LIMIT THEOREMS FOR LOCAL AND OCCUPATION TIMES
OF RANDOM WALKS AND BROWNIAN MOTION ON A SPIDER

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Abstract
A simple random walk and a Brownian motion are considered on a spider that is a collection of half lines (we call them legs) joined at the origin. We give a strong approximation of these two objects and their local times. For fixed number of legs we establish limit theorems on local and occupation times in $n$ steps.

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1 Introduction

Consider the following collection of half lines on the complex plane

$$\text{SP}(N) := \{v_N(x, j), x \geq 0, j = 1, 2, \ldots, N\},$$

where, with $i = \sqrt{-1}$,

$$v_N(x, j) = x \exp \left( \frac{2\pi i (j-1)}{N} \right).$$

We will call $\text{SP}(N)$ a spider with $N$ legs. Also,

$$v_N(0) := v_N(0, 1) = v_N(0, 2) = \ldots = v_N(0, N)$$

is called the body of the spider, and $L_j := \{v_N(x, j), x > 0\}$ is the $j$-th leg of the spider.

The number of legs $N$ is fixed, so we will often suppress $N$ in the notation and, instead of $v_N(x, j)$ or $v_N(0)$, we will simply write $v(x, j)$ or $v(0) = 0$, whenever convenient.

In this paper we consider a random walk $S_n$, $n = 0, 1, 2, \ldots$, on $\text{SP}(N)$ that starts from the body of the spider, i.e., $S_0 = v_N(0) = 0$, with the following transition probabilities:

$$P(S_{n+1} = v_N(1, j)|S_n = v_N(0)) = p_j, \quad j = 1, \ldots, N,$$

with

$$\sum_{j=1}^{N} p_j = 1,$$

and, for $r = 1, 2, \ldots, j = 1, \ldots, N$,

$$P(S_{n+1} = v_N(r + 1, j)|S_n = v_N(r, j)) = P(S_{n+1} = v_N(r - 1, j)|S_n = v_N(r, j)) = \frac{1}{2}. $$

The thus defined random walk $S_n$ on $\text{SP}(N)$ will be called random walk on spider (RWS) or simply spider walk in this paper.

Observe that the particular case $N = 2$, $p_1 = p_2 = 1/2$ corresponds to the simple symmetric random walk $S(n)$, $n = 0, 1, 2, \ldots$, on the line. The spider walk $S_n$ can be constructed from $S(n)$, $n = 0, 1, \ldots$, as follows. Consider the absolute value $|S(n)|$, $n = 1, 2, \ldots$, that consists of infinitely many excursions from zero, and let $G_1, G_2, \ldots$, denote the excursion intervals. Put these excursions, independently of each other, on leg $j$ of the spider with probability $p_j$, $j = 1, 2, \ldots, N$. What we obtain this way is the first $n$ steps of the spider walk $S_\cdot$, created from the first $n$ steps of the random walk $S(\cdot)$. Let $\gamma_i, i = 1, 2, \ldots$, be i.i.d. random variables with

$$P(\gamma_i = j) = p_j, \quad j = 1, 2, \ldots, N,$$

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that are independent from the simple symmetric random walk $S(\cdot)$. Consequently, one can also define the just introduced spider walk as $S_0 = S(0) = 0$,

$$S_n := \sum_{m=1}^{\infty} I\{n \in G_m\}v_N(|S(n)|, \gamma_m), \; n = 1, 2, \ldots,$$

if $S(n) \neq 0$, and $S_n = v_N(0)$ if $S(n) = 0$.

In view of this definition, and the notations already used, in the sequel $S_n$ will stand for a spider walk and $S(n)$ for a simple symmetric random walk on the line, with respective probabilities denoted by $P$ and $P$.

The limit process is the so-called Brownian motion on spider (BMS), or simply Brownian spider, a version of Walsh’s Brownian motion (cf Walsh [28], and Introduction of Csáki et al. [10]), that can be constructed from a standard Brownian motion $\{B(t), t \geq 0\}$ on the line as follows. The process $\{|B(t)|, t \geq 0\}$ has a countable number of excursions from zero, and denote by $J_1, J_2, \ldots$, a fixed enumeration of its excursion intervals away from zero. Then, for any $t > 0$ for which $B(t) \neq 0$, we have that $t \in J_m$ for one of the values of $m = 1, 2, \ldots$. Let $\kappa_m, m = 1, 2, \ldots$, be i.i.d. random variables, independent of $B(\cdot)$ with

$$P(\kappa_m = j) = p_j, \; j = 1, 2, \ldots, N.$$

We now construct the Brownian spider $\{B(t), t \geq 0\}$ by putting the excursion whose interval is $J_m$ to the $\kappa_m$-th leg of the spider $\text{SP}(N)$. Hence we can define the Brownian spider $B(\cdot)$ by

$$B(t) := \sum_{m=1}^{\infty} I\{t \in J_m\}v_N(|B(t)|, \kappa_m), \; \text{if} \; B(t) \neq 0, \quad (1.1)$$

and

$$B(t) := v_N(0) = 0, \; \text{if} \; B(t) = 0.$$

In [10] we investigated the heights of the spider walk $S_n$ on the legs of the spider. We also established a strong approximation as follows.

**Theorem 1.1** On a rich enough probability space one can define a BMS $\{B(t), t \geq 0\}$ and an RWS $\{S_n, n = 0, 1, 2, \ldots\}$, both on $\text{SP}(N)$, and both selecting their legs with the same probabilities $p_j, j = 1, 2, \ldots, N$, so that, as $n \to \infty$, we have

$$|S_n - B(n)| = O(n^{1/4}(\log n)^{1/2}(\log \log n)^{1/4}) \; \text{a.s.}$$

In this paper we keep $N$ and $p_1, \ldots, p_N$ fixed and consider limit theorems concerning local and occupation times on the legs, as the number of steps $n$ tends to infinity.
2 Local times

Throughout we use the notation $I\{A\}$ for the indicator of an event $A$ in the brackets, i.e., $I\{A\} = 1$ if $A$ occurs and $I\{A\} = 0$ otherwise.

