Dedicated to Professor E.B. Davies on the occasion of his 60th birthday

Abstract

We prove Rellich and improved Rellich inequalities that involve the distance function from a hypersurface of codimension $k$, under a certain geometric assumption. In case the distance is taken from the boundary, that assumption is the convexity of the domain. We also discuss the best constant of these inequalities.

**AMS Subject Classification:** 35J20 (35P20, 35P99, 26D10, 47A75)

**Keywords:** Rellich inequality, best constants, distance function

1 Introduction

The classical Rellich inequality states that for $p > 1$

$$\int_0^\infty |u''|^p dt \geq \left( \frac{p-1}{p} \right)^p \int_0^\infty \frac{|u|^p}{t^2p} dt,$$

for all $u \in C_c^\infty((0, \infty))$. A multi-dimensional version of (1.1) for $p = 2$ is also classical and states that for any $\Omega \subset \mathbb{R}^N$, $N \geq 5$, there holds

$$\int_{\Omega} (\Delta u)^2 dx \geq \frac{N^2(N-4)^2}{16} \int_{\Omega} \frac{u^2}{|x|^4} dx,$$

for all $u \in C_c^\infty(\Omega)$.

Davies and Hinz [DH] generalized (1.2) and showed that for any $p \in (1, N/2)$ there holds

$$\int_{\Omega} |\Delta u|^p dx \geq \left( \frac{(p-1)N|N-2p|}{p^2} \right)^p \int_{\Omega} \frac{|u|^p}{|x|^{2p}} dx, \quad u \in C_c^\infty(\Omega \setminus \{0\}).$$

---

"Department of Mathematics, University of Ioannina, 45110 Ioannina, Greece

†Department of Mathematics, University of Crete, 71409 Heraklion, Greece and Institute of Applied and Computational Mathematics, FORTH, 71110 Heraklion, Greece"
Inequality (1.1) has also been generalized to higher dimensions in another direction, where the singularity involves the distance \( d(x) = \text{dist}(x, \partial \Omega) \). Owen [O] proved among other results that if \( \Omega \) is bounded and convex then

\[
\int_\Omega (\Delta u)^2 \, dx \geq \frac{9}{16} \int_\Omega \frac{u^2}{d(x)^4} \, dx, \quad u \in C^\infty_c(\Omega).
\]  

(1.4)

Recently, an improved version of (1.2) has been established in [TZ]. Among several other results they showed that for a bounded domain \( \Omega \) in \( \mathbb{R}^N \), \( N \geq 5 \), there holds

\[
\int_\Omega (\Delta u)^2 \, dx \geq \frac{N^2(N - 4)^2}{16} \int_\Omega \frac{u^2}{|x|^4} \, dx + \left(1 + \frac{N(N - 4)}{8}\right) \sum_{i=1}^{\infty} \int_\Omega \frac{u^2}{|x|^4} X_i^2 X_{i+1}^2 \ldots X_N^2 \, dx,
\]  

as well as

\[
\int_\Omega (\Delta u)^2 \, dx \geq \frac{N^2}{4} \int_\Omega \frac{|\nabla u|^2}{|x|^2} \, dx + \frac{1}{4} \sum_{i=1}^{\infty} \int_\Omega \frac{|\nabla u|^2}{|x|^2} X_i^2 X_{i+1}^2 \ldots X_N^2 \, dx,
\]  

for all \( u \in C^\infty_c(\Omega \setminus \{0\}) \). Here \( X_k \) are iterated logarithmic functions; see (1.6) for precise the definition.

Rellich inequalities have various applications in the study of fourth-order elliptic and parabolic PDE’s; see e.g. [DH, O, B]. Improved Rellich inequalities are useful if critical potentials are additionally present. As a simplest example, one obtains information on the existence of solution and asymptotic behavior for the equation \( u_t = -\Delta^2 + V \) for critical potentials \( V \). Corresponding problems for improved Hardy’s inequalities have recently attracted considerable attention: see [BV, BM, BFT1] and references therein.

Our aim in this paper is to obtain sharp improved versions of inequalities (1.3) and (1.4), where additional non-negative terms are present in the respective right-hand sides. At the same time we obtain some new improved Rellich inequalities which are new even at the level of plain Rellich inequalities; these involve the distance to a surface \( K \) of intermediate codimension.

**Statement of results**

Before stating our theorems let us first introduce some notation. We denote by \( \Omega \) a domain in \( \mathbb{R}^N \), \( N \geq 2 \). For the sake of simplicity all functions considered below are assumed to be real-valued; in relation to this we note however that minor modifications of the proofs or a suitable application of [D, Lemma 7.5] can yield the validity of Theorems 1-3 below for complex-valued functions \( u \). We let \( K \) be a closed, piecewise smooth surface of codimension \( k \), \( k = 1, \ldots, N \). We do not assume that \( K \) is connected but only that it has finitely many connected components. In the case \( k = N \) we assume that \( K \) is a finite union of points while in the case \( k = 1 \) we assume that \( K = \partial \Omega \). We then set

\[
d(x) = \text{dist}(x, K),
\]

and assume that \( d(x) \) is bounded in \( \Omega \).

We define recursively

\[
X_1(t) = (1 - \log t)^{-1}, \quad t \in (0, 1],
\]

\[
X_i(t) = X_1(X_{i-1}(t)), \quad i = 2, 3, \ldots, t \in (0, 1].
\]  

(1.6)
These are iterated logarithmic functions that vanish at an increasingly slow rate at \( t = 0 \) and satisfy \( X_t(1) = 1 \). Given an integer \( m \geq 1 \) we define
\[
\eta_m(t) = \sum_{i=1}^{m} X_1(t) \cdots X_i(t), \quad \zeta_m(t) = \sum_{i=1}^{m} X_i^2(t) \cdots X_i^2(t). \quad (1.7)
\]

We note that \( \lim_{t \to 0} \eta_m(t) = \lim_{t \to 0} \zeta_m(t) = 0 \). Now, it has been shown \([\text{BFT2}]\) that both series in (1.7) converge for any \( t \in (0,1) \). This allows us to also introduce the functions \( \eta_\infty \) and \( \zeta_\infty \) as the infinite series.

We fix a parameter \( s \in \mathbb{R} \) and we assume that the following inequality holds in the distributional sense:
\[
p \neq k + s, \quad (k + s - p)(d\Delta d - k + 1) \geq 0 \quad \text{in} \ \Omega \setminus K.
\]

For a detailed discussion of this condition we refer to \([\text{BFT1}]\). Here we simply note that it is satisfied in the following two important cases: (i) it is satisfied as an equality if \( k = N \) and \( K \) consists of single point and (ii) it is also satisfied if \( K = \partial \Omega \) (so \( k = 1 \), \( s + 1 - p < 0 \) and \( \Omega \) is convex.

