Inverse problems in the multidimensional hyperbolic equation with rapidly oscillating absolute term

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Abstract. The paper is devoted to the development of the theory of inverse problems for evolution equations with terms rapidly oscillating in time. A new approach to setting such problems is developed for the case in which additional constraints are imposed only on several first terms of the asymptotics of the solution rather that on the whole solution. This approach is realized in the case of a multidimensional hyperbolic equation with unknown absolute term.

Mathematics Subject Classification. Primary 35B40, 35R30; Secondary 35L10, 35L15.

Keywords: multidimensional hyperbolic equation, rapidly oscillating absolute term, asymptotics of solution, inverse problem.

Introduction

We consider some problems of recovering rapidly oscillating in time absolute term from certain data on a partial asymptotics of the solution. Hence we study some of the coefficient inverse problems. The theory of inverse problems was the subject of many monographs (see, e.g. [15]–[17]) and papers (see, e.g. [15]–[17]). But there are almost no problems with rapidly oscillating data in the classical theory of inverse problems.

This paper as paper [18] was motivated by the paper [17], in which inverse problems for the one-dimensional wave equation with unknown absolute term was posed and solved. In [17] right-hand side represented in the form \( f(x)r(t) \), where \( r \) is unknown. An additional condition in [17] was the value of \( q(t) \) of the solution at a fixed point \( x = x_0 \). In [18] we have the same form of right-hand side of multidimensional hyperbolic equation, but the unknown term rapidly oscillate: \( r = r(t, \omega t), \omega \gg 1 \). This brings up the question, should we impose an additional condition on the whole solution, as in [17]. In paper [18] it was established that the additional condition may be imposed only on several first coefficients of the asymptotics of the solution rather than on the whole solution. In the present paper following inverse problem are solved: 1) \( f \) is unknown; 2) \( f \) and fast component of \( r \) are unknown.

In conclusion, we mention that, problems with data rapidly oscillating in time model many physical (and other) processes (in particular, related to high-frequency mechanical, electromagnetic, and other actions on
a medium) see, for example, [19]–[22]. The inverse problems with such specificity have been studied in [18], [25], [26] by us.

1 Principal symbols

Let $\Omega$ denote a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}$. Its boundary. We denote the open cylinder $\Omega \times (0, T) \subset \mathbb{R}^{n+1}$ by $Q_T$, its closure $\overline{Q}_T$. Consider the following hyperbolic initial boundary-value problem with a large parameter $\omega$:

$$\frac{\partial^2 u}{\partial t^2} = Lu + f(x,t) r(t, \omega t), (x,t) \in Q_T,$$

(1.1)

$$u|_{t=0} = 0, \quad \frac{\partial u}{\partial t}|_{t=0} = 0,$$

(1.2)

$$u|_{x \in S} = 0,$$

(1.3)

All functions are real. We consider that the symmetric differential expression $Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left[ a_{ij}(x) \frac{\partial u}{\partial x_j} \right] - c(x)u -$ (1.4)

is defined in $\Omega$ and satisfies the ellipticity condition, so that

$$a_{ij}(x) = a_{ji}(x), \quad \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \gamma \sum_{i=1}^{n} \xi_i^2, \text{ where } \gamma = \text{const} > 0,$$

(1.5)

for all $x \in \Omega$ and any real vector $\xi = (\xi_1, \xi_2, ..., \xi_n)$.

We shall assume that the function $r(t, \tau)$ is defined and is continuous on the set $D = \{(t, \tau) : (t, \tau) \in [0, T] \times [0, \infty)\}$ and $2\pi$-periodic in $\tau$. Let us represent it as the sum:

$$r(t, \tau) = r_0(t) + r_1(t, \tau),$$

where $r_0(t)$ – is the mean value of $r(t, \tau)$ over $\tau$:

$$r_0(t) = \langle r(t, \cdot) \rangle = \langle r(t, \tau) \rangle_\tau = \frac{1}{2\pi} \int_{0}^{2\pi} r(t, \tau) d\tau.$$

2 The auxilary results

2.1 The results of V.A. Il’in [27]

Let us consider the problem

$$\frac{\partial^2 u}{\partial t^2} = Lu + F(x,t), (x,t) \in Q_T,$$

(2.1)
\[ u|_{t=0} = \varphi(x), \quad \frac{\partial u}{\partial t}|_{t=0} = \psi(x), \quad (2.2) \]

\[ u|_{x \in S} = 0, \quad (2.3) \]

Let domain \( \Omega \), the coefficients of the expression \( L \), right-hand side \( F \) and initial conditions \( \varphi, \psi \) satisfy the following conditions.

I. \( \Omega \) is bounded connected domain in \( \mathbb{R}^n \), \( n \in \mathbb{N} \), contained, together with its boundary \( S \), in an open domain \( C \in \mathbb{R}^n \).

