Stationary States of the Generalized Jackson Networks

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Abstract
We consider Jackson Networks on general countable graphs and with arbitrary service times. We find natural sufficient conditions for existence and uniqueness of stationary distributions. They generalise these obtained earlier by Kelbert, Kontsevich and Rybko.

Keywords: Jackson Networks, countable Markov chains, uniqueness of the stationary distribution, mean-field, double semi-stochastic matrices.

1 Introduction
This is a paper about open queuing networks. In our previous papers [RST], [RS2] we were considering closed “mean field” queuing systems. The closedness of the system means that the customers are never leaving the system,
while new customers never come to it. The “mean field” condition means
that the network of $N$ servers forms a complete graph, and after being served
the customer is allowed to go for his next service to any of $N$ servers with
uniform probability $\frac{1}{N}$. In addition, the random service time $\eta$ is the same
for all customers and for all servers.

In the present paper we will consider open networks, when customers are
leaving the system after several steps, while outside customers are coming for
service. We will show that under general conditions of not being overloaded
such systems always satisfy the Poisson Hypothesis (PH). We will relax the
mean-field symmetry of our system. (It is the largest symmetry possible,
corresponding to the action of the permutation group $S_N$.) Namely, we will
allow different values for the probabilities to go to different servers, as well
as different service times, $\eta_i$, depending on server.

The rough idea why PH always holds for our class of open systems is the
following. As we know from [RS2], the reason for the possible violation of
PH is that the memory about the initial state of the system is preserved, to
some degree. Since, however, every customer of the open system spends in
it only a finite average time, the memory of the initial state fades away as
the number of the customers initially present in the system goes to zero with
time.

Here we will study one special class of networks, which are called Jackson
Networks. The Jackson Network – JN – is defined by the transition matrix $P$;
the servers have exponential service times. The Generalized Jackson Network
– GJN – is defined by the transition matrix $P$ and arbitrary service times.
For the finite JN and GJN the conditions of existence and uniqueness of the
stationary distribution are known. For the infinite JN such conditions were
obtained by Kelbert, Kontsevich and Rybko, [KKR]. For the infinite GJN
they were not known.

In this paper we find sufficient conditions of existence and uniqueness of
the stationary distribution for the mean-field limits of Generalized Jackson
Networks (both finite and infinite), i.e. for the corresponding Non-Linear
Markov Processes.
2 Open systems

2.1 Finite number of server groups

In the present section we consider the simplest case of the open system we can treat. To define it we need to have a Markov chain \( \mathfrak{M}_m \) with \( m + 1 \) states, the number \( m \geq 1 \) being the number of different types of servers in our network.

2.1.1 Markov chain

Let \( P \) be a transition matrix of finite Markov chain \( \mathfrak{M}_m \) with \( m + 1 \) states 1, 2, ..., \( m, \infty \), where the last state \( \infty \) is absorbing. We assume that all the matrix elements \( p_{ij} > 0 \) for \( 1 \leq i, j \leq m \), and that the probability \( p_{i\infty} \) to get from the state \( i \) to \( \infty \) is positive for at least one \( i \):

\[
p_{i\infty} = 1 - \sum_{j=1}^{m} p_{ij} > 0. \tag{1}
\]

2.1.2 The servers network

Let \( Nm \) be the total number of servers in our queuing network. We assume further that to every group \( i = 1, \ldots, m \) the random service time, \( \eta_i \), is assigned.

Once a customer starts her service at a server in the \( i \)-th group, it lasts a random time \( \eta_i \). After that time the customer leaves the server. Then with probability \( p_{i\infty} \) the customer leaves the network, while she goes to the type \( j \) server with probability \( p_{ij} \). Within the group \( j \) she chooses one of its \( N \) servers uniformly, with probability \( \frac{1}{N} \). If it is occupied, the customer goes into the queue, being the last in it.

In addition, to every server of the \( i \)-th type there is assigned an inflow of external customers, which is Poisson flow with the constant rate \( v_i \). Assuming that thus defined Markov process is ergodic, we denote by \( \pi_N \) its stationary distribution, which is a measure on \( \Omega^{Nm} \).

Clearly, in order to have the ergodicity of the network, we need some sort of condition that the system is not overloaded. (In fact, it is sufficient for the ergodicity, see [FR].) Let the vector \( V = (v_1, \ldots, v_m) \). We have to assume that the vector

\[
\bar{V} = V + VP + VP^2 + \ldots, \tag{2}
\]
which solves the equation on $\Lambda$:

$$\Lambda = V + \Lambda P,$$  \hfill (3)

satisfy for all $i$ the relation

$$\mathbb{E}(\eta_i) \bar{V}_i < 1.$$  \hfill (4)

(Note that under our hypothesis (1) the equation (3) always has exactly one solution, which is given by the (convergent) series (2).)

