Some new formulas for Appell series over finite fields ∗

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Abstract

In 1987 Greene introduced the notion of the finite field analogue of hypergeometric series. In this paper we give a finite field analogue of Appell series and obtain some transformation and reduction formulas. We also establish the generating functions for Appell series over finite fields.

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1 Introduction

We first introduce some notation. Let $p$ denote a prime and $r$ a positive integer. Set $q = p^r$ and denote $\mathbb{F}_q$ as the finite field of $q$ elements. Let $\mathbb{F}_q^*$ be the group of multiplicative characters of $\mathbb{F}_q^*$. We extend the domain of all characters $\chi$ of $\mathbb{F}_q^*$ to $\mathbb{F}_q$ by defining $\chi(0) = 0$ including trivial character $\varepsilon$, and denote $\overline{\chi}$ as the inverse of $\chi$. For a more detailed introduction to characters, see [10, Chapter 8] and [5]. We note that although [5] contains proofs of most results involving characters, care must be taken for cases involving the trivial character $\varepsilon$, which is defined to be 1 at 0 in this reference.

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Following Bailey [4], the classical generalized hypergeometric series is defined as

\[ \sum_{k=0}^{\infty} \frac{(a_0)_k(a_1)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_s)_k} \frac{z^k}{k!}, \]

(1.1)

where \((a)_k = a(a+1) \cdots (a+k-1)\).

In 1987, Greene [9] developed the theory of hypergeometric functions over finite fields and he also proved a number of transformation and summation identities for hypergeometric series over finite fields that are analogues of those in the classical case. For characters \(A, B, C \in \hat{F}_q\) and \(x \in F_q\), Greene [9] defined

\[ 2F_1 \left[ \begin{array}{c} A, B \\ C \end{array} ; x \right] \]

as the finite field analogue of the integral representation of Gauss hypergeometric series [1, 4]:

\[ \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b}(1-tx)^{-a} \frac{dt}{t(1-t)}. \]

And he also introduced

\[ \binom{A}{B}^G = \frac{B(-1)}{q} J(A, B), \]

where \(J(\cdot, \cdot)\) denotes the classical Jacobi sum. In fact, \(\binom{A}{B}^G\) could be regarded as the finite field analogue of the binomial coefficient. For the sake of simplicity, we define

\[ \binom{A}{B} = B(-1)J(A, B) = q \binom{A}{B}^G \]

and

\[ 2F_1 \left[ \begin{array}{c} A, B \\ C \end{array} ; x \right] = q 2F_1 \left[ \begin{array}{c} A, B \\ C \end{array} ; x \right]^G. \]

(1.2)

Now we restate two theorems of Greene in our notation.

**Theorem 1.1.** [9, Theorem 3.6] For characters \(A, B, C \in \hat{F}_q\) and \(x \in F_q\) we have

\[ 2F_1 \left[ \begin{array}{c} A, B \\ C \end{array} ; x \right] = \frac{1}{q-1} \sum_{\chi} \binom{A\chi}{\chi} \binom{B\chi}{C\chi} \chi(x). \]

**Theorem 1.2.** [9, Theorem 4.9] For characters \(A, B, C \in \hat{F}_q\) we have

\[ 2F_1 \left[ \begin{array}{c} A, B \\ C \end{array} ; 1 \right] = A(-1) \binom{B}{AC}. \]
Theorem 1.2 is the finite field analogue of Gauss’s evaluation [1, 4]:

\[ \text{$_2F_1 \left[ \frac{a, b}{c}; 1 \right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$} \]

For more information about the finite field analogue of (1.1), please consult, for example, [8, 11].

There are many double hypergeometric functions which are important in the field of hypergeometric functions [7, 14]. Appell’s four functions [2, 3, 12], which are shown as follows, may be the most famous and well-known functions:

\[
\begin{align*}
F_1(a; b, b'; c; x, y) &= \sum_{m \geq 0} \sum_{n \geq 0} \frac{(a)_{m+n}(b)_m(b')_n}{m!n!(c)_{m+n}} x^m y^n, & |x|, |y| < 1, \\
F_2(a; b, b'; c, c'; x, y) &= \sum_{m \geq 0} \sum_{n \geq 0} \frac{(a)_{m+n}(b)_m(b')_n}{m!n!(c)c'_{m+n}} x^m y^n, & |x| + |y| < 1, \\
F_3(a, a'; b, b'; c; x, y) &= \sum_{m \geq 0} \sum_{n \geq 0} \frac{(a)_m(a')_n(b)_m(b')_n}{m!n!(c)c'_{m+n}} x^m y^n, & |x|, |y| < 1, \\
F_4(a; b; c, c'; x, y) &= \sum_{m \geq 0} \sum_{n \geq 0} \frac{(a)_{m+n}(b)_{m+n}}{m!n!(c)c'_{m+n}} x^m y^n, & |x|^\frac{1}{2} + |y|^\frac{1}{2} < 1.
\end{align*}
\]