Our first theorem involves the functions \( \eta_m = \eta_m(d(x)/D) \) and \( \zeta_m = \zeta_m(d(x)/D) \), \( x \in \Omega \), for a large enough parameter \( D > 0 \). In any case \( D \) will be large enough so that the quantity \( 1 + \alpha \eta_m + \beta \eta_m^2 + \gamma \zeta_m \) is positive in \( \Omega \). We also set
\[
H = \frac{k + s - p}{p}. \quad (1.8)
\]

**Theorem 1 (weighted improved Hardy inequality)** Let \( p > 1 \) and \( m \in \mathbb{N} \cup \{\infty\} \).

Let \( \Omega \) be a domain in \( \mathbb{R}^N \) and \( K \) a piecewise smooth surface of codimension \( k \), \( k = 1, \ldots, N \). Suppose that \( p \neq k + s \), that \( \sup_{x \in \Omega} d(x) < \infty \) and that
\[
(k + s - p)(d\Delta d - k + 1) \geq 0 \quad \text{in} \ \Omega \setminus K. \quad (1.9)
\]

Also, let \( \alpha, \beta, \gamma \in \mathbb{R} \) be fixed. Then there exists a positive constant \( D_0 \geq \sup_{x \in \Omega} d(x) \) such that for any \( D \geq D_0 \) and all \( u \in C_c^\infty(\Omega \setminus K) \) there holds
\[
\int_\Omega d^s(1 + \alpha \eta_m + \beta \eta_m^2 + \gamma \zeta_m)|\nabla u|^p dx \geq |H|^p \int_\Omega d^{s-p}|u|^p dx + \\
+ |H|^p \alpha \int_\Omega d^{s-p}\eta_m|u|^p dx + \left( |H|^p \beta + \frac{|H|^p H^2 H\alpha}{2} \right) \int_\Omega d^{s-p}\eta_m^2 |u|^p dx \\
+ \left( \frac{p - 1}{2p} |H|^{p-2} + \frac{|H|^p H^2 H\alpha}{2} + |H|^p \gamma \right) \int_\Omega d^{s-p}\zeta_m |u|^p dx,
\]

where \( \eta_m = \eta_m(d(x)/D) = \sum_{i=1}^{m} X_1(d(x)/D) \cdots X_i(d(x)/D) \) and \( \zeta_m = \zeta_m(d(x)/D) = \sum_{i=1}^{m} X_i^2(d(x)/D) \cdots X_i^2(d(x)/D) \).

We note that the special case \( s = \alpha = \beta = \gamma = 0 \) has been proved in \([\text{BFT2}]\).

To state our next theorem we define the constant
\[
Q = \frac{(p - 1)k(k - 2p)}{p^2}. \quad (1.10)
\]
Theorem 2 (improved Rellich inequality I) Let \( p > 1 \). Let \( \Omega \) be a domain in \( \mathbb{R}^N \) and \( K \) a piecewise smooth surface of codimension \( k \), \( k = 1, \ldots, N \). Suppose that 
\[
\sup_{x \in \Omega} d(x) < \infty. \quad \text{Suppose also that } k > 2p \text{ and that }
\]
d\Delta d - k + 1 \geq 0 \quad \text{in } \Omega \setminus K
in the distributional sense. Then there exists a positive constant \( D_0 \geq \sup_{x \in \Omega} d(x) \) such that for any \( D \geq D_0 \) and all \( u \in C^\infty_c(\Omega \setminus K) \) there holds
\[
\int_\Omega |\Delta u|^p dx \geq Q^p \int_\Omega \frac{|u|^p}{d^{2p}} dx + \frac{p-1}{2p^2} |Q|^{p-2} \left\{ k^2 (p-1)^2 + (k-2p)^2 \right\} \sum_{i=1}^\infty \int_\Omega \frac{|u|^p}{d^{2p}} X_i X_2 \ldots X_i dx,
\]
where \( X_j = X_j(d(x)/D) \).

It is remarkable that the geometric assumption of this Theorem 2 only involves \( \Delta d \), as in the case of Theorem 1, and not higher-order derivatives of \( d \) as one might expect.

The above theorem does not cover the important case \( k = 1 \) which corresponds to \( d(x) = \text{dist}(x, \partial \Omega) \). This is done in the following theorem for the case \( p = 2 \).

Theorem 3 (improved Rellich inequality II) Let \( \Omega \) be convex and such that \( d(x) := \text{dist}(x, \partial \Omega) \) is bounded in \( \Omega \). Then there exists a positive constant \( D_0 \geq \sup_{x \in \Omega} d(x) \) such that for any \( D \geq D_0 \) and all \( u \in C^\infty_c(\Omega) \) there holds
\[
\begin{align*}
(i) \quad \int_\Omega (\Delta u)^2 dx & \geq \frac{1}{4} \int_\Omega \frac{\nabla u^2}{d^2} dx + \frac{1}{4} \sum_{i=1}^\infty \int_\Omega \frac{\nabla u_i^2}{d^2} X_i X_2 \ldots X_i dx, \\
(ii) \quad \int_\Omega (\Delta u)^2 dx & \geq \frac{9}{16} \int_\Omega \frac{u^2}{d^2} dx + \frac{5}{8} \sum_{i=1}^\infty \int_\Omega \frac{u_i^2}{d^2} X_i X_2 \ldots X_i dx,
\end{align*}
\]
where \( X_j = X_j(d(x)/D) \).

In our last theorem we prove the optimality of the constants appearing in Theorems 2 and 3 above. In a similar manner one can prove the optimality of the constants in Theorem 1; we omit the proof since it follows very closely the proof of [BF12, Proposition 3.1]. Anyway, we note that in some particular cases the optimality of Theorem 1 follows indirectly from the optimality of Theorems 2 and 3, which we do prove. In relation to Theorem 4 see also the remark at the end of the paper.

