Semi-simple actions of the Higman-Thompson groups $T_n$ on finite-dimensional CAT(0) spaces

Motoko Kato

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Abstract
In this paper, we study isometric actions on finite-dimensional CAT(0) spaces for the Higman–Thompson groups $T_n$, which are generalizations of Thompson’s group $T$. It is known that every semi-simple action of $T$ on a complete CAT(0) space of finite covering dimension has a global fixed point. After this result, we show that every semi-simple action of $T_n$ on a complete CAT(0) space of finite covering dimension has a global fixed point. In the proof, we regard $T_n$ as ring groups of homeomorphisms of $S^1$ introduced by Kim, Koberda and Lodha, and use general facts on these groups.

Keywords Thompson’s groups · CAT(0) spaces · Fixed point properties · Groups of homeomorphisms of the circle

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1 Introduction

Thompson’s groups $F$, $T$ and $V$ are finitely presented groups of homeomorphisms of the unit interval $[0, 1]$, the circle and the Cantor space, respectively. These groups were originally studied by Richard Thompson in the 1960s, and are known to have many interesting properties. In particular, $T$ and $V$ were the first examples of finitely presented simple groups. There are various generalizations of Thompson’s groups, for example, the Higman–Thompson groups $F_n$, $T_n$ and $V_n$. These groups are “$n$-adic” versions of $F$, $T$ and $V$, where $n$ is a natural number greater than 2 ($F_2$, $T_2$ and $V_2$ coincide with $F$, $T$ and $V$, respectively).

We treat actions of these groups on CAT(0) spaces: geodesic spaces whose geodesic triangles are not fatter than the comparison triangles in the Euclidean plane. We study the
existence of fixed points of group actions on CAT(0) spaces. Such properties, which we call fixed point properties, are related to many other properties of groups. For example, a group is said to have Serre’s property FA if every cellular action on every simplicial tree (that is, a 1-dimensional CAT(0) cube complex) has a global fixed point. If a group has Serre’s property FA, then the group does not split as an amalgamated product or an HNN extension [12]. For every $k \geq 2$, a group is said to have property FA$_k$ if every isometric action of the group on a complete CAT(0) space of covering dimension $k$ has a global fixed point [5, 13]. Property FA$_k$ is related to representations of groups: if a group has FA$_{k-1}$, then the group is of integral $k$-representation type [5]. Although it is generally hard to check whether a specific group has such fixed point properties, $T$ and $V$ are known to serve as examples of groups with these properties:

- $T$ and $V$ have Serre’s property FA [7].
- $T$, $V$, $T_n$ and $V_n$ act properly and isometrically on infinite dimensional CAT(0) cube complexes [6, 9].
- For $T$, $V$, $V_n$ and some other generalizations of $V$, every semi-simple isometric action on a complete CAT(0) space of finite covering dimension has a global fixed point ([10], see also [8] for the case of finite dimensional CAT(0) cube complexes). In other words, these groups have property FA$_k$ for semi-simple actions for every $k \in \mathbb{N}$.

On the last result [10], we consider semi-simple isometric action on a complete CAT(0) space of finite covering dimension, in the case of $T_n$.

We regard each $T_n$ as a ring group. For $n \in \mathbb{Z}_{\geq 3}$, an $n$-ring group is generated by a sequence of $n$ orientation preserving homeomorphisms of $S^1$ whose supports form a “ring” of open intervals in $S^1$. Further, it is requested that subgroups generated by two of the generators are isomorphic to $F$ when the supports have nonempty intersection. Ring groups are $S^1$-versions of chain groups. For $n \in \mathbb{Z}_{\geq 2}$, an $n$-chain group is generated by a sequence of orientation preserving homeomorphisms of $\mathbb{R}$ whose supports form a “chain” of open intervals. It is also requested that subgroups generated by two of the generators are isomorphic to $F$ when the supports have nonempty intersection. In particular, every 2-chain group is isomorphic to $F$. By taking subsequences of generators, we may consider chain subgroups of chain and ring groups. In particular, by considering 2-chain subgroups, we can find many subgroups of chain and ring groups which are isomorphic to $F$. Chain and ring groups are considered by Kim, Koberda and Lodha. In [11], they studied various general properties of chain groups. To the author’s knowledge, general properties of ring groups are not studied yet.

Our main theorems can be stated as follows.

**Proposition 1** Let $n \geq 2$. There is a generating set of $T_n$ as an $(n^2 + n)$-ring group with a minimal $n$-chain subgroup.

**Theorem 1** Let $m \geq 6$. Let $G_F$ be an $m$-ring group. Assume that $G_F$ has a minimal $m'$-chain subgroup with $2 \leq m' \leq m - 4$. Let $H$ be a finitely generated subgroup of $[G_F, G_F]$. Then for every semi-simple isometric action of $G_F$ on a complete CAT(0) space of finite covering dimension, elements of $H$ have a common fixed point.

**Corollary 1** For every $n \geq 2$, every semi-simple isometric action of $T_n$ on a complete CAT(0) space of finite covering dimension has a global fixed point. In other words, $T_n$ has property FA$_k$ for semi-simple actions for every $k \in \mathbb{N}$.

The outline of the proof of the main theorems are as follows. In order to prove Proposition 1, we explicitly construct a generating set of $T_n$ as a ring group. Corollary 1 is a consequence of
Proposition 1 and Theorem 1. In order to prove Theorem 1, we first observe that elements of the commutator subgroup of $F$ always admit fixed points (Lemma 8). Since $H$ is generated by conjugates of some copies of the commutator subgroup of $F$ in $G_F$, we get a finite generating set of $H$ consisting of elements that always admit fixed points. Next, we exchange $H$ with a larger subgroup with a finite generating set consisting of elements of the “small” supports. To deal with a technical difficulty in this step, we use the assumption on the minimal chain subgroup (Lemma 9). By the small support condition, we may apply general facts on semi-simple isometries of complete CAT(0) spaces of finite covering dimension (Theorem 2 and Theorem 3), and show that these generators have a common fixed point.

Let me compare above proof with the proof in the case of $T$ in [10], where we do not use the idea of ring groups. In [10, Theorem 1.1], a sufficient condition for the fixed point property is given by a sequence of subgroups with no nontrivial homomorphisms to $\mathbb{R}$. Such a sequence of subgroups is constructed with isometric copies of the subgroup of $F$ consisting of elements which are trivial in neighborhoods of 0 and 1 [10, Corollary 3.3]. Such a subgroup of $F$ coincides with the commutator subgroup of $F$, which is simple [4, Theorems 4.4 and 4.5]. It follows that the subgroup does not have nontrivial homomorphisms to $\mathbb{R}$. If we try to literally generalize this proof in the case of $T_n$, we consider the subgroup of $F_n$ consisting of elements which are trivial in neighborhoods of 0 and 1. However, when $n \geq 3$, such a subgroup has a nontrivial homomorphism to $\mathbb{Z}^{n-2}$, and thus to $\mathbb{R}$ (2, Proof of Lemma 4.12), where $F_n$ is denoted by $F_{n,1}$ and the subgroup is denoted by $F_{n,1}^0$. On the other hand, when we consider $T_n$ as a ring group as above, we may attribute the discussion on $T_n$ to the discussion on 2-chain subgroups of $T$, which is isomorphic to $F$. Then we may avoid technical difficulties in $n$-adic case.

The rest of this paper is organized as follows. In Sect. 2, we recall precise definitions of $F$, $T$, $F_n$, $T_n$, chain groups and ring groups. We introduce basic facts on chain and ring groups, mainly from [11]. We show Proposition 1. In Sect. 3, we discuss fixed point properties for Thompson’s groups, chain and ring groups. We start with the case of $F$ (Lemma 8). We next discuss the case of chain groups (Thereom 4). On these results, we show Theorem 1 and then Corollary 1. The part of the proof that requires the assumption on the minimal chain subgroup is discussed independently as a lemma (Lemma 9). We also treat the case of $F_n$ for independent interests (Corollary 2).

2 Thompson’s groups, chain groups and ring groups

We gather some notations and conventions. For a group $G$, we denote the commutator subgroup by $[G, G]$. We fix an orientation on $S^1$. We define an open interval in $S^1$ as a subset of the form $(a, b) = \{ t \in S^1 \mid a < t < b \}$, where $a \neq b \in S^1$ and $<$ is the cyclic order on $S^1$. For an open interval $J$ in $S^1$, we write $a = \partial_- J$ and $b = \partial_+ J$. We also fix a basepoint on $S^1$, and identify $S^1$ with $[0, 1]/\{0 = 1\}$ so that 0 = 1 corresponds to the basepoint. Through this identification, we identify subsets of $[0, 1]$ with subsets of $S^1$. We also identify every homeomorphism of $[0, 1]$ with the homeomorphism of $S^1$ that fix the basepoint.

