Gravitational Deflection of Light in Lemaitre-Tolman-Bondi Metric

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We calculate the gravitational deflection of light around a region of tiny density with a homogeneous cosmic background, which can be described by Lemaitre-Tolman-Bondi metric. The deflection angle is very different with it in Schwarzschild metric. This means the background metric (or the expanding universe) do can have significant influence on the local gravity effect in general relativity under some extreme conditions. Our calculation also gives a set of nontrivial analytic solutions of the geodesic equation in a dynamic spacetime.

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I. INTRODUCTION

In astronomy, the calculation of gravitational lensing is based on the deflection angle of light in Schwarzschild metric. The background of Schwarzschild metric is actually a Minkowski metric. Here we want to see whether the deflection angle would be changed if we change the background from Minkowski metric to Friedmann-Lemaitre-Robertson-Walker (FLRW) metric. That is to calculate the deflection angle around an overdense region with a homogeneous cosmic background, which can be described by Lemaitre-Tolman-Bondi (LTB) metric [1–3]

\[ ds^2 = -c^2dt^2 + \frac{A^2(r,t)}{1 - Kr^2 + \varepsilon k(r)}dr^2 + A^2(r,t)d\Omega^2, \] (1)

where \( \dot{r} \equiv \frac{dr}{dt} \) (notice that the meaning of this symbol will be changed after Eq.(23)). \( Kr^2 \) means the background curvature term corresponds to FLRW metric, and \( \varepsilon k(r) \) is the curvature term produced by inhomogeneous matter density. In order to show the influence of the background metric, we assume the lens density is much less than the background density (otherwise a compact lens would be act as the Schwarzschild metric [4]). Technically, we denote \( \varepsilon \) as a dimensionless infinitesimal in math and let \( \varepsilon = 1 \) after we get the final solution. The dimension of \( A(r,t) \) is m, and \( r \) is dimensionless.

In this paper, we only study the flat background case, i.e. \( K = 0 \) in Eq.(1). The energy-momentum tensor \( T_{\mu\nu} = \text{diag}(\rho c^4, 0, 0, 0) \). Substituting Eq.(1) into Einstein equation \( G_{\mu\nu} = 8\pi GT_{\mu\nu}/c^4 \), we obtain two independent equations

\[ \ddot{A} + 2\dot{A} + \varepsilon kc^2 = 0, \] (2)

\[ \frac{\ddot{A}}{A^2} + 2\frac{\dot{A}}{A} \dot{A} + \frac{\varepsilon k c^2}{A} = 8\pi G\rho, \] (3)

where \( \dot{\equiv} \frac{d}{dt} \). Three boundary conditions are used to determine the parameters in \( A(r, t) \), that is \( A(r, t) = 0 \), \( A(r, t_0) = a_0r \), and \( \rho(r, t_0) \).

This paper is organized as follows. In section II, we solve the metric with series expansion method. Section III provides the solution of the geodesic equation in the above metric. The deflection angle of light was calculated in section IV. Finally, conclusions are presented in section V.

II. METRIC

Eq.(2) is a differential equation depends on \( t \), and \( k \) can be regard as a constant. It is not hard to find the series solution with initial condition \( A(r, t) = 0 \)

\[ A(r, t) = Ct^{2/3} \left( 1 - \frac{9}{20} \frac{\varepsilon k c^2 t^{2/3}}{C^2} + O(\varepsilon^2) \right), \] (4)

where \( C \) is an integral constant. We will omit the \( O(\varepsilon^2) \) term hereafter. By direct replacement of \( A(r, t_0) = a_0r \) into Eq.(4) one finds

\[ C = a_0r_{t_0}^{-2/3} \left( 1 + \frac{9}{20} \frac{\varepsilon k c^2 t_{0}^{2/3}}{a_0^3 r_{t_0}^{3/2}} \right). \] (5)

