On the longest common subsequence of independent random permutations invariant under conjugation

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Abstract

Bukh and Zhou [2016] conjectured that the expectation of the length of the longest common subsequence of two i.i.d random permutations of size $n$ is greater than $\sqrt{n}$. We prove in this paper that there exists a universal constant $n_1$ such that their conjecture is satisfied for any pair of i.i.d random permutations of size greater than $n_1$ with distribution invariant under conjugation. We prove also that asymptotically, this expectation is at least of order $2\sqrt{n}$ which is the asymptotic behaviour of the uniform setting. More generally, in the case where the laws of the two permutations are not necessarily the same, we give a lower bound for the expectation. In particular, we prove that if one of the permutations is invariant under conjugation and with a good control of the expectation of the number of its cycles, the limiting fluctuations of the length of the longest common subsequence are of Tracy-Widom type. This result holds independently of the law of the second permutation.

Keywords: Random permutations, longest increasing subsequence, longest common subsequence, Tracy-Widom distribution.

1 Introduction and main results

Let $\mathfrak{S}_n$ be the symmetric group, namely the group of permutations of $\{1, \ldots, n\}$. Given $\sigma \in \mathfrak{S}_n$, $(\sigma(i_1), \ldots, \sigma(i_k))$ is a subsequence of $\sigma$ of length $k$ if $i_1 < i_2 < \cdots < i_k$. We denote by $LCS(\sigma_1, \sigma_2)$ the length of the longest common subsequence (LCS) of two permutations.

In the sequel of this article, we consider two sequences of random permutations $(\sigma_{1,n})_{n \geq 1}$ and $(\sigma_{2,n})_{n \geq 1}$ with joint distribution $\mathbb{P}$ and associated expectation $\mathbb{E}$ such that $\sigma_{1,n}$ and $\sigma_{2,n}$ are independent and supported on $\mathfrak{S}_n$. The study of the LCS of independent random

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permutations was initiated by Houdré and Işlak [2014]. Recently Houdré and Xu [2018]
showed that for i.i.d random permutations
\[ E(\text{LCS}(\sigma_{1,n}, \sigma_{2,n})) \geq \sqrt{n}. \]
It is conjectured by [Bukh and Zhou, 2016] that for i.i.d random permutations,
\[ E(\text{LCS}(\sigma_{1,n}, \sigma_{2,n})) \geq \sqrt{n}. \]
In this article, we obtain asymptotic bounds in the scale of \( \sqrt{n} \) in the case where the law
of at least one of the two permutations is invariant under conjugation. We say that the law
of \( \sigma_n \) is invariant under conjugation if for any \( \hat{\sigma} \in S_n \), \( \hat{\sigma} \circ \sigma_n \circ \hat{\sigma}^{-1} \) is equal in distribution
to \( \sigma_n \).

1.1 LCS of two independent random permutations with distribution invariant under conjugation

In Theorem 1, we give an asymptotic lower bound for the LCS of two independent random
permutations. Under a good control of the number of fixed points, we give a better bound in
Proposition 2. Finally, as an application of Proposition 2, we give an asymptotically optimal
lower bound for i.i.d random permutations with distributions invariant under conjugation
in Corollary 3.

**Theorem 1.** Assume that for any \( n \geq 1 \), \( \sigma_{1,n} \) and \( \sigma_{2,n} \) are independent and their distributions are invariant under conjugation. Then
\[
\liminf_{n \to \infty} \frac{E(\text{LCS}(\sigma_{1,n}, \sigma_{2,n}))}{\sqrt{n}} \geq 2 \sqrt{\theta} \simeq 0.564,
\]
where \( \theta \) is the unique solution of \( G(2 \sqrt{x}) = \frac{2 + x}{12} \),
\[
G := [0, 2] \to \left[ 0, \frac{1}{2} \right],
\]
\[
x \mapsto \int_{-1}^{1} \left( \Omega(s) - \left| s + \frac{x}{2} \right| - \frac{x}{2} \right) \, ds,
\]
and
\[
\Omega(s) := \begin{cases} 
\frac{2}{\pi} (s \arcsin(s) + \sqrt{1 - s^2}) & \text{if } |s| < 1 \\
|s| & \text{if } |s| \geq 1.
\end{cases}
\]

The function \( \Omega \) appears as the Vershik-Kerov-Lagan-Shepp limit shape. For more de-
tails, one can see Figure 2 and Lemma 9. We will prove this result in Subsection 2.3 by
comparison with the uniform distribution on \( S_n \) and the uniform distribution on the set of
involutions.
Under a good control of the number of fixed points, we obtain a better bound.

**Proposition 2.** Let \(0 \leq \alpha \leq 2\). Assume that for any \(n \geq 1\), \(\sigma_{1,n}\) and \(\sigma_{2,n}\) are independent and their distributions are invariant under conjugation.

- If
  \[
  \lim_{n \to \infty} \max\left(\mathbb{P}(\sigma_{1,n}(1) = 1), \mathbb{P}(\sigma_{2,n}(1) = 1)\right) = 0,
  \]
  then
  \[
  \liminf_{n \to \infty} \frac{\mathbb{E}(\text{LCS}(\sigma_{1,n}, \sigma_{2,n}))}{\sqrt{n}} \geq 2.
  \]

- If
  \[
  \liminf_{n \to \infty} \sqrt{n}\mathbb{P}(\sigma_{1,n}(1) = 1)\mathbb{P}(\sigma_{2,n}(1) = 1) \geq \alpha,
  \]
  then
  \[
  \liminf_{n \to \infty} \frac{\mathbb{E}(\text{LCS}(\sigma_{1,n}, \sigma_{2,n}))}{\sqrt{n}} \geq \alpha.
  \]

Consequently, we obtain the following result for i.i.d random permutations.

**Corollary 3.** Assume that for any \(n \geq 1\), \(\sigma_{1,n}\) and \(\sigma_{2,n}\) are two independent and identically distributed random permutations with distribution invariant under conjugation. Then

\[
\liminf_{n \to \infty} \frac{\mathbb{E}(\text{LCS}(\sigma_{1,n}, \sigma_{2,n}))}{\sqrt{n}} \geq 2.
\]

We conjecture that we can get rid of (2) and (4); the stability under conjugation is sufficient to obtain (3) which is equivalent to replace \(2\sqrt{\theta}\) by 2 in Theorem 1. We will prove Proposition 2 and Corollary 3 in Subsection 2.2. The idea of the proof is to study the longest increasing subsequence of \(\sigma_{1,n}^{-1} \circ \sigma_{2,n}\), knowing that under a good control of the number of fixed points of the two permutations, the number of cycles of \(\sigma_{1,n}^{-1} \circ \sigma_{2,n}\) is sufficiently small to compare it with the uniform distribution.

### 1.2 LCS of two independent random permutations where one of the distributions is invariant under conjugation

When \(\sigma_{2,n}\) is not invariant under conjugation, we give an asymptotic lower bound of \(\frac{\mathbb{E}(\text{LCS}(\sigma_{1,n}, \sigma_{2,n}))}{\sqrt{n}}\) in Theorem 4. Moreover, we prove in Proposition 5 that under a good control of the number of cycles of \(\sigma_{1,n}\), \(\lim_{n \to \infty} \frac{\mathbb{E}(\text{LCS}(\sigma_{1,n}, \sigma_{2,n}))}{\sqrt{n}} = 2\) and under a stronger control, we have Tracy-Widom fluctuations for \(\text{LCS}(\sigma_{1,n}, \sigma_{2,n})\).
Theorem 4. Assume that for any $n \geq 1$, $\sigma_{1,n}$ and $\sigma_{2,n}$ are independent and the law of $\sigma_{1,n}$ is invariant under conjugation. Then

$$\liminf_{n \to \infty} \frac{\mathbb{E}(LCS(\sigma_{1,n},\sigma_{2,n}))}{\sqrt{n}} \geq G^{-1} \left( \liminf_{n \to \infty} \frac{\mathbb{E}(\#(\sigma_{1,n}))}{2n} \right),$$

where $\#(\sigma)$ is the number of cycles of $\sigma$ and $G$ is defined in (1).

