Kupczynski’s contextual setting-dependent parameters offer no escape from Bell-CHSH

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ABSTRACT
In a sequence of papers, Marian Kupczynski has argued that Bell’s theorem can be circumvented if one takes correct account of contextual setting-dependent parameters describing measuring instruments. We show that this is not true. Taking account of such parameters in the way he suggests, the Bell-CHSH inequality can still be derived. Violation thereof by quantum mechanics cannot be easily explained away: quantum mechanics and local realism (including Kupczynski’s expanded concept of local realism) are not compatible with one another. Further inspection shows that Kupczynski is actually falling back on the detection loophole. Since 2015, numerous loophole-free experiments have been performed, in which the Bell-CHSH inequality is violated, so despite any other possible imperfections of such experiments, Kupczynski’s escape route for local realism is not available.

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1 INTRODUCTION
Marian Kupczynski (“MK”) is the author of a thought-provoking paper published (2020) in the journal Frontiers in Physics [1] and entitled “Is the Moon There If Nobody Looks: Bell Inequalities and Physical Reality”. The work is built around a mathematical claim by MK which is actually false, and MK’s reasoning around this claimed assertion is also false. The proof of that is the main content of our present paper. He moreover perpetuates misunderstandings of Bell’s theorem which came up almost immediately after Bell’s first proof of his theorem was published. They are still widespread despite Bell’s published first replies [2] to his critics, and his later more detailed published accounts of his theorem [3]. All of this seems to justify publication of a “Comment” on the paper [1].

Kupczynski’s paper starts with a very extensive abstract, which makes it a very convenient starting point for the reader of the present paper. For the reader’s convenience, we reproduce MK’s abstract here, separated into three sections. First comes an introduction:

Bell-CHSH inequalities are trivial algebraic properties satisfied by each line of an $N \times 4$ spreadsheet containing $\pm 1$ entries, thus it is surprising that their violation in some experiments allows us to
speculate about the existence of non-local influences in nature and casts doubt on the existence of the objective external physical reality. Such speculations are rooted in incorrect interpretations of quantum mechanics and in a failure of local realistic hidden variable models to reproduce quantum predictions for spin polarization correlation experiments (SPCE). In these models, one uses a counterfactual joint probability distribution of only pairwise measurable random variables \((A, A', B, B')\) to prove Bell-CHSH inequalities.

Next, MK explains what is for him the heart of the matter:

In SPCE, Alice and Bob, using 4 incompatible pairs of experimental settings, estimate imperfect correlations between clicks registered by their detectors. Clicks announce the detection of photons and are coded by ±1. Expectations of corresponding random variables – \(E(AB), E(AB'), E(A'B),\) and \(E(A'B')\) – are estimated and compared with quantum predictions. These estimates significantly violate CHSH inequalities. Since variables \((A, A')\) and \((B, B')\) cannot be measured jointly, neither \(N \times 4\) spreadsheets nor a joint probability distribution of \((A, A', B, B')\) exist, thus Bell-CHSH inequalities may not be derived. Nevertheless, imperfect correlations between clicks in SPCE may be explained in a locally causal way, if contextual setting-dependent parameters describing measuring instruments are correctly included in the description.

Finally, he presents his own metaphysical conclusions regarding quantum interpretations and the foundations of quantum mechanics:

The violation of Bell-CHSH inequalities may not therefore justify the existence of a spooky action at the distance, super-determinism, or speculations that an electron can be both here and a meter away at the same time. In this paper we review and rephrase several arguments proving that such conclusions are unfounded. Entangled photon pairs cannot be described as pairs of socks nor as pairs of fair dice producing in each trial perfectly correlated outcomes. Thus, the violation of inequalities confirms only that the measurement outcomes and ‘the fate of photons’ are not predetermined before the experiment is done. It does not allow for doubt regarding the objective existence of atoms, electrons, and other invisible elementary particles which are the building blocks of the visible world around us.

