Exponential Attractor for One-Dimensional Self-Organizing Target-Detection Model

By

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Abstract. Okaie et al. [8] utilized the Keller-Segel model for mobile bionanosensor networks for target tracking. They introduced a mathematical formulation and described numerical results. In this paper, we would like to study analytically their model. We first construct a unique local solution for model equations. Second, we establish a priori estimates for local solutions to obtain a global solution. Finally, after constructing a non-autonomous dynamical system, we will show existence of exponential attractors.

Key Words and Phrases. Attraction-repulsion chemotaxis model, Non-autonomous problem, Exponential attractors.

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1. Introduction

We are concerned with the initial value problem for the following attraction-repulsion chemotaxis model:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= a_1 \frac{\partial^2 u}{\partial x^2} - \frac{\partial}{\partial x} \left[ u \left( \frac{\partial}{\partial x} \chi_1(v) - \frac{\partial}{\partial x} \chi_2(w) \right) \right] \quad \text{in } I \times (0, \infty), \\
\frac{\partial v}{\partial t} &= a_2 \frac{\partial^2 v}{\partial x^2} + g_1 T(x,t) u - dv \quad \text{in } I \times (0, \infty), \\
\frac{\partial w}{\partial t} &= a_3 \frac{\partial^2 w}{\partial x^2} + g_2 u - hw \quad \text{in } I \times (0, \infty), \\
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial w}{\partial x} &= 0 \quad \text{on } \partial I \times (0, \infty), \\
u(x,0) &= u_0(x), \quad v(x,0) = v_0(x), \quad w(x,0) = w_0(x) \quad \text{in } I,
\end{align*}
\]

in the unit open interval \( I = (0, 1) \).

This mathematical model has been introduced by Okaie et al. [8]. They considered a mobile bionanosensor network designed for target tracking in molecular environments. In their modeling, they are inspired by Keller-Segel.
model [3]. The network consists of bioparticles and two types of signaling molecules: attractants for a group of bioparticles to move toward targets, and repellents to spread over the environment.

The unknown functions \( u = u(x, t) \), \( v = v(x, t) \), and \( w = w(x, t) \) denote density of bioparticles, concentration of chemical attractants, and concentration of chemical repellents, respectively, in \( I \) at time \( t \). The bioparticles are motile in response to the gradients of \( \chi_1(v) \) and \( \chi_2(w) \), where \( \chi_1(v) \) and \( \chi_2(w) \) are sensitivity functions of bioparticles to chemical attractants and chemical repellents. The term \(-\partial / \partial x [u(\partial \chi_1(v) / \partial x - \partial \chi_2(w) / \partial x)]\) denotes the nonlinear advection which is affected by chemical attractants and chemical repellents. Bioparticles move preferentially towards higher (resp. lower) concentration of chemical attractants (resp. repellents). The term \( g_1 T(x, t) u \) denotes that bioparticles produce chemical attractant when they find the target \( T(x, t) \). On the other hand, bioparticles always release chemical repellents by a production rate \( g_2 u \). The terms \(-w_1 / C_0 \) and \(-w_2 / C_0 \) denote decay rates of chemical attractants and repellents. The unknown functions \( u \), \( v \), and \( w \) satisfy the Neumann boundary conditions at \( x = 0, 1 \).

In nature, many organisms are known to exhibit complex aggregation behaviors and coordinations. These can be seen in schools of fish, honey bee colonies, and mounds built by termites and so on (see [13]). These seemingly intelligent behaviors are a result of relatively simple interactions between individuals in the swarm and their environment. In particular, Keller-Segel [3] introduced a mathematical formulation for chemotaxis by diffusion-advection equations. There have been significant works investigating the complexities of cell-cell communication among bacteria (e.g. [1, 7, 11]). Among them, some attraction-repulsion chemotaxis models have been proposed in [10, 6]. In these years attraction-repulsion chemotaxis models were studied by many mathematicians (e.g. [4, 5, 12]). However, few researchers have handled the case where a production rate of chemical attractants depends on position \( x \) and time \( t \), that is, (1.1).

We assume that \( \chi_1(v) \) and \( \chi_2(w) \) are real smooth functions for \( 0 \leq v < \infty \) and \( 0 \leq w < \infty \), respectively, satisfying the conditions

\[
\sup_{0 \leq v < \infty} \left| \frac{d^i \chi_1(v)}{dv^i} \right| < \infty \quad \text{and} \quad \sup_{0 \leq w < \infty} \left| \frac{d^i \chi_2(w)}{dw^i} \right| < \infty \quad \text{for} \quad i = 1, 2.
\]

We also assume that \( 0 \leq T(x, t) \leq 1 \) and \( T(\cdot, t) \) is in \( H^1(I) \) for any \( t \in [0, \infty) \) and satisfies the two conditions:

\[
(1.3) \quad \sup_{0 \leq t < \infty} \| T(x, t) \|_{H^1} < \infty,
\]

\[
(1.4) \quad \| T(x, t_1) - T(x, t_2) \|_{H^1} \leq c |t_1 - t_2|, \quad \forall t_1, t_2 \in [0, \infty),
\]
with some constant $\alpha \geq 0$. The initial functions $u_0(x)$, $v_0(x)$, and $w_0(x)$ are nonnegative in $I$. Furthermore, $a_i$ ($i = 1, 2, 3$), $g_i$ ($i = 1, 2$), $d$, and $h$ are positive ($> 0$) constants.

In this paper, we first construct a unique local solution to (1.1). Second, we establish a priori estimates for local solutions to obtain a global solution. Finally, after constructing a non-autonomous dynamical system generated by (1.1), we will show existence of exponential attractors.

2. Abstract formulation

Let us formulate (1.1) as the Cauchy problem for a non-autonomous semilinear abstract equation:

$$\begin{cases}
\frac{dU}{dt} + AU = F(t, U), & 0 < t < \infty, \\
U(0) = U_0,
\end{cases}$$

in $X$. Here, we set the underlying space $X$ as

$$X = \left\{ U = \begin{pmatrix} u \\ v \\ w \end{pmatrix}; u \in L_2(I), v \in H^1(I) \text{ and } w \in H^1(I) \right\},$$

$X$ being equipped with the norm

$$\|U\| = \|u\|_{L_2} + \|v\|_{H^1} + \|w\|_{H^1}.$$

Thanks to such a setting, the nonlinear advection term $-\partial/\partial x[u(\partial x_1(v)/\partial x - \partial x_2(w)/\partial x)]$ can be treated as a lower term. That is, we can formulate the quasilinear problem (1.1) as a semilinear problem of the form (2.1).

