Relativistic resonance and decay phenomena

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Abstract. The exact relation $\tau = \hbar/\Gamma$ between the width $\Gamma$ of a resonance and the lifetime $\tau$ for the decay of this resonance could not be obtained in standard quantum theory based on the Hilbert space or Schwartz space axiom in non-relativistic physics as well as in the relativistic regime. In order to obtain the exact relation, one has to modify the Hilbert space axiom or the Schwartz space axiom and choose new boundary conditions based on the Hardy space axioms in which the space of the states and the space of the observables are described by two different Hardy spaces. As consequences of the new Hardy space axioms, one obtains, instead of the symmetric time evolution for the states and the observables, asymmetrical time evolutions for the states and observables which are described by two semi-groups. A relativistic resonance obeying the exponential time evolution can be described by a relativistic Gamow vector, which is defined as superposition of the exact out-plane wave states with a Breit-Wigner energy distribution of the width $\Gamma$.

1. Introduction
Resonance and decaying processes have been studied for a long time in scattering experiments. Although resonance and decaying state are studied in two different processes, resonance in the scattering process and decaying state in the decay process, they are believed to be two aspects of the same phenomenon. It means that the lifetime of the decaying state is mathematically inverse width of the resonance. This belief came first in non-relativistic scattering theory [1] using the Hilbert space axiom for which an approximate relation between lifetime of the decaying state and the width of resonance was obtained. Khalfin [2] showed that in the standard quantum theory (based on Hilbert space axiom), there exists no vector with the exponential time evolution, i.e. it is impossible to obtain the exact relation between the width of the resonance and the lifetime of the decaying state and the width of resonance was obtained. Khalfin [2] showed that in the standard quantum theory (based on Hilbert space axiom), there exists no vector with the exponential time evolution, i.e. it is impossible to obtain the exact relation between the width of the resonance and the lifetime of the decaying state in the quantum theory using the Hilbert space axiom. Such a vector, called Gamow vector, with an exponential time evolution property was introduced by Gamow [3]. However, the Gamow vector has not been favored by physicists since there are no complex eigenvalues for a self-adjoint Hamiltonian $H$ of physical system in the conventional quantum physics.

The time evolution of the solutions of the dynamical equations (Schrödinger equation for the state in the Schrödinger picture or Heisenberg equation for the observable in the Heisenberg picture) extends over $-\infty < t < \infty$. This time-symmetric evolution is a result of the famous mathematical theorem (Stone-von Neumann theorem [4]) for the Hilbert space. The time-symmetric evolution of the observable (in the Heisenberg picture) or the state (in the Schrödinger picture) leads to the result, that the Born probability to detect the observable in the state exists for all time, $-\infty < t < \infty$. This means that the detectors can detect the observables (or products of decay) for all the time, even before the apparatus are turned on (e.g. an accelerator.
in the scattering experiment). This is in contrast to the causality condition for the scattering experiment since the detector can not detect anything until the experiment is prepared and the apparatus (e.g. accelerator) are turned on.

In order to obtain the unified theory of resonances and decaying states, one has to modify the Hilbert space and also the Schwartz space boundary condition for the space of the states and the space of the observables, and use a new boundary condition, called Hardy space axioms, which distinguishes between the space of the states and the space of the observables. The space of states and the space of observables are mathematically described by two different Hardy spaces, Hardy space on the lower and on the upper complex energy plane second sheet of the $S$-matrix, respectively. The result of the Hardy space axioms is that, one can get an exact relation between the width of the resonance and the lifetime of corresponding decaying state. Furthermore, one predicts that the probability to detect an observable in a state exists only after a finite time $t_0(=0)$ at which the state has been prepared. In relativistic regime, the states and the observables can be represented by the two different Poincaré semi-groups, one is the forward light cone semi-group for the observables and another is the backward light cone semi-group for the states.

2. Overview about Resonance and Decaying State

A resonance is defined as an intermediate state in scattering processes, e.g.:

$$a + b \rightarrow R \rightarrow c + d + \cdots$$  \hspace{1cm} (1)

In the scattering process (1), two beams of particles, i.e., $a$ and $b$, are accelerated through an accelerator and collide (for example, head-on collisions in electron-positron or proton-anti proton experiments), or just one beam of particles $a$ is accelerated by the accelerator and hits the target $b$ which is at rest. The resonance $R$ is formed after the collisions and then decays into the products $c, d, \cdots$ which are registered by detectors.

