Examples of twisted cyclic cocycles from covariant differential calculi

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Abstract

For two covariant differential \( \ast \)-calculi, the twisted cyclic cocycle associated with the volume form is represented in terms of commutators \([\mathcal{F}, \rho(x)]\) for some self-adjoint operator \(\mathcal{F}\) and some \(\ast\)-representation \(\rho\) of the underlying \(\ast\)-algebra.

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0 Introduction

Twisted cyclic cocycles arise under certain assumptions from covariant differential calculi on quantum groups or quantum spaces [KMT]. Let \(\Gamma = \bigoplus_k \Gamma^k\) be a differential calculus on an algebra \(\mathcal{X}\). We assume the existence of a volume form \(\omega \in \Gamma^\wedge\) (that is, for each \(n\)-form \(\eta \in \Gamma^{\wedge n}\), there is a unique element \(\pi(\eta) \in \mathcal{X}\) such that \(\eta = \pi(\eta)\omega\), of an algebra automorphism \(\sigma_1\) of \(\mathcal{X}\) for which \(\omega x = \sigma_1(x)\omega\), and of a twisted trace \(h\) on \(\mathcal{X}\) (that is, there is an algebra automorphism \(\sigma_2\) of \(\mathcal{X}\) such that \(h(xy) = h(\sigma_2(y)x)\) for all \(x, y \in \mathcal{X}\)). The twisted cyclic cocycle \(\tau_{\omega,h}\) on \(\mathcal{X}\) associated with \(\omega\) and \(h\) is defined by

\[
\tau_{\omega,h}(x_0, x_1, \ldots, x_n) = h(\pi(x_0 dx_1 \wedge \cdots \wedge dx_n)), \quad x_0, x_1, \ldots, x_n \in \mathcal{X}.
\]

Precise definitions and assumptions are given below.
The purpose of this paper is to investigate two examples of such twisted cyclic cocycles \( \tau_{\omega,h} \). The first example, discussed in Section 2, is the \( \ast \)-algebra \( \mathcal{X}_0 \) with generators \( z, z^\ast \) and defining relation \( z^\ast z - q^2 zz^\ast = (1 - q^2)\alpha \), where \( 0 < q < 1 \) and \( \alpha = 0, 1, -1 \). We study the twisted cyclic cocycle associated with the distinguished differential calculus on this \( \ast \)-algebra. The second example, treated in Section 3, concerns left-covariant differential calculi on Hopf algebras. First we develop the general framework of representing differential forms in terms of commutators and then we carry out the details for Woronowicz’ 3D-calculus on the quantum group \( SU_q(2) \). Our main technical tool is to construct an appropriate commutator representation \( dx \sim i [F, \rho(x)] \) of the corresponding first order \( \ast \)-calculus, where \( \rho \) is a \( \ast \)-representation of \( \mathcal{X} \) and \( F \) is a self-adjoint operator on a Hilbert space. In both examples, the cocycle \( \tau_{\omega,h} \) is then described in the form

\[
\tau_{\omega,h}(x_0, x_1, \ldots, x_n) = \text{Tr} A \gamma_q \rho(x_0)[F, \rho(x_1)] \cdots [F, \rho(x_n)],
\]

where \( A \) is a certain density operator and \( \gamma_q \) is a grading operator (see Theorems 2.1 and 3.3 below for details).

1 Twisted cyclic cohomology

Suppose that \( \mathcal{X} \) is a complex algebra and \( \sigma \) is an algebra automorphism of \( \mathcal{X} \). Let \( \varphi \) be an \((n + 1)\)-linear form on \( \mathcal{X} \). The \( \sigma \)-twisted coboundary operator \( b_\sigma \) and the \( \sigma \)-twisted cyclicity operator \( \lambda_\sigma \) on \( \mathcal{X} \) are defined by

\[
(b_\sigma \varphi)(x_0, \ldots, x_n) = \sum_{j=0}^{n-1} (-1)^j \varphi(x_0, \ldots, x_j x_{j+1}, \ldots, x_n)
\]

\[
+ (-1)^n \varphi(\sigma(x_n)x_0, x_1, \ldots, x_{n-1}),
\]

\[
(\lambda_\sigma \varphi)(x_0, \ldots, x_n) = (-1)^n \varphi(\sigma(x_n), x_0, \ldots, x_{n-1}),
\]

where \( x_0, \ldots, x_n \in \mathcal{X} \). An \((n + 1)\)-form \( \varphi \) is called a \( \sigma \)-twisted cyclic \( n \)-cocycle if \( b_\sigma \varphi = 0 \) and \( \lambda_\sigma \varphi = \varphi \).

