Testing whether muon neutrino flavor mixing is maximal

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(Dated: November 10, 2018)

PACS numbers: 13.15.+g, 14.60.Lm, 14.60.Pq

The small difference between the survival probabilities of muon neutrino and antineutrino beams, traveling through earth matter in a long baseline experiment such as MINOS, is shown to be an important measure of any possible deviation from maximality in the flavor mixing of those states.

PACS numbers: 13.15.+g, 14.60.Lm, 14.60.Pq

How really close to maximal \(1\) is the flavor mixing of muon neutrinos and antineutrinos discovered in atmospheric neutrino experiments \(2\)? We propose in this Letter a way of probing the deviation, if any, from this maximality. The idea is to measure the small difference \(\Delta P_{\mu\mu} = P_{\mu\mu} - P_{\bar{\mu}\bar{\mu}}\), with \(P_{\mu\mu} = P[\nu_\mu(0) \to \nu_\mu(L)]\) and \(P_{\bar{\mu}\bar{\mu}} = P[\bar{\nu}_\mu(0) \to \bar{\nu}_\mu(L)]\) representing respective survival probabilities of muonic neutrion and antineutrino beams after passage through a distance \(L\) in earth matter. Suppose we neglect the subdominant oscillations due to the smaller mass difference relevant to the solar neutrino problem, but retain a nonzero mixing between the first and the third family neutrinos. Working to the linear perturbative order in the earth matter effect, we are then able to show that \(\Delta P_{\mu\mu}\) is proportional to \(|U_{e3}|^2(1 - 2|U_{\mu3}|^2)\), \(U\) being the Pontecorvo-Maki-Nakagawa-Sakata neutrino flavor mixing matrix, with a computable proportionality coefficient. Thus the effect is linear in the deviation \(1/\sqrt{2} - |U_{\mu3}|\) from maximal mixing.

Much has been learnt recently about neutrino masses and mixing angles from solar \(3\), atmospheric \(2\), reactor \(4\, 5\) and long baseline \(6\) studies \(7\). We know now that the squared mass difference between one pair of neutrinos comprising \(\nu_\mu\) and \(\nu_\tau\), mixed nearly maximally \((\theta_{23} \simeq 45^\circ)\) \(8\), is \(|\delta m_{21}^2| \sim 2 \times 10^{-3}\) \(eV^2\). We also know that the squared mass difference between another pair of neutrinos, involving \(\nu_e\) mixed by a large angle \((\theta_{12} \simeq 30^\circ)\) with a nearly equal combination of \(\nu_\mu\) and \(\nu_\tau\), is \(\delta m_{23}^2 \sim 7 \times 10^{-5}\) \(eV^2\). Furthermore, the mixing between the third possible pair, characterized by the angle \(\theta_{13}\), is known to be quite restricted, as elaborated below.

Yet longer baseline experiments with \(\nu, \bar{\nu}\) beams/superbeams promise to be the wave of the future in neutrino physics. MINOS \(9\) and off-axis NUMI \(10\) are forthcoming experiments and will be followed by JPARC \(11\), CNGS \(12\) and other efforts. Many theoretical and phenomenological analyses \(13\) of physics issues pertaining to long baseline experiments have been carried out meanwhile. However, the focus of recent studies has largely been on appearance experiments: specifically, the ‘golden’ \((\nu_e \to \nu_\mu)\) and ‘silver’ \((\nu_\mu \to \nu_\tau)\) channels and their synergy in probing leptonic \(CP\) violation as well as the mixing between the first and third family neutrinos. Less attention has been paid to survival probabilities, the originally measured quantities \(14\) in atmospheric neutrino studies. This is since they do not yield direct information on those aspects. However, it is the survival probabilities for \(\nu_\mu\) and \(\bar{\nu}_\mu\) beams/superbeams, specifically their difference, which should help determine the deviation from maximality, if any, of the flavor mixing of muon neutrinos.