A resonance $R$ is characterized by a mass $M_R$ and a width $\Gamma_R$. The mass $M_R$ and width $\Gamma_R$ of the resonance $R$ are experimentally measured by fitting the partial cross section $\sigma_j(W)$ to the Breit-Wigner line shape with a slowly varying background $B(W)$. The partial cross section $\sigma_j(W)$ is a function of the center-of-mass energy:

$$\sigma_j(W) \sim \frac{r}{W - (M_R - i\Gamma_R/2)} + B(W)^2.$$  \hspace{1cm} (2)

Here, $W$ is the square root of the center-of-mass energy $s$, $W = \sqrt{s}$.

The width $\Gamma_R$ of a resonance $R$ can theoretically be related by the decay rate $\Gamma$ of resonance $R$ to final states $f$ using quantum field theory [5]:

$$\Gamma = \frac{1}{2m} \sum_f \int d\Pi_f |\mathcal{M}(R \rightarrow f)|^2.$$  \hspace{1cm} (3)

Here, $d\Pi_f$ is the relativistically invariant phase space of the final states $f$ and $\mathcal{M}(R \rightarrow f)$ is a scattering amplitude of resonance $R$ which decays into the final states $f$. The accuracy of the value of decay rate depends on the approximation used in calculation of scattering amplitude $\mathcal{M}(R \rightarrow f)$, i.e., the order in the Feynman diagrams used to calculate the matrix $\mathcal{M}(R \rightarrow f)$.

A decaying state $D$ appears in decay process:

$$D \rightarrow \eta + \eta' \cdots$$  \hspace{1cm} (4)

In the decay experiment, the decay products $\eta, \eta', \cdots$, are counted by detectors which may be constructed to observe some specific particles.
A decaying state $D$ is characterized by a mass $m_D$ and a lifetime $\tau_D$. The lifetime $\tau_D$ of the decaying state $D$ is experimentally obtained by fitting the counting rate $N_\eta/N(t)$ of the product $\eta$ in the time interval $\Delta t_i$ to the exponential law for a partial decay rate of product $\eta$, $R_\eta(t)$:

$$R_\eta(t) \approx \frac{1}{N} \frac{\Delta N_\eta(t_i)}{\Delta t_i} \sim e^{-t/\tau}.$$  \hspace{1cm} (5)

Here, $N$ is the total number of decay products of the decaying state $D$, i.e., it is sum of all partial decay product $\eta$, $\eta', \cdots$ which are counted by the detectors:

$$N = \sum_{\text{all } \eta \text{ decay products}} N_\eta.$$ \hspace{1cm} (6)

The observed partial decay rate $\frac{1}{N} \frac{\Delta N_\eta(t_i)}{\Delta t}$ is fitted by (5) to the exponential time evolution $e^{-t/\tau}$. From this fit, the lifetime $\tau_D$ of the decaying state $D$ is determined. Physicists believe that the resonance and decaying state are the same entities, i.e, they are two different aspects of the same phenomenon and the lifetime $\tau$ can be calculated as inverse width $\Gamma$, and vice versa:

$$\tau = \frac{\hbar}{\Gamma}.$$ \hspace{1cm} (7)

The relation (7) between the lifetime $\tau$ and the width $\Gamma$ was first obtained in non-relativistic quantum physics in the zero order of probability to detect the observable $\psi$ in the state $\phi(t)$ (by Goldberger and Watson) using Weisskopf-Wigner methods [1]:

$$|\langle \psi | \phi(t) \rangle|^2 = \mathcal{P}_{\phi(t)}(\psi) \sim e^{-\frac{\Gamma t}{\tau}} (1 + \frac{\Gamma t}{\tau} + \frac{3\Gamma^2 t^2}{2\tau^2} + \frac{\Gamma^3 t^3}{3\tau^3} + \cdots) \sim e^{-\frac{\Gamma t}{\tau}} = e^{-\frac{t}{\tau}} \text{ where } \tau \approx \hbar/\Gamma.$$ \hspace{1cm} (8)

There is a problem in the standard (Hilbert space) quantum theory that the exact relation between the lifetime $\tau$ and the width $\Gamma$ in (7) can not be obtained because the exponential time evolution is not possible in the Hilbert space (Khalfin’s theorem) [2]. In contrast to this theorem based on the Hilbert space, experimental data show that the decay probability of quantum systems is in good agreement with the exponential law, if the decaying state can be isolated from the background, e.g, in $K^0_L$-decay experiment in the relativistic regime [6].