Twisted cyclic \( n \)-cycles occur in the study of differential calculi on algebras (see [KMT] and [KR]). Let \( \Gamma^\wedge = \oplus_{k=0}^\infty \Gamma^{\wedge k} \) be a differential calculus on \( \mathcal{X} \) with differentiation \( d : \Gamma^{\wedge k} \to \Gamma^{\wedge (k+1)} \). We assume that there is an \( n \)-form \( \omega \in \Gamma^{\wedge n} \) such that for each \( \eta \in \Gamma^{\wedge n} \) there exists a unique element \( \pi(\eta) \in \mathcal{X} \) such that \( \eta = \pi(\eta) \omega \). Moreover, we assume that there is an algebra automorphism \( \sigma_1 \) of \( \mathcal{X} \) such that

\[
\omega x = \sigma_1(x)\omega, \quad x \in \mathcal{X}.
\]
Suppose further that $h$ is a linear functional on $\mathcal{X}$ and $\sigma_2$ is an algebra automorphism of $\mathcal{X}$ satisfying

$$h(x_1 x_2) = h(\sigma_2(x_2)x_1), \quad x_1, x_2 \in \mathcal{X}, \quad (3)$$

$$h(\pi(dx_1 \wedge \cdots \wedge dx_n)) = 0, \quad x_1, \ldots, x_n \in \mathcal{X}. \quad (4)$$

Define $\sigma = \sigma_2 \circ \sigma_1$ and

$$\tau_{\omega,h}(x_0, x_1, \ldots, x_n) := h(\pi(x_0dx_1 \wedge \cdots \wedge x_n)), \quad x_0, \ldots, x_n \in \mathcal{X}. \quad (5)$$

Then, using the Leibniz rule and assumption (2) and (3), it is not difficult to check that $b_\sigma \tau_{\omega,h} = 0$. Using in addition (4), we obtain $\lambda_\sigma \tau_{\omega,h} = \tau_{\omega,h}$. (Note that (4) is crucial for the proof of the relation $\lambda_\sigma \tau_{\omega,h} = \tau_{\omega,h}$.) Thus, $\tau_{\omega,h}$ is a $\sigma$-twisted cyclic $n$-cycle on $\mathcal{X}$. We call $\tau_{\omega,h}$ the twisted cyclic $n$-cycle associated with the $n$-form $\omega$ of the differential calculus $\Gamma^\wedge$ and the functional $h$.

Let us briefly discuss the preceding assumptions on the calculus. Assumptions (3) and (4) are fulfilled in most interesting cases, for instance, if $\Gamma^\wedge$ is a left or right covariant differential calculus on a compact quantum group Hopf algebra $\mathcal{X}$ and $h$ is the Haar state of $\mathcal{X}$. The existence of a form $\omega \in \Gamma^\wedge n$ as above is satisfied for many, but not all, covariant differential calculi on Hopf algebras. It is satisfied for the standard bicovariant differential calculi on the quantum groups $GL_q(k)$ and $SL_q(k)$ (with $n = k^2$ and $\sigma_1 = \text{id}$, see [Sch]). The paper [H] contains a list of left covariant differential calculi on $O(SL_q(2))$ satisfying the assumptions with $n = 3$.

2 Quantum disc, complex quantum plane, and stereographic projection of the Podles’ sphere

In this section, let $\mathcal{X}_0$ be the unital $*$-algebra with generators $z$, $z^*$ and defining relation

$$z^*z - q^2zz^* = \alpha(1 - q^2), \quad (6)$$

where $q$ and $\alpha$ are fixed real numbers such that $0 < q < 1$ and $\alpha \in \{0, 1, -1\}$. There is a distinguished first order differential $*$-calculus $(\Gamma_0, d)$ on $\mathcal{X}_0$ with $\mathcal{X}_0$-bimodule structure given by

$$dz z = q^2zd\bar{z}, \quad dz z^* = q^{-2}z^*d\bar{z}, \quad dz^* z = q^2zdz^*, \quad dz^* z^* = q^{-2}z^*dz^*. \quad (7)$$

These simple relations have been found in [S2].

3
All three cases $\alpha = 0, 1, -1$ give interesting quantum spaces. In the case $\alpha = 0$, we get the quantum complex plane on which the Hopf $\ast$-algebra $\mathcal{U}_q(e_2)$ acts. For $\alpha = 1$, it gives the quantum disc algebra $[KL][SSV]$ which is a $\mathcal{U}_q(su_{1,1})$-module $\ast$-algebra. For $\alpha = -1$, the $\ast$-algebra $\mathcal{X}_0$ is a left $\mathcal{U}_q(su_{2})$-module $\ast$-algebra which can be interpreted as the stereographic projection of the standard Podles’ quantum sphere. To be more precise, the left action of $\mathcal{U}_q(su_{2})$ on the Hopf $\ast$-algebra $\mathcal{O}(SU_q(2))$ of the quantum SU(2) group extends to an action on the Ore extension $\hat{\mathcal{O}}(SU_q(2))$ with respect to the Ore set $\{b^r c^s ; r, s \in \mathbb{N}_0\}$. Then the left $\mathcal{U}_q(su_{2})$-module $\ast$-algebra $\mathcal{X}_0$ can be identified with the $\ast$-subalgebra of $\hat{\mathcal{O}}(SU_q(2))$ generated by the element $z := ac^{-1}$. In all three cases, the differential calculus $(\Gamma_0, d)$ on $\mathcal{X}_0$ is covariant with respect to the action of the corresponding Hopf $\ast$-algebras.

