Two-species Bose–Einstein condensate in an optical lattice: analytical approximate formulae

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Received 24 September 2015, revised 19 January 2016
Accepted for publication 20 January 2016
Published 22 February 2016

Abstract
Employing a general variational method and perturbation theory, we derived explicit solutions for the description of one-dimensional two species Bose–Einstein condensates confined by a harmonic trap potential in an optical lattice. We consider the system of two coupled Gross–Pitaevskii equations (GPE) and derive explicit expressions for the chemical potentials and wavefunctions in terms of the atom–atom interaction parameters and laser intensity. We have compared our results with the numerical solutions of the GPE and performed a quantitative analysis for the both considered methods. We underline the importance of the obtained explicit solutions to characterize the density profile or degree of miscibility of the two components.

Keywords: Bose–Einstein condensates, variational method, chemical potentials

(Some figures may appear in colour only in the online journal)

1. Introduction

Multiple Bose–Einstein condensates (BEC) of different atomic species have been realized in recent years. A mixture of alkali atoms of $^{87}$Rb in two different hyperfine internal spin states [1], atoms $^{23}$Na with a superposition of spinor condensates [2], combination of $^{41}$K–$^{87}$Rb [3], $^{87}$Rb–$^{85}$Rb [4], $^{87}$Rb–$^{133}$Cs [5], and gases of rare atomic species $^{168}$Yb–$^{174}$Yb [6], have been employed to produce two species BEC. These quantum degenerate mixtures allow the study of several intriguing phenomena such as the dynamics of the superfluid system [2, 3], the
production of heteronuclear polar molecules [7] and the miscibility or immiscibility of the two quantum fluids [8], among other effects. Two-species BEC loaded in a optical lattice have also been explored [9, 10]. A similar system but of Fermi–Bose quantum gas mixture in a three-dimensional optical lattice was implemented to study the interfering paths of the bosonic wave function scattered by the presence of fermionic atoms [11]. These results have led to intense theoretical and mathematical studies on the properties of the two-coupled Gross–Pitaevskii equations. The basis of this research lies on the knowledge of the dependence of the chemical potentials as functions of the interparticle interactions and the spatial density probability [12]. A fascinating experimental realization to study the one-dimensional (1D) transport properties of ultracold fermionic and bosonic atoms in a periodic potential has been reported in [12].

From the theoretical point of view there are several studies for the description of two-species Bose condensates. Typically, numerical approaches or Thomas–Fermi approximation are employed to calculate the chemical potential and the ground state wave functions [13]. Collective excitations for a two-component BEC are the focus of the [14] and [15], where superfluidity, phase separation condition and quantum fluctuation have been addressed.

In [16] the mixture of 1D two interacting condensates modeled by the Bose–Hubbard Hamiltonian and by using the quantum Monte Carlo numerical simulations is analyzed. Theoretical analysis of the 1D two component BEC problem becomes an important reservoir to mimic different physical effects of the condensed matter physics (see for example [17–19]), including the magnetic properties of the bosonic mixtures with tunable interspecies interactions [20]. Also, as will be shown below, we can take advantage of analytical results for the study of quantum effects and predictions for cold atoms researches.

Assuming a ‘cigar-shaped’ type for the BEC [21, 22] of a gas composed by two kinds of bosons loaded in an optical lattice, we can consider the following system of 1D GP equations:

\[ \mathbf{L}_0 \Phi + [\mathbf{L}_q - \nu] \Phi = 0, \]

where \( \mathbf{L}_0 \) and \( \mathbf{L}_q \) are, respectively, the operators

\[ \begin{bmatrix}
- \frac{\hbar^2}{2m_1} \frac{d^2}{dx^2} + \frac{1}{2} m_1 \omega_1^2 x^2 & 0 \\
0 & - \frac{\hbar^2}{2m_2} \frac{d^2}{dx^2} + \frac{1}{2} m_2 \omega_2^2 x^2
\end{bmatrix}, \]

\[ \begin{bmatrix}
\lambda_1 |\Phi_1|^2 - V_L \cos^2 \left( \frac{2\pi x}{d} \right) & \lambda_3 \Phi_1 \Phi_2 \\
\lambda_3 \Phi_1 \Phi_2 & \lambda_2 |\Phi_2|^2 - V_L \cos^2 \left( \frac{2\pi x}{d} \right)
\end{bmatrix} \]

and

\[ \Phi = \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix}, \quad \nu = \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix}. \]

Here, \( \omega_i > 0 \) denotes the harmonic trap frequencies where for simplicity we consider the same for both condensates, i.e., \( \omega_1 = \omega_2 = \omega, m_i > 0 \), and \( \nu_i \) are, respectively, the mass and chemical potential for the specie \( i \) (\( i = 1 \) and 2), \( V_L > 0 \) and \( d > 0 \) the intensity and laser wavelength, \( \lambda_i \) takes into account the self-interaction term for the \( i \)th species, and \( \lambda_{ij} \), the interaction between unlike particles of the species 1 and 2. In this system, the complex function \( \Phi_i(x) \) is known [21] as the macroscopic wavefunction or order parameter of the \( i \)th
component and is defined as the expectation value of the corresponding field operator, namely \( \Phi_i(x) = \langle \Phi_i(x) \rangle \). The functions \( \Phi_i \) satisfy the normalization conditions

\[
\int_\mathbb{R} |\Phi_i(x)|^2 \, dx = N_i, \quad i = 1, 2, \tag{5}
\]

where \( N_i \) denotes the number of atoms of the \( i \)th species.

It is worth to notice that in some situations, as in the case of spinor condensates, where one produces confinement of an atomic cloud of an element in different spin states \([21, 23]\), the condition (5) must be substituted by

\[
\int_\mathbb{R} |\Phi_1(x)|^2 \, dx + \int_\mathbb{R} |\Phi_2(x)|^2 \, dx = N, \quad N = N_1 + N_2.
\]

We can rewrite the system (1) in its dimensionless form, by considering, for instance, \( l = \sqrt{\hbar/(m_1 \omega)} \), \( x = l \xi \), and \( \Phi_i(x) = \psi_i(\xi)/\sqrt{l} \), \( i = 1, 2 \), in which case we have

\[
\mathcal{L}_0 \Psi + [\mathcal{L}_f - \mu] \Psi = 0, \tag{6}
\]

where \( \mathcal{L}_0 \) and \( \mathcal{L}_f \) are respectively the operators

\[
\begin{bmatrix}
- \frac{1}{2} \frac{d^2}{d\xi^2} + \frac{1}{2} \xi^2 & 0 \\
0 & - \frac{a_2}{2} \frac{d^2}{d\xi^2} + \frac{1}{2a_2} \xi^2
\end{bmatrix}
\]

\[
\begin{bmatrix}
\lambda_1 |\psi_1|^2 - V_0 \cos^2(\alpha \xi) & \lambda_3 |\psi_1\psi_2| \\
\lambda_3 |\psi_1\psi_2| & \lambda_2 |\psi_2|^2 - V_0 \cos^2(\alpha \xi)
\end{bmatrix}
\]

\[
\Psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \quad \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}. \tag{7}
\]

