Regularity theory for time-fractional advection-diffusion-reaction equations

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Abstract

We investigate the behavior of the time derivatives of the solution to a linear time-fractional, advection-diffusion-reaction equation, allowing space- and time-dependent coefficients as well as initial data that may have low regularity. Our focus is on proving estimates that are needed for the error analysis of numerical methods. The nonlocal nature of the fractional derivative creates substantial difficulties compared with the case of a classical parabolic PDE. In our analysis, we rely on novel energy methods in combination with a fractional Gronwall inequality and certain properties of fractional integrals.

Keywords: Fractional PDE, regularity analysis, energy arguments, fractional Gronwall inequality

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1. Introduction

This paper is the sequel to a study [1] of existence and uniqueness of the weak solution to a time-fractional PDE of the form

$$\partial_t u - \nabla \cdot (\kappa \nabla \partial_t^{1-\alpha} - F \partial_t^{1-\alpha} - G)u + (a \partial_t^{1-\alpha} + b)u = g$$

for $x \in \Omega$ and $0 < t \leq T$, subject to the boundary and initial conditions

$$u(x,t) = 0 \quad \text{for } x \in \partial \Omega \text{ and } 0 \leq t \leq T, \quad (2)$$
$$u(x,0) = u_0(x) \quad \text{for } x \in \Omega. \quad (3)$$

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Various special cases of this problem occur in descriptions of subdiffusive transport, with the parameter \( \alpha \) arising from a continuous-time, random walk model [2, section 3.4] in which the waiting-time distribution is a power law decaying like \( t^{-1-\alpha} \) as \( t \to \infty \) [3, 4, 5, 6, 7, 8]. For more details, see our related paper [1].

Our purpose here is to derive estimates for the derivatives of \( u \), motivated by their crucial role in the error analysis of numerical methods [9, 10, 11, 12, 13, 14, 15, 16] for applications included in the class [1] of time-fractional problems. For the basic fractional diffusion equation, given by the special case \( \bar{F} = \bar{G} = \bar{u} = 0 \) and \( a = b = 0 \), the solution admits a series representation via separation of variables, which, in combination with the asymptotics of the Mittag-Leffler function, yields bounds on the time derivatives of \( u \) in various spatial norms [17, 18]. One may also represent the solution in terms of a fractional resolvent [19, 20, 21]. These simple approaches no longer work in the general case, and the analysis that follows relies instead on the tools used in our study [1] of well-posedness: energy methods and a fractional Gronwall inequality.

We assume that \( 0 < \alpha < 1 \) and that the spatial domain \( \Omega \subseteq \mathbb{R}^d \) \((d \geq 1)\) is bounded and Lipschitz. The coefficients \( \bar{F}, \bar{G}, a, b \), as well as the source term \( g \), may depend on \( x \) and \( t \), but the generalized diffusivity \( \kappa = \kappa(x) \) may depend only on \( x \). Our theory requires that for appropriate \( m \geq 1 \),

\[
\bar{F}, \bar{G} \in C^{m+1}([0, T]; W^m_\kappa(\Omega)^d) \quad \text{and} \quad a, b \in C^m([0, T]; L_\kappa(\Omega)),
\]

where \( W^k_\kappa(\Omega) \) denotes the Sobolev space of functions with all partial derivatives up to and including order \( k \) belonging to \( L_\kappa(\Omega) \). The generalized diffusivity is permitted to be a bounded, \( d \times d \) matrix-valued function, that is, \( \kappa \in L_\kappa(\Omega; \mathbb{R}^{d \times d}) \). In addition, we ensure that the spatial operator \( \nu \mapsto -\nabla \cdot (\kappa \nu) \) is uniformly elliptic on \( \Omega \) by assuming \( \kappa(x) \) is symmetric and positive-definite with its minimal eigenvalue bounded away from zero. The fractional time derivative is understood in the Riemann–Liouville sense, that is \( \partial_t^{1-\alpha} v(x, t) = \frac{d}{dt} \mathscr{I}^\alpha v(x, t) \) with \( \mathscr{I}^\alpha \) the fractional integral given by

\[
\mathscr{I}^\alpha v(x, t) = \int_0^t \omega_\alpha(t-s)v(x,s)ds \quad \text{where} \quad \omega_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha).
\]

Let \( \langle \cdot , \cdot \rangle \) denote the inner product in \( L_2(\Omega) \) or \( L_2(\Omega)^d \). The weak solution \( u \) of [1] is defined by the condition

\[
\langle u(t), v \rangle + \langle (\kappa \mathscr{I}^\alpha \nabla u)(t), \nabla v \rangle - \langle (B_1 u)(t), \nabla v \rangle + \langle (B_2 u)(t), v \rangle = \langle f(t), v \rangle, \tag{5}
\]

for all \( v \in H^1_0(\Omega) \), where \( f(t) = u_0 + \mathscr{I}^{1-\alpha} g(t) \) and

\[
B_1 \phi(t) = \mathscr{I}^{1}(\bar{F} \partial_t^{1-\alpha} \phi)(t) + \mathscr{I}^{1}(\bar{G} \phi)(t), \\
B_2 \phi(t) = \mathscr{I}^{1}(a \partial_t^{1-\alpha} \phi)(t) + \mathscr{I}^{1}(b \phi)(t). \tag{6}
\]

To see why, take the inner product of [1] with \( v \), apply the first Green identity and integrate in time (making use of the initial condition). We proved in our previous paper [1, Theorems 4.1 and 4.2] that the above problem is well-posed in the following sense.
Theorem 1. Assume that the coefficients satisfy (4) for \( m = 1 \), that the source term \( g \) satisfies \( \|g(t)\| \leq Mt^{n-1} \) for \( 0 < t \leq T \), where \( M \) and \( \eta \) are positive constants, and that the initial data \( u_0 \in L^2(\Omega) \). Then, problem (1) has a unique weak solution \( u \in L^2((0,T);L^2(\Omega)) \) that satisfies (5), and is such that

1. The restriction \( u : (0,T) \rightarrow L^2(\Omega) \) is continuous.
2. If \( 0 < t \leq T \), then \( u(t) \in H^1_0(\Omega) \) with \( \|u(t)\| + t^{\alpha/2}\|\nabla u(t)\| \leq C(\|u_0\| + Mt^\eta) \).
3. \( \mathcal{S}^\alpha u, B_2u \in C([0,T];L^2(\Omega)) \) and \( \mathcal{S}^\alpha \nabla u, B_1u \in C([0,T];L^2(\Omega)^d) \).
4. If \( t = 0 \), then \( \mathcal{S}^\alpha u = B_2u = 0, \mathcal{S}^\alpha \nabla u = B_1u = 0 \) and \( u(0) = u_0 \).
5. If \( t \rightarrow 0 \), then \( \langle u(t),v \rangle \rightarrow \langle u(0),v \rangle \) for each \( v \in L^2(\Omega) \).

Sakamoto and Yamamoto [22, Corollary 2.6] show, for example, that if \( g = 0 \) then the solution of (7) satisfies a bound of the form \( \|\partial_t^m u\| \leq C t^{-m}\|u_0\| \), where \( \cdot \| \) denotes the norm in \( L^2(\Omega) \). Mu, Ahmad and Huang [23] obtain analogous estimates using weighted H"older norms. Recently, Le et al. [24] studied (1) for the case \( \mathcal{G} = 0 \) and \( a = b = 0 \), with \( \tilde{F} = \tilde{F}(x,t) \). One of their regularity results [24, Theorem 7.3] gives the bound \( \|\partial_t^m u\| \leq C t^{-m+1/2}\|u_0\|_{H^2(\Omega)} \) when \( g = 0 \), subject to the restriction \( 1/2 < \alpha < 1 \).

The next section gathers together some technical preliminaries needed for our analysis, which uses delicate energy arguments, a fractional Gronwall inequality and several properties of fractional integrals to prove a priori estimates for the weak solution \( u \) of (1). In Section 3, we estimate the derivatives of \( u \) and \( \nabla u \) with respect to time assuming \( u_0 \in L^2(\Omega) \). For example, Corollary 10 shows that if \( g(t) \equiv 0 \) then, with \( m \geq 1 \) such that (4) holds,

\[
\|\partial_t^m u\| + t^{\alpha/2}\|\partial_t^m \nabla u\| + t^{-\alpha}\|\partial_t^{m-\alpha} u\| + t^{-\alpha/2}\|\partial_t^{m-\alpha} \nabla u\| \leq C t^{-m}\|u_0\|
\]

for \( 0 < t \leq T \). Unlike a classical parabolic PDE, the fractional problem (1) exhibits only limited spatial smoothing as \( t \) increases [17], and in Section 4, we investigate the consequences of more regular initial data. For example, Theorems 12 and 13 show that when \( g(t) \equiv 0 \) and \( u_0 \in H^\mu(\Omega) \) for \( 0 \leq \mu \leq 2 \), and under additional assumptions on \( \kappa \) and \( \Omega \),

\[
\|\partial_t^m u\| + t^{-\alpha}\|\partial_t^{m-\alpha} u\| + t^{\alpha}\|\partial_t^m u\|_{H^\mu(\Omega)} + t^{-\alpha/2}\|\partial_t^{m-\alpha} \nabla u\| \leq C t^{-m+\alpha\mu/2}\|u_0\|_{H^\mu}
\]

The paper concludes with an Appendix containing three technical lemmas.