Since we are relying on the results of the paper [RS1], we have to impose some conditions on the service time distributions. We will assume the following properties: for each $i$

1. the density function $p_i(t)$ of $\eta$ is positive on $t \geq 0$ and uniformly bounded from above;

2. $p_i(t)$ satisfies the following strong Lipschitz condition: for some $C < \infty$ and for all $t \geq 0$

$$|p_i(t + \Delta t) - p_i(t)| \leq C p_i(t) |\Delta t|,$$

provided $t + \Delta t > 0$ and $|\Delta t| < 1$;

3. introducing the random variables

$$\eta_i \bigg|_\tau = \left( \eta_i - \tau \bigg| \eta_i > \tau \right), \tau \geq 0,$$

we suppose that for some $\delta > 0$, $M_{\delta, \tau} < \infty$

$$\mathbb{E}\left(\eta_i \bigg|_\tau\right)^{2+\delta} < M_{\delta, \tau}.$$  

Of course, this condition holds once

$$M_{\delta} \equiv \mathbb{E}(\eta_i)^{2+\delta} < \infty.$$

4. the probability density $p_i(t)$ is differentiable in $t$, with $p_i'(t)$ continuous. Moreover, let us introduce the functions $p_{i, \tau}(t)$, which are the densities of the random variables $\eta_i \bigg|_\tau$, i.e.

$$p_{i, \tau}(t) = \frac{p_i(t + \tau)}{\int_0^\infty p_i(t + \tau) \, dt}.$$  

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We need that the function $p_{i,\tau}(0)$ is bounded uniformly in $\tau \geq 0$ (and in $i$ – for the infinite network) while the function $\frac{d}{d\tau}p_{i,\tau}(0)$ is continuous and bounded uniformly in $\tau \geq 0$;

5. the limits $\lim_{\tau \to \infty} p_{i,\tau}(0), \lim_{\tau \to \infty} \frac{d}{d\tau}p_{i,\tau}(0)$ exist and are finite.

In what follows we will always assume all these properties, unless stated otherwise.

### 2.1.3 Weak PH

In the limit as $N \to \infty$ we have a convergence to a system of Non-Linear Markov Processes (NLMP), under proviso (4) above that the network is not overloaded. Informally, this process can be described as follows. It is the evolution of the collection of measures $\mu_1(t), \mu_2(t), ..., \mu_m(t)$, which describe the states of the nodes 1, 2, ..., $m$. Each of the measures $\mu_i(t)$ is a probability distribution on possible queues at the corresponding node $i$ at time $t$. According to our service rules each state $\mu_i(t)$ generates an exit (non-Poissonian in general) flow from the node $i$, having the rate function $b_i(t)$. On the other hand, the inflows to every node are Poissonian with the rate functions $\lambda_i(t) = v_i + \sum_{j=1}^{m} b_j(t) p_{ji}$. For more details the reader should consult [BRS] and [RS1].

The weak PH is the following statement:

**Theorem 1** In the limit $N \to \infty$ the network described in Sect. 2.1.2 has the following properties:

1. the (total) flows of customers to different servers become independent;
2. the (total) flow of customers to any server of $i$-th type, $i = 1, ..., m$, tends to a Poisson flow with the rate function $\lambda_i(t)$ (which depends on the initial state of our system);
3. the (non-Poissonian) limiting flow of customers from any server of $i$-th type, $i = 1, ..., m$, has rate function $b_i(t)$ (also depending on the initial state), and for every $i$ we have

$$\lambda_i(t) = v_i + \sum_{j=1}^{m} b_j(t) p_{ji}. \tag{5}$$

The statement holds provided the properties 4 and 5 of the service time distributions are valid. We need no conditions on the initial state, and we do not suppose the underload relation (4).
The above result is easily obtained by the technique of the paper [BRS] in the case of continuous distributions $\eta_i$. (In fact, our situation is even simpler, since in [BRS] we deal with the network where the servers can exchange their positions.) The easier case of discrete random variables $\eta_i$ can be treated along the lines of Appendix I of [RS2].

