CHOLESKY DECOMPOSITION OF POSITIVE SEMIDEFINITE MATRICES
OVER COMMUTATIVE SEMIRINGS

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ABSTRACT. We extend the definition of positive semidefiniteness to matrices over
commutative semirings. We prove that every symmetric positive semidefinite
strongly invertible matrix over a commutative semiring has a Cholesky decom-
position.

1. INTRODUCTION

The Cholesky decomposition $A = LL^T$ of a positive semidefinite matrix $A$ over
the field of real or complex numbers, where $L$ is a lower triangular matrix, is one
of the fundamental tools in matrix computations. The standard algorithm for its
computation dates from the early part of the previous century and it is one of the
most numerically stable of all matrix algorithms. The Cholesky decomposition
exists for every positive semidefinite matrix. When assumed $A$ and $L$ are both
positive definite, the Cholesky decomposition $A = LL^T$ is unique.

However, not much is known about the Cholesky decomposition of matrices
over semirings. The theory of semirings has many applications in optimization
theory, automatic control, models of discrete event networks and graph theory
(see e.g. [1, 2, 7, 10]).

The theory of invertible matrices over semirings differs from those over reals.
The invertible matrices over semirings with no non-zero additively invertible el-
ements were characterized in [3]. We follow [4] and say that a matrix is strongly
invertible if all its leading principal submatrices are invertible. Strongly invert-
ible matrices over semirings were first investigated in [8]. The author showed
that a matrix over a semiring is strongly invertible if and only if it has an LU de-
composition. When applicable over reals, the Cholesky decomposition is roughly
twice as efficient as the $LU$ decomposition for solving systems of linear equations
(see for example [6]).

In this paper, we study the Cholesky decomposition of strongly invertible ma-
trices over semirings. In order to construct the Cholesky decomposition, we
define the notion of a positive semidefinite matrix. Since positive elements in
semirings are not well defined, we substitute the real case assumption of $x^T M x$
to be a positive element with the assumption of $x^T M x$ to be a square. We show
that every symmetric positive semidefinite strongly invertible matrix $A$ has a

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Cholesky decomposition $A = LL^T$ and $L$ is a symmetric strongly invertible matrix (see Theorem 3.4). We also show that not every symmetric positive semidefinite invertible (but not necessarily strongly invertible) matrix has a Cholesky decomposition (see Example 3.1). However, for some special classes of semirings we prove that a symmetric matrix is positive semidefinite and strongly invertible if and only if it has a Cholesky decomposition (see Corollary 3.6).

2. Preliminaries

A semiring is a set $S$ equipped with binary operations $+$ and $\cdot$ such that $(S, +)$ is a commutative monoid with identity element 0, and $(S, \cdot)$ is a monoid with identity element 1. In addition, operations $+$ and $\cdot$ are connected by distributivity and 0 annihilates $S$. A semiring is commutative if $ab = ba$ for all $a, b \in S$.

Throughout the paper we assume that $S$ is a commutative semiring. The simplest example of a commutative semiring is the binary Boolean semiring $B$, the set $\{0, 1\}$ in which $1 + 1 = 1 \cdot 1 = 1$. Some other examples of semirings include set of nonnegative integers (or reals) with the usual operations of addition and multiplication, distributive lattices, tropical semirings, dioïds, fuzzy algebras, inclines and bottleneck algebras. (See e.g. [5].)

An element $a \in S$ is called invertible if there exists an element $b \in S$ such that $ab = 1$. Such an element $b$ is called the inverse of $a$ in $S$ and denoted by $a^{-1}$. It is easily proved that the inverse of $a$ in $S$ is unique. Let $U(S)$ denote the set of all invertible elements in $S$. Then $U(S)$ forms a group with respect to the multiplication in $S$.

An element $a \in S$ is called additively invertible if $a + b = 0$ for some element $b$ in $S$. Such an element $b$ is unique and denoted by $-a$. Let $V(S)$ denote the set of all additively invertible elements in $S$. It is clear that $V(S) = S$ if and only if $S$ is a ring and that $V(S) = \{0\}$ if and only if 0 is the only additively invertible element in $S$.