2. Preliminaries and notations

This section introduces some notations and states some technical results that will be used in our subsequent regularity analysis. As in our recent paper [1], we define the
quadratic operators \( \mathcal{L}_1^\mu \) and \( \mathcal{L}_2^\mu \), for \( \mu \geq 0 \) and \( 0 \leq t \leq T \), by

\[
\mathcal{L}_1^\mu(\phi, t) = \int_0^t \langle \phi, \mathcal{J}^\mu \phi \rangle \, ds \quad \text{and} \quad \mathcal{L}_2^\mu(\phi, t) = \int_0^t \| \mathcal{J}^\mu \phi \|^2 \, ds. \tag{8}
\]

These operators coincide when \( \mu = 0 \), so we write \( \mathcal{L}^0 = \mathcal{L}_1^0 = \mathcal{L}_2^0 \). We recall the following positivity property [25, Theorem 2]

\[
\mathcal{L}_1^\mu(\phi, T) \geq 0 \quad \text{for} \ 0 \leq \mu \leq 1. \tag{9}
\]

The next four lemmas establish key inequalities satisfied by \( \mathcal{L}_1^\mu \) and \( \mathcal{L}_2^\mu \).

**Lemma 2** ([13, Lemma 3.2]). If \( 0 < \alpha < 1 \) and \( \epsilon > 0 \), then

\[
\left| \int_0^t \langle \phi, \mathcal{J}^\alpha \psi \rangle \, ds \right| \leq \frac{\mathcal{L}_1^\alpha(\phi, t)}{4\epsilon(1-\alpha)^2} + \epsilon \mathcal{L}_1^\alpha(\psi, t), \tag{10}
\]

\[
\mathcal{L}_2^\alpha(\phi, t) \leq \frac{2\alpha}{1-\alpha} \mathcal{L}_1^\alpha(\phi, t), \tag{11}
\]

\[
\mathcal{L}_2^\alpha(\phi, t) \leq 2^\alpha \mathcal{L}_2^0(\phi, t). \tag{12}
\]

**Lemma 3** ([1, Lemmas 2.2 and 2.3]). If \( 0 < \alpha \leq 1 \), then for \( \phi \in L_2((0, t), L_2(\Omega)) \),

\[
\mathcal{L}_2^\alpha(\phi, t) \leq 2 \int_0^t \omega_\alpha(t-s) \mathcal{L}_1^\alpha(\phi, s) \, ds \quad \text{and} \quad \mathcal{J}^{1-\alpha}(\| \mathcal{J}^\alpha \phi \|^2) \leq 2 \mathcal{L}_1^\alpha(\phi, t). \]

Furthermore, if \( \phi \in W_1^1((0, t); L_2(\Omega)) \) and \( \phi(0) = \mathcal{J}^\alpha \phi'(0) = 0 \), then

\[
\| \phi(t) \|^2 \leq 2 \omega_{2-\alpha}(t) \mathcal{L}_1^\alpha(\phi', t). \]

**Lemma 4** ([13, Lemma 3.1]). If \( 0 \leq \mu \leq \nu \leq 1 \), then \( \mathcal{L}_2^\nu(\phi, t) \leq 2^{2(\nu-\mu)} \mathcal{L}_2^\mu(\phi, t) \).

We will make essential use of the following fractional Gronwall inequality.

**Lemma 5** ([26, Theorem 3.1]). Let \( \beta > 0 \) and \( T > 0 \). Assume that \( a \) and \( b \) are non-negative and non-decreasing functions on the interval \([0, T]\). If \( q : [0, T] \to \mathbb{R} \) is an integrable function satisfying

\[
0 \leq q(t) \leq a(t) + b(t) \mathcal{J}^\beta q(t) \quad \text{for} \ 0 \leq t \leq T,
\]

then

\[
q(t) \leq a(t) E_\beta(b(t)t^\beta) \quad \text{for} \ 0 \leq t \leq T.
\]

Let \( \mathcal{M}^j \) denote the operator of pointwise multiplication by \( t^j \), that is,

\[
(\mathcal{M}^j \phi)(t) = t^j \phi(t),
\]

and note the commutator properties (for any integer \( j \geq 1 \) and any real \( \mu \geq 0 \))

\[
\partial_t^j \mathcal{M} - \mathcal{M} \partial_t^j = j \partial_t^{j-1}, \quad \partial_t \mathcal{M}^j - \mathcal{M} \partial_t^j = j \mathcal{M}^{j-1}, \quad \mathcal{M} \mathcal{J}^\mu - \mathcal{J}^\mu \mathcal{M} = \mu \mathcal{J}^{\mu+1}. \tag{13}
\]

The following identities then follow by induction on \( m \).
Lemma 6. For $0 \leq q \leq m$ and $\mu \geq 0$, there exist constant coefficients $a_{j}^{m,q}$, $b_{j}^{m,q}$, $c_{j}^{m,\mu}$ and $d_{j}^{m,\mu}$ such that

$$
\partial_{t}^{q} \mathcal{M}^{m} = \mathcal{M}^{m} \partial_{t}^{q} + \sum_{j=1}^{q} a_{j}^{m,q} \mathcal{M}^{m-j} \partial_{t}^{q-j},
$$

(14)

$$
\mathcal{M}^{m} \partial_{t}^{q} = \partial_{t}^{q} \mathcal{M}^{m} + \sum_{j=1}^{q} b_{j}^{m,q} \partial_{t}^{q-j} \mathcal{M}^{m-j},
$$

(15)

$$
\mathcal{D}^{\mu} \mathcal{M}^{m} = \mathcal{M}^{m} \mathcal{D}^{\mu} + \sum_{j=1}^{m} c_{j}^{m,\mu} \mathcal{M}^{m-j} \mathcal{D}^{\mu+j},
$$

(16)

$$
\mathcal{M}^{m} \mathcal{D}^{\mu} = \mathcal{D}^{\mu} \mathcal{M}^{m} + \sum_{j=1}^{m} d_{j}^{m,\mu} \mathcal{D}^{\mu+j} \mathcal{M}^{m-j}.
$$

(17)

For later reference, we set $a_{0}^{m,q} = b_{0}^{m,q} = c_{0}^{m,\mu} = d_{0}^{m,\mu} = 1$ and

$$
a_{j}^{m,q} = a_{q-j}^{m,q}, \quad b_{j}^{m,q} = b_{q-j}^{m,q}, \quad c_{j}^{m,\mu} = c_{m-j}^{m,\mu}, \quad d_{j}^{m,\mu} = d_{m-j}^{m,\mu}.
$$

When $\mu = 0$ the formulas involving $\mathcal{D}^{\mu}$ become redundant, and we see that $c_{j}^{m,0} = 0 = d_{j}^{m,0}$ for $1 \leq j \leq m$. Likewise, $\omega_{j}(t) = 0$ for $0 \leq j \leq m$ since $\Gamma(z)$ has a pole at $z = -j$. We conclude this section by noting that if $m \geq 1$ and $\mu \geq 0$, then

$$
\partial_{t}^{m} \mathcal{D}^{\mu} \phi(t) = \mathcal{D}^{\mu} \partial_{t}^{m} \phi(t) + \sum_{j=0}^{m-1} \omega_{j}(t) \partial_{t}^{q-j} \phi(0) \quad \text{for } \phi \in W_{1}^{m}((0,t];L_{2}(\Omega)),
$$

(18)

which amounts to a restatement of the relation between the Riemann–Liouville and Caputo fractional derivatives.

