On the local form of static plane symmetric spacetimes in the presence of matter

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Abstract

For any configuration of a static plane symmetric distribution of matter along a spacetime, there are coordinates where the metric can be set explicitly as a function of the energy density and pressures. It satisfies Einstein equations as long as we require conservation of the energy-momentum tensor, which is a single ordinary differential equation for self-gravitating hydrostatic equilibrium. As a direct application, a general solution is given when the pressures are linearly related to the energy density, recovering, in special cases, most of the known solutions of static plane symmetric Einstein equations.

Keywords: exact solutions, plane symmetry, static spacetimes

1. Introduction

The gravitational field in the vicinity of a large and isolated distribution of matter is approximately static if the interval of time considered is small compared to the characteristic time over which the system changes. As the matter distribution does not suffer any substantial change except in only one direction, it is also plane symmetric. This is the case, for example, when we are dealing with Newtonian gravitation close to the surface of a planet. In this case, define the height from the ground as $z$, the gravitational potential as $\phi$ and the atmospheric mass distribution and pressure as $\rho$ and $p$, respectively. They are approximately functions of $z$ only. Besides an appropriate equation of state relating the latter thermodynamic variables, there is the hydrostatic equilibrium [1]

$$\frac{dp}{dz} = -\rho \frac{d\phi}{dz},$$

with the Newtonian potential a solution of the Poisson equation $\frac{d^2\phi}{dz^2} = \frac{1}{\Sigma} \rho$. Here the units are such that $8\pi G = 1$ and $c = 1$. 

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The gravitational equation is simple enough to be integrated and to give \( \phi \) as
\[
\frac{d}{dz} \left( \frac{1}{\rho} \frac{dp}{dz} \right) = -\frac{1}{2} \rho.
\]

To set the stage for the relativistic generalization of the situation above, we start with a four dimensional Lorentzian manifold \( M \) allowing an isometric action of the transformation group of the Euclidean plane with 2-dimensional space-like orbits. Let \( e_1 \) and \( e_2 \) stand for the induced killing vector fields corresponding to translations in orthogonal directions. We also assume there is another killing vector field \( e_0 \) that is time-like and is orthogonal to and commuting with \( e_1 \) and \( e_2 \). Choosing a curve \( \gamma(z) \) such that \( \dot{\gamma}(z) \equiv 0 \) is orthogonal to \( e_i \), \( i = 0, 1, 2 \), and spreading it around using the fluxes of the three killing vector fields, we define a coordinate system where \( e_0 = \frac{\partial}{\partial z} \), \( e_1 = \frac{\partial}{\partial x} \), \( e_2 = \frac{\partial}{\partial y} \) and the metric is
\[
dx^2 = g_{00}(z) \, dt^2 + g_{11}(z) \left( dx^2 + dy^2 \right) + g_{33}(z) \, dz^2.
\]

We also assume that the energy-momentum tensor is equally symmetric, such that in these coordinates it is diagonal:
\[
(T^\mu_\nu) = \text{diag} \left\{ \rho(z), -p_{\|}(z), -p_{\|}(z), -p(z) \right\}.
\]

Note that the energy density \( \rho = T_{00} \), the pressure parallel to the plane of symmetry \( p_{\|} = -T_1^1 = -T_2^2 \) and the pressure orthogonal to it \( p = -T_3^3 \) behave as functions under a coordinate change of the type \( u = u(z) \). This implies they are all defined globally in such a spacetime. To simplify our analysis, we set the pressure difference \( \delta p \) as
\[
\delta p = p - p_{\|}.
\]

Such a scheme defines what we mean by a static plane symmetric spacetime (see also [2], ch 15).

Unlike Newtonian physics, where the gravitational part is played by one potential and one linear Poisson equation, in the relativistic analogue we deal with three unknown functions playing the role of the potential, \( g_{00} \), \( g_{11} \) and \( g_{33} \), and three non-linear independent Einstein equations, which are far from being easily integrated (see [3], for instance). Indeed, the component \( g_{33} \) is spurious, as we can make it an equal unit in a simple coordinate change \( dz^2 = g_{33}(z) \, dz^2 \).

Although we may identify matter as a fluid, in the relativistic approach it is natural to consider a more complex and general structure of the energy-momentum tensor. Therefore there is no need to consider an isotropic stress, that is, \( \delta p = 0 \). As an example, for an electric charged plane [3] we have \( \delta p = 2p = -2\rho \). As a result, we have two complementary equations of state\(^1\). This makes our system well defined: we have five independent unknown functions, \( g_{00}, g_{11}, \rho, p \) and \( \delta p \), related by five independent equations: three of them coming from the gravitational part and the rest from the sort of matter present in the system. In this paper, our main result is to give \( g_{00}, g_{11} \) and \( g_{33} \) in terms of \( \rho, p \) and \( \delta p \) such that they satisfy the Einstein equations, as we explain below.