The linear operator $A$ is given by $A = \text{diag}\{A_1, A_2, A_3\}$ in $X$. The operator $A_1 = -a_1\partial^2/\partial x^2 + 1$ is a positive definite self-adjoint operator of $L_2(I)$ with domain $\mathcal{D}(A_1) = H^2_N(I) = \{u \in H^2(I); du/dx(0) = du/dx(1) = 0\}$. In addition, the two operators $A_2 = -a_2\partial^2/\partial x^2 + d$ and $A_3 = -a_3\partial^2/\partial x^2 + h$ are positive definite self-adjoint operators of $H^1(I)$ with domain $\mathcal{D}(A_2) = \mathcal{D}(A_3) = H^3_N(I) = \{u \in H^3(I); du/dx(0) = du/dx(1) = 0\}$. Therefore, the domain of $A$ is given by

$$\mathcal{D}(A) = \left\{ U = \begin{pmatrix} u \\ v \\ w \end{pmatrix}; u \in H^2_N(I), v \in H^3_N(I) \text{ and } w \in H^3_N(I) \right\},$$

and $A$ is also a positive definite self-adjoint operator of $X$. Then, according to [14, Theorems 1.35 and 16.1], we know the following characterization of the
domains of the fractional powers:

\[
\mathcal{D}(A^\eta) = \left\{ U = \begin{pmatrix} u \\ v \\ w \end{pmatrix} : u \in H_N^{2\eta}(I), \ v \in H_N^{1+2\eta}(I) \text{ and } w \in H_N^{1+2\eta}(I) \right\}
\]

with norm equivalence, where the exponent \( \eta \) is fixed in such a way that \( 3/4 < \eta < 1 \). The definition of the Sobolev spaces \( H^s(I) \) for nonintegral orders \( s \geq 0 \) is given in [14, Chapter 1], and the definition of the fractional powers \( A^\eta \) of a sectorial operator is given in [14, Chapter 2].

In addition, the nonlinear operator \( F : (0, \infty) \times \mathcal{D}(A^\eta) \rightarrow X \) is given by

\[
F(t, U) = \begin{pmatrix} u - \frac{\partial}{\partial x} \left[ u \left( \frac{\partial}{\partial x} x_1(\text{Re} \ v) - \frac{\partial}{\partial x} x_2(\text{Re} \ w) \right) \right] \\
g_1 T(x, t) u \\
g_2 u \\
g_3 u \end{pmatrix}, \quad U = \begin{pmatrix} u \\ v \\ w \end{pmatrix}.
\]

Finally, we set the space of initial values by

\[
\mathcal{K} = \left\{ U = \begin{pmatrix} u \\ v \\ w \end{pmatrix} : 0 \leq u \in L_2(I), \ 0 \leq v \in H^1(I) \text{ and } 0 \leq w \in H^1(I) \right\}.
\]

3. Local solutions

3.1. Construction of local solutions

Consider the Cauchy problem:

\[
\begin{aligned}
\frac{dU}{dt} + AU &= F(t, U), \quad s < t < \infty, \\
U(s) &= U_s,
\end{aligned}
\]

in \( X \) with initial time \( s \geq 0 \). In order to construct local solutions to (3.1), we utilize the theory of non-autonomous semilinear abstract evolution equations. According to [14, pages 199 and 200], it is sufficient to prove that \( F(t, U) \) satisfies the Lipschitz condition:

\[
\|F(t_1, U_1) - F(t_2, U_2)\| \\
\leq \phi(\|U_1\| + \|U_2\|) \\
\times \{ \|A^\eta(U_1 - U_2)\| + (\|A^\eta U_1\| + \|A^\eta U_2\|)[|t_1 - t_2| + \|U_1 - U_2\|]\},
\]

\((t_1, U_1), (t_2, U_2) \in (s, \infty) \times \mathcal{D}(A^\eta),\)

where \( \phi(\cdot) \) is some increasing continuous function.
Proposition 3.1. $F(t, U)$ satisfies (3.2) with some $\phi(\cdot)$ which does not depend on the initial time $s$. 

Proof. Since it holds for all $\theta \geq 0$ that 
\[
\max\{|\Re u|_{H^\theta}, |\Im u|_{H^\theta}\} \leq |u|_{H^\theta} \leq |\Re u|_{H^\theta} + |\Im u|_{H^\theta}, \quad u \in H^0(\Omega),
\]
it is sufficient to prove (3.2) in the case where $U_1$ and $U_2$ are real valued. 

Since $\chi_i(\cdot)$ ($i = 1, 2$) are smooth functions, we see from \cite[(A.1)]{14} that 
\[
||\chi_i(v)||_{H^1} \leq \phi(||v||_{H^1}), \quad v \in H^1(\Omega),
\]
and 
\[
||\chi_i(v_1) - \chi_i(v_2)||_{H^1} \leq \phi(||v_1||_{H^1} + ||v_2||_{H^1})||v_1 - v_2||_{H^1}, \quad v_1, v_2 \in H^1(\Omega).
\]

In addition, we obtain that 
\[
|||\frac{d^2}{dx^2} \chi_i(v)|||_{L^\infty} = ||\chi''_i(v) \left(\frac{dv}{dx}\right)^2 + \chi'_i(v) \frac{d^2v}{dx^2}||_{L^\infty}
\]
\[
\leq \phi(||v||_{H^1}) (||v||_{H^2}^2 + ||v||_{H^{1+2\eta}}^2)
\]
\[
\leq \phi(||v||_{H^1}) (||v||_{H^1}^{1/\eta} ||v||_{H^{1+2\eta}}^{2-1/\eta} + ||v||_{H^{1+2\eta}}^2)
\]
\[
\leq \phi(||v||_{H^1}) ||v||_{H^{1+2\eta}}, \quad v \in H^{1+2\eta}(\Omega).
\]

Let $U_1 = ^t(u_1, w_1)$ and $U_2 = ^t(u_2, w_2) \in \mathcal{D}(A^\eta)$, then 
\[
||F(t_1, U_1) - F(t_2, U_2)|| \leq ||u_1 - u_2||_{L^2} + ||\frac{d}{dx} \left[u_1 \frac{d}{dx} (\chi_1(v_1) - \chi_1(v_2))\right]||_{L^2}
\]
\[
+ ||\frac{d}{dx} \left[(u_1 - u_2) \frac{d}{dx} \chi_1(v_2)\right]||_{L^2}
\]
\[
+ ||\frac{d}{dx} \left[u_1 \frac{d}{dx} (\chi_2(w_1) - \chi_2(w_2))\right]||_{L^2}
\]
\[
+ ||\frac{d}{dx} \left[(u_1 - u_2) \frac{d}{dx} \chi_2(w_2)\right]||_{L^2}
\]
\[
+ g_1 ||T(x, t_1)u_1 - T(x, t_2)u_2||_{H^1} + g_2 ||u_1 - u_2||_{H^1}.
\]

Here, we easily obtain estimates for the first and last terms of the right hand side.