Next we pass to an appropriate Hilbert space representation of the $\ast$-algebra $\mathcal{X}_0$. It acts on an orthonormal basis $\{e_n ; n \in \mathbb{Z}\}$ of $\mathcal{H} = l^2(\mathbb{Z})$ and $\{e_n ; n \in \mathbb{N}_0\}$ of $\mathcal{H} = l^2(\mathbb{N}_0)$ for $\alpha = 0$ and $\alpha = \pm 1$, respectively, by the following formulas:

\[
\begin{align*}
\alpha = 0 : & \quad ze_n = q^{2(n+1)}e_{n+1}, \quad z^* e_n = q^{2n}e_{n-1}, \\
\alpha = 1 : & \quad ze_n = (1 - q^{2(n+1)})^{1/2}e_{n+1}, \quad z^* e_n = (1 - q^{2n})^{1/2}e_{n-1}, \\
\alpha = -1 : & \quad ze_n = (q^{-2n} - 1)^{1/2}e_{n-1}, \quad z^* e_n = (q^{-2(n+1)} - 1)^{1/2}e_{n+1}.
\end{align*}
\]

Let $\mathcal{X}$ be the $\ast$-algebra $L^+(\mathcal{D})$ of all linear operators $T$ on $\mathcal{D} = \text{Lin}\{e_n ; n \in \mathbb{Z}\}$ resp. $\mathcal{D} = \text{Lin}\{e_n ; n \in \mathbb{N}_0\}$ such that $T\mathcal{D} \subseteq \mathcal{D}$ and $T^*\mathcal{D} \subseteq \mathcal{D}$. We identify the $\ast$-algebra $\mathcal{X}_0$ with the corresponding $\ast$-subalgebra of $\mathcal{X}$.

Set $\beta = 1$ for $\alpha = 1$ and $\beta = -1$ for $\alpha = 0, -1$. Put $y := \beta(\alpha - zz^*)$. Note that $y^{-1} \in \mathcal{X}$. Let $\mathcal{X}_c$ be the set of all elements $x \in \mathcal{X}$ such that $y^k xy^l$ is bounded for all $k, l \in \mathbb{Z}$. Then $\mathcal{X}_c$ is a $\ast$-subalgebra of $\mathcal{X}$ and, moreover, a $\mathcal{X}_0$-bimodule. Notice that the closures of all operators $y^k xy^l$, $x \in \mathcal{X}_c$, are of trace class. We define an algebra automorphism $\sigma$ of $\mathcal{X}$ by $\sigma(x) = yxy^{-1}$, $x \in \mathcal{X}$. Since $y \geq 0$,

\[
h(x) := \text{Tr} y^{-1}x, \quad x \in \mathcal{X}_c,
\]

(8)

is a positive linear functional on $\mathcal{X}_c$ which obviously satisfies

\[
h(x_1 x_2) = h(\sigma(x_2) x_1), \quad x_1, x_2 \in \mathcal{X}_c.
\]

It can be shown that the action of the Hopf $\ast$-algebra $\mathcal{U}_q(e_2)$, $\mathcal{U}_q(su_{1,1})$ resp. $\mathcal{U}_q(su_{2})$ on $\mathcal{X}_0$ extends to $\mathcal{X}$ such that $\mathcal{X}$ and $\mathcal{X}_c$ are left module $\ast$-algebras and $h$ is an invariant functional on $\mathcal{X}_c$ (cf. [KW]). We do not carry out the details, because we will not need this fact in what follows.
Next we extend the first order differential calculus \((\Gamma_0, d)\) on \(X_0\) to the larger \(*\)-algebra \(X\) by using a commutator representation (see [S1]). Define an operator \(F\) on \(H \oplus H\) and a \(*\)-homomorphism \(\rho\) of \(X\) into the \(*\)-algebra \(L^+(\mathcal{D} \oplus \mathcal{D})\) by

\[
F = (1 - q^2)^{-1} \begin{pmatrix} 0 & z^* \\ z & 0 \end{pmatrix}, \quad \rho(x) = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, \quad x \in X.
\]

Then there is an injective linear map

\[
\Gamma_0 \ni x_1 dx_2 \mapsto \rho(x_1)[iF, \rho(x_2)] \in L^+(\mathcal{D} \oplus \mathcal{D}), \quad x_1, x_2 \in X_0.
\]

For simplicity, we identify \(x_1 dx_2\) with \(\rho(x_1)[iF, \rho(x_2)]\), so the differentiation of the calculus \((\Gamma_0, d)\) on \(X_0\) is given by the commutator with the operator \(iF\). Defining \(x_1 dx_2 := \rho(x_1)[iF, \rho(x_2)]\) for \(x_1, x_2 \in X\), the calculus \((\Gamma_0, d)\) is extended to a first order \(*\)-calculus \((\Gamma, d)\) on the \(*\)-algebra \(X\). Since \([z^*, z] = (1-q^2)(\alpha-zz^*) = (1-q^2)\beta y\) by (6), we have

\[
dz = \begin{pmatrix} 0 & 0 \\ i\beta y & 0 \end{pmatrix}, \quad dz^* = \begin{pmatrix} 0 & -i\beta y \\ 0 & 0 \end{pmatrix},
\]

so that

\[
dz x = \sigma(x)dz, \quad dz^* x = \sigma(x)dz^*, \quad x \in X. \quad (9)
\]

Since \(y\) is invertible, \(\{dz, dz^*\}\) is a left module basis of \(\Gamma\). Thus, for any \(x \in X\), there are uniquely determined elements \(\partial_x(z), \partial_x^*(x) \in X\) such that

\[
dx = \partial_x(z)dz + \partial_x^*(x)dz^*.
\]

