INTEGRAL COMPARISONS OF NONNEGATIVE POSITIVE DEFINITE FUNCTIONSON LOCALLY COMPACT ABELIAN GROUPS

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Abstract. In this paper, we discuss the following general questions. Let \( \mu, \nu \) be two regular Borel measures of finite total variation. When do we have a constant \( C \) satisfying that
\[
\int f \, d\nu \leq C \int f \, d\mu
\]
whenever \( f \) is a continuous nonnegative positive definite function? How the admissible constants \( C \) can be characterized and what is the best value?

First we discuss the problem in locally compact Abelian groups and then apply the results to the case when \( \mu, \nu \) are the traces of the usual Lebesgue measure over centered and arbitrary intervals, respectively. This special case was earlier investigated by Shapiro, Montgomery, Halász and Logan. It is a close companion of the more familiar problem of Wiener, as well.

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Let $\mu, \nu$ be two regular Borel measures of finite total variation. In this paper we discuss the following general question:

**Problem 1.** Under what conditions do we have a constant $C$ satisfying

\begin{equation}
\int f \, d\nu \leq C \int f \, d\mu
\end{equation}

whenever $f$ is any continuous nonnegative positive definite function vanishing at infinity?

This question was posed by Halász (in oral communication) in the context of the similar problem of Shapiro [24] and of Montgomery [22] on the real line. However, observe that this formulation is also an interpretation of the question originally not formulated precisely as regards the type of measures and the function spaces for the integrands. We can extend the question to arbitrary locally compact Abelian (abbreviated as LCA) groups without any difficulty, so this will be our setup in the main part of the paper.

In fact, there are four motivations for the present work. First, we would like to clarify matters regarding a recent paper [7], which in its main part, rediscovered an old estimate of Logan [20]. Second, we aim to prove a duality conjecture formulated in the same paper [7]. Third, we answer the closely related but much more general question in Problem [1]. Fourth, we extend the investigations from $\mathbb{R}$, where these investigations took place so far, to general LCA groups. In this regard we note that the closely related, yet much better known Wiener problem has been investigated in the generality of $d$-dimensional Euclidean spaces and torii in [10], but – to the best of our knowledge – the topic has not been discussed in general LCA groups yet.

To describe the origins of the problem in focus one has to start with the better known Wiener problem of estimating the integral of $|f|^p$ over an arbitrary interval $(a - \delta, a + \delta)$ by the integral of the same over the centered interval $(-\delta, \delta)$ for arbitrary (say continuous) positive definite functions $f$ on the torus $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. The original Wiener question was in fact posed for the estimation of the full integral over the whole period (the whole torus) by the integral over the centered interval $(-\delta, \delta)$ – but these are essentially equivalent problems. The interpretation of the original Wiener question becomes of essence when we pass on to the real line, where the integral over the whole real line might be unbounded, and thus Wiener’s problem “has a negative answer in general”, if we interpret it as the comparison of the full integral and the integral over a finite centered interval. This is the interpretation what e.g. the recent study [10] takes. Instead, we interpret the problem as the comparison of finite integrals (integrals over compact sets) in order to have a meaningful extension to $\mathbb{R}$ and other noncompact cases.

The Wiener question itself has been described and popularized only later by Shapiro [24], while Wiener himself published only his investigations with the additional focus to series with gaps [28]. It is well-known and a relatively easy fact that for $p = 2$, whence also for $p \in 2\mathbb{N}$, the answer to Wiener’s problem is positive, while it is a highly nontrivial matter that for other values of $p$ the question has a negative answer [27] [24] [2].

From the classical proof of Wiener’s result (of the positive case) it is clear that the crucial property is that the integrand must be *doubly positive* (i.e. both positive definite and nonnegative itself) instead of being taken to absolute value square. We will denote this double positivity condition by $f \gg 0$ (while positive definiteness itself is denoted by $f \gg 0$, as usual). Shapiro pointed out that with this assumption the estimation statement holds for all integer values of $p$, not only for $p \in 2\mathbb{N}$. (Shapiro was discussing the case of $\mathbb{T}$ only.) The price we pay is that then one must assume at the outset that $f \gg 0$ and also $f \geq 0$. From this the positive answer cases of the Wiener problem follow by noting that $f \gg 0$ implies that $|f|^2 \gg 0$. 

1. Introduction
Later Logan claimed having found the exact solution to Shapiro’s extremal problem of finding the best constant $S(\delta)$ in this question for the torus $T$, see the turn of pages 369 and 370 in [20]. However, we have found no subsequent mention of this extremal problem in his further publications.

Wiener’s problem has a considerable literature (especially for $T$) but for more on the history and development of the Wiener problem itself we refer to [23, 20, 22] as our focus here is the question of integrals of doubly positive functions only. This question somehow remained in the shadow of the better known Wiener problem. After Shapiro mentioned this variant of the problem there seems to be no direct mention until the paper of Logan [20].

In the case where $d\nu = \chi_{[a-T,a+T]}dx$ and $d\mu = \chi_{[-1,1]}dx$ Logan found the upper estimate

\[(2) \quad C \leq C(T) = \frac{1}{2} \left( \frac{[2T]+1}{[2T]+1-T} \right)^2 \]

in the setting of Dirichlet polynomials $P(t) = \sum_k a_k e^{i\lambda_k t}$ with finite sums and different real exponents $\lambda_k$. In [7] we obtained the same upper bound, in the setting of positive definite nonnegative functions. Although the proof in [20] relies on ad-hoc ideas, our proof provides a more direct approach to this sort of problems, with a simple idea. Namely, we observe that if $h$ is a positive definite, say continuous function satisfying

\[ h \leq h_C := C\chi_{[-1,1]} - \chi_{[a-T,a+T]} - \chi_{[-a-T,-a+T]} \]

then for every continuous $f \gg 0$, we have

\[ 0 \leq \int_{-\infty}^{+\infty} f \cdot h \cdot dx \leq \int_{-\infty}^{+\infty} f \cdot h_C \cdot dx = C \int_{-1}^{1} f \cdot dx - \int_{a-T}^{a+T} f \cdot dx - \int_{-a-T}^{-a+T} f \cdot dx \]

meaning that the constant $C/2$ is admissible. We conjectured that this approach is in principle optimal, that is, the following should hold true.

**Conjecture 1.** The constant

\[ \frac{1}{2} \inf \left\{ C : \exists h \text{ continuous positive definite such that } h \leq C\chi_{[-1,1]} - \chi_{[a-T,a+T]} - \chi_{[-a-T,-a+T]} \right\} \]

is the best among the admissible ones.

While we will prove this conjecture in §8, we are still unable to decide if the estimate (2), already obtained in [7] with a somewhat different construction, can be improved or if it is sharp. It is certainly sharp for $2T \in \mathbb{N}$, as is shown already by Logan, but the possibly minor difference between the estimate and the exact constant escapes all our attempts to catch it.

The main result of the present paper is the following answer to Problem [11].

**Theorem 1.** Let $\mu, \nu$ be two (bounded, regular, Borel) real measures on the LCA group $G$. Then (1) holds for all continuous functions $f \gg 0$ if and only if $C\mu - \nu$ can be decomposed as

\[ C\mu - \nu = \sigma + \tau + o, \]

where $\sigma$ is a nonnegative real measure, $\tau$ is a real measure of positive type and $o$ is an odd measure.

This result is somewhat abstract and it is far from easy to check the characterization provided by it. A moment’s thought may convince the reader that the condition on the decomposability of the measure $C\mu - \nu$ is a sufficient one for the inequality to hold. Yet, the result is neither trivial nor useless, as we hope to demonstrate in the following. In particular, as a nontrivial application of Theorem [11] in §8 we will be able to prove Conjecture [11] as well.
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2. An overview of harmonic analysis on LCA groups

2.1. Radon measures. Let $\mathbb{F}$ be either $\mathbb{C}$ or $\mathbb{R}$. For a locally compact (Hausdorff) topological space $X$ let us denote by $C(X, \mathbb{F})$ the set of all continuous $\mathbb{F}$-valued functions on $X$. Furthermore, denote by $C_c(X, \mathbb{F})$, $C_0(X, \mathbb{F})$ and $\mathcal{C}_b(X, \mathbb{F})$ the subset of those functions of $C(X, \mathbb{F})$ which have compact support, vanish at infinity and are bounded, respectively. The inclusions

$C_c(X, \mathbb{F}) \subseteq C_0(X, \mathbb{F}) \subseteq \mathcal{C}_b(X, \mathbb{F}) \subseteq C(X, \mathbb{F})$

are obvious. If $\mathcal{C}_b(X, \mathbb{F})$ is equipped with the supremum norm $\| \cdot \|_\infty$ it forms a normed space $(\mathcal{C}_b(X, \mathbb{F}), \| \cdot \|_\infty)$. It is well-known that the closure of $C_c(X, \mathbb{F})$ is $C_0(X, \mathbb{F})$ in this supremum norm, while the latter is a closed subspace in $\mathcal{C}_b(X, \mathbb{F})$ (and a proper one exactly when $X$ is not compact). Further, $C(X, \mathbb{F})$ might be equipped with the locally uniform convergence topology $U_{loc}$. This makes $C(X, \mathbb{F})$ a topological vector space $(C(X, \mathbb{F}), U_{loc})$. The closure of $C_c(X, \mathbb{F})$ with respect to $U_{loc}$ is already $C(X, \mathbb{F})$ – whence the same holds for the intermediate spaces, too.

Radon measure is one of the most fundamental concepts in abstract harmonic analysis. In fact, there are plenty of different terminologies in the literature. In this paper we adopt the following one:

(i) A linear functional $T : C_c(X, \mathbb{C}) \to \mathbb{C}$ is termed to be a (complex) Radon measure if for any $K \subseteq X$ compact set (the symbol $\subseteq$ stands for compact inclusion throughout the paper) there exists an $L > 0$ such that $|T(f)| \leq L \|f\|$ holds whenever $\text{supp } f \subseteq K$.

(ii) A Radon measure $T$ is said to be a real Radon measure if for any $f \in C_c(X, \mathbb{R})$ we have $T(f) \in \mathbb{R}$.

(iii) A Radon measure is said to be a positive Radon measure if for any continuous compactly supported function $f \geq 0$ we have $T(f) \geq 0$.

Note that Radon measures in general are assumed to be continuous (or bounded) functionals on neither $(C_c(X, \mathbb{F}), \| \cdot \|_\infty)$ nor $(C_c(X, \mathbb{F}), U_{loc})$.

The family of all $\mathbb{F}$-valued Radon measures will be denoted by $R(X, \mathbb{F})$.

(iv) An element of the dual space $(C_c(X, \mathbb{F}), \| \cdot \|_\infty)'$ (or equivalently, of $(C_0(X, \mathbb{F}), \| \cdot \|_\infty)'$) is called a bounded Radon measure.

The celebrated Riesz representation theorem (see, e.g., [23, Appendix E4]) determines the connection between the above abstract notion of Radon measures and the standard measure theoretic one. Namely, the elements of $(C_c(X, \mathbb{F}), \| \cdot \|_\infty)'$ are exactly of the form

$$f \mapsto \int f \, d\mu \quad (f \in C_c(X, \mathbb{F}))$$

with some regular complex Borel measure $\mu$ of finite total variation. Just as in (3), the dual space $(C_c(X, \mathbb{F}), U_{loc})'$ could be identified with the set of compactly supported Radon measures [6, 4.10.1 Theorem]. We introduce the following notations:

(i) $M(X, \mathbb{F})$: the set of bounded Radon measures, or, equivalently, $\mathbb{F}$-valued Borel measures of finite total variation;

(ii) $M_+(X)$: the set of bounded positive Radon measures, or, equivalently, nonnegative Borel measures of finite total variation;
(iii) $M_c(X, F)$: the set of bounded Radon measures with compact support, or, equivalently, $F$-valued Borel measures of compact support.

Note that the class of general (possibly unbounded) Radon measures can also be characterized in a measure-theoretic way. Let the symbol $\mathcal{B}$ stand for the Borel sigma-algebra with $\mathcal{B}_0$ being its subset of Borel sets with compact closure. Then $R(X, F)$ is the family of set functions on $\mathcal{B}_0$, the trace (restriction) of which to any compact set is a regular Borel measure.

In the sequel both complex and real valued functions and measures show up frequently. To make our notation easier, we drop the notation $C$ if the corresponding functions or measures are assumed to be complex valued, and the real valued counterparts are denoted by underlying the same symbols. For instance, we write $C(X)$, $M(X)$ and $\mathcal{C}(X), \mathcal{M}(X)$, accordingly.

2.2. Haar measure on LCA groups. Let $G$ be an LCA group. If $E \subseteq G$ is a Borel set, then $\chi_E$ denotes its characteristic function. A Radon measure $\mu$ is said to be translation invariant if $\mu(f(x + g)) = \mu(f(x))$ holds for any $f \in C_c(G)$ and $g \in G$. This condition is equivalent to assuming that the representing measure satisfies $\mu(E + g) = \mu(E)$ for any Borel set $E \in \mathcal{B}_0$ and $g \in G$. A nonzero translation invariant Radon measure is called Haar measure which is unique up to a harmless normalization in any LCA group, see [8, §2.2] for a full proof. This unique Haar measure will be denoted by $\lambda$. As a direct consequence of uniqueness, we also have $\lambda(E) = \lambda(-E)$ for all Borel measurable set $E$, see [23, 1.1.4].

2.3. Fourier transform. A continuous map $\gamma$ from the LCA group $G$ into the complex unit circle $\mathbb{T}$ satisfying

$$\gamma(x + y) = \gamma(x)\gamma(y) \quad (x, y \in G)$$

is called a character of $G$. The set of all characters of $G$ forms a group (with pointwise multiplication) which is called the dual group of $G$ and is denoted by $\hat{G}$. The dual group $\hat{G}$ when it is equipped with the locally uniform convergence topology is also an LCA group. Moreover, the Pontryagin-van Kampen duality theorem (see [15, (24.8)] or [23, 1.7.2. Theorem]) tells us that the mapping from $G$ to $\hat{\hat{G}}$ defined as

$$g \mapsto \left( \gamma \mapsto \gamma(g), \quad (\forall \gamma \in \hat{G}) \right)$$

provides a homeomorphic group isomorphism between $G$ and $\hat{\hat{G}}$. Hence the characters of $\hat{G}$ are exactly the so-called "point value evaluation functionals" at any fixed $g \in G$, by what we mean the functions $\gamma \mapsto \gamma(g) \ (\gamma \in \hat{G})$.