Then

\[ A(r, t) = a_0r_{t_0}^{-2/3} \left( 1 + \frac{9(1 - \xi_t) \varepsilon k c^2 t_{0}^{2/3}}{20 a_0^3 r_{t_0}^{3/2}} \right), \] (6)

where \( \xi_t \equiv t^{2/3}/t_{0}^{2/3} \). Assuming the matter density at present equals

\[ \rho(r, t_0) = \rho_0 + \varepsilon \rho_e f \left( \frac{r}{r_e} \right), \] (7)

where \( \rho_0 \), \( \rho_e \), and \( r_e \) is positive constant, and \( f(x) \) can be any function. \( \rho_0 \) corresponds to the background density of the universe, while \( \rho_e \) and \( r_e \) represent the amplitude and size of the matter density fluctuation, respectively. In the beginning, we denote \( \varepsilon \) as a dimensionless infinitesimal in math, this corresponds to \( \rho_e / \rho_0 \ll 1 \) in physics. We denote \( \xi_t \equiv t/r_e \) in the following. By direct replacement of Eq.(6–7) into Eq.(3) we find 8\( \pi G\rho_0 t_{0}^{4/3} = 4/3 \) and
the differential equation of $k(r)$

$$\frac{3c^2}{5a_0^2} \frac{rk'}{r^2} + k = 8\pi G \rho_c f(\xi_r).$$

(8)

Integration of Eq.(8) yields

$$k(r) = \frac{40\pi G \rho_c a_0^2 r^3}{3c^2} F(\xi_r),$$

(9)

where

$$F(x) = \frac{1}{x} \int x^2 f(x) \, dx.$$  

(10)

There still exist an unknown integral constant in the indefinite integration of Eq.(10). But in order to let $F(x)$ keeps meaningful at $x = 0$, this integral constant can be determined and $F(x)$ can be rewritten as

$$F(x) = \frac{1}{x} \int_0^x x^2 f(x) \, dx.$$  

(11)

The above integral actually is $\int_0^x x^2 f(y) \, dy$. To declare the integral notion in the following, we change the variable from $y$ to $x$ here. Eq.(6), Eq.(9) and Eq.(11) give a general solution of Einstein equation under LTB metric, which satisfy three boundary conditions $A(r, t_0) = 0$, $A(r, t_0) = a_0 r$ and $r(0) = \rho_0 + \varepsilon \rho_c f(\xi_r)$.

The integral in Eq.(11) corresponds to the total fluctuation mass. If we calculate the high order term of $A(r, t)$, we would find $A(r, t)$ is dependent on $f(r)$ just through the form of $F(r)$. So the influence of the density perturbation to the outside is simply dependent on the total fluctuation mass. This property is similar to the Schwarzschild metric.

Assuming

$$f(x) = \begin{cases} 1 & x \leq 1, \\ 0 & x > 1. \end{cases}$$

(12)

Notice that $x < 1$ means $r < r_c$. We only consider the outside region in the following, i.e. $x > 1$. From Eq.(11) we obtain

$$F(x) = \frac{1}{3x}.  

(13)

Then, Eq.(9) becomes

$$k(r) = \frac{40\pi G \rho_c a_0^2 r^3}{9c^2 r},$$

(14)

So far we find a simple metric to describe the outside region

$$ds^2 = -c^2 dt^2 + B(r, t) dr^2 + A^2(r, t) d\Omega^2,$$

(15)

and

$$B(r, t) = a_0 \xi_t \left(1 + \varepsilon \frac{r^3(1 - \xi_t)}{4r^3}\right),$$

(16)

where we have redefine $\varepsilon \equiv 8\pi G \rho_c t^2$ as a real dimensionless infinitesimal in physics ($\varepsilon \ll 1$ because $8\pi G \rho_0 t^2 = 4/3$ and $\rho_c/\rho_0 \ll 1$). The metric described by Eq.(15) is correct up to $O(\varepsilon)$, so the high order term is also omitted in the following.

III. GEODESIC

We cannot derive the equation of orbit directly as we did in Schwarzschild metric [6], because the metric described by Eq.(15) is dependent on time. In order to calculate the deflection angle in this metric, we have to solve the geodesic equation. By the way, analytic solutions for the geodesic equations are only known for some static or stationary metric up to now (see [6–12] and relevant references).