Comparing the latter and the expressions for \(dz\) and \(dz^*\), we get

\[
\partial_x(z) = (1 - q^2)^{-1}\beta[z^*, x]y^{-1}, \quad \partial_x^*(x) = -(1 - q^2)^{-1}\beta[x, z^*]y^{-1}. \quad (10)
\]

Let us describe the higher order differential calculus \(\Gamma^k = \oplus_k \Gamma^k\) on \(X\) associated with the first order calculus \((\Gamma, d)\). The non-zero 2-form \(\omega := q^{-2}y^{-2}dz^* \wedge dz\) is a left module basis of \(\Gamma^k\) and we have \(\Gamma^k = \{0\}\) if \(k \geq 3\). (Incidentally, \(\omega\) is invariant with respect to the action of Hopf \(*\)-algebras mentioned above.) From (9) and the definition of \(\sigma\), we get \(\omega x = x\omega\) for all \(x \in X\), that is, condition (2) is satisfied with \(\sigma_1\) being the identity. Moreover, since \(dz \wedge dz = dz^* \wedge dz^* = 0\) and \(dz^* \wedge dz = -q^2dz \wedge dz^*\) in \(\Gamma^k\) by (7), it follows that

\[
x_0dx_1 \wedge dx_2 = x_0(q^2\partial_x^*(x_1)\sigma(\partial_x(x_2)) - \partial_x(x_1)\sigma(\partial_x^*(x_2)))y^2\omega
\]
for $x_0, x_1, x_2 \in \mathcal{X}$. Therefore, using (5), (8) and (10), we compute

$$
\tau_{\omega,h}(x_0, x_1, x_2) = h(x_0(q^2 \partial_z(x_1) \sigma(\partial_z(x_2))) - \partial_z(x_1) \sigma(\partial_z(x_2))) y^2
= (1 - q^2)^{-2} \text{Tr} y^{-1}x_0(-q^2[z, x_1][z^*, x_2] + [z^*, x_1][z, x_2])
= \text{Tr} \gamma_q \rho(y^{-1}) \rho(x_0)[\mathcal{F}, \rho(x_1)][\mathcal{F}, \rho(x_2)]
$$

(11)

for $x_0, x_1, x_2 \in \mathcal{X}_c$, where $\gamma_q$ denotes the “grading operator”

$$
\gamma_q = \begin{pmatrix} q^2 & 0 \\ 0 & -1 \end{pmatrix}.
$$

Finally, we have to show that condition (4) is satisfied. This could be done by using the invariance of $\omega$ under the Hopf algebra action, but we prefer to check it directly. Using the commutation relations $y^{-1}z = q^{-2}zy^{-1}, y^{-1}z^* = q^2z^*y^{-1}$ and (6), we compute

$$
\text{Tr} y^{-1}([z^*, x_1][z, x_2] - q^2[z, x_1][z^*, x_2])
= \text{Tr} y^{-1}(-x_1 z^* z x_2 + q^2 x_1 z z^* x_2) - \text{Tr} y^{-1}(z^* x_1 x_2 z - q^2 z x_1 x_2 z^*)
+ \text{Tr} y^{-1}(z^* x_1 x_2 z - q^2 x_1 z z^* x_2) + \text{Tr} y^{-1}(x_1 z^* x_2 z - q^2 z x_1 z^* x_2)
= \text{Tr} y^{-1}(-x_1 x_2 \alpha(1 - q^2)) - \text{Tr} (q^2 z^* y^{-1} x_1 x_2 z - z y^{-1} x_1 x_2 z^*)
+ \text{Tr} (q^2 z^* y^{-1} x_1 x_2 z - q^2 y^{-1} x_1 z x_2 z^*) + \text{Tr} (y^{-1} x_1 z^* x_2 z - z y^{-1} x_1 x_2 z^*)
= 0
$$

(12)

for $x_1, x_2 \in \mathcal{X}_c$. The last equality follows from the trace property since the closures of the operators $y^{-1}x_1, z y^{-1}x_1, z^* y^{-1}x_1$ are of trace class and $x_2, x_2 z, z_2, x_2 z^*, z^* x_2, z^* x_2, z_2 x_2 z^*$ are bounded by the definition of $\mathcal{X}_c$. As

$$
h(\pi(dx_1 dx_2)) = \tau_{\omega,h}(1, x_1, x_2) = \text{Tr} \gamma_q \rho(y^{-1})[\mathcal{F}, \rho(x_1)][\mathcal{F}, \rho(x_2)] = 0
$$

by (11) and (12), condition (4) holds. Therefore, $\tau_{\omega,h}$ is a $\sigma$-twisted cyclic 2-cocycle.

We summarize the preceding in the following theorem.