Here, \( a_2 = m_1/m_2 \), \( \lambda_i = \hbar c/\hbar, \quad (i = 1, 2, 3) \), \( V_0 = V_4/\hbar \omega \), \( \alpha = 2 l \pi/d \), and \( \mu_j = \nu_j/\hbar \omega \) \( (j = 1, 2) \). For the system (6), the energy functional can be cast as

\[
E(\psi_1, \psi_2) = E_1(\psi_1) + E_2(\psi_2) + \frac{\lambda_3}{2} \int_\mathbb{R} |\psi_1(\xi)|^2 |\psi_2(\xi)|^2 d\xi, \tag{10}
\]

with

\[
E_1(\psi) = \frac{1}{4} \int_\mathbb{R} |\psi'(\xi)|^2 \, d\xi + \frac{1}{4} \int_\mathbb{R} |\xi^2 \psi(\xi)|^2 d\xi 
+ \frac{\lambda_1}{4} \int_\mathbb{R} |\psi(\xi)|^4 d\xi - \frac{V_0}{2} \int_\mathbb{R} \cos^2(\alpha \xi) |\psi(\xi)|^2 d\xi,
\]

\[
E_2(\psi) = \frac{a_2}{4} \int_\mathbb{R} |\psi'(\xi)|^2 d\xi + \frac{1}{4a_2} \int_\mathbb{R} |\xi^2 \psi(\xi)|^2 d\xi 
+ \frac{\lambda_2}{4} \int_\mathbb{R} |\psi(\xi)|^4 d\xi - \frac{V_0}{2} \int_\mathbb{R} \cos^2(\alpha \xi) |\psi(\xi)|^2 d\xi.
\]

Therefore, the partial Fréchet derivatives of \( E \) are

\[
\partial_1 E = E_1'(\psi_1) + \lambda_3 |\psi_2(\xi)|^2 \psi_1(\xi), \tag{11}
\]

\[
\partial_2 E = E_2'(\psi_2) + \lambda_3 |\psi_1(\xi)|^2 \psi_2(\xi). \tag{12}
\]
The minimum of the energy $E(\psi_1, \psi_2)$ under the restrictions
\[ \int_{\mathbb{R}} |\psi_1(\xi)|^2 \, d\xi = N_1, \]
\[ \int_{\mathbb{R}} |\psi_2(\xi)|^2 \, d\xi = N_2, \]
the Lagrange conditions for some constants $\mu_i/2$ ($i = 1, 2$)
\[ \partial_1 E = \mu_1 \psi_1(\xi), \quad \partial_2 E = \mu_2 \psi_2(\xi). \]

Notice that (13) coincides with (6).

In previous works [24–27], we have presented different methods to express the chemical potential $\mu$ and the order parameter $\psi(\xi)$ as function of the interaction parameter $\lambda$ for the 1D Gross–Pitaevskii equation. In the present paper, we adapt two of these methods (the generalized variational approach [27] and perturbation theory) for the system (6), by considering the vector chemical potential $\mu$ as function of the atom–atom interaction strength of each component $\lambda_1$, $\lambda_2$ and the interaction between both species, $\lambda_3$.

The paper is organized as follows: in section 2 we present the mathematical framework of the variational problem formulation, which characterizes the condensate as ground state solution for the system (6), as well as its equivalent integral representation. We also report an exact representation $\psi(\xi)$ which is based our variational approach described in section 3. In section 4 we develop the perturbation method valid for two coupled GP equations. Section 5 is devoted to present the results of these two approaches comparing with the exact numerical solution of the system (6). Also, final conclusions are delivered showing the range of validity of both considered methods, with respect to parameter values employed for the description of two-species Bose–Einstein condensate in an optical lattice.

2. General mathematical framework

In this section we establish the functional framework for the mathematical analysis of existence, regularity and stability of ground state solutions for the system (6). There is a great amount of mathematical work on these questions, some of which is mentioned in the references below. The eigenvalue problem (6) has an intrinsic mathematical interest, but the ground state solutions (i.e., stationary states solutions of minimal energy) play important role for condensates. By stationary state we mean solution of the evolution equation
\[ i \frac{\partial \Psi}{\partial t} = [\mathcal{L}_0 + \mathcal{L}_d] \Psi, \]
of the form
\[ \Psi(t, \xi) = \begin{bmatrix} \exp(-i\mu_1 t) \psi_1(\xi) \\ \exp(-i\mu_2 t) \psi_2(\xi) \end{bmatrix}. \]

2.1. Existence of ground states and their stability

We consider the following minimization problem
\[ E_{\text{min}}(\lambda) = \min \{ E(\Psi); \Psi \in \Sigma \}, \]
where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, $\Psi = (\psi_1, \psi_2)$,
\[ \Sigma = \left\{ \Psi \in \Xi; \int_{\mathbb{R}} |\psi_1(\xi)|^2 \, d\xi = N_1, \int_{\mathbb{R}} |\psi_2(\xi)|^2 \, d\xi = N_2 \right\}. \]
and \( \Xi = \mathcal{V} \times \mathcal{V} \), where
\[
\mathcal{V} = \left\{ \psi \in H^1(\mathbb{R}) ; \int_{\mathbb{R}} \left[ |\psi'(\xi)|^2 + \xi^2 |\psi(\xi)|^2 \right] d\xi < \infty \right\}
\]
and \( H^1(\mathbb{R}) \) is the standard Sobolev space.

Although the solutions of equation (6) are in general complex valued functions, we can restrict ourselves to just the real valued ones. This can easily be justified because any solution of this system satisfies the following inequality [28]: there exists \( 0 < \delta \leq 1 \) and \( C(\delta) > 0 \) such that
\[
|\Psi(\xi)|^2 + |\Psi'(\xi)|^2 \leq C(\delta) \exp(-\delta \xi^2), \quad \forall \xi \in \mathbb{R}.
\] (16)

Indeed, assuming that
\[
\Psi(\xi) = \begin{bmatrix} \psi_{1R}(\xi) + i \psi_{1I}(\xi) \\ \psi_{2R}(\xi) + i \psi_{2I}(\xi) \end{bmatrix}
\] (17)
the exponential decay (16) and a simple calculus gives
\[
\psi_{1R}' \psi_{1I} - \psi_{1I}' \psi_{1R} = \frac{d}{d\xi} \left( \frac{\psi_{1R}}{\psi_{1I}} \right) |\psi_{1I}|^2 = 0.
\]

