GENERATING SERIES FOR NETWORKS OF CHEN–FLIESS SERIES

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ABSTRACT. Consider a set of single-input, single-output nonlinear systems whose input-output maps are described only in terms of convergent Chen–Fliess series without any assumption that finite dimensional state space models are available. It is shown that any additive or multiplicative interconnection of such systems always has a Chen–Fliess series representation that can be computed explicitly in terms of iterated formal Lie derivatives.

CONTENTS

1. Introduction 1
2. Preliminaries 2
3. Formal Realizations 4
4. Formal Representations 10
5. Networks of Chen–Fliess Series 12
6. Conclusions and Future Work 16
Acknowledgments 16
References 16

1. INTRODUCTION

The study of interconnections of nonlinear control systems is normally posed in a state space setting. Issues like controllability, observability and synchronization are natural to consider in this context [2, 20]. The goal of this paper is to consider networks of nonlinear systems described only in terms of Chen–Fliess series without any assumption that finite dimensional state space models are available [4, 5]. Such models are useful in the context of system identification as relatively few parameters need to be estimated to yield an accurate approximation of the input-output map [8, 18]. On the other hand, it is not automatically evident that any interconnection of such systems has a Chen–Fliess series representation. Dynamic output feedback systems, for example, where both the plant and controller have Chen–Fliess series representations have been shown to always have such a representation [6, 7]. The prove relies on the contraction mapping theorem applied in the ultrametric space of noncommutative formal power series. While a perfectly valid approach, it does not scale easily to complex networks. So in this paper an entirely different approach is taken based on the notion of a universal control system due to Kawski and Sussmann [13]. The idea is relatively straightforward in that networks of universal control systems are synthesized leading to the notion of a formal realization evolving on an n-fold direct product of formal Lie
groups. Then the generating series for any input-output pair in the network is described using the notion of a formal representation, a type of infinite dimensional analogue of differential representations that are common in nonlinear control theory \cite{12, 15}. It should be stated, however, that this does not prove that the resulting Chen–Fliess series converges in any sense. The tools used here are purely formal and algebraic. As is often the case when working with Chen–Fliess series, the algebra and the analytic issues can be considered separately with the former providing the setting for the latter, which is actually quite convenient \cite{17}. In particular, it will be shown any additive or multiplicative interconnection of a set of convergent single-input, single-output Chen–Fliess series always has a Chen–Fliess series representation that can be computed explicitly in terms of iterated formal Lie derivatives. The problem of determining convergence of the network’s generating series will be deferred to future work.

The paper is organized as follows. The next section establishes the notation and terminology of the paper. Section 3 presents the concept of a formal realization. Formal representations are described in the subsequent section. The main results of the paper along with several examples are given in Section 5. The conclusions are summarized in the final section, as well as directions for future research.

2. Preliminaries

An alphabet \( X = \{x_0, x_1, \ldots, x_m\} \) is any nonempty and finite set of noncommuting symbols referred to as letters. A word \( \eta = x_i \cdots x_k \) is a finite sequence of letters from \( X \). The number of letters in a word \( \eta \), written as \( |\eta| \), is called its length. The empty word, \( \emptyset \), is taken to have length zero. The collection of all words having length \( k \) is denoted by \( X^k \). Define \( X^* = \bigcup_{k \geq 0} X^k \), which is a monoid under the concatenation product. Any mapping \( c : X^* \to \mathbb{R}^\ell \) is called a formal power series. Often \( c \) is written as the formal sum \( c = \sum_{\eta \in X^*} \langle c, \eta \rangle \eta \), where the coefficient \( \langle c, \eta \rangle \in \mathbb{R}^\ell \) is the image of \( \eta \in X^* \) under \( c \). The support of \( c \), \( \text{supp}(c) \), is the set of all words having nonzero coefficients. The set of all noncommutative formal power series over the alphabet \( X \) is denoted by \( \mathbb{R}^\ell \langle \langle X \rangle \rangle \). The subset of series with finite support, i.e., polynomials, is represented by \( \mathbb{R}^\ell \langle X \rangle \). Each set is an associative \( \mathbb{R} \)-algebra under the concatenation product and an associative and commutative \( \mathbb{R} \)-algebra under the shuffle product, that is, the bilinear product uniquely specified by the shuffle product of two words

\[
(x_i \eta) \shuffle (x_j \xi) = x_i (\eta \shuffle (x_j \xi)) + x_j ((x_i \eta) \shuffle \xi),
\]

where \( x_i, x_j \in X, \eta, \xi \in X^* \) and with \( \eta \shuffle \emptyset = \emptyset \shuffle \eta = \eta \) \cite{4}. For any letter \( x_i \in X \), let \( x_i^{-1} \) denote the \( \mathbb{R} \)-linear left-shift operator defined by \( x_i^{-1}(\eta) = \eta' \) when \( \eta = x_i \eta' \) and zero otherwise. It acts as derivation on the shuffle product. The Lie bracket \([x_i^{-1}, x_j^{-1}] = x_i^{-1} x_j^{-1} - x_j^{-1} x_i^{-1}\) also acts as a derivation on the shuffle product. Finally, the left-shift operator is defined inductively for higher order shifts via \( (x_i \eta)^{-1} = \eta^{-1} x_i^{-1} \), where \( \eta \in X^* \).

For \( p \in \mathbb{R} \langle X \rangle \), let \( p^{-1} := \sum_{\eta \in X^*} \langle p, \eta \rangle \eta^{-1} \).

Given any \( c \in \mathbb{R}^\ell \langle \langle X \rangle \rangle \) one can associate a causal \( m \)-input, \( \ell \)-output operator, \( F_c \), in the following manner. Let \( p \geq 1 \) and \( t_0 < t_1 \) be given. For a Lebesgue measurable function \( u : [t_0, t_1] \to \mathbb{R}^m \), define \( \|u\|_p = \max\{\|u_i\|_p : 1 \leq i \leq m\} \), where \( \|u_i\|_p \) is the usual \( L_p \)-norm.
for a measurable real-valued function, \( u_t \), defined on \([t_0, t_1]\). Let \( L_p^m[t_0, t_1] \) denote the set of all measurable functions defined on \([t_0, t_1]\) having a finite \( \| \cdot \|_p \) norm and \( B_p^m(R)[t_0, t_1] := \{ u \in L_p^m[t_0, t_1] : \| u \|_p \leq R \} \). Assume \( C[t_0, t_1] \) is the subset of continuous functions in \( L_p^m[t_0, t_1] \). Define inductively for each word \( \eta = x_i\bar{\eta} \in X^* \) the map \( E_{\eta} : L_p^m[t_0, t_1] \to C[t_0, t_1] \) by setting 
\( E_{\emptyset}[u] = 1 \) and letting
\[
E_{x_i\bar{\eta}}[u](t, t_0) = \int_{t_0}^{t} u_t(\tau) E_{\bar{\eta}}[u](\tau, t_0) d\tau,
\]
where \( x_i \in X, \bar{\eta} \in X^* \), and \( u_0 = 1 \). The Chen–Fliess series corresponding to \( c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle \) is
\[
y(t) = F_c[u](t) = \sum_{\eta \in X^*} \langle c, \eta \rangle E_{\eta}[u](t, t_0)
\]
[4]. If there exist real numbers \( K_c, M_c > 0 \) such that
\[
|\langle c, \eta \rangle| \leq K_c M_c^{|\eta|!} |\eta|!, \quad \forall \eta \in X^*,
\]
then \( F_c \) constitutes a well defined mapping from \( B_p^m(R)[t_0, t_0 + T] \) into \( B_q^l(S)[t_0, t_0 + T] \) for sufficiently small \( R, T > 0 \) and some \( S > 0 \), where the numbers \( p, q \in [1, \infty] \) are conjugate exponents, i.e., \( 1/p + 1/q = 1 \) [9]. (Here, \(|z| := \max_i |z_i| \) when \( z \in \mathbb{R}^\ell \)). The set of all such locally convergent series is denoted by \( \mathbb{R}_LC^\ell\langle\langle X\rangle\rangle \), and \( F_c \) is referred to as a Fliess operator.

