Probability of radiation of twisted photons by axially symmetric bunches of particles

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Abstract. In most cases, the twisted photons generated directly by charged particles in undulators and laser waves are produced by bunches of particles and not by one charged particle. However, up to now, the theoretical studies of such a radiation were mainly based on description of radiation produced by one charged particle. In the present paper, we investigate the effect of a finite width of a particle bunch on radiation of twisted photons. The general formulas connecting the radiation probability distribution of twisted photons produced by bunches of identical particles with the radiation probability distribution of twisted photons generated by one particle are obtained for axially symmetric bunches. The bunch is called axially symmetric if it is axially symmetric with respect to the detector axis at some instant of time and all the particles in the bunch move along parallel trajectories. The general sum rules for the probability of radiation of twisted photons by axially symmetric bunches are established. In particular, we prove that the projection of the average total angular momentum of radiated twisted photons per particle in the bunch does not depend on the radial profile of the bunch. The uniform, Gaussian, and exponential radial bunch profiles are considered in detail. The radiation of axially symmetric bunches in ordinary and crystalline undulators is investigated. The selection rules for radiation of twisted photons by one particle in undulators are violated when the finite width of the particle bunch is taken into account. We find the condition when this violation is marginal. The form of the radiation probability distribution of twisted photons becomes universal for wide incoherent axially symmetric particle bunches. We completely describe these universal distributions.

1 Introduction

The twisted photons are the states of the electromagnetic field that possess a definite energy, unique projections of the momentum and of the total angular momentum onto the detector axis, and a well defined helicity [1–6]. These states constitute a complete set, and any free electromagnetic field can be represented as a collection of twisted photons. It is important that such decomposition is not only a mathematical property, but it can be performed by suitably designed detectors and observed experimentally [7–11]. Even in the X-ray spectral range the twisted photons can be directly detected [12]. Nowadays, the wave packets of twisted photons sufficiently narrow in quantum numbers are widely used in science and technology (see for reviews [13–16]).

One of the bright and sufficiently pure sources of twisted photons are undulators and undulator-like devices, for example, FELs and laser waves. The generation of narrow wave packets of twisted photons by such devices was predicted theoretically [4, 5, 17–27] and confirmed experimentally [26, 28–30]. Recently, we have found a general formula for the probability of radiation of twisted photons by classical currents [31]. This formula allowed us to describe completely the radiation produced by one point charged particle in undulators and wigglers, but the effects caused by a finite size of a particle bunch were completely neglected. In contrast to radiation of plane-wave photons, the radiation probability of twisted photons is not invariant under translations of a source that are perpendicular to the detector axis [31]. Therefore, a finiteness of the particle bunch results in nontrivial modification of the radiation probability of twisted photons even in the case of incoherent radiation. In the present paper, we investigate the effect of finiteness of the
transverse bunch size on the form of the probability of radiation produced by axially symmetric bunches (see the precise definition below). In general, we establish that, in comparison with the radiation created by one point particle, the finite transverse size of a particle bunch leads to broadening of the radiation probability distribution over the projections $m$ of the photon total angular momentum onto the detector axis.

In sect. 2, we start with the general formulas for the probability of radiation of twisted photons produced by a bunch of identical particles, which is axially symmetric with respect to the detector axis at some instant of time and all the trajectories of particles in the bunch are obtained from one trajectory by a parallel transport. For brevity, we call such a bunch as axially symmetric. Such round particle beams possess a high bunch-by-bunch luminosity and are created, for example, in the electron-positron collider VEPP-2000, Novosibirsk [32]. Of course, the assumption that the particles move along parallel trajectories is only an approximation. It is fulfilled, for example, when, on the radiation formation scale, the identical particles move in the electromagnetic field invariant under the translations perpendicular to the detector axis and the interaction between these particles is negligible. The assumption that the bunch is axially symmetric allows us to obtain simple formulas relating the radiation probability for a bunch to the radiation probability for one point particle. These formulas involve the interference factors depending on the radial distribution of particles in the bunch. Several exact sum rules for the probability distributions and interference factors are found. In particular, it is shown that the radiation produced by an axially symmetric bunch and the radiation produced by one particle possess the same projections of the total angular momentum onto the detector axis per photon. This property holds for both coherent and incoherent radiations of a bunch. As for incoherent radiation, this sum rule is valid for bunches with zero dipole moment with respect to the center of the bunch and not only for axially symmetric bunches, provided the particles move along parallel trajectories. Then we derive a simple general condition when the one-particle answer for radiation probability can be used to describe radiation produced by a bunch. As an example, we obtain the interference factors for bunches with simple radial profiles: uniform, Gaussian, and exponential. Then we consider the generalization of formulas to the case when the symmetry axis of the bunch is tilted from the detector axis by a small angle. In that case, the formula for incoherent radiation remains intact, while the formula for coherent radiation is modified and the new interference factor arises. We find the explicit expression for this factor for bunches with simple radial profiles mentioned above. In conclusion of sect. 2, the transition from the bunch of particles to a continuous flow is discussed.