We define
\[
J_0[u] = \int_\Omega |\Delta u|^p dx - |Q|^p \int_\Omega \frac{|u|^p}{d^{2p}} dx
\]
and for \( m \in \mathbb{N} \),
\[
J_m[u] = \int_\Omega |\Delta u|^p dx - |Q|^p \int_\Omega \frac{|u|^p}{d^{2p}} dx - \frac{p-1}{2p^3} |Q|^{p-2} \left\{ k^2 (p-1)^2 + (k-2p)^2 \right\} \sum_{i=1}^m \int_\Omega \frac{|u|^p}{d^{2p}} X_i X_2 \ldots X_i dx.
\]

Our next theorem reads:
In this section we give the proof of Theorem 1. We note that in the special case 2 Series expansion for weighted Hardy inequality
The sharpness of Theorem 3 (i) follows implicitly from the sharpness of 3 (ii).

It follows in particular that all constants in Theorem 2 and Theorem 3 (ii) are sharp.

We recall that Young’s inequality to obtain

\[ \int_{\Omega} \nabla \psi |\nabla u|^p dx \leq \int_{\Omega} |\nabla \psi| |\nabla u|^{p-1} |\nabla u| dx. \]

We set for simplicity \( \psi = (1 + \alpha \eta_m + \beta \eta_m^2 + \gamma \zeta_m) \). If \( T \) is a vector field in \( \Omega \), then, for any \( u \in C_c^\infty(\Omega \setminus K) \) we first integrate by parts and then use Young’s inequality to obtain

\[ \int_{\Omega} \text{div} T |u|^p dx \leq p \int_{\Omega} |T||\nabla u||u|^{p-1} dx \]

\[ \leq \int_{\Omega} d^s \psi |\nabla u|^p dx + (p-1) \int_{\Omega} d^{s-1} |T|^{\frac{p}{p-1}} |\nabla \psi|^{\frac{p}{p-1}} |\nabla u|^p dx, \]

and thus conclude that

\[ \int_{\Omega} d^s |\nabla u|^p dx \geq \int_{\Omega} (\text{div} T - (p-1)d^{s-1} |T|^{\frac{p}{p-1}} |\nabla \psi|^{\frac{p}{p-1}}) |\nabla u|^p dx. \] (2.2)

We recall that \( H = (k + s - p)/p \) and define

\[ T(x) = H|H|^{p-2} d^s + p(x) \nabla d(x) \left( 1 + \frac{p-1}{pH} \right) \eta_m (d(x)/D) + B\eta_m^2 (d(x)/D) \]

where \( D \geq \sup_{\Omega} d(x) \) and \( B \in \mathbb{R} \) is a free parameter to be chosen later. In any case, once \( B \) is chosen, \( D \) will be large enough so that the quantity \( 1 + \frac{p-1}{pH} \eta_m (d(x)/D) + B\eta_m^2 (d(x)/D) \) is positive on \( \Omega \). Note that \( T \) is singular on \( K \), but since \( u \in C_c^\infty(\Omega \setminus K) \) all previous calculations are legitimate. In view of (2.2), to prove the theorem it is enough
to show that there exists \( D_0 \geq \sup_\Omega d(x) \) such that for \( D \geq D_0 \)

\[
\text{div} T - (p - 1) \frac{\partial}{\partial r} \left| T \right|^{\frac{p}{p - 1}} \psi^{-\frac{1}{p - 1}} - d^{\delta - p} \left( |H|^p + |H|^p \alpha \eta_m \right) + \left( |H|^p \beta + \frac{|H|^p - H\alpha}{2} \right) \eta_m^2 + \left( \frac{p - 1}{2p} |H|^{p - 2} + \frac{|H|^p - H\alpha}{2} + |H|^p \gamma \right) \zeta_m \geq 0
\]

(2.3)

for all \( x \in \Omega \).

To compute \( \text{div} T \) we shall need to differentiate \( \eta_m(d/D) \). For this we note that easily implies

\[
\eta_m'(t) = \frac{1}{t} \left( X_1^2 + (X_1^2 X_2 + X_1^2 X_2^3) + \ldots + (X_1^2 X_2 \ldots X_m + \ldots + X_1^2 \ldots X_m) \right),
\]

from which follows that

\[
t \eta_m'(t) = \frac{1}{2} \zeta_m(t) + \frac{1}{2} \eta_m^2(t).
\]

(2.4)

We also define \( \theta_m \) on \((0, 1)\) by

\[
\zeta_m(t) = \frac{\theta_m(t)}{t},
\]

and, for simplicity, we set \( A = \alpha + (p - 1)/(pH) \) so that \( T = |H|^{p - 2} H d^{s + 1 - p} \nabla d (1 + A\eta_m + B\eta_m^2) \). We think of \( \eta_m \) as an independent variable, which we may assume to be small by taking \( D \) large enough. Simple computations together with assumption \((1.9)\) and the fact that \(|\nabla d| = 1\) give

\[
\text{div} T = d^{\delta - p} \left( |p| H |^p + p |H|^p A \eta_m + (p |H|^p B + |H|^{p - 2} H\frac{A}{2} \eta_m^2 + |H|^{p - 2} H \frac{A}{2} \zeta_m + + |H|^{p - 2} H B \left( \frac{3}{4} \eta_m + \eta_m \zeta_m \right) \right) \geq d^{\delta - p} \left( |p| H |^p + p |H|^p A \eta_m + (p |H|^p B + |H|^{p - 2} H\frac{A}{2} \eta_m^2 + |H|^{p - 2} H \frac{A}{2} \zeta_m + + |H|^{p - 2} H B \left( \frac{3}{4} \eta_m + \eta_m \zeta_m \right) \right)
\]

(2.5)

Moreover, since \(|\nabla d| = 1\), Taylor’s expansion gives

\[
|T|^{\frac{p}{p - 1}} = |H|^p d^{\frac{(s + 1 - p)}{p - 1}} \left( 1 + A\eta_m + B\eta_m^2 \right)^{\frac{p}{p - 1}}
\]

\[
= |H|^p d^{\frac{(s + 1 - p)}{p - 1}} \left\{ 1 + \frac{pA}{p - 1} \eta_m + \left( \frac{pB}{p - 1} + \frac{pA^2}{2(p - 1)^2} \right) \eta_m^2 + \left( \frac{p - 1}{6(p - 1)^2} - \frac{p(p - 2)A^3}{6(p - 1)^3} \right) \eta_m^3 + O(\eta_m^4) \right\}
\]

(2.7)
and also
\[
\psi^{-\frac{1}{p-1}} = 1 - \frac{\alpha}{p-1} \eta_m + \left( -\frac{\beta}{p-1} + \frac{p\alpha^2}{2(p-1)^2} \right) \eta_m^2 - \frac{\gamma}{p-1} \zeta_m + \frac{p\alpha^2}{2(p-1)^2} \eta_m^3 + \frac{p\alpha\gamma}{(p-1)^2} \eta_m \zeta_m + O(\eta_m^4). \tag{2.8}
\]

