Interior dynamics of fatou sets

Mi Hu

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Abstract
In this paper, we investigate the precise behavior of orbits inside attracting basins. Let \( f \) be a holomorphic polynomial of degree \( m \geq 2 \) in \( \mathbb{C} \), \( \mathcal{A}(p) \) be the basin of attraction of an attracting fixed point \( p \) of \( f \), and \( \Omega_i (i = 1, 2, \cdots) \) be the connected components of \( \mathcal{A}(p) \). Assume \( \Omega_1 \) contains \( p \) and \( \{ f^{-1}(p) \} \cap \Omega_1 \neq \{ p \} \). Then there is a constant \( C \) so that for every point \( z_0 \) inside any \( \Omega_i \), there exists a point \( q \in \bigcup_k f^{-k}(p) \) inside \( \Omega_i \) such that \( d_{\Omega_i}(z_0, q) \leq C \), where \( d_{\Omega_i} \) is the Kobayashi distance on \( \Omega_i \). In paper Hu (Dynamics inside parabolic basins, 2022), we proved that this result is not valid for parabolic basins.

Keywords Basin of attraction · Blaschke products · Kobayashi metric

Mathematics Subject Classification Primary 37F10 · Secondary 30F45

1 Introduction
A general goal in discrete dynamical systems is to qualitatively and quantitatively describe the possible dynamical behaviour under the iteration of maps satisfying certain conditions.

Let \( \hat{\mathbb{C}} = \mathbb{C} \cup \{ \infty \} \), \( f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) be a nonconstant holomorphic map, and \( f^n : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) be its \( n \)-fold iterate. In complex dynamics, two crucial disjoint invariant sets are associated with \( f \), the Julia set and the Fatou set [11], which partition the sphere \( \hat{\mathbb{C}} \). The Fatou set of \( f \) is defined as the largest open set where the family of iterates is locally normal. In other words, for any point \( z \in \hat{\mathbb{C}} \), there exists some neighborhood \( U \) of \( z \) so that the sequence of iterates of the map restricted to \( U \) forms a normal family, so the iterates are well-behaved. The complement of the Fatou set is called the Julia set. The connected components of the Fatou set of \( f \) are called Fatou components. A Fatou component \( \Omega \subset \hat{\mathbb{C}} \) of \( f \) is invariant if \( f(\Omega) = \Omega \). For \( z \in \hat{\mathbb{C}} \), the set \( \{ z_n \} = \{ z_1 = f(z_0), z_2 = f^2(z_0), \cdots \} \) is called the orbit of the point \( z = z_0 \). If \( z_N = z_0 \) for some integer \( N \), we say that \( z_0 \) is a periodic point of \( f \). If \( N = 1 \), then \( z_0 \) is a fixed point of \( f \).
At the beginning of the 20th century, Fatou [5–7] classified all possible invariant Fatou components of rational functions on the Riemann sphere. He proved that there are just three possibilities:

1. (attracting case) $\Omega$ contains a fixed point $p$, and the orbit of every point in $\Omega$ converges to $p$.
2. (parabolic case) $\partial\Omega$ contains a fixed point $p$, and the orbit of every point in $\Omega$ converges to $p$.
3. (rotation domain) $\Omega$ is conformally equivalent to a disk or an annulus, and the map is conjugate to an irrational rotation.

The classification of Fatou components was completed in the 1980s when Sullivan proved that every Fatou component of a rational map of degree larger than one is preperiodic, i.e., there are $n, m \in \mathbb{N}$ such that $f^{n+m}(\Omega) = f^m(\Omega)$. For more details and results, we refer the reader to [12].

However, there has been no detailed study until now of the more precise behavior of orbits inside the Fatou set. For example, let $A(p) := \{z \in \mathbb{C}; f^n(z) \to p\}$ be the basin of attraction of an attracting fixed point $p$. One can ask when $z_0$ is close to $\partial A(p)$, what orbits $\{z_n\}$ going from $z_0$ to near the attracting fixed point $p$ look like, or how many iterations are needed to reach this. One application arises from Newton’s method in [1]. It is of practical interest to know how many times Newton’s method must be iterated to get the desired approximation of the root.

These kinds of questions are the main topics of this paper. In the second section, we study the dynamics of holomorphic polynomials on the attracting basins and prove our main Theorem A and Theorem B:

**Theorem A** Suppose $f(z)$ is a polynomial of degree $N \geq 2$ on $\mathbb{C}$, $\Omega$ is the immediate attracting basin of $f(z)$, and $p$ is an attracting fixed point inside $\Omega$, $\{f^{-1}(p)\} \cap \Omega \neq \{p\}$. Then there is a constant $C$ such that for every point $z_0 \in \Omega$, there exists a point $q \in \cup_k f^{-k}(p)$, $k \geq 0$ so that $d_\Omega(z_0, q) \leq C$, $d_\Omega$ is the Kobayashi distance on $\Omega$.

To prove Theorem A, we use conjugation to consider the orbits of Blaschke products on the unit disk instead of considering the orbits of polynomials on the attracting basin. This implies that Theorem A is still valid for the attracting basin of $\infty$ if this basin is simply connected, see Corollary 4 in the following section. Furthermore, we also generalized Theorem A to the following Theorem B when $\Omega$ is the whole basin of attraction.