In particular, if $\lim_{n \to \infty} \mathbb{E}\left(\frac{\#(\sigma_{1,n})}{n}\right) = 0$, we have

$$\liminf_{n \to \infty} \frac{\mathbb{E}(LCS(\sigma_{1,n},\sigma_{2,n}))}{\sqrt{n}} \geq 2.$$

Proposition 5. Assume that for any $n \geq 1$, $\sigma_{1,n}$ and $\sigma_{2,n}$ are independent and the law of $\sigma_{1,n}$ is invariant under conjugation.

- If $\frac{\#(\sigma_{1,n})}{\sqrt{n}} \overset{\mathbb{P}}{\to} 0$, then $\forall s \in \mathbb{R}$,

$$\lim_{n \to \infty} \mathbb{P}\left( \frac{LCS(\sigma_{1,n},\sigma_{2,n}) - 2\sqrt{n}}{\sqrt{n}} \leq s \right) = F_2(s),$$

where $F_2$ is the cumulative distribution function of the Tracy-Widom distribution.

- If $\frac{\#(\sigma_{1,n})}{\sqrt{n}} \overset{\mathbb{P}}{\to} 0$, then $\frac{LCS(\sigma_{1,n},\sigma_{2,n})}{\sqrt{n}} \overset{\mathbb{P}}{\to} 2$.

- If $\lim_{n \to \infty} \mathbb{E}\left(\frac{\#(\sigma_{1,n})}{\sqrt{n}}\right) = 0$, then $\lim_{n \to \infty} \frac{\mathbb{E}(LCS(\sigma_{1,n},\sigma_{2,n}))}{\sqrt{n}} = 2$.

Note that in Theorem 4 and in Proposition 5, we do not have any assumption on the distribution of $\sigma_{2,n}$. The proof in Subsection 2.4 is based on a coupling argument between $\sigma_{1,n}$ and a uniform permutation.

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2 Proof of results

2.1 General tools

Given $\sigma \in \mathfrak{S}_n$ and $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, the subsequence $(\sigma(i_1), \ldots, \sigma(i_k))$ is an increasing subsequence of $\sigma$ if $\sigma(i_1) < \cdots < \sigma(i_k)$. We denote by $\ell(\sigma)$ the length of the
longest increasing subsequence of $\sigma$.
For example, for the permutation
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 1 & 4 \end{pmatrix},$$
we have $\ell(\sigma) = 2$. The study of the longest common subsequence is strongly related to the notion of longest increasing subsequence. More precisely, we have the following.

**Proposition 6.** Let $\sigma_1, \sigma_2 \in S_n$.

$$LCS(\sigma_1, \sigma_2) = \ell(\sigma_1^{-1} \circ \sigma_2) = \ell(\sigma_2^{-1} \circ \sigma_1).$$

**Proof.** It is clear that the length of the longest common subsequence is invariant under left composition. Consequently

$$LCS(\sigma_1, \sigma_2) = LCS(\sigma_1^{-1} \circ \sigma_1, \sigma_1^{-1} \circ \sigma_2) = LCS(Id_n, \sigma_1^{-1} \circ \sigma_2).$$

Observe that by definition, the subsequences of $Id_n$ are the increasing subsequences which concludes the proof. \qed

We will use in the remainder of this paper the Robinson–Schensted correspondence [Robinson, 1938, Schensted, 1961] or the Robinson–Schensted–Knuth correspondence [Knuth, 1970]. We denote by

$$\lambda(\sigma) = \{\lambda_i(\sigma)\}_{i \geq 1},$$

the shape of the image of $\sigma$ by this correspondence. We will not include here detailed description of the algorithm. For further reading, we recommend [Sagan, 2001, Chapter 3].

We denote by

$$I_1(\sigma) : = \{s \subset \{1, 2, \ldots, n\}; \forall i, j \in s, (i - j)(\sigma(i) - \sigma(j)) \geq 0\},$$

$$I_{k+1}(\sigma) : = \{s \cup s', s \in I_k, s' \in I_1\}.$$

The link to the longest increasing subsequence is given by the following result.

**Lemma 7.** [Greene, 1974] For any permutation $\sigma \in S_n$,

$$\max_{s \in I_1(\sigma)} |s| = \sum_{k=1}^{i} \lambda_k(\sigma).$$

In particular,

$$\ell(\sigma) = \max_{s \in I_1(\sigma)} |s| = \lambda_1(\sigma).$$
Let $L_{\lambda(\sigma)}$ be the height function of $\lambda(\sigma)$ rotated by $\frac{3\pi}{4}$ and extended by the function $x \mapsto |x|$ to obtain a function defined on $\mathbb{R}$.
For example, if $\lambda(\sigma) = (7, 5, 2, 1, 1, 0)$, then the associated function $L_{\lambda(\sigma)}$ is represented by Figure 1.

The image of the uniform permutation by the Robinson-Schensted correspondence is known as the Plancherel measure. Its typical shape was studied separately by Logan and Shepp [1977] and Vershik and Kerov [1977]. Stronger results have been proved by Vershik and Kerov [1985]. In 1993, Kerov studied the limiting fluctuations but did not publish his results. One can see [Ivanov and Olshanski, 2002] for further details.

To prove our results, we will use the Markov operator $T$ defined on $\mathfrak{S}_n$ and associated to the stochastic matrix $\left[\frac{1_{A_{\sigma_1}}(\sigma_2)}{\text{card}(A_{\sigma_1})}\right]_{\sigma_1, \sigma_2 \in \mathfrak{S}_n}$ where

$$A_{\sigma} = \begin{cases} \{\sigma\} & \text{if } \#(\sigma) = 1 \\ \{\rho \in \mathfrak{S}_n, \sigma^{-1} \circ \rho = (i_1, i_2) \circ (i_1, i_3) \cdots \circ (i_1, i_{\#(\sigma)}) \text{ and } \#(\rho) = 1\} & \text{if } \#(\sigma) > 1 \end{cases}.$$ 

We recall that $\#(\sigma)$ is the number of cycles of $\sigma$. $T$ is then the Markov operator mapping a permutation $\sigma$ to a permutation uniformly chosen among the permutations obtained by merging the cycles of $\sigma$ using transpositions having all a common point. Note that $A_{\sigma}$ is not empty since any choice of one point in each cycle gives a possible $(i_1, i_2, \ldots i_{\#(\sigma)})$ and a correspondent permutation $\rho$. 

Figure 1: $L_{(7,5,2,1,1,0)}$
Lemma 8. For any permutation $\sigma$,

- Almost surely,
  \[
  |\ell(T(\sigma)) - \ell(\sigma)| \leq \#(\sigma).  \tag{6}
  \]

- More generally, almost surely,
  \[
  \max_{i \geq 1} \left| \sum_{k=1}^i (\lambda_k(\sigma) - \lambda_k(T(\sigma))) \right| \leq \#(\sigma).  \tag{7}
  \]

Moreover, for any random permutation $\sigma_n$ invariant under conjugation on $\mathfrak{S}_n$, the law of $T(\sigma_n)$ is the uniform distribution on permutations with a unique cycle.

Note that the uniform distribution on permutations with a unique cycle is also known as the Ewens’s distribution with parameter 0. We denote it by $Ew(0)$.

Proof. The law of $T(\sigma_n)$ is clearly invariant under conjugation. Indeed, let $\sigma, \rho \in \mathfrak{S}_n$.

\[
P(T(\sigma_n) = \sigma) = 1_{\#(\sigma)=1} \sum_{\delta \in \mathfrak{S}_n} 1_{\sigma \in A_\delta} \frac{P(\sigma_n = \delta)}{\text{card}(A_\delta)}
\]

\[
= 1_{\#(\sigma)=1} \sum_{\delta \in \mathfrak{S}_n} 1_{\rho \circ \sigma \circ \rho^{-1} \in A_{\rho \circ \delta \circ \rho^{-1}}} \frac{P(\rho \circ \sigma_n \circ \rho^{-1} = \delta)}{\text{card}(A_\delta)}
\]

\[
= 1_{\#(\sigma)=1} \sum_{\delta \in \mathfrak{S}_n} 1_{\rho \circ \sigma \circ \rho^{-1} \in A_\delta} \frac{P(\rho \circ \sigma_n \circ \rho^{-1} = \delta)}{\text{card}(A_\delta)}
\]

\[
= 1_{\#(\rho \circ \sigma \circ \rho^{-1})=1} \sum_{\delta \in \mathfrak{S}_n} 1_{\rho \circ \sigma \circ \rho^{-1} \in A_\delta} \frac{P(\sigma_n = \delta)}{\text{card}(A_\delta)}
\]

\[
= P(T(\sigma_n) = \rho \circ \sigma \circ \rho^{-1}).
\]

Moreover, by construction, almost surely, $\#(T(\sigma_n)) = 1$. Consequently, the law of $T(\sigma_n)$ is $Ew(0)$.

