On convergence of a $q$-random coordinate constrained algorithm for non-convex problems

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Abstract

We propose a random coordinate descent algorithm for optimizing a non-convex objective function subject to one linear constraint and simple bounds on the variables. Although it is common use to update only two random coordinates simultaneously in each iteration of a coordinate descent algorithm, our algorithm allows updating arbitrary number of coordinates. We provide a proof of convergence of the algorithm. The convergence rate of the algorithm improves when we update more coordinates per iteration. Numerical experiments on large scale instances of different optimization problems show the benefit of updating many coordinates simultaneously.

Keywords: random coordinate descent algorithm, convergence analysis, densest $k$-subgraph problem, eigenvalue complementarity problem.

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1 Introduction

The prominence of so called “Big Data” has given rise to new challenges for the field of optimization. Algorithms that aim to optimize large scale optimization problems should provide good solutions in reasonable time, be memory efficient and well scalable.

Well suited methods for optimizing high-dimensional functions are coordinate descent (CD) methods. Coordinate descent algorithms solving lasso penalized regression trace back to [9]. In its most basic form, a CD method iteratively minimizes the objective function, by updating only a (strict) subset of variables per iteration. Variants of CD methods are distinguished by the selection procedure of the variables to be updated (e.g., cyclic, greedy, random), and consequently the method to update selected variables. Interested readers are referred to surveys [26, 31] for more details on CD methods. Coordinate descent methods find applications in various areas, including image denoising [17], sensor network localization [21], machine learning, such as Support Vector Machines [10] and penalized regression [3].

Global convergence of random CD methods for convex objective functions was proved by Nesterov [18]. Nesterov [18] considered both constrained and unconstrained convex optimization problems, and provided an accelerated variant for the unconstrained case. These results were generalized in [23], where Richtárik and Takáč extended the analysis of [18] to include objective functions that are the sum of a smooth convex function and a simple nonsmooth block-separable convex function. In [19], a class of huge-scale convex optimization problems with sparse subgradients has been considered. The proposed method works well on problems with uniform sparsity of corresponding linear operators.

Despite these results for convex optimization problems, there are few developments on solving large-scale non-convex problems. Patrascu and Necula [22] introduced two 2-random coordinate
descent algorithms for large scale structured non-convex optimization problems. The objective functions they consider are consisting of two terms where one is non-convex and smooth, and the other one is convex. One of the algorithms from [22] considers unconstrained problems, while the other one singly linearly constrained problems. Both algorithms are designed to update two block coordinates in each iteration. Cristofari [8] proposed an algorithm for optimization of a non-convex function subject to one linear constraint and simple bounds on the variables. The algorithm from [8] iteratively chooses a pair of coordinates according to the almost cycling strategy, i.e., one variable is selected in a cyclic manner, and the other one not. This algorithm has deterministic convergence properties, which is not the case for random CD methods. Two random coordinate descent-based algorithms for solving a non-convex problem that allow updating two or more coordinates simultaneously, in each iteration, are introduced in [27]. Extensive numerical tests in [27] show better performance of the algorithms when more than two variables are simultaneously updated.

Motivated by the algorithms from [27], we propose a random coordinate descent algorithm for minimization of a non-convex objective function subject to one linear constraint and bounds on the variables. In each iteration, our q-random coordinate constrained descent (q-RCCD) algorithm randomly selects q (q ≥ 2) variables, with uniform probability. The q-RCCD algorithm updates those variables based on the optimization of a convex approximation of the objective. In particular, updates of variables are based on a projected gradient method. For this purpose, we assume q-coordinate Lipschitz continuity of the gradient of the objective function. This assumption differs from the assumption on the gradient of the objective from [27]. Moreover, we provide a proof of convergence of the q-RCCD algorithm, using techniques similar to those employed in [22]. The rate of convergence for the expected values of an appropriate stationarity measure of the q-RCCD algorithm, coincides with the convergence rate of the algorithm from [22] when q = 2 and blocks are of size one. However, the performance of the q-RCCD algorithm improves for q larger than two. This implies that we improve on the work of Patrascu and Nechita [22] when block sizes are of size one. We test the q-RCCD algorithm for solving large scale instances of the densest k-subgraph (DkS) problem and the eigenvalue complementarity (EiC) problem. Our numerical results show the benefit of updating more than two coordinates simultaneously. Additionally, we compare the q-RCCD algorithm with an Alternating Direction Method of Multipliers (ADMM), see e.g., [4], the projected gradient method (PGM) [6], and the deterministic 2-coordinate descent algorithm by Beck [1]. Our results clearly show that the q-RCCD algorithm is superior for most of the considered instances.

The rest of this paper is organized as follows. We formally introduce the q-RCCD algorithm and the considered class of optimization problems in Section 2. We define a stationarity measure in Section 3, which we use to prove the convergence of the q-RCCD algorithm in Section 4. Numerical results are presented in Section 5.

2 A q-random coordinate constrained descent algorithm

Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be a non-convex differentiable function, and \( a = (a_i) \in \mathbb{R}^n \) and \( b \in \mathbb{R} \) given vectors. Further, let \( e = (e_i) \in \mathbb{R}^n \) and \( g = (g_i) \in \mathbb{R}^n \) be vectors such that \( e \leq g \). We consider the following non-convex optimization problem with one linear constraint and bounded variables:

\[
\min \left\{ f(x) : a^\top x = b, \, e_i \leq x_i \leq g_i, \, i \in [n] \right\}.
\]

(2.1)

Here \([n]\) denotes the set \( \{1, \ldots, n\} \). If \( a_i \neq 0 \) for all \( i \in [n] \), then without loss of generality, one may assume that \( a_i = 1 \) for all \( i \in [n] \) by a simple rescaling argument, see e.g., [1, Section 2.1].

In many applications, function \( f \) is of the form \( f(x) = x^\top Qx + c^\top x \) where \( Q = (q_{ij}) \) is an indefinite matrix and \( c = (c_i) \) a given vector. Examples are the continuous densest subgraph problem (see Section 5.1), the continuous quadratic knapsack problem, support vector machine training, the Chebyshev center problem, etc.

We denote the feasible set of the optimization problem (2.1) by \( P \). That is

\[
P = \left\{ x \in \mathbb{R}^n : a^\top x = b, \, e_i \leq x_i \leq g_i, \, i \in [n] \right\}.
\]

(2.2)
The first order necessary optimality condition for local solutions of (2.1) are given in the sequel.

**Necessary Optimality Condition.** If \( x^* \in P \) is a local minimum of (2.1), then there exist a \( \lambda^* \in \mathbb{R} \), and vectors \( \gamma^*, \delta^* \in \mathbb{R}^n \) with

\[
\nabla f(x^*) + \lambda^* a - \gamma^* + \delta^* = 0, \quad (2.3)
\]

\[
a^\top x^* = b, \quad (2.4)
\]

\[
g_j^* - x_j^* \geq 0, \; j \in [n] \quad (2.5)
\]

\[
x_j^* - e_j \geq 0, \; j \in [n] \quad (2.6)
\]

\[
\gamma_j^*(e_j - x_j) = 0, \; j \in [n] \quad (2.7)
\]

\[
\delta_j^*(g_j - x_j^*) = 0, \; j \in [n]. \quad (2.8)
\]

We refer to a vector \( x^* \) satisfying (2.3)–(2.8) as a stationary point. We aim to devise an iterative algorithm with a feasible descending random direction that converges to a stationary point of (2.1).

Before we present our algorithm, we state the following assumption that applies to the whole paper.

**Assumption 1.** Let \( q \in \mathbb{N}, q \geq 2 \) be given. The function \( f \) has \( q \)-coordinate Lipschitz continuous gradient at \( x \in \mathbb{R}^n \), i.e., for any \( J \subseteq [n] \) with \( |J| = q \), there exists a constant \( L_J > 0 \) such that

\[
\| \nabla f(x + s_J) - \nabla f(x) \| \leq L_J \| s_J \|, \quad (2.9)
\]

where \( \| \cdot \| \) denotes the Euclidean norm, \( s_J \in \mathbb{R}^n \) is a vector with elements \( s_j \) for \( j \in J \), and zeros for \( j \notin J \). Analogously, \( \nabla f(x) \in \mathbb{R}^n \) denotes the vector consisting of \( \frac{\partial f}{\partial x_j} \) for \( j \in J \) and zeros for \( j \notin J \).

To simplify notation, we also use \( \nabla_J f(x) \) for the projection of the gradient onto the subspace \( \mathbb{R}^q \) identified by \( J \), when it is clear from the context what we mean.

By Assumption 1 and from [20, Lemma 1.2.3], we have

\[
|f(x + s_J) - f(x) - \langle \nabla_J f(x), s_J \rangle| \leq \frac{L_J}{2} \| s_J \|^2, \quad \forall x \in \mathbb{R}^n, s_J \in \mathbb{R}^n,
\]

from where it follows that

\[
f(x + s_J) \leq f(x) + \langle \nabla_J f(x), s_J \rangle + \frac{L_J}{2} \| s_J \|^2, \quad \forall x \in \mathbb{R}^n, s_J \in \mathbb{R}^n. \quad (2.10)
\]

We use the quadratic approximation of \( f \) in (2.10) to obtain descending directions of our algorithm. In particular, in each iteration of our \( q \)-random coordinate constrained descent algorithm we update \( q \geq 2 \) random coordinates by exploiting the right hand side of (2.10).