We assume light move in the equatorial plane, i.e. set $\theta = \pi/2$. The time component of geodesic equation is

$$c^2 \frac{d^2t}{d\lambda^2} + B\frac{dr}{d\lambda}^2 + A\frac{d\varphi}{d\lambda}^2 = 0.$$  

(18)

The metric is independent of $\varphi$, so the momentum $p_\varphi$ is conserved and this gives

$$A^2 \frac{d\varphi}{d\lambda} = L,$$  

(19)

where $L$ is a constant. The last differential equation can be given by $ds^2 = 0$,

$$-c^2 \frac{dt}{d\lambda}^2 + B^2 \frac{dr}{d\lambda}^2 + A^2 \frac{d\varphi}{d\lambda}^2 = 0.$$  

(20)

Our mission is to find a group of $\{t(\lambda), r(\lambda), \varphi(\lambda)\}$, which satisfy Eq.(18–20) up to $\varepsilon$ order.

It is not hard to find the zero order solutions of Eq.(18–20),

$$t(\lambda) = t_0(\lambda + 1)^{3/5},$$  

(21)

$$r(\lambda) = \sqrt{\frac{9c^2 t_0^2}{a_0^2}[(\lambda + 1)^{1/5} - 1]^2 + R^2},$$  

(22)

$$\varphi(\lambda) = \arctan \left(\frac{3c t_0}{a_0 R}(1 - (\lambda + 1)^{1/5})\right),$$  

(23)

where $R$ is a positive constant and $R > r_c$. In the above solutions, we fixed a free parameter originate from the affine parameter subjectively. We denote $\xi \equiv \frac{d}{d\lambda}$ in the following. The initial conditions correspond to the above solutions are

$$t(0) = t_0, r(0) = R, \varphi(0) = 0, r'(0) = 0.$$  

(24)
where the first three is the initial position, and the last one means the direction of the initial velocity. Eq.(23) also means the light move anticlockwise, and $L = -3a_0Rct_0/5$ can be obtained by Eq.(19).

Now we calculate the first order term of $t(\lambda)$, $r(\lambda)$ and $\varphi(\lambda)$. We denote $\xi_\lambda \equiv (\lambda + 1)^{1/5}$ in the following and assume

$$t(\lambda) = t_0\xi_\lambda^3 + \varepsilon t_0 g_t(\lambda),$$

$$r(\lambda) = \sqrt{\frac{g_{t_0}^2 t_0^2}{a_0^5}(\xi_\lambda - 1)^2 + R^2} + \varepsilon g_r(\lambda),$$

$$\varphi(\lambda) = \arctan\left(\frac{3ct_0}{a_0 R}(1 - \xi_\lambda)\right) + \varepsilon g_\varphi(\lambda),$$

$$L = -\frac{3a_0 Rct_0}{5} + \varepsilon L_1,$$

where $L_1$ is a constant. We still take the initial conditions in Eq.(24) and this gives $g_t(0), g_r(0), g_\varphi(0) = 0$ and $g_r'(0) = 0$. Taking a Taylor expansion of $g_t(\lambda)$ and $g_\varphi(\lambda)$, i.e. set $g_t(\lambda) = b_1\lambda + \cdots$ and $g_\varphi(\lambda) = b_2\lambda + \cdots$, and using series expansion method to solve the differential equations Eq.(19–20), we obtain

$$b_1 = -b_2\frac{a_0 R}{ct_0}, \quad b_2 = \frac{L_1}{a_0^2 R^2}.$$  

$L_1$ can be determined subjectively because of the free parameter in affine parameter. We set $L_1 = 0$ in the following, and this gives the other two initial conditions $g_t'(0) = 0$ and $g_r'(0) = 0$.