The local time of an RWS $S_n$ on $\SP(N)$ is defined as

$$\xi((r, j), n) := \sum_{i=1}^{n} I\{S_i = v(r, j)\}, \quad r = 1, 2, \ldots, j = 1, 2, \ldots, N,$$

$$\xi(0, n) := \sum_{i=1}^{n} I\{S_i = v(0)\},$$

$n = 1, 2, \ldots$.

The local time of a BMS is defined by

$$\eta((x, j), t) := \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{0}^{t} I\{B(s) \in (v(x - \varepsilon, j), v(x + \varepsilon, j))\} ds, \quad x > 0,$$

$$\eta(0, t) := \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{0}^{t} \sum_{j=1}^{N} I\{B(s) \in (0, v(\varepsilon, j))\} ds.$$

Note that by the constructions given in Section 1 we have

$$\xi(0, n) = \xi(0, n) \quad \text{and} \quad \eta(0, n) = \eta(0, n),$$

where $\xi(0, n)$ and $\eta(0, n)$, are the local times at zero of the simple symmetric random walk $S(\cdot)$ and standard Brownian motion $B(\cdot)$, respectively, that are used in the constructions.

First we consider the case $N = 2$, i.e., the so called skew random walk (SRW) and skew Brownian motion (SBM). In this case we let $p_1 = p, \; p_2 = 1 - p = q$.

2.1 Skew random walk and skew Brownian motion

The skew random walk (SRW) $\{S_n^\ast, n = 0, 1, 2, \ldots \}$ with parameter $p$ is the particular case of an RWS with $N = 2$. It is a Markov chain on $\mathbb{Z}$ with transition probabilities

$$P(0, 1) = p, \; P(0, -1) = q = 1 - p, \; P(x, x + 1) = P(x, x - 1) = 1/2, \; x = \pm 1, \pm 2, \ldots$$

The skew Brownian motion (SBM) $B^\ast(\cdot)$ with parameter $p$ can be defined as follows (cf. [1]). Let $B(\cdot)$ be a standard Brownian motion on the line, and let $J_1, J_2, \ldots$ be the excursion intervals from zero of $|B(\cdot)|$. Let $\delta_i, \; i = 1, 2, \ldots$, be i.i.d. random variables, independent of $B(\cdot)$, with $P(\delta_1 = 1) = p = 1 - P(\delta_1 = -1)$. Then

$$B^\ast(t) := \sum_{m=1}^{\infty} I\{t \in J_m\}\delta_m|B(t)|, \quad \text{if} \quad B^\ast(t) \neq 0, \quad (2.1)$$
\[ B^*(t) := 0, \quad \text{if} \quad B(t) = 0. \]  

(2.2)

The skew Brownian motion was introduced by Itô and McKean [19] (cf. also the Introduction of Harrison and Shepp [17]). Weak invariance principle between skew random walk and skew Brownian motion was established by Harrison and Shepp [17] and Cherny et al. [5]. We note that throughout this paper we use the above introduced respective abbreviations SRW and SBM. For further results concerning SRW and SBM, we refer to Revuz and Yor [27], Lejay [22], [23], Appuhamillage et al. [1], Hajri [16] and references given in these papers.

We note in passing that the above definition of SBM is a special case of BMS as in (1.1) with \( N = 2, \quad p_1 = p, \quad p_2 = 1 - p = q, \) so that, in this case, the first one of the two legs is the positive half-line, and the second one is the negative half-line. In particular, we thus have

\[ v_2(|B(t)|, \kappa_m) = \delta_m |B(t)|. \]

The local time of SRW is defined by

\[ \xi^*(k, n) := \sum_{i=1}^n I\{S^*_i = k\}, \quad k = 0, \pm 1, \pm 2, \ldots, \quad n = 1, 2, \ldots, \]

and that of an SBM is defined by

\[ \eta^*(x, t) := \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t I\{B^*(s) \in (x - \varepsilon, x + \varepsilon)\} \, ds. \]  

(2.3)

Various results for local times are given in Walsh [28], Burdzy and Chen [4], Lyulko [24], Gairat and Shcherbakov [15].

Here we first give a joint strong invariance principle for SBM and SRW and their local times.

**Theorem 2.1** A probability space with an SBM \( \{B^*(t), \ t \geq 0\} \) and an SRW \( \{S^*_n, \ n = 0, 1, 2, \ldots\} \) on it with the same parameter \( p \), can be so constructed that, as \( n \to \infty \), we have

\[ |S^*_n - B^*(n)| = O(n^{1/4}(\log n)^{1/2}(\log \log n)^{1/4}) \quad \text{a.s.} \]  

(2.4)

and

\[ \max_{1 \leq k \leq n} \sup_{x \in \mathbb{Z}} |\xi^*(x, k) - \eta^*(x, k)| = O(n^{1/4}(\log n)^{1/2}(\log \log n)^{1/4}) \quad \text{a.s.} \]  

(2.5)

**Proof.** The statement of (2.4) is a special case of Theorem 1.1 when \( N = 2 \), with \( p_1 = p, \quad p_2 = 1 - p = q \).

