THE NERON-TATE PAIRING AND ELLIPTIC K3 SURFACES

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Abstract. In this paper, we demonstrate a connection between the group structure and Neron-Tate pairing on elliptic curves in an elliptic fibration with section on a K3 surface, and the structure of the ample cone for the K3 surface. Part of the result can be thought of as a case of the specialization theorem.

Introduction

In an earlier work [Bar11], we described a hyperbolic cross section of the ample cone for a class of K3 surfaces, and came up with the Poincaré ball model in Figure 1. If $X = X(\mathbb{Q})$ is a K3 surface in this class, then $X$ is fibered by elliptic curves, all of which are in the divisor class represented by the divisor $[E]$ in Figure 1. If we unfold the Poincaré ball into the Poincaré upper half space with $[E]$ the point at infinity, we get Figure 2. The Euclidean structure of the boundary at infinity is rather captivating, and is related to the group structure of the elliptic curves in the fibration.

Let the Euclidean lattice we see in Figure 2 be generated by the horizontal translation $t_1$ and diagonal translation $t_2$. The fibration has a section $O$ with divisor class $[O]$. The plane $[O] \cdot x = 0$ is a face of the ample cone, and is represented by the circle so noted in Figure 2. Since $O$ is a smooth rational curve, $[O] \cdot [O] = -2$, and because it is a section, $[O] \cdot [E] = 1$. That is, for every elliptic curve $E$ in the

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divisor class \([E]\), there is a unique point \(O_E\) in the intersection of \(E\) with the curve \(O\). The section gives us a natural way of defining the zero element \(O_E\) on every elliptic curve in the fibration.

The translates \(D_1 = t_1([O])\) and \(D_2 = t_2([O])\) give two more divisors with self intersection \(-2\). Since the corresponding faces \((D_i \cdot x = 0)\) are walls of the ample cone, they both are represented with \(-2\) curves \([\text{Kov94}]\). Since \(t_i\) fixes \(E\), we have \(D_i \cdot [E] = t_i([O]) \cdot [E] = |O| \cdot t_i^{-1}[E] = |O| \cdot |E| = 1\), so both are sections. For fixed \(E\) in the fibration \([E]\), let \(Q_{i,E} = D_i \cap E\), where we have abused notation by letting \(D_i\) represent both a divisor and the unique \(-2\) curve in the divisor class. Let us use the points \(Q_{i,E}\) (as \(E\) varies in \([E]\)) to define automorphisms \(\tau_i \in \text{Aut}(X/Q)\) by

\[\tau_i(P) = P + Q_{i,E},\]

where \(E\) is the fiber that contains \(P\). Then, as we will see in this paper, \(\tau_i \cdot x = t_i\). That is, the translation \(t_i\) represents addition by \(Q_{i,E}\) on the elliptic curve \(E\).

Let \(v_i = D_i - [O]\). Then \(t_i\) restricted to the Euclidean plane (the boundary of the hyperbolic space) would appear to be translation in the direction \(v_i\), and this is indeed the case as we will see.

Let \(D\) be an ample divisor and \(h_D\) a Weil height on \(X\) associated to \(D\). For a fixed \(E\) in \([E]\), we define the canonical height \(\hat{h}\) on \(E\) using \(h_D\) restricted to \(E\). The Neron-Tate pairing on a pair of points \(P_1\) and \(P_2\) in the group \(Q_{1,E} \mathbb{Z} \oplus Q_{2,E} \mathbb{Z}\) is
given by
(1) \langle P_1, P_2 \rangle = \hat{h}(P_1 + P_2) - \hat{h}(P_1) - \hat{h}(P_2).

Let us define the height of \( E \) to be \( h(E) = h_D(O_E) \). Then, as we will see in this paper,
\[ \langle Q_i, E, Q_j, E \rangle = h(E)([E] \cdot D)v_i \cdot v_j + O(1). \]

The error term represented by \( O(1) \) is independent of \( E \). (We trust that there is no confusion between the notation for the section \( O \) and the big Oh notation.) In particular
\[ \lim_{h(E) \to \infty} \frac{1}{h(E)([E] \cdot D)} \langle Q_i, E, Q_j, E \rangle = v_i \cdot v_j, \]
which gives the relative geometry of the lattice in Figure 2.

This phenomenon is true in general.

