\), define \(\sigma\) as follows. If \(e = v_1u_1 \in E(G)\) with \(|\supp(D_{u_1})| = 3\), then let \(\sigma(u_1) = u_2\) where \(u_2 \in [u_1]_D\) is the unique vertex such that \(\sigma(v_1)u_2 \in E(G)\). Then we must have \(\sigma(v_2) = u_3\) where \(u_3\) is the unique vertex with \(\sigma(v_2)u_3 \in E(G)\). On the other hand, if \(|\supp(D_{u_1})| = 1\), then let \(\sigma\) act as the identity on \(u_1\).

Let this process, where vertex classes induce orderings on their adjacent vertex classes, propagate outwards. If we reach a situation where a vertex class with one vertex induces an order on a vertex class with
three vertices, pick some arbitrary ordering on those three vertices and define $\sigma$ accordingly. We will show that the order chosen does not matter, and that this process provides us with our desired automorphism.

**Proposition 3.6.** The map $\sigma$ is a cyclic automorphism of order 3 that does not fix any edge of $G$.

**Proof.** Since our graph $G$ is connected, the propagation process induces an order on each vertex class in $G$. The only case in which $\sigma$ fails to be an automorphism is if the induced orderings are incompatible with each other. Suppose that we have a vertex class $[v]_D$ with one ordering induced by an adjacent class $[u]_D$ and another ordering induced by an adjacent class $[w]_D$. It is clear that $[u]_D \neq [w]_D$ and that $|[v]_D| = 3$. However, this implies that there are two paths from our original vertex class to $[v]_D$. If we consider the divisor $D_v$ with $[v]_D \in \text{supp}(D)$, it is clear that by beginning Dhar’s burning algorithm at some vertex in $[v]_D$, the whole graph must burn (because we have at least two burning edges entering each vertex in $[v]_D$). This is a contradiction because we assumed $r(D) \geq 1$.

Notice also that if $e = uv \in E(G)$, then $\sigma^{-1}(uv) = \sigma^{-1}(v) \in E(G)$ because $\sigma^{-1}(u) = \sigma^2(u)$ and $\sigma^{-1}(v) = \sigma^2(v)$. Hence, we have shown that $\sigma$ is an automorphism. By definition, $\sigma$ is cyclic and we have already demonstrated that $\sigma$ has order 3. Finally, we see that $\sigma$ does not fix any edge of $G$ because we have already shown that we cannot have an edge between two equivalence classes with one vertex each (recall that the third edge case in Figure 3.2 is impossible).

Now consider the quotient morphism $\varphi : G \rightarrow G/\sigma$ defined in the following way:

1. If $v \in V(G)$, then $\varphi(v) = [v]_D$.
2. If $e = xy \in E(G)$ and $\varphi(x) = \varphi(y)$, then $\varphi(e) = [x]_D = [y]_D$.
3. If $e = xy \in E(G)$ and $\varphi(x) \neq \varphi(y)$, then $\varphi(e) = [e]_D$ (where $[e]_D$ has endpoints $[x]_D$ and $[y]_D$).

**Proposition 3.7.** The quotient morphism $\varphi : G \rightarrow G/\sigma$ is harmonic and non-degenerate. Moreover, $G/\sigma$ is a tree.

**Proof.** If we pick a vertex $[v]_D \in V(G/\sigma)$, there exists at least one edge $[e]_D \in E(G/\sigma)$ such that $[v]_D \in [e]_D$ (otherwise, our image graph would be disconnected, violating the definition of a morphism). Since $\sigma$ is an automorphism, for each vertex $v \in V(G)$ such that $v \in [v]_D$, there exists some edge $e \in E(G)$ such that $v \in e$ and $\varphi(e) = [e]_D$. Hence, $\varphi$ is non-degenerate. To show that $\varphi$ is harmonic, fix a vertex $v \in V(G)$ and consider all edges $[e]_D \in E(G/\sigma)$ such that $\varphi(v) = [v]_D \in [e]_D$. If $|[v]_D| = 3$, then $|[e]_D| = 1$, then $|[e]_D| = [E(v)] = D_v(v) = 3$, which is also independent of our choice of $e$. Hence, $\varphi$ is harmonic.