**Theorem 2.1** In the above notation, $\tau_{\omega,h}$ is a $\sigma$-twisted cyclic 2-cycle on the $*$-algebra $\mathcal{X}_c$ and we have

$$
\tau_{\omega,h}(x_0, x_1, x_2) = \text{Tr} \gamma_q \rho(y^{-1}) \rho(x_0)[\mathcal{F}, \rho(x_1)][\mathcal{F}, \rho(x_2)], \quad x_0, x_1, x_2 \in \mathcal{X}_c.
$$
3 The 3D-Calculus on the Quantum Group $SU_q(2)$

In this section, we use some facts on covariant differential calculi on Hopf algebras which can be found e.g. in [KS, Chapter 14]. Suppose that $(\Gamma, \Delta)$ is a finite dimensional left-covariant first order differential calculus on a Hopf algebra $A$ and $\Gamma^\wedge = \oplus_j \Gamma^{\wedge^j}$ is a differential calculus on $A$ such that $\Gamma^{\wedge 1} = \Gamma$. As in Section 1, we assume that there are a natural number $n$ and a $n$-form $\omega \in \Gamma^{\wedge n}$ such that for each $\xi \in \Gamma^{\wedge n}$ there exists a unique element $\pi(\xi) \in A$ such that $\xi = \pi(\xi)\omega$.

Our first aim is to express $\pi(x_0dx_1 \wedge \cdots \wedge dx_n)$ in terms of commutators in the cross product algebra $A \ltimes A^\circ$.

Let $\{\omega_1, \ldots, \omega_m\}$ be a basis of the vector space $\Gamma_{inv}$ of left-invariant elements of $\Gamma$. Then there exist functionals $X_k, f^i_k, k, l = 1, \ldots, m$, of the Hopf dual $A^\circ$ of $A$ such that

$$\text{d}x = \sum_k (X_k \circ x)\omega_k, \quad \omega_k x = \sum_j (f^k_j \circ x)\omega_j$$

$$\Delta(X_k) = \varepsilon \otimes X_k + \sum_j X_j \otimes f^i_j, \quad \Delta(f^k_i) = \sum_j f^k_j \otimes f^i_j,$$  \hspace{1cm} (13) \hspace{1cm} (14)

where $f^i_j \circ x := \langle x, f_j^i \rangle x^i$ is the left action of $f \in A^\circ$ on $x \in A$. Recall that the cross product algebra $A \ltimes A^\circ$ is the algebra generated by the two subalgebras $A$ and $A^\circ$ with respect to the cross relation

$$fx = (f^1 \circ x)f^2 \equiv \langle f^1, x \rangle f^2, \quad x \in A, \ f \in A^\circ. \hspace{1cm} (15)$$

We shall use the notation $(\xi, l) := (k_1, \ldots, k_r, l), \omega_\xi := \omega_{k_1}\omega_{k_2} \cdots \omega_{k_r}, f_{\xi}^i := f_{j_1}^{k_1} \cdots f_{j_r}^{k_r}$ for multi-indices $\xi = (k_1, \ldots, k_r)$ and $j = (j_1, \ldots, j_r)$, where $k_i, j_i, l \in \{1, \ldots, m\}$. Let $x_0, x_1, \ldots, x_n \in A$. From (13), we deduce

$$x_0dx_1 \wedge \cdots \wedge dx_n$$

$$= \sum_{\xi, l_i} x_0(X_{\xi} \circ x_1)(f^{(\xi)}_{l_1}X_{l_1} \circ x_2)(f^{(\xi)}_{l_2}X_{l_2} \circ x_3) \cdots (f^{(\xi)}_{l_{n-1}}X_{l_{n-1}} \circ x_n)\omega_{(\xi, l_{n-1}, l_n)},$$

where the summation is over all multi-indices $\xi = (k_1, \ldots, k_i)$ and numbers $l_i \in \{1, \ldots, m\}$. Furthermore, Equations (14) and (15) give

$$[X_k, x] = \sum_j (X_j \circ x)f^i_j, \quad f^k_j x = \sum_l (f^k_l \circ x)f^i_j, \quad x \in A. \hspace{1cm} (17)$$
Using these relations, a straightforward induction argument shows that

\[ [X_{j_1}, x_1][X_{j_2}, x_2] \cdots [X_{j_n}, x_n] = \sum_{\{i_1, \ldots, i_n\}} (X_{l_1} x_1)(f_{l_1}^{j_1} X_{l_2} x_2) \cdots (f_{l_{n-2}}^{j_{n-2}} X_{l_{n-1}} x_{n-1}) (f_{l_{n-1}}^{j_{n-1}} X_{l_n} x_n) \]

for \( j_1, \ldots, j_n \in \{1, \ldots, m\} \). Since \( \sum_i f_i^{j} S(f_i^{j}) = \varepsilon(f_i^{j}) = \delta_i^{j} \), we get

\[ \sum_{i=(j_1, \ldots, j_n)} [X_{j_1}, x_1] \cdots [X_{j_n}, x_n] S(f_{l_1}^{j_1}) \cdots S(f_{l_n}^{j_n}) = \sum_{i=(j_1, \ldots, j_n)} (X_{l_1} x_1)(f_{l_1}^{j_1} X_{l_2} x_2) \cdots (f_{l_{n-2}}^{j_{n-2}} X_{l_{n-1}} x_{n-1}) (f_{l_{n-1}}^{j_{n-1}} X_{l_n} x_n). \] Combining (16) and (18) proves the assertion of the following lemma.

