A stochastic turbidostat model with Ornstein-Uhlenbeck process: dynamics analysis and numerical simulations

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Abstract Many turbidostat models are affected by environmental noise due to various complicated and uncertain factors, and Ornstein-Uhlenbeck process is a more effective and precise way. We formulate a stochastic turbidostat system incorporating Ornstein-Uhlenbeck process in this paper and develop dynamical behavior for the stochastic model, which includes the existence and uniqueness of globally positive equilibrium, sufficient conditions of the extinction, the existence of a unique stationary distribution and an expression of density function of quasi-stationary distribution around the positive solution of the deterministic model.

Keywords Stochastic turbidostat model · Ornstein-Uhlenbeck process · Extinction · Stationary distribution · Density function

1 Introduction

It is reported that oil occupies a crucial position in social development, while oil spills occur frequently, causing serious pollution to the ecological environment. For example, on April 20, 2010, the oil spill in the Gulf of Mexico in the USA caused great concern from the international community and affected the survival of many marine organisms. In order to recover the polluted marine organisms, microbial remediation is adopted. It has the advantages of no secondary pollution, short time, low cost and so on. It is, therefore, extremely urgent to cultivate microorganisms. We know that the chemostat, composed of three parts: nutrition container, culture container and collection container, is a device for cultivating microorganisms. After continuous improvement and optimization,
now most people use turbidostat instead of chemostat to cultivate microorganisms, because it has a better negative feedback mechanism [1–5]. Thus, in this paper, we consider the dynamic behavior for the microorganisms in the turbidostat model.

To the best of our knowledge, the turbidostat model may inevitably be affected by environmental noise due to uncertain factors such as temperature, pH, growth factor [4–6]. Recently, Yu et al. [4] indicated that stochastic model with discrete delay and random perturbation can be used to explore and study the dynamics of microorganism. Yu et al. [6] discussed dynamical behavior of a multispecies turbidostat model with the white noise. The maximum growth rate, however, will not always increase or decrease; it will always fluctuate between the average value. The mean reversion refers to the trend that stock prices will return to value center with high probability whether it is higher or lower than value center (or mean). The mean-reverting Ornstein-Uhlenbeck process, therefore, better describes the maximum subjected to disturbance interventions [7,8,10]. For instance, Zhang et al. [7] formulated the stochastic chemostat system incorporates with Monod-Haldane response function and Ornstein-Uhlenbeck process and analyzed dynamical behavior of the system. They obtained the sufficient conditions for the extinction and persistence of SIS system with color noise (i.e., mean-reverting Ornstein-Uhlenbeck process) in [8].

In this context, the main contributions are as follows: (i) The mean-reverting Ornstein-Uhlenbeck process is considered in the turbidostat system. (ii) Dynamics analysis and numerical simulations for a stochastic turbidostat system are obtained. There are, in addition, two questions to be solved with regard to this paper: (i) What are the sufficient conditions for extinction and permanence for the turbidostat system incorporating Ornstein-Uhlenbeck process? (ii) How to control the speed of reversion and the intensity of volatility to promote the growth of microorganisms?

To answer the above-mentioned questions, we present the following systems and results: the stochastic turbidostat model incorporating the mean-reverting Ornstein-Uhlenbeck process is formulated, and basic concepts are given in Section 2. In Section 3, we discuss the main results with regard to dynamics of the system, including the existence and uniqueness of globally positive equilibrium, the sufficient conditions for extinction of microorganisms, asymptotic stability and density function of the stochastic system. In Section 4, some numerical examples are used to support our theoretical results. We make a brief summary of this paper in the last part.

2 The model formulation and basic concepts

2.1 The model formulation

Assuming that microorganisms grow in the form of Contois, Yao et al. [3] investigated the following turbidostat system with time delay:

\[
\begin{align*}
\frac{dS}{dt} &= (S_0 - S(t))(d + kx(t)) - \frac{1}{\gamma} \frac{mS(t)x(t)}{ax(t) + S(t)} dt, \\
\frac{dx}{dt} &= x(t) \left[-(d + kx(t)) + \frac{mS(t - \tau)}{ax(t - \tau) + S(t - \tau)} \right] dt,
\end{align*}
\]

(1)

where \(S(t)\) and \(x(t)\) are the concentration of nutrient and microorganism, respectively. In system (1), assume that all parameters are positive and their biological significance is shown in Table 1.

For the convenience of discussion, the dimensionless variables are introduced, let

\[
S = \tilde{S}S_0, \quad x = \tilde{x}\gamma S_0, \quad k = \frac{\tilde{k}}{\gamma S_0}, \quad a = \tilde{a},
\]

\[
m = \tilde{m}, \quad t = \tilde{t},
\]

simplify symbols by omitting the checks, model (1) becomes:

| Parameters | Biological meanings |
|------------|---------------------|
| \(S_0\)  | The input concentration of the nutrient |
| \(d\)    | The flow volume     |
| \(\gamma\)| The yield constant   |
| \(d + kx(t)\)| The dilution rate   |
| \(m\)    | The maximum growth rate |
| \(a\)    | The half saturation constant |
| \(\tau\) | The time delay of the growth response of the microorganism |
A stochastic turbidostat model

