Student Teachers’ Proof Schemes on Proof Tasks Involving
Inequality: Deductive or Inductive?

A H Rosyidi¹, A W Kohar²

¹,²Department of Mathematics, Universitas Negeri Surabaya, Indonesia

Email: ¹abdulharis@unesa.ac.id

Abstract. Exploring student teachers’ proof ability is crucial as it is important for improving the quality of their learning process and help their future students learn how to construct a proof. Hence, this study aims at exploring at the proof schemes of student teachers in the beginning of their studies. Data were collected from 130 proofs resulted by 65 Indonesian student teachers on two proof tasks involving algebraic inequality. To analyse, the proofs were classified into the refined proof schemes level proposed by Lee (2016) ranging from inductive, which only provides irrelevant inferences, to deductive proofs, which consider addressing formal representation. Findings present several examples of each of Lee’s level on the student teachers’ proofs spanning from irrelevant inferences, novice use of examples or logical reasoning, strategic use examples for reasoning, deductive inferences with major and minor logical coherence, and deductive proof with informal and formal representation. Besides, it was also found that more than half of the students’ proofs coded as inductive schemes, which does not meet the requirement for doing the proof for the proof tasks examined in this study. This study suggests teacher educators in teacher colleges to reform the curriculum regarding proof learning which can accommodate the improvement of student teachers’ proving ability from inductive to deductive proof as well from informal to formal proof.

1. Introduction
At university level, constructing formal mathematical proof is a vital skill expected to be mastered by undergraduate students at mathematics courses. This is crucial as undergraduate students are demanded to have such skill as the highest level of proving so that they lead to perceiving that proof is an essential component of mathematics which is arguably what differentiate mathematics from other disciplines [1]. However, Clark and Lovric [2] explored numerous challenges that undergraduate students experienced when making the transition to a more formal proof construction at university-level mathematics. This transition requires undergraduate students to change the schemes of reasoning used. For instances, they need to shift from informal to formal language or representation, shift from inductive to deductive inference, understand and apply the theorem, reason from mathematical definitions, justify trueness of propositions using a valid mathematical argument, and to make connections between mathematical objects.

The proof reasoning as discussed above was agreed by experts as an important skill the undergraduate students require to construct a proof [3]. Such reasoning, more particularly, were also needed by undergraduate students who are student teachers. This is due to need of a teacher to have a deep understanding of the nature and the role of proof for the need of instructional practice [1]. In addition, knowledge about mathematical proof is considered as one of the essential components of subject matter
knowledge which should be mastered by mathematics teachers [1]. By having this knowledge, as Jones [1] argued, they would have an extremely secure subject knowledge base of mathematical proof if they are to teach it accurately and confidently.

However, many of the proof-related studies on undergraduate students, including for those who were student teachers, indicate their weaknesses regarding knowledge of proof construction. As evidence, they have been observed to experience mathematical proof nearly procedural, regarding the construction and writing of a proof as an algorithm to follow rather than a creative process for solving a problem [4,5]. Such finding was scrutinized to be caused by the finding that they usually used non-formal reasoning when studying at secondary level while at the university level they start to learn formal reasoning in the mathematics courses they took. This influence they way they perceive and construct mathematical proof at the beginning of their studies in undergraduate level. It is clearly important for a student teacher of mathematics to be able to construct, understand, and validate formal mathematical arguments [6], yet research indicates that a lot of students eventually do not succeed in developing skills of mathematical proving, not only in the beginning of their studies, but also by the end of their undergraduate programs. As such, capturing the student teacher’s proving schemes in the beginning of their studies is required as one of tools of evaluation for the improvement of courses programs offered in the student teacher curriculum at university level.

This present study concerns on exploring the proof schemes resulted by student teachers who were studying in the first semester of their undergraduate studies in a mathematics education program. The topic examined to the student teachers was inequality since it was offered in Calculus course they took in the first semester.

