RIEMANNIAN GEOMETRY ON LOOP SPACES

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ABSTRACT. A Riemannian metric on a manifold $M$ induces a family of Riemannian metrics on the loop space $LM$ depending on a Sobolev space parameter $s$. In Part I, we compute the Levi-Civita connection for these metrics for $s \in \mathbb{Z}^+$. The connection and curvature forms take values in pseudodifferential operators (ΨDOs), and we compute the top symbols of these forms. In Part II, we develop a theory of Wodzicki-Chern-Simons classes $CS^W_K \in H^{2k-1}(LM^{2k-1})$, for $K = (k_1, ..., k_\ell)$ a partition of $2k-1$, using the Wodzicki residue on ΨDOs. We use $CS^W_5$ to prove that certain actions on $S^2 \times S^3$ are not smoothly homotopic, and that $\pi_1(\text{Diff}(S^2 \times S^3))$ is infinite.

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References
1. Introduction

The loop space \( LM \) of a manifold \( M \) appears frequently in mathematics and mathematical physics. In this paper, using an infinite dimensional version of Chern-Simons theory, we develop a computable theory of secondary characteristic classes on the tangent bundle to loop spaces. We apply these secondary classes to distinguish circle actions on \( S^2 \times S^3 \), and we prove that \( \pi_1(\text{Diff}(S^2 \times S^3)) \) is infinite.

The theory of primary characteristic classes on certain infinite rank bundles was treated via Chern-Weil theory in [20], and depends on the choice of a trace on the algebra of nonpositive order classical pseudodifferential operators (\( \Psi \text{DOs} \)). While these classes can be nonzero for some traces [16], the Chern classes vanish on mapping spaces Maps\((N, M)\) for the trace used in this paper. Thus we are forced to consider secondary classes.

In finite dimensions, a characteristic class on a manifold \( M \) can give rise to two types of secondary classes: (i) if a characteristic form of degree \( 2k \) vanishes for a specific connection \( \nabla \), there is an associated “absolute” Chern-Simons form \( CS(\nabla) \in H^{2k-1}(M, \mathbb{R}/\mathbb{Z}) \); (ii) if the characteristic forms are equal for two connections \( \nabla_0, \nabla_1 \), there is an associated “relative” Chern-Simons class \( CS(\nabla_0, \nabla_1) \in H^{2k-1}(M, \mathbb{R}) \). For technical reasons, it is easier to work with the relative classes in our context.

Since Chern-Weil and Chern-Simons theory are geometric, it is necessary to understand connections and curvature on loop spaces. We work with Sobolev spaces of highly differentiable loops, as this is a Hilbert manifold, although we could work with the Fréchet manifold of smooth loops. A Riemannian metric \( g \) on \( M \) induces a family of metrics \( g^s \) on \( LM \) parametrized by a Sobolev space parameter \( s \geq 0 \), where \( s = 0 \) gives the usual \( L^2 \) metric, and the smooth case is a kind of limit as \( s \to \infty \). Thus we think of \( s \) as a regularizing parameter, and pay attention to the parts of the theory which are independent of \( s \).

In Part I, we compute the connection and curvature for the Levi-Civita connection for \( g^s \) for \( s \in \mathbb{Z}^+ \). These forms take values in zeroth order pseudodifferential operators (\( \Psi \text{DOs} \)) acting on a trivial bundle over \( S^1 \), as first shown by Freed for loop groups [10]. We calculate the principal and subprincipal symbols for the connection and curvature forms. In contrast, there is no Levi-Civita connection for nonintegral values of \( s \) in a precise sense, but we can define a family of modified connections which vary continuously in \( s \).

In Part II, we develop a theory of Chern-Simons classes on loop spaces. The structure group for the Levi-Civita connection for \((LM, g^s)\) is the set of invertible zeroth order \( \Psi \text{DOs} \), so we need invariant polynomials on the corresponding Lie algebra. The naive choice is the standard polynomials \( \text{Tr}(\Omega^k) \) of the curvature \( \Omega = \Omega^s \), where \( \text{Tr} \) is the operator trace. However, \( \Omega^k \) is zeroth order and hence not trace class, and in any case the operator trace is impossible to compute in general. Instead, as in [20] we use the Wodzicki residue, the only trace on the full algebra of \( \Psi \text{DOs} \). Following Chern-Simons [6] as much as possible, we build a theory of Wodzicki-Chern-Simons (WCS)
classes, which gives classes in $H^{2k-1}(LM^{2k-1})$ associated to partitions of $k$. The main difference from the finite dimensional theory is the absence of a Narasimhan-Ramanan universal connection theorem. As a result, we do not have a theory of differential characters [4]. In contrast to the operator trace, the Wodzicki residue is locally computable, so we can write explicit expressions for the WCS classes. In particular, we can see how the WCS classes depend on the Sobolev parameter $s$, and hence define a “regularized” or $s$-independent WCS classes. The local expression also yields some vanishing results for WCS classes. More importantly, we produce a nonvanishing WCS class on $L(S^2 \times S^3)$. This leads to the topological results described in the first paragraph.

The paper is organized as follows. Part I treats the family of metrics $g^s$ on $LM$ associated to $(M, g)$. §2 discusses connections associated to $g^s$. After some preliminary material, we compute the Levi-Civita connection for $s = 0$ (Lemma 2.1), $s = 1$ (Theorem 2.2), and $s \in \mathbb{Z}^+$ (Theorem 2.10). In contrast, the Levi-Civita connection does not exist in a precise sense for nonintegral $s$ (Theorem 2.12). Nevertheless, by removing the “bad terms” from this calculation, we can construct a family of connections, called the $H^s$ connections, which depend continuously on $s$ (Proposition 2.13). These connections allow us to track how the geometry of $LM$ depends on $s$.

Both the Levi-Civita and $H^s$ connections have connection and curvature forms taking values in ΨDOs of order zero. In §3, we compute the symbols of these forms needed in Part II. In §4, we compare our results to Freed’s on loop groups [10].

Part II covers Wodzicki-Chern-Simons classes. In §5, we review the finite dimensional construction of Chern and Chern-Simons classes, and use the Wodzicki residue to define Wodzicki-Chern (WC) and WCS classes (Definition 5.1). We prove the necessary vanishing of the WC classes for mapping spaces (and in particular for $LM$) in Proposition 5.2. In Theorem 5.5, we give the explicit local expression for the relative WCS class $CS^{W}_{2k-1}(g) \in H^{2k-1}(LM^{2k-1})$ associated to the trivial partition of $k$. We then define the regularized or $s$-independent WCS class. In Theorem 5.6, we give a vanishing result for WCS classes.

In particular, the WCS class which is the analogue of the classical dimension three Chern-Simons class vanishes on loop spaces of 3-manifolds, so we look for nontrivial examples on 5-manifolds. In §6 we use a Sasaki-Einstein metric constructed in [12] to produce a nonzero WCS class $CS^{W}_5 \in H^5(L(S^2 \times S^3))$. We prove $CS^{W}_5 \neq 0$ by an exact computer calculation showing $\int C [\alpha \pi] CS^{W}_5 \neq 0$, where $[\alpha \pi] \in H_5(LM)$ is a cycle associated to a simple circle action on $S^2 \times S^3$. From this nonvanishing, we conclude both that the circle action is not smoothly homotopic to the trivial action and the $\pi_1(\text{Diff}(S^2 \times S^3))$ is infinite. We expect other similar results in the future.

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Part I. The Levi-Civita Connection on the Loop Space $LM$
In this part, we compute the Levi-Civita connection on $LM$ associated to a Riemannian metric on $M$ and a Sobolev parameter $s \in \mathbb{Z}^+$. In §2, the main results are Lemma 2.1, Theorem 2.2, and Theorem 2.10 which compute the Levi-Civita connection for $s = 0$, $s = 1$, and $s \in \mathbb{Z}^+$, respectively. For $s \not\in \mathbb{Z}^+$, the Levi-Civita connection does not exist (Theorem 2.12), but there are closely related connections (Proposition 2.13) which depend continuously on $s$. In §3, we compute the relevant symbols of the connection one-forms and the curvature two-forms. In §4, we compare our results with work of Freed [10] on loop groups.

2. The Levi-Civita Connection for Sobolev Parameter $s \geq 0$

This section covers background material and computes the Levi-Civita connection on $LM$ for integer Sobolev parameter $s$. In §2.1, we review material on $LM$, and in §2.2 we review pseudodifferential operators and the Wodzicki residue. In §2.3, we give the main computation for the Levi-Civita connections for $s = 0$, $s = 1$. In §2.4, we extend this computation to $s \in \mathbb{Z}^+$. In §2.5, we show that the Levi-Civita connection does not exist for $s \not\in \mathbb{Z}^+$. In §2.6, we modify the Levi-Civita connection for $s \in \mathbb{Z}^+$ to a torsion free connection, called the $H^s$ connection, which extends to all $s \geq 0$. In this sense, the $H^s$ connection behaves better than the Levi-Civita connection. In §2.7, we discuss how the geometry of $LM$ forces an extension of the structure group of $LM$ from a gauge group to a group of bounded invertible $\Psi$DOs.

2.1. Preliminaries on $LM$.

Let $(M, \langle \cdot, \cdot \rangle)$ be a closed oriented Riemannian $n$-manifold with loop space $LM = C^\infty(S^1, M)$ of smooth loops. $LM$ is a smooth infinite dimensional Fréchet manifold, but it is technically simpler to work with the smooth Hilbert manifold of loops in some Sobolev class $s \gg 0$, as we now recall. For $\gamma \in LM$, the formal tangent space $T_\gamma LM$ is $\Gamma(\gamma^*TM)$, the space of smooth sections of the pullback bundle $\gamma^*TM \rightarrow S^1$. For some $s > 1/2$, we complete $\Gamma(\gamma^*TM \otimes \mathbb{C})$ with respect to the Sobolev inner product

$$\langle X, Y \rangle_s = \frac{1}{2\pi} \int_0^{2\pi} \langle (1 + \Delta)^s X(\alpha), Y(\alpha) \rangle_{\gamma(\alpha)} d\alpha, \ X, Y \in \Gamma(\gamma^*TM).$$

Here $\Delta = D^*D$, with $D = D/d\gamma$ the covariant derivative along $\gamma$. (We use this notation instead of the classical $D/dt$ to keep track of $\gamma$.) We need the complexified pullback bundle $\gamma^*TM \otimes \mathbb{C}$, denoted from now on just as $\gamma^*TM$, in order to apply the pseudodifferential operator $(1 + \Delta)^s$. The construction of $(1 + \Delta)^s$ is reviewed in §2.2. We denote this completion by $H^s(\gamma^*TM)$.

A small real neighborhood $U_\gamma$ of the zero section in $H^s(\gamma^*TM)$ is a coordinate chart near $\gamma$ in the space of $H^s$ loops via the pointwise exponential map

$$\exp_\gamma : U_\gamma \rightarrow LM, \ X \mapsto \left(\alpha \mapsto \exp_{\gamma(\alpha)} X(\alpha)\right). \quad (2.1)$$

The differentiability of the transition functions $\exp_{\gamma_1} \cdot \exp_{\gamma_2}$ is proved in [7] and [11, Appendix A]. Here $\gamma_1, \gamma_2$ are close loops in the sense that a geodesically convex
neighborhood of $\gamma_1(\theta)$ contains $\gamma_2(\theta)$ and vice versa for all $\theta$. Since $\gamma^* TM$ is (non-
canonical) isomorphic to the trivial bundle $R = S^1 \times \mathbb{C}^n \to S^1$, the model space for $LM$ is the set of $H^s$ sections of this trivial bundle.

The complexified tangent bundle $TLM$ has transition functions $d(\exp_{\gamma_1}^{-1} \circ \exp_{\gamma_2})$. Under the isomorphisms $T\gamma_i LM \simeq R \simeq T\gamma_2 LM$, the transition functions lie in the gauge group $G(R)$, so this is the structure group of $TLM$.

2.2. Review of $\Psi$DO Calculus.

We recall the construction of classical pseudodifferential operators ($\Psi$DOs) on a closed manifold $M$ from \[13, 22\], assuming knowledge of $\Psi$DOs on $\mathbb{R}^n$.

A linear operator $P : C^\infty(M) \to C^\infty(M)$ is a $\Psi$DO of order $d$ if for every open chart $U \subset M$ and functions $\phi, \psi \in C^\infty_c(U)$, $\phi P \psi$ is a $\Psi$DO of order $d$ on $\mathbb{R}^n$, where we do not distinguish between $U$ and its diffeomorphic image in $\mathbb{R}^n$. Let $\{U_i\}$ be a finite cover of $M$ with subordinate partition of unity $\{\phi_i\}$. Let $\psi_i \in C^\infty_c(U_i)$ have $\psi_i \equiv 1$ on $\text{supp}(\phi_i)$ and set $P_i = \psi_i P \phi_i$. Then $\sum_i \phi_i P \psi_i$ is a $\Psi$DO on $M$, and $P$ differs from $\sum_i \phi_i P \psi_i$ by a smoothing operator, denoted $P \sim \sum_i \phi_i P \psi_i$. In particular, this sum is independent of the choices up to smoothing operators. All this carries over to $\Psi$DOs acting on sections of a bundle over $M$.

An example is the $\Psi$DO $(1 + \Delta - \lambda)^{-1}$ for $\Delta$ a positive order nonnegative elliptic $\Psi$DO and $\lambda$ outside the spectrum of $1 + \Delta$. In each $U_i$, we construct a parametrix $P_i$ for $A_i = \psi_i(1 + \Delta - \lambda)\phi_i$ by formally inverting $\sigma(A_i)$ and then constructing a $\Psi$DO with the inverted symbol. By \[11\ App. A], $B = \sum_i \phi_i P_i \psi_i$ is a parametrix for $(1 + \Delta - \lambda)^{-1}$. Since $B \sim (1 + \Delta - \lambda)^{-1}$, $(1 + \Delta - \lambda)^{-1}$ is itself a $\Psi$DO. For $x \in U_i$, by definition

$$\sigma((1 + \Delta - \lambda)^{-1})(x, \xi) = \sigma(P)(x, \xi) = \sigma(\phi P \phi)(x, \xi),$$

where $\phi$ is a bump function with $\phi(x) = 1$ [13 p. 29]; the symbol depends on the choice of $(U_i, \phi_i)$.

The operator $(1 + \Delta)^s$ for $\text{Re}(s) < 0$, which exists as a bounded operator on $L^2(M)$ by the functional calculus, is also a $\Psi$DO. To see this, we construct the putative symbol $\sigma_i(1 + \Delta)^s \phi_i$ in each $U_i$ by a contour integral $\int \lambda^s \sigma_i[(1 + \Delta - \lambda)^{-1}] d\lambda$ around the spectrum of $1 + \Delta$. We then construct a $\Psi$DO $Q_i$ on $U_i$ with $\sigma(Q_i) = \sigma_i$, and set $Q = \sum_i \phi_i Q_i \psi_i$. By arguments in \[22\], $(1 + \Delta)^s \sim Q$, so $(1 + \Delta)^s$ is a $\Psi$DO.

Recall that the Wodzicki residue of a $\Psi$DO $P$ on sections of a bundle $E \to M^n$ is

$$\text{res}^w(P) = \int_{S^*M} \text{tr } \sigma_{-n}(P)(x, \xi) d\xi dx,$$

where $S^*M$ is the unit cosphere bundle for some metric. The Wodzicki residue is independent of choice of local coordinates, and up to scaling is the unique trace on the algebra of $\Psi$DOs if $\dim(M) > 1$ (see e.g. \[9\] in general and \[21\] for the case $M = S^1$).

The Wodzicki residue will be used in Part II to define characteristic classes on $LM$. In our particular case, the operator $P$ will be an $\Psi$DO of order $-1$ acting on
sections of a bundle over $S^1$ (see [5,10]), so $\sigma_{-1}(P)$ is globally defined. Of course, 
\[ \int_{S^1} \text{tr} \sigma_{-1}(P) d\xi d\theta = 2 \int_{S^1} \text{tr} \sigma_{-1}(P) d\theta. \]
It is easy to check that this integral, which strictly speaking involves a choice of cover of $S^1$ and a partition of unity, equals the usual $2 \int_0^{2\pi} \text{tr} \sigma_{-1}(P) d\theta$.

2.3. The Levi-Civita Connection for $s = 0, 1$.

The $H^s$ metric makes $LM$ a Riemannian manifold. The corresponding Levi-Civita connection on $LM$, if it exists, is determined by the six term formula
\[ 2\langle \nabla^s_Y X, Z \rangle_s = X\langle Y, Z \rangle_s + Y\langle X, Z \rangle_s - Z\langle X, Y \rangle_s \]
\[ + \langle [X, Y], Z \rangle_s + \langle [Z, X], Y \rangle_s - \langle [Y, Z], X \rangle_s. \]

In particular, the Levi-Civita connection exists if for fixed $X, Y$, each term $T_i$ on the right hand side defines a continuous linear functional $T_i : H^s(\gamma^*TM) \to \mathbb{C}$ in $Z$. If so, $T_i(Z) = \langle T_i(X, Y), Z \rangle_s$ for a unique $T'(X, Y) \in H^s(\gamma^*TM)$, and $\nabla^s_Y X = \frac{1}{2} \sum_i T_i$.

We first discuss local coordinates on $LM$. For motivation, recall that
\[ [X, Y]^a = X(Y^a)\partial_a - Y(X^a)\partial_a \equiv \delta_X(Y) - \delta_Y(X) \]
in local coordinates on a finite dimensional manifold. Note that $X^i\partial_i Y^a = X(Y^a) = (\delta_X Y)^a$ in this notation.