For any given edge $[e]_D \in E(G/\sigma)$,

$$|\varphi^{-1}([e]_D)| = \sum_{v \in [e]_D} |\{e \in E(G) : v \in e, \varphi(e) = [e]_D\}| = \sum_{v \in [e]_D} D_v(v),$$

for any choice of $[v]_D$ such that $[v]_D \in [e]_D$. Thus, $\varphi$ is a degree 3 morphism.

We will now show that $G/\sigma$ is a tree. We define the pullback map $\varphi^* : \text{Div}(G') \rightarrow \text{Div}(G)$ associated to a harmonic morphism $\varphi : G \rightarrow G'$ as

$$\varphi^*(D')(v) = m_{\varphi(v)} \cdot D'(\varphi(v)).$$

Note that for any given vertices $x, y \in V(G/\sigma)$, we have

$$\varphi^*((x)) = \sum_{v \in V(G), \varphi(v) = x} m_{\varphi(v)}(v) \cdot v = \sum_{u \in [x]_D} D_u(v) \cdot v \sim \sum_{v' \in [y]_D} D_{\varphi(v')}(v') \cdot (v') = \sum_{v' \in V(G), \varphi(v') = y} m_{\varphi(v')}(v') \cdot (v') = \varphi^*((y)).$$

By Theorem 4.13 from [3], the induced homomorphism $\varphi^* : \text{Jac}(G') \rightarrow \text{Jac}(G)$ is injective. Since $\varphi^*((x)) \sim \varphi^*((y))$, we find that $(x) \sim (y)$ which implies, by Lemma 1.1 of [3], that $G/\sigma$ is a tree.

We are now ready to prove Theorem 3.2.

**Proof of Theorem 3.2.** The implication (1) $\implies$ (2) follows from Proposition 3.6. Note that (2) $\implies$ (3) is also an easy consequence of the proof of Proposition 3.7. If we assume that $\sigma$ is an automorphism which does not fix any edge of $G$, then defining $\varphi$ in the manner described above gives a non-degenerate harmonic morphism of degree 3.
Finally, we want to show that (3) \( \Rightarrow \) (1). Suppose that there exists a non-degenerate harmonic morphism \( \varphi : G \to T \) such that \( \deg(\varphi) = k \) and \( T \) is a tree. Fix \( x_0 \in T \) and let

\[
D = \varphi^*((x_0)) = \sum_{v \in V(G), \varphi(v) = x_0} m_{\varphi}(v) \cdot (v).
\]

It is clear that \( D \) is effective and by Lemma 2.13 in [3], \( \deg(D) = k \). We claim that \( r(D) \geq 1 \). Pick \( x \in G \). Since \( T \) is a tree, by Lemma 1.1 from [3], \( (\varphi(x)) \sim (x_0) \). Now, by Proposition 4.2 (again from [3]),

\[
D = \varphi^*((x_0)) \sim \varphi^*((\varphi(x))) = \sum_{v \in V(G), \varphi(v) = \varphi(x)} m_{\varphi}(v) \cdot (v)
\]

\[
= m_{\varphi}(x) \cdot (x) + \sum_{v \in V(G) - \{x\}, \varphi(v) \neq \varphi(x)} m_{\varphi}(v) \cdot (v)
\]

\[
= m_{\varphi}(x) \cdot (x) + E,
\]

where \( E \) is an effective divisor. Notice that because \( \varphi \) is non-degenerate, \( m_{\varphi}(x) > 0 \) so \( D \sim c \cdot (x) + E \) for each \( x \in V(G) \) with \( c \in \mathbb{Z}_{\geq 0} \). Hence, we find that \( r(D) \geq 1 \). In the context of our theorem, we use \( k = 3 \), so this means that gon(\( G \)) \( \leq 3 \). By Lemma 2.7, it follows that gon(\( G \)) = 3. \( \square \)

