One-relator groups with torsion are coherent

Larsen Louder and Henry Wilton

May 31, 2018

Abstract

We show that any one-relator group $G = F/\langle \langle w \rangle \rangle$ with torsion is coherent – i.e., that every finitely generated subgroup of $G$ is finitely presented – answering a 1974 question of Baumslag in this case.

1 Introduction

Definition 1.1. A group $G$ is coherent if every finitely generated subgroup of $G$ is finitely presentable.

A well known question of Baumslag asks whether every one-relator group $F/\langle \langle w \rangle \rangle$ is coherent [Bau74, p. 76]. It is a curious feature of one-relator groups that the case with torsion, in which the relator $w$ is a proper power, is often better behaved than the general case; most famously, one-relator groups with torsion are always hyperbolic [New68], and Wise proved that one-relator groups with torsion are residually finite, indeed linear [Wis12]. In this note we answer Baumslag’s question affirmatively for one-relator groups with torsion.

Theorem 1.2. If $G$ is a one-relator group with torsion – that is, $G \cong F/\langle \langle w^n \rangle \rangle$, for $n > 1$ – then $G$ is coherent.

When this manuscript was completed, we learned from Daniel Wise that he has also proved Theorem 1.2 [Wis].

Our proof is a surprisingly straightforward consequence of Wise’s $w$-cycles conjecture (Theorem 3.1), which was proved independently by the authors [LW17] and by Helfer–Wise [HW16].

The outline of the proof is as follows. We realize $G$ as the fundamental group of a compact, aspherical orbicomplex $X$. Since one-relator groups are virtually torsion-free, there is a finite-sheeted covering map $X_0 \twoheadrightarrow X$ so that $G_0 = \pi_1 X_0$ is torsion free. We then use the $w$-cycles conjecture to show that, whenever $Y \twoheadrightarrow X_0$ is an immersion from a compact two-complex without free faces, the number of two-cells in $Y$ is bounded by the number of generators of $\pi_1(Y)$. In the final step, a folding argument expresses an arbitrary finitely generated subgroup $H$ of $G_0$ as a direct limit of fundamental groups of 2-complexes with boundedly many 2-cells, and we deduce that $H$ is finitely presented.

*Supported by EPSRC Standard Grant EP/L026481/1.
2 One-relator orbicomplexes

Let \( F \) be a finitely generated free group, and \( G = F/\langle \langle w^n \rangle \rangle \) a one-relator group, where \( w \in F \) is not a proper power. In the usual way, \( F \) can be realized as the fundamental group of some finite topological graph \( \Gamma \), and \( w \) by a continuous map \( w : S^1 \rightarrow \Gamma \). (Since we are only interested in \( w \) up to conjugacy, we ignore base points.) Let \( D_n \subseteq \mathbb{C} \) be the closed unit disk equipped with a cone point of order \( n \) at the origin. The orbicomplex \( X = \Gamma \cup w D_n \) provides a natural model for \( G \), in the sense that \( G \) is the (orbifold) fundamental group of \( X \). We call \( X \) a one-relator orbicomplex. (There is a much more general theory of orbicomplexes – see, for instance, [Hae91] or [BH99, Chapter III. C] – but the one-relator orbicomplexes defined here are sufficient for our purposes.) When \( n = 1 \), \( X \) is a one-relator complex.

A map of 2-complexes is a morphism if it sends \( n \)-cells homeomorphically to \( n \)-cells, for \( n = 0, 1, 2 \). A morphism of 2-complexes \( Y \rightarrow Z \) is an immersion if it is a local injection; in this case, we write \( Y \hookrightarrow Z \). If \( Y \) is a 2-complex and \( X \) is the one-relator orbicomplex defined above, a continuous map \( Y \rightarrow X \) is a morphism if it sends vertices to vertices, edges homeomorphically to edges, and restricts, on each 2-cell, to the standard degree-\( n \) map \( p_n : D_1 \rightarrow D_n \) given by \( p_n(z) = z^n \). A morphism \( Y \rightarrow X \) is an immersion if it is locally injective away from the cone points in the 2-cells (again, we write \( Y \hookrightarrow X \)), and a covering if it is locally a homeomorphism except at the cone points. The next definition plays a crucial role in our argument.

**Definition 2.1 (Degree).** If \( f : Y \hookrightarrow X \) is an immersion of two-dimensional (orbi)complexes, then the degree of \( f \), denoted by \( \text{deg} f \), is the minimum number of preimages of a generic point in a 2-cell of \( X \). That is: if \( X \) is a 2-complex, then \( \text{deg} f \) is the minimum number of preimages of any point in the interior of a 2-cell of \( X \); and if \( X \) is a one-relator orbicomplex with 2-cell \( D_n \), then \( \text{deg} f \) is the number of preimages of any point in the interior of \( D_n \) except 0.

