EXTENDED OBJECTS WITH EDGES

Riccardo Capovilla\textsuperscript{(1)} and Jemal Guven\textsuperscript{(2)}

\textsuperscript{(1)} Departamento de Física
Centro de Investigacion y de Estudios Avanzados del I.P.N.
Apdo Postal 14-740, 07000 México, D. F., MEXICO
capo@fis.cinvestav.mx

\textsuperscript{(2)} Instituto de Ciencias Nucleares
Universidad Nacional Autónoma de México
Apdo. Postal 70-543, 04510 México, D.F., MEXICO
jemal@nuclecu.unam.mx

Abstract

We examine, from a geometrical point of view, the dynamics of a relativistic extended object with loaded edges. In the case of a Dirac-Nambu-Goto [DNG] object with DNG edges, the worldsheet $m$ generated by the parent object is, as in the case without boundary, an extremal timelike surface in spacetime. Using simple variational arguments, we demonstrate that the worldsheet of each edge is a constant mean curvature embedded timelike hypersurface on $m$, which coincides with its boundary, $\partial m$. The constant is equal in magnitude to the ratio of the bulk to the edge tension. The edge, in turn, exerts a dynamical influence on the motion of the parent through the boundary conditions induced on $m$, specifically that the traces of the projections of the extrinsic curvatures of $m$ onto $\partial m$ vanish.

PACS: 11.27.+d
I. INTRODUCTION

The lowest order phenomenological action describing the dynamics of a relativistic extended object, or membrane, is proportional to the area of its worldsheet, \( m \), and is known as the Dirac-Nambu-Goto [DNG] action (For relevant examples, see [1, 2, 3] and, in the related statistical mechanical context, [4]). The corresponding equations of motion of a closed object (without boundary) are completely described by the worldsheet diffeomorphism covariant system of non-linear second order hyperbolic partial differential equations:

\[
K^i = 0 .
\]

Here \( K^i \) is the trace of the \( i \)th extrinsic curvature of \( m \) embedded in spacetime, one for each co-dimension of the embedding. In particular, the classical dynamics is entirely independent of the tension of the membrane, \( \mu_0 \).

In this paper we focus on the modification required to this geometrical description when massive edges are admitted. Such edges may consist of several disconnected components. Concrete examples consist of a segment of string with monopoles attached to its ends (a disconnected boundary) or a domain wall bounded by a string. The former is relevant in hadron physics as an effective description of color flux tubes in QCD[5]. The dynamics of such systems is also relevant in cosmology because objects of this type could have been generated if the early universe underwent an appropriate sequence of phase transitions[1, 6].

The key observation is that each edge worldsheet is itself an embedded hypersurface in the worldsheet of the parent membrane, which coincides with the boundary of the parent worldsheet. The edges are thus treated as membranes themselves, one dimension lower than the parent membrane. The parent worldsheet is the spacetime where the edges live. Since the parent membrane has a dynamics of its own, however, this is no longer a fixed, prescribed background spacetime. In the lowest order approximation, the edges will also be described by a DNG action of the appropriate dimension with its own characteristic tension, \( \mu_b \) (or mass, if pointlike). The edge worldsheet \( \partial m \) will then satisfy[7]

\[
\mu_b k = -\mu_0 ,
\]

where \( k \) is the trace of the extrinsic curvature of \( \partial m \) embedded in \( m \).

There are two tractable approximations. In the limit that the mass of the edge tends to zero the null boundary dynamics associated with the theory of open membranes is recovered[2]. This limit is the one adopted for the open string action of string theory in its most ambitious form as a theory of everything. On the other hand, in the limit that the edge tension goes to infinity, \( \mu_b \to \infty \), the edges themselves become extremal surfaces of the background spacetime, and the membrane interpolates accordingly. In particular, if the background spacetime is flat, the edges can be assumed fixed. This approximation is frequently exploited in the string approximation of the inter quark potential[5].