Let us estimate the second term. Obviously,
\[
\frac{d}{dx} \left[ u_1 \frac{d}{dx} (\chi_1(v_1) - \chi_1(v_2)) \right] = \frac{du_1}{dx} \frac{d}{dx} (\chi_1(v_1) - \chi_1(v_2)) + u_1 \frac{d^2}{dx^2} (\chi_1(v_1) - \chi_1(v_2)).
\]

Due to (3.4),
\[
\left\| \frac{du_1}{dx} \frac{d}{dx} (\chi_1(v_1) - \chi_1(v_2)) \right\|_{L^2} \leq \left\| \frac{du_1}{dx} \right\|_{L^\infty} \left\| \chi_1(v_1) - \chi_1(v_2) \right\|_{H^1}
\leq \phi(\|v_1\|_{H^1} + \|v_2\|_{H^1})\|u_1\|_{H^{\infty}} \|v_1 - v_2\|_{H^1}
\leq \phi(\|U_1\| + \|U_2\|)\|A^\eta U_1\| \|U_1 - U_2\|.
\]

On the other hand, by using the same techniques as in (3.4), we have
\[
\left\| u_1 \frac{d^2}{dx^2} (\chi_1(v_1) - \chi_1(v_2)) \right\|_{L^2} \leq \left\| u_1 \chi_1''(v_1) \right\|_{L^2} \left\| \frac{d^2}{dx^2} \left( \frac{d^2}{dx^2} \chi_1(v_1) \right) \right\|_{L^2} + \left\| u_1 \chi_1''(v_1) \right\|_{L^2} \left\| \frac{d^2}{dx^2} \left( \frac{d^2}{dx^2} \chi_1(v_1) \right) \right\|_{L^2}
\leq \left\| u_1 \chi_1''(v_1) \right\|_{L^2} \left\| \frac{d}{dx} \frac{d^2}{dx^2} \chi_1(v_1) \right\|_{L^\infty}
\leq \phi(\|U_1\| + \|U_2\|)\{\|A^\eta (U_1 - U_2)\| + (\|A^\eta U_1\| + \|A^\eta U_2\|)\|U_1 - U_2\|}\}
\]

By (3.3) and (3.5), the third term in the right hand side of (3.6) is estimated by
\[
\left\| \frac{d}{dx} \left[ (u_1 - u_2) \frac{d}{dx} \chi_1(v_2) \right] \right\|_{L^2} \leq \left\| \frac{d^2}{dx^2} \chi_1(v_2) \right\|_{L^\infty} \|u_1 - u_2\|_{L^2} + \left\| \frac{d}{dx} \chi_1(v_2) \right\|_{L^2} \left\| \frac{d}{dx} (u_1 - u_2) \right\|_{L^\infty}
\leq \phi(\|U_2\|)(\|A^\eta U_2\| \|U_1 - U_2\| + \|A^\eta (U_1 - U_2)\|).
The similar techniques are available to estimate the forth and fifth terms of the right hand side of (3.6).

Finally, from (1.3) and (1.4), we conclude that

\[ kT(x, t_1)u_1 - T(x, t_2)u_2 \leq \| T(x, t_1) \|_{H^1} \| u_1 - u_2 \|_{H^1} + \| T(x, t_1) - T(x, t_2) \|_{H^1} \]

\[ \leq C(\| A_1^{\theta}(U_1 - U_2) \| + \| A_1^{\theta}U_2 \| ||t_1 - t_2||) . \]

Note that this \( C \) does not depend on the initial time \( s \). Therefore, we verify the desired estimate (3.2).

\[ \text{Theorem 3.1. Let } 0 \leq s < \infty. \text{ For any } U_s \in X, \text{ there exists a unique local solutions to (3.1) in the function space:} \]

\[ (3.7) \quad 0 \leq U \in \mathcal{C}([s, s + T_{U_s}]; \mathcal{D}(A)) \cap \mathcal{C}([s, s + T_{U_s}]; X) \cap \mathcal{C}^1([s, s + T_{U_s}]; X), \]

where \( T_{U_s} \) is determined by the norm \( \| U_s \| \) alone. In addition,

\[ (3.8) \quad (t - s)\| A(U)(t) \| + \| U(t) \| \leq C_{U_s}, \quad s < t \leq s + T_{U_s}, \]

where \( C_{U_s} \) is determined by the norm \( \| U_s \| \) alone. In particular, \( T_{U_s} \) and \( C_{U_s} \) do not depend on the initial time \( s \).

\[ \text{Proof. Thanks to [14, Theorem 4.4], we conclude that for any initial value } U_s \in X, \text{ (3.1) possesses a unique local solution in the function space:} \]

\[ U \in \mathcal{C}([s, s + T_{U_s}]; \mathcal{D}(A)) \cap \mathcal{C}([s, s + T_{U_s}]; X) \cap \mathcal{C}^1([s, s + T_{U_s}]; X) \]

with the norm estimate (3.8).

We notice that \( U(t) \) is real valued. Indeed, the complex conjugate \( \overline{U(t)} \) of \( U(t) \) is also a local solution of (3.1) with the same initial value \( U_s \). So, the uniqueness of solution implies that \( \overline{U(t)} = U(t) \); hence, \( U(t) \) must be real valued. In addition, it is easy to verify by the truncation method that \( u_s \geq 0, \)

\( v_s \geq 0 \) and \( w_s \geq 0 \) imply that the local solution to (3.1) also satisfies \( u(t) \geq 0, \)

\( v(i) \geq 0, \) and \( w(t) \geq 0 \) for every \( s < t \leq s + T_{U_s} \) (cf. [14, Section 12]).

\[ \text{3.2. Lipschitz continuity of solutions in initial data} \]

We verify Lipschitz continuity of solutions in the initial data. Let \( 0 < R < \infty \). Let \( X_R = X \cap B^X(0; R) \), where \( B^X(0; R) \) denotes a closed ball of \( X \) centered at 0 with radius \( R \). Then, there is an interval \( [s, s + T_R] \) on which (3.1) has a unique local solution for any \( U_s \in X_R \), where \( T_R > 0 \) is determined by \( R \) alone. Due to [14, Theorem 4.5], we have
(3.9) \[(t - s)^\eta \|A^\eta [U_1(t) - U_2(t)]\| + \|U_1(t) - U_2(t)\| \leq C_R \|U_1^1 - U_2^1\|, \quad s < t \leq s + T_R,\]

where \(U_1(t)\) (resp. \(U_2(t)\)) is a local solution to (3.1) for initial data \(U_1^1 \in X_R\) (resp. \(U_2^1 \in X_R\)).