\(^1\) By an equation of state we mean any one coming from the matter source, which could be in the form of an algebraic equation, as a differential equation coming from a lagrangian, etc.
Define the metric
\[ ds^2 = e^{2\phi} dt^2 - e^{4(z_1 - z)} \int_{z}^{w} \frac{F(w)}{(z_1 - z)^2} (dw^2 + dy^2) - \frac{4(z_2 - z_1)(z - z_1) F(z)^2}{(z - z_2)^2} \frac{1}{p} \frac{dz}{dz^2}, \] (7)
where \( \phi = \ln(g_{00})^2 \) is the generalized gravitational potential given by \( \frac{d\phi}{dz} = F(z) \) and
\[ F(z) = \frac{z - z_2}{z - z_1} \left( \frac{p}{(z - z_1)(\rho - p) + (z_2 - z_1)(\rho + 7p - 4bp)} \right). \] (8)

We also assume that \( F(z) \) is continuous and different from zero in a neighborhood of the origin. Thus, close to \( z = 0 \) neither \( p = 0 \) nor \( 2p = bp \). The arbitrary constants \( z_1 \) and \( z_2 \) are chosen in order to have a well defined \( F(0) \) and \( (z_2 - z_1)p_0 < 0 \), as \( p_0 \) is the value of \( p \) at \( z = 0 \).

The conservation of the energy-momentum tensor (5) in a spacetime with the metric above, \( \nabla_{\mu} T_{\mu}^{\nu} = 0 \), turns out to be a generalization of the hydrostatic equilibrium relation
\[ \frac{dp}{dz} = \left( \rho + p + \frac{4(z_2 - z_1)}{z - z_2} \frac{dp}{dz} \right) \frac{d\phi}{dz}. \] (9)
This is commonly found in the literature for an unknown \( \phi \) and \( dp = 0 \) ([4], section 5.4). If we apply the definition of \( \phi \) (equation (8)), we arrive at the relativistic self-gravitating hydrostatic equilibrium equation
\[ \frac{dp}{dz} = - \left( \frac{(z - z_2)(\rho + p) + 4(z_2 - z_1) dp}{(z - z_1)(\rho - p) + (z_2 - z_1)(\rho + 7p - 4bp)} \right) \frac{p}{z - z_1}. \] (10)

The strange form of the metric in equation (7) is justified as we prove that it, together with the energy-momentum tensor in equation (5), satisfies Einstein equations under the hypothesis of the conservation represented by (10).

In order to illustrate the whole scheme above, let us first investigate the Newtonian limit of \( \phi \), as it is defined by the derivative of the function \( F(z) \). In this case we have a positive pressure satisfying \( p \ll \rho \) and \( bp = 0 \). Defining the proper distance coordinate \( z' \) by
\[ \frac{dz'}{dz} = \frac{2 F(z)}{z - z_2} \sqrt{\frac{(z_2 - z_1)(z - z_1)}{p}}, \] (11)
we ensure that the distance from a point to the plane \( z = 0 \) is given by its \( z' \) coordinate. This clearly emulates the idea of the ‘\( z' \) coordinate in Newtonian theory as the ‘height from the ground’. Therefore we should investigate derivatives of \( \phi \) with respect to \( z' \) instead of \( z \). Hence, we have for \( p \ll \rho \):
\[ \frac{d^2 \phi}{dz'^2} = \frac{dz}{dz'} \frac{d}{dz} \left( \frac{dz'}{dz} \frac{dF(z)}{dz'} \right) = \left( \frac{z + z_2 - 2z_1}{2(z - z_2)} \right) \frac{\rho}{4(z_2 - z_1)} \left[ 1 + \frac{(z - z_2)}{2(z - z_1)} \left( \frac{(z - z_1) dp}{p dz - 1} \right) \right]. \] (12)

From the conservation equation (10), we derive
\[ \frac{(z - z_1) dp}{p dz} = - \frac{z - z_2}{z + z_2 - 2z_1} + \alpha \left( \frac{p}{\rho} \right), \] (13)
which, going back to equation (12), tells us that
\[ p \ll \rho \Rightarrow \frac{d^2\phi}{dz^2} = \frac{\rho}{2}. \]  

This establishes the Newtonian limit of the generalized gravitational potential \( \phi \). Conveniently choosing the constants \( z_1 \) and \( z_2 \), it will be written in the form of (2).