2. Student Teachers’ Proof Schemes on Inequality Task

The classifications of proof schemes as many scholars proposed were based on an underpinning thought that asserted undergraduates’ mathematical proving evolved from being inductive-oriented toward deductive-oriented [4]. Miyazaki [7] classified proof schemes from deductive scheme to inductive scheme into four types: Proof A, Proof B, Proof C, and Proof D. Proof A is the type of proof which deductive reasoning is involved and a functional language is used in the course of doing a proof, while Proof B is the type of proof in which deductive reasoning is involved and other languages, drawings, and movable objects are used in the course of doing a proof. While Proof C is the type of proof in which inductive reasoning is involved and other languages, drawings, and movable objects are used, Proof D is the type of proof which inductive reasoning is involved and a functional language is used. Lee [8] further refined such schemes into two types of schemes in terms of deductive-proof construction for true proposition and proof-by-counterexample construction for false propositions. Regarding the former schemes, Lee’s [8] schemes present a hierarchical six-level of proof schemes ranging from inductive, which only provides irrelevant inferences, to deductive proofs, which consider addressing formal representation. Table 1 summarizes such schemes.

| Level | Characteristics | Description of students’ proofs |
|-------|------------------|--------------------------------|
| 0     | Irrelevant or minimal engagement in inferences | Do not know how to prove or appeal to conviction in external knowledge sources Unable to relate the antecedent and the consequent with examples |
| 1     | Novice use of examples or logical reasoning | Conclude the implication is true (or false incorrectly) using one or more examples or an incorrect example or counterexample Falsify the implication “if P then Q” through erroneous logical reasoning, e.g., “if not P then not Q” Derive a mathematical property unrelated to the implication |
2. **Strategic use of examples for reasoning**
   - Generate examples based on random or case-based sampling, or extreme cases. Uses mathematical properties inferred from generated examples to make conclusions.

3. **Use of deductive inferences with major flaws in logical coherence and validity**
   - Deduce relevant mathematical properties but missing one or two deductive inferences for proving the implication.
   - Deduce the implication to be true for some cases of the antecedent but omit other cases.

4. **Use of deductive inferences with minor flaws in logical coherence and validity**
   - Generate a chain of deductive inferences to justify conclusions but one or two inferences may be interpreted as inductive due to insufficient substantiation or as writing errors.
   - Displays disorganization in the chain of deductive inferences.

5. **Construction of informal deductive proofs**
   - Generate a chain of deductive inferences to justify the implication using informal justifications.

6. **Construction of deductive proofs using formal representations**
   - Generate a chain of deductive inferences to justify the implication using formal representations.

According to Lee [8], level 0 means someone is unable to provide any inferences to the proof he/she creates, level 1 and 2 indicates inductive inferences with different level of attainment, while live 3,4,5,6 shows deductive inferences with the different progress of proof. Table 1 also describes what makes distinguish between one another level as seen in the description of students’ proof column.

### 3. Method

We involved 65 student teachers who were studying mathematics in the first semester of their study at a mathematics education program in our university. The data were collected from their written responses on two proof-related tasks involving algebraic inequality. Thus, we had 130 responses (65 × 2 task responses). Figure 1 shows the proof-related tasks examined to the student teachers. While Lee’s [8] research instruments concerned on the complete proposition which required participants to justify the trueness of such proposition and prove the reasoning of proofs they created, the task we developed in this study required participants to (1) choose correct option/s to complete an incomplete proposition and prove their justification, as well as to (2) solve an inequality task involving arbitrary two different real number and justify whether its solution is same with a similar inequality task. For task no 2, some students worked on (x − a)(x + b) > 0, while the others worked on (x − a)(x − b) < 0.

| 1. If a and b are two different real numbers, then... |
|-----------------------------------------------------|
| (i) (a − b)(b − a) < 0 |
| (ii) (a − b)(b − a) = 0 |
| (iii) (a − b)(b − a) > 0 |
| You may choose more than one statements when you think they are correct. Support your option/s by giving complete arguments. |

| 2. If a and b are two different real numbers, solve the inequality below (x − a)(x + b) > 0 |
|---------------------------------------------------------------|
| Then, determine whether its solution will be same as the solution of the inequality below. |
| (x − a) > 0 |
| (x + b) > 0 |
| Give your reasons. |

**Figure 1.** The proof-related tasks

Data were analyzed by coding each of responses into an appropriate level of proof as proposed by Lee [8] as seen in table 1. To describe, we exemplify the responses of student teachers’ proof schemes on those two tasks by presenting examples of each of the levels.
4. Results and Discussion