**Theorem B** Suppose $f(z)$ is a polynomial of degree $N \geq 2$ on $\mathbb{C}$, $p$ is an attracting fixed point of $f(z)$, $\Omega_1$ is the immediate basin of attraction of $p$, $\{f^{-1}(p)\} \cap \Omega_1 \neq \{p\}$, $\mathcal{A}(p)$ is the basin of attraction of $p$, $\Omega_i$ ($i = 1, 2, \cdots$) are the connected components of $\mathcal{A}(p)$. Then there is a constant $\tilde{C}$ so that for every point $z_0$ inside any $\Omega_i$, there exists a point $q \in \cup_k f^{-k}(p)$ inside $\Omega_i$ such that $d_{\Omega_i}(z_0, q) \leq \tilde{C}$, where $d_{\Omega_i}$ is the Kobayashi distance on $\Omega_i$.

Note that Theorem A and Theorem B are no longer valid for parabolic basins, see details in [9].

## 2 Dynamics of holomorphic polynomials inside attracting basins

Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a holomorphic map and $\Omega$ be an invariant Fatou component. It follows from the classification above in Sect. 1 that we have a complete understanding of the long-term behaviour of all orbits in $\Omega$, but how precisely the iterates of a point move inside $\Omega$?
is presently still unknown. This section will more precisely describe how iterates of a point move inside an invariant Fatou component and the whole basin of attraction. Then we give our main results in Theorem A and Theorem B.

2.1 The Kobayashi metric

**Definition 1** Let \( \hat{\Omega} \subset \mathbb{C} \) be a domain. We choose a point \( z \in \hat{\Omega} \) and a vector \( \xi \) which is tangent to the plane at the point \( z \). Let \( \Delta \) denote the unit disk in the complex plane. We define the **Kobayashi metric** \([10]\)

\[
F_{\hat{\Omega}}(z, \xi) := \inf \{ \lambda > 0 : \exists f : \Delta \xrightarrow{\text{hol}} \hat{\Omega}, f(0) = z, \lambda f'(0) = \xi \}.
\]

Let \( \gamma : [0, 1] \to \hat{\Omega} \) be a piecewise smooth curve. The **Kobayashi length** of \( \gamma \) is defined to be

\[
L_{\hat{\Omega}}(\gamma) = \int_{\gamma} F_{\hat{\Omega}}(z, \xi)|dz| = \int_0^1 F_{\hat{\Omega}}(\gamma(t), \gamma'(t))|\gamma'(t)|dt.
\]

For any two points \( z_1 \) and \( z_2 \) in \( \hat{\Omega} \), the **Kobayashi distance** between \( z_1 \) and \( z_2 \) is defined to be

\[
d_{\hat{\Omega}}(z_1, z_2) = \inf \{ L_{\hat{\Omega}}(\gamma) : \gamma \text{ is a piecewise smooth curve connecting } z_1 \text{ and } z_2 \}.
\]

Note that \( d_{\hat{\Omega}}(z_1, z_2) \) is defined where \( z_1, z_2 \) are in the same connected component of \( \hat{\Omega} \). Let \( d_E(z_1, z_2) \) denote the Euclidean metric distance for any two points \( z_1, z_2 \in \Delta \).

We know that if \( \hat{\Omega} = \Delta \), then the Kobayashi metric is the same as the Poincaré metric (see page 9 in [4] and sections 0 and 3 in [10]). And

\[
F_{\Delta}(z, \xi) = \frac{|\xi|}{1 - |z|^2}.
\]

Note that it is common to call \( F_{\hat{\Omega}} \) a “Poincaré metric” for any \( \hat{\Omega} \subset \mathbb{C} \). However, for generalization to higher dimensions, and also for the basin of \( \infty \), we call it a “Kobayashi metric” even in dimension one to be more consistent.

**Proposition 1** *(The distance decreasing property of the Kobayashi Metric [10])* Suppose \( \Omega_1, \Omega_2 \) are domains in \( \mathbb{C} \), \( z, \omega \in \Omega_1, \xi \in \mathbb{C} \), and \( f : \Omega_1 \to \Omega_2 \) is holomorphic. Then

\[
F_{\Omega_2}(f(z), f'(z)\xi) \leq F_{\Omega_1}(z, \xi), \quad d_{\Omega_2}(f(z), f(\omega)) \leq d_{\Omega_1}(z, \omega).
\]

**Corollary 2** Suppose \( \Omega_1 \subseteq \Omega_2 \subseteq \mathbb{C} \). Then for any \( z, \omega \in \Omega_1 \) and \( \xi \in \mathbb{C} \), we have

\[
F_{\Omega_2}(z, \xi) \leq F_{\Omega_1}(z, \xi), \quad d_{\Omega_2}(z, \omega) \leq d_{\Omega_1}(z, \omega).
\]

