Further Comments on the Symmetric Subtraction of the Nonlinear Sigma Model

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Abstract

Recently a perturbative theory has been constructed, starting from the Feynman rules of the nonlinear sigma model at the tree level in the presence of an external vector source coupled to the flat connection and of a scalar source coupled to the nonlinear sigma model constraint (flat connection formalism).

The construction is based on a local functional equation, which overcomes the problems due to the presence (already at one loop) of non chiral symmetric divergences. The subtraction procedure of the divergences in the loop expansion is performed by means of minimal subtraction of properly normalized amplitudes in dimensional regularization.

In this paper we complete the study of this subtraction procedure by giving the formal proof that it is symmetric to all orders in the loopwise expansion. We provide further arguments on the issue that, within our subtraction strategy, only two parameters can be consistently used as physical constants.

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1 Introduction

A quantum field theory based on the Feynman rules of the nonlinear sigma model is plagued not only by the presence of an infinite number of superficially divergent amplitudes but also by the fact that the divergences are not chiral invariant. These difficulties are present already at the one loop level, as has been widely discussed in the existing literature [1]-[7]. Recently the construction of a perturbative theory for the nonlinear sigma model has been proposed by using a local functional equation for the generating functional of the 1 PI amplitudes [8]. The equation stems from the invariance under local chiral transformations of the Haar measure in the path integral. This formulation overcomes the difficulty due to the lack of chiral symmetry of the divergences. The subtraction of the divergences is performed in dimensional regularization by using minimal subtraction. In the present work we discuss this subtraction procedure and give the formal proof that it is symmetric to all orders in the loopwise expansion.

We use the notion of “symmetric subtracted” theory when the perturbation series: i) can be made finite by the subtraction of the infinities in a local fashion and ii) the defining functional equation is not spoiled by the introduction of counterterms. Since the defining functional equation induces transformations on the vertex functional in a natural way (see Section 3) and the counterterms must have definite invariance properties under these transformations, we use the adjective “symmetric” in order to indicate the whole of these essential properties.

The construction of the perturbative series starts from the Feynman rules of the nonlinear sigma model. The radiative corrections are regularized by continuation in the dimensions. The strategy by which the divergences are removed in the limit $D = 4$ makes use of two important properties of the functional equation, that are duly discussed in Ref. [8], [9] and [10]: i) hierarchy ii) Weak Power Counting (WPC). As summarized in Section 2 the functional equation has a rigid hierarchy structure in the loop expansion: all amplitudes involving the pion fields (descendant amplitudes) are derived from those involving only insertions of the flat connection ($F_{a \mu}$) and the order parameter (the constraint $\phi_0$), the ancestor amplitudes. The important consequence of this fact comes from the second property: the WPC. At each order of the loop expansion the number of divergent ancestor amplitudes is finite, since the superficial divergence of a graph is ($N_J$ and $N_K$ are the number of flat connection and order parameter insertions)

$$(D - 2)n + 2 - N_J - 2N_{K_0}. \quad (1)$$

The proof of this result is recalled in Appendix A. Thus at each loop order
all amplitudes are made finite by a finite number of subtractions. Moreover the WPC remains valid only if one does not introduce terms of higher dimensions in the tree-level Feynman rules. These facts suggest our subtraction strategy: if one finds a way to perform subtractions without introducing free parameters for higher dimensions counterterms in the tree level Feynman rules, then one gets a consistent theory with a finite number of physical parameters. The subtraction strategy is suggested by the functional equation itself. The violation of the equation at $n$-th order, when the counterterms are introduced up to order $n-1$, has simple dimensional properties when the scale parameter is varied. Then a simple pole removal (minimal subtraction) automatically restores the functional equation.

In previous works [8], [9] and [10] we have discussed this point by means of some non trivial examples and of formal arguments. In this work we present the proofs of the necessary steps for its implementation.

Under the assumption of the validity of the local functional equation and of the WPC the tree-level Feynman rules turn out to be unique. These results depend on the form of the parametrization of the group elements in terms of fields and consequently on the particular form of the transformation on the fields. Moreover the issue of the number of independent physical parameters within this perturbative framework can be addressed. We confirm that only the vacuum expectation value of the order parameter and the scale of the radiative correction enter in the final expression of the subtracted amplitudes.

In the case of the nonlinear sigma model the theory is defined through the effective action $\Gamma$ which has to obey a nonlinear local functional equation. At the one loop level the counterterms $\hat{\Gamma}^{(1)}$ obey a linearized form of the same equation. These counterterms have a particular feature: they are not present in the vertex functional $\Gamma^{(0)}$ at the tree level. Therefore the question arises whether they can be inserted back into the tree-level vertex functional. The answer is negative. Some of them do not obey the nonlinear defining functional equation. Others modify in a substantial way the unperturbed space of states (by introducing ghost states associated to kinetic terms in $\Box^2$). Finally there are some that could be introduced in the vertex functional $\Gamma^{(0)}$ at the tree level, since they obey the defining local functional equation, but they would spoil the WPC.

That is, the tree-level vertex functional is uniquely fixed by the symmetry and the WPC.

Then one can try to assign free parameters to the counterterms at the one loop level. Despite this is mathematically allowed, we argue that this strategy is not sustainable from the physical point of view, since parameters should enter in the zero loop vertex functional $\Gamma^{(0)}$. We stress this fundamental point: the expansion parameters of the classical action might differ from
the physical parameters of the zero loop vertex functional. The presence of a vacuum state that induces a reshuffling of the perturbative expansion (spontaneous symmetry breaking) is one example where such a distinction is essential.

After we have excluded free parameters in association to the counterterms, the question remains of the number of independent parameters in the symmetric subtraction scheme we are proposing. One parameter is present in $\Gamma^{(0)}$; for instance, the vacuum expectation value of $\phi_0$. However an extra mass parameter can be introduced in order to perform dimensional subtraction. We argue that this parameter has the very important role of fixing the scale of the radiative corrections. One can formulate the model in such a way that the dimensional subtraction scale appears as a front factor of the whole set of Feynman rules. The final consequence of this physical requirement is that our subtraction procedure for the nonlinear sigma model depends on two parameters, e.g. the v.e.v. of the spontaneous breakdown and the dimensional subtraction scale.

The present paper is devoted to a detailed illustration of the above mentioned facts. The finding of a symmetric subtraction scheme for the nonlinear sigma model which is consistent to all orders in the loopwise expansion allows us to investigate explicitly one example of a nonrenormalizable theory that can be consistently subtracted (i.e. symmetrically and locally).

The discussion is illustrated at the one loop level, but the necessary tools for the extension at higher order are also provided. In particular we discuss for any order in the loop expansion the equation obeyed by counterterms and the consistency of the subtraction procedure.

