The Desargues and Pappus properties in finite quantum systems

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Abstract. The Desargues and Pappus properties in projective geometry, are studied in the context of Quantum Physics. The Desargues property implies that two experiments each of which involves two successive projective measurements, have selective correlations. The Pappus property in a quantum context, is also discussed. The work links Projective Geometry to Quantum Physics and Quantum Information.

1. Introduction
After the fundamental work of Birkhoff and von Neumann[1], quantum logic has been studied extensively in the literature,[2, 3, 4, 5, 6]. In this paper we show that ideas from quantum logic lead to novel approaches in the study of quantum correlations, which might play an important role in modern quantum technologies.

Quantum logic is described with the Birkhoff-von Neumann lattice of the closed subspaces of the Hilbert space, with the operations of conjunction, disjunction and complementation. In the case of the infinite-dimensional Hilbert space $H_{osc}$ of the harmonic oscillator, the Birkhoff-von Neumann lattice $L(H_{osc})$ is orthomodular. In the case of $d$-dimensional Hilbert space $H(d)$ [7, 8] considered here, all subspaces are closed, and the lattice of subspaces $L(d)$ is a modular orthocomplemented lattice [9, 10, 11, 12, 13]. The modular orthocomplemented lattice $L(d)$ obeys modularity (which is a weak version of distributivity), but the orthomodular lattice $L(H_{osc})$ violates modularity. Orthomodularity is a weaker concept than orthocomplemented modularity.

Refs [14, 15, 16, 17, 18] have pointed out that the lattices describing finite quantum systems, are a special case of modular orthocomplemented lattices, because they have extra stronger properties. One such property is the Desargues property which is fundamental in Projective Geometry[19]. The analogue of this geometrical property in the context of both classical and quantum Physics has been discussed in [20], and it leads to the novel concept of selective correlations. In this paper we review briefly some aspects of this work and bring into a quantum context, another fundamental property of projective geometry, described by the Pappus theorem.
Quantum Mechanics of finite quantum systems is a special case of modular orthocomplemented lattices, in the sense that it obeys both the Desargues and Pappus properties, and many other generalisations of these properties. It is an interesting open problem to find all extra properties that we need to add into modular orthocomplemented lattices, in order to get quantum mechanical structures for finite quantum systems.

In section II we discuss briefly modular orthocomplemented lattices, in order to define the notation. In section III we give a ‘dictionary’ between the terminology of projective geometry and the terminology of quantum physics. In section IV we discuss the Desargues property in a quantum context (this has been discussed in more detail in [20]). Section V is the novel part of this paper, and discusses the Pappus property. We conclude in section VI with a discussion of our results.

2. The modular orthocomplemented lattice \( \mathcal{L}(d) \)
We consider a quantum system with states in a \( d \)-dimensional Hilbert space \( H(d) \)[7, 8]. In the set of subspaces of \( H(d) \), we define the conjunction (logical AND) and disjunction (logical OR) as\[9, 10, 11, 12, 13\]
\[ H_1 \wedge H_2 = H_1 \cap H_2; \quad H_1 \vee H_2 = \text{span}(H_1 \cup H_2). \] (1)
The set of subspaces of \( H(d) \) with these operations is a lattice \( \mathcal{L}(d) \). The \( I = H(d) \) and also the \( O = H(0) \) (the zero-dimensional subspace that contains only the zero vector), are both elements of \( \mathcal{L}(d) \).

\( \mathcal{L}(d) \) is a modular orthocomplemented lattice. Modularity is a weak version of distributivity and it states that
\[ H_1 \prec H_3 \rightarrow H_1 \vee (H_2 \wedge H_3) = (H_1 \vee H_2) \wedge H_3. \] (2)
The orthocomplement of \( H_1 \) (logical NOT operation) is another subspace which we denote as \( H_1^{\perp} \), with the properties
\[ H_1 \wedge H_1^{\perp} = O; \quad H_1 \vee H_1^{\perp} = I; \quad (H_1^{\perp})^{\perp} = H_1 \]
\[ (H_1 \wedge H_2)^{\perp} = H_1^{\perp} \vee H_2^{\perp}; \quad (H_1 \vee H_2)^{\perp} = H_1^{\perp} \wedge H_2^{\perp}. \] (3)
We will use the notation \( \Pi(H) \) for the projector to the subspace \( H \). Then
\[ \Pi(H_1) + \Pi(H_1^{\perp}) = 1; \quad \Pi(H_1)\Pi(H_1^{\perp}) = 0; \quad \dim(H_1) + \dim(H_1^{\perp}) = d. \] (4)

