Special homogeneous curves

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Abstract

We classify all special homogeneous curves. A special homogeneous curve $\mathcal{H}$ consists of connected components of the hyperbolic points in the level set $\{h = 1\}$ of a homogeneous polynomial $h$ in two real variables of degree at least three, and admits a transitive group action of a subgroup $G \subset \text{GL}(2)$ on $\mathcal{H}$ that acts via linear coordinate change.

Keywords: affine differential geometry, centro-affine curves, special real geometry, real algebraic curves, homogeneous spaces

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1 Introduction and main results

In this work we classify a certain class of homogeneous real curves. We study hyperbolic homogeneous polynomials $h : \mathbb{R}^2 \to \mathbb{R}$ of degree $\tau \geq 3$, so that the action of their respective linear automorphism group $G^h \subset \text{GL}(2)$ acts transitively on at least one connected component of $\{h = 1\} \cap \{\text{hyperbolic points of } h\}$. Hyperbolicity means that there exists a point $p \in \{h > 0\}$, such that $-\partial^2 h_p$ is of Minkowski type. In dimension two, $-\partial^2 h_p$ is required to have precisely one positive and one negative eigenvalue. Two polynomials are called equivalent if they are related by a linear transformation, and similarly two connected components of $\{h = 1\} \cap \{\text{hyperbolic points of } h\}$ for a given polynomial $h$ are called equivalent if they are related by a linear transformation. Throughout this work we will refer to such homogeneous curves as special homogeneous curves. The reason for the term special will become clear momentarily. Note that for $\tau = 2$ it is a well-known fact that there exists precisely one such polynomial up to equivalence, namely $h = x^2 - y^2$. The level set $\{h = 1\}$ is the two-sheeted hyperbola and consists of two connected components that exclusively contain hyperbolic points and are related by a reflection. For $\tau = 3$ and $\tau \geq 4$, we are in the realm of projective special real (PSR) curves and generalized projective special real (GPSR) curves, respectively [CHM, L1]. In higher dimensions, PSR and GPSR manifolds are defined analogously. PSR curves have been completely classified in [CHM] and there exists up to equivalence precisely one hyperbolic homogeneous cubic polynomial $h = x^2 y$, such that $\{h = 1\} \cap \{\text{hyperbolic points of } h\}$ is a homogeneous space. Similar to the case $\tau = 2$, $\{x^2 y = 1\}$ contains only hyperbolic points and the two connected components are also related by a reflection. For $\tau \geq 4$, that is GPSR
Theorem 1.1. Let \{h = 1\} ∩ \{hyperbolic points of h\} \[L3\]. Up to equivalence there exist exactly two hyperbolic homogeneous quartic polynomials

\[
h_1 = x^4 - x^2y^2 + \frac{1}{4}y^4, \quad h_2 = x^4 - x^2y^2 + \frac{2\sqrt{2}}{3\sqrt{3}}xy^3 - \frac{1}{12}y^4,
\]
such that the respective positive level sets contain a homogeneous curve. The set \{h = 1\} contains four equivalent connected components and \{h = 1\} contains two equivalent connected components, all of which consisting only of hyperbolic points. For degree \(\tau \geq 5\) there is no known general classification, and to our knowledge there has also not been any specific result of hyperbolic quartics in higher dimensions. However, in an extensive work by A.B. Korshagin and D.A. Weinberg \[L3\] the authors successfully classify all isotopy types of affine and projective quartic curves. For degree of the polynomials greater than four, there are no classification results. Hoping to get to, even in the long term, a complete classification of hyperbolic
homogeneous polynomials in arbitrary degree is most likely not realistic. Even in the cubic case, the corresponding moduli space is only slightly better understood when restricting the global geometry of the corresponding PSR manifolds [L1, L2]. We aim nonetheless for a classification of all special homogeneous spaces. The next step after this work is the classification of special homogeneous surfaces. We are currently cautiously optimistic to obtain such a result in a reasonable amount of time.

