HIGHLY TWISTED PLAT Diagrams

NIR LAZAROVICH, YOAV MORIAH, AND TALI PINSKY

Abstract. We prove that the knots and links in the infinite set of 3-highly twisted 2m-plats, with \( m \geq 2 \), are all hyperbolic. This should be compared with a result of Futer-Purcell for 6-highly twisted diagrams. While their proof uses geometric methods our proof is achieved by showing that the complements of such knots or links are unannular and atoroidal. This is done by using a new approach involving an Euler characteristic argument.

1. INTRODUCTION

The prevailing feeling among low dimensional topologists is that “most” links \( \mathcal{L} \) in \( S^3 \) are hyperbolic. That means that the open manifold \( S^3 \setminus \mathcal{L} \) can be endowed with a complete hyperbolic metric of sectional curvature \( -1 \). Being hyperbolic is a property of the manifold with far reaching consequences. However, proving that a specific link \( \mathcal{L} \) is hyperbolic turns out to be not trivial. This is especially true if the link \( \mathcal{L} \) is “heavy duty”, i.e., has a very large crossing number. See for example [8].

The question of when can one decide if the complement of a link in \( S^3 \) is a hyperbolic manifold from its projection diagram has been of interest for a long time. Just to give three examples: The first result in this direction is by Hatcher and Thurston who proved that complements of 2-bridge knots which have at least two twist regions (they are not torus knots or links) are hyperbolic, see [3]. The second is Menasco’s result [7] that a non-split prime alternating link which is not a torus link is hyperbolic. Later Futer and Purcell proved in [2], among other results, that every link with a 6-highly twisted irreducible diagram and which has at least two twist regions is hyperbolic. Their result is obtained by applying Marc Lackenby’s 6-surgery theorem, see [4], to the corresponding fully augmented links. Our main theorem is:

**Theorem 1.1.** Let \( L \) be a 3-highly twisted 2m-plat, \( m \geq 2 \), with at least three twist regions, then \( L \) is hyperbolic.

Every link \( \mathcal{L} \) in \( S^3 \) has a plat projection [1]. Assume that \( \mathcal{L} \) has a 2m-plat projection \( D(\mathcal{L}) = D_P(\mathcal{L}) \) in some plane \( P \), for some \( m \geq 2 \). Note that every 2m-plat projection defines a knot or link projection with \( m \geq b(\mathcal{L}) \) bridges, and every \( m \)-bridge knot or link has a 2m-plat projection (see [1] p. 24]). Here, \( b(\mathcal{L}) \) denotes the bridge number of \( \mathcal{L} \) as defined in the next section. It follows from Theorem 1.1 that not all links have

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a 3-highly twisted plat diagram. The subset of links that do is a “large” subset in a sense that can be made precise, see the discussion in [6]. In Theorem 1.1 we weaken the conditions imposed in [2] on \( L \) from 6-highly twisted to 3-highly twisted, at the price of requiring that the diagram of \( L \) be a plat. It seems that the techniques developed here using the Euler characteristic, with some additional work, might be adequate for more general diagrams. Thus, we would like to make the following conjecture:

**Conjecture 1.2.** Let \( L \) be a link in \( S^3 \) with a link diagram which is prime, twist-reduced, 3-highly twisted and has least two twist regions, then the link \( L \) is hyperbolic.

2. **Preliminaries**

2.1. **Bubbles and twist regions.** Given a projection of a link \( L \) onto a plane \( P \), surround each crossing in the projection diagram by a small 3-ball \( B \). Denote the collection of these 3-balls by \( \mathcal{B} \). Then \( L \) is isotopic to a link \( L' \) that is embedded in \( P \cup \partial \mathcal{B} \). For a single crossing we refer to \( \partial B \) as a **bubble**. Note that \( P \) divides each bubble into two hemispheres denoted by \( \partial B^+ \) and \( \partial B^- \). Denote the union of all the \( \partial B^± \) by \( \mathcal{B}^± \) respectively. Denote the two disjoint 2-spheres \( (P \setminus \mathcal{B}) \cup \mathcal{B}^± \) by \( P^± \) respectively. Each of \( P^± \) bounds a 3-ball \( H^± \) in \( S^3 \setminus L \).

A **twist region** \( T \) in \( L \) is a “cube” \( D \times [-\varepsilon, +\varepsilon] \) where \( D \) is a maximal disk in \( P \) so that and \((T, T \cap L)\) is a trivial integer 2-tangle. For example, in Figure 1, a box labeled \( a_{i,j} \) indicates a twist region with \( a_{i,j} \) crossings, where \( a_{i,j} \) can be positive, negative, or zero. In the example in the figure, \( a_{1,1} = a_{2,2} = -3 \), and all other \( a_{i,j} = -4 \).

2.2. **Plats.** Let \( b \) be an element in the braid group \( \mathcal{B}_{2m} \) on \( 2m-1 \) generators \( \{\sigma_1, \ldots, \sigma_{2m-1}\} \). We require that \( b \) is written as a concatenation of sub-words as follows:

\[
    b = b_1 \cdot b_2 \cdot \ldots \cdot b_{n-1},
\]

where \( n \) is odd, and where \( b_i \) has the following properties:

1. When \( i \) is odd, \( b_i \) is a product of all \( \sigma_j \) with \( j \) even. Namely:

\[
    b_i = \sigma_{2}^{a_{i,1}} \cdot \sigma_{4}^{a_{i,2}} \cdot \ldots \cdot \sigma_{2m-2}^{a_{i,m-1}}
\]

2. When \( i \) is even, \( b_i \) is a product of all \( \sigma_j \) with \( j \) odd. Namely:

\[
    b_i = \sigma_{1}^{a_{i,1}} \cdot \sigma_{3}^{a_{i,2}} \cdot \ldots \cdot \sigma_{2m-1}^{a_{i,m}}
\]

Consider the geometric braid on \( 2m \) strings corresponding to the element \( b \). At the top of \( b_1 \), connect each pair of strands \( \{1, 2\}, \{3, 4\}, \ldots \{2m-1, 2m\} \), (ordered from left to right) by a small unknotted arc. Similarly connect the same pairs of strands at the bottom of \( b_n \).

The obtained knot or link is a **2m-plat**. The number \( m \) is the **width** of the plat and \( n \), is the **length** of the plat. The braid \( b \) will be called the **underlying braid** of the plat.

For any knot or link \( L \subset S^3 \) there is an \( m \in \mathbb{N} \) so that \( L \) has a 2m-plat projection, on some projection plane \( P \) as indicated in Figure 1. This follows from the fact that
any knot or link can be presented as a closed braid (proved by Alexander in 1923 see [1, p. 23]) and then strands of the braid closure can be pulled across the braid diagram; for example see [1, p. 24]. The number \( m \) is by no means unique. If \( m = br(\mathcal{L}) \), where \( br(\mathcal{L}) \) is the bridge number of \( \mathcal{L} \) (see Schubert [9]), then the \( 2m \)-plat is a minimal plat for \( \mathcal{L} \).

Given a knot or link in a \( 2m \)-plat projection then the twist regions corresponding to the \( \sigma_{i,j}^{a_{i,j}} \) are called twist boxes. The twist boxes are arranged in a configuration which is “almost” a matrix: There are \( n \in \mathbb{N} \) rows indexed by \( i \in \{1, \ldots, n\} \). For odd \( i \)'s there are \( m-1 \) columns and rows with an even \( i \) have \( m \) columns. Denote the crossing number in each twist box by \( t_{i,j} = a_{i,j} \).

**Definition 2.1.** A \( 2m \)-plat will be called \( c \)-highly twisted if \( |t_{i,j}| \geq c \) for some constant \( c \), for all \( i,j \). Similarly, a knot or link that admits a \( c \)-highly twisted plat projection will be called a \( c \)-highly twisted knot or link.

**2.3. Diagram regions.** Every knot of link diagram \( D(L) \) in a projection plane \( P \subset S^3 \) defines a planar 4-regular graph. The complementary surfaces of \( P \setminus D(L) \) can be colored black and white so that two surfaces adjacent to an edge are colored in different colors.

A \( 2m \)-plat diagram \( D(L) \) determines a template diagram \( T \) in \( P \) where each twist region is replaced by a rectangle. Thinking of these rectangles as vertices of valency 4
in the graph determined by $\mathcal{T}$, one can color the complementary regions of the graph in a black and white checkerboard manner.

**Definition 2.2** (Lackenby [5]). A link diagram $D(L)$ is *prime* if any simple closed curve in $P$ intersecting $D(L)$ in two points bounds a subdiagram with no crossings.

A link diagram $D(L)$ is *twist-reduced* if any simple closed curve in $P$ which intersects the edges of $\mathcal{T}$ transversally in four points composed of two pairs each of which is adjacent to a crossing of $D(L)$, bounds a subdiagram which is the diagram of an integer 2-tangle.

**Remark 2.3.** Let $L$ be a 3-highly twisted $2m$-plat with $m \geq 3$. Then one can observe directly that the corresponding diagram $D(L)$ is prime and twist-reduced.

**Remark 2.4.** An arc in $P$ connecting two regions of different colors must cross an edge of the graph or go through a rectangle. Hence an arc on $P^\pm$ in the $2m$-plat diagram connecting two regions of different colors must intersect $L$ or meet a bubble (by which we mean that the arc intersects $\partial B$ in an arc for some bubble $B$).

**Definition 2.5.** Given two regions in $A, B \in P/\cup_{\mathcal{T}}$ define the distance $d(A, B)$ between $A$ and $B$ to be the minimal number of color changes over all arcs between $A$ and $B$. Similarly, if $a, b$ are points in regions $A, B$ respectively, then define the distance $d(a, b)$ to be the distance $d(A, B)$.

