ROTATIONAL ELASTICITY

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Summary

We consider an infinite 3-dimensional elastic continuum whose material points experience no displacements, only rotations. This framework is a special case of the Cosserat theory of elasticity. Rotations of material points are described mathematically by attaching to each geometric point an orthonormal basis which gives a field of orthonormal bases called the coframe. As the dynamical variables (unknowns) of our theory we choose the coframe and a density. We write down the general dynamic variational functional for our rotational theory of elasticity, assuming our material to be physically linear but the kinematic model geometrically nonlinear. Allowing geometric nonlinearity is natural when dealing with rotations because rotations in dimension 3 are inherently nonlinear (rotations about different axes do not commute) and because there is no reason to exclude from our study large rotations such as full turns. The main result of the paper is an explicit construction of a class of time-dependent solutions which we call plane wave solutions; these are travelling waves of rotations. The existence of such explicit closed form solutions is a nontrivial fact given that our system of Euler–Lagrange equations is highly nonlinear.

In the last section we consider a special case of our rotational theory of elasticity which in the stationary setting (harmonic time dependence and arbitrary dependence on spatial coordinates) turns out to be equivalent to a pair of massless Dirac equations.

1. Introduction

We work in 3-dimensional Euclidean space and view it as an elastic continuum whose material points can experience no displacements, only rotations, with rotations of different material points being totally independent. Rotations of material points of the 3-dimensional elastic continuum are described mathematically by attaching to each geometric point an orthonormal basis. This gives a field of orthonormal bases called the coframe.

The purpose of our paper is to develop a theory of elasticity on rotations, i.e. a theory of elasticity in which the coframe plays the role of the dynamical variable (unknown quantity). Recall that in classical elasticity the vector field of displacements is the dynamical variable.

Our motivation for studying such a seemingly exotic problem comes from three main sources.

The first source is Cosserat elasticity. In 1909 the Cosserat brothers proposed a theory of elasticity (1) which generalised classical elasticity by giving each material point rotational degrees of freedom. Cosserat elasticity has since become an accepted part of solid mechanics, though for most real life materials effects resulting from rotations of material points are small compared to effects resulting from displacements. From a purely mathematical point of view classical elasticity and rotational elasticity are two limit cases of Cosserat elasticity.
One of these limit cases, classical elasticity, has been extensively studied so it seems natural to examine now the other limit case.

The second source is teleparallelism (= absolute parallelism = fernparallelismus), a subject promoted by A. Einstein and É. Cartan (2, 3, 4) in the late 1920s. The idea of rotating material points lies at the heart of teleparallelism and can easily be traced back to Cosserat elasticity: when in 1922 Cartan started developing what eventually became modern differential geometry he acknowledged (5) that he drew inspiration from the ‘beautiful’ work of the Cosserat brothers. The relationship between Cosserat elasticity and teleparallelism is examined in detail in the review paper (6).

The third source is the theory of liquid crystals and, in particular, the concept of an Ericksen fluid. According to (7), in a liquid crystal one can observe ‘orientation waves which propagate, inducing little or no motion of the fluid’ and Ericksen’s mathematical model is the natural way of describing this phenomenon. The only difference between Ericksen’s model and ours is that in Ericksen’s model one attaches to each geometric point a single unit vector rather than an orthonormal basis, as we do.

Our paper has the following structure. In Section 2 we define our dynamical variables (unknowns of our theory), in Section 3 we write down the kinetic energy and Section 4 we write down the potential energy. The Lagrangian of rotational elasticity is written down in Section 5. In Section 6 we reformulate our model in the language of spinors and in Section 7 we discuss the corresponding Euler–Lagrange equation. In Section 8 we construct an explicit class of solutions which we call plane wave solutions; this construction is summarised in Theorem 8.3 which is the main result of our paper. Finally, in Section 9 we compare our model with the massless Dirac equation.

2. Setting the playing field

We work in Euclidean space \( \mathbb{R}^3 \) equipped with Cartesian coordinates \( x^\alpha, \alpha = 1, 2, 3 \), and standard Euclidean metric. We denote time by \( x^0 \). Partial differentiation in \( x^0 \) and \( x^\alpha \), \( \alpha = 1, 2, 3 \), is denoted by \( \partial_0 \) and \( \partial_\alpha \) respectively.

The coframe \( \theta \) is a triple of orthonormal covector fields \( \theta^j, j = 1, 2, 3 \), in \( \mathbb{R}^3 \). Each covector field \( \theta^j \) can be written more explicitly as \( \theta^j_\alpha \) where the tensor index \( \alpha = 1, 2, 3 \) enumerates the components. The orthonormality condition for the coframe can be represented as a single tensor identity

\[
\delta_{jk} \theta^j \otimes \theta^k
\]  

(2.1)

where \( \delta \) is the Kronecker delta and \( g = g_{\alpha\beta} = \delta_{\alpha\beta} \) is the Euclidean metric. For the sake of clarity we repeat formula (2.1) giving tensor indices explicitly and performing summation over frame indices explicitly: \( \delta_{\alpha\beta} = \theta^{1}_\alpha \theta^1_\beta + \theta^{2}_\alpha \theta^2_\beta + \theta^{3}_\alpha \theta^3_\beta \) where \( \alpha \) and \( \beta \) run through the values 1, 2, 3. We view the identity (2.1) as a kinematic constraint: the covector fields \( \theta^j \) are chosen so that they satisfy (2.1), which leaves us with three real degrees of freedom at every point of \( \mathbb{R}^3 \).

We work only with coframes which have positive orientation, i.e. which satisfy the condition

\[
\det \theta^{j}_\alpha = +1 > 0.
\]  

(2.2)

If one views \( \theta^{j}_\alpha \) as a \( 3 \times 3 \) real matrix-function, then conditions (2.1) and (2.2) mean that this matrix-function is special orthogonal. Thus, the coframe can be thought of as a field of special orthogonal matrices.
As dynamical variables in our model we choose the coframe \( \vartheta \) and a positive density \( \rho \). Our coframe and density are functions of Cartesian coordinates \( x^\alpha, \alpha = 1, 2, 3 \), as well as of time \( x^0 \). At a physical level, making the density \( \rho \) a dynamical variable means that we view our continuum more like a fluid rather than a solid. In other words, we allow the material to redistribute itself so that it finds its equilibrium density distribution. Observe that the total number of real dynamical degrees of freedom contained in the coframe \( \vartheta \) and positive density \( \rho \) is four, exactly as in a two-component complex-valued spinor field.

Note that there is nothing wrong in taking a prescribed density (as opposed to a density which is a dynamical variable): the theory one gets is very similar to the one described in the current paper and most formulae carry through with minimal changes.

Below is the list of the main assumptions on which our model will be based.

**Assumption 1: our model is geometrically nonlinear.** This means that we do not linearise rotations and we do not linearise the density. In other words, we allow our material points to experience full turns and we allow our density to experience changes comparable to the density itself.

**Assumption 2: our material is physically linear.** This means that our potential energy is chosen to be quadratic in torsion (the latter serves as the measure of rotational deformations, see subsection 4.1). Note that physical linearity does not contradict geometric nonlinearity: locally (in space and time) material points “do not know” that they may eventually experience full rotations and the density “does not know” that it may eventually experience a change comparable to its current value.

**Assumption 3: our material is homogeneous and isotropic.** Homogeneity means that physical properties of the material are the same at all points of our continuum and isotropy means that there are no preferred directions.

**Assumption 4: our model is invariant under rigid rotations of the coframe.** By a rigid rotation of the coframe we understand the transformation

\[
\vartheta^j \mapsto O^j_k \vartheta^k
\]

where \( O^j_k \) is a constant special orthogonal matrix. The thinking here is that when we attach an orthonormal basis to each geometric point of our continuum there is no reason to associate one particular direction with \( \vartheta^1 \), another with \( \vartheta^2 \) and a third with \( \vartheta^3 \). What matters is how these directions change when we move from one point to another, i.e. how orthonormal bases at different points differ relative to each other. A rigid rotation of the coframe means that we simultaneously rotate all our orthonormal bases by the same angle around the same axis. We view rigid rotations of the coframe as gauge transformations and assume that our model does not feel them. See also (8) for a detailed exposition of gauge theory for problems similar to the ones considered in our paper.

3. **Kinetic energy**

Kinetic energy is given by the formula

\[
K(x^0) = c^{\text{kin}} \int \|\omega\|^2 \rho \, dx^1 \, dx^2 \, dx^3
\]

where \( c^{\text{kin}} \) is some positive constant and \( \omega \) is the (pseudo)vector of angular velocity

\[
\omega = \frac{1}{2} \ast (\delta_{jk} \vartheta^j \wedge \partial_0 \vartheta^k).
\]
Here $\wedge$ is the exterior product and $\star$ is the Hodge star (A.1).

