ON A FOURTH ORDER EQUATION DESCRIBING SINGLE-COMPONENT FILM MODELS

MARTINA MAGLIOCCA

ABSTRACT. We study existence results for a fourth order problem describing single-component film models assuming initial data in Wiener spaces.

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1. INTRODUCTION

The aim of this work consists in analyzing, from a mathematical point of view, a specific physical model which describes controlled solid-state dewetting processes. The importance of this type of processes is due to their several applications.

We here focus on the single-component film with uniform composition of the bulk and the surface. The study of this particular problem has been inspired by the paper proposed by M. Khenner [11]. In his work, M. Khenner developed a theoretical model which describes the redistribution of the alloy components in the bulk of the film and on its surface during the dewetting process. A detailed physical description and some applications of this problems are proposed in [11, Sections II.A & II.C]. More precisely, the interested reader may find the derivation of this kind of problems in [11, Section II.A], where the binary alloy model considers two different atomic species, while [11, Section II.C] contains the 1-dimensional evolution PDE (with only one atomic specie) that we are going to study.

Moreover, M. Khenner underlined the link between controlled solid-state dewetting processes and new technologies based on nonlinear optics, plasmonics, photovoltaics and photocatalysis. Some concrete examples are contained in the works of P. Heger et al [10] and R. Santbergen et al [16] as far as optics is concerned, H. Liao et al [12], and S. Fafard et al [8] regarding, respectively, nanomedicine and applied physic.

The problem in object is given by the following Mullins type equation:

\[
\partial_t u = -VMg_0 u_{xxxx} - VM \left( -\frac{c_1}{d + u} + \frac{2c_2}{(d + u)^2} \right) u_{xx} - \left( \frac{c_1}{d + u} - \frac{2c_2}{(d + u)^2} \right)_{xx}, \tag{1.1}
\]

where \( u = u(t, x) > 0 \) represents the height of the film surface. The complete list of the physical parameters involved in (1.1) is contained in [11, Table I] but, for the reader’s convenience, we indicate here their most common values.

The terms \(-c_1/(u + d)\) and \(c_2/(u + d)^2\) take into account the wetting interaction between the film surface and the substrate. The term \(-c_1d/(u + d)\) (resp. \(c_2d^2/(u + d)^2\)) describes the long-range van der Waals-type attraction (resp. the short-range repulsion). The constants \(c_i\) are connected to the Hamaker constant \(A_H\) through the relation \(c_i \sim A_H/12\pi d^2\).

As far as (1.1) is concerned, the main result provided by M. Khenner [11, Section II.C] is the linear stability analysis of (1.1). He thus exploited a linearization argument about \( u(t, x) = u(0, x) \),

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being $u(0, x) > 0$ the film thickness immediately after the film deposition and at the beginning of the annealing phase. He proved that a film of uniform thickness $u$ is stable (resp. unstable) if $u(0, x) > (\leq) d(3c_2/c_1 - 1)$.

Our current goal is proving existence and regularity results for equation of (1.1) type in higher dimensions. The technique we are going to adopt is inspired to the one used in a work of R. Granero-Belinchón with the author [9]. Here, we analysed several fourth order problems modelling the growth of crystal surfaces, among which

$$\partial_t u = \Delta e^{-\alpha u} \quad \text{in} \quad (0, T) \times \mathbb{T}^N.$$  \hfill (1.2)

In short, we begin manipulating (1.2) to obtain its quasilinear version

$$\partial_t u = -\Delta^2 u + \mathcal{N}(t, x, u, \ldots, \Delta^2 u).$$

In this way, we have the harmonic heat equation plus some nonlinear terms depending also on $\Delta^2 u$. We thus pass to the Fourier formulation and then the proof reduces to proving a suitable a priori estimates and compactness results.

A similar approach is contained in the work of J.-G. Liu & B. Strain [13], where they set the equation (1.2) in $\mathbb{R}^N$ and make use of the Fourier transform. This technique works also with several problems arising in Fluid Dynamics. For instance, it has been previously employed to study different type of problems: for instance, we quote the work of G. Bruell & R. Granero-Belinchón regarding the evolution of thin films in Darcy and Stokes flows [3], by D. Córdoba and F. Gancedo [7], P. Constantin, D. Córdoba, F. Gancedo, Rodriguez-Piazza, & B. Strain [6] for the Muskat problem (see also [6] and [16]), by Burczak & R. Granero-Belinchón [4] to analyze the Keller-Segel system of PDE with diffusion given by a nonlocal operator and by H. Bae, R. Granero-Belinchón & O. Lazar [2] to prove several global existence results (with infinite $L^p$ energy) for nonlocal transport equations.

### 1.1. Notations and basic tools.

We set us in the torus $\mathbb{T}^N = [-\pi, \pi]^N, N \geq 1$, and assume initial data $u_0 > 0$ in the Wiener space $A^0(\mathbb{T}^N)$, being

$$A^p(\mathbb{T}^N) = \left\{ u(x) \in L^1(\mathbb{T}^N) : \|u\|_{A^p} := \sum_{k \in \mathbb{Z}^N} |k|^\alpha |\hat{u}(k)| < \infty \right\}.$$  \hfill (1.3)

We also introduce the space of Radon measures from an interval $[0, T]$ to a Banach space $X$, $\mathcal{M}(0, T; X)$.

For a generic function $f(t, x)$, we will often write $f(x)$ (resp. $f(t)$) when referring to $f(\cdot, x)$ (resp. $f(t, \cdot)$). We write

$$f_{ij} = \frac{\partial f}{\partial x_j} \quad j = 1, \ldots, N,$$

and we adopt Einstein convention for summation.

We name $c$ positive constants which may vary line to line during our incoming proofs.

We will make largely use of the interpolation inequality [3, Lemma 2.1]:

$$\|u\|_{A^p} \leq \|u\|_{A^0}^{1-\theta}\|u\|_{A^q}^\theta \quad \text{for} \quad 0 \leq p \leq q, \quad \theta = \frac{p}{q}$$  \hfill (1.3)
1.2. The dimensionless problem. The aim of this Section is rewriting the equation (1.1) for $N \geq 1$ and in its dimensionless form. We will represent with $\tilde{\tau}$ all the dimensionless terms involved. Because of physical reasons, we assume $u(0, x) = u_0(x) > 0$ in the whole paper.

Let

$$G(u) = -\frac{c_1d}{d + u} + \frac{c_2d^2}{(d + u)^2}.$$  

Then, equation (1.1) in higher dimension reads

$$\partial_t u = -VMG_0\Delta^2 u - VM\Delta \left( G(u)\Delta u - G'(u) \right).$$

We observe that

$$\int_{\mathbb{T}^N} \partial_t u \, dx = 0,$$

from which we deduce the mass conservation

$$\int_{\mathbb{T}^N} u(t, x) \, dx = \int_{\mathbb{T}^N} u_0(x) \, dx,$$

so it is natural to consider solutions of the type

$$u(t, x) = \|u_0\|_{L^1} \nu(t, x) \quad \text{with} \quad \int_{\mathbb{T}^N} \nu(t, x) \, dx = 0.$$  

We set the dimensionless variables and the unknown

$$\bar{\tau} = \frac{x}{d}, \quad \tilde{\tau} = \frac{VMG_0}{d^4} t, \quad \bar{\tau}(\bar{\tau}, \tilde{\tau}) = \frac{\nu(t, x)}{\|u_0\|_{L^1} + d}.$$  

We also define the dimensionless parameters

$$\bar{c}_1 = \frac{d}{d + \|u_0\|_{L^1} g_0}c_1, \quad \bar{c}_2 = \frac{d^2}{(d + \|u_0\|_{L^1})^2 g_0}c_2,$$

and the function

$$\bar{G}(\tau) = -\frac{\bar{c}_1}{1 + \bar{\tau}} + \frac{\bar{c}_2}{(1 + \bar{\tau})^2}.$$  

Then the dimensionless equation (in compact form) reads

$$\partial_\tau \bar{\tau} = -\Delta^2 \bar{\tau} - \Delta \bar{\tau} \left( \bar{G}(\tau)\Delta \bar{\tau} - \bar{G}'(\tau) \right). \quad (1.4)$$