## 2 The Nonlinear Sigma Model

The $D$-dimensional classical action of the nonlinear sigma model in the flat connection formalism \cite{8} is

\[
\Gamma^{(0)} = \frac{m_D^2}{8} \int d^D x \left( F^\mu_a - J^\mu_a \right)^2 + \int d^D x K_0 \phi_0
\]  

(2)

where $m_D = m_D^{D-1}$. The flat connection is

\[
F^\mu = F^\mu_a \tau_a = \frac{i}{g} \Omega \partial_\mu \Omega^\dagger
\]

\[
\Omega = \frac{1}{m_D} (\phi_0 + ig\tau_a \phi_a)
\]  

(3)

\footnote{Here the external current $\tilde{J}_\mu$ of Ref. \cite{8} has been rescaled by a factor $-\frac{m_D^2}{4}$ and a harmless $J^2$ has been introduced in the effective action.}
where $\vec{J}_\mu$ is the background connection and $K_0$ is the source of the constraint $\phi_0$ of the nonlinear sigma model

$$\phi_0^2 + g^2 \phi_0^2 = m_D^2.$$  

(4)

$\Gamma^{(0)}$ obeys a $D$-dimensional local functional equation associated to the local chiral transformations induced by left multiplication on $\Omega$ by SU(2) matrices

$$U(\omega) \simeq 1 + \frac{i}{2} g_\tau a_\omega.$$  

(5)

The following equation is required to be valid for the effective action on the basis of a path integral formulation of the model

$$- \partial_\mu \frac{\delta \Gamma}{\delta J_\mu^a} - g \epsilon_{abc} J_\mu^b \frac{\delta \Gamma}{\delta J_\mu^a} + g \frac{1}{2} \epsilon_{abc} \phi_0 \frac{\delta \Gamma}{\delta \phi_b} + g^2 \frac{1}{2} \phi_a K_0 + \frac{1}{2} \frac{\delta \Gamma}{\delta K_0} \frac{\delta \Gamma}{\delta \phi_a} = 0.$$  

(6)

The equation is local (no x-integration). The generating functional of the Green functions obeys the corresponding equation

$$\left( \partial_\mu \frac{\delta \Gamma}{\delta J_\mu^a} + g \epsilon_{abc} J_\mu^b \frac{\delta \Gamma}{\delta J_\mu^a} + \frac{1}{2} g \epsilon_{abc} K_0 \frac{\delta \Gamma}{\delta K_0} - \frac{1}{2} g^2 \phi_0 \frac{\delta \Gamma}{\delta \phi_a} + \frac{1}{2} \frac{\delta \Gamma}{\delta K_0} \frac{\delta \Gamma}{\delta \phi_a} \right) Z = 0.$$  

(7)

The naïve Feynman rules given implicitly in eq. (2) yield amplitudes that solve eqs. (6) and (7). This property has been conjectured in Ref. [8] and it is proved in Appendix B.

The most general solution to eq.(6) in the loopwise expansion has been characterized by cohomological methods in [11].

The spontaneous breakdown of the global chiral symmetry is fixed by the boundary condition

$$\left. \frac{\delta \Gamma}{\delta K_0} \right|_{\vec{J}_\mu = \vec{F}_\mu = 0} = m_D.$$  

(8)

It will be required that these equations ((6), (7)and (8)) remain valid also for the subtracted amplitudes (symmetric subtraction).

The non linearity of the equation (6) is responsible for many peculiar facts. In particular by eq.(8) $\frac{\delta \Gamma}{\delta K_0}$ is invertible as a formal power series. Therefore by using eq.(6) all amplitudes involving the $\vec{\phi}$ fields (descendants) can be derived from those of $\vec{F}_\mu$ and $\phi_0$ (ancestors), i.e. the functional derivatives with respect to $\vec{J}_\mu$ and $K_0$ (hierarchy).
The tree level amplitudes are fixed by the conditions
\[
\frac{\delta^2 \Gamma^{(0)}}{\delta J^\mu_a(x) \delta J^\nu_b(y)} = \frac{m_D^2}{4} g_{\mu\nu} \delta_{ab} \delta_D(x - y)
\]
\[
\frac{\delta^2 \Gamma^{(0)}}{\delta K_0(x) \delta K_0(y)} = 0
\]
\[
\frac{\delta^2 \Gamma^{(0)}}{\delta K_0(x) \delta J^\nu_b(y)} = 0.
\] (9)

The dependence of the solution from the parameter \(g\) is somehow peculiar. Given an unsubtracted solution \(\Gamma[J, K_0, \phi, m_D, g]\) of equation (6) with the boundary conditions (8) and (9), one can check that
\[
\Gamma[g^{-1}J, g^{-1}K_0, g^{-1}\phi, g m_D, g]
\] (10)
obeys the same equations with \(g = 1\). Thus \(g\) can be removed by a redefinition of the mass scale parameter \(m_D \rightarrow g m_D\) (together with \(J \rightarrow g^{-1}J\) and \(K_0 \rightarrow g^{-1}K_0\)), i.e. for unsubtracted vertex functional one has
\[
\Gamma[g^{-1}J, g^{-1}K_0, g^{-1}\phi, g m_D, g] = \Gamma[J, K_0, \phi, m_D, 1].
\] (11)

However the situation changes if one wants to define the theory at \(D = 4\). Subtraction of poles is needed and, together with this, a scale parameter in the definition of the Feynman amplitudes is necessary. At one loop level the dependence of the subtracted amplitudes from \(\ln m\) (in \(D = 4\)) does not allow the complete removal of \(g\). Thus, at least at the one loop level, the introduction of \(g\) is equivalent to use an extra mass scale in the dimensional subtraction and accounts for variants of the minimal subtraction.

There is another interesting rescaling strategy, i.e. consider
\[
\Gamma[g^{-1}J, K_0, g^{-1}\phi, m_D, g].
\] (12)

This vertex functional satisfies the eq. (9) with \(g = 1\) and eq. (8) unchanged. But eq. (9) becomes
\[
\frac{\delta^2 \Gamma^{(0)}}{\delta J^\mu_a \delta J^\nu_b} = \frac{m^2}{4 g^2} g_{\mu\nu} \delta_{ab} = \left(\frac{m}{g}\right)^2 \frac{m^{(D-4)}}{4} g_{\mu\nu} \delta_{ab}
\] (13)
i.e., again, we have a new mass parameter
\[
v = \frac{m}{g}.
\] (14)
The discussion on the rôle of the parameter \(g\) will be resumed and expanded in Sec. 7.

The Feynman rules provided by eq. (2) give rise to a perturbative expansion governed by the WPC theorem [9]. The superficial degree of divergence of a \(n\)-th loop amplitude involving \(N_J\) insertions of the flat connection and \(N_{K_0}\) insertions of the nonlinear sigma model constraint is (see Appendix A)

\[
\delta = (D - 2)n + 2 - N_J - 2N_{K_0}.
\]  

(15)

3 Subtractions at \(D = 4\). One Loop

Subtractions at \(D = 4\) are performed in dimensional regularization. At one loop level the counterterms (obtainable from the pole parts of the vertex functional \(\Gamma^{(1)}\)) obey a linearized form of the local equation eq. (16)

\[
S_a(\hat{\Gamma}^{(1)}) = \left[ \frac{1}{2} \frac{\delta \Gamma^{(0)}}{\delta \phi_a} \frac{\delta}{\delta K_0} + \frac{1}{2} \frac{\delta \Gamma^{(0)}}{\delta \phi_0} \frac{\delta}{\delta \phi_a} + g \frac{\epsilon_{abc}}{2} \phi_c \frac{\delta}{\delta \phi_b} \right. \\
- \partial_{\mu} \frac{\delta}{\delta J_{\mu}^a} - g \epsilon_{abc} J_{\mu}^b \frac{\delta}{\delta J_{\mu}^c} \left. \right] \hat{\Gamma}^{(1)} = 0.
\]  

(16)

It is easy to trace in eq. (16) the transformations induced through the dependence on \(\vec{J}_\mu\) and \(\vec{\phi}\). Further properties can be derived by introducing the Grassmann parameter \(\omega_a\) and the nilpotent operator [9]

\[
s \omega_a = -\frac{g}{2} \epsilon_{a i j k} \omega_j \omega_k.
\]  

(18)

We consider the Legendre transform

\[
S_0 = \Gamma^{(0)} \big|_{K_0=0}.
\]

(19)

where \(S_0 = \Gamma^{(0)} \big|_{K_0=0}\). One gets

\[
s K_{\omega'} = -\frac{g}{2} \epsilon_{\omega' a k} \omega_k K_a + \frac{g^2}{2} \omega_{\omega'} K_0
\]  

(20)
and

\[ s \, K_0 = -\frac{1}{2} \omega_a K_a. \]  \hfill (21)