### 2.1.4 Strong PH

The Strong Poisson Hypothesis, which is formulated below, is the statement about the asymptotic independence of our network from its initial state. The Strong Poisson Hypothesis means the validity of the following two theorems:

**Theorem 2** Suppose the underload relation (1) holds. There exist values $\hat{\lambda}_i, \hat{b}_i$, depending only on the rates $v_i$, service times $\eta_i$ and the matrix $P$, such that for any initial state $\kappa = \otimes \kappa_i$ of our network the limiting behavior of the functions $\lambda_i(t), b_i(t)$ as $t \to \infty$, does not depend on the initial state $\kappa$ of the system, and moreover

\[
\lambda_i(t) \to \hat{\lambda}_i, \quad b_i(t) \to \hat{b}_i \quad \text{as} \quad t \to \infty.
\]

Let $\nu_i$ be the stationary distribution of the stationary Markov process on a single server $\mathcal{N}_i$, corresponding to the stationary Poisson input flow with constant rate $\hat{\lambda}_i$, and service time $\eta_i$. (Such a distribution exists provided the server $\mathcal{N}_i$ is not overloaded.) Define $\hat{\pi}_N$ to be the product state on $\Omega^{Nm}$:

\[
\hat{\pi}_N = \prod_{i=1}^{m} \nu_i \otimes \ldots \otimes \nu_i.
\]

**Theorem 3** The set of limit points of the family $\pi_N$ contains at most one point, which coincides with the limit $\lim_{N \to \infty} \hat{\pi}_N$.

Before giving the proof of the theorems we will explain why, in contrast with the closed network case, we do not need to impose any condition on the initial state of our network, when it is open. It is based on the following statement.

**Claim 4** Let $\eta$ and $\xi_i, \ i = 1, 2, \ldots$ be independent random variables, and $\eta$ be an integer random variable. Then

\[
\Xi = \sum_{i=1}^{\eta} \xi_i
\]
also is a random variable. In particular, for every $\varepsilon > 0$ there exists a value $T(\varepsilon) < \infty$ such that

$$\Pr(\Xi > T(\varepsilon)) < \varepsilon.$$  

Proof. Trivial. ■

That claim explains that no matter what initial condition is chosen for our network, after some time, depending on the condition, it is almost forgotten. Now we will give the proof of the above theorems.

Proof. The flattening relation (6) for the rates $\lambda_i(t)$, $b_i(t)$ is proven in the same way as in [RS1]. First we recall the basic relation (26) of [RS1] between (arbitrary) input rate $\bar{\lambda}_i(t)$ and the corresponding output rate $\bar{b}_i(t)$ of the server $N_i$:

$$\bar{b}_i(t) = (1 - \varepsilon(t)) \left[ \bar{\lambda}_i * q_{\bar{\lambda}_i,t} \right](t) + \varepsilon(t) Q(t). \quad (7)$$

Here the functions $\varepsilon(t)$, $Q(t)$ and the family $q_{\bar{\lambda}_i,t}$ of probability densities are functionals of the initial state of $N_i$ and of the rate function $\bar{\lambda}_i$; the function $Q(t)$ is uniformly bounded in $t$, the function $\varepsilon(t)$ goes to zero as $t \to \infty$, while each of the densities $q_{\bar{\lambda}_i,t}(\cdot)$ has its support on positive semi-axis and depends on the function $\bar{\lambda}_i(\tau)$ only via its restriction to $\{\tau \leq t\}$. For more details the reader can go to [RS1], Sect. 8.

To apply the relation (7) we need to have some information on the regularity properties of the kernels $q_{\bar{\lambda}_i,t}$. In the situation of the closed systems, studied in [RS1], we were using the uniform compactness of the family $\{q_{\bar{\lambda}_i,t}, t > 0\}$: the integrals

$$\int_0^K q_{\bar{\lambda}_i,t}(\tau) \, d\tau \to 1 \quad (8)$$

as $K \to \infty$, uniformly in $t$. In the present situation it will be sufficient for our purposes to establish the following weaker property: there exists a function $K(t)$, such that

$$\int_0^{K(t)} q_{\bar{\lambda}_i,t}(\tau) \, d\tau \to 1, \text{ with } t - K(t) \to \infty \text{ as } t \to \infty. \quad (9)$$

We will show that the above property indeed holds, provided our network is underloaded, see (14) or (4) below.