3. Regularity of the weak solution

Our aim in this section is to estimate higher-order time derivatives of $u$ assuming appropriate bounds on the higher-order time derivatives of $f$ (and hence, ultimately, of $g$), as well as sufficient smoothness of the coefficients in $f$. We will not attempt to prove the existence of the higher-order derivatives of $u$, which could be done by estimating the corresponding derivatives of the projected solution $u_{X}$ from our earlier paper [4], corresponding to a finite dimensional subspace $X = X_{n} \subset H_{0}^{1}(\Omega)$, and then taking appropriate limits as $n \to \infty$. For the remainder of the paper, we assume that [4] holds, and that

$$
\|g^{(j-1)}(t)\| = O(t^{\alpha-j}) \quad \text{as } t \to 0, \text{ for } 1 \leq j \leq m.
$$

It follows that the existence and uniqueness of the weak solution $u$ are guaranteed by Theorem [4]. Henceforth, $C$ will denote a generic constant that may depend on the coefficients in [4], the spatial domain $\Omega$, the time interval $[0,T]$, the fractional exponent $\alpha$,
the parameter \( \eta \), and the integer \( m \) in (4). Also, we rescale the time variable if necessary so that the minimum eigenvalue of \( \kappa \) satisfies
\[
\lambda_{\text{min}}(\kappa(x)) \geq 1 \quad \text{for } x \in \Omega.
\] (19)

For brevity, we introduce some more notations. Let
\[
(B_{\psi}^t \phi)(t) = \psi(t) \mathcal{F}^t \phi(t) \quad \text{for } 0 \leq \mu \leq 1.
\] (20)

Integrating by parts and recalling (6), we find that
\[
\vec{B}_1 = B_1^{\alpha} \vec{F} + B_1^{\beta} \vec{G} \quad \text{and} \quad B_2 = B_2^{\alpha} + B_2^{\beta}.
\] (21)

Generalizing (20), for \( j \in \{0, 1, 2, \ldots\} \) we put
\[
B_{\psi, j}^\mu \phi(t) = \partial_j t (M_j I^{\alpha} (\kappa \partial_1^{1-\mu} \phi)) = (M_j B_{\psi}^\mu \phi)(j)(t),
\]
and generalizing (8) we put
\[
Q_{\psi, j}^\mu i = Q_{\psi, j}^\mu i (M_j \phi), \quad 0 \leq t \leq T \quad \text{and} \quad i \in \{1, 2\},
\]
with \( Q_{\psi, j}^\mu 1 = Q_{\psi, j}^\mu 2 = Q_{\psi, j}^\mu 0 \). The next result relies on Lemma 15 from the Appendix.

Lemma 7. For \( 0 < t \leq T \) and for \( m \geq 1 \),
\[
\mathcal{L}_1^\mu \alpha^m (u, t) + \mathcal{L}_2^\mu \alpha^m (\nabla u, t) \leq C \sum_{j=0}^m \mathcal{L}_{1}^{0, j}(f, t),
\]
and
\[
\mathcal{L}_1^\mu 0, m (u, t) + \mathcal{L}_2^\mu 0, m (\nabla u, t) \leq C \sum_{j=0}^m \mathcal{L}_{1}^{0, j}(f, t).
\]

Proof. Since \( (\mathcal{F}^\mu \nabla u)(0) = 0 \) by part 4 of Theorem 1
\[
\int_0^t \langle \kappa \nabla \partial_1^{1-\alpha} u(s), \nabla v \rangle ds = \langle \kappa \int_0^t (\mathcal{F}^\mu \nabla u)'(s) ds, \nabla v \rangle = \langle \kappa (\mathcal{F}^\mu \nabla u)(t), \nabla v \rangle,
\]
and by the identity in (17),
\[
\mathcal{M}^\mu \mathcal{F}^\alpha \nabla u = \mathcal{F}^\mu \mathcal{M}^\alpha \nabla u + \sum_{j=0}^{m-1} d_j^{m, \alpha} \mathcal{F}^{\alpha + m-j} \mathcal{M}^j \nabla u.
\]

Thus, multiplying both sides of (5) by \( t^m \) yields
\[
\langle \mathcal{M}^\mu u, v \rangle + \langle \kappa \mathcal{F}^\mu \mathcal{M}^\mu \nabla u, \nabla v \rangle + \sum_{j=1}^m d_j^{m, \alpha} \langle \kappa \mathcal{F}^{\alpha + m-j} \mathcal{M}^j \nabla u, \nabla v \rangle
\]
\[
= \langle \mathcal{M}^\mu B_1^{\alpha} u + \mathcal{M}^\mu B_1^{\beta} u, \nabla v \rangle - \langle \mathcal{M}^\mu B_2^{\alpha} u + \mathcal{M}^\mu B_2^{\beta} u, v \rangle + \langle \mathcal{M}^\mu f, v \rangle.
\]
for \( v \in H^1_0(\Omega) \). We have

\[
\partial^m_s \mathcal{I}^m \mathcal{M}^s \nabla u = \partial^1_s \mathcal{I}^m \mathcal{M}^s \nabla u = \partial^1_s \mathcal{I}^m \mathcal{M}^s \nabla u = \mathcal{I}^m \mathcal{M}^s \nabla u,
\]

where the final step follows by (18) because

\[
\partial^1_s (\mathcal{M}^s \nabla u) = 0 \quad \text{for } 0 \leq i \leq j - 1 \leq m - 1.
\]

Likewise, \( \partial^m_s \mathcal{I}^m \mathcal{M}^s \nabla u = \mathcal{I}^m \mathcal{M}^s \nabla u \) because

\[
\partial^1_s (\mathcal{M}^s \nabla u) = 0 \quad \text{for } 0 \leq j \leq m - 1,
\]

and therefore

\[
\langle \partial^m_s \mathcal{M}^s \nabla u, v \rangle + \langle \mathcal{I}^m \mathcal{M}^s \nabla u, \nabla v \rangle = \langle B^m_s + B^1_m, \nabla v \rangle - \langle \partial^m_s \mathcal{M}^s \nabla f, v \rangle = (22)
\]

We let \( \mathcal{E}(u) = 2\|B^m_s\|^2 + 2\|B^1_m\|^2 + 2\|B^1_m\|^2 \), and conclude using the Cauchy–Schwarz inequality that

\[
\langle \partial^m_s \mathcal{M}^s \nabla u, v \rangle + \langle \mathcal{I}^m \mathcal{M}^s \nabla u, \nabla v \rangle \leq \mathcal{E}(u) + C \sum_{j=0}^{m-1} \| \mathcal{I}^m \mathcal{M}^s \nabla u \|^2
\]

Choosing \( v = \mathcal{I}^m \mathcal{M}^s \nabla u \), integrating over the time interval \((0, t)\) and using (19), we have

\[
\mathcal{Q}_1^m(u, t) + \frac{1}{2} \mathcal{Q}_2^m(u, t) \leq \int_0^t \mathcal{E}(u) ds + C \sum_{j=0}^{m-1} \mathcal{Q}_1^j(u, t)
\]

\[
+ \int_0^t \mathcal{Q}_1^m(u, t) + \int_0^t (\mathcal{I}^m \mathcal{M}^s \nabla u, \mathcal{I}^m \mathcal{M}^s \nabla u) ds,
\]

and by the Cauchy-Schwarz inequality and Lemma\[3\]

\[
\mathcal{I}^{1-\alpha}(\| \mathcal{I}^m (\mathcal{M}^s \nabla u) \|^2(t) \leq 2 \mathcal{Q}_1^m (\mathcal{I}^m \mathcal{M}^s \nabla u, t).
\]

Thus,

\[
\int_0^t (\mathcal{I}^m \mathcal{M}^s \nabla f, \mathcal{I}^m \mathcal{M}^s \nabla u) ds \leq \int_0^t \| \mathcal{I}^m \mathcal{M}^s \nabla f \| \| \mathcal{I}^m \mathcal{M}^s \nabla u \| ds
\]

\[
\leq C \left( \int_0^t (t-s)^\alpha \| \mathcal{I}^m \mathcal{M}^s \nabla f \|^2 ds \right)^{1/2} \left( \int_0^t (t-s)^{-\alpha} \| \mathcal{I}^m \mathcal{M}^s \nabla u \|^2 ds \right)^{1/2}
\]

\[
\leq C \left( t^\alpha \mathcal{Q}_2^m(f, t) \right)^{1/2} \left( \mathcal{I}^{1-\alpha}(\| \mathcal{I}^m (\mathcal{M}^s \nabla u) \|^2(t) \right)^{1/2}
\]

\[
\leq C t^\alpha \mathcal{Q}_2^m(f, t) + \frac{1}{2} \mathcal{Q}_1^m(u, t),
\]

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implying that the function \( q_m(t) = \mathcal{Q}_1^m(u,t) + \mathcal{Q}_2^m(\nabla u,t) \) satisfies

\[
q_m(t) \leq 2 \int_0^t \mathcal{E}(u) \, ds + C \sum_{j=0}^{m-1} \mathcal{Q}_2^{j+1}(\nabla u,t) + C t^\alpha \mathcal{Q}_0^m(f,t).
\]

By Lemma 15

\[
\mathcal{Q}_0^0(B_F^{\alpha,m}u,t) + \mathcal{Q}_0^0(B_u^{\alpha,m}u,t) \leq C \sum_{j=0}^m \mathcal{Q}_2^{j+1}(u,t) \quad (23)
\]

and, applying Lemma 4 with \( \nu = 1 \) and \( \mu = \alpha \),

\[
\mathcal{Q}_0^0(B_G^{1,m}u,t) + \mathcal{Q}_0^0(B_0^{1,m}u,t) \leq C \sum_{j=0}^m \mathcal{Q}_2^{j+1}(u,t) \leq C t^{2(\alpha-1)} \sum_{j=0}^m \mathcal{Q}_2^{j+1}(u,t) \quad (24)
\]

for \( m \geq 0 \). By combining the above estimates,

\[
q_m(t) \leq C \mathcal{Q}_2^m(u,t) + C \sum_{j=0}^{m-1} q_j(t) + C t^\alpha \mathcal{Q}_0^m(f,t).
\]

Consequently, we conclude (recursively) that

\[
q_m(t) \leq C \sum_{j=0}^m \mathcal{Q}_2^{j+1}(u,t) + C t^\alpha \sum_{j=0}^m \mathcal{Q}_0^{j+1}(f,t),
\]

so, by applying the first inequality in Lemma 3 with \( \phi = (\mathcal{H}^1 u)^{(j)} \),

\[
q_m(t) \leq Ct^\alpha \sum_{j=0}^m \mathcal{Q}_0^{j+1}(f,t) + C \sum_{j=0}^m \int_0^t \omega(t-s) q_j(s) \, ds.
\]

Therefore, a repeated application of Lemma 5 yields the first desired estimate.