In writing the formula for kinetic energy (3.1) we think of each material point as a uniform ball possessing a moment of inertia and without a preferred axis of rotation.

We give for reference a more explicit version of the formula for angular velocity (3.2):

$$\omega_\alpha = \frac{1}{2} \sum_{j=1}^3 \left( \vartheta^j_2 \partial_0 \vartheta^j_3 - \vartheta^j_3 \partial_0 \vartheta^j_2 \right) \left( \vartheta^j_3 \partial_0 \vartheta^j_1 - \vartheta^j_1 \partial_0 \vartheta^j_3 \right) \left( \vartheta^j_1 \partial_0 \vartheta^j_2 - \vartheta^j_2 \partial_0 \vartheta^j_1 \right). \quad (3.3)$$

4. Potential energy

4.1 Measuring rotational deformations

In order to write down the formula for the potential energy we need to measure deformations caused by rotations of the coframe. More specifically, we need to measure deformations caused by the fact that at different points the coframe is oriented differently. Obvious candidates for a measure of deformations are the rank two tensors

$$K^j := \partial \vartheta^j, \quad j = 1, 2, 3, \quad (4.1)$$

or, in more explicit form, $K^j_{\alpha\beta} := \partial_\alpha \vartheta^j_\beta$. The problem is that taken separately the three rank two tensors (4.1) are not invariant under rigid rotations of the coframe (2.3). The natural way of forming a truly invariant object is to make one rank three tensor out of the three rank two tensors $K^j$ according to the formula

$$K := \delta_{jk} \vartheta^j \otimes K^k = \delta_{jk} \vartheta^j \otimes \partial \vartheta^k. \quad (4.2)$$

The rank three tensor $K$ is invariant under rigid rotations of the coframe (2.3) and, moreover, the individual rank two tensors $K^j$ can be recovered from $K$ as $K^j_{\gamma\delta} = \vartheta^j_\alpha K^\alpha_{\gamma\delta}$ so there is no loss of information.

Let us examine the symmetries of the tensor $K$. Observe that formula (2.1) implies

$$0 = \partial_\alpha g_{\beta\gamma} = \partial_\alpha (\delta_{jk} \vartheta^j_\beta \vartheta^k_\gamma) = \delta_{jk} (\partial_\alpha \vartheta^j_\beta) \vartheta^k_\gamma + \delta_{jk} \vartheta^j_\beta (\partial_\alpha \vartheta^k_\gamma) = K^k_{\alpha\beta\gamma} + K^k_{\alpha\gamma\beta}$$

which means that the rank three tensor $K$ is antisymmetric in the first and third indices,

$$K^k_{\gamma\alpha\beta} = -K^k_{\beta\alpha\gamma}. \quad (4.3)$$

Now, let us introduce another rank three tensor

$$T := \delta_{jk} \vartheta^j \otimes d\vartheta^k \quad (4.4)$$

where $d$ stands for the exterior derivative. The tensor (4.4) is obviously antisymmetric in the second and third indices

$$T^\alpha_{\beta\gamma} = -T^\alpha_{\gamma\beta} \quad (4.5)$$

and is expressed via our original deformation tensor (4.2) as

$$T^\alpha_{\beta\gamma} = K^\alpha_{\beta\gamma} - K^\alpha_{\gamma\beta}. \quad (4.6)$$
Formulae (4.6) and (4.3) imply

\[ T_{\alpha\beta\gamma} = K_{\alpha\beta\gamma} + K_{\beta\gamma\alpha}, \]
\[ T_{\gamma\alpha\beta} = K_{\gamma\alpha\beta} + K_{\alpha\beta\gamma}, \]
\[ T_{\beta\gamma\alpha} = K_{\beta\gamma\alpha} + K_{\gamma\alpha\beta}, \]

where the last two identities were obtained from the first one by a cyclic relabelling of tensor indices. Adding up the first and second identities and subtracting the third one we get

\[ K_{\alpha\beta\gamma} = \frac{1}{2} (T_{\alpha\beta\gamma} + T_{\gamma\alpha\beta} - T_{\beta\gamma\alpha}) = \frac{1}{2} (T_{\alpha\beta\gamma} + T_{\beta\alpha\gamma} + T_{\gamma\alpha\beta}) \]  \hspace{1cm} (4.7)

(here we also used (4.5)). Note that the argument carried out above is a rephrasing of the standard argument that for a metric compatible affine connection contortion can be expressed via torsion, see subsection 7.2.6 in (9).

Formulae (4.6) and (4.7) show that the tensors \( K \) and \( T \) are expressed via each other so either of them can be used as a measure of rotational deformations. We choose to use the tensor \( T \) because it has a clear geometric meaning: it is the torsion of the teleparallel connection generated by the coframe \( \vartheta \), see Appendix A of (10) for a concise exposition. An additional advantage of using the tensor \( T \) is that the definition (4.4) of this tensor does not require the use of covariant derivatives so it works when the metric \( g \) appearing in formula (2.1) is not assumed to be Euclidean. The latter was important for Einstein and Cartan who arrived at the mathematical model similar to the one described in in our paper coming from general relativity. Recall that in general relativity the metric plays the role of dynamical variable so for someone with a relativistic background assuming the metric to be Euclidean (i.e. space to be flat) is unnatural.

Starting from Einstein’s works (4) torsion is traditionally used as a measure of deformations when modelling elastic continua with rotations. We shall follow this tradition and construct our potential energy as a function(al) of \( T \). However, before writing down the formula for potential energy we will simplify matters by using the fact that we are working in 3D (our previous arguments were dimension-independent).

Applying the Hodge star (A.1) in the second and third indices we switch from the original torsion tensor \( T \) to the tensor

\[ \tilde{T}_{\alpha\beta} := \frac{1}{2} T_{\alpha}^\delta \varepsilon_{\gamma} \delta_{\beta}. \]  \hspace{1cm} (4.8)

Of course, the tensor \( T \) can be recovered from \( \tilde{T} \) as

\[ T_{\alpha\beta\gamma} = \tilde{T}_{\alpha}^\delta \varepsilon_{\delta\beta\gamma}. \]  \hspace{1cm} (4.9)

Formulae (4.8) and (4.9) show that the tensors \( T \) and \( \tilde{T} \) are expressed via each other so either of them can be used as a measure of rotational deformations. We choose to use the tensor \( \tilde{T} \) because it has lower rank, two instead of three.

Formulae (4.4) and (4.8) imply

\[ \tilde{T} = \delta_{jk} \vartheta^j \otimes *d\vartheta^k = \delta_{jk} \vartheta^j \otimes \text{curl} \vartheta^k. \]  \hspace{1cm} (4.10)
We see that $^*T$ is a rank two tensor without any symmetries and with arbitrary trace. This is the tensor we will be using as a measure of rotational deformations when writing down the formula for potential energy. The tensor $^*T$ is sometimes called the dislocation density tensor (11).

We give for reference a more explicit version of formula (4.10):

$$^*T_{\alpha\beta} = 3 \sum_{j=1}^{3} \begin{pmatrix} \vartheta_j^1 \partial_2 \vartheta_j^3 - \vartheta_j^1 \partial_3 \vartheta_j^2 & \vartheta_j^1 \partial_3 \vartheta_j^2 - \vartheta_j^1 \partial_2 \vartheta_j^3 \\ \vartheta_j^2 \partial_3 \vartheta_j^1 - \vartheta_j^2 \partial_1 \vartheta_j^3 & \vartheta_j^2 \partial_1 \vartheta_j^3 - \vartheta_j^2 \partial_3 \vartheta_j^1 \\ \vartheta_j^3 \partial_1 \vartheta_j^2 - \vartheta_j^3 \partial_2 \vartheta_j^1 & \vartheta_j^3 \partial_2 \vartheta_j^1 - \vartheta_j^3 \partial_1 \vartheta_j^2 \end{pmatrix}.$$  (4.11)

### 4.2 Irreducible decomposition of rotational deformations

Recall the logic of classical linear elasticity (12): after identifying the measure of deformation one decomposes it into irreducible pieces. We follow this logic by decomposing the tensor $^*T$ into irreducible pieces. The construction presented below is similar to (12), the only difference being that instead of a symmetric rank two tensor, strain, we deal with a rank two tensor, $^*T$, without any symmetries.

