Navier-Stokes blow-up rates in certain Besov spaces whose regularity exceeds the critical value by \( \epsilon \in [1, 2] \)

Joseph P. Davies\(^1\) and Gabriel S. Koch\(^2\)

Abstract

For a solution \( u \) to the Navier-Stokes equations in spatial dimension \( n \geq 3 \) which blows up at a finite time \( T > 0 \), we prove the blowup estimate \( \|u(t)\|_{\dot{B}^{\epsilon/2}_{p,q}(\mathbb{R}^n)} \gtrsim \|\varphi\|_{\dot{B}^{\epsilon/2}_{p,q}(\mathbb{R}^n)} (T - t)^{-\epsilon/2} \) for all \( \epsilon \in [1, 2) \) and \( p,q \in [1, \infty) \), where \( \varphi \) is the scaling-critical regularity, and \( \varphi \) is the cutoff function used to define the Littlewood-Paley projections. For \( \epsilon = 2 \), we prove the same type of estimate but only for \( q = 1 \): \( \|u(t)\|_{\dot{B}^{\epsilon/2}_{p,1}(\mathbb{R}^n)} \gtrsim \|\varphi\|_{\dot{B}^{\epsilon/2}_{p,1}(\mathbb{R}^n)} (T - t)^{-1} \) for all \( p \in [1, \infty) \). Under the additional restriction that \( p,q \in [1, 2] \) and \( n = 3 \), these blowup estimates are implied by those first proved by Robinson, Sadowski and Silva (J. Math. Phys., 2012) for \( p = q = 2 \) in the case \( \epsilon \in (1, 2) \), and by McCormick, Olson, Robinson, Rodrigo, Vidal-López and Zhou (SIAM J. Math. Anal., 2016) for \( p = 2 \) in the cases \( (\epsilon, q) = (1, 2) \) and \( (\epsilon, q) = (2, 1) \).

1. Introduction

According to the physical theory, if an incompressible viscous Newtonian fluid occupies the whole space \( \mathbb{R}^n \) in the absence of external forces, then the velocity \( U(\tau, x) \) and kinematic pressure \( \Pi(\tau, x) \) of the fluid at time \( \tau > 0 \) and position \( x \in \mathbb{R}^n \) satisfy the Navier-Stokes equations

\[
\begin{align*}
\partial_t U - \nu \Delta U + (U \cdot \nabla) U + \nabla \Pi &= 0, \\
\nabla \cdot U &= 0,
\end{align*}
\]

where the coefficient \( \nu > 0 \) is the kinematic viscosity of the fluid. By considering the rescaled quantities

\[
t = \nu \tau, \quad u(t, x) = \nu^{-1} U(\nu^{-1} t, x), \quad \varpi(t, x) = \nu^{-2} \Pi(\nu^{-1} t, x),
\]

whose physical dimensions are powers of length alone, we may rewrite the Navier-Stokes equations in the standardised form

\[
\begin{align*}
\partial_t u - \Delta u + (u \cdot \nabla) u + \nabla \varpi &= 0, \\
\nabla \cdot u &= 0.
\end{align*}
\]

(1.1)

At the formal level, if \( (u, \varpi) \) satisfy (1.1) then \( \varpi \) may be recovered from \( u \) by the formula

\[
\varpi = (-\Delta)^{-1} \nabla^2 : (u \otimes u).
\]

(1.2)

Writing \( \Lambda \) to denote the physical dimension of length, the quantities \( x, t, u, \varpi \) have physical dimensions

\[
[x] = \Lambda, \quad [t] = \Lambda^2, \quad [u] = \Lambda^{-1}, \quad [\varpi] = \Lambda^{-2};
\]

this is related to the fact that the standardised Navier-Stokes equations are preserved under the rescaling

\[
x_\lambda = \lambda x, \quad t_\lambda = \lambda^2 t, \quad u_\lambda(t_\lambda, x_\lambda) = \lambda^{-1} u(t, x), \quad \varpi_\lambda(t_\lambda, x_\lambda) = \lambda^{-2} \varpi(t, x).
\]

(1.3)

Definition 1.1. For \( T \in (0, \infty) \), a function \( u : (0, T) \times \mathbb{R}^n \to \mathbb{R}^n \) is said to be a \textit{regular solution} to the standardised Navier-Stokes equations on \( (0, T) \) if

\begin{enumerate}
\item[(i)] \( u \) is smooth on \( (0, T) \times \mathbb{R}^n \), with every derivative belonging to \( C((0, T); L^2(\mathbb{R}^n)) \);
\item[(ii)] \( (u, \varpi) \) satisfy (1.1), with \( \varpi \) being given by (1.2).
\end{enumerate}

Definition 1.2. For \( T \in (0, \infty) \), a regular solution \( u \) on \( (0, T) \) is said to blow up at (rescaled) time \( T \) if \( u \) doesn’t extend to a regular solution on \( (0, T') \) for any \( T' > T \).

