Abstract
In this work we propose a formal system for fuzzy algebraic reasoning. The sequent calculus we define is based on two kinds of propositions, capturing equality and existence of terms as members of a fuzzy set. We provide a sound semantics for this calculus and show that there is a notion of free model for any theory in this system, allowing us (with some restrictions) to recover models as Eilenberg-Moore algebras for some monad. We will also prove a completeness result: a formula is derivable from a given theory if and only if it is satisfied by all models of the theory. Finally, leveraging results by Milius and Urbat, we give HSP-like characterizations of subcategories of algebras which are categories of models of particular kinds of theories.

1 Introduction
One of the most fruitful and influential lines of research of Logic in Computer Science is the algebraic study of computation. After Moggi’s seminal work [15] showed that notions of computation can be represented as monads, Plotkin and Power [19] approached the problem using operations and equations, i.e., Lawvere theories. Since then, various extensions of the notion of Lawvere theory have been introduced in order to accommodate an ever increasing number of computational notions within this framework; see, e.g., [20, 9, 18], and more recently [3, 4] for quantitative algebraic reasoning for probabilistic computations.

Along this line of research, in this work we study algebraic reasoning on fuzzy sets. Algebraic structures on fuzzy sets are well known since the seventies (see e.g., [21, 14, 1, 17]). Fuzzy sets are very important in computer science, with applications ranging from pattern recognition to decision making, from system modeling to artificial intelligence. So, it is natural to ask if it is possible to use an approach similar to above for fuzzy algebraic reasoning.

In this paper we answer positively to this question. We propose a sequent calculus based on two kind of propositions, one expressing equality of terms and the other the existence of a term as a member of a fuzzy set. These sequents have a natural interpretation in categories of fuzzy sets endowed with operations. This calculus is sound and complete for such a semantics: a formula is satisfied by all the models of a given theory if and only if it is derivable from it.

It is possible to go further. Both in the classical and in the quantitative settings there is a notion of free model for a theory; we show that is also true for theories in our formal system for fuzzy sets. In general the category of models of a given theory will not be equivalent to the category of Eilenberg-Moore algebras for the induced monad, but we will show that this equivalence holds for theories with sufficiently simple axioms. Finally we will use the techniques developed in [15] to prove two results analogous to Birkhoff’s theorem.
Synopsis. In Section §2 we recall the category $\text{Fuz}(H)$ of fuzzy sets over a frame $H$. Section §3 introduces the syntax and the rules of fuzzy theories. Then, in Section §4 we introduce the notions of algebras for a signature and of models for a theory; in this section we will also show that the calculus proposed is sound and complete. Section §5 is devoted to free models and it is shown that if a theory is basic then its category of models arose as the category of Eilenberg-Moore algebras for a monad on $\text{Fuz}(H)$. In Section §6 we use the results of §5 to prove two HSP-like theorems for our calculus. Finally, Section §7 draws some conclusions and directions for future work. Appendix A contains two detailed derivations used in the paper.

2 Fuzzy sets

In this section we will recall the definition and some well-known properties of the category of fuzzy sets over a frame $H$ (i.e. a complete Heyting algebra [10]).

► Definition 2.1 (23 [24]). Let $H$ be a frame. A $H$-fuzzy set is a pair $(A, \mu_A)$ consisting in a set $A$ and a membership function $\mu_A : A \to H$. The support of $\mu_A$ is the set supp$(A,\mu_A)$ of elements $x \in A$ such that $\mu_A(x) \neq \bot$. An arrow $f : (A,\mu_A) \to (B,\mu_B)$ is a function $f : A \to B$ such that $\mu_A(x) \leq \mu_B(f(x))$ for all $x \in A$.

We denote by $\text{Fuz}(H)$ the category of $H$-fuzzy sets and their arrows. We will often drop the explicit reference to the frame $H$ when there is no danger of confusion.

► Proposition 2.2. For any frame $H$, the forgetful functor $\mathcal{V} : \text{Fuz}(H) \to \text{Set}$ has both a left and a right adjoint $\nabla$ and $\Delta$ on any set $X$ with the function constantly equal to the bottom and the top element of $H$, respectively.

Proof. If $\nabla(X)$ and $\Delta(X)$ are, respectively $(X,e_\bot)$ and $(X,e_\top)$, where $e_\bot$ and $e_\top$ are the functions $X \to H$ constant in $\bot$ and $\top$, then for any $X \in \text{Set}$, id$_X : \mathcal{V}(\Delta(X)) = X \to X = \mathcal{V}(\nabla(X))$ is the counit of $\mathcal{V} \dashv \Delta$ and the unit of $\nabla \dashv \mathcal{V}$.

► Definition 2.3. Let $e : A \to B$ and $m : C \to D$ be two arrows in a category $C$, we say that $m$ has the left lifting property with respect to $e$ if for any two arrows $f : A \to C$ and $g : B \to D$ such that $m \circ f = g \circ e$ there exists a unique $k : B \to C$ with $m \circ k = g$.

A strong monomorphism is an arrow $m$ which has the left lifting property with respect to all epimorphisms.

► Proposition 2.4. Let $f : (A,\mu_A) \to (B,\mu_B)$ be an arrow of $\text{Fuz}(H)$, then:

1. $f$ is a monomorphism iff it is injective; $f$ is an epimorphism iff it is surjective;
2. $f$ is a strong monomorphism iff it is injective and $\mu_B(f(x)) = \mu_A(x)$ for all $x \in A$;
3. $f$ is a split epimorphism iff for any $b \in B$ there exists $a_b \in f^{-1}(b)$ s.t. $\mu_B(b) = \mu_A(a_b)$.

Let us start with the following observation.

► Remark 2.5. In any category $C$, if $m : A \to B$ is a strong monomorphism then it is a monomorphism. Suppose that $m \circ f = m \circ g$ for some $f$ and $g$, then we have the square

$$
\begin{array}{ccc}
C & \xrightarrow{f} & A \\
\downarrow{id_C} & & \downarrow{m} \\
C & \xrightarrow{m \circ g} & B
\end{array}
$$

thus there exists a unique $k$ such that $m \circ k = m \circ g$ but both $f$ and $g$ satisfy this equation.
Remark 2.6. Notice that the previous remark implies that for any square as in the definition of strong monomorphism, \( k \circ e = f \).

Proof of Proposition 2.4. 1. \( \Psi : \text{Fuz}(H) \to \text{Set} \) is faithful, so it reflects monomorphisms and epimorphisms, since it is both a left and a right adjoint it preserves them too, hence 1 follows.

2. Let us show the two implications.

\[ \Rightarrow \] Injectivity follows from Remark 2.5, let \((D, \mu_D) := (f(A), \mu_B|f(A))\) be the set-theoretic image of \( f \) endowed with the restriction of \( \mu_B \). We have a factorization of \( f \) as \( m \circ e \) where \( m \) is a monomorphism and \( e \) an epimorphism, so we have the square

\[
\begin{array}{ccc}
(A, \mu_A) & \xrightarrow{id_{(A, \mu_A)}} & (A, \mu_A) \\
\downarrow e & & \downarrow f \\
(D, \mu_D) & \xrightarrow{m} & (B, \mu_B)
\end{array}
\]

and its diagonal filling \( k : (D, \mu_D) \to (A, \mu_A) \). For any \( x \in A \):

\[ \mu_B(f(x)) = \mu_B(m(e(x))) = \mu_B(e(x)) \geq \mu_A(k(e(x))) = \mu_A(x) \]

and we get the thesis.

\[ \Leftarrow \] Any square

\[
\begin{array}{ccc}
(C, \mu_C) & \xrightarrow{g} & (A, \mu_A) \\
\downarrow e & & \downarrow f \\
(D, \mu_D) & \xrightarrow{m} & (B, \mu_B)
\end{array}
\]

induces the square

\[
\begin{array}{ccc}
C & \xrightarrow{g} & A \\
\downarrow e & & \downarrow f \\
D & \xrightarrow{m} & B
\end{array}
\]

in \( \text{Set} \), which, by point 1 and 2, has a diagonal filling \( k : D \to A \). Now:

\[ \mu_D(x) \leq \mu_B(m(x)) = \mu_B(f(k(x))) = \mu_A(k(x)) \]

and we can conclude that \( k \) is an arrow of \( \text{Fuz}(H) \).

- If \( f \) is split then there exists a right inverse \( e : (B, \mu_B) \to (A, \mu_A) \) and we can put \( a_b := e(b) \). The other direction follows noticing that \( b \mapsto a_b \) is a splitting of \( f \).

Definition 2.7 ([11]). A proper factorization system on a category \( C \) is a pair \((\mathcal{E}, \mathcal{M})\) given by two classes of arrows such that:

- \( \mathcal{E} \) and \( \mathcal{M} \) are closed under composition;
- every isomorphism belongs to both \( \mathcal{E} \) and \( \mathcal{M} \);
- every \( e \in \mathcal{E} \) is an epimorphism and every \( m \in \mathcal{M} \) is a monomorphism;
- every \( m \in \mathcal{M} \) has the left lifting property with respect to every \( e \in \mathcal{E} \);
- every arrow of \( C \) is equal to \( m \circ e \) for some \( m \in \mathcal{M} \) and \( e \in \mathcal{E} \).
Lemma 2.8. For any frame $H$, $\text{Fuz}(H)$ has all products. Moreover the classes of epimorphisms and strong monomorphisms form a proper factorization system on it.

Proof. $\Delta(1)$ is the terminal object by Proposition 2.2. Given a family $\{(A_i, \mu_i)\}$ of fuzzy sets, their product is constructed taking their product in $\text{Set}$ equipped with

$$\mu : \prod_{i \in I} A_i \to H \quad (a_i)_{i \in I} \mapsto \prod_{i \in I} \mu_i(a_i)$$

The last part of the thesis follows at once noticing that every arrow $f : (A, \mu_A) \to (B, \mu_B)$ factors as

$$(A, \mu_A) \xrightarrow{c} (f(A), \mu_B|_{f(A)}) \xrightarrow{m} (B, \mu_B)$$

and applying Proposition 2.4.

Remark 2.9. Completeness and the existence of both adjoints to $\mathcal{V}$ can be deduced directly from the fact that $\text{Fuz}(H)$ is topological over $\text{Set}$ [23 Prop. 71.3].

3 Fuzzy Theories

In this section we introduce the syntax and logical rules of fuzzy theories. The first step is to introduce an appropriate notion of signature. Differently from usual signatures, in fuzzy theories constants cannot be seen simply as 0-arity operations, because, as we will see in Section 4, these are interpreted as fuzzy morphisms from the terminal object, and these correspond only to elements whose membership degree is $\top$.

Definition 3.1. A signature $\Sigma = (O, \text{ar}, C)$ is a set $O$ of operations with an arity function $\text{ar} : O \to \mathbb{N}_+$ and a set $C$ of constants. Signatures form a category $\text{Sign}$ in which an arrow $\Sigma_1 = (O_1, \text{ar}_1, C_1) \to \Sigma_2 = (O_2, \text{ar}_2, C_2)$ is a pair $F = (F_1, F_2)$ of functions: $F_1 : O_1 \to O_2$ and $F_2 : C_1 \to C_2$ with the property that $\text{ar}_2 \circ F_1 = \text{ar}_1$.

An algebraic language $L$ is a pair $(\Sigma, X)$ where $\Sigma$ is a signature and $X$ a set. The category $\text{Lng}$ of algebraic languages is just $\text{Sign} \times \text{Set}$.

Example 3.2. The signature of semigroups $\Sigma_S$ in which $O = \{\cdot\}$, $\text{ar}(\cdot) = 2$ and $C = \emptyset$.

Example 3.3. The signature of groups $\Sigma_G$ is equal to $\Sigma_S$ plus an operation $(-)^{-1}$ of arity 1 and a constant $e$.

Given a language $L$ we can inductively apply the operations to the set of variables to construct terms, and once terms are built we can introduce equations.

Definition 3.4. Given a language $L = (\Sigma, X)$, the set $L$-Terms is the smallest set s.t.

- $X \cup C \subseteq L$-Terms;
- if $f \in O$ and $t_1, \ldots, t_{\text{ar}(f)} \in L$-Terms then $f(t_1, \ldots, t_{\text{ar}(f)}) \in L$-Terms.

Proposition 3.5. There exists a functor $\text{Terms} : \text{Lng} \to \text{Set}$ sending $L$ to $L$-Terms.