Given Fliess operators \( F_c \) and \( F_d \), where \( c, d \in \mathbb{R}_LC^\ell\langle\langle X\rangle\rangle \), the parallel and product connections satisfy \( F_c + F_d = F_{c+d} \) and \( F_c F_d = F_{c\circ d} \), respectively [4]. When Fliess operators \( F_c \) and \( F_d \) with \( c \in \mathbb{R}_LC^\ell\langle\langle X\rangle\rangle \) and \( d \in \mathbb{R}_LC^m\langle\langle X\rangle\rangle \) are interconnected in a cascade fashion, the composite system \( F_c \circ F_d \) has the Fliess operator representation \( F_{c\circ d} \), where the composition product of \( c \) and \( d \) is given by
\[
c \circ d = \sum_{\eta \in X^*} \langle c, \eta \rangle \psi_d(\eta)(1)
\]
[3]. Here \( 1 \) denotes the monomial \( \emptyset \), and \( \psi_d \) is the continuous (in the ultrametric sense) algebra homomorphism from \( \mathbb{R}^\ell\langle\langle X\rangle\rangle \) to the vector space endomorphisms on \( \mathbb{R}^\ell\langle\langle X\rangle\rangle \), \( \text{End}(\mathbb{R}^\ell\langle\langle X\rangle\rangle) \), uniquely specified by \( \psi_d(x_i, \eta) = \psi_d(x_i) \circ \psi_d(\eta) \) with \( \psi_d(x_i)(e) = x_0(d_i w e), i = 0, 1, \ldots, m \) for any \( e \in \mathbb{R}^\ell\langle\langle X\rangle\rangle \), and where \( d_i \) is the \( i \)-th component series of \( d \). By definition, \( \psi_d(\emptyset) \) is the identity map on \( \mathbb{R}^\ell\langle\langle X\rangle\rangle \). It is sometimes useful to associate a unique alphabet with each operator. For example, let \( X = \{ x_0, x_1, \ldots, x_m \} \) and \( \bar{X} = \{ \bar{x}_0, \bar{x}_1, \ldots, \bar{x}_\bar{m} \} \). If \( c \in \mathbb{R}_LC^\ell\langle\langle \bar{X}\rangle\rangle \) and \( d \in \mathbb{R}_LC^m\langle\langle X\rangle\rangle \), then the cascade connection \( F_c \circ F_d \) has the generating series in \( \mathbb{R}^\ell\langle\langle X\rangle\rangle \)
\[
c \circ d = \sum_{\bar{n} \in \bar{X}^*} \langle c, \bar{n} \rangle \psi_d(\bar{n})(1),
\]
where now \( \psi_d(x_i) : \mathbb{R}^\ell\langle\langle X\rangle\rangle \to \mathbb{R}^\ell\langle\langle X\rangle\rangle, e \mapsto x_0(d_i w e), i = 0, 1, \ldots, \bar{m} \). In this case, the letters in \( X \) are identified with the inputs of \( F_d \), and the letters of \( \bar{X} \) are identified with the inputs of \( F_c \). There is a natural isomorphism between \( x_0 \) and \( \bar{x}_0 \) since both symbols correspond to the unity input \( (\bar{u}_0 = u_0 = 1) \).
Example 2.1. Suppose \( X = \{ x_0, x_1 \} \) and \( \tilde{X} = \{ \tilde{x}_0, \tilde{x}_1 \} \). Let \( c = \tilde{x}_1 \tilde{x}_1 \) and \( d = x_1 \). The generating series for the series interconnected system, \( c \circ d = \tilde{x}_1 \tilde{x}_1 \circ x_1 \), can be computed directly from (3) as

\[
c \circ d = \langle c, \tilde{x}_1 \tilde{x}_1 \rangle \psi_d(\tilde{x}_1 \tilde{x}_1)(1) = \psi_d(\tilde{x}_1 \circ \tilde{x}_1)(1) = x_0(x_1 \mathcal{W}(x_1(\mathcal{W}1))) = x_0 x_0 x_1 + 2x_0^2 x_1^2.
\]

\( \square \)

3. Formal Realizations

For any finite \( T > 0, u \in \mathcal{L}_m^m[0,T] \) and fixed \( t \in [0,T] \), one can associate the formal power series in \( \mathbb{R}\langle\langle X \rangle\rangle \)

\[
P[u](t) = \sum_{\eta \in X^*} \eta E_\eta[u](t,0),
\]

which is usually called a Chen series. If, for example, \( u_i(t) = \alpha_i \in \mathbb{R}, i = 1, 2, \ldots, m \) on \([0,T]\) then \( P[u](0) = 1 \) and

\[
\frac{d}{dt} P[u](t) = \sum_{\eta \in X^*} \eta \frac{d}{dt} E_\eta[u](t,0)
\]

\[
= \sum_{\eta \in X^*} \sum_{i=0}^{m} \eta u_i(t) E_{x_i^{-1}(\eta)}[u](t,0)
\]

\[
= \sum_{\eta \in X^*} \sum_{i=0}^{m} \alpha_i x_i \eta E_\eta[u](t,0)
\]

\[
= \left( \sum_{i=0}^{m} \alpha_i x_i \right) P[u](t).
\]

It follows directly that

\[
\frac{d^n}{dt^n} P[u](0) = \left( \sum_{i=0}^{m} \alpha_i x_i \right)^n, \quad n \geq 0,
\]

and, therefore

\[
P[u](t) = \sum_{n=0}^{\infty} \left( \sum_{i=0}^{m} \alpha_i x_i \right)^n \frac{t^n}{n!} = \exp \left( t \sum_{i=0}^{m} \alpha_i x_i \right).
\]

In general, \( P[u] \) is the solution to the formal differential equation

\[
\frac{d}{dt} P[u] = \left( \sum_{i=0}^{m} x_i u_i \right) P[u], \quad P[u](0) = 1,
\]

so that \( P[u] \) is always the exponential of some Lie element over \( X \). That is, if \( \mathcal{L}(X) \) is the free Lie algebra generated by \( X \), then any \( d \in \mathbb{R}\langle\langle X \rangle\rangle \) is a Lie series if it can be written in the form \( d = \sum_{n \geq 1} p_n \), where each polynomial \( p_n \in \mathcal{L}(X) \) has support residing in \( X^n \). The set of all Lie series will be denoted by \( \hat{\mathcal{L}}(X) \). An exponential Lie series is any series
$e = \exp(d)$, where $d$ is a Lie series. In general, \cite{4} has a solution of the form $P[u] = \exp(U)$ with $U(t) \in \hat{L}(X)$ for fixed $t \geq 0$. As a consequence of the Baker–Campbell–Hausdorff formula, the set of all exponential Lie series forms a group, $\mathcal{G}(X)$, under the Cauchy product with unit $1$.

