Galton-Watson processes with random generating times

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Abstract

A new class of branching model, Galton-Watson processes with random generating times, is considered in this paper. Unlike the traditional case, we consider each particle can generate \( k \) times with probability \( h_k \), where \( k \) can be infinite, and producing \( m \) new particles with probability \( p_m \) after each splitting. These are Galton-Watson processes with countably many types in which particles of type \( i \) may only give descendants to type \( i + 1 \) and type 1. It is revealed that the extinction probability \( q \) of such model is the smallest roots of the equation \( s = h(g(s)) \) in \([0, 1]\), where \( h(s) \) and \( g(s) \) are p.g.f of \( \{h_k, k = 0, 1, 2, \ldots\} \) and \( \{p_k, k = 0, 1, 2, \ldots\} \) respectively. Moreover, the reciprocal of the analogue of Perron-Frobenius eigenvalue in infinity many types case is actually the extinction probability of a continuous-time branching process, from which the ergodic properties are discussed.

Keywords: Galton-Watson process; random generating times; extinction probability.

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1. Introduction

A Galton-Watson process with random generating times is a discrete time stochastic process \( \{Z_n; n = 0, 1, 2, \ldots\} \) on the nonnegative integers. This process is controlled by two series \( \{h_k; k = 0, 1, 2, \ldots\} \) and \( \{p_k; k = 0, 1, 2, \ldots\} \), with \( p_k, h_k \geq 0, \sum p_k = 1 \) and \( \sum h_k = 1 \).

The process can be regarded as representing an evolving population of particles. It starts at time 0 with \( Z_0 \) particles, after a unit of time, each particle decide to give birth or die according to how many times the particle has already split and the probability series \( \{h_k\} \). If the particle remains alive, then it will give birth independently to a random number of offsprings according to the probability law \( \{p_k\} \). These newborns and still alive particles both constitute the first generation members \( Z_1 \) and these go on to produce the second generations in the same way and so on, this process is obviously not markovian.

Naturally, if we think of the particle which have already split \( i \) times as type \( i + 1 \) particle, then this model is actually the branching processes with countably many types. Obviously, when \( h_1 = 1 \), process \( \{Z_n^{(1)}; n = 0, 1, 2, \ldots\} \) is actually the classical Galton-Watson process. In the countably many types case, the analogue of the perron-Frobenius eigenvalue often refers to the convergence radius of the power series, \( \sum_{k \geq 0} r^k (M^k)_{ij} \), where \( M \) is the mean
progeny matrix of the process. There are two different extinction: global extinction \((q)\) and partial extinction \((\hat{q})\), and it is possible for the population of each type to become extinct almost surely \((\hat{q} = 1)\) while the whole group explodes on the average \((q < 1)\).

Branching processes with countably many types have already been much investigated. Moya considers a general type space and proves the extinction probability is a solution of \(s = f(s)\), where \(f(\cdot)\) is the progeny generating function of such process. S. Hautphenne and G. Latouche give a sufficient condition for the appearance of the above situation in terms of truncated Galton-Watson processes and discuss the connection between the convergence norm of the mean progeny matrix and the extinction criteria in irreducible and reducible case. P. Braunsteins and S. Hautphenne consider a so-called lower Hessenberg branching processes with countably many types, which restrict individuals of type \(j\) to give birth to type \(j \leq i + 1\) only. They prove the existence of a continuum of fixed points of the progeny generating function and show that the minimum of such continuum is \(q\) and the maximum is \(\hat{q}\). Our main result is a more explicit extinction criteria of the goal processes and a more efficient way to calculate the convergence norm from which the ergodic properties can be obtained.

Therefore, Galton-Watson processes with random generating times can be defined as a lower Hessenberg branching processes with countably many types. Let \(\mathbb{N} = \{0, 1, 2, 3, \ldots\}\) and consider a such process \(\{Z_n = (Z^{(1)}_n, Z^{(2)}_n, Z^{(3)}_n, \ldots)\}_{n \in \mathbb{N}}\), where \(Z^{(l)}_n\) represents the number of individuals type \(l\) particle at the \(n\)th generation, and \(l\) in the typeset \(\chi = \{1, 2, 3, \ldots\}\). The transition function is defined below

\[
P(i, j) = P(Z_{n+1} = j | Z_n = i) = \text{the coefficient of } s^j \text{ in } [f(s)]^i,
\]

where \(f(s) = (f^{(1)}(s), f^{(2)}(s), \ldots)\) and

\[
f^{(i)}(s) = \sum_{j \in \mathbb{N}^\infty} p^{(i)}_j \prod_{k \in \chi} s_k^j, \quad s \in [0, 1]^\chi
\]

