The general communication tree embedding problem is the problem of mapping a set of communicating terminals, represented by a graph $G$, into the set of vertices of some physical network represented by a tree $T$. In the case where the vertices of $G$ are mapped into the leaves of the host tree $T$ the underlying tree is called a routing tree and if the internal vertices of $T$ are forced to have degree 3, the host tree is known as layout tree. Different optimization problems have been studied in the class of communication tree problems such as well-known minimum edge dilation and minimum edge congestion problems. In this report we study the less investigate measure i.e. tree length, which is a representative for average edge dilation (communication delay) measure and also for average edge congestion measure. We show that finding a routing tree $T$ for an arbitrary graph $G$ with minimum tree length is an NP-Hard problem.

1 Definitions and Introductory Points

Consider a group of terminals communicating via a finite network $G = (V, E)$, where the set of vertices (finite set $V$) and edges (finite set $E$), respectively represent the collection of terminals and their direct communication paths. We show $|V|$ by $n$ and $|E|$ as $m$.

The general communication tree embedding problem is the problem of mapping the set of terminals into the set of vertices of some physical network represented by a tree $T$. Accordingly, the two vertices $v, u \in V(G)$ that are directly connected via $e \in E(G)$, are connected indirectly via some path $P_T(v, u)$ in $T$. In the case where the vertices of $G$ are mapped to the leaves of the host tree, the underlying tree is called a routing tree. In this report we mostly focus on the case where the internal vertices of the host tree have degree 3 (known as tree layout problem). We denote the sets of leaf nodes and internal nodes of tree $T$ respectively by $V_L(T)$ and $V_I(T)$.

For a graph $G$ and a communication tree $T$ for $G$, there are different measures defined in literature. In following we define the two measures that we are intrusted in this report. For a comprehensive list of measures, an interested reader can refer to [3].

**Definition 1.1 (Edge Dilation).** Consider a graph $G$ and a communication tree $T$ and a bijection $\varphi$ from vertices of $G$ to leaf nodes of $T$. The dilation $\lambda(uv, T, \varphi, G)$ of an edge $\{u, v\} \in E(G)$ is the distance between $\varphi(u)$ and $\varphi(v)$ in $T$. $\Box$

We represent the distance of two vertices $\{u, v\}$ in a graph $G$ with $d_G(u, v)$

**Definition 1.2 (Edge Congestion).** Give a graph $G$ and a communication tree $T$ and and a bijective mapping $\varphi: V(G) \to V_L(T)$. The congestion $\delta(xv, T, \varphi, G)$ and of an edge $\{x, y\} \in E(T)$ is the the number of edges in $\{u, v\} \in E(G)$ that in $T$, the path $P_T(\varphi(v), \varphi(u))$ traverse trough $\{x, y\}$. $\Box$

$^1$We try to use the term node in case of trees as opposed to the term vertex, which we use for general graphs.
Based on the definition of the communication tree for a graph $G$, removal of every edge $\{x, y\} \in E(T)$ partitions the set of vertices of $G$ into two component. Hence every edge of tree corresponds to a cut in $G$. Therefore the congestion of $\{x, y\} \in E(T)$ is the size of the cut it corresponds to.

Several optimization problems can be defined based on these two measures. Minimum tree layout dilation is the problem of finding a tree layout for a given graph $G$ such that the maximum edge dilation is minimized, where the maximum is taken over all edge of $G$. In [7] it is shown that the problem of finding a tree layout with minimum dilation is NP-hard, when the layout tree is rooted.

Similarly, given a graph $G$, in minimum tree layout congestion problem the goal is to find a tree layout $T$ such that the maximum edge congestion is minimized. In [8] Seymour and Thomas show that the minimum tree layout congestion problem is polynomially solvable for the case of planer graphs, and is NP-hard when considering general graphs. In this report we study the minimum tree layout length problem (shortly called Min Tree Length), formally defined as it follows.

**Definition 1.3 (Minimum tree layout length).** Consider the finite undirected graph $G = (V, E)$. The minimum tree layout length problem is the problem of finding layout tree $T$ and a bijective mapping $\varphi : V(G) \rightarrow V_L(T)$ such that $L(T, \varphi, G) = \sum_{\{u,v\} \in E(G)} \lambda(uv, T, \varphi, G)$ is minimized.

It is not hard to see that $\sum_{\{u,v\} \in E(G)} \lambda(uv, T, \varphi, G) = \sum_{\{x,y\} \in E(T)} \delta(xv, T, \varphi, G)$. Hence, in the rest of this report we may use them interchangeably.

Accordingly, in the communication graph embedding problems, the dilation of an edge $\{u, v\} \in E(G)$ abstractly represent the communication delay between vertices $u$ and $v$. Similarly the congestion of an edge $e \in E(T)$ is a representative for the traffic on the physical link $e$. Hence tree length measure corresponds to the average delay between the vertices of $G$ and also to the average edge congestion of the host tree.

## 2 Minimum Length of Tree Layout

In the special case of tree layout problem, the underlying host graph is a tree $T$ where the degree of every node is either 1 or 3 and the vertices of $G$ are being mapped to leaves of $T$. In this section we study minimum tree layout length. We show that Min Tree Length problem is NP-hard for multi-graphs and later on we show the problem stays NP-hard when restricted to the class of simple graphs.

### 2.1 Min Tree Length of Complete Graphs

Consider the complete graph $G = (V, E)$ where $\forall u, v \in V(G), \{u, v\} \in E(G)$. It is not hard to see that a layout tree $T$ is a solution for the Min Tree Layout problem for $G$, iff $|V_L(T)| = n$ (and hence $|V(T)| = 2n - 1$), and the summation of distance of leaf nodes of $T$ is minimized. We denote the summation of distances of leaves of a tree $T$ by:

$$\sigma_{LL}(T) = \frac{1}{2} \sum_{x,y \in V_L(T)} d_T(x, y)$$

Leaf to leaf distance summation measure $\sigma_{LL}$ is very similar to the definition of Wiener index (proposed by chemist Wiener [10]), which is the summation of distances of all vertices of a given graph $G$ as represented in following equation.

$$\sigma(G) = \frac{1}{2} \sum_{u,v \in V(G)} d_G(u, v)$$

---

2By multi-graph we refer to finite graphs with possibility of parallel edges and no loop.
Wiener index is widely studied both in mathematical and chemical literature. In [2] Fischermann et al. study Wiener index of trees. In this work authors represent the structure of the family of the trees that have minimum (or maximum) Wiener index among all the trees of the same order with maximum node degree $\Delta \geq 3$. Due to similarity of $\sigma_{LL}$ measure and Wiener index, in the rest of this subsection we borrow some of the notations and definitions from [2] in order to study $\sigma_{LL}$ for trees with maximum degree $\Delta$.

**Definition 2.1 ($\mathcal{T}(R, \Delta)$ tree family).** Consider integers $\Delta$ and $R \in \{\Delta, \Delta - 1\}$. For a given $n \in \mathbb{N}$, the family $\mathcal{T}(R, \Delta)$ of trees with $n$ nodes has a unique member $T$ up to isomorphism, defined using a planar embedding as it follows.

Let $M_k(R, \Delta) \leq n < M_{k+1}(R, \Delta)$ where:

$$M_k(R, \Delta) = \begin{cases} 1 & k = 0 \\ 1 + R + R \times (\Delta - 1) + \ldots + R \times (\Delta - 1)^{k-1} & k \geq 1 \end{cases}$$

Figure 1 depicts the embedding of tree $T$ with the following properties:

1. all nodes of $T$ lie on some line $L_i$ for $0 \leq i \leq k+1$
2. exactly one node $v$ lies on the line $L_0$ which has $\min\{n - 1, R\}$ children on line $L_1$
3. for $i = 1$ to $k - 1$ every node on line $L_i$ is connect to $\Delta - 1$ nodes on line $L_{i+1}$ and one node on line $L_{i-1}$
4. the only line that may be incomplete, is line $L_{k+1}$. Let $n - M_k(R, \Delta) = m \times (\Delta - 1) + r$ for $0 \leq r < \Delta - 1$, where $n - M_k(R, \Delta)$ is the number of remaining nodes on line $L_{k+1}$. Also let $\{v_1, v_{m+1}\}$ be the set of $m$ left most nodes on line $L_k$ where $v_{m+1}$ is the right most one in the set. Each of $v_1, \ldots v_m$ nodes is connected to $\Delta - 1$ nodes on line $L_{k+1}$, while $v_{m+1}$ is connected to $r$ nodes from line $L_{k+1}$ (see figure 1).