As a consequence of Theorem 3.2, we can easily determine the geometric gonalities of certain trivalent graphs. For example, the 3-cube graph \( Q_3 \) illustrated in Figure 3.3 is trivalent and 3-edge-connected. It can be computationally verified that gon(\( Q_3 \)) = 4. Hence, by Theorem 3.2, we know that ggon(\( Q_3 \)) \( \geq 4 \). Figure 3.3 depicts a non-degenerate harmonic morphism on \( Q_3 \) of degree 4 so we have ggon(\( Q_3 \)) = 4.

![Figure 3.3. The 3-cube Q_3 with gonality 4](image)

Furthermore, the second condition in Theorem 3.2 is useful for computing divisorial gonalities. For example, consider the Frucht graph in Figure 3.4, which is the smallest trivalent graph with no nontrivial automorphisms [13]. It is 3-edge-connected, and thus has gonality at least 4. It can be computationally verified that the divisor depicted in Figure 3.4 is indeed a winning divisor, so the Frucht graph has divisorial gonality 4.

Since the condition of being 3-edge-connected is relatively strong, we might wonder whether a weaker condition, such as being trivalent, is sufficient for Theorem 3.2 to hold. The next result shows that this is not the case.

**Proposition 3.8.** If \( G \) is a simple, bridgeless trivalent graph that is not 3-edge-connected, then gon(\( G \)) \( \geq 4 \).

**Proof.** Since \( G \) is bridgeless and not 3-edge-connected, it must be exactly 2-edge-connected. This means that there exists some way to partition \( G \) into two subgraphs, \( H_1 \) and \( H_2 \), connected by exactly two edges, as illustrated in Figure 3.5. Suppose for the sake of contradiction that there exists \( D \in \text{Div}_+(G) \) with \( \deg(D) = 3 \) and \( r(D) = 1 \). Then there exists some divisor \( D' \sim D \) such that \( D' \) has exactly two chips on
Figure 3.4. The Frucht graph with gonality 4

$H_1$ and one chip on $H_2$: we must be able to move at least one chip onto both subgraphs, and since there are only two edges connecting the subgraphs, we can move at most two chips in a single subset firing move.

Figure 3.5. Simple, trivalent, exactly 2-edge-connected graph

Let $v_1, v_2 \in H_1$ and $v_3 \in H_2$ be the vertices such that $\text{supp}(D') = \{v_1, v_2, v_3\}$. We will first consider the case where $v_1 \neq v_2$. Suppose that removing $v_1$ and $v_2$ from $G$ disconnects the graph into at least two connected components. Let $H_3$ be one of the connected components which does not contain $v_3$. A trivalent 2-edge-connected graph is also 2-vertex-connected. This implies that there exists at least one edge incident to both $v_1$ and some vertex in $H_3$, and that the same holds for $v_2$. Since each vertex is trivalent, by symmetry, we have at most two edges connecting each vertex in $\{v_1, v_2\}$ with vertices in $H_3$.

First, suppose that there exists exactly one edge incident to $v_1$ and one edge incident to $v_2$ entering $H_3$, as illustrated in the top graph in Figure 3.6. Then, there exist two vertices $v'_1, v'_2 \in H_3$ which are the endpoints of these edges. We know that $v'_1 \neq v'_2$ because $G$ is 2-vertex-connected. Fire onto $H_3$, moving the two chips from $\{v_1, v_2\}$ onto $\{v'_1, v'_2\}$. Suppose that we can continue firing in this manner, i.e., moving chips onto two vertices which are each connected by exactly one edge to the rest of the graph. Since our graph is finite, this process must terminate at some point. If we are able to hit all vertices in $H_3$, we have a contradiction because this implies that at least two vertices in $H_3$ are not trivalent. Hence, before hitting all of the vertices in $H_3$, we reach a state where we are no longer able to fire from our subset of two vertices without inducing debt. (Notice that we cannot fire from either vertex separately either, because this would imply the existence of a bridge.)