Proof. For any $F = ((F_1, F_2), h) : (\Sigma_1, X) \to (\Sigma_2, Y)$ we can define $\text{Terms}(F)$ by induction:

- for any $x \in X$, $\text{Terms}(F)(x) := g(x)$;
- for any $c \in C_1$, $\text{Terms}(F)(c) := F_2(c)$;
The set \( \Lambda \) we put substituting \( \phi \) from \( \Phi \) theory in the language \( \text{Seq} \). On the other hand, identities are preserved and an easy induction shows that composition is respected.

\[ \phi \in \Gamma \quad \Gamma \vdash \phi \quad \frac{\Gamma \vdash \phi}{\Gamma \cup \Delta \vdash \phi} \quad \text{WEAK} \]
\[ \frac{\Gamma \vdash s \equiv s\quad \Gamma \vdash s \equiv t\quad \Gamma \vdash t \equiv s\quad \Gamma \vdash t \equiv u}{\Gamma \vdash s \equiv t\quad \Gamma \vdash t \equiv u} \quad \text{SYM} \]
\[ \frac{\sigma : X \rightarrow \text{L-Terms}}{\frac{\Gamma[\sigma] \vdash \psi[\sigma]}{\text{SUB}}} \]
\[ \frac{\Gamma \vdash E(l, t)}{\Gamma \vdash E(\bot, t)} \quad \text{INF} \]
\[ \frac{S \subset H}{\Gamma \vdash E(\text{sup}(S), t)} \quad \text{SUP} \]
\[ \frac{\Gamma \vdash E(l, t)}{\Gamma \vdash E(l, l')} \quad \text{MON} \]
\[ \frac{\Gamma \vdash E(t \land l', t)}{\Gamma \vdash E(l, t)} \quad \text{EXP} \]
\[ \frac{\Gamma \vdash E(\inf \{l_i\}_{i=1}^n, f(t_1, \ldots, t_{ar(f)}))}{\Gamma \vdash E(l, s)} \quad \text{FUN} \]

\[ \{\Gamma \vdash \phi \mid \phi \in \Phi\} \quad \Phi \vdash \psi \quad \text{CUT} \]
\[ \frac{\Gamma \vdash \psi}{\Gamma \vdash \psi} \quad \text{TRANS} \]

for any \( f \in O \) and \( t_1, \ldots, t_{ar(f)} \in \text{L-Terms} \) it is \( \text{ar}_2(F_1(f)) = \text{ar}_1(f) \), so we can define

\[ \text{Terms}(F)(f(t_1, \ldots, t_{ar(f)})) := F_1(f)(\text{Terms}(F)(t_1), \ldots, \text{Terms}(F)(t_{ar(f)})) \]


\[ \text{Definition 3.6 (Formulae).} \quad \text{For any language } \mathcal{L} \text{ we define the sets } \text{Eq} (\mathcal{L}) \text{ of equations as the product } \text{Eq} (\mathcal{L}) := \text{L-Terms} \times \text{L-Terms} \text{ and the set } \mathcal{M} (\mathcal{L}) \text{ of membership propositions as } \mathcal{M} (\mathcal{L}) := H \times \text{L-Terms}. \text{ We will write } s \equiv t \text{ for } (s, t) \in \text{Eq} (\mathcal{L}) \text{ and } E(l, t) \text{ for } (l, t) \in \mathcal{M} (\mathcal{L}). \text{ The set } \text{Form} (\mathcal{L}) \text{ of formulae is then defined as } \text{Eq} (\mathcal{L}) \cup \mathcal{M} (\mathcal{L}). \]

Clearly, a proposition \( s \equiv t \) means “\( s \) and \( t \) are equivalent and hence interchangeable”; on the other hand, \( E(l, t) \) intuitively means “the degree of existence of \( t \) is at least \( l \)”.

\[ \text{Definition 3.7 (Sequent ant fuzzy theory).} \quad \text{A sequent } \Gamma \vdash \psi \text{ is an element } (\Gamma, \psi) \text{ of } \text{Seq} (\mathcal{L}) := \mathcal{P} (\text{Form} (\mathcal{L})) \times \text{Form} (\mathcal{L}), \text{ where } \mathcal{P} \text{ is the (covariant) power set functor. A fuzzy theory in the language } \mathcal{L} \text{ is a subset } \Lambda \subset \text{Seq} (\mathcal{L}) \text{ and we will use } \text{Th} (\mathcal{L}) \text{ for the set } \mathcal{P} (\text{Seq} (\mathcal{L})). \]

\[ \text{Notation.} \quad \text{We will write } \vdash \phi \text{ for } \emptyset \vdash \phi. \]

For any function \( \sigma : X \rightarrow \text{L-Terms} \) and \( t \in \text{L-Terms} \) we denote \( t[\sigma] \) the term obtained substituting \( \sigma(x) \) to any occurrence of \( x \) in \( t \). Moreover, for any formula \( \phi \in \text{Form} (\mathcal{L}) \) we define \( \phi[\sigma] \) as \( t[\sigma] \equiv s[\sigma] \) if \( \phi \) is \( t \equiv s \) or as \( E(l, t[\sigma]) \) if \( \phi \) is \( E(l, t) \). Finally, given \( \Gamma \subset \mathcal{P} (\text{Form} (\mathcal{L})) \) we put \( \Gamma[\sigma] := \{ \phi[\sigma] \mid \phi \in \Gamma \} \).

\[ \text{Definition 3.8.} \quad \text{For any } \mathcal{L}, \text{ the fuzzy sequent calculus is given by the rules in Figure 1.} \]

Given a fuzzy theory \( \Lambda \), its deductive closure \( \Lambda^+ \) is the smallest subset of \( \text{Seq} (\mathcal{L}) \) which contains \( \Lambda \) and it is closed under the rules of fuzzy sequent calculus. A sequent is derivable from \( \Lambda \) (or simply derivable if \( \Lambda = \emptyset \)) if it belongs to \( \Lambda^+ \). We will write \( \vdash^\Lambda \phi \) if \( \phi \in \Lambda^+ \).

Finally we say that two theories \( \Lambda \) and \( \Theta \) are deductively equivalent if \( \Lambda^+ = \Theta^+ \).

The next result shows how maps between languages are lifted to theories.

\[ \text{Proposition 3.9.} \quad \text{For any } F : \mathcal{L}_1 \rightarrow \mathcal{L}_2 \text{ arrow of } \text{Lng}: \]
1. there exists a Galois connection \( F_* \dashv F^* \) between \( (\text{Th} (\mathcal{L}_1), \subset) \) and \( (\text{Th} (\mathcal{L}_2), \subset) \);
2. \( F_*(A_1^\downarrow) \subset (F_*(A_1))^\downarrow \) and \( (F^*(A_2))^\uparrow \subset F^*(A_2^\uparrow) \) for any \( A_1 \in \text{Th} (\mathcal{L}_1) \) and \( A_2 \in \text{Th} (\mathcal{L}_2) \).
Proof. 1. For any formula $\psi$ we can define (using Proposition 3.5):

$$\text{Form}(F)(t \equiv s) := \text{Terms}(F)(t) \equiv \text{Terms}(F)(s)$$

$$\text{Form}(F)(E(l, t)) := E(l, \text{Terms}(F)(t))$$

getting a function $\text{Form}(F) : \text{Form}(L_1) \rightarrow \text{Form}(L_2)$. We can extend it to sequents defining

$$\text{Seq}(F)(\Phi \vdash \psi) := \{ \text{Form}(F)(\phi) \mid \phi \in \Phi \} \vdash \text{Form}(F)(\psi)$$

Now it is enough to take $F^*$ and $F^*$ as the image and the preimage of $\text{Seq}(F)$ respectively.

2. First of all notice that the two inequalities are equivalent because $F^* \dashv F^*$. Now, the first one holds since if a sequent $\Gamma \vdash \phi$ follows from the set of sequents $\{ \Gamma_i \vdash \phi_i \}_{i \in I}$ by the application of one of the rules of the fuzzy sequent calculus then the same rule can be applied to $\{ \text{Seq}(F)(\Gamma_i \vdash \phi_i) \}_{i \in I}$ to get $\text{Seq}(F)(\Gamma \vdash \psi)$. ◀

Usually, logics enjoy the so-called “deduction lemma”, which allows us to treat elements of a theory on a par with assumptions in sequents. In fuzzy theories, this holds in a slightly restricted form, as proved next.

Lemma 3.10 (Partial deduction lemma). Let $\Lambda$ be in $\text{Th}(L)$ and $\Gamma \in \mathcal{P}(\text{Form}(L))$, let also $\Lambda[\Gamma]$ be the theory $\Lambda \cup \{ \emptyset \vdash \phi \mid \phi \in \Gamma \}$. Then the following are true:

1. $\Gamma \cup \Delta \vdash \psi$ in $\Lambda \vdash \psi$ implies $\Delta \vdash \psi$ in $(\Lambda[\Gamma])^+$;
2. if $\Delta \vdash \psi$ is derivable from $\Lambda[\Gamma]$ without using rule SUB then $\Gamma \cup \Delta \vdash \psi$ is in $\Lambda^+$.\r

Proof. 1. By hypothesis $\Gamma \cup \Delta \vdash \psi$ is in $\Lambda^+$ then, since $\Lambda \subset \Lambda[\Gamma]$, it is also in $(\Lambda[\Gamma])^+$. Now, for any $\phi \in \Gamma$ and $\theta \in \Delta$ rules $\text{WEAK}$ and $\text{A}$ give

$$\frac{\vdash \phi}{\Delta \vdash \phi} \quad \text{WEAK} \quad \frac{\Delta \vdash \theta}{\Lambda} \quad \text{A}$$

so $\{ \Delta \vdash \phi \mid \phi \in \Gamma \cup \Delta \}$ is contained in $(\Lambda[\Gamma])^+$ and we can apply $\text{Cut}$ to get the thesis:

$$\frac{\{ \Delta \vdash \phi \mid \phi \in \Gamma \cup \Delta \}}{\Delta \vdash \psi} \quad \text{Cut}$$

2. Let us proceed by induction on a derivation of $\Delta \vdash \psi$ from $\Lambda[\Gamma]$.

- If $\Delta \vdash \psi \in \Lambda[\Gamma]$ then or $\Delta \vdash \psi \in \Lambda$ and we are done, or $\psi \in \Gamma$ and we can conclude since $\Gamma \cup \Delta \vdash \phi$ is in $\Lambda^+$ by rules $\text{A}$ and $\text{WEAK}$
- If $\Delta \vdash \psi$ follows from the application of one of the rules $\text{A}$, $\text{INF}$ or $\text{REFL}$ then it belongs to the closure of any theory, by $\text{WEAK}$ this is true even for $\Gamma \cup \Delta \vdash \psi$ which, in particular, it belongs to $\Lambda^+$.
- Suppose that $\Delta \vdash \psi$ comes from an application of $\text{WEAK}$, then there exists $\Psi$ and $\Phi$ such that $\Psi \cup \Phi = \Delta$ and $\Psi \vdash \phi$ is in $(\Lambda[\Gamma])^+$. By inductive hypothesis we have that

$$\frac{\Gamma \cup \Psi \vdash \psi}{\Gamma \cup \Psi \cup \Phi \vdash \psi} \quad \text{WEAK}$$

is a derivation of $\Gamma \cup \Delta \vdash \psi$ from $\Lambda$.
Finally we get the theory of left ideal $\Lambda_{LI}$ if we add the axioms (one for any $l \in L$):

$$E(l, y) \vdash E(l, x \cdot y)$$

Similarly the theory $\Lambda_{RI}$ of right ideal is obtained from the axioms:

$$E(l, x) \vdash E(l, x \cdot y)$$

Finally we get the theory of (bilateral) ideal $\Lambda_I$ taking the union of the above theories.

Example 3.12 ([21] [12]). Let $\Sigma_G$ be the signature of groups and $X$ a countable set. The theory $\Lambda_G$ of fuzzy groups is simply the usual theory of groups, i.e that given by the sequents (using infix notation)

$$\vdash (x \cdot y) \cdot z \equiv x \cdot (y \cdot z)$$

We get the theory $\Lambda_N$ of normal fuzzy groups ([14]) if we add the axioms:

$$E(l, x) \vdash E(l, y \cdot (x \cdot y^{-1}))$$

Proposition 3.13. The sequent $E(l, x) \vdash E(l, e)$ is derivable from $\Lambda_G$ and $E(l, y \cdot (x \cdot y^{-1})) \vdash E(l, x)$ is derivable from $\Lambda_N$.

Proof. The two derivations are shown in Appendix A. In the second derivation we have used the fact that $\vdash y \equiv (y^{-1})^{-1}$ and $\vdash x \equiv y \cdot ((y^{-1} \cdot (x \cdot y)) \cdot y^{-1})$ are derivable in the theory of groups (this can be shown as in the case of ordinary group theory) and we have substituted $y^{-1} \cdot (x \cdot y)$ for $x$ in the first application of SUB and $y^{-1}$ for $y$ in the second one. ◀
4 Fuzzy algebras and semantics

In this section we provide a sound and complete semantics to the syntax and sequents introduced in Section 3. The first step is to define the semantic counterpart of a signature.

Definition 4.1. Given a signature $\Sigma$, a $\Sigma$-fuzzy algebra $A := ((A, \mu_A), \Sigma^A)$ is a fuzzy set $(A, \mu_A)$ and a collection $\Sigma^A = \{f^A : f \in O\} \cup \{c^A : c \in C\}$ of arrows:

$$f^A : (A, \mu_A)^{\sharp(f)} \to (A, \mu_A) \quad c^A : (1, c_\bot) \to (A, \mu_A)$$

where $c_\bot$ is the constant function in $\bot$. A morphism of $\Sigma$-fuzzy algebras $A \to B$ is an arrow $g : (A, \mu_A) \to (B, \mu_B)$ such that $g \circ c^A = c^B$ and $f^B \circ g^\sharp(f) = g \circ f^A$ for every $c \in C$ and $f \in O$. We will write $\Sigma\text{-Alg}$ for the resulting category of $\Sigma$-fuzzy algebras.

Remark 4.2. We will not distinguish between a function from the singleton and its value.

Definition 4.3. Let $L = (\Sigma, X)$ be a language and $A = ((A, \mu_A), \Sigma^A)$ be a $\Sigma$-algebra.