Following the approach of Kawski and Sussmann in \cite{13,16}, $\mathcal{G}(X)$ can be viewed as a formal Lie group with $\hat{L}(X)$ as its corresponding Lie algebra\footnote{Certain aspects of this framework can also be found in \cite{11,10}.}. A commutative algebra of real-valued functions on $\mathcal{G}(X)$ is defined using the shuffle algebra on the $\mathbb{R}$-vector space $\mathbb{R}_{LC} \langle \langle X \rangle \rangle$. Specifically, for any fixed $c \in \mathbb{R}_{LC} \langle \langle X \rangle \rangle$ define $f_c : \mathcal{G}(X) \to \mathbb{R}$ by

$$z \mapsto f_c(z) = \sum_{\eta \in X^*} \langle c, \eta \rangle \langle z, \eta \rangle =: \langle c, z \rangle,$$

so that via Friedrich’s criterion

$$f_c(z)f_d(z) = \langle c, z \rangle \langle d, z \rangle = \langle c \shuffle d, z \rangle = f_{c \shuffle d}(z).$$

Convergence follows from the fact that the shuffle product is known to preserve local convergence\footnote{The authors of \cite{13,16} defined their algebra on $\mathbb{R}(X)$, which entirely avoids the convergence issue, but here $\mathbb{R}_{LC} \langle \langle X \rangle \rangle$ is more suitable for the applications to follow.}

Of ten, elements $c \in \mathbb{R}_{LC} \langle \langle X \rangle \rangle$ are group-like, that is, for $c, d \in \mathbb{R}_{LC} \langle \langle X' \rangle \rangle$ one has $\langle c \shuffle d, z \rangle = \langle c \otimes d, \Delta_{\shuffle} z \rangle = \langle c \otimes d, z \otimes z \rangle = \langle c, z \rangle \langle d, z \rangle$. Here $\Delta_{\shuffle}$ is the unshuffle coproduct dualizing the shuffle product. On the other hand, elements $p \in \hat{L}(X)$ are primitive, i.e., $\Delta_{\shuffle} p = p \otimes 1 + 1 \otimes p$ such that $\langle c \shuffle d, p \rangle = \langle c, p \rangle \langle d, 1 \rangle + \langle c, 1 \rangle \langle d, p \rangle$. Moreover,
\[ \Delta_\omega p z = \Delta_\omega p \Delta_\omega z \] yields \( \langle c \omega d, p z \rangle = \langle c, p z \rangle \langle d, z \rangle + \langle c, z \rangle \langle d, p z \rangle . \) However, in this work a Hopf algebraic approach has been suppressed in favor of a purely Lie theoretic presentation.

For any \( p \in \hat{\mathcal{L}}(X) \), the mapping
\[ V_p : \mathcal{G}(X) \to T_z \mathcal{G}(X), \ z \mapsto V_p(z) := p z \]
is a formal right-invariant vector field on \( \mathcal{G}(X) \). Here \( \mathcal{X} \) will denote the set of all such right-invariant vector fields. In addition, the formal Lie derivative is defined to be the mapping
\[ \ell_p : \mathbb{R}^{\ell \mathcal{G}}(\langle X \rangle) \to \mathbb{R}^{\ell \mathcal{G}}(\langle X \rangle), \ c \mapsto \ell_p c := \hat{p}^{-1} c \]
so that
\[ \ell_p c(z) = \langle \ell_p c, z \rangle = \langle \hat{p}^{-1} c, z \rangle = \langle c, p z \rangle = V_p(z)(c), \]
and, in particular,
\[ \ell_p(c \omega d)(z) = \langle \ell_p(c \omega d), z \rangle = \langle c \omega d, p z \rangle = (\ell_p c(z)) d(z) + c(z) \ell_p d(z), \]
which is just an alternative form of (6).

Finally, note that \([\ref{6}]\) can be written componentwise as
\[ y_k(t) = \langle c_k, z(t) \rangle, \ k = 1, 2, \ldots, \ell, \]
where \( c_k \in \mathbb{R}^{\ell \mathcal{G}}(\langle X \rangle) \) denotes the \( k \)-th component of \( c \in \mathbb{R}^{\ell \mathcal{G}}(\langle X \rangle) \) and \( z(t) = P[u](t) \). This leads to the following definition.

**Definition 3.1.** For any \( c \in \mathbb{R}^{\ell \mathcal{G}}(\langle X \rangle) \), the formal realization of the Fliess operator \( y = F_c[u] \) is

\begin{align}
(7a) & \quad \dot{z} = \sum_{i=0}^{m} x_i z u_i, \ z(0) = 1 \\
(7b) & \quad y_k = \langle c_k, z \rangle, \ k = 1, 2, \ldots, \ell.
\end{align}

Observe that
\[ L_{x_i} c_k(1) = x_i^{-1} c_k(1) = \langle x_i^{-1} c_k, 1 \rangle = \langle c_k, x_i \rangle \]
\[ L_{x_j} L_{x_i} c_k(1) = x_j^{-1} x_i^{-1} c_k(1) = \langle x_j^{-1} x_i^{-1} c_k, 1 \rangle = \langle c_k, x_i x_j \rangle, \]
so that the coefficients of \( c_k \) can always be written in terms of formal Lie derivatives as
\[ \langle c_k, \eta \rangle = \langle c_k, x_i_1 \cdots x_n \rangle = L_{x_i_1} \cdots L_{x_n} c_k(1) =: L_\eta c_k(1). \]

The notion of a formal realization in Definition 3.1 is now extended by taking a finite number of direct products of \( \mathcal{G}(X) \), i.e., \( \mathcal{G}^\ell(X) := \mathcal{G}(X) \times \mathcal{G}(X) \times \cdots \times \mathcal{G}(X) \), where \( \mathcal{G}(X) \) appears \( n \) times. For any \( \hat{c} = c_1 \otimes \cdots \otimes c_n \in \mathbb{R}^{\otimes \ell \mathcal{G}}(\langle X \rangle) \) define
\[ f_{\hat{c}} : \mathcal{G}^n(X) \to \mathbb{R} \]
\[ z \mapsto (c_1 \otimes \cdots \otimes c_n)(z_1, \ldots, z_n) = \langle c_1, z_1 \rangle \cdots \langle c_n, z_n \rangle. \]
A commutative algebra on the \( \mathbb{R} \)-vector space of all such real-valued functions on \( G^n(X) \) is given by defining

\[
f_{\hat{c}}(z) f_{\hat{d}}(z) = [(c_1, z_1) \cdots (c_n, z_n)] [(d_1, z_1) \cdots (d_n, z_n)]
\]

\[
= (c_1 \mathfrak{w} d_1, z_1) \cdots (c_n \mathfrak{w} d_n, z_n)
\]

\[
= : (\hat{c} \mathfrak{w} \hat{d})(z_1, z_2, \ldots, z_n)
\]

\[
= f_{\hat{c} \mathfrak{w} \hat{d}}(z).
\]

As earlier, \( f_{\hat{c}}(z) \) will often be abbreviated as \( \hat{c}(z) \). The Lie algebra of \( G^n(X) \), denoted by \( \widehat{\mathcal{L}}^n(X) \), is similarly defined as the \( n \)-fold direct product of the Lie algebra \( \widehat{\mathcal{L}}(X) \) for \( G(X) \).