To get the relation (9) we need to revert to the definition of the densities $q_{\bar{\lambda}_i,t}$, which is used in the course of the proof of the Theorem 3 of [RS1]. It is
quite complicated, but we need only some properties of it. According to this
definition, \( q_{\bar{\lambda},t} \) is the density of a certain random variable \( \xi_{\bar{\lambda},t} \geq 0 \), having
the following property:

Let a realization \( \{x_1, x_2, ..., x_n\} \subset [0, t] \) of the Poisson random field with
rate function \( \bar{\lambda}_i(\tau) \), \( \tau \in [0, t] \), as well as the (unordered) sequence \( \{l_1, l_2, ..., l_{n+1}\} \)
are given, where \( l_k \) are iid random variables with distribution \( \eta_i \). Under this
condition the random variable \( \xi_{\bar{\lambda},t} \) takes its values in the set

\[
L(l_1, l_2, ..., l_{n+1}) = \left\{ \sum_{k \in A} l_k : A \subset \{1, 2, ..., n+1\}, A \neq \emptyset \right\} \subset \mathbb{R}^1,
\]

with probabilities, depending on the sample \( \{x_1, x_2, ..., x_n\} \). Therefore the
(unconditioned) random variable \( \xi_{\bar{\lambda},t} \) is dominated from above by the random
variable

\[
\sum_{k=1}^{\chi_{\bar{\lambda},t+1}} l_k, \]

where \( l_k \) are iid random variables with distribution \( \eta_i \), while \( \chi_{\bar{\lambda},t} \) is a Poisson
random variable with parameter \( \int_0^t \bar{\lambda}_i(\tau) \, d\tau \). So to get (9) it is enough to
show that under the conditions (14) or (11) below we have that for all \( t \)

\[
\mathbb{E}(\chi_{\bar{\lambda},t}) \leq c t \mathbb{E}(\eta_i),
\]

with \( c < 1 \).

Let us derive the “flattening” relations (6), assuming (9) and (7). Applying (7) to rate functions \( \lambda_j(t) \), \( b_j(t) \), and using (9) we obtain the relations

\[
b_j^+ = \limsup_{t \to \infty} b_j(t) \leq \lambda_j^+ = \limsup_{t \to \infty} \lambda_j(t),
\]

\[
b_j^- = \liminf_{t \to \infty} b_j(t) \geq \lambda_j^- = \liminf_{t \to \infty} \lambda_j(t).
\]

(The property (3) is needed only for (12) .) Indeed, the relation (7) is telling
us that the function \( b(t) \) is the result of averaging of \( \lambda(\cdot) \) over a segment
\([t - K(t), t] \) with some probabilistic kernel, while the left-end point of the
segment \( t - K(t) \to \infty \), as \( t \to \infty \).

Introducing now the vectors \( \mathcal{L} = \{ \lambda_j^+ - \lambda_j^- \} \), \( \mathcal{B} = \{ b_j^+ - b_j^- \} \), \( j = 1, ..., m \), we have that \( \mathcal{L} \geq \mathcal{B} \) coordinate-wise. Applying \( \limsup_{t \to \infty} \) and
\( \liminf_{t \to \infty} \) to both sides of (5) we get

\[
\mathcal{L} \leq \mathcal{B} \mathcal{P}.
\]
Therefore
\[ \mathcal{L} \leq \mathcal{L} P, \]
and so for all \( n > 0 \)
\[ \mathcal{L} \leq \mathcal{L} P^n. \]
Due to condition \( \mathcal{I} \), for every \( x = (x_1, \ldots, x_m) \) with \( x_i \geq 0 \)
\[ ||xp^n||_{L^1} \leq c ||x||_{L^1}, \tag{13} \]
for all \( n \geq m \) with some \( c < 1 \). Thus, we conclude that \( \mathcal{L} = 0 \). That proves \( \mathcal{I} \).

To have the existence of the stationary measures and their convergence, as well as the regularity \( \mathcal{E} \), we need the system to be underloaded. In case when we have just one type of servers, i.e. \( i = 1 \), this condition is simply the requirement that
\[ \mathbb{E}(\eta_1) \frac{v_1}{1 - p} < 1. \tag{14} \]
Here \( p = p_{11} < 1 \) is the probability that the client will return back to the server after being served. The ratio \( \frac{v_1}{1 - p} \equiv v_1 + v_1p + v_1p^2 + \ldots \) has the following meaning: it is the mean number of the customers who will visit a given server \( \mathcal{N}_1 \) and who are the descendants of clients who first entered the network during the unit time interval \([0, 1]\). (The flow of these customers is not a Poisson flow.) Note that some of these customers visit \( \mathcal{N}_1 \) only very late in time. Therefore the condition \( \text{(14)} \) implies that the average number of clients per given server times the average service time is less than 1, since it is bounded from above by \( \mathbb{E}(\eta_1) \frac{v_1}{1 - p} \). Therefore \( \mathcal{E} \) holds.

The existence of the stationary measures under \( \mathcal{I} \) is a result of the paper [FR]. Let us derive the relation \( \mathcal{E} \). It is almost immediate after what was said in the two preceding paragraphs. Indeed, the expectation \( \mathbb{E}(\chi_{\lambda,t}) \) is nothing else as the mean value of the customers visiting a given server \( \mathcal{N}_i \) during the time interval \([0,t]\). (Moreover, now they even form a Poisson flow!) As was explained above, the estimate \( \mathbb{E}(\chi_{\lambda,t}) \leq \bar{V}_t \) holds. (In fact, the inequality is strict, since some customers will come to the service much later than \( t \).) Together with \( \mathcal{I} \) it implies \( \mathcal{E} \).