Then

\[
\overline{K}_0 \equiv K_0 \phi_0 + K_a \phi_a \\
= K_0 \phi_0 - \phi_a \frac{\delta}{\delta \phi_a} S_0 + g^2 \frac{K_0}{\phi_0} \phi_a \phi_a \\
= \frac{m_D^2 K_0}{\phi_0} - \phi_a \frac{\delta}{\delta \phi_a} S_0
\]

is invariant under \( s \):

\[ s \, \overline{K}_0 = 0. \]  \hfill (23)

In terms of the background connection \( J_{a\mu} \) and of the flat connection

\[ F^\mu_a = \frac{2}{m_D^2} \left( \phi_0 \partial^\mu \phi_a - \partial^\mu \phi_0 \phi_a + g \epsilon_{abc} \partial^\mu \phi_b \phi_c \right) \]  \hfill (24)

the invariant solutions of the linearized functional equation which enter at the one loop level read \cite{9}

\[
\mathcal{I}_1 = \int d^D x \left[ D_\mu (F - J)_\nu \right] \left[ D_\nu (F - J)^\mu \right]_a, \\
\mathcal{I}_2 = \int d^D x \left[ D_\mu (F - J)_\mu \right] \left[ D_\nu (F - J)^\nu \right]_a, \\
\mathcal{I}_3 = \int d^D x \epsilon_{abc} \left[ D_\mu (F - J)_a \right] \left( F^\mu_b - J^\mu_b \right) \left( F^\nu_c - J^\nu_c \right), \\
\mathcal{I}_4 = \int d^D x \left( \frac{m_D^2 K_0}{\phi_0} - \phi_a \frac{\delta S_0}{\delta \phi_a} \right)^2, \\
\mathcal{I}_5 = \int d^D x \left( \frac{m_D^2 K_0}{\phi_0} - \phi_a \frac{\delta S_0}{\delta \phi_a} \right) \left( F^\mu_b - J^\mu_b \right)^2, \\
\mathcal{I}_6 = \int d^D x \left( F^\mu_a - J^\mu_a \right)^2 \left( F^\nu_b - J^\nu_b \right)^2, \\
\mathcal{I}_7 = \int d^D x \left( F^\mu_a - J^\mu_a \right) \left( F^\nu_a - J^\nu_a \right) \left( F_{b\mu} - J_{b\mu} \right) \left( F_{b\nu} - J_{b\nu} \right), \]  \hfill (25)

where \( D_\mu \) denotes the covariant derivative w.r.t. \( F_{ab} \):

\[ D_{ab} = \delta_{ab} \partial_\mu + g \epsilon_{abc} F_{c\mu}. \]  \hfill (26)
By dimensional arguments one expects that at one loop the counterterms
(the $1/(D-4)$ pole parts) are linear combinations of $I_1 \ldots I_7$. In Ref. [9] the
linear combination is explicitly evaluated. On these grounds other solutions
of eq. (16) are excluded, e.g.

$$
\int d^D x \bar{K}_0.
$$

(27)

4 Subtraction at $D = 4$. Higher Loops

We now discuss the subtraction procedure at higher loops. The content of
this Section has been the subject of conjectures and explicit examples in
References [8], [9] and [10]. Here we try to present an organic formulation.
The proofs are given in Appendix [11], [12] and [13].

At higher loops the counterterms obey a more complex equation, since
the lower order terms contribute to a non homogeneous term

$$
S_a(\hat{\Gamma}^{(n)}) = \left[ \frac{1}{2} \delta \Gamma^{(0)} \frac{\delta}{\delta \phi_a} \delta \bar{K}_0 + \frac{1}{2} \frac{\delta \Gamma^{(0)}}{\delta \phi_a} \frac{\delta}{\delta \phi_a} + \frac{1}{2} \epsilon_{abc} \phi_c \frac{\delta}{\delta \phi_a} \delta \bar{K}_0 \delta \phi_b 
- \partial^\mu \frac{\delta}{\delta J_a^\mu} - g \epsilon_{abc} J_b^\mu \frac{\delta}{\delta J_c^\mu} \right] \hat{\Gamma}^{(n)} = \frac{1}{2} \sum_{j=1}^{n-1} \frac{\delta \hat{\Gamma}^{(j)}}{\delta \bar{K}_0} \frac{\delta \hat{\Gamma}^{(n-j)}}{\delta \phi_a}.
$$

(28)

The above equation is valid provided that the subtractions are performed in
such a way that eqs. (6) and (7) are preserved. By standard arguments ([12],
[13]) one can show the validity of the consistency condition

$$
s \int d^D x \omega_a \left( \sum_{j=1}^{n-1} \frac{\delta \hat{\Gamma}^{(j)}}{\delta \phi_a} \frac{\delta \hat{\Gamma}^{(n-j)}}{\delta \bar{K}_0} \right) = 0
$$

(29)

under the assumption that eq. (28) is recursively fulfilled up to order $n - 1$.
For the discussion presented in the next Sections it is worth to outline the
arguments that lead to eq. (28). Consider the formal perturbative expansion
of the generating functional of the Feynman amplitudes, where the counter-
terms $\hat{\Gamma}^{(j)}$ have been introduced

$$
Z[\bar{K}, K_0, \bar{K}] = \exp \left( i \left( \Gamma^{(0)} + \sum_{j \geq 1} \hat{\Gamma}^{(j)} \right) \right)_{\phi_a = -i \frac{\delta}{\delta \phi_a}}
$$

$$
= \exp \frac{1}{2} \int d^D x \ d^D y \ K_b(x) D_F(x - y) K_b(y).
$$

(30)

We introduce a shorthand notation

$$
\hat{\Gamma} = \Gamma^{(0)} + \sum_{j \geq 1} \hat{\Gamma}^{(j)}.
$$

(31)
In the Appendix B we give a diagrammatic proof of the following relation, which essentially shows the validity of the Quantum Action Principle à la Breitenlohner and Maison \[14\]

\[
\begin{align*}
&\left(-\partial_\mu \frac{\delta}{\delta J_\mu^a} - g\epsilon_{abc}J_\mu^b \frac{\delta}{\delta J_\mu^c} + \frac{1}{2} g^2 K_0 \frac{\delta}{\delta K_a} - \frac{1}{2} K_a \frac{\delta}{\delta K_0} - \frac{1}{2} g\epsilon_{abc}K_b \frac{\delta}{\delta K_c}\right) Z = \\
i\left(-\partial_\mu \frac{\hat{\Gamma}}{\delta J_\mu^a} - g\epsilon_{abc}J_\mu^b \frac{\hat{\Gamma}}{\delta J_\mu^c} - \frac{1}{2} g^2 K_0 \phi_a + \frac{1}{2} \delta \hat{\Gamma} \delta \phi_a + g\frac{1}{2} \epsilon_{abc} \phi_c \frac{\delta \hat{\Gamma}}{\delta \phi_b} + \frac{1}{2} \delta \hat{\Gamma} \delta \phi_a \frac{\delta \hat{\Gamma}}{\delta \phi_b}\right) \cdot Z,
\end{align*}
\]

where the dot indicates the insertion of the local operators. Eq. (32) shows the connection between eqs. (6) and (7) and eq. (28). Equation (32) for generic $D$ is valid also without counterterms. In this case it shows that the amplitudes constructed with the naïve Feynman rules generated from $\Gamma^{(0)}$ are solutions of eqs. (6) and (7) \[8\].