The topics treated in this work are additionally motivated by special Kähler geometry [F, ACD] and by the study of the geometry of Kähler cones [DP, W, M]. In special Kähler geometry, affine and projective special Kähler manifolds are studied. The so-called supergravity r-map allows one to explicitly construct such manifolds from connected special real manifolds [CHM]. But for hyperbolic homogeneous polynomials of degree at least four, no straightforward generalisation of that construction exists. One ansatz to find a good candidate is to study what it should do with special homogeneous spaces, as it might then be of more algebraic nature and should be required to preserve the symmetries in some sense. In the geometry of Kähler cones, hyperbolic homogeneous polynomials and their positive level sets appear as follows. Given a compact Kähler manifold $X$ of complex dimension $\tau \geq 3$, the real homogeneous polynomial

$$h : H^{1,1}(X, \mathbb{R}) \to \mathbb{R}, \quad [\omega] \mapsto \int_X \omega^\tau$$

is hyperbolic. In particular all points in the Kähler cone $\mathcal{K} \subset H^{1,1}(X, \mathbb{R})$, that is the subset of classes in $H^{1,1}(X, \mathbb{R})$ containing a Kähler metric, are hyperbolic points of $h$ by the Hodge-Riemann bilinear relations, so $\mathcal{K} = \{ h = 1 \} \cap \mathcal{K}$ is a GPSR manifold. It is in general not known which hyperbolic polynomials can be constructed in this way. A reasonable ansatz would be to try and find a compact Kähler manifold leading to polynomials corresponding to special homogeneous curves. We are currently working on that problem and hope to extend possible results to special homogeneous surfaces once they have been successfully classified.

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2 Preliminaries

We start by giving rigorous definitions of the considered objects and technical tools needed to prove our results.

Definition 2.1. A homogeneous polynomial $h : \mathbb{R}^{n+1} \to \mathbb{R}$ is called hyperbolic if there exists $p \in \{ h > 0 \}$, such that $-\partial^2 h_p$ is of Minkowski type. Such a point $p$ is called hyperbolic point of $h$. Two homogeneous polynomials $h, \overline{h}$ are called equivalent if they are related by a linear coordinate change, i.e. there exists $A \in \text{GL}(n+1)$, such that $A^* h = \overline{h}$. A hypersurface $\mathcal{H}$ contained in the level set $\{ h = 1 \}$ of a hyperbolic homogeneous polynomials of degree $\tau \geq 3$ is called projective special real (PSR) manifold for $\tau = 3$ and generalised projective special real (GPSR) manifold for $\tau \geq 4$. Two (G)PSR manifolds $\mathcal{H}$ and $\overline{\mathcal{H}}$ are called equivalent if there exists $A \in \text{GL}(n+1)$, such that $A(\overline{\mathcal{H}}) = \mathcal{H}$.

As in the introduction, we will refer to homogeneous (G)PSR manifolds as special homogeneous spaces, respectively special homogeneous curves in the one-dimensional case.
Remark 2.2. For two equivalent (G)PSR manifolds \( \mathcal{H} \subset \{ h = 1 \} \) and \( \overline{\mathcal{H}} \subset \{ \overline{h} = 1 \} \) with \( A(\overline{\mathcal{H}}) = \mathcal{H} \) for some \( A \in \text{GL}(n+1) \), the corresponding defining polynomials are automatically equivalent via \( A^t h = \overline{h} \). The converse does in general not hold true, which can be seen by restricting a given (G)PSR manifold to an open subset that is not the entire (G)PSR manifold.