We consider the case of $\tau = 0$, since $S(t) + x(t) = 1$, model (2) is simplified to analyze a one-dimensional system as follows:

$$
\begin{align*}
\frac{dx(t)}{dt} & = x(t) \left[ -(1 + kx(t)) + \frac{m(1 - x(t))}{ax(t) + 1 - x(t)} \right] dt, \\
\frac{dm(t)}{dt} & = \alpha(m - m(t)) dt + \beta dB(t),
\end{align*}
$$

(3)

In the process of cultivating microorganisms, the maximum growth rate is inevitably affected by some random factors. Therefore, in recent years, the study of stochastic microorganism systems has received extensive attention. The general method is to establish a stochastic microorganism system by introducing white noise. However, in this paper, we will discuss the stochastic turbidostat system with mean-reverting Ornstein-Uhlenbeck process. For system (3), we assume that $m$ is the Ornstein-Uhlenbeck process in a randomly varying environment, which is in the following form:

$$
\frac{dm(t)}{dt} = \alpha(\bar{m} - m(t)) dt + \beta d B(t),
$$

(4)

where $\alpha$ and $\beta$ denote the speed of reversion, the intensity of volatility, respectively. $\bar{m}$ represents the average regression level of $m$, and $B(t)$ is standard Brownian motion. $\alpha > 0$, $\beta > 0$, and $\bar{m} > 0$. In financial economics, the form of Ornstein-Uhlenbeck process was first analyzed by Dixit and Pindyck in [9], also known as Dixit & Pindyck model. Therefore, the novel stochastic turbidostat system with mean-reverting Ornstein-Uhlenbeck process becomes:

$$
\begin{align*}
\frac{dx(t)}{dt} & = x(t) \left[ -(1 + kx(t)) + \frac{m(1 - x(t))}{ax(t) + 1 - x(t)} \right] dt, \\
\frac{dm(t)}{dt} & = \alpha(\bar{m} - m(t)) dt + \beta dB(t).
\end{align*}
$$

2.2 Basic concepts

The three definitions and two lemmas are as follows: Define a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ and

$$
\Theta = \{(x, m) \in \mathbb{R}_+ \times \mathbb{R} : 0 < x < 1, 
-\infty < m < +\infty\}.
$$

Next, let the Lebesgue measure space $L^1 = L^1(\mathbb{X}, \Sigma_1, \hat{m})$, where $\mathbb{X} = \Theta$, $\Sigma_1$ be a $\sigma$-algebra of Borel subsets of $\mathbb{X}$ and $\hat{m}$ be a Lebesgue measure. Let $\mathbb{D}$ contain all densities, i.e.,

$$
\mathbb{D} = \{g \in L^1 : g \geq 0, \|g\| = 1\},
$$

where $\|\cdot\|$ is $L^1$-norm. A linear mapping $P : L^1 \rightarrow L^1$ is called a Markov operator if $P(\mathbb{D}) \subset \mathbb{D}$ [12].

**Definition 2.1** ([12]). If there is a kernel $\tilde{k} : \mathbb{X} \times \mathbb{X} \rightarrow [0, \infty)$ satisfying

$$
\int_{\mathbb{X}} \tilde{k}(x, y) \hat{m}(dx) = 1, \quad \forall y \in \mathbb{X},
$$

then

$$
P g(x) = \int_{\mathbb{X}} \tilde{k}(x, y) g(y) \hat{m}(dy),
$$

$P$ is known an integral Markov operator.

If a family of Markov operators $\{P(t)\}_{t \geq 0}$ satisfies:

1. $P(0) = Id$ (Id is the identity matrix);
2. $P(t + s) = P(t) P(s)$ for every $s, t \geq 0$;
3. $\forall g \in L^1$, the function $t \mapsto P(t) g$ is continuous for the $L^1$-norm;

then it is known as a Markov semigroup.

**Definition 2.2** ([12]). The Markov semigroup $\{P(t)\}_{t \geq 0}$ is asymptotically stable if there is an invariant density $g^*$ satisfies

$$
lim_{t \rightarrow \infty} \|P(t) g - g^*\| = 0, \quad \forall g \in \mathbb{D}.
$$

**Definition 2.3** ([12]). If

$$
lim_{t \rightarrow \infty} \int_{A_1} P(t) g(x) \hat{m}(dx) = 0, \quad \forall g \in \mathbb{D}, A_1 \in \Sigma_1,
$$

then $\{P(t)\}_{t \geq 0}$ is called as sweeping.

**Lemma 2.4** ([12]). Let $\{P(t)\}_{t \geq 0}$ be an integral Markov semigroup with the kernel $\tilde{k}(t, x, y)$ and

$$
\int_{\mathbb{X}} \tilde{k}(t, x, y) \hat{m}(dx) = 1 \text{ for all } y \in \mathbb{X} \text{ and } t > 0.
$$

If

$$
\int_0^\infty P(t) g(x) dt > 0, \quad \forall g \in \mathbb{D} \text{ a.e.}
$$

Then $\{P(t)\}_{t \geq 0}$ is asymptotically stable or is sweeping concerning compact sets.
Lemma 2.5 ([11]). For the equation:
\[ \Lambda^2 + A\Sigma + \Sigma A^T = 0, \]
where \( \Sigma \) is a symmetric matrix, \( A = (a_{ij})_{3 \times 3} \) and the diagonal matrix \( \Lambda = \text{diag}(\sigma_1, \sigma_2, \sigma_3)(\sigma_i > 0, i = 1, 2, 3) \), if the characteristic equation of \( A \) is \( \varphi_A(\lambda) = \lambda^3 + \eta_1\lambda^2 + \eta_2\lambda + \eta_3 \), which satisfies
\[ \eta_1 > 0, \quad \eta_3 > 0, \quad \eta_1\eta_2 - \eta_3 > 0, \]
then \( \Sigma \) is positive definite matrix.

