Difference between AdS and dS spaces : wave equation approach

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Abstract

We study the wave equation for a massive scalar field in three-dimensional AdS-black hole and dS (de Sitter) spaces to find what is the difference and similarity between two spaces. Here the AdS-black hole is provided by the $J = 0$ BTZ black hole. To investigate its event (cosmological) horizons, we compute the absorption cross section, quasinormal modes, and study the AdS(dS)/CFT correspondences. Although there remains an unclear point in defining the ingoing flux near infinity of the BTZ black hole, quasinormal modes are obtained and the AdS/CFT correspondence is confirmed. However, we do not find quasinormal modes and thus do not confirm the assumed dS/CFT correspondence. This difference between AdS-black hole and dS spaces is very interesting, because their global structures are similar to each other.

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I. INTRODUCTION

Recently an accelerating universe has proposed to be a way to interpret the astronomical data of supernova [1–3]. The inflation is employed to solve the cosmological flatness and horizon puzzles arisen in the standard cosmology. Combining this observation with the need of inflation leads to that our universe approaches de Sitter geometries in both the infinite past and the infinite future [4–6]. Hence it is important to study the nature of de Sitter (dS) space and the assumed dS/CFT correspondence [7–9]. However, there exist some difficulties in studying de Sitter space with a positive cosmological constant $\Lambda_{dS} > 0$. i) There is no spatial (timelike) infinity and global timelike Killing vector. Thus it is not easy to define conserved quantities including mass, charge and angular momentum appeared in asymptotically dS space. ii) The dS solution is absent from string theories and thus we do not get a definite example to test the assumed dS/CFT correspondence. iii) It is hard to define the $S$-matrix because of the presence of the cosmological horizon. For the AdS-black hole with a negative cosmological constant $\Lambda_{AdS} < 0$ \footnote{Here we wish to distinguish the pure AdS space and the AdS-black hole ($J = 0$ BTZ black hole) space. The latter is obtained by identifying points in the pure AdS space. Also the AdS-black hole space does not have globally timelike Killing vector but the AdS space has it. In this work we are interested mainly in the AdS-black hole.}, all of three difficulties mentioned in dS space seem to be resolved even though lacking for a globally timelike Killing vector [4]. There exists a spatial (timelike) infinity and hence the region outside the event horizon is noncompact. Many AdS solutions are arisen from string theory or $M$-theory. There is no notion of an $S$-matrix in asymptotically AdS space, but one has the correlation functions of the boundary CFT. The correlators of the boundary CFT may provide the $S$-matrix elements of the $\Lambda \rightarrow 0$ theory.

We remind the reader that the cosmological horizon in dS space is very similar to the event horizon of the balck hole in the sense that one can define its thermodynamic quantities of a temperature and an entropy using the same way as was done for the black hole [10]. Two important quantities in studying the black hole are the Bekenstein-Hawking entropy and the absorption cross section (greybody factor). The former relates to the intrinsic property of the black hole itself, while the latter relates to the effect of spacetime curvature on the propagation of the perturbed wave. In other words, the computation of the absorption cross section is based on the solution to the wave equation. Explicitly the greybody factor for the black hole arises as scattering of a wave off the gravitational potential surrounding the horizon [11]. The low-energy $s$-wave greybody factor for a massless scalar wave has a universality such that it is equal to the area of the horizon for all spherically symmetric black holes [12]. Also the greybody factor measures the Hawking radiation in a semiclassical way. The entropy for the cosmological horizon was discussed in literature [13] and the absorption of a scalar wave in the cosmological horizon was investigated in [14].

Here we focus on the consequences of the wave equation approach in both AdS-black hole and dS spaces. In addition to the greybody factor, these include quasinormal modes [15], and AdS(dS)/CFT correspondences [16]. The quasinormal modes of a scalar field were studied in the background of black holes. The associated complex frequencies describe the time-decay...
of the scalar perturbation in the black hole background. The radiation associated with these modes is expected to be seen with the gravitational wave detectors in the near future. Most fields propagating in the pure AdS space can be expanded in ordinary real normal modes [17]. That is, the negative cosmological constant provides an effective confining (bounded) box and solutions exist with discrete real frequencies. However, if a black hole is present in AdS space, the fields fall into the AdS-black hole and decay continuously. In this case frequencies become complex, showing characteristic of the black hole which describe the decay of the perturbation outside the event horizon.

Also the quasinormal modes of the AdS-black hole have a direct interpretation in terms of the dual CFT. Using the AdS/CFT correspondence, a black hole in AdS space corresponds to a thermal state in the CFT approximately. Perturbing the black hole by a bulk scalar field corresponds to perturbing the thermal state in the CFT by making use of the corresponding operator $O$. The timescale for the decay of a scalar is given by the imaginary part ($\omega_I$) of quasinormal frequencies. The decay of scalar perturbation describes the return to thermal equilibrium in the CFT. So we can obtain a prediction for the thermalization timescale in the two-dimensional CFT. In the three-dimensional AdS-black hole (BTZ black hole), there is a precise quantitative agreement between quasinormal frequencies and location of the poles of the Fourier transform for the retarded correlation function of $O$: $D^{ret}(x, x') = i\theta(t - t') < [O(x), O(x')] >_T$ [18]. The set of poles characterizes the decay of thermal perturbation on the CFT side. We call this the thermal AdS/CFT correspondence. As a result, the thermal AdS/CFT correspondence is well established beyond our expectation. We study this connection to understand the cosmological horizon in dS space and to test whether or not the thermal dS/CFT correspondence can be realized.

In this work we investigate the consequences of the wave equation approach for a massive scalar in the background of both the AdS-black hole and dS spaces. As a toy model of the AdS space with the event horizon, the $J = 0$ BTZ black hole is introduced [19]. The BTZ black hole is a rotating black hole in three dimensions, as is shown in its other name of Kerr-AdS black hole. This is a counterpart of the Kerr-dS solution. However, these rotating solutions have the singularity at $r = 0$. In order to avoid the singularity, we consider the non-rotating ($J = 0$) BTZ black hole with the event horizon and dS space with the cosmological horizon only.

At this stage we wish to point out a difference between the cosmological horizon in de Sitter space and the event horizon in the Schwarzschild black hole. The cosmological horizon is usually assumed to be in thermal equilibrium with perturbations [20]. This implies that the cosmological horizon not only absorbs radiation, but also emits radiation previously emitted by itself at the same rate, keeping the curvature radius $l$ fixed. The two of black hole and heat bath will be in thermal equilibrium if the box is bounded because the black hole (radiation in box) have a negative (positive) specific heat. As an example, the eternal black hole in AdS space (AdS-Schwarzschild black hole in four and higher dimensions) [21] was introduced because AdS space is considered as an effective confining box. If the box is unbounded, the black hole evaporates completely as the Schwarzschild black hole does. In this sense the de Sitter cosmological horizon is similar to the event horizon of AdS-black hole [22]. The organization of this paper is as follows. In section II we briefly review two examples of quantum mechanics. Section III is devoted to studying the consequences of the wave equation approach in the AdS-black hole space. We study the dS wave equation
along the AdS-black hole in section IV. We summarize our results in section V: what is the difference and similarity between AdS-black hole and dS spaces in the wave equation approach.