II. Coefficients \( a_{ij}(x) \) and \( c(x) \) ensure existence of full orthonormal in \( L_2(\Omega) \) system classic eigenfunctions of problem

\[ \begin{cases} 
Lu = \lambda u, \\
u|_S = 0.
\end{cases} \]

To do this, since \( \text{[27]} \) it suffices to provide further conditions. Functions \( a_{ij}(x), c(x) \) can be continued to domain \( C \) so that \( a_{ij} \in C^{1+\mu}(C), c \in C^{\mu}(C), \mu \geq 0 \). Moreover, \( a_{ij} \in C^{[\frac{3}{2}]+2}(\overline{\Omega}), c \in C^{[\frac{1}{2}]+1}(\overline{\Omega}) \). Let \( y_m, \lambda_m, m = 1, 2, ... \), denote eigenfunctions and eigenvalues noted above. We shall assume that \( \{\lambda_m\} \) is nondecreasing sequence: \( 0 < \lambda_1 \leq \lambda_2 \leq ... \)

III. Initial functions \( \varphi \in C^{[\frac{3}{2}]+3}(\overline{\Omega}), \psi \in C^{[\frac{3}{2}]+2}(\overline{\Omega}) \) and \( \varphi|_{x \in S} = L\varphi|_{x \in S} = ... = L^{[\frac{3}{2}]+3}\varphi|_{x \in S} = 0, \psi|_{x \in S} = L\psi|_{x \in S} = ... = L^{[\frac{3}{2}]+3}\psi|_{x \in S} = 0 \). Let \( \varphi_m, \psi_m \) denote the coefficients of the Fourier expansion of functions \( \varphi(x), \psi(x) \) in the basis of \( y_m \).

IV. The right-hand side \( F \in C([0,T], C^{[\frac{3}{2}]+2}(\overline{\Omega})) \), \( F|_{x \in S} = Lf|_{x \in S} = ... = L^{[\frac{3}{2}]+3}f|_{x \in S} = 0 \). Let \( F_m(t) \) denote the coefficients of the Fourier expansion of function \( F(x,t) \) in the basis of \( y_m \).

**Theorem 1.** (\( V.A. \) Il’\( \text{in} \)) If conditions I–IV hold, the series

\[ u(x,t) = \sum_{m=1}^{\infty} y_m(x) \left[ \varphi_m \cos \sqrt{\lambda_m}t + \frac{\psi_m}{\sqrt{\lambda_m}} \sin \sqrt{\lambda_m}t \right] + \sum_{m=1}^{\infty} y_m(x) \frac{1}{\sqrt{\lambda_m}} \int_0^t F_m(\tau) \sin \sqrt{\lambda_m}(t-\tau) d\tau \quad (2.4) \]

and the series \( u_t, u_{tt} \) obtained by single and double differentiation of \( (2.4) \) with respect to \( t \) are converge uniformly in \( \overline{Q_T} \). The series \( u_{x_i}, u_{tx_i}, u_{x_j} \), obtained by single and double differentiation of \( (2.4) \) with respect to any two variables are converge uniformly in any domain that is strictly contained in \( Q_T \). At the same time, \( (2.4) \) is classic solution of \( (2.1) - (2.3) \).

1Recall that a domain is said to be normal if the Dirichlet problem for the Laplace equation in this domain is solvable for continuous boundary function.

2Here and in what follows, we use results of \( \text{[27]} \) in classical terms (see \( \text{[27]} \) Remark 3, p. 114 of the
**Lemma 1.** If conditions I, II, III hold, bilinear series for eigenfunctions
\[ \sum_{m=1}^{\infty} \frac{y_m^2(x)}{\lambda_m^{n+1}} \] is converges uniformly in \( \Omega \), bilinear series
\[ \sum_{m=1}^{\infty} \frac{\partial y_m(x)}{\lambda_m^{n+1}} \] and bilinear series
\[ \sum_{m=1}^{\infty} \frac{\partial^2 y_m(x)}{\lambda_m^{n+1}} \] are converge uniformly in any domain that is strictly contained in \( \Omega' \subset \Omega \).

**Lemma 2.** Let coefficients \( a_{ij}(x) \) continuous there together with their derivatives up to orde r \( k \), and \( c(x) \) continuous there together with its derivatives up to order \( k - 1 \). We shall assume that function \( \Phi(x), x \in \Omega \) satisfies following conditions:
1) \( \Phi \in C^{k+1} (\Omega) \),
2) \( \Phi|_{x \in S} = L\Phi|_{x \in S} = ... = L^{[\frac{n}{2}]} \Phi|_{x \in S} = 0 \).