### 3.3. Regularity property of local solutions

We can verify some regularity property of local solutions which is. Consider the Cauchy problem:

\[
\begin{aligned}
\frac{dU}{dt} + AU &= G(t, U), \quad s < t < \infty, \\
U(s) &= U_s,
\end{aligned}
\]

in the underlying space

\[Y = \left\{ U = \begin{pmatrix} u \\ v \\ w \end{pmatrix}; u \in H^1(I), v \in H^2_N(I) \text{ and } w \in H^2_N(I) \right\}.
\]

In (3.10), the linear operator \(A\) is regarded as a positive definite self-adjoint operator of \(Y\) with domain

\[D(A) = \left\{ U = \begin{pmatrix} u \\ v \\ w \end{pmatrix}; u \in H^3_N(I), v \in H^4_N(I) \text{ and } w \in H^4_N(I) \right\},
\]

where \(H^4_N(I) = \{ u \in H^4(I); du/dx(0) = 0 \text{ and } d^3u/dx^3(0) = 0 \}\). In addition, the nonlinear operator \(G: (s, \infty) \times D(A^\eta) \to Y\) is given by

\[G(t, U) = \begin{pmatrix} u - \frac{\partial}{\partial x} \left( u \frac{\partial}{\partial x} x_1(\text{Re} v) - \frac{\partial}{\partial x} x_2(\text{Re} w) \right) \\ g_1 T(x, t)u \\ g_2 u \end{pmatrix}, \quad U = \begin{pmatrix} u \\ v \\ w \end{pmatrix}.
\]

By the same techniques as for the proof of Proposition 3.1, we can verify the following Lipschitz condition:

\[
\|G(t_1, U_1) - G(t_2, U_2)\|_Y \leq \phi(\|U_1\|_Y + \|U_2\|_Y) \\
\times \{\|A^\eta(U_1 - U_2)\|_Y + (\|A^\eta U_1\|_Y + \|A^\eta U_2\|_Y)\|t_1 - t_2| + \|U_1 - U_2\|_Y\}, \quad (t_1, U_1), (t_2, U_2) \in (s, \infty) \times D(A^\eta),
\]
where $\phi(\cdot)$ is some increasing continuous function. Therefore, we obtain the following theorem.

**Theorem 3.2.** Let $0 \leq s < \infty$. For any $U_s \in Y$, (3.10) possesses a unique local solution in the function space:

$$U \in C((s, s + T_{U_s}); D(A)) \cap C([s, s + T_{U_s}]; Y) \cap C^1((s, s + T_{U_s}); Y),$$

where $T_{U_s}$ is determined by the norm $\|U_s\|_Y$ alone.

### 4. Global solutions

This section is devoted to showing the global existence of solutions. For $U_0 = \iota(u_0, v_0, w_0) \in K$, let $U = \iota(u, v, w)$ denote local solutions of (2.1) in the function space (3.7), i.e.,

$$0 \leq U \in C((0, T_U); D(A)) \cap C([0, T_U]; X) \cap C^1((0, T_U); X),$$

where $[0, T_U]$ denotes the interval of existence of each $U = \iota(u, v, w)$. We build up a priori estimates for the local solutions.

**Proposition 4.1.** There exist a continuous increasing function $p(\cdot)$ and some positive exponent $\gamma > 0$ such that the estimate

$$\|U(t)\| \leq p(e^{-\gamma t}\|U_0\| + \|u_0\|_{L_1}), \quad 0 \leq t \leq T_U,$$

holds for any local solution $U$ of (2.1) in (4.1), $p(\cdot)$ and $\gamma$ being independent of $T_U$.

**Proof.** In the proof, the notations $p(\cdot)$ and $C$ stand for some continuous increasing functions and some constants, respectively, which are determined by the initial constants in (1.1) and by $I$ in a specific way in each occurrence. We divide the proof into five steps.

**Step 1.** Integrate the first equation of (1.1) in $I$. Then, in view of $u \geq 0,$

$$\frac{d}{dt} \|u\|_{L_1} = 0,$$

hence,

$$\|u(t)\|_{L_1} = \|u_0\|_{L_1}, \quad 0 \leq t \leq T_U.$$

**Step 2.** Let us consider the linear problem:

$$\begin{cases}
  \frac{d}{dt} v + A_2 v = g_1 T(x, t) u, & 0 < t \leq T_U, \\
  v(0) = v_0
\end{cases}$$

(4.4)
in $H^1(I)'$, where $H^1(I)'$ is the dual space of $H^1(I)$. In (4.4), $A_2$ is regarded as a positive definite self-adjoint operator of $H^1(I)'$ with domain $H^1(I)$. The operator $A_2$ generates an analytic semigroup $e^{-tA_2}$ on $H^1(I)'$ with the estimate

$$
\|e^{-tA_2}\|_{\mathcal{L}(H^1(I)')} \leq Ce^{-dt}, \quad 0 \leq t < \infty.
$$

Then, the fractional power of $A_2$ satisfies

$$
\mathcal{D}(A_2^\theta) = [H^1(I)', H^1(I)]_\theta = H^{1-2\theta}(I)', \quad 0 \leq \theta < \frac{1}{2}
$$

with norm equivalence.

Meanwhile, since

$$
\|g_1 T(x,t)u(t) - g_1 T(x,s)u(s)\|_{(H^1)'}
\leq C\|T(x,t)u(t) - T(x,s)u(t)\|_{L_2} + \|T(x,s)u(t) - T(x,s)u(s)\|_{L_2}
\leq C\|u(t)\|_{L_2}\|T(x,t) - T(x,s)\|_{H^1} + \|T(x,s)\|_{H^1}\|u(t) - u(s)\|_{L_2},
$$

we observe from (1.3), (1.4), and (4.1) that

$$
g_1 T(x,t)u(t) \in \mathcal{C}([0, T_U]; H^1(I)'), \quad \mathcal{C}^0(0, T_U; H^1(I)') \cap \mathcal{C}^1((0, T_U); H^1(I)').
$$

Then, according to [14, Theorem 3.4], there exists a unique local solution $v$ to (4.4) in the function space:

$$
v \in \mathcal{C}((0, T_U]; \mathcal{H}^1(I)) \cap \mathcal{C}([0, T_U]; H^1(I)', \mathcal{C}^1((0, T_U); H^1(I)').
$$