Let \( x^m \) be a feasible solution of problem (2.1) in iteration \( m \) of the \( q \)-RCCD algorithm. Let \( J_m = \{ j^1_m, \ldots, j^q_m \} \subseteq [n] \) with \( |J_m| = q \) be a set of random coordinates that needs to be updated simultaneously in step \( m \). Our \( q \)-random coordinate constrained update in \( m \)-th iteration is as follows:

\[
d_{J_m}(x^m) := \begin{cases} 
    u^m_j(x^m) - x^m_j & \text{if } j \in J_m, \\
    0 & \text{if } j \not\in J_m,
\end{cases} \quad (2.11)
\]

where \( u^m(x^m) \) is the optimal solution of a convex optimization problem. To simplify notation, we sometimes write \( u^m \) instead of \( u^m(x^m) \), and \( d_{J_m} \) instead of \( d_{J_m}(x^m) \). In view of (2.10), we define \( u^m(x^m) \) as follows:

\[
u^m(x^m) := \arg\min_{u \in \mathbb{R}^n} \sum_{j \in J_m} \frac{\partial f(x^m)}{\partial x_j} (\bar{u}_j - x^m_j) + \frac{L_J}{2} \sum_{j \in J_m} (\bar{u}_j - x^m_j)^2
\]

\[
s.t. \sum_{j \in J_m} a_j \bar{u}_j = b - \sum_{j \notin J_m} a_j x^m_j = \sum_{j \in J_m} a_j x^m_j
\]

\[
e_j \leq \bar{u}_j \leq g_j, \quad \forall j \in J_m.
\]

\[3\]
Let us consider (2.12) in more detail. The objective function can be rewritten as

$$\arg\min_{\bar{u} \in \mathbb{R}^q} 2\left\langle \frac{1}{L_{J_m}} \nabla J_m f(x^m) - x^m, \bar{u} \right\rangle + \bar{u}^T \bar{u} = \arg\min_{\bar{u} \in \mathbb{R}^q} \left\| \bar{u} - \left( x^m - \frac{1}{L_{J_m}} \nabla J_m f(x^m) \right) \right\|^2.$$ 

This shows that

$$u^m(x^m) = \left( x^m - \frac{1}{L_{J_m}} \nabla J_m f(x^m) \right) \mathcal{P},$$

(2.13)

where $(\cdot)_\mathcal{P}$ denotes the orthogonal projection onto the feasible set of (2.12). As this set is convex, computing such a projection is equivalent to solving a convex quadratic programme, which can be done in polynomial time (see [13]). However, some specific vectors $a$, $e$ and $g$, allow for the use of more efficient projection algorithms, as we will see in Section 5.

We update $x^m$ as follows:

$$x^{m+1} = x^m + d_{J_m}(x^m),$$

(2.14)

Remark 2.1. The objective in (2.12) differs from the objective in [27], where the following objective is used:

$$\sum_{j \in J_m} \frac{\partial f(x^m)}{\partial x_j} (\bar{u}^m_j - x^m_j) + \sum_{j \in J_m} L_j \left( \bar{u}^m_j - x^m_j \right)^2,$$

as well as an assumption that $f$ is coordinatewise Lipschitz continuous with constants $L_j$ for $j \in J_m$.

Let us summarize the $q$-RCCD algorithm. For fixed $q$ such that $2 \leq q < n$, the $q$-random coordinate constrained descent algorithm is as follows. The algorithm starts with a feasible solution for (2.1). Then, chooses $q$ coordinates randomly by uniform distribution on $[n]$. For this subset, a $q$-coordinate Lipschitz constant is calculated that satisfies (2.9) and the auxiliary quadratic convex optimization problem (2.12) is solved. If no stopping criteria is satisfied, the algorithm calculates a new feasible solution of (2.1) by using (2.14), and continues. The algorithm may stop if the difference in two consecutive objective values is less than a pre-specified tolerance. For more details on the algorithm see the pseudo-code given by Algorithm 1.

In light of (2.13), the $q$-RCCD algorithm can be considered as a combination of coordinate descent and PGM [6]. In fact, in the extreme case $q = n$, the $q$-RCCD algorithm is deterministic and equivalent to PGM with constant stepsize $1/L_J$ (note that $L_J$ is constant when $q = n$).

**Algorithm 1:** The $q$-RCCD algorithm for the non-convex problem (2.1)

1. **Input:** A feasible initial solution $x^0$ of (2.1), and parameter $q \in \mathbb{N}, q \geq 2$.
2. **Initialize:** $m \leftarrow 0$.
3. **while** stopping criteria not satisfied **do**
   4. Select $J_m \subseteq [n], |J_m| = q$ coordinates randomly by uniform distribution on $[n]$.
   5. Calculate appropriate $L_{J_m}$.
   6. Compute (2.13) and use (2.11) to obtain $d_{J_m}(x^m) \in \mathbb{R}^n$.
   7. Update $x^{m+1}$ by using (2.14).
   8. $m \leftarrow m + 1$.
4. **Output:** $f(x^{m-1})$.

Observe that the vector $u^m \in \mathbb{R}^q$ is an optimal solution of the convex optimization problem
in (2.12) if and only if there exist $\lambda \in \mathbb{R}$, and $\gamma, \delta \in \mathbb{R}^q$ such that

$$\frac{\partial f(x^m)}{\partial x_j} + L_{J_m}(u^m_j - x^m_j) + \lambda a_j - \gamma_j + \delta_j = 0, \quad j \in J_m$$

(2.15)

$$\sum_{j \in J_m} a_j u^m_j = b - \sum_{j \notin J_m} a_j x^m_j$$

(2.16)

$$e_j \leq u^m_j \leq g_j, \quad j \in J_m$$

(2.17)

$$\gamma_j (u^m_j - e_j) = 0, \quad j \in J_m$$

(2.18)

$$\delta_j (g_j - u^m_j) = 0, \quad j \in J_m.$$  

(2.19)

Note that for $x^m \in P$, see (2.2), a feasible solution $u^m$ for (2.12) leads to

$$\langle a_{J_m}, u^m - x^m \rangle = \sum_{j \notin J_m} a_j u^m_j - \sum_{j \in J_m} a_j x^m_j = b - \sum_{j \in J_m} a_j x^m_j = 0,$$

(2.20)

where $a_{J_m}$ (resp., $x^m_{J_m}$) is the sub-vector of $a$ (resp., $x^m$) with indices in $J_m$. Note that we omit subscript $m$ on $J_m$ in $x^m_{J_m}$, and write $x^m_{J_m}$. By abuse of notation, we write (2.20) as

$$\langle a_{J_m}, d_{J_m} \rangle = 0.$$

### 2.1 Notation

We define the set of all index sets of size $2 \leq q < n$ as

$$J_q := \{ J : J \subseteq [n], |J| = q \},$$

(2.21)

and often omit the subscript $q$ for brevity. We assume that any two index sets $J, J' \in J_q$ have identical probability to be selected. We set

$$z := |J_q| = \binom{n}{q}.$$

(2.22)

By our previous assumption, each index set $J \in J_q$ can be chosen with probability $1/\mu$. Note that for $|J| = q$ and fixed $i$ we have $\mu = \binom{n-1}{q-1}$ numbers of different $J$’s such that each of them contains element $i$. Therefore, given Lipschitz constants $L_J > 0, J \in J_q$, we define the vector $\Gamma = (\Gamma_j) \in \mathbb{R}^n$ where:

$$\Gamma_i := \frac{1}{\mu} \sum_{J : i \in J} L_J, \quad i \in [n].$$

(2.23)

We further define the diagonal matrix

$$D_\Gamma := \operatorname{Diag}(\Gamma) \in \mathbb{R}^{n \times n}.$$ 

(2.24)

Here Diag is the operator that maps a vector into the diagonal matrix whose diagonal elements correspond to the elements of the input vector. Further, we define the following pair of primal-dual norms:

$$\|x\|_\Gamma = (x^\top D_\Gamma x)^\frac{1}{2} = \|D_\Gamma^{\frac{1}{2}} x\|, \quad \forall x \in \mathbb{R}^n,$$

$$\|y\|_\Gamma^* = (y^\top D_\Gamma^{-1} y)^\frac{1}{2}, \quad \forall y \in \mathbb{R}^n.$$ 

We denote by $\mathbf{1}$ the vector of all-ones. For comparison purpose of our method with the one of [22], we follow a similar notation and methodology.
3 Stationarity measure

In this section we introduce a function whose optimal solution is used to identify a stationary point of optimization problem (2.1).

Consider the polyhedron

$$S(x) = \{ s \in \mathbb{R}^n : a^T s = 0, \ e \leq s + x \leq g \},$$

for some $x \in \mathbb{R}^n$, and the corresponding local polyhedron with respect to $J \subseteq [n]$, i.e.,

$$S_J(x) = \{ s \in S(x) : s_j = 0, \ \forall j \notin J \}.$$  

(3.1)

(3.2)

Note that for any vector $s \in S(x)$ we have $a^T (x + s) = b$ when $x \in P$, see (2.2). Thus, any vector $s \in S_J(x)$ where $|J| = q$ plays the role of a feasible update direction in each iterate of Algorithm 1.

For any $x \in P$, $\alpha > 0$, and vector $\Gamma \in \mathbb{R}^n$, we define the convex function $\psi_{\alpha \Gamma} (\cdot ; x) : \mathbb{R}^n \to \mathbb{R}$, as follows:

$$\psi_{\alpha \Gamma} (s;x) := f(x) + \langle \nabla f(x), s \rangle + \frac{\alpha}{2} \| s \|^2.$$  

(3.3)

Note that when $x$ is a stationary point of (2.1) and $s = 0$, all terms of (3.3) vanish but the first one. Thus, $\psi_{\alpha \Gamma} (s;x)$ plays the role of an overestimator of the function $f(\cdot)$ in the vicinity of a stationary point, see (2.10).

Let us consider the following minimization problem:

$$\min_{s \in S(x)} \psi_{\alpha \Gamma} (s;x).$$  

(3.4)

Since $\psi_{\alpha \Gamma} (\cdot ; x)$ is a convex function, $s \in S(x)$ is a global optimum of the minimization problem (3.4) if and only if there exist $\lambda, \gamma, \delta \in \mathbb{R}$ such that

$$\nabla f(x) + \alpha D_T \cdot s + \lambda a - \gamma + \delta = 0,$$

(3.5)

$$\gamma^T (s + x - e) = 0, \ \delta^T (g - s - x) = 0, \ a^T s = 0, \ e \leq s + x \leq g.$$  

(3.6)

Thus,

$$d_{\alpha \Gamma} (x) := \arg\min_{s \in S(x)} \psi_{\alpha \Gamma} (s;x) = \frac{1}{\alpha} D_T^{-1} (- \nabla f(x) - \lambda a + \gamma + \delta),$$

(3.7)

provided that $d_{\alpha \Gamma} (x) \in S(x)$ and satisfies (3.6).

Let $x \in P$ be a feasible solution of the non-convex optimization problem (2.1). We introduce the following stationarity measure, see also (2.22).

$$M_q (x, \Gamma) = \| d_{\alpha \Gamma} (x) \|_{\Gamma},$$

(3.8)

where $\mu = \binom{n-1}{q-1}$, $\Gamma$ is defined in (2.23), and $d_{\alpha \Gamma} (x)$ is given in (3.7).

**Lemma 3.1.** Let $2 \leq q < n$, and $\Gamma \in \mathbb{R}^n_+ \ be \ defined \ as \ in \ (2.23). \ Then, \ x^* \in P \ is \ a \ stationary \ point \ of \ non-convex \ optimization \ problem \ (2.1) \ if \ and \ only \ if \ M_q (x^*, \Gamma) = 0.$

**Proof.** Suppose that $M_q (x^*, \Gamma) = 0$. Then, it follows from (3.6) that $x^*$ is a stationary point for optimization problem (2.1), see also (2.23) and (2.28).