Let $M$ be either $\mathbb{R}$ or $S^1$. We write Homeo$^+(M)$ for the group of orientation preserving homeomorphisms of $M$. For $f \in$ Homeo$^+(M)$, we write supp$(f)$ for the set-theoretic support of $f$: supp$(f) = \{ t \in M \mid f(t) \neq t \}$. For $S \subset$ Homeo$^+(M)$, we write supp$(S)$ for the union of supp$(s)$ for all $s \in S$. We note that supp$(f)$ and supp$(S)$ are open subsets of $M$. 

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2.1 Thompson’s groups $F$, $T$ and the Higman–Thompson groups $F_n$, $T_n$

In this section, we recall definitions and properties of the Higman–Thompson groups. We start with definitions of Thompson’s groups $F$ and $T$, following [4]. $F$ is a group of piecewise affine homeomorphisms from the unit interval $[0, 1]$ to itself, differentiable with derivatives of powers of 2 on finitely many standard dyadic intervals. Here, a standard dyadic interval is a subinterval of $[0, 1]$ of the form $[a/2^b, (a + 1)/2^b]$, where $a, b$ are nonnegative integers. $T$ is a group of piecewise affine homeomorphisms from the circle $S^1$ to itself that preserve the set of dyadic rationals and that are differentiable with derivatives of powers of 2 on finitely many standard dyadic intervals. We note that $F$ is identified with the subgroup of $T$ consisting of elements that fix the basepoint of $S^1$.

Following [4], we represent each element of Thompson’s groups by a pair of finite rooted binary trees with the same number of leaves. When we represent elements of $T \setminus F$, we denote the image of the leftmost leaf of the domain tree by a small circle in the range tree.

As a generalization of $F$ and $T$, we can consider $n$-adic versions of $F$ and $T$. For every natural number $n \geq 2$, we define standard $n$-adic intervals as intervals of the form $[a/n^b, (a + 1)/n^b]$ in $[0, 1]$. The ($n$-adic) Higman–Thompson group $F_n$ (resp. $T_n$) is a group of homeomorphisms of $[0, 1]$ (resp. $S^1$), differentiable with derivatives of powers of $n$ on finitely many standard $n$-adic intervals (cf. Proposition 4.4 of [2]). By definition, $F_2$ and $T_2$ coincide with $F$ and $T$. We note that $F_n$ is identified with the subgroup of $T_n$ consisting of elements that fix the basepoint of $S^1$.

We identify $n$-adic divisions of $[0, 1]$, that is, divisions of $[0, 1]$ obtained by repeatedly dividing the interval into $n$ equal parts, with finite rooted $n$-ary trees. With this identification, we represent each element of the $n$-adic Higman–Thompson group by a pair of finite rooted $n$-ary trees with the same number of leaves. An $n$-caret in an $n$-ary tree, which is shown in Fig. 1, corresponds to a division of an interval. Leaves of carets are ordered from left to right. When we represent elements of $T_n \setminus F_n$, we denote the image of the leftmost leaf of the domain tree by a small circle in the range tree.

For later use, we list some facts on $F_n$ and $T_n$.

Lemma 1 Let $n \geq 2$. Let $I = [a/n^b, (a + 1)/n^b]$ be a standard $n$-adic interval in $[0, 1]$. If $f \in F_n$ is affine on $I = [a/n^b, (a + 1)/n^b]$, then $f(I)$ is a standard $n$-adic interval of the form $[a'/n^{b'}, (a' + 1)/n^{b'}]$, where $a \equiv a' \mod (n - 1)$. Conversely, if $I' = [a'/n^{b'}, (a' + 1)/n^{b'}]$ is a standard $n$-adic interval in $[0, 1]$ with $a \equiv a' \mod (n - 1)$, then there exists $f \in F_n$ such that $f|_I$ is affine and that $f(I) = I'$.

For a proof, see Lemmas 1.1.3 and 1.2.1 of [3], for example.

Lemma 2 [2, Section 4C] For every natural number $n \geq 2$, $F_n$ is generated by $x_{n,i}$ ($1 \leq i \leq n$), where

\[ \text{Springer} \]
When I in subgroups which are isomorphic to the whole group, with arbitrarily small supports. Let \( F_n \) phisms. When we treat \( F_n \) note that every \( \phi \) isomorphism \( \phi \) a piecewise affine homeomorphism \( \psi \) for \( n \geq 2 \). For every standard \( n \)-adic interval \( I \), let 

\[
F_n(I) = \{ f \in F_n \mid \text{supp}(f) \subset I \} < F_n.
\]  

(2.4)

Figure 2 shows tree pairs that represent generators of \( F_n \) and \( T_n \).

A useful property of the Higman–Thompson groups is their “self-similarity”: there exist subgroups which are isomorphic to the whole group, with arbitrarily small supports. Let \( n \geq 2 \). For every standard \( n \)-adic interval \( I \), let 

\[
F_n(I) = \{ f \in F_n \mid \text{supp}(f) \subset I \} < F_n.
\]  

(2.4)

Note that every \( F_n(I) \) is isomorphic to \( F_n \) by means of a piecewise affine conjugation. With a piecewise affine homeomorphism \( \phi_I : I \rightarrow [0, 1] \) such that all singularities are in \( \mathbb{Z}[1/n] \) and that the derivative at any non-singular point is a power of \( n \), we have an isomorphism \( \psi_{n,I} : F_n(I) \rightarrow F_n \) defined by 

\[
(\psi_{n,I}(f))(x) = (\phi_I f (\phi_I)^{-1})(x) \quad (x \in I).
\]  

(2.5)

When \( I \) is obtained by an \( n \)-adic division of \( [0, 1] \), \( \phi_I : I \rightarrow [0, 1] \) is the affine homeomorphism from \( I \) to \( [0, 1] \) ([4, Lemma 4.4] for the case of \( n = 2 \)). We call \( \psi_{n,I} \) a canonical isomorphism from \( F_n(I) \) to \( F_n \).

We often consider the natural action of \( F_n \) (resp. \( T_n \)) on \([0, 1] \) (resp. \( S^1 \)) by homeomorphisms. When we treat \( F_n(I) \), we also consider the action of \( F_n(I) \) on \( I \) which is induced by the natural action of \( F_n \) on \([0, 1] \). Let \( G \) be a group of homeomorphisms of \( \mathbb{R} \). We say \( G \) is minimal if every orbit is dense in \( \text{supp}(G) \).
Lemma 3 Let $n \geq 2$. Let $I$ be a standard $n$-adic interval. The natural action of $F_n(I)$ on $I$ is minimal.

Proof We show that the natural action of $F_n$ on $[0, 1]$ is minimal. We take $x, y \in (0, 1)$ and $\varepsilon > 0$ arbitrarily. We fix standard $n$-adic intervals $I_x = [a/n^b, (a + 1)/n^b]$ around $x$ and $I_y = [a'/n^{b'}, (a' + 1)/n^{b'}]$ around $y$ in $(0, 1)$. Taking $b' \in \mathbb{N}$ sufficiently large, we may assume that $|1/n^{b'}| < \varepsilon$. We further divide $I_y$ into $n$ subintervals $I_{y,i} = [(na' + i)/n^{b'+1}, (na' + i + 1)/n^{b'+1}]$ ($0 \leq i \leq n - 1$). We take $i \in \{0, \ldots, n - 1\}$ such that $na' + i \equiv a \mod n$. By Lemma 1, there exists $f \in F_n$ such that $f(I_x) = I_{y,i}$. Since $f(x)$ and $y$ are in $I_y$, $|f(x) - y| < \varepsilon$. It follows that the orbit $F_n \cdot x$ is dense in $[0, 1]$. It is clear that the action of $F_n(I)$ on $I$ is also minimal.

\[ \square \]

2.2 Chain groups and ring groups

In this section, we treat another generalization of $F$, defined by Kim, Koberda and Lodha [11]. Let $m \geq 2$ be a natural number. An $m$-chain is an ordered $m$-tuple of open intervals $(J_1, \ldots, J_m)$ in $\mathbb{R}$, satisfying that

- $J_i \cap J_{i+1}$ is nonempty and connected for every $i \in \{1, \ldots, m - 1\}$, and
- $J_i \cap J_j = \emptyset$ if $|i - j| \geq 2$.