II. TWO EXAMPLES IN QUANTUM MECHANICS

In order to obtain the absorption cross section, we have to calculate the outgoing/ingoing fluxes in the AdS-black hole and dS space. However, at the timelike infinity of the AdS-black hole space, there remains some ambiguities to define the outgoing/ingoing fluxes. To estimate it clearly, let us review the flux computation in the potential barrier in quantum mechanics. Also there exists some problems in computation of the flux in dS space. For this purpose, we introduce the wave propagation under the potential step with $0 < E < V_0$. This gives rise to the classical picture of what goes on: the total reflection occurs due to the potential step.

A. Plane wave under potential step

Let us start with a wave propagation under a potential step with height $V_0 > E$ shown in Fig.1. Here we introduce the Schrödinger equation with $\hbar = 1, m = 1/2$ as

$$-\frac{d^2 \Psi(x)}{dx^2} + V(x)\Psi(x) = E\Psi(x).$$

(2.1)

The solution to the Schrödinger equation is given by

$$\Psi_I(x) = Te^{qx}, \quad \text{region I}$$

(2.2)

$$\Psi_{II}(x) = e^{-ikx} + Re^{ikx}, \quad \text{region II}$$

(2.3)

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.pdf}
\caption{The upper portion of the figure shows the potential step $V(x)$. The total energy $E$ is indicated by the dashed line. The lower part shows the absolute square of the wave function $|\Psi_I(x)|$, and $|\Psi_{II}(x)|$. In region II, we have a standing wave pattern, which arises because the reflected wave interferes with the incident wave. Also the wave penetrates into the classically forbidden region I.}
\end{figure}
with \( k = \sqrt{E} \), \( q = \sqrt{V_0 - E} \). Considering the continuity of \( \Psi(x) \) and \( \Psi'(x) \) at \( x = 0 \) leads to

\[
1 + R = T, \quad (2.4)
\]
\[
\imath k (1 - R) = -qT. \quad (2.5)
\]

The above relations take the form

\[
R = \frac{1 - \imath \tan \theta}{1 + \imath \tan \theta} = e^{-2\imath \theta} \left( |R|^2 = 1 \right), \quad (2.6)
\]
\[
T = 2 e^{-i \theta} \cos \theta \quad (2.7)
\]

with \( \tan \theta = q/k \). In region I the wave function is given by

\[
\Psi_I(x) = 2 e^{-i \theta} \cos \theta e^{qx}, \quad |\Psi_I(x)|^2 = 4 \cos^2 \theta e^{2qx}. \quad (2.8)
\]

and in region II, it takes

\[
\Psi_{II}(x) = 2 e^{-i \theta} \cos(kx - \theta), \quad |\Psi_{II}(x)|^2 = 4 \cos^2(kx - \theta). \quad (2.9)
\]

Since the density of incident wave is unity, its flux \( (\mathcal{F}_{\text{inc}}) \) is equal to \(-2\sqrt{E} < 0\). The reflected flux \( \mathcal{F}_{\text{ref}} \) is given by \( 2\sqrt{E} > 0 \) and thus there is no net flux in region II: \( \mathcal{F}_{\text{inc}} + \mathcal{F}_{\text{ref}} = 0 \). According to the flux conservation, we expect that there is no flux in region I. As a check, one finds that \( \mathcal{F}_I = \frac{1}{2} \left[ \Psi_I' \right] \left| \Psi_I \right|^2 = 0 \). Even though the probability density of finding a particle between \( x \) and \( x + dx (x < 0) \) is not zero, its flux is zero. This means that the quantum mechanical picture reduces to the classical picture of the total reflection.

A plane wave moving under a potential step of height \( (V_0 > E) \) corresponds to a toy model for the total reflection with \( |R|^2 = 1 \) and \( R = e^{-2i \theta} \) (non-zero phase). A similar situation occurs in a scalar wave propagating under de Sitter space.

**B. Plane wave under potential barrier**

In this section we clarify the flux ambiguity by considering a plane wave propagation under the potential barrier shown in Fig.2. The solution to the Schrödinger equation leads to

\[
\Psi_I(x) = e^{ikx} + Re^{-ikx}, \quad \text{region I} \quad (2.10)
\]
\[
\Psi_{II}(x) = Ae^{qx} + Be^{-qx}, \quad \text{region II} \quad (2.11)
\]
\[
\Psi_{III}(x) = Te^{ikx}, \quad \text{region III} \quad (2.12)
\]

with \( k = \sqrt{E} \), \( q = \sqrt{V_0 - E} \). Imposing the boundary condition on the wave function and its derivative at \( x = 0 \) and \( x = a \), one finds four relations:

\[
1 + R = A + B, \quad (2.13)
\]
\[
1 - R = -i \rho (A - B), \quad (2.14)
\]
\[
A e^\theta + B e^{-\theta} = T e^{i \delta}, \quad (2.15)
\]
\[
A e^\theta - B e^{-\theta} = \frac{T}{\rho} e^{i \delta}, \quad (2.16)
\]
FIG. 2. The upper portion of the figure shows the potential barrier \( V_0 > E \) with the total energy \( E \) denoted by the dashed line. The lower portion shows the absolute square of the wave function: \( |\Psi_I(x)|^2 \), \( |\Psi_{II}(x)|^2 \), \( |\Psi_{III}(x)|^2 \). We note the transmitted wave in region III and the exponentially decreasing function inside the barrier. In region I, we have an imperfect standing wave pattern. Because of the conservation of the flux in whole region, one finds \( \mathcal{F} \neq 0 \) anywhere.

where \( \rho = q/k \), \( \theta = qa \), \( \delta = ka \). From the above relations, we obtain relevant quantities

\[
A = \frac{T}{2} e^{(i\delta - \theta)} (1 + \frac{i}{\rho}),
\]

\[
B = \frac{T}{2} e^{(i\delta + \theta)} (1 - \frac{i}{\rho}),
\]

where

\[
T = \frac{e^{i\delta}}{\cosh \theta + \frac{i}{2}(\rho - \frac{1}{\rho})\sinh \theta}.
\]

Here we focus on computing the flux at region II. At the first sight, we might have thought that \( \mathcal{F}_{II} = 0 \), since the wave function \( \Psi_{II}(x) \) does not take a form of travelling wave. According to the flux conservation, we expect that \( \mathcal{F}_I = \mathcal{F}_{II} = \mathcal{F}_{III} \), otherwise waves would be created or destroyed. Actually one has a non-zero flux \([23]\)

\[
\mathcal{F}_{II} = 4q \text{Im}[AB^*].
\]

This shows that if the coefficient \( A \) or \( B \) is zero, or if they are both real (in general, if they are complex but have the same phase), the flux \( \mathcal{F}_{II} \) is indeed zero. In the previous subsection, the coefficient corresponding to \( A \) is zero and thus its flux is zero. In general potential barrier problem, neither \( A \) nor \( B \) is zero, and their relative phase gives a non-zero flux. However, we note that it is not easy to determine whether the value of the flux \( \mathcal{F}_{II} \) in Eq.(2.20) is positive or negative. A similar situation to this case occurs in the scalar wave propagation near infinity in the BTZ black hole background.
III. WAVE EQUATION IN ADS-BLACK HOLE SPACE