The Bell-CHSH inequalities certainly can be seen as trivial algebraic properties of some very elementary mathematical structures. However, what MK calls speculations are actually arguments, and they are arguments which start with assumptions. MK’s general objections were already answered by John Bell himself in Bell (1975) [2], and the assumptions behind the Bell-CHSH inequalities have been clearly stated by Bell (1981) [3]. We will come back to these matters later.

Our focus will be on what MK sees as the content of Bell’s theorem, and summarizes as “the failure of local realistic hidden variable models to reproduce quantum predictions for spin polarization correlation experiments (SPCE)”. MK’s concept of “local realistic hidden variables models” is a restricted one, built on Bell’s original ideas (themselves inherited from EPR), in which perfect anti-correlation is used to argue for realism. Thanks to perfect anti-correlation, the principle of locality is furthermore used to argue that in the idealised EPR-B experiment, all correlation and all randomness must derive from hidden variables “originally located at the source”. MK appears to depart from this by introducing further hidden variables “located in the measurement devices” and with probability distributions which may depend on the local setting. He does not realise that allowing contextual setting-dependent parameters does not actually extend the sets of probability distributions of possible measurements allowed in the standard and
apparently restrictive “local realistic hidden variable models”. His superficially broader framework also
cannot reproduce the standard quantum predictions for spin polarization correlation experiments. The
correlations allowed within his extended model necessarily satisfy the Bell-CHSH inequalities.

The fact is that MK’s framework also enables the construction of joint probability distributions of
measurement outcomes under different possible settings. And it enables exactly the same collection of
joint probability distributions to be realised as those enabled by the old framework. The possibility of
extracting probability distributions of outcomes of feasible experiments from probability distributions of
joint outcomes of infeasible experiments is called “counterfactual definiteness” by some. Unfortunately,
the phrase “counterfactual variables” leads to the objection that the concept is unphysical. We know from
quantum mechanics that one cannot measure spin in different directions simultaneously, hence it makes no
physical sense whatsoever to introduce such variables. We will comment on this debate towards the end of
this paper. For now, we just remark that it is irrelevant to our primary claim: the Bell-CHSH inequalities do
hold in Kupczynski’s framework. It therefore cannot reproduce the standard predictions for the EPR-B
model. It cannot explain how Bell inequalities are violated in real experiments.

As John Bell himself once remarked, proofs of what is impossible often demonstrate little more than their
authors’ own lack of imagination. MK confuses the mathematical issue of existence of joint probability
distributions with given lower dimensional marginal distributions, with the metaphysical issue of the
existence of counterfactual variables.

Our paper also contains a discussion of further issues in MK’s paper. On some matters, we find ourselves
able to agree with MK, for the completely opposite reason! Specifically, because of Bell’s theorem, we
agree with MK that violation of Bell inequalities is strong indication that “entangled photon pairs cannot
be described as pairs of socks nor as pairs of fair dice” and also that “the measurement outcomes are not
predetermined”.

Our findings have consequences for a sequence of earlier papers by MK [4, 5, 6] all presenting the same
ideas in more or less detail. We chose to focus on [1] as being the most recent comprehensive exposition
of MK’s results and ideas. Similar ideas have been developed by several other authors, in particular Karl
Hess, Hans de Raedt, Andrei Khrennikov, Theo Nieuwenhuizen. Their works are all cited in [1] and to
keep this paper brief, we will not present and critique their contributions (in as much as they follow an
approach similar to MK) here.