Let $\sigma$ be a permutation. By definition of $\ell(\sigma)$, there exists $i_1 < i_2 < \cdots < i_{\ell(\sigma)}$ such that $\sigma(i_1) < \cdots < \sigma(i_{\ell(\sigma)})$. Let $\rho = \sigma \circ (j_1, j_2) \circ (j_1, j_3) \cdots \circ (j_1, j_{\#(\sigma)})$ be a permutation with a unique cycle and $i'_1, i'_2, \ldots, i'_m$ be the same sequence as $i_1, i_2, \ldots, i_{\ell(\sigma)}$ after removing $j_1, j_2, \ldots, j_{\#(\sigma)}$ if needed. We have $\ell(\sigma) - \#(\sigma) \leq m$ and $\sigma(i'_1) < \cdots < \sigma(i'_m)$. Knowing that $\forall i \notin \{j_1, j_2, \ldots, j_{\#(\sigma)}\}$, $\rho(i) = \sigma(i)$, so that

\[
\rho(i'_1) < \cdots < \rho(i'_m).
\]

Therefore, $m \leq \ell(\rho)$ and

\[
\ell(\sigma) - \ell(\rho) \leq \#(\sigma).
\]
We can obtain the reverse inequality in (6) using the same techniques. Similarly, to prove (7), let \( l \geq 1 \) and \( \{i_1, i_2, \ldots, i_{\sum_{k=1}^l \lambda_k(\sigma)}\} \in \mathcal{F}_l(\sigma) \). Let \( i'_1, i'_2, \ldots, i'_m \) be the same sequence as \( i_1, i_2, \ldots, i_{\ell(\sigma)} \) after removing \( j_1, j_2, \ldots, j_{\#(\sigma)} \) if needed. We have \( \{i'_1, i'_2, \ldots, i'_m\} \in \mathcal{F}_l(\rho) \) and we conclude as in the proof of (6).

For more details, one can see [Kammoun, 2018]. We used the same techniques of proof with a different Markov operator. Here, the bound is better thanks to the use of the same point \( i_1 \) to merge cycles.

**Lemma 9.** [Kammoun, 2018, Theorem 1.8] Assume that the distribution of \( \sigma_n \) is \( Ew(0) \). Then for all \( \varepsilon > 0 \),

\[
\lim_{n \to \infty} \mathbb{P}\left( \sup_{s \in \mathbb{R}} \left| \frac{1}{\sqrt{2n}} L_{\lambda(\sigma_n)} \left( s \sqrt{2n} \right) - \Omega(s) \right| < \varepsilon \right) = 1,
\]

where we recall that

\[
\Omega(s) := \begin{cases} 
\frac{2}{\pi} (s \arcsin(s) + \sqrt{1 - s^2}) & \text{if } |s| < 1 \\
|s| & \text{if } |s| \geq 1
\end{cases}.
\]

For the remainder of this paper, we will refer to this limiting shape as the Vershik-Kerov-Logan-Shepp shape. See Figure 2\(^1\). This convergence is closely related to the Wigner’s semi-circular law. For further details, one can see [Kerov, 1993a,b, 1999].

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Figure 2: Illustration of the Vershik-Kerov-Logan-Shepp convergence

\(^1\)This figure is generated by DPPy [Gautier, Bardenet, and Valko, 2018]
Corollary 10. Assume that the distribution of $\sigma_n$ is $Ew(0)$. Then for any $0 \leq \gamma \leq 2$, for any $\varepsilon > 0$,
\[
P\left(\sum_{i=1}^{n}(\frac{\lambda_i(\sigma_n) - \gamma \sqrt{n}}{\sqrt{2n}}) > 2G(\gamma) - \varepsilon\right) \to 1.
\]

Proof. This is a direct application of Lemma 9. One can see that $\sum_{i=1}^{n}(\frac{\lambda_i(\sigma_n) - \gamma \sqrt{n}}{\sqrt{2n}})$ is the area of the region delimited by the curves of the functions $x \mapsto |x|$, $x \mapsto \gamma + x$ and $x \mapsto \frac{L_{\lambda(\sigma)}(s\sqrt{2n})}{\sqrt{2n}}$, see Figure 3. By construction, this area is equal to
\[
\int_{-\infty}^{\infty} \left(\frac{L_{\lambda(\sigma)}(s\sqrt{2n})}{\sqrt{2n}} - \frac{|s + \gamma| - \gamma}{2}\right)_+ ds.
\]
By Lemma 9,
\[
\int_{-1}^{1} \left(\frac{L_{\lambda(\sigma)}(s\sqrt{2n})}{\sqrt{2n}} - \frac{|s + \gamma| - \gamma}{2}\right)_+ ds \xrightarrow{P} G(\gamma).
\]
We can conclude then that
\[
\sum_{i=1}^{n}(\frac{\lambda_i(\sigma) - \gamma \sqrt{n}}{\sqrt{2n}})_+ = 2 \int_{-\infty}^{\infty} \left(\frac{L_{\lambda(\sigma)}(s\sqrt{2n})}{\sqrt{2n}} - \frac{|s + \gamma| - \gamma}{2}\right)_+ ds \geq 2 \int_{-1}^{1} \left(\frac{L_{\lambda(\sigma)}(s\sqrt{2n})}{\sqrt{2n}} - \frac{|s + \gamma| - \gamma}{2}\right)_+ ds \xrightarrow{P} 2G(\gamma).
\]
This yields (8). \qed
Figure 3: $\lambda = (7, 2, 2, 1, 0)$ and $\gamma = 1$

Note that it is not difficult to prove that
\[
\sum_{i=1}^{n}(\lambda_i(\sigma_n) - \gamma\sqrt{n})_+ \xrightarrow{p} 2G(\gamma).
\]
We skip the proof here as we only need (8) in the sequel.

**Corollary 11.** For any permutation $\sigma$, for any $\alpha \geq 0$, almost surely,
\[
\left| \sum_{i=1}^{\infty}(\lambda_i(\sigma) - \alpha\sqrt{n})_+ - \sum_{i=1}^{\infty}(\lambda_i(T(\sigma)) - \alpha\sqrt{n})_+ \right| \leq \#(\sigma).
\]

**Proof.** We prove first that
\[
\sum_{i=1}^{\infty}(\lambda_i(\sigma) - \alpha\sqrt{n})_+ - \sum_{i=1}^{\infty}(\lambda_i(T(\sigma)) - \alpha\sqrt{n})_+ \leq \#(\sigma).
\]
If $\lambda_1(\sigma) \leq \alpha\sqrt{n}$, the inequality is trivial as the right hand side is non-negative and the left hand side is non-positive. Otherwise, let $k := \max\{j \geq 1, \lambda_j(\sigma) > \alpha\sqrt{n}\}$. We have
\[
\sum_{i=1}^{\infty}(\lambda_i(\sigma) - \alpha\sqrt{n})_+ = \sum_{i=1}^{k}(\lambda_i(\sigma) - \alpha\sqrt{n})_+ + \sum_{i=k+1}^{\infty}(\lambda_i(\sigma_n) - \alpha\sqrt{n})_+ \\
= \sum_{i=1}^{k}(\lambda_i(\sigma) - \alpha\sqrt{n}),
\]
and
\[
\sum_{i=1}^{\infty} (\lambda_i(T(\sigma)) - \alpha \sqrt{n})_+ \geq \sum_{i=1}^{k} (\lambda_i(T(\sigma)) - \alpha \sqrt{n})_+ \geq \sum_{i=1}^{k} (\lambda_i(T(\sigma)) - \alpha \sqrt{n}).
\]

Using (7), we obtain
\[
\sum_{i=1}^{\infty} (\lambda_i(\sigma) - \alpha \sqrt{n})_+ - \sum_{i=1}^{\infty} (\lambda_i(T(\sigma)) - \alpha \sqrt{n})_+ \leq \sum_{i=1}^{k} \lambda_i(\sigma) - \lambda_i(T(\sigma)) \leq \#(\sigma).
\]

The reverse inequality is obtained by exchanging the role of $\sigma$ and $T(\sigma)$.