**Definition 2.6.** The regions in $P/\cup_{\mathcal{T}}$ are composed of quadrilaterals, triangles, four bigons and a single unbounded region. The bigon regions are called the *corners* of the template $\mathcal{T}$.

**Definition 2.7.** We say that an arc of $L$ connects two regions if its endpoints are contained in the closure of the two regions respectively (note that the same arc can connect different pairs of regions).

**Definition 2.8.** Given a knot or link projection with the over-crossings and under-crossings indicated, a bridge\(^1\) does not contain any under-crossings, i.e., it is a subarc of $L \cap P^*$. Note that in a plat, a bridge can pass over at most two crossings.

**Observation 2.9.** If $L$ is 2-highly twisted, and $\alpha$ a bridge which connects two regions of the diagram $D(L)$ then:

1. If $\alpha$ passes over two crossings, then $\alpha$ is the unique bridge connecting the same regions that passes over two crossings.
2. If $\alpha$ passes over two crossings and the two regions that are connected by $\alpha$ are at distance at least 2, then $\alpha$ is the unique bridge connecting them.

\(^1\)Note that the standard definition of a bridge requires the arc to be maximal and contain at least one over-crossing.
Note that the only case of an arc $\alpha$ which passes over two crossings and connects regions of distance less than 2 occurs on the boundaries of the corner bigon regions of the diagram. In this case, there is a second arc which does not pass over any crossings which connects the same regions.

3. Surfaces in link complements.

3.1. Normal position. We are interested in studying compact surfaces $S$ properly embedded in $S^3 \setminus \mathcal{N}(L)$. If $\partial S \neq \emptyset$ we extend $S$ by shrinking the neighborhood $\mathcal{N}(L)$ radially. This determines a map $i: S \to S^3$, whose image we denote by $S$ as well, which is an embedding on $S \setminus \partial S$ and $i(\partial S) \subseteq L$.

**Lemma 3.1.** Let $S \subset S^3 \setminus \mathcal{N}(L)$ be a proper surface with no meridional boundary components, and let $(T,t)$ be a twist region. Then, up to isotopy, and $S \cap T$ is a disjoint union of disks $D \subset (T,t)$ of one of the following three types:

- **Type 0:** $D$ separates the two strings of $t$.
- **Type 1:** $\partial D$ decomposes as the union of two arcs $\alpha \cup \beta$ such that $\alpha \subset t$ and $\beta \subset \partial T$.
- **Type 2:** $\partial D$ decomposes as the union of four arcs $\alpha_1 \cup \beta_1 \cup \alpha_2 \cup \beta_2$ where $\alpha_i \subset t_i$ and $\beta_i \subset \partial T$.

Moreover, the isotopy decreases the number of bubbles that $S$ meets, and if no component of $\partial S$ is a meridian, then we may further assume that $i|_{\partial L}: \partial S \to L$ is a covering map.

**Proof.** If no component of $\partial S$ is a meridian, we may assume that up to isotopy $i|_{\partial L}: \partial S \to L$ is a covering map.

The twist region $(T,t)$ is a trivial 2-tangle. The complement $T \setminus \mathcal{N}(t)$ can be identified with $P \times [0,1]$ where $P$ is a twice holed disk. Let $E$ be the disk $\alpha \times [0,1]$ where $\alpha$ is the simple arc connecting the two holes of $P$. Up to a small isotopy, we may assume that $S$ intersects $E$ transversely. Since the bubbles in $T$ are in some neighborhood of $E$, we may assume that $S$ meets a bubble if it does so in $E$. The intersection $S \cap E$ comprises of simple closed curves and arcs. All curves and arcs except those connecting $\alpha \times \{0\}$ to $\alpha \times \{1\}$ can be eliminated by an isotopy pushing $S$ off $T$. This isotopy decreases the number of bubbles $S$ meets. The number of bubbles the resulting surface meets equals the number of such arcs times the number of twist in the twist box.

Up to isotopy, we may also assume that $S$ intersects $P \times \{\frac{1}{2}\}$ transversely. Hence, $S \cap (P \times \{\frac{1}{2}\})$ is a collection of simple closed curves and arcs. By pushing $S$ outwards towards the boundary of the disk $P$, one can assume that each component of $S \cap (P \times \{\frac{1}{2}\})$ is of the following form:

- (0) An arc connecting the boundary of the disk $P$ to itself separating the holes, and intersecting $\alpha$ once.
Figure 2. The possible three types of intersection of $S$ with a twist box.

(1) An arc connecting a hole to the boundary of the disk and not intersecting $\alpha$.
(2) An arc connecting the two holes and not intersecting $\alpha$.

Thus, $S \cap (P \times [\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon])$ is a collection of disks of types (0), (1) or (2) as stated. By an ambient isotopy, we can stretch the slab $P \times [\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon]$ to $P \times [0, 1] = T$. The number of bubbles the resulting surface meets equals the number of arcs of type (0) times the number of twist in the twist box. The arcs of type (0) are in one-to-one correspondence with the arcs of $S \cap E$.

The fact that $i : \partial S \to L$ is a covering map was not affected by the isotopies above. □

Definition 3.2. A surface $S \subset S^3 \setminus \mathcal{N}(L)$ is in normal position if its extension intersects each twist region as specified in Lemma 3.1 and $i : \partial S \to L$ is a covering map. In particular, $S$ has no meridional boundary components.

Corollary 3.3. Let $S \subset S^3 \setminus \mathcal{N}(L)$ be a proper incompressible surface, and let $(T, t)$ be a twist region. Then, up to isotopy, each component of the intersection $S \cap T \cap P^\pm$ looks as in Figure 2.

3.2. Curves of intersection. Let $S \subset S^3 \setminus \mathcal{N}(L)$ be a surface in normal position. We would like to study the surface $S$ through its curves of intersection with the planes $P^\pm$. However, disks of Type (2) cause some technical complications. In order to simplify the situation we consider the surface $\hat{S}$ obtained by removing those disks from $S$. Explicitly: Let $\mathcal{T}$ be the union of all twist boxes of the plat $L$. Consider the collection of disks $\mathcal{D}$ of Type (2) which occur as intersections $S \cap \mathcal{T}$. We may assume that $\partial \mathcal{D} \subset P \cup L$, and that the subsurface $\hat{S} = S \setminus \mathcal{D}$ is transversal to $P^\pm$.

Define $\mathcal{C}^+ = \partial i^{-1}(\hat{S} \cap H^\pm)$ and $\mathcal{C}^- = \partial i^{-1}(\hat{S} \cap H^-)$. Now define $\mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^-$. As each of $P^\pm$ is a 2-sphere, $\hat{S} \cap H^\pm$ is a collection of subsurfaces of $\hat{S}$, the boundary of which are simple closed curves $c \subset S$. For $c \in \mathcal{C}^+$, denote by $S_c$ the component of $\hat{S} \cap H^+$ so that $c \subset \partial S_c$, and respectively for $c \in \mathcal{C}^-$. Although formally defined as the preimage under $i$, we think of curves in $\mathcal{C}^+$ as curves on $P^+$, as such they are disjoint outside $L$.

Definition 3.4. For $c \in \mathcal{C}$,
We say that a curve \( c \in \mathcal{C} \) passes through a bubble \( B \) if \( c \cap (\partial B \setminus L) \neq \emptyset \).

An intersection point on \( c \in \mathcal{C} \) is an endpoint of the intervals in \( c \setminus L \).

Denote by \( \#_b(c) \) the number of bubbles (with multiplicities) through which \( c \) passes.

Denote by \( \#_I(c) \) the number of intersection points along \( c \).

Define \( \#(c) = \#_b(c) + \#_I(c) \).

Define \( \mathcal{C}_{i,j} = \{ c \in \mathcal{C} \mid \#_b(c) = i, \#_I(c) = j \} \).

### 3.3. Taut surfaces.

**Definition 3.5.** Given an incompressible surface \( S \subset S^3 \setminus \mathcal{N}(L) \) we define a lexicographic complexity of \( S \) as follows:

\[
\text{Com}(S) = (|\mathcal{C}|, \sum_{c \in \mathcal{C}} \#_b(c), \sum_{c \in \mathcal{C}} \#_I(c))
\]  

(3.1)

Recall that a properly embedded surface \( S \) in a 3-manifold \( M \) is called essential if it is either a 2-sphere which does not bound a 3-ball, or it is incompressible, boundary incompressible and not boundary parallel.

**Lemma 3.6.** Let \( S \subset S^3 \setminus \mathcal{N}(L) \) be an essential surface in normal position. Assume that either

(i) \( S \) is an essential 2-sphere, and minimizes complexity among all essential 2-spheres, or

(ii) \( S \) is not a 2-sphere, the link \( L \) is not split (i.e., \( S^3 \setminus L \) is irreducible), and \( S \) minimizes complexity in its isotopy class.

Then, for all \( c \in \mathcal{C} \) we have:

1. \( S_c \cong \mathbb{D}^2 \).
2. \( \#_I(c) \) is even.
3. If \( \#_I(c) \leq 2 \) then \( \#_b(c) > 0 \).
4. If \( \#_I(c) = 0 \) then \( \#_b(c) \) is even.
5. The curve \( c \) does not pass twice through the same bubble on the same side of \( L \).
6. The curve \( c \) does not contain an arc, as depicted in Figure 3, that passes through exactly one bubble and has two intersection points with an edge of \( L \) that emanates from the bubble on both sides of the edge.
7. The curve \( c \) does not contain an arc, as depicted in Figure 4, that is contained in a twist box, and passes through exactly one bubble and one intersection point.
8. There is no arc of \( c \cap L \) such that a small extension of the arc along \( c \) has endpoints in the same region.