2.2 Main results about orbits inside an attracting basin

Let \( f : \mathbb{C} \to \mathbb{C} \) be a polynomial of degree \( N \geq 2 \) and \( \Omega \) be an immediate basin of attraction that contains an attracting fixed point \( p \) of \( f \). Then \( \Omega \) is a connected component of \( \mathcal{A}(p) \). In addition, an immediate basin of attraction of a holomorphic polynomial is simply connected. We can apply the Riemann mapping theorem to conjugate the immediate basin of attraction to the unit disk and send the attracting fixed point \( p \) to 0. Furthermore, this conjugation
preserves the metric on $\Omega$ and is equivalent to the metric on the unit disk (see Sect. 3 in [10]). Hence we should study the proper holomorphic self-maps of the unit disk. Moreover, the proper maps on the unit disk can be written as $g(z) = e^{i\theta} \prod_{j=1}^{m} \frac{z-a_j}{1-\overline{a_j}z}$ for $e^{i\theta}$ on the unit circle $\partial \Delta$ (see Lemma 15.5 on page 163 in [11]), and $m \geq 2$ since there is at least one critical point inside $\Omega$ (see Theorem 2.2 on page 59 in [3]), and constants $|a_j| < 1$ with at least one of the $a_j = 0$ since $f$ has an attracting fixed point $p$ which is sent to 0. Note that the degree of $g$ depends on how many critical points are inside $\Omega$, so $2 \leq m \leq N$. Therefore, instead of considering the orbits of polynomials on the attracting basin with an attracting fixed point at $p$, we only need to consider the orbits of Blaschke products on the unit disk with an attracting fixed point at 0.

First, we discuss the simplest case when all $a_j = 0$. Then for $g = e^{i\theta} z^m$, we have the following theorem.

**Theorem 3** Suppose $g(z) = e^{i\theta} z^m$, $m \geq 2$, we pick a point $\hat{p} \in \Delta \setminus \{0\}$. Then there exists a constant $C_0 > 0$ such that for every point $z_0 \in \Delta$, there exists $q \in \cup_k g^{-k}(\hat{p})$, $k \geq 0$ satisfying $d_\Delta(z_0, q) \leq C_0$, where $d_\Delta$ is the Kobayashi distance on the unit disk $\Delta$.

**Proof** It is easy to describe the dynamics of $g(z) = e^{i\theta} z^m$, $m \geq 2$ since all its iterates can be written very explicitly as $g^n(z) = e^{i\frac{kn-mt}{m}} z^m$. Let the point $\hat{p} = \rho e^{i\theta_0}$, $0 < \rho < 1$, then the inverse images of $\hat{p}$ are

$$T_k := s e^{i\phi}, s = \rho^{1/m^k}, \phi = \frac{2j\pi + \theta_0 - \frac{1-m^k}{1-m} \theta}{m^k}, j = 0, \ldots, m^k - 1.$$ 

We need to show that all points are at most a uniformly bounded distance from this sequence in the Kobayashi metric of the disk.

The $k$th preimages under this map of an annulus of the form $\{\rho \leq |z| \leq \rho^{1/m}\}$ can be represented as $m^k$ essentially disjoint ‘boxes’ each consisting of a sector of the annulus $\{\rho^{1/m^k} \leq |z| \leq \rho^{1/m^{k+1}}\}$ with angle $2\pi/m^k$. Each of these boxes must contain a preimage of the given point $\hat{p}$, for a suitable choice of $\rho$. The length of any path in such a box w.r.t. the Kobayashi metric (on the unit disk, it is equal to the Poincaré metric) is bounded above by its Euclidean length multiplied by a simple upper bound for the density of the Kobayashi metric in the box. This gives an upper bound for the box diameter w.r.t. the Kobayashi metric of

$$\left(1 - \rho^{1/m^k}\right) + \frac{2\pi}{m^k} \frac{1}{1 - \rho^{1/m^{k+1}}}.$$ (1)

Then the estimates

$$1 - \frac{1}{2} x \geq e^{-x} \geq 1 - x, \text{ for } 0 \leq x \leq 1,$$

with $x = \frac{\ln(1/\rho)}{m^k}$, $\frac{\ln(1/\rho)}{m^{k+1}}$, for $k$ sufficiently large, gives

$$\frac{\ln(1/\rho)}{2m^{k+1}} \leq 1 - \rho^{1/m^{k+1}} \leq 1 - \rho^{1/m^k} \leq \frac{\ln(1/\rho)}{m^k},$$

which, using (1), shows that these boxes have a uniformly bounded diameter. □

**Corollary 4** Let $f$ be a polynomial of degree $m \geq 2$, considered as a map on $\mathbb{C}$. Suppose that $\Omega(\infty)$ is the basin of $\infty$ and is simply connected. If $p \in \Omega \setminus \{\infty\}$, then there exists a constant $C_0 > 0$ such that for every point $z_0 \in \Omega$, there exists $q \in \cup_k g^{-k}(p)$, $k \geq 0$ satisfying $d_\Omega(z_0, q) \leq C_0$, where $d_\Omega$ is the Kobayashi distance on $\Omega$. 