Substituting Eq.(19–20) into Eq.(18) to eliminate $\frac{dt}{d\lambda}$ and $\frac{d\varphi}{d\lambda}$, and then substituting Eq.(25–28) into the result, we find the differential equation about $g_t(\lambda)$

$$(\lambda + 1)^2 g_t'' + \frac{4(\lambda + 1)g_t'}{5} - \frac{6g_t}{25} = S(\lambda),$$

where

$$S(\lambda) = \frac{3(\lambda + 1)}{25\Gamma^3} r_0^3 \left(\frac{3}{2\Gamma^2} - 1\right),$$

$$\Gamma = \sqrt{\frac{(\xi_\lambda - 1)^2}{\beta^2} + 1},$$

$$\beta = \frac{a_0 R}{3ct_0},$$

Luckily, this differential equation is independent of $g_r(\lambda)$ and $g_\varphi(\lambda)$. Eq.(30) is an Euler differential equation. Considering the initial conditions $g_t(0), g_t'(0) = 0$, we obtain

$$g_t(\lambda) = \xi_\lambda^3 \int_0^\lambda S(\lambda) d\lambda - \frac{1}{\xi_\lambda} \int_0^\lambda S(\lambda) d\lambda.$$  

Substituting Eq.(19) into Eq.(20) to eliminate $\frac{d\xi_\lambda}{d\lambda}$, and then substituting Eq.(25–28) into the result, we find the differential equation about $g_r(\lambda)$

$$g_r' - \frac{g_r}{5\xi_\lambda^2(\xi_\lambda - 1)\Gamma^2} = Q(\lambda),$$

where

$$Q(\lambda) = \frac{\Gamma g_r' R}{3\xi_\lambda^2(\xi_\lambda - 1)} - \frac{2(\Gamma^2 - 2)g_t R}{15\Gamma \xi_\lambda^2(\xi_\lambda - 1)} - \frac{(\xi_\lambda - 1)^2 r_0^3}{21\Gamma^2 \xi_\lambda^2 R^2} - \frac{(2\Gamma^2 - 4)(\xi_\lambda + 1)r_0^3}{20\Gamma^4 \xi_\lambda^2 R^2}.$$  

The general solution of $g_r(\lambda)$ satisfy $g_r(0) = 0$ automatically. Considering the initial condition $g_r'(0) = 0$, we obtain

$$g_r(\lambda) = \frac{\xi_\lambda - 1}{\Gamma} \int_0^\lambda Q(\lambda) d\lambda.$$  

After some calculation, we obtain

$$g_r(\lambda) = \frac{\xi_\lambda - 1}{\Gamma R^2} \left[ 15\beta(3\beta^4 - 24\beta^2 + 8) \frac{\xi_\lambda^2}{4\xi_\lambda^2} + 5\beta^2 \frac{\xi_\lambda^2}{4} \cdot \ln(\Gamma + \frac{\xi_\lambda - 1}{\beta}) \right] - \frac{5\xi_\lambda^3}{\Gamma},$$

$$g_r(\lambda) = \frac{5\xi_\lambda^3}{\Gamma R^2} \left[ 15\beta(3\beta^4 - 24\beta^2 + 8) \frac{8\xi_\lambda^3}{4\xi_\lambda^3} + 5\beta^2 \frac{8\xi_\lambda^3}{8\xi_\lambda^3} \cdot \ln(\Gamma + \frac{\xi_\lambda - 1}{\beta}) \right],$$

$$g_r(\lambda) = \frac{\xi_\lambda - 1}{\Gamma R^2} \left[ 15\beta(3\beta^4 - 24\beta^2 + 8) \frac{8\xi_\lambda^3}{4\xi_\lambda^3} + 5\beta^2 \frac{8\xi_\lambda^3}{8\xi_\lambda^3} \cdot \ln(\Gamma + \frac{\xi_\lambda - 1}{\beta}) \right].$$
Substituting Eq. (25–28) into Eq. (19), we can find a simple differential equation about \( g_\varphi (\lambda) \), which satisfy \( g_\varphi (0) = 0 \) automatically. With the initial condition \( g_\varphi (0) = 0 \), we obtain

\[
g_\varphi (\lambda) = \frac{2}{5\beta} \int_0^\lambda \frac{2g_\rho}{3\Gamma^2\xi_\lambda^3} + \frac{g_\varphi}{\Gamma^2\xi_\lambda^4} + \frac{(1 - \xi_\lambda^4)r_3}{4\Gamma^2\xi_\lambda^3R^3} d\lambda. \tag{40}
\]