**Corollary 12.** For any $\alpha < 2$, there exist $\beta > 0$ and $n_\alpha > 0$ such that for any $n > n_\alpha$, for any random permutation $\sigma_n$ invariant under conjugation satisfying $E(\#(\sigma_n)) < n\beta$, we have

\[
E(\ell(\sigma_n)) \geq \alpha \sqrt{n}.
\]

**Proof.** This is a direct application of Corollary 10 and Corollary 11. Let $\alpha < \gamma < 2$, $\varepsilon > 0$ and $\beta > 0$ such that $1 - \frac{\beta}{\gamma} - \varepsilon > \frac{\alpha}{\gamma}$. By Corollary 10, we obtain the existence of $n_\alpha$ such that for any $n > n_\alpha$,

\[
P\left(\frac{\sum_{i=1}^{n} (\lambda_i(T(\sigma_n)) - \gamma \sqrt{n})_+}{n} > G(\gamma)\right) > 1 - \varepsilon.
\]

Since $\{\ell(\sigma) > k\}$ is equivalent to $\{\sum_{i=1}^{\infty} (\lambda_i(\sigma) - k)_+ > 0\}$ and by Markov inequality, we obtain

\[
E(\ell(\sigma)) \geq \gamma \sqrt{n}P(\ell(\sigma) \geq \gamma \sqrt{n}) \geq \gamma \sqrt{n}\left(\frac{\sum_{i=1}^{n} (\lambda_i(T(\sigma_n)) - \gamma \sqrt{n})_+}{n} > G(\gamma), \frac{\#(\sigma_n)}{n} < G(\gamma)\right) \geq \gamma \sqrt{n}\left(1 - \frac{\beta}{G(\gamma)} - \varepsilon\right) \geq \alpha \sqrt{n}.
\]

**Lemma 13.** Let $\sigma \in S_n$ and $\rho \in A_{\sigma}$, then

\[
\ell(\sigma) \geq \sup \left\{ k \in \mathbb{N}, \sum_{i=1}^{\infty} (\lambda_i(\rho) - k)_+ \geq \#(\sigma) \right\}
\]

and

\[
\ell(\rho) \geq \sup \left\{ k \in \mathbb{N}, \sum_{i=1}^{\infty} (\lambda_i(\sigma) - k)_+ \geq \#(\sigma) \right\}.
\]
Proof. By the equivalence between \( \{ \ell(\sigma) > k \} \) and \( \{ \sum_{i=1}^{\infty} (\lambda_i(\sigma) - k)_+ > 0 \} \), this a direct application of Corollary 11.

\[
\begin{align*}
2.2 \quad \text{Proof of Proposition 2 and Corollary 3} \\
\end{align*}
\]

To prove Proposition 2 and Corollary 3, we distinguish two cases. For the first case, we suppose that the number of fixed points is large enough. We use the fact that for a given permutation, the length of the longest increasing subsequence is bigger than the number of fixed points. For the second case, when the number of fixed points is controlled, we prove in Lemma 14 that the number of cycles of \((\sigma_1,n)^{-1} \circ \sigma_2,n\) is sufficiently controlled to use Corollary 12. In both cases, we can conclude by Proposition 6.

Lemma 14. For any \( k \geq 2 \), there exists \( C, C' > 0 \) such that for any \( n \geq 1 \), for any independent random permutations \( \sigma_1,n \) and \( \sigma_2,n \) with distributions invariant under conjugation,

\[
\mathbb{P}(c_1((\sigma_1,n)^{-1} \circ \sigma_2,n) = k) \leq \frac{C}{n} + C'(\mathbb{P}(\sigma_1,n(1) = 1) + \mathbb{P}(\sigma_2,n(1) = 1)),
\]

where \( c_m(\sigma) \) is the length of the cycle of \( \sigma \) containing \( m \).

To prove this result, we will introduce some new objects. To a couple of permutations, we will associate a couple of graphs.

We denote by \( G^n_k \) the set of oriented simple graphs with vertices \( \{1, 2, \ldots, n\} \) and having exactly \( k \) edges.

For example, \( G^2_1 = \left\{ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1 \quad 2 \\
1 \quad 2
\end{array}
\end{array} \\
\end{array} \right\} \).

Given \( g \in G^n_k \), we denote by \( E_g \) the set of its edges and by \( A_g := [1_{(i,j) \in E_g}]_{1 \leq i,j \leq n} \) its adjacency matrix. A connected component of \( g \) is called trivial if it does not have any edge and a vertex \( i \) of \( g \) is called isolated if \( E_g \) does not contain any edge of the form \((i, j)\) or \((j, i)\). We say that two oriented simple graphs \( g_1 \) and \( g_2 \) are isomorphic if one can obtain \( g_2 \) by changing the labels of the vertices of \( g_1 \). In particular, if \( g_1, g_2 \in G^n_k \) then \( g_1, g_2 \) are isomorphic if and only if there exists a permutation matrix \( \sigma \) such that \( A_{g_1}\sigma = \sigma A_{g_2} \). Let \( g \in G^n_k \), we denote by \( \tilde{g} \) the graph obtained from \( g \) after removing isolated vertices. Let \( \mathcal{R} \) be the equivalence relation such that \( g_1 \mathcal{R} g_2 \) if \( \tilde{g}_1 \) and \( \tilde{g}_2 \) are isomorphic. We denote by \( \hat{G}_k := \bigcup_{n \geq 1} G^n_k / \mathcal{R} \) the set of equivalence classes of \( \bigcup_{n \geq 1} G^n_k \) for the relation \( \mathcal{R} \).
For example, \( \mathcal{R} = \{ \circ, \triangle, \diamond \} \) and \( \mathcal{G}_1 = \{ [\bullet - \circ], [\circ - \bullet], [\circ - \diamond] \} \).

Let \( n \) be a positive integer and \( \sigma_1, \sigma_2 \in \mathcal{G}_n \). Let \( k_m := c_m(\sigma_1^{-1} \circ \sigma_2) \) with \( (i_1^m = m, i_2^m, \ldots, i_k^m) \) be the cycle of \( \sigma_1^{-1} \circ \sigma_2 \) containing \( m \) and \( j_1^m := \sigma_2(i_1^m) \). In particular, \( i_1^m, i_2^m, \ldots, i_k^m \) are pairwise distinct and \( j_1^m, j_2^m, \ldots, j_k^m \) are pairwise distinct. We denote by \( G^m_1(\sigma_1, \sigma_2) \in \mathcal{G}^m_k \) the graph such that \( E_{G^m_1(\sigma_1, \sigma_2)} = \{(i_1^m, j_k^m)\} \cup \left( \bigcup_{l=1}^{k_m} \{ (i_l^m, j_l^m) \} \right) \). We denote also by \( G^m_2(\sigma_1, \sigma_2) \in \mathcal{G}^m_k \) the graph such that \( E_{G^m_2(\sigma_1, \sigma_2)} = \bigcup_{l=1}^{k_m} \{ (i_l^m, j_l^m) \} \). In particular, \( G^m_1(\sigma_1, \sigma_2) \) and \( G^m_2(\sigma_1, \sigma_2) \) have the same set of non-isolated vertices. For \( i \in \{1, 2\} \), let \( \hat{G}^m_i(\sigma_1, \sigma_2) \) be the equivalence class of \( G^m_i(\sigma_1, \sigma_2) \).

For example, if

\[
\sigma_1 = \left( \begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
5 & 3 & 2 & 1 & 4
\end{array} \right)
\quad \text{and} \quad
\sigma_2 = \left( \begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
2 & 3 & 5 & 1 & 4
\end{array} \right),
\]

we obtain \( E_{G^m_1(\sigma_1, \sigma_2)} = \{(1, 5), (3, 2)\} \), \( E_{G^m_2(\sigma_1, \sigma_2)} = \{(1, 2), (3, 5)\} \),

\[
\begin{array}{c}
\text{\( G^m_1(\sigma_1, \sigma_2) = \)} \end{array}
\]

\[
\begin{array}{c}
\text{\( G^m_2(\sigma_1, \sigma_2) = \)} \end{array}
\]

Finally, given \( g \in \mathcal{G}^n_k \), we denote by

\[
\mathcal{G}_{n,g} := \{ \sigma \in \mathcal{G}_n; \forall (i, j) \in E_g, \sigma(i) = j \}.
\]

It is not difficult to prove the two following lemmas.