**Proof.** Let \( S \subset S^3 \setminus \mathcal{N}(L) \) be an essential surface satisfying (i) or (ii). Note that in both cases, compressing along a disk \( D \subset S^3 \setminus \mathcal{N}(L) \) with \( D \cap S = \partial D \) results either in two
essential spheres, or a surface in the same isotopy class of $S$. Thus, by the assumption on $S$, surfaces obtained by such a compression cannot have lower complexity.

(1) Since $S$ is essential, each subsurface $S_c$ must be planar, as otherwise it contains a non-trivial compression disk. If $S_c$ has more than one boundary component then compressing along a disk in $H^+$ or $H^-$ whose boundary separates boundary components of $S_c$ will result in a surface with fewer intersections with $P$ in contradiction to the choice of $S$.

(2) By definition, $\#I(c)$ is the number of endpoints of arcs in $c \setminus L$. Since each arc has two endpoints, $\#I(c)$ is even.

(3) By (2) $\#I(c)$ is either two or zero. If $\#I(c) = 0$ and $\#b(c) = 0$ then $c$ bounds a disk on $P \setminus L$. Compressing $S$ along this disk reduces the number of intersections with $P$.

If $\#I(c) = 2$ and $\#b(c) = 0$ then $c$ bounds a disk $D$ in $P$ such that $\partial D = \alpha \cup \beta$, where $\alpha$ is an arc in $L$ and $\beta$ is an arc in $P$. Since both $\alpha$ and $\beta$ do not pass through bubbles $\partial D$ bounds a disk in $P \setminus L$. By choosing an innermost such $D$ we may assume that $D \subset P \setminus (L \cup S)$ and thus $D$ is a boundary compression disk. Compressing $S$ along $D$ we get an isotopic surface with less intersections with $P$ which is a contradiction.

(4) Consider the colors of complementary regions of $P \setminus T$ which the curve $c$ intersects. If $\#I(c) = 0$ every change of colors, of these regions along $c$, accounts for one bubble that $c$ meets. Since $c$ is a closed curve the total number of color changes is even, and correspondingly $\#b(c)$ is even.

(5) This claim follows directly from Lemma 1(ii) of [7].

(6) In this case there is an ambient isotopy of the surface $S$ which pushes the disk bounded by the curve $c$ through the bubble thus reducing the number of bubbles met by $S$ by one. The isotopy is indicated by the arrow in Figure 3. The surface $S$ is assumed to be in normal form, by Definition 3.2. This contradicts the assumption on the choice of $S$ as minimizing the complexity as in Definition 3.5.

(7) The proof in this case is the same as in case (6), using the isotopy described by Figure 4. Note that as a result of the isotopy we see two more intersection points with $L$ but one less bubble.

(8) If such an arc $\alpha \subseteq c \cap L$ then by pushing $S$ through $P$ in a neighborhood of $\alpha$ we reduce the number of intersection points by 2, in contradiction to the minimal complexity of $S$. □

**Definition 3.7.** A surface $S$ is *taut* if it satisfies the conditions specified in Lemma 3.6.

**Remark 3.8.** Note that if $S$ is taut then $C_{0,0} = C_{0,2} = C_{i,2k+1} = C_{2k+1,0} = \emptyset$ for all $i, k \in \mathbb{N} \cup \{0\}$.

**Remark 3.9.** From now on we assume that the surface $S$ is taut.
4. Euler Characteristic and curves of intersection

4.1. Distributing Euler characteristic among curves. For each curve $c \in \mathcal{C}$ we will define the contribution of $c$, and show that the Euler characteristic of $S$ can be computed by summing up the contributions of curves $c \in \mathcal{C}$.

**Definition 4.1.** The contribution $\chi_+(c)$ of a curve $c \in \mathcal{C}$ is defined by

$$\chi_+(c) = \frac{\chi(S_c)}{\partial S_c} - \frac{1}{4} \#(c).$$

Note that if $S$ is taut, then $S_c \cong D^2$, and therefore $\chi_+(c) = 1 - \frac{1}{4} \#(c)$.

**Lemma 4.2.** If $S \subset S^3 \setminus \mathcal{N}(L)$ is taut then $\chi(S) = \sum_{c \in \mathcal{C}} \chi_+(c)$.

**Proof.** The union of the collection of all the curves $c \in \mathcal{C}$ on $S$ is an embedded graph $X$ of $S$. Let $X^0$ and $X^1$ denote the vertex and the edge sets of $X$ respectively. The vertices of the graph are the points in $S$ corresponding to the points on $c$ which are on $P \cap \mathcal{B}$ (i.e., where $c$ meets a bubble) or $P \cap L$ (i.e., an intersection point of $c$). The graph $X$ partitions $S$ into disk regions of three types:

1. the subsurfaces $S_c \subset \widehat{S} \cap H^+$ for $c \in \mathcal{C}^+$,
2. the regions $R \subset \widehat{S} \cap B$ where $B \in \mathcal{B}$ is a 3-ball bounded by a bubble, or
3. regions $D \subset S$ corresponding to Type (2) disks.

In case (3), the regions $D$ are disks whose boundary consists of two arcs on $L$ and two edges of $X$. By collapsing each such disk $D$ to one of the edges in $X$ we get a homotopic
We compute how each $c \in C$ contributes to each of the summands in (4.1):

**The vertices of $X$.** Every curve $c \in C$ passes through $2\#b(c)$ vertices of $X^0$ in the interior of $S$ (because it goes in and out of a bubble). Further, it goes through $\#I(c)$ vertices of $X^0$ in $\partial S$. Moreover, each of these vertices belongs to two curves $c \in C$. Hence,

$$|X^0| = \sum_{c \in C} \left( \#b(c) + \frac{1}{2} \#I(c) \right).$$

(4.2)

The edges of $X$. Every curve $c \in C$ passes through $2\#b(c) + \#I(c)$ edges in $X^1$. Hence,

$$|X^1 \cap \partial S| = \sum_{c \in C} \frac{1}{2} \#I(c).$$

(4.3)

Each edge in $X^1 \cap \text{int}(S) \setminus B$ accounts for two vertices in $X^0$. So the number of these edges is equal to

$$|X^1 \cap \text{int}(S) \setminus B| = \frac{1}{2} |X^0| = \frac{1}{2} \sum_{c \in C} \left( \#b(c) + \frac{1}{2} \#I(c) \right).$$

Adding these contributions together gives

$$|X^1| = \sum_{c \in C} \left( \frac{3}{2} \#b(c) + \frac{3}{2} \#I(c) \right).$$

(4.5)

**Regions $S' \subset S \cap H^\ast$.** To every curve $c \in C$ there is a surface $S_c \subset S \cap H^\ast$, and each such surface is associated to $|\partial S_c|$ curves $c \in C$. Thus,

$$\sum_{S' \subset S \cap H^\ast} \chi(S') = \sum_{c \in C} \frac{\chi(S_c)}{|\partial S_c|}.$$ 

(4.4)
Summing over all of the above we get,

\[
\chi(S) = |X^0| - |X^1| + \sum_{S' \subseteq S \cap H^+} \chi(S') + \sum_{R \subseteq S \cap B} \chi(R)
\]

\[
= \sum_{c \in C} \left( \frac{\chi(S_c)}{|\partial S_c|} - \frac{1}{4}(\#_b(c) + \#_I(c)) \right)
\]

\[
= \sum_{c \in C} \chi_+(c).
\]

\[\square\]

4.2. Redistributing positive Euler characteristic among curves. In order to prove that \(L\) is hyperbolic, we would eventually be interested in surfaces \(S\) with non-negative Euler characteristic, namely spheres, annuli and tori. It would thus be useful to distribute the Euler characteristic of \(S\) in such a way that each summand is non-positive. This would rule out 2-spheres, and show that each summand must be zero for \(S\) to be a torus or an annulus.

Lemma 4.2 shows that the Euler characteristic of \(S\) is the sum of the contributions of curves in \(C\). However, some curves might have positive contributions. Our strategy would be to “pass” the contributions of curves with \(\chi_+ > 0\) to “neighbouring” curves with \(\chi_+ < 0\). This will be done by defining \(\chi'(c)\) in Definition 4.7 and proving in Lemmas 4.8 and 4.9 respectively that \(\chi(S) = \sum \chi'(c)\) and \(\chi'(c) \leq 0\).

Our search for “neighbouring” curves will use the following definition:

**Definition 4.3.**

1. Let \(c \in C\), two subarcs \(\alpha_1, \alpha_2\) of \(c\) whose endpoints are not in \(B \cup L\) are equivalent if they are isotopic in \(c\) so that the endpoints of each arc in the isotopy are not in \(B \cup L\). Note that for equivalent arcs \(\alpha_1, \alpha_2\), we have \(#_b(\alpha_1) = #_b(\alpha_2)\) and \(#_I(\alpha_1) = #_I(\alpha_2)\). We will use the term subarc of \(c\) (or simply arc) to indicate its equivalence class. Two arcs \(\alpha, \alpha'\) are disjoint if they are subarcs of different curves or if they have disjoint representatives. Alternatively, two arcs \(\alpha, \alpha'\) are non-disjoint, if they are subarcs of the same curve, and they overlap in a bubble or an intersection point.

2. Two curves (or subarcs of curves) \(c, c' \in C\) are said to be opposite along an arc \(\alpha\), or simply opposite, if \(c \neq c'\) and \(\alpha \subset c \cap c' \setminus L\). Note that if \(c, c'\) are opposite then one of them is in \(P^+\) and the other in \(P^-\).

**Definition 4.4.** Denote by \(C_{>0}\) (resp. \(C_{=0}, C_{\leq 0}, C_{<0}\)) the collection of all \(c \in C\) such that \(\chi_+(c) > 0\) (resp. \(= 0, \leq 0, < 0\)).