This paper is organized in the following manner: in section 2 we show that the metric (7) and the energy-momentum tensor (5) satisfy the Einstein equations, that is, \( R_{\mu}^\nu - \frac{1}{2} R \delta_\mu^\nu = T_{\mu}^\nu \), provided conservation (10) is attained. Furthermore, the cases where \( p = 0 \) or \( 2p = 2p = \delta p \) are treated separately, since \( F(z) \) vanishes or is ill-defined there. After setting this general framework, in section 3 we study some special solutions, mainly those with linear equations of state \( p = (\gamma - 1) \rho \) and \( \delta p = \epsilon \rho \), where \( \gamma \) and \( \epsilon \) are arbitrary constants. In this context, general formulas for their solutions are given in five different types. Exemplifying them, the perfect fluid \( (\epsilon = 0) \), the charged plane \( (\gamma = 0, \epsilon = -2) \) and the vacuum with cosmological constant \( (\gamma = \epsilon = 0) \) are obtained and associated with some known formulas in the literature. We finish the paper with some final remarks on the singular nature of such spacetimes.

2. The local form of the metric

Our main result is given in the following theorem.

**Theorem 1.** The metric given in equation (7) and the energy-momentum tensor in equation (5) satisfy the Einstein equations in the neighbourhood of \( z = 0 \), provided the latter attains the self-gravitating hydrostatic equilibrium given in equation (10).

**Proof.** We proceed straightforwardly in computing the Einstein tensor for the metric (7). Thus, the non-vanishing components of the Levi-Civita connection are

\[ \Gamma_{\ell z}^z = F; \quad \Gamma_{xz}^z = \Gamma_{yz}^z = 2 \frac{(z_2 - z_1) F}{z - z_2}; \]

\[ \Gamma_{zz}^z = \frac{e^{2\phi}(z - z_2)^2 p}{4 (z - z_1)(z_2 - z_1) F}; \quad \Gamma_{xt}^z = \Gamma_{yy}^z \]

\[ \Gamma_{yy}^z = \frac{(z - z_2) p}{2 (z - z_1) F} e^{4(z - z_1)} \int_0^z dw \frac{F}{w^2}; \]

\[ \Gamma_{zz}^z = \frac{1}{2} \frac{d}{dz} \ln \left( \frac{4(z_2 - z_1)(z - z_1)(F)^2}{(z - z_2)^2} \right). \]

Through the rest of our proof we assume \( \nabla_\mu T_{\mu}^\nu = 0 \), which is equivalent to equation (9). Therefore, during the calculation of the curvature \( R_{\mu
u}^{\lambda\rho} = g^{\mu\rho}(\partial_{\nu} \Gamma_{\lambda\rho}^{\lambda\mu} - \partial_{\lambda} \Gamma_{\mu\rho}^{\lambda\mu} + \ldots) \) we apply the identities

\[ \frac{p' F}{F} = - \left( \rho + p + \frac{4(z_2 - z_1)}{z - z_2} \delta p \right). \]  

\[ (15) \]
and
\[
\frac{p}{F} = \frac{z - z_1}{z - z_2} ((z - z_1)(\rho - p) + (z_2 - z_1)(\rho + 7p - 4\delta p)),
\]
(16)
the last one coming from the definition of \(F(z)\). Then we obtain for its independent components
\[
R_i^j = R_j^i = \frac{(z - z_2) p}{2(z - z_1)}; \quad R_{i\bar{z}} = \frac{1}{2}(\rho + p - 2\delta p) + \frac{z_2 - z_1}{z - z_1} p
\]
(17)
and
\[
R_{\bar{j}z} = \frac{z_2 - z_1}{z - z_1} p; \quad R_{\bar{j}\bar{z}} = R_{\bar{z}j} = -\frac{1}{2}\left(\rho + \frac{z_2 - z_1}{z - z_1} p\right).
\]
(18)
For the Ricci tensor \(R_{ij} = R_{j\bar{i}}\) and the scalar curvature we have
\[
R_i^i = \frac{1}{2}(\rho + 3p - 2\delta p); \quad R_\bar{j}^\bar{j} = R_j^j = -\frac{1}{2}(\rho - p)
\]
\[
R_{\bar{z}z} = -\frac{1}{2}(\rho - p + 2\delta p); \quad R = -\rho + 3p - 2\delta p.
\]
Hence, the Einstein equations hold for \(ds^2\), as can be readily verified.

It is not true that \(ds^2\) in equation (7) can be well defined for any kind of matter. In order to be complete, the following theorem explains when such a scheme is not possible.

