On Hiemenz flow of Maxwell incompressible
viscoelastic medium

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Abstract. Steady axisymmetric flow of a viscoelastic incompressible fluid near the critical point of the plane surface is considered. Model of Maxwellian fluid with upper convected derivative in the rheological constitutive law is used. Velocity and extra stresses field are discussed for various Weissenberg numbers. The subclass of solutions in a stationary case is completely described. The asymptotic solutions for small Weissenberg numbers are compared to the solutions of the complete nonlinear problem. It is found that the fluid elasticity decreases the boundary layer thickness.

1. Introduction

Stagnation regions exist on all blunt bodies moving in a fluid. The results of studies about stagnation flows of viscoelastic media are of great technical importance, for example, in the prediction of skin-friction. The recent state of the mathematical theory of viscoelastic fluids dynamics is presented by Zvyagin and Turbin in [2]. Features of incompressible viscoelastic Maxwell model consist that the corresponding system of the quasilinear equations has both real, and imaginary characteristics [3]. Non-stationary flow of viscous incompressible fluid near a critical point on solid surface is considered in work of Petrova, Pukhnachev and Frolovskaya [4]. It is shown that depending on flow at the initial moment, the solution can collapse on a finite time. Several analytical solutions for the velocity and stresses fields of viscoelastic Maxwell fluid in a non-stationary two-dimensional flow were constructed in work of Meleshko, Moshkin and Pukhnachev [7]. Example of partially invariant solution for a case with rotational convected derivative was considered by Pukhnachev [10].

2. Problem statement

The system of the equations describing axisymmetric motion of the incompressible viscoelastic Maxwell media is considered. Media is characterize by density, $\rho$, viscosity, $\mu$, and relaxation time, $\tau$. A fluid impinges on a wall and flows away radially in all directions. To solve the problem we use cylindrical coordinates $(r, z, \theta)$. Assume that the wall is at $z = 0$, the stagnation point is at the origin and that the flow is in the direction of the negative $z$-axis. The flow is assumed axisymmetric therefore there is no dependence on axial coordinate. Independent variables are cylindrical coordinates $(r, z, \theta)$ and time $t$, and unknown functions are projections $u$ and $w$ of velocity vector $\vec{v}$ on the axes of coordinates $r$, $z$, respectively, pressure $p$ and components $S_{rr} = A$, $S_{rz} = B$, $S_{zz} = C$, $S_{\theta\theta} = \Theta$ of extra stress tensor $\vec{\tau}$ that determined by the rheological equation.
Components \( S_{r\theta} = S_{z\theta} = 0 \) owing to axial symmetry of a flow. The continuity equation and the momentum equation of an incompressible continuous medium have the form

\[
\nabla \cdot \vec{v} = 0, \quad (1)
\]
\[
\rho(\vec{v}_t + \vec{v} \cdot \nabla \vec{v}) = -\nabla p + \text{div} S. \quad (2)
\]

We use the rheological constitutive law in the form with upper convective derivative [1]:

\[
\tau \left( \frac{\partial S}{\partial t} + \vec{v} \cdot \nabla S - \nabla \vec{v} \cdot S - S \cdot \nabla \vec{v} \right) + S = 2\mu D, \quad (3)
\]

depicting strain rate tensor. System (1)–(3) describes the motion of an incompressible viscoelastic Maxwell medium in the general three-dimensional case. Below, we consider two-dimensional motions. System (1)–(3) in the cylindrical coordinates \((r, z)\) has the following form

\[
(ur)_r + rw_z = 0, \quad (4)
\]
\[
\rho(u_t + uw_r + uw_z) = -p_r + A_r + B_z + (A - \Theta)/r, \quad (5)
\]
\[
\rho(w_t + uw_r + wz_z) = -p_z + B_r + C_z + B/r, \quad (6)
\]
\[
A_t + uA_r + wA_z - 2(Au_r + Bu_z) + \tau^{-1}A = 2\mu\tau^{-1}u_r, \quad (7)
\]
\[
B_t + uB_r + wB_z - (Cu_z + Aw_z - Bu/r) + \tau^{-1}B = \mu\tau^{-1}(u_z + w_r), \quad (8)
\]
\[
C_t + uC_r + wC_z - 2(Bw_r + Cw_z) + \tau^{-1}A = 2\mu\tau^{-1}w_z, \quad (9)
\]
\[
\Theta_t + u\Theta_r + w\Theta_z - 2\Theta u/r + \tau^{-1}\Theta = 2\mu\tau^{-1}u/r. \quad (10)
\]

