BILINEAR CONTROL OF A DEGENERATE HYPERBOLIC EQUATION

P. CANNARSA, P. MARTINEZ, AND C. URBANI

ABSTRACT. We consider the linear degenerate wave equation, on the interval (0, 1)
\[
w_{tt} - (x^\alpha w)_x = p(t)\mu(x)w,
\]
with bilinear control \( p \) and Neumann boundary conditions. We study the controllability of this nonlinear control system, locally around a constant reference trajectory, the “ground state”.

Under some classical and generic assumption on \( \mu \), we prove that there exists a threshold value for time, \( T_0 = \frac{1}{\alpha} \), such that the reachable set is
- a neighborhood of the ground state if \( T > T_0 \),
- contained in a \( C^1 \)-submanifold of infinite codimension if \( T < T_0 \).
- a \( C^1 \)-submanifold of codimension 1 if \( \alpha \in (0,1) \), and a neighborhood of the ground state if \( \alpha \in (1,2) \) if \( T = T_0 \), the case \( \alpha = 1 \) remaining open.

This extends to the degenerate case the work of Beauchard [6] concerning the bilinear control of the classical wave equation (\( \alpha = 0 \)), and adapts to bilinear controls the work of Alabau-Boussouira, Cannarsa and Leugering [1] on the degenerate wave equation where additive control are considered. Our proofs are based on a careful analysis of the spectral problem, and on Ingham type results, which are extensions of the Kadec’s \( \frac{1}{4} \) theorem.

1. Introduction

1.1. The context and the problem we study.

Degenerate partial differential equations appear in many domains, in particular physics, climate dynamics, biology, economics (see, e.g., [11, 16, 20]). Control of degenerate parabolic equations is, by now, a fairly well-developed subject (see, for instance, [10, 11, 12, 13]), but very few results are available in the case of degenerate hyperbolic equations. To our best knowledge, a class of degenerate wave equations has been studied from the point of view of control theory in [1], where boundary control is studied using HUM and multiplier methods, and in [32, 33, 34], where locally distributed control are considered.

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On the other hand, in many applications, one is naturally led to use bilinear controls, as such controls are more realistic than additive ones to govern the evolution of certain systems (see [2, 3, 9, 14, 17, 18, 19] for parabolic equations.). For example, [26] mentions in particular

- the linearized nuclear chain reaction
  \[ u_t = a^2 \Delta u + v(x, t)u, \]
  where \( u(x, t) \) is the neutron density at point \( x \) at time \( t \) and \( v \) is the bilinear control that models the effect of the "control rods";
- the approximate controllability of the rod made of a smart material
  \[ u_{tt} + u_{xxxx} + v(t)u_{xx} = 0, \]
  with hinged ends, where \( u \) is the displacement of the beam, and \( v \) is related to the magnitude of the electric field that heats the beam in order to control the vibrations.

This is why, in this paper, we address a bilinear control problem for the equation

\[
\begin{align*}
  w_{tt} - (x^\alpha w_x)_x &= p(t)\mu(x)w, & x \in (0, 1), t \in (0, T), \\
  (x^\alpha w_x)(x = 0) &= 0, & t \in (0, T), \\
  w_x(x = 1) &= 0, & t \in (0, T), \\
  w(x, 0) &= w_0(x), & x \in (0, 1), \\
  w_t(x, 0) &= w_1(x), & x \in (0, 1).
\end{align*}
\]

(1. 1)

Here,

- \( \alpha \in [0, 2) \) is the degeneracy parameter (\( \alpha = 0 \) for the classical wave equation and \( \alpha \in (0, 2) \) in the degenerate case),
- \( p \in L^2(0, T) \) is a multiplicative control,
- \( \mu \) is an admissible potential (and a key feature will be to analyze to which class \( \mu \) has to belong in order to prove a controllability result.)

Let us recall that the action of bilinear controls is weaker than the one of additive controls, in the sense that, with bilinear controls, one cannot expect the same kind of controllability results that can be proved with additive controls. This fact is described by the negative result obtained by Ball-Marsden-Slemrod in [5], where it is shown that the attainable set of any abstract linear system, subject to a bilinear control, has a dense complement.

For the Schrödinger equation, and then for the classical wave equation, attainability results with bilinear controls were obtained by Beauchard and Laurent in [7] and Beauchard in [6]. In particular, it is proved in [6] that,

- starting from the "ground state" (which is the constant state associated to the first eigenvalue, 0), the solution is more regular than expected, namely
  \( (w(T), w_t(T)) \in H^3(0, 1) \times H^2(0, 1) \),
- generically with respect to \( \mu \) (under a suitable condition relating \( \mu \) and the eigenvalues and eigenfunctions of the Laplacian operator with Neumann boundary conditions), the wave equation is locally controllable along the ground state with respect to the \( H^3(0, 1) \times H^2(0, 1) \) topology, in time \( T > 2 \) with controls in \( L^2((0, T), \mathbb{R}) \).
The goal of our paper is to extend these results to the degenerate case \( \alpha \in (0, 2) \). We obtain the following results:

- we show that the solution starting from the ground state is more regular than expected, that is,
  \[
  (w(T), w_t(T)) \in H^3_\alpha(0, 1) \times H^2_\alpha(0, 1),
  \]
  where the spaces \( H^2_\alpha(0, 1) \) and \( H^3_\alpha(0, 1) \) are the classical weighted Sobolev spaces associated with the degenerate elliptic operator that appears in (1.1) (see Proposition 3.3);
- generically with respect to \( \mu \) in some suitable Banach space, we prove that any target, which is close to the ground state in the \( H^3_\alpha(0, 1) \times H^2_\alpha(0, 1) \) topology, is reachable in time
  \[
  T > \frac{4}{2 - \alpha}
  \]
  with controls in \( L^2((0, T), \mathbb{R}) \) (see Theorem 3.1);
- when
  \[
  0 < T < \frac{4}{2 - \alpha},
  \]
  the set of reachable targets close to the ground state in the \( H^3_\alpha(0, 1) \times H^2_\alpha(0, 1) \) topology is contained in a \( C^1 \)-manifold of infinite dimension and of infinite codimension (see Theorem 3.3).
- and when
  \[
  T = \frac{4}{2 - \alpha},
  \]
  we prove that the situation is different for \( \alpha \in [0, 1) \) and \( \alpha \in [1, 2) \) (see Theorem 3.2):
  - if \( \alpha \in [0, 1) \), the set of reachable targets close to the ground state in the \( H^3_\alpha(0, 1) \times H^2_\alpha(0, 1) \) topology is a \( C^1 \)-manifold of codimension 1,
  - and if \( \alpha \in (1, 2) \) (except for a countable set of \( \alpha \), where a more precise analysis would have to be performed), any target close to the ground state in the \( H^3_\alpha(0, 1) \times H^2_\alpha(0, 1) \) topology is reachable, as in the case
    \[
    T > \frac{4}{2 - \alpha}.\]
  Note that in particular the case \( \alpha = 1 \) remains an open and interesting case.

Our approach follows the strategy proposed by Beauchard [6] in the nondegenerate case. However, it is worth noting that several new difficulties appear in the degenerate case:

- the spectral analysis: as often happens concerning degenerate elliptic operators, the eigenvalues and eigenfunctions of the associated spectral problem are given in terms of Bessel functions, see Propositions 3.1 when \( \alpha \in (0, 1) \) and Proposition 3.2 when \( \alpha \in (1, 2) \). The spectral analysis here is entirely new;
- the unboundness of the eigenfunctions: in the work by Beauchard [6], a key point is that the eigenfunctions (cosine functions) are bounded. We will prove that for \( \alpha > 0 \), the eigenfunctions are no longer bounded (see Lemmas 5.1 and 5.2) and this fact brings new difficulties that need to be overcome;
- the role of the potential \( \mu \) since the solutions of degenerate wave equations are less regular with respect to those of the classical wave equation, the
potential \( \mu \) will play a crucial role and this explains the extra conditions this function has to satisfy;

- the moment problem: we obtain
  
  - positive local controllability results when
    \[
    T > \frac{4}{2 - \alpha},
    \]
    using Ingham’s arguments,
  
  - “negative” local controllability results when
    \[
    0 < T \leq \frac{4}{2 - \alpha}
    \]

  combining general results on the families of exponentials \((e^{i\omega n t})_{n \in \mathbb{Z}}\) in \(L^2(0, T)\) (see [4]) with the gap property satisfied by the eigenvalues of the problem.

  As for the critical case \( T = T_0 \) is concerned, we prove some elementary but useful extensions of the classical Kadec’s \( \frac{1}{4} \) Theorem ([31] Theorem 1.14 p. 42), that allow us to treat this special case (see Lemmas 8.1 and 8.4).

To conclude, let us observe that, as the reader may have noticed, we have assumed the degeneracy exponent \( \alpha \) to be in the interval \([0, 2)\): this restriction is partly due to our method, but, on the other hand, it is known that problem (1.1), with (additive) boundary control, fails to be controllable for \( \alpha \geq 2 \) (see [1]).

1.2. Plan of the paper.

- In section 2 we recall the functional setting and the well-posedness results for the degenerate wave equation.
- In section 3 we state our main results:
  - the analysis of the eigenvalue problem associated to (1.1), see Proposition 3.1 for \( \alpha \in [0, 1) \) and Proposition 3.2 for \( \alpha \in [1, 2) \),
  - a hidden regularity result of the value map, which is the fundamental observation before studying the bilinear control problem, see Proposition 3.3
  - a positive bilinear control result, see Theorem 3.1 when \( T > \frac{1}{2 - \alpha} \),
  - a bilinear control result, see Theorem 3.2 when \( T = \frac{4}{2 - \alpha} \),
  - a “negative” bilinear control result, see Theorem 3.3 when \( T < \frac{4}{2 - \alpha} \).
- In section 4 we prove the well posedness results.
- In section 5 we prove Propositions 3.1 and 3.2
- In section 6 we prove Proposition 3.3
- In section 7 we prove Theorem 3.1
- In section 8 we prove Theorem 3.2
- In section 9 we prove Theorem 3.3

2. Functional setting and well-posedness

2.1. Functional setting for \( \alpha \in [0, 1) \).
For $0 \leq \alpha < 1$, we consider

\[(2.1) \quad H^1_\alpha(0, 1) := \left\{ u \in L^2(0, 1), \quad u \text{ absolutely continuous in } [0, 1], x^{\alpha/2}u_x \in L^2(0, 1) \right\}, \]

and

\[(2.2) \quad H^2_\alpha(0, 1) := \left\{ u \in H^1_\alpha(0, 1), x^\alpha u_x \in H^1(0, 1) \right\}. \]

$H^1_\alpha(0, 1)$ is endowed with the natural scalar product

\[
\forall f, g \in H^1_\alpha(0, 1), \quad (f, g) = \int_0^1 (x^\alpha f_x g_x + fg) \, dx.
\]

The operator $A : D(A) \subset L^2(0, 1) \to L^2(0, 1)$ will be defined by

\[(2.3) \quad \begin{cases} \forall u \in D(A), & Au := (x^\alpha u_x)_x, \\ D(A) := \{ u \in H^2_\alpha(0, 1), (x^\alpha u_x)(0) = 0, u_x(1) = 0 \}. \end{cases} \]

Then, the following results hold:

**Proposition 2.1.** Let $\alpha \in [0, 1)$. Then,

a) $H^1_\alpha(0, 1)$ is a Hilbert space,

b) $A : D(A) \subset L^2(0, 1) \to L^2(0, 1)$ is a self-adjoint negative operator with dense domain.

2.2. **Functional setting for $\alpha \in [1, 2)$.**

For $1 \leq \alpha < 2$, we consider the following spaces:

\[(2.4) \quad H^1_\alpha(0, 1) := \left\{ u \in L^2(0, 1), \quad u \text{ locally absolutely continuous in } (0, 1], x^{\alpha/2}u_x \in L^2(0, 1) \right\}, \]

and

\[(2.5) \quad H^2_\alpha(0, 1) := \{ u \in H^1_\alpha(0, 1) \mid x^\alpha u_x \in H^1(0, 1) \}. \]

The operator $A : D(A) \subset L^2(0, 1) \to L^2(0, 1)$ will be defined by

\[
\begin{cases} \forall u \in D(A), & Au := (x^\alpha u_x)_x, \\ D(A) := \{ u \in H^2_\alpha(0, 1), (x^\alpha u_x)(0) = 0, u_x(1) = 0 \}. \end{cases}
\]

Then, the following results hold:

**Proposition 2.2.** Let $\alpha \in [1, 2)$. Then,

a) $H^1_\alpha(0, 1)$ is a Hilbert space,

b) $A : D(A) \subset L^2(0, 1) \to L^2(0, 1)$ is a self-adjoint negative operator with dense domain.

We deduce that, for any $\alpha \in [0, 2)$, $A$ is the infinitesimal generator of an analytic semigroup of contractions $e^{tA}$ on $L^2(0, 1)$. 

2.3. Well posedness of the problem.
Consider the non-homogeneous problem
\[ \begin{cases} w_{tt} - (x^\alpha w_x)_x = p(t)\mu(x)w + f(x, t), & x \in (0, 1), t \in (0, T), \\ (x^\alpha w_x)(x = 0) = 0, & t \in (0, T), \\ w_{tt}(x = 1) = 0, & t \in (0, T), \\ w(x, 0) = w_0(x), & x \in (0, 1), \\ w_t(x, 0) = w_1(x), & x \in (0, 1). \end{cases} \]
(2.6)

In order to recast (2.6) into the a first order problem, we introduce
\[ W := \begin{pmatrix} w \\ w_t \end{pmatrix}, \quad W_0 := \begin{pmatrix} w_0 \\ w_1 \end{pmatrix}, \quad F(x, t) := \begin{pmatrix} 0 \\ f(x, t) \end{pmatrix}, \]
the state space
\[ \mathcal{X} := H^1_\alpha(0, 1) \times L^2(0, 1), \]
and the operators
\[ A := \begin{pmatrix} 0 & \text{Id} \\ \alpha & 0 \end{pmatrix}, \quad D(A) := D(A) \times H^1_\alpha(0, 1), \]
(2.7)
and
\[ B := \begin{pmatrix} 0 & 0 \\ \mu & 0 \end{pmatrix}, \quad D(B) := H^1_\alpha(0, 1) \times L^2(0, 1). \]
(2.8)

So, problem (2.6) can be rewritten as
\[ \begin{cases} W'(t) = AW(t) + p(t)BW(t) + F(t), \\ W(0) = W_0. \end{cases} \]
(2.9)

We also introduce the space
\[ V^{(1, \infty)}_\alpha := \{ \mu \in H^1_\alpha(0, 1), x^{\alpha/2} \mu_x \in L^\infty(0, 1) \}. \]
(2.10)

We can now state the well-posedness result for problem (2.9).

**Proposition 2.3.** Let \( T > 0, p \in L^2(0, T) \) and \( f \in L^2((0, T), H^1_\alpha(0, 1)) \). Assume that
\[ \mu \in V^{(1, \infty)}_\alpha := \begin{cases} H^1_\alpha(0, 1) & \text{if } \alpha \in [0, 1), \\ V^{(1, \infty)}_\alpha & \text{if } \alpha \in [1, 2). \end{cases} \]
(2.11)

Then, for all \( W_0 \in D(A) \), there exists a unique classical solution of (2.9), i.e. a function
\[ W \in C^0([0, T], D(A)), \]
such that the following equality holds in \( D(A) \): for every \( t \in [0, T] \),
\[ W(t) = e^{At}W_0 + \int_0^t e^{A(t-s)}(BW(s) + F(s)) \, ds. \]
(2.12)

Moreover, there exists \( C = C(\alpha, T, p) > 0 \) such that \( W \) satisfies
\[ \|W\|_{C^0([0, T], D(A))} \leq C \left( \|W_0\|_{D(A)} + \|F\|_{L^2(0, T; D(A))} \right). \]
(2.13)
3. Main results

3.1. Preliminary result: Spectral problem.
We investigate the eigenvalues and eigenfunctions of the operator

\[-Au := -(x^\alpha u_x)_x, \quad x \in (0, 1)\]

with Neumann boundary conditions:

\[(x^\alpha u_x)(x = 0) = 0, \quad \text{and} \quad u_x(x = 1).\]

Hence, we look for solutions \((\lambda, \Phi)\) of the problem

\[
\begin{cases}
- (x^\alpha \Phi')' = \lambda \Phi, & x \in (0, 1), \\
\Phi'(1) = 0.
\end{cases}
\]

(3. 1)

The difference with the spectral analysis of [12, 13] is in the boundary condition at the point \(x = 1\) which leads to new difficulties.

3.1.a. Eigenvalues and eigenfunctions when \(\alpha \in [0, 1)\).

**Proposition 3.1.** Given \(\alpha \in [0, 1)\), set

\[
\kappa_\alpha := \frac{2 - \alpha}{2}, \quad \nu_\alpha := \frac{1 - \alpha}{2 - \alpha},
\]

and consider the Bessel function \(J_{-\nu_\alpha}\) of negative order \(-\nu_\alpha\) and of first kind, and the positive zeros \((j_{-\nu_\alpha+1,n})_{n \geq 1}\) of the Bessel function \(J_{-\nu_\alpha+1}\).

Then, the set of solutions \((\lambda, \Phi)\) of problem (3. 1) is

\[
S = \{ (\lambda_{\alpha,n}, \rho \Phi_{\alpha,n}), n \in \mathbb{N}, \rho \in \mathbb{R} \},
\]

where

- for \(n = 0\),

\[
\lambda_{\alpha,0} = 0, \quad \Phi_{\alpha,0}(x) = 1,
\]

- for \(n \geq 1\),

\[
\lambda_{\alpha,n} = \kappa_\alpha^2 j_{-\nu_\alpha+1,n}^2, \quad \Phi_{\alpha,n}(x) = K_{\alpha,n} x^{\frac{1+\nu_\alpha}{2}} J_{-\nu_\alpha}(j_{-\nu_\alpha+1,n} x^{\frac{2-\alpha}{2}}),
\]

where the positive constant \(K_{\alpha,n}\) is chosen such that \(\|\Phi_{\alpha,n}\|_{L^2(0,1)} = 1\).

Moreover, the sequence \((\Phi_{\alpha,n})_{n \geq 0}\) forms an orthonormal basis of \(L^2(0,1)\). Additionally, the sequence \((\sqrt{\lambda_{\alpha,n+1}} - \sqrt{\lambda_{\alpha,n}})_{n \geq 1}\) is decreasing and

\[
\sqrt{\lambda_{\alpha,n+1}} - \sqrt{\lambda_{\alpha,n}} \to \frac{2 - \alpha}{2} \pi \quad \text{as} \quad n \to \infty.
\]

3.1.b. Eigenvalues and eigenfunctions when \(\alpha \in [1, 2)\).

**Proposition 3.2.** Given \(\alpha \in [1, 2)\), set

\[
\kappa_\alpha := \frac{2 - \alpha}{2}, \quad \nu_\alpha := \frac{\alpha - 1}{2 - \alpha},
\]

and consider the Bessel function \(J_{\nu_\alpha}\) of positive order \(\nu_\alpha\) and of first kind, and the positive zeros \((j_{\nu_\alpha+1,n})_{n \geq 1}\) of the Bessel function \(J_{\nu_\alpha+1}\).

Then, the set of solutions \((\lambda, \Phi)\) of problem (3. 1) is

\[
S = \{ (\lambda_{\alpha,n}, \rho \Phi_{\alpha,n}), n \in \mathbb{N}, \rho \in \mathbb{R} \},
\]

where
• for \( n = 0 \),

\[ \lambda_{\alpha,0} = 0, \quad \Phi_{\alpha,0}(x) = 1, \]

• for \( n \geq 1 \),

\[ \lambda_{\alpha,n} = \kappa_{\alpha}^2 j_{\nu_{\alpha}+1,n}, \quad \Phi_{\alpha,n}(x) = K_{\alpha,n} x^{1-\nu_{\alpha}} J_{\nu_{\alpha}}(j_{\nu_{\alpha}+1,n} x^{2-\nu_{\alpha}}), \]

where the positive constant \( K_{\alpha,n} \) is chosen such that \( \| \Phi_{\alpha,n} \|_{L^2(0,1)} = 1 \).

Moreover, the sequence \( (\Phi_{\alpha,n})_{n \geq 0} \) forms an orthonormal basis of \( L^2(0,1) \). Additionally, the sequence \((\sqrt{\lambda_{\alpha,n+1}} - \sqrt{\lambda_{\alpha,n}})_{n \geq 1}\) is decreasing and

\[ \sqrt{\lambda_{\alpha,n+1}} - \sqrt{\lambda_{\alpha,n}} \to \frac{2 - \alpha}{2} \pi \quad \text{as} \quad n \to \infty. \]

3.2. Hidden regularity.

3.2.a. Some notations.

Let us start by introducing some notation which will be used in the proofs of our results. To avoid possible problems generated by the eigenvalue 0, we define

\[ \lambda_{\alpha,n}^* := \begin{cases} 1 & \text{for } n = 0, \\ \lambda_{\alpha,n} & \text{for } n \geq 1. \end{cases} \]

It will be useful to introduce the following intermediate Sobolev spaces for any \( s > 0 \):

\[ H_s((0)) := D((-A)^{s/2}) = \left\{ \psi \in L^2(0,1), \sum_{k=0}^{\infty} (\lambda_{\alpha,k}^*)^s \langle \psi, \Phi_{\alpha,k} \rangle_{L^2(0,1)}^2 < \infty \right\}, \]

equipped with the norm

\[ \| \psi \|_{H_s((0))} := \left( \sum_{k=0}^{\infty} (\lambda_{\alpha,k}^*)^s \langle \psi, \Phi_{\alpha,k} \rangle_{L^2(0,1)}^2 \right)^{1/2}. \]

We also define the following spaces

\[ V_{\alpha}^2((0)) := \{ \mu \in H_{\alpha}^2(0,1), x^{\alpha/2} \mu_x \in L^\infty(0,1) \}, \]

and

\[ V_{\alpha}^2((0),\infty) := \{ \mu \in H_{\alpha}^2(0,1), x^{\alpha/2} \mu_x \in L^\infty(0,1), (x^{\alpha} \mu_x)_x \in L^\infty(0,1) \}, \]

and the closed subspace of \( H_{\alpha}^2(0,1) \)

\[ V_{\alpha}^{2,0} := \{ w \in H_{\alpha}^2(0,1), (x^{\alpha} w_x)(0) = 0 \}. \]

Given \((w_0, w_1) \in H_{\alpha}^1(0,1) \times L^2(0,1) \) and \( p \in L^2(0,T) \), we will denote by \( w^{(w_0,w_1;p)} \) the solution of (1.1), associated to the initial conditions \( w_0, w_1 \) and control \( p \). In particular, when \((w_0, w_1) = (1, 0) \) and \( p = 0 \), we note that the constant function equal to 1 satisfies (1.1), hence

\[ w^{(1,0);0}(1, 0) = 1, \quad w^{(1,0);0}_t(1, 0) = 0. \]

In the following, we will be interested in the regularity of the solution of (1.1) starting from the ground state \((1, 0)\), that is, the solution \( w^{(1,0);p} \) (or, more simply,
3.2.b. A hidden regularity result.

We consider the solution \( w \) of (3.13) and, if \( p \in L^2(0,T) \) and \( \mu \in V^\alpha_1 \), thanks to Proposition 2.3 we know that

\[
(w(T), w_t(T)) \in D(A) \times H^3_{(0)}(0,1).
\]

Before stating Theorem 3.1, we will prove the following result, which extends the regularity result of [6, Theorem 3] to the degenerate case:

**Proposition 3.3.** Let \( T > 0 \) and

\[
\mu \in V^\alpha_2 := \begin{cases} V^\alpha_2(2,\infty) & \text{if } \alpha \in [0,1), \\ V^\alpha_2(2,\infty,\infty) & \text{if } \alpha \in [1,2). \end{cases}
\]

Then, for all \( p \in L^2(0,T) \), the solution \( w^{(p)} \) of (3.13) has the following additional regularity

\[
(w^{(p)}(T), w_t^{(p)}(T)) \in H^3_{(0)} \times D(A).
\]

Moreover, the map

\[
\Theta_T : L^2(0,T) \to H^3_{(0)} \times D(A), \quad \Theta_T(p) := (w^{(p)}(T), w_t^{(p)}(T))
\]

is of class \( C^1 \).

3.3. Main controllability results.

Because of the negative result contained in [5], one could not expect any controllability property to hold in the spaces \( H^2_{(0)}(0,1) \times H^3_{(0)}(0,1) \). However, since the multiplication operator \( Bu := \mu u \) does not preserve the space \( H^3_{(0)}(0,1) \), the chance to achieve controllability results in \( H^3_{(0)}(0,1) \times H^2_{(0)}(0,1) \) is still open. For this purpose, we will need additional assumptions on the admissible potential \( \mu \). Furthermore, we observe that controllability properties will depend on a threshold value for the controllability time because of the finite speed of propagation, as it always happens for hyperbolic equations.

3.3.a. Threshold value of \( T \) and the admissible potentials \( \mu \).

We will show that the value

\[
T_0 := \frac{4}{2 - \alpha}
\]

is the threshold time for controllability. Let us define the following subclass of admissible potentials \( \mu \)

\[
V^{(adm)} := \{ \mu \in V^\alpha_2, \text{ such that } \exists c > 0, \forall n \geq 0, \quad |\langle \mu, \Phi_{\alpha,n} \rangle|_{L^2(0,1)} \geq \frac{c}{\lambda_n^2} \}.
\]

We observe that the space \( V^{(adm)} \) is non empty. Indeed, in the proposition that follows we exhibit an admissible potential \( \mu \in V^{(adm)} \):
Proposition 3.4. The function \( \mu : \mu(x) = x^{2-\alpha} \) belongs to \( V^{(\text{adm})} \). Moreover the space \( V^{(\text{adm})} \) is dense in \( V^2_\alpha \).

We refer to the recent work of Urbani [29] (Chap 5) for sufficient conditions for building polynomials functions that fulfill the last condition in (3. 18) are given.

3.3.b. Controllability result for \( T > T_0 \).

Theorem 3.1. Given \( \alpha \in [0, 2) \), let \( \mu \in V^{(\text{adm})} \) (defined in (3. 18)) and \( T > T_0 \).

Then, there exists a neighbourhood \( \mathcal{V}(1,0) \) of \( (1,0) \) in \( H^3_{(0)}(0,1) \times D(A) \) such that, for all \( (w^f_0, w^f_1) \in \mathcal{V}(1,0) \), there exists a unique \( p^f \in L^2(0,T) \) close to 0 such that

\[
(w(p^f)(T), w_t(p^f)(T)) = (w^f_0, w^f_1).
\]

Moreover, the application

\[
\Gamma_{\alpha,T} : \mathcal{V}(1,0) \to L^2(0,T), \quad (w^f_0, w^f_1) \mapsto p^f
\]

is of class \( C^1 \).

3.3.c. Controllability result when \( T = T_0 \).

Theorem 3.2. Let \( \mu \in V^{(\text{adm})} \) (defined in (3. 18)) and

\[
T = T_0.
\]

Then,

- for \( \alpha \in [0, 1) \), the reachable set is locally a \( C^1 \)-submanifold of \( H^3_{(0)}(0,1) \times D(A) \) of codimension 1,
- for \( \alpha \in (1, 2) \) and \( \frac{1}{2-\alpha} \notin \mathbb{N} \), the reachable set is a whole neighborhood of \( (0,1) \) in \( H^3_{(0)}(0,1) \times D(A) \).

What happens for \( \alpha = 2 - \frac{1}{k} \), \( k \in \mathbb{N}^* \) (in particular for \( \alpha = 1 \)) is still an open problem. These values are the points where the nature of the set \( \{ e^{i\omega_{\alpha,n,t}}, n \in \mathbb{Z} \} \) changes. Indeed,

- for \( \alpha \in [0, 1) \), \( \{ e^{i\omega_{\alpha,n,t}}, n \in \mathbb{Z} \} \) is a Riesz basis of \( L^2(0,T_0) \),
- for \( \alpha \in (1, \frac{3}{2}) \), \( \{ e^{i\omega_{\alpha,n,t}}, n \in \mathbb{Z} \} \) has a deficiency equal to 1 in \( L^2(0,T_0) \),
- for \( \alpha \in (\frac{3}{2}, 2) \), \( \{ e^{i\omega_{\alpha,n,t}}, n \in \mathbb{Z} \} \) has a deficiency equal to 2 in \( L^2(0,T_0) \),

and so on. A detailed analysis is given in Lemma 8.5 that derives from the Kadec’s Theorem (see Lemma 8.4).

We would like to draw the attention to the different nature of the reachable set for the weak (\( \alpha \in [0,1) \)) and the strong (\( \alpha \in [1,2) \)) degeneracy: while in the first case it is a submanifold of codimension 1, in the latter case it is a complete neighborhood of \( (1,0) \) (except for the aforementioned particular values of \( \alpha \)).

3.3.d. The controllability result when \( T < \frac{4}{2-\alpha} \).

Theorem 3.3. Let \( \mu \in V^{(\text{adm})} \) (defined in (3. 18)) and

\[
T < T_0.
\]

Then the reachable set is locally contained in a \( C^1 \)-submanifold of \( H^3_{(0)}(0,1) \times D(A) \) of infinite dimension and of infinite codimension.
4. Functional setting: proof of Propositions 2.1, 2.2 and 2.3

4.1. Proof of Propositions 2.1

4.1.a. Integration by parts.

Let us prove the following integration by parts formula.

Lemma 4.1. Let \( \alpha \in [0, 1) \), then

\[
(4.1) \quad \forall f, g \in H^2_\alpha(-1, 1), \quad \int_{-1}^{1} (x^\alpha f'(x))' g(x) \, dx = -\int_{-1}^{1} x^\alpha f'(x) g'(x) \, dx.
\]

Proof of Lemma 4.1. If \( f \in H^2_\alpha(0, 1) \), then

\[
F(x) := x^\alpha f'(x) \in H^1(0, 1).
\]

Let \( g \in H^2_\alpha(0, 1) \), and \( \varepsilon \in (0, 1) \). Decompose

\[
\int_{-1}^{1} F'(x) g(x) \, dx = \int_{-\varepsilon}^{\varepsilon} F'(x) g(x) \, dx + \int_{\varepsilon}^{1} F'(x) g(x) \, dx.
\]

Then, since \( g \in H^2_\alpha(0, 1) \subset H^1(\varepsilon, 1) \), the classical integration by parts formula gives

\[
\int_{\varepsilon}^{1} F'(x) g(x) \, dx = [F(x) g(x)]_{\varepsilon}^{1} - \int_{\varepsilon}^{1} F(x) g'(x) \, dx
\]

\[
= [F(x) g(x)]_{\varepsilon}^{1} - \int_{\varepsilon}^{1} (x^{\alpha/2} f'(x)) (x^{\alpha/2} g'(x)) \, dx.
\]

Now, since \( x^{\alpha/2} f' \) and \( x^{\alpha/2} g' \) belong to \( L^2(0, 1) \), we have

\[
\int_{\varepsilon}^{1} F(x) g'(x) \, dx \to \int_{0}^{1} F(x) g'(x) \, dx \quad \text{as} \quad \varepsilon \to 0,
\]

and since \( F' \) and \( g \) belong to \( L^2(0, 1) \), we get

\[
\int_{0}^{\varepsilon} F'(x) g(x) \, dx \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]

It remains to study the boundary terms: first, because of Neumann boundary condition at 1, we have

\[
[F(x) g(x)]_{\varepsilon}^{1} = -F'(\varepsilon) g(\varepsilon).
\]

We note that \( F(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \), and \( g \) is absolutely continuous on \([0, 1]\), hence

\[
[F(x) g(x)]_{\varepsilon}^{1} = -F'(\varepsilon) g(\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]

\[\square\]

4.1.b. Proof of Proposition 2.2

Part a) of Proposition 2.1 is well known, see e.g. [12]. Let us prove Part b).