\(p^{(i)}_{j_1, j_2, j_3, \ldots}\) is the probability of a type \(i\) parent particle produces \(j_1\) particles of type 1, \(j_2\) particles of type 2, \(j_n\) particles of type \(n\) and so on. Actually, from the description of the model, one can write out the specific expression of \(p^{(i)}_j\). Before it, set

\[
q_k = \begin{cases} 
\sum_{i=k}^{\infty} h_i / \sum_{i=k-1}^{\infty} h_i, & \text{if } \sum_{i=k-1}^{\infty} h_k \neq 0 \\
0, & \text{if } \sum_{i=k-1}^{\infty} h_k = 0,
\end{cases} \quad (1.1)
\]

provide \(\prod_{i=1}^{0} q_i = 1\), it can be show that

\[
h_k = (1 - q_{k+1}) \prod_{i=1}^{k} q_i. \quad (1.2)
\]

From equation (1.2), after \(k - 1(k > 0)\) times splitting, the particle dies with probability \(1 - q_k\), or it will continue to give birth after a certain time with probability \(q_k\). Then,

\[
p^{(i)}_{j_1, j_2, j_3, \ldots} = \begin{cases} 
1 - q_i, & j_1, j_2, \ldots = 0 \\
q_i p_k, & j_1 = k, j_{i+1} = 1, j_{\text{else}} = 0 \\
0, & \text{others},
\end{cases}
\]
We can also calculate the mean progeny matrix \( M = \{m_{ij}; i, j = 1, 2, 3, \ldots \} \) with
\[
m_{ij} = \begin{cases} 
m q_i, & \text{if } j = i \\
q_i, & \text{if } j = i + 1 \\
0, & \text{else.}
\end{cases}
\]
where \( m_{ij} = E Z_{ij}^{(i)} \) represents the expected number of type \( j \) offspring of a single type \( i \) particle in one generation.

If the splitting times are bounded (i.e. there exist \( k > 0, h_i = 0 \) for all \( i \geq k \)), then these model degenerate to the classical multi-type Galton-Watson processes by the expression of \( p_j \). To avoid trivialities, we make the

**Basic Assumption 1:** \( q_i \neq 0 \) for all \( i > 0 \).

**Basic Assumption 2:** The progeny matrix \( M \) is all elementwise finite which equals to \( m < \infty \),

**Basic Assumption 3:** \( \lim_{k \to \infty} h_k = 0 \) which is equal to \( \lim_{k \to \infty} \inf_{i \leq k} q_i < 1 \).

In the second section, we give a more efficient way to calculate the convergence norm of \( M \) and discuss the ergodic properties. In section 3, we give an explicit extinction criteria and the property of extinction probability. In the last section, we give some different perspectives to treat this model and show the connection with other processes.

### 2. Convergence norm and ergodic properties

A branching process \( \{Z_n; n = 0, 1, 2, \ldots \} \) is said to be irreducible if and only if its mean progeny matrix \( M \) is irreducible, and \( M \) is said to be irreducible if for any index \( i, j \) there exists an integer \( n \in \mathbb{N} \) such that \( (M)^n \) is irreducible. In this model, It can be proved that \( M \) is irreducible under the Basic Assumption 1. Actually if there exist some positive integer \( m \), such that \( q_m = 0 \), which indicates \( h_k = 0 \) for all integer \( k \geq m \), then this model degenerates to the finitely many types case and of which \( M \) is also irreducible.