The equations of motion (1) and (2) are not complete as they stand. What is missing is a statement about the dynamical feedback that the edges have on the parent object
spanning them. (This simple fact was overlooked in Ref. [7].) This is implemented in the form of constraints on the extrinsic geometry of \( m \) at the boundary. Specifically, one obtains that,

\[
\mathcal{H}^{ab} K^i_{ab} = 0, \tag{3}
\]

where \( K^i_{ab} \) is the \( i \)th extrinsic curvature of \( m \) embedded in spacetime, \( \mathcal{H}^{ab} \) is the projection operator in \( m \) onto \( \partial m \), and this equation is to be evaluated on \( \partial m \). These constraints must be implemented as boundary conditions on Eqs. (1). Suppose we had failed to implement these conditions. Then any given initial conditions on the boundary which are tangent to the worldsheet of a closed solution to Eqs. (1) would simply generate a timelike hypersurface on this worldsheet. The only feedback it would have on the parent membrane would be to determine the limits of the truncation of the closed solution. On the contrary, the boundary conditions (3) generally place stringent conditions on the motion of the membrane. They are, however, vacuous when the membrane is totally geodesic, \( K^i_{ab} = 0 \). Such is the case of a planar worldsheet describing, for example, a non-rotating string or a planar disc of membrane in Minkowski space.

For the case of a string with massive ends, the boundary equation of motion (2) and boundary conditions (3) can be cast in a particularly attractive form. If the trajectory of an end is parametrized with proper time \( \tau \), then Eqs. (2) and (3) can be combined to give:

\[
\mu_b D_\tau \left( \frac{dX^\mu}{d\tau} \right) = -\mu_0 \eta^\mu, \tag{4}
\]

where \( \eta^\mu \) is the inward normal to the boundary of the string worldsheet, and \( D_\tau := (dX^\mu/d\tau)D_\mu \) is the projection onto the end worldline of the spacetime covariant derivative. The acceleration of an end is therefore of constant magnitude and directed into the string worldsheet.

This paper is organized as follows: In sect.II we provide a summary of the relevant mathematical formalism. In particular, we discuss the connection between the hierarchy of embeddings: \( \partial m \) in \( m \), \( m \) in spacetime, and the direct embedding of the edges in spacetime. In order to simplify our presentation we confine our attention to the case of an extremal membrane, described by the DNG action, with edges described by a DNG action of one lower dimension. In sect.III we derive the complete equations of motion for this system, Eqs. (1), (2) and (3). What is remarkable is just how efficient variational principles are in isolating the appropriate boundary conditions (3). We conclude in Sect. IV with a brief discussion that focuses on a rotating string with massive ends.

II. KINEMATICS

To begin with, consider an oriented timelike worldsheet \( m \) of dimension \( D \), which corresponds to the trajectory of the membrane in an \( N \)-dimensional spacetime \( \{ M, g_{\mu\nu} \} \). The worldsheet \( m \) is described by the embedding (3).
\[ x^\mu = X^\mu(\xi^a), \] (5)

where \( x^\mu \) are coordinates on \( M \), and \( \xi^a \) coordinates on \( m \) \( (\mu, \nu, \cdots = 0, \cdots, N - 1, \text{ and} \ a, b, \cdots = 0, \cdots, D - 1) \). The \( D \) vectors,

\[ e_a := X_\mu^a \partial_\mu, \] (6)

form a basis of tangent vectors to \( m \), at each point of \( m \). The Lorentzian metric induced on the worldsheet is then given by,

\[ \gamma_{ab} = e_\mu^a e_\nu^b g_{\mu\nu}. \] (7)

Note that in statistical mechanics applications we are interested in a Euclidean “spatial” metric \( g_{\mu\nu} \), and, of course, the induced metric \( \gamma_{ab} \) is Euclidean as well.

Let the spacetime vectors \( n^\mu_i \) denote the \( i \)th unit normal to the worldsheet \( (i, j, \cdots = 1, \cdots, N - D) \), defined, up to a local \( O(N - D) \) rotation, with

\[ g_{\mu\nu} e_\mu^a n_\nu^i = 0, \quad g_{\mu\nu} n_\mu^i n_\nu^j = \delta^{ij}. \] (8)

Normal indices are raised and lowered with \( \delta^{ij} \) and \( \delta_{ij} \), respectively, whereas tangential indices are raised and lowered with \( \gamma^{ab} \) and \( \gamma_{ab} \), respectively.