Then for \( \Phi \) inequality of Bessel type holds true:
\[ \sum_{m=1}^{\infty} \Phi_m^2 \lambda_m^{k+1} \leq \left\{ \begin{array}{l} \int_{\Omega} \left[ \sum_{i,j=1}^{n} a_{ij} \frac{\partial}{\partial x_i} (L^{[\frac{n}{2}]} \Phi) \frac{\partial}{\partial x_j} (L^{[\frac{n}{2}]} \Phi) + c(L^{[\frac{n}{2}]} \Phi)^2 \right] dx, k = - , \\
\int_{\Omega} \left[ L^{[\frac{n}{2}]} \Phi \right]^2 dx, k = - . \end{array} \right. \]

### 2.2 The problem 1

The direct problem 1. The three-term asymptotics

Consider problem \( \text{[1.1]-[1.3]} \), where domain \( \Omega \), elliptic differential expression \( L \) are the same as in Theorem 1.

Concerning the function \( f(x,t) \) defined at \( (x,t) \in Q_T \), we assume that there exist continuous functions \( f, Lf, f_t, f_{tt}, f_{ttt} \) and \( Lf_t \), such that all of them belong to the space of functions \( C_{t,x}^{[\frac{n}{2}]+2} (Q_T) \) and, moreover,
\[ f|_{x \in S} = Lf|_{x \in S} = ... = L^{[\frac{n}{2}]}f|_{x \in S} = 0. \]

For brevity, we refer to functions \( r \) with these properties as functions of class \( F_1 \).

We shall assume that the function \( r(t, \tau) \) is defined and is continuous on the set \( D = \{ (t, \tau) : (t, \tau) \in [0,T] \times [0, \infty) \} \) and \( 2\pi \) -periodic in \( \tau \). As in Sec 1 let us represent \( r \) as the sum of slow and oscillating components:
\[ r(t, \tau) = r_0(t) + r_1(t, \tau); \]

(Russian original]). In [27], such classical versions are not stated explicitly, but when referring to results of [27], we always mean their classical versions.
we shall assume that $r_0 \in C([0, T])$, and the functions $r_1, r_{tt}, r_{ttt}$ and $r_{tttt}$ belong to the class $C(D)$. We denote function $r$ with such properties as function of class $R_1$.

In the present paper, by a solution of problem (1.1)-(1.3) we mean its classical solution, i.e., a function $u \in C(\overline{Q_T})$, which has continuous derivatives $u_t \in C(\overline{Q_T}), u_{ttt} \in C(Q_T), i, j = 1, n, and satisfies relations (1.1)–(1.3). Under our assumptions, the solution of problem (1.1)–(1.3), exists and is unique according to the Theorem 1.

Below we define functions and constants needed in what follows:

\[ \rho_0(t, \tau) = \int_0^\tau \left( \int_0^p r_1(t, s)ds - \left\langle \int_0^\tau \int_0^p r_1(t, s)ds \right\rangle \right) dp - \left\langle \int_0^\tau \left( \int_0^p r_1(t, s)ds - \left\langle \int_0^\tau r_1(t, s)ds \right\rangle \right) dp \right\rangle \]

\[ \rho_1(t, \tau) = \left\langle \int_0^\tau \rho_0(t, s)ds \right\rangle - \int_0^\tau \rho_0(t, s)ds. \]

\[ b_{1,m} = -\rho_0(0, 0)f_m(0), \]

\[ d_m = -\rho_0(0, 0)f_m(0), \]

\[ b_{2,m} = -(2\rho_1(0, 0) + \rho_0(0, 0))f_m'(0) - (2\rho_{tt}(0, 0) + \rho_0(0, 0))f_m(0), \]

where the $f_m(t)$ are the coefficients of the Fourier expansion of $f(x, t)$ in the basis of $y_m$.

Let us represent the solution of problem (1.1)-(1.3) in the form:

\[ u_\omega(x, t) = U_\omega(x, t) + W_\omega(x, t), \omega \gg 1, \]

\[ U_\omega(x, t) = u_0(x, t) + \omega^{-1}u_1(x, t) + \omega^{-2}[u_2(x, t) + v_2(x, t, \omega t)] \omega \gg 1, \]

\[ u_0(x, t) = \sum_{m=1}^\infty \frac{y_m(x)}{\sqrt{\lambda_m}} \int_0^t f_m(s)r_0(s) \sin \sqrt{\lambda_m}(t - s)ds, \]

\[ u_1(x, t) = \sum_{m=1}^\infty \frac{b_{1,m}}{\sqrt{\lambda_m}} y_m(x) \sin \sqrt{\lambda_m}t, \]

\[ v_2(x, t, \tau) = f(x, t)\rho_0(t, \tau), \]

\[ u_2(x, t) = \sum_{m=1}^\infty y_m(x) \left( d_m \cos \sqrt{\lambda_m}t + \frac{b_{2,m}}{\sqrt{\lambda_m}} \sin \sqrt{\lambda_m}t \right). \]