First, we note that \( D(A) \) is dense in \( X \), since it contains all the functions of class \( C^\infty \), compactly supported in \((0, 1)\).

We derive from Lemma 4.1 that

\[
\forall f \in D(A), \quad \langle Af, f \rangle = \int_{0}^{1} (|x|^\alpha f'(x))^2 f(x) \, dx = -\int_{0}^{1} x^\alpha f'(x)^2 \, dx \leq 0,
\]

therefore \( A \) is dissipative.
In order to show that $A$ is symmetric, we apply Lemma 4.1 twice to obtain that

$$\forall f, g \in D(A), \quad (Af, g) = \int_0^1 (x^\alpha f'(x))' g(x) \, dx = -\int_0^1 x^\alpha f'(x) g'(x) \, dx$$

$$= -\int_0^1 (x^\alpha g'(x)) f'(x) \, dx = \int_0^1 (x^\alpha g'(x))' f(x) \, dx = \langle f, Ag \rangle.$$  

Finally, we check that $I - A$ is surjective. Let $f \in L^2(0, 1)$. Then, by Riesz theorem, there exists one and only one $u \in H^1_0(0, 1)$ such that

$$\forall v \in H^1_0(0, 1), \quad \int_0^1 (uv + x^\alpha u'v') = \int_0^1 f v.$$  

In particular, the above relation holds true for all $v$ of class $C^\infty$, compactly supported in $(0, 1)$. Thus, $x \mapsto x^\alpha u'$ has a weak derivative given by

$$(x^\alpha u')' = -(f - u).$$  

Since $f - u \in L^2(0, 1)$, we obtain that $(x^\alpha u')' \in L^2(0, 1)$. Hence, $u \in H^2_0(0, 1)$. Now, choosing first $v$ of class $C^\infty$ compactly supported in $[\frac{1}{2}, 1]$, but not equal to 0 at the point $x = 1$, we derive that

$$\int_0^1 f v = \int_0^1 (uv + x^\alpha u'v')$$

$$= \int_0^1 uv + [x^\alpha u'v]'_0 - \int_0^1 (x^\alpha u')'v = [x^\alpha u'v]_0 + \int_0^1 (u - (x^\alpha u')) v,$$

therefore $u'(1)v(1) = 0$ that implies $u'(1) = 0$. In the same way, by choosing $v$ of class $C^\infty$ compactly supported in $[0, \frac{1}{2}]$, but not equal to 0 at the point $x = 0$, we obtain that $(x^\alpha u'(0) = 0$. Thus $u \in D(A)$, and $(I - A)u = f$. So, the operator $I - A$ is surjective. This concludes the proof of Proposition 2.1, part b). \qed

4.2. Proof of Propositions 2.2

4.2.a. Integration by parts.

Let us prove the following integration by parts formula:

**Lemma 4.2.** Let $\alpha \in [1, 2)$, then

$$\forall f, g \in H^2_0(0, 1), \quad \int_0^1 (x^\alpha f'(x))' g(x) \, dx = -\int_0^1 x^\alpha f'(x) g'(x) \, dx.$$  

**Proof of Lemma 4.2.** If $f \in H^2_0(0, 1)$, then

$$F(x) := x^\alpha f'(x) \in H^1(0, 1).$$

Let $g \in H^2_0(0, 1)$, and $\varepsilon \in (0, 1)$. Decompose

$$\int_0^1 F'(x)g(x) \, dx = \int_0^\varepsilon F'(x)g(x) \, dx + \int_\varepsilon^1 F'(x)g(x) \, dx.$$  

Since $g \in H^2_0(0, 1) \subset H^1(0, 1)$, the classical integration by parts formula gives

$$\int_0^1 F'(x)g(x) \, dx = \int_0^\varepsilon F'(x)g(x) \, dx + [F(x)g(x)]_\varepsilon^1 - \int_\varepsilon^1 F(x)g'(x) \, dx.$$

$$\int_0^1 F'(x)g(x) \, dx = \int_0^1 \left( \int_0^x \frac{d}{dy} \left( \int_0^y f(z) \, dz \right) \right) \, dy = \int_0^1 \frac{d}{dy} \left( \int_0^y f(z) \, dz \right) \, dy = \int_0^1 \frac{d}{dy} \left( \frac{y^\alpha}{\alpha} f'(y) \right) \, dy = \frac{1}{\alpha} \int_0^1 y^\alpha f'(y) \, dy = \frac{1}{\alpha} \int_0^1 f'(y) y^\alpha \, dy = \frac{1}{\alpha} \
$$
To prove equation (4.2), we have to let $\varepsilon \to 0$ in the above identity. First, note that
\[
\int_{\varepsilon}^{1} F(x)g'(x) \, dx = \int_{\varepsilon}^{1} (x^{\alpha} f'(x)) g'(x) \, dx = \int_{\varepsilon}^{1} (x^{\alpha/2} f'(x)) (x^{\alpha/2} g'(x)) \, dx,
\]
and since $x \mapsto x^{\alpha/2} f'(x)$ and $x \mapsto x^{\alpha/2} g'(x)$ belong to $L^2(0, 1)$, we have that
\[
\int_{\varepsilon}^{1} (x^{\alpha/2} f'(x)) (x^{\alpha/2} g'(x)) \, dx \to \int_{0}^{1} (x^{\alpha/2} f'(x)) (x^{\alpha/2} g'(x)) \, dx \quad \text{as} \quad \varepsilon \to 0.
\]
Hence,
\[
\int_{\varepsilon}^{1} F(x)g'(x) \, dx \to \int_{0}^{1} F(x)g'(x) \, dx \quad \text{as} \quad \varepsilon \to 0.
\]
Moreover, since $F'$ and $g$ belong to $L^2(0, 1)$, we get that
\[
\int_{0}^{\varepsilon} F'(x)g(x) \, dx \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]
It remains to study the boundary terms: first, because of Neumann boundary conditions at 1, we have $F(1) = 0$ and $g$ has a finite limit as $x \to 1$. Therefore,
\[
[F(x)g(x)]_{\varepsilon}^{1} = F(\varepsilon)g(\varepsilon) - F(1)g(1) = F(\varepsilon)g(\varepsilon).
\]
Now, we note that
\[
\forall x \in (0, 1), \quad (F(x)g(x))' = F'(x)g(x) + F(x)g'(x) = F'(x)g(x) + (x^{\alpha/2} f'(x))(x^{\alpha/2} g'(x)),
\]
and, since $F'$, $g$, $x^{\alpha/2} f'$, $x^{\alpha/2} g'$ belong to $L^2(0, 1)$, we obtain that $(Fg)' \in L^1(0, 1)$. Thus, $Fg$ is absolutely continuous on $(0, 1)$ and it has a limit as $x \to 0$. This means that there exists $L$ such that
\[
F(x)g(x) \to L \quad \text{as} \quad x \to 0^+.
\]
We claim that $L = 0$. Indeed:

- function $x \mapsto x^{\alpha} f'(x)$ belongs to $H^1(0, 1)$, hence it has a limit as $x \to 0^+$:
  \[
x^{\alpha} f'(x) \to \ell \quad \text{as} \quad x \to 0^+;
  \]
- if $\ell \neq 0$,
  \[
x^{\alpha/2} f'(x) \sim \frac{\ell}{x^{\alpha/2}} \quad \text{as} \quad x \to 0^+.
  \]
However, since $\alpha \geq 1$, we have that $\frac{\ell}{x^{\alpha/2}} \notin L^2(0, 1)$, so $\ell = 0$;
- moreover,
  \[
  \forall x \in (0, 1), \quad x^{\alpha} f'(x) = \int_{0}^{x} (s^{\alpha} f'(s))' \, ds
  \]
and using the Cauchy-Schwarz inequality, one has
  \[
  \forall x \in (0, 1), \quad |x^{\alpha} f'(x)| \leq C \sqrt{x};
  \]
- finally,
  \[
  \forall x \in (0, 1), \quad |x^{\alpha} f'(x)g(x)| \leq C \sqrt{x}|g(x)|,
  \]
  thus,
  \[
  \forall x \in (0, 1), \quad |F(x)g(x)| \leq C \sqrt{x}|g(x)|.
  \]
If \( L \neq 0 \), then for \( x \) sufficiently close to 0 we have
\[
|g(x)| \geq \frac{CL}{2\sqrt{x}},
\]
which is in contradiction with \( g \in L^2(0, 1) \). Therefore, \( L = 0 \).

This implies that
\[
[F(x)g(x)]' = -F(\varepsilon)g(\varepsilon) \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0^+.
\]
This concludes the proof of Lemma 4.2. \( \square \)

4.2.b. Proof of Proposition 2.2

Part a) of Proposition 2.2 is well known, see e.g. [13]. Therefore, we prove Part b). The strategy of the proof is similar to the one of Proposition 2.1, part b), and relies on the integration by parts formula given in Lemma 4.1. We only need to check the boundary conditions. As already noted, since \( u \in H^1_0(0, 1) \), this implies that \( x^\alpha u'(x) \rightarrow 0 \) as \( x \rightarrow 0 \). Hence the boundary condition is satisfied at \( x = 0 \).

Taking now \( v \) of class \( C^\infty \), not equal to 0 at the point \( x = 1 \), we derive that
\[
\int_0^1 fv = \int_0^1 (uv + x^\alpha u'v')
\]
\[
= \int_0^1 uv + [x^\alpha u']_0^1 - \int_0^1 (x^\alpha u')'v = [x^\alpha u']_0^1 + \int_0^1 (u - (x^\alpha u'))'v,
\]
thus \( u'(1)v(1) = 0 \), and therefore \( u'(1) = 0 \). We obtain that \( u \in D(A) \), and \((I - A)u = f\). So, the operator \( I - A \) is surjective. This concludes the proof of Proposition 2.2 part b). \( \square \)

4.3. Proof of Proposition 2.3

First, let us prove the following regularity result.

**Lemma 4.3.** Let \( \mu \in V^1_\alpha \). Then, the operator \( B \) defined in (2.8) satisfies
\[
B \in \mathcal{L}_c(D(A), D(A)).
\]

**Proof of Lemma 4.3** We have to prove that
\[
w_0 \in D(A) \implies \mu w_0 \in H^1_\alpha(0, 1)
\]
and that there exists \( C > 0 \) such that
\[
\|\mu w_0\|_{H^1_\alpha(0, 1)} \leq C\|w_0\|_{D(A)}.
\]
We distinguish the cases \( \alpha \in [0, 1) \) and \( \alpha \in [1, 2) \).

- \( \alpha \in [0, 1) \): we can decompose \( w_0' \) as follows \( w_0' = (x^{\alpha/2}w_0')(x^{-\alpha/2}) \). Since \( w_0 \in H^1_\alpha(0, 1) \), we deduce that \( w_0' \in L^1(0, 1) \) and thus \( w_0 \in L^\infty(0, 1) \). The same holds for \( \mu \) because \( V^1_\alpha = H^1_\alpha(0, 1) \) for \( \alpha \in [0, 1) \). Hence, \((\mu w_0)' = \mu w_0 + \mu w_0' \in L^1(0, 1) \) and therefore \( \mu w_0 \) is absolutely continuous on \([0, 1]\). Furthermore, we have that \( x^{\alpha/2}(\mu w_0)' = (x^{\alpha/2}\mu')w_0 + \mu(x^{\alpha/2}w_0') \in L^2(0, 1) \) and so we infer that \( \mu w_0 \in H^1_\alpha(0, 1) \). Finally, there exists \( C > 0 \) such that
\[
\forall w \in H^1_\alpha(0, 1), \quad \|w\|_{L^\infty(0, 1)} \leq C\|w\|_{H^1_\alpha(0, 1)},
\]
and this implies that (4.3) holds.
• $\alpha \in [1, 2)$: first, we observe that $\mu \in V^{1, \infty}(0, 1)$ implies that $|\mu_x| \leq \frac{C}{\alpha - 1}$. Therefore, we get that $\mu_x \in L^1(0, 1)$, and so $\mu \in L^\infty(0, 1)$ and $\mu w_0 \in L^2(0, 1)$. Moreover, $x^{\alpha / 2}(\mu w_0)' = (x^{\alpha / 2} \mu_x)w_0 + (x^{\alpha / 2} w_0')\mu$ and since $x^{\alpha / 2} \mu_x \in L^\infty(0, 1)$ and $w_0 \in L^2(0, 1)$, we have that $(x^{\alpha / 2} \mu_x)w_0 \in L^2(0, 1)$. Furthermore, since $x^{\alpha / 2} w_0' \in L^2(0, 1)$ and $\mu \in L^\infty(0, 1)$, we deduce that $(x^{\alpha / 2} w_0')\mu \in L^2(0, 1)$ and hence $x^{\alpha / 2}(\mu w_0)' \in L^2(0, 1)$. By reasoning as in the case $\alpha \in [0, 1)$, we deduce that (4. 3) is verified.

\[\square\]

**Proof of Proposition 2.3.** We prove the existence and uniqueness of the solution of problem (2. 9) by a fixed point argument. We consider the map 

$$K : C^0([0, T], D(A)) \rightarrow C^0([0, T], D(A))$$

defined by

\[\forall t \in [0, T], \quad K(W)(t) := e^{At}W_0 + \int_0^t e^{A(t-s)} (p(s)BW(s) + F(s)) \, ds.\]

We first prove that $K$ is well-defined, which means that it maps $C^0([0, T], D(A))$ into itself. We observe that, for any $W \in C^0([0, T], D(A))$, $BW \in C^0([0, T], D(A))$ and thus $pBW \in L^2((0, T), D(A))$. Hence, it is possible to apply the classical result of existence of strict solutions (see, for instance, [8, Proposition 3.3]) and deduce that $K(W) \in C^0([0, T], D(A))$. Moreover, for any $W_1, W_2 \in C^0([0, T], D(A))$, it holds that

\[\|K(W_1)(t) - K(W_2)(t)\|_{D(A)} = \left\| \int_0^t e^{A(t-s)} p(s)(BW_1(s) - BW_2(s)) \, ds \right\|_{D(A)} \leq C_1 \int_0^t |p(s)|\|BW_1(s) - BW_2(s)\|_{D(A)} \, ds \leq C_1 C_B \|p\|_{L^1(0, T)} \|W_1 - W_2\|_{C^0([0, T], D(A))}.\]

Suppose $C_1 C_B \|p\|_{L^1(0, T)} < 1$. Then $K$ is a contraction and therefore has a unique fixed point. Furthermore, we have that

\[\|W\|_{C^0([0, T], D(A))} \leq \sup_{t \in [0, T]} \|e^{At}W_0 + \int_0^t e^{A(t-s)} (p(s)BW(s) + F(s)) \, ds\|_{D(A)} \leq C_1 \left( \|W_0\|_{D(A)} + \int_0^T |p(s)| \|BW(s)\|_{D(A)} + \|F(s)\|_{D(A)} \, ds \right) \leq C_1 \left( \|W_0\|_{D(A)} + C_B \|W\|_{C^0([0, T], D(A))} \|p\|_{L^1(0, T)} + \sqrt{T} \|F\|_{L^2(0, T), D(A)} \right).\]

Therefore,

\[\|W\|_{C^0([0, T], D(A))} \leq C_1 \frac{C_B}{1 - C_1 C_B \|p\|_{L^1(0, T)}} (\|W_0\|_{D(A)} + \sqrt{T} \|F\|_{L^2(0, T), D(A)}).\]

We have thus obtained the conclusion under the extra hypothesis that $p$ satisfies $C_1 C_B \|p\|_{L^1(0, T)} < 1$. In the general case, it is sufficient to represent $[0, T]$ as the union of a finite family of sufficiently small subintervals where we can repeat the above argument in each one.
Equivalently, (2.13) can be proved by Gronwall’s Lemma, obtaining:

\[(4.6) \quad \|W\|_{C^0([0,T],D(A))} \leq C_1 \left( \|W_0\|_{D(A)} + \sqrt{T} \|F\|_{L^2(0,T;D(A))} \right) e^{C_1 \|p\|_{L^1(0,T)}}.\]

\[\square\]

5. Spectral problem: proof of Propositions 3.1 and 3.2

5.1. A classical change of variables.

First, we note that if \((\lambda, \Phi)\) solves (3.1), then \(\lambda \geq 0\): indeed, multiplying by \(\Phi\), we obtain

\[
\int_0^1 \Phi^2 = \int_0^1 (x^\alpha \Phi')' \Phi = \left[ - (x^\alpha \Phi') \Phi \right]_0^1 + \int_0^1 x^\alpha (\Phi')^2 = \int_0^1 x^\alpha (\Phi')^2.
\]

Moreover, if \(\lambda = 0\), then \(x \mapsto x^\alpha \Phi'\) is constant and by imposing the boundary conditions we find that it is actually equal to 0. Thus, the constant functions are the ones and only ones associated to the eigenvalue \(\lambda = 0\).

We now investigate the positive eigenvalues: if \(\lambda > 0\), we introduce the function \(\psi\) defined by the relation

\[
\Phi(x) = x^{\frac{1-\alpha}{2-\alpha}} \psi \left( \frac{2}{2-\alpha} \sqrt{\lambda x^{\frac{2-\alpha}{2}}} \right),
\]

and the associated new space variable

\[
y = \frac{2}{2-\alpha} \sqrt{\lambda x^{\frac{2-\alpha}{2}}},
\]

(see, e.g., [13, 21]). After some classical computations, we obtain that \(\psi\) satisfies the following problem:

\[
(5.1) \quad \begin{cases}
y^2 \psi''(y) + y \psi'(y) + \left( y^2 - \left( \frac{1-\alpha}{2-\alpha} \right)^2 \right) \psi(y) = 0, & y \in (0, \frac{2}{2-\alpha} \sqrt{\lambda}), \\
y \psi'(y) + \frac{1-\alpha}{2-\alpha} y^{\frac{2-\alpha}{2}} \psi(y) \rightarrow 0 \text{ as } y \rightarrow 0, \\
\sqrt{\lambda} \psi' \left( \frac{2}{2-\alpha} \sqrt{\lambda} \right) + \frac{1-\alpha}{2-\alpha} \psi' \left( \frac{2}{2-\alpha} \sqrt{\lambda} \right) = 0.
\end{cases}
\]

The first equation in (5.1) is the Bessel equation of order \(\nu = \frac{1-\alpha}{2-\alpha}\) (see [28] or [30]).

In the following, we solve (3.1) using (5.1) and distinguishing several cases

1. \(\alpha \in [0,1)\),
2. \(\alpha \in (1,2)\) and \(\nu_\alpha = \frac{\alpha-1}{2-\alpha} \notin \mathbb{N}\),
3. \(\alpha \in (1,2)\) and \(\nu_\alpha = \frac{\alpha-1}{2-\alpha} \in \mathbb{N}^\ast\),
4. \(\alpha = 1\).

Our study will be based on well-known properties of Bessel functions, and it is similar to the one in [13], the only difference, which makes the analysis interesting, lying in the boundary condition at point \(x = 1\). In every case, the strategy is

1. to exhibit a basis of the vector space of dimension 2 of the solution of the first equation (5.1), and this is where Bessel functions appear, and where the distinction of the different cases is necessary,
2. to take into account the functional setting of the problem, in order to eliminate some possible solutions,
3. finally, to impose the boundary conditions at points \(x = 0\) and \(x = 1\), in order to determine the eigenvalues and the associated eigenfunctions.
5.2. The case $\alpha \in [0,1)$.

5.2.a. The study of the ODE.

In this section we assume that $\alpha \in [0,1)$. In this case, we have

$$\nu_\alpha := \frac{1 - \alpha}{2 - \alpha} \in \left(0, \frac{1}{2}\right].$$

The ODE we need to solve is

(5.2) $$y^2 \psi''(y) + y \psi'(y) + (y^2 - \nu^2) \psi(y) = 0, y \in I \subset (0, +\infty)$$

with $\nu = \nu_\alpha$ and $I = \left(0, \frac{\alpha}{2 - \alpha} \sqrt{\lambda}\right)$. The above equation is called Bessel’s equation for functions of order $\nu$. The fundamental theory of ordinary differential equations establishes that the solutions of (5.2) generate a vector space $S_\nu$ of dimension 2. Consider the Bessel function of order $\nu$ and of the first kind $J_\nu$:

(5.3) $$J_\nu(y) := \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \nu + 1)} \left(\frac{y}{2}\right)^{2m+\nu} = \sum_{m=0}^{\infty} c_{\nu,m}^+ y^{2m+\nu}, \quad y \geq 0$$

and $J_{-\nu}$:

(5.4) $$J_{-\nu}(y) := \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m - \nu + 1)} \left(\frac{y}{2}\right)^{2m-\nu} = \sum_{m=0}^{\infty} c_{\nu,m}^- y^{2m-\nu}, \quad y > 0.$$ 

Since $\nu \notin \mathbb{N}$, the two functions $J_\nu$ and $J_{-\nu}$ are linearly independent and therefore the pair $(J_\nu, J_{-\nu})$ forms a fundamental system of solutions of (5.2), (see [30, section 3.1, (8), p. 40], [30, section 3.12, eq. (2), p. 43] or [28, eq. (5.3.2), p. 102])). Hence,

(5.5) $$\begin{cases} y^2 \psi''(y) + y \psi'(y) + (y^2 - \nu^2) \psi(y) = 0, \quad y \in I \\ \Rightarrow \quad \exists C_+, C_- \in \mathbb{R}, \quad \begin{cases} \psi(y) = C_+ J_\nu_\alpha(y) + C_- J_{-\nu_\alpha}(y), \quad y \in I. \end{cases} \end{cases}$$

Thus, going back to the original variables, we obtain that

(5.6) $$\begin{cases} - (x^\alpha \Phi')' = \lambda \Phi, \quad x \in (0,1) \\ \Rightarrow \quad \exists C_+, C_- \in \mathbb{R}, \quad \begin{cases} \Phi(x) = C_+ \Phi_+(x) + C_- \Phi_-(x), \quad x \in (0,1), \end{cases} \end{cases}$$

with

(5.7) $$\begin{align*} \Phi_+(x) &= x^{\frac{1-\alpha}{2}} J_{\nu_\alpha} \left(\frac{2}{2 - \alpha} \sqrt{\lambda x^{2 - \alpha}}\right) = x^{\frac{1-\alpha}{2}} \sum_{m=0}^{\infty} c_{\nu_\alpha,m}^+ \left(\frac{2}{2 - \alpha} \sqrt{\lambda x^{2 - \alpha}}\right)^{2m+\nu_\alpha} \\
&= \sum_{m=0}^{\infty} c_{\nu_\alpha,m}^+ \left(\frac{2}{2 - \alpha} \sqrt{\lambda}\right)^{2m+\nu_\alpha} x^{1-\alpha(2-\alpha)m} = \sum_{m=0}^{\infty} \tilde{c}_{\alpha,\lambda,m} x^{1-\alpha(2-\alpha)m} \end{align*}$$
and
\[(5.8)\]
\[\Phi_-(x) = x^{1-\alpha} J_{\nu_\alpha} \left( \frac{2}{2-\alpha} \sqrt{\lambda x^{\frac{2}{\alpha}}} \right) = x^{1-\alpha} \sum_{m=0}^{\infty} c_{\nu_\alpha,m} \left( \frac{2}{2-\alpha} \sqrt{\lambda x^{\frac{2}{\alpha}}} \right)^{2m-\nu_\alpha} \]
\[= \sum_{m=0}^{\infty} c_{\nu_\alpha,m} \left( \frac{2}{2-\alpha} \sqrt{\lambda} \right)^{2m-\nu_\alpha} x^{(2-\alpha)m} = \sum_{m=0}^{\infty} \tilde{c}_{\alpha,\lambda,m} x^{(2-\alpha)m} \]

5.2.b. Information given by the functional setting.

Note that \(\Phi_+^0(x) \to 0\) as \(x \to 0^+\), hence \(\Phi_+ \in L^2(0,1)\). Moreover,
\[\Phi_+'(x) \sim \tilde{c}_{\alpha,\lambda,0}^+ \frac{1-\alpha}{x^{\alpha}} \quad \text{as} \quad x \to 0^+,\]
therefore, \(\Phi_+\) is absolutely continuous on \([0,1]\). Furthermore,
\[x^{\alpha/2} \Phi_+'(x) \sim \tilde{c}_{\alpha,\lambda,0}^+ \frac{1-\alpha}{x^{\alpha/2}} \quad \text{as} \quad x \to 0^+,\]
thus, \(\Phi_+ \in H^1_{\alpha}(0,1)\). Finally,
\[(x^\alpha \Phi_+)'(x) \to 0 \quad \text{as} \quad x \to 0^+,\]
and we deduce that \(\Phi_+ \in H^2_\alpha(0,1)\).

In the same way, one easily checks that \(\Phi_- \in H^2_\alpha(0,1)\).

5.2.c. Information given by the boundary condition at \(x = 0\).

An eigenfunction must satisfy additionally the boundary conditions. In particular, we should have that \(x^\alpha \Phi'(x) \to 0\) as \(x \to 0^+\). We observe that
\[x^\alpha \Phi'_-(x) \to 0 \quad \text{as} \quad x \to 0,\]
while
\[x^\alpha \Phi'_-(x) \to \tilde{c}_{\alpha,\lambda,0}^+(1-\alpha) \neq 0 \quad \text{as} \quad x \to 0.\]

Therefore, we conclude that
\[\begin{cases} 
- (x^\alpha \Phi')' = \lambda \Phi, & x \in (0,1) \\
(x^\alpha \Phi')(0) = 0
\end{cases} \quad \Rightarrow \quad \exists C_- \in \mathbb{R}, \quad \begin{cases} 
\Phi(x) = C_- \Phi_-(x), \\
x \in (0,1),
\end{cases}\]

5.2.d. Information given by the boundary condition at \(x = 1\).

As regards the boundary condition at \(x = 1\), \(\Phi\) has to solve \(\Phi'(1) = 0\). We compute \(\Phi'_-\):
\[\Phi'_-(x) = \frac{1-\alpha}{2} x^{\frac{1}{2} - \frac{\alpha}{2}} J_{\nu_\alpha} \left( \frac{2}{2-\alpha} \sqrt{\lambda x^{\frac{2}{\alpha}}} \right) + x^{\frac{1}{2} - \frac{\alpha}{2}} \sqrt{\lambda} x^{-\alpha/2} J'_{\nu_\alpha} \left( \frac{2}{2-\alpha} \sqrt{\lambda x^{\frac{2}{\alpha}}} \right).\]
and we deduce that the following relation must hold
\[(5.9) \quad \frac{1-\alpha}{2} J_{\nu_\alpha} \left( \frac{2}{2-\alpha} \sqrt{\lambda} \right) + \sqrt{\lambda} J'_{\nu_\alpha} \left( \frac{2}{2-\alpha} \sqrt{\lambda} \right) = 0.\]
This is the equation that characterizes the eigenvalues \( \lambda \). Multiplying by \( \frac{2}{2 - \alpha} \), (5.9) becomes

\[
(5.10) \quad 2 \left( \frac{2}{2 - \alpha} \right) J_{-\nu}(\alpha \sqrt{\lambda}) + \frac{2}{2 - \alpha} \sqrt{\lambda} J'_{-\nu}(\alpha \sqrt{\lambda}) = 0.
\]

Introducing the variable

\[
X_\lambda = \frac{2}{2 - \alpha} \sqrt{\lambda},
\]

we have

\[
(5.11) \quad \nu \alpha J_{-\nu}(X_\lambda) + \frac{2}{2 - \alpha} \sqrt{\lambda} J'_{-\nu}(X_\lambda) = 0.
\]

We now consider the following well-known relation (see [30, p. 45, formula (4)])

\[
(5.12) \quad z J''_{\nu}(z) - \nu J_{\nu}(z) = z J_{\nu+1}(z).
\]

from which we deduce that

\[
\nu \alpha J_{-\nu}(X_\lambda) + \frac{2}{2 - \alpha} \sqrt{\lambda} J'_{-\nu}(X_\lambda) = 0.
\]

Thus, equation (5.10) is equivalent to

\[
(5.13) \quad X_\lambda J'_{-\nu+1}(X_\lambda) = 0,
\]

which implies that

\[
(5.14) \quad J_{-\nu+1}(X_\lambda) = 0.
\]

Thus, the possible values for \( X_\lambda \) are the positive zeros of \( J_{-\nu+1} \):

\[
\exists n \geq 1 : \frac{2}{2 - \alpha} \sqrt{\lambda} = X_\lambda = j_{-\nu+1,n}.
\]

We obtain that the eigenvalues of (5.1) have the following form:

\[
\exists n \geq 1 : \lambda_n = \kappa_{\alpha}^2 j_{-\nu+1,n}^2.
\]

Vice-versa, given \( n \geq 1 \), consider

\[
\lambda_n := \kappa_{\alpha}^2 j_{-\nu+1,n}^2 \quad \text{and} \quad \Phi_n(x) = \frac{j_{-\nu+1,n}^{2 - \alpha}}{2} J_{-\nu}(j_{-\nu+1,n} x).\]

From the previous analysis we deduce that \( \Phi_n \in H^2_{\alpha}(0,1) \) and that \( (\lambda_n, \Phi_n) \) solves (5.1).

Finally, the proof of (5.4) follows directly from [27, p. 135]: since \(-\nu+1 \geq \frac{1}{2}\), the sequence \((j_{-\nu+1,n+1} - j_{-\nu+1,n})_{n \geq 1}\) is nonincreasing and

\[
j_{-\nu+1,n+1} - j_{-\nu+1,n} \to \pi \quad \text{as} \quad n \to \infty.
\]

This concludes the proof of Proposition 5.1.

5.2.e. Additional information on the eigenvalues and eigenfunctions.

We are going to prove useful properties of the eigenfunctions for \( \alpha \in [0,1) \).