Figure 3 shows an example of a 4-chain of intervals. An $m$-prechain group $G_{\mathcal{F}}$ is a group generated by $\mathcal{F} = \{f_i\}_{1 \leq i \leq m} \subset \text{Homeo}^+(\mathbb{R})$ such that the sequence $(\text{supp}(f_i))_{1 \leq i \leq m}$ is an $m$-chain. An $m$-chain group is an $m$-prechain group such that $(f_i, f_{i+1})$ is isomorphic to $F$ for every $1 \leq i \leq m - 1$. A group is a chain group if it is an $m$-chain group for some $m$. A chain subgroup of a chain group $G_{\mathcal{F}}$ is a subgroup generated by elements of $\mathcal{F}$ with consecutive indices. The family of chain groups is broad; in fact, there exist uncountably many isomorphism types of 3-chain groups (Theorem 1.7 of [11]). It is known that $F_n$ is a “stabilization” of $n$-prechain groups. That is, for an $n$-prechain group $G_{\mathcal{F}}$ with $\mathcal{F} = \{f_i\}_{1 \leq i \leq n}$, for all but finitely many $N \in \mathbb{N}$, the subgroup $\langle \{f_i^N\}_{1 \leq i \leq n} \rangle$ of $G_{\mathcal{F}}$ is a chain group and is isomorphic to $F_n$ [11, Proposition 1.10].

Let $m \geq 3$ be an integer. An $m$-ring is an ordered $m$-tuple of open intervals $(J_1, \ldots, J_m)$ in $S^1$, satisfying that

- $J_i \cap J_{i+1}$ is nonempty and connected for every $i$ modulo $m$, and
- $J_i \cap J_j = \emptyset$ if $2 \leq |i - j| \leq m - 2$.

An $m$-ring group $G_{\mathcal{F}}$ is a subgroup of $\text{Homeo}^+(S^1)$ generated by $\mathcal{F} = \{f_i\}_{1 \leq i \leq m} \subset \text{Homeo}^+(S^1)$ such that
For the natural action of $G$, exactly one of the following holds (Theorem 1.3 of [11]):

- The action of $G$ is minimal. In this case, $[G, G]$ is simple.
- The closure of some $G$-orbit is perfect and totally disconnected. In this case, $G$ canonically surjects onto an $n$-chain group that acts minimally.

In the remainder of this section, we will introduce basic facts on chain and ring groups that will be used throughout subsequent sections.

**Lemma 4** [11, Lemma 3.1] Let $G_\mathcal{F}$ be a 2-prechain group with respect to $\mathcal{F} = \{f_1, f_2\}$. Suppose that $f_2 f_1 (\partial_- \supp(f_2)) \geq \partial_+ \supp(f_1)$. Then $\{f_1, f_2\} \cong F$.

**Lemma 5** [11, Lemma 3.6 (3), (4)] Let $m \geq 2$. Let $G_\mathcal{F}$ be an $m$-prechain group with respect to $\mathcal{F} = \{f_i\}_{1 \leq i \leq m}$.

1. For the natural action of $G_\mathcal{F}$ on $\supp(G_\mathcal{F})$, every $G_\mathcal{F}$-orbit is a $[G_\mathcal{F}, G_\mathcal{F}]$-orbit.
2. Let $x$ be a boundary point of $\supp(f)$ of some $f \in \mathcal{F}$. For every proper compact subset $I \subset \supp(G_\mathcal{F})$ and every open neighborhood $J$ of $x$, there exists an element $g \in [G_\mathcal{F}, G_\mathcal{F}]$ such that $g(I) \subset J$. In particular, the natural action of $[G_\mathcal{F}, G_\mathcal{F}]$ on $\supp(G_\mathcal{F})$ is locally CO-transitive.
3. If $G_\mathcal{F}$ is minimal, then for every proper compact subset $I$ and every nonempty open subset $J$ of $\supp(G_\mathcal{F})$, there exists an element $g \in [G_\mathcal{F}, G_\mathcal{F}]$ such that $g(I) \subset J$. That is, the natural action of $[G_\mathcal{F}, G_\mathcal{F}]$ on $\supp(G_\mathcal{F})$ is CO-transitive.
4. For $m \geq 3$, if $G_\mathcal{F}$ has a minimal prechain subgroup, then $G_\mathcal{F}$ itself is minimal.
5. For every $f \in \{G_\mathcal{F}, G_\mathcal{F}\}$, $\supp(f)$ is contained in a closed interval in $\supp(G_\mathcal{F})$.

**Proof** (1) and (2) are Lemma 3.6 (3) and (4) of [11], respectively.

(3) By (1), if $G_\mathcal{F}$ is minimal, then $[G_\mathcal{F}, G_\mathcal{F}]$ is also minimal. Then (3) follows from (2) and Lemma 2.6 of [11], which states that a minimal locally CO-transitive group action is CO-transitive.

(4) Suppose that there exists a minimal prechain subgroup $G_{\mathcal{F}'}$ of $G_\mathcal{F}$. Let $\mathcal{F}' = \{f_i\}_{m_1 \leq i \leq m_2}$. We show that $G_\mathcal{F}$ is minimal, that is, for every $x \in \supp(G_\mathcal{F})$, the $G_\mathcal{F'}$-orbit of $x$ is dense in $\supp(G_\mathcal{F})$.

We fix $x \in \supp(G_\mathcal{F})$ arbitrarily. By (2), for every $y \in \supp(G_\mathcal{F})$ and $\varepsilon > 0$, there exists $g_1 \in G_\mathcal{F}$ such that

$$g_1(x) \in (\partial_+ \supp(f_{m_1}) - \varepsilon, \partial_+ \supp(f_{m_1}) + \varepsilon).$$

In addition, there exists $g_2 \in G_\mathcal{F}$ such that

$$g_2((y - \varepsilon, y + \varepsilon)) \subset (\partial_+ \supp(f_{m_1}) - \varepsilon, \partial_+ \supp(f_{m_1}) + \varepsilon).$$

We may assume that the $\varepsilon$-neighborhood of $\partial_+ \supp(f_{m_1})$ is in $\supp(G_\mathcal{F})$. Since $G_{\mathcal{F}'}$ is minimal, by (3), there exists $g_3 \in G_{\mathcal{F}'}$ such that

$$g_3(g_1(x)) \in g_2((y - \varepsilon, y + \varepsilon)).$$
Then, \( g_2^{-1} g_3 g_1(x) \in (y - \varepsilon, y + \varepsilon) \). Hence the orbit of \( x \) is dense in \( \text{supp}(G_x) \).

(5) Note that the commutator subgroup is normally generated by commutators of generators. Therefore, in the case of prechain groups, \([G_x, G_x]\) is normally generated by \([f_i, f_{i+1}] = f_i f_{i+1}^{-1} f_i^{-1} f_{i+1}^{-1} (1 \leq i \leq m - 1)\). For every \( i \in \{1, \ldots, m - 1\} \), we can check that \( \text{supp}([f_i, f_{i+1}]) \) is contained in a closed interval \([\partial_{+} \text{supp}(f_{i+1}), f_2(\partial_{+} \text{supp}(f_i))]\) in \( \text{supp}(G_x) \). It follows that for every \( f \in [G_x, G_x] \), \( \text{supp}(f) \) is contained in a closed interval \( \text{supp}(G_x) \).

Lemma 6  Let \( m \geq 4 \). Let \( G_x \) be an \( m \)-ring group with respect to \( \mathcal{F} = \{f_i\}_{1 \leq i \leq m} \). Then, for all \( m' \geq m \), \( G_x \) can be described as an \( m' \)-ring group. That is, there exists a generating set \( \mathcal{F}' = \{f_j'\}_{1 \leq j \leq m'} \) of \( G_x \) which satisfy the assumptions in the definition of the ring group. Moreover, when \( m \geq 5 \), we can take such \( \mathcal{F}' \) so that \( f_j' = f_j \) for \( (m - 4) \) consecutive indices \( j \).

Proof Although the proof is similar to the corresponding result for chain groups (Proposition 1.5 of [11]), we add some arguments needed for ring groups. It is enough to show the case where \( m' = m + 1 \).

First, we give a new generating set for \( G_x \) as an \((m+1)\)-ring group. Let \( \mathcal{F} = \{f_1, \ldots, f_m\} \). We exchange generators with conjugates of the original ones. Let

\[
\begin{align*}
    f_{m-2}' &= f_{m-N} f_{m-2} f_{m-1}^{-1}, \\
    f_m' &= (f_{m-2} f_m)^N f_{m-N} (f_{m-2} f_m)^{-N}, \\
    f_{m-1}' &= (f_m')^{-1} f_{m-1} f_m, \\
    f_{m+1}' &= f_m
\end{align*}
\]

for some \( N \in \mathbb{N} \). We further define \( f_j' \) (\( 1 \leq i \leq m - 3 \)) by

\[
    f_j' = \begin{cases} 
    f_m N f 1 f_m N & (j = 1) \\
    f_j & \text{(otherwise),}
\end{cases}
\]

and let \( \mathcal{F}' = \{f_j'\}_{1 \leq j \leq m+1} \). By the construction of \( \mathcal{F}' \), \( G_{x'} = G_x \) as a subgroup of \( \text{Homeo}^+(S^1) \). If \( N \) is sufficiently large, \( (\text{supp}(f_j'))_{2 \leq j \leq m+1} \) is a chain of intervals and \( \langle f_j', f_j' \rangle = F \) for \( 2 \leq j \leq m \) [11, Theorem 4.7].