2 KUPCZYNSKI’S FRAMEWORK

Kupczynski [1] incorporates contextual hidden variables, standing for random disturbances arising in
the measurement apparatus and dependent on the local measurement setting, as follows. Consider an
experiment in which Alice and Bob’s settings will be x and y. To begin with, hidden variables (\lambda_1, \lambda_2)
with some arbitrary joint probability mass function \( p(\lambda_1, \lambda_2) \), not depending on the local settings x and
y chosen by the experimenters, are transmitted from the source to the two measurement stations. At
Alice’s station and Bob’s station, independently of one another, and independently of (\lambda_1, \lambda_2), local hidden
variables \( \lambda_x \) and \( \lambda_y \) are created with probability mass functions \( p_x(\lambda_x) \) and \( p_y(\lambda_y) \). The measurement
outcome on Alice’s side is then \( A_x(\lambda_1, \lambda_x) \), and similarly on Bob’s side, \( B_y(\lambda_1, \lambda_y) \). The functions \( A_x \)
and \( B_y \) depend in any way whatever on x and y respectively; even the domains of these functions can vary.
The sets of possible outcomes of \( \lambda_x \) and \( \lambda_y \) may depend on x and y respectively. Now repeat this
story with, instead of \( x, y \), settings \( x, y' \), then \( x', y \), then \( x', y' \). In this way, Kupczynski has defined
the four expectation values \( E(A_x B_y) \), \( E(A_{x'} B_y) \), \( E(A_x B_{y'}) \), \( E(A_{x'} B_{y'}) \) of interest, on four “dedicated”
hidden variable spaces, which he moreover states are “disjoint”. Therefore, he is unable to define certain “counterfactual” expectations which are used in his proof in the non-contextual case of the Bell-CHSH inequalities. Does this mean that the inequalities need not hold? His argument that they could be violated is based on the huge number of free parameters which his model allows. However, he does not actually specify any particular instantiation of all those parameters which does the job. He does claim that other authors did already do just that.

But how did they do that? Here, a somewhat hidden feature of Kupczynski’s framework comes into play. Kupczynski’s measurement outcomes actually take three values: −1, +1, and 0. This can be mathematically reduced to binary outcome values ±1 by interpreting his measurement functions A, B as representing the conditional expectation values of the outcomes at each measurement station, conditional on the hidden variables carried from the source, and given the settings. His outcome “zero” would then stand for: “no particle was detected; toss an independent fair coin to get a numerical outcome ±1”. This does not change the fact that, as we will show in the next section, the CHSH inequality does hold for the correlations produced by the Kupczynski model. After some “redefining” of various mathematical variables one ends up in the standard setup with outcomes ±1 and 0, (or as we just remarked, just ±1) and binary settings x, x′ for Alice; y, y′ for Bob. The standard simple proof of the CHSH inequality applies.

However, thanks to the initial presence of the further possible outcome 0, the correlations generated by MK’s context-dependent local hidden variables model are not the ones usually of interest to the experimenter. The experimenter, typically trusting quantum mechanics and interested in the quantum mechanical physics revealed in his or her experiment, not interested in refuting local realism, would compute post-selected correlations obtained when trials with zero outcomes are discarded. The problem with this is that, as is very well known, it is possible to “fake” quantum correlations in an entirely local realistic way, provided one tolerates a sufficiently large rejection rate. The experimenter interested in rigorously testing local realism must take account of that. MK knows this too. He has a section entitled “Subtle relationship of probabilistic models with experimental protocols” in which he repeats what is well known, that (under the model of local hidden variables) the correlations E(AxB|Ax ≠ 0, By ≠ 0) need not satisfy the CHSH inequalities. Curiously, in this section MK also reveals that he does know that Bell inequalities hold for his framework of “probabilistic local causality”; see the appendix to this paper. All in all, the paper is full of self-contradictions.

MK is aware that in recent experiments these loopholes appear to have been avoided. Two of the 2015 “loophole-free” experiments (NIST, Vienna) exploited the fact that technological improvements had given the experimenters detection rates above the critical threshold of 66.67%. The other two experiments (Delft, Munich) were actually three party experiments using the technology of entanglement swapping in order to create “event-ready detectors”. In the first case the crucial theoretical ingredient was provided by the classic results of Eberhard (1993) [7], in the second case by the experimental set-up described by John Bell in his famous paper “Bertlmann’s socks and the nature of reality” (Bell, 1981) [3]. The important thing about all those four loophole free experiments, was that “zero outcomes” did not occur at all! No outcomes were discarded! Admittedly, this is a subtle issue with respect to the Delft and Munich three party experiments, in which one studies the correlations between Alice and Bob’s outcomes conditional on their settings and on a third player, Carol’s, outcome. The spatio-temporal arrangement of the experiment should be such that Carol’s outcome cannot influence Alice and Bob’s settings or outcomes without action faster than the speed of light.