We defined the family $\mathcal{T}(R, \Delta)$ for the general case of trees with maximum degree $\Delta$, while we focus on the case where the degree of every internal node is $\Delta = 3$, but all the results presented in the rest of this subsection extend to all trees with arbitrary max degree $\Delta \geq 3$. 

![Figure 1: Planar embedding of $\mathcal{T}$ represented as the embedding of nodes on $k + 2$ lines $L_i$ for $0 \leq i \leq k + 1$.](image)
**Lemma 2.2.** Consider tree $T \in \mathcal{Ξ}(\mathcal{R}, \Delta)$ of order $n$, and assume $M_k(\mathcal{R}, \Delta) < n < M_{k+1}(\mathcal{R}, \Delta)$ for $k \geq 1$. Let $\overline{T}$ be an arbitrary tree of order $n$ constructed from tree $\overline{T}_0 \in \mathcal{Ξ}(\mathcal{R}, \Delta)$, with $M_k(\mathcal{R}, \Delta)$ nodes, by attaching $n - M_k(\mathcal{R}, \Delta)$ nodes to the leaf nodes of $\overline{T}_0$ (which lie on the line $L_k$). Then it is the case that either $\sigma_{LL}(\overline{T}) > \sigma_{LL}(T)$ or $\overline{T}$ is isomorphic with $T$.

**Proof.** We prove this lemma using induction on the height of tree $T$, as it follows.

Consider tree $L$ mapped to the right most node of $\overline{T}$, constructed from $L$. Node $v$ is connected to $\mathcal{R}$ subtrees $\{T_1, \ldots, T_R\}$. Based on the assumption of induction every subtrees $T_i$ is a member of the family $\mathcal{Ξ}(\Delta - 1, \Delta)$ of height $k$ or $k - 1$. Since $T$ and $\overline{T}$ are not isomorph, there are at least two subtrees $T_i$ and $T_j$ for $i \neq j$ where are incomplete on line $L_k$. Formally speaking $\exists 1 \leq i \neq j \leq \mathcal{R}, T_i, T_j \in \mathcal{Ξ}(\Delta - 1, \Delta)$ and $|V(T_i)| - M_{k-1}(\Delta - 1, \Delta) = r_i > 0 \land |V(T_j)| - M_{k-1}(\Delta - 1, \Delta) = r_j > 0$. Without loss of generality we assume $i = 1$ and $j = 1$ and also $|V(T_1)| \leq |V(T_2)|$. Figure 2 abstractly represents tree $T$.

![Figure 2](image1.png)

(a) Initial tree $\overline{T}$.

Figure 2: Figure 2a represents tree $\overline{T} \in \mathcal{Ξ}(\mathcal{R}, \Delta)$ of height $k + 1$, constructed from tree $\overline{T}_0 \in \mathcal{Ξ}(\mathcal{R}, \Delta)$ of height $k$. Node $v$ is connected to at least two subtrees $T_1, T_2 \in \mathcal{Ξ}(\Delta - 1, \Delta)$ of height $k$ which are incomplete on line $L_k$.

Alternative tree $\overline{T}'$ depicted in figure 2b is constructed by relocating some leaf nodes of subtree $\overline{T}_1$ that are on line $L_k$. As you see in this construction $r_2$ left most leaves of $\overline{T}_1$ are relocated to complete subtree $\overline{T}_2$. In this example the number of leaf nodes of $\overline{T}_1$ on $L_k$, is large enough to complete the subtree $\overline{T}_2$.

Let $L_l(T)$ denote the set of nodes of tree $T$ on line $l$. We present an alternative tree $\overline{T}'$, by relocating some leaf nodes of $L_k(\overline{T}_1)$ (in order from left to right) to complete the line $L_k$ of $\overline{T}_2$ (in order from right to left). Let $L \subseteq L_k(T_1)$ be the set of leaf nodes of $\overline{T}_1$ on line $L_k$, candidate for relocation. Figure 2b depicts tree $\overline{T}'$ constructed from $T$.

In the alternative tree $\overline{T}'$, consider a bijective mapping $\varphi_0$ from the nodes of $\overline{T}_1$ to nodes of subtree $\overline{T}_2$, where $\overline{T}_2$ is the modified version of $T_2$ in $T$. More specifically, mapping $\varphi_0$ is a reflection form $T_1$ to $T_2$, which reflects nodes of $L_l(\overline{T}_1)$ to the nodes of $L_{l-1}(\overline{T}_2)$ for $2 \leq l \leq k$, such that the left most node on line $L_l$ of $\overline{T}_1$ is mapped to the right most node of $\overline{T}_2$ on line $L_l$. Accordingly $\varphi_0$ maps every leaf node of $\overline{T} = L_k(\overline{T}_1) - L$ (the set of remaining leaf nodes of $\overline{T}_1$ on line $L_k$) to one leaf node of $\overline{T}_2$ on line $L_k$. On the other hand every leaf node of $L_{k-1}(\overline{T}_1)$ is mapped to one node (leaf or internal) of $\overline{T}_2$ on line $L_{k-1}$.

From $\varphi_0$ we construct a bijective mapping $\varphi$ from leaf nodes of $\overline{T}_1$ to sets of leaf nodes of $\overline{T}_2$. For every $w \in V_L(T)$, $\varphi(w) = \{\varphi_0(w)\}$ if $\varphi_0(w) \in V_L(T_2)$ is a leaf node node, otherwise $\varphi_0(w)$ is an internal node of $\overline{T}_2$ on line $L_{k-1}$, $\varphi$ maps $w$ to the set of direct children of $\varphi_0(w)$ on line $L_k$.

Using the bijective mapping $\varphi$, we analyze the change in value of $\sigma_{LL}$ in the process of constructing $\overline{T}'$ from $\overline{T}$ as it follows.

1. Clearly the internal summation of distances of nodes in $\mathcal{L}$ stays unchanged.

\[3\text{In this case } |\varphi(w)| \leq \Delta - 1.\]
2. For every leaf node \( w_1 \in \mathcal{L}, w_2 \in V_L(T) - V_L(T_1) \cup V_L(T_2), \) it is the case that \( d_T(w_1, w_2) = d_{\overline{T}}. \) Hence, the summation of distances among nodes in \( \mathcal{L} \) and leaf nodes of \( V_L(T) - V_L(T_1) \cup V_L(T_2) \) also does not change.