We want to recover a more handy version of (1.4). To this aim, we pass to the Einstein notation and compute

$$\partial_\tau \bar{\tau} = -\bar{\nu}_{\bar{n} \bar{i} \bar{j} \bar{k}} - \left( \bar{G}(\tau)\bar{\nu}_{\bar{n} \bar{i} \bar{j}} - \bar{G}'(\tau) \right)_{\bar{i} \bar{j}}$$

$$= -\bar{\nu}_{\bar{n} \bar{i} \bar{j} \bar{k}} - \left( \bar{G}(\tau)\bar{\nu}_{\bar{n} \bar{i} \bar{j}} + \bar{G}'(\tau) \right)_{\bar{i} \bar{j}}$$
$$+ \left[ \bar{G}'(\tau)\bar{\nu}_{\bar{i} \bar{j}} + \bar{G}''(\tau)\bar{\nu}_{\bar{i} \bar{j} \bar{j}} \right]. \quad (1.5)$$

We want to isolate all the terms which involve both $\Delta$- and $\Delta^2$- . We take advantage of the following formula:

$$(1 + y)^{-m} = \sum_{r \geq 0} (-1)^r \binom{m + r - 1}{m - 1} y^r,$$

in order to rewrite the $p$-th derivative of $\bar{G}$ as

$$\left( \bar{G}(\tau) \right)^{(p)} = (-1)^p p! \left[ -\frac{\bar{c}_1}{(1 + \bar{\tau})^{1+p}} + (p + 1)\frac{\bar{c}_2}{(1 + \bar{\tau})^{2+p}} \right]$$

$$= (-1)^p p! \left[ -\bar{c}_1 \sum_{r \geq 0} (-1)^r \binom{p + r}{p} \bar{\tau}^r + (p + 1)\bar{c}_2 \sum_{r \geq 0} (-1)^r \binom{p + r + 1}{p + 1} \bar{\tau}^r \right].$$

In this way, the terms in (1.5) assume the form

$$\bar{G}(\tau)\bar{\nu}_{\bar{n} \bar{i} \bar{j} \bar{k}} = (-\bar{c}_1 + \bar{c}_2)\bar{\nu}_{\bar{n} \bar{i} \bar{j}} + \sum_{r \geq 1} (-1)^r \bar{\lambda}_r^r,$$

$$\bar{G}'(\tau) \left[ 2\bar{\nu}_{\bar{i} \bar{j}} \bar{\nu}_{\bar{n} \bar{i} \bar{j}} + \bar{\nu}_{\bar{n} \bar{i} \bar{j}} \bar{\nu}_{\bar{n} \bar{i} \bar{j}} \right] = \sum_{r \geq 0} (-1)^r \bar{\lambda}_r^r,$$
Remark 1.1. We say that a function \( v \) is a weak solution of \( (P) \) if

\[
\begin{align*}
\forall \varphi \in W^{1,1}(0, T; L^1(\mathbb{T}^N)) \cap L^2(0, T; W^{4,1}(\mathbb{T}^N)) \\
\text{and verifies the following weak formulation:}
\int_{\mathbb{T}^N} v_0 \varphi(0) \, dx + \int_{0}^{T} \int_{\mathbb{T}^N} \nabla v \cdot \nabla \varphi + (G(v) \Delta v - G'(v)) \Delta \varphi \, dx \, dt = 0
\end{align*}
\]

for every \( \varphi \in W^{1,1}(0, T; L^1(\mathbb{T}^N)) \cap L^1(0, T; W^{4,1}(\mathbb{T}^N)) \cap L^2(0, T; H^2(\mathbb{T}^N)) \).

We begin presenting our existence results.

Theorem 2.2 (Existence and regularity results with \( A^0 \) data). Assume that the parameters \( c_1, c_2 \) verify

\[
c_2 > \max \left\{ c_1 - 1, \frac{c_1}{3} \right\},
\]

Let \( v_0 \in A^0(\mathbb{T}^N) \) such that

\[
\|v_0\|_{A^0} < 1.
\]

Defined the values \( \delta_1 = \delta_1(\|v_0\|_{A^0}) \) as

\[
\delta_1 = \left\|v_0\right\|_{A^0} \frac{c_1}{1 - \left\|v_0\right\|_{A^0}} + \frac{c_2 (2 - \left\|v_0\right\|_{A^0}) + 3c_1}{2 (1 - \left\|v_0\right\|_{A^0})^2} + 2 (1 - \left\|v_0\right\|_{A^0})^3 + \frac{3c_1 \left\|v_0\right\|_{A^0}}{(1 - \left\|v_0\right\|_{A^0})^4},
\]

2. Main results

In what follows, we omit the \( \gamma \) superscript.

We now focus on the dimensionless problem

\[
\begin{align*}
\partial_t v &= -\Delta^2 v - \Delta (G(v) \Delta v - G'(v)) \quad \text{in} \ (0, T) \times \mathbb{T}^N, \\
v(0, x) &= v_0(x) \quad \text{in} \ \mathbb{T}^N,
\end{align*}
\]

being \( G : (0, \infty) \to \mathbb{R} \) continuous and defined as

\[
G(v) = -\frac{c_1}{1 + v} + \frac{c_2}{(1 + v)^2}, \quad c_i > 0.
\]

The notion of solution we are going to consider is given below.

Definition 2.1. We say that a function \( v \) is a weak solution of \( (P) \) if

\[
v \in L^\infty((0, T) \times \mathbb{T}^N) \cap L^2(0, T; H^2(\mathbb{T}^N))
\]

and verifies the following weak formulation:

\[
\begin{align*}
-\int_{\mathbb{T}^N} v_0 \varphi(0) \, dx + \int_{0}^{T} \int_{\mathbb{T}^N} \nabla v \cdot \nabla \varphi + (G(v) \Delta v - G'(v)) \Delta \varphi \, dx \, dt = 0
\end{align*}
\]

for every \( \varphi \in W^{1,1}(0, T; L^1(\mathbb{T}^N)) \cap L^1(0, T; W^{4,1}(\mathbb{T}^N)) \cap L^2(0, T; H^2(\mathbb{T}^N)) \).
\[
\delta_2 = \|v_0\|_{A^0} \left[ 2c_1 \frac{3 - 3\|v_0\|_{A^0} + \|v_0\|^2_{A^0}}{(1 - \|v_0\|_{A^0})^3} \\
+ 3! \frac{c_2(4 - 6\|v_0\|_{A^0} + 4\|v_0\|^2_{A^0} - \|v_0\|^4_{A^0}) + c_1 + 4\|v_0\|_{A^0}}{(1 - \|v_0\|_{A^0})^4} \right],
\]

we also require \(\|v_0\|_{A^0}\) small enough in order to have
\[
D_1 = 1 + c_2 - c_1 - \delta_1 > 0 \quad \text{and} \quad D_2 = 2(3c_2 - c_1) - \delta_2 > 0.
\]

Then, there exist at least one global weak solution to equation (P) such that
\[
v \in L^\infty((0, T) \times \mathbb{T}^N) \cap M(0, T; W^{4,\infty}(\mathbb{T}^N)) \cap L^2(0, T; H^2(\mathbb{T}^N))
\]
and
\[
v \in L^2(0, T; W^{r,\infty}(\mathbb{T}^N)) \cap L^4(0, T; W^{1,\infty}(\mathbb{T}^N)) \quad \text{for} \quad r = 1, 2, 3,
\]
for any \(T > 0\). Furthermore, we also have the continuity regularity
\[
v \in C([0, T]; L^2(\mathbb{T}^N)),
\]
and the exponential decay
\[
\|v(t)\|_{L^\infty} \leq \|v_0\|_{A^0} \exp(-D_1 t) \quad \forall t \in (0, T).
\]

Note that the function \(G\) defined in (2.1) is a continuous bounded function for \(v \geq 0\) and it may be negative. In the following result we will need \(1 + G(v)\) to be strictly positive. It is for this reason that we modify (2.2) as (2.10).

**Theorem 2.3** (Regularity results with \(A^0(\mathbb{T}^N) \cap H^2(\mathbb{T}^N)\) data). Assume that the parameters \(c_i\) verify
\[
c_2 > \max \left\{ c_1 - 1, \frac{c_1}{3}, \frac{c_1^2}{4} \right\}.
\]

Let \(v_0 \in A^0(\mathbb{T}^N) \cap H^2(\mathbb{T}^N)\) satisfy
\[
\|v_0\|_{A^0} < 1,
\]
and the smallness condition in (2.5) (see also (2.3), (2.4)). Then, the solution also verifies
\[
v \in C([0, T]; H^2(\mathbb{T}^N)) \cap L^2(0, T; H^4(\mathbb{T}^N)),
\]
\[
\|v(t)\|_{H^2} + C_1 \int_0^T \|v(t)\|_{H^4}^2 \, dt \leq C_2,
\]
where \(C_1 = C_1(c_1, c_2)\) and \(C_2 = C_2(\|v_0\|_{A^0})\).