MK goes on to discuss various already known weaknesses of those experiments. Indeed, the 2015 experiments were not perfect. Since then, they have been replicated and sharpened; the interest of
experimentalists has moved on to possible technological applications of loophole-free experiments (device independent quantum key distribution), where new challenges have to be met. That is all another story.

We return briefly to the metaphysical implications of the Bell-CHSH inequalities, and discussion of the physical assumptions behind them, after presenting the central mathematical content of this paper. We will show explicitly how one can derive Bell-CHSH inequalities for Kupczynski’s model. Contextual setting-dependent parameters don’t allow one to escape the Bell-CHSH inequalities.

3 THE MATHEMATICAL DETAILS

Here we will continue to use MK’s notation from \[1\]. MK says about his framework: “counterfactual expectations \(E(A_\lambda A_{\lambda'}), E(B_\lambda B_{\lambda'}), E(A_\lambda A_{\lambda'} B_{\lambda} B_{\lambda'})\) do not exist and Bell and CHSH inequalities may not be derived”. He hereby refers to the usual CHSH inequalities for the four expectations \(E(A_\lambda B_y), E(A_\lambda B_{y'}), E(A_{\lambda'} B_y), E(A_{\lambda'} B_{y'})\). The context is a Bell-type experiment in which Alice chooses between settings \(x\) and \(x'\), and Bob chooses between settings \(y\) and \(y'\). MK talks about four different Kolmogorov probability models for the four sub-experiments (one setting choice for Alice and one for Bob). Here are his expressions for the four expectation values of interest, where I have amplified his already long formulas by inserting part of the definition of the four underlying sample spaces \(\Lambda_{xy}, \Lambda_{xy'}, \Lambda_{x'y}, \Lambda_{x'y'}\).

\[
E(A_\lambda B_y) = \sum_{(\lambda_1, \lambda_2, \lambda_x, \lambda_y)\in \Lambda_{xy}} A_\lambda(\lambda_1, \lambda_x) B_y(\lambda_2, \lambda_y)p_x(\lambda_x)p_y(\lambda_y)p(\lambda_1, \lambda_2),
\]

\[
E(A_\lambda B_{y'}) = \sum_{(\lambda_1, \lambda_2, \lambda_x, \lambda_{y'})\in \Lambda_{xy'}} A_\lambda(\lambda_1, \lambda_x) B_{y'}(\lambda_2, \lambda_{y'})p_x(\lambda_x)p_{y'}(\lambda_{y'})p(\lambda_1, \lambda_2),
\]

\[
E(A_{\lambda'} B_y) = \sum_{(\lambda_1, \lambda_2, \lambda_{x'}, \lambda_y)\in \Lambda_{x'y}} A_{\lambda'}(\lambda_1, \lambda_{x'}) B_y(\lambda_2, \lambda_y)p_{x'}(\lambda_{x'})p_y(\lambda_y)p(\lambda_1, \lambda_2),
\]

\[
E(A_{\lambda'} B_{y'}) = \sum_{(\lambda_1, \lambda_2, \lambda_{x'}, \lambda_{y'})\in \Lambda_{x'y'}} A_{\lambda'}(\lambda_1, \lambda_{x'}) B_{y'}(\lambda_2, \lambda_{y'})p_{x'}(\lambda_{x'})p_{y'}(\lambda_{y'})p(\lambda_1, \lambda_2).
\]

These four equations are a complicated way to say the following: with settings \(x, y\), hidden variables \((\lambda_1, \lambda_2)\) with some arbitrary joint probability mass function \(p(\lambda_1, \lambda_2)\) (independent of \(x\) and \(y\)) are transmitted from the source to the two measurement stations. At Alice’s station and Bob’s station, independently of one another, local hidden variables \(\lambda_x\) and \(\lambda_y\) are created with probability mass functions \(p_x(\lambda_x)\) and \(p_y(\lambda_y)\). The measurement outcome on Alice’s side is \(A_\lambda(\lambda_1, \lambda_x)\), and similarly on Bob’s side, \(B_{\lambda}(\lambda_1, \lambda_y)\). Now repeat this story with, instead of \(x, y\), settings \(x, y'\), then \(x', y\), then \(x', y'\). Kupczynski has defined these expectation values on four “dedicated” hidden variable spaces, which he moreover states are “disjoint”. Therefore, he says, he is unable to defined certain “counterfactual” expectations which his proof in the non-contextual case used, and hence he can’t derive Bell and CHSH inequalities.

