Electric field of a point-like charge in a strong magnetic field

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Abstract. We describe the potential produced by a point electric charge placed into a constant magnetic field, so strong that the electron Larmour length is much shorter than its Compton length. The standard Coulomb law is modified due to the vacuum polarization by the external magnetic field. Only mode-2 photons mediate the static interaction. The corresponding vacuum polarization component, taken in the one-loop approximation, grows linearly with the magnetic field. Thanks to this fact a scaling regime occurs in the limit of infinite magnetic field, where the potential is determined by a universal function, independent the magnetic field. The scaling regime implies a short-range character of interaction in the Larmour scale, expressed as a Yukawa law. On the contrary, the electromagnetic interaction regains its long-range character in a larger scale, characterized by the Compton length. In this scale the tail of the Yukawa potential follows an anisotropic Coulomb law: it decreases as the distance from the charge increases, slower along the magnetic field and faster across. The equipotential surface is an ellipsoid stretched along the magnetic field. As a whole, the modified Coulomb potential is a narrower function than the standard Coulomb function, the narrower the stronger the field. The singular behavior in the vicinity of the charge remains unsuppressed by the magnetic field. These results may be useful for studying atomic spectra in super-strong magnetic fields of several Schwinger’s characteristic values.

1. Introduction

The fact that the vacuum, in which an external magnetic field is present, is an optically anisotropic medium, has been known, perhaps, since the time when the nonlinearity of quantum electrodynamics was first recognized: in a nonlinear theory electromagnetic fields do interact with one another, provided the strength of at least one of them is sufficiently strong (of the order of or larger than $m^2/e$, where $m$ and $e$ are electron
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mass and charge, respectively.) If the external field is strong, other fields interact with it, the result of the interaction depending upon the direction specified by the external field, hence the anisotropy. Depending on the wave amplitude, the electromagnetic waves propagation in this medium may be considered as a nonlinear process [1], including the transformation of one photon into two [2] or more photons, or taken in the linear approximation with respect to the amplitude. In the latter case, the second-rank polarization tensor is responsible for the properties of the medium. In the kinematic domain where the photon absorption processes like electron-positron pair creation are not allowed, the polarization tensor is symmetric and real, and the medium is transparent and birefringent‡. In the absorption domain the medium is dichroic [4]. The limit of low frequency and momentum belongs to the transparency domain and corresponds to a constant anisotropic dielectric permeability of the medium. In this limit the polarization tensor may be obtained by differentiations with respect to the fields of an effective Lagrangian, calculated on the class of constant external electric and magnetic fields. For small values of these fields [5] and for extremely large [4] fields the polarization operator was in this way considered using the Heisenberg-Euler effective Lagrangian calculated [6] within the one-loop approximation. The knowledge of this limit is useful for studying the dielectric screening of the fields that are (almost) static and (almost) constant in space. For more general purposes, however, this limit is not sufficient, and one should calculate the polarization tensor directly, using the Feynman diagram technique of the Furry picture in the external magnetic field. On the photon mass shell, i.e. when the photon energy, \( k_0 \), and 3-momentum, \( \mathbf{k} \), are related by the free vacuum dispersion law \( k_0^2 = \mathbf{k}^2 \), such calculations were done by S.Adler [7] and D.H.Constantinescu [8]. The results obtained are appropriate for handling the photon propagation in weakly dispersive medium, when the dispersion law does not essentially deviate from its vacuum shape. The polarization operator for the case of general relation between the photon mass and momentum was calculated by I.A.Batalin and A.E.Shabad [2], W.Tsai [10], V.N.Baier et al. [11], D.V.Melrose and R.J.Stoneham [12]. This gave the possibility of studying the photon propagation [13] under the conditions where the deviation from the vacuum dispersion law may be very strong either due to the phenomenon of the cyclotron resonance in the vacuum polarization [14] - this phenomenon is responsible for the effect of photon capture by a magnetic field [15], [16], [17] - or due to magnetic fields, much larger than the characteristic value \( B_0 = m^2/e \simeq 4.4 \times 10^{13} \text{G} \) [18], [19], [20], [21], or due to the both circumstances [22].

Now we start an investigation of problems of electro- and magneto-statics in the presence of a strong external magnetic field in the vacuum. For this purpose expressions for the dielectric permeability of the magnetized vacuum obtained from the Heisenberg-Euler Lagrangian are applicable only as far as the fields slowly varying in the space

‡ If the scheme is extended to electron-positron plasma in a magnetic field [3], the polarization tensor is no longer symmetric, but remains Hermitian in the transparency domain. The absorption mechanism in this case, apart from the pair creation, includes also the inverse Cherenkov and inverse cyclotron radiation of the plasma electrons.
are concerned. Otherwise, the spatial dispersion becomes important. For this reason, when considering the electric field produced by a point-like electric charge in the present paper, we refer again to the polarization tensor calculated off-shell in ([9])-([13]), taking it in the static limit $k_0 = 0$, but keeping the dependence on $k$.

Using the tensor decomposition of the polarization operator and the photon Green function, established in ([9], [13]) in an approximation-independent way, we find that photons of only one polarization mode (mode-2 in the nomenclature of these references) may be carriers of electrostatic force. This is in agreement with the fact that the electromagnetic field of these photons is, in the static limit, purely electric and longitudinal. The photons of the other two modes mediate in this limit the magneto-static field of constant currents.