When the number of types is finite and \( M \) is irreducible, it is well-known that the extinction probability \( q < 1 \) if and only if the Perron-Frobenius eigenvalue, or in another word the spectral radius of \( M \) is strictly great than 1. The replacement to the spectral radius as the extinction criteria in countably many types case is known as the convergence norm \( \rho \) of \( M \). Let \( \gamma \) be the convergence radius of the power series \( \sum_{k \geq 0} r^{k}(M^k)_{ij} \) which does not rely on \( i, j \) when \( M \) is irreducible. Then
\[
\rho = \gamma^{-1} = \lim_{k \to \infty} \left\{ (M^k)_{ij} \right\}^{1/k},
\]
note that \( \rho \) is equivalent with the spectral radius of \( M \) in finitely many types case. Serik Sagitov \cite{5} give a natural classification according to the asymptotic properties of \( M^{(n)} = (m_{ij}^{(n)})_{i,j=1}^{\infty} \) as \( n \to \infty \).

**Definition 2.1.** An irreducible matrix \( M \) is called \( \gamma \)-recurrent or \( \gamma \)-transient depending on the divergence or the convergence of the series \( \sum_{k \geq 0} \gamma^{k}(M^k)_{ij} \). A \( \gamma \)-recurrent matrix \( M \) is called \( \gamma \)-positive if \( \lim_{k \to \infty} \gamma^{k}(M^k)_{ij} > 0 \) for some pair \( (i, j) \) and then for all pairs \( (i, j) \), and
it is called \( \gamma \)-null if this limit is zero. Furthermore, a Galton-Watson process with countably many types is said to be \emph{transient} \{positive recurrent,null recurrent\} if its mean progeny matrix is \( \gamma \)-transient \{\( \gamma \)-positive,\( \gamma \)-null\}.

The vector \( v = \{v^{(i)}\}_{i>0} \), \( u^T = \{u^{(i)}\}_{i>0} \) is called \( \gamma \)-invariant measure and \( \gamma \)-invariant vector of the matrix \( M \) respectively if \( u, v > 0 \) and
\[
\gamma vM = v, \quad \gamma Mu = u.
\]

where we write \( v > 0 \) to indicate that \( v_i \geq 0 \) for all \( i > 0 \) but at least one strictly greater. Denote the \((k \times k)\) northwest corner truncation matrix of \( M \) by \( M^{(k)} \), its maximal eigenvalue by \( k \rho \), and the corresponding left and right eigenvalue by \( k v, k u \).

It is observed that \( q = \tilde{q} < 1 \) if \( \rho > 1 \), and \( q \leq \tilde{q} = 1 \) if \( \rho \leq 1 \) in irreducible case[5]. Moreover, the behavior of \( Z_n \) is much related to \( \rho^n \) as \( n \to \infty \). \( \rho \) plays a key role in investigating Galton-Watson processes with countably many types, but there is not easy to evaluate \( \rho \). For this model, we give a more efficient way to compute the convergence norm of \( M \).

**Lemma 2.1.** \( k_{+1} \rho > k \rho \) for all irreducible matrix \( M^{(k)} \), and \( k \rho \uparrow \rho \).

**Lemma 2.2.** \( M \) is \( \gamma \)-positive if and only if the \( \gamma \)-invariant measure and \( \gamma \)-invariant vector satisfy
\[
v u = \sum_{i>0} u^{(i)} v^{(i)} < \infty.
\]

The proof of above two lemma can be found in Seneta[Theorem6.8 and Theorem6.4][6].

**Theorem 2.1.** If \( ml \leq 1, \rho = 1 \), if \( ml > 1, \rho > 1 \) and \( \rho^{-1} \) is the unique solution of
\[
B(s) = \sum_{k=0}^{\infty} b_k s^k = 0 \text{ in } (0,1), \text{ where } B(s) \text{ is the generating function of a continuous-time branching process with}
\]
\[
b_0 = 1, \ b_1 = -(1 + m - mh_0), \ b_k = mh_{k-1}(k > 1).
\]

**Proof.** The characteristic polynomial of \( M^{(k)} \) is
\[
f^{(k)}(\lambda) = |\lambda E - M^{(k)}| = \begin{vmatrix}
\lambda - mq_1 & -q_1 & 0 & \cdots & 0 \\
-mq_2 & \lambda & -q_2 & \cdots & 0 \\
-mq_3 & 0 & \lambda & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-mq_{k-1} & 0 & 0 & \cdots & -q_{k-1} \\
-mq_k & 0 & 0 & \cdots & \lambda
\end{vmatrix}
\]
\[
= (-mq_k)(-1)^{k+1}(-q_1(-q_2)\cdots(-q_{k-1})) + \lambda(-1)^{2k}f^{(k-1)}(\lambda)
\]
\[
= \lambda f^{(k-1)}(\lambda) - mq_1q_2\cdots q_k
\]
\[
= \lambda^k - m(q_1\cdots q_k + q_1\cdots q_{k-1}\lambda + \cdots + q_1q_2\lambda^{k-2} + q_1\lambda^{k-1}).
\]
Easy to know

