Negative group delay for Dirac particles travelling through a potential well

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The properties of group delay for Dirac particles travelling through a potential well are investigated. A necessary condition is put forward for the group delay to be negative. It is shown that this negative group delay is closely related to its anomalous dependence on the width of the potential well. In order to demonstrate the validity of stationary-phase approach, numerical simulations are made for Gaussian-shaped temporal wave packets. A restriction to the potential-well’s width is obtained that is necessary for the wave packet to remain distortionless in the travelling. Numerical comparison shows that the relativistic group delay is larger than its corresponding non-relativistic one.

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I. INTRODUCTION

The question of how much time it takes quantum particles to tunnel through a potential barrier has been controversial for these decades [1,2,3]. Theoretical investigations [4,5,6,7,8] and experimental researches [9,10,11,12,13,14,15] show that the group delay for some kinds of barriers, also known as the phase time in the literature, which describes the motion of a wave packet peak [14], has the well-known superluminality. In addition, the faster-than-light propagation was also predicted [15] and experimentally verified [16,17,18] for light pulses through anomalous dispersion media. In a previous paper [19], Li and Wang have elaborated the superluminal and even negative properties of the group delay for quantum particles travelling through a potential well, instead of tunnelling through a potential barrier. This counterintuitive phenomenon resulted from the interference of multi-reflected waves in the potential well was demonstrated in a microwave analogy experiment [20]. Recently, the concept of negative group delay has been extended to microelectronics [21].

Most of the theoretical works on tunnelling times in quantum mechanics rely on Schrödinger’s non-relativistic theory. Such a theory has a potential deficiency in accurately addressing the question of causality [22,23]. For this reason, a few authors [22,24,25] extended the analysis of the tunnelling to Dirac’s fully relativistic quantum theory. Leavens and Aers [24] used the stationary-state method to analyze the Larmor-clock transmission times for single barriers and resonant double-barriers. Krekora et al. [22] solved numerically the time-dependent Dirac equation for a quantum wave packet tunnelling through a potential barrier. And Petrillo and Janner [25] studied the dynamics of wave-packet tunnelling through a barrier. In a recent work [23], we discussed an energy-transfer associated traversal time for Dirac particles tunnelling through a potential barrier.

The purpose of this paper is to investigate the properties of the group delay for Dirac particles travelling through a potential well. It is shown that it behaves superluminal and even negative in much the same way as in the non-relativistic quantum mechanics [19]. The negativity of the group delay is closely related to its anomalous dependence on the width of the potential well around transmission resonances. In order to demonstrate the validity of the stationary-phase approximation, numerical simulations are made for Gaussian-shaped temporal wave packets. A restriction to the width of the potential well is given that is necessary for the wave packet to remain distortionless in the travelling. Finally, a numerical comparison shows that the Dirac’s relativistic group delay is larger than its corresponding non-relativistic one.

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II. RELATIVISTIC GROUP DELAY AND ITS NON-RELATIVISTIC LIMIT

Consider Dirac particles of precisely defined incident energy $E$ and of helicity $+1$, travelling through a one-dimensional rectangular potential well $-V_0\Theta(z)\Theta(a-z)$ (with $V_0$ positive, representing the depth of the potential well). Let the incident wave function be

$$\psi_{in}(z) = \begin{pmatrix} 1 \\ 0 \\ \frac{hc}{E+\mu c^2} \\ 0 \end{pmatrix} e^{ikz}, (z < 0),$$

(1)

where $k = (E^2 - \mu^2c^4)^{1/2}/hc$, $\mu$ is the mass of incident particles, and $c$ is the speed of light in vacuum. Then Dirac equation and boundary conditions give for the transmitted wave function

$$\psi_{tr}(z) = F \begin{pmatrix} 1 \\ 0 \\ \frac{hc}{E+\mu c^2} \\ 0 \end{pmatrix} e^{ik(z-a)}, (z > a),$$

(2)

where the transmission coefficient $F = e^{i\phi}/f$ is determined by the following complex number,

$$fe^{i\phi} = \cos k'a + (i/2)(\chi + 1/\chi) \sin k'a,$$

so that

$$\phi = \text{int} \left( \frac{k'a}{\pi} + \frac{1}{2} \right) \pi + \tan^{-1} \left[ \frac{1}{2} \left( \chi + \frac{1}{\chi} \right) \tan k'a \right],$$

