Universality of $1/Q$ corrections to jet-shape observables rescued

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Abstract

We address the problem of potential non-universality of the leading $1/Q$ power corrections to jet shapes emerging from the non-inclusive character of these observables. We consider the thrust distribution as an example and analyse the non-inclusive contributions which emerge at the two-loop level. Although formally subleading in $\alpha_s$, they modify the existing naïve one-loop result for the expected magnitude of the power term by a factor of order unity. Such a promotion of a subleading correction into a numerical factor is natural since the non-perturbative power terms are explicitly proportional to powers of the QCD scale $\Lambda$ which can be fixed precisely only at the two-loop level. The “jet-shape scaling factor” depends on the observable but remains perturbatively calculable. Therefore it does not undermine the universal nature of $1/Q$ power corrections, which remain expressible in terms of the universal running coupling and universal soft-gluon emission.

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1 Introduction

The analysis of power-behaved contributions to infrared and collinear safe characteristics of hard QCD processes has recently been developed as a method to quantify non-perturbative effects [1–8]. It has been recognised that the leading power corrections to infrared/collinear safe quantities are determined by an incomplete cancellation of soft-gluon contributions. A QCD process at a hard scale $Q \gg \Lambda$ is determined by the small space-time intervals between the quarks and gluons involved. In spite of that, at the level of power suppressed corrections, the cross section of such a process may acquire a contribution from a soft gluon which travels a finite distance $1 \text{fm} \gg 1/Q$ and thus is sensitive to the non-perturbative interaction domain.

The sensitivity of a given observable to soft-gluon radiation can be studied perturbatively to determine the power $n$ of the expected leading $Q^{-n}$ correction. Such theoretical input is valuable on its own, especially for jet physics which deals with intrinsically Minkowskian quantities and therefore typically receives no hints from the Euclidean-based OPE technology [9]. For example, an analysis of large-distance soft-gluon effects predicts that the properties of jets assembled with the use of the (once standard) JADE jet finder are contaminated by confinement effects at the level of $Q^{-1}$ corrections, while the Durham ($k_{\perp}$) algorithm produces jets whose rates will depart from perturbative predictions only at the $Q^{-2}$ level [1].

More challenging is the prediction of the magnitude of expected power corrections. For a single observable this would be impossible given the present state of the art. Nevertheless it is natural to expect that soft-gluon universality implies proportionality between the magnitudes of power-behaved terms in different observables [2]. Along these lines one aims to predict the relative sizes of power corrections to observables belonging to the same class, linking together, for example, jet mass and thrust distributions [3] with $1/M_{Q}$ effects in heavy quark energy spectra [4,5] ($n = 1$) or power corrections to the Drell-Yan $K$-factor [2,6] with those in the Gross-Llewellyn-Smith DIS sum rule and $e^{+}e^{-}$ fragmentation functions [7] ($n = 2$).

However, some doubt has been expressed as to the universality of power corrections to jet shapes because the latter are not fully inclusive with respect to final-state branching of the soft gluon under focus. To the best of our knowledge this question was first considered by Nason and Seymour in [8].

The notion of soft-gluon universality contains two ingredients. First, it exploits the universal character of the soft-radiation matrix element in the spirit of Low’s theorem [10]. Secondly, it requires universality of the gluon radiation intensity (running coupling).

The running QCD coupling in hard processes emerges after inclusive integration over the gluon decay products, resulting typically in $\alpha_{s}(k^2_{\perp})$ with $k_{\perp}$ the gluon transverse momentum [11]. In QED it suffices to consider the virtual photon self-energy blobs (“bubbles”), while in the QCD context there are also the corrections to the gluon emission vertex. The dispersive representation has been used in [7] to give meaning to the QCD running coupling at low scales [4,12].

The value of a jet-shape variable depends, however, on the kinematics of all final state particles (hadrons, partons). As long as the observable is collinear safe the (quasi-)collinear
gluon decays, which are the main contributors to the running $\alpha_s$, do not affect the observable and therefore can be treated inclusively. However, in the “large-angle” gluon decay region, the inclusive interplay between real and virtual gluon loops gets broken because a real decay may correspond to a value of the observable different from that for the case of a virtual loop. These contributions seem to undermine the above-mentioned universality.

In the case of the thrust distribution, for example, the naïve inclusive treatment applies as long as the decay products fall into the same hemisphere. Then the contribution to thrust, $T$, can be attributed to the parent gluon, and integration over the virtuality of the latter results in the running coupling in a standard way.

The existence of the part of the decay phase space in which the two gluon offspring partons (quarks or gluons) happen to fly into opposite hemispheres, has two consequences. First, this region has to be treated separately as its contribution to $T$ depends on details of the gluon decay matrix element. At the same time the effective intensity of emission of the parent gluon as a whole gets modified. This is due to the fact that to reconstruct the running coupling one sums inclusively over the gluon decay products, while in this case one modifies the final state phase space. Thus the running coupling (its “spectral density”) loses part of its standard support.

It has been argued that such a non-inclusive contribution is higher order in $\alpha_s$. Its quark-antiquark (“Abelian”) part was studied numerically in [8] and found to be small.

In this paper we analyse the question of (non-)universality of the leading power corrections to jet shapes and thrust in particular. We find that the naïve inclusive treatment has to be modified by two effects. The first is due to non-inclusive configurations of the parent gluon decay (non-inclusive correction). The second is due to an incomplete compensation between real and virtual logarithmic terms coming from the soft-gluon splitting into gluons (inclusive correction). This correction is absent in the contribution from the gluon splitting into a quark-antiquark pair. We show that both non-inclusive and inclusive corrections, although formally subleading in $\alpha_s$, get promoted to a factor of order unity in the magnitude of the $1/Q$ power correction. As a consequence, previously obtained results based on the naïve inclusive treatment of an observable $j$ acquire a “rescaling factor” $(1+r_j)$ which is calculable but depends on the observable under consideration.

In this sense one can still consider power terms in jet observables as universal, i.e. based on the universal coupling and the universal soft-gluon radiation pattern, although their relative magnitudes differ from the naïve expectations based on one-loop off-shell soft-gluon matrix elements.

For thrust values close to 1, only soft-gluon radiation is essential. For $T$ well below 1, one has contributions both from multiple soft-gluon emission and from rare $N$-jet configurations ($N > 2$). The first contribution has been found [3] to be dominant at least down to $T = 2/3$, the minimum $T$ for any three-jet configuration. As far as power corrections are concerned, it was shown in [8] that the order $\alpha_s(Q^2)$ three-jet configuration does not develop a $1/Q$ power correction.

Therefore to analyse the thrust distribution at two loops we take into consideration [13] particle ensembles consisting of the primary quark-antiquark pair accompanied by partons
originating from multiple emission of primary soft gluons and employ the existing all-order resummed perturbative prediction [14].

We extract the leading \(1/Q\) power term in the radiator, namely the exponent of the Mellin-transformed \(T\)-distribution. We do this in the framework of the treatment of non-perturbative power corrections to infrared/collinear safe observables introduced in [7], based on the notion of an infrared-finite strong coupling which differs from the perturbative form in the infrared region.

At the two-loop level, non-inclusiveness of the thrust reveals itself and the characteristic QCD scale \(\Lambda\) becomes precisely defined through the fixing of the scheme for the running coupling. To this end we choose the CMW renormalisation scheme [15], in which \(\alpha_s\) is defined as the intensity of soft-gluon radiation.

Section 2 contains the necessary ingredients for the two-loop analysis of the radiator of the thrust distribution: kinematics, structure of the soft-gluon emission followed by gluon splitting into \(qq\) and gluon pairs, scheme fixing, and thrust resummation. In section 3 we recall the dispersive method of [7] and its application to the thrust distribution in the naïve inclusive approach [3]. Section 4 is devoted to the exact treatment of the problem, the analysis and calculation of the inclusive and non-inclusive contributions to the thrust rescaling factor \((1 + r_T)\). Section 5 contains a discussion of the results and the prospects for the future.