**Lemma 5.1.** Let \( \alpha \in [0,1) \) and \( n \geq 1 \). Then, \( \Phi_{\alpha,n} \) has finite limits as \( x \to 0^+ \) and \( x \to 1^- \), and satisfies

\[
(5.15) \quad |\Phi_{\alpha,n}(1)| = \sqrt{2 - \alpha},
\]

and

\[
(5.16) \quad \Phi_{\alpha,n}(0) \sim c_{\nu,0} \sqrt{\frac{(2 - \alpha)\pi}{2}} (j_{-\nu+1,n})^{\frac{1}{2} - \nu} \quad \text{as} \quad n \to +\infty.
\]
Proof of Lemma 5.1. First, we note that \( j_{-\nu_0+1,n} \) is not a zero of \( J_{-\nu_0} \):

\[
\forall \alpha \in [0,1), \forall n \geq 1, \quad J_{-\nu_0}(j_{-\nu_0+1,n}) \neq 0.
\]

Indeed, if \( J_{-\nu_0}(j_{-\nu_0+1,n}) = 0 \), we derive from (5.12) that \( J'_{-\nu_0}(j_{-\nu_0+1,n}) = 0 \), and then the Cauchy problem satisfied by \( J_{-\nu_0} \) would imply that \( J_{-\nu_0} \) is constantly equal to zero.

We also derive from (5.12) that

\[
J'_{-\nu_0}(j_{-\nu_0+1,n}) = \frac{\nu_0}{j_{-\nu_0+1,n}} J_{-\nu_0}(j_{-\nu_0+1,n}).
\]

We compute the value of the constants \( K_{\alpha,n} \) that appear in (3.3)

\[
1 = K_{\alpha,n}^2 \int_0^1 x^{1-\alpha} J_{-\nu_0} \left( j_{-\nu_0+1,n} x^{\frac{2-\alpha}{\nu_0}} \right)^2 dx.
\]

Thanks to the change of variables \( y = x^{\frac{2-\alpha}{\nu_0}} \), we get

\[
1 = K_{\alpha,n}^2 \int_0^{j_{-\nu_0+1,n}} y J_{-\nu_0} \left( j_{-\nu_0+1,n} y \right)^2 dy,
\]

and applying [28] formula (5.14.5) p.129, we obtain

\[
1 = K_{\alpha,n}^2 \left( \frac{J'_{-\nu_0}(j_{-\nu_0+1,n})^2 + \left( 1 - \frac{\nu_0^2}{j_{-\nu_0+1,n}} \right) J_{-\nu_0}(j_{-\nu_0+1,n})^2}{2 - \alpha} \right).
\]

Therefore,

\[
K_{\alpha,n} = \left( \frac{2 - \alpha}{J'_{-\nu_0}(j_{-\nu_0+1,n})^2 + \left( 1 - \frac{\nu_0^2}{j_{-\nu_0+1,n}} \right) J_{-\nu_0}(j_{-\nu_0+1,n})^2} \right)^{1/2},
\]

and using (5.18), we obtain a simple expression for \( K_{\alpha,n} \):

\[
\forall \alpha \in [0,1), \forall n \geq 1, \quad K_{\alpha,n} = \frac{\sqrt{2 - \alpha}}{|J_{-\nu_0}(j_{-\nu_0+1,n})|}.
\]

Thus, from (5.3) we deduce the value of \( |\Phi_{\alpha,n}(1)| \) given in (5.15), and also the value of \( \Phi_{\alpha,n}(0) \). Indeed, from (5.4), we obtain that for all \( n \geq 1 \), \( \Phi_{\alpha,n} \) has a finite limit as \( x \to 0^+ \), and that

\[
\Phi_{\alpha,n}(0) = \frac{\sqrt{2 - \alpha}}{|J_{-\nu_0}(j_{-\nu_0+1,n})|} c_{\alpha,0}^- (j_{-\nu_0+1,n})^{-\nu_0}.
\]

Moreover, using the following classical asymptotic development ([28] formula (5.11.6) p. 122): \( J_\nu(z) = \sqrt{\frac{2}{\pi z}} \cos(z - \frac{\nu \pi}{2} - \frac{\pi}{4})(1 + O(\frac{1}{z^2})) + O(\frac{1}{z}) \) as \( z \to \infty \), we obtain that

\[
J_\nu(z) = \frac{2}{\pi z} \cos^2(z - \frac{\nu \pi}{2} - \frac{\pi}{4}) + O(\frac{1}{z^2}) \quad \text{as} \quad z \to \infty.
\]

(5.21)

(5.22)

(5.20)

(5.20)
Applying the latter formula with \( \nu + 1 \), we get
\[
z J_{\nu+1}(z)^2 = \frac{2}{\pi} \cos^2\left(z - \frac{(\nu + 1)\pi}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{z}\right) = \frac{2}{\pi} \sin^2\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{z}\right).
\]
Therefore,
\[
z J_{\nu}(z)^2 + z J_{\nu+1}(z)^2 = \frac{2}{\pi} + O\left(\frac{1}{z}\right),
\]
which gives that
\[
(5.23) \quad z J_{\nu}(z)^2 + z J_{\nu+1}(z)^2 \to \frac{2}{\pi} \quad \text{as} \quad z \to +\infty.
\]
So, we deduce that
\[
j_{-\nu+1,n}j_{-\nu}(j_{-\nu+1,n})^2 + j_{-\nu+1,n}j_{-\nu+1}(j_{-\nu+1,n})^2 \to \frac{2}{\pi} \quad \text{as} \quad n \to +\infty.
\]
Hence,
\[
(5.24) \quad J_{-\nu}(j_{-\nu+1,n})^2 \sim \frac{2}{\pi j_{-\nu+1,n}} \quad \text{as} \quad n \to +\infty,
\]
and then, combining (5.24) with (5.20) we finally obtain (5.10).

5.3. The case \( \alpha \in [1,2) \).

5.3.a. Analysis of the ODE for \( \alpha \in (1,2) \) and \( \nu_{\alpha} = \frac{\alpha-1}{2-\alpha} \notin \mathbb{N} \).

In this section we want to solve problem (3.1), hence (5.1), for \( \alpha \in (1,2) \). As recalled before, when \( \nu_{\alpha} \notin \mathbb{N} \), \( J_{\nu_{\alpha}} \) and \( J_{-\nu_{\alpha}} \) form a fundamental system of solutions of (5.2) (with \( \nu = \nu_{\alpha} \)). Therefore, (5.5) and (5.6) still hold. However, the difference lies in the functions \( \Phi_{+} \) and \( \Phi_{-} \); indeed, in this case we have
\[
(5.25) \quad \Phi_{+}(x) = x^{\frac{1}{2-\alpha}} J_{\nu_{\alpha}} \left(\frac{2}{2-\alpha} \sqrt{x} \frac{2}{2-\alpha} \right) = x^{\frac{1}{2-\alpha}} \sum_{m=0}^{\infty} c_{\nu_{\alpha},m}^{+} \left(\frac{2}{2-\alpha} \sqrt{x} \frac{2}{2-\alpha} \right)^{2m+\nu_{\alpha}}
\]
\[
= \sum_{m=0}^{\infty} c_{\nu_{\alpha},m}^{+} \left(\frac{2}{2-\alpha} \sqrt{x} \frac{2}{2-\alpha} \right)^{2m+\nu_{\alpha}} x^{(2-\alpha)m} = \sum_{m=0}^{\infty} c_{\alpha,\lambda,m}^{+} x^{(2-\alpha)m}
\]
and
\[
(5.26) \quad \Phi_{-}(x) = x^{\frac{1}{2-\alpha}} J_{-\nu_{\alpha}} \left(\frac{2}{2-\alpha} \sqrt{x} \frac{2}{2-\alpha} \right) = x^{\frac{1}{2-\alpha}} \sum_{m=0}^{\infty} c_{\nu_{\alpha},m}^{-} \left(\frac{2}{2-\alpha} \sqrt{x} \frac{2}{2-\alpha} \right)^{2m-\nu_{\alpha}}
\]
\[
= \sum_{m=0}^{\infty} c_{\nu_{\alpha},m}^{-} \left(\frac{2}{2-\alpha} \sqrt{x} \frac{2}{2-\alpha} \right)^{2m-\nu_{\alpha}} x^{(2-\alpha)m} = \sum_{m=0}^{\infty} c_{\alpha,\lambda,m}^{-} x^{1-\alpha+(2-\alpha)m}.
\]

5.3.b. Information given by the functional setting for \( \nu_{\alpha} = \frac{\alpha-1}{2-\alpha} \notin \mathbb{N} \).

We note that
\[
\Phi_{+}(x) \to \tilde{c}_{\alpha,\lambda,0}^{+} \quad \text{as} \quad x \to 0^{+},
\]
hence, \( \Phi_{+} \in L^{2}(0,1) \). Moreover,
\[
x^{\alpha/2} \Phi'_{+}(x) \sim \tilde{c}_{\alpha,\lambda,1}^{+} (2-\alpha) x^{1-\frac{\alpha}{2}} \quad \text{as} \quad x \to 0^{+},
\]
which implies that \( \Phi_{+} \in H^{1}_{\lambda}(0,1) \). Furthermore, we have that
\[
(x^{\alpha} \Phi'_{+})(x) \to \tilde{c}_{\alpha,\lambda,1}^{+} (2-\alpha) \quad \text{as} \quad x \to 0^{+},
\]
The possible values for $X$.

5.3.c. Information given by the boundary condition at $x = 0$.

and we deduce that $Φ_+ \notin H^1(0, 1)$, and, in particular, $Φ_- \notin H^2(0, 1)$. Therefore, $C_- = 0$ and (5.6) yields

$$Φ(x) = C_+ Φ_+(x), \quad x \in (0, 1).$$

5.3.c. Information given by the boundary condition at $x = 0$ for $ν_α = \frac{α}{2 - α} \notin N$.

Observe that in this case $(x^αΦ(x))(0) = 0$, and therefore the boundary condition at $x = 0$ is automatically satisfied.

5.3.d. Information given by the boundary condition at $x = 1$ for $ν_α = \frac{α}{2 - α} \notin N$.

In order to be an eigenfunction, $Φ$ has to solve the second boundary condition: $Φ'(1) = 0$. We recall that

$$Φ'(x) = \frac{1 - α}{2} x^{-1 - α} J_{ν_α} \left( \frac{2}{2 - α} \sqrt{λx^{2 - α}} \right) + \frac{1 - α}{2} \sqrt{λ} x^{-α} J_{ν_α} \left( \frac{2}{2 - α} \sqrt{λx^{2 - α}} \right).$$

Hence, if $Φ$ is an eigenfunction, $C_+ \neq 0$ and $Φ'(1) = 0$, and so

$$\frac{1 - α}{2} J_{ν_α} \left( \frac{2}{2 - α} \sqrt{λ} \right) + \sqrt{λ} J'_{ν_α} \left( \frac{2}{2 - α} \sqrt{λ} \right) = 0.$$

This is the equation that characterizes the eigenvalues $λ$. Multiplying by $\frac{2}{2 - α}$, (5.28) becomes

$$\frac{2}{2 - α} \left( \frac{1 - α}{2} J_{ν_α} \left( \frac{2}{2 - α} \sqrt{λ} \right) + \sqrt{λ} J'_{ν_α} \left( \frac{2}{2 - α} \sqrt{λ} \right) \right) = 0.$$

Introducing once again the variable

$$X_λ = \frac{2}{2 - α} \sqrt{λ},$$

equation (5.29) can be rewritten as

(5.30) $- ν_α J_{ν_α}(X_λ) + X_λ J'_{ν_α}(X_λ) = 0.$

Using again (5.12), we have

$$- ν_α J_{ν_α}(X_λ) + X_λ J'_{ν_α}(X_λ) = X_λ J_{ν_α}(X_λ).$$

Thus, (5.28) becomes

(5.31) $X_λ J_{ν_α + 1}(X_λ) = 0,$

or, equivalently

(5.32) $J_{ν_α + 1}(X_λ) = 0.$

The possible values for $X_λ$ are the positive zeros of $J_{ν_α + 1}$:

$$\exists n \geq 1, \quad \frac{2}{2 - α} \sqrt{λ} = X_λ = j_{ν_α + 1, n}.$$

The above identity provides the following expression for the eigenvalues

$$\exists n \geq 1, \quad λ_n = k_α^2 j_{ν_α + 1, n}.$$

Vice-versa, given $n \geq 1$, consider

$$λ_n := k_α^2 j_{ν_α + 1, n} \quad \text{and} \quad Φ_n(x) = x^{\frac{2 - α}{2}} J_{ν_α + 1, n} \left( j_{ν_α + 1, n} x^{\frac{2 - α}{2}} \right).$$
From the previous analysis we deduce that \( \Phi_n \in H^2_\alpha(0,1) \) and that \((\lambda_n, \Phi_n)\) solves (3.1).

Finally, the proof of (3.7) follows directly from \([27, \text{p. 135}]\): since \(\nu_{\alpha} + 1 \geq 1 > \frac{1}{2}\), the sequence \((j_{\nu_{\alpha}+1,n+1} - j_{\nu_{\alpha}+1,n})_{n \geq 1}\) is decreasing

\[
j_{\nu_{\alpha}+1,n+1} - j_{\nu_{\alpha}+1,n} \to \pi \quad \text{as} \quad n \to \infty.
\]

This concludes the proof of the related part of Proposition 3.2. \(\square\)

### 5.3.e. The main changes for \(\nu_{\alpha} = \alpha - \frac{1}{2} - \alpha \in \mathbb{N}\).

In this case, it has been proved in \([13]\) that (5.27) remains true (with \(\Phi^+\) defined in (5.25), the only difference is that now the fundamental system of the solutions of (5.2) (with \(\nu = \nu_{\alpha}\)) involves \(J_\nu\) and \(Y_\nu\), the Bessel’s function of order \(\nu\) and of second kind (see \([30, \text{section 3.54, eq. (1)-(2), p. 64}]\) or \([28, \text{eq. (5.4.5)-(5.4.6), p. 104}]\)). Studying the behavior as \(x \to 0\), one obtains once again (5.27) (we refer to \([13]\) for the details). Then, one can conclude the proof of Proposition 3.2 by reasoning as for the case \(\nu_{\alpha} \notin \mathbb{N}\). \(\square\)

### 5.3.f. Additional information on the eigenvalues and eigenfunctions.

We are going to prove the following results, that will be useful in the following:

**Lemma 5.2.** Let \(\alpha \in [1,2)\). Then, the \(\Phi_{\alpha,n}\) has finite limit as \(x \to 0^+\) and \(x \to 1^-\), and satisfies

\[
\forall n \geq 1, \quad |\Phi_{\alpha,n}(1)| = \sqrt{2 - \alpha},
\]

and

\[
\Phi_{\alpha,n}(0) \sim c^{+}_{\nu_{\alpha},0} \sqrt{\frac{(2 - \alpha)\pi}{2}} (j_{\nu_{\alpha}+1,n})^{2+\nu_{\alpha}} \quad \text{as} \quad n \to +\infty
\]

(where the coefficient \(c^{+}_{\nu_{\alpha},0}\) is defined in (5.3)). In particular, the sequence \((\Phi_{\alpha,n}(0))_{n \geq 1}\) is unbounded.

**Proof of Lemma 5.2.** First, we note that \(j_{\nu_{\alpha}+1,n}\) is not a zero of \(J_{\nu_{\alpha}}:\)

\[
\forall \alpha \in [1,2), \forall n \geq 1, \quad J_{\nu_{\alpha}}(j_{\nu_{\alpha}+1,n}) \neq 0.
\]

Indeed, if \(J_{\nu_{\alpha}}(j_{\nu_{\alpha}+1,n}) = 0\), we derive from (5.12) that \(J'_{\nu_{\alpha}}(j_{\nu_{\alpha}+1,n}) = 0\), and then the Cauchy problem satisfied by \(J_{\nu_{\alpha}}\) would imply that \(J_{\nu_{\alpha}}\) is constantly equal to zero.

We also deduce from (5.2) that

\[
J'_{\nu_{\alpha}}(j_{\nu_{\alpha}+1,n}) = \frac{\nu_{\alpha}}{j_{\nu_{\alpha}+1,n}} J_{\nu_{\alpha}}(j_{\nu_{\alpha}+1,n}), \quad \forall n \geq 1.
\]

With the same strategy adopted in Lemma 5.1 we compute the value of \(K_{\alpha,n}\) that appears in (3.6) and we find that

\[
\forall \alpha \in [1,2), \forall n \geq 1, \quad K_{\alpha,n} = \frac{\sqrt{2 - \alpha}}{|J_{\nu_{\alpha}}(j_{\nu_{\alpha}+1,n})|}.
\]

Therefore, we obtain from (5.6) the value of \(|\Phi_{\alpha,n}(1)|\) given in (5.33), and the value of \(\Phi_{\alpha,n}(0)\). Indeed, using (5.3), we have

\[
\Phi_{\alpha,n}(0) = \frac{\sqrt{2 - \alpha}}{|J_{\nu_{\alpha}}(j_{\nu_{\alpha}+1,n})|} c^{+}_{\nu_{\alpha},0} (j_{\nu_{\alpha}+1,n})^{\nu_{\alpha}},
\]
and, in particular, function $\Phi_{\alpha,n}$ has a finite limit as $x \to 0$. Moreover, using \cite{5.23}, we obtain

$$j_{\nu_{\alpha,n}}(j_{\nu_{\alpha,n}}+1)^2 + j_{\nu_{\alpha,n}+1}(j_{\nu_{\alpha,n}+1})^2 \to \frac{2}{\pi} \text{ as } n \to +\infty.$$ 

Hence,

$$(5.39) \quad j_{\nu_{\alpha,n}}^2 \sim \frac{2}{\pi j_{\nu_{\alpha,n}}} \text{ as } n \to +\infty,$$

and, combining \eqref{5.39} with \eqref{5.20}, we deduce \eqref{5.34}. \hfill \Box

6. Proof of Proposition 3.3

Let us give a more precise formulation of Proposition 3.3:

**Proposition 6.1.** Let $\mu \in V^2_{\alpha}(0,1)$ (defined in \cite{3.14}). Then,

a) for all $p \in L^2(0,T)$, the solution $w^{(p)}$ of \eqref{3.13} has the following additional regularity

$$(6.1) \quad (w^{(p)}(T), w_t^{(p)}(T)) \in H^3_{(0)}(0,1) \times D(A),$$

so, the map

$$(6.2) \quad \Theta_T : L^2(0,T) \to H^3_{(0)}(0,1) \times D(A), \quad \Theta_T(p) := (w^{(p)}(T), w_t^{(p)}(T))$$

is well-defined,

b) given $p \in L^2(0,T)$, $\Theta_T$ is differentiable at $p$, and $D\Theta_T(p) : L^2(0,T) \to H^3_{(0)}(0,1) \times D(A)$ is a continuous linear application and satisfies

$$D\Theta_T(p) \cdot q = (W^{(p,q)}(T), W_t^{(p,q)}(T)),$$

where $W^{(p,q)}$ is the solution of

$$(6.3) \quad \begin{cases}
W_{tt}^{(p,q)} - (x^\alpha W_x^{(p,q)})_x = p(t)\mu(x)W^{(p,q)} + q(t)\mu(x)w^{(p)}, & x \in (0,1), t \in (0,T), \\
(x^\alpha W_x^{(p,q)})(x = 0, t) = 0, & t \in (0,T), \\
W_x^{(p,q)}(x = 1, t) = 0, & t \in (0,T), \\
W_t^{(p,q)}(x,0) = 0, & x \in (0,1), \\
W_t^{(p,q)}(x,0) = 0, & x \in (0,1),
\end{cases}$$

c) the map $\Theta_T$ is of class $C^1$.

The proof of Proposition 6.1 is based on several steps, the first one consists in analyzing the eigenvalues and eigenfunctions of the operator $A$.

6.1. Eigenvalues and eigenfunctions of $A$.

Let us give the following preliminary result.

**Lemma 6.1.** Consider, for all $n \in \mathbb{Z}$,

$$(6.4) \quad \omega_{\alpha,n} = \begin{cases}
-\sqrt{\lambda_{\alpha,n}} & n \leq -1, \\
0 & n = 0, \\
\sqrt{\lambda_{\alpha,n}} & n \geq 1,
\end{cases}$$
and

\[ \Psi_{\alpha,n} = \begin{cases} 
\frac{\Phi_{\alpha,|n|}}{-i\sqrt{\lambda_{\alpha,|n|}\Phi_{\alpha,|n|}}}, & n \leq -1, \\
\Phi_{\alpha,0} = 1, & n = 0, \\
\frac{\Phi_{\alpha,n}}{i\sqrt{\lambda_{\alpha,n}\Phi_{\alpha,n}}}, & n \geq 1.
\end{cases} \tag{6.5} \]

Then, \{\omega_{\alpha,n}\}_{n \in \mathbb{Z}} and \{\Psi_{\alpha,n}\}_{n \in \mathbb{Z}} fulfill

\[ \forall n \in \mathbb{Z}, \quad \mathcal{A}\Psi_{\alpha,n} = i\omega_{\alpha,n}\Psi_{\alpha,n}. \tag{6.6} \]

**Proof of Lemma 6.1.** A direct computation shows that for all \( n \geq 0 \) we have that

\[
\mathcal{A}\Psi_{\alpha,n} = \begin{pmatrix} 0 & \text{Id} \\
A & 0 \end{pmatrix} \begin{pmatrix} \Phi_{\alpha,|n|} \\
i\sqrt{\lambda_{\alpha,|n|}\Phi_{\alpha,|n|}} \end{pmatrix} = \begin{pmatrix} i\sqrt{\lambda_{\alpha,|n|}\Phi_{\alpha,|n|}} \\
-i\sqrt{\lambda_{\alpha,|n|}\Phi_{\alpha,|n|}} \end{pmatrix} = i\omega_{\alpha,n}\Psi_{\alpha,n},
\]

and, for \( n \leq -1 \),

\[
\mathcal{A}\Psi_{\alpha,n} = \begin{pmatrix} 0 & \text{Id} \\
A & 0 \end{pmatrix} \begin{pmatrix} \Phi_{\alpha,|n|} \\
i\sqrt{\lambda_{\alpha,|n|}\Phi_{\alpha,|n|}} \end{pmatrix} = \begin{pmatrix} -i\sqrt{\lambda_{\alpha,|n|}\Phi_{\alpha,|n|}} \\
-i\sqrt{\lambda_{\alpha,|n|}\Phi_{\alpha,|n|}} \end{pmatrix} = i\omega_{\alpha,n}\Psi_{\alpha,n},
\]

which are exactly the identities in \((6.4)\).

In order to rigorously compute the eigenvalues and eigenfunctions of \( \mathcal{A} \), one has to introduce the natural extension of \( \mathcal{A} \) to the complex valued functions:

\[
\forall (f,h),(g,k) \in D(\mathcal{A}), \quad \tilde{\mathcal{A}}\begin{pmatrix} f(x) + ig(x) \\
h(x) + ik(x) \end{pmatrix} := \mathcal{A}\begin{pmatrix} f(x) \\
h(x) \end{pmatrix} + i\mathcal{A}\begin{pmatrix} g(x) \\
k(x) \end{pmatrix},
\]

and then we can investigate the spectral problem

\[ \tilde{\mathcal{A}}\tilde{\Psi} = \tilde{\omega}\tilde{\Psi} \quad \text{with} \quad \tilde{\Psi}(x) = \begin{pmatrix} f(x) + ig(x) \\
h(x) + ik(x) \end{pmatrix}, \quad (f,h),(g,k) \in D(\mathcal{A}). \tag{6.7} \]

**Lemma 6.2.** The set of solutions (\( \tilde{\omega}, \tilde{\Psi} \)) of problem \((6.7)\) is

\[ \tilde{\mathcal{S}} = \{ (i\omega_{\alpha,n}, \tilde{\rho} \Psi_{\alpha,n}), n \in \mathbb{Z}, \tilde{\rho} \in \mathbb{C} \}, \]

where \( \omega_{\alpha,n} \) is defined in \((6.4)\), and \( \Psi_{\alpha,n} \) is defined in \((6.5)\).

**Proof of Lemma 6.2.** Let \( \tilde{\omega} = a + ib \). Then, we have

\[ \tilde{\mathcal{A}}\begin{pmatrix} f + ig \\
h + ik \end{pmatrix} = \begin{pmatrix} h + ik \\
Af + iAg \end{pmatrix}, \quad \tilde{\omega}\begin{pmatrix} f + ig \\
h + ik \end{pmatrix} = \begin{pmatrix} (af - bg) + i(ag + bf) \\
(ah - bk) + i(ak + bh) \end{pmatrix}. \]
Thus, \( [5, 7] \) reduces to the following system

\[
\begin{aligned}
h &= af - bg \\
k &= ag + bf \\
Af &= ah - bk \\
Ag &= ak + bh.
\end{aligned}
\]

We deduce that

\[
\begin{aligned}
Af &= a(af - bg) - b(ag + bf) = (a^2 - b^2)f - 2abg \\
Ag &= a(ag + bf) + b(af - bg) = (a^2 - b^2)g + 2abf.
\end{aligned}
\]

To solve the above system, we have to distinguish several cases:

- \( \bar{\omega} \neq 0 \): in this case we have that \( Af = 0 = Ag \). Thus, \( f \) and \( g \) are constant: \( f = c_1, \ g = c_2 \) (with \( c_1, c_2 \in \mathbb{R} \)) and \( h = 0 = k \). Therefore,

\[
\bar{\omega} = 0, \ \text{and} \ \bar{\Psi} = \left( \begin{array}{c} f + ig \\ 0 \end{array} \right) = (c_1 + ic_2) \left( \begin{array}{c} 1 \\ 0 \end{array} \right);
\]

- \( \bar{\omega} \neq 0 \) and \( f = 0 \): we must have \( g \neq 0 \) (otherwise it would imply \( h = 0 = k \)). So, we get that \( ab = 0 \) and \( g \) is an eigenfunction of \( A \) associated to the eigenvalue \( a^2 - b^2 \). However, since it must hold that \( a = 0 \) or \( b = 0 \), and \( A \) is non-positive, the only possibility is that \( a = 0 \), and therefore for any \( n \geq 1 \), \( b^2 = \lambda_n, \) and \( g = c\Phi_{\alpha,n} \) (for some \( c \in \mathbb{R} \)). Moreover, we have that \( h = -bg \) and \( k = 0 \). Thus, in this case we have two possible sets of solutions:

\[
\bar{\omega} = a + ib = i\sqrt{\lambda}, \ \text{and} \ \bar{\Psi} = \left( \begin{array}{c} i\Phi \Phi_{\alpha,n} \\ -\sqrt{\Phi_{\alpha,n}}, i\Phi \Phi_{\alpha,n} \end{array} \right) = ic \left( \begin{array}{c} \Phi \Phi_{\alpha,n} \\ i\Phi \Phi_{\alpha,n} \end{array} \right);
\]

or

\[
\bar{\omega} = a + ib = -i\sqrt{\lambda}, \ \text{and} \ \bar{\Psi} = \left( \begin{array}{c} i\Phi \Phi_{\alpha,n} \\ \sqrt{\Phi_{\alpha,n}}, i\Phi \Phi_{\alpha,n} \end{array} \right) = ic \left( \begin{array}{c} \Phi \Phi_{\alpha,n} \\ -i\Phi \Phi_{\alpha,n} \end{array} \right);
\]

- \( \bar{\omega} \neq 0 \) and \( g = 0 \): similarly to the previous case, we deduce that \( a = 0 \), \( b^2 = \lambda_n \) and \( f = c\Phi_{\alpha,n} \) for every \( n \geq 1 \) (for some \( c \in \mathbb{R} \)). Furthermore, \( h = 0 \) and \( k = bf \). Hence, also in this case, we have two possible sets of solutions:

\[
\bar{\omega} = a + ib = i\sqrt{\lambda}, \ \text{and} \ \bar{\Psi} = \left( \begin{array}{c} c\Phi \Phi_{\alpha,n} \\ i\sqrt{\Phi_{\alpha,n}}, c\Phi \Phi_{\alpha,n} \end{array} \right) = c \left( \begin{array}{c} \Phi \Phi_{\alpha,n} \\ i\Phi \Phi_{\alpha,n} \end{array} \right);
\]

or

\[
\bar{\omega} = a + ib = -i\sqrt{\lambda}, \ \text{and} \ \bar{\Psi} = \left( \begin{array}{c} c\Phi \Phi_{\alpha,n} \\ -\sqrt{\Phi_{\alpha,n}}, c\Phi \Phi_{\alpha,n} \end{array} \right) = c \left( \begin{array}{c} \Phi \Phi_{\alpha,n} \\ -i\Phi \Phi_{\alpha,n} \end{array} \right);
\]

- \( \bar{\omega} \neq 0 \), \( f \neq 0 \neq g \) and \( g \) is colinear with \( f \): let \( g = rf \) (with some \( r \in \mathbb{R} \)). Then \( f \) is an eigenfunction of \( A \), and

\[
(a^2 - b^2) - 2abr = (a^2 - b^2) + \frac{2ab}{r}.
\]

Thus,

\[
2ab(r + \frac{1}{r}) = 0.
\]

Since \( r \in \mathbb{R} \), then \( ab = 0 \), and we can reason as in previous cases: so, we get that \( a = 0 \) (from the non-positivity of \( A \)), and for every \( n \geq 1 \), \( b^2 = \lambda_n \),
and \( f = c\Phi_{\alpha,n} \) (for some \( c \in \mathbb{R} \)). Moreover, we obtain that \( g = c\Phi_{\alpha,n} \), \( h = 0 - bg \) and \( k = bf \), and this leads to two possibilities:

\[
\omega = i\sqrt{\lambda_{\alpha,n}}, \quad \text{and} \quad \tilde{\Psi} = c(1 + ir) \begin{pmatrix} \Phi_{\alpha,n} \\ i\sqrt{\lambda_{\alpha,n}}\Phi_{\alpha,n} \end{pmatrix};
\]

or

\[
\omega = -i\sqrt{\lambda_{\alpha,n}}, \quad \text{and} \quad \tilde{\Psi} = c(1 + ir) \begin{pmatrix} \Phi_{\alpha,n} \\ -i\sqrt{\lambda_{\alpha,n}}\Phi_{\alpha,n} \end{pmatrix};
\]

- \( \omega \neq 0 \) and \( f \) and \( g \) are free: in this case we note that the plane generated by \( f \) and \( g \) is stable under the action of \( A \). Let us denote by \( P_{f,g} \) the plane generated by \( f \) and \( g \), and by \( A_{f,g} \) the restriction of \( A \) to this plane. \( A_{f,g} \) is an endomorphism, and it is symmetric (and non-positive). Since

\[
A_{f,g} = \begin{pmatrix} a^2 - b^2 & -2ab \\ 2ab & a^2 - b^2 \end{pmatrix} = (a^2 - b^2) \text{Id}_{P_{f,g}} + 2ab \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]

the matrix that defines the action of \( A_{f,g} \) on the basis composed by \( f \) and \( g \) is

\[
\text{Mat}(A; f, g) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

So, since \( A_{f,g} \) is symmetric, \( \text{Mat}(A_{f,g}; f, g) \) is diagonalizable. This would imply that

\[
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

would also be diagonalizable, which is a contradiction.