3. We show that for every \( w_1 \in \mathcal{L} \) the summation of distances of \( w_1 \) from leaf nodes of \( V_L(T_2) \cup V_L(T_1) - \{ w_1 \} \) is greater than the summation of distances of \( \varphi(w_1) \) from leaf nodes of \( V_L(T_1') \cup V_L(T_2') - \{ \varphi w_1 \}. \)

**Notation 2.3.** Let \( \mathcal{L}_1, \mathcal{L}_2 \subset V(T), \) for an arbitrary tree \( T. \) By \( \sigma_T(\mathcal{L}_1, \mathcal{L}_2) \) we denote the summation of distances of nodes of \( \mathcal{L}_1 \) and \( \mathcal{L}_2. \) Formally speaking:

\[
\sigma_T(\mathcal{L}_1, \mathcal{L}_2) = \frac{1}{2} \sum_{v \in \mathcal{L}_1} \sum_{w \in \mathcal{L}_2} d_T(v, u)
\]

For every \( w_2 \in V_L(T_1') - \mathcal{L}: \)

- If \( w_2 \) is on line \( \mathbf{L}_k, \) it is the case that \( d_T(w_1, w_2) = d_{\overline{T}}(\varphi(w_1), \varphi(w_2)). \) Therefore:

\[
(1) \quad \sigma_T(\{ w_1 \}, \{ w_2 \} \cup \varphi(w_2)) = \sigma_T(\{ \varphi(w_1) \}, \{ w_2 \} \cup \varphi(w_2))
\]

- If \( w_2 \) is on line \( \mathbf{L}_{k-1} \) and \( \varphi(w_2) \) maps \( w_2 \) to a leaf node of \( T'_2 \) on line \( \mathbf{L}_{k-1}, \) similar to previous case we have:

\[
(2) \quad \sigma_T(\{ w_1 \}, \{ w_2 \} \cup \varphi(w_2)) = \sigma_T(\{ \varphi(w_1) \}, \{ w_2 \} \cup \varphi(w_2))
\]

- Otherwise \( w_2 \) is on line \( \mathbf{L}_{k-1} \) and \( \varphi(w_2) \) maps \( w_2 \) to a non-empty set of leaf node of \( T'_2 \) on line \( \mathbf{L}_k. \) Assume the size of this set is \( 1 \leq \nabla \leq \Delta. \) Since for some nodes \( w_2 \) where \( |\varphi(w_2)| = \nabla = \Delta - 1 > 1, \) then for such \( w_2, \) the following equally holds:

\[
(3) \quad \sigma_T(\{ w_1 \}, \{ w_2 \} \cup \varphi(w_2)) - \sigma_T(\{ \varphi(w_1) \}, \{ w_2 \} \cup \varphi(w_2)) = \\
(d_T(w_1, w_2) + 2k \times \nabla) - (2k + (d_T(\varphi(w_1), \varphi_0(w_2)) + 1) \times \nabla) = \\
(d_T(w_1, w_2) + 2k \times \nabla) - (2k + (d_T(w_1, w_2) + 1) \times \nabla) = \\
(2k - d_T(w_1, w_2)) \times (\nabla - 1) - d_T(w_1, w_2) \geq \\
2k - 2d_T(w_1, w_2) > 0.
\]

Putting the results of previous cases and equations 1, 2 and 3 we conclude that \( \sigma_{LL}(\overline{T}) > \sigma_{LL}(\overline{T'}). \) If \( T'_1 \in \mathcal{X}(\mathcal{R}, \Delta), \) then based on the assumption of induction, replacing the subtree \( T'_1, \) with subtree \( T''_1 \in \mathcal{X}(\mathcal{R}, \Delta) \) of the same order, results in tree \( T''_1, \) where \( \sigma_{LL}(\overline{T''}) < \sigma_{LL}(\overline{T'}). \)

Using the same approach and continuing with \( \overline{T''}, \) in a sequence of leaf node relocations, we can construct the final tree \( \overline{T}, \) such that in \( \overline{T}, \) node \( v \) on line \( \mathbf{L}_0 \) has exactly one incomplete subtree \( \overline{T}_{i} \) where \( \overline{T}_{i} \in \mathcal{X}(\Delta - 1, \Delta). \) More specifically, \( \forall 1 \leq l < i, \) all leaf nodes of \( \overline{T}_{l} \) lie on line \( \mathbf{L}_l \) and \( \forall i < l \leq \Delta, \) all leaf nodes of \( \overline{T}_{l} \) lie on line \( \mathbf{L}_{k-1}, \) i.e. \( \overline{T} \in \mathcal{X}(\mathcal{R}, \Delta). \)

**Notation 2.4.** Given an arbitrary tree \( T, \) we define the planar line embedding of \( T \) similar to the approach in the definition 2.1. Starting from a designated \( v \in V(T), \) we embed \( T \) on lines of plane, where \( v \) lies on line \( \mathbf{L}_0 \) and direct neighbours of \( v \) are placed on line \( \mathbf{L}_1. \) Similarly all the nodes in distance \( d \) from \( v \) are placed on line \( \mathbf{L}_d. \) Also for \( u \in V(T) \) on line \( \mathbf{L}_l, \) the subtree rooted at \( u, \) where all its nodes are on lines \( \mathbf{L}_l \) for \( l' \geq l \) is denoted by \( T_u. \) Formally speaking \( w \in V(T_u) \) iff \( u \) is on the shortest path from \( w \) to \( v \) on line \( \mathbf{L}_0. \)
**Notation 2.5.** Consider a tree $T$ of order $n$ and a node $v \in V(T)$ of degree $\Delta$. Removing $v$ form $T$ partitions $T$ into a set of subtrees $\{T_1, \ldots, T_k\}$. We call node $v$ central if $\forall 1 \leq i \leq \Delta, |V(T_i)| \leq \frac{n}{2}$. The set of central nodes of tree $T$ is represented by $C_T$.

**Theorem 2.6.** Every arbitrary tree $T$ has at least one and at most two central nodes. In other word $1 \leq |C_T| \leq 2$.

For proof see [9]. Using this theorem, we proof the main result of this subsection as we present in the theorem 2.7

**Theorem 2.7.** Consider tree $T$ with maximum node degree $\Delta$. where $\sigma_{LL}(T) \leq \sigma_{LL}(T')$ for every tree $T'$ (that $|V_L(T')| = |V_L(T)|$). Then in the planar line embedding of $T$ with central node $v \in C_T$ on fixed line $L_0$, it is the case that:

- $T \in \mathcal{I}(\Delta, \Delta)$
- $T_u \in \mathcal{I}(\Delta - 1, \Delta)$ for $u \neq v$

**Proof.** The proof of this theorem is carried out using induction on the height of planar line embedding of tree $T$. It is not hard to check the correctness of theorem for trees of height 1 and 2. Let $T$ be a graph of height $h \geq 3$, and let $u_1, \ldots, u_\Delta$ be the direct neighbours of $v$ on line $L_0$ and $T_1, \ldots, T_\Delta$ respectively be their corresponding subtrees.

**Case 1:** $\exists w_1, w_2 \in V_L(T), d_T(w_1, v) \geq d_T(w_2, v) + 2$. Based on the assumption of induction $w_1$ and $w_2$ can not be on the same subtree. Without loss of the generality assume $w_1$ and $w_2$ are the two leaf nodes with maximum distance and $w_1 \in T_1$ and $w_2 \in T_2$. Let $T_{11}, T_{12}, \ldots, T_{\Delta-1} \subset T_1$ be subtrees respectively with roots $u_{11}, \ldots, u_{1,\Delta-1}$ connected to $u_1$ (nodes $u_1, \ldots, u_{1,\Delta-1}$ lie on line $L_2$). Also assume $w_1 \in V_L(T_{11})$.

**Case 1.1:** $|V_L(T_{11})| > |V_L(T_2)|$. We construct an alternative tree $\overline{T}$ by removing edges $\{u_1, u_{11}\}$ and $\{v, u_2\}$ and introducing two new edges $\{v, u_{11}\}$ and $\{u_1, u_2\}$. The structures of initial tree $T$ and the alternative tree $\overline{T}$ are represented in figure 3. One can verify that the following equation 4 correctly represents the relation

![Diagram](Figure 3: Representation of initial tree $T$ and its modified version $\overline{T}$.)
between $\sigma_{LL}(T)$ and $\sigma_{LL}(\overline{T})$.

4. \[ \sigma_{LL}(T) - \sigma_{LL}(\overline{T}) = \]
\[
= |V_L(T_{11})| \times (\sum_{2 \leq i \leq 1} |V_L(T_{1i})| - \sum_{3 \leq i \leq \Delta} |V_L(T_i)|)
+ |V_L(T_2)| \times (\sum_{3 \leq i \leq \Delta} |V_L(T_i)| - \sum_{2 \leq i \leq \Delta - 1} |V_L(T_{1i})|)
= (|V_L(T_{11})| - |V_L(T_2)|) \times (\sum_{3 \leq i \leq \Delta - 1} |V_L(T_{1i})| - \sum_{3 \leq i \leq \Delta} |V_L(T_i)|)
\]

It is the case that $\sum_{2 \leq i \leq \Delta - 1} |V_L(T_{1i})| < \sum_{3 \leq i \leq \Delta} |V_L(T_i)|$ otherwise it must be the case that $|V(T_1)| > \sum_{2 \leq i \leq \Delta} |V(T_i)| > \frac{|V(T)|}{2}$, which is in contradiction with the centrality of node $v$. Therefore $\sigma_{LL}(T) - \sigma_{LL}(\overline{T}) > 0$, which contradict the optimality of $T$.