2 Two-loop analysis of thrust distribution

2.1 Thrust: kinematics

Consider a QCD hard process in which the final state consists of a leading quark-antiquark pair with momenta \(p\) and \(\bar{p}\) together with a system of \(N\) partons (quarks and gluons) with momenta \(k_1, \ldots, k_N\). It is convenient to cast the cross section in terms of the Sudakov (light-cone) variables defined with respect to the thrust axis of the event. To this end we introduce two light-like vectors

\[ P^\mu, \bar{P}^\mu; \quad P^2 = \bar{P}^2 = 0; \quad P^\mu + \bar{P}^\mu = Q^\mu, \quad 2(P\bar{P}) = Q^2 \equiv 1, \]

and represent the parton 4-momenta as

\[ k_i^\mu = \beta_i P^\mu + \alpha_i \bar{P}^\mu + (k_{i\perp})^\mu, \quad \alpha_i \beta_i = k_{i\perp}^2 \]

The leading quark momenta become

\[ p = (1 - \sum_{i=1} \beta_i - \beta_p) P + \alpha_p \bar{P} + p_\perp, \]
\[ \bar{p} = (1 - \sum_{i=1} \alpha_i - \alpha_p) \bar{P} + \beta_p P + \bar{p}_\perp, \]
Adding the normalised longitudinal momenta of the secondary partons, \( k_i \parallel \frac{Q}{Q} = |\alpha_i - \beta_i|/2 \), with those of the two quarks, for the thrust value we obtain

\[
T = \frac{1}{2} \sum_i |\alpha_i - \beta_i| + \frac{1}{2} [ (1 - \sum_i (\beta_i - \beta_p)) - \alpha_p ] + \frac{1}{2} [ (1 - \sum_i (\alpha_i - \alpha_p)) - \beta_p ] \\
= 1 - \sum_i \min\{\alpha_i, \beta_i\} - (\alpha_p + \beta_p).
\]

This expression implies that \( p \) and \( \bar{p} \) momenta belong to opposite hemispheres, which is the dominant configuration (the correction is numerically small and relatively suppressed at least as \((1 - T)\) when \( T \) is large).

The values of the “small components” of the quark momenta, \( \alpha_p \) and \( \beta_p \) depend on quark transverse momenta with respect to the thrust axis through the on-mass-shell conditions, \( p^2 = \bar{p}^2 = 0 \). Transverse quark recoils are typically of order \( p^2 \perp \sim \bar{p}^2 \perp \sim \sqrt{N \cdot k^2 \perp} \), with \( k \perp \) characteristic transverse momentum of a secondary parton. We have kept in the estimate a (logarithmic) enhancement factor due to the parton multiplicity \( N \).

As we shall see the leading power correction comes from the region \( \alpha_i \sim \beta_i \) of the order of the transverse momentum (in units of \( Q \)). Therefore in (2.1) we shall neglect \( \alpha_p \) and \( \beta_p \) which are of the order of the transverse momentum squared.

### 2.2 Gluon splitting

**Two-body variables and phase space.** The two-parton phase space is

\[
d\Gamma_2(k_1, k_2) = (4\pi)^4 \frac{d^4k_1 d^4k_2}{(2\pi)^6} \delta(k_1^2) \delta(k_2^2) = \frac{4}{\pi^2} \left( \frac{1}{2} \frac{d\alpha_1}{\alpha_1} d^2k_{1\perp} \right) \left( \frac{1}{2} \frac{d\alpha_2}{\alpha_2} d^2k_{2\perp} \right).
\]

Introducing the relative fraction of \( \alpha \)-components, \( z \), and the invariant mass squared of the two-parton system \( m^2 \) we write:

\[
\alpha \equiv \alpha_1 + \alpha_2, \quad \alpha_1 = z\alpha, \quad \alpha_2 = (1 - z)\alpha,
\]

\[
m^2 = (k_1 + k_2)^2 = (\alpha_1\beta_2 + \alpha_2\beta_1) - 2\vec{k}_{1\perp} \cdot \vec{k}_{2\perp} = z(1 - z) \left[ \frac{\vec{k}_{1\perp}}{z} - \frac{\vec{k}_{2\perp}}{1 - z} \right]^2.
\]

In terms of the scaled parton transverse momenta,

\[
\vec{q}_1 = \frac{\vec{k}_{1\perp}}{z}, \quad \vec{q}_2 = \frac{\vec{k}_{2\perp}}{1 - z}, \quad \vec{q} = \vec{q}_1 - \vec{q}_2,
\]

the phase space takes the form

\[
d\Gamma_2(k_1, k_2) = \frac{d\alpha}{\alpha} dz \frac{d^2q_1}{\pi} \frac{d^2q_2}{\pi} dm^2 \delta \left( \frac{m^2}{z(1 - z)} - (\vec{q}_1 - \vec{q}_2)^2 \right).
\]

For later use we also define

\[
\vec{k}_{1\perp} = \vec{k}_{1\perp} + \vec{k}_{2\perp} = z\vec{q}_1 + (1 - z)\vec{q}_2; \\
\beta = \beta_1 + \beta_2 = \frac{k^2_{1\perp} + m^2}{\alpha}, \quad \beta_1 = \frac{z}{\alpha} q^2_1; \quad \beta_2 = \frac{1 - z}{\alpha} q^2_2.
\]
Matrix elements. The squared matrix elements for the production of two gluons or a quark pair with 4-momenta $k_1, k_2$ read, in the soft limit ($Q k_i \ll Q^2$ (see [13])

$$M^2(k_1, k_2) = 4 C_F^2 W_1 W_2 + 4 C_F \left( M_{gg}^2 + M_{qq}^2 \right)$$

$$4 \left( M_{gg}^2 + M_{qq}^2 \right) = C_A \left( 2 S + H_g \right) + n_f H_q.$$  

(2.4)

The first term, proportional to $C_F^2$, describes the independent (Abelian) emission of two soft gluons off the hard $q \bar{q}$, where

$$W_i \equiv W(k_i) = \frac{p \bar{p}}{(pk_i)(k_i \bar{p})} = \frac{2}{k_{i \perp}^2}$$

is the standard dipole radiation cross section.

The second (non-Abelian) term, proportional to $C_F C_A$, contains two pieces. The $S$ term describes the leading infrared contributions which is singular in the ratio of gluon energies:

$$S \equiv W_{12} + W_{21} - W_1 W_2,$$

where

$$W_{ij} \equiv \frac{(p \bar{p})}{(pk_i)(k_j \bar{p})} = \frac{4}{m^2} \frac{1}{\alpha_i \beta_j}$$

stands for the ordered 2-dipole emission.

$H_g$ and $H_q$ are responsible for parton configurations with energies of the same order. The “hard” parts of the two-gluon ($H_g$) and the two-quark ($H_q$) contributions read

$$H_g \equiv -SR + J^2 - 4 \frac{W(k)}{(k_1 k_2)} , \quad (2.5a)$$

$$H_q \equiv -J^2 + 2 \frac{W(k)}{(k_1 k_2)} . \quad (2.5b)$$

Here

$$J^2 = 2 \left( \frac{(p k_1)(k_2 \bar{p}) - (p k_2)(k_1 \bar{p})}{(k_1 k_2)(p k)(k \bar{p})} \right)^2 = 8 \left( \frac{\alpha_1 \beta_2 - \alpha_2 \beta_1}{m^2 \alpha \beta} \right)^2 ,$$

$$R = \frac{(p k_1)(k_2 \bar{p}) + (p k_2)(k_1 \bar{p})}{(p k)(k \bar{p})} = \frac{\alpha_1 \beta_2 + \alpha_2 \beta_1}{\alpha \beta} ,$$

(2.6)

and $W(k) = (p \bar{p})/(p k)(k \bar{p})$ is the dipole emission of $k = k_1 + k_2$.