Conversely, suppose that $x^*$ is a stationary point for problem (2.1). Since $x^*$ is a stationary point, it follows from (2.3) that there exist $\lambda^* \in \mathbb{R}$, $\gamma^*, \delta^* \in \mathbb{R}^n_+$ such that

$$\nabla f(x^*) + \lambda^* a - \gamma^* + \delta^* = 0.$$  

The solution of (3.4) for $x = x^*$ and $\alpha := \mu = \binom{n-1}{q-1}$, is denoted by $d_{\mu \Gamma} (x^*) \in S(x^*)$. From (3.5) it follows that there exists a $\lambda \in \mathbb{R}$ and $\gamma, \delta \in \mathbb{R}^n_+$ such that

$$\nabla f(x^*) + \mu D_T \cdot d_{\mu \Gamma} (x^*) + \lambda a - \gamma + \delta = 0.$$  

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By subtracting the previous two equations, we obtain
\[ \mu D\Gamma \cdot d_{\mu\Gamma}(x^*) + (\lambda - \lambda^*)a + (\gamma^* - \gamma) - (\delta^* - \delta) = 0. \]
Taking the inner product with \(d_{\mu\Gamma}(x^*)\) and using that \(a^T d_{\mu\Gamma}(x^*) = 0\), we obtain
\[ \mu ||d_{\mu\Gamma}(x^*)||_\Gamma^2 + d_{\mu\Gamma}(x^*)^T (\gamma^* - \delta^* - \gamma \Gamma + \delta) = 0. \] (3.9)
From the optimality conditions \(3.6\) we have
\[ \gamma^T (d_{\mu\Gamma}(x^*) + x^* - c) = \delta^T (d_{\mu\Gamma}(x^*) + x^* - g) = 0, \]
and thus
\[ \gamma^T d_{\mu\Gamma}(x^*) = \gamma^T (e - x^*) \quad \text{and} \quad \delta^T d_{\mu\Gamma}(x^*) = \delta^T (g - x^*). \] (3.10)
Substituting \(3.10\) in \(3.9\) and using that \(e \leq d_{\mu\Gamma}(x^*) + x^* \leq g\) yields
\[ \mu ||d_{\mu\Gamma}(x^*)||_\Gamma^2 = \gamma^T d_{\mu\Gamma}(x^*) - \delta^T d_{\mu\Gamma}(x^*) - \gamma^T d_{\mu\Gamma}(x^*) + \delta^T d_{\mu\Gamma}(x^*) \]
\[ = \gamma^T (e - x^*) - \delta^T (x^* - g) - \gamma^T d_{\mu\Gamma}(x^*) + \delta^T d_{\mu\Gamma}(x^*) \]
\[ \leq 0 + 0 - \gamma^T (e - x^*) + \delta^T (g - x^*) \]
\[ = 0, \]
from where it follows that \(||d_{\mu\Gamma}(x^*)||_\Gamma = M_q(x^*, \Gamma) = 0\).

We prove another lemma related to \(d_{\alpha\Gamma}(x)\) and function \(\psi\).

**Lemma 3.2.** Let \(x \in P\), \(\psi_{\alpha\Gamma}(:; x)\) as in \(3.3\), and \(d_{\alpha\Gamma}(x)\) according to \(3.7\). Then
\[ \nabla \psi_{\alpha\Gamma}(d_{\alpha\Gamma}(x); x)^T d_{\alpha\Gamma}(x) \leq 0. \]

**Proof.** From \(3.5\), with \(s = d_{\alpha\Gamma}(x)\), and \(\nabla \psi_{\alpha\Gamma}(s; x) = \nabla f(x) + \alpha D_{\Gamma} \cdot s\) we have
\[ \nabla \psi_{\alpha\Gamma}(d_{\alpha\Gamma}(x); x) = -\lambda a + \gamma - \delta. \]
Taking the inner product with \(d_{\alpha\Gamma}(x)\), and using that \(a^T d_{\alpha\Gamma}(x) = 0\), yields
\[ \nabla \psi_{\alpha\Gamma}(d_{\alpha\Gamma}(x); x)^T d_{\alpha\Gamma}(x) = \gamma^T d_{\alpha\Gamma}(x) - \delta^T d_{\alpha\Gamma}(x). \] (3.11)
Nonpositivity of this expression then follows from substituting the first two equations from \(3.6\) in \(3.11\), where \(s = d_{\alpha\Gamma}(x)\).

**4 Convergence analysis of the \(q\)-RCCD algorithm**

In this section we provide a convergence analysis of the \(q\)-RCCD algorithm and its convergence rate. We first define several terms and present known results that are needed later in the section.

**Definition 4.1.** If the function \(h\) is twice continuously differentiable, then it is strongly convex with parameter \(\rho\) if and only if \(\nabla^2 h(x) \succeq \rho I\) for all \(x\) in the domain. An equivalent condition is
\[ h(x_1) \geq h(x_2) + \nabla h(x_2)^T (x_1 - x_2) + \frac{\rho}{2} ||x_1 - x_2||^2, \]
for every \(x_1, x_2\) in the domain of \(h\).
Let $K \subseteq \mathbb{R}^n$ be an arbitrary subspace. The set of indices corresponding to the nonzero coordinates in the vector $x \in K$ is called the support of $x$, and denoted by $\text{supp}(x)$. For $s, s' \in \mathbb{R}^n$, we say that $s'$ is conformal to $s$ if

$$\text{supp}(s') \subseteq \text{supp}(s), \quad s_j s'_j \geq 0, \quad \forall j.$$ \hfill (4.1)

The second condition in (4.1) states that the nonzero components of $s'$ and the corresponding components of $s$ have the same signs. A nonzero vector $s$ is an elementary vector of $K$ if $s \in K$ and there is no nonzero $s' \in K$ that is conformal to $s$ and $\text{supp}(s') \neq \text{supp}(s)$. Any two elementary vectors $s$ and $s'$ of $K$ with identical support must be (nonzero) scalar multiples of each other. Let us state the following well known result.

**Proposition 4.2** \hfill (2.4). Let $x$ be a nonzero vector in a subspace $K$ of $\mathbb{R}^n$ (or even generally in $\mathbb{R}^n$). Then, there exist elementary vectors $x_1, \ldots, x_r$ of $K$ such that $x = x_1 + \cdots + x_r$. They can be chosen such that each is conformal to $x$ and $r$ does not exceed the dimension of $K$ or the number of elements in $\text{supp}(x)$.

The following lemma provides an analogue of Proposition 4.2 for the specific polyhedron $S(x)$, instead of general subspaces.

**Lemma 4.3.** Let $x \in P$ and $2 \leq q < n$. Any $s \in S(x)$ can be written as

$$s = \sum_{j \in J_q} s_j,$$

where each $s_j \in S_j(x)$, contains at most two nonzero entries, and conformal to $s$.

**Proof.** Let $x \in P$, $s \in S(x)$ and

$$K = \{u \in \mathbb{R}^n : a^\top u = 0\}.$$

Vector $s$ is contained in subspace $K$ and thus, by Proposition 4.2, it follows that $s$ can be written as $s = \sum_{i=1}^r s^i$, where each $s^i$ is an elementary vector of $K$ and conformal to $s$. By the conformity property, the entries of these vectors satisfy

$$0 \leq s^i_j \leq s_i \quad \forall i \text{ such that } s_i \geq 0,$$

$$s_i \leq s^i_j \leq 0 \quad \forall i \text{ such that } s_i \leq 0.$$

As $s \in S(x)$, its entries satisfy

$$e_i - x_i \leq s_i \leq g_i - x_i.$$

Since $x \in P$ (and thus, $e \leq x \leq g$), we have $e_i - x_i \leq 0 \leq g_i - x_i$. Therefore,

$$0 \leq s^i_j \leq s_i \leq g_i - x_i \quad \forall i \text{ such that } s_i \geq 0,$$

$$e_i - x_i \leq s_i \leq s^i_j \leq 0 \quad \forall i \text{ such that } s_i \leq 0.$$

\hfill (4.2)

Moreover, $s^i \in K$, which implies $a^\top s^i = 0$. Combined with (4.2), this shows that $s^i \in S(x)$, see (3.1). Furthermore, each $s^i$ is an elementary vector of $K$, which implies that $s^i$ has at most two nonzero entries. This shows that

$$s^i \in S_j(x), \text{ for some } J \in J_2,$$

which proves the lemma for $q = 2$. For any $q \geq 2$, the lemma follows from noting that $|J_q| > |J_2|$, which proves the existence of injective function $h : J_2 \to J_q$. Then, (4.3) can be transformed in

$$s^i \in S_{h(J)}(x), \text{ for some } h(J) \in J_q.$$
We remark that, for any 2 vectors $s_J$ and $s_{J'}$ appearing in the decomposition of Lemma 4.3, we have
\[ s_J^T s_{J'} \geq 0, \]
which follows from the conformality property.

In the convergence analysis, we use properties of separable functions. We say that a convex function $h : \mathbb{R}^n \to \mathbb{R}$ is additively separable if
\[ h(x) = \sum_{j=1}^{n} h_j(x_j), \quad \forall x \in \mathbb{R}^n, \]
where $h_j : \mathbb{R} \to \mathbb{R}$, $j \in [n]$ are convex functions. For example, the function $\|x\|^2$ is additively separable since $\|x\|^2 = \sum_{j=1}^{n} x_j^2$.

**Lemma 4.4.** [28, Lemma 6.1] Let $h$ be a convex, additively separable function. For any $x$, $x + d$ from the domain of $h$, let $d$ be expressed as $d = \sum_{t=1}^{r} d^t$, for some $r \geq 1$ and nonzero $d^t \in \mathbb{R}^n$ conformal to $d$ for all $t \in [r]$. Then
\[ h(x + d) - h(x) \geq \sum_{t=1}^{r} (h(x + d^t) - h(x)). \]

Note that vectors $d^t$, $t \in [r]$ in Lemma 4.4 do not have to be elementary. We need the following result in the proof of Proposition 4.6.