**IV. THE DEFLECTION ANGLE**

When \( \lambda \to +\infty \), \( g_t (\lambda) \) and \( g_r (\lambda) \) can be simplified to

\[
g_t (\lambda) = -\frac{3\beta^2r_3^3}{10R^3}\xi_\lambda^3, \quad g_r (\lambda) = -\frac{\beta r_3^3}{10R^3}\xi_\lambda^3. \tag{41}
\]

Both \( g_t (\lambda) \) and \( g_r (\lambda) \) is proportional to its correspond zero order value. So the solution of \( t(\lambda) \) and \( r(\lambda) \) is valid even when \( \lambda \) approach infinity. But \( g_t (\lambda) \) and \( g_r (\lambda) \) will be divergent when \( \lambda = -1 \) (corresponds \( t = 0 \) in zero order solution), and this make the analytic solution of \( t(\lambda) \) and \( r(\lambda) \) invalid near \( \lambda = -1 \). So we only consider the deflection angle from now to the infinite future with the initial condition Eq. (24), and then we need to calculate \( g_\varphi (+\infty) \). Substituting Eq. (38) and Eq. (39) into Eq. (40), we obtain

\[
g_\varphi (+\infty) = -\left(\frac{\beta}{2} + 3\beta^2\right)\frac{r_3^3}{R^3}. \tag{42}
\]

We then use Eq. (42) and the initial value Eq. (38) to find the deflection angle

\[
\Delta \varphi = \varepsilon g_\varphi (+\infty) = \frac{r_3}{a_0 R} \left(\frac{ct_0}{2a_0 R} + 1\right), \tag{43}
\]

where the Schwarzschild radius \( r_s = 2GM/c^2 \), and the total fluctuation mass \( M = \frac{4}{3}\pi \rho_c (a_0 r_s)^3 \). Using \( M = \frac{4}{3}\pi \rho_c (a_0 r_s)^3 \) to calculate the total fluctuation mass is reasonable, because we already assume \( \varepsilon \ll 1 \). Notice that the deflection angle in Schwarzschild metric in this situation is \( \Delta \varphi = r_s/R \). So the background metric do can have significant influence on the local gravitational effect. There is one thing should be noted that Eq. (43) is valid when \( \rho_c \ll \rho_0 \), so this result cannot be used to analyze the real gravitational lensing events.

Eq. (43) shows \( \Delta \varphi \sim \varepsilon \beta \) if \( \beta \ll 1 \) and \( R \sim r_s \). This also can be seen from the metric directly. Light bending mainly occurred in the center region when \( \beta \ll 1 \), i.e. \( t \sim t_0 \) with the initial conditions Eq. (24). Taking a Taylor expansion of \( \xi_t \) at \( t = t_0 \) gives \( (\xi_t - 1) \approx 2(t - t_0)/(3t_0) \). Considering the motion of light, we know \( c(t - t_0) \approx a_0 R \tan \varphi \). We can set \( r, R \sim r_s \) and \( \tan \varphi \sim 1 \) when the light travel near the center region. With these approximations, we know \( 5a_0^2r_3^3/(18c^2t_0^2R^3) \sim \beta^2 \) and \( r_3^3(\xi_t - 1)/(2r_3^3) \sim \beta \) in Eq. (17). So the “potential” along the light path is proportional to \( \varepsilon \beta \), and this will cause a same order influence to the deflection angle.

**V. CONCLUSIONS**

In this Letter, we have analyzed the gravitational deflection of light in LTB metric. We assume the density perturbation in the center is much less than the background density of the universe, which allows us to calculate the metric of the universe by series expansion method. The surprising thing is that the geodesic equation can be solved analytically in this metric. Then the deflection angle from now to the infinite future was calculated through these solutions. We find that the deflection angle in LTB metric is very different with it in Schwarzschild metric, which means the background metric (or the expanding universe) do can have considerable influence on the local gravity effect.

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