Decomposing the rank two tensor $^*T$ into irreducible pieces means the following. We fix a point in $\mathbb{R}^3$ and at this point look at all real rank two tensors $P$. Such tensors can be viewed as elements of a real 9-dimensional vector space $V$ equipped with inner product

$$(P, Q)_{V} := P_{\alpha\beta} Q^{\alpha\beta}$$  (4.12)

and corresponding norm

$$\|P\|_V = \sqrt{(P, P)_{V}} = \sqrt{P_{\alpha\beta} P^{\alpha\beta}}.$$  (4.13)

Let us now examine what happens when we rotate our Cartesian coordinate system $x^\alpha$, i.e. when we perform a linear change of coordinates preserving the metric $g_{\alpha\beta}$ and orientation. The components of our tensors $P_{\alpha\beta}$ change in a particular way under rotations of the coordinate system, so we get an action of the group $SO(3)$ on the vector space $V$. Looking for irreducible pieces of torsion means identifying subspaces of $V$ which are invariant under the action of the group $SO(3)$, i.e. which map into themselves, and which do not contain smaller nontrivial invariant subspaces.

In our case the invariant subspaces are obvious. These are

- the 1-dimensional subspace of real rank two tensors proportional to the metric,
- the 3-dimensional subspace of real antisymmetric rank two tensors and
- the 5-dimensional subspace of real symmetric trace-free rank two tensors.

These three subspaces are clearly irreducible and mutually orthogonal in the inner product (4.12).

Our rank two tensor $^*T$ can now be written as a sum of three irreducible pieces

$$^*T = ^*T^{\text{ax}} + ^*T^{\text{vec}} + ^*T^{\text{ten}}.$$  (4.14)
where

\[ T_{\alpha\beta}^{ax} := \frac{\gamma^{\gamma}}{3} g_{\alpha\beta}, \] (4.15)

\[ T_{\alpha\beta}^{vec} := \frac{T_{\alpha\beta} - T_{\beta\alpha}}{2}, \] (4.16)

\[ T_{\alpha\beta}^{ten} := T_{\alpha\beta} - T_{ax}^{ax} - T_{vec}^{vec} = \frac{T_{\alpha\beta} + T_{\beta\alpha}}{2} - \frac{\gamma^{\gamma}}{3} g_{\alpha\beta}. \] (4.17)

We label the irreducible pieces (4.15), (4.16) and (4.17) by the adjectives \textit{axial}, \textit{vector} and \textit{tensor} respectively, which is terminology traditional in alternative theories of gravity (6).

4.3 Formula for potential energy

Following the logic of classical linear elasticity (12) we now write down the explicit formula for potential energy:

\[ P(x^0) = \int \left( c^{ax} \| T^{ax} \|_V^2 + c^{vec} \| T^{vec} \|_V^2 + c^{ten} \| T^{ten} \|_V^2 \right) \rho \, dx_1 \, dx_2 \, dx_3 \] (4.18)

where \( c^{ax}, c^{vec} \) and \( c^{ten} \) are some nonnegative constants (elastic moduli), not all zero, and \( \| \cdot \|_V \) is the norm (4.13). Comparing our formula (4.18) with formula (4.3) from (12) we see a difference between classical and rotational elasticity: classical elasticity involves two elastic moduli whereas rotational elasticity involves three. The extra elastic modulus \( c^{vec} \) is needed because the tensor \( \bar{T} \) which we use as measure of rotational deformations is not necessarily symmetric.

Formula (4.18) is the one traditionally used in teleparallelism. This formula already appears in the original papers of Einstein (4), though for some reason\(^\dagger\) Einstein did not include the axial term \( c^{ax} \| T^{ax} \|^2 \). Subsequent authors always used three terms, see, for example, formula (26) in (6).

4.4 Simplifying the formula for potential energy

Let us introduce the (pseudo)scalar

\[ f := T^{ax}_{\alpha\alpha}, \] (4.19)

and the vector

\[ v_{\alpha} := T^{\beta\gamma}_{\alpha\beta\gamma}. \] (4.20)

Formulæ (4.13)–(4.17), (4.19) and (4.20) imply

\[ \| T^{ax} \|_V^2 = \frac{1}{3} f^2, \] (4.21)

\[ \| T^{vec} \|_V^2 = \frac{1}{2} \| v \|_V^2, \] (4.22)

\(^\dagger\) The reason could be that Einstein was primarily interested in providing a geometric interpretation of electromagnetism and might have felt that the axial term would not contribute to the electromagnetic field.
\[ \|\hat{T}^{\text{ten}}\|^2 = \|\hat{T}\|^2 - \frac{1}{3} f^2 - \frac{1}{2} \|v\|^2. \]  

(4.23)

Substituting (4.11) into (4.19) and (4.20) we get more explicit formulae for \( f \) and \( v \):

\[ f = \sum_{j=1}^{3} \left( \vartheta_j^{1} \partial_{1} \vartheta_j^{3} - \vartheta_j^{3} \partial_{1} \vartheta_j^{1} + \vartheta_j^{1} \partial_{2} \vartheta_j^{3} - \vartheta_j^{3} \partial_{2} \vartheta_j^{1} + \vartheta_j^{2} \partial_{3} \vartheta_j^{1} - \vartheta_j^{1} \partial_{3} \vartheta_j^{2} \right), \]  

(4.24)

\[ v_\alpha = \sum_{j=1}^{3} \left( \vartheta_j^{2} \partial_{1} \vartheta_j^{1} - \vartheta_j^{1} \partial_{2} \vartheta_j^{2} - \vartheta_j^{2} \partial_{3} \vartheta_j^{1} + \vartheta_j^{1} \partial_{3} \vartheta_j^{2} - \vartheta_j^{1} \partial_{2} \vartheta_j^{3} + \vartheta_j^{3} \partial_{1} \vartheta_j^{2} \right). \]  

(4.25)

Substituting formulae (4.19)–(4.21) into formula (4.18) we get

\[ P(x_0) = \int \left( \frac{c^{\text{ax}} - c^{\text{ten}}}{3} f^2 + \frac{c^{\text{vec}} - c^{\text{ten}}}{2} \|v\|^2 + c^{\text{ten}} T_{\alpha\beta}^{\ast} T_{\alpha\beta}^{\ast} - c^{\text{kin}} \|\omega\|^2 \right) \rho \, dx^1 dx^2 dx^3. \]  

(4.26)

The advantage of writing potential energy in the form (4.26) is that the geometric quantities \( f, v \) and \( \hat{T} \) appearing in this formula have relatively compact explicit representations (4.24), (4.25) and (4.11).

5. Lagrangian of rotational elasticity

We combine our potential energy (4.26) and kinetic energy (3.1) in forming the action (variational function) of dynamic rotational elasticity

\[ S(\vartheta, \rho) = \int (P(x_0) - K(x_0)) \, dx^0 = \int L(\vartheta, \rho) \, dx^0 dx^1 dx^2 dx^3 \]  

(5.1)

where

\[ L(\vartheta, \rho) = \left( \frac{c^{\text{ax}} - c^{\text{ten}}}{3} f^2 + \frac{c^{\text{vec}} - c^{\text{ten}}}{2} \|v\|^2 + c^{\text{ten}} T_{\alpha\beta}^{\ast} T_{\alpha\beta}^{\ast} - c^{\text{kin}} \|\omega\|^2 \right) \rho \]  

(5.2)

is the Lagrangian density. Recall that the geometric quantities \( f, v, \hat{T} \) and \( \omega \) appearing in formula (5.2) are defined by formulae (4.24), (4.25), (4.11) and (3.3) respectively.

Our construction of the action (5.1) out of potential and kinetic energies is Newtonian: compare with classical linear elasticity or even the harmonic oscillator in classical mechanics. An alternative approach is the relativistic one which boils down to rewriting the formula for potential energy in Lorentzian signature in dimension 1+3, with this “extended” potential energy becoming the action. The Newtonian and relativistic approaches are different which can be seen, for example, from the fact that the relativistic approach always imposes a unique velocity of wave propagation (speed of light) whereas with the Newtonian approach one expects to get at least two distinct wave velocities.

Starting with Einstein, most authors working in the subject of teleparallelism adopt the relativistic approach. We shall, however, stick with the Newtonian approach (5.1).
6. Reformulating the problem in the language of spinors

Our field equations (Euler–Lagrange equations) are obtained by varying the action (5.1) with respect to the coframe \( \vartheta \) and density \( \rho \). Varying with respect to the density \( \rho \) is easy: this gives the field equation \( \frac{\epsilon_{\alpha\beta\gamma}}{2} f_\alpha - \frac{\epsilon_{\alpha\beta}}{2} \rho_v \| v \|^2 + \epsilon_{\alpha\beta\gamma} T_{\alpha\beta} T^{\alpha\beta} - \epsilon_{\alpha\beta} \| \omega \|^2 = 0 \) which is equivalent to \( L(\vartheta, \rho) = 0 \). Varying with respect to the coframe \( \vartheta \) is more difficult because we have to maintain the kinematic constraint (2.1).