Earth matter directly affects only neutrinos with electronlike flavor. However, this gets induced into the other neutrino flavors indirectly through mixing cum oscillation effects. For neutrinos of muonic flavor, there are two sources of such an occurrence: (1) mixing between the electronlike flavor and the third physical neutrino through the factor \(|U_{e3}|^2\) and (2) subdominant oscillations between electronlike and muonlike neutrino flavors, driven by \(\delta m_{21}^2\). It is generally known, and we will show later, that the effect of (2) is quite small for the baseline length of MINOS \(9\). This means that one can take the earth matter effect in such an experiment to be due to (1) only. We also assume that the actual value of \(|U_{e3}|^2\) is not as small as one order of magnitude less than its current upper bound 0.05 – 0.07 \(15\), so that our effect will be measurable. Moreover, we shall treat this effect to the lowest perturbative order \(16\) in \(A = \sqrt{2}G_F N_e\), \(N_e\) being the average electron density of the earth. This will also be justified later. On the other hand, if \(|U_{e3}|^2\) turns out to be smaller by more than an order of magnitude than its current upper bound, the content of our Letter will prove empty.

The effective Hamiltonian for neutrino oscillation in
matter can be written as

$$H = U \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta_{21} & 0 \\ 0 & 0 & \Delta_{31} \end{pmatrix} U^\dagger + \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (1)$$

Here $\Delta_{ij} = (m_i^2 - m_j^2)/(2E_0)$, $m_i$ and $E_0$ being the mass of the $i$th (physical) neutrino and the beam energy respectively. For simplicity, let us work in the uniform

earth density approximation, though our results extend to the more general variable density case, as shown by using the evolution operator formalism \[17\]. When $\Delta_{31}$ is neglected in comparison with $\Delta_{31}$, the effects of the solar neutrino mixing angle $\theta_{12}$ and of the CP violating phase $\delta$ in $U$ become inconsequential. Then, in the standard parametrization and with $c_{ij} \equiv \cos \theta_{ij}$, $s_{ij} \equiv \sin \theta_{ij}$, $H$ takes the form

$$H \approx \text{diag.}(A, 0, 0) + \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Delta_{31} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & -s_{13} \\ 0 & 1 & 0 \\ s_{13} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & -s_{23} \\ 0 & s_{23} & c_{23} \end{pmatrix}. \quad (2)$$

The RHS of (2) can be diagonalized by the unitary matrix $\bar{U}$, yielding the eigenvalues $\lambda_{1,2,3}$, given by

$$l_2 = 0, \quad \text{(3a)}$$

$$l_{1,3} = \frac{1}{2}(\Delta_{31} + A \mp B), \quad \text{(3b)}$$

where

$$B = \sqrt{\Delta_{31}^2 + A^2 - 2\Delta_{31}A \cos 2\theta_{13}}. \quad (4)$$

Moreover, $\bar{U}$ can be written as

$$\bar{U} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{\theta_m} & 0 & s_{\theta_m} \\ 0 & 1 & 0 \\ -s_{\theta_m} & 0 & c_{\theta_m} \end{pmatrix} \begin{pmatrix} s_{23} \theta_m & c_{23} \theta_m & 0 \\ -c_{23} \theta_m & s_{23} \theta_m & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (5)$$

where the angle $\theta_m$ is related to $\theta_{13}$ and $A$ by,

$$\tan 2\theta_m = \frac{\Delta_{31} \sin 2\theta_{13}}{\Delta_{31} \cos 2\theta_{13} - A}. \quad (6)$$

The $\nu_\mu$ survival probability, after propagation through a distance $L$ in matter, is

$$P_{\mu\mu} = 1 - 4 \left( |\bar{U}_{\mu1}|^2 |\bar{U}_{\mu2}|^2 \sin^2 \frac{l_2 - l_1}{2} L + |\bar{U}_{\mu1}|^2 |\bar{U}_{\mu3}|^2 \sin^2 \frac{l_3 - l_1}{2} L + |\bar{U}_{\mu2}|^2 |\bar{U}_{\mu3}|^2 \sin^2 \frac{l_3 - l_2}{2} L \right)$$

$$= 1 - 4 \left( s_{23}^2 c_{23} \theta_m \sin^2 \Delta_{31} + \frac{A - B}{4} L \right)$$

$$+ \frac{4 s_{23}^2 \theta_m^2 c_{\theta_m}^2}{2} \sin^2 \frac{BL}{2}$$

$$+ \frac{4 s_{23}^2 \theta_m \sin^2 \Delta_{31} + \frac{A + B}{4} L}{4}. \quad (7)$$