We note that the conclusion of (2.5) was established in Csörgő and Horváth [13] in the case of a simple symmetric random walk, i.e., when \( p = 1/2 \). We are to conclude (2.5) in the present context via following the lines of the proof in [13]. Let \( \{B^*(t), \ t \geq 0\} \) be an SBM and define \( \tau_0 = 0, \tau_1 = \inf\{t : t > 0, |B^*(t)| = 1\}, \tau_n = \inf\{t : t > \tau_{n-1}, |B^*(t) - B^*(\tau_{n-1})| = 1\}, \ n = 2, 3, \ldots \) Using
these Skorokhod stopping times, in view of (1.1) and the proof of Theorem 1.1 in [8], it is easy to see that
\[ S_0^* = 0, \quad S_n^* = \sum_{i=1}^{n} X_i^*, \quad n = 1, 2, \ldots, \text{ with } X_i^* = B^*(\tau_i) - B^*(\tau_{i-1}), i = 1, 2, \ldots, \text{ is an SRW.} \]

We note in passing that, along these lines, we can again conclude the statement of (2.4).

Proving now (2.5), similarly, as in [13], put
\[ a_i(x) = \eta^*(x, \tau_{\nu(i)+1}) - \eta^*(x, \tau_{\nu(i)}), \]
where \( \nu(1) = \min\{k : k \geq 0, B_k^* = x\} \), \( \nu(n) = \min\{k : k > \nu(n-1), B_k^* = x\}, n \geq 2. \)

It follows from Borodin and Salminen [3], p. 173, (3.3.2), that for any fixed \( x \in \mathbb{Z} \), the random variables \( \{a_i(x), i \geq 1\} \) are i.i.d. with exponential density function \( e^{-y}, y \geq 0 \). Thus, for further use, for \( i = 1, 2, \ldots, \) and \( x \in \mathbb{Z} \), we have
\[
P(a_i(x) \geq y) = e^{-y}, \quad y \geq 0. \tag{2.6}
\]

Similarly to (2.3) in [13], we observe that, for all \( n \geq 1 \), we have
\[
P\left( \left| \sum_{i=1}^{\xi^*(x,n)} a_i(x) - \eta^*(x, \tau_n) \right| \leq \eta^*(0, \tau_1) + a_{\xi^*(x,n)}(x) \text{ for all } x \in \mathbb{Z} \right) = 1. \tag{2.7}
\]

Having (2.6), we can find positive constants \( C_1, C_2 \) so that
\[
P\left( \max_{1 \leq i \leq n} \max_{-n \leq x \leq n} a_i(x) > C_2 \log n \right) \leq \frac{C_1}{n^2}
\]
and, therefore, as \( n \to \infty \), by the Borel-Cantelli Lemma,
\[
\max_{1 \leq i \leq n} \max_{-n \leq x \leq n} a_i(x) = O(\log n) \quad \text{a.s.} \tag{2.8}
\]

In view of (2.7) and (2.8) we conclude
\[
\max_{-n \leq x \leq n} \left| \sum_{i=1}^{\xi^*(x,n)} a_i(x) - \eta^*(x, \tau_n) \right| = O(\log n) \quad \text{a.s.} \tag{2.9}
\]

Observe that by the law of the iterated logarithm (LIL) of Kesten [20] for the maximum in \( x \in \mathbb{Z} \) of the local time of the simple symmetric random walk \( S(\cdot) \), we have
\[
\limsup_{n \to \infty} (2n \log \log n)^{-1/2} \max_{x \in \mathbb{Z}} \xi(x, n) = 1 \quad \text{a.s.} \tag{2.10}
\]

Consequently, for the maximum local time of \( S_n^* \), we have
\[
\limsup_{n \to \infty} (2n \log \log n)^{-1/2} \max_{x \in \mathbb{Z}} \xi^*(x, n) \leq 2 \quad \text{a.s.} \tag{2.11}
\]
Also, similarly to (2.8) in [13], on using Theorem 15 and Lemma 5 of Petrov [25], Section 3.3, we can find positive constants \( C_3, C_4 \) so that, for each fixed \( k \) we have

\[
P \left( \max_{-k \leq x \leq k} \left| \sum_{i=1}^{k} (a_i(x) - 1) \right| > C_3 (k \log k)^{1/2} \right) \leq \frac{C_4}{k^2}. \tag{2.12}
\]

Applying now (2.12) with the Borel-Cantelli Lemma, and then (2.11), we obtain

\[
\limsup_{n \to \infty} n^{-1/4} (\log \log n)^{-1/4} (\log n)^{-1/2} \max_{-n \leq x \leq n} \left| \xi^*(x,n) \sum_{i=1}^{\infty} a_i(x) - \xi^*(x,n) \right| \leq C_5 \text{ a.s.} \tag{2.13}
\]

with some constant \( C_5 \).

It is well known that, in terms of the Skorokhod stopping times \( \{\tau_i, i \geq 0\} \), the random variables \( \{\tau_{i,-1}, i \geq 1\} \) are i.i.d. random variables with mean 1 and variance 1. Thus, by the LIL for partial sums, we have

\[
\limsup_{n \to \infty} (2n \log \log n)^{-1/2} |\tau_n - n| = 1 \quad \text{a.s.}
\]

The latter LIL combined with Theorem 3 of Csáki et al. [8] with \( g(n) = 4(n \log \log n)^{1/2} \) yields

\[
\limsup_{n \to \infty} n^{-1/4} (\log \log n)^{-1/4} (\log n)^{-1/2} \sup_{x} |\eta^*(x,n \pm g(n)) - \eta^*(x,n)| \leq C_6 \text{ a.s.} \tag{2.14}
\]

with an appropriate constant \( C_6 \).