An assignment is simply a function $\iota : X \to A$. We define the evaluation in $A$ with respect to $\iota$ as the function $(-)^{\iota,A} : L\text{-Terms} \to A$ by induction:

$$x^{\iota,A} := \iota(x) \text{ if } x \in X;$$
$$c^{\iota,A} := c^A \text{ if } c \in C;$$
$$(f(t_1, \ldots, t_\varphi(f)))^{\iota,A} := f^A(t_1^{\iota,A}, \ldots, t_\varphi(f)^{\iota,A}) \text{ if } f \in O.$$

Proposition 4.4. Let $A$ be a $\Sigma$-algebra. Given a function $\sigma : X \to L\text{-Terms}$ and an assignment $\iota : X \to A$ define $\iota\sigma : X \to A$ as the assignment sending $x$ to $(\sigma(x))^{\iota,A}$. Then $A \models \iota_\sigma \phi$ if and only if $A \models \iota_\iota \phi$.

Proof. This follows at once noticing that $(t[\sigma])^{\iota,A} = t^{\iota,\iota}$ holds for every term $t$.

Definition 4.5. $A$ satisfies $\phi \in \text{Form}(L)$ with respect to $\iota$, and we write $A \models_\iota \phi$, if $\phi$ is $E(l, t)$ and $l \leq \mu_A(t^{\iota,A})$ or if $\phi$ is $t \equiv s$ and $t^{\iota,A} = s^{\iota,A}$.

$A$ satisfies $\phi$ if $A \models_\iota \phi$ for all $\iota : X \to A$, and we write $A \models \phi$.

Given $\Gamma \subseteq \text{Form}(L)$, $A \models \Gamma (A \models_\iota \Gamma)$ means $A \models \phi$ if $A \models_\iota \phi$ for every $\phi \in \Gamma$. Finally, given a sequent $\Gamma \proves \phi$ we say that $A$ satisfies it with respect to $\iota$ and we will write $A \satisfies _{\iota,A} \phi$ if $A \models_\iota \phi$ for any $\phi \in \Gamma$; if this happens for all assignments $\iota$ we say that $A$ satisfies the sequent and we will write $A \satisfies \Gamma$.

We say that a $\Sigma$-fuzzy algebra $A$ is a model of a fuzzy theory $\Lambda \in \text{Th}(L)$ if it satisfies all the sequents in it.

We will write $\text{Mod}(\Lambda)$ for the full subcategory of $\Sigma\text{-Alg}$ given by the models of $\Lambda$.

Clearly $\Sigma\text{-Alg} = \text{Mod}(\emptyset)$. For any $\Lambda \in \text{Th}(L)$ there exist two forgetful functors $U_\Lambda : \text{Mod}(\Lambda) \to \text{Fuz}(L)$ and $\Psi_\Lambda : \text{Mod}(\Lambda) \to \text{Set}$. We will write $U_\Sigma$ and $\Psi_\Sigma$ for $U_\emptyset$ and $\Psi_\emptyset$.

Proposition 4.6. For any signature $\Sigma$, $\Psi_\Sigma$ has a left adjoint $\text{Fuz}^{\text{Set}} : \text{Set} \to \text{Mod}(\Lambda)$.

Proof. For any set $X$ take the language $L_X$ and define $\text{Fuz}^{\text{Set}}(X) := (\nabla(L_X\text{-Terms}), \Sigma^{\text{Set}}(X))$ where $\Sigma^{\text{Set}}(X) := c$ and

$$f^{\text{Set}}(X) : \nabla(L_X\text{-Terms})^{\varphi(f)} \to \nabla(L_X\text{-Terms}) \quad (t_1, \ldots, t_\varphi(f)) \mapsto f(t_1, \ldots, t_\varphi(f))$$

for any $f \in O$. Now, it is immediate to see that for any $\iota : X \to \Psi_\Sigma(A)$ the evaluation $(-)^{\iota,A}$ is the unique morphism of $\Sigma\text{-Alg}$ that composed with the inclusion $X \to L_X\text{-Terms}$ gives back $\iota$. □
We now provide two technical results about interpretations. The first describes how interpretations are moved along morphisms of algebras.

**Proposition 4.7.** Let \( \mathcal{L} = (\Sigma, X) \) be a language, \( \Lambda \in \text{Th}(\mathcal{L}) \) and \( A = ((A, \mu_A), \Sigma^A) \), \( B = ((B, \mu_B), \Sigma^B) \) be two \( \Sigma \)-algebras. Let also \( f : A \to B \) be a morphism between them, then:

1. \( A \) is a model of \( \Lambda \) if and only if it is a model of \( \Lambda^r \);
2. \( f \circ (\neg)^A = (\neg)^B \circ f \) for every assignment \( \iota : X \to A \);
3. for any assignment \( \iota : X \to A, A \models \phi \) entails \( B \models f \circ \iota \phi \);
4. if \( \mathcal{U}_L(f) \) is a strong monomorphism in \( \text{Fuz}(H) \) and \( \iota : X \to A \) is an assignment then, for any formula \( \phi, A \models \phi \) if and only if \( B \models f \circ \iota \phi \);
5. if \( \mathcal{U}_L(f) \) is a strong monomorphism in \( \text{Fuz}(H) \) and \( B \in \text{Mod}(\Lambda) \) then \( A \in \text{Mod}(\Lambda) \).

**Proof.** 1. The implication from the right to the left is obvious since \( \Lambda \subseteq \Lambda^r \). The other follows from soundness.
2. This follows at once by structural induction.
3. This follows from the previous point: if \( \phi \) is \( t \equiv s \) then
   \[
   t^{B,f \circ \iota} = f(f^{A,\iota}) = f(s^{A,\iota}) = s^{B,f \circ \iota}
   
   \] otherwise
   \[
   l \leq \mu_A(t^{A,\iota}) \leq \mu_B(f(t^{A,\iota})) = \mu_B(t^{B,f \circ \iota})
   
   \]
4. We have to show only the implication from the right to the left. Let us proceed by cases remembering that by Proposition 2.4 \( \mathcal{U}_L(f) \) is a strong monomorphism if and only if it is injective and \( \mu_A(a) = \mu_B(f(a)) \) for any \( a \in A \).
   - \( \phi = t \equiv s, A \models f \circ \iota t \equiv s \) if and only if \( t^{A,\iota} = s^{A,\iota} \), by injectivity of \( f \) this is equivalent to \( f(t^{A,\iota}) = f(s^{A,\iota}) \).
   - \( \phi = \mathcal{E}(l,t) \). As before, \( A \models \phi \) if and only if \( \mu_A(t^{A,\iota}) \geq l \) but
   \[
   \mu_A(t^{A,\iota}) = \mu_B(f(t^{A,\iota})) = \mu_B(t^{B,f \circ \iota})
   
   \]
   and thus \( A \models \phi \) is equivalent to \( B \models f \circ \iota \phi \).
5. Let \( \Gamma \vdash \phi \) be in \( \Lambda \) and suppose \( A \models \phi, \Gamma \) for some assignment \( \iota : X \to B \), by the previous point this implies \( B \models f \circ \iota \phi, \Gamma \), so, by hypothesis \( B \models f \circ \iota \phi \) and we can conclude again using point 3.

We can also move interpretations and theories along morphisms of signatures.

**Definition 4.8.** For any \( F : \Sigma_1 \to \Sigma_2 \) arrow of \( \text{Sign} \) and any \( A = ((A, \mu_A), \Sigma^A) \in \Sigma_2\text{-Alg} \), we define \( r_F(A) = \left( (A, \mu_A), \Sigma_{\text{w}(F)}^A \right) \in \Sigma_1\text{-Alg} \) putting, for any \( f \in O_1 \)

\[
\begin{align*}
    f_{\text{w}(F)} : (A, \mu_A) &\to (A, \mu_A) \\
    (a_1, \ldots, a_{\text{w}(f)}) &\mapsto F_2(f)(A^1, \ldots, A_{\text{w}(f)})
\end{align*}
\]

and \( \text{e}_{\text{w}(F)}(A) := F_3(c)^A \) for every \( c \in C_1 \).

**Lemma 4.9.** Let \( \mathcal{L}_1 = (\Sigma_1, X) \) and \( \mathcal{L}_2 = (\Sigma_2, Y) \) and \( F = ((F_1, F_2), g) : \mathcal{L}_1 \to \mathcal{L}_2 \), then:

1. there exists a functor \( r_F : \Sigma_2\text{-Alg} \to \Sigma_1\text{-Alg} \) sending \( A \) to \( r_F(A) \);
2. \( f_{\text{w}(F)}(A) \circ g = (\text{Terms}(F(t))^{A^{\iota}})_{\text{w}(f)} \) for any assignment \( \iota : Y \to A \) and \( t \in \mathcal{L}_1\text{-Terms} \);
3. for any assignment \( \iota : Y \to A, r_F(A) \models \phi \) if and only if \( A \models \text{Form}(F)(\phi) \);
4. If \( X = Y \) and \( g = id_X \) then \( r_F \) restricts to a functor \( r_{F, A} : \text{Mod}(\Lambda) \to \text{Mod}(F^*(\Lambda)) \).
Fuzzy algebraic theories

10

Proof. 1. Let \( h : A \to B \) be a morphism of \( \Sigma_2\)-Alg then \( h \circ u^A = u^B \circ g_{\alpha_2(h)} \) and \( d^B = h \circ d^A \) for all \( u \in \Omega_2 \) and \( d \in C_2 \). In particular this is true when \( u = F_2(f) \) and \( d = F_3(c) \) for \( f \in O_1 \) and \( c \in C_1 \) but these are exactly the conditions making \( h \) into an homomorphism \( r_F(A) \to r_F(B) \).

2. This follows by induction:
   - for any \( x \in X \)
     \[ x^{r_F(A),\text{log}} = \iota(g(x)) = (\text{Terms}(F)(x))^{A,i} \]
   - and for any constant \( c \in C_1 \)
     \[ c^{r_F(A),\text{log}} = c^{r_F(A)} = F_3(c)^A = (\text{Terms}(F)(c))^A = (\text{Terms}(F)(c))^{A,i} \]
   - for any \( f \in O_1 \):
     \[ (f(t_1, \ldots, t_{ar(f)}))^{r_F(A),\text{log}} = f^{r_F(A)} \left( t_1^{r_F(A),\text{log}}, \ldots, t_{ar(f)}^{r_F(A),\text{log}} \right) \]
     \[ = f^{r_F(A)} \left( (\text{Terms}(F)(t_1))^{A,i}, \ldots, (\text{Terms}(F)(t_{ar(f)}))^{A,i} \right) \]
     \[ = F_1(f)^A (\text{Terms}(F)(t_1))^{A,i}, \ldots, (\text{Terms}(F)(t_{ar(f)}))^{A,i} \]
     \[ = (F_1(f) ((\text{Terms}(F)(t_1)), \ldots, (\text{Terms}(F)(t_{ar(f)}))))^{A,i} \]
     \[ = \text{Terms}(F)(f(t_1, \ldots, t_{ar(f)}))^{A,i} \]

3. This follows immediately by point 2:
   \[ r_F(A) \models_{\text{log}} (t \equiv s) \iff t^{r_F(A),\text{log}} = s^{r_F(A),\text{log}} \]
   \[ \iff (\text{Terms}(F)(t))^{A,i} = (\text{Terms}(F)(s))^{A,i} \]
   \[ \iff A \models_{\iota} \text{Form}(F)(t \equiv s) \]
   \[ r_F(A) \models_{\text{log}} E(l, t) \iff l \leq \mu_A(t^{r_F(A),\text{log}}) \]
   \[ \iff l \leq \mu_A(\text{Terms}(F)(t))^{A,i} \]
   \[ \iff A \models_{\iota} \text{Form}(F)(E(l, t)) \]

4. Let \( \Gamma \vdash \psi \) be in \( F^*(\Lambda) \), by the previous point, for any assignment \( \iota : X \to A \)
   \[ r_F(A) \models_{\text{log}} \Gamma \iff A \models_{\iota} \{ \text{Form}(F)(\phi) \}_{\phi \in \Gamma} \]
   Since \( \{ \text{Form}(F)(\phi) \}_{\phi \in \Gamma} \vdash \text{Form}(F)(\psi) \) is in \( \Lambda \) we have \( A \models_{\iota} \text{Form}(F)(\psi) \) but, using again point 3, this implies \( r_F(A) \models_{\iota} \phi \). □

Example 4.10. The models for \( \Lambda_S \), \( \Lambda_{LI} \), \( \Lambda_{RI} \) and \( \Lambda_I \) (Example 3.11) are precisely the structures defined in [17], while the models for \( \Lambda_G \) (Example 3.12) are precisely the fuzzy groups as in [21] and those of \( \Lambda_N \) are the structures called normal fuzzy subgroups in [2] [14] [13].

Soundness. Now we can proceed proving the soundness of the rules in Figure 1.

Lemma 4.11. Let \( \mathcal{L} = (\Sigma, X) \) be a language and \( A = ((A, \mu_A), \Sigma^A) \) a \( \Sigma \)-algebra, then:
1. for any assignment \( \iota : X \to A \) and rule
\[
\frac{\{ \Psi \vdash \xi \}_{i \in I}}{\Psi \vdash \xi} \quad R
\]
different from \( \text{Sub} \), if \( \Psi \vdash_{A,i} \xi \) for all \( i \in I \) then \( \Psi \vdash_{A} \xi \) too;

2. for any \( \sigma : X \to \mathcal{L}\)-Terms, if \( \Gamma \vdash_{A} \psi \) then \( \Gamma[\sigma] \vdash_{A} \psi[\sigma] \).

