About calculation of traces of Dirac $\gamma$-matrices contracted with massless vectors in Minkowski space

Alexander L. Bondarev\textsuperscript{1} and Alexander R. Roslik\textsuperscript{2}

National Scientific and Educational Center of Particle and High Energy Physics of the Belarusian State University
M.Bogdanovich str.,153, Minsk 220050, Belarus
\textsuperscript{1}e-mail: bondarev@hep.by
\textsuperscript{2}e-mail: roslik@hep.by

Abstract
A new method for calculation of traces of Dirac $\gamma$-matrices contracted with massless vectors in Minkowski space is discussed

1 Introduction
Calculation of Dirac $\gamma$-matrices traces is standard procedure in high energy physics computation. However the expressions for the traces of more than ten $\gamma$-matrices are too long.

Application of Chisholm [1] identities and Kahane [2] algorithm leads to short expressions for traces in 4-spacetime. More compact formulae can be obtained through the new method proposed in [3] and based on the properties of orthonormal bases in the Minkowski space and isotropic tetrads (see e.g. [4]-[5]) constructed from the vectors of these bases.

The purpose of the article is further simplification of the formulae presented in [3] for the case of Dirac $\gamma$-matrices contracted with massless vectors. This task is important today because most of analytical calculations in high energy physics are performed at massless approximation (see, for example, [6]).

The article is arranged in the following way. The new method of trace computation for arbitrary vectors is shortly described in Section 2. In Section 3 one explores application of the traces calculation method for Dirac $\gamma$-matrices contracted with massless vectors. The formulae obtained there are good enough to realize an algorithm for any numerical calculation.

Note, that the method of calculation of the traces from the [3] is included in the system of analytical computation ALHEP [8], and this article contains the proposals for further improvement of the similar programs.
In this paper we use the Feynman metrics:

\( \mu = 0, 1, 2, 3, \quad a^\mu = (a_0, \vec{a}), \quad a_\mu = (a_0, -\vec{a}), \quad (ab) = a_\mu b^\mu = a_0 b_0 - \vec{a} \vec{b}, \)

and a sign of the Levi-Civita tensor is fixed as

\( \varepsilon_{0123} = +1. \)

Orientation of the orthonormal basis vectors in Minkowski space

\( (l_A l_B) = g_{AB}, \quad (A, B = 0, 1, 2, 3) \)

is constrained by

\( \varepsilon_{\mu \nu \rho \lambda} l_\mu^0 l_\nu^1 l_\rho^2 l_\lambda^3 = +1. \)

Isotropic tetrads \( q_\pm, e_\pm \) are constructing in the following way:

\[
q_+ = \frac{l_0 + l_1}{\sqrt{2}}, \quad q_- = \frac{l_0 - l_1}{\sqrt{2}}, \quad (q_+ q_-) = 1; \quad e_+ = \frac{l_2 + il_3}{\sqrt{2}}, \quad e_- = \frac{l_2 - il_3}{\sqrt{2}}, \quad (e_+ e_-) = -1, \quad e_\pm^* = e_\pm; \\
(q_+ e_+) = (q_- e_-) = (q_- e_+) = (q_- e_-) = 0. \tag{1}
\]

2 The new method of trace calculation for Dirac \( \gamma \)-matrices contracted with arbitrary vectors

The new method to calculate traces of Dirac \( \gamma \)-matrices contracted with arbitrary vectors \( a_i \ (i = 1, \cdots, 2n) \) in Minkowski space is presented in [3]. So the following expressions were obtained:

\[
\frac{1}{2} \text{Tr}[(1 - \gamma^5)\hat{a}_1 \hat{a}_2] = F_1(a_1, a_2) + F_3(a_1, a_2); \tag{2}
\]

\[
\frac{1}{2} \text{Tr}[(1 - \gamma^5)\hat{a}_1 \hat{a}_2 \hat{a}_3] = \\
= F_1(a_1, a_2)F_1(a_3, a_4) + F_2(a_1, a_2)F_3(a_3, a_4) + F_3(a_1, a_2)F_3(a_3, a_4) + F_4(a_1, a_2)F_2(a_3, a_4); \tag{3}
\]

where \( F_1, F_2, F_3, F_4 \) are some functions (see below) of vectors \( a_i \).

