Emergence of classical structures from the quantum vacuum

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After a quantum phase transition the quantum vacuum can break up to form classical topological defects. We examine this process for scalar field models with $Z_2$ symmetry for different quench rates for the phase transition. We find that the number density of kinks at late times universally scales as $C m^{1/2} t^{-1/2}$ where $m$ is a mass scale in the model and $C \approx 0.22$; it does not depend on the quench timescale in contrast to the Kibble-Zurek scaling for thermal phase transitions. A subleading correction $\propto t^{-3/2}$ to the kink density depends on the details of the phase transition.

Quantum field theories generally contain small quantum excitations around a true vacuum that we call particles and large classical structures called solitons that interpolate between different degenerate vacua. Often the solitons have a topological character and are then also known as topological defects of which kinks, domain walls, strings, and magnetic monopoles are all examples. As first proposed by Kibble [1], these structures can be formed during a phase transition. A more quantitative estimate of their number density formed after a thermal phase transition is given by the Kibble-Zurek proposal [2-7] that has been tested in various systems such as liquid crystals [8, 9], superfluids [10-14], superconductors [15-17] and other systems involving liquid crystal light valves [18] and ultra-cold quantum gases [19], with conflicting conclusions. The Kibble-Zurek proposal is based on imposing a physically motivated cutoff on the growing correlation length prior to the thermal phase transition and then matching the pre-phase transition correlation length to the post-phase transition correlation length. In this paper we will be concerned with a quantum phase transition and we will solve for the full quantum dynamics relevant to defect formation.

To describe our approach we start with the $\lambda \phi^4$ model for a real, scalar field $\phi$,

$$L = \frac{1}{2} (\partial_{\mu} \phi)^2 - \frac{1}{2} m_2(t) \phi^2 - \frac{\lambda}{4} \phi^4.$$  

(1)

To study the production of kinks in this model we imagine that the mass parameter $m_2(t)$ has an externally controlled time dependence,

$$m_2(t) = -m^2 \tanh \left( \frac{t}{\tau} \right),$$  

(2)

where $\tau$ is the “quench time scale”. For $t < 0$, the model has a unique vacuum at $\phi = 0$, while for $t > 0$, there are two degenerate vacua $\phi = \pm m / \sqrt{\lambda}$. In the $t \to \infty$ limit where $m_2 = -m^2$, the model has a double well potential and it admits static classical kink and anti-kink solutions

$$\phi_{\pm}(x) = \pm \frac{m}{\sqrt{\lambda}} \tanh \left( \frac{mx}{\sqrt{2}} \right).$$  

(3)

These non-perturbative solutions interpolate between the two degenerate vacua of the model over a spatial width $\sim 1/m$. They are topological defects characterized by a topological charge, positive for a kink and negative for an anti-kink. In fact the topological charge classifies kinks and anti-kinks according to the nature of the sign change occurring in the field profile: negative to positive for a kink, and vice-versa for an anti-kink.

Since all we are interested in is changes in the sign of the field, we can simplify our model to eliminate the $\lambda \phi^4$ term in (1). Then the free field model

$$L = \frac{1}{2} (\partial_{\mu} \phi)^2 - \frac{1}{2} m_2(t) \phi^2$$  

(4)

also has $Z_2$ symmetry that is spontaneously broken after the quench and thus has topological kinks. Without the $\lambda \phi^4$ term, the kink width is not stabilized and becomes smaller with time. However this does not affect the kink number density which is what interests us. Hence the model in (4) captures the essential physics of defect formation.