In sect. 3, we apply the general formulas to describe the radiation of twisted photons by axially symmetric bunches of charged particles in undulators and wigglers. We show that the selection rules fulfilled for the forward radiation generated by one particle in undulators [17, 24–26, 31, 33] become violated when the radiation produced by a bunch of such particles is considered. However, if the spreading of radiation probability distribution over $m$ is small, this violation is marginal. We find the condition when broadening of the distribution over $m$ is negligible and so is the violation of the selection rules. Recall that the forward radiation of twisted photons by one charged particle in helical undulators obeys the selection rule $m = \chi n$, where $n$ is the harmonic number and $\chi = \pm 1$ is the chirality of the undulator. As for planar undulators, the selection rule for radiation by one charged particle says that $m + n$ is an even number. As the second example, we consider the radiation of twisted photons by crystalline undulators [34, 35]. The possibility of creation of radiation with large angular momentum by channeling of charged particles was pointed out in [36] (see also [37]). Here we study a relatively soft radiation generated by particles moving in a specially designed crystal. The spreading of radiation probability distribution over $m$ is very large in this case and the shape of this distribution becomes universal. Thus, in considering the undulator radiation of twisted photons, we examine their radiation in a wide range of energies from the THz domain to the hard X-rays. In conclusion section, we summarize the results.

We use the notation and conventions adopted in [31]. In particular, $\hbar = c = 1$ and $e^2 = 4\pi\alpha$, where $\alpha \approx 1/137$ is the fine structure constant.

## 2 General formulas

### 2.1 Translations of the source

In order to obtain the radiation amplitude of a twisted photon produced by an axially symmetric bunch of particles, it is convenient to derive the general formula for the change of radiation amplitude under translations of the source by the vector $a = (a_1, a_2, a_3)$. As soon as one finds such a relation, the radiation amplitude by an axially symmetric bunch of particles is obtained from the radiation amplitude by one charged particle since the currents corresponding to different particles in the bunch are obtained by translations. The radiation amplitude by a particle bunch in the first Born approximation is the sum of one-particle radiation amplitudes over all the particles in the bunch. In the case of incoherent addition of amplitudes, the sum of radiation probabilities corresponding to different particles in the bunch should be taken.

Suppose that the translations are realized in the Fock state space by a unitary operator $\hat{V}(a)$. Then

$$\hat{V}(a) \hat{A}(x) \hat{V}^{-1}(a) = \hat{A}(x + a),$$

(1)
where $\hat{A}_i(x)$ is the operator of the electromagnetic potential in the Coulomb gauge. Substituting the expansion of the operator $\hat{A}_i$ in terms of the mode functions (eq. (18) in [31]) into this expression, we obtain

$$\hat{V}(a)\hat{c}_\alpha\hat{V}^{-1}(a) = A_{\alpha\beta}(a)\hat{c}_\beta,$$

where $\psi_{\alpha}(x)$ are the mode functions of a twisted photon (eq. (17) in [31]). The coefficients $A_{\alpha\beta}(a)$ can easily be found with the aid of the addition theorem (eq. (A6) in [31]) for the Bessel functions (see also [38])

$$j_m(k_\perp(x_+ + a_+), k_\perp(x_- + a_-)) = \sum_{n=-\infty}^{\infty} j_{m-n}(k_\perp a_+, k_\perp a_-) j_n(k_\perp x_+, k_\perp x_-).$$

Using this formula, we deduce

$$A_{\alpha\beta}(a) \equiv A(s, m, k_\perp; s', m', k'_\perp; a) = e^{ik_3a_3} j_{m'-m}(k_\perp a_+, k_\perp a_-) \delta_{ss'} \frac{2\pi}{L} \delta(k_3 - k'_3) \frac{\pi}{R} \delta(k_\perp - k'_\perp).$$

Inasmuch as

$$\hat{V}(a)\hat{S}_{T/2-\rightarrow T/2}[j^\mu(t, x)]\hat{V}^{-1}(a) = \hat{S}_{T/2-\rightarrow T/2}[j^\mu(t, x - a)],$$

the amplitude of creation of a twisted photon by the current $j^\mu(t, x - a)$ is given by (see eq. (30) in [31])

$$A(a; s, m, k_3, k_\perp) = \sum_{n=-\infty}^{\infty} e^{-ik_3a_3} j_{m-n}(k_\perp a_+, k_\perp a_-) A(0; s, n, k_3, k_\perp).$$

Below, we shall use this formula to derive the probability of radiation of twisted photons by a bunch of identical charged particles moving along parallel trajectories. The details of derivation of the general formula for the probability of radiation of twisted photons can be found in [31].

### 2.2 Radiation by a bunch of particles

Let the distribution of identical particles in the bunch be axially symmetric with respect to the detector axis at some instant of time and be described by the function $f(r/\sigma)$, where $\sigma$ characterizes the width of the bunch. The normalization condition looks as

$$2\pi \int_0^\infty dr f(r/\sigma) = N,$$

where $N$ is the number of particles. Suppose that the particle trajectories pass one into another by shifts by the vectors of the form $a = (a_1, a_2, 0)$, i.e., $a_3 = 0$.