We may write the squared matrix elements in the following form:

$$M_{gg}^2 + M_{qq}^2 = \frac{M^2}{m^2 (z q_1^2 + (1 - z) q_2^2)} ,$$

(2.7)

where

$$M^2 = 2 C_A \left( 2 S + H_g \right) + 2 n_f H_q .$$

(2.8)

The distributions determining $M^2$ are functions of the dimensionless ratios $u_i^2$ with

$$\bar{u}_i = \frac{\bar{q}_i}{|q|} .$$
Namely,

\[ 2S = \frac{zu_1^2 + (1-z)u_2^2}{z(1-z)} \left( \frac{1}{u_1^2} + \frac{1}{u_2^2} - \frac{1}{u_1^2 u_2^2} \right), \quad (2.9a) \]

\[ \mathcal{H}_g = -2 + z(1-z) \frac{(u_1^2 - u_2^2)^2}{zu_1^2 + (1-z)u_2^2} - \frac{u_1^2 + u_2^2}{2} \left[ \frac{1}{u_1^2} + \frac{1}{u_2^2} - \frac{1}{u_1^2 u_2^2} \right], \quad (2.9b) \]

\[ \mathcal{H}_q = 1 - z(1-z) \frac{(u_1^2 - u_2^2)^2}{zu_1^2 + (1-z)u_2^2}. \quad (2.9c) \]

The collinear limit \((q^2 \propto m^2 \to 0)\) corresponds to taking \(u_1^2 \approx u_2^2 \to \infty\) while keeping fixed the modulus of the vector difference,

\[ (\vec{u}_1 - \vec{u}_2)^2 = 1, \]

see (2.2). Observing (see Appendix A) that the azimuthal average of the ratio

\[ \left\langle \frac{(u_1^2 - u_2^2)^2}{zu_1^2 + (1-z)u_2^2} \right\rangle \to 2, \]

we derive the parton splitting functions

\[ 2C_A(2S + \mathcal{H}_g) \to 4C_A \left( \frac{1}{z(1-z)} - 2 + z(1-z) \right), \quad (2.10a) \]

\[ 2n_f \mathcal{H}_q \to 2n_f (1 - 2z(1-z)). \quad (2.10b) \]

Note that the latter expression is twice the standard \(g \to q\bar{q}\) splitting probability because thrust as an inclusive observable does not differentiate between quarks and antiquarks, and the symmetry factor \(1/2!\) will be included in the thrust trigger-function, see (2.23) below.

### 2.3 Gluon decay and the \(\beta\)-function

The \(\beta\)-function is related to the integrated two-parton production probability. Keeping fixed the invariant mass and total transverse momentum of the pair, and integrating \(M^2\) over the azimuthal angle \(\phi\) with respect to \(\vec{k}_\perp\) and over the relative momentum fraction \(z\), it is straightforward to derive (see Appendix A)

\[ \frac{1}{2!} \int_0^1 dz \int_0^{2\pi} \frac{d\phi}{2\pi} \left( M_{gg}^2 + M_{qq}^2 \right) = \frac{1}{m^2(k_\perp^2 + m^2)} \left( -\beta_0 + 2C_A \ln \frac{k_\perp^2(k_\perp^2 + m^2)}{m^4} \right), \quad (2.11) \]

where

\[ \beta_0 = \frac{11}{3} N_c - \frac{2}{3} n_f. \quad (2.12) \]

The logarithmic singularity at \(m^2 \to 0\) comes from the soft singularity in the \(g \to gg\) matrix element (splitting function) and is cancelled by the corresponding virtual correction to the one-gluon radiation vertex. The latter can be written as an integral over an intermediate gluon virtuality \(\mu^2\),

\[ \chi = -\int_0^{\infty} \frac{d\mu^2 k_\perp^2}{\mu^2(k_\perp^2 + \mu^2)} \left\{ 2C_A \ln \frac{k_\perp^2(k_\perp^2 + \mu^2)}{\mu^4} + s \left( \frac{\mu^2}{k_\perp^2} \right) \left( \frac{\alpha_s}{4\pi} \right)^2 \right\}. \quad (2.13) \]
Here we have chosen the form of the logarithmic term so as to cancel exactly the corresponding real emission contribution in (2.11). The mismatch function $s(x)$ vanishes at the origin, $s(0) = 0$, and depends on the scheme one adopts for calculating the divergent vertex correction.

### 2.4 Fixing the scheme for $\alpha_s$

To fix the scheme for defining the perturbative expansion parameter $\alpha_s$, we have to consider first the relation between particle production at order $\alpha_s^2$ and the coupling. To this end we address the non-Abelian contribution to an inclusive quantity such as the quark anomalous dimension.

The inclusive probability for producing a soft gluon or a two-parton system with gluon quantum numbers reads

$$dw = 4C_F \frac{d\alpha}{\alpha} \frac{dk_{\perp}^2}{k_{\perp}^2} \gamma(\alpha_s),$$  

where the “anomalous dimension” $\gamma$ becomes

$$\gamma = \frac{\alpha_0}{4\pi} + \chi(k_{\perp}^2) + \int_0^\infty \frac{dm^2 k_{\perp}^2}{m^2(k_{\perp}^2 + m^2)} \left( \frac{\alpha_s}{4\pi} \right)^2 \left\{ -\beta_0 + 2C_A \ln \frac{k_{\perp}^2(k_{\perp}^2 + m^2)}{m^4} \right\}. \quad (2.15)$$

The singular logarithmic term cancels against that in $\chi(k_{\perp}^2)$, and we arrive at

$$\gamma = \frac{\alpha_0}{4\pi} + c_s \left( \frac{\alpha_s}{4\pi} \right)^2 + \int_0^\infty \frac{dm^2 k_{\perp}^2}{m^2(k_{\perp}^2 + m^2)} \left( \frac{\alpha_s(m^2)}{4\pi} \right)^2 (-\beta_0). \quad (2.16)$$

Here $c_s$ is a constant given by the integral of the scheme-dependent piece of $R_{12}$, see (2.13):

$$c_s = \int_0^\infty \frac{dx}{x(1 + x)} s(x).$$

First we observe that, to the necessary accuracy, $\gamma$ satisfies,

$$k_{\perp}^2 \frac{d}{dk_{\perp}^2} \gamma = -\beta_0 \left( \frac{\alpha_s(k_{\perp}^2)}{4\pi} \right)^2 + \ldots \simeq k_{\perp}^2 \frac{d}{dk_{\perp}^2} \frac{\alpha_s(k_{\perp}^2)}{4\pi}.$$

This means that we can choose $\gamma$ to represent the running coupling $\alpha_s$. Equation (2.16) then becomes a dispersive relation for $\alpha_s(k_{\perp}^2)$, in which the combination of $\alpha_0$ and $c_s\alpha_s^2$ defines the coupling constant at $k_{\perp}^2 = 0$:

$$\gamma \equiv \frac{\alpha_s(k_{\perp}^2)}{4\pi} = \frac{1}{4\pi} \left\{ \alpha_s(0) + \int_0^\infty \frac{dm^2 k_{\perp}^2}{m^2(k_{\perp}^2 + m^2)} \rho_s(m^2) \right\} = -\frac{1}{4\pi} \int_0^\infty \frac{dm^2}{k_{\perp}^2 + m^2} \rho_s(m^2), \quad (2.17)$$

where $\rho_s(m^2)$ is the spectral density, to be discussed later.