To describe the static field, produced in the magnetized vacuum by a point electric charge, we confine ourselves to considering the asymptotically large magnetic fields $B \gg B_0$, such that the electron Larmour length $(eB)^{-1/2}$ is much less than the electron Compton length $m^{-1}$. Therefore, two different scales occur in the problem: the Larmour scale and the Compton scale. A simplifying expression for the mode-2 eigenvalue of the polarization operator is used, valid for such fields. It was first obtained by Yu.M.Loskutov and V.V.Skobelev [23] within a special two-dimensional technique intended for large fields, and by A.E.Shabad [18], and D.B.Melrose and R.J.Stoneham [12] as the asymptote of the mode-2 eigenvalue calculated ([9])-([13]) in the one-loop approximation. (See [22] for the detailed derivation of the large-field asymptotic behavior.) The most important, now well known, fact about this asymptotic behavior (see, e.g., the monographs [24], [19], [25]) is that the mode-2 eigenvalue contains a term linearly growing with the magnetic field. It is sometimes expected that this term - it appears in the denominator of the photon propagator and hence of the expression for the potential - should lead to suppression of the interaction mediated by mode-2 photons. In a different problem we already had an opportunity to show that this is not exactly the case [26], [27]. We demonstrate in the present paper that this term is crucial for the possibility that a scaling regime might be achieved in the limit of an infinite magnetic field, characterized by a comparatively simple universal function, independent of the magnetic field. This function gives the potential of a point charge in the energy units of $(eB)^{1/2}$ as a dimensionless function of the space coordinates in the units of the electron Larmour length $(eB)^{-1/2}$. Except for the closest vicinity of the source charge, its form coincides with the Yukawa law (see Eq. ([28]) below) with the dimensionless mass parameter $2\alpha/\pi$ (which is the "effective photon mass" discussed in [28], measured in the units $(eB)^{1/2}$). Thus, this mass governs the exponential decrease of the potential away from its source. As one moves farther from the source, the Yukawa decrease ceases, and the potential tail approaches the anisotropic Coulomb shape of the form of Eq. ([35]). The law of decreasing along the magnetic field is unaffected by the latter, the decrease is the fastest in the direction orthogonal to the magnetic field. Therefore, the linearly growing term in the mode-2 eigenvalue of the polarization operator leads, first, to the faster decrease of the potential in the direction across the magnetic field.
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for large distances, and, second, to its steeper shape for small distances. This may be recognized as suppression, indeed. On the other hand, the long-distance behavior along the magnetic field, as well as the standard Coulomb singularity near the source do not sense the magnetic field at all, no matter how strong it is.

In Section 2 we give an approximation-independent derivation for the form of the potential of a point charge in terms of the mode-2 component of the photon propagator. In Section 3 we consider the short-distance behavior of the potential. Three terms of its asymptotic expansion near the source are explicitly written. The universal function for the scaling infinite-field limit is obtained as a one-fold integral. In Section 4 the anisotropic Coulomb law is obtained for large distances by considering different mathematical means, appropriate for different remote regions in the space. The analytical results for the potential are supplemented by its shapes drawn by a computer, presented in Figures 1 - 5.

2. General representation for the static potential of a point-like charge

Electromagnetic 4-vector potential produced by the 4-current $j_\nu(y)$ is

$$A_\mu(x) = \int D_{\mu\nu}(x - y) j^\nu(y) d^4y, \quad \mu, \nu = 0, 1, 2, 3. \quad (1)$$

Here $x$ and $y$ are 4-coordinates, and $D_{\mu\nu}(x - y)$ is the photon Green function in a magnetic field in the coordinate representation. The metric in the Minkowski space is defined so that $\text{diag } g_{\mu\nu} = (1, -1, -1, -1)$. Eq. (1) defines the 4-vector potential with the arbitrariness of a free-field solution, including the gauge arbitrariness. If the photon Green function is chosen as causal, only the gauge arbitrariness remains.

The current of a point-like static charge $q$, placed in the point $y = 0$ is

$$j^\nu(y) = q \delta_\nu^0 \delta^3(y). \quad (2)$$

Hence

$$A_\mu(x) = q \int_{-\infty}^{\infty} D_{\mu0}(x_0 - y_0, x) dy_0$$

$$= q \int_{-\infty}^{\infty} D_{\mu0}(x_0 + y_0, x) dy_0 = q \int_{-\infty}^{\infty} D_{\mu0}(y_0, x) dy_0. \quad (3)$$

This 4-vector potential is also static.

Where there is no magnetic field, and the photon propagator is free and taken in the Feynman gauge (with the pole handled in a causal way)

$$D_{\mu\nu}(x - y) = D_{\mu\nu}^C(x - y) \equiv \frac{g_{\mu\nu}}{i4\pi^2(x - y)^2}, \quad (4)$$

only the zeroth component of the 4-vector potential is present:

$$A_0^C(x) = q \int_{-\infty}^{\infty} D_{\mu0}^C(y_0, x) dy_0 = \frac{q}{i4\pi^2} \int_{-\infty}^{\infty} \frac{dy_0}{y_0^2 - x^2} = \frac{1}{4\pi} \frac{q}{|x|}. \quad (5)$$

This is the Coulomb potential in the Heaviside-Lorentz system of units used throughout.
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Let there be an external magnetic field $B$ directed along axis 3 in the Lorentz frame where the charge $q$ is at rest in the origin $x = 0$, and no external electric field exists in this frame. Call this frame special. Define the Fourier transform as

$$D_{\mu\nu}(x) = \frac{1}{(2\pi)^4} \int \exp(ikx) D_{\mu\nu}(k) \, d^4k, \quad \mu, \nu = 0, 1, 2, 3. \quad (6)$$

Then (3) becomes

$$A_\mu(x) = \frac{q}{(2\pi)^4} \int \exp(i(k_0y_0 - kx)) D_{\mu0}(k) \, d^4k \, y_0 = \frac{q}{(2\pi)^3} \int D_{\mu0}(0, k) \exp(-ikx) d^3k. \quad (7)$$

The four 4-eigenvectors $b^{(a)\nu}_\nu$, $a = 1, 2, 3, 4$, of the polarization tensor $\Pi_{\mu\nu}$ take in the special frame (and arbitrary normalization) the form - the components are counted downwards as $\nu = 0, 1, 2, 3$ -

$$b^{(1)}_\nu = k^2 \begin{pmatrix} 0 \\ k_1 \\ k_2 \\ 0 \end{pmatrix}_\nu + k^2 \begin{pmatrix} k_0 \\ k_1 \\ k_2 \\ k_3 \end{pmatrix}_\nu, \quad b^{(2)}_\nu = \begin{pmatrix} k_3 \\ 0 \\ 0 \\ k_0 \end{pmatrix}_\nu,$$

$$b^{(3)}_\nu = \begin{pmatrix} 0 \\ k_2 \\ -k_1 \\ 0 \end{pmatrix}_\nu, \quad b^{(4)}_\nu = \begin{pmatrix} k_0 \\ k_1 \\ k_2 \\ k_3 \end{pmatrix}_\nu. \quad (8)$$