**Proof.**

1. The thesis is vacuous for rule \( A \) and it is trivial for \( \text{Refl}, \text{Sym} \) and \( \text{Trans} \). For rule \( \text{Weak} \) if \( A \vdash_{i}, \Gamma \cup \Delta \) then \( A \vdash_{i}, \Gamma \) and so, since \( \Gamma \vdash_{A,i} \phi \) this implies \( A \vdash_{i} \phi \). The thesis for \( \text{Inf} \) follows from the fact that \( \bot \) is the bottom element, for \( \text{Cong} \) and \( \text{Fun} \) it follows since \( \mu_{A} \) and \( f^{A} \in \Sigma^{A} \) are functions for any \( f \in O \). We’re left with four rules.

- **Cut.** For any assignment \( \iota \), if \( A \vdash_{i}, \Gamma \) then, since \( \Gamma \vdash_{A,i} \phi_{i} \) for any \( i \in I \) we have \( A \vdash_{i} \{ \phi_{i} \}_{i \in I} \), but \( \{ \phi_{i} \}_{i \in I} \vdash_{A,i} \psi \) and this implies \( A \vdash_{i} \psi \).

- **Sup.** Let \( \iota : X \to A \) be such that \( A \vdash_{i}, \Gamma \), then \( \Gamma \vdash_{A,i} E(l,t) \) for all \( l \in \mathcal{L} \) implies that \( \mu_{A}(t^{A}) \) is an upper bound of \( \mathcal{L} \), so the thesis follows.

- **Mon.** As before let \( \iota \) be an assignment for which \( A \) satisfies all the formulae in \( \Gamma \), then, since \( \Gamma \vdash_{A,i} E(l,t) \) we have that \( l \leq \mu_{A}(t^{A,i}) \), so, for any \( l' \in H \), \( l' \land l \leq \mu_{A}(t^{A,i}) \) holds too.

- **Exp.** For any \( \iota \) such that \( A \vdash_{i}, \Gamma \), \( \Gamma \vdash_{A,i} E(l,t_{i}) \) entails \( l_{i} \leq \mu_{A}(t_{i}^{A,i}) \) for any \( 1 \leq i \leq a(f) \), therefore:
\[
\bigwedge_{i=1}^{a(f)} l_{i} \leq \bigwedge_{i=1}^{a(f)} \mu_{A}(t_{i}^{A,i}) = \mu_{A}(t^{A,i}_{1}, \ldots, t^{A,i}_{a(f)}) \leq \mu_{A}(f^{A}(t^{A,i}_{1}, \ldots, t^{A,i}_{a(f)}))
\]
and we are done.

2. Let \( \iota : X \to A \) an assignment such that \( A \vdash_{i}, \phi_{i}[\sigma] \) for any \( \phi_{i} \in \Gamma \). By Proposition \( 4.4 \) this means that \( A \vdash_{i_{\sigma}} \Gamma \) and so, by hypothesis, \( A \vdash_{i_{\sigma}} \psi \) which, again by Proposition \( 4.4 \) entails \( A \vdash_{i} \psi[\sigma] \).

**Corollary 4.12** (Soundness). If a \( \Sigma \)-algebra satisfies all the premises of a rule of the fuzzy sequent calculus then it satisfies also its conclusion.

**Remark 4.13.** Let us see why the deduction lemma (Lemma \( 3.10 \)) cannot be extended to rule \( \text{Sub} \). Take \( \Sigma \) to be the empty set, \( X = \{ x, y, z \} \) and \( H = \{ 0, 1 \} \). Notice that \( \Sigma\text{-Alg} = \text{Fuz}(H) \). We have the derivation
\[
\vdash x \equiv y \\
\vdash x \equiv z \\
\text{Sub}
\]
If the deduction lemma held for \( \text{Sub} \), \( x \equiv y \vdash x \equiv z \) would be in \( \emptyset^{o} \), hence satisfied by any fuzzy set, but \( (H, \text{id}_{H}) \) with \( \iota : X \to H \) sending \( x \) and \( y \) to \( 0 \) and \( z \) to \( 1 \) is a counterexample.

**Remark 4.14.** Let us take \( \Sigma = \emptyset \) and \( H = \{ 0, 1 \} \) and \( X = \{ x, y, z \} \) and the derivation as in Remark \( 4.13 \). Now, a fuzzy set \( (A, \mu_{A}) \) satisfies \( \vdash_{i} x \equiv y \) if and only if \( \iota(x) = \iota(y) \), thus, if we take \( (H, \text{id}_{H}) \) and the assignment \( \iota \) of the previous example, then \( (H, \text{id}_{H}) \vdash_{i} x \equiv y \) but it does not satisfy \( x \equiv z \) with respect to \( \iota \).
Completeness  Now we prove that the calculus we have provided in Section 3 is complete. Let us start with the following observation.

Remark 4.15. For any \( \Lambda \in \text{Th}(\mathcal{L}) \) the relation \( \sim_\Lambda \) given by all \( t \) and \( s \) such that \( \vdash_\Lambda t \equiv s \), is an equivalence relation on \( \mathcal{L} \)-Terms.

Using this equivalence, we can define the model of terms, as done next.

Definition 4.16. Let \( \mathcal{L} = (\Sigma, X) \) be a language and \( \Lambda \in \text{Th}(\mathcal{L}) \), we define \( \text{Terms}(\Lambda) \) to be the quotient of \( \mathcal{L} \)-Terms by \( \sim_\Lambda \), moreover, by rule \( \text{FUN} \), the function

\[
\hat{\mu} : \mathcal{L} \text{-Terms} \to H \quad t \mapsto \sup \{ l \in H \mid \vdash_\Lambda E(l, t) \}
\]

induces a function \( \mu_\Lambda : \text{Terms}(\Lambda) \to H \). For any \( f \in O \) and \( c \in C \) putting \( cT_\Lambda := [c] \) and

\[
f^{T_\Lambda} : \text{Terms}(\Lambda)^{ar(f)} \to \text{Terms}(\Lambda) \quad ([t_1], \ldots, [t_{ar(f)}]) \mapsto [f(t_1, \ldots, t_{ar(f)})]
\]

gives us a \( \Sigma \)-algebra \( T_\Lambda = ((\text{Terms}(\Lambda), \mu_\Lambda), \Sigma T_\Lambda) \), called the \( \Sigma \)-algebra of terms in \( \Lambda \). The canonical assignment is the function \( \iota_{\text{can}} : X \to \text{Terms}(\Lambda) \) sending \( x \) to its class \([x]\).

Remark 4.17. Rule \( \text{CONG} \) assures us that \( f^{T_\Lambda} \) is well defined while \( \text{EXP} \) implies that it is an arrow of \( \text{Fuz}(H) \).

Lemma 4.18. Let \( \mathcal{L} = (\Sigma, X) \) be a language and \( \Lambda \in \text{Th}(\mathcal{L}) \), then:

1. for any \( \phi \in \text{Form}(\mathcal{L}) \) the following are equivalent:
   a. \( T_\Lambda \models \phi \),
   b. \( T_\Lambda \models_{\iota_{\text{can}}} \phi \),
   c. \( \vdash_\Lambda \phi \);

2. for any assignment \( \iota : X \to \text{Terms}(\Lambda) \) and formula \( \phi \), \( T_\Lambda \models_\iota \phi \) if and only if \( \vdash_\Lambda \phi_\sigma \) for one (and thus any) section \( \sigma \) of the quotient \( \mathcal{L} \)-Terms \( \to \text{Terms}(\Lambda) \);

3. \( T_\Lambda = ((\text{Terms}(\Lambda), \mu_\Lambda), \Sigma T_\Lambda) \) is a model of \( \Lambda \).

Proof. Let us start with a technical result.

Proposition 4.19. Let \( \mathcal{L} = (\Sigma, X) \) be a language, \( \Lambda \in \text{Th}(\mathcal{L}) \), and \( \sigma : \text{Terms}(\Lambda) \to \mathcal{L} \)-Terms a section of the quotient \( \mathcal{L} \)-Terms \( \to \text{Terms}(\Lambda) \). The equation \( t_{\Lambda, \iota}^{T_\Lambda} = [t_\sigma \circ \iota] \) holds for any assignment \( \iota : X \to \text{Terms}(\Lambda) \) and \( t \in \mathcal{L} \)-Terms. In particular \( t_{\Lambda, \iota}^{T_\Lambda} = [t] \).

Proof. For constants it is obvious, let us proceed by induction.

For any \( x \in X \), \( \sigma(x) \) is a representative of \( \iota(x) \), so:

\[
x_{T_\Lambda, \iota} = \iota(x) = [\sigma(x)]/x = [x_\sigma \circ \iota]
\]

For any \( f \in O \) and \( t_1, \ldots, t_{ar(f)} \in \mathcal{L} \)-Terms:

\[
(f(t_1, \ldots, t_{ar(f)}))_{T_\Lambda, \iota}^{T_\Lambda} = f^{T_\Lambda}(t_1_{T_\Lambda, \iota}^{T_\Lambda}, \ldots, t_{T_\Lambda, \iota}^{T_\Lambda}) = f^{T_\Lambda}(t_1_{T_\Lambda, \iota}^{T_\Lambda}, \ldots, t_{T_\Lambda, \iota}^{T_\Lambda})
\]

\[
= f^{T_\Lambda}([t_1_{\sigma \circ \iota}], \ldots, [t_{ar(f)}_{\sigma \circ \iota}]) = [f(t_1_{\sigma \circ \iota}, \ldots, t_{ar(f)}_{\sigma \circ \iota})][\sigma \circ \iota]
\]

Now we can proceed with the proof of Lemma 4.18.

1. Let us show the three implications.
exists and it is unique up to isomorphism.

To give the definition of free models (Definition 5.8) we need some preliminary constructions.

The free fuzzy algebra on a fuzzy set

Let \( \Sigma \) be a \( \Sigma \)-algebra and \( f : (B, \mu_B) \to \mathcal{U}_\Sigma(A) \) an arrow in \( \mathbf{Fuz}(H) \), a \( \Sigma \)-algebra generated by \( f \) in \( A \) is a morphism \( \epsilon : (B, \mu_B)_{A,f} \to A \) such that:

- \( \mathcal{U}_\Sigma(\epsilon) \) is strong mono;
- there exists \( f : (B, \mu_B) \to (B, \mu_B)_{A,f} \) such that \( \mathcal{U}_\Sigma(\epsilon) \circ f = f \);
- if \( g : C \to A \) is a morphism such that \( \mathcal{U}_\Sigma(g) \) is strong monomorphism and \( \mathcal{U}_\Sigma(g) \circ h = f \) for some \( h \) then there exists a unique \( k : (B, \mu_B)_{A,f} \to C \) such that \( g \circ k = \epsilon \).

Let \( \Sigma \) be a \( \Sigma \)-algebra and \( f : (B, \mu_B) \to \mathcal{U}_\Sigma(A) \). \( (B, \mu_B)_{A,f} \) exists and it is unique up to isomorphism.

Proof. This is straightforward: if \( A = ((A, \mu_A), \Sigma^A) \) it is enough to consider as carrier set \( \mathcal{U}_\Sigma((B, \mu_B)_{A,f}) \) the smallest subset of \( A \) such that:
Fuzzy algebraic theories

\[ f(B) \subset \mathcal{U}_\Sigma((B, \mu_B)_{A,f}); \]
\[ \{ c^A \}_{c \in C} \subset \mathcal{U}_\Sigma((B, \mu_B)_{A,f}); \]
\[ \text{if } g \in O \text{ and } b_1, \ldots, b_{\alpha(g)} \in \mathcal{U}((B, \mu_B)_{A,f}) \text{ then } g^A(b_1, \ldots, b_{\alpha(g)}) \in \mathcal{U}_\Sigma((B, \mu_B)_{A,f}). \]

with the restriction of \( \mu_A \) as membership degree function. As \( \Sigma \)-algebra structure we can take the restriction of \( \Sigma^A \). By Proposition 2.4 the inclusion \( i \) is a strong monomorphism and construction an arrow of \( \Sigma \text{-Alg} \). Let \( g : C \to A \) and \( h \) be as in the definition, if we consider the inclusion \( i : (f(B), \mu_{A,(f(B))}) \to (A, \mu_A) \), by the left lifting property we obtain

\[ \begin{array}{ccc}
(B, \mu_B) & \xrightarrow{h} & \mathcal{U}_\Sigma(C) \\
\tilde{f} & \xrightarrow{k} & \mathcal{U}_\Sigma(g) \\
(f(B), \mu_{A,(f(B))}) & \xrightarrow{i} & (A, \mu_A)
\end{array} \]

and we can define \( k \) by induction as:

\[ k(b) = \tilde{k}(b) \text{ if } b \in f(B); \]
\[ k(c^A) = c^C \text{ if } c \in C; \]
\[ k(t^A(b_1, \ldots, b_{\alpha(t)})) = t^C(k(b_1), \ldots, k(b_{\alpha(t)})) \text{ if } t \in O \text{ and } b_1, \ldots, b_{\alpha(t)} \in (B, \mu_B)_{A,f}. \]