**Theorem 1.** Let \( X/k \) be a K3 surface defined over a number field \( k \) with an elliptic fibration \([E]\) and a section \( O \). Let \( \Gamma = \{ \sigma : \sigma \in \text{Aut}(X/k) \} \) be the pull back of the group of automorphisms on \( X \). Suppose \( \Gamma_{[E]} \), the stabilizer of \([E]\), has an Abelian subgroup \( G = t_1\mathbb{Z} \oplus \cdots \oplus t_{\rho-2}\mathbb{Z} \cong \mathbb{Z}^{\rho-2} \) of maximal rank. Let \( D \) be an ample divisor on \( X \) and \( h_D \) a Weil height associated to \( D \). Let \( D_i \) represent both the divisor class \( t_i([O]) \) and the unique \(-2\) curve in \( D_i \). For any elliptic curve \( E \) in the fibration \([E]\), let
\[
\begin{align*}
O_E &= O \cap E \\
Q_i, E &= D_i \cap E \\
v_i &= D_i - [O] \\
h(E) &= h_D(O_E).
\end{align*}
\]

Define \( \tau_i : X \to X \) by \( \tau_i(P) = P + Q_i, E \), where \( E \) is the elliptic curve in \([E]\) that contains \( P \). Finally, let \( \langle , \rangle \) be the Neron-Tate pairing on \( E \) for any \( E \in [E] \). Then
\[ \tau_i^* = t_i \]
and
(2) \[ \lim_{h(E) \to \infty} \frac{1}{h(E)([E] \cdot D)} \langle Q_i, E, Q_j, E \rangle = v_i \cdot v_j. \]

Furthermore, the map \( t_i \) is translation by \( v_i \) when viewed in a Poincaré upper half hyperspace model with \([E]\) the point at infinity.

Some of the preceding, and in particular Eq. (2), appears in works by Silverman and Tate [Tat83, Sil83] (see also [Sil94, Section III.11]). Our approach and the geometric interpretation, via pictures of the ample cone, appear to be novel. Lemmas 5 and 6 are also notable and possibly novel.

**Remark 1.** The notation of this example was chosen so as to be consistent with a suitable notation for this paper, and differs significantly from the notation used in [Bar11]. For those who might be interested, \([E]\) is denoted by \( D_1 \) in [Bar11], \([O]\) by \( D_4 \), \( t_1 = ST_2ST_2 \), and \( t_2 = T_2T_4 \).
1. Background

1.1. K3 surfaces. Let $X$ be a K3 surface defined over a number field $k$. Let $\text{Pic}(X)$ be its Picard group and let $\{D_1, \ldots, D_\rho\}$ be a basis over $\mathbb{Z}$, where $\rho$ is the Picard number. Let $J = [D_i \cdot D_j]$ be the intersection matrix. By the Hodge Index Theorem, $J$ has signature $(1, \rho - 1)$, so is a Lorentz product on $\text{Pic}(X) \otimes \mathbb{R}$, and hence there is an underlying hyperbolic structure. Let $D$ be an ample divisor and define the light cone to be

$$\mathcal{L} = \{x \in \text{Pic}(X) \otimes \mathbb{R} : x \cdot x > 0, x \cdot D > 0\}.$$ 

Let

$$\mathcal{H} = \{x \in \mathcal{L} : x \cdot x = 1\}.$$ 

For two points $A$ and $B$ in $\mathcal{L}$, let us define a distance $|AB|$ by $|A||B| \cosh |AB| = A \cdot B$, where $|x| = \sqrt{x \cdot x}$. Then the set $\mathcal{H}$ equipped with this distance is a model of hyperbolic geometry $\mathbb{H}^{\rho - 1}$. At times, it will be convenient to identify $\mathcal{H}$ with $\mathcal{L}/\mathbb{R}^+$, equipped with the same metric. The boundary $\partial \mathcal{H} = \partial \mathcal{L}/\mathbb{R}^+$ is the usual compactification of $\mathbb{H}^{\rho - 1}$ and is congruent to $S^{\rho - 2}$.

Let $K$ be the ample cone for $X$. A cross section of $K$ is a polyhedron with possibly an infinite number of faces. Each face is a plane through the origin, so forms a hyperplane in $\mathcal{H}$. This is what the rendering in Figure 1 represents: Every circle on the sphere represents a plane in the Poincaré ball model of $\mathbb{H}^2$, and $K/\mathbb{R}^+$ is the region bounded by all these hyperbolic planes.

Let

$$O(R) = \{T \in M_{2 \times 2}(R) : T^t J T = J\},$$ 

and

$$O^+(R) = \{T \in O(R) : T \mathcal{L} = \mathcal{L}\}.$$ 

Then $O^+(\mathbb{R})$ is the group of isometries on $\mathcal{H}$. Let $O'' \leq O^+(\mathbb{Z})$ be the group of symmetries of $K$ in $O^+(\mathbb{Z})$. If $\sigma \in \text{Aut}(X/k)$, then its pullback $\sigma^*$ clearly preserves $K$, has integer entries, and preserves the intersection pairing. We therefore have a natural homomorphism

$$\Phi : \text{Aut}(X/k) \to O''$$

$$\sigma \mapsto \sigma^*.$$ 

For $k$ sufficiently large, the map $\Phi$ has a finite kernel and co-kernel [PSS71].