Finally, one easily verifies that these necessary conditions are also sufficient. This concludes the proof of Lemma 6.2. \( \square \)

### 6.2. Two integral expressions for the solution of \((3.13)\).

Since the family \( \{\Phi_{\alpha,n}\}_{n \in \mathbb{N}} \) is an orthonormal basis of \( L^2(0,1) \), we can decompose the solution \( w(t) \) of \((3.13)\) under the form

\[
w(t, x) = \sum_{n=0}^{\infty} w_n(t) \Phi_{\alpha,n}(x).
\]

We decompose in the same way the nonlinear term

\[
r(x, t) := p(t)\mu(x)w(t, x) = \sum_{n=0}^{\infty} r_n(t) \Phi_{\alpha,n}(x)
\]

with

\[
r_n(t) = \langle p(t)\mu(\cdot)w(\cdot, t), \Phi_{\alpha,n} \rangle_{L^2(0,1)}.
\]

So, \((3.13)\) implies that the sequence \( (w_n(t))_{n \geq 0} \) satisfies

\[
\begin{align*}
& \begin{cases}
w''_0(t) = r_0(t), \\
w_0(0) = 1,
\end{cases} \quad \text{and} \quad \forall n \geq 1, \quad \begin{cases}
w''_n(t) + \lambda_{\alpha,n}w_n(t) = r_n(t), \\
w_n(0) = 0,
\end{cases}
\end{align*}
\]

We obtain that

\[
w_0(t) = 1 + \int_0^t r_0(s)(t - s) \, ds \quad \text{and} \quad w_n(t) = \int_0^t r_n(s) \sin \sqrt{\lambda_{\alpha,n}(t - s)} \, ds.
\]
Hence, the solution of (3.13) can be written as

\[(6.9) \quad w^{(p)}(x, t) = \left(1 + \int_0^t r_0(s)(t - s)ds\right) + \sum_{n=1}^{\infty} \left(\int_0^t r_n(s)\frac{\sin \sqrt{\lambda_{\alpha,n}(t - s)}}{\sqrt{\lambda_{\alpha,n}}}ds\right) \Phi_{\alpha,n}(x),\]

and

\[(6.10) \quad w_t^{(p)}(x, t) = \left(\int_0^t r_0(s)ds\right) + \sum_{n=1}^{\infty} \left(\sqrt{\lambda_{\alpha,n}} \int_0^t r_n(s)\frac{\cos \sqrt{\lambda_{\alpha,n}(t - s)}}{\sqrt{\lambda_{\alpha,n}}}ds\right) \Phi_{\alpha,n}(x),\]

or, equivalently,

\[
\begin{pmatrix}
  w^{(p)}(x, t) \\
  w_t^{(p)}(x, t)
\end{pmatrix}
= \left(\begin{array}{c}
w_0(t) \\
w'_0(t)
\end{array}\right) + \sum_{n=1}^{\infty} \left(\begin{array}{c}
\int_0^t r_n(s)\frac{\sin \sqrt{\lambda_{\alpha,n}(t - s)}}{\sqrt{\lambda_{\alpha,n}}}ds \Phi_{\alpha,n}(x) \\
\sqrt{\lambda_{\alpha,n}} \int_0^t r_n(s)\frac{\cos \sqrt{\lambda_{\alpha,n}(t - s)}}{\sqrt{\lambda_{\alpha,n}}}ds \Phi_{\alpha,n}(x)
\end{array}\right).
\]

Now, manipulating the above formula and we get

\[
\begin{pmatrix}
w^{(p)}(x, t) \\
w_t^{(p)}(x, t)
\end{pmatrix} - \left(\begin{array}{c}
w_0(t) \\
w'_0(t)
\end{array}\right) = \sum_{n=1}^{\infty} \frac{1}{2i \sqrt{\lambda_{\alpha,n}}} \left(\begin{array}{c}
\int_0^t r_n(s)(e^{i\sqrt{\lambda_{\alpha,n}(t - s)}} - e^{-i\sqrt{\lambda_{\alpha,n}(t - s)}})ds \Phi_{\alpha,n}(x) \\
\int_0^t r_n(s)(e^{i\sqrt{\lambda_{\alpha,n}(t - s)}} + e^{-i\sqrt{\lambda_{\alpha,n}(t - s)}})ds i \sqrt{\lambda_{\alpha,n}} \Phi_{\alpha,n}(x)
\end{array}\right)
\]

\[
= \sum_{n=1}^{\infty} \frac{1}{2i \sqrt{\lambda_{\alpha,n}}} \left(\begin{array}{c}
\int_0^t r_n(s)e^{-i\sqrt{\lambda_{\alpha,n} s}}ds \left(\Phi_{\alpha,n}(x) i \sqrt{\lambda_{\alpha,n}} \Phi_{\alpha,n}(x)\right) e^{i\sqrt{\lambda_{\alpha,n} t}} \\
\int_0^t r_n(s)e^{i\sqrt{\lambda_{\alpha,n} s}}ds \left(-i \sqrt{\lambda_{\alpha,n}} \Phi_{\alpha,n}(x)\right) e^{-i\sqrt{\lambda_{\alpha,n} t}}
\end{array}\right)
\]

\[
= \sum_{n \in \mathbb{Z}, n \neq 0} \frac{1}{2i \omega_{\alpha,n}} \left(\int_0^t r_{|n|}(s)e^{-i\omega_{\alpha,n} s}ds \left(\Phi_{\alpha,n}(x) i \omega_{\alpha,n} \Phi_{\alpha,n}(x)\right) e^{i\omega_{\alpha,n} t},\right.
\]

that can be expressed more compactly as

\[(6.11) \quad \begin{pmatrix}
w^{(p)}(x, t) \\
w_t^{(p)}(x, t)
\end{pmatrix} = \begin{pmatrix}
1 + \int_0^t r_0(s)(t - s)ds \\
\int_0^t r_0(s)ds
\end{pmatrix} + \sum_{n \in \mathbb{Z}, n \neq 0} \frac{1}{2i \omega_{\alpha,n}} \left(\int_0^t r_{|n|}(s)e^{-i\omega_{\alpha,n} s}ds \Psi_{\alpha,n}(x)e^{i\omega_{\alpha,n} t},\right.
\]

To lighten the notation, we rewrite (6.11) as

\[(6.12) \quad \begin{pmatrix}
w^{(p)}(x, T) \\
w_t^{(p)}(x, T)
\end{pmatrix} = g^{(p)}_0(T) + \sum_{n \in \mathbb{Z}, n \neq 0} \frac{1}{2i \omega_{\alpha,n}} g^{(p)}_n(T) \Psi_{\alpha,n}(x)e^{i\omega_{\alpha,n} T},\]
For this purpose, we will prove 

\( \Gamma_0^{(p)}(T) = \left( 1 + \int_0^T r_0(s)(T-s) \, ds \right) \left( \begin{array}{c} \gamma_{00}^{(p)}(T) \\ \gamma_{01}^{(p)}(T) \end{array} \right), \)

and 

\( \forall n \in \mathbb{Z}^*, \quad \gamma_n^{(p)}(T) = \int_0^T r_{|n|}(s) e^{-i\omega_{\alpha,n}s} \, ds, \)

where we recall that \( r_{|n|} \) is defined in (6.8).

Formula (6.12) shows the role of the functions \((x, t) \mapsto \Psi_{\alpha,n}(x) e^{i\omega_{\alpha,n}t}, \) which are solution of the homogeneous equation

\[(\Psi_{\alpha,n}(x) e^{i\omega_{\alpha,n}t}_t = A(\Psi_{\alpha,n}(x) e^{i\omega_{\alpha,n}t}).\]

6.3. A sufficient condition to prove Proposition 6.1, part a).

From Proposition 2.3, we already know that \((w(T), w_t(T)) \in D(A) \times H_0^1(0,1).\)

To prove the hidden regularity result, it is useful to consider the expression (6.12).

We have that

\[ w^{(p)}(T) = \sum_{n=1}^{\infty} \frac{1}{2i\omega_{\alpha,n}} \left( \gamma_n^{(p)}(T) e^{i\omega_{\alpha,n}T} - \gamma_{-n}^{(p)}(T) e^{-i\omega_{\alpha,n}T} \right) \Phi_{\alpha,n}(x), \]

hence, \( w^{(p)}(T) \in H_0^3(0,1) \) if and only if

\[ \sum_{n=1}^{\infty} \lambda_{\alpha,n}^3 \left| \frac{1}{2i\omega_{\alpha,n}} \left( \gamma_n^{(p)}(T) e^{i\omega_{\alpha,n}T} - \gamma_{-n}^{(p)}(T) e^{-i\omega_{\alpha,n}T} \right) \right|^2 < \infty. \]

Moreover,

\[ w_t^{(p)}(T) = \sum_{n=1}^{\infty} \frac{1}{2} \left( \gamma_n^{(p)}(T) e^{i\omega_{\alpha,n}T} + \gamma_{-n}^{(p)}(T) e^{-i\omega_{\alpha,n}T} \right) \Phi_{\alpha,n}(x), \]

thus, \( w_t^{(p)}(T) \in H_0^2(0,1) \) if and only if

\[ \sum_{n=1}^{\infty} \lambda_{\alpha,n}^2 \left| \frac{1}{2} \left( \gamma_n^{(p)}(T) e^{i\omega_{\alpha,n}T} + \gamma_{-n}^{(p)}(T) e^{-i\omega_{\alpha,n}T} \right) \right|^2 < \infty. \]

Therefore,

\[ \sum_{n \in \mathbb{Z}} \lambda_{\alpha,|n|}^2 \left| \gamma_n^{(p)}(T) \right|^2 < \infty \]

\[ \implies (w^{(p)}(T), w_t^{(p)}(T)) \in H_0^3(0,1) \times D(A). \]

In what follows, we prove that

\[ \sum_{n \in \mathbb{Z}} \lambda_{\alpha,|n|}^2 \left| \gamma_n^{(p)}(T) \right|^2 < \infty, \]

or, equivalently, using definition (6.14) of \( \gamma_n^{(p)}(T) \) and definition (6.8) of \( r_n, \) that

\[ \sum_{n \in \mathbb{Z}} \lambda_{\alpha,|n|}^2 \left| \int_0^T \langle p(s) \mu(x) w^{(p)}(\cdot, s), \Phi_{\alpha,|n|} \rangle_{L^2(0,1)} e^{-i\omega_{\alpha,n}s} \, ds \right|^2 < \infty. \]

For this purpose, we will prove
• A general convergence result (see Lemma 6.3 in section 6.4):
\[
\sum_{n=1}^{\infty} \lambda_{\alpha,n}^2 \left| \int_0^T \langle p(s) g(\cdot, s), \Phi_{\alpha,n} \rangle_{L^2(0,1)} e^{i\omega_{\alpha,n} s} ds \right|^2 < \infty
\]
if \( g \in C^0([0,T], V^2_{\alpha}(0,1)) \),
• A regularity result (see Lemma 6.7 in section 6.5.a):
\[
\mu \in V^2_{\alpha}(0,1), w \in C^0([0,T], D(A)) \implies \mu w \in C^0([0,T], V^{2,0}_{\alpha}(0,1)).
\]

Then, (6.16) will easily follow, see section 6.5.b. (The intermediate lemmas 6.4-6.6 are useful to prove Lemma 6.3.)

6.4. A general regularity result.
In this section a fundamental role will be played by the space \( V^{2,0}_{\alpha}(0,1) \) defined in (3.12).

Lemma 6.3. Let \( T > 0, p \in L^2(0,T), g \in C^0([0,T], V^{2,0}_{\alpha}(0,1)) \). Consider the sequence \( (S_n^{(p,g)})_{n \geq 1} \) defined by

\[
(6.17) \quad \forall n \geq 1, \quad S_n^{(p,g)} = \int_0^T p(s) \langle g(\cdot, s), \Phi_{\alpha,n} \rangle_{L^2(0,1)} e^{i\sqrt{\lambda_{\alpha,n}} s} ds.
\]

Then, \( (S_n^{(p,g)})_{n \geq 1} \) satisfies

\[
(6.18) \quad \sum_{n=1}^{\infty} \lambda_{\alpha,n}^2 |S_n^{(p,g)}|^2 < \infty,
\]
and moreover, there exists a constant \( C(\alpha, T) > 0 \) independent of \( p \in L^2(0,T) \) and of \( g \in C^0([0,T], V^{2,0}_{\alpha}(0,1)) \) such that

\[
(6.19) \quad \left( \sum_{n=1}^{\infty} \lambda_{\alpha,n}^2 |S_n^{(p,g)}|^2 \right)^{1/2} \leq C(\alpha, T) \|p\|_{L^2(0,T)} \|g\|_{C^0([0,T], V^{2,0}_{\alpha}(0,1))}.
\]

Proof of Lemma 6.3. We proceed as in [6], and the properties of the space \( V^{2,0}_{\alpha}(0,1) \) will be crucial to overcome some new difficulties. (Note that \( V^{2,0}_{\alpha}(0,1) = H^2_{\alpha}(0,1) \) for \( \alpha \in [1,2) \).)

First, we note that

\[
(6.20) \quad S_n^{(p,g)} = \int_0^T p(s) \langle g(\cdot, s), \Phi_{\alpha,n} \rangle_{L^2(0,1)} e^{i\sqrt{\lambda_{\alpha,n}} s} ds
\]

\[
= \int_0^T p(s) \langle g(\cdot, s), -A \Phi_{\alpha,n} \rangle_{L^2(0,1)} e^{i\sqrt{\lambda_{\alpha,n}} s} ds
\]

\[
= -\frac{1}{\lambda_{\alpha,n}} \int_0^T p(s) \langle g(\cdot, s), x^n \Phi'_{\alpha,n} \rangle_{L^2(0,1)} e^{i\sqrt{\lambda_{\alpha,n}} s} ds.
\]
Then, integrating by parts, we have
\[
\langle g(\cdot, s), (x^\alpha \Phi_{\alpha,n})' \rangle_{L^2(0,1)} = \int_0^1 g(x, s) (x^\alpha \Phi_{\alpha,n}(x))' \, dx \\
= [g(x, s) x^\alpha \Phi_{\alpha,n}(x)]_0^1 - \int_0^1 g'(x, s) x^\alpha \Phi_{\alpha,n}'(x) \, dx \\
= [g(x, s) x^\alpha \Phi_{\alpha,n}'(x)]_0^1 - [x^\alpha g'(x, s) \Phi_{\alpha,n}(x)]_0^1 + \int_0^1 (x^\alpha g'(x, s))' \Phi_{\alpha,n}(x) \, dx.
\]

Using the above expression of the scalar product in (6.20), we get
\[
(6.21) \quad -\lambda_{\alpha,n} S_n^{(p,g)} = S_n^{(1)} - S_n^{(2)} + S_n^{(3)},
\]
with
\[
(6.22) \quad \forall i \in \{1, 2, 3\}, \quad S_n^{(i)} = \int_0^T h_n^{(i)}(s) e^{i\sqrt{\lambda_{\alpha,n}} s} \, ds,
\]
and the associated functions
\[
(6.23) \quad h_n^{(1)}(s) = p(s) [g(x, s) x^\alpha \Phi_{\alpha,n}(x)]_{x=0}^{x=1},
\]
\[
(6.24) \quad h_n^{(2)}(s) = p(s) [x^\alpha g_x(x, s) \Phi_{\alpha,n}(x)]_{x=0}^{x=1},
\]
\[
(6.25) \quad h_n^{(3)}(s) = p(s) ((x^\alpha g_x)_x, \Phi_{\alpha,n})_{L^2(0,1)}.
\]

To conclude the proof we appeal to the following results.

6.4.a. The term $S_n^{(1)}$ associated to $h^{(1)}$.

**Lemma 6.4.** Let $T > 0$, $p \in L^2(0,T)$ and $g \in C^0([0,T], V_{\alpha}^{(2,0)}(0,1))$. Then, function $h_n^{(1)}$ defined in (6.23) satisfies
\[
\forall s \in [0,T], \quad h_n^{(1)}(s) = 0.
\]

**Proof of Lemma 6.4.** Since $g(\cdot, s) \in H_0^2(0,1)$, then $g(\cdot, s) \in H^1(1/2,1)$ and has a finite limit as $x \to 1$. Hence, thanks to the Neumann boundary condition at $x = 1$ for $\Phi_{\alpha,n}$, we have
\[
g(x, s) x^\alpha \Phi_{\alpha,n}'(x) \to 0 \quad \text{as } x \to 1.
\]

When $x \to 0$, we have to distinguish the cases of weak and strong degeneracy:

- $\alpha \in [0,1)$: first, we notice that $g(\cdot, s)$ has a finite limit as $x \to 0$. Indeed,
\[
g_x(x, s) = (x^{\alpha/2} g_x(x, s)) x^{-\alpha/2},
\]
and since $x \to x^{\alpha/2} g_x(x, s)$ and $x \to x^{-\alpha/2}$ belong to $L^2(0,1)$, then $g_x(\cdot, s) \in L^1(0,1)$, which implies that $g(\cdot, s)$ has a finite limit as $x \to 0$. Therefore,
\[
g(x, s) x^\alpha \Phi_{\alpha,n}'(x) \to 0 \quad \text{as } x \to 0
\]

- $\alpha \in [1,2)$: observe that $g$ can be unbounded as $x \to 0$. However, the series of $\Phi_{\alpha,n}$ obtained thanks to (5.25) gives that
\[
\exists C_{\alpha,n} : |x^\alpha \Phi_{\alpha,n}(x)| \leq C_{\alpha,n} x, \quad \forall x \in (0,1).
\]

We claim that $x \to x g(x, s)$ has a finite limit as $x \to 0$. Indeed,
\[
(x g(x, s))_x = g(x, s) + x g_x(x, s) = g(x, s) + x^{\alpha/2} g_x(x, s) x^{1-\alpha/2},
\]
and since \( g(\cdot, s) \in L^2(0, 1), x \mapsto x^{\alpha/2}g_x(x, s) \in L^2(0, 1) \) and \( x \mapsto x^{1-\alpha/2} \in L^\infty(0, 1) \), we have that \( (xg(x, s))_x \in L^1(0, 1) \). Therefore \( x \mapsto xg(x, s) \) has a finite limit as \( x \to 0 \):

\[
\exists \ell(s), \quad xg(x, s) \to \ell(s) \quad \text{as} \quad x \to 0.
\]

However, since \( g(\cdot, s) \in L^2(0, 1) \), we get that \( x \mapsto \frac{\ell(s)}{x} \in L^2(0, 1) \), which is possible only if \( \ell(s) = 0 \). Thus,

\[
xg(x, s) \to 0 \quad \text{as} \quad x \to 0,
\]

and so

\[
g(x, s)x^{\alpha}\Phi'_{\alpha,n}(x) \to 0 \quad \text{as} \quad x \to 0.
\]

\[ \square \]

6.4.b. The term \( S^{(2)}_\alpha \) associated to \( h^{(2)} \).

**Lemma 6.5.** Let \( T > 0 \), \( p \in L^2(0, T) \) and \( g \in C^0([0, T], V^{(2,0)}_\alpha(0, 1)) \). Then, function \( h^{(2)}_\alpha \) defined in (6.24) belongs to \( L^2(0, T) \) and there exists \( C(\alpha, T) > 0 \) independent of \( p \in L^2(0, T) \) and \( g \in C^0([0, T], V^{(2,0)}_\alpha(0, 1)) \) and of \( n \geq 1 \) such that

\[
\|h^{(2)}_\alpha\|_{L^2(0, T)} \leq C(\alpha, T) \|p\|_{L^2(0, T)} \|g\|_{C^0([0, T], V^{(2,0)}_\alpha(0, 1))}.
\]

Furthermore,

\[
\sum_{n=1}^\infty |S^{(2)}_n|^2 < \infty,
\]

and there exists \( C_2(\alpha, T) > 0 \) independent of \( p \in L^2(0, T) \) and \( g \in C^0([0, T], V^{(2,0)}_\alpha(0, 1)) \) such that

\[
\sum_{n=1}^\infty |S^{(2)}_n|^2 \leq C_2(\alpha, T)^2 \|p\|^2_{L^2(0, T)} \|g\|^2_{C^0([0, T], V^{(2,0)}_\alpha(0, 1))}.
\]

**Proof of Lemma 6.5.** We recall that

\[
h^{(2)}_\alpha(s) = p(s)(x^\alpha g_x(x, s)\Phi_{\alpha,n}(x))|_{x=1} - p(s)(x^\alpha g_x(x, s)\Phi_{\alpha,n}(x))|_{x=0}.
\]

Using the definition of \( V_\alpha^{(2,0)}(0, 1) \) and respectively (5.16) when \( \alpha \in [0, 1) \) and (5.34) when \( \alpha \in [1, 2) \) in (6.24), we obtain that

\[
(x^\alpha g_x(x, s)\Phi_{\alpha,n}(x))|_{x=0} = 0.
\]

Therefore, from (6.16) and (5.34), we have

\[
|h^{(2)}_\alpha(s)| = |p(s)(x^\alpha g_x(x, s)\Phi_{\alpha,n}(x))|_{x=1} - |p(s)(x^\alpha g_x(x, s))|_{x=1} = \sqrt{2-\alpha} |p(s)(x^\alpha g_x(x, s))|_{x=1}.
\]

Moreover, since \( g(\cdot, s) \in H^2_\alpha(0, 1) \), then \( x \mapsto x^\alpha g_x(x, s) \) belongs to \( H^1(0, 1) \). By the continuous injection of \( H^1(0, 1) \) into \( C^0([0, 1]) \), there exists a positive constant \( C_\infty \) such that

\[
|g(\cdot, s)|_{H^2_\alpha(0, 1)} \leq C_\infty \|g\|_{C^0([0, T], V^{(2,0)}_\alpha(0, 1))}.
\]

Therefore, we get

\[
|h^{(2)}_\alpha(s)| \leq C_\infty \sqrt{2-\alpha} |p(s)| \|g\|_{C^0([0, T], V^{(2,0)}_\alpha(0, 1))}, \quad \forall n \geq 1.
\]
hence, \( h_n^{(2)} \in L^2(0, T) \) and

\[ (6.30) \quad \exists C'_n > 0 : \| h_n^{(2)} \|_{L^2(0, T)} \leq C'_n \| p \|_{L^2(0, T)} \| g \|_{C^0([0, T], V_{\text{reg}}^{(2, 0)}(0, 1))}, \forall n \geq 1. \]

This proves (6.26).

Now, we prove (6.27) and (6.28). These results follow from (6.26) and from classical results of Ingham type (we refer, in particular, to [7, Proposition 19, Theorem 6 and Corollary 4]). We have seen in Proposition 3.1, when \( \alpha \) is of finite support and complex values, it holds that there exist \( \alpha \) such that, for every sequence \( (c_n)_{n \geq 1} \) with finite support and complex values, it holds that

\[ \sqrt{\lambda_{\alpha, n+1}} - \sqrt{\lambda_{\alpha, n}} \to \frac{2 - \alpha}{2\pi} \quad \text{as} \quad n \to \infty. \]

Furthermore, a stronger gap condition holds

\[ \forall \alpha \in [0, 2), \quad \sqrt{\lambda_{\alpha, n+1}} - \sqrt{\lambda_{\alpha, n}} \geq \frac{2 - \alpha}{2\pi}, \]

hence we are allowed to apply a general result of Ingham (see, e.g., [23, Theorem 4.3], generalized by Haraux [22], see also [7, Theorem 6]), and we derive that given

\[ T_1 > T_0 : \frac{2\pi}{2 - \alpha} = \frac{4}{2 - \alpha}, \]

there exist \( C_1(\alpha, T_1), C_2(\alpha, T_1) > 0 \) such that, for every sequence \( (c_n)_{n \geq 1} \) with finite support and complex values, it holds that

\[ (6.31) \quad C_1 \sum_{n=1}^{\infty} |c_n|^2 \leq \int_0^{T_1} \sum_{n=1}^{\infty} c_n e^{i\sqrt{n\lambda_{\alpha, n}}t} \| g \|_{L^2(0, T)}^2 \leq C_2 \sum_{n=1}^{\infty} |c_n|^2. \]

Therefore, if \( T_1 > T_0, (6.31) \) implies that the sequence \( (e^{i\sqrt{n\lambda_{\alpha, n}}t})_{n \geq 1} \) is a Riesz basis of \( \text{Vect} \{ e^{i\sqrt{n\lambda_{\alpha, n}}t}, n \geq 1 \} \subset L^2(0, T_1) \) (see [7, Proposition 19, point (2)]). So, for all \( T > 0 \), there exists a positive constant \( C_I(\alpha, T) \) such that

\[ (6.32) \quad \forall f \in L^2(0, T), \quad \sum_{n=1}^{\infty} \int_0^{T} |f(t) e^{i\sqrt{n\lambda_{\alpha, n}}t} dt|^2 \leq C_I(\alpha, T) \| f \|_{L^2(0, T)}^2 \]

by applying [7, Proposition 19, point (3)] for \( T > T_0 \), or by extending \( f \) by 0 on \( (T, T_0) \) for \( T \leq T_0 \), see also [7, Corollary 4]). We can now conclude the proof of Lemma 6.5. First, we note from (6.29) that

\[ \int_0^{T} h_n^{(2)}(t) e^{i\sqrt{n\lambda_{\alpha, n}}t} dt = \int_0^{T} p(t) (x^\alpha g_x(x, t) \Phi_{\alpha, n}(x))_{|x=1} e^{i\sqrt{n\lambda_{\alpha, n}}t} dt \]

\[ = \sqrt{2 - \alpha} \int_0^{T} p(t) g_x(1, t) e^{i\sqrt{n\lambda_{\alpha, n}}t} dt, \]

and then we can apply (6.32) to the function \( t \mapsto p(t) g_x(1, t) \) (which is independent of \( n \)), and we obtain that

\[ \sum_{n=1}^{\infty} |S_n^{(2)}|^2 = \sum_{n=1}^{\infty} \int_0^{T} h_n^{(2)}(s) e^{i\sqrt{n\lambda_{\alpha, n}}s} ds \leq (2 - \alpha) \sum_{n=1}^{\infty} \int_0^{T} p(t) g_x(1, t) e^{i\sqrt{n\lambda_{\alpha, n}}t} dt \]

\[ \leq (2 - \alpha) C_I(\alpha, T) \| p \|_{L^2(0, T)} \| g \|_{C^0([0, T], V_{\text{reg}}^{(2, 0)}(0, 1))}^2, \]

This concludes the proof of Lemma 6.5. \( \square \)
6.4.c. The term $S^{(3)}_n$ associated to $h^{(3)}$.

Finally, we analyze $h^{(3)}$, and we prove the following

**Lemma 6.5.** Let $T > 0$, $p \in L^2(0,T)$ and $g \in C^0([0,T], V\alpha^{(2,0)}(0,1))$. Then, the sequence $(S^{(3)}_n)_{n \geq 1}$ satisfies

\[
\sum_{n=1}^{\infty} |S^{(3)}_n|^2 < \infty,
\]

and there exists $C_3(\alpha, T) > 0$ independent of $p \in L^2(0,T)$ and of $g \in C^0([0,T], V\alpha^{(2,0)}(0,1))$ such that

\[
\sum_{n=1}^{\infty} |S^{(3)}_n|^2 \leq C_3(\alpha, T)^2 \|p\|^2_{L^2(0,T)} \|g\|^2_{C^0([0,T], V\alpha^{(2,0)}(0,1))}.
\]

**Proof of Lemma 6.5.** First, let us prove that $\sum_n |S^{(3)}_n|^2 < \infty$. Notice that

\[
|S^{(3)}_n|^2 \leq \left( \int_0^T |h^{(3)}_n(s)|^2 \, ds \right)^2 \leq \|p\|^2_{L^2(0,T)} \left( \int_0^T \left| \left( x^\alpha g_x \right)_x \varphi_{\alpha,n} \right|^2 \, ds \right),
\]

hence,

\[
\sum_{n=1}^{\infty} |S^{(3)}_n|^2 \leq \|p\|^2_{L^2(0,T)} \left( \int_0^T \sum_{n=1}^{\infty} \left| \left( x^\alpha g_x \right)_x \varphi_{\alpha,n} \right|^2 \, ds \right)
\]

\[
\leq \|p\|^2_{L^2(0,T)} \left( \int_0^T \left| \left( x^\alpha g_x \right)_x \right|^2 \, ds \right)
\]

\[
\leq \|p\|^2_{L^2(0,T)} \int_0^T \|g\|^2_{C^0([0,T], V\alpha^{(2,0)}(0,1))} \, dt.
\]

This concludes the proof of Lemma 6.5. □

The proof of Lemma 6.5 follows directly from (6.21) and Lemmas 6.4, 6.6 and 6.6. □

6.5. **Proof of Proposition 6.1, part a.**

In this section we prove that Lemma 6.3 implies that (6.16) holds true, and then from (6.13) we deduce that $(w^{(p)}(T), u^{(p)}(T)) \in H^{3}_0(0, 1) \times D(A)$, which is the aim in point a) of Proposition 6.1.

6.5.a. A regularity result.

**Lemma 6.7.** If $\mu \in V_\alpha^2(0,1)$ (defined in (3.14)) and $w \in C^0([0,T], D(A))$, then $\mu w \in C^0([0,T], V_\alpha^{(2,0)}(0,1))$. Moreover, there exists $C(\alpha, T) > 0$, independent of $\mu \in V_\alpha^2$ and $w \in C^0([0,T], D(A))$, such that

\[
\|\mu w\|^2_{C^0([0,T], V_\alpha^{(2,0)}(0,1))} \leq C(\alpha, T) \|\mu\|_{V_\alpha^2} \|w\|^2_{C^0([0,T], D(A))}.
\]

**Proof of Lemma 6.7.** We distinguish the cases $\alpha \in [0,1)$ and $\alpha \in [1,2)$.

- $\alpha \in [0,1)$: let $\mu \in V_\alpha^{(2,\infty)}(0,1)$ and $w \in V_\alpha^{(2,0)}(0,1)$. As we have already shown, $\mu \in H_\alpha^2(0,1)$ implies that $\mu \in L^\infty(0,1)$, and so $\mu w \in L^2(0,1)$. Moreover, we have that $(\mu w)_x = \mu_x w + \mu w_x \in L^1(0,1)$ because $\mu, w \in L^\infty(0,1)$.