**Lemma 4.5.** Let $L_I$ be defined as in Assumption 1. $S(x)$ (resp., $S_J(x)$) defined as in (3.1) (resp., (3.2)), and $x \in P$, see (2.2). Assume that $s \in S(x)$ is expressed as $s = \sum_{J \in \mathcal{J}} s_J$ where $s_J \in S_J(x)$ is conformal to $s$. Then
\[ \sum_{J \in \mathcal{J}} L_J \|s_J\|^2 \leq \sum_{J \in \mathcal{J}} T_J s_J. \]

**Proof.** Define $h(x) := \|x\|^2$. From Lemma 4.4 we have
\[ \| \sum_{J \in \mathcal{J}} \sqrt{T_J} s_J \|^2 \geq \sum_{J \in \mathcal{J}} \| 0 + \sqrt{T_J} s_J - 0 \|^2 = \sum_{J \in \mathcal{J}} L_J \|s_J\|^2. \]

Formally, Lemma 4.3 requires all the terms $\sqrt{T_J} s_J$ appearing in the decomposition to be nonzero, which might not be the case here. To resolve this, one can sum over a subset of $\mathcal{J}$ given by $\{J \in \mathcal{J} | s_J \neq 0\}$.

Let us introduce the set
\[ \xi^k = \{J_0, J_1, \ldots, J_k\}, \]
where $J_i \in \mathcal{J}_q$ ($i = 0, 1, \ldots, k$) are random selections obtained from Algorithm 1.

**Proposition 4.6.** Let $\{x^m\}$ be a sequence generated by Algorithm 1 using the uniform distribution. Then the following inequality holds for all $m$:
\[ E \left[ \psi_{L_{j_m}}(d_{j_m}(x^m); x^m) | x^{m-1} \right] \leq \left( 1 - \frac{1}{2} \right) f(x^m) + \frac{1}{2} \psi_{\mu r}(d_{\mu r}(x^m); x^m), \]
where $z = \binom{n}{q}$, $\mu = \binom{n-1}{q-1}$ and $\psi_{\mu r}(\cdot ; x^m)$ is defined in (3.3).

**Proof.** (See also [22, Lemma 8].) For the sake of brevity, we omit $m$ and $q$ from the notations. Therefore, the current point is denoted by $x$, $L_{j_m}$ is replaced by $L_J$, $\xi^{m-1}$ by $\xi$, and $\mathcal{J}_q$ by $\mathcal{J}$.

From the definition of $\psi_{\mu r}(\cdot ; x)$, see (3.3), and the definition of $d_J$, see (2.11), it follows:
\[ \psi_{L_{j_m}}(d_J(x); x) \leq \psi_{L_{j_m}}(s_J; x), \forall s_J \in S_J(x). \quad (4.5) \]
Now, by taking expectation on both sides of \( (4.5) \) w.r.t. random variable \( J \in \mathcal{J} \) (see (2.21)) conditioned on \( \xi \), it follows:

\[
E[\psi_{L,1}(d_J(x);x)|\xi] \leq f(x) + \frac{1}{z} \left[ \sum_{J \in \mathcal{J}} \langle \nabla_J f(x), s_J \rangle + \sum_{J \in \mathcal{J}} \frac{L_J}{2} \|s_J\|^2 \right],
\]

where \( s_J \in S_J(x) \), and \( z \) is defined in (2.22). In particular, \( (4.6) \) holds for \( s_J \in S_J(x) \), \( J \in \mathcal{J} \) such that each \( s_J \) is conformal to \( d_{\mu^J}(x) \) and \( d_{\mu^J}(x) = \sum_{J \in \mathcal{J}} s_J \), see (4.7). Note that such decomposition exists by Lemma 4.3.

From the definition of \( \Gamma_J \) in (2.23), it follows that

\[
L_J \leq \mu \min_{i \in \mathcal{J}} \{ \Gamma_i \}.
\]

Combining the definition of \( D_T \) in (2.24), inequalities (4.3) and (4.7), and Lemma 4.5, we find

\[
\sum_{J \in \mathcal{J}} L_J \|s_J\|^2 = \sum_{J \in \mathcal{J}} \|\sqrt{L_J} s_J\|^2 \leq \sum_{J \in \mathcal{J}} \sum_{J \in \mathcal{J}} D_{\mu^J}^2 \|s_J\|^2 = \mu \|d_{\mu^J}(x)\|^2.
\]

Substituting this inequality in \( (4.3) \) yields

\[
E[\psi_{L,1}(d_J(x);x)|\xi] \leq f(x) + \frac{1}{z} \left[ \langle \nabla f(x), d_{\mu^J}(x) \rangle + \frac{\mu}{2} \|d_{\mu^J}(x)\|^2 \right] = \left(1 - \frac{1}{z}\right)f(x) + \frac{1}{z} \left[ \psi_{\mu^J}(d_{\mu^J}(x);x) \right].
\]

The following lemma shows that the optimal solution of the convex optimization subproblem (2.12) provides a descending direction for the non-convex minimization problem (2.1).

**Lemma 4.7.** Let \( \{x^m\} \) be a sequence generated by Algorithm 2. For the update step \( d_{f_m} \), defined in (2.11), it holds that

\[
f(x^m + d_{f_m}) \leq f(x^m).
\]

**Proof.** For the sake of brevity, we omit \( m \) from the notations. Therefore, the current point is denoted by \( x \), the update step is \( d_J \), and \( J_m \) is replaced by \( J \) and \( L_J \) by \( L_J \).

Let \( \gamma, \delta \in \mathbb{R}_+^d \) satisfy (2.18) and (2.19). We rewrite (2.15) in a vector notation as follows:

\[
L_J d_J + \nabla_J f(x) + \lambda a_J - \gamma + \delta = 0,
\]

where \( d_J \) (resp., \( a_J \)) is the projection of \( d \) (resp., \( a \)) onto the subspace \( \mathbb{R}^d \) identified by \( J \). Inner vector product of (1.8) with \( d_J \) leads to

\[
L_J \langle d_J, d_J \rangle + \langle \nabla_J f(x), d_J \rangle + \lambda \langle a_J, d_J \rangle - \langle \gamma, d_J \rangle + \langle \delta, d_J \rangle = 0.
\]

Considering (2.20) we have that \( \langle a_J, d_J \rangle = 0 \). From the equations (2.18) and (2.19) we obtain

\[
\langle \gamma, d_J \rangle = \langle \gamma, u_J - x_J \rangle = \langle \gamma, u_J - e_J \rangle - \langle \gamma, x_J - e_J \rangle = -\langle \gamma, x_J - e_J \rangle,
\]

\[
\langle \delta, d_J \rangle = \langle \delta, u_J - x_J \rangle = \langle \delta, u_J - g_J \rangle + \langle \delta, g_J - x_J \rangle = \langle \delta, g_J - x_J \rangle,
\]

respectively. Therefore,

\[
L_J \|d_J\|^2 + \langle \nabla_J f(x), d_J \rangle + \langle \gamma, x_J - e_J \rangle + \langle \delta, g_J - x_J \rangle = 0.
\]

Lastly, the substitution of (1.9) in (2.19), with \( d_J \) instead of \( s_J \), leads to

\[
f(x + d_J) \leq f(x) - \frac{L_J}{2} \|d_J\|^2 - \langle \gamma, x_J - e_J \rangle - \langle \delta, g_J - x_J \rangle \leq f(x),
\]

which completes the proof.
Lemma 4.7 shows that the objective value of Algorithm 1 is not larger than the objective value in the current iteration.

For the proof of convergence, we need the following well-known property of supermartingales.

Theorem 4.8. (See, e.g., Theorem 11.5) Let \( (X_n)_{n \geq 1} \) be a supermartingale such that \( \sup_n E[|X_n|] < +\infty \). Then \( \lim_{n \to \infty} X_n = X^* \) exists almost surely (a.s.).

The following two theorems present convergence properties of Algorithm 1.

Theorem 4.9. Let the objective function \( f(x) \) of problem 2.1 satisfy Assumption 1 and let the sequence \( \{x^m\} \) be generated by Algorithm 1 using the uniform distribution. Then, the sequence \( \{f(x^m)\} \) converges to a random variable \( f \) a.s., and the sequence of random variables \( \{M^J(x^m, \Gamma)\} \) converges to 0 almost surely.

Proof. We have,

\[
f(x^{m+1}) \leq f(x^m) - \frac{L_J}{2} \|d_{J_m}(x^m)\|^2 - \langle \gamma, x^m_J - e_{J_m} \rangle - \langle \delta, g_{J_m} - x^m_{\nabla J_m} \rangle, \tag{4.11}\]

\( \forall m \geq 0, \) see (4.10). This inequality shows that the objective function decreases in each step of Algorithm 1 meaning that the algorithm is descending. Applying expectation conditioned on \( \xi^{m-1} \) we obtain

\[
E[f(x^{m+1})|\xi^{m-1}] \leq E[f(x^m)|\xi^{m-1}] - E\left[\frac{L_J}{2} \|d_{J_m}(x^m)\|^2 + \langle \gamma, x^m_{\nabla J_m} - e_{J_m} \rangle + \langle \delta, g_{J_m} - x^m_{\nabla J_m} \rangle |\xi^{m-1}\right].
\]

Considering the fact that \( E[|f(x^{m+1})| |\xi^{m-1}] < \infty \), since \( \{x^m\} \) is in the bounded set \( \mathcal{P} \) and \( f \) is differentiable, the martingale convergence Theorem 4.8 states that \( \{f(x^m)\} \) converges to a random variable \( f \) a.s. when \( m \to \infty \). Due to the almost sure convergence of the sequence \( \{f(x^m)\} \), it can be immediately observed that

\[
\lim_{m \to \infty} (f(x^m) - f(x^{m+1})) = 0 \text{ a.s.}
\]

Considering this, it follows from (4.11) that

\[
\lim_{m \to \infty} \langle \gamma, x^m_J - e_{J_m} \rangle + \langle \delta, g_{J_m} - x^m_{\nabla J_m} \rangle \leq \lim_{m \to \infty} - \frac{L_J}{2} \|d_{J_m}(x^m)\|^2 \text{ a.s.}
\]

The fact that \( \langle \gamma, x^m_J - e_{J_m} \rangle + \langle \delta, g_{J_m} - x^m_{\nabla J_m} \rangle \) is nonnegative, and the right hand side is nonpositive implies

\[
\lim_{m \to \infty} \langle \gamma, x^m_J - e_{J_m} \rangle + \langle \delta, g_{J_m} - x^m_{\nabla J_m} \rangle = 0, \lim_{m \to \infty} \|d_{J_m}(x^m)\| = \|x^{m+1} - x^m\| = 0 \text{ a.s.} \tag{4.12}
\]