Therefore, \( \psi_{1R} = \beta \psi_{1I} \) for some real constant \( \beta \neq 0 \). The same holds for second component of \( \Psi \), which gives us \( \psi_{2R} = \gamma \psi_{2I} \) for some constant \( \gamma \). Hence, the function
\[
U(\xi) = \begin{bmatrix} \sqrt{1 + \beta^2} \psi_{1R}(\xi) \\ \sqrt{1 + \gamma^2} \psi_{2R}(\xi) \end{bmatrix}
\] (18)
is a real solution of (6) and (17) is given by
\[
\Psi(\xi) = \begin{bmatrix} \frac{1}{\sqrt{1 + \beta^2}} + \frac{i \beta}{\sqrt{1 + \beta^2}} & \psi_{1R}(\xi) \\ \frac{1}{\sqrt{1 + \gamma^2}} + \frac{i \gamma}{\sqrt{1 + \gamma^2}} & \psi_{2R}(\xi) \end{bmatrix}
\] (19)

The existence of a minimal energy solution is a consequence of the Gagliardo–Nirenberg inequality (see theorem 1.3.7 in [29]), which in 1D allows us to show that the energy functional \( E \) is bounded by below on the manifold \( \Sigma \), for all values of \( \lambda_i \in \mathbb{R}, i = 1, 2, 3 \). With arguments of convexity, we can show that the (real) solution of minimal energy is unique provided that \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) are positive. Moreover, since the system (6) has the properties of conservation of energy and mass (i.e., the number of particles), we can prove the orbital stability [27, 30] of ground states.

On the other hand, the space \( \mathcal{H} = L^2(\mathbb{R}) \times L^2(\mathbb{R}) \) is a Hilbert space if one considers the usual inner product
\[
(\Psi|\Phi)_{\mathcal{H}} = \int_{\mathbb{R}} \psi_1(\xi) \phi_1(\xi) d\xi + \int_{\mathbb{R}} \psi_2(\xi) \phi_2(\xi) d\xi
\]
and the differential operator
\[
L_0 : D(L_0) \subset \mathcal{H} \to \mathcal{H}
\]
is self-adjoint and maximal monotone [31]. So, it is invertible and we can rewrite the equation (6) as
\[ \Psi = \mathcal{L}_0^{-1} (\mu - \mathcal{L}_1) \Psi. \]  

Since \( D(\mathcal{L}_0) \subset \Xi \) and \( \Xi \) is compactly embedded in \( \mathcal{H} \) \cite{27}, \( \mathcal{L}_0^{-1} \) is a compact integral operator.

### 2.2. Exact formulae

We assume that, for each \( \lambda \in \mathbb{R}^3 \), we can choose \( \Psi_\lambda \in \Sigma \) such that the map \( \lambda \mapsto \Psi_\lambda \) is a differentiable manifold in \( \Xi \). Then, we have

\[
\frac{\partial E_{\text{min}}}{\partial \lambda_1} (\lambda) = \left( E_1' (\psi_{1 \lambda}) : \frac{\partial}{\partial \lambda_1} \psi_{1 \lambda} \right) + \frac{1}{4} \| \psi_{1 \lambda} \|^4 + \left( E_2' (\psi_{2 \lambda}) : \frac{\partial}{\partial \lambda_1} \psi_{2 \lambda} \right) + \lambda_3 \left( |\psi_{2 \lambda}|^2 \psi_{1 \lambda} : \frac{\partial}{\partial \lambda_1} \psi_{1 \lambda} \right) + \lambda_3 \left( |\psi_{1 \lambda}|^2 \psi_{2 \lambda} : \frac{\partial}{\partial \lambda_1} \psi_{2 \lambda} \right) = \mu_1 (\lambda) \frac{\partial}{\partial \lambda_1} \| \psi_{1 \lambda} \|^2 + \mu_2 (\lambda) \frac{\partial}{\partial \lambda_1} \| \psi_{2 \lambda} \|^2 + \frac{1}{4} \| \psi_{1 \lambda} \|^4.
\]

Since \((\psi_{1 \lambda}, \psi_{2 \lambda}) \in \Sigma\) implies

\[
\frac{\partial}{\partial \lambda_1} \| \psi_{1 \lambda} \|^2 = \frac{\partial}{\partial \lambda_1} \| \psi_{2 \lambda} \|^2 = 0,
\]

we get

\[
\frac{\partial E_{\text{min}}}{\partial \lambda_1} (\lambda) = \frac{1}{4} \| \psi_{1 \lambda} \|^4.
\]

**Mutatis mutandis**, we have

\[
\frac{\partial E_{\text{min}}}{\partial \lambda_2} (\lambda) = \frac{1}{4} \| \psi_{2 \lambda} \|^4
\]

and with the same arguments, we obtain

\[
\frac{\partial E_{\text{min}}}{\partial \lambda_3} (\lambda) = \frac{1}{2} \| \psi_{1 \lambda} \psi_{2 \lambda} \|^2.
\]

Therefore

\[
\nabla E_{\text{min}}(\lambda) = \left( \frac{1}{4} \| \psi_{1 \lambda} \|^4, \frac{1}{4} \| \psi_{2 \lambda} \|^4, \frac{1}{2} \| \psi_{1 \lambda} \psi_{2 \lambda} \|^2 \right)
\]

and

\[
E_{\text{min}}(\lambda) = E_{\text{min}}(0) + \int_0^1 \nabla E_{\text{min}}(\lambda(s)) \cdot \frac{d}{ds} \lambda(s) \, ds,
\]

for any smooth path \( \lambda(s) \) in \( \mathbb{R}^3 \) joining the points \((0, 0, 0)\) and \((\lambda_1, \lambda_2, \lambda_3)\). In particular, for the linear path \( \lambda(s) = s \lambda = (s \lambda_1, s \lambda_2, s \lambda_3) \), \( 0 \leq s \leq 1 \), for which we have the following formula

\[
E_{\text{min}}(\lambda) = E_{\text{min}}(0, 0, 0) + \frac{1}{4} \int_0^1 \left( \| \psi_{1 \lambda} \|^4 \lambda_1 + \| \psi_{2 \lambda} \|^4 \lambda_2 + 2 \| \psi_{1 \lambda} \psi_{2 \lambda} \|^2 \lambda_3 \right) \, ds. \tag{21}
\]

The chemical potentials \( \mu_1 \) and \( \mu_2 \) as function of the parameter \( \lambda \) can be easily calculated by multiplying the first equation of \((6)\) by \( \psi_{1 \lambda} \), the second by \( \psi_{2 \lambda} \) and taking the integral over \( \mathbb{R} \). By this calculation, we get
\[ \mu_1(\lambda) = \frac{2}{N_1} \left( E_1(\psi_{1,k}) + \frac{\lambda_1}{4} \| \psi_{1,k} \|^2 + \frac{\lambda_2}{2} \| \psi_{1,k} \psi_{2,k} \|^2 \right). \]  

\[ \mu_2(\lambda) = \frac{2}{N_2} \left( E_2(\psi_{2,k}) + \frac{\lambda_2}{4} \| \psi_{2,k} \|^2 + \frac{\lambda_1}{2} \| \psi_{1,k} \psi_{2,k} \|^2 \right). \]  

