Relative Frequency and Probability in the Everett Interpretation of Heisenberg-Picture Quantum Mechanics

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Abstract

The existence of probability in the sense of the frequency interpretation, i.e., probability as “long term relative frequency,” is shown to follow from the dynamics and the interpretational rules of Everett quantum mechanics in the Heisenberg picture. This proof is free of the difficulties encountered in applying to the Everett interpretation previous results regarding relative frequency and probability in quantum mechanics. The ontology of the Everett interpretation in the Heisenberg picture is also discussed.

Key words: relative frequency, probability, Everett interpretation, Heisenberg picture, Born rule

1 Introduction

The Everett interpretation of quantum mechanics [1] posits that all physical phenomena can be described by unitary time evolution. In the standard Copenhagen interpretation of quantum mechanics (see, e.g., [2]), the phenomenon of probability arises by virtue of “reduction of the wavefunction,” an explicitly nonunitary type of time evolution which, when measurements are made, supplants the unitary evolution that otherwise occurs. A challenge for the Everett interpretation is, therefore, to show that it predicts the existence of probability in the context of completely unitary time evolution.

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Probability, whether in a quantum-mechanical or classical context, has two characteristic aspects\cite{3}. One of these is \textit{randomness}, the fact that the result of one repetition of an experiment cannot in general be predicted with certainty. Everett quantum mechanics predicts and explains the existence of randomness in a straightforward manner. Suppose an experimenter, Alice, performs a quantum experiment to measure an observable with two possible outcomes, “up” and “down.” After Alice has performed the experiment and measured the observable, there exist two noninteracting copies of Alice, each having the same memories as the other except with regard to the result of the experiment—we may call them “Alice-who-saw-up” and “Alice-who-saw-down.” To the question of which Alice is the “real Alice,” the unequivocal answer is, “both are.” Thus, in the Everett interpretation, the question “which result will Alice see” does not even in principle have an answer. (See discussion and references in \cite{4}).

The other characteristic aspect of probability referred to above is the \textit{regularity} of the relative frequency of the results of an experiment when it is repeated a large number of times or, equivalently, when a large ensemble of identical experiments are performed. “Whenever we say that the probability of an event $E$ with respect to an experiment $\mathcal{E}$ is equal to $P$, the concrete meaning of this assertion [is] the following: In a long series of repetitions of $\mathcal{E}$, it is practically certain that the [relative] frequency of $E$ will be approximately equal to $P \ldots$ This statement will be referred to as the frequency interpretation of the probability $P$ \cite[pp. 148-9]{3}.” Or, more succinctly, probability is “the theoretical value of long range relative frequency \cite[p. 58]{5}.”

These two properties are equally true of probability whether the setting is quantum or classical. In the case of quantum-mechanical probability, there is another key property: namely, the Born rule for the value of the probability. The possible results of a measurement corresponding to an operator $\hat{a}$ are the eigenvalues $\alpha_i$ of $\hat{a}$,

$$\hat{a}|\alpha_i\rangle = \alpha_i|\alpha_i\rangle, \quad i = 1, 2, \ldots, n.$$  \hspace{1cm} (1)

The probability $P(\alpha_i)$ of measuring the result $\alpha_i$ when, immediately before the measurement, the system being measured is in the state

$$|\psi_0\rangle = c_1|\alpha_1\rangle + \ldots + c_n|\alpha_n\rangle,$$  \hspace{1cm} (2)

where the $c_i$’s are complex numbers satisfying

$$\sum_{i=1}^{n} |c_i|^2 = 1,$$  \hspace{1cm} (3)

is given by the Born rule

$$P(\alpha_i) = |c_i|^2.$$  \hspace{1cm} (4)

(We assume here and in the remainder of this paper that the $\alpha_i$ are nondegenerate.)

A proof has been presented by Hartle\cite{6} that probability, as characterized by the above-mentioned properties, exists in unitary quantum mechanics without having to be incorporated via wavefunction reduction. (See also \cite{7}.) This proof makes use of the interpretive rule of quantum mechanics which states that if a system is in an eigenstate corresponding the eigenvalue $\alpha_i$ of the operator $\hat{a}$, then a measurement of the observable $\hat{a}$ on the system
will with certainty yield the result \( \alpha_i \). Hartle makes use of operators \( \hat{f}_N(\alpha_i) \), corresponding to the measurement of the relative frequency of the result \( \alpha_i \) in an ensemble of \( N \) identical systems (i.e., the fraction of systems in the ensemble in which the result \( \alpha_i \) is obtained upon measurement) \[6, 8\], and shows that, in the limit \( N \to \infty \), the state vector of the entire ensemble

\[
|\psi_N\rangle = |\psi(1)\rangle|\psi(2)\rangle \ldots |\psi(N)\rangle
\]  

(where each \( |\psi(\nu)\rangle \) is of the form (2) with the same \( c_i \)'s) approaches an eigenstate of the relative frequency operators \( \hat{f}_N(\alpha_i) \) with respective eigenvalues given by the Born rule (4):

\[
\lim_{N \to \infty} \hat{f}_N(\alpha_i)|\psi_N\rangle = |c_i|^2 \lim_{N \to \infty} |\psi_N\rangle.
\]

Farhi, Goldstone and Gutmann\[9\] prove a similar result working from the outset in the \( N = \infty \) Hilbert space.

As has been pointed out by Kent\[10\] and by Zurek\[11\], these proofs do not demonstrate that the phenomenon of probability exists in the Everett version of quantum mechanics. That is because, in the Everett interpretation, the state vector (5) is the state vector of the entire “multiverse” \[12\] and describes all the Everett branches at once. The state vector (5), using (2), is equal to

\[
|\psi_N\rangle = \sum_{i^{(1)}=1}^{n} \ldots \sum_{i^{(N)}=1}^{n} c_1^{r_{i^{(1)}}} \ldots c_n^{r_{i^{(N)}}}|i^{(1)}, \ldots, i^{(N)}\rangle,
\]

where

\[
r_{i} = r_{i}(i^{(1)}, \ldots, i^{(N)}) = \sum_{p=1}^{n} \delta_{i,i^{(p)}}
\]

is the number of factors of eigenvalue-\( i \) states in a given term in (7), and where each of the \( n^N \) states

\[
c_1^{r_{i^{(1)}}} \ldots c_n^{r_{i^{(N)}}}|i^{(1)}, \ldots, i^{(N)}\rangle = c_1^{r_{\alpha_{(1)}}} \ldots c_n^{r_{\alpha_{(N)}}}|\alpha_{(1)}\rangle \ldots |\alpha_{(n)}\rangle,
\]

is termed a “branch” of the overall state vector (7). In the Everett interpretation, as usually formulated in terms of state vectors, each branch (9) is deemed to correspond to a distinct physical reality. If an observer \( O \) measures each of the \( N \) systems in the ensemble, the state vector describing the observer and the ensemble is, after the measurement has taken place,

\[
|O, \psi_N\rangle = \sum_{i^{(1)}=1}^{n} \ldots \sum_{i^{(N)}=1}^{N} c_1^{r_{i^{(1)}}} \ldots c_n^{r_{i^{(N)}}}|O; i^{(1)}, \ldots, i^{(N)}\rangle|i^{(1)}, \ldots, i^{(N)}\rangle.
\]