**3. Existence and regularity results with \(v_0 \in A^0(\mathbb{T}^N)\)**

In order to prove our existence results, we will make use of the following approximating problem
\[
\left\{ \begin{array}{ll}
\partial_t v_n = -\Delta^2 v_n - \Delta \left( G_n(v_n) \Delta v_n - G_n'(v_n) \right) & \text{in} \quad (0, T) \times \mathbb{T}^N, \\
v_n(0, x) = v_0(x) & \text{in} \quad \mathbb{T}^N,
\end{array} \right. \quad (P_n)
\]
with
\[
G_n(v_n) = -c_1 \sum_{r=0}^n (-1)^r v_n^r + c_2 \sum_{r=0}^n (-1)^r (r + 1) v_n^r.
\]

**Remark 3.1** (On the approximating problem). The solutions \(\{v_n\}_n\) to \((P_n)\) can be constructed through the Faedo-Galerkin method.

Another possibility, is using a Rothe type approximation scheme (see, for instance, [21, Section 2], [19, Theorem 4.3]) whose main ingredient is discretizing in the time variable. The next steps are a fixed point argument to prove the existence of solutions of this new elliptic problem, and finding suitable estimates which do not depend on the discretization parameter to recover the time variable. The approximating solutions \(\{v_n\}_n\) belong to \(C^1(0, T; A^0(\mathbb{T}^N)) \cap L^1(0, T; A^4(\mathbb{T}^N))\). The rigorous proof of this claim can be done by following these steps.

We consider the regularized problem
\[
\left\{ \begin{array}{ll}
\partial_t v_n^{(k)} = -\partial_k \Delta^2 \partial_k \Delta v_n^{(k)} - \partial_k \Delta \left( G_n(v_n^{(k)}) \Delta v_n^{(k)} - G_n'(v_n^{(k)}) \right) & \text{in} \quad (0, T) \times \mathbb{T}^N, \\
v_n^{(k)}(0, x) = v_0(x) & \text{in} \quad \mathbb{T}^N,
\end{array} \right. \quad (P_k)
\]
being \( \delta_k \) mollifiers. Then, following the arguments in [14, Chapter 3], we obtain that \( v_n^{(k)} \) belongs to \( C^1(0,T; L^0(T_n)) \). It can be proved that \( v_n^{(k)} \in L^1(0,T; A^4(T_n)) \) reasoning as in Proposition 3.2. Now, we only need a suitable \( k \)-uniform estimates to take the limit in this parameter, and then deducing that \( \{ v_n \}_n \subset C^1(0,T; A^0(T_n)) \cap L^1(0,T; A^4(T_n)) \).

**Proposition 3.2** (A priori estimates in Wiener spaces). Let the parameters \( c_i \) verify (2.2). Assume that \( v_0 \in A^0(T_n) \) satisfies

\[
\| v_0 \|_{A^0} < 1,
\]

and the smallness condition in (2.5) (see also (2.3), (2.4)). Then, every sequence \( \{ v_n \}_n \) of solutions of (P_n) is uniformly bounded in

\[
W^{1,1}(0,T; A^0(T_n)) \cap L^1(0,T; A^4(T_n)).
\]

Furthermore,

\[
\| v_n(t) \|_{A^0} \leq \exp (- (D_1 + D_2) t) \| v_0 \|_{A^0},
\]

for \( D_i \) defined in (3.6)–(3.7).

**Proof.** We omit the subscript \( n \), so we write \( v \) when referring to \( v_n \).

We are going to work with the quasilinear version of the main equation in (P_n), i.e.

\[
\partial_t v = -(1 + c_2 - c_1) v_{nij} + 2(3c_2 - c_1) v_{ij} - \sum_{r=0}^n (-1)^r (N_r^c - N_r^s) - \sum_{r=0}^n (-1)^r (N_r^s + N_r^c - N_r^c),
\]

for

\[
N_r^c = [-c_1 + (r + 1)c_2] v^r v_{nij},
\]

\[
N_r^s = [c_1(r + 1) - c_2(r + 2)(r + 1)] v^r (2v_{ij} v_{nij} + v_{ii} v_{ii}),
\]

\[
N_r^2 = [-c_1(r + 2)(r + 1) + c_2(r + 3)(r + 2)(r + 1)] v^r v_{ij} v_{ij},
\]

\[
N_r^3 = [c_1(r + 3)(r + 2)(r + 1) - c_2(r + 4)(r + 3)(r + 2)(r + 1)] v^r v_{ij} v_{ij}.
\]

**Step 1: The Fourier series of (3.4).** We begin computing the Fourier series of (3.4). Read in terms of the \( k \)-th Fourier coefficient, we have:

\[
\partial_t \tilde{v} = -(1 + c_2 - c_1)|k|^4 \tilde{v} - 2(3c_2 - c_1)|k|^2 \tilde{v}
\]

\[
- \sum_{r=1}^n (-1)^r \left( \tilde{N}_r^c - \tilde{N}_r^s \right) - \sum_{r=0}^n (-1)^r \left( \tilde{N}_r^s + \tilde{N}_r^c - \tilde{N}_r^c \right),
\]

for

\[
\tilde{N}_r^c(k) = [-c_1 + (r + 1)c_2] \sum_{a_1 \in \mathbb{Z}^N} \ldots \sum_{a_r \in \mathbb{Z}^N} \tilde{\beta}(a^r) \prod_{s=1}^{r-1} \tilde{\beta}(a^s - a^{s+1}) |k - a^1|^4 \tilde{\beta}(k - a^1),
\]

\[
\tilde{N}_r^s(k) = [c_1(r + 1) - c_2(r + 2)(r + 1)] \sum_{a_1 \in \mathbb{Z}^N} \ldots \sum_{a_r \in \mathbb{Z}^N} \tilde{\beta}(a^r + 1) \prod_{s=2}^{r-1} \tilde{\beta}(a^s - a^{s+1})
\]

\[
\times \left[ 2(k_j - a^1_j) \tilde{\beta}(k - a^1)|a^1 - a^{s+1}|^2(\tilde{\beta}(a^1 - a^2) + |k - a^1|^2 \tilde{\beta}(k - a^1)|a^1 - a^2|^2 \tilde{\beta}(a^1 - a^2) \right],
\]

\[
\tilde{N}_r^2(k) = [-c_1(r + 2)(r + 1) + c_2(r + 3)(r + 2)(r + 1)] \sum_{a_1 \in \mathbb{Z}^N} \ldots \sum_{a_r \in \mathbb{Z}^N} \tilde{\beta}(a^r + 1)
\]

\[
\times \prod_{s=3}^{r+1} \tilde{\beta}(a^s - a^{s+1}) (k_j - a^1_j) \tilde{\beta}(k - a^1)(a^1_j - a^2_j) \tilde{\beta}(a^1 - a^2)|a^1 - a^2|^2 \tilde{\beta}(a^2 - a^3),
\]

\[
\tilde{N}_r^3(k) = [-c_1(r + 2)(r + 1) + c_2(r + 3)(r + 2)(r + 1)]
\]
\[
\times \sum_{a_1 \in \mathbb{Z}_N} \cdots \sum_{a_r \in \mathbb{Z}_N} \partial(a^r) \prod_{s=1}^{r-1} \partial(a^s - a^{s+1}) |k - a^1|^2 \partial(k - a^1),
\]

\[
\hat{N}_4(k) = [c_1(r+3)(r+2)(r+1) - c_2(r+4)(r+3)(r+2)(r+1)] \sum_{a_1 \in \mathbb{Z}_N} \cdots \sum_{a^{r+1} \in \mathbb{Z}_N} \partial(a^{r+1})
\]

\[
\times \prod_{s=2}^{r} \partial(a^s - a^{s+1})(k_j - a^1) \partial(k - a^1)(a^1_j - a^2_j) \partial(a^1 - a^2).
\]