In spatial dimension \( n \geq 3 \), it remains unknown whether there exist regular solutions which blow up. Since Leray’s seminal paper [3], prospective blowing-up solutions have been studied using homogeneous norms:

Definition 1.3. A norm \( \| \cdot \|_X \) (defined on a subspace \( X \subseteq \mathcal{S}'(\mathbb{R}^n) \) which is closed under dilations of \( \mathbb{R}^n \)) is said to be homogenous of degree \( \alpha = \alpha(X) \) if, under the rescaling \( x_\lambda = \lambda x \) and \( f_\lambda(x_\lambda) = f(x) \), we have \( \|f_\lambda\|_X \approx \lambda^\alpha \|f\|_X \) for all \( \lambda \in (0, \infty) \) and \( f \in X \).
If \( \| \cdot \|_X \) is homogeneous of degree \( \alpha \), then under the scaling (1.3) of the Navier-Stokes equations we have \( \| u(t) \|_X \approx X \lambda^{\alpha-1} \| u(t) \|_X \), so the quantity \( \| u \|_X \) has physical dimension \( \Lambda^{\alpha-1} \). In the context of the Navier-Stokes equations, the homogeneous norm \( \| \cdot \|_X \) is said to be subcritical if \( \alpha(X) < 1 \), critical if \( \alpha(X) = 1 \), and supercritical if \( \alpha(X) > 1 \). If \( \| \cdot \|_X \) is subcritical, then the blowup estimate

\[
\text{u blows up at time } T \Rightarrow \| u(t) \|_X \gtrsim_X (T-t)^{-1-\alpha(X)/2} \tag{1.4}
\]

does not make sense.

We investigate (1.3) in the context of the homogeneous Besov norms

\[
\| f \|_{B^{p,q} s (\mathbb{R}^n)} := \left\| j \mapsto 2^{js} \left\| \mathcal{F}^{-1} \varphi (2^{-j} \xi) \mathcal{F} f \right\|_{L^p (\mathbb{R}^n)} \right\|_{\ell^q (\mathbb{Z})}
\]

for \( s \in \mathbb{R} \), \( p, q \in [1, \infty] \), where \( \mathcal{F} \) is the Fourier transform and \( \varphi \) is a cut-off function satisfying certain properties. Amongst other things, for any \( p, p_1, q, q_1 \in [1, \infty] \) and \( \delta, \gamma \in \mathbb{R} \) the Besov norms satisfy

\[
\| f \|_{B^{p_1+\delta,q_1} s (\mathbb{R}^n)} \gtrsim_n \| f \|_{B^{p,q} s (\mathbb{R}^n)} \quad \text{if } p_1 \leq p_2, q_1 \leq q_2,
\]

\[
\| f \|_{B^{p_1,q_1} s (\mathbb{R}^n)} \gtrsim \| f \|_{L^p (\mathbb{R}^n)} \| f \|_{B^{p_1,q_1} s (\mathbb{R}^n)},
\]

\[
\| f \|_{B^{p,q} s (\mathbb{R}^n)} \approx \| f \|_{H^s (\mathbb{R}^n)},
\]

and \( \| \cdot \|_{B^{p,q} s (\mathbb{R}^n)} \) is homogeneous of degree \( \alpha(B^{p,q} s (\mathbb{R}^n)) = 3 - s \). In the context of Navier-Stokes we define

\[
s_p = s_p(n) := -1 + \frac{n}{p},
\]

so the norm \( \| \cdot \|_{B^{p,q} s_p (\mathbb{R}^n)} \) is critical for \( \epsilon = 0 \), and subcritical for \( \epsilon > 0 \).

It is known that if a regular solution \( u \) blows up at a finite time \( T > 0 \), then

\[
\| u(t) \|_{B^{3-1-\epsilon^{-1},\infty} s_p (\mathbb{R}^n)} \gtrsim (T-t)^{-1-\epsilon^{-1} / 2} \quad \text{for all } t \in (0, T), \epsilon \in (0, 1),
\]

\[
\| u(t) \|_{L^\infty (\mathbb{R}^n)} \gtrsim (T-t)^{-1/2} \quad \text{for all } t \in (0, T).
\]

By virtue of (1.8) and (1.9), the left-hand side of (1.8) may be replaced by \( \| u(t) \|_{B^{3-1-\epsilon^{-1},\infty} s_p (\mathbb{R}^n)} \) for any \( p, q \in [1, \infty] \), while the left-hand side of (1.8) may be replaced by \( \| u(t) \|_{B^{3,1+p} p_{p,q} (\mathbb{R}^n)} \) for any \( p \in [1, \infty] \).