Assuming the coherent addition of radiation amplitudes, we find the total radiation amplitude,

$$\int da_1 da_2 f(|a_1|/\sigma) A(a; s, m, k_3, k_\perp) = 2\pi \int_0^\infty dr f(r/\sigma) J_0(k_\perp r) A(0; s, m, k_3, k_\perp).$$

Integrating in (8) over the azimuth angle, only one term at $n = m$ in the sum (6) survives. The probability of coherent radiation by such a bunch of particles becomes

$$dP_f^R(s, m, k_3, k_\perp) = N^2 \varphi^2(k_\perp \sigma) dP_f(s, m, k_3, k_\perp),$$

where

$$\varphi(k_\perp \sigma) := \frac{2\pi}{N} \int_0^\infty dr f(r/\sigma) J_0(k_\perp r),$$

$$dP_f(s, m, k_3, k_\perp)$$

is the probability of radiation of twisted photons by one particle moving along the trajectory with the parameter $a = 0$. The interference factor obeys the normalization condition $\varphi(0) = 1$. The average projection of the total angular momentum onto the detector axis and the projection of the total angular momentum per photon take the form

$$dJ^R_{\perp f}(s, k_3, k_\perp) = \sum_{m=-\infty}^{\infty} m dP_f^R(s, m, k_3, k_\perp) = N^2 \varphi^2(k_\perp \sigma) dJ_{31}(s, k_3, k_\perp),$$

$$\ell^R_f(s, k_3, k_\perp) = dJ^R_{\perp f}(s, k_3, k_\perp)/dP_f^R(s, k_3, k_\perp) = \ell_1(s, k_3, k_\perp),$$

where $dJ_{31}(s, k_3, k_\perp)$ and $\ell_1(s, k_3, k_\perp)$ are the respective quantities for the radiation produced by one particle.
The coherent addition of radiation amplitudes of different particles occurs when the wavelength of radiated photons is larger or of the order of the size of a bunch of particles. Besides, the formulas for coherent radiation can be used in semiclassical description of radiation created by a wave packet of a charged particle. Some properties of coherent radiation of twisted photons, such as its infrared asymptotics and the selection rules for symmetrical sources, were investigated in [39]. The studies of scattering processes of particles with the wave functions localized in space and possessing nontrivial phases can be found, for example, in [40–51], where, in particular, the dependence of process probabilities on the impact parameter was investigated.

As far as incoherent radiation is concerned, the radiation probabilities are summed up, rather than the amplitudes. Therefore,

\[
dP_{nc}^f(s, m, k_3, k_\perp) = \int da_1 da_2 f(a_\perp)|A(a_\perp)|^2 = \sum_{n, n'}^\infty \int da_1 da_2 f(|a_\perp|/\sigma) \times A(0, s, n, k_3, k_\perp) A^*(0, s, n', k_3, k_\perp) j_{m-n}(k_\perp a_+, k_\perp a_-) j_{m-n'}(k_\perp a_+, k_\perp a_-).
\]

(11)

Notice that, in the case of incoherent radiation, the parameter \(a_3\) can be nonzero for different trajectories. The following sum rule is fulfilled:

\[
\sum_{m=-\infty}^\infty dP_{nc}^f(s, m, k_3, k_\perp) = N \sum_{m=-\infty}^\infty dP_f(s, m, k_3, k_\perp),
\]

(12)

which is valid for any bunch of particles moving along parallel trajectories. This property stems from the addition theorem (see eq. (A6) in [31])

\[
\sum_{m=-\infty}^\infty j_{m-n}(k_\perp a_+, k_\perp a_-) j_{m-n'}(k_\perp a_+, k_\perp a_-) = j_{n-n'}(0, 0) = \delta_{n,n'}.
\]

(13)

A less formal explanation of this property is that, having summed over \(m\), the probability of photon radiation becomes invariant with respect to translations of the current \(j^\mu(x)\). Then the summed radiation probabilities are the same for all the particles in the bunch and simply add up in the case of incoherent radiation.

Taking into account the relation

\[
\sum_{m=-\infty}^\infty m j_{m-n}(k_\perp a_+, k_\perp a_-) j_{m-n'}(k_\perp a_+, k_\perp a_-) = n\delta_{nn'} + \frac{1}{2} (k_\perp a_+ \delta_{n,n'+1} + k_\perp a_- \delta_{n,n'-1}),
\]

(14)

we obtain

\[
dJ_{3f}^{nc}(s, k_3, k_\perp) = N dJ_{31}(s, k_3, k_\perp)
\]

\[
+ \text{Re} \sum_{n=-\infty}^\infty A(0, s, n, k_3, k_\perp) A^*(0, s, n-1, k_3, k_\perp) \int da_1 da_2 f(a_\perp) k_\perp a_+.
\]

(15)

If the bunch dipole moment defined with respect to the center of the bunch at the initial instant of time is zero, which is valid, for example, for an axially symmetric bunch, then the last term vanishes. The last term also vanishes for the forward radiation of undulators and the forward radiation of helical and planar wigglers for any profile \(f(a_\perp)\). This is a consequence of the selection rules fulfilled in these cases [31]. Then we have the another sum rule

\[
dJ_{3f}^{nc}(s, k_3, k_\perp) = N dJ_{31}(s, k_3, k_\perp), \quad \ell_{3f}^{nc}(s, k_3, k_\perp) = \ell_3(s, k_3, k_\perp).
\]

(16)

Relations (10) and (16) indicate that the angular momentum per photon is a rather robust characteristic of the radiation produced by axially symmetric bunches.