Using (2.6), (2.7) and (2.8) we see that the LHS of (2.8) is greater than or equal to \(d^{s-p}\) times a linear combination of powers of \(\eta_m, \zeta_m\) and \(\theta_m\) plus \(O(\eta_m^4)\). Recalling that \(A = \alpha + (p-1)/(pH)\), we easily see that the constant term and the coefficients of \(\eta_m, \eta_m^2\) and \(\zeta_m\) vanish, independently of the choice of the parameter \(B\). The remaining two coefficients, that is the coefficients of \(\eta_m^3\) and \(\eta_m \zeta_m\) are, respectively,
\[
\frac{(p-1)\alpha}{2pH^2} + \frac{\beta}{H} + \frac{(p-2)(p-1)}{6p^2H^3}, \quad B + \gamma.
\]

Since \(\zeta_m \leq \eta_m^2 \leq m \zeta_m\), we conclude that taking \(B\) to be large and positive (if \(H > 0\)) or large and negative (if \(H < 0\)), inequality (2.8) is satisfied provided \(\eta_m\) is small enough, which amounts to \(D\) being large enough. This completes the proof of the theorem. //

In the proof of Theorem 2, we are going to use the last theorem in the following special case which corresponds to taking \(p = 2\) and \(s = -2q + 2\):

**Special case.** Assume that \(k \neq 2q\) and that \((k-2q)(d\Delta d - k+1) \geq 0\) on \(\Omega \setminus K\). Then for \(D\) large enough there holds
\[
\int_{\Omega} d^{-2q+2}(1 + \alpha \eta_m + \beta \eta_m^2 + \gamma \zeta_m) |\nabla u|^2 dx \geq \frac{(k-2q)^2}{4} \int_{\Omega} d^{-2q} u^2 dx + \int_{\Omega} \frac{(k-2q)^2 \alpha}{4} d^{-2q} \eta_m u^2 dx + \left( \frac{(k-2q)^2 \beta}{4} + \frac{(k-2q) \alpha}{4} \right) \int_{\Omega} d^{-2q} \eta_m^2 u^2 dx + \left( \frac{k-2q}4 \gamma \right) \int_{\Omega} d^{-2q} \zeta_m u^2 dx \tag{2.9}
\]
for all \(u \in C^\infty_c(\Omega \setminus K)\).

## 3 The improved Rellich inequality

In this section we are going to prove Theorems 2 and 3 as well as the corresponding optimality theorem. We begin with the following lemma where, we note, \(\Delta \phi\) is to be understood in the distributional sense.

**Lemma 5** For any locally bounded function \(\phi\) with \(|\nabla u| \in L^p_{\text{loc}}(\Omega \setminus K)\) we have
\[
\int_{\Omega} |\Delta u|^p dx \geq p(p-1) \int_{\Omega} \phi |u|^{p-2} |\nabla u|^2 dx - \int_{\Omega} \left( \Delta \phi + (p-1) |\phi|^{\frac{p-1}{p-2}} \right) |u|^p dx, \quad (3.10)
\]
for all \(u \in C^\infty_c(\Omega \setminus K)\).

**Proof.** Given \(u \in C^\infty_c(\Omega \setminus K)\) we have
\[
- \int_{\Omega} \Delta \phi |u|^p dx = p \int_{\Omega} \nabla \phi \cdot (|u|^{p-2} u \nabla u) dx
\]
\[
\begin{align*}
&= -p \int_{\Omega} \phi |u|^{p-2} u \Delta u \, dx - p(p-1) \int_{\Omega} \phi |u|^{p-2} |\nabla u|^2 \, dx \\
&\leq p \left( \frac{p-1}{p} \int_{\Omega} \phi |u|^p \, dx + \frac{1}{p} \int_{\Omega} |\Delta u|^p \, dx \right) - p(p-1) \int_{\Omega} \phi |u|^{p-2} |\nabla u|^2 \, dx.
\end{align*}
\]

which is (3.10).

**Proof of Theorem 2.** Let \( m \in \mathbb{N} \) be fixed and let \( \eta_m \) and \( \zeta_m \) be as in (1.7). We apply (3.10) with \( \phi(x) = \lambda d(x)^{-2p+2}(1 + \alpha \eta_m + \beta \eta_m^2) \), \( \lambda > 0 \), where, as always, \( \eta_m = \eta_m(d(x)/D) \) and \( D \) is yet to be determined. We thus obtain

\[ \int_{\Omega} |\Delta u|^p \, dx \geq T_1 + T_2 + T_3 \tag{3.11} \]

where

\[ T_1 = p(p-1) \int_{\Omega} \phi |u|^{p-2} |\nabla u|^2 \, dx, \]
\[ T_2 = - \int_{\Omega} \Delta \phi |u|^p \, dx, \]
\[ T_3 = -(p-1) \int_{\Omega} \phi |u|^{p-2} |\nabla u|^2 \, dx. \]

To estimate \( T_1 \) we set \( v = |u|^{p/2} \) and apply (2.9) for \( q = p \),

\[ T_1 = \frac{4(p-1)}{p} \int_{\Omega} d^{-2p+2} \left( 1 + \alpha \eta_m + \beta \eta_m^2 \right) |\nabla v|^2 \, dx \]
\[ \geq \frac{4(p-1)}{p} \int_{\Omega} d^{-2p} \left\{ \frac{(k - 2p)^2}{4} \alpha \eta_m + \left( \frac{k - 2p}{4} \alpha \eta_m + \frac{k - 2p}{4} \beta \eta_m^2 \right) \right\} |u|^p \, dx \tag{3.12} \]