The lower index designates a derivative with respect to corresponding coordinate. The continuity equation is carried out if to put

\[
w = -2f(z,t), \quad u = rf(z,t). \quad (11)
\]

Eliminating pressure by cross differentiation and using the equation (11) one get

\[
\rho(rf_{tz} + rf_z^2 - 2rf f_{zz})_z = A_{xz} + B_{zz} + (A_z - \Theta_z)/r - B_{xr} - C_{rz} - (B/r)r, \quad (12)
\]
\[
A_t + rf_A r - 2(Af_z + r f_{zz}) + \tau^{-1}A = 2\mu\tau^{-1}f_z,
\]
\[
B_t + rf_B r - 2f B_z - (rC f_{zz} - B f_z) + \tau^{-1}B = \mu\tau^{-1}f_{zz},
\]
\[
C_t + rf_C r - 2f C_z + 4f f_z + \tau^{-1}C = -4\mu\tau^{-1}f_z,
\]
\[
\Theta_t + rf_\Theta r - 2f f_z + \tau^{-1}\Theta = 2\mu\tau^{-1}f_z.
\]

System (12) has a wide class of solutions, where \(A, B, C\) and \(\Theta\) are polynomials of \(r\). Let us restrict ourselves to solutions of the form

\[
A = r^2 a(z, t) + d(z, t), \quad B = rb(z, t), \quad C = c(z, t); \quad \Theta = \theta(z, t). \quad (13)
\]

Notice that Pan-Thien and Tanner [5, 6] came to the solution of the form (13) by heuristic way. Solutions near a critical point considered in papers [5, 6] can be derived in a sense from concept of partially invariant solutions introduced by Ovsyannikov [8]. The group-theoretic properties of the solution in such form were considered by Meleshko and Pukhnachev [10]. It was shown that (13) is the invariant solution of partially invariant submodel of the Navier-Stokes equations.
Substituting (13) in the equations (12) we obtain the system of the Maxwell equations with upper convected derivative in constitutive equations (an analog of Hiemenz flow for the viscous media)

\[ \rho(f_t + f'^2 - 2ff'') = b' + 3a + \rho \varphi(t), \]  
\[ \tau(a_t - 2fa' + 2bf'') + a = 0, \]  
\[ \tau(b_t - 2fb' + 2bf' - cf'') + b = \mu f'', \]  
\[ \tau(c_t - 2fc' + 4cf') + c = -4\mu f', \]  
\[ \tau(d_t - 2fd' - 2df') + d = 2\mu f', \]  
\[ \tau(\theta_t - 2f\theta' - 2f'\theta) + \theta = 2\mu f'. \]  

Here \( \varphi(t) \) is an arbitrary function of time, the prime symbol (') is used to designate a derivative with respect to \( z \). Let's notice that equations (18) and (19) have the same structure, therefore \( \theta = d \) for the corresponding boundary and initial conditions. The equations (18) and (19) are solved independently after functions \( f, a, b \) and \( c \) are found.

3. Stationary flow

The stationary solution of a system (14)–(19) represents some interest. Substituting \( \rho \varphi(t) = \rho k^2 \) we obtained system of the equations similar to the Hiemenz equations for a case of viscous incompressible liquid. Without loss of generality we assume \( \rho = 1 \) and performed the affine transformation to represent system in dimensionless form. By analogy to a case of a viscous fluid [11] we define new variables and unknown functions

\[ y = \alpha z, \quad f(z) = \hat{f}(y), \quad b(z) = \hat{b}(y), \quad c(z) = \hat{c}(y), \quad a(z) = \hat{a}(y), \]

where \( \alpha^2F^2 = k^2, \quad F = \sqrt{\mu k}, \quad \alpha B = k^2, \quad C = \mu k \). Omitting sign at dimensionless quantities, we rewrite the system (14)–(19)

\[ f'^2 - 2ff'' = b' + 3a + 1, \]
\[ W_i(-2fc' + 4cf') + c = -4f', \]
\[ W_i(-2fb' + 2bf' - cf'') + b = f'', \]
\[ W_i(-2fa' + 2bf'') + a = 0, \]
\[ W_i(-2fd' - 2df') + d = 2f', \]

where \( W_i = k\tau \) is Weissenberg number, prime symbol denotes differentiation with respect to \( y \). Equations (20)–(24) have to satisfy the following boundary conditions