For more details about Appell series, please refer to [6, 13, 15]. The $F_1$ function has an integral representation in terms of a single integral [4, 9.3(4)]:

\[
F_1(a; b, b'; c; x, y) = \frac{\Gamma(c)}{(a)\Gamma(c-a)} \int_0^1 u^{-a-1}(1-u)^{c-a-1}(1-ux)^{-b}(1-uy)^{-b'} du,
\]

where $0 < \text{Re}(a) < \text{Re}(c)$. Motivated by Greene [9] we attempt to give a finite field analogue of $F_1(a; b, b'; c; x, y)$. Although the other three series have integral representations with a single integral (involving products of $\text{$_2F_1$}$’s) and simple double integral representations, we cannot give simple and beautiful analogues of them easily.

**Definition 1.1.** For any multiplicative characters $A, B, B', C \in \mathbb{F}_q$ and any $x, y \in \mathbb{F}_q$, we define

\[
\mathbb{F}_1(A; B, B'; C; x, y) = \varepsilon(xy)AC(-1) \sum_u A(u)\overline{AC}(1-u)\overline{B}(1-ux)\overline{B'}(1-uy).
\]

In this definition we have dropped the constant $\frac{\Gamma(c)}{(a)\Gamma(c-a)}$ in order to obtain simpler results. The factor $\varepsilon(xy)AC(-1)$ is chosen so as to lead to a better expression in terms of binomial coefficients. In the following theorem we give an another representation for $\mathbb{F}_1(A; B, B'; C; x, y)$ involving binomial coefficients.
Theorem 1.3. For characters \( A, B, B', C \in \hat{F}_q \) and \( x, y \in F_q \) we have

\[
\mathbb{F}_1(A; B, B'; C; x, y) = \frac{1}{(q-1)^2} \sum_{\chi, \lambda} \left( \frac{A \chi \lambda}{C \chi \lambda} \right) \left( \frac{B \lambda}{\chi} \right) \left( \frac{B' \chi}{\lambda} \right) \chi(x) \lambda(y).
\]

From Definition 1.1 we get the following corollary immediately.

Corollary 1.1. For characters \( A, B, B', C \in \hat{F}_q \) and \( x, y \in F_q \) we obtain

\[
\mathbb{F}_1(A; B, B'; C; x, y) = F_1(A; B', B; C, y, x),
\]

\[(1.3)\]

\[
\mathbb{F}_1(A; B, B'; C; x, x) = \frac{1}{2} \mathbb{F}_1\left[ B', \frac{A}{C}; x \right].
\]

\[(1.4)\]

The main goal of this paper is to give some transformation and reduction formulas and generation functions for \( \mathbb{F}_1(A; B, B'; C; x, y) \). Several examples of such formulas are

\[
\mathbb{F}_1(A; B, B'; C; x, y) = C(-1)B(1-x)B'(1-y)\mathbb{F}_1\left( \frac{AC}{B}; B, B'; C; \frac{x}{x-1}, \frac{y}{y-1} \right),
\]

\[
\varepsilon(x-y)\mathbb{F}_1(A; B, B'; BB'; x, y) = \varepsilon(xy)\overline{A}(1-x) \mathbb{F}_1\left[ B', \frac{A}{BB'}; y-x \right]
\]

\[
- \varepsilon(y-x)\overline{B}(-x)\overline{B}'(-y)
\]

and

\[
\frac{1}{q-1} \sum_{\theta} \left( \frac{A \overline{C} \theta}{\theta} \right) \mathbb{F}_1(A \theta; B, B'; C; x, y) \theta(t)
\]

\[
= \varepsilon(t)\overline{A}(1-t)\mathbb{F}_1\left( A; B, B'; C; \frac{x}{1-t}, \frac{y}{1-t} \right) - \varepsilon(xy)\overline{AC}(-t)\overline{B}(1-x)\overline{B}'(1-y)
\]

as given in Theorem 3.2, Corollary 3.2 and Theorem 4.1, respectively.