Also,

\[
\xi^*(x,n) = 0 \quad \text{if } |x| > n, \quad \text{and} \quad \limsup_{n \to \infty} \eta^*(x,n) = 0 \quad \text{a.s.} \tag{2.15}
\]

As a consequence of (2.8), (2.13), (2.14), we now conclude

\[
\limsup_{n \to \infty} n^{-1/4} (\log \log n)^{-1/4} (\log n)^{-1/2} \sup_{x \in \mathbb{Z}} |\xi^*(x,n) - \eta^*(x,n)| \leq C_5 + C_6 \quad \text{a.s.} \tag{2.16}
\]

that, in turn, implies (2.5), and hence also concludes the proof of Theorem 2.1. \( \square \)

**Remark.** In [13] Csörgő and Horváth show that, using the Skorokhod embedding scheme, the rate of convergence in their Theorem 1 is best possible. The latter result is based on their conclusion that, using the Skorokhod embedding in case of the simple symmetric random walk, one has

\[
\limsup_{n \to \infty} n^{-1/4} (\log \log n)^{-1/4} (\log n)^{-1/2} \max_{1 \leq k \leq n} |\xi(0,k) - \eta(0,k)| = C
\]

with some positive constant \( C \). Since the local time of an SRW \( S_n^* \) at zero and that of an SBM \( B^*(t) \) respectively coincide with those of a simple symmetric random walk and standard Brownian motion, the above conclusion also implies that the rate of approximation in (2.5) of our Theorem 2.1 is also best possible when using the Skorokhod embedding scheme.
This invariance principle is suitable for proving so-called first order limit theorems for local times. For example, in view of having the LIL for the maximal local time of an SRW as in (2.11), we can conclude the same LIL for the maximal local time of an SBM. Establishing the exact constant for any one of these two LIL’s, the same constant would be inherited by the other one in hand.

For the so-called second order limit theorems we introduce the following iterated Brownian motion, or iterated Wiener process. Let $W(t)$, $t \geq 0$ be a standard Wiener process on the line, and let $\tilde{\eta}(t)$ a Wiener local time at zero, independent of $W(\cdot)$. Put

$$Z(t) = W(\tilde{\eta}(t)), \quad t \geq 0.$$ 

In the sequel we call $Z(t)$ iterated Brownian motion (IBM). For fixed real number $x \neq 0$ we define $c(\cdot)$, for throughout use from now on, as

$$c(x) := \begin{cases} 
2p, & x > 0, \\
2q, & x < 0. 
\end{cases}$$

The next strong invariance principles deal with second order limit theorems for the local times in hand, at fixed location for large times.

**Theorem 2.2** For any fixed integer $x \neq 0$, a probability space with an SRW \{\textstyle S_n^*, n = 0, 1, \ldots \}, and an IBM \{\textstyle Z_1(t), t \geq 0 \}$ on it can be so constructed that for the local times of SRW, as $n \to \infty$, we have

$$\xi^*(x, n) - c(x, n)\xi^*(0, n) = (c(x)(4|x| - 1) - c^2(x))^{1/2}Z_1(n) + O(n^{1/8}), \quad \text{a.s.}$$

**Theorem 2.3** For any fixed real number $x \neq 0$, a probability space with an SBM \{\textstyle B^*(t), \quad t \geq 0 \}, and an IBM \{\textstyle Z_2(t), t \geq 0 \}$, can be so constructed that for the local times of SBM, as $t \to \infty$, we have

$$\eta^*(x, t) - c(x, t)\eta^*(0, t) = 2(c(x)|x|)^{1/2}Z_2(t) + O(t^{1/8}), \quad \text{a.s.}$$

**Proof of Theorem 2.2.** Define

$$\rho_0 = 0, \quad \rho_k = \min\{i > \rho_{k-1} : S_i^* = 0\}, \quad k = 1, 2, \ldots,$$

the times of returns of SRW to the origin. Then, for fixed integer $x \neq 0$ we have

$$P(\xi^*(x, \rho_1) = 0) = 1 - \frac{c(x)}{2|x|},$$

$$P(\xi^*(x, \rho_1) = m) = \frac{c(x)}{4x^2} \left(\frac{2|x|-1}{2|x|}\right)^{m-1}, \quad m = 1, 2, \ldots$$

The two formulas above are simple consequences of Theorem 9.7 of Révész [26, page102], where the above distributions are given for a simple symmetric random walk. Our formula is modified, as
the skew random walk has unequal probabilities for stepping right or left from the origin, yielding $c(x)$ in our formula. Accordingly, one can obtain

$$E(\xi^*(x, \rho_1)) = c(x), \quad \text{Var}\,\xi^*(x, \rho_1) = c(x)(4|x| - 1) - c^2(x).$$

Now Theorem 2.2 follows from Theorem 3.1 of Csáki and Csörgő [7]. To see this, one should consult the latter paper for appropriate definitions in order to check the conditions of Theorem 3.1 in [7].

**Proof of Theorem 2.3.** To prove Theorem 2.3, consider the skew Brownian motion $B^*(\cdot)$ as a diffusion process with scale function

$$s(x) = \frac{x}{c(x)}, \quad x \neq 0,$$

and speed measure

$$m(dx) = 2c(x)dx, \quad x \neq 0,$$

We note that in Itô and McKea [19], and also in Borodin and Salmi [3], Appendix 1.12, the scale function is given as twice of the above scale function and the speed measure as half of the above speed measure, but this is equivalent to the quantities given above, in that they yield the usual scale and speed for Brownian motion, i.e., for $p = 1/2$.

To apply Theorem 3.2 of Csáki and Salminen [12], we note that the local time $L_t^x$ is defined there with respect to the speed measure, while $\eta^*(x, t)$ is defined in (2.3) with respect to the Lebesgue measure. One can easily see that

$$L_t^x = \frac{1}{2c(x)} \eta^*(x, t), \quad x \neq 0,$$

$$L_t^0 = \frac{1}{2} \eta^*(0, t).$$

Let $\tau := \min\{s : \eta^*(0, s) = 1\}$. Using the formula (2.18) in [12] for $t = 1/2$, we obtain for $\beta > 0$,

$$E\left(\exp\left(-\beta \eta^*(x, \tau)\right)\right) = \exp\left(-\frac{c(x)\beta}{1 + 2\beta|x|}\right), \quad x \neq 0,$$

from which

$$E(\eta^*(x, \tau)) = c(x), \quad \text{Var}(\eta^*(x, \tau)) = 4c(x)|x|, \quad x \neq 0.$$

Now Theorem 2.3 follows from Theorem 3.2 of Csáki and Salminen [12]. To see this, one should consult the latter paper for appropriate definitions in order to check the conditions of its Theorem 3.2.