Having calculated

\[
\text{Tr}[(1 - \gamma^5)\hat{a}_1 \cdots \hat{a}_{2n-1} \hat{a}_{2n}],
\]

we may obtain an expression for

\[
\text{Tr}[(1 - \gamma^5)\hat{a}_1 \cdots \hat{a}_{2n-1} \hat{a}_{2n} \hat{a}_{2n+1} \hat{a}_{2n+2}]
\]
from the previous one by the following replacement:

\[ F_1(a_{2n-1}, a_{2n}) \to F_1(a_{2n-1}, a_{2n})F_1(a_{2n+1}, a_{2n+2}) + F_2(a_{2n-1}, a_{2n})F_4(a_{2n+1}, a_{2n+2}) , \]

\[ F_3(a_{2n-1}, a_{2n}) \to F_3(a_{2n-1}, a_{2n})F_3(a_{2n+1}, a_{2n+2}) + F_4(a_{2n-1}, a_{2n})F_2(a_{2n+1}, a_{2n+2}) , \]

\[ F_2(a_{2n-1}, a_{2n}) \to F_1(a_{2n-1}, a_{2n})F_2(a_{2n+1}, a_{2n+2}) + F_2(a_{2n-1}, a_{2n})F_3(a_{2n+1}, a_{2n+2}) , \]

\[ F_4(a_{2n-1}, a_{2n}) \to F_3(a_{2n-1}, a_{2n})F_4(a_{2n+1}, a_{2n+2}) + F_4(a_{2n-1}, a_{2n})F_1(a_{2n+1}, a_{2n+2}) . \] (4)

The expression for

\[ \text{Tr}[(1 - \gamma^5) \hat{a}_1 \cdots \hat{a}_{2n-1} \hat{a}_{2n}] \]

obtained through the method will contain \( 2^n \) terms.

\[ \text{Tr}[(1 + \gamma^5) \hat{a}_1 \hat{a}_2 \cdots \hat{a}_{2n}] = (\text{Tr}[(1 - \gamma^5) \hat{a}_1 \hat{a}_2 \cdots \hat{a}_{2n}])^* . \] (5)

The functions mentioned above have the following forms:

\[ F_1(a_i, a_j) = 2[(a_i q_-)(a_j q_+)-(a_i e_+)(a_j e_-)] = \]

\[ = (a_i a_j) + G \left( \begin{array}{ll} a_i & a_j \\ l_0 & l_1 \end{array} \right) + i \ G \left( \begin{array}{ll} a_i & a_j \\ l_2 & l_3 \end{array} \right) = \] (6)

\[ = \frac{1}{4} \text{Tr}[(1 - \gamma^5) \hat{q}_+ \hat{q}_- \hat{a}_i \hat{a}_j] = - \frac{1}{4} \text{Tr}[(1 - \gamma^5) \hat{e}_- \hat{e}_+ \hat{a}_i \hat{a}_j] , \]

\[ F_3(a_i, a_j) = 2[(a_i q_+)(a_j q_-)-(a_i e_-)(a_j e_+)] = \]

\[ = (a_i a_j) - G \left( \begin{array}{ll} a_i & a_j \\ l_0 & l_1 \end{array} \right) - i \ G \left( \begin{array}{ll} a_i & a_j \\ l_2 & l_3 \end{array} \right) = \] (7)

\[ = \frac{1}{4} \text{Tr}[(1 - \gamma^5) \hat{q}_+ \hat{q}_- \hat{a}_i \hat{a}_j] = - \frac{1}{4} \text{Tr}[(1 - \gamma^5) \hat{e}_+ \hat{e}_- \hat{a}_i \hat{a}_j] , \]

\[ F_2(a_i, a_j) = 2[(a_i e_+)(a_j q_-)-(a_i q_-)(a_j e_+)] = 2 \ G \left( \begin{array}{ll} a_i & a_j \\ e_+ & q_- \end{array} \right) = \]

\[ = - G \left( \begin{array}{ll} a_i & a_j \\ l_0 & l_2 \end{array} \right) + i \ G \left( \begin{array}{ll} a_i & a_j \\ l_1 & l_3 \end{array} \right) + G \left( \begin{array}{ll} a_i & a_j \\ l_2 & l_1 \end{array} \right) - i \ G \left( \begin{array}{ll} a_i & a_j \\ l_0 & l_3 \end{array} \right) = \] (8)