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Proof Using Theorem 9.5 in [11], we know that $\Omega(\infty)$ is conformally isomorphic to the exterior of the closed unit disk $\Delta$ under an isomorphism

$$\Phi : \Omega(\infty) \rightarrow \hat{\mathbb{C}} \setminus \Delta,$$

which conjugates $f$ on $\Omega(\infty)$ to the $m$th power map $w \mapsto w^m$.

Then by Theorem 3, the proof is done.

Next, we generalize Theorem 3 to general Blaschke products $g(z) = e^{i\theta} \prod_{j=1}^{n} \frac{z-a_j}{1-\overline{a_j}z}$, with some $a_j \neq 0$. However, there is at least one of the $a_j = 0$ since 0 is a fixed point of $g$. In this case, $\hat{p} \in \Delta \setminus \{0\}$ in Theorem 3 can now be replaced by 0.

Theorem 5 Suppose

$$g(z) = e^{i\theta} z^{m_1} \prod_{l=m_1+1}^{m} \frac{z-a_l}{1-\overline{a_l}z}, \quad m \geq 2, \quad 1 \leq m_1 < m, \quad a_l \neq 0, \quad a_l \in \Delta, \quad e^{i\theta} \in \partial \Delta.$$ 

There exists a constant $C_0$ such that for every point $z_0 \in \Delta$, there exists $q \in \cup_k g^{-k}(0)$ satisfying $d_{\Delta}(z_0, q) \leq C_0$.

Proof First of all, if $|z_0| \leq \rho < 1$ for some constant $\rho$ to be chosen later, then we choose $q = 0$. And by the formula (on page 21 in [11]) for the Kobayashi metric on the disk, we have

$$d_{\Delta}(z_0, q) = d_{\Delta}(0, \frac{z_0 - q}{1 - z_0\overline{q}}) = \ln \frac{1 + |z_0|}{1 - |z_0|} \leq \ln \frac{1 + \rho}{1 - \rho} := C_0'.$$

Therefore, there is a constant $C_0'$ such that for every point $z_0 \in \Delta$ and $|z_0| \leq \rho$, there exists a $q = 0$ satisfying $d_{\Delta}(z_0, q) \leq C_0'$.

Secondly, if $|z_0| > \rho$, then we will show that there still exists a point $q \in \cup_k g^{-k}(0)$ such that $d_{\Delta}(z_0, q)$ is uniformly bounded even when $z_0$ is very close to $\partial \Delta$.

Let us first remind the reader why the preimages of 0 are dense in the Euclidean distance on the boundary of the unit disk intersecting the unit disk, i.e., the set $\cup_{n \rightarrow \infty} \{g^{-n}(0)\}$ clusters at every point in $\partial \Delta \cap \Delta$. First of all, we know that the Blaschke product $g$ carries $\Delta$ onto $\Delta$ and $\hat{\mathbb{C}} \setminus \Delta$ onto $\hat{\mathbb{C}} \setminus \Delta$, and the Julia set of $g$ is the unit circle. See Theorem 1.8 on page 57 and the corresponding example on page 58 in [3]. Second, by Theorem 4.10 on page 47 in [11], we know that for any $\hat{z}_0 \in \partial \Delta$, there exists a sequence $\{g^{-n}(0)\}$ that converges to $\hat{z}_0$. Therefore, the preimages of 0 are dense in $\partial \Delta \cap \Delta$.

Note that this does not finish the proof because the Kobayashi distance from $\{g^{-n}(0)\}$ to points near the boundary might still be arbitrarily large. We still need to show that this is not the case.

Before continuing with the proof of Theorem 5, we have the following two lemmas for the orbits near the boundary of the unit disk.

Lemma 6 $|g'(z)| > 1$ for all $z \in \partial \Delta$.

Proof Lemma 3.4.3 on page 46 in [8] shows that $g'(z) \neq 0$ for all $z \in \partial \Delta$.

However, $g(z) = e^{i\theta} z^{m_1} \prod_{l=m_1+1}^{m} \frac{z-a_l}{1-\overline{a_l}z}$. Then $a_l = 0$ when $l = 1, 2, \ldots, m_1$. Hence

$$|g'(z)| = m_1 + \sum_{l=m_1+1}^{m} \frac{|z-a_l|^2}{|z|^2} > m_1 \geq 1,$$
for all $z \in \partial \Delta$. We refer the readers to the proof of lemma 3.4.3 in [8] for more details.

Therefore, we can choose $r_0 \in (0, 1)$ so that $|g'(z)| > 1 + \epsilon$ for any $z$ inside the annulus $A := \{z : r_0 < |z| < 1\}$ and some fixed $\epsilon > 0$.

**Lemma 7** $|g^{-1}(z)| > |z|$ for all $z \in A$.

**Proof** See the Schwarz lemma and Lemma 6 above.

We continue with the proof of Theorem 5. We consider the inverse orbit of 0 near the boundary of the unit disk precisely, especially we investigate the inverse orbit inside $A$.