If the bunch is axially symmetric, the integral over the azimuth angle in (11) is readily evaluated:

\[
\int_0^{2\pi} d\varphi j_{m-n}(k_\perp a_+, k_\perp a_-) j_{m-n'}(k_\perp a_+, k_\perp a_-) = 2\pi \delta_{n,n'} j_{m-n}^2(k_\perp |a_+|),
\]

(17)

where \(\varphi = \text{arg} a_+\). Then the radiation probability of twisted photons created by such a bunch of particles takes the form

\[
dP_{nc}^f(s, m, k_3, k_\perp) = N \sum_{n=-\infty}^\infty f_{m-n}(k_\perp \sigma) dP_f(s, n, k_3, k_\perp),
\]

\[
f_m(k_\perp \sigma) := \frac{2\pi}{N} \int_0^{\infty} drr f(r/\sigma) J_m^2(k_\perp r).
\]

(18)
It is clear that \( f_m(0) = \delta_{m,0} \) and \( f_m(x) = f_{-m}(x) \). The small argument expansion reads as
\[
f_m(x) = \frac{2\pi\sigma^2}{N} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(2|m| + k + 1)_k}{I^2(|m| + k + 1)} \left( \frac{x}{2} \right)^{2|m|+2k} \int_0^{\infty} dr r^{2|m|+2k+1} f(r).
\] (19)

For \( x \gg \max(1, |m|) \), we have the asymptotics
\[
f_m(x) \approx \frac{1}{\pi x} \frac{2\pi\sigma^2}{N} \int_0^{\infty} dr f(r).
\] (20)

This asymptotics does not depend on \( m \). The sum rule (12) leads to the normalization condition for the interference factor
\[
\sum_{m=-\infty}^{\infty} f_m(x) = 1.
\] (21)

Using these properties, one can check once again the validity of general formulas (16) for axially symmetric bunches.

If \( dP_1(m) \sim \delta_{m,m_0} \), which holds, for example, for a charged particle moving along an ideal helix [17, 24–26, 31], then
\[
dP_{1c}(s, m, k_3, k_\perp) = N f_{m-m_0}(k_\perp) dP_1(s, m_0, k_3, k_\perp).
\] (22)

In the dipole approximation, the forward radiation created by one particle is mainly concentrated at \( m = \pm 1 \) [31]. In that case,
\[
dP_{1c}(s, m, k_3, k_\perp) \approx N f_{m-1}(k_\perp) dP_1(s, 1, k_3, k_\perp) + N f_{m+1}(k_\perp) dP_1(s, -1, k_3, k_\perp).
\] (23)

As we can see, the finite width of a particle bunch results in spreading of the incoherent radiation distribution over \( m \) in comparison with the radiation probability of twisted photons produced by one point particle.

If \( f(r/\sigma) \) is small for \( r \gg \sigma \), one can neglect the spreading of distribution over \( m \) when \( k_\perp \sigma \ll 1 \). (24)

It was shown in [31,39] that, in the ultrarelativistic case, the major part of twisted photons is radiated with
\[
n_\perp := k_\perp/k_0 \approx \lambda/\gamma,
\] (25)

where
\[
\lambda = \max(1, K), \quad K = \beta_\perp \gamma,
\] (26)

and \( K \) is the undulator strength parameter, \( \beta_\perp > 0 \) is a typical value of the velocity component perpendicular to the detector axis, and \( \gamma \) is the Lorentz factor of a radiating particle. Then the spreading of distribution over \( m \) caused by a finite transverse size of the bunch can be neglected, provided that
\[
k_0 \sigma \lambda \ll \gamma.
\] (27)

Roughly, this estimate says that one can use the one-particle radiation probability distribution to describe the radiation of twisted photons when the wavelength of radiated photons is larger than the transverse size of a bunch, divided by the Lorentz factor. For example, for the forward radiation produced by charged particles in undulators, we have (see, e.g., eq. (117) in [31])
\[
\frac{2n\omega \gamma \sigma \lambda}{1 + K^2 + \lambda^2} \ll 1,
\] (28)

where \( n \) is a harmonic number and \( \omega \) is a circular oscillation frequency of particles in the undulator. In sect. 3, we shall consider the radiation by particle bunches in undulators in more detail.

### 2.3 Interference factors

Let us find the explicit expressions for the interference factors for simple profile functions \( f(r/\sigma) \).

a) Uniform distribution
\[
f(r/\sigma) = \frac{N}{\pi \sigma^2} \theta(\sigma - r).
\] (29)

Then
\[
\varphi(x) = 2J_1(x)/x.
\] (30)
Fig. 1. The interference factors $f_m(x)$ for the different bunch profiles.