\[ = \frac{1}{4} \text{Tr}[(1 - \gamma^5) \hat{q}_- \hat{e}_+ \hat{a}_i \hat{a}_j] , \]

\[ F_4(a_i, a_j) = 2[(a_i e_-)(a_j q_+)-(a_i q_+)(a_j e_-)] = 2 \ G \left( \begin{array}{ll} a_i & a_j \\ e_- & q_+ \end{array} \right) = \]

\[ = - G \left( \begin{array}{ll} a_i & a_j \\ l_0 & l_2 \end{array} \right) + i \ G \left( \begin{array}{ll} a_i & a_j \\ l_1 & l_3 \end{array} \right) - G \left( \begin{array}{ll} a_i & a_j \\ l_2 & l_1 \end{array} \right) + i \ G \left( \begin{array}{ll} a_i & a_j \\ l_0 & l_3 \end{array} \right) = \] (9)

\[ = \frac{1}{4} \text{Tr}[(1 - \gamma^5) \hat{q}_+ \hat{e}_- \hat{a}_i \hat{a}_j] , \]
where $G$ are Gram determinants.

There is an example of $F_i$ functions for some orthonormal basis. Let’s to fix the last as follow:

$$l_i^\mu = (1, 0, 0, 0), \quad l_i^\nu = (0, 1, 0, 0), \quad l_i^\rho = (0, 0, 1, 0), \quad l_3^\mu = (0, 0, 0, 1).$$

Then the isotropic tetrads can be expressed as

$$q^\mu_\pm = \frac{1}{\sqrt{2}}(1, \pm 1, 0, 0), \quad (aq^\pm) = \frac{1}{\sqrt{2}}(a_0 \mp a_x),$$

$$e^\mu_\pm = \frac{1}{\sqrt{2}}(0, 0, 1, \pm i), \quad (ae^\pm) = -\frac{1}{\sqrt{2}}(a_y \pm ia_z).$$

At last

$$F_1(a_i, a_j) = [ (a_i)_{0}(a_j)_{x} - (a_i)_{x}(a_j)_{0} ] + i [ (a_i)_{y}(a_j)_{z} - (a_i)_{z}(a_j)_{y} ] ,$$

$$F_3(a_i, a_j) = [ (a_i)_{0}(a_j)_{y} - (a_i)_{y}(a_j)_{0} ] - i [ (a_i)_{x}(a_j)_{z} - (a_i)_{z}(a_j)_{x} ] ,$$

$$F_2(a_i, a_j) = [ (a_i)_{0}(a_j)_{y} - (a_i)_{y}(a_j)_{0} ] + i [ (a_i)_{x}(a_j)_{z} - (a_i)_{z}(a_j)_{x} ] +$$

$$+ [ (a_i)_{x}(a_j)_{y} - (a_i)_{y}(a_j)_{x} ] + i [ (a_i)_{0}(a_j)_{y} - (a_i)_{y}(a_j)_{0} ] ,$$

$$F_4(a_i, a_j) = [ (a_i)_{0}(a_j)_{y} - (a_i)_{y}(a_j)_{0} ] + i [ (a_i)_{x}(a_j)_{z} - (a_i)_{z}(a_j)_{x} ] -$$

$$- [ (a_i)_{x}(a_j)_{y} - (a_i)_{y}(a_j)_{x} ] - i [ (a_i)_{0}(a_j)_{y} - (a_i)_{y}(a_j)_{0} ] .$$

For clarity one can writes expressions of traces in the following form:

$$\frac{1}{2} \mathrm{Tr} [(1 - \gamma^5)\hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4 \cdots \hat{a}_{2n-1}\hat{a}_{2n}] =$$

$$= \mathrm{Tr} \left( \begin{array}{cc} F_3(a_1, a_2) & F_4(a_1, a_2) \\ F_2(a_1, a_2) & F_1(a_1, a_2) \end{array} \right) \cdot \left( \begin{array}{cc} F_3(a_3, a_4) & F_4(a_3, a_4) \\ F_2(a_3, a_4) & F_1(a_3, a_4) \end{array} \right) \cdots \left( \begin{array}{cc} F_3(a_{2n-1}, a_{2n}) & F_4(a_{2n-1}, a_{2n}) \\ F_2(a_{2n-1}, a_{2n}) & F_1(a_{2n-1}, a_{2n}) \end{array} \right).$$

It’s obvious that the trace of $2n$ Dirac $\gamma$-matrices is reduced to trace of $n$ matrices with $2 \times 2$ dimension.