**Theorem 2.** If \(p = 0\) or \(2\rho = 2p = \delta p\) in the neighborhood of \(z = 0\), then one of the following holds, with \(ds^2\) satisfying the Einstein equations:

1. \(i.\) \(p = \rho = 0\), \(\delta p = \delta p(z)\). Defining \(\varphi = -\int_{z_0}^{z} dw\frac{w - z}{\delta p + (w - z)^2}\), we have

\[
ds^2 = e^{2\varphi} dr^2 - \left(\frac{d\varphi'}{z - z_1}\right)^2.
\]

2. \(ii.\) \(p = 0\), \(\rho = 4\delta p\), \(\delta p = \delta p(z)\). Defining \(\varphi = \frac{1}{3}\int_{z_0}^{z} dw\frac{w - z}{\delta p + (w - z)^2}\), we have

\[
ds^2 = e^{2\varphi} dr^2 - e^{-4\varphi}\left(dx^2 + dy^2\right) - \left(\frac{3\varphi'}{z - z_1}\right)^2.
\]

3. \(iii.\) \(\rho = p\), \(\delta p = 2p\). There are constants \(\alpha\) and \(\beta\) such that

\[
p = \frac{4\alpha\beta}{3(1 + (\alpha + \beta)z)^2}.
\]
(21)
If \(\alpha + \beta \neq 0\),
\[
ds^2 = (1 + (\alpha + \beta)z)^{\frac{2}{3}\left(\frac{3\alpha - \beta}{\alpha + \beta}\right)} \left(\frac{3\alpha - \beta}{\alpha + \beta}\right) (dx^2 + dy^2) - dz^2.
\]
(22)
and if \(\alpha + \beta = 0\),
\[
ds^2 = e^{\frac{2}{3}\alpha z} dr^2 - e^{-\frac{2}{3}\alpha z}\left(dx^2 + dy^2\right) - dz^2.
\]
(23)
In the special case \( p = 0 \), we obtain, for \( \beta = 0 \), the Minkowski metric described by an observer with a uniform acceleration \( \alpha \) ([5]) or, for \( \alpha = 0 \), the Taub-Levi-Civita vacuum solution ([3]).

**Proof.** If we take the metric (4) in coordinates such that \( du = \sqrt{-g_{11}} \, dz \) and define

\[
\xi(u) = \frac{3}{4} \frac{d}{du} \ln |g_{11}| \quad \text{and} \quad \psi(u) = \frac{d}{du} \left( \frac{1}{2} \ln g_{00} + \frac{1}{4} \ln |g_{11}| \right),
\]

such that

\[
dx^2 = e^{\xi} \int_{u}^{u'} (3\psi + \xi) \, du^2 - e^{\psi} \int_{u}^{u'} \xi (dx^2 + dy^2) - du^2,
\]

then the Einstein equations turn into

\[
G_t^t = T_t^t : - \frac{4}{3} (\xi' + \xi^2) = \rho \tag{26}
\]

\[
G_x^x = T_x^x : - 4G_x^x = T_x^x : 4(\psi' + \psi^2) = \rho + 4p - 4\delta p \tag{27}
\]

\[
G_u^u = T_u^u : - \frac{4}{3} \xi \psi = - p. \tag{28}
\]

If \( p = 0 \), from equation (28) we conclude that \( \xi = 0 \) or \( \psi = 0 \).

If \( p = 0 \) and \( \xi = 0 \), then \( \rho = 0 \) as equation (26) demands. Defining the coordinate function \( z = \psi + z_1 \), for an arbitrary constant \( z_1 = \sqrt{4\delta p(0)} \), from equation (27) we obtain

\[
dz = - \left( \delta p + (z - z_1)^2 \right) du,
\]

implying formula (19). This proves (i).

If \( p = 0 \) and \( \psi = 0 \), then \( \rho = 4\delta p \) follows from equation (27). As before, defining the coordinate function \( z = \xi + z_1 \) and using equation (26) we get equation (20), hence proving (ii).

If \( \rho = p \) and \( \delta p = 2p \), from equations (26)–(28) we obtain the following system of ODEs:

\[
\xi' + \xi^2 + \psi \xi = 0 \quad \psi' + \psi^2 + \psi \xi = 0.
\]

Its general solution is, after we set \( z = u \),

\[
\psi = \frac{\alpha}{1 + (\alpha + \beta) z} \quad \xi = \frac{\beta}{1 + (\alpha + \beta) z}.
\]

Returning to these expressions in the previous formulas, we prove (iii).

\( \square \)

3. Exact solutions

To illustrate the theorems above, we give a series of exact solutions for the static plane symmetric Einstein equations.