3. Variational approach

We consider the following trial functions:

\[ \psi_k(\xi) = \sqrt{N_k} \left( \frac{2\pi}{2\pi} \right)^{1/4} \exp(-\tau_k \xi^2), \quad k = 1, 2. \]  

By calculating the energy \( E \) with these functions, we get:

\[ E(\psi_1, \psi_2) = \sum_{k=1}^{2} N_k \left( a_k \psi_{1,k} + \frac{1}{16\sigma_k} + \frac{\lambda_1 N_k \tau_1^2}{4\sqrt{\pi}} - \frac{V_0}{4}(1 + e^{-\alpha/2\tau_1}) \right) \]

\[ + \frac{\lambda_2 N_1 N_2}{\lambda_2 \lambda_1} \left( \frac{\tau_2}{\tau_1 + \tau_2} \right)^{1/2}, \]

where, to simplify the notation, we introduced \( a_1 = 1 \). So, by denoting \( f(\tau_1, \tau_2) = E(\psi_1, \psi_2) \), it is easy to see that \( f(\tau_1, \tau_2) \) is bounded by below. Indeed, if \( \lambda_2 \geq 0 \), we have

\[ f(\tau_1, \tau_2) \geq \sum_{k=1}^{2} N_k \left( a_k \psi_{1,k} + \frac{1}{16\sigma_k} + \frac{\lambda_1 N_k \tau_1^2}{4\sqrt{\pi}} - \frac{V_0}{4}(1 + e^{-a/2\tau_1}) \right) \]

and the conclusion is evident. Otherwise, notice that

\[ \left( \frac{\tau_2}{\tau_1 + \tau_2} \right)^{1/2} \leq \frac{1}{2} \sqrt{\tau_1 + \tau_2} \leq \frac{\sqrt{2}}{2} \max \{ \sqrt{\tau_1}, \sqrt{\tau_2} \} \]

\[ \leq \frac{\sqrt{2}}{2} (\sqrt{\tau_1} + \sqrt{\tau_2}), \quad \forall \tau_1, \tau_2 > 0 \]

from which we get

\[ f(\tau_1, \tau_2) \geq \sum_{k=1}^{2} N_k \left[ a_k \psi_{1,k} + \frac{1}{16\sigma_k} + \frac{\lambda_1 N_k \tau_1^2}{2\sqrt{\pi} N_k} \right] - \frac{V_0}{4}(1 + e^{-a/2\tau_1}). \]

Hence, \( f(\tau_1, \tau_2) \) reaches its minimum at some \( \tau_k(\lambda_1, \lambda_2, \lambda_3), (k = 1, 2) \) which are necessarily solutions of the algebraic system \((i \neq j)\):

\[ a_i^2 \tau_i^2 + \lambda_i N_i a_i \tau_i^{3/2} + \frac{2\lambda_3 N_1 N_2 a_i}{\sqrt{2\pi N_i}} \left( \frac{\tau_j}{\tau_i + \tau_j} \right)^{3/2} - \frac{V_0 \alpha^2 a_i}{2} e^{-\alpha/2\tau_i} = \frac{1}{4}, \quad i \neq j. \]  

These are the equations to be solved in order to obtain \( \tau_k(\lambda) \) and \( \tau_j(\lambda) \) which will be used in the formulas of \( \mu_{\text{app},1}(\lambda) \) and \( \mu_{\text{app},2}(\lambda) \) (see below). Notice that if \( \lambda_i \neq \lambda_j \), the respective roots are different even in the case \( V_0 = 0, N_1 = N_2 = N \) and \( m_1 = m_2 \). Indeed, by subtracting the first equation from the second one in (25), we obtain:

\[ \tau_2^2 - \tau_1^2 = \frac{N}{2\sqrt{\pi}} [\lambda_1 \tau_1^{3/2} - \lambda_2 \tau_2^{3/2}] \]

and we see that, if \( \tau_1 = \tau_2 \), then \( \lambda_1 \tau_1^{3/2} - \lambda_2 \tau_2^{3/2} = 0 \), which implies that \( \lambda_1 = \lambda_2 \).
By choosing, \( \sigma_i = \sqrt{a_i \pi} \), \( i = 1, 2 \), the equations (25) can be written as

\[
\sigma_i^4 + \frac{\Lambda_i}{2\sqrt{\pi}} \sigma_i^3 + \frac{\sqrt{2} \Lambda_j a_i \sqrt{a_j}}{\sqrt{\pi}} \left( \frac{\sigma_i^2 \sigma_j^2}{a_i^2 + a_j^2} \right)^{3/2} - \frac{V_0 \alpha^2 \sigma_i^4 e^{-\alpha^2 \sigma_i^2 / 2 \sigma_i^2}}{2} = \frac{1}{4}, \quad i \neq j,
\]

where, for \( i \neq j \), \( \Lambda_i = \lambda_i N_i / \sqrt{a_i} \) and \( \Lambda_j = \lambda_j N_j / \sqrt{a_j} \).

3.1. Approximate formulae

Let \( \sigma_1(\Lambda) \) and \( \sigma_2(\Lambda) \) with \( \Lambda = (\Lambda_1, \Lambda_2, \Lambda_{12}, \Lambda_{21}) \), the solution of the system (26).

Using equations (22) and (23), a direct calculation gives:

\[
\mu_{\text{app},1}(\Lambda) = \frac{\sigma_1^2(\Lambda)}{2} + \frac{1}{8\sigma_1^2(\Lambda)} + \frac{\Lambda_1}{2\sqrt{\pi}} \sigma_1(\Lambda) - \frac{V_0}{2} (1 + e^{-\alpha^2 / 2 \sigma_1^2(\Lambda)})
\]
\[
+ \frac{\sqrt{2}}{2 \sqrt{\pi}} \left( \frac{\sigma_1^2(\Lambda) \sigma_2^2(\Lambda)}{a_2 \sigma_1^2(\Lambda) + \sigma_2^2(\Lambda)} \right)^{1/2},
\]

\[
\mu_{\text{app},2}(\Lambda) = \frac{\sigma_2^2(\Lambda)}{2} + \frac{1}{8\sigma_2^2(\Lambda)} + \frac{\Lambda_2}{2\sqrt{\pi}} \sigma_2(\Lambda) - \frac{V_0}{2} (1 + e^{-\alpha^2 / 2 \sigma_2^2(\Lambda)})
\]
\[
+ \frac{\sqrt{2}}{2 \sqrt{\pi}} \left( \frac{\sigma_1^2(\Lambda) \sigma_2^2(\Lambda)}{a_1 \sigma_1^2(\Lambda) + \sigma_2^2(\Lambda)} \right)^{1/2}.
\]