  Thus, $\mu w$ is absolutely continuous on $[0,1]$. Furthermore, $x^{\alpha/2}(\mu w)_x = (x^{\alpha/2}\mu_x)w + (x^{\alpha/2}w_x)\mu \in L^2(0,1)$. Therefore, $x^{\alpha/2}(\mu w)_x = (x^{\alpha/2}\mu_x)w + (x^{\alpha/2}w_x)\mu \in L^2(0,1)$ because $x^{\alpha/2}\mu_x, x^{\alpha/2}w_x \in L^2(0,1)$ and $\mu, w \in L^\infty(0,1)$. We observe that $(x^{\alpha}(\mu w)_x)_x = (x^{\alpha}\mu_x)w + (x^{\alpha}w_x)\mu + (x^{\alpha-1}\mu)_x w$.


hence, we deduce that we have

We use this development in (3.13) to find the equation satisfied by the supposed

Thus \( \mu w \in V_{1}^{(2,0)}(0,1) \). We conclude that if \( \mu \in V_{1}^{(2,\infty)}(0,1) \) and \( w \in C^{0}([0,T],D(A)) \), then \( \mu w \in C^{0}([0,T],V_{1}^{(2,0)}(0,1)) \).

- \( \alpha \in [1,2) \): we observe that \( \mu \in V_{1}^{(2,\infty,\infty)}(0,1) \) implies that \( |\mu| \leq \frac{C}{x^{\alpha/2}} \), hence \( \mu \in L^{1}(0,1) \). Therefore, \( \mu \in L^{\infty}(0,1) \) and so \( \mu w \in L^{2}(0,1) \). Moreover, \( x^{\alpha/2}(\mu w)_{x} = (x^{\alpha/2} \mu)_{x} + (x^{\alpha/2} w)_{x} \mu \in L^{2}(0,1) \) because \( x^{\alpha/2} \mu \in L^{\infty}(0,1) \), and \( x^{\alpha/2} w \in L^{2}(0,1) \). Furthermore, since \( x^{\alpha/2} w \in L^{2}(0,1) \) and \( \mu \in L^{\infty}(0,1) \), we have \( (x^{\alpha/2} w)_{x} \mu \in L^{2}(0,1) \). Hence \( x^{\alpha/2}((\mu w)_{x})_{x} = (x^{\alpha/2} \mu)_{x} w + (x^{\alpha/2} w)_{x} \mu + 2x^{\alpha/2} \mu w \). Since \( x^{\alpha/2} \mu \in L^{\infty}(0,1) \), we have \( x^{\alpha/2} \mu_{x} \in L^{2}(0,1) \) and \( x^{\alpha/2} \mu_{x} \mu \in L^{2}(0,1) \). Moreover, since \( x^{\alpha/2} x \mu w \in L^{2}(0,1) \) and \( \mu \in L^{\infty}(0,1) \), it holds that \( (x^{\alpha/2} x w)_{x} \mu \in L^{2}(0,1) \). Concerning the term \( 2x^{\alpha} \mu_{x} w \), we note that \( \mu \in V_{1}^{(2,\infty,\infty)}(0,1) \) implies that \( |x^{\alpha/2} \mu_{x} w| \leq Cx^{\alpha/2} |w_{x}| \in L^{2}(0,1) \). Therefore, \( \mu w \in H_{V_{1}}^{2}(0,1) \). Finally, \( \mu w \in H_{V_{1}}^{2}(0,1) \), with \( \alpha \in [1,2) \), implies that \( x^{\alpha} \mu w_{x}^{2} \in L^{2}(0,1) \).

So, we have proved that \( \mu w \in V_{1}^{(2,0)}(0,1) \). And, if \( \mu \in V_{1}^{(2,\infty,\infty)}(0,1) \) and \( w \in C^{0}([0,T],D(A)) \), then \( \mu w \in C^{0}([0,T],V_{1}^{(2,0)}(0,1)) \). This concludes the proof of Lemma 6.7.

\[ \square \]

6.5.b. Proof of Proposition 6.1, part a).

Using formula (6.14), Lemma 6.4 (with \( w = w^{(p)} \)) and Lemma 6.6, we obtain that (6.16) holds true and then (6.15) shows that \( (w^{(p)}(T), w^{(p)}(T)) \in H_{V_{1}}^{3}(0,1) \times D(A) \), which proves the validity of point a) of Proposition 6.1.

\[ \square \]

6.6. Proof of Proposition 6.1, part b).

In Proposition 6.1, part a), we have proved that \( \Theta_{T} \) maps \( L^{2}(0,T) \) into \( H_{V_{1}}^{3}(0,1) \times D(A) \). Now we show that \( \Theta_{T} \) is differentiable at every \( p \in L^{2}(0,T) \). Let \( p_{0}, q \in L^{2}(0,T) \). Then, consider \( w^{(p_{0})} \) solution of (3.13) with \( p = p_{0} \), and \( w^{(p_{0}+q)} \) solution of (3.13) with \( p = p_{0} + q \).

Formally, let us write a limited development of \( w^{(p_{0}+q)} \) with respect to \( q \):

\[ w^{(p_{0}+q)} = w^{(p_{0})} + W_{1}(q) + \cdots \]

We use this development in (3.13) to find the equation satisfied by the supposed first order term \( W_{1}(q) \): denoting

\[ Pw := w_{tt} - (x^{a} w_{x})_{x}, \]

we have

\[ P(w^{(p_{0})} + W_{1}(q) + \cdots) = (p_{0}(t) + q(t)) \mu(x) \left( w^{(p_{0})} + W_{1}(q) + \cdots \right), \]

hence, we deduce that \( W_{1}(q) \) is solution of

\[ PW_{1}(q) = p_{0}(t) \mu(x) W_{1}(q) + q(t) \mu(x) w^{(p_{0})}, \]
which is the motivation to consider $W_1(q)$ as the solution of (6.3) with $p = p_0$, that is,

$$W_1(q) = W^{(p_0,q)}.$$

So, we introduce

$$v^{(p_0,q)} := w^{(p_0,q)} - w^{(p_0)} - W^{(p_0,q)},$$

which allows us to write

$$\Theta_T(p_0 + q) = \Theta_T(p_0) + (W^{(p_0,q)}(T), W_t^{(p_0,q)}(T)) + (v^{(p_0,q)}(T), v_t^{(p_0,q)}(T)).$$

We are going to prove the following lemmas.

**Lemma 6.8.** The application

$$L^2(0,T) \to H^1_{(0)}(0,1) \times D(A)$$

$$q \mapsto (W^{(p_0,q)}(T), W_t^{(p_0,q)}(T))$$

is well-defined, linear and continuous.

and

**Lemma 6.9.** The application

$$L^2(0,T) \to H^1_{(0)}(0,1) \times D(A)$$

$$q \mapsto (v^{(p_0,q)}(T), v_t^{(p_0,q)}(T))$$

is well-defined, and satisfies

$$\frac{\|v^{(p_0,q)}(T), v_t^{(p_0,q)}(T)\|_{H^1_{(0)}(0,1) \times D(A)}}{\|q\|_{L^2(0,T)}} \to 0, \quad \text{as} \quad \|q\|_{L^2(0,T)} \to 0.$$

Then, we conclude that $\Theta_T$ is differentiable at $p_0$ and that

$$D\Theta_T(p_0) \cdot q = (W^{(p_0,q)}(T), W_t^{(p_0,q)}(T)).$$

**6.6.a. Proof of Lemma 6.8.**

First, we prove that $(W^{(p_0,q)}(T), W_t^{(p_0,q)}(T)) \in H^1_{(0)}(0,1) \times D(A)$. We observe that problem (6.3) is well-posed. Indeed, from Proposition 2.3 we deduce that $w^{(p_0)} \in C^1([0,T], D(A))$, and by applying Lemma 5.5 we obtain that $\mu w^{(p_0)} \in C^0([0,T], V^2_{\alpha}(0,1))$. Therefore, $\mu w^{(p_0)} \in C^0([0,T], H_{\alpha}^1(0,1))$. Furthermore, we have that $q \mu w^{(p_0)} \in L^2(0,T; H_{\alpha}^1(0,1))$. Thus, we can apply Proposition 2.3 to (6.3) (taking $f = q \mu w^{(p_0)}$), obtaining that

$$(W^{(p_0,q)}, W_t^{(p_0,q)}) \in C^0([0,T], D(A) \times H_{\alpha}^1(0,1)).$$

Moreover, (2.13) gives that

$$\frac{\|W^{(p_0,q)}\|_{C^0([0,T], D(A))} + \|W_t^{(p_0,q)}\|_{C^0([0,T], H^1_{\alpha}(0,1))}}{\|q\|_{L^2(0,T)}} \leq C(T) \|\mu w^{(p_0)}\|_{L^2(0,T; H_{\alpha}^1(0,1))}.$$

We can now decompose $(W^{(p_0,q)}(T), W_t^{(p_0,q)}(T))$ as follows: denoting

$$R_n(s) = \langle p_0(s)\mu(\cdot)W^{(p_0,q)}(\cdot, s) + q(s)\mu(\cdot)w^{(p_0)}(\cdot, s), \Phi_{\alpha,n}(\cdot) \rangle_{L^2(0,1)},$$
we have

\[
(6.40) \quad \left( W^{(p_0,q)}(x,T), W^{(p_0,q)}_t(x,T) \right) = I_0^{(p_0,q)}(T) + \sum_{n \in \mathbb{Z}^*} \frac{1}{2i\omega_{\alpha,n}} \gamma_{n}^{(p_0,q)}(T) \Phi_{\alpha,n}(x) e^{i\omega_{\alpha,n} T},
\]

with

\[
(6.41) \quad I_0^{(p_0,q)}(T) = \left( \int_0^T R_0(s)(T-s) \, ds, \int_0^T R_0(s) \, ds \right) = \left( \begin{array}{c} \gamma_{00}^{(p_0,q)}(T) \\ \gamma_{01}^{(p_0,q)}(T) \end{array} \right),
\]

and

\[
(6.42) \quad \forall n \in \mathbb{Z}^*, \quad \gamma_{n}^{(p_0,q)}(T) = \int_0^T R_n(s) e^{-i\omega_{\alpha,n} s} \, ds.
\]

Moreover, the following implication holds true

\[
(6.43) \quad \sum_{n \in \mathbb{Z}} \lambda_{n}^{2} |\gamma_{n}^{(p_0,q)}(T)|^2 < \infty \quad \implies \quad (W^{(p_0,q)}(T), W^{(p_0,q)}_t(T)) \in H^3(0,1) \times D(A).
\]

Therefore, to show that \((W^{(p_0,q)}(T), W^{(p_0,q)}_t(T)) \in H^3(0,1) \times D(A)\), we have to prove the convergence of the above series. We decompose as follows: \(\forall n \neq 0\)

\[
(6.44) \quad \gamma_{n}^{(p_0,q)}(T) = \int_0^T p_0(s) \langle \mu(\cdot)W^{(p_0,q)}(\cdot), \Phi_{\alpha,n}(\cdot) \rangle_{L^2(0,1)} e^{-i\omega_{\alpha,n} s} \, ds
\]

\[
+ \int_0^T q(s) \langle \mu(\cdot)u^{(p_0)}(\cdot), \Phi_{\alpha,n}(\cdot) \rangle_{L^2(0,1)} e^{-i\omega_{\alpha,n} s} \, ds
\]

\[
= : \gamma_{n}^{(p_0,\mu,W^{(p_0,q)})}(T) + \gamma_{n}^{(q,\mu,u^{(p_0)})}(T).
\]

We apply Lemma [6.3] first choosing \(p = p_0\) and \(g = \mu W^{(p_0,q)}\). Since \(p_0 \in L^2(0,T)\) and \(\mu W^{(p_0,q)} \in C^0([0,T], V_{\alpha}^{(2,0)}(0,1))\) we obtain

\[
(6.45) \quad \sum_{n \in \mathbb{Z}^*} \lambda_{n}^{2} |\gamma_{n}^{(p_0,\mu,W^{(p_0,q)})}(T)|^2 < \infty,
\]

and furthermore

\[
(6.46) \quad \left( \sum_{n \in \mathbb{Z}^*} \lambda_{n}^{2} |\gamma_{n}^{(p_0,\mu,W^{(p_0,q)})}(T)|^2 \right)^{1/2}
\]

\[
\leq C(\alpha, T) \|p_0\|_{L^2(0,T)} \|\mu W^{(p_0,q)}\|_{C^0([0,T], V_{\alpha}^{(2,0)}(0,1))}
\]

\[
\leq C'(\alpha, T) \|p_0\|_{L^2(0,T)} \|\mu\|_{V_{\alpha}^2} \|W^{(p_0,q)}\|_{C^0([0,T], D(A))}
\]

\[
\leq C''(\alpha, T) \|p_0\|_{L^2(0,T)} \|\mu\|_{V_{\alpha}^2} \|q\|_{L^2(0,T)} \|u^{(p_0)}\|_{C^0([0,T], H_{\alpha}^1(0,1))}
\]

\[
\leq C'''(\alpha, T) \|p_0\|_{L^2(0,T)} \|q\|_{L^2(0,T)} \|\mu\|_{V_{\alpha}^2} \|u^{(p_0)}\|_{C^0([0,T], D(A))},
\]

In the same way, we apply Lemma [6.3] with \(p = q\) and \(g = \mu u^{(p_0)}\) and we get

\[
(6.47) \quad \sum_{n \in \mathbb{Z}^*} \lambda_{n}^{2} |\gamma_{n}^{(q,\mu,u^{(p_0)})}(T)|^2 < \infty.
\]
and, moreover,

\[
\left( \sum_{n \in \mathbb{Z}^*} \lambda_{\alpha, n}^2 |\gamma_n^{(q, \mu, w^{(p_0)})}(T)|^2 \right)^{1/2} 
\]

(6.48)

\[
\leq C(\alpha, T) \|q\|_{L^2(0,T)} \|\mu w^{(p_0)}\|_{C^0([0,T], V_{\mu,0}^2(0,1))} 
\leq C'(\alpha, T) \|q\|_{L^2(0,T)} \|\mu\|_{V_{\mu,0}^2(0,1)} \|w^{(p_0)}\|_{C^0([0,T], D(A))}.
\]

We have proved that

\[
\sum_{n \in \mathbb{Z}^*} \lambda_{\alpha, n}^2 |\gamma_n^{(q, \mu, w^{(p_0)})}(T)|^2 < \infty,
\]

so, from (6.43) we have that

\[
(W^{(p_0, q)}(T), W_t^{(p_0, q)}(T)) \in H^3_{(0)}(0,1) \times D(A),
\]

and furthermore

\[
\left\| \begin{pmatrix} W^{(p_0, q)}(\cdot, T) \\ W_t^{(p_0, q)}(\cdot, T) \end{pmatrix} - \Gamma_0^{(p_0, q)}(T) \right\|_{H^3_{(0)}(0,1) \times D(A)} 
\leq C\left( \sum_{n \in \mathbb{Z}^*} \lambda_{\alpha, n}^2 |\gamma_n^{(q, \mu, w^{(p_0)})}(T)|^2 \right)^{1/2} 
\leq C''(\alpha, T) \|p_0\|_{L^2(0,T)} \|q\|_{L^2(0,T)} \|\mu\|_{V_{\mu,0}^2(0,1)} \|w^{(p_0)}\|_{C^0([0,T], D(A))} 
+ C'(\alpha, T) \|q\|_{L^2(0,T)} \|\mu\|_{V_{\mu,0}^2(0,1)} \|w^{(p_0)}\|_{C^0([0,T], D(A))}.
\]

However, \(\Gamma_0^{(p_0, q)}(T)\) is independent of \(x\), hence

\[
\left\| \Gamma_0^{(p_0, q)}(T) \right\|_{H^3_{(0)}(0,1) \times D(A)} = \left\| \Gamma_0^{(p_0, q)}(T) \right\|_{L^2(0,1) \times L^2(0,1)} \leq C \int_0^T |R_0(s)| \, ds,
\]

and

\[
|R_0(s)| \leq |p_0(s)| \left( \|\mu W^{(p_0, q)}(s, \Phi_{\alpha,0})\|_{L^2(0,1)} + |q(s)| \left( \|\mu w^{(p_0)}(s, \Phi_{\alpha,0})\|_{L^2(0,1)} \right) \right) 
\leq |p_0(s)| \left( \|\mu W^{(p_0, q)}(s)\|_{L^2(0,1)} + |q(s)| \left( \|\mu w^{(p_0)}(s)\|_{L^2(0,1)} \right) \right) 
\leq C|p_0(s)| \|\mu\|_{V_{\mu,0}^2(0,1)} \|W^{(p_0, q)}\|_{C^0([0,T], D(A))} 
+ C|q(s)| \|\mu\|_{V_{\mu,0}^2(0,1)} \|w^{(p_0)}\|_{C^0([0,T], D(A))} 
\leq C'|p_0(s)| \|\mu\|_{V_{\mu,0}^2(0,1)} \|q\|_{L^2(0,T)} \|w^{(p_0)}\|_{C^0([0,T], D(A))} 
+ C|q(s)| \|\mu\|_{V_{\mu,0}^2(0,1)} \|w^{(p_0)}\|_{C^0([0,T], D(A))}.
\]

Thus

\[
\|R_0\|_{L^1(0,T)} \leq C(\alpha, T, \mu, w^{(p_0)}) \|q\|_{L^2(0,T)}.
\]

and therefore

\[
\left\| \begin{pmatrix} W^{(p_0, q)}(\cdot, T) \\ W_t^{(p_0, q)}(\cdot, T) \end{pmatrix} \right\|_{H^3_{(0)}(0,1) \times D(A)} \leq C(\alpha, T, \mu, w^{(p_0)}) \|q\|_{L^2(0,T)}.
\]

Hence, we have proved that the application \(q \mapsto (W^{(p_0, q)}(T), W_t^{(p_0, q)}(T))\) is continuous. This concludes the proof of Lemma 6.8. \(\square\)
6.6.b. Proof of Lemma 6.9

Function \( v_t^{(p_0-q)} \), defined in (6.36), is the classical solution of

\[
\begin{align*}
&v_t^{(p_0-q)} - (x^\alpha v_x^{(p_0-q)})_x = p_0(t)\mu(x)v^{(p_0-q)} + q(t)\mu(x)(w^{(p_0+q)} - w^{(p_0)}), \\
&(x^\alpha v_x^{(p_0-q)})(x = 0, t) = 0, \\
&v_x^{(p_0-q)}(x = 1, t) = 0, \\
v^{(p_0-q)}(x, 0) = 0, \\
v_t^{(p_0-q)}(x, 0) = 0,
\end{align*}
\]

(6.49)

that is actually a problem similar to (6.3) with \( p = p_0 \) and \( w^{(p_0+q)} - w^{(p_0)} \) that replaces \( wp \). Then, the linear control system (6.49) is well-posed, and

\[
(v^{(p_0-q)}, v_t^{(p_0-q)}) \in C^0([0, T], D(A) \times H^1_0(0, 1)).
\]

So, (2.13) gives that

(6.50)

\[
||v^{(p_0-q)}||_{C^0([0, T], D(A))} + ||v_t^{(p_0-q)}||_{C^0([0, T], H^1_0(0, 1))}
\leq C||q\mu(w^{(p_0+q)} - w^{(p_0)})||_{L^2(0, T; H^1_0(0, 1))}
\leq C||q||_{L^2(0, T)}||\mu(w^{(p_0+q)} - w^{(p_0)})||_{C^0([0, T], H^1_0(0, 1))}
\leq C||q||_{L^2(0, T)}||v_t^{(p_0-q)}||_{L^2(0, 1)}||w^{(p_0+q)} - w^{(p_0)}||_{C^0([0, T], H^1_0(0, 1))}.
\]

We can decompose \((v^{(p_0-q)}(T), v_t^{(p_0-q)}(T))\) as follows: denoting

(6.51)

\[
z_n(s) = \langle p_0(s)\mu(\cdot)v^{(p_0-q)}(\cdot, s) + q(s)\mu(\cdot)(w^{(p_0+q)} - w^{(p_0)})(\cdot, s), \Phi_\alpha,|n| \rangle_{L^2(0, 1)},
\]

we have

(6.52)

\[
\left( \begin{array}{c} v^{(p_0-q)}(x, T) \\ v_t^{(p_0-q)}(x, T) \end{array} \right) = E_0^{(p_0-q)}(T) + \sum_{n \in \mathbb{Z}^*, \lambda^2_n < \infty} \frac{1}{2i\omega_n} \varepsilon_n^{(p_0-q)}(T) \Psi_\alpha,|n|(x) e^{i\omega_n x T},
\]

with

(6.53)

\[
E_0^{(p_0-q)}(T) = \left( \begin{array}{c} \int_0^T z_0(s)(T - s) \, ds \\ \int_0^T z_0(s) \, ds \end{array} \right),
\]

and

(6.54)

\[
\forall n \in \mathbb{Z}^*, \quad \varepsilon_n^{(p_0-q)}(T) = \int_0^T z_n(s) e^{-i\omega_n x s} \, ds.
\]

As showed in section 6.33

(6.55)

\[
\sum_{n \in \mathbb{Z}} \lambda_n^2 |\varepsilon_n^{(p_0-q)}(T)|^2 < \infty \implies (v^{(p_0-q)}(T), v_t^{(p_0-q)}(T)) \in H^3_0(0, 1) \times D(A),
\]

Thus, to ensure that \((v^{(p_0-q)}(T), v_t^{(p_0-q)}(T)) \in H^3_0(0, 1) \times D(A)\), we have to prove the convergence of the above series. We observe that

\[
p_0(t)\mu(x)v^{(p_0-q)} + q(t)\mu(x)(w^{(p_0+q)} - w^{(p_0)})
= (p_0(t) + q(t))\mu(x)v^{(p_0-q)} + q(t)\mu(x)W^{(p_0-q)},
\]
therefore, we decompose $\varepsilon_n^{(p_0,q)}(T)$ as follows: $\forall n \neq 0$

$$\varepsilon_n^{(p_0,q)}(T) = \int_0^T (p_0(s) + q(s)) (\mu(\cdot)v_n^{(p_0,q)}(\cdot,s), \Phi_{\alpha,n}) L^2(0,1) e^{-i\omega_n s} ds$$

(6. 56)

$$= \int_0^T q(s) (\mu(\cdot)W^{(p_0,q)}(\cdot,s), \Phi_{\alpha,n}) L^2(0,1) e^{-i\omega_n s} ds + \gamma_n^{(p_0+q,\mu,v^{(p_0,q)})}(T)$$

Applying twice Lemma 6.3, we obtain

(6. 57)

$$\left(\sum_{n=1}^{\infty} \lambda^2_{\alpha,n} |\gamma_n^{(p_0+q,\mu,v^{(p_0,q)})}(T)|^2 \right)^{1/2}$$

$$\leq C(\alpha, T) \|p_0 + q\|_{L^2(0,T)} \|\mu v_n^{(p_0,q)}\|_{C^0([0,T], V^{(2,0)}_\alpha(0,1))}$$

$$\leq C'(\alpha, T) \|p_0 + q\|_{L^2(0,T)} \|\mu\|_{V^2_\alpha(0,1)} \|v_n^{(p_0,q)}\|_{C^0([0,T], D(A))}$$

$$\leq C''(\alpha, T) \|p_0 + q\|_{L^2(0,T)} \|\mu\|_{V^2_\alpha(0,1)} \|v_n^{(p_0,q)} - w_n^{(p_0)}\|_{C^0([0,T], D(A))}$$

and

(6. 58)

$$\left(\sum_{n=1}^{\infty} \lambda^2_{\alpha,n} |\gamma_n^{(q,\mu,W^{(p_0,q)})}(T)|^2 \right)^{1/2}$$

$$\leq C(\alpha, T) \|q\|_{L^2(0,T)} \|\mu W^{(p_0,q)}\|_{C^0([0,T], V^{(2,0)}_\alpha(0,1))}$$

$$\leq C'(\alpha, T) \|q\|_{L^2(0,T)} \|\mu\|_{V^2_\alpha(0,1)} \|W^{(p_0,q)}\|_{C^0([0,T], D(A))}$$

$$\leq C''(\alpha, T) \|q\|_{L^2(0,T)} \|\mu\|_{V^2_\alpha(0,1)} \|w^{(p_0)}\|_{C^0([0,T], D(A))}$$

Thus, we have proved that

$$\sum_{n \in \mathbb{Z}} \lambda^2_{\alpha,n} |\varepsilon_n^{(p_0,q)}(T)|^2 < \infty,$$

which implies that

$$(u^{(p_0,q)}(T), v_t^{(p_0,q)}(T)) \in H^3(0,1) \times D(A).$$

Furthermore,

$$\left\| \left( \begin{array}{c} v^{(p_0,q)}(\cdot,T) \\ v_t^{(p_0,q)}(\cdot,T) \end{array} \right) - E_0^{(p_0,q)}(T) \right\|_{H^3(0,1) \times D(A)}$$

$$\leq C \left( \sum_{n \in \mathbb{Z}} \lambda^2_{\alpha,n} |\varepsilon_n^{(p_0,q)}(T)|^2 \right)^{1/2}$$

$$\leq C'(\alpha, T) \|p_0 + q\|_{L^2(0,T)} \|q\|_{L^2(0,T)} \|\mu\|_{V^2_\alpha(0,1)} \|w^{(p_0+q)} - w^{(p_0)}\|_{C^0([0,T], D(A))}$$

$$+ C''(\alpha, T) \|q\|_{L^2(0,T)} \|\mu\|_{V^2_\alpha(0,1)} \|w^{(p_0)}\|_{C^0([0,T], D(A))}.$$
is solution of
\[
\begin{align*}
  (u^{(p_0,q)}_t)(x) - x^\alpha u^r(x) = p_0(t)\mu(x)u^{(p_0,q)} + q(t)\mu(x)u^{(p_0+q)}, \\
  (x^\alpha u^r(x)) = 0, \\
  u^{(p_0,q)}(x) = 0, \\
  u^{(p_0,q)}(0,0) = 0,
\end{align*}
\]

(6. 60)

hence, \(P_2\) implies that
\[
\|u^{(p_0+q)} - u^{(p_0)}\|_{C^0([0,T], D(A))} = \|u^{(p_0,q)}\|_{C^0([0,T], D(A))} \\
\leq C\|q\mu u^{(p_0+q)}\|_{L^2(0,T; H^1_2(\Omega))} \\
\leq C\|q\|_{L^2(0,T)}\|\mu\|_{V^2_0} \|w^{(p_0+q)}\|_{C^0([0,T], D(A))}.
\]

So, we get
\[
\left\|
\begin{pmatrix} v^{(p_0,q)}(\cdot, T) \\ v^{(p_0,q)}_t(\cdot, T) \end{pmatrix}
\right\|_{\mathcal{H}_0^5(\Omega_1) \times D(A)} \\
\leq C'\|q\|_{L^2(0,T)} \left(\|p_0 + q\|_{L^2(0,T)} \|\mu\|_{V^2_0} \|w^{(p_0+q)}\|_{C^0([0,T], D(A))} \\
+ \|\mu\|_{V^2_0} \|w^{(p_0)}\|_{C^0([0,T], D(A))}\right).
\]

However, \(E_0^{(p_0,q)}(T)\) is independent of \(x\), therefore, we deduce that
\[
\|E_0^{(p_0,q)}(T)\|_{\mathcal{H}_0^5(\Omega_1) \times D(A)} = \|E_0^{(p_0,q)}(T)\|_{L^1(0,1) \times L^2(0,1)} \leq C \int_0^T |z_0(s)| ds,
\]

and
\[
|z_0(s)| \leq |p_0(s)| \|\mu v^{(p_0,q)}(s, \Phi_{\alpha_0})\|_{L^2(0,1)} + |q(s)| \|\mu(w^{(p_0+q)} - w^{(p_0)})\|_{L^2(0,1)} \\
\leq |p_0(s)| \|\mu v^{(p_0,q)}(s)\|_{L^2(0,1)} + |q(s)| \|\mu(w^{(p_0+q)} - w^{(p_0)})\|_{L^2(0,1)} \\
\leq C|p_0(s)| \|\mu\|_{V^2_0} \|v^{(p_0,q)}\|_{C^0([0,T], D(A))} \\
+ C|q(s)| \|\mu\|_{V^2_0} \|w^{(p_0+q)} - w^{(p_0)}\|_{C^0([0,T], D(A))} \\
\leq C'|p_0(s)| \|\mu\|_{V^2_0} \|q\|_{L^2(0,T)} \|w^{(p_0+q)} - w^{(p_0)}\|_{C^0([0,T], H^1_2(\Omega_1))} \\
+ C|q(s)| \|\mu\|_{V^2_0} \|q\|_{L^2(0,T)} \|w^{(p_0+q)}\|_{C^0([0,T], D(A))} \\
\leq C'|p_0(s)| \|\mu\|_{V^2_0} \|q\|_{L^2(0,T)} \|w^{(p_0+q)}\|_{C^0([0,T], D(A))} \\
+ C|q(s)| \|\mu\|_{V^2_0} \|q\|_{L^2(0,T)} \|w^{(p_0+q)}\|_{C^0([0,T], D(A))}.
\]

Hence, we have showed that
\[
\|z_0\|_{L^1(0,T)} \leq C(\alpha, \mu, w^{(p_0)}) \|q\|_{L^2(0,T)}^2
\]

and so,
\[
\left\|
\begin{pmatrix} v^{(p_0,q)}(\cdot, T) \\ v^{(p_0,q)}_t(\cdot, T) \end{pmatrix}
\right\|_{\mathcal{H}_0^5(\Omega_1) \times D(A)} \leq C(\alpha, \mu, w^{(p_0)}) \|q\|_{L^2(0,T)}^2.
\]

Thus, \([6. 37]\) is satisfied and this concludes the proof of Lemma \([6.9]\) \(\square\).
Lemma 7.1. \[ \text{The key point of the proof of Theorem 3.1 is the following result.} \]
\[ \Theta_T \text{ is surjective, and} \]
\[ \square \]
We leave the proof of (6.61) to the reader.

Proceeding as in section 6.6.a it is easy to verify that there exists \( C(\alpha, T, \mu, p_0) > 0 \) such that for any \( \tilde{p}, q \in L^2(0,T) \)

\[
\| \Theta_T(p_0 + \tilde{p}) - \Theta_T(p_0) \|_{L^2(0,T), H^3_0} \rightarrow 0, \quad \| \tilde{p} \|_{L^2(0,T)} \rightarrow 0.
\]

Thus, \( \Theta_T \) is continuous and this concludes the proof of Proposition 6.1, part c). We leave the proof of (6.61) to the reader. \( \square \)

7. Reachability for \( T > T_0 \): proof of Theorem 3.1

The proof follows from the classical inverse mapping theorem applied to the function \( \Theta_T : L^2(0,T) \rightarrow H^3_0 \times D(A) \) at the point \( p_0 = 0 \). First, we recall that \( \Theta_T(p_0 = 0) = (1,0) \). In the following we study \( \Theta_T(0) \).

7.1. Surjectivity of \( \Theta_T(0) \): study of the associated moment problem.

The key point of the proof of Theorem 3.1 is the following result.