**Lemma 3.1** For \( x_0, x_1, \ldots, x_n \in A \), we have

\[ \pi(x_0 dx_1 \wedge \cdots \wedge dx_n) = \sum_{j_1, \ldots, j_n} x_0 [X_{j_1}, x_1] \cdots [X_{j_n}, x_n] S(f_{l_1}^{j_1}) \cdots S(f_{l_n}^{j_n}) \pi(\omega_{k_1} \cdots \omega_{k_n}). \]

As in Section 2 we want to express the twisted cyclic cocycle \( \tau_{\omega,h} \) in terms of commutators \([F, \rho(x)]\), where \( F \) is a self-adjoint operator and \( \rho \) is a representation of \( A \) on a Hilbert space. It seems reasonable to look for an operator \( F \) of the form \( F = \sum_j X_j \otimes \eta_j \), where \( \eta_1, \ldots, \eta_m \) are appropriate matrices. We shall carry out this for Woronowicz’ 3D-calculus [W] on the quantum group \( SU_q(2) \). Let \( A \) be the Hopf \( \ast \)-algebra \( O(SU_q(2)) \) with usual generators \( a, b, c, d \) and let \( U_q(su_2) \) be the Hopf \( \ast \)-algebra with generators \( E, F, K, K^{-1} \), relations

\[ KK^{-1} = K^{-1}K = 1, \quad KE = qEK, \quad FK = qKF, \]

involution \( E^* = F, K^* = K \), comultiplication

\[ \Delta(E) = E \otimes K + K^{-1} \otimes E, \quad \Delta(F) = F \otimes K + K^{-1} \otimes F, \quad \Delta(K) = K \otimes K, \]

counit \( \varepsilon(E) = \varepsilon(F) = \varepsilon(K - 1) = 0 \) and antipode \( S(K) = K^{-1}, S(E) = -qE, S(F) = -q^{-1}F \). There is a non-degenerate dual pairing of these Hopf \( \ast \)-algebras given on generators by \( \langle K^{\pm 1}, d \rangle = \langle K^{\pm 1}, a \rangle = q^{\pm 1/2}, \langle E, c \rangle = \langle F, b \rangle = 1 \) and zero otherwise. For the Haar state \( h \) on \( O(SU_q(2)) \), we have

\[ h(x_1 x_2) = h(\sigma_2(x_2) x_1) \quad \text{with} \quad \sigma_2(x) = K^{-2}c(x) K^{-2}. \]
We will need the following facts on the 3D-calculus $(\Gamma, d)$ (see [W] and [KS Subsections 14.1.3 and 14.3.3]). The quantum tangent space of $(\Gamma, d)$ is spanned by the functionals

\[ X_0 = q^{-1/2}FK, \quad X_1 = (1 - q^{-2})^{-1}(1 - K^4), \quad X_2 = q^{1/2}EK. \]

The functionals \( f^i_j \) are \( f^0_0 = f^2_2 = K^2, f^1_1 = K^4 \) and zero otherwise. The basis elements \( \omega_0, \omega_1, \omega_2 \) of \( \Gamma_{\text{inv}} \) satisfy the relations

\[ \omega_0^2 = \omega_1^2 = \omega_2^2 = 0, \quad \omega_1 \omega_0 = -q^4 \omega_0 \omega_1, \quad \omega_2 \omega_0 = -q^2 \omega_0 \omega_2, \quad \omega_2 \omega_1 = -q^4 \omega_1 \omega_2. \]

The corresponding higher order calculus satisfies the assumptions stated at the beginning of this section with \( n = 3 \) and the volume form \( \omega := \omega_0 \omega_1 \omega_2 \). Further, we have

\[ \omega a = q^{-4} a \omega, \quad \omega b = q^4 b \omega, \quad \omega c = q^{-4} \omega, \quad \omega d = q^4 d \omega \]

so that \( \omega x = \sigma_1(x) \omega \) with \( \sigma_1(x) = K^{q^2 x} \). Hence the automorphism \( \sigma = \sigma_2 \circ \sigma_1 \) is \( \sigma(x) = K^{6q^2 x} K^{-q^2 x}, x \in \mathcal{O}(SU_q(2)) \). From the commutation relations of the basis elements \( \omega_i \), we derive

\[ \pi(\omega_1 \omega_0 \omega_2) = \pi(\omega_0 \omega_2 \omega_1) = -q^4, \quad \pi(\omega_2 \omega_1 \omega_0) = -q^{10}, \quad \pi(\omega_1 \omega_1 \omega_0) = \pi(\omega_2 \omega_0 \omega_1) = q^6. \]

All other elements \( \pi(\omega_1 \omega_j \omega_k) \) are zero. Inserting these facts in (16), we obtain the following explicit expression for the \( \sigma \)-twisted cyclic cocycle \( \tau_{\omega, h} \):

\[
\tau_{\omega, h}(x_0, x_1, x_2, x_3) = h(\pi(x_0 dx_1 \wedge dx_2 \wedge dx_3)) \\
= h(x_0(x_0 x_1)(K^2 X_{1p} x_2)(K^6 X_{2p} x_3) - q^4(K^2 X_{2p} x_2)(K^4 X_{1p} x_3) \\
+ x_0(x_1 x_1)(K^4 X_{2p} x_2)(K^6 X_{0p} x_3) - q^4(K^4 X_{0p} x_2)(K^6 X_{2p} x_3) \\
+ x_0(x_2 x_1)(K^6 X_{0p} x_2)(K^4 X_{1p} x_3) - q^{10}(K^2 X_{1p} x_2)(K^6 X_{0p} x_3))).
\]

