The central configuration of the planar \((N+1)\)-body problem with a regular \(N\)-polygon for homogeneous force laws

Liang Ding\(^1,2\) · Jinlong Wei\(^3\) · Shiqing Zhang\(^4\)

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Abstract
This paper deals with the planar \((N+1)\)-body problem for homogeneous force laws, where \(N\) positive masses are located at the vertices of a regular \(N\)-polygon centered on the \((N+1)\)-th positive mass, and we prove that the \(N+1\) bodies form a central configuration if and only if the positive masses located at the vertices of the regular \(N\)-polygon are equal to each other.

Keywords \((N+1)\)-body problem · Central configuration · Regular \(N\)-polygon

1 Introduction and main result

For the planar \((N+1)\)-body problem with a general homogeneous potential \(V_\alpha\), the motion of positive masses \(m_j\) at positions \(x_j(t)\) are described by

\[
  \ddot{x}_j = \frac{\partial(-V_\alpha(x))}{\partial x_j}, \quad j = 1, 2, \ldots, N+1,
\]

where

\[
  V_\alpha(x) = -\sum_{1 \leq k < j \leq N+1} \frac{m_km_j}{|x_k(t) - x_j(t)|^\alpha}, \quad \text{constant } \alpha > 0,
\]

and \(x(t) = (x_1(t), x_2(t), \ldots, x_{N+1}(t))^T\). At a given time \(t = t_0\), a configuration

\[
  x(t_0) = \{(x_1(t_0), x_2(t_0), \ldots, x_{N+1}(t_0))^T \in (\mathbb{R}^2)^{N+1} : x_k(t_0) \neq x_j(t_0) \text{ when } k \neq j\},
\]

is called a central configuration if there is some constant \(\kappa\) such that \(x(t_0)\) satisfies the following algebraic equations

\[
  \frac{\partial(-V_\alpha(x))}{\partial x_j} \big|_{t=t_0} = \kappa m_j (x_j(t_0) - c_0), \quad j = 1, 2, \ldots, N+1,
\]

where \(c_0 = \frac{\sum_{j=1}^{N+1} m_j x_j(t_0)}{\sum_{j=1}^{N+1} m_j}\) represents the mass center of the \(N+1\) bodies at the initial time \(t_0\).

The study of central configurations is a very important and interesting topic in celestial mechanics with a long and varied history, see Wintner (1941). Until now, the complete understanding of central configuration remains elusive, and it is still an actively researched area, of special note, a well-known problem in this direction is for given positive masses, are there only finitely many central configurations, and the problem is highlighted by Smale (1998) as one of his important 18 mathematical problems for the 21st century. In this paper, we study the inverse problem for (1.3) by focusing on the central configuration with some high symmetry in which \(N\) positive masses are located at the vertices of a regular \(N\)-polygon, and the \((N+1)\)-th mass is located at the geometric center of the regular \(N\)-polygon (see Fig. 1). It is well-known that \(V_1\) (i.e. \(\alpha = 1\)) is the Newtonian potential, and the corresponding motion is the classical planar Newtonian \((N+1)\)-body problem. For the limiting case that \(\alpha = 0\,
we can define $V_0(x(t)) = \sum_{1 \leq j \leq N+1} m_j m_j \log |x(t) - x_j(t)|$, which corresponds to an $(N+1)$-vortex problem, and here we only mention some previous works which have special bearing on this paper. For the 3-body $(N = 3)$ problem with the Newtonian potential, Lagrange (1772) established the well known ‘equilateral-triangle central configuration’. For the $N$-body $(N \geq 4)$ problem with the Newtonian potential, Perko and Walter (1985) proved that if $N$ positive masses are located at the vertices of a regular $N$-polygon (see Fig. 2), then they can form a regular polygonal central configuration if and only if all the masses are equal to each other. For the $N$-vortex $(N \geq 4)$ problem with the potential $V_0$, Celli et al. (2011) obtained the same result. Later, Li and Wang (2013) considered the planar central configuration of Newtonian $(N-1)$-body problem with the potential $V_1$, and they proved that if $N$ positive masses are located at the vertices of a regular $N$-polygon, then the $N$-1 masses did not form a central configuration for any value of masses (see Fig. 3). For the planar $(N+1)$-body $(N \geq 3)$ problem with the homogeneous potential $V_\alpha$ (arbitrary $\alpha > 0$), if $N$ equal masses $m_i$ ($i = 1, 2, \ldots, N$) are located at the vertices of a regular $N$-polygon and $m_{N+1}$ is located at the center of the polygon (see Fig. 1), Arribas et al. (2006) obtained that the $N+1$ bodies can form a central configuration. For other related works on regular polygonal central configurations, we refer to Ding et al. (2022), Marchesin and da Paixão (2014), Marchesin (2017, 2019), Wang (2019), Zhao and Chen (2013, 2015), Zhang and Zhou (2003), Zhu (2005).