**Lemma 15.** If \( m_1 \in \{j^m_l; 1 \leq l \leq k_{m_2}\} \), then \( G^{m_1}_1(\sigma_1, \sigma_2) = G^{m_2}_1(\sigma_1, \sigma_2) \) and \( G^{m_1}_2(\sigma_1, \sigma_2) = G^{m_2}_2(\sigma_1, \sigma_2) \).

**Proof.** If \( m_1 \in \{j^m_l; 1 \leq l \leq k_{m_2}\} \), then there exists \( 1 \leq l \leq k_{m_1} \) such that \( (\sigma_1^{-1} \circ \sigma_2)^l(m_1) = m_2 \). Consequently, \( k_{m_1} = k_{m_2} \), \( (i_1^{m_1}, i_2^{m_1}, \ldots, i_{k_{m_1}}^{m_1}) = (i_1^{m_2}, i_2^{m_2}, \ldots, i_{k_{m_2}}^{m_2}) \) and \( (j_1^{m_1}, j_2^{m_1}, \ldots, j_{k_{m_2}}^{m_1}) = (j_1^{m_2}, j_2^{m_2}, \ldots, j_{k_{m_2}}^{m_2}) \) and we can check easily that

\[
G^{m_1}_1(\sigma_1, \sigma_2) = G^{m_2}_1(\sigma_1, \sigma_2) \quad \text{and} \quad G^{m_1}_2(\sigma_1, \sigma_2) = G^{m_2}_2(\sigma_1, \sigma_2).
\]

\[\square\]
Lemma 16. Let \( g_1, g_2 \in \mathbb{G}_k^n \). Assume that there exists \( \rho \in \mathcal{S}_n \) such that \( A_{g_2} \rho = \rho A_{g_1} \). If \( \rho \) has a fixed point on any non-trivial connected component of \( g_1 \), then \( \mathcal{S}_{n,g_1} \cap \mathcal{S}_{n,g_2} = \emptyset \) or \( A_{g_1} = A_{g_2} \).

Proof. Let \( \rho \in \mathcal{S}_n \) be a permutation having a fixed point on any non-trivial connected component of \( g_1 \) such that \( A_{g_2} \rho = \rho A_{g_1} \). Assume that \( A_{g_1} \neq A_{g_2} \). There exists necessarily \((i, j) \in E_{g_1} \) such that \( \rho(i) = i \) and \( \rho(j) \neq j \) or \( \rho(j) = j \) and \( \rho(i) \neq i \). In the first case, \( \mathcal{S}_{n,g_1} \cap \mathcal{S}_{n,g_2} \subseteq \{ \sigma \in \mathcal{S}_n; \sigma(i) = j, \sigma(i) = \rho(j) \} = \emptyset \). In the second case, \( \mathcal{S}_{n,g_1} \cap \mathcal{S}_{n,g_2} \subseteq \{ \sigma \in \mathcal{S}_n; \sigma(i) = j, \sigma(\rho(i)) = j \} = \emptyset \).

The following result is immediate.

Corollary 17. For any graph \( g \in \mathbb{G}_k^n \) having \( p \) non-trivial connected components and \( v \) non-isolated vertices, for any random permutation \( \sigma_n \) with distribution invariant under conjugation on \( \mathcal{S}_n \),

\[
\mathbb{P}(\sigma_n \in \mathcal{S}_{n,g}) \leq \frac{1}{(n-p)(v-p)!}.
\]

Proof. If there exist \( i, j, l \), with \( j \neq l \) such that \( \{(i, j) \cup (i, l)\} \subseteq E_g \) or \( \{(j, i) \cup (l, i)\} \subseteq E_g \) then \( \mathcal{S}_{n,g} = \emptyset \). Therefore, if \( \mathcal{S}_{n,g} \neq \emptyset \), then non-trivial connected components of \( g \) having \( w \) vertices are either cycles of length \( w \) or isomorphic to \( g_w \), where \( A_{g_w} = \{ 1 \leq i, j \leq w \} \). For example, \( g_5 = \begin{array}{c} 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \end{array} \). Let \( g \in \mathbb{G}_k^n \) such that \( \mathcal{S}_{n,g} \neq \emptyset \). Fix \( p \) vertices \( x_1, x_2, \ldots, x_p \) each belonging to a different non-trivial connected components of \( g \). Let \( \{x_1, x_2, \ldots, x_p, \ldots, x_v\} \) be the set of non-isolated vertices of \( g \). Let

\[
F = \{(y_i)_{p+1 \leq i \leq n}; y_i \in \{1, 2, \ldots, n\} \setminus \{x_1, \ldots, x_p\} \text{ pairwise distinct}\}.
\]

Given \( y = (y_i)_{p+1 \leq i \leq n} \in F \), we denote by \( g_y \in \mathbb{G}_k^n \) the graph isomorphic to \( g \) obtained by fixing the labels of \( x_1, x_2, \ldots, x_p \) and by changing the labels of \( x_i \) by \( y_i \) for \( p+1 \leq i \leq v \). Since non trivial connected components of \( g \) of length \( w \) are either cycles or isomorphic to \( g_w \), if \( y \neq y' \in F \), then \( g_y \neq g_y' \) and by Lemma 16, \( \mathcal{S}_{n,g_y} \cap \mathcal{S}_{n,g_y'} = \emptyset \). Since \( \sigma_n \) is invariant under conjugation, we have \( \mathbb{P}(\sigma_n \in \mathcal{S}_{n,g_y}) = \mathbb{P}(\sigma_n \in \mathcal{S}_{n,g_y'}) = \mathbb{P}(\sigma_n \in \mathcal{S}_{n,g}) \). Therefore,

\[
\mathbb{P}(\sigma_n \in \mathcal{S}_{n,g}) = \frac{\sum_{y \in F} \mathbb{P}(\sigma_n \in \mathcal{S}_{n,g_y})}{\text{card}(F)} = \frac{\mathbb{P}(\sigma_n \in \cup_{y \in F} \mathcal{S}_{n,g_y})}{\text{card}(F)} \leq \frac{1}{\text{card}(F)} = \frac{1}{(n-p)(v-p)!}.
\]

We will now prove Lemma 14.
Proof of Lemma 14. Note that \( \hat{G}_k \) is finite. Therefore, it is sufficient to prove that for any \( \hat{g}_1, \hat{g}_2 \in \hat{G}_k \) having the same number of vertices, there exist two constants \( C_{\hat{g}_1, \hat{g}_2} \) and \( C'_{\hat{g}_1, \hat{g}_2} \) such that for any integer \( n \),

\[
\mathbb{P}((\hat{g}_1^{1}(\sigma_1, n, \sigma_2, n), \hat{g}_2^{1}(\sigma_1, n, \sigma_2, n)) = (\hat{g}_1, \hat{g}_2)) \leq \frac{C_{\hat{g}_1, \hat{g}_2}}{n} + C'_{\hat{g}_1, \hat{g}_2}(\mathbb{P}(\sigma_1(n) = 1) + \mathbb{P}(\sigma_2(n) = 1)).
\]

Let \( \hat{g}_1, \hat{g}_2 \in \hat{G}_k \) be two unlabeled graphs having respectively \( p_1 \) and \( p_2 \) connected component and \( v \leq 2k \) vertices. Let \( B_{\hat{g}_1, \hat{g}_2} \) be the set of couples \( (g_1, g_2) \in (G_k^m)^2 \) having the same non-isolated vertices such that 1 is a non-isolated vertex of both graphs and, for \( i \in \{1, 2\} \), the equivalence class of \( g_i \) is \( \hat{g}_i \).