On the other hand, if there exist at least two edges incident to either $v_1$ or $v_2$ entering $H_3$ and at least one edge incident to the other vertex, we are already at a state where we cannot fire onto $H_3$ without inducing debt; see the bottom graph in Figure 3.6. For both of these cases, choose a vertex $v_0 \in H_3$ (in the first case, choose a vertex in the subgraph we are unable to fire onto). Since we are unable to fire without inducing debt, if we begin Dhar’s burning algorithm at $v_0$, at least one of our two vertices has fewer chips than edges incident to $H_3$. Hence, everything in $H_3$ must burn, including at least one of the two vertices with chips. This forces the other vertex with a chip to burn as well. Since we have only one other vertex with exactly one chip, this implies that the whole graph burns. We initially assumed that $r(D) = 1$, so this is a contradiction. Thus, in the case where $v_1 \neq v_2$, removing the set $\{v_1, v_2\}$ cannot disconnect the graph.

In the case where $v_1 = v_2$, it is clear that we cannot remove $v_1$ and disconnect the graph because our graph is 2-vertex-connected. Now, choose a vertex $v'_0 \in H_2$ such that $v'_0 \neq v_3$ (such a vertex exists due to trivalence). If we begin Dhar’s burning algorithm at $v'_0$, we find that the entirety of $H_2$ must burn, since there exists only one vertex with a single chip in $H_2$. The fire then spreads across the two edges incident to $H_1$. Since removing $v_1$ and $v_2$ does not disconnect the graph, the fire must burn every vertex in $H_1$ except
possibly \( v_1 \) and \( v_2 \). However, because our graph is simple, there exists at most one edge between \( v_1 \) and \( v_2 \), implying that each must have at least two incident burning edges. Hence, the whole graph burns, implying that \( r(D) < 1 \). Again, this is a contradiction. We conclude that the gonality of the graph is at least 4. \( \square \)

This result does not extend to multigraphs. Figure 3.7 depicts an example of a graph which is bridgeless, trivalent, and not 3-edge-connected, but has gonality 3.

![Figure 3.7. Multigraph G with gon(G) = 3](image)

Corollary 3.9. If \( G \) is a simple, bridgeless trivalent graph that is not 3-edge-connected, then \( \text{ggon}(G) \neq 3 \).

Proof. By Proposition 3.8, \( G \) does not have gonality 3. Hence, by Theorem 3.2, there exists no non-degenerate harmonic morphism of degree 3 from \( G \) to a tree. \( \square \)

4. Multigraphs of Gonality Three

Our result on simple, 3-edge-connected graphs also extends to multigraphs, with the caveat that we no longer have a condition similar to statement (2) in Theorem 3.2. We also need to consider the edge case when \( |V(G)| = 3 \); this was not an issue for simple graphs because there are no simple 3-edge-connected graphs on 3 vertices.

Theorem 4.1. If \( G \) is a 3-edge-connected combinatorial multigraph, then the following are equivalent:

1. \( G \) has gonality 3.
2. There exists a non-degenerate harmonic morphism \( \varphi : G \to T \) where \( \deg(\varphi) = 3 \) and \( T \) is a tree, or \( |V(G)| = 3 \).

Proof. We will first show that (1) \( \implies \) (2). Let \( G \) be a graph of gonality 3. First note that if \( |V(G)| \leq 3 \), we know, by the proof of Proposition 3.1, that gonality 3 is only if \( |V(G)| = 3 \).