Let $Y$ be a vector field on $LM$, and let $X$ be a tangent vector at $\gamma \in LM$. The local variation $\delta_X Y$ of $Y$ in the direction of $X$ at $\gamma$ is defined as usual: let $\gamma(\epsilon, \theta)$ be a family of loops in $M$ with $\gamma(0, \theta) = \gamma(\theta)$. Fix $\theta$, and let $(x^a)$ be coordinates near $\gamma(\theta)$. We call these coordinates manifold coordinates. Then
\[ \delta_X Y^a(\gamma)(\theta) \equiv \frac{d}{d\epsilon} \bigg|_{\epsilon=0} X^a(\gamma(\theta)). \]

Note that $\delta_X Y^a = (\delta_X Y)^a$ by definition.

Remark 2.1. Having $(x^a)$ defined only near a fixed $\theta$ is inconvenient. We can find coordinates that work for all $\theta$ as follows. For fixed $\gamma$, there is an $\epsilon$ such that for all $\theta$, $\exp_{\gamma(\theta)} X$ is inside the cut locus of $\gamma(\theta)$ if $X \in T_{\gamma(\theta)} M$ has $|X| < \epsilon$. Fix such an $\epsilon$. Call $X \in H^s(\gamma^*TM)$ short if $|X(\theta)| < \epsilon$ for all $\theta$. Then
\[ U_\gamma = \{ \theta \mapsto \exp_{\gamma(\theta)} X(\theta)|X \text{ is short} \} \subset LM \]
is a coordinate neighborhood of $\gamma$ parametrized by $\{ X : X \text{ is short} \}$.

We know $H^s(\gamma^*TM) \simeq H^s(S^1 \times \mathbb{R}^n)$ noncanonically, so $U_\gamma$ is parametrized by short sections of $H^s(S^1 \times \mathbb{R}^n)$ for a different $\epsilon$. In particular, we have a smooth diffeomorphism $\beta$ from $U_\gamma$ to short sections of $H^s(S^1 \times \mathbb{R}^n)$.

Put coordinates $(x^a)$ on $\mathbb{R}^n$, which we identify canonically with the fiber $\mathbb{R}^n_\theta$ over $\theta$ in $S^1 \times \mathbb{R}^n$. For $\eta \in U_\gamma$, we have $\beta(\eta) = (\beta(\eta)^1(\theta), ..., \beta(\eta)^n(\theta))$. As with finite dimensional coordinate systems, we will drop $\beta$ and just write $\eta = (\eta(\theta)^a)$. These coordinates work for all $\eta$ near $\gamma$ and for all $\theta$. The definition of $\delta_X Y$ above carries over to exponential coordinates.
We will call these coordinates \textit{exponential coordinates}. \hfill (2.4)

(2.4) continues to hold for vector fields on \(LM\), in either manifold or exponential coordinates. To see this, one checks that the coordinate-free proof that \(L_X Y (f) = [X, Y] (f)\) for \(f \in C^\infty (M)\) (e.g. [24, p. 70]) carries over to functions on \(LM\). In brief, the usual proof involves a map \(H(s, t)\) of a neighborhood of the origin in \(\mathbb{R}^2\) into \(M\), where \(s, t\) are parameters for the flows of \(X, Y\), resp. For \(LM\), we have a map \(H(s, t, \theta)\), where \(\theta\) is the loop parameter. The usual proof uses only \(s, t\) differentiations, so \(\theta\) is unaffected. The point is that the \(Y_i\) are local functions on the \((s, t, \theta)\) parameter space, whereas the \(Y^i\) are not local functions on \(M\) at points where loops cross or self-intersect.

We first compute the \(L^2\) \((s = 0)\) Levi-Civita connection invariantly and in manifold coordinates.

\textbf{Lemma 2.1.} Let \(\nabla^{LC}\) be the Levi-Civita connection on \(M\). Let \(ev_\theta : LM \rightarrow M\) be \(ev_\theta (\gamma) = \gamma(\theta)\). Then \(D_X Y (\gamma)(\theta) \overset{\text{def}}{=} (ev_\theta^* \nabla^{LC})_X Y (\gamma)(\theta)\) is the \(L^2\) Levi-Civita connection on \(LM\). In manifold coordinates,

\[(D_X Y)^a (\gamma)(\theta) = \delta_X Y^a (\gamma)(\theta) + \Gamma^a_{bc} (\gamma(\theta)) X^b (\gamma)(\theta) Y^c (\gamma)(\theta).\] \hfill (2.5)

\textit{Proof.} \(ev_\theta^* \nabla^{LC}\) is a connection on \(ev_\theta^* TM \rightarrow LM\). We have \(ev_{\theta, X}(X) = X(\theta)\). If \(U\) is a coordinate neighborhood on \(M\) near some \(\gamma(\theta)\), then on \(ev_\theta^{-1}(U)\),

\[
(ev_\theta^* \nabla^{LC})_X Y^a (\gamma)(\theta) = (\delta_X Y)^a (\gamma)(\theta) + ((ev_\theta^* \omega^{LC}_X) Y)^a (\theta) = (\delta_X Y^a (\gamma)(\theta) + \Gamma^a_{bc} (\gamma(\theta)) X^b (\gamma)(\theta) Y^c (\gamma)(\theta).\]

Since \(ev_\theta^* \nabla^{LC}\) is a connection, for each fixed \(\theta, \gamma\) and \(X \in T_\gamma LM\), \(Y \mapsto (ev_\theta^* \nabla^{LC})_X Y (\gamma)\) has Leibniz rule with respect to functions on \(LM\). Thus \(D\) is a connection on \(LM\).

\(D\) is torsion free, as from the local expression, \(D_X Y - D_Y X = \delta_X Y - \delta_Y X = [X, Y]\).

To show that \(D_X Y\) is compatible with the \(L^2\) metric, first recall that for a function \(f\) on \(LM\), \(D_X f = \delta_X f = \frac{d}{d\varepsilon}|_{\varepsilon=0} f(\gamma(\varepsilon, \theta))\) for \(X(\theta) = \frac{d}{d\varepsilon}|_{\varepsilon=0} \gamma(\varepsilon, \theta)\). (Here \(f\) depends
only on $\gamma$. Thus (suppressing the partition of unity, which is independent of $\varepsilon$)

\[
D_X\langle X, Y \rangle_0 = \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \int_{S^1} g_{ab}(\gamma(\varepsilon, \theta)) Y^a(\gamma(\varepsilon, \theta)) Z^b(\gamma(\varepsilon, \theta)) d\theta \\
= \int_{S^1} \partial_c g_{ab}(\gamma(\varepsilon, \theta)) X^c Y^a(\gamma(\varepsilon, \theta)) Z^b(\gamma(\varepsilon, \theta)) d\theta \\
+ \int_{S^1} g_{ab}(\gamma(\varepsilon, \theta))(\delta_X Y)^a(\gamma(\varepsilon, \theta)) Z^b(\gamma(\varepsilon, \theta)) d\theta \\
+ \int_{S^1} g_{ab}(\gamma(\varepsilon, \theta)) Y^a(\gamma(\varepsilon, \theta))(\delta_X Z)^b(\gamma(\varepsilon, \theta)) d\theta \\
= \int_{S^1} \Gamma^e_{ca} g_{eb} X^c Y^a Z^b + \Gamma^e_{cb} g_{ae} X^c Y^a Z^b \\
+ g_{ab}(\delta_X Y)^a Z^b + g_{ab}(\delta_X Z)^b d\theta \\
= \langle D_X Y, Z \rangle_0 + \langle Y, D_X Z \rangle_0.
\]

\[\square\]

**Remark 2.2.** The local expression for $D_X Y$ also holds in exponential coordinates. More precisely, let $(e_1(\theta), \ldots, e_n(\theta))$ be a global frame of $\gamma^*TM$ given by the trivialization of $\gamma^*TM$. Then $(e_i(\theta))$ is also naturally a frame of $T_X T_{\gamma(\theta)}M$ for all $X \in T_{\gamma(\theta)}M$. We use $\exp_{\gamma(\theta)}$ to pull back the metric on $M$ to a metric on $T_{\gamma(\theta)}M$:

$$ g_{ij}(X) = (\exp_{\gamma(\theta)}^* g)(e_i, e_j) = g(d(\exp_{\gamma(\theta)} x(e_i)), d(\exp_{\gamma(\theta)} x(e_j)))_{\exp_{\gamma(\theta)} x}. $$

Then the Christoffel symbols $\Gamma^a_{bc}(\gamma(\theta))$ are computed with respect to this metric. For example, the term $\partial_c g_{bc}$ means $e_c g(e_a, e_b)$, etc. The proof that $D_X Y$ has the local expression (2.5) then carries over to exponential coordinates.

The $s = 1$ Levi-Civita connection on $LM$ is given as follows.

**Theorem 2.2.** The $s = 1$ Levi-Civita connection $\nabla = \nabla^1$ on $LM$ is given at the loop $\gamma$ by

$$ \nabla_X Y = D_X Y + \frac{1}{2}(1 + \Delta)^{-1}[-\nabla_{\dot{\gamma}}(R(X, \dot{\gamma})Y) - R(X, \dot{\gamma})\nabla_{\dot{\gamma}} Y \\
- \nabla_{\dot{\gamma}}(R(Y, \dot{\gamma})X) - R(Y, \dot{\gamma})\nabla_{\dot{\gamma}} X \\
+ R(X, \nabla_{\dot{\gamma}} Y)\dot{\gamma} - R(\nabla_{\dot{\gamma}} X, Y)\dot{\gamma}]. $$

We prove this in a series of steps.

**Proposition 2.3.** The Levi-Civita connection for the $s = 1$ metric is given by

$$ \nabla_X^1 Y = D_X Y + \frac{1}{2}(1 + \Delta)^{-1}[D_X, 1 + \Delta] Y + \frac{1}{2}(1 + \Delta)^{-1}[D_Y, 1 + \Delta] X + A_X Y, $$
where we assume that $A_X Y$ is well-defined by
\[-\frac{1}{2} ([D_Z, 1 + \Delta] X, Y)_0 = \langle A_X Y, Z \rangle_1.\]

**Proof.** By Lemma 2.1,
\[
X(Y, Z)_1 = X((1 + \Delta) Y, Z)_0 = \langle D_X ((1 + \Delta) Y), Z \rangle_0 + \langle (1 + \Delta) Y, D_X Z \rangle_0
\]
\[
Y(X, Z)_1 = \langle D_Y ((1 + \Delta) X), Z \rangle_0 + \langle (1 + \Delta) X, D_Y Z \rangle_0
\]
\[
-Z(X, Y)_1 = -\langle D_Z ((1 + \Delta) X), Y \rangle_0 - \langle (1 + \Delta) X, D_Z Y \rangle_0
\]
\[
\langle [X, Y], Z \rangle_1 = (1 + \Delta) (\delta_X Y - \delta_Y X), Z \rangle_0 = (1 + \Delta) (D_X Y - D_Y X), Z \rangle_0
\]
\[
\langle [Z, X], Y \rangle_1 = (1 + \Delta) (D_Z X - D_X Z), Y \rangle_0
\]
\[
-\langle [Y, Z], X \rangle_1 = -((1 + \Delta) (D_Y Z - D_Z Y), X \rangle_0.
\]

The six terms on the left hand side sum up to twice the $s = 1$ Levi-Civita connection $\nabla = \nabla^1$. After some cancellations, we get
\[
2\langle \nabla X Y, Z \rangle_1 = \langle D_X ((1 + \Delta) Y), Z \rangle_0 + \langle D_Y ((1 + \Delta) X), Z \rangle_0
\]
\[
+ (1 + \Delta) (D_X Y - D_Y X), Z \rangle_0 - \langle D_Z ((1 + \Delta) X), Y \rangle_0
\]
\[
+ (1 + \Delta) D_Z X, Y \rangle_0
\]
\[
= \langle (1 + \Delta) D_X Y, Z \rangle_0 + \langle [D_X, 1 + \Delta] Y, Z \rangle_0
\]
\[
+ (1 + \Delta) D_Y X, Z \rangle_0 + \langle [D_Y, 1 + \Delta] X, Z \rangle_0
\]
\[
+ (1 + \Delta) (D_X Y - D_Y X), Z \rangle_0 - \langle [D_Z, 1 + \Delta] X, Y \rangle_0
\]
\[
= 2\langle D_X Y, Z \rangle_1 + (1 + \Delta)^{-1} [D_X, 1 + \Delta] Y, Z \rangle_1
\]
\[
+ (1 + \Delta)^{-1} [D_Y, 1 + \Delta] X, Z \rangle_1 - \langle A_X Y, Z \rangle_1.
\]
\]

Now we compute the bracket terms in the Proposition. We have $[D_X, 1 + \Delta] = [D_X, \Delta]$. Also,
\[
0 = \dot{\gamma} (X, Y)_0 = \langle \nabla \dot{\gamma} X, Y \rangle_0 + \langle X, \nabla \dot{\gamma} Y \rangle_0,
\]
so
\[
\Delta = \nabla^* \nabla \dot{\gamma} = -\nabla^2 \dot{\gamma}.
\]

**Lemma 2.4.** $[D_X, \nabla \dot{\gamma}] Y = R(X, \dot{\gamma}) Y$.

**Proof.** Note that $\gamma^\nu, \dot{\gamma}^\nu$ are locally defined functions on $S^1 \times LM$. Let $\tilde{\gamma} : [0, 2\pi] \times (-\varepsilon, \varepsilon) \to M$ be a smooth map with $\tilde{\gamma}(\theta, 0) = \gamma(\theta)$, and $\frac{\partial}{\partial \tau}|_{\tau = 0} \tilde{\gamma}(\theta, \tau) = Z(\theta)$. Since $(\theta, \tau)$ are coordinate functions on $S^1 \times (-\varepsilon, \varepsilon)$, we have
\[
Z(\dot{\gamma}^\nu) = \delta_Z (\dot{\gamma}^\nu) = \delta_Z^2 (\dot{\gamma}^\nu) = \frac{\partial}{\partial \tau} \left|_{\tau = 0} \left( \frac{\partial}{\partial \theta} (\dot{\gamma}(\theta, \tau)^\nu) \right) \right.
\]
\[
= \frac{\partial}{\partial \theta} \frac{\partial}{\partial \tau} \left|_{\tau = 0} \dot{\gamma}(\theta, \tau)^\nu = \partial_\theta Z^\nu \equiv \dot{Z}^\nu.
\]

We compute
\[(D_X \nabla_{\dot{\gamma}} Y)^a = \delta_X (\nabla_{\dot{\gamma}} Y)^a + \Gamma_{bc}^a X^b \nabla_{\dot{\gamma}} Y^c \]
\[= \delta_X (\dot{\gamma}^j \partial_j Y^a + \Gamma_{bc}^a \dot{\gamma}^b Y^c) + \Gamma_{bc}^a X^b (\dot{\gamma}^j \partial_j Y^c + \Gamma_{ef}^c \dot{\gamma}^e Y^f) \]
\[= \dot{X}^j \partial_j Y^a + \dot{\gamma}^j \partial_j \delta_X Y^a + \partial_M \Gamma_{bc}^a X^m \dot{\gamma}^b Y^c + \Gamma_{bc}^a \dot{X}^b Y^c + \Gamma_{bc}^a \dot{\gamma}^b \delta_X Y^c + \Gamma_{bc}^a \Gamma_{ef}^c X^b \dot{\gamma}^e Y^f. \]
\[(\nabla_{\dot{\gamma}} D_X Y)^a = \dot{\gamma}^j (\partial_j (D_X Y)^a + \Gamma_{bc}^a \dot{\gamma}^b (D_X Y)^c) \]
\[= \dot{\gamma}^j \partial_j (\delta_X Y^a + \Gamma_{bc}^a X^b Y^c) + \Gamma_{bc}^a \dot{\gamma}^b (\delta_X Y^c + \Gamma_{ef}^c X^e Y^f) \]
\[= \dot{\gamma}^j \partial_j \delta_X Y^a + \dot{\gamma}^j \partial_j \Gamma_{bc}^a X^b Y^c + \Gamma_{bc}^a \dot{X}^b Y^c + \Gamma_{bc}^a \dot{\gamma}^b \delta_X Y^c + \Gamma_{bc}^a \Gamma_{ef}^c \dot{\gamma}^b X^e Y^f. \]
Therefore
\[(D_X \nabla_{\dot{\gamma}} Y - \nabla_{\dot{\gamma}} D_X Y)^a = \partial_m \Gamma_{bc}^a X^m \dot{\gamma}^b Y^c - \partial_j \Gamma_{bc}^a \dot{\gamma}^j X^b Y^c + \partial_m \Gamma_{bc}^a X^m \dot{\gamma}^b Y^c + \Gamma_{bc}^a \Gamma_{ef}^c \dot{\gamma}^b X^e Y^f \]
\[= (\partial_j \Gamma_{bc}^a - \partial_b \Gamma_{jc}^a + \Gamma_{jc}^e \Gamma_{be}^e) \dot{\gamma}^b X^j Y^c + R_{jbc}^a X^j \dot{\gamma}^b Y^c, \]
so
\[D_X \nabla_{\dot{\gamma}} Y - \nabla_{\dot{\gamma}} D_X Y = R(X, \dot{\gamma} Y). \]
\[\square \]

**Corollary 2.5.** At the loop $\gamma$, $[D_X, \Delta] Y = -\nabla_{\dot{\gamma}} (R(X, \dot{\gamma}) Y) - R(X, \dot{\gamma}) \nabla_{\dot{\gamma}} Y$. In particular, $[D_X, \Delta]$ is a zeroth order operator.