A. Wave equation

We start with the wave propagation for a massive scalar field with mass \( m \)

\[
(\nabla^2_{\text{BTZ}} - m^2)\Phi_{\text{BTZ}} = 0 \tag{3.1}
\]

in the background of the \( J = 0 \) BTZ black hole whose line element is given by [19]

\[
ds^2_{\text{BTZ}} = -\left( -M + \frac{r^2}{l^2} \right) dt^2 + \left( -M + \frac{r^2}{l^2} \right)^{-1} dr^2 + r^2 d\phi^2, \tag{3.2}
\]

where \( M \) is the mass of the black hole and \( l \) is the curvature radius of AdS space. Hereafter we set \( M = l = 1 \) for simplicity unless otherwise stated. The above metric is singular at the event horizon of \( r = r_{EH} = 1 \), which divides space into four regions. Its global structure (Penrose diagram) is shown in Fig.3. There are two regions with \( 0 \leq r < r_{EH} \) which correspond to the interior region of the \( J = 0 \) BTZ black hole. Two regions with \( r_{EH} \leq r \leq \infty \) correspond to the exterior of the black hole. A timelike infinity of \( r = \infty \) appears but there does not exist any spacelike infinity in AdS-black hole space. A timelike Killing vector \( \frac{\partial}{\partial t} \) is future-directed only in the shaded diamond. This means that there is no globally defined timelike Killing vector in AdS-black hole space. To obtain the greybody factor, we have to obtain a definite wave propagation as time evolves. Hence we confine ourselves to the shaded region (SR). This means that our working space is noncompact, in contrast to the case of dS space. A scalar with mass \( m^2 \geq -1 \) including the tachyon with \( m^2 = -3/4 \) may be allowed for AdS space. A massless scalar with \( m^2 = 0 \) is special and it would be treated separately. Hence we are interested in the massive scalar propagation with \( m^2 \geq 0 \).

Assuming a mode solution

![Fig. 3. The global structure of the J = 0 BTZ black hole spacetime. The causally connected region (shaded region : SR) is noncompact. Hence we expect that the AdS/CFT correspondence which is usually defined at r = \infty is established in the SR.](image-url)
the radial part of the wave equation is

\[(r^2 - 1)f''_\ell(r) - \left(1 - 3r\right)f'_\ell(r) + \left(\frac{\omega^2}{r^2 - 1} - \frac{\ell^2}{r^2} - m^2\right)f_\ell(r) = 0, \quad (3.4)\]

where the prime (') denotes the differentiation with respect to its argument. Hereafter one should distinguish \(\ell\) of the angular momentum number from \(l\) of the curvature radius of AdS space.

B. Potential analysis

We observe from Eq.(3.4) that it is not easy to see how a scalar wave propagates in the exterior of the \(J = 0\) BTZ black hole. In order to do that, we must transform the wave equation into the Schrödinger-like equation by introducing a tortoise coordinate \(r^*\) \cite{24,25}. Then we can get asymptotic waveforms in regions of \(r^* = -\infty\) and \(r^* = 0\) through a potential analysis. For our purpose, we introduce

\[r^* = \frac{1}{2} \ln \left[\frac{r - 1}{r + 1}\right], \quad r = -\coth r_* \quad (3.5)\]

to transform Eq.(3.4) into the Schrödinger-like equation with the energy \(E = \omega^2\)

\[\frac{d^2}{dr^*^2}f_\ell + V_{BTZ}(r)f_\ell = Ef_\ell. \quad (3.6)\]

Here the BTZ potential is given by \cite{19}

\[V_{BTZ}(r) = (r^2 - 1)\left[\frac{3}{4} + m^2 + \frac{\ell^2}{r^2} + \frac{1}{4r^2}\right]. \quad (3.7)\]

From Eq.(3.5) we confirm that \(r^*\) is a tortoise coordinate such that \(r^* \to -\infty (r \to 1)\), whereas \(r^* \to 0 (r \to \infty)\). We express the BTZ potential as a function of \(r^*\)

\[V_{BTZ}(r^*) = \left[(\frac{3}{4} + m^2) \coth^2 r^* - m^2 + \ell^2 - \frac{1}{2} -(\ell^2 + \frac{1}{4}) \tanh^2 r^*\right]. \quad (3.8)\]

We observe that for \(m^2 > 0\), \(V_{BTZ}(r^*)\) decreases exponentially to zero as one approaches the event horizon \((r^* \to -\infty, r \to r_{EH} = 1)\), while it goes infinity as one approaches infinity\((r^* \to 0, r \to \infty)\). A typical form of \(V_{BTZ}(r^*)\) is shown in Fig.4. We thus conjecture that travelling waves appear near the event horizon of \(r^* = -\infty\). But it is not easy to develop a genuine travelling wave at infinity. One finds an approximate equation near \(r^* = 0\)

\[\frac{d^2 f_{\ell,0}}{dr^*^2} - \left(\frac{3}{4} + m^2\right) \frac{f_{\ell,0}}{r^*^2} = 0. \quad (3.9)\]

This gives us a solution
The shape of $V_{\text{BTZ}}(r^*)$ with $m^2 = 1$, $\ell^2 = 0$, $E = 10$. The ingoing wave ($\leftarrow$) and the outgoing wave ($\rightarrow$) are developed near the event horizon of $r^* = -\infty$. $r^* = 0$ corresponds to infinity of $r = \infty$. At this point, it is subtle to define the ingoing flux.

$$f_{\ell,0}(r^*) = A_{\text{BTZ}} \,e^{(1+\sqrt{1+m^2})\ln r^*} + B_{\text{BTZ}} \,e^{(1-\sqrt{1+m^2})\ln r^*}. \quad (3.10)$$

For $m^2 > 0$, the first term corresponds to a normalizable mode at $r = \infty$ ($r^* = 0$), while the second term is a non-normalizable mode.

On the other hand, near the event horizon $r_{EH} = 1$ ($r^* = -\infty$) one obtains an approximate equation because $V_{\text{BTZ}}(r^*) \to 0$ as $r^* = -\infty$

$$\frac{d^2}{dr^2}f_{\ell,-\infty} + \omega^2 f_{\ell,-\infty} = 0 \quad (3.11)$$

which gives us a travelling wave solution

$$f_{\ell,-\infty}(r^*) = C_{\text{BTZ}} \,e^{-i\omega r^*} + D_{\text{BTZ}} \,e^{i\omega r^*}, \quad (3.12)$$

where considering $e^{-i\omega t}$, the first term is the ingoing wave ($\leftarrow$), and the second is the outgoing wave ($\rightarrow$).