Among them there is only one, whose zeroth component survives the substitution $k_0 = 0$. It is $b^{(2)}_\nu$. This implies that out of the four ingredients of the general decomposition of the photon propagator

$$D_{\mu\nu}(k) = \sum_{a=1}^{4} D_a(k) \frac{b^{(a)\mu}_\nu b^{(a)\nu}}{(b^{(a)})^2},$$

$$D_a(k) = \begin{cases} -(k^2 + \kappa_a(k))^{-1}, & a=1,2,3, \\ \text{arbitrary,} & a=4 \end{cases}, \quad (9)$$

where $\kappa_a(k)$ are scalar eigenvalues of the polarization tensor:

$$\Pi_{\mu\nu}(k) b^{(a)\mu}_\nu = \kappa_a(k) b^{(a)\nu}_\nu, \quad \kappa_4(k) = 0. \quad (10)$$

only the term with $a = 2$, $D_2(k)b^{(2)\mu}_\mu b^{(2)\nu}_\nu/(b^{(2)})^2$, participates in (7), i.e. only mode-2 (virtual) photon may be a carrier of electro-static interaction, and not photons of modes 1,2, nor the purely gauge mode 4. Bearing in mind that $(b^{(2)})^2 = k^2_3 - k^2_0$, we have

$$A_0(x) = \frac{q}{(2\pi)^3} \int \frac{e^{-ikx} d^3k}{k^2 - \kappa_2(0, k^2_3, k^2_1)}, \quad A_{1,2,3}(x) = 0. \quad (11)$$

Here $k^2_\perp = k^2_1 + k^2_2$. Thus, the static charge gives rise to electric field only, as it might be expected. The gauge arbitrariness in the choice of the photon propagator $D_4(k)b^{(4)\mu}_\mu b^{(4)\nu}_\nu = D_4(k)k_\mu k_\nu$ indicated in (9) has no effect in (11). Certainly, the potential (11) is defined up to gauge transformations.
The result that only mode-2 photons mediate electrostatic interaction may be understood, if we inspect electric and magnetic components of the fields of the eigenmodes obtained from their 4-vector potentials \( \mathbf{e}^{(a)} = k_0 \mathbf{b}^{(a)} - k b_0^{(a)}, \mathbf{h}^{(a)} = \mathbf{k} \times \mathbf{b}^{(a)} \). These are

\[
\mathbf{e}^{(1)} = -\frac{k_\perp}{k_\perp} k_0, \quad \mathbf{h}^{(1)} = \left( \frac{k_\perp}{k_\perp} \times \mathbf{k}_3 \right), \tag{12}
\]

\[
\mathbf{e}^{(2)} = k_\perp k_3, \quad \mathbf{e}_3^{(2)} = \frac{k_3}{k_3} (k_3^2 - k_0^2), \quad \mathbf{h}^{(2)} = -k_0 \left( \frac{k_\perp}{k_\perp} \times \frac{k_3}{k_3} \right), \tag{13}
\]

\[
\mathbf{e}^{(3)} = -k_0 \left( \frac{k_\perp}{k_\perp} \times \frac{k_3}{k_3} \right), \quad \mathbf{h}_\perp^{(3)} = -\frac{k_\perp}{k_\perp} k_3, \quad \mathbf{h}_3^{(3)} = \frac{k_3}{k_3} k_\perp. \tag{14}
\]

Here the cross stands for the vector product, and the boldfaced letters with subscripts 3 and \( \perp \) denote vectors along the directions, parallel and perpendicular to the external magnetic field, resp.

The photon energy and momenta here are not, generally, connected by any dispersion law. Therefore, we can discuss polarizations of virtual photons - carriers of the interaction - basing on Eqs. (12)-(14). The electric field \( \mathbf{e} \) in mode 1 is parallel to \( \mathbf{k}_\perp \), in mode 2 it lies in the plane containing the vectors \( \mathbf{k}, \mathbf{B} \), in mode 3 it is orthogonal to this plane, i.e. mode 3 is always transversely polarized, \( \mathbf{e}^{(3)} \mathbf{k} = 0 \). For the special case of the virtual photon propagation transverse to the external magnetic field, \( k_3 = 0 \), (this reduces to the general case of propagation under any angle \( \theta \neq 0 \) by a Lorentz boost along the external magnetic field), mode 2 is transversely polarized, \( \mathbf{e}^{(2)} \mathbf{k} = 0 \), as is always the case for mode 3. Mode 1 for transverse propagation, \( k_3 = 0 \), is longitudinally polarized, \( \mathbf{e}^{(3)} \times \mathbf{k} = 0 \), and its magnetic field is zero. The lowest-lying cyclotron resonance of the vacuum polarization [14], the one that corresponds to the threshold \( k_0^2 - k_3^2 = 4m^2 \) of creation of the pair of electron and positron in the lowest Landau state each, belongs to mode 2. It gives rise to the photon capture effect with the photon turning into a free [15] or bound [16, 17] electron-positron pair. Another consequence of the cyclotron resonance is that a real photon of mode 2 undergoes the strongest refraction in the large magnetic field limit [22] even if its frequency is far beyond the pair production threshold.

In the static limit \( k_0 = 0 \) the magnetic field in mode 2 disappears, \( \mathbf{h}^{(2)} = 0 \), while its electric field is collinear with \( \mathbf{k} \), \( \mathbf{e}^{(2)} = \mathbf{k} \). It becomes a purely longitudinal virtual photon. Unlike mode 2, in modes 1 and 3 in the static limit \( k_0 = 0 \) only the magnetic fields survive: \( \mathbf{e}^{(1,2)} = 0, \mathbf{h}^{(1)} = k_\perp \times \mathbf{B}, \mathbf{h}_3^{(3)} = -k_\perp k_3, \mathbf{h}_3^{(3)} = k_\perp k_3, \mathbf{h}^{(1,3)} \mathbf{k} = 0 \). (Here normalizations are arbitrary and kept fixed only within the same mode). A virtual mode-1 photon carries magneto-static interaction. It is responsible for magnetic field produced by a current flowing through a straight-linear conductor oriented along the external magnetic field. In accordance with the above formula for \( \mathbf{h}^{(1)} \) its magnetic field is orthogonal both to \( \mathbf{B} \) and to the radial direction in the transverse plane \( k_\perp \), along which the magnetic field of the current decreases. The mode-3 photon contributes as an interaction carrier in the problem of a magneto-static field produced by a solenoid with
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its axis along axis 3. In the present paper, however, we do not consider magneto-static problems.