3. A dictionary between Projective Geometry and Quantum Physics
The following ‘dictionary’ shows the analogy between Projective Geometry and Quantum Physics [20]
It is used to ‘translate’ the Desargues and Pappus theorems in projective geometry, into the language of Quantum Physics.

- We call points the one-dimensional subspaces, lines the two-dimensional subspaces of \( H(d) \), and planes the three-dimensional subspaces of \( H(d) \). A point \( h \) (denoted with lower case latters) contains a single quantum state, and \( \text{Tr}[\Pi(h)] = \dim(h) = 1 \). A line \( H \) (denoted with upper case latters) contains two quantum states and all their superpositions, and \( \text{Tr}[\Pi(H)] = \dim(H) = 2 \).
Three points $h_1, h_2, h_3$ are on the same line, if
\[ \dim(h_1 \lor h_2 \lor h_3) = 2, \]  
(5)
or equivalently if the rank of any matrix that contains three generic vectors from $h_1, h_2, h_3$ (which we denote as $(h_1, h_2, h_3)$), is
\[ \text{rank}(h_1, h_2, h_3) = 2. \]  
(6)

- The disjunction of two points $h_1 \lor h_2$, is a line through the points $h_1, h_2$. Physically, $h_1 \lor h_2$ contains all the superpositions of the two quantum states corresponding to $h_1$ and $h_2$.
- If $H_1, H_2$ are two-dimensional subspaces of $H(d)$ such that $\dim(H_1 \lor H_2) = 3$, the conjunction $H_1 \land H_2$ is a point at the intersection of the lines $H_1, H_2$.
- If $H_1 = h_1^\perp$, is the relative orthocomplement of $h_1$ with respect to some subspace $H_0$, then $H_1$ and $h_1$ represent the negation of each other (relative logical NOT within the $H_0$). In terms of projectors $\Pi(H_1) + \Pi(h_1) = \Pi(H_0)$ and $\Pi(H_1)\Pi(h_1) = 0$.

4. The Desargues property in a quantum context

The Desargues property in the context of projective geometry is shown in fig.1. If the three lines $AA', BB', CC'$, intersect at the same point $O$, then the three points which are the intersections of the lines $(AB, A'B')$, $(AC, A'C')$, $(BC, B'C')$ are on the same line. This is translated in a quantum context as follows.

Let $h_1, h_2, h_3$ be distinct one-dimensional subspaces of $H(d)$ (points). In the generic case that $\dim(h_1 \lor h_2 \lor h_3) = 3$, we say that the points $h_1, h_2, h_3$ form a triangle. Also let $(h'_1, h'_2, h'_3)$ be another triangle on the same plane, i.e.,
\[ h_1 \lor h_2 \lor h_3 = h'_1 \lor h'_2 \lor h'_3 = H_0; \quad \dim(H_0) = 3. \]  
(7)
We consider the lines
\[ H_{ij} = h_i \lor h_j; \quad H'_{ij} = h'_i \lor h'_j; \quad i \neq j, \]  
(8)
and the points at the intersection of a side in the first triangle with the corresponding side in the second triangle
\[ b_k = H_{ij} \land H'_{ij} = (h_i \lor h_j) \land (h'_i \lor h'_j); \quad \{i, j, k\} = \{1, 2, 3\}. \]  
(9)

$b_k$ contains a state which is both a superposition of the two quantum states contained in $h_i$ and $h_j$, and also a superposition of the two quantum states contained in $h'_i$ and $h'_j$.