For any $\mu \in M(G)$, the symbol $\hat{\mu}$ denotes the Fourier transform of $\mu$, that is,

$$\hat{\mu} \in \mathcal{C}(C); \quad \hat{\mu}(\gamma) = \int_{G} \gamma d\mu \quad (\gamma \in \hat{G}).$$

The Fourier transform of $\hat{\mu}$ is bounded by $\|\mu\|$ and is uniformly continuous on $\hat{G}$, see [23, 1.1.3 Theorem]. The inverse Fourier transform of a measure $\nu \in M(\hat{G})$ is defined by

$$(4) \quad \hat{\nu}(x) = \int_{\hat{G}} \gamma(x) d\nu(\gamma) \quad (x \in G).$$

\(^1\)Recall that our terminology comes from measure theory instead of probability theory where the term "characteristic function" (of a probability distribution) refers to Fourier transform, while the characteristic function of a set is called its "indicator function".
The following formula is a version of the so-called Plancherel theorem and in fact it is an easy consequence of Fubini’s theorem: if \( \nu \in M(G) \) and \( \sigma \in M(\hat{G}) \), then

\[
\nu(\sigma) := \int_{\hat{G}} \sigma d\nu = \int_{\hat{G}} \int_{G} \gamma(x) d\sigma(\gamma) = \int_{\hat{G}} \hat{\nu} d\sigma =: \sigma(\hat{\nu}).
\]

2.4. Convolution of Radon measures and functions. First we define the convolution of two Radon measures \( \mu, \nu \) on the LCA group by the formula

\[
(\mu \ast \nu)(\varphi) = \int_{G} \varphi(x + y) d\mu(x) d\nu(y) \quad (\varphi \in C_c(G))
\]

whenever the double integral exists. This condition fulfills for example in the following two important cases \([6, 4.19.2-4]\):

(i) at least one of the Radon measures \( \mu \) or \( \nu \) has compact support;

(ii) the Radon measures \( \mu, \nu \) are bounded.

By identifying absolutely continuous measures with their Radon-Nikodym derivatives the concept of convolution is extended to functions. Let us calculate the convolution of Radon measures when a reference Haar measure \( \lambda \) is fixed on \( G \) and \( d\mu = f d\lambda \) with some \( f \in L^1(G) \). Then by an easy application of Fubini’s theorem we have

\[
(\mu \ast \nu)(\varphi) = \int_{G} \varphi(x + y) f(x) d\lambda(x) d\nu(y) = \int_{G} \left( \int_{G} \varphi(x) f(x - y) d\lambda(x) \right) d\nu(y) = \int_{G} \varphi(x) \left( \int_{G} f(x - y) d\nu(y) \right) d\lambda(x) \equiv \rho(\varphi)
\]

where \( \rho \) is the absolutely continuous measure associated to the Radon-Nikodym derivative \( d\rho/d\lambda = \int_{G} f(x - y) d\nu(y) \). Hence one may define the convolution of any \( f \in L^1(G) \) and a Radon measure \( \nu \) on \( G \) as

\[
(f \ast \nu)(x) := \int_{G} f(x - y) d\nu(y)
\]

whenever, for example, either (i) or (ii) is satisfied for \( d\mu = f d\lambda \) and \( d\nu \). Note that the above definition is also proper when \( f \in L^1_{\text{loc}}(G) \) and the measure \( \nu \) has compact support. In the case where both \( \mu \) and \( \nu \) are absolutely continuous with Radon-Nikodym derivatives \( f, g \in L^1(G) \), respectively, we have

\[
(f \ast g)(x) = \int_{G} f(x - y) g(y) d\lambda(y).
\]

Again, the definition \([6]\) is also proper for \( f \in L^1_{\text{loc}}(G) \) and \( g \in L^1(G) \) with compact support. In particular if \( f \in L^1_{\text{loc}}(G) \) and \( g \in C_c(G) \), then \( f \ast g \in C(G) \). If \( f \) is locally uniformly integrable (that is, there exists some open set \( U \) with compact closure such that \( \int_{U+x} |f| \leq C \) for all \( x \in G \)), then \( f \ast g \in C_b(G) \), and it is uniformly continuous, too.

The convolution makes the set \( L^1(G) \) a commutative Banach algebra possessing the property \( \hat{f \ast g} = \hat{f} \ast \hat{g} \) \([23, Theorem 1.2.4]\) for any \( f, g \in L^1(G) \). Similarly, for \( \nu, \mu \in M(G) \) the convolution \( \mu \ast \nu \) belongs to \( M(G) \) and we also have \( \hat{\mu \ast \nu} = \hat{\mu} \ast \hat{\nu} \) \([23, Theorem 1.3.3]\).
As for general Borel functions \( f \) and \( g \), their convolution also could be defined by the formula \([10]\), at least in the pointwise sense for those \( x \in G \) for which the integral
\[
\int_G |f(x - y)g(y)|d\lambda(y)
\]
exists. In the particular case when \( f, g \in L^2(G) \) the convolution \( f \ast g \) is defined everywhere on \( G \) and \( f \ast g \in C_0(G) \) \([23\text{ Theorem 1.1.6(d)}]\).

3. Positive definite functions, functions and measures of positive type

3.1. Positive definite functions. Here, we single out some basic notions and results on positive definite functions which will be needed throughout the sequel.

On a LCA group \( G \) a function \( f \) is called positive definite (denoted by \( f \gg 0 \)) if the inequality
\[
\sum_{j=1}^{n} \sum_{k=1}^{n} c_j c_k f(x_j - x_k) \geq 0
\]
holds true for all choices of \( n \in \mathbb{N} \), \( c_j \in \mathbb{C} \) and \( x_j \in G \) for \( j = 1, \ldots, n \). First of all, we warn the reader that here (in accordance with for instance \([14, 15]\)) by default the term 'function' means that it takes finite values everywhere on its domain. For example, in the paper of Logan \([20]\) it is not so, as there locally integrable functions come into picture which are defined only almost everywhere (or, equivalently up to equivalence classes modulo a measure zero set).

We shall frequently use the following well-known and immediate consequences of the definition. For an arbitrary \( f \gg 0 \) we have (see e.g. \([23\text{ §1.4.1]}\)):

- (p1) the value \( f(0) \) is a nonnegative real number;
- (p2) the function \( f \) is necessarily bounded in absolute value by \( f(0) \);
- (p3) the continuity of \( f \) all over \( G \) is equivalent to that at 0;
- (p4) \( f(x) = \tilde{f}(-x) \) holds for all \( x \in G \) \([23\text{ p. 18, Eqn (2)}]\), whence
- (p5) the support of \( f \) is necessarily symmetric and the condition \( \text{supp} f \subseteq \Omega \) implies \( \text{supp} f \subseteq \Omega \cap (-\Omega) \).

We will use the following further notations:

- (i) \( \mathcal{P} \): the cone of nonnegative continuous functions, that is, \( \mathcal{P} := \{ f \in \mathcal{C}(G) : f \geq 0 \} \);
- (ii) \( \mathcal{P}_0 \): the nonnegative cone of \( C_0(G) \), that is, \( \mathcal{P}_0 := \mathcal{P} \cap C_0(G) \);
- (iii) \( \mathcal{P}_c \): the nonnegative cone of \( \mathcal{C}_c(G) \), that is, \( \mathcal{P}_c := \mathcal{P} \cap \mathcal{C}_c(G) \);
- (iv) \( \mathcal{D}, \mathcal{D}_c \): the family of continuous complex and real valued positive definite functions, i.e.,
  \[ \mathcal{D} := \{ f \in \mathcal{C}(G) : f \gg 0 \}, \quad \mathcal{D}_c := \{ f \in \mathcal{C}_c(G) : f \gg 0 \}; \]
- (v) \( \mathcal{D}_0, \mathcal{D}_{0c} \): the cone of all positive definite elements of \( C_0(G), \mathcal{C}_0(G) \), resp., that is,
  \[ \mathcal{D}_0 := \mathcal{D} \cap C_0(G), \quad \mathcal{D}_{0c} := \mathcal{D} \cap \mathcal{C}_0(G); \]
- (vi) \( \mathcal{D}_c, \mathcal{D}_{c0} \): the cone of all positive definite elements of \( \mathcal{C}_c(G), \mathcal{C}_c(G) \), resp., that is,
  \[ \mathcal{D}_c := \mathcal{D} \cap \mathcal{C}_c(G), \quad \mathcal{D}_{c0} := \mathcal{D} \cap \mathcal{C}_0(G). \]

Let us see some fundamental examples for positive definite functions. Characters play a fundamental role whenever LCA groups are considered. So it is of relevance that all characters \( \gamma \in \hat{G} \) of a LCA group \( G \) are positive definite. To see this, one need to use only the multiplicative property of the characters to get
\[
\sum_{j=1}^{n} \sum_{k=1}^{n} c_j c_k \gamma(x_j - x_k) = \sum_{j=1}^{n} \sum_{k=1}^{n} c_j \gamma(x_j) c_k \gamma(x_k) = \left| \sum_{j=1}^{n} c_j \gamma(x_j) \right|^2 \geq 0
\]
for all choices of $n \in \mathbb{N}$, $c_j \in \mathbb{C}$ and $x_j \in G$ for $j = 1, \ldots, n$.

Together with any $f \gg 0$ also the functions $f^*$ (where $f^*(x) := f(-x)$) and $\overline{f}$ are positive definite functions. It is a nontrivial fact that if $f, g \gg 0$, then so is $fg$. This assertion follows from Schur's theorem (see [13, §85, Theorem 2] or [15, (D.12) Lemma, Appendix D, pp. 683–684]) concerning the Hadamard product (that is, the entrywise product) of positive semidefinite matrices. By that result, if $A = [a_{jk}]_{j=1,\ldots,n}^{k=1,\ldots,n}$ and $B = [b_{jk}]_{j=1,\ldots,n}^{k=1,\ldots,n}$ are positive semidefinite matrices (in the usual linear algebraic sense), then so is their Hadamard product $[a_{jk}b_{jk}]_{j=1,\ldots,n}^{k=1,\ldots,n}$.

In particular, if $f \gg 0$ and $\gamma \in \hat{G}$ is a character, then $\gamma f$ and $\overline{\gamma f}$ are both positive definite.

To obtain further examples, assume that $f$ and $g$ are positive definite functions on $G$. One easily checks that if $\alpha, \beta > 0$ are arbitrary positive constants, then $\alpha f + \beta g \gg 0$.

It is also a simple matter to check that the inverse Fourier transform of any $\nu \in M_+(\hat{G})$ is positive definite. As it was noted above, it is continuous, as well. According to the celebrated Bochner-Weil theorem (cf. [6, 10.3.8 Theorem] or [23, 1.4.3. Theorem]) the converse statement is also true. Namely, a continuous function $f \in \mathcal{C}(G)$ is positive definite if and only if $f$ can be obtained as the inverse Fourier transform of a bounded positive Radon measure $\nu \in M_+(\hat{G})$.

Another fundamental way to obtain positive definite continuous functions is to take "convolution squares" in the following sense.

**Lemma 1.** [23, §1.4.2(a)] Let $f \in L^2(G)$ be arbitrary. Then the "convolution square" $f \ast \overline{f}$ exists and it is a continuous positive definite function. Moreover, we have $f \ast \overline{f} \in \mathcal{D}_0$, too. Furthermore, if $\text{supp} \ f \subseteq G$, then we also have $f \ast \overline{f} \in \mathcal{D}_c$.

In general, positive definite continuous functions cannot be represented as "convolution squares". However, this still holds true for $f \in \mathcal{D}_c$.

**Lemma 2.** [15, p. 309, (33.24) (a)] Let $f \in \mathcal{D} \cap L^1(G)$ be arbitrary. Then there exists a so-called "Boas-Kac square-root" (or, in another words, "convolution square-root") $g \in L^1(G)$ satisfying $g \ast g = f$. Furthermore, if $f \in \mathcal{D}_c$, then we also have $g \in G$.

### 3.2. Functions and measures of positive type

The above discussed concept of positive definiteness is due to Toeplitz [23] on $\mathbb{T}$ and Matthias [21] on $\mathbb{R}$. However, in the literature another sort of closely related function class appeared. Essentially following Godement [9], but adopting the later terminology of Folland [8], a function $f \in L^1_{loc}(G)$ will be called a *function of positive type* if it defines a positive 2-homogeneous functional on the Banach algebra $\mathcal{C}_c(G)$ in the sense that it holds

$$\int_G f(u \ast \overline{u}) \ d\lambda \geq 0 \quad (\forall u \in \mathcal{C}_c(G))$$

or, equivalently

$$u \ast \overline{u} \ast f(0) \geq 0 \quad (\forall u \in \mathcal{C}_c(G)).$$

The set of functions of positive type will be denoted by $\mathcal{D}^*$. Note the distinction between the classes of positive definite *functions*, defined finitely everywhere and satisfying [17], and "functions" of *positive type*, defined only almost everywhere in accordance with [8].

As a matter of fact, one can define functions of positive type *with respect to a given class $\mathcal{F}$* of functions, playing the role of $\mathcal{C}_c(G)$ above. It seems that this idea was first analyzed by Cooper [31] on the real line. For instance in the particular case $G = \mathbb{R}$ it is well-known since [33] that positive type with respect to $\mathcal{C}_c(G)$ and $L^2(G)$ are the same. Next we discuss this in the slightly more general settings of measures below.

In a similar fashion, a Radon measure $\mu$ is said to be a *measure of positive type* with respect to a given class of suitable functions $\mathcal{F}$ whenever

$$\int_G (u \ast \overline{u}) \ d\mu \geq 0 \quad (\forall u \in \mathcal{F})$$

or, equivalently

$$u \ast \overline{u} \ast \mu(0) \geq 0 \quad (\forall u \in \mathcal{F}).$$
Here, we implicitly assume that the above expressions are well-defined (that is, the corresponding integrals exist at least in the extended sense) and \( \mathcal{F} \) matches the class \( \mathcal{F}^*: = \{ f^*: f \in \mathcal{F} \} \). In our applications these conditions are always satisfied.

Denote by \( \mathcal{M}(\mathcal{F}) \) the family of all measures of positive type with respect to \( \mathcal{F} \). For the most important particular cases we introduce the notations \( \mathcal{M} := \mathcal{M}(\mathcal{C}_c(G)) \) and \( \mathcal{M}^2 := \mathcal{M}(L^2(G)) \). If no ambiguity occurs, we take the freedom to write \( \mu \gg 0 \) equivalently to \( \mu \in \mathcal{M} \). However, we need to emphasize the differences between the definitions of positive definiteness and positive type, in general.

**Lemma 3.** Let \( F \subseteq \mathbb{C} \) be a closed set and \( \mu \in M(G) \) arbitrary. If \( \mu(u \ast \tilde{u}) \in F \) holds for every \( u \in \mathcal{C}_c(G) \), then we necessarily have \( \mu(g \ast \tilde{g}) \in F \) for all \( g \in L^2(G) \), too.

**Proof.** Choose an \( \varepsilon > 0 \) and let \( g \in L^2(G) \) be arbitrary. Note that \( g \ast \tilde{g} \in \mathcal{C}_0(G) \) and thus \( \mu(g \ast \tilde{g}) \) exists. Let us take a function \( u \in \mathcal{C}_c(G) \) such that \( \|u - g\|_2 < \varepsilon \); such a \( u \) indeed exists, by the density of \( \mathcal{C}_c(G) \) in \( L^2(G) \).