Note that, in view of the Theorem 1 the series (2.11)-(2.14) converge uniformly and absolutely.
Theorem 2. The solution \( u_\omega(x,t) \) of problem (1.1)-(1.3) can be expressed in the form (2.9)-(2.14), where

\[
\|W_\omega(x,t)\|_{C(\Omega_T)} = o(\omega^{-2}), \omega \to \infty.
\] (2.15)

The inverse problem 1

Suppose that the function \( f(x,t) \) in the initial boundary-value problem (1.1)-(1.3) is the function of class \( F_1 \) and the function \( r \in R_1 \) is unknown. Choose a point \( x^0 \in \Omega \) at which \( f(x^0,t) \neq 0, t \in [0,T] \), and functions \( \varphi_0(t) \) and \( \chi(t,\tau) \) satisfying the conditions:

\[
\varphi_0 \in C^1([0,T]), \varphi_0(0) = 0, \varphi_0'(0) = 0; \quad \chi \in C^{3,2}(D),
\]

where the function \( \chi(t,\tau) \) is 2\( \pi \)-periodic in \( \tau \) and has zero mean \( \langle \chi(t,\cdot) \rangle = 0 \). Consider the functions \( \varphi_1(t) \) and \( \varphi_2(t) \) defined by

\[
\varphi_1(t) = \sum_{m=1}^{\infty} b_{1,m} y_m(x^0) \sin \sqrt{\lambda_m} t, \quad \varphi_2(t) = \sum_{m=1}^{\infty} y_m(x^0) \left( d_m \cos \sqrt{\lambda_m} t + \frac{b_{2,m}}{\sqrt{\lambda_m}} \sin \sqrt{\lambda_m} t \right),
\] (2.16) (2.17)

where the \( b_{1,m}, b_{2,m} \) and \( d_m \) are the same as in (3.12)-(3.13), but \( \rho_0(t,\tau) \) is now defined by

\[
\rho_0(t,\tau) = \frac{1}{f(x^0,t)} \left( \int_0^\tau \left( \int_0^p \chi_{ss}(t,s)ds - \int_0^\tau \chi_{ss}(t,\tau) d\tau \right) dp - \int_0^\tau \left( \int_0^p \chi_{ss}(t,s)ds - \int_0^\tau \chi_{ss}(t,s) d\tau \right) d\tau \right).
\]

The inverse problem 1 is to find a function \( r \in R_1 \) for which the solution \( u_\omega(x,t) \) of problem (1.1)-(1.3) satisfies the condition

\[
\|u_\omega(x^0,t) - \left[ \varphi_0(t) + \frac{1}{\omega} \varphi_1(t) + \frac{1}{\omega^2} \left( \varphi_2(t) + \chi(t,\omega t) \right) \right]\|_{C([0,T])} = o(\omega^{-2}), \omega \to \infty.
\] (2.18)

Theorem 3. For any pair of functions \( \chi, \varphi_0 \) and point \( x^0 \) satisfying the conditions specified above inverse problem 1 is uniquely solvable.

Remark. Finding the function \( r_0 \) reduces to solving a Volterra equation of the second kind

\[
f(x^0,t)r_0(t) + \int_0^t K(t,s)r_0(s) ds = \varphi_0''(t),
\] (2.19)
\[ K(t, s) = -\sum_{m=1}^{\infty} \sqrt{\lambda_m} f_m(s) \sin \left( \sqrt{\lambda_m} (t-s) \right) y_m(x^0). \]

Function \( r_1 \) calculated by
\[ r_1(t, \tau) = \frac{1}{f(x^0, t)} \frac{\partial^2}{\partial \tau^2} \chi(t, \tau), \quad (2.20) \]

Remark. The Theorem 2 and 3 can be found together with their proof in paper [18].

2.3 The lemma Krasnoselʼskii et al. [28, Sec. 22.1]

Suppose that \( \Omega \) is bounded connected domain in \( \mathbb{R}^n \) and \( S \) its boundary. We denote \( k_0 > \frac{n}{2} \) is natural value such that \( S \in C^{2k_0} \) and functions \( b_{ij}, d \in C^{2k_0-2}(\Omega) \). Moreover, boundary smoothness meant in the same manner as in [29, Theorem 15.2]. In space \( L_2(\Omega) \) consider elliptic differential operator
\[ L_0 u = \sum_{i,j=1}^{n} b_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} - d(x)u, \quad u \in D(L_0) \equiv \hat{W}^2_2(\Omega), \quad (2.21) \]

where \( \hat{W}^2_2(\Omega) \) is closure in \( W^2_2(\Omega) \) of set of smooth finite in \( \Omega \) function. We shall assume that coefficient \( d(x) \) is so large that \( L_0 \) is invertible operator. Results of [29] imply the estimate
\[ \left\| L_0^{k_0} u \right\|_{L_2} \geq c \left\| u \right\|_{W^{2k_0}_2}, \quad u \in D(L_0^{k_0}), c - \text{positive value.} \quad (2.22) \]

We assume that the domain \( \Omega \) satisfies Sobolev’s imbedding Theorem:
\[ \left\| u \right\|_{C^l(\Omega)} \leq c \left\| u \right\|_{W^s_2(\Omega)}, \quad u \in W^s_2(\Omega), \quad (2.23) \]

where \( s - l > \frac{n}{2}, c \) is positive value. Classic condition for this Theorem is that \( \Omega \) is star domain.