The formal tangent space at the unit \( 1_n := (1, \ldots, 1) \), \( T_{1_n} G^n(X) \), is identified with \( \widehat{\mathcal{L}}^n(X) \) via the one-parameter subgroup \( H(t) := (\exp(t p_1), \exp(t p_2), \ldots, \exp(t p_n)) \), \( p = (p_1, p_2, \ldots, p_n) \in \widehat{\mathcal{L}}^n(X) \) so that \( \dot{H}(0) = p \). For any fixed \( p \in \widehat{\mathcal{L}}^n(X) \), there is a corresponding tangent vector at \( 1_n \) represented by the linear functional

\[
V_p(1_n) : \mathbb{R}^\otimes_n \langle \langle X \rangle \rangle \rightarrow \mathbb{R}, \ \hat{c} \mapsto \frac{d}{dt} (\hat{c} \circ H(t)) |_{t=0}.
\]

Observe that

\[
V_p(1_n)(\hat{c}) = \frac{d}{dt} \langle (c_1, \exp(tp_1)) \cdots (c_i, \exp(tp_i)) \cdots (c_n, \exp(tp_n)) \rangle |_{t=0}
\]

\[
= \sum_{i=1}^{n} \langle c_i, 1 \rangle \cdots \langle c_i, p_i 1 \rangle \cdots \langle c_n, 1 \rangle
\]

satisfies the Leibniz rule:

\[
V_p(1_n)(\hat{c} \mathfrak{w} \hat{d}) = \sum_{i=1}^{n} \langle c_i \mathfrak{w} d_1, 1 \rangle \cdots \langle c_i \mathfrak{w} d_i, p_i 1 \rangle \cdots \langle c_n \mathfrak{w} d_n, 1 \rangle
\]

\[
= \sum_{i=1}^{n} \langle c_i \mathfrak{w} d_1, 1 \rangle \cdots \langle p_i^{-1}(c_i \mathfrak{w} d_i), 1 \rangle \cdots \langle c_n \mathfrak{w} d_n, 1 \rangle
\]

\[
= \sum_{i=1}^{n} \langle c_i \mathfrak{w} d_1, 1 \rangle \cdots \langle p_i^{-1}(c_i) \mathfrak{w} d_i, 1 \rangle \cdots \langle c_n \mathfrak{w} d_n, 1 \rangle +
\]

\[
= \sum_{i=1}^{n} \langle c_i \mathfrak{w} d_1, 1 \rangle \cdots \langle c_i \mathfrak{w} p_i^{-1}(d_i), 1 \rangle \cdots \langle c_n \mathfrak{w} d_n, 1 \rangle
\]

\[
= V_p(1_n)(\hat{c}) \hat{d}(1_n) + \hat{c}(1_n) V_p(1_n)(\hat{d}).
\]

The tangent space at \( z \in G^n(X) \), denoted \( T_z G^n(X) \), is defined via right translation to be the vector space of linear functionals

\[
V_p(z) : \mathbb{R}^\otimes_n \langle \langle X \rangle \rangle \rightarrow \mathbb{R}
\]

\[
\hat{c} \mapsto \sum_{i=1}^{n} \langle c_1, z_1 \rangle \cdots \langle c_i, p_i z_i \rangle \cdots \langle c_n, z_n \rangle
\]

so as to satisfy

\[
V_p(z)(\hat{c} \mathfrak{w} \hat{d}) = V_p(z)(\hat{c}) \hat{d}(z) + \hat{c}(z) V_p(z)(\hat{d}).
\]
For any $p \in \mathcal{L}^n(X)$, the mapping
\[ V_p : \mathcal{G}^n(X) \to T_x \mathcal{G}^n(X), \quad z \mapsto (p_1 z_1, \ldots, p_n z_n) \]
is a formal right-invariant vector field on $\mathcal{G}^n(X)$. Here $\mathcal{X}^n$ will denote the set of all such right-invariant vector fields. In this context, the formal Lie derivative is defined to be the mapping
\[ L_p : \mathbb{R}^\otimes_n \langle \langle X \rangle \rangle \to \mathbb{R}^\otimes_n \langle \langle X \rangle \rangle \]
so that
\[ L_p \hat{c}(z) = \left( \sum_{i=1}^n c_1 \otimes \cdots \otimes p_i^{-1}(c_i) \otimes \cdots \otimes c_n \right)(z_1, \ldots, z_n) \]
(9)
\[ = \sum_{i=1}^n \langle c_1, z_1 \rangle \cdots \langle c_i, p_i z_i \rangle \cdots \langle c_n, z_n \rangle \]
(10)
and directly
\[ L_p(\hat{c} \omega \hat{d})(z) = (L_p \hat{c}(z)) \hat{d}(z) + \hat{c}(z)L_p \hat{d}(z). \]

In this generalized setting, a set of $n$ systems with state $z = (z_1, z_2, \ldots, z_n)$ evolves on the group $\mathcal{G}^n(X)$ according to the formal state equations
\[ \dot{z}_j = \sum_{i=0}^m x_i z_j u_{ij}, \quad z_j(0) = 1, \]
where $u_{ij} \in L_2[0,T]$ and $u_{0j} = 1$ for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$. Define $\ell$ outputs $y_k = \hat{c}_k(z)$, where $\hat{c}_k \in \mathbb{R}^\otimes_n \langle \langle X \rangle \rangle$, $k = 1, 2, \ldots, \ell$. Therefore, the corresponding input-output map $u \mapsto y$ takes an $m \times n$ matrix of inputs to $\ell$ outputs. Consider now the situation where a network is formed by allowing each system input to be interconnected to some function of the system’s outputs and a new external input $v_{ij}$ to yield a new input-output map $v \mapsto y$, for example, $u_{ij} = \hat{d}_{ij}(z) + v_{ij}$, where $\hat{d}_{ij} \in \mathbb{R}^\otimes_n \langle \langle X \rangle \rangle$. In this case, the state equations for the interconnected system become
\[ \dot{z}_j = x_0 z_j + \sum_{i=1}^m x_i \hat{d}_{ij}(z) z_j + x_i z_j v_{ij}, \quad z_j(0) = 1. \]
Note, in particular, the appearance of state dependent vector fields $p_j z_j$ with $p_j(t) = \sum_{i=1}^m x_i \hat{d}_{ij}(z(t)) \in \mathcal{L}(X)$. The solution to $\dot{z}_j = p_j z_j$, $z_j(0) = 1$ has the form $z_j(t) = \exp(U_j(t))$, where $U_j(t) \in \mathcal{L}(X)$. The corresponding tangent vector at $z(t)$ is
\[ V_{p(t)}(z(t)) : \mathbb{R}^\otimes_n \langle \langle X \rangle \rangle \to \mathbb{R} \]
\[ \dot{c} \mapsto \frac{d}{dt}(\dot{c} \circ z(t)) = \sum_{j=1}^n \langle c_1, z_1(t) \rangle \cdots \langle c_j, p_j(t) z_j(t) \rangle \cdots \langle c_n, z_n(t) \rangle \]
Substituting \( p_j(t) = \sum_{i=1}^{m} x_i \hat{d}_{ij}(z(t)) \) on the right-hand side above, where \( \hat{d}_{ij}(z(t)) = \langle d_{ij}^{(1)}, z_1(t) \rangle \cdots \langle d_{ij}^{(n)}, z_n(t) \rangle \), gives