The vectors \( \{ e_a, n_i \} \) form a basis for spacetime vectors adapted to the situation of interest here.

The worldsheet projection of the spacetime covariant derivatives is defined by \( D_a := e_\mu^a D_\mu \), where \( D_\mu \) is the (torsionless) covariant derivative compatible with \( g_{\mu\nu} \). The classical Gauss-Weingarten equations (see \[9, 10\]) are given by,

\[ D_a e_\nu^b = \gamma^{\nu c} e_\mu^c - K_{\mu i} n_\mu^i, \] (9)

\[ D_a n_\mu^i = K_{\mu i} e_\nu^b + \omega^{ij}_{ab} n_\nu^j. \] (10)

The \( \gamma^{\nu c} = \gamma_{\nu c}^{ab} \) are the connection coefficients compatible with the worldsheet metric \( \gamma_{ab} \). The quantity \( K_{\mu i} \) is the \( i \)th extrinsic curvature of the worldsheet defined by

\[ K_{\mu i} = -g_{\mu\nu} n_\nu^i D_a e_\nu^b = K_{\mu i}. \] (11)

The extrinsic geometry of \( m \) is determined by \( K_{\mu i} \), and by the extrinsic twist potential, \( \omega_{ij} \) associated with the covariance under normal frame rotations. (see \[e.g. \ [1], [10]\]).

Not every specification of the intrinsic and of the extrinsic geometry is necessarily consistent with some embedding. There are integrability conditions, the Gauss-Codazzi, Codazzi-Mainardi, and Ricci equations, which must be satisfied by the intrinsic and extrinsic geometry, for an embedding to exist. We will return to these equations below in the context of the boundary.
We turn now to the definition of the intrinsic and extrinsic geometry of the worldsheet boundary $\partial m$. We treat $\partial m$ as a timelike surface of dimension $D - 1$, described by the embedding in the worldsheet $m$,

$$\xi^a = \chi^a(u^A) \quad (12)$$

where $A, B, ... = 0, 1, \ldots, D - 2$, and $u^A$ are coordinates on $\partial m$.

The definition of the extrinsic and intrinsic geometry of the worldsheet boundary provides a special case of the discussion given above for an arbitrary worldsheet. In order to establish our notation, we repeat it, specializing to the case of co-dimension one. The $D - 2$ vectors $\epsilon_A := \chi^a_A \partial_a$ are tangent to the boundary worldsheet $\partial m$. The metric induced on $\partial m$ is then,

$$h_{AB} = \gamma_{ab} \chi^a_A \chi^b_B. \quad (13)$$

The normal to $\partial m$ is defined by

$$\gamma_{ab} \eta^a \epsilon^b_A = 0, \quad \gamma_{ab} \eta^a \eta^b = 1. \quad (14)$$

The Gauss-Weingarten equations take the form:

$$\nabla_A \epsilon^a_B = \gamma_{AB}^C \epsilon^a_C - k_{AB}^I \eta^I, \quad (15)$$

$$\nabla_A \eta^a = k_{AB}^I \epsilon^a_B + \omega_{AI}^J \eta^J. \quad (16)$$

where $\nabla_A = \epsilon^a_A \nabla_a$ is the gradient along the tangential basis $\{\epsilon^a\}$, $\gamma_{AB}^C$ are the connection coefficients compatible with the boundary worldsheet metric $h_{AB}$, and $k_{AB} = k_{BA}$ is the edge worldsheet extrinsic curvature associated with the embedding of $\delta m$ in $m$. For a co-dimension one embedding, the extrinsic geometry is determined completely by the extrinsic curvature, and the Ricci integrability conditions are vacuous.