In the asymptotic limit of high magnetic field \( B \gg k_3^2, \) \( B \gg m^2/e \equiv B_0 \) the eigenvalue \( \kappa_2(0, k) \), as calculated within the one-loop approximation \[9\], \[13\], with the accuracy of terms that grow with \( B \) only as its logarithm and slower is \[18\]

\[
\kappa_2(0, k_3^2, k_\perp^2) = -\frac{2\alpha_b m^2}{\pi} \exp \left(-\frac{k_\perp^2}{2m^2b}\right) T \left(\frac{k_3^2}{4m^2}\right),
\]

(15)

where \( b = B/B_0 \) and

\[
T(y) = y \int_0^1 \frac{(1 - \eta^2)d\eta}{1 + y(1 - \eta^2)} = 1 - \frac{1}{2} \ln \frac{\sqrt{1 + y} + \sqrt{y}}{\sqrt{1 + y} - \sqrt{y}}.
\]

(16)

Note that \( T(y \to 0) \sim 2y/3, T(\infty) = 1. \)

Other eigenvalues, \( \kappa_{1,3} \), do not contain the coefficient \( b \) that provides the linear increase of \( \kappa_2 \) with the magnetic field.

Expression for \( \kappa_2 \) is to be used in (11) or, equivalently, in the expression

\[
A_0(x) = \frac{q}{2(2\pi)^2} \int_0^\infty J_0(k_\perp x_\perp) \left( \int_{-\infty}^\infty \frac{e^{-ik_3 x_3} dk_3}{k_\perp^2 + k_3^2 - \kappa_2(0, k_3^2, k_\perp^2)} \right) dk_\perp^2
\]

(17)

obtained from (11) by integration over the angle between the 2-vectors \( k_\perp \) and \( x_\perp \), which are projections of \( k \) and \( x \) onto the plane transverse to the magnetic field. In (17) \( x_\perp = \sqrt{x_1^2 + x_2^2} \) and \( J_0 \) is the Bessel function of order zero.

The nontrivial - other than the leading asymptote \( \sim k_3^2 \) near the point \( k_3 = k_\perp = 0 \) - dependence of the polarization operator eigenvalue \[15\] on the photon momentum components \( k_3, k_\perp \) is the spatial dispersion. We shall see in Section 3 that it is important in the vicinity of the charge, where the field has a large gradient. As for the anisotropic behavior far from the charge, to be studied in Section 4, only the above asymptote is essential, inferable also from the Heisenberg-Euler Lagrangian.

3. Short-distance behavior

3.1. Asymptotic expansion

To consider the behavior of the potential near its point-like source let us add to and subtract from (17) the standard Coulomb potential \[5\] in the form

\[
A_0^C(x) = \frac{q}{(2\pi)^3} \int \frac{e^{-ikx}d^3k}{k^2}
\]

\[
= \frac{q}{2(2\pi)^2} \int_0^\infty J_0(k_\perp x_\perp) \left( \int_{-\infty}^\infty \frac{e^{-ik_3 x_3} dk_3}{k_\perp^2 + k_3^2} \right) dk_\perp^2 = \frac{1}{4\pi} \frac{q}{\sqrt{x_1^2 + x_3^2}}
\]

(18)

so that

\[
A_0(x) = A_0^C(x) - \Delta A_0(x),
\]

(19)
where

$$\Delta A_0(x) = \frac{q}{2(2\pi)^2} \int_0^\infty J_0(k_{\perp} x_{\perp}) \int_{-\infty}^\infty \left( \frac{e^{-ik_3 x_3}}{k_{\perp}^2 + k_3^2} - \frac{e^{-ik_3 x_3}}{k_{\perp}^2 + k_3^2 - \kappa_2(0, k_3^2, k_{\perp}^2)} \right) dk_3 dk_{\perp}^2. \quad (20)$$

Note that the function $\Delta A_0(x_3, x_{\perp})$ is an entire function of $x_{\perp}$, since the exponential in $|x_{\perp}|$ provides convergence of the integral $|x_{\perp}|$ for any complex value of this variable. Keeping quadratic terms in the power series expansion of $J_0(k_{\perp} x_{\perp})$ and $\exp(-ik_3 x_3)$ in $|x_{\perp}|$ we obtain the first three terms of the asymptotic expansion of the potential $|x_{\perp}|$ near the origin $x_3 = x_{\perp} = 0$

$$A_0(x) \sim \frac{q}{4\pi} \left( \frac{1}{|x|} - 2m(C - (2mx_{\perp})^2C_{\perp} - (2mx_3)^2C_{||}) \right), \quad (21)$$

where $C$, $C_{\perp}$ and $C_{||}$ are dimensionless positive constants depending on the external field:

$$C = \frac{2\pi}{qm} \Delta A_0(0) =$$

$$\frac{abm}{\pi^2} \int_0^\infty T \left( \frac{k_3^2}{4m^2} \right) \int_0^\infty \frac{\exp\left(-\frac{k_3^2}{2m^2} \right)}{(k_{\perp}^2 + k_3^2)(k_{\perp}^2 + k_3^2 - \kappa_2(0, k_3^2, k_{\perp}^2))} dk_3, \quad (22)$$

$$C_{\perp} = \frac{ab}{16m^2\pi^2} \int_0^\infty T \left( \frac{k_3^2}{4m^2} \right) \int_0^\infty \frac{k_{\perp}^2 \exp\left(-\frac{k_{\perp}^2}{2m^2} \right)}{(k_{\perp}^2 + k_3^2)(k_{\perp}^2 + k_3^2 - \kappa_2(0, k_3^2, k_{\perp}^2))} dk_3, \quad (23)$$

$$C_{||} = \frac{ab}{8m^2\pi^2} \int_0^\infty T \left( \frac{k_3^2}{4m^2} \right) \int_0^\infty \frac{k_{\perp}^2 \exp\left(-\frac{k_{\perp}^2}{2m^2} \right)}{(k_{\perp}^2 + k_3^2)(k_{\perp}^2 + k_3^2 - \kappa_2(0, k_3^2, k_{\perp}^2))} dk_3, \quad (24)$$