Euler’s homogeneous function theorem implies that \( dh_p(p) = \tau h(p) \) for all homogeneous polynomials of degree \( \tau \geq 3 \). Hence, the position vector field \( \xi \in \mathfrak{X}(\mathbb{R}^{n+1}) \) is transversal to any given (G)PSR manifold. We can thus study (G)PSR manifolds in the setting of centro-affine geometry. The centro-affine fundamental form \( g \) of a centro-affine hypersurface \( \mathcal{H} \) is defined to be the unique symmetric \((0, 2)\)-tensor fulfilling the centro-affine \( \text{Gauß} \) equation

\[
D_X Y = \nabla_{\nabla}^c a X Y + g(X, Y)\xi
\]

for all \( X, Y \in \mathfrak{X}(\mathcal{H}) \). In the above equation, \( D \) denotes the canonical flat connection on \( \mathbb{R}^{n+1} \), and \( \nabla^c \) denotes the \emph{centro-affine connection} which is also uniquely determined by the centro-affine \( \text{Gauß} \) equation. Note that the centro-affine connection and the Levi-Civita connection induced by \( g \), assuming \( g \) being non-degenerate, do in general not coincide. For a reference on affine and centro-affine geometry see \([NS]\). In the case of (G)PSR manifolds \( \mathcal{H} \subset \{ h = 1 \} \), the centro-affine fundamental form is a Riemannian metric and of the form \( g = -\frac{1}{\tau} \partial^2 h|_{T\mathcal{X} \times T\mathcal{X}} \) \([CNS, \text{Prop. 1.3}]\). The Riemannian property follows from the hyperbolicity condition. To see this, observe that for all \( p \in \mathcal{H} \), \( \ker(dh_p) \) is orthogonal to \( \xi_p \) with respect to the Lorentzian metric \(-\partial^2 h \) on \( \mathbb{R}_{>0} \cdot \mathcal{H} \). This follows, again, from Euler’s theorem for homogeneous functions. Since \(-\partial^2 h(\xi, \xi) = -\tau(\tau - 1)h \) is negative on \( \mathcal{H} \), we deduce that \( g \) is indeed positive definite by the hyperbolicity of every point in \( \mathcal{H} \).

Definition 2.3. For a given homogeneous polynomial \( h : \mathbb{R}^{n+1} \rightarrow \mathbb{R} \) we denote by \( G^h \subset \text{GL}(n+1) \) the linear automorphism group of \( h \).

Note that for any (G)PSR manifold \( \mathcal{H} \subset \{ h = 1 \} \), elements in \( G^h \) are isometries with respect to the centro-affine fundamental form.

Definition 2.4. Let \( \mathcal{H} \subset \{ h = 1 \} \) be a (G)PSR manifold that is closed in the ambient space \( \mathbb{R}^{n+1} \), and denote \( U = \mathbb{R}_{>0} \cdot \mathcal{H} \). Then \( \mathcal{H} \) is said to have \emph{regular boundary behaviour} if

\begin{enumerate}[(i)]
  \item \( dh_p \neq 0 \),
  \item \(-\partial^2 h|_{T_p(\partial U \setminus \{0\}) \times T_p(\partial U \setminus \{0\})} \geq 0 \) and \( \dim \ker \left( -\partial^2 h|_{T_p(\partial U \setminus \{0\}) \times T_p(\partial U \setminus \{0\})} \right) = 1 \),
\end{enumerate}

for all \( p \in \partial U \setminus \{0\} \). If condition (i) is violated, independently of the validity of condition (ii), \( \mathcal{H} \) is called \emph{singular at infinity}.

In the case of closed PSR manifolds, having regular boundary behaviour is equivalent to condition (i) in Definition 2.4, cf. \([L1, \text{Thm. 4.12}]\), that is closed PSR manifolds have regular boundary behaviour if and only if they are not singular at infinity. In the case of special homogeneous curves, we will prove the following.

Lemma 2.5. Special homogeneous curves are singular at infinity and thereby have non-regular boundary behaviour.

For the proof of the above lemma see Section 3.

Remark 2.6. Special homogeneous spaces are closed in the ambient space \( \mathbb{R}^{n+1} \). This follows from their completeness as Riemannian manifolds with respect to their centro-affine fundamental form. In fact, any geodesically complete (G)PSR manifold \( (\mathcal{H}, g) \) is closed in the ambient space \( \mathbb{R}^{n+1} \) \([CNS, \text{Prop. 1.8}]\). Whether the converse statement holds also true in general is an open
question for $\deg(h) \geq 4$. For PSR manifolds the converse does indeed hold true [CNS, Thm. 2.5]. Furthermore, it also holds true for all GPSR manifolds with regular boundary behaviour [CNS, Thm. 1.18].