Motivated by the above mentioned works, a natural question to ask is the following:

**Question** For the planar $(N+1)$-body configuration with the homogeneous potential $V_\alpha$ (arbitrary $\alpha > 0$) that $N$ non-equal positive masses $m_i$ are located at the vertices of a regular $N$-polygon and the $(N+1)$-th mass $m_{N+1}$ is located at the geometric center of the polygon (see Fig. 1), can we obtain the existence of the central configuration?

In this paper for $j = 1, 2, \ldots, N$, we suppose that the coordinates of $x_j(t_0)$ are the $N$-th roots of unity $e^{i\theta_j} = e^{2\pi i} = 1$. Moreover, we denote $q_j = e^{-i\frac{2\pi j}{N}}$, and assume that the $(N+1)$-th mass particle is located at the geometric center of the regular $N$-polygon, and the coordinate of $x_{N+1}(t_0)$ is 0 (see Fig. 1). The main result is described as follows.

**Theorem 1.1** For the $(N+1)$-body problem with any $\alpha > 0$ and any $N \geq 3$, the configuration made of a regular $N$-polygon with positive masses $m_1, m_2, \ldots, m_N$ located at the vertices and $m_{N+1}$ located at the geometric center of the polygon is a central configuration if and only if $m_1 = m_2 = \cdots = m_N$.

**Remark 1.1** For the $(N+1)$-body problem with homogeneous force laws, Arribas et al. (2006) obtained that if $m_1 = m_2 = \cdots = m_N > 0$, then the $N+1$ bodies can form a central configuration, but no analysis on the values of the mass $m_{N+1}$. In Theorem 1.1, we show that no matter what positive value $m_{N+1}$ takes, the $N+1$ bodies can form a central configuration.

**Remark 1.2** Li and Wang (2013) proved the non-existence of a central configuration for the planar Newtonian $(N-1)$-body problem $(\alpha = 1)$ with $N$-1 positive masses are located at the vertices of a regular $N$-polygon (see Fig. 3). For the general planar $N$-body problem (arbitrary $\alpha > 0$) with $N$ positive masses are located at the vertices of a regular $N$-polygon, Wang (2019) obtained the necessary condition of the existence of the planar central configuration (see Fig. 2). For the spatial Newtonian $2N$-body problem $(\alpha = 1)$, Wang and Li (2015) proved the necessary condi-
tion of the existence of the spatial twisted central configuration formed by two paralleled regular $N$-polygons ($N \geq 3$) with the distance was $h > 0$ and the twist angle was $\theta = 0$ (see Fig. 4). Also for the spatial Newtonian $2N$-body problem ($\alpha = 1$), Ding et al. (2022) obtained the necessary and sufficient conditions for the existence of the twisted central configuration with the twist angle was $\theta = 0$ or $\theta = \pi / N$ (see Fig. 4). Now in Theorem 1.1, for the planar $(N+1)$-body problem with the homogeneous potential $V_\alpha$ (arbitrary $\alpha > 0$), we obtain the necessary and sufficient conditions for the existence of the planar central configuration (see Fig. 1).

As a corollary of Theorem 1.1, we have

**Corollary 1.1** For the central configuration of the planar $(N+1)$-body ($N \geq 3$) problem with the homogeneous potential $V_\alpha$ (arbitrary $\alpha > 0$), if $N$ positive masses are located at the vertices of the regular $N$-polygon, and the $(N+1)$-th positive mass is located at the geometric center of the regular $N$-polygon, then the geometric center of the $N+1$ bodies is just the mass center of the $N+1$ bodies.

### 2 Some lemmas

Before giving the proof of Theorem 1.1, we need some properties about the eigenvalues and eigenvectors for circulant matrices. Let us first introduce the definition of circulant matrices.

**Definition 2.1** [Pages 65-66, Marcus and Minc (1964)] A matrix $D = (d_{jk})_{N \times N}$ ($1 \leq j, k \leq N$) is circulant if $d_{jk} = d_{j-1,k-1}$ where $d_{0,k} = d_{N,k}$ and $d_{0,0} = d_{N,N}$.

**Lemma 2.1** [Page 303, Perko and Walter (1985)] For any circulant matrix $D$, both the eigenvalues $\mu_{\alpha,j}(D)$ and the corresponding eigenvectors $\xi_j$ have the same forms, more precisely,

$$
\mu_{\alpha,j}(D) = \sum_{k=1}^{N} d_{1,k} q_{j-1}^{k-1}, \quad \xi_j = (q_{j-1}, q_{j-2}^{2}, \ldots, q_{j-1}^{N})^T,
$$

$$
q_{j-1} = e^{i\theta_j} = e^{2\pi i j / N}, \quad q_{j-1}^s = (q_{j-1})^s,
$$

(2.1)

where $j = 1, 2, \ldots, N$ and $s = 1, 2, \ldots, N$.