Every one-relator group is virtually torsion free [FKS72], and it follows that the orbicomplex \( X \) is finitely covered by a genuine 2-complex.

**Theorem 2.2 (Unwrapped covers of one-relator complexes).** Let \( G = F/\langle \langle w^n \rangle \rangle \) be a one-relator group and \( X \) the orbicomplex defined above. Then there is a finite-sheeted covering map \( X_0 \hookrightarrow X \) where \( X_0 \) is a compact, connected 2-complex.

We emphasize that complex \( X_0 \) in the above theorem is a 2-complex, not just an orbicomplex. That is, the covering map \( X_0 \rightarrow X \) restricts to \( p_n \) on each 2-cell. We will call such a cover \( X_0 \) unwrapped.
3 A bound on the number of 2-cells

A two-complex \( Y \) is reducible if it has a free face. Writing

\[
\partial Y : \coprod S^1 \to Y^{(1)}
\]

for the disjoint union of the attaching maps of the 2-cells, this means that there is an edge \( e \) of the 1-skeleton \( Y^{(1)} \) such that \( \partial^{-1}_Y(e) \) consists of a single edge in \( \coprod S^1 \). If \( e \) is such an edge and \( Y' \) is the 2-complex obtained by collapsing the face of \( Y \) incident at \( e \), then the natural inclusion map \( Y' \hookrightarrow Y \) is a homotopy equivalence, and induces an isomorphism on fundamental groups. Of course, if \( Y \) is not reducible it is called irreducible.

The main theorem of [LW17] (or [HW16]) can be restated as a result about immersions to one-relator orbicomplexes, as follows.

**Theorem 3.1.** Let \( X \) be a one-relator orbicomplex, \( Y \) a finite 2-complex and \( f : Y \hookrightarrow X \) an immersion. If \( Y \) is irreducible then

\[
\chi(Y) + \deg f \leq 0.
\]

**Proof.** This is precisely [LW17, Theorem 1.2], with \( \Gamma = X^{(1)} \), \( \Gamma' = Y^{(1)} \), \( \rho \) the restriction of \( f \) to \( \Gamma \), \( \Lambda = \omega \), and \( S \) the disjoint union of the boundaries of the 2-cells of \( Y \). In this case, \( \Lambda' \) is the coproduct of the attaching maps of the 2-cells of \( Y \) and \( \sigma \) is the induced map from the boundaries of the 2-cells of \( Y \) to the boundary of the 2-cell of \( X \), and the result follows. \( \square \)

Here, we apply Theorem 3.1 to relate the number of 2-cells of \( Y \) to the rank of its fundamental group. (By the rank of a group, we mean the minimal cardinality of a generated set.)

**Corollary 3.2.** Let \( f : Y \hookrightarrow X \) be an immersion from a finite 2-complex \( Y \) to a one-relator orbicomplex \( X \) that factors through an unwrapped cover of \( X \). Then

\[
\chi(Y) + (n-1)|\{2\text{-cells in } Y\}| \leq 0.
\]

In particular,

\[
|\{2\text{-cells in } Y\}| \leq \frac{\text{rk}(\pi_1(Y)) - 1}{n-1} < \text{rk}(\pi_1(Y))
\]

if \( n > 1 \).

**Proof.** By Theorem 3.1 \( \chi(Y^{(1)}) + \deg(f) \leq 0 \). Since \( f \) factors through an unwrapped cover, \( f \) restricts to \( p_n \) on each 2-cell of \( Y \). It follows that

\[
\deg f = n|\{2\text{-cells in } Y\}|
\]

since \( X \) is one-relator.

The Euler characteristic of \( Y \) is the Euler characteristic of \( Y^{(1)} \) plus the number of two-cells in \( Y \), so Theorem 3.1 implies the first assertion. The second assertion now follows from the first, using the trivial fact that

\[
1 - \text{rk}(\pi_1 Y) \leq \chi(Y)
\]

for any connected 2-complex \( Y \). \( \square \)
In the case with torsion, Corollary 3.2 gives a bound on the number of 2-cells of an immersion in terms of the rank of the fundamental group. In order to make a connection to arbitrary finitely generated subgroups of $G$, we use folding, along the lines of [LW18 §§6.4].

4 Folding

Our folding argument starts with [LW18 Lemma 6.11]; we include the proof, for the reader’s convenience.

Lemma 4.1. Any combinatorial map of finite 2-complexes $A \to B$ factors as

$$A \to C \hookrightarrow B$$

where $A \to C$ is surjective and $\pi_1$-surjective.