For the role it will play in the sequel, it is useful also to contrast this description with the description of the boundary $\delta m$, or, which is the same thing, of the edge worldsheet, embedded directly in spacetime,

$$x^\mu = X^\mu(u^A), \quad (17)$$

with tangents $e^\mu_A := e^\mu_a \epsilon^a_A$. This corresponds to the map composition $X^\mu(\xi^a(u^A)) = X^\mu(u^A)$. The induced metric is exactly as before, Eq. (13). The spacetime normals to $m$ are also normal to $\partial m$ in spacetime. With $\eta^\mu := e^\mu_a \eta_a$, these vectors complete the normal basis which we label $n^I := \{\eta, n^a\}$. We will use the index 0 to denote the direction along $\eta^\mu$. It should not be confused with a timelike index. We can now write down the corresponding Gauss-Weingarten equations ($D_A := e^\mu_A D_\mu$),

$$D_A e^\mu_B = \gamma_{AB}^C e^\mu_C - K_{AB}^I n^\mu_I, \quad (18)$$

$$D_A n^\mu_I = K_{AB}^I e^\mu_B + \omega_{AI}^J n^\mu_J. \quad (19)$$
With respect to this adapted basis, it is simple to check that $K_{i}^{AB} = \epsilon_{A}^{a} \epsilon_{B}^{b} K_{i}^{ab}$, and $K_{i}^{0AB} = k_{i}^{AB}$. In addition, $\omega_{Aij} = \epsilon_{A}^{a} \omega_{aij}$. The boundary inherits the extrinsic curvature and twist of the worldsheet. However, note that $\omega_{A,i0} = \eta^{a} \epsilon_{A}^{b} K_{abi}$. Thus, there is the possibility that the boundary worldsheet might have a non trivial twist (associated with its embedding in spacetime) though the parent worldsheet does not. In particular, this might be the case when the parent worldsheet is embedded as a hypersurface in spacetime. In the case of a one-dimensional boundary, however, the extrinsic twist will be pure gauge.

It is also instructive to examine the hierarchy of integrability conditions which emerges in these alternative embeddings of the boundary. On one hand, we have the Gauss-Codazzi, Codazzi-Mainardi, and Ricci integrability conditions associated with the embedding of $m$ in spacetime:

\[ R_{abcd} = R_{abcd} - K_{ac}^{i} K_{bd}^{i} + K_{ad}^{i} K_{bc}^{i} , \]  

\[ R_{abc} = \nabla_{a} K_{bc} - \nabla_{b} K_{ac} , \]  

and

\[ R_{abij} = \Omega_{abij} - K_{ac}^{i} K_{b}^{j} + K_{bc}^{i} K_{a}^{j} . \]  

Here $\nabla_{a}$ is the covariant derivative associated with the extrinsic twist potential $\omega_{aij}$, and $\Omega_{abij}$ is its curvature. The left hand side of these equations denote the contraction of the background spacetime Riemann tensor $R^{\mu}_{\nu\rho\sigma}$ with the basis $\{e_{a}, n^{i}\}$. $R_{abcd}$ is the Riemann tensor of the worldsheet covariant derivative $\nabla_{a}$.

We also have the Gauss-Codazzi and Codazzi-Mainardi integrability conditions associated with the embedding of $\partial m$ in $m$:

\[ R_{ABCD} = R_{ABCD} - k_{AC} k_{BD} + k_{AD} k_{BC} , \]  

and

\[ R_{ABCD}^{d} = D_{A} k_{BC} - D_{B} k_{AC} . \]  

The left hand side of these equations denote the contraction of the worldsheet Riemann tensor $R^{a}_{bcd}$ with the basis $\{e_{A}, \eta\}$. We use the notation $R_{ABCD}$ to denote the Riemann tensor of the boundary covariant derivative $D_{A}$.