Next, we confirm that \( \langle \text{supp}(f_j') \rangle_{1 \leq j \leq m+1} \) is a chain of intervals. According to the proof of [11, Theorem 4.7], \( \langle \text{supp}(f_j') \rangle_{2 \leq j \leq m+1} \) is a chain of intervals. We confirm that \( \langle \text{supp}(f_j') \rangle_{j=m,m+1,1,2} \) is a chain of intervals. In the following, we write \( < \) for the induced order on some open interval in \( S^1 \). Since \( f_m' = (f_{m-1}) f_{m-2} f_m N \) and \( f_1' = (f_1) f_m N \),

\[
\begin{align*}
    \partial_{+} \text{supp}(f_m') &= (f_{m-2} f_m)^N f_m N \partial_{+} \text{supp}(f_m) = (f_{m-2} f_m)^N \partial_{+} \text{supp}(f_{m-1}) \\
    &= f_m N \partial_{+} \text{supp}(f_{m-1}) \quad < f_m N \partial_{+} \text{supp}(f_1) = \partial_{+} \text{supp}(f_1').
\end{align*}
\]

Here, the third equality follows from an observation that \( f_m^n(\partial_{+} \text{supp}(f_{m-1})) \in \text{supp}(f_m) \) for all \( n \in \mathbb{Z} \) and \( f_{m-2}(x) = x \) for all \( x \in \text{supp}(f_m) \). Moreover, since \( f_m \) fixes \( \text{supp}(f_m) \), both \( \partial_{+} \text{supp}(f_m') \) and \( \partial_{-} \text{supp}(f_1') \) are in \( \text{supp}(f_m) = \text{supp}(f_{m+1}') \). Therefore,

\[
\partial_{-} \text{supp}(f_{m+1}') < \partial_{+} \text{supp}(f_{m}') < \partial_{-} \text{supp}(f_1') < \partial_{+} \text{supp}(f_{m+1}).
\]

Since \( f_2' \) is either \( f_2 (m \geq 5) \) or \( f_3^N f_2 f_3^N (m = 4) \), we have \( \partial_{-} \text{supp}(f_2') = \partial_{-} \text{supp}(f_2) \) in each case. Therefore,

\[
\partial_{+} \text{supp}(f_{m+1}') < \partial_{-} \text{supp}(f_2') < \partial_{+} \text{supp}(f_1').
\]

It follows that \( \langle \text{supp}(f_j') \rangle_{j=m,m+1,1,2} \) is a chain of intervals.
Finally, we show that \(\langle f_j', f_{j+1}' \rangle = F\) for \(1 \leq j \leq m + 1\), where indices are considered modulo \((m + 1)\). With sufficiently large \(N\), \(\langle f_j', f_{j+1}' \rangle = F\) for \(2 \leq j \leq m\) [11, Theorem 4.7]. The remaining cases are \(\langle f_j', f_{j+1}' \rangle\) for \(j = m + 1\) and 1. For \(j = m + 1\),

\[
\langle f_{m+1}', f_1' \rangle = \langle f_m, f_m^N f_1 f_m^{-N} \rangle = f_m^N \langle f_m, f_1 \rangle f_m^{-N} \cong \langle f_m, f_1 \rangle \cong F.
\]

For \(j = 1\), when \(m \geq 5\), we have \(f_2' = f_2\) and thus

\[
\langle f_1', f_2' \rangle = \langle f_m^N f_1 f_m^{-N}, f_2 \rangle = (f_m^N f_1 f_m^{-N}, f_m^N f_2 f_m^{-N}) = f_m^N \langle f_1, f_2 \rangle f_m^{-N} \cong \langle f_1, f_2 \rangle \cong F.
\]

Here, since \(\text{supp}(f_2') = \text{supp}(f_2)\) and \(\text{supp}(f_m)\) are disjoint, \(f_m^N f_2 f_m^{-N} = f_2'.\) Similarly, when \(m = 4\), we have \(f_2' = f_3^{-N} f_2 f_3^N\) and thus

\[
\langle f_1', f_2' \rangle = \langle f_4^N f_1 f_4^{-N}, f_2' \rangle = (f_4^N f_1 f_4^{-N}, f_4^N f_2' f_4^{-N}) \cong \langle f_1, f_2' \rangle = \langle f_1, f_3^{-N} f_2 f_3^N \rangle = \langle f_3^{-N} f_1 f_3^N, f_3^{-N} f_2 f_3^N \rangle \cong \langle f_1, f_2 \rangle \cong F.
\]

Here, since \(\text{supp}(f_2')\) and \(\text{supp}(f_4)\) are disjoint, \(f_2' = f_4^{-N} f_2' f_4^N\). Since \(\text{supp}(f_1)\) and \(\text{supp}(f_3)\) are disjoint, \(f_1 = f_3^{-N} f_1 f_3^N\).

Therefore, \(G_{\mathcal{K}}\) is an \((m + 1)\)-ring group which is isomorphic to \(G_{\mathcal{K}}\). Note that \(f_j' = f_j\) for \(2 \leq j \leq m - 3\), and that the generators can be renumbered cyclically.

\(\square\)

### 2.3 The Higman–Thompson groups \(T_n\) are ring groups

In this section, we show Proposition 1. First, we construct generating sets for \(F_n\) as a chain group. In Lemma 7, we give two generating sets for each \(F_n\). The first generating set, denoted by \(\{f_i\}_{1 \leq i \leq n}\), can be constructed immediately from the standard generating set in Lemma 2. The second generating set, \(\{g_j\}_{1 \leq j \leq n^2 + n - 1}\), is a modification of the first one. With the second generating set, some subgroups of \(F_n\) of the form \(F_n(I)\) appears as chain subgroups of \(F_n\). We note that \(F_n\) is minimal (Lemma 3). Finally, by extending the second generating set, we regard \(T_n\) as a ring group with a minimal chain subgroup. The existence of a minimal chain subgroup will be used in the proof of Corollary 1, which will be treated in Sect. 3.

**Lemma 7** Let \(n \geq 2\).

1. There exists a generating set \(\{f_i\}_{1 \leq i \leq n}\) of \(F_n\) such that
   1-i) \((\text{supp}(f_1), \ldots, \text{supp}(f_n))\) is an \(n\)-chain, and
   1-ii) \((f_i, f_{i+1})\) is isomorphic to \(F\) for every \(i \in \{1, \ldots, n - 1\}\).

   In particular, every \(F_n\) is an \(n\)-chain group.

2. There exists a generating set \(\{g_j\}_{1 \leq j \leq n^2 + n - 1}\) such that
   2-i) \((\text{supp}(g_1), \ldots, \text{supp}(g_{n^2 + n - 1}))\) is an \((n^2 + n - 1)\)-chain,
   2-ii) \((g_j, g_{j+1})\) is isomorphic to \(F\) for every \(j \in \{1, \ldots, n^2 + (n - 1) - 1\}\), and
   2-iii) for every \(k \in \{1, \ldots, n\}\), a subgroup \(F_n([((k - 1)/n, k/n)]\) defined in (2.4) coincides with the subgroup generated by \(\{g_j\}_{1+(k-1)(n+1) \leq j \leq n+(k-1)(n+1)}\).

   In particular, all \(F_n([((k - 1)/n, k/n)]\) can be described as chain subgroups of \(F_n\) simultaneously.
Proof (1) First, we define \( \{f_i\}_{1 \leq i \leq n} \) and show that it is a generating set of \( F_n \). Let
\[
 f_i = x_{n,i+1}^{i-1}, \quad f_n = x_{n,n},
\]
where \( \{x_{n,i}\}_{1 \leq i \leq n} \) is a generating set of \( F_n \) defined by (2.1) and (2.2) in Lemma 2 (see also Fig. 2). Figure 4 shows tree pairs that represent \( f_i \) \( (1 \leq i \leq n) \). It follows that
\[
 x_{n,i} = f_n f_{n-1} \cdots f_i \quad (1 \leq i \leq n),
\]
and thus \( \{f_i\}_{1 \leq i \leq n} \) is another generating set of \( F_n \). Figure 4 shows tree pairs that represent \( f_i \) \( (1 \leq i \leq n) \).