In general, the levels of student teachers’ responses can be described as a progression through four phases: attempts that failed to relate the antecedent and the consequent (level 0), proofs that were based on examples or logical reasoning errors (levels 1–2), proofs that were based on incomplete deductive inferences (levels 3–4), and proofs that were based on coherent deductive inferences (level 5–6). Table 2 shows the frequency distribution of student teachers’ deductive-proof construction schemes on the inequality task.

| Levels | Task 1 | % | Task 2 | % | Total | % |
|--------|--------|---|--------|---|-------|---|
| 0      | 3      | 4.6% | 8      | 12.3% | 11    | 8.5% |
| 1      | 30     | 46.2% | 9      | 13.8% | 41    | 31.5% |
| 2      | 14     | 21.5% | 12     | 18.5% | 26    | 20.0% |
| 3      | 5      | 7.7% | 12     | 18.5% | 17    | 13.1% |
| 4      | 6      | 9.2% | 12     | 18.5% | 18    | 13.8% |
| 5      | 3      | 4.6% | 7      | 10.8% | 8     | 6.2% |
| 6      | 4      | 6.2% | 5      | 7.7% | 9     | 6.9% |
| Total  | 65     |     | 65     |     | 130   | 100.0% |

The distribution of student teachers’ responses as shown in table 2 indicates that most of the responses, which is 41 out of 130, were indicated in level 1 (the use of novice examples), while only 9 responses (6.9%) were indicated in level 6 (formal representation of deductive inference). Regarding deductive and inductive inferences, generally, there were 67 inductive responses (level 1, 2) and 52 deductive responses (level 3, 4, 5, 6). It means the student teachers performed inductive inferences more frequently than deductive inferences.

In the next discussion, we provide each of examples of the students’ responses for each level. Figure 2 shows a collective examples of responses for task 1, while figure 3 shows examples for task 2.

- **a) Level 0**
  I think all of the options are correct because real numbers consist of integer, whole numbers, natural number, fraction, etc. If \( a \) and \( b \) are different then those three options satisfies

- **b) Level 1**
  So, if any numbers are substituted into the inequality, it will have a negative number

- **c) Level 2**
  Options (i) and (iii) are both correct because if \( a, b \) are different real numbers, then \((a-b) \neq 0 \) and \((b-a) \neq 0\).
  So, \((a-b)(b-a) \neq 0\).

- **d) Level 3**
If \( a = 5, b = 2 \), then \((5 - 2)(2 - 5) < 0 \) (negative)
If \( a = 5, b = -2 \), then \((5 + 2)(-2 - 5) < 0 \) (negative)
If \( a = -5, b = -2 \), then \((-5 - 2)(-2 + 5) < 0 \) (negative)
If \( a = -5, b = -2 \), then \((-5 - 2)(2 + 5) < 0 \) (negative)
All are correct

Accordingly, \((a - b)(b - a) > 0 \) and \((a - b)(b - a) < 0 \)


\emph{e)} Level 4

I selected (i) since multiplication of those two factors will always less than 0. Let \( a > b \), then \((a - b) > 0 \) and \((b - a) < 0 \)
Obtained that \((a - b)(b - a) < 0 \)
Let Let \( a < b \), then \((b - a) > 0 \) and \((a - b) < 0 \)
Obtained that \((a - b)(b - a) < 0 \)

\emph{f)} Level 5

\[
(a - b)(b - a) = -a^2 + 2ab - b^2
\]
For \( a = + \) and \( b = + \), then \(-(-)^2 + 2(+)(+) - (+)^2 = -
\]
For \( a = + \) and \( b = - \), then \(-(-)^2 + 2(+)(-) - (-)^2 = -
\]
For \( a = - \) and \( b = + \), then \(-(-)^2 + 2(-)(+) - (+)^2 = -
\]
For \( a = - \) and \( b = - \), then \(-(-)^2 + 2(-)(-) - (-)^2 = -
\]
Then, \((a - b)(b - a) < 0 \)

\emph{g)} Level 6

\[
(a - b)(b - a) < 0
\]
\[
-(a - b)(a - b) < 0
\]
\[
-(a - b)^2 < 0
\]
(If negative is multiplied by the square of any number, the product is always negative as well, \( a \) and \( b \) are two different numbers. This is correct)
\[-(a - b)^2 = 0 \text{. This happens if } a \text{ and } b \text{ are same numbers}
\]
\[-(a - b)^2 > 0 \text{. This is impossible since it always be negative.}\]