Before we describe the subtraction procedure, it is worth to illustrate a further equation for the 1PI generating functionals $\Gamma^{(n)}$. We sort the 1PI functionals according to the total power in $\hbar$ of the counterterms present in the Feynman integrals. Let us then define by $\Gamma^{(n,k)}$ the n-loop 1PI functional where the power of $\hbar$ of the counterterms is a fixed $k \leq n$. Then the vertex function at n-loop is

\[
\Gamma^{(n)} = \sum_{j=0}^{n} \Gamma^{(n,j)}. \tag{33}
\]

In Appendix C we prove the following equation \((n > 0)\)

\[
\begin{align*}
&\left(-\partial_\mu \frac{\delta}{\delta J_\mu^a} - g\epsilon_{abc}J_\mu^b \frac{\delta}{\delta J_\mu^c} + \frac{1}{2} g^2 K_0 \frac{\delta}{\delta K_a} - \frac{1}{2} K_a \frac{\delta}{\delta K_0} - \frac{1}{2} g\epsilon_{abc}K_b \frac{\delta}{\delta K_c}\right) \Gamma^{(n,k)} + \\
&\frac{1}{2} \sum_{n' = 1}^{n-1} \sum_{j = \min(k,n')} \frac{\Gamma^{(n',j)} \Gamma^{(n-n',k-j)}}{\delta K_0 \delta \phi_a} = 0. \tag{34}
\end{align*}
\]

The consistency condition

\[
\int d^D x \omega_a(x) \left( \sum_{n' = 1}^{n-1} \sum_{j = \max(0,k-n')} \frac{\Gamma^{(n',j)} \Gamma^{(n-n',k-j)}}{\delta K_0 \delta \phi_a} \right) = 0 \tag{35}
\]
is valid also in this case, but the proof will be omitted. Eq. (34) is a very powerful tool for investigations over the validity of the functional equation (6), since it allows the study of the counterterms by introducing a grading on them. Consider the 1PI generating functional where counterterms $\hat{\Gamma}^{(j)}$ have been introduced up to $n-1$ loops in such a way to fulfill the local functional equation and to remove the poles up to order $n-1$. Then at $n$-loops poles in $D-4$ are present and moreover we expect a violation of eq. (6). In Appendix D we show that the breaking is given by the following equation

\[
-\partial^\mu \frac{\delta \Gamma^{(n)}}{\delta J^\mu_a} - g \epsilon_{abc} J^\mu_b \frac{\delta \Gamma^{(n)}}{\delta J^\mu_c} + \frac{1}{2} g \epsilon_{abc} \phi_c \frac{\delta \Gamma^{(n)}}{\delta \phi_b} \delta \Gamma^{(n)} \\
+ \frac{1}{2} \delta \Gamma^{(0)} \frac{\delta \Gamma^{(n)}}{\delta K_0} \delta \phi_a + \frac{1}{2} \delta \Gamma^{(0)} \frac{\delta \Gamma^{(n)}}{\delta K_0} \delta \phi_a + \frac{1}{2} \sum_{j=1}^{n-1} \frac{\delta \Gamma^{(n-j)}}{\delta K_0} \delta \phi_a \\
+ \frac{1}{2} \sum_{j=1}^{n-1} \frac{\delta \Gamma^{(n-j)}}{\delta K_0} \delta \phi_a = \frac{1}{2} \sum_{j=1}^{n-1} \delta \hat{\Gamma}^{(j)} \frac{\delta \Gamma^{(n-j)}}{\delta \phi_a}.
\] (36)

Since the bilinear terms have no poles in $D-4$, the procedure of minimal subtraction yields $n$-loop counterterms that obey a non homogeneous linearized equation. These $n$-th order counterterms obey then eq. (28).

Our strategy of subtraction of infinities is based on eq. (36). If we properly normalize the amplitudes, the breaking term in eq. (36) contains only poles in $D-4$ (no finite parts!). Thus minimal subtraction for the properly normalized amplitudes removes the breaking term and therefore yields a recursive and consistent procedure based only on the parameters $v$ and $m$, i.e. on the vev and the scale of dimensional regularization. This subtraction procedure is presented in full details in Appendix D.

### 5 Parameters Fixing

In this Section we show that we cannot introduce at the tree level new Feynman vertices associated to the one-loop counterterms if we want to produce a sensible and consistent theory.

Minimal subtraction is of course a very interesting option in order to make finite the perturbative series. The conjecture that this subtraction algorithm is symmetric (i.e. eq. (4) is stable) is supported by some general arguments (given in Sec. 4) and by an explicit example in Ref. [10]. Appendix D gives the final proof that the conjecture is indeed correct. Thus this theory can be tested by experiments.
A frequent objection to the present proposal of making finite a nonrenormalizable theory is that one needs seven parameter-fixing appropriate measures in order to evaluate the coefficients of $I_1 \ldots I_7$. This objection is legitimate if the above mentioned invariants are action-like. As one should do in power counting renormalizable theories, according to algebraic renormalization [15]-[18]. Here the situation is more involved. This is evident if we paraphrase the problem in the following way. Can we introduce at the tree level the seven invariants with arbitrary coefficients and treat them as bona fide interaction terms intervening in the loop expansion as the original one provided in $\Gamma^{(0)}$ of eq. (2)? The answer to this question is in general negative. If one allows this modification of the unperturbed effective action, the one loop corrections will be modified by extra terms generated by the newly introduced Feynman rules, thus bringing to a never ending story.

In particular the introduction at tree level of the vertices described by the invariants in eq.(25) implies new Feynman rules which invalidate the weak power-counting [9] (with the exception of the combination

$$2(I_1 - I_2) - 4I_3 + I_6 - I_7 = \int d^Dx G_{a\mu
u} G^{\mu\nu}_a,$$

(37)

which however depends only on the field strength squared of the external source $J_{a\mu}$ and thus does not modify the Feynman rules for the pions).

The superficial degree of divergence of the ancestor amplitudes is not any more given by eq. (15). As a direct consequence of the violation of the weak power-counting, already at one loop the number of divergent ancestor amplitudes is infinite.

A closer look to $I_1 \ldots I_7$ shows that there are also other reasons that forbid the use of some of these invariants as unperturbed effective action terms. $I_1, I_2$ can be introduced into $\Gamma^{(0)}$ without breaking eq. (6). However unless they appear in the combination $I_1 - I_2$ (see eq.(37)) they modify the spectrum of the unperturbed states (by introducing negative norm states) through kinetic terms with four derivatives. $I_4, I_5$ cannot be introduced into $\Gamma^{(0)}$ because they violate eq. (6).

6 Finite Subtractions

After we excluded the possibility of introducing in the tree level effective action the invariants $I_1 \ldots I_7$, there is still the possibility to use them for a finite, in principle arbitrary, renormalization strictly at one loop level. I.e. in
the book keeping of the Feynman rules one could enter new terms

\[ \hbar \sum_j \lambda_j \int d^D x \mathcal{I}_j(x), \]  

(38)

where we have explicitly exhibited the \( \hbar \) factor in order to remind that these vertices are of first order in \( \hbar \) expansion. \( \lambda_j \) are arbitrary real parameters.

More explicitly we can tell the story in the following way. The subtraction of the poles in \( D - 4 \) requires a series of counterterms of the form \( (38) \) where the coefficients carry the pole factor \( 1/(D - 4) \). Then the option to use these extra degrees of freedom as free parameters can be explored since it is mathematically allowed.

In the case of a power counting renormalizable theory the fixing of the finite parts of the symmetric counterterms can be seen as a way to introduce the renormalization by a reset of the parameters entering into the classical action. The situation is clearly different in the present case, since the invariants \( \mathcal{I}_1, \ldots, \mathcal{I}_7 \) are not action-like and therefore the additional parameters \( \lambda_j \) can be introduced only as independent quantum corrections.

The meaning of this latter procedure seems to us rather unclear from the physical point of view. If one wishes to introduce new independent parameters one should do so at tree level (notice that the local functional equation is non-anomalous). If one requires in addition the WPC, then there is no more freedom left and one ends up with the tree-level Feynman rules encoded in eq. (2).

7 Normalization of the amplitudes

In minimal subtraction we use pure pole subtraction in order to make the theory finite in \( D = 4 \). Even with this clear cut strategy, still there is some freedom left connected to the presence of \( g \) or equivalently to the use of a second scale parameter in the Feynman rules in dimensional renormalization. Here we would like to give a formulation of this choice that has some appeal.