Another problem in standard quantum theory is that Born’s probability to detect an observable $A(t) = |\psi(t)\rangle\langle\psi(t)|$ in the state $\phi$:

$$\mathcal{P}_\phi(A(t)) = \text{Tr}(|\psi(t)\rangle\langle\psi(t)||\phi\rangle\langle\phi|) = |\langle \psi(t) | \phi \rangle|^2 = |\langle e^{-iHt} \psi(0) | \phi \rangle|^2, \text{ } -\infty < t < +\infty$$ \hspace{1cm} (9)

is theoretically predicted for all time $-\infty < t < +\infty$. This prediction (9) is a mathematical consequence (by Stone-von Neumann theorem [4]) of the Hilbert space boundary condition for the solutions of the Schrödinger equation for the state $\phi$ or of the Heisenberg equation for the observable. However, the prediction (9) violates the causality, which states that a state $\phi$ must be prepared first (e.g, by an accelerator) before the observable $\psi(t)$ (for decay products) can be measured (e.g, by detectors) in the scattering experiment.

### 3. Hardy space boundary condition for the quantum system

The causality principle suggests that in the scattering experiment, the detectors are only able to detect the observables (e.g, decay products) after the state has been prepared (e.g after the accelerator is turned on and the beam has hit the fixed target, or two accelerated beams are guided to make head-on collision). This means that the Born probability to detect an observable...
\[ \Lambda(t) = |\psi(t)\rangle\langle\psi(t)| \] in the state \( \phi \) can experimentally not be for all time \(-\infty < t < +\infty\), but only for the time \( t \) after the finite time \( t_0 \) at which the state has been prepared, i.e., for \( t \geq t_0 \).

The experimental Born probability to detect an observable \( \psi(t) \) in the state \( \phi \):

\[ \mathcal{P}_\phi(\psi(t))) = |\langle\psi(t)|\phi\rangle|^2 = |\langle e^{-iH(t-t_0)}\psi(t_0)|\phi\rangle|^2, \] (10)

exists only for the time \( t \) after the time \( t_0 \), \( t_0 \leq t < +\infty \). The finite time \( t_0 \) can mathematically be set as zero, \( t_0 = 0 \), then the probability (10) can be rewritten as

\[ \mathcal{P}_\phi(\psi(t))) = |\langle\psi(t)|\phi\rangle|^2 = |\langle e^{-iHt}\psi|\phi\rangle|^2, \] for the time \( 0 \leq t < +\infty \). (11)

From (11), it is clear that the time evolution of the observable \( \psi(t) \) can not be described by the time-symmetric (unitary) group which is a consequence of the Stone-von Neumann theorem for the solutions of the Heisenberg equation using the Hilbert space or the Schwartz space axiom. Hence, the time evolution needs to be given by the semi-group with the finite starting time \( t_0 \) (which can be chosen as \( t_0 = 0 \)):

\[ \psi(t) = e^{-iHt}\psi, \] for the time \( 0 \leq t < +\infty \). (12)

Theoretically, this semi-group time evolution (12) can not be obtained from the conventional quantum mechanics using the Hilbert space axiom or the Schwartz space axiom. But the semi-group time evolution like (12) can be obtained using the Hardy space axioms [7].

The Hardy space axiom asserts that:

The space of in-states \( \{\phi^+\} \) is the Hardy space \( \Phi_- \) of the lower complex \( s \)-plane \( \mathbb{C}_- \) on second sheet of \( S \)-matrix:

\[ \{\phi^+\} = \Phi_- \subset \mathcal{H} \subset \Phi^X_- . \] (13a)

Space of observables \( \{\psi^-\} \) is the Hardy space \( \Phi_+ \) of the upper complex \( s \)-plane \( \mathbb{C}_+ \) on second sheet of \( S \)-matrix:

\[ \{\psi^-\} = \Phi_+ \subset \mathcal{H} \subset \Phi^X_+ . \] (13b)

Here, we use the superscriptions \(+\) in \( \phi^+ \) to describe the prepared in-states and \(-\) in \( \psi^- \) to describe the registered observables. But the subscripts \(-\) and \(+\) in \( \Phi_{\pm} \) to describe the Hardy space of the lower and upper complex \( s \)-plane \( \mathbb{C}_{\pm} \) on the second sheet of \( S \)-matrix, respectively.

These Hardy spaces \( \Phi_{\pm} \) in (13) are mathematically represented by the smooth Hardy functions rather than the \( L^2 \)-integrable functions of the invariance mass \( s = p^\mu p_\mu \) used in the Hilbert space. The Hardy space axiom requires that the functions of \( s = p^\mu p_\mu \) in the upper and lower half-plane of the second sheet of the \( s \)-surface are smooth Hardy functions [8]:

- for the observable \( \psi^- \in \Phi^+ \iff \psi^-(s) = \langle \tilde{p} j_3[sj], \eta | \psi^- \rangle \in \tilde{\mathcal{S}} \cap \mathcal{H}^2_{\pm}|_{\mathbb{R}_{s_0}} \otimes \tilde{\mathcal{S}}(\mathbb{R}^3) \), (14a)