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\[ \text{This is a further simplification for the form of the metric appearing in [6].} \]
3.1. Simple fluids: general solutions

Suppose that matter satisfies equations of state in the form

\[ p = (\gamma (\rho) - 1) \rho \quad \text{and} \quad \frac{dp}{d\rho} = \epsilon (\rho) \rho, \]  

(32)

In this case, the configuration of the system is completely determined by solving the simple first order ordinary differential equation (ODE) of hydrostatic equilibrium for \( \rho(z) \):

\[
\left( \frac{d\gamma}{d\rho} + \gamma - 1 \right) \frac{d\rho}{dz} = - \left( \frac{(z - z_0) \gamma + (z_2 - z_1) (4\epsilon - \gamma)}{(z - z_1) (2 - \gamma) + (z_2 - z_1) (7\gamma - 4 - 6)} \right) (\gamma - 1) \rho. 
\]  

(33)

Even though this ODE is not exactly solvable, any approximation method for its solution is reflected instantaneously in the metric components, as demanded by equations (7), (8) and (32). This is a great simplification in the analysis of such systems.

In the following we give all exact solutions for constants \( \gamma \) and \( \epsilon \). They are divided in five types, the first one being the most general. As we will see throughout this section, various physical situations fall within them.

Type I: \( \gamma = 1; \gamma = 2; \gamma = 4 \epsilon + 6 \).

This is the generic situation. Equation (33) is easily integrated as

\[
\frac{\rho}{\rho_0} = \left( 1 - \frac{z}{z_1} \right)^{\frac{4}{\gamma - 4 \epsilon - 6}} \left( 1 - \frac{z}{\alpha z_1} \right)^{\frac{4(1 - \gamma)}{(2 - \gamma)(\gamma - 4 \epsilon - 6)}} 
\]  

(34)

where \( \rho_0 = \rho(0) \) and

\[
\alpha = 1 - \left( 1 - \frac{z}{z_1} \right) \left( \frac{7\gamma - 4 \epsilon - 6}{\gamma - 2} \right). 
\]  

(35)

Applying it in equation (7), we arrive at the expression for the metric

\[
dz^2 = \left( 1 - \frac{z}{z_1} \right)^{\frac{4(1 - \gamma)}{(2 - \gamma)(\gamma - 4 \epsilon - 6)}} \left( 1 - \frac{z}{\alpha z_1} \right)^{\frac{4(1 - \gamma)}{(2 - \gamma)(\gamma - 4 \epsilon - 6)}} dz^2 
- \left( 1 - \frac{z}{z_1} \right)^{\frac{4(1 - \gamma)}{(2 - \gamma)(\gamma - 4 \epsilon - 6)}} \left( 1 - \frac{z}{\alpha z_1} \right)^{\frac{4(1 - \gamma)}{(2 - \gamma)(\gamma - 4 \epsilon - 6)}} (dx^2 + dy^2) 
- \frac{4 (1 - \gamma) (z_2 - z_1)}{\alpha^2 (2 - \gamma)^2 \rho_0 z_1^2} \rho_0 \frac{z_1}{z_1}^{\frac{1}{\gamma - 4 \epsilon - 6}} 
\times \left( 1 - \frac{z}{\alpha z_1} \right)^{\frac{4(5\gamma^2 - 2 \gamma - 11 \gamma + 2 \epsilon + 6)}{(2 - \gamma)(\gamma - 4 \epsilon - 6)}} \right) dz^2 
\]  

(36)

with \( (\gamma - 1) \rho_0 (z_2 - z_1) z_1 < 0 \).

Type II: \( \gamma = 1 \).

In this case we have \( p = 0 \), which restricts the solutions to fall in one of the three classes considered in theorem 2. The first one encompasses the non-physical behaviour \( p = \rho = 0 \) and \( \frac{dp}{d\rho} = \rho \gamma \), giving the solution represented in equation (19). The second states that \( \epsilon = \frac{1}{2} \) is a necessary condition if \( p = 0 \) and \( \rho \neq 0 \). This imposes the metric to be as in equation (20), and further information on the energy density must be provided. The third
one refers to the two possibilities for the vacuum \( p = \rho = \delta p = 0 \): the Minkowski and Taub-Levi-Civita solutions.

Type III: \( \gamma = 2; \epsilon = 2 \).

Here we have stiff matter \( (p = \rho) \) with anisotropies. Solving equation (33), we get

\[
\frac{\rho}{\rho_0} = \left(1 - \frac{\xi}{z_1}\right)^{\frac{1}{\gamma-2}} e^{-\frac{\gamma}{\gamma-2} \frac{z}{z_1}}.
\]  

(37)

Regarding \( \rho_0 (z_2 - z_1) z_1 < 0 \), the metric in equation (7) becomes

\[
ds^2 = \left(1 - \frac{z}{z_1}\right)^{\frac{\gamma-1}{\gamma-2}} e^{\frac{\gamma-1}{\gamma-2} \left(1 - \frac{z_2}{z_1}\right)} \left(1 - \frac{z}{z_1}\right)^{-1} \left[dr^2 - \frac{4 (z_2 - z_1)(2 - \epsilon)^2}{z_1} \right].
\]  

(38)

Type IV: \( \gamma = 2; \epsilon = 2 \).