\emph{Figure 2.} Student teachers’ responses on task 1

At level 0, some proof schemes referred to justifications given in textbooks, such as arguing that it is based on the student teachers’ memory when learning inequality at secondary level. Others, as shown in figure 2a, show irrelevant argument which failed to relate the antecedent and the consequent (the chosen option). At level 1, the proof schemes show novice examples, such as examining the trueness of the proposition after the option/s were selected by substituting random numbers \( (a = 1, b = 2) \) to the inequality and conclude that the proposition is correct. At level 2, the proof schemes were observed to have more sophisticated examples since some of the student
teachers consider giving case-based sampling to examine their justification. Figure 2c shows that the proof provided all the possibilities of pairs of positive-negative numbers and examined those numbers to the inequality to conclude that the proposition is correct. At level 3, the proofs were labelled as deductive inferences although many incomplete chain of inferences were found. The student teacher on the response shown in figure 2d, for example, consider only one of the cases of possibilities. In other words, instead of considering the the existance of trichotomy property of real number \((a = b, a > b, or a < b)\) which might have different cases for each of the inequalities offered in the options, the student teacher assumed that there were two possibilities of two different numbers \((a \neq b)\), namely \(a > b, or a < b\), and claimed that those two possibilities were both correct without examining the correctness of each. At level 4, the student teachers constructed almost complete deductive inferences. The characteristic of this level can be observed from the existance of missing step of claim which is not supported by valid arguments. For task 1, we found that the student teacher at figure 2e categorized his/her inferences for the cases of \(a > b\) and \(b > a\), then by using an algebraic operation, the student teacher claimed that \((a - b)(b - a) < 0\). However, this claim was not supported by any arguments such as arguing why \((a - b)(b - a) < 0\). At level 5, a proof demonstrating logically valid and coherent chain of deductive inferences would incorporate deductive inferences about examining each of cases for the value of \(a\) and \(b\) which is negative or positive. Figure 2f indicates that for all possibilities of \(a\) and \(b\), the results of \((a - b)(b - a)\) is always negative. Thus, \((a - b)(b - a) < 0\). However, it is not indicated that this proof include any formal representation. This is why this proof is categorized at level 5. At level 6, proofs are more deductive and of course are valid. Figure 2g shows that the student teacher used symbolic representations which met relevant theorems of inequality.
The solution of those two inequalities are same. For every \( x \), with \( a = 1 \) and \( b = 2 \) both give solution of \( x < 1 \) and \( x > 1 \) (indicated by the number line).

The solutions are same, obtained from finding the zero makers first, namely \((x - a)\) and \((x + b)\) resulting \((a)\) and \((-b)\). Then, make a number line to put those two zero makers. Examine the correctness of interval by arbitrary numbers, usually 0 (to make it easy). It will give the solution.

\[(x - a) (x - b)\] will be positive, if:
- \((x - a)\) is positive, if:
- \((x - b)\) is positive, if:
- \((x - a)\) is
- \((x - b)\) is
- So, those two inequalities have the same solution. The difference is only about \( x - b = 0 \) for the first, whereas \( x - b \neq 0 \). However, it still remaining same because te sign is >, not ≥.

