Random Matrix Theory and QCD$_3$

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Abstract

We suggest that the spectral properties near zero virtuality of three dimensional QCD, follow from a Hermitean random matrix model. The exact spectral density is derived for this family of random matrix models both for even and odd number of fermions. New sum rules for the inverse powers of the eigenvalues of the Dirac operator are obtained. The issue of anomalies in random matrix theories is discussed.
In four dimensions, the spontaneous breakdown of chiral symmetry in QCD is characterized by the order parameter \( \langle \bar{q}q \rangle \) that relates to the spectral density of the QCD Dirac operator near zero virtuality, but many level spacings away from the origin \( \Pi \). In order to analyze the dynamics of the order parameter, it is therefore natural to study not only the asymptotic limit of the spectrum near zero virtuality, but also the way this limit is approached as the thermodynamic limit is taken. More concretely, we will analyze the spectrum near zero virtuality at finite space-time volume.

The spectrum of the Dirac operator fluctuates over the ensemble of gauge fields. This raises the question whether these fluctuations may be independent of the particular dynamics of the system. As we know from the study of quantum chaos and universal conductance fluctuations and from experimentally measured spectra of compound nuclei \( \Pi \), correlations between levels of the order of several level spacing are universal. Because of this they can be described by a random matrix model which has only the symmetries of the system as input. Our conjecture is that the same is true for the correlations of the eigenvalues of the Dirac operator near zero virtuality \( \Pi, \Pi, \Pi \).

Since chirality is not defined in odd space-time dimensions, the issue of spontaneous breaking is more subtle in three dimensions. However, we may still consider the possibility of a spontaneous breaking of global flavor symmetry and parity. Three dimensional QCD (\( QCD_3 \)) maybe of interest to four dimensional QCD at high temperature \( \Pi \) and to quantum antiferromagnetism \( \Pi \). It may also be used to describe certain disordered condensed matter systems \( \Pi \).

In this letter we would like to argue that the spontaneous breaking of flavor and/or parity in three dimensional QCD is related to the behavior of the spectral density near zero virtuality. Using the universality arguments presented above, we will derive the so called microscopic spectral density \( \Pi, \Pi \), from random matrix theory. However, in order to formulate the correct model we have to analyze the symmetries of the underlying theory. The outcome will be a Hermitean random matrix model which satisfies the 3-dimensional analogues of the four dimensional Leutwyler-Smilga sum rules \( \Pi \). The way the anomaly shows up in random matrix theories will be briefly discussed.
Consider $QCD_3$ with $N_f$ flavors,

$$\mathcal{L} = -\frac{1}{4} \text{Tr} F^2 + \sum_{a=1}^{N_f} \bar{q}_a \left( \hat{D} + m_a \right) q_a$$  \hspace{1cm} (1)

where $\hat{D} = \gamma_\mu D_\mu$ is the covariant derivative. The $q_a$ are two component spinors in the fundamental representation of $SU(N_c)$ Color and spin indices have been suppressed. In the limit where all the masses are equal $m_a = m$, there is a global $U(N_f)$ symmetry of which the $U(1)$ part relates to the baryon number, and is conserved independent of $m_a$. In three dimensions, the gamma matrices are Pauli matrices. There is no distinction between left and right handed fermions so that $U(N_f)$ does not have the chiral structure of four dimensions.

In three dimensions, the parity operation is a reflection about one axis, say $x_1$. This operation is implemented by $\gamma_0 \gamma_2$ on the spinors. As a result $\bar{q} \rightarrow -\bar{q}$ under parity. For massless quarks, (1) is invariant under parity. However, the Dirac operator for a given color field is not. The total symmetry group of (1) is $SU(N_c) \times U(N_f) \times Z_2$ at the classical level. If the quark masses are arranged in pairs of opposite signs but equal magnitude, then (1) is still parity preserving. Indeed, for $N_f$ even, a rearrangement of (1) gives

$$\mathcal{L} = -\frac{1}{4} \text{Tr} F^2 + \sum_{a=1}^{N_f/2} \bar{q}_a D q_a + m \left( \sum_{a=1}^{N_f/2} \bar{q}_a q_a - \sum_{b=1}^{N_f/2} \bar{q}_b q_b \right)$$  \hspace{1cm} (2)

which is $SU(N_c) \times (U(N_f/2) \times U(N_f/2)) \times Z_2$ at the tree level. Here, the discrete $Z_2$ is the product of the usual parity ($P_1$) with the interchange $a \rightarrow b$ ($E$).