Thanks to the exponential factor the integrals over $k_{\perp}^2$ here are fast converging. The resulting functions decrease for large $k_3$ as $\sim 1/k_3^4$, so the remaining integrals over $k_3$ in $(22), (23), (24)$ converge, bearing in mind that $T$ is a bounded function. The values of the coefficients $(22), (23), (24)$ calculated for four values of the magnetic field $b = 10^4$, $b = 10^5$, $b = 10^6$ and $b = 10^{10}$ are, respectively, $C = 2.21, 9.08, 31.37, 32.70 \times 10^2$, $C_{\perp} = 75.9, 2.58 \times 10^3, 8.38 \times 10^4, 8.49 \times 10^9$, $C_{||} = 174.3, 5.55 \times 10^3, 1.76 \times 10^5, 1.67 \times 10^{11}$.

In Figs. 1 and 2 functions $A_0(x_3, 0)$ and $A_0(0, x_{\perp})$ are drawn by a computer using $(17)$ for three values of the magnetic field: $b = 10^4$, $b = 10^5$ and $b = 10^6$. In agreement with the above analysis the curves in Figs. 1 2 approach the Coulomb law $q/4\pi|x_3|$ or $q/4\pi x_{\perp}$ as $x_3 \to 0$ or $x_{\perp} \to 0$, respectively. In Fig. 3 the ratio of the modified Coulomb law $(17)$ to the standard Coulomb law $(3)$ at $x_3 = 0$ is plotted against the transversal coordinate $x_{\perp}$ for the magnetic field $B = 10^4B_0$. It deviates from unity as we move away from the origin $x_{\perp} = 0$. The dotted line is the interpolation corresponding to the potential taken in the Yukawa form $(q/4\pi x_{\perp}) \exp(-2mCx_{\perp})$ with $C$ defined by Eq. $(22)$ to be in this case $C = 2.21$. This interpolation is in agreement with the first two terms of the asymptotic expansion $(21)$. The Yukawa interpolation is good within the range $0 < x_{\perp} < 0.1(2m)^{-1} = 5L_B$. 
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3.2. Scaling regime

It is remarkable to note that in the infinite-magnetic-field limit, potential (13), measured in the inverse Larmour length $L_B^{-1} = \sqrt{eB}$ units becomes a universal, magnetic-field-independent function of coordinates measured in Larmour units $L_B$. To establish this scaling regime let us make the change of variables in the integral (20) $\tilde{k}_3 = k_3 L_B$, $\tilde{k}_\perp = k_\perp L_B$ and define the new dimensionless coordinates $x_3 = \tilde{x}_3 L_B$, $x_\perp = \tilde{x}_\perp L_B$. After that, we may substitute unity in place of the function $T(\tilde{k}_3^2/4m^2L_B^2)$, since it is the value of this bounded function in the limit $(eB/4m^2) = 1/4m^2L_B^2 = \infty$. Then Eq. (20) becomes

$$\Delta A_0(x) \simeq \frac{q}{2(2\pi)^2L_B} \int_0^\infty J_0(\tilde{k}_\perp \tilde{x}_\perp) \int_{-\infty}^\infty \left( \frac{e^{-i\tilde{k}_3 \tilde{x}_3}}{\tilde{k}_3^2 + \tilde{k}_3^2} - \frac{e^{-i\tilde{k}_3 \tilde{x}_3}}{\tilde{k}_3^2 + \tilde{k}_3^2 + 2\alpha \pi \exp \left( -\frac{k^2}{2} \right)} \right) d\tilde{k}_3 d\tilde{k}_\perp. \quad (25)$$

The $\tilde{k}_3$-integration here can be performed by calculating the residues in the poles on the imaginary axis. Finally one gets

$$\Delta A_0(x) \simeq \frac{q}{4\pi L_B} \int_0^\infty J_0(\tilde{k}_\perp \tilde{x}_\perp) \left( e^{-\tilde{k}_\perp |\tilde{x}_3|} - \frac{\tilde{k}_\perp e^{-|\tilde{x}_3|\sqrt{\tilde{k}_\perp^2 + 2\alpha}} \exp \left( -\frac{k^2}{4} \right)}{\sqrt{\tilde{k}_\perp^2 + 2\alpha \pi \exp \left( -\frac{k^2}{4} \right)}} \right) d\tilde{k}_\perp. \quad (26)$$
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This simple representation can be further simplified if $x_3$ or $x_\perp$ are large in the Larmour scale: $|\tilde{x}_3| \gg 1$, or $|\tilde{x}_\perp| \gg 1$. In this case the integration in (26) is restricted to the domain $\tilde{k}_\perp^2 \ll 1$ where the exponential $\exp(-\tilde{k}_\perp^2/2)$ should be taken as unity. Then (26) is reduced to

$$\Delta A_0(x) \simeq \frac{q}{4\pi L_B} \frac{1 - \exp\left\{-\left(\frac{2\alpha}{\pi}\right)^{\frac{3}{2}} \sqrt{\tilde{x}_\perp^2 + \tilde{x}_3^2}\right\}}{\sqrt{\tilde{x}_\perp^2 + \tilde{x}_3^2}},$$

which implies the Yukawa law for (19)

$$A_0(x) \simeq A_Y(x) = \frac{q}{4\pi L_B} \frac{\exp\left\{-\left(\frac{2\alpha}{\pi}\right)^{\frac{3}{2}} \sqrt{\tilde{x}_\perp^2 + \tilde{x}_3^2}\right\}}{\sqrt{\tilde{x}_\perp^2 + \tilde{x}_3^2}} = \frac{q}{4\pi} \frac{\exp\left\{-\left(\frac{2\alpha b}{\pi}\right)^{\frac{3}{2}} m|x|\right\}}{|x|}. \quad (28)$$