\[
L_{p(t)} \hat{c}(z(t)) = \sum_{j=1}^{n} \langle c_j, z_1(t) \rangle \cdots \langle c_j, p_j(t) z_j(t) \rangle \cdots \langle c_n, z_n(t) \rangle
\]

\[
= \sum_{j=1}^{n} \langle c_j, z_1(t) \rangle \cdots \sum_{i=1}^{m} \hat{d}_{ij}(z(t)) \langle c_j, x_i z_j(t) \rangle \cdots \langle c_n, z_n(t) \rangle
\]

\[
= \sum_{i=1}^{m} \sum_{j=1}^{n} \langle c_1 \mathbf{w} d_{ij}^{(1)}, z_1(t) \rangle \cdots \langle c_j \mathbf{w} d_{ij}^{(j)}, z_j(t) \rangle \cdots \langle c_n \mathbf{w} d_{ij}^{(n)}, z_n(t) \rangle
\]

\[
=: \hat{c}'(z(t)).
\]

In this way, a second Lie derivative can now be computed directly using (11), thus circumventing the difficult task of explicitly composing time-varying vector fields. Henceforth, all such state dependent Lie series will be written as \( p(z) \). No other type of state dependent series will appear in this paper. In this context, a generalization of Definition 3.1 is presented.

**Definition 3.2.** Let \( V_i \in \mathcal{X}^n, i = 0, 1, \ldots, m \) with

\[
V_i : \mathcal{G}^n(X) \to T_x \mathcal{G}^n(X)
\]

\[
z = (z_1, \ldots, z_n) \mapsto V_i(z) = (V_{i1}(z) z_1, \ldots, V_{in}(z) z_n),
\]

where \( V_{ij}(z) \in \hat{\mathcal{L}}(X) \). The \( j \)-th component of the corresponding state equation on \( \mathcal{G}^n(X) \) is

\[
\dot{z}_j = \sum_{i=0}^{m} V_{ij}(z) z_j u_{ij}, \quad z_j(0) = z_{j0}.
\]

Given \( \hat{c}_k \in \mathbb{R}^n_{\text{LC}}(\langle X \rangle), k = 1, 2, \ldots, \ell \), the \( k \)-th output equation is defined to be

\[
y_k = \hat{c}_k(z).
\]

Collectively, \((V, z_0, \hat{c})\) is a formal realization on \( \mathcal{G}^n(X) \) of the formal input-output map \( u \mapsto y \).

For convenience the integer \( n \) will be referred to here as the dimension of the realization, though this is a misnomer as the underlying group \( \mathcal{G}(X) \) is not finite dimensional, therefore neither is the state \( z \). The following example illustrates how the concept naturally arises when Chen–Fliess series are composed.

**Example 3.1.** Reconsider the systems \( y_2 = F_c[u_2] \) and \( y_1 = F_d[u_1] \) in Example 2.1 using the same alphabet \( X = \{x_0, x_1\} \) for both series. Each has a formal realization of the form given in Definition 3.1. Setting \( u_2 = y_1 \) so that \( y_2 = F_c \circ F_d[u_1] \) yields a formal realization of dimension two:

\[
\dot{z}_1 = x_0 z_1 + x_1 z_1 u_1, \quad z_1(0) = 1
\]

\[
\dot{z}_2 = (x_0 + x_1 (d, z_1)) z_2, \quad z_2(0) = 1
\]
\( y_2 = \langle 1, z_1 \rangle \langle c, z_2 \rangle \).

(Note that \( \langle 1, z_1 \rangle = 1 \).) Therefore,
\[
V_0(z) = \begin{bmatrix} x_0 z_1 \\ (x_0 + x_1 (d, z_1)) z_1 \end{bmatrix}, \quad V_1(z) = \begin{bmatrix} x_1 z_1 \\ 0 \end{bmatrix},
\]
and \( \hat{c} = 1 \otimes c \). Observe that the composition \( F_c \circ F_d = F_{cd} \) introduces in the second component of the tangent vector \( V_0(z) \) a \( z_1 \) dependence. The aim is to express \( c \circ d \) directly in terms of \( (V_1, 1_2, \hat{c}) \). This leads to the notion of a formal representation of a series as presented in the next section. It can be viewed as a generalization of (8).

\[\Box\]

4. Formal Representations

The following definition is a formal analog of a differential representation as appears, for example, in [12, 15].

**Definition 4.1.** A formal representation of a series \( d \in \mathbb{R} \langle\langle X \rangle\rangle \) is any triple \((\mu, z_0, \hat{c})\), where
\[
\mu : X^* \to X^n, \quad x_i \mapsto V_i
\]
defines a monoid homomorphism, \( z_0 \in G^n(X) \), and \( \hat{c} \in \mathbb{R}^{\otimes n} \langle\langle X \rangle\rangle \), so that for any word \( \eta = x_{i_k} x_{i_{k-1}} \cdots x_{i_1} \in X^* \)
\[\langle d, \eta \rangle = L_{\mu(\eta)} \hat{c}(z_0) := L_{\mu(x_{i_1})} L_{\mu(x_{i_2})} \cdots L_{\mu(x_{i_k})} \hat{c}(z_0).\]

By definition, \( \langle d, \emptyset \rangle = L_0 \hat{c}(z_0) := \hat{c}(z_0) \). The integer \( n \geq 1 \) will be called the dimension of the representation.

**Example 4.1.** For the trivial case where \( n = 1 \), \( \mu(x_i) = x_i \), \( z_0 = 1 \), and \( d = \hat{c} = c \) it is immediate that (14) reduces to (8) with \( \ell = 1 \). \(\Box\)

The following lemma provides a sufficient condition under which formal representations are always well defined.