This corresponds to choosing the CMW renormalisation scheme \[15\] in which $\alpha_s$ is defined as the intensity of soft-gluon radiation. In another scheme, e.g. \[\overline{\text{MS}}\], the anomalous dimension in the soft limit would contain both $\alpha_s$ and $\alpha_s^2$ terms.
2.5 Thrust: resummation

For a $q\bar{q} + N$-parton final state the thrust distribution becomes

$$\frac{d\sigma}{dT} = \int d\Gamma_N M^2(q_1, q_2, \ldots q_N) \delta(1 - T - \sum_{i=1}^{N} \min\{\alpha_i, \beta_i\}) ,$$

(2.18)

with $M$ the $N$-parton matrix element and $d\Gamma_N$ the corresponding $N$-body phase space. In the soft approximation $(1 - T \ll 1)$ the final partons are either relatively soft gluons emitted off the quark-antiquark line (primary gluons) or their decay products. The matrix element for an ensemble of primary gluons is very simple in the soft approximation, as it is given by the product of independent radiation amplitudes (essentially an Abelian pattern). By taking a Mellin representation for the $\delta$-function,

$$\delta(1 - T - \sum_{i=1}^{N} \min\{\alpha_i, \beta_i\}) = \int d\nu \frac{1}{2\pi i} \exp\{\nu(1 - T)\} \prod_{i=1}^{N} \exp\{-\nu \min\{\alpha_i, \beta_i\}\} ,$$

the thrust distribution takes the form

$$\frac{1}{\sigma} \frac{d\sigma}{dT} = \int \frac{d\nu}{2\pi i} e^{\nu(1-T)} \exp\{\mathcal{R}(\nu)\} .$$

(2.19)

At next-to-leading order, the “radiator” contains two terms, $\mathcal{R}(\nu) = \mathcal{R}_1 + \mathcal{R}_2$, due respectively to one- and two-parton contributions.

One-gluon contribution. The one-gluon contribution, $\mathcal{R}_1$, is given by the following formal expression:

$$\mathcal{R}_1 = 8C_F \int_0^1 \frac{d\alpha}{\alpha} \int \frac{dk_{\perp}^2}{k_{\perp}^2} \left( e^{-\nu\alpha} - 1 \right) \Theta(k_{\perp}^2 - \alpha^2) \cdot \left( \frac{\alpha_s(0)}{4\pi} + \chi(k_{\perp}^2) \right) ,$$

(2.20)

where the $\Theta$-function selects the smaller longitudinal component ($\alpha < \beta = k_{\perp}^2/\alpha$). The term with $\exp(-\nu\alpha)$ corresponds to the case in which the gluon is emitted, while $-1$ corresponds to the case in which the gluon is in a virtual loop. The function $\chi(k_{\perp}^2)$ is the order $\alpha_s^2$ contribution that describes the $k_{\perp}$-dependent one-loop vertex correction to one-gluon emission, given in (2.13).

Using the expression for the virtual correction, (2.13), we can split $\mathcal{R}_1$ into two contributions

$$\mathcal{R}_1 = \mathcal{R}_{11} + \mathcal{R}_{12} ,$$

where

$$\mathcal{R}_{11} = 8C_F \int_0^1 \frac{d\alpha}{\alpha} \int \frac{dk_{\perp}^2}{k_{\perp}^2} \left( e^{-\nu\alpha} - 1 \right) \Theta(k_{\perp}^2 - \alpha^2) \cdot \frac{\alpha_s(0)}{4\pi} ,$$

(2.21a)

$$\mathcal{R}_{12} = -16C_F C_A \int_0^1 \frac{d\alpha}{\alpha} \left( e^{-\nu\alpha} - 1 \right) \int_0^{\infty} \frac{dk_{\perp}^2}{\mu^2} \ln \frac{k_{\perp}^2}{\mu^2} \left( \frac{\alpha_s(0)}{4\pi} \right)^2 \Theta(k_{\perp}^2 - \alpha) .$$

(2.21b)

The weird factor $\alpha_s(0)$ in (2.21a), which stands for the two-loop value of the on-mass-shell coupling, is there simply because a real gluon has been emitted. The contribution $\mathcal{R}_{12}$ is also ill-defined.
As we have seen above, $\mathcal{R}_{11}$, is cancelled by the contribution from two-parton gluon splitting in the collinear limit in (2.11), resulting in a finite answer in which $\alpha_s(0)$ gets replaced by the running $\alpha_s$.

The second contribution, $\mathcal{R}_{12}$, taken together with the logarithmic piece $\mathcal{R}_{22}$ of the two-parton emission in $\mathcal{R}_2$, defined below, produces a finite inclusive correction to the answer.

**Two-parton emission contribution.** The second part of the radiator describes gluon decay into two quarks or into two gluons. The latter is described by the non-Abelian part of the $q\bar{q}gg$ matrix element $M_{gg}^2$ of (2.4), proportional to $C_F C_A$, see [13]. We have

$$\mathcal{R}_2 = 4C_F \int d\Gamma_2 \left( \frac{\alpha_s(m^2)}{4\pi} \right)^2 \left( M_{gg}^2(k_1, k_2) + M_{qq}^2(k_1, k_2) \right) \Omega_T. \quad (2.22)$$

Here essential higher-order virtual corrections have been taken into account, leading to the running of the coupling with the invariant mass of the virtual gluon, $|\alpha_s(-m^2)|^2 = \alpha_s^2(m^2)(1+O(\alpha_s^2))$.

The “thrust trigger-function” $\Omega_T$ in (2.22) determines the contribution to the thrust of any two-parton configuration and is given by

$$\Omega_T = \frac{1}{2!} \left[ \exp(-\nu \min\{\alpha_1, \beta_1\}) - \nu \min\{\alpha_2, \beta_2\} - 1 \right]. \quad (2.23)$$

We note that having included the symmetry factor 1/2! for both gluons and quarks we have implicitly defined $M_{qq}^2(k_1, k_2)$ to describe the probability of finding either a quark or an antiquark with momentum $k_1$.

**Splitting the answer.** Now we are ready to assemble the pieces of the radiator into three contributions each of which is explicitly finite.

The “thrust trigger-function” $\Omega_T$ in (2.22) depends on the azimuthal angle between the partons, $\phi$, and does not allow the integration in (2.11) to be performed. It can however be done, in a naive (“inclusive”) approximation in which one employs a simplified version of the $\Omega$-factor, namely,

$$\Omega_T^{\text{simp}} = \frac{1}{2!} \left[ \exp(-\nu \min\{\alpha, \beta\}) - 1 \right], \quad (2.24)$$

where $\alpha = \alpha_1 + \alpha_2$ and $\beta = \beta_1 + \beta_2$. This corresponds to the contribution to the thrust from the parent gluon, disregarding the possibility that its offspring may go into opposite hemispheres. The regions $\alpha > \beta$ and $\beta > \alpha$ contribute equally, so that it is sufficient to consider only the latter:

$$\Omega_T^{\text{simp}} \Rightarrow (e^{-\nu \alpha} - 1) \Theta(k_{\perp}^2 + m^2 - \alpha^2). \quad (2.25)$$

Using $\vec{q}$ and $\vec{k}_{\perp}$ as integration variables, we obtain

$$\mathcal{R}_2^{\text{simp}} = 8C_F \int \frac{d\alpha}{\alpha} \int dK_{\perp} \int dm^2 \frac{\alpha_s^2(m^2)}{2! (4\pi)^2} \int_0^1 dz \int_0^{2\pi} d\phi \left( M_{gg}^2 + M_{qq}^2 \right) \Omega_T^{\text{simp}}, \quad (2.26)$$
with \( \phi \) the angle between \( \vec{q} \) and \( \vec{k}_\perp \).