Now we develop a commutator representation \([\mathcal{F}, \rho(x)]\) of the 3D-calculus. We slightly modify the construction from [SII Section 3]. Let \( \mathcal{H} \) be the Hilbert space completion of \( \mathcal{O}(SU_q(2)) \) with respect to the inner product \( (x, y) := h(y^* x), x, y \in \mathcal{O}(SU_q(2)) \). There is a \(*\)-representation of the left cross product \(*\)-algebra \( \mathcal{O}(SU_q(2)) \times \mathcal{U}_q(su_2) \) on the domain \( \mathcal{O}(SU_q(2)) \) such that \( x \in \mathcal{O}(SU_q(2)) \) acts by left multiplication and \( f \in \mathcal{U}_q(su_2) \) acts by the left action \( \circ \). Let \( \mathcal{C} \) be the closure
of the Casimir operator $C = FE + (q - q^{-1})^{-2}(qK^2 + q^{-1}K^{-2} - 2)$. Let $\zeta(z)$ denote the holomorphic function

$$\zeta(z) = \sum_{n=1}^{\infty} n[n/2]_q^{-2z}[n]_q, \quad z \in \mathbb{C}, \quad \text{Re} \ z > 1.$$  

**Lemma 3.2** If $z \in \mathbb{C}$, $\text{Re} \ z > 1$, and $x \in \mathcal{O}(SU_q(2))$, then the closure of $C^{-z}K^2x$ and $C^{-z}K^{-6}xK^8$ are of trace class and

$$h(x) = (z)^{-1}\text{Tr} C^{-z}K^2x = \zeta(z)^{-1}\text{Tr} C^{-z}K^{-6}xK^8.$$

**Proof.** The first equality is essentially [SW, Theorem 5.7]. Since we use the left crossed product algebra instead of the right crossed product algebra in [SW], the operator $K^{-2}$ in [SW, Theorem 5.7] has to be replaced by $K^2$. The proof of [SW, Theorem 5.7] assures that $C^{-z}K^2x$ is of trace class for all $x \in \mathcal{O}(SU_q(2))$. Given an $x \in \mathcal{O}(SU_q(2))$, there is a $y \in \mathcal{O}(SU_q(2))$ such that $xK^8 = K^8y$. Hence $C^{-z}K^{-6}xK^8 = C^{-z}K^2y$ is of trace class. The second equality follows by a slight modification of the proof of [SW, Theorem 5.7].

Let $\gamma_q, \eta_0, \eta_1, \eta_2$ be complex square matrices of the same dimension such that

$$\text{Tr} \ \gamma_q \eta_0 \eta_j \eta_k = \pi(\omega_i \omega_j \omega_k), \quad i, j, k \in \{0, 1, 2\}, \quad \text{(19)}$$

and $\eta_0^* = \eta_2, \eta_1^* = \eta_1$. For instance, take

$$\gamma_q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & q^6 & 0 \\ 0 & 0 & 0 & q^6 \end{pmatrix}, \quad \eta_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \eta_1 = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & \alpha_3 & 0 \\ 0 & 0 & 0 & \alpha_4 \end{pmatrix},$$

and $\eta_2 = \eta_0^*$, where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are real numbers such that

$$\alpha_3 + \alpha_4 = 1, \quad \alpha_1 + \alpha_2 = -q^4, \quad q^6(\alpha_1^2 + \alpha_2^2) + \alpha_3^2 + \alpha_4^2 = 0.$$

Let $\rho$ denote the *-representation of $\mathcal{O}(SU_q(2))$ on $\mathcal{H} \otimes \mathbb{C}^4$ given by $\rho(x) := x \otimes I$. For simplicity, we write $C^{-z}$ for $C^{-z} \otimes I$, $K$ for $K \otimes I$ and $\gamma_q$ for $I \otimes \gamma_q$. Since $X_0^* = X_2, X_1^* = X_1$ and $\eta_0^* = \eta_2, \eta_1^* = \eta_1$, the operator $F := \sum_{j=0}^{2} X_j \otimes \eta_j$ is
self-adjoint on the Hilbert space $\mathcal{H} \otimes \mathbb{C}^4$. Set $\Omega_k := \sum_{j=0}^{2} f_{j}^{k} \otimes \eta_j$. From (17), we obtain for $x \in \mathcal{O}(SU_q(2))$

$$\Omega_k \rho(x) = \sum_{j} f_{j}^{k} x \otimes \eta_j = \sum_{j,l} (f_{l}^{k} x) f_{j}^{l} \otimes \eta_j = \sum_{l} (f_{l}^{k} x) \Omega_l,$$

$$[\mathcal{F}, \rho(x)] = \sum_{k} [X_k, x] \otimes \eta_k = \sum_{k,j} (X_j x) f_{k}^{j} \otimes \eta_k = \sum_{j} (X_j x) \Omega_j.$$