Case 1.2: $|V_L(T_{11})| \leq |V_L(T_2)|$. Hence, $d_T(w_1, v) = d_T(w_2, v) + 2$ and $w_1 \in V_L(T_{11})$ is located on line $L_k$ and $w_2 \in V_L(T_2)$ lies on line $L_{k-2}$. Also non of subtrees $T_{11}$ and $T_2$ can be complete respectively on lines $L_k$ and $L_{k-1}$.

Let $L_l(T)$ denote the set of nodes of tree $T$ on line $l$. Similar to the proof of lemma 2.2, we present an alternative tree $\overline{T}$, by relocating some leaf nodes of $L_k(T_{11})$ (in order from left to right) to complete the line $L_{k-1}$ of $T_2$ (in order from right to left). Let $L \subseteq L_k(T_{11})$ be the set of leaf nodes of $T_{11}$ on line $L_k$, candidate for relocation. Based on an exact reasoning as in lemma 2.2 (case 3), which omit, it can be inferred that the summation of distance of leaf nodes in $L$ from leaf nodes of $T_{11} \cup T_2$ reduces going from $T$ to $\overline{T}$. Formally it can be deduced that:

5. \[ \sigma_T(L, V_L(T_{11})) + \sigma_T(L, V_L(T_2)) > \sigma_{\overline{T}}(\overline{L}, V_L(\overline{T}_{11})) + \sigma_{\overline{T}}(\overline{L}, V_L(\overline{T}_2)) \]

Where $\overline{T}_{11}$ and $\overline{T}_2$ respectively correspond to $T_{11}$ and $T_2$ after relocating leaf nodes of $L$ (represented by $\overline{L}$ in $\overline{T}$).

On the other hand, relocating $L$, increases the distance of every leaf node in $L$ from leaf node of $T_{12}, \ldots, T_{1,\Delta-1}$ by 1 unit, while it decreases the distance of every node of $L$ from every leaf node of $T_3, \ldots, T_\Delta$. Since we assumed that $w_1 \in V_L(T_{11})$ and $w_2 \in V_L(T_2)$ have the maximum distance among all leaf nodes, then:

6. \[ |V_L(T_{12})| + \ldots + |V_L(T_{1,\Delta-1})| < |V_L(T_3)| + \ldots + |V_L(T_\Delta)| \]

From equations 5 and 6 we conclude the following contradictory result:

7. \[ \sigma_{LL}(T) - \sigma_{LL}(\overline{T}) = \]
\[
= \sigma_T(L, V_L(T_{11})) - \sigma_{\overline{T}}(\overline{L}, V_L(\overline{T}_{11})) +
\sigma_T(L, V_L(T_2)) - \sigma_{\overline{T}}(\overline{L}, V_L(\overline{T}_2)) +
\sigma_T(L, V_L(T) - V_L(T_{11}) \cup V_L(T_2)) - \sigma_{\overline{T}}(\overline{L}, V_L(\overline{T}) - V_L(\overline{T}_{11}) \cup V_L(\overline{T}_2))
> 0
\]

Case 2: $\forall w_1, w_2 \in V_L(T), |d_T(w_1, v) - d_T(w_2, v)| < 2$. Since based on the assumption of induction $T_1, \ldots, T_\Delta \in \mathcal{T}(\Delta - 1, \Delta)$, then $T$ can be constructed from some tree $T_0 \in \mathcal{T}(\mathcal{R}, \Delta)$ of order $|V(T_0)| = M_k(\mathcal{R}, \Delta)$, by attaching $n - M_k(\mathcal{R}, \Delta)$ nodes to the leaf nodes of $T_0$. Therefore, based on lemma 2.2 $T$ is optimal iff $T \in \mathcal{T}(\Delta, \Delta)$.

Corollary 2.8. Consider the complete graph $G$ of order $n = |V(G)|$. Tree $T$ is an optimal tree layout for $G$ iff $|V_L(T)| = n$ and $T \in \mathcal{T}(3, 3)$. 

7
Example 2.9. Let $G$ be a complete graph of order $n = 2^l$ for some $l > 1$. Based on the result of theorem 2.7, a layout tree $T$ for $G$ (of order $2n - 1$) has minimum value $\sigma_{LL}$ (and accordingly is a solution for Min Layout Length) iff it is isomorphic to some tree with a structure similar to the tree in figure 7.

![Two planar embeddings of an optimal layout tree](image)

Figure 4: Two planar embeddings of an optimal layout tree $T$ of size $2n - 1$, corresponding to the complete graph $G$ with $n = 2^l$ vertices.

2.2 Min Tree Length of Multi-Graphs

Graph $G$ is a multi-graph if either it is a simple undirected graph, or it can be constructed from a simple undirected graph by adding parallel edges. In our main result of this report we show that the Min Tree Length problem is NP-hard for class of multi-graphs. Finally we show that this result can be extended to the class of simple graphs.

Definition 2.10 (Equal Size 4-Clique Cover). Given graph $G$, Equal Size 4-Clique Cover problem is the problem of partitioning $V(G)$ into four disjoint subsets $V_1, \ldots, V_4$ s.t. $G(V_i)$ is a clique of size $\frac{n}{4}$, for $1 \leq i \leq 4$.

Lemma 2.11. Equal Size 4-Clique Cover problem is NP-complete, even for the class of graphs of order $2^l$ vertices for some $l \in \mathbb{N}$.

For proof you can refer to [].

Theorem 2.12. Min Tree Length problem is NP-hard for the class of multi-graphs.

Proof. The correctness of the theorem can be represented using a polynomial reduction from Equal Size 4-Clique Cover.

Consider an arbitrary graph $G$, as an instance input of Equal Size 4-Clique Cover problem, where $\vert V(G) \vert = 2^l$ for some $l \in \mathbb{N}$. Let $G'$ be the multi-graph obtained from $G$ by introducing $M = m \times (2n - 2)$ parallel edges between every two vertices $u, v \in V(G)$. Notice that every tree layout $T$ for graph $G$ has $2n - 2$ edges where the congestion of each edge is less than $m = \vert E(G) \vert$. Considering graph $G$ as an vertex induced subgraph of $G'$, then $G' = G \cup \bar{G}$, where $\bar{G}$ is another vertex induced subgraph of $G'$. Subgraph $\bar{G}$ is complete multi-graph. Hence for every tree layout $T$ and $\phi : V(G) \rightarrow V_L(T)$ for $G'$ we have:

\[
L(T, \phi, G') = L(T, G) + L(T, \phi, \bar{G})
\]

From corollary 2.8 we know that a layout tree $\bar{T}$ for $\bar{G}$ is optimal iff $\bar{T} \in \mathcal{T}(3, 3)$. Also for every layout tree $T \in \mathcal{T}(3, 3)$, it is the case that $L(T, \bar{G}) \geq L(\bar{T}, \bar{G}) + M$. On the other hand for $n > 2$ and every layout tree $T$ where $\vert V_L(T) \vert = n$, it is always the case that $L(T, G) < M$. 