Let the matrix $A_\alpha$ be defined as follows:

$$
A_\alpha = (a_{k,j}), \quad a_{k,j} = \begin{cases} \frac{1 - q_{j-1}^k}{|1 - q_{j-1}|^{\alpha+2}}, & k \neq j, \\ 0, & k = j. \end{cases}
$$

Then $A_\alpha$ is a circulant matrix. By Lemma 2.1 we have the following results.

**Lemma 2.2** [Corollary 3 and Lemma 5, Wang (2019)] The eigenvalues of matrix $A_\alpha$ have the property that for any $\alpha > 0$ and any $N \geq 4$, we have $\mu_{\alpha,k} > 0$ except that $\mu_{\alpha, N+1} = 0$ for odd $N$.

**Lemma 2.3** [Page 305, Perko and Walter (1985)] The eigenvectors $\xi_j$ ($j = 1, 2, \ldots, N$) of any $N \times N$ circulant matrix forms a basis of $\mathbb{C}^N$.

**Lemma 2.4** [Pages 65, Marcus and Minc (1964)] Let $(\tilde{\xi}_k)^T$ be the conjugate transpose of $\tilde{\xi}_k$. Then

\[
(\tilde{\xi}_k)^T \xi_j = \begin{cases} N, & k = j, \\ 0, & k \neq j. \end{cases}
\]

(2.2)

Moreover, we introduce another useful lemma.

**Lemma 2.5** [Lemma 1, Wang (2019)] Let $q_j$ be given in Sect. 1, then

$$
\frac{1}{N} \sum_{j=1}^{N-1} \frac{1 - q_j}{|1 - q_j|^{\alpha+2}} = \frac{1}{\sum_{j=1}^{N-1} \frac{1}{|1 - q_j|^{\alpha+2}}}. 
$$

Based upon Lemma 2.5 we have

**Lemma 2.6** \(\frac{1}{N} \sum_{1 \leq k < j \leq N} \frac{1}{|q_k - q_j|^{\alpha}} = \sum_{1 \leq j < N} \frac{1}{|1 - q_j|^{\alpha+2}}\).

**Proof** One computes that

$$
2 \sum_{1 \leq k < j \leq N} \frac{1}{|q_k - q_j|^{\alpha}} = \sum_{1 \leq k \neq j \leq N} \frac{1}{|q_k - q_j|^{\alpha}}.
$$
Employing (2.3) and (2.4), we see that
\[
2 \sum_{1 \leq k < j \leq N} \left| q_k - q_j \right|^a = \sum_{1 \leq k \neq j \leq N} \frac{1}{ \left| q_k - q_j \right|^a }
= N \sum_{1 \leq j \leq N-1} \frac{1}{ \left| q_j \right|^a - 2 \cos \frac{2j \pi}{N} \left| q_j \right|^a }
= N \sum_{1 \leq j \leq N-1} \frac{1}{ \left| q_j \right|^a \left( 2 \cos \frac{2j \pi}{N} \right) ^a },
\]
and it implies that
\[
\frac{1}{N} \sum_{1 \leq k < j \leq N} \frac{1}{ \left| q_k - q_j \right|^a } = \sum_{1 \leq j \leq N-1} \frac{1}{ \left( 2 \cos \frac{2j \pi}{N} \right) ^a }.
\]
Then employing Lemma 2.5, we complete the proof. \( \square \)

### 3 Proof of Theorem 1.1

By (1.3), we have
\[
\frac{\partial (-V_\alpha(x))}{\partial x_j} |_{x_0} = \alpha \sum_{k \neq j \leq N+1} \frac{m_km_j (x_k(t_0) - x_j(t_0))}{|x_k(t_0) - x_j(t_0)|^{a+2}}
= \kappa m_j (x_j(t_0) - c_0), \quad j = 1, 2, \ldots, N+1,
\]
which implies that a configuration
\[
x(t_0) = (x_1(t_0), x_2(t_0), \ldots, x_{N+1}(t_0))^T \in (\mathbb{R}^2)^{N+1} : \\
x_k(t_0) \neq x_j(t_0) \text{ when } k \neq j,
\]
is called a central configuration if there exists some constant \( \lambda = -\kappa/\alpha \) such that
\[
\sum_{k \neq j \leq N+1} \frac{m_km_j (x_k(t_0) - x_j(t_0))}{|x_k(t_0) - x_j(t_0)|^{a+2}} \\
= -\lambda m_j (x_j(t_0) - c_0), \quad j = 1, 2, \ldots, N+1.
\]

We choose the geometric center of the regular \( N \)-polygon as the origin of our coordinate system. The proof Theorem 1.1 is divided into two parts.

**Part 1.** We prove that for the \((N+1)\)-body central configuration that \( N \) masses \( m_i \) \((i = 1, 2, \ldots, N)\) are located at the vertices of a regular \( N \)-polygon and \( m_{N+1} \) is located at the geometric center of the polygon, then \( m_1 = m_2 = \cdots = m_N \).