At the quantum level, global anomalies can crop in. For an odd number of flavors, parity is either broken explicitly by quark masses, or radiatively in the massless limit, with the appearance of a Chern-Simons term. For an even number of flavors a rearrangement of the masses in doublets as in (2) causes the anomaly to vanish. In this case, parity is a good symmetry even at the quantum level. The parity broken phase is a screening phase, with free triality. The quarks are heavy and carry fractional statistics. This phase is related to the flux phase of quantum antiferromagnets as discussed by Wiegman [8]. The parity symmetric phase maybe confining, with some analogy with four dimensional QCD.

Could it be that flavor and/or parity symmetry are spontaneously broken in $QCD_3$?
In so far, there is no lattice simulation to support that 1. We suspect that for an even number of flavors \( U(N_f) \times Z_2 \) breaks spontaneously to \( U(N_f/2) \times U(N_f/2) \times Z_2 \) while for an odd number of flavors \( Z_2 \) is radiatively broken by the anomaly in the massless case, and explicitly in the massive case. By a generalization of the Vafa-Witten theorem \[13\] to QCD\(_3\) it follows that the absolute values of all condensates are equal. As follows from the derivation of the Banks-Casher relation, the quark mass and the condensate for each flavor have the same sign if the symmetry is broken \textit{spontaneously}. Therefore, we can introduce the order parameter

\[ |\Sigma| = \frac{1}{N_f} \sum_{a=1}^{N_f} |<\bar{q}_a q_a>|. \tag{3} \]

Throughout, we will study the spectrum of the Dirac operator at finite space-time volume. In the chiral limit the spectrum is well defined and contains information on the structure of the vacuum, including the way the flavor symmetry is broken. In the large \( N_c \) limit, a rerun of the Coleman-Witten argument \[14\] for QCD\(_3\) suggests that in the even case the symmetry is broken according to \( U(N_f) \rightarrow U(N_f/2) \times U(N_f/2) \). This is confirmed below using an effective Lagrangian method in the saddle point approximation. In this spirit and for even \( N_f \), the condensates will be combined in pairs of opposite sign.

To understand the generic behaviour of the Dirac spectrum near zero virtuality, we note that the massless Euclidean Dirac operator \( i\hat{D}[A] \) in an arbitrary background field is Hermitean and does not commute with \( P_1E \). By analogy with four dimensions, the random matrix model with the symmetries of (2) is

\[ Z(m) = \int \mathcal{D}TP(T) \prod_{a=1}^{N_f} \det(iT + m_a), \tag{4} \]

where the Haar measure is over \( N \times N \) hermitean matrices, and the \( m_j \) are the eigenvalues of the mass matrix. The weight distribution \( P(T) \) is chosen to be Gaussian consistent with no additional input but the symmetries of the system. We note that the fermion determinants make the integrand not necessarily positive. This ensemble with the determinant replaced by its absolute value is also known as the generalized gaussian ensemble \[12\].

\[ ^1 \text{However, lattice calculation have been performed for QED}_3 \[11\], and a nonzero value of } \langle \bar{q}q \rangle \text{ was found.} \]
The order parameter (3) follows from
\[ \Sigma = - \lim_{m \to 0} \lim_{N \to \infty} \frac{1}{N} \frac{d}{dm} \log Z \tag{5} \]

If we were to define the continuum "spectral density" by
\[ \rho_C(\lambda) = \lim_{m \to 0} \lim_{N \to \infty} \langle \frac{1}{N} \sum_n \delta(\lambda - \lambda_n) \rangle \tag{6} \]

where the expectation value is over the partition function (2), then (7) can be rewritten as
\[ \Sigma = i\pi \rho_C(0) - P \int d\lambda \frac{\rho_C(\lambda)}{\lambda} \tag{7} \]

For even spectra, the principle value part vanishes.