For calculation of expressions like this

$$\mathrm{Tr}(\gamma^0\hat{a}_1\hat{a}_2 \cdots \hat{a}_{2n+1}) \mathrm{Tr}(\gamma^0\hat{b}_1\hat{b}_2 \cdots \hat{b}_{2m+1}),$$

where summing over index $\rho$ is supposed, one can use the Fiertz transform:

$$[(1 \pm \gamma^5)\gamma^i]\delta^{ij}[(1 \mp \gamma^5)\gamma^k]_{kl} = 2(1 \pm \gamma^5)_{kl}^\mu(1 \mp \gamma^5)^{ij}_k,$$

(14)
So one achieves the following expressions:

\[
\text{Tr}[(1 \pm \gamma^5)\gamma^\rho a_1 a_2 \cdots a_{2n+1}] \cdot \text{Tr}[(1 \mp \gamma^5)\gamma^\rho \hat{b}_1 \hat{b}_2 \cdots \hat{b}_{2m+1}] = \]

\[
= 4 \text{Tr}[(1 \mp \gamma^5)\hat{a}_1 \hat{a}_2 \cdots \hat{a}_{2n+1}\hat{b}_1 \hat{b}_2 \cdots \hat{b}_{2m+1}],
\]

\[
\text{Tr}[(1 \pm \gamma^5)\gamma^\rho \hat{a}_1 \hat{a}_2 \cdots \hat{a}_{2n+1}] \cdot \text{Tr}[(1 \mp \gamma^5)\gamma^\rho \hat{b}_2 \cdots \hat{b}_{2m+1}] = \]

\[
= \text{Tr}[(1 \pm \gamma^5)\gamma^\rho \hat{a}_1 \hat{a}_2 \cdots \hat{a}_{2n+1}] \cdot \text{Tr}[(1 \mp \gamma^5)\gamma^\rho \hat{b}_{2m+1} \cdots \hat{b}_1] = \]

\[
= 4 \text{Tr}[(1 \mp \gamma^5)\hat{a}_1 \hat{a}_2 \cdots \hat{a}_{2n+1}\hat{b}_{2m+1} \cdots \hat{b}_1],
\]

\[
\text{Tr}(\gamma^\rho \hat{a}_1 \hat{a}_2 \cdots \hat{a}_{2n+1}) \cdot \text{Tr}(\gamma^\rho \hat{b}_1 \hat{b}_2 \cdots \hat{b}_{2m+1}) = 2 \text{Tr}[\hat{a}_1 \hat{a}_2 \cdots \hat{a}_{2n+1}(\hat{b}_1 \hat{b}_2 \cdots \hat{b}_{2m+1} + \hat{b}_{2m+1} \cdots \hat{b}_1)].
\]

### 3 Trace calculation of Dirac $\gamma$-matrices contracted with massless vectors

#### 3.1 Application of massless vectors

In the method of trace calculation, what was briefly explained in Section 2, vectors $a_i$ contracted with Dirac $\gamma$-matrices are arbitrary. But there is a significant simplification of expressions for traces when vectors $a_i$ are massless (see e.g. [3], [7]).