To show the second estimate, choose \( \nu = \partial_t^m \mathcal{H}^m u \) in (22) and obtain

\[
\|\partial_t^m \mathcal{H}^m u\| + (\kappa \mathcal{H}^m \partial_t^m \mathcal{H}^m \nabla u, \partial_t^m \mathcal{H}^m \nabla u) = -(Eu, \partial_t^m \mathcal{H}^m u) - \sum_{j=0}^{m-1} \partial_t^{m-j} (\kappa \mathcal{H}^m \partial_t^j \mathcal{H}^m \nabla u, \partial_t^m \mathcal{H}^m \nabla u) + (\partial_t^m \mathcal{H}^m f, \partial_t^m \mathcal{H}^m u),
\]

where \( Eu = \nabla \cdot B_F^{\alpha,m}u + B_0^{\alpha,m}u + \nabla \cdot B_G^{1,m}u + B_b^{1,m}u \). The first and the last terms on the right-hand side are bounded by \( |Eu|^2 + \|\partial_t^m \mathcal{H}^m f\|^2 + \frac{1}{2} \|\partial_t^m \mathcal{H}^m u\|^2 \) so, after integrating in time, using (12) and applying (10) (for a sufficiently large \( \varepsilon \)),

\[
\frac{1}{2} \mathcal{Q}_0^m(u,t) + \mathcal{Q}_1^m(\nabla u,t) \leq \int_0^t \|Eu(s)\|^2 \, ds + \mathcal{Q}_0^m(f,t) + \frac{1}{2} \mathcal{Q}_1^m(\nabla u,t) + C \sum_{j=0}^{m-1} \mathcal{Q}_2^{j+1}(\nabla u,t).
\]
Since $\nabla \cdot (\tilde{F} \partial_t^{1-\alpha} u) = (\nabla \cdot \tilde{F}) \partial_t^{1-\alpha} u + \tilde{F} \cdot \nabla \partial_t^{1-\alpha} u$, we see that

$$\nabla \cdot B^0 \partial_t^{1-\alpha} u = \partial_t^{1-\alpha} \mathcal{Q} (\nabla \cdot (\tilde{F} \partial_t^{1-\alpha} u)) = B^0 \partial_t^{1-\alpha} u + B^1 \partial_t^{1-\alpha} u,$$

and therefore, applying Lemma 15 followed by Lemma 4.

$$\int_0^t \| E_\alpha(u) \|^2 \, ds \leq 4 \left( \mathcal{Q} (\nabla \cdot B^0 \partial_t^{1-\alpha} u, t) + \mathcal{Q} (B^0 \partial_t^{1-\alpha} u, t) \right)$$

$$\leq C \sum_{j=0}^m \left( \mathcal{Q}^j (u, t) + \mathcal{Q}^j (\nabla u, t) \right).$$

Hence, the function $q_m(t) = \mathcal{Q}^0 (u, t) + \mathcal{Q}^1 (\nabla u, t)$ satisfies

$$q_m(t) \leq 2 \mathcal{Q}^0 (f, t) + C \sum_{j=0}^{m-1} \mathcal{Q}^j (\nabla u, t) + C \sum_{j=0}^m \left( \mathcal{Q}^j (u, t) + \mathcal{Q}^j (\nabla u, t) \right),$$

and so, using (11) and (12), it follows that

$$q_m(t) \leq 2 \mathcal{Q}^0 (f, t) + C \sum_{j=0}^{m-1} q_j(t) + C \left( \mathcal{Q}^m (u, t) + \mathcal{Q}^m (\nabla u, t) \right).$$

By the first inequality in Lemma 3 and (12),

$$\mathcal{Q}^m (u, t) + \mathcal{Q}^m (\nabla u, t) \leq C \int_0^t \omega_t (t - s) q_m(s) \, ds,$$

and thus by Lemma 5

$$q_m(t) \leq C \mathcal{Q}^0 (f, t) + C \sum_{j=0}^{m-1} q_j(t).$$

Applying this inequality recursively gives

$$q_m(t) \leq C \sum_{j=0}^m \mathcal{Q}^0 (f, t),$$

which completes the proof.

We can now show pointwise bounds for the norms in $L_2(\Omega)$ of the time derivatives of $u$ and $\nabla u$.

**Theorem 8.** For $m \geq 1$ and $0 < t \leq T$,

$$\| (\partial_t^m u)(t) \|^2 + \| \partial_t^m (\nabla u)(t) \|^2 \leq Ct^{-1} 2m \sum_{j=0}^{m-1} \mathcal{Q}^j (f, t).$$
Proof. Since $\mathcal{M} \partial_t^m = \partial_t^m \mathcal{M} - m \partial_t^{m-1}$, we see using (15) (and setting $\tilde{b}_{m, m-1} = 0$) that

$$\mathcal{M}^{m+1} \partial_t^m = \mathcal{M}^m \partial_t^m \mathcal{M} - m \mathcal{M}^m \partial_t^{m-1} = \sum_{j=1}^{m+1} (\mathcal{M}^m - m \tilde{b}_{j-1}^{m-1}) \partial_t^{j-1} \mathcal{M},$$

and hence

$$\| (\mathcal{M}^{m+1} \partial_t^m u) (t) \|^2 \leq C \sum_{j=1}^{m+1} \| (\partial_t^{j-1} \mathcal{M}^j u) (t) \|^2. \quad (25)$$

Using the second inequality in Lemma 3 with $\phi = \partial_t^{j-1} \mathcal{M}^j u$ and the first bound in Lemma 7 we get

$$\| (\partial_t^{j-1} \mathcal{M}^j u) (t) \|^2 \leq C t^{1-\alpha} \mathcal{P}_1 (\partial_t^j \mathcal{M}^j u, t) \leq C t \sum_{I=0}^{J} \mathcal{P}^0_j (f, t)$$

and so

$$\| (\partial_t^m u) (t) \|^2 = t^{-2m-2} \| (\mathcal{M}^{m+1} \partial_t^m u) (t) \|^2 \leq C t^{-1-2m} \sum_{j=0}^{m+1} \mathcal{P}^0_j (f, t).$$

Applying the same argument to $\nabla u$ in place of $u$, and using the second bound in Lemma 7 the result follows.

Next, we estimate fractional time derivatives of $u$ and $\nabla u$. These bounds will later help in our study of spatial regularity, and reflect the presence of the fractional time derivative in (1).

**Theorem 9.** For $m \geq 1$ and $0 < t \leq T$,

$$\| (\partial_t^{m-\alpha} u) (t) \|^2 + t^\alpha \| (\partial_t^{m-\alpha} \nabla u) (t) \|^2 \leq C t^{-1-2(m-\alpha)} \sum_{j=0}^{m+1} \mathcal{P}^0_j (f, t).$$

**Proof.** Using the inequality (25),

$$\| (\mathcal{M}^m \partial_t^{m-\alpha} u) (t) \|^2 = \| (\mathcal{M}^m \partial_t^{m-1} \partial_t^{-\alpha} u) (t) \|^2 \leq C \sum_{j=1}^{m} \| (\partial_t^{j-1} \mathcal{M}^j \partial_t^{1-\alpha} u) (t) \|^2,$$

and using (15) and (18) (with $m = 1$),

$$\mathcal{M} \partial_t^{1-\alpha} u = \mathcal{M} \partial_t \mathcal{I}^\alpha u = \mathcal{M} (\mathcal{I}^\alpha \partial_t u + 0) \omega_{\alpha}\omega = \mathcal{I}^\alpha \mathcal{M} u + \alpha \mathcal{I}^{\alpha+1} \partial_t u + 0) \omega_{\alpha} = \mathcal{I}^\alpha \mathcal{M} u + \alpha \mathcal{I}^{\alpha} (u - u(0)) + \alpha u(0) \omega_{\alpha+1} = \mathcal{I}^{\alpha} (\mathcal{M} u + \alpha u).$$