This technical difficulty can be overcome by switching to a different dynamical variable. Namely, it is known (13) that in dimension 3 a coframe \( \vartheta \) and a (positive) density \( \rho \) are equivalent to a 2-component complex-valued spinor field \( \xi = \xi^a = \left( \xi^1, \xi^2 \right) \) modulo the sign of \( \xi \). The explicit formulae establishing this equivalence are

\[
\rho = \xi^a \sigma_{ab} \xi^b, \quad (6.1)
\]

\[
(\vartheta^1 + i \vartheta^2)_\alpha = \rho^{-1} \epsilon^a \sigma_{ob} \xi^a \sigma_{cd} \xi^d, \quad (6.2)
\]

\[
\vartheta^3_\alpha = \rho^{-1} \xi^a \sigma_{ab} \xi^b. \quad (6.3)
\]

Here \( \sigma \) are Pauli matrices and \( \epsilon \) is “metric spinor” (see (A.2)–(A.4)), the free tensor index \( \alpha \) runs through the values 1, 2, 3, and the spinor summation indices run through the values 1, 2 or 1, 2. The advantage of switching to a spinor field \( \xi \) is that there are no kinematic constraints on its components, so the derivation of field equations becomes straightforward.

We give for reference more explicit versions of formulae (6.1)–(6.3):

\[
\rho = \bar{\xi}^1 \xi^1 + \bar{\xi}^2 \xi^2, \quad (6.4)
\]

\[
(\vartheta^1 + i \vartheta^2)_\alpha = (\bar{\xi}^1 \xi^1 + \bar{\xi}^2 \xi^2)^{-1} \begin{pmatrix} (\xi^1)^2 - (\xi^2)^2 \\ i((\xi^1)^2 + i(\xi^2)^2) \\ -2\xi^1 \xi^1 \end{pmatrix}, \quad (6.5)
\]

\[
\vartheta^3_\alpha = (\bar{\xi}^1 \xi^1 + \bar{\xi}^2 \xi^2)^{-1} \begin{pmatrix} i\bar{\xi}^2 \xi^1 + \bar{\xi}^1 \xi^2 \\ i\bar{\xi}^2 \xi^1 - i\bar{\xi}^1 \xi^2 \\ \bar{\xi}^1 \xi^1 - \bar{\xi}^2 \xi^2 \end{pmatrix}. \quad (6.6)
\]

Let us rewrite the geometric quantities \( f, v, T \) and \( \omega \) appearing in formula (5.2) in terms of the spinor field \( \xi \). The spinor representation of angular velocity \( \omega \) was derived in (13):

\[
\omega_\alpha = i \frac{\bar{\xi}^a \sigma_{ab} \partial_b \xi^b - \xi^b \sigma_{ab} \partial_b \bar{\xi}^a}{\xi^c \sigma_{cd} \xi^d} \quad (6.7)
\]

or, more explicitly,

\[
\omega_\alpha = \frac{1}{\bar{\xi}^1 \xi^1 + \bar{\xi}^2 \xi^2} \begin{pmatrix} i\bar{\xi}^2 \partial_0 \xi^1 + i\bar{\xi}^1 \partial_0 \xi^2 \\ -\bar{\xi}^2 \partial_0 \xi^1 + \bar{\xi}^1 \partial_0 \xi^2 \\ i\bar{\xi}^1 \partial_0 \xi^1 - i\bar{\xi}^2 \partial_0 \xi^2 \end{pmatrix} + \text{c.c.} \quad (6.8)
\]

where the “c.c.” stands for “complex conjugate term”. The spinor representation of the
tensor $\tilde{T}$ is derived in Appendix B, see formula (B.1) or its more explicit version (B.7). Substituting (B.1) or (B.7) into (4.19) and (4.20) we arrive at spinor representations for the (pseudo)scalar $f$ and vector $v$:

\[
f = -2i \frac{\bar{\xi} \sigma^{ab} \partial_{\alpha} \xi_{b} - \xi^{b} \sigma^{ab} \partial_{\alpha} \bar{\xi}_{a}}{\xi^{c} \sigma_{0cd} \xi^{d}}, \tag{6.9}
\]

\[
v_{\alpha} = -i \varepsilon_{\beta\gamma\alpha} \frac{\bar{\xi}^{a} \sigma^{b} \partial_{\gamma} \xi_{b} - \xi^{b} \sigma^{b} \partial_{\gamma} \bar{\xi}_{a}}{\xi^{c} \sigma_{0cd} \xi^{d}} \tag{6.10}
\]

where $\partial^{\alpha} := g^{\alpha\beta} \partial_{\beta} = \partial_{\alpha}$, or, more explicitly,

\[
f = \frac{2}{\xi^{1} \xi^{1} + \xi^{2} \xi^{2}} (-i \xi^{1} \partial_{1} \xi^{2} - i \xi^{2} \partial_{1} \xi^{1} - \xi^{1} \partial_{2} \xi^{2} + \xi^{2} \partial_{2} \xi^{1} - i \xi^{1} \partial_{3} \xi^{1} + i \xi^{2} \partial_{3} \xi^{2}) + \text{c.c.}, \tag{6.11}
\]

\[
v_{\alpha} = \frac{1}{\xi^{1} \xi^{1} + \xi^{2} \xi^{2}} \left( \frac{i \xi^{1} \partial^{2} \xi^{1} - i \xi^{2} \partial^{2} \xi^{2} - \xi^{1} \partial^{3} \xi^{1} + \xi^{2} \partial^{3} \xi^{2}}{i \xi^{1} \partial^{3} \xi^{1} + i \xi^{2} \partial^{3} \xi^{1}} \right) + \text{c.c.} \tag{6.12}
\]

Note that formula (6.9) is a rephrasing of formula (B.5) from (13).

From now on we write our action (5.1) and Lagrangian density (5.2) as $S(\xi)$ and $L(\xi)$ rather than $S(\theta, \varphi)$ and $L(\theta, \varphi)$, thus indicating that we have switched to spinors. The explicit formula for $L(\xi)$ is obtained by substituting formulae (6.1), (6.9), (6.10), (B.1) and (6.7) into (5.2). The nonvanishing spinor field $\xi$ is the new dynamical variable and it will be varied without any constraints.

7. Euler–Lagrange equation

Let us perform a formal variation of our spinor field $\xi \rightarrow \xi + \delta \xi$, where $\delta \xi : \mathbb{R} \times \mathbb{R}^{3} \rightarrow \mathbb{C}^{2}$ is an arbitrary (infinitely) smooth function with compact support. Then, after integration by parts, the variation of our action can be written as

\[
\delta S = \int (F_{a} \delta \bar{\xi}^{a} + \bar{F}_{a} \delta \xi^{a}) \, dx^{0} \, dx^{1} \, dx^{2} \, dx^{3} \tag{7.1}
\]

where $F$ is a dotted spinor field uniquely determined by the undotted spinor field $\xi$. The Euler–Lagrange for our unknown spinor field $\xi$ is, therefore,

\[
F = 0. \tag{7.2}
\]

The map

\[
\xi \rightarrow F \tag{7.3}
\]

defines a nonlinear second order partial differential operator in the variables $x^{0}$ (time) and $x^{\alpha}, \, \alpha = 1, 2, 3$ (Cartesian coordinates).

We shall refrain from writing down the Euler–Lagrange equation (7.2) explicitly. The reason for this is that in the current paper we are interested in finding a particular class of solutions for which the procedure is much simpler.
Note that for the special case of a purely axial material, i.e. material with
\[ c^{\text{vec}} = 0, \quad c^{\text{ten}} = 0, \] (7.4)
the Euler–Lagrange equation (7.2) was written down explicitly in (13). The calculations in (13) were carried out under the additional assumption
\[ c^{\text{kin}} = \frac{4}{3} c^{\text{ax}}, \] (7.5)
which can always be achieved by rescaling time \( x^0 \).

8. Plane wave solutions
We seek solutions of the form
\[ \xi(x) = e^{-ip \cdot x} \zeta, \] (8.1)
where \( \zeta \neq 0 \) is a constant (complex) spinor and \( p \) is a constant real covector. Here we use relativistic notation, incorporating time \( x^0 \) into our coordinates. This means that \( x = (x^0, x^1, x^2, x^3) \) and \( p = (p_0, p_1, p_2, p_3) \); bold type indicates that we are working in (1+3)-dimensional spacetime. The number \( |p_0| \) is the wave frequency and the covector \( (p_1, p_2, p_3) \) is the wave vector in original 3-dimensional Euclidean space. The 4-component covector \( p = (p_0, p_1, p_2, p_3) \) has the meaning of relativistic 4-momentum.

Throughout this section as well as the next one we assume that
\[ p_0 \neq 0 \] (8.2)
which means that we are not interested in static (time-independent) solutions. The sign of \( p_0 \) can be arbitrary.