- for the in-state \( \phi^+ \in \Phi^- \iff \phi^+(s) = \langle \tilde{p} j_3[sj], \eta | \phi^+ \rangle \in \tilde{\mathcal{S}} \cap \mathcal{H}^2_{\pm}|_{\mathbb{R}_{s_0}} \otimes \tilde{\mathcal{S}}(\mathbb{R}^3) \). (14b)

Here, the space \( \tilde{\mathcal{S}} \cap \mathcal{H}^2_{\pm}|_{\mathbb{R}_{s_0}} \) are not the spaces of \( L^2 \)-integrable energy wave functions but are the intersection between the Schwartz function space \( \tilde{\mathcal{S}} \) and Hardy class function \( \mathcal{H}^2_\pm \) of the lower and upper complex energy plane (for the analytic \( S \)-matrix), respectively. \( \mathbb{R}_{s_0} \) means that the value of \( s = p_\mu p^\mu \) is restricted to the physical values, \( s_0 = (m_1 + m_2 + \cdots + m_n)^2 < s < \infty \). The space \( \tilde{\mathcal{S}}(\mathbb{R}^3) \) is the Schwartz function space of the space-components of the 4-velocity vector \( \tilde{p} = p/s \).
4. Representations of Relativistic Resonance and Decaying State

A relativistic particle of the invariant mass square $s = m_p^2$ and spin $j_p$ is described by the unitary irreducible representation (UIR) of the Poincaré group $P$ labeled by invariant mass squared $s = m_p^2$ and spin $j_p$ and other quantum numbers $\eta_p$ (species quantum number, isospin, lepton number, etc.) [9]:

$$ \mathcal{P} = \{(\Lambda, x) \mid \Lambda \in SO(3,1), \det \Lambda = +1, \Lambda_0^0 \geq 0\}. \quad (15) $$

Instead of using the momentum eigenvectors $|p_j3[sj]\rangle$ as basis vectors of the irreducible representation space of $[s,j]$, one can as well use the three-space-components of the 4-velocity $\hat{p} = p/s$ and label the eigenvectors as $|\hat{p} j3[sj]\rangle$. These labels can also be used for the basis vectors of semi-group which will be discussed later. The 4-velocity kets $|\hat{p} j3[sj]\rangle$ are the eigenkets of a conjugate operator $M^\times$ of the mass operator $M = (\hat{p}_\mu P^\mu)^{1/2}$ with eigenvalue $\sqrt{s}$, a conjugate operator $\hat{P}_\mu^\times$ of the 4-velocity operator $\hat{p}_\mu = P_\mu M^{-1}$ with eigenvalue $\hat{p}_\mu$, a total angular momentum operator (spin) $j$ where the eigenvalue of $-\hat{\omega}^2 = -\hat{\omega}^\mu \hat{w}_\mu$ is $j(j+1)$, and a third-component of the total angular momentum operator $U(L(\hat{p}))\hat{w}_3 U^{-1}(L(\hat{p}))$ with eigenvalue $j_3$ [10]

$$ M^\times |\hat{p} j3[sj]\rangle, \eta) = \sqrt{s} |\hat{p} j3[sj]\rangle, \eta) , \quad \hat{P}_\mu^\times |\hat{p} j3[sj]\rangle, \eta) = \hat{p}_\mu |\hat{p} j3[sj]\rangle, \eta) , \quad -\hat{\omega}^2 |\hat{p} j3[sj]\rangle, \eta) = j(j+1) |\hat{p} j3[sj]\rangle, \eta) , \quad U(L(\hat{p}))\hat{w}_3 U^{-1}(L(\hat{p})) |\hat{p} j3[sj]\rangle, \eta) = j_3 |\hat{p} j3[sj]\rangle, \eta) . \quad (16) $$

Here $s = p_\mu p^\mu$ is invariant mass squared, $\hat{p}_\mu = p_\mu / \sqrt{s}$ is 4-velocity with $\hat{p}^\alpha = \hat{E}(\hat{p}) = \sqrt{1 + \hat{p}^2}$, and space-components $\hat{p} \in \mathcal{R}^3$, the Pauli-Lubanski operator is $\hat{\omega} = \epsilon^\mu\nu\rho\sigma \hat{P}_\mu J_{\rho\sigma}$, and $\eta$ is particle species number. Both operator $\hat{P}_\mu^\times$ and $M^\times$ are conjugate operators of $\hat{P}_\mu$ and $M$, respectively, in the dual Schwartz space $\Phi^\times$ which are unique extensions of $\hat{P}_\mu^\dagger$ and $M^\dagger$ in the Hilbert space $\mathcal{H}$ to the dual Schwartz space $\Phi^\times$ (continuous antilinear functionals) of the Schwartz space $\Phi$.