Thus, by (17),

$$\mathcal{M}^j \partial_t^{1-\alpha} u = \mathcal{M}^{j-1} \mathcal{I}^\alpha (\mathcal{M} u + \alpha u) = \sum_{\ell=0}^{j-1} \frac{\mathcal{M}^\ell \partial_t^{-\alpha} \mathcal{I}^{\alpha+1-\ell} \mathcal{M}^{j-1-\ell}}{\mathcal{I}^{\alpha+1-\ell}} (\mathcal{M} u + \alpha u).$$
We have

\[ \partial_t^{j-1} \mathcal{M}^{\alpha + j-1-\ell} \mathcal{M} (\mathcal{M} u' + \alpha u) = \partial_t^{j-1} \mathcal{M}^{\alpha + j-1-\ell} \mathcal{M} (\mathcal{M} u' + \alpha u) = \partial_t^{j-1} \partial_t^{\alpha} \mathcal{M}^{\ell} (\mathcal{M} u' + \alpha u), \]

where we used the identity (13) (with \( m = 1 \)) and the fact that \( \partial_t^\ell \mathcal{M} (\mathcal{M} u' + \alpha u)(0) = 0 \) for \( 0 \leq i \leq \ell - 1 \). Hence,

\[ \| (\partial_t^{j-1} \mathcal{M}^{\alpha} \partial_t^{1-\alpha} u)(t) \| \leq \left\| \sum_{l=0}^{j-1} \partial_t^{j-1} \partial_t^{\alpha} \mathcal{M}^{\ell} (\mathcal{M} u' + \alpha u)(t) \right\| \leq C \sum_{l=0}^{j-1} \| \mathcal{M}^\alpha \phi_l(t) \| \]

where \( \phi_l = \partial_t^{\ell} \mathcal{M}^{\ell} (\mathcal{M} u' + \alpha u) \). Using (14),

\[ \phi_l = \sum_{r=0}^{\ell} \partial_t^{r} \mathcal{M}^{\ell} \partial_t^{\ell} (\mathcal{M} u' + \alpha u) = \sum_{r=0}^{\ell} \partial_t^{r} \mathcal{M}^{\ell} (\mathcal{M} \partial_t^{\ell} u' + i \partial_t^{\ell} u' + \alpha \partial_t^{\ell} u) \]

and so, by Theorem 8

\[ \| \phi_l(t) \|^2 \leq C \sum_{r=0}^{\ell+1} \| (\mathcal{M}^\alpha \partial_t^{\ell} u)(t) \|^2 \leq C \sum_{r=0}^{\ell+1} \sum_{j=0}^{r-1} \sum_{i=0}^{j+1} \| \omega^{0,i} (f,t) \| \leq C \sum_{r=0}^{\ell+2} \sum_{i=0}^{r} \| \omega^{0,i} (f,t) \| \quad (27) \]

Since \( \| \phi_l(t) \| \leq C \omega_{1/2}(t) \psi_l(t) \) where \( \psi_l(t) = \sqrt{\sum_{r=0}^{\ell+1} \sum_{i=0}^{r} \| \omega^{0,i} (f,t) \|} \) is nondecreasing, we see that \( \| \mathcal{M}^\alpha \phi_l(t) \| \leq C \omega_{1/2}(t) \psi_l(t) \). Therefore,

\[ \| (\partial_t^{j-1} \mathcal{M}^{\alpha} \partial_t^{1-\alpha} u)(t) \|^2 \leq C \sum_{l=0}^{j-1} \| \mathcal{M}^\alpha \phi_l(t) \|^2 \leq C \sum_{l=0}^{j-1} (t^{(\alpha + 1/2) - 1})^2 \psi_l(t)^2 \]

\[ \leq C t^{2\alpha - 1} \sum_{l=0}^{j-1} \| \omega^{0,i} (f,t) \|, \]

and the desired bound for \( \| (\partial_t^{m-\alpha} u)(t) \|^2 \) follows at once from (26).

Replacing \( u \) with \( \nabla u \) in the preceding argument, we have

\[ \| (\mathcal{M}^\alpha \partial_t^{m-\alpha} \nabla u)(t) \|^2 \leq C \sum_{j=0}^{m} \| (\partial_t^{j-1} \mathcal{M}^{\alpha} \partial_t^{1-\alpha} \nabla u)(t) \|^2 \leq C \sum_{j=0}^{m} \sum_{l=0}^{j-1} \| \mathcal{M}^\alpha \phi_l(t) \|^2 \]

where, this time, \( \phi_l = \partial_t^{\ell} \mathcal{M}^{\ell} (\mathcal{M} \nabla u' + \alpha \nabla u) \) and hence

\[ \| \phi_l(t) \| \leq C \omega_{1/2}(t) \psi_l(t) \].

It follows that \( \| \mathcal{M}^\alpha \psi_l \| \leq C \omega_{1/2}(t) \psi_l(t) \) and therefore \( t^\alpha \| (\mathcal{M}^\alpha \partial_t^{m-\alpha} \nabla u)(t) \|^2 \) is bounded by

\[ Ct^\alpha \sum_{l=0}^{m} \| \mathcal{M}^\alpha \phi_l(t) \|^2 \leq C t^\alpha \sum_{l=0}^{m-1} (t^{(\alpha + 1/2) - 1})^2 \psi_l(t)^2 \leq C t^{2\alpha - 1} \sum_{l=0}^{m+1} \| \omega^{0,i} (f,t) \|, \]

as required.

The following simplified bounds are perhaps more immediately useful. \( \square \)
Corollary 10. Let $m \geq 1$ and suppose that $g : (0, T] \rightarrow L^2(\Omega)$ is $C^m$ with
\[ \|g^{(j)}(t)\| \leq M t^{\eta - 1-j} \quad \text{for } 0 \leq j \leq m \text{ and some } \eta > 0. \] (28)

Then
\[ \| (\partial_t^m u)(t) \| + t^{\alpha/2} \| (\partial_t^m \nabla u)(t) \| \leq C t^{-m}(\| u_0 \| + M t^{\eta}) \]
and
\[ \| (\partial_t^{m-\alpha} u)(t) \| + t^{\alpha/2} \| (\partial_t^{m-\alpha} \nabla u)(t) \| \leq C t^{-m}(\| u_0 \| + M t^{\eta}). \]

Proof. Since $f^{(j)}(t) = g^{(j-1)}(t)$ for $1 \leq j \leq m + 1$, (14) implies that
\[ \| (\mathcal{M} f)^{(j)}(t) \| \leq C M t^{\eta} \text{ for } 1 \leq j \leq m + 1, \]
with $\| f(t) \| \leq \| u_0 \| + M \eta^{-1} t^\eta$. Thus,
\[ \mathcal{Q}^{0,j}(f, t) \leq C M t^{2\eta + 1} \text{ for } 1 \leq j \leq m + 1, \]
with $\mathcal{Q}^0(f, t) \leq C t(\| u_0 \| + M t^{\eta})^2$, so
\[ t^{-1-2m} \sum_{j=0}^{m+1} \mathcal{Q}^{0,j}(f, t) \leq C t^{-2m}(\| u_0 \| + M t^{\eta})^2 \]
and the result follows from Theorems 8 and 9. \hfill \Box

4. More regular initial data

We will now investigate further the relation between the regularity of $u$ and that of the initial data $u_0$. In particular, Theorem 13 below extends Corollary 11 and proves a bound used in an error analysis of a finite element discretization of the fractional Fokker–Planck equation [13]. The fractional PDE (1) can be rewritten as
\[ u' - \nabla \cdot (\kappa \partial_t^{1-\alpha} \nabla u) = h \quad \text{for } x \in \Omega \text{ and } 0 < t < T, \]
where $h = g - \nabla \cdot (\tilde{F} \partial_t^{1-\alpha} u + \tilde{G} u) - (a \partial_t^{1-\alpha} u + bu)$. We can therefore apply known results for the fractional diffusion equation to establish the following bounds in the norm $\| v \|_\mu = \| A^{\mu/2} v \|$ of the fractional Sobolev space $H^\mu(\Omega)$, where $A^{\mu/2}$ is defined via the spectral representation of $A v = -\nabla \cdot (\kappa \nabla v)$ using the Dirichlet eigenfunctions on $\Omega$ [17, 27]. The results of this section require $H^2$-regularity for the Poisson problem, and to ensure this property we make the additional assumptions [28, Theorems 2.2.2.3 and 3.2.1.2]
\[ \kappa \text{ is Lipschitz on } \Omega \quad \text{and} \quad \Omega \text{ is } C^{1,1} \text{ or convex.} \] (29)
It follows that $H^1(\Omega) = H^1_0(\Omega)$ and $H^2(\Omega) = H^2(\Omega) \cap H^1_0(\Omega)$. We also require that $g$ satisfies (28). Our first result does not assume any additional smoothness of $u_0$.