Observe that \( \mu((g - u) \ast \tilde{u}) = \mu((g - u) \ast \tilde{g}) + \mu(u \ast (g - u)) \) where the terms on the right hand side can be estimated by the Cauchy-Schwarz inequality as

\[
|\mu((g - u) \ast \tilde{g})| \leq \|\mu\| \| (g - u) \ast \tilde{g} \|_\infty \leq \|\mu\| \|g\|_2 \cdot \varepsilon
\]

and similarly

\[
|\mu(u \ast (g - u))| \leq \|\mu\| \|u \ast (g - u)\|_\infty \leq \|\mu\| \|g\|_2 + \varepsilon \cdot \varepsilon.
\]

Thus for any given function \( g \in L^2(G) \) the distance \( |\mu(g \ast \tilde{g}) - \mu(u \ast \tilde{u})| \) can be made arbitrarily small. Therefore, any neighborhood of \( \mu(g \ast \tilde{g}) \) intersects to \( F \). Since \( F \) is closed, \( \mu(g \ast \tilde{g}) \) belongs to \( F \). The proof is complete. \( \square \)

The forthcoming lemma tells us that there is no difference between considering the classes \( \mathcal{C}_c(G) \) or \( L^2(G) \), as long as we restrict our attention to bounded Radon measures.

**Lemma 4.** For a bounded Radon measure \( \mu \) the property (\( \text{p6} \)), that is \( \mu \) being of positive type, is equivalent to the same condition assumed for all weights \( u \in L^2(G) \). In other words, we have \( \mathcal{M} = \mathcal{M}^2 \).

**Proof.** The inclusion \( \mathcal{M} \supseteq \mathcal{M}^2 \) is immediate because \( \mathcal{C}_c(G) \subseteq L^2(G) \). To prove the converse, it suffices to use Lemma 3 for \( F = [0, +\infty) \). \( \square \)

Recall that if \( \mu \) is a measure in \( M(G) \), then its converse \( \hat{\mu} \) is defined by

\[
\hat{\mu}(E) := \hat{\mu}(-E) \quad (E \in \mathcal{B})
\]

which is equivalent to assuming \( \hat{\mu}(f) := \overline{\mu(f^*)} = \mu(\overline{f}) \) for all \( f \in \mathcal{C}_c(G) \), as

\[
\hat{\mu}(f) := \int_G f(x) d\mu(-x) = \int_G f(-x) d\mu(x) = \int_G f(-x) d\mu(x).
\]

In analog with (\( \text{p4} \)) and (\( \text{p5} \)) for positive definite functions, we have the following properties.

\( \text{p6} \) \( \hat{\mu} = \mu \quad (\forall \mu \in \mathcal{M}) \), whence

\( \text{p7} \) the support of \( \mu \) is necessarily symmetric and \( \text{supp} \mu \subseteq \Omega \) entails \( \text{supp} \mu \subseteq \Omega \cap (-\Omega) \).

The property (\( \text{p6} \)) might be folklore as the analogous statement for positive definite functions. However, we did not find a reference for it. Thus, for the sake of completeness we give two independent proofs of the property (\( \text{p6} \)). Moreover, for further use we verify (\( \text{p6} \)) for unbounded measures, too. Recall that in this latter case measures are well-defined only for sets in \( \mathcal{B}_0 \).

The next auxiliary statement – the approximation lemma for constant one locally uniformly – is a fundamental property of LCA groups, see e.g. [9] or [23].
Lemma 5. Let $K \subseteq G$ be arbitrary. Then for any $\varepsilon > 0$ there exists $g \gg 0$, $g \in C_c(G)$ with $g \geq 0$, $g|_K \geq 1$ and $\|g\|_\infty = g(0) \leq 1 + \varepsilon$.

Proof. Although, the statement is well-known and follows e.g. from [19, Lemma 2], we present another short proof here.

We may assume $0 \in K$ at the outset (replacing $K$ by $K \cup \{0\}$, if necessary). According to [23, 2.6.7 Theorem on p. 52] or [13, (31.96) Theorem] we have some set $V \in B_0$ (in fact, an open set) satisfying $\mu(V - K) \leq (1 + \varepsilon)\mu(V)$. Consider the function $h := \chi_{-V - K} \ast \chi_{-K + V}$. It is easy to see that together with $V$ also $K - V$ is open, while in view of continuity of addition $K - V \in B_0$, too. Thus the occurring characteristic functions are in $L^2(G)$. Therefore, in virtue of Lemma [1] we conclude $h \in D_c$. Apparently, we have

$$\|h\|_\infty = h(0) = \int_G \chi_{-V + K}(-y)\chi_{V - K}(y) \, dy = \mu(V - K) \leq (1 + \varepsilon)\mu(V),$$

while for any $x \in K$ we infer that

$$h(x) = \int_G \chi_{-V + K}(x - y)\chi_{V - K}(y) \, dy \geq \int_G \chi_{-V + x}(x - y)\chi_{V}(y) \, dy = \mu(V).$$

Taking $g := \mu(V)^{-1}h$, we obtain the desired function. \hfill \Box

Proof of property (p6). Consider the measure $\nu := \mu - \tilde{\mu}$ together with its Fourier transform $\hat{\nu}$. What we have is that $\nu(g \ast \tilde{g}) = 0$ for all $g \in L^2(G)$. Indeed, if $f := g \ast \tilde{g} \in C_0(G)$, then $\hat{f} = f$. Further we also have $\mu(f) = \mu(f)$ because $\mu(f) \geq 0$. Thus, $\nu(f) = \mu(f) - \tilde{\mu}(f) = \mu(f) - \mu(\tilde{f}) = 0$, as asserted. Since $\{g \ast \tilde{g} : g \in L^2(G)\}$ matches the class $\{g \ast g : g \in L^2(G)\}$ we infer

$$\int_G g \ast g d\nu = 0 \quad (g \in L^2(G)) \quad (10)$$

The $L^2$-Fourier transform of $g$ being $\widehat{g}$, we want to show first that the function $g \ast \tilde{g} \in C_0(G)$ is the usual $L^1$-inverse Fourier transform of the $L^1_+(\widehat{G})$ function $h := |\tilde{g}|^2$. Indeed, the $L^2$-Fourier transform is an isometry, whence $\widehat{g} \in L^2(\widehat{G})$ and so $h := |\tilde{g}|^2 \in L^1_+(\widehat{G})$. Note that $\widehat{g} = \overline{\tilde{g}}$, by [13, (23.10.) Theorem (iv)]. Moreover, according to [13, (31.29) Theorem] the inverse Fourier transform of $h = \widehat{g}$ is $\tilde{g}$ (which is identical to the inverse Fourier transform of the nonnegative measure $d\sigma := hd\lambda_{\widehat{G}}$), is exactly $g \ast \tilde{g}$.

Now, it follows from (10) and (5) that

$$0 = \int_G \overline{\hat{\nu}}d\sigma = \int_G \overline{\hat{\nu}}d\sigma = \int_G \hat{\nu}d\lambda_{\widehat{G}},$$

where $\sigma \equiv \hat{h} = g \ast \tilde{g}$, $d\sigma = hd\lambda_{\widehat{G}}$, $h = |\tilde{g}|^2$. So the integral on $\widehat{G}$ of the function $\hat{\nu} \overline{\hat{h}}$ vanishes for all $h \in L^1_+(\widehat{G})$, yielding $\hat{\nu} = 0$ first $\lambda_{\widehat{G}}$-a.e., and then by continuity everywhere. Therefore, $\nu$ is the zero measure and the proof is complete when $\mu$ is bounded.

Let us turn to the unbounded case. Consider the measure $d\nu = w \ast \tilde{w}d\mu$ where $w \in C_c(G)$ is an arbitrary but fixed weight function. Taking into account that $w \in C_c(G)$, the measure $\nu$ is apparently a bounded Radon measure in $M(G)$. For any $u \in C_c(G)$, the function $(u \ast \tilde{u})(w \ast \tilde{w})$ is integrable (with respect to the Haar measure) and positive definite, by the aforementioned Schur’s product theorem. Referring to Lemma [2] we infer that there exists a Boac-Kac square...
root $v$ such that $v \ast \tilde{v} = (u \ast \tilde{u})(w \ast \tilde{w})$ with $\text{supp } v \subseteq G$. The calculation

$$\int_G u \ast \tilde{u} dv = \int_G (u \ast \tilde{u})(w \ast \tilde{w}) d\mu = \int_G v \ast \tilde{v} d\mu \geq 0$$

shows that the bounded measure $\nu$ also satisfies $\nu(u \ast \tilde{u}) \geq 0$ for any $u \in C_c(G)$, that is, $\nu \in M$. By the first part of the proof, it follows that $\nu = \nu$. Now, take an $E \in B_0$. An application of Lemma 5 ensures that for any $\varepsilon > 0$, we can choose a function $w$ such that $w \ast \tilde{w} \in [1, 1 + \varepsilon]$ on $E$ and thus on $-E$, too. This yields

$$\mu(E) \leq \nu(E) = \tilde{\nu}(E) = \nu(-E) = \int_{-E} w \ast \tilde{w} d\mu = \int_E (w \ast \tilde{w})^* d\mu^* = \int_E w \ast \tilde{w} d\tilde{\mu} \leq (1 + \varepsilon)\mu(E)$$

and, very similarly, we get that

$$\tilde{\mu}(E) = \mu(-E) \leq \int_{-E} w \ast \tilde{w} d\mu = \nu(-E) = \nu(E) = \int_E w \ast \tilde{w} d\mu \leq (1 + \varepsilon)\mu(E).$$

Since $\varepsilon$ could be arbitrary, the latter two inequalities yield that $\mu(E) = \tilde{\mu}(E)$ holds for all $E \in B_0$. The proof is complete. \hfill \Box

We present another short proof, which was communicated to us by Halász in the setting $G = \mathbb{R}$.

**Proposition 1.** Assume that $\mu$ is a Radon measure with the property that

$$\int_G (u \ast \tilde{u})(x) d\mu(x) \in \mathbb{R} \quad (\forall u \in C_c(G)).$$

Then $\mu(E) = \tilde{\mu}(E)$ holds for all $E \in B_0$.

**Proof.** The assumption, applied to $\gamma \cdot u$ in place of $u$, implies that

$$\exists \int_G \gamma(x)(u \ast \tilde{u})(x) d\mu(x) = 0 \quad (u \in C_c(G), \ \gamma \in \hat{G}).$$

Since

$$2i \exists \int_G \gamma(x)(u \ast \tilde{u})(x) d\mu(x) = \int_G \gamma(x)(u \ast \tilde{u})(x) d\mu(x) - \int_G \gamma(x)(u \ast \tilde{u})(x) d\tilde{\mu}(x)$$

and $(u \ast \tilde{u})(x) = (u \ast \tilde{u})(-x)$, we easily obtain

$$\int_G \gamma(x)(u \ast \tilde{u})(x)(d\mu - d\tilde{\mu})(x) = 0 \quad (u \in C_c(G), \ \gamma \in \hat{G})$$

meaning that the Fourier transform of the measure $d\nu_u := (u \ast \tilde{u})(d\mu - d\tilde{\mu})$ vanishes. Now, choose a compact set $K := K_u$ containing the open set $\Omega_u = \{x \in G : (u \ast \tilde{u})(x) \neq 0\}$. The restriction of $\mu$ to any compact set, and in particular to $K$, is finite. This implies that for any fixed $u$ the measure $\nu_u$ defines a bounded Radon measure. According to the uniqueness theorem for Fourier transform [23, 1.3.6. Theorem], we have $d\nu_u = (u \ast \tilde{u})(d\mu - d\tilde{\mu}) = 0$ and thus $d\mu = d\tilde{\mu}$ on $\Omega_u$. For any given set $E$ in $B_0$, by an appropriate choice of the function $u$ the set $\Omega_u$ might be taken to cover $E$. Therefore, $\mu(E) = \tilde{\mu}(E)$ holds for every $E \in B_0$. \hfill \Box
Next, we are concerned with verifying that the classes
\[ M^* := \{ \mu \in M(G) : \int f d\mu \geq 0 \ (\forall f \in D) \} \]
\[ M_0^* := \{ \mu \in M(G) : \int f d\mu \geq 0 \ (\forall f \in D_0) \} \]
\[ M_1^* := \{ \mu \in M(G) : \int f d\mu \geq 0 \ (\forall f \in D \cap L^1(G)) \} \]
coincide with the above class \( M = M^2 \). The following (weaker) observation is immediate.

**Corollary 1.** A bounded Radon measure is of positive type if and only if \( \mu(f) \geq 0 \) for all \( f \in D \cap L^1(G) \). That is, \( M = M_1^* \).

**Proof.** Clearly, \( u \ast \tilde{u} \in D \cap L^1(G) \) for any \( u \in C_c(G) \), whence \( M \supseteq M_1^* \). Conversely, any element \( f \) of \( D \cap L^1(G) \) arises from a convolution square \( f = g \ast \tilde{g} \) with some \( g \in L^2(G) \), by Lemma 2, so that \( M_1^* \supseteq M^2 \).

**Corollary 2.** A bounded Radon measure is of positive type if and only if \( \mu(f) \geq 0 \) for all \( f \in D \). In other words, \( M^* = M \).

**Proof.** As \( M_1^* \supseteq M^* \) is obvious, in view of Corollary 1 it suffices to prove the converse inclusion \( M^* \subseteq M \). So take any \( \mu \in M^* \). Let \( f \in D \) be arbitrary. Assume, as we may, \( \|f\|_{\infty} = 1 \). Let now \( \delta > 0 \) be fixed and \( K \subseteq G \) be a compact set such that \( |\mu|(G \setminus K) < \delta \). Furthermore, for the given value of \( \delta > 0 \) and for the above defined compact set \( K \subseteq G \), let \( k \) be an approximation of the identically 1 function \( 1 \), provided by Lemma 5. Apparently, we have
\[
\int f d\mu = \int_{G \setminus K} (1 - k) f d\mu + \int_{G} (1 - k) f d\mu + \int_{G} k f d\mu
\]
where, using \( \|f\|_{\infty} = 1 \), the first two terms can be estimated as
\[
\int_{G \setminus K} (1 - k) f d\mu \leq |\mu|(G \setminus K) < \delta, \quad \int_{K} (1 - k) f d\mu \leq \|\mu\| \delta.
\]
It follows that for any given \( \varepsilon > 0 \) and then suitable \( \delta > 0 \), \( K \subseteq G \) and \( k \approx 1 \), it holds
\[
\left| \int_{G} f d\mu - \int_{G} k f d\mu \right| < \varepsilon.
\]
Since \( k f \in D \cap L^1(G) \), using \( \mu \in M_1^* \) entails \( \int_{G} k f d\mu \geq 0 \). Therefore, from the last displayed inequality we conclude that for any \( \varepsilon > 0 \) the distance of \( \int f d\mu \) from \([0, +\infty)\) is \( < \varepsilon \). Therefore, \( \int f d\mu \geq 0 \) holds, too. It follows that \( \mu(f) \geq 0 \) for all \( f \in D \), as wanted.