From (4.12), we also conclude that

\[
\lim_{m \to \infty} \langle \gamma, x^m_J - e_{J_m} \rangle + \langle \delta, g_{J_m} - x^m_{\nabla J_m} \rangle |\xi^{m-1}\rangle \to 0, \text{ a.s.}
\]

Let us now prove the second statement. From Proposition 4.6 we obtain a sequence which bounds \( \psi \mu \Gamma(d_{\mu \Gamma}; x) \) from below as follows:

\[
z E\left[\psi_{L_m^J} (d_{J_m}(x^m); x^m) |\xi^{m-1}\right] - (z - 1) f(x^m) \leq \psi \mu \Gamma(d_{\mu \Gamma}(x^m); x^m).
\]

Further, it follows from Lemma 4.3 that any \( s \in S(x) \) has a conformal realization given as \( s = \sum_J s_J \), where \( s_J \in S_J(x) \) are conformal to \( s \) and have at most two nonzero elements. By exploiting this
and Jensen’s inequality, we derive an upper bound for $\psi_{\mu\Gamma}(d_{\mu\Gamma}(x^m); x^m)$ as follows:

$$
\psi_{\mu\Gamma}(d_{\mu\Gamma}(x^m); x^m) = \min_{s \in S(x^m)} \left[ f(x^m) + \langle \nabla f(x^m), s \rangle + \frac{1}{2}\|s\|_{\mu\Gamma}^2 \right] \\
= \min_{s \in S(x^m)} \left[ f(x^m) + \langle \nabla f(x^m), \sum_j s_j \rangle + \frac{1}{2}\|\sum_j s_j\|_{\mu\Gamma}^2 \right] \\
= \min_{\bar{s}_j \in S_j(x^m)} \left[ f(x^m) + \frac{1}{z^2}\|\sum_j (\nabla f(x^m), \bar{s}_j) + \frac{1}{2}\|\sum_j \bar{s}_j\|_{\mu\Gamma}^2 \right] \\
\leq \min_{\bar{s}_j \in S_j(x^m)} \left[ f(x^m) + \frac{1}{z^2}\|\sum_j (\nabla f(x^m), \bar{s}_j) + \frac{1}{2}\|\sum_j \bar{s}_j\|_{\mu\Gamma}^2 \right] \\
= E\left[ \psi_{\mu\Gamma}(d_{\mu\Gamma}(x^m); x^m)|\xi^{m-1}\right],
$$

where $\bar{s}_j = zs_j$, see \eqref{eq:222}.

Let us summarize the previous results. In particular, bounds on $\psi_{\mu\Gamma}(d_{\mu\Gamma}(x^m); x^m)$ are given below

$$
\begin{align*}
zE\left[ \psi_{L,m,1}(d_{\mu\Gamma}(x^m); x^m)|\xi^{m-1}\right] - (z - 1)f(x^m) & \leq \psi_{\mu\Gamma}(d_{\mu\Gamma}(x^m); x^m) \\
& \leq E\left[ \psi_{\mu\Gamma}(d_{\mu\Gamma}(x^m); x^m)|\xi^{m-1}\right]
\end{align*}
$$

Recall that $d_{\mu\Gamma}(x^m) = u^m - x^m \to 0$ a.s. Therefore $E[\psi_{\mu\Gamma}(d_{\mu\Gamma}(x^m); x^m)|\xi^{m-1}]$ converges to $\bar{f}$ a.s. when $m \to \infty$. Clearly, sequences of lower and upper bounds of $\psi_{\mu\Gamma}(d_{\mu\Gamma}(x^m); x^m)$ converge to $\bar{f}$ and therefore $\psi_{\mu\Gamma}(d_{\mu\Gamma}(x^m); x^m)$ converges to $\bar{f}$ a.s. for $m \to \infty$.

Note that the function $\psi_{\mu\Gamma}(s; x)$ is strongly convex in the variable $s$ with parameter $\mu$ w.r.t. norm $\| \cdot \|_{\Gamma}$, see \textbf{Definition 4.1}. This follows from $\nabla^2\psi_{\mu\Gamma} = \mu D_{\Gamma} > 0$. Therefore, $d_{\mu\Gamma}(x)$ is the unique minimizer and the following inequality holds:

$$
\psi_{\mu\Gamma}(s; x) \geq \psi_{\mu\Gamma}(d_{\mu\Gamma}(x); x) + \nabla \psi_{\mu\Gamma}(d_{\mu\Gamma}(x); x)\top (s - d_{\mu\Gamma}(x)) + \frac{\mu}{2}\|s - d_{\mu\Gamma}(x)\|^2_{\Gamma}, \quad \forall x, s \in \mathbb{R}^n,
$$

which leads to

$$
\psi_{\mu\Gamma}(0; x^m) \geq \psi_{\mu\Gamma}(d_{\mu\Gamma}(x^m); x^m) - \nabla \psi_{\mu\Gamma}(d_{\mu\Gamma}(x^m); x^m)\top d_{\mu\Gamma}(x^m) + \frac{\mu}{2}\|d_{\mu\Gamma}(x^m)\|^2_{\Gamma} \\
\geq \psi_{\mu\Gamma}(d_{\mu\Gamma}(x^m); x^m) + \frac{\mu}{2}\|d_{\mu\Gamma}(x^m)\|^2_{\Gamma},
$$

where the last inequality follows from $\nabla \psi_{\mu\Gamma}(d_{\mu\Gamma}(x^m); x^m)\top d_{\mu\Gamma}(x^m) \leq 0$, see \textbf{Lemma 3.2}. Since $\psi_{\mu\Gamma}(0; x^m) = f(x^m)$, it follows that $\psi_{\mu\Gamma}(0; x^m)$ converges to $\bar{f}$ a.s. when $m \to \infty$. Thus, both sequences $\psi_{\mu\Gamma}(0; x^m)$ and $\psi_{\mu\Gamma}(d_{\mu\Gamma}(x^m); x^m)$ converge to $\bar{f}$ a.s. for $m \to \infty$.

From the previous discussion and (4.13) it follows that

$$
\bar{f} \geq \bar{f} + \frac{\mu}{2}\|d_{\mu\Gamma}(x^m)\|^2_{\Gamma} \quad \text{a.s.,}
$$

which results in $\|d_{\mu\Gamma}(x^m)\|^2_{\Gamma} \leq 0$, from where it follows that the sequence $M_q(x^m, \Gamma) = \|d_{\mu\Gamma}(x^m)\|^2_{\Gamma}$ converges to 0 a.s. when $m \to \infty$.

We use the result of \textbf{Theorem 4.9} to prove the following theorem.

\textbf{Theorem 4.10.} Assume that the objective function $f(x)$ of problem \eqref{prob:2.1} satisfies \textbf{Assumption 1} and let the sequence $\{x^m\}$ be generated by \textbf{Algorithm 7} using the uniform distribution. Then any accumulation point of the sequence $\{x^m\}$ is a stationary point for \eqref{prob:2.1}.

\textbf{Proof.} Assume that the entire sequence $\{x^m\}$ is convergent, and let $\bar{x}$ be the limit point of this sequence. Using the fact that $d_{\mu\Gamma}(x^m)$, see \eqref{eq:37}, is the minimizer of $\psi_{\mu\Gamma}(\cdot; x^m)$, see \eqref{eq:33}, we have

$$
\psi_{\mu\Gamma}(d_{\mu\Gamma}(x^m); x^m) = f(x^m) + \langle \nabla f(x^m), d_{\mu\Gamma}(x^m) \rangle + \frac{\mu}{2}\|d_{\mu\Gamma}(x^m)\|^2_{\Gamma} \leq f(x^m) + \langle \nabla f(x^m), s \rangle + \frac{\mu}{2}\|s\|^2_{\Gamma}
$$

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for all $s \in S(\mathbf{x}^m)$. Taking the limit $m \to \infty$, we obtain
\[
f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), d_{\mu^{\Gamma}}(\mathbf{x}) \rangle + \frac{\mu}{2} \|d_{\mu^{\Gamma}}(\mathbf{x})\|^2 \leq f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), s \rangle + \frac{\mu}{2} \|s\|^2,
\]
for all $s \in S(\mathbf{x})$. Thus, $\psi_{\mu^{\Gamma}}(d_{\mu^{\Gamma}}(\mathbf{x}); \mathbf{x}) \leq \psi_{\mu^{\Gamma}}(s; \mathbf{x})$ for all $s \in S(\mathbf{x})$. This inequality accompanied with the a.s. convergence of $\{d_{\mu^{\Gamma}}(\mathbf{x}^m)\}$ to zero (see Theorem 4.10) implies that $d_{\mu^{\Gamma}}(\mathbf{x}) = 0$ is the minimum of (3.3) for $x = \mathbf{x}$ and thus $M_q(\mathbf{x}, \Gamma) = 0$. From Lemma 3.1 it follows that $\mathbf{x}$ is a stationary point of non-convex optimization problem (2.1). \hfill\square

The following theorem provides the convergence rate for Algorithm 1.

**Theorem 4.11.** Let the objective function $f(x)$ of problem (2.1) satisfy Assumption 1. Then, Algorithm 1, based on the uniform distribution, generates a sequence $\{\mathbf{x}^m\}$ satisfying the following convergence rate for the expected values of the stationarity measure:
\[
\min_{0 \leq k \leq m} E \left[ (M_q(\mathbf{x}^k, \Gamma))^2 \right] \leq \frac{2n(f(x^0) - f^*)}{q(m + 1)}, \quad \forall m \geq 0,
\]
where $f^*$ is the optimal value for non-convex optimization problem (2.1), $M_q(x, \Gamma)$ is given in (3.3), and $\Gamma$ in (2.28).

**Proof.** For the sake of brevity, we omit $m$ from the notations. Therefore, the current point is denoted by $x$, $L_{\mu^{\Gamma}}$ is replaced by $L_J$, and $\xi^m$ by $\xi$.