3 Dynamical behavior

Firstly, we will give the theorem of the existence and uniqueness of the global positive equilibrium for system (4).

3.1 Existence and uniqueness of globally positive equilibrium

**Theorem 3.1** For any initial value \((x(0), m(0)) \in \Theta\), there exists a unique solution of system (4) and \((x(t), m(t)) \in \Theta \) for all \( t \geq 0 \) almost surely.

**Proof** We follow the method mentioned in [13] to prove the theorem. On account of the coefficients of model (4) are local Lipschitz continuous, \( \forall (x(0), m(0)) \in \Theta \), there exists a unique local solution \((x(t), m(t))\) on \([0, \tau_\epsilon]\), where \( \tau_\epsilon \) is the explosion time [14]. To show that \((x(t), m(t))\) is global, we need to deduce \( \tau_\epsilon = \inf\{t : \max(x(t), m(t)) = \infty\} \) a.s. Let the stopping time \( \tau^+\):
\[ \tau^+ = \inf\{t \in [0, \tau_\epsilon) : \min\{\ln x(t), m(t)\} \leq 1 \text{ or } \max\{\ln x(t), m(t)\} \geq 1\}, \]
where \( \inf \emptyset = \infty (\emptyset \text{ is the empty set}) \). We have \( \tau^+ \leq \tau_\epsilon \) by the definition of \( \tau^+ \). If we prove that \( \tau^+ = \infty \) a.s., then \( \tau_\epsilon = \infty \) and \((x(t), m(t)) \in \Theta \) a.s. for all \( t \geq 0 \). If \( \tau^+ < \infty \), for \( T > 0 \), then \( P(\tau^+ < T) > 0 \).

Define the \( C^2 \) function \( U : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R} \)
\[ U(x, m) = -\ln x - \ln(1 - x) + \frac{1}{2\alpha} (m - \tilde{m})^2, \]
Application of Itô’s formula [14], for \( \omega \in \{\tau^+ < T\} \), and for all \( t \in [0, \tau_\epsilon) \),
\[ dU(x, m) = LU dt + \frac{(m - \tilde{m})\beta}{\alpha} dB(t), \]
where \( LU(x, m) \) is defined by
\[ LU(x, m) \leq 1 + kx - \frac{m(1 - x)}{ax + 1 - x} + \frac{x}{1 - x} \left[ - (1 + kx) + \frac{m(1 - x)}{ax + 1 - x} \right] - (m - \tilde{m})^2 + \frac{\beta^2}{2\alpha} \leq 1 + k - \frac{(m - \tilde{m} + \bar{m})(1 - x)}{ax + 1 - x} + \frac{mx}{ax + 1 - x} - (m - \tilde{m})^2 + \frac{\beta^2}{2\alpha} \leq 1 + k + \max\{|m - \bar{m}| - (m - \tilde{m})^2\} + \bar{m} + \frac{|m|}{a} + \frac{\beta^2}{2\alpha} := H_1 > 0. \]
where \( H_1 \) is a positive constant. The other proofs are similar to [13]. Here we omit it.

In this part, we obtain the sufficient conditions for extinction of the microorganism \( x(t) \).

3.2 Extinction

**Theorem 3.2** Let \((x(t), m(t)) \) be the solution of system (4) with given initial value \((x(0), m(0)) \in \Theta \). If \( \bar{m} - 1 + \frac{\beta}{\sqrt{2\alpha}} < 0 \), then
\[ \lim_{t \to \infty} x(t) = 0 \text{ a.s.} \]
Namely, the microorganism \( x(t) \) in system (4) will extinct with probability one.

**Proof** By using Itô’s formula,
\[ d(\ln x + \frac{c_1(m - \tilde{m})^2}{2\alpha}) = \left[ -1 - kx + \frac{m(1 - x)}{ax + 1 - x} - c_1|m - \bar{m}|^2 + \frac{c_1\beta^2}{2\alpha} \right] dt + \frac{c_1(m - \tilde{m})\beta}{\alpha} dB(t) \]
\[ = \left[ -1 - kx + \frac{m(1 - x)}{ax + 1 - x} - c_1|m - \bar{m}|^2 \right] dt + \frac{c_1(m - \tilde{m})\beta}{\alpha} dB(t) \]
\[ \leq (\bar{m} - 1 + |m - \bar{m}| - c_1|m - \bar{m}|^2 + \frac{c_1\beta^2}{2\alpha}) dt + \frac{c_1(m - \tilde{m})\beta}{\alpha} dB(t) \]
\[ \leq (\bar{m} - 1 + \frac{1}{4c_1} + \frac{c_1\beta^2}{2\alpha}) dt + \frac{c_1(m - \tilde{m})\beta}{\alpha} dB(t), \]
\[ = \left[ -1 - kx + \frac{m(1 - x)}{ax + 1 - x} - c_1|m - \bar{m}|^2 \right] dt + \frac{c_1(m - \tilde{m})\beta}{\alpha} dB(t) \]
\[ \leq (\bar{m} - 1 + |m - \bar{m}| - c_1|m - \bar{m}|^2 + \frac{c_1\beta^2}{2\alpha}) dt + \frac{c_1(m - \tilde{m})\beta}{\alpha} dB(t). \]
where we choose \( c_1 = \frac{\sqrt{2 \beta}}{2 \alpha} \) and obtain