Our Euler–Lagrange equation (7.2) is highly nonlinear so it is by no means obvious that one can seek solutions in the form of plane waves (8.1). Fortunately (and miraculously) this is the case. In order to see this, we rewrite our Euler–Lagrange equation (7.2) in equivalent form
\[ e^{ip \cdot x} F = 0. \] (8.3)
Note that the sign in the exponent in (8.3) is opposite to that in (8.1). We have

LEMMA 8.1. If the spinor field \( \xi \) is a plane wave (8.1) then the left-hand side of equation (8.3) is constant, i.e. it does not depend on \( x \).

Of course, Lemma 8.1 can be equivalently reformulated as follows: the nonlinear partial differential operator (7.3) maps a plane wave (8.1) into a plane wave with the same relativistic 4-momentum \( p \).

The proof of Lemma 8.1 is quite technical and is given in Appendix C.

Lemma 8.1 justifies separation of variables, i.e. it reduces the study of the nonlinear partial differential equation (7.2) for the unknown spinor field \( \xi \) to the study of the rational algebraic equation (8.3) for the unknown constant spinor \( \zeta \). We suspect that the underlying group-theoretic reason for our nonlinear partial differential equation (7.2) admitting separation of variables is the fact that our model is U(1)-invariant, i.e. it is invariant under the multiplication of the spinor field \( \xi \) by a complex constant of modulus 1. Hence, it is feasible
that one could prove Lemma 8.1, as well as Lemma 8.2 stated further down in this section, without performing the laborious calculations presented in Appendix C.

We are now faced with the task of writing down the LHS of equation (8.3) explicitly and with minimal calculations. To this end we address a seemingly different issue: we examine what happens when we substitute our plane wave (8.1) into our Lagrangian density \( L(\xi) \), rather than the Euler–Lagrange equation (7.2).

Substituting formula (8.1) into formulæ (6.1), (6.9), (6.10), (B.1) and (6.7) we get

\[
\rho = \zeta^a \sigma_{ab} \zeta^b, \quad (8.4)
\]

\[
f = -\frac{4\zeta^a \sigma_{ab} p_0 \zeta^b}{\zeta^c \sigma_{cd} \zeta^d}, \quad (8.5)
\]

\[
v_\alpha = -\frac{2\varepsilon_{\beta\gamma} \zeta^a \sigma^b \zeta^c \sigma_{cd} \zeta^d}{\zeta^e \sigma_{ef} \zeta^f}, \quad (8.6)
\]

\[
\vartheta_{\alpha\beta} = 2\zeta^a \sigma_{ab} p_0 \zeta^b - \zeta^a \sigma^b \zeta^c \sigma_{cd} \zeta^d g_{\alpha\beta}, \quad (8.7)
\]

\[
\omega_\alpha = \frac{2\zeta^a \sigma_{ab} p_0 \zeta^b}{\zeta^c \sigma_{cd} \zeta^d}, \quad (8.8)
\]

or, more explicitly,

\[
\rho = \frac{4}{\zeta^1 \zeta^1 + \zeta^2 \zeta^2} \left( p_1(-\zeta^1 \zeta^2 - \zeta^2 \zeta^1) + ip_2(\zeta^1 \zeta^2 - \zeta^2 \zeta^1) + p_3(-\zeta^1 \zeta^1 + \zeta^2 \zeta^2) \right), \quad (8.9)
\]

\[
f = \frac{4}{\zeta^1 \zeta^1 + \zeta^2 \zeta^2} \left( \frac{p_1(-\zeta^1 \zeta^2 - \zeta^2 \zeta^1) + ip_2(\zeta^1 \zeta^2 - \zeta^2 \zeta^1) + p_3(-\zeta^1 \zeta^1 + \zeta^2 \zeta^2)}{\zeta^1 \zeta^1 + \zeta^2 \zeta^2} \right), \quad (8.10)
\]

\[
v_\alpha = \frac{2}{\zeta^1 \zeta^1 + \zeta^2 \zeta^2} \left( \begin{array}{c}
p_2 \left( \zeta^1 \zeta^1 - \zeta^2 \zeta^2 \right) + ip_3 \left( \zeta^1 \zeta^2 - \zeta^2 \zeta^1 \right) \\
p_3 \left( \zeta^1 \zeta^2 + \zeta^2 \zeta^1 \right) + p_1 \left( -\zeta^1 \zeta^1 + \zeta^2 \zeta^2 \right) \\
 ip_1 \left( -\zeta^1 \zeta^1 + \zeta^2 \zeta^2 \right) + p_2 \left( -\zeta^1 \zeta^2 - \zeta^2 \zeta^1 \right) \end{array} \right), \quad (8.11)
\]

\[
\begin{pmatrix}
\vartheta_{11} \\
\vartheta_{12} \\
\vartheta_{13} \\
\vartheta_{21} \\
\vartheta_{22} \\
\vartheta_{23} \\
\vartheta_{31} \\
\vartheta_{32} \\
\vartheta_{33}
\end{pmatrix} = \frac{2}{\zeta^1 \zeta^1 + \zeta^2 \zeta^2} \left( \begin{array}{c}
p_2 \left( \zeta^1 \zeta^2 - \zeta^2 \zeta^1 \right) + p_3 \left( -\zeta^1 \zeta^1 + \zeta^2 \zeta^2 \right) \\
ip_2 \left( -\zeta^1 \zeta^2 + \zeta^2 \zeta^1 \right) + p_3 \left( -\zeta^1 \zeta^1 + \zeta^2 \zeta^2 \right) \\
p_1 \left( \zeta^1 \zeta^1 - \zeta^2 \zeta^2 \right) + p_2 \left( \zeta^1 \zeta^2 + \zeta^2 \zeta^1 \right) \\
p_1 \left( \zeta^1 \zeta^1 - \zeta^2 \zeta^2 \right) + p_2 \left( -\zeta^1 \zeta^2 + \zeta^2 \zeta^1 \right) \\
p_3 \left( \zeta^1 \zeta^2 + \zeta^2 \zeta^1 \right) \\
ip_3 \left( -\zeta^1 \zeta^2 + \zeta^2 \zeta^1 \right) \\
 p_3 \left( -\zeta^1 \zeta^1 + \zeta^2 \zeta^2 \right) + p_1 \left( -\zeta^1 \zeta^2 + \zeta^2 \zeta^1 \right) \\
 p_3 \left( -\zeta^1 \zeta^1 + \zeta^2 \zeta^2 \right) + p_1 \left( \zeta^1 \zeta^2 + \zeta^2 \zeta^1 \right) \\
 p_1 \left( -\zeta^1 \zeta^1 + \zeta^2 \zeta^2 \right) + ip_2 \left( \zeta^1 \zeta^2 - \zeta^2 \zeta^1 \right) \end{array} \right), \quad (8.12)
\]

\[
\omega_\alpha = \frac{2p_0}{\zeta^1 \zeta^1 + \zeta^2 \zeta^2} \left( \begin{array}{c}
-i\zeta^1 \zeta^2 - i\zeta^2 \zeta^1 \\
-\zeta^1 \zeta^1 + \zeta^2 \zeta^2 \\
 i\zeta^1 \zeta^1 + i\zeta^2 \zeta^2 \\
 i\zeta^1 \zeta^1 + i\zeta^2 \zeta^2 \\
-i\zeta^1 \zeta^2 - i\zeta^2 \zeta^1 \\
\end{array} \right), \quad (8.13)
\]
Substituting formulae (8.4)–(8.8) or their more explicit versions (8.9)–(8.13) into formula (5.2) we arrive at a Lagrangian density \( L(\zeta; p) \) which does not depend on \( x \). The self-contained formula for this Lagrangian density, written in terms of 4-momentum \( p \) and 4-current

\[
j_\alpha := \bar{\zeta} \bar{\sigma}_{\alpha\dot{a}} \zeta^b, \tag{8.14}
\]

is

\[
L(\zeta; p) = \frac{2}{j_0} (c^{\text{vec}} + c^{\text{ten}}) \| p \|^2 \| j \|^2 \frac{4}{j_0} \left( \frac{4}{3} c^{\text{ax}} - \frac{1}{2} c^{\text{vec}} + \frac{1}{6} c^{\text{ten}} \right) (j \cdot p)^2 - \frac{4}{j_0} \epsilon^{\text{kin}} p_0 \| j \|^2. \tag{8.15}
\]

Here we view our 4-covectors as \( p = (p_0, p) \) and \( j = (j_0, j) \), where \( p \) and \( j \) are 3-covectors.