$P_{\mu\mu}$ can be obtained from $P_{\mu\mu}$ simply by changing $A$ to $-A$ \[19\]. Moreover, in vacuum with $A = 0$ and hence $l_1 = 0 = l_2$ and $l_3 = \Delta_{31}$ as well as $\bar{U} = U$, we have

$$P_{\mu\mu}^{\text{vac}} = P_{\mu\mu} = 1 - 4|U_{\mu3}|^2(1 - |U_{\mu3}|^2) \sin^2 \frac{\Delta_{31} L}{2}. \quad (8)$$

The oscillation probability for a $\nu_\mu$/$\bar{\nu}_\mu$ traveling in vacuum, namely $1 - P_{\mu\mu}^{\text{vac}}$, is maximal, corresponding to maximal mixing, when $|U_{\mu3}| = 1/\sqrt{2}$. Though a careful measurement of $P_{\mu\mu}^{\text{vac}}$ may yield a value of $|U_{\mu3}|^2$ slightly different from 1/2, one here faces the difficulty of measuring a small term of order $(1/\sqrt{2} - |U_{\mu3}|)^2$. The vacuum term being quadratic in the deviation from maximality of muon neutrino flavor mixing contrasts strikingly with the matter effect term being linear in the said deviation.

Let us now consider \[17\], keeping terms only to linear order in $A$. The $A^2$ terms will be shown to cancel from our effect and the corrections will be shown to be of $O([U_{e3} A]^2)$. Note that, for the MINOS baseline length and energies, $|A/\Delta_{31}| = O(10^{-1})$. It is useful to note that

$$s_{23}^2 = s_{23}^2(1 + 2A \Delta_{31}^{-1} c_{13}^2) + O(A^2), \quad (9a)$$

$$c_{23}^2 = c_{23}^2(1 - 2A \Delta_{31}^{-1} s_{13}^2) + O(A^2), \quad (9b)$$

$$\sin^2 \Delta_{31} + \frac{A - B}{4} L = O(A^2), \quad (9c)$$

$$\sin^2 B L = \sin^2 \frac{\Delta_{31} L}{2} - \frac{AL}{2} \cos 2\theta_{13} \sin(\Delta_{31} L) + O(A^2), \quad (9d)$$

$$\sin^2 \frac{\Delta_{31} + A + B}{4} L = \sin^2 \frac{\Delta_{31} L}{2} + \frac{AL}{2} s_{13} \sin(\Delta_{31} L) + O(A^2). \quad (9e)$$

Utilizing (9a-e) in (7), we obtain

$$P_{\mu\mu} = P_{\mu\mu}^{\text{vac}} + 2s_{23}^2 c_{13}^2 s_{13}^2 (c_{23}^2 - s_{23}^2 \cos 2\theta_{13}). \quad (10)$$
while the thin lines are obtained by the exact numerical solution of the equation of motion of the neutrinos traveling in matter. The comparison between our analytic results and the exact numerical solution is shown for three different values of $|U_{e3}|^2$. For the exact numerical solution we have used $\delta m^2_{21} = 7 \times 10^{-5} \text{ eV}^2$ and $\sin^2 \theta_{12} = 0.3$.

FIG. 1: $\Delta P_{\mu\mu}$ as a function of the neutrino energy. The thick lines show the approximate analytic form given in Eq. (10) while the thin lines are obtained by the exact numerical solution of Eq. (10) as a measure of $|U_{e3}|^2$. The accuracy in the determination of $\delta m^2_{21}$ is quite small, thus enhancing our confidence in the use of $|U_{e3}|^2$. The comparison between our analytic results and the exact numerical solution is shown for three different values of $|U_{e3}|^2$. For the exact numerical solution we have used $\delta m^2_{21} = 7 \times 10^{-5} \text{ eV}^2$ and $\sin^2 \theta_{12} = 0.3$.

leading to

$$\Delta P_{\mu\mu} = 4|U_{e3}|^2|U_{\mu3}|^2(1 - 2|U_{\mu3}|^2)$$

$$A \left[ 4 \Delta_{31}^{-1} \sin^2 \frac{\Delta_{31} L}{2} - L \sin(\Delta_{31} L) \right] + O(A^2).$$