**C. Flux calculation**

Up to now we obtain the approximate solutions near $r^* = -\infty, 0$. In order to obtain the solution which is valid for whole region outside the black hole, we solve equation (3.4) explicitly. We wish to transform it into a hypergeometric equation using $z = (r^2 - 1)/r^2$. Our working space remains unchanged as $0 \leq z \leq 1$ covering the exterior of the BTZ black hole. This equation takes a form

$$z(1-z)f''_\ell(z) + (1-z)f'_\ell(z) + \frac{1}{4}\left(\frac{\omega^2}{z} - \frac{\ell^2}{1-z} - m^2\right)f_\ell(z) = 0. \quad (3.13)$$

Here one finds two poles at $z = 0, 1(r = 1, \infty)$ and so makes a further transformation to cancel these by choosing an ingoing solution at $z = 0(r = r_{EH})$
\[ f(z) = z^\alpha (1 - z)^\beta w(z), \quad \alpha = -\frac{i\omega}{2}, \quad \beta = \frac{1}{2}(1 - \sqrt{1 + m^2}). \] (3.14)

Then we obtain the hypergeometric equation

\[ z(1 - z)w''(z) + [c - (a + b + 1)]w'(z) - ab w(z) = 0, \] (3.15)

where \( a, b \) and \( c \) are given by

\[ a = \frac{1}{2}(i\ell - i\omega + 1 - \sqrt{1 + m^2}), \quad b = \frac{1}{2}(-i\ell - i\omega + 1 - \sqrt{1 + m^2}), \quad c = 1 + -i\omega. \] (3.16)

The ingoing solution near \( z = 0 \) to Eq.(3.13) is chosen as\(^2\)

\[ f(z) = C_{BTZ} z^\alpha (1 - z)^\beta F(a, b, c; z) \] (3.17)

which is in accordance with Eq.(3.12) with \( D_{BTZ} = 0 \). Now we are in a position to calculate an ingoing flux\(^3\) at \( z = 0(r = 1, r^* = -\infty) \) which shows

\[ \mathcal{F}_m(z = 0) = 2\frac{2\pi i}{i}[f^* z \partial_z f - f z \partial_z f^*]|_{z=0} = -4\pi \omega |C_{BTZ}|^2. \] (3.18)

To obtain the flux at infinity \( z = 1(r = \infty, r^* = 0) \), we use a formula:

\[ F(a, b, c; z) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} F(a, b, a + b - c + 1; 1 - z) \]

\[ + \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} (1 - z)^{c-a-b} F(c - a, c - b, -a - b + c + 1; 1 - z) \] (3.19)

with

\[ c - a = \frac{1}{2}[-i\ell - i\omega + 1 + \sqrt{1 + m^2}] \neq a^*, \]

\[ c - b = \frac{1}{2}[i\ell - i\omega + 1 + \sqrt{1 + m^2}] \neq b^*, \]

\[ c - a - b = \sqrt{1 + m^2} \neq (a + b - c)^*. \] (3.20)

We note here that by comparing Eq.(3.16) with Eq.(3.20), \( c - a - b, c - a, c - b \) are not complex conjugates of \( a + b - c, a, b \) respectively. This is because we consider only the case of \( m^2 > 0 \). Using \( 1 - z \approx r^2(\approx 1/r^2) \) near infinity, one finds from Eqs.(3.17) and (3.19) the following form:

\(^2\)In the neighborhood of the event horizon, the two linearly independent solution to Eq.(3.15) are given by \( w(z) = F(a, b, c; z) \) and \( w(z) = z^{1-c} F(a - c + 1, b - c + 1, 2 - c; z) \) [26]. But the latter corresponds to the outgoing wave and thus we choose the former for our purpose.

\(^3\)In this work, one defines “ingoing flux” as negative whereas “outgoing flux” is defined to be positive.
\[ f_{0 \to 1} \equiv f_{\text{nor}} + f_{\text{non}} = H_{\text{nor}} e^{(1+\sqrt{1+m^2}) \ln r^*} + H_{\text{non}} e^{(1-\sqrt{1+m^2}) \ln r^*} \]
\[ = H_{\text{nor}} e^{-(1+\sqrt{1+m^2}) \ln r} + H_{\text{non}} e^{-1+\sqrt{1+m^2}) \ln r}, \]

where the explicit forms of normalizable and nonnormalizable amplitudes are given by

\[ H_{\text{nor}} = C_{\text{BTZ}} E_{\text{nor}}, \quad E_{\text{nor}} = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}, \]
\[ H_{\text{non}} = C_{\text{BTZ}} E_{\text{non}}, \quad E_{\text{non}} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \]

Then we can match Eq.(3.10) with Eq.(3.21) near infinity to yield

\[ A_{\text{BTZ}} = H_{\text{nor}}, \quad B_{\text{BTZ}} = H_{\text{non}}. \]

Finally we calculate the flux near \( z = 1 \) \((r = \infty)\) as

\[ \mathcal{F}(z = 1) = \frac{2\pi i}{\pi} \left[ f_{0 \to 1}^*(r^3 \partial_r f_{0 \to 1}) - f_{0 \to 1}^*(r^3 \partial_r f_{0 \to 1}^*) \right]_{r=\infty} \]
\[ = 8\pi \sqrt{1 + m^2} \text{Im}[H_{\text{nor}}^* H_{\text{non}}] \]

which is similar to the case of potential barrier in Eq.(2.20). This shows that if the coefficient \( E_{\text{nor}} \) or \( E_{\text{non}} \) is zero, or if they are both real (in general, if they are complex but have the same phase), the flux is indeed zero. In the present case, neither \( E_{\text{nor}} \) nor \( E_{\text{non}} \) is zero, and both belong to complex. Their relative phase gives a non-zero flux. However we have to say that it is not easy to determine whether the value of the flux \( \mathcal{F}(z = 1) \) in Eq.(3.26) is positive or negative.

**D. Absorption cross section**

Up to now we do not insert the curvature radius \( l \) of AdS space. The correct absorption coefficient can be recovered when replacing \( \omega(m) \) with \( \omega l(ml) \) inspired from \( r \to r/l \). An absorption coefficient by the event horizon is defined by

\[ A_{\text{BTZ}} = \frac{\mathcal{F}_{\text{in}}(z = 0)}{\mathcal{F}_{\text{in}}(z = 1)}, \]

where \( \mathcal{F}_{\text{in}}(z = 0) \) is given by Eq.(3.18). However it is not easy to separate the ingoing flux from \( \mathcal{F}(z = 1) \) in Eq.(3.26). Hence we may use \( \mathcal{F}(z = 1) \) instead of \( \mathcal{F}_{\text{in}}(z = 1) \) for a further study. We follow the conventional approach to obtain the absorption cross section in black hole physics [14]. The absorption cross section in three dimensions is formally defined by

\[ \sigma_{\text{abs}}^{\text{BTZ}} = \frac{A_{\text{BTZ}}}{\omega} = -\frac{l}{2\sqrt{1 + (ml)^2} \text{Im}[E_{\text{nor}}^* E_{\text{non}}]} \]