The problem now is that of a quantum field interacting with a classical background. As discussed in [20-22], the solution to the quantum problem can be written in terms of the solution of a classical problem but in higher dimensions. More specifically, let us discretize a compactified space of size $L$ by $N$ lattice points $i = 1, \ldots, N$, with lattice spacing $a = L/N$. The resulting lattice-point-dependent Heisenberg picture field operators $\hat{\phi}_i$ describe a set of $N$, quadratically coupled, simple harmonic oscillators and we can write

$$\hat{\phi}_i = Z_{ij} \hat{a}_j + Z_{ij} \hat{a}_j^\dagger,$$  

(5)

where $\hat{a}_j$ and $\hat{a}_j^\dagger$ are the annihilation and creation operators associated with the quantum variable $\hat{\phi}_i$ at $t = -\infty$ when the potential is upright and time-independent. The $N \times N$ complex matrix $Z$ satisfies the classical equation,

$$\ddot{Z} + \Omega_2(t) Z = 0$$  

(6)
In our particular application, since $m_Z$ diagonalize the problem is translationally invariant. We can thus their (after the quench when $m_Z > 0$ and becomes an inverted quadratic after the quench when $m_Z < 0$. The model has a $Z_2$ symmetry that is spontaneously broken and hence has kinks.

where the matrix $\Omega_2$ is given by

$$[\Omega_2]_{ij} = \begin{cases} +2/a^2 + m_Z(t), & i = j \\ -1/a^2, & i = j \pm 1 \text{ (mod } N) \\ 0, & \text{otherwise}. \end{cases}$$

In our particular application, since $m_Z(t)$ does not depend on the lattice point, all matrices are circulant (i.e. their $(i, j)$ element only depends on $i - j$ (mod $N$)) and the problem is translationally invariant. We can thus diagonalize $Z$ to work in momentum space. This leads to

$$Z_{jl} = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} c_k(t) e^{-i(j-l)2\pi k/N},$$

where the mode coefficients $c_k$ satisfy

$$\dot{c}_k + \left[ \frac{4}{a^2} \sin^2 \left( \frac{\pi k}{N} \right) + m_Z(t) \right] c_k = 0.$$  

The quantum state (in Heisenberg picture) will be chosen to be the vacuum state long before the phase transition i.e. at $t = -\infty$ when $m_Z(t) = +m^2$. In practice this is achieved by choosing an initial time $t_0 \ll -\tau$ and setting up the following initial conditions for the mode coefficients:

$$c_k(t_0) = \frac{-i}{\sqrt{2aN}} \left[ \frac{4}{a^2} \sin^2 \left( \frac{\pi k}{N} \right) + m_Z(t_0) \right]^{-1/4},$$

$$\dot{c}_k(t_0) = \frac{1}{\sqrt{2aN}} \left[ \frac{4}{a^2} \sin^2 \left( \frac{\pi k}{N} \right) + m_Z(t_0) \right]^{1/4}.$$  

While it may be easier to solve for the quantum dynamics in momentum space by computing the $c_k$ mode coefficients, the kinks are defined as zeros of the field in physical space. It is therefore useful to determine the physical space wavefunctional $\Psi[\{\phi_i\}, t]$ for the model in Eq. (4) by solving the corresponding Schrödinger equation:

$$i \frac{\partial \Psi}{\partial t} = -\frac{1}{2a} \sum_{i=1}^{N} \frac{\partial^2 \Psi}{\partial \phi_i^2} + \frac{a}{2} \phi^T \Omega_2(t) \phi \Psi.$$  

We find

$$\Psi[\{\phi_i\}, t] = \frac{1}{(2\pi)^{N/2}} \times \exp \left[ -\frac{1}{2} \int_{0}^{t} dt' \text{Tr} M(t') + \frac{i a}{2} \phi^T M(t) \phi \right],$$  

where $M \equiv \tilde{Z} Z^{-1}$ and we have introduced the column vector $\phi$ such that $\phi^T \equiv (\phi_1, \ldots, \phi_N)$. Using the constraint $Z^T \tilde{Z} - \tilde{Z}^T Z = i/a \[20\], this gives a probability distribution for the field,

$$P[\{\phi_i\}, t] = \frac{1}{\sqrt{\text{det}(2\pi K)}} e^{-\phi^T K^{-1} \phi/2},$$  

where $K = ZZ^T$ is the covariance matrix of the field $\phi$. Notice that $K$ is real and symmetric \[20\] (as can be verified by using \[[6]\] and the initial conditions).