**Step 2: A priori estimates in** \( L^1(0, T; A^4(\mathbb{T}^N)) \cap L^\infty(0, T; A^0(\mathbb{T}^N)) \).

Since

\[
\partial_t |\partial(t, k)| = Re \left( \overline{\partial(t, k)} \partial_t \partial(t, k) / |\partial(t, k)| \right),
\]

we estimate the time derivative of the \( A^0(\mathbb{T}^N) \) semi-norm as

\[
\frac{d}{dt} \| u(t) \|_{A^0} \leq -(1 + c_2 - c_1) \| u(t) \|_{A^1} - 2(3c_2 - c_1) \| u(t) \|_{A^2} + \sum_{r \geq 1} (\| N^a_0(t) \|_{A^0} + \| N^a_1(t) \|_{A^0})
\]

\[
+ \sum_{r \geq 0} (\| N^a_1(t) \|_{A^0} + \| N^a_2(t) \|_{A^0} + \| N^a_3(t) \|_{A^0}).
\]

(3.5)

We make use of Tonelli’s Theorem and the interpolation inequality (1.3), getting

\[
\| N^a_0(t) \|_{A^0} \leq [c_1 + (r + 1)c_2] \| u(t) \|_{A^0} \| u(t) \|_{A^1},
\]

\[
\| N^a_1(t) \|_{A^0} \leq [c_1(r + 1) + c_2(r + 2)(r + 1)] \| u(t) \|_{A^0} \| u(t) \|_{A^2} + \sum_{r \geq 1} (\| N^a_0(t) \|_{A^0} + \| N^a_1(t) \|_{A^0})
\]

\[
+ \sum_{r \geq 0} (\| N^a_1(t) \|_{A^0} + \| N^a_2(t) \|_{A^0} + \| N^a_3(t) \|_{A^0}).
\]

We now deal with the power series in \( r \). To this aim, we recall that, for any \( 0 < w < 1 \), it holds that

\[
\sum_{r \geq 0} w^r = \frac{1}{1 - w},
\]

\[
\sum_{r \geq 0} \prod_{j=1}^{m} (r + j) w^r = \left( \sum_{r \geq 0} w^{r+m} \right)^{(m)} = \left( \frac{w^m}{1 - w} \right)^{(m)} = \frac{m!}{(1 - w)^{m+1}},
\]

\[
\sum_{r \geq 1} \prod_{j=1}^{m} (r + j) w^r = \left( \frac{w^{m+1}}{1 - w} \right)^{(m)} = m! \frac{1 - (1 - w)^{m+1}}{(1 - w)^{m+1}}.
\]

We use the above computations with \( w = \| u(t) \|_{A^0} \) in order to bound the sums of \( \| N^a_1(t) \|_{A^0} \) as

\[
\sum_{r \geq 1} \| N^a_1(t) \|_{A^0} \leq \| u(t) \|_{A^0} \left( \frac{c_1}{1 - \| u(t) \|_{A^0}} + \frac{c_2(2 - \| u(t) \|_{A^0})}{(1 - \| u(t) \|_{A^0})^2} \right) \| u(t) \|_{A^1},
\]

\[
\sum_{r \geq 0} \| N^a_2(t) \|_{A^0} \leq 3 \| u(t) \|_{A^0} \left( \frac{c_1}{1 - \| u(t) \|_{A^0}} + \frac{2c_2}{(1 - \| u(t) \|_{A^0})^2} \right) \| u(t) \|_{A^1},
\]

\[
\sum_{r \geq 1} \| N^a_3(t) \|_{A^0} \leq \| u(t) \|_{A^0} \left( \frac{2c_1}{1 - \| u(t) \|_{A^0}} + \frac{3c_2}{(1 - \| u(t) \|_{A^0})} \right) \| u(t) \|_{A^1},
\]

\[
\sum_{r \geq 1} \| N^a_4(t) \|_{A^0} \leq \| u(t) \|_{A^0} \left( 3 - 3 \| u(t) \|_{A^0} + \| u(t) \|_{A^0} \right) \| u(t) \|_{A^1},
\]

\[
+ 3c_2 \left( \frac{4 - 6 \| u(t) \|_{A^0} + 4 \| u(t) \|_{A^0}^2 - \| u(t) \|_{A^0}^3}{(1 - \| u(t) \|_{A^0})^3} \right) \| u(t) \|_{A^2},
\]

\[
\sum_{r \geq 0} \| N^a_5(t) \|_{A^0} \leq \| u(t) \|_{A^0} \left( \frac{3c_1}{1 - \| u(t) \|_{A^0}} + \frac{4c_2}{(1 - \| u(t) \|_{A^0})} \right) \| u(t) \|_{A^2}.
\]
We gather the above estimates in two groups w.r.t. the $A^4(T^N)$ and the $A^2(T^N)$ semi-norms:

\[
\sum_{r \geq 1} \|N_r(t)\|_{A^0} + \sum_{\ell=1}^2 \sum_{r \geq 0} \|N_{r,\ell}(t)\|_{A^0} \leq \|v(t)\|_{A^0} \left[ \frac{c_1}{1 - \|v(t)\|_{A^0}} + \frac{c_2 (2 - \|v(t)\|_{A^0})}{(1 - \|v(t)\|_{A^0})^2} + 3 \left( \frac{c_1}{1 - \|v(t)\|_{A^0}} \right)^2 + \frac{2c_2}{(1 - \|v(t)\|_{A^0})^3} \right]
\]

\[
+ \|v(t)\|_{A^0} \left( \frac{2c_1}{(1 - \|v(t)\|_{A^0})^2} + \frac{3c_2}{(1 - \|v(t)\|_{A^0})^3} \right) \|v(t)\|_{A^4} + \delta_1(\|v(t)\|_{A^0})\|v(t)\|_{A^4}.
\]

\[
\sum_{r \geq 1} \|N_r(t)\|_{A^2} + \sum_{r \geq 0} \|N_{r,\ell}(t)\|_{A^2} \leq \|v(t)\|_{A^2} \left( \frac{2c_1}{1 - \|v(t)\|_{A^2}} + \frac{3c_2}{(1 - \|v(t)\|_{A^2})^2} + \frac{3c_2}{(1 - \|v(t)\|_{A^2})^3} \right)
\]

\[
+ \|v(t)\|_{A^2} \left( \frac{2c_1}{(1 - \|v(t)\|_{A^2})^2} + \frac{3c_2}{(1 - \|v(t)\|_{A^2})^3} \right) \|v(t)\|_{A^1} + \frac{4c_2}{(1 - \|v(t)\|_{A^2})^4} \|v(t)\|_{A^2}.
\]

where $\delta_i(\|v(t)\|_{A^0}) = \delta_i(t)$ are

\[
\delta_1(t) = \|v(t)\|_{A^0} \left( \frac{c_1}{1 - \|v(t)\|_{A^0}} + \frac{c_2 (2 - \|v(t)\|_{A^0}) + 3c_1}{(1 - \|v(t)\|_{A^0})^2} + \frac{3c_2 + c_1 \|v(t)\|_{A^0}}{(1 - \|v(t)\|_{A^0})^3} + \frac{3c_2 \|v(t)\|_{A^0}}{(1 - \|v(t)\|_{A^0})^4} \right),
\]

\[
\delta_2(t) = \|v(t)\|_{A^2} \left( \frac{2c_1}{1 - \|v(t)\|_{A^2}} + \frac{3c_2}{(1 - \|v(t)\|_{A^2})^2} + \frac{3c_2 + c_1 \|v(t)\|_{A^2}}{(1 - \|v(t)\|_{A^2})^3} + \frac{4c_2}{(1 - \|v(t)\|_{A^2})^4} \right).
\]

We come back to the main inequality (3.5) so that, thanks to the above computations, we obtain

\[
\frac{d}{dt} \|v(t)\|_{A^0} \leq -(1 + c_2 - c_1 - \delta_1(t))\|v(t)\|_{A^1} - (2(3c_2 - c_1) - \delta_2(t))\|v(t)\|_{A^2}.
\]

We are going to prove that, thanks to the smallness condition (2.5), we also have that

\[
1 + c_2 - c_1 - \delta_1(t) > 0,
\]

\[
2(3c_2 - c_1) - \delta_2(t) > 0,
\]

for small values of $t$. Indeed, since $\delta_i(t) = \delta_i(\|v(t)\|_{A^0})$ are increasing functions in $\|v(t)\|_{A^0}$, then

\[
\delta_i(t) \leq \delta_i(0) = \delta_i \quad \forall t \leq t^*,
\]

where

\[
t^* = \sup \{ \tau : \|v(t)\|_{A^0} \leq \|v_0\|_{A^0} \quad \forall t \leq \tau \}.
\]

This implies that, for $t \leq t^*,$

\[
1 + c_2 - c_1 - \delta_1(t) \geq 1 + c_2 - c_1 - \delta_1 = D_1 > 0,
\]

\[
2(-c_1 + 3c_2) - \delta_2(t) \geq 2(-c_1 + 3c_2) - \delta_2 = D_2 > 0.
\]

Then we have

\[
\frac{d}{dt} \|v(t)\|_{A^0} + D_2 \|v(t)\|_{A^2} + D_1 \|v(t)\|_{A^1} \leq 0 \quad t \leq t^*,
\]

so we deduce that $v \in L^1(0, t^*; A^4(T^N)) \cap L^\infty(0, t^*; A^0(T^N))$. We claim that this regularity holds for every time. If not, namely $t^* < \infty$, then we would have that

\[
\frac{d}{dt} \|v(t)\|_{A^0} \bigg|_{t=t^*} < 0
\]

which leads to a contradiction.