The latter fact can be established if we return to the previous representation (25), which can then be traced back to (19), (20) and finally to (17) with

$$-\kappa_2(0, \infty, 0) = \frac{2\alpha}{\pi L_B^2} = m^2 \frac{2\alpha b}{\pi} \equiv M^2 \quad (29)$$

substituted for $-\kappa_2(0, k_3^2, k_\perp^2)$ in the denominator. Here $M$ is the "effective photon mass" of Ref. [28]. The deviation of (26) from (27) when $\tilde{x}_\perp$ and $\tilde{x}_3$ are both small is not important in the sum (19) against the background of the first term $A_0^C(x)$, singular in the origin $\tilde{x}_\perp = \tilde{x}_3 = 0$. Therefore, the Yukawa law (28) is fulfilled in the vicinity of

![Figure 2. Modified Coulomb potential $A_0(0, x_\perp)$ plotted along the axis $x_\perp$ passing through the charge $q$ transversely to the magnetic field. Dashed lines correspond to three values of the magnetic field (from left to right): $B = 10^6 B_0$, $B = 10^5 B_0$ and $B = 10^4 B_0$. Solid line is the standard Coulomb law $A_0^C(0, x_\perp) = q/4\pi x_\perp$. Dashed lines approach asymptotically the solid line at the left edge of the figure and the short solid lines $A_0(0, x_\perp) = q/4\pi x_\perp$ at the right edge. The abscissa represents the distance across the magnetic field in the units $(2\ m)^{-1}$. The ordinate represents the potential in the units $qm/2\pi$.]

This simple representation can be further simplified if $x_3$ or $x_\perp$ are large in the Larmour scale: $|\tilde{x}_3| \gg 1$, or $|\tilde{x}_\perp| \gg 1$. In this case the integration in (26) is restricted to the domain $\tilde{k}_\perp^2 \ll 1$ where the exponential $\exp(-\tilde{k}_\perp^2/2)$ should be taken as unity. Then (26) is reduced to

$$\Delta A_0(x) \simeq \frac{q}{4\pi L_B} \frac{1 - \exp\left\{-\left(\frac{2\alpha}{\pi}\right)^{\frac{3}{2}} \sqrt{\tilde{x}_\perp^2 + \tilde{x}_3^2}\right\}}{\sqrt{\tilde{x}_\perp^2 + \tilde{x}_3^2}},$$

which implies the Yukawa law for (19)

$$A_0(x) \simeq A_Y(x) = \frac{q}{4\pi L_B} \frac{\exp\left\{-\left(\frac{2\alpha}{\pi}\right)^{\frac{3}{2}} \sqrt{\tilde{x}_\perp^2 + \tilde{x}_3^2}\right\}}{\sqrt{\tilde{x}_\perp^2 + \tilde{x}_3^2}} = \frac{q}{4\pi} \frac{\exp\left\{-\left(\frac{2\alpha b}{\pi}\right)^{\frac{3}{2}} m|x|\right\}}{|x|}. \quad (28)$$

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Figure 3. Ratio of the modified Coulomb potential \( A_0(0,x_\perp)/A_0^C(0,x_\perp) \) drawn as a solid line in function of the transverse coordinate \( x_\perp \) for the magnetic field value \( B = 10^4 B_0 \). The dashed line is the constant \( (x_\perp/x'_\perp) = (1 + \alpha B/3\pi B_0)^{-1/2} = 0.338 \) from the law (35). The dotted line is the Yukawa interpolation. The abscissa is the same as in Fig. 2.

the charge. By comparing (27) with the second term of the expansion (21) we deduce that for strong fields

\[
C \sim \frac{M}{2m} = \sqrt{\frac{\alpha b}{2\pi}}
\]

asymptotically. For the four values of the external field \( b = 10^4, 10^5, 10^6, 10^{10} \) the values of \( C \) calculated following (31) are: 3.41, 10.78, 34.01 and 34.08 \( \times 10^2 \). We face the coincidence with its values listed in previous Subsection, the better the higher the field.

The Yukawa law (28) establishes the short-range character of the static electromagnetic forces in the Larmour scale. To avoid a misunderstanding, stress that the genuine photon mass understood as its rest energy is always exactly equal to zero as a consequence of the gauge invariance reflected in the approximation-independent relation \( \kappa_\alpha(0,0,0) = 0 \). Correspondingly, the potential, produced by a static charge, should be long-range for sufficiently large distances. This is the case, indeed. The point is that the scaling regime (26), as it follows from its derivation, is valid in the limit \( eB/4m^2 = \infty \). In this limit the ratio of the Compton length \( (2m)^{-1} \) to the Larmour length \( L_B = (eB)^{-1/2} \) is infinite. Therefore, Eqs. (26), (28) cannot be extended to the distances of Compton scale and larger, where the long-range character of the electromagnetic interaction is restored, as we shall see in the next Section.
4. Large-distance behavior

4.1. Large \(x_\perp\) in Larmor scale

For large transverse distances the term linearly growing with the magnetic field leads to suppression of the static potential in the transverse direction.

To be more precise, consider the region

\[ x_\perp \gg \frac{m^{-1}}{\sqrt{2b}} = \frac{L_B}{\sqrt{2}} \]  

(31)

Once the Bessel function \(J_0\) in oscillates and decreases for large values of its argument \(k_\perp x_\perp\), the main contribution into the integral over \(k_\perp^2\) in comes from the integration variable domain \(k_\perp^2 \ll 2m^2b\), and the dependence upon \(k_\perp^2\) in \(\kappa_2\) may thus be disregarded. Then the \(k_\perp^2\)-integration in can be explicitly performed to give

\[ A_0(x_3, x_\perp) \simeq \frac{2q}{(2\pi)^2} \int_0^\infty \mathcal{K}_0 \left( x_\perp \sqrt{k_3^2 - \kappa_2(0, k_3^2, 0)} \right) \cos(k_3 x_3)dk_3, \]  