To estimate \( T_2 \) we first note that

\[ \nabla \phi = \lambda d^{-2p+1} \left\{ -2(p-1)(1 + \alpha \eta_m + \beta \eta_m^2) + \frac{\alpha}{2} (\eta_m^2 + \zeta_m^2) + \beta (\eta_m^3 + \eta_m \zeta_m) \right\} \nabla d \]

and hence compute

\[ -\Delta \phi = \lambda d^{-2p} (-2p + 1 + d \Delta d) \left\{ 2(p-1)(1 + \alpha \eta_m + \beta \eta_m^2) - \frac{\alpha}{2} (\eta_m^2 + \zeta_m) - \beta (\eta_m^3 + \eta_m \zeta_m) \right\} + \lambda d^{-2p} \left\{ (p-1) \alpha (\eta_m^2 + \zeta_m) - \beta (\eta_m^3 + \eta_m \zeta_m) - \frac{\alpha}{2} (\eta_m^3 + \eta_m \zeta_m + \eta_m + \eta_m^2) + O(\eta_m^4) \right\} \]
Using the geometric assumption \( d\Delta d - k + 1 \geq 0 \) and collecting similar terms we conclude that

\[
T_2 \geq \lambda \int_{\Omega} d^{-2p} \left\{ 2(p-1)(k-2p) + \frac{k + 4p - 2}{2} \alpha \eta_m + \right. \\
+ \left( (k-2p)(k-2p)\beta + \frac{k + 4p - 2}{2} \alpha \right) \eta_m^2 \\
+ \left. \frac{k + 4p - 2}{2} \alpha \zeta_m - \left( (k-2p+1)\beta + \frac{\alpha}{2} \right) (\eta_m^3 + \eta_m \zeta_m) - \frac{\alpha}{2} \theta_m + O(\eta_m^4) \right\} dx. 
\]  

(3.13)

From Taylor’s theorem we have

\[
(1 + \alpha \eta_m + \beta \eta_m^2) \frac{\partial \beta}{\partial \eta_m} = 1 + \frac{p \alpha}{p-1} \eta_m + \left( \frac{p \beta}{p-1} + \frac{p \alpha^2}{2(p-1)^2} \right) \eta_m^2 + \\
+ \left( \frac{p \alpha \beta}{(p-1)^2} + \frac{p(2-p)\alpha^3}{6(p-1)^3} \right) \eta_m^3 + O(\eta_m^4) 
\]

from which follows that

\[
T_3 = - (p-1)|\lambda| \frac{\beta}{\partial \eta_m} \int_{\Omega} d^{-2p} \left\{ 1 + \frac{p \alpha}{p-1} \eta_m + \left( \frac{p \beta}{p-1} + \frac{p \alpha^2}{2(p-1)^2} \right) \eta_m^2 + \\
+ \left( \frac{p \alpha \beta}{(p-1)^2} + \frac{p(2-p)\alpha^3}{6(p-1)^3} \right) \eta_m^3 + O(\eta_m^4) \right\} dx. 
\]  

(3.14)

Using the above estimates on \( T_1, T_2 \) and \( T_3 \) and going back to (3.11) we obtain the inequality

\[
\int_{\Omega} |\Delta u|^p dx \geq \int_{\Omega} d^{-2p} V |u|^p dx 
\]

(3.15)

where the potential \( V \) has the form

\[
V(x) = r_0 + r_1 \eta_m + r_2 \eta_m^2 + r_2' \zeta_m + r_3 \eta_m^3 + r_3' \eta_m \zeta_m + r_3'' \theta_m. 
\]

We compute the coefficients \( r_i, r_i' \) by adding the corresponding coefficients from (3.12), (3.13) and (3.14). We ignore for now the coefficients of the third-order terms. For the others we find

\[
r_0 = (p-1) \left( \frac{k(k-2p)}{p} \lambda - |\lambda| \frac{\beta}{\partial \eta_m} \right) \\
r_1 = \frac{(p-1)k(k-2p)\alpha \lambda - p \alpha |\lambda|}{p} \\
r_2 = \frac{pk - 2k + 2p}{2p} \alpha \lambda + \frac{(p-1)k(k-2p)\beta \lambda - (p-1)\left( \frac{p \beta}{p-1} + \frac{p \alpha^2}{2(p-1)^2} \right) |\lambda|}{p} \\
r_2' = \left( \frac{p-1}{p} + \frac{pk - 2k + 2p}{2p} \alpha \right) \lambda 
\]

We now make a specific choice for \( \alpha \) and \( \lambda \). We recall that \( Q = (p-1)(k-2p)/p^2 \), and choose

\[
\lambda = Q^{p-1}, \quad \alpha = \frac{(p-1)(k-2p)}{p^2 Q}. 
\]
We then have $r_0 = Q^p$, $r_1 = r_2 = 0$, irrespective of the value of $\beta$. We also have

$$r'_2 = \left(\frac{p - 1}{p}Q^{p-2}\left(Q + \frac{(pk - 2k + 2p)^2}{2p^2}\right)\right).$$

Substituting these values in (3.15) we thus obtain

$$\int_{\Omega} |\Delta u|^p dx \geq Q^p \int_{\Omega} -2p|u|^p dx + \frac{p - 1}{p}Q^{p-2}\left(Q + \frac{(pk - 2k + 2p)^2}{2p^2}\right) \int_{\Omega} -2p \zeta_m|u|^p dx$$

$$+ \int_{\Omega} -2p(r_3\eta_m + r'_3\eta_m \zeta_m + r''_3 \theta_m + O(\eta_m^4))|u|^p dx$$

We still have not imposed any restriction on $\beta$. We now observe that $r'_2$ and $r''_3$ are independent of $\beta$, while $r_3 = c_1\beta + c_2$ with $c_1 = Q^{p-1}/(2k/p - 2k + 2p - 3) < 0$. Hence, since the functions $\eta_m$, $\eta_m \zeta_m$ and $\theta_m$ are comparable in size to each other, the integral is made positive by choosing $\beta$ to be large and negative and $\eta_m$ small enough, which amounts to $D$ being large enough. Hence we have proved that for $D \geq D_0$ there holds

$$\int_{\Omega} |\Delta u|^p dx \geq Q^p \int_{\Omega} |u|^p dx + \left(\frac{p - 1}{p}Q^{p-1} + \frac{p - 1}{2p}Q^{p-2}R^2\right) \sum_{i=1}^m \int_{\Omega} |u|^p \frac{X_i^2 X_2^2 \ldots X_i^2}{dx},$$

where $X_j = X_j(d(x)/D)$. This concludes the proof of the theorem. //

**Remark.** Let us mention here that in the proofs of Theorems 1 and 2 we did not use at any point the assumption that $k$ is the codimension of the set $K$. Indeed, a careful look at the two proofs shows that $K$ can be any closed set such that $\text{dist}(x, K)$ is bounded in $\Omega$ and for which the condition $d\Delta d - k + 1 \geq 0$ or $\leq 0$ is satisfied; the proof does not even require $k$ to be an integer. Of course, the natural realizations of these conditions are that $K$ is smooth and $k = \text{codim}(K)$. However, the argument also applies in the case where $K$ is a union of sets of different codimensions; see [BFT1].