Explicitly, \( E_{\text{nor}} \) and \( E_{\text{non}} \) are given by
\[
E^*_{\text{nor}} = \frac{\Gamma(1 + i\omega l)\Gamma(-\sqrt{1 + (ml)^2})}{\Gamma((i\omega l - i\ell + 1 - \sqrt{1 + (ml)^2})/2)\Gamma((i\omega l + i\ell + 1 - \sqrt{1 + (ml)^2})/2)},
\]

\[
E_{\text{non}} = \frac{\Gamma(1 - i\omega l)\Gamma(\sqrt{1 + (ml)^2})}{\Gamma((i\omega l - i\ell + 1 + \sqrt{1 + (ml)^2})/2)\Gamma((-i\omega l + i\ell + 1 + \sqrt{1 + (ml)^2})/2)}. 
\]

For \((ml)^2 = -1\), one finds that \(E^*_{\text{nor}} E_{\text{non}} \to |E_{\text{nor}}|^2\) (real) and thus \(\mathcal{F}(z = 1) \to 0\) irrespective of the factor \(\sqrt{1 + (ml)^2}\). For a massless scalar propagation with \(m^2 = 0\), one has

\[
E^*_{\text{nor}} = \frac{\Gamma(1 + i\omega l)\Gamma(-1)}{\Gamma((i\omega l - i\ell)/2)\Gamma((i\omega l + i\ell)/2)}, 
\]

\[
E_{\text{non}} = \frac{\Gamma(1 - i\omega)\Gamma(1)}{\Gamma((-i\omega l - i\ell)/2 + 1)\Gamma((-i\omega l + i\ell)/2 + 1)}. 
\]

Here we find that \(E^*_{\text{nor}} E_{\text{non}} \to \text{real}\) and thus \(\mathcal{F}(z = 1) \to 0\), although there exists a divergent term of \(\Gamma(-1) = \pi/\sin[2\pi]\) in \(E^*_{\text{nor}}\). We do not consider the massless scalar propagation because of this divergent term. Even for the \(s(\ell = 0)\)-wave scalar propagation, it is difficult to split \(\mathcal{F}(z = 1)\) into \(\mathcal{F}_{\text{in}}(z = 1)\) and \(\mathcal{F}_{\text{out}}(z = 1)\). For the case of \(m^2 > 0\), one cannot find the correct ingoing flux from Eq.(3.26) to compute the absorption cross section. This mainly arises from the fact that one cannot find an asymptotically flat region which is necessary for defining the scattering process well. In general, we need a spatial infinity to obtain an absorption cross section in the wave equation approach. Here “spatial infinity” means \(r = \infty\) in asymptotically flat space, while “spatial (timelike) infinity” denotes \(r = \infty\) in asymptotically AdS space. We note that the previous calculations [27,25] based on the wave equation approach in the BTZ black hole belong to the approximate ones which are required to match with the results of the CFT. Hence we conclude that the absorption cross section is not derived from the bulk scalar propagation in AdS-black hole space, but it can be obtained from the dual CFT correlation function [18] as the scattering matrix does [4].

E. Quasinormal modes

Even though we encounter some difficulties in calculating the absorption cross section in the BTZ black hole background, we can calculate the quasinormal modes. In asymptotically flat space like in the Schwarzschild black hole [15], these modes are usually defined as the solutions which are purely ingoing wave \((\Phi \sim e^{-i\omega(t+r^*)})\) at the event horizon and which purely outgoing \((\Phi \sim e^{-i\omega(t-r^*)})\) near infinity (see Fig.5). The initial ingoing wave from infinity is not allowed.

In asymptotically AdS space [17], these modes are defined as the solutions which are purely ingoing wave \((\Phi_{\text{BTZ}} \sim e^{-i\omega(t+r^*)})\) at the event horizon and which vanish \((\Phi_{\text{BTZ}} \sim 0)\) at infinity (see Fig.6). This comes from the boundary condition that the wave function is zero at infinity because its potential is infinite at infinity. Because one cannot find any travelling wave at infinity, we can extend this definition to the flux boundary condition in AdS-black hole space: the ingoing flux \((\mathcal{F}_{\text{in}}(z = 0) < 0)\) at the event horizon and no flux \((\mathcal{F}(z = 1) = 0)\) at infinity. Then there are two ways to obtain \(\mathcal{F}(z = 1) = 0\) with the
complex frequency. First one chooses $E_{\text{non}} = 0$ to get the quasinormal modes. These impose the restriction as

$$c - a = -n, \quad c - b = -n,$$

where $n = -1, -2, \cdots$ (negative integers). Considering Eq.(3.30) leads to

$$\omega_1 = -\frac{\ell}{l} - 2\frac{i}{l}(n + \frac{1}{2} + \sqrt{\frac{1 + (ml)^2}{2}}),$$

$$\omega_2 = \frac{\ell}{l} - 2\frac{i}{l}(n + \frac{1}{2} + \sqrt{\frac{1 + (ml)^2}{2}}).$$

Second from $E_{\text{nor}}^* = 0$, one finds $a = -n, b = -n$ which lead to

$$\omega_3 = -\frac{\ell}{l} + 2\frac{i}{l}(n + \frac{1}{2} - \sqrt{\frac{1 + (ml)^2}{2}}),$$

$$\omega_4 = \frac{\ell}{l} + 2\frac{i}{l}(n + \frac{1}{2} - \sqrt{\frac{1 + (ml)^2}{2}}).$$

When decomposing the quasinormal frequencies into real and imaginary parts : $\omega = \omega_R - i\omega_I$, $\omega_I$ should be positive for all quasinormal frequencies because those modes decay with time. Hence the true quasinormal frequencies are given by Eqs.(3.34) and (3.35) but not by Eqs.(3.36) and (3.37). We note that in calculating quasinormal modes in AdS space, one has to use a definition of the zero-flux at infinity. Someone uses the Dirichlet boundary condition on the wave function at infinity which eventually leads to $E_{\text{non}} = 0$. A single term of either normalizable one or nonnormalizable one leads to zero-flux at infinity. But requiring the Dirichlet condition (no nonnormalizable term) at infinity leads to a correct definition of quasinormal frequencies.

### F. AdS/CFT correspondence

The complex quasinormal frequencies describe the decay of the massive scalar perturbation in the background of AdS-black hole. These depend on the parameters of AdS-black

![FIG. 5. Shape of potential $V_{\text{Sch}}(r^*)$ and its quasinormal modes of a massless scalar in the Schwarzschild black hole background. The ingoing wave ($\leftarrow$) at the event horizon and the outgoing wave ($\rightarrow$) at infinity denote quasinormal modes.](image-url)
FIG. 6. The quasinormal modes in AdS space. These are denoted by purely ingoing modes (←) at the event horizon.

hole \((M, l)\) as well as the parameters of perturbed field \((m, \ell)\). In terms of the AdS/CFT correspondence, an off-equilibrium configuration in the AdS-black hole space \((J = 0 \text{ BTZ black hole})\) is related to an off-equilibrium thermal state in the boundary conformal field theory. The timescale for the decay of scalar perturbation is given by the imaginary part of the quasinormal frequencies \((\omega_I)\). Using the AdS/CFT correspondence, one can obtain a prediction of the timescale for return to equilibrium of its dual CFT. Actually there is a precise agreement between quasinormal modes on the bulk side and the location of the poles for the retarded correlation function describing the linear response of an perturbing operator \(\mathcal{O}\) on the CFT side [18]. This provides a new quantitative test for the thermal AdS/CFT correspondence. Further, expressing \(\omega_I \sim 2/l\) as \(\omega_I \sim \frac{r_{EH}}{\gamma} \frac{1}{r_{EH}}\), where \(\gamma\) is the Choptuik scaling parameter, it suggests a deeper connection between the critical phenomena of the BTZ black hole thermodynamics and quasinormal modes [28]. According to the thermal AdS/CFT correspondence, this leads to the boundary CFT interpretation of Choptuik scaling (a universal scaling behavior).