In addition, we give a Bochner-type characterization of measures of positive type by their Fourier transform which too will be useful. To do so, let us introduce the class of measures
\[ M_\wedge : = M_\wedge^+(G) := \{ \mu \in M(G) : \hat{\mu} \geq 0 \}. \]

We summarize the results of the subsection as follows.

**Theorem 2.** We have \( M = M^2 = M^* = M_0^* = M_1^* = M_\wedge \).

**Proof.** The identities \( M = M^2 = M^* = M_1^* \) have been already clarified.

The identity \( M_0^* = M \) is now in fact trivial. Indeed, the inclusion \( M^* \subseteq M_0^* \) follows immediately from \( D_0 \subseteq D \), while the inclusion \( M_0^* \subseteq M \) can be obtained by observing that for any \( u \in C_c(G) \) we have \( u \ast \tilde{u} \in C_c(G) \subseteq C_0(G) \), and so \( u \ast \tilde{u} \in D_0 \), too.
To establish the equivalence $\mathcal{M}_\mu^+ = \mathcal{M}$, let first $\mu \in \mathcal{M}_\mu^+$ be fixed. Note that the condition $\mu(f) \geq 0$ for all $f \in \mathcal{D}$ is equivalent to that of $\mu(\overline{f}) \geq 0$ for all $f \in \mathcal{D}$. Now take any $f \in \mathcal{D}$ and represent it as $f = \sigma$ with some $\sigma \in M(\hat{G})_+$, according to Bochner’s theorem. Then referring to [3] we have $\mu(\overline{f}) = \mu(\overline{\sigma}) = \overline{\sigma(\mu)} = \sigma(\overline{\mu}) \geq 0$, that is $\mu \in \mathcal{M}_\mu^+$. This shows $\mathcal{M}_\mu^+ \subseteq \mathcal{M}^* = \mathcal{M}$.

As for $\mathcal{M} \subseteq \mathcal{M}_\mu^+$, we should only note that characters are continuous positive definite functions, that is $\hat{G} \subseteq \mathcal{D}$ entailing $\mathcal{M}^* \subseteq \mathcal{M}_\mu^+$. 

The class of functions of positive type are intimately connected to the class of positive definite functions. As it is noted in [8, 3.35 Proposition], for a continuous function $f$ the following conditions are equivalent:

(i) $f$ is positive definite – in the sense of (7);
(ii) $f$ is an (essentially) bounded, by what we mean that it is an element of $L^\infty(G)$, function of positive type – in the sense of (8);
(iii) $f$ satisfies (8) for any $u \in L^1(G)$.

Indeed, as $\mathcal{C}_c(G)$ is dense in $L^1(G)$ (iii) is equivalent to assuming the condition only to functions $u \in \mathcal{C}_c(G)$, whence (ii) and (iii) are equivalent. By [8, 3.21 Corollary], every bounded function of positive type agrees locally almost everywhere with a continuous function. From this one deduces easily that (ii) is equivalent to (i).

We shall also need even more, namely, the following fact [9, Proposition 4].

**Lemma 6.** A continuous function of positive type is necessarily positive definite.

Note that here the function in question is not assumed to be bounded – the lack of any boundedness assumption is (in part) made up for by the local boundedness at zero, valid for any continuous function. From here to arrive at the same conclusion than before for a priori bounded functions, one needs to incorporate a version of the Gelfand-Raikov theorem which will also be formulated later as Theorem 9.

We conclude this subsection by noting that a measurable positive definite function is necessarily continuous locally almost everywhere [15, (32.12) Theorem].

### 3.3. Positive definiteness and type "in the real sense". In the sequel we will need a restricted notion of positive definiteness and positive type for the real analysis treatment of duality. It would be natural to consider real valued positive definite functions and measures. However, the structural content of positive definiteness or type essentially refers to complex coefficients or weights, and restricting the conditions to only real coefficients or weights is not equivalent to the restriction of positive definite functions or measures of positive type to assume real values only. More precisely, we have the following statements.

**Proposition 2.** A function $f$ is positive definite and real valued if and only if it satisfies (7) for all real values of the coefficients $c_j$ and it is even.

Similarly, a measure $\mu$ is of positive type and real valued if and only if it satisfies (8) for arbitrary real valued weight functions $u \in \mathcal{C}_c(G)$ and it is even.

**Proof.** Since positive definiteness of a function $f : G \rightarrow \mathbb{C}$ is characterized by the positive semi-definiteness (in the linear algebraic sense) of the matrix $[f(x_j - x_k)]_{j,k=1}^{1,...,n}$ for all $n$-tuple $(x_1, \ldots, x_n)$, the first statement follows from the following equivalence:

$$\langle Az, z \rangle \geq 0 \ (z \in \mathbb{C}^n) \quad \iff \quad \langle Ay, y \rangle \geq 0 \ (y \in \mathbb{R}^n), \ A = A^T$$

where $A$ is an $n \times n$ real matrix and $^T$ stands for transposition.

To see this equivalence, assume first that $\langle Ay, y \rangle \geq 0$ holds for all $y \in \mathbb{R}^n$ and $A$ is symmetric. Then for the vector $z := x + iy$ with $x, y \in \mathbb{R}^n$ we calculate (using that for a symmetric real
matrix \( A \) we necessarily have \( \langle Ax, y \rangle = \langle Ay, x \rangle \) for all \( x, y \in \mathbb{R}^n \)

\[
\langle Az, z \rangle = \langle A(x + iy), x + iy \rangle = \langle Ax, x \rangle + i\langle Ay, x \rangle + \langle Ax, iy \rangle + i\langle Ay, iy \rangle
\]

\[
= \langle Ax, x \rangle + i\langle Ay, x \rangle - i\langle Ax, y \rangle + \langle Ay, y \rangle = \langle Ax, x \rangle + \langle Ay, y \rangle \geq 0,
\]

which verifies the above necessity.

Second, to see the converse we need only note that a self-adjoint and real matrix is necessarily symmetric, while the condition of nonnegativity for real arguments \( y \in \mathbb{R}^n \) is contained in the same condition assumed for all complex vectors \( z \in \mathbb{C}^n \).

Next, we turn to the analogous assertion for measures of positive type. Here we need to recall that for any \( u \in C_c(G) \) the convolution square \( f := u \ast \tilde{u} \) is positive definite, and it satisfies \( \tilde{f} \equiv f \), by property (p4). So assume first that \( \mu \) is an even measure of positive type in the real sense, and let \( w = u + iv \in C_c(G) \) be a complex weight function. Then we have

\[
\int w \ast \tilde{w} d\mu = \int u \ast \tilde{u} d\mu + \int v \ast \tilde{v} d\mu + i \int (-u \ast \tilde{v} + v \ast \tilde{u}) d\mu \geq 0,
\]

for the first two integrals need to be nonnegative by assumption, while the function \( -u \ast \tilde{v} + v \ast \tilde{u} \) under the last integral sign is odd, hence is orthogonal to the even measure \( \mu \). That is, \( \mu \gg 0 \). Conversely, assume now that \( \mu \gg 0 \) is real-valued. Then (p6) gives evenness of \( \mu \), while validity of the integral condition \( \int u \ast \tilde{u} d\mu \geq 0 \) for real valued weights \( u \in C_c(G) \) follows from the same, assumed for arbitrary complex weights.

In accordance with the above, the classes of real functions and measures satisfying the corresponding positive definiteness conditions \([7]\) and \([8]\) just for real coefficients or weights are larger than the real valued ones of positive definite functions or measures of positive type, respectively. To facilitate this distinction, we thus introduce here a few corresponding notations.

(i) \( \mathcal{N} \): the family of (bounded, regular) real Radon measures of positive type in the real sense, that is, \( \mathcal{N} := \{ \mu \in M(G) : \int u \ast \tilde{u} d\mu \geq 0 \ (\forall u \in C_c(G)) \} \); (vii) \( \mathcal{N}^2 \): the family of (bounded, regular) real Radon measures of positive type in the real sense with respect to \( L^2(G) \), that is, \( \mathcal{N}^2 := \{ \mu \in \overline{M(G)} : \int u \ast \tilde{u} d\mu \geq 0 \ (\forall u \in L^2(G)) \} \).

Further we introduce

(ix) \( \mathcal{N}_* := \{ \mu \in \overline{M(G)} : \int f d\mu \geq 0 \ (\forall f \in \mathcal{D}^\ast) \}; \)

(x) \( \mathcal{N}_0^\ast := \{ \mu \in \overline{M(G)} : \int f d\mu \geq 0 \ (\forall f \in \mathcal{D}_0^\ast) \}; \)

(xi) \( \mathcal{N}_1^\ast := \{ \mu \in \overline{M(G)} : \int f d\mu \geq 0 \ (\forall f \in \mathcal{D}_1^\ast) \}; \)

(xii) \( \mathcal{N}_+ := \{ \mu \in \overline{M(G)} : \Re \tilde{\mu}(\gamma) \geq 0 \}. \)

Note that any real measure \( \mu \in \overline{M(G)} \) can be written as

\[
\mu = \nu + o, \quad \text{where} \quad \nu := \frac{1}{2} (\mu + \tilde{\mu}) = \frac{1}{2} (\mu + \mu^*) , \quad o := \frac{1}{2} (\mu - \tilde{\mu}) = \frac{1}{2} (\mu - \mu^*)
\]

are the even and the odd components of \( \mu \), respectively. In this decomposition the second versions with *-s hold only for real valued measures, so that \( \nu \) and \( o \) are not necessarily even resp. odd for arbitrary complex measures \( \mu \), but for real ones they are. So any element of \( \overline{M(G)} \) is decomposed in a unique way as the sum of an even and an odd part – according to (11). Thus we can write \( \overline{M(G)} = \mathcal{E} + \mathcal{O} \), where \( \mathcal{E} := \{ \nu \in \overline{M(G)} : \nu = \nu^* \} \) is the subspace of even real measures and \( \mathcal{O} := \{ o \in \overline{M(G)} : o = -o^* \} \) is the subspace of odd real measures, and these subspaces are complementing ones in the sense of uniqueness of the representation. According to (p6), measures of positive type satisfy \( \tilde{\mu} = \mu \), whence for real measures of positive type \( \mu^* = \mu \) and so they are even – that is, \( \mathcal{M} \subseteq \mathcal{E} \).

We conclude the section with the "real sense analogue" of Theorem 2.

**Theorem 3.** We have \( \mathcal{N} = \mathcal{N}^2 = \mathcal{N}_* = \mathcal{N}_0^\ast = \mathcal{N}_1^\ast = \mathcal{N}_+ = \mathcal{M} + \mathcal{O} \).
Proof. The inclusions \( C_c(G) \subseteq L^2(G) \) and \( L^2(G) \ast L^2(G) \subseteq C_0(G) \subseteq C(G) \) combined with Lemma 1 entail that \( \mathcal{N} \supseteq \mathcal{N}' \supseteq \mathcal{N}_0' \supseteq \mathcal{N}^* \). Similarly, we also have \( \mathcal{N} \supseteq \mathcal{N}_1' \supseteq \mathcal{N}^* \) for \( C_0(G) \ast C_0(G) \subseteq C(G) \cap L^1(G) \subseteq C(G) \).

Denote \( \tilde{\mathcal{N}} := \{ \mu \in \mathcal{M}(G) : \mu + \tilde{\mu} \in \mathcal{M} \} \). Let us check first that \( \tilde{\mathcal{N}} = \mathcal{M} + \mathcal{O} \). Indeed, if \( \mu \in \mathcal{M}(G) \) and \( \mu = \nu + o \) is a decomposition with \( \nu \in \mathcal{M} \) and \( o \in \mathcal{O} \), then \( \nu \) is necessarily \( \frac{1}{2}(\mu + \tilde{\mu}) \) and so \( \frac{1}{2}(\mu + \tilde{\mu}) \in \mathcal{M} \) proving \( \nu \in \mathcal{N} \). Conversely, if \( \mu \in \mathcal{M}(G) \) and \( \mu + \tilde{\mu} \in \mathcal{M} \) then \( \mu = \nu + o \) with \( \nu := \frac{1}{2}(\mu + \tilde{\mu}) \in \mathcal{M} \) and \( o := \frac{1}{2}(\mu - \tilde{\mu}) \in \mathcal{O} \), hence \( \mu \in \mathcal{M} + \mathcal{O} \).

Using Theorem 2 it follows that \( \mathcal{M} = \mathcal{M}_+^* \subseteq \mathcal{N}^* \). Also, if \( o \) is an odd measure, then for any even \( f \in C(G) \) – and so in particular for any real-valued continuous positive definite function \( f \) – the integral \( o(f) = \int_C f d o \) vanishes. So we find \( \mathcal{M} \subseteq \mathcal{N}^* \) and \( \mathcal{O} \subseteq \mathcal{N}^* \) which lead us to \( \mathcal{M} + \mathcal{O} \subseteq \mathcal{N}^* \).

Up to here we have proved that \( \mathcal{M} + \mathcal{O} = \tilde{\mathcal{N}} \subseteq \mathcal{N}^* \subseteq \mathcal{N}' \subseteq \mathcal{N}_0', \mathcal{N}_1', \mathcal{N} \). Therefore, it remains to show \( \mathcal{N} \supseteq \mathcal{N}_1' \subseteq \mathcal{N} \).

Now, consider the inclusion that \( \mathcal{N}_1' \subseteq \mathcal{N} \). For any real measure \( \mu \in \mathcal{M}(G) \) an easy calculation yields for any \( \gamma \in \hat{G} \) that
\[
\overline{\mu}(\gamma) = \int_{\hat{G}} \gamma d\mu = \int_{\hat{G}} \gamma^* d\mu^* = \int_{\hat{G}} \overline{\gamma} d\overline{\mu} = \widehat{\overline{\mu}}(\gamma).
\]

Therefore, \( 2\Re \widehat{\overline{\mu}}(\gamma) = \widehat{\overline{\mu}}(\gamma) + \overline{\widehat{\overline{\mu}}(\gamma)} = \mu + \overline{\mu}(\gamma) \), that is, \( \Re \widehat{\overline{\mu}} = \nu \) with \( \nu \) being the even part of \( \mu \) in \( \mathcal{N} \). So if \( \mu \in \mathcal{N}_1' \), then we find that the even component \( \nu \) satisfies \( \nu = \Re \widehat{\overline{\mu}} \geq 0 \), that is, \( \nu \in M_+(G) \). According to Theorem 2 we have, however, \( M^*_+(G) = \mathcal{M} \), so that \( \nu \in \mathcal{M} \), hence \( \nu \in \mathcal{N}_1' \), too. This proves \( \nu \in \mathcal{N} \), that is, \( \mathcal{N}_1' \subseteq \mathcal{N} \), as wanted.