When we evaluate the counterterms, by starting from the vertex functional in \( D \) dimensions, we automatically make a statement on their finite parts. Thus from a generic amplitude in \( D \) dimensions involving \( n \) external currents \( J \)

\[ \Gamma[J_1 \cdots J_n|D] \]  

(39)

the counterterm is obtained by using the normalized function

\[ \frac{m^2}{m_D^2} \Gamma[J_1 \cdots J_n|D] = \frac{1}{m^{(D-4)}} \Gamma[J_1 \cdots J_n|D]. \]  

(40)
Its pole part in $D = 4$ fixes the counterterms. For example, the single pole part in $\Gamma[J_1 \cdots J_n, |D]$ is removed by the counterterm mechanism

$$\Gamma[J_1 \cdots J_n, |D] = \frac{m^{(D-4)}}{(D-4)} \lim_{D' \to 4} \left( (D' - 4)m^{-(D'-4)}\Gamma[J_1 \cdots J_n, |D'] \right). \quad (41)$$

The normalization used in eq. (40) is needed in order to produce the correct dimensions of the counterterms in eq. (41). Similarly one proceeds with $K_0$.

The normalized function is

$$\frac{m^{D-4}}{m^{(\frac{D}{2}-2)}} \Gamma[K_0 \cdots K_{0n}, |D] = m^{(1-\frac{n}{2})(D-4)}\Gamma[K_0 \cdots K_{0n}, |D]. \quad (42)$$

Eq. (40) show that the parameter $g$ can be removed only in unsubtracted amplitudes (i.e. at $D \neq 4$). At the one loop level the replacement $m_D \to g m_D (m \to mg^2)$ leads to a $\ln g$ dependence of the amplitude in $D = 4$.

In fact eq. (40) becomes

$$\left(m g^2 m_D\right)^{(4-D)} \Gamma[J_1 \cdots J_n, |D]. \quad (43)$$

Thus the minimal subtraction introduces in this case a new mass scale $mg$.

The formulation with two parameters takes a particularly elegant form if we suppress $g$ in favor of a second mass scale and moreover we assign to $K_0$ a dimension that is $D$-independent; i.e. in a way that the normalization factor for the subtraction of the poles is identical both for $\vec{J}^\mu$ and $K_0$. To achieve this normalization we perform a transformation similar to eq. (12)

$$\Gamma\left[g^{-1} \vec{J}, \left(\frac{gm}{m_D}\right)^{-1} K_0, \left(\frac{gm}{m_D}\right) g^{-1} \vec{\phi}, m_D, g\right]. \quad (44)$$

Thus one gets

$$\begin{align*}
- \partial^\mu \frac{\delta \Gamma}{\delta J_\mu^a} - \epsilon_{abc} J_\mu^b \frac{\delta \Gamma}{\delta J_\mu^c} &+ \frac{1}{2} \epsilon_{abc} \frac{\delta \Gamma}{\delta \phi_b} - \frac{1}{2} \phi_a K_0 + \frac{1}{2} \delta K_0 \frac{\delta \Gamma}{\delta \phi_a} = 0, \\
\frac{\delta^2 \Gamma^{(0)}}{\delta J_\mu^a \delta J_\nu^b} = &\frac{v^2}{4} m^{(D-4)} g_{\mu\nu} \delta_{ab} \quad (46)
\end{align*}$$

and

$$\frac{\delta \Gamma}{\delta K_0} \bigg|_{\vec{\phi} = J_\mu = K_0 = 0} = \left(\frac{m_D}{gm}\right) m_D = vm^{(D-4)}. \quad (47)$$

Note that $v$ cannot be removed by a rescaling $\vec{\phi} \to v \vec{\phi}$ and $K_0 \to 1/v K_0$. In fact the dependence on $v$ remains in eq. (46).
This amounts to formally perform the path-integral according to ($D\Omega$ denotes the invariant Haar measure over SU(2))

$$Z[\vec{J}, K_0, \vec{K}] = \int D\Omega \exp i \int d^D x \left\{ m^{(D-4)} \left[ \frac{v^2}{8} (F_\mu^a - J_\mu^a)^2 + K_0 \phi_0 + K_a \phi_a \right] \right\}$$

(48)

with

$$F_\mu^a = F_\mu^a \frac{1}{2} \tau_a = i \Omega \partial_\mu \Omega^\dagger$$

$$\Omega = \frac{1}{vm^{(D-4)}} (\phi_0 + i \tau_a \phi_a)$$

(49)

and

$$\phi_0^2 + \phi_j^2 = v^2 m^{2(D-4)}.$$ 

(50)

By this choice the dimensions of $\vec{J}_\mu$ and $K_0$ are equal to one and three respectively. The evaluation of the counterterms is then the same (independently from the number of $\vec{J}_\mu$ and of $K_0$), via simple pole subtraction of the normalized functions as in eq. (40)

$$\Gamma = \left( \frac{1}{m} \right)^{(D-4)} \Gamma[J_1 \cdots J_n K_{01} \cdots K_{0n'} | D].$$

(51)

Then the full set of Feynman rules is

$$\hat{\Gamma} = \int d^D x \left\{ m^{(D-4)} \left( \frac{v^2}{8} (F_\mu^a - J_\mu^a)^2 + \sum_{j=1}^{M} M^{(j)} \right) + K_0 \phi_0 \right\},$$

(52)

where the $M^{(j)}$ are the local counterterms containing the pole parts in $D = 4$. The (non trivial) finite parts of the subtractions are governed by the sole front factor $m^{-(D-4)}$ in eq. (51). The resulting amplitudes depend on the parameters $v$ and $m$. The last one is not present in the classical action at $D = 4$: it sneaks in as a scale of the radiative corrections.

A similar mechanism has a renowned antecedent in the theory of Lamb shift\cite{19}, where the radiative corrections due to the excited state transitions need a ultraviolet cut-off which is not present at the lowest level of the theory of the Hydrogen atom.

A comment is in order here. In the NLSM a shift in $m$ cannot be compensated by a shift in $v$. Therefore $m$ has to be treated as a second independent
free parameter (in addition to \( v \)) to be determined through the fit with the experimental data.

The question whether this subtraction can be performed by means of other regularization schemes has been considered. Only limited results have been achieved. One difficulty is related to the fact that all these schemes spoil the defining local functional equation. By using the renormalized linear sigma model in the limit of large coupling constant one can get, after subtraction of divergent terms, the nonlinear sigma model we are proposing (one loop has been checked in ref. [20]). This requires a fine tuning in the finite subtractions and consequently there is no evidence for a particular advantageous choice in the finite subtraction as in dimensional regularization. In order to study this issue it is very useful to consider the most general solution allowed by the linearized homogeneous functional equation. At one loop this means seven arbitrary coefficients associated to the invariants reported in Sect. 3 (see eq. [25]). The same pattern is present in other regularization procedures as Pauli-Villars.

8 Conclusions

In this paper it has been proven that the subtraction procedure based on the flat connection formalism is indeed consistent (i.e. local and symmetric) to all orders in the loop expansion.

This subtraction scheme is based on some novel technical tools. First the use of the vertex functional instead of the action, in order to formulate the theory. A functional equation is then the defining instrument. This equation has some essential properties: hierarchy and weak power counting. These properties are the guiding tracks for the construction of a perturbative expansion in the number of loops. The restoration of the functional equation at every order at \( D = 4 \) suggests a subtraction procedure which defines the theory itself. The procedure is based on minimal subtraction in dimensional regularization.

The resulting theory depends on \( m \) and \( g \) (or \( v \) if one uses two scales as in Section 7). We have also shown that in the subtraction procedure a mass parameter enters as a scale of the radiative corrections. In Section 7 we formulated the symmetrically subtracted nonlinear sigma model in such a way that the second parameter enters as a common front factor of the whole Feynman rules (counterterms included).

It is a remarkable fact that the tree-level Feynman rules are uniquely fixed by the local functional equation and the weak power-counting.