Next, we confirm (1-i). By (2.8) and (2.9),
\[
 \text{supp}(f_i) = \left( \frac{i-1}{n}, \frac{i+1}{n} \right) \quad (1 \leq i \leq n-2),
\]
\[
 \text{supp}(f_{n-1}) = \left( 1 - \frac{2}{n}, 1 - \frac{1}{n} + \frac{1}{n^2} \right), \quad \text{supp}(f_n) = \left( 1 - \frac{1}{n}, 1 \right).
\]
Therefore, \( \text{supp}(f_i) \) \( (1 \leq i \leq n) \) form a chain of open intervals.

Finally, we confirm (1-ii). We use Lemma 4. We have
\[
 f_{i+1} f_i \left( \frac{i+1}{n} \right) = \frac{i+2}{n} - \frac{2}{n^2} \geq \frac{i+1}{n} \quad (1 \leq i \leq n-3),
\]
\[
 f_{n-1} f_{n-2} \left( 1 - \frac{2}{n} \right) = 1 - \frac{1}{n} + \frac{1}{n^2} - \frac{1}{n^3} \geq 1 - \frac{1}{n},
\]
\[
 f_n f_{n-1} \left( 1 - \frac{1}{n} \right) \geq 1 - \frac{1}{n^2} \geq 1 - \frac{1}{n} + \frac{1}{n^2}.
\]
By Lemma 4, \( \langle f_i, f_{i+1} \rangle \) \( (1 \leq i \leq n-1) \) are isomorphic to \( F \). It follows that \( F_n \) is an \( n \)-chain group for every \( n \geq 2 \).

(2) First, we define \( \{g_j\}_{1 \leq j \leq n^2+1} \) and show that it is a generating set of \( F_n \). Recall that for a standard \( n \)-adic interval \( I \) obtained by an \( n \)-adic division of \([0, 1], \phi_I : I \rightarrow [0, 1] \) is the affine map and that \( \psi_{n,I} : F_n(I) \rightarrow F_n \) is the canonical isomorphism (see (2.5)). For every \( k \in \{1, \ldots, n\} \), let
\[
 g_{k,i} = \psi_{n,[(k-1)/n,k/n]}(f_i) \quad (1 \leq i \leq n), \quad (2.10)
\]
Fig. 5 The construction of $h_k$

where $\{f_i\}_{1 \leq i \leq n}$ is the generating set of $F_n$ defined in (1). We further define elements $\{h_k\}_{1 \leq k \leq n-1}$ in $F_n$ by

$$h_k = \left( \psi_{n,\lfloor k/n \rfloor, k/n} \left( x_{n,1} \right)^{-1} \right) \circ f_k \circ \left( \psi_{n,\lfloor (k-1)/n \rfloor, k/n} \left( x_{n,1} \right)^{-1} \right). \quad (2.11)$$

Figure 5 shows tree pairs that represent $h_k$. Using these elements, we define a sequence of elements $(g_1, g_2, \ldots, g_{n^2 + (n-1)})$ by

$$\left( g_{1,1}, \ldots, g_{1,n}, h_{1,1}, \ldots, h_{k-1,1}, g_{k,1}, \ldots, g_{k,n}, h_k, \ldots, h_{n-1,1}, g_{n,1}, \ldots, g_{n,n} \right). \quad (2.12)$$

It follows that

$$\psi_{n,\lfloor k/n \rfloor, \lfloor (k+1)/n \rfloor} \left( x_{n,1}^{-1} \right) \in \langle \{g_{k+1,i}\}_{1 \leq i \leq n} \rangle.$$ 

Similarly,

$$\psi_{n,\lfloor (k-1)/n \rfloor, k/n} \left( x_{n,1}^{-1} \right) \in \langle \{g_{k,i}\}_{1 \leq i \leq n} \rangle.$$ 

Then, by (2.11),

$$f_k \in \langle \{g_{k,i}\}_{1 \leq i \leq n}, h_k \rangle$$

for every $k \in \{1, \ldots, n-1\}$. In addition,

$$f_n = x_{n,n} \in F_n([n - 1/n, 1]) = \langle \{g_{n,i}\}_{1 \leq i \leq n} \rangle.$$ 

Therefore, $\{g_j\}_{1 \leq j \leq n^2 + (n-1)}$ generates $F_n$. 
Next, we confirm (2-i). By (2.5) and (2.10),
\[ \text{supp}(g_{k,i}) = (\phi((k-1)/n,k/n))^{-1} (\text{supp}(f_i)) \]
for every \( k \) and \( i \). In other words, intervals \( (\text{supp}(g_{k,i}))_{1 \leq i \leq k} \) on \([ (k-1)/n, k/n ] \) are affine copies of the intervals \( (\text{supp}(f_i))_{1 \leq i \leq k} \) on \([0,1] \). By (1), \( (\text{supp}(f_i))_{1 \leq i \leq k} \) is a chain that covers \((0,1)\). It follows that \( (\text{supp}(g_{k,i}))_{1 \leq i \leq k} \) is a chain that covers \((k/n, (k+1)/n)\). Moreover, we have
\[ \text{supp}(h_k) = \left( \frac{k}{n} - \frac{1}{n^2}, \frac{k}{n} + \frac{1}{n^2} \right). \]

Note that
\[ \partial_- \text{supp}(h_k) < \partial_+ \text{supp}(g_{k,n}) = \partial_- \text{supp}(g_{k+1,1}) = \frac{k}{n}. \]
Note also that
\[ \partial_- \text{supp}(h_k) \geq \partial_+ \text{supp}(g_{k,n-1}) = \frac{k}{n} - \frac{1}{n^2} + \frac{1}{n^3}, \]
and
\[ \partial_+ \text{supp}(h_k) = \partial_- \text{supp}(g_{k+1,2}). \]

Then \( (\text{supp}(g_i))_{1 \leq i \leq n^2+(n-1)} \) is an \( (n^2 + (n-1)) \)-chain.

Next, we confirm (2-ii). Through isomorphisms \( \psi_{n,[(k-1)/n,k/n]} \), (2-ii) corresponds to (1-ii), except for the cases of \( \langle g_{k-1,n}, h_k \rangle \) and \( \langle h_k, g_{k+1,2} \rangle \) \( (k \in \{1, \ldots, n-1\}) \). Then it is sufficient to show that \( \langle g_{k-1,n}, h_k \rangle \) and \( \langle h_k, g_{k+1,2} \rangle \) are isomorphic to \( F \) for every \( k \in \{1, \ldots, n-1\} \). We have
\[ h_k g_{k-1,n} (\partial_- \text{supp}(h_k)) = h_k g_{k-1,n} \left( \frac{k}{n} - \frac{1}{n^3} \right) = \frac{k}{n} + \frac{2}{n^2} - \frac{1}{n^3} \]
\[ \geq \frac{k}{n} = \partial_+ \text{supp}(g_{k-1,n}). \]
By Lemma 4, \( \langle g_{k-1,n}, h_k \rangle \simeq F \). Similarly,
\[ g_{k+1,2} h_k (\partial_- \text{supp}(g_{k+1,2})) = g_{k+1,2} h_k \left( \frac{k}{n} \right) = \frac{k}{n} + \frac{2}{n^2} - \frac{2}{n^3} \]
\[ > \frac{k}{n} + \frac{1}{n^2} - \frac{1}{n^3} = \partial_+ \text{supp}(h_k). \]

Then \( \langle h_k, g_{k+1,2} \rangle \simeq F \).

Finally, we confirm (2-iii). This is immediate from the definition of \( g_j \). In fact, for every \( k \in \{1, \ldots, n\} \), \( \{g_{k,i}\}_{1 \leq i \leq k} \) generates \( F_n([ (k-1)/n, k/n ] ) \) since \( \psi_{n,[(k-1)/n,k/n]} \) is a group isomorphism. Together with (2-i) and (2-ii), when we regard \( F_n \) as a chain group with the generating set \( \{g_{j}\}_{1 \leq j \leq n^2+n} \), each \( F_n([ (k-1)/n, k/n ] ) \) is a chain subgroup of \( F_n \).

\textbf{Proof (Proof of Proposition 1)} Let \( n \geq 2 \). It is sufficient to construct a generating set \( \{t_j\}_{1 \leq j \leq n^2+n} \) of \( T_n \) such that

(i) \( (\text{supp}(t_1), \ldots, \text{supp}(t_{n^2+n})) \) is an \( (n^2 + n) \)-ring,
(ii) \( \langle t_j, t_{j+1} \rangle \) is isomorphic to \( F \) for every \( j \) mod \( (n^2 + n) \),
(iii) \( F_n \) coincides with the subgroup generated by \( \{t_j\}_{1 \leq j \leq n^2+n-1} \), and
(iv) \( F_n([(k - 1)/n, k/n]) \) coincides with the subgroup generated by

\[
\{t_j\}_{1+(k-1)(n+1) \leq j \leq n+(k-1)(n+1)}
\]

for every \( k \in \{1, \ldots, n\} \).