1.2. The Euclidean structure of $\partial \mathcal{L}/\mathbb{R}^+$. Let $\mathbb{R}^{1, \rho - 1}$ be a Lorentz space equipped with the Lorentz product $\cdot$ (which may be thought of as $\text{Pic}(X) \otimes \mathbb{R}$ equipped with the intersection pairing). The superscript $1, \rho - 1$ is the signature of the Lorentz product; that is, it has one positive eigenvalue and $\rho - 1$ negative eigenvalues. Let us distinguish a $D$ with $D \cdot D > 0$ and define the light cone $\mathcal{L}$ as above. Let us fix $E \in \partial \mathcal{L}$ and define

$$\partial \mathcal{H}_E := (\partial \mathcal{L} \setminus E \mathbb{R}^+)/\mathbb{R}^+.$$ 

For any $A \in \partial \mathcal{L} \setminus E \mathbb{R}^+$, let $\bar{A}$ be its equivalence class in $\partial \mathcal{H}_E$. For any $\bar{A}$ and $\bar{B} \in \partial \mathcal{H}_E$, let us define

$$|\bar{A} \bar{B}|_E := \sqrt{\frac{2A \cdot B}{(A \cdot E)(B \cdot E)}}.$$
Lemma 2. The function $|\bar{A}\bar{B}|_E$ defines a Euclidean metric on $\partial\mathcal{H}_E$. Furthermore, if $\gamma$ preserves the Lorentz product and $\gamma E = E$, then $\gamma$ is a Euclidean isometry on $\partial\mathcal{H}_E$.

Proof. We first note that $|\bar{A}\bar{B}|_E$ is invariant under scalar multiples of $A$ or $B$, so it is well defined on $\partial\mathcal{H}_E$.

Let us define the space perpendicular to $E$ to be

$$V^\perp_E := \{x \in \mathbb{R}^{1,\rho-1} : x \cdot E = 0\}.$$  

Note that $E \in V^\perp_E$.

If $x \in V^\perp_E$ and $x \cdot x = 0$, then the space spanned by $x$ and $E$ is in $\partial\mathcal{L}$. Since $\partial\mathcal{L}$ is a cone, it contains no two-dimensional subspaces, so $x$ must be a scalar multiple of $E$. Thus, $V^\perp_E$ is tangent to $\partial\mathcal{L}$. Now suppose $P \in \partial\mathcal{L}$ but is not a multiple of $E$. Then $P$ and $D$ are on the same side of $V^\perp_E$, so $P \cdot E$ and $D \cdot E$ share the same sign. That is, $P \cdot E > 0$. Without loss of generality, we may scale $P$ with a positive scalar so that $P \cdot E = 1$.

Since $P \notin V^\perp_E$, the set $\{P, V^\perp_E\}$ spans $\mathbb{R}^{1,\rho-1}$. For an arbitrary $A \in \partial\mathcal{L}$, let us write

$$A = a_P P + a_E E + a,$$

where

$$a \in V^\perp_{E,P} := \{x \in \mathbb{R}^{1,\rho-1} : x \cdot E = x \cdot P = 0\}.$$  

As with $P$, we may scale $A$ so that $A \cdot E = 1$. Note that $A \cdot E = a_P$, so we now have

$$(3) \quad A = P + a_E E + a.$$  

Since $A \cdot A = 0$, we get

$$0 = A \cdot A = (P + a_E E + a) \cdot (P + a_E E + a) = 2a_E + a \cdot a,$$

so

$$a_E = -\frac{a \cdot a}{2}.$$  

Let us now calculate $|\bar{A}\bar{B}|_E$:

$$|\bar{A}\bar{B}|_E = \sqrt{2(P - \frac{a \cdot a}{2} E + a) \cdot (P - \frac{b \cdot b}{2} E + b) \over (A \cdot E)(B \cdot E)}$$

$$= \sqrt{-a \cdot a - b \cdot b + 2a \cdot b}$$

$$= \sqrt{-(a - b) \cdot (a - b)}.$$  

We note that $V^\perp_{E,P}$ is the intersection of two tangent spaces to the light cone, so $a$ and $b$ are in a space where the Lorentz product is negative definite. Thus $|\bar{A}\bar{B}|_E$ is a Euclidean metric.