The number density of zeros ($n_Z$) is obtained by counting the number of sign changes of $\phi$. To compute an explicit formula, we first define the quantum operator

$$\hat{n}_Z \equiv \frac{1}{L} \sum_{j=1}^{N} \frac{1}{4} \left[ \text{sgn}(\hat{\phi}_j) - \text{sgn}(\hat{\phi}_{j+1}) \right]^2,$$

where, because of the periodicity of the lattice, $\hat{\phi}_{N+1} = \hat{\phi}_1$. The number density of zeros is then simply given by the quantum average of this operator. After using translational invariance (and in particular the fact that $K^{-1}$ is circulant), it reads:

$$n_Z = \langle \hat{n}_Z \rangle = \frac{N}{2L} \left[ 1 - \langle \text{sgn}(\tilde{\phi}_1 \tilde{\phi}_2) \rangle \right].$$

The expectation value in \[16\] can now be written as

$$\langle \text{sgn}(\tilde{\phi}_1 \tilde{\phi}_2) \rangle = \frac{1}{\sqrt{\text{det}(2\pi K)}} \times \sum_{\text{quads.}} \int d\phi_1 \ldots d\phi_N \text{sgn}(\phi_1 \phi_2) e^{-\phi^T K^{-1} \phi/2},$$

where the sum is over the four quadrants in the $(\phi_1, \phi_2)$ plane. The coefficient, $\text{sgn}(\phi_1 \phi_2)$ is +1 for quadrants with $\phi_1 \phi_2 > 0$ and -1 for quadrants with $\phi_1 \phi_2 < 0$. Consider the integral in the first quadrant,

$$I_1 = \int_{0}^{\infty} d\phi_1 \int_{0}^{\infty} d\phi_2 \int_{-\infty}^{\infty} d\phi_3 \ldots d\phi_N e^{-\phi^T K^{-1} \phi/2}.$$  

\[18\]
To perform the integration, we write

$$K^{-1} = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix},$$

(19)

where $A$ a $2 \times 2$ matrix, $B$ a $2 \times (N - 2)$ matrix, and $C$ an $(N - 2) \times (N - 2)$ matrix. (The matrices $A$ and $C$ are symmetric.) Next, we also use the vectors $\chi$ and $\xi$ such that $\chi^T = (\phi_1, \phi_2)$ and $\xi^T = (\phi_3, \ldots, \phi_N)$. Then,

$$\phi^T K^{-1} \phi = (\xi + C^{-1} B^T \chi)^T C (\xi + C^{-1} B^T \chi) + \chi^T (A - BC^{-1} B^T) \chi,$$

and we can perform the integration over $\xi$ first using

$$\int d^{N-2} \xi \ e^{-\langle \xi+C^{-1}B^T\chi \rangle^TC(\xi+C^{-1}B^T\chi)/2} = \frac{(2\pi)^{(N-2)/2}}{\sqrt{\det(C)}}.$$

We thus obtain

$$I_1 = \frac{(2\pi)^{(N-2)/2}}{\sqrt{\det(C)}} \int_0^\infty d\phi_1 \int_0^\infty d\phi_2 \ e^{-\chi^T A \chi/2}$$

$$= \frac{(2\pi)^{(N-2)/2}}{\sqrt{\det(C)\det(A)}} \left[ \frac{\pi}{2} - \tan^{-1} \left( \frac{A_{12}'}{\sqrt{\det(A')}} \right) \right],$$

(21)

where $A'$, the Schur complement of $C$, is defined by $A' = A - BC^{-1} B^T$.

The integral over the third quadrant in the $\phi_1 \phi_2$ plane, $I_3$, is seen to be equal to $I_1$ after the change of variables $\chi \rightarrow -\chi$. The integrals over the second and fourth quadrants, $I_2$ and $I_4$, are similarly equal, and are related to $I_1$. To see this we note that the integral over the second quadrant reduces to the one over the first quadrant by the transformation $\phi_1 \rightarrow -\phi_1$. This transformation is alternatively implemented by changing $A'_{12}$ to $-A'_{12}$ and not changing anything else. Hence the integral over the second quadrant is simply

$$I_2 = \frac{(2\pi)^{(N-2)/2}}{\sqrt{\det(C)\det(A')}} \left[ \frac{\pi}{2} + \tan^{-1} \left( \frac{A'_{12}}{\sqrt{\det(A')}} \right) \right].$$