We conclude this step deducing that the uniform boundedness $v \in L^1(0, T; A^4(T^N)) \cap L^\infty(0, T; A^0(T^N))$ holds up to $T \leq \infty.$
**Step 3:** The regularity $W^{1,1}(0,T;A^0(T^N))$.

The continuity regularity follows from the inequality
\[
\frac{d}{dt} \|v(t)\|_{A^0} \leq c (D_2 \|v(t)\|_{A^2} + D_1 \|v(t)\|_{A^1}) \quad \forall t > 0.
\]

**Step 4:** The decay estimate. The inequality in (3.8) for $t \leq \infty$, the fact that $\|v\|_{A^\alpha} \leq \|v\|_{A^\beta}$ for $\alpha \leq \beta$ and Gronwall’s inequality provide us with the exponential decay
\[
\|v(t)\|_{A^0} \leq \exp \left( - (D_1 + D_2) t \right) \|v_0\|_{A^0} \quad \forall t > 0.
\]

We collect the compactness results we need to prove the existence of solutions as Definition 2.1 in the following Proposition.

**Proposition 3.3** (Compactness results). Let the parameters $c_i$ verify (2.2). Assume that $v_0 \in A^0(T^N)$ satisfies
\[
\|v_0\|_{A^0} < 1,
\]
and the smallness condition in (2.5) (see also (2.3), (2.4)). Then, there exists a function $v$ such that, up to subsequences, every approximating sequence of solutions $\{v_n\}_n$ of $(P_n)$ verifies
\[
\begin{align*}
&v_n \to v \quad \text{a.e.} \quad (0,T) \times T^N, \\
&v_n \rightharpoonup^* v \quad \text{in} \quad L^\infty((0,T) \times T^N), \\
&v_n \rightharpoonup^* v \quad \text{in} \quad M(0,T;W^{4,\infty}(T^N)), \\
&v_n \to v \quad \text{in} \quad L^2(0,T;H^2(T^N)).
\end{align*}
\]

**Proof.** We take advantage of (3.2) and the Banach-Alaoglu Theorem to deduce (3.10)-(3.11).

Since $\|f\|_{W^{\alpha,\infty}} \leq \|f\|_{A^\alpha}$ for every $\alpha \geq 0$, the weak convergence (3.12) directly follows interpolating between $0 < 2 < 4$ (see the interpolation inequality in (1.3)) and by the uniform bounds in (3.2):
\[
\int_0^T \|v_n(t)\|_{L^2}^2 \, dt \leq c \int_0^T \|v_n(t)\|_{A^2}^2 \, dt \leq c \|v_0\|_{L^\infty(A^0)} \int_0^T \|v_n(t)\|_{A^4} \, dt < \infty.
\]
The a.e. convergence (3.9) is a consequence of the above ones.

The following convergence result follows from Propositions 3.2 and 3.3, and it will allow us to complete the proof of Theorem 2.2.

**Corollary 3.4** (Further regularity results). Let the parameters $c_i$ verify (2.2). Assume that $v_0 \in A^0(T^N)$ satisfies
\[
\|v_0\|_{A^0} < 1,
\]
and the smallness condition in (2.5) (see also (2.3), (2.4)). Then, up to subsequences, we have that every approximating sequence of solutions $\{v_n\}_n$ of $(P_n)$ verifies
\[
\begin{align*}
&v_n \rightharpoonup^* v \quad \text{in} \quad L^{\frac{2}{r}}(0,T;W^{r,\infty}(T^N)) \quad \text{for} \quad r = 1,2,3, \\
&v_n \to v \quad \text{in} \quad L^2(0,T;H^r(T^N)), \quad 0 \leq r < 2.
\end{align*}
\]

Furthermore,
\[
v \in C([0,T];L^2(T^N)).
\]

**Proof.** The interpolation inequality (1.3) applied with $0 < r < 4$ gives us
\[
\int_0^T \|v_n(t)\|_{A^r}^r \, dt \leq \|v_n(t)\|_{L^\infty(A^0)}^{\frac{r}{2}} \int_0^T \|v_n(t)\|_{A^4} \, dt < \infty
\]
thanks, again, to (3.2). Since $\|f\|_{W^{\alpha,\infty}} \leq \|f\|_{A^\alpha}$ for every $\alpha \geq 0$, we recover the $*$-weak convergence in (3.14).

We now focus on (3.15). To obtain this convergence, we claim that we only need to prove that
\[
\partial_t v_n \to \partial_t v \quad \text{in} \quad L^2(0,T;H^{-2}(T^N)).
\]
Indeed, recalling (3.12) and invoking classical compactness results (see, for instance, [17, Corollary 4]), then we get
\[
v_n \to v \quad \text{in} \quad L^2(0,T;L^2(T^N)),
\]
and this convergence leads to (3.15), since interpolation inequalities and (3.12) imply
\[
\|\nu_n - \nu\|_{L^2(H^\alpha)}^2 \leq \|\nu_n - \nu\|_{L^2(H^\beta)}^2 \|\nu_n - \nu\|_{L^2(L^2)}^2 < c \|\nu_n - \nu\|_{L^2(L^2)}^2.
\]
Then, we are left with the proof of (3.17). Thanks to the Riesz Theorem, we know that
\[
\|\partial_t \nu_n(t)\|_{H^{-2}} = \sup_{\varphi \in H^2(T^N)} |\langle \partial_t \nu_n(t), \varphi \rangle_{H^2}|.
\]
Problem (P_n) implies
\[
\left| \int_{T^N} \partial_t \nu_n(t) \varphi \, dx \right| \leq \int_{T^N} (1 + G_n(\nu_n)) \Delta \nu_n(t) \Delta \varphi \, dx + \int_{T^N} G_n'(\nu_n(t)) \Delta \varphi \, dx \\
\leq \left(1 + \max_n |G_n(\nu_n)| \right) \|\nu_n(t)\|_{H^2} \|\varphi\|_{H^2} + \max_n |G_n'(\nu_n)| \|\varphi\|_{H^2} \\
\leq \max \left\{1 + \max_n |G_n(\nu_n)|, \max_n |G_n'(\nu_n)| \right\} (1 + \|\nu_n(t)\|_{H^2}) \|\varphi\|_{H^2}.
\]
We claim that \(G_n(\nu)\) and \(G_n'(\nu)\) are bounded uniformly in \(n\). We briefly detail the proof of the boundedness of \(G_n(\nu)\), being the one of \(G_n'(\nu)\) analogous. By definition of \(G_n(\nu)\) in (3.1), we have that
\[
|G_n(\nu)| \leq c_1 n \|\nu\|^r + c_2 \sum_{r=0}^n (r + 1) \|\nu\|^r \leq c_1 \frac{1}{1 - |\nu|} + c_2 \left(\|\nu\| \sum_{r=0}^n |\nu|^r \right).
\]
We now recall that we have assumed \(\|\nu_0\|_{A^0} < 1\) and also that, by Proposition 3.2 (Step 4), we have \(\|\nu(t)\|_{A^0} \leq \|\nu_0\|_{A^0} < 1\). Then, we are allowed to bound \(|G_n(\nu)|\) as
\[
|G_n(\nu)| \leq \frac{c_1}{1 - \|\nu_0\|_{A^0}} + \frac{c_2}{(1 - \|\nu_0\|_{A^0})^2} < \infty.
\]
Reasoning in the same way, we deduce the boundedness of \(|G_n'(\nu)|\).

We thus get that
\[
\|\partial_t \nu_n(t)\|_{H^{-2}}^2 \leq c (1 + \|\nu_n(t)\|_{H^2})^2
\]
and the desired bound follow by (3.12).

Finally, the continuity regularity (3.16) is due to (3.13) and (3.19). \(\square\)

We are now ready to prove Theorem 2.2.