In the proof of Part 1, we divide the proof into two cases.

**Case 1:** \( N = 3 \).
Observing that (3.1) is equivalent to
\[
\sum_{1 \leq k \leq N, k \neq j} \frac{m_k m_j (q_k - q_j)}{|q_k - q_j|^{\alpha + 2}} + \frac{m_4 m_j (x_4(t_0) - q_j)}{|x_4(t_0) - q_j|^{\alpha + 2}} = -\lambda m_j (q_j - c_0),
\]
\[j = 1, 2, 3, \quad \sum_{1 \leq k \leq N} \frac{m_m m_4}{|q_k - x_4(t_0)|^{\alpha + 2}} (q_k - x_4(t_0)) = -\lambda m_4 (x_4(t_0) - c_0),\]
\[j = 4. \tag{3.2}\]
Notice that \(x_4(t_0) = 0\), \(q_3 = 1\) and \(q_{-k} = q_{-k+3s}\) \((k = 1, 2, 3, s \in \mathbb{Z})\), then from (3.2) we obtain
\[
\begin{pmatrix}
0 & \frac{1 - q_1}{1 - q_1|^{\alpha + 2}} & \frac{1 - q_2}{1 - q_2|^{\alpha + 2}} & 1 \\
\frac{1 - q_1}{1 - q_1^{\alpha + 2}} & 0 & \frac{1 - q_2}{1 - q_2^{\alpha + 2}} & 1 \\
q_1 & \frac{1 - q_2}{1 - q_2^{\alpha + 2}} & 0 & 1 \\
q_2 & q_3 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
m_1 \\
m_2 \\
m_3 \\
m_4
\end{pmatrix}
= \begin{pmatrix}
\lambda - \lambda c_0 q_2 \\
\lambda - \lambda c_0 q_1 \\
\lambda - \lambda c_0 \\
\lambda c_0
\end{pmatrix}.
\tag{3.3}
\]
Therefore,
\[
\left[\frac{1 - q_1}{1 - q_1|^{\alpha + 2}} m_1 + \frac{1 - q_2}{1 - q_2^{\alpha + 2}} m_2 + m_4\right] + (m_1 q_1 + m_2 q_2 + m_3 q_3) = [\lambda - \lambda c_0] + \lambda c_0 = \lambda.
\tag{3.4}
\]
Since \(q_1 = e^{j2\pi/3}\) and \(q_2 = e^{j2\pi/3}\), we conclude that \(Re(q_1) = Re(q_2)\) and \(Im(q_1) = -Im(q_2) = \sqrt{3}/2\). Thus, \(1/|1 - q_1|^3 = 1/|1 - q_2|^3\). Note that \(m_1, m_2, m_3, m_4 \in \mathbb{R}\), then combining (3.4), we have
\[
-\frac{\sqrt{3}}{2} \left[\frac{1}{1 - q_1|^{\alpha + 2}} (m_1 - m_2) + \frac{\sqrt{3}}{2} i (m_1 - m_2) = 0,
\]
which implies that \(m_1 = m_2 = m\).

Next we will prove that \(m_3 = m\). By (3.3) and \(q_3 = 1\), we see that
\[
\begin{cases}
(m_3 - m) \frac{1 - q_2}{1 - q_2|^{\alpha + 2}} + (m - m_3) \frac{1 - q_1}{1 - q_1|^{\alpha + 2}} = \lambda c_0 (q_1 - q_2), \\
q_1 m_1 + q_2 m_2 + m_3 = \lambda c_0.
\end{cases}
\tag{3.5}
\]
From the representations of \(q_1\) and \(q_2\), then \(q_1^2 = q_2\) and \(q_2^2 = q_1\). It follows from the second identity in (3.5) that
\[
\lambda c_0 (q_1 - q_2) = [m q_1 + m q_2 + m_3] (q_1 - q_2) = (m - m) (q_1 - q_2) = (m_3 - m) (q_1 - q_2).
\]
By the first part of system (3.5), it follows that
\[
(m_3 - m) \frac{1 - q_2}{1 - q_2|^{\alpha + 2}} + (m - m_3) \frac{1 - q_1}{1 - q_1|^{\alpha + 2}} = -m_3 (q_1 - q_2) \]
\[
= (m_3 - m) (q_1 - q_2),
\]
\[
= (m_3 - m) \left[\frac{1}{1 - q_1|^{\alpha + 2}} - 1\right] (q_1 - q_2)
\]
\[
= (m_3 - m) \left[\frac{1}{(\sqrt{3})^{\alpha + 2}} - 1\right] \sqrt{3} i = 0.
\]
Hence \(\alpha > 0\) yields \(m_3 = m\), which implies that we complete the proof of Part 1 for the case of \(N = 3\).