**Proof.**
\[[D_X, \Delta] Y = (D_X \nabla_{\dot{\gamma}} Y + \nabla_{\dot{\gamma}} D_X Y) \]
\[= -\nabla_{\dot{\gamma}} (D_X \nabla_{\dot{\gamma}} Y + R(X, \dot{\gamma}) \nabla_{\dot{\gamma}} Y) + \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} D_X Y \]
\[= -\nabla_{\dot{\gamma}} (R(X, \dot{\gamma}) Y) - R(X, \dot{\gamma}) \nabla_{\dot{\gamma}} Y. \]
\[\square \]

Now we complete the proof of Theorem 2.2. In the proof, we justify that $A_X Y$ in Proposition 2.3 exists.

**Proof of Theorem 2.2.** By Proposition 2.3 and Corollary 2.5, we have
\[\nabla_X Y = D_X Y + \frac{1}{2} (1 + \Delta)^{-1} [D_X, 1 + \Delta] Y + (X \leftrightarrow Y) + A_X Y \]
\[= D_X Y + \frac{1}{2} (1 + \Delta)^{-1} (-\nabla_{\dot{\gamma}} (R(X, \dot{\gamma}) Y) - R(X, \dot{\gamma}) \nabla_{\dot{\gamma}} Y) + (X \leftrightarrow Y) + A_X Y, \]
where $(X \leftrightarrow Y)$ denotes the previous term with $X$ and $Y$ switched.
The curvature tensor satisfies
\[-\langle Z, R(X,Y)W \rangle = \langle R(X,Y)Z, W \rangle = \langle R(Z,W)X, Y \rangle\]
pointwise, so
\[
\langle A_X Y, Z \rangle_1 = -\frac{1}{2} \langle [D_Z, 1 + \Delta]X, Y \rangle_0 = -\frac{1}{2} \langle (-\nabla_{\dot{\gamma}} (R(Z, \dot{\gamma}) X) - R(Z, \dot{\gamma}) \nabla_{\dot{\gamma}} X, Y) \rangle_0
\]
\[= -\frac{1}{2} \langle R(Z, \dot{\gamma}) X, \nabla_{\dot{\gamma}} Y \rangle_0 + \frac{1}{2} \langle R(Z, \dot{\gamma}) \nabla_{\dot{\gamma}} X, Y \rangle_0
\]
\[= -\frac{1}{2} \langle R(X, \nabla_{\dot{\gamma}} Y) Z, \dot{\gamma} \rangle_0 + \frac{1}{2} \langle R(\nabla_{\dot{\gamma}} X, Y), Z, \dot{\gamma} \rangle_0
\]
\[= \frac{1}{2} \langle Z, R(X, \nabla_{\dot{\gamma}} Y) \dot{\gamma} \rangle_0 - \frac{1}{2} \langle Z, R(\nabla_{\dot{\gamma}} X, Y) \dot{\gamma} \rangle_0
\]
\[= \frac{1}{2} \langle Z, (1 + \Delta)^{-1} (R(X, \nabla_{\dot{\gamma}} Y) \dot{\gamma} - R(\nabla_{\dot{\gamma}} X, Y) \dot{\gamma}) \rangle_1.
\]
Thus \(A_X Y = \frac{1}{2} (1 + \Delta)^{-1} (R(X, \nabla_{\dot{\gamma}} Y) \dot{\gamma} - R(\nabla_{\dot{\gamma}} X, Y) \dot{\gamma})\). For \(X, Y\) smooth and in \(H^1 = H^1(\gamma^* TM)\), \(A_X Y \in H^2 \subset H^1\), since \(R\) is zeroth order. It follows that \(A_X Y\) takes \(H^1\) to \(H^1\) continuously as a map on either \(X\) or \(Y\). Thus taking the \(H^1\) inner product with \(A_X Y\) is a continuous linear functional on \(H^1\).

**Remark 2.3.** Locally on \(LM\), we should have \(D_X Y = \delta_X^L M Y + \omega_X^L M (Y)\). Now \(\delta_X^L M Y\) can only mean \(\frac{d}{d\epsilon} \bigg|_{\epsilon=0} \frac{d}{d\tau} \bigg|_{\tau=0}\gamma(\epsilon, \tau, \theta)\), where \(\gamma(0, 0, \theta) = \gamma(\theta)\), \(d\epsilon|_{\epsilon=0} \gamma(\epsilon, 0, \theta) = X(\theta)\), \(d\tau|_{\tau=0} \gamma(\epsilon, \tau, \theta) = Y_{\gamma(\epsilon, 0, \cdot)}(\theta)\). In other words, \(\delta_X^L M Y\) equals \(\delta_X Y\). Since \(D_X Y^a = \delta_X Y^a + \Gamma^a_{bc}(\gamma(\theta))\), the connection one-form for the \(L^2\) Levi-Civita connection on \(LM\) is given by
\[
\omega_X^L M (Y)^a (\gamma(\theta)) = \Gamma^a_{bc}(\gamma(\theta)) X^b Y^c = \omega_X^M (Y)^a (\gamma(\theta)).
\]
By this remark, we get

**Corollary 2.6.** The connection one-form \(\omega = \omega^1\) for \(\nabla^1\) in exponential coordinates is
\[
\omega_X (Y)(\gamma)(\theta) = \omega_X^M (Y)(\gamma(\theta)) + \frac{1}{2} \left\{ (1 + \Delta)^{-1} [-\nabla_{\dot{\gamma}} (R(X, \dot{\gamma}) Y) - R(X, \dot{\gamma}) \nabla_{\dot{\gamma}} Y
\right.
\]
\[-\nabla_{\dot{\gamma}} (R(Y, \dot{\gamma}) X) - R(Y, \dot{\gamma}) \nabla_{\dot{\gamma}} X
\]
\[+ R(X, \nabla_{\dot{\gamma}} Y) \dot{\gamma} - R(\nabla_{\dot{\gamma}} X, Y) \dot{\gamma} \right\}(\theta). \tag{2.8}
\]

### 2.4. The Levi-Civita Connection for \(s \in \mathbb{Z}^+\)

For \(s > 0\), the proof of Prop. 2.3 extends directly to give

**Lemma 2.7.** The Levi-Civita connection for the \(H^s\) metric is given by
\[
\nabla_X^s Y = D_X Y + \frac{1}{2} (1 + \Delta)^{-s} [D_X, (1 + \Delta)^s] Y + \frac{1}{2} (1 + \Delta)^{-s} [D_Y, (1 + \Delta)^s] X + A_X Y,
\]
where
\[
-\frac{1}{2}\langle [D_Z, (1 + \Delta)^s]X, Y \rangle_0 = \langle A_X Y, Z \rangle_s, \tag{2.9}
\]
provided that for each \(X, Y \in H^s\), \(Z \mapsto -\frac{1}{2}\langle [D_Z, (1 + \Delta)^s]X, Y \rangle_0\) is a continuous linear functional on \(H^s\).

We now compute the bracket terms.

**Lemma 2.8.** For \(s \in \mathbb{Z}^+\), at the loop \(\gamma\),
\[
[D_X, (1 + \Delta)^s]Y = \sum_{k=1}^{s} (-1)^k \left( \sum_{j=0}^{2k-1} \nabla^j_\gamma (R(X, \dot{\gamma}) \nabla^{2k-1-j}_\gamma Y) \right). \tag{2.10}
\]

**Proof.** The sum over \(k\) comes from the binomial expansion of \((1 + \Delta)^s\), so we just need an inductive formula for \([D_X, \Delta^s]\). The case \(s = 1\) is Proposition 2.3. For the induction step, we have
\[
[D_X, \Delta^s] = D_X \Delta^{s-1} \Delta - \Delta^s D_X \\
= \Delta^{s-1} D_X \Delta + [D_X, \Delta^{s-1}] \Delta - \Delta^s D_X \\
= \Delta^s D_X + \Delta^{s-1} [D_X, \Delta] + [D_X, \Delta^{s-1}] \Delta - \Delta^s D_X \\
= \Delta^{s-1} (-\nabla^1_\gamma (R(X, \dot{\gamma}) Y) - R(X, \dot{\gamma}) \nabla Y) \\
- \sum_{j=0}^{2s-3} (-1)^{s-1} \nabla^j_\gamma (R(X, \dot{\gamma}) \nabla^{2k-1-j}_\gamma Y) \\
= (-1)^{s-1} (-\nabla^{2s-1}_\gamma (R(X, \dot{\gamma}) Y) - (-1)^{s-1} \nabla^{2s-2}_\gamma (R(X, \dot{\gamma}) \nabla Y) \\
+ \sum_{j=0}^{2s-3} (-1)^{s} \nabla^j_\gamma (R(X, \dot{\gamma}) \nabla^{2k-1-j}_\gamma Y).
\]

We now show that \(A_X Y\) exists for \(s \in \mathbb{Z}^+\).

**Lemma 2.9.** For \(s \in \mathbb{Z}^+\) and fixed \(X, Y \in H^s\), the left hand side of (2.9) is a continuous linear map from \(Z \in H^s\) to \(H^s\). Thus \(A_X Y \in H^s\) is well defined.
Proof. For smooth $X, Y \in H^s$ and $j, 2k - 1 - j \in \{0, 1, \ldots, 2s - 1\}$, a typical term in
(2.10) is
\[
\langle \nabla^j_{\hat{\gamma}}(R(Z, \dot{\gamma})\nabla^{2k-1-j}_\gamma X, \nabla^j Y) \rangle_0 = (-1)^j \langle R(Z, \dot{\gamma})\nabla^{2k-1-j}_\gamma X, \nabla^j Y \rangle_0
\]
\[
= (-1)^j \int_{S^1} g_{\ell} R(Z, \dot{\gamma})\nabla^{2k-1-j}_\gamma X (\nabla^j Y)^\ell d\theta
\]
\[
= (-1)^j \int_{S^1} g_{\ell} Z^k R_{k\ell m} \nabla^{2k-1-j}_\gamma X (\nabla^j Y)^m d\theta
\]
\[
= (-1)^j \int_{S^1} g_{\ell} Z^k R_{k\ell m} \nabla^{2k-1-j}_\gamma X (\nabla^j Y)^m d\theta
\]
\[
= (-1)^j \int_{S^1} g_{\ell} Z^k R_{k\ell m} \nabla^{2k-1-j}_\gamma X (\nabla^j Y)^m d\theta
\]
\[
= (-1)^j \langle Z, R^{i\ell r}_{k\ell m} \nabla^{2k-1-j}_\gamma X (\nabla^j Y)^\ell \partial_t \rangle_0
\]
\[
= (-1)^j \langle Z, R^{i\ell r}_{k\ell m} \nabla^{2k-1-j}_\gamma X (\nabla^j Y)^\ell \partial_t \rangle_0
\]
\[
= (-1)^j \langle Z, (1 + \Delta)^{-s} R(\nabla^{2k-1-j}_\gamma X, \nabla^j Y) \rangle_0
\]
\[
= (-1)^j \langle Z, (1 + \Delta)^{-s} R(\nabla^{2k-1-j}_\gamma X, \nabla^j Y) \rangle_0
\]
\[
(\text{In the integrals and inner products, the local expressions are in fact globally defined}
\]
\[
\text{one-forms on } S^1, \text{ resp. vector fields along } \gamma, \text{ so we do not need a partition of unity.)}
\]
\[
(1 + \Delta)^{-s} R(\nabla^{2k-1-j}_\gamma X, \nabla^j Y) \dot{\gamma} \text{ is of order at least one in either } X \text{ or } Y, \text{ so this term}
\]
\[
\text{is in } H^{s+1} \subset H^s. \text{ Thus the last inner product is well defined.}
\]

By (2.9), (2.10), we get
\[
A_X Y = \sum_{k=1}^{s} (-1)^k \left( \begin{array}{c} s \\ k \end{array} \right) \sum_{j=0}^{2k-1} (-1)^j \langle (1 + \Delta)^{-s} R(\nabla^{2k-1-j}_\gamma X, \nabla^j Y) \dot{\gamma} \rangle.
\]

This gives:

**Theorem 2.10.** For $s \in \mathbb{Z}^+$, the Levi-Civita connection for the $H^s$ metric at the loop
\[
\gamma \text{ is given by}
\]
\[
\nabla^X Y(\gamma) = D_X Y(\gamma) + \frac{1}{2} (1 + \Delta)^{-s} \sum_{k=1}^{s} (-1)^k \left( \begin{array}{c} s \\ k \end{array} \right) \sum_{j=0}^{2k-1} \nabla^j_{\hat{\gamma}} (R(X, \dot{\gamma}) \nabla^{2k-1-j}_\gamma Y)
\]
\[
+ (X \leftrightarrow Y)
\]
\[
\sum_{k=1}^{s} (-1)^k \left( \begin{array}{c} s \\ k \end{array} \right) \sum_{j=0}^{2k-1} (-1)^j \langle (1 + \Delta)^{-s} R(\nabla^{2k-1-j}_\gamma X, \nabla^j Y) \dot{\gamma} \rangle.
\]

2.5. **The Levi-Civita Connection for** $s \in \mathbb{R}^+ \setminus \mathbb{Z}^+$.

In this subsection, we show that the results for $s \in \mathbb{Z}^+$ do not carry over to nonintegral $s$. 
By Lemma 2.7, we have to examine if \( \langle [D_Z, (1 + \Delta)^s]X, Y \rangle_0 \) is a continuous linear functional on \( Z \in H^s \).

**Lemma 2.11.** (i) For \( \text{Re}(s) \neq 0 \), \( [D_Z, (1 + \Delta)^s]X \) is a \( \Psi DO \) of order \( 2s - 1 \) in \( X \).

(ii) \( \sigma_{2s-k}([D_Z, (1 + \Delta)^s]) \) has \( k \) \( \theta \)-derivatives of \( Z \).

**Proof.** (i) For \( f : LM \rightarrow \mathbb{C} \), we get \([D_Z, (1 + \Delta)^s]fX = f[D_Z, (1 + \Delta)^s]X\), since \([f, (1 + \Delta)^s] = 0\). Therefore, \([D_Z, (1 + \Delta)^s]X\) depends only on \( X |_{\gamma} \).

By Lemma 2.1, \( D_Z = \delta_Z + \Gamma \cdot Z \) in shorthand exponential coordinates. The Christoffel symbol term is zeroth order and \((1 + \Delta)^s\) has scalar leading order symbol, so \([\Gamma \cdot Z, (1 + \Delta)^s]\) has order \( 2s - 1 \).

From the integral expression for \((1 + \Delta)^s\), it is immediate that

\[
[D_Z, (1 + \Delta)^s]X = (\delta_Z(1 + \Delta)^s)X + (1 + \Delta)^s \delta_Z X - (1 + \Delta)^s \delta_Z X = (\delta_Z(1 + \Delta)^s)X.
\]

\( \delta_Z(1 + \Delta)^s \) is a limit of differences of \( \Psi DOs \) on bundles isomorphic to \( \gamma^*TM \). Since the algebra of \( \Psi DOs \) is closed in the Fréchet topology of all \( C^k \) seminorms of symbols and smoothing terms on compact sets, \( \delta_Z(1 + \Delta)^s \) is a \( \Psi DO \).

Assume \( \text{Re}(s) < 0 \). As in the construction of \((1 + \Delta)^s\), we will compute what the symbol asymptotics of \( \delta_Z(1 + \Delta)^s \) should be, and then construct an operator with these asymptotics. From the functional calculus for unbounded operators, we have

\[
\delta_Z(1 + \Delta)^s = \delta_Z \left( \frac{i}{2\pi} \int_{\Gamma} \lambda^s (1 + \Delta - \lambda)^{-1} d\lambda \right)
\]

\[
= \frac{i}{2\pi} \int_{\Gamma} \lambda^s \delta_Z (1 + \Delta - \lambda)^{-1} d\lambda
\]

\[
= -\frac{i}{2\pi} \int_{\Gamma} \lambda^s (1 + \Delta - \lambda)^{-1} (\delta_Z \Delta)(1 + \Delta - \lambda)^{-1} d\lambda,
\]

where \( \Gamma \) is a contour around the spectrum of \( 1 + \Delta \), and the hypothesis on \( s \) justifies the exchange of \( \delta_Z \) and the integral. The operator \( A = (1 + \Delta - \lambda)^{-1} \delta_Z \Delta(1 + \Delta - \lambda)^{-1} \) is a \( \Psi DO \) of order \(-3\) with top order symbol

\[
\sigma_{-3}(A)(\theta, \xi) = (\xi^2 - \lambda)^{-\frac{1}{2}} \delta^\ell_k(2Z^i \partial_i \Gamma^{\mu k}_\nu \dot{\gamma}^\nu - 2\Gamma^{\mu k}_\nu \dot{Z}^\nu)\xi(\xi^2 - \lambda)^{-\frac{1}{2}} \delta^\mu_j
\]

Thus the top order symbol of \( \delta_Z(1 + \Delta)^s \) should be

\[
\sigma_{2s-1}(\delta_Z(1 + \Delta)^s)(\theta, \xi) = \frac{-i}{2\pi} \int_{\Gamma} \lambda^s (-2Z^i \partial_i \Gamma^{\mu \ell}_\nu \dot{\gamma}^\nu - 2\Gamma^{\mu \ell}_\nu \dot{Z}^\nu)\xi(\xi^2 - \lambda)^{-\frac{1}{2}} d\lambda
\]

\[
= \frac{i}{2\pi} \int_{\Gamma} s \lambda^s (2Z^i \partial_i \Gamma^{\mu \ell}_\nu \dot{\gamma}^\nu - 2\Gamma^{\mu \ell}_\nu \dot{Z}^\nu)\xi(\xi^2 - \lambda)^{-\frac{1}{2}} d\lambda
\]

\[
= s(-2Z^i \partial_i \Gamma^{\mu \ell}_\nu \dot{\gamma}^\nu - 2\Gamma^{\mu \ell}_\nu \dot{Z}^\nu)\xi(\xi^2 - \lambda)^{s-1}.
\]
Similarly, all the terms in the symbol asymptotics for $A$ are of the form $B_j^i \xi^n(\xi^2 - \lambda)^m$ for some matrices $B_j^i = B_j^i(n, m)$. This produces a symbol sequence $\sum_{k \in \mathbb{Z}^+} \sigma_{2s-k}$, and there exists a ΨDO $P$ with $\sigma(P) = \sum \sigma_{2s-k}$. (As in [2.2], we first produce operators $P_i$ on a coordinate cover $U_i$ of $S^1$, and then set $P = \sum_i \phi_i P_i \psi_i$.) The construction depends on the choice of local coordinates covering $\gamma$, the partition of unity and cutoff functions as above, and a cutoff function in $\xi$; as usual, different choices change the operator by a smoothing operator. Standard estimates show that $P - \delta_Z(1 + \Delta)^s$ is a smoothing operator, so $\delta_Z(1 + \Delta)^s$ is a ΨDO of order $2s - 1$.