Adapting the energy methods of [1, 5, 6], we will prove the following blowup estimates in the case \( \epsilon \in [1, 2] \):

**Theorem 1.4.** Let \( n \geq 3 \) and \( T \in (0, \infty) \). If \( u \) is a regular solution (see Definition (2.2)) to the standardised Navier-Stokes equations on \( (0, T) \) which satisfies \( \lim_{r \to 0} \| u(t) \|_{B^{3-1/2,\infty} (\mathbb{R}^n)} = \infty \), then

\[
\| u(t) \|_{B^{3-1/2,\infty} (\mathbb{R}^n)} \gtrsim (T-t)^{-1/2} \quad \text{for all } t \in (0, T), \epsilon \in [1, 2], p, q \in \left[ \frac{1}{2}, \frac{n}{n-\epsilon} \right]
\]

and

\[
\| u(t) \|_{B^{3/2+\epsilon,\infty} (\mathbb{R}^n)} \gtrsim (T-t)^{-1} \quad \text{for all } t \in (0, T), p \in [1, \infty).
\]

Under the additional restrictions that \( p, q \in [1, 2] \) and \( \alpha = 3 \), the blowup estimate (1.10) is implied by the blowup estimate for \( H^{s_p} (\mathbb{R}^3) \), which was proved in the case \( \epsilon = 1 \) by McCormick et al. [4]. Under the additional restrictions that \( p \in [1, 2] \) and \( \alpha = 3 \), the blowup estimate (1.11) is implied by the blowup estimate for \( B^{3/2} (\mathbb{R}^3) \), which was proved by McCormick et al. [4].

The rest of this paper is organised as follows. In section 2 we recall some standard properties of Besov spaces, using [1] as our main reference. In section 3 we prove some commutator estimates, adapting the ideas of [11, Lemma 2.100]. In section 4 we prove Theorem 1.4. We will henceforth use the abbreviations \( L^p = L^p (\mathbb{R}^n) \), \( H^s = H^s (\mathbb{R}^n) \), \( B^{s,p,q} = B^{s,p,q} (\mathbb{R}^n) \) and \( \mathcal{L} = \mathcal{L} (\mathbb{Z}) \).

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3We will give more detailed definitions in section 2. Choosing a different function \( \varphi \) yields an equivalent norm.

4These estimates follow from the local theory and regularity properties of mild solutions with initial data \( f \), where the existence time is bounded below by a constant multiple of \( \| f \|_{L^2 (\mathbb{R}^n)} \) \( \| f \|_{L^\infty (\mathbb{R}^n)} \). Estimate (1.9) comes from Lemaré [3], while the local theory for initial data in Besov spaces is discussed in Lemaré-Rieusset’s book [2]. We aim to give a more detailed account of the regularity properties of such solutions in an upcoming paper.

5Note that this is a natural assumption for a solution blowing up at a finite time \( T \) in view of (1.3) with \( \epsilon = \frac{1}{2} \). In fact, to prove the parts of (1.10) - (1.11) with \( \epsilon \neq 1 \), one may replace this assumption with \( \lim_{r \to 0} \| u(t) \|_{L^\infty (\mathbb{R}^n)} = \infty \) (coming from (1.8)) by replacing (1.9) in the proof with the estimate

\[
\| u(t) \|_{L^\infty (\mathbb{R}^n)} \leq \| u(t) \|_{B^{3-1/2,\infty} (\mathbb{R}^n)} \| u(t) \|^{\lambda}_{B^{3/2+\epsilon,\infty} (\mathbb{R}^n)} \quad \text{with } \lambda = \frac{\epsilon - 1}{\epsilon - 1 + \frac{\epsilon}{2}}.
\]
2. Besov spaces

Lemma 2.1. ([11], Proposition 2.10). Let \( \mathcal{B} \) be the annulus \( B(0,8/3) \setminus \overline{B}(0,3/4) \). Then the set \( \overline{\mathcal{B}} = B(0,2/3) + \mathcal{B} \) is an annulus, and there exist radial functions \( \chi \in \mathcal{D}(B(0,4/3)) \) and \( \varphi \in \mathcal{D}(\mathcal{C}) \), taking values in \([0,1]\), such that