### 2.2 Infinite number of server groups

In this subsection we explain the changes needed in order to extend the results of the previous Section to the case of countably many servers. As above, it will be built on the Markov chain \( \mathcal{M} \), which now will be countable.
2.2.1 Markov chain

We will need some condition of transience of $\mathcal{M}$, analogous to (1). The condition (1) was used to derive the relation (13), which means that the only invariant measure of our chain is zero measure. The proper analog of it in the present context follows. It turns out to be the condition of vanishing of $L^\infty$-invariant measures (they do not need to be probability measures). The corresponding sufficient conditions on the transition matrix are given by Theorems 5, 7 or 8 below.

Let $\mathcal{M}$ be a countable irreducible Markov chain with the states $i = 1, 2, ... \cup \infty$. Let $P = \{p_{ij} \geq 0\}$ be the transition matrix, $\sum_j p_{ij} \leq 1$, while

$$p_{i\infty} = 1 - \sum_j p_{ij} \geq 0$$

be the probabilities to go from $i$ to $\infty$. The state $\infty$ is absorbing.

The measure $\lambda = \{\lambda (i) \geq 0\}$ is called $L^\infty$-measure, if $\lambda (i) \leq C$, for some $C > 0$, uniformly in $i$. The measure $\lambda$ is called invariant for the Markov chain $\mathcal{M}$, if

$$\lambda = \lambda P,$$

i.e. if $\lambda (j) = \sum_i \lambda (i) p_{ij}$. For example, the measure $\lambda = 0$ is invariant.

The first such transience condition was obtained in [KKR], and it applies only to double semi-stochastic matrices $P$. We recall that $P$ is called double semi-stochastic, if for all $j$ we have

$$\sum_i p_{ij} \leq 1.$$

In this case there exists one special – ‘maximal’ – invariant measure $\lambda^*$ of $\mathcal{M}$. It is constructed as follows. Consider the dual Markov chain $\mathcal{M}^*$ on $1, 2, ... \cup \infty$, with transition probabilities $p^*_{ij} = p_{ji}$, and with

$$p^*_{i\infty} = 1 - \sum_j p^*_{ij} \geq 0, \quad p^*_{\infty\infty} = 1,$$

i.e. $\infty$ is the absorbing state. Define the measure $\lambda^*$ by

$$\lambda^* (i) = \text{Pr} \{\text{the chain } \mathcal{M}^*, \text{ started from } i, \text{ never gets to } \infty\}.$$  \hspace{1cm} (15)

Clearly, $\lambda^* (i) = \sum_j p^*_{ij} \lambda^* (j) \equiv \sum_j \lambda^* (j) p_{ji}$, so $\lambda^*$ is an invariant measure for $\mathcal{M}$. 

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The following theorem gives sufficient condition for the zero measure to be the only invariant measure.

**Theorem 5** (see [KKR].) Suppose that the transition matrix $P$ of the irreducible Markov chain $M$ is double semi-stochastic. Then the following two properties are equivalent:

1. the invariant measure $\lambda^*$ (15) of the Markov chain $M$ is zero.
2. zero measure is the only invariant $L^\infty$-measure of $M$.

The following proof is simpler than the original one, see [KKR], and easily leads to generalizations, which follow.

**Proof.** Let us write a formula for the measure $\lambda^*$. To this end denote by $\gamma_0^{(0)}(i)$ the function
\[
\gamma_0^{(0)}(i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases},
\]
and put
\[
\gamma_0^{(n)} = P\gamma_0^{(n-1)}. \tag{17}
\]
$\gamma$-s are column-vectors, $\gamma_j^{(1)}$ being the $j$-th column of $P$. The value $\gamma_j^{(n)}(i)$ is the probability for $M^n$, started at $j$, to be in $i$ after $n$ steps. Therefore the sum $\sum_i \gamma_j^{(n)}(i)$ is the probability that $M^n$, started at $j$, is not at $\infty$ after $n$ steps. Evidently,
\[
\sum_i \gamma_j^{(n)}(i) = e P^n \gamma_j^0,
\]
where (the measure) $e$ is given by $e(i) \equiv 1$. The probabilities $\lambda^*(j)$ are just the limits
\[
\lambda^*(j) = \lim_{n \to \infty} e P^n \gamma_j^0.
\]
They exist because for every $j$ the sequence $e P^n \gamma_j^0$, $n = 0, 1, 2, \ldots$ is non-increasing.