Uniqueness of the tree-level Feynman rules has some important conse-
quences. In fact at each order of the perturbative series one can introduce finite renormalizations by using the appropriate local solutions of eq. (28). Yet these renormalizations cannot be inserted back into the tree-level vertex functional. Thus they cannot be interpreted as additional physical parameters in the loopwise expansion.

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A Weak Power Counting

Consider amplitudes involving only the flat connection $F_{a\mu}$ and the order parameter field $\phi_0$. The number of loops $n$ is given by

$$n = I - \sum_{j} V^J_j - \sum_{k} V^K_0 - \sum_{l} V^\phi_l + 1$$

(53)

where $I$ is the number of internal $\vec{\phi}$-lines, $V^J_j$ the number of vertices with one $J$ and $j$ $\vec{\phi}$, $V^K_0$ the number of vertices with one $K_0$ and $k$ $\vec{\phi}$, $V^\phi_l$ the number of vertices with $l$ $\vec{\phi}$.

The superficial degree of divergence is

$$\delta = nD - 2I + \sum_{j} V^J_j + 2\sum_{l} V^\phi_l$$

$$= nD + \sum_{j} V^J_j - 2\left(n + \sum_{j} V^J_j + \sum_{k} V^K_0 - 1\right)$$

$$= n(D - 2) + 2 - N_J - 2N_{K_0}$$

(54)

where

$$N_J \equiv \sum_{j} V^J_j, \quad N_{K_0} \equiv \sum_{k} V^K_0.$$  

(55)

It should be remarked that the superficial degree of divergence does not depend on the number of $\vec{\phi}$ self-interaction vertices.
B Perturbative Solutions of the Functional Equation

In this Appendix we provide a diagrammatic proof of eq. (32). We follow a technique suggested in Ref. [8], Section 13. The framework is given by dimensional regularization, where the Feynman rules are given by a formal series of local operators as in eq. (31). The propagator and the vertices are originated from the partition of \( \hat{\Gamma} \) into a free bilinear term \( \Gamma^{(0)}_{\text{BIL}} \) and the rest \( \hat{\Gamma}_{\text{INT}} \) which yields the interaction and the counterterms.

Consider the following operation

\[
\left( -\partial^\mu \frac{\delta}{\delta J^\mu_a} - g\epsilon_{abc} J^\mu_b \frac{\delta}{\delta J^\mu_c} + \frac{1}{2} g^2 K_0 \frac{\delta}{\delta K_a} - \frac{1}{2} K_a \frac{\delta}{\delta K_0} - \frac{1}{2} g\epsilon_{abc} K_0 \frac{\delta}{\delta K_c} \right) Z
\]

on the generating functional

\[
Z[J, K_0, \vec{K}] = \int D[\Omega] \exp i \left( \hat{\Gamma}[J, K_0, \phi] + \int d^Dx K_a \phi_a \right)
\]

(56)

The functional derivatives are local insertions according to the formalism of path integral

\[
\frac{\delta Z}{\delta J^\mu_a(x)} = i \frac{\delta \hat{\Gamma}}{\delta J^\mu_a(x)} \cdot Z \equiv i \int D[\Omega] \frac{\delta \hat{\Gamma}}{\delta J^\mu_a(x)} \exp i \left( \hat{\Gamma} + \int d^Dy K_a \phi_a \right)
\]

\[
\frac{\delta Z}{\delta K_a(x)} = i \phi_a(x) \cdot Z \equiv i \int D[\Omega] \phi_a(x) \exp i \left( \hat{\Gamma} + \int d^Dy K_a \phi_a \right)
\]

(57)

Continuation in \( D \) dimensions guarantees the validity of eqs. (58). Let us consider

\[
\left[ \frac{\delta}{\delta K_{a_1}(x_1)} \cdots \frac{\delta}{\delta K_{a_n}(x_n)} \left( \frac{\delta \hat{\Gamma}}{\delta \phi_a(x)} \cdot Z \right) \right]_{R=0} = i^n \left( \frac{\delta \hat{\Gamma}}{\delta \phi_a(x)} \phi_{a_1}(x_1) \cdots \phi_{a_n}(x_n) \cdot Z \right)_{R=0}
\]

(59)

where the last step is a consequence of the equation of motion for \( \vec{\phi} \). In fact
one finds

\[
\left. \left( \frac{\delta \hat{\Gamma}}{\delta \phi_a(x)} \phi_{a_1}(x_1) \cdots \phi_{a_n}(x_n) \cdot Z \right) \right|_{k=0}
\]

\[
= \int d^D z \frac{\delta^2 \Gamma^{(0)}_{\text{BIL}}}{\delta \phi_a(x) \delta \phi_b(z)} \left( \phi_b(z) \phi_{a_1}(x_1) \cdots \phi_{a_n}(x_n) \cdot Z \right) \bigg|_{k=0}
\]

\[
+ \left. \left( \frac{\delta \hat{\Gamma}_{\text{INT}}}{\delta \phi_a(x)} \phi_{a_1}(x_1) \cdots \phi_{a_n}(x_n) \cdot Z \right) \right|_{k=0}
\]

(60)

Finally we perform the relevant contractions

\[
= i \sum_{j=1}^n \delta_{aa_j} \delta(x - x_j) \left( \phi_{a_1}(x_1) \cdots \hat{\phi}_{a_j}(x_j) \cdots \phi_{a_n}(x_n) \cdot Z \right) \bigg|_{k=0}
\]

\[
+ i \int d^D z \frac{\delta^2 \Gamma^{(0)}_{\text{BIL}}}{\delta \phi_a(x) \delta \phi_b(z)} \int d^D z' \langle T(\phi_b(z)\phi_{b'}(z')) \rangle \left( \frac{\delta \hat{\Gamma}_{\text{INT}}}{\delta \phi_a(x)} \phi_{a_1}(x_1) \cdots \phi_{a_n}(x_n) \cdot Z \right) \bigg|_{k=0}
\]

\[
+ \left. \left( \frac{\delta \hat{\Gamma}_{\text{INT}}}{\delta \phi_a(x)} \phi_{a_1}(x_1) \cdots \phi_{a_n}(x_n) \cdot Z \right) \right|_{k=0}
\]

\[
= i \sum_{j=1}^n \delta_{aa_j} \delta(x - x_j) \left( \phi_{a_1}(x_1) \cdots \hat{\phi}_{a_j}(x_j) \cdots \phi_{a_n}(x_n) \cdot Z \right) \bigg|_{k=0}
\]

(61)

Eq. (59) shows that

\[
\frac{\delta \hat{\Gamma}}{\delta \phi_a(x)} \cdot Z = -K_a(x)Z.
\]

(62)

In order to apply the result of eq. (62) to the expression in eq. (56), one has to consider the situation where

\[
x = x_j.
\]

(63)

These contributions have to be neglected since the massless propagator for coinciding points is zero in dimensional regularization

\[
D_F(0) = \frac{i}{(2\pi)^D} \int \frac{d^D k}{k^2 + i\epsilon} = 0.
\]

(64)
Then eq. \(58\) shows together with eq. \(62\) that
\[
\left(-\partial^\mu \frac{\delta}{\delta J_\mu^a} - g\epsilon_{abc} J_\mu^b \frac{\delta}{\delta J_\mu^c} + \frac{1}{2} g^2 K_0 \frac{\delta}{\delta K_a} - \frac{1}{2} K_a \frac{\delta}{\delta K_0} - \frac{1}{2} g\epsilon_{abc} K_0 \frac{\delta}{\delta K_c}\right) Z = i \left(-\partial^\mu \frac{\delta \hat{\Gamma}}{\delta J_\mu^a} - g\epsilon_{abc} J_\mu^b \frac{\delta \hat{\Gamma}}{\delta J_\mu^c} + \frac{1}{2} g^2 K_0 \frac{\delta \hat{\Gamma}}{\delta \phi_a} + \frac{1}{2} K_a \frac{\delta \hat{\Gamma}}{\delta \phi_0} + \frac{1}{2} g\epsilon_{abc} \phi_a \frac{\delta \hat{\Gamma}}{\delta \phi_b}\right) Z
\]
i.e. one gets eq. \(32\).