(3)

int(\cdot) stands for the integer part of involved number, $k' = [(E + V_0)^2 - \mu^2c^4]^{1/2}/hc$. Noting that the real parameter $\chi$ is defined as

$$\chi = \frac{kE + V_0 + \mu c^2}{k'E + \mu c^2},$$

and has the property that $0 < \chi < 1$. Obviously, $\phi$ is the phase shift of transmitted wave $[2]$ at $z = a$ with respect to the incident wave $[1]$ at $z = 0$. The transmission probability $T$ is a periodical function of the width $a$ of the potential well,

$$T = \frac{1}{f^2} = \frac{4\chi^2}{4\chi^2 + (\chi^2 - 1)^2 \sin^2 k'a}.$$

(4)

The group delay, defined as the derivative of the phase shift $\phi$ with respect to particle’s energy $E$ $[3, 14]$, is given by

$$\tau_\phi = h \frac{\partial \phi}{\partial E} = \frac{T}{2\chi h k'c} \left[ (1 + \chi^2)(E + V_0) - (1 - \chi^2)\frac{\mu V_0(2E + V_0)}{h^2k^2} \sin 2k'a \right] \frac{a}{c}. $$

(5)

For the following comparisons, let us first give its non-relativistic limit. By non-relativistic limit it is meant that the speed of incident particles is much smaller than $c$, so that their kinetic energy $E' = E - \mu c^2$ is much smaller than their rest energy $\mu c^2$, $E' \ll \mu c^2$. It is also meant that the interaction energy $V_0$ satisfies $V_0 \ll \mu c^2$. In this limit, we have $k \approx \sqrt{2\mu E'/h}$, $k' \approx (2\mu(E' + V_0))^{1/2}/hc$, and $\chi \approx k/k'$. Collecting all these, we get

$$\tau_\phi \approx \tau'_\phi = \frac{2\mu a k^2(k^2 + k'^2)/k_0^3 - \sin 2k'a/2k'a}{4k^2k'^2/k_0^4 + \sin^2 k'a},$$

(6)

which is expected from Schrödinger’s non-relativistic theory $[10]$, where $k_0 \approx \sqrt{2\mu V_0/h}$. 

III. NEGATIVE PROPERTY OF THE GROUP DELAY

In this section, we discuss the negative property of the group delay. It is seen from Eq. (5) that when inequality

$$(1 + \chi^2)(E + V_0) < (1 - \chi^2)\frac{\mu V_0 (2E + V_0)}{\hbar^2 k^2} \sin 2k' a$$

holds, the group delay is negative, $\tau_\phi < 0$. Since $\sin 2k' a / 2k' a < 1$, the above inequality leads to the following necessary condition for the group delay to be negative,

$$(1 + \chi^2)(E + V_0) < (1 - \chi^2)\mu V_0 (2E + V_0) \frac{\hbar^2 k^2}{k^2}$$

which can be expressed as a restriction to the total energy of incident particles as follows,

$$E < E_t \equiv \frac{\mu c^2}{2} \left\{ \frac{V_0}{\mu c^2} + \left[ \left( \frac{V_0}{\mu c^2} \right)^2 - \left( \frac{1}{3} \right)^2 \right]^{1/2} \right\}^{1/3}$$

$$+ \mu c^2 \left\{ \frac{V_0}{\mu c^2} - \left[ \left( \frac{V_0}{\mu c^2} \right)^2 - \left( \frac{1}{3} \right)^2 \right]^{1/2} \right\}^{1/3}$$

This means that when the energy of incident particles satisfies Eq. (8), that is to say, the energy of incident particles $E$ is less than a threshold energy $E_t$, one can always find a width $a$ of the potential well at which the group delay is negative. Of course, $E_t$ in Eq. (8) is always real and larger than $\mu c^2$. In the case of $V_0 < 2\mu c^2 / 3\sqrt{3}$, $E_t$ can be rewritten as

$$E_t = \mu c^2 \left\{ \frac{V_0}{2\mu c^2} + i \left[ \left( \frac{1}{3} \right)^3 - \left( \frac{V_0}{2\mu c^2} \right)^2 \right]^{1/2} \right\}^{1/3} + \text{c.c.}$$

In the non-relativistic limit, $V_0 \ll \mu c^2$, we obtain $E_t \approx \mu c^2 + V_0 / 2$. Therefore, the necessary condition (8) now reduces to $E' < V_0 / 2$, as observed previously in Schrödinger’s theory [19].