Next we state some consequences of the strong approximations given in Theorems 2.2 and 2.3. First, we have the following Dobrushin [14]-type theorems, where $\xrightarrow{d}$ means convergence in distribution.
Corollary 2.1 For fixed integer \( x \neq 0 \) we have, as \( n \to \infty \),
\[
\frac{\xi^*(x, n) - c(x)\xi^*(0, n)}{(c(x)(4|x| - 1) - c^2(x))^{1/2}n^{1/4}} \overset{d}{\to} U \sqrt{|V|},
\]
where \( U \) and \( V \) are independent standard normal random variables.

Corollary 2.2 For any fixed real number \( x \neq 0 \) we have, as \( t \to \infty \),
\[
\frac{\eta^*(x, t) - c(x)\eta^*(0, t)}{(4c(x)|x|)^{1/2}t^{1/4}} \overset{d}{\to} U \sqrt{|V|},
\]
where \( U \) and \( V \) are independent standard normal random variables.

Strassen-type theorems for the IBM, that also imply limsup results, are given in Csáki et al. [9], [11], and Hu et al. [18]. A Chung-type liminf result is shown in Hu et al. [18]. Applying these results, we also have the following corollaries to Theorems 2.2 and 2.3 respectively.

Corollary 2.3 For any fixed integer \( x \neq 0 \), the set of functions
\[
g_n(s) = \frac{\xi^*(x, [sn]) - c(x)\xi^*(0, [sn])}{(c(x)(4|x| - 1) - c^2(x))^{1/2}2^{5/4}3^{-3/4}n^{1/4}(\log \log n)^{3/4}}, \quad 0 \leq s \leq 1,
\]
is relatively compact in \( C[0,1] \) and the set of its limit points, as \( n \to \infty \), is the set of functions \( f(s), 0 \leq s \leq 1, \) with \( f(0) = 0 \), that are absolutely continuous with respect to Lebesgue measure and
\[
\int_0^1 |f(s)|^{3/4} \, ds \leq 1.
\]

Consequently,
\[
\limsup_{n \to \infty} \frac{\xi^*(x, n) - c(x)\xi^*(0, n)}{(c(x)(4|x| - 1) - c^2(x))^{1/2}2^{5/4}3^{-3/4}n^{1/4}(\log \log n)^{3/4}} = 1
\]
amost surely. Also,
\[
\liminf_{n \to \infty} \frac{(\log \log n)^{3/4}}{(c(x)(4|x| - 1) - c^2(x))^{1/2}2^{5/4}3^{-3/4}n^{1/4}(\log \log n)^{3/4}} \max_{0 \leq k \leq n} |\xi^*(x, k) - c(x)\xi^*(0, k)| = \left( \frac{3\pi^2}{8} \right)^{3/4}
\]
amost surely.

Corollary 2.4 For any fixed real number \( x \neq 0 \) the set of functions
\[
g_t(s) = \frac{\eta^*(x, st) - c(x)\eta^*(0, st)}{2(c(x)|x|^{1/2}2^{5/4}3^{-3/4}t^{1/4}(\log \log t)^{3/4}}}, \quad 0 \leq s \leq 1,
\]
is relatively compact in $C[0,1]$ and the set of its limit points, as $t \to \infty$, is the set of functions $f(s)$, $0 \leq s \leq 1$, with $f(0) = 0$, that are absolutely continuous with respect to Lebesgue measure and
\[
\int_0^1 |f(s)|^{3/4} \, ds \leq 1.
\]

Consequently,
\[
\limsup_{t \to \infty} \frac{\eta^*(x,t) - c(x)\eta^*(0,t)}{2(c(x)|x|)^{1/2}2^{3/4}3^{3/4}t^{1/4}(\log \log t)^{3/4}} = 1
\]
almost surely. Also,
\[
\liminf_{t \to \infty} \frac{(\log \log t)^{3/4}}{2(c(x)|x|)^{1/2}t^{1/4}} \max_{0 \leq s \leq t} |\eta^*(x,s) - c(x)\eta^*(0,s)| = \left(\frac{3\pi^2}{8}\right)^{3/4}
\]
almost surely.

### 2.2 Local time on spider

First we give a joint invariance principle for the Brownian spider and the random walk on spider $\text{SP}(N)$, and their local times, an analogue of Theorem 2.1.

**Theorem 2.4** On a rich enough probability space one can define a Brownian spider $\{B(t), t \geq 0\}$ and random walk on the spider $\{S_n, n = 0, 1, 2, \ldots\}$, both on $\text{SP}(N)$, and both selecting their legs with the same probabilities $p_j$, $j = 1, 2, \ldots, N$, so that, as $n \to \infty$, we have
\[
|S_n - B(n)| = O(n^{1/4}(\log n)^{1/2}(\log \log n)^{1/4}) \quad \text{a.s.}
\]
and
\[
\sup_{(r,j)} |\xi((r,j), n) - \eta((r,j), n)| = O(n^{1/4}(\log n)^{1/2}(\log \log n)^{1/4}) \quad \text{a.s.,}
\]
where $\sup$ is taken for $r = 0, 1, 2, \ldots, j = 1, 2, \ldots, N$.