(22)

Putting together the contributions of all the four quadrants, we get

$$\langle \text{sgn} (\phi_1 \phi_2) \rangle = -\frac{2}{\pi} \tan^{-1} \left( \frac{A'_{12}}{\sqrt{\det(A')}} \right),$$

(23)

where we made use of the fact that $A'$ is the Schur complement of $C$ which implies

$$\det(A')\det(C) = \det(K^{-1}) = \frac{1}{\det(K)}.$$

This in turn gives us the number density of zeros as

$$n_Z = \frac{N}{2L} \left[ 1 + \frac{2}{\pi} \tan^{-1} \left( \frac{A'_{12}}{\sqrt{\det(A')}} \right) \right].$$

(24)

We can make this formula more explicit by noticing that $(A')^{-1}$ is equal to the $2 \times 2$ upper left block of the matrix $K$. This can be seen via a block LDU decomposition of $K^{-1}$. Using [8] we can then write

$$(A')^{-1} = K_{2\times2} = \alpha \mathbf{1} + \beta \sigma_x,$$

(25)

where $\sigma_x$ is the first Pauli spin matrix and

$$\alpha = \sum_{k=1}^N |c_k|^2, \quad \beta = \sum_{k=1}^N |c_k|^2 \cos(2\pi k/N).$$

(26)

Therefore

$$A' = \frac{1}{\alpha^2 - \beta^2} (\alpha \mathbf{1} - \beta \sigma_x),$$

(27)

and

$$\det(A') = \frac{1}{\alpha^2 - \beta^2}, \quad A'_{12} = \frac{\beta}{\alpha^2 - \beta^2}.$$

(28)

We now have all the pieces needed to evaluate the number density of zeros in [24] which can be written as

$$n_Z = \frac{N}{2L} \left[ 1 - \frac{2\theta}{\pi} \right],$$

(29)

where

$$\sin \theta \equiv \frac{\beta}{\alpha} = \frac{\sum_{k=1}^N |c_k|^2 \cos(2\pi k/N)}{\sum_{k=1}^N |c_k|^2}.$$

(30)

Not all field zeros, however, correspond to kinks. Some of the zeros simply correspond to field oscillations that come in and out of existence. They are spurious or virtual kinks and we eliminate them from our counting by restricting the summations in [30] to modes that aren’t oscillating. So the number density of kinks, $n_K$, is

$$n_K = \frac{N}{2L} \left[ 1 - \frac{2}{\pi} \sin^{-1} \left( \frac{\sum_{|k| \leq k_c} |c_k|^2 \cos(2\pi k/N)}{\sum_{|k| \leq k_c} |c_k|^2} \right) \right],$$

(31)

where, as seen in [9], the time-dependent cut-off mode for $t > 0$ is defined by

$$\sin \left( \frac{\pi k_c(t)}{N} \right) = \frac{a}{2} \sqrt{|m_2(t)|},$$

(32)

and $c_{-k} \equiv c_{N-k}$ for $0 \leq k \leq N - 1$.

We can now numerically solve [9] and use [31] to obtain the number density of kinks as a function of time. The only scale in the problem is the mass so we work in units of $1/m$ by setting $m = 1$. We also choose $L = 6400$ and $N = 12800$, which are both large enough to accurately describe the continuum, infinite space limit of the

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1 This is similar to the situation in inflationary cosmology where only non-oscillating super-horizon modes lead to density perturbations.
FIG. 2. Log-log plot of $\langle n_K \rangle$ versus time for $\tau = 0.1$ (Purple), 0.5 (Red), 1.0 (Green), 5.0 (Orange), 10.0 (Blue). The black dashed line shows the exhibited power law at late times, i.e. $t^{-1/2}$.

FIG. 3. Differences between the average kink density for different values of $\tau$, $n_K(t, \tau_1) - n_K(t, \tau_2)$ vs. time, for $\tau_1 = 0.1$; $\tau_2 = 0.5$ (Blue), 1.0 (Red), 5.0 (Purple), 10.0 (Green). The black dashed line shows the exhibited power law, i.e. $t^{-3/2}$.