Notice that injectivity of \( g \) implies that \( k \) is actually well-defined, in fact, proceeding by induction:

\[ \text{if } x \in B \text{ is such that } f(x) = c^A \text{ for some } c \in C \text{ then} \]
\[ g(b(x)) = \epsilon(f(x)) = c^A = g(c^C) \]
\[ \text{so} \]
\[ k(f(x)) = h(x) = c^C = k(c^A) \]
\[ \text{if } x \in B \text{ is such that } f(x) = t^A(b_1, \ldots, b_{\alpha(t)}) \text{ for some } t \in O \text{ and } b_1, \ldots, b_{\alpha(t)} \in (B, \mu_B)_{A,f} \text{ then} \]
\[ g(b(x)) = \epsilon(f(x)) = t^A(b_1, \ldots, b_{\alpha(t)}) \]
\[ = t^A(g(k(b_1)), \ldots, g(k(b_{\alpha(t)}))) = g(t^C(b_1, \ldots, b_{\alpha(t)})) \]
\[ \text{so } h(x) = t^C(b_1, \ldots, b_{\alpha(t)}) \text{ and we are done}; \]
\[ \text{if } c \in C \text{ is such that } c^A = t^A(b_1, \ldots, b_{\alpha(t)}) \text{ for some } t \in O \text{ and } b_1, \ldots, b_{\alpha(t)} \in (B, \mu_B)_{A,f} \text{ then} \]
\[ g(c^C) = c^A = t^A(b_1, \ldots, b_{\alpha(t)}) = t^A(g(k(b_1)), \ldots, g(k(b_{\alpha(t)}))) = g(t^C(b_1, \ldots, b_{\alpha(t)})) \]
\[ \text{so } c^C = t^C(b_1, \ldots, b_{\alpha(t)}) \text{ and we are done again}; \]
\[ \text{if } s^A(b'_1, \ldots, b'_{\alpha(s)}) = t^A(b_1, \ldots, b_{\alpha(t)}) \text{ for some } t, s \in O \text{ and } b'_1, \ldots, b'_{\alpha(s)}, b_1, \ldots, b_{\alpha(t)} \in (B, \mu_B)_{A,f} \text{ then} \]
\[ g(c^C(k(b'_1), \ldots, k(b'_{\alpha(s)}))) = s^A(b'_1, \ldots, b'_{\alpha(s)}) = t^A(b_1, \ldots, b_{\alpha(t)}) \]
\[ = t^A(g(k(b_1)), \ldots, g(k(b_{\alpha(t)}))) = g(t^C(b_1, \ldots, b_{\alpha(t)})) \]
\[ \text{so } s^C(k(b'_1), \ldots, k(b'_{\alpha(s)})) = t^C(b_1, \ldots, b_{\alpha(t)}) \text{ and even in this case } k \text{ is well defined.} \]
Moreover, for any \( t \in (B, \mu_B)_{A,f} \), if \( \mathcal{U}(\mathcal{C}) = (D, \mu_D) \) then
\[
\mu_D(k(t)) = \mu_A(g(k(t))) = \mu_A(i(t))
\]
Uniqueness follows at once by induction.

- **Remark 5.3.** Proposition 4.7 implies that, given a model \( A = ((A, \mu_A), \Sigma^A) \) of a theory \( \Lambda \in \text{Th}(\mathcal{L}) \), and a morphism \( f : (B, \mu_B) \to (A, \mu_A) \), the \( \Sigma \)-algebra \( \langle B, \mu_B \rangle_{A,f} \) is in \( \text{Mod}(\Lambda) \).

- **Proposition 5.4.** Let \( A \) be a \( \Sigma \)-algebra and \( f : (B, \mu_B) \to \mathcal{U}_\Sigma(A) \), then, for any other \( \Sigma \)-algebra \( \mathcal{C} \) and \( h : (B, \mu_B) \to \mathcal{U}_\Sigma(\mathcal{C}) \) there exists at most one \( k : (B, \mu_B)_{A,f} \to \mathcal{C} \) such that \( k \circ f = h \).

**Proof.** Let \( k \) and \( k' \) as in the thesis, we can proceed by induction:

- \( k(f(x)) = k'(f(x)) \) for any \( x \in B \) by hypothesis;
- \( k(c^{(B, \mu_B)_{A,f}}) = k'(c^{(B, \mu_B)_{A,f}}) \) since \( k \) and \( k' \) are morphism of \( \Sigma \)-Alg;
- \( k(g^{(B, \mu_B)_{A,f}}(b_1, \ldots, b_{\text{ar}(g)})) = g^c(k(b_1), \ldots, k(b_{\text{ar}(g)})) \), by inductive hypothesis this is equal to \( g^c(k'(b_1), \ldots, k'(b_{\text{ar}(g)})) \) and we conclude using again the fact that \( k' \) is an arrow of \( \Sigma \)-Alg.

The next definition explains how to extend a theory in a given language with the data of a fuzzy set.

- **Definition 5.5.** Let \( (M, \mu_M) \) be a fuzzy set, \( \mathcal{L} = (\Sigma, X) \) a language with \( \Sigma = (O, \text{ar}, C) \). We define \( \Sigma[M, \mu_M] \) to be \( (O, \text{ar}, C \cup M) \) and \( \mathcal{L}_{(M, \mu_M)} \) to be \( (\Sigma[M, \mu_M], X) \). We have an obvious morphism \( 1 : \mathcal{L} \to \mathcal{L}_{(M, \mu_M)} \) given by the identities and the inclusion \( \text{id}: C \to C \cup M \).

  For any \( \Lambda \in \text{Th}(\mathcal{L}) \) we define \( \Lambda[M, \mu_M] \in \mathcal{L}_{(M, \mu_M)} \) as \( \mathcal{L}_{(\Lambda)}(\Lambda) \cup (M, \mu_M) \) where \( (M, \mu_M) = \{ t \in E(l, m) \mid m \in M, l \in L \text{ and } \mu_M(m) \geq l \} \).
Definition 5.8. For any language $L$, $\Lambda \in \text{Th}(L)$ and $(M, \mu^M) \in \text{Fuz}(H)$ we define the free model $F^\Lambda(M, \mu^M)$ of $\Lambda$ generated by $(M, \mu^M)$ to be $\mathfrak{r}_L \Lambda(M, \mu^M)(\langle M, \mu^M \rangle_{\text{Th}}, \eta_{(M, \mu^M)})$. We set $T^\Lambda(M, \mu^M)$ to be $\mathcal{U}_L(F^\Lambda(M, \mu^M))$.

Now it is enough to check that the free model just defined actually provides the left adjoint.

Theorem 5.9. For any language $L$ and $\Lambda \in \text{Th}(L)$ the functor $\mathcal{U}_L : \text{Mod}(\Lambda) \rightarrow \text{Fuz}(L)$ has a left adjoint $F^\Lambda$.

Proof. By construction $\eta_{(M, \mu^M)}$ factors through $\eta_{(M, \mu^M)} : (M, \mu^M) \rightarrow T^\Lambda(M, \mu^M)$ which sends $m$ to $[m]$. Let now $g : (M, \mu^M) \rightarrow \mathcal{U}_L(\mathcal{B})$ be another arrow in $\text{Fuz}(H)$, we can turn $\mathcal{B}$ into a $\Sigma[M, \mu^M]$-algebra $B^g$ setting $m^{B^g}$ to be $g(m)$ for any $m \in M$.

Let us show that $B^g$ is a model of $\Lambda[M, \mu^M]$. Surely it is a model of $\Lambda$ since $B$ is, let $\vdash E(l, m)$ be a sequent in $(M, \mu^M)$, then for any assignment $\nu : V \rightarrow B$:

$$B^g \models_\nu E(l, m) \iff l \leq \mu_B(m^{B^g}) \iff \mu_\Lambda \left( \frac{E^\Lambda(M, \mu^M)}{\mathcal{U}_L(\mathcal{B})} \right) = l \leq \mu_B(g(m))$$

but $g$ is an arrow of $\text{Fuz}(H)$ so $\mu_B(g(m)) \geq \mu^M(m)$ and we are done.

Now, since $B^g$ is a model of $\Lambda[M, \mu^M]$, we can take $\bar{g}$ to be the image through $\mathfrak{r}_L \Lambda[M, \mu^M]$ of the unique arrow $\langle M, \mu^M \rangle_{\text{Th}}(\eta_{(M, \mu^M)}) \rightarrow B^g$, by construction

$$\bar{g}(\eta_{(M, \mu^M)}(m)) = \bar{g}([m]) \in \mu^{B^g} = g(m)$$

so $\mathcal{U}_L(\bar{g}) \circ \eta_{(M, \mu^M)} = g$. Uniqueness follows from Proposition 5.7.

Definition 5.10. Given a theory $\Lambda \in \text{Th}(L)$, its associated monad $T^\Lambda : \text{Fuz}(H) \rightarrow \text{Fuz}(H)$ is the composite $\mathcal{U}_L \circ F^\Lambda$.

Remark 5.11. If $\Lambda$ is the empty theory (in any language), then, by Proposition 4.6 the composition $F^\emptyset \circ \mathcal{U}_\Sigma$ gives us a functor isomorphic to $F^\emptyset_{\text{Set}}$.

Notation. We will denote by $F^\emptyset$ with $F^\Sigma$ and with $T^\Sigma$ the monad $T^\emptyset = \mathcal{U}_\Sigma \circ F^\emptyset$.

In this setting we can provide a result similar to Lemma 4.18.

Lemma 5.12. For any language $L = (\Sigma, X)$ we define the natural assignment $\nu_{\text{nat}}$ as the adjoint to the unit $\nabla(X) \rightarrow T^\Lambda(\nabla(X))$. Then $F^\Lambda(\nabla(X)) \models_{\nu_{\text{nat}}} \phi$ if and only if $\vdash \Lambda \phi$.

Proof. The implication from the right to the left follows immediately since $F^\Lambda(\nabla(X))$ is a model for $\Lambda$. By adjointness he canonical assignment $\nu_{\text{can}}$ induces an arrow $\nabla(X) \rightarrow \mathcal{U}_L(\nabla(X))(T^\Lambda(\nabla(X)))$, which, in turn, induces a morphism $e : F^\Lambda(\nabla(X)) \rightarrow T^\Lambda(\nabla(X))$ of $\Sigma$-algebras such that, as function between sets, $e \circ \nu_{\text{nat}} = \nu_{\text{can}}$. Recalling that $I$ is the arrow $(\Sigma, X) \rightarrow (\Sigma[\nabla(X)], X)$ and using Proposition 4.7 Lemma 4.9 and Lemma 4.18

$$F^\Lambda(\nabla(X)) \models_{\nu_{\text{nat}}} \phi \iff \mathfrak{r}_L(\nabla(X))(T^\Lambda(\nabla(X))) \models_{\nu_{\text{nat}}} \phi$$

$$\iff \mathfrak{r}_L(\nabla(X)) \models_{\nu_{\text{nat}}} \phi$$

$$\iff \nabla(X) \models_{\nu_{\text{nat}}} \phi$$

$$\iff T^\Lambda(\nabla(X)) \models_{\nu_{\text{can}}} \phi$$

$$\iff \vdash \Lambda(\nabla(X)) \phi$$

Now, by definition $\nabla(X)$ is equal to $\{ \vdash E(1, x) \mid x \in X \}$, therefore $(\Lambda[\nabla(X)])^\dagger = \Lambda^\dagger$ and we get the thesis.
5.2 Eilenberg-Moore algebras and models

In this section we will compare the category \textbf{Mod}(\Lambda) of models of some \( \Lambda \in \text{Th}(\mathcal{L}) \) and \textbf{Alg}(T_{\Lambda}) of Eilenberg-Moore algebras for the corresponding monad \( T_{\Lambda} \). First of all we recall the following classic lemma ([7, Prop. 4.2.1] and [12, Theorem VI.3.1]).

\begin{lemma}
Let \( L : \mathcal{C} \to \mathcal{D} \) be a functor with right adjoint \( R \) and let \( T = R \circ L \) be its associated monad, then there exists a comparison functor \( K : \mathcal{D} \to \textbf{Alg}(T) \) such that \( \mathcal{U}_T \circ K = R \), where \( \mathcal{U}_T : \textbf{Alg}(T) \to \mathcal{C} \) is the forgetful functor. \( K \) sends \( D \) in \( (R(D), R(\epsilon_D)) \), where \( \epsilon \) is the counit of the adjunction.
\end{lemma}

As a consequence, for any theory \( \Lambda \) we have a functor from \textbf{Mod}(\Lambda) to \textbf{Alg}(T_{\Lambda}). We want to construct an inverse of such functor.

\begin{definition}
Let \( \Lambda \) be in \( \text{Th}(\mathcal{L}) \) and \( \xi : T_{\Lambda}(M, \mu_M) \to (M, \mu_M) \) an object of \textbf{Alg}(T_{\Lambda}), we define its associated algebra \( \mathcal{H}(\xi) = (M, \mu_M, \Sigma^{\mathcal{H}(\xi)}) \) putting
\[
c^{\mathcal{H}(\xi)} := \xi(c_{\mathcal{L}, (X, \mu_X)}), \quad f^{\mathcal{H}(\xi)} := \xi \circ f_{\mathcal{L}, (X, \mu_X)} \circ \eta_{(M, \mu_M)}
\]
for every \( c \in C \) and \( f \in O \).
\end{definition}

\begin{lemma}
For any Eilenberg-Moore algebra \( \xi : T_{\Lambda}(M, \mu_M) \to (M, \mu_M) \), \( \xi \) itself is an arrow \( \mathcal{H}(\xi) : L_{\Lambda}(X, \mu_X) \to \mathcal{H}(\xi) \) of \( \Sigma\text{-Alg} \).
\end{lemma}