For $\text{Re}(s) > 0$, motivated by differentiating $(1 + \Delta)^{-s} \circ (1 + \Delta)^s = \text{Id}$, we set
\[
\delta_Z(1 + \Delta)^s = -(1 + \Delta)^{s} \circ \delta_Z(1 + \Delta)^{-s} \circ (1 + \Delta)^s.
\]
This is again a ΨDO of order $2s - 1$.

(ii) The covariant derivative along $\gamma$ on $Y \in \Gamma(\gamma^* TM)$ is given by
\[
\frac{DY}{d\gamma} = (\gamma^* \nabla M)_{\partial_b}(Y) = \partial_b Y + (\gamma^* \omega^M)(\partial_\theta)(Y)
\]
where $\nabla^M$ is the Levi-Civita connection on $M$ and $\omega^M$ is the connection one-form in exponential coordinates on $M$. For $\Delta = (\frac{D}{d\gamma})^s \frac{D}{d\gamma}$, an integration by parts using the formula $\partial_t g_{\text{ar}} = \Gamma_{\ell t} g_{\text{rn}} + \Gamma_{\ell r} g_{\text{tn}}$ gives
\[
(\Delta Y)^k = -\partial^2_k Y^k - 2\Gamma^k_{\nu \mu} \dot{\gamma}^\nu \partial_\nu Y^\mu - (\partial_\theta \Gamma^k_{\nu \delta} \dot{\gamma}^\nu + \Gamma^k_{\nu \delta} \dot{\gamma}^\nu + \Gamma^k_{\nu \mu} \Gamma^\mu_{\varepsilon \delta} \dot{\gamma}^\varepsilon \dot{\gamma}^\mu) Y^\delta.
\]
Therefore, by (2.7)
\[
((\delta_Z \Delta) Y)^k = (-2Z^i \partial_i \Gamma^k_{\nu \mu} \dot{\gamma}^\nu - 2\Gamma^k_{\nu \mu} \dot{Z}^\nu) Y^\mu + \left(\dot{Z}^i \partial_i \Gamma^k_{\nu \delta} \dot{\gamma}^\nu + Z^i \partial_i \Gamma^k_{\nu \delta} \dot{\gamma}^\nu + \Gamma^k_{\nu \delta} \dot{\gamma}^\nu \right)
\]
\[
+ Z^i \partial_i \Gamma^k_{\nu \delta} \dot{\gamma}^\nu + \Gamma^k_{\nu \delta} \dot{\gamma}^\nu + Z^i \partial_i \Gamma^k_{\nu \mu} \Gamma^\mu_{\varepsilon \delta} \dot{\gamma}^\varepsilon \dot{\gamma}^\mu + \Gamma^k_{\nu \mu} Z^i \partial_i \Gamma^\mu_{\varepsilon \delta} \dot{\gamma}^\varepsilon \dot{\gamma}^\mu
\]
\[
+ \Gamma^k_{\nu \mu} \Gamma^\mu_{\varepsilon \delta} \dot{\gamma}^\varepsilon \dot{\gamma}^\mu + \Gamma^k_{\nu \mu} \Gamma^\mu_{\varepsilon \delta} \dot{\gamma}^\varepsilon \dot{\gamma}^\mu \right) Y^\delta.
\]
Thus $((\delta_Z \Delta) Y)^k$ is a second order differential operators in $Z$ and a first order differential operator in $Y$. It follows from (2.12) that $\delta_Z(1 + \Delta)^s Y$ is a second order differential operator in $Z$. As in [2.13], a symbol calculus calculation shows that $\sigma_{-k}(A)$ has $k - 2$ derivatives of $Z$. It follows that for $\text{Re}(s) < 0$, $\sigma_{2s-k}(\delta_Z(1 + \Delta)^s)$ has at most $k$ derivatives of $Z$, and the same result then holds for $\text{Re}(s) > 0$.

\[\square\]

**Remark 2.4.** (i) For $s \in \mathbb{Z}^+$, $\delta_Z(1 + \Delta)^s$ differs from the usual definition by a smoothing operator.

(ii) For all $s$, the proof shows that $\sigma(\delta_Z(1 + \Delta)^s) = \delta_Z(\sigma((1 + \Delta)^s))$.

We now determine if there exists $A_X Y \in H^s$ such that $-\frac{1}{2}([D_Z, (1 + \Delta)^s]X, Y)_0 = (A_X Y, Z)_s$ for each $X, Y, Z \in H^s$. As above, we write $D_Z = \delta_Z + \Gamma \cdot \dot{Z}$, and consider each term separately.
For the $\Gamma \cdot Z$ term, we have

$$\langle [\Gamma \cdot Z, (1 + \Delta)^s]X, Y \rangle_0 = \int_{S^1} g_{ab}(\Gamma \cdot Z(1 + \Delta)^s X)^a Y^b - \int_{S^1} g_{ab}((1 + \Delta)^s \Gamma \cdot Z \cdot X)^a Y^b,$$

where we have omitted the partition of unity. The first term on the right hand side of (2.15) equals

$$\int_{S^1} g_{ab} (\Gamma_{dc} Z^d (1 + \Delta)^s X)^a Y^b = \int_{S^1} g_{de} Z^d g_{ab} g^{ef} \Gamma^a_{fc} ((1 + \Delta)^s X)^c Y^b$$

$$= \langle Z, g_{ab} g^{ef} \Gamma^a_{fc} ((1 + \Delta)^s X)^c Y^b \partial_e \rangle_0$$

$$= \langle Z, (1 + \Delta)^{-s} [g_{ab} g^{ef} \Gamma^a_{fc} ((1 + \Delta)^s X)^c Y^b \partial_e] \rangle_s.$$

Since the components of $(1 + \Delta)^s X$ and $Y$ are in $H^{-s}_{\text{loc}}$ and $H^s_{\text{loc}}$ respectively, the product $((1 + \Delta)^s X)^c Y^b$ is in $H^{-s}_{\text{loc}}$. Thus $(1 + \Delta)^{-s} [g_{ab} g^{ef} \Gamma^a_{fc} ((1 + \Delta)^s X)^c Y^b \partial_e] \in H^s$, so the inner products in (2.16) are well defined. A similar argument shows that the second term on the right hand side of (2.15) can be rewritten as an $H^s$ inner product with $Z$.

We now consider the term $[\delta_Z, (1 + \Delta)^s]X = \delta_Z(1 + \Delta)^s X$ (by (2.11)). By Lemma 2.11 $\delta_Z(1 + \Delta)^s$ is a $\Psi$DO with explicitly computable symbol $\sigma \sim \sum_{k \in \mathbb{Z},} \sigma_{2s-k}$, and $\sigma_{2s-k}$ contains at most $k$ derivatives of $Z$. Fix $\ell$ and let $P = P_{Z,\ell}$ be a $\Psi$DO with symbol $\sum_{k=1}^{\ell} \sigma_{2s-k}$. Here $P = \sum_i \phi_i P_i \psi_i$ with respect to some cover $\{(a_i, b_i)\}$ of $S^1$. $Q = \delta_Z(1 + \Delta)^s - P$ is an operator of order at most $2s - \ell - 1$. Suppressing the $i$ dependence of the symbols, we have

$$\int_{S^1} g_{ab} \delta_Z (1 + \Delta)^s X^a Y^b$$

$$= \int_{S^1} g_{ab} P(X)^a Y^b + \int_{S^1} g_{ab} Q(X)^a Y^b$$

$$= \sum_i \int_{(a_i, b_i)} \phi_i g^{(i)}_{ab} \left( \sum_j \int_{T^*(a_i, b_i)} e^{i(\theta - \theta') \cdot \xi} \phi_j(\theta) \cdot  \sum_{k=1}^{\ell} (\sigma_{2s-k})_c^a(\theta, \xi) \psi_j(\theta') X^c(\theta') d\xi d\theta ' \right) Y^b d\theta$$

$$+ \sum_i \int_{(a_i, b_i)} \phi_i g^{(i)}_{ab} (QX)^a Y^b.$$
θ-derivatives act on φ, replace this term with φ/φ, noting that this function extends by zero to \( \{ \phi_i = 0 \} \). This gives the expression

\[
\sum_i \int_{(a_i, b_i)} \phi_i \frac{\phi^{(k)}}{\phi_i} g_{ab} \left( \sum_j \int_{T^*(a_j, b_j)} e^{i(\theta - \theta') \cdot \xi} Z^d A_{de}^a(\theta, \xi) \phi_j(\theta) \psi_j(\theta') X^c(\theta') d\xi d\theta' \right) Y^b d\theta
\]

\[
= \sum_i \int_{(a_i, b_i)} \phi_i \frac{\phi^{(k)}}{\phi_i} g_{de} g_{er} g_{ab} \left( \sum_j \int_{T^*(a_j, b_j)} e^{i(\theta - \theta') \cdot \xi} Z^d A_{rc}^a(\theta, \xi) \phi_j(\theta) \psi_j(\theta') X^c(\theta') d\xi d\theta' \right) Y^b d\theta,
\]

which locally is the \( L^2 \) inner product of \( Z = Z^d \partial_d \) with

\[
\frac{\phi^{(k)}}{\phi_i} g_{er} g_{ab} \left( \sum_j \int_{T^*(a_j, b_j)} e^{i(\theta - \theta') \cdot \xi} A_{rc}^a(\theta, \xi) \phi_j(\theta) \psi_j(\theta') X^c(\theta') d\xi d\theta' \right) Y^b \partial_c.
\]

This is the (local) \( H^s \) inner product of \( Z \) with

\[
(1 + \Delta)^{-s} \left[ \frac{\phi^{(k)}}{\phi_i} g_{er} g_{ab} \left( \sum_j \int_{T^*(a_j, b_j)} e^{i(\theta - \theta') \cdot \xi} A_{rc}^a(\theta, \xi) \phi_j(\theta) \psi_j(\theta') X^c(\theta') d\xi d\theta' \right) Y^b \partial_c \right].
\]

This term is a \( \Psi DO \) of order \( -2s \) as an operator on \( Y \).

There are similar expressions if \( k \) \( \theta \)-derivatives act on \( g_{ab} \), on \( e^{i\theta \cdot \xi} \), or on \( A_{de}^a(\theta, \xi) \phi_j(\theta) \), and in each case we get terms which are \( \Psi DOs \) of order \( -2s \) as an operator on \( Y \). Finally, if \( k \) \( \theta \)-derivatives act on \( Y^b \), we get the local \( H^s \) inner product of \( Z \) with

\[
(1 + \Delta)^{-s} \left[ g_{er} g_{ab} \left( \sum_j \int_{T^*(a_j, b_j)} e^{i(\theta - \theta') \cdot \xi} A_{rc}^a(\theta, \xi) \phi_j(\theta) \psi_j(\theta') X^c(\theta') d\xi d\theta' \right) Y^b \partial_c \right],
\]

which is of order \( -2s + k \) as an operator on \( Y \).

In general, each of the \( k \) integrations by parts will act as in one of the previous cases, so the final result can be written as an \( H^s \) inner product with \( Z \).

In summary, for fixed \( \ell \), \( A_X Y \), if it exists, is the sum of two operators, \((1 + \Delta)^{-s} P'_{X, \ell} \) and \((1 + \Delta)^{-s} Q'_{X, \ell} \). Here \((1 + \Delta)^{-s} P'_{X, \ell} \) is of order at least \(-2s + \ell \) and defined by

\[
\int_{S^1} g_{ab} P_{Z, \ell}(X)^a Y^b = \int_{S^1} g_{ab} P'_{X, \ell}(Y)^a Z^b = \langle (1 + \Delta)^{-s} P'_{X, \ell}(Y), Z \rangle_s. \tag{2.18}
\]

By a similar argument, \((1 + \Delta)^{-s} Q'_{X, \ell} \) is defined by

\[
\int_{S^1} g_{ab} Q_{Z, \ell}(X)^a Y^b = \int_{S^1} g_{ab} Q'_{X, \ell}(Y)^a Z^b = \langle (1 + \Delta)^{-s} Q'_{X, \ell}(Y), Z \rangle_s. \tag{2.19}
\]
Proof. For are torsion free connections on $TLM$ produces new connections provided the right hand side exists. In particular, $A$ for all $\gamma$.

Definition 2.1.\[\text{Theorem 2.12.}\]

For general, so the right hand side of (2.20) is not defined. However, for $\ell > 2s + 1$ and $Y \in H^s$, $(1 + \Delta)^{-s}(P' + Q')X,\ell(Y) \notin H^s$ in general, so the right hand side of (2.20) is not defined.

This proves that there is no Levi-Civita connection in this case. To be precise, let $H^s(TLM)$ consist of the smooth sections $Y = Y_\gamma$ of $TLM$ such that $Y_\gamma \in H^s(\gamma^*TM)$ for all $\gamma$.

Theorem 2.12. For $s \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, there is no Levi-Civita connection for the $H^s$ metric on $LM$, in the sense that there exists no map $\nabla^s : H^s(\gamma^*TM) \times H^s(TLM) \rightarrow H^s(TLM)$, $(X,Y) \mapsto \nabla^s_X Y$, which is linear over $C^\infty(LM)$ in $X$ and a derivation in $Y$, and such that

\[
2\langle \nabla^s_X Y, Z \rangle_s = X\langle Y, Z \rangle_s + Y\langle X, Z \rangle_s - Z\langle X, Y \rangle_s + \langle [X, Y], Z \rangle_s + \langle [Z, X], Y \rangle_s - \langle [Y, Z], X \rangle_s.
\]

2.6. Connections on LM that Vary Continuously in s.

We now find a family of connections on $LM$ that depend continuously on $s$. By the previous subsection, this family cannot consist of Levi-Civita connections for the $H^s$ metric, but they can be used to construct Chern-Simons invariants in Part II.

Lemma 2.13. The family of operators

\[
\nabla^s_X Y = D_X Y + \frac{1}{2}(1 + \Delta)^{-s}[D_X, (1 + \Delta)^s]Y + \frac{1}{2}(1 + \Delta)^{-s}[D_Y, (1 + \Delta)^s]X
\]

are torsion free connections on $TLM$ for each $s > 0$.

$\nabla^{1,s}$ is just the connection in Lemma 2.7 with the $A_X Y$ term dropped.

Proof. For $f \in C^\infty(LM)$, $(1 + \Delta)^{-s}[D_X, (1 + \Delta)^s]Y = f(1 + \Delta)^{-s}[D_X, (1 + \Delta)^s]Y$, since $(1 + \Delta)^{-s}f = f(1 + \Delta)^{-s}$ for fixed $\gamma$. Also, $(1 + \Delta)^{-s}[D_X, (1 + \Delta)^s]fY = f(1 + \Delta)^{-s}[D_X, (1 + \Delta)^s]Y$, since $(1 + \Delta)^{-s}X(f) = X(f)(1 + \Delta)^{-s}$. Thus $(X,Y) \mapsto (1 + \Delta)^{-s}[D_X, (1 + \Delta)^s]Y$ is in $\Lambda^1(LM, \text{End}(TLM))$, as is $(1 + \Delta)^{-s}[D_Y, (1 + \Delta)^s]X$. $D_X Y$ is a connection on $TLM$, so adding these elements of $\Lambda^1(LM, \text{End}(TLM))$ produces new connections $\nabla^{1,s}$.

Since $\frac{1}{2}(1 + \Delta)^{-s}[D_X, (1 + \Delta)^s]Y + \frac{1}{2}(1 + \Delta)^{-s}[D_Y, (1 + \Delta)^s]X$ is symmetric in $X$ and $Y$, we get

\[
\nabla^{1,s}_X Y - \nabla^{1,s}_Y X = D_X Y - D_Y X = [X, Y].
\]

Definition 2.1. $\nabla^{1,s}$ is called the $H^s$ connection.
2.7. Extensions of the Frame Bundle of LM. In this subsection we discuss the choice of structure group for the $H^s$ and Levi-Civita connections on $LM$.

Let $\mathcal{H}$ be the Hilbert space $H^{s_0}(\gamma^*TM)$ for a fixed $s_0$ and $\gamma$. Let $GL(\mathcal{H})$ be the group of bounded invertible linear operators on $\mathcal{H}$; inverses of elements are bounded by the closed graph theorem. $GL(\mathcal{H})$ has the subset topology of the norm topology on $B(\mathcal{H})$, the bounded linear operators on $\mathcal{H}$. $GL(\mathcal{H})$ is an infinite dimensional Banach Lie group, as a group which is an open subset of the infinite dimensional Hilbert manifold $B(\mathcal{H})$ [19, p. 59], and has Lie algebra $B(\mathcal{H})$. Let $\Psi DO^{\leq 0}$, $\Psi DO^*_0$ denote the algebra of classical $\Psi DO$s of nonpositive order and the group of invertible zeroth order $\Psi DO$s, respectively, where all $\Psi DO$s act on $\mathcal{H}$. Note that $\Psi DO^*_0 \subset GL(\mathcal{H})$.