The partition function (4) can be evaluated if we recall that the general decomposition for Hermitian matrices is \( T = U \Lambda U^\dagger \), where \( \Lambda \) is a diagonal matrix. Using the eigenvalues and eigenangles of \( T \) as new integration variables, the partition function can be rewritten as
\[ Z(m) = \int \prod_k d\lambda_k \prod_{k<l} |\lambda_k - \lambda_l|^2 \prod_{a=1}^{N_f} (i\lambda_k + m_a) \exp\left(-\frac{N\Sigma^2}{2} \sum_k \lambda_k^2\right) \tag{8} \]

The one-point function \( \rho(\lambda) \) is defined as the integral over all eigenvalues in this partition function except one. Its normalization can be expressed through \( Z = \int d\lambda \rho(\lambda) \). From (8) it follows immediately that \( \rho(-\lambda) = (-1)^{N_f} \rho(\lambda) \). For an even number of flavors \( \rho(\lambda) \) is positive in the chiral limit and can be interpreted as the spectral density of the Dirac operator. For an odd number of flavors \( \rho(\lambda) \) is still an even function for even \( N \) but is not necessarily positive definite. For odd \( N \) and an odd number of flavors it is an odd function. In this case \( Z(0) = 0 \) and \( Z(m) \sim \partial_m Z|_{m=0} \). In the calculation of the condensate (see (2)) the derivative of the partition function cancels and to leading order in \( m \) we obtain \( \Sigma = \lim_{m \to 0} \lim_{N \to \infty} (-1)^{N/N}(-1)/Nm = 0 \). Thus, chiral symmetry remains unbroken in this limit.

Before discussing the spectral density related to (4), we will derive the finite volume partition function in the static limit, as for \( QCD_4 \) [3]). This is achieved by rewriting the fermion determinant as Grassmann integrals, and averaging over \( T \). The result is a four-fermion interaction. The effective partition function is obtained after bosonisation and
a saddle point approximation. To leading order in $1/N$ only the saddle point with the eigenvalues of $\sigma$ in opposite pairs contributes \cite{15}. The resulting partition function has a 'hyperbolic' symmetry \cite{15} and is given by
\begin{equation}
Z(\mathcal{M}) = \int_{U \in SU(N_f)} DU \exp(N\Sigma \text{Tr} \mathcal{M} U^I U^\dagger),
\end{equation}
where we have extended the integral over the coset $U(N_f)/U(N_f/2) \times U(N_f/2)$ to $SU(N_f)$, and $I = \text{diag}(1_{N_f/2}, -1_{N_f/2})$. Equivalently, the result \eqref{9} could also be arrived at using general symmetry arguments \cite{6}.

The linear term in $\mathcal{M}$ in the expansion of $Z(\mathcal{M})$ vanishes. The integrals over $SU(N_f)$ that occur in the terms of $O(M^2)$ are well known. We find
\begin{equation}
Z(\mathcal{M}) = Z(0) \left[ 1 + \sum_{k} \frac{N_f^2}{N_f - 1} \left( \frac{1}{N_f} \text{Tr} \mathcal{M}^2 - \frac{1}{N_f^2} \text{Tr}^2 \mathcal{M} \right) + \cdots \right].
\end{equation}
On the other hand, the $QCD_3$ partition function can be expanded as
\begin{equation}
Z(m) = Z(0) \left\{ 1 + i \text{Tr} \mathcal{M} \sum_{k} \frac{1}{\lambda_k} + \frac{1}{2} (\text{Tr} \mathcal{M}^2 - \text{Tr}^2 \mathcal{M}) \sum_{k} \frac{1}{\lambda_k^2} - \frac{1}{2} \text{Tr}^2 \mathcal{M} \sum_{k \neq l} \frac{1}{\lambda_k \lambda_l} + \cdots \right\},
\end{equation}
resulting in the sum rules
\begin{equation}
\frac{1}{N_f^2} \sum_{k} \frac{1}{\lambda_k^2} = \frac{N_f}{N_f - 1} \Sigma^2 \quad \text{and} \quad \frac{1}{N_f^2} \sum_{k \neq l} \frac{1}{\lambda_k \lambda_l} = -\frac{\Sigma^2}{N_f + 1}.
\end{equation}
Note that the average over the second term in \eqref{11} vanishes, in agreement with the effective partition function.