The rest of this paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.3. In Section 3 we give several transformation and reduction formulas for \( \mathbb{F}_1(A; B, B'; C; x, y) \). Section 4 is devoted to the generating functions for \( \mathbb{F}_1(A; B, B'; C; x, y) \).
2 Proof of Theorem 1.3

First we list the following three propositions which will be used frequently without indication in this paper.

Proposition 2.1. [9, (2.6) and (2.15)] If $A, B, C \in \hat{\mathbb{F}}_q$ then

$$\left( \begin{array}{c} A \\ B \end{array} \right) = \left( \begin{array}{c} A \\ AB \end{array} \right),$$

$$\left( \begin{array}{c} C \\ A \end{array} \right) \left( \begin{array}{c} A \\ B \end{array} \right) = \left( \begin{array}{c} C \chi \\ AB \end{array} \right) - (q - 1)(B(-1)\delta(A) - AB(-1)\delta(BC)),$$

Proposition 2.2. [9, (2.10)] (Binomial Theorem) For character $A$ of $\hat{\mathbb{F}}_q$ and $x \in \mathbb{F}_q$

$$\overline{A}(1 - x) = \delta(x) + \frac{1}{q - 1} \sum_{\chi} \left( \begin{array}{c} A \chi \\ \chi \end{array} \right) \chi(x),$$

where the sum ranges over all multiplicative characters of $\hat{\mathbb{F}}_q$ and $\delta$ is the function

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases}$$

Proposition 2.3. [10, p. 89-90] (Orthogonal Relations) If $\chi \in \hat{\mathbb{F}}_q$ and $t \in \mathbb{F}_q$ then

$$\sum_{\chi} \chi(t) = (q - 1)\delta(t - 1)$$

and

$$\sum_{t} \chi(t) = (q - 1)\delta(\chi),$$

where, for characters, we define $\delta(\varepsilon) = 1$ and $\delta(\chi) = 0$ for all $\chi \neq \varepsilon$.

Proof of Theorem 1.3: From the binomial theorem over finite fields we know that

$$B(1 - ux) = \delta(ux) + \frac{1}{q - 1} \sum_{\chi} \left( \begin{array}{c} B \chi \\ \chi \end{array} \right) \chi(ux).$$

Since $\varepsilon(x)A(u)\delta(ux) = 0$ and $\varepsilon(y)A(u)\delta(uy) = 0$ for all $x, y \in \mathbb{F}_q$ then

$$\mathbb{F}_1(A; B, B'; C; x, y) = \varepsilon(xy)AC(-1) \sum_u A(u)\overline{AC}(1 - u)(\delta(ux) + \frac{1}{q - 1} \sum_{\chi} \left( \begin{array}{c} B \chi \\ \chi \end{array} \right) \chi(ux))$$
This completes the proof.

3 Transformation and reduction formulas for $\mathbb{F}_1$

As we know, only some special hypergeometric series have summation formulas. So studying their transformation and reduction formulas is becoming more important and it is also true over finite fields. In this section we obtain some transformation and reduction formulas for $\mathbb{F}_1(A; B, B'; C; x, y)$.

Just by some simple calculation we will obtain several reduction formulas immediately, which can be considered as the finite field analogues of

$$F_1(a; b, 0; c; x, y) = \binom{a}{b/c; x}$$

and

$$F_1(a; 0, b'; c; x, y) = \binom{a}{b'/c; y},$$

respectively.