3.2. Properties of the wavefunction and the minimal energy

As it was pointed out in equation (16), each component of \( \Psi(\xi) \) in equation (9) behaves as a Gaussian as \( \xi \to \pm \infty \), for all values of \( \Lambda, \mu \) and \( V_0 \). In a general way, this behaviour justifies the selection of the trial function (24). Nevertheless, as it is achieved in figure 1, the variation of the wavefunction of one species with respect to the optical lattice intensity, \( V_0 = V_L / h \omega \)
and the reduced wavelength, $\alpha^{-1} = d/(2 \pi \hbar)$ cannot be accounted by a Gaussian trial function (24). The strong variation of the optical lattice potential $U(\xi) = -V_0 \cos^2(\alpha \xi)$ with respect to $\alpha$ and $V_0$, keep off the contribution of the monotonic behavior of the harmonic potential $\xi^2$ to order parameter. Thus, the variational approach presented here does not allow good results in the case $V_0 \neq 0$ is large enough. Indeed, by an effective numerical solution of the 1D Gross–Pitaevskii equation we obtain the order parameter $\psi(\xi)$ as shown in figure 1. On the other hand, if we consider the equivalent formula of (25) for the one component BEC, we obtain [27]

$$\sigma^4 + \frac{\lambda}{4\sqrt{\pi}} \sigma^3 - \frac{V_0 \alpha^2}{4} e^{-\alpha^2/2\sigma^2} = \frac{1}{4}.$$  (29)

For $\lambda \geq 0$ fixed, the function $\sigma(V_0)$ implicitly defined by equation (29) satisfies the differential equation

$$\frac{d\sigma}{dV_0} = \frac{\alpha^2 e^{-\alpha^2/2\sigma^2}}{16\sigma^3 + \frac{3\lambda}{\sqrt{\pi}} \sigma^2 - \frac{V_0 \alpha^4}{\sigma^3} e^{-\alpha^2/2\sigma^2}}.$$  (30)

which shows that it is increasing and blows up for a certain $V_0$ large enough.

The choice of a test function that takes into account the variation shown in the figure will be treated in a future publication.

Also, the presence of two-species introduces an effective interaction of the unlike particle, which is described in our model by the coefficient $\Lambda_3$. The effect of the $\Lambda_3 |\psi_i|^2 \psi_i$ term on the condensates is to attract ($\Lambda_3 < 0$) or to repel ($\Lambda_3 > 0$) the cloud probability densities $|\psi_i(\xi)|^2$. Thus, in the case where we are dealing with a strong repulsive interaction, the maximum of the density probability lies at $\xi = 0$. Notice that the nature of our trial functions does not take into account the present peculiarity of two-species BEC. In section 5 below we present a brief discussion of this effect.

4. Perturbation theory

Following the result of equation (20), we can write the system of coupled integral equations

$$\Psi(\xi) = \int_{-\infty}^{\infty} G(\xi, \xi') [\mu - \mathcal{L}_I] \Psi(\xi') d\xi',$$  (31)

where the kernel

$$G(\xi, \xi') = \begin{pmatrix} G_1(\xi, \xi') & 0 \\ 0 & G_2(\xi, \xi') \end{pmatrix}$$  (32)

is the solution of the differential equations $\mathcal{L}_I G(\xi, \xi') = i\hbar (\xi - \xi')$ and $I$ the identity matrix. In the spectral representation we have the Green function [32]

$$G_i(\xi, \xi') = \sum_{n=0}^{\infty} \frac{\varphi_i(n\hbar)}{(n + 1/2)^{1/2}} \varphi_i'(n\hbar), i = 1, 2,$$  (33)

with $l_i = \sqrt{\alpha_i}$ and $\varphi_i(\xi)$ is the harmonic oscillator wavefunction [33]. Thus, inserting $G(\xi, \xi')$ in (31) we get
\[ \Psi(\xi) = \begin{bmatrix} \psi_1(\xi) \\ \psi_2(\xi) \end{bmatrix} = \sum_{n=0}^{\infty} \frac{\sqrt{N_1} C_n \varphi_n(\xi/l_1)}{\sqrt{N_2} D_n \varphi_n(\xi/l_2)}, \tag{34} \]

where the vectors \( \mathbf{C} = (C_1, C_2, \ldots) \) and \( \mathbf{D} = (D_1, D_2, \ldots) \) are given by

\[
C_n[D_n] = \frac{1}{\left(n + \frac{1}{2}\right)} \int_{-\infty}^{\infty} \left[ (\mu_{1[2]} - \lambda_{1[2]} |\psi_{1[2]}(\xi')|^2 + V_0 \cos^2(\alpha \xi')) \\
+ \lambda_3 |\psi_{2[1]}(\xi')|^2 \varphi_n(\xi'/l_{1[2]}) \psi_{1[2]}(\xi') \right] d\xi'. \tag{35} \]

To satisfy equations (34) and (35), the vector coefficients \( \mathbf{C} \) and \( \mathbf{D} \) must fulfill the nonlinear system of equations

\[ 0 = [\Delta(\mu_2) + \Lambda_1 \mathbf{C} \cdot \mathbf{T} \cdot \mathbf{C} + \Lambda_2 \mathbf{D} \cdot \mathbf{S}(l_r) \cdot \mathbf{D} - V_0 \mathbf{P}(\alpha \sqrt{\alpha_l})] \mathbf{C}, \tag{36} \]

\[ 0 = \left[ \Delta(\mu_2) + \Lambda_3 \mathbf{D} \cdot \mathbf{T} \cdot \mathbf{D} + \Lambda_{21} \mathbf{C} \cdot \mathbf{S}\left(\frac{1}{l_r}\right) \cdot \mathbf{C} - V_0 \mathbf{P}(\alpha \sqrt{\alpha_l}) \right] \mathbf{D}, \tag{37} \]

where \( l_r = \sqrt{\alpha_l/\alpha_l^2} \), \( \Delta_{mn} = (n + 1/2 - \mu_1) \delta_{mn} \), \( \mathbf{T} \) and \( \mathbf{P}(\alpha) \) are matrices given elsewhere [26] and \( \mathbf{S}(z) \) is defined in the appendix A.