- noslip conditions on solid boundary \( y = 0 \),
  \[ f(0) = f'(0) = 0; \]
- correspondence with case of nonviscous flow \( u = kr, \quad v = -2kz \)
  \[ f' \to 1, \quad y \to \infty; \]
- stress vanishing as \( y \to \infty \)
  \[ b \to 0, \quad a \to 0, \quad y \to \infty; \]
- asymptotic of the equations (21), (24) at infinity
  \[ c \to c_\infty = -\frac{4}{4W_i + 1}, \quad y \to \infty, \]
  \[ d \to -\frac{2}{2W_i - 1}, \quad y \to \infty. \]
3.1. Asymptotic solution for small $W_i$

The solution of a system of equations (20)–(24) at small values of the Weissenberg number $W_i$, which corresponds to small values of the relaxation time $\tau$, has some practical interest. The formal asymptotic solution of problem can be constructed as a power series in parameter $W_i$,

$$f = f_0 + W_if_1 + \ldots, \quad c = c_0 + W_ic_1 + \ldots,$$

$$b = b_0 + W_ib_1 + \ldots, \quad a = a_0 + W_ia_1 + \ldots, \quad d = d_0 + W_id_1 + \ldots.$$

Zero terms (the initial solutions) are defined from system,

$$f_0^2 - 2f_0f''_0 + 3a_0 + b'_0 + 1; \quad c_0 = -4c_\infty^{-1}f''_0, \quad a_0 = 0; \quad b_0 = f''_0,$$

with boundary conditions

$$f_0(0) = f'_0(0) = 0, \quad f'_0(y) \to 1, \quad y \to \infty,$$

This boundary value problem coincides with the equation of an axisymmetric flow of viscous fluid near a critical point [11]. First-order terms correspond to system

$$2f_0f'_0 - 2(f_1f''_0 + f_0f''_1) = 3a_1 + b'_1, \quad a_1 - 2a'_0f_0 + 2b_0f''_0 = 0, \quad -2b'_0 + 2b_0f'_0 - c_\infty c_0f''_0 + b_1 = f''_1, \quad -2c'_0f_0 + 4c_0f'_0 + c_1 = -4c_\infty^{-1}f''_1, \quad f_1(0) = f'_1(0) = 0, \quad f'_1(y) \to 0, \quad y \to \infty.$$

Functions $d$ and $\theta$ can be found as a solution of the following equations

$$d_0 = 2f'_0, \quad -4f''_0f_0 - 4f'_0f'' + d_1 = 2f'_1, \quad d_1 \to 4, \quad y \to \infty.$$

Boundary value problem is easily solved by standard subroutine from “MATLAB” or similar instrument.

3.2. Numerical solution of problem

To extend the asymptotic solutions to any values of $W_i$ a numerical solution of the equations (20)–(23) is necessary. Rewrite equations (20)–(23)

$$f'^2 - 2ff'' = \tau_{\infty}f'' + (b' - \tau_{\infty}f''') + 3a + 1, \quad W_if'c' - (41f' + 1)/2c = 2c_\infty f', \quad W_ifb' - (2W_if' + 1)/2b = -(1 + W_\infty c_\infty/2c)f''', \quad W_ifa' - 1/2a = 1/2W_ibf'''.$$

Notice that the third derivative is added to the first equation. This derivative compensated by the third derivative with a minus sign in parentheses. It is made to organize iterative process with parameter $\tau_{\infty}$.

Take into account that for $f$ there is a boundary value problem it is convenient to write down the first equation as a system of two equations of the second and first order. Designating by $\eta = f$, $\psi = f'$ we obtain

$$\eta'^2 - 2\eta\psi' = \tau_{\infty}\psi'' + (b' - \tau_{\infty}\psi''') + 3a + 1, \quad \eta' = \psi,$$
with boundary conditions

\[ \psi(0) = 0, \quad \psi(\infty) = 1, \quad \eta(0) = 0, \]

\[ \psi^{s,k}\psi^{s,k-1} - \eta^{s,k-1}(\psi')^{s,k} = \tau_i(\psi')^{s,k} + (b' - \tau_i\psi')^{s,k-1} + 3a^{s-1} + 1, \]

\[ \psi^{s,k}(0) = \varepsilon, \quad \psi^{s,k}(\infty) = 1, \quad (\eta')^{s,k} = \psi^{s,k}, \]

\[ \eta^{s,k}(0) = 0, \quad k = 1, \ldots, K. \]