Suppose now that $\lambda^* = 0$. That means that $\lim_{n \to \infty} (e P^n)(j) = 0$ for every $j$. If $h \geq 0$ is an invariant $L^\infty$ measure, $h = hP$, then for some $C$ we have $h \leq Ce$. But then evidently $h(j) \leq C (e P^n)(j)$, so $h$ has to be zero. ■

**Corollary 6** Let $P$ satisfies the conditions of the previous theorem, and $k \geq 0$ be any $L^\infty$ measure. Then $k P^n \to 0$ weakly, i.e. $(k P^n)(j) \to 0$ for every $j$ (though not uniformly in $j$).
Proof. The proof is contained in the proof of the theorem. Note that in fact we have proven a stronger result, which generalizes the above theorem to the class of stochastic matrices $P$, which, instead of being double-stochastic, have their columns summable.

**Theorem 7** Suppose that $P$ is a stochastic matrix, such that for every $j$ the function $\gamma_j^{(n)}$, defined by relations (16), (17) belongs to $L^1$ once $n \geq n_0$. Suppose moreover that the limits

$$\lambda^* (j) = \lim_{n \to \infty} \left\| \gamma_j^{(n)} \right\|_{L^1}$$

exist.

If $\lambda^* (j) = 0$ for all $j$, then for every $L^\infty$ measure $k \geq 0$ we have $kP^n \to 0$ pointwise. So in particular zero is the only invariant measure.

Still stronger statement holds as well.

**Theorem 8** Suppose that $P$ is a matrix with non-negative entries, $p_{ij} \geq 0$, such that for all $j$ the functions $\gamma_j^{(n)}$, defined by relations (16), (17), belong to $L^1$ once $n \geq n_0$. Suppose that the convergence

$$\left\| \gamma_j^{(n)} \right\|_{L^1} \to 0 \text{ as } n \to \infty$$

holds for just one value $j = j_0$, and suppose also that all the corresponding matrix elements $p_{ij_0}$ are positive. Then we have the convergence

$$\left\| \gamma_j^{(n)} \right\|_{L^1} \to 0 \text{ as } n \to \infty$$

for all other values of $j$, so in particular all the conclusions of the preceding Theorem holds.

Proof. In the notation of the proof of the theorem the vectors $eP^n$ are well defined once $n \geq n_0$. Indeed, $(eP^n) (j) \equiv \left\| \gamma_j^{(n)} \right\|_{L^1}$. By our assumption we have $eP^n\gamma_j^{(0)} \to 0$ as $n \to \infty$. But that implies immediately that also $eP^n\gamma_{j_0}^{(1)} \to 0$ as $n \to \infty$. Evidently,

$$eP^n\gamma_{j_0}^{(1)} = \sum_i p_{ij_0} eP^n\gamma_i^{(0)}.$$

Since $p_{ij_0} > 0$ for all $i$, the convergence $\sum_i p_{ij_0} eP^n\gamma_i^{(0)} \to 0$ as $n \to \infty$ implies that $eP^n\gamma_i^{(0)} \to 0$ for every $i$. The claim of the Corollary is still valid.
2.2.2 The servers network

We suppose that the network consists of $Nm$ servers of $m = m(N)$ types, with $m = m(N) \to \infty$ as $N \to \infty$. Again, to every type $i$ the random service time, $\eta_i$, is assigned. The probability of going from server of type $i$ to type $j$ is $p_{ij}$, while the probability of leaving the system is $\tilde{p}_{i\infty} = 1 - \sum_j p_{ij}$. Within the type $j$ the customer chooses the server uniformly. The notations $v_i$, $\pi_N$ have the same meaning as above.

2.2.3 Weak PH

In the limit as $N \to \infty$ we again have a convergence to a system of Non-Linear Markov Processes (NLMP), see [BRS]. All the claims of the Section (2.1.3) remains true. The only difference is that now we have infinitely many NLMP-s.