But we can do all that! And we can even use his original proof in the non-contextual case to get those inequalities! Here is just one of many ways.

Take as sample space a set of tuples \(\lambda = (\lambda_1, \lambda_2, \lambda_x, \lambda_{x'}, \lambda_y, \lambda_{y'})\). This space is just the Cartesian product of the spaces whose existence Kupczynski already hypothesized. Take as probability mass function on this space the product \(p(\lambda_1, \lambda_2)p_x(\lambda_x)p_{x'}(\lambda_{x'})p_y(\lambda_y)p_{y'}(\lambda_{y'})\). Finally, define new measurement functions \(A(\lambda, x) = A_x(\lambda_1, \lambda_x), B(\lambda, y) = B_y(\lambda_2, \lambda_y)\) where \(x\) can be replaced by \(x'\) and/or \(y\) by \(y'\). Now compute
E(A_xB_y), also with x replaced by x' and/or y by y'. It is immediately clear that the four new expectation values of products have exactly the same values as those just exhibited of Kupczynski’s. We can now go back to Kupczynski’s own earlier traditional derivation of Bell-CHSH. There is no barrier to running through the usual proof since all four expectations of products are defined on the same probability space.

There are more efficient constructions. As one learns in courses on Monte Carlo simulation, one can define a discrete random variable with an arbitrary probability distribution as a function of a single uniformly distributed random variable on the unit interval [0, 1]. Thus one could define \( \lambda_x \) and \( \lambda'_x \) as functions of a single uniformly distributed random variable \( U_1 \) and of a second argument \( x \) or \( x' \); similarly define \( \lambda_y \) and \( \lambda'_y \) as functions of a single, independent, uniformly distributed random variable \( U_2 \) and of a second argument \( y \) or \( y' \). We just add to our original \( (\lambda_1, \lambda_2) \) two independent random variables \( U_1, U_2 \) and redefine our measurement functions in the obvious way. In this way, we can accomodate any number of setting choices for Alice and Bob without introducing more “contextual” randomness into the two measurement functions. This is an important insight. Contextual randomness does not need randomness dependent on the setting. The setting dependence can be passed into the deterministic part of the model.

Finally a remark. MK mysteriously says “counterfactual expectations \( E(A_xA_{x'}) \), \( E(B_yB_{y'}) \), \( E(A_xA_{x'},B_yB_{y'}) \) do not exist”. We do not understand the purpose of this remark. The Bell-CHSH inequalities follow from the existence of a joint probability distribution of four random variables \( (A_x, A_{x'}, B_y, B_{y'}) \) which contain as bivariate marginals, the joint probability distributions of \( (A_x, B_y) \), \( (A_{x'}, B_y) \), \( (A_x, B_{y'}) \), \( (A_{x'}, B_{y'}) \). A typical proof of the Bell-CHSH inequalities next considers the random variable \( A_xB_y - A_{x'}B_y - A_xB_{y'} + A_{x'}B_{y'} \) which can hereby be constructed. In the case that outcomes take values in \( \{-1, +1\} \), one shows that this random variable takes values in \( \{-2, +2\} \). In the superficially more general case with outcomes in \( [-1, +1] \) one shows that it takes values in \( [-2, +2] \). In either case, the expectation value lies in \( [-2, +2] \) thus yielding two one-sided Bell-CHSH inequalities for \( E(A_xB_y) - E(A_{x'}B_y) - E(A_xB_{y'}) + E(A_{x'}B_{y'}) \). Is it so confusing to give the same names to some of a number of random variables defined on five different probability spaces.