This expression vanishes when

$$x = k_\perp \sigma = \alpha_1^{(n)},$$

where $\alpha_1^{(n)}$ are the zeros of the Bessel function $J_1(x)$, i.e., the coherent bunch of charged particles does not radiate the twisted photons with $k_\perp$ satisfying (31). For $x \gtrsim 3$, the corresponding factor in the probability of coherent radiation (9) drops as $1/x^3$.

As for incoherent radiation, the interference factor is written as

$$f_m(x) = J_2|m| x - 2|m| x J_1|m| x J_{|m|+1}(x) + J_2|m| x J_{|m|+1}(x).$$

(32)

This expression drops as $2/\pi x$ as $x$ tends to infinity.

b) Gaussian bunch

$$f(r/\sigma) = \frac{N}{2\pi \sigma^2} e^{-r^2/(2\sigma^2)}.$$  
(33)

In this case, the coherent interference factor becomes

$$\varphi(x) = e^{-x^2/2}.$$  
(34)

This expression is a monotonically decreasing function of $x = k_\perp \sigma$ and nonvanishing. For $x \gtrsim 3$, the coherent radiation (9) is virtually absent. The incoherent interference factor takes the form

$$f_m(x) = e^{-x^2} I_{|m|}(x^2),$$

(35)

where $I_{\nu}(z)$ is the modified Bessel function of the first kind. For $x$ large, the interference factor decreases as $1/(\sqrt{2\pi x})$.

c) Exponential profile (see, e.g., [51])

$$f(r/\sigma) = \frac{N}{2\pi \sigma^2} e^{-r/\sigma}.$$  
(36)

The interference factor for coherent radiation is

$$\varphi(x) = (1 + x^2)^{-3/2}.$$  
(37)

For $x \gtrsim 3$, the corresponding factor entering into (9) declines as $1/x^6$. The interference factor for incoherent radiation reads as

$$f_m(x) = 2 \Gamma(|m| + 3/2) \pi^{1/2} |m|! x^{2|m|} x^{F(|m| + 1/2, |m| + 3/2; 2|m| + 1; -4x^2)}.$$  
(38)

For large $x$, this quantity drops as $1/(\pi x)$. The plots of the functions $f_m(x)$ are presented in fig. 1.

We see that for all the three cases considered above the radiation produced by different points of the coherent bunch interferes destructively when $k_\perp \sigma \gtrsim 3$. The coherent radiation is strongly suppressed in this case. This property seems to be a general one for a particle bunch of a general profile localized near the bunch axis.
2.4 Radiation at a small angle

The above formulas can be easily generalized to the case when the detector axis is directed at a small angle \( \theta \) to the axis of an axially symmetric bunch. In that case, choosing properly the system of coordinates, the trajectories of particles in the bunch are obtained from one trajectory by means of the translations of the form

\[
a_+ = \tau_1 \cos \theta + i \tau_2, \quad a_3 = \tau_1 \sin \theta,
\]

where \( \tau_{1,2} \in \mathbb{R} \) are the transformation parameters.

For incoherent radiation, the probability to detect the twisted photon is given by

\[
dP_{\text{inc}}(s,m,k_3,k_\perp) = \int d\tau_1 d\tau_2 f(|\tau_+|/\sigma) |A(a(\tau); s,m,k_3,k_\perp)|^2
\]

\[
= \sum_{n,n'=\pm} \int d\tau_1 d\tau_2 f(|\tau_+|/\sigma)
\times A(0; s,n,k_3,k_\perp)A^*(0; s,n',k_3,k_\perp) j_{m-n}(k_\perp a_+o(\tau),k_\perp a_-o(\tau)) j_{m-n'}(k_\perp a_+o(\tau),k_\perp a_-o(\tau)).
\]

If

\[
k_\perp \sigma \theta^2/2 \ll 1, \quad \theta^2 \ll 1,
\]

then, using the expansions,

\[
a_\pm = \tau_\pm - \tau_1 \theta^2/2 + \cdots,
\]

\[
j_k(k\perp a_+,k\perp a_-) = j_k(k\perp a_+o(\tau),k\perp a_-o(\tau)) + \frac{k_1 \tau_1 \theta^2}{4} [j_{k+1}(k\perp a_+o(\tau),k\perp a_-o(\tau)) - j_{k-1}(k\perp a_+o(\tau),k\perp a_-o(\tau))] + \cdots,
\]

we conclude that one can set \( \cos \theta = 1 \) in expression (39) for \( a_\pm \), provided \( f(r/\sigma) \) is small for \( r \gtrsim \sigma \). As a result, integral (40) is reduced to (11), and we revert to formula (18). Thus, in the case of incoherent radiation of twisted photons by axially symmetric bunches with the symmetry axis tilted from the detector axis by a small angle, one can still use formula (18).