\[
d(ln x + \frac{c_1 (m - \bar{m})^2}{2 \alpha}) \leq (\bar{m} - 1 + \frac{\beta}{\sqrt{2 \alpha}})dt + \frac{c_1 (m - \bar{m}) \beta}{\alpha} dB(t),
\]

integrating it from 0 to \( t \) and dividing by \( t \), we obtain

\[
\ln x(t) - \ln x(0) \leq \ln x(0) + \frac{c_1 (m(0) - \bar{m})^2}{2 \alpha} + (\bar{m} - 1)
+ \frac{\beta}{\sqrt{2 \alpha}} + \frac{1}{\alpha} \int_0^t (m(s) - \bar{m})dB(s).
\]

since \( \bar{m} - 1 + \frac{\beta}{\sqrt{2 \alpha}} < 0 \), \( \lim_{t \to \infty} \frac{c_1 (m(0) - \bar{m})^2}{2 \alpha} + \bar{m} - 1 \)

\[
= \bar{m} - 1 + \frac{\beta}{\sqrt{2 \alpha}} < 0 \text{ a.s.,}
\]

therefore,

\[
\lim_{t \to \infty} x(t) = -\infty \text{ a.s.}
\]

Namely, the microorganism \( x(t) \) will be extinct with probability one. The proof is complete. \( \square \)

The purpose of this part is to discuss asymptotic stability of system (4).

3.3 Asymptotic stability

For any \( B \in \Sigma_1 \),

\[
P(t, x, m, B) = \text{Prob}(x(t, m(t)) \in B \mid (x(0), m(0)) = (x_0, m_0)),
\]

where \((x(0), m(0)) = (x_0, m_0)\) is the initial condition for the solution \((x(t), m(t))\) of stochastic system (4) and \(P(t, x, m, B)\) is the transition probability function.

If the distribution of \((x(t), m(t))\) is absolutely continuous for the Lebesgue measure and \( t > 0 \), then the density \( U(t, x, m) \) of \((x(t), m(t))\) satisfies the Fokker-Planck equation [15]:

\[
\frac{\partial U}{\partial t} = \frac{\beta^2}{2} \frac{\partial^2 U}{\partial m^2} - \frac{\partial (f_1(x, m)U)}{\partial x} - \frac{\partial (f_2(x, m)U)}{\partial m}.
\]

(6)

with

\[
f_1(x, m) = -x \left[ (1 + kx) - \frac{m(1 - x)}{ax + 1 - x} \right]
\]

\[
f_2(m) = \alpha(m - m).
\]

A Markov semigroup with regard to Eq. (6) is described in this paper. Set \( P(t)V(x, m) = U(t, x, m), \forall V \in \mathbb{D} \). Since the operator \( P(t) \) is a contraction on \( \mathbb{D} \), it can extend into a contraction on \( L^1 \). \( \{P(t)\}_{t \geq 0} \) have the formation of a Markov semigroup. \( \mathcal{A} \) denotes the infinitesimal generator of \( \{P(t)\}_{t \geq 0} \), namely,

\[
\mathcal{A}V = \frac{\beta^2}{2} \frac{\partial^2 (m^2 V)}{\partial m^2} - \frac{\partial (f_1(x, m)V)}{\partial x} - \frac{\partial (f_2(m)V)}{\partial m}.
\]

(7)

The adjoint operator of \( \mathcal{A} \) is as follows:

\[
\mathcal{A}^*V = \frac{\beta^2}{2} \frac{\partial^2 (m^2 V)}{\partial m^2} + \frac{\partial (f_1(x, m)V)}{\partial x} + \frac{\partial (f_2(m)V)}{\partial m}.
\]

Theorem 3.3 Let \((x(t), m(t))\) be a solution of (4) with \((x(0), m(0)) \in \Theta \). The distribution of \((x(t), m(t))\) has a density \( U(t, x, m) \), \( \forall t > 0 \). If \( \bar{m} - 1 - \frac{\beta}{\sqrt{2 \alpha}} > 0 \), then there is a unique density \( U^*(x, m) \) satisfying

\[
\lim_{t \to \infty} \int_{\Theta} [U(t, x, m) - U^*(x, m)] \, dx \, dm = 0.
\]

The proofs of Theorem 3.3 require the following four steps:

Step 1. According to Hörmander condition [15], the kernel function of the diffusion process \((x(t), m(t))\) is absolutely continuous, that is, \( \tilde{k} \in C^\infty((0, \infty) \times \Theta) \).