If the counterterms obey the equation
\[
-\partial^\mu \frac{\delta \hat{\Gamma}}{\delta J_\mu^a} - g\epsilon_{abc} J_\mu^b \frac{\delta \hat{\Gamma}}{\delta J_\mu^c} + \frac{1}{2} g^2 K_0 \frac{\delta \hat{\Gamma}}{\delta \phi_a} + \frac{1}{2} K_a \frac{\delta \hat{\Gamma}}{\delta \phi_0} + \frac{1}{2} g\epsilon_{abc} \phi_a \frac{\delta \hat{\Gamma}}{\delta \phi_b} = 0
\]
then eq. \(7\) is valid order by order. Thus eq. \(28\) has to be imposed on the counterterms.

It should be stressed that no special requirements are imposed on \(\hat{\Gamma}\). In particular the counterterms might be absent. In this case eq. \(65\) proves that the construction of the perturbative series in \(D\) dimension based on the Feynman rules of the nonlinear sigma model (without subtractions) yields a solution of the functional equation \(6\). In fact \(\Gamma^{(0)}\) obeys eq. \(66\).

## C Grading by the Counterterms

This Appendix is devoted to the proof of eq. \(34\). In Appendix \(B\) we proved that the functional equation \(7\) for the Feynman amplitudes is satisfied in \(D\) dimensions, if the Feynman rules, collectively denoted by \(\hat{\Gamma}\) in eq. \(31\), obey the equation \(28\). The index \(j\) in eq. \(31\) denotes the grading in the \(\hbar\) expansion.

One can change the Feynman rules by using a real parameter \(\rho\)
\[
\hat{\Gamma}_\rho = \Gamma^{(0)} + \sum_{j \geq 1} \rho^j \hat{\Gamma}^{(j)}
\]
and eq. \(28\) is still valid. By using \(\hat{\Gamma}_\rho\) one gets a new set of Feynman amplitudes generated by \(Z[\rho, J, K_0, K]\) which obey eq. \(7\). The corresponding 1PI \(\Gamma[\rho, J, K_0, \phi]\) satisfies eq. \(13\). By construction only \(\Gamma[\rho, J, K_0, \phi]\big|_{\rho=1}\) has a finite limit for \(D \to 4\).
For practical calculations it is useful to exploit the fact that \( \Gamma[\rho, \vec{J}, K_0, \vec{\phi}] \) is a solution of the functional equation (6) for any real \( \rho \). As in reference [10] we introduce the notation

\[
\Gamma^{(n)}_\rho = \sum_{j=0}^{n} \Gamma^{(n,j)}_\rho \rho^j,
\]

(68)

where the exponent of \( \rho \) counts the total power of \( \hbar \) of the counterterms. By inserting this expression in eq. (6) one gets (for \( n > 0 \))

\[
\sum_{k=0}^{n} \rho^k \left( -\partial^\mu \frac{\delta}{\delta J_\mu^a} - g\epsilon_{abc} J_\mu^b \frac{\delta}{\delta J_\mu^c} + \frac{1}{2} g\epsilon_{abc} \phi_\epsilon \frac{\delta}{\delta \phi_b} \right) \Gamma^{(n,k)} + \frac{1}{2} \sum_{n'=0}^{n} \sum_{j=0}^{n'} \sum_{j'=0}^{j} \rho^{j+j'} \frac{\delta \Gamma^{(n-n',j)}_\rho}{\delta K_0} \frac{\delta \Gamma^{(n',j')}_\rho}{\delta \phi_a} = 0 .
\]

(69)

Since \( \rho \) is an arbitrary parameter, one gets

\[
\left( -\partial^\mu \frac{\delta}{\delta J_\mu^a} - g\epsilon_{abc} J_\mu^b \frac{\delta}{\delta J_\mu^c} + \frac{1}{2} g\epsilon_{abc} \phi_\epsilon \frac{\delta}{\delta \phi_b} \right) \Gamma^{(n,k)} + \frac{1}{2} \sum_{n'=0}^{n} \sum_{j=0}^{\min(n',k)} \frac{\delta \Gamma^{(n',j)}_\rho}{\delta K_0} \frac{\delta \Gamma^{(n-n',k-j)}_\rho}{\delta \phi_a} = 0 ,
\]

(70)

i.e. equation (69).

### D The Subtraction Procedure

In this Appendix we describe in details how the subtraction of the divergences are performed in order to take the limit \( D \to 4 \). The naturalness of the procedure has induced us to propose it as a rule in the construction of a physical theory tout court.

We use the Feynman rules in eq. (48), where \( g \) has been traded by \( v \) according to eq. (14). This choice of parameters has the advantage that we can keep trace of the dimensions of the counterterms in terms of powers of \( m \), the scale of the radiative corrections.

After rescaling

\[
\phi_a \to m^{(D-4)} \phi_a
\]

(71)
we get the tree level vertex functional
\[
m^{(D-4)} \int d^D x \left\{ \frac{1}{2} \left( \partial_\mu \phi_0 \partial^\mu \phi_0 + \partial_\mu \phi_a \partial^\mu \phi_a - \frac{v^2}{2} F^\mu_a J_{a\mu} + \frac{v^2}{4} J^2 \right) + K_0 \phi_0 \right\}
\]
\[
= m^{(D-4)} \int d^D x \left\{ \frac{1}{2} \left( \partial_\mu \phi_a \partial^\mu \phi_a + \frac{\phi_a \partial_\mu \phi_a \partial^\mu \phi_a}{\phi_0^2} - \frac{v^2}{2} F^\mu_a J_{a\mu} + \frac{v^2}{4} J^2 \right) + \frac{v^2}{2} F^\mu_a J_{a\mu} \right\}
\]
(72)
with
\[
\phi_0 = \sqrt{v^2 - \vec{\phi}^2}
\]
\[
\Omega = \frac{1}{v} (\phi_0 + i \tau_a \phi_a)
\]
\[
F^\mu_a = \frac{2}{v^2} (\phi_0 \partial^\mu \phi_a - \partial^\mu \phi_0 \phi_a + \epsilon_{abc} \partial^\mu \phi_b \phi_c).
\]
(73)

Then the \(\vec{\phi}\) propagator has a factor \(m^{-(D-4)}\), while every vertices \(J - \vec{\phi}^j\), \(K_0 - \vec{\phi}^k\) and \(\vec{\phi}\) (see the notations in Appendix A) has a factor \(m^{(D-4)}\).

### D.1 One Loop

The one loop 1PI amplitudes have total power of \(m\) equal to zero. Since the counterterms for ancestor amplitudes in eq. (52) are of the form
\[
\hat{\Gamma}^{(1)} = m^{(D-4)} \int d^D x M^{(1)}[J, K_0](x)
\]
(74)

the dimensional subtraction has to be performed on the normalized vertex functional for the ancestor amplitudes
\[
\int d^D x M^{(1)}[J, K_0](x) = -\frac{1}{D-4} \lim_{D^\prime \to D} (D^\prime - 4) \frac{1}{m^{(D^\prime-4)}} \Gamma^{(1)}[J, K_0]. \quad (75)
\]

If we look at the defining functional equation at one loop [45]
\[
S_a(\Gamma^{(1)}) = \left[ \frac{1}{2m^{(D-4)}} \frac{\delta \Gamma^{(0)}}{\delta \phi_a} \frac{\delta}{\delta \phi_0} + \frac{1}{2m^{(D-4)}} \frac{\delta \Gamma^{(0)}}{\delta K_0} \frac{\delta}{\delta \phi_a} + \frac{1}{2} \epsilon_{abc} \phi_c \frac{\delta}{\delta \phi_b} - \partial_\mu \frac{\delta}{\delta J^\mu_a} - \epsilon_{abc} J^\mu_b \frac{\delta}{\delta J^\mu_c} \right] \Gamma^{(1)} = 0
\]
(76)
(where the rescaling (71) has been accounted for), one sees that the same normalization and pole subtraction as in eq. (75) should be used for amplitudes involving only one external $\vec{\phi}$. In fact this can be seen on purely dimensional grounds by counting the powers of $m$ for an arbitrary one-loop graph after the rescaling in eq. (71).