\[ q_1 q_2 \cdots q_k = \sum_{j=k}^{\infty} h_j, \]

then

\[
f^{(k)}(\lambda) = \lambda^k - m\left(\sum_{j=k}^{\infty} h_j + \cdots + \sum_{j=2}^{\infty} h_j \lambda^{k-2} + \sum_{j=1}^{\infty} h_j \lambda^{k-1}\right)\]
\[
= \lambda^k - m\left(\sum_{j=k}^{\infty} h_j \sum_{i=0}^{k-1} \lambda^i + \sum_{j=1}^{\infty} h_j \sum_{i=1}^{k-1} \lambda^{k-i}\right)\]
\[
= \lambda^k - m((1 - h_0) \sum_{i=0}^{k-1} \lambda^i - \sum_{j=1}^{\infty} h_j \sum_{i=0}^{k-j-1} \lambda^i). \quad (2.2)
\]

From (2.2),

\[
f^{(k)}(1) = 1 - m(k(1 - h_0) - \sum_{j=1}^{k-1} h_j(k - j))\]
\[
= 1 - m\left(\sum_{j=k}^{\infty} k h_j + \sum_{j=1}^{\infty} j h_j\right),\]

which implies \( f^{(k)}(1) \downarrow (1 - ml) \) even if \( ml \) may be infinite. There are three cases, \( ml > 1 \), \( ml = 1 \) and \( ml < 1 \).

On the case \( ml > 1 \), then there exists \( N > 0 \), when \( k > N \)

\[
f^{(k)}(1) = 1 - m\left(\sum_{j=k}^{\infty} k h_j + \sum_{j=1}^{k-1} j h_j\right) < 0,\]

Also, for all \( k > N \) and assume \( \lambda \neq 1 \),

\[
f^{(k)}(\lambda) > \lambda^k - m(1 + \lambda + \lambda^2 + \cdots + \lambda^{k-1})\]
\[
= \lambda^k - m\frac{\lambda^k - 1}{\lambda - 1}\]
\[
= \frac{\lambda^{k+1} - (1 + m)\lambda^k + m}{\lambda - 1}.\]

hence for all \( \lambda' \geq 1 + m \), \( f^{(k)}(\lambda') > 0 \). Therefore, for every \( k > N \), there always exist at least one root of the equation \( f^{(k)}(\lambda) = 0 \) in \((1, 1+m)\) and there are no roots in \([1+m, \infty)\).
Remark 2.1. When $\rho \downarrow \rho \downarrow 0$ is the root of $B(s) = 0$ in $(1, 1 + m)$ for all $k > N$, by equation (2.2)

\[
f^{(k)}(\lambda) = \lambda^k - m(1 - h_0)\lambda^k - 1 + m \sum_{j=1}^{k-1} h_j \frac{\lambda^{k-j} - 1}{\lambda - 1}
\]

\[
= \frac{\lambda^{k+1} - (1 + m)\lambda^k + m \sum_{j=0}^{k-1} h_j \lambda^{k-j} + m - m \sum_{j=0}^{k-1} h_j}{\lambda - 1}
\]

\[
= \frac{\lambda^{k+1}}{\lambda - 1} - (1 + m)\lambda^{-1} + m \sum_{j=0}^{k-1} h_j \lambda^{-j} + \frac{1}{\lambda - 1}(m - m \sum_{j=0}^{k-1} h_j).
\] (2.3)

As $k \uparrow \infty$, $\frac{1}{\lambda - 1}(m - m \sum_{j=0}^{k-1} h_j) \downarrow 0$, $\frac{\lambda^{k+1}}{\lambda - 1} \uparrow \infty$ and there always exist roots of $f^{(k)}(\lambda) = 0$ in $(1, 1 + m)$ for all $k > N$, further, by Lemma 2.1, $k \rho \uparrow \rho$, thus $\rho$ must satisfy $B(\frac{1}{\rho}) = 0$, where

\[
B(s) = 1 - (1 + m)s + m \sum_{j=0}^{\infty} h_j s^{j+1} = \sum_{j=0}^{\infty} b_j s^j.
\]