The above leads to the following result:

Lemma 3. For any integer \( |r| \in [0, 2k_0 - \frac{n}{2}] \) operator \( D^r L_0^{-k_0} \) continuously acts from \( L_2(\Omega) \) to \( C^{2k_0-r-\frac{n}{2}}(\Omega) \), where \( D^r u = \frac{\partial^r u}{\partial x_{i_1}^{r_1} \cdots x_{i_n}^{r_n}}, r = (r_1, \ldots, r_n) \) is multi-index with length \( |r| = r_1 + \cdots + r_n \).

Lemm 3 can be found in [28, 22.2] without specialization of some requirements to coefficients and boundary.

3 The main results

3.1 The problem 2

The direct problem 2. The main term of asymptotics
Let as in Sec 2.2 Ω and operator $L$ satisfies Theorem 1 conditions.

Let us consider the problem (1.1)-(1.3). From this point onward function $f(x, t) = f(x), x \in \Omega$. We also assume that $f \in C[\frac{\Omega}{2}] + 2(\Omega), f|_{x \in S} = Lf|_{x \in S} = L^2f|_{x \in S} = \ldots = L^{\frac{n+2}{2}}f|_{x \in S} = 0. \quad (3.1)$

Let us denote the class of such functions by $F_2$.

We shall also assume that function $r(t, \tau)$ is defined and is continuous on the set $D = \{(t, \tau) : (t, \tau) \in [0, T] \times [0, \infty)\}$ and $2\pi$-periodic in $\tau$. As above let represent it as the sum:

$r(t, \tau) = r_0(t) + r_1(t, \tau), \quad (3.2)$

where $r_0$ is slow component and $r_1$ is oscillating component. Let us assume that $r_0 \in C([0, T]), r_1 \in C(D)$.

**Theorem 4.** The following asymptotic formula holds

$$\|u_\omega - u_0\|_{C(\Omega)} = o(1), \omega \to \infty, \quad (3.2)$$

where $u_\omega$ is solution of problem (1.1)-(1.3).

The inverse problem 2

Consider the problem (1.1)-(1.3) in domain $\Omega$ with boundary $S \in C^2[\frac{\Omega}{2}] + 4$. Let coefficients of expression $L$ belong to the following Holder classes:

$$a_{ij} \in C^3[\frac{\Omega}{2}] + 6(\Omega), c \in C^3[\frac{\Omega}{2}] + 5(\Omega), \alpha \in (0, 1), c(x) \geq 0, x \in \Omega. \quad (3.3)$$

We shall assume that function $r(t, \tau)$ is known, satisfies the Theorem 3 conditions, and, moreover, $r_0 \in C([0, T])$. Suppose there exist a point $t_0 \in (0, T]$ such that

$$|r_0(t_0)| > |r_0(0)|. \quad (3.4)$$

Let $R_2$ denote the class of functions $r$ satisfy conditions above. We assume that function $f$ is unknown and belong to the class $F_2$.

Following lemm holds, where

$$\Lambda_m(t) \equiv \int_0^t r_0(s) \sin \sqrt{\lambda_m}(t - s)ds, t \in [0, T].$$

**Lemm 4.** For any function $r \in R_2$ there exist values $c_0 > 0$ and $m_0 \in \mathbb{N}$ such that for every number $m \geq m_0$ we have $\Lambda_m(t_0) > \frac{c_0}{\lambda_m}$. 

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For brevity, we shall assume that set \( M_0 \equiv \{ m : \Lambda_m(t_0) = 0 \} = \emptyset \).

Concerning the system (1.1)–(1.3) with unknown function \( f \), we supplement the problem with function \( \psi \) such that

\[
\psi \in C^{\frac{3}{2}}([\frac{T}{2}], \Omega), \quad \psi|_{x \in S} = L\psi|_{x \in S} = L^2\psi|_{x \in S} = \ldots = L^{\frac{3}{2}+3}\psi|_{x \in S} = 0. \tag{3.5}
\]

The inverse problem 2 is to find function \( f \in F_2 \) for which the solution \( u_\omega(x, t) \) of problem (1.1)–(1.3) satisfies the condition:

\[
\|u_\omega(x, t_0) - \psi(x)\|_{C([0, T])} = o(1), \omega \to \infty. \tag{3.6}
\]

**Theorem 5.** Let functions \( r_0, \psi \) and point \( t_0 \) satisfying the conditions specified above. Then inverse problem 2 is uniquely solvable. At the same time, the function \( f(x) \) calculated by \( f(x) = \sum_{m=1}^{\infty} f_m y_m(x) \), \( f_m = \frac{\psi_m}{\Lambda_m} \).