As is well known, satisfaction of the complete set of eight one-sided Bell-CHSH inequalities together with the restriction of matching expectations of single variables, and the assumption that all random variables take values in \( \{-1, +1\} \), is a necessary and sufficient condition for the existence of the afore-mentioned coupling. That result is due to Fine (1982). Boole gave a general methodology for deriving results of this kind in his 1854 book The Laws of Thought.

4 WHAT JOHN BELL ALREADY SAID ABOUT THIS, AND WHAT MK DID GET RIGHT

As we said earlier, Bell (1975) answered critics of his work, explaining lucidly how many of those criticisms were based on simple misunderstandings. At the end of his paper he writes

The objection of de la Peña, Cetto, and Brody is based on a misinterpretation of the demonstration of the theorem. In the course of it reference is made to \( A(a', \mu), B(b', \mu) \) as well as to \( A(a, \mu), B(b, \mu) \). These authors say “Clearly, since \( A, A', B, B' \) are all evaluated for the same \( \mu \), they must refer to four measurements carried out on the same electron–positron pair. We can suppose, for instance, that \( A' \) is obtained after \( A \), and \( B' \) after \( B' \). But by no means. We are not at all concerned with sequences of measurements on a given particle, or of pairs of measurements on a given pair of particles. We are
concerned with experiments in which for each pair the ‘spin’ of each particle is measured once only.

The quantities $A(a', \mu), B(b', \mu)$ are just the same functions $A(a, \mu), B(b, \mu)$ with different arguments.

We think it is absolutely clear that what Bell is saying here, is that his hypothesis of local hidden variables entails that functions $A(\cdot, \cdot)$ and $B(\cdot, \cdot)$ exist, together with a probability distribution of some variable called $\mu$ with a probability distribution $\rho$ which does not depend on the settings, such that the “observed” joint probability distributions of pairs of variables which can be observed together, match the probability distributions predicted by the triple $(A, B, \rho)$: two functions and one probability measure. As we have previously shown, there does exist such a triple which exactly reproduces any of the sets of four probability distributions generated by Kupczynski’s model.

A later paper, Bell (1981) [3], makes absolutely clear what the hidden variable $\mu$ is supposed to stand for, only it is now denoted by $\lambda$.

It is notable that in this argument nothing is said about the locality, or even localizability, of the variable $\lambda$. These variables could well include, for example, quantum mechanical state vectors, which have no particular localization in ordinary space time.

Earlier in the same paper he writes

It seems reasonable to expect that if sufficiently many such causal factors can be identified and held fixed [RDG: together, this they form $\lambda$] . . . $\lambda$ denotes any number of other variables that might be relevant the residual fluctuations will be independent. . . . That is to say we suppose that there are variables $\lambda$, which, if only we knew them, would allow decoupling of the fluctuations.

Bell goes on to derive the CHSH inequalities, where now, his functions $A$ and $B$ would be the conditional expectations of the outcome value at each measurement location, conditional on the local setting and the sources of correlation. Thus Bell’s $A(a, \lambda)$ would be Kupczynski’s $\sum_{\lambda x} A_x(\lambda_1, \lambda x)$ with $\lambda \equiv (\lambda_1, \lambda_2)$ and $a \equiv x$.

Very early on, Bell (1971) [3] had discussed the issue of placing hidden variables also in the measuring devices. MK actually devotes a large part of a whole section of [1] to this issue, see the Appendix.

5 CONCLUSIONS

MK writes: a “... root of quantum non-locality is Bell’s insistence that the violation of Bell-type inequalities in SPCE would mean that a locally causal description of these experiments is impossible”, then quoting Bell’s famous summary “in a theory in which parameters are added to quantum mechanics to determine the results of individual measurements, without changing the statistical predictions, there must be a mechanism whereby the setting of one measuring device can influence the reading of another instrument, however remote. Moreover, the signal involved must propagate instantaneously, so that such a theory could not be Lorentz invariant.” MK adds “Bell’s statement is correct only if one is talking about an ideal EPRB which does not exist,” and “imperfect correlations in SPCE may be explained in a locally causal way if instrument parameters are correctly included in a probabilistic model”. We have shown that this claim is false. What MK calls a probabilistic model is simply, mathematically, a special case of the classical local hidden variables model of Bell. Bell inequalities hold. MK attempts to escape from this by invoking the detection loophole, but experiments have now closed the detection loophole. Naturally, there is always
room to do better experiments still, removing other imperfections, but we feel that it is wishful thinking to suppose this will never happen.