**Lemma 4.1.** Given \((\mu, z_0, \hat{c})\), if for each \( x_i \in X \) \( [\mu(x_i)]_{i_j} (z) := V_{ij}(z) z_j \) has \( V_{ij}(z) \) being some Lie polynomial in \( L(X) \), then there exists a well defined \( d \in \mathbb{R} \langle\langle X \rangle\rangle \) satisfying (14).

**Proof:** If \((\mu, z_0, \hat{c})\) is a formal representation of \( d \) then necessarily for any \( \eta = x_{i_1} \cdots x_{i_k} \in X^* \)
\[
\langle d, x_{i_k} \cdots x_{i_1} \rangle = L_{\mu(x_{i_1})} L_{\mu(x_{i_2})} \cdots L_{\mu(x_{i_k})} \hat{c}(z_0),
\]
where each \( V_{ij}(z) \) is a Lie polynomial. Therefore, each Lie derivative can be written as a polynomial in functions of the form \( \langle e, p_i z_i \rangle \) with \( p_i \in L(X) \), \( i = 1, 2, \ldots, n \), and \( e \in \mathbb{R}_{LC} \langle\langle X \rangle\rangle \), implying that \( d \) is well defined, in fact, locally finite [1]. \(\Box\)
Example 4.2. Continuing Examples 2.1 and 3.1, the claim is that $c \circ d$ has a formal representation $(\mu, 1_2, \hat{c})$, where $\mu$ is defined in terms of the vector fields $V_0$ and $V_1$ in Example 3.1 and $\hat{c} = 1 \otimes c$. Note that both vector fields satisfy the condition in Lemma 4.1. As an example, it is verified that

$$
\langle x_0^2 c \circ x_1^2, c \circ d \rangle = L_{\mu(x_0^2 x_1^2)} \hat{c}(1) = L_{V_1} L_{V_1} L_{V_0} \hat{c}(1) = 2.
$$

First apply (11) (suppressing all $t$ dependence)

$$
L_{V_0} \hat{c}(z) = \langle c, V_{02}(z) z_2 \rangle = \langle x_1^2, (x_0 + x_1(x_1, z_1)) z_2 \rangle.
$$

Now, regarding the $z_1$ dependence of $V_{02}(z)$, use the procedure described in the previous section to get

$$
L_{V_0} \hat{c}(z) = \langle x_1, z_1 \rangle \langle x_1, z_2 \rangle = (x_1 \otimes x_1)(z_1, z_2) = \hat{c}'(z).
$$

Apply (11) a second time gives:

$$
L_{V_1} L_{V_0} \hat{c}(z) = L_{V_1} \hat{c}'(z) = \langle x_1, V_{01}(z) z_1 \rangle \langle x_1, z_2 \rangle + \langle x_1, z_1 \rangle \langle x_1, V_{02}(z) z_2 \rangle
$$

$$
= \langle x_1, x_0 z_1 \rangle \langle x_1, z_2 \rangle + \langle x_1, z_1 \rangle \langle x_1, (x_0 + x_1(x_1, z_1)) z_2 \rangle
$$

$$
= \langle x_1, z_1 \rangle^2 \langle 1, z_2 \rangle
$$

$$
= \langle x_1 \otimes x_1, z_1 \rangle \langle 1, z_2 \rangle
$$

$$
= (2x_1^2 \otimes 1)(z_1, z_2) = \hat{c}''(z).
$$

Continuing in this fashion,

$$
L_{V_1} L_{V_1} L_{V_0} \hat{c}(z) = L_{V_1} \hat{c}''(z) = \langle 2x_1, z_1 \rangle \langle 1, z_2 \rangle
$$

$$
= (2x_1 \otimes 1)(z_1, z_2) = \hat{c}'''(z)
$$

and

$$
L_{V_1} L_{V_1} L_{V_0} L_{V_0} \hat{c}(z) = L_{V_1} \hat{c}'''(z) = \langle 21, z_1 \rangle \langle 1, z_2 \rangle.
$$

Therefore, $\langle x_0^2 x_1^2, c \circ d \rangle = L_{V_1} L_{V_1} L_{V_0} \hat{c}(1) = 2$ as anticipated.

The proposition in the previous example is established in the following theorem.

Theorem 4.1. If $d \in \mathbb{R} \langle \langle X \rangle \rangle$ has a well defined formal representation $(\mu, z_0, \hat{c}_k)$, then the input-output map $u \mapsto y_k$ of the corresponding formal realization (12)-(13) has a Chen–Fliess series representation with generating series $d$.

Proof: Without loss of generality, assume there is a single output so that the subscript on $\hat{c}_k$ and $y_k$ can be dropped. Likewise, assume $n = 1$ so the index on the state can be omitted. Since $\dot{z}(t)$ is a tangent vector at $z(t) \in G(X)$ for any $t \geq 0$, it follows directly from (10) that

$$
\dot{z}(t)(\hat{c}) = \sum_{i=0}^{m} V_i(z(t))(\hat{c})u_i(t) = \sum_{i=0}^{m} L_{V_i} \hat{c}(z(t))u_i(t).
$$
Integrating both sides on \([0, t]\) gives
\[
\dot{c}(z(t)) = \dot{c}(z_0) + \sum_{i=0}^{m} \int_0^t L_{V_i} \dot{c}(z(\tau)) u_i(\tau) \, d\tau.
\]
Noting that \(y(t) = \dot{c}(z(t))\) and substituting the entire right-hand side for \(\dot{c}(z(\tau))\) inside the integral gives
\[
y(t) = \dot{c}(z_0) + \sum_{i=0}^{m} \int_0^t L_{V_i} \dot{c}(z(\tau)) u_i(\tau) \, d\tau + \sum_{i_1, i_2=0}^{m} \int_0^t L_{V_{i_1}} \left[ \int_0^{\tau_1} L_{V_{i_2}} \dot{c}(z(\tau_2)) u_{i_2}(\tau_2) \, d\tau_2 \right] u_{i_1}(\tau_1) \, d\tau_1.
\]
Continuing in this way yields
\[
y(t) = \sum_{\eta \in X^*} L_{\mu(\eta)} \dot{c}(z_0) E_{\eta}[u](t) = \sum_{\eta \in X^*} \langle d, \eta \rangle E_{\eta}[u](t),
\]
which proves the theorem.

5. Networks of Chen–Fliess Series

In this section specific types of networks of Chen–Fliess series are considered for which both Lemma 4.1 and Theorem 4.1 apply. To avoid a barrage of indices, the component systems are assumed to be single-input, single-output. There is, however, no technical reason for avoiding the multivariable case. A variety of different configurations are possible. The following is perhaps the simplest.

Definition 5.1. A set of \(m\) single-input, single-output Chen–Fliess series mapping \(u_i \mapsto y_i\) with generating series \(c_i \in \mathbb{R}\langle\langle X_i \rangle\rangle\), where \(X_i = \{x_0, x_i\}\), and weighting matrix \(M \in \mathbb{R}^{m \times m}\) is said to be additively interconnected if \(u_i = v_i + \sum_{j=1}^{m} M_{ij} y_j\), \(i = 1, 2, \ldots, m\).