We also consider the lines that join a vertex in the first triangle with a vertex in the second triangle:
\[ H_i = h_i \lor h'_i. \]  
(10)
$H_i$ contains all the superpositions of the two quantum states corresponding to $h_i$ and $h'_i$.

If $\dim(H_1 \land H_2 \land H_3) = 1$ (the lines $H_1, H_2, H_3$ intersect at the same point), then $\dim(h_1 \lor h_2 \lor h_3) = 2$ (the points $h_1, h_2, h_3$ are on the same line). This can be verified with the following two experiments:
Figure 1. The Desargues property in projective geometry

Figure 2. The Pappus property in projective geometry
On a state $|s\rangle$ we perform measurement with the projector $\Pi(H_1 \land H_2)$, and if the outcome is ‘yes’ we get the state 

$$|s_1\rangle = N_1 \Pi(H_1 \land H_2)|s\rangle.$$  (11)

Here $N_1$ is a normalization constant. We then perform a second measurement with the projector $\Pi(H_3)$, and if the second outcome is ‘yes’, we get the state 

$$|s_2\rangle = N_2 \Pi(H_3) \Pi(H_1 \land H_2)|s\rangle.$$  (12)

We show this schematically:

$$|s\rangle \xrightarrow{\Pi(H_1 \land H_2)} |s_1\rangle \xrightarrow{\Pi(H_3)} |s_2\rangle.$$  (13)

On the same state $|s\rangle$ we perform measurement with the projector $\Pi(h_3)$ and if the outcome is ‘yes’ we get the state 

$$|t_1\rangle = N_3 \Pi(h_3)|s\rangle.$$  (14)

We then perform a second measurement with the projector $\Pi(h_1 \lor h_2)$, and if the second outcome is ‘yes’ we get the state 

$$|t_2\rangle = N_4 \Pi(h_1 \lor h_2) \Pi(h_3)|s\rangle.$$  (15)

We show this schematically:

$$|s\rangle \xrightarrow{\Pi(h_3)} |t_1\rangle \xrightarrow{\Pi(h_1 \lor h_2)} |t_2\rangle.$$  (16)

Desargues property says that there is the following selective correlation between these two experiments. If $\Pi(H_3) \Pi(H_1 \land H_2) = \Pi(H_1 \land H_2)$ and therefore $|s_1\rangle = |s_2\rangle$ in the first experiment, then $\Pi(h_1 \lor h_2) \Pi(h_3) = \Pi(h_3)$ and therefore $|t_1\rangle = |t_2\rangle$ in the second experiment (and vice-versa). In the general case, the two outputs are not correlated.

Examples of this have been given in [20].

5. The Pappus property in a quantum context

The Pappus property in the context of projective geometry is shown in fig. 2. The points $A, B, C$ are on one line, and the points $A', B', C'$ are on another line. Then the three points which are the intersections of the lines $(AB', A'B), (AC', A'C), (BC', B'C)$ are on the same line. This is translated in a quantum context as follows.

Let $h_1, h_2, h_3$ be three points on the same line, and $h'_1, h'_2, h'_3$ be three more points on another line. Then 

$$\dim(h_1 \lor h_2 \lor h_3) = \dim(h'_1 \lor h'_2 \lor h'_3) = 2.$$  (17)

We consider the lines $h_1 \lor h'_2, h'_1 \lor h_2$, and their intersection point $\delta_1 = (h_1 \lor h'_2) \land (h'_1 \lor h_2)$. Similarly, we consider the lines $h_1 \lor h'_2, h'_1 \lor h_2$ and their intersection point $\delta_2 = (h_1 \lor h'_2) \land (h'_1 \lor h_2)$, and also
the lines $h_1 \lor h'_2$, $h'_1 \lor h_2$ and their intersection point $\mathfrak{F}_3 = (h_1 \lor h'_2) \land (h'_1 \lor h_2)$. Then the points $\mathfrak{F}_1$, $\mathfrak{F}_2$, $\mathfrak{F}_3$ are on the same line, i.e.,

$$\dim(\mathfrak{F}_1 \lor \mathfrak{F}_2 \lor \mathfrak{F}_3) = 2,$$

or equivalently

$$\text{rank}(\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3) = 2. \tag{19}$$