Remark 2.5. The inclusions of $\Psi DO^*_0$, $\Psi DO^{\leq 0}$ into $GL(\mathcal{H}), B(\mathcal{H})$ are trivially continuous in the subset topology. For the Fréchet topology on $\Psi DO^{\leq 0}$, the inclusion is continuous as in [16].

We recall the relationship between the connection one-form $\theta$ on the frame bundle $FN$ of a manifold $N$ and local expressions for the connection on $TN$. For $U \subset N$, let $\chi : U \to FN$ be a local section. A metric connection $\nabla$ on $TN$ with local connection one-form $\omega$ determines a connection $\theta$ on each fiber, and (i) $\theta_{FN}$ is the Maurer-Cartan one-form on each fiber, and (ii) $\theta_{FN}(Y_u) = \omega(X_p)$, for $Y_u = \chi^*X_p$ [23, Ch. 8, Vol. II], or equivalently $\chi^*\theta_{FN} = \omega$.

This applies to $N = LM$. The frame bundle $FLM \to LM$ is constructed as in the finite dimensional case. The fiber over $\gamma$ is isomorphic to the gauge group $G$ of $\mathcal{R}$ and fibers are glued by the transition functions for $TLM$. Thus the frame bundle is topologically a $G$-bundle.

However, by Corollaries 2.5 and 2.6 the $H^s$ connection one-form

\[
\omega^1,s_X Y = \omega^M_X Y + \frac{1}{2} (1 + \Delta)^{-s} [D_X, (1 + \Delta)^s] Y + \frac{1}{2} (1 + \Delta)^{-s} [D_Y, (1 + \Delta)^s] X
\]

takes values in $\Psi DO^{\leq 0}$. The curvature two-form $\Omega^1,s = d_{LM} \omega^1,s + \omega^1,s \wedge \omega^1,s$ also takes values in $\Psi DO^{\leq 0}$. (Here $d_{LM} \omega^1,s(X,Y)$ is defined by the Cartan formula for the exterior derivative.) These forms should take values in the Lie algebra of the structure group. Thus we should extend the structure group to the Fréchet Lie group $\Psi DO^*_0$, since its Lie algebra is $\Psi DO^{\leq 0}$. This leads to an extended frame bundles, also denoted $FLM$. The transition functions are unchanged, since $G \subset \Psi DO^*_0$. Thus $(FLM, \theta^s)$ as a geometric bundle (i.e. as a bundle with connection $\theta^s$ associated to $\nabla^1,s$) is a $\Psi DO^*_0$-bundle. Similar remarks hold for the Levi-Civita connection if $s \in \mathbb{Z}^+$. In summary, for the $H^s$ connections we have

\[
\begin{array}{ccc}
\mathcal{G} & \to & FLM \\
\downarrow & & \downarrow \\
LM & & LM
\end{array}
\quad \quad \quad
\begin{array}{ccc}
\Psi DO^*_0 & \to & (FLM, \theta^s) \\
\downarrow & & \downarrow \\
LM & & LM
\end{array}
\]
Remark 2.6. If we extend the structure group of the frame bundle with connection from $\Psi DO^*_\theta$ to $GL(\mathcal{H})$, the frame bundle becomes trivial by Kuiper’s theorem. Thus there is a potential loss of information if we pass to the larger frame bundle.

The situation is similar to the following examples. Let $E \to S^1$ be the $GL(1, \mathbb{R})$ (real line) bundle with gluing functions (multiplication by) $1$ at $1 \in S^1$ and $2$ at $-1 \in S^1$. $E$ is trivial as a $GL(1, \mathbb{R})$-bundle, with global section $f$ with $\lim_{\theta \to -\pi^+} f(e^{i\theta}) = 1$, $f(1) = 1$, $\lim_{\theta \to -\pi^-} f(e^{i\theta}) = 1/2$. However, as a $GL(1, \mathbb{Q})^+$-bundle, $E$ is nontrivial, as a global section is locally constant. As a second example, let $E \to M$ be a nontrivial $GL(n, \mathbb{C})$-bundle. Embed $\mathbb{C}^n$ into a Hilbert space $\mathcal{H}$, and extend $E$ to an $GL(\mathcal{H})$-bundle $\mathcal{E}$ with fiber $\mathcal{H}$ and with the same transition functions. Then $\mathcal{E}$ is trivial.

3. Local Symbol Calculations

In this section, we compute the $0$ and $-1$ order symbols of the connection one-form and the curvature two-form of the $s=1$ Levi-Civita connection. We also compute the $0$ and $-1$ order symbols of the connection one-form for the $H^s$ connection, and the $0$ order symbol of the curvature for the $H^s$ connection. These results are used in the calculations of Wodzicki-Chern-Simons classes in §6. The formulas show that the $s$-dependence of these symbols is linear, which will be used to define regularized Wodzicki-Chern-Simons classes (see Definition 5.2).

3.1. Connection and Curvature Symbols for $s=1$.

Using Corollary 2.6, we can compute these symbols easily.

Lemma 3.1. (i) At $\gamma(\theta)$, $\sigma_0(\omega_X)_b^a = (\omega_X^M)_b^a = \Gamma^a_{cb} X^c$.

(ii) $\frac{1}{i|\xi|^{-2}} \sigma_{-1}(\omega_X) = \frac{1}{2}(-2R(X, \dot{\gamma}) - R(\cdot, \dot{\gamma})X + R(X, \cdot)\dot{\gamma}).$ 

Equivalently,

$\frac{1}{i|\xi|^{-2}} \sigma_{-1}(\omega_X)_b^a = \frac{1}{2}(-2R_{cda}^a - R_{bdc}^a + R_{cbd}^a)X^c\gamma^d.$

Proof. (i) For $\sigma_0(\omega_X)$, the only term in (2.8) of order zero is the Christoffel term.

(ii) For $\sigma_{-1}(\omega_X)$, label the last six terms on the right hand side of (2.8) by (a), ..., (f). By Leibniz rule for the tensors, the only terms of order $-1$ come from: in (a), $-\nabla_\gamma (R(X, \dot{\gamma})Y) = -R(X, \dot{\gamma})\nabla_\gamma Y +$ lower order in $Y$; in (b), the term $-R(X, \dot{\gamma})\nabla_\gamma Y$;

in (c), the term $-R(\nabla_\gamma Y, \dot{\gamma})X$; in (e), the term $R(X, \nabla_\gamma Y)\dot{\gamma}$.

For any vectors $Z, W$, the curvature endomorphism $R(Z, W) : TM \to TM$ has

$R(Z, W)_b^a = R_{cda}^a Z^c W^d.$

Also, since $(\nabla_\gamma Y)^a = \frac{d}{d\sigma} Y^a$ plus zeroth order terms, $\sigma_1(\nabla_\gamma) = i\xi \cdot Id$. Thus in (a) and (b), $\sigma_1(-R(X, \dot{\gamma})\nabla_\gamma)_b^a = -R_{cda}^a X^c\gamma^d.$
For (c), we have \(-R(\nabla_X Y, \gamma)X = -R_{cdb}^a (\nabla_X Y) c \xi^d X^b \partial_a\), so the top order symbol is \(-R_{cdb}^a \xi^d X^b = -R_{bdc}^a \xi^d X^c\).

For (e), we have \(R(X, \nabla_Y Y) \gamma = R_{cdb}^a X^c (\nabla_Y Y)^d \xi^b \partial_a\), so the top order symbol is \(R_{cdb}^a X^c \xi^b = R_{cub}^a X^c \xi^d\).

Since the top order symbol of \((1 + \Delta)^{-1}\) is \(|\xi|^{-2}\), adding these four terms finishes the proof. 

We now compute the top symbols of the curvature tensor. \(\sigma_1(\Omega)\) involves the covariant derivative of the curvature tensor on \(M\), but fortunately this symbol will not be needed in Part II.

**Lemma 3.2.** (i) \(\sigma_0(\Omega(X, Y))_b^a = R^M(X, Y)_b^a = R_{cdb}^a X^c Y^d\).

(ii) \[
\frac{1}{i|\xi|^{-2}} \sigma_1(\Omega(X, Y))_b^a = \frac{1}{2} (\nabla_X [-2R(Y, \gamma) - R(\cdot, \gamma)Y + R(Y, \cdot)\gamma])
- (X \leftrightarrow Y)
- [-2R([X, Y], \gamma) - R(\cdot, \gamma)[X, Y] + R([X, \cdot] \gamma)]
\]

Equivalently, in exponential normal coordinates on \(M\) centered at \(\gamma(\theta)\),

\[
\frac{1}{i|\xi|^{-2}} \sigma_1(\Omega(X, Y))_b^a = \frac{1}{2} X([-2R_{cub}^a - R_{bdc}^a + R_{cub}^a \gamma^d] Y^c - (X \leftrightarrow Y)
= \frac{1}{2} X([-2R_{cub}^a - R_{bdc}^a + R_{cub}^a \gamma^d] Y^c - (X \leftrightarrow Y)
+ \frac{1}{2} [-2R_{cub}^a - R_{bdc}^a + R_{cub}^a] X^d Y^c - (X \leftrightarrow Y)
\]  

**Proof.** (i) \[
\sigma_0(\Omega(X, Y))_b^a = \frac{1}{2} \sigma_0(d\omega + \omega \wedge \omega)(X, Y)_0^a
= [(d\sigma_0(\omega) + \sigma_0(\omega) \wedge \sigma_0(\omega))(X, Y)]_b^a
= [(d\omega^M + \omega^M \wedge \omega^M)(X, Y)]_b^a
= R^M(X, Y)_b^a = R_{cub}^a X^c Y^d.
\]

(ii) Since \(\sigma_0(\omega_X)\) is independent of \(\xi\), after dividing by \(i|\xi|^{-2}\) we have \[
\sigma_{-1}(\Omega(X, Y))_b^a = (d\sigma_{-1}(\omega)(X, Y))_b^a + \sigma_0(\omega_X)_c^a \sigma_{-1}(\omega_Y)_b^c + \sigma_{-1}(\omega_X)_c^a \sigma_0(\omega_Y)_b^c
- \sigma_0(\omega_Y)_c^a \sigma_{-1}(\omega_X)_b^c + \sigma_{-1}(\omega_Y)_c^a \sigma_0(\omega_X)_b^c.
\]

As an operator on sections of \(\gamma^* TM\), \(\Omega^L - \Omega^M\) has order \(-1\) so \(\sigma_{-1}(\Omega^L) = \sigma_{-1}(\Omega^M)\) is independent of coordinates. In normal coordinates at a fixed
point \( \gamma(\theta) \), \( \sigma_0(\omega_X) = \sigma_0(\omega_Y) = 0 \), so

\[
\sigma_{-1}(\Omega(X,Y))_b^a = (d\sigma_{-1}(\omega)(X,Y))_b^a
\]

\[
= X(\sigma_{-1}(\omega_X))_b^a - Y(\sigma_{-1}(\omega_X))_b^a - \sigma_{-1}(\omega_{[X,Y]})_b^a
\]

\[
= \frac{1}{2} X[(-2R_{cda}) - R_{bdc} + R_{cbd}] Y^c \dot{\gamma}^d - (X \leftrightarrow Y)
\]

\[
- \frac{1}{2} (-2R_{cda} - R_{bdc} + R_{cbd}) [X,Y]^c \dot{\gamma}^d.
\]

The terms involving \( X(Y^c) - Y(X^c) - [X,Y]^c \) cancel (as they must, since the symbol two-form cannot involve derivatives of \( X \) or \( Y \)). Thus

\[
\sigma_{-1}(\Omega(X,Y))_b^a = \frac{1}{2} X[(-2R_{cda} - R_{bdc} + R_{cbd}) Y^c \dot{\gamma}^d] - (X \leftrightarrow Y).
\]

This gives the first coordinate expression in (3.1). The second expression follows from \( X(\dot{\gamma}^d) = \dot{X}^d \) (see (2.7)).

To convert from the coordinate expression to the covariant expression, we follow the usual procedure of changing ordinary derivatives to covariant derivatives and adding bracket terms. For example,

\[
\nabla_X(R(Y, \dot{\gamma})) = (\nabla_X R)(Y, \dot{\gamma}) + R(\nabla_X Y, \dot{\gamma}) + R(Y, \nabla_X \dot{\gamma})
\]

\[
= X^i R_{cda} Y^c \dot{\gamma}^d + R(\nabla_X Y, \dot{\gamma}) + R_{cda} Y^c (\nabla_X \dot{\gamma})^d.
\]

In exponential normal coordinates at \( \gamma(\theta) \), we have \( X^i R_{cda} = X^i \partial_i R_{cda} = X(R_{cda}) \) and \( (\nabla_X \dot{\gamma})^d = X(\dot{\gamma}^d) \). Thus

\[
\nabla_X(R(Y, \dot{\gamma})) - (X \leftrightarrow Y) - R([X,Y], \dot{\gamma}) = X(R_{cda} \dot{\gamma}^d) Y^c - (X \leftrightarrow Y).
\]

The other terms are handled similarly. \( \Box \)

3.2. Connection and Curvature Symbols for general \( s \).

The noteworthy feature of these computations is the linear dependence of \( \sigma_{-1}(\omega^{1,s}) \) on \( s \).

Let \( g \) be the Riemannian metric on \( M \).

**Lemma 3.3.** (i) At \( \gamma(\theta) \), \( \sigma_0(\omega^{1,s}_X)_b^a = (\omega^M_X)_b^a = \Gamma^a_{cb} X^c. \)

(ii) \( \sigma_0(\Omega^{1,s}(X,Y))_b^a = R^M(X,Y)_b^a = R_{cda} X^c Y^d. \)

(iii) \( \frac{1}{i!} \sigma_{-1}(\omega^{1,s}_X)_b^a = sT(X, \dot{\gamma}, g), \) where \( T(X, \dot{\gamma}, g) \) is tensorial and independent of \( s \).

**Proof.** (i) By Lemma 2.11, the only term of order zero in Lemma 2.13 is \( \omega^M_X \).

(ii) The proof of Lemma 3.2(ii) carries over.

(iii) By Lemma 2.13, we have to compute \( \sigma_{-1} \) for \( \frac{1}{2} [D_X, (1 + \Delta)^s] \) and for \( \frac{1}{2} [D_X, (1 + \Delta)^s] X \).

Write \( D_X = \delta_X + \Gamma \cdot X \) in shorthand. Since \( (1 + \Delta)^s \) has scalar leading order symbol, \([\Gamma \cdot X, (1 + \Delta)^s] \) has order \( 2s - 1 \). Thus we can compute \( \sigma_{2s-1}([\Gamma \cdot X, (1 + \Delta)^s]) \) in...
any coordinate system. For fixed $\theta$, at the center point $\gamma(\theta)$ of exponential normal coordinates, the Christoffel symbols vanish. Thus $\sigma_{-1}(\frac{1}{2}[\Gamma \cdot X, (1 + \Delta)^s]) = 0$.

By (2.11), (2.13), $\sigma_{-1}(\frac{1}{2}[\delta \cdot X, (1 + \Delta)^s])$ is $s$ times a tensorial expression in $X, \dot{g}, \gamma, g$, since

$$\partial_i \Gamma_{\ell j} = \frac{1}{3} \left( R_{ij\ell} + R_{ij\ell} \right) \text{ in normal coordinates.}$$

Thus $\sigma_{-1}(\frac{1}{2}(1 + \Delta)^{-s}[\delta \cdot X, (1 + \Delta)^s])$ is $s$ times this tensorial expression.

The argument for $\sigma_{-1}(\frac{1}{2}(1 + \Delta)^{-s}[\delta \cdot X, (1 + \Delta)^s])$ is similar. The term with $\Gamma$ vanishes, and the term $[\delta \cdot (1 + \Delta)^s]$ is treated by (2.14) and the functional calculus as in (2.13).

4. The Loop Group Case

In this section, we relate our work to Freed’s work on based loop groups $\Omega G \ [10]$. We find a particular representation of the loop algebra that controls the order of the curvature of the $H^1$ metric on $\Omega G$.

$\Omega G \subset LG$ with base point e.g. $e \in G$ has tangent space $T_0 \Omega G = \{X \in T_0 LG : X(0) = X(2\pi) = 0\}$ in some Sobolev topology. Instead of using $D^2/d\theta^2$ to define the Sobolev spaces, the usual choice is $\Delta_{S^1} = -d^2/d\theta^2$ coupled to the identity operator on the Lie algebra $g$. Since this operator has no kernel on $T_0 \Omega M, 1 + \Delta$ is replaced by $\Delta$. These changes in the $H^s$ inner product do not alter the spaces of Sobolev sections, but they do change the Levi-Civita connection. In any case, for $X, Y, Z$ left invariant vector fields, the first three terms on the right hand side of (2.3) vanish. Under the standing assumption that $G$ has a left invariant, Ad-invariant inner product, one obtains

$$2\nabla^{(s)} X = [X, Y] + \Delta^{-s} [X, \Delta^s Y] + \Delta^{-s} [Y, \Delta^s X] \ [10].$$

It is an interesting question to compute the order of the curvature operator as a function of $s$. For based loops, Freed proved that this order is at most $-1$. In [15], it is shown that the order of $\Omega^s$ is at most $-2$ for all $s \neq 1/2, 1$ on both $\Omega G$ and $LG$, and is exactly $-2$ for $G$ nonabelian. For the case $s = 1$, we have a much stronger result.