In the following theorem, let \(e_i \in \mathbb{R}_{LC}^{m}\langle\langle X \rangle\rangle\) denote the series with the \(i\)-th component series being the monomial \(1\), and the remaining components are the series having all coefficients equal to zero. In addition, given \(c_j \in \mathbb{R}_{LC}\langle\langle X \rangle\rangle\), define \(\hat{c}_j = 1 \otimes \cdots \otimes 1 \otimes c_j \otimes 1 \cdots \otimes 1\), where \(c_j\) appears in the \(j\)-th position.

Theorem 5.1. The input-output map \(v \mapsto y\) of any additive interconnection of \(m\) single-input, single-output Chen–Fliess series with generating series \(c_i \in \mathbb{R}\langle\langle X_i \rangle\rangle\) has a well defined generating series \(d \in \mathbb{R}^m\langle\langle X \rangle\rangle\), where \(d_j\) has the formal representation \((\mu, 1, \hat{c}_j)\) with \(\mu\) defined in terms of the vector fields
\[
V_0(z) = \begin{bmatrix} x_0 z_1 \\ x_0 z_2 \\ \vdots \\ x_0 z_n \end{bmatrix} + \text{diag}(x_1, \ldots, x_m)M \begin{bmatrix} \langle c_1, z_1 \rangle z_1 \\ \langle c_2, z_2 \rangle z_2 \\ \vdots \\ \langle c_m, z_m \rangle z_n \end{bmatrix},
\]
and \(V_i(z) = x_i z_i e_i\) for \(i = 1, 2, \ldots, m\).
Proof: It is straightforward to show that the set of interconnected Chen–Fliess series constitutes an \( m \) input, \( m \) output system with formal realization given by the vector fields as shown. Therefore, the claim follows directly from Lemma 4.1 and Theorem 4.1 with \( \mu(x_i) = V_i, i = 0, 1, \ldots m, z_0 = 1_m, \) and \( \hat{c}_j \in \mathbb{R}^m \langle \langle X \rangle \rangle \).

**Figure 1.** Single system additively interconnected

**Example 5.1.** A single system additively interconnected with itself as shown in Figure 1 would correspond to propositional output feedback, i.e., \( u = v + My \) (dropping all subscripts). Thus, the corresponding representation is given by

\[
V_0(z) = (x_0 + x_1 M \langle c, z \rangle) z, \quad V_1(z) = x_1 z,
\]

\( z_0 = 1_1 = 1 \), and \( \hat{c} = c \). For a unity feedback system, i.e., \( M = 1 \), applying \([14]\) gives the following generating series for the closed-loop system:

\[
\langle d, 1 \rangle = c(1) = \langle c, 1 \rangle \\
\langle d, x_1 \rangle = L_{V_1} c(1) = \langle c, x_1 \rangle \\
\langle d, x_0 \rangle = L_{V_0} c(1) = \langle c, x_0 \rangle + \langle c, x_1 \rangle \langle c, 1 \rangle \\
\langle d, x_0^2 \rangle = L_{V_1} L_{V_0} c(1) = \langle c, x_0^2 \rangle \\
\langle d, x_1 x_0 \rangle = L_{V_0} L_{V_1} c(1) = \langle c, x_1 x_0 \rangle + \langle c, x_1 \rangle \langle c, 1 \rangle \\
\langle d, x_0 x_1 \rangle = L_{V_1} L_{V_0} c(1) = \langle c, x_0 x_1 \rangle + \langle c, x_1 \rangle \langle c, 1 \rangle + \langle c, x_0^2 \rangle \langle c, 1 \rangle \\
\langle d, x_0^2 \rangle = L_{V_1} L_{V_0} c(1) = \langle c, x_1 \rangle \langle c, x_0 \rangle + \langle c, x_0 x_1 \rangle \langle c, 1 \rangle + \langle c, x_0^2 \rangle \langle c, 1 \rangle + \langle c, x_0 x_1 \rangle \langle c, 1 \rangle + \langle c, x_0^2 \rangle \langle c, 1 \rangle + \langle c, x_0 x_1 \rangle \langle c, 1 \rangle \\
\vdots
\]

These expressions are consistent with those in [6], where \( d = S(-c) \), and \( S \) is the antipode of the output feedback Hopf algebra.

\[\vdots\]

**Figure 2.** Two systems additively interconnected
Example 5.2. Consider two additively interconnected systems as shown in Figure 2 where $M_{ij} = 0$ when $i = j$. Setting $M_{ij} = 1$ for $i \neq j$ gives a representation of $d_j$ specified by

$$V_0(z) = \begin{bmatrix} (x_0 + x_1 \langle c_2, z_2 \rangle) z_1 \\ (x_0 + x_2 \langle c_1, z_1 \rangle) z_2 \end{bmatrix}, \quad V_i(z) = x_i z_i e_i, \quad i = 1, 2,$$

$z_0 = 1_2$, and $\hat{c}_j$. For example, the generating series $d_1$ for the mapping $v \mapsto y_1$ is:

$$\langle d_1, 1 \rangle = \hat{c}_1(1_2) = \langle c_1, 1 \rangle$$

$$\langle d_1, x_1 \rangle = L_{V_1} \hat{c}_1(1_2) = \langle c_1, x_1 \rangle$$

$$\langle d_1, x_2 \rangle = L_{V_2} \hat{c}_1(1_2) = 0$$

$$\langle d_1, x_0 \rangle = L_{V_0} \hat{c}_1(1_2) = \langle c_1, x_0 \rangle + \langle c_1, x_1 \rangle \langle c_2, 1 \rangle$$

$$\langle d_1, x_1^2 \rangle = L_{V_1} L_{V_1} \hat{c}_1(1_2) = \langle c_1, x_1^2 \rangle$$

$$\langle d_1, x_1 x_2 \rangle = L_{V_2} L_{V_1} \hat{c}_1(1_2) = 0$$

$$\langle d_1, x_2 x_1 \rangle = L_{V_2} L_{V_2} \hat{c}_1(1_2) = 0$$

$$\langle d_1, x_2^2 \rangle = L_{V_2} L_{V_2} \hat{c}_1(1_2) = 0$$

$$\langle d_1, x_1 x_0 \rangle = L_{V_0} L_{V_2} \hat{c}_1(1_2) = \langle c_1, x_1 x_0 \rangle + \langle c_1, x_1^2 \rangle \langle c_2, 1 \rangle$$

$$\langle d_1, x_0 x_1 \rangle = L_{V_1} L_{V_0} \hat{c}_1(1_2) = \langle c_1, x_0 x_1 \rangle + \langle c_1, x_1^2 \rangle \langle c_2, 1 \rangle$$

... and similarly for $d_2$ corresponding to the map $v \mapsto y_2$. Unlike the first example, for networks with more than one system, there is at present no known alterative algebraic method against which to compare these results.

\[ \blacksquare \]

![Figure 3. Three systems additively interconnected](image)