8
Therefore, a layout tree $T$ for $G'$ is optimal iff $T$ and $\tilde{T}$ are isomorphic. In other words $T$ for $G'$ is optimal iff $T \in \mathcal{S}(3,3)$. Hence the Min Tree Length problem for $G'$ reduces to the problem of finding an optimal bijection $\varphi$ form vertices of $G$ to leaf nodes of $\tilde{T} \in \mathcal{S}(3,3)$, such that the summation of edge dilations for all $\{u,v\} \in E(G)$ is minimized. Formally speaking:

$$
\text{(9) } \arg\min_{(T,\varphi)} L(T,\varphi,G') = (\tilde{T}, \arg\min_{\varphi} L(\tilde{T},\varphi,G))
$$

Let $\overline{G}$ denote the complement of graph $G$. One can easily check that:

$$
\text{(10) } L(\tilde{T},\varphi,G) = L(\tilde{T},\varphi,K_n) - L(\tilde{T},\varphi,G)
$$

Where $K_n$ is a complete graph of size $n$. Therefore:

$$
\text{(11) } \arg\min_{\varphi} L(\tilde{T},\varphi,G) = \arg\max_{\varphi} L(\tilde{T},\varphi,G) = \arg\max_{\{u,v\} \in E(\overline{G})} \lambda(u,v,\tilde{T},\varphi,G)
$$

Also based on the structure of $\tilde{T}$, $\exists c_1, c_2 \in C_{\overline{T}}$ (in figure 4 shown by double lined circles). Since $n = 2^l$ for $l \in \mathbb{N}$, removing central nodes $c_1$ and $c_2$ partition leaf nodes of $\tilde{T}$ into 4 equally sized partitions $L_1, \ldots, L_4$.

Finally, we can deduce that the original graph $G$ can be partitioned into 4 complete sub-graphs of size $\frac{n}{4}$, iff there exist bijection $\varphi$ such that $\forall \{u,v\} \in E(\overline{G}), \varphi(u) \in L_i \land \varphi(v) \in L_j$ for $1 \leq i \neq j \leq 4$. In other words, graph $G$ can be partitioned into 4 complete sub-graphs $G_1, \ldots, G_4$ of size $\frac{n}{4}$, iff $(\tilde{T},\varphi)$ is a solution for Min Tree Length of $G'$, where $\varphi$ maps vertices of $G_i$ to leaf nodes of $L_i$ for $1 \leq i \neq j \leq 4$. Which infers the NP-hardness of Min Tree Length problem for the multi-graphs.

\[\square\]

### 2.3 Min Tree Length of Simple Graphs

Finite graph $G'$ is simple, if for $u \neq v \in V(G')$ there is at most one edge $\{u,v\} \in E(G')$ and $\forall u \in V(G), \{u,u\} \notin E(G)$. Consider complete multi-graph $G$, where every two vertices $u$ and $v$ are connected via $l$ parallel edges. Having multi-graph $G$ one can obtain a simple graph $G$' by subdividing every edge $\{u,v\} \in E(G)$ and introducing a new vertex $x$ of degree 2.

Consider a tree layout $T'$ and bijective mapping $\varphi': V(G') \rightarrow VL(T')$ for the simple graph $G'$. For every $x \in V(G')$, $\varphi'(x) \in VL(T')$ is directly connected to some internal node $w \in V(T')$. Removing $w$ results in the set of three subtree $\{T'_{u,1}, \varphi'(u), T'_{w,2}\}$. For every $x \in V(G')$, by $T'_{1}(x)$ and $T'_{2}(x)$ we refer respectively to sub-tree $T'_{u,1}$ and $T'_{u,2}$, where $w$ is the direct neighbour of $\varphi'(x)$ in $T'$. It is easy to see that $\sigma(\{\varphi'(x),w\}, T', \varphi', G')$ is equal to the degree of $x$ in $G$, and also if $x$ is of degree 2 then the congestion of the two edges connecting $x$ to $T'_{u,1}$ and $T'_{u,2}$ is equal iff $\varphi'(u) \in V(T'_{u,1})$ and $\varphi'(v) \in T'_{u,2}$, where $u$ and $v$ are the direct neighbors of $x$ in $G$.

Consider tree layout $T$ and bijective mapping $\varphi: V(G) \rightarrow VL(T)$ for the multi-graph $G$. Starting from tree $T$, we can constructed a new tree $T'$ for the simple graph $G'$ by subdividing some edge of $T$ and introducing sub-tree layouts containing only vertices of degree 2. In other words, $T'$ is constructed from $T$ by introducing $m = |E(G)|$ internal node and $m$ leaf nodes (each corresponding to one vertex of $G'$ of degree 2). Figures 5a and 5b respectively depict the multi-graph $G$ and its corresponding tree layout $T$, while in figures 5b and 6b you can see simple graph $G'$ obtained from $G$ and one possible tree layout $T'$ for simple graph $G$' constructed from $T$.

**Lemma 2.13.** Let $G$ be a complete multi-graph where $\forall u,v \in V(G)$ there exist $l$ parallel edge $\{u,v\} \in E(G)$. Let $T$ and $\varphi$ be arbitrary tree layout and the corresponding mapping for $G$. Also assume $G'$ is the simple graph obtained by subdividing every edge of $T$ and introducing sub-tree layouts containing only vertices of degree 2. In other words, $T'$ is constructed from $T$ by subdividing each parallel edge $\{u,v\} \in E(G)$. We show the class of all possible layout trees for $G'$, constructed from $T$ by $\mathfrak{T}(T)$. Consider tree layout $T' \in \mathfrak{T}(T)$, where $\forall T'' \in \mathfrak{T}(T)$, $LA(T'',G) \leq LA(T'',G)$. Then it is the case that $T'$ is constructed by subdividing only edges of $T_0$ that each one is adjacent to a leaf node (which we call external edges).
Figure 5: Simple graph $G'$ obtained from $G$ by subdividing every edge of $G$ and introducing a vertex of degree 2.

Proof. We know that for every edge $e \in E(T), \sigma(e, T, G) \geq l \times (n - 1)$ where $n = V(G)$ and equality holds only if $e$ is adjacent to a leaf node $w \in V_L(T)$. Now consider an arbitrary $T'' \in \mathfrak{T}(T)$, obtained by subdividing at least one internal edge $e \in E(T)$; it is easy to check that $\forall e \in E(T), \sigma(e, T, G) \geq l \times (n - 2)$. As oppose to tree layout $T''$ for $G'$, we suggest tree layout $T'$ constructed by relocating the subtree (possibly more than one subtree) that subdivides $e$ (and hence removing all the subdivisions of $e$) and creating a new subdivision (possibly more than one subdivision) in an external edge $\{w, w'\}$ (assume $w \in V_L(T)$). We choose an external edge $\{w, w'\}$ such that for the every leaf node $\pi$ of the relocated subtree, $\{\varphi^{-1}(\pi), \varphi^{-1}(w)\} \in E(G')$. Hence based on the construction of $T'$ form $T''$, the following equation holds, which in turn evidences the correctness of the lemma.

$$LA(T'', G') - LA(T', G') > (l \times (n - 2)) - (l \times (n - 1)) > 0$$

From the result of lemma 2.13 one can infer the fact that given layout tree $T$ and mapping $\varphi$ for complete multi-graph $G$, an optimal tree layout (based on $T$) for simple graph $G'$ can be constructed by subdividing only external edges of $T$ and introducing sub-tree layouts corresponding to the new vertices of degree 2. But it does not provide any information regarding the exact structure of the optimal tree. In what follows and without providing all the details of the proof, we present the structure of the optimal layout tree for $G'$, constructed from $T$. Note that for the sake of the main theorem in this subsection, we do not need to know the exact structure of the three layout with minimum tree length.

The simple graph $G'$ contains exactly $l \times \frac{n(n-1)}{2}$ vertices of degree 2. Every vertex $v \in G'$ of degree $l \times (n - 1)$ is directly connected to $l \times (n - 1)$ vertices of degree 2. The optimal tree layout $T'$ obtains from $T$, by subdividing every external edge $\{w, w'\} \in E_E(T)$ (where $w \in V_L(T)$) exactly once with a sub-tree layout containing $l \times (n - 1)$ leaf nodes. Every leaf node of this subtree correspond to one vertex $x \in V(G')$ with degree 2 where $x$ is directly connected to $w$.