- Suppose that \( \hat{g}_1 \) and \( \hat{g}_2 \) do not contain any loop i.e no edges of type \((i, i)\). Then \( p_1 \leq \frac{v}{2} \) and \( p_2 \leq \frac{v}{2} \). Consequently,

\[
\mathbb{P}((\hat{g}_1^{1}(\sigma_1, n, \sigma_2, n), \hat{g}_2^{1}(\sigma_1, n, \sigma_2, n)) = (\hat{g}_1, \hat{g}_2))
\]

\[
= \sum_{(g_1, g_2) \in B_{\hat{g}_1, \hat{g}_2}^{n}} \mathbb{P}((\hat{g}_1^{1}(\sigma_1, n, \sigma_2, n), \hat{g}_2^{1}(\sigma_1, n, \sigma_2, n)) = (g_1, g_2))
\]

\[
\leq \sum_{(g_1, g_2) \in B_{\hat{g}_1, \hat{g}_2}^{n}} \mathbb{P}(\sigma_1(n) \in S_{n, g_1}, \sigma_2(n) \in S_{n, g_2})
\]

\[
= \sum_{(g_1, g_2) \in B_{\hat{g}_1, \hat{g}_2}^{n}} \mathbb{P}(\sigma_1(n) \in S_{n, g_1})\mathbb{P}(\sigma_2(n) \in S_{n, g_2})
\]

\[
\leq \sum_{(g_1, g_2) \in B_{\hat{g}_1, \hat{g}_2}^{n}} \frac{1}{(v-p_1)!} \frac{1}{(v-p_2)!}
\]

\[
= \text{card}(B_{\hat{g}_1, \hat{g}_2}^{n})
\]

\[
\leq \frac{(n-1)!}{v^2}
\]

\[
\leq C_{g_1, g_2} n^{v-1-(v-p_1+p_2)} = C_{g_1, g_2} n^{p_1+p_2-v-1} \leq \frac{C_{g_1, g_2}}{n}.
\]

- Suppose that \( \hat{g}_1 \) contains a loop. By Lemma 15, if \( \hat{g}_1^{m}(\sigma_1, \sigma_2) = \hat{g}_1 \), then there exists \( j \) a fixed point of \( \sigma_1 \) such that \( k_j = k \) and \( j \in \{i^m_l, 1 \leq l \leq k\} \). Thus, almost surely,

\[
\sum_{i=1}^{n} 1_{\hat{g}_1^{1}(\sigma_1, n, \sigma_2, n) = \hat{g}_1} \leq k \text{card} \{ i \in \text{fix}(\sigma_1, n); k_i = k \} \leq k \text{card}(\text{fix}(\sigma_1, n)),
\]
where \( \text{fix}(\sigma) \) is the set of fixed points of \( \sigma \). Consequently, since \( \sigma_{1,n} \) is invariant under conjugation,

\[
\mathbb{P}\left( \left( \hat{G}_1^1(\sigma_{1,n}, \sigma_{2,n}), \hat{G}_2^1(\sigma_{1,n}, \sigma_{2,n}) \right) = (\hat{g}_1, \hat{g}_2) \right) \leq \mathbb{P}\left( \hat{G}_1^1(\sigma_{1,n}, \sigma_{2,n}) = \hat{g}_1 \right) = \frac{\sum_{i=1}^{\mathbb{E}(\text{fix}(\sigma_{1,n}))} \mathbb{P}\left( \hat{G}_i^1(\sigma_{1,n}, \sigma_{2,n}) = \hat{g}_1 \right)}{n} \leq k \frac{\mathbb{E}(\text{card}(\text{fix}(\sigma_{1,n})))}{n} = k \mathbb{P}(\sigma_{1,n}(1) = 1).
\]

Similarly, if \( \hat{g}_2 \) contains a loop, then

\[
\mathbb{P}\left( \left( \hat{G}_1^1(\sigma_{1,n}, \sigma_{2,n}), \hat{G}_2^1(\sigma_{1,n}, \sigma_{2,n}) \right) = (\hat{g}_1, \hat{g}_2) \right) \leq k \mathbb{P}(\sigma_{2,n}(1) = 1).
\]

We will now prove Proposition 2.

\textbf{Proof of Proposition 2.} Under the condition of Proposition 2,

- Assume that

\[
\liminf_{n \to \infty} \frac{\mathbb{E}(\text{LCS}(\sigma_{1,n}, \sigma_{2,n}))}{\sqrt{n}} \geq \liminf_{n \to \infty} \frac{\mathbb{E}(\text{card}(\text{fix}(\sigma_{1,n} \circ \sigma_{2,n}^{-1})))}{\sqrt{n}} \geq \liminf_{n \to \infty} \frac{\mathbb{E}(\text{card}(\text{fix}(\sigma_{1,n})))}{\sqrt{n}} \geq \mathbb{P}(\sigma_{1,n}(1) = 1) \mathbb{P}(\sigma_{2,n}(1) = 1) \geq \alpha.
\]

- Assume that

\[
\lim_{n \to \infty} \max(\mathbb{P}(\sigma_{1,n}(1) = 1), \mathbb{P}(\sigma_{2,n}(1) = 1)) = 0. \tag{9}
\]

In this case,

\[
\mathbb{P}\left( \sigma_{1,n}^{-1} \circ \sigma_{2,n}(1) = 1 \right) = \sum_{i=1}^{n} \mathbb{P}(\sigma_{1,n}(1) = i) \mathbb{P}(\sigma_{2,n}(1) = i) = \mathbb{P}(\sigma_{1,n}(1) = 1) \mathbb{P}(\sigma_{2,n}(1) = 1) + (1 - \mathbb{P}(\sigma_{1,n}(1) = 1))(1 - \mathbb{P}(\sigma_{2,n}(1) = 1)) \frac{n-1}{n-1} = o(1).
\]
For any random permutation $\sigma_n \in S_n$ invariant under conjugation,

$$
E(\#(\sigma_n)) = E\left( \sum_{i=1}^{n} \frac{1}{c_i(\sigma_n)} \right) = \sum_{i=1}^{n} E\left( \frac{1}{c_i(\sigma_n)} \right) = nE\left( \frac{1}{c_1(\sigma_n)} \right),
$$

and for $n_\beta := \lfloor \frac{1}{\beta} \rfloor + 1$, with the same $\beta$ as in Corollary 12,

$$
\frac{E(\#(\sigma_n))}{n} = \sum_{k=1}^{\infty} \frac{1}{k} \mathbb{P}(c_1(\sigma_n) = k)
\leq \mathbb{P}(c_1(\sigma_n) = 1) + \sum_{k=2}^{n_\beta} \mathbb{P}(c_1(\sigma_n) = k) + \frac{1}{n_\beta + 1} \sum_{k=n_\beta + 1}^{\infty} \mathbb{P}(c_1(\sigma_n) = k)
\leq \mathbb{P}(\sigma_n(1) = 1) + \sum_{k=2}^{n_\beta} \mathbb{P}(c_1(\sigma_n) = k) + \frac{1}{n_\beta + 1}.
$$

Consequently, under (9), by Lemma 14, we have

$$
\frac{E(\#(\sigma_{1,n} \circ \sigma_{2,n}^{-1}))}{n} \leq \frac{1}{n_\beta + 1} + o(1) < \beta + o(1).
$$

Hence, we obtain Proposition 2 thanks to Corollary 12.

\[ \square \]

Proof of Corollary 3. This is a direct application of Proposition 2. In fact, if

$$
\mathbb{P}(\sigma_{1,n}(1) = 1) \geq \frac{\sqrt{2}}{\sqrt{n}}
$$

then

$$
\lim_{n \to \infty} \sqrt{n} \mathbb{P}(\sigma_{1,n}(1) = 1)\mathbb{P}(\sigma_{2,n}(1) = 1) \geq 2.
$$

Otherwise,

$$
\lim_{n \to \infty} \max(\mathbb{P}(\sigma_{1,n}(1) = 1), \mathbb{P}(\sigma_{2,n}(1) = 1)) = 0.
$$

\[ \square \]

2.3 Proof of Theorem 1

By observing that if $\sigma_{1,n}$ and $\sigma_{2,n}$ are independent random permutations with distribution invariant under conjugation then $\sigma_{1,n}^{-1} \circ \sigma_{2,n}$ is invariant under conjugation, proving Theorem 1 is equivalent to prove the following.
Theorem 18. For any sequence of random permutations \( \{\sigma_n\}_{n \geq 1} \) invariant under conjugation,

\[
\liminf_{n \to \infty} \frac{\mathbb{E}(\ell(\sigma_n))}{\sqrt{n}} \geq 2\sqrt{\theta}.
\]

The argument will be by comparison with the uniform measure on \( \mathfrak{S}_n \) and the uniform measure on the set of involutions. We will use the uniform permutation on \( \mathfrak{S}_n \) if we have a few number of cycles. Otherwise, we will use the uniform measure on the set of involution since it has approximately \( \frac{n}{2} \) cycles with high probability. In this section, we denote by \( \mathfrak{S}_n^2 := \{ \sigma \in \mathfrak{S}_n, \sigma \circ \sigma = Id_n \} \) the set of involution of \( \mathfrak{S}_n \). If \( \sigma_n \) is distributed according to the uniform distribution on \( \mathfrak{S}_n^2 \), the distribution of \( \lambda(\sigma_n) \) on the set of Young diagrams \( \mathcal{Y}_n \) is known as the Gelfand distribution. In particular, we have the following results.