Example 5.3. Consider three additively interconnected systems as shown in Figure 3 where again $M_{ij} = 0$ when $i = j$, and the output branches have been suppressed. For the case where $M_{ij} = 1$ when $i \neq j$, a representation of $d_j$ is given by

$$V_0(z) = \begin{bmatrix} (x_0 + x_1 \langle c_2, z_2 \rangle + x_1 \langle c_3, z_3 \rangle) z_1 \\ (x_0 + x_2 \langle c_1, z_1 \rangle + x_2 \langle c_3, z_3 \rangle) z_2 \\ (x_0 + x_3 \langle c_1, z_1 \rangle + x_3 \langle c_2, z_2 \rangle) z_3 \end{bmatrix}$$

$$V_i(z) = x_i z_i e_i, \quad i = 1, 2, 3,$$
$z_0 = 1_3$, and $\hat{c}_j$. For example, the generating series $d_1$ for the mapping $v \mapsto y_1$ is:

\[
\langle d_1, 1 \rangle = \hat{c}_1(1_3) = \langle c_1, 1 \rangle \\
\langle d_1, x_1 \rangle = L_{V_1} \hat{c}_1(1_3) = \langle c_1, x_1 \rangle \\
\langle d_1, x_2 \rangle = L_{V_2} \hat{c}_1(1_3) = 0 \\
\langle d_1, x_3 \rangle = L_{V_3} \hat{c}_1(1_3) = 0 \\
\langle d_1, x_0 \rangle = L_{V_0} \hat{c}_1(1_3) = \langle c_1, x_0 \rangle + \langle c_1, x_1 \rangle \langle c_2, 1 \rangle + \langle c_1, x_1 \rangle \langle c_3, 1 \rangle \\
\langle d_1, x_1^2 \rangle = L_{V_1} L_{V_1} \hat{c}_1(1_3) = \langle c_1, x_1^2 \rangle \\
\langle d_1, x_1 x_2 \rangle = L_{V_2} L_{V_1} \hat{c}_1(1_3) = 0 \\
\langle d_1, x_1 x_3 \rangle = L_{V_3} L_{V_1} \hat{c}_1(1_3) = 0 \\
\langle d_1, x_1 x_0 \rangle = L_{V_0} L_{V_1} \hat{c}_1(1_2) = \langle c_1, x_1 x_0 \rangle + \langle c_1, x_1^2 \rangle \langle c_2, 1 \rangle + \langle c_1, x_1 x_1 \rangle \langle c_3, 1 \rangle \\
\langle d_1, x_0 x_1 \rangle = L_{V_1} L_{V_0} \hat{c}_1(1_2) = \langle c_1, x_0 x_1 \rangle + \langle c_1, x_1^2 \rangle \langle c_2, 1 \rangle + \langle c_1, x_1 x_1 \rangle \langle c_3, 1 \rangle \\
\vdots
\]

and similarly for $d_i$ corresponding to the map $v \mapsto y_i$, $i = 2, 3$.

Free from the bonds of linearity, other types of interconnections are also possible as considered next.

**Definition 5.2.** A set of $m$ single-input, single-output Chen–Fliess series mapping $u_i \mapsto y_i$ with generating series $c_i \in \mathbb{R}\langle\langle X_i \rangle\rangle$, where $X_i = \{x_0, x_i\}$, and weighting matrix $M \in \mathbb{R}^{m \times m}$ is said to be **multiplicatively interconnected** if $u_i = v_i \prod_{j=1}^m M_{ij} y_j$, $i = 1, 2, \ldots, m$.

**Theorem 5.2.** Every input-output map $v \mapsto y$ of any multiplicative interconnection of $m$ single-input, single-output Chen–Fliess series with generating series $c_i \in \mathbb{R}\langle\langle X_i \rangle\rangle$ has a well defined generating series $d \in \mathbb{R}^m\langle\langle X \rangle\rangle$, where $d_j$ has the formal representation $(\mu, 1_m, \hat{c}_j)$ with $\mu$ defined in terms of the vector fields

\[
V_0(z) = \begin{bmatrix}
x_0 z_1 \\
x_0 z_2 \\
\vdots \\
x_0 z_n
\end{bmatrix}, \quad V_i(z) = x_i \prod_{j=1}^m M_{ij} \langle c_j, z_j \rangle z_i e_i.
\]

**Proof:** The proof is perfectly analogous to that of Theorem 5.1. \qed

**Example 5.4.** Reconsider the single system network in Example 5.1 except now multiplicatively interconnected, that is, $u = v M y$ (again dropping all subscripts). The corresponding representation is given by

\[
V_0(z) = x_0 z, \quad V_1(z) = x_1 M \langle c, z \rangle z,
\]

$z_0 = 1$, and $\hat{c} = c$. Setting $M = 1$ and applying (14) gives the following generating series for the closed-loop system:

\[
\langle d, 1 \rangle = c(1) = \langle c, 1 \rangle
\]
\[\langle d, x_1 \rangle = LV_1 c(1) = \langle c, x_1 \rangle \langle c, 1 \rangle\]
\[\langle d, x_0 \rangle = LV_0 c(1) = \langle c, x_0 \rangle\]
\[\langle d, x_1^2 \rangle = LV_1 LV_1 c(1) = \langle c, x_1^2 \rangle \langle c, 1 \rangle + \langle c, x_1 \rangle \langle c, x_1 \rangle \langle c, 1 \rangle\]
\[\langle d, x_1 x_0 \rangle = LV_0 LV_1 c(1) = \langle c, x_1 x_0 \rangle \langle c, 1 \rangle + \langle c, x_1 \rangle \langle c, x_0 \rangle\]
\[\langle d, x_0 x_1 \rangle = LV_1 LV_0 c(1) = \langle c, x_0 x_1 \rangle \langle c, 1 \rangle\]
\[\langle d, x_3 \rangle = LV_1 LV_1 LV_1 c(1) = \langle c, x_3 \rangle \langle c, 1 \rangle + 4 \langle c, x_1^2 \rangle \langle c, x_1 \rangle \langle c, 1 \rangle \langle c, 1 \rangle + \langle c, x_1 \rangle \langle c, x_1 \rangle \langle c, x_1 \rangle \langle c, 1 \rangle \langle c, 1 \rangle\]
\[\langle d, x_0^2 \rangle = LV_0 LV_0 c(1) = \langle c, x_0^2 \rangle\]

Consider the particular case where \(c = \sum_{k \geq 0} k! x_1^k\). Applying the formulas above gives the closed-loop generating series
\[d = 1 + x_1 + 3x_1^2 + 15x_1^3 + \cdots,\]
which is consistent with what was computed in [7, Example 4.10] using the antipode of the output affine feedback Hopf algebra.

6. Conclusions and Future Work

Using the concept of a formal realization and a formal representation, it was shown that any additive or multiplicative interconnection of a set of convergent single-input, single-output Chen–Fliess series always has a Chen–Fliess series representation whose generating series can be computed explicitly in terms of iterated formal Lie derivatives. This of course does not exhaust the list of possible network topologies for which this method is suitable. For example, there can be mixtures of additive and multiplicative nodes in a given network. There is also no technical barrier to applying the methodology in the full multivariable setting. Finally, the issue of convergence of the network’s generating series needs to be addressed in every case.

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