where the factor

\[
|O; i^{(1)}, \ldots, i^{(N)}\rangle
\]

in each branch

\[
c_1^{r_{i^{(1)}}} \ldots c_n^{r_{i^{(N)}}}|O; i^{(1)}, \ldots, i^{(N)}\rangle|i^{(1)}, \ldots, i^{(N)}\rangle
\]

corresponds to a distinct perception on the part of the observer, correlated with definite measurement results for each of the systems in the ensemble.
What must be shown to establish the frequency interpretation of probability and the Born rule in Everett quantum mechanics is that an observer performing a relative frequency measurement on an ensemble of independent identical systems, in the limiting case that the number of systems making up the ensemble becomes infinite, will always (never) have the perception of measuring a relative frequency equal to (different from) the Born-rule probability (4). In terms of the branches (12), what must be shown is that, in the limit \( N \to \infty \), the observer in each branch perceives the Born-rule relative frequency. The complete state vector (10) does not correspond to an Everett branch in which an observer has experienced specific measurement results for each of the systems in the ensemble—else it would be of the form (12)—so demonstrating that the complete state vector is an eigenfunction of the relative frequency operators \( \hat{f}_N(\alpha_i) \) with the corresponding Born-rule eigenvalues (4) does not make the case that must be made.

DeWitt[13] and Okhuwa[14] demonstrate that the state vector \( |\chi_N\rangle \) which is the sum of all of the branches experiencing non-Born relative frequencies has zero norm in the limit of an infinitely-large ensemble,

\[
\lim_{N \to \infty} \langle \chi_N | \chi_N \rangle = 0.
\]

(13)

Since experience inheres in individual branches, it is not relevant to the task of characterizing which experiences are or are not possible that some branches can or cannot be grouped together so that the norm of their sum does or does not vanish. (It is true that the norm of a branch corresponding to any particular outcome of measurement of the entire ensemble will go to zero as the size of the ensemble, and hence the number of possible outcomes, goes to infinity. But this is as true for branches in which Born-rule relative frequencies are experienced as it is for branches in which relative frequencies differ from the Born rule (4).)

Hartle and Farhi, Goldstone and Gutmann work in the Schrödinger picture. DeWitt and Okhuwa work in the Heisenberg picture, but focus on states rather than operators. Recently, Deutsch and Hayden [15] and the author [16, 17] have shown that the issue of locality in Everett quantum theory is clarified when one works in the Heisenberg picture and focuses on the dynamics of operators. In this paper I show that such an approach clarifies as well the analysis of the origin of probability. In Sec. 2 below I review the formalism and interpretive rules of Heisenberg-picture Everett quantum mechanics, and introduce a rule which corresponds to the interpretation given to (13) by DeWitt and Okhuwa but which is free of the above-mentioned difficulties. In Sec. 3 I show that these rules, applied to a physically-realizable relative-frequency-measuring device—one of finite resolution, as described by Graham[8]—lead to the conclusion that the device will, to within the limits of its resolution, observe a result consistent with the Born rule (4) in the limiting case of an infinitely-large ensemble. Sec. 4 contains a summary of the proof. Sec. 5 discusses the ontology of Everett quantum mechanics in the Heisenberg picture, and comments on some other frequency-based approaches to probability in the Everett interpretation.\(^1\)

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\(^1\)Approaches to probability in Everett or other interpretations of quantum mechanics which use concepts of probability other than the frequency interpretation, such as ignorance[4, 18], decision theory[19, 20, 21, 22], and Bayesian inference[23], are outside the scope of this paper.
2 Measurement in Everett Quantum Mechanics

2.1 Schrödinger Picture

An observer $O$ measures an observable $\hat{g}$ pertaining to a system $S$. The eigenspectrum of $\hat{g}$ will be taken to finite, discrete and nondegenerate:

$$\hat{g}|S;\gamma_i\rangle = \gamma_i|S;\gamma_i\rangle, \quad i = 1, \ldots, M.$$  \hfill (14)

The state space of $O$ is spanned by $L + 1$ eigenstates of the operator $\hat{b}$ corresponding to the observer’s state of belief regarding the result of the measurement:\footnote{"Observer" here can refer as well to a machine as to a human being, with “state of belief” denoting results stored in a memory device.}

$$\hat{b}|O;\beta_i\rangle = \beta_i|O;\beta_i\rangle, \quad i = 0, \ldots, L,$$  \hfill (15)

where $\beta_0$ corresponds to the state of ignorance prevailing before a measurement has been made. We do not assume that the measurement produces a one-to-one mapping from the $\gamma_i$'s to the $\beta_i$'s, $i > 0$, but allow for the possibility that several different $\gamma_i$'s might correspond to the same $\beta_i$:

$$\{\beta_i\} = \{\beta(\gamma_j)\}, \quad i = 1, \ldots, L, \quad j = 1, \ldots, M, \quad L \leq M,$$  \hfill (16)

$$\beta_i \neq \beta_j, \quad i \neq j$$  \hfill (17)

with the values of $\beta_i$ otherwise arbitrary.

We will always consider situations in which the state before the measurement has occurred, $|\psi(t_0)\rangle$, is of the form

$$|\psi(t_0)\rangle = |O;\beta_0\rangle|S;\psi\rangle;$$  \hfill (18)

i.e., $O$ is in a state of ignorance and $S$ is an arbitrary state the observables of which are uncorrelated with those of $O$.

If before the measurement $S$ is in the eigenstate of $\hat{\gamma}$ with eigenvalue $\gamma_i$, then the state vector describing both $O$ and $S$ before the measurement is the product state

$$|O;\beta_0\rangle|S;\gamma_i\rangle.$$  \hfill (19)

By virtue of the interpretation of operator eigenstates as states in which measurement will definitely obtain the corresponding eigenvalue as a result, the action on the initial state (19) of the unitary operator $\hat{U}$ corresponding to $O$ measuring $S$ must be to produce the state in which $\hat{g}$ definitely has the value $\gamma_i$ and $\hat{b}$ definitely has the value $\beta(\gamma_i)$:

$$\hat{U}|O;\beta_0\rangle|S;\gamma_i\rangle = |O;\beta(\gamma_i)\rangle|S;\gamma_i\rangle.$$  \hfill (20)

From this result and the linearity of quantum mechanics, we conclude that

$$\hat{U} = \sum_{i=1}^{L} \tilde{u}_i \otimes \tilde{P}_i$$  \hfill (21)
where \( \hat{P}_i^S \) is the projection operator which projects out those states corresponding to \( \beta_i \):

\[
\hat{P}_i^S = \sum_{j|\beta_j(\gamma_j) = \beta_i} \hat{P}_j^S, \quad i = 1, \ldots, L,
\]  

(22)

\[
\hat{P}_j^S = |S; \gamma_j\rangle\langle S; \gamma_j|, \quad j = 1, \ldots, M.
\]  

(23)

The action of \( \hat{u}_i \) in the state space of \( \mathcal{O} \) is

\[
\hat{u}_i|\mathcal{O}; \beta_0\rangle = |\mathcal{O}; \beta_i\rangle, \quad i = 1, \ldots, L.
\]  

(24)

The action of \( \hat{u}_i \) on the other states of \( \mathcal{O} \), \( |\mathcal{O}; \beta_i\rangle \) with \( i > 0 \), will not play a role in subsequent analysis. (For an example of the complete action of \( \hat{u}_i \) in the state space of an observer with \( L = 2 \) see [16, Sec. 4.1].)