For odd $N_f$ it is not possible to organize the saddle points in opposite pairs. Because of the Jacobian and to leading order in $1/N$ the saddle points occur at $(N_f+1)/2$ eigenvalues with $\pm1/\Sigma$ and $(N_f - 1)/2$ eigenvalues with $\mp1/\Sigma$. Both sets of saddle points cannot be transformed into each other by a unitary transformation. However, all saddle point in one set are connected by a unitary transformation. The sum of the two saddle points leads to the partition function
\begin{equation}
Z(\mathcal{M}) = \int_{U \in SU(N_f)} DU \cosh(N\Sigma \text{Tr} \mathcal{M} U^I U^\dagger),
\end{equation}
for even $N$, and $\cosh \rightarrow \sinh$ for odd $N$. The matrix $I$ is now $\text{diag}(1_{(N_f+1)/2}, -1_{(N_f-1)/2})$.

The finite volume partition function is therefore based on the coset $U(N_f)/U((N_f-1)/2) \times U((N_f+1)/2)$. Sum rules follow from comparison to the $QCD_3$ partition function
\begin{equation}
\frac{1}{N_f^2} \sum_{k} \frac{1}{\lambda_k^2} = \frac{\Sigma^2}{N_f} \quad \text{and} \quad \frac{1}{N_f^2} \sum_{k \neq l} \frac{1}{\lambda_k \lambda_l} = -\frac{\Sigma^2}{N_f},
\end{equation}
for even values of $N$.

With these results in mind, we now turn to the evaluation of the spectral density using random matrix theory. The microscopic spectral density and its correlation functions will be the master formulae for all sum rules. The construction of the spectral density, can be achieved with the help of the orthogonal polynomial method from random matrix theory (see for example [7]). For even $N_f$ we proceed as for the Gaussian Unitary Ensemble [4]. The result for the spectral density can be written as

$$\rho(\lambda) = \sum_{k=0}^{N-1} \frac{1}{r_k} P_k(\lambda) P_k(\lambda) \lambda^{N_f} \exp(-a^2 \lambda^2),$$

where the $P_k$ are orthogonal polynomials that satisfy

$$\int_{-\infty}^{\infty} d\lambda P_k(\lambda) P_l(\lambda) \lambda^{N_f} \exp(-a^2 \lambda^2) = r_k \delta_{kl},$$

and $r_k$ is a normalization factor. For even $N_f$ these polynomials can be expressed in terms of the generalized Laguerre polynomials

$$P_{2k}(\lambda) = L_k^{N_f-1} \left( a^2 \lambda^2 \right)$$

and

$$P_{2k+1}(\lambda) = \lambda L_k^{N_f+1} \left( a^2 \lambda^2 \right).$$

Using (16) and (17) the normalization factors are determined readily.

The spectral density follows immediately from (15). It can be written as the sum of two terms. Both sums can be evaluated with the help of the Christoffel-Darboux formula. The result is straightforward, and yields an exact analytical expression for the level density. Its microscopic limit

$$\rho_S(z) = \lim_{N \to \infty} \frac{1}{N} \rho \left( \frac{z}{N} \right).$$

follows from the asymptotic properties of the generalized Laguerre polynomials. The result for even $N_f$ is [12]

$$\rho_S(z) = \frac{\sum z^2}{4} \left[ J_{N_f-1}^2(z) - J_{N_f+1}^2(z) \right] + \frac{J_{N_f+3}^2(z) - J_{N_f-3}^2(z)}{2}. \quad (19)$$

It reproduces the first sum rule of (12).