Finally, if $\gamma$ preserves the Lorentz product and $\gamma E = E$, then $\gamma$ clearly preserves the metric $|\bar{A}\bar{B}|_E$, so is a Euclidean isometry on $\partial\mathcal{H}_E$. $\square$

The Euclidean structure outlined above is the one we are used to; that is to say, it is the $(\rho - 2)$-dimensional Euclidean structure of the boundary of the Poincaré upper-half space model of $\mathbb{H}^{\rho-1}$ with $E$ the point at infinity.
The Euclidean space \( \partial H_E \) can also be represented by \( V^\perp_{E,P} \) in a natural way, via the identification

\[
\phi : \quad \partial H_E \rightarrow V^\perp_{E,P}
\]

\[
\bar{A} \mapsto \frac{a}{A \cdot E},
\]

where \( A \) is any representative of \( \bar{A} \) and \( A = a_P P + a_E E + a \), with \( a \in V^\perp_{E,P} \). Note that

\[
a = A - A \cdot E E \cdot P P - A \cdot P E.
\]

We can use this subspace to build a Poincaré upper half-space model of \( \mathbb{H}^{\rho-1} \). We let \((x, z) \in V^\perp_{E,P} \times \mathbb{R}^+ \) and equip this set with the arclength element

\[
ds^2 = \frac{-dx \cdot dx + dz^2}{z^2}.
\]

The negative sign arises because the Lorentz product is negative definite on \( V^\perp_{E,P} \).

**Lemma 3.** Let \( U = wP + vE + u \in H \) where \( u \in V^\perp_{E,P} \). The map

\[
\Phi : \quad H \rightarrow V^\perp_{E,P} \times \mathbb{R}^+
\]

\[
U \mapsto \left( \frac{u}{w}, \frac{1}{w} \right)
\]

is an isomorphism of hyperbolic spaces.

**Proof.** We prove this by demonstrating that the arclength element \( ds' \) on \( H \) induced by the Lorentz product maps to the Poincaré arclength element \( ds \). Let \( \Phi(U) = (x, z) \). Then

\[
x = \frac{u}{w} \quad z = \frac{1}{w}
\]

\[
dx = \frac{du}{w} - \frac{udw}{w^2} \quad dz = -\frac{dw}{w^2}.
\]

Thus,

\[
ds^2 = -du \cdot du + \frac{2(u \cdot du)dw}{w} - \frac{(u \cdot u)dw^2}{w^2} + \frac{dw^2}{w^2}.
\]

Using \( 2vw + u \cdot u = 1 \), we get

\[
ds^2 = -du \cdot du + \frac{2u \cdot du dw}{w} + \frac{2vwdw^2}{w^2}.
\]

The arclength element induced by the Lorentz product satisfies

\[
(ds')^2 = -dU \cdot dU = -2vwdw - du \cdot du,
\]

where the minus sign comes from the signature \((1, \rho - 1)\) of our Lorentz product.

We use \( 2vw + u \cdot u = 1 \) to solve for \( dv \):

\[
2vdw + 2wdw + 2u \cdot du = 0
\]

\[
-dv = \frac{vdw + u \cdot du}{w},
\]

and plugging this into our formula for \((ds')^2\):

\[
(ds')^2 = \frac{2vdw^2}{w} + \frac{2(u \cdot du)dw}{w} - du \cdot du = ds^2,
\]

as desired. \( \square \)
Thus for some $k$ field free part of $E$

$P = [\log \text{arithmic height of } P]$

The Neron-Tate Pairing.

1.3. Remark 3

Let $v \in V^\perp(E,P)$. The map

$$T_v(x) = x - \left( x \cdot v + \frac{1}{2}(x \cdot E)(v \cdot v) \right) E + (x \cdot E)v$$

is in $O^+$, fixes $E$, and acts as translation by $v$ in $V^\perp \cong \partial H_E$.

Proof. It is a straightforward calculation to verify that $T_v(x) \cdot T_v(y) = x \cdot y$, and that $T_v(E) = E$. If $A \in \partial H_E$, then $A \cdot A = 0$, and after some calculation, one finds $\phi(T_v(A)) = \phi(A) + v$. Hence the action of $T_v$ on $V^\perp \cong \partial H_E$ is translation by $v$. $\Box$

Note that $((O) + [E]) \cdot ([O] + [E]) = 0$ and $([O] + [E]) \cdot [E] = 1$, so we may choose $P = [O] + [E]$. With this choice, $t_i = T_v$ for some $v \in V^\perp[O],[E]$. Note that

$$D_i = t_i([O]) = T_v([O]) = [O] + cE + v$$

for some $c$. We can isolate $c$ by noting $[O] \cdot v = ([O] + [E]) \cdot v = 0$ so $D_i \cdot [O] = -2 + c$. Thus $v = D_i - [O] - (2 + D_i \cdot [O])E$. Thus $t_i = T_v$. Note that $v_i$ and $v$ differ by a multiple of $E$. While Lemma 4 uses $v \in V^\perp$, it is straightforward to verify that $T_v = T_v + aE$ for any $a$. Thus, $t_i = T_v$, as desired.