**Proposition 4.1.** The curvature of the Levi-Civita connection for the $H^1$ inner product on $\Omega G$ associated to $-\frac{d^2}{d\theta^2} \otimes \operatorname{Id}$ is a $\Psi DO$ of order $-\infty$.

**Proof:** We give two proofs.

By [10], the $H^1$ curvature operator $\Omega = \Omega^1$ satisfies

$$\langle \Omega(X, Y) Z, W \rangle_1 = \left( \int_{S^1} [Y, \dot{Z}], \int_{S^1} [X, \dot{W}] \right)_g - \langle X \leftrightarrow Y \rangle,$$

where the inner product is the Ad-invariant form on the Lie algebra $g$. We want to write the right hand side of this equation as an $H^1$ inner product with $W$, in order to recognize $\Omega(X, Y)$ as a $\Psi DO$. 

Let \( \{e_i\} \) be an orthonormal basis of \( \mathfrak{g} \), considered as a left-invariant frame of \( TG \) and as global sections of \( \gamma^*TG \). Let \( c^k_{ij} = ([e_i, e_j], e_k)_\theta \) be the structure constants of \( \mathfrak{g} \). (The Levi-Civita connection on left invariant vector fields for the left invariant metric is given by \( \nabla_X Y = \frac{1}{2}[X, Y] \), so the structure constants are twice the Christoffel symbols.) For \( X = X^i e_i = X^i(\theta) e_i, Y = Y^j e_j \), etc., integration by parts gives

\[
\langle \Omega(X, Y)Z, W \rangle_1 = \left( \int_{S^1} \dot{Y}^i Z^j d\theta \right) \left( \int_{S^1} \dot{X}^\ell W^m d\theta \right) c^k_{ij} c^n_{\ell m} \delta_{kn} - (X \leftrightarrow Y).
\]

Since

\[
\int_{S^1} c^n_{\ell m} \dot{X}^\ell W^m = \int_{S^1} \left( \delta^{mc}_{\ell c} \dot{X}^\ell e_m, W^b e_b \right)_\theta = \left\langle \Delta^{-1}(\delta^{mc}_{\ell c} \dot{X}^\ell e_m), W \right\rangle_1,
\]

we get

\[
\langle \Omega(X, Y)Z, W \rangle_1 = \left\langle \left[ \int_{S^1} \dot{Y}^i Z^j \right] c^k_{ij} \delta_{kn} \delta^{ms} c^n_{\ell s} \Delta^{-1}(\dot{X}^\ell e_m), W \right\rangle_1 - (X \leftrightarrow Y)
\]

\[
= \left\langle \left[ \int_{S^1} a^k_j(\theta, \theta') Z^j(\theta') d\theta' \right] e_k, W \right\rangle_1,
\]

with

\[
a^k_j(\theta, \theta') = \dot{Y}^i(\theta') c^r_{ij} \delta_{rn} \delta^{ms} c^n_{\ell s} \left( \Delta^{-1}(\dot{X}^\ell e_m) \right)^k(\theta) - (X \leftrightarrow Y). \tag{4.1}
\]

We now show that \( Z \mapsto \left( \int_{S^1} a^k_j(\theta, \theta') Z^j(\theta') d\theta' \right) e_k \) is a smoothing operator. Applying Fourier transform and Fourier inversion to \( Z^j \) yields

\[
\int_{S^1} a^k_j(\theta, \theta') Z^j(\theta') d\theta' = \int_{S^1 \times \mathbb{R} \times S^1} a^k_j(\theta, \theta') e^{i(\theta - \theta') \cdot \xi} Z^j(\theta') d\theta' d\xi d\theta' = \int_{S^1 \times \mathbb{R} \times S^1} \left[ a^k_j(\theta, \theta') e^{-i(\theta - \theta') \cdot \xi} \right] e^{i(\theta - \theta') \cdot \xi} Z^j(\theta') d\theta' d\xi d\theta',
\]

so \( \Omega(X, Y) \) is a \( \Psi \text{DO} \) with symbol

\[
b^k_j(\theta, \xi) = \int_{S^1} a^k_j(\theta, \theta') e^{i(\theta - \theta') \cdot \xi} d\theta', \tag{4.2}
\]

with the usual mixing of local and global notation.

For fixed \( \theta \), (4.2) contains the Fourier transform of \( \dot{Y}^i(\theta') \) and \( \dot{X}^i(\theta') \), as these are the only \( \theta' \)-dependent terms in (4.1). Since the Fourier transform is taken in a local chart with respect to a partition of unity, and since in each chart \( \dot{Y}^i \) and \( \dot{X}^i \) times the partition of unity function is compactly supported, the Fourier transform of \( a^k_j \) in each chart is rapidly decreasing. Thus \( b^k_j(\theta, \xi) \) is the product of a rapidly decreasing function with \( e^{i\theta \cdot \xi} \), and hence is of order \(-\infty\).

We now give a second proof. For all \( s \),

\[
\nabla_X Y = \frac{1}{2}[X, Y] - \frac{1}{2}\Delta^{-s}[\Delta^s X, Y] + \frac{1}{2}\Delta^{-s}[X, \Delta^s Y].
\]
Thus the symbol of (1) is $\sigma((1)) = \frac{1}{2}X^\varepsilon e^{\mu}_\varepsilon$. Abbreviating $(\xi^2)^{-s}$ by $\xi^{-2s}$, we have

\[
\sigma((2)) = \frac{1}{2} c^a_{\varepsilon\mu} \left[ \xi^{-2s} \Delta^s X^\varepsilon - \frac{2s}{\ell} \xi^{-2s-1} \partial_\theta \Delta^s X^\varepsilon \right] 
+ \sum_{\ell=2}^\infty \frac{(-2s)(2s-1)\ldots(2s-\ell+1)}{i^\ell \ell!} \xi^{-2s-\ell} \partial_\theta^\ell \Delta^s X^\varepsilon
\]

\[
= \frac{1}{2} c^a_{\varepsilon\mu} \left[ \xi^{-2s} \Delta^s X^\varepsilon + \sum_{\ell=1}^\infty \frac{(-2s)(2s-1)\ldots(2s-\ell+1)}{i^\ell \ell!} \xi^{-\ell} \partial_\theta^\ell X^\varepsilon \right].
\]

Thus

\[
\sigma(\nabla_X)^a = \frac{1}{2} c^a_{\varepsilon\mu} \left[ 2X^\varepsilon - \xi^{-2s} \Delta^s X^\varepsilon + \frac{2s}{\ell} \xi^{-2s-1} \partial_\theta \Delta^s X^\varepsilon \right]
- \sum_{\ell=2}^\infty \frac{(-2s)(2s-1)\ldots(2s-\ell+1)}{i^\ell \ell!} \xi^{-2s-\ell} \partial_\theta^\ell \Delta^s X^\varepsilon
+ \sum_{\ell=1}^\infty \frac{(-2s)(2s-1)\ldots(2s-\ell+1)}{i^\ell \ell!} \xi^{-\ell} \partial_\theta^\ell X^\varepsilon.
\]

(4.3)

Set $s = 1$ in (4.3), and replace $\ell$ by $\ell - 2$ in the first infinite sum. Since $\Delta = -\partial_\theta^2$, a little algebra gives

\[
\sigma(\nabla_X)^a = c^a_{\varepsilon\mu} \sum_{\ell=0}^\infty \frac{(-1)^\ell}{i^\ell} \partial_\theta^\ell X^\varepsilon \xi^{-\ell} = \text{ad} \left( \sum_{\ell=0}^\infty \frac{(-1)^\ell}{i^\ell} \partial_\theta^\ell X \xi^{-\ell} \right). \tag{4.4}
\]

Denote the infinite sum in the last term of (4.4) by $W(X, \theta, \xi)$. The map $X \mapsto W(X, \theta, \xi)$ takes the Lie algebra of left invariant vector fields on $LG$ to the Lie algebra $\mathfrak{g}[[\xi^{-1}]]$, the space of formal $\Psi$DOs of nonpositive integer order on the trivial bundle $S^1 \times \mathfrak{g} \to S^1$, where the Lie bracket on the target involves multiplication of power series and bracketing in $\mathfrak{g}$. We claim that this map is a Lie algebra homomorphism. Assuming this, we see that

\[
\sigma(\Omega(X, Y)) = \sigma([\nabla_X, \nabla_Y] - \nabla_{[X,Y]}) \sim \sigma([[\text{ad} W(X), \text{ad} W(Y)] - \text{ad} W([X,Y])])
= \sigma(\text{ad}([[W(X), W(Y)]) - \text{ad} W([X,Y])) = 0,
\]

which proves that $\Omega(X, Y)$ is a smoothing operator.

To prove the claim, set $X = x^a e^{i\theta} e_a, Y = y^b e^{i\theta} e_b$. Then

\[
W([X,Y]) = W(x^a y^m e^{i(n+m)\theta} e_{ab} e_k) = \sum_{\ell=0}^\infty \frac{(-1)^\ell}{i^\ell} c^k_{ab} \partial_\theta^\ell x^a y^m e^{i(n+m)\theta} \xi^{-\ell} e_k
\]

\[
[W(X), W(Y)] = \sum_{\ell=0}^\infty \sum_{p+q=\ell} \frac{(-1)^{p+q}}{i^{p+q}} \partial_\theta^p x^a e^{i\theta} \partial_\theta^q y^b e^{i\theta} \xi^{-(p+q)} c^k_{ab} e_k.
\]
and these two sums are clearly equal.

It would be interesting to understand how the map $W$ fits into the representation theory of the loop algebra $L\mathfrak{g}$.

**Part II. Characteristic Classes on $LM$**

In this part, we construct a general theory of Chern-Simons classes on certain infinite rank bundles including the frame/tangent bundle of loop spaces, following the construction of primary characteristic classes in [20]. The primary classes vanish on the tangent bundles of loop spaces, which forces the consideration of secondary classes. The key ingredient is to replace the ordinary matrix trace in the Chern-Weil theory of invariant polynomials on finite dimensional Lie groups with the Wodzicki residue on invertible bounded $\Psi$DOs.

As discussed in the Introduction, there are absolute and relative versions of Chern-Simons theory. We use the relative version, which assigns an odd degree form to a pair of connections. In particular, for $TLM$, we can use the $L^2$ and $s = 1$ Levi-Civita connections to form Wodzicki-Chern-Simons (WCS) classes associated to a metric on $M$.

In §5, we develop the general theory of Wodzicki-Chern and WCS classes for bundles with structure group $\Psi\text{DO}_0^*$, the group of invertible classical zeroth order pseudodifferential operators. We show the vanishing of the Wodzicki-Chern classes of $LM$ and more general mapping spaces. As in finite dimensions, we show the existence of WCS classes in $H^n(LM, \mathbb{C})$ if $\text{dim}(M) = n$ is odd (Definition 5.1) and give the local expression for the WCS classes associated to the Chern character (Theorem 5.5). In Theorem 5.6, we prove that the Chern character WCS class vanishes if $\text{dim}(M) \equiv 3 \pmod 4$. In §6, we associate to every circle action $a : S^1 \times M^n \rightarrow M^n$ an $n$-cycle $[a]$ in $LM$. For a specific metric on $S^2 \times S^3$ and a specific circle action $a$, we prove via exact computer calculations that the WCS class is nonzero by integrating it over $[a]$. Since the corresponding integral for the cycle associated to the trivial action is zero, $a$ cannot be homotoped to the trivial action. We use this result to prove that $\pi_1(\text{Diff}(S^2 \times S^3))$ is infinite.

Throughout this part, $H^*$ always refers to de Rham cohomology for complex valued forms. By [2], $H^*(LM) \simeq H^*_\text{sing}(LM, \mathbb{C})$.

## 5. Chern-Simons Classes on Loop Spaces

We begin in §5.1 with a review of Chern-Weil and Chern-Simons theory in finite dimensions, following [6].

In §5.2, we discuss Chern-Weil and Chern-Simons theory on a class of infinite rank bundles including the frame bundles of loop spaces. As in §2.7, the geometric structure group of these bundles is $\Psi\text{DO}_0^*$, so we need a trace on the Lie algebra $\Psi\text{DO}_{\leq 0}$ to define invariant polynomials. There are two types of traces, one given by
taking the zeroth order symbol and one given by the Wodzicki residue \([17], [21]\). Here we only consider the Wodzicki residue trace.

Using this trace, we generalize the usual definitions of Chern and Chern-Simons classes in de Rham cohomology. In particular, given a \(U(n)\)-invariant polynomial \(P\) of degree \(k\), we define a corresponding WCS class \(CS_P^W \in H^{2k-1}(LM)\) if \(\dim(M) = 2k - 1\). We are forced to consider these secondary classes, because the Wodzicki-Chern classes of mapping spaces \(\text{Maps}(N,M)\) vanish. In Theorem 5.5, we give an exact expression for the WCS classes associated to the Chern character. In Theorem 5.6, we show that these WCS classes in \(H^{4k+3}(LM)\) vanish; in contrast, in finite dimensions, the Chern-Simons classes associated to the Chern character vanish in \(H^{4k+1}(M)\).

5.1. Chern-Weil and Chern-Simons Theory for Finite Dimensional Bundles. We first review the Chern-Weil construction. Let \(G\) be a finite dimensional Lie group with Lie algebra \(g\), and let \(G \to F \to M\) be a principal \(G\)-bundle over a manifold \(M\). Set \(g^k = g \otimes \cdots \otimes g\) and let

\[ I^k(G) = \{ P : g^k \to \mathbb{C} \mid P \text{ symmetric, multilinear, Ad-invariant} \} \]

be the degree \(k\) Ad-invariant polynomials on \(g\).

**Remark 5.1.** For classical Lie groups \(G\), \(I^k(G)\) is generated by the polarization of the Newton polynomials \(\text{Tr}(A^\ell)\), where \(\text{Tr}\) is the usual trace on finite dimensional matrices.

For \(\phi \in \Lambda^\ell(F, g^k), \ P \in I^k(G)\), set \(P(\phi) = P \circ \phi \in \Lambda^\ell(F)\).

**Theorem 5.1** (The Chern-Weil Homomorphism \([14]\)). Let \(F \to M\) have a connection \(\theta\) with curvature \(\Omega_F \in \Lambda^2(F, g)\). For \(P \in I^k(G)\), \(P(\Omega_F)\) is a closed invariant real form on \(F\), and so determines a closed form \(P(\Omega_M) \in \Lambda^{2k}(M)\). The Chern-Weil map

\[ \oplus_k I^k(G) \to H^*(M), \ P \mapsto [P(\Omega_M)] \]

is a well-defined algebra homomorphism, and in particular is independent of the choice of connection on \(F\).

The proof depends on:

- (The commutativity property) For \(\phi \in \Lambda^\ell(F, g^k)\),

\[ d(P(\phi)) = P(d\phi). \quad (5.1) \]

- (The infinitesimal invariance property) For \(\psi_i \in \Lambda^\ell_i(F, g), \phi \in \Lambda^1(F, g)\) and \(P \in I^k(G)\),

\[ \sum_{i=1}^k (-1)^{\ell_1 + \cdots + \ell_i} P(\psi_1 \wedge \cdots \wedge [\psi_i, \phi] \wedge \cdots \psi_l) = 0. \quad (5.2) \]
$[P(\Omega_M)]$ is called the characteristic class of $P$. For example, the characteristic class associated to $\text{Tr}(A^k)$ is the $k$th component of the Chern character of $F$.

Part of the theorem’s content is that for any two connections on $F$, $P(\Omega_1) - P(\Omega_0) = dCS_P(\theta_1, \theta_0)$ for some odd form $CS_P(\nabla_1, \nabla_0)$. Explicitly,

$$CS_P(\theta_1, \theta_0) = \int_0^1 P(\theta_1 - \theta_0, \Omega_t, ..., \Omega_t) \, dt$$

(5.3)

where

$$\theta_t = t\theta_0 + (1-t)\theta_1, \quad \Omega_t = dt + \theta_t \wedge \theta_t$$

[5] Appendix.

Remark 5.2. For $F \xrightarrow{\pi} M, \pi^* F \rightarrow F$ is trivial. Take $\theta_1$ to be the flat connection on $\pi^* F$ with respect to a fixed trivialization. Let $\chi$ denote the connection $\chi^* \theta_1$ on $F$, where $\chi$ is the global section of $\pi^* F$. For any other connection $\theta_0$ on $F$, $\theta_t = t\theta_0, \Omega_t = t\Omega_0 + (t^2 - t)\theta_0 \wedge \theta_0$. Assume an invariant polynomial $P$ takes values in $\mathbb{R}$. Then we obtain the formulas for the transgression form $TP(\Omega_1)$ on $F$: for

$$\phi_t = t\Omega_1 + \frac{1}{2}(t^2 - t)[\theta, \theta], \quad TP(\theta) = l \int_0^1 P(\theta \wedge \phi_t^{k-1}) dt,$$

(5.4)

d$TP(\theta) = P(\Omega_1) \in \Lambda^2(F)$ [6]. $TP(\Omega_1)$ pushes down to an $\mathbb{R}/\mathbb{Z}$-class on $M$, the absolute Chern-Simons class.

As usual, these formulas carry over to connections $\nabla = d + \omega$ on vector bundles $E \rightarrow M$ in the form

$$CS_P(\nabla_1, \nabla_0) = \int_0^1 P(\omega_1 - \omega_0, \Omega_t, ..., \Omega_t) \, dt,$$

(5.5)

since $\omega_1 - \omega_0$ and $\Omega_t$ are globally defined forms.