For $N_f$ odd it is not possible to construct a set of orthogonal polynomials satisfying (16). For example, the zeroth order polynomial cannot be orthogonal to the first order polynomial. Therefore, in the Vandermonde determinant, we replace the $2k - 1$'th row
by the difference of the $2k'$th and the $2k-1'$th row and the $2k'$th row by the sum of
the $2k-1'$th and the $2k'$th row. The polynomials in the $2k-1'$th and the $2k'$th row
therefore have the same degree. In this case the level density is again given by (15) but
the orthogonal polynomials $P_{2k-1}$ and $P_{2k}$ have the same degree. The construction
of these polynomials is straightforward

$$P_{2k-1} = (1 - \lambda a) L_{k-1}^{N_f}(\lambda^2 a^2) \quad \text{and} \quad P_{2k} = (1 + \lambda a) L_{k}^{N_f}(\lambda^2 a^2).$$  \hspace{1cm} (20)

Repeating the steps leading (19) we arrive at the microscopic spectral density

$$\rho_S(z) = (-1)^{N_f/2} \sum_{z} \left( J_{N_f/2}(z) - J_{N_f/2+1}(z) J_{N_f/2-1}(z) \right), \hspace{1cm} (21)$$

which reproduces the first spectral sum rule of (14). The case of odd $N$ and odd $N_f$
will be discussed elsewhere.

Finally, we would like to comment on the issue of the anomaly. In three dimensions,
$\det(i\hat{D}_3[A])$ is noninvariant under an adiabatic switch-on or -off of a large gauge trans-
formation \[16\]. The gauge noninvariance is usually followed by a spectral flow in the
fermionic spectrum with level crossing at zero. This means an overall change in the sign
of the fermion determinant. Thus the anomaly and the phase of the fermion determinant
are related. Indeed, consider the case when one eigenvalue say $\lambda$, crosses 0. The relevant
factor in our case that determines the change in phase is given by $\prod_{a=1}^{N_f} \left( \lambda + im_a \right)$. When
the crossing is completed with $|m_a| \ll |\lambda|$, the determinant acquires the extra phase

$$\exp(\pm i \pi \sum_{a=1}^{N_f} \text{sgn} \ m_a).$$ \hspace{1cm} (22)

This result is to be contrasted with the continuum result \[16\]

$$\exp(\pm i \pi \sum_{a=1}^{N_f} \text{sgn} \ m_a \ W_{cs}[A]),$$ \hspace{1cm} (23)

where $W_{cs}[A]$ is the Chern-Simons action in Euclidean space. The imaginary character
of (23) in Euclidean space, follows from the fact that $W_{cs}$ is $T$-odd. The sign ambiguity
in our case, corresponds to the sign ambiguity left by the Pauli-Villars regulator in the
continuum. Thus, the net sign effect in random matrix theory amounts to a Chern-Simons
term.
To conclude: we have argued that on the basis of universality arguments that the spectral density of QCD$_3$ near zero virtuality follows from a hermitean random matrix model. We have explicitly constructed the microscopic spectral densities for an even and odd number of flavors. The resulting sum rules are in agreement with the expected results following from an effective Lagrangian formulation based solely on symmetry arguments. For an even number of flavors, we have suggested using the effective Lagrangian formulation, that $U(N_f)$ is likely to be maximally broken to $U(N_f/2) \times U(N_f/2)$. We have also shown that the Chern-Simons term has a natural explanation in the context of random matrix theory.

Finally, we want to note that the random matrix model used in this work is based on the assumption that flavor symmetry is broken spontaneously. It is this assumption that leads to the finite volume static partition function as quoted in the text. Since, in general, odd dimensions do not sustain semiclassical physics, this leads us to the questions: what is the mechanism behind the spontaneous breaking of flavor symmetry? Could it be that this mechanism is also responsible for the spontaneous breaking of chiral symmetry in four dimensions? These points deserve further investigation.

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