Novel Finite Temperature Conductivity in Quantum Hall Systems

Sudhansu S. Mandal, S. Ramaswamy and V. Ravishankar

Department of Physics, Indian Institute of technology, Kanpur – 208 016, INDIA

Abstract

We study quantum Hall systems (mainly the integer case) at finite temperatures and show that there is a novel temperature dependence even for a pure system, thanks to the ‘anomalous’ nature of generators of translation. The deviation of Hall conductivity from its zero temperature value is controlled by a parameter $T_0 = \pi \rho / m^* N$ which is sample specific and hence the universality of quantization is lost at finite temperatures.

PACS numbers: 73.40.Hm, 11.15.Bt
Quantum Hall (QH) systems have proved to be a rich source of exploring many interesting and unexpected features of quantized gauge theories. The Hall conductivity which is quantized is topological \[1\] in nature; it appears (in the integer system that we mainly study here) as the coefficient of Chern-Simons (CS) term which is induced by first order quantum corrections. Further it is exact \[2\] – with no higher order corrections, and it does not suffer any renormalization. It is also known that the Hall conductivity can be looked upon as a manifestation of chiral anomaly \[3,4,5,6,7\], inherited by the effectively planar system from the parent three dimensional system.

Recall that in the Landau level problem at hand, the gauge transformations get mixed up with the Euclidean transformations in such a manner that the associated group is no more the (2+1) Euclidean group \(E_3\), as one would normally have for a pure system. Rather, it is the group \(M_3\) of magnetic translations which is a proper subgroup of \(E_3\). It is recognized that the transition from \(E_3 \rightarrow M_3\) is crucial. The generators of translation (or equivalently, the operators for the centre of the orbit) do not commute \[8\]. In his studies on the closely related CS superconductivity (CSS), Fradkin \[9\] has designated this feature as ‘anomalous’ and has drawn detailed and explicit comparison with the well-known Schwinger – Anderson mechanism \[10,11\] which is a proper field theoretic anomaly.

Here we do not attempt to rewrite the above mentioned non-commutativity in the standard language of field theoretic anomaly. However, we do believe in the essential correctness of Fradkin’s analogy, and as an explicit consequence we shall show that such an ‘anomaly’ is responsible for a novel temperature evolution of Hall conductivity \(\sigma_H\) even for a pure system. In this context we may recall that it is standard lore \[12\] that the presence of impurities, apart from its crucial role in stabilizing the quantization (in form of plateaus) is further required to destroy translational invariance in the system. It is believed that without such a breaking QH effect (QHE) would be trivial. Note that according to this argument, a uniform distribution of impurities would still be insufficient to give temperature dependence to Hall conductivity. We show that the Maxwell gauge interactions that are at play here belie such a naive expectation.
Such a temperature dependence has been noticed in the allied albeit rather academic example of CSS [13,14]. Even for QH systems, Bellisard et. al. [15], have made rough estimates of the temperature dependence of $\sigma_H$ for a pure system. We believe that this paper presents, for the first time, a complete finite temperature (FT) analysis. Further, we also hope that the results obtained here will be verified experimentally.

Consider a system of (weakly) interacting electrons in two space dimensions in the presence of a uniform external magnetic field of strength $B$, confined to the direction perpendicular to the plane. The strength is fine tuned such that $N$ Landau levels (LL) are exactly filled. In the presence of sufficiently high magnetic field (as is relevant to our case), the spins of the fermions would be ‘frozen’ in the direction of magnetic field. Therefore, one may treat the fermions as spinless. The study of such a spinless system can be accomplished with the Lagrangian density,

$$
\mathcal{L} = \psi^* iD_0 \psi - \frac{1}{2m^*} |D_k \psi|^2 + \psi^* \mu \psi - e A_0^{in} \rho + \frac{1}{2} \int d^3 x' A_0^{in}(x) V^{-1}(x - x') A_0^{in}(x').
$$