\[
    \left\{
        \begin{array}{ll}
            \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j} \xi) = 1 & \forall \xi \in \mathbb{R}^n, \\
            \sum_{j \geq 0} \varphi(2^{-j} \xi) = 1 & \forall \xi \in \mathbb{R}^n \setminus \{0\}, \\
            |j - j'| \geq 2 \Rightarrow \text{supp } \varphi(2^{-j} \cdot) \cap \text{supp } \varphi(2^{-j'} \cdot) = \emptyset, \\
            j \geq 1 \Rightarrow \text{supp } \chi \cap \text{supp } \varphi(2^{-j-1} \cdot) = \emptyset, \\
            |j - j'| \geq 5 \Rightarrow 2^{-j} \mathcal{C} \cap 2^{-j'} \mathcal{C} = \emptyset, \\
            1/2 \leq \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j} \xi) \leq 1 & \forall \xi \in \mathbb{R}^n, \\
            1/2 \leq \sum_{j \geq 0} \varphi(2^{-j} \xi) \leq 1 & \forall \xi \in \mathbb{R}^n \setminus \{0\}.
        \end{array}
    \right.
\]

We fix \( \chi, \varphi \) satisfying Lemma 2.1. For \( j \in \mathbb{Z} \) and \( u \in S' \), we define

\[
    \hat{S}_j u := \chi(2^{-j} D) u = \mathcal{F}^{-1} \chi(2^{-j} \xi) \mathcal{F} u, \\
    \hat{\Delta}_j u := \varphi(2^{-j} D) u = \mathcal{F}^{-1} \varphi(2^{-j} \xi) \mathcal{F} u.
\]

Lemma 2.2. For any \( j, j' \in \mathbb{Z} \) and \( u, v \in S' \), we have

\[
    |j - j'| \geq 2 \Rightarrow \hat{\Delta}_j \hat{\Delta}_{j'} u = 0, \\
    |j - j'| \geq 5 \Rightarrow \hat{\Delta}_j \left( \hat{S}_{j-1} u \hat{\Delta}_{j'} v \right) = 0.
\]

Proof. This is a consequence of Lemma 2.1. In particular, the implication \( |j - j'| \geq 2 \Rightarrow \hat{\Delta}_j \hat{\Delta}_{j'} u = 0 \) follows from the implication \( |j - j'| \geq 2 \Rightarrow \text{supp } \varphi(2^{-j} \cdot) \cap \text{supp } \varphi(2^{-j'} \cdot) = \emptyset \), while the implication \( |j - j'| \geq 5 \Rightarrow \hat{\Delta}_j \left( \hat{S}_{j-1} u \hat{\Delta}_{j'} v \right) = 0 \) follows from the implication \( |j - j'| \geq 5 \Rightarrow 2^{-j} \mathcal{C} \cap 2^{-j'} \mathcal{C} = \emptyset \).

We recall the following useful properties:

Lemma 2.3. ([11], Lemmas 2.1-2.2, Remark 2.11). Let \( \rho \) be a smooth function on \( \mathbb{R}^n \setminus \{0\} \) which is positive homogeneous of degree \( \lambda \in \mathbb{R} \). Then for all \( j \in \mathbb{Z} \), \( u \in S' \), \( t \in (0, \infty) \) and \( 1 \leq p \leq q \leq \infty \) we have

\[
    \| \hat{S}_j u \|_{L^p} \vee \| \hat{\Delta}_j u \|_{L^p} \lesssim_\rho \| u \|_{L^p}, \quad (2.1)
\]

\[
    \| \rho(D) \hat{\Delta}_j u \|_{L^p} \lesssim_\rho 2^{j \lambda} 2^{\frac{j}{2}} \| \hat{\Delta}_j u \|_{L^p}, \quad (2.2)
\]

\[
    \| \hat{\Delta}_j u \|_{L^p} \lesssim_\rho 2^{-j} \| \nabla \hat{\Delta}_j u \|_{L^p}. \quad (2.3)
\]

One can give meaning to the decomposition \( u = \sum_{j \in \mathbb{Z}} \hat{\Delta}_j u \) in view of the following lemma:

Lemma 2.4. ([11], Propositions 2.12-2.14) If \( u \in S' \), then \( \hat{S}_j u \uparrow \uparrow^{j \to \infty} \) \( u \) in \( S' \). Define

\[
    S'_h := \left\{ u \in S' : \| \hat{S}_j u \|_{L^\infty} \uparrow \uparrow^{j \to \infty} 0 \right\},
\]

so if \( u \in S'_h \) then \( u = \sum_{j \in \mathbb{Z}} \hat{\Delta}_j u \) in \( S' \).