### 2.2 Heisenberg Picture

We will take the initial-time constant Heisenberg-picture state vector to be the before-measurement state (18), and we will distinguish Heisenberg-picture operators by explicit time arguments. At the initial time, \( t_0 \), the Heisenberg-picture operators are equal to the corresponding Schrödinger-picture operators:

\[
\hat{g}(t_0) = \hat{g},
\]  

(25)

\[
\hat{b}(t_0) = \hat{b}.
\]  

(26)

At time \( t_1 \), after the measurement has taken place,

\[
\hat{g}(t_1) = \hat{U}^\dagger \hat{g} \hat{U},
\]  

(27)

\[
\hat{b}(t_1) = \hat{U}^\dagger \hat{b} \hat{U}.
\]  

(28)

Using (21)-(23), (27) and (28), we find that

\[
\hat{g}(t_1) = \hat{g},
\]  

(29)

\[
\hat{b}(t_1) = \sum_{i=1}^{L} \hat{i}_i \otimes \hat{P}_i^S
\]  

(30)

where

\[
\hat{i}_i = \hat{u}_i^\dagger \hat{b} \hat{u}_i,
\]  

(31)

so

\[
\hat{i}_i|\mathcal{O}; \beta_0\rangle = \beta_i|\mathcal{O}; \beta_0\rangle.
\]  

(32)

The fact that at time \( t_1 \) the operator \( \hat{b}(t) \) takes the form (30), a sum of operators in the state space of \( \mathcal{O} \) each of which is “labeled” with a factor acting in the state space of \( S \), is taken to indicate that the state of awareness of \( \mathcal{O} \) at that time can be regarded, for all practical purposes, as split into \( L \) noninteracting copies, with copy \( i \) perceiving measurement result \( \beta_i \). We will term this “interpretive rule 1.”

6
To this interpretive rule we adjoin the following “interpretive rule 2”: At any time \( t \) only those copies of \( O \) exist which have nonzero values for \( W_i(t) \),
\[
W_i(t) \neq 0, \quad (33)
\]
where the “weight” \( W_i(t) \) is defined as the matrix element of the label factor between the initial state (18) and its adjoint; in the present example,
\[
W_i(t) \equiv \langle \psi(t_0)|\hat{P}_i\hat{S}|\psi(t_0) \rangle = \langle S; \psi|\hat{P}_i\hat{S}|S; \psi \rangle. \quad (34)
\]
The condition
\[
W_i(t) = 0, \quad (35)
\]
indicating that no observer-copy exists at time \( t \) who perceives measurement result \( \beta_i \) will play a role analogous to the condition (13) in the approaches of DeWitt and Okhuwa—with the distinction that condition (35), for each value of \( i \), explicitly refers to a single Everett copy of an observer.

3 The Frequency Interpretation of Probability and the Born Rule in Everett Quantum Mechanics

3.1 State Spaces of Systems and Observers

We consider an ensemble of \( N \) identical physical systems \( S^{(p)}, p = 1, \ldots, N \). The state space of \( S^{(p)} \) is spanned by the two eigenstates of an operator \( \hat{a}^{(p)} \) which acts nontrivially only in that state space:
\[
\hat{a}^{(p)}|S^{(p)};\alpha_{i^{(p)}}\rangle = \alpha_{i^{(p)}}|S^{(p)};\alpha_{i^{(p)}}\rangle, \quad i^{(p)} = 1, 2, \quad p = 1, \ldots, N, \quad (36)
\]
\[
\alpha_1 \neq \alpha_2. \quad (37)
\]
To each system \( S^{(p)} \) there corresponds an observer/measuring device \( O^{(p)} \), the state space of which is spanned by the three eigenvectors of an operator \( \hat{b}^{(p)} \) which acts nontrivially only in that state space:
\[
\hat{b}^{(p)}|O^{(p)};\beta_{i^{(p)}}\rangle = \beta_{i^{(p)}}|O^{(p)};\beta_{i^{(p)}}\rangle, \quad i^{(p)} = 0, 1, 2, \quad p = 1, \ldots, N, \quad (38)
\]
\[
\beta_i \neq \beta_j, \quad i \neq j. \quad (39)
\]
The observer \( O^{(p)} \) interacts with the system \( S^{(p)} \) in such a manner as to determine the value of the observable \( \hat{a}^{(p)} \), as described in the previous section. The eigenvalue \( \beta_0 \) indicates the state of ignorance, while \( \beta_1 \) and \( \beta_2 \) correspond, respectively, to \( O^{(p)} \) having measured \( \alpha_1 \) or \( \alpha_2 \).

After these \( N \) measurement interactions have taken place—it is immaterial whether they take place simultaneously or in any arbitrary order since, by virtue of the fact that each measurement interaction affects a distinct system-observer pair, the operators corresponding to the different interactions commute—an additional observer \( F \) interacts with
all of the $O^{(p)}$ so as to determine the relative frequency of the result $\beta_1$ among the observers. Since there are $N$ observers, the possible values of a measurement of the relative frequency are the $N + 1$ numbers $0, 1/N, 2/N, \ldots, 1$. As $N \to \infty$ the number of possible values for the relative frequency grows without bound, and the difference between adjacent values shrinks to zero.

As emphasized in this context by Graham[8], the resolution of any real measuring device is finite. Suppose $F$ queries each observer in sequence and in the end computes the relative frequency. If the physical device $F$ occupies a finite volume and has at its disposal a finite amount of energy, it will be able to record at most a finite number of measurements. Suppose, for example, it uses the directions of spins of neutrons to record the responses of each observer $O^{(p)}$. As more neutrons are added to the finite volume available to $F$, more energy is required to pack the neutrons into the volume as a result of degeneracy pressure. An upper limit on how much information can be recorded in this manner is set by the number of neutrons which, when located within the finite available volume, would have sufficient mass to produce a black hole with an event horizon surrounding the volume. (The black-hole limit would, of course, apply as well to any information-storage scheme involving bosonic degrees of freedom. See, e.g., [24] and references therein.) Of course, existing information-storage devices have capacities well below these limits[24]. As $N \to \infty$, $F$ will have to either stop recording new information from the $O^{(p)}$'s or discard old information. The situation is qualitatively no different if $F$ updates the value of the relative frequency after interacting with each observer. As $N \to \infty$ the number of digits required to record any possible number of the form (integer from 0 through $N) \times 1/N$ will grow until it outstrips the available memory.