5.2. Chern-Weil and Chern-Simons Theory for $\Psi DO^*_0$-Bundles. Let $E \rightarrow M$ be an infinite rank bundle over a paracompact Banach manifold $M$, with the fiber of $E$ modeled on a fixed Sobolev class of sections of a finite rank hermitian vector bundle $E \rightarrow N$, and with structure group $\Psi DO^*(E)$. For such $\Psi DO^*_0$-bundles, we can produce primary and secondary characteristic classes once we choose a trace on $\Psi DO_{\leq 0}(E)$.

These traces were classified in [17], although there are additional traces in our special case $N = S^1$ [21]. Roughly speaking, the traces fall into two classes, the leading order symbol trace [20] and the Wodzicki residue. In this paper, we consider only the Wodzicki residue, and refer to [16] for the leading order symbol trace.

For simplicity, we mainly restrict to the generating invariant polynomials $P_k(A) = A^k$, and only consider $E = TLM$, which we recall is the complexified tangent bundle. We will work with vector bundles rather than principal bundles.
Definition 5.1. (i) The $k$th Wodzicki-Chern (WC) form of a $\Psi DO_0^*$-connection $\nabla$ on $TLM$ with curvature $\Omega$ is
\[
t^W_k(\Omega)(\gamma) = \frac{1}{k!} \int_{S^*S^1} \text{tr} \sigma_-(\Omega^k) \, d\xi dx. \tag{5.6}
\]
Here we recall that for each $\gamma \in LM$, $\sigma_-(\Omega^k)$ is a $2k$-form with values in endomorphisms of a trivial bundle over $S^*S^1$.

(ii) The $k$th Wodzicki-Chern-Simons (WCS) form of two $\Psi DO_0^*$-connections $\nabla_0, \nabla_1$ on $TLM$ is
\[
C^W_{2k-1}(\nabla_1, \nabla_0) = \frac{1}{k!} \int_0^1 \int_{S^*S^1} \text{tr} \sigma_-(((\omega_1 - \omega_0) \wedge (\Omega_t)^{k-1}) \, dt \tag{5.7}
\]
\[
= \frac{1}{k!} \int_0^1 \text{res}_w[(\omega_1 - \omega_0) \wedge (\Omega_t)^{k-1}] \, dt.
\]

(iii) The $k$th Wodzicki-Chern-Simons form associated to a Riemannian metric $g$ on $M$, denoted $C^W_{2k-1}(g)$, is $C^W_{2k-1}(\nabla_1, \nabla_0)$, where $\nabla_0, \nabla_1$ refer to the $L^2$ and $s = 1$ Levi-Civita connections on $LM$, respectively.

(iv) Let $\Sigma = \{\sigma\}$ be the group of permutations of $\{1, ..., k\}$. Let $I : 1 \leq i_1 < ... < i_\ell = k$ be a partition of $k$ (i.e. with $i_0 = 0$, $\sum_{j=1}^k (i_j - i_{j-1}) = k$). For the symmetric, $U(n)$-invariant, multilinear form on $u(n)$
\[
P_I(A_1, A_2, ..., A_k) = \frac{1}{k!} \sum_{\sigma} \text{tr}(A_{\sigma(1)} \cdot ... \cdot A_{\sigma(i_1)}) \text{tr}(A_{\sigma(i_1+1)} \cdot ... \cdot A_{\sigma(i_2)})
\]
\[
\cdot ... \cdot \text{tr}(A_{\sigma(i_{\ell-1})} \cdot ... \cdot A_{\sigma(k)}),
\]
define the symmetric, $\Psi DO_0^*$-invariant, multilinear form on $\Psi DO_{\leq 0}$ by
\[
P^W_I(B_1, ..., B_k) = \frac{1}{k!} \sum_{\sigma} \left( \int_{S^*S^1} \text{tr} \sigma_-(B_{\sigma(1)} \cdot ... \cdot B_{\sigma(i_1)})
\right.
\]
\[
\cdot \int_{S^*S^1} \text{tr} \sigma_-(B_{\sigma(i_1+1)} \cdot ... \cdot B_{\sigma(i_2)})
\]
\[
\cdot ... \cdot \int_{S^*S^1} \text{tr} \sigma_-(B_{\sigma(i_{\ell-1})} \cdot ... \cdot B_{\sigma(k)}) \right).
\]
The Wodzicki-Chern form associated to $P_I$ for a $\Psi DO_0^*$-connection on $TLM$ with curvature $\Omega$ is
\[
c^W_{P_I}(\Omega) = P^W_I(\Omega, \Omega, ..., \Omega) = \frac{1}{k!} \int_{S^*S^1} \text{tr} \sigma_-(\Omega^{k_1}) \cdot \int_{S^*S^1} \text{tr} \sigma_-(\Omega^{k_2}) \cdot ... \cdot \int_{S^*S^1} \text{tr} \sigma_-(\Omega^{k_\ell})
\]
\[
= \frac{k_1!k_2! \cdot ... \cdot k_\ell!}{k!} c^W_{k_1}(\Omega)c^W_{k_2}(\Omega) \cdot ... \cdot c^W_{k_\ell}(\Omega),
\]
where $k_1 = i_1 - i_0, k_2 = i_2 - i_1, ..., k_\ell = i_\ell - i_{\ell-1}$.
Setting $K = (k_1, \ldots, k_ℓ)$, we also denote $c_k^W(Ω)$ by $c_k^W(Ω)$.

(v) Let $∇_0, ∇_1$ be $ΨDO^*_0$-connections on $TLM$ with connection forms $ω_0, ω_1$, respectively. The Wodzicki-Chern-Simon form associated to that $k$th claim that there is a normalization which gives classes with integral periods. Note

\[
\int_0^1 P_1^W(ω_1 - ω_0, Ω_1, \ldots, Ω_ℓ)dt.
\]

In (iv) and (v), we do not bother with a normalizing constant, since we do not claim that there is a normalization which gives classes with integral periods. Note that the $k$th WCS class is associated to $P_k(A_1, \ldots, A_k) = \text{tr}(A_1 \cdots A_k)$, i.e. the partition $K = (k)$, or in other words to the polynomial giving the $k$th component of the Chern character.

As in finite dimensions, $c_k^W(∇)$ is a closed $2k$-form, with de Rham cohomology class $c_k(LM)$ independent of $∇$, as $c_k^W(Ω_1) - c_k^W(Ω_0) = dCS_{2k-1}^W(∇_1, ∇_0)$.

**Remark 5.3.** It is an interesting question to determine all the $ΨDO^*_0$-invariant polynomials on $ΨDO_{≤0}$. As above, $U(n)$-invariant polynomials combine with the Wodzicki residue (or the other traces on $ΨDO_{≤0}$) to give $ΨDO^*_0$-polynomials, but there may be others.

The tangent space $TLM$, and more generally mapping spaces Maps$(N, M)$ with $N$ closed have vanishing Wodzicki-Chern classes. Here we take a Sobolev topology on Maps$(N, M)$ for some large Sobolev parameter, so that Maps$(N, M)$ is a paracompact Banach manifold. We denote the de Rham class of $c_k^W(Ω)$ for a connection on $E$ by $c_{P_1}(E)$.

**Proposition 5.2.** Let $N, M$ be closed manifolds, and let Maps$_f(N, M)$ denote the component of a fixed $f : N \rightarrow M$. Then the cohomology classes $c_{P_1}(\text{Maps}_f(N, M))$ of $T\text{Maps}(N, M)$ vanish.

**Proof.** For $TLM$, the $L^2$ connection in Lemma 2.1 has curvature $Ω$ which is a multiplication operator. Thus $σ_{-1}(Ω)$ and hence $σ_{-1}(Ω^i)$ are zero, so the WC forms $c_{P_1}(Ω)$ also vanish.

For $n ∈ N$ and $h : N \rightarrow M$, let $ev_n : \text{Maps}_f(N, M)$ be $ev_n(h) = h(n)$. Then $D_XY(h)(n) \overset{\text{def}}{=} (ev_h^x \nabla^{LC,M})X Y(h)(n)$ is the $L^2$ Levi-Civita connection on Maps$(N, M)$. As in Lemma 2.1, the curvature of $D$ is a a multiplication operator. Details are left to the reader. □

**Remark 5.4.** (i) These mapping spaces fit into the framework of the Families Index Theorem in the case of a trivial fibration $Z \rightarrow M \overset{π}{\rightarrow} B$ of closed manifolds. Given a finite rank bundle $E \rightarrow M$, we get an associated infinite rank bundle $E = π_∗E \rightarrow B$. For the fibration $N \rightarrow N \times \text{Maps}(N, M) \rightarrow \text{Maps}(N, M)$ and $E = ev^*TM$, $E$ is $T\text{Maps}(N, M)$. A connection $∇$ on $E$ induces a connection $∇^E$ on $E$ defined by

\[
(∇^E_Z s)(b)(z) = (ev^*θ^u_{(Z,0)}u_s)(b, z).
\]
Here \( u_s(b, z) = s(b)(z) \). The curvature \( \Omega^E \) satisfies

\[
\Omega^E(Z, W)u_s(b, z) = (ev^* \Omega)((Z, 0), (W, 0))u_s(b, z).
\]

This follows from

\[
\Omega^E(Z, W)u_s(b, z) = [\nabla_Z^E \nabla_W^E - \nabla_W^E \nabla_Z^E - \nabla_{[Z, W]}^E]u_s(b, z).
\]

Thus the connection and curvature forms take values in multiplication operators, and so \( c^W_k(E) = 0 \).

If the fibration is nontrivial, the connection on \( E \) depends on the choice of a horizontal complement to \( TZ \) in \( TM \), and the corresponding connection and curvature forms take values in first order differential operators.

(ii) In finite dimensions, odd Chern forms of complexified real bundles like \( TMaps(N, M) \) vanish, because the form involves a composition of an odd number of skew-symmetric matrices. In contrast, odd WC forms involve terms like \( \sigma_{-1}(\Omega^1) \wedge \Omega^M \wedge ... \wedge \Omega^M \), where \( \Omega^1 \) is the curvature of the \( s = 1 \) Levi-Civita connection. By Lemma 3.2(ii), \( \sigma_{-1}(\Omega^1) \) is not skew-symmetric as an endomorphism. Thus it is not obvious that the odd WC forms vanish.

Similarly, in finite dimensions the Chern-Simons form for the odd Chern classes of complexified real bundles vanish, but this need not be the case for WCS forms. In fact, we will produce nonvanishing WCS classes associated to \( c^W_3(TLM^5) \) in §6.

In finite dimensions, Chern classes are topological obstructions to the reduction of the structure group and geometric obstructions to the existence of a flat connection. Wodzicki-Chern classes for \( \PsiDO^*_0 \)-bundles are also topological and geometric obstructions, but the geometric information is a little more refined due to the grading on the Lie algebra \( \PsiDO_{\leq 0} \).

**Proposition 5.3.** Let \( E \rightarrow B \) be an infinite rank \( \PsiDO^*_0 \)-bundle, for \( \PsiDO^*_0 \) acting on \( E \rightarrow N^n \). If \( E \) admits a reduction to the gauge group \( \mathcal{G}(E) \), then \( c^W_k(E) = 0 \) for all \( k \), and hence \( c^W_{P_l}(E) = 0 \) for all \( P_l \). If \( E \) admits a \( \PsiDO^*_0 \)-connection whose curvature has order \( -k \), then \( c_\ell(E) = 0 \) for \( \ell \geq [n/k] \).

**Proof.** If the structure group of \( E \) reduces to the gauge group, there exists a connection one-form with values in \( \text{Lie}(\mathcal{G}) = \text{End}(E) \), the Lie algebra of multiplication operators. Thus the Wodzicki residue of powers of the curvature vanishes, so the Wodzicki-Chern classes vanish. For the second statement, the order of the curvature is less than \( -n \) for \( \ell \geq [n/k] \), so the Wodzicki residue vanishes in this range. \( \square \)

However, we do not have examples of nontrivial WC classes; cf. [16], where it is conjectured that these classes always vanish.

The relative WCS form is not difficult to compute.
Proposition 5.4. Let $\sigma$ be in the group of permutations of $\{1, \ldots, 2k-1\}$. Then

$$CS_{2k-1}^W(g)(X_1, \ldots, X_{2k-1}) = \frac{2}{(2k-1)!} \sum_{\sigma} \text{sgn}(\sigma) \int_{S^1} \text{tr}[( -2R(X_{\sigma(1)}, \dot{\gamma}) - R(\cdot, \dot{\gamma})X_{\sigma(1)} + R(X_{\sigma(1)}, \cdot)\dot{\gamma})
\cdot (\Omega^M)^k(X_{\sigma(2)}, \ldots X_{\sigma(2k-1)})].$$

Proof.

Thus

$$CS_{2k-1}^W(g) = \int_0^1 \int_{S^*S^1} \text{tr} \sigma_-(\omega_1 - \omega_0) \wedge (\sigma_0(\Omega_t))^k \ dt. \quad (5.10)$$

Moreover,

$$\sigma_0(\Omega_t) = td(\sigma_0(\omega_0)) + (1 - t)d(\sigma_0(\omega_1))
+ (t\sigma_0(\omega_0) + (1 - t)\sigma_0(\omega_1)) \wedge (t\sigma_0(\omega_0) + (1 - t)\sigma_0(\omega_1))
= d\omega^M + \omega^M \wedge \omega^M
= \Omega^M.$$ 

Therefore

$$CS_{2k-1}^W(g) = \int_0^1 \int_{S^*S^1} \text{tr}[\sigma_-(\omega_1) \wedge (\Omega^M)^k] \ dt, \quad (5.11)$$

since $\sigma_-(\omega_0) = 0$. We can drop the integral over $t$. The integral over the $\xi$ variable contributes a factor of 2: the integrand has a factor of $|\xi|^2\xi$, which equals $\pm 1$ on the two components of $S^*S^1$. Since the fiber of $S^*S^1$ at a fixed $\theta$ consists of two points with opposite orientation, the “integral” over each fiber is $1 - (-1) = 2$. Thus

$$CS_{2k-1}^W(g)(X_1, \ldots, X_{2k-1}) = \frac{2}{(2k-1)!} \sum_{\sigma} \text{sgn}(\sigma) \int_{S^1} \text{tr}[( -2R(X_{\sigma(1)}, \dot{\gamma}) - R(\cdot, \dot{\gamma})X_{\sigma(1)} + R(X_{\sigma(1)}, \cdot)\dot{\gamma})
\cdot (\Omega^M)^k(X_{\sigma(2)}, \ldots X_{\sigma(2k-1)})].$$

by Lemma 3.1 \hfill \square

This produces odd classes in the de Rham cohomology of the loop space of an odd dimensional manifold.

Theorem 5.5. (i) Let $\dim(M) = 2k - 1$ and let $P$ be a $U(n)$-invariant polynomial of degree $k$. Then $c_P^W(\Omega) \equiv 0$ for any $\Psi$DO$^*$-connection $\nabla$ on $TLM$. Thus $CS_P^W(\nabla_1, \nabla_0)$ is closed and defines a class $[CS_P^W(\nabla_1, \nabla_0)] \in H^{2k-1}(LM)$. In particular, we can define $[CS_P^W(g)] \in H^{2k-1}(LM)$ for a Riemannian metric $g$ on $M$. 
Theorem 5.6. For $\dim(M) = 2k - 1$, the $k^{\text{th}}$ Wodzicki-Chern-Simons form $CS_{2k-1}^{W}(g)$ simplifies to

$$CS_{2k-1}^{W}(g)(X_{1}, ..., X_{2k-1}) = \frac{2}{(2k-1)!} \sum_{\sigma} \text{sgn}(\sigma) \int_{S^{1}} \text{tr}[-R(\cdot, \gamma)X_{\sigma(1)} + R(X_{\sigma(1)}, \cdot)\gamma] \cdot (\Omega^{M})^{k-1}(X_{\sigma(2)}, ..., X_{\sigma(2k-1)})].$$

(5.13)

Proof. (i) Let $\Omega$ be the curvature of a $2k$-manifold. This is a 2-form on $M$, and hence vanishes.

(ii) Since $R(X_{1}, \gamma) \cdot (\Omega^{M})^{k}(X_{2}, ..., X_{2k-1}) = [i_{\gamma} \text{tr}(\Omega^{k})](X_{1}, ..., X_{2k-1}) = \text{tr}(\Omega^{k})(\gamma, X_{1}, ..., X_{2k-1})$, the first term on the right hand side of (5.12) vanishes on a $(2k-1)$-manifold.

Remark 5.5. There are several variants to the construction of relative WCS classes.

(i) If we define the transgression form $Tc_{k}(\nabla)$ with the Wodzicki residue replacing the trace in (5.4), it is easy to check that $Tc_{k}(\nabla)$ involves $\sigma_{-1}(\Omega)$. For $\nabla$ the $H^{1}$ connection, this WCS class vanishes. For $\nabla$ the $H^{s}$ connection, $s > 0$, $\sigma_{-1}(\Omega)$ involves the covariant derivative of the curvature of $M$ (cf. Lemma 3.2 for $s = 1$). Thus the relative WCS class is easier for computations than the absolute class $[Tc_{k}(\nabla)]$.

(ii) If we define $CS_{k}^{W}(g)$ using the $H^{1}$ connection instead of the $s = 1$ connection, then we omit the term involving $R(X_{\sigma(1)}, \cdot)\gamma$ from (5.9) as well. For the $H^{1}$ connection omits the $A_{X}Y$ term in Proposition 2.3 and hence omits the last two terms in Theorem 2.2 and the last term in Lemma 3.1.