Our key result is Eq. (33). It tells us how the quantum vacuum breaks up into classical solitons after a quench. Further, it shows that the result at late times is universal and does not depend on the quench timescale even though there is some dependence closer to the time of the phase transition. In the Kibble-Zurek proposal for kinks produced during a thermal phase transition, the kink density immediately after the phase transition depends on the quench timescale and is proportional to $\tau^{-1/4}$ in certain systems [23]. This is to be contrasted with the $\tau^{-1/3}$ fall off in Fig. 4.

We have also cross-checked our results by numerically discretized model. Notice that, thanks to the physical cutoff we placed on the mode sums, there are no UV divergences.

In Fig. 2 we show our results for several different values of the quench parameter $\tau$. The remarkable feature of this plot is that all the curves have the same late time behavior which we can determine to be a $t^{-1/2}$ power law. In fact, we can take differences for different values of $\tau$, $\Delta n_K(t, \tau_1, \tau_2) \equiv n_K(t, \tau_1) - n_K(t, \tau_2)$ (see Fig. 3), and these follow a $t^{-3/2}$ power law. Thus at late times we can write

$$n_K(t) = C \sqrt{\frac{m}{t}} + O\left(t^{-3/2}\right), \quad (33)$$

where we get $C \approx 0.22$ from our numerical solution. Note that $C$ is independent of $\tau$. At early times (i.e. immediately after the phase transition), $n_K$ increases from 0 to a maximum value $(n_K)_{\text{max}}$ within a time $t_{\text{max}}$, before decreasing again. This is to be expected: the phase transition triggers the creation of kinks with randomly distributed positions and velocities, which later start annihilating with each other. In Figs. 4 and 5 we plot $(n_K)_{\text{max}}$ and $t_{\text{max}}$ respectively, as a function of the quench parameter $\tau$. This confirms the intuitive expectation according to which the faster the phase transition (smaller $\tau$), the more kinks are produced and the quicker they start annihilating.

FIG. 4. Log-Log plot of the maximum average kink density $(n_K)_{\text{max}}$ vs. $\tau$. For larger values of $\tau$ the power law manifested is $\sim \tau^{-0.33}$.

FIG. 5. Log-Log plot of the time at which maximum average kink density $(n_K)_{\text{max}}$ occurs vs. $\tau$. For larger values of $\tau$ the power law exhibited is $\sim \tau^{-0.34}$. 
estimating $n_K$ as the inverse of the correlation length that is extracted from the covariance matrix $K$, as well as analytically by studying the limiting case of a sudden phase transition with $\tau = 0$. We plan to present these cross-checks in a separate publication \cite{24}. Similar analyses for a sudden thermal quench have been done in \cite{25,26} where a $t^{-1/2}$ scaling of defect density was also observed but the dependence on quench timescale was not studied.

In our approach we have realized that the emergence of topological defects is sensitive only to the field dynamics near the unstable point, $\phi = 0$, on the symmetry breaking potential. The creation of a defect depends on whether the field at close-by spatial points rolls towards different vacua. The shape of the potential for large $|\phi|$ does not play a role in the direction of rolling and hence in the formation of a defect. The potential for large $\phi$ only plays a role in the characteristics of the defect, like its width and energy. Hence our analysis has zoomed in on the creation of topological structures and should apply to all models with $\mathbb{Z}_2$ symmetry breaking.

The analysis we have done in this paper can be generalized to higher dimensions to discuss global vortex formation in two spatial dimensions and global monopole formation in three spatial dimensions, since the models for these can be truncated to free fields in time dependent backgrounds. The introduction of gauge fields, however, will lead to new interactions that will be more difficult to analyze.

Finally it is worth mentioning that there is a deeper foundational question in this problem that we have studied. Our initial state is a translationally invariant quantum vacuum, while the final state involves classical kinks with definite positions and velocities. The translational symmetry is preserved when averages are taken over an ensemble of kink realizations, but each realization of the kinks breaks translational symmetry. As in Schrödinger’s cat, the classical kinks materialize and break translational symmetry only when there is a detector that detects them.