The integral in (2.26) has been calculated above in (2.11). It consists of two terms

\[ R_{21}^{\text{simp}} = R_{21} + R_{22}, \]

\[ R_{21} = 8C_F \int \frac{d\alpha}{\alpha} \left(e^{-\nu \alpha} - 1\right) \int \frac{dm^2 dk^2_\perp}{m^2(k^2_\perp + m^2)} \cdot (-\beta_0) \left(\frac{\alpha_s}{4\pi}\right)^2 \Theta(k^2_\perp + m^2 - \alpha^2); \]
\[ R_{22} = 16C_F C_A \int \frac{d\alpha}{\alpha} \left(e^{-\nu \alpha} - 1\right) \int \frac{dm^2 dk^2_\perp}{m^2(k^2_\perp + m^2)} \ln \frac{k^2_\perp(k^2_\perp + m^2)}{m^4} \left(\frac{\alpha_s}{4\pi}\right)^2 \Theta(k^2_\perp + m^2 - \alpha^2). \]

Thus we may represent the answer as a sum of three finite terms:

\[ R = R^{(0)} + R^{(i)} + R^{(n)}, \] (2.28)

where

\[ R^{(0)} = R_{11} + R_{21}, \]
\[ R^{(i)} = R_{12} + R_{22}, \]
\[ R^{(n)} = R_2 - R_{21}^{\text{simp}}. \] (2.29c)

In the first contribution (2.29a) \( R_{11} \) involves \( \alpha_s(0) \) which is cancelled by \( R_{21} \) to give the standard “naive” answer, as we shall see in the next section. The second contribution, we shall refer to as an “inclusive correction”. Here both \( R_{12} \) and \( R_{22} \) contain \( m^2 \to 0 \) logarithmic singularities which, however, are damped by the difference of the two simplified “thrust trigger functions”

\[ \Omega^{\text{simp}}_T - \Omega^{\text{simp}}_T(m^2 = 0) \propto \Theta(k^2_\perp + m^2 - \alpha^2) - \Theta(k^2_\perp - \alpha^2). \]

Similarly, the non-inclusive correction (2.29c) is also finite as it involves the difference of the true and the simplified trigger functions,

\[ \delta \Omega_T = \Omega_T - \Omega^{\text{simp}}_T, \] (3.30)

which also vanishes as \( m^2 \to 0 \).

Now we are ready to recall the dispersive method and apply it to the calculation of (2.29), starting from the main term \( R^{(0)} \).

### 3 Naive power correction for \( T \) in the dispersive method

#### 3.1 Dispersive method (recollection)

In the approach of [7] the dispersive representation for \( \alpha_s \) is implemented, in which the dispersive variable \( m^2 \) acts as a “gluon mass” in the matrix elements (see 2.17):

\[ \frac{\alpha_s(k^2)}{k^2} = - \int_0^\infty \frac{dm^2}{k^2(k^2 + m^2)} \rho_s(m^2) = \int_0^\infty \frac{dm^2}{(m^2 + k^2)^2} \alpha_{\text{eff}}(m^2). \] (3.1)
At one-loop level one has
\[ \rho_s(m^2) = \frac{d}{d \ln m^2} \alpha_{\text{eff}}(m^2) = -\frac{\beta_0}{4\pi} \alpha_s^2(m^2) + \ldots \]  
(3.2)

The perturbative answer, \( \mathcal{O}_j(Q^2) \), for an observable \( j \) is then formulated in terms of the \( m^2 \) integral
\[ \mathcal{O}_j(Q^2) = \int_0^\infty \frac{dm^2}{m^2} \alpha_{\text{eff}}(m^2) \cdot \dot{F}_j(m^2/Q^2), \quad \dot{F}_j = -\frac{d}{d \ln m^2} F_j, \]  
(3.3)

with \( F \) the “characteristic function” one derives by computing the \( m^2 \)-dependent soft-gluon radiation matrix element. For an infrared/collinear safe observable the \( m^2 \) integral converges and is mainly determined by the region \( m^2 \sim Q^2 \), the typical hard scale of the process. This region reproduces the standard one-loop perturbative prediction
\[ \mathcal{O}_j^{\text{PT}} = \alpha_{\text{eff}}(Q^2) F(1) \approx \alpha_s(Q^2) F(1). \]

The genuine non-perturbative component of the answer, \( \delta \mathcal{O}_{\text{NP}} \), is triggered by the non-perturbative component of the effective coupling \( \delta \alpha_{\text{eff}}(m^2) \). Imposing the ITEP/OPE restriction \( \alpha_s(k^2) \) in the ultraviolet region receive no power correction \( k^2 - p \) due to the large-distance interaction domain, one derives from (3.1) that the integer moments of the non-perturbative effective coupling vanish:
\[ \int_0^\infty \frac{dm^2}{m^2} \delta \alpha_{\text{eff}}(m^2) (m^2)^i = 0, \quad i = 1, 2, \ldots p. \]

Thus, the power corrections are obtained from the non-analytic \( m^2 \to 0 \) behaviour of the one-loop matrix element (characteristic function). For example, the behaviour \( \delta \mathcal{O}_{\text{NP}}^T \sim Q^{-1} \) in the thrust distribution originates from
\[ \dot{F}_T(m^2/Q^2) \to \frac{2C_F}{\pi} \sqrt{\frac{m^2}{Q^2}}, \quad m^2 \to 0. \]  
(3.4)

The magnitude of the power correction is then expressed in terms of the \( m \)-moments of an “effective charge” \( \delta \alpha_{\text{eff}}(m^2) \) describing the intensity of non-perturbative gluon radiation:
\[ \delta \mathcal{C}_j^{\text{NP}}(Q^2) = \int_0^\infty \frac{dm^2}{m^2} \delta \alpha_{\text{eff}}(m^2) \dot{F}_j(m^2/Q^2) \approx C_j \frac{A_{n,q}}{Q^n}, \]  
(3.5)

where
\[ A_{2p,q} = \frac{C_F}{2\pi} \int_0^\infty \frac{dm^2}{m^2} \delta \alpha_{\text{eff}}(m^2) \cdot (m^2)^p \ln^q m^2 \]  
(3.6)

(for \( p \) half-integer or \( q > 0 \)). In particular, for the case of thrust from (3.4) we deduce: \( p = \frac{1}{2}, \quad q = 0, \) and the observable-dependent constant \( C_j \) is \( C_T = 4 \).

### 3.2 Inclusive treatment of the thrust distribution

In an inclusive approximation one employs the simplified version of the \( \Omega \)-factor (2.24). The contribution \( \mathcal{R}_{21} \) (2.27a), proportional to \( \beta_0 \), is the only one which has been dealt with in the past \( 3 \). Making use of (3.2) it can be written as
\[ \mathcal{R}_{21} = \frac{2C_F}{\pi} \int \frac{d\alpha}{\alpha} \left( e^{-\nu \alpha} - 1 \right) \int_0^\infty \frac{dm^2}{m^2} \frac{d\alpha_{\text{eff}}(m^2)}{dm^2} \int dk_+^2 \frac{\Theta(k_+^2 + m^2 - \alpha^2)}{k_+^2 + m^2}. \]  
(3.7)
To reconstruct the known result we integrate by parts. Since $\alpha_{\text{eff}}(0) = \alpha_s(0)$, the surface term cancels $R_{11}$, given in (2.21a), and we obtain

$$\mathcal{R}^{(0)} = R_{11} + R_{21} = \frac{2C_F}{\pi} \int \frac{d\alpha}{\alpha} \left( e^{-\nu \alpha} - 1 \right) \int_0^\infty \frac{dm^2}{m^2} \alpha_{\text{eff}}(m^2) \Theta(m^2 - \alpha^2). \tag{3.8}$$

The normal perturbative prediction is obtained by integrating over $\alpha$ between $1/\nu \ll 1$ and 1. In order to examine the non-perturbative contribution, one substitutes for $\alpha_{\text{eff}}$ its non-perturbative part, $\delta \alpha_{\text{eff}}$, which is concentrated at small $m^2$. This implies that $\alpha$ is small, allowing us to expand the exponential term,

$$\delta \mathcal{R}^{(0)} = -\nu \frac{2C_F}{\pi} \int_0^\infty \frac{dm^2}{m^2} m \delta \alpha_{\text{eff}}(m^2) \equiv -\nu 4A_{1,0}, \tag{3.9}$$

resulting in a non-perturbative shift of the thrust distribution, proportional to the first moment of $\delta \alpha_{\text{eff}}$.

$$\frac{d\sigma(T)}{dT} = \frac{d\sigma^{(PT)}(T - \Delta T)}{dT}, \quad \Delta T^{(0)} = \frac{4A_{1,0}}{Q}. \tag{3.10}$$

Taking into account subleading $\alpha_s^2$-effects, the naive inclusive prediction (3.10) gets modified by the thrust rescaling factor $r_T$:

$$\Delta T = \frac{4A_{1,0}}{Q} (1 + r_T). \tag{3.11}$$

This factor originates from the real and virtual two-loop matrix element and depends on the observable under consideration.