Finally, there are the Gauss-Codazzi, Codazzi-Mainardi, and Ricci integrability conditions associated with the direct embedding of the boundary in spacetime:

\[ R_{ABCD} = R_{ABCD} - K_{AC}^{I} K_{BD}^{I} + K_{AD}^{I} K_{BC}^{I} , \]  

\[ R_{ABCI} = \bar{D}_{A} K_{BC}^{I} - \bar{D}_{B} K_{AC}^{I} , \]  

and
\[ R_{ABIJ} = \Omega_{ABIJ} - K_{ACI}K_B^C J + K_{BCI}K_A^C J. \]  

\( \tilde{\nabla}_A \) is the twist covariant derivative associated with \( \omega_A^{IJ} \). We note that consistency between Eq. (22) and (27) implies

\[
\Omega_{ABij} = \epsilon_A^b \epsilon_B^b \Omega_{abij}, \\
\Omega_{ABi0} = \epsilon_C^c \left[ \epsilon_A^a K_{aci}k_B^c - \epsilon_B^b K_{bcj}k_A^c \right].
\]

### III. Extremal Objects with Loaded Edges

The dynamics of the membrane is specified by the choice of an appropriate phenomenological action, constructed with scalars built with the quantities that characterize the intrinsic and extrinsic geometry of the membrane worldsheet. In the presence of edges, one needs also to specify some dynamical rule for the edges themselves. We choose the DNG action for the membrane, and the same action for its edges.

The action we consider is

\[ S = S_0 + S_b, \]  

where

\[ S_0[X, \chi] = -\mu_0 \int_m d^D \xi \sqrt{-\gamma}, \]  

\[ S_b[\chi, X] = -\mu_b \int_{\partial m} d^{D-1} u \sqrt{-h}, \]

\( \mu_0 \) is the membrane tension, \( \mu_b \) is the tension of the edge membrane, \( \gamma \) the determinant of the membrane worldsheet metric \( \gamma_{ab} \), and \( h \) is the determinant of the boundary worldsheet metric \( h_{AB} \). This action is a functional both of the embedding \( X^\mu \) of \( m \) in \( M \), and of the embedding \( \chi^a \) of \( \partial m \) in \( m \). There may well be many disconnected edges. To avoid clutter we ascribe the one tension \( \mu_b \) to all.

To derive the equations of motion arising from the action (28), consider first a variation of the embedding of \( m \), \( X^\mu \to X^\mu + \delta X^\mu \). The displacement is assumed to vanish on two spacelike hypersurfaces of \( m \), which play the role of initial and final times.

We decompose the displacement with respect to the spacetime basis \( \{ e^a, n^i \} \), as

\[ \delta X = \Phi^a e_a + \Phi^i n_i. \]  

We now have that under this displacement, the intrinsic metric change is \[ \mu_0, \mu_b \]  

\[ \delta X_\gamma_{ab} = 2K_{aci}^i \Phi_i + \nabla_a \Phi_b + \nabla_b \Phi_a. \]
The variation of the membrane action $S_0$ gives,

$$
\delta_X S_0 = -\mu_0 \int_m d^D \xi \sqrt{\gamma} \gamma^{ab} \left[ K_{ab} \Phi_i + \nabla_a \Phi_b \right]
$$

$$
= -\mu_0 \int_m d^D \xi \sqrt{\gamma} \left[ K^i \Phi_i + \nabla_a \Phi^a \right]
$$

$$
= -\mu_0 \int_m d^D \xi \sqrt{-\gamma} K^i \Phi_i - \mu_0 \int_{\partial m} d^{D-1} u \sqrt{-h} \eta^b \Phi_b .
$$

(33)

The last line obtains from the preceding one by applying Stokes’ theorem to the second term. Here $\eta^a$ is the outward pointing normal to $\partial m$ introduced in Eq.(14). We find that only the normal projection of the variation, $\Phi_i$, contributes to the equations of motion of the membrane, this is generally true regardless of the form of the action $S_0$ so long as it is constructed in a worldsheet ($m$) diffeomorphism invariant way. There is no boundary term associated with $\Phi_i$. This is not, however, generally true, it is an artifact of extremal dynamics.

The tangential variation gives only a boundary term. This is a consequence of the fact that tangential deformations correspond modulo a displacement of the boundary to infinitesimal worldsheet diffeomorphisms. This is why we could ignore such variations in our study of objects without boundary.