**Case 2:** \(N \geq 4\).

Since the \((N+1)\)-th mass is located at the geometric center of the regular \(N\)-polygon, we have \(x_{N+1}(t_0) = 0\) and it suggests that
\[
\begin{aligned}
c_0 &= \sum_{j=1}^{N+1} m_j x_j (t_0) = m_{N+1} x_{N+1}(t_0) + \sum_{j=1}^{N} m_j q_j \\
&= \frac{\sum_{j=1}^{N+1} m_j}{\sum_{j=1}^{N} m_j}.
\end{aligned}
\tag{3.6}
\]
By (3.6), \(x_{N+1}(t_0) = 0\) and \(|q_k| = |e^{i\theta_k}| = 1\) with \(k = 1, 2, \ldots, N\), we gain that system (3.1) is equivalent to
\[
\begin{cases}
\sum_{k \neq j} \frac{m_k}{|q_k - q_j|^{\alpha + 2}} (q_k - q_j) = -m_{N+1} q_j \\
\sum_{k \neq j} \frac{m_k}{|q_k - q_j|^{\alpha + 2}} (q_k - q_j) = -\lambda \left[ q_j - \sum_{k=1}^{N} \frac{m_k q_k}{|q_j - q_k|^{\alpha + 2}} \right], \quad j \neq N + 1, \\
\sum_{k=1}^{N} m_k q_k = \lambda \sum_{k=1}^{N} m_k q_k, \quad j = N + 1.
\end{cases}
\tag{3.7}
\]
From the second equation in (3.7), then \(\lambda = \sum_{k=1}^{N+1} m_k\) or \(\sum_{k=1}^{N} m_k q_k = 0\). In the following, we will discuss two subcases.

**Case 2.1:** \(\lambda = \sum_{k=1}^{N+1} m_k\).

By the first equation in (3.7), we have
\[
\sum_{k \neq j} \frac{q_k - q_j}{|q_k - q_j|^{\alpha + 2}} m_k
\]
Multiplying both sides of (3.8) by $q_{-j}$, we acquire
\[
\sum_{k \neq j, 1 \leq k \leq N} \frac{1 - q_{k-j}^{-1}}{|q_k - q_j|^{a+2}} m_k = \sum_{k=1}^{N} m_k - \sum_{k=1}^{N} m_k q_{k-j}, \quad j \neq N + 1,
\]
then it follows from $|q_k - q_j|^{a+2} = 2^{a+2} |\sin(k - j) \pi / N|^{a+2} = |1 - q_{k-j}^{-1}|^{a+2}$ that
\[
\sum_{k \neq j, 1 \leq k \leq N} \frac{1 - q_{k-j}^{-1}}{|1 - q_k^{-1}|^{a+2}} m_k = \sum_{k=1}^{N} m_k - \sum_{k=1}^{N} m_k q_{k-j}, \quad j \neq N + 1.
\]

Let the circulant matrix $A_\alpha$ be given by (2.2). With the help of (3.9) and Lemma 2.1, we see that
\[
A_\alpha (m_1, m_2, \ldots, m_N)^T = \left( \sum_{k=1}^{N} m_k \right) \xi_1 - \left( \sum_{k=1}^{N} m_k q_k \right) (q_{-1}, q_2^2, \ldots, q_{-N}^2),
\]
where eigenvector $\xi_1$ is defined in (2.1).

Observing that $q_{j}^{-1} = e^{-j \theta / N} = e^{-j \frac{2 \pi j}{N}}$ and $q_{j}^{N-1} = (e^{j \theta})^{N-1} = e^{j \frac{2 \pi (N-1) j}{N}}$, thus $q_{-j} = q_{j}^{N-1}$. Therefore, (3.10) is equivalent to
\[
A_\alpha (m_1, m_2, \ldots, m_N)^T = \left( \sum_{k=1}^{N} m_k \right) \xi_1 - \left( \sum_{k=1}^{N} m_k q_k \right) \xi_N,
\]
where eigenvectors $\xi_1$ and $\xi_N$ are defined in (2.1).

On the other hand, from Lemma 2.3, there exist $\beta_1, \beta_2, \ldots, \beta_N \in \mathbb{C}$ such that
\[
(m_1, m_2, \ldots, m_N)^T = \beta_1 \xi_1 + \beta_2 \xi_2 + \cdots + \beta_N \xi_N,
\]
where eigenvectors $\xi_2, \ldots, \xi_{N-1}$ are defined in (2.1).

