Saturation of Gluon Density at Small $x$\textsuperscript{a}

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Computing parton distribution functions in QCD is a formidable task. It is intrinsically non-perturbative and thus hard to calculate from first principles. On the other hand, knowledge of these distribution functions is required in any dynamical process involving hadrons. This note describes a new approach to the problem. It is shown how this new approach leads to saturation of the gluon density at small $x$.

1 Introduction

In perturbation theory, one considers evolution of soft gluon density in $x, Q^2$ or both. In the double log approximation of DGLAP formalism, terms of the form $\alpha \ln(x) \ln(Q^2)$ are summed. This is done by considering the so-called ladder diagrams where both longitudinal and transverse momenta are ordered.

For very small values of $x$ it may be more appropriate to sum terms of the form $\alpha \ln(x)$. This is done in BFKL formalism where one has ordering in longitudinal momenta while assuming that everything is happening at more or less the same transverse area. It is important to realize that both DGLAP and BFKL are evolution equations and as such require initial parton distributions as input.

Regardless of whether BFKL or DGLAP is more appropriate for description of soft gluons, they both have one outstanding feature in common: they both predict a sharp rise of soft gluon density at very small values of $x$ and thus would eventually lead to violation of unitary bound on physical cross sections.

However, the above picture of harder partons splitting into softer ones with no further interaction among them is perhaps a bit too naive. Physically we know that at very small values of $x$, the parton density will be high and they will spatially overlap. The effects of parton recombination, screening etc., will therefore be important. This will eventually lead to saturation of soft gluon density at small $x$.

It may be more natural to consider soft gluons not as quasi-free partons but rather as classical fields with large amplitudes. After summing the effects of high density into the classical fields one can do perturbation theory in the background of these large classical fields.

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2 McLerran-Venugopalan Model

In the McLerran-Venugopalan model of small $x$ gluons, one considers a large nucleus traveling with very high velocities and thus highly Lorentz contracted. It therefore has a high valence quark surface density. This approach can also be used for hadrons at sufficiently small values of $x$ where the parton density is large. The color charge density $\rho$ is the only dimensionfull parameter in the problem. Therefore the strong coupling constant $\alpha_s$, as a function of $\rho$, will be small for large charge densities so that one can use weak coupling methods. Since the coupling is weak, the valence quarks do not lose an appreciable fraction of their momenta and stay on straight line trajectories and are therefore static sources of color charge. This is the no-recoil approximation. Also, due to the high density of glue, one can treat them as classical charges. This corresponds to taking a higher dimensional representation of the color algebra. Treating the valence charges as classical leads to computing expectation values with a Gaussian distribution of the form

$$\int [d\rho] \exp\left\{ \frac{-1}{2\mu^2} \int d^2 x_i \rho^2(x_i) \right\}$$

where $\mu^2$ is the average color charge squared per unit area.

Alternatively, one can solve the Yang-Mills equations in the presence of this static color charge. Once a solution is found, one can then compute the distribution function with the above Gaussian weight. It was shown in [1] that this leads to a distribution function of the Weizsäcker-Williams form for soft gluons. Quantum corrections to the classical results were also computed and similar to standard perturbation theory, it was found that there are potentially large logs of ratio of longitudinal momenta $x$. This led to a longitudinal structure for the color charge density $\rho$. Then the problem to consider was to solve the classical equation of motion in the presence of this rapidity dependant charge density.