(1)

Here $D_\nu = \partial_\nu - ie(A_\nu + A_0^{in} \delta_{\nu,0})$ (where $A_\nu$ is the external Maxwell gauge field and $A_0^{in}$ is identified as internal scalar potential), $\mu$ is the chemical potential, and $m^*$ and $\rho$ are the effective mass and the mean density of electrons respectively. The fourth term in Eq.(1) describes the charge neutrality of the system. Finally, $V^{-1}(x - x')$ represents the inverse of the instantaneous charge interaction potential (in the operator sense). The above Lagrangian density is equivalent to the usual interaction term with quartic form of fermi fields, which can be obtained by an integration of $A_0^{in}$ field in Eq. (1). Note also that the electrons interact with each other via $1/r$ or some other short range potential, i.e., the internal dynamics is governed by the (3+1)-dimensional Maxwell Lagrangian as is appropriate for the medium.

The procedure for evaluating the FT properties of the system with the above Lagrangian density is standard. We do not discuss the details here since they have been presented in the allied context of CSS elegantly by Randjbar-Daemi, Salam and Strathdee [13], and has been extensively used [14]: in brief, we construct the partition function ($\beta = 1/T$ being the inverse temperature),
\[
Z = \int [dA_0^{\text{in}}][d\psi][d\psi^*] \exp \left[-\int_0^\beta d\tau \int d^2r \mathcal{L}^{(E)} \right],
\]
(2)

which on integration over the fermionic fields, (by fixing of the saddle point at the uniform background magnetic field \( B \)), factors into \( Z = Z_B Z_I \). Here \( \mathcal{L}^{(E)} \) is the Euclidean version of \( \mathcal{L} \) in Eq.(1). The background part of the partition function is given by \( (1/A) \ln Z_B = \rho_l \sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} \ln[\epsilon_n - \mu + i\omega_j] \). Here \( \epsilon_n = (n + 1/2)\omega_c \) (we have chosen the unit \( \hbar = c = 1 \)) is the energy corresponding to \( n \)-th LL, where \( \omega_c = (e/m^*)B \) is the cyclotron frequency. \( \rho_l = m^*\omega_c/2\pi \) is the degeneracy per unit area in each level, and \( A \) is the area of the system. Finally, \( \omega_j = (2j + 1)\pi/\beta \) is the fermionic Matsubara frequency. The corresponding thermodynamic potential is obtained as \( (\Omega / A) = -\left( \rho_l / \beta \right) \sum_{n=0}^{\infty} \ln (1 + \exp[-\beta(\epsilon_n - \mu)]) \), from which all the properties for the system in the background field can be inferred.

Writing the partition function corresponding to the external probe as \( Z_I = \int [dA_0^{\text{in}}] \exp[-S_{\text{eff}}] \) (where we have expanded the fermionic determinant up to quadratic terms in powers of the fields \( A_0^{\text{in}} \) and the external probe \( A_{\nu} \) around the background field), we identify \( S_{\text{eff}} \) with the one-loop effective action which is obtained as

\[
S_{\text{eff}} = \frac{1}{2} \int d^3x \int d^3x' (A_{\mu} + A_{\mu}^{\text{in}} \delta_{\mu 0}) \Pi^{\mu\nu}(x, x')(A_{\nu} + A_{\nu}^{\text{in}} \delta_{\nu 0})
- \frac{1}{2} \int d^3x \int d^3x' A_0^{\text{in}}(x)V^{-1}(x - x')A_0^{\text{in}}(x') \quad \text{(3)}
\]