We next prove Theorem 3.

**Proof of Theorem 3.** We note that the convexity of $\Omega$ implies that $\Delta d \leq 0$ on $\Omega$ in the distributional sense [EG, Theorem 6.3.2]. Now, let $u \in C^\infty_c(\Omega)$ be given. Applying Theorem 1 (with $k = 1$, $p = 2$, $s = 0$ and $\alpha = \beta = \gamma = 0$) to the partial derivatives $u_{x_i}$ we have

$$\int_{\Omega} (\Delta u)^2 dx = \sum_{i=1}^n \int_{\Omega} |\nabla u_{x_i}|^2 dx$$

$$\geq \sum_{i=1}^n \left\{ \frac{1}{4} \int_{\Omega} \frac{u_{x_i}^2}{d^2} dx + \frac{1}{4} \int_{\Omega} u_{x_i}^2 \zeta_m dx \right\}$$

$$= \frac{1}{4} \int_{\Omega} \frac{|\nabla u|^2}{d^2} (1 + \zeta_m) dx,$$

for $D$ large enough, where $\zeta_m = \zeta_m(d(x)/D)$. Applying Theorem 1 once more (this time with $k = 1$, $p = 2$, $s = -2$, $\alpha = \beta = 0$ and $\gamma = 1$) we obtain

$$\int_{\Omega} \frac{|\nabla u|^2}{d^2} (1 + \zeta_m) dx \geq \frac{9}{4} \int_{\Omega} \frac{u^2}{d^4} dx + \frac{5}{2} \int_{\Omega} \frac{u^2}{d^4} \zeta_m dx.$$  (3.17)
Combining (3.16) and (3.17) we obtain
\[ \int_{\Omega} (\Delta u)^2 dx \geq \frac{9}{16} \int_{\Omega} u^2 dx + \frac{5}{8} \int_{\Omega} \frac{u^2}{d^4} \zeta_m dx, \]
which is the stated inequality.

We next give the proof of Theorem 4. We recall that Ω is a domain in \( \mathbb{R}^N \) and that \( K \) is a piecewise smooth surface of codimension \( k \) such that \( K \cap \Omega \neq \emptyset \), unless \( k = 1 \) in which case \( K = \partial \Omega \). All the calculations below are local, in a small ball of radius \( \delta \), and indeed, it would be enough to assume that \( K \) has a smooth part. We also note that for \( k = N \) (distance from a point) the subsequent calculations are substantially simplified, whereas for \( k = 1 \) (distance from the boundary) one should replace \( B_\delta \) by \( B_\delta \cap \Omega \). This last change entails some minor modifications, the arguments otherwise being the same.

**Proof.** We shall only give the proof of (ii) since the proof of (i) is much simpler. For the proof we shall use some of the ideas and tools developed in [BFT2]. All our analysis will be local, say, in a fixed ball of \( B(x_0, \delta) \) where \( x_0 \in K \) and \( \delta \) is small, but fixed throughout the proof. We therefore fix a smooth, non-negative function \( \phi \) such that \( \phi(x) = 1 \) on \( \{|x - x_0| < \delta/2\} \) and \( \phi(x) = 1 \) on \( \{|x - x_0| > \delta\} \). For given \( \epsilon_0, \epsilon_1, \ldots, \epsilon_m > 0 \) we then define the function
\[
u = \phi d^{-k+2+\epsilon_0} \frac{1}{p} X_1^{-1+\epsilon_1} X_2^{-1+\epsilon_2} \cdots X_m^{-1+\epsilon_m} =: \phi v.
\]
A standard argument using cut-off functions shows that \( u \) belongs in \( W^{2p}_0(\Omega \setminus K) \) and therefore is a legitimate test-function for the infimum above. We intend to see how \( J_{m-1}[u] \) behaves as the \( \epsilon_i \)'s tend to zero. We shall not be interested in terms that remain bounded for small values of the \( \epsilon_i \)'s. To distinguish such terms we shall need the following fact, cf. [BFT2] (3.8]): we have
\[
\int_{\Omega} \phi^p |d^{k+\beta_0} X_1^{1+\beta_1} (d/D) \cdots X_m^{1+\beta_m} (d/D)| dx < \infty \iff
\begin{align*}
\beta_0 > 0 \\
\text{or } \beta_0 = 0 \text{ and } \beta_1 > 0 \\
\text{or } \beta_0 = \beta_1 = 0 \text{ and } \beta_2 > 0 \\
\text{or } \beta_0 = \beta_1 = \cdots = \beta_{m-1} = 0 \text{ and } \beta_m > 0.
\end{align*}
\]
Now, we have \( \Delta u = \phi \Delta v + 2 \nabla \phi \cdot \nabla v + v \Delta \phi \) and hence, using the inequality
\[
|a + b|^p \leq |a|^p + c(|a|^{p-1}|b| + |b|^p),
\]
we have
\[
\begin{align*}
\int_{\Omega} |\Delta u|^p dx & \leq \int_{\Omega} \phi^p |\Delta v|^p dx + c \left\{ \int_{\Omega} (|\Delta v| |\nabla v| + |v| |\Delta \phi|)^p + (|\nabla \phi| |\nabla v| + |v| |\Delta \phi|)^p \right\} \\
& \leq \int_{\Omega} \phi^p |\Delta v|^p dx + \int_{\Omega} \left(|\Delta v|^{p-1} (|\nabla v| + |v|) + (|\nabla v| + |v|)^p\right) dx.
\end{align*}
\]
where $X(2.1)$ one easily sees that expression in the braces equals

$\int_{\Omega} |\Delta u|^p dx = \int_{\Omega} \phi^p |\Delta v|^p dx + O(1), \quad (3.20)$

where the $O(1)$ is uniform in all the $\epsilon_i$'s.