The above system is an infinite generalized eigenvalue problem for \( \mu_i \) \( (i = 1, 2) \) and the vector coefficients \( \mathbf{C} \) and \( \mathbf{D} \). An efficient algorithm for solving equations (36)–(37) is presented in [25]. Nevertheless, it is very useful to carry with explicit expressions for \( \mu_i \) and \( \psi_i \) in terms of the leading parameters \( \mathbf{A} \) and \( V_0 \). Assuming that the contribution of the nonlinear terms and the optical potential appearing in the system (36)–(37) are small enough in comparison with that of the harmonic potentials, allows that the vector solutions \( \mathbf{C} \) and \( \mathbf{D} \) can be sought as Taylor polynomials of the parameters \( \mathbf{A} \) and \( V_0 \). Up to second order terms, and solving simultaneously the system (36)–(37), it is possible to show that the chemical potentials are given by

\[
\mu_{\text{per},1} = \frac{1}{2} + \frac{\Lambda_1}{\sqrt{2}\pi} + \frac{\Lambda_{12}}{\sqrt{\pi(1 + l_r^2)}} - \frac{V_0}{2} \left[ 1 + \exp(-\alpha^2 a_1) \right] - 0.033106 \Lambda_{12}^2,
\]

\[
+ \frac{\sqrt{2} \Lambda_{12}}{\pi \sqrt{1 + l_r^2}} \left[ 2 \Lambda_1 f(l_r) + \Lambda_{12} \left( \frac{1}{l_r^2} \right) \right] + \frac{\Lambda_{12}}{\pi (1 + l_r^2)} \left[ \Lambda_{12} g(l_r) + 2 l_r \Lambda_{12} g \left( \frac{1}{l_r} \right) \right]
\]

\[
+ \frac{\exp(-\alpha^2 a_1) V_0}{\sqrt{2} \pi} \left[ \Lambda_{12} \left( \frac{1}{l_r^2} \right) h \left( \frac{l_r^2 \alpha^2 a_1}{1 + l_r^2} \right) \right] + \Lambda_t h \left( \frac{\alpha^2 a_1}{2} \right)
\]

\[
+ \frac{\exp(-\alpha^2 a_1) V_0 \Lambda_{12}}{\sqrt{2} \pi (1 + l_r^2)} \left[ \frac{\alpha^2 a_1}{1 + l_r^2} \right] - \frac{\exp(-2\alpha^2 a_1) V_0^2}{4} \text{ch}(2\alpha^2 a_1).
\tag{38} \]

Functions \( f(z) \), \( g(z) \) and \( h(z) \) and \( \text{ch}(z) \) are defined in appendix B.

Finally, the dimensionless order parameter, \( \psi_1 \), considering corrections up to the first order in \( \Lambda_1, \Lambda_{12}, \) and \( V_0 \), can be expressed as
as the term $\lambda_{12}$ is de

g (0) as the order parameter for the second species, and $\lambda_{12}$ decreases if $l_{r}$ is almost constant as a function of the self-interaction term of the

we obtain similar expressions by just changing $l_{r} \leftrightarrow 1/l_{r}$ in equations (38)–(40).

5. Discussion of the results and conclusions

In the following we present our results and discuss the reliability of the two implemented methods of solution. It will be useful to compare the obtained analytical expressions with direct numerical calculations. This comparison allows to find ranges of values of the parameters $\lambda_{1}, \lambda_{2}, \lambda_{12}$ and $V_{0}$, where the variational approach and perturbation method can be implemented for the description and predictions of the properties of the cigar-shape 1D two-species BEC. For the numerical evaluation of the system (20) we choose a finite difference method described in [26].

5.1. Chemical potentials

First, we analyze the case when the intensity of optical lattice is turned off, $V_{0} = 0$. Figure 2 shows the reduced chemical potential $\mu_{1}$ as a function of the dimensionless nonlinear term $\lambda_{1}$ for the following values of the inter-species $\lambda_{12} = \pm 0.5, \pm 1, \pm 1.5$. In the calculation we have fixed $\lambda_{2} = 1$, $\lambda_{21} = \pm 0.5$, and $l_{r} = 1$. Variational approach calculations given by equations (27) and (28) are indicated by dashed lines, while the perturbation approach, using equation (38), is symbolized by solid lines. Symbols represent the results obtained by direct numerical evaluation of equation (6). Taking as reference the particular limit of one component, where $\lambda_{3} = 0$, as it is shown in figure 2, we observe that the influence of the inter-species interaction on the chemical potential is to increase $\mu_{1}$ as the term $\lambda_{12} > 0$ increases, while the opposite result is achieved, i.e., $\mu_{1}$ decreases if $\lambda_{12} < 0$ decreases.

The small difference seen in the figure between the perturbation theory with respect to the variational and numerical solutions for $\lambda_{1} > 0$ lies in the range of validity of equation (38). In [27] it is shown that the perturbation theory for one component reproduces quite well the chemical potential if $|\lambda_{1}| \lesssim 2$. In the present case, the inter-species interaction plays the role as an effective nonlinear term given by $\lambda_{1} |\psi_{1}|^{2} + \lambda_{12} |\psi_{2}|^{2}$. Hence, the range of validity of equation (38) as function of $\lambda_{1} > 0$ is reduced if $\lambda_{12} > 0$. The opposite can be argued if $\lambda_{12} < 0$, i.e., the function $\mu_{1}(\lambda_{1})$ given by (38) match the variational and numerical calculations in a large range of values of $\lambda_{1} > 0$. Similar arguments can be performed for the various combination of values of the parameters considered in figure 2.

In figure 3 we checked the influence of several species, $l_{r} = 0.5, 1, \text{and} 2$, on $\mu_{1}$ and $\mu_{2}$ as function of $\lambda_{1}$ without optical lattice, $\lambda_{2} = 1$, $\lambda_{21} = 0.5$, and $\lambda_{12} = 1$. As might be expected, the chemical potential $\mu_{2}$ is almost constant as a function of the self-interaction term of the first species $\lambda_{1}$. We note that for $l_{r} > 1$ the value of the chemical potential $\mu_{1}$ ($\mu_{2}$) is reduced...
increased), while the opposite is obtained if \( \lambda_2 < 1 \). This result is explained by the fact that the effective inter-species \( \lambda_3 \) \(|\psi_j|^2\) depends on the mass ratio \( \lambda_1 \) (see equations (6), (27), (28) and (38)).

It can be seen that the variational approach fits very well the numerical calculations, but the perturbation theory presents some differences as \( \lambda_1 > 0 \) (\( \lambda_1 < 0 \)) increases (decreases). The same argument, as it is given in the analysis of figure 2, we can argue for the dependence

**Figure 2.** Dependence of the reduced chemical potential \( \mu_{\text{red},i} = \mu_i / \hbar \omega \) on the dimensionless self-interaction parameter \( \lambda_i \) for the inter-particle term \( \lambda_{12} = \pm 0.5, \pm 1.0, \pm 1.5 \) and \( \lambda_{21} = \pm 0.5 \). Values of \( V_0 = 0 \), \( \lambda_2 = 1 \) and \( \lambda_3 = 1 \) are fixed. Dashed and solid lines represent the analytical results from equations (27) and (38), respectively. Symbols correspond to the numerical solution of equation (6). For the sake of comparison, the limit of one component (\( \lambda_3 = 0 \)) using equation (27) is shown.