Paradoxically, precisely because of Bell’s theorem and the experimental violation of Bell-CHSH inequalities in rigorously performed experiments, we also tend to the same metaphysical conclusions as MK: violation of Bell inequalities is strong indication that “entangled photon pairs cannot be described as pairs of socks nor as pairs of fair dice”, and “the measurement outcomes are not predetermined”.

CONFLICT OF INTEREST STATEMENT
The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

AUTHOR CONTRIBUTIONS
RDG initiated this project and wrote an initial draft which he shared with JLP. It turned out that JLP had already written a critique on another work of Kupczynski together with Rodney Franco. Discussions led to many changes and to the appendix connecting the results to further material in [1]. Both authors stand behind all the content of the entire paper.

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6 APPENDIX: SOME LOGICAL ISSUES
In this appendix, we present yet another proof that MK’s “contextual approach” confirms the validity of the Bell inequality. Ironically the proof is already contained in MK’s paper in the section “Subtle relationship of probabilistic models with experimental protocols”.

In that section of his paper, MK explains how Bell derived his inequality averaging first over the measuring devices’ hidden variables [3]. We reproduce the equations of Bell’s approach for convenience

\[
E(A_x B_y) = \sum_{\lambda_1, \lambda_2, \lambda_x, \lambda_y} A_x(\lambda_1, \lambda_x) B_y(\lambda_2, \lambda_y) p_x((\lambda_x)p_y(\lambda_y)p(\lambda_1, \lambda_2)
\]

\[
= \sum_{\lambda_1, \lambda_2} \sum_{\lambda_x} A_x(\lambda_1, \lambda_x) p_x(\lambda_x) \sum_{\lambda_y} B_y(\lambda_2, \lambda_y) p_y(\lambda_y) p(\lambda_1, \lambda_2)
\]

(1)

Setting

\[
\overline{A}_x(\lambda_1) = \sum_{\lambda_x} A_x(\lambda_1, \lambda_x) p_x(\lambda_x)
\]

(2)

\[
\overline{A}_y(\lambda_2) = \sum_{\lambda_y} A_y(\lambda_1, \lambda_y) p_y(\lambda_y)
\]

(3)
We get

$$E(A_x B_y) = \sum_{\lambda_1, \lambda_2} \overline{A}_x(\lambda_1) \overline{B}_y(\lambda_2) \rho(\lambda_1, \lambda_2)$$  \hspace{1cm} (4)$$

where $|\overline{A}_x| \leq 1$ and $|\overline{A}_y| \leq 1$. Analogous results follows for $E(A'_x B_y)$, $E(A_x B'_y)$, and $E(A'_x B'_y)$, therefore the derivation with hidden variables proceeds as usual.

According to MK, while Bell’s equations (1), (2), (3), and (4) describe an impossible to implement experiment, his contextual approach corresponds to real experiments. However, he recognizes that Bell’s formulas give the same results as his “contextual model”

Although the expectations calculated using the Equations (11–14) and (19–22) have the same values, the two sets of formulas describe different experiments.

In [1], the above equations (11–14) correspond to MK’s contextual model while (19–22) describe Bell’s model. Kupczynski explicitly states that the calculated expectations of Bell and his contextual model have the same values.

The obvious conclusion is that even if we assume that MK is correct and Bell’s formulas are “impossible to implement”, since they give the same result as his “contextual model”, MK’s model describing actual experiments also satisfies the Bell inequality. Thus MK’s contextual model confirms the interpretation of most experts in the field. The empirical results prove that local realism is untenable.

Section “Joint Probabilites” in Ref. [9] discusses other logical issues regarding Kupczynski’s claims on the existence of joint probabilities.

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