Proof. Folding shows that the map of 1-skeletons factors as

$$A^{(1)} \to C^{(1)} \hookrightarrow B^{(1)}$$

where $A^{(1)} \to C^{(1)}$ is surjective and $\pi_1$-surjective. We now construct $C$ by pushing the attaching maps of the 2-cells of $A$ forward to $C^{(1)}$ and identifying any 2-cells with the same image in $B$ and equal boundary maps. The resulting map $A \to C$ is surjective and $\pi_1$-surjective. It remains to check that the natural map $C \to B$ is an immersion.

Since $C \to B$ is combinatorial, it can only fail to be locally injective at a point $c \in C$ if two higher-dimensional cells incident at $c$ have the same image in $B$. The map of 1-skeleta is an immersion, so this can only occur if two 2-cells $e_1, e_2$ in $C$, incident at $c$, have the same image in $B$. Because the attaching maps of $e_1$ and $e_2$ agree at $c$ and $C^{(1)} \to B^{(1)}$ is an immersion, it follows that the attaching maps of $e_1$ and $e_2$ agree everywhere. Therefore, $e_1$ and $e_2$ are equal in $C$ by construction.

We apply Lemma 4.1 to express a covering space of $X$ as a convenient direct limit (cf. [LW18 Lemma 6.13]).

Lemma 4.2. Let $X$ be a 2-complex, $G = \pi_1(X)$, and $H \leq G$ a finitely generated subgroup. Then there is a sequence of $\pi_1$-surjective immersions of compact, connected two-complexes

$$Y_0 \hookrightarrow Y_1 \hookrightarrow \ldots \hookrightarrow Y_i \hookrightarrow \ldots X$$

with the following properties.

(i) Each $Y_i$ has no free faces.

(ii) $H = \lim_{i \to \infty} \pi_1 Y_i$

(iii) The number of edges of $Y_i$ that are not incident at a 2-cell is uniformly bounded.
Proof. Let $Y_0$ be a finite, connected graph equipped with an immersion $Y_0 \hookrightarrow X$ so that $\pi_1 Y_0$ surjects $H$. Let $\{r_1, r_2, \ldots \}$ be representatives of an enumeration of the conjugacy classes of $\ker(\pi_1 Y_1 \to \pi_1 X)$, thinking of each as realized by a morphism $r_i : S^1 \to Y_0$. We construct immersions $Y_i \to X$ inductively to satisfy items (i) and (iii), and so that $\{r_1, \ldots, r_i\} \subseteq \ker(\pi_1 Y_i \to X)$.

By induction, we therefore suppose that $Y_i$ has been constructed. We may realize $r_{i+1}$ as a loop in $Y_i$ by composing with the map $Y_0 \to Y_i$. Let $E_{i+1} \to X$ be a van Kampen diagram for $r_{i+1}$, and let $Z_{i+1} = Y_i \cup_{r_{i+1}} E_{i+1}$. The immersion $Y_i \hookrightarrow X$ naturally factors as $Y_i \to Z_{i+1} \to X$. We now apply Lemma 4.1 to the map $Z_{i+1} \to X$, which yields $Z_{i+1} \to Y_{i+1} \hookrightarrow X$ for some 2-complex $Y_{i+1}$. Let $Y_{i+1} \subseteq Y_{i+1}'$ be the result of collapsing free faces. Since the preimage of a free face under an immersion is also a free face, and $Y_i$ has no free faces, it follows that $Y_i \hookrightarrow X$ factors through $Y_{i+1}$.

Item (i) is satisfied by construction. Item (ii) follows immediately from the fact that $\{r_1, \ldots, r_i\} \subseteq \ker(\pi_1 Y_i \to X)$. Finally, any edge of $Y_i$ not incident at a 2-cell is the image of an edge of $Y_0$; this proves item (iii).

We are now ready to prove our main result.

Proof of Theorem 1.2. Realize $G$ as the fundamental group of a one-relator orbicomplex $X$. Let $H$ be a finitely generated subgroup of $G$. Let $G_0 \leq G$ be a torsion-free subgroup of finite index, corresponding to the unwrapped cover $X_0 \hookrightarrow X$ provided by Theorem 2.2. Since a finite extension of a finitely presented group is finitely presented, we may replace $H$ by $H \cap G_0$, and so assume that $H \leq G_0$. Consider the sequence of immersions $Y_0 \hookrightarrow Y_1 \hookrightarrow \cdots \hookrightarrow Y_i \hookrightarrow \cdots X_0$ provided by Lemma 4.2 (taking $X_0$ for $X$). By Corollary 3.2, the number of 2-cells of each $Y_i$ is bounded. Each 2-cell of $Y_i$ is a copy of the unique 2-cell of $X$, hence has boundary of bounded length. Combining this with item (iii) of Lemma 4.2, we see that the number of 1-cells (and hence also 0-cells) of $Y_i$ is also bounded. Since $X_0$ is finite, there are only finitely many combinatorial types of immersions $Y_i \hookrightarrow X_0$. Because $Y_i \hookrightarrow X_0$ factors through $Y_{i+1} \hookrightarrow X_0$, there is a subsequence $Y_{i_1} \hookrightarrow Y_{i_2} \hookrightarrow \cdots \hookrightarrow Y_{i_j} \hookrightarrow \cdots X_0$ so that each map $Y_{i_j} \hookrightarrow Y_{i_{j+1}}$ is a homeomorphism; therefore, $H = \lim_{\to} \pi_1 Y_{i_j}$ is finitely presented, as required.