For massless vectors the particular equation is true:

\[
a_i^2 = (a_i l_0)^2 - (a_i l_1)^2 - (a_i l_2)^2 - (a_i l_3)^2 = 0,
\]

i.e.

\[
(a_i l_0)^2 - (a_i l_1)^2 = (a_i l_2)^2 + (a_i l_3)^2,
\]

or

\[
(a_i q_+)(a_i q_-) = (a_i e_+)(a_i e_-).
\]

Thus the functions $F_k(a_i, a_j)$ become to be like the function $F_1(a_i, a_j)$ multiplied by some factor:

\[
F_2(a_i, a_j) = 2[(a_i e_+)(a_j q_-) - (a_i q_-)(a_j e_+)] =
\]

\[
= 2[(a_i e_+)\frac{(a_j e_-)(a_j q_-)}{(a_j q_+)} - (a_i q_-)(a_j e_+)(a_j q_+)] = -\frac{(a_j e_+)}{(a_j q_+)}2[(a_i q_-)(a_j q_+)(a_i e_+)(a_j e_-)],
\]

(21)
that is

\[ F_2(a_i, a_j) = -\frac{(a_j e_+) F_1(a_i, a_j)}{(a_j q_+)} . \]  

(22)

In the similar way

\[ F_3(a_i, a_j) = -\frac{(a_i e_-)}{(a_i q_-)} \cdot \frac{(a_j e_+)}{(a_j q_+)} F_1(a_i, a_j) = -\frac{(a_i e_-)}{(a_i q_-)} F_2(a_i, a_j) = -\frac{(a_j e_+)}{(a_j q_+)} F_4(a_i, a_j) . \]  

(23)

\[ F_4(a_i, a_j) = \frac{(a_i e_-)}{(a_i q_-)} F_1(a_i, a_j) . \]  

(24)

Formulae of the traces become to be simpler by far.

Through the identity

\[ (1 \pm \gamma^5)qQ(1 \pm \gamma^5)q = \text{Tr}[(1 \pm \gamma^5)qQ] (1 \pm \gamma^5)q, \]  

(25)

which is valid for any massless vector \( q \) and any operator \( Q \), one can obtain

\[ \frac{1}{2} \text{Tr}[(1 - \gamma^5)\hat{a}_1 \hat{a}_2 \hat{a}_3 \cdots \hat{a}_{2n}] = \frac{1}{4(a_1 q_-)} \text{Tr}[(1 + \gamma^5)\hat{q}_- \hat{a}_1 \hat{a}_2 \hat{a}_3 \cdots \hat{a}_{2n} \hat{a}_1] = \]

\[ = \frac{1}{4(a_1 q_-)} \cdot \frac{1}{4(a_3 q_-)} \cdot \text{Tr}[(1 + \gamma^5)\hat{q}_- \hat{a}_1 \hat{a}_2 \hat{a}_3(1 + \gamma^5)\hat{q}_- \hat{a}_3 \cdots \hat{a}_{2n} \hat{a}_1] = \]

\[ = \frac{\text{Tr}[(1 + \gamma^5)\hat{q}_- \hat{a}_1 \hat{a}_2 \hat{a}_3] \cdot \text{Tr}[(1 + \gamma^5)\hat{q}_- \hat{a}_3 \hat{a}_4 \hat{a}_5] \cdots \text{Tr}[(1 + \gamma^5)\hat{q}_- \hat{a}_{2n-1} \hat{a}_{2n} \hat{a}_1]}{4(a_1 q_-) \cdot 4(a_3 q_-) \cdots 4(a_{2n-1} q_-)} . \]  

(26)

Further

\[ \frac{1}{4(a_i q_-)} \text{Tr}[(1 + \gamma^5)\hat{q}_- \hat{a}_i \hat{a}_j \hat{a}_k] = \frac{1}{16(a_i q_-)(a_j q_+)} \text{Tr}[(1 + \gamma^5)\hat{q}_- \hat{a}_i \hat{a}_j (1 - \gamma^5)\hat{q}_+ \hat{a}_j \hat{a}_k] = \]

\[ = \frac{1}{32(a_i q_-)(a_j q_+)} \cdot \text{Tr}[(1 + \gamma^5)\hat{q}_- \hat{a}_i \hat{a}_j \hat{q}_+ (1 + \gamma^5)\hat{q}_- \hat{q}_+ \hat{a}_j \hat{a}_k] = \]

\[ = \frac{1}{32(a_i q_-)(a_j q_+)} \cdot \text{Tr}[(1 - \gamma^5)\hat{q}_+ \hat{q}_- \hat{a}_i \hat{a}_j] \cdot \text{Tr}[(1 + \gamma^5)\hat{q}_- \hat{q}_+ \hat{a}_j \hat{a}_k] = \frac{F_1(a_i, a_j) F_3^*(a_j, a_k)}{2(a_i q_-)(a_j q_+)} , \]  