### 3.2 The inverse problem 3

In this section we consider again problem (1.1)–(1.3). Assume that coefficients of operator \( L \) satisfying to conditions (3.3), domain boundary \( S \in C^{\frac{3}{2}}([\frac{T}{2}], \Omega) \).

Let function \( f \) and \( r \) belong to \( F_3 \) and \( R_3 \) respectively:

\[
F_3 : f, Lf \in C^{\frac{3}{2}}([\frac{T}{2}], \Omega), f|_{x \in S} = Lf|_{x \in S} = \ldots = L^{\frac{3}{2}+4}\!f|_{x \in S} = 0;
\]

\[
R_3 : r(t, \tau) \text{ is } 2\pi\text{-periodic in } \tau. \text{ As above let represent it as the sum:}
\]

\[
r(t, \tau) = r_0(t) + r_1(t, \tau),
\]

where

\[
r_0 \in C^1([0, T]); r_1, r_{1t}, r_{1tt}, r_{1ttt} \in C(D).
\]

We shall assume that function \( r_0 \) is known, and functions \( f \) and \( r_1 \) are unknown. For brevity, as in Sec. 3.1 suppose that set \( M_0 \equiv \{ m : \Lambda_m(t_0) = 0 \} = \emptyset \). Choose a 2\pi-periodic with zero mean in second variable \( \chi(t, \tau), \chi \in C^{3,2}(D), D = [0, T] \times [0, \infty) \), and function \( \psi \in C^{\frac{3}{2}}([\frac{T}{2}], \Omega) \) satisfying the conditions

\[
\psi|_{x \in S} = L\psi|_{x \in S} = L^2\psi|_{x \in S} = \ldots = L^{\frac{3}{2}+4}\!\psi|_{x \in S} = 0. \tag{3.7}
\]

And let \( x^0 \in \Omega \) is a point at which \( \tilde{f}(x^0) \neq 0 \), where

\[
\tilde{f}(x) = \sum_{m=1}^{\infty} \tilde{f}_m y_m(x), \quad \tilde{f}_m = \frac{\psi_m}{\Lambda_m} \tag{3.8}
\]

Consider the functions \( \varphi_0(t), \varphi_1(t), \varphi_2(t) \), defined as follows. Function \( \varphi_0(t) \) is solution of Cauchy problem

\[
\begin{cases}
\varphi''_0(t) = \tilde{f}(x^0) r_0(t) + \int_0^t K(t, s) r_0(s) \, ds, \\
\varphi_0(0) = \varphi'_0(0) = 0,
\end{cases} \tag{3.9}
\]

\[
\varphi(t, \tau, x) = \tilde{f}(x) \varphi_0(t) + \int_0^t \tilde{f}(x) K(t, s) r_0(s) \, ds,
\]

where

\[
\tilde{f}(x) = \sum_{m=1}^{\infty} \tilde{f}_m y_m(x), \quad \tilde{f}_m = \frac{\psi_m}{\Lambda_m}.
\]
where

\[ K(t, s) = - \sum_{m=1}^{\infty} \sqrt{\lambda_m} \tilde{f}_m \sin \sqrt{\lambda_m}(t - s) y_m(x^0). \]

Functions \( \varphi_1, \varphi_2 \) satisfying the conditions

\[ \varphi_1(t) = \sum_{m=1}^{\infty} \tilde{b}_{1,m} y_m(x^0) \sin \sqrt{\lambda_m} t, \tag{3.10} \]

\[ \varphi_2(t) = \sum_{m=1}^{\infty} y_m(x^0) \left( \tilde{d}_m \cos \sqrt{\lambda_m} t + \frac{\tilde{b}_{2,m}}{\sqrt{\lambda_m}} \sin \sqrt{\lambda_m} t \right), \tag{3.11} \]

where

\[ \tilde{b}_{1,m} = - \rho_0 \tau(0,0) \tilde{f}_m, \tag{3.12} \]

\[ \tilde{d}_m = - \rho_0(0,0) \tilde{f}_m, \tag{3.13} \]

\[ \tilde{b}_{2,m} = -(2\rho_1(0,0) + \rho_0(0,0)) \tilde{f}_m. \tag{3.14} \]

The inverse problem 3 is to find a functions \( f \) and \( r_1 \) such that \( f \in F_3, r_1 \) is \( 2\pi \)-periodic in \( \tau \) and, moreover, \( r_1, r_{11}, r_{111}, r_{1111} \in C(D) \) for which the solution \( u_\omega(x, t) \) of problem (1.1)-(1.3) satisfies the conditions

\[ \| u_\omega(x^0, t) - \left[ \varphi_0(t) + \frac{1}{\omega} \varphi_1(t) + \frac{1}{\omega^2} \varphi_2(t) + \chi(t, \omega t) \right] \|_{C([0,T])} = o(\omega^{-2}), \tag{3.15} \]

\[ \| u_\omega(x, t_0) - \psi(x) \|_{C(\Omega)} = o(1), \omega \to \infty. \tag{3.16} \]