2.2.4 Strong PH

Strong PH holds here as well, in the sense of Section (2.1.4). The proof of the claim (6) proceeds in the same way as there. The contraction property (13) of the operator $P$ is now ensured by the Theorems [7, 8] see Corollary [6]. In order to use the contraction property we need to know in advance that the network is underloaded. That is, we need to know that the initial state is forgotten after some time, and that the inflow rates are not too high. The conditions on inflow rates are very similar to those of the Theorem [2]. Namely, let $\mathbf{V} = \{v_i\}$ be the vector of the (constant) inflow rates; we need that the vector

$$\mathbf{\bar{V}} = \mathbf{V} + \mathbf{VP} + \mathbf{VP}^2 + \ldots$$

satisfies for each $i$ the inequality

$$\mathbb{E}(\eta_i) \mathbf{\bar{V}}_i < 1.$$  (19)

Let $\boldsymbol{\kappa}_i$ be the initial states of our network. The only condition we need is that at every node we have a finite random queue, which means (tautologically) that the probability of infinite queue is 0. The reason is that for the networks defined by the matrix $P$ satisfying the conditions of Theorem [5, Theorem 7] or Theorem [8] the network becomes underloaded after some finite time, provided (19) holds. It is proven in [KKR] for networks satisfying the conditions of
Theorem 5 and having exponential service times. In the general case the proof is the same. So the initial state does not play any role in the asymptotic state of our network, as it was the case for the finite networks.

The convergence property (18) hold, for example, for any $V$ from $L^1$ and any $P$ satisfying Theorem 5, since it can be shown that in this situation the matrix $P^*$ is evidently transient.

Another example is if $V$ is from $L^\infty$ and the set supp ($V$) is non-massive for the chain $M^*$, defined for $P$ satisfying Theorem 5. We recall that a set $A$ is called non-massive for the chain $M^*$, if the probability that the chain never hits $A$ is positive. For the proof see [DY], Sect. 1.8.

We conclude by stating the main theorem of this section.

**Theorem 9** Consider the network, defined by the matrix $P$, satisfying the conditions of either Theorem 5, Theorem 7 or Theorem 8. Suppose the input rates satisfy the relations (18), (19).

There exist values $\hat{\lambda}_i, \hat{b}_i$, depending only on the rates $v_i$, service times $\eta_i$ and the matrix $P$, such that for any initial state $\kappa = \otimes \kappa_i$ of our network with finite queues the limiting behavior of the functions $\lambda_i(t), b_i(t)$ as $t \to \infty$, does not depend on the initial state $\kappa$ of the system, and moreover for every $i$

$$\lambda_i(t) \to \hat{\lambda}_i, \ b_i(t) \to \hat{b}_i \text{ as } t \to \infty.$$ 

**Proof.** After all the remarks made above, the proof goes in the same way as for the Theorem 2 and is therefore omitted.

## 3 Closed systems

Here we again consider the same situation as in the Section 2, but now the chain $M_m$ has $m$ states, with transition matrix $P$, and all the exit probabilities $p_{i \infty}$ are zero. That is, our network is closed; all exterior flow rates $v_i$ are then equal to zero. We assume additionally that all matrix elements $p_{ij} > 0$, so the chain is ergodic. Again we will study the mean-field type model, where we interconnect $N$ copies of our network. For every $N$ we will have fixed number of customers, $K$, and we will consider the limit when $N, K \to \infty$, so that $\frac{K}{N_m} \to \rho$.

For every $N$ we have some initial conditions, and we suppose that as $N \to \infty$, they converge to the limit, which will be the initial condition
Κ = ⊗ Κᵢ for the Non-Linear Markov Process. Now, in contrast with the open systems, we need to impose some restrictions on Κ in order to have strong PH. It is the same condition which appeared already in [RS1] – the finiteness of the expected service times \( S(Κᵢ) \). They are defined as follows.

Consider the function

\[
R_{ηᵢ}(τ) = E\left( ηᵢ \mid τ \right),
\]

which is the expected (remaining) time of the service of the client, who already spent the time \( τ \) in the server. For a queue \( ω = (n, τ) \), containing \( n \) clients, one of whom is already served for the time \( τ \), we define its expected service time, \( S(ω) \), by

\[
S(ω) = \begin{cases} 
0 & \text{for } ω = 0, \\
(n - 1) E(ηᵢ) + R_{ηᵢ}(τ) & \text{for } ω = (n, τ), \text{with } n > 0.
\end{cases}
\]

We then define the expected service time \( S(Κ) \) as \( E_μ(S(ω)) \).

Note that if the total expected service time \( ∑_i S(Κᵢ) \) is finite, then so is the total expected number of clients in our network.

In the following we will sketch the proof of the Strong PH for the above setting. The main ideas are contained in [RS1].