Let $D_0 = g^{-1}(\Delta \setminus A)$, and choose $r_1 > r_0$ such that $D_0 \subset B_0 = \{z : |z| \leq r_1 < 1\}$. Let $D_1 = g^{-1}(\overline{B}_0)$, then by Lemma 7, we can choose $r_2 > r_1$ such that $D_1 \subset \overline{B}$ where $B = \{z : |z| < r_2 < 1\}$. Let $B_1 := \{z : r_1 < |z| < r_2\}$, then for any $z_0 \in \Delta \setminus B$, there exists a point $z'_0 := g^{M_1}(z_0) \in B_1$, $z'_1 := g^{M_1-1}(z_0) \in \Delta \setminus B$ for some integer $M_1$ (see figure 1). Since the preimages of 0 are dense in the boundary of the unit disk, we can calculate all preimages of 0 and choose some integer $M_2$ such that $q_0 := g^{-M_2}(0) \in B_1$, $q_1 := g^{-1}(q_0) \in \Delta \setminus B$.

Now, we need to choose a sector $A^\pm$ such that $z'_0, q_0 \in A^\pm$, but $z'_0, q_0$ are neither on $\partial A^\pm$ nor very close to $\partial A^\pm$: first, choose a sector $S$ with angle $\frac{\pi}{2}$ and one boundary line of $S$ passes through 0 and $z'_0$ such that $q_0 \notin S$. Secondly, choosing a sector $\tilde{S}$ inside $S$ with angle $\frac{\pi}{2}$ such that $\tilde{S}$ and $S$ have the same angle bisector region of $A$ not including $z'_0$ and $q_0$. Then we let $A^\pm := A \setminus (\tilde{S} \cap A)$ (see the whole green area, including the red area, in Fig. 2).

Since we know precisely the orbit $\{z_n\}$, we can choose the inverse map to be the branch of $g^{-1}(z)$ on $A^\pm$ such that $g^{-1}(z'_0) = g^{-1}(g^{M_1}(z_0)) = g^{M_1-1}(z_0) = z'_1$ sending $q_0$ to $q_1$. In addition, there are no critical points inside $A$ since $|g'(z)| > 1 + \epsilon$ and $0 < |(g^{-1}(z))'| < \frac{1}{1+\epsilon}$ for all $z \in A$. Then the branch of the inverse map $g^{-1}(z)$ which we choose is well defined in $A^\pm$. Inductively, we can, in the same way, define $g^{-n}(z)$ in $A^\pm$. We obtain that the inverse map $g^{-n}(z) : A^\pm \to A$ is analytic. And the annulus $\Delta \setminus B$ contains $z'_1, z'_2, z'_3, \ldots, z'_n$ and $q_1, q_2, q_3, \ldots, q_n$. © Springer
Next, we estimate the Kobayashi distance. Choosing a sector \( \bar{S} \subset S \) with an angle \( \pi/4 \), then let \( B_1^+ = B_1 \setminus (\bar{S} \cap B_1) \subset A^+ \) (see the red area in Fig. 2), then \( B_1^+ \subset A \) including \( z_0', q_0 \). Hence there is a constant \( C_0 \) such that

\[
d_{B_1^+}(z_0', q_0) \leq C_0. \tag{3}
\]

In addition, by Corollary 2, we have

\[
d_{A^+}(z_0', q_0) \leq d_{B_1^+}(z_0', q_0). \tag{4}
\]

Inductively, \( g^{-M_1}(z) : A^+ \to A \) sends the points \( z_0', q_0 \in A^+ \) to the points \( z_0, q_{M_1} \in A \), respectively. The Kobayashi distance is decreasing by Proposition 1, then we obtain

\[
d_{A}(z_0, q_{M_1}) \leq d_{A^+}(z_0', q_0). \tag{5}
\]

Since \( A \subset \Delta \), using Corollary 2 once again, we know

\[
d_{\Delta}(z_0, q_{M_1}) \leq d_{A}(z_0, q_{M_1}). \tag{6}
\]

Therefore, from Eqs. (3) to (6), there is a constant \( C_0 \) and a point \( q := q_{M_1} \in \cup_k g^{-k}(0) \) such that

\[
d_{\Delta}(z_0, q) \leq d_{A}(z_0, q) = d_{A}(z_0, q_{M_1}) \leq d_{A^+}(z_0', q_0) \leq d_{B_1^+}(z_0', q_0) \leq C_0, \tag{7}
\]

for some integer \( k \). Choosing \( \rho = r_2 \), we finally prove the theorem.

Remark: An alternative approach would be to use that \( g \) near the unit circle is conformally conjugate to a power map in an annulus lying in the unit disc and with the unit circle as a boundary component. More precisely, on page 5 in [2], Bergweiler and Morosawa showed that there exists a compact set \( E \subset \Delta, \rho_0 \in (0, 1) \) and a homeomorphism \( \phi : \{\rho_0 < |z| < 1\} \to \Delta \setminus E \) such that \( g(\phi(z)) = \phi(z^m) \) for \( \{\rho_0^{1/m} < |z| < 1\} \). Moreover, \( \phi \) is also analytic.
on \(|\rho_0 < |z| < 1|\) by Theorem VI.5.1 in [3]. Then our Theorem 5 holds by Theorem 3 for \(|z_0| > \rho\) using the localization theorem for the Kobayashi metric as in Wold [13].