We next define the function

$g(t) = \epsilon_0 + (-1 + \epsilon_1)X_1 + (-1 + \epsilon_2)X_1X_2 + \cdots (-1 + \epsilon_m)X_1X_2 \cdots X_m, \quad t > 0,$

where $X_i = X_i(t/D)$. We shall always think of $g(d(x))$ as a small quantity. Recalling (2.1) one easily sees that

$$\frac{d\eta_m}{dt} = \frac{1}{t} \sum_{1 \leq i \leq j \leq m} (-1 + \epsilon_j)X_i^2 \cdots X_i X_{i+1} \cdots X_j$$

$$=: \frac{h(t)}{t}. \quad (3.21)$$

Also, for any $\beta$ there holds

$$\frac{d}{dt} \left( t^{\beta+\epsilon_0} X_1^{-1+\epsilon_1} X_2^{-1+\epsilon_2} \cdots X_m^{-1+\epsilon_m} \right) = t^{\beta+\epsilon_0} X_1^{-1+\epsilon_1} X_2^{-1+\epsilon_2} \cdots X_m^{-1+\epsilon_m} \left( \frac{k-p}{p} - d\Delta d - \frac{g}{p} \right) \left( \frac{k-2p}{p} - \frac{g}{p} \right) \frac{h}{p}. \quad (3.22)$$

Applying (3.22) first for $\beta = k - 2p$, then for $\beta = k - p$ and using (3.21) we obtain

$$\Delta v = d^{k+\epsilon_0} X_1^{-1+\epsilon_1} X_2^{-1+\epsilon_2} \cdots X_m^{-1+\epsilon_m} \left( \left( \frac{k-p}{p} - d\Delta d - \frac{g}{p} \right) \left( \frac{k-2p}{p} - \frac{g}{p} \right) \frac{h}{p} \right),$$

where, here and below, we use $g$, $h$ and $X_i$ to denote $g(d(x))$, $h(d(x))$ and $X_i(d(x))/D$.

Now, by [AS] Theorem 3.2 we have $d\Delta d = k - 1 + O(d)$ as $d(x) \to 0$. Hence, the expression in the braces equals

$$Q + \frac{R}{p} g - \frac{1}{p^2} g^2 - \frac{1}{p} h + O(d) \quad \text{as} \quad d(x) \to 0,$$

where $R = (2k - pk - 2p)/p$. The $O(d)$ gives a bounded contribution by an application of (3.19) – as was done earlier. Hence (3.20) gives

$$\int_{\Omega} |\Delta u|^p dx = \int_{\Omega} \phi^p d^{-k+\epsilon_0} X_1^{-1+\epsilon_1} \cdots X_m^{-1+\epsilon_m} \left| Q + \frac{R}{p} g - \frac{1}{p^2} g^2 - \frac{1}{p} h \right|^p dx + O(1). \quad (3.23)$$

To estimate this we take the Taylor’s expansion of $|Q + t|^p$ about $t = 0$. We obtain

$$\int_{\Omega} |\Delta u|^p dx = \int_{\Omega} \phi^p d^{-k+\epsilon_0} X_1^{-1+\epsilon_1} \cdots X_m^{-1+\epsilon_m} \left\{ |Q|^p + |Q|^{p-2} R g + \right.$$

$$+ \left( -\frac{1}{p} |Q|^{p-2} Q + \frac{p-1}{2p} |Q|^{p-2} R^2 \right) g^2 -$$

$$- |Q|^{p-2} Q \zeta_m + O(g^3) + O(gh) + O(h^2) \right\} dx + O(1). \quad (3.24)$$
Using (3.19) once again, it is not difficult to see that the terms $O(g^3)$, $O(gh)$ and $O(h^2)$ give a contribution that is bounded uniformly in the $\epsilon_i$’s and can therefore be dropped. At this point, and in order to simplify the notation, we introduce some auxiliary quantities. For $0 \leq i \leq j \leq m$ we define

$$A_0 = \int_{\Omega} \phi^p d^{-k+\epsilon_0} X_1^{-1+\epsilon_1} \cdots X_m^{-1+\epsilon_m} dx$$
$$A_i = \int_{\Omega} \phi^p d^{-k+\epsilon_0} X_1^{-1+\epsilon_1} \cdots X_i^{-1+\epsilon_i} X_{i+1}^{-1+\epsilon_{i+1}} \cdots X_m^{-1+\epsilon_m} dx$$
$$\Gamma_{0j} = \int_{\Omega} \phi^p d^{-k+\epsilon_0} X_1^{\epsilon_1} \cdots X_i^{\epsilon_i} X_{i+1}^{-1+\epsilon_{i+1}} \cdots X_m^{-1+\epsilon_m} dx$$
$$\Gamma_{ij} = \int_{\Omega} \phi^p d^{-k+\epsilon_0} X_1^{\epsilon_1} \cdots X_i^{\epsilon_i} X_{i+1}^{\epsilon_{i+1}} \cdots X_j^{\epsilon_j} X_{j+1}^{-1+\epsilon_{j+1}} \cdots X_m^{-1+\epsilon_m} dx,$$

with the convention that $\Gamma_{ii} = A_i$. It is then easily seen that

$$\int_{\Omega} \phi^p d^{-k+\epsilon_0} X_1^{-1+\epsilon_1} \cdots X_m^{-1+\epsilon_m} g dx = \epsilon_0 A_0 - \sum_{i=1}^{m} (1 - \epsilon_i) \Gamma_{0i}$$
$$\int_{\Omega} \phi^p d^{-k+\epsilon_0} X_1^{-1+\epsilon_1} \cdots X_m^{-1+\epsilon_m} g^2 dx = \epsilon_0^2 A_0 + \sum_{i=1}^{m} (1 - \epsilon_i)^2 A_i - 2 \epsilon_0 \sum_{i=1}^{m} (1 - \epsilon_i) \Gamma_{0i} +
+ 2 \sum_{i<j} (1 - \epsilon_i)(1 - \epsilon_j) \Gamma_{ij}$$
$$\int_{\Omega} \phi^p d^{-k+\epsilon_0} X_1^{-1+\epsilon_1} \cdots X_m^{-1+\epsilon_m} h dx = - \sum_{i=1}^{m} (1 - \epsilon_i) A_i - \sum_{i<j} (1 - \epsilon_j) \Gamma_{ij}.$$  

(Here and below $\sum_{i<j}$ means $\sum_{1 \leq i < j \leq m}$.) Let us also define the constant

$$P = -\frac{1}{p} |Q|^{p-2} Q + \frac{p-1}{2p} |Q|^{p-2} R^2.$$  

Going back to (3.24) and noting that

$$\int_{\Omega} \frac{|u|^p}{d^{2p}} dx = A_0$$
$$\sum_{i=1}^{m-1} \int_{\Omega} \frac{|u|^p}{d^{2p}} X_1^2 \cdots X_i^2 dx = \sum_{i=1}^{m-1} A_i.$$

we obtain

$$J_{m-1}[u] = |Q|^{p-2} QR \left( \epsilon_0 A_0 - \sum_{i=1}^{m} (1 - \epsilon_i) \Gamma_{0i} \right) +
+ P \left( \epsilon_0^2 A_0 + \sum_{i=1}^{m} (1 - \epsilon_i)^2 A_i - 2 \epsilon_0 \sum_{i=1}^{m} (1 - \epsilon_i) \Gamma_{0i} + 2 \sum_{i<j} (1 - \epsilon_i)(1 - \epsilon_j) \Gamma_{ij} \right)
+ |Q|^{p-2} Q \left( \sum_{i=1}^{m} (1 - \epsilon_i) A_i + \sum_{i<j} (1 - \epsilon_j) \Gamma_{ij} \right) - G \sum_{i=1}^{m-1} A_i + O(1).$$  