In fact, with such a generating set, each \( T_n \) is regarded as an \((n^2 + n)\)-ring group with an \( n\)-chain subgroup \( F_n([1 - \frac{1}{n}, 1]) \), which is minimal (Lemma 3).

First, we define \( \{t_j\}_{1 \leq j \leq n^2+n} \subset T_n \) by adding one element to \( \{g_j\}_{1 \leq j \leq n^2+n-1} \) in Lemma 7 (2). Let

\[
t_j = \begin{cases} 
g_j & (1 \leq j \leq n^2 + n - 1), 
\gamma_n g_{n+1} \gamma_n^{-1} & (j = n^2 + n),
\end{cases}
\]

(2.13)

where \( \gamma_n \) is defined by (2.3) in Lemma 2 (see also Fig. 2). Figure 6 shows tree pairs that represent \( t_{n^2+n} \). We denote the image of the leftmost leaf of the domain tree by a small circle in the range tree.

Figure 4 shows tree pairs that represent \( t_{n^2+n} \). We denote the image of the leftmost leaf of the domain tree by a small circle in the range tree (Fig. 6).

Next, we show that \( \{t_j\}_{1 \leq j \leq n^2+n} \) generates \( T_n \). Recall that \( T_n \) is generated by \( F_n \cup \{\gamma_n\} \) in Lemma 2. It is sufficient to show that \( \gamma_n \in <F_n \cup \{t_{n^2+n}\}> \). Let \( I = [a, b] \) be a standard \( n\)-adic interval such that \( \gamma_n|I \) is affine and that \( \gamma_n(I) \subset (0, 1) \). By Lemma 1, we take \( f \in F_n \) such that \( f|\gamma_n(I) \) is affine and that \( f(\gamma_n(I)) = I \). Then \( \text{supp}(f \gamma_n) \subset S^1 \setminus I \). We take \( f' \in F_n \) such that \( f'(I) \supset [1/n^2, 1 - 1/n^2] \) (see Lemmas 3 and 5 (3)). Then \( \text{supp}(f'(f \gamma_n)f'^{-1}) \subset S^1 \setminus [1/n^2, 1 - 1/n^2] \). Since

\[
t_{n^2+n}([1 - 1/n^2, 1 - 1/n^2 + 1/n^3]) = S^1 \setminus [1/n^2, 1 - 1/n^2],
\]

it follows that

\[
\text{supp}(t_{n^2+n}^{-1}f'(f \gamma_n)(t_{n^2+n}^{-1}f')^{-1}) \subset [1 - 1/n^2, 1 - 1/n^2 + 1/n^3].
\]

Let \( f'' = t_{n^2+n}^{-1}f'(f \gamma_n)(t_{n^2+n}^{-1}f')^{-1} \). We have \( \gamma_n = f^{-1}f''t_{n^2+n}f''t_{n^2+n}^{-1}f' \), where \( f, f', f'' \in F_n \). Therefore, \( \gamma_n \in (F_n \cup \{t_{n^2+n}\}) \).

Finally, we confirm that \( \{t_j\}_{1 \leq j \leq n^2+n} \) satisfies four requirements. Since (iii) and (iv) are related only to \( t_{n^2+n} \), \( 1 \leq j \leq n^2 + n - 1 \), which corresponds to \( g_j \), (iii) and (iv) are satisfied by Lemma 7 (2). We confirm (i) and (ii). By definition of \( t_j \), (i) and (ii) correspond to (2-i) and (2-ii) in Lemma 7, except as related to \( t_{n^2+n} \). Namely, it is sufficient to confirm that

(i)’ \( (t_{n^2+n-2}, t_{n^2+n-1}, t_{n^2+n}, t_1, t_2) \) is a 5-ring, and
(ii) \( \langle t_{n^2+n-1}, t_{n^2+n} \rangle \) and \( \langle t_{n^2+n}, t_1 \rangle \) are isomorphic to \( F \).

Since
\[
y_n|_{[0,1/n]} = (\phi_{[1-1/n,1]}^{-1})^{-1} \phi_{[0,1/n]},
\]
for \( j \in \{1, \ldots, n\}, \)
\[
y_n t_j y_n^{-1} = ((\phi_{[1-1/n,1]}^{-1})^{-1} \phi_{[0,1/n]}) t_j ((\phi_{[1-1/n,1]}^{-1})^{-1} \phi_{[0,1/n]})^{-1}
= ((\phi_{[1-1/n,1]}^{-1})^{-1} \phi_{[0,1/n]}) g_j ((\phi_{[1-1/n,1]}^{-1})^{-1} \phi_{[0,1/n]})^{-1}
= \phi_{[1-1/n,1]}^{-1} (\phi_{[0,1/n]} g_j \phi_{[0,1/n]})^{-1} \phi_{[1-1/n,1]}
= (\phi_{[1-1/n,1]}^{-1})^{-1} f_j \phi_{[1-1/n,1]} = t_j \phi_{-(n+1)+(n^2+n)}.
\]
Similarly, \( y_n t_1 y_n^{-1} = t_1 \) and \( y_n t_{n+1} y_n^{-1} = t_2 \). By definition, \( y_n t_{n+1} y_n^{-1} = t_{n^2+n} \).

Therefore,
\[
\langle t_{n^2+n-2}, t_{n^2+n-1}, t_{n^2+n}, t_1, t_2 \rangle
\]
are the conjugates of \( t_{n-1}, t_n, t_{n+1}, t_{n+2}, t_{n+3} \) by \( y_n \), respectively in \( T_n \). Then (i)’ and (ii)’ come down to when \( t_{n^2+n} \) is not relevant. \( \square \)

3 Fixed point properties

In this section, we show the main results: Theorem 1 and Corollary 1. First, we cite general facts on group actions on finite-dimensional CAT(0) spaces: Theorems 2 [1, Proposition 3.4] and 3 [10, Theorem 1.1]. These theorems are used in the proof of the following lemmas and theorems. Next, we discuss fixed point properties in the case of \( F \) (Lemma 8). Using Lemma 8, we discuss the case of of chain groups (Theorem 4). We add Corollary 2 for independent interests. Using Lemma 8 and Theorem 4, we discuss the case of ring groups (Theorem 1). The part of the proof that requires the assumption on the minimal chain subgroup is discussed independently as a lemma (Lemma 9). Finally, we discuss the case of \( T_n \) (Corollary 1), which is a consequence of Theorem 1 and Proposition 1.

The Lebesgue covering dimension of a topological space \( X \) is the minimum \( k \in \mathbb{N} \) such that every finite open cover of \( X \) has a refinement in which no point of \( X \) belongs to more than \( (k + 1) \) elements. In this paper, we say a topological space \( X \) is \( k \)-dimensional if the Lebesgue covering dimension of \( X \) is \( k \). We say a space is finite dimensional if the Lebesgue covering dimension of the space is finite.

We say that an isometry \( \gamma \) of a metric space \( (X, d) \) is semi-simple if there exists \( x \in X \) such that \( d(x, \gamma(x)) = \inf \{d(y, \gamma(y)) \mid y \in X\} \). An isometric group action is called semi-simple if every group element acts as a semi-simple isometry.

In the following, we assume that all group actions are isometric, and that all metric spaces are complete.

**Theorem 2** [1, Proposition 3.4] Let \( X \) be a \( k \)-dimensional CAT(0) space. Let \( k_1, \ldots, k_l \) be positive integers such that \( 0 < k < \sum_{i=1}^{l} k_i \). Let \( S_1, \ldots, S_l \) be finite subsets of \( \text{Isom}(X) \). For all \( i \neq j \in \{1, \ldots, l\} \), suppose that

- \( s_i \) and \( s_j \) commute for all \( s_i \in S_i \) and \( s_j \in S_j \), and that
S_i is a conjugate of S_j in Isom(X).

If each k_i-element subset of S_i has a fixed point in X for every i ∈ {1, ..., l}, then for every i ∈ {1, ..., l}, elements of S_i have a common fixed point.

Theorem 2 is a variation of Helly’s Theorem for convex subsets in Euclidean spaces.

Theorem 3 [10, Theorem 1.1] Let k ∈ ℤ. Let G be a group acting faithfully on a set A. Let g ∈ G be an element satisfying the following conditions: there exists a sequence of subgroups ⟨g⟩ = H_0 < H_1 < ... < H_k < H_k+1 = G with elements g_i ∈ H_{i+1} (1 ≤ i ≤ k) such that for every 1 ≤ i ≤ k,

- every homomorphism from H_i to ℜ is trivial, and
- g_i(supp(H_i)) ∩ supp(H_i) = Ø in A.