Lemma 3.4 For every \((x_0, m_0) \in \mathbb{R} \) and \( t > 0 \), the transition probability function \( P(t, x_0, m_0, B) \) has a continuous kernel function \( \tilde{k}(t, x, m; x_0, m_0) \) related to Lebesgue measure.
Proof Let \( \mathbf{p}(y), \mathbf{b}(y) \in \mathbb{R}^2_+ \) be vector fields, and the Lie bracket \([\mathbf{p}, \mathbf{b}]\) is also a vector field, which is expressed by
\[
[p, b]_j(y) = \sum_{i=1}^{2} (u_i \frac{\partial b_j}{\partial y_i} - b_i \frac{\partial u_j}{\partial y_i}), \quad j = 1, 2.
\]
Let
\[
\mathbf{p}(x, m) = \begin{pmatrix}
-x \left[ (1+kx) - \frac{m(1-x)}{ax+1-x} \right] \\
\alpha(m - m)
\end{pmatrix}
\]
and
\[
\mathbf{b}(x, m) = \begin{pmatrix}
0 \\
\beta
\end{pmatrix}.
\]
Then Lie bracket \([\mathbf{p}, \mathbf{b}]\) is
\[
[p, b] = \sigma \begin{pmatrix}
-\beta(1-x) \\
ax + 1 - x
\end{pmatrix}^T
\]
and we get
\[
\| [p, b] \| = \begin{vmatrix}
0 & -\frac{\beta(1-x)}{ax+1-x} \\
\beta & 0
\end{vmatrix} = -\frac{\beta^2(1-x)}{ax+1-x} \neq 0.
\]
Therefore, \( \mathbf{b} \) and \([\mathbf{p}, \mathbf{b}]\) are linearly independent on \( \Theta \), that is, the vector \( \mathbf{b} \) and \([\mathbf{p}, \mathbf{b}]\) span the space \( \Theta \) for each \((x, m) \in \Theta\). Based on the Hörmander theorem [15], the transition probability function \( \mathbb{P}(t, x_0, m_0, B) \) exists a continuous kernel function \( \tilde{k}(t, x, m; x_0, m_0) \), i.e., \( \tilde{k} \in C^\infty((0, \infty) \times \Theta) \). ♦

Step 2. Based on the support theorems [17], the kernel function \( \tilde{k} > 0 \) on \( \Theta \).

Lemma 3.5 For every \((x_1, m_1) \in \Theta \) and \((x_2, m_2) \in \Theta \), there exists a \( T > 0 \) such that \( \tilde{k}(T, x_2, m_2; x_1, m_1) > 0 \).

Proof We may use the method mentioned in [17] to verify \( \tilde{k} > 0 \). Firstly, we fix a point \((x_0, m_0) \in \mathbb{R} \) and a function \( \phi \in L^2([0, T]; \mathbb{R}) \) and obtain the following integral system:
\[
x_\phi(t) = x_0 + \int_0^t f_1(x_\phi(s), m_\phi(s)) ds, \\
m_\phi(t) = m_0 + \int_0^t [f_2(m_\phi(s)) + \beta \phi] ds,
\]
where \( f_1(x, m) \) and \( f_2(m) \) as shown in (6). The Fréchet derivative of the function \( h \mapsto N_{\phi+h}(T) : L^2([0, T]; \mathbb{R}) \to \mathbb{R} \) is expressed by \( D_{x_0, m_0; \phi} \), where \( N_{\phi+h} = (x_{\phi+h}, m_{\phi+h})^T \). If \( \text{rank}(D_{x_0, m_0; \phi}) = 2 \) for some \( \phi \in L^2([0, T]; \mathbb{R}) \), then \( \tilde{k}(T, x, m; x_0, m_0) > 0 \) for \( N = N_{\phi}(T) \) holds. Let
\[
\Psi(t) = f'(N_{\phi}(t)) + \phi \mathbf{g}'(N_{\phi}(t)),
\]
where \( f' \) is the Jacobians of \( [f_1(x, m), f_2(m)]^T \) and \( \mathbf{g} = [0, \beta]^T \). Set matrix function \( Q(t, t_0) \) satisfies
\[
\frac{dQ(t, t_0)}{dt} = \Psi(t) Q(t, t_0).
\]
Then
\[
Q(t, s) = I + \Psi(T)Q(T-s) + o(T-s).
\]
Therefore,
\[
D_{x_0, m_0; \phi} = \epsilon \mathbf{g} - \frac{1}{2} \epsilon^2 \Psi(T) \mathbf{g} + o(\epsilon^2),
\]
then compute
\[
\Psi(T) \mathbf{v} = \begin{pmatrix}
-(1+kx) + \frac{m(1-x)}{ax+1-x} & \frac{1-x}{ax+1-x} - \alpha \\
0 & -\alpha
\end{pmatrix} \begin{pmatrix}
0 \\
\beta
\end{pmatrix} = \begin{pmatrix}
\frac{\beta(1-x)}{ax+1-x} \\
-\alpha \beta
\end{pmatrix},
\]
thus
\[
|\Psi(T) \mathbf{v}| = \begin{vmatrix}
0 & \frac{\beta(1-x)}{ax+1-x} \\
\beta & -\alpha \beta
\end{vmatrix} = \frac{\beta^2(1-x)}{ax+1-x} \neq 0,
\]
therefore, \( \mathbf{v} \) and \( \Psi(T) \mathbf{v} \) are linearly independent. So \( \text{rank}(D_{N_{\phi}; \phi}) = 2 \).