The 4-velocity kets $|\hat{p} j3[sj]\rangle, \eta)$ are elements of the dual space $\Phi^\times$. The Schwartz space $\Phi$ is a dense subspace of Hilbert space $\mathcal{H}$ and then one obtains the Schwartz space triplet (or Rigged Hilbert Space (RHS)) $\Phi \subset \mathcal{H} \subset \Phi^\times$.

The normalization of free 4-velocity eigenkets is chosen as

$$ \langle \hat{p} j3[sj], \eta | \hat{p} j3'[sj'], \eta' \rangle = 2 \hat{E}(\hat{p}) \delta(\hat{p} - \hat{p}') \delta(s - s') \delta_{j3,j3'} \delta_{\eta,\eta'} , \quad (17) $$

where $\hat{E}(\hat{p}) = E(p)/\sqrt{s} = \sqrt{1 + \hat{p}^2} = \gamma = \frac{1}{\sqrt{1 - v^2}}$.

If the relativistic particle system consists of 2-particles, which each of which is represented by an unitary irreducible representation (UIR) of Poincaré group labeled by the mass squared $s_i$, spin $j_i$, $i = 1,2$ (and by the other quantum numbers collectively denoted by quantum number $\eta_i$ (including particle species numbers)), the direct product space of the 2-particle system $\mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2$, can be reduced into a direct sum of two UIR spaces of different mass and spin, $\mathcal{H}_{12} = \mathcal{H}_{12}^\eta(s_1,j_1) \oplus \mathcal{H}_{12}^\eta(s_2,j_2)$ [10] [11].

In the center of mass system of two decay products of masses $m_1$ and $m_2$, any vector $\phi \in \Phi \subset \mathcal{H} \subset \Phi^\times$ can be expanded as a continuous linear superposition of the 4-velocity kets which is analogy to the Dirac’s basis vector expansion for momentum kets (mathematically given by Nuclear Spectral Theorem for Rigged Hilbert Space (RHS) [7]):

$$ \phi = \sum_{j3j} \int_{(m_1 + m_2)^2}^\infty ds \int \frac{d^3\hat{p}}{2\hat{p}} |\hat{p} j3[sj]\rangle, \eta) \langle \hat{p} j3[sj]\rangle, \eta|\phi) . \quad (18) $$
The interacting in- and out-states basis kets $|\hat{p} j_3[sj]\eta, n^\pm\rangle$ are obtained from the free-interaction basis kets by the Moeller wave operators $\Omega^\pm$ [12]:

$$|\hat{p} j_3[sj], \eta^\pm\rangle = \Omega^\pm|\hat{p} j_3[sj], \eta\rangle.$$  \hfill (19)

The in-state $|\hat{p} j_3[sj], \eta^+\rangle$ and out-state $|\hat{p} j_3[sj], \eta^-\rangle$ basis kets are elements of dual spaces: $|\hat{p} j_3[sj], \eta^+\rangle \in \Phi^+_x$ of the Hardy spaces $\Phi^+_x$, respectively. By the Hardy space axiom of quantum theory, these Hardy spaces $\Phi^+_x$ contain the spaces of the prepared in-state $\phi^+$ and detected or observed “out-state” $\psi^-$: $\phi^+ \in \Phi^-$ and $\psi^- \in \Phi^+$, respectively. We restrict the representation space of $[s, j]$ to the case in which 4-velocity $\hat{p}_\mu = p_\mu/\sqrt{s}$ is real while the mass $\sqrt{s}$ and therewith 4-momentum $p^\mu$ are complex. This is a restriction of generality but may be sufficient to accommodate to resonance and decay phenomena. This is called “minimally complex” representation for the 4-velocity ket [10].

In the minimally complex representation, the kets $|\hat{p} j_3[sj], \eta^\pm\rangle$ are eigenkets of the conjugate operator $\hat{P}_\mu^\pm$ of the 4 velocity operator $\hat{P}_\mu = p_\mu M^{-1}$ with real eigenvalue $\hat{p}_\mu = p_\mu/m$ and the conjugate operator $M^\times$ of the mass operator $M = (p_\mu p^\mu)^{1/2}$ with the complex eigenvalue $\sqrt{s}$, respectively:

$$M^\times |\hat{p} j_3[sj], \eta^\pm\rangle = \sqrt{s} |\hat{p} j_3[sj], \eta^\pm\rangle,$$

$$\hat{P}_\mu^\pm |\hat{p} j_3[sj], \eta^\pm\rangle = \hat{p}_\mu |\hat{p} j_3[sj], \eta^\pm\rangle.$$  \hfill (20)

The operator $M^\times$ and $\hat{P}_\mu^\pm$ are conjugate operators of $M$ and $\hat{P}_\mu$ in the dual spaces $\Phi^x_\pm$ of the Hardy spaces $\Phi^x_\pm$, respectively, which are unique extensions of $M^\dagger$ and $\hat{P}_\mu^\dagger$ in the Hilbert space $\mathcal{H}$ to the dual spaces $\Phi^x_\pm$.