For \( s \in \mathbb{R} \) and \( p, q \in [1, \infty] \), we define the Besov seminorm

\[
    \| u \|_{\dot{B}^{s}_{p,q}} := \left\| j \mapsto 2^{jn} \| \hat{\Delta}_j u \|_{L^p} \right\|_q \text{ for } u \in S',
\]

and the Besov space

\[
    \dot{B}^{s}_{p,q} := \left\{ u \in S'_h : \| u \|_{\dot{B}^{s}_{p,q}} < \infty \right\},
\]

so that \( (\dot{B}^{s}_{p,q}, \| \cdot \|_{\dot{B}^{s}_{p,q}}) \) is a normed space [11] Proposition 2.16]. Lemma 2.3 and Lemma 2.4 yield the inequalities

\[
    \| u \|_{\dot{B}^{s}_{p,q}} \lesssim_{n} \| u \|_{\dot{B}^{s}_{p,q}} \text{ for } p_1 \leq p_2, q_1 \leq q_2, s \in \mathbb{R}, u \in S', \quad (2.4)
\]

\[
    \| u \|_{\dot{B}^{s}_{p,q}} \lesssim_{\rho} \| u \|_{L^p} \text{ for } u \in S', \quad (2.5)
\]

\footnote{We adopt the convention that \( \mathcal{F} f(\xi) = \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x) \, dx \) for \( f \in \mathcal{S} \). We recall that the Fourier transform of a compactly supported distribution is a smooth function.}

\footnote{Note that \( \hat{S}_{j-1} u \) is spectrally supported on \( 2^j B(0,2/3) \), while \( \hat{\Delta}_j v \) is spectrally supported on \( 2^j \mathcal{C} \), so by properties of convolution we have that \( \hat{S}_{j-1} u \hat{\Delta}_j v \) is spectrally supported on \( 2^j \mathcal{C} \).

\footnote{For example, if \( Fu \) is locally integrable near \( \xi = 0 \), then \( u \in S'_h \). We remark that the condition \( u \in S'_h \) is independent of our choice of \( \varphi \).}

\footnote{Choosing a different function \( \varphi \) yields an equivalent seminorm [11] Remark 2.17].}
Lemma 2.7. \[(2.13)\]

We also have the interpolation inequalities

\[
\|u\|_{L^p} \leq \|u\|_{B^p_{r,1}} \quad \text{for } u \in \mathcal{S}_r^c.
\]

\

\[
\|u\|_{B^p_{r,1} + (1 - \lambda)B^q_{r,2}} \leq \|u\|_{B^p_{r,1}} + \|u\|_{B^q_{r,2}} \quad \text{for } \lambda \in (0, 1), \ u \in \mathcal{S}.
\]

\[
\|u\|_{B^p_{r,1} + (1 - \lambda)B^q_{r,2}} \leq \frac{1}{\Lambda(1 - \lambda)(s_2 - s_1)} \|u\|_{B^p_{r,1}} \|u\|_{B^q_{r,2}} \quad \text{for } \lambda \in (0, 1), \ s_1 < s_2, \ u \in \mathcal{S}.
\]

Note: If \((h)\) hypothesis of convergence is satisfied, and assume that the series \(\sum_{j \in \mathbb{Z}} \) converges in \( \mathcal{S} \) to some \( u \in \mathcal{S} \), then

\[
\|u\|_{B^p_{r,q}} \leq C_n^{1+|s|} \|u\|_{L^p} \|v\|_{B^q_{r,2}}.
\]

Note: If \((s, p, q)\) satisfy the condition

\[
s < \frac{n}{p}, \quad \text{or} \quad s = \frac{n}{p} \quad \text{and} \quad q = 1,
\]

then the hypothesis of convergence is satisfied, and \( u \in \mathcal{S}_r^c \).

A useful consequence of Lemma 2.5 is that if \( u \in \mathcal{S} \) satisfies \( \|u\|_{B^p_{r,q}} < \infty \) for some \((s, p, q)\) satisfying \((2.9)\), then \( u \in \mathcal{S}_r^c \).

If \( u \in B^0_{p,1} \) and \( v \in B^q_{r,2} \) with \( \frac{1}{p_1} + \frac{1}{p_2} \leq 1 \), then the series \( uv = \sum_{(j, j') \in \mathbb{Z}^2} \hat{u}_j \hat{j}'_v \) converges absolutely in \( L^{p_1+p_2} \), which justifies the Bony decomposition

\[
uv = T_v u + T_u v + \hat{R}(u, v),
\]

\[
T_v u = \sum_{j \in \mathbb{Z}} \hat{S}_{j-1} u \hat{j}_v,
\]

\[
\hat{R}(u, v) = \sum_{j \in \mathbb{Z}} \sum_{|j' - j| \leq 1} \hat{S}_{j'} u \hat{j}_v.
\]

We will require the following estimates for the operators \( T \) and \( \hat{R} \):