Remark 2. The map $T_v$ was derived by first considering an arbitrary $A = a_P P + a_E E + a \in \partial L$ with $a \in V^\perp$. We note that $T_v(A) \cdot E = A \cdot T_v^{-1} E = A \cdot E = a_P$, so $T_v(a_P^{-1} A) = P + a_P^{-1} E + a$. We use $T_v(A) \cdot T_v(A) = 0$ to solve for $a_P^{-1}$. Finally, we note that $a \cdot v = A \cdot v$ and gather together the components of $A$ to get the formula for $T_v$.

Remark 3. It is straightforward to verify $T_v^m(x) = T_{mv}(x)$, and that $T_v \circ T_w = T_{wv} \circ T_v$.

1.3. The Neron-Tate Pairing. Let $E$ be an elliptic curve defined over a number field $k$. Then $E(k) \cong E_{tor} \times \mathbb{Z}$, and there exists a basis $\{P_1, ..., P_r\}$ to the torsion-free part of $E(k)$. For a point $P \in E(k)$, let $H(P)$ be its naive height. The logarithmic height of $P$ is $h(P) = \log(H(P))$, and the canonical height is

$$\hat{h}(P) = \lim_{n \to \infty} \frac{h([n]P)}{n^2}.$$

The canonical height has several nice properties:

\begin{align*}
(4) \quad h(P) &= \hat{h}(P) + O(1) \\
(5) \quad \hat{h}([n]P) &= n^2 \hat{h}(P) \\
(6) \quad \hat{h}(P + Q) + \hat{h}(P - Q) &= 2\hat{h}(P) + 2\hat{h}(Q).
\end{align*}

From Eq. (4) and (5), it follows that $\hat{h}(P) = 0$ if and only if $P \in E_{tor}$. We define the Neron-Tate pairing to be

$$\langle P, Q \rangle = \hat{h}(P + Q) - \hat{h}(P) - \hat{h}(Q).$$

It is a nice exercise (using Eq. (5)) to show that the Neron-Tate pairing is a bilinear form. That is, $\langle P, Q \rangle = \langle Q, P \rangle$ and

$$\langle [m]P + Q, R \rangle = m \langle P, R \rangle + \langle Q, R \rangle.$$
1.4. **Vector Heights.** Given a basis \( D = \{D_1, ..., D_\rho\} \) of \( \text{Pic}(X) \), let us define a dual basis \( D^* = \{D^*_1, ..., D^*_\rho\} \) such that

\[
D_i \cdot D^*_j = \delta_{ij},
\]

where \( \delta_{ij} \) is the Kronecker-delta symbol (\( \delta_{ij} = 1 \) if \( i = j \), \( \delta_{ij} = 0 \) if \( i \neq j \)). For each \( D_i \), let us pick a Weil height \( h_{D_i} \) with respect to the divisor \( D_i \), and define the vector height

\[
h(P) : X \to \text{Pic}(X) \otimes \mathbb{R}
\]

\[
P \mapsto \sum_{i=1}^\rho h_{D_i}(P)D^*_i.
\]

The vector height has a couple of nice properties [Bar03]. For any Weil height \( h_D \) associated to the divisor \( D \), we have

\[
h_D(P) = h(P) \cdot D + O(1),
\]

where the constant implied by the \( O(1) \) is independent of \( P \), but may depend on \( D \). Also, for any \( \sigma \in \text{Aut}(X) \),

\[
h(\sigma P) = \sigma_* h(P) + O(1),
\]

where again the constant implied by the \( O(1) \) is independent of \( P \) but may depend on \( \sigma \).

2. **The main result**

2.1. **The automorphisms of \( X \) that fix the fibers in \([E]\).** Let

\[
\sigma_0 : \quad X \to X
\]

\[
P \mapsto -P
\]

where the operation is on the unique elliptic curve \( E \in [E] \) that contains \( P \). Then \( \sigma_0 \in \text{Aut}(X) \) and its pullback \( \sigma_0^* \) acts linearly on \( \text{Pic}(X) \). The following describes the action of \( \sigma_0^* \):

**Lemma 5.** The pullback \( \sigma_0^* \) of \( \sigma_0 \) has eigenvectors \([E]\) and \([O]\) associated to the eigenvalue \( \lambda = 1 \), and acts as multiplication by \(-1\) on \( V^\perp[E],[O] \).