So, we take the state space of $F$ to be spanned by $\nu + 2$ eigenvectors:

$$\hat{f}|F; \phi_i\rangle = \phi_i|F; \phi_i\rangle, \quad i = 0, \ldots, \nu + 1,$$

(40)

where, for $i > 0$, the eigenvalue $\phi_i$ is one of the possible outputs of $F$ after it has interacted with all of the $O^{(p)}$s,

$$\phi_i = (i - 1)/\nu, \quad i = 1, \ldots, \nu + 1.$$

(41)

$\phi_0$ will be taken to indicate a state of ignorance. For this purpose the only requirement is that $\phi_0$ not equal any of the other eigenvalues (41); it will be convenient to assign it the value

$$\phi_0 = -1/\nu.$$

(42)

### 3.2 Measurement Interactions

#### 3.2.1 Measurement of $S^{(p)}$ by $O^{(p)}$

The unitary operator corresponding to the interaction between $S^{(p)}$ and $O^{(p)}$, following Sec. 2.1 above, is

$$\hat{U}^{(p)} = \sum_{i=1}^{2} \hat{u}_i^{(p)} \otimes \hat{P}_{iS^{(p)}}, \quad p = 1, \ldots, N,$$

(43)

where

$$\hat{P}_{iS^{(p)}} = |S^{(p)}; \alpha_i\rangle\langle S^{(p)}; \alpha_i|, \quad i = 1, 2, \quad p = 1, \ldots, N,$$

(44)
and
\[
\hat{u}_i^{(p)} |\mathcal{O}(p); \beta_0 \rangle = |\mathcal{O}(p); \beta_i \rangle, \quad i = 1, 2, \quad p = 1, \ldots, N. \tag{45}
\]

From (43), (44) and the unitarity of $\hat{U}^{(p)}$ it follows that
\[
\hat{u}_i^{(p)\dagger} \hat{u}_i^{(p)} = 1, \quad i = 1, 2, \quad p = 1, \ldots, N. \tag{46}
\]

### 3.2.2 Measurement of $\mathcal{O}(p)$ by $F$

Define the relative frequency function for the result $\beta_{i(p)} = \beta_1$,
\[
f(\beta_{i(1)}, \ldots, \beta_{i(N)}) = (1/N) \sum_{p=1}^{N} \delta_{i(p),1}, \quad i^{(p)} \neq 0 \quad \forall \quad p. \tag{47}
\]

The possible values of this function are
\[
f_l = l/N, \quad l = 0, \ldots, N. \tag{48}
\]

Define also the finite-resolution relative frequency function $\phi(\beta_{i(1)}, \ldots, \beta_{i(N)})$ to be that $\phi_i$, $i = 1, \ldots, \nu + 1$, which is closest in value to $f(\beta_{i(1)}, \ldots, \beta_{i(N)})$:
\[
\phi(\beta_{i(1)}, \ldots, \beta_{i(N)}) = \arg \min_{\phi_i} |\phi_i - f(\beta_{i(1)}, \ldots, \beta_{i(N)})|, \quad i^{(p)} \neq 0 \quad \forall \quad p \tag{49}
\]

(smaller $\phi_i$ in case of a tie)

where the $\phi_i$’s are as given in (41). It will also prove convenient to define
\[
\tilde{\phi}(\beta_{i(1)}, \ldots, \beta_{i(N)}) = \phi(\beta_{i(1)}, \ldots, \beta_{i(N)}), \quad i^{(p)} \neq 0 \quad \forall \quad p \tag{50}
\]

\[
= \phi_0 \quad \text{otherwise} \tag{51}
\]

where $\phi_0$ is as given in (42).

The unitary operator corresponding to the measurement of all of the $\mathcal{O}(p)$’s by $F$ can then be written as
\[
\hat{U}_F = \sum_{k=0}^{\nu+1} \hat{u}_k^F \otimes \hat{P}_k^{\mathcal{O}(p)}, \tag{52}
\]

where
\[
\hat{u}_k^F |F; \phi_0 \rangle = |F; \phi_k \rangle, \quad k = 0, \ldots, \nu + 1, \tag{53}
\]

Note that $\hat{u}_0^F$ acts as the identity on the ignorant state $|F; \phi_0 \rangle$. $\hat{P}_k^{\mathcal{O}(p)}$ is the projection operator which projects out those states corresponding to $\phi_k$:
\[
\hat{P}_k^{\mathcal{O}(p)} = \sum_{i^{(1)}=0}^{2} \cdots \sum_{i^{(N)}=0}^{2} \delta_{\phi_k(\beta_{i(1)}, \ldots, \beta_{i(N)})} \otimes_{p=1}^{N} \hat{P}_{i^{(p)}}^{\mathcal{O}(p)} \tag{54}
\]

where
\[
\hat{P}_{i^{(p)}}^{\mathcal{O}(p)} = |\mathcal{O}(p); \beta_{i^{(p)}} \rangle \langle \mathcal{O}(p); \beta_{i^{(p)}} |. \tag{55}
\]
Using (41), (42) and (49)-(51) we see that, for any given set of values for \(i^{(1)}, \ldots, i^{(N)}\), the quantity \(\nu\tilde{\phi}(\beta_{i^{(1)}}, \ldots, \beta_{i^{(N)}})\) is equal to one of the values \(-1, 0, 1, \ldots, \nu\). Therefore

\[
\delta_{\nu\tilde{\phi}(\beta_{i^{(1)}}, \ldots, \beta_{i^{(N)}}, k-1)} \delta_{\nu\tilde{\phi}(\beta_{i^{(1)}}, \ldots, \beta_{i^{(N)}}, l-1)} = \delta_{k,l} \nu\tilde{\phi}(\beta_{i^{(1)}}, \ldots, \beta_{i^{(N)}}, k-1)
\]

(56)

and

\[
\sum_{k=0}^{\nu+1} \delta_{\nu\tilde{\phi}(\beta_{i^{(1)}}, \ldots, \beta_{i^{(N)}}, k-1)} = 1.
\]

(57)

Using these we verify that

\[
\tilde{P}_k \tilde{P}_l = \delta_{k,l} \tilde{P}_k,
\]

(58)

and

\[
\sum_{k=0}^{\nu+1} \tilde{P}_k = 1.
\]

(59)

From (52), (58) and the unitarity of \(\hat{U}_F\) it follows that

\[
\hat{u}_k F = \hat{u}_k F = 1,
\]

(60)

3.2.3 Complete Measurement Transformation Operator

The unitary operator corresponding to all the \(O^{(p)}\)'s measuring their \(S^{(p)}\)'s, followed by \(F\) measuring all the \(O^{(p)}\)'s and determining the relative frequency of \(\beta_1\) observations, is

\[
\hat{U} = \hat{U}_F \hat{U}_O,
\]

(61)

where

\[
\hat{U}_O = \otimes_{p=1}^{N} \hat{U}^{(p)}.
\]

(62)

3.3 Post-Measurement Heisenberg-Picture Operators

3.3.1 \(\hat{a}^{(p)}(t_1)\)

After measurement,

\[
\hat{a}^{(p)}(t_1) = \hat{U}^{(p)} \hat{a}^{(p)} \hat{U},
\]

(63)

or, using (61),

\[
\hat{a}^{(p)}(t_1) = \hat{U}_O^{\dagger} \hat{U}_F^{\dagger} \hat{a} \hat{U}_F \hat{U}_O.
\]

(64)

From (52)-(55) we see that \(\hat{U}_F\) doesn’t act in the state space of \(S^{(p)}\), so

\[
[\hat{U}_F, \hat{a}^{(p)}] = 0,
\]

(65)

and (64) can be written as

\[
\hat{a}^{(p)}(t_1) = \hat{U}_O^{\dagger} \hat{a}^{(p)} \hat{U}_O = \left( \otimes_{q=1}^{N} \hat{U}^{(q)} \right) \hat{a} \left( \otimes_{r=1}^{N} \hat{U}^{(r)} \right)
\]