We note that the boundary contribution to Eq.(34) is (minus $\mu_0$ times) the change in the worldsheet volume under a normal deformation of the boundary, $\delta \chi^a = \eta^b \Phi_b \eta^a$. As we will see, it will contribute to the equations of motion of the edge. The projections of $\Phi_b$ onto $\partial m$ do not contribute.

Before considering the variational analysis of $S_b$, we comment briefly on the case of an open membrane with a massless boundary. The normal term will vanish whenever $\gamma(\eta, \eta) = 0$ or the boundary is null. Physically, no momentum may cross the surface. This can only occur if the boundary is a null surface, moving at each point at the speed of light [2]. In the textbook treatment this is arranged by demanding that the normal projection of the worldsheet derivative of the embedding function vanish on the boundary. (For a geometric treatment of such boundary conditions, see [13].)

Let us consider now the variation of the edge action under the infinitesimal variation in the worldsheet (32). This variation is transmitted to the geometry of the boundary through its effect on $\gamma_{ab}$, given by Eq.(32). We have

$$
\delta_X S_b = -\frac{1}{2} \mu_b \int_{\partial m} d^{D-1} u \sqrt{-h} h^{AB} \delta \chi^A \chi^B
$$

$$
= -\mu_b \int_{\partial m} d^{D-1} u \sqrt{-h} \mathcal{H}^{ab} (K_{ab} \Phi_i + \nabla_a \Phi_b) ,
$$

(35)

where we have introduced the projector onto $\partial m$, $\mathcal{H}^{ab} := h^{AB} \chi_A^a \chi_B^b$ and the fact that $\delta_X h_{AB} = \chi_A^a \chi_B^b \delta \chi^a \gamma_{ab}$. 

7
The vanishing of the variation of the total action $S$ under arbitrary normal deformations, $\Phi_i$, gives the equations of motion for the membrane, Eq. (1), $K^i = 0$. $\Phi_i$ is not fixed on the boundary, so there is a boundary term appearing in Eq. (33) to contend with. It vanishes whenever, Eq. (3), $H^{ab}K_{ab}^i = 0$, evaluated on $\partial m$, is satisfied. The variational principle has therefore also provided the natural boundary conditions on the embedding $X$. We will discuss the interpretation of these conditions below.

The vanishing of the variation of $S$, under arbitrary tangential deformations, $\Phi_a$ with support on the boundary, gives the equation of motion for the boundary (2),(if $\mu_b \neq 0$)

$$k = -\mu_0/\mu_b.$$  

To see this we note that

$$H^{ab}\nabla_a \Phi_b = D_A \Phi^A + k\eta^a \Phi_a,$$  

where we have exploited the fact that $k = H^{ab} \nabla_a \eta_b$ and we define $\Phi_A = \gamma_{ab} \Phi^a \eta^b_A$, so that

$$\delta \chi S_b = -\mu_b \int_{\partial m} D^{-1} u \sqrt{-h} (H^{ab}K_{ab}^i \Phi_i + D_A \Phi^A + k\eta^a \Phi_a).$$  

The first term appearing on the right hand side of Eq. (36) is a divergence — corresponding to an infinitesimal boundary diffeomorphism, $\delta \chi^A = \Phi^A$. In the case of a smooth physical boundary this term does not contribute (the boundary of a boundary is zero). The latter term appearing on the right hand side of Eq. (36), however, adds to the surface term appearing in Eq. (34) to give Eq. (2).

We have not had to vary the action with respect to the boundary embedding to obtain Eq. (2). For completeness, and consistency, let us now consider the variation in $S$ induced by a displacement of the boundary worldsheet $\partial m$,

$$\delta \chi = \Psi \eta + \Psi^A \epsilon_A.$$  

We obtain

$$\delta \chi S = -\mu_0 \int_{\partial m} D^{-1} u \sqrt{-h} \Psi - \mu_b \int_{\partial m} D^{-1} u \sqrt{-h} h^{AB}K_{AB}\Psi,$$  

modulo the same divergence appearing in Eq. (36). We again reproduce Eq. (2), nothing new is obtained. It is worth noting that this variation does not pick up Eq. (3).