Considering (2.10) for $d_J$ yields:
\[
f(x + d_J) \leq f(x) + \langle \nabla f(x), d_J \rangle + \frac{L_J}{2} \|d_J\|^2.
\]
Let $x^+ := x + d_J$ and take expectation conditioned on $\xi$ for the above inequality:
\[
E[f(x^+)|\xi] \leq E[f(x) + \langle \nabla f(x), d_J \rangle + \frac{L_J}{2} \|d_J\|^2 |\xi] = E[\psi_{L_J}(d_J(x); x)|\xi].
\]
Combining the previous inequality and Proposition 4.6 we obtain
\[
E[f(x^+)|\xi] \leq \left(1 - \frac{1}{2}\right) f(x) + \frac{1}{2} \psi_{\mu^{\Gamma}}(d_{\mu^{\Gamma}}(x); x),
\]
from which it follows:
\[
E[f(x^+)|\xi] \leq \left(1 - \frac{1}{2}\right) E[f(x)|\xi] + \frac{1}{2} E[\psi_{\mu^{\Gamma}}(d_{\mu^{\Gamma}}(x); x)|\xi].
\]
Thus,
\[
E[f(x)|\xi] - E[f(x^+)|\xi] \geq \frac{1}{2} \left( E[f(x)|\xi] - E[\psi_{\mu^{\Gamma}}(d_{\mu^{\Gamma}}(x); x)|\xi] \right)
\geq \frac{1}{2} \left( E[\psi_{\mu^{\Gamma}}(0; x)|\xi] - E[\psi_{\mu^{\Gamma}}(d_{\mu^{\Gamma}}(x); x)|\xi] \right)
\geq \frac{\mu}{2} E\left[\|d_{\mu^{\Gamma}}(x)\|^2 \right] - \frac{q}{2n} E\left[\|d_{\mu^{\Gamma}}(x)\|^2 \right].
\]
The last inequality is an implication of $\mu$-strongly convexity of $\psi_{\mu^{\Gamma}}(d_{\mu^{\Gamma}}(x); x)$, see (4.13). Getting back to the notation dependent on iterations $m$ and summing up w.r.t. entire history, we obtain:
\[
\frac{q}{2n} \sum_{k=0}^{m} E[(M_q(\mathbf{x}^k, \Gamma))^2] \leq f(x^0) - f^*,
\]
from where the theorem follows. \hfill\square

**Remark 4.12.** Theorem 4.11 shows that for $q = 2$, the convergence rate of Algorithm 1 coincides with the convergence of Algorithm 2-RCD [22, Theorem 5]. However, for $q > 2$ our algorithm converges faster than the algorithm from [22]. Note that the algorithm from [22] allows for choosing blocks of coupled variables. For this comparison, we assume that these blocks only contain single variables.
5 Numerical experiments

In this section we present numerical results for solving the densest $k$-subgraph problem and the eigenvalue complementarity problem. Numerical results are performed on an Intel i7-1165G7 2.80GHz processor with 4 cores, and 16GB RAM.

In the following numerical experiments, we verify whether the computed points $x^*$ are stationary. Theoretically, by using Lemma 3.1 this can be done by checking whether $M_\mu(x^*, \Gamma) = ||d_\mu(x^*)|| = 0$. However, in practice, $n$ can be large, which prohibits the computation of $\mu = (\frac{n-1}{q-1})$, and consequently, the computation of $M_\mu(x^*, \Gamma)$. We therefore consider another, related value.

A point $x^* \in P$, see (2.2), is a stationary point of $f$ if $(\nabla f(x^*), x - x^*) \geq 0$ for all $x \in P$ (see e.g., [2] Proposition 2.1.2). We define the function $\tilde{M} : P \to \mathbb{R}_+$, as

$$\tilde{M}(x^*) := - \min_{x \in P} \langle \nabla f(x^*), x - x^* \rangle$$

(5.1)

Note that $\tilde{M}$ can be computed by solving a linear programme on $n$ variables. Moreover, $\tilde{M}(x^*) = 0$ implies that $x^*$ is stationary.

One can download our codes from the following link:

https://github.com/LMSinjorgo/q-RCCD_algorithm

5.1 The densest $k$-subgraph problem

Let $G$ be an undirected graph with $n$ vertices and $k \in \mathbb{N}$ a given number such that $3 \leq k \leq n-2$. The densest $k$-subgraph problem is the problem of finding a subgraph of $G$ with $k$ vertices and the maximum number of edges. The D$k$S problem is known to be NP-hard. The problem plays a role in analyzing web graphs and social networks, but also in computational biology and cryptography. There does not exist a polynomial time approximation scheme for the D$k$S problem in general graphs [12]. Exact approaches for solving the D$k$S problem report solutions for instances only up to 160 vertices, see e.g., [14]. Thus, obtaining good bounds for the problem is a crucial step for solving large scale instances.

The D$k$S problem can be formulated as follows:

$$\max \left\{ x^\top A x : \sum_{i=1}^{n} x_i = k, \ x_i \in \{0, 1\}, \ i \in [n] \right\},$$

(5.2)

where $A$ is the adjacency matrix of $G$. We consider here the continuous relaxation of the D$k$S problem, i.e.,

$$\max \left\{ x^\top A x : \sum_{i=1}^{n} x_i = k, \ 0 \leq x_i \leq 1, \ i \in [n] \right\},$$

(5.3)

and use the $q$-RCCD algorithm to compute stationary points for various instances.

The next lemma shows that, in case of the D$k$S problem, one can easily compute a Lipschitz constant in (2.9).

Lemma 5.1. Let $f(x) = x^\top A x$, where $A$ is the adjacency matrix of a graph. Then, the Lipschitz condition (2.9) is satisfied for $L_J = 2||A_J||$, where $A_J$ is the $|J| \times |J|$ principal submatrix of $A$, with columns and rows indexed by $J$.

Proof. Let $J \subseteq [n]$, and $\nabla_J f$ and $s_J$ be defined as in (2.9). For any $J \subseteq [n]$, we have $\nabla_J f(x) = (2Ax)_J$, and so $\nabla_J f$ is linear. Thus,

$$||\nabla_J f(x + s_J) - \nabla_J f(x)|| = ||\nabla_J f(s_J)|| \leq 2||A_J|| ||s_J||.$$

Remark 5.2. Because of norm equivalency in $\mathbb{R}^n$, it is not important to use a particular norm $||A_J||_p$ in the calculation. Since $A$ is the adjacency matrix of a simple graph, $A_J$ is symmetric which implies that $||A_J||_\infty = ||A_J||_1 = \Delta$, where $\Delta$ denotes the largest degree in the subgraph. We note that, for large subgraphs, computing $\Delta$ is significantly cheaper than computing $||A_J||_2$.  

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To compute projections onto the feasible set of \( (5.3) \), see (2.13), we use the algorithm proposed in \cite{22}, which runs in \( O(q^2) \).

We first test the performance of the \( q \)-RCCD algorithm on the Erdős-Rényi graphs and the Erdős-Rényi graphs with planted subgraphs. Then, we test our algorithm on graphs from the literature. In the Erdős-Rényi graph \( G_p(n) \) with \( n \) vertices, each edge is generated independently of other edges with probability \( p \in (0, 1] \). The graph obtained by planting a complete subgraph with \( m \) vertices in \( G_p(n) \) is known as the Erdős-Rényi graph with a planted subgraph and denoted by \( P_p^m(n) \).

In Table 1 for graph \( G_{0.5}(2048) \) and \( k = 30 \), we show the performance of the \( q \)-RCCD algorithm, for \( q \in \{ 2, 50, 100, 200, 500, 750 \} \) and 500, 1000, 5000, 10000 iterations. We report the obtained lower bound on (5.3) and the required computation time in seconds, averaged over 30 different runs of the \( q \)-RCCD algorithm. As starting vector, we take \( x = (k/n)1_n \). Observe that the bounds are stronger for higher values of \( q \), at the expense of greater computation times. Furthermore, smaller values of \( q \) require more iterations to obtain similar bounds obtained by higher values of \( q \). For example, for \( q = 100 \) and \( 10^4 \) iterations the objective value is 724.411, and for \( q = 200 \) and only \( 5 \cdot 10^3 \) iterations the objective value is 724.874. Thus, the two cases lead to similar bounds while the number of iterations times \( q \) is the same, at \( 10^6 \).

In Table 2 we test the \( q \)-RCCD algorithm for graph \( P_{0.3}^{100}(4096) \). For each \( q \in \{ 200, 300, 400, 500 \} \), we run the \( q \)-RCCD algorithm for 750 and 1000 iterations for the \( P_{0.3}^{100}(4096) \) graph. Note that \( P_{0.3}^{100}(4096) \) attains an optimal objective value of 9900 with high probability. In the table, we report the minimum, median, mean and maximum determined objective values averaged over 100 different runs of the \( q \)-RCCD algorithm. We also report the average value of \( \bar{M}(x) \), see (5.1), using the \( x \) computed at the final iteration, and the average time (in seconds) for each \( q \) and two iteration numbers. Observe that with relatively little computational effort, we are able to determine near optimal bounds for \( q = 500 \) in 750 iterations. Moreover, for \( q = 500 \) and 1000 iterations, the \( q \)-RCCD algorithm attains the optimal value of 9900 in every run, while the average value of the stationarity measure \( \bar{M}(x) \) is 3.1E−06.

In Table 3 we compare the \( q \)-RCCD algorithm with three different numerical methods for solving the DkS problem on 12 graphs. In this table we set \( k = 200 \). The methods we compare are: the ADMM, the PGM, and a version of the 2-Random Coordinate Descent (2-RCD) from \cite{22}. The ADMM for the DkS problem is proposed by Bombina and Ames \cite{22}, and we use the code provided by the authors.\(^1\) The PGM is the deterministic version of the \( q \)-RCCD algorithm, i.e., when \( q = n \). The 2-RCD algorithm, with parameter \( B \in \mathbb{N} \), implemented here differs from the \( q \)-RCCD algorithm in only the variable selection method: given the parameter \( B \in \mathbb{N} \), we divide the \( n \) variables in blocks, each of size \( \lfloor B/n \rfloor \). In every iteration, two blocks are picked, uniformly at random, and the \( 2B \) variables in these blocks are updated in the same way as the \( q \)-RCCD algorithm. For the \( q \)-RCCD algorithm and graphs with edge density of approximate order \( 10^{-0.1} \), we set \( q = 100 \). For all other graphs we set \( q = 1500 \). For the 2-RCD algorithm, we chose \( B = 10 \) for every graph because the preliminary results show that for this \( B \) the algorithm provides the highest possible values.

Each method runs for a set time (10, 60 or 120 seconds). We report the objective value, denoted Obj. We also display a lower bound on the optimal value of the DkS problem, denoted LB, which is obtained as follows: given the nonbinary \( x \in \mathbb{R}^n \) returned by one of the four algorithms, we set the \( k \) highest values in \( x \) to 1, and the rest to 0. This produces a binary vector feasible to (5.2), and we use this feasible vector to compute a lower bound LB. Since ADMM returns a matrix \( X \in \mathbb{R}^{n \times n} \), we perform the same operation on the \( n \)-dimensional diagonal of \( X \). Additionally, for all methods other than ADMM, we report the value of \( \bar{M}(x) \), see (5.1), using the nonbinary \( x \) obtained at the final iteration.\(^2\) For the \( q \)-RCCD and 2-RCD algorithms, the reported values Obj, LB and \( \bar{M} \) are averaged over three runs.