Lemma 7.1. Assume that (5.19) and (3.18) are satisfied. Then, the linear application

\[ \Theta_T(0) : L^2(0,T) \rightarrow H^3_0 \times D(A) \]

\[ q \mapsto (W^{(0,q)}(T), W_t^{(0,q)}(T)) \]

is surjective, and

\[ \Theta_T(0) : \text{Vect} \{ 1, t, \cos \sqrt{\lambda_{\alpha,n} t}, \sin \sqrt{\lambda_{\alpha,n} t}, n \geq 1 \} \rightarrow H^3_0 \times D(A) \]

is invertible.

Proof of Lemma 7.1. Since \( w(0) = 1 \), (6.3) implies that \( W^{(0,q)} \) is solution of the following linear problem

\[
\begin{cases}
W^{(0,q)}_{tt} - (x^\alpha W^{(0,q)}_x)_x = q(t)\mu(x), & x \in (0,1), t \in (0,T), \\
(x^\alpha W^{(0,q)}_x)(x = 0, t) = 0, & t \in (0,T), \\
W^{(0,q)}(x = 1, t) = 0, & t \in (0,T), \\
W^{(0,q)}(x,0) = 0, & x \in (0,1), \\
W^{(0,q)}(x,0) = 0, & x \in (0,1).
\end{cases}
\]

(7.1)

Following the procedure of section 6.2 we introduce

\[
r_n(s) = \langle q(s)\mu, \Phi_{\alpha,n} \rangle_{L^2(0,1)} = \mu_{\alpha,n} q(s) \quad \text{with} \quad \mu_{\alpha,n} = \langle \mu, \Phi_{\alpha,n} \rangle_{L^2(0,1)}.
\]

(7.2)
and we have can express the solution of (7.1) at time $T$, $(W^{(0,q)}(T), W_t^{(0,q)}(T))$ as follows

$$
W^{(0,q)}(x, T) = \int_0^T r_0(s)(T - s) ds + \sum_{n=1}^{\infty} \left( \int_0^T r_n(s) \frac{\sin \sqrt{\lambda_{\alpha,n}}(T - s)}{\sqrt{\lambda_{\alpha,n}}} ds \right) \Phi_{\alpha,n}(x),
$$

and

$$
W_t^{(0,q)}(x, T) = \int_0^T r_0(s) ds + \sum_{n=1}^{\infty} \left( \sqrt{\lambda_{\alpha,n}} \int_0^T r_n(s) \frac{\cos \sqrt{\lambda_{\alpha,n}}(T - s)}{\sqrt{\lambda_{\alpha,n}}} ds \right) \Phi_{\alpha,n}(x).
$$

To prove Lemma 7.1, we choose any pair $(Y^f, Z^f) \in H^3_{(0)} \times D(A)$, and we want to show that there exists $q \in L^2(0, T)$ such that

$$(7.5) \quad (W^{(0,q)}(T), W_t^{(0,q)}(T)) = (Y^f, Z^f).$$

Introducing the Fourier coefficients of the target state

$$Y_{\alpha,n}^f = \langle Y^f, \Phi_{\alpha,n} \rangle_{L^2(0,1)} \quad \text{and} \quad Z_{\alpha,n}^f = \langle Z^f, \Phi_{\alpha,n} \rangle_{L^2(0,1)},$$

we can decompose $(Y^f, Z^f)$ as follows

$$Y^f(x) = \sum_{n=0}^{\infty} Y_{\alpha,n}^f \Phi_{\alpha,n}(x) = Y_{\alpha,0}^f + \sum_{n=1}^{\infty} Y_{\alpha,n}^f \Phi_{\alpha,n}(x)$$

and

$$Z^f(x) = \sum_{n=0}^{\infty} Z_{\alpha,n}^f \Phi_{\alpha,n}(x) = Z_{\alpha,0}^f + \sum_{n=1}^{\infty} Z_{\alpha,n}^f \Phi_{\alpha,n}(x).$$

We derive from (7.3) and (7.4) that (7.6) is satisfied if and only if

$$
(7.6) \quad \begin{cases} 
\int_0^T r_0(s) ds = Z_{\alpha,0}^f, \\
\int_0^T r_n(s) \frac{\cos \sqrt{\lambda_{\alpha,n}}(T - s)}{\sqrt{\lambda_{\alpha,n}}} ds = Z_{\alpha,n}^f, \quad \text{for all } n \geq 1, \\
\int_0^T r_n(s) \frac{\sin \sqrt{\lambda_{\alpha,n}}(T - s)}{\sqrt{\lambda_{\alpha,n}}} ds = Y_{\alpha,n}^f, \quad \text{for all } n \geq 1, \\
\int_0^T r_0(s)(T - s) ds = Y_{\alpha,0}^f.
\end{cases}
$$

Introducing the function

$$Q(s) := q(T - s),$$

(7.6) becomes

$$
(7.7) \quad \begin{cases} 
\mu_{\alpha,0} \int_0^T Q(t) dt = Z_{\alpha,0}^f, \\
\mu_{\alpha,n} \int_0^T Q(t) \cos \sqrt{\lambda_{\alpha,n}} t dt = Z_{\alpha,n}^f, \quad \text{for all } n \geq 1, \\
\mu_{\alpha,n} \int_0^T Q(t) \sin \sqrt{\lambda_{\alpha,n}} t dt = \sqrt{\lambda_{\alpha,n}} Y_{\alpha,n}^f, \quad \text{for all } n \geq 1, \\
\mu_{\alpha,0} \int_0^T Q(t) t dt = Y_{\alpha,0}^f.
\end{cases}
$$

System (7.7) is usually called moment problem. Observe that (3.18) implies that the coefficients $\mu_{\alpha,n}$ are different from 0 for all $n \geq 0$, which is a necessary condition to solve (7.7).
Let us introduce
\[
\begin{align*}
A_{\alpha,0}^f := & \frac{Z_{\alpha,0}^f}{\mu_{\alpha,0}}, \\
A_{\alpha,n}^f := & \frac{Z_{\alpha,n}^f}{\mu_{\alpha,n}}, \quad \text{for all } n \geq 1, \\
B_{\alpha,0}^f := & \sqrt{\lambda_{\alpha,0}} Y_{\alpha,0}^f, \\
B_{\alpha,n}^f := & \sqrt{\lambda_{\alpha,n}} Y_{\alpha,n}^f,
\end{align*}
\]
(7. 8)
and
\[
\begin{align*}
c_{\alpha,0} : t \in (0, T) & \mapsto 1, \\
c_{\alpha,n} : t \in (0, T) & \mapsto \cos \sqrt{\lambda_{\alpha,n}} t \quad \text{for all } n \geq 1, \\
s_{\alpha,n} : t \in (0, T) & \mapsto \sin \sqrt{\lambda_{\alpha,n}} t \quad \text{for all } n \geq 1, \\
s_{\alpha,0} : t \in (0, T) & \mapsto t,
\end{align*}
\]
(7. 9)
in such a way that (7.10) can be written as
\[
\begin{align*}
\langle Q, c_{\alpha,0} \rangle_{L^2(0,T)} &= A_{\alpha,0}^f, \\
\langle Q, c_{\alpha,n} \rangle_{L^2(0,T)} &= A_{\alpha,n}^f \quad \text{for all } n \geq 1, \\
\langle Q, s_{\alpha,n} \rangle_{L^2(0,T)} &= B_{\alpha,n}^f \quad \text{for all } n \geq 1, \\
\langle Q, s_{\alpha,0} \rangle_{L^2(0,T)} &= B_{\alpha,0}^f.
\end{align*}
\]
(7. 10)
We are going to prove that system (7.10) has (at least) a solution $Q$ in two steps:
\begin{itemize}
\item we prove that the reduced system
\[
\begin{align*}
\langle Q, c_{\alpha,0} \rangle_{L^2(0,T)} &= A_{\alpha,0}^f, \\
\langle Q, c_{\alpha,n} \rangle_{L^2(0,T)} &= A_{\alpha,n}^f \quad \text{for all } n \geq 1, \\
\langle Q, s_{\alpha,n} \rangle_{L^2(0,T)} &= B_{\alpha,n}^f \quad \text{for all } n \geq 1
\end{align*}
\]
(7. 11)
has at least a solution $Q_\alpha$,
\item using $Q_\alpha$, we construct a solution $Q$ of the full system (7.10).
\end{itemize}
We use results stated in \cite{7} Appendix (see also \cite{15}).

\textbf{Step 1: Existence of a solution of the reduced system (7.11).} We consider the space
\[
E_\alpha := \text{Vect } \{c_{\alpha,0}, c_{\alpha,n}, s_{\alpha,n}, n \geq 1\},
\]
which is a closed subspace of $L^2(0,T)$. To solve the reduced system, we use the following characterization of Riesz Basis (see (7 Prop. 19)):
the family
\[
\{c_{\alpha,0}, c_{\alpha,n}, s_{\alpha,n}, n \geq 1\}
\]
is a Riesz basis of $E_\alpha$ if and only if there exist $C_1(\alpha, T), C_2(\alpha, T) > 0$ such that, for all $N \geq 1$ and for any $(a_n)_{0 \leq n \leq N}, (b_n)_{1 \leq n \leq N}$ it holds that
\[
C_1 \left( a_0^2 + \sum_{n=1}^{N} a_n^2 + b_n^2 \right) \leq \int_0^T \left| S^{(\alpha,b)}(t) \right|^2 dt \leq C_2 \left( a_0^2 + \sum_{n=1}^{N} a_n^2 + b_n^2 \right),
\]
(7. 12)
where
\[
S^{(\alpha,b)}(t) = a_0 c_{\alpha,0}(t) + \sum_{n=1}^{N} a_n c_{\alpha,n}(t) + b_n s_{\alpha,n}(t).
\]
(7. 13)
We observe that (7.12) holds true as a consequence of Ingham theory: indeed, by expressing $\cos y$ and $\sin y$ as
\[
\cos y = \frac{e^{iy} + e^{-iy}}{2}, \quad \text{and} \quad \sin y = \frac{e^{iy} - e^{-iy}}{2i},
\]
and
we have that
\[ S^{(a,b)}(t) = a_0 e^{i\omega_{\alpha,0} t} + \sum_{n=1}^{N} a_n e^{i \omega_{\alpha,n} t} + b_n e^{i \omega_{\alpha,n} t - \frac{\alpha}{2} t} + \sum_{n=-N}^{N} d_n e^{i \omega_{\alpha,n} t}, \]
with
\[
\begin{cases}
  d_0 = a_0, \\
  d_n = \frac{a_n}{2} + \frac{b_n}{2}, & \text{for } n \geq 1, \\
  d_n = \frac{a_n}{2} - \frac{b_n}{2}, & \text{for } n \leq -1.
\end{cases}
\]

Since \(\omega_{\alpha,n+1} - \omega_{\alpha,n} > 0\) for all \(n \in \mathbb{Z}\) and
\[
|n| \geq 2, \quad \omega_{\alpha,n+1} - \omega_{\alpha,n} \geq \frac{2 - \alpha}{2 \pi},
\]
we can apply a general result of Haraux [22] (see also [7, Theorem 6]) that ensures that if
\[
T > \frac{2 \pi}{2 - \alpha},
\]
then, there exist \(C_1^{(f)}, C_2^{(f)} > 0\) independent of \(N\) and of the coefficients \((d_n)_{-N \leq n \leq N}\), such that
\[
(7.14) \quad C_1^{(f)} \sum_{n=-N}^{N} |d_n|^2 \leq \int_{0}^{T} \left| \sum_{n=-N}^{N} d_n e^{i \omega_{\alpha,n} t} \right|^2 dt \leq C_2^{(f)} \sum_{n=-N}^{N} |d_n|^2.
\]

Since
\[
\sum_{n=-N}^{N} |d_n|^2 = a_0^2 + \frac{a_0^2 + b_0^2}{2} + \sum_{n=1}^{N} |d_n|^2,
\]
(7.14) implies that (7.12) holds true and so the family \(\{c_{\alpha,0}, c_{\alpha,n}, s_{\alpha,n}, n \geq 1\}\) is a Riesz basis of \(E_\alpha\). Thus, the application \(F : E_\alpha \to \ell^2(\mathbb{N})\):
\[
F(f) = (\langle f, c_{\alpha,0} \rangle_{L^2(0,T)}, \langle f, c_{\alpha,1} \rangle_{L^2(0,T)}, \langle f, c_{\alpha,1} \rangle_{L^2(0,T)}, \langle f, c_{\alpha,2} \rangle_{L^2(0,T)}, \cdots)
\]
is an isomorphism (see, e.g., [7, Proposition 20]). We note that
\[
Y^f \in H_3^0(0,1) \quad \Rightarrow \quad \sum_{n=0}^{\infty} \lambda_{\alpha,n}^3 |Y^f_n|^2 < \infty,
\]
and
\[
Z^f \in D(A) \quad \Rightarrow \quad \sum_{n=0}^{\infty} \lambda_{\alpha,n}^2 |Z^f_n|^2 < \infty,
\]
and then (3.18) ensures us that
\[
|A_{\alpha,0}^f|^2 + \sum_{n=1}^{\infty} |A_{\alpha,n}^f|^2 + |B_{\alpha,n}^f|^2 < \infty.
\]
Therefore, there exists a unique \(Q_\alpha \in E_\alpha\) such that
\[
F(Q_\alpha) = (A_{\alpha,0}^f, A_{\alpha,1}^f, B_{\alpha,1}^f, A_{\alpha,2}^f, \cdots).
\]
Since \( TQ \) solves (7.15). However, \( (\ldots) \) and different from \( Q \) fact that \( \tilde{a} \) is continuous.

\[ p \] denote \( \rho \) and, moreover, the application

\[ \text{Thus,} \]

\[ 0 \rightarrow \tilde{E}_c \text{ and, moreover, the application} \]

\[ \text{Step 2: Existence of a solution of the full system (7.10).} \]

We claim that \( \tilde{s}_{\alpha,0} \notin E_{\alpha} \): indeed, if \( t \mapsto t \) was the limit of a sequence of linear combinations of \( c_{\alpha,0}, \) \( c_{\alpha,n}, \) and \( s_{\alpha,n}, \) the same would be true for the function \( t \mapsto t^2, \) by integration.

Then, by integrating further, also \( t \mapsto t^3 \) would be the limit of a sequence of linear combinations of \( c_{\alpha,0}, \) \( c_{\alpha,n}, \) and \( s_{\alpha,n}. \) Thus, by iterating this procedure, we deduce that all the polynomials could be written in this form. Therefore, \( L^2(0,T) \) would be equal to \( E_{\alpha} \) and \( (7.15) \) would have a unique solution. However, this is not the case: define \( T_0 := \frac{T_0}{2} \) and \( Q, \) smooth, compactly supported in \((0, \frac{T_0}{2})\) and different from \( Q_{\alpha} \) on that interval. Now, consider the following problem

\[ \begin{cases}
\langle \tilde{Q}_{\alpha}, c_{\alpha,0} \rangle_{L^2(0, T)} = A^f_{\alpha,0} - \langle q_{\alpha}, s_{\alpha,0} \rangle_{L^2(0, T)}, \\
\langle \tilde{Q}_{\alpha}, c_{\alpha,n} \rangle_{L^2(0, T)} = A^f_{\alpha,n} - \langle q_{\alpha}, s_{\alpha,n} \rangle_{L^2(0, T)} \quad \text{for all } n \geq 1, \\
\langle \tilde{Q}_{\alpha}, s_{\alpha,n} \rangle_{L^2(0, T)} = B^f_{\alpha,n} - \langle q_{\alpha}, s_{\alpha,n} \rangle_{L^2(0, T)} \quad \text{for all } n \geq 1.
\end{cases} \]

Since \( T - \frac{T_0}{2} > T_0 \), and the sequences \( (\langle q_{\alpha}, c_{\alpha,n} \rangle_{L^2(0, T)} )_{n} \) and \( (\langle q_{\alpha}, s_{\alpha,n} \rangle_{L^2(0, T)} )_{n} \) are square-summable (by integration by parts), there exists a solution \( Q_{\alpha} \in L^2(0, T) \) of (7.16). So, the function

\[ Q^*_{\alpha} := \begin{cases}
q_{\alpha} \text{ on } (0, \frac{T_0}{2}), \\
\tilde{Q}_{\alpha} \text{ on } (\frac{T_0}{2}, T)
\end{cases} \]

solves (7.15). However, \( Q^*_{\alpha} \neq Q_{\alpha} \) on \((0, \frac{T_0}{2})\), which is contradiction with the fact that \( Q_{\alpha} \) is the unique solution of (7.15). Therefore \( \tilde{s}_{\alpha,0} \notin E_{\alpha} \), and if we denote \( p^\perp_{\alpha,0} \) the orthogonal projection of \( \tilde{s}_{\alpha,0} \) on \( E_{\alpha} \), then \( \tilde{s}_{\alpha,0} - p^\perp_{\alpha,0} \neq 0 \), and

\[ Q^\perp_{\alpha} := \frac{\tilde{s}_{\alpha,0} - p^\perp_{\alpha,0}}{||\tilde{s}_{\alpha,0} - p^\perp_{\alpha,0}||_{L^2(0, T)}} \]

is orthogonal to \( E_{\alpha} \), and furthermore

\[ \langle Q^\perp_{\alpha}, \tilde{s}_{\alpha,0} \rangle_{L^2(0, T)} = 1. \]

Thus,

\[ Q := Q_{\alpha} + B_{\alpha,0}Q^\perp_{\alpha} \]

solves (7.10). Moreover,

\[ \|Q\|_{L^2(0, T)}^2 = \|Q_{\alpha}\|_{L^2(0, T)}^2 + \|B^f_{\alpha,0}Q^\perp_{\alpha}\|_{L^2(0, T)}^2 \leq C \sum_{n=0}^{\infty} |A^f_{\alpha,n}|^2 + |B^f_{\alpha,n}|^2, \]

which completes the proof of Lemma (7.11).
7.2. Proof of Theorem 3.1 (inverse mapping argument).

We define the space

\[ F_\alpha := \text{Vect}\ \{1, t, \cos \sqrt{\lambda_{\alpha,n}} t, \sin \sqrt{\lambda_{\alpha,n}} t, n \geq 1\} \]

Then, the restriction of \( \Theta_T \) to \( F_\alpha \)

\[ \Theta_{\alpha,T} : F_\alpha \to H^3_0(0,1) \times D(A), \]

\[ p \mapsto \Theta_{\alpha,T}(p) := \Theta_T(p) \]

is \( C^1 \) (Proposition 6.1) and \( D\Theta_{\alpha,T}(0) \) is invertible (Lemma 7.1). Thus, the inverse mapping theorem ensures that there exists a neighborhood \( \mathcal{V}(0) \subset F_\alpha \) and a neighborhood \( \mathcal{V}(1,0) \subset H^3_0(0,1) \times D(A) \) such that

\[ \Theta_{\alpha,T} : \mathcal{V}(0) \to \mathcal{V}(1,0) \]

is a \( C^1 \)-diffeomorphism. Hence, given \( (w^f_0, w^f_1) \in \mathcal{V}(1,0) \), we choose \( p^f := \Theta_{\alpha,T}^{-1}(w^f_0, w^f_1) \), and so the solution of (3.13) with \( p = p^f \) satisfies

\[ (w(T), w_t(T)) = \Theta_T(p^f) = \Theta_T(\Theta_{\alpha,T}^{-1}(w^f_0, w^f_1)) = (w^f_0, w^f_1). \]

This concludes the proof of Theorem 3.1. \( \square \)

8. Proof of Theorem 3.2: Reachability for \( T = T_0 \)

8.1. Proof of Theorem 3.2 first part: Reachability for \( T = T_0 \) and \( \alpha \in [0,1) \).

The proof follows from classical arguments concerning families of exponentials \([4]\) in the space \( L^2(0,T) \) and the strategy of Beauchard \([6]\):

- we study the solvability of the moment problem (7.6) (or equivalently (7.7)),
- we conclude using the inverse mapping theorem.

8.1.a. Main tools to study the solvability of the moment problem (7.7).

In order to use classical results on complex exponentials, we can transform (7.7) into the following system

\[
\begin{align*}
\mu_{\alpha,0} \int_0^T Q(t) dt &= Z^f_{\alpha,0}, \\
\mu_{\alpha,n} \int_0^T Q(t) e^{-i \sqrt{\lambda_{\alpha,n}} n t} dt &= Z^f_{\alpha,n} - i \sqrt{\lambda_{\alpha,n}} Y^f_{\alpha,n}, \quad \text{for all } n \geq 1, \\
\mu_{\alpha,n} \int_0^T Q(t) e^{i \sqrt{\lambda_{\alpha,n}} n t} dt &= Z^f_{\alpha,n} + i \sqrt{\lambda_{\alpha,n}} Y^f_{\alpha,n}, \quad \text{for all } n \geq 1, \\
\mu_{\alpha,0} \int_0^T Q(t) t dt &= Y^f_{\alpha,0}.
\end{align*}
\]

By introducing the notation

\[
\begin{align*}
C^f_{\alpha,0} := \frac{Z^f_{\alpha,0}}{\mu_{\alpha,0}} \\
C^f_{\alpha,n} := \frac{Z^f_{\alpha,n} - i \sqrt{\lambda_{\alpha,n}} Y^f_{\alpha,n}}{\mu_{\alpha,n}} \quad \text{for all } n \geq 1, \\
C^f_{\alpha,-n} := \frac{Z^f_{\alpha,n} + i \sqrt{\lambda_{\alpha,n}} Y^f_{\alpha,n}}{\mu_{\alpha,n}} \quad \text{for all } n \geq 1,
\end{align*}
\]

and the natural scalar product in \( L^2(0,T;\mathbb{C}) \)

\[ \forall f, g \in L^2(0,T;\mathbb{C}), \quad (f,g)_{L^2(0,T;\mathbb{C})} = \int_0^T f(t) \overline{g(t)} dt, \]
can be written as
\[
\begin{aligned}
\left\{ & \langle Q, e^{i\omega_\alpha t} \rangle_{L^2(0,T;\mathbb{C})} = C^f_{\alpha,0}, \\
& \langle Q, e^{i\omega_\alpha n t} \rangle_{L^2(0,T;\mathbb{C})} = C^f_{\alpha,n} \text{ for all } n \geq 1, \\
& \langle Q, e^{i\omega_\alpha n t} \rangle_{L^2(0,T;\mathbb{C})} = C^f_{\alpha,n} \text{ for all } n \leq -1, \\
& \langle Q, \hat{s}_{\alpha,0} \rangle_{L^2(0,T;\mathbb{C})} = B^f_{\alpha,0}, \n\right.
\end{aligned}
\]  
(8. 3)

(where \( \hat{s}_{\alpha,0} \) and \( B^f_{\alpha,0} \) have been defined in (7. 8) and (7. 9)). We are going to study first the solvability of the subsystem composed by the first three equations, that is, the following moment problem
\[
\forall n \in \mathbb{Z}, \quad \langle Q, e^{i\omega_\alpha n t} \rangle_{L^2(0,T;\mathbb{C})} = C^f_{\alpha,n}.
\]  
(8. 4)

8.1.b. Main solvability results for \( \alpha \in [0,1) \).

**Lemma 8.1.** Let \( \alpha \in [0,1) \). Then, the sequence \( (e^{i\omega_\alpha n t})_{n \in \mathbb{Z}} \) is a Riesz basis in \( L^2(0,T_0) \).

From the previous Lemma we deduce the following result.

**Lemma 8.2.** Let \( \alpha \in [0,1) \) and \( T = T_0 \). Then, the moment problem (8. 4) has one and only one solution \( Q \in L^2(0,T_0;\mathbb{R}) \).

Lemma 8.2 will imply the following result.

**Lemma 8.3.** Let \( \alpha \in [0,1) \) and \( T = T_0 \). There exists a closed hyperplane of \( H^3_0(\alpha) \times D(A) \), denoted by \( P^f_{\alpha} \) and defined in (8. 8), such that the moment problem (8. 3) has a solution if and only if \((Y^f, Z^f) \in P^f_{\alpha} \).

8.1.c. Proof of Lemma 8.1.

The proof follows from the Kadec’s \( 1 \) Theorem ([23], [31, Theorem 1.14 p. 42]). First, we note that the sequence \( (\omega_{\alpha,n})_{n \in \mathbb{Z}} \) is odd, that is \( \omega_{\alpha,-n} = -\omega_{\alpha,n} \), and
\[
\forall n \geq 1, \quad \omega_{\alpha,n} = \kappa_n j_{\nu,n} + \frac{2-\alpha}{2} \frac{j_{\nu,n}}{\pi n}.
\]

Mac Mahon’s formula (see [30, p. 506]) provides the following asymptotic development of \( j_{\nu,n} \) as \( n \to \infty \)
\[
j_{\nu,n} = \pi (n + \frac{\nu}{2} - \frac{1}{4}) + O(\frac{1}{n}).
\]
Hence, we have
\[
\omega_{\alpha,n} = \frac{2-\alpha}{2} \pi \left( n + \frac{1}{2(2-\alpha)} - \frac{1}{4} \right) + O(\frac{1}{n}) = \frac{2-\alpha}{2} \pi \left( n + \frac{\alpha}{4(2-\alpha)} \right) + O(\frac{1}{n}).
\]
Therefore, we deduce that
\[
\frac{2}{\pi(2-\alpha)} \omega_{\alpha,n} \to n + \frac{\alpha}{4(2-\alpha)} + O(\frac{1}{n}) \quad \text{as } n \to +\infty,
\]
and, in particular,
\[
\frac{2}{\pi(2-\alpha)} \omega_{\alpha,n} - n \to \frac{\alpha}{4(2-\alpha)} \quad \text{as } n \to +\infty.
\]
Since \( \alpha \in [0,1) \), it holds that
\[
\frac{\alpha}{4(2-\alpha)} = \frac{1}{4} \frac{\alpha}{2-\alpha} < \frac{1}{4}.
\]
thus, we gather that for any \( L \in \left( \frac{1}{2}, \frac{\alpha}{2} - \frac{1}{2} \right) \) there exists \( N_0 \) such that

\[
\forall n \geq N_0, \quad \left| \frac{2}{\pi(2 - \alpha)} \omega_{\alpha,n} - n \right| \leq L < \frac{1}{4}.
\]

Since the sequence \( \left( \frac{2}{\pi(2 - \alpha)} \omega_{\alpha,n} \right)_{n \in \mathbb{Z}} \) is odd, the above bound holds also for negative indices, hence

\[
\forall |n| \geq N_0, \quad \left| \frac{2}{\pi(2 - \alpha)} \omega_{\alpha,n} - n \right| \leq L < \frac{1}{4}.
\]

Then, consider the sequence \( (\tilde{\omega}_{\alpha,n})_{n \in \mathbb{Z}} \) defined by

\[
\tilde{\omega}_{\alpha,n} := \begin{cases} n, & \forall |n| < N_0, \\ \frac{2}{\pi(2 - \alpha)} \omega_{\alpha,n}, & \forall |n| \geq N_0. \end{cases}
\]

The new sequence \( (\tilde{\omega}_{\alpha,n})_{n \in \mathbb{Z}} \) satisfies

\[
\forall n \in \mathbb{Z}, \quad |\tilde{\omega}_{\alpha,n} - n| \leq L < \frac{1}{4},
\]

and we can apply Kadec’s \( \frac{1}{4} \) Theorem, which implies that the sequence \( (e^{i\tilde{\omega}_{\alpha,n}\tau})_{n \in \mathbb{Z}} \) is a Riesz basis in \( L^2(-\pi, \pi; \mathbb{C}) \) (where \( \tau \) is the variable in \( (-\pi, \pi) \)), see [31, Theorem 1.14 p. 42]. Thanks to the change of variables

\[
[-\pi, \pi] \rightarrow [0, T_0], \quad \tau \mapsto t = \frac{\tau T_0}{\pi} + \frac{T_0}{2},
\]

we obtain that the sequence \( (e^{i\tilde{\omega}_{\alpha,n}\tau})_{n \in \mathbb{Z}} \) is a Riesz basis in \( L^2(0, T_0; \mathbb{C}) \). Indeed, for any \( g \in L^2(0, T_0; \mathbb{C}) \), we define

\[
f(\tau) := g \left( \frac{T_0}{2\pi} \tau + \frac{T_0}{2} \right) \in L^2(-\pi, \pi; \mathbb{C}),
\]

which can be developed using the Riesz basis \( (e^{i\tilde{\omega}_{\alpha,n}\tau})_{n \in \mathbb{Z}} \)

\[
f(\tau) = \sum_{n \in \mathbb{Z}} c_n e^{i\tilde{\omega}_{\alpha,n}\tau}, \quad \text{with} \quad A_0 \sum_{n \in \mathbb{Z}} |c_n|^2 \leq \|f\|^2 \leq B_0 \sum_{n \in \mathbb{Z}} |c_n|^2,
\]

where \( A_0 \) and \( B_0 \) are suitable positive constants. Going back to the original time interval \( (0, T_0) \), we obtain

\[
g(t) = f \left( \frac{t - \frac{T_0}{2}}{\frac{T_0}{2}} \right) = \sum_{n \in \mathbb{Z}} c_n e^{i\tilde{\omega}_{\alpha,n}2\pi \frac{t - \frac{T_0}{2}}{T_0}} = \sum_{n \in \mathbb{Z}} \tilde{c}_n e^{i\tilde{\omega}_{\alpha,n}2\pi \frac{\omega_{\alpha,n}}{2} t}
\]

where \( \tilde{c}_n = c_n e^{-i\tilde{\omega}_{\alpha,n}\pi} \). We further deduce that there exist two positive constants \( A_1 \) and \( B_1 \) such that

\[
A_1 \sum_{n \in \mathbb{Z}} |\tilde{c}_n|^2 \leq \|g\|^2 \leq B_1 \sum_{n \in \mathbb{Z}} |\tilde{c}_n|^2,
\]

because \( |\tilde{c}_n| = |c_n| \) and \( (e^{i\tilde{\omega}_{\alpha,n}\tau})_{n \in \mathbb{Z}} \) is a Riesz basis of \( L^2(-\pi, \pi; \mathbb{C}) \).

We notice that

\[
\forall |n| \geq N_0, \quad \tilde{\omega}_{\alpha,n} \frac{2\pi}{T_0} = \omega_{\alpha,n} \frac{2\pi}{\frac{T_0}{2} - \alpha} = \frac{2}{\pi(2 - \alpha)} \omega_{\alpha,n} \frac{\pi(2 - \alpha)}{2} = \omega_{\alpha,n},
\]
and since modifying a finite number of terms does not affect the fact of being a Riesz basis (Lemma II.4.11 p. 105), we deduce that \((e^{it\omega_{\alpha,n}})_{n\in\mathbb{Z}}\) is a Riesz basis in \(L^2(0,T_0;\mathbb{C})\).