It is instructive to compare the equations of motion describing the dynamics of an isolated boundary (imagine the spanning membrane removed) with Eq. (2). The dynamics is now simply extremal and we have (in the notation of sect.II) $h^{AB}K_{AB}^i = 0$, or alternatively $k = 0$ and $H^{ab}K_{ab}^i = 0$. The former differs from (2) in the manner we would expect. The latter set of equations, however, reproduce the boundary conditions given by Eqs. (3). The departure from extremality when the boundary is spanned by a membrane occurs along the normal which is tangent to the membrane worldsheet.

The boundary conditions (3) are still not exactly the standard (Robin) kind of boundary condition we are accustomed to handle. It is worthwhile therefore to demonstrate explicitly that they are sensible boundary conditions on Eqs. (1). We note that
\[ K_{ab}^i = -n_i^a \left( \nabla_a \nabla_b X^\mu + \Gamma_{ab}^\mu X_a X_b^\beta \right), \quad (40) \]

so that Eq.(3) reads

\[ n_i^a \left[ (\Delta - \eta^a \eta^b \nabla_a \nabla_b) X^\mu + \Gamma_{ab}^\mu \mathcal{H}_{\alpha\beta} X_\alpha X_\beta \right] = 0. \quad (41) \]

We now exploit the fact that the Laplacian, \( \Delta \Psi \), of any worldsheet scalar (such as \( X^\mu \)) can be decomposed as

\[ \Delta \Psi = D^A D_A \Psi + (\eta^a \nabla_a)^2 \Psi + k \eta^a \nabla_a \Psi, \quad (42) \]

and the fact that

\[ \eta^a \eta^b \nabla_a \nabla_b \Psi = (\eta^a \nabla_a)^2 \Psi, \quad (43) \]

to express Eq.(41) in the alternative form

\[ n_i^a [D^A D_A X^\mu + \Gamma_{ab}^\mu \mathcal{H}_{\alpha\beta}] = 0. \quad (44) \]

where we have defined \( \mathcal{H}_{\alpha\beta} := \mathcal{H}^{ab} X_\alpha X_\beta \). In this form, the equations Eq.(3) involve only derivatives of \( X^\mu \) along \( \partial m \), and thus it provides sensible boundary conditions for Eq. (1).

Moreover, the form (44) of the boundary conditions suggests to reexpress the edge equations of motion, Eq. (2), as

\[ \eta^a \eta^b \nabla_a \nabla_b \Psi = (\eta^a \nabla_a)^2 \Psi, \quad (43) \]

to express Eq.(11) in the alternative form

\[ n_i^a [D^A D_A X^\mu + \Gamma_{ab}^\mu \mathcal{H}_{\alpha\beta}] = -\frac{\alpha_0}{\mu_b}, \quad (45) \]

and we can now combine Eqs.(14) and (15) as

\[ D^A D_A X^\mu + \Gamma_{ab}^\mu \mathcal{H}_{\alpha\beta} = -\frac{\alpha_0}{\mu_b} \eta^\mu. \quad (46) \]

This equations exhibit clearly the effect of the spanning membrane on the dynamics of the edges, via the driving term on the right hand side. In the case of a string, with proper time \( \tau \) along the trajectory of a boundary point, Eq.(16) reduces to (\( D_x = (dX^\mu/d\tau) D_\mu \))

\[ \mu_b D_\tau \left( \frac{dX^\mu}{d\tau} \right) = -\mu_0 \eta^\mu, \quad (47) \]

so that the acceleration of a boundary point \( D_\tau(dX^\mu/d\tau) \) is constant in magnitude and directed into \( m \).

**IV. DISCUSSION**

Consider the example of a rigidly rotating string bounded by point particles. It is clear that there is no solution of Eq.(3) corresponding to a straight *non-rotating* segment.
of string with massless ends. Energy conservation would imply that such a configuration has a fixed proper length which is inconsistent with the nullity of the ends. With masses loading the ends, however, a solution exists because energy can be transferred from the string to its boundary. The monopoles are accelerated towards each other by the constant force provided by the tension in the string, the string collapses to a singularity.