Proposition 19. [Méliot, 2011, Theorem 1] If \( \sigma_n \) is distributed according to the uniform distribution on \( \mathfrak{S}_n^2 \), then

\[
\lim_{n \to \infty} \mathbb{P}\left( \sup_{s \in \mathbb{R}} \left| \frac{1}{\sqrt{2n}} L_{\lambda(\sigma_n)} \left( s \sqrt{2n} \right) - \Omega(s) \right| < \varepsilon \right) = 1.
\]

Proposition 20. [Flajolet and Sedgewick, 2009, Page 692, Proposition IX.19] If \( \sigma_n \) is distributed according to the uniform distribution on \( \mathfrak{S}_n^2 \) then

\[
\lim_{n \to \infty} \frac{\mathbb{E}(\text{card}(\text{fix}(\sigma_n))))}{\sqrt{n}} = 1.
\]

We will now prove the following.

Corollary 21. If \( \sigma_n \) is invariant under conjugation and supported on \( \mathfrak{S}_n^2 \) then

\[
\liminf_{n \to \infty} \frac{\mathbb{E}(\ell(\sigma_n))}{\sqrt{n}} \geq 2.
\]

Idea of the proof. If \( \frac{\mathbb{E}(\text{card}(\text{fix}(\sigma_n))))}{\sqrt{n}} \geq 2 \) the result is trivial. Otherwise, the technique of proof is identical to that of Corollary 12. Going back to Lemma 8, we replace \( A_\sigma \) by

\[
A_\sigma' := \{ \rho \in \mathfrak{S}_n; \sigma = \rho \circ (i_1, i_2) \circ \cdots \circ (i_{\text{card}(\text{fix}(\sigma))}, i_{\text{card}(\text{fix}(\sigma))}), \text{fix}(\rho) = \emptyset \}
\]

if \( n \) is even and by

\[
A_\sigma' := \{ \rho \in \mathfrak{S}_n; \sigma = \rho \circ (i_1, i_2) \circ \cdots \circ (i_{\text{card}(\text{fix}(\sigma))}, i_{\text{card}(\text{fix}(\sigma))}, \text{fix}(\rho)) = 1 \}
\]

if \( n \) is odd. We denote by \( T' \) the Markov operator on \( \mathfrak{S}_n^2 \) associated to the stochastic matrix \( \left[ \frac{1}{\text{card}(A_{\sigma_1})} \right]_{\sigma_1, \sigma_2 \in \mathfrak{S}_n^2} \). That means that we merge couples of fixed points to obtain the uniform distribution on permutations having only cycles of length 2 when \( n \) is even and having an additional fixed point when \( n \) is odd. Similarly to that we did in Lemma 8, for any permutation \( \sigma \), we have the following.
- Almost surely, 

$$|\ell(T'(\sigma)) - \ell(\sigma)| \leq \text{card}(\text{fix}(\sigma)).$$

- More generally, almost surely, 

$$\max_{i \geq 1} \left| \sum_{k=1}^{i} (\lambda_k(\sigma) - \lambda_k(T'(\sigma))) \right| \leq \text{card}(\text{fix}(\sigma)).$$

Moreover, if $\sigma_n$ is invariant under conjugation, the law of $T'(\sigma_n)$ does not depend on the law of $\sigma_n$.

Consequently, Corollary 21 follows using the same techniques as in the proof of Corollary 12.

Corollary 22. Let $\{\sigma_n\}_{n \geq 1}$ be a sequence of random permutations each one being invariant under conjugation. Assume that there exists a sequence $(\beta_n)_{n \geq 1}$ such that 

$$\lim_{n \to \infty} \beta_n = +\infty,$$

and for any $n \geq 1,$

$$\mathbb{P}(\text{card}(\text{fix}(\sigma_n^2)) > \beta_n) = 1.$$ 

Then

$$\liminf_{n \to \infty} \frac{\mathbb{E}(\ell(\sigma_n))}{\sqrt{\text{card}(A)}} \geq 2.$$

Proof. Giving $A \subset \mathbb{N}$ finite, we denote by $\mathcal{S}_A$ (resp. $\mathcal{S}_A^2$) the set of permutations (respect involutions) of $A$. A random permutation $\sigma_A$ supported on $\mathcal{S}_A$ is called invariant under conjugation if for any $\sigma \in \mathcal{S}_A$, $\sigma \circ \sigma_A \circ \sigma^{-1}$ is equal in distribution to $\sigma_A$.

Fix $\varepsilon > 0$. By Corollary 21, there exists $n_0$ such that for any $A \subset \mathbb{N}$ with $n_0 < \text{card}(A) < +\infty$, for any random permutation $\hat{\sigma}_A$ supported on $\mathcal{S}_A^2$ invariant under conjugation,

$$\mathbb{E}(\ell(\hat{\sigma}_A)) \geq 2 - \varepsilon.$$

Let $\sigma_n$ be a random permutation invariant under conjugation and $\rho_n$ be the restriction of $\sigma_n$ on $\text{fix}(\sigma_n^2)$. In particular, almost surely $\ell(\rho_n) \leq \ell(\sigma_n)$. One can see that for any $A \subset \{1, 2, \ldots, n\}$ such that $\mathbb{P}(\text{fix}(\sigma_n^2) = A) > 0$, for any $\hat{\sigma}_1, \hat{\sigma}_2 \in \mathcal{S}_A$,

$$\mathbb{P}(\rho_n = \hat{\sigma}_1 | \text{fix}(\sigma_n^2) = A) = \mathbb{P}(\rho_n = \hat{\sigma}_2 \circ \hat{\sigma}_1 \circ \hat{\sigma}_2^{-1} | \text{fix}(\sigma_n^2) = A).$$

(10)
Consequently, if $\beta_n > n_0$,

$$
\frac{\mathbb{E}(\ell(\sigma_n))}{\sqrt{\beta_n}} = \sum_{|A| > \beta_n, \mathbb{P}(\text{fix}(\sigma_n^2) = A) > 0} \frac{\mathbb{E}(\ell(\sigma_n)|\text{fix}(\sigma_n^2) = A)}{\sqrt{\beta_n}} \mathbb{P}(\text{fix}(\sigma_n^2) = A) \\
\geq \sum_{|A| > \beta_n, \mathbb{P}(\text{fix}(\sigma_n^2) = A) > 0} (2 - \varepsilon) \sqrt{\frac{\text{card}(A)}{\beta_n}} \mathbb{P}(\text{fix}(\sigma_n^2) = A) \\
\geq \sum_{|A| > \beta_n, \mathbb{P}(\text{fix}(\sigma_n^2) = A) > 0} (2 - \varepsilon) \mathbb{P}(\text{fix}(\sigma_n^2) = A) = 2 - \varepsilon.
$$

This yields Corollary 22. \hfill \square

**Lemma 23.** For any permutation $\sigma \in \mathcal{S}_n$,

$$\text{card}(\text{fix}(\sigma^2)) \geq 6\#(\sigma) - 3\text{card}(\text{fix}(\sigma)) - 2n. $$

**Proof.** We denote by $\#_k(\sigma)$ the number of cycles of $\sigma$ of length $k$. We have

$$\sum_{k \geq 1} k\#_k(\sigma) = n \quad \text{and} \quad \sum_{k \geq 1} \#_k(\sigma) = \#(\sigma).$$