Our proof above actually used only basic knowledge of complex analysis to explore how orbits go in annuli lying in the unit disc and with the unit circle as their outer boundaries. And we also showed how big \(\rho\) needs to be chosen exactly. Moreover, we showed the exact behavior in which the iterations of a point move inside the immediate attraction basin. This understanding is crucial and beneficial for proving Theorem B later on, particularly concerning the orbits between different components of the whole basin of attraction.

Now, we can obtain our main theorems as follows:

**Theorem A** Suppose \(f(z)\) is a polynomial of degree \(N \geq 2\) on \(\mathbb{C}\), \(\Omega\) is the immediate attracting basin of \(f(z)\), and \(p\) is an attracting fixed point inside \(\Omega\), \(\{f^{-1}(p)\} \cap \Omega \neq \{p\}\). Then there is a constant \(C\) such that for every point \(z_0 \in \Omega\), there exists a point \(q \in \cup_k f^{-k}(p)\), \(k \geq 0\) so that \(d_\Omega(z_0, q) \leq C\). \(d_\Omega\) is the Kobayashi distance on \(\Omega\).

This theorem would imply that the orbit of \(z_0\) behaves like the orbit of some preimage of the fixed point \(p\). More precisely, the orbit of \(z_0\) is shadowed by the orbit of the point \(f^{-k}(p)\) for some positive integer \(k\). Hence it would be enough to consider only these preimage points \(f^{-k}(p)\). But then, instead of considering the forward orbit of the preimage of the fixed point \(p\), we can equivalently study the backward orbit of the fixed point \(p\). Note that this is very useful for practical purposes because these inverse orbits can be color plotted.

**Proof** Since \(\Omega\) is an immediate attracting basin of \(f(z)\), \(p\) is an attracting fixed point inside \(\Omega\), we know that \(f(\Omega) = \Omega\) and \(\Omega\) is simply connected. By the Riemann mapping theorem, there exists a conformal mapping \(\phi\) such that we can conjugate \(f\) on \(\Omega\) to \(g = e^{i\theta} \prod_{j=1}^m \frac{z-a_j}{1-a_jz}\) on the unit disk, i.e., \(\phi \circ f = g \circ \phi\). And \(g\) is a proper self-map of the unit disk \(\Delta\), with an attracting fixed point at the origin. We refer the reader to Lemma 15.5 on page 163 in [11] for more details. In addition, for any two points \(a, b \in \Omega\), we have \(d_\Omega(a, b) = d_\Delta(\phi(a), \phi(b))\) by the Riemann mapping theorem (see the Remark on page 8 in [10]). Then this theorem is true because of Theorem 5.

In the remainder of this section, we will prove that Theorem A still holds when \(\Omega\) is the whole basin of attraction.

**Theorem B** Suppose \(f(z)\) is a polynomial of degree \(N \geq 2\) on \(\mathbb{C}\), \(p\) is an attracting fixed point of \(f(z)\), \(\Omega_1\) is the immediate basin of attraction of \(p\), \(\{f^{-1}(p)\} \cap \Omega_1 \neq \{p\}\), \(A(p)\) is the basin of attraction of \(p\), \(\Omega_i (i = 1, 2, \ldots)\) are the connected components of \(A(p)\). Then there is a constant \(\tilde{C}\) so that for every point \(z_0\) inside any \(\Omega_i\), there exists a point \(q \in \cup_k f^{-k}(p)\) inside \(\Omega_i\) such that \(d_{\Omega_i}(z_0, q) \leq \tilde{C}\), where \(d_{\Omega_i}\) is the Kobayashi distance on \(\Omega_i\).

**Proof** Since \(f\) is a polynomial with an attracting fixed point \(p \in \Omega_1\), there are two possible cases.

1. \(A = \Omega_1\). Then this theorem is essentially as same as Theorem A.
2. \(A\) has at least two connected components, then \(A\) has infinitely many connected components: Suppose \(\Omega_2\) is another connected component of \(f\) which is distinct from \(\Omega_1\) and \(f(\Omega_2) = \Omega_1\). We know \(f(\Omega_1) = \Omega_1\), and then there must be a third component \(\Omega_3\) which can be mapped to \(\Omega_2\), and so on. It implies that \(A\) has infinitely many connected components.
For the second situation, let us first consider the orbit between two connected components, \( \Omega_1 \) and \( \Omega_2 \), where \( f(\Omega_2) = \Omega_1 \). If the start point \( z_0 \) is inside \( \Omega_1 \), then the proof is done by Theorem A. If the start point \( z_0 \) is inside \( \Omega_2 \), then there is a point \( \hat{z}_0 := f(z_0) \in \Omega_1 \). By the above Theorem A, we know that there is a constant \( C \) such that for every point \( \hat{z}_0 \in \Omega_1 \), there exists a point \( \hat{q} \in \bigcup_k f^{-k}(p) \), \( k \geq 0 \) in \( \Omega_1 \) so that \( d_{\Omega_1}(\hat{z}_0, \hat{q}) \leq C \), \( d_{\Omega_1} \) is the Kobayashi distance on \( \Omega_1 \). Next, we need to show that there is a constant \( C' \) such that for every point \( z_0 \in \Omega_2 \), there exists a point \( q := f^{-1}(\hat{q}) \in \bigcup_k f^{-k}(p) \), \( k \geq 0 \) in \( \Omega_2 \) so that \( d_{\Omega_2}(z_0, q) \leq C' \). Note that \( f \) has only finitely many critical points. If \( \Omega_2 \) contains no critical points, then \( f : \Omega_2 \to \Omega_1 \) is an isometry, hence, we can choose \( C' = C \). We only need to show the existence of \( C' \) for the cases when \( \Omega_2 \) contains critical points of \( f \).