Actually, it can be seen that $B(1) = 0$, $b_i \geq 0 (i \neq 1)$ and $b_1 < 0$, then $B(s)$ can be regarded as the probability generating function of one continuous-time branching process. Meanwhile

\[
B'(1) = -(1 + m) + m \sum_{j=0}^{\infty} (j + 1)h_j = ml - 1 > 0,
\]

which explains $\rho$ is the unique solution of the equation $B(s) = 0$ in $(1, \infty)$.

On the case $ml = 1$, then $f^{(k)}(1) \downarrow 1$, $B'(1) = 0$ which indicates there is no solution of $B(s) = 0$ in $(1, \infty)$, hence that $\rho = 1$.

When $ml < 1$, it is easy to know $\rho \leq 1$. Recall

\[
f^{(k)}(\lambda) = \lambda^k - m(q_1 \cdots q_k + q_1 \cdots q_{k-1} \lambda + \cdots + q_1 q_2 \lambda^{k-2} + q_1 \lambda^{k-1}),
\]

then

\[
\frac{\partial f^{(k)}(\lambda)}{\partial \lambda} = k\lambda^{k-1} - m(q_1 q_2 \cdots q_{k-1} + 2q_1 q_2 \cdots q_{k-2} \lambda + \cdots + (k - 1)q_1 \lambda^{k-2})
\]

\[
= f^{(k-1)}(\lambda) + \lambda f^{(k-2)}(\lambda) + \cdots + \lambda^{k-2} f(\lambda) + \lambda^{k-1}.
\]

As $k \to \infty$, Since $f^{(k)}(1) \downarrow (1 - ml) \neq 0$, $\partial f^{(k)}(1)/\partial \lambda \uparrow \infty$. Also, for all $\epsilon > 0$, $f^{(k)}(1 - \epsilon) \downarrow 0$ obviously. It implies when $ml < 1$, the biggest real roots of $f^{(k)}(\lambda) = 0$ tend to 1, that is, $k \rho \uparrow \rho = 1$. The proof is finally completed. \hfill \Box

Remark 2.1. When $ml < 1$, it is observed that $f^{(k)}(\rho) \to 0$, which indicate the $\gamma$-invariant measure or vector do not exist; This method of calculating $\rho$ is also available when the type is finite. Actually when $ml < 1$, the reciprocal of Perron-Frobenius eigenvalue $1/\rho$ is the root of $B(s) = 0$ in $(1, \infty)$. 
Theorem 2.2. Under the three basic assumption, if \( ml \geq 1 \), the Galton-Watson processes with random generating times \( \{ Z_n \}_{n \geq 0} \) is positive recurrent, if \( ml < 1 \), \( \{ Z_n \}_{n \geq 0} \) is transient.

Proof. From Seneta[6], we know for a \( \gamma \)-recurrent matrix \( T \), there always exists \( \gamma \)-invariant measure and vector. \( M \) is thus \( \gamma \)-transient when \( ml < 1 \) by Remark 2.1, and then \( \{ Z_n \}_{n \geq 0} \) is transient.

When \( ml \geq 1 \), without loss of generality, let \( u^{(1)} = 1 \), \( v^{(1)} = 1 \), then by \( Mu = \rho u \) and \( vM = \rho v \), one can present that

\[
\begin{align*}
  u^{(k)} &= \frac{f^{(k-1)}(\rho)}{q_1 q_2 \cdots q_{k-1}}, \\
  v^{(k)} &= \frac{q_1 q_2 \cdots q_{k-1}}{\rho^{k-1}}.
\end{align*}
\]  

(2.4)

When \( ml > 1 \), set

\[
B^{(k)}(s) := 1 - (1 + m)s + \sum_{j=1}^{k} mh_{j-1} s^j, \quad (k > 1)
\]

by equation (2.3),

\[
f^{(k)}(\lambda) = \frac{\lambda^{k+1}}{\lambda - 1} B^{(k)}\left(\frac{1}{\lambda}\right) + \frac{m - m \sum_{j=0}^{k-1} h_j}{\lambda - 1}.
\]

then \( vu \) is given by

\[
v u = 1 + \sum_{i>0} \frac{\rho}{\rho - 1} B^{(i)}\left(\frac{1}{\rho}\right) + \sum_{i>0} \frac{m - m \sum_{j=0}^{i-1} h_j}{\rho^i (\rho - 1)}.
\]