---

4 Edge $e \in E(T)$ is external if it is adjacent to a leaf node of $T$, and internal otherwise. Let $E_I(T)$ and $E_E(T)$ respectively represent the set of internal and external edges of $T$

5 For the sake of the main theorem in this section, we assume $l$ is an even integer.
\( \sigma = 6 \quad \sigma = 6 \quad \sigma = 6 \quad \sigma = 6 \)

(a) An arbitrary tree layout \( T \) for \( G \).

\( \sigma = 8 \quad \sigma = 4 \quad \sigma = 6 \quad \sigma = 6 \)

(b) One possible tree layout \( T' \) for \( G' \), constructed from \( T \). As you can see, \( T' \) is obtained by subdividing some edges of tree \( T \) (possibly several times) and introducing sub-tree layouts containing only vertices of degree 2.

Figure 6: Tree layout \( T' \) for simple graph \( G' \), constructed from initial tree layout \( T \). The congestion of every edge is include by that edge.

Let \( T \) be an subtree of tree layout \( T' \) where its leaf nodes correspond to only vertices of degree 2 in \( G' \). Since no two distinct vertices \( x, y \in V(G') \) of degree 2 are neighbors, the summation of congestions of edges of a subtree \( T \) (with a fixed number of nodes) is minimum only when \( T \) is a complete rooted binary tree with no node of degree 2. The root node of \( T \) is directly connected to the internal node subdividing edge \( \{w, w'\} \in E_T(T) \). In the suggested optimal tree \( T' \) constructed from \( T \), for every newly introduced edge \( e \), we have \( \sigma(T', G') \leq l \times (n - 1) \).

Figure 7 depicts the structure of an optimal layout tree for \( G' \) constructed from the initial layout tree \( T \). Accordingly the value of optimal layout \( T' \) for \( G' \), based on the initial tree layout \( T \) for \( G \), is equal to:

\[
LA(T', G') = LA(T, G) + n \times (T + l \times (n - 1))
\]

Where the constant \( T \) is the summation of all edges’ congestion of an optimal subtree, containing \( \frac{l}{2} \times (n - 1) \) leaf nodes, such that every leaf node correspond to a vertex of degree 2 in \( G' \). Also constant \( l \times (n - 1) \) is the congestion of every external edge in \( T \). As you can see, the constant part of this equation does not depend on the structure of the initial tree layout \( T \). Hence consider two tree layouts \( T' \) and \( T'' \) for \( G' \), respectively optimally obtained from tree layouts \( T_0 \) and \( T_0'' \) for complete multi-graph \( G \). Then \( LA(T', G') < LA(T'', G') \) iff \( LA(T_0', G) < LA(T_0'', G) \).

**Corollary 2.14.** Let \( T \in \mathcal{X}(3, 3) \) and \( \varphi : V(G) \to V_L(T) \) be the solution of the Min Tree Length problem for the complete multi-graph \( G \), where every two distinct vertices are connected via \( l \) parallel edges. Assume \( G' \)
is the simple graph obtained from $G$ by subdividing every edge with a vertex of degree 2. Also, let layout $T'$ and bijective mapping $\varphi' : V(G') \rightarrow V_L(T')$ be the optimal solution of Min Tree Length problem for graph $G'$. Then it is the case that:

- $\forall v \in V(G') \cap V(G)$, $\varphi'(v) = \varphi(v)$
- $T'$ is constructed from $T$ by subdividing every external edge $\{w, w\}'$ of $T$ using sub-tree layout $T_w$, containing $\frac{1}{2} \times (n - 1)$ leaf nodes, where
  - for every leaf node $\overline{x} \in V_L(T_w)$, $\varphi'^{-1}(\overline{x}) = x \in V(G')$, where $x$ is directly connected to $\varphi'^{-1}(w)$ in $G'$.

Facilitating the result of corollary 2.14, we conclude this section by showing that the Min Tree Length problem stays NP-hard even for the class of simple graphs.

**Theorem 2.15.** Min Tree Length problem is NP-hard for the class of simple graphs.

**Proof.** This theorem can be proven using a similar approach as we used in the proof of theorem 2.12 by a reduction form Equal Size 4-Clique Cover problem.

Hence given graph $G$, as an instance input of Equal Size 4-Clique Cover problem, we construct multi-graph, by introducing $M = m \times (2n - 2)$ parallel edges between every two vertices $u, v \in V(G)$. In the next step we obtain a simple graph $G'$ by subdividing every newly introduced edges.

Considering graph $G$ as an vertex induced subgraph of $G'$, then $G' = G \cup \overline{G}$, where $\overline{G}$ is also a vertex induced subgraph of $G'$. $\overline{G}$ is the simple graph obtained from a complete multi-graph $G_0$, by subdividing every edge. Hence for every tree layout $T$ and $\varphi : V(G) \rightarrow V_L(T)$ for $G'$ we have:

$$L(T, \varphi, G') = L(T, \varphi', G) + L(T, \varphi, \overline{G})$$

Where $\varphi'$ is partially defined from $\varphi$, in other words, $\varphi'(v) = \varphi(v)$ for every $v \in V(G)$.

From corollary 2.14 we know that a layout tree $\overline{T}$ for $\overline{G}$ is optimal iff $\overline{T}$ is optimally constructed from a tree layout $T_0 \in \mathcal{X}(3, 3)$ for $G_0$. Also for every layout tree $T''$ optimally constructed from some tree layout $T_i \in \mathcal{X}(3, 3)$, it is the case that $L(T'', \overline{G}) \geq L(\overline{T}, \overline{G}) + M$. On the other hand for $n > 2$ and every layout tree $T$ where $|V_L(T)| = n$, it is always the case that $L(T, G) < M$.

Optimal tree $\overline{T}$ for $G'$ has a similar structure to the structure of $T_0 \in \mathcal{X}(3, 3)$, in the sense that:
• the leaf nodes can be partitioned into the 4 subtrees $T_1, \ldots, T_4$ of the same size and isomorphic structure,

• every subtree $T_i$ contains $n/4$ leaf nodes, corresponding to the vertices of $G$, where

$$\forall w_1, w_2 \in T_i, w_2 \in T_j \text{ for } i \neq j, \text{ where } \varphi^{-1}(w_1), \varphi^{-1}(w_2), \varphi^{-1}(w_3) \in V(G), \text{ it is the case that } d_{T}(w_1, w_2) < d_{T}(w_1, w_3).$$

In other words, the distance of every two leaf nodes $w_1, w_2$ (corresponding to vertices of $G$) in the same subtree $T_i$ is less than the distance of every two leaf nodes that belong to two distinct subtrees $T_i$ and $T_j$.

Hence the Min Tree Length problem for $G'$ reduces to the problem of finding an optimal bijection form vertices of $G$ to the leaf nodes of $T$ (that correspond to the vertices of $G$), such that the summation of edge dilations for all $\{u, v\} \in E(G)$ is minimized. Therefore, similar the proof of theorem 2.11 and omitting the details, it can be inferred that $G$ is an positive instance of 4-Clique Cover problem, iff using the optimal tree layout $T$ for $G'$, vertices of $G$ can be partitioned into 4 complete graphs of size $n/4$, which indicates the NP-hardness of Min Tree Length problem for the class of simple graphs.

3 Layout Tree Problem in Relation with Graph Reassembling

In this section we study the relation between tree layout problem and graph reassembling problem as defined in [6]. Graph reassembling problem plays a key role in the efficiency of main programs in earlier work on a domain-specific language (DSL) for the design of flow networks [1, 4, 5].

Consider a simple graph graph $G = (V, E)$ (not necessarily connected), partitioned into the set of $|V|$ one-vertex components by cutting every edge into into two halves. Reassembling of the graph $G$ corresponds to the problem of finding the sequence of edge reconnections $\Theta$ that minimizes two measures that depend on the edge-boundary degrees of assembled components in the intermediate steps of reassembling $G$. The first step of $\Theta$ initiates with the set of one-vertex components, and the final step results in the initial graph $G$. The optimization goal of the graph reassembling can be either minimizing the maximum edge-boundary degree encountered during the reassembling process, which is called the $\alpha$-measure of the reassembling, or the sum of all edge-boundary degrees, denoted by $\beta$-measure.