Proof of Theorem 2.2. Let \(\varphi \in W^{1,1}(0, T; L^1(T^N)) \cap L^1(0, T; W^{k,1}(T^N)) \cap L^2(0, T; H^2(T^N))\). We begin proving that the limit in \(n\) of
\[
-\int_{T^N} \nu_0 \varphi(0) \, dx + \int_{(0,T) \times T^N} -\nu_0 \partial_t \varphi + \nu_2 \Delta^2 \varphi + (G(\nu_n) \Delta \nu_n - G'(\nu_n)) \Delta \varphi \, dx \, dt = 0
\]
gives us the weak formulation in Definition 2.1. To this aim, we use the boundedness and convergence results obtained in Propositions 3.2 and 3.3. We have that the convergence
\[
\int_{(0,T) \times T^N} -\nu_0 \partial_t \varphi + \nu_2 \Delta^2 \varphi \, dx \, dt \rightarrow \int_{(0,T) \times T^N} -u \partial_t \varphi + u \Delta^2 \varphi \, dx \, dt
\]
holds thanks to (3.10) and since \(\partial_t \varphi, \Delta^2 \varphi \in L^1((0, T) \times T^N)\). Furthermore, since \(G, G'\) are continuous functions and \(\nu_n \rightarrow \nu\) a.e., then
\[
\int_{(0,T) \times T^N} (G(\nu_n) \Delta \nu_n - G'(\nu_n)) \Delta \varphi \, dx \, dt \rightarrow \int_{(0,T) \times T^N} (G(\nu) \Delta u - G'(\nu)) \Delta \varphi \, dx \, dt
\]
being \(\nu_n\) uniformly bounded in \(L^2(0, T; H^2(T^N))\) and \(\Delta \varphi \in L^2((0, T) \times T^N)\).

The regularities in (2.6), (2.7) and (2.8) follows combining Proposition 3.3 and Corollary 3.4.

The exponential decay in (2.9) is due to the decay in (3.3) and the weakly-*lower semicontinuity of the norm, which implies
\[
\|\nu(t)\|_{L^\infty} \leq \lim_{n \rightarrow \infty} \|\nu_n(t)\|_{A^0} \leq \exp\left(-(D_1 + D_2) t \right) \|\nu_0\|_{A^0}.
\]
\(\square\)
4. Regularity Results with \( v_0 \in A^0(\mathbb{T}^N) \cap H^2(\mathbb{T}^N) \)

We first prove some inequalities we will use during the incoming section.

**Lemma 4.1.** We have that the following estimates hold:

\[
\begin{align*}
\| \nabla w \|_{L^2}^2 & \leq c \| w \|_{H^2} \| w \|_{A^0}, \\
\| \nabla w \|_{L^4}^4 & \leq c \| w \|_{H^2} \| w \|_{A^0}, \\
\| \nabla w \|_{L^2}^2 & \leq c \| w \|_{L^2} \| w \|_{A^2}.
\end{align*}
\]

**Proof.** We recall that \( \| f \|_{W^{\alpha,\infty}} \leq \| f \|_{A^\alpha} \) for \( \alpha \geq 0 \), and proceed with the proofs integrating by parts. We begin with (4.1):

\[
\| \nabla w \|_{L^2}^2 = \int_{\mathbb{T}^N} w_{ij} w_{ij} \, dx = -\int_{\mathbb{T}^N} w_{i} w_{ij} w_{j} \, dx \leq c \| w \|_{L^\infty} \| w \|_{H^2} \leq c \| w \|_{A^0} \| w \|_{H^2}.
\]

We now prove both the estimates (4.2)-(4.3) regarding the \( W^{1,4}(\mathbb{T}^N) \) norm of \( w \). Again, we integrate by parts obtaining

\[
\| \nabla w \|_{L^4}^4 = \int_{\mathbb{T}^N} w_{ij} w_{ij} w_{ij} w_{ij} \, dx = -\int_{\mathbb{T}^N} (w_{i} w_{ij} w_{j}) w \, dx = -\int_{\mathbb{T}^N} (2w_{i} w_{ij} w_{j} + w_{i} w_{ij} w_{ij}) w \, dx.
\]

Then, thanks also to Hölder’s inequality, we estimate either as

\[
-\int_{\mathbb{T}^N} (2w_{i} w_{ij} w_{j} + w_{i} w_{ij} w_{ij}) w \, dx \leq c \| w \|_{A^0} \| w \|_{H^2} \| \nabla w \|_{L^2}^2,
\]

or

\[
-\int_{\mathbb{T}^N} 2w_{i} w_{ij} w_{j} + w_{i} w_{ij} w_{ij} \, dx \leq c \| w \|_{L^2} \| w \|_{A^2} \| \nabla w \|_{L^2}^2,
\]

and the proof is concluded. \( \square \)

**Proposition 4.2 (A priori estimates in Sobolev spaces).** Let the parameters \( c_i \) verify (2.2). Assume that \( v_0 \in A^0(\mathbb{T}^N) \cap H^2(\mathbb{T}^N) \) satisfies

\[
\| v_0 \|_{A^0} < 1,
\]

and the smallness condition in (2.5) (see also (2.3), (2.4)). Then, every approximating sequence of solutions \( \{ v_n \}_n \) of (Pn) is uniformly bounded in

\[
\{ v_n \}_n \in L^\infty(0,T; H^2(\mathbb{T}^N)) \cap L^2(0,T; H^4(\mathbb{T}^N)).
\]

**Proof.** Again, we omit the \( n \)-subscript and the dependence of \( v \) on \( t, x \).

We multiply the equation in (Pn) by \( \Delta^2 v \) and integrate in space, getting

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^N} |\Delta v|^2 \, dx = -\int_{\mathbb{T}^N} |\Delta^2 v|^2 \, dx - \int_{\mathbb{T}^N} \Delta (G_n(v)\Delta v - G'_n(v)) \Delta^2 v \, dx.
\]

Using the Einstein notation, we split the last integral in (4.4) as

\[
-\int_{\mathbb{T}^N} (G_n(v)v_{ii})_{\ell \ell} v_{ijkk} \, dx = -\int_{\mathbb{T}^N} G_n(v)v_{ii\ell\ell} v_{ijkk} \, dx
\]

\[
-\int_{\mathbb{T}^N} (G_n''(v)v_{ii\ell} v_{ii} + G_n'(v)v_{ii\ell \ell} v_{ii}) v_{ijkk} \, dx
\]

\[
-2 \int_{\mathbb{T}^N} G_n'(v)v_{ii\ell} v_{ijkk} \, dx,
\]

\[
\int_{\mathbb{T}^N} (G_n'(v))_{\ell \ell} v_{ijkk} \, dx = \int_{\mathbb{T}^N} (G_{n''}(u)v_{i\ell} v_{i\ell} + G_n'(v)v_{i\ell \ell}) v_{ijkk} \, dx.
\]

Then, setting

\[
I_1 = -\int_{\mathbb{T}^N} (G_n''(v)v_{ii\ell} v_{ii} + G_n'(v)v_{ii\ell \ell} v_{ii}) v_{ijkk} \, dx,
\]

\[
I_2 = -2 \int_{\mathbb{T}^N} G_n'(v)v_{ii\ell} v_{ijkk} \, dx,
\]

\[
I_3 = \int_{\mathbb{T}^N} (G_{n''}(v)v_{i\ell} v_{i\ell} + G_n'(v)v_{i\ell \ell}) v_{ijkk} \, dx,
\]

we rearrange (4.4) as

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^N} |v_{ij}|^2 \, dx = -\int_{\mathbb{T}^N} (1 + G_n(v))|v_{ijkk}|^2 \, dx + I_1 + I_2 + I_3.
\]

(4.5)
We now focus on the first integral in the r.h.s.
We claim that, for $n$ large, we have

$$1 + G_n(v) > 0.$$ 

Since $G_n(v) \to G(v)$ for $n \to \infty$, it is sufficient to check whether $1 + G(v) > 0$. Computing $G'(v)$, we find that $G(v)$ attains a global minimum in $v = 2c_2/c_1 - 1$, and this leads to

$$1 + \min_v G(v) = 1 - \frac{c_1^2}{4c_2^2} > 0, \quad \text{i.e.} \quad c_2 > \frac{c_1}{2}$$

which, jointly with (2.2), gives assumption (2.10).

Then, thanks to the previous remarks, we have

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^N} |v|_{ij}^2 \, dx \leq |I_1| + |I_2| + |I_3|. \quad \text{(4.6)}$$

In the following computations, we need $G_n(v)$, and its derivatives as well, to be bounded. We omit this boundedness result and refer to the proof of Corollary 3.4 for more details.