The current correlation functions \( \Pi^{\mu\nu}(x, x') \equiv \delta \langle j^\mu(x) \rangle / \delta A_{\nu}(x') \), where \( j^\mu \) is the fermionic current, and \( A_{\nu} \) is the sum of all the gauge fields, have to be determined at the saddle point. Using Galilean and gauge invariance, we write (in the momentum space)

\[
\Pi^{\mu\nu}(\omega, \mathbf{q}) = \Pi_0(\omega, \mathbf{q})(q^2 g^{\mu\nu} - q^\mu q^\nu) + (\Pi_2 - \Pi_0)(\omega, \mathbf{q})
\times (q^2 \delta^{ij} - q^i q^j)\delta^{\mu i} \delta^{\nu j} + i\Pi_1(\omega, \mathbf{q}) q^\nu \quad \text{(4)}
\]

At FT, \( \Pi_0 \) acquires a pole at \( \mathbf{q}^2 = 0 \), i.e., \( \Pi_0 = \bar{\Pi}_0 + \Gamma / \mathbf{q}^2 \). Note that this pole exists in the limit \( \omega = 0, \mathbf{q}^2 \to 0 \). On the other hand, for \( \mathbf{q}^2 = 0, \omega \to 0, \Gamma \equiv 0 \) (no pole exists). However, the limits do commute as far as other form factors are concerned. The FT responses of the system are driven by the temperature behaviour of these form factors.
As we are interested in the low energy response of the system, it is sufficient that we evaluate the form factors at $\omega = 0$, $q^2 = 0$ (keeping in mind the above mentioned singularity of $\Pi_0$). Therefore we obtain

$$\bar{\Pi}_0(0,0) = e^2 \frac{\pi}{4} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{f_m - f_n}{\epsilon_n - \epsilon_m} [(n+1)\delta_{m,n+1} + n\delta_{m,n-1} - (2n+1)\delta_{m,n}]$$

$$= e^2 \frac{\pi}{4\omega_c} \sum_{n=0}^{\infty} f_n - e^2 \beta \sum_{n=0}^{\infty} (2n+1)f_n(1-f_n),$$  \hspace{1cm} (5a)

$$\Pi_1(0,0) = \bar{\Pi}_0 \omega_c, \hspace{1cm} (5b)$$

$$\Pi_2(0,0) = e^2 \omega_c \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{f_m - f_n}{\epsilon_n - \epsilon_m} [3(n+1)^2\delta_{m,n+1} + 3n^2\delta_{m,n-1}$$

$$- (n+1)(n+2)\delta_{m,n+2} - n(n-1)\delta_{m,n-2} - (2n+1)^2\delta_{m,n}]$$

$$= e^2 \frac{\pi}{4m^*} \sum_{n=0}^{\infty} (2n+1)f_n - e^2 \omega_c \sum_{n=0}^{\infty} (2n+1)^2f_n(1-f_n)$$ \hspace{1cm} (5c)

$$\Gamma = e^2 m^* \omega_c \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{f_m - f_n}{\epsilon_n - \epsilon_m} \delta_{mn} \hspace{1cm}$$

$$= e^2 m^* \beta \omega_c \sum_{n=0}^{\infty} f_n(1-f_n), \hspace{1cm} (5d)$$

with $f_n = [1 + \exp(\beta(\epsilon_n - \mu))]^{-1}$.

For each of the form factors (except $\Gamma$) the first term in Eq. (5) contributes for all values of $\omega$ and is continuous at $\omega = 0$. On the other hand, the second term in the expressions for $\bar{\Pi}_0$, $\Pi_1$ and $\Pi_2$, as well as $\Gamma$, has a singular dependence on $\omega$, i.e., it is non-vanishing only at $\omega = 0$. Indeed, the former non-singular contribution is caused by the inter LL transitions, while the latter singular ones owe their existence to the intra LL transitions in the virtual process shown in Fig. 1. It is significant that the entire temperature dependence of $\Pi_1$, and hence Hall conductivity $\sigma_H$ as obtained below, is singular. We observe that this is a direct consequence of the ‘anomaly’ in the translation generators at hand; for, but for the infinite degeneracy in the LL, this novel temperature dependence in $\Pi_1$ would not survive in the thermodynamic limit.