By (3.11) and (3.12), there exist eigenvalues $\mu_1, \mu_2, \ldots, \mu_N \in \mathbb{C}^N$ such that
\[
A_\alpha (\beta_1 \xi_1 + \beta_2 \xi_2 + \cdots + \beta_N \xi_N) = \beta_1 A_\alpha \xi_1 + \beta_2 A_\alpha \xi_2 + \cdots + \beta_N A_\alpha \xi_N
\]
\[
= \mu_1 \beta_1 \xi_1 + \mu_2 \beta_2 \xi_2 + \cdots + \mu_N \beta_N \xi_N
\]
\[
= \left( \sum_{k=1}^{N} m_k \right) \xi_1 - \left( \sum_{k=1}^{N} m_k q_k \right) \xi_N.
\]
From Lemma 2.4, we have that eigenvectors $\xi_1, \xi_2, \ldots, \xi_N$ are linearly independent. Then combining (3.13), we have $\mu_1, \mu_2, \mu_3 = \cdots = \mu_N, \mu_{N-1} \beta_{N-1} = 0$. With the help of Lemma 2.2, we have $\mu_{a, k} 
eq 0$ except that $\mu_{a, \frac{N+1}{2}} = 0$, which implies that
\[
\beta_k = 0, \quad \text{where } k = 2, 3, \ldots, N-1 \text{ and } k \neq \frac{N+1}{2}.
\]
If $N$ is even, combining (3.12) and (3.14), it yields that
\[
(m_1, m_2, \ldots, m_N)^T - \beta_1 \xi_1 = \beta_N \xi_N,
\]
where $\xi_1 = (1, 1, \ldots, 1)^T$ and $\xi_N = (q_{N-1}, q_{N-2}^2, \ldots, q_{N-1}^{N-1})$. We set $\beta_1 = a_1 + ib_1$ and $\beta_N = a_2 + ib_2$. Notice that $(m_1, m_2, \ldots, m_N)^T$ is a real vector, then employing (3.15) we see that $(a_2 + ib_2)q_j + ib_1$ are real numbers for $0 \leq j \leq N-1$. Observing that
\[
(a_2 + ib_2)q_j + ib_1
\]
\[
= (a_2 + ib_2)e^{j \frac{2 \pi}{N}} + ib_1
\]
\[
= a_2 \cos \left( \frac{2 \pi j}{N} \right) - b_2 \sin \left( \frac{2 \pi j}{N} \right)
\]
\[
+ i \left[ b_2 \cos \left( \frac{2 \pi j}{N} \right) + a_2 \sin \left( \frac{2 \pi j}{N} \right) + b_1 \right],
\]
therefore,
\[
b_2 \cos \left( \frac{2 \pi j}{N} \right) + a_2 \sin \left( \frac{2 \pi j}{N} \right) + b_1 = 0, \quad j = 0, 1, \ldots, N-1.
\]
By choosing $j = 0$ and $j = N/2$, we conclude that
\[
b_2 + b_1 = 0, \quad -b_2 + b_1 = 0.
\]
Thus $b_2 = b_1 = 0$. By (3.15) it follows that
\[
(m_1 - a_1, m_2 - a_1, \ldots, m_N - a_1)^T = a_2 \xi_N,
\]
which implies that $a_2 e^{2 j \frac{\pi}{N}}$ are real numbers for $j = 0, 1, \ldots, N-1$. Since $N \geq 4$, then $a_2 = 0$. Hence $(m_1, m_2, \ldots, m_N)^T = a_1 \xi_1$, and so $m_1 = m_2 = \cdots = m_N$.

If $N$ is odd, by (3.12) and (3.14), we have
\[
(m_1, m_2, \ldots, m_N)^T = \beta_1 \xi_1 + \beta_N \xi_N + \beta_{N+1} \xi_{N+1}.
\]
So
\[
(m_1, m_2, \ldots, m_N)^T - \beta_1 \xi_1 = \beta_N \xi_N + \beta_{N+1} \xi_{N+1},
\]
The central configuration of the planar \((N+1)\)-body problem with a regular \(N\)-polygon for homogeneous force laws

where

\[
\frac{\xi}{N^2} = (q_{\frac{N+1}{2}} - q_{\frac{N+1}{2} - 1}, \ldots, q_{\frac{N}{2} + 1} - q_{\frac{N}{2} - 1})^T.
\]

We set \(\beta_1 = a_1 + ib_1, \beta_N = a_2 + ib_2\) and \(\beta_{N+1} = a_3 + ib_3\). Then by \((3.16)\), we know that \((a_2 + ib_2)q_{N-j} + (a_3 + ib_3)q_{\frac{N}{2} - j + 1} + ib_1\) are real numbers for \(1 \leq j \leq N\). Notice that

\[
(a_2 + ib_2)q_{N-j} + (a_3 + ib_3)q_{\frac{N}{2} - j + 1} + ib_1
\]

\[= (a_2 + ib_2)e^{\frac{2\pi j(N-j)}{N}} + (a_3 + ib_3)e^{\frac{2\pi j(N-j)}{N}} + ib_1\]

\[= a_2 \cos\left(\frac{2j\pi}{N}\right) + a_2 \sin\left(\frac{2j\pi}{N}\right) + (-1)^j (a_3 \cos\left(\frac{j\pi}{N}\right) - a_3 \sin\left(\frac{j\pi}{N}\right)) + b_1,
\]

therefore,

\[b_2 \cos\left(\frac{2j\pi}{N}\right) - a_2 \sin\left(\frac{2j\pi}{N}\right) + (-1)^j (b_3 \cos\left(\frac{j\pi}{N}\right) - a_3 \sin\left(\frac{j\pi}{N}\right)) + b_1 = 0, \quad j = 1, 2, \ldots, N.
\]