Fig. 1 shows a typical example of the dependence of $\tau_\phi$ on the width $a$, where the well depth $V_0 = 0.4\mu c^2 > 2\mu c^2 / 3\sqrt{3}$ ($E_t = 1.16\mu c^2$), the total energy $E = 1.01\mu c^2 < E_t$, and $a$ is re-scaled to be $k' a$. For comparison, Fig. 1 also shows the periodical dependence of transmission probability $T$ on $a$ under the same conditions. It is interesting to note that the oscillation of the group delay with respect to $a$ is closely related to the periodical occurrence of transmission resonances at $k' a = m\pi$ ($m = 1, 2, 3, \ldots$).

![Graph showing dependence of group delay on potential-well’s width](image)

**FIG. 1:** Dependence of the group delay (in the unit of $\hbar/\mu c^2$) on potential-well’s width $a$, where $E = 1.01\mu c^2$, $V_0 = 0.4\mu c^2$, and $a$ is re-scaled to be $k' a$. Here the periodical dependence of transmission probability $T$ on $a$ is also depicted by dashed curve.

On the one hand, at resonances, the group delay becomes

$$\tau_\phi |_{k' a = m\pi} = \frac{1}{2} \left[ \left( \frac{1}{\chi} + \frac{1}{\chi} \right) \frac{E + V_0 a}{\hbar k' c} \frac{a}{c} > \frac{a}{c} \right]$$

(10)
which is proportional to $a$. Its corresponding group velocity is less than the velocity of light in vacuum, $c$. On the other hand, the derivative of group delay $\tau_\phi$ with respect to $a$ is, at resonances,

$$\frac{\partial \tau_\phi}{\partial a} \bigg|_{k'a=m\pi} = \frac{1}{2\chi\hbar c^2} \left[ (1 + \chi^2)(E + V_0) - (1 - \chi^2)\frac{\mu V_0(2E + V_0)}{\hbar^2 k'^2} \right].$$

(11)

When the necessary condition $\chi$ is satisfied, it is negative. This shows that the group delay depends anomalously on the width around resonance points. In other words, it decreases with increasing the width of the potential well at resonances, as displayed clearly in Fig. 1.

In connection with the negative characteristics of the group delay, the phase shift (3) shows a quantum-like behavior with respect to the potential-well’s width. It can be seen on the width around resonance points. In other words, it decreases with increasing the width of the potential well at resonances, as displayed clearly in Fig. 1.

In connection with the negative characteristics of the group delay, the phase shift (3) shows a quantum-like behavior with respect to the potential-well’s width. Fig. 2 indicates such a behavior, where $E = 1.01\mu c^2$ and $V_0 = 0.4\mu c^2$.

This quantum-like behavior is also related to the periodical occurrence of the transmission resonances. It can be seen from Eq. (3) and Fig. 2 that at resonances, $k'a = m\pi$, the phase shift becomes $\phi = k'a$ and changes rapidly around here with respect to $a$. However, when the width is far from the resonance points as in the middle between two adjacent resonance points, the phase shift changes slowly.