For \( s = 1 \) we required \( (b')^0 = 0, \quad (\psi')^0 = 0, \quad (a)^0 = 0 \) and as a result we obtain the solution of Hiemenz \((W_i = 0)\). For \( k = 1 \) we assume \( \psi^{s,0} = \psi^{s-1}, \quad \eta^{s,0} = \eta^{s-1} \). Iterations on nonlinearity are carried out to convergence (in principle it is possible to be limited several iterations not to reach convergence)

\[ \|\psi^{s,\tilde{K} - 1}\| + \|\eta^{s,\tilde{K} - 1}\| \leq \varepsilon_s, \]

where \( \varepsilon_s \) sufficiently small. Further we denote \( \psi^s = \psi^{s,\tilde{K}}, \quad \eta^s = \eta^{s,\tilde{K}} \). The remained three equations of the first order can be solved consistently

\[ W_i\eta^s(c')^s - [4W_i\psi^s + 1]/2c^s = 2c_{\infty}(\psi')^s, \quad c^s(\infty) = 1. \]

\[ W_i\eta^s(b')^s - [2W_i\psi^s + 1]/2b^s = -[1 + W_i c_{\infty} c'] (\psi')^s, \quad b^s(\infty) = 0, \]

\[ W_i\eta^s(a')^s - 1/2a^s = 1/2W_i b'(\psi')^s, \quad a^s(\infty) = 0. \]

Computations are carried out on a nonuniform grid which is uniform on interval \([0, \sigma]\) and further a grid evenly increases a step

\[ \Omega_h = \left\{ \zeta_i = (i - 1) * h, \quad i = 1, \ldots, M; \quad h = \sigma / M; \quad \zeta(i + 1) = \zeta(i)q, \quad i = M, \ldots, N \right\}. \]

Conditions at \( \infty \) shift out on the boundary \( \zeta = \zeta_{\infty}, \quad \zeta_{\infty} = \zeta_{N + 1} \). Tomas algorithm is used to solve the first equation of a system (27), which is tridiagonal system. The second equation is integrated with use of Euler method

\[ \left( \psi_{i+1} - y_i \right) / h_{i+1} - \left( \psi_i - \psi_{i-1} \right) / h_i = \eta_i \psi_{i+1} - \psi_i / h_{i+1} - \tilde{\psi}_i y_i = -1 - (b^s - \tau_i(\psi')^s) + 2a^s, \]

\[ \psi(1) = 0, \quad \psi(N + 1) = 1, \quad i = 1, \ldots, N, \]

\[ \eta_{i+1} - \eta_i / h_{i+1} = \psi_{i+1} + \psi_i / 2, \quad \eta(1) = 0. \]

Functions with “tick” undertake from the previous iteration. The solution of a system of the equations (28) is obtained after convergence of iterative process Cauchy’s problem is solved on the interval \([0, \zeta_{\infty}]\). The solution is obtained from right to left, i.e. an integration is obtained from \( \zeta_{\infty} \) to zero with the data set at the right-hand end of the interval

\[ W_i c_{i+1}/h_{i+1} - (1 + 4W_i \psi_{i+1/2}) / 2c_i = 2c_{\infty}^{-1} \psi_{i+1/2}^{-1}, \quad c_{N + 1} = 1, \]

\[ W_i b_{i+1}/h_{i+1} - [W_i \psi_{i+1/2} + 1] / 2b_i = -[1 + W_i c_{\infty} c_{i+1/2}] / 2\psi_{i+1/2}, \quad b_{N + 1} = 0, \]

\[ W_i a_{i+1}/h_{i+1} - a_i / 2a_i = 1/2W_i b_i \psi_{i+1/2}, \quad a_{N + 1} = 0. \]
Derivatives are defined using a finite-difference approximations. The convergence of the general iterative process was reached provided condition

\[ \| (b')^{s+1} - (b')^s \| \leq \varepsilon_g, \]

where \( \varepsilon_g \) is sufficiently small.

4. Results and discussions
In figures 1 and 2 we display the components \( a, b, \) and \( c \) of extra stress tensor at value of \( W_i = 0.01 \). Figure 1 shows results of asymptotic solution. Figure 2 shows results of numerical solution of the equations (20)–(23). Comparison of these figures states that the graphics of the components of extra stress tensor at small Weissenberg number \( W_i = 0.01 \) is almost identical. Sufficiently good qualitative and quantitative correspondence of the solutions of the complete system and the asymptotic solution on Weissenberg number show validity of the perturbation solutions for small values of Weissenberg number. Dependence on \( y \) function \( f(y) \) and its first derivative at different values of Weissenberg number \( W_i = 0.1, 0.3, 0.5, 0.8 \) are shown in figures 3 and 4. Arrows show the direction of the Weissenberg number increasing. From figure 4 it is visible that velocity reaches the asymptotic value faster with increasing Weissenberg number. It is characterized by an decreasing of a boundary layer width near solid wall.
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