Theorem 11. Assume (28) and (29). If $u_0 \in L^2(\Omega)$, then
\[ t^\mu \| u^{(m)}(t) \|_\mu \leq C \| u_0 \| t^{-\mu \alpha/2} + C M t^{\eta - \mu \alpha/2} \text{ for } 0 \leq \mu \leq 2 \text{ and } 0 < t \leq T. \]
Theorem 12. Assume [28] and (25). If $0 \leq \mu \leq 2$ and $u_0 \in H^\mu(\Omega)$, then
\[
\| u(t) - u_0 \| + t^{\mu/2} \| \nabla (u(t) - u_0) \| \leq C \| u_0 \| t^{\alpha \mu/2} + M t^n,
\]
and for $m \geq 1$,
\[
\| u^{(m)}(t) \| + t^{\mu/2} \| \nabla u^{(m)}(t) \| \leq C r^{-m} \left( \| u_0 \| t^{\alpha \mu/2} + M t^n \right)
\]
with
\[
\| \partial_t^{m-\alpha} u(t) \| + t^{\mu/2} \left( \| \partial_t^{m-\alpha} \nabla u(t) \| \right) \leq C r^{-m} \left( \| u_0 \| t^{\alpha \mu/2} + M t^n \right).
\]

Proof. Introduce the solution operator $u(t) = \mathcal{U}(u_0, g, t)$. By linearity, $u = u_1 + u_2$ where $u_1(t) = \mathcal{U}(u_0(t), 0, t)$ and $u_2(t) = \mathcal{U}(0, g, t)$. In view of Corollary [10] it suffices to consider $u_1$. Let $w(t) = u_1(t) - u_0$ so that $w(0) = 0$, and suppose to begin with that $u_0 \in \mathring{H}^2(\Omega)$. Using [4], we find that
\[
\langle w(t), v \rangle + \langle \mathcal{K}(\mathcal{H}^\alpha \nabla w)(t), \nabla v \rangle - \langle (\mathcal{B}_1 w)(t), \nabla v \rangle + \langle (\mathcal{B}_2 w)(t), v \rangle = \langle \rho(t), v \rangle,
\]
for $m \geq 0$ and for $0 \leq \mu \leq 2$, with
\[
\| h^{(j)}(s) \| \leq \| \partial_t^j g(s) \| + C \sum_{\ell=0}^j \left( \| \partial_t^{\ell+1-\alpha} \nabla u(s) \| + \| \partial_t^\ell u(s) \| \right).
\]
Corollary [10] shows that $\| h^{(j)}(s) \|$ is bounded by
\[
M s^{\eta j} + C \sum_{\ell=0}^j \left( s^{\alpha / 2 - \ell - 1} + s^{-\alpha / 2 - \ell} + s^{\alpha / 2 - 1} + s^{-\ell} \right) \| u_0 \| + M s^{\eta j}
\]
so $s^j \| h^{(j)}(s) \| \leq C \| u_0 \| s^{\alpha / 2 - 1} + M s^{\eta j}$ and hence
\[
\int_0^t (t-s)^{-\mu \alpha / 2} s^j \| h^{(j)}(s) \| ds \leq C \| u_0 \| (s_\| \mu_\| \alpha_\| / \| 2_\| * \| \omega_\| / \| 2_\| ) (t) + C M (s_\| \mu_\| \alpha_\| / \| 2_\| * \| \omega_\| / \| 2_\| ) (t)
\]
\[
\leq C \| u_0 \| (1-\mu \| \mu_\| / \| 1_\| + M t^n - \mu \| \alpha_\| / \| 2_\| ),
\]
completing the proof. \(\square\)
where \( \rho(t) = \mathcal{S}^\alpha \nabla \cdot (\kappa \nabla u_0) - \nabla \cdot \vec{B}u_0 - \vec{B}_2u_0 \). Since \( (\mathcal{S}^\alpha u_0)'(t) = u_0 \omega \zeta(t) \), and recalling the definitions \( \mathcal{S} \), we have

\[
\rho'(t) = (\nabla \cdot (\kappa \nabla u_0) - \nabla \cdot (\vec{F}(t)u_0) - a(t)u_0) \omega \zeta(t) - \nabla \cdot (\vec{G}(t)u_0) - b(t)u_0,
\]

so \( \|\rho^{(m)}(t)\| \leq C\|u_0\|2^{\alpha-m} \). Therefore, by Corollary \( 10 \)

\[
\|w^{(m)}(t)\| + t^{\alpha/2}\|\nabla w^{(m)}(t)\| \leq Ct^{-m}\left(\|w(0)\| + \|u_0\|2^{\alpha}\right) = C\|u_0\|2^{\alpha-m},
\]

which proves the result for integer-order time derivatives in the case \( \mu = 2 \). Similarly, for the fractional-order time derivatives,

\[
\|\partial_{t}^{m-\alpha} w(t)\| + t^{\alpha/2}\|\partial_{t}^{m-\alpha} \nabla w(t)\| \leq Ct^{2\alpha-m}\left(\|w(0)\| + \|u_0\|2^{\alpha}\right) = Ct^{2\alpha-m}\|u_0\|2,
\]

completing the proof for \( \mu = 2 \). Since Corollary \( 10 \) also implies the case \( \mu = 0 \), the result follows for \( 0 < \mu < 2 \) by interpolation.

With the help of Lemma \( 16 \) from the Appendix, we can generalize Theorem \( 11 \) as follows.

**Theorem 13.** Assume \( 28 \) and \( 29 \). If \( 0 \leq \mu \leq 2 \) and \( u_0 \in H^\mu(\Omega) \), then

\[
\|u^{(m)}(t)\|_2 \leq Ct_{-m}\left(\|u_0\|_2t^{-(2-\mu)\alpha/2} + Mt^\eta\right) \quad \text{for } 0 < t \leq T.
\]

**Proof.** We know from Theorem \( 11 \) that

\[
\|u^{(m)}(t)\|_2 \leq Ct_{-m}\left(\|u_0\|_2 + Mt^\eta\right), \tag{30}
\]

so there is nothing to prove if \( u_0 = 0 \). Thus, by linearity, we may assume that \( g(t) \equiv 0 \) and so \( M = 0 \). Integrating \( 11 \) in time, we see that

\[
\begin{align*}
\rho &= \mathcal{S}^{1-\alpha}\partial_{t}^{1-\alpha}(u_0 - \nabla \cdot \vec{B}_1u - B_2u).
\end{align*}
\]

Since \( -\nabla \cdot (\kappa \nabla u^{(m)}) = \rho^{(m)} \) in \( \Omega \), with \( u^{(m)}(t) = 0 \) on \( \partial \Omega \) for \( 0 \leq t \leq T \), it follows by \( H^2 \)-regularity for the Poisson problem that

\[
\|u^{(m)}(t)\|_2 \leq C\|\rho^{(m)}(t)\|. \tag{31}
\]

The identity \( 18 \) (with \( m = 1 \)) implies that

\[
\begin{align*}
\rho &= \mathcal{S}^{1-\alpha}\partial_{t}^{1-\alpha}(u_0 - \nabla \cdot \vec{B}_1u - B_2u) \\
&= -\mathcal{S}^{1-\alpha}u' - \mathcal{S}^{1-\alpha}(\nabla \cdot (\vec{F}_{\partial}^{1-\alpha}u + \vec{G}_u) + a\partial_{t}^{1-\alpha}u + bu),
\end{align*}
\]

and Lemma \( 16 \) and Theorem \( 12 \) (with \( \mu = 2 \)) imply that

\[
\begin{align*}
\|\partial_{t}^{m} \mathcal{S}^{1-\alpha}u'(t)\| &\leq C \max_{0 \leq s \leq t} \sum_{j=0}^{m} s^{1-\alpha+1+j} \|u^{(j+1)}(s)\| \\
&\leq C \max_{0 \leq s \leq t} s^{1-\alpha}(\|u_0\|2s^{\alpha}) = Ct\|u_0\|_2.
\end{align*}
\]
Similarly, showing that

\[ \text{lem} \]

\[
\begin{align*}
\tau_{m+1} || \partial_t^m \mathcal{G}^{1-\alpha} (\nabla \cdot (F \partial_t^{1-\alpha} u)) (t) || \\
\leq C \max_{0 \leq s \leq t} \sum_{j=0}^{m} s^{1-\alpha+1+j} \left( \| \partial_t^j (F \partial_t^{1-\alpha} u) \| + \| \partial_t^j (F \partial_t^{1-\alpha} \nabla u) \| \right) \\
= C \max_{0 \leq s \leq t} \sum_{j=0}^{m} s^{1-\alpha+1+j} \left( \| \partial_t^j \partial_t^{1-\alpha} u \| + \| \partial_t^j \partial_t^{1-\alpha} \nabla u \| \right) \\
\leq C \max_{0 \leq s \leq t} s^{1-\alpha+1} \left( \| u_0 \| + s^{\alpha/2} \| \nabla u_0 \| \right) \\
\leq C t^{1+\alpha/2} \| u_0 \| _2.
\end{align*}
\]