**Theorem 3.1.** For $A, B, B', C \in \hat{\mathbb{F}}_q$ and $x, y \in \mathbb{F}_q$ we have

$$\mathbb{F}_1(A; B, \varepsilon; C; x, y) = \varepsilon(y) \binom{B}{A/C; x} - \varepsilon(x) \overline{AC}(1 - y)B(y - x),$$

(3.1)

$$\mathbb{F}_1(A; \varepsilon, B'; C; x, y) = \varepsilon(x) \binom{B'}{A/C; y} - \varepsilon(y) \overline{AC}(1 - x)B'(x - y).$$

(3.2)
Proof. First we prove (3.1). Obviously, (3.1) holds when \( y = 0 \). For \( y \neq 0 \) we have

\[
\mathbb{F}_1(A; B, \varepsilon; C; x, y) = \varepsilon(x)AC(-1) \sum_{u \neq 1/y} A(u)\overline{A}C(1 - u)\overline{B}(1 - ux)\varepsilon(1 - uy)
\]

\[
= \varepsilon(x)AC(-1) \sum_{u \neq 1/y} A(u)\overline{A}C(1 - u)\overline{B}(1 - ux)
\]

\[
= \varepsilon(x)AC(-1) \left( \sum_{u \neq 1/y} A(u)\overline{A}C(1 - u)\overline{B}(1 - ux) - A(1/y)\overline{A}C(1 - 1/y)\overline{B}(1 - x/y) \right)
\]

\[
= 2\mathbb{F}_1 \left[ B, \frac{A}{C}; x \right] - \varepsilon(x)\overline{A}C(1 - y)\overline{B}(y - x).
\]

Combing (1.3) with (3.1) we can get (3.2) immediately. This completes the proof. \( \square \)

In [13] Schlosser stated the following three transformation formulas for \( F_1 \).

\[
F_1(a; b, b'; c; x, y) = (1 - x)^{-b}(1 - y)^{-b'}F_1(c - a; b, b'; c; \frac{x}{x-1}, \frac{y}{y-1}), \tag{3.3}
\]

\[
F_1(a; b, b'; c; x, y) = (1 - x)^{-a}F_1 \left( a; -b - b' + c, b'; c; \frac{x}{x-1}, \frac{y-x}{1-x} \right) \tag{3.4}
\]

and

\[
F_1(a; b, b'; c; x, y) = (1 - x)^{c-a-b}(1 - y)^{-b'}F_1 \left( c - a; c - b - b', b'; c; x, \frac{x-y}{1-y} \right). \tag{3.5}
\]

In the following theorem we give finite field analogues of (3.3)-(3.5).

**Theorem 3.2.** For characters \( A, B, B' \) and \( C \) of \( \overline{\mathbb{F}}_q \) and \( x, y \in \mathbb{F}_q \) we have

\[
\mathbb{F}_1(A; B, B'; C; x, y) = C(-1)\overline{B}(1 - x)\overline{B'}(1 - y)\mathbb{F}_1 \left( \overline{AC}; B, B'; C; \frac{x}{x-1}, \frac{y}{y-1} \right), \tag{3.6}
\]

\[
\varepsilon(x - y)\mathbb{F}_1(A; B, B'; C; x, y) = \varepsilon(y)\overline{A}(1 - x)\mathbb{F}_1 \left( A; B\overline{B'C}, B'; C; \frac{x}{x-1}, \frac{y-x}{1-x} \right), \tag{3.7}
\]

\[
\varepsilon(x - y)\mathbb{F}_1(A; B, B'; C; x, y) = \varepsilon(y)C(-1)\overline{AC}(1 - x)\overline{B'}(1 - y)
\]

\[
\times \mathbb{F}_1 \left( \overline{AC}; B\overline{B'C}, B'; C; x, \frac{x-y}{1-y} \right) \tag{3.8}
\]

Proof. First we prove (3.6). We have

\[
C(-1)\overline{B}(1 - x)\overline{B'}(1 - y)\mathbb{F}_1 \left( \overline{AC}; B, B'; C; \frac{x}{x-1}, \frac{y}{y-1} \right)
\]
\[
\varepsilon \left( \frac{xy}{(x-1)(y-1)} \right) AC(-1) \overline{B}(1-x) \overline{B'}(1-y) \\
\times \sum_u AC(u) A(1-u) \overline{B} \left( 1 - \frac{ux}{x-1} \right) \overline{B'} \left( 1 - \frac{uy}{y-1} \right) \\
= \varepsilon(xy) AC(-1) \sum_u AC(u) A(1-u) \overline{B}((u-1)x + 1) \overline{B'}((u-1)y + 1) \\
= \varepsilon(xy) AC(-1) \sum_u A(v) \overline{AC}(1-v) \overline{B}(1-vx) \overline{B'}(1-vy) \\
= \varepsilon \left( \frac{1}{x} \right)_q AC(-1) \sum_u A(u) \overline{AC}(1-u) \overline{B}(1-u) \overline{B'}(1-u)
\]