Observe that \( B^{(i)}\left(\frac{1}{\rho}\right) \uparrow 0 \) and \( (m - m \sum_{j=0}^{i-1} h_j) \downarrow 0 \) as \( i \to \infty \), therefore

\[
v u < \sum_{i>0} \frac{1}{\rho^i (\rho - 1)} < \infty,
\]

which implies \( M \) is \( \gamma \)-positive and \( \{ Z_n \}_{n \geq 0} \) is positive recurrent by Lemma2.2.

When \( ml = 1 \), \( vu \) is given by

\[
v u = 1 + \sum_{i>0} f^{(i)}(1) = 1 + \sum_{i>0} (1 - mq_1 - mq_1 q_2 - \cdots - mq_1 q_2 \cdots q_i).
\]

Denote the limit \( \lim_{k \to \infty} \inf_{i \leq k} q_i \) by \( \alpha \), for all \( k > 0 \), let

\[
q'_k = \begin{cases} \alpha, & \text{if } q_k > \alpha \\ q_k, & \text{if else.} \end{cases}
\]

Therefore, \( \lim_{k \to \infty} q'_k = \alpha \) and for all \( i > 0 \),

\[
f^{(i)}(1) = 1 - mq_1 - mq_1 q_2 - \cdots - mq_1 q_2 \cdots q_i \leq 1 - mq'_1 - mq'_1 q'_2 - \cdots - mq'_1 q'_2 \cdots q'_i =: f^{(i)'}(1).
\]
By O’Stolz theorem,
\[
\lim_{i \to \infty} \frac{f^{(i)}(1)}{f^{(i-1)}(1)} = \lim_{i \to \infty} \frac{f^{(i+1)}(1) - f^{(i)}(1)}{f^{(i)}(1) - f^{(i-1)}(1)} = \lim_{i \to \infty} \frac{mq_i' \cdots q_{i+1}'}{mq_1' \cdots q_i'} = \alpha.
\]
and \(\alpha < 1\) by Basic Assumption 3. It implies the series \(\sum_{i>0} f^{(i)}(1)\) converges and \(\mathbf{w}\) converges consequently, which indicates that \(\{Z_n\}_{n \geq 0}\) is \(\gamma\)-positive on the case \(ml = 1\) and the proof is complete. □

3. Extinction criteria

In this section, we discuss the extinction criteria and the property of the extinction probability.

Since it is possible that every type of individuals eventually disappear while the whole population explodes in countably many types case [3], we refer to the event that the whole population becomes extinct as globe extinction probability and denote its probability vector by \(\mathbf{q}\). Partial extinction and \(\bar{\mathbf{q}}\) is used to indicate the other event and its probability vector. The total population size at \(n\)th generation is \(|Z_n| := \sum_{l \in \chi} Z_n^{(l)}\), let \(e_i := (0, \ldots, 1, \ldots)\), with the 1 in the \(i\)th component. Then
\[
q^{(i)} = P_{n \to \infty} |Z_n| = 0|Z_0 = e_i],
\]
\[
\bar{q}^{(i)} = P_{\forall l \in \chi : n \to \infty} Z_n^{(l)} = 0|Z_0 = e_i].
\]

Lemma 3.1. Both \(\mathbf{q}\) and \(\bar{\mathbf{q}}\) are the solution of \(f(s) = s\) in \([0,1]^{\chi}\), and \(0 \leq \mathbf{q} \leq \bar{\mathbf{q}} \leq 1\).