Moreover, $v$ is necessarily given by the formula

$$
v(t) = e^{-tA_2}v_0 + g_1 \int_0^t e^{-(t-\tau)A_2}T(x,\tau)u(\tau)d\tau, \quad 0 \leq t \leq T_U.
$$

By the similar arguments, we obtain that

$$
w(t) = e^{-tA_2}w_0 + g_2 \int_0^t e^{-(t-\tau)A_2}u(\tau)d\tau, \quad 0 \leq t \leq T_U.
$$

**Step 3.** Let us estimate $v(t)$. It follows from (4.7) that

$$
A_2v(t) = e^{-tA_2}A_2v_0 + g_1 \int_0^t A_2^{7/8} e^{-(t-\tau)/2}A_2e^{-((t-\tau)/2)A_2}A_2^{1/8}[T(x,\tau)u(\tau)]d\tau.
$$

Note that $L^1(I) \subset H^{3/4}(I)'$ with

$$
\| \cdot \|_{(H^{3/4})'} \leq C\| \cdot \|_{L_1}.
$$
Then, by (4.5) and (4.6),
\[\|A_2v(t)\|_{(H^1)'} \leq C \left[ e^{-dt} \|A_2v_0\|_{(H^1)'} + \int_0^t (t - \tau)^{-7/8} e^{-(d/2)(t-\tau)} \|T(x, \tau)u(\tau)\|_{L_1} d\tau \right].\]

Therefore, we obtain by (1.3) and (4.3) that
\[
\|v(t)\|_{H^1} \leq C[e^{-dt}\|v_0\|_{H^1} + \|u_0\|_{L_1}], \quad 0 \leq t \leq T_U.
\]

By the similar arguments, we obtain from (4.8) that
\[
\|w(t)\|_{H^1} \leq C[e^{-ht}\|w_0\|_{H^1} + \|u_0\|_{L_1}], \quad 0 \leq t \leq T_U.
\]

**Step 4.** We shall use the notation
\[p_1(U) = p(\|v\|_{H^1} + \|w\|_{H^1} + \|u\|_{L_1}), \quad U = t(u, v, w) \in X.\]

Multiply the second equation of (1.1) by \(2\partial^2 v / \partial x^2\) and integrate the product in \(I\). Then, by (1.3),
\[
\frac{d}{dt} \|v\|_{L_2}^2 + 2\|v\|_{L_2}^2 + 2a_2 \|\partial^2 v / \partial x^2\|_{L_2}^2 \leq -2 \int_I T(x, t) u \partial^2 v / \partial x^2 \, dx \leq a_2 \|\partial^2 v / \partial x^2\|_{L_2}^2 + C\|u\|_{L_2}^2.
\]

Meanwhile, by Gagliardo-Nirenberg’s inequality,
\[
\|u\|_{L_2} \leq C\|u\|_{H^1}^{1/3}\|u\|_{L_1}^{2/3} \leq \xi_1 \left( \|u\|_{L_2} + \left\| \frac{\partial u}{\partial x} \right\|_{L_2} \right) + C_\xi \|u\|_{L_1}
\]

with any \(0 < \xi_1 < 1\), hence,
\[
\|u\|_{L_2} \leq \frac{\xi_1}{1 - \xi_1} \left\| \frac{\partial u}{\partial x} \right\|_{L_2} + \frac{C_\xi}{1 - \xi_1} \|u\|_{L_1}.
\]

Therefore, we have
\[
\frac{d}{dt} \left\| \frac{\partial v}{\partial x} \right\|_{L_2}^2 + 2d \left\| \frac{\partial v}{\partial x} \right\|_{L_2}^2 + 2a_2 \left\| \partial^2 v / \partial x^2 \right\|_{L_2}^2 \leq C\xi \|u_0\|_{L_1}^2
\]

with any \(0 < \xi_2 < 1\).

It is the same for \(w(t)\). Hence,
\[
\frac{d}{dt} \left\| \frac{\partial w}{\partial x} \right\|_{L_2}^2 + 2h \left\| \frac{\partial w}{\partial x} \right\|_{L_2}^2 + 2a_3 \left\| \partial^2 w / \partial x^2 \right\|_{L_2}^2 \leq C\xi \|u_0\|_{L_1}^2
\]

with any \(0 < \xi_3 < 1\).
In the meantime, multiply the first equation of (1.1) by $2u$ and integrate the product in $I$. Then,

$$\frac{d}{dt} \|u\|_{L^2}^2 + 2a_1 \|\partial_x u\|_{L^2}^2 = 2 \int_I u \left( \frac{\partial u}{\partial x} \right) \left( \frac{\partial}{\partial x} (\chi_1(v) - \chi_2(w)) \right) dx \leq a_1 \|\partial_x u\|_{L^2}^2 + C \int_I u^2 \left( \frac{\partial}{\partial x} (\chi_1(v) - \chi_2(w)) \right)^2 dx.$$ 

Here, on account of (4.9) and (4.10),

$$\int_I u^2 \left( \frac{\partial}{\partial x} (\chi_1(v) - \chi_2(w)) \right)^2 dx \leq C \|u\|_{L^4}^2 \left( \|\partial_x v\|_{L^4}^2 + \|\partial_x w\|_{L^4}^2 \right) \leq C \|u\|_{H^1} \|u\|_{L^2} \left( \|\partial_x v\|_{H^1}^{1/2} \|\partial_x v\|_{L^2}^{3/2} + \|\partial_x w\|_{H^1}^{1/2} \|\partial_x w\|_{L^2}^{3/2} \right) \leq \zeta \left( \|u\|_{H^1}^2 + \|\partial_x v\|_{L^2}^2 + \|\partial_x w\|_{L^2}^2 \right) + C \zeta p_1(U_0),$$

with any $\zeta > 0$. Therefore, due to (4.11), we obtain that

$$\frac{d}{dt} \|u\|_{L^2}^2 + \|u\|_{L^2}^2 + (a_1 - \zeta) \|\partial_x u\|_{L^2}^2 - \zeta \left( \|\partial_x v\|_{H^1}^2 + \|\partial_x w\|_{L^2}^2 \right) \leq C \zeta p_1(U_0).$$

Now, we sum up this, (4.12), and (4.13). Then, by choosing $\zeta_1, \zeta_2, \text{ and } \zeta_3$ properly,

$$\frac{d}{dt} \left[ \|u\|_{L^2}^2 + \|\partial_x v\|_{L^2}^2 + \|\partial_x w\|_{L^2}^2 \right] + \delta \left[ \|u\|_{L^2}^2 + \|\partial_x v\|_{L^2}^2 + \|\partial_x w\|_{L^2}^2 \right] \leq p_1(U_0),$$

with $\delta = \min\{1, 2d, 2h\}$. Thus, we arrive at the estimate

$$\|u(t)\|_{L^2} \leq e^{-\delta t} \|u_0\|_{L^2} + \|v_0\|_{H^1} + \|w_0\|_{H^1} + p_1(U_0), \quad 0 \leq t \leq T_U.$$  \hspace{1cm} (4.14)