(66)
using (62). From (43) and (44) we see that \( \hat{U}(q) \) only acts nontrivially on \( \hat{a}^{(p)} \) for \( q = p \), so, using (43), (67) becomes

\[
\hat{\alpha}^{(p)}(t_1) = \left( \sum_{j(p)=1}^2 \hat{u}_{j(p)}^{(p)\dagger} \otimes \hat{P}_{j(p)}^{S(p)} \right) \hat{\alpha}^{(p)} \left( \sum_{j(p)=1}^2 \hat{u}_{j(p)}^{(p)} \otimes \hat{P}_{j(p)}^{S(p)} \right).
\]

Since

\[
\hat{\alpha}^{(p)} = \sum_{k(p)=1}^2 \alpha_{k(p)} \hat{P}_{k(p)}^{S(p)},
\]

it follows from (68), (44) and (46) that \( \hat{\alpha}^{(p)}(t) \) is unchanged by the measurement process,

\[
\hat{\alpha}^{(p)}(t_1) = \hat{\alpha}^{(p)}.
\]

### 3.3.2 \( \hat{b}^{(p)}(t_1) \)

After measurement,

\[
\hat{b}^{(p)}(t_1) = \hat{U} \hat{b}^{(p)} \hat{U}.
\]

Using (61),

\[
\hat{b}^{(p)}(t_1) = \hat{U}_{\mathcal{O}} \hat{\bar{U}}^\dagger \hat{b}^{(p)} \hat{U}_{\mathcal{F}} \hat{U}_{\mathcal{O}},
\]

or

\[
\hat{b}^{(p)}(t_1) = \hat{U}_{\mathcal{O}} \hat{\bar{X}}_b^{(p)} \hat{U}_{\mathcal{O}},
\]

where the intermediate quantity \( \hat{\bar{X}}_b^{(p)} \) is defined as

\[
\hat{\bar{X}}_b^{(p)} = \hat{U}_{\mathcal{F}} \hat{\bar{b}}^{(p)} \hat{U}_{\mathcal{F}}.
\]

From (74) and (52),

\[
\hat{\bar{X}}_b^{(p)} = \left( \sum_{k=0}^{\nu+1} \hat{u}_k^\dagger \hat{P}_{k}^{\bar{\mathcal{O}}} \right) \hat{\bar{b}}^{(p)} \left( \sum_{l=0}^{\nu+1} \hat{u}_l^\dagger \hat{P}_{l}^{\bar{\mathcal{O}}} \right)
\]

\[
= \sum_{k,l=0}^{\nu+1} \beta_k \beta_l \hat{P}_{k}^{\bar{\mathcal{O}}} \hat{P}_{l}^{\bar{\mathcal{O}}} \hat{\bar{X}}_b^{(p)}
\]

since

\[
\hat{\bar{b}}^{(p)} = \sum_{i=0}^2 \beta_i \hat{P}_{i}^{\bar{\mathcal{O}}^{(p)}}.
\]

Using (54)-(56)

\[
\hat{P}_{k}^{\bar{\mathcal{O}}} \hat{P}_{i}^{\bar{\mathcal{O}}^{(p)}} \hat{P}_{l}^{\bar{\mathcal{O}}} = \delta_{k,l} \sum_{i^{(1)}=0}^2 \cdots \sum_{i^{(N)}=0}^2 \delta_{i^{(p)},i} \delta_{i^{(\phi(\beta_{i^{(1)},\ldots,\beta_{i^{(N)})}),k-1}}} \otimes_{q=1}^N \hat{P}_{i^{(q)}}^{\bar{\mathcal{O}}^{(q)}}.
\]

Using (78), (56) and (77) in (76),

\[
\hat{\bar{X}}_b^{(p)} = \hat{\bar{b}}^{(p)}.
\]
Using (79), (43), (44), (46) and (62) in (73),
\[
\hat{b}^{(p)}(t_1) = \sum_{i=1}^{2} \hat{b}_i^{(p)} \otimes \hat{P}_i^{S(p)},
\] (80)
where
\[
\hat{b}_i^{(p)} = \hat{u}_i^{(p)\dagger} \hat{b}_i^{(p)} \hat{u}_i^{(p)}.
\] (81)

### 3.3.3 \(\hat{f}(t_1)\)

After measurement,
\[
\hat{f}(t_1) = \hat{U}^\dagger \hat{f} \hat{U}.
\] (82)

Using (61),
\[
\hat{f}(t_1) = \hat{U}_\mathcal{O}^\dagger \hat{U}_\mathcal{F}^\dagger \hat{f} \hat{U}_\mathcal{F} \hat{U}_\mathcal{O},
\] (83)
or
\[
\hat{f}(t_1) = \hat{U}_\mathcal{O}^\dagger \hat{X}_f \hat{U}_\mathcal{O},
\] (84)
where the intermediate quantity \(\hat{X}_f\) is defined as
\[
\hat{X}_f = \hat{U}_\mathcal{F}^\dagger \hat{f} \hat{U}_\mathcal{F}.
\] (85)

Using (52) and (58), (85) becomes
\[
\hat{X}_f = \sum_{k=0}^{\nu+1} \hat{u}_k^F \otimes \hat{P}_k^{\mathcal{O}}.
\] (86)

Using this in (84),
\[
\hat{f}(t_1) = \sum_{k=0}^{\nu+1} \hat{f}_k \otimes \hat{L}_k,
\] (87)
where
\[
\hat{f}_k = \hat{u}_k^F \hat{f} \hat{u}_k^F, \quad k = 0, \ldots, \nu + 1,
\] (88)
and
\[
\hat{L}_k = \sum_{i^{(1)}=1}^{2} \ldots \sum_{i^{(N)}=1}^{2} \left( \left( \otimes_{p=1}^{N} \hat{u}_i^{(p)\dagger} \right) \hat{P}_k \left( \otimes_{q=1}^{N} \hat{u}_i^{(q)} \right) \right) \otimes_{r=1}^{N} \hat{P}_i^{S(r)} \otimes_{r=1}^{N} \hat{P}_i^{S(r)}, \quad k = 0, \ldots, \nu + 1.
\] (89)

### 3.4 Initial State

For the constant Heisenberg-picture state we take the product state in which \(\mathcal{F}\) and the \(O^{(p)}\)'s are ignorant and each of the \(S^{(p)}\)'s is in a superposition of \(|S^{(p)}; \alpha_1\rangle\) and \(|S^{(p)}; \alpha_2\rangle\) with the same coefficients:
\[
|\psi(t_0)\rangle = |\mathcal{F}; \phi_0\rangle \prod_{p=1}^{N} |O^{(p)}; \beta_0\rangle \prod_{q=1}^{N} |S^{(q)}; \psi_0\rangle,
\] (90)
where
\[ |S^{(q)}; \psi_0\rangle = c_1^{(q)} |S^{(q)}; \alpha_1\rangle + c_2^{(q)} |S^{(q)}; \alpha_2\rangle, \quad q = 1, \ldots, N, \quad (91) \]
with
\[ c_1^{(q)} = c_1, \quad c_2^{(q)} = c_2 \quad \forall \ q, \quad (92) \]
\[ |c_1|^2 + |c_2|^2 = 1. \quad (93) \]

### 3.5 Weights

Using (90), the definition of the weight in Sec. 2.2 and the results of Secs. 3.3.1 - 3.3.3 we see that the weights associated with \( \hat{a}^{(p)} \) both before and after measurement, and with \( \hat{b}^{(p)} \) and \( \hat{f} \) before measurement, are simply unity. All of these operators are of the form (30) only in a trivial sense.