**Proof.** The proof can be reduced to the case $N = 2$. Namely, for fixed $j$, consider the SRW and SBM with $p = p_j$, $q = 1 - p_j$. For this $j$, Theorem 2.4 follows from Theorem 2.1. Repeating the proof for all $j = 1, \ldots, N$, yields the proof of Theorem 2.4. \qed

Concerning the so-called second order limit theorems for the local time, we have the following results corresponding respectively to Theorems 2.2 and 2.3, on replacing $c(x)$ by $2p_j$.

**Theorem 2.5** For any fixed positive integer $x$ and fixed $j = 1, 2, \ldots, N$, a probability space with an RWS $S_n$ and an IBM $Z_3(t)$ can be so constructed that, as $n \to \infty$, we have
\[
\xi((x,j), n) - 2p_j\xi(0,n) = (2p_j(4x - 1) - 4p_j^2)^{1/2}Z_3(n) + o(n^{1/8}) \quad \text{a.s.}
\]
Theorem 2.6 For any fixed positive real number $x$ and fixed $j = 1, 2, ..., N$, a probability space with an BMS $B(t)$ and an IBM $Z_A(t)$ can be so constructed that, as $t \to \infty$, we have

$$\eta((x, j), t) - 2p_j \eta(0, t) = (8p_j x)^{1/2} Z_A(t) + o(t^{1/8}) \text{ a.s.}$$

Similarly to Corollaries 2.1-2.4, we have

Corollary 2.5 For any fixed integer $x > 0$ we have, as $n \to \infty$,

$$\frac{\xi((x, j), n) - 2p_j \xi(0, n)}{(2p_j(4x - 1) - 4p_j^2)^{1/2} n^{1/4}} \to U \sqrt{|V|},$$

where $U$ and $V$ are independent standard normal random variables.

Corollary 2.6 For any fixed real number $x > 0$ we have, as $t \to \infty$,

$$\frac{\eta((x, j), t) - 2p_j \eta(0, t)}{(8p_j x)^{1/2} n^{1/4}} \to U \sqrt{|V|},$$

where $U$ and $V$ are independent standard normal random variables.

Corollary 2.7 For any fixed integer $x > 0$ the set of functions

$$g_n(s) = \frac{\xi((x, j), [sn]) - 2p_j \xi(0, [sn])}{(2p_j(4x - 1) - 4p_j^2)^{1/2} 2^{5/4} 3^{-3/4} n^{1/4} (\log \log n)^{3/4}} \quad 0 \leq s \leq 1,$$

is relatively compact in $C[0, 1]$ and the set of its limit points, as $n \to \infty$, is the set of functions $f(s)$, $0 \leq s \leq 1$, with $f(0) = 0$, that are absolutely continuous with respect to Lebesgue measure and

$$\int_0^1 |f(s)|^{3/4} ds \leq 1.$$

Consequently,

$$\limsup_{n \to \infty} \frac{\xi(x, n) - 2p_j \xi(0, n)}{(2p_j(4x - 1) - 4p_j^2)^{1/2} 2^{5/4} 3^{-3/4} n^{1/4} (\log \log n)^{3/4}} = 1$$

almost surely. Also,

$$\liminf_{n \to \infty} \frac{(\log \log n)^{3/4}}{(2p_j(4x - 1) - 4p_j^2)^{1/2} n^{1/4}} \max_{0 \leq k \leq n} |\xi((x, j), k) - 2p_j \xi(0, k)| = \left( \frac{3\pi^2}{8} \right)^{3/4}$$

almost surely.
Corollary 2.8 For any fixed real number $x > 0$ the set of functions
\[ g_t(s) = \frac{\eta(x, j, st) - 2p_j\eta(0, st)}{(8p_jx)^{1/2}2^{5/4}3^{-3/4}t^{1/4}(\log \log t)^{3/4}}; \quad 0 \leq s \leq 1, \]
is relatively compact in $C[0,1]$ and the set of its limit points, as $t \to \infty$, is the set of functions $f(s), 0 \leq s \leq 1$, with $f(0) = 0$, that are absolutely continuous with respect to Lebesgue measure and
\[ \int_0^1 |f(s)|^{3/4} ds \leq 1. \]
Consequently,
\[ \limsup_{t \to \infty} \frac{\eta(x, t) - 2p_j\eta(0, t)}{(8p_jx)^{1/2}2^{5/4}3^{-3/4}t^{1/4}(\log \log t)^{3/4}} = 1 \]
almost surely. Also,
\[ \liminf_{t \to \infty} \frac{(\log \log t)^{3/4}}{(8p_jx)^{1/2}t^{1/4}} \max_{0 \leq s \leq t} |\eta((x, j), s) - 2p_j\eta^*(0, s)| = \left( \frac{3\pi^2}{8} \right)^{3/4} \]
almost surely.

3 Occupation times.

In this section we consider the time the RWS, or the BMS, spends on particular legs. First we give the definitions of the occupation times. Recall that $L_j$ denotes the leg $j$ of the spider $SP(N)$. Let $S_j, j = 0, 1, \ldots$, be a random walk on the spider $SP(N)$, with probabilities $p_1, \ldots, p_N$. The occupation time on leg $j$ in $n$ steps is defined by
\[ T(j, n) := \sum_{k=1}^n I\{S_k \in L_j\} + I\{S_{k-1} \in L_j, S_k = 0\}. \]
The occupation time of the Brownian spider $B(s), s \geq 0$, up to time $t$ is defined by
\[ Z(j, t) := \int_0^t I\{B(s) \in L_j\} ds. \]

By the second statement of Theorem 2.4 and the classical LIL, we have the following strong invariance principle.