Hence, by (13), there is an injective linear mapping $\mathcal{I} : \Gamma \rightarrow \mathcal{L}(\mathcal{O}(SU_q(2)) \otimes \mathbb{C}^4)$ such that $\mathcal{I}(x y) = \rho(x) i[\mathcal{F}, \rho(y)]$ for all $x, y \in \mathcal{O}(SU_q(2))$. This means that the pair $(\mathcal{F}, \rho)$ gives a faithful commutator representation $[S1]$ of the 3D-calculus. Further, we compute

$$\text{Tr}_{\mathcal{H} \otimes \mathbb{C}^4} C^{-z} K^{-6} \gamma_q \rho(x_0) [\mathcal{F}, \rho(x_1)] [\mathcal{F}, \rho(x_2)] [\mathcal{F}, \rho(x_3)]$$

$$= \text{Tr}_{\mathcal{H} \otimes \mathbb{C}^4} C^{-z} K^{-6} \left( \sum_{i,j,k} x_0 [X_i, x_1] [X_j, x_2] [X_k, x_3] \otimes \gamma_q \eta_i \eta_j \eta_k \right)$$

$$= \text{Tr}_{\mathcal{H}} C^{-z} K^{-6} \left( \sum_{i,j,k} x_0 [X_i, x_1] [X_j, x_2] [X_k, x_3] \pi(\omega_i \omega_j \omega_k) \right)$$

$$= \text{Tr}_{\mathcal{H}} C^{-z} K^{-6} \left( \sum_{i,j,k,l,r,s} x_0 [X_i, x_1] [X_j, x_2] [X_k, x_3] S(f_{(i,j,k)}^{(l,r,s)}) \pi(\omega_i \omega_j \omega_k) \right) K^8$$

$$= \text{Tr}_{\mathcal{H}} C^{-z} K^{-6} \pi(x_0 dx_1 \wedge dx_2 \wedge dx_3) K^8$$

$$= \zeta(z) h(\pi(x_0 dx_1 \wedge dx_2 \wedge dx_3)) = \zeta(z) \tau_{\omega,h}(x_0, x_1, x_2, x_3).$$

Here the first equality is the definition of $\mathcal{F}$ and the second equality follows from (19). For the third equality we used the fact that $f_{(i,j,k)}^{(l,r,s)} = K^8$ for all indices $i, j, k, l, r, s \in \{0, 1, 2\}$ for which $\pi(\omega_i \omega_j \omega_k) \neq 0$. (This is certainly not true for other calculi.) The fourth equality is the assertion of Lemma 3.1 while the fifth follows from Lemma 3.2. Remind that $\pi(x_0 dx_1 \wedge dx_2 \wedge dx_3)$ belongs to $\mathcal{O}(SU_q(2))$. The last equality is the definition of the cocycle $\tau_{\omega,h}$.

Summarizing the preceding, we have proved the following

**Theorem 3.3** Retaining the foregoing notation, the $\sigma$-twisted cyclic cocycle $\tau_{\omega,h}$ associated with the volume form $\omega$ of the 3D-calculus on the Hopf $*$-algebra $\mathcal{O}(SU_q(2))$ is given by

$$\tau_{\omega,h}(x_0, x_1, x_2, x_3)$$

$$= \zeta(z)^{-1} \text{Tr}_{\mathcal{H} \otimes \mathbb{C}^4} C^{-z} K^{-6} \gamma_q \rho(x_0) [\mathcal{F}, \rho(x_1)] [\mathcal{F}, \rho(x_2)] [\mathcal{F}, \rho(x_3)],$$

where $x_0, x_1, x_2, x_3 \in \mathcal{O}(SU_q(2))$ and $z \in \mathbb{C}$, $\text{Re} \ z > 1$.  

11
References

[C] Connes, A., *Noncommutative Geometry*, Academic Press, San Diego, 1994.

[H] Heckenberger, I., *Classification of left-covariant differential calculi on the quantum group SL_q(2)*, J. Algebra **237** (2001), 203–237.

[KL] Klimek, S. and A. Lesniewski, *A two-parameter deformation of the unit disc*, J. Funct. Anal. **115** (1993), 1–23.

[KR] Khalkhali, M. and B. Rangipour, *Invariant cyclic homology*, [math.KT/0207118](http://arxiv.org/abs/math.KT/0207118).

[KS] Klimyk, K. A. and K. Schmüdgen, *Quantum Groups and Their Representations*, Springer, Berlin, 1997.

[KW] Kürsten, K.-D. and E. Wagner, *Invariant integration theory on non-compact quantum spaces: Quantum (n, 1)-matrix ball*, in preparation.

[KMT] Kustermans, J., G. J. Murphy and L. Tuset, *Differential calculi over quantum groups and twisted cyclic cocycles*, J. Geom. Phys. 44 (2003), 570–594.

[S1] Schmüdgen, K., *Commutator representations of covariant differential calculi on quantum groups*, Lett. Math. Phys. **59** (2002), 95–106.

[S2] Schmüdgen, K., *Covariant differential calculi on quantum spaces*, NTZ-preprint 24, Leipzig, 1991.

[Sch] Schüler, A., *Differential Hopf algebras on quantum groups of type A*, J. Algebra **214** (1999), 479–518.

[SW] Schmüdgen, K. and E. Wagner, *Hilbert space representations of cross product algebras*, J. Funct. Anal. **200** (2003), 451–493.

[SSV] Shklyarov, D., S. Sinel’shchikov and L. L. Vaksman, *On function theory in quantum disc: Integral representations*, [math.QA/9908015](http://arxiv.org/abs/math.QA/9908015).

[W] Woronowicz, S. L., *Twisted SU(2) group. An example of a noncommutative differential calculus*, Publ. RIMS Kyoto Univ. **23** (1987), 117–181.