On the integration of Shapley-Scarf housing markets

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Relevance

The Shapley-Scarf model is one of the simplest and most widely used models in matching: resource allocation without using money. Every agent has one house, and can exchange it for a different one.

Applications: kidney exchange and school choice.
Motivation

Kidney exchange and school choice markets tend to be partitioned into smaller markets, and this fragmentation causes inefficiencies.

Agarwal et al. (2019, AER): most kidney exchanges in the US are conducted locally, despite the existence of centralized clearinghouses, which could increase the number of transplants by up to 63 percent.

Same fragmentation in school choice by districts, with strong opposition for integration.
Contribution

After markets integrate:

1. How many people become better off?

2. How large are their welfare gains in terms of house ranking?

We perform both worst- and average-case analysis.

Assumption: the core allocation is chosen before and after the merge.
Main Results

1. Almost all agents* may be harmed by integration and losses can be up to 50% of the size of their preferences (tight, stark difference).

2. Agents from all markets obtain expected gains from integration, in particular from small markets (exact computation).

3. When markets have the same size, less than 50% of agents are harmed if each market has less than $8 \pi$ agents.
Related Literature

• Frieze and Pittel (1995): “Probabilistic analysis of an algorithm in the theory of markets in indivisible goods”, The Annals of Applied Probability.

• Knuth (1996): “An exact analysis of stable allocation”, Journal of Algorithms.

• Ortega (2018): “Social integration in two-sided matching markets”. Journal of Mathematical Economics.
Model
Shapley-Scarf markets
Shapley-Scarf markets

Top Trading Cycles (TTC)
Extended Shapley-Scarf markets

Market 1
(each market has a different number of agents)

Market 2
Extended Shapley-Scarf markets

\[ N \text{ Total number of agents} \]

\[ n_j \text{ Number of agents in market } j \]

\[ k \text{ Number of different markets} \]
Example
Example 1: An EHM with $N = 7$, $k = 2$, $C_1 = \{a, b, c\}$ and $C_2 = \{d, e, f, g\}$. The integrated (resp. segregated) core allocation appears in a diamond (resp. circle).
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|   | a | b | c |   | d | e | f | g |
|---|---|---|---|---|---|---|---|---|
|   |   |   |   |   | d | c | a |   |
|   | b | a | d |   | a | f | g | d |
|   |   |   |   |   | e | a | a | a |

\[
\begin{array}{c}
\text{d} \\
\text{b} \\
\uparrow \\
\text{g} \\
\text{b} \\
\uparrow \\
\end{array} \\
\begin{array}{c}
\text{c} \\
\downarrow \\
\text{d} \\
\downarrow \\
\text{c} \\
\downarrow \\
\end{array} \\
\begin{array}{c}
\text{b} \\
\uparrow \\
\text{g} \\
\uparrow \\
\end{array} \\
\begin{array}{c}
\text{c} \\
\downarrow \\
\text{g} \\
\downarrow \\
\end{array}
\]

\[
\begin{array}{c}
+1 \\
-4 \\
-2 \\
+1 \\
-6 \\
-5 \\
-3 \\
\end{array}
\]
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| $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ |
|-----|-----|-----|-----|-----|-----|-----|
| $\Diamond$ | $\Box$ | $\Box$ | $\Diamond$ | $\Box$ | $\Box$ | $\Box$ |
| $\Box$ | $\Diamond$ | $\Box$ | $\Box$ | $\Diamond$ | $\Box$ | $\Box$ |
| $+$ | $-4$ | $-2$ | $+$ | $-6$ | $-5$ | $-3$ |

Total losses

$-\frac{18}{7^2} = 37\%$
Worst-case Results

1. $n - k$ agents may be harmed by integration - tight.

2. Losses can be up to (tight)

$$\frac{-n^2 + n + k^2 + k}{2n^2}$$

For $n \to \infty$, average losses assymptotically become $1/2$ of the length of agents‘ preference lists.
Average-case Results

The worst-case results were obtained with very particular preferences, doesn’t tell us much about what is likely to happen.

We look at random markets, in which each possible preference order is chosen uniformly at random.
Average-case Results

1. The expected **gains** for an agent of market $j$ equal

$$\frac{(N+1)[(n_j+1)H_{n_j} - n_j]}{n_j(n_j+1)N} - \frac{(N+1)H_n - N}{N^2}$$

Where $H_n$ is the $n$-th Harmonic number.

**Conclusion:** everybody benefits from market integration in expectation, particularly if $n_j$ is small.
Average-case Results

The second result is a (not tight) bound on how many people are harmed by integration.

2. The number of people harmed by integration in market $j$ are smaller than $n_j - \sqrt{2 \pi n_j} + O(\log n_j)$

**Intuition:** In each cycle generated by TTC in the segregated allocation, at least one person must be unharmed by integration.
Example

Exp. Gains

60
30
10

Harmed ≤

41
16
2

Not tight!
Corollary

If every market has the same number of agents, the percentage of agents harmed by integration is 50% or less if $n_j < 8 \pi$.

The above result also holds if preferences are dual-dictatorships, meaning that at each step of TTC at most two people are in the top of everybody’s preferences in their market – i.e. preferences are highly correlated.
Conclusion

Integrating one-sided markets can go terribly wrong (much worse than in two-sided markets), but this is unlikely to happen.

Open questions

Other assumptions on which assignment is chosen before/after.
 Improve our bounds.
 Specific preference domains.
Thanks

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