Lemma 2.6. \( (17), \) Theorem 2.47. Suppose that \( s = s_1, s_2, p = \frac{p_1 p_2}{p_1 + p_2} \) and \( q = \frac{q_1 q_2}{q_1 + q_2} \). Let \( u, v \in \mathcal{S} \), and assume that the series \( \sum_{j \in \mathbb{Z}} \hat{S}_{j-1} u \hat{j}_v \) converges in \( \mathcal{S} \) to some \( \hat{T}_u v \in \mathcal{S} \). Then

\[
\|\hat{T}_u v\|_{B^p_{r,q}} \leq C n^{1+|s|} \|u\|_{L^p} \|v\|_{B^q_{r,2}}.
\]

Note: If \( (s, p, q) \) satisfy \((2.10)\), and the right hand side of either \((2.10)\) or \((2.11)\) is finite, then the hypothesis of convergence is satisfied, and \( \hat{T}_u v \in \mathcal{S}_r^c \).

Lemma 2.7. \( (17), \) Theorem 5.52. Suppose that \( s = s_1, s_2, p = \frac{p_1 p_2}{p_1 + p_2} \) and \( q = \frac{q_1 q_2}{q_1 + q_2} \). Let \( u, v \in \mathcal{S} \), and assume that the series \( \sum_{j \in \mathbb{Z}} \hat{S}_{j-1} u \hat{j}_v \) converges in \( \mathcal{S} \) to some \( \hat{R}(u, v) \in \mathcal{S} \). Then

\[
\|\hat{R}(u, v)\|_{B^p_{r,q}} \leq C n^{1+|s|} \|u\|_{L^p} \|v\|_{B^q_{r,2}}.
\]

Note: If \( (s, p, q) \) satisfy \((2.10)\), and the right hand side of \((2.12)\) is finite, or if \( (s, p, q) \) satisfy \((2.9)\) and the right hand side of \((2.13)\) is finite, then the hypothesis of convergence is satisfied, and \( \hat{R}(u, v) \in \mathcal{S}_r^c \).
3. Commutator estimates

In this section, we will adapt the proof of [1] Lemma 2.100 to prove the commutator estimates in the following proposition, which will be crucial to the proof of Theorem 1.4.

**Proposition 3.1.** For \( v, f \in \cap_{r \in [0, \infty)} \dot{H}^r \) and \( j \in \mathbb{Z} \), define \(^\text{10}\)

\[
R_j = [v \cdot \nabla, \Delta_j]f = [v_k, \Delta_j] \nabla_k f
\]

where \([\cdot, \cdot]\) denotes the commutator \([A, B] = AB - BA\), and suppose that \( s = s_1 + s_2, p = \frac{2p}{p-1} \) and \( q = \frac{2q}{q-1} \) (\( s_j \in \mathbb{R} \) and \( p_j, q_j \in [1, \infty) \)). Then we have the decomposition \( R_j = \sum_{i=1}^{6} R_j^i \) with \(^\text{11}\)

\[
R_j^1 = [\dot{T}_{v_k}, \Delta_j] \nabla_k f, \quad R_j^2 = \dot{T}_{\nabla_k} \Delta_j, f v_k, \quad R_j^3 = -\Delta_j \dot{T}_{\nabla_k} f v_k,
\]

\[
R_j^4 = \dot{R}(v_k, \nabla_k \Delta_j f), \quad R_j^5 = -\nabla_k \dot{R}(v_k, f), \quad R_j^6 = \dot{\Delta}_j \dot{R}(\nabla_k v_k, f)
\]

which satisfy the estimates

\[
\left\| j \mapsto 2^{js} \| R_j^i \|_{L^p} \right\|_{L^q} \lesssim \| \nabla v \|_{L^{p_1}} \| f \|_{L^{p_2}}, \quad j \mapsto 2^{j^2} \| R_j^i \|_{L^p} \lesssim \| \nabla v \|_{L^{p_1}} \| f \|_{L^{p_2}} \quad \text{if } s_1 < 0, \\
\left\| j \mapsto 2^{js} \| R_j^i \|_{L^p} \right\|_{L^q} \lesssim \| \nabla v \|_{L^{p_1}} \| f \|_{L^{p_2}} \quad \text{if } s_1 > 0, \\
\left\| j \mapsto 2^{j^2} \| R_j^i \|_{L^p} \lesssim \| \nabla v \|_{L^{p_1}} \| f \|_{L^{p_2}} \quad \text{if } s_2 < 1, \\
\left\| j \mapsto 2^{j^2} \| R_j^i \|_{L^p} \lesssim \| \nabla v \|_{L^{p_1}} \| f \|_{L^{p_2}} \quad \text{if } s > 0,
\right.
\]

where the implied constants depend on \( \varphi, s_1, s_2, p_1, p_2 \).