We view the 4-momentum \( p \) as a parameter and the constant spinor \( \zeta \neq 0 \) as the dynamical variable. The Lagrangian density \( L(\zeta; p) \) is a smooth function of Re \( \zeta \) and Im \( \zeta \), so varying \( \zeta \) we get

\[
\delta L = \bar{G}_a \delta \bar{\zeta}_a + \bar{G}_a \delta \zeta^a \tag{8.16}
\]

where \( G \) is a dotted constant spinor expressed via the partial derivatives of \( L(\zeta; p) \) with respect to Re \( \zeta \) and Im \( \zeta \). It is natural to ask the question: what is the relationship between the spinor field \( F \) appearing in formula (7.1) and the constant spinor \( G \) appearing in formula (8.16)? The answer is given by

**Lemma 8.2.** If the spinor field \( \xi \) is a plane wave (8.1) then \( G = e^{i p \cdot x} F \).

The proof of Lemma 8.2 is presented in Appendix C.

Lemma 8.2 reduces the construction of plane wave solutions of rotational elasticity to finding the critical, with respect to \( \zeta \), points of the function \( L(\zeta; p) \). Varying (8.15), we arrive at the following equation for critical points:

\[
\frac{4}{j_0} (c^{\text{vec}} + c^{\text{ten}}) \| p \|^2 j^\alpha \sigma_{\alpha\dot{a}} \zeta^b - \frac{2}{j_0} (c^{\text{vec}} + c^{\text{ten}}) \| p \|^2 \| j \|^2 \sigma_{0\dot{a}} \zeta^b
\]

\[
+ \frac{8}{j_0} \left( \frac{4}{3} c^{\text{ax}} - \frac{1}{2} c^{\text{vec}} + \frac{1}{6} c^{\text{ten}} \right) (j \cdot p) p^\alpha \sigma_{\alpha\dot{a}} \zeta^b - \frac{4}{j_0} \left( \frac{4}{3} c^{\text{ax}} - \frac{1}{2} c^{\text{vec}} + \frac{1}{6} c^{\text{ten}} \right) (j \cdot p)^2 \sigma_{0\dot{a}} \zeta^b
\]

\[
- \frac{8}{j_0} \epsilon^{\text{kin}} p_0 j^\alpha \sigma_{\alpha\dot{a}} \zeta^b + \frac{4}{j_0} \epsilon^{\text{kin}} p_0 \| j \|^2 \| j \|^2 \sigma_{0\dot{a}} \zeta^b = 0. \tag{8.17}
\]

Recall that the 4-current \( j = (j_0, j) \) appearing in the above equation is defined in accordance with formula (8.14).

It now remains to find the 4-momenta \( p \) and spinors \( \zeta \neq 0 \) which satisfy equation (8.17). We carry out the analysis of equation (8.17) assuming that

\[
\rho = j_0 = \bar{\zeta}^1 \zeta^1 + \bar{\zeta}^2 \zeta^2 = 1. \tag{8.18}
\]

Condition (8.18) is a normalisation of the density: general plane wave solutions differ from those satisfying condition (8.18) by a real scaling factor. Furthermore, we assume that

\[
\zeta^b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{8.19}
\]
Indeed, any spinor $\zeta$ satisfying condition (8.18) can be transformed into the spinor (8.19) by the linear transformation
$$
\zeta \mapsto U\zeta
$$
where $U$ is a special ($\det U = 1$) unitary matrix. The transformation (8.20) leads to a rotation of the spatial part of the 4-current (8.14), so plane wave solutions with general $\zeta$ differ from those with $\zeta$ of the form (8.19) by a rotation. Substituting (8.17), (8.19), (A.3), (A.4) and (8.13) into (8.16) we get
$$
4 \left( c^{\text{vec}} + c^{\text{ten}} \right) \|p\|^2 \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} c^{\text{vec}} + \frac{1}{6} c^{\text{ten}} \end{pmatrix} \left( \begin{pmatrix} p_3 \\ p_1 + ip_2 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} c^{\text{eq}} - \frac{1}{2} c^{\text{vec}} + \frac{1}{6} c^{\text{ten}} \end{pmatrix} \frac{p_3}{2} \right) - 4 c^{\text{kin}} \begin{pmatrix} p_0^2 \\ 0 \end{pmatrix} = 0,
$$
or, equivalently,
$$
2 \left( c^{\text{vec}} + c^{\text{ten}} \right) \left( \|p\|^2 \right) + 4 \left( \frac{1}{3} c^{\text{ax}} - \frac{1}{2} c^{\text{vec}} + \frac{1}{6} c^{\text{ten}} \right) \left( \begin{pmatrix} p_3 \\ 2p_3(p_1 + ip_2) \end{pmatrix} - \begin{pmatrix} p_0^2 \\ 0 \end{pmatrix} \right) = 0. \quad (8.21)
$$
Put
$$
v_1 := \sqrt{\frac{4c^{\text{ax}} + 2c^{\text{ten}}}{3c^{\text{kin}}}}, \quad v_2 := \sqrt{\frac{c^{\text{vec}} + c^{\text{ten}}}{2c^{\text{kin}}}}. \quad (8.22)
$$
Note that because we assumed our three elastic moduli to be nonnegative and not all zero, our $v_1$ and $v_2$ are nonnegative and not both zero. Using (8.22) we can now rewrite equation (8.21) in more compact form
$$
v_2^2 \left( \|p\|^2 \right) + (v_1^2 - v_2^2) \left( \begin{pmatrix} p_3^2 \\ 2p_3(p_1 + ip_2) \end{pmatrix} - \begin{pmatrix} p_0^2 \\ 0 \end{pmatrix} \right) = 0. \quad (8.23)
$$

The analysis of equation (8.23) is elementary and the outcome is summarised in the following theorem, which is the main result of our paper.

**Theorem 8.3.** Plane wave solutions of rotational elasticity can, up to rescaling and rotation, be explicitly written down in the form (8.1), (8.19) with arbitrary nonzero $p_0$ and $p = (p_1, p_2, p_3)$ determined as follows.

- If $v_1 > 0$ and $v_2 > 0$ and $v_1 \neq v_2$ then we have two possibilities:
  - $p = \left( 0, 0, \pm \frac{p_0}{v_1} \right)$ (type 1 wave), or
  - $p = \left( \frac{|p_0|}{v_2} \cos \varphi, \frac{|p_0|}{v_2} \sin \varphi, 0 \right)$ where $\varphi \in \mathbb{R}$ is arbitrary (type 2 wave).
- If $v_1 > 0$ and $v_2 > 0$ and $v_1 = v_2$ then $p$ is an arbitrary 3-vector satisfying $\|p\| = \frac{|p_0|}{v_1}$. 
- If $v_1 > 0$ and $v_2 = 0$ then $p = \left( 0, 0, \pm \frac{p_0}{v_1} \right)$. 
- If $v_1 = 0$ and $v_2 > 0$ then $p = \left( \frac{|p_0|}{v_2} \cos \varphi, \frac{|p_0|}{v_2} \sin \varphi, 0 \right)$ where $\varphi \in \mathbb{R}$ is arbitrary.
Theorem 8.3 shows that rotational elasticity, like classical linear elasticity, produces two distinct types of plane wave solutions. We call these solutions type 1 and type 2 and they propagate with velocities $v_1$ and $v_2$ respectively, with $v_1$ and $v_2$ given by formulae (8.22).

However, unlike with classical linear elasticity, in rotational elasticity the two wave velocities, $v_1$ and $v_2$, are not ordered, i.e. we do not know a priori which one, $v_1$ or $v_2$, is bigger. The reason the two wave velocities are not ordered is because rotational elasticity has three elastic moduli compared to the two elastic moduli of classical linear elasticity. The "extra" elastic modulus is $c^{	ext{vec}}$, the one associated with the antisymmetric part of the rank two tensor $\tilde{T}$ which we use as a measure of rotational deformations. If we set $c^{	ext{vec}} = 0$, we end up with the inequality $v_1 \geq \sqrt{\frac{4}{3}} v_2$ similar to the well known inequality from classical linear elasticity, see formula (22.5) in (12).

9. The massless Dirac equation

In this section we consider a purely axial material (7.4), (7.5). Substituting (7.4) and (7.5) into (8.22) we get $v_1 = 1$ and $v_2 = 0$, so a purely axial material supports only type 1 waves. Throughout this section we also retain the assumption (8.2).

Note that a purely axial material has the remarkable property that its potential energy is invariant under conformal rescalings of the spatial metric by an arbitrary positive scalar function. We do not elaborate on this issue in the current paper because we chose to work with a specific (standard Euclidean) metric. The appropriate arguments are presented in Section 2 of (13).