IV. WAVE EQUATION IN DE SITTER SPACE

A. dS wave equation

We start with the wave equation for a massive scalar field [29,14]

\[
(\nabla^2_{dS} - m^2)\Phi_{dS} = 0
\]

in the background of three dimensional de Sitter space

\[
d_{dS}^2 = -(1 - \frac{r^2}{l^2})dt^2 + \left(1 - \frac{r^2}{l^2}\right)^{-1}dr^2 + r^2 d\phi^2.
\]

Here \(l^2\) is the curvature radius of de Sitter space and hereafter we set \(l = 1\) for simplicity unless otherwise stated. The above metric is singular at the cosmological horizon \(r = r_{CH} = \ldots\)
FIG. 7. The global structure of de Sitter spacetime. This diagram looks like that of the $J = 0$ BTZ black hole except replacing $r = 0 (r = \infty)$ here by $r = \infty (r = 0)$ in Fig. 3. But the difference is that the causally connected region (shaded SD) in this figure is compact, whereas the causally connected region (shaded region: SR) in Fig. 3 is noncompact. Hence we expect that the assumed dS/CFT correspondence which is usually defined in the FT or PT including $r = \infty$ is not established in the SD where is causally disconnected to the FT and PT.

1 which divides space into four regions, as is shown in Fig. 7. There are two regions with $0 \leq r \leq 1$ which correspond to the causal diamonds of observers at the north and south poles: northern diamond (ND) and southern diamond (SD). An observer at $r = 0$ is surrounded by a cosmological horizon at $r = 1$. Two regions with $1 < r \leq \infty$ containing the future-null infinity $I^+$ and past-null infinity $I^-$ are called future triangle (FT) and past triangle (PT), respectively. A timelike Killing vector $\frac{\partial}{\partial t}$ is future-directed only in the southern diamond. To obtain the greybody factor, we have to obtain a definite wave propagation as time evolves. Hence we confine ourselves to the southern diamond (shaded region). This means that our working space here is compact, in contrast to the AdS-black hole.

In connection with the assumed dS/CFT correspondence, one may classify the mass-squared $m^2 \geq 0$ into three cases [9]: $m^2 \geq 1$, $0 < m^2 < 1$, $m^2 = 0$. For a massive scalar with $m^2 \geq 1$, one has a non-unitary CFT. A scalar with mass $0 < m^2 < 1$ can be related to a unitary CFT. A massless scalar with $m^2 = 0$ is special and it would be treated separately. Here we consider only $m^2 (0 < m^2 < 1)$ as a parameter at the beginning. Assuming a mode solution

$$\Phi_{dS}(r, t, \phi) = \tilde{f}_\ell(r)e^{-i\omega t}e^{i\ell\phi}, \quad (4.3)$$

Eq.(4.1) leads to the differential equation for $r$ [29,30]

$$(1 - r^2)\tilde{f}_\ell''(r) + \left(\frac{1}{r} - 3r\right)\tilde{f}_\ell'(r) + \left(\frac{\omega^2}{1 - r^2} - \frac{\ell^2}{r^2} - m^2\right)\tilde{f}_\ell(r) = 0, \quad (4.4)$$

where the prime (') denotes the differentiation with respect to its argument. Apparently this equation can be obtained from equation (3.4) by replacing $\ell^2, m^2$ by $-\ell^2, -m^2$, respectively. However, an important thing is that the working space is different: the working region of AdS space (SR) is from $1 \leq r \leq \infty$, while that of dS space (SD) is $0 \leq r \leq 1$. That is, there is no notion of spatial (timelike) infinity in the SD of dS space.
B. dS potential analysis

We observe from Eq.(4.4) that it is not easy to find how a scalar wave propagates in the southern diamond. In order to do that, we must transform the wave equation into the Schrödinger-like equation using a tortoise coordinate \( r^\ast \) [14]. Then we can get wave forms in asymptotic regions of \( r^\ast \to \pm \infty \) through a potential analysis. We introduce \( r^\ast = g(r) \) with \( g'(r) = 1/r(1-r^2) \) to transform Eq.(4.4) into the Schrödinger-like equation with the energy \( E = \omega^2 \)

\[
-\frac{d^2}{dr^*2} \tilde{f}_\ell + V_{dS}(r) \tilde{f}_\ell = E \tilde{f}_\ell
\]

with the dS potential

\[
V_{dS}(r) = \omega^2 + r^2(1-r^2)\left[m^2 + \ell^2 + \frac{\omega^2}{1-r^2}\right].
\]

Here, in contrast to the BTZ black hole, \( E = \omega^2 \) is not singled out as the energy term of the Schrödinger equation because the dS potential also includes this term.

Considering \( r^\ast = g(r) = \int g'(r)dr \), one finds

\[
r^\ast = \ln r - \frac{1}{2} \ln \left[(1+r)(1-r)\right], \quad e^{2r^\ast} = \frac{r^2}{1-r^2}, \quad r^2 = \frac{e^{2r^\ast}}{1+e^{2r^\ast}}. \tag{4.7}
\]

From the above we confirm that \( r^\ast \) is a tortoise coordinate such that \( r^\ast \to -\infty (r \to 0) \), whereas \( r^\ast \to \infty (r \to 1) \). We can express the potential as a function of \( r^\ast \) explicitly

\[
V_{dS}(r^\ast) = \omega^2 + \frac{e^{2r^\ast}}{(1+e^{2r^\ast})^2} \left[m^2 + \frac{1+e^{2r^\ast}}{e^{2r^\ast}} \ell^2 - (1+e^{2r^\ast})\omega^2\right]. \tag{4.8}
\]

For \( m^2 = 1, \ell^2 = 0, E = \omega^2 = 0.01 \), the shape of this takes a potential barrier (\( \sim \)) located at \( r^\ast = 0 \). On the other hand, for all non-zero \( \ell \), one finds the potential step (\( \sim \)) with its height \( \omega^2 + \ell^2 \) on the left-hand side of \( r^\ast = 0 \). All potentials \( V_{dS}(r^\ast) \) decrease exponentially to zero as \( r^\ast \) increases on the right-hand side. Roughly speaking, the outline of \( V_{dS}(r^\ast) \) shown in Fig.8 is similar to the potential step in Fig.1. We conjecture that travelling waves appear near the cosmological horizon of \( r^\ast = \infty \). But near the coordinate origin of \( r^\ast = -\infty (r = 0) \), it is not easy to develop a travelling wave. Near \( r = 0 (r^\ast = -\infty) \) one finds the approximate equation

\[
\frac{d^2}{dr^*2} \tilde{f}_{\ell,-\infty} - \ell^2 \tilde{f}_{\ell,-\infty} = 0. \tag{4.9}
\]

This gives us a solution

\[
\tilde{f}_{\ell,-\infty}(r^\ast) = A_{dS}e^{\ell r^\ast} + B_{dS}e^{-\ell r^\ast} \tag{4.10}
\]

which is equivalently rewritten by making use of Eq.(4.7) as

\[
\tilde{f}_{\ell,r=0}(r) = A_{dS}r^\ell + \frac{B_{dS}}{r^\ell}. \tag{4.11}
\]
FIG. 8. The shape of $V_{\text{dS}}(r^*)$ with $m^2 = 1$, $\ell^2 = 1$, $E = 10$. The ingoing wave (the outgoing wave) near the cosmological horizon ($r^* = \infty$) are indicated by $\leftarrow$ ($\rightarrow$) respectively. $r^* = -\infty$ corresponds to the origin of coordinate $r = 0$.