As a consequence of (20), the kets $|\hat{p} j_3[sj], \eta^\pm\rangle$ are also the eigenkets of the conjugate operators $(p^\mu)^x$ of the exact 4-momentum operator $p^\mu$ with the eigenvalue $\hat{p}^\mu \sqrt{s}$:

$$(p^\mu)^x |\hat{p} j_3[sj], \eta^\pm\rangle = \hat{p}^\mu \sqrt{s} |\hat{p} j_3[sj], \eta^\pm\rangle,$$  \hfill (21)

where $\hat{p}^\mu = (\hat{p}, \hat{p}^3) = (\gamma, \gamma v)$.

Working in the rest frame of the system ($\hat{p} = 0$), the interacting in- and out-kets $|\hat{p} j_3[sj]\eta, n^\pm\rangle$ can be described as formal solutions of the Lippmann-Schwinger equation. Therefore, in analogy to the Lippmann-Schwinger equation in non-relativistic scattering, the “solution” of the Lippmann-Schwinger equation [13] in the relativistic case is

$$|0 j_3[sj]\eta, n^\pm\rangle = \left(1 + \frac{1}{s - H \pm i\epsilon} V\right) |0 j_3[sj]\eta, n\rangle.$$  \hfill (22)

The in- and out-kets in the lab frame can then be obtained from the rest frame kets by the rotation-free Lorentz “boost transformation, $\mathcal{U}(L(\hat{p}))$:

$$|\hat{p} j_3[sj], \eta^\pm\rangle = (\mathcal{U}(L(\hat{p})))^x |0 j_3[sj]\eta, n^\pm\rangle.$$  \hfill (23)

Here $L(\hat{p})$ is the rotation-free Lorentz boost which is a function of the real 4-velocity $\hat{p}^\mu$. The conjugate operator $(\mathcal{U}(L(\hat{p})))^x$ acts on the dual space $\Phi^x_\pm$ and $\mathcal{U}^\dagger$ is the restriction of the unitary operator $\mathcal{U}^\dagger$ to the Hardy spaces $\Phi^x_\pm$.

Corresponding to the two distinct Hardy spaces $\Phi^x_\pm$ for the out-observables $\psi^-$ and in-states $\phi^+$ (according to the Hardy space axiom (13)), there are two semi-groups of the Poincaré transformation. One is into the forward light cone for the Hardy space $\Phi^x_+$ of the upper complex s-plane $\mathbb{C}^+_+$ on second sheet of $S$-matrix [14]:

$$\mathcal{P}^+ = \{(\Lambda, x) \mid \Lambda \in SO(3,1), \det \Lambda = +1, \Lambda^0_0 \geq 0, x^2 = t^2 - x \geq 0, t \geq 0\}.$$  \hfill (24a)
Another semi-group of the Poincaré transformation is into the backward light cone for the Hardy space $\Phi_-$ of the lower complex $s$-plane $\mathbb{C}_+$ on second sheet of $S$-matrix:

$$\mathcal{P}_- = \{ (\Lambda, x) \mid \Lambda \in SO(3,1), \det\Lambda = +1, \Lambda_0^0 \geq 0, x^2 = t^2 - x \geq 0, t \leq 0 \}.$$ (24b)

The transformations of the 4-velocity kets $|\hat{p}_j\hat{s}_j[s]j, \eta \rangle \in \Phi_\pm$ under inhomogeneous Lorentz transformation $(\Lambda, x)$ are given by the operators:

$$\mathcal{U}_\pm^\Lambda (\Lambda, x) |\hat{p}_j\hat{s}_j[s]j, \eta \rangle = e^{-i p.x} \sum_{j_3} D_{j_3, j_3}^j \left( W(\Lambda^{-1}, p) \right) |\Lambda^{-1} \hat{p}_j\hat{s}_j[s]j, \eta \rangle \text{ for } t \geq 0, x^2 \geq 0, \quad (25a)$$

$$\mathcal{U}_\pm^\Lambda (\Lambda, x) |\hat{p}_j\hat{s}_j[s]j, \eta^+ \rangle = e^{-i p.x} \sum_{j_3} D_{j_3, j_3}^j \left( W(\Lambda^{-1}, p) \right) |\Lambda^{-1} \hat{p}_j\hat{s}_j[s]j, \eta^+ \rangle \text{ for } t \leq 0, x^2 \geq 0. \quad (25b)$$

Here, the subscripts $\pm$ of the operators $\mathcal{U}_\pm^\Lambda$ represent the transformations of the 4-velocity kets $|\hat{p}_j\hat{s}_j[s]j, \eta \rangle \pm |\hat{p}_j\hat{s}_j[s]j, \eta^+ \rangle$ under the corresponding semi-groups of Poincaré transformation into the forward light cone (25a) and into the backward light cone (25b), respectively.