Similarly, \( \tau_{m+1} || \partial_t^m \mathcal{G}^{1-\alpha} (a \partial_t^{1-\alpha} u) || \leq C t^{1+\alpha/2} \| u_0 \| _2, \) whereas

\[
\begin{align*}
\tau_{m+1} || \partial_t^m \mathcal{G}^{1-\alpha} (\nabla \cdot (\G u)) (t) || \\
\leq C \max_{0 \leq s \leq t} \left( s^{1-\alpha} \left( \| u(s) \| + \| \nabla (u - u_0)(s) \| + \| \nabla u_0 \| \right) \\
+ \sum_{j=1}^{m} s^{1-\alpha+1+j} \left( \| u^{(j)}(s) \| + \| \nabla u^{(j)}(s) \| \right) \right) \\
\leq C \max_{0 \leq s \leq t} \left( s^{1-\alpha} \left( \| u_0 \| + s^{-\alpha/2} \| \nabla u_0 \| \right) \\
+ s^{1-\alpha} \left( 1 + s^{-\alpha/2} \| \nabla u_0 \| \right) \right) \leq C t^{2-\alpha} \| u_0 \| _2
\end{align*}
\]

and \( \tau_{m+1} || \partial_t^m \mathcal{G}^{1-\alpha} (bu) || \leq C t^{2-\alpha} \| u_0 \| _2. \) Thus,

\[ \tau_{m+1} || \rho^{(m)} (t) || \leq C (t + t^{1+\alpha/2} + t^{2-\alpha}) \| u_0 \| _2, \]

showing that \( \tau_m || \rho^{(m)} (t) || \leq C \| u_0 \| _2 \) and therefore, by \( \text{lem} \) and \( \text{lem} \),

\[ \| u^{(m)} (t) \| _2 \leq C t^{m-\alpha} \| u_0 \| \quad \text{and} \quad \| u^{(m)} (t) \| _2 \leq t^{-m} \| u_0 \| _2, \]

which proves the result in the cases \( \mu = 0 \) and \( \mu = 2 \). The case \( 0 < \mu < 2 \) then follows by interpolation. \( \square \)

Appendix A. Further technical lemmas

The following identity uses the notation from Lemma \( \text{lem} \).

Lemma 14. Let \( \mu > 0 \) and \( 1 \leq q \leq m \). If, for \( m - q + 1 \leq j \leq m \),

\[ \mathcal{M}^j \phi \in W^j_{1,1-\mu} (0, T) \]

with

\[ (\partial_t^k \mathcal{M}^j \phi)(0) = 0 \quad \text{for} \quad 0 \leq k \leq j - (m - q) - 1, \]

then

\[ \partial_t^m \mathcal{M}^j \phi = \sum_{j=0}^{m-q} \partial_t^j \mathcal{M}^{j+m-q-j} \mathcal{M}^j \phi + \sum_{j=m-q+1}^m \partial_t^j \mathcal{M}^{j+m-q-j} \mathcal{M}^j \phi. \]
Proof. By (17),
\[
\mathcal{M}^m \mathcal{I}^\mu = \sum_{j=0}^{m-q} d^m_j \mathcal{I}^{\mu+m-j} \mathcal{M}^j + \sum_{j=m-q+1}^{m} d^m_j \mathcal{I}^{\mu+m-j} \mathcal{M}^j.
\]
If \(0 \leq j \leq m-q\), then \(m-q-j \geq 0\) so \(\partial^q_i \mathcal{I}^{\mu+m-j} = \partial^q_i \mathcal{I}^{\mu} \mathcal{I}^{\mu+m-q-j} = \mathcal{I}^{\mu+m-q-j}\).
Therefore,
\[
\partial^q_i \sum_{j=0}^{m-q} d^m_j \mathcal{I}^{\mu+m-j} \mathcal{M}^j \phi = \sum_{j=0}^{m-q} d^m_j \mathcal{I}^{\mu+m-q-j} \mathcal{M}^j \phi \quad \text{for} \ \phi \in L_1(0,T).
\]
If \(m-q+1 \leq j \leq m\) then \(j-(m-q) \geq 1\) so
\[
\partial^q_i \mathcal{I}^{\mu+m-j} = \partial^q_i (\mu-(m-q) \mathcal{I}^{\mu+m-j} \mathcal{M}^j \phi = \partial^q_i (\mu)
\]
and thus
\[
\partial^q_i \sum_{j=m-q+1}^{m} d^m_j \mathcal{I}^{\mu+m-j} \mathcal{M}^j \phi = \sum_{j=m-q+1}^{m} d^m_j \mathcal{I}^{\mu+m-q-j} \mathcal{M}^j \phi.
\]
By (18),
\[
\partial^q_i (\mu \mathcal{M}^j \phi = \mathcal{I}^{\mu} \partial^q_i \mathcal{M}^j \phi + \sum_{k=0}^{j-(m-q)-1} (\partial^q_i \mathcal{M}^j \phi)(0) \omega_{j-k},
\]
and our hypotheses on \(\phi\) ensure that all terms in the sum over \(k\) vanish. \(\square\)

The next lemma was used in the proof of Lemma 7.

Lemma 15. Let \(\psi \in W^{2m-1}_\infty((0,T);L_\infty(\Omega)^d)\) for some \(m \geq 1\) and let \(\mu \geq 0\). Then,
\[
\mathcal{S}^{\phi,m}(B^\psi_\phi,t) \leq C \sum_{j=0}^{m} \partial^\mu \mathcal{M}^j (\phi,t) \quad \text{for} \ 0 \leq t \leq T \quad \text{and} \ \phi \in C^m_{\alpha}.
\]

Proof. We integrate by parts \(m\) times to obtain
\[
B^\mu_\psi \phi = \mathcal{I}^1(\partial^\mu_1 \mathcal{M}^j \phi) = \sum_{i=0}^{m-1} (-1)^i \psi^{(i)} \mathcal{I}^{\mu+i} \phi + (-1)^m \mathcal{I}^1(\psi^{(m)} \mathcal{I}^{\mu+m-1} \phi),
\]
and so
\[
B^\mu_\psi \phi = (\mathcal{M}^m \mathcal{I}^\mu \phi)^{(m)} = \sum_{i=0}^{m} (-1)^i \mathcal{I}^m \phi, \tag{A.1}
\]
where
\[
\mathcal{I}^m \phi = \begin{cases} 
\partial^m_\mathcal{M} (\psi^{(i)} \mathcal{M}^i \phi) & \text{for} \ 0 \leq i \leq m-1, \\
\partial^m_\mathcal{M} (\psi \mathcal{M}^m \mathcal{I}^1(\mathcal{M}^{m-1} \phi) & \text{for} \ i = m.
\end{cases}
\]
If \(0 \leq i \leq m-1\), then
\[
\mathcal{I}^m \phi = \partial^m_\mathcal{M} (\psi^{(i)} \mathcal{M}^i \phi) = \sum_{q=0}^{m} \binom{m}{q} \psi^{(i+m-q)} \partial^q_\mathcal{M} \mathcal{I}^1 \mathcal{M}^m \mathcal{I}^{\mu+i} \phi
\]
and
Thus, for $0 \leq \text{so}$ and for $m \leq \text{our assumption on } \psi$

By Lemma 14,

\[ \frac{\partial q}{\partial t} \left( \sum_{j=0}^{m} \ldots \right) \]

and by 16,

\[ \mathcal{I}^\mu \mathcal{M}^j = \sum_{k=0}^{j} c_k^j \mathcal{M}^{j-k} \mathcal{I}^\mu + k \quad \text{for } \mu > 0 \text{ and } j \geq 0, \]

with

\[ \frac{\partial q}{\partial t} \left( \sum_{r=0}^{q} \ldots \right) \]

Thus, for $0 \leq j \leq m - q$,

\[ \mathcal{I}^\mu i + m - q - j \mathcal{M}^j = \sum_{k=0}^{j} c_k^j \mathcal{M}^{j-k} \mathcal{I}^\mu + i + m - q - j - k \]

and for $m - q + 1 \leq j \leq m$,

\[ \mathcal{I}^\mu j - (m - q) \mathcal{M}^j = \sum_{r=0}^{j-\ldots} \ldots \]

so

\[ \left\| \left( \frac{\partial q}{\partial t} \mathcal{M}^m \mathcal{I}^\mu + i \phi \right)(t) \right\|^2 \leq C \sum_{j=0}^{m-q} \left\| \left( \mathcal{I}^\mu i + m - q - j \mathcal{M}^j \phi \right)(t) \right\|^2 \]

\[ + \sum_{j=m-q+1}^{m} \ldots \]

\[ \leq C \sum_{j=0}^{m-q} \sum_{k=0}^{j} \ldots \]

\[ + \sum_{j=m-q+1}^{m} \ldots \]

\[ \ldots \]
Integrating in time, since \( \mathcal{L}^0(\mathcal{M}^j \mathcal{G}^\mu \phi, t) \leq t^{2j} \mathcal{L}_2^\mu (\phi, t) \), we see that

\[
\mathcal{L}^0(\partial_t^q \mathcal{M}^m \mathcal{G}^\mu + i \phi, t) \leq C \sum_{j=0}^{m-q} \sum_{k=0}^{j} t^{2(j-k)} \mathcal{L}_2^\mu + i + m - j + k (\phi, t) \\
+ C \sum_{j=m-q+1}^{m} \sum_{r=0}^{j-(m-q)-m-q} t^{2(m-q-k)} \mathcal{L}_2^\mu + i + k r (\phi, t)
\]

and therefore, by Lemma 4

\[
\mathcal{L}^0(\partial_t^q \mathcal{M}^m \mathcal{G}^\mu + i \phi, t) \leq C t^{2(i+m-q)} \sum_{r=0}^{q} \mathcal{L}_2^\mu r (\phi, t). \quad (A.3)
\]