The reassembling sequence $\Theta$ for graph $G$ correspond to a unique binary tree $B$ (called binary reassembling tree), where the set of leaf nodes is bijectively related to the set of one-vertex components and the root node correspond to the reassembled graph $G$. Every internal node $w \in V_I(B)$, as the root of the a subtree $B_w$, correspond to the vertex induced sub-graph of $G$ comprising the set of vertices represented by leaf nodes of $B_w$.

Hence, the $\alpha$-optimal reassembling problem for graph $G$ is the problem of finding a rooted binary tree $B$ and a bijective mapping from vertices of $G$ to leaf node of $B$, such that the maximum edge congestion of $B$ is minimized. Similarly the $\beta$-optimal reassembling problem is the problem of finding a rooted binary tree $B$ and a bijective mapping from vertices of $G$ to leaf node of $B$, where tree length of $B$ is minimized. It is easy to see that all the result for the minimum tree layout congestion problem can be directly inferred for the case where the underlying tree is rooted. On the other hand the same statement can not immediately be inferred for minimum tree layout length problem. Therefore in this section we study the Min Tree Length problem where the host graph is a rooted binary tree (Min Rooted Tree Length problem for short).

**Lemma 3.1.** Min Tree Length problem is NP-hard for the class of graphs $\mathcal{G}_{\nabla=1}$ where for every member $G \in \mathcal{G}_{\nabla=1}$, $G$ is connected and exists $v \in V(G)$ of degree 1 (i.e. $\mathcal{G}_{\nabla=1}$ is a the class of graphs with min degree $\nabla = 1$).

**Proof.** One dimidiate result of the methods that are used in proofs of theorems 2.12 and 2.15 is that the Min Tree Length problem stays NP-hard even for the class of congested graphs. A graph $G$ with minimum vertex degree $\nabla$ is congested if for every tree layout $T$ for $G$ it is the case $\sigma(e, T, G) \geq \nabla$ for every edge $e \in E(T)$.

---

The graphs that are used in both proof are clearly congested graphs.
Consider congested graph $G$ with min degree $\nabla$. If $\nabla = 1$ we are done, otherwise let $v \in V(G)$ be a vertex with degree $\nabla$ and $G'$ be the graph constructed by augmenting $G$ with a new vertex $u$ and edge $\{u, v\}$. Also let tree layout $T'$ and mapping $\varphi'$ be a solution for Min Tree Length problem of $G'$ where $\{\varphi'(v), w\}$ is edge incident to leaf node $\varphi'(v)$ (with congestion $\nabla$). It is not hard to verify that it must be the case that $\{\varphi'(u), w\} \in E(T')$.

Let $T$ be a layout tree for $G$, obtained from $T'$ after removing vertices $\varphi'(u)$ and $w$ (and their incident edges) and introducing edge $\{\varphi'(v), w'\}$. Where $w'$ is the third neighbor of $w$ in $T'$. Hence the following equality holds.

$$LA(T', \varphi', G') = LA(T, \varphi, G) + 2$$

Where $\varphi(v) = \varphi'(v)$ for every $v \in V(G)$.

**Claim 3.2.** Tree layout $T$ and along side with the bijective mapping $\varphi$ is a solution for Min Tree Length problem of $G$.

Assume there exist Tree layout $\tilde{T}$ and bijective mapping $\tilde{\varphi}$ such that $LA(\tilde{T}, \tilde{\varphi}, G) < LA(T, \varphi, G)$. Hence one can construct a tree layout $\tilde{T}'$ and bijective mapping $\tilde{\varphi}'$ for $G'$ by subdividing the edge incident to leaf node $\tilde{\varphi}(v)$ and introducing internal node $w$ which is directly connected to leaf node $\tilde{\varphi}'(u)$. Therefore:

$$LA(\tilde{T}', \tilde{\varphi}', G') = LA(\tilde{T}, \tilde{\varphi}, G) + 2 < LA(T', \varphi', G')$$

Which contradicts the assumption that $T'$ and $\varphi'$ are a solution for Min Tree Length problem of $G'$.

Using the result of lemma 3.1 it can be shown that Min Rooted Tree Length is NP-hard for the class of non necessary connected graphs. Later on we extend the this result to the class of connected graphs.

**Theorem 3.3.** Consider the finite undirected graph $G$ (non necessarily connected). The problem of finding a bijective mapping from the vertices of $G$ so some rooted binary tree $T$ with minimum length is not polynomially solvable unless $P=NP$.

**Proof.** We proof this theorem using the NP-hardness of Min Tree Length problem for the class of $G_{\nabla=1}$. Hence given a graph $G \in G_{\nabla=1}$, we obtain a disconnected graph $\overline{G} = G \cup G_0$ where $G_0 = (\{v\}, \{\})$ is a one-vertex graph. Assume rooted binary tree $\overline{T}$ and bijective mapping $\overline{\varphi}$ are the solution for the Min Rooted Tree Length problem of the augmented graph $\overline{G}$.

**Claim 3.4.** (I) $w_1 = \overline{\varphi}(v)$ is directly connected to the root node $r$ of $\overline{T}$, and

(II) let $w_2$ be the second direct neighbor of the root of $\overline{T}$. Node $w_2$ subdivides an edge (or equivalently, is connected to two edges) with congestion 1. Figure 8 depicts the claimed structure of $\overline{T}$.

![Figure 8: The structure of rooted tree layout for augmented graph $\overline{G}$](image-url)
Proof of I. Assume the opposite holds. Therefore \( w_1 = \varphi(v) \) is directly connected to an internal node of \( w' \in V_I(T) \), where \( w' \) is incident to two edge \( e_1, e_2 \in E(T) \) such that \( \sigma(e_1, T, \varphi, G) = \sigma(e_2, T, \varphi, G) > 0 \). We construct an alternative rooted tree layout \( T' \) from \( T \) as described following. We remove edge \( \{w_1, w'\} \) and \( w' \) and introducing an new edge, replacing \( e_1 \) and \( e_2 \). Also we introduce a new root node \( r' \) and connect it to \( w_1 \) and \( r \). Based on the construction process of \( T' \), it is the case that \( LA(T', G) - LA(T', G) = \sigma(e_2, T, \varphi, G) > 0 \), which contradicts the assumption of optimality of \( T \).

Proof of II. Clearly exists edge \( e \in E(\bar{T}) \) such that \( \sigma(e, \bar{T}, \varphi, \bar{G}) = 1 \). Hence assuming that \( w_2 \) subdivides an edge with congestion greater that 1, similar to the approach in the proof of II, one can obtain an alternative rooted tree layout \( T' \) with the contradictory property \( LA(T', \bar{G}) < LA(T, \bar{G}) \).

Claim 3.5. Graph \( G \in G_{k=1} \) has tree length less than \( k \), iff the augmented graph \( \bar{G} \) has a rooted tree layout with tree length less than \( k + 1 \). More specifically given an rooted tree layout \( T \) and bijective mapping \( \varphi \) for augmented tree \( \bar{G} \) (with the structure presented in claim 3.4 and figure 8), removing nodes \( r, w_1 \) and \( w_2 \) and joining the two edges of congestion 1 incident to \( w_2 \), obtains an optimal tree layout \( T \) and mapping \( \varphi \) for graph \( G \) (where \( \varphi(u) = \varphi(u) \) for every \( u \in V(G) \)).