Then for every semi-simple action of G on a k-dimensional CAT(0) space, g has a fixed point.

In the following, we argue fixed point properties of ring groups. We start with a lemma for Thompson’s group F. The proof can be found in the proof of [10, Corollary 3.3], but we summarize the proof for the readers’ convenience.

Lemma 8 For every f ∈ [F, F] and every semi-simple action of F on a finite-dimensional CAT(0) space, f has a fixed point.

Proof We fix k ∈ ℤ and a semi-simple action of F on a k-dimensional CAT(0) space X arbitrarily. Let \{I_i\}_{1 ≤ i ≤ k} be a sequence of standard dyadic intervals in (0, 1) such that each I_i is contained in the interior of I_{i+1}.

Without loss of generality, we may assume that supp(f) is contained in the interior of I_1. Indeed, if not, we take a closed interval I ⊇ supp(f) (see Lemmas 5 (5) and 7 (1)), and an element g ∈ F that maps I in the interior of I_1 (see Lemma 1 (2)). Then supp(gfg^{-1}) = g supp(f) is contained in I_1. When gfg^{-1} fixes a point x ∈ X, f also fixes a point g(x) ∈ X.

For every 1 ≤ i ≤ k, let H_i be the commutator subgroup of F(I_i) = {h ∈ F | supp(h) ⊂ I_i} < F, and let H_{k+1} = F. Through the canonical isomorphism ψ_{n,I_i} : F(I_i) → F, every H_i is isomorphic to the commutator subgroup of F and thus nonabelian and simple. It follows that \{H_i\}_{1 ≤ i ≤ k} satisfies the first assumption in Theorem 3. We take elements \{g_i\}_{1 ≤ i ≤ k} such that g_i ∈ H_{i+1} and g_i(I_i) ∩ I_i = Ø (see Lemmas 3 and 5 (3)). By construction, \{H_i\}_{1 ≤ i ≤ k+1} and \{g_i\}_{1 ≤ i ≤ k} satisfy the second assumption in Theorem 3. Therefore, according to Theorem 3, f fixes a point in X. □

Theorem 4 Let m ≥ 2. Let G_Τ be an m-chain group with respect to the generating set Τ = {f_i}_{1 ≤ i ≤ m}. Let H be a finitely generated subgroup of [G_Τ, G_Τ]. Then for every semi-simple action of G_Τ on a finite-dimensional CAT(0) space, elements of H have a common fixed point.

Proof We fix k ∈ ℤ and a semi-simple action of G_Τ on a k-dimensional CAT(0) space X arbitrarily. Let

\[ H_i := (f_i, f_{i+1}) \quad (1 ≤ i ≤ m - 1). \]

First, we construct a finite subset S' of G_Τ such that

- H < (S'), and
- for every s ∈ S', there exists i ∈ {1, ..., m - 1} such that supp(s') is contained in a closed interval in supp(H_i).
Let 

\[ c_i := [f_i, f_{i+1}] \quad (1 \leq i \leq m - 1). \]

Since \([G_F, G_F]\) is generated by conjugates of \([c_i]_{1 \leq i \leq m-1}\), there exists a finite generating set \(S = [c'_j]_{1 \leq j \leq n}\) of \(H\) consisting of conjugates of \(c_i\). By Lemma 5 (5), every \(\text{supp}(c_i)\) is contained in a closed interval in \(\text{supp}(H_1)\). Since each \(c'_j\) is a conjugate of some \(c_i\), \(\text{supp}(c'_j)\) is contained in a closed interval \([a_j, b_j]\) in \((0, 1)\). By assumptions on \(F\), there exists \(N_j \in \mathbb{N}\) such that for every \(i \in [1, \ldots, m - 2]\),

\[ f_i^{N_j} \cdots f_2^{N_j} f_1^{N_j}(a_j) > \partial_- \text{supp}(f_{i+1}). \]

Applying Lemma 5 (1) for each chain subgroup \(H_i\), there exists \(g_{i,j} \in [H_i, H_i]\) such that

\[ g_{i,j} \left( f_{i-1}^{N_j} \cdots f_2^{N_j} f_1^{N_j}(a_j) \right) = f_i^{N_j} \left( f_{i-1}^{N_j} \cdots f_2^{N_j} f_1^{N_j}(a_j) \right). \]

Again by Lemma 5 (5), every \(\text{supp}(g_{i,j})\) is contained in a closed interval in \(\text{supp}(H_i)\). Let

\[ g_j := g_{j,m-2} \cdots g_{j,1}. \]

Since \(g_j\) is order preserving,

\[ \partial_- \text{supp}(f_{m-1}) < g_j(a_j) < g_j(b_j) < \partial_+ \text{supp}(f_m). \]

It follows that

\[ \text{supp}(g_j c'_j g_j^{-1}) = g_j(\text{supp}(c'_j)) \subset [g_j(a_j), g_j(b_j)] \subset \text{supp}(H_{m-1}). \]

Let

\[ S' := \{ g_j c'_j g_j^{-1} \}_{1 \leq j \leq l} \cup \{ g_{j,i} \}_{1 \leq j \leq l, 1 \leq i \leq m-2}. \tag{3.1} \]

This \(S'\) satisfies the requirements.

Next, we show that every element of \(S'\) has a fixed point in \(X\). Applying Lemma 8 to the induced action of \(H_i \cong F\) on \(X\), we see that every element of \([H_i, H_i]\) fixes a point in \(X\). Therefore, all \(c_i\) and \(g_{j,i}^{-1}\) has a fixed point. It follows that every \(g_j c'_j g_j^{-1}\) has a fixed point, since it is a conjugate of some \(c_i\).

Finally, we show that elements of \(S'\) have a common fixed point in \(X\). By the second requirement for \(S'\), the support of every \((k+1)\)-element subset \(S'_k\) of \(S'\) is contained in a closed interval in \(\text{supp}(G_F)\). By Lemma 5 (3), we can take \((k+1)\) elements \(h_1, \ldots, h_{k+1}\) of \(G_F\) which map \(\text{supp}(S'_k)\) into \((k+1)\) disjoint open subsets of \(\text{supp}(G_F)\). We apply Theorem 2 by setting \(k_1 = \cdots = k_l = 1\) and \(S_l\) to be the conjugate of \(S'_k\) by \(h_i\) for every \(i \in [1, \ldots, k+1]\). It follows that elements of \(S'\) have a common fixed point.

**Corollary 2** Let \(n \geq 2\). Let \(H\) be a finitely generated subgroup of \([F_n, F_n]\). Then for every semi-simple action of \(F_n\) on a finite-dimensional CAT(0) space, elements of \(H\) have a common fixed point.

**Remark 2** For any semi-simple action of a chain group \(G_F\) on any finite-dimensional CAT(0) space, \(G_F\) does not necessarily have a global fixed point. In fact, \(G_F\) does not admit Serre’s property FA. Property FA for a group \(G\) is equivalent to the three conditions: \(G\) is finitely generated, \(G\) does not surject on \(\mathbb{Z}\), and \(G\) is not an amalgamated product [12]. We fix a map \(w : G_F \to \bigcup_{l \in \mathbb{N}} (\mathcal{F} \cup \mathcal{F}^{-1})^l\), which maps each element \(g \in G_F\) to a finite word in \(\mathcal{F} \cup \mathcal{F}^{-1} = \{ f_i^\pm \}_{1 \leq i \leq m}\) which represents \(g\). Let

\[ p_1(g) = (\text{the number of } f_1 \text{ in } w(g)) - (\text{the number of } f_1^{-1} \text{ in } w(g)). \]
By observing the action of $g$ on a neighborhood of $0 \in [0, 1]$, we can check that $p_1(g)$ does not depend on the choice of $w$. It follows that $p_1 : G_{\mathcal{F}} \rightarrow \mathbb{Z}$ is a well-defined surjection, and thus $G_{\mathcal{F}}$ does not have property FA.

**Lemma 9** Let $m, m', G_{\mathcal{F}}$, and $H$ be those in the statement of Theorem 1. Let

$$\mathcal{F} = \{f_i\}_{1 \leq i \leq m},$$

and

$$\mathcal{F}' := \{f_i\}_{1 \leq i \leq m'}.$$

Without loss of generality, we assume that the subgroup generated by $\mathcal{F}'$, which we write $G_{\mathcal{F}'}$, is minimal. Let

$$H_i := \langle f_i, f_{i+1} \rangle \quad (1 \leq i \leq m),$$

where indices are considered modulo $m$.