In what follows we analyse and compute the thrust rescaling factor. In our framework, it consists of two components: an inclusive $r_T^{(i)}$ and a non-inclusive $r_T^{(n)}$ corrections.

### 4 Exact treatment

#### 4.1 Inclusive correction

As we have seen above, the logarithmic terms in two-particle production $\mathcal{R}_{22}$ (2.27b) and the vertex correction $\mathcal{R}_{12}$ (2.21b) cancel in the fully inclusive anomalous dimension.

In a less inclusive quantity, like thrust, the cancellation is no longer complete: divergences cancel but a finite correction remains. Indeed, in the real emission contribution the kinematical restriction involves the mass of the pair and reads

$$\Theta\left(\sqrt{k_{\perp}^2 + m^2} - \alpha\right).$$

In the virtual correction, however, we deal with a real gluon restricted simply by

$$\Theta\left(\sqrt{k_{\perp}^2} - \alpha\right).$$
As a result the finite correction which emerges reads
\[
\mathcal{R}^{(i)} = 16 C_F C_A \int \frac{d\alpha}{\alpha} \left( e^{-\nu \alpha} - 1 \right) \int \frac{dk_2^2 dm^2}{m^2 (k_2^2 + m^2)} \left( \frac{\alpha_s(m^2)}{4\pi} \right)^2 \ln \frac{k_2^2 (k_2^2 + m^2)}{m^4} \left[ \Theta \left( \sqrt{k_2^2 + m^2} - \alpha \right) - \Theta \left( k_2 - \alpha \right) \right].
\] (4.1)

Expanding the exponent and performing the \(\alpha\)-integration we arrive at
\[
\mathcal{R}^{(i)} = -16 C_F C_A \nu \int \frac{dm^2}{m^2} \left( \frac{\alpha_s(m^2)}{4\pi} \right)^2 \int_{0}^{\infty} \frac{dk_2^2}{k_2^2 + m^2} \left( \sqrt{k_2^2 + m^2 - k_\perp} \right) \ln \frac{k_2^2 (k_2^2 + m^2)}{m^4},
\]
where we have extended the \(k_\perp^2\)-integration to infinity since it converges. Now we use (3.2) and integrate by parts to arrive at
\[
\mathcal{R}^{(i)} = -8 C_F C_A \nu \int \frac{dm^2}{m^2} \alpha_{\text{eff}}(m^2) \left\{ 2m c^{(i)} \right\} \] (4.2)

with
\[
c^{(i)} = 2 \int_{0}^{\infty} \frac{xdx}{1 + x^2} \ln \left[ x^2 (1 + x^2) \right] = 3.299. \] (4.3)

Substituting \(\delta \alpha_{\text{eff}}\) for \(\alpha_{\text{eff}}\) we get the power correction
\[
\delta \mathcal{R}^{(i)} = -\nu \cdot c^{(i)} \frac{2 C_F C_A}{\beta_0} \int \frac{dm^2}{m^2} m \delta \alpha_{\text{eff}}(m^2). \] (4.4)

Recalling the definition of the non-perturbative moment, (3.6), the relative correction to the shift in the thrust distribution (3.10) becomes
\[
\Delta T^{(i)} = \frac{4 A_{1.0}}{Q} \cdot r_T^{(i)}, \quad r_T^{(i)} = \frac{C_A}{\beta_0} c^{(i)} = 1.100 \, (0.900) \quad \text{for} \quad n_f = 3 \, (0). \] (4.5)

Notice that the inclusive correction is of the same magnitude as the naïve term.

### 4.2 Non-inclusive correction

Since the leading power correction originates from the region of large gluon radiation angles, \(\alpha \sim \beta \sim m, \ k_\perp < m\), the kinematic region of offspring partons moving into opposite hemispheres will give an essential correction.

This correction is taken into account by \(\mathcal{R}^{(n)}\) in (2.29c), given by the expression (2.22) with the thrust trigger-function \(\Omega_T\) replaced by \(\delta \Omega_T\) (2.30). In order to obtain the first power correction it suffices to expand the exponents in (2.23) to first order as above. Straightforward calculation results in
\[
\int_{0}^{\infty} \frac{d\alpha}{\alpha} \delta \Omega \simeq -\nu \tilde{\Omega}_T, \quad \tilde{\Omega}_T(q_1, q_2) = z q_1 + (1 - z) q_2 - \sqrt{z q_1^2 + (1 - z) q_2^2}. \] (4.6)

The expression for \(\mathcal{R}^{(n)}\) becomes
\[
\mathcal{R}^{(n)} = -\nu \frac{4 C_F}{2\pi} \int dm^2 \left( \frac{\alpha_s(m^2)}{4\pi} \right)^2 \int dk_2^2 \int_{0}^{1} dz \int \frac{d\phi}{2\pi} \left( M_{gg}^2 + M_{qq}^2 \right) \tilde{\Omega}_T. \] (4.7)
Now we use (3.2) and integrate by parts. Taking into account that at \( m = 0 \)
\[
\tilde{\Omega}_T(q_1 = q_2) = 0,
\]
the surface term vanishes and we get
\[
\mathcal{R}^{(n)} = -\nu \frac{2C_F}{\pi} \int dm^2 \alpha_{\text{eff}}(m^2) \frac{d}{dm^2} \left\{ \int dk_\perp^2 \int dz \int \frac{d\phi}{2\pi} m^2 \left( M_{gg}^2 + M_{qq}^2 \right) \tilde{\Omega}_T \right\} (4.8)
\]
The expression in the curly brackets has dimensions of mass. \( Q^2 \) enters only in the upper limit of the \( k_\perp^2 \) integration which is convergent in the ultra-violet region. Therefore we integrate up to infinity so that the result does not depend on \( Q \) and is given by
\[
\{ (4.8) \} = 2 \tau_T^{(n)} m + \mathcal{O} \left( m^2 / Q \right),
\]
with \( \tau_T^{(n)} \) a number still to be calculated. In the \( m \rightarrow 0 \) limit this leads to
\[
\delta \mathcal{R}^{(n)} = -\frac{4A_{10}}{Q} \tau_T^{(n)} . \quad (4.9)
\]
As a result, the non-perturbative shift in the thrust distribution (3.11) acquires a contribution
\[
\Delta T^{(n)} = \frac{4A_{10}}{Q} \tau_T^{(n)} . \quad (4.10)
\]
In what follows we construct the relevant matrix element and compute the non-inclusive component of the thrust rescaling factor \( \tau_T^{(n)} \).