(32)

where \(\mathcal{K}_0\) is the McDonald function of order zero, and

\[ \kappa_2(0, k_3^2, 0) = -\frac{2\alpha b}{\pi} m^2 T \left( \frac{k_3^2}{4m^2} \right). \]  

(33)

As the McDonald function \(\mathcal{K}_0\) decreases exponentially when its argument increases, only small values of the square root contribute into integral, and the \(k_3\)-integration domain in it is restricted to the interval \(k_3^2 \ll 4m^2\), wherein

\[ T \left( \frac{k_3^2}{4m^2} \right) \simeq \frac{k_3^2}{6m^2}. \]  

(34)

Then the potential form becomes (we use Eq. 6.671.14 of the reference book [29])

\[ A_0(x_3, x_\perp) \simeq \frac{2q}{(2\pi)^2} \int_0^\infty \mathcal{K}_0 \left( x_\perp \sqrt{1 + \frac{\alpha b}{3\pi}} \right) \cos(k_3 x_3)dk_3 = \]

\[ = \frac{1}{4\pi} \frac{q}{\sqrt{(x'_\perp)^2 + x_3^2}}, \quad x'_\perp = x_\perp \left( 1 + \frac{\alpha b}{3\pi} \right)^{1/2}, \quad x'_\perp > x_\perp. \]  

(35)

Eq. (35) is an anisotropic Coulomb law, according to which the attraction force decreases with distance from the source along the transverse direction faster than along the magnetic field (remind that \(b = (B/B_0) \gg 1\)), but remains long-range. The equipotential surface is an ellipsoid stretched along the magnetic field. The electric field of the charge \(\mathbf{E} = -\nabla A_0(x_3, x_\perp)\) is a vector with the components

\[ \frac{q}{2\pi}(x_3^2 + \beta^2 x_\perp^2)^{-3/2}(x_3, \beta^2 x_\perp), \]  

where \(\beta = (1 + \alpha b/3\pi)^{1/2}\). It is not directed towards the charge, but makes an angle \(\phi\) with the radius-vector \(\mathbf{r}\),

\[ \cos \phi = (\frac{x_3^2 + \beta^2 x_\perp^2}{x_3^2 + x_\perp^2} - \frac{1}{2})^{-1/2}. \]

In the limit of infinite magnetic field, \(\beta = \infty\), the electric field of the point charge is directed normally to the axis \(x_3\). This regime corresponds to the dielectric permeability of the vacuum independent of the frequency, with its dependence on \(k\) (spatial dispersion) being reduced solely to that upon the angle in the space (cf, [22]).
The result (35) is in agreement with the curves in Figs.2, 3 drawn for \( x_3 = 0 \). The curves in Fig.2 approach the standard Coulomb law when \( x_\perp \to 0 \) in accordance with \( 21 \), as explained above, rather sharply fall down, following the Yukawa law \( 28 \) in the Larmour range \( x_\perp \sim L_B \) to reach the asymptotic long-range regime \( A_0(0, x_\perp) \approx q/4\pi x_\perp^3 \) for larger \( x_\perp \) in the region \( 31 \).

4.2. Large \( x_3 \)

It remains to consider the remote coordinate region of large \( x_3 \), complementary to \( 31 \).

To this end we apply the residue method to the inner integral over \( k_3 \) in \( 17 \). Using the integral representation \( 16 \) the function \( \kappa_2 \) \( 15 \) may be, for a fixed positive value of \( k_\perp^2 \), analytically continued from the real values of the variable \( k_3 \) into the whole complex plane of it, cut along two fragments of the imaginary axis. In the lower half-plane the cut runs from \( \text{Im} \, k_3 = -2m \) down to \( \text{Im} \, k_3 = -\infty \), while in the upper half-plane it extends within the limits \( 2m \leq \text{Im} \, k_3 \leq \infty \). Other singularities of the \( k_3 \)-integrand in \( 17 \) are poles yielded by zeros of the denominator, i.e., solutions of the equation (associated with the photon dispersion equation)

\[
k_3^2 + \frac{k_\perp^2}{k_3^2} - \kappa_2(0, k_3^2, k_\perp^2) = 0.
\]

As \( k_\perp \) varies within the limits \( (0, \infty) \) two roots of this equation \( k_3^{\pm} = \pm iK(k_\perp) \) move along the imaginary axis from the point \( K(0) = 0 \) to the points \( k_3^{\pm} = \pm i2m \) \( 14 \), \( 13 \), \( 22 \). There is yet another branch of the solution to equation \( 36 \), corresponding to the photon absorption via the \( \gamma \to e^+e^- \)-decay, but the corresponding poles lie in the nonphysical sheet of the described complex plane, behind the cuts, and will not be of importance for the consideration below.

Let us consider positive values of \( x_3 \). Negative values can be handled in an analogous way. Turning the positive part of integration path \( 0 \leq k_3 \leq \infty \) clockwise to the lower half-plane by the angle \( \pi/2 \), and the negative part \( -\infty \leq k_3 \leq 0 \) counterclockwise by the same angle, and referring to the fact that the exponential \( \exp(-ik_3x_3) \) decreases, for \( x_3 > 0 \), in the lower half-plane of \( k_3 \) as \( |k_3| \to \infty \) so that the integrals over the remote arcs may be omitted, we get a representation for the inner integral in \( 17 \)

\[
\int_{-\infty}^{\infty} \frac{e^{-ik_3x_3} \, dk_3}{k_3^2 + k_3^2 - \kappa_2(0, k_3^2, k_\perp^2)} = i \int_{2m}^{\infty} e^{-|k_3|x_3} \Delta(|k_3|^2, k_\perp^2) \, dk_3 - 2\pi e^{-(K(k_\perp^2)x_3)} \text{Res}(k_3^2),
\]