5.1. Example

In $H(3)$ we consider the one-dimensional subspaces $(h_1, h_2, h_3)$ that contain the vectors

$$h_1 = \begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix}; \quad h_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}; \quad h_3 = h_1 + 3h_2 = \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix}. \tag{20}$$

In order to simplify the notation, we represent a space with a ‘generic vector’ that it contains. Since $h_3 = h_1 + 3h_2$ the points $h_1, h_2, h_3$ are on the same line, and Eq.(17) holds.

We also consider the one-dimensional subspaces $(h'_1, h'_2, h'_3)$ that contain the vectors

$$h'_1 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}; \quad h'_2 = \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix}; \quad h'_3 = h'_1 + 2h'_2 = \begin{pmatrix} -1 \\ 9 \\ 6 \end{pmatrix}. \tag{21}$$

Since $h'_3 = h'_1 + 2h'_2$ the points $h'_1, h'_2, h'_3$ are on the same line.

Then we calculate the lines (two-dimensional spaces that depend on two parameters)

$$h_1 \lor h'_2 = \begin{pmatrix} a_1 - b_1 \\ -5a_1 + 3b_1 \\ 2a_1 + 2b_1 \end{pmatrix}; \quad h'_1 \lor h_2 = \begin{pmatrix} A_1 + B_1 \\ 3A_1 + B_1 \\ 2A_1 + B_1 \end{pmatrix},$$

$$h_1 \lor h'_3 = \begin{pmatrix} a_1 - b_1 \\ -5a_1 + 9b_1 \\ 2a_1 + 6b_1 \end{pmatrix}; \quad h'_1 \lor h_3 = \begin{pmatrix} A_1 + 4B_1 \\ 3A_1 - 2B_1 \\ 2A_1 + 5B_1 \end{pmatrix},$$

$$h_2 \lor h'_3 = \begin{pmatrix} a_1 - b_1 \\ a_1 + 9b_1 \\ a_1 + 6b_1 \end{pmatrix}; \quad h'_2 \lor h_3 = \begin{pmatrix} -A_1 + 4B_1 \\ 3A_1 - 2B_1 \\ 2A_1 + 5B_1 \end{pmatrix}. \tag{22}$$

From this we find the points

$$\begin{pmatrix} (h_1 \lor h'_2) \land (h'_1 \lor h_2) = \begin{pmatrix} 5 \\ -17 \\ -6 \end{pmatrix} \\ (h_1 \lor h'_3) \land (h'_1 \lor h_3) = \begin{pmatrix} 25 \\ -149 \\ 2 \end{pmatrix} \\ (h_2 \lor h'_3) \land (h'_2 \lor h_3) = \begin{pmatrix} -30 \\ 80 \\ 47 \end{pmatrix}. \tag{23}$$
These points belong to the same line, because we have checked numerically that

$$\text{rank} \begin{pmatrix} 5 & 25 & -30 \\ -17 & -149 & 80 \\ -6 & 2 & 47 \end{pmatrix} = 2.$$  \hspace{1cm} (24)

6. Discussion
Quantum Mechanics of finite quantum systems is a special case of modular orthocomplemented lattices. We have discussed two of the many extra properties that we need to add into modular orthocomplemented lattices, in order to get quantum mechanical structures for finite quantum systems. They are the Desargues property, and the Pappus property, which are familiar from projective geometry.

The Desargues property can be interpreted physically in terms of selective quantum correlations. The Pappus property is a constraint that does not hold in the case of a general modular orthocomplemented lattice, but does hold in quantum mechanics of finite quantum systems.

The work discusses ‘quantumness’ from the angle of lattice theory and projective geometry. There are deep links between quantum mechanics and projective geometry which require further study, and the ‘dictionary’ in section II is helpful in this direction. In the present work we discussed the Desargues and Pappus properties.

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