5. Relativistic Gamow Vector and Time Asymmetric Evolution

The probability to detect the out-observable $\psi^-$ in the in-state $\phi^+$ can be given in term of reduced $S$-matrix element or j-th partial $S$-matrix $S_j^{\eta\eta'} (s)$:

$$(\psi^-, \phi^+) = \sum_j \int_{s_0}^{s_\infty} ds \sum_{j_3} \int \frac{d^3p}{2p} \langle \psi^- | \hat{p}_j\hat{s}_j[s]j, \eta^- \rangle S_j^{\eta\eta'} (s) \langle \hat{p}_j\hat{s}_j[s]j, \eta' | \phi^+ \rangle.$$ (26)

Here, $(\psi^-, \phi^+) j$ is the j-th partial probability which responds to the j-th partial $S$-matrix $S_j^{\eta\eta'} (s)$:

$$(\psi^-, \phi^+) = \int_{s_0}^{s_\infty} ds \sum_{j_3} \int \frac{d^3p}{2p} \langle \psi^- | \hat{p}_j\hat{s}_j[s]j, \eta^- \rangle S_j^{\eta\eta'} (s) \langle \hat{p}_j\hat{s}_j[s]j, \eta' | \phi^+ \rangle.$$ (27)

The j-th partial S-matrix $S_j^{\eta\eta'} (s)$ in (26) or (27) is then connected to the scattering amplitude of the j-th partial wave $a_j(s)$ of the scattering process:

$$S_j^{\eta\eta'} (s) = \begin{cases} 2 \ i \ a_j(s) + 1 & \text{for elastic scattering } \eta = \eta' \\ 2 \ i \ a_j(s) & \text{for reaction from } \eta' \text{ into channel } \eta \end{cases}$$ (28)

In $S$-matrix theory [15], a resonance is defined as a first order pole on 2nd sheet of the j-th partial S-matrix $S_j(s)$ at the complex value $s_R$. For simplicity, we assume that there is only one first-order pole $s_R = M_R - i \Gamma / 2$ of j-th partial S-matrix $S_j(s)$ which corresponds to one resonance in the scattering process (1). In other words, there is just one resonance created in the collisions in the scattering experiment. Then, we can expand the j-th partial S-matrix $S_j(s)$ as a series of the first-order pole $s_R = M_R - i \Gamma / 2$:

$$S_j(s) = \frac{R}{s - s_R} + R_0 + R_1(s - s_R) + \cdots$$ (29)

Therefore, the j-th partial S-matrix of the probability can be expressed in terms of a contour integral $C$ around the pole $s_R$ of $S_j(s)$ and a background which does not depend on the pole $s_R$:

$$(\psi^-, \phi^+) = \int_{s_0}^{s_\infty} ds \sum_{j_3} \int \frac{d^3p}{2p} \langle \psi^- | \hat{p}_j\hat{s}_j[s]j, \eta^- \rangle S_j^{\eta\eta'} (s) \langle \hat{p}_j\hat{s}_j[s]j, \eta' | \phi^+ \rangle$$

$$= \sum_{j_3} \int \frac{d^3p}{2p} \int_{s_0}^{s_\infty} ds \langle \psi^- | \hat{p}_j\hat{s}_j[s]j, \eta^- \rangle S_{II}(s) \langle \hat{p}_j\hat{s}_j[s]j, \eta' | \phi^+ \rangle +$$

$$+ \sum_{j_3} \int \frac{d^3p}{2p} \int_{s_0}^{s_\infty} ds \langle \psi^- | \hat{p}_j\hat{s}_j[s]j, \eta^- \rangle \frac{R}{s - s_R} \langle \hat{p}_j\hat{s}_j[s]j, \eta' | \phi^+ \rangle +$$

$$= \langle \psi^- | \psi^\eta \rangle + \sum_{j_3} \int \frac{d^3p}{2p} \int_{s_0}^{s_\infty} ds \langle \psi^- | \hat{p}_j\hat{s}_j[s]j, \eta^- \rangle \frac{R}{s - s_R} \langle \hat{p}_j\hat{s}_j[s]j, \eta' | \phi^+ \rangle .$$ (30)
Here, the term
\[ |\phi^{(b)}| = \sum_{j,s} \int_{-\infty}^{\infty} \frac{d^5p}{2p^0} \int_{s_0}^{s} ds \ |\hat{p} j_3[sj]\rangle |\eta\rangle S_{II}(s) |\hat{p} j_3[sj]\rangle |\eta\rangle |\phi^{(a)}|, \tag{31} \]
does not relate to the resonance pole \( s_R \) and therefore represents the non-resonant background term in the scattering experiment.