8.1.d. **Proof of Lemma 8.2.**
Since \((e^{it\omega_{\alpha,n}})_{n\in\mathbb{Z}}\) is a Riesz basis in \(L^2(0,T_0;\mathbb{C})\), there exists one and only one biorthogonal sequence: \((\sigma_m(t))_{m\in\mathbb{Z}}\) satisfying
\[
\langle \sigma_m, e^{it\omega_{\alpha,n}} \rangle_{L^2(0,T_0;\mathbb{C})} = \int_0^{T_0} \sigma_m(t)e^{-it\omega_{\alpha,n}} dt = \delta_{mn}.
\]

Taking the conjugate, we obtain that
\[
\int_0^{T_0} \overline{\sigma_m(t)} e^{it\omega_{\alpha,n}} dt = \delta_{mn}.
\]
Recalling that \(\omega_{\alpha,n} = -\omega_{\alpha,-n}\), we have
\[
\langle \sigma_m, e^{it\omega_{\alpha,-n}} \rangle_{L^2(0,T_0;\mathbb{C})} = \delta_{mn} = \langle \sigma_m, e^{it\omega_{\alpha,n}} \rangle_{L^2(0,T_0;\mathbb{C})},
\]
which implies that
\[
\forall m \in \mathbb{Z}, \quad \sigma_m = \sigma_{-m}.
\]
Now, using once again that \((e^{it\omega_{\alpha,n}})_{n\in\mathbb{Z}}\) is a Riesz basis in \(L^2(0,T_0;\mathbb{C})\), the moment problem \((8.4)\) has one and only one solution, given by \((8.6)\)
\[
Q(t) = \sum_{m \in \mathbb{Z}} C^f_{\alpha,m} \sigma_m(t).
\]
It remains to verify that \(Q\) takes its values in \(\mathbb{R}\): taking the conjugate, we have
\[
\overline{Q}(t) = \sum_{m \in \mathbb{Z}} C^f_{\alpha,m} \sigma_m(t) = \sum_{m \in \mathbb{Z}} C^f_{\alpha,-m} \sigma_{-m}(t) = Q(t).
\]
Hence \(Q \in L^2(0,T_0;\mathbb{R})\). \(\square\)

8.1.e. **Proof of Lemma 8.3.**
We have proved in Lemma 8.2 that the subsystem \((8.4)\) admits a unique solution \(Q \in L^2(0,T_0;\mathbb{R})\), given by \((8.6)\). Therefore, the moment problem \((8.3)\) is satisfied if and only if the solution \(Q\) given by \((8.6)\) satisfies also the last equation in \((8.3)\).
Since \((e^{it\omega_{\alpha,n}})_{n\in\mathbb{Z}}\) is a Riesz basis in \(L^2(0,T_0;\mathbb{C})\), there exists a unique sequence \((\beta_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})\) such that
\[
\tilde{s}_{\alpha,0}(t) = \sum_{m \in \mathbb{Z}} \beta_m e^{it\omega_{\alpha,m}} \quad \text{in } L^2(0,T_0;\mathbb{C}).
\]
Since \(\tilde{s}_{\alpha,0}\) is real-valued, we have that
\[
\sum_{m \in \mathbb{Z}} \beta_m e^{it\omega_{\alpha,m}} = \tilde{s}_{\alpha,0}(t) = \overline{\tilde{s}_{\alpha,0}(t)} = \sum_{m \in \mathbb{Z}} \overline{\beta_m} e^{-it\omega_{\alpha,m}} = \sum_{m \in \mathbb{Z}} \overline{\beta_{-m}} e^{it\omega_{\alpha,m}},
\]
from which we deduce that
\[
\forall m \in \mathbb{Z}, \quad \overline{\beta_m} = \beta_{-m}.
\]
Thus,
\[
\langle Q, \tilde{s}_{\alpha,0} \rangle_{L^2(0,T_0;\mathbb{C})} = \langle Q, \sum_{m \in \mathbb{Z}} \beta_m e^{it\omega_{\alpha,m}} \rangle_{L^2(0,T_0;\mathbb{C})} = \sum_{m \in \mathbb{Z}} \overline{\beta_m} \langle Q, e^{it\omega_{\alpha,m}} \rangle_{L^2(0,T_0;\mathbb{C})} = \sum_{m \in \mathbb{Z}} \overline{\beta_m} C^f_{\alpha,m}.
\]
Hence, the solution $Q$ given by (8.6) solves (8.3) if and only if
\[
B^I_{\alpha,0} = \sum_{m \in \mathbb{Z}} \beta_m C^I_{\alpha,m},
\]
or, equivalently, if and only if
\[
(8.7) \quad \frac{Y^I_{\alpha,0}}{\mu_{\alpha,0}} = \beta_0 \frac{Z^I_{\alpha,0}}{\mu_{\alpha,0}} + \sum_{m \geq 1} \left( 2(\text{Re} \, \beta_m) \frac{Z^I_{\alpha,m}}{\mu_{\alpha,m}} - 2(\text{Im} \, \beta_m) \omega_{\alpha,m} \frac{Y^I_{\alpha,m}}{\mu_{\alpha,m}} \right),
\]
where the relation of $B^I_{\alpha,0}$ and $C^I_{\alpha,m}$ with $(Y^I, Z^I)$ are given in (8.8) and (8.9).

We now introduce the closed hyperplane $P^I_\alpha$ of $H^3(0) \times D(A)$ defined by
\[
(8.8) \quad P^I_\alpha := \{(Y^I, Z^I) \in H^3(0) \times D(A) \text{ such that } (8.3) \text{ is satisfied}\}.
\]
Therefore (8.3) has a solution if and only if $(Y^I, Z^I) \in P^I_\alpha$.

8.1.f. Proof of Theorem 3.2 first part (inverse mapping argument).

As in section 7.2, we consider the application
\[
(8.9) \quad \Theta_{T_0} : L^2(0, T_0) \to H^3_{(0)} \times D(A), \quad \Theta_{T_0}(p) := (w^{(p)}(T_0), w_t^{(p)}(T_0)).
\]
We recall that $P^I_\alpha$ is the closed hyperplane of $H^3_{(0)} \times D(A)$ defined by (8.8).

From the previous section it follows that the application $D\Theta_{T_0}$ satisfies
\[
D\Theta_{T_0}(0)(L^2(0, T_0)) \subset P^I_\alpha.
\]
Indeed, if $(Y^I, Z^I) = (W^{(0,q)}(T_0), W_t^{(0,q)}(T_0))$, then the moment problem is satisfied and the Fourier coefficients of $Y^I$ and $Z^I$ satisfy (8.7), hence $(Y^I, Z^I) \in P^I_\alpha$.

Moreover,
\[
D\Theta_{T_0}(0) : L^2(0, T_0) \to P^I_\alpha,
\]
is invertible. In fact, it follows from Lemma 8.2 and formula (8.6).

Now consider $(Y^\perp, Z^\perp) \neq 0$ and orthogonal to $P^I_\alpha$: this allows us to decompose the space $H^3_{(0)} \times D(A)$ into
\[
H^3_{(0)} \times D(A) = P^I_\alpha \oplus (P^I_\alpha)\perp
\]
where $(P^I_\alpha)\perp = \mathbb{R}(Y^\perp, Z^\perp)$ is one dimensional. Consider the associated orthogonal projections $\text{proj}_{P^I_\alpha}$ and $\text{proj}_{(P^I_\alpha)\perp}$. Any $(Y, Z) \in H^3_{(0)}(0, 1) \times D(A)$ can be decompose as
\[
(Y, Z) = \text{proj}_{P^I_\alpha}(Y, Z) + \text{proj}_{(P^I_\alpha)\perp}(Y, Z) \quad \text{with} \quad \begin{cases} \text{proj}_{P^I_\alpha}(Y, Z) \in P^I_\alpha, \\ \text{proj}_{(P^I_\alpha)\perp}(Y, Z) \in (P^I_\alpha)\perp. \end{cases}
\]
The application
\[
\hat{\Theta}_{\alpha, T_0} : L^2(0, T_0) \to P^I_\alpha, \quad \hat{\Theta}_{\alpha, T_0}(q) = \text{proj}_{P^I_\alpha}(\Theta_{T_0}(q))
\]
satisfies
\[
D\hat{\Theta}_{\alpha, T_0}(0) : L^2(0, T_0) \to P^I_\alpha, \quad D\hat{\Theta}_{\alpha, T_0}(0) = \text{proj}_{P^I_\alpha}(D\Theta_{T_0}(0)).
\]
Hence $D\hat{\Theta}_{\alpha, T_0}(0) : L^2(0, T_0) \to P^I_\alpha$ is invertible, and therefore the inverse mapping theorem implies that there exists a neighborhood $\mathcal{V}(0) \subset L^2(0, T_0)$ and a neighborhood $\mathcal{V}(\text{proj}_{P^I_\alpha}((1, 0))) \subset P^I_\alpha$ such that
\[
\hat{\Theta}_{\alpha, T_0} : \mathcal{V}(0) \subset L^2(0, T_0) \to \mathcal{V}(\text{proj}_{P^I_\alpha}((1, 0))) \subset P^I_\alpha
\]
is a $C^1$-diffeomorphism. Therefore,

$$\Theta_{T_0} (V(0)) = \{(Y, Z) + \text{proj}_{P^T_0}^+ (\Theta_{T_0} (\tilde{\Theta}^{-1}_{\alpha,T_0} (Y, Z))) , (Y, Z) \in \text{proj}_{P^T_0}^- ((1, 0))\},$$

which means that $\Theta_{T_0} (V(0))$ is the graph of the application

$$\text{proj}_{P^T_0}^- (1, 0) \rightarrow (P^T_0)^\perp, \ (Y, Z) \mapsto \text{proj}_{P^T_0}^- (\Theta_{T_0} (\tilde{\Theta}^{-1}_{\alpha,T_0} (Y, Z))),$$

hence $\Theta_{T_0} (V(0))$ is a submanifold of codimension 1. This concludes the proof of Theorem 3.3 in the case $T = T_0$ and $\alpha \in [0, 1).$ \hfill $\square$

8.2. Proof of Theorem 3.2 second part: Reachability when $T = T_0$ and $\alpha \in [1, 2).$

When $\alpha \in [1, 2)$, we derive from (8.5) that

$$\frac{2}{\pi (2 - \alpha)} \omega_{\alpha,n} - n \rightarrow \frac{\alpha}{4 (2 - \alpha)} \quad \text{as} \quad n \rightarrow \infty.$$ 

However, we notice that

$$\alpha \in [1, 2) \quad \implies \quad \frac{\alpha}{4 (2 - \alpha)} \geq \frac{1}{4}.$$ 

This fact represents the main difference with respect to the analysis of the solvability of moment problem (8.3) of section 8.1 (Lemma 8.1-8.3).

8.2.a. Main solvability results when $\alpha \in [1, 2).$

In this section will prove an extension of the Kadec’s $\frac{1}{4}$ Theorem ([25], [31, Theorem 1.14 p. 42]). Our results are similar to those of [24, Theorem F p. 149], however, thanks to our assumptions, we are able to give very simple statements and proofs.

Lemma 8.4. Consider an odd sequence of real numbers $(x_n)_{n \in \mathbb{Z}}$, that is $x_{-n} = -x_n$, such that there exist $k \in \mathbb{N}^*, \delta \in (0, \frac{1}{4})$ and $N_0 \geq 0$ for which

$$(8.10) \quad n \geq N_0 \quad \implies \quad \left| x_n - n - \frac{k}{2} \right| \leq \frac{1}{4} - \delta.$$

Then, choosing distinct real numbers $x'_1, \ldots, x'_k \notin \{x_n, n \in \mathbb{Z}\}$, the set

$$\{e^{ix'_1 t}, \ldots, e^{ix'_k t}\} \cup \{e^{ix_n t}, n \in \mathbb{Z}\}$$

is a Riesz basis of $L^2(-\pi, \pi)$.

As a consequence of the above result, we deduce the following Lemma.

Lemma 8.5. Let $\alpha \in [1, 2)$. Assume that

$$(8.11) \quad \frac{1}{2 - \alpha} \notin \mathbb{N}.$$ 

Denote by $k_\alpha$ the integer part of $\frac{1}{2 - \alpha}$. Then, the set $\{e^{i\omega_{\alpha,n} t}, n \in \mathbb{Z}\}$ can be complemented by $k_\alpha$ exponentials to form a Riesz basis of $L^2(0, T_0)$.

Lemma 8.5 implies the following result.

Lemma 8.6. Let $\alpha \in [1, 2)$ satisfy (8.11) and let $T = T_0$. Then, moment problem (8.3) has a unique solution

$$Q \in L^2(0, T_0; \mathbb{R}) \in F_\alpha := \text{Vect} \{\tilde{s}_{\alpha,0}, e^{i\omega_{\alpha,n} t}, n \in \mathbb{Z}\}.$$
8.2.b. Proof of Lemma [8.4]

We are going to prove it for \( k = 1 \) and \( k = 2 \), and then the other cases are easily deduced.

**Case \( k \) even.** We consider \( k = 2 \), however the method applies similarly for all \( k \) even. For \( k = 2 \), the assumption reads as: there exist \( \delta \in (0, \frac{1}{4}) \) and \( N_0 \geq 0 \) such that

\[
\text{if } \forall n \in \mathbb{Z}, \quad x_n \in \mathbb{R}, \quad x_n \geq N_0 \implies |x_n - (n + 1)| \leq \frac{1}{4} - \delta.
\]

Then, let us consider the following sequence

\[
\forall n \in \mathbb{Z}, \quad x_n^{(\text{mod})} = \begin{cases} 
  x_{n-1} & \text{if } n \geq N_0 + 1, \\
  n & \text{if } |n| \leq N_0, \\
  x_{n+1} & \text{if } n \leq -N_0 - 1.
\end{cases}
\]

We claim that

\[
\forall n \in \mathbb{Z}, \quad |x_n^{(\text{mod})} - n| \leq \frac{1}{4} - \delta.
\]

Indeed, (8.14) is straightforward for any \( |n| \leq N_0 \). Moreover, if \( n \geq N_0 + 1 \) we get that

\[
|x_n^{(\text{mod})} - n| = |x_{n-1} - n| \leq \frac{1}{4} - \delta
\]

thanks to (8.12). Finally, if \( n \leq -N_0 - 1 \),

\[
|x_n^{(\text{mod})} - n| = |x_{n+1} - n| = |-x_{n+1} + |n|| = |x_{n+1} - |n|| \leq \frac{1}{4} - \delta
\]

once again using (8.12). Then (8.14) is satisfied. We deduce from the Kadec’s \( \frac{1}{4} \) Theorem ([25, 31, Theorem 1.14 p. 42]) that the set \( \{e^{ix_n^{(\text{mod})}t}, n \in \mathbb{Z}\} \) is a Riesz basis of \( L^2(-\pi, \pi) \). However, we can reorder the family \( \{e^{ix_n^{(\text{mod})}t}, n \in \mathbb{Z}\} \) as follows:

\[
\{e^{ix_n^{(\text{mod})}t}, n \in \mathbb{Z}\} = \{e^{ix_n^{(\text{mod})}t}, n \leq -N_0 - 1\} \cup \{e^{ix_n^{(\text{mod})}t}, |n| \leq N_0\} \cup \{e^{ix_n^{(\text{mod})}t}, n \geq N_0 + 1\} = \{e^{ix_m^{\text{mod}}t}, m \leq -N_0\} \cup \{e^{ix_m^{\text{mod}}t}, |n| \leq N_0\} \cup \{e^{ix_m^{\text{mod}}t}, m \geq N_0\} = \{e^{ix_m^{\text{mod}}t}, |m| \geq N_0\} \cup \{e^{ix_n^{(\text{mod})}t}, |n| \leq N_0 - 1\} \cup \{e^{ix_n^{(\text{mod})}t}, |n| = N_0\}.
\]

In order to keep the property to be a Riesz basis, we are allowed to modify a finite number of the elements of the family \( \{e^{ix_n^{(\text{mod})}t}, n \in \mathbb{Z}\} \) (see [4, Lemma II.4.11 p. 105]), if we do not consider twice the same element. Therefore, we can transform the set of \( 2N_0 + 1 \) elements

\[
\{e^{ix_n^{(\text{mod})}t}, |n| \leq N_0 - 1\} \cup \{e^{ix_n^{(\text{mod})}t}, |n| = N_0\}
\]

into

\[
\{e^{ix_n^{\text{mod}}t}, |n| \leq N_0 - 1\} \cup \{e^{ix_n^{\text{mod}}t}, |n| = N_0\} \cup \{e^{ix_0^{\text{mod}}t}, e^{ix_0^{\text{mod}}t}\}
\]

with \( x_0' \neq x_0'' \) and \( x_0', x_0'' \notin \{x_n, n \in \mathbb{Z}\} \). Thus,

\[
\{e^{ix_m^{\text{mod}}t}, m \in \mathbb{Z}\} \cup \{e^{ix_0^{\text{mod}}t}, e^{ix_0^{\text{mod}}t}\}
\]

is a Riesz basis of \( L^2(-\pi, \pi) \). Therefore Lemma [8.4] is proved when \( k = 2 \) and when \( k \) is even, with the same method. \( \square \)
Case $k$ odd. In the same way, we treat the case $k = 1$, which can be easily extended for any $k$ odd. For $k = 1$, the assumption reads as: that there exist $\delta \in (0, \frac{1}{4})$ and $N_0 \geq 0$ such that

$$n \geq N_0 \implies \left| x_n - n - \frac{1}{2} \right| \leq \frac{1}{4} - \delta.$$  

(8. 15)

Consider the following sequence

$$\forall n \in \mathbb{Z}, \quad x_n^{(\text{mod})} = \begin{cases} 
  x_n - \frac{1}{2} & \text{if } n \geq N_0, \\
  n & \text{if } -N_0 \leq n \leq N_0 - 1, \\
  x_{n+1} - \frac{1}{2} & \text{if } n \leq -N_0 - 1.
\end{cases}$$

(8. 16)

We claim that

$$\forall n \in \mathbb{Z}, \quad \left| x_n^{(\text{mod})} - n \right| \leq \frac{1}{4} - \delta.$$  

(8. 17)

Indeed, for $-N_0 \leq n \leq N_0 - 1$ (8. 17) is trivially true. Moreover, for $n \geq N_0$ we have that

$$\left| x_n^{(\text{mod})} - n \right| = \left| x_n - n - \frac{1}{2} \right| \leq \frac{1}{4} - \delta$$

thanks to (8. 15). Finally, for $n \leq -N_0 - 1$,

$$\left| x_n^{(\text{mod})} - n \right| = \left| x_{n+1} - n - \frac{1}{2} \right| = \left| x_{n+1} - x_n \right| + \left| n - \frac{1}{2} \right| = \left| x_{n+1} - x_n \right| + \left| n - \frac{1}{2} \right| \leq 1 - \frac{1}{4} - \delta$$

using that $\{x_n\}_{n \in \mathbb{Z}}$ is odd and, once again, thanks to (8. 15). Then, (8. 17) is satisfied. We deduce from the Kadec’s $\frac{1}{4}$ Theorem ([25], [31, Theorem 1.14 p. 42]) that the set $\{e^{ix_n^{(\text{mod})}t}, n \in \mathbb{Z}\}$ is a Riesz basis of $L^2(-\pi, \pi)$. Now, we shift this basis. To this purpose, we observe that if $f \in L^2(-\pi, \pi)$, then $g : t \mapsto g(t) = f(t)e^{-it/2}$ is still a function of $L^2(-\pi, \pi)$. Hence, it can be decomposed as

$$f(t)e^{-it/2} = \sum_{n \in \mathbb{Z}} c_n e^{ix_n^{(\text{mod})}t} \quad \text{with} \quad A \sum_{n \in \mathbb{Z}} |c_n|^2 \leq \|g\|_{L^2(-\pi, \pi)}^2 \leq B \sum_{n \in \mathbb{Z}} |c_n|^2,$$

since $\{e^{ix_n^{(\text{mod})}t}, n \in \mathbb{Z}\}$ is a Riesz basis of $L^2(-\pi, \pi)$. Therefore, we have that

$$f(t) = \sum_{n \in \mathbb{Z}} c_n e^{ix_n^{(\text{mod})}t + \frac{1}{2}it} \quad \text{with} \quad A \sum_{n \in \mathbb{Z}} |c_n|^2 \leq \|f\|_{L^2(-\pi, \pi)}^2 \leq B \sum_{n \in \mathbb{Z}} |c_n|^2.$$

Hence, the set $\{e^{ix_n^{(\text{mod})}t + \frac{1}{2}it}, n \in \mathbb{Z}\}$ is another Riesz basis of $L^2(-\pi, \pi)$ and can be rewritten as

$$\{e^{ix_n^{(\text{mod})}t + \frac{1}{2}it}, n \in \mathbb{Z}\} = \{e^{ix_n^{(\text{mod})}t}, n \leq -N_0 - 1\} \cup \{e^{ix_n^{(\text{mod})}t + \frac{1}{2}it}, -N_0 \leq n \leq N_0 - 1\} \cup \{e^{ix_n^{(\text{mod})}t}, n \geq N_0\} = \{e^{ix_m^{(\text{mod})}t}, |m| \geq N_0\} \cup \{e^{ix_n^{(\text{mod})}t}, -N_0 \leq n \leq N_0 - 1\}.$$  

The last set on the right-hand side of the above formula contains $2N_0$ elements which can be modified without changing the Riesz basis property as follows

$$\{e^{ix_m^{(\text{mod})}t}, |m| \leq N_0 - 1\} \cup \{e^{ix_0^{(\text{mod})}t}\},$$
where \( x_0' \notin \{ x_n, n \in \mathbb{Z} \} \). Therefore, Lemma 8.3 is proved for \( k = 1 \) and, similarly, for any \( k \) odd.

8.2.c. Proof of Lemma 8.5

We know from (8.5) that

\[
\frac{2}{\pi(2-\alpha)} \omega_{\alpha,n} - n \to \frac{\alpha}{4(2-\alpha)} =: \ell_{\alpha} \quad \text{as } n \to +\infty.
\]

Hence, we introduce

\[
\forall n \in \mathbb{Z}, \quad x_n := \frac{2}{\pi(2-\alpha)} \omega_{\alpha,n}.
\]

Since \( \frac{1}{2-\alpha} \notin \mathbb{N} \), we can decompose it into the sum of its integer part \( k_{\alpha} \) and its fractional part \( \theta_{\alpha} \):

\[
\frac{1}{2-\alpha} = \left[ \frac{1}{2-\alpha} \right] + \left\{ \frac{1}{2-\alpha} \right\} = k_{\alpha} + \theta_{\alpha} \quad \text{where } \theta_{\alpha} \in (0, 1).
\]

Then, we can rewrite \( \ell_{\alpha} \) as

\[
\ell_{\alpha} = \frac{\alpha}{4(2-\alpha)} = \frac{1}{4}(k_{\alpha} + \theta_{\alpha}) \left( 2 - \frac{1}{k_{\alpha} + \theta_{\alpha}} \right) = \frac{1}{4}(2k_{\alpha} + 2\theta_{\alpha} - 1) = \frac{k_{\alpha}}{2} + \frac{2\theta_{\alpha} - 1}{4}.
\]

Hence,

\[
x_n - n - \frac{k_{\alpha}}{2} \to \ell_{\alpha} - \frac{k_{\alpha}}{2} = \frac{2\theta_{\alpha} - 1}{4} \in \left( \frac{1}{4}, \frac{3}{4} \right),
\]

and (8.10) is satisfied with \( k = k_{\alpha} \). Therefore, the set \( \{ e^{i\alpha t}, n \in \mathbb{Z} \} \) can be complemented by \( k_{\alpha} \) exponentials to form a Riesz basis of \( L^2(-\pi, \pi) \). Consequently, the set \( \{ e^{i\omega_{\alpha,n} t}, n \in \mathbb{Z} \} \) can be complemented by \( k_{\alpha} \) exponentials to form a Riesz basis of \( L^2(0, T_0) \) (as we have seen in the proof of Lemma 8.1). This concludes the proof of Lemma 8.5.

8.2.d. Proof of Lemma 8.6

As a consequence of Lemma 8.5, the set of solutions of the moment problem (8.3) is an affine space generated by a vectorial space of dimension \( k_{\alpha} \). To solve the whole moment problem (8.3), it is sufficient to note that

\[
\tilde{s}_{\alpha,0} \notin \text{Vect} \{ e^{i\omega_{\alpha,n} t}, n \in \mathbb{Z} \}:
\]

indeed, if this was not the case, then \( \text{Vect} \{ e^{i\omega_{\alpha,n} t}, n \in \mathbb{Z} \} \) would contain all the polynomials (integrating, as in step 2 of section 7.1). Hence, it would contain \( L^2(0, T_0) \). However, this is contradiction with the fact that \( \text{Vect} \{ e^{i\omega_{\alpha,n} t}, n \in \mathbb{Z} \} \) is of codimension \( k_{\alpha} \) in \( L^2(0, T_0) \). Therefore, proceeding as in step 2 of section 7.1, we deduce that there exists a unique solution in \( \text{Vect} \{ \tilde{s}_{\alpha,0}, e^{i\omega_{\alpha,n} t}, n \in \mathbb{Z} \} \) of moment problem (8.3). Note that this solution is real-valued: indeed, we derive from (8.3) that

\[
\int_0^T Q(t) \cos \omega_{\alpha,n} t \, dt, \int_0^T Q(t) \sin \omega_{\alpha,n} t \, dt, \int_0^T Q(t) \, dt \in \mathbb{R}.
\]

Thus, denoting by \( Q_2 \) the imaginary part of \( Q \), we have that

\[
\int_0^T Q_2(t) \cos \omega_{\alpha,n} t \, dt = \int_0^T Q_2(t) \sin \omega_{\alpha,n} t \, dt = \int_0^T Q_2(t) \, dt = 0.
\]
Hence, the following conditions
\[
\begin{align*}
\int_0^T Q_2(t) e^{-i\omega_{\alpha,n} t} \, dt &= 0 \quad \text{for all } n \in \mathbb{Z}, \\
\int_0^T Q_2(t) t \, dt &= 0,
\end{align*}
\]
implies that \( Q_2 \) is orthogonal to \( \text{Vect}\{\tilde{s}_{\alpha,0}, e^{i\omega_{\alpha,n} t}, n \in \mathbb{Z}\} \). However, \( 2iQ_2 = Q - \overline{Q} \in \text{Vect}\{\tilde{s}_{\alpha,0}, e^{i\omega_{\alpha,n} t}, n \in \mathbb{Z}\} \). Therefore, \( Q_2 = 0 \) and \( Q \) is real-valued. \( \square \)

8.2.e. **Proof of Theorem 3.2 second part (inverse mapping argument).**

Thanks to the previous results, we can conclude the proof Theorem 3.2 with the same procedure explained in section 7.2. \( \square \)

9. **Proof of Theorem 3.3: Reachability for \( T < T_0 \)**

9.1. **Proof of Theorem 3.3 when \( \alpha \in [0, 1) \).**

We recall that in Lemma 8.1 we have proved that \( (e^{i\omega_{\alpha,n} t})_{n \in \mathbb{Z}} \) is a Riesz basis of \( L^2(0, T_0) \).

As in sections 7 and 8.1.b, the proof of Theorem 3.3 is divided in two steps:

- we first study the solvability of moment problem (8.1),
- then, we conclude by an inverse mapping argument.

9.1.a. **The moment problem (8.3) is overdetermined when \( T < T_0 \).**

Let \( T < T_0 \). Following [4, p. 100], we introduce
\[
\forall r > 0, \quad n(r) := \text{card}\{n \in \mathbb{Z}, |\omega_{\alpha,n}| < r\}.
\]

We recall from Propositions 3.1 and 3.2 that the sequence \( (\omega_{\alpha,n+1} - \omega_{\alpha,n})_n \) is nonincreasing and goes to \( \kappa_\alpha\pi \) as \( n \to \infty \). Hence, we deduce that
\[
\forall n \geq 0, \quad \omega_{\alpha,n} \geq \kappa_\alpha\pi(n - 1).
\]

Therefore,
\[
n - 1 \geq \frac{r}{\kappa_\alpha\pi} \quad \implies \quad \omega_{\alpha,n} \geq r.
\]

This gives that
(9.1)
\[
n(r) \leq 2\frac{r}{\kappa_\alpha\pi} + 1,
\]
where the factor 2 comes from the negatives indices.

On the other hand, given \( \varepsilon > 0 \), there exists \( n_0 \geq 0 \) such that
\[
\forall n \geq n_0, \quad \kappa_\alpha\pi \leq \omega_{\alpha,n+1} - \omega_{\alpha,n} \leq \kappa_\alpha\pi + \varepsilon.
\]

Hence,
\[
\forall n \geq n_0, \quad \omega_{\alpha,n_0} + \kappa_\alpha\pi(n - n_0) \leq \omega_{\alpha,n} \leq \omega_{\alpha,n_0} + (\kappa_\alpha\pi + \varepsilon)(n - n_0).
\]

Thus, given \( r > 0 \)
\[
0 \leq n < n_0 + \frac{r - \omega_{\alpha,n_0}}{\kappa_\alpha\pi + \varepsilon} \quad \implies \quad \omega_{\alpha,n} < r.
\]

We derive that for all \( r \) large enough
(9.2)
\[
n(r) \geq 2\left(n_0 + \frac{r - \omega_{\alpha,n_0}}{\kappa_\alpha\pi + \varepsilon}\right) - 1,
\]

Then, we obtain from (9.1) and (9.2) that
(9.3)
\[
\lim_{r \to +\infty} \frac{n(r)}{r} = \frac{2}{\kappa_\alpha\pi} = \frac{4}{(2 - \alpha)\pi} = \frac{T_0}{\pi}.
\]
Since $T < T_0$, we have
\[
\limsup_{r \to +\infty} \frac{n(r)}{r} > \frac{T}{\pi},
\]
and so we can apply [4, Corollary II.4.2 p. 100] and we obtain that the family \( \{ e^{i\omega_n t}, n \in \mathbb{Z} \} \) is not minimal in \( L^2((0, T), \mathbb{C}) \).

9.1.b. How much overdetermined the moment problem (8.3) is when $T < T_0$.

As proved in Lemma 8.1, \( \{ e^{i\omega_n t}, n \in \mathbb{Z} \} \) is a Riesz basis of \( L^2((0, T_0), \mathbb{C}) \). From Horváth-Jóó [23] (see also [4, Theorem II.4.16 p. 107]) we deduce that there exists a subfamily \( \{ e^{i\omega_n t}, n \in \mathbb{Z} \} \) which is a Riesz basis of \( L^2((0, T), \mathbb{C}) \). Then, consider
\[
\forall r > 0, \quad n_{\varphi}(r) := \text{card} \{ n \in \mathbb{Z}, |\omega_{\alpha,n}| < r \}.
\]
Since \( \{ e^{i\omega_n t}, n \in \mathbb{Z} \} \) is minimal in \( L^2((0, T), \mathbb{C}) \), we derive from [4, Corollary II.4.2 p. 100], that
\[
\limsup_{r \to +\infty} \frac{n_{\varphi}(r)}{r} \leq \frac{T}{\pi},
\]
hence,
\[(9.4) \quad \liminf_{r \to +\infty} \frac{n(r) - n_{\varphi}(r)}{r} \geq \frac{T_0 - T}{\pi}.
\]
Since \( n(r) - n_{\varphi}(r) = \text{card} \{ n \in \mathbb{Z}, |\omega_{\alpha,n}| < r \text{ and } n \notin \text{Im } \varphi \} \), the asymptotic behaviour (9.4) gives an idea of how much overdetermined the moment problem (8.3) is.