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\begin{thebibliography}{99}
  \bibitem{1} T. Kibble, J. Phys. A 9, 1387 (1976).
  \bibitem{T. Kibble, Phys. Rept. 67, 183 (1980).}
  \bibitem{T. Kibble, Nature 317, 505 (1985).
  \bibitem{W. Zurek, Acta Phys. Polon. B 24, 1301 (1993).
  \bibitem{W. Zurek, Phys. Rept. 276, 177 (1996) arXiv:cond-mat/9607135
  \bibitem{T. Vachaspati, Kinks and domain walls: An introduction to classical and quantum solitons (Cambridge University Press, 2010).
  \bibitem{T. Kibble, Physics Today 60, 47 (2007).
  \bibitem{I. Chuang, R. Durrer, N. Turok, and B. Yurke, Science 251, 1336 (1991).
  \bibitem{https://science.sciencemag.org/content/251/4999/1336.full.pdf
  \bibitem{M. J. Bowick, L. Chandar, E. A. Schiff, and A. M. Srivastava, Science 263, 943 (1994).
  \bibitem{https://science.sciencemag.org/content/263/5149/943.full.pdf
  \bibitem{P. C. Hendry, N. S. Lawson, R. A. M. Lee, P. V. E. McClintock, and C. D. H. Williams, Nature 368, 315 (1994) https://www.nature.com/articles/368315a0.pdf
  \bibitem{M. E. Dodd, P. C. Hendry, N. S. Lawson, P. V. E. McClintock, and C. D. H. Williams, Phys. Rev. Lett. 81, 3703 (1998).
  \bibitem{V. M. H. Ruutu, V. B. Eltsov, A. J. Gill, T. W. B. Kibble, M. Krusius, Y. G. Makhlin, B. Plaas, G. E. Volovik, and W. Xu, Nature 382, 334 (1996) https://www.nature.com/articles/382334a0.pdf
  \bibitem{C. Buerle, Y. M. Bunkov, S. N. Fisher, H. Godfrin, and G. R. Pickett, Nature 382, 332 (1996) https://www.nature.com/articles/382332a0.pdf
  \bibitem{V. Eltsov, M. Krusius, and G. Volovik (Elsevier, 2005) pp. 1 - 137.
  \bibitem{R. Monaco, J. Mygind, and R. J. Rivers, Phys. Rev. Lett. 89, 080603 (2002).
  \bibitem{R. Carmi, E. Polturak, and G. Koren, Phys. Rev. Lett. 84, 4966 (2000).
  \bibitem{A. Maniv, E. Polturak, and G. Koren, Phys. Rev. Lett. 91, 197001 (2003).
  \bibitem{S. Ducci, P. L. Ramazza, W. González-Vijas, and F. T. Arecchi, Phys. Rev. Lett. 83, 5210 (1999).
  \bibitem{J. Beugnon and N. Navon, Journal of Physics B: Atomic, Molecular and Optical Physics 50, 022002 (2017).
  \bibitem{T. Vachaspati and G. Zahariade, (2018). arXiv:1807.10282 [hep-th]
  \bibitem{J. Ollé, O. Pujolas, T. Vachaspati, and G. Zahariade, (2019). arXiv:1904.12962 [hep-th]
  \bibitem{M. Mukhopadhyay and T. Vachaspati, Phys. Rev. D 100, 096018 (2019) arXiv:1907.03762 [hep-th]
  \bibitem{P. Laguna and W. H. Zurek, Phys. Rev. Lett. 78, 2519 (1997) arXiv:gr-qc/9607041 [gr-qc]
  \bibitem{M. Mukhopadhyay, T. Vachaspati, and G. Zahariade, (work in progress).
  \bibitem{D. Boyanovsky, Phys. Rev. E 48, 767 (1993).
  \bibitem{D. Boyanovsky, H. de Vega, and R. Holman, NATO Sci. Ser. C 549, 139 (2000). arXiv:hep-ph/9903534
\end{thebibliography}