In the above two equations, the first terms correspond to normalizable modes at $r = 0$ ($r^* = -\infty$), while the second terms are non-normalizable, singular modes. On the other hand, near the cosmological horizon $r_c = 1(r^* = \infty)$ one obtains a differential equation which is irrespective of $\ell$

$$\frac{d^2}{dr^*_2} f_{\ell,\infty} + \omega^2 f_{\ell,\infty} = 0. \quad (4.12)$$

This has a solution

$$f_{\ell,\infty}(r^*) = C_{\text{ds}} e^{-i\omega r^*} + D_{\text{ds}} e^{i\omega r^*}. \quad (4.13)$$

The first/second waves in Eq.(4.13) together with $e^{-i\omega t}$ imply the ingoing ($\leftarrow$)/outgoing ($\rightarrow$) waves across the cosmological horizon. These are shown in Fig. 8. This picture is based on the observer confined in the southern diamond.

### C. dS flux calculation

In order to solve equation (4.4) explicitly, we first transform it into a hypergeometric equation using $z = r^2$. Here the working space still remains unchanged as $0 \leq z \leq 1$ covering the southern diamond. This equation takes a form

$$z(1-z)\tilde{f}_{\ell}''(z) - (2z - 1)\tilde{f}_{\ell}'(z) + \frac{1}{4} \left( \frac{\omega^2}{1-z} - \frac{\ell^2}{z} - m^2 \right) \tilde{f}_{\ell}(z) = 0. \quad (4.14)$$

Here one finds two poles at $z = 0, 1(r = 0, 1)$ and so makes a further transformation to cancel these by choosing a normalizable solution at $z = 0(r = 0)$

$$\tilde{f}_{\ell}(z) = z^\alpha (1-z)^\beta \tilde{w}(z), \quad \alpha = \frac{\ell}{2}, \quad \beta = \frac{i\omega}{2}. \quad (4.15)$$

Then we obtain the hypergeometric equation
\[ z(1 - z)\ddot{w}(z) + [c - (a + b + 1)z]\dot{w}(z) - ab \dot{w}(z) = 0 \] (4.16)

where \(a, b\) and \(c\) are given by
\[ a = \frac{1}{2}(\ell + i\omega + h_+), \quad b = \frac{1}{2}(\ell + i\omega + h_-), \quad c = \ell + 1 \] (4.17)

with
\[ h_\pm = 1 \pm \sqrt{1 - m^2}. \] (4.18)

One regular solution near \(z = 0\) to Eq. (4.14) is given by [26]
\[ \tilde{f}_+(z) = A_{dS} z^{\ell/2}(1 - z)^{i\omega/2} F(a, b, c; z) \] (4.19)

with an unknown constant \(A_{dS}\). In addition, there is the other solution with a logarithmic singularity at \(z = 0\) as \(\tilde{f}_-(z) = \tilde{A}_{dS} z^{\ell/2}(1 - z)^{i\omega/2}[F(a, b, c; z) \ln z + \cdots]\). However, both solutions have vanishing flux at \(z = 0\) because the relevant part \((z^{\ell/2})\) is not complex but real.

Now we are in a position to calculate an outgoing flux at \(z = 0(r = 0, r^* = -\infty)\) which is defined as
\[ \tilde{F}_{out}(z = 0) = 2\frac{2\pi}{\ell} [\tilde{f}_+ z \partial_z \tilde{f}_+ - \tilde{f}_+ z \partial_z \tilde{f}_+] |_{z = 0}. \] (4.20)

For any kind of real functions near \(z = 0(r = 0)\) including \(\tilde{f}_\pm\), the outgoing \((\rightarrow)\) flux is obviously given by
\[ \tilde{F}_{out}(z = 0) = 0. \] (4.21)

This means that if a wave form is real near \(z = 0\), one cannot find any non-zero flux. We choose a regular solution of \(\tilde{f}_+(z)\) for further calculation. To obtain a flux at the horizon of \(z = 1(r = 1)\), we first use a formula of Eq.(3.19) with
\[ c - a = b^*, \quad c - b = a^*, \quad c - a - b = (a + b - c)^*. \] (4.22)

Using \(1 - z \approx e^{-2r^*}\) near \(z = 1\), one finds from Eq.(4.19) the following form:
\[ \tilde{f}_{+, 0 \rightarrow 1} \equiv \tilde{f}_{in} + \tilde{f}_{out} = \tilde{H}_{\omega, \ell} e^{-i\omega r^*} + \tilde{H}_{-\omega, \ell} e^{i\omega r^*} \] (4.23)

where
\[ \tilde{H}_{-\omega, \ell} = \tilde{H}_{\omega, \ell}^* = A_{dS} \alpha_{-\omega, \ell}, \quad \alpha_{-\omega, \ell} = \frac{\Gamma(1 + \ell)\Gamma(i\omega)2i\omega}{\Gamma[(\ell + i\omega + h_+)^2/2)]\Gamma[(\ell + i\omega + h_-)^2/2)]}. \] (4.24)

Then we match Eq.(4.13) with Eq.(4.23) to yield \(C_{dS} = \tilde{H}_{\omega, \ell}\) and \(E_{dS} = \tilde{H}_{-\omega, \ell}\) near the cosmological horizon. Finally we calculate its outgoing \((\rightarrow)\) flux at \(z = 1(r^* = \infty)\) as
\[ \tilde{F}_{out}(z = 1) = 2\frac{2\pi}{\ell} [\tilde{f}_{out}^* \dot{r} \tilde{f}_{out} - \tilde{f}_{out} \dot{r} \tilde{f}_{out}^*] |_{r^* = \infty} = 4\pi \omega A_{dS}^2 |\alpha_{-\omega, \ell}|^2. \] (4.25)

On the other hand, the ingoing \((\leftarrow)\) flux is given by
\[ \tilde{F}_{in}(z = 1) = 2\frac{2\pi}{\ell} [\tilde{f}_{in}^* \dot{r} \tilde{f}_{in} - \tilde{f}_{in} \dot{r} \tilde{f}_{in}^*] |_{r^* = \infty} = -4\pi \omega A_{dS}^2 |\alpha_{\omega, \ell}|^2. \] (4.26)