Now we give the proof of (3.7). Obviously, it holds when \( x = 0 \). For \( x \neq 0 \), substituting \( u = \frac{v(1-x)}{1-vx} \) into the right hand of (3.7) we have

\[
\varepsilon(y) \overline{A}(1-x) F_1 \left( A; BB'C, B'; C; \frac{x}{x-1}, \frac{y-x}{1-x} \right) \\
= \varepsilon(xy) \varepsilon(y-x) \overline{A}(1-x) AC(-1) \\
\times \sum_u A(u) \overline{AC}(1-u) \overline{BB'C} \left( 1 - \frac{ux}{x-1} \right) \overline{B'} \left( 1 - \frac{u(y-x)}{1-x} \right) \\
= \varepsilon(xy) \varepsilon(y-x) AC(-1) \sum_u A(v) \overline{AC}(1-v) \overline{B}(1-vx) \overline{B'}(1-vy) \\
= \varepsilon(y-x) F_1(A; B, B'; C; x, y).
\]

The proof of (3.8) is similar to (3.7), but with the substitution \( u = \frac{1-v}{1-vx} \) on the right hand side. This completes the proof. \( \square \)

Combining Theorem 3.2 with (1.3) we obtain the following theorem.

**Theorem 3.3.** For characters \( A, B, B', C \) of \( \hat{\mathbb{F}}_q \) and \( x, y \in \mathbb{F}_q \) then

\[
\varepsilon(x-y) F_1(A; B, B'; C; x, y) = \varepsilon(x) \overline{A}(1-y) F_1 \left( A; B, BB'C; C; \frac{x-y}{1-y}, \frac{y}{y-1} \right) 
\]

and

\[
\varepsilon(x-y) F_1(A; B, B'; C; x, y) = \varepsilon(x) C(-1) \overline{B}(1-x) \overline{AB'C}(1-y) \\
\times F_1 \left( AC; B, BB'C; C; \frac{y-x}{1-x}, y \right)
\]

hold.
Proof. The proofs of (3.9) and (3.10) are similar so the proof of (3.10) is skipped. We have

\[ \varepsilon(x-y)F_1(A;B,B';C;x,y) = \varepsilon(x-y)F_1(A;B',B;C;x,y) \]

\[ = \varepsilon(x)A(1-y)F_1\left(A;BB'C,B;\frac{y-x}{y-1},\frac{x-y}{1-y}\right) \]

Then the proof is completed. \(\square\)

Remark 1. Theorem 3.3 gives finite field analogues of the following two transformation formulas \[13, (27, 29)\]:

\[ F_1(a;b,b';c;x,y) = (1-y)^{-a}F_1\left(a;b,c-b-b';c;\frac{x-y}{1-y}x\right) \]

and

\[ F_1(a;b,b';c;x,y) = (1-x)^{-b}(1-y)^{c-a-b'}F_1(c-a;b,c-b-b';c;\frac{y-x}{1-x},y) \].

Putting \(B' = \varepsilon\) into (3.6) and using Theorem 3.1 we obtain the following analogue of the well-known Pfaff-Kummer transformation of \(2F_1\):

\[ _2F_1\left[ a, b \begin{array}{c} \frac{x+y}{c} \end{array} ; x \right] = (1-x)^{-b}F_1\left[ c-a, b \begin{array}{c} x \frac{x+y}{c} \end{array} ; x \right] \]. \hspace{1cm} (3.11)

Corollary 3.1. For \(x \in \mathbb{F}_q \setminus \{1\}\) we have

\[ _2F_1\left[ B, A \begin{array}{c} \frac{x+y}{C} \end{array} ; x \right] = C(-1)\overline{B}(1-x)\overline{C}\overline{F}\overline{1}\left[ B, A \begin{array}{c} \frac{x+y}{C} \end{array} ; x \right] \].

Remark 2. In \[9\] Greene also obtained the analogue of (3.11) which is read as

\[ _2F_1\left[ A, B \begin{array}{c} \frac{x+y}{C} \end{array} ; x \right] = C(-1)\overline{A}(1-x)\overline{C}\overline{F}\overline{1}\left[ A, B \begin{array}{c} \frac{x+y}{C} \end{array} ; x \right] \]

and the above equation extends the domain of Corollary 3.1 to \(\mathbb{F}_q\).