Proof. It is easy to see that \( LA(T, G) = LA(T, G) + 1 \). Now assume there exist a tree layout \( T' \) and bijective mapping \( \varphi' \) for \( G \) such that \( LA(T', \varphi', G) < LA(T, \varphi, G) \). On the other hand for every tree layout \( T' \) for \( G \), exists \( e \in E(T') \) where \( \sigma(e, T', G) = 1 \). From \( T' \) (and \( \varphi' \) ), an alternative rooted tree layout \( T' \) (and bijective mapping \( \varphi' \) ) can be obtained by introducing three new nodes, node \( r \) designated as the root of \( T' \), leaf node \( w_1 \), directly connected to \( r \) (where \( \varphi'(w_1) = v \) ) and internal node \( v \) which subdivides edge \( e \). It can be verified that \( LA(T', \varphi', G) = LA(T', \varphi', G) + 1 \). Hence the following contradictory result concludes the proof of this theorem:

\[
LA(T', \varphi', G) < LA(T, \varphi, G) + 1 = LA(T, \varphi, G)
\]

\( \Box \)

4 Tree Length of Routing Trees

In the previous sections we focused on the problem of embedding vertices of an input graph \( G \) into leaf nodes of a host tree \( T \), where the degree of every internal node of \( T \) is 3, known as tree layout problem. In this section we extend some results to the general routing tree problems. In this problem the vertices of the source graph \( G \) are being embedded into the leaf nodes of some communication tree \( T \) with fixed maximum degree \( \Delta \).

Definition 4.1 (Minimum Routing Tree Length). Given graph \( G \) and integer \( \Delta \), Minimum Routing Tree Length problem (Min Routing Length for short) is the problem of finding tree \( T \) with maximum degree \( \Delta \) and a bijective mapping \( \varphi: V(G) \to V_L(T) \), such that \( LA(T, \varphi, G) \) is minimized. \( \Box \)

Proof of our final result on Min Routing Length problem is build on some intermediate result as presented in what follows.

Definition 4.2 (Fixed Size k-Clique Cover). Consider graph \( G = (V, E) \) and \( k \) positive integers \( n_1, \ldots, n_k \) where \( \sum_{1 \leq i \leq k} n_i = n \). Fixed Size k-Clique Cover problem is the problem of partitioning \( V \) into \( k \) disjoint subsets \( V_1, \ldots, V_k \) s.t. \( G(V_i) \) is a clique of size \( n_i \), for \( 1 \leq i \leq k \). \( \Box \)

Lemma 4.3. Fixed Size k-Clique Cover is NP-complete.

Proof. Similar to the proof of NP-completeness of Fixed Size k-Clique Cover problem as a variation of graph \( k \)-colorability problem. \( \Box \)
**Definition 4.4 (Equal Size k-Clique Cover).** Given graph $G = (V, E)$ where $|V| = k \times l$ for some $l \in \mathbb{N}$, Equal Size $k$-Clique Cover problem is the problem of partitioning $V$ into $k$ disjoint subsets $V_1, \ldots, V_k$ s.t. $G(V_i)$ is a clique of size $\frac{n}{k}$, for $1 \leq i \leq k$.

**Lemma 4.5.** Equal Size $k$-Clique Cover problem is NP-complete.

**Proof.** Similar to the proof of NP-completeness of Equal Size $k$-Clique Cover problem.

Consider the class of graphs where for every graph $G$ in this class, $\exists l \in \mathbb{N}$ such that $|V(G)| = k \times (k - 1)^l$. Equal Size $k$-Clique Cover problem stays NP-complete for this class. Concisely in what follows, a polynomial reduction from Equal Size $k$-Clique Cover problem for general graphs is presented. Given graph $G = (V, E)$ where $|V| = k \times n'$ for $n' \in \mathbb{N}$, one can obtain graph $G'$ by augmenting $G$ with $k$ complete components $C_1, \ldots, C_k$ of size $(k - 1)^l - n'$, where $l$ is the smallest integer such that $(k - 1)^l \geq n'$. Also in $G'$ every newly introduces vertex $v \in V(C_1) \cup \ldots \cup V(C_k)$ is connected to every vertex $u \in V(G)$. It is not hard to check that $|V(G')| = k \times (k - 1)^l$ and more importantly $G$ is a positive instance of Equal Size $k$-Clique Cover problem if $G'$ is a positive instance of Equal Size $k$-Clique Cover problem.

In the rest of this section we only consider graph of order $|V(G)| = k \times (k - 1)^l$. Obviously all the harness results for this class immediately extend to the class of general sized graphs.

**Theorem 4.6.** Given multi-graph $G$ and integer $\Delta$, the problem of finding a routing tree $T$ and bijective mapping $\varphi : V(G) \to V_L(T)$ with minimum tree length is NP-hard.

**Proof.** Similar to the proof of theorem 2.12 it can be shown that the Equal Size $k$-Clique Cover problem is not harder than Minimum Routing Tree Length problem.

Hence, consider graph $G$ as the input of the Equal Size $k$-Clique Cover problem where $|V(G)| = k \times (k - 1)^l$ for some $l \in \mathbb{N}$. Let $G'$ be the multi-graph obtained from $G$ by introducing $M = m \times (2n - 2)$ parallel edges between every two vertices $u, v \in V(G)$. Therefore $G' = G \cup \tilde{G}$, where complete multi-graph $\tilde{G}$, is a vertex induced subgraph of $G'$. Using similar reasoning as in the proof of theorem 2.12 $\tilde{G}$ dictates the structure of optimal routing tree for $G'$. In other words, the problem of finding optimal routing tree $T$ and mapping $\varphi$ for $G'$ reduces to the problem of finding mapping $\varphi$ from vertices of original graph $G$ to the leaf nodes of a fixed-structure tree $T$ such that $LA(T, \varphi, G)$ is minimized. Based on corollary 2.8 $T \in \mathcal{T}(k, k)$.

Removing the only central node of $T \in \mathcal{T}(k, k)$ partitions $T$ into $k$ subtrees $T_1, T_2, \ldots, T_k \in \mathcal{T}(k, k)$ of the same order and $(k - 1)^l$ leaf nodes. As explained with more details in the proof of theorem 2.12 it can be inferred that $G$ is a positive instance of the Equal Size $k$-Clique Cover problem (in other words vertices of $G$ can be partitioned into $k$ equal sized complete sub-graphs $G_1, G_2, \ldots, G_k$) iff given routing tree $T$ and mapping $\varphi$ as the solution for Min Routing Length problem (with $\Delta = k$), $\varphi$ maps vertices of $G_i$ to leaf nodes of $T_i$ for $1 \leq i \leq k$.

**References**

[1] Azer Bestavros and Assaf Kfoury. A Domain-Specific Language for Incremental and Modular Design of Large-Scale Verifiably-Safe Flow Networks. In Proc. of IFIP Working Conference on Domain-Specific Languages (DSL 2011), EPTCS Volume 66, pages 24–47, Sept 2011.

[2] Miranca Fischermann, Arne Hoffmann, Dieter Rautenbach, László Székely, and Lutz Volkmann. Wiener index versus maximum degree in trees. Discrete Applied Mathematics, 122(1):127–137, 2002.

[3] Jordi Petit i Silvestre. Layout problems. PhD thesis, Ph. D. thesis, Universitat Politecnica de Catalunya, Barcelona, 25 May, 2001.

[4] Assaf Kfoury. The Denotational, Operational, and Static Semantics of a Domain-Specific Language for the Design of Flow Networks. In Proc. of SBLP 2011: Brazilian Symposium on Programming Languages, Sept 2011.
[5] Assaf Kfoury and Saber Mirzaei. A different approach to the design and analysis of network algorithms. 2013.

[6] Assaf Kfoury and Saber Mirzaei. Efficient reassembling of graphs, part 1: The linear case. Tech Report of CS Dept., Boston University, 2015.

[7] Burkhard Monien. The complexity of embedding graphs into binary trees. In Fundamentals of computation theory, pages 300–309. Springer, 1985.

[8] Paul D. Seymour and Robin Thomas. Call routing and the ratcatcher. Combinatorica, 14(2):217–241, 1994.

[9] Yossi Shiloach. A minimum linear arrangement algorithm for undirected trees. SIAM Journal on Computing, 8(1):15–32, 1979.

[10] Harry Wiener. Structural determination of paraffin boiling points. Journal of the American Chemical Society, 69(1):17–20, 1947.