Since \( \Omega_1 \) and \( \Omega_2 \) are simply connected, by the Riemann mapping theorem, there are two biholomorphic maps, \( \psi_1 : \Omega_1 \to \Delta \) and \( \psi_2 : \Omega_2 \to \Delta \). Then \( f \) is conjugate with \( g = \psi_1 \circ f \circ \psi_2^{-1} \), which is a proper self-map of the unit disk \( \Delta \). Hence \( g \) is a Blaschke product. Then these three points \( z_0 \in \Omega_2, \hat{z}_0 = f(z_0) \in \Omega_1, \hat{q} \in \Omega_1 \) are sent to \( \Delta \), we denote them by \( Z_0, \hat{Z}_0, \hat{Q} \in \Delta \), respectively, and \( d_{\Delta}(\hat{Z}_0, \hat{Q}) = d_{\Omega_1}(\hat{z}_0, \hat{q}) < C \). Therefore, it is equivalent to prove that there exists a point \( Q = g^{-1}(\hat{Q}) \in \Delta \) and a constant \( C' \) (independent of \( z_0 \)) such that \( d_{\Delta}(Z_0, Q) < C' \).

We know that \( g \) has finitely many critical points. We can choose a disk \( \Delta(0, r_0) = \{ z : |z| \leq r_0 < 1 \} \) including all critical points of \( g \). We denote \( D_0 = g(\Delta(0, r_0)) \) and choose a disk \( \Delta(0, R_0) = \{ z : |z| \leq R_0 < 1 \} \) such that \( D_0 \subseteq \Delta(0, R_0) \), then let \( D_1 := \Delta \setminus \Delta(0, R_0) \), see the following Fig. 3. Then there are four cases for distributing \( \hat{Z}_0 \) and \( \hat{Q} \).

Case 1: if \( \hat{Z}_0, \hat{Q} \in \Delta(0, R_0) \), we choose a disk \( \Delta(0, r_1) = \{ z : |z| \leq r_1 < 1 \} \) such that \( g^{-1}(\Delta(0, R_0)) \subseteq \Delta(0, r_1) \), then \( Z_0 \subseteq \Delta(0, r_1) \) and there exists a point \( Q = g^{-1}(\hat{Q}) \in \Delta(0, r_1) \). Hence \( d_{\Delta}(Z_0, Q) \leq C' \) for some uniform constant \( C' \).

Case 2: if \( \hat{Z}_0 \in \Delta(0, R_0), \hat{Q} \in D_1 \), then there exists a disk \( \Delta(0, R_1) = \{ z : |z| \leq R_1 < 1 \} \) including \( \hat{Q} \) and \( \hat{Z}_0 \) since \( d_{\Delta}(\hat{Z}_0, \hat{Q}) \leq C \). Hence there is a disk \( \Delta(0, r_2) = \{ z : |z| \leq r_2 < 1 \} \) so that \( g^{-1}(\Delta(0, R_1)) \subseteq \Delta(0, r_2) \). Then letting \( Q \) be any point of \( g^{-1}(\hat{Q}) \) inside \( \Delta(0, r_2) \). Hence \( d_{\Delta}(Z_0, Q) \leq C'' \) for some uniform constant \( C'' \).

Case 3: if \( \hat{Z}_0 \in D_1, \hat{Q} \in \Delta(0, R_0) \), this situation is the same as case 2.

Case 4: if \( \hat{Z}_0, \hat{Q} \in D_1 \). We know that \( g^{-1}(z) \) is locally holomorphic from a subset of \( D_1 \) to \( \Delta \). Then we can choose the branch of the inverse map of \( g \) such that \( Z_0 = g^{-1}(\hat{Z}_0) \in \Delta \setminus \Delta(0, r_0) \), then there exists a point \( Q := g^{-1}(\hat{Q}) \in \Delta \setminus \Delta(0, r_0) \). Next, we need to show
that the Kobayashi distance \( d_\Delta(Z_0, Q) \) is still uniformly bounded. The way to prove this is similar to the proof of Theorem 5.

If \( r_0 \leq |Z_0| \leq \rho < 1 \), then using Eq. (2), we conclude that there is a constant \( C' \) such that \( d_\Delta(Z_0, Q) \leq C' \). But if either \( |Z_0| > \rho \) or \( |Q| > \rho \), we can prove that \( d_\Delta(Z_0, Q) \) is still uniformly bounded as follows.