This is the third class in theorem 2, where \( \rho = p \) and \( \delta p = 2 \rho \).

Type V: \( \gamma = 1; \gamma = 2; 7 \gamma = 4 \epsilon + 6 \).

In this last class of solutions the equations of state turn into \( p = (\gamma - 1) \rho \) and \( 4 \delta p = 7 p + \rho \). We have

\[
\frac{\rho}{\rho_0} = \left(1 - \frac{z}{z_1}\right)^{\frac{\gamma-1}{\gamma-2}} e^{-\frac{\gamma-1}{\gamma-2} \left(1 - \frac{z_2}{z_1}\right)} \left(1 - \frac{z}{z_1}\right)^{-1} \frac{1}{\pi},
\]  

(39)

and for \( z_1 \) and \( z_2 \) such that \( (\gamma - 1) \rho_0 (z_2 - z_1) z_1 < 0 \), equation (7) tells us that

\[
ds^2 = \left(1 - \frac{z}{z_1}\right)^{\frac{\gamma-1}{\gamma-2}} e^{-\frac{\gamma-1}{\gamma-2} \left(1 - \frac{z_2}{z_1}\right)} \left(1 - \frac{z}{z_1}\right)^{-1} \left[dr^2 - \frac{4 (z_2 - z_1)(2 - \epsilon)^2}{z_1} \right].
\]  

(40)

3.2. Perfect fluids \( (\delta p = 0) \)

The general barotropic perfect fluid satisfies \( p = p(\rho) \) and \( \delta p = 0 \). If we define the \( \rho \)-dependent parameter \( \gamma = 1 + \frac{\rho_0}{\rho} \), then the self-gravitating behaviour of the fluid is determined by solving equation (33) with \( \epsilon = 0 \) and applying equation (7) afterwards to obtain the metric. Indeed there is one exception: this general procedure cannot be applied to the dust solution \( p = 0 \). But in this case, if we examine each item in theorem 2 under the hypothesis \( \delta p = 0 \), we readily verify that no other solution but the vacuum is permitted. This fact has been known at least since the 1950s ([7]).

For a constant \( \gamma \), we have the Collins solutions ([11, 12]). In this case matter obeys the equation of state \( p = (\gamma - 1) \rho \). As discussed in the last paragraph, \( \gamma = 1 \) implies \( p = \rho = 0 \). If \( \rho \neq 0, \rho \neq p \) and \( 7 p + \rho \neq 0 \), then the solutions for \( \rho(z) \) and \( ds^2 \) are type I.
with $\epsilon = 0$. If $p \neq 0$ and $\rho = p$, then the solution is type III with $\epsilon = 0$. From the equation $7p + \rho = 0$ we obtain a type V solution. To define specific values of $\gamma$, the reader might consult the rich literature on the subject ([2] Section 15.7.1, [7–10]). In the following, we exemplify our approach in two distinct examples: the Collins solutions ([11, 12]) and the Tabensky-Taub solution ([8]).

The generic perfect fluid solution is given by formula (36) with $\gamma = 1$, $\gamma = 2$, $\gamma = \frac{5}{2}$ and $\epsilon = 0$:

$$
\text{ds}^2 = \left(1 - \frac{z}{z_1}\right)^{\frac{2(1-\gamma)}{\gamma+6}}\left(1 - \frac{z}{\alpha z_1}\right)^{\frac{4(1-\gamma)(3\gamma-2)}{12\gamma(\gamma+6)}} \text{dr}^2
$$

$$
- \left(1 - \frac{z}{z_1}\right)^{\frac{4(1-\gamma)}{\gamma+6}}\left(1 - \frac{z}{\alpha z_1}\right)^{\frac{4(1-\gamma)}{12\gamma(\gamma+6)}} (dx^2 + dy^2)
$$

$$
- \frac{4}{\alpha^2} (\gamma - 2)(\gamma - 6) \rho_0 \frac{z_1}{z_1} \left(1 - \frac{z}{z_1}\right)^{-\frac{8\gamma+6}{\gamma+6}}\left(1 - \frac{z}{\alpha z_1}\right)^{-\frac{4(5\gamma^2 - 11\gamma + 6)}{12\gamma(\gamma+6)}},
$$