**Proof.** Note that \( \sigma_0^2 \) is the identity on \( X \), so \( (\sigma_0^*)^2 = 1 \). Hence the minimal polynomial for \( \sigma_0^* \) divides \( \lambda^2 - 1 \), so \( \sigma_0^* \) is diagonalizable over \( \mathbb{Q} \) (thinking of \( \sigma_0^* \) acting on \( \text{Pic}(X) \otimes \mathbb{Q} \)) with eigenvalues \( \lambda = \pm 1 \). Since \( \sigma_0 E = E \) for any \( E \in [E] \), and \( \sigma_0 O = O \), both \([E]\) and \([O]\) are eigenvectors with associated eigenvalue \( \lambda = 1 \). The space \( V^\perp[E],[O] \) perpendicular to \( \text{span}\{[E],[O]\} \) is invariant under the action of \( \sigma_0^* \). To see this, suppose \( v \cdot [E] = 0 \). Then

\[
\sigma_0^* v \cdot [E] = v \cdot \sigma_0^* [E] = v \cdot [E] = 0.
\]

We can therefore complete a basis of eigenvectors (over \( \mathbb{Q} \)) with eigenvectors in \( V^\perp[E],[O] \). Suppose there exists an eigenvector \( w \in V^\perp[E],[O] \) with eigenvalue \( \lambda = 1 \). Without loss of generality (by multiplying by a suitable integer), we may assume \( w \) is an integral linear combination of \( \{v_1, ..., v_{\rho-2}\} \). (This is where we use that \( \Gamma_E \) has an Abelian subgroup of maximal rank.) Then

\[
T_{mw} \in \Gamma_E
\]
for all $m \in \mathbb{Z}$. Let $D_{mw} = T_{mw}([O])$. Then $D_{mw} \cdot x = 0$ is a face of the ample cone, so $D_{mw}$ represents a $-2$ curve on $X$ for all integers $m$. Further, by Lemma 3

$$D_{mw} = T_{mw}([O]) = [O] + c_{mw}E + m\omega,$$

so $D_{mw}$ is an eigenvector of $\sigma_0^m$ with eigenvalue $\lambda = 1$. That is, $\sigma_0^m D_{mw} = D_{mw}$, and hence $\sigma_0 (D_{mw}) = D_{mw}$, where we are abusing notation as before and letting $D_{mw}$ represent both a divisor class and the unique $-2$ curve on $X$ that represents the class. Let

$$Q_{mw,E} = D_{mw} \cap E \in X,$$

for any $E \in [E]$. Then

$$\sigma_0 (Q_{mw,E}) = \sigma_0 (D_{mw} \cap E) = \sigma_0 (D_{mw}) \cap \sigma_0 (E) = \sigma_0 (D_{mw}) \cap E = Q_{mw,E}.$$

But from the definition of $\sigma_0$,

$$\sigma_0 (Q_{mw,E}) = -Q_{mw,E}.$$

Hence, $2Q_{mw,E} = 0$. There exists a generic fiber $E$ where the $-2$ curves $D_{mw}$ intersect $E$ at infinitely many points, so on this fiber we have found an infinite number of points of order 2, a contradiction. Thus, there is no eigenvector in $V^\perp [E], [O]$ associated to the eigenvalue $\lambda = 1$, so $V^\perp [E], [O]$ is the eigenspace associated to $\lambda = -1$.

Let

$$\sigma_i : X \to X$$

$$P \mapsto Q_i,E - P,$$

where $E$ is the unique fiber in $[E]$ that contains $P$. Then $\sigma_i \in \text{Aut}(X)$.

**Lemma 6.** The pullback $\sigma_i^*$ of $\sigma_i$ has eigenvectors $[E]$ and $[O] + D_i$ associated to $\lambda = 1$, and is $-1$ on the perpendicular space $V^\perp [E], [O] + D_i$.

**Proof.** The proof is similar to that of Lemma 3. While $\sigma_i E = E$ as before, we note that $\sigma_i O = D_i$ and $\sigma_i D_i = O$, so $\sigma_i^* ([O] + D_i) = [O] + D_i$. As before, the perpendicular space $V^\perp [E], [O] + D_i$ is invariant under $\sigma_i^*$, so we can complete a basis of eigenvectors with elements in this space. We assume there exists an eigenvector $w \in V^\perp [E], [O] + D_i$ (over $\mathbb{Q}$) with associated eigenvalue $\lambda = 1$. Let us write

$$w = w_O [O] + w_E [E] + w',$$

where $w' \in V^\perp [E], [O]$. Then

$$0 = w \cdot [E] = w_O$$

$$w \cdot [O] = w_E,$$

so

$$w' = w - (w \cdot [O])[E].$$

Thus, $w'$ is in the eigenspace for $\lambda = 1$. As before, by multiplying by a suitable integer, we may take $w'$ to be an integer linear combination of $\{v_1, \ldots, v_{\rho - 2}\}$. Thus, as before, $T_{mv'} \in \Gamma_E$ for all integers $m$. $D_{mv'} = T_{mv'}([O])$ is represented by a $-2$ curve on $X$, $\sigma_i^* D_{mv'} = D_{mv'}$, and therefore

$$\sigma_i (Q_{mv',E}) = Q_{mv',E}.$$

But

$$\sigma_i (Q_{mv',E}) = Q_i - Q_{mv',E},$$
so we get $2Q_{m,w',E} = Q_i$ for all integers $m$. For any $E$, there are at most four solutions to $2P = Q_i$, but for a generic fiber (all but finitely many), the points $Q_{m,w',E}$ form an infinite set. Thus, no such $w$ can exist, so $V^\perp_{[E],[O]+D_i}$ is the $(\lambda = -1)$-eigenspace for $\sigma^*_i$. \[\square\]

We are now ready to prove the first assertion of Theorem 7.