**Step 5.** Let $t \in [0, T_U]$ be fixed. Then, from (4.9) and (4.10),

$$\|v\left( \frac{t}{2} \right)\|_{H^1} + \|w\left( \frac{t}{2} \right)\|_{H^1} \leq C \left[ e^{-\delta(t/2)} \|v_0\|_{H^1} + e^{-\delta(t/2)} \|w_0\|_{H^1} + \|u_0\|_{L^1} \right].$$  \hspace{1cm} (4.15)
By regarding \( U(t/2) \) as an initial value, we obtain from (4.14) that
\[
\|u(t)\|_{L_2} \leq e^{-\delta t/2} \left[ \left\| u \left( \frac{t}{2} \right) \right\|_{L_2} + \left\| v \left( \frac{t}{2} \right) \right\|_{H^1} + \left\| w \left( \frac{t}{2} \right) \right\|_{H^1} \right] + p \left( \left\| v \left( \frac{t}{2} \right) \right\|_{H^1} + \left\| w \left( \frac{t}{2} \right) \right\|_{H^1} + \|u_0\|_{L_1} \right).
\]
Then, due to (4.14) and (4.15),
\[
(4.16) \quad \|u(t)\|_{L_2} \leq p(e^{-\gamma t}\|u_0\|_{L_2} + \|v_0\|_{H^1} + \|w_0\|_{H^1} + \|u_0\|_{L_1}).
\]
In this way, by (4.9), (4.10), and (4.16), we have established the desired a priori estimate (4.2). \( \square \)

It is now possible to construct a global solution to (1.1) by the standard arguments.

**Theorem 4.1.** For any initial value \( U_0 \in \mathcal{X} \), there exists a unique global solution \( U = \mathcal{L}(u,v,w) \) of (1.1) in the function space:
\[
(4.17) \quad 0 \leq U \in C(\mathbb{R}; \mathcal{T}(A)) \cap \mathcal{C}(\mathbb{R}; X) \cap \mathcal{C}^1((0, \infty); X),
\]

**Proof.** Let \( U_0 \in \mathcal{X} \) be initial value. Let us start with the local solution \( U(t) \) to (2.1) on \([0, T_{U_0}]\) obtained in Theorem 3.1 with \( s = 0 \). Set \( U_s = U(s) \) for any \( s \in (0, T_{U_0}) \). On account of Proposition 4.1, we see that \( \|U_s\| \leq C_{U_0} \), where \( C_{U_0} = p(||U_0|| + \|u_0\|) \) and \( p(\cdot) \) is the function in Proposition 4.1.

We here consider the Cauchy problem (3.1). Then, by using Theorem 3.1 again, there exists a unique local solution \( V(t) \) on an interval \([s, s + t_{U_0}]\), where a length \( t_{U_0} > 0 \) is determined by \( C_{U_0} \) alone. Note that \( t_{U_0} \) does not depend on the time \( s \). Here, put
\[
\tau = \begin{cases} 
\frac{t_{U_0}}{2} & \text{if } t_{U_0} < T_{U_0}, \\
\frac{T_{U_0}}{2} & \text{if } T_{U_0} \leq t_{U_0},
\end{cases}
\]
and \( s = T_{U_0} \). Then, by the uniqueness of solution, we have \( U(t) = V(t) \) for \( T_{U_0} - \tau \leq t \leq T_{U_0}; \) this means that we have constructed a local solution \( V(t) \) to (2.1) on the interval \([0, T_{U_0} + \tau]\). Note that \( U(t) \) and \( V(t) \) have the same initial value \( U_0 \). Since we can choose the same constant \( C_{U_0} \) regardless of \( U(t) \) or \( V(t) \) due to Proposition 4.1, we can continue this extension procedure unlimitedly. Each time, the local solution is extended over the fixed length \( \tau \) of interval. So, by finite times, the extended interval can cover any bounded interval \([0, T]\). \( \square \)
For \( U_0 \in \mathcal{X} \), let \( U(t; U_0) \) be its global solution of (1.1) in (4.17). From Proposition 4.1, we obtain the estimate

\[
\|U(t; U_0)\| \leq p(e^{-\gamma t}\|U_0\| + \|u_0\|_{L_1}), \quad 0 \leq t < \infty.
\]

This jointed with (3.8) provides the following stronger dissipative estimate.

**Theorem 4.2.** For \( U(t; U_0) \), it holds that

\[
\|AU(t; U_0)\| \leq (1 + t^{-1})\tilde{p}(e^{-\gamma t}\|U_0\| + \|u_0\|_{L_1}), \quad 0 < t < \infty,
\]

with some other continuous increasing function \( \tilde{p}(\cdot) \).

**Proof.** In the proof, the continuous increasing function \( \tilde{p}(\cdot) \) may vary from line to line. Let \( s \in [0, \infty) \) and consider (3.1) with initial value \( U_s = U(s; U_0) \). We then apply Theorem 3.1 to this problem to conclude that there exists \( \tau > 0 \) such that

\[
\|AU(t; U_0)\| \leq (t - s)^{-1}\tilde{p}(\|U(s; U_0)\|), \quad s < t \leq s + \tau.
\]

Note that \( \tau \) depends only on \( \|U(s; U_0)\| \) and hence due to (4.18) only on \( p(\|U_0\|) \). First, applying this with \( s = 0 \), we see that

\[
\|AU(t; U_0)\| \leq t^{-1}\tilde{p}(\|U_0\|), \quad 0 < t \leq \tau.
\]

Second, taking \( s = t - \tau \), we have

\[
\|AU(t; U_0)\| \leq \tau^{-1}\tilde{p}(\|U(t - \tau; U_0)\|)
\leq \tau^{-1}\tilde{p}(p(e^{-\gamma(t-\tau)}\|U_0\| + \|u_0\|_{L_1})), \quad \tau < t < \infty.
\]

Since \( \tau \) depends only on \( p(\|U_0\|) \), we obtain that

\[
\|AU(t; U_0)\| \leq \tilde{p}(e^{-\gamma t}\|U_0\| + \|u_0\|_{L_1}), \quad \tau < t < \infty.
\]

Combining (4.20) and (4.21), we conclude the desired estimate (4.19).