We begin estimating the $I_1$ term using the $L^4$-gradient estimate in (4.2) with $w = v$, getting

$$|I_1| \leq c \left( \int_{\mathbb{T}^N} |v_{i\ell}| \, |\nabla v_{i\ell}| \, |v_{ijkk}| \, dx + \int_{\mathbb{T}^N} |v_{i\ell}| \, |\nabla v_{ijkk}| \, dx \right)$$

$$\leq c \left( \|\Delta^2 v\|_{A^0} \|\nabla v\|_{H^2}^2 + \|\Delta^2 v\|_{A^0} \|v\|_{H^2}^2 \right)$$

$$\leq c \left( \|v\|_{A^4} \|\nabla v\|_{A^0}^2 + \|v\|_{A^4} \|v\|_{H^2}^2 \right)$$

$$\leq c(1 + \|v\|_{A^0}^2) \|v\|_{A^4} \|v\|_{H^2}^2. \quad \text{(4.7)}$$

As far as $I_3$ is concerned, we use (4.1) with $w = v$ and estimate as

$$|I_3| \leq c \left( \int_{\mathbb{T}^N} |v_{i\ell}| \, |\nabla v_{i\ell}| \, |v_{ijkk}| \, dx + \int_{\mathbb{T}^N} |v_{i\ell}| \, |\nabla v_{ijkk}| \, dx \right)$$

$$\leq c \left( \|\Delta^2 v\|_{A^0} \|\nabla v\|_{H^2}^2 + \|\Delta^2 v\|_{A^0} \|v\|_{H^2}^2 \right)$$

$$\leq c \left( \|v\|_{A^4} \|\nabla v\|_{A^0}^2 + \|v\|_{A^4} \|v\|_{H^2}^2 \right)$$

$$\leq c(1 + \|v\|_{A^0}^2) \|v\|_{A^4} \|v\|_{H^2}^2.$$

We now deal with the $I_2$ term. We want an estimate of the type (4.7). In order to do so, we proceed with a chain of integration by parts aimed at rewriting $I_2$ in a handier form. We have

$$I_2 = \int_{\mathbb{T}^N} G_n'(v) v_{i\ell} v_{ni\ell} v_{ijkk} \, dx$$

$$= - \int_{\mathbb{T}^N} (G_n'(v) v_{i\ell} v_{ni\ell} v_{ijkk})_{,k} \, v_{ijkk} \, dx$$

$$= - \int_{\mathbb{T}^N} G_n''(v) v_{k} v_{i\ell} v_{ni\ell} v_{ijkk} \, dx - \int_{\mathbb{T}^N} G_n'(v) v_{i\ell} v_{ni\ell} v_{ijkk} \, dx - \int_{\mathbb{T}^N} G_n'(v) v_{i\ell} v_{ni\ell} v_{ijkk} \, dx.$$

We follow dealing with the last integral in the above equality:

$$- \int_{\mathbb{T}^N} G_n''(v) v_{i\ell} v_{ni\ell} v_{ijkk} \, dx = \int_{\mathbb{T}^N} (G_n'(v) v_{i\ell} v_{ijkk})_{,i} \, v_{ni\ell} \, dx$$

$$= \int_{\mathbb{T}^N} G_n''(v) v_{i\ell} v_{ijkk} v_{ni\ell} \, dx + \int_{\mathbb{T}^N} G_n'(v) v_{i\ell} v_{ijkk} v_{ni\ell} \, dx$$

$$+ \int_{\mathbb{T}^N} G_n'(v) v_{i\ell} v_{ijkk} v_{ni\ell} \, dx.$$

We are in the same situation as before. Then

$$\int_{\mathbb{T}^N} G_n'(v) v_{i\ell} v_{ijkk} v_{ni\ell} \, dx = - \int_{\mathbb{T}^N} (G_n'(v) v_{i\ell} v_{ni\ell} v_{ijkk})_{,j} \, v_{ijkk} \, dx$$

$$= - \int_{\mathbb{T}^N} G_n''(v) v_{ij} v_{i\ell} v_{ni\ell} v_{ijkk} \, dx - \int_{\mathbb{T}^N} G_n'(v) v_{i\ell} v_{ijkk} v_{ni\ell} \, dx$$

$$- \int_{\mathbb{T}^N} G_n'(v) v_{i\ell} v_{ni\ell} v_{ijkk} \, dx.$$
We finally can say that

\[-\int_{T^N} G''_n(v) v_{i\ell jk} v_{jki} \, dx = -\frac{1}{2} \int_{T^N} G'_n(v) v_{i\ell} \left( v_{jki} v_{jki} \right)_{\ell} \, dx \]

\[= \frac{1}{2} \int_{T^N} \left( G'_n(v) v_{i\ell} \right) v_{jki} v_{jki} \, dx \]

\[= \frac{1}{2} \int_{T^N} G'_n(v) v_{i\ell} v_{i\ell} v_{jki} v_{jki} \, dx + \frac{1}{2} \int_{T^N} G'_n(v) v_{i\ell} v_{i\ell} v_{jki} v_{jki} \, dx.\]

Resuming

\[I_2 = -\int_{T^N} G''_n(v) v_{i\ell jk} v_{jki} v_{jki} \, dx - \int_{T^N} G'_n(v) v_{i\ell i\ell} v_{jki} \, dx + \int_{T^N} G'_n(v) v_{i\ell} v_{i\ell} v_{jki} v_{jki} \, dx \]

\[+ \frac{1}{2} \int_{T^N} G'_n(v) v_{i\ell} v_{i\ell} v_{jki} v_{jki} \, dx + \frac{1}{2} \int_{T^N} G'_n(v) v_{i\ell} v_{i\ell} v_{jki} v_{jki} \, dx.\]

Since the last expression of \(I_2\) is made up of integrals involving only first and third or second and third order derivatives, we just focus on the first two integrals in the r.h.s.. The inequalities in (4.2) with \(w = v, (4.3)\) with \(w = \Delta u\) and the fact that \(\|f\|_{W^{2,\infty}} \leq \|f\|_{A^4}\) yield to

\[\int_{T^N} G''_n(v) v_{i\ell jk} v_{jki} v_{jki} \, dx \leq c \int_{T^N} |v_{i\ell}| |v_{i\ell}| |v_{jki}| \, dx \]

\[\leq \|\nabla u\|_{L^4}^2 \|\nabla \Delta u\|_{L^4} \]

\[\leq c \|v\|_{A^4}^2 \|v\|_{A^4}^2,\]

\[\int_{T^N} G'_n(v) v_{i\ell i\ell} v_{jki} \, dx \leq c \int_{T^N} |v_{i\ell}| |v_{i\ell}| |v_{jki}| \, dx \]

\[\leq c \|v\|_{H^2}^2 \|\nabla \Delta u\|_{L^4} \]

\[\leq c \|v\|_{H^2}^2 \|v\|_{A^4}.\]

Then, we estimate \(I_2\) as

\[I_2 \leq c (1 + \|v\|_{A^4}) \|v\|_{H^2}^2 \|v\|_{A^4}\]

and, consequently, the main inequality in (4.6) becomes

\[\frac{d}{dt} \|v\|_{H^2}^2 \leq c (1 + \|v\|_{A^4}) \|v\|_{A^4} \left( 1 + \|v\|_{H^2}^2 \right).\]

Standard computations lead to

\[\|v(t)\|_{H^2}^2 \leq e^{(1 + \|v\|_{L^\infty(A^0)}) \int_0^t \|v(s)\|_{A^4} \, ds} \]

\[\times \left[ \|v_0\|_{H^2}^2 + C(1 + \|v\|_{L^\infty(A^0)}) \int_0^t \|v(s)\|_{A^4} e^{-\left(1 + \|v\|_{L^\infty(A^0)}\right) \int_0^s \|v(\tau)\|_{A^4} d\tau} \, ds \right] \]

\[\leq e^{(1 + \|v\|_{L^\infty(A^0)}) \int_0^t \|v(s)\|_{A^4} \, ds} \left[ \|v_0\|_{H^2}^2 + C(1 + \|v\|_{L^\infty(A^0)}) \int_0^t \|v(s)\|_{A^4} \, ds \right] \]

\[\leq c (\|v_0\|_{A^0}),\]

where the last inequality is due to Theorem 2.2 (see (3.8) and comments below). In particular, we have deduced that \(v\) is uniformly bounded in \(L^\infty(0, T; H^2(T^N))\).