(27)

and finally from (26), (27) it follows that

\[ \frac{1}{2} \text{Tr}[(1 - \gamma^5)\hat{a}_1 \hat{a}_2 \hat{a}_3 \cdots \hat{a}_{2n}] = \frac{F_1(a_1, a_2) F_3^*(a_2, a_3) \cdots F_1(a_{2n-1}, a_{2n}) F_3^*(a_{2n}, a_1)}{2^n(a_1 q_-)(a_2 q_+)(a_3 q_-) \cdots (a_{2n} q_+)} . \]  

(28)

From (22), (23) we have

\[ \frac{F_1(a_i, a_j) F_3^*(a_j, a_k)}{(a_j q_+)} = -\frac{F_2(a_i, a_j) F_4^*(a_j, a_k)}{(a_j q_-)} , \]  

(29)
and (28) takes the form
\[ \frac{1}{2} \text{Tr}[(1 - \gamma^5) \hat{a}_1 \hat{a}_2 \hat{a}_3 \cdots \hat{a}_{2n}] = (-1)^n \cdot \frac{F_2(a_1, a_2) F_2^* (a_2, a_3) \cdots F_2(a_{2n-1}, a_{2n}) F_2^* (a_{2n}, a_1)}{2^n (a_1 q_-) (a_2 q_-) \cdots (a_{2n} q_-)}. \] (30)

Note that
\[ \text{Tr}[(1 - \gamma^5) \hat{a}_1 \hat{a}_2 \hat{a}_3 \cdots \hat{a}_{2n}] = \text{Tr}[(1 + \gamma^5) \hat{a}_2 \hat{a}_3 \cdots \hat{a}_{2n} \hat{a}_1] = \left( \text{Tr}[(1 - \gamma^5) \hat{a}_2 \hat{a}_3 \cdots \hat{a}_{2n} \hat{a}_1] \right)^*, \] (31)
then (28) leads to the one more expression of traces
\[ \frac{1}{2} \text{Tr}[(1 - \gamma^5) \hat{a}_1 \hat{a}_2 \hat{a}_3 \cdots \hat{a}_{2n}] = \frac{F_3(a_1, a_2) F_1^* (a_2, a_3) \cdots F_3(a_{2n-1}, a_{2n}) F_1^* (a_{2n}, a_1)}{2^n (a_1 q_+)(a_2 q_-) \cdots (a_{2n} q_-)}. \] (32)

At last an identity [see (23), (24)]
\[ \frac{F_3(a_i, a_j) F_1^* (a_j, a_k)}{(a_j q_-)} = -\frac{F_4(a_i, a_j) F_4^* (a_j, a_k)}{(a_j q_+)}, \] (33)
provide for
\[ \frac{1}{2} \text{Tr}[(1 - \gamma^5) \hat{a}_1 \hat{a}_2 \hat{a}_3 \cdots \hat{a}_{2n}] = (-1)^n \cdot \frac{F_4(a_1, a_2) F_4^* (a_2, a_3) \cdots F_4(a_{2n-1}, a_{2n}) F_4^* (a_{2n}, a_1)}{2^n (a_1 q_+)(a_2 q_-) \cdots (a_{2n} q_-)}. \] (34)

The expressions for traces (28), (30), (32), (34) are equivalent. Since in the case of massless vectors
\[ |F_1(a_i, a_j)| = 2 \sqrt{(a_i a_j)(a_i q_-)(a_j q_+)}, \]
\[ |F_3(a_i, a_j)| = 2 \sqrt{(a_i a_j)(a_i q_+)(a_j q_-)}, \] (35)
\[ |F_2(a_i, a_j)| = 2 \sqrt{(a_i a_j)(a_i q_-)(a_j q_-)}, \]
\[ |F_4(a_i, a_j)| = 2 \sqrt{(a_i a_j)(a_i q_+)(a_j q_+)}, \]
then
\[ \frac{1}{2} \text{Tr}[(1 - \gamma^5) \hat{a}_1 \hat{a}_2 \hat{a}_3 \cdots \hat{a}_{2n}] = 2^n \sqrt{(a_1 a_2)(a_2 a_3) \cdots (a_{2n} a_1)}, \] (36)
(see also [4]).