**Theorem 7.** The map $t_i \in \Gamma_E$ is the push forward of $\tau_i \in \text{Aut}(X)$.

**Proof.** We note that $\tau_i = \sigma_i \circ \sigma_0$. Thus $\tau_i^* = \sigma_0^* \circ \sigma_i^*$, so

$$\tau_{i*} = (\sigma_0^* \circ \sigma_i^*)^{-1} = \sigma_i^* \circ \sigma_0^*,$$

where we have used that $\sigma_i^2 = \sigma_0^2 = id$. We use Lemmas 5 and 6 to calculate $\tau_{i*}$. Given $A \in \text{Pic}(X) \otimes \mathbb{Q}$, let us write

$$A = a_O[O] + a_E[E] + a,$$

where $a \in V^\perp_{[E],[O]}$. We can write

$$a = \frac{1}{2}a \cdot ([O] + D_i) + a',$$

where $a' \in V^\perp_{[E],[O]+D_i}$, from which it follows

$$\sigma_i^* \circ \sigma_0^* A = A + a_O(D_i - [O]) - a \cdot ([O] + D_i)[E].$$

Noting that $a_O = A \cdot E$ and substituting $v_i$, we get

$$\tau_{i*}(A) = A + (A \cdot E)v_i + (2 + D_i \cdot [O] - a \cdot ([O] + D_i))[E].$$

Because $\Gamma$ is discrete and $t_i$ has infinite order in $\Gamma_E$, it must be a translation on $\mathcal{E}'$. Thus $t_i = T_v$ for some $v \in V^\perp_{[E],[O]}$. Using $D_i = t_i([O])$ and Lemma 5, we conclude $v = v_i$. Thus,

$$t_i(A) = A + (A \cdot E)v_i + \left(A \cdot v_i + \frac{1}{2}(A \cdot [E])(v_i \cdot v_i)\right)[E].$$

We can verify directly that the coefficients of $[E]$ in $\tau_{i*}(A)$ and $t_i(A)$ are equal, or we can observe that both $\tau_{i*}$ and $t_i$ are isometries, so

$$\tau_{i*}(A) \cdot D_i = A \cdot \tau_{i*}^{-1}(D_i) = A \cdot [O]$$

$$t_i(A) \cdot D_i = a \cdot t_i^{-1}(D_i) = A \cdot [O].$$

Thus

$$\tau_{i*}(A) \cdot D_i = t_i(A) \cdot D_i,$$

and since $[E] \cdot D_i = 1 \neq 0$, we get that the coefficients of $[E]$ are equal. Hence, $\tau_{i*} = t_i$, as claimed. \[\square\]

2.2. **The Neron-Tate Pairing.** In this section, we use vector heights to calculate $\langle Q_{i,E}, Q_{j,E} \rangle$. Let us choose the basis

$$D = \{[E], [O] + [E], v_1, ..., v_{p-2}\}.$$

Then the dual basis is

$$D^* = \{[O] + [E], [E], v_1^*, ..., v_{p-2}^*\}.$$
where \( \text{span}\{v_1^*, \ldots, v_{\rho-2}^*\} = V^\perp[E],[O] \). Let us define a projection of \( X \) onto the section \( O \) by

\[
\pi : \quad X \to O
\]

\[
P \to O_E
\]

where \( E \) is the unique fiber that contains \( P \). Let us define a logarithmic height \( h \) on the section \( O \). Note that the pull back \( \pi^* \) of a point in \( O \) is a fiber in \( [E] \), so \( h \circ \pi \) is a Weil height with respect to \([E]\). Let us use \( h_{[E]} = h \circ \pi \) in our definition of the vector height, so \( h(P) \cdot [E] = h_{[E]}(P) \). Now suppose \( \sigma \in \langle \sigma_0, \sigma_1, \ldots, \sigma_{\rho-2} \rangle \). Since \( \sigma \) fixes \( E \) for every \( E \in [E] \), we know

\[
h_{[E]}(\sigma P) = h_{[E]}(P).
\]

On the other hand,

\[
h_{[E]}(\sigma P) = h(\sigma P) \cdot [E] = (\sigma_* h(P)) + \mathbf{O}(1) \cdot [E]
\]

\[
= \sigma_* h(P) \cdot [E] + \mathbf{O}(1) \cdot [E]
\]

\[
= h(P) \cdot \sigma^*[E] + \mathbf{O}(1) \cdot [E]
\]

\[
= h(P) \cdot [E] + \mathbf{O}(1) \cdot [E]
\]

\[
= h_{[E]}(P) + \mathbf{O}(1) \cdot [E].
\]

Thus, the error term \( \mathbf{O}(1) \) for \( \sigma \) lies in \( V^\perp[E] \), since it satisfies \( \mathbf{O}(1) \cdot [E] = 0 \).