The parity and time reversal violating form factor $\Pi_1$ which is the coefficient of CS term in the effective action has interesting properties. It is purely topological. Moreover, it does not get renormalized by the higher order calculation of correlation function according to
Coleman-Hill theorem \[2\]. Therefore, $\Pi_1$ in Eq. (5) is exact.

A straightforward linear response analysis from Eqs. (3-5) yields the Hall conductivity to be

$$\sigma_H = \Pi_1(0,0) = \frac{e^2}{2\pi} \sum_{n=0}^{\infty} f_n - \frac{e^2}{4\pi} \beta \omega_c \sum_{n=0}^{\infty} (2n + 1) f_n (1 - f_n),$$

subject to the condition $\lim_{q^2 \to 0} V(|q|)q^2 = 0$. In other words, for any short ranged potential, Hall conductivity is exactly the parity and time reversal violating form factor. Note that the electron-electron interaction does not otherwise play any major role in this case. Therefore the emergence of $\sigma_H$ is also purely topological and exact. Note that the diagonal conductivity vanishes by virtue of the purity of the system.

At $T = 0$, $\sigma_H$ is quantized to the value $\nu(e^2/2\pi)$, where the filling fraction $\nu = N$ (an integer). The quantization is ‘universal’, i.e., it does not depend on the microscopic details of the system. Since QHE has been observed at very low temperatures, a low temperature expansion of $\sigma_H$ should suffice. In that case, it is analytically evaluated as a perturbation in $\exp[-\beta \omega_c/2]$ (see Refs. 13 and 14 for details of calculation) and is found to be

$$\sigma_H(T) = \frac{e^2}{2\pi} N (1 - 4y),$$

where $y = (T_0/T) \exp[-T_0/T]$, with $T_0 = \pi \rho/m^* N$. Bellissard et al. [15] have also discussed the temperature dependent Hall conductivity using Kubo formula for the sample of infinite relaxation time. To compare their results with our exact result, we note that they only make an approximate estimation of the change in $\sigma_H$ where they get the correct exponential term, but miss the crucial prefactor multiplying it. From the expression (5), it is clear that the novel temperature dependence is indeed accompanied by corresponding deviation from universality of quantization in virtue of its dependence on the parameter $T_0$ which is the only sample specific parameter that enters the analysis. In fact, at any temperature, although we cannot evaluate $\sigma_H(T)$ analytically, it is easy to check that the Hall defect

$$\mathcal{R} = \left| \frac{\sigma_H(T) - \sigma_H(0)}{\sigma_H(0)} \right|$$

(8)
is a function of the dimensionless variable $T_0/T$. This type of temperature dependence and the specific form of $T_0$ is a reflection of the fundamental energy scale $\omega_c$.

If we fix the value of $\mathcal{R}$, the temperature $T_\mathcal{R}$ at which the defect $\mathcal{R}$ would occur for QHE follows a simple expression

$$\frac{T_0}{T_\mathcal{R}} = C,$$  

(9)

where the constant $C$ is independent of the sample. For example, the value of $C$ at $\mathcal{R} = 10^{-n}$ is approximately given by $0.325 + 0.776(n + 1)$ for $3 \leq n \leq 8$. Fig. 2 shows how $T_\mathcal{R}$ depends on $\mathcal{R}$ for $\nu = 1$ and 2 over a range 0.01 ppm to 0.1% for a specific choice of $\rho/m^*$.

We have considered integer QHE here for simplicity’s sake. We report that a similar analysis holds for fractional QHE within the composite fermion model [16]. The analysis in this case involves other aspects such as the mean field ansatz, which we shall defer to a different paper.

Before we conclude, we observe that there is already a wealth of experimental information available on FT QHE for both integer and fractional case [17,18,19] showing the deviation from quantization at the central value of $B$. We therefore believe that it is not impossible to verify experimentally the effect predicted here. Admittedly the contribution from impurity dominates over the one at hand [20]. However we hope that $\sigma_H(T)$ will be measured by varying the disorder (keeping other parameters fixed). (See [21] for one such experiment). One may then extrapolate the result to zero impurity concentration, or, if the impurity contribution is well understood, merely subtract that part to extract the required temperature dependence.