For our classification we will bring all considered polynomials to a certain form. In order to do so, we need the following result.

**Proposition 2.7.** Let $\mathcal{H}$ be a closed connected (G)PSR manifold. Then for all $p \in \mathcal{H}$

$$(\mathbb{R}_{>0} \cdot \mathcal{H}) \cap (p + T_p \mathcal{H}) \subset \mathbb{R}^{n+1}$$

is convex and precompact.

**Proof.** Follows from [CNS, Cor. 1.11, Lem. 1.14]. \qed

Since special homogeneous curves are automatically closed, the above in particular applies to each of their connected components.

### 3 Proofs of Theorem 1.1 and Lemma 2.5

**Proof of Theorem 1.1.** Using Proposition 2.7 we obtain that $\{h = 0\}$ contains at least two lines. Thus we can without loss of generality assume that $\{x = 0\} \subset \{h = 0\}$ and $\{y = 0\} \subset \{h = 0\}$. Using that special homogeneous curves are necessarily closed in their ambient space we might further assume that $h$ is hyperbolic on $\{x > 0, y > 0\}$ and that the connected component of $\{h = 1\} \cap \{x > 0, y > 0\}$ is a special homogeneous curve. Then $h$ is of the form

$$h = xyP,$$

where $P$ is a homogeneous polynomial in $x$ and $y$ of degree $\deg(h) - 2$. Then there exists a connected one-dimensional subgroup $G$ of $GL(2)$ acting transitively on $\{h = 1\} \cap \{x > 0, y > 0\}$. Note that $G \subset G^h$. This means that the action of $G$ must leave both $\{x = 0\}$ and $\{y = 0\}$ invariant. Hence, $G$ has an infinitesimal generator of the form

$$a = \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}$$

for some $r, s \in \mathbb{R}$ with $rs \neq 0$. By writing $h$ in the form

$$h = \sum_{k=1}^{\tau-1} f_k x^{\tau-k} y^k,$$

$f_k \in \mathbb{R}$ for all $1 \leq k \leq \tau - 1$, we find that $a$ is an infinitesimal symmetry of $h$ if and only if

$$\left( \sum_{k=1}^{\tau-1} (\tau - k) f_k x^{\tau-k} y^k \right) r + \left( \sum_{k=1}^{\tau-1} k f_k x^{\tau-k} y^k \right) s = 0.$$

The above is equivalent to $((\tau - k)r + ks) f_k = 0$ for all $1 \leq k \leq \tau - 1$. Since $k \mapsto \frac{k}{k+r}$ is strictly monotonously increasing in $k > 0$, we deduce that $f_k \neq 0$ for at most one $k \in \{1, \ldots, \tau - 1\}$. If all $f_k$ vanish identically, $h \equiv 0$, which is not a hyperbolic polynomial, and by the assumption that $h$ is positive on $\{x > 0, y > 0\}$ we obtain after rescaling $x$ or $y$ with a positive factor if necessary that $h$ is equivalent to $h = x^{\tau-k} y^k$ for some $k \in \{1, \ldots, \tau - 1\}$. After possibly
switching variables, we can further assume that \( k \in \{1, \ldots, \lceil \frac{r}{2} \rceil \} \). We now need to show that each such \( h \) is actually hyperbolic on \( \{ x > 0, y > 0 \} \). To do so, consider for \( t \in [0,1] \)

\[
-\partial^2 h_{(1-t)} = - \left( \begin{array}{c}
(t-k)(\tau-k-1)t^{\tau-k-2}(1-t)^k \\
(t-k)kt^{\tau-k-1}(1-t)^{k-1}
\end{array} \right) \frac{1}{k(k-1)t^{\tau-k}(1-t)^{k-2}},
\]

\[
\det \left( -\partial^2 h_{(1-t)} \right) = (t-k)k(1-\tau)t^{2(\tau-k-1)(1-t)^2(1-t)^{2(k-1)}}.
\]

One sees that \( \det \left( -\partial^2 h_{(1-t)} \right) \) is negative for all \( t \in (0,1) \), meaning by dimensional reason and homogeneity of \( h \) that \( -\partial^2 h \) has one negative and one positive eigenvalue on \( \{ x > 0, y > 0 \} \).