**Remark 3.2.** Write \( A_1, A_2, A_3, A_5, A_6 \) to denote the constraints

\[
A_1 : s_1 \leq 0, \quad A_2 : s_1 \geq -1, \quad A_3 : s_2 \leq 1, \quad A_5 : s \geq -1, \quad A_6 : s \geq 0.
\]

For \( i = 1, 2, 3, 5, 6 \), a simple modification of our arguments yields the estimates

\[
\left\| j \mapsto 2^{js} \| R_j^i \|_{L^p} \lesssim \| \nabla v \|_{L^{p_1}} \| f \|_{L^{p_2}} \quad \text{if } q = 1 \text{ and } A_i \quad \text{holds,}
\]

but we will not need these estimates when proving Theorem 1.4.

As in [1] Lemma 2.100, to prove Proposition 3.1 we will rely on the following lemma:

**Lemma 3.3.** ([1], Lemma 2.97). Let \( \theta \in C^1(\mathbb{R}^n) \) be such that \( \int_{\mathbb{R}^n} (1 + |\xi|) |\mathcal{F} \theta(\xi)|\,d\xi < \infty \). Then for any \( a \in C^1(\mathbb{R}^n) \) with \( \nabla a \in L^p(\mathbb{R}^n) \), any \( b \in L^q(\mathbb{R}^n) \), and any \( \lambda \in (0, \infty) \), we have

\[
\| \theta(\lambda^{-1}D) a b \|_{L^q(\mathbb{R}^n)} \lesssim \lambda^{-\lambda} \| \nabla a \|_{L^p(\mathbb{R}^n)} \| b \|_{L^q(\mathbb{R}^n)}.
\]

**Remark 3.4.** If we take \( \theta = \varphi \) and \( \lambda = 2^j \), then Lemma 3.3 yields the estimate

\[
\| [\Delta_j, a b] \|_{L^q(\mathbb{R}^n)} \lesssim \varphi \lambda^{-\lambda} \| \nabla a \|_{L^p(\mathbb{R}^n)} \| b \|_{L^q(\mathbb{R}^n)}.
\]

**Proof of Proposition 3.1.** The decomposition \( R_j = \sum_{i=1}^{6} R_j^i \) comes from applying the Bony decomposition; the very strong regularity assumption \( v, f \in \cap_{r \in [0, \infty)} \dot{H}^r \) is more than sufficient to address any confluence issues that may arise. In the following computations, we write \( (c_j)_{j \in \mathbb{Z}} \) to denote a sequence satisfying \( \| (c_j) \|_{L^1} \leq 1 \), and the constants implied by the notation \( \lesssim \) depend on \( \varphi, s_1, s_2, p_1, p_2 \).

**Bounds for \( 2^{js} \| R_j^i \|_{L^p} \).** By Lemma 3.3 we have

\[
R_j^1 = \sum_{|j-j'| \leq 4} [\dot{S}_{j'-1} v_k, \Delta_j] \nabla_k \Delta_{j'} f,
\]

so by Remark 3.4 and Lemma 2.9 we have

\[
2^{js} \| R_j^1 \|_{L^p} \lesssim \sum_{|j-j'| \leq 4} 2^{js} 2^{j-j'} \| \nabla \dot{S}_{j'-1} v_k \|_{L^p} \| \dot{\Delta}_{j'} f \|_{L^p}, \quad (3.1)
\]

\(^{10}\)We apply the summation convention to the index \( k \).

\(^{11}\)Note that \( R_j^6 = 0 \) whenever \( \nabla \cdot v = 0 \).
By (2.1), we deduce that
\[ 2^{js} \| R^j_L \|_{L^p} \lesssim \sum_{|j-j'| \leq 4} 2^{js} 2^{j'-j} \| \nabla v \|_{L^{p_1}} \| \Delta_{j'} f \|_{L^{p_2}} \lesssim c_j \| \nabla v \|_{L^{p_1}} \| f \|_{B_{p_2,q}^{s_1}}. \]

On the other hand, if \( s_1 < 0 \) then (3.1) implies that
\[
2^{js} \| R^j_L \|_{L^p} \lesssim \sum_{|j-j'| \leq 4} 2^{js} 2^{j'-j} \| \nabla \Delta_{j'} v \|_{L^{p_1}} \| \Delta_{j'} f \|_{L^{p_2}}
\]
\[
\quad \lesssim \sum_{|j-j'| \leq 4} 2^{(j'-j)+1} 2^{j'-j} \| \nabla \Delta_{j'} v \|_{L^{p_1}} 2^{j'-j} \| \Delta_{j'} f \|_{L^{p_2}}
\]
\[
\quad \lesssim c_j \| \nabla v \|_{B_{p_1,q_1}^{s_1}} \| f \|_{B_{p_2,q_2}^{s_2}},
\]
where we used the inequality \( \| (\alpha * \beta) \gamma \|_{L^r} \leq \| \alpha \|_{L^r} \| \beta \|_{L^r} \| \gamma \|_{L^r} \leq \| \alpha \|_{L^r} \| \beta \|_{L^r} \| \gamma \|_{L^r} \) in the last line.