The ADMM algorithm requires computing a singular value decomposition (SVD) at each iteration. In Table 3, entries for which the SVD led to a memory error, or for which the SVD could not be computed in the allotted maximum time, are marked with *.

Seven of the 12 tested graphs in Table 3 are taken from \cite{16, 25}, and more information on those

\(^1\) Code available at https://github.com/pbombina/admmDSM

\(^2\) In case \( B \) does not divide \( n \), we set \( B \) as the closest divisor of \( n \).

\(^3\) Due to numerical imprecision, \( \bar{M} \) is sometimes computed as a negative number.
Table 1: Results averaged over 30 runs on the graph $G_{0.5}(2048) \ (k = 30)$

| $q$ | 2   | 50  | 100 | 200 | 500 | 750 | Iter.   |
|-----|-----|-----|-----|-----|-----|-----|---------|
| Obj.| 452.413 | 630.422 | 641.349 | 654.440 | 662.429 | 669.300 | $\frac{1}{2} \cdot 10^4$ |
| Time (s) | 0.028 | 0.070 | 0.136 | 0.292 | 0.845 | 1.598 |         |
| Obj.| 454.781 | 676.842 | 689.236 | 700.214 | 704.653 | 710.191 | $10^3$ |
| Time (s) | 0.055 | 0.117 | 0.215 | 0.475 | 1.434 | 2.836 |         |
| Obj.| 464.721 | 713.992 | 720.316 | 724.874 | 728.748 | 728.534 | $5 \cdot 10^3$ |
| Time (s) | 0.112 | 0.299 | 0.635 | 1.605 | 5.749 | 12.093 |         |
| Obj.| 469.149 | 725.579 | 724.411 | 729.379 | 732.162 | 731.733 | $10^4$ |
| Time (s) | 0.162 | 0.494 | 1.106 | 2.941 | 10.913 | 23.371 |         |

Table 2: Results averaged over 100 runs on the graph $P_{100}^{10}(4096) \ (k = 100)$

| $q$ | 200 | 300 | 400 | 500 | Iter. |
|-----|-----|-----|-----|-----|-------|
| Obj. Min. | 9849.015 | 9886.895 | 9890.530 | 9899.955 | 750 |
| Median | 9888.472 | 9899.841 | 9899.998 | 9900.000 |       |
| Mean | 9884.331 | 9898.664 | 9899.803 | 9899.999 |       |
| Max. | 9899.932 | 9899.994 | 9900.000 | 9900.000 |       |
| Avg. $M$ | 1.6E+01 | 1.3E+00 | 2.0E−01 | 7.1E−04 |       |
| Avg. T. (s) | 1.119 | 1.902 | 2.895 | 4.087 |       |
| Obj. Min. | 9853.434 | 9889.105 | 9892.962 | 9900.000 | 1000 |
| Median | 9896.287 | 9899.971 | 9900.000 | 9900.000 |       |
| Mean | 9892.447 | 9899.697 | 9899.927 | 9900.000 |       |
| Max. | 9899.963 | 9900.000 | 9900.000 | 9900.000 |       |
| Avg. $M$ | 7.6E+00 | 3.0E−01 | 7.3E−02 | 3.1E−06 |       |
| Avg. T. (s) | 1.491 | 2.524 | 3.835 | 5.420 |       |

graphs can be found in these references. For these seven graphs, we removed isolated vertices and self-loops. Among the other five graphs are three Erdős-Rényi graphs and two Erdős-Rényi graphs with planted subgraphs. For each graph, the number of vertices, edges, and edge-densities, are given by $|V|, |E|$ and $D := |E| / (|V|^2 / 2)$ respectively.

Table 3 demonstrates the superior performance of the $q$-RCCD algorithm for solving the D$k$S problem compared to the other algorithms. The $q$-RCCD algorithm provides the best objective values and lower bounds for most graphs. The very small values of $\tilde{M}$, which are obtained for all non-Erdős-Rényi graphs, indicate that the $q$-RCCD algorithm converges to a stationary point. For the Erdős-Rényi graphs (with possible planted subgraphs), a clear evidence of stationarity might be obtained by running the $q$-RCCD algorithm for more than two minutes.
Table 3: Performance of different methods for solving the D&S problem ($k = 200$)

| $G$          | $|V|$ | $|E|$ | $D$ | ADMM | PGM | q-RCCD | 2-RCD ($B = 10$) | T. (s) |
|--------------|------|------|-----|------|------|--------|------------------|-------|
|              |      |      |     | Obj. | LB   | $M$    | Obj.             |       |
| C1000-9      | 1000 | 450079 | 9.0E-01 | 5025.2 | 100   | 37911.7 | 37911.7 | 1.4E-10 | 60    |
|              |      |       |      | 3025.2 | 100   | 37911.7 | 37911.7 | 1.4E-10 | 60    |
| CA-AstroPh   | 18771 | 198050 | 1.1E-03 | 4908.8 | 501.0 | 34216.0 | 34216.0 | 1.4E-01 | 60    |
|              |      |       |      | 4908.8 | 501.0 | 34216.0 | 34216.0 | 1.4E-01 | 60    |
| CA-CondMat   | 23133 | 93439 | 3.5E-04 | 2854.6 | 454.1 | 43150.1 | 43150.1 | 1.4E-01 | 60    |
|              |      |       |      | 2854.6 | 454.1 | 43150.1 | 43150.1 | 1.4E-01 | 60    |
| $G_{0.75}(10000)$ | 10000 | 37491205 | 7.5E-01 | 30484.3 | 5140.1 | 31015.7 | 31015.7 | 7.0E-01 | 60    |
|              |      |       |      | 30484.3 | 5140.1 | 31015.7 | 31015.7 | 7.0E-01 | 60    |
| $G_{0.7}(8192)$ | 8192 | 23485837 | 7.0E-01 | 30484.3 | 5140.1 | 31015.7 | 31015.7 | 7.0E-01 | 60    |
|              |      |       |      | 30484.3 | 5140.1 | 31015.7 | 31015.7 | 7.0E-01 | 60    |
| $G_{0.8}(8192)$ | 8192 | 26840496 | 8.0E-01 | 30484.3 | 5140.1 | 31015.7 | 31015.7 | 7.0E-01 | 60    |
|              |      |       |      | 30484.3 | 5140.1 | 31015.7 | 31015.7 | 7.0E-01 | 60    |
| $P_{100}^{0.5}(4096)$ | 4096 | 4193301 | 5.0E-01 | 27800.2 | 27801.0 | 27801.0 | 27801.0 | 7.0E-01 | 60    |
|              |      |       |      | 27800.2 | 27801.0 | 27801.0 | 27801.0 | 7.0E-01 | 60    |
| $P_{100}^{0.5}(3812)$ | 8192 | 11747692 | 3.5E-01 | 27800.2 | 27801.0 | 27801.0 | 27801.0 | 7.0E-01 | 60    |
|              |      |       |      | 27800.2 | 27801.0 | 27801.0 | 27801.0 | 7.0E-01 | 60    |
| Wiki-Vote    | 7115 | 100762 | 4.0E-03 | 14194.0 | 12860.0 | 14194.0 | 14194.0 | 7.0E-01 | 60    |
|              |      |       |      | 14194.0 | 12860.0 | 14194.0 | 14194.0 | 7.0E-01 | 60    |
| brock800-1   | 800 | 207505 | 6.5E-01 | 20952.8 | 20952.8 | 20952.8 | 20952.8 | 7.0E-01 | 60    |
|              |      |       |      | 20952.8 | 20952.8 | 20952.8 | 20952.8 | 7.0E-01 | 60    |
| p2p-Gnutella4 | 10876 | 39994 | 6.8E-04 | 20952.8 | 20952.8 | 20952.8 | 20952.8 | 7.0E-01 | 60    |
|              |      |       |      | 20952.8 | 20952.8 | 20952.8 | 20952.8 | 7.0E-01 | 60    |
| p2p-Gnutella9 | 8114 | 26013 | 7.9E-04 | 3240.0 | 3240.0 | 3240.0 | 3240.0 | 7.0E-01 | 60    |
|              |      |       |      | 3240.0 | 3240.0 | 3240.0 | 3240.0 | 7.0E-01 | 60    |
5.2 The eigenvalue complementarity problem

We study the performance of the q-RCCD algorithm on the eigenvalue complementarity (EiC) problem. Patrascu and Necoara [22] also tested their 2-random coordinate descent algorithm on the EiC problem. The numerical results in [22] show that the 2-RCD algorithm provides better performance in terms of objective and computational times than the efficient algorithm for the EiC problem from [13]. We also compare our algorithm with PGM and the deterministic 2-coordinate descent algorithm from [1].

The EiC problem is an extension of the classical eigenvalue problem whose non-convex logarithmic formulation is as follows (see [11]):

$$
\begin{align*}
\max_{x \in \mathbb{R}^n} & \quad f(x) = \ln \left( \frac{x^\top Ax}{x^\top Bx} \right) \\
\text{s.t.} & \quad 1^\top x = 1, \ x \geq 0,
\end{align*}
$$

(5.4)

where we assume that both $x^\top Ax$ and $x^\top Bx$ are positive for all feasible $x$. For some $J \subseteq [n]$ and any matrix $A$, we define $A_J$ as the submatrix of $A$ generated from the rows and columns of $A$ indicated by $J$. Then, as proven in [22, Lemma 9], function $f$ from (5.4), with $A$ and $B$ elements of (5.7), satisfies Lipschitz condition (2.9) for

$$
L_J = 2 \left( \frac{\|A_J\|_{x^\top Ax}}{x^\top Ax} + \frac{\|B_J\|_{x^\top Bx}}{x^\top Bx} \right) \leq 2n \left( \frac{\|A_J\|}{\min_{i \in [n]} A_{ii}} + \frac{\|B_J\|}{\min_{i \in [n]} B_{ii}} \right).
$$

(5.5)

In the algorithm from [22], the above upper bound on $L_J$ is taken as Lipschitz constant. In contrast, in our algorithm we take $L_J$ as Lipschitz constant, and take for $\| \cdot \|$ the 1-norm (that is, the largest absolute column sum).

It is not difficult to verify that

$$
\nabla f(x) = 2 \left( \frac{1}{x^\top Ax} A - \frac{1}{x^\top Bx} B \right) x.
$$

(5.6)

Note that for the update (2.13), we only need $\nabla_J f$, and so it is not necessary to compute all the entries of $Ax$ and $Bx$ in (5.6), but only those indicated by $J$.