By the positivity of \( h \) on the latter set we have thus shown that for all \( k \in \{1, \ldots, \lceil \frac{r}{2} \rceil \} \), every point in \( \{ x > 0, y > 0 \} \) is a hyperbolic point of \( h \). For \( \tau \) and \( k \) fixed, we find that a possible choice for the infinitesimal generator \( a \) of \( G \) is given by \( r = k, s = k - \tau \).

It remains to show that for two distinct allowed choices for \( k \) the corresponding polynomials are not equivalent. The only linear transformations that leave \( \{ x = 0 \} \) and \( \{ y = 0 \} \) invariant are compositions of diagonal transformations and \( \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \). By the allowed range for \( k \) we can exclude transformations switching \( x \) and \( y \), so that only diagonal transformations are allowed. Any such transformation acts on \( x^{\tau-k}y^k \) via rescaling. Hence, two polynomials of the form \( h = x^{\tau-k}y^k \) cannot be equivalent for different choices of \( k \). The number of inequivalent connected special homogeneous curves with \( \deg(h) = \tau \) is thus \( \lfloor \frac{r}{2} \rfloor \).

Next, we will determine the number of connected components of each set \( \{ h = 1 \} \). For \( h = x^{\tau-k}y^k, k \in \{1, \ldots, \lceil \frac{r}{2} \rceil \} \), observe that \( \{ h = 1 \} \) has two connected components if either \( \tau \) is odd, or \( \tau \) is even and \( k \) is odd. If \( \tau \) is even and \( k \) is even, \( h \) is non-negative and \( \{ h = 1 \} \) has four connected components. See Figure 1 for examples of plots of \( \{ h = 1 \} \) in each case.

One quickly checks that the connected components of \( \{ h = 1 \} \) are equivalent independently of the choice of \( \tau \) and \( k \) via combinations of \( x \to -x, y \to -y \) and \( x \leftrightarrow y \), and thus in particular contain exclusively hyperbolic points. In order to determine the automorphism groups \( G^h \), the only case that one has to be careful with is \( k \) even and \( \tau = 2k \). In that case we obtain an additional symmetry given by \( (x,y) \to (-x,-y) \) which does not commute with e.g. \( x \to -x \). Hence, in these cases, \( G^h \) is a semi-direct product of \( \mathbb{R} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \) and \( \mathbb{Z}_2 \), the last factor acting by switching \( x \) and \( y \).

\[ \square \]

**Remark 3.1.** As mentioned in the introduction, it is in general an open and most likely very difficult problem to classify all special homogeneous spaces of dimension higher than one. For special homogeneous curves we have used that the corresponding sets \( \{ h = 0 \} \) are easy to control. For higher dimensional spaces it is however much more difficult to control the real projective algebraic varieties \( \{ h = 0 \} \), even in the comparatively most likely easiest of the open
cases, deg(h) = 4. Note however that the classification of homogeneous PSR manifolds of any
dimension in [DV] did not require complete control over \{h = 0\}.

Proof of Lemma 2.5. For any \( h = x^{\tau-k}y^k \) as in Theorem 1.1, we have \( dh = (\tau - k)x^{\tau-k-1}y^k \, dx + kx^{\tau-k}y^{k-1} \, dy \). One of the connected components of \( \{h = 1\} \) is contained in, and spans, \{x > 0, y > 0\}. By \( \tau - k \geq 2 \), \( dh \) vanishes identically on \( \{x = 0\} \). This shows that every special homogeneous curve is singular at infinity.

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