Finally, we aim to prove the inclusion \( \mathcal{N} \subseteq \mathcal{N}_1' \). Note that for any Radon measure, we always have for arbitrary weights \( u \in C_c(G) \) that
\[
\widetilde{\mu}(u \ast \tilde{u}) = \int (u \ast \tilde{u}) d\mu^* = \int (u \ast \tilde{u})^* d\overline{\mu} = \int (u \ast \tilde{u}) d\mu = \mu(u \ast \tilde{u})
\]
for \( (u \ast \tilde{u}) = u \ast \tilde{u} \), always. So in particular for any \( \mu \in \mathcal{N} \) also \( \mu \in \mathcal{N} \). Therefore, even \( \nu := \frac{1}{2}(\mu + \tilde{\mu}) \in \mathcal{N} \). However, \( \nu \) is also even, whence by the second part of Proposition 2, which says that \( \mathcal{M} = \mathcal{N} \cap \mathcal{E} \) – we find \( \nu \in \mathcal{M} \). Thus we are led to \( \nu \in \mathcal{M} \), and so in view of Theorem 2 \( \nu \in M_+(G) \), too. That is, we get \( \nu \geq 0 \). However, \( \widehat{\nu} = \Re \widehat{\overline{\nu}} \) for any \( \mu \in \mathcal{M}(G) \), whence this implies \( \mu \in \mathcal{N}_1' \), concluding the proof. 

4. The dual cone of the cone of positive and positive definite functions

First let us gather some basic facts concerning dual cones in Banach spaces. Assume that \( E \) is a real Banach space with dual space \( E' \). If \( K \subseteq E \) is a set, then the cone generated by \( K \) will be denoted by \( \text{Cone}(K) \). For any set \( S \subseteq E \) the dual cone of \( S \) is denoted by \( S^+ \) and defined as
\[
S^+ = \{ \varphi \in E' : \varphi(x) \geq 0 \quad (\forall x \in S) \}.
\]
Note that \( S^+ \) is always a closed cone.

In this section, our goal is to characterize the dual cone of \( P_0 \cap D_0 \) in the real Banach space \( M(G) = (C_0(G), \| \cdot \|_\infty)' \). A convex cone is subject to the appropriate version of the Krein-Milman or Choquet theorem, so its closure consists of limits of linear combinations of its extreme points. However, describing the extreme points of \( P_0 \cap D_0 \) is still an open problem even for the most immediate case of \( G = \mathbb{R} \). For more on this problem, attributed to Choquet, the interested reader can consult with [17].
In the lights of the above, one may expect that the dual cone of $\mathcal{P}_0 \cap \mathcal{D}_0$ is even less easy to describe. Our major tool to the solution is an intersection formula on the dual cones of the intersection of two cones. Basically, what we are after is an intersection formula stating $(A \cap B)^+ = A^+ + B^+$, the point being that from the generally true statement $(A \cap B)^+ = A^+ + B^+$ we would like to get rid of the closure.

Such theorems are known to hold particularly when the intersection of the cones is large enough or one of the cones has a nonempty interior. However, the cones we are considering here will not provide us such easy criteria for an intersection formula to hold, as neither $\mathcal{P}_0$ nor $\mathcal{D}_0$ has a nonempty interior in $(C_0(G), \| \cdot \|_\infty)$. Indeed, for any $f \in \mathcal{P}_0$ the $\varepsilon$-neighbourhood contains ultimately negative functions because $f$ vanishes at infinity. Further, it is even more obvious that for any $f \in \mathcal{D}_0$ one finds $C_0(G)$ functions arbitrarily close to it in $\| \cdot \|_\infty$ but not admitting the property (p4) of positive definite functions. The same way, (p2) fails for arbitrarily close functions, too. To circumvent the difficulty, we invoke a lesser known version of the intersection formula which gives the same conclusion under somewhat weaker hypothesis. The precise formulation of the corresponding statement [18, Lemma 2.2.] (see also [16, Section 15.D]) reads as follows.

**Lemma 7.** Assume that $A$ and $B$ are closed convex sets in a real Banach space $E$. If $0 \in A \cap B$ and $\text{Cone}(B - A)$ is a closed subspace of $E$, then in $E'$ we have

$$ (A \cap B)^+ = A^+ + B^+. $$

It seems highly non-trivial that the conditions of Lemma 7 hold in the current case when we consider the real Banach space $E = C_0(G)$ and its subsets $A = \mathcal{P}_0$, $B = \mathcal{D}_0$. This is our point with the next lemma.

**Lemma 8.** For any $f \in C_0(G)$, there exists $F \in \mathcal{D}_0$ such that $f \leq F$. In other words, $\mathcal{D}_0 - \mathcal{P}_0 = C_0(G)$.

**Proof.** Assume, as we may, $\|f\|_\infty = 1$. Define the subsets

$$ K_n := \{ x \in G : |f(x)| > 2^{-n} \}. $$

Therefore, $K_0 = \emptyset$ and all the sets $K_n$ are relatively compact in view of ”$\lim_\infty f = 0$”. Thus, $K_n$ ($n \in \mathbb{N}$) is an increasing sequence from $\mathcal{B}_0$ with

$$ H_n := K_n \setminus K_{n-1} = \{ x \in G : 2^{-n} < |f(x)| \leq 2 \cdot 2^{-n} \}. $$

Consequently, on $H_n$ we have $f \leq 2^{1-n} g_n$ with the function $g_n$ constructed for $\varepsilon = 1$ and the set $K_n$ by means of Lemma 5. Note that

$$ \bigcup_{n=1}^\infty H_n = \bigcup_{n=1}^\infty K_n = \{ x \in G : f(x) \neq 0 \}. $$

Recall that we also have $g_n \geq 0$. Therefore,

$$ f \leq F := \sum_{n=1}^\infty 2^{1-n} g_n $$

on the whole $G$ where the series converges normally and thus uniformly. So we find that $F \in C_0(G)$. Moreover, $F \in \mathcal{D}_0$ holds, too. \hfill \Box

**Corollary 3.** We have $(\mathcal{P}_0 \cap \mathcal{D}_0)^+ = \mathcal{P}_0^+ + \mathcal{D}_0^+$ in $\mathcal{M}(G)$.

**Proof.** Clearly, $\mathcal{P}_0$ and $\mathcal{D}_0$ are closed cones in $C_0(G)$, $0 \in \mathcal{D}_0 \cap \mathcal{P}_0$. By Lemma 8 we have $\mathcal{D}_0 - \mathcal{P}_0 = C_0(G)$ which is obviously a closed subspace of itself. Therefore, the dual cone intersection formula in Lemma 7 applies. \hfill \Box
Now we are in a position to present the main result of the section, the point of which is to arrive at a closure-free description.

**Theorem 4.** The dual of the cone of real valued continuous nonnegative positive definite functions vanishing at infinity is the Minkowski sum of the cone of nonnegative (bounded) Borel measures in $M(G)$, the cone of measures of positive type in $M(G)$, and the family $O = O(G, \mathbb{R})$ of (real, bounded) odd measures. That is, we have

$$(P_0 \cap D_0)^+ = M_+(G) + M + O.$$ 

**Proof.** According to Corollary 3 it suffices to characterize the dual cones of the sets $P_0$ and $D_0$ themselves. For the first, $P_0^+ = M_+(G)$ is well-known and easy to check. As for the second, note that with the notation of the previous section the dual cone $D_0^+$ is exactly the same as $N_0^*$. Thus, the result follows immediately from Theorem 3.

**5. Shapiro-type extremal problems on LCA groups**

In Section 3 we have introduced various function spaces and corresponding notions of positive-definite functions or functions of positive type: $D_0, D_0, D_0$ and $D_0^*$. In what follows we define certain extremal quantities on these function classes.

**Definition 1.** Let $U \subseteq B_0$ be a symmetric neighborhood of 0 and $k \in \mathbb{N}$. Denote $kU := \bigoplus_{j=0}^k U$. We define the extremal quantities

$$(13)\quad Q(U, k) := \sup_{\substack{0 \leq f \in D \atop f \neq 0 \text{ loc a.e.}}} \frac{\int_{kU} f d\lambda}{\int_U f d\lambda}; \quad Q_c(U, k) := \sup_{\substack{0 \leq f \in D_0 \atop f \neq 0 \text{ loc a.e.}}} \frac{\int_{kU} f d\lambda}{\int_U f d\lambda}; \quad Q^*(U, k) := \sup_{\substack{0 \leq f \in D_0^* \atop f \neq 0 \text{ loc a.e.}}} \frac{\int_{kU} f d\lambda}{\int_U f d\lambda}. $$

In case of $\mathbb{R}$ or even $\mathbb{T}$ (with e.g. some small enough $\delta$ and $U := (-\delta, \delta)$), analogous questions can be posed for any dilate $kU$ of $U$ with any $k > 0$. However, in general LCA groups there may not exist dilates. So, considering $kU$ is the next best thing to imitate the nature of the problems for $\mathbb{R}$ and $\mathbb{T}$. Dilates are considered for the Wiener problem even on $\mathbb{R}^d$ and $\mathbb{T}^d$ in [10] and the most important base sets occurring there are balls and cubes, which are centrally symmetric convex bodies, so in these cases $kU$ in the dilate sense equals to the above defined $kU$ in the multiple self-addition sense. In some other cases (for example, in cases of starlike domains) there is an asymptotic equivalence with $kU \sim k(\text{con } U)$ in higher dimensions.$^2$ However, in general LCA groups we cannot even define $\text{con } U$, whence there is no obvious way to compare our definition with the one in [10] analyzed in $\mathbb{R}^d$ and $\mathbb{T}^d$.

**Definition 2.** Let $U, V \subseteq B_0$ with $U$ a symmetric neighborhood of 0. We define the extremal quantities

$$(14)\quad S(U, V) := \sup_{\substack{0 \leq f \in E \atop f \neq 0 \text{ loc a.e.}}} \frac{\int_V f d\lambda}{\int_U f d\lambda}; \quad S_c(U, V) := \sup_{\substack{0 \leq f \in D_0 \atop f \neq 0 \text{ loc a.e.}}} \frac{\int_V f d\lambda}{\int_U f d\lambda}; \quad S^*(U, V) := \sup_{\substack{0 \leq f \in D_0^* \atop f \neq 0 \text{ loc a.e.}}} \frac{\int_V f d\lambda}{\int_U f d\lambda}. $$

In particular, $Q(U, k) := S(U, kU)$ etc.

$^2$Recall that according to Caratheodory’s theorem in dimension $d$ for every convex combinations $v \in V := \text{con } U$ there exists $d + 1$ elements of $U$ and nonnegative coefficients $\alpha_j \geq 0$ summing to 1 such that $\sum_{j=1}^{d+1} \alpha_j u_j = v$. Then it is easy to show that $kU^* \subseteq kV \subseteq (k + d)U^*$ with $U^* := \{au : 0 \leq \alpha \leq 1, u \in U\}$ being the “starlike hull” of $U$. It means that for large $k$ the deviation remains bounded.
**Definition 3.** Let $U \in \mathcal{B}_0$ be a symmetric neighborhood of 0 and $g \in G$ be arbitrary. We define the extremal quantities

$$
T(U, g) := \sup_{0 \leq f \in \mathcal{D}_{f \neq 0} \text{ loc a.e.}} \frac{\int_{U+g} f d\lambda}{\int_{U} f d\lambda}; \quad T_c(U, g) := \sup_{0 \leq f \in \mathcal{D}_{f \neq 0} \text{ loc a.e.}} \frac{\int_{U+g} g f d\lambda}{\int_{U} f d\lambda}; \quad T^*(U, g) := \sup_{0 \leq f \in \mathcal{D}_{f \neq 0} \text{ loc a.e.}} \frac{\int_{U+g} f d\lambda}{\int_{U} f d\lambda}.
$$

That is, $T(U, g) = S(U, g + U)$ etc.

These might be called the Shapiro-type extremal problems on $\mathcal{D}, \mathcal{D}_c$ and $\mathcal{D}^\ast$, respectively. If we replace the nonnegativity assumption $f \geq 0$ on the integrands by simply considering arbitrary $f \gg 0$ but writing $|f|^2$ in the integrands, then we obtain the Wiener type extremal quantities. Any Shapiro-type extremal quantity is at least as large as the corresponding Wiener-type extremal quantity because for any $f \gg 0$ the function $|f|^2$ is always a doubly positive function.

Several variants of these extremal quantities appear in the literature. A basic one is the analogous quantity with $\mathcal{D}_0$ replacing $\mathcal{D}$ – we may write $Q_0(U, V), S_0(U, V), T_0(U, V)$ for the arising extremal quantities.

If the conditions are $f(x) = \sum_{j=1}^{m} a_j \gamma_j(x)$ with $m \in \mathbb{N}$, $\gamma_j \in \hat{G}$ and $a_j \geq 0$, that is, if the class is the family of the trigonometric polynomials from $\mathcal{D}$, then we may term the corresponding special case the second Montgomery-Logan-Shapiro type question. The Montgomery-type question was originally posed for Dirichlet polynomials on $\mathbb{R}$. In fact, according to the Bochner-Weil theorem any $f \in \mathcal{D}$ is the inverse Fourier transform of a nonnegative Radon measure $\nu \in M_+(\hat{G})$, and the Montgomery-type case corresponds to the restriction of occurring measures to nonnegative measures having finite support:

$$
\nu \in M^\#(\hat{G}) := M_+(\hat{G}) \cap M^\#(\hat{G})
$$

where $M^\#(\hat{G}) := \{\nu \in M(\hat{G}) : \# \text{ supp } \nu < \infty\}$. So we may set

$$
\mathcal{D}^\# := \{f \in C(G) : f = \hat{\nu}, \nu \in M^\#(\hat{G})\}.
$$

Accordingly, we may denote the corresponding extremal problems (extended over functions $0 \leq f \in \mathcal{D}^\#$) by $Q^\#(U, k), S^\#(U, V)$ and $T^\#(U, g)$, respectively. If again here we consider integrals of $|f|^2$ for all $f \in \mathcal{D}^\#$, then we obtain the original “second Montgomery-type question” – it was posed for the comparison of the integrals (over a centered interval on $\mathbb{R}$ vs. over some arbitrary interval) of the absolute value square of a Dirichlet polynomial having nonnegative coefficients.

Further, it is possible here to involve other restrictions on the representing measures occurring in the Bochner type representation of the continuous positive definite functions $f \in \mathcal{D}$ which we consider. If we take all atomic measures $\nu$ (of not necessarily finite support), we obtain the class $\mathcal{D}^\ast$, and the respective problems $Q^\ast(U, k), S^\ast(U, V)$ and $T^\ast(U, g)$; and we can as well restrict considerations to Rajchman measures (getting $\mathcal{D}^R\ast$) and absolutely continuous measures (getting $\mathcal{D}^{ac}$). Although these (and other variants) do occur in the literature, we do not pursue these variants any further because of a good reason explained below.

**Theorem 5.** Let $U, V \in \mathcal{B}_0$ with $U$ a symmetric neighborhood of 0. Then $S_c(U, V) = S^\ast(U, V)$.

**Corollary 4.** All these extremal problems are equivalent, that is, we have $S_c(U, V) = S_0(U, V) = S(U, V) = S_1(U, V) = S^\ast(U, V) = S^\#(U, V) = S^R(U, V) = S^\#(U, V) = S^{ac}(U, V)$.