Corollary 3.2. For characters \(A, B, B'\) of \(\hat{\mathbb{F}}_q\) and \(x, y \in \mathbb{F}_q\) we have

\[ \varepsilon(x-y)F_1(A;B,B';BB';x,y) = \varepsilon(xy)\overline{A}(1-x)\overline{F}\overline{1}\left[ B', A \begin{array}{c} \frac{x+y}{C} \end{array} ; \frac{x-y}{1-x} \right] \]

\[ - \varepsilon(y-x)\overline{B}(1-y)\overline{F}\overline{1}\left[ B'B, A \begin{array}{c} \frac{x-y}{C} \end{array} ; \frac{x-y}{1-x} \right] \]
and

\[ \varepsilon(x - y) F_1(A; B, B'; BB'; x, y) = \varepsilon(xy) BB'(-1) \overline{AB}'(1 - x) \overline{B}(1 - y) \]

\[ \times \mathbf{2}_1 F_1 \left[ B', \frac{ABB'}{BB'; x - y}; 1 - y \right] - \varepsilon(x - y) \overline{B}(-x) \overline{B}(-y). \]

Proof. Putting \( C = BB' \) into (3.7) and (3.8), respectively, and applying Theorem 3.1, we can obtain the desired result. \( \square \)

Remark 3. Corollary 3.2 gives finite field analogues of the following reduction formulas (see, [13, 26a]):

\[ F_1(a; b, b'; b + b'; x, y) = (1 - x) \delta x \]

\[ 2 F_1 \left[ \frac{b + b' - a}{b + b'; 1 - x} \right]. \]

Theorem 3.4. For characters \( A, B, B', C \) of \( \hat{F}_q \) and \( x, y \in \mathbb{F}_q \setminus \{0, 1\} \) then

\[ \mathbf{F}_1(A; B, B'; C; x, y) = BB'(-1) \mathbf{F}_1(A; B, B'; ABB'C; 1 - x, 1 - y). \]

Proof. Setting \( u = \frac{\nu}{\nu - 1} \) in the definition of \( \mathbf{F}_1(A; B, B'; C; x, y) \) and applying \( \varepsilon(x) = \varepsilon(1 - x) \) we have

\[ \mathbf{F}_1(A; B, B'; C; x, y) = \varepsilon(xy) AC(-1) \sum_u A(u) \overline{AC}(1 - u) \overline{B}(1 - ux) \overline{B}(1 - uy) \]

\[ = \varepsilon((1 - x)(1 - y)) C(-1) \sum_v A(v) BB'C(1 - v) \overline{B}(1 - v(1 - x)) \overline{B}(1 - v(1 - y)) \]

\[ = BB'(-1) \mathbf{F}_1(A; B, B'; ABB'C; 1 - x, 1 - y). \]

The proof is completed. \( \square \)

If we set \( y = 1 \) and \( B' = \varepsilon \) in Theorem 3.4 and combine this with (1.5) and the identity \( \varepsilon(x) = \varepsilon(1 - x) + \delta(1 - x) - \delta(x) \) we obtain Theorem 4.4 from [9]:

Corollary 3.3. For characters \( A, B, C \) of \( \hat{F}_q \) and \( x \in \mathbb{F}_q \) we have

\[ 2 \mathbf{F}_1 \left[ B, \frac{A}{C}; x \right] = B(-1) \mathbf{2}_1 F_1 \left[ B, \frac{A}{ABC'; 1 - x} \right] \]

\[ + B(-1) \left( \frac{A}{BC} \right) \delta(x - 1) \delta(x). \]
4 Generating Functions for $\mathbb{F}_1$

Generating functions play an important role in many fields of Mathematics. In this section we obtain two generating functions for $\mathbb{F}_1(A; B, B'; C; x, y)$.