### 4.3 Evaluation of \( \tau_T^{(n)} \)

We are now in a position to discuss the determination of the non-inclusive thrust rescaling factor \( \tau_T \). From (4.8), we have that
\[
\tau_T^{(n)} = \frac{1}{m} \int \frac{d^2k_\perp \ d\phi}{\pi} \frac{M^2}{2\pi} \frac{\tilde{\Omega}_T(q_1, q_2)}{4\beta_0} . \quad (4.11)
\]
To express the integration measure in terms of \( u_1 \) and \( u_2 \) we write
\[
\frac{d^2k_\perp \ d\phi}{\pi} = \frac{d^2q_1 \ d^2q_2}{\pi} \delta(q^2 - (q_1 - q_2)^2) = \frac{d^2q_1 \ d^2q_2}{\pi} \delta(q^2 - (\tilde{q}_1 - \tilde{q}_2)^2) = q_1 dq_1 q_2 dq_2 \frac{2}{\pi} d\phi \delta(q^2 - q_1^2 - q_2^2 + 2q_1 q_2 \cos \phi) = \frac{4q^2}{\pi} \frac{u_1 du_1 u_2 du_2}{2u_1 u_2 |\sin \phi|}.
\]
Using the \( \delta \)-function condition to express \( \sin \phi \) we arrive at
\[
\frac{d^2k_\perp \ d\phi}{\pi} = \frac{4q^2}{\pi} \frac{u_1 du_1 u_2 du_2}{\sqrt{J}} , \quad (4.12)
\]
where the Jacobian factor \( J(u_1, u_2) \) is
\[
J = ((u_1 + u_2)^2 - 1) \ (1 - (u_1 - u_2)^2) .
\]
Making use of the fact that $\tilde{\Omega}$ in (4.6) satisfies

$$\tilde{\Omega}_T(q_1, q_2) = q \tilde{\Omega}_T(u_1, u_2) = \frac{m}{\sqrt{z(1-z)}} \tilde{\Omega}_T(u_1, u_2),$$

(4.13)

we finally obtain

$$r_T^{(n)} = \frac{1}{\pi \beta_0} \frac{1}{z \sqrt{1-z}} \int_0^1 \frac{dz}{\sqrt{z(1-z)}} \int_0^\infty u_1 du_1 \int_0^\infty u_2 du_2 \frac{1}{\sqrt{J}}$$

$$\times \frac{\mathcal{M}^2(u_1, u_2)}{zu_1^2 + (1-z)u_2^2} \tilde{\Omega}_T(u_1, u_2),$$

(4.14)

where the integrals run over the region $J \geq 0$. This region looks simpler in terms of $u_\pm = u_1 \pm u_2$:

$$\int \frac{du_1 du_2}{\sqrt{J}} = \frac{1}{2} \int_{-1}^1 \frac{du_-}{\sqrt{1 - u_-^2}} \int_{-1}^1 \frac{du_+}{\sqrt{u_+^2 - 1}}.$$  

(4.15)

**Convergence of $r_T^{(n)}$.** It is necessary to check that the $u_+ \to \infty$ and $z(1-z) \to 0$ regions do not lead to divergences of the $r_T^{(n)}$ integral. This is shown in Appendix B.

Numerical integration results in the value

$$r_T^{(n)} = \frac{2}{\beta_0} \frac{1}{\beta_0} \left[ -1.227 C_A + 0.365 C_A - 0.052 n_f \right],$$  

(4.16)

where the three terms originate respectively from the soft gluon, hard gluon and quark matrix elements (2.9). This gives

$$r_T^{(n)} = -0.710 (-0.470), \quad \text{for } n_f = 3 (0).$$  

(4.17)

We have that the non-inclusive correction alone would lower the “naive” expectation for the factor in the leading $1/Q$ power correction to thrust by about 30%. However, assembling the inclusive and non-inclusive corrections to the thrust rescaling factor we finally obtain

$$r_T = r_T^{(i)} + r_T^{(n)} = \frac{1}{\beta_0} \left[ 1.575 C_A - 0.104 n_f \right] = 0.490 (0.430) \quad \text{for } n_f = 3 (0).$$  

(4.18)

The conclusion is that the thrust rescaling factor $1 + r_T$ increases the naive value by 50%. The most important correction is the inclusive one (4.13).

### 5 Discussion and conclusions

Within the dispersive approach of power corrections to perturbative QCD predictions, and to jet-shape observables in particular, are expressed in terms of moments of the non-perturbative component of the effective coupling, $\delta \alpha_{\text{eff}}(m^2)$.

To fix the absolute normalisation for the leading power corrections to jet-shape observables a two-loop analysis has to be carried out: since power corrections are proportional to
powers of the QCD scale $\Lambda$, it is necessary to fix the latter to control the magnitude of the non-perturbative correction.

In this paper we have examined the thrust distribution in the high-thrust region to two-loop accuracy to extract the $1/Q$ power contribution. To do this it suffices to consider radiation of soft gluons followed by their two-parton decays.

To this end we have used the physical CMW scheme \[15\] in which $\alpha_s$ is defined as the strength of inclusive soft-gluon radiation (the magnitude of the singular part of the two-loop quark anomalous dimension).

Given $\Delta T^{(0)}$ as the result of the naïve inclusive one-loop treatment, we have found that the rescaling factor $1 + r_T$ for the thrust power correction, $\Delta T = (1 + r_T)\Delta T^{(0)}$ is rather large. This large correction comes mainly from the gluon splitting into two gluons.

The correction $r_T$ originates from two sources. A positive term $r_T^{(i)}$ (the “inclusive correction”) comes from an incomplete compensation of the logarithmic contributions in soft-gluon splitting, due to the kinematical difference between virtual (massless) and real (massive) gluon radiation \[1.1\]. This gives the most important correction (see 4.3). A negative term $r_T^{(n)}$ (the “non-inclusive correction”) is due to the kinematical region of the 4-parton phase space in which the gluon decay products fly into opposite hemispheres \[4.7\]. The contribution to this last term coming from the decay of the primary gluon into a quark-antiquark pair proves to be positive and numerically very small. This agrees with the result of the analysis of Nason and Seymour \[8\].

It is the region of large gluon emission angle, $\theta \simeq \pi/2$, that triggers the non-analyticity in the gluon virtuality $m^2$ and thus power corrections. This makes opposite-hemisphere configurations of the decay products quite common. But it so happens that the most important correction is the inclusive one.

Since the moments of $\delta\alpha_{\text{eff}}(m^2)$ are not calculable given the present state of the art, they must be determined phenomenologically. Therefore to be of any practical value, the programme of calculating power corrections at two loops must be extended to other observables driven by the same (first) moment as the thrust. In a forthcoming publication we will generalise the present analysis to energy-energy correlations and jet-broadening in $e^+e^-$-annihilation \[10\].

For a general observable $j$ we will have a rescaling factor $1 + r_j$, where $r_j$ originates from the primary soft-gluon emission and its two-parton decay convoluted with the trigger function specific to the observable $j$. Since $r_j$ depends on the observable, one could argue that universality of the leading power correction is violated. On the other hand, we would prefer to consider universality as still being valid. Indeed, the ingredients of universality, which are the universality of soft-gluon radiation and universality of the coupling, remain intact in the two-loop analysis. Moreover, only at the two-loop level can universality be given a precise meaning, since only at this level can one define the QCD scale and thus the coupling. The fact that the rescaling factor is observable dependent does not break universality, since $r_j$ remains under perturbative control.

Let us stress that the two-loop analysis has led to a term $r_T$ in the rescaling factor which
does not contain any small parameter and is just a number. One may wonder whether higher-loop calculations will give corrections of the same order as \( r_j \). We argue that this is not the case and that terms higher than two-loops give "subleading" contributions, i.e. with additional powers of the perturbative coupling.