The same identifications hold for the extremal quantities $Q$ and $T$, too.

---

3The first problem is the analogous question with Hardy-Littlewood type majorization. We do not deal with this type of more general question here.
Remark 1. This is interesting for Logan [20], when constructing lower estimates for the extremal
constants $S^*([-1, 1], [-T, T])$ on $\mathbb{R}$ emphasizes the role of unfamiliar behavior of singular positive
definite functions (i.e. $f \in \mathcal{D}^*$), as opposed to familiar behavior of continuous positive definite
functions (i.e. $f \in \mathcal{D}$). Now it seems that even if finding lower estimates may be simpler in the
wider class of $\mathcal{D}^*$ -- e.g. manipulations with special functions obtained by singular series becomes
possible -- but there can be no difference regarding the extremal values.

Lemma 9. Let $f \in L^1_{\text{loc}}(G)$ be any function. Then for any function $\varphi \in \mathcal{C}_c(G)$ we have that
$f \ast \varphi \in \mathcal{C}(G) \cap L^1_{\text{loc}}(G)$. Furthermore, for any $V \in \mathcal{B}_0$ and any $\varepsilon > 0$ there exists a doubly positive
function $\varphi \in \mathcal{C}_c(G)$ such that $\|f \ast \varphi - f\|_{L^1(V)} < \varepsilon$. Moreover, in this last statement we can take $\varphi$ to be a constant multiple of the convolution square of the characteristic function of any
sufficiently small neighborhood $W$ of 0.

Proof. This is only a slight variant of the more familiar standard case when $f \in L^1(G)$, see e.g. [23] 1.1.8. Theorem .

As $f \in L^1_{\text{loc}}(G)$, we also have $f \in L^1(V')$ whenever $V' \in \mathcal{B}_0$. So, let us take a fixed open $V'$ with
compact closure but with $\overline{V} \Subset V'$. Then according to the known case of summable functions
applied to $g := f|_{V'}$ there exists a neighborhood $U$ of the origin such that whenever $u$ is a
nonnegative weight function with support in $U$ and total mass $\int_{G} u = 1$, then $\|g - g \ast u\|_{L^1(G)} < \varepsilon$.

Now we want to construct a function $u$ which is a positive definite convolution square. To
this end, we take a sufficiently small open neighborhood $W$ of 0 (specified later) and define
$\varphi := (\chi_W \ast \chi_{W})/\lambda(W)$. According to Lemma [1] we have $\varphi \in \mathcal{D}_x$. Moreover, it is easy to see that
$$\varphi(x) = \frac{1}{\lambda(W)} \int \chi_W(x - y) \tilde{\chi}_W(y) dy = 0$$
unless $x \in W - W$. Also, by construction $\varphi \geq 0$ and $\int_{G} \varphi = 1$. Therefore, $\varphi$ makes an admissible
weight function $u$ provided that $W - W \subseteq U$ which is one condition on $W$ to be satisfied
when choosing it. Such open neighborhoods of 0 exist by the continuity of the group operation
$(x, y) \mapsto x - y$. So, if $W_0$ is one such neighborhood, then we will restrict ourselves to consider $W \subseteq W_0$.

Next we aim to ascertain that on $V$ replacing $f$ by $g := f|_{V'}$ does not change anything, i.e.,
$$f(x) - f \ast \varphi(x) = g(x) - g \ast \varphi(x) \quad (\forall x \in V).$$

Note that this will also show that $f \ast \varphi(x)$ is continuous (on $V$, but as $V$ was arbitrary, in fact
everywhere), as $g \ast \varphi(x)$ is obviously such because $g \in L^1(G)$ and $\varphi \in \mathcal{C}_c(G)$.

Clearly, for the first term $f$ itself we have $f = f|_{V'}$ when $V \subseteq V'$. As for the second term, one
has to note that
$$f \ast \varphi(x) = \int_{G} f(x - y) \varphi(y) dy = \int_{W - W} f(x - y) \varphi(y) dy$$
for $\varphi(y) = 0$ if $y \notin W - W$. It follows that in case $x - W + W \subseteq V'$ we also have
$$f \ast \varphi(x) = \int_{W - W} f(x - y) \varphi(y) dy = \int_{W - W} f|_{V'}(x - y) \varphi(y) dy = g \ast \varphi(x).$$

We want this for all $x \in V$, whence our second requirement for $W$ is that $V + W - W \subseteq V'$.

It remains to see the standard fact that this is indeed satisfied for some open neighborhood
$W'$ of 0. As $V'$ is open, certainly for any $x \in V'$ the inclusion
$$x + Z - Z \subseteq V'$$
holds true with some open neighborhood \( Z = Z_x \) of 0, by the continuity of the group operation. In particular, these sets form a cover of \( \overline{V} \), i.e.,

\[
\overline{V} \subseteq \bigcup_{x \in \overline{V}} (x + Z_x - Z_x).
\]

By compactness, there exists a finite subcover satisfying

\[
\overline{V} \subseteq \bigcup_{j=1}^{n} (x_j + Z_{x_j} - Z_{x_j}).
\]

Taking \( W' := \cap_{j=1}^{n} Z_{x_j} \), we finally find for any \( x \in V \) that with some \( j \) it holds \( x \in (x_j + Z_{x_j} - Z_{x_j}) \) and thus

\[
x - W' + W' \subseteq x_j + Z_{x_j} - Z_{x_j} \subseteq V'.
\]

Finally, if \( W \) is taken as any open neighborhood contained in \( W_0 \cap W' \), then all the above requirements are fulfilled. Moreover, \( \| f - f \ast \varphi \|_{L^1(V)} = \| g - g \ast \varphi \|_{L^1(V)} < \varepsilon \) holds, as desired. \( \square \)

Before proceeding, let us point out a useful aspect of the above construction: the convolution of \( f \in L^1_{\text{loc}}(G) \) and \( \varphi \in C_c(G) \) is continuous, always. Indeed, to prove \( (f \ast \varphi)|_O \in C(G) \) for an arbitrary open set \( O \in B_0 \), we can take \( \Omega := O + \text{supp} \varphi \in B_0 \) and note that \( (f \ast \varphi)|_O = (f|_\Omega \ast \varphi)|_O \), while the convolution of an integrable and of a compactly supported continuous function is clearly continuous.

We also need the following auxiliary lemma.

**Lemma 10.** If \( f \in D^* \) and \( f \neq 0 \) locally almost everywhere, then we necessarily have \( \int_O f > 0 \) for all \( O \in B_0 \).

**Proof.** First observe that if \( f \) is a function of positive type and \( \varphi \in D_c \), then \( f \ast \varphi \) is a continuous function of positive type. To see this, take a compactly supported Boas-Kac square root \( g \) provided by Lemma 2. Then \( \varphi = g \ast \tilde{g} \), whence for any weight function \( u \in C_c(G) \) we have

\[
u \ast \tilde{u} \ast f \ast \varphi(0) = u \ast \tilde{u} \ast f \ast g \ast \tilde{g}(0) = u \ast g \ast \tilde{u} \ast \tilde{g} \ast f(0) = (u \ast g) \ast \tilde{u} \ast \tilde{g} \ast f(0) \geq 0
\]

because \( w := u \ast g \) is compactly supported together with \( u \) and \( g \), and a little thought shows \( w \in C_c(G) \), too. This means that \( f \ast \varphi \) is a function of positive type, and the continuity follows from the fact that the convolution of a locally integrable function and a compactly supported continuous function is continuous. Hence in virtue of Lemma 6 the function \( f \ast \varphi \) is positive definite, too.

Now, assume for contradiction that there is an \( O \in B_0 \) such that \( \int_O f = 0 \). Then

\[
(f \ast \varphi)(0) = \int_O f(x) \varphi(-x) \leq \| \varphi \|_{\infty} \int_O f = 0
\]

which yields \( f \ast \varphi \equiv 0 \) whenever \( \varphi \in D_c \), supported in \( O \); so in particular when \( \varphi \) is a constant multiple of the convolution square of a characteristic function of a small enough neighborhood of \( 0 \). So it follows from Lemma 9 that for all \( \varepsilon > 0 \) and for any open \( V \in B_0 \) the estimate \( \| f \|_{L^1(V)} < \varepsilon \) holds, that is, \( f \equiv 0 \) a.e. on \( V \), whence locally a.e. on \( G \). \( \square \)

Now we are in a position to present the proof of the main result of the section.

**Proof of Theorem 5.** Let \( U, V \in B_0 \) be given. We divide the proof into two parts, the first being \( S(U, V) = S^*(U, V) \) while the second is \( S(U, V) = S_c(U, V) \).
First we intend to show that for any \(0 \leq f \in \mathcal{D}^*\) with \(f \neq 0\) on the given \(U\) (which by Lemma 10 entails \(f_U > 0\)), and any \(\varepsilon > 0\) there exists another function \(0 \leq f_1 \in \mathcal{D}\) with the property \(\|f - f_1\|_{L^1(U \cup V)} < \varepsilon\). If having proved this, we get
\[
\frac{\int_U f \cdot f_1}{\int_U f} \geq \frac{\int_U f - \varepsilon}{\int_U f + \varepsilon}
\]
yielding that the supremum on the left, that is, \(S(U, V)\) is at least as large as the quantity
\[
\frac{\int_V f - \varepsilon}{\int_U f + \varepsilon}.
\]
Taking supremum on the right (with respect to \(\varepsilon > 0\) first) implies that \(S(U, V) \geq S^*(U, V)\). The other direction \(S(U, V) \leq S^*(U, V)\) being obvious we will thus infer \(S(U, V) = S^*(U, V)\).

So, for the proof of the existence of such a function \(f_1\), we apply Lemma 5 to the set \(U \cup V\) taking \(f_1 := f \ast \varphi\). Then with an appropriate \(\varphi\) we get \(\|f - f \ast \varphi\|_{L^1(U \cup V)} < \varepsilon\). Note that according to what was said preceding Lemma 10, we have \(f_1 = f \ast \varphi \in \mathcal{C}(G)\). Furthermore, it is easy to see that \(f \in \mathcal{D}^*\) implies that also \(f \ast \chi_W \ast \overline{\chi_W} \in \mathcal{D}_c^*\), whence \(f_1 \in \mathcal{D}^*_c\), too. But once \(f_1\) is both continuous and of positive type, it is also positive definite in view of Lemma 6. That is, we have \(f_1 \in \mathcal{C}(G) \cap \mathcal{D}_c^* = \mathcal{D}\). As \(\varphi \geq 0\), it is obvious that \(f \geq 0\) entails \(f_1 \geq 0\), too. This concludes the proof of the first part.

To start the second part, now we prove that any \(0 \leq f_1 \in \mathcal{D}\) can be approximated uniformly on the compact set \(K := U \cup V\) by a nonnegative \(f_2 \in \mathcal{D}_c\) with an error as small as it is desired. For this we just need a continuous positive definite compactly supported function \(k \geq 0\) which approximates the constant 1 function \(1\) uniformly within an arbitrarily fixed error \(\varepsilon > 0\) on \(K\). Such a function, even lying strictly above \(1\) on \(K\), is provided by Lemma 5 because \(k \geq 0\) implies
\[
k(z) \leq k(0) = 1 + \varepsilon
\]
for all \(z \in G\). We can then take \(f_2 := f_1 \cdot k\), which is again compactly supported (as its support is a subset of \(\text{supp} \ k\)), continuous and also positive definite, being the product of two positive definite functions. It follows that
\[
\|f_1 - f_1 \cdot k\|_{L^\infty(K)} \leq \|f_1\|_{L^\infty(K)} \cdot 1 - k\|_{L^\infty(K)} = \varepsilon \cdot \|f_1\|_{L^\infty(K)}
\]
and the function \(f_2 \geq 0\) provides arbitrarily good uniform approximation to \(f_1\) on \(K\).

To conclude the proof of the second part, let us choose any \(\eta > 0\) and an approximant \(0 \leq f_2 \in \mathcal{D}_c\) satisfying \(\|f_1 - f_2\|_{L^\infty(K)} < \eta\). With this we find
\[
\int_K |f_1 - f_2| \leq \eta \cdot \lambda(K).
\]
As \(\lambda(K) < \infty\) and \(\eta > 0\) is arbitrarily small, we get an arbitrarily close \(L^1(U \cup V)\) approximation as above implying \(S(U, V) = S^*_c(U, V)\), as before.

Combining this with the first part \(S(U, V) = S^*(U, V)\) yields the full statement. \(\square\)

6. Proof of Theorem 1

Let \(\mu, \nu\) be two Radon measures on the LCA group \(G\). Now we return to Problem 1, that is, find conditions to the inequality
\[
(16) \quad \int_G fd\nu \leq C \int_G fd\mu
\]
to hold for all nonnegative continuous positive definite functions $f$ with compact support. Here we assume that the occurring Radon measures are finite: $\mu, \nu \in M(G)$, or, in other words $\mu, \nu$ belong to the dual of $C^0_0(G)$.

We start with extending (16) to all nonnegative continuous positive definite functions vanishing at infinity. This follows from the fact that if for any $0 \leq f \in D_0$ and the approximant $k$ of the constant one function $1$ from Lemma 9 satisfy $f \cdot k \in P \cap D_0$ and so by condition (16) we have

$$\int_G (f \cdot k) d[C\mu - \nu] \geq 0,$$

then the last displayed inequality holds true for the uniform limit of $fk$ and thus for $f$ itself.

Next we may extend (16) to all continuous nonnegative positive definite functions $f$. These functions do not belong to the space $C^0_0(G)$, the dual of which contains the measures $\mu, \nu$, yet, as positive definite functions are uniformly bounded, we have no trouble to estimate, given any $\varepsilon > 0$, the integrals of any such function with respect to $\mu$ and $\nu$ within an error $\varepsilon$. Indeed, if $\eta > 0$ is chosen, and $K \subseteq G$ is a sufficiently large compact set, then $|\mu|(G \setminus K) < \eta$ and $|\nu|(G \setminus K) < \eta$, whence

$$\left| \int_{G \setminus K} f d[C\mu - \nu] \right| < \| f \|_\infty (|C| + 1) \varepsilon < \varepsilon$$

for appropriately chosen small $\eta$.

So, choose $k \in D_0$ to be an approximation of the constant 1 function $1$ satisfying $1 \leq k \leq 1 + \delta$ on $K$ and of course $0 \leq k \leq 1 + \delta$ on $G$ everywhere. Such a function exists for any choice of $\delta > 0$ in view of Lemma 5. With this function we find that

$$\left| \int_K (f \cdot k - f) d[C\mu - \nu] \right| \leq \delta \| f \|_\infty (|C|\|\mu\| + \|\nu\|) < \varepsilon$$

whenever $\delta$ is small enough. On the other hand

$$\left| \int_{G \setminus K} f \cdot (1 - k) d[C\mu - \nu] \right| \leq \| f \|_\infty \eta (|C| + 1),$$

as above, whence this part stays below $\varepsilon$. Combining these we arrive at the inequality

$$\left| \int_G f \cdot (1 - k) d[C\mu - \nu] \right| < 2\varepsilon.$$

Hence

$$\int_G f d[C\mu - \nu] \geq \int_G (f \cdot k) d[C\mu - \nu] - 2\varepsilon \geq -2\varepsilon$$

because $f \cdot k \in P \cap D_0$ and inequality (16) was assumed to hold for such functions. Finding the same with arbitrary $\varepsilon > 0$ proves that (16) holds for all $0 \leq f \in D_0$ too.