Note that if \(S\) is taut, then \(C_{>0} = C_{2,0} \cup C_{1,2}\) and \(C_{\leq 0} = C_{4,0} \cup C_{2,2} \cup C_{0,4}\). We begin by studying the curves in \(C_{>0}\):

Curves in \(C_{2,0}\). Let \(c \in C_{2,0}\). The two possible configurations for \(c\) are shown in Figure 5.

In case (a), opposite to \(c\) there are two arcs \(\kappa'\) and \(\kappa''\) shown in Figure 5 passing through two and four bubbles respectively. Let \(K_{4,0}\) denote the collection of all such
arcs $\kappa''$ opposite to some $c \in C_2$. (In this and following discussion, the subscript 4,0 indicates that the arcs in $K_{4,0}$ pass through four bubbles and zero intersection points.)

In case (b), opposite to $c \in C_{2,0}$ there are two arcs, $\kappa'$ and $\kappa''$ shown in Figure 5. Each of the arcs $\kappa'$ and $\kappa''$ passes through three bubbles. Let $K_{3,0}$ denote the collection of all such $\kappa', \kappa''$ which are opposite to some $c \in C_{2,0}$ as described above.

Each arc $\kappa' \in K_{3,0}$ is part of some closed curve $c' \in \mathcal{C}$. We would like to distinguish those $\kappa'$ for which $c' \in C_{4,0}$ as those have $\chi_+(c') = 0$. We denote the subcollection of all $\kappa' \in K_{3,0}$ for which $c' \notin C_{4,0}$ by $\widehat{K}_{3,0} \subset K_{3,0}$.

Let $\kappa' \in K_{3,0} \setminus \widehat{K}_{3,0}$. Then $\kappa'$ is a subarc of a curve $c'$ with $\#_b(c') = 4$, which is opposite to some $c \in C_{2,0}$. Note that this case can only occur in the “corners” of the plat, as shown in Figure 9 for the top left “corner”. Let $\widehat{\kappa}$ be the arc shown in Figure 9. Let $\widehat{K}$ denote the collection of all $\widehat{\kappa}$ constructed in this way.

Curves in $C_{1,2}$. Schematically, a curve $c \in C_{1,2}$ must be as shown in Figure 6.

**Remark 4.5.** As $c$ emanates from $c \cap L$ into regions of different colors, $c \cap L$ cannot pass over a crossing which is not the top or bottom in its twist box.

Similarly, opposite to a curve $c \in C_{1,2}$ there is an arc $\kappa$ passing through two bubbles and one intersection point, shown in Figure 6. Let $K_{2,1}$ be the collection of all such $\kappa$ opposite to some $c \in C_{1,2}$.

Define $K = \widehat{K}_{3,0} \cup K_{4,0} \cup \widehat{K} \cup K_{2,1}$.

**Figure 5.** $c \in C_{2,0}$ and the opposite arcs $\kappa'$ and $\kappa''$.

**Figure 6.** $c \in C_{1,2}$ and the opposite arc $\kappa$.
Lemma 4.6. Arcs in $\mathcal{K}$ are pairwise disjoint.

Proof. By definition (see Definition 4.3), two arcs are not-disjoint if they overlap in a bubble or an intersection point.

One can check that the intersection of an arc $\kappa$ with a bubble in a twist box determines its intersection with the entire twist box, and that $\kappa$ must pass through either the top or bottom bubble in this twist box. Because the plat is 3-highly twisted, an arc $\kappa$ cannot pass through both the top and bottom bubbles of the twist box. The part of the arc leaving the twist box determines the curve $c \in \mathcal{C}_{2,0} \cup \mathcal{C}_{1,2} \cup \mathcal{C}_{4,0}$ opposite to it.

Thus, two arcs $\kappa_1, \kappa_2$ which overlap in a bubble determine the same opposite curve $c$, hence they are identical.

Similarly, if two arcs $\kappa_1, \kappa_2$ overlap in an intersection point, they are opposite to the same curve $c \in \mathcal{C}_{1,2}$, and they are identical. □

Recall that our goal is to redistribute the Euler characteristic among curves so that each will contribute non-positively. The quantity $\chi'$ defined below is the sought for redistribution.

Definition 4.7. Let $c' \in \mathcal{C}_{3,0}$. Let $n_3$ (resp. $\bar{n}_3$; $n_4$; $\bar{n}$; $n_{2,1}$) be the number of subarcs $\kappa \in \mathcal{K}_{3,0}$ (resp. $\mathcal{K}_{3,0}; \mathcal{K}_{4,0}; \mathcal{K}; \mathcal{K}_{2,1}$) in $c'$. We associate to $c'$ the following quantity

$$\chi'(c') = \chi_+(c') = \frac{1}{4} \bar{n}_3 + \frac{1}{2} n_4 + \frac{1}{2} \bar{n} + \frac{1}{4} n_{2,1}.$$ 

The next lemma shows that $\chi'$ is a redistribution of the Euler characteristic of $S$ among curves in $\mathcal{C}_{3,0}$.

Lemma 4.8. $\chi(S) = \sum_{c' \in \mathcal{C}_{3,0}} \chi'(c')$.

Proof. Since by Lemma 4.2 $\chi(S) = \sum_{c \in \mathcal{C}} \chi_+(c)$, it remains to prove that

$$\sum_{c \in \mathcal{C}} \chi_+(c) = \sum_{c' \in \mathcal{C}_{3,0}} \chi'(c').$$

Subtracting $\sum_{c' \in \mathcal{C}_{3,0}} \chi_+(c')$ from both sides and recalling that $\mathcal{C}_{3,0} = \mathcal{C} \setminus \mathcal{C}_{4,0}$, we have to show

$$\sum_{c \in \mathcal{C}_{4,0}} \chi_+(c) = \sum_{c' \in \mathcal{C}_{3,0}} (\chi'(c') - \chi_+(c')).$$

The left hand side is simply $\frac{1}{2} |\mathcal{C}_{2,0}| + \frac{1}{2} |\mathcal{C}_{1,2}|$ since $\mathcal{C}_{4,0} = \mathcal{C}_{2,0} \cup \mathcal{C}_{1,2}$ and

$$\chi_+(c) = \begin{cases} \frac{1}{2} & \text{if } c \in \mathcal{C}_{2,0} \\ \frac{1}{4} & \text{if } c \in \mathcal{C}_{1,2} \end{cases}.$$ 

By the definition of $\chi'$, the right hand side gives $\frac{1}{2} |\mathcal{K}_{3,0}| + \frac{1}{2} |\mathcal{K}_{4,0}| + \frac{1}{4} |\mathcal{K}| + \frac{1}{4} |\mathcal{K}_{2,1}|$. Note that for every arc $\kappa \in \mathcal{K}_{3,0} \setminus \mathcal{K}_{3,0}$ there exists a unique arc $\bar{\kappa} \in \mathcal{K}$. Hence we have $|\mathcal{K}_{3,0} \setminus \mathcal{K}_{3,0}| = |\mathcal{K}|$. Therefore, the sum becomes $\frac{1}{2} |\mathcal{K}_{3,0}| + \frac{1}{2} |\mathcal{K}_{4,0}| + \frac{1}{4} |\mathcal{K}_{2,1}|$.

Since every curve in $\mathcal{C}_{2,0}$ is opposite to either two arcs in $\mathcal{K}_{3,0}$, or one arc in $\mathcal{K}_{4,0}$ we get $|\mathcal{C}_{2,0}| = \frac{1}{2} |\mathcal{K}_{3,0}| + |\mathcal{K}_{4,0}|$ which, after dividing by 2, gives

$$\frac{1}{2} |\mathcal{C}_{2,0}| = \frac{1}{4} |\mathcal{K}_{3,0}| + \frac{1}{2} |\mathcal{K}_{4,0}|.$$ (4.6)
Similarly, every curve in $C_{1,2}$ is opposite to one arc in $K_{2,1}$ and so $|C_{1,2}| = |K_{2,1}|$ which, after dividing by 4, gives
\[
\frac{1}{4}|C_{1,2}| = \frac{1}{4}|K_{2,1}|
\] (4.7)
Adding together (4.6) and (4.7) completes the proof.

The next lemma shows that indeed $\chi'$ is non-positive.

**Lemma 4.9.** $\chi'(c') \leq 0$ for all $c' \in C_{< 0}$.

**Proof.** Let $c' \in C_{< 0}$ and let $\tilde{n}_3, n_4, \tilde{n}, n_{2,1}$ be as in Definition 4.7 of $\chi'(c')$. Then we have,
\[
\begin{align*}
\chi'(c') &= \chi_+(c') + \frac{1}{4}\tilde{n}_3 + \frac{1}{2}n_4 + \frac{1}{4}\tilde{n} + \frac{1}{4}n_{2,1} \\
&\leq (1 - \frac{1}{4}\#(c')) + \frac{1}{4}\tilde{n}_3 + \frac{1}{2}n_4 + \frac{1}{4}\tilde{n} + \frac{1}{4}n_{2,1} \\
&\leq 1 + \sum_{c' \in K_{3,0}} (\frac{1}{4} - \frac{1}{4}\#(\kappa)) + \sum_{c' \in K_{4,0}} (\frac{1}{2} - \frac{1}{4}\#(\kappa)) \\
&\quad + \sum_{c' \in K_{2,1}} (\frac{1}{4} - \frac{1}{4}\#(\kappa)) \\
&= 1 - \frac{1}{2}(\tilde{n}_3 + n_4 + \tilde{n} + n_{2,1}).
\end{align*}
\] (4.8)

Where the last equality follows since each one of the summands is $-\frac{1}{2}$. We divide into cases depending on the sum $n = \tilde{n}_3 + n_4 + \tilde{n} + n_{2,1}$.