![FIG. 2: The quantum-like behavior of phase shift (3) with respect to the width of potential well, $a$, where $E = 1.01\mu c^2$ and $V_0 = 0.4\mu c^2$.](image)

In addition, it is also indicated from Eq. (5) that the group delay depends not only on the potential-well’s width $a$, but also on the incident energy $E$ and the depth of the potential well $V_0$. To see the latter more clearly, we convert it to a dimensionless form. Denoting $\alpha = E/\mu c^2$, $\beta = V_0/\mu c^2$, and $\gamma = h/\alpha \mu c$, we have

$$\frac{\tau_\phi}{\tau_0} = \frac{T}{2\chi (\alpha + \beta)^2 - 1} \left[ (1 + \chi^2)(\alpha + \beta) - (1 - \chi^2)\frac{\beta(2\alpha + \beta)\sin 2k'a}{\alpha^2 - 1} \right],$$

(12)

where $\tau_0 = h/\alpha \mu c$, $k'a = [(\alpha + \beta)^2 - 1]^{1/2}/\gamma$. When the kinetic energy of incident particles $E'$ is small enough that $E'/\mu c^2 \rightarrow 0$ with the depth of the potential well remaining finite, we have $\alpha \rightarrow 1$, so that $\chi \rightarrow 0$. In this limit, the group delay (12) has the following form,

$$\lim_{\alpha \rightarrow 1} \frac{\tau_\phi}{\tau_0} = -\sqrt{\frac{\beta + 2}{(\alpha^2 - 1)}\cot k'a}.$$

which approaches negative infinite when $\cot k'a > 0$. Fig. 3 shows such a dependence of the group delay on the incident energy $E$, where $\beta = 0.4$ and $\gamma = 0.01$. A strange phenomenon occurs here that for a given potential well, the absolute value of the negative group delay becomes larger when decreasing the incident kinetic energy. Of course, the transmission probability in this limit tends to zero in the following way,

$$\lim_{\alpha \rightarrow 1} T = \frac{4\chi^2}{4\chi^2 + \sin^2 k'a}$$

(13)

so that very few particles can travel through the potential well at this negative group velocity.

In order to demonstrate the validity of the above stationary-phase method in this problem, we proceed to numerical simulations of the group delay for a Gaussian-shaped wave packet. The incident wave function of Dirac particles is assumed to be, at $z = 0$,

$$\Psi_{in}(t)|_{z=0} = \begin{pmatrix} 1 \\ 0 \\ \frac{\hbar k_0 c}{E_0 + \mu c} \\ 0 \end{pmatrix} \exp(-t^2/2\gamma^2 - iE_0 t/\hbar),$$

(14)
FIG. 3: Dependence of the group delay (in the unit of $\hbar/\mu c^2$) on the incident energy $\alpha$, where $\beta = 0.4$ and $\gamma = 0.01$.

which has the Fourier integral of the following form,

$$
\Psi_{in}(t)|_{z=0} = \frac{1}{\sqrt{2\pi}} \int A(E)\psi(E_0) \exp(-iEt/\hbar)dE,
$$

(15)

where $k_0 = (E_0^2 - \mu^2c^4)^{1/2}/\hbar c$, the spinor $\psi(E_0)$ is $[1, 0, h\hbar c/(E_0 + \mu c^2), 0]^T$, the energy spectral distribution $A(E)$ with the central energy $E_0$ is given by

$$
A(E) = (w/\hbar) \exp\left[-(w^2/2\hbar^2)(E - E_0)^2\right],
$$

and $w$ is the temporal width of the wave packet. The transmitted wave function takes the following form,

$$
\Psi_{tr}(z, t) = \frac{1}{\sqrt{2\pi}} \int F(E)A(E)\psi(E_0) \exp\{i[k(z-a) - Et/\hbar]\}dE,
$$

(16)

The numerically calculated group delay, $\tau_{\phi}^N$, is defined here by

$$
|\Psi_{tr}(z = a, \tau_{\phi}^N)|^2 = \max\{|\Psi_{tr}(z = a, t)|^2\}.
$$

(17)

Since the incident wave is of perfect Gaussian shape, the integral limit in Eq. (15) and hence in Eq. (16) should be from $-\infty$ to $+\infty$. But the energy of incident particles must be larger than $\mu c^2$. So the real integral in numerical simulations is taken to be from $\mu c^2$ to $+\infty$.

Calculations show that the stationary-phase approximation for the group delay is in good agreement with the numerical result, especially when the energy spectral distribution is sharp. In Fig. 3 we show such a comparison between theoretical and numerical results, where $E_0 = 1.01\mu c^2$, $V_0 = 0.4\mu c^2$, and the temporal width $w = 300\tau_0$. For the chosen temporal width, the corresponding energy spreading $\Delta E = h/2w = \mu c^2/600$ is narrow enough that the integral in Eq. (16) can be performed from $\mu c^2$ to $2\mu c^2$ without changing the result significantly.