As we have clarified above, we can consider

$$\int_G f d[C\mu - \nu] \geq 0$$

whether for all $f \in P \cap D_0$ or $f \in P \cap D$ or $f \in P \cap D$. The following result is a strengthened version of Theorem 4.

**Theorem 6.** Let $\mu, \nu \in M(G)$ be arbitrary and $C \in \mathbb{R}$ be any constant. Then the following statements are equivalent:

1. $\int_G f d[C\mu - \nu] \geq 0$ for all $f \in P \cap D_0$.
2. $\int_G f d[C\mu - \nu] \geq 0$ for all $f \in P \cap D$.
3. $\int_G f d[C\mu - \nu] \geq 0$ for all $f \in P \cap D$.
4. $\int_G f d[C\mu - \nu] \geq 0$ for all $f \in P \cap D_0$.
(i) The inequality 16 is satisfied for all $f \in \mathcal{D}_0 \cap \mathcal{P}$.
(ii) The inequality 16 is satisfied for all $f \in \mathcal{D}_0 \cap \mathcal{P}$.
(iii) The inequality 16 is satisfied for all $f \in \mathcal{D}_0 \cap \mathcal{P}$.
(iv) $C\mu - \nu \in \mathcal{D}_0^+ + \mathcal{D}_0^- = \mathcal{M}_+ + \mathcal{M} + \mathcal{O}$.

**Proof.** That all functions of the three function classes (i)–(iii) satisfy 16 with exactly the same measures $C\mu - \nu$ is already clarified above. Remaining in $\mathcal{C}_0(G)$, this means that the functional $C\mu - \nu \in \mathcal{M}_+(G)$ assumes nonnegative values on functions of the intersection of the convex cones $\mathcal{P}_0$ and $\mathcal{D}_0$, that is, $C\mu - \nu \in (\mathcal{P}_0 \cap \mathcal{D}_0)^+$. Description of this dual cone was established in Theorem 3 whence the assertion. □

**Proposition 3.** $(\mathcal{P}_0 \cap \mathcal{D}_0)^+ \cap -(\mathcal{P}_0 \cap \mathcal{D}_0)^+ = \mathcal{O}$.

**Proof.** A bounded odd measure is orthogonal to $\mathcal{P}_0 \cap \mathcal{D}_0$, whence it belongs to both cones $\mathcal{P}_0 \cap \mathcal{D}_0$ and $-(\mathcal{P}_0 \cap \mathcal{D}_0)$. Therefore, $\mathcal{O}$ belongs to their intersection, too.

Conversely, let $\sigma$ be a bounded measure belonging to the intersection of these two cones. Taking $\sigma = \omega + \tau + o$ to be any decomposition of the measure $\sigma$ as an element of $\mathcal{P}_0^+ + \mathcal{M} + \mathcal{O}$, furnished by Theorem 3, we get in view of $\int_G \omega = 0$ that

$$\hat{\sigma}(0) = \hat{\omega}(0) + \hat{\tau}(0) + \hat{o}(0) = \int_G d\omega + \int_G d\tau \geq 0,$$

as for $\omega \geq 0$ we clearly have $\hat{\omega}(0) \geq 0$, and $\tau \in \mathcal{M} \subset \mathcal{M}$ is equivalent to $\hat{\tau}(\gamma) \geq 0$ $(\forall \gamma \in \hat{G})$, as it was clarified above in Theorem 2.

In a similar fashion, we can decompose $-\sigma = \alpha + \beta + \rho$ with some $\alpha \in \mathcal{P}_0^+$, $\beta \in \mathcal{M}$ and $\rho \in \mathcal{O}$. This yields $-\hat{\sigma}(0) \geq 0$ which entails that $\hat{\sigma}(0) = 0$. Therefore, taking into account that both $\hat{\omega}(0) \geq 0$ and $\hat{\tau}(0) \geq 0$, we get from $0 = \hat{\sigma}(0) = \hat{\omega}(0) + \hat{\tau}(0)$ that $\hat{\omega}(0) = \hat{\tau}(0) = 0$. By the same way, we obtain $\hat{o}(0) = \hat{\beta}(0) = 0$. However, $\hat{\omega}(0) = \omega(G)$ means that $\omega = 0$ in view of our a priori knowledge of $\omega \in \mathcal{M}_+(G)$. Similarly, we also conclude that $\alpha = 0$.

Thus $\sigma = \tau + o = -\beta - \rho$ where $\tau, \beta$ are (real) bounded measures of positive type. It follows that $\tau + \beta = -o - \rho \in \mathcal{O}$ is an odd measure, while $\tau + \beta \equiv 0$. However, from the second part of Proposition 2, it follows that the real valued measure $\tau + \beta$ of positive type must be even. As the even measure $\tau + \beta$ can be equal to the odd measure $-o - \rho$ only if both sides are 0 (identically zero measure), we conclude that $\tau + \beta = 0$.

Finally, it remains to see why both $\tau$ and $\beta$ must be 0. Taking Fourier transforms, we find $\hat{\tau}, \hat{\beta} \equiv 0$ (for $\tau, \beta \in \mathcal{M} = \mathcal{M}^\perp$), and their sum is $\hat{\tau} + \hat{\beta} = \hat{0} \equiv 0$. So, indeed, $\hat{\tau} \equiv 0, \hat{\beta} \equiv 0$. As above, by the uniqueness of Fourier transform it follows that $\tau = 0$ and $\beta = 0$.

Lastly, $\sigma = \omega + \tau + o = 0 + 0 + o$, whence $\sigma$ is an odd measure, as claimed. □

Next we collect certain properties of the admissible constants $C$.

**Proposition 4.** For arbitrary $\mu, \nu \in \mathcal{M}(G)$ the set

$$A(\mu, \nu) := \{C : C\mu - \nu \in \mathcal{D}_0^+ + \mathcal{D}_0^\perp\}$$

possesses the following properties:

(i) $A(\mu, \nu)$ is a closed subinterval of $\mathbb{R}$;
(ii) $0 \in A(\mu, \nu) \iff \int_G f d\nu \leq 0 \ (\forall f \in \mathcal{P}_0 \cap \mathcal{D}_0) \iff \nu \in \mathcal{P}_0^- + \mathcal{D}_0^- = -(\mathcal{P}_0 \cap \mathcal{D}_0)^+$;
(iii) $A(-\mu, \nu) = -A(\mu, \nu)$;
(iv) If $A(\mu, \nu) \neq \emptyset$, then we have $\mu \in \mathcal{P}_0^+ + \mathcal{D}_0^+$ $\iff$ $\sup A(\mu, \nu) = +\infty$;
(v) $A(\mu, \nu) = \mathbb{R} \iff \nu \in -(\mathcal{P}_0^+ + \mathcal{D}_0^+)$ and $\mu \in \mathcal{O}$;
(vi) If $\mu \notin \mathcal{P}_0^+ + \mathcal{D}_0^+$ and $\mu \notin -(\mathcal{P}_0^+ + \mathcal{D}_0^+)$, then $A(\mu, \nu)$ is bounded.
Theorem 7. The atomic measure $C\mu$ regarding the occurring measures in the representation of

$$
\int_G fd[C\mu - \nu] = C \int_G fd\mu - \int_G fd\nu \quad (f \in \mathcal{P}_0 \cap D_0),
$$

directly implying that for any given $f \in \mathcal{P}_0 \cap D_0$

\begin{equation}
\frac{1}{C} \int_G fd\nu \leq \int_G fd\mu \quad (f \in \mathcal{P}_0 \cap D_0).
\end{equation}

Taking $C \to \infty$ on the left-hand side (which is possible for $\sup A(\mu, \nu) = \infty$ was assumed), gives us $\int_G fd\mu \geq 0$, so that $\mu \in (\mathcal{P}_0 \cap D_0)^+$ and thus $\mu \in \mathcal{P}_0^+ + D_0^+$, again by Theorem 4. This verifies the necessity.

For the sufficiency, we note that if $\mu \in \mathcal{P}_0^+ + D_0^+$, then so is $\kappa \mu$ with any $\kappa > 0$. Hence $C \in A(\mu, \nu)(\neq \emptyset)$ implies $C + \kappa \in A(\mu, \nu)$ from which $[C, \infty) \subset A(\mu, \nu)$ and $\sup A(\mu, \nu) = \infty$ follows.

To verify (v), assume first that $C > 0$ and $C \in A(\mu, \nu)$. Then we have for any given $f \in \mathcal{P}_0 \cap D_0$ the validity of (17), and thus taking the limit $C \to +\infty$, yields again that $\int_G fd\mu \geq 0$ holds for all $f \in \mathcal{P}_0 \cap D_0$. Similarly, taking negative values of $C \in A(\mu, \nu)$ into account, we conclude that $\int_G fd\mu \leq 0$ is satisfied for all $f \in \mathcal{P}_0 \cap D_0$. Therefore, by Proposition 3 we have $\mu \in \mathcal{O}$. Further, $0 \in A(\mu, \nu)$ means $-\nu \in \mathcal{P}_0^+ + D_0^+$, that is, $\nu \in -(\mathcal{P}_0^+ + D_0^+)$. The converse is obvious.

Property (vi) follows from (iii) and (iv), noting that in case $A(\mu, \nu) = \emptyset$ we have nothing to prove. \hfill \Box

7. A specialization – the case of atomic measures

When both $\mu$ and $\nu$ are atomic measures, in our problem we can make further specializations regarding the occurring measures in the representation of $C\mu - \nu$.

**Theorem 7.** The atomic measure $\sigma = \sigma_{at}$ lies in $\mathcal{P}_0^+ + D_0^+$ if and only if there exists

(i) a nonnegative atomic even measure $\omega_{at} \geq 0$;

(ii) an atomic real measure of positive type $\rho_{at} \gg 0$;

(iii) an atomic real odd measure $\sigma = o_{at}$

such that $\sigma_{at} = \omega_{at} + \rho_{at} + o_{at}$.

We already know the existence of some decompositions. The novelty here is that all the components can be taken atomic, too.

For the continuation of the proof we need some more background materials about almost periodic functions and mean values on LCA groups.

Let $f$ be a function on $G$. For any $g \in G$ denote by $T_g f$ the $g$-translate of $f$ defined by $T_g f(x) := f(g + x)$. The function $f$ is said to be *almost periodic* if for all $\varepsilon > 0$ there exists a finite set $\{g_j : j = 1, \ldots, n\}$ such that the translates $\{T_g f : g = 1, \ldots, n\}$ constitute an $\varepsilon$-net among all the translates $\{T_g f : g \in G\}$ of $f$ (in the uniform norm metric).

We will use the fact that a translation invariant mean value functional $\mathcal{M} := \mathcal{M}_G$ exists on the set of all almost periodic functions on any locally compact group $G$ \cite{14} (18.8 Theorem). Hence also on $\hat{G}$ which will be denoted by $\mathcal{M}_G$. Note that this mean value operation is a bounded positive linear functional on the set of all almost periodic functions and is normalized so that $\mathcal{M}(1) = 1$. Furthermore, this mean value functional is unique \cite{14} (18.9 Theorem). Also, it vanishes on
all characters $\chi$ not identically one (for any $z \in G$ with $\chi(z) \neq 1$ translation invariance entails $\mathbb{M}(\chi(z)) = \mathbb{M}(\chi(z + x)) = \chi(z)\mathbb{M}(\chi(x))$ i.e. $(1 - \chi(z))\mathbb{M}(\chi) = 0$).

In fact, the mean value functional extends to a much larger class than the set of all almost periodic functions, e.g. at least to the class of so-called weakly almost periodic functions which simultaneously contains the following classes of functions:

(i) almost periodic functions;
(ii) positive definite functions;
(iii) all $C_0(G)$ functions.

The crucial reference in this direction is the classical work of Eberlein [4]. It is most useful to recall some results of Eberlein which will be needed.

Let $\sigma := \sigma_{at}$ be an atomic bounded Radon measure. Then $\sigma_{at}$ is of the form $\sigma_{at} = \sum_{j=1}^{\infty} a_j \delta_{x_j}$ (with $x_j$ running over a countable subset in $G$, and $\delta_x$ denoting the Dirac measure concentrated at $x$), so that its Fourier transform is

$$\hat{\sigma}(\gamma) = \sigma_{at}(\gamma) = \int_G \gamma(x) d\sigma_{at}(x) = \sum_{j=1}^{\infty} a_j \gamma(x_j).$$

Note that $\|\sigma_{at}\| = \sum_{j=1}^{\infty} |a_j| < \infty$ implies that the series expansion of $\hat{\sigma}_{at}$ is normally (whence absolutely and uniformly) convergent and thus defines a continuous function. Therefore, $\sigma_{at} \in \mathcal{C}_b(\widehat{G})$. Furthermore, it is necessarily an almost periodic function, see [11] (18.3) Theorem (iv)].

This statement can be reversed due to the following result of Eberlein [5] Theorem 3].

**Lemma 11** (Eberlein). *The Fourier transform of a bounded Radon measure is almost periodic if and only if the measure itself is a bounded atomic measure.*

We have seen that the atomic masses determine a convergent series representation of the Fourier transform. In what follows, we explain that conversely, the mass of the atomic component $\sigma(\{x_0\})$ of $\sigma$ at the arbitrary but fixed point $x_0 \in G$ (and thus the whole atomic measure itself) can be reconstructed using the mean value functional as

$$\sigma_{at}(\{x_0\}) = \sigma(\{x_0\}) = \mathbb{M}_G(\gamma(x_0)\hat{\sigma}(\gamma)).$$

First, for any $\sigma \in \mathcal{M}(G)$ the mean value functional $\mathbb{M}_G$ can be applied to $\gamma(x_0)\hat{\sigma}(\gamma)$, as decomposing $\sigma = \sigma_+ - \sigma_-$ with both $\sigma_+, \sigma_- \in \mathcal{M}_+(G)$, we find that $\sigma_+, \hat{\sigma}_-$, together with their products with the positive definite character $\gamma \mapsto \gamma(x_0)$ on $\widehat{G}$, are continuous positive definite functions on $\widehat{G}$, to which $\mathbb{M}_G$ is certainly extended.

Recall that a measure $\mu$ is called *continuous*, if for all singletons $\sigma(\{x_0\}) = 0$. Clearly, taking $\sigma_{at} := \sum_{x_j} \sigma(\{x_j\})\delta_{x_j}$ with all the points $x_j$ having nonzero mass is an atomic measure, and the left over remainder $\sigma - \sigma_{at}$ is a continuous measure. It remains to establish that the measure of a singleton $\{x_0\}$ equals to the mean value in (18). However, let us point out that Eberlein has proved the following even stronger result (cf. [4] Theorem 15.2] and [5] Theorem 1]), as well.