For a subset $T \subseteq S$, let $T^n$ denote the set of all vectors of size $n$ over $T$. We denote by $M_n(S)$ the set of all $n \times n$ matrices over $S$. For $A \in M_n(S)$, we denote by $A_{ij}$ the $(i, j)$-entry of $A$, and denote by $A^T$ the transpose of $A$. It is easy to see that $M_n(S)$ forms a (noncommutative) semiring with respect to the matrix addition and the matrix multiplication. We denote the $m \times n$ zero matrix by $0_{m,n}$ and the $n \times n$ identity matrix by $I_n$. If clear from the context, we omit the subscript denoting the size of the matrix.

Over real (and complex) numbers a symmetric (Hermitian) matrix $A$ has the Cholesky decomposition $A = LL^T$ for a lower triangular matrix $L$ if and only if $A$ is positive semidefinite. Since the notion of a positive semidefinite matrix is defined by utilizing the concept of positive real numbers, we need to introduce a similar concept to the commutative semiring setting. Therefore, we define the set of squares of a commutative semiring $S$ as

$$Q(S) = \{a \in S; a = b^2 \text{ for some } b \in S\}.$$  

For $S = \mathbb{R}$, the set $Q(S)$ coincides with nonnegative real numbers. Now, we can define the positive semidefinite matrices over commutative semirings.
Definition 2.1. The matrix $M \in M_n(S)$ is positive semidefinite if $x^T M x \in Q(S)$ for all $x \in S^n$.

The construction of the Cholesky decomposition will utilize the notion of the Schur complement of a matrix. For $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_n(\mathbb{R})$, the Schur complement of an invertible submatrix $A$ of $M$ is defined as $M/A = D - CA^{-1}B$. For the theory of the Schur complement over real matrices, we refer the reader to [9]. Since the subtraction is generally not possible in commutative semirings, we shall only define the Schur complement of the leading $1 \times 1$ submatrix of $M$.

Definition 2.2. Let
\[
M = \begin{bmatrix} a & b^T \\ b & C \end{bmatrix} \in M_n(S)
\]
where $a \in U(S)$, $b \in V(S)^{n-1}$ and $C \in M_{n-1}(S)$. We define
\[
M/a = C + a^{-1}(-b)b^T
\]
to be the Schur complement of $a$ of matrix $M$.

The next technical lemma is straightforward.

Lemma 2.3. If $a \in U(S)$, $b \in V(S)^{n-1}$ and $C \in M_{n-1}(S)$, then the matrix
\[
\begin{bmatrix} 1 & 0 \\ a^{-1}b & I \end{bmatrix}
\]
is invertible and
\[
\begin{bmatrix} a & 0 \\ 0 & M/a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a^{-1}(-b) & I \end{bmatrix} \begin{bmatrix} a & b^T \\ b & C \end{bmatrix} \begin{bmatrix} 1 & 0 \\ a^{-1}(-b) & I \end{bmatrix}^T.
\]

Proof. Observe that $\begin{bmatrix} 1 & 0 \\ a^{-1}b & I \end{bmatrix}^{-1} = \begin{bmatrix} a^{-1} & 0 \\ -b & I \end{bmatrix}$. The second statement of the lemma is a straightforward calculation. $\blacksquare$

The next lemma shows that the Schur complement preserves the positive semidefiniteness.

Lemma 2.4. If $a \in U(S)$, $b \in V(S)^{n-1}$, $C \in M_{n-1}(S)$ and
\[
M = \begin{bmatrix} a & b^T \\ b & C \end{bmatrix} \in M_n(S)
\]
is positive semidefinite, then $a$ and $M/a$ are positive semidefinite.

Proof. Suppose that $M$ is positive semidefinite. Choose $e = [1 \ 0 \ldots 0]^T \in S^n$ and an arbitrary $y \in S^{n-1}$, and observe that $a = e^T M e \in Q(S)$ and
\[
y^T (M/a) y = [0 \ y^T] \begin{bmatrix} a & 0 \\ 0 & M/a \end{bmatrix} \begin{bmatrix} 0 \\ y \end{bmatrix}.
\]
Define
\[
x = \begin{bmatrix} 1 & a^{-1}(-b)^T \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 \\ y \end{bmatrix}
\]
and observe that by Lemma 2.3
\[
y^T (M/a) y = x^T M x \in Q(S).
\]
3. Cholesky decomposition

First, we demonstrate that not every symmetric positive semidefinite invertible matrix has a Cholesky decomposition.