The counterterms obey the equation

$$S_a(\hat{\Gamma}^{(1)}) = \left[ \frac{1}{2m(D-4)} \frac{\delta \Gamma^{(0)}}{\delta \phi_a} \frac{\delta}{\delta K_0} + \frac{1}{2m(D-4)} \frac{\delta \Gamma^{(0)}}{\delta K_0} \frac{\delta}{\delta \phi_a} + \frac{1}{2} \epsilon_{abc} \phi_c \frac{\delta}{\delta \phi_b} \frac{\delta}{\delta \phi_b} \right] \hat{\Gamma}^{(1)} = 0.$$  \hspace{1cm} (77)

### D.2 Two Loops

Once the counterterms at one loop have been introduced, the two-loop amplitudes need a further subtraction in order to take the limit $D \to 4$.

This problem can be described from different points of view. We find it illuminating to use of the grading in the counterterms as expressed in eq. (34) and discussed in Appendix C. Let $\Gamma^{(2,1)}$ be the vertex functional at two loops containing the counterterms of first order $\hat{\Gamma}^{(1)}$. Then eq. (34), after the rescaling given in eqs. (44) and (71), reads

$$\left( -\frac{\partial^\mu}{\partial \delta J_a^\mu} - \epsilon_{abc} J_b^\mu \frac{\delta}{\delta J_c^\mu} + \frac{1}{2} \epsilon_{abc} \phi_c \frac{\delta}{\delta \phi_b} \frac{\delta}{\delta \phi_b} \right) \hat{\Gamma}^{(1)} = 0.$$  \hspace{1cm} (78)

On the other side, $\Gamma^{(2,0)}$ obeys the functional equation in $D$ dimensions

$$\left( -\frac{\partial^\mu}{\partial \delta J_a^\mu} - \epsilon_{abc} J_b^\mu \frac{\delta}{\delta J_c^\mu} + \frac{1}{2} \epsilon_{abc} \phi_c \frac{\delta}{\delta \phi_b} \frac{\delta}{\delta \phi_b} \right) \Gamma^{(2,0)} + \frac{1}{2m(D-4)} \frac{\delta \Gamma^{(0)}}{\delta K_0} \frac{\delta \Gamma^{(1,0)}}{\delta \phi_a} = 0.$$  \hspace{1cm} (79)
Then $\Gamma^{(2,0)} + \Gamma^{(2,1)}$, the vertex functional at two loop with only first order counterterms, obeys the equation

\begin{equation}
\left(-\partial^\mu \frac{\delta}{\delta J^\mu_a} - \epsilon_{abc} J^\mu_b \frac{\delta}{\delta J^\mu_c} + \frac{1}{2} \epsilon_{abc} \phi_c \frac{\delta}{\delta \phi_b} + \frac{1}{2m^{(D-4)}} \frac{\delta \Gamma^{(0)}}{\delta \phi_a} \frac{\delta}{\delta \phi_a} + \frac{1}{2m^{(D-4)}} \frac{\delta \Gamma^{(0)}}{\delta \phi_a} \frac{\delta}{\delta \phi_a}\right) \left(\Gamma^{(2,0)} + \Gamma^{(2,1)}\right) + \frac{1}{2m^{(D-4)}} \frac{\delta \Gamma^{(0)}}{\delta \phi_a} \frac{\delta}{\delta \phi_a} \frac{1}{2m^{(D-4)}} \frac{\delta \Gamma^{(0)}}{\delta \phi_a} \frac{\delta}{\delta \phi_a} = \frac{1}{2m^{(D-4)}} \frac{\delta \hat{\Gamma}^{(1)}}{\delta \phi_a} \frac{\delta \hat{\Gamma}^{(1)}}{\delta \phi_a},
\end{equation}

which agrees with eq. (80). Two comments are in order for eq. (80):

i) After we normalize the amplitudes and subtract the pole parts as described in eq. (75), the breaking term in eq. (80) disappears.

ii) Had we chosen to perform a further finite renormalization at one loop by using the local invariant solutions of the linearized equation in eq. (25), the pure pole structure of the breaking term in eq. (80) would have been destroyed. Consequently no criterion would be left at our disposal in order to choose the subtraction at two loops.

### D.3 n Loops

A last straightforward step is necessary in order to complete the recursive procedure of subtraction. We use again eq. (34) in order to find the breaking of the defining functional equation for $\Gamma^{(n)}$ once counterterms up to order $n-1$ have been introduced

\begin{equation}
\Gamma^{(n)} = \sum_{k=0}^{n-1} \Gamma^{(n,k)}.
\end{equation}

The grading in the counterterms is very useful in subtracting out the remaining singularities at $D = 4$. In fact one can spot how the $n$-th order
counterterms enter in an essential way. By using the identity (84)

\[
\left(-\partial^{\mu} \frac{\delta}{\delta J_a^{\mu}} - \epsilon_{abc} J_b^{\mu} \frac{\delta}{\delta J_c^{\mu}} + \frac{1}{2} \epsilon_{abc} \phi_c \frac{\delta}{\delta \phi_b} + \frac{1}{2 m^{(D-4)}} \frac{\delta \Gamma^{(0)}}{\delta K_0} \frac{\delta}{\delta \phi_a} + \frac{1}{2 m^{(D-4)}} \frac{\delta \Gamma^{(0)}}{\delta K_0} \frac{\delta}{\delta \phi_a} \right) + 1
\]

\[
+ \frac{1}{2 m^{(D-4)}} \left( \sum_{k=0}^{n-1} \Gamma^{(n,k)} \right)
\]

\[
+ \frac{1}{2 m^{(D-4)}} \sum_{n'=1}^{n-1} \frac{\delta}{\delta K_0} \left( \sum_{j=0}^{n'} \Gamma^{(n',j)} \right) \frac{\delta}{\delta \phi_a} \left( \sum_{j'=0}^{n-n'} \Gamma^{(n-n',j')} \right)
\]

\[
= \frac{1}{2 m^{(D-4)}} \sum_{n'=1}^{n-1} \frac{\delta}{\delta K_0} \left( \Gamma^{(n',n')} \right) \frac{\delta}{\delta \phi_a} \left( \Gamma^{(n-n',n-n')} \right).
\]  \hspace{1cm} (82)

Since, by definition \( \Gamma^{(k,k)} = \hat{\Gamma}^{(k)} \), one gets the breaking term of eq. (83). Thus the recursive removal of the divergences is consistent.

Let us give a closer look at the subtraction procedure. We perform the subtraction of the pole parts on the normalized ancestor amplitudes. The power \( \nu \) in the factor \( m^{\nu(D-4)} \) present in any \( n \)-th order ancestor amplitude can be evaluated as in Appendix A

\[
\nu = - I + \sum_j V_j^J + \sum_k V_k^{K_0} + \sum_l V_l^{\phi}
\]

\[
= 1 - n,
\]  \hspace{1cm} (84)

where \( n \) is the number of loops. Thus \( \Gamma^{(n)}[J,K_0] \) behaves as \( m^{(1-n)(D-4)} \) and the normalized amplitudes as \( m^{-n(D-4)} \). Consequently the removal of the poles for the normalized amplitude \( m^{-(D-4)} \Gamma^{(n)}[J,K_0] \) corresponds to a nontrivial choice in the subtraction in dimensional regularization.

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