The regularity \(L^2(0, T; H^4(T^N))\) follows integrating in time the inequality in (4.5) and taking into account the estimates on the time-dependent terms \(I_1, I_2, I_3\):

\[\frac{1}{2} \|v(t)\|_{H^2}^2 + \int_{(0,t) \times T^N} |\Delta^2 u|^2 \, dx \, ds \leq \int_0^t I_1(s) + I_2(s) + I_3(s) \, ds + \frac{1}{2} \|v_0\|_{H^2}^2 \]

\[\leq c (1 + \|v\|_{L^\infty(A^0)}) \left( 1 + \|v\|_{L^\infty(H^2)} \right) \int_0^t \|v(s)\|_{A^4} \, ds + \frac{1}{2} \|v_0\|_{H^2}^2 \]

\[\leq c (\|v_0\|_{A^0}).\]
Proposition 4.3 (Compactness results). Let the parameters verify (2.10). Assume that \( v_0 \in A^0(\mathbb{T}^N) \cap H^2(\mathbb{T}^N) \) satisfies
\[
\|v_0\|_{A^0} < 1,
\]
and the smallness condition in (2.5) (where \( \delta_1, \delta_2 \) have been defined in (2.3), (2.4)). Then, up to subsequences, we have that every approximating sequence of solutions \( \{v_n\}_n \) of \((P_n)\) verifies
\[
v_n \to v \quad \text{in} \quad L^2(0, T; H^r(\mathbb{T}^N)), \quad 0 \leq r < 4.
\]
Furthermore, the limit function \( v \) satisfies
\[
v \in C([0, T]; H^2(\mathbb{T}^N)).
\]

Proof. Hölder’s inequality implies
\[
\int_0^t \|v_n(s) - v(s)\|^2_{H^r} \, ds \leq \int_0^t \|v_n(s) - v(s)\|^\frac{2}{1+2r} \|v_n(s) - v(s)\|^\frac{4-2r}{1+2r} \, ds
\]
which converges to zero by (3.18), then (4.9) follows.

The continuity regularity (4.10) can be proved as in Corollary 3.4, namely using that
\[
\Delta v \in L^2(0, T; H^2(\mathbb{T}^N))
\]
from Proposition 4.2, and then proving that
\[
\partial_t \Delta v \in L^2(0, T; H^{-2}(\mathbb{T}^N)).
\]

Proof of Theorem 2.3. The regularity (2.11) follows from Propositions 4.2 and 4.3.

The energy inequality in (2.12) is a consequence of (4.8).

\[\square\]

5. Final remarks

5.1. On the uniqueness. The techniques previously employed do not provide us with enough regularity to prove uniqueness results. Indeed, we just have \( v \in M(0, T; W^{4,\infty}(\mathbb{T}^N)) \) but we would need
\[
v \in L^1(0, T; A^4(\mathbb{T}^N))
\]
in order to get a comparison result between sub and supersolution of \((P)\) reasoning as in Proposition 3.2. A possible approach is the one exploited by J.-G. Liu and R. Strain in [13]: here, under the assumption of medium size data belonging to \( A^2 \), the authors first proved the uniqueness of solutions to
\[
\partial_t u = \Delta e^{-\Delta} u \quad \text{in} \quad (0, T) \times \mathbb{R}^N,
\]
and then they took advantage of this information to obtain uniqueness in \( A^0(\mathbb{R}^N) \). We explicitly point out that the uniqueness class does not coincide with the existence class.

5.2. A fixed point technique. Another possible way to approach problems of \((P)\) type is the one proposed by D. M. Ambrose in [1]. Here, the author proves existence and analyticity results for a fourth order problem which describes crystal growth surfaces through fixed point techniques. The equation in object is (1.2). The main contributions provided by [1] concern both the proof of the analyticity and the improvement of the smallness condition on the \( A^0 \) semi-norm of the initial datum w.r.t. [9], where the same problem is studied.

Hence, due to the nature of the model in \((P)\), it could be interesting to try by means of similar arguments as in [1].

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REFERENCES

[1] D.M. Ambrose, The radius of analyticity for solutions to a problem in epitaxial growth on the torus, Bulletin of the London Mathematical Society, 51(5) (2018), pp. 877–886.

[2] H. Bae, R. Granero-Belinchón and O. Lazar, Global existence of weak solutions to dissipative transport equations with nonlocal velocity, Networks and Heterogeneous Media, 14(3) (2019), pp. 471–487.

[3] G. Bruell and R. Granero-Belinchón, On the thin film Muskat and the thin film Stokes equations, Journal of Mathematical Fluid Mechanics, 21(2) (2019), pp. 1–31.

[4] J. Burczak and R. Granero-Belinchón, On a generalized doubly parabolic Keller–Segel system in one spatial dimension, Mathematical Models and Methods in Applied Sciences, 26(01) (2016), pp. 111–160.

[5] C.H. Chiu, Stable and uniform arrays of self-assembled nanocrystalline islands, Physical Review B, 69(16) (2004), 165413.

[6] P. Constantin, D. Córdoba, F. Gancedo, L. Rodríguez-Piazza and R.M. Strain, On the Muskat problem: global in time results in 2D and 3D, American Journal of Mathematics, 138(6) (2016), pp. 1455–1494.

[7] D. Córdoba and F. Gancedo, Contour dynamics of incompressible 3-D fluids in a porous medium with different densities, Communications in Mathematical Physics, 273(2) (2007), pp. 445–471.

[8] S. Fafard, Z.R. Wasilewski, C.N. Allen, K. Hinzer, J.P. McCaffrey, and Y. Feng, Lasing in quantum-dot ensembles with sharp adjustable electronic shells, Applied physics letters, 75(7) (1999), pp. 986–988.

[9] R. Granero-Belinchón and M. Magliocca, Global existence and decay to equilibrium for some crystal surface models, Discrete & Continuous Dynamical Systems-A, 39(4) (2019), 2101-2131.

[10] P. Heger, O. Stenzel and N. Kaiser, Metal island films for optics. In Advances in optical thin films, International Society for Optics and Photonics, 5250 (2004), pp. 21–28.

[11] M. Khenner, Modeling solid-state dewetting of a single-crystal binary alloy thin films, Journal of Applied Physics, 123(3) (2018), 034302.

[12] H. Liao, C.L. Nehl, and J.H. Hafner, Biomedical applications of plasmon resonant metal nanoparticles, Nanomedicine, (2006), pp. 201–208.

[13] J.G. Liu and R.M. Strain, Global stability for solutions to the exponential PDE describing epitaxial growth, Interfaces and Free Boundaries, 21(1) (2019), pp. 61–86.

[14] A. Majda & A. Bertozzi, Vorticity and incompressible flow. Cambridge texts in applied mathematics, 2002.

[15] Y.R. Niu, K.L. Man, A. Pavlovska, E. Bauer and M.S. Altman, Fe on W (001) from continuous films to nanoparticles: Growth and magnetic domain structure, Physical Review B, 95(6) (2017), 064404.

[16] R. Santberg, T.L. Temple, R. Liang, A.H.M. Smets, R.A.C.M.M. Van Swaaij and M. Zeman, Application of plasmonic silver island films in thin-film silicon solar cells, Journal of Optics, 14(2) (2012), 024010.

[17] J. Simon, Compact sets in the space L^p(0,T;B), Annali di Matematica pura ed applicata, 146(1) (1986), pp. 65–96.

[18] H.L. Skriver and N.M. Rosengaard, Surface energy and work function of elemental metals, Physical Review B, 46(11) (1992), 7157.

[19] O. Stein and M. Winkler, Amorphous molecular beam epitaxy: global solutions and absorbing sets, European Journal of Applied Mathematics, 2005, vol. 16, no 6, p. 767-798.

[20] Z. Suo and Z. Zhang, Epitaxial films stabilized by long-range forces, Physical Review B, 58(8) (1998), 5116.

[21] M. Winkler, Global solutions in higher dimensions to a fourth-order parabolic equation modeling epitaxial thin-film growth, Zeitschrift für angewandte Mathematik und Physik, 2011, vol. 62, no 4, p. 575-608.

(M. Magliocca) DEPARTMENT OF MATHEMATICAL ANALYSIS, FACULTY OF MATHEMATICS, UNIVERSITY OF SEVILLA, C/TARFIA s/n, CAMPUS REINA MERCEDES, SEVILLA 41012, SPAIN., MMAGLIOCCA@US.ES