Equation (10) is the key result. It shows that the linear $A$ term in $\Delta P_{\mu\mu}$ is proportional to $1 - 2|U_{\mu3}|^2$ which is the deviation from maximality of the $\nu_\mu$ flavor mixing in vacuum, as evident from Eq. (8). We may also note that the $O(A^2)$ terms cancel between $P_{\mu\mu}$ and $P_{\mu\bar{\nu}}$. Moreover, the corrections involve $O(|U_{e3}A|^2)$ terms since the lowest order $A$ terms in the transition amplitude [20] come with coefficients $|U_{e3}|^2$ and $U_{e3}$. Thus the $A^3$ terms are further suppressed.

Our two significant approximations, i.e. ignoring $\delta m^2_{21}$-driven subdominant oscillations and the $O((|U_{e3}A|^2)$ terms in Eq. (10), were for the convenience of analytical calculations. A numerical code has been developed [21] to calculate $\Delta P_{\mu\mu}$, treating both the subdominant and the earth matter effect exactly. In Fig. 1 the thin (thick) lines show the $\Delta P_{\mu\mu}$ of exact numerical evaluation (of Eq. (10)) against $E_\nu$ with the other parameters specified in the figure labels. The differences are quite small, thus enhancing our confidence in the use of Eq. (10) as a measure of $|U_{e3}|^2(1 - 2|U_{\mu3}|^2)$. Interestingly, even for $|U_{e3}|^2 = 0.01$ when the magnitude of $\Delta P_{\mu\mu}$ is a lot less, these differences are small. The explanation why subdominant oscillations are suppressed is that the $A\Delta_{21}$ term in the transition amplitude [20] comes with a coefficient proportional to $U_{e3}$. We have also checked numerically that, while the location (in $E_\nu$) of the $\Delta P_{\mu\mu}$ peak is sensitive to variations in $\delta m^2_{21}$, its magnitude is not.

The sensitivity of MINOS to $\Delta P_{\mu\mu}$ is best discussed in terms of $\Delta N_{\nu}$, the difference – due to the deviation of $|U_{\mu3}|^2$ from maximality – between the number of expected neutrino and antineutrino events. For an anticipated MINOS $\nu_\mu$ exposure [22] of $16 \times 10^{20}$ primary protons on target (p.o.t), this has been plotted in Fig. 2 along with the $1\sigma$ statistical errorbars for the parameter values shown in the labels. Given the detection cross-section for $\nu_\mu$'s to be about half that of $\nu_\mu$'s, we have assumed twice as much exposure for $\bar{\nu}_\mu$'s as for $\nu_\mu$'s. The plot demonstrates the feasibility of such a measurement.

In conclusion, we have shown how the measurement of $\Delta P_{\mu\mu}$, in a long baseline experiment such as MINOS, can probe the deviation $1/\sqrt{2} - |U_{\mu3}|$ from maximality of the flavor mixing of the muon neutrino. The exclusion (with errors taken into account) of a vanishing value of $\Delta P_{\mu\mu}$ will simultaneously demonstrate $U_{e3} \neq 0 \neq \frac{1}{\sqrt{2}} - |U_{\mu3}|$. The accuracy in the determination of $|U_{e3}|(\frac{1}{\sqrt{2}} - |U_{\mu3}|)$ will depend more sensitively on the measurement error in $\Delta P_{\mu\mu}$ than on the then uncertainty in the knowledge
of $\delta m_{31}^2$.

We thank Yuval Grossman, Sanjib Mishra, Hitoshi Murayama, Sandip Pakvasa and Sergey Petcov for helpful discussions and Michele Frigerio for correcting an error.

In a three-generation framework, which is essential to our analysis, one can define maximal flavor mixing for the muon neutrino as the situation when, traveling in vacuum, it converts its flavor with the maximum probability. If the latter is calculated neglecting the subdominant oscillation due to $\delta m_{21}^2$, as done in Ref. [2], one finds $|U_{\mu 3}| = 1/\sqrt{2}$, see our later discussion after Eq. (5). Now if $|U_{e 3}| = \epsilon, |U_{\mu 3}| = 1/\sqrt{2}(1-\epsilon^2)^{1/2}$.

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