From \(N \geq 4\) and \(N\) is odd, we have \(N \geq 5\). By choosing \(j = N, N-1, N-2, 2, 1\), we derive the following linear equations

\[
\begin{aligned}
b_2 + b_3 + b_1 &= 0, \\
b_2 \cos\left(\frac{2\pi}{N}\right) + a_2 \sin\left(\frac{2\pi}{N}\right) - b_3 \cos\left(\frac{2\pi}{N}\right) - a_3 \sin\left(\frac{2\pi}{N}\right) + b_1 &= 0, \\
b_2 \cos\left(\frac{2\pi}{N}\right) + a_2 \sin\left(\frac{2\pi}{N}\right) + b_3 \cos\left(\frac{2\pi}{N}\right) + a_3 \sin\left(\frac{2\pi}{N}\right) + b_1 &= 0, \\
b_2 \cos\left(\frac{2\pi}{N}\right) - a_2 \sin\left(\frac{2\pi}{N}\right) + b_3 \cos\left(\frac{2\pi}{N}\right) - a_3 \sin\left(\frac{2\pi}{N}\right) + b_1 &= 0, \\
b_2 \cos\left(\frac{2\pi}{N}\right) - a_2 \sin\left(\frac{2\pi}{N}\right) - b_3 \cos\left(\frac{2\pi}{N}\right) + a_3 \sin\left(\frac{2\pi}{N}\right) + b_1 &= 0.
\end{aligned}
\]

(3.17)

We deduce from \((3.17)\) that

\[
b_1 = b_2 = b_3 = a_2 = a_3 = 0.
\]

(3.18)

By \((3.16), (3.18)\), \(\beta_N = a_2 + ib_2\) and \(\beta_{N+1} = a_3 + ib_3\), we arrive at \((m_1, m_2, \ldots, m_N)^T = a_1 \xi_1\), and so \(m_1 = m_2 = \cdots = m_N\).

Altogether, if \(\lambda = \sum_{k=1}^{N+1} m_k\), we conclude that \(m_1 = m_2 = \cdots = m_N\).

Case 2.2. \(\sum_{k=1}^{N} m_k q_k = 0\).

Recall from the first part of system (3.7), we have

\[
\sum_{\substack{k \neq j \atop 1 \leq k \leq N}} \frac{q_k - q_j}{|q_k - q_j|^2} m_k = (-\lambda + m_{N+1})q_j, \quad j \neq N + 1.
\]

(3.19)

Multiplying both sides of (3.19) by \(q_{-j}\), we acquire

\[
\sum_{\substack{k \neq j \atop 1 \leq k \leq N}} \frac{1 - q_k - j}{|q_k - q_j|^2} m_k = \lambda - m_{N+1}, \quad j \neq N + 1,
\]

it then follows from \(|q_k - q_j|^2 = 2\alpha^2 |\sin((k - j)\pi/N)|\) that

\[
\sum_{\substack{k \neq j \atop 1 \leq k \leq N}} \frac{1 - q_k - j}{|q_k - q_j|^2} m_k = \lambda - m_{N+1}, \quad j \neq N + 1,
\]

and it is equivalent to

\[
A_\alpha (m_1, m_2, \ldots, m_N)^T = (\lambda - m_{N+1}) \xi_1.
\]

(3.20)

Employing (3.12) and (3.20), there exist eigenvalues \(\mu_{\alpha, 1}, \mu_{\alpha, 2}, \ldots, \mu_{\alpha, N} \in \mathbb{C}\) such that

\[
A_\alpha (\beta_1 \xi_1 + \beta_2 \xi_2 + \cdots + \beta_N \xi_N) = A_\alpha \beta_1 \xi_1 + A_\alpha \beta_2 \xi_2 + \cdots + A_\alpha \beta_N \xi_N
\]

\[= \mu_{\alpha, 1} \beta_1 \xi_1 + \mu_{\alpha, 2} \beta_2 \xi_2 + \cdots + \mu_{\alpha, N} \beta_N \xi_N
\]

\[= (\lambda - m_{N+1}) \xi_1.
\]

Then similar to the procedure of Case 2.1, we are able to obtain \(m_1 = m_2 = \cdots = m_N\).

Part 2. Under the assumption that the \((N+1)\)-th \((N \geq 3)\) mass is located at the geometric center of the regular \(N\)-polygon, we prove that if \(m_1 = m_2 = \cdots = m_N\), then no matter what positive value \(m_{N+1}\) takes, all the \(N+1\) bodies can form a central configuration.