The case of coherent radiation is a little more sophisticated. The radiation amplitude is written as

\[
\int d\tau_1 d\tau_2 f(|\tau_+|/\sigma) A(a(\tau); s,m,k_3,k_\perp) = \int_0^\infty dr f(r/\sigma) \sum_{n=-\infty}^\infty A(0; s,n,k_3,k_\perp)
\times \int_0^{2\pi} d\varphi e^{-i|k_3 r| \sin \theta \cos \varphi + (m-n) \arg a_+o(\tau)} J_{m-n}(k_\perp a_+o(\tau)),
\]

where \( r := |\tau_+| \) and \( \varphi \) is the azimuth angle in the \((\tau_1, \tau_2)\) plane. Under assumptions (41), we can take \( a_+ = \tau_+ \) in expression (43). Then the integral over \( \varphi \) is the Bessel function, and we have

\[
\int d\tau_1 d\tau_2 f(|\tau_+|/\sigma) A(a(\tau); s,m,k_3,k_\perp) = N \sum_{n=-\infty}^\infty \varphi_{m-n}(k\perp \sigma,k_3 \sigma \sin \theta) A(0; s,n,k_3,k_\perp),
\]

where the interference factor takes the form

\[
\varphi_m(k\perp \sigma,k_3 \sigma \sin \theta) := i^{-m} \frac{2\pi}{N} \int_0^\infty dr f(r/\sigma) J_m(k_3 r \sin \theta) J_m(k\perp r).
\]

Obviously,

\[
\varphi_m(x,y) = \varphi_m(y,x), \quad \varphi_m(x,y) = (-1)^m \varphi_{-m}(x,y), \quad \varphi_m(x,0) = \delta_{m,0} \varphi(x),
\]

\[
\varphi_m(x,x) = i^{-m} f_m(x), \quad \sum_{m=-\infty}^\infty t^m \varphi_m(x,y) = \varphi(\sqrt{(x+iy)/(x-iy)}) = \varphi(x,t).
\]

The last formula provides the generating function for the interference factors \( \varphi_m(x,y) \) and, as a particular case, for \( f_m(x) \). We see that, formally, all the interference factors for both coherent and incoherent radiation can be obtained from the coherent interference factor \( \varphi(x) \).
The probability of coherent radiation of twisted photons becomes

$$\frac{dP}{ds}(s, m, k_\perp, k_\parallel) = N^2 \left| \sum_{n=-\infty}^{\infty} \varphi_{m-n}(k_\perp \sigma, k_\parallel \sin \theta) A(0; s, n, k_3, k_\perp) \right|^2. \tag{47}$$

If $f(r/\sigma)$ decreases rapidly for $r \gg \sigma$, and

$$k_3 \theta \ll 1, \tag{48}$$

then, up to contributions of the order $f(10)$, we come back to formulas (8), (9).

In the dipole approximation, the radiation amplitude of a twisted photon is mainly concentrated at $m = \pm 1$. In that case, only the two terms with $n = \pm 1$ are left in the sum (47). When the particles move along an ideal helix with the axis coinciding with the detector axis, a single term survives in the sum (47). It corresponds to $|n| = n_0$, where $n_0$ is the harmonic number, and the sign of $n$ is determined by the helix chirality. The examples of radiation probability distributions of twisted photons recorded at a small angle and created by axially symmetric particle bunches are given in fig. 2.

Let us present the explicit expressions for $\varphi_m(x, y)$ for the bunch profiles considered above [52]:

\begin{enumerate}
    \item $\varphi_m(x, y) = 2^{-m} x J_{|m|}(y) J_{|m|-1}(x) - y J_{|m|}(x) J_{|m|-1}(y)$,
    \item $\varphi_m(x, y) = i^{-m} e^{-(x^2+y^2)/2} I_{|m|}(xy)$,
    \item $\varphi_m(x, y) = -\frac{2}{\pi} i^{-m} Q^{1/2}_{|m|-1/2}(1+x^2+y^2)/(2xy)/(1+(x+y)^2)(1+(x-y)^2)$,
\end{enumerate}

where $x := k_\perp \sigma$, $y := k_\parallel \sin \theta$, and the associated Legendre function of the second kind, $Q^{1/2}_{|m|-1/2}(x)$, is defined in such a way that it is real and analytic if $x > 1$. In cases b) and c), the interference factors decline rapidly to zero out of the diagonal $x = y$, i.e., for

$$k_\perp / k_3 \neq \theta, \tag{50}$$

provided $m \neq 0$. This implies that if $x \neq y$, then only the term with $n = m$ survives in the sum (47), and the spreading of radiation probability distribution over $m$ is negligible. In case a), the modulus of $\varphi_m(x, y)$ also reaches the maximal value at $x = y$, but its decrease out of the diagonal is not so drastic. For $m = 0$, besides the diagonal,
expressions (49) are not small in the region \( x \lesssim 1, y \lesssim 1 \). Even on the diagonal, where \( \varphi(x, x) = i^{-m} f_m(x) \), the quantity \(|\varphi_0(x, x)| \gg |\varphi_m(x, x)|\) for \( m \neq 0 \) and \( x \lesssim 1 \) (see fig. 1), and so the spreading of the corresponding one-particle radiation probability distribution is negligible in this case too (see fig. 2). A large spreading over \( m \) occurs only when \( x \approx y \gtrsim 1 \).

Notice that the expression b) turns into (34) when not only \( y \ll 1 \), but \( xy \ll 1 \). At first glance, this contradicts the statement made below formula (48). However, this controversy is resolved if one observes that, in the case at hand, the exact expression b) is very small for large \( x \). The error arising in passing from (47) to (9) is negligibly small. The plots of functions \( \varphi_m(x, y) \) are given in fig. 3.