When the string rotates, the massive ends experience a centrifugal acceleration. Our non-relativistic intuition suggests that stable bound states exist. In particular, circular orbits with a fixed radius, $R$ (corresponding to a fixed string length) and fixed angular velocity $\omega$ exist. These orbits are constrained by the requirement that $\omega R \leq 1$. The corresponding worldsheet of the string is simply a truncation at this radius of the circular timelike helicoid of a rigidly rotating string with massless ends. Geometrically, this is possible because the boundary conditions Eq.(3) are automatically satisfied when $\omega$ and $R$ are constants.

In the higher dimensional case of a membrane bounded by a string, a non-trivial interplay between the tension in the membrane and that in the boundary is possible. These forces might operate in opposite directions. This is the case for a circular hole in a planar sheet of membrane. The tension on the circle tends to restore the membrane, that in the membrane to self destruction. There is clearly a critical radius determining which one will prevail. This competition is expected to play a role in topology changing processes.

In a subsequent publication we examine perturbation theory pointing out, in particular, how we must modify the treatment in [11] or [10] when dynamical boundaries are taken into account [14]. In [15] the analysis undertaken here for DNG extended objects is generalized to arbitrary phenomenological actions, both for the membrane and for the boundary.
Acknowledgements

We gratefully acknowledge support from CONACyT grant no. 211085-5-0118PE.

References

[1] A. Vilenkin, Phys. Rep. 121 (1985) 263; E.P.S. Shellard and A. Vilenkin Cosmic Strings and Other Topological Defects (Cambridge Univ. Press, Cambridge, 1995).

[2] M.B. Green, J.H. Schwarz, and E. Witten, Superstring Theory Vol. 1 (Cambridge University Press, Cambridge, 1987).

[3] The Role of Extended Objects in Particle Physics and Cosmology, Proceedings of the Trieste Conference on Super-membranes and Physics in 2 + 1 dimensions, Trieste 1989, ed. by A. Aurilia, M.J. Duff, C.N. Pope and E. Sezgin (World Scientific, Singapore, 1990).

[4] Statistical Mechanics of Membranes and Surfaces, Proceedings of the Jerusalem Winter School for Theoretical Physics, Vol. 5, ed. by D. Nelson, T. Piran and S. Weinberg (World Scientific, Singapore, 1989); L. Peliti in Fluctuating Geometries in Statistical Mechanics and Field Theory, eds. F. David and P. Ginsparg, Les Houches LXII (cond-mat/9501076).

[5] B.M. Barbashov and V.V. Nesterenko Introduction to Relativistic String Theory (World Scientific, Singapore, 1990); P.A. Collins, J.F.L. Hopkinson, and R.W. Tucker Nucl. Phys. B 100 157 (1975).

[6] T. Kibble, in Formation and Evolution of Cosmic Strings, ed. by G.W. Gibbons, S.W. Hawking, and T. Vachaspati (Cambridge University Press, 1990).

[7] R. Capovilla and J. Guven, Rev. Mex. Fís. 41 765 (1995).

[8] C. Lanczos, The Variational Principles of Mechanics (University of Toronto Press).

[9] L.P. Eisenhart, Riemannian Geometry (Princeton Univ. Press, Princeton, 1947); M. Dajczer, Submanifolds and Isometric Immersions (Publish or Perish, Houston, Texas, 1990); B. Carter, Journal of Geometry and Physics 8 52 (1992).

[10] R. Capovilla and J. Guven, Phys. Rev. D51 6736 (1995).

[11] J. Guven, Phys. Rev. D48 5562 (1993).

[12] R. Capovilla and J. Guven Class. Quant. Grav. 12 L107 (1995).
[13] D.H. Hartley and R.W. Tucker, in *Geometry of Low Dimensional Manifolds: 1* London Mathematical Society Lecture Note Series *150*, ed. S.K. Donaldson and C.B. Thomas (Cambridge University Press, Cambridge, 1990).

[14] R. Capovilla and J. Guven (1996), in preparation.

[15] R. Capovilla and J. Guven (1996), in preparation.