In fact, if both \((3.2)\) and \((3.7)\) hold for \(N \geq 3\), then all the \(N+1\) bodies form a central configuration. Hence it suffice to prove \((3.2)\) and \((3.7)\) for \(N \geq 3\). Note that \(m_1 = m_2 = \cdots = m_N\) and the \((N+1)\)-th mass is located at the geometric center of the regular \(N\)-polygon, and the mass center \(c_0 = x_{N+1}(t_0) = 0, \) the second parts of \((3.2)\) and \((3.7)\) hold. It remains to check the first parts of \((3.2)\) and \((3.7)\).

Firstly, by the definition of central configuration in \((1.3)\), then

\[
\kappa = \frac{\lambda}{\alpha} = \frac{V_0(x)}{\alpha f(x)}|_{x = t_0},
\]

(3.21)
where the homogeneous potential \( V_\alpha (x(t_0)) \) and the moment of the inertia \( I(x(t_0)) \) are given by

\[
\begin{cases}
V_\alpha (x(t_0)) = -\sum_{1 \leq k < j \leq N+1} \frac{m_km_j}{|x_k(t_0) - x_j(t_0)|^\alpha}, \\
I(x(t_0)) = \sum_{1 \leq j \leq N+1} m_j |x_j(t_0) - c_0|^2.
\end{cases}
\tag{3.22}
\]

In fact, by (3.1), \( m_1 = m_2 = \cdots = m_N, \ c_0 = x_{N+1}(t_0) = 0 \) and

\[
\partial (-V_\alpha (x)) |_{x=0} = \alpha \sum_{1 \leq k < j \leq N+1} \frac{m_km_j}{|x_k(t_0) - x_j(t_0)|^\alpha},
\]

\( j = 1, 2, \ldots, N + 1, \)

we deduce that \( \lambda \) must satisfy

\[
\frac{1}{\alpha} \frac{\partial V_\alpha (x)}{\partial x_j} |_{x=0} = \lambda m_j x_j(t_0), \quad j = 1, 2, \ldots, N + 1. \tag{3.23}
\]

By the fact that the potential \( V_\alpha \) is homogeneous of degree \(-\alpha\), we have

\[
(\{x_1(t_0), x_2(t_0), \ldots, x_N(t_0), x_{N+1}(t_0)\})_{1 \times (N+1)}
\]

\[
\times \begin{pmatrix}
\frac{\partial V_\alpha (x)}{\partial x_1} |_{x=0} \\
\frac{\partial V_\alpha (x)}{\partial x_2} |_{x=0} \\
\vdots \\
\frac{\partial V_\alpha (x)}{\partial x_{N+1}} |_{x=0}
\end{pmatrix}_{(N+1) \times 1}
\]

\[
= -\alpha V_\alpha (x)|_{x=0},
\]

Then combining with (3.23), \( \kappa = -\lambda/\alpha \) and (1.3), we obtain (3.21).

Secondly, we denote the mass of \( m_k \) (1 \( \leq k \leq N \)) by \( m \) and assume \( m_{N+1} = bm \) where both \( m \) and \( b \) are arbitrary positive constants. By \( c_0 = x_{N+1}(t_0) = 0 \), (3.21), (3.22) and \( x_j(t_0) = q_j \) (\( j = 1, 2, \ldots, N \)), we have

\[
\lambda = -\alpha \kappa = -\frac{V_\alpha(x)}{I(x)}|_{x=0} = m \sum_{1 \leq j \leq N} \frac{1}{|q_j|^2 + bm}.
\]

Let \( N = 3 \) in (3.24), we have

\[
\sum_{1 \leq j \leq 3} m_j m_k (q_k - q_j) \left| \frac{1}{|q_j|^2 + bm} \right| \leq \sum_{1 \leq j \leq 3} m_j m_k (q_k - q_j) \left| \frac{1}{|q_j|^2} \right|,
\]

\[
= -m q_j (m \sum_{1 \leq j \leq 3} \frac{1}{|q_j|^2 + bm}) = -\lambda m q_j,
\]

which means that the first part of (3.2) hold for \( N = 3 \). Similarly, we prove the first part of (3.7) for \( N \geq 4 \). Thus for any \( m > 0 \) and any \( b > 0 \), the \((N+1)\)-body configuration made of a regular \( N \)-polygon with positive equal-mass \( m_1 = m_2 = \cdots = m_N = m \) located at the vertices and an arbitrary positive mass \( m_{N+1} = bm \) located at the geometric center of the polygon is a central configuration. \( \blacksquare \)

Remark 3.1 In the proof of Case 2.1 in Part 1, the assumption that \( N \geq 4 \) is needed to obtain (3.18), and that is why we should divide the proof of Theorem 1.1 into two cases: \( N = 3 \) and \( N \geq 4 \).

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Declarations

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