5. Exponential attractors

5.1. Dynamical system

Let us construct a non-autonomous dynamical system determined from (1.1). For this purpose, we consider, for any \(-\infty < s < \infty\), the Cauchy problem:

\[
\begin{aligned}
\frac{dU}{dt} + AU &= F(t, U), \quad s < t < \infty, \\
U(s) &= U_s,
\end{aligned}
\]
in $X$. Here, the nonlinear operator $F : R \times \mathcal{D}(A^q) \rightarrow X$ is defined by

$$F(t, U) = \begin{cases} F(0, U), & -\infty < t < 0, \\ F(t, U), & 0 \leq t < \infty. \end{cases}$$

Since $F(t, U)$ satisfies (3.2) in $R \times \mathcal{D}(A^q)$, (5.1) possesses, for any $U_t \in \mathcal{H}$, a unique global solution $U$ in the function space:

$$0 \leq U \in C((s, \infty); \mathcal{D}(A)) \cap C([s, \infty); X) \cap C^1((s, \infty); X).$$

Let $U(\cdot, s; U_s)$ denote the global solution to (5.1) for initial function $U_s \in \mathcal{H}$ with initial time $s \in R$. We set

$$U(t, s)U_s = U(t, s; U_s)$$

as a family of nonlinear operators $U(t, s)$ acting on $\mathcal{H}$ defined for $(t, s) \in A = \{(t, s); -\infty < s \leq t < \infty \}$. Then, the dissipative estimates (4.18) and (4.19) are rewritten as

$$\|U(t, s; U_s)\| \leq p(e^{-\gamma(t-s)}\|U_s\| + \|u_s\|_{L^1}), \quad s \leq t < \infty,$$

and

$$\|AU(t, s; U_s)\| \leq (1 + (t-s)^{-1})p(e^{-\gamma(t-s)}\|U_s\| + \|u_s\|_{L^1}), \quad s < t < \infty.$$}

For simplicity, $p(\cdot)$ is used instead of $\tilde{p}(\cdot)$.

Since it is clear from the uniqueness of solutions that $U(s, s) = I$ for $s \in R$ and $U(t, s) = U(t, r) \circ U(r, s)$ for $(t, r), (r, s) \in A$, $U(t, s)$ is an evolution operator acting on $\mathcal{H}$.

**Proposition 5.1.** $U(t, s)$ is a continuous evolution operator on $\mathcal{H}$, i.e., the mapping $G : A \times \mathcal{H} \rightarrow \mathcal{H}$, where $G(t, s; U_0) = U(t, s)U_0$, is continuous.

**Proof.** Let $0 < R < \infty$ and $0 < T < \infty$ be arbitrarily fixed. We notice from (5.2) that $\|U(t, s)U_0\| \leq p(2R)$ for any $0 \leq t - s < \infty$ provided $U_0 \in \mathcal{H}_R$. For simplicity of notation, we rewrite $p(2R)$ to $p(R)$. By applying (3.9) with radius $p(R)$, we see that

$$\|U(t, s)U_0 - U(t, s)V_0\| \leq C_{p(R)}\|U_0 - V_0\|, \quad U_0, V_0 \in \mathcal{H}_{p(R)},$$

provided that $0 \leq t - s \leq T_{p(R)}$.

Let next $T_{p(R)} \leq t - s \leq 2T_{p(R)}$. Then,

$$\|U(t, s)U_0 - U(t, s)V_0\|$$

$$= \|U(t, t - T_{p(R)})U(t - T_{p(R)}, s)U_0 - U(t, t - T_{p(R)})U(t - T_{p(R)}, s)V_0\|$$

$$\leq C_{p(R)}\|U(t - T_{p(R)}, s)U_0 - U(t - T_{p(R)}, s)V_0\|$$

$$\leq C_{p(R)}^2\|U_0 - V_0\|.$$
Repeating these arguments, we see that
\[ \|U(t,s)U_0 - U(t,s)V_0\| \leq C_{R,T} \|U_0 - V_0\| \]
for \(0 \leq t - s \leq T\) with some constant \(C_{R,T} > 0\).

On the other hand, we observe that \(U(t,s)U_0\) satisfies the integral equation
\[ U(t,s)U_0 = e^{-(t-s)A}U_0 + \int_s^t e^{-(t-\tau)A}F(\tau, U(\tau,s)U_0)\,d\tau, \quad s < t < \infty. \]

We can then verify that \(U(t,s)U_0\) is continuous for \((t,s)\) with values in \(X\).

Therefore, as \(((t_1,s_1), U_1) \to ((t_0,s_0), U_0)\),
\[ \|G(t_1,s_1; U_1) - G(t_0,s_0; U_0)\| \leq \|G(t_1,s_1; U_1) - G(t_1,s_1; U_0)\| + \|G(t_1,s_1; U_0) - G(t_0,s_0; U_0)\| \to 0. \]

Hence, \((U(t,s), X, X)\) becomes a non-autonomous dynamical system determined from (2.1).

5.2. Exponential attractors

We proceed to construct an exponential attractor. Efendiev, Yamamoto and Yagi [2] introduced a version of exponential attractor for non-autonomous dynamical systems. In their paper, a family \(\{\mathcal{M}(t)\}_{t \in \mathbb{R}}\) of subsets of \(X\) is called an exponential attractor for \((U(t,s), X)\) if:

(i) Each \(\mathcal{M}(t)\) is a compact set of \(X\) and its fractal dimension is finite and uniformly bounded, i.e., \(\sup_{t \in \mathbb{R}} \dim \mathcal{M}(t) < \infty\).

(ii) It is positively invariant, i.e., \(U(t,s)\mathcal{M}(s) \subset \mathcal{M}(t)\) for all \((t,s) \in \Delta\).

(iii) There exist an exponent \(\alpha > 0\) and two monotone functions \(Q\) and \(\tau\) such that, for any bounded subset \(B \subset X\),
\[ h(U(t,s)B, \mathcal{M}(t)) \leq Q(\|B\|_X) e^{-\alpha(t-s)}, \]
\[ s \in \mathbb{R}, s + \tau(\|B\|_X) \leq t < \infty. \]

Furthermore, they gave a sufficient condition for constructing exponential attractors. Let us use the sufficient condition, namely, we show that there exists a family of closed bounded subset \(\mathcal{X}(t), t \in \mathbb{R}\), of \(X\) having the properties (1)~(5) in [2, Section 2]:

1. The diameter \(\|\mathcal{X}(t)\|\) of \(\mathcal{X}(t)\) is uniformly bounded, i.e., \(\sup_{t \in \mathbb{R}} \|\mathcal{X}(t)\| < \infty\).