(iii) If we define $CS_{k}^{W}(g)$ using the $H^{s}$ connection instead of the $H^{1}$ connection, the WCS class is multiplied by $s$ by Lemma 3.3. Therefore we can remove the dependence of the WCS class on the artificial parameter $s$ by setting $s = 1$.

Definition 5.2. The regularized $k^{\text{th}}$ WCS class associated to a Riemannian metric $g$ on $M$ is $CS_{k}^{W,\text{reg}}(g) \overset{\text{def}}{=} CS_{k}^{W}(\nabla^{1,1}, \nabla_{0})$, where $\nabla^{1,1}$ is the $H^{1}$ connection and $\nabla_{0}$ is the $L^{2}$ Levi-Civita connection.

By (ii), the $H^{1}$ connection one-form is $\sigma_{-1}(\omega_{X}^{1,1}) = i[\xi|^{-1} \xi [-R(X, \gamma) - R(\cdot, \gamma)]$. By Theorem 5.5, the regularized WCS class is therefore the cohomology class of

$$CS_{k}^{W,\text{reg}}(g)(X_{1}, ..., X_{2k-1}) = \frac{2}{(2k-1)!} \sum_{\sigma} \text{sgn}(\sigma) \int_{S^{1}} \text{tr}[-R(\cdot, \gamma)X_{\sigma(1)} \cdot (\Omega^{M})^{k-1}(X_{\sigma(2)}, ..., X_{\sigma(2k-1)})].$$

We conclude this section with a vanishing result that does not have a finite dimensional analogue.

Theorem 5.6. The $k^{\text{th}}$ WCS class $CS_{k}^{W}(g)$ vanishes if $\dim(M) \equiv 3 \pmod{4}$. 

Proof. Let dim$(M) = 2k - 1$. Since $\Omega^M$ takes values in skew-symmetric endomorphisms, so does $(\Omega^M)^{k-1}$ if $k$ is even, i.e. if dim$(M) \equiv 3 \pmod{4}$. The term $-R(\cdot, \gamma) X_{\sigma(1)} + R(X_{\sigma(1)}, \cdot) \gamma$ in (5.13) is a symmetric endomorphism. For in normal coordinates, this term is $(-R_{bdca} + R_{cbda}) X_{\cdot} \sigma(1) + R(X_{\cdot} \sigma(1), \cdot) \gamma^d \equiv A_{ab}$, say, so the curvature terms in $A_{ab} - A_{ba}$ are

$$-R_{bdca} + R_{cbda} + R_{adcb} - R_{cadb} = -R_{bdca} + R_{cbda} + R_{ebad} - R_{dbca} = -R_{bdca} + R_{cbda} - R_{ebda} + R_{bdca} = 0.$$ 

Thus the integrand in (5.13) is the trace of a symmetric endomorphism composed with a skew-symmetric endomorphism, and so vanishes. □

Example 5.7. We contrast Theorem 5.6 with the situation in finite dimensions. Let dim$(M) = 3$. The only invariant monomials of degree two are $\text{tr}(A_1 A_2)$ and $\text{tr}(A_1) \text{tr}(A_2)$ (corresponding to $c_2$ and $c_1^0$, respectively).

For $M$, $\text{tr}(A_1 A_2)$ gives rise to the classical Chern-Simons invariant for $M$. However, the Chern-Simons class associated to $\text{tr}(A_1) \text{tr}(A_2)$ involves $\text{tr}(\omega_1 - \omega_0) \text{tr}(\Omega_t)$, which vanishes since both forms take values in skew-symmetric endomorphisms.

In contrast, on $LM$ we know that the WCS class $CS^W_3$ associated to $\text{tr}(A_1 A_2)$ vanishes. The WCS associated to $\text{tr}(A_1) \text{tr}(A_2)$ involves $\text{tr} \sigma^{-1}(\omega_1 - \omega_0) \text{tr}\Omega_t$, which does not imply that the terms in their symbol expansions are skew-symmetric. In fact, a calculation using Lemma 3.1 shows that $\sigma^{-1}(\omega_1)$ is not skew-symmetric. Thus the WCS class associated to $\text{tr}(A_1) \text{tr}(A_2)$ may be nonzero.

6. An Application of Wodzicki-Chern-Simons Classes to Circle Actions

In this section we use WCS classes to distinguish different $S^1$ actions on $M = S^2 \times S^3$. We use this to conclude that $\pi_1(\text{Diff}(M), \text{id})$ is infinite.

Recall that $H^*(LM)$ denotes de Rham cohomology of complex valued forms. In particular, integration of closed forms over homology cycles gives a pairing of $H^*(LM)$ and $H_*(LM, \mathbb{C})$.

For any closed oriented manifold $M$, let $a_0, a_1 : S^1 \times M \to M$ be two smooth actions. Thus

$$a_i(0, m) = m, \ a_i(\theta, a(\psi, m)) = a_i(\theta + \psi, m).$$

Definition 6.1. (i) $a_0$ and $a_1$ are smoothly homotopic if there exists a smooth map

$$F : [0, 1] \times S^1 \times M \to M, \ F(0, \theta, m) = a_0(\theta, m), \ F(1, \theta, m) = a_1(\theta, m).$$

(ii) $a_0$ and $a_1$ are smoothly homotopic through actions if $F(t, \cdot, \cdot) : S^1 \times M \to M$ is an action for all $t$.

We can rewrite an action in two equivalent ways.
Lemma 6.1. $a_0$ is smoothly homotopic to $a_1$ through actions iff $[a_0^D] = [a_1^D] \in \pi_1(\text{Diff}(M), id)$.

Proof. ($\Rightarrow$) Given $F$ as above, set $G : [0,1] \times S^1 \to \text{Diff}(M)$ by $G(t,\theta)(m) = F(t,\theta,m)$. We have $G(0,\theta)(m) = a_0(\theta, m) = a^D(\theta)(m)$, $G(1,\theta)(m) = a_1(\theta, m) = a_1^D(\theta)(m)$. $G(t,\theta) \in \text{Diff}(M)$, because

$$G(t, -\theta)(G(t,\theta)(m)) = F(t, -\theta, F(t,\theta, m)) = F(t,0, m) = m.$$  

(This uses that $F(t, \cdot, \cdot)$ is an action.) Since $F$ is smooth, $G$ is a continuous (in fact, smooth) map of $\text{Diff}(M)$. Thus $a_0^D, a_1^D$ are homotopic as elements of $\text{Maps}(S^1, \text{Diff}(M))$, so $[a_0^D] = [a_1^D]$.

($\Leftarrow$) Let $G : [0,1] \times S^1 \to \text{Diff}(M)$ be a continuous homotopy from $a_0^D(\theta) = G(0,\theta)$ to $a_1^D(\theta) = G(1,\theta)$ with $G(t,0) = id$ for all $t$. It is possible to approximate $G$ arbitrarily well by a smooth map, since $[0,1] \times S^1$ is compact. Set $F : [0,1] \times S^1 \times M \to M$ by $F(t,\theta,m) = G(t,\theta)(m)$. $F$ is smooth. Note that $F(0,\theta,m) = G(0,\theta)(m) = a_0^D(\theta)(m) = a_0(\theta, m)$, and $F(1,\theta,m) = a_1(\theta, m)$. Thus $a_0$ and $a_1$ are smoothly homotopic. \hfill $\square$

There are similar results for $a^L$.

Lemma 6.2. $a_0$ is smoothly homotopic to $a_1$ iff $a_0^L, a_1^L : M \to LM$ are smoothly homotopic.

Proof. Let $F$ be the homotopy from $a_0$ to $a_1$. Set $H : [0,1] \times M \to LM$ by $H(t,m)(\theta) = F(t,\theta,m)$. Then $H(0,m)(\theta) = F(0,\theta,m) = a_0(\theta, m) = a_0^L(m)(\theta)$, $H(1,m)(\theta) = a_1^L(m)(\theta)$, so $H$ is a homotopy from $a_0^L$ to $a_1^L$. It is easy to check that $H$ is smooth.

Conversely, if $H : [0,1] \times M \to LM$ is a smooth homotopy from $a_0^L$ to $a_1^L$, set $F(t,\theta,m) = H(t,m)(\theta)$. \hfill $\square$
Corollary 6.3. If $a_0$ is smoothly homotopic to $a_1$, then $[a_0^L] = [a_1^L] \in H_n(LM, \mathbb{Z})$.

Proof. By the last Lemma, $a_0^L$ and $a_1^L$ are homotopic. Thus $[a_0^L] = a_{0,*}^L[M] = a_{1,*}^L[M] = [a_1^L]$. □

This yields a technique to use WCS classes to distinguish actions and to investigate $\pi_1(Diff(M), id)$. From now on, “homotopic” means “smoothly homotopic.”

Proposition 6.4. Let $\dim(M) = 2k - 1$. Let $a_0, a_1 : S^1 \times M \to M$ be actions.

(i) If $\int_{[a_0]} CS_{2k-1}^W \neq \int_{[a_1]} CS_{2k-1}^W$, then $a_0$ and $a_1$ are not homotopic through actions, and $[a_0^P] \neq [a_1^P] \in \pi_1(Diff(M), id)$.

(ii) If $\int_{[a_1]} CS_{2k-1}^W \neq 0$, then $\pi_1(Diff(M), id)$ is infinite.

Proof. (i) By Stokes’ Theorem, $[a_0^L] \neq [a_1^L] \in H_n(LM, \mathbb{C})$. By Corollary 6.3, $a_0$ and $a_1$ are not homotopic, and hence not homotopic through actions. By Lemma 6.1 $[a_0^P] \neq [a_1^P] \in \pi_1(Diff(M), id)$.

(ii) Let $a_n$ be the $n$th iterate of $a_1$, i.e. $a_n(\theta, m) = a_1(n\theta, m)$.

We claim that $\int_{[a_n]} CS_{2k-1}^W = n \int_{[a_1]} CS_{2k-1}^W$. By (5.9), every term in $CS_{2k-1}^W$ is of the form $\int_{S^1} \hat{\gamma}(\theta)f(\theta)$, where $f$ is a periodic function on the circle. Each loop $\gamma \in a_1^L(M)$ corresponds to the loop $\gamma(n\cdot) \in a_n^L(M)$. Therefore the term $\int_{S^1} \hat{\gamma}(\theta)f(\theta)$ is replaced by

$$\int_{S^1} \frac{d}{d\theta} \gamma(n\theta)f(n\theta)d\theta = n \int_{0}^{2\pi} \hat{\gamma}(\theta)f(\theta)d\theta.$$ 

Thus $\int_{[a_n]} CS_{2k-1}^W = n \int_{[a_1]} CS_{2k-1}^W$. By (i), the $[a_n^L] \in \pi_1(Diff(M), id)$ are all distinct. □

Remark 6.1. If two actions are homotopic through actions, the $S^1$ index of an equivariant operator of the two actions is the same. (Here equivariance means for each action $a_t, t \in [0,1]$.) In contrast to Proposition 6.4 (ii), the $S^1$ index of an equivariant operator cannot distinguish actions on odd dimensional manifolds, as the $S^1$ index vanishes. This can be seen from the local version of the $S^1$ index theorem [3, Thm. 6.16]. For the normal bundle to the fixed point set is always even dimensional, so the fixed point set consists of odd dimensional submanifolds. The integrand in the fixed point submanifold contribution to the $S^1$-index is the constant term in the short time asymptotics of the appropriate heat kernel. In odd dimensions, this constant term is zero.

In [18], we interpret the $S^1$ index theorem as the integral of an equivariant characteristic class over $[a^L]$.

We now apply these methods to a Sasaki-Einstein metric on $S^2 \times S^3$ constructed in [12] to prove the following:

Theorem 6.5. (i) There is an $S^1$ action on $S^2 \times S^3$ that is not smoothly homotopic to the trivial action.

(ii) $\pi_1(Diff(S^2 \times S^3), id)$ is infinite.
The content of (i) is that although the $S^1$-orbit $\gamma_x$ through $x \in S^2 \times S^3$ is contractible to $x$, the contraction cannot be constructed to be smooth in $x$.

**Proof.** According to [12], the locally defined metric

$$
\begin{align*}
g &= \frac{1-cy}{6} (d\theta^2 + \sin^2 \theta d\phi^2) + \frac{1}{w(y)q(y)} dy^2 + \frac{q(y)}{9} [d\psi^2 - \cos \theta d\phi^2] \\
& \quad + w(y) \left[ d\alpha + \frac{ac - 2y + y^2c}{6(a-y^2)} [d\psi - \cos \theta d\phi] \right]^2,
\end{align*}
$$

(6.1)

with

$$
w(y) = \frac{2(a-y^2)}{1-cy}, \quad q(y) = \frac{a-3y^2+2cy^3}{a-y^2},
$$

is a family of Sasaki-Einstein metrics on a coordinate ball in the variables $(\phi, \theta, \psi, y, \alpha)$. Here $a$ and $c$ are constants, and we can take $a \in (0, 1], c = 1$. For $p, q$ relatively prime, $q < p,$ and satisfying $4p^2 - 3q^2 = n^2$ for some integer $n$, and for $a = a(p, q) < 1$, the metric extends to a 5-manifold $Y^{p,q}$ which has the coordinate ball as a dense subset. In this case, $(\phi, \theta, \psi, y)$ are spherical coordinates on $S^2 \times S^2$ with a nonstandard metric, and $\alpha$ is the fiber coordinate of an $S^1$-fibration $S^1 \to Y^{p,q} \to S^2 \times S^2$. $Y^{p,q}$ is diffeomorphic to $S^2 \times S^3$, and has first Chern class which integrates over the two $S^2$ factors to $p$ and $q$ [12, §2]. The coordinate ranges are $\phi \in (0, 2\pi), \theta \in (0, \pi), \psi \in (0, 2\pi), \alpha \in (0, 2\pi \ell)$, where $\ell = \ell(p, q)$, and $y \in (y_1, y_2)$, with the $y_i$ the two smaller roots of $a - 3y^2 + 2y^3 = 0$. $p$ and $q$ determine $a, \ell, y_1, y_2$ explicitly [12] (3.1), (3.4), (3.5), (3.6).

For these choices of $p, q$, we get an $S^1$-action $a_1$ on $Y^{p,q}$ by rotation in the $\alpha$-fiber. We claim that for e.g. $(p, q) = (7, 3)$,

$$
\int_{[a_1^*]} CS^5_W (g) \neq 0.
$$

(6.2)

By Proposition 6.4(iii), this implies $\pi_1(\text{Diff}(S^2 \times S^3), id)$ is infinite. Since the trivial action $a_0$ has $\int_{[a_0^*]} CS^5_W (g) = 0$ (by the proof of Proposition 6.4(ii) with $n = 0$), $a_0$ and $a_1$ are not smoothly homotopic by Proposition 6.4(i). Thus showing (6.2) will prove the theorem.

Set $M = S^2 \times S^3$. Since $a_1^T : M \to LM$ has degree one on its image,

$$
\int_{[a_1^*]} CS^5_W (g) = \int_M a_1^L* CS^5_W (g).
$$

(6.3)

For $m \in M$,

$$
a_1^L* CS^5_W (g)_m = f(m) d\phi \wedge d\theta \wedge dy \wedge d\psi \wedge d\alpha
$$
for some \( f \in C^\infty(M) \). We determine \( f(m) \) by explicitly computing \( a^L_{1,*}(\partial_\phi), \ldots, a^L_{1,*}(\partial_\alpha) \), (e.g. \( a^L_{1,*}(\partial_\phi)(a^L(m))(t) = \partial_\phi|_{a(m,t)} \)), and noting
\[
\begin{align*}
    f(m) &= f(m)d\phi \wedge d\theta \wedge dy \wedge d\psi \wedge d\alpha(\partial_\phi, \partial_\theta, \partial_y, \partial_\psi, \partial_\alpha) \\
    &= a^L_{1,*}CS^W_5(g)(\partial_\phi, \ldots, \partial_\alpha) \quad (6.4) \\
    &= CS^W_5(g)a^L_{1,*}(\partial_\phi, \ldots, a^L_{1,*}(\partial_\alpha)).
\end{align*}
\]

Since \( CS^W_5(g) \) is explicitly computable from the formulas in §3, we can compute \( f(m) \) from (6.4). Then \( \int_{[a^L]} CS^W_5(g) = \int_M f(m)d\phi \wedge d\theta \wedge dy \wedge d\psi \wedge d\alpha \) can be computed as an ordinary integral in the dense coordinate space.

Via this method, in the Mathematica file \texttt{ComputationsChernSimonsS2xS3.pdf} at \url{http://math.bu.edu/people/sr/} \( \int_{[a^L]} CS^W_5(g) \) is computed as a function of \((p,q)\).

For example, \((p,q) = (7,3)\),
\[
\int_{[a^L]} CS^W_5(g) = -\frac{1849\pi^4}{22050}.
\]

This formula is exact; the rationality up to \( \pi^4 \) follows from \( 4p^2 - 3q^2 \) being a perfect square, as then the various integrals computed in (6.3) with respect to our coordinates are rational functions evaluated at rational endpoints. In particular, (6.2) holds. \( \square \)

**Remark 6.2.** For \( a = 1 \), the metric extends to the closure of the coordinate chart, but the total space is \( S^5 \) with the standard metric. \( \pi_1(\text{Diff}(S^5)) \) is torsion \[8\]. By Proposition 6.4(ii), \( \int_{[a^L]} CS^W_5 = 0 \) for any circle action on \( S^5 \). In the formulas in the Mathematica file, \( \int_{[a^L]} CS^W_5 \) is proportional to \((-1 + a)^2\), which vanishes at \( a = 1 \). This gives a check of the validity of the computation.

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