**Theorem 6.** Let functions \( r_0, \psi, \chi \) and points \( x^0, t_0 \) satisfying the conditions specified above. Then inverse problem 3 uniquely solvable. At the same time, the function \( f(x) = \tilde{f}(x) \) calculated by (3.8), and

\[ r_1(t,\tau) = (f(x^0))^{-1} \frac{\partial^2}{\partial \tau^2} \chi(t, \tau). \tag{3.17} \]

### 4 Proof of the main results

**Proof of the Theorem 4**

Consider the function

\[ W_\omega(x, t) = u_\omega(x, t) - u_0(x, t) = \sum_{m=1}^{\infty} f_m y_m(x) \int_0^t \sin \sqrt{\lambda_m}(t - s) r_1(s, \omega s) ds, \tag{4.1} \]
Note that, in view of Lemmas 1,2 and Cauchy-Schwarz inequality, the series in right-hand side of (4.1) converges uniformly with respect to \( t \in [0,T] \). Represent \( W_\omega(x,t) \) in the form

\[
W_\omega(x,t) = \sum_{m=1}^{m_0} \frac{f_m y_m(x)}{\lambda_m} \int_{0}^{t} \sin \sqrt{\lambda_m(t-s)} r_1(s,\omega s) \, ds + \sum_{m=m_0+1}^{\infty} \frac{f_m y_m(x)}{\lambda_m} \int_{0}^{t} \sin \sqrt{\lambda_m(t-s)} r_1(s,\omega s) \, ds \equiv S_{\omega,1} + S_{\omega,2}, m_0 \in \mathbb{N}.
\]

Let \( \epsilon \) is arbitrary value. Taking into account uniform convergence of the series (4.1), we take number \( m_0 \) sufficiently large such that for all \( m,m \geq m_0 \), and \( \omega > 0 \)

\[
\|S_{\omega,2}\|_{C(\Omega)} < \frac{\epsilon}{2}.
\]  

(4.2)

For the estimation of \( S_{\omega,1} \) choose \( \delta > 0 \) so small that

\[
\int_{0}^{\delta} \sin \sqrt{\lambda_m(t-s)} r_1(s,\omega s) \, ds < \frac{\epsilon}{2m_0 s_0},
\]  

(4.3)

where \( s_0 = \max_{1 \leq i \leq m_0} \left| \frac{\int_{0}^{\delta} y_i(s) \, ds}{\lambda_i} \right\|_{C(\Omega)} \). Further, considering \( t \in (s,T] \), we divide the interval \([\delta,t]\) into \( k \) equal parts \([t_j, t_{j+1})\), \( j = 0, k-1 \), and apply the relation

\[
\int_{\delta}^{t} \sin \sqrt{\lambda_m(t-s)} r_1(s,\omega s) \, ds = \\
\sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \sin \sqrt{\lambda_m(t-s)} r_1(s,\omega s) \, ds - \int_{t_j}^{t_{j+1}} \sin \sqrt{\lambda_m(t-t_j)} r_1(t_j,\omega s) \, ds + \\
\sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \sin \sqrt{\lambda_m(t-t_j)} r_1(t_j,\omega s) \, ds = S_1 + S_2.
\]

Choose \( k = k(t) \) so large that

\[
|S_1| < \frac{\epsilon}{4m_0 s_0}
\]  

(4.4)

for all \( m : m < m_0 \) and \( \omega > 0 \).

Further, in view of equality \( (r_1(t,\tau))_\tau = 0 \), we choose \( \omega_0 \) sufficiently large that

\[
|S_2| < \frac{\epsilon}{4m_0 s_0}
\]  

(4.5)

for given \( k, t \in [0,T] \), and any \( \omega > \omega_0 \).
Since inequalities (4.4), (4.5) there exist number $\omega_0 > 0$ such that
\[ |S_{\omega,1}| < \frac{\varepsilon}{2} \quad (4.6) \]

for any $\omega > \omega_0$. Relations (4.2), (4.6) imply the relation (3.2). This completes the proof of Theorem 4.

**Proof of the Lemma 4.**

Choose $t_0$ that $|r_0(t_0)| > |r_0(0)|$ and apply the relation
\[ \Lambda_m(t_0) = \int_0^{t_0} \frac{\sin \sqrt{\lambda_m}(t_0 - s)}{\sqrt{\lambda_m}} r_0(s) ds = \frac{r_0(t_0) - r_0(0) \cos \sqrt{\lambda_m} t_0}{\lambda_m} + \int_0^{t_0} \frac{\cos \sqrt{\lambda_m}(t_0 - s)}{\lambda_m} r_0'(s) ds. \]

Taking into account the condition (3.4), note that $|r_0(t_0)| \neq |r_0(0) \cos \sqrt{\lambda_m} t_0|$ for all $m \in \mathbb{N}$. Thus there exist positive values $c_0$ and $m_0$ such that
\[ |\Lambda_m(t_0)| > \frac{c_0}{\lambda_m} \]

for $m > m_0$. The Lemma is proved.