- **Case 0.** $n = 0$. We have $\chi'(c') = \chi_+(c')$. But since $c' \notin C_{> 0}$ we have $\chi_+(c') \leq 0$ and we are done. (Note that this case includes the case $\#_6(c') = 4$ and $n_3 = 1$. See Remark 4.10)

- **Case 1.** $n = 1$. That is, $c'$ contains a single subarc $\kappa \in \tilde{K}_{3,0} \cup K_{4,0} \cup \tilde{K} \cup K_{2,1}$. We further divide into sub-cases:

  - **Case 1.a.** $\kappa \in \tilde{K}_{3,0}$. Clearly $\#_6(c') \geq \#_4(\kappa) = 3$. By definition of $\tilde{K}_{3,0}$, $\#_4(c') = 4$. Then, either $\#_6(c') \geq 6$ or $c'$ has at least two intersection points with $K$. Therefore, $\#(c') \geq 5$ which gives

    \[
    \chi'(c') = \chi_+(c') + \frac{1}{4} \leq (1 - \frac{5}{4}) + \frac{1}{4} = 0.
    \]

  - **Case 1.b.** $\kappa \in K_{4,0}$ so $\#_6(\kappa) = 4$. The distance between the endpoints of $\kappa$ is 2 in the dual graph, so the closed curve $c'$ that contains $\kappa$ must contain two additional bubbles or two additional intersection points. Thus, $\#(c') \geq 6$, which gives

    \[
    \chi'(c') = \chi_+(c') + \frac{1}{2} \leq (1 - \frac{6}{4}) + \frac{1}{2} = 0.
    \]

  - **Case 1.c.** $\kappa \in \tilde{K}$. Here again $\#_6(\kappa) = 3$. This happens in one of the cases described in Figure 9. The endpoints of $\tilde{K}$ are at distance $\geq 2$. Hence, the closed curve $c'$ containing $\kappa$ must have two additional bubbles or two additional intersection points. It follows that $\#(c') \geq 5$, which gives

    \[
    \chi'(c') = \chi_+(c') + \frac{1}{4} \leq (1 - \frac{5}{4}) + \frac{1}{4} = 0.
    \]

  - **Case 1.d.** $\kappa \in K_{2,1}$ so $\#_6(\kappa) = 2$ and $\#_1(\kappa) = 1$. The curve $c'$ must contain an additional intersection point. Let $\alpha$ be the subarc of $c'$ between the two intersection points. Let $\kappa^*$ be a small continuation of $\kappa \cup \alpha$ along $c'$.
Claim. The endpoints of $\kappa^*$ are at distance greater or equal to 1.

Proof. The arc $\alpha$ cannot contain more than one overpass: Otherwise, $\alpha$ passes over two crossings which occur in different twist boxes. This implies that the curve $c \in C_{1,2}$ opposite to $\kappa$ is such that $c \cap L$ is contained in one of the twist boxes, and connects the regions to its left and right, which must have the same color. This contradicts the fact that $c \setminus (c \cap L)$ passes through exactly one bubble.

If $\alpha$ does not contain an overpass the regions containing the endpoints of $\kappa^*$ have different colors, thus the endpoints of $\kappa^*$ are at distance greater or equal to 1.

If $\alpha$ contains one overpass then if the endpoints of $\kappa^*$ can be connected by an arc containing no intersection points or bubbles then the union of $\kappa^*$ and the arc bound a subdiagram of $D(L)$ which contradicts the assumption that $L$ is twist-reduced. □

It follows from the claim that $\#(c') \geq \#(\kappa^*) + 1 = 4 + 1 = 5$, which gives

$$\chi'(c') = \chi_+(c') + \frac{1}{4} \leq (1 - \frac{5}{4}) + \frac{1}{4} = 0.$$

Case 2. $n \geq 2$. In this case we are done by inequality (4.8). □

Remark 4.10. Note that, when $n_3 = 1$ and $c' \in C_{4,0}$, i.e., when $c'$ contains a subarc in $K_{3,0} \setminus 3_0$ then $\chi'(c') = \chi_+(c') = 0$. In this case, the positive contribution of the unique $c \in C_{2,0}$ opposite $c'$ is counted not by $\chi'(c')$ but by $\chi'(_c)$ where $c$ is the curve opposite to $c'$ containing the corresponding arc $\bar{k}$.

The proof above also gives the following lemma:

Lemma 4.11. The curve $c' \in C_{50}$ satisfies $\chi'(c') = 0$ exactly in the following cases:

(0) $c'$ does not contain a subarc in $K$ and $\#(c') = 4$. (it might have $n_3 = 1$)

(1) $c'$ contains exactly one subarc $\kappa$ of $K$ and satisfies either:

(a) $\#(c') = 5$ and $\kappa \in 3_0$.
(b) $\#(c') = 6$ and $\kappa \in 4_0$.
(c) $\#(c') = 5$ and $\kappa \in 3_1$.
(d) $\#(c') = 5$ and $\kappa \in 2_1$. 

In all cases, \( \#(c') = \#(\kappa) + 2. \)

(2) \( c' \) is the union of exactly two arcs of \( K. \)

As an immediate corollary to Lemma 4.9 we get the following:

**Corollary 4.12.** The link \( L \) is non-split.

**Proof.** Assume for contradiction that \( S^3 \setminus L \) contains an essential sphere. Let \( S \) be an essential sphere with least complexity among all essential spheres. By Lemma 3.6, \( S \) is taut. By Lemma 4.8, \( \chi(S) = \sum_{c' \in C_{\geq 0}} \chi'(c') \). Thus it follows from Lemma 4.9 that \( \chi(S) \leq 0 \), which is a contradiction. \( \square \)

## 5. Analysing the curves

As stated in the introduction our goal is to prove the following:

**Theorem 5.1.** There are no essential tori or annuli in \( S^3 \setminus \mathcal{N}(L) \).

In what follows, assume that \( S \) is a taut essential torus or annulus. By Lemma 4.8 \( \sum_{c' \in C_{\geq 0}} \chi'(c') = \chi(S) = 0 \). By Lemma 4.9 each summand \( \chi'(c') \leq 0 \), and thus must be equal to 0. It follows that all the curves \( c' \in C_{\geq 0} \) are as classified in Lemma 4.11.

The proof of Theorem 5.1 will proceed by analysing each case of Lemma 4.11 separately, and showing that \( S \) must be boundary parallel. Before we give the proof we first need to define some notation which will be used below.

**Definition 5.2.** The projection of a twist region in the diagram is a rectangle called a twist box, we refer to its edges as top/bottom/left/right as obvious from Figure 8.

**Figure 8.** A crossing and turning arc in a twist box

**Lemma 5.3.** There are no arcs in \( K_{3,0} \setminus \tilde{K}_{3,0} \). In particular \( \tilde{K} = \emptyset \).
Proof. Assume in contradiction that there exists an arc $\kappa' \in K_{3,0} \setminus \bar{K}_{3,0}$. By assumption, the curve $c'$ containing $\kappa'$ is in $C_{4,0}$. Let $c \in C_{2,0}$ be the curve opposite to $\kappa'$, let $\bar{\kappa}$ be the corresponding arc in $\bar{K}$ opposite to $c'$, and let $\bar{c}$ be the curve containing $\bar{\kappa}$. Without loss of generality, we may assume that $c \subset P^+$. This implies that $\kappa', c' \subset P^-$ and $\bar{\kappa}, \bar{c} \subset P^+$. The possible configurations of $c, c', \kappa', \bar{\kappa}$ are depicted in Figure 9. Note that under the assumptions above, the sign of the twists in the figure must be as depicted.

![Figure 9](image)

The curve $\bar{c}$ must satisfy one of the cases of Lemma 4.11. Note that only Cases (1c) and (2) are applicable.

**Case (1c).** In this case $\#_b(\bar{c}) = 3$ and $\#_I(\bar{c}) = 2$. The complementary subarc $\beta = \bar{c} \setminus \bar{\kappa}$ must have two intersection points and no bubbles. We rule out each of the cases. In Case (a) of Figure 9, the arc $\beta$ meets $L$ and travels along it in a single arc in $P^+$ which cannot go through an underpass. As can be seen from Figure 9, no such arc exists. In Case (b), the arc $\bar{\kappa}$ ends in a twist box, and so the complementary arc $\beta$ must pass through a bubble. Cases (c) and (d) are ruled out similarly to Cases (a) and (b) respectively.

**Case (2).** In this case the complementary arc $\beta$ is in $\bar{K}$. The curve $\kappa$ cannot be in $K_{4,0}$ or $K_{2,1}$ as otherwise the curve $c$ would be in $C_{7,0}$ or $C_{5,1}$. However these sets are empty by Lemma 3.6.
In Case (a), the endpoints of $\beta$ are the same as those of $\bar{\kappa}$. If $\beta \in \bar{\mathcal{K}}$ then it is one of the arcs depicted in Figure 9. Hence, it must also be as in Case (a) and must run parallel to $\bar{\kappa}$. But this would imply that $\bar{\kappa}$ passes through the same bubble twice on the same side of $L$, in contradiction to tautness of $S$. If $\beta \in \mathcal{K}_{3,0}$, then there is a corresponding subarc $\alpha$ in $L \cap P^+$ depicted in Figure 10 that connects the same regions as $\beta$. As Figure 9 is a precise depiction of the possibilities, one can readily check that there is no such arc.