**Theorem 4.1.** For $A, B, B', C \in \mathbb{F}_q$ and $x, y, t \in \mathbb{F}_q$ we have

$$
\frac{1}{q-1} \sum_\theta \left( \frac{AC\theta}{\theta} \right) \mathbb{F}_1(A\theta; B, B'; C; x, y)\theta(t)
= \varepsilon(t)A(1-t)\mathbb{F}_1\left( A; B, B'; C; \left\frac{x}{1-t}, \frac{y}{1-t} \right\right) - \varepsilon(xy)AC(-t)B(1-x)B'(1-y).
$$

**Proof.** Replacing $u$ by $v(1-t)$ and using

$$
\overline{AC} \left( 1 + \frac{vt}{1-v} \right) = \delta \left( \frac{vt}{1-v} \right) + \frac{1}{q-1} \sum_\theta \left( \frac{AC\theta}{\theta} \right) \theta\left( \frac{-vt}{1-v} \right)
$$

we obtain

$$
\varepsilon(t)A(1-t)\mathbb{F}_1\left( A; B, B'; C; \left\frac{x}{1-t}, \frac{y}{1-t} \right\right)
= \varepsilon(t)\varepsilon\left( \frac{xy}{(1-t)^2} \right) AC(-1) \sum_u \left( A(u)\overline{AC}(1-u)B \left( 1 - \frac{ux}{1-t} \right) \right) B' \left( 1 - \frac{uy}{1-t} \right) \overline{A}(1-t)
= \varepsilon(t)\varepsilon(xy)AC(-1) \sum_v \left( A(v)\overline{AC}(1-v + vt)B(1-vx)B'(1-vy) \right)
= \varepsilon(t)\varepsilon(xy)AC(-1) \left( \sum_{v \neq 1} \left( A(v)\overline{AC}(1-v)B(1-vx)B'(1-vy) \right) \overline{AC} \left( 1 + \frac{vt}{1-v} \right) \right)
+ \overline{AC}(t)B(1-x)B'(1-y)
= \frac{1}{q-1} \sum_\theta \left( \frac{AC\theta}{\theta} \right) \varepsilon(xy)AC\theta(-1) \sum_v \left( A\theta(v)\overline{A}\theta C(1-v)B(1-vx)B'(1-vy) \right) \theta(t)
+ \varepsilon(xy)\overline{AC}(-t)B(1-x)B'(1-y)
= \frac{1}{q-1} \sum_\theta \left( \frac{AC\theta}{\theta} \right) \mathbb{F}_1(A\theta; B, B'; C; x, y)\theta(t) + \varepsilon(xy)\overline{AC}(-t)B(1-x)B'(1-y).
$$

This completes the proof. \hfill \Box

**Remark 4.** Theorem 4.1 could be regarded as the finite field analogue of [6]

$$
\sum_{k=0}^\infty \left( \binom{a+k-1}{k} \right) F_1(a+k; b, b'; c; x, y)t^k = (1-t)^{-a} F_1\left( a; b, b'; c; \frac{x}{1-t}, \frac{y}{1-t} \right), \quad (|t| < 1).
$$
**Theorem 4.2.** For $A, B, B', C \in \mathbb{F}_q$ and $x, y, t \in \mathbb{F}_q$ we obtain
\[
\frac{1}{q-1} \sum_{\theta} \left( B \theta \right) \mathbb{F}_1(A; B \theta, B'; C; x, y) \theta(t) = \varepsilon(t) \overline{B}(1-t) \mathbb{F}_1 \left( A; B, B'; C; \frac{x}{1-t}, y \right)
\]
\[- \varepsilon(y) \overline{B}(-t) B' \mathcal{C}(x) \overline{AC}(1-x) \overline{B}(x-y) .
\]