5 Groups with good stackings

The results of [LW17] apply equally well to a class of groups which is rather larger than the class of one-relator groups.
Definition 5.1 (Stacking). Let $X$ be a 2-dimensional orbicomplex and let
$$\Lambda : \coprod S^1 \to X^{(1)}$$
be the coproduct of the attaching maps of the 2-cells. A stacking of $X$ is a lift of $\Lambda$ to an embedding
$$\hat{\Lambda} : S^1 \equiv \coprod S^1 \hookrightarrow X^{(1)} \times \mathbb{R};$$
write $\hat{\Lambda}(x) = (\Lambda(x), h(x))$. A stacking is called good if, for each component $S$ of the domain of $\Lambda$, there is a point $a \in S$ so that $h(a) \geq h(x)$ for all $x \in S$ with $\Lambda(a) = \Lambda(x)$, and there is also a point $b \in S$ so that $h(b) \leq h(x)$ for all $x \in S$ with $\Lambda(b) = \Lambda(x)$.

The results of [LW17] apply to the fundamental groups of orbicomplexes with good stackings.

Definition 5.2. We say that a group has a good stacking if it is the fundamental group of a compact, 2-dimensional orbicomplex that admits a good stacking. We say it has a good branched stacking if it has a good stacking, and every 2-cell has a cone point of index at least 2.

Every one-relator group admits a good stacking [LW17, Lemma 3.4], which is branched if the group has torsion. Corollary 3.2 applies to groups with a good branched stacking. The proof of Theorem 1.2 applies verbatim to groups with good branched stackings, except that groups with good branched stackings are not known to admit unwrapped covers – that is, the analogue of Theorem 2.2 is unknown.

However, we conjecture that Wise’s proof that one-relator groups with torsion are residually finite goes through for groups with branched good stackings.

Conjecture 5.3. If $G$ has a good branched stacking then $G$ is hyperbolic, and has a virtual quasiconvex hierarchy, in the sense of [Wis12].

By the results of [Wis12], Conjecture 5.3 would imply that every such group is virtually torsion-free, and hence coherent by the same proof as Theorem 1.2.

References

[Bau74] Gilbert Baumslag, Some problems on one-relator groups, Proceedings of the Second International Conference on the Theory of Groups (Australian Nat. Univ., Canberra, 1973) (Berlin), Lecture Notes in Math., vol. 372, Springer, 1974, pp. 75—-81.

[BH99] Martin R. Bridson and André Haefliger, Metric spaces of non-positive curvature, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999.
[FKS72] J. Fischer, A. Karrass, and D. Solitar, *On one-relator groups having elements of finite order*, Proc. Amer. Math. Soc. 33 (1972), 297–301. MR 0311780

[Hae91] André Haefliger, *Complexes of groups and orbihedra*, Group theory from a geometrical viewpoint (Trieste, 1990), World Sci. Publ., River Edge, NJ, 1991, pp. 504–540. MR 1170375

[HW16] Joseph Helfer and Daniel T. Wise, *Counting cycles in labeled graphs: the nonpositive immersion property for one-relator groups*, Int. Math. Res. Not. IMRN (2016), no. 9, 2813–2827.

[LW17] Larsen Louder and Henry Wilton, *Stackings and the W-cycles Conjecture*, Canad. Math. Bull. 60 (2017), no. 3, 604–612.

[LW18] Larsen Louder and Henry Wilton, *Negative immersions for one-relator groups*, arXiv:1803.02671, 2018.

[New68] B. B. Newman, *Some results on one-relator groups*, Bull. Amer. Math. Soc. 74 (1968), 568–571.

[Wis] Daniel T. Wise, *Coherence, local-indicability and non-positive immersions*, Preprint.

[Wis12] D. T. Wise, *The structure of groups with a quasi-convex hierarchy*, Preprint, http://goo.gl/3ctNvX, April 2012.

L. Louder, Department of Mathematics, University College London, Gower Street, London WC1E 6BT, UK
E-mail address: l.louder@ucl.ac.uk

H. Wilton, DPMMS, Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WB, UK
E-mail address: h.wilton@maths.cam.ac.uk