Working in light-cone gauge $A^+=0$ and using light cone notation $x^\pm \sim (t \pm z)$ and $y \sim \ln(x^-)$, we have

$$D_i \frac{d}{dy} A_i = g^2 \rho(y, x_i).$$

One can write a formal solution to this equation in terms of $\rho$ and then compute the correlation function $G_{ii}^{\text{soft}} \sim \langle A_i^a A_i^a \rangle > \rho$ where now both $\rho$ and $\mu^2$ depend on $y$. The result is

$$G_{ii}^{\text{soft}} = \frac{4(N_c^2 - 1)}{N_c x_t} \left[ 1 - (x_t^2 \Lambda_{QCD}^2)^{\frac{4}{3N c}} \chi(y, Q^2) x_t^4 \right]$$

where $\chi(y, Q^2)$ is the Sudakov form factor.
where
\[ \chi(y, Q^2) = \int_y^{\infty} dy' \mu^2(y', Q^2) \] (4)
is the average color charge squared per unit area at rapidity \( y \) and resolution scale \( Q^2 \).

3 RG Equation for \( \chi \)

Next we need to determine \( \chi(y, Q^2) \). We start with the valence charge density and then include hard gluons into the valence charge step by step. In other words, we integrate out higher \( x \) gluons perturbatively. This generates an effective Lagrangian and renormalizes the charge density. Iterating this procedure one can derive a RG equation for \( \chi \). Diagramatically this can be represented as

\[ \frac{d \chi}{dy \, dQ^2} = \ldots \]

In the first diagram, the propagator of the hard field is in the background of classical fields. This propagator was computed in 3. In the second diagram the hard field propagates in the background of soft modes. This is where Sudakov and non-Sudakov form factors are expected to show up. There are also diagrams where the hard and soft modes are mixed.

As a first approximation, we will use the following ladder diagram on the right hand side of our RG equation where the vertices are eikonalized. It is known that \( \ln(x) \) terms come from eikonal vertices.

\[ \text{Figure 2: eikonalized ladder diagram} \]

Solid and dashed lines are hard and soft gluons respectively. With this
approximation, the RG equation becomes

\[ \frac{d^2 \chi}{dydQ^2} \sim G^{\alpha\alpha}_{ii}(y,Q^2). \]  

(5)

It should be emphasized that the right hand side is still a non-linear functional of \( \chi(y,Q^2) \).

4 Solution to RG Equation

In the high \( Q^2 \) region \( (Q^2 \gg \alpha^2 \chi) \), it can be shown that our RG equation reduces to the standard DGLAP evolution equation. Assuming \( \alpha \) to be independent of \( Q^2 \), the approximate solution is

\[ \chi \sim \exp \left( 2\sqrt{N_c \alpha_s} \frac{\pi y}{\ln Q^2} \right). \]  

(6)

In the intermediate \( Q^2 \) region \( (Q^2 \sim \alpha^2 \chi) \), the effects of the background fields as well as the soft fields are expected to be important quantitatively. However, here we are interested in the qualitative behavior of our RGE. Assuming that, as in BFKL, the transverse phase space is a constant, we get the following BFKL like behavior

\[ \chi \sim \exp \left( \# \frac{N_c \alpha_s}{\pi} y \right). \]  

(7)

Finally in the low \( Q^2 \) region \( (Q^2 \ll \alpha^2 \chi) \), our RG equation saturates and the solution is of the form

\[ \chi = \chi_0 + \kappa(y_0 - y)Q^2 \]  

(8)

where \( \kappa \) is a slowly varying function of \( Q^2 \). This leads to saturation of the gluon distribution function at small \( x \). As a result, cross sections computed with this distribution function would satisfy unitarity bounds. To illustrate this saturation, let us consider the behavior of \( \chi(y,Q^2) \) at some fixed \( Q^2 \) as \( x \) decreases. In the high transverse momentum region \( (k_t^2 \gg \alpha^2 \chi) \), the distribution function will grow as some power of \( x \). As \( x \) gets smaller, \( \alpha^2 \chi \) will eventually become larger than \( Q^2 \) and we will be in the saturation region (the region between \( \Lambda_{QCD}^2 \) and \( \alpha^2 \chi \)) where the distribution function is a slowly varying function of \( k_t^2 \). This is shown below.
Notice that this saturation is due to shrinking of the transverse phase space as indicated above.

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