2.5 Transition to a continuous flow

We implicitly assumed above that the bunch of particles can be replaced by a continuous flow, i.e., one can pass from the summation over particles to the integration over a continuous distribution. This transition is justified in the case when the radiation by a particle wave packet or a coherent bunch of quantum particles are considered in the semiclassical approximation. In this approximation, the radiation is such as if it were generated by the current of a charged fluid with the properties (the charge density and the velocity) determined by the wave function. Then the radiation produced is coherent, i.e., one should use the formulas for coherent radiation. As for the bunch of particles, it can be replaced by a continuous flow in the case when

\[
k_{\perp} |\Delta a_+| \ll 1,
\]

where \( |\Delta a_+| \) is a distance between neighboring particles. This is valid for both coherent and incoherent radiation generated by particle bunches.

3 Undulator radiation

3.1 Ordinary undulators

As an example of applications of the general formulas obtained above, we consider the radiation of twisted photons by a bunch of charged particles in undulators. The radiation of twisted photons by one charged particle in an undulator was thoroughly investigated in [17–21, 24–26, 31]. We will use the notation and the results presented in sect. 5 of [31] (see also [53]).
Fig. 4. Top line: (0) The probability of forward radiation of a twisted photon at the first harmonic by one electron in the planar undulator with the parameters: the electron Lorentz factor $\gamma = 500$, the undulator period $\lambda_0 = 10$ cm, the number of undulator sections $N_s = 40$, the strength of the magnetic field $H = 100$ G, the undulator strength parameter $K = 6.6 \times 10^{-2}$. The photons are radiated in the middle ultraviolet spectral range. (a) The forward radiation probability per electron by an axially symmetric incoherent bunch with the uniform radial distribution of particles. (b) The same but for the Gaussian radial distribution of particles in the bunch. (c) The same but for the exponential radial distribution of particles in the bunch. Middle line: The same as in the top line but for an ideal helical wiggler with $\gamma = 100$, $\lambda_0 = 10$ cm, $N_s = 40$, $K = 5$. The fifth harmonic is considered. The photons are radiated in the THz spectral range. The forward radiation produced by one particle obeys the selection rule $m = n$, where $n$ is the harmonic number. The finite width of a particle bunch results in spreading of the distribution over $m$. Bottom line: The same as in the middle line but for the planar wiggler with the parameters $\gamma = 100$, $\lambda_0 = 10$ cm, $N_s = 40$, $K = 5$. The third harmonic is considered. The photons are radiated in the THz spectral range. The forward radiation produced by one particle in the planar wiggler obeys the selection rule $n + m$ is an even number [31]. The finite width of a particle bunch leads to violation of this selection rule. Nevertheless, the peaks at $m = \{-3, 1, 3\}$ are still pronounced.

From the physical point of view, the most interesting case is the incoherent addition of radiation amplitudes of separate particles in the bunch. It is this situation which is commonly realized in experiments. The analysis carried out in the previous section indicate that the spreading of distributions of radiated twisted photons over $m$ is small when (27) is satisfied. In that case, the one-particle results provide a good approximation for the radiation by particle bunches. In figs. 2, 4, the distributions of twisted photons over $m$ radiated by an electron bunch with a round cross section $\sigma = 125 \mu$m are presented. Such bunches of electrons and positrons are created, for example, in the electron-positron collider VEPP-2000 [32]. Despite the spreading of distributions, the projection of the angular momentum per photon $l$ remains the same as for the one-particle distribution (see (16) and figs. 2, 4). The selection rules holding for the forward radiation produced by one particle in the helical and planar undulators are violated for the radiation generated by a bunch of particles. However, if $k_x \sigma$ is small, their existence for a one-particle distribution can be tracked out from the plots in fig. 4. As is seen from fig. 2, the periodicity in $m$ for undulator radiation at an angle $\gamma$ is a more robust property.
Fig. 5. Left: The dependence of probability of radiation of twisted photons produced in the crystalline undulator with $K = 0.23$ on the photon energy for the different observation angles. The photon energy $k_0 = 0.15 \text{m} = 76.6 \text{keV}$ used on the right panel is depicted by a dotted vertical line. Right: The probability per electron of incoherent radiation of twisted photons in the crystalline undulator at $k_0 = 76.5 \text{keV})$ for the different axially symmetric bunch profiles. The projection of the total angular momentum per photon $\ell$ is the same for all the profiles. The thick line is a curve given by (53) for the Gaussian bunch. The inset: The one-particle radiation probability distribution of twisted photons over $m$.