**Proof of the Theorem 5.**

Choose $t_0$ that $|r_0(t_0)| > |r_0(0)|$. We assume that the function $f \in F_2$ is found. It follows from Theorem 4 and conditions (??), (3.6) that
\[ \sum_{m=1}^{\infty} f_m y_m(x) \Lambda_m(t_0) = \sum_{m=1}^{\infty} \psi_m y_m(x). \]

For $\Lambda_m \neq 0, m \in \mathbb{N}(M_0 = \emptyset)$ we obtain
\[ f(x) = \sum_{m=1}^{\infty} f_m y_m(x), f_m = \frac{\psi_m}{\Lambda_m(t_0)}. \]

It remains to show that $f$ belongs to class $F_2$.

In the first place we shall show that function $f \in C^2[\Omega]$. Let us consider the series
\[ L^{[\Omega]} f(x) = \sum_{m=1}^{\infty} \frac{\psi_m}{\Lambda_m(t_0)} L^{[\Omega]} y_m(x) = \sum_{m=1}^{m_0} \frac{\psi_m}{\Lambda_m(t_0)} L^{[\Omega]} y_m(x) + \sum_{m=m_0+1}^{\infty} \frac{\psi_m}{\Lambda_m(t_0)} L^{[\Omega]} y_m(x) = Y_1 + Y_2, \]

In view of Lemmas 1, 2, 4 and CauchySchwarz inequality, series $Y_2$ may be estimate as follows
\[ \|Y_2\|_{L_2(\Omega)} \leq \frac{1}{c_0} \left( \sum_{m=m_0+1}^{\infty} \frac{y_m^2(x)}{\lambda_m^{m+1}} \right)^{1/2} \left( \sum_{m=m_0+1}^{\infty} \psi_m^2 \lambda_m^{m+7} \right)^{1/2}, \]

where $c_0$ and $m_0$ are the same as in Lemma 4.
Further, let $g$ denote the function $g(x) = L^{[\frac{n}{2}]}_{\frac{n}{2}} f(x), g \in L_2(\Omega)$. Thus

$$f = L^{-[\frac{n}{2}]}_{\frac{n}{2}} g.$$ 

As in the Lemma consider $D^{[\frac{n}{2}]}_{\frac{n}{2}}$ is the derivative of order $[\frac{n}{2}] + 2$, and then apply it to the function $f$, we obtain

$$D^{[\frac{n}{2}]}_{\frac{n}{2}} f = D^{[\frac{n}{2}]}_{\frac{n}{2}} L^{-[\frac{n}{2}]}_{\frac{n}{2}} g.$$ 

From the Lemma it follows that function $D^{[\frac{n}{2}]}_{\frac{n}{2}} f$ is continuous.

Note that, since proved smoothness of the function $f$ and properties of the eigenfunctions $y_m(x)$ it follows that for founded function $f(x)$ conditions (3.1) are hold. This completes the proof of Theorem 5.

**Proof of the Theorem 6.**

Let the hypotheses of current theorem holds. Then according to Theorem the inverse problem 2 with given functions $r_0, \psi$ and point $t_0$ is uniquely solvable, and function $\tilde{f}$ calculable by (3.8) is the inverse problem 2 solution. Providing similar to Theorem reasoning we obtain that $f \in F_3$.

Further, consider system (1.1)-(1.3) with $f(x, t) = \tilde{f}(x)$, and also the inverse problem 1 with given functions $\chi, \varphi_i, i = 0, 1, 2$ and point $x^0$. In view condition (3.9), the function $r_0(t)$ satisfies Volterra equation of the second kind

$$\varphi''_0(t) = \tilde{f}(x^0) r_0(t) + \int_0^t K(t, s) r_0(s) ds,$$

$$K(t, s) = - \sum_{m=1}^{\infty} \sqrt{\lambda_m} \tilde{f}_m \sin \sqrt{\lambda_m} (t - s) y_m(x^0).$$

From theorem it follows that the inverse problem 1 with given data is uniquely solvable, moreover, its solution may be represented in form $r(t, \tau) = r_0(t) + r_1(t, \tau)$, where $r_1$ calculated by (3.17). Because of the conditions on function $r_0$ the inverse problem 1 solution $r$ belongs to the class $R_3$.

Hence pair of functions $\tilde{f}, r_1$ is solution of the inverse problem 3. This completes the proof of this Theorem.
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