\includegraphics{figure10}

Figure 10. The arc $\alpha$ parallel to $\kappa \in \mathcal{K}_{3,0}$

In Case (b), $\bar{\kappa}$ ends in the “middle” of a twist box after passing through its top bubble. The arc $\beta = \bar{\kappa} \setminus \bar{\kappa}$ continuing $\bar{\kappa}$ must pass through the bubble immediately below. The arc of $\beta$ intersecting this bubble cannot be contained in any arc in $\mathcal{K}$. Hence, this case is impossible. This finishes the proof of the claim. □

Lemma 5.4. If $c'$ contains a subarc $\kappa$ of $\mathcal{K}_{4,0}$, then it is of the form shown in Figure 11.

\includegraphics{figure11}

Figure 11. The unique form of a curve in $S \cap P^+$ containing an arc $\kappa$ in $\mathcal{K}_{4,0}$.

Proof. Let $\kappa$ be the subarc of $c'$ in $\mathcal{K}_{4,0}$, with endpoints $a$ and $b$. As $c$ must have $\chi'(c') = 0$, it follows from Lemma 4.11 that the complementary arc of $\kappa$ in $c'$ is either in $\mathcal{K}_{4,0}$, or passes through exactly two bubbles, or through two intersection points with the link $L$.

The distance between $a$ and $b$ is 2: It is clearly smaller or equal to 2. It must be even as the regions have the same color and if it is 0 the diagram would not be twist-reduced. So, by Lemma 2.9 the only subarc $\alpha \subset L \cap P^+$ connecting the region containing $a$ to the
region containing $b$ is the bridge adjacent to $\kappa$. See Figure 11. The complement of $\kappa$ in $c'$ cannot be an arc of $\mathcal{K}_{4,0}$ since any arc of $\mathcal{K}_{4,0}$ connecting these regions must follow $\alpha$ from the same side, contradicting the assumption that $S$ is taut. The complement of $\kappa$ in $c'$ cannot contain exactly two intersection points, as the arc spanning them must be $\alpha$, which again results in a contradiction to the tautness of $S$. Thus, the complement of $\kappa$ in $c'$ must contain two bubbles. The only possible such arc is an arc on the other side of $\alpha$ adjacent to $\kappa$, as in Figure 11. □

Lemma 5.5. If $c'$ contains a subarc $\kappa \in \mathcal{K}_{3,0}$ then it is of the form shown in Figure 12.

Proof. Let $c'$ be a curve containing a subarc $\kappa \in \mathcal{K}_{3,0}$. Let $a, b$ be the endpoints of $\kappa$ as in Figure 10. As $\chi'(c') = 0$, it follows from Lemma 4.9 that the complementary subarc $\beta = c' \setminus \kappa$ is either in $\mathcal{K}_{3,0}$ (in which case $c' \in C_{6,0}$), or contains exactly two intersections and no bubbles (in which case $c' \in C_{3,2}$).

If $\beta \in \mathcal{K}_{3,0}$, let $\alpha$ be the subarc of $L \cap P^+$ connecting the regions containing $a$ and $b$, adjacent to $\kappa$, as in Figure 10. Similarly, let $\alpha'$ be the subarc of $L \cap P^+$ connecting the regions containing $a$ and $b$, adjacent to $\beta$. As each of $\alpha, \alpha'$ passes over two crossings and the uniqueness implied by Lemma 4.9, $\alpha = \alpha'$. Since $S$ is taut, $\kappa$ and $\beta$ must be two different sides of $\alpha$, resulting in the configuration depicted in Figure 12.

Next suppose $\beta$ has two intersection points and no bubbles. Note that $\kappa$ must pass through two bubbles in one twist box and through a single bubble in another. Let $T$ be the second twist box. The arc $\beta$ starts from $b$ and first meets $L$ at some intersection point $p$. The two endpoints of $\beta$ must belong to regions of different colors since its complement in $c'$ passes through three bubbles. In particular $\beta$ cannot connect the two regions to the right and left of the twist box $T$. It follows that the point $p$ cannot be any of the points depicted by small empty squares in Figure 13(a), as otherwise the arc $c' \cap L \notin \beta$ would be one of the arcs in $P^+ \cap L \cap T$ connecting its right and left sides. The point $p$ also cannot be the point depicted by a small empty circle as otherwise $S$ is not taut.

Assume that $c'$ is a curve on $P^+$. Consider the curve $\tilde{c}$ opposite to $c'$ at $b$ on $P^-$. See Figure 13(b). The curve $\tilde{c}$ must pass through two bubbles in $T$ and through the intersection point $p$. Denote a point on $\tilde{c}$ just beyond this second bubble by $q$ (see
Figure 13. A possible configuration of $c' \in C_{3,2}$ on $P^+$ that has an arc of $K_{3,0}$: (a) The side $P^+$, the twist box $T$ and the forbidden positions for the point $p$. (b) The side $P^-$, the opposite curve $\tilde{c}$, and one of the possible positions for the point $p$.

Figure 13(b)). The points $b$ and $q$ are on opposite sides of the twist box $T$. The curve $\tilde{c}$ does not contain any $\eta$ in $K$: If it does, then $\eta$ will not pass through the two bubbles or intersection point specified above; thus, $\#(\tilde{c}) \geq \#(\eta) + 3$ in contradiction to Lemma 4.11.

Therefore, as $\tilde{c}$ passes through at least two bubbles and two intersection points and contains no arc of $K$, by Lemma 4.11 it must be $\tilde{c} \in C_{2,2}$. Thus, $\tilde{c} \cap L$ is an arc of $L \cap P^-$ connecting the regions containing $b, q$ and ending at the point $p$. Since $\tilde{c} \cap L$ connects the two regions to the left and right of a twist box it must be an arc that passes through the twist box. Since $\tilde{c}$ is opposite to $c'$, the point $p$ cannot be any of the points depicted by small empty squares and circles in Figure 13(a). This leaves only one possible such arc, depicted by a black dotted line in Figure 13(b). However, if $\tilde{c}$ contains such an arc, this would imply that $S$ is not taut in contradiction to the assumption on $S$.

\[\square\]

**Lemma 5.6.** There is no curve $c' \in C$ containing a subarc $\kappa$ in $K_{2,1}$

Note that, in general, if $c \in C_{1,2}$ then the arc $\alpha = c \cap L$ is a bridge. If $\alpha$ contains no crossings then it corresponds to Figure 3, which would imply that $S$ is not taut. Thus, $\alpha$ passes through either one or two crossings.

**Proof.** Assume, for contradiction, that $c' \in C$ is a curve containing a subarc $\kappa \in K_{2,1}$. By Lemma 4.11 there are two cases: Either $c'$ is a union of two arcs in $K_{2,1}$, or $c' \in C_{3,2}$ and it contains one arc in $K_{2,1}$.

Assume first that $c'$ is the union of exactly two arcs $\kappa_1, \kappa_2 \in K_{2,1}$. It follows that $c' \in C_{4,2}$, and can be viewed as the union of three arcs $\alpha', \beta_1, \beta_2$, where $\alpha' = c' \cap L$ and $\beta_1, \beta_2$ are subarcs of $\kappa_1, \kappa_2$ respectively each passing through two bubbles. Let $c_i, i = 1, 2$, be the curve in $C_{1,2}$ opposite to $\kappa_i$, and let $\alpha_i = c_i \cap L$. 
Assume first that $\alpha'$ does not pass over any crossing. Thus, $\beta_1, \beta_2$ emanate from $\alpha'$ into adjacent regions with opposite colors. Each of them passes through two bubbles, and hence they cannot meet to form a closed curve.

Next assume that $\alpha'$ passes over two crossings. Then, $\alpha_1$ (and $\alpha_2$) passes over a crossing which is not the top or bottom in its twist box, contradicting Remark 4.5. See Figure 14.

Finally, assume $\alpha'$ goes through a single crossing. By Remark 4.5 $\alpha_1$ (and $\alpha_2$) must pass over the top or bottom crossings of a twist box. The only possible configuration is if $\alpha'$ passes over the middle crossing of a twist box with three crossings. The fact that $c_1$ and $c_2$ pass through bubbles at the top or bottom of two other twist boxes, determines the orientations of the two adjacent twist boxes as depicted in Figure 15. However, as can be readily checked, a plat diagram $L$ cannot contain a subdiagram as in the figure.

Now, assume that $c' \in C_{3,2}$ and contains exactly one arc $\kappa \in K_{2,1}$. Let $c \in C_{1,2}$ be the curve opposite to $\kappa$.

Assume first that $\alpha = c \cap L$ passes through two crossings. Since $c \setminus \alpha$ passes through exactly one bubble, the regions containing the endpoints of $\alpha$ are at distance 1. Such a bridge $\alpha$ exists only in the corners of the diagram, in one of the configurations depicted.
in Figure 16. The corresponding curve $c'$ in each of the possible configurations, contains an arc $\eta$ as in the figure, which prolongs $\kappa$. The endpoints of $\eta$ are at distance greater than one, and hence $c' \setminus \eta$ cannot pass through only one bubble, in contradiction to the assumption.

Next assume that $\alpha$ passes through one crossing. Then let $\eta$ be small prolongation of $\kappa \cup (c' \cap L)$ in $c'$. The arc $c' \cap L$ contains zero, one or two crossings.

If it contains two crossings then the arc $c \cap L$ passes over a crossing which is not the top or bottom of its twist box, in contradiction to Remark 4.5 similar to Figure 14.