In addition, we claim that there exist a control function \( \phi \) and \( T > 0 \) such that \( X_{\phi}(0) = X_0, X_{\phi}(T) = X \) for any two points \( X_0 \in \mathbb{R}^2_+ \) and \( X \in \mathbb{R}^2_+ \) holds, then system (8) becomes:
\[
x_{\phi}(t) = f_1(x_{\phi}, m_{\phi}), \\
m'_{\phi}(t) = f_2(m_{\phi}) + \beta \phi.
\]
Firstly, we seek out a positive constant \( T \) and a \( C^2 \)-function \( x_\phi : [0, T] \to (0, +\infty) \), which satisfies
\[
m_{\phi}(t) = (x_{\phi}'(t) + (1+kx_{\phi}(t))x_{\phi}(t)) \left( \frac{ax_{\phi}(t) + 1 - x_{\phi}(t)}{x_{\phi}(t)(1-x_{\phi}(t))} \right) > 0,
\]
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and the boundary value conditions are satisfied as follows: $x_0(0) = x_0$, $x_0(T) = x$, $m_0(0) = m_0$, $m(T) = m$. Finally, from the second equation of (9), there is a continuous control function $\phi$:

$$\phi = \frac{1}{\beta}[m_\phi'(t) - \alpha(\bar{m} - m_\phi(t))].$$

This completes the proof. \(\square\)

**Step 3.** The Markov semigroup is asymptotically stable or is sweeping concerning compact sets by Lemma 2.4.

**Lemma 3.6** The Markov semigroup $\{P(t)\}_{t \geq 0}$ is asymptotically stable or is sweeping concerning compact sets.

**Proof** From Lemma 3.4, we know that $\{P(t)\}_{t \geq 0}$ is an integral Markov semigroup with respect to a continuous density $\tilde{k}(t, x, m)$ for $t > 0$. And by Lemma 3.5, we know for every $g \in \mathbb{D}$,

$$\int_0^\infty P(t) dt > 0, \text{ a.s. on } \mathbb{R}^2_+,$$

since $\tilde{k}(t, x, m) > 0$ and $P(t)g = \int_{\mathbb{R}^2_+} \tilde{k}(t, x)g(x)\tilde{m}(dx)$ with $x = (x, m)$. Thus, on the basis of Lemma 2.4, we get the result of the above lemma. \(\square\)

**Step 4.** For the sake of excluding sweeping, we prove the existence of the Khasminskii function.

**Lemma 3.7** The Markov semigroup $\{P(t)\}_{t \geq 0}$ is asymptotically stable if $\bar{m} - 1 - \frac{\beta}{\sqrt{2\alpha}} > 0$.

**Proof** According to Lemma 3.6, the semigroup $\{P(t)\}_{t \geq 0}$ satisfies the Foguel alternative. For excluding sweeping, we design a Khasminskii function $V$ [16] and a closed set $\Gamma \in \Theta$ such that

$$\sup_{(x, m) \in \Theta \setminus \Gamma} \mathcal{A}V < 0.$$

Define a nonnegative $C^2$-function $V$:

$$V(x, m) = M(-\ln x + \frac{c_2(m - \bar{m})^2}{2\alpha}) - \ln(1 - x) + \frac{(m - \bar{m})^2}{2\alpha} := MV_1 + V_2 + V_3,$$

where $M > 0$ and $c_2 > 0$ will be determined later.

First, we compute

$$\mathcal{A}V_1 = 1 + kx - \frac{m(1 - x)}{ax + 1 - x} - c_1 |m - \bar{m}|^2 + \frac{c_1\beta^2}{2\alpha} \leq 1 + kx - \bar{m} - \frac{\bar{m}(1 - x)}{ax + 1 - x} + |m - \bar{m}| - c_1 |m - \bar{m}|^2 + \frac{c_1\beta^2}{2\alpha} \leq 1 - \bar{m} + kx + \frac{\bar{m} \max}{ax + 1 - x} + \frac{1}{4c_1} \alpha \leq -\left(\bar{m} - 1 - \frac{\beta}{\sqrt{2\alpha}}\right) + kx + \frac{\bar{m} \max}{ax + 1 - x},$$

$$\mathcal{A}V_2 = \frac{x}{1 - x} - \frac{kx^2}{1 - x} + \frac{mx}{ax + 1 - x} \leq -\frac{x}{1 - x} - \frac{kx^2}{1 - x} + \frac{mx}{ax + 1 - x} \leq -\frac{1 - x}{1 - x} + \frac{|m|}{a} = 1 - \frac{1}{1 - x} + \frac{|m|}{a},$$

and

$$\mathcal{A}V_3 \leq -(m - \bar{m})^2 + \frac{\beta^2}{2\alpha}.$$  

where we take $c_2 = \beta\sqrt{\frac{\alpha}{2}}$, combine (11), (12) and (13) and get

$$\mathcal{A}V = \mathcal{A}V_1 + \mathcal{A}V_2 + \mathcal{A}V_3 \leq M\left(-\frac{m - 1 - \frac{\beta}{\sqrt{2\alpha}}}{\sqrt{2\alpha}} + kx + \frac{\bar{m} \max}{ax + 1 - x}\right) - \frac{1}{1 - x} + (m - \bar{m})^2 \leq -2 + M\left(kx + \frac{\bar{m} \max}{ax + 1 - x}\right) - \frac{1}{1 - x} - (m - \bar{m})^2 \equiv G(x, m),$$

where $-M\left(m - 1 - \frac{\beta}{\sqrt{2\alpha}}\right) + \frac{\beta^2}{2\alpha} + 1 + \frac{|m|}{a} \leq -2$. Define a bounded closed set

$$\Gamma = \{(x, m) \in \Theta \mid \epsilon \leq x \leq 1 - \epsilon, |m| \leq \frac{1}{\epsilon}\},$$

where $0 < \epsilon < 1$ is sufficient small real numbers.
Then we have
\[
G(x, m) \leq \begin{cases} 
G(0, m) \leq -2, & \text{as } x \to 0^+, \\
G(1, m) \to -\infty, & \text{as } x \to 1^-, \\
x^{\infty}\to -\infty, & \text{as } m \to +\infty, \\
x^{\infty}\to -\infty, & \text{as } m \to -\infty.
\end{cases}
\]
All in all, there exists a closed set \( \Gamma \in \Theta \) such that
\[
sup_{(x, m) \in \Theta \setminus \Gamma} \sigma^*V \leq -1 < 0.
\]