After measurement, \( \hat{b}^{(p)} \) and \( \hat{f} \) are nontrivially in the form (30). The weight associated with \( \hat{b}^{(p)} \) before measurement, is, according to (80),
\[ W_{b,i}(t_1) = \langle \psi(t_0)|\hat{P}_{S_i}|\psi(t_0)\rangle, \quad i = 1, 2, \quad p = 1, \ldots, N. \quad (94) \]
Using (44) and (90)-(92), we find
\[ W_{b,i}(t_1) = |c_i|^2, \quad i = 1, 2, \quad p = 1, \ldots, N, \quad (95) \]
which is nonzero for any value of \( N \) and for both values of \( i \), unless either \( c_1 = 0 \) or \( c_2 = 0 \).

From (87), the weight for \( \hat{f}_k \) is
\[ W_{f,k}(t_1) = \langle \psi(t_0)|\hat{L}_{k}|\psi(t_0)\rangle. \quad (96) \]
Using (89), (90)-(92), (45), (54), and (55),
\[ W_{f,k}(t_1) = \sum_{i(1)=1}^{2} \cdots \sum_{i(N)=1}^{2} \delta_{\nu\phi(\beta_{i(1)}),\ldots,\beta_{i(N)},k-1} \prod_{r=1}^{N} |c_{i(r)}|^2, \quad k = 0, \ldots, \nu + 1. \quad (97) \]
From (41), (42), and (49)-(51), we see that, for any value of \( N \), \( W_{f,k}(t_1) \) vanishes for \( k = 0, \)
\[ W_{f,0}(t_1) = 0 \quad \forall \ N. \quad (98) \]

So, at \( t_1 \), there is no Everett copy of the relative-frequency observer \( \mathcal{F} \) who has the perception of being in a state of ignorance. This is of course what we expect to find, given that we have defined the evolution operators \( \hat{U}^{(p)} \) to measure the states of the \( S^{(p)} \)'s without error, and the evolution operator \( \hat{U}_\mathcal{F} \) to accurately determine the \( O^{(p)} \)'s measurements and compute the appropriate finite-resolution frequency \( \phi_i \).

For \( k = 1, \ldots, \nu + 1, \)
\[ W_{f,k}(t_1) = \sum_{i(1)=1}^{2} \cdots \sum_{i(N)=1}^{2} \delta_{\nu\phi(\beta_{i(1)}),\ldots,\beta_{i(N)},k-1} \prod_{r=1}^{N} |c_{i(r)}|^2. \quad (99) \]
Using (49)-(51), (57), (92), (93) and the binomial theorem, we verify that
\[ \sum_{k=1}^{\nu+1} W_{f,k}(t_1) = 1. \] (100)

Using (41) and (47)-(51) in (99), we obtain
\[ W_{f,k}(t_1) = \sum_{l \mid 0 \leq l \leq N, N(\phi_k-(1/2\nu))<l\leq N(\phi_k+(1/2\nu))} p_{N,l} \] (101)
where
\[ p_{N,l} = \frac{N!}{l!(N-l)!}|c_1|^{2l}|c_2|^{2(N-l)}. \] (102)

To evaluate (101) in the limit \( N \to \infty \) we make use of Bernoulli’s law of large numbers. From [25, p.195], for example, we have, in our notation,
\[ \lim_{N \to \infty} S_N = 0, \] (103)
where
\[ S_N = \sum_{l \mid 0 \leq l \leq N, |l-N|c_1^2| > N\epsilon} p_{N,l}. \] (104)
(The value of \( \epsilon \) in (104) is independent of \( N \).) This is true for every positive number \( \epsilon \). From (102) we see that
\[ p_{N,l} \geq 0 \forall N, l. \] (105)
Therefore, in (104), we can, for each \( N \), replace the sum over all integers \( l \) between 0 and \( N \) inclusive satisfying \( |l-N|c_1^2| > N\epsilon \) with the sum over an arbitrary subset \( \{l\}_N \) of these integers. That is,
\[ \lim_{N \to \infty} \tilde{S}_N = 0, \] (106)
where
\[ \tilde{S}_N = \sum_{\{l\}_N \mid 0 \leq l \leq N, |l-N|c_1^2| > N\epsilon} p_{N,l}, \] (107)
since \( |\tilde{S}_N| \leq |S_N| \forall N \).

Suppose \( |c_1|^2 \) is closer to the finite resolution relative frequency \( \phi_{k'} \) than to any other \( \phi_k, k = 1, \ldots, \nu + 1 \),
\[ k' = \arg \min_k |\phi_k - |c_1|^2|. \] (108)
For concreteness say that \( |c_1|^2 \) is less than \( \phi_{k'} \),
\[ |c_1|^2 = \phi_{k'} - \Delta, \] (109)
where
\[ 0 < \Delta < \frac{1}{2\nu}. \] (110)
(See Fig. 1.) Consider now the expression (101) for the weight \( W_{f,k''}(t_1) \), where
\[ k'' = k' - 1. \] (111)
Figure 1: Finite resolution relative frequencies $\phi_{k''}$, $\phi_{k'}$ and Born-rule probability $|c_1|^2$ in the case that $|c_1|^2 < \phi_{k'}$.

The values of $l$ which enter into the sum in (101), for $k = k''$, are bounded above by

$$l_{ub} = N \left( \phi_{k''} + \frac{1}{2\nu} \right).$$

Therefore, using (41), (109), (111) and (112), the values that the quantity $|l - N|c_1|^2|$ can take for any $l$ appearing in the sum in $W_{f,k''}(t_1)$ are bounded below by

$$|l_{ub} - N|c_1|^2| = N \left( \frac{1}{2\nu} - \Delta \right).$$

So, for any positive $\epsilon$ such that

$$\epsilon < \left( \frac{1}{2\nu} - \Delta \right)$$

the sum in (101) is of the form (107), implying, by (106),

$$\lim_{N \to \infty} W_{f,k''}(t_1) = 0.$$  (115)

For $k \neq k''$ or $k'$, allowed values of $l$ in the sum in the expression (101) for $W_{f,k}(t_1)$ are bounded away from $N|c_1|^2$ even more strongly, so the same argument applies, as it does also in the case that $|c_1|^2 > \phi_{k'}$. So, we conclude that

$$\lim_{N \to \infty} W_{f,k}(t_1) = 0, \quad k \neq k'. $$  (116)

The above analysis does not apply to $W_{f,k'}(t_1)$ since, for fixed $\epsilon$, the sum will include values of $l$ approaching closer to $N|c_1|^2$ than $N\epsilon$ for $N$ sufficiently large, no matter how small a fixed value of $\epsilon$ is used. From (100) and (116) we see that

$$\lim_{N \to \infty} W_{f,k'}(t_1) = 1.$$  (117)