**Lemma 12** (Eberlein). *If the bounded Radon measure $\mu \in \mathcal{M}(G)$ has the decomposition $\mu = \mu_{at} + \nu$, where $\mu_{at} = \sum_{j=1}^{\infty} a_j \delta_{x_j}$ is the atomic component of $\mu$ and $\nu$ is the continuous component of $\mu$ then $\mathbb{M}_G(|\mu|) = \mathbb{M}_G(|\mu_{at}|) + \mathbb{M}_G(|\nu|) = \sum_{j=1}^{\infty} |a_j|^2$ and $\mathbb{M}_G(|\nu|) = 0$.*

From here (18) follows immediately using the ”orthogonality relations” (that is, $\mathbb{M}(1) = 1$ and $\mathbb{M}(\chi) = 0$ for any character not identically one) and standard manipulations with Parseval formula.

We will now discuss how Lemma 12 entails Lemma 11 more precisely, its nontrivial part stating that $\mu$ is atomic provided its Fourier transform is almost periodic.
As a special case of Lemma \[\text{(12)}\], we already know that the mean square value of the Fourier transform of a bounded Radon measure vanishes if and only if the measure is continuous. Consider an arbitrary measure \(\mu\) with almost periodic Fourier transform \(\hat{\mu}\). Decomposing as in the Lemma, the Fourier transform \(\hat{\mu}_{\text{at}}\) of the atomic component \(\mu_{\text{at}}\) is of course almost periodic, whence so is the difference \(\hat{\mu} - \hat{\mu}_{\text{at}} = \hat{\nu}\) – the Fourier transform of the continuous part of \(\mu\) –, too. Recall that if the mean square value of an almost periodic function is zero, then the function itself is identically zero (see e.g. \[\text{(4)}\] or \[\text{(14, (18.8) Theorem (i), (ii))}\]). Hence Lemma \[\text{(12)}\] entails that in this case \(\nu\) is identically zero. Whence \(\mu = \mu_{\text{at}}\) is atomic.

In the proof of Theorem \[\text{(7)}\] we also need the following useful observation.

**Lemma 13.** If \(0 \leq \phi \in C_b(G)\) and \(\phi = \phi_{\text{ap}} + \phi_{\text{a}}\) is a decomposition of \(\phi\) to an almost periodic part \(\phi_{\text{ap}}\) and a complementing small part \(\phi_{\text{a}}\) satisfying \(M(|\phi_{\text{a}}|^2) = 0\), then we necessarily have \(\phi_{\text{ap}} \geq 0\).

Note that in the conditions of the Lemma we do not assume that e.g. \(\phi \gg 0\) (which need not be true) but we assume the very existence of the above decomposition and nonnegativity of \(\phi\).

**Proof.** For any \(w \in \mathbb{R}\) define \(w_+ := \max(w, 0)\) and \(w_- := \min(w, 0)\). Now if \(a \geq 0\) and \(b \in \mathbb{R}\), then \((a - b)_- \leq b_+\) holds true. Using this, we can write

\[
0 \leq M((\phi_{\text{ap}})_-)^2 = M((\phi - \phi_{\text{a}})_-)^2 \leq M((\phi_{\text{a}})_+)^2 \leq M(|\phi_{\text{a}}|^2) = 0,
\]

whence the function \((\phi_{\text{ap}})_-\) has zero mean square, that is, the nonnegative function \(|(\phi_{\text{ap}})_-|^2\) has zero mean.

Note that once \(f\) is an almost periodic function on \(G\), then so is its negative part \(f_-\). Indeed, we have for any two translates and any point \(x \in G\) the inequality

\[
|T_g f_-(x) - T_h f_-(x)| \leq |T_g f(x) - T_h f(x)|,
\]

because for any two real numbers \(u, v \in \mathbb{R}\) we have \(|u_- - v_-| \leq |u - v|\). This furnishes that even the corresponding uniform norms satisfy

\[
\|T_g f_- - T_h f_-\|_\infty \leq \|T_g f - T_h f\|_\infty.
\]

Therefore, if the translates of \(f\) by a set \(\{g_j : j = 1, \ldots, n\}\) constitute an \(\varepsilon\)-net of all the translates of \(f\), we necessarily have that the translates of \(f_\) by the same set \(\{g_j : j = 1, \ldots, n\}\) form an \(\varepsilon\)-net for the set of all the translates of \(f_-\). Thus existence of a finite subset of translates constituting an \(\varepsilon\)-net for any given \(\varepsilon > 0\) for \(f\) entails the same property even for the translates of \(f_-\). This is equivalent to almost periodicity.

Continuity of \(f\) is trivially inherited by taking \(f_-\). So we get that \((\phi_{\text{ap}})_-\) is a continuous almost periodic function, whence so is \(|(\phi_{\text{ap}})_-|^2\). We found that \(|(\phi_{\text{ap}})_-|^2\) is a continuous nonnegative almost periodic function on \(G\) having zero mean value. As it was noted above, this implies that it is identically zero, in other words, \(\phi_{\text{ap}} = (\phi_{\text{ap}})_+ \geq 0\).

**Proof of Theorem \[\text{(7)}\]** Let \(\sigma = \omega + \tau\) be any decomposition of \(\sigma\) as an element of \(\mathcal{P}^+_0 + \mathcal{D}^+_0 = M_+(G) + \mathcal{N}\). Recall that according to Theorem \[\text{(3)}\] the measure \(\tau\) is of positive type in the real sense if and only if \(\Re \hat{\tau} \geq 0\). We start by decomposing the occurring measures to their atomic and continuous parts:

\[
\sigma = \sigma_{\text{at}}, \quad \omega = \omega_{\text{at}} + \omega_c, \quad \tau = \tau_{\text{at}} + \tau_c.
\]

Being the atomic part of a nonnegative measure, \(\omega_{\text{at}} \geq 0\). By the uniqueness of the decomposition, the continuous parts are opposite (as \(0 = \omega_c + \tau_c\)) and the atomic parts satisfy \(\omega_{\text{at}} + \tau_{\text{at}} = \sigma_{\text{at}} = \sigma\). It remains to prove that \(\tau_{\text{at}} \in \mathcal{D}^+_0 = \mathcal{N}\) holds, too. To do so, let us consider the respective Fourier transforms \(\hat{\tau} = \hat{\tau}_{\text{at}} + \hat{\tau}_c\). The occurring Fourier transforms are in
Theorem 9. Let \( \mu \) be a Radon measure of positive type on \( G \). Assume that there exists a neighborhood \( U \) of 0, and a real number \( K \geq 0 \), such that for all weight functions \( 0 \leq u \in C_c(G) \) and \( \text{supp} \ u \subseteq U \), it holds

\[
(19) \quad u \star \tilde{u} \star \mu(0) \leq K \left( \int_G u d\lambda \right)^2.
\]

Then \( \mu \) is absolutely continuous with a continuous positive definite Radon-Nikodym derivative \( \Phi \) satisfying \( \|\Phi\|_\infty \leq K \).
For full precision we note that the uniform norm estimate is not contained in the cited version but it obviously follows from continuity of $\Phi$ and $\|\Phi\|_\infty = \Phi(0)$ (which is property (p2) of positive definite functions from $\mathcal{D}$). Indeed, for any $\varepsilon > 0$ there exist a neighborhood $V$ of 0 such that $\Phi|_V \geq C := \Phi(0) - \varepsilon$; then for any open $W \in \mathcal{B}_0$ such that $W - W - W \subseteq U \cap V$, and for the continuous nonnegative function $u := \chi_W * \tilde{\chi}_W$ with symmetric support $S := \text{supp} \ u \subset W - W$ with $S - S \subseteq U \cap V$ we get that

$$K \left( \int_G u \, d\lambda \right)^2 \geq u \star \tilde{u} \star \mu(0) = \int \int u(-y - z) \tilde{u}(z) \Phi(y) \, d\lambda(y) \, d\lambda(z)$$

$$= \int \int_{-s-s-s} u(-y - z) \tilde{u}(z) \Phi(-y) \, d\lambda(y) \, d\lambda(z) \geq \int \int_{s-s-s} u(-y - z) u(-z) \, C \, d\lambda(y) \, d\lambda(z) = C \left( \int_G u \right)^2$$

proving $C \leq K$, that is, $\Phi(0) - \varepsilon \leq K$ which is equivalent to the assertion.

**Proof of Theorem** It suffices to prove $S(U, V) = \sigma(U, V)$.

First let us check the part $S(U, V) \leq \sigma(U, V)$. If $C \in \sigma(U, V)$ is an admissible constant such that $h_C$ stays above a continuous positive definite function $g$, then we necessarily have for any $0 \leq f \in \mathcal{D}_c$ that

$$0 \leq \int f g \leq \int f g + \int f(h_C - g) = \int f h_C = C \int_U f - \int_V f - \int f.$$

That is, using evenness of $f \in \mathcal{D}_c$ (which is property (p4) in the real-valued case), $\int_V f / \int_U f \leq \frac{1}{2} C$ obtains. First taking supremum over $0 \leq f \in \mathcal{D}_c$ and then taking infimum over all such constants $C$ we thus infer $S_c(U, V) \leq \sigma(U, V)$. However, we have proved as part of Theorem 5 that $S_c(U, V) = S(U, V)$, whence we even have $S(U, V) \leq \sigma(U, V)$.

Second, we turn to the converse inequality $S(U, V) \geq \sigma(U, V)$. So now let us take an admissible constant $c$ from the definition of $S(U, V)$, i.e. a constant $c$ with the property $\int_V f \leq c \int_U f$ ($\forall 0 \leq f \in \mathcal{D}$). In other words, $c$ is subject to the condition $\int_G f \mu / \int_U f \geq 0$ ($0 \leq f \in \mathcal{D}$) with the absolutely continuous measures $\mu := \lambda|_U = \chi_U \, d\lambda$, $\nu_1 := \lambda|_V = \chi_V \, d\lambda$. Taking into consideration that all $f \in \mathcal{D}$ is even, we can as well consider the symmetric measures: $\mu := \lambda|_U = \chi_U \, d\lambda$ (for $U$ was symmetric), and $\nu_2 := \nu_1 := \lambda|_{-V} = \chi_{-V} \, d\lambda$. Adding, we even find with $C := 2c$ the inequality $\int_G f \mu / \int_U f \geq 0$ ($0 \leq f \in \mathcal{D}$) with $\mu := \lambda|_U = \chi_U \, d\lambda$, $\nu := \nu_1 + \nu_2 = \lambda|_V + \lambda|_{-V} = (\chi_V + \chi_{-V}) \, d\lambda$, too.

The general result in Theorem 5 tells us that this holds for all $0 \leq f \in \mathcal{D}$ if and only if the measure

$$\rho := C \mu - \nu = \mathcal{P}_6^+ + \mathcal{D}_6^+ = M_+(G) + \mathcal{M} + \mathcal{O}.$$

Therefore, if the constant $C$ is admissible in the above sense of $S(U, V)$, then the measure $\rho$ has the representation $\rho = \omega + \tau + o$ where $\omega \in \mathcal{P}_6^+ = M_+(G)$ is nonnegative, $\tau \in \mathcal{D}_6^+ = \mathcal{M}$ is real and of positive type, and $o \in \mathcal{O}$ is real and odd.

We have that $\rho$ is absolutely continuous with Radon-Nikodym derivative $h_C \in L^\infty(G)$. Moreover, $K := \|h_C\|_\infty \leq C + 2$ also holds. Clearly, if $0 \leq u \in \mathcal{C}_c(G)$ is any weight function, then we have

$$u \star \tilde{u} \star \rho(0) = \int \int_G u(-y) u(-z) h_C(y - z) \, dydz \leq K \left( \int_G u \right)^2.$$

For the nonnegative measure $\omega$ and the nonnegative weight function $u$ we obviously have $u \star \tilde{u} \star \omega(0) \geq 0$. For the odd measure $o$, however, we have $u \star \tilde{u} \star o(0) = 0$, because the convolution
square \( u \ast \tilde{u} \) is positive definite and real, whence even. As \( \rho = \omega + \tau + o \) with \( \omega \geq 0 \), the above yields

\[
u \ast \tilde{u} \ast \tau(0) \leq u \ast \tilde{u} \ast \rho(0) \leq K \left( \int_G u \right)^2.
\]

Here, \( \tau \in \mathcal{M} \) is a measure of positive type, whence the Gelfand-Raikov Theorem applies. This furnishes that even \( \tau \) is absolutely continuous with Radon-Nikodym derivative \( \phi \in \mathcal{D} \) and \( \| \phi \|_\infty \leq K \). As a result, \( \omega + o = \rho - \tau \) is absolutely continuous, too.

Observe, further, that \( \rho := h_C d\lambda \) was even, and \( \tau \in \mathcal{M} \) is even, too. That is, the difference \( \rho - \tau \in \mathcal{E} \). Now the representation \( \omega + o \) is not necessarily unique, so we cannot state just for any representation in this form that \( o = 0 \), but at this point we can rewrite this side as \( \omega + o = 2 \left[ (\omega + o) + (\omega + o)^* \right] + 2 \left[ (\omega + o) - (\omega + o)^* \right] \), where the first, even part is \( \omega_1 := \frac{1}{2} \left[ (\omega + o) + (\omega + o)^* \right] = \frac{1}{2} [\omega + \omega^*] + \frac{1}{2} [o + o^*] = \frac{1}{2} [\omega + \omega^*] \) because \( o \) was odd. At this point, however, we can indeed observe that \( \omega_1 \), together with \( \omega \), is nonnegative, while it is also even. The other, odd component is \( o_1 := \frac{1}{2} [ (\omega + o) - (\omega + o)^* ] = \frac{1}{2} [\omega - \omega^*] + o \).

Recalling that \( \omega + o = \omega_1 + o_1 \) equals to the even measure \( \rho - \tau \), it follows that \( o_1 = \rho - \tau - \omega_1 \) is both even and odd, whence is null, and we finally find that \( \omega_1 = \rho - \tau \) with both sides are even, and, as a result of the absolute continuity of the right hand side, the same holds for \( \omega_1 \), too. So its Radon-Nikodym derivative \( \psi \) is even and it satisfies

\[0 \leq \psi \in L^\infty(G) \cap L^1(G).
\]

Thus we find that \( h_C = \psi + \phi \) with \( \psi \geq 0 \) and \( \phi \in \mathcal{D} \), or equivalently, \( h_C \geq \phi \), which is exactly the condition occurring in the definition of the quantity \( \sigma(U, V) \).

It means that whenever \( c \) was an admissible value for \( S(U, V) \), \( C := 2c \) was found to be an admissible value for the extremal problem \( \sigma(U, V) \), too. Whence \( S(U, V) \geq \sigma(U, V) \).

As the converse inequality have been already shown, the proof is complete. \( \square \)

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