To compute (2.13), we use the algorithm proposed in [7], for projecting onto the (scaled) simplex. This algorithm has worst case complexity $O(q^3)$. However, its complexity is observed linear in practice, see [7].

We test the $q$-RCCD algorithm on the EiC problem, for symmetric matrices $A$ and $B$ from the following set:

$$
\begin{align*}
\{ & A \in \mathbb{R}^{n \times n} : A_{ij} = A_{ji} \geq 0, \ A_{ii} = 0.001 + |Z_i| \ \forall i, j \in [n], \\
& Z_i \text{ standard normal random variable } \forall i \in [n] \}
\end{align*}
$$

(5.7)

Specifically, we generate $A$ and $B$ from the set (5.7), such that their nonzero off-diagonal entries are randomly drawn uniform variables on the range $(0, 1]$. This ensures that $x^\top Ax > 0$ and $x^\top Bx > 0$, for all feasible $x$. Hence, $f(x)$, $L_J$ and $\nabla f(x)$, see (5.4), (5.5) and (5.6), are well defined for all feasible $x$. We denote by $d$ the density of matrices (i.e., the number of nonzeros divided by $n^2$), which we vary across the numerical experiments.

In Table 4, we take $n = 10^5$ and $d = 10^{-4}$, and report the objective values attained by the $q$-RCCD algorithm, averaged over 50 runs, for different values of $q$ and the number of iterations. We report the determined lower bound of (5.3) as $\ell(x)$ and the required computation time in seconds. Note that, while the number of iterations times $q$ is constant per row, the required computation time is not. Based on Table 4, larger values of $q$, i.e., $q \geq 20$, yield both better bounds and lower computation times. This is a result of the low density $d$: since $d$ is low, computing gradients of size $q$, with $q$ large, is relatively cheap, and so the results improve when taking larger gradient vectors (even the PGM, with $q = n$, performs well on such sparse matrices). When $q$ is small, each iteration requires less time, but also brings little improvement to the objective value. Since every iteration of the $q$-RCCD algorithm incurs considerable overhead from indexing matrices $A$ and $B$ based on $J$, a
Table 4: The $q$-RCCD algorithm for the EiC problem averaged over 50 runs ($n = 10^5$, $d = 10^{-4}$)

| $q$ | 2   | 5   | 20  | 50  | 100 | 200 | Iter. |
|-----|-----|-----|-----|-----|-----|-----|-------|
| Bound | 46.323 | 56.467 | 73.887 | 74.130 | 72.251 | 68.403 | $1 \cdot 10^7/q$ |
| Time (s) | 124.130 | 73.017 | 23.055 | 11.461 | 7.705 | 5.984 |
| Bound | 73.526 | 84.044 | 112.827 | 110.987 | 105.747 | $2 \cdot 10^7/q$ |
| Time (s) | 239.199 | 144.536 | 45.873 | 22.951 | 15.376 | 11.981 |
| Bound | 91.455 | 103.666 | 141.461 | 137.785 | 140.309 | 132.169 | $3 \cdot 10^7/q$ |
| Time (s) | 353.920 | 215.922 | 68.906 | 34.461 | 23.076 | 17.974 |

large value of $q$ performs better. A large $q$ ensures that you perform less iterations in the same time as a small $q$, and thus lose less overhead time. Moreover, the lower number of iterations compared to small $q$, is compensated by the fact that iterations for large $q$ improve the objective value more, compared to small $q$.

Next, we investigate the convergence properties of the $q$-RCCD algorithm towards a stationary point. For this purpose, we generate two matrices $A$ and $B$ from (5.7), with $n = 3000$, and density $d = 10^{-3}$. We set $q = 750$ and run the $q$-RCCD algorithm for $7.5 \cdot 10^5$ iterations, while tracking the value of $\log_{10}(\tilde{M})$, see (5.1), during these iterations. We repeat this procedure 50 times, and present the averaged results in Figure 1. The figure clearly demonstrate convergence of the algorithm towards a stationary point.

![Figure 1: Convergence of the $q$-RCCD algorithm ($n = 3000$, $d = 10^{-3}$, $q = 750$)](image)

Lastly, in Table 5 we investigate the performance of the $q$-RCCD algorithm compared to PGM, 2-RCD and the deterministic 2-coordinate descent algorithm by Amir Beck [1], abbreviated as AB2CD. At each iteration, the AB2CD algorithm picks the two coordinates that, when updated, decrease a specific stationarity measure the most. Let us briefly present the approach used in AB2CD for the index selection, specifically for the maximization problem (5.4). AB2CD computes first $\text{argmax}_i (\nabla f(x)_i)$, for the current solution $x$. Then AB2CD constructs another vector that depends on $\max_i (\nabla f(x)_i)$, and computes the index corresponding to the largest element of this new vector. In the EiC problem, both these vectors are dense and $n$-dimensional.

We test these algorithms on six different sets of matrices $A$ and $B$ from (5.7), with varying sizes $n$ and densities $d$, which refers to the density of both $A$ and $B$. The number of nonzeros in matrix is denoted $\text{nnz}$, and is given by $n^2d$. Note that the tested matrices are much more dense than those

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[^1]: It can occur that $\tilde{M} = 0$, in which case the logarithm is not defined. To circumvent this, we instead track the value of $\log_{10}(\max\{\tilde{M}, 10^{-16}\})$. 

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Table 5: Objective values of different methods for the large scale EiC problem \((q = 50, B = 25)\)

| \(\log_{10}(n)\) | \(d\) | \(\text{nnz}(:, n^d)\) | AB2CD | PGM | \(q\)-RCCD | 2-RCD | T. (min) |
|---|---|---|---|---|---|---|---|
| 5 | \(\frac{1}{2} \cdot 10^{-2}\) | \(\frac{1}{2} \cdot 10^8\) | 852.80 | 79.57 | 494.29 | 286.62 | 15 |
| | | | 852.80 | 120.43 | 734.96 | 426.91 | 30 |
| | | | 852.80 | 178.42 | 952.72 | 588.05 | 60 |
| 5 | \(10^{-2}\) | \(10^8\) | 2202.94 | 34.00 | 240.89 | 195.39 | 15 |
| | | | 2202.94 | 53.30 | 383.76 | 296.83 | 30 |
| | | | 2202.94 | 80.81 | 590.80 | 405.25 | 60 |
| 5 | \(2 \cdot 10^{-2}\) | \(2 \cdot 10^8\) | 1295.42 | 9.99 | 147.98 | 88.59 | 15 |
| | | | 1295.42 | 19.60 | 253.13 | 142.84 | 30 |
| | | | 1295.42 | 30.95 | 387.43 | 241.92 | 60 |
| 6 | \(10^{-4}\) | \(10^8\) | 1.39 | 14.44 | \textbf{49.53} | 44.52 | 15 |
| | | | 2.56 | 26.24 | \textbf{80.49} | 70.41 | 30 |
| | | | 5.60 | 43.11 | \textbf{136.95} | 114.85 | 60 |
| 6 | \(10^{-3}\) | \(10^9\) | 1.03 | 1.06 | \textbf{5.86} | 4.87 | 15 |
| | | | 1.15 | 1.13 | \textbf{12.19} | 8.78 | 30 |
| | | | 1.56 | 1.30 | \textbf{23.16} | 15.99 | 60 |
| 7 | \(10^{-5}\) | \(10^9\) | 1.00 | 1.40 | \textbf{7.45} | 6.94 | 15 |
| | | | 1.01 | 1.98 | \textbf{15.26} | 14.38 | 30 |
| | | | 1.04 | 3.73 | \textbf{25.75} | 23.78 | 60 |

from numerical results in [22]. We report the attained objective values as \(e(x)\), see (5.4), per method at 15, 30 and 60 minutes. For the \(q\)-RCCD and 2-RCD algorithms, we average the objective values over five runs, and have set \(q = 50\) and blocksize \(B = 25\), which seemed to attain high objective values based on preliminary numerical tests. We display the results in Table 5 and mark the best objective value in boldface.

For \(n = 10^5\), AB2CD performs well: it converges to a stationary point in 15 minutes, but it is unable to escape this local maximum. In the same instances, the convergence of \(q\)-RCCD to stationary points is slower. However, due to its random element, the \(q\)-RCCD algorithm may find more favorable (with higher objective) stationary points. This allows the \(q\)-RCCD algorithm to outperform the AB2CD in the first instance at 60 minutes.

For all three instances with \(n = 10^5\), the \(q\)-RCCD algorithm is unable to converge to any stationary point within the time limit. This claim is justified as follows: the output of our algorithm shows that the objective values of the \(q\)-RCCD algorithm at minute 60, was noticeably higher compared to the objective values at minute 59 (average increase of 1.01% over all \(n = 10^5\) instances).

When \(n = 10^6\) or \(n = 10^7\), \(q\)-RCCD is clearly the superior algorithm. The iterations of the AB2CD and PGM algorithms become too costly due to their operations on dense \(n\)-dimensional vectors. The computational cost of 2-RCD is similar to that of \(q\)-RCCD. However, it is still consistently outperformed by the \(q\)-RCCD algorithm, as the data shows.

6 Conclusions

In this paper, we propose a random coordinate descent algorithm for minimizing a non-convex objective function subject to one linear constraint and bounds on the variables, see Algorithm 1. The \(q\)-RCCD algorithm randomly selects \(q\) \((q \geq 2)\) variables and updates them based on the solution of an appropriate convex optimization problem, see (2.12). We prove that any accumulation point of the sequence generated by the \(q\)-RCCD algorithm, using the uniform distribution, is a stationary point for (2.1), see Theorem 4.10. The convergence rate for the expected values of the stationarity measure is given in Theorem 4.11. The convergence rates of the 2-RCCD algorithm and the algorithm from [22], for the case that blocks are of size one, coincide. The convergence rate of the \(q\)-RCCD algorithm improves for \(q > 2\).

We test the \(q\)-RCCD algorithm for solving the DkS problem and the EiC problem in Section 5.
Extensive numerical results show that the quality of bounds improves as $q$ increases, and that the $q$-RCCD algorithm converges towards a stationary point. In both these problems, the $q$-RCCD algorithm compares favourably with the other algorithms. In particular, Table 5 demonstrates that for large scale EiC problems ($n \geq 10^6$), the $q$-RCCD algorithm significantly outperforms AB2CD and PGM, and also improves on the 2-RCD algorithm.

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