**Figure 3.** Student teachers’ responses on task 2

Figure 3 a shows that the student teachers at level 0 did not use the idea of solving an inequality task correctly since the student teacher seemed mixed up the idea of solving an equation and an inequality. At level 1, we found some student teachers inferred that the implication (the solution of those two inequalities are same) is true based on some examples of numbers for \( a, b, \) and \( x \) substituted into the
inequalities. See figure 3b. A little different from figure 3b, figure 3c indicates that the student teachers at level 2 solved for $x$ by firstly substituting both $a$ and $b$ with certain numbers and verified the other inequality with those numbers to prove that both the two inequalities have the same solutions. Thus, they took a case-based sampling for $x$. At level 3, the student teachers, as seen in figure 3d, no longer deduced by testing some numbers. Instead, they tried to solve each of the two inequalities by keeping $x$, $a$, and $b$ as variables. However, it seemed that they did not consider carefully about the possibilities of values and the position of $a$ and $b$ in the number line they drew. Thus, they made major flaws to validate their inferences. At level 4, the student teachers made logical deductive inferences by arguing that if the open interval represented by the number line shown in figure 3e are examined with arbitrary numbers, then both the two inequalities would have same solutions. However, such claims were not supported by some arguments. Also, the positions of $a$ and $(-b)$ in the number line is interchangeable as $a$ can be more than $(-b)$ and vice versa. As such, the proof is not completed due to insufficient information related to the proof of the student teachers' claims. At level 5, on the other hand, the student teachers have completed deductive inferences by showing some informal representations. This is shown by Figure 3f pointing out that the proofs were presented in terms of case-based inferences. The student teachers categorized their inferences by considering the probable sign of both factors $(x - a)$ and $(x + b)$ and examined the value of the product of such two factors whether it is positive or negative. In this case, however, the student teachers used daily language, instead of mathematical representation using symbols, to prove their claims. At level 6, the difference is that the students teachers used mathematical symbols to represent all of their idea. Thus, the student teachers at this level applied deductive inferences as well as formal representations. See figure 3g for the detail work.

The prospective teachers in our study were the student teachers who start learning mathematics in a more formal way since they were still in the beginning in their studies in college when they were involved as participants. In accordance with this fact, Bاسترک [9] associates the reason for the difficulties experienced by the prospective teachers in performing proof with the fact that they suddenly come across proof in universities. His finding reporting that the difficulties experienced by them are because students in his country do not perform proof at an elementary school and high school levels are similar with what happened in the mathematics curriculum. Therefore, it is suggested that students must encounter proof in their educational lives before beginning their college education. Furthermore, to suggest on how proof is learned in college courses, Güler & Dikici [10] suggest to organized courses about proof by firstly giving samples → theorem → proof or samples → proof → theorem formats with the help of thoroughly selected examples and in a way that allows for understanding theorems, instead of only definition → theorem → proof or theorem → proof → examples. Furthermore, they also suggested that activities oriented towards different methods of proof must be performed in the courses of mathematical proof regarding how to begin the proof in order to increase the proof performances of the prospective teachers. More specifically, because we found many prospective teachers experienced difficulties mainly at the use of definition when proving, it is also suggested that the activities recommended by Güler & Dikici [10] would also be able to give attention on how prospective teachers develop their skill in composing proof by considering the limitation properties of symbols/variables they selected as well as how to use those kinds of functional language properly.

5. Conclusions
We would highlight that more than half of the students' proofs were coded as inductive inferences, which does not meet the requirement for doing proof, which is requiring deductive inferences, for the proof tasks examined in this study. This study suggests teacher educators in teacher colleges to reform the curriculum regarding proof learning which can accommodate the improvement of student teachers’ proving ability from inductive to deductive proof as well from informal to formal proof.

6. References
[1] Jones K 1997 Student teachers' conceptions of mathematical proof Mathematics Education Review 9 21-32
[2] Clark M and Lovric M 2009 Understanding secondary–tertiary transition in mathematics *International Journal of Mathematical Education in Science and Technology* **40** 755-776

[3] Blanton M, Stylianou D, David M 2003 The Nature of scaffolding in Undergraduate Transition to Formal Proof *In: Pateman N editors Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education Honolulu* **2** 113-120

[4] Harel G and Sowder L 1998 Students’ proof schemes: Results from exploratory studies Research in collegiate mathematics education **3** 234-283

[5] Knuth E J 2002 Secondary school mathematics teachers' conceptions of proof *Journal for research in mathematics education* **37** 39-405

[6] Selden A and Selden J 2003 Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem? *Journal for research in mathematics education* **4** 3-36

[7] Miyazaki M 2000 Levels of proof in lower secondary school mathematics *Educational Studies in Mathematics* **41** 47-68

[8] Lee K 2016 Students’ proof schemes for mathematical proving and disproving of propositions *The Journal of Mathematical Behavior* **41** 26-44

[9] Basturk S 2010 First-year secondary school mathematics students’ conceptions of mathematical proofs and proving *Educational Studies* **36** 283-298

[10] Güler G and Dikici R 2014 Examining prospective mathematics teachers’ proof processes for algebraic concepts *International Journal of Mathematical Education in Science and Technology* **45** 475-497