2. It is positively invariant, i.e., \(U(t,s)\mathcal{X}(s) \subset \mathcal{X}(t)\) for all \((t,s) \in \Delta\).
(3) It is absorbing in the sense that there is a monotone function \( \sigma \) such that
\[
\forall B \subset \mathcal{X} \text{ bounded, } U(t, s)B \subset \mathcal{X}(t), \quad s \in \mathbb{R}, s + \sigma(\|B\|) \leq t < \infty.
\]

(4) There is \( \tau^* > 0 \) such that, for every \( s \in \mathbb{R} \), \( U(\tau^* + s, s) \) is a compact perturbation of contraction on \( \mathcal{X}(s) \) in the sense that
\[
\|U(\tau^* + s, s)U_0 - U(\tau^* + s, s)V_0\| \\
\leq \delta \|U_0 - V_0\| + \|K(s)U_0 - K(s)V_0\|, \quad U_0, V_0 \in \mathcal{X}(s),
\]
where \( \delta \) is a constant such that \( 0 \leq \delta < 1/2 \) and where \( K(s) \) is an operator from \( \mathcal{X}(s) \) into another Banach space \( Z \) which is embedded compactly in \( X \) and satisfies a Lipschitz condition
\[
\|K(s)U_0 - K(s)V_0\|_Z \leq L_1 \|U_0 - V_0\|, \quad U_0, V_0 \in \mathcal{X}(s),
\]
with some constant \( L_1 > 0 \) independent of \( s \).

(5) For any \( s \in \mathbb{R} \) and any \( \tau \in [0, \tau^*] \), the Lipschitz condition
\[
\|U(\tau + s, s)U_0 - U(\tau + s, s)V_0\| \leq L_2 \|U_0 - V_0\|, \quad U_0, V_0 \in \mathcal{X}(s),
\]
holds with some constant \( L_2 > 0 \) independent of \( s \) and \( \tau \).

In the present case, the norm \( \|u(t)\|_{L_1} = \|u\|_{L_1} \) is conserved for every \( t \in [s, \infty) \). Therefore, no compact set can attract every solution of (5.1) (cf. [9, Section 6]). This suggests that, for each \( \|u_0\|_{L_1} = 1 > 0 \), we have to consider a space of initial values of the form \( \mathcal{X}^I = \{u, v, w \in \mathcal{X}; \|u\|_{L_1} = 1\} \). In the same way as above, for any \( U_0 \in \mathcal{X}^I \), (5.1) possesses a unique global solution in the function space:
\[
U \in C((s, \infty); \mathcal{D}(A)) \cap C([s, \infty); \mathcal{X}^I) \cap C^1((s, \infty); \mathcal{X}^I).
\]
Moreover, we can construct the non-autonomous dynamical system \((U(t, s), \mathcal{X}^I, X)\) determined from (2.1).

Let us construct an exponential attractor for \((U(t, s), \mathcal{X}^I, X)\).

**Theorem 5.1.** There exists an exponential attractor \( \mathcal{M}(t) \) for \((U(t, s), \mathcal{X}^I, X)\).

**Proof.** In view of the dissipative estimate (5.3), we consider a subset
\[
\mathcal{B} = \mathcal{X}^I \cap \overline{B}^{\mathcal{D}(A)}(0; 2\rho(1 + l))
\]
This \( \mathcal{B} \) is a compact set of \( X \) and is a bounded subset of \( \mathcal{D}(A) \). Since \( \mathcal{B} \) is also a bounded subset of \( \mathcal{X}^I \), we observe from (5.3) that there exists a time \( t_{\mathcal{B}} \) such that \( U(t, s)\mathcal{B} \subset \mathcal{B} \) for every \( t \geq t_{\mathcal{B}} + s \), where \( t_{\mathcal{B}} \) is independent of \( s \). We here
set, for each \( t \in \mathbb{R} \), that
\[
\mathcal{X}(t) = \bigcup_{-\infty < s \leq t} U(t, s) \mathcal{B} = \bigcup_{t-t_s \leq s \leq t} U(t, s) \mathcal{B}.
\]
Let us see that the family \( \mathcal{X}(t), \ t \in \mathbb{R} \), fulfills all the desired conditions. It is clear that \( \mathcal{X}(t) \subset \mathcal{K}^1 \). In addition, since a mapping \( g: [t-t_s, t] \times \mathcal{B} \to \mathcal{K}^1 \) such that \( g(s, U_0) = U(t, s)U_0 \) is continuous and the subset \( [t-t_s, t] \times \mathcal{B} \) is compact, its image \( g([t-t_s, t] \times \mathcal{B}) = \mathcal{X}(t) \) is also compact. Hence, the condition (1) is fulfilled.

By the definition of \( \mathcal{X}(t) \),
\[
\mathcal{X}(s) = \bigcup_{-\infty < r \leq s} U(s, r) \mathcal{B}.
\]
For each \( r \) and \( t \) such that \( -\infty < r \leq s \leq t < \infty \), it follows that \( U(t, s) \circ U(s, r) \mathcal{B} = U(t, r) \mathcal{B} \subset \mathcal{X}(t) \). Hence, \( U(t, s) \mathcal{X}(s) \subset \mathcal{X}(t) \), i.e., (2) is valid.

Consider any bounded subset \( B \) of \( \mathcal{K}^1 \). Thanks to (5.3), there exists a time \( t_B \) such that \( U(t, s)B \subset \mathcal{B} \) for every \( t \geq t_B + s \). Since \( \mathcal{B} \subset \mathcal{X}(t) \), this means that the condition (3) is valid.

In the same way as in the proof of [2, Proposition 5.1], we prove that the union \( \bigcup_{t \in \mathbb{R}} \mathcal{X}(t) \) is a bounded subset of \( \mathcal{D}(A) \). Hence, there is \( R > 0 \) such that \( \bigcup_{t \in \mathbb{R}} \mathcal{X}(t) \subset \mathcal{K}^1 = \mathcal{K}^1 \cap \mathcal{B}^1(0; \mathcal{R}) \). We here set \( Z = \mathcal{D}(A^\mathbb{R}) \). Then, (3.9) shows that the condition (4) is valid provided \( \tau^* = T_R, \delta = 0, \) and \( K(s) = U(s + T_R, s) \). The estimate provides also the Lipschitz condition of (5).

Therefore, we have verified that all the conditions (1)~(5) are fulfilled. Hence, [2, Theorem 2.1] provides existence of exponential attractors \( \{ \mathcal{M}(t) \}_{t \in \mathbb{R}} \) for \((U(t, s), \mathcal{K}^1, X)\). □

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