\textbf{Proof}. Let us start with the following observation.

\begin{claim}
Let \( \Lambda \) be a theory in the language \( \mathcal{L} \) and \( \hat{\mu} \) the multiplication of \( T_{\Lambda} \), then, for any \( g : (A, \mu_A) \to (B, \mu_B) \) and operation \( f \) the following diagrams commutes:
\[
\begin{aligned}
&T_{\Lambda}(T_{\Lambda}(M, \mu_M))^{\mathcal{H}(f)} \ar[d]_{\hat{\mu}^{\mathcal{H}(f)}} \ar[r]^{f_{\mathcal{L}_{\Lambda}(T_{\Lambda}(M, \mu_M))}} & T_{\Lambda}(T_{\Lambda}(M, \mu_M)) \ar[d]_{\hat{\mu}(M, \mu_M)} \ar[r]^{c_{\mathcal{L}_{\Lambda}(T_{\Lambda}(M, \mu_M))}} & T_{\Lambda}(T_{\Lambda}(M, \mu_M)) \ar[dl]_{\hat{\mu}(M, \mu_M)} \ar[r]^{c_{\mathcal{L}_{\Lambda}(T_{\Lambda}(M, \mu_M))}} & T_{\Lambda}(T_{\Lambda}(M, \mu_M)) \\
&T_{\Lambda}(M, \mu_M) \ar[d]_{T_{\Lambda}(g)^{\mathcal{H}(f)}} \ar[r]^{f_{\mathcal{L}_{\Lambda}(T_{\Lambda}(M, \mu_M))}} & T_{\Lambda}(M, \mu_M) \ar[d]_{T_{\Lambda}(g)} \ar[r]^{c_{\mathcal{L}_{\Lambda}(T_{\Lambda}(M, \mu_M))}} & T_{\Lambda}(M, \mu_M) \ar[dl]_{c_{\mathcal{L}_{\Lambda}(T_{\Lambda}(M, \mu_M))}} \ar[r]^{c_{\mathcal{L}_{\Lambda}(T_{\Lambda}(M, \mu_M))}} & T_{\Lambda}(M, \mu_M)
\end{aligned}
\]
\end{claim}

\textbf{Proof}. \( T_{\Lambda}(g) = U_{\Lambda}(\mathcal{F}_{\Lambda}(g)) \) and \( \hat{\mu}_{(A, \mu_A)} = U_{\Lambda}(c_{\mathcal{L}_{\Lambda}(g), (A, \mu_A)}) \) where \( c : \mathcal{F}_{\Lambda} \circ U_{\Lambda} \to \text{id}_{\textbf{Mod}(\Lambda)} \) is the counit ([12] proposition 4.2.1 or [12, chapter VI]). Now the thesis follows since both \( \mathcal{F}_{\Lambda}(g) \) and \( c_{\mathcal{L}_{\Lambda}(g), (A, \mu_A)} \) are arrows of \( \Sigma\text{-Alg} \).

We can now come back to the proof of our lemma: \( \xi \circ c_{\mathcal{L}_{\Lambda}(M, \mu_M)} = c^{\mathcal{H}(\xi)} \) while the other condition is equivalent to commutativity of the outer rectangle in the diagram:
In general $\mathcal{H}(\xi)$ is not a model of $\Lambda$, but we can restrict to a class of theories such this holds. As in [4][13], we consider theories whose sequents’ premises contain only variables.

**Definition 5.17.** A theory $\Lambda \in \text{Th}(\mathcal{L})$ is basic\(^1\) if, for any sequent $\Gamma \vdash \phi$ in it, all the formulae in $\Gamma$ contain only variables.

**Example 5.18.** Fuzzy groups, fuzzy normal groups, fuzzy semigroups and left, right, bilateral ideals (Examples 3.11 and 3.12) are all examples of basic theories.

**Lemma 5.19.** $\mathcal{H}(\xi)$ is a model of $\Lambda$ for any basic theory $\Lambda \in \text{Th}(\mathcal{L})$ and Eilenberg-Moore algebra $\xi : T_\Lambda(M, \mu_M) \to (M, \mu_M)$.

**Proof.** We can start by observing that if $\Gamma \vdash \phi$ is in $\Lambda$ and $t : X \to M$ is an assignment such that $\mathcal{H}(\xi) \models \Gamma$ then $\mathcal{F}_\Lambda(M, \mu_M) \models_{\eta_{(M, \mu_M)}} \phi$. Indeed, for every $\psi$ in $\Gamma$, we have two cases:

- $\psi$ is $x \equiv y$. Since $t(x) = t(y)$ we can easily conclude.
- $\psi$ is $E(l, x)$. The thesis follows at once since the membership degree of $\eta_{(M, \mu_M)}(t(x))$ in $T_\Lambda(M, \mu_M)$ is greater than $\mu_M(t(x))$.

Therefore, we know that $\mathcal{F}_\Lambda(M, \mu_M) \models_{\eta_{(M, \mu_M)}} \phi$. Let us split again the two cases.

- $\phi$ is $t \equiv s$. In this case, $t\mathcal{F}_\Lambda(M, \mu_M)_{\eta_{(M, \mu_M)}} = s\mathcal{F}_\Lambda(M, \mu_M)_{\eta_{(M, \mu_M)}}$, point 2 of Proposition 4.7 and the fact that $\xi$ is an Eilenberg-Moore algebra thus imply:

\[
\begin{align*}
\Delta^\mathcal{H}(\xi) & = \Delta^\mathcal{H}(\xi) \circ \eta_{(M, \mu_M)} \circ t = \xi \left( t\mathcal{F}_\Lambda(M, \mu_M)_{\eta_{(M, \mu_M)}} \right) \\
& = \xi \left( s\mathcal{F}_\Lambda(M, \mu_M)_{\eta_{(M, \mu_M)}} \right) = \Delta^\mathcal{H}(\xi) \circ \eta_{(M, \mu_M)} = \Delta^\mathcal{H}(\xi)
\end{align*}
\]

- $\phi$ is $E(l, t)$. This means that $l \leq \mu_M \left( t\mathcal{F}_\Lambda(M, \mu_M)_{\eta_{(M, \mu_M)}} \right)$, hence, using again Lemma 5.15 and Proposition 4.7

\[
\begin{align*}
l \leq & \mu_M \left( t\mathcal{F}_\Lambda(M, \mu_M)_{\eta_{(M, \mu_M)}} \right) \\
& \leq \mu_M \left( \xi \left( t\mathcal{F}_\Lambda(M, \mu_M)_{\eta_{(M, \mu_M)}} \right) \right) \\
& = \mu_M \left( \Delta^\mathcal{H}(\xi) \right)
\end{align*}
\]

and we can conclude.

\(^1\) In [3] such theories are called *simple*. 
Theorem 5.20. For any basic theory $\Lambda \in \text{Th}(\mathcal{L})$, the functor $K : \text{Mod}(\Lambda) \to \text{Alg}(T_\Lambda)$ has an inverse $H : \text{Alg}(T_\Lambda) \to \text{Mod}(\Lambda)$ sending $\xi : T_\Lambda(M, \mu_M) \to (M, \mu_M)$ to $H(\xi)$.

Proof. Let $\xi : T_\Lambda(M, \mu_M) \to (M, \mu_M)$, $\xi' : T_\Lambda(N, \mu_N) \to (N, \mu_N)$ be two Eilenberg-Moore algebras and $g : (M, \mu_M) \to (N, \mu_N)$ an arrow between them, we claim that $g$ itself is a morphism of $\text{Mod}(\Lambda)$. Let $f \in O$ and $c \in C$, we have diagrams:

Commutativity of (1) and (4) follows from Claim 5.16 that of (3) from naturality of $\eta$, (2) and (5) from the fact that $g$ is an arrow of $\text{Alg}(T_\Lambda)$. So $H$ is a functor. Since both $H$ and $K$ are the identity on arrows, it is enough to show that they are the inverse of each other on objects. Let $\xi : T_\Lambda(M, \mu_M) \to (M, \mu_M)$ be in $\text{Alg}(T_\Lambda)$, then Lemma 5.15 implies that $\xi$ is a morphism of $\text{Mod}(\Lambda)$ such that $U_\Lambda(\xi) \circ \eta(M, \mu_M) = \text{id}_{(M, \mu_M)}$, but there is only one such morphism, namely the component of the counit in $H(\xi)$, so $U_\Lambda(H(\xi)) = \xi$ and $K \circ H = \text{id}_{\text{Alg}(T_\Lambda)}$. On the other hand, let $A = ((A, \mu_A), \Sigma^A) \in \text{Mod}(\Lambda)$, and consider, for any $f \in O$ and $c \in C$, the diagrams:

Commutativity of (1) and (3) follows since each component of $\epsilon$ is an arrow of $\text{Mod}(\Lambda)$, that of (2) since $\epsilon$ is the counit. So we can deduce now that $f^H(U_\Lambda(\epsilon_A)) = f^A$ and $c^H(U_\Lambda(\epsilon_A)) = c^A$ from which we can deduce that $H \circ K = \text{id}_{\text{Mod}(\Lambda)}$.

Corollary 5.21. For any basic theory $\Lambda \in \text{Th}(\mathcal{L})$, $\text{Alg}(T_\Lambda)$ and $\text{Mod}(\Lambda)$ are isomorphic, and thus equivalent, categories.
6 Equational axiomatizations

In this section we prove two results for our calculus analogous to the classic HSP theorem \([15]\), using the results by Milius and Urbat \([15]\).

The abstract framework Let us start recalling the tools introduced in \([15]\), adapted to our situation. In the following we will fix a tuple\(^2\)(\(C, (\mathcal{E}, \mathcal{M}), X\)) where \(C\) is a category, \((\mathcal{E}, \mathcal{M})\) is a proper factorization system on \(C\) and \(X\) is a class of objects of \(C\).

- Definition 6.1. An object \(X\) of \(C\) is projective with respect to an arrow \(f : A \rightarrow B\) if for any \(h : X \rightarrow B\) there exists a \(k : X \rightarrow A\) such that \(f \circ k = h\). We define \(\mathcal{E}_X\) as the class of \(e \in \mathcal{E}\) such that for every \(X \in X\), \(X\) is projective with respect to \(e\). A \(\mathcal{E}_X\)-quotient is just an arrow in \(\mathcal{E}_X\).

In the rest of the section, we assume that \((C, (\mathcal{E}, \mathcal{M}), X)\) satisfies the following requirements:

- \(C\) has all (small) products;
- for any \(X \in X\), the class \(X \downarrow C\) of all \(e \in \mathcal{E}\) with domain \(X\) is essentially small, i.e. there is a set \(J \subset X \downarrow C\) such that for any \(e : X \rightarrow C \in X \downarrow C\) there exists \(e' : X \rightarrow D \in J\) and an isomorphism \(\phi\) such that \(\phi \circ e = e';\)
- for every object \(C\) of \(C\) there exists \(e : X \rightarrow C\) in \(\mathcal{E}_X\) with \(X \in X\).

- Definition 6.2. An \(X\)-equation is an arrow \(e \in X \downarrow C\) with \(X \in X\). We say that an object \(A\) of \(C\) satisfies \(e : X \rightarrow C\), and we write \(A \models_X e\), if for every \(h : X \rightarrow A\) there exists \(q : C \rightarrow A\) such that \(q \circ e = h\). Given a class \(\mathcal{E}\) of \(X\)-equations, we define \(\mathcal{V}(\mathcal{E})\) as the full subcategory of \(C\) given by objects that satisfy \(e\) for every \(e \in \mathcal{E}\). A full subcategory \(\mathcal{V}\) is \(X\)-equationally presentable if there exists \(\mathcal{E}\) such that \(\mathcal{V} = \mathcal{V}(\mathcal{E})\).

- Remark 6.3. The definition of equation in \([15]\) Def. 3.3] is given in terms of suitable subclasses of \(X \downarrow C\). However in our setting Milius and Urbat’s definition reduces to ours (cfr. \([15]\) Remark 3.4).

- Theorem 6.4 \([15]\) Th. 3.15, 3.16]. A full subcategory \(\mathcal{V}\) of \(C\) is \(X\)-equationally presentable if and only if it is closed under \(\mathcal{E}_X\)-quotients, \(\mathcal{M}\)-subobjects and (small) products.

Application to fuzzy algebras In order to apply the results above to \(\Sigma\text{-Alg}\), we need to define the required inputs, i.e., to specify a factorization system and a class of \(\Sigma\)-algebras.

- Lemma 6.5. For any signature \(\Sigma\) the classes \(\mathcal{E}_\Sigma := \{e\text{ arrow of }\Sigma\text{-Alg} | \mathcal{U}_\Sigma(e)\text{ is epi}\}\) and \(\mathcal{M}_\Sigma := \{m\text{ arrow of }\Sigma\text{-Alg} | \mathcal{U}_\Sigma(m)\text{ is strong mono}\}\) form a proper factorization system on \(\Sigma\text{-Alg}\).