Our aim is to compare our model with the linear partial differential equation (or, more precisely, system of two linear partial differential equations)

$$i(\pm\sigma^0_{ab}\partial_0 + \sigma^\alpha_{ab}\partial_\alpha)\xi^b = 0. \quad (9.1)$$

Here $\sigma$ are Pauli matrices (A.3), (A.4), the free spinor index $\dot{a}$ runs through the values $\dot{1}, \dot{2}$, summation is carried out over the tensor index $\alpha = 1, 2, 3$ as well as the spinor index $b = 1, 2,$ and $\xi$ is the unknown spinor field. We give for reference a more explicit version of equation (9.1):

$$i\left(\mp\partial_0 + \partial_3, \partial_1 \mp i\partial_2, \mp\partial_0 - \partial_3\right)\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0. \quad (9.2)$$

Equation (9.1) is called the massless Dirac equation or the Weyl equation. This equation is the accepted mathematical model for a massless neutrino field. The two choices of sign in (9.1) give two versions of the Weyl equation which differ by time reversal. Thus, we have a pair of Weyl equations.

We want to compare plane wave solutions (see (8.1)) of our model with those of the Weyl equation. As both models are invariant under the rescaling of the spinor field by a positive real constant as well as the rotations of Euclidean 3-space, it is sufficient to compare plane wave solutions for $\zeta$ of the form (8.19). Substituting (8.1) and (8.19) into (9.1) or its more explicit version (9.2) we get $p = (0, 0, \pm p_0)$ which is exactly what Theorem 8.3 gives us. Thus, we have established

Theorem 9.1. In the case of a purely axial material a plane wave spinor field is a solution of rotational elasticity if and only if it is a solution of one of the two Weyl equations (9.1).
It turns out that, in fact, a much stronger result holds. Consider a spinor field of the form

$$\xi(x^0, x^1, x^2, x^3) = e^{-ip_0x^0} \eta(x^1, x^2, x^3).$$  \hspace{1cm} (9.3)

We will call spinor fields of the form (9.3) \textit{stationary}. In considering stationary spinor fields what we are doing is separating out time only as opposed to separating out all the variables.

The following result generalises Theorem 9.1.

**Theorem 9.2.** In the case of a purely axial material a nonvanishing stationary spinor field is a solution of rotational elasticity if and only if it is a solution of one of the two Weyl equations (9.1).

Theorem 9.2 was proved in (13) and the proof is quite delicate. It involves an argument which reduces a nonlinear second order partial differential equation of a particular type to a pair of linear first order partial differential equations, which is, effectively, a form of integrability. An abstract self-contained version of this argument is given in Appendix B of (14).

10. **Acknowledgments**

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**APPENDIX A**

*Notation*

Our notation follows (10, 13, 14, 15). The only difference with (10, 15) is that in the latter the spacetime metric has opposite signature. In (13, 14) the signature is the same as in the current paper, i.e. the 3-dimensional spatial metric has signature + + + .

We use Greek letters for tensor (holonomic) indices and Latin letters for frame (anholonomic) indices. We identify differential forms with covariant antisymmetric tensors.

We define the action of the Hodge star on a rank \( r \) antisymmetric tensor \( R \) as

\[
(\ast R)_{\alpha r+1...\alpha 3} := (r!)^{-1} R^{\alpha 1...\alpha r} \varepsilon_{\alpha 1...\alpha 3}
\]

where \( \varepsilon \) is the totally antisymmetric quantity, \( \varepsilon_{123} := +1 \).

We use two-component complex-valued spinors (Weyl spinors) whose indices run through the values 1, 2 or \( \dot{1}, \dot{2} \). Complex conjugation makes the undotted indices dotted and vice versa.

We define the “metric spinor”

\[
\epsilon_{ab} = \epsilon_{\dot{a} \dot{b}} = \epsilon^{ab} = \epsilon^{\dot{a} \dot{b}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

and choose Pauli matrices

\[
\sigma_{0\dot{a} \dot{b}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -\sigma_{\dot{a} \dot{b}},
\]

\[
\sigma_{1\dot{a} \dot{b}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma^{1 \dot{a} \dot{b}}, \quad \sigma_{2\dot{a} \dot{b}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma^{2 \dot{a} \dot{b}}, \quad \sigma_{3\dot{a} \dot{b}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma^{3 \dot{a} \dot{b}}.
\]

Here the first index enumerates rows and the second enumerates columns.

**APPENDIX B**

*Spinor representation of torsion*

We show in this appendix that the tensor \( T_{\alpha \dot{b}} \) defined by formula (4.10) (it is the Hodge dual, in the last pair indices, of the torsion tensor) is expressed via the spinor field \( \xi \) as

\[
T_{\alpha \dot{b}} = \frac{i}{\xi^a} \sigma_{\dot{a} \dot{b}} \partial_\alpha \xi^b - \xi^b \sigma_{\dot{a} \dot{b}} \partial_\alpha \xi^a - (\xi^a \sigma^\gamma_{ab} \partial_\gamma \xi^b - \xi^b \sigma^\gamma_{ab} \partial_\gamma \xi^a) \gamma_{a\dot{b}}.
\]

Note that formula (B.1) is invariant under the rescaling of our spinor field by an arbitrary positive scalar function.

Formula (B.1) is proved by direct substitution of formulae (6.2) and (6.3) into (4.10). In order to simplify calculations we observe that the expressions in the left- and right-hand sides of formula
(B.1) have an invariant nature, hence it is sufficient to prove formula (B.1) at a point at which the spinor field takes the value \( \xi^a = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \). Then at this point we have

\[
\vartheta^\beta = \delta^\beta, \quad (B.2)
\]

\[
[\partial_\alpha(\vartheta^1 + i\vartheta^2)]_\beta = \begin{pmatrix} \partial_\alpha \xi^1 - \partial_\alpha \xi^2 \\ i\partial_\alpha \xi^1 - i\partial_\alpha \xi^2 \\ -2\partial_\alpha \xi^2 \end{pmatrix}, \quad [\partial_\alpha \vartheta^3]_\beta = \begin{pmatrix} \partial_\alpha \xi^2 + i\partial_\alpha \xi^1 \\ -i\partial_\alpha \xi^2 + i\partial_\alpha \xi^1 \\ 0 \end{pmatrix}, \quad (B.3)
\]

where \( \alpha = 1, 2, 3 \). Note that formulae (B.3) imply

\[
[curl(\vartheta^1 + i\vartheta^2)]_\beta = \begin{pmatrix} -2\partial_\alpha \xi^2 - \partial_\alpha (i\xi^1 - i\xi^2) \\ 2\partial_\alpha \xi^2 + \partial_\alpha (\xi^1 - \xi^2) \\ \partial_\alpha (i\xi^1 - i\xi^2) - \partial_\alpha (\xi^1 - \xi^2) \end{pmatrix}, \quad (B.4)
\]

\[
[curl \vartheta^3]_\beta = \begin{pmatrix} -\partial_\alpha (-i\xi^2 + i\xi^3) \\ \partial_\alpha (\xi^2 + \xi^3) \\ \partial_\alpha (-i\xi^2 + i\xi^3) - \partial_\alpha (\xi^2 + \xi^3) \end{pmatrix}. \quad (B.5)
\]

We now rewrite formula (4.10) in the form

\[
\tilde{T} = \frac{1}{2}(\vartheta^1 - i\vartheta^2) \otimes \text{curl}(\vartheta^1 + i\vartheta^2) + \frac{1}{2}(\vartheta^1 + i\vartheta^2) \otimes \text{curl}(\vartheta^1 - i\vartheta^2) + \vartheta^3 \otimes \text{curl} \vartheta^3. \quad (B.6)
\]

Substituting formulae (B.2), (B.4) and (B.5) into formula (B.6) we get

\[
\tilde{T}_{\alpha\beta} = \begin{pmatrix} -\partial_2(\xi^2 + \bar{\xi}^2) - i\partial_3(\xi^1 - \bar{\xi}^1) & \partial_1(\xi^2 + \bar{\xi}^2) & i\partial_1(\xi^1 - \bar{\xi}^1) \\ i\partial_2(\xi^2 - \bar{\xi}^2) & -i\partial_1(\xi^2 - \bar{\xi}^2) - i\partial_3(\xi^1 - \bar{\xi}^1) & \partial_3(\xi^2 + \bar{\xi}^2) \\ i\partial_3(\xi^2 - \bar{\xi}^2) & \partial_3(\xi^2 + \bar{\xi}^2) & -i\partial_1(\xi^2 - \bar{\xi}^2) - \partial_2(\xi^2 + \bar{\xi}^2) \end{pmatrix}
\]

which coincides with the RHS of formula (B.1). This completes the proof.