If $c' \cap L$ contains one crossing, the curve $c$ bounds a disk $D$ on the plane, containing a segment of $L$ which connects two twist boxes. The five possibilities for such a curve $c$ are depicted in Figure 17. The subarc $\eta$ of $c'$ is the arc depicted in blue in the figure. In cases (a) and (b), the endpoints of $\eta$ are in regions with the same color, contradicting the assumption that $c \setminus \eta$ passes through a single bubble. In case (d) and (e), the arc $\kappa$ cannot be continued to an arc $\eta$ such that $\eta \cap L$ has one crossing because of a conflict in orientation, hence (d) and (e) do not occur. We are left with case (c). In this case $\eta$ can be closed to form $c'$ as shown in figure. However, the arc $\gamma$ opposite to $c'$, at the bottom of the figure, passes through two bubbles and two intersection points, none of which belong to an arc of $K_{2,1}$. By Lemma 4.11, the curve containing $\gamma$ must close up with no further bubbles or intersection points, which is impossible.

Thus, suppose $c' \cap L$ contains no crossings. In this case, the endpoints of $\eta$, shown in blue in Figure 18, are in the same two regions as the endpoints of the maximal bridge (dashed) passing above the curve $c$. As $c'$ contains one additional bubble the endpoints of the bridge are at distance 1 and thus the bridge is at a corner of the plat. In each corner, there are two possible such bridges, and on each of these the curve $c$ can be oriented in two ways. Thus, altogether there are four possibilities that are depicted in Figure 19.

In each of these possibilities, there is an arc $\tau$ opposite to $c'$, so that $\tau$ has three or four bubbles and its endpoints are at distance at least two. Thus, the closed curve containing $\tau$ cannot be in $K$ and thus must have $\chi' < 0$.

**Definition 5.7.** If a curve enters and exists a twist box from the left or right side edges then we say that it crosses the twist box. Otherwise, if it enters from a side edge and exists from a top/bottom edge or vice versa we say that it turns at the twist box.
Figure 17. The five possibilities for a curve $c \in \mathcal{C}_{1,2}$ (in red) and the subarc $\eta$ of the curve $c'$ (in blue) opposite to $c$ such that $c \cap L$ passes under one crossings. Note that case (a) can occur also on the top boundary of the plat.

Figure 18. The bridge of a curve $c \in \mathcal{C}_{1,2}$ such that $c \cap L$ has one crossing.

Figure 19. The four possibilities for a curve $c \in \mathcal{C}_{1,2}$ with a bridge passing through zero crossings.

Lemma 5.8. There are no curves in $\mathcal{C}_{2,2}$.

Proof. For any curve $c$ in $\mathcal{C}_{2,2}$. Consider the subarcs $\alpha = c \cap L$ and $\beta = c \setminus \alpha$. 
The two endpoints of a small extension of \( \alpha \) on \( c \) must be contained in regions of the same color as the rest of \( c \) contains exactly two bubbles. As \( S \) is taut, these points cannot be in the same region. Thus the arc \( \alpha \) passes over at least one crossing of the projection diagram.

**Claim 5.9.** If \( C_{2,2} \) is nonempty then there exists a curve \( c \in C_{2,2} \) such that \( \alpha \) passes over one crossing, and \( \beta \) crosses one twist box.

**Proof of claim.**

**Case 1.** Assume first, that \( \alpha \) passes over two crossings. Following \( \beta \) from one of its endpoints, \( p, q \), we meet a twist box either from its side or its top/bottom.

If, following \( \beta \) from either endpoint, we meet a twist box from its top/bottom, then \( \beta \) meets two different twist boxes. Consider the curve \( c' \) opposite to \( c \) sharing the arc of \( \beta \) between the bubbles. The curve \( c' \) crosses the same two twist boxes. The diagram is twist-reduced and thus \( c' \notin C_{4,0} \) as otherwise it would bound a reducing subdiagram. Therefore, we must have \( \chi'(c') < 0 \), a contradiction.

Thus, following \( \beta \) from one of its endpoints, say \( p \), \( \beta \) meets a twist box \( T \) from the side. We claim that one of the two curves opposite to \( c \) containing \( p \) or \( q \) must cross \( T \): Let \( c' \) be a curve opposite to \( c \) which contains \( p \). If \( c' \) crosses \( T \), we are done. Thus assume that \( c' \) turns at \( T \) and exists through, say, its bottom. This implies that \( \beta \) must cross \( T \) and pass through its two bottom bubbles. Therefore, the curve \( c'' \) opposite to \( c \) containing \( q \) crosses \( T \) and passes through its second and third bubbles (counted from the bottom).

Let \( c' \) be a curve opposite to \( c \) containing an endpoint of \( \beta \) and crossing \( T \), as in the previous paragraph. As \( c' \) passes through at least two bubbles and has at least two intersection points, it must be in \( C_{2,2} \) by Lemma 4.11 and Lemma 5.6. Since \( c \cap L \) passes over two crossings, the arcs \( c \cap L \) and \( c' \cap L \) are as the arcs \( \alpha' \) and \( \alpha_1 \) in Figure 14. In particular, \( c' \cap L \) passes over one crossing and therefore, \( c' \) is the required curve.

**Case 2.** Now assume that \( \alpha \) passes over one crossing. Repeating the argument above we conclude that there is a curve \( c' \in C_{2,2} \) opposite to \( c \) that crosses a twist box. If \( \alpha' = c' \cap L \) passes over two crossings, we are back to Case 1. Otherwise, it passes over one crossing and we are done.

Let \( c \) be a curve as in the claim. A small pushout of \( c \) bounds a subdiagram of \( L \) as in the definition of a twist-reduced diagram (Definition 2.2). Since \( L \) is twist-reduced, the subdiagram must be contained in a twist box \( T \). Thus, both \( \alpha \) and \( \beta \) are contained in \( T \). Since \( S \) passes through all bubbles in \( T \), and every other bridge of \( T \), there exists an innermost curve such that both \( \alpha \) and \( \beta \) pass through the same bubble. However, this implies that \( S \) is not taut in contradiction.

**Lemma 5.10.** Every curve \( c' \in C_{4,0} \) is of one of the forms depicted in Figure 20.

**Proof.** Assume by contradiction that \( c' \in C_{4,0} \) and it is not of the form in Figure 20. The curve \( c' \) passes through four bubbles which are divided between at most four twist
boxes. Assume that \( c' \) only turns at twist boxes. Consider one of the turns of \( c' \) at a twist box \( T \). Such a curve has a curve \( c'' \) opposite to it which crosses the twist box. The curve \( c'' \) must also be in \( C_{4,0} \) as it cannot be of any of the types formerly investigated: \( C_{2,2} \) is empty and furthermore \( c'' \) cannot have an arc in \( \mathcal{K} \) since \( c'' \) is opposite to an arc in \( C_{4,0} \) and not to \( C_{5,0} \).

In addition, \( c'' \) cannot be of one of the forms in Figure 20. Otherwise one of the arcs \( c' \cap c'' \) continues in \( c' \) to cross a twist box, in contradiction to the assumption that \( c' \) only turns. By replacing \( c' \) by \( c'' \), if need be, we assume from now on that

(i) the curve \( c' \) is a curve in \( C_{4,0} \) which crosses a twist box, and
(ii) \( c' \) is not of one of the forms in Figure 20

Let \( c' \) be an innermost such curve. Let \( D \) be the innermost disc bounded by \( c' \). Consider the intersection of \( D \) with the twist boxes meeting \( c' \). Since the diagram is twist-reduced, the curve \( c' \) cannot cross two different twist boxes, this leaves us with three cases:

Case 0. The curve \( c' \) passes twice through the same bubble. Then it is of the desired form, in contradiction to the assumption.

Case 1. The intersection of one of these twist boxes with \( D \) contains a (projection of a) bubble that is not met by \( c' \). Let \( B, B' \) be adjacent bubbles of a twist box so that \( B \) is contained in \( D \) and \( c' \) passes through \( B' \), see Figure 21. There must be a curve \( c'' \), on the same side of \( P \) as \( c' \), passing through the bubbles \( B \) and \( B' \), as follows from Figure 2. Since \( c' \) is innermost, and the curve \( c'' \) crosses the twist box. By the previous lemmas, namely Lemmas 4.11, 5.3, 5.5, 5.4, and 5.6, it is in one of the forms shown in Figures 11, 12 or 20. In all of the cases, \( c'' \) contains an arc outside the twist box, connecting \( B \) and \( B' \). There are two cases to consider, either \( c' \) crosses the twist box, or turns at the twist box, see Figure 21. If \( c' \) crosses, the curve \( c^* \) opposite to \( c' \) (and \( c'' \)), shown in Figure 21 passes through four bubbles, it must therefore close up without passing through any additional bubbles, which would imply that \( c' \) passes through the same bubble \( B' \) twice, in which case we are done by case 0. If \( c' \) turns, then the curve \( c^* \), passes three times through two bubbles in the twist box, and in order to close up, must turn at another twist box at a bubble \( B^* \), it follows that \( c' \) passes through the bubble \( B^* \) twice. In both cases, we are done by Case 0.
We are left with the following case:

**Case 2.** The disc $D$ does not contain a bubble in a twist box that $c'$ meets, and $c'$ does not pass twice through the same bubble. Therefore, $c'$ must cross once some twist box, which we denote by $U$, and turn at two other twist boxes which we denote by $V$ and $W$ (see Figure 22). Let $B, B'$ be the two adjacent bubbles in $U$ through which $c'$ crosses. As we are not in Case 1, the innermost disk bounded by $c'$ contains no bubbles of $U, V$ or $W$. Thus, one of the bubbles $B, B'$, say $B$, is extremal in the twist box $U$, meaning its the first bubble in $U$ counted from the top or bottom. Let $\tau$ be the subarc of $c'$ exiting $B'$, and connecting $U$ to one of $V, W$, say $V$. If $\tau$ enters $V$ from the side, then opposite to $\tau$ there is a curve $c^*$ crossing the twist boxes $U$ and $V$, which we have shown can not exist because $L$ is twist-reduced. Thus, $\tau$ must enter $V$ from the bottom or the top. A similar argument shows that the subarc $\eta$ of $c'$ connecting $V$ to $W$, must enter $W$ from the top or the bottom. Thus, the subarc $\mu$ of $c'$ connecting $U$ and $W$, emerges from the side of $U$ and ends in the side $W$ as depicted in Figure 22.