One can then define a relativistic Gamow ket (or 4-velocity eigenket at a pole \( s = s_R \)) as superposition of the exact “out-plane waves” \( |\hat{p} j_3[sj]\rangle |\eta\rangle \) with the Breit-Wigner energy distribution \( \frac{1}{s - s_R} \) [16]:
\[ \psi_{s_R}^G = \hat{p} j_3[sRj]|\eta\rangle = \frac{i}{2\pi} \int_{-\infty}^{\infty} ds \ |\hat{p} j_3[sj]\rangle |\eta\rangle \frac{1}{s - s_R}. \tag{32} \]

The relativistic Gamow kets \( \psi_{s_R}^G \) are transformed by the Poincaré semi-group into the forward light cone \( \mathcal{P}_+ \) (24a). Moreover, the relativistic Gamow kets are eigenvectors of the 4-momentum operators \( P^\mu \) with complex eigenvalues \( \sqrt{s_R} \):
\[ P^\mu \ |\hat{p} j_3[sRj]\rangle |\eta\rangle = \sqrt{s_R} \hat{p}_\mu \ |\hat{p} j_3[sRj]\rangle |\eta\rangle \quad \text{with} \quad \hat{p}_\mu = (\hat{p}^0, \hat{p}^j) = (\gamma, \gamma v). \tag{33} \]

Since the relativistic Gamow kets \( \psi_{s_R}^G \) are transformed under the Poincaré semi-group of forward light cone \( \mathcal{P}_+ \), the space-time translation \((I, x)\) of the Gamow ket \( \psi_{s_R}^G(t) \) is then given by:
\[ \mathcal{U}_x(I, x) |\hat{p} j_3[sRj]\rangle |\eta\rangle = e^{-i\hat{p}_x I} |\hat{p} j_3[sRj]\rangle |\eta\rangle = e^{-i\gamma \sqrt{s_R(t-x^2)}} |\hat{p} j_3[sRj]\rangle |\eta\rangle \quad \text{for} \quad t \geq 0, x^2 \geq 0. \tag{34} \]

Working in the rest frame \( \hat{p} = 0 \), a time evolution of the relativistic Gamow kets \( \psi_{s_R}^G(t') \) at time \( t' = t\gamma \) (proper time) is given by
\[ \psi_{s_R}^G(t') = e^{-iH\gamma t'/\hbar} |\hat{p} = 0, j_3[sRj]\rangle |\eta\rangle = e^{-i\sqrt{s_R} t'} |\hat{p} = 0, j_3[sRj]\rangle |\eta\rangle = e^{-iM_R t'/\hbar} e^{-(\Gamma_R/2)t'/\hbar} |\hat{p} = 0, j_3[sRj]\rangle |\eta\rangle \tag{35} \]
and is valid only for \( t' \geq 0 \). The probability to detect an observable \( \psi^-(t) \) in the Gamow ket \( \psi_{s_R}^G(t) \) in the rest frame is then given by
\[ \langle |\psi^-(t)| |\hat{p} = 0, j_3[sRj]\rangle |\eta\rangle \rangle^2 = \langle |e^{iHt'/\hbar} |\hat{p} = 0, j_3[sRj]\rangle |\eta\rangle \rangle^2 = e^{-\Gamma_R t'/\hbar} |\psi^- | |\hat{p} = 0, j_3[sRj]\rangle |\eta\rangle \rangle^2, \tag{36} \]
and only for \( t \geq 0 \) because of (34). Hence, we get for the Gamow state an exact exponential decay with a lifetime \( \tau \) which is exactly given by the width \( \Gamma = \Gamma_R \) of the corresponding scattering resonance:
\[ \tau = \frac{\hbar}{\Gamma_R}. \tag{37} \]

Summarizing: The resonances in the relativistic scattering process or unstable, decaying states in relativistic regime are described by the relativistic Gamow kets \( \psi_{s_R}^G \) defined in (32). These Gamow kets \( \psi_{s_R}^G \) are associated with the first-order pole of the \( S \)-matrix at \( s_R = M_R - i\Gamma_R/2 \). As a result of (35), the time evolution of the relativistic Gamow kets \( \psi_{s_R}^G \) is not given by an unitary group for which the time \( t \) extends for all time \( -\infty < t < \infty \), but by the semi-group (34) for which the time \( t \) extends over \( 0 \leq t < \infty \).
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