In the case that $|c_1|^2$ is precisely equidistant from two adjacent $\phi_k$'s,

$$|c_1|^2 = \phi_k^< + \frac{1}{2\nu} = \phi_k^> - \frac{1}{2\nu},$$

$$k^> = k^< + 1,$$

the argument above goes through for all $k \neq k^<$ or $k^>$ to show that

$$\lim_{N \to \infty} W_{f,k}(t_1) = 0, \quad k \neq k^<$ or $k^>.$$  (120)
Using (100), we therefore conclude

$$\lim_{N \to \infty} (W_{f,k<}(t_1) + W_{f,k>}(t_1)) = 1. \tag{121}$$

In the case that $|c_1|^2 = |c_2|^2 = 1/2$ and $\nu$ is odd, the sums (101) for $W_{f,k<}(t_1)$ and for $W_{f,k>}(t_1)$ are term-by-term identical except for one term:

$$W_{f,k<}(t_1) = T(N) + T^< (N), \tag{122}$$

$$W_{f,k>}(t_1) = T(N) + T^> (N), \tag{123}$$

where

$$T(N) = \sum_{l \mid N \left(\frac{\nu-2}{2\nu}\right) < l < \frac{N}{2}} \frac{N!}{2^N l!(N-l)!}, \tag{124}$$

$$T^<(N) = \frac{N!}{2^N \left(\left[\frac{N}{2}\right]!\right)^2}, \quad N \text{ even}, \tag{125}$$

$$= 0 \quad \text{otherwise}, \tag{126}$$

and

$$T^>(N) = \frac{N!}{2^N \left[N \left(\frac{\nu-2}{2\nu}\right)!\right] \left[N \left(\frac{\nu+2}{2\nu}\right)!\right]}, \quad N \left(\frac{\nu+2}{2\nu}\right) \text{ is an integer > 0} \tag{127}$$

$$= 0 \quad \text{otherwise}. \tag{128}$$

From Stirling’s formula,

$$\lim_{N \to \infty} T^<(N) = \lim_{N \to \infty} T^>(N) = 0. \tag{129}$$

Using (121)-(123) and (129),

$$\lim_{N \to \infty} T(N) = 1/2. \tag{130}$$

(The limit of a sum of sequences is the sum of the limits of those sequences; see, e.g., [26, p. 49]). We conclude from (122), (123), (129) and (130) that

$$\lim_{N \to \infty} W_{f,k<}(t_1) = \lim_{N \to \infty} W_{f,k>}(t_1) = 1/2 \neq 0. \tag{131}$$

So, in the limiting case of an infinitely large ensemble of identical systems measured by an observer $F$ performing a real (finite-resolution) relative-frequency measurement there is, after measurement, either a single Everett copy of $F$ who perceives that finite-resolution relative frequency $\phi_k$ closest to the Born-rule value $|c_1|^2$, or two Everett copies of $F$ who respectively perceive one of the two $\phi_k$’s closest to $|c_1|^2$.  

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4 Summary

Premises:

1. Definition of measurement situation:
   (a) The initial state of observer and observed system is of the uncorrelated form (18).
   (b) The interaction between observer and system leaves the Heisenberg-picture operator corresponding to the state of belief of the observer in the form (30).
   (c) The observer has finite resolution, i.e., is capable of perceiving a discrete finite set of possible outcomes.

2. Interpretation of operators after measurement:
   (a) Each of the terms in (30) corresponds to a copy of the observer with a distinct perception of the outcome of the measurement; with the proviso that
   (b) Only those copies with nonzero weight (see Sec. 2.2) exist.

Conclusions:

1. After the measurement of relative frequency, in the limiting case of an infinitely large ensemble of measured systems, the only existing Everett copies of an observer measuring relative frequency will perceive results equal to the Born rule values, to within available resolution.

2. If one defines probability in accord with the frequency interpretation (see Sec. 1), then probability exists in Everett quantum mechanics, in that both randomness and, given any particular experiment (as specified by the initial state), an essentially unique limiting value for the measured relative frequency of an outcome in an ensemble of identical experiments, exist.

5 Discussion

The above analysis has been limited to “measurement situations.”

It may be possible to generalize it to apply to a broader class of situations. However, it is not necessary that it be possible to generalize it to apply to all possible physical situations. It is a hallmark of quantum mechanics that in many situations, for many observables, probability simply is not defined[27, 28]. (Page argues that probabilities need only be defined in the context of situations involving conscious perceptions[29, 30, 31].)

The ontology of the Everett interpretation in the Heisenberg picture is different from that in the Schrödinger picture. Consider the scenario of Sec. 3 above, and, for the moment,
the case of finite \( N \). The Everett interpretation in the Schrödinger picture would describe the situation after all the measurements have been made in terms of \( 2^N \) branches of the state vector. In the Heisenberg picture, there are two copies of each of the \( N \) systems \( \mathcal{S}^{(p)} \), two copies of each of the \( N \) observers \( \mathcal{O}^{(p)} \), and \( \nu + 1 \) copies of \( \mathcal{F} \). The relative frequency observer \( \mathcal{F} \) has interacted with all of the \( \mathcal{O}^{(p)} \)'s, yet there is only a limited sense in which any particular Everett copy of \( \mathcal{F} \) can be said to be “associated with” a particular copy of one of the \( \mathcal{O}^{(p)} \)'s.

There is no requirement that such associations exist in the formalism, either from a logical or physical point of view or in order that it accord with our experience. The perception of a fact regarding the result of a measurement (or information of any other sort) must be embodied in the state of some physical system \([32, 33]\]; more facts require more systems, or more states of a given system. In the scenario of Sec. 3 above there exist perceptions of the outcomes of measurements of each of the \( \mathcal{S}^{(p)} \)'s as well as perceptions of outcomes of relative-frequency measurements. To the extent that copies of one observer/measuring device can be associated with copies of another observer/measuring device or physical system, such associations are not necessarily one-to-one. In the present example, different copies of \( \mathcal{F} \) may be thought of as “sharing” those copies of the \( \mathcal{O}^{(p)} \)'s which are consistent with the relative frequencies measured by the former.

Rather than a set of “parallel” or “foliated” universes\([34]\), the situation after measurement interactions have taken place has here a more complicated structure. E.g., if the perceptions of having measured \( \mathcal{S}^{(1)}, \ldots, \mathcal{S}^{(N)} \) are stored in respective cell assemblies \( \mathcal{O}^{(1)}, \ldots, \mathcal{O}^{(N)} \) in the experimenter’s brain, and the awareness of the computation of the relative frequency is stored in a cell assembly \( \mathcal{F} \), then the experimenter after having made the measurements may be visualized as a “Siamese (\( \nu + 1 \))-tuplet,” members of which are distinguished by different copies of \( \mathcal{F} \) but share in common copies of the \( \mathcal{O}^{(p)} \)'s consistent with their differing perceived relative-frequency values.

Stein \([35]\), in criticizing Geroch’s presentation \([36]\) of the Everett interpretation, comments that “quantum mechanics will indeed predict with high probability that the statistics of the outcomes of these experiments [on ensembles of identical systems] will deviate little from a specifiable set of [relative] frequencies. But how, in the ordinary practice of physics, do we go about checking this prediction? We do so by performing the experiments and noting and counting their outcomes.” Consistent with this criticism, Farhi, Goldstone and Gutmann allow that the relative frequency measurements in their analysis must be the outputs of a device specifically constructed to measure relative frequency without obtaining any information about the results of measurements of the individual systems in the ensemble. In the present analysis, the individual outcomes have most certainly been noted, by the explicit interactions with the \( \mathcal{O}^{(p)} \)'s, as well as counted by \( \mathcal{F} \).