Based on [16], the existence of a Khasminskii function shows that \( \{ P(t) \}_{t \geq 0} \) exclude sweeping from the set \( \Gamma \in \Theta \), i.e., \( \{ P(t) \}_{t \geq 0} \) is asymptotically stable. \( \square \)

### 3.4 Density function

Let \( u = x - x^* \) and \( v = m - \bar{m} \), then the corresponding linearized system of system (4) is:

\[
\begin{aligned}
\dot{u} &= (-a_{11} u + a_{12} v)dt, \\
\dot{v} &= -a_{22} vdt + \beta dB(t),
\end{aligned}
\]

where \( 0 < a < 1, a_{11} = 1 + 2kx^* + \bar{m}(a - 1)(x^*)^2 + 2m(x^*)^3 - \bar{m} > 0, a_{12} = (x^*(1-x^*))/a > 0, a_{22} = \alpha > 0. \) And let \( \Phi(u, v) \) be the density function of \((u, v)\), which can be expressed approximately by the Fokker-Planck equation:

\[
0 = \frac{\beta^2}{2} \frac{\partial^2 \Phi}{\partial v^2} - \frac{\partial}{\partial u} \left[ (-a_{11} u + a_{12} v) \Phi \right] - \frac{\partial}{\partial v} \left[ (-a_{22} v) \Phi \right].
\]

In the following, the analytic form of the density function of linear system (16) is presented.

**Theorem 3.8** If \( 0 < a < 1 \) and \( \bar{m} - 1 - \frac{\beta}{\sqrt{2a}} > 0 \), then the distribution of the solution \((u, v)\) of system (16) has a density function \( \Phi(u, v) \):

\[
\Phi(u, v) = (2\pi)^{-1/2} \left| \Sigma \right|^{-1/2} e^{-\frac{1}{2}(u, v) \Sigma^{-1}(u, v)^T},
\]

which corresponding to the marginal probability density function

\[
\Psi(u) = \frac{1}{\sqrt{2\pi \sigma_1}} e^{-\frac{u^2}{2\sigma_1^2}},
\]

where \( \Sigma^{-1} \) and \( \sigma_1 \) are defined as below. \( \Sigma \) represents the covariance matrix of \((u, v)\) and satisfies

\[
\Lambda^2 + A \Sigma + \Sigma A^T = 0.
\]

**Proof** By Lemma 2.5, we have

\[
\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix} \text{ and } A = \begin{pmatrix} -a_{11} & a_{12} \\ 0 & -a_{22} \end{pmatrix},
\]

where we know that \( \beta > 0 \) and the characteristic equation of \( A \) is

\[
\varphi_A(\lambda) = \lambda^2 + \eta_1 \lambda + \eta_0,
\]

with

\[
\eta_1 = a_{11} + a_{22} > 0, \\
\eta_2 = a_{11} a_{22} > 0.
\]

Direct calculation

\[
\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix},
\]

where

\[
\sigma_{22} = \frac{\beta^2}{2a} > 0, \sigma_{12} = \frac{a_{12} \sigma_{22}}{a_{11} + \alpha} > 0, \sigma_{11} = \frac{a_{12} \sigma_{12}}{a_{11}} > 0.
\]

Thus, \( \Sigma \) is positive definite matrix and the solution \((u, v)\) has a normal density function, which can be expressed by:

\[
\Phi(u, v) = Ce^{-\frac{1}{2}(u, v) \Sigma^{-1}(u, v)^T},
\]

where \( C \) is a positive constant satisfying \( \int_{\Theta} \Phi(u, v)du = 1, \Sigma^{-1} = \Sigma^*, \) and then

\[
\Sigma^* = \begin{pmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{21} & \sigma_{11} \end{pmatrix}.
\]

![Fig. 1](image.png) The path of \( x(t) \) for stochastic system (4) (blue line) and its corresponding deterministic model (red line) with initial parameters \((x(0), m(0)) = (0.8, 0.5)\)
\(|\Sigma| = \sigma_{11}\sigma_{22} - \sigma_{12}^2\). So the marginal probability density function is:

\[
\Psi(u) = \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{u^2}{2\sigma_1^2}},
\]

where \(\sigma_1 = \frac{\sigma_{22}}{\sigma_{11}\sigma_{22} - \sigma_{12}^2}\). The proof is complete. \(\square\)

4 Numerical Simulations

In this part, the solution of system (4) is simulated by MATLAB to verify our theoretical results. Using the Milstein’s method [18], the following discretization equation of (4) is:

\[
\left\{
\begin{array}{l}
x_{i+1} = x_i + x_i \left[-(1 + kx_i) + \frac{m(1 - x_i)}{\alpha x_i + 1 - x_i}\right] \Delta t,
\end{array}
\right.
\]

\[
\left\{
\begin{array}{l}
m_{i+1} = m_i + [\alpha(\bar{m} - m_i)] \Delta t + \beta \sqrt{\Delta t}\xi_i + \frac{1}{2}\beta^2(\xi_i^2 - 1) \Delta t,
\end{array}
\right.
\]

where \(\xi_i^2\) \((i = 1, 2, \ldots)\) are independent Gaussian random variables \(N(0, 1)\). Choose appropriate parameters and five numerical examples are provided.