3.2 Crystalline undulators

As the second example, we consider the radiation of twisted photons by crystalline undulators [35]. The trajectory of the electron in such an undulator is given approximately by

$$x(t) = -a \cos(\omega t), \quad y(t) = 0, \quad z(t) = \beta_\| (t - L/2), \quad t \in [0, L],$$

(52)

where $\beta_\| = (1 - \gamma^{-2})^{1/2}$, $\gamma = 10^3$, $\omega = 2\pi\beta_\|\lambda_0^{-1}$, $N_s = 15$, $L = 2\pi N_s/\omega$, $\lambda_0 = 23 \mu\text{m}$, $a = 10d$, $d = 0.1 \text{nm}$. The parameters are taken from [34]. We suppose that the transverse size of the electron bunch is characterized by $\sigma = 125 \mu\text{m}$, and the number of particles in the bunch $N = 10^{10}$. In that case, the energy of radiated photons is large and $k_\| \sigma \gg 1$.

If $k_\| \sigma \gg 1$, then $f_m(x)$ depends weakly on $m$ in the region where $f_m(x)$ is not exponentially suppressed. Therefore, if the one-particle radiation distribution over $m$ is nonvanishing only in the region $|m| \ll k_\| \sigma$, as, for example, in the case of the forward dipole radiation (23), then (18) implies, approximately,

$$dP_{nc}^1(s, m, k_3, k_\perp) \approx N f_m(k_\perp \sigma) \sum_{n=-\infty}^{\infty} dP_1(s, n, k_3, k_\perp) = N f_m(k_\perp \sigma) dP_1(s, k_3, k_\perp).$$

(53)

Thus, in this case, the distribution over $m$ of twisted photons radiated by a bunch of charged particles is universal up to a common factor. The approximation (53) is rough and does not comply with the exact sum rule (16). Nevertheless, (53) reproduces the shape of the curve $dP(m)$ quite well (see fig. 5). One can secure the compliance with the sum rule (16) by making the replacement $f_m(k_\perp \sigma) \rightarrow f_{m-m_0}(k_\perp \sigma)$ in (53), where $m_0$ is found from the requirement

$$\sum_{m=-\infty}^{\infty} m f_{m-m_0}(k_\perp \sigma) = \ell_1(s, k_3, k_\perp),$$

(54)

and $\ell_1$ is given in (16). In general, $m_0 = m_0(s, k_3, k_\perp)$.

4 Conclusion

Let us briefly recapitulate the results. We obtained the general formulas (9), (18), and (47) relating the radiation probability of twisted photons produced by axially symmetric bunches of particles to the same quantity for one particle. The general sum rules (10), (12), and (16) were established. In particular, we found that the projection of the total angular momentum per photon radiated by axially symmetric particle bunches does not depend on the bunch profile and coincides with the same quantity for the radiation created by one point particle. The relations between the radiation probabilities by particle bunches and the one-particle radiation probability involve the interference factors.
We proved some general properties of these interference factors and obtained their explicit form for simple radial bunch profiles. In general, the finiteness of particle bunch width leads to spreading of the radiation probability of twisted photons over \( m \). We found condition (24) when this spreading is marginal and the one-particle radiation probability distribution is an adequate approximation for description of radiation by particle bunches.

In physical terms, the violation of condition (24) means that the radiation amplitude becomes highly sensitive to the transverse structure of the bunch as the transverse wavelength of radiation becomes smaller than the typical transverse size of the bunch. If the transverse particle distribution is homogeneous on the scales of order \( k_\perp^{-1} \) and larger, then the radiation amplitudes of a twisted photon produced by different particles in the bunch interfere destructively, the coherent interference factor is suppressed, and so is the corresponding radiation probability. In the case of incoherent addition of amplitudes, the finiteness of the transverse size of a particle bunch leads to the appearance of radiated twisted photons with the projection of the total angular momentum that differs from the one-particle value by the quantity of order \( k_\perp d \), where \( d \) is the additional transverse distance from the given charged particle to the reference charged particle moving along the center of the bunch. Therefore, the bunch of particles with typical transverse size \( \sigma_\perp \) radiates incoherently the twisted photons with the projections of the total angular momentum \( |m - m_0| \lesssim k_\perp \sigma_\perp \), where \( m_0 \) is the projection of the total angular momentum of a twisted photon radiated by the reference charged particle.

The developed general theory was applied to undulator radiation. The two types of undulators were considered: the ordinary undulator and the crystalline one. We showed that the selection rules fulfilled for the forward undulator radiation \([17, 24–26, 31]\) become violated when the radiation by a bunch of particles is considered. However, when condition (24) is satisfied, this violation is negligible. The projection of the total angular momentum per photon is, of course, the same as for the one-particle radiation since it is independent of the axially symmetric bunch profile. It was shown in \([31]\) that the radiation probability distribution over \( m \) possesses a periodic structure when the radiation of twisted photons by one particle in undulator is considered at a small angle to the undulator axis. In the present paper, we showed that this property holds for the radiation by particle bunches as well and is less sensitive to the bunch size than the selection rules mentioned above. The plots of the probability of radiation of twisted photons by particle bunches in ordinary undulators are presented in figs. 2, 4.

As for crystalline undulators, the twisted photons produced by them are much harder than those created in ordinary undulators. Then it follows from condition (24) that the one-particle radiation probability distribution over \( m \) is subjected to a drastic broadening when a finiteness of a bunch width is taken into account. In that case, the shape of the radiation probability distribution over \( m \) becomes universal (see fig. 5), but the projection of the total angular momentum per photon remains the same as for the one-particle radiation.

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