**Figure 3.** Dimensionless chemical potentials \( \mu_1 \) and \( \mu_2 \) as a function of \( \lambda_1 \) for several species \( \lambda_2 = 0.5, 1.0 \) and 2.0. The same nomenclature as in figure (2) are employed. In the calculation we sorted \( V_0 = 0 \), \( \lambda_2 = 1 \), \( \lambda_{12} = 1 \) and \( \lambda_{21} = 0.5 \).
Nevertheless, this analysis has to be taken with caution. The presence of the functions $f(z)$ and $g(z)$ in equation (38) establishes different ranges of validity for $\mu_i$ as a function of $l_r$. Notice that $f(z) < 0$ for $z > 0$, while $g(z) < 0$ ($g(z) > 0$) for $z > 1$ ($z < 1$) (see appendix B).

5.2. Influence of the optical lattice

Figure 4 shows the behavior of the chemical potential as function of $\Lambda_1$ for several values of the laser intensity $V_0$, the reduced wavelength $\alpha$ and the $\Lambda_{12}$ parameter. Solid lines represent the calculation following equation (38), dashed lines the variational approach as given by equation (27) with $\Lambda_{12} = 0$. Symbols correspond to the numerical solution of equation (6) for $\Lambda_{12} = 0$. From figure 4 it can be seen that equation (27) does not match with the perturbation calculations neither numerical solutions. As $V_0$ increases, the variational approach becomes worse, reflecting the choice of the trial functions (24) we have employed to calculate the energy. In connection with the perturbation theory, the agreement is satisfactory for any $V_0$ less than 200, where a small deviation from the exact numerical results is achieved. As it is
expected, the influence of the unlike interspecies interaction is to increase the chemical potential (the opposite is obtained if \( \Lambda_{12} < 0 \), not shown in the figure).

5.3. Miscibility of the two species

A central issue for a description of the properties of multi species is the evaluation of the order parameter as a function of particle–particle and interspecies interaction. The control of the unlike particle interaction by Feshbach resonance [4] allows to tune the miscibility or not of these structures [8] and the challenge to create ultracold polar molecules.

Figure 5 displays the spatial distribution density \( |\psi_{\text{per},1}(\xi)|^2 \) as a function of the dimensionless parameters \( \Lambda_{12} \) (panels (a) and (b)) and \( \Lambda_1 \) (panel (c)). From figures 5(a) and (b) we observe the influence of one species over another. The condensate is more delocalized as the inter-species parameter \( \Lambda_{12} \) increases. Also, as the mass of the second species increases, the probability density \( |\psi_{\text{per},1}(\xi)|^2 \) spreads on the space and the maximum of the wavefunction decreases. The opposite is observed for the attractive interaction when \( \Lambda_{12} < 0 \), i.e., the density profile becomes more confined at \( \xi \approx 0 \) as \( \Lambda_{12} \) decreases. Moreover, a stronger
localization occurs as the parameter $l_1$ increases. In other words, the system with large mass difference presents a more effective attraction between both components, which means that it favors the miscibility among both species if $\Lambda_{12} < 0$. A comparison between attractive and repulsive dimensionless nonlinear parameter $\Lambda_1$ is sorted in panel (c) of the figure. As $\Lambda_1$ increases from 0 to 3, the density is spread in space. Also, for $\Lambda_1$ large enough, the maximum of $|\psi_{\text{per},1}(\xi)|^2$ is displaced by the particle–particle repulsive interaction. In the case of attractive interaction, $\Lambda_1 < 0$, the maximum of the order parameter $\psi_{\text{per},1}(\xi)$ lies at the origin.

For sake of clarity, in panel (d) we show the influence of the interaction $\Lambda_{12}$ on the density profile $|\psi_{\text{per},i}(\xi)|^2$ ($i = 1, 2$). Notice that the ground state is modulated by the repulsive interaction induced by the species 2 and the maximum of density probability is shifted to $\xi \neq 0$ as $\Lambda_{12}$ increases. From the physical point of view this results are clear, the species 2 is expelled off the origin by the first condensate. The mutual repulsion between the two-species affect the spatial localization of density profile As we stated above, this effect is driven not only by the values of $\Lambda_{12}$, but also by the ratio of the masses involved in the two condensates (see equation (40)).

The density distributions results of figure 5 indicate in a general way the degree of the immiscibility or phase separation of binary condensate due to the interspecies repulsion. In our case the structure is symmetric and it is related with the ratio of number of particles $N_1/N_2$. These results are in complete concordance with recent experimental reported observations for the $^{87}\text{Rb}$–$^{133}\text{Cs}$ binary condensates [5]. The trial wavefunctions (24) cannot take into account these behaviors over the spatial distribution as a function of $\Lambda_{12}$, since they are a priori located at the origin.

In conclusion, we have derived simple explicit expressions for the chemical potentials and order parameters in the case of two species of non-homogeneous BEC, where the system is loaded in a harmonic trap potential. We generalize the variational method for the case of two coupled GP equations, showing that the obtained closed analytical expressions for $\mu_i$ ($i = 1, 2$) represent very good solutions for any values of the vector $\Lambda$ if $V_0 = 0$. Also, employing the perturbation theory we are able to get analytical solutions for $\mu_i$ and the order parameter components $\psi_i$ as functions of the dimensionless vector $\Lambda$. By comparison with the numerical solutions we found the range of validity of the equation (38). By the calculations we show the strong dependence of $\mu_i$ and $\psi_i$ on the strengths $\Lambda_1, \Lambda_2, \Lambda_{12}, \Lambda_{21}$ and $V_0$. This study gives a very useful result establishing the universal range where each solution can be easily implemented. In particular, the dependence of the order parameter $\psi_i$ on $\Lambda_i$ and $\Lambda_{ij}$ ($i \neq j$) allows to study the immiscibility of two given species. We should note that the variational model here developed can be extended to a cubic–quintic model [22] and allows to explore the influence of quintic nonlinear terms on the ideal 1D two coupled pure cigar-like shape system.

Acknowledgments

This work was partially supported by the UFRJ and SECITI-DF/CLAF. C T-G wishes to acknowledge the hospitality of the Instituto de Matemática, UFRJ.
Appendix A. Matrix elements

The fourth dimensional matrix $S(l_i)$ is defined as

$$S_{mnpl}(l_i) = \frac{1}{\pi^{2n+m+l+p}n!m!l!p!} \int_{-\infty}^{\infty} \exp[-(1 + l_i^2)z^2]H_n(z),$$

with $H_n(z)$ the Hermitian polynomials [34]. The matrix elements $S_{mnpl}(l_i)$ have the following properties:

(i) $l_i S_{mnpl}(l_i) = S_{plmn}(1/l_i)$;

(ii) $S_{2m;0,0}(l_i)$, $S_{0,0,m}(l_i)$ and $S_{km;0,0}(l_i)$ are equal to [35]

$$S_{2m;0,0}(l_i) = \frac{(-1)^m \sqrt{(2m)!}}{\pi (1 + l_i^2)^{2m}!} \frac{l_i^2}{(1 + l_i^2)^m}.$$  

$$S_{0,0,m}(l_i) = \frac{(-1)^m \sqrt{(k+m)!}}{\pi \sqrt{k!m!}! 2^{m+1}(k+m)!} \left( \frac{l_i^2}{1 + l_i^2} \right)^{\frac{m(k+m+1)}{2}}.$$  

$$S_{km;0,0}(l_i) = \frac{(-1)^m \Gamma \left( \frac{k+m+1}{2} \right)}{\pi \sqrt{k!m!}! 2^{m+1}(k+m)!} \left( \frac{l_i^2}{1 + l_i^2} \right)^{\frac{m(k+m+1)}{2}} F \left( -k-m, \frac{1-k-m}{2}, \frac{1+2l_i^2}{2l_i^2} \right).$$

with $\Gamma(z)$ the Gamma function [34] and $F(\alpha, \beta; z)$ the confluent hypergeometric function [35].