(3.25)

Now, by [BFT2] p184,

$$\epsilon_0^2 - 2 \epsilon_0 \sum_{i=1}^{m} (1 - \epsilon_i) \Gamma_{0i} = \sum_{i=1}^{m} (\epsilon_i - \epsilon_i^2) A_i + \sum_{i<j} (2 \epsilon_i - 1)(1 - \epsilon_j) \Gamma_{ij} + O(1).$$
For the sake of simplicity, we set
\[ G = \frac{p-1}{2p^3} |Q|^{p-2} \left\{ k^2(p-1)^2 + (k - 2p)^2 \right\}. \]
One then easily sees that \( P + |Q|^{p-2}Q = G \). Hence, collecting similar terms,
\[
J_{m-1}[u] = |Q|^{p-2}QR(\epsilon_0 A_0 - \sum_{j=1}^{m} (1 - \epsilon_j) \Gamma_{0j}) - G \left( \sum_{i=1}^{m} \epsilon_i A_i - \sum_{i<j} (1 - \epsilon_j) \Gamma_{ij} \right) + GA_m + O(1),
\]
where the \( O(1) \) is uniform for small \( \epsilon \)'s.

Up to this point the parameters \( \epsilon_0, \epsilon_1, \ldots, \epsilon_m \) where positive. We intend to take limits as they tend to zero in that order. Due to (3.18), as \( \epsilon_0 \to 0 \) all terms have finite limits except those involving \( A_0 \) and \( \Gamma_{0j} \) which, when viewed separately, diverge. However a simple argument involving an integration by parts (see [BFT 2, (3.9)]) shows that
\[
\epsilon_0 A_0 - \sum_{j=1}^{m} (1 - \epsilon_j) \Gamma_{0j} = O(1) \quad (3.26)
\]
uniformly in \( \epsilon_0, \ldots, \epsilon_m \). Hence, letting \( \epsilon_0 \to 0 \) we conclude that
\[
J_{m-1}[u] = -G \left( \sum_{i=1}^{m} \epsilon_i A_i - \sum_{i<j} (1 - \epsilon_j) \Gamma_{ij} \right) + GA_m + O(1) \quad (\epsilon_0 = 0)
\]
Now – as was the case with (3.26) – an integration by parts shows that (see [BFT 2, (3.9)]) if \( \epsilon_0 = \epsilon_1 = \ldots = \epsilon_{i-1} = 0 \), then
\[
\epsilon_i A_i - \sum_{j=1}^{m} (1 - \epsilon_j) \Gamma_{ij} = O(1). \quad (3.27)
\]
We now let \( \epsilon_1 \to 0 \). Again, all terms have finite limits except those involving \( A_1 \) and \( \Gamma_{1j} \) which diverge. Using (3.27) we see that when combined these terms stay bounded in the limit \( \alpha_1 \to 0 \). We proceed in this way and after letting \( \epsilon_{m-1} \to 0 \) we are left with
\[
J_{m-1}[u] = G(1 - \epsilon_m) A_m + O(1) \quad (\epsilon_0 = \ldots = \epsilon_{m-1} = 0).
\]
Let us denote by \( G' \) the infimum in the left-hand side of part (ii) of Theorem 4. We have thus proved that
\[
G' \leq \frac{G(1 - \epsilon_m) A_m + O(1)}{A_m} \quad (\epsilon_0 = \ldots = \epsilon_{m-1} = 0).
\]
Letting now \( \epsilon_m \to 0 \) we have \( A_m \to +\infty \) (by 3.18), and thus conclude that \( G' \leq G \), as required.

**Remark.** Slightly modifying the above argument one can also prove the optimality of the power \( X^2_m \) of the improved Rellich inequalities (1.11) and (1.13). Namely, for any \( \epsilon > 0 \) there holds
\[
\inf_{u \in C^0_c(\Omega \setminus K)} \frac{J_{m-1}[u]}{\int_{\Omega} |u|^p x_1^2 \ldots X_{m-\epsilon} X_{m-\epsilon} dx} = 0.
\]

**Acknowledgment** We acknowledge partial support by the RTN European network Fronts–Singularities, HPRN-CT-2002-00274.
References

[AS] Ambrosio L. and Soner H.M. Level set approach to mean curvature flow in arbitrary codimension. *J. Diff. Geometry* **43** (1996) 693-737.

[B] Barbatis G. Boundary decay estimates for solutions of fourth-order elliptic equations. In preparation.

[BFT1] Barbatis G., Filippas S., and Tertikas A. A unified approach to improved $L^p$ Hardy inequalities. *Trans. Amer. Math. Soc.* **356** (2004) 2169-2196.

[BFT2] Barbatis G., Filippas S., and Tertikas A. Series expansion for $L^p$ Hardy inequality. *Indiana Univ. Math. J.* **52** (2003) 171-190.

[BM] Brezis H. and Marcus M. Hardy’s inequalities revisited. Ann. Scuola Norm. Pisa **25** (1997) 217-237.

[BV] Brezis H. and Vázquez J.-L. Blow-up solutions of some nonlinear elliptic problems. *Rev. Mat. Univ. Comp. Madrid* **10** (1997) 443-469.

[D] Davies E.B. One-parameter semigroups. LMS Monographs, Academic Press, London 1980.

[DH] Davies E.B. and Hinz A.M. Explicit constants for Rellich inequalities in $L^p(\Omega)$. *Math. Z.* **227** (1998) 511-523.

[EG] Evans L.C. and Gariepy R.F. Measure theory and fine properties of functions. CRC Press 1992.

[O] Owen M.P. The Hardy-Rellich inequality for polyharmonic operators. *Proc. Royal Soc. Edinburgh* **129A** (1999) 825-839.

[TZ] Tertikas A. and Zographopoulos N. Best constants in the Hardy-Rellich inequalities and related improvements. Preprint 2004.

[T] Tidblom J. $L^p$ Hardy inequalities in general domains. Preprint 2003.