To see this observe that the \((\ell+2)\)-loop matrix element would lead to the appearance of an \( m^2\)-integral of \( \alpha_s^{\ell+2}(m^2) \). By using (3.2) and performing the analysis as before, one finds that the corresponding contribution to the non-perturbative power correction will be proportional to \( \alpha_s^{\ell+2}\alpha_{\text{eff}} \) and thus will be down, with respect to the two-loop result, by an extra power \( \ell \) of the coupling. Strictly speaking, such a perturbative argument is far from being perfect since the coupling here enters at a small scale. Whether the higher order contributions are actually small depends on the numerical value of the "perturbative" effective coupling \( \alpha_{\text{eff}}/\pi \) at low scales. The large value of the two-loop correction \( r_T \) indicates that \( \delta\alpha_{\text{eff}} \) is smaller than was indicated by the naive treatment.

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A Two-parton production cross section, (2.11)

Using the reduced squared matrix element \( M^2 \) in (2.7), we can write

\[
\frac{1}{2!} \int_0^1 dz \int_0^{2\pi} d\phi \left( M_{gg}^2 + M_{gq}^2 \right) = \frac{1}{m^2(k_{\perp}^2 + m^2)} \int_0^1 dz \int_0^{2\pi} d\phi \frac{1}{2\pi} \left\{ C_A \left( 2S + \mathcal{H}_g \right) + n_f \mathcal{H}_q \right\},
\]

where the distributions \( 2S, \mathcal{H}_g \) and \( \mathcal{H}_q \) are given in (2.9) as functions of \( \vec{u}_i = \vec{q}_i/q \) with

\[
\vec{q}_1 = \vec{k}_\perp + (1-z)\vec{q}, \quad \vec{q}_2 = \vec{k}_\perp - z\vec{q}.
\]

Performing the integration over \( \phi \), the angle between \( \vec{k}_\perp \) and \( \vec{q} \), we obtain

\[
\int_0^{2\pi} d\phi \frac{1}{2\pi q_1^2} = \frac{1}{q_1^2} = \frac{1}{|k_{\perp}^2 - (1-z)^2 q^2|},
\]

\[
\int_0^{2\pi} d\phi \frac{1}{2\pi q_1^2 q_2^2} = \frac{1}{m^2 + k_{\perp}^2} \left\{ \frac{1-z}{q_1^2} + \frac{z}{q_2^2} \right\},
\]

\[
\int_0^{2\pi} d\phi \frac{q_1^2}{2\pi q_2^2} = 1 + \left\{ \frac{q_1^2}{q_2^2} \right\} - \frac{2}{z} B(z),
\]

where

\[
B(z) \equiv \Theta(z^2 q^2 - k_{\perp}^2) = \Theta(z - \frac{k_{\perp}^2}{k_{\perp}^2 + m^2}).
\]
Similar expressions hold for $\langle 1/q_2^2 \rangle$ and $\langle q_2^2/q_1^2 \rangle$. By using (2.9) we find

\[
\frac{1}{2!} \langle 2S \rangle = \frac{1 - B(z) - B(1 - z)}{z(1 - z)},
\]

\[
\frac{1}{2!} \langle H_g \rangle = -2 + z(1 - z) + \frac{B(z)}{2z} + \frac{B(1 - z)}{2(1 - z)} + \frac{m^2}{k_1^2 + m^2} \frac{1 - 6z(1 - z)}{2},
\]

\[
\frac{1}{2!} \langle H_q \rangle = \frac{1}{2} - z(1 - z) - \frac{m^2}{k_2^2 + m^2} \frac{1 - 6z(1 - z)}{2}.
\]

The last two terms in $\langle H_g \rangle$ and $\langle H_q \rangle$ vanish upon $z$-integration and we find

\[
\int_0^1 dz \int_0^{2\pi} d\phi \frac{2\pi}{n_f \langle H_g \rangle} = 2 \int_0^1 dz \left\{ C_A (-2 + z(1 - z)) + \frac{n_f}{2} (1 - 2z(1 - z)) \right. \\
\left. + \frac{C_A}{z} (1 - B(1 - z) - \frac{1}{2} B(z)) + \frac{C_A}{1 - z} \left( 1 - B(z) - \frac{1}{2} B(1 - z) \right) \right\} \\
= -\beta_0 + 2C_A \ln \frac{k^2 (k_1^2 + m^2)}{m^4}. \tag{A.3}
\]

This gives (2.11).

## B Convergence of $r_T^{(n)}$

To check that the integral (4.14) determining $r_T^{(n)}$ converges, we first consider the large-$u_+$ behaviour of the integrand. The reduced matrix element (2.8) has a finite limit at $u_+ = \infty$. The integration measure in (4.15), together with the ratio $u_1 u_2 / (z u_1^2 + (1 - z) u_2^2)$, behaves like $d u_+ / u_+$. The function $\tilde{\Omega}$ that triggers the non-inclusive correction to thrust, (4.6), can be cast as

\[
\tilde{\Omega}(u_1, u_2) = \frac{1}{2} \left\{ u_+ + (2z - 1) u_- - \sqrt{u_+^2 + u_-^2 + 2 u_+ u_- (2z - 1)} \right\} \\
= \frac{2z(1 - z) u_-^2}{u_+ + (2z - 1) u_- + \sqrt{(u_+ + (2z - 1) u_-)^2 + 4z(1 - z) u_-^2}}. \tag{B.1}
\]

The crucial point is that it vanishes for large $u_+$,

\[
\tilde{\Omega}(u_1, u_2) \xrightarrow{u_+ \to \infty} - \frac{z(1 - z) u_-^2}{u_+}, \tag{B.2}
\]

thus ensuring convergence.

By inspecting (B.2) we conclude that $\tilde{\Omega}$ also vanishes at $z(1 - z) = 0$ since

\[
\tilde{\Omega}(u_1, u_2) \xrightarrow{z(1 - z) \to 0} \frac{z(1 - z) u_-^2}{u_+ + (2z - 1) u_-}. \tag{B.3}
\]

In general, this is sufficient to compensate for the soft singularity of the matrix elements

\[
\mathcal{M}^2 \propto \frac{1}{z(1 - z)}. \tag{18}
\]
Special care should be taken when analysing the edge of the phase space which corresponds to one of the partons being collinear with the quark direction:

\[ u_+ - 1 \sim 1 - |u_-| \sim \sqrt{z(1-z)} \ll 1 , \]

where (B.3) vanishes only as \( \sqrt{z(1-z)} \).

For \( z \to 0 \) the potentially singular region is \( u_\pm \to 1 \), which can be explored by writing

\[ u_+ = 1 + (\rho \cos \psi)^2, \quad u_- = 1 - (\rho \sin \psi)^2 . \]

For \( z \) and \( \rho \) small we have

\[ zu_1^2 + (1-z)u_2^2 = \frac{1}{4} \rho^4 + z + O \left( z\rho^2 \right) . \]

This combination appears in the denominators of the phase space and thus generates singularities for \( \rho \to 0 \) and \( z \to 0 \).

Consider first the region \( \rho^2 > \sqrt{z} \). Here the integrand of \( r^{(n)}_I \) has the form

\[ dI \sim \sqrt{z} \, dz \, \frac{d\rho^2}{\rho^4} \Theta \left( \rho^2 - \sqrt{z} \right) \, d\psi \, M^2 . \]  

(B.4)

For the soft reduced matrix element \( S \) one has

\[ S \sim \frac{2\rho^2 (\cos^2 \psi - \sin^2 \psi) + O(\rho^4)}{z} \Rightarrow O \left( \rho^4 \right) \frac{1}{z} , \]

where the first term is cancelled upon \( \psi \) integration. The integration over \( \rho \) is convergent for \( \rho \to 0 \) and gives a contribution with the integrable singularity \( 1/\sqrt{z} \).

For the hard reduced matrix elements \( H_g \) and \( H_q \) the dominant contribution in this region is a constant. Thus the \( \rho \) integration gives a contribution regular in \( z \).

Finally, the integration over \( \rho \) in the region \( \rho^2 < \sqrt{z} \) leads to a contribution which, after angular integration, is regular in \( z \).
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