Hence, recalling (A.2),

\[
\mathcal{L}^0(\mathcal{P}_m^m \phi, t) \leq C \sum_{q=0}^{m} \mathcal{L}^0(\partial_t^q \mathcal{M}^m \mathcal{G}^\mu + i \phi, t) \leq Ct^{2i} \sum_{r=0}^{m} \mathcal{L}_2^\mu r (\phi, t), \quad 0 \leq i \leq m - 1. \quad (A.4)
\]

It remains to estimate \( \mathcal{P}_m^m \phi = \partial_t^m \mathcal{M}^m \mathcal{G}^1 (\psi^m \mathcal{G}^\mu + m - 1 \phi) \). Taking \( q = m \) and \( \mu = 1 \) in Lemma 4 gives

\[
\partial_t^m \mathcal{M}^m \mathcal{G}^1 = \bar{d}_0^m \mathcal{G}^1 + \sum_{j=1}^{m} \bar{d}_j^m \mathcal{G}^1 \partial_t^j \mathcal{M}^j,
\]

and so

\[
\mathcal{P}_m^m \phi = \bar{d}_0^m \mathcal{G}^1 (\psi^m \mathcal{G}^\mu + m - 1 \phi) + \sum_{j=1}^{m} \bar{d}_j^m \mathcal{G}^1 \partial_t^j \mathcal{M}^j (\psi^m \mathcal{G}^\mu + m - 1 \phi).
\]

Thus,

\[
\mathcal{L}^0(\mathcal{P}_m^m \phi, t) \leq C \mathcal{L}^0(\mathcal{G}^1 (\psi^m \mathcal{G}^\mu + m - 1 \phi), t) \\
+ C \sum_{j=1}^{m} \sum_{q=0}^{j} \mathcal{L}^0(\partial_t^q \mathcal{M}^j \mathcal{G}^\mu + m - 1 \phi, t),
\]

and since

\[
\| (\mathcal{G}^1 (\psi^m \mathcal{G}^\mu + m - 1 \phi)(t) \|^2 \leq \left( \int_0^t \| \psi^m(s) \|^2 ds \right) \left( \int_0^t \| \mathcal{G}^\mu + m - 1 \phi)(s) \|^2 ds \right) \leq C t \mathcal{L}_2^{\mu + m - 1} (\phi, t)
\]

we have, by Lemma 4

\[
\mathcal{L}^0(\mathcal{G}^1 (\psi^m \mathcal{G}^\mu + m - 1 \phi), t) \leq C t^2 \mathcal{L}_2^{\mu + m - 1} (\phi, t) \leq C t^{2m} \mathcal{L}_2^\mu (\phi, t).
\]
Finally, using (A.3) with \( m \) replaced by \( j \) and with \( i \) replaced by \( m - 1 \),

\[
\mathcal{Q}^0(\mathcal{F}_m^t \phi, t) \leq C_{2m} \mathcal{Q}_2^\mu(\phi, t) + C \sum_{j=k}^{m} \sum_{q=0}^{j-1} t^{2(m-j-q)} \sum_{r=0}^{j} \mathcal{Q}_2^{\mu, r}(\phi, t)
\]

\[
\leq C_{2m} \sum_{q=0}^{m-1} \mathcal{Q}_2^{\mu, r}(\phi, t).
\]

The result now follows from (A.1) and (A.4). \( \square \)

We used the following result in the proof of Theorem 13.

**Lemma 16.** If \( m \geq 0 \), \( \psi \in W_m^\infty((0, T); L_\infty(\Omega)^d) \) and

\[ \mathcal{M}^k \psi \in W_1^0(0, T) \quad \text{for } 0 \leq k \leq m + 1, \]

with

\[ (\partial_\psi^q, \mathcal{M}^k \psi)(0) = 0 \quad \text{for } 1 \leq q \leq k - 1 \text{ and } 1 \leq k \leq m + 1, \]

then

\[ t^{m+1} \left\| \partial_\psi^m, \mathcal{M}^\mu(\psi \phi)(t) \right\| \leq C \max_{0 \leq s \leq t} \sum_{j=0}^{m} \left\| \mathcal{M}^{j+1} \mathcal{M}^j \mathcal{M}^\mu(\psi \phi)(s) \right\| \quad \text{for } 0 < t \leq T. \]

**Proof.** By (A.5),

\[
\left\| \mathcal{M}^{m+1} \partial_\psi^m, \mathcal{M}^\mu(\psi \phi) \right\| = \left\| \sum_{j=0}^{m} \partial_\psi^{m+m} \mathcal{M}^{j+1} \mathcal{M}^j \mathcal{M}^\mu(\psi \phi) \right\|
\]

\[
\leq C \sum_{j=0}^{m} \left\| \partial_\psi^j \mathcal{M}^{j+1} \mathcal{M}^\mu(\psi \phi) \right\|,
\]

and in turn,

\[
\partial_\psi^j \mathcal{M}^{j+1} \mathcal{M}^\mu(\psi \phi) = \sum_{k=0}^{j+1} \partial_\psi^{j+1, k} \partial_\psi^j \mathcal{M}^\mu(\psi \phi).
\]

Since \( \partial_\psi^j \mathcal{M}^\mu(\psi \phi) = \partial_\psi^j(\partial_\psi^k \mathcal{M}^\mu(\psi \phi)) = \partial_\psi^k(\partial_\psi^{j+1, k-j} \mathcal{M}^\mu(\psi \phi)) = \partial_\psi^k \mathcal{M}^{j+1} \mathcal{M}^\mu(\psi \phi) \) for \( 0 \leq k \leq j + 1 \),

\[
\left\| \partial_\psi^j \mathcal{M}^{j+1} \mathcal{M}^\mu(\psi \phi)(t) \right\| \leq \sum_{k=0}^{j+1} \left\| \partial_\psi^{j+1, k} \mathcal{M}^\mu(\psi \phi) \right\|
\]

\[
= C \sum_{k=0}^{j+1} \left\| \partial_\psi^{k+1, k} \mathcal{M}^\mu(\psi \phi) \right\| = C \sum_{k=0}^{j+1} \left\| \mathcal{M}^{k+1} \mathcal{M}^k(\psi \phi)(t) \right\| \quad \text{(A.6)}
\]

where, in the last step, we used the fact that \( \partial_\psi^q \mathcal{M}^k(\psi \phi)(0) = 0 \) for \( 0 \leq q \leq k - 1 \). We have

\[
\partial_\psi^j \mathcal{M}^k(\psi \phi) = \partial_\psi^j(\psi \mathcal{M}^k \phi) = \sum_{q=0}^{k} \binom{k}{q} \psi^{(k-q)} \partial_\psi^q \mathcal{M}^k \phi
\]

\[
= \sum_{q=0}^{k} \binom{k}{q} \psi^{k-q} \sum_{r=0}^{q} \partial_\psi^r \mathcal{M}^{k-(q-r)} \partial_\psi^r \phi,
\]

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and hence

\[ \| \mathcal{G}^\mu \partial^k_t \mathcal{M}^k(\psi \phi)(t) \| \leq C \sum_{q=0}^{k} \sum_{r=0}^{q} \| \mathcal{G}^{\mu+1} \mathcal{M}^{k-q+r} \partial^r_t \phi(t) \| \]

\[ = C \sum_{q=0}^{k} \sum_{r=0}^{q} \int_0^t \omega_{\mu+1}(t-s)s^{k-q-\mu-1} \| s^{r+\mu+1} \phi^{(r)}(s) \| ds \]

\[ \leq C \sum_{r=0}^{k} \left( \max_{0 \leq s \leq t} \| s^{r+\mu+1} \phi^{(r)}(s) \| \right) \sum_{q=0}^{k} \left( \omega_{\mu+1} * \omega_{k-q-\mu} \right)(t). \]

Since \((\omega_{\mu+1} * \omega_{k-q-\mu})(t) = \omega_{k-q+1}(t) \leq Ct^{k-q}\), the result now follows from (A.5) and (A.6).

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