In the Copenhagen interpretation the weight, say \( W_{f,i}(t_1) \), is of course just the probability that the unique outcome of the measurement of \( \hat{f} \) is \( f_i \). The weight differs from Vaidman’s \([4, \text{ Sec. 3.5}] \) “measure of existence of a world,” \( \mu_i \). The latter is equal to the norm-squared of the Schrödinger-picture branch in which a definite outcome for the measurement of \( \hat{f} \) as well as definite outcomes for measurements of all the \( \hat{a}^{(p)} \)'s and \( \hat{b}^{(p)} \)'s have occurred; i.e., to the joint probability for all of these outcomes. Vaidman also introduces the concept of a measure of existence for “I,” “the sum of measures of existence of all differ-
ent worlds in which I exist...the measure of existence of my perception world[4, Sec. 3.5]."

If we approximate the totality of my perceptions by my perception of the single fact of the result of the relative frequency measurement being \( f \), then \( W_{f,i}(t) \) is equal to this measure. If we enlarge our model of “I” or “my perception world” to include my perceptions of the measurements of the individual physical systems \( S^{(p)} \), then, before deciding on an appropriate measure for “I,” we must address the question of how individual perceptions meld together to yield our subjective sense of unified conscious self-awareness. This is a question, as-yet-unsolved, of neuroscience[37, p. 464], and as such involves neural properties which can be described adequately by classical physics[38]. As seen above, the lack of an answer to this question does not in stand in the way of constructing locally-realistic quantitative models of quantum measurement situations, their outcomes and (using infinite ensembles) the probabilities of these outcomes.\(^4\)

Regarding the \( N \to \infty \) limit: The number of post-measurement Everett copies of \( F \) for finite \( N \) is \( \nu + 1 \); for infinite \( N \) it is one or two (see Sec. 3.5). If we consider the \( N \) measurements of the \( \mathcal{O}^{(p)} \)’s to be made sequentially, with \( F \) recomputing a value of \( \phi \) after each additional \( \mathcal{O}^{(p)} \)’s measurement (this would require a different type of interaction than that described in Sec 3), then the weights for the non-Born-rule copies of \( F \) decrease with each successive measurement, and these copies “finally” (at \( N = \infty \)) vanish altogether from existence.

One can never actually experiment with an infinitely-large ensemble,\(^5\) so one will never encounter this “mass extinction” of Everett copies. Even if it did occur for a physically-realizable size of the ensemble, it would not be the only situation in which dynamics dictates that an Everett copy with a certain perception cease to exist. Vaidman [18, Sec. 3] discusses a sentient neutron which enters an interferometer containing at its entrance a beam splitter which splits the trajectory of an incoming neutron wave-packet into two separate trajectories. The exit of the interferometer contains another beam splitter which coherently recombines the two internal trajectories so that the neutron emerges in a single definite direction. Immediately before the time at which the neutron wave packets within the interferometer pass through the second beam splitter, there are two Everett copies of the neutron, one perceiving “moving up,” the other perceiving “moving down.” Immediately after the time at which the wave packets pass through the beam splitter, only a single Everett copy remains, perceiving (say) “moving up.”

The analysis which I have presented here introduces additional structure into the theory

\(^4\)As a candidate for an operator corresponding to “unified conscious self-awareness” one might consider, following the pattern of Secs. 2 and 3, an additional operator acting on a \( (3^N (\nu + 2) + 1) \)-dimensional eigenspace with nondegenerate eigenvalues, one of which indicates a state of ignorance and the remaining \( 3^N (\nu + 2) \) of which correspond (via measurement interactions with the \( \mathcal{O}^{(p)} \)’s and \( F \) taking place subsequent to \( U_O \) and \( U_F \)) to the \( 3^N (\nu + 2) \) distinct sets of eigenvalues of the \( \tilde{b}^{(p)} \)’s and \( \hat{\tau} \). (This operator could be defined for finite \( N \) only.) A nonzero weight for a post-measurement Everett copy of this operator would be equal to the measure of existence of a world, \( \mu_i \), with specific values for the results of measurements of \( \tilde{a}^{(1)}, \ldots, \tilde{a}^{(N)} \), and appropriate corresponding values for the results of measurements of \( \tilde{b}^{(1)}, \ldots, \tilde{b}^{(N)} \) and \( \hat{\tau} \).

\(^5\)This fact does not pose any problems in using the frequency interpretation of probability as we have above. Although we cannot experiment with an \( N \to \infty \) ensemble, this limiting case is well-defined within the quantum formalism, which, as we have seen, gives an unambiguous answer as to what would happen if we \textit{could} do such experiments.
beyond the basic Heisenberg-picture formalism; namely, the notion of existence/nonexistence introduced through the second interpretive rule in Sec. 2.2. Other frequency-related approaches to probability in the Everett interpretation introduce different sorts of additional structure into the theory, such as the ideas of associating a continuous infinity of copies\cite{39} or minds\cite{40, 41} with each branch of the state vector. This approach can also be applied to Heisenberg-picture copies\cite{16}. The number of copies in a branch of the state vector is taken to be proportional to the Born-rule relative frequency, so at any time the fraction of the (infinite) total number of copies which perceive any particular measurement outcome will be exactly equal to the probability as given by the Born rule. It is an additional assumption to say that this state of affairs, or, indeed, the corresponding state of affairs in any other approach based on counting the number of outcomes\footnote{Weissman’s outcome-counting approach\cite{42} explicitly modifies the dynamics of the theory, adding a nonlinear process.} (branches, copies, minds, ...), is equivalent to the phenomenon of probability. Why is it of consequence to me what other copies of me perceive? In the Everett interpretation in the Heisenberg picture, the number of these continuously-infinite copies or minds will be proportional to the weight. However, if one introduces the notion of the weight, one can then consider ensembles of many such measurements, thereby obtaining the conclusions obtained in this paper and rendering the \textit{ad hoc} introduction of continuous infinities of copies, minds etc. superfluous.

Zurek\cite[Sec. 4]{11} uses the fact that the density matrix of a quantum system subject to environmental decoherence evolves into a form which is for all practical purposes identical to the density matrix describing a classical statistical ensemble of quantum systems to argue that the probabilistic interpretation of the latter can be applied to the former. (See also\cite[Sec. 20]{43}.) Since quantum mechanics is the more fundamental theory, it seems preferable to derive probabilistic behavior purely from within the quantum theory without invoking classical concepts at the outset.

Graham’s analysis of probability in the Everett interpretation employs two of the features which are crucial to the success of the present approach. He explicitly includes the dynamics of a device to measure relative frequency, and emphasizes that, on physical grounds, such a device must have finite resolution. However, he also introduces the notion that such a device, as part of a “two step” measuring process,\footnote{It may seem that the present approach also involves two steps, the measurements by the $O^{(p)}$’s and then the measurement by $F$. But we could have considered a finite-resolution relative-frequency measuring device coupled directly to the $S^{(p)}$’s and reached the same conclusions. The $O^{(p)}$’s were included to demonstrate that, as discussed above, the relative frequency being measured is the familiar quantity compatible with measurement of individual outcomes.} must first enter a state of thermodynamic equilibrium after it has made its measurements, and then must be measured by an additional observer before Everett branches emerge in such a way that the vast majority of them correspond to perceptions of Born-rule relative frequencies. As shown above, these additional concepts are not necessary to demonstrate the existence of Born-rule-consistent probability in the Heisenberg-picture Everett interpretation.
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