**Theorem 3.1** On the probability space of Theorem 2.4 we have, as $n \to \infty$,
\[ \sup_{1 \leq j \leq N} |T(j, n) - Z(j, n)| = O(n^{3/4+\varepsilon}) \quad a.s. \]
Since the occupation time is usually of order $n$, using this strong invariance, it suffices to prove strong theorems either for RWS or BMS.

The occupation time of RWS on a particular leg of $\text{SP}(N)$ can be considered as having only two legs, one to which the walker proceeds with probability $p_j$ from the origin, and another one to which it goes with probability $1 - p_j$. Based on this idea, we consider the following setup.

Let $N = 2$, $p = p_1$, and $q = p_2 = 1-p$. This corresponds to a skew random walk $\{S^*_i, i = 0, 1, \ldots\}$ with transition probabilities

$$
P(0, 1) = p, \quad P(0, -1) = q = 1 - p, \quad P(x, x + 1) = P(x, x - 1) = 1/2, \quad x \neq 0.
$$

Let $T^*(1, n)$ be the occupation time of the positive half-line of a skew random walk, i.e.,

$$
T^*(1, n) = \sum_{i=1}^{n} (I\{S^*_i > 0\} + I\{S^*_i = 0, S^*_i - 1 > 0\}).
$$

The limiting distribution of $T^*(1, n)$ can be obtained from the density of occupation time on the positive side of skew Brownian motion on one leg, for which we have

$$
P\left( \frac{Z^*(1, t)}{t} \in dx \right) = \frac{pq}{\pi \sqrt{x(1-x)(p^2(1-x) + q^2x)}} \, dx.
$$

The latter formula is given in Lamperti [21] as a limiting distribution of occupation time of certain discrete time processes, including SRW. For this formula and related results, see also Appuhamillage et al. [1].

For the joint distribution of $\{Z(j, t)/t, j = 1, \ldots, N\}$, Barlow et al. [2] (see also Lejay [22]) have shown the following equality in distribution

$$
\left\{ \frac{Z(j, t)}{t}, \quad j = 1, \ldots, N \right\} \overset{d}{=} \left\{ \frac{p_j U_j}{\sum_{k=1}^{N} p_k U_k}, \quad j = 1, \ldots, N \right\},
$$

where $U_1, \ldots, U_N$ are independent one-sided stable 1/2 random variables. This gives also the joint limiting distribution of $\{T(j, n)/n, j = 1, \ldots, N\}$.

Now let

$$
T_M(n) = \max_{1 \leq j \leq N} T(j, n), \quad T_m(n) = \min_{1 \leq j \leq N} T(j, n).
$$

and

$$
Z_M(t) = \max_{1 \leq j \leq N} Z(j, t), \quad Z_m(t) = \min_{1 \leq j \leq N} Z(j, t).
$$

For the limsup of $T_M, Z_M$ and liminf of $T_m, Z_m$, we show that the Chung-Erdős [6] result for simple symmetric walk remains valid.
Theorem 3.2 Let \( f(x) \) be a positive nondecreasing function for which \( \lim_{x \to \infty} f(x) = \infty \), \( x/f(x) \) is nondecreasing and \( \lim_{x \to \infty} x/f(x) = \infty \).

Let
\[
I(f) := \int_1^\infty \frac{dx}{x(f(x))^{1/2}} < \infty.
\]

Then
\[
P\left( T_M(n) > n \left(1 - \frac{1}{f(n)}\right) \text{ i.o. as } n \to \infty \right) = 0 \text{ or } 1, \\
P\left( Z_M(t) > t \left(1 - \frac{1}{f(t)}\right) \text{ i.o. as } t \to \infty \right) = 0 \text{ or } 1, \\
P\left( T_m(n) < \frac{n}{f(n)} \text{ i.o. as } n \to \infty \right) = 0 \text{ or } 1
\]
and
\[
P\left( Z_m(t) < \frac{t}{f(t)} \text{ i.o. as } t \to \infty \right) = 0 \text{ or } 1
\]
according as \( I(f) \) converges or diverges.

Proof of Theorem 3.2. We will only prove the third statement, the proofs for the others go similarly. The limiting distribution of \( T(i, n) \) for fixed \( i \) is the same as that of skew random walk, which is a general arcsin law (see Lamperti [21] or Watanabe, [29]). For small \( x \) it is of the same order as that of the arcsin law. Therefore,
\[
c_1 \sqrt{x} \leq P(T(i, n) \leq nx) \leq c_2 \sqrt{x}
\]
with some positive constants \( c_1 \) and \( c_2 \). As
\[
P(T(1, n) \leq nx) \leq P(T_m(n) \leq nx) = P\left( \bigcup_{i=1}^N \{T(i, n) \leq nx\} \right) \leq N \left( \max_{i \leq N} P(T(i, n) \leq nx) \right),
\]
it follows that for small \( x \) we have
\[
c_3 \sqrt{x} \leq P(T_m(n) \leq nx) \leq c_4 \sqrt{x}
\]
with positive constants \( c_3, c_4 \). Based on the proof of Chung-Erdős [6], we can prove the theorem as follows.
Convergent part: Let $n_k = 2^k$. Then we know that for a positive nondecreasing function $f(x)$, the integral $I(f) = \int_1^\infty \frac{dx}{x(f(x))^{1/2}}$ converges if and only if the sum $\sum_{k=1}^\infty \frac{1}{(f(2^k))^{1/2}}$ converges.