### 3.2 Calculation of traces in case of appearance the type $0 \overline{0}$ uncertainty

The appearance of uncertainty type $0 \overline{0}$ is possible during numerical calculation by reason of denominators presence in the formulae (28), (30), (32), (34).
For example, when the formula (34) is being used and orthonormal basis is chosen according to (10), the uncertainty will appear, if
\[ a_i = \text{const} \cdot q_+ \]
i.e. if
\[ (a_i)_0 = (a_i)_x . \]

The simplest way to avoid the appearing of uncertainties in such points of phase space is to use for calculation here another basis vectors, because obtained formulas are correct for any orthonormal basis and numerical results received are independent from it’s choice. But such approach results in unnecessary complicating of the computer program.

There are three solutions of this problem:

1. At the points of phase space, where denominators are 0, one should use common formulae from Section 2 (see e.g. [8]).
2. One can choose an arbitrary 4-vector \( t \) normalized by condition \( t^2 = 1 \) to perform the identical transformation of the initial expression:
\[
\frac{1}{2} \text{Tr}[(1 - \gamma^5)\hat{a}_1\hat{a}_2\hat{a}_3\cdots\hat{a}_{2n}] = \frac{1}{2} \text{Tr}[(1 + \gamma^5)t\hat{a}_1t\hat{a}_2t\hat{a}_3t\cdots t\hat{a}_{2n}t] =
\]
\[
= \frac{1}{2} \text{Tr}[(1 + \gamma^5)\hat{a}_1'\hat{a}_2'\hat{a}_3'\cdots\hat{a}_{2n}'] = \frac{1}{2} \text{Tr}[(1 - \gamma^5)\hat{a}_2'\hat{a}_3'\cdots\hat{a}_{2n}'\hat{a}_1'] ,
\]
where
\[
a_i' = -a_i + 2(a_i t) t .
\]

Thus
\[
(a_i')^2 = (a_i)^2 - 4(a_i t)^2 + 4(a_i t)^2 t^2 = (a_i)^2 = 0 ,
\]
so one can use the formulae from Section 3.1 to calculate the transformed expression. At the same time the denominators take forms
\[
(a_i' q_{\pm}) = -(a_i q_{\pm}) + 2(a_i t)(t q_{\pm}) ,
\]
where the second term in (10) is positive.

3. It is possible to transform the assumption formula. Let us suppose that (34) is using for calculation and for \( i = 2s \)
\[
(a_{2s} q_+) = 0 .
\]
In this situation one can replace [see (33)]
\[
\frac{F_4(a_{2s-1}, a_{2s}) F_4^*(a_{2s}, a_{2s+1})}{(a_{2s} q_+)} \rightarrow \frac{-F_5(a_{2s-1}, a_{2s}) F_4^*(a_{2s}, a_{2s+1})}{(a_{2s} q_-)} .
\]

Notice that in this case
\[
(a_{2s} q_-) \neq 0 .
\]
4 Conclusion

The simple and compact formulae are proposed to calculate traces of Dirac $\gamma$-matrices contracted with massless vectors. These formulae may be easily implemented as a simple yet efficient computer algorithm.

References

[1] J.S.R. Chisholm, Nuovo Cim. 30 (1963) 426.

[2] J. Kahane, J. Math. Phys. 9 (1968) 1732.

[3] A.L. Bondarev, Nucl. Phys. B 733 (2006) 48; E-print arXiv: hep-ph/0504223.

[4] V.I. Borodulin, R.N. Rogalyov, S.R. Slabospitsky, Preprint IHEP-95-90 (Protvino, 1995); E-print arXiv: hep-ph/9507456.

[5] V.V. Andreev, Phys. Rev. D 62 (2000) 014029; E-print arXiv: hep-ph/0101140.

[6] R. Gastmans and T.T. Wu, The Ubiquitous Photon: Helicity Method for QED and QCD (Clarendon Press, Oxford, 1990).

[7] S.M. Sikach, IP ASB preprint no.658 (Minsk, 1992).

[8] ALHEP, http://www.hep.by/alhep
V. Makarenko, E-print arXiv:0704.1839v1 [hep-ph].