Assume for now that \( |V(G)| > 3 \). Since gonality 3, there exists a divisor \( D \in \text{Div}_+(G) \) such that \( \deg(D) = 3 \) and \( r(D) = 1 \). Notice that the proof of Lemma 3.3 does not require our graph to be simple. Hence, for a vertex \( v \in V(G) \), we can again define \( D_v \) to be the unique divisor such that \( v \in \text{supp}(D_v) \). Define the same equivalence relation \( \sim_D \) where \( v_1 \sim_D v_2 \) if and only if \( v_1 \in \text{supp}(D_{v_2}) \) and \( v_2 \in \text{supp}(D_{v_1}) \). The notation \([v]_D\) again refers to the equivalence class associated to \( v \) under \( \sim_D \).
Define the quotient morphism \( \varphi : G \rightarrow G/\sim_D \) in the same way as in the proof of Theorem 3.2. We will now show that \( \varphi \) is a non-degenerate harmonic morphism of degree 3. Lemma 3.5 also holds for multigraphs, so we have

\[
m_\varphi(v) = |\{ e \in E(G) : v \in e, \varphi(e) = [e]_D \}| = D_v(v),
\]

for each \([e]_D \in E(G/\sim_D)\) such that \([v]_D \in [e]_D\). The assumption that \(|V(G)| > 3\) ensures that we have at least one edge between vertices in different equivalence classes. Since \(D_v(v) > 0\) for each \([v]_D\), this means that our morphism is non-degenerate. Furthermore, since \(m_\varphi(v)\) does not depend on our choice of \([e]_D \in E(G/\sim_D)\), this also shows that our morphism is harmonic. Hence,

\[
\deg(\varphi) = \sum_{v \in [e]_D} |\{ e \in E(G) : v \in e, \varphi(e) = [e]_D \}| = \sum_{v \in [e]_D} D_v(v) = 3.
\]

Using the same technique as in the proof of Proposition 3.7, we will show that \( \varphi(G) = G/\sim_D \) is a tree. Define the same pullback map \( \varphi^* \) as in the proof of Theorem 3.2. If we consider two vertices \( x, y \in \varphi(G) \), we again see that

\[
\varphi^*((x)) = \sum_{v \in V(G), \varphi(v) = x} m_\varphi(v) \cdot (v) = \sum_{v \in [e]_D} D_v(v) \cdot (v) \sim \sum_{v' \in [y]_D} D_{v'}(v') \cdot (v') = \sum_{v' \in V(G), \varphi(v') = y} m_\varphi(v') \cdot (v') = \varphi^*((y)).
\]

We conclude that \( \varphi(G) \) is a tree.

To show that (2) \( \implies \) (1), first suppose that \(|V(G)| = 3\). Then, by Lemma 2.7

\[
\text{gon}(G) \geq \min\{\eta(G), |V(G)|\} = 3
\]

and by Lemma 2.10 \( \text{gon}(G) \leq 3 \). On the other hand, if there exists a non-degenerate harmonic morphism of degree 3 from \( G \) to a tree, then the same proof provided for Theorem 3.2 shows that there exists a divisor \( D \) with \( \deg(D) = 3 \) and \( r(D) \geq 1 \). By Lemma 2.7 it follows that \( \text{gon}(G) = 3 \).

In fact, this criterion can be applied to certain graphs with bridges, assuming that they become 3-edge-connected after contracting these bridges. This is due to the following proposition, which comes as an immediate consequence of Corollary 5.10 in [3] on rank-preservation under bridge contraction.

**Proposition 4.2.** If \( G \) is a graph and \( G' \) is the graph obtained by contracting every bridge of \( G \), then \( \text{gon}(G) = 3 \) if and only if \( \text{gon}(G') = 3 \).

5. **Constructing Graphs of Gonality 3**

In [7], Chan presents the following construction for all trivalent, 2-edge-connected graphs of gonality 2. Choose a tree \( T \) where each vertex \( v \in V(T) \) satisfies \( \text{val}(v) \leq 3 \).