We give for reference a more explicit version of formula (B.1):

\[
\begin{pmatrix} T_{11} \\ T_{12} \\ T_{13} \\ T_{12} \star \\ T_{21} \\ T_{22} \star \\ T_{23} \\ T_{31} \\ T_{32} \star \\ T_{33} \end{pmatrix} = \frac{1}{\xi^1 \xi^2 + \xi^3 \xi^2} \begin{pmatrix} \xi^2 \partial_2 \xi^1 - \xi^2 \partial_3 \xi^2 + i\xi^2 \partial_3 \xi^1 - i\xi^1 \partial_3 \xi^2 \\ \xi^1 \partial_3 \xi^2 - \xi^2 \partial_3 \xi^1 \\ i\xi^2 \partial_3 \xi^1 - i\xi^2 \partial_3 \xi^2 \\ i\xi^2 \partial_2 \xi^1 + i\xi^3 \partial_2 \xi^2 \\ i\xi^2 \partial_2 \xi^1 - i\xi^2 \partial_2 \xi^2 \\ i\xi^2 \partial_3 \xi^2 + i\xi^3 \partial_3 \xi^1 \\ i\xi^2 \partial_3 \xi^1 - i\xi^2 \partial_3 \xi^2 \\ i\xi^2 \partial_3 \xi^1 + i\xi^2 \partial_3 \xi^2 \\ \xi^3 \partial_1 \xi^2 - \xi^3 \partial_1 \xi^2 \\ -i\xi^3 \partial_2 \xi^1 - i\xi^2 \partial_2 \xi^1 + \xi^2 \partial_1 \xi^1 - \xi^2 \partial_1 \xi^2 \\ -i\xi^3 \partial_1 \xi^2 - i\xi^2 \partial_1 \xi^1 + \xi^2 \partial_1 \xi^1 - \xi^2 \partial_1 \xi^2 \end{pmatrix} + \text{c.c.} \quad (B.7)
\]
APPENDIX C

Separation of variables

In this appendix we prove Lemmata 8.1 and 8.2. Note that it would suffice to prove Lemma 8.2 only, because Lemma 8.1 follows from Lemma 8.2. However, we prove Lemma 8.1 first for the sake of clarity of exposition.

Let us arrange the (pseudo)scalar $f$, the three components of the vector $v_{\alpha}$, the nine components of the tensor $\hat{T}_{\alpha\beta}$ and the three components of the (pseudo)vector $\omega_{\alpha}$ into one 16-component “vector” $V_{J}$, $J=1,\ldots,16$. Then our Lagrangian density (5.2) can be written as

$$L(\xi) = \rho \sum_{J=1}^{16} A_{J}V_{J}^{2}$$  \hfill (C.1)

where the $A_{J}$ are some real constants. Put $W_{J} := \sqrt{\rho}V_{J}$. Then formula (C.1) takes the form

$$L(\xi) = \sum_{J=1}^{16} A_{J}W_{J}^{2}.$$  \hfill (C.2)

According to formulae (6.9), (6.10), (B.1), (6.7) and (6.1) the components of the “vector” $W$ are expressed via the spinor field $\xi$ as

$$W_{J} = i\bar{\xi}^{a}B_{\gamma}^{ab}\partial_{\alpha}\xi^{b} - \xi^{b}B_{\gamma}^{ab}\partial_{\alpha}\bar{\xi}^{a} \over (\xi^{c}\sigma_{0cd}\xi^{d})^{1/2} \over (C.3)$$

where $B_{\gamma}^{ab}$ are some constants and summation is carried out over the spinor indices $\dot{a}=1,2$, $\dot{b}=1,2$, and over $\alpha=0,1,2,3$. Here we use bold type to indicate relativistic notation, when time $x^{0}$ is viewed as one of the coordinates in (1+3)-dimensional spacetime.

Note that for given $J$ and $\alpha$ the $2 \times 2$ matrices $B_{\gamma}^{ab}$ are Hermitian. This is because each of these matrices is a linear combination with real coefficients of the Pauli matrices $\sigma^{\beta}_{ab}$, $\beta=1,2,3$.

Our action is $S(\xi) = \int L(\xi)$ where for the sake of brevity we dropped $dx^{0}dx^{1}dx^{2}dx^{3}$. Substituting (C.3) into (C.2) and varying the spinor field $\xi$ we get

$$\delta S(\xi) = 2 \int \sum_{J=1}^{16} A_{J} \left[ i(W_{J})^{a}B_{\gamma}^{ab}\partial_{\alpha}\xi^{b} - \xi^{b}B_{\gamma}^{ab}\partial_{\alpha}\bar{\xi}^{a} \over (\xi^{c}\sigma_{0cd}\xi^{d})^{1/2} \over - (\bar{\xi}^{a}\sigma_{0\alpha\dot{a}}\xi^{\dot{a}}W_{J}) \right] W_{J} + \text{c.c.}$$

where we wrote down explicitly the terms with $\delta \bar{\xi}$ and incorporated the terms with $\delta \xi$ into the “c.c.” (complex conjugate term). Integrating the term with $\partial_{\alpha}\bar{\xi}^{a}$ by parts and taking out the common factor $\delta \bar{\xi}$ we rewrite the above formula as

$$\delta S(\xi) = 2 \int (\delta \bar{\xi}) \sum_{J=1}^{16} A_{J} \left[ i(W_{J})^{a}B_{\gamma}^{ab}\partial_{\alpha}\xi^{b} - \xi^{b}B_{\gamma}^{ab}\partial_{\alpha}\bar{\xi}^{a} \over (\xi^{c}\sigma_{0cd}\xi^{d})^{1/2} \over - (\bar{\xi}^{a}\sigma_{0\alpha\dot{a}}\xi^{\dot{a}}W_{J}) \right] + \text{c.c.}$$

Comparing this formula with formula (7.1) we conclude that the spinor field $F$ appearing in the latter is given by formula

$$F_{\dot{a}} = 2 \sum_{J=1}^{16} A_{J} \left[ i(W_{J})^{a}B_{\gamma}^{ab}\partial_{\alpha}\xi^{b} - \xi^{b}B_{\gamma}^{ab}\partial_{\alpha}\bar{\xi}^{a} \over (\xi^{c}\sigma_{0cd}\xi^{d})^{1/2} \over - (\bar{\xi}^{a}\sigma_{0\alpha\dot{a}}\xi^{\dot{a}}W_{J}) \right] + \text{c.c.}$$  \hfill (C.4)

where the $W_{J}$ are, in turn, given by formula (C.3).
If we now substitute the plane wave (8.1) into formula (C.3) we get
\[ W_J = \frac{2\bar{\zeta}^a B^a_J \sigma_0 \zeta^b}{(\bar{\zeta}^a \sigma_0 \zeta^d)^{1/2}}. \] (C.5)

Note that the above \( W_J \) are constant (do not depend on \( x \)), which simplifies the next step: substituting (8.1) into (C.4) and dividing through by the common factor \( e^{-i p \cdot x} \) we get
\[ e^{i p \cdot x} F_a = 2 \sum_{J=1}^{16} A_J \left[ \frac{2W_J B^a_J \sigma_0 \zeta^b}{(\bar{\zeta}^a \sigma_0 \zeta^d)^{1/2}} - \frac{W_J^2 \sigma_0 \zeta^b}{2\bar{\zeta}^a \sigma_0 \zeta^d} \right]. \] (C.6)

The remarkable feature of formula (C.6) is that its RHS is constant, i.e. it does not depend on \( x \). This completes the proof of Lemma 8.1.

Let us now substitute the plane wave (8.1) directly into our Lagrangian density (C.2). Our Lagrangian density takes the form
\[ L(\zeta; p) = \sum_{J=1}^{16} A_J W_J^2 \] (C.7)
where the \( W_J \) are given by formula (C.5). The Lagrangian density (C.7) does not depend on \( x \). The dynamical variable in this Lagrangian density is the constant 2-component complex spinor \( \zeta \), whereas the relativistic 4-momentum \( p \) plays the role of a parameter. Varying the spinor \( \zeta \) we get
\[ \delta L(\zeta; p) = 2 \sum_{J=1}^{16} A_J \left( \frac{2(\delta \bar{\zeta}^a) B^a_J \sigma_0 \zeta^b}{(\bar{\zeta}^a \sigma_0 \zeta^d)^{1/2}} - \frac{(\delta \bar{\zeta}^a) \sigma_0 \zeta^b}{2\bar{\zeta}^a \sigma_0 \zeta^d} W_J \right) W_J + \text{c.c.} \]
Comparing this formula with formula (8.16) we conclude that the constant spinor \( G \) appearing in the latter is given by formula
\[ G_a = 2 \sum_{J=1}^{16} A_J \left[ \frac{2W_J B^a_J \sigma_0 \zeta^b}{(\bar{\zeta}^a \sigma_0 \zeta^d)^{1/2}} - \frac{W_J^2 \sigma_0 \zeta^b}{2\bar{\zeta}^a \sigma_0 \zeta^d} \right]. \] (C.8)

It remains to observe that the right-hand sides of formulae (C.6) and (C.8) are the same. This completes the proof of Lemma 8.2.