First of all, we have the balance relation:

\[
λᵢ(t) = ∑_j b_j(t) p_{ji}.
\]  

(20)

We also have for all \( i = 1, 2, ..., m \) :

\[
b_i(t) = (λᵢ * q_{λᵢ,t})(t) + ε_i(t),
\]

(21)

where \( ε_i(t) \to 0 \) as \( t \to ∞ \). (This is the analog of the relation (7)). The case \( m = 1 \) is the one treated in [RS1]. The relation (21) boils then down to

\[
λ(t) = (λ(·) * q_{λ(·),t})(t) + ε(t).
\]

(22)

In [RS1] we were able to show that some apriori properties of the function \( λ \) imply corresponding properties of the stochastic kernels \( q_{λ,t} \), which in turn imply the convergence of the function \( λ(t) \) to the limit value as \( t \to ∞ \). The properties of the kernels needed are the following:
1. for every $\varepsilon > 0$ there exists a value $K(\varepsilon)$, such that

$$\int_0^{K(\varepsilon)} q_{\lambda(\cdot),t}(x) \, dx \geq 1 - \varepsilon$$

(23)

uniformly in $t, \lambda(\cdot)$.

2. For every $T$ the (monotone continuous) function

$$F_T(\delta) = \inf_{x \geq X(T)} \inf_{D \subset [0,T]} \int_D q_{\lambda(\cdot),t}(x) \, dx$$

(24)

is positive once $\delta > 0$, for some choice of the function $X(T) < \infty$ – compare with relations (99), (100) of [RS1].

They were derived in [RS1] from the fact that in this situation the probability that the node is empty becomes positive after some (long) time. This property, in turn, follows from the fact that the mean number of clients is conserved in the NLMP.

For $m > 1$ the situation is a bit more complex. To treat it, let us introduce the families $\Gamma_{i,n}$ of the trajectories of the chain $\mathfrak{M}_m$, $i = 1, 2, \ldots, m$, $n \geq 1$. The family $\Gamma_{i,n}$ consists of all loops $\gamma(t) \in \{1, \ldots, m\}$, $t = 0, 1, \ldots, n$, such that $\gamma(0) = \gamma(n) = i$, while $\gamma(k) \neq i$ for $k = 1, \ldots, n - 1$. For $\gamma \in \Gamma_{i,n}$ let us define

$$p(\gamma) = \prod_{k=1}^n p_{\gamma(k-1)\gamma(k)}.$$ 

Then, for all $i$ we have

$$\sum_n \sum_{\gamma \in \Gamma_{i,n}} p(\gamma) = 1,$$

since the last sum is precisely the probability of the event to return to the state $i$, starting from it. It follows that for all $i$

$$\lambda_i(t) = \sum_n \sum_{\gamma \in \Gamma_{i,n}} p(\gamma) \int \cdots \int q_{\gamma(n-1):\lambda_{\gamma(n-1)},t}(x_1)$$

$$q_{\gamma(n-2):\lambda_{\gamma(n-2)},t-x_1}(x_2) \ldots q_{\gamma(0):\lambda_{\gamma(0)},t-x_1-\ldots-x_n}(x_n)$$

$$\lambda_i(t-x_1-\ldots-x_n) \, dx_1 \ldots dx_n.$$
Indeed, from (20) and (21) we have, ignoring the $\varepsilon$-term, that

$$\lambda_i(t) = \sum_j (\lambda_j * q_{i,\lambda_j, t})(t) p_{ji}.$$  \(26\)

Let us fix an index $i$, write this expression for all $j \neq i$ and insert the resulting representations in (26). Iterating this procedure we get (25). The advantage of (25) over (21) is that it expresses each function $\lambda_i(t)$ via itself at earlier times, as is the case in relation (22), via convolution with the stochastic kernels

$$Q_{\{\lambda_1, ..., \lambda_m\}, t}(X) = \sum_{\gamma} p(\gamma) Q_{\{\lambda_1, ..., \lambda_m\}, \gamma, t}(X),$$

where

$$Q_{\{\lambda_1, ..., \lambda_m\}, \gamma, t}(X) =$$

$$= \int ... \int q_{\gamma(n-1), \lambda_{\gamma(n-1)}, t}(x_1) q_{\gamma(n-2), \lambda_{\gamma(n-2)}, t-x_1}(x_2) \times$$

$$\times ... q_{\gamma(0), \lambda_{\gamma(0)}, t-x_1-...-x_{n-1}}(X - x_1 - ... - x_{n-1}) dx_1 ... dx_{n-1};$$

these kernels, however, do depend on all the functions $\{\lambda_1, ..., \lambda_m\}$ at earlier times. To extract the relaxation properties of the functions $\lambda_i$ from the representation (25) we again need to check that the kernels $Q$ have the properties 1, 2 listed above. Now we do not have the property that the mean number of clients at every node is conserved in the NLMP. However in our closed network we have the fact that the mean number of clients in the whole network is conserved, and that implies that at every node the mean number of clients is bounded, which property is sufficient for our purposes, as the analysis in [RS1] shows.

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