1. Duplicate \( T \), making two copies \( T_1 \) and \( T_2 \).
2. For each vertex \( v_1 \in T_1 \) with \( \text{val}(v_1) \leq 2 \), connect it to the matching vertex in \( T_2 \) with \( 3 - \text{val}(v_1) \) edges.

Every graph constructed in this way is called a ladder. By [7, Theorem 4.9], each graph arising from this construction has gonality 2, and every 2-edge-connected trivalent graph of gonality 2 with genus at least 3 comes from such a construction.

We now provide a similar construction for graphs of gonality 3. In contrast to the results of [7], not every graph of gonality 3 arises from this construction.

**Proposition 5.1.** Let \( T \) be an arbitrary tree that is not a single vertex. Construct a graph \( \mathcal{T}(T) \) as follows (see Figure 5.1):

1. Duplicate \( T \) twice, for a total of three copies of \( T \). Call these copies \( T_1, T_2, \) and \( T_3 \).
2. For each vertex \( v_1 \in T_1 \) with \( \text{val}(v_1) \leq 2 \) and its corresponding vertices \( v_2 \in T_2 \) and \( v_3 \in T_3 \), connect each pair of vertices with an edge so that all three vertices are connected in a 3-cycle.

Then \( \text{gon}(\mathcal{T}(T)) = 3 \).
Proof. It is clear that the morphism $\varphi : \mathcal{T}(T) \to T$ which maps corresponding triples of vertices $\{v_1, v_2, v_3\}$ to each other is a non-degenerate harmonic morphism. Notice that arguing that $(3) \implies (1)$ in the proof of Theorem 3.2 does not require 3-edge-connectivity. Hence, there exists a divisor $D$ on $\mathcal{T}(T)$ such that $\deg(D) = 3$ and $r(D) \geq 1$. Since $\mathcal{T}(T)$ is simple and every vertex is at least trivalent, by Lemma 2.9 we have $\text{gon}(\mathcal{T}(T)) = 3$. □

Figure 5.1. Construction of $\mathcal{T}(T)$

We can extend this result to include certain multigraphs. Notice that we can add arbitrary edges between corresponding triples of vertices (which are already connected via a 3-cycle) while retaining a graph of gonality 3. This is because we still have the same non-degenerate harmonic morphism (the added edges are simply contracted) and because treewidth of a multigraph is equivalent to the treewidth of the underlying simple graph. From here, we apply Lemma 2.9 which gives us the desired gonality.

One can also generalize this construction to create graphs of gonality $k \geq 3$. Make $k$ copies of a tree $T$ that has at least two vertices. For each vertex $v$ of $T$ with $\text{val}(v) \leq k - 1$, connect all the $k$ copies of $v$ to each other with $\binom{k}{2}$ edges. Call the resulting graph $\mathcal{T}(T)$. Our construction guarantees that each vertex has valence $k$, so $\text{gon}(\mathcal{T}(T)) \geq k$. There is a natural harmonic morphism of degree $k$ from $\mathcal{T}(T)$ to $T$, which by the argument from $(3) \implies (1)$ in the proof of Theorem 3.2 shows that $\text{gon}(\mathcal{T}(T)) \leq k$. We thus have $\text{gon}(\mathcal{T}(T)) = k$.

6. Future Directions

All results in this paper only hold for combinatorial graphs, as opposed to metric graphs, which have lengths associated to their edges. The first natural generalization of our work would be to determine the extent to which our results generalize to metric graphs. The work by [7] on hyperelliptic graphs was done in the setting of metric graphs so some of our results may extend via similar arguments.

Another natural question would be that of algorithmically testing whether or not a graph has gonality 3. In general, computing the divisorial gonality of a graph is NP-hard [15], but it is possible to check if a graph has gonality 2 in polynomial time [5]. The next step would be either to develop an efficient algorithm for determining if a graph has gonality 3, or to prove that this problem is computationally hard. The criteria we present in Theorem 3.2 may be useful for this endeavor.

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