where we have used formula (35) to write $z_2 - z_1$ in terms of $\alpha$. If we apply the substitutions

$$
t' = (1 - \beta) \frac{1}{n+7} t; \quad u = \frac{1 - \frac{z}{z_1}}{1 - \beta \frac{z}{z_1}}; \quad \beta = \frac{1}{\alpha}; \quad \eta = \frac{1}{\gamma - 1},
$$

we obtain

$$
\text{ds}^2 = u^{-\frac{2}{n+7}} (1 - \beta u)^{-\frac{2}{n+7}} \text{d}t'^2 - u^{-\frac{4}{n+7}} \left(dx^2 + dy^2\right)
$$

$$
- \frac{4}{\eta} \beta \frac{1 - \beta u}{\eta - 1} \left(1 - \beta u\right)^{-\frac{2}{n+7}} \left(1 - \frac{z}{z_1}\right)^{-\frac{4\eta}{n+7}} \text{d}u^2
$$

As the constant $\rho_0$ is properly redefined, this emulates Saravi’s formula (31) in [12].

For stiff matter $p = \rho$ ($\gamma = 2$) we have a type III solution. Hence, as we put $\epsilon = 0$ in equation (38), we get

$$
\text{ds}^2 = \left(1 - \frac{z}{z_1}\right)^{\frac{1}{4}} e^{-\beta \frac{z}{z_1}} \text{dr}^2 - \left(1 - \frac{z}{z_1}\right)^{\frac{1}{4}} \left(dx^2 + dy^2\right)
$$

$$
- \frac{\beta}{4} \frac{1}{\rho_0} \left(1 - \frac{z}{z_1}\right)^{-\frac{5}{4}} e^{-\beta \frac{z}{z_1}} \text{d}z^2
$$

where we have set $\beta = \frac{1}{4} (1 - \frac{z}{z_1})^{-1}$ and $\rho_0 \beta > 0$. Defining the positive constants

$$
\alpha = \left(\frac{\rho_0 e^{-\frac{1}{\beta} \frac{1}{z_1}}}{\beta}\right)^{\frac{1}{3}} \quad \text{and} \quad \kappa = \alpha^2 \sqrt{\beta \rho_0}
$$

and the new variables

$$
t' = \frac{e^{-\frac{\beta}{\sqrt{\alpha}}}}{\sqrt{\alpha}} t; \quad x' = \alpha x; \quad y' = \alpha y; \quad z' = \frac{1}{\alpha^2} \left(1 - \frac{z}{z_1}\right)^{\frac{1}{4}}
$$
the metric (44) turns into the Tabensky-Taub solution ([8]):

\[ ds^2 = \frac{g^{\alpha \gamma}}{\sqrt{c^2 - z^2}} \left( \frac{1}{\alpha} (\frac{d\sigma}{\alpha \left( z - \rho_0 \right)})^2 - \frac{1}{\alpha} \left( \frac{d\tau}{\alpha \left( z - \rho_0 \right)} \right)^2 \right) \left( dx^2 + dy^2 \right). \] (47)

### 3.3. The charged plane

The solution of the Einstein–Maxwell problem for a charged plane ([3, 13]) is obtained as we set \( \gamma = 0 \) and \( \epsilon = -2 \) in the type I solution in equation (36):

\[ ds^2 = \left( 1 - \frac{z}{\alpha} \right) \left( 1 - \frac{z}{\alpha \left( z_1 - \rho_0 \right)} \right)^{-2} dr^2 - \left( 1 - \frac{z}{\alpha \left( z_1 - \rho_0 \right)} \right)^{-2} \left( 1 - \frac{z}{\alpha \left( z_1 - \rho_0 \right)} \right)^{2} \left( dx^2 + dy^2 \right) \] (48)

with \( \alpha = 2 - \frac{z}{\alpha \left( z_1 - \rho_0 \right)} \neq 0 \) and \( 1 - \alpha \rho_0 > 0 \). If we define the new coordinate

\[ z' = \frac{z}{b \sigma \left( \alpha \left( z_1 - \rho_0 \right) \right)}; \quad \sigma = \frac{1}{1 - \alpha}, \quad \rho_0 = b \sigma^2, \] (49)

then the metric (48) is in coordinates of a uniform electric field:

\[ ds^2 = \left( 1 + b \sigma z' \left( 1 + \sigma z' \right) \right) dr^2 - \left( 1 + \sigma z' \right)^{-2} \left( dx^2 + dy^2 \right) \] (50)

The term \( \left( 1 + \sigma z' \right)^{-2} \) preceding \( dx^2 + dy^2 \) in equation (50) corrects a typographical error in formula (59) from [3].