Proof. From the independence of individuals and probability decomposition of the first spitting, one can show that \(\bar{\mathbf{q}}\) and \(\mathbf{q}\) satisfy the equation \(f(s) = s\). When the type is finite, globe extinction and partial extinction is equivalent. However, when the type is infinite, by Fatou’s Lemma:
\[
\lim_{n \to \infty} |Z_n| = \lim_{n \to \infty} \sum_{i=1}^{\infty} Z_n^{(l)} \geq \sum_{i=1}^{\infty} \lim_{n \to \infty} Z_n^{(l)},
\]
thus,
\[
P_{n \to \infty} |Z_n| = 0|Z_0 = e_i] \leq P_{\forall l \in \chi : n \to \infty} Z_n^{(l)} = 0|Z_0 = e_i],
\]
which means \(\mathbf{q} \leq \bar{\mathbf{q}}\). □

Lemma 3.2. Assume \(M\) is irreducible, if \(\rho > 1\), then \(\bar{\mathbf{q}} = \mathbf{q} < 1\), and if \(\rho \leq 1\), then \(\mathbf{q} \leq \bar{\mathbf{q}} = 1\). □
The proof can be seen in [3].

**Theorem 3.1.** The extinction probability $\tilde{q} = q$ all the time, and $q^{(1)}$ is the smallest root of $K(s) = s$ in $[0, 1]$, where

$$K(s) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} h_{j}p(j, i)s^{i}, \quad (3.1)$$

$p(j, i)$ is the $j$th convolution of $p(1, i)$. Moreover, $q = 1$ if and only if $ml \leq 1$.

**Proof.** Let $g(s), h(s)$ be the probability generating function of $\{p_{k}, k = 0, 1, 2, \ldots\}, \{h_{k}, k = 0, 1, 2, \ldots\}$. From Lemma 3.1, $q$ and $\tilde{q}$ are solutions of $f(s) = s$, then write it componentwise

$$\begin{cases} 
1 - q_{1} + q_{1}s_{2}(\sum_{i=0}^{\infty} p_{1}s_{1}^{i}) = s_{1} \\
1 - q_{2} + q_{2}s_{3}(\sum_{i=0}^{\infty} p_{1}s_{1}^{i}) = s_{2} \\
1 - q_{3} + q_{3}s_{4}(\sum_{i=0}^{\infty} p_{1}s_{1}^{i}) = s_{3}.
\end{cases} \quad (3.2)$$

Substituting (3.3) into (3.2) yields

$$1 - q_{1} + q_{1}g(s_{1})(1 - q_{2} + q_{2}s_{3}g(s_{1})) = s_{1}, \quad (3.5)$$

similarly substituting (3.4) into (3.5)

$$1 - q_{1} + q_{1}g(s_{1})(1 - q_{2} + q_{2}g(s_{1})(1 - q_{3} + q_{3}s_{4}g(s_{1}))) = s_{1}. \quad (3.6)$$

Iterating to infinity obtains

$$\sum_{j=0}^{\infty} h_{j}(\sum_{i=0}^{\infty} p_{1}s_{1}^{i})^{j} = s_{1},$$

that is

$$K(s_{1}) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} h_{j}p(j, i)s_{1}^{i} = h(g(s_{1})) = s_{1}.$$

Observing that

$$K(s)^{m} = (h(g(s)))^{m} = \sum_{j=0}^{\infty} h_{j}^{m*}(\sum_{i=0}^{\infty} p_{1}s_{1}^{i})^{j} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} h_{j}^{m*}p(j, i)s_{i}^{i}.$$
set $r(m, i) := \sum_{j=0}^{\infty} h_j^m p(j, i)$, then

$$K(s)^m = (\sum_{j=0}^{\infty} r(1, j) s^j)^m = \sum_{j=0}^{\infty} r(m, j) s^j.$$ 

Therefore, $K(s)$ is a probability generating function of one branching process\[1\], which indicate that there are solutions of $K(s) = s$. It is known when $K'(1) \leq 1$, there only exists one root $s = 1$ in $[0, 1]$ which implies $q^{(1)} = \tilde{q}^{(1)} = 1$, and $\tilde{q} = q = 1$ from the expression of $s = f(s)$. When $K'(1) > 1$, there exists $0 < q < 1$ satisfies $K(q) = q$, and by Theorem 2.1 and Lemma 3.2, we know $\tilde{q} = q < 1$. Also, Zucca\[8\] points out $\lim_{n \to \infty} Z_n^{(l)} = 0$ for all types $l$ if and only if there exist at least one type become extinct regardless of the initial type, which indicates $q^{(1)} = \tilde{q}^{(1)} = q$. Actually $K'(1) = \sum_{i=0}^{\infty} s^i = \sum_{j=0}^{\infty} h_j s^j = \sum_{i=0}^{\infty} s^i = m l$.