Thus one finds that \(\tilde{F}_{out}(z = 1) + \tilde{F}_{in}(z = 1) = 0\) because of \(|\alpha_{\omega, \ell}|^2 = |\alpha_{-\omega, \ell}|^2\).
D. dS absorption cross section

Up to now we do not insert the curvature radius \( l \) of dS space. The correct absorption coefficient can be recovered when replacing \( \omega(m) \) with \( \omega(lm) \). An absorption coefficient by the cosmological horizon is defined formally by

\[
\tilde{A}_{dS} = \frac{\tilde{F}_{\text{out}}(z=1)}{\tilde{F}_{\text{out}}(z=0)},
\]

(4.27)

It is found that there is no absorption of a scalar wave in de Sitter space in a semiclassical way. This means that de Sitter space is usually stable and in thermal equilibrium with the scalar perturbation, unlike the AdS-black hole. The cosmological horizon not only absorbs radiation (scalar perturbation) but also emits that previously absorbed by itself at the same rate, keeping the curvature radius \( l \) of de Sitter space fixed. It can be proved by the relation of \( \tilde{F}_{\text{out}}(z=1) + \tilde{F}_{\text{in}}(z=1) = 0 \) and \( \tilde{F}_{\text{out}}(z=0) = 0 \). This picture coincides with the plane wave propagation of the energy \( E \) under the potential step with \( 0 < E < V_0 \) which shows the classical picture of what goes on, as is shown in Sec. II-A.

E. dS quasinormal modes

Because one cannot find any travelling wave at \( r = 0 \), we assume the definition of quasinormal modes to the flux boundary condition in dS space: the outgoing flux \( (\tilde{F}_{\text{out}}(z = 1) > 0) \) at the cosmological horizon and the zero-flux \( (\tilde{F}(z = 0) = 0) \) at \( r = 0 \). See Fig.9 for the assumed quasinormal modes. Now let us check whether or not these modes exist in dS space. One may choose \( \tilde{H}_{\omega,\ell} = 0 \) to get the quasinormal modes. This imposes the restriction as

\[
c - a = -n, \quad c - b = -n,
\]

(4.28)

where \( n = -1, -2, \ldots \) (negative integers). Considering Eq.(3.30) leads to

![FIG. 9. The assumed quasinormal modes (--- >) in de Sitter space. However, these are hard to exist in dS space.](image-url)
\[ \omega_1 = -2i \left( n + \frac{1}{2}(l+1) - \frac{\sqrt{1-(ml)^2}}{2} \right), \quad (4.29) \]
\[ \omega_2 = -2i \left( n + \frac{1}{2}(l+1) + \frac{\sqrt{1-(ml)^2}}{2} \right). \quad (4.30) \]

However, this condition of Eq. (4.28) leads through Eq. (4.22) to \( b^* = -n, a^* = -n \) which implies that \( \mathcal{F}_{\text{out}}(z = 1) = 0 \) because its dependence of \( |\alpha-\omega,\ell|^2 \). Also their complex conjugates of \( \omega_1^*, \omega_2^* \) which comes from \( \mathcal{H}_{-\omega,\ell} = 0 \) are not the quasinormal modes because their outgoing fluxes are zero and \( \omega_I < 0 \). Hence we cannot define the quasinormal modes in dS space\(^4\). This result is consistent with the picture of stable cosmological horizon because the presence of quasinormal frequencies implies that the dS scalar wave is loosing its energy continuously into the cosmological horizon.

**F. dS/CFT correspondence**

We find that one cannot obtain the purely outgoing flux near the cosmological horizon. Hence there is no decay of a bulk scalar into the cosmological horizon. Hence we do not obtain quasinormal frequencies. Because the cosmological horizon is stable, there is no off-equilibrium configuration by the scalar perturbation. In turn there is no perturbation of the boundary thermal states in CFT. The thermal dS/CFT correspondence\(^5\) is not established in the wave equation approach.

**V. SUMMARY**

We study the wave equation for a massive scalar in three-dimensional AdS-black hole and dS spaces. Our results are summarized in TABLE. For the AdS-black hole, the absorption cross section is not obtained directly from the bulk wave equation approach because the space is asymptotically not flat but anti de Sitter. Instead, one could obtain it from the dual CFT [18,27]. On the other hand, quasinormal frequencies are found and the AdS/CFT correspondence is established.

For dS space with the cosmological horizon, one finds that the absorption cross section is zero. By analogy of the quantum mechanics with the potential step, it is evident that

\(^4\)Authors in [30] have found quasinormal modes in dS space. However, their calculation was based on the FT and PT where are outside the SD in Fig. 7. These regions are causally disconnected to an observer at \( r = 0 \) in the SD. Actually it is difficult to define the wave propagation well in the FT and PT. In this sense their computations are different from ours in the SD.

\(^5\)This can be considered as an extension of the assumed dS/CFT correspondence to study dS space [9]. The assumed dS/CFT correspondence can be checked somewhere by calculation of entropy [13] and conserved quantities [8] in the FT or PT where is causally disconnected to an observer in the SD. But one cannot confirm this thermal correspondence by making use of the wave equation approach in the SD of dS Space.
TABLE I. Comparison of the AdS-black hole with de Sitter space in the wave equation approach.

| Physical consequences | AdS-black hole | dS space |
|-----------------------|----------------|----------|
| Absorption cross section | Not found but it can be obtained from its dual CFT. | 0 |
| Quasinormal modes | Well-defined and it is consistent with the CFT results. | Not defined. |
| Bulk/boundary correspondence | Thermal AdS/CFT correspondence is realized. | Thermal dS/CFT correspondence is not realized. |

there is no absorption of a scalar wave in de Sitter space. This means that de Sitter space is usually stable and it is in thermal equilibrium with the scalar perturbation. Explicitly the cosmological horizon not only absorbs scalar waves but also emits those previously absorbed by itself at the same rate. As a result, de Sitter space keeps the curvature radius $l$ fixed. This can be proved by the flux conservation of $\mathcal{F}_\text{out}(z = 1) + \mathcal{F}_\text{in}(z = 1) = 0$ and $\mathcal{F}(z = 0) = 0$. It is identical to that in the wave propagation of the energy $E$ under the potential step with $0 < E < V_0$, which shows the classical picture of what goes on. This contrasts to the case of the AdS-black hole. The quasinormal modes are not defined because one cannot find the purely outgoing flux at the cosmological horizon. Also the thermal dS/CFT correspondence is not realized in the wave equation approach.

Finally we wish to mention two important results. First we clarify that the absorption cross section is not derived from the bulk scalar wave propagation in the background of the AdS-black hole. But this could be obtained from the dual CFT through the AdS/CFT correspondence. It implies that the shaded region of AdS-black hole space in Fig.3 is too small to find the greybody factor. Second, thermal properties of the cosmological horizon are different from those of the event horizon of the AdS-black hole. The cosmological horizon is in thermal equilibrium with the scalar perturbation whereas the event horizon of the $J = 0$ BTZ black hole is not in equilibrium with the scalar perturbation because the scalar waves decay into the horizon continuously.

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