As pointed out in the optical analog, for an incident wave packet of energy spreading $\Delta E$, the corresponding spreading of $k'a$ should be much smaller than $\pi$, the period of $|F|$, in order that the stationary-phase approximation is valid. With the energy spreading $\Delta E = h/2w$, this leads to the following restriction to the width of the potential well,

$$
a \ll 2\pi w \frac{\partial E}{\hbar \partial k'}.\quad \text{(18)}
$$

It is noted that $\partial E/\hbar \partial k'$ is nothing but the group velocity of particles in the region of potential well. By introducing a characteristic length $L$ which is defined as $w\partial E/\hbar \partial k'$, the above restriction is simplified to be $a \ll 2\pi L$. With this restriction, the temporal wave packet can travel through the potential well with negligible distortion.

All the above discussions show that the group delay in Dirac’s relativistic theory can be superluminal and even negative in much the same way as in Schrödinger’s non-relativistic theory. In this sense, the superluminal and negative properties is not an artifact due to the deficiency of non-relativistic quantum theory. Rather, it is a real effect. In the next section, we investigate the difference between the group delays in relativistic and non-relativistic theories.
FIG. 4: Comparison of theoretical and numerical results of group delays (in the unit of $\hbar/\mu c^2$) with respect to $a$, where $E_0 = 1.01\mu c^2$, $V_0 = 0.4\mu c^2$, $w = 300\tau_0$ and $a$ is re-scaled to $k'a$. The theoretical result for the group delay (5) is depicted by the real curve, and the dash curve corresponds to the numerical result for the group delay defined by (17).

IV. RELATIVISTIC EFFECT ON THE GROUP DELAY

As mentioned in Sec.II, when the incident kinetic energy $E' \ll \mu c^2$ and the potential-well’s depth $V_0 \ll \mu c^2$, the group delay (5) tends to its non-relativistic limit (6). To show their differences, we rewrite the non-relativistic group delay in the following dimensionless form,

$$\tau'_{\phi} = \frac{k'a}{\tau_0} \frac{(\alpha - 1)(2\alpha + \beta - 2) - \beta^2 \sin 2k'/2k'}{4(\alpha - 1)(\alpha + \beta - 1) + \beta^2 \sin^2 k'}. \quad (19)$$

where $\alpha$, $\beta$, and $\gamma$ are the same as before, $k'a = [2(\alpha + \beta - 1)]^{1/2}/\gamma$ in the non-relativistic quantum mechanics.

In order to address the relativistic effect on the group delay, we draw in Fig. 5 the dependence of relativistic and non-relativistic group delays on the width of the potential well for $\alpha = 1.01$ and $\beta = 0.2$ ($V_0 = 0.2\mu c^2 < 2\mu c^2/3\sqrt{2}$), where the relativistic group delay is shown by solid curve, the corresponding non-relativistic one is shown by dashed curve, and the width of potential well is re-scaled by their own $k'$ to be $k'a$.

FIG. 5: Comparison of relativistic and non-relativistic group delays (in the unit of $\hbar/\mu c^2$) for $\alpha = 1.01$ and $\beta = 0.2$. The real curve denotes the relativistic group delay (12), and the dashed curve corresponds to the non-relativistic group delay (19).

From Fig. 5 we see that the relativistic group delay for travelling through a potential well is larger than the corresponding non-relativistic one. This is in agreement with the result of Leavens and Aers [24], who observed that the local mean velocity for transmitted particles is reduced due to relativistic effect for resonant double barriers where the group delay is always larger than zero. The difference between the two group delays are very small for the low energy and potential-well depth. It is expected that the agreement between the group delays in relativistic and non-relativistic theory is obtained when $E' \ll \mu c^2$ and $V_0 \ll \mu c^2$.
V. CONCLUSIONS