Thus

$$n + 2\text{card}(\text{fix}(\sigma)) + \#_2(\sigma) = 3\text{card}(\text{fix}(\sigma)) + 3\#_2(\sigma) + \sum_{k \geq 3} k\#_k(\sigma)$$

$$\geq 3 \sum_{k \geq 1} \#_k(\sigma) = 3\#(\sigma).$$

Consequently,

$$\#_2(\sigma) \geq 3\#(\sigma) - n - 2\text{card}(\text{fix}(\sigma)).$$

Finally,

$$\text{card}(\text{fix}(\sigma^2)) = \text{card}(\text{fix}(\sigma)) + 2\#_2(\sigma) \geq 6\#(\sigma) - 3\text{card}(\text{fix}(\sigma)) - 2n. $$

\hfill \square

We will now prove Theorem 18.
Proof. In this proof, we use the following convention. Let $A, B \subset S_\infty$ and $f : S_\infty \to \mathbb{R}$. If $P(\sigma_n \in A) = 0$, we assign $P(\sigma_n \in B | \sigma_n \in A) = 0$ and $E(f(\sigma_n) | \sigma_n \in A) = 0$. We have

$\E(\ell(\sigma_n)) = \E\left(\ell(\sigma_n) \bigg| \#(\sigma_n) \leq \frac{(2 + \theta)n}{6}\right) \P\left(\#(\sigma_n) \leq \frac{(2 + \theta)n}{6}\right) + \E\left(\ell(\sigma_n) \bigg| \#(\sigma_n) > \frac{(2 + \theta)n}{6}\right) \P\left(\#(\sigma_n) > \frac{(2 + \theta)n}{6}\right)$.

Since the condition on the number of cycles is invariant under conjugation, it is sufficient to prove Theorem 18 in the two particular cases.

- Assume that almost surely $\#(\sigma_n) \leq \frac{(2 + \theta)n}{6}$. By Lemma 13, for any $0 < \gamma < 2$,

$\P\left(\frac{\ell(\sigma_n)}{\sqrt{n}} > \gamma\right) \geq \P\left(\sum_{i=1}^{n} (\lambda_i(T(\sigma_n)) - \gamma \sqrt{n} + \frac{2 + \theta}{6})\right)$.

As $T(\sigma_n)$ is distributed according to the $Ew(0)$, by choosing $\gamma = 2\sqrt{\theta} - \varepsilon$ for some $\varepsilon > 0$ in Corollary 10, we can conclude that the right hand side goes to 1 as $n$ goes to infinity.

- Assume that almost surely $\#(\sigma_n) > \frac{(2 + \theta)n}{6}$. We can write,

$\E(\ell(\sigma_n)) = \E(\ell(\sigma_n) | \text{card}(\text{fix}(\sigma_n)) \geq 2\sqrt{n\theta}) \P(\text{card}(\text{fix}(\sigma_n)) \geq 2\sqrt{n\theta})$

$+ \E(\ell(\sigma_n) | \text{card}(\text{fix}(\sigma_n)) < 2\sqrt{n\theta}) \P(\text{card}(\text{fix}(\sigma_n)) < 2\sqrt{n\theta})$.

Clearly, if $\P(\text{card}(\text{fix}(\sigma_n)) \geq 2\sqrt{n\theta}) > 0$, then

$\E(\ell(\sigma_n) | \text{card}(\text{fix}(\sigma_n)) \geq 2\sqrt{n\theta}) \geq 2\sqrt{n\theta}$.

Moreover, under the condition $\text{card}(\text{fix}(\sigma_n)) < 2\sqrt{n\theta}$, we have by Lemma 23, almost surely,

$\text{card}(\text{fix}(\sigma_n)) > \theta n - 6\sqrt{n\theta}$.

We can then conclude by Corollary 22 that if $\P(\text{card}(\text{fix}(\sigma_n)) < 2\sqrt{n\theta}) > 0$, then

$\liminf_{n \to \infty} \frac{\E(\ell(\sigma_n) | \text{card}(\text{fix}(\sigma_n)) < 2\sqrt{n\theta})}{\sqrt{n\theta - 6\sqrt{n\theta}}}$

$\geq 2$.

Thus, if $\P(\text{card}(\text{fix}(\sigma_n)) < 2\sqrt{n\theta}) > 0$, then

$\liminf_{n \to \infty} \frac{\E(\ell(\sigma_n) | \text{card}(\text{fix}(\sigma_n)) < 2\sqrt{n\theta})}{\sqrt{n\theta}}$ $\geq 2\sqrt{\theta}$. 

\[\square\]
2.4 Proof of Theorem 4 and Proposition 5.

The proofs of Theorem 4 and Proposition 5 are based on the following observation.

**Lemma 24.** For any permutations \( \sigma_1, \sigma_2 \), almost surely,
\[
|LCS(\sigma_1, \sigma_2) - LCS(T(\sigma_1), \sigma_2)| \leq \#(\sigma_1).
\]

The proof is identical to that of Lemma 8.

**Corollary 25.** Assume that the law of \( \tilde{\sigma}_{1,n} \) is \( Ew(0) \) and \( \tilde{\sigma}_{1,n} \) and \( \sigma_{2,n} \) are independent. Then
\[
\lim_{n \to \infty} \mathbb{P} \left( \frac{LCS(\tilde{\sigma}_{1,n}, \sigma_{2,n}) - 2\sqrt{n}}{\sqrt{n}} \leq s \right) = F_2(s),
\]
\[
\lim_{n \to \infty} \mathbb{E} \left( \frac{LCS(\tilde{\sigma}_{1,n}, \sigma_{2,n})}{\sqrt{n}} \right) = 2 \quad \text{and} \quad \frac{LCS(\tilde{\sigma}_{1,n}, \sigma_{2,n})}{\sqrt{n}} \xrightarrow{d} 2.
\]

**Proof.** Note that if \( \sigma_{1,n} \) is distributed according the uniform distribution, one can see that the independence between \( \sigma_{1,n} \) and \( \sigma_{2,n} \) implies that \( \sigma_{1,n}^{-1} \circ \sigma_{2,n} \) follows also the uniform distribution. In this case,
\[
\lim_{n \to \infty} \mathbb{P} \left( \frac{LCS(\sigma_{1,n}, \sigma_{2,n}) - 2\sqrt{n}}{\sqrt{n}} \leq s \right) = \lim_{n \to \infty} \mathbb{P} \left( \frac{\ell(\sigma_{1,n}) - 2\sqrt{n}}{\sqrt{n}} \leq s \right) = F_2(s), \tag{11}
\]
\[
\lim_{n \to \infty} \mathbb{E} \left( \frac{LCS(\sigma_{1,n}, \sigma_{2,n})}{\sqrt{n}} \right) = \lim_{n \to \infty} \mathbb{E} \left( \frac{\ell(\sigma_{1,n})}{\sqrt{n}} \right) = 2, \tag{12}
\]
and
\[
\frac{LCS(\sigma_{1,n}, \sigma_{2,n})}{\sqrt{n}} \xrightarrow{d} \frac{\ell(\sigma_{1,n})}{\sqrt{n}} \xrightarrow{P} 2. \tag{13}
\]

The second equality of (11) is due to Baik, Deift, and Johansson [1999] and the second equality of (12) and the convergence of (13) are due to Vershik and Kerov [1977]. Hence, one can conclude by Lemma 24 since \( \mathbb{E}(\#(\sigma_{1,n})) = \log(n) + O(1) \) and \( LCS(\tilde{\sigma}_{1,n}, \sigma_{2,n}) \) is equal in distribution to \( LCS(T(\sigma_{1,n}), \sigma_{2,n}) \).

Using again Lemma 24, Corollary 25 imply Proposition 5 since \( T(\sigma_{1,n}) \) is distributed according to \( Ew(0) \).

**Sketch of proof of Theorem 4.** Using the same technique as in Corollary 10, we can prove that for any \( \varepsilon > 0 \),
\[
\mathbb{P} \left( \frac{LCS(\sigma_{1,n}, \sigma_{2,n})}{\sqrt{n}} > G^{-1} \left( \frac{\#(\sigma_{1,n})}{2n} + \varepsilon \right) - \varepsilon \right) \to 1.
\]

Consequently,
\[
\liminf_{n \to \infty} \frac{\mathbb{E}(LCS(\sigma_{1,n}, \sigma_{2,n}))}{\sqrt{n}} \geq \mathbb{E} \left( G^{-1} \left( \liminf_{n \to \infty} \frac{\#(\sigma_{1,n})}{2n} \right) \right).
\]

Since \( G^{-1} \) is convex, we can conclude using Jensen’s inequality. \( \square \)
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