Acknowledgements: We thank S. D. Joglekar and J. K. Bhattacharjee for helpful discussions.
REFERENCES

* Present Address: Department of Physics, Birla Institute of Technology and Science, Pilani –333 031, India.

[1] J. Avron, R. Seiler and B. Simon, Phys. Rev. Lett. 51, 51 (1983); Q. Niu and D. J. Thouless, Phys. Rev. A 17, 2453 (1983); Q. Niu, D. J. Thouless and Y.-S. Wu, Phys. Rev. B 31, 3372 (1985); M. Kohmoto, Ann. Phys. (NY) 160, 343 (1985).

[2] S. Coleman and B. Hill, Phys. Lett. B 159, 184 (1985).

[3] R. Jackiw, Phys. Rev. D 29, 2375 (1984).

[4] M. H. Freidman, J. B. sokoloff, A. Widom and Y. N. Srivastava, Phys. Rev. Lett. 52, 1587 (1984).

[5] K. Ishikawa, Phys. Rev. Lett. 53, 1615 (1984).

[6] R. J. Hughes, Phys. Lett. B 148, 215 (1984).

[7] A. Widom, M. H. Friedman and Y. N. Srivastava, Phys. Rev. B 31, 6588 (1985).

[8] Y.-H. Chen, F. Wilczek, E. Witten and B. I. Halperin, in Int. J. Mod. Phys. B 3, 1001 (1989), designate this feature as violation of fact.

[9] E. Fradkin, Phys. Rev. B 42, 570 (1990).

[10] J. Schwinger, Phys. Rev. 128, 2425 (1962).

[11] P. W. Anderson, Phys. Rev. 130, 439 (1963).

[12] See e.g., The Quantum Hall Effect, ed. by R. E. Prange and S. M. Girvin (Springer-Verlag, 1990), pp 11–13.

[13] S. Randjbar-Daemi, A. Salam and J. Strathdee, Nucl. Phys. B 340, 403 (1990).

[14] S. S. Mandal, S. Ramaswamy and V. Ravishankar, Mod. Phys. Lett. B 8, 561 (1994); Int. J. Mod. Phys. B 8, 3095 (1994) and the references quoted there.
[15] J. Bellissard, A. van Elst and H. S.-Blades, J. Math. Phys. 35, 5373 (1994).

[16] J. K. Jain, Phys. Rev. Lett. 63, 199 (1989); Phys. Rev. B 41, 7653 (1990).

[17] K. Yoshihiro, J. Kinoshita, K. Inagaki, C. Yamanouchi, J. Moriyama and S. Kawaji, Physica B 117, 706 (1983).

[18] M. E. Cage, B. F. Field, R. F. Dziuba, S. M. Girvin, A. C. Gossard and D. C. Tsui, Phys. Rev. B 30, 2286 (1984).

[19] A. M. Chang, P. Berglund, D. C. Tsui, H. L. Stormer and J. C. M. Hwang, Phys. Rev. Lett. 53, 997 (1984).

[20] To be published; See B. Huckestein, Rev. Mod. Phys. 67, 357 (1995) and references cited therein for a detailed discussion of the contribution of the impurties using the scaling theory.

[21] J. E. Furneaux, S. V. Kravchenko, W. E. Mason, G. E. Bowker and V. M. Pudalov, Phys. Rev. B 51, 17227 (1995).
FIG. 1. Vacuum polarization diagram for the virtual process.

FIG. 2. The temperatures $T_R$ as a function of Hall defect $\mathcal{R}$ for (a) $\nu = 1$ and (b) $\nu = 2$ for a typical value of $\rho/m^* = 20$ cm$^{-1}$.