We choose a sector \( D_+^1 \) inside \( D_1 \) such that \( \hat{Z}_0, \hat{Q} \in D_+^1 \), but \( \hat{Z}_0, \hat{Q} \) are neither on \( \partial D_+^1 \) nor very close to \( \partial D_+^1 \). The way to choose \( D_+^1 \) is the same as choosing \( A_+ \) in the proof of Theorem 5. Then we have

\[
d_\Delta(Z_0, Q) \leq d_{D_+^1}(\hat{Z}_0, \hat{Q})
\]
since \( g^{-1}(z) : D_+^1 \rightarrow \Delta \) is holomorphic.

Now, we need to show that \( d_{D_+^1}(\hat{Z}_0, \hat{Q}) \) is bounded by some constant. In Wold’s paper [13] (Theorem 3.4), he proved that

\[
F_{D_+^1}(z, \xi_1) - F_\Delta(z, \xi_2) = O(\delta(z)),
\]
where \( \delta \) denotes the boundary distance. Then

\[
d_{D_+^1}(\hat{Z}_0, \hat{Q}) \leq d_\Delta(\hat{Z}_0, \hat{Q}) + d_E(\gamma(t), \partial \Delta) \leq C + \int_0^1 O \left( \left| \gamma(t) - \frac{\gamma'(t)}{\left| \gamma'(t) \right|} \right| \right) |\gamma'(t)| dt
\]

\[
\leq C + \int_0^1 |\gamma'(t)| dt \leq C + |\hat{Z}_0 - \hat{Q}| < C + 2,
\]
where \( \gamma(t) \) is a suitable curve from \( \hat{Z}_0 \) to \( \hat{Q} \). Hence there exists a constant \( C' := C + 2 \) such that

\[
d_\Delta(Z_0, Q) \leq d_{D_+^1}(\hat{Z}_0, \hat{Q}) < C'.
\]

Therefore, this theorem is true for all these four cases, i.e., there is a constant \( \tilde{C} \) so that there exists a point \( \hat{q} = f^{-1}(\hat{q}) \in \Omega_2 \) such that \( d_{\Omega_2}(z_0, \hat{q}) \leq \tilde{C} \).

Let us continuously consider the orbit between more connected components for the case (2), i.e., \( A \) has infinitely many connected components. Suppose the starting point \( z_0 \) is inside some connected component \( \Omega_{i_1}, i_1 = 3, 4, 5, \ldots \). Then there is a positive integer \( N_0 \) such that \( \hat{z}_0 := f^{N_0}(z_0) \in \Omega_1 \). Note that if \( N_0 = 1 \), it is the same as the orbit between two connected components, so the rest is to consider when \( N_0 \geq 2 \). By Theorem A, we have that there is a constant \( C \) such that for every point \( \hat{z}_0 \in \Omega_1 \), there exists a point \( \hat{q} \in \bigcup_k f^{-k}(p), k \geq 0 \).
in $\Omega_1$ so that $d_{\Omega_1}(\hat{z}_0, \hat{q}) \leq C$. Then we only need to show that there is a point $q \in \Omega_{i_1}$ such that $d_{\Omega_{i_1}}(z_0, q)$ is uniformly bounded. When finding $q$, which is some point of iterating the inverse of $\hat{q} \in \Omega_1$, we need to be careful in dealing with the critical points when it appears in the inverse orbit.

We know that $f$ has finitely many critical points, so there are only finitely many $\Omega_i$ containing critical points. Let $\Omega_{i_2}$ be a connected component satisfies $f(\Omega_{i_2}) = \Omega_1$ and $f^{-N_{i_2}}(\Omega_{i_2}) = \Omega_{i_2}$. If there are some critical points in $\Omega_{i_2}$, then we do the same procedure as above to find that there is a point $f^{-1}(\hat{q})$ such that $d_{\Omega_{i_2}^i}(f^{-1}(\hat{z}_0), f^{-1}(\hat{q}))$ is bounded by some constant $C'$. If there are no critical points inside $\Omega_{i_2}$, then the Kobayashi metric is an isometry, thus we can choose $C' = C$, see the following Fig. 4. Inductively, after $N_0$ times of iterating the inverse orbit of $\hat{q}$, we definitely can find a constant so that for every point $z_0 \in \Omega_{i_1}$, there exists a point $q := f^{-N_0}(\hat{q}) \in \Omega_{i_1}$ such that $d_{\Omega_{i_1}}(z_0, q)$ is uniformly bounded. Note that the constant we want to find only increases when there are critical points in the component $\Omega_i$ by $f^{-1}$ from a previous component, but there are only finitely many $\Omega_i$ containing critical points, it will only increase finitely many times. Hence, the uniform constant we need can be very large but it always exists.

Therefore, no matter how many connected components $A(p)$ has, there is a constant $\hat{C}$ so that for every point $z_0$ inside any $\Omega_i$, there exists a point $q \in \bigcup_k f^{-k}(p)$ inside $\Omega_i$ such that $d_{\Omega_i}(z_0, q) \leq \hat{C}$, where $d_{\Omega_i}$ is the Kobayashi distance on $\Omega_i$.

\( \square \)

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