Using the above relations it is possible to get equations (38)–(40).

Appendix B. Functions

The functions introduced in equation (38) are defined as:

$$f(z) = \ln \left[ \frac{1}{2} + \frac{1}{2} \sqrt{\frac{2+z^2}{2(1+z^2)}} \right].$$ 

$$g(z) = \ln \left[ \frac{1}{2} + \frac{1}{2} \sqrt{\frac{1+2z^2}{(1+z^2)^2}} \right].$$ 

$$h(z) = \text{Ei}(z) - \mathcal{C} - \ln z; \quad \text{ch}(z) = \text{Chi}(z) - \mathcal{C} - \ln z,$$ 

where $\text{Ei}(z) = \int_{-\infty}^{x} \frac{\exp(x)}{x} \, dx$ is the exponential integral, $\text{Chi}(z)$ the cosine hyperbolic integral, and $\mathcal{C}$ the Euler’s constant.

In equation (40) the $\mathcal{G}(\xi; z)$ is given by

$$\mathcal{G}(\xi; z) = \frac{\exp(-\xi^2/2)}{z \sqrt{\pi} \sqrt{1/2}} \int_{1}^{1/z} \frac{\exp \left[ -\xi z(1-\gamma^2) \right] - 1}{1-\gamma^2} \, dy.$$  

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References

[1] Myatt C J, Burt E A, Ghrist R W, Cornell E A and Wieman C E 1997 Phys. Rev. Lett. 78 586
[2] Stenger J, Inouye S, Stamper-Kurn D M, Miesner H-J, Chikkatur A and Ketterle W 1998 Nature 396 345
[3] Modugno G, Modugno M, Riboli F, Roati G and Inguscio M 2002 Phys. Rev. Lett. 89 190404
[4] Thalhammer G, Barontini G, De Sarlo L, Catani J, Minardi F and Inguscio M 2008 Phys. Rev. Lett. 100 210402
[5] McCarron D J, Cho H W, Jenkin D L, Köppinger M P and Cornish S L 2011 Phys. Rev. A. 84 011603(R)
[6] Sugawa S, Yamazaki R, Sh Taie and Takahashi Y 2011 Phys. Rev. A. 84 011610(R)
[7] Damski B, Santos L, Tiemann E, Lewenstein M, Kotochigova S, Julienne P and Zoller P 2003 Phys. Rev. Lett. 90 110401
[8] Papp S B, Pino J M and Wieman C E 2008 Phys. Rev. Lett. 101 040402
[9] Catani J, De Sarlo L, Barontini G, Minardi F and Inguscio M 2008 Phys. Rev. A 77 011603(R)
[10] Altman E, Hofstetter W, Demler E and Lukinet M D 2003 New J. Phys. 5 113
[11] Cipolatti R, Kavian O 2001 Cipolatti R and Kavian O 2001 Phys. Rev. A. 67 043615
[12] Kuklov A, Prokof’ev N and Svistunov B 2004 Phys. Rev. Lett. 92 050402
[13] Ott H, de Mirandes E, Ferlaino F, Roati G, Modugno G and Inguscio M 2004 Phys. Rev. Lett. 92 160601
[14] Ho Tin-Lun and Shenoy V B 1996 Phys. Rev. Lett. 77 3276
[15] Pu H and Bigelow N P 1998 Phys. Rev. Lett. 80 1130
[16] Saito H, Kawaguchi Y and Ueda M 2009 Phys. Rev. Lett. 102 230403
[17] Linder J and Sudbø A 2009 Phys. Rev. A 79 063610
[18] Suthar K, Roy A and Angom D 2015 Phys. Rev. A 91 043615
[19] Roscilde T and Cirac J Ignacio 2007 Phys. Rev. Lett. 98 190402
[20] Wilkinson S R, Bharucha C F, Madison K W, Niu Q and Raizen M G 1996 Phys. Rev. Lett. 76 4512
[21] Cataliotti F S, Burger S, Fort C, Maddaloni P, Minardi F, Trombettoni A, Smerzi A and Inguscio M 2001 Science 293 843
[22] Ott H, de Mirandes E, Ferlaino F, Roati G, Modugno G and Inguscio M 2007 Phys. Rev. Lett. 98 190402
[23] Dalmonte M, Ercolessi E, Mattioli M, Ortolani F and Vodola D 2013 Eur. Phys. J. Spec. Top. 217 15
[24] Carretero-González R, Frantzeskakis D J and Kevrekidis P G 2008 Nonlinearity 21 R139
[25] Trallero-Giner C, Cipolatti R and Liew T C H 2013 Eur. Phys. J. D 67 143
[26] Saito H, Kawaguchi2 Y and Ueda M 2009 Phys. Rev. Lett. 102 230403
[27] Trallero-Giner C, Drake J, Lopez-Richard V, Herrero C Trallero- and Birman Joseph L 2006 Phys. Rev. Lett. A 354 115
[28] Trallero-Giner C, Drake-Perez J C, López-Richard V and Birman Joseph L 2008 Physica D 237 2342
[29] Trallero-Giner C, Lopez-Richard V, Chung M-Ch and Buchleitner A 2009 Phys. Rev. A 79 063211
[30] Cipolatti R, Gendiar J L and Trallero-Giner C 2012 Physica D 241 755
[31] Cipolatti R and Kavian O 2001 J. Differ. Equa. 176 223
[32] Cazenave T 2003 Semilinear Schrödinger Equations (Courant Lecture Notes in Mathematics vol 10) (Providence, RI: American Mathematical Society)
[33] Hajaiej H 2014 J. Math. Anal. Appl. 420 195
[34] Brezis H 1983 Analyse Fonctionnelle, Théorie et Applications (Paris: Masson)
[35] Morse P M and Feshbach H 1953 Methods of Theoretical Physics (New York: McGraw-Hill)
[36] Landau L D and Lifshitz E M 1977 Quantum Mechanics (UK: Pergamon)
[37] Abramowitz M and Stegun I (ed) 1972 Handbook of Mathematical Functions (New York: Dover)
[38] Gradshteyn I S and Ryzhik I M 1980 Tables of Integrals, Series and Products (New York: Academic)