In summary, we have investigated the negative property of the group delay for Dirac particles travelling through a quantum potential well. A necessary condition \( E' < V_0/2 \) is given for the group delay to be negative, which is a restriction on the energy of incident particles and reduces to \( E' < V_0/2 \) in non-relativistic limit. The relation of the negativity of the group delay with its anomalous dependence on the width of the potential well around transmission resonances is discussed. In order to demonstrate the validity of the stationary-phase approach, numerical simulations are made for a Gaussian-shaped temporal wave packet. A restriction to the width of the potential well is given that is necessary for the wave packet to remain distortionless in the travelling. Comparison of the relativistic and non-relativistic group delays is also made. It is found that the value of the relativistic group delay is larger than that of the non-relativistic one. It should be pointed out that the negative group delay discussed here is not at odds with the principle of causality. When a wave packet travels through a potential well, the boundary effect leads to a reshaping of the edge of the packet which results in an effective acceleration of the wave packet. This phenomenon is similar to but different from the negative group-velocity propagation performed by Wang et al. \[18\] where the gain-assisted anomalous dispersion plays the role. We hope that this work may stimulate further experimental researches in electronic domains.

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[1] E.H. Hauge and J.A. Støvneng, Rev. Mod. Phys. 61, 917 (1989).
[2] G. Nimtz and W. Heitmann, Prog. Quantum Electron. 21, 81 (1997).
[3] R.Y. Chiao and A.M. Steinberg, Tunneling Times and Superluminality, Ed. E. Wolf, Progress in Optics Vol. XXXVII (Elsevier B.V. Science, Amsterdam, 1997), p. 345.
[4] L.A. MacColl, Phys. Rev. 40, 621 (1932).
[5] T.E. Hartman, J. Appl. Phys. 33, 3427 (1962).
[6] M. Büttiker and R. Landauer, Phys. Rev. Lett. 49, 1739 (1982).
[7] Th. Martin and R. Landauer, Phys. Rev. A 45, 2611 (1992).
[8] A.M. Steinberg and R.Y. Chiao, Phys. Rev. A 49, 3283 (1994).
[9] C.K. Carniglia and L. Mandel, J. Opt. Soc. Am. 61, 1035 (1971); 61, 1423 (1971).
[10] A. Enders and G. Nimtz, J. Phys. I (France) 2, 1693 (1992); 3, 1089 (1993).
[11] A.M. Steinberg, P.G. Kwiat, and R.Y. Chiao, Phys. Rev. Lett. 71, 708 (1993).
[12] Ch. Spielmann, R. Szipöcs, A. Stingl, and F. Krausz, Phys. Rev. Lett. 73, 2308 (1994).
[13] Ph. Balcou and L. Dutriaux, Phys. Rev. Lett. 78, 851 (1997).
[14] E.P. Wigner, Phys. Rev. 98, 145 (1955).
[15] C.G.B. Garrett and D.E. McCumber, Phys. Rev. A 1, 305 (1970).
[16] S. Chu and S. Wong, Phys. Rev. Lett. 48, 738 (1982).
[17] B. Segard and B. Macke, Phys. Lett. A 109, 213 (1985).
[18] L.J. Wang, A. Kuzmich and A. Dogariu, Nature 406, 277 (2000)
[19] C.-F. Li and Q. Wang, Phys. Lett. A 275, 287 (2000).
[20] R.-M. Vetter, A. Haibel, and G. Nimtz, Phys. Rev. E 63, 046701 (2001).
[21] D. Solli, R.Y. Chiao, and J.M. Hickmann, Phys. Rev. E 66, 056601 (2002).
[22] P. Krekora, Q. Su, and R. Grobe, Phys. Rev. A 63, 032107 (2001).
[23] C.-F. Li and X. Chen, Ann. Phys. (Leipzig) 11, 916 (2002).
[24] C.R. Leavens and G.C. Aers, Phys. Rev. B 40, 5387 (1989).
[25] V. Petriillo and D. Janner, Phys. Rev. A 67, 012110 (2003).
[26] X.-H. Huang and C.-F. Li, Europhys. Lett. 63, 28 (2003).
[27] Y. Japha and G. Kurizki, Phys. Rev. A 53, 586 (1996).