**Example 3.1.** Let $S = \mathbb{Z}_2 \times B$, where $B$ is the Boolean semiring, and

$$M = \begin{bmatrix} (1,0) & (0,1) \\ (0,1) & (1,0) \end{bmatrix} \in M_2(S).$$

Note that $M^2 = I$, so $M$ is invertible. Observe that $Q(S) = S$ and thus $M$ is positive semidefinite, but one can easily see that there does not exist a lower triangular matrix $L$, such that $LL^T = M$.

We will therefore use a stronger condition than invertibility, namely the strong invertibility. The matrix $A \in M_n(S)$ is strongly invertible if all the leading principal submatrices of $A$ are invertible. Note that the matrix $M$ in Example 3.1 is invertible, but not strongly invertible.

The main result of this paper will show that every symmetric positive semidefinite strongly invertible matrix has a Cholesky decomposition $A = LL^T$, which is unique up to a right sided multiplication of $L$ by a diagonal matrix $D$ such that $D^2 = I$.

The next lemma proves that every strongly invertible matrix has a Schur complement. It is a straightforward corollary of [8, Lemma 2.4].

**Lemma 3.2.** Let $a \in S$, $b \in S^{n-1}$ and $C \in M_{n-1}(S)$. If $M = \begin{bmatrix} a & b^T \\ b & C \end{bmatrix}$ is a strongly invertible matrix, then $b$ is additively invertible. Hence, if $M$ is strongly invertible, then there exists the Schur complement $M/a$.

Next, we prove that the set of strongly invertible matrices is invariant under taking the Schur complement.

**Lemma 3.3.** Let $a \in S$, $b \in S^{n-1}$ and $C \in M_{n-1}(S)$. If $M = \begin{bmatrix} a & b^T \\ b & C \end{bmatrix}$ is a strongly invertible, then $M/a$ is strongly invertible.

**Proof.** Choose $k$, $1 \leq k \leq n - 1$, and let $\tilde{M}_k$ be the leading principal submatrix of $M/a$ of size $k$ and $\tilde{M}_k$ the leading principal submatrix of $M$ of size $k + 1$. Observe that $\tilde{M}_k = \begin{bmatrix} a & \hat{b}^T \\ b & \hat{C} \end{bmatrix}$, where $\hat{b} \in S^{k-1}$ consists of the first $k - 1$ components of the vector $b$ and $\hat{C} \in M_{k-1}(S)$ is the leading principal submatrix of $C$. Furthermore, $\tilde{M}_k$ is the Schur complement of $\tilde{M}_k$. Since $M$ is strongly invertible, we conclude that the matrix $\tilde{M}_k$ is invertible, and thus by Lemma [2.3] it follows that $\begin{bmatrix} a & 0 \\ 0 & \tilde{M}_k/a \end{bmatrix}$ is invertible. This implies that $\tilde{M}_k \in U(M_k(S))$, hence $M/a$ is strongly invertible.

**Theorem 3.4** (Cholesky decomposition). Let $M \in M_n(S)$ be a symmetric positive semidefinite strongly invertible matrix. Then $M = LL^T$ for a strongly invertible lower triangular matrix $L \in M_n(S)$. 

Furthermore, the above decomposition \( M = LL^T \) is unique up to right sided multiplication of \( L \) by a diagonal matrix \( D \in M_n(S) \), such that \( D^2 = I \).