Proof. Let \(A = ((A, \mu_A), \Sigma^A)\) and \(B = ((B, \mu_B), \Sigma^B)\) two \(\Sigma\)-algebras with a morphism \(g : A \rightarrow B\) between them. \(\mathcal{U}_\Sigma(g)\) factors as \(m \circ e\) where \(e : (A, \mu_A) \rightarrow (g(A), \mu_{B|\phi(A)})\) and \(m : (g(A), \mu_{B|\phi(A)}) \rightarrow (B, \mu_B)\) is the usual epi-monomorphism factorization of \(f\) on \(Fuz(H)\) (cfr. Lemma 2.8). Notice that \(e^B = g(e^A) \in g(A)\) for all \(c \in C\) and \(f^B = g(a_1), \ldots, g(a_{ar(g)}) = g(f^A(a_1), \ldots, a_{ar(g)}) \in g(A)\) for every \(f \in C\) and \(g(a_1), \ldots, g(a_{ar(g)}) \in g(A)\) so \(\Sigma^B\) restricts to a \(\Sigma\)-algebra structure on \((g(A), \mu_{B|\phi(A)})\) and it is clear that with this choice \(e\) and \(m\) becomes morphisms of \(\Sigma\text{-Alg}\). We have now to show the left lifting property. Given \(e \in \mathcal{E}_\Sigma,\)

\[m \in \mathcal{M}_\Sigma\] and \(g\) and \(h\) such that \(m \circ g = h \circ e\) we can apply \(\mathcal{U}_\Sigma\) and get a square.

\(^2\) In their work Milius and Urbat additionally require a full subcategory of \(C\) and a fixed class of cardinals, but we will not need this level of generality.
which, by Lemma 2.8, has a diagonal filling $k : \mathcal{U}_\Sigma(A) \to \mathcal{U}_\Sigma(B)$. Let us show that $k$ is a morphism of $\Sigma$-$\text{Alg}$. For any $c \in C$ we have $c^A = e(c^A)$ and for every $f \in O$ and $x_1, \ldots, x_{\text{ar}(f)} \in \mathcal{U}_\Sigma(A')$ there exist $a_1, \ldots, a_{\text{ar}(f)} \in \mathcal{U}_\Sigma(A)$ such that $e(a_i) = x_i$ for $1 \leq i \leq \text{ar}(f)$, so $k(c^A) = k(e(c^A)) = g(c^A) = e^B$ and
\[
\begin{align*}
    k\left(f^A(x_1, \ldots, x_{\text{ar}(f)})\right) &= k\left(f^A(e(a_1), \ldots, e(a_{\text{ar}(f)}))\right) = k(e(f^A(a_1, \ldots, a_{\text{ar}(f)}))) \\
    &= g(f^A(a_1, \ldots, a_{\text{ar}(f)})) = f^B(g(a_1), \ldots, g(a_{\text{ar}(f)})) \\
    &= f^B(k(e(a_1)), \ldots, k(e(a_{\text{ar}(f)}))) = f^B(k(x_1), \ldots, k(x_{\text{ar}(f)}))
\end{align*}
\]
Finally, properness follows at once since $\mathcal{U}_\Sigma$ is faithful and so reflects epis and monos. ▶

**Definition 6.6.** We define the following two classes of $\Sigma$-algebras:
\[
    \Sigma_0 := \{f^{\text{Set}}_\Sigma(X) \mid X \in \text{Set}\} \quad \Sigma_E := \{f_\Sigma(X, \mu_X) \mid (X, \mu_X) \in \text{Fuz}(H)\}
\]
We will use $\mathcal{E}_{\Sigma, \Sigma_0}$ (resp., $\mathcal{E}_{\Sigma, \Sigma_E}$) for the class of $e \in \mathcal{E}$ such that every $X \in \Sigma_0$ (resp. $X \in \Sigma_E$) is projective with respect to $e$.

**Remark 6.7.** $\Sigma_0 = \{f_\Sigma(X, \mu_X) \mid \text{supp}(X, \mu_X) = \emptyset\}$.

We have now all the ingredients needed to use the results recalled above.

**Lemma 6.8.** With the above definitions:

1. $\Sigma_0, \Sigma_E \subseteq \Sigma$;
2. $\mathcal{E}_{\Sigma_0} = \{e \in \mathcal{E} \mid \mathcal{U}_\Sigma(e) \text{ is split}\}$;
3. $(\Sigma$-$\text{Alg}, (\mathcal{E}_\Sigma, \mathcal{M}_\Sigma), \Sigma_0)$ and $(\Sigma$-$\text{Alg}, (\mathcal{E}_\Sigma, \mathcal{M}_\Sigma), \Sigma_E)$ satisfy the conditions of our settings.

**Proof.** Let us start adapting the usual notion of congruence to our set environment.

**Definition 6.9.** Given a $\Sigma$-algebra $A = ((A, \mu_A), \Sigma^A)$, a fuzzy congruence on $A$ is a pair $(\sim, \mu)$ where

- $\sim$ is a congruence: i.e. an equivalence relation such that, for any $f \in O$, if $a_i \sim b_i$ for $1 \leq i \leq \text{ar}(f)$ then $f^A(a_1, \ldots, a_{\text{ar}(f)}) \sim f^A(b_1, \ldots, b_{\text{ar}(f)})$;
- $\mu : A \to H$ is a function such $\mu(a) = \mu(b)$ whenever $a \sim b$;
- for any $f \in O$,
\[
    \bigwedge_{i=1}^{\text{ar}(f)} \mu(a_i) \leq \mu(f^A(a_1, \ldots, a_{\text{ar}(f)}))
\]
- $\mu_A(a) \leq \mu(a)$ for every $a \in A$.

**Proposition 6.10.** Let $A = ((A, \mu_A), \Sigma^A)$ be a $\Sigma$-algebra, then:

(a) if $\{\{\sim_i, \mu_i\} \mid i \in I\}$ is a family of fuzzy congruences then $\bigcap_{i \in I} \sim_i$ with $\mu : A \to H$ the pointwise infimum of $\{\mu_i\} \mid i \in I$ is a fuzzy congruence;
(b) for every fuzzy congruence \((\sim, \mu)\) there exists an epimorphism: \(e_{(\sim, \mu)} : A \to B\) such that 
\(\mu_B(b) = \mu(a)\) for any \(a \in e^{-1}(b)\) and \(e_{(\sim, \mu)}(a) = e_{(\sim, \mu)}(b)\) if and only if \(a \sim b\);

(c) for every epimorphism \(e : A \to B\) there exists a fuzzy congruence \((\sim_e, \mu_e)\) on \(A\) such that 
\(e \leq e_{(\sim, \mu)}\) and \(e_{(\sim, \mu)} \leq e\) in \(\mathcal{A} \downarrow \Sigma\text{-Alg}\).

**Proof.**

(a) This is straightforward.

(b) Define \(B = ((B, \mu_B), \Sigma^B)\) setting \(B := A/\sim, \mu_B([a]) := \mu(a)\) and, for any \(f \in O\)

\[
\begin{align*}
    f^A([a_1], \ldots, [a_{\sigma(f)}]) &:= [f^B([a_1], \ldots, [a_{\sigma(f)}])] \\

\end{align*}
\]

Since \((\sim, \mu)\) is a fuzzy congruence all these objects are well defined, the fact that \(f^B\) is an arrow of fuzzy sets follows from the second condition on \(\mu\), while the last condition entails that the projection on the quotient is an arrow of \(\textbf{Fuz}(H)\).

(c) Put a \(a \sim b\) if and only if \(e(a) = e(b)\) and \(\mu_e(a) := \mu_B(e(a))\). Since \(e\) is a morphism of \(\Sigma\text{-Alg}\) we get the first and the last condition in the definition of a fuzzy congruence, while the second one follows since

\[
\begin{align*}
    \mu_e(f^A(a_1, \ldots, a_{\sigma(f)})) &= \mu_B(e(f^A(a_1, \ldots, a_{\sigma(f)}))) = \mu_B(f^B(e(a_1), \ldots, e(a_{\sigma(f)}))) \\
    &\geq \bigwedge_{i=1}^{\sigma(f)} \mu_B(e(a_i)) = \bigwedge_{i=1}^{\sigma(f)} \mu_e(a_i)
\end{align*}
\]

Now, it is immediate to see that the function sending the equivalence class \([a]\) of \(a \in A\) to \(e(b)\) induces an isomorphism of \(\Sigma\text{-Alg}\) witnessing the thesis. \(\blacksquare\)

So equipped we can turn back to the proof of Lemma 6.8

1. Let \(e : A \to B\) be an arrow in \(\mathcal{E}_\Sigma\) and let \(h : \mathcal{F}_\Sigma^\text{Set}(X) \to B\) be any morphism of \(\Sigma\text{-Alg}\). By point 2 of Proposition 4.6 \(e\) is surjective so for any \(x \in X\) we can take a \(a_x \in e^{-1}(h(\eta_X(x)))\), where \(\eta\) is the unit of the adjunction \(\mathcal{F}_\Sigma^\text{Set} \dashv \mathcal{V}_\Sigma\) of Proposition 4.6 and define \(k : X \to A\) mapping \(x\) to \(a_x\), where \(\mathcal{A} = ((A, \mu_A), \Sigma^A)\). By adjointness, from \(\tilde{k}\) we get \(k : \mathcal{F}_\Sigma^\text{Set}(X) \to A\) and

\[
\begin{align*}
    (e \circ k) \circ \eta_X &= e \circ (k \circ \eta_X) = e \circ \tilde{k} = h \circ \eta_X
\end{align*}
\]

so \(e \circ k = h\).

2. Let \(e : A \to B\) be in \(\mathcal{E}_\Sigma\) such that \(\mathcal{U}_\Sigma(e)\) is split and let \(s\) be a section in \(\textbf{Fuz}(H)\), then, for any \(h : \mathcal{F}_\Sigma(X, \mu_X) \to B\) we can consider the arrow \(s \circ h \circ \eta_{(X, \mu_X)}\), which, by adjointness provides a \(k : \mathcal{F}_\Sigma(X, \mu_X) \to A\), moreover:

\[
\begin{align*}
    (e \circ k) \circ \eta_{(X, \mu_X)} &= e \circ (k \circ \eta_{(X, \mu_X)}) = e \circ (s \circ h \circ \eta_{(X, \mu_X)}) \\
    &= (e \circ s) \circ (h \circ \eta_{(X, \mu_X)}) = h \circ \eta_{(X, \mu_X)}
\end{align*}
\]

so \(k\) is the wanted lifting. On the other hand, if \(e\) is in \(\mathcal{E}_{\Sigma^X}\), we can take the diagram:
Moreover, Λ

Proof. Let us proceed by steps.

3. Definition 6.11. Let

Lemma 6.12. Let

where ε₆ is the component of the counit ε : Σ₂ ↠ id₅₆ and k its lifting. Taking

U₅₆(k) o η₅₆(B) we get the desired section of U₅₆(e).

3. Let us proceed by steps.

- **Fuz(H)** has all products by Lemma 2.8
- X | C is essentially small by point 3 of Proposition 6.10
- For any fuzzy set (X, µₓ) we can consider the identity idₓ(X, µₓ) : (X, µₓ) → (X, µₓ) and the counit ε(x, µₓ) : Σ(X) → (X, µₓ) of the adjunction Σ := Σ₂ of Proposition 2.2. They induce arrows e₀ : Σ₂(X, µₓ) → (X, µₓ) and ε₆ : Σ₂(X, µₓ) → (X, µₓ) such that U₅₆(e₀) o η₅₆(X) = ε(x, µₓ) and U₅₆(e₆) o η₅₆(X) = id(x, µₓ). So U₅₆(e₆) is split and, since ε(x, µₓ) is surjective, point 2 of Proposition 2.4 allows us to conclude that U₅₆(e₀) is an epimorphism.

We want now to translate formulae of our sequent calculus into X₀ₗ- and Xₖₑ-equations. To this end, we have to restrict to two classes of theories, which we introduce next.

**Definition 6.11.** Let L = (Σ, X) be a language, a Λ ∈ Th(L) is said to be:

- unconditional (138, App. B.5) if any sequent in Λ is of the form ⊢ φ for some formula φ;
- of type E if any sequent in Λ is of the form {E(l, x_i)} ⊢ φ for some formula φ, \{x_i\}_{i∈I} ⊆ X and \{l_i\}_{i∈I} ⊆ H.

**Lemma 6.12.** For any signature Σ and Xₖₑ-equation e : Σ₂(X, µₓ) → B there exists a type E theory Λₑ such that, for every Σ-algebra A, A ⊨ Λₑ e if and only if A ∈ Mod(Λₑ). Moreover, \( |Γ| ≤ |supp(X, µₓ)| \) for any \( Γ ⊢ φ ∈ Λₑ \).

**Proof.** Let Lₑ be (Σ, X). We define Γₓ := \{E(µₓ(x), x) | x ∈ supp(X, µₓ)\} and Λₑ ∈ Th(L) as \( Λ₁ₑ ∪ Λ₂ₑ \) where

\[
Λ₁ₑ := \{Γₓ ⊢ E(l, l) \mid l ≤ µₓ(e([l]))\} \quad Λ₂ₑ := \{Γₓ ⊢ [s] ⊨ [l] \mid v([l]) = v([s])\}
\]

and \((B, µₓ) ∈ \mathcal{U₅₆}(B)\). Let us show the two implications.