**Proof.** Obviously, the above equation holds when $x = 0$ or $y = 0$. So we only need to consider the case $xy \neq 0$. Since
\[
\left( B \theta \right) \left( B \chi \theta \right) = \left( B \theta \right) \left( B \chi \theta \right)
\]
\[- \left( B \chi \theta \right) \left( B \chi \theta \right) - (q-1)(\theta(-1) \delta(B \theta) - B(-1) \delta(B \chi)),
\]
then combining the binomial theorem over finite fields we obtain
\[
\frac{1}{q-1} \sum_{\theta} \left( B \theta \right) \mathbb{F}_1(A; B \theta, B'; C; x, y) \theta(t) - \varepsilon(t) \overline{B}(1-t) \mathbb{F}_1 \left( A; B, B'; C; \frac{x}{1-t}, y \right)
\]
\[= \frac{1}{(q-1)^2} \sum_{\chi, \lambda, \theta} \left( \left( B \chi \theta \right) \left( B \chi \theta \right) - \left( B \chi \theta \right) \left( B \chi \theta \right) \left( A \chi \lambda \right) \left( B' \lambda \right) \left( C \chi \lambda \right) \left( \lambda \right) \right) \chi(x) \lambda(y) \theta(t)
\]
\[= \frac{1}{(q-1)^2} \sum_{\chi, \lambda, \theta} (B(-1) \delta(B \chi) - \theta(-1) \delta(B \theta)) \left( A \chi \lambda \right) \left( B' \lambda \right) \left( C \chi \lambda \right) \left( \lambda \right) \chi(x) \lambda(y) \theta(t)
\]
\[= - \overline{B}(-t) \sum_{\chi, \lambda} \left( A \chi \lambda \right) \left( B' \lambda \right) \left( C \chi \lambda \right) \chi(x) \lambda(y)
\]
\[= - \overline{B}(-t) \sum_{\chi, \lambda} \left( B' \lambda \right) \chi(y) \lambda(x) \sum_{\chi} \left( A \chi \lambda \right) \chi(x)
\]
\[= - \overline{B}(-t) \sum_{\chi, \lambda} \left( B' \lambda \right) \lambda(y/x) (\overline{AC}(1-x) - \delta(x))
\]
\[= - \overline{B}(-t) \sum_{\chi, \lambda} \left( B' \lambda \right) \lambda(y/x) (\overline{AC}(1-x) - \delta(y/x))
\]
\[= - \overline{B}(-t) B' \mathcal{C}(x) \overline{AC}(1-x) \overline{B}(x-y).
\]
This completes the proof of this theorem. \[\square\]

From Theorems 4.1 and 4.2 we can derive two generating functions for $2 \mathbb{F}_1$ which are shown in the following theorem.

**Theorem 4.3.** For $A, B, C \in \mathbb{F}_q$ and $x, y, t \in \mathbb{F}_q$ we have
\[
\frac{1}{q-1} \sum_{\theta} \left( A \theta \right) \mathbb{F}_1 \left( B, A \theta, C; x \right) \theta(t) = \varepsilon(t) \overline{A}(1-t) \mathbb{F}_1 \left( B, A, C; \frac{x}{1-t} \right)
\]
\[- \varepsilon(x) A(1-t) \overline{AC}(t) \overline{B}(1-x)
\]
\[
= - \varepsilon(x) A(1-t) \overline{AC}(t) \overline{B}(1-x).
\]
and
\[
\frac{1}{q-1} \sum_{\theta} \left( \frac{B\theta}{\theta} \right) \, _{2}\bar{F}_{1} \left[ \begin{array}{c} B\theta, \ A \\ C \end{array} ; x \right] \theta(t) = \varepsilon(t)B(1-t) \, _{2}\bar{F}_{1} \left[ \begin{array}{c} B, \ A \\ C \end{array} ; \frac{x}{1-t} \right] \]
\[
\quad - B(-t)AC(1-x)C(x). \tag{4.2}
\]

Proof. First we prove (4.1). Setting \( y = 1 \) and \( B' = \varepsilon \) into Theorem 4.1 and applying (1.5) we have
\[
\frac{1}{q-1} \sum_{\theta} \left( \frac{AC\theta}{\theta} \right) \, _{2}\bar{F}_{1} \left[ \begin{array}{c} B, \ A\theta \\ C \end{array} ; x \right] \theta(t) = \varepsilon(t)A(1-t) \, _{1}F_{1} \left( A; B, \varepsilon; C; \frac{x}{1-t} \right)
\]
\[
= \varepsilon(x)tA(1-t)AC(-1) \sum_{u} A(u)AC(1-u)B \left( 1 - \frac{ux}{1-t} \right) \varepsilon \left( 1 - \frac{u}{1-t} \right)
\]
\[
= \varepsilon(x)tA(1-t)AC(-1) \sum_{u \neq 1-t} A(u)AC(1-u)B \left( 1 - \frac{ux}{1-t} \right) - \varepsilon(x)AC(-t)B(1-x)
\]
\[
= \varepsilon(t)A(1-t) \, _{2}\bar{F}_{1} \left[ \begin{array}{c} B, \ A \\ C \end{array} ; \frac{x}{1-t} \right] - \varepsilon(x)AC(-t)B(1-x).
\]
The proof of (4.2) is similar, setting \( y = 1 \) and \( B' = \varepsilon \) in Theorem 4.2. The details are omitted. \( \square \)

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