**Proof.** Suppose \( M = \begin{bmatrix} a & \tilde{b}^T \\ b & C \end{bmatrix} \in M_n(S) \), where \( a \in S \), \( b \in S^{n-1} \) and \( C \in M_{n-1}(S) \), \( a \) is positive semidefinite and strongly invertible. We will prove the existence of \( L \) by the induction on \( n \). For \( n = 1 \), the statement is clear. Suppose \( n > 1 \). Since \( M \) is strongly invertible, we have \( a \in U(S) \). By Lemmas 2.4, 3.2 and 3.3, the Schur complement \( M/a \in M_{n-1}(S) \) exists and is positive semidefinite as well as strongly invertible. Thus by the induction hypothesis, there exists a strongly invertible lower triangular matrix \( K \in M_{n-1}(S) \) such that \( M/a = KK^T \). Moreover, Lemma 2.4 implies that \( a \) is positive semidefinite and thus by definition, there exists \( k \in S \), such that \( a = k^2 \). Note that \( a \in U(S) \) also implies \( k \in U(S) \). If, \( L = \begin{bmatrix} k & 0 \\ k^{-1}b & K \end{bmatrix} \), then \( LL^T = M \). Since \( K \) is a lower triangular strongly invertible matrix, it follows directly that \( L \) is strongly invertible.

Moreover, if \( M = LL^T = \tilde{L}\tilde{L}^T \), where \( L \) and \( \tilde{L}^T \) are invertible lower triangular matrices, then

\[
L^{-1}\tilde{L} = L^T(\tilde{L}^T)^{-1} = (\tilde{L}^{-1}L)^T.
\]

This implies that \( L^{-1}\tilde{L} = (\tilde{L}^{-1}L)^T \) is a diagonal matrix. We denote \( D = L^{-1}\tilde{L} \) and so \( L = \tilde{L}D \) and \( \tilde{L} = LD \), where \( D^2 = I \). Obviously, for any diagonal matrix \( D \), such that \( D^2 = I \), we have \((LD)(LD)^T = LL^T\). \( \blacksquare \)

The next example shows that in general, the lower triangular matrix \( L \) from Theorem 3.4 cannot be chosen to be positive semidefinite.

**Example 3.5.** Let \( S = \mathbb{Z}_2[x]/(x^3) \) be a (semi)ring. Observe that \( Q(S) = \{0, 1, x^2, 1 + x^2\} \). Choose

\[
M = \begin{bmatrix} 1 & 0 \\ 0 & 1 + x^2 \end{bmatrix} \in M_2(S).
\]

Note that \( M \) is strongly invertible. Since \( 1 + x^2 \in Q(S) \) and \( (a + b)^2 = a^2 + b^2 \) for all \( a, b \in S \), \( M \) is positive semidefinite. One can check that for

\[
L = \begin{bmatrix} 1 & 0 \\ 0 & 1 + x \end{bmatrix} \in M_2(S)
\]

we have \( LL^T = M \), but \( L \) is not positive semidefinite, since \( 1 + x \notin Q(S) \). Now, if \( D \) is any diagonal matrix with \( D^2 = I \), then the diagonal entries of \( D \) are either equal to 1 or \( 1 + x^2 \). Since both of these two elements are from \( Q(S) \) and \( 1 + x \notin Q(S) \), the matrix \( LD \) cannot be positive semidefinite.

The next corollary shows that in some special commutative semirings, we can obtain a characterization of positive semidefinite strongly invertible matrices by their Cholesky decomposition.

**Corollary 3.6.** Let \( S \) be a commutative semiring such that \( Q(S) + Q(S) \subseteq Q(S) \) and let \( M \in M_n(S) \) be a symmetric matrix. Then \( M \) is positive semidefinite and strongly invertible if and only if \( M = LL^T \) for a lower triangular strongly invertible matrix \( L \in M_n(S) \).
Proof. If $M$ is a symmetric matrix that is also positive semidefinite and strongly invertible, then $M$ has a Cholesky decomposition by Theorem 3.4.

Assume now that $M \in M_n(S)$ is a symmetric matrix, and that there exists a strongly invertible matrix $L \in M_n(S)$ such that $M = LL^T$. If $x \in S^n$, then $x^T M x = (L^T x)^T (L^T x) \in \sum Q(S) \subseteq Q(S)$, hence $M$ is positive semidefinite. Since the leading principal submatrices of $L$ and $L^T$ are invertible, it follows that any leading principal submatrix of $M$ is a product of two invertible matrices, hence invertible. Therefore $M$ is strongly invertible. ■

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