**Lemma 8.** Suppose \( u, v \in V^\perp[E] \). Then

\[
|u \cdot v| \leq ||u||||v||,
\]

where \( ||u|| = \sqrt{u \cdot u} \).

**Proof.** Let us write \( u = u_E[E] + u' \), etc., with \( u' \in V^\perp[E],[O] \). Then

\[
u \cdot v = u' \cdot v',
\]

from which the result follows, since \( \cdot \) is negative definite on \( V^\perp[E],[O] \).

Let \( v \in v_1 \mathbb{Z} + \ldots + v_{\rho-2} \mathbb{Z} \). Let \( \tau_v \in \text{Aut}(X) \) be the canonical automorphism with \( \tau_v = T_v \). (That is, \( \tau_v(P) = P + Q_{v,E} \) where \( E \) is the fiber in \([E]\) that contains \( P \), \( Q_{v,E} = E \cap D_v \) and \( D_v = T_v([O]) \).) Then

\[
h(\tau_v P) = T_v h(P) + \mathbf{O}(1),
\]

where the error term is bounded. Let us decompose the error terms into two parts, \( \mathbf{O}(1) = \mathbf{O}'(1) + O(1)[E] \), where \( \mathbf{O}'(1) \in V^\perp[E],[O] \). Let \( M \) bound both \( ||\mathbf{O}'(1)|| \) and \( |O(1)| \). (Note that \( M \) depends on \( v \), but not \( P \).) We drop the prime notation \( \mathbf{O}'(1) \) in the following:

**Lemma 9.** For fixed \( v \in v_1 \mathbb{Z} + \ldots + v_{\rho-2} \mathbb{Z} \),

\[
h(\tau_v^n P) = T_v^n h(P) + \mathbf{O}(n) + O(n^2)[E],
\]

where \( \mathbf{O}(n) \in V^\perp[E],[O] \), \( ||\mathbf{O}(n)|| \) is bounded by \( Mn \) and the scalar error term \( O(n^2) \) is bounded by \( M||v||n^2 \).
Proof (by induction on \( n \)). The base case is clear. Consider

\[
\hat{h}(P) = \lim_{n \to \infty} \frac{h(D(n)[Q_{v,E}]n^2)}{n^2} = \frac{h(D(n)[O(n)] + O(n^2)[E])}{n^2} + O(1) \cdot D
\]

\[
= \lim_{n \to \infty} \frac{1}{n^2} (T_v^n h(P) + O(n) + O(n^2)[E]) \cdot D
\]

\[
= \lim_{n \to \infty} \frac{1}{n^2} (T_v^n h(\tau_v^n O_E) + O(n) + O(n^2)[E]) \cdot D
\]

\[
= \lim_{n \to \infty} \frac{1}{n^2} \left( h(\tau_v^n O_E) - \left( (h(\tau_v^n O_E) \cdot [E]) - \frac{1}{2} (h(O_E) \cdot [E]) (n^2) \cdot [E] \right) + (h(\tau_v^n O_E) \cdot [E]) \cdot D + O(n + O(n^2)[E]) \right) \cdot D
\]

\[
= \frac{1}{2} (h(\tau_v^n O_E) \cdot [E]) \cdot D + O(1)
\]

The error term \( O(1) \) is bounded by \( M \| v \| \cdot [E] \cdot D \), and is independent of our choice of fiber \( E \). This result is similar to [Tat83, Corollary 1].

Finally, we calculate the Neron-Tate pairing:

\[
\langle Q_{v,E}, Q_{w,E} \rangle = \hat{h}(Q_{v,E} + Q_{w,E}) - \hat{h}(Q_{v,E}) - \hat{h}(Q_{w,E})
\]

\[
= \hat{h}(Q_{v+w,E}) - \hat{h}(Q_{v,E}) - \hat{h}(Q_{w,E})
\]

\[
= \frac{1}{2} h(E) ([E] \cdot D) (v + w) \cdot (v + w) - v \cdot v - w \cdot w + O(1)
\]

\[
= h(E) ([E] \cdot D) (v \cdot w) + O(1).
\]

In particular,

\[
\lim_{h(E) \to \infty} h(E) ([E] \cdot D) (Q_{v,E}, Q_{w,E}) = v \cdot w.
\]

These last two results are similar to [Sil94, Theorem 11.1 and Corollary 11.3.1].

This completes all the pieces of Theorem 1.

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