Example 4.1 For stochastic system (4), we choose the following parameters:

\[a = 0.09,\ k = 0.6,\ \beta = 0.05,\ \alpha = 0.96,\ \bar{m} = 0.5,\]

Fig. 2 The path of \(x(t)\) for stochastic model (4) (blue line) and its corresponding deterministic model (red line) with initial parameters \((x(0), m(0)) = (0.8, 0.5)\) under different reversion speed \(\alpha = 0.66, 0.96, 1.26\) and 1.56.
where
\[ \bar{m} - 1 + \frac{\beta}{\sqrt{2\alpha}} = 0.5 - 1 - \frac{0.05}{\sqrt{2} \times 0.96} = -0.4639 < 0, \]
which meets the condition in Theorem 3.2. In deterministic model (3) and stochastic system (4), Figure 1 indicates that the population \( x(t) \) will eventually extinct, which is in accord with the result in Theorem 3.2.

**Example 4.2** Under the condition in Theorem 3.2, for the sake of analyzing the effect of the speed of reversion on the microorganism, we fix \( \beta = 0.05 \) first, choose the different reversion speed \( \alpha = 0.66, 0.96, 1.26 \) and 1.56. From Figure 2, with the speed of reversion \( \alpha \) rise, the number of microorganism is less, which the result consistent with the result that the speed of reversion can accelerate the extinction of \( x(t) \).

**Example 4.3** In addition, we assume that the speed of reversion \( \alpha = 0.96 \) keeps unchanged and changed the intensity of volatility \( \beta \). Figure 3 demonstrates that, contrary to the law of the speed of reversion, the smaller the value of \( \beta \), the faster the extinction of the microorganism.

**Example 4.4** In the following, we take
\[ a = 0.09, \ k = 0.6, \ \beta = 0.005, \ \alpha = 0.96, \ \bar{m} = 1.24, \]
\[ \]
where $\bar{m} - 1 - \frac{\beta}{\sqrt{2\alpha}} > 0$, which the condition in Theorem 3.3 holds. In the left-hand side of Figure 4, it shows that there exists a stationary distribution. Compared with (a) and (c), keeping $\beta = 0.005$ fixed, the speed of reversion $\alpha$ has little effect on the persistence of $x(t)$. In the right-hand side of Figure 4, keeping all the parameters fixed, the distribution of $x(t)$ is around the deterministic steady state and positively skewed with $\alpha = 0.36$ and 0.96, respectively.

**Example 4.5** For indicating the effect of the intensity of volatility on the stationary distribution, we choose the intensity of volatility $\beta = 0.01$ and 0.03, keeping $\alpha = 0.96$ fixed, which meets the condition in Theorem 3.3. Compared with (a) and (c) of Figure 5, we observe that the fluctuation of microorganism $x(t)$ becomes larger near the positive equilibrium state with the strength of volatility increasing. Moreover, the density distribution scope becomes larger in the right hand of Figure 5.
5 Conclusion

The main goal of this paper is to investigate the dynamic behavior of the stochastic turbidostat system with mean-reverting Ornstein-Uhlenbeck process. Our theoretical results and numerical experiment results indicate that: (1) The extinction rate of the microorganism $x(t)$ can be accelerated with the increase of the speed of reversion $\alpha$; it can also cause the rapid death of $x(t)$ as the intensity of volatility $\beta$ decreases. (2) The sufficient conditions for the existence and uniqueness of stationary distribution depend on the speed of rever-

sion and the intensity of volatility. Therefore, the mean-reverting Ornstein-Uhlenbeck process has significant influence on the dynamics of the stochastic turbidostat system. For answering the questions mentioned in the introduction, we list main results as follows:

(i) Based on Theorem 3.2, the microorganism population $x(t)$ of model (4) will be extinct if $\bar{m} - 1 + \frac{\beta}{\sqrt{2a}} < 0$.

(ii) By Theorem 3.3, there is a unique stationary distribution if $\bar{m} - 1 - \frac{\beta}{\sqrt{2a}} > 0$.

(iii) By Theorem 3.8, system (16) has a density function $\Phi(u, v)$ if $0 < \alpha < 1$ and $\bar{m} - 1 - \frac{\beta}{\sqrt{2a}} > 0$. 

Fig. 5 Figures on the left hand show the path of $x(t)$ for stochastic turbidostat model (4) (blue line) and its corresponding deterministic model (red line). Figures on the right hand present histograms of the probability density function of $x(100)$ with $\beta = 0.01$ and 0.03.
Some other interesting issues concerning this study deserve further discussion. In this paper, the dynamics of turbidostat system with mean-reverting Ornstein-Uhlenbeck process is analyzed. Other population systems with mean-reverting Ornstein-Uhlenbeck process, such as phytoplankton zooplankton system, are worth studying.

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Author contributions DJ designed the research and methodology. XM wrote the original draft. All authors read and approved the final manuscript.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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