where \( \text{Res}(k_3^2) \) designates the residue of the expression \( D_2(0, -|k_3|^2, k_\perp^2) = (k_\perp^2 - |k_3|^2 - \kappa_2(0, -|k_3|^2, k_\perp^2))^{-1} \) in the pole \( k_3^{-} = -iK(k_\perp^2) \), while \( \Delta(|k_3|^2, k_\perp^2) = D_2(0, -|k_3|^2 + i0, k_\perp^2) - D_2(0, -|k_3|^2 - i0, k_\perp^2) \) is the cut discontinuity. It was explained above that \( 0 < K(k_\perp^2) < 2m \) everywhere but in the limit \( k_\perp \to \infty \), where \( K = 2m \). Consequently the residue term in \( 37 \) dominates over the cut-discontinuity term everywhere in the \( k_3 \)-integration domain in the outer integral in \( 17 \), except for the region near \( k_\perp = \infty \). In this limit, however, \( \kappa_2 \) disappears due to the exponential in
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Eq. (16), together with the cut discontinuity, since the latter is only due to the branching points in the function (16). Therefore, keeping the residue term in (37) as the leading one, we neglect the contribution that decreases with large longitudinal distance at least as fast as $\exp(-2m|x_3|)$. In this way we come to the asymptotic representation of the potential (17) in the region of large longitudinal distances $|x_3| \gg (2m)^{-1}$ (the negative $x_3$ at this step are also included - to treat them the fragments of integration path were rotated in the directions opposite to the above)

$$A_0(x) \simeq \frac{q}{8\pi} \int_0^\infty \frac{J_0(k_\perp x_\perp)}{K(k_\perp^2)(1 + H(-K^2(k_\perp^2), k_\perp^2))} e^{-(K(k_\perp^2)x_3)} dk_\perp^2,$$  \hspace{1cm} (38)

where

$$H(k_3^2, k_\perp^2) = \frac{2\alpha b m^2}{\pi} \exp(-\frac{k_\perp^2}{2m^2b}) \frac{d}{dk_3^2} T\left(\frac{k_3^2}{4m^2}\right).$$  \hspace{1cm} (39)

Here $K^2(k_\perp^2)$ is the solution of equation (36) in the negative region of the variable $k_3^2$ - see [22] for its form. $K(\infty) = 2m$, $K(0) = 0$. $T$ is given by (16).

Due to the exponential factor in the integrand of (38), for large $x_3$ the main contribution comes from the integration region of $k_\perp$ that provides minimum to the function $K(k_\perp)$. The minimum value of $K(k_\perp)$ is zero. It is achieved in the point $k_\perp = 0$ - a manifestation of the fact that the photon mass defined as its rest energy is strictly equal to zero owing to the gauge invariance: $\kappa_a(k_0 = k_3 = k = 0) = 0$. In view of (33) and (34), near the point $k_\perp = 0$ the dispersion equation (36) has the solution $K(k_\perp) = k_\perp/\sqrt{1 + \alpha b/3\pi}$. Simultaneously, near the minimum point $1 + H(0, 0) = 1 + \alpha b/3\pi$. With these substitutions and the use of 6.661.1 of [29], Eq. (38) becomes again the anisotropic Coulomb law (q/4\pi)(x_\perp^2 + x_3^2)^{1/2}. We have, therefore, established its validity everywhere in the region remote from the center, irrespectively of the direction. In agreement with this result the curves in Figure 1 for $A_0(x_3, 0)$ are approaching the Coulomb law q/4\pi|x_3| as $x_3$ grows. To see how fast they are doing this, set $x_\perp = 0$ in the difference (20). By replacing the exponential in the expression for $\kappa_2$ (15) by unity we overestimate $|\Delta A_0(x_3, 0)|$. Then the integration over $k_\perp^2$ can be explicitly performed to yield the inequality

$$|\Delta A_0(x_3, 0)| < \left| \frac{q}{8\pi^2} \int_{-\infty}^{\infty} e^{-ix_3k_3} \ln \left(1 + \frac{2\alpha b m^2}{\pi k_3^2} T\left(\frac{k_3^2}{4m^2}\right)\right) dk_3\right|. \hspace{1cm} (40)$$

The only singularities of the integrand of (40) as a function of the complex variable $k_3$ are (inverse square root) branchings on the imaginary axis in the points $\pm i2m$ owing to the analytical properties of the function $T(y)$ (16). Note that the argument of the logarithm in (40) may disappear only on the second sheet of the complex plane. By turning the integration path to the lower half-plane, when $x_3$ is positive, or to the upper half-plane otherwise, we reduce (40) to an integral of the cut discontinuity multiplied by $\exp(x_3\text{Im}k_3)$ along the imaginary axis of $k_3$, the minimum integration value of $|\text{Im}k_3|$ being $2m$. Therefore the difference (20) between the potential $A_0(x_3, 0)$ and its large-$x_3$ asymptote $q/4\pi|x_3|$ decreases in Figure 1 at least as fast as $\exp(-2m|x_3|)$.

Eq. (38) was used for computer calculation with the large value $x_3 = 10m^{-1}$. It has lead to the curve shown in Figs 4, 5. In the region (31) it agrees with the result (35),
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Figure 4. Modified Coulomb potential \( A_0(2\pi', q m) \) plotted against the transverse coordinate \( x_\perp \) with the longitudinal coordinate fixed at the large value \( x_3 = 10(2m)^{-1} \). The dashed line corresponds to the magnetic field value \( B = 10^4 B_0 \). Solid line is the standard Coulomb law \( A_0^C(x_3, x_\perp) = qm/2\pi((2mx_\perp)^2 + 100)^{-1/2} \). The dashed line is indistinguishable from the anisotropic Coulomb law \( A_0(x_3, x_\perp) = \) in the scale of the drawing. The coordinate axes are the same as in Fig. 2.

Figure 5. The same as Fig. 4 but viewed at a detailed scale near \( x_\perp = 0 \). The thin solid line is the anisotropic Coulomb law \( qm/2\pi((0.338 \cdot 2mx_\perp)^2 + 100)^{-1/2} \) valid in that region \( (L_B = 0.02(2m)^{-1} \) for \( b = 10^4 \)). In practice and (35) are the same.
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5. Conclusion

The modification of the Coulomb law should affect, first of all, the field of an atomic nucleus, placed in a magnetic field. Properties of matter (including individual atoms) at the surface of strongly magnetized neutron stars and various physical processes (such as radiation of particles) where the electric field of particles is important (for a review on physics of strongly magnetized neutron stars, see [30]) may become sensitive to the present modification of the Coulomb law.

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