First we show that if $I(f) < \infty$, then

$$\sum_{k=1}^\infty P\left(T_m(n_{k-1}) \leq \frac{n_k}{f(n_k)}\right) < \infty. \quad (3.2)$$

We have

$$P\left(T_m(n_{k-1}) \leq \frac{2^k}{f(2^k)}\right) = P\left(T_m(2^{k-1}) \leq 2 \cdot \frac{2^{k-1}}{f(2^{k-1})} \cdot \frac{f(2^{k-1})}{f(2^k)}\right) \leq P\left(T_m(2^{k-1}) \leq 2 \cdot \frac{2^{k-1}}{f(2^{k-1})}\right) \leq c_2 \sqrt{\frac{2}{f(2^{k-1})^{1/2}}}.$$

where we used that $f(\cdot)$ is nondecreasing. By assumption, we now have that (3.2) is convergent. Thus, for $k$ big enough,

$$T_m(n_{k-1}) \geq \frac{n_k}{f(n_k)} \quad a.s.$$

Now for $n_{k-1} \leq n \leq n_k$, we have for $n$ big enough that

$$T_m(n) \geq T_m(n_{k-1}) \geq \frac{n_k}{f(n_k)} \geq \frac{n}{f(n)}$$

Divergent part: First we show that the theorem is valid for a skew random walk. To this end suppose that $N = 2$, and $p_1 = p \leq 1/2$ and $p_2 = q = 1 - p$. Clearly if $p = 1/2$ there is nothing to prove, in view of the original Chung-Erdős theorem. That is to say, we know for the time $T(1, n)$ spent on the positive side (spent on the first leg), we have $P(T(1, n) < n/f(n) \text{ i.o.}) = 1$, whenever $I(f) = \infty$. If $p < 1/2$, then we start with a simple symmetric walk again, and keep each excursion spent on the positive side with probability $2p$ and, with probability $1 - 2p$, we flip it to the negative side. Thus we get a skew random walk with $p_1 = p < 1/2$. Denote the time spent on the positive side of this walk by $T^p(1, n)$. By construction, $T^p(1, n) \leq T(1, n)$, so $P(T^p(1, n) < n/f(n) \text{ i.o.}) = 1$, whenever $I(f) = \infty$. Similarly, if we have a skew random walk with $p_1 = p > 1/2$, we start again with a simple symmetric walk, but now apply the Chung-Erdős theorem for the time spent on the negative side (call it leg two). Then we have $P(T(2, n) < n/f(n) \text{ i.o.}) = 1$, whenever $I(f) = \infty$.

Now construct the new skew random walk by keeping each excursion on leg two with probability $2q$, and flipping them to the first leg with probability $1 - 2q$. As before, $T^p(2, n) \leq T(2, n)$, hence $P(T^p(2, n) < n/f(n) \text{ i.o.}) = 1$, whenever $I(f) = \infty$. Consequently, for a skew random walk $T_{pq}^*(n) = \min(T^p(1, n), T^p(2, n))$, we have $P(T(1, n) < n/f(n) \text{ i.o.}) = 1$, whenever $I(f) = \infty$. Now for the spider random walk, it is enough to observe that selecting the leg (or one of the legs) with $p^* := \min_{1 \leq i \leq N} p_i$ and putting all the excursions from the remaining legs to a second leg, we create a
skew random walk for which \( P(T_p(1,n) < n/f(n) \text{ i.o.}) = 1 \), whenever \( I(f) = \infty \). Clearly \( T_m(n) \leq T^{ps}(1,n) \), so the theorem is now proved. \( \Box \).

Recall the respective definitions of \( T_M, Z_M \) and \( T_m, Z_m \), given right before Theorem 3.2. For the liminf of \( T_M, Z_M \), and the limsup of \( T_m, Z_m \), we have the following results.

**Theorem 3.3**

\[
\lim_{n \to \infty} \frac{T_M(n)}{n} = \lim_{n \to \infty} \frac{T_m(n)}{n} = \frac{1}{N} \text{ a.s.} \tag{3.3}
\]

and, similarly,

\[
\lim_{t \to \infty} \frac{Z_M(t)}{t} = \lim_{t \to \infty} \frac{Z_m(t)}{t} = \frac{1}{N} \text{ a.s.} \tag{3.4}
\]

**Proof of Theorem 3.3.** We show this result for \( Z_m \) and \( Z_M \). The strong invariance result of Theorem 3.1 implies the conclusion for \( T_m \) and \( T_M \). Recalling (3.1), we know that \( U_j, j = 1, 2, ..., N \), are independent stable \( 1/2 \) random variables, implying that the events

\[
\left\{ \frac{1 - \varepsilon}{N} \leq \min_{1 \leq j \leq N} \frac{p_j U_j}{\sum_{k=1}^{N} p_k U_k} \leq \frac{1}{N} \right\}
\]

and

\[
\left\{ \frac{1}{N} \leq \max_{1 \leq j \leq N} \frac{p_j U_j}{\sum_{k=1}^{N} p_k U_k} \leq \frac{1 + \varepsilon}{N} \right\}
\]

have positive probability for \( 0 < \varepsilon < 1 \).

Hence for some \( \alpha > 0 \) and all \( t > 0 \)

\[
P\left( \frac{1}{N} \leq \frac{Z_M(t)}{t} \leq \frac{1 + \varepsilon}{N} \right) \geq \alpha
\]

and

\[
P\left( \frac{1 - \varepsilon}{N} \leq \frac{Z_m(t)}{t} \leq \frac{1}{N} \right) \geq \alpha.
\]

Consequently, the events

\[
\frac{1}{N} \leq \liminf_{t \to \infty} \frac{Z_M(t)}{t} \leq \frac{1 + \varepsilon}{N} \tag{3.5}
\]

and

\[
\frac{1 - \varepsilon}{N} \leq \limsup_{t \to \infty} \frac{Z_m(t)}{t} \leq \frac{1}{N} \tag{3.6}
\]

have positive probability, independently of \( t \), for all \( 0 < \varepsilon < 1 \). It follows from the 0-1 law that the respective events as in (3.5) and (3.6) hold true infinitely often almost surely as \( t \to \infty \). Since \( \varepsilon \) is arbitrary, (3.4) follows. \( \Box \)
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