**Figure 22.** The only possible configuration in Case 2

Let $c^*$ be the curve opposite to $c'$ along $\mu$. The curve $c^*$ passes through three bubbles, one in $U$ and two in $W$, and must close after an additional turn. By the checkerboard coloring, the additional turn must be at the bubble $B^*$ at the bottom of $W$, as shown in Figure 22. Opposite to $c^*$ at $B^*$, there is another curve $c^{**}$ which passes through
By the previous lemmas, as in case 1, $c^{**}$ in $C_{4,0}$. If it is in $C_{4,0}$, one can check that in order to close up $c^{**}$ must cross $U$ parallel to $c'$. Iterating this argument gives an infinite collection of nested curves intersecting $B$ and $B^*$, which is a contradiction. This finishes the proof of the lemma. □

6. No tori and no annuli

In this section we prove Theorem 5.1 and Theorem 1.1.

**Lemma 6.1.** If $S$ is a taut torus or annulus and if all the curves in $C_{\leq 0}$ are of the form depicted in Figures 11, 12 and 20 then $S$ is a boundary parallel torus.

**Proof.** Let $S$ intersect $P^\pm$ only in curves as in Figures 11, 12 and 20. The surface $S$ is obtained by capping each curve in $P^\pm$ by the disk it bounds in $H^\pm$ respectively. Thus, $S$ is a torus following one of the components of $L$. That is, $S$ is boundary parallel. □

**Corollary 6.2.** The link complement $S^3 \setminus \mathcal{N}(L)$ does not contain essential tori. In particular, the link $L$ is prime.

**Proof.** Let $S$ be an essential torus in $S^3 \setminus \mathcal{N}(L)$. By Lemma 3.6, we may assume that $S$ is taut. By Lemmas 5.3, 5.4, 5.5 and 5.10 all the curves in $\mathcal{C}$ are of the form depicted in Figures 11, 12 and 20. Hence, by Lemma 6.1, the torus $S$ is boundary parallel which contradicts the choice of $S$. □

**Proposition 6.3.** The link complement $S^3 \setminus \mathcal{N}(L)$ does not contain essential annuli.

**Proof.** Let $S$ be an incompressible and boundary incompressible annulus. Since by Corollary 6.2 $L$ is prime, we may assume that $\partial S$ has no meridional component. By Lemma 3.6 we may assume that $S$ is taut.

Assume first that $S$ passes through all twist boxes in disks of Type (0) or (2) only, as in Lemma 3.1. Therefore, as $S$ is an annulus, there must be a twist box $T$ such that $S$ meets $T$ in a Type (2) disk. When $S$ emerges from $T$ it intersects $P$ in a curve $c \in \mathcal{C}$. By Lemmas 5.6 and 5.8 such a curve must be in $C_{0,4}$. The curve $c$ cannot meet a twist box as otherwise this twist box will meet $S$ in a disk of Type (1). Consider the disk $D$ in $P$ bounded by $c$. Either int($D$) $\cap$ $L$ = $\emptyset$ in which case $S$ is not taut, or the diagram $D(L)$ is not prime, a contradiction.

Therefore, there is a twist box $T$ which intersects $S$ in a disk of Type (1). Since $\partial S$ contains a string of $T$, there is a curve $c$ in $\mathcal{C}$ which passes through the “middle” of the twist box $T$—i.e., the intersection $c \cap L$ contains a bridge which does not meet the top and bottom of $T$. Let $\alpha$ be a small continuation of this bridge in $c$. Lemmas 5.6 and 5.8 rule out curves containing $K_{2,1}$ and curves in $C_{2,2}$. Thus, $c$ must be in $C_{0,4}$, and must contain another bridge. Let $\beta$ be a small continuation of that bridge in $c$. The arcs $\alpha$ and $\beta$ belong to different components of $\partial S$: Otherwise, the arc on $c$ connecting $\alpha$ and $\beta$ together with $\partial S$ bounds a disk in $S$. An innermost curve $c \in \mathcal{C}$ in this disk
must have only two intersection points with $L$, which is a contradiction to Lemma 3.6. The endpoints of $\alpha$ in $c$ belong to regions of the same color, hence the same holds for $\beta$. Note that a bridge in a plat passes at most two crossings. We divide the proof into cases depending on the number of crossings $\beta$ passes over.

Case 0: If $\beta$ does not pass over any crossing. Then, as the endpoints of $\beta$ belong to regions of the same color, they must belong to the same region, contradicting the assumption that $S$ is taut.

Case 1: If $\beta$ passes over one crossing. A small pushout of the curve $c$ bounds a sub-diagram of $L$ containing the two crossings points over which $\alpha$ and $\beta$ pass. Since we assumed that $L$ is twist-reduced these two crossing points must belong to the same twist box $T$. Let $n$ be the number of bridges of $T$ in-between $\alpha$ and $\beta$. We further divide into sub-cases according to $n$.

Sub-case 1.0: $n = 0$. First, if $\alpha$ and $\beta$ meet the same bridge of $L$ then $c$ bounds a boundary compression disk for $S$, which is a contradiction. So, assume that $\alpha$ and $\beta$ meet adjacent bridges of $S$. The annulus $S$ must thus spiral in-between the strands of $L \cap T$. Thus we obtain a disk 2 (as in Lemma 3.1) and hence, by the definition of $C^+$ and $C^-$ as in the beginning of §3.2 this curve does not appear in $C$.

Sub-case 1.1: $n = 1$. The tangle $L \cap T$ has two components $\lambda_1, \lambda_2$. Let $l_1, l_2$ be the corresponding components of $L$ (possibly $l_1 = l_2$). Because $n = 1$ the arcs $\alpha$ and $\beta$ meet the same string of $L \cap T$, say $\lambda_1$. Hence the two boundary components of $S$ are contained in the same component $l_1$ of $L$. If $l_1 = l_2$, then there exists a curve $c' \in C_{0.4}$ which meets the bridge in-between $\alpha$ and $\beta$, and this curve must be as in Case 1.0. Thus, we may assume that $l_1 \neq l_2$. Consider the disk $\Delta$ as depicted in Figure 23(a). Its interior intersects $L$ in a single point in $l_2$, and its boundary is the union of an arc on $S$ and an arc on $l_1$. The manifold $N(S) \cup N(l_1)$ has two torus boundary components $U$ and $V$. See Figure 23(b). Let $U$ be the torus that meets $\Delta$. Let $U_-$ be the component of $S^3 \setminus U$ containing $l_2$, and let $U_+$ be the other component. The torus $U$ is incompressible in $U_+$, as such a compression must be on $\Delta$ and $\Delta$ meets $l_2$ once. It is also incompressible in $U_-$, as if such a compression disk exists then since it cannot intersect $l_1$, it gives a compression of the annulus $S$, in contradiction to the incompressibility of $S$.

By Corollary 6.2, $U$ must be boundary parallel to either $\partial N(l_2)$ or $\partial N(l_1)$. If $U$ is parallel to $\partial N(l_2)$, then since $l_1$ is parallel to a curve in $U$ crossing $\Delta$ once, then there exists an annulus $A \subset S^3 \setminus L$ whose boundary is $l_1 \cup l_2$. The annulus $A$ is incompressible, since otherwise $l_1 \cup l_2$ would be a 2-component unlink that is not linked with $L$, i.e., $L$ is split, contradicting Corollary 4.12. The annulus $A$ is trivially boundary-incompressible because the boundary components of $A$ are on two different components of $L$. If we run the argument for $A$ instead of $S$, Case 1.1 cannot occur because the boundary components of $A$ are on two different components of $L$.

If $U$ is parallel to $\partial N(l_1)$, the intersection $\Delta \cap U$ is a curve on $U$ which meets the meridian of $N(l_1)$ exactly once: as if it meets it more than once, then the union $N(\Delta) \cup N(l_1)$ determines a once-punctured non-trivial lens space contained in $S^3$.,
which is impossible. Thus, $\partial \Delta$ which is parallel to $\Delta \cap U$ is also parallel to $l_1$. Therefore, the arcs $\partial \Delta \setminus l_1 \subset S$ and $l_1 \setminus \partial \Delta \subset L$ bound a disk. Since the arc $\partial \Delta \setminus l_1$ connects different components of $S$ it is an essential arc, and the disk is a boundary compression for $S$, a contradiction.

Sub-case 1.2: $n \geq 2$. As the boundary of the annulus $S$ must pass through every other bridge in $T$, there must be another curve of $C_{0,4}$ in between $\alpha$ and $\beta$. By choosing an innermost such curve we are back in one of the previous cases.

Case 2: If $\beta$ passes over two crossings, then since the endpoints of $\beta$ have the same color they are at distance 2. Therefore, we are in Case 2 of Observation 2.9. By the uniqueness of such arcs, $\alpha = \beta$. However, since $\alpha$ is a bridge in the “middle” of a twist box, it has only one crossing, a contradiction.

Theorem 5.1 immediately follows from Corollary 6.2 and Proposition 6.3.

Proof of Theorem 1.1. If $m = 2$ then $L$ is a 2-bridge knot/link which is not a torus knot/link, since there are at least two twist regions, hence it is atoroidal and unannular by 3. If $m \geq 3$ then the theorem follows directly from Theorem 5.1. In both case apply Thurston’s result (see 10) that a link complement which contains no essential annuli or tori is hyperbolic.

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