### 3.4. Solutions with a cosmological constant \( \Lambda \)

We could have gone even further, inserting a cosmological constant \( \Lambda \) from the beginning: just take \( \rho + \Lambda \) and \( p - \Lambda \) instead of \( \rho \) and \( p \), respectively. As an example, consider the family of vacuum solutions with cosmological constant \( \Lambda \neq 0 \) ([2, 14, 15]), which in this case turns out to be a type I solution with \( \rho = -p = \Lambda \) and \( \partial \rho = 0 \), that is, \( \gamma = 0 \) and \( \epsilon = 0 \).

Applying equation (36) with \( \alpha - 1 \Lambda > 0 \), we get

\[ ds^2 = \left( 1 - \frac{z}{\alpha \left( z_1 - \rho_0 \right)} \right)^{3} \left( 1 - \frac{z}{\alpha \left( z_1 - \rho_0 \right)} \right)^{-2} \left( dx^2 + dy^2 \right) \] (51)

If \( \Lambda > 0 \), we define the coordinate \( z' \) through

\[ \frac{z}{z_1} = 1 - (\alpha - 1) \tan^2(\alpha z'), \quad a = \frac{\sqrt{3\Lambda}}{2}. \] (52)

Except for rescaling the coordinates \( t, x, y \) with suitable constant parameters, the metric (51) in the \( z' \)-coordinate representation is
This is the vacuum solution presented by Novitný and Horský ([2, 15]).
A similar formula holds for $\Lambda < 0$ if we change the trigonometric functions to hyperbolic ones.

4. Final remarks

In this work we have shown that any static plane symmetric metric can be given, at least locally, as a simple function of the matter content of a spacetime. This is not a new feature in General Relativity, as one can see tracing back the Oppenheimer-Volkoff equation of hydrostatic equilibrium for static spherically symmetric perfect fluids (see [16], section 23.5).

As a new improvement, we have shown in this paper that the remaining conservation equation of the former is a single ordinary differential equation on the energy and pressures, instead of the integral-differential equation of the latter. Furthermore, we have not restricted the physical characteristics of matter beyond the necessary adjustments to fit the symmetry.

Although studying the singular nature of plane symmetric spacetimes is not the aim of this work, a few remarks on it are suitable. Hence, since the existence of singularities is often detected as a blow up of the curvature or Kretschmann scalars in a point, we can use the formulas in the proof of theorem 1 to relate them to the matter parameters. Therefore we obtain for the curvature scalar

$$ R = -\rho + 3p - 2\delta p, $$

whereas for the Kretschmann scalar

$$ K = R_{\mu\nu}^\kappa R_{\rho\sigma}^\lambda = 2\left(\frac{z - z_2}{z - z_1}\right)^2 + \left(\rho + p - 2\delta p + 2\frac{z_2 - z_1}{z - z_1}p\right)^2 + 4\left(\frac{z_2 - z_1}{z - z_1}p\right)^2 + 2\left(\rho + \frac{z_2 - z_1}{z - z_1}p\right)^2. $$

The expressions (54) and (55) give us some clues of where look for a singularity: in a point where the matter content blows up or in the plane $z = z_1$, as long as the pressure close to it behaves like $p \sim (z - z_1)^{\alpha}$, $\alpha < 1$. As noted in [12], for ordinary matter (perfect fluid with $1 < \gamma < 2$) a singularity appears in $z = z_1$, even though no matter is present there, for $p(z_1) = \rho(z_1) = 0$. This is readily verified in formula (34) with $\epsilon = 0$.

Concerning the special configurations of energy and pressures in theorem 2, little can be said for the condition $p = 0$ and $\delta p = 0$, since in such case the spacetime geometry will depend on the arbitrary function $\delta p(z)$, as is clear in the formulas (19) and (20). On the other hand, if $2\rho = 2p = \delta p = 0$, it is straightforward to see in (21) that $z = (\alpha + \beta)^{-1}$ is a singular plane, since $R$ blows up there. For the vacuum ($\alpha \beta = 0$) with $\alpha + \beta = 0$, we have a Rindler coordinate horizon in the case $\beta = 0$, or a true singularity in the Taub-Levi-Civita solution when $\alpha = 0$. The latter is well studied in [3]. For $\alpha + \beta = 0$, we observe a constant energy density, and no hint of singular behavior is apparent in the metric (23).

The results obtained so far purport to be valuable in the foundations of General Relativity. As a particular interest, grasping the relationship between the quantum Casimir effect and gravitation was the starting point of this work [17]. From this viewpoint, we have just set the groundwork where the problem may lay, and grasping it is a long, deep and uncertain journey. Investigations are under way.
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