⇒ Any \( ψ : X → A \) such that \( A ⊨ Λₑ \) defines an arrow \( i(X, µₓ) → \mathcal{U₅₆}(A) \). By adjointness we have a homomorphism \( hᵢ : Σ₂(X, µₓ) → A \) hence, by hypothesis, there exists \( qᵦ : B → A \) such that \( qᵦ o e = hᵢ \). Now, notice that (see Theorem 5.9 and Proposition 5.7) \( hᵢ([l]) = t^{A,e} \). Take a sequent \( Γₓ ⊨ ψ \) in \( Λₑ \), we have two cases, depending on ψ.

- If \( ψ = E(l, l) ∈ Λ₁ₑ \), we have

\[
l ≤ µₓ(e([l])) ≤ µₓ(qᵦ(e([l]))) = µ₅₆(hᵢ([l])) = t^{A,e}
\]

therefore \( A ⊨ ψ \).

- If \( ψ = [s] ⊨ [l] ∈ Λ₂ₑ \) then

\[
t^{A,e} = hᵢ([l]) = qᵦ(e([l])) = qᵦ(e([s])) = hᵢ([l]) = s^{A,e}
\]

even in this case \( A ⊨ ψ \).
Fuzzy algebraic theories

\( \Leftarrow \) Take \( h : \mathcal{F}_\Sigma(X, \mu_X) \to A \), \( \mathcal{U}_\Sigma(h) \circ \eta_\Sigma(X) \) is an arrow \( (X, \mu_X) \to \mathcal{U}_\Sigma(A) \), so forgetting the fuzzy set structure too gives us an assignment \( \iota_h : X \to A \) such that \( A \models _{\iota_h} \Gamma_X \). As before \( h([t]) = t^{A, \iota_h} \) for every \( [t] \in \mathcal{F}_\Sigma(X, \mu_X) \). Since \( A \in \text{Mod}(\Lambda_e) \) we have

\[
\begin{align*}
t^{A, \iota_h} &= s^{A, \iota_h} \quad \text{for all terms } t \text{ and } s \text{ such that } e([t]) = e([s]);
\end{align*}
\]

\[
\begin{align*}
l &\leq \mu_A(t^{A, \iota_h}) \quad \text{for all } t \text{ such that } l \leq \mu_B(e([l])).
\end{align*}
\]

So, the function \( q : B \to A \) which sends \( b \in B \) to \( h([l]) \) for some \( [l] \in e^{-1}(b) \), provides us with an arrow \( \mathcal{U}_\Sigma(B) \to \mathcal{U}_\Sigma(A) \) such that \( q \circ e = h \). Now:

\[
\begin{align*}
q(e^B) &= q(e(c^{\mathcal{F}_\Sigma(X, \mu_X)}_{\Sigma}))) = q(f^B(e(c_1), \ldots, (e(c_{ar(f)})))) \\
q &= h(c^{\mathcal{F}_\Sigma(X, \mu_X)}_{\Sigma}) = c^A \\
&= h(f^{\mathcal{F}_\Sigma(X, \mu_X)}_{\Sigma}(c_1, \ldots, c_{ar(f)})) = f^{A}(h(c_1), \ldots, h(c_{ar(f)})) \\
&= f^{A}(q(e(c_1)), \ldots, q(e(c_{ar(f)}))) = f^{A}(q(b_1), \ldots, q(b_{ar(f)}))
\end{align*}
\]

so we can conclude that \( q \) is an arrow of \( \Sigma\text{-Alg} \) and we are done. \( \blacktriangle \)

**Corollary 6.13.** For any signature \( \Sigma \) and \( \mathcal{X}_0 \)-equation \( e : \mathcal{F}_\Sigma^{\text{Set}}(X) \to B \) there exists an unconditional theory \( \Lambda_e \) such that, for any \( \Sigma \)-algebra \( A \), \( A \models _{\Lambda_e} e \) if and only if \( A \in \text{Mod}(\Lambda_e) \).

Finally, from the results above we can easily conclude HSP-like results for \( \Sigma\text{-Alg} \).

**Theorem 6.14.** Let \( V \) be a full subcategory of \( \Sigma\text{-Alg} \), then

1. \( V \) is closed under epimorphisms, (small) products and strong monomorphisms if and only if there exists a class of unconditional theories \( \{ \Lambda_e \} \in \Xi \) such that \( A \in V \) if and only if \( A \in \text{Mod}(\Lambda_e) \) for all \( e \in \Xi \).
2. \( V \) is closed under split epimorphisms, (small) products and strong monomorphisms if and only if there exists a class of type \( \Xi \) theories \( \{ \Lambda_e \} \in \Xi \) such that \( A \in V \) if and only if \( A \in \text{Mod}(\Lambda_e) \) for all \( e \in \Xi \).

**Proof.** This is straightforward in light of Theorem 6.4, Lemma 6.12 and Corollary 6.13. \( \blacktriangle \)

**Remark 6.15.** We cannot arrange the collection \( \{ \Lambda_e \} \in \Xi \) into a unique theory since a proper class of variables is needed to write down all the necessary sequents. A possible way to deal with this issue is to fix two Grothendieck universes \([22]\) \( U_1 \subset U_2 \) and modify the definition of language allowing for a possible class (i.e., an element of \( U_2 \)) of variables. All the proof of this paper can be repeated verbatim in this context carefully distinguishing between fuzzy sets (i.e., those defined on an element of \( U_1 \)) and fuzzy classes (i.e., those defined on an element of \( U_2 \)). Then the algebras of terms will be a fuzzy class in general but it is possible to show, using the explicit construction, that \( T_\Lambda(X, \mu_X) \) is a fuzzy set if \( X \in U_1 \) and so we can retain all the results of Section 5.

The issue mentioned in the previous remark can be avoided if the family \( \{ \Lambda_e \} \in \Xi \) satisfies a boundedness property about the premises of the sequents belonging to each \( \Lambda_e \).

**Definition 6.16.** Given a cardinal \( \kappa \) we say that a \( \mathcal{X}_e \)-equation \( e : \mathcal{F}_{\Sigma}(X, \mu_X) \to B \) is \( \kappa \)-supported if \( |\text{support}(X, \mu_X)| < \kappa \).

**Proposition 6.17.** Let \( V = \mathcal{V}(\Xi) \) be an \( \mathcal{X}_e \)-equational defined subcategory of \( \Sigma\text{-Alg} \) and suppose every \( e \in \Xi \) is \( \kappa \)-supported, then there exists a theory \( \Lambda \in \text{Th}(\mathcal{L}) \), where \( \mathcal{L} = (\Sigma, \kappa) \), such that \( V = \text{Mod}(\Lambda) \).
Proof. For any \( e : \mathcal{F}_\Sigma(X_e, \mu X_e) \to \mathcal{B}_e \) in \( \mathbb{E} \) we can fix an injection \( i_e : \text{supp}(X_e, \mu X_e) \to \kappa \) and an extension let \( \bar{i}_e : X \to \kappa \) of it, fix also morphisms \( \Gamma : \mathcal{L}_e \to \mathcal{L} \) given by \((\text{id}, \bar{i}_e)\). Let now \( \{\Lambda_e\}_{e \in \mathbb{E}} \) be the collection of theories given by Corollary \([6,13]\) and Theorem \([6,14]\) since each \( \Lambda_e \in \text{Form}(\mathcal{L}_e) \) we can define \( A \) as \( \bigcup_{e \in \mathbb{E}} \Gamma^*_e(\Lambda_e) \). We have to show that \( A \in \text{Mod}(\Lambda) \) if and only if \( A \in \text{Mod}(\Lambda) \).

\[ \Rightarrow \] Let \( \text{Form}(\Gamma^*)(\Gamma_X)^e \vdash \text{Form}(\Gamma^*)(\psi) \) be a sequent in \( \Lambda \) and let \( \iota : \kappa \to A \) an assignment such that \( A \models \iota \). By point 3 of Lemma \([4,9]\) this implies \( A \models \iota \circ i \circ \iota \) therefore \( A \models \iota \circ i \circ \iota \) and we conclude applying lemma \([4,9]\) again.

\[ \Leftarrow \] If \( \mathcal{U}_{\Sigma}(A) = (\emptyset, \mu H) \), \((\mu H \text{ being the empty map } \emptyset \to H)\) then there are no assignment \( \kappa \to A \) and so \( A \) is in \( \text{Mod}(\Lambda) \). In the other cases let \( \Gamma_X \vdash \psi \) be in \( \Lambda_e \) and \( \iota : X_e \to A \) such that \( A \models \iota \circ \iota \), since \( A \neq \emptyset \) there exists \( \bar{i} : \kappa \to A \) such that \( \bar{i} \circ i \circ i = \iota \) as in the previous point Lemma \([4,9]\) implies \( A \models \Gamma^*(\Gamma_X)^e \), so \( A \models \Gamma^*(\Gamma_X)^e \) and again this is equivalent to \( A \models \psi \).

Corollary 6.18. \( V \) is closed under epimorphisms, (small) products and strong monomorphisms if and only if there exists a language \( \mathcal{L} \) and an unconditional theory \( \Lambda \in \text{Th}(\mathcal{L}) \) such that \( V = \text{Mod}(\Lambda) \).

7 Conclusions and future work

In this paper we have introduced a fuzzy sequent calculus to capture equational aspects of fuzzy sets. While equalities are captured by usual equations, information contained in the membership function is captured by membership proposition of the form \( E(l, t) \), to be interpreted as “the membership degree of \( t \) is at least \( l \)”. We have used a natural concept of fuzzy algebras to provide a sound and complete semantics for such calculus in the sense that a formula is satisfied by all the models of a given theory if and only if it is derivable from it using the rules of our calculus.

As in the classical and quantitative contexts, there is a notion of free model of a theory \( \Lambda \) and thus an associated monad \( T_\Lambda \) on the category \( \textbf{Fuz}(H) \). However, in general Eilenberg-Moore algebras for such monads are not equivalent to models of \( \Lambda \), but we have shown that this equivalence holds if \( \Lambda \) is basic. In this direction it would be interesting to better understand the categorical status of our approach, investigating possible links between our notion of fuzzy theory and \( \textbf{Fuz}(H) \)-Lawvere theories as introduced in full generality by Nishizawa and Power in \([13]\). A difference between the two approaches is that for us arities are simply finite sets, while following \([13]\) a \( \textbf{Fuz}(H) \)-Lawvere theory arities would be given by finite fuzzy sets.

Finally, using the results provided in \([15]\) we have proved that, given a signature \( \Sigma \), subcategories of \( \Sigma \)-Alg which are closed under products, strong monomorphisms and epimorphic images correspond precisely with categories of models for unconditional theories, i.e. theories axiomatised by sequents without premises. Moreover, using the same results, we have also proved that the categories of models of theories of type \( \mathbb{E} \), i.e. those whose axioms’ premises contain only membership propositions involving variables, are exactly those subcategories closed under products, strong monomorphisms and split epimorphisms.

Our category \( \textbf{Fuz}(H) \) of fuzzy sets has crisp arrows and crisp equality: arrows are ordinary functions between the underlying sets and equalities can be judged to be either true or false. A way to further “fuzzifying” concepts is to use the topos of \( H \)-sets over the frame \( H \) introduced in \([8]\): this is equivalent to the topos of sheaves over \( H \) and contains \( \textbf{Fuz}(H) \) as a (non full) subcategory. By construction, equalities and functions are “fuzzy”. It would be interesting to study an application of our approach to this context. A promising
feature is that in an $H$-set the membership degree function is built-in as simply the equality relation, so it would not be necessary to distinguish between equations and membership propositions. Even more generally, we can replace $H$ with an arbitrary quantale $V$ and consider the category of sets endowed with a “$V$-valued equivalence relation” [6].

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A Derivations for Proposition 3.13

\[ E(l, x) \vdash E(l, x) \]

\[ E(l, x) \vdash E(l, x) \quad \text{A} \]

\[ E(l, x) \vdash E(l, x^{-1}) \quad \text{EXP} \]

\[ E(l, x) \vdash E(l, x^{-1}) \quad \text{EXP} \]

\[ E(l, x) \vdash E(l, e) \quad \text{FUN} \]

\[ \vdash e \equiv x \cdot x^{-1} \]

\[ \frac{E(l, x) \vdash E(l, x)}{E(l, x) \vdash E(l, x \cdot x^{-1})} \quad \text{EXP} \]

\[ E(l, x) \vdash E(l, e) \quad \text{FUN} \]

\[ \vdash y \equiv (y^{-1})^{-1} \]

\[ \frac{E(l, x) \vdash E(l, y \cdot (x \cdot y^{-1}))}{E(l, x) \vdash E(l, y^{-1} \cdot (x \cdot y^{-1}))} \quad \text{SUB} \]

\[ E(l, y^{-1} \cdot (x \cdot y^{-1})) \vdash E(l, x) \quad \text{SUB} \]

\[ E(l, (y^{-1})^{-1} \cdot (x \cdot y^{-1})) \vdash E(l, x) \quad \text{SUB} \]

\[ E(l, y \cdot (x \cdot y^{-1})) \vdash E(l, x) \quad \text{FUN} \]