9.1.c. Solvability of the moment problem.

We consider (8.1), or equivalently (8.3). First, assume that (8.3) has a solution \( Q \in L^2(0, T; \mathbb{R}) \). This implies that
\[(9.5) \quad \langle Q, e^{i\omega_{\alpha,n}(t)} \rangle_{L^2(0,T;\mathbb{C})} = C_{\alpha,\varphi}(n) \quad \text{for all } n \in \mathbb{Z}.
\]
Now, consider \( m \notin \text{Im } \varphi \). Then, \( e^{i\omega_{\alpha,m} t} \) can be decomposed as follows
\[
e^{i\omega_{\alpha,m} t} = \sum_{n \in \mathbb{Z}} \Omega_{\alpha,\varphi}(n) e^{i\omega_{\alpha,n}(t)} \quad \text{in } L^2(0, T; \mathbb{C}).
\]
Therefore, we have that
\[
C_{\alpha,m}^f = \langle Q, e^{i\omega_{\alpha,m}(t)} \rangle_{L^2(0,T;\mathbb{C})} = \langle Q, \sum_{n \in \mathbb{Z}} \Omega_{\alpha,\varphi}(n) e^{i\omega_{\alpha,n}(t)} \rangle_{L^2(0,T;\mathbb{C})}
\[
= \sum_{n \in \mathbb{Z}} \Omega_{\alpha,\varphi}(n) \langle Q, e^{i\omega_{\alpha,n}(t)} \rangle_{L^2(0,T;\mathbb{C})} = \sum_{n \in \mathbb{Z}} \Omega_{\alpha,\varphi}(n) C_{\alpha,\varphi}(n).
\]
In the same way, \( \hat{s}_{\alpha,0} \) can be decomposed as follows
\[
\hat{s}_{\alpha,0} = \sum_{n \in \mathbb{Z}} \hat{s}_{\alpha,\varphi}(n) e^{i\omega_{\alpha,n}(t)} \quad \text{in } L^2(0, T; \mathbb{C}),
\]
which implies that
\[
B_{\alpha,0}^f = \sum_{n \in \mathbb{Z}} \hat{s}_{\alpha,\varphi}(n) C_{\alpha,\varphi}(n).
\]
Hence,

\[ Q \text{ solution of } (8.3) \implies \begin{cases} C^f_{\alpha,m} = \sum_{n \in \mathbb{Z}} \tilde{\Omega}^{(m)}_{\alpha,\varphi(n)} C^f_{\alpha,\varphi(n)} & \forall m \notin \text{Im } \varphi, \\ B^f_{\alpha,0} = \sum_{n \in \mathbb{Z}} S_{\alpha,\varphi(n)} C^f_{\alpha,\varphi(n)}. \end{cases} \]

This leads to consider the space (9.6)

\[ H^f_{\alpha} := \{(Y^f, Z^f) \in H^2_{(0)} \times D(A), \begin{cases} C^f_{\alpha,m} = \sum_{n \in \mathbb{Z}} \tilde{\Omega}^{(m)}_{\alpha,\varphi(n)} C^f_{\alpha,\varphi(n)} & \forall m \notin \text{Im } \varphi, \\ B^f_{\alpha,0} = \sum_{n \in \mathbb{Z}} S_{\alpha,\varphi(n)} C^f_{\alpha,\varphi(n)} \end{cases} \} \]

where the relations between \((Y^f, Z^f)\) and \(B^f_{\alpha,0}, C^f_{\alpha,m}\), \(m \geq 1\) are given in (7.8) and (8.2). Thus, we have proved that (9.7)

\[ D\Theta_T(0)(L^2(0, T; \mathbb{R}) \subset H^f_{\alpha}. \]

Now, let us prove the following reverse inclusion.

**Lemma 9.1.** Let \(\alpha \in [0, 1), T < T_0\) and \(H^f_{\alpha}\) be defined in (9.6). Then, the following identity holds (9.8)

\[ D\Theta_T(0)(L^2(0, T; \mathbb{R}) = H^f_{\alpha}. \]

**Proof of Lemma 9.1.** Since we already proved (9.7), it is sufficient to prove that \(H^f_{\alpha} \subset D\Theta_T(0)(L^2(0, T; \mathbb{R}))\). Let \((Y^f, Z^f) \in H^f_{\alpha}\). Since \((e^{i\omega_{\alpha,\varphi(n)} t})_{n \in \mathbb{Z}}\) is a Riesz basis of \(L^2(0, T)\), the moment problem (4.5) has one and only one solution \(Q\) (which can be expressed using the unique biorthogonal family to \((e^{i\omega_{\alpha,\varphi(n)} t})_{n \in \mathbb{Z}}\)). Then, for all \(m \notin \text{Im } \varphi\), we have

\[ \langle Q, e^{i\omega_{\alpha,\varphi(n)} t} \rangle_{L^2(0, T; \mathbb{C})} = \sum_{n \in \mathbb{Z}} \tilde{\Omega}^{(m)}_{\alpha,\varphi(n)} e^{i\omega_{\alpha,\varphi(n)} t} \langle Q(e^{i\omega_{\alpha,\varphi(n)} t})_{L^2(0, T; \mathbb{C})} = \sum_{n \in \mathbb{Z}} \tilde{\Omega}^{(m)}_{\alpha,\varphi(n)} C^f_{\alpha,\varphi(n)} = C^f_{\alpha,0}, \]

where the last equality derives from the fact that \((Y^f, Z^f) \in H^f_{\alpha}\). In the same way, we get

\[ \langle Q, \tilde{s}_{\alpha,0} \rangle_{L^2(0, T; \mathbb{C})} = \sum_{n \in \mathbb{Z}} S_{\alpha,\varphi(n)} \langle Q(e^{i\omega_{\alpha,\varphi(n)} t})_{L^2(0, T; \mathbb{C})} = \sum_{n \in \mathbb{Z}} S_{\alpha,\varphi(n)} C^f_{\alpha,\varphi(n)} = B^f_{\alpha,0}. \]

Hence, \(Q\) solves the whole moment problem (8.3). It remains to prove that \(Q\) is real-valued: this follows easily from the fact that

\[ \forall n \geq 0, \begin{cases} \int_0^T Q(t) e^{-i\omega_{\alpha,n} t} dt = C^f_{\alpha,n} = \frac{Z^f_{\alpha,n} - i \sqrt{\lambda_{\alpha,n}} Y^f_{\alpha,n}}{\mu_{\alpha,n}}, \\ \int_0^T Q(t) e^{i\omega_{\alpha,n} t} dt = C^f_{\alpha,n} = \frac{Z^f_{\alpha,n} + i \sqrt{\lambda_{\alpha,n}} Y^f_{\alpha,n}}{\mu_{\alpha,n}}. \end{cases} \]

By adding (subtracting) one to each other the above equations, we obtain

\[ \forall n \geq 0, \begin{cases} \int_0^T Q(t) \cos \omega_{\alpha,n} t dt = \frac{2 Z^f_{\alpha,n}}{\mu_{\alpha,n}}, \\ \int_0^T Q(t) 2i \sin \omega_{\alpha,n} t dt = \frac{2i \sqrt{\lambda_{\alpha,n}} Y^f_{\alpha,n}}{\mu_{\alpha,n}}. \end{cases} \]
Thus, the real part of $Q$ solves (7.7), and its imaginary part $Q_2$ satisfies

$$
\forall n \geq 0, \quad \int_0^T Q_2(t) \cos \omega_{\alpha,n} t \, dt = 0 = \int_0^T Q_2(t) \sin \omega_{\alpha,n} t \, dt.
$$

Therefore,

$$
\forall n \in \mathbb{Z}, \quad \langle Q_2, e^{i \omega_{\alpha,n} t} \rangle_{L^2(0,T;\mathbb{C})} = 0 = \langle Q_2, e^{i \omega_{\alpha,n} t} \rangle_{L^2(0,T;\mathbb{C})}.
$$

which implies $Q_2 = 0$ since $(e^{i \omega_{\alpha,n} t})_{n \in \mathbb{Z}}$ is a Riesz basis of $L^2(0,T)$. So, we have proved that $Q$ is real-valued and this completes the proof of (9.8). □

We conclude by proving the following

**Lemma 9.2.** $H^I_{\alpha}$ is a closed vectorial space of $H^3_0 \times D(A)$ of infinite dimension and infinite codimension.

**Proof of Lemma 9.2.** Let us consider

$$
\forall m \notin \text{Im } \varphi, \quad L^m_T : H^3_0 \times D(A) \to \mathbb{C}, \quad L^m_T(Y^f, Z^f) := C^f_{\alpha,m} - \sum_{n \in \mathbb{Z}} \Omega_{\alpha,\varphi(n)}^{(m)} C^{f}_{\alpha,\varphi(n)},
$$

and

$$
\ell^0_T : H^3_0 \times D(A) \to \mathbb{C}, \quad \ell^0_T(Y^f, Z^f) := B^f_{\alpha,0} - \sum_{n \in \mathbb{Z}} S_{\alpha,\varphi(n)} C^f_{\alpha,\varphi(n)}.
$$

Observe that $\ell^0_T$ and $L^m_T$ are linear continuous forms, and

$$
H^I_{\alpha} = \text{Ker } \ell^0_T \cap (\cap_{m \notin \text{Im } \varphi} \text{Ker } L^m_T).
$$

Hence, $H^I_{\alpha}$ is a closed vectorial space of $H^3_0 \times D(A)$, and is of infinite dimension. To prove that $H^I_{\alpha}$ has infinite codimension, we use the fact that $\mathbb{Z} \setminus (\text{Im } \varphi)$ is infinite (see section 9.1.b). Then, fix $N \geq 1$, and $n_1, \ldots, n_N \notin (\text{Im } \varphi \cup \{0\})$, and consider

$$
L^{n_1,\ldots,n_N}_T : H^3_0 \times D(A) \to \mathbb{C}^N, \quad L^{n_1,\ldots,n_N}_T(Y^f, Z^f) := \left( L^{n_1}_T(Y^f, Z^f), \ldots, L^{n_N}_T(Y^f, Z^f) \right).
$$

Observe that

$$
L^{n_1,\ldots,n_N}_T(\Phi_{\alpha,n_1}, 0) = -\frac{i \omega_{\alpha,n_1}}{\mu_{\alpha,n_1}} \epsilon_1, \quad \ldots, \quad L^{n_1,\ldots,n_N}_T(\Phi_{\alpha,n_N}, 0) = -i \frac{\omega_{\alpha,n_N}}{\mu_{\alpha,n_N}} \epsilon_N,
$$

where $\epsilon_1, \ldots, \epsilon_N$ is the canonical basis of $\mathbb{C}^N$. Hence, $L^{n_1,\ldots,n_N}_T$ is surjective, and its kernel is of codimension $N$. Since this is true for all $N \geq 1$, this implies that $H^I_{\alpha}$ is a closed vectorial space of $H^3_0 \times D(A)$ of infinite codimension. □

9.1.d. **Proof of Theorem 3.3 for $\alpha \in [0,1)$ (inverse mapping argument).**

The proof follows the scheme of section 8.1.d, replacing the hyperplane $P^I_\alpha$ by $H^I_{\alpha}$. Therefore, $\Theta_T(\mathcal{V}(0))$ turns out to be a submanifold of infinite dimension and infinite codimension. □
9.2. Proof of Theorem 3.3: modifications when $\alpha \in [1, 2)$.

When $\alpha \in (1, 2)$ and $\frac{1}{2-\alpha} \notin \mathbb{N}$, we replace Lemma 8.1 by Lemma 8.5 to obtain that $\{e^{i\omega_n t}\}_{n \in \mathbb{Z}}$, complemented by a finite number of exponentials $\{e^{i\omega_0 t}, \ldots, e^{i\omega_\kappa t}\}$, is a Riesz basis of $L^2(0, T_0)$. Then, one can extract a subfamily of $\{e^{i\omega_n t}\}_{n \in \mathbb{Z}} \cup \{e^{i\omega_0 t}, \ldots, e^{i\omega_\kappa t}\}$ that will be a Riesz basis of $L^2(0, T)$. Replacing the possible elements coming from the finite set of exponentials $\{e^{i\omega_0 t}, \ldots, e^{i\omega_\kappa t}\}$ by the same number of exponentials of the original family $\{e^{i\omega_n t}\}_{n \in \mathbb{Z}}$, we have a subfamily of $\{e^{i\omega_n t}\}_{n \in \mathbb{Z}}$ that is a Riesz basis of $L^2(0, T)$, and then we can complete the proof as in section 9.1.

In a more general way, without assuming $\frac{1}{2-\alpha} \notin \mathbb{N}$, we consider $x \in \mathbb{R}$, $r > 0$ and define

$$N(x, r) := \text{card} \{\omega_{n, x} \leq \omega_{n, x} < x + r\}.$$ 

Thanks to (8.5), one can prove that

$$\frac{N(x, r)}{r} \to \frac{T_0}{2\pi} \quad \text{as } r \to +\infty$$

uniformly with respect to $x \in \mathbb{R}$. Indeed, assume first that we are in the simplest case:

$$\forall n \geq 1, \quad \omega_{n, x} = \frac{\pi}{2}(2-\alpha) + \frac{\alpha}{2(2-\alpha)}n + \frac{\pi\alpha}{8},$$

and so the gap $\omega_{n+1, x} - \omega_{n, x}$ is constant with respect to $n \geq 1$:

$$\forall n \geq 1, \quad \omega_{n+1, x} - \omega_{n, x} = \frac{\pi}{2}(2-\alpha).$$

Then it is clear that any "window" $[x, x + r]$ contains $N(x, r)$ terms, where $N(x, r)$ behaves as $\frac{r^2}{(2-\alpha)^2}$. More precisely, there exists some computable $N_0$ such that

$$\forall x \in \mathbb{R}, x > 0, \quad \frac{2}{\pi(2-\alpha)}r - N_0 \leq N(x, r) \leq \frac{2}{\pi(2-\alpha)}r + N_0.$$ 

Then, in this case we have

$$\forall x \in \mathbb{R}, x > 0, \quad \frac{2}{\pi(2-\alpha)}r - N_0 \leq N(x, r) \leq \frac{2}{\pi(2-\alpha)}r + N_0,$$

and therefore

$$\frac{N(x, r)}{r} \to \frac{2}{\pi(2-\alpha)} \quad \text{as } r \to +\infty \quad \text{uniformly with respect to } x \in \mathbb{R}.$$

Hence, for this particular case, (9, 9) is satisfied.

The general case is not as simple as the one assumed in (9, 10). However, (8, 5) implies that, given $\varepsilon_1 > 0$ small, there exists $N_1 \geq 0$ such that

$$\forall n \geq N_1, \quad \omega_{n, x} \in \left(\frac{\pi}{2}(2-\alpha)n + \frac{\pi\alpha}{8} - \varepsilon_1, \frac{\pi}{2}(2-\alpha)n + \frac{\pi\alpha}{8} + \varepsilon_1\right).$$

So, if $x \geq x_1 := \frac{2}{\pi(2-\alpha)}N_1 + \frac{\pi\alpha}{8} + \varepsilon_1$, the situation is very close to the simple one studied before, and there exists $N_2$ such that

$$\forall x \geq x_1, \forall r > 0, \quad \frac{2}{\pi(2-\alpha)}r - N_2 \leq N(x, r) \leq \frac{2}{\pi(2-\alpha)}r + N_2.$$ 

By symmetry, the situation is similar if $x + r \leq -x_1$. And finally, if the window $[x, x + r]$ intersects $[-x_1, x_1]$, then only the location of $\omega_{n, x_1}, \ldots, \omega_{n, -x_1}$ can
modify the counting number \(N(x,r)\), and thus there exists \(N_3\) (depending only on \(N_1\) and \(N_2\)) such that

\[
\forall x \in \mathbb{R}, \forall r > 0, \quad \frac{2}{\pi(2-\alpha)} r - N_3 \leq N(x,r) \leq \frac{2}{\pi(2-\alpha)} r + N_3.
\]

Therefore the conclusion (9.9) follows also for the general case.

Now that (9.9) is proved, we deduce from (9.9) and from [4, Theorem II.4.18 p. 109] that, given \(T < T_0\), the family \(\{e^{i\omega \cdot x}\} \in \mathbb{Z}\) contains a subfamily that forms a Riesz basis of \(L^2(0,T)\). And then, once again, we can proceed as in section 9.1 to prove Theorem 8.3.

10. PROOF OF PROPOSITION 3.4

First we check that

\[
\mu(x) = x^{2-\alpha}
\]
satisfies all the regularity assumptions:

- \(\alpha \in [0,1)\): first, we observe that \(\mu'(x) = (2-\alpha)x^{1-\alpha} \in L^1(0,1)\). Hence, \(\mu\) is absolutely continuous on \([0,1)\). Moreover, \(x^{\alpha/2}\mu'(x) = (2-\alpha)x^{1-\frac{\alpha}{2}} \in L^2(0,1)\). Thus, \(\mu \in H^1_0(0,1)\). Furthermore, \(x^\alpha \mu'(x) = (2-\alpha)x \in H^1(0,1)\) and so \(\mu \in H^2_0(0,1)\). Finally, we observe that \(x^{\alpha/2}\mu'(x) = (2-\alpha)x^{1-\frac{\alpha}{2}} \in L^\infty(0,1)\) that implies that \(\mu \in V^2(2,\infty)(0,1)\):

- \(\alpha \in [1,2)\): in this case it is easy to check that \(\mu \in L^2(0,1)\). Moreover, \(x^{\alpha/2}\mu'(x) = (2-\alpha)x^{1-\frac{\alpha}{2}} \in L^2(0,1)\) and therefore \(\mu \in H^1_0(0,1)\). Furthermore, \(x^\alpha \mu'(x) = (2-\alpha)x \in H^1(0,1)\). Thus, we have that \(\mu \in H^2_0(0,1)\). Finally, we note that \(x^{\alpha/2}\mu'(x) = (2-\alpha)x^{1-\frac{\alpha}{2}} \in L^\infty(0,1)\), and \((x^\alpha \mu')' = 2-\alpha \in L^\infty(0,1)\). Hence, \(\mu \in V^{2,\infty}(2,\infty)(0,1)\).

We have showed that the regularity assumptions are satisfied. It remains to check the validity of (3.18). Direct computations show that

\[
\langle \mu, \Phi_{\alpha,n} \rangle_{L^2(0,1)} = \int_0^1 x^{2-\alpha} dx = \frac{1}{3-\alpha},
\]

and, for all \(n \geq 1\), we develop the scalar product as follows

\[
\langle \mu, \Phi_{\alpha,n} \rangle_{L^2(0,1)} = \int_0^1 \mu(x)\Phi_{\alpha,n} dx = \frac{1}{\lambda_{\alpha,n}} \int_0^1 \mu(x)\lambda_{\alpha,n}\Phi_{\alpha,n} dx
\]

\[
= \frac{1}{\lambda_{\alpha,n}} \int_0^1 \mu(x)(-x^\alpha \Phi'_{\alpha,n}) dx
\]

\[
= \frac{1}{\lambda_{\alpha,n}} \left( [-x^\alpha \mu(x)\Phi'_{\alpha,n}(x)]_0^1 + \int_0^1 x^\alpha \mu'(x)\Phi_{\alpha,n}(x) \right).
\]

Recalling that \(\mu(x) = x^{2-\alpha}\), we obtain

\[
\int_0^1 x^\alpha \mu'(x)\Phi_{\alpha,n}(x) = (2-\alpha) \int_0^1 x^{\alpha-1}\Phi_{\alpha,n}(x)
\]

\[
= (2-\alpha)\int_0^1 x\Phi_{\alpha,n}(x) dx - (2-\alpha)\int_0^1 \Phi_{\alpha,n}(x) dx
\]

\[
= (2-\alpha)\int_0^1 x\Phi_{\alpha,n}(x) dx - (2-\alpha)\langle \Phi_{\alpha,0}, \Phi_{\alpha,n} \rangle_{L^2(0,1)}.
\]
Finally, once again from Lemmas 5.1 and 5.2 we have that
\[ | \Phi_{\alpha,n}(x) | \]
therefore
\[ \alpha,n \]
We have also proved (Lemmas 5.1 and 5.2) that \( \Phi \)
Finally, since \((x_0 \alpha, \Phi_{\alpha,n}) = \sqrt{2 - \alpha} \) that
yields
\[ |(2 - \alpha)|x_{\alpha,n}(x) | \leq (2 - \alpha)^{3/2} \]
and
\[ |\langle \mu, \Phi_{\alpha,n} \rangle_{L^2(0,1)}| = \frac{(2 - \alpha)^{3/2}}{\lambda_{\alpha,n}}. \]
Hence, (3. 18) is satisfied.

Now, let us prove that the set of functions \( \mu \) satisfying (3. 18) is dense in \( V^2_\alpha \). By integrating by parts, we get
\[
\langle \mu, \Phi_{\alpha,n} \rangle_{L^2(0,1)} = \frac{1}{\lambda_{\alpha,n}} \left( [-x^\alpha \mu(x) \Phi_{\alpha,n}'(x)]_{0}^{1} + \int_{0}^{1} x^\alpha \mu'(x) \Phi_{\alpha,n}(x) \, dx \right)
\]
\[ = \frac{1}{\lambda_{\alpha,n}} \left( [-x^\alpha \mu(x) \Phi_{\alpha,n}'(x)]_{0}^{1} + [x^\alpha \mu'(x) \Phi_{\alpha,n}(x)]_{0}^{1} - \int_{0}^{1} (x^\alpha \mu'(x)) L_{\alpha,n} \Phi_{\alpha,n}(x) \, dx \right). \]

Then, since \( \mu \in L^\infty(0,1) \), we have
\[ [-x^\alpha \mu(x) \Phi_{\alpha,n}'(x)]_{0}^{1} = 0. \]
Moreover, since \( x^\alpha \mu' \in L^\infty(0,1) \) and \( \Phi_{\alpha,n} \) has a finite limit as \( x \to 0 \), we deduce
\[ x^\alpha \mu'(x) \Phi_{\alpha,n}(x) \to 0 \quad \text{as} \quad x \to 0. \]
Thus, we obtain that
\[ [x^\alpha \mu'(x) \Phi_{\alpha,n}(x)]_{0}^{1} = \mu'(1) \Phi_{\alpha,n}(1). \]
Finally, since \( (x^\alpha \mu'(x))' \in L^2(0,1) \), we get
\[ \int_{0}^{1} (x^\alpha \mu'(x))' \Phi_{\alpha,n}(x) \, dx = \langle (x^\alpha \mu'(x))', \Phi_{\alpha,n} \rangle_{L^2(0,1)} \to 0, \quad \text{as} \quad n \to \infty. \]
So, recalling that \(|\Phi_{\alpha,n}(1)| = \sqrt{2 - \alpha}\), we infer
\[ \mu \in V^2_\alpha(0,1) \quad \Longrightarrow \quad |\lambda_{\alpha,n} \langle \mu, \Phi_{\alpha,n} \rangle_{L^2(0,1)}| \to \sqrt{2 - \alpha} |\mu'(1)|, \quad \text{as} \quad n \to \infty. \]
We define the spaces
\[ V_n := \begin{cases} 
\{ \mu \in V^2_\alpha(0,1), \langle \mu, \Phi_{\alpha,n} \rangle_{L^2(0,1)} \neq 0 \} & \text{for } n \geq 0, \\
\{ \mu \in V^2_\alpha(0,1), \mu'(1) \neq 0 \} & \text{for } n = -1,
\end{cases} \]
and

\[ V_2^\alpha := \cap_{n=1}^{\infty} V_n. \]

Every \( V_n \) is open and dense in \( V_2^\alpha \). Indeed, consider \( \mu \in V_2^\alpha(0,1) \) such that \( \mu \notin V_n \) for some \( n \geq -1 \), and define

\[ \tilde{\mu}_\varepsilon(x) := \mu(x) + \varepsilon x^{2-\alpha} \]

where \( \varepsilon \in \mathbb{R}^* \). Then, if \( n \geq 0 \), we have

\[ \langle \tilde{\mu}_\varepsilon, \Phi_{a,n} \rangle_{L^2(0,1)} = \varepsilon \langle x^{2-\alpha}, \Phi_{a,n} \rangle_{L^2(0,1)} \neq 0, \]

and if \( n = -1 \), we have

\[ \tilde{\mu}'_\varepsilon(1) = \varepsilon (2 - \alpha) \neq 0. \]

Therefore, \( \tilde{\mu}_\varepsilon \in V_n \) and it is close to \( \mu \) in \( V_2^\alpha \) if \( \varepsilon \) is sufficiently small. This proves that \( V_n \) is dense in \( V_2^\alpha \). Thus, \( V_2^\alpha \) is the intersection of a sequence of open and dense subsets and, thanks to Baire Theorem, it is dense in \( V_2^\alpha \). \( \square \)

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References

[1] F. Alabau-Boussouira, P. Cannarsa, and G. Leugering. Control and stabilization of degenerate wave equations. *SIAM J. Control Optim.*, 55(3):2052–2087, 2017.

[2] F. Alabau-Boussouira, P. Cannarsa, and C. Urbani. Exact controllability to eigensolutions for evolution equations of parabolic type via bilinear control. *arXiv preprint arXiv:2105.05732*, 2021.

[3] F. Alabau-Boussouira, P. Cannarsa, and C. Urbani. Super exponential stabilizability of evolution equations of parabolic type via bilinear control. *J. Evol. Equ.*, 21(1):941–967, 2021.

[4] S. A. Avdonin and S. A. Ivanov. *Families of exponentials*. Cambridge University Press, Cambridge, 1995. The method of moments in controllability problems for distributed parameter systems, Translated from the Russian and revised by the authors.

[5] J. M. Ball, J. E. Marsden, and M. Slemrod. Controllability for distributed bilinear systems. *SIAM J. Control Optim.*, 20(4):575–597, 1982.

[6] K. Beauchard. Local controllability and non-controllability for a 1D wave equation with bilinear control. *J. Differential Equations*, 250(4):2064–2098, 2011.

[7] K. Beauchard and C. Laurent. Local controllability of 1D linear and nonlinear Schrödinger equations with bilinear control. *J. Math. Pures Appl. (9)*, 94(5):520–554, 2010.

[8] A. Bensoussan, G. da Prato, M.C. Delfour, and S.K. Mitter. *Representation and control of infinite dimensional systems*. Systems & Control: Foundations & Applications. Birkhäuser Boston, Inc., Boston, MA, second edition, 2007.

[9] P. Cannarsa, G. Floridia, and A.Y. Khapalov. Multiplicative controllability for semilinear reaction-diffusion equations with finitely many changes of sign. *J. Math. Pures Appl. (9)*, 108(4):425–458, 2017.

[10] P. Cannarsa, P. Martinez, and J. Vancostenoble. Carleman estimates for a class of degenerate parabolic operators. *SIAM J. Control Optim.*, 47(1):1–19, 2008.

[11] P. Cannarsa, P. Martinez, and J. Vancostenoble. *Global Carleman estimates for degenerate parabolic operators with applications*, volume 239. 2016.

[12] P. Cannarsa, P. Martinez, and J. Vancostenoble. The cost of controlling weakly degenerate parabolic equations by boundary controls. *Math. Control Relat. Fields*, 7(2):171–211, 2017.

[13] P. Cannarsa, P. Martinez, and J. Vancostenoble. The cost of controlling strongly degenerate parabolic equations. *ESAIM Control Optim. Calc. Var.*, 26:Paper No. 2, 50, 2020.

[14] P. Cannarsa and C. Urbani. Superexponential stabilizability of degenerate parabolic equations via bilinear control. In *International Conference on Inverse Problems*, pages 31–45. Springer, 2018.

[15] A. Duca. Controllability of bilinear quantum systems in explicit times via explicit control fields. *Internat. J. Control*, 94(3):724–734, 2021.
[16] C.L. Epstein and R. Mazzeo. *Degenerate diffusion operators arising in population biology*, volume 185 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2013.

[17] G. Floridia. Approximate controllability for nonlinear degenerate parabolic problems with bilinear control. *J. Differential Equations*, 257(9):3382–3422, 2014.

[18] G. Floridia. Nonnegative multiplicative controllability for semilinear multidimensional reaction-diffusion equations. *Minimax Theory Appl.*, 6(2):341–352, 2021.

[19] G. Floridia, C. Nitsch, and C. Trombetti. Multiplicative controllability for nonlinear degenerate parabolic equations between sign-changing states. *ESAIM Control Optim. Calc. Var.*, 26:Paper No. 18, 34, 2020.

[20] M. Ghil. Climate stability for a Sellers-type model. *J. Atmospheric Sci.*, 33(1):3–20, 1976.

[21] M. Gueye. Exact boundary controllability of 1-D parabolic and hyperbolic degenerate equations. *SIAM J. Control Optim.*, 52(4):2037–2054, 2014.

[22] A. Haraux. Séries lacunaires et contrôle semi-interne des vibrations d’une plaque rectangulaire. *J. Math. Pures Appl. (9)*, 68(4):457–465 (1990), 1989.

[23] M. Horváth and I. Jóó. On Riesz bases. II. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.*, 33:267–271 (1991), 1990.

[24] I. Jóó. On Riesz bases. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.*, 31:141–153 (1989), 1988.

[25] M.I. Kadec. The exact value of the Paley-Wiener constant. *Dokl. Akad. Nauk SSSR*, 155:1253–1254, 1964.

[26] A.Y. Khapalov. Controllability of partial differential equations governed by multiplicative controls, volume 1995 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2010.

[27] V. Komornik and P. Loreti. *Fourier series in control theory*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2005.

[28] N.N. Lebedev. *Special functions and their applications*. Prentice-Hall, Inc., Englewood Cliffs, N.J., English edition, 1965. Translated and edited by Richard A. Silverman.

[29] C. Urbani. *Bilinear control of evolution equations*. Thesis. Gran Sasso Science Institute & Sorbonne Université, 2020.

[30] G.N. Watson. *A treatise on the theory of Bessel functions*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1995. Reprint of the second (1944) edition.

[31] R.M. Young. *An introduction to nonharmonic Fourier series*, volume 93 of *Pure and Applied Mathematics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1980.

[32] M. Zhang and H. Gao. Null controllability of some degenerate wave equations. *J. Syst. Sci. Complex.*, 30(5):1027–1041, 2017.

[33] M. Zhang and H. Gao. Persistent regional null controllability of some degenerate wave equations. *Math. Methods Appl. Sci.*, 40(16):5821–5830, 2017.

[34] M. Zhang and H. Gao. Interior controllability of semi-linear degenerate wave equations. *J. Math. Anal. Appl.*, 457(1):10–22, 2018.