Let $I$ be a closed interval in $S^1$ such that the closure of $S^1 \setminus I$ is contained in the support of a chain subgroup $G$ of $G_{\mathcal{F}}$. Then there exists a sequence of elements $g_1, \ldots, g_l \in [G_{\mathcal{F}'}, G_{\mathcal{F}'}, G_{\mathcal{F}'}] \cup \bigcup_{m' \leq i \leq m} [H_i, H_i]$ such that

$$(g_l \cdots g_1)(I) \subset \text{supp}(H_m).$$

**Proof** As in the proof of Theorem 4, we apply Lemma 5 (2) repeatedly for chain subgroups of $G$, and get $h_1', \ldots, h_l' \in [G_{\mathcal{F}'}, G_{\mathcal{F}'}, G_{\mathcal{F}'}]$ such that

$$(h_l' \cdots h_1)(cl(S^1 \setminus I)) \subset \text{supp}(G_{\mathcal{F}'}),$$

where $cl(S^1 \setminus I)$ is the closure of $S^1 \setminus I$. By the minimality of $G_{\mathcal{F}'}$, we may apply Lemma 5 (3): for any $\varepsilon > 0$, there exists $h \in [G_{\mathcal{F}'}, G_{\mathcal{F}'}]$ such that

$$h^{-1}([\partial_- \text{supp}(f_1) + \varepsilon, \partial_+ \text{supp}(f_{m'}) - \varepsilon]) \subset (h_l' \cdots h_1)(S^1 \setminus I_j).$$

Therefore,

$$(hh_l' \cdots h_1)(I) \subset S^1 \setminus [\partial_- \text{supp}(f_1) + \varepsilon, \partial_+ \text{supp}(f_{m'}) - \varepsilon].$$

For sufficiently small $\varepsilon > 0$,

$$S^1 \setminus [\partial_- \text{supp}(f_1) + \varepsilon, \partial_+ \text{supp}(f_{m'}) - \varepsilon] \subset \text{supp}((f_{m'+1}, \ldots, f_m, f_1)).$$

and thus

$$(hh_l' \cdots h_1)(I) \subset \text{supp}((f_{m'+1}, \ldots, f_m)).$$

Again, we apply Lemma 5 (2) repeatedly for chain subgroups of $(f_{m'+1}, \ldots, f_m)$ as in the proof of Theorem 4, and get a sequence of elements $h_{l''}^1, \ldots, h_{l''}^l$ of $\bigcup_{m' \leq i \leq m} [H_i, H_i]$ such that

$$(h_l' \cdots h_1 hh_l' \cdots h_1)(I) \subset \text{supp}(H_m).$$

Therefore,

$$(g_1, \ldots, g_l) := (h_1', \ldots, h_l', h, h', \ldots, h_{l''}^l) \quad \text{andindicesareconsideredmodulo} m.$$
Fig. 7 A construction of the sequence of elements whose composition maps \( I \) in \( \text{supp}(H_m) \), where \( m = 4 \) and \( m' = 2 \) satisfies the requirements. We show an example of the construction of the sequence of elements in Fig. 7. \( \square \)

**Proof of Theorem 1** Let \( m, m', G, \mathcal{F}, \) and \( H \) be those in the statement of Theorem 1. We fix \( k \in \mathbb{N} \) and a semi-simple action of \( G_{\mathcal{F}} \) on a \( k \)-dimensional CAT(0) space \( X \) arbitrarily. Let

\[
\mathcal{F} = \{ f_i \}_{1 \leq i \leq m},
\]

and

\[
\mathcal{F}' := \{ f_i \}_{1 \leq i \leq m'}.
\]

Without loss of generality, we assume that the subgroup generated by \( \mathcal{F}' \), which we write \( G_{\mathcal{F}'} \), is minimal. According to Lemma 6, we may exchange generators \( f_{m'+1}, \ldots, f_m \) and assume that \( m \gg k, m' \). Let

\[
H_i := \langle f_i, f_{i+1} \rangle \quad (1 \leq i \leq m),
\]

where indices are considered modulo \( m \).

First, we construct a finite subset \( S' \) of \( G_{\mathcal{F}} \) such that

- \( H < \langle S' \rangle \), and
- for every \( s' \in S' \), \( \text{supp}(s') \) is contained in a closed interval in \( \text{supp}(G_{\mathcal{F}'}) \), or there exists \( i \in \{ m', \ldots, m \} \) such that \( \text{supp}(s') \) is contained in a closed interval in \( \text{supp}(H_i) \).

Let

\[
c_i := [f_i, f_{i+1}] \quad (1 \leq i \leq m),
\]

where indices are considered modulo \( m \). Since \( [G_{\mathcal{F}}, G_{\mathcal{F}}] \) is generated by conjugates of \( \{ c_i \}_{1 \leq i \leq m} \), there exists a finite generating set \( S = \{ c'_j \}_{1 \leq j \leq n} \) of \( H \) consisting of conjugates of \( c_i \). We claim that, for every \( j \), there exists a sequence of elements

\[
g_{j,1}, \ldots, g_{j,l_j} \in [G_{\mathcal{F}'}, G_{\mathcal{F}}] \cup \bigcup_{m' \leq i \leq m} [H_i, H_i]
\]

such that

\[
\text{supp} \left( (g_{j,i_j} \cdots g_{j,1})c'_j(g_{j,i_j} \cdots g_{j,1})^{-1} \right) \subset \text{supp}(H_{i_j})
\]

for some \( i_j \in \{ 1, \ldots, m \} \). Since \( c'_j \in [G_{\mathcal{F}}, G_{\mathcal{F}}] \), there exists a closed interval \( I_j \) in \( S^1 \) such that

\[
\text{supp}(c'_j) \subset I_j
\]
for every $j$ (Lemma 5 (5)). Note that either $I_j$ or the closure of $S^1 \setminus I_j$ is contained in the support of a chain subgroup of $G_x$, which we name $G_j$. If $S^1 \setminus I_j$ is in the support of a chain subgroup, the claim follows from Lemma 9. We note that this is the only point where we use the minimality of the chain subgroup $G_x$. Suppose that $I_j$ is in the support of a chain subgroup. As in the proof of Theorem 4, we apply Lemma 5 (2) repeatedly for chain subgroups of $G_j$, and get $g_{j,1}, \ldots, g_{j,l_j} \in [G_x, G_x] \cup \bigcup_{m' \leq i \leq m} [H_i, H_i]$ such that

$$g_{j,1} \cdots g_{j,l_j}(I_j) \subset \text{supp}(H_{ij})$$

for some $i_j$. Therefore,

$$\text{supp}\left((g_{j,l_j} \cdots g_{j,1})c'_j(g_{j,l_j} \cdots g_{j,1})^{-1}\right) = (g_{j,l_j} \cdots g_{j,1})\left(\text{supp}\left(c'_j\right)\right)$$

for some $i_j$. Hence the claim follows. Let

$$S' = \{(g_{j,l_j} \cdots g_{j,1})c'_j(g_{j,l_j} \cdots g_{j,1})^{-1}\}_{1 \leq j \leq n} \cup \{g_{j,i_j}\}_{1 \leq l_j \leq l_j, 1 \leq i \leq n} \quad (3.2)$$

This $S'$ satisfies the requirements.

Next, we show that every element of $S'$ has a fixed point in $X$. Applying Lemma 8 to the induced action of $H_i \cong F$ on $X$, we see that every element of $[H_i, H_i]$ has a fixed point. It follows that their conjugates $g_{j,l_j} \cdots g_{j,1}c'_j$ also have fixed points. Applying Theorem 4 to the induced action of $G_x$ on $X$, we see that every element of $[G_x, G_x]$ has a fixed point.

Finally, we show that elements of $S'$ have a common fixed point in $X$. Since $m >> k, m'$, the support of every $(k + 1)$-element subset $S_k'$ of $S'$ is contained in a closed interval in the support of an $(m - 1)$-chain subgroup $L$ of $G_x$. By applying Lemma 5 (3) to $L$, we can take $(k + 1)$ elements of $G_x$ which map supp($S_{k+1}'$) into $(k + 1)$ disjoint open subsets of supp($L$). By applying Theorem 2 to $S'$, it follows that elements of $S'$ have a common fixed point. \hfill $\square$

**Proof of Corollary 1** By Proposition 1, each $T_n$ is an $(n^2 + n)$-ring group with a minimal $n$-chain subgroup. For every $n \geq 3$, $T_n$ is simple [2, Theorem 4.15]. In particular, $[T_n, T_n] = T_n$ is finitely generated. Then we can apply Theorem 1 for $G_x = T_n$ and $H = [T_n, T_n] = T_n$. \hfill $\square$

**Data availability** Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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