The proof is completed. □ □

4. More on the process

I. Through out the previous four sections, we deal with the model in the terminology of the Galton-Watson processes with countably many types, however, it can be considered from another angle. In the genealogy of a family, everyone has his own place by seniority in the hierarchy, and if we label each particle with seniority instead of type in this model, then the genealogy of the first particle(if we assume the model begin with a single particle) is of interests and it is also a way to deal with such model. Let $\{X_n; n = 0, 1, 2\ldots\}$ be the associated hierarchy process, for which rank $n$ particles are the daughter particle of the rank $(n-1)'s$. It covers the ordinary Galton-Watson processes when $h_1 = 1$ and $n$ means exactly the time. In fact, $\{X_n; n = 0, 1, 2\ldots\}$ is a time homogeneous controlled branching process\[7\] defined inductively by $X_0 = 1$ and for $n = 1, 2, 3, \ldots$,

$$X_n = \sum_{i=1}^{\phi(X_{n-1})} \xi_i,$$

where $\sum_{i=1}^{0} = \xi_i(i > 0)$ are i.i.d. with common probability distribution $\{p_k; k = 0, 1, 2, \ldots\}$ and $\{\phi(k), k = 0, 1, 2, \ldots\}$ is a sequence of non-negative integer-valued random variables which are independent from $\xi_i$. Moreover, $\{\phi(k), k = 0, 1, 2, \ldots\}$ satisfies

$$\mathbb{E}[s^{\phi(k)}] = (h(s))^k, \quad (4.1)$$

where $h(s)$ is the probability generating function of $\{h_k; k = 0, 1, 2, \ldots\}$. It is not difficult to show that the hierarchy process $\{X_n, n = 0, 1, 2\ldots\}$ and the Galton-Watson process with countably many types $\{Z_n\}_{n \geq 0}$ share the same extinction criteria. It implies that for time homogeneous controlled branching processes, if the control random variable $\phi(k)$ satisfies
then the extinction probability of such processes $q = 1$ if and only if $g'(1)h'(1) \leq 1$ where $g(s)$ is p.g.f of random variable $\xi_i$ for all $i > 0$.

**II.** The number of $n$th generation particles only relies on the $(n-1)$th’s in ordinary Galton-Watson processes, however, there are processes that the number of $n$th generation particles depends on all the generations before $n$, which can be handled by the branching processes with random generating times. Denote $\{\alpha_k; k = 0, 1, 2, \ldots\}$ be the sequence of influential parameter with $\alpha_i \in [0, 1]$ for all $i > 0$, the branching mechanism be $\circ \beta$, the offspring variable distribution $\{p_k; k = 0, 1, 2, \ldots\}$, and the corresponding process be $\{Y_n\}_{n \geq 0}$. Assume $Y_0 = 1$, then for all $n > 0$

$$Y_n \overset{D}{=} (\alpha_1 Y_{n-1} + \alpha_2 Y_{n-2} + \cdots + \alpha_n Y_0) \circ \beta,$$

where $\overset{D}{=} \text{means they have the same probability distribution}$. From the equation above, the number of particles which give birth may not be integer. For instance, if $k_n = \lfloor k_n \rfloor + l_n$, where $\lfloor k_n \rfloor$ is the integer part of $k_n$ and $l_n < 1$, then equation $Y_n \overset{D}{=} k_n \circ \beta$ is equivalent with

$$Y_n = \sum_{i=1}^{\lfloor k_n \rfloor} \xi_i + \xi_0,$$

where $\xi_i (i > 0)$ are offspring random variables with common distribution $\{p_k; k = 0, 1, 2, \ldots\}$ and $\xi_0$ satisfies

$$p(\xi_0 = k) = \begin{cases} \frac{l_n p_k}{1 - l_n (1 - p_0)}, & k > 0 \\ 1 - l_n (1 - p_0), & k = 0. \end{cases}$$

It can be proven that the process $\{Y_n\}_{n \geq 0}$ is identical with the process $\{Z_n^{(1)}\}_{n \geq 0}$, and $\{Z_n\}_{n \geq 0}$ is a Galton-Watson process with countably many types with $q_i = \alpha_i$ for all $i \geq 0$.

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