**Bounds for** \( 2^{js} \| R^j_L \|_{L^p} \). By Lemma 2.3, we have
\[
R^j_L = \sum_{j' \geq j+1} \tilde{S}_{j'-1} \nabla \Delta_{j'} f \Delta_{j'} v_k.
\]
so by Lemma 2.3, we have
\[
2^{js} \| R^j_L \|_{L^p} \lesssim \sum_{j' \geq j+1} 2^{js} 2^{j'-j} \| \nabla \Delta_{j'} v \|_{L^{p_1}} \| \Delta_{j'} f \|_{L^{p_2}}.
\]

If \( s_1 > -1 \), then we deduce that
\[
2^{js} \| R^j_L \|_{L^p} \lesssim \sum_{j' \geq j+1} 2^{(j'-j)(s_1+1)} 2^{j'-j} \| \nabla \Delta_{j'} v \|_{L^{p_1}} 2^{j'-j} \| \Delta_{j'} f \|_{L^{p_2}}
\]
\[
\quad \lesssim c_j \| \nabla v \|_{B_{p_1,q_1}^{s_1}} \| f \|_{B_{p_2,q_2}^{s_2}},
\]
where we used the inequality \( \| (\alpha * \beta) \gamma \|_{L^r} \leq \| \alpha \|_{L^r} \| \beta \|_{L^r} \| \gamma \|_{L^r} \leq \| \alpha \|_{L^r} \| \beta \|_{L^r} \| \gamma \|_{L^r} \) in the last line.

**Bounds for** \( 2^{js} \| R^j_L \|_{L^p} \). By Lemma 2.3, we have
\[
R^j_L = -\sum_{|j-j'| \leq 4} \Delta_{j'} \left( \tilde{S}_{j'-1} \nabla k \Delta_{j'} v_k \right)
\]
\[
\quad = -\sum_{|j-j'| \leq 4} \Delta_{j'} \left( \tilde{S}_{j'-1} \nabla k \Delta_{j'} v_k \right),
\]
so by Lemma 2.3, we have
\[
2^{js} \| R^j_L \|_{L^p} \lesssim \sum_{|j-j'| \leq 4} 2^{js} 2^{j'-j} \| \nabla \Delta_{j'} v \|_{L^{p_1}} \| \Delta_{j'} f \|_{L^{p_2}}.
\]

If \( s_2 < 1 \), then we deduce that
\[
2^{js} \| R^j_L \|_{L^p} \lesssim \sum_{j' \leq j-2} 2^{js} 2^{j'-j} \| \nabla \Delta_{j'} v \|_{L^{p_1}} 2^{j'-j} \| \nabla \Delta_{j'} \tilde{S}_{j'-1} \nabla k \Delta_{j'} v_k \|_{L^{p_1}} \| \Delta_{j'} f \|_{L^{p_2}}
\]
\[
\quad \lesssim c_j \| \nabla v \|_{B_{p_1,q_1}^{s_1}} \| \nabla \Delta_{j'} \tilde{S}_{j'-1} \nabla k \|_{B_{p_1,q_1}^{s_1}} \| f \|_{B_{p_2,q_2}^{s_2}},
\]
where we used the inequality \( \| (\alpha * \beta) \gamma \|_{L^r} \leq \| \alpha \|_{L^r} \| \beta \|_{L^r} \| \gamma \|_{L^r} \leq \| \alpha \|_{L^r} \| \beta \|_{L^r} \| \gamma \|_{L^r} \) in the last line.

**Bounds for** \( 2^{js} \| R^j_L \|_{L^p} \). Defining \( \tilde{S}_{j'} = \sum_{|\nu| \leq 1} \tilde{S}_{j'-\nu} \), by Lemma 2.3, we have
\[
R^j_L = \sum_{|j-j'| \leq 2} \Delta_{j'} v_k \nabla \Delta_{j'} f
\]
so by Lemma 2.3 and the inequality \( \| \alpha \beta \|_{L^r} \leq \| \alpha \|_{L^r} \| \beta \|_{L^r} \), we have
\[
2^{js} \| R^j_L \|_{L^p} \lesssim c_j \| \nabla v \|_{B_{p_1,q_1}^{s_1}} \| f \|_{B_{p_2,q_2}^{s_2}}.
\]

**Bounds for** \( 2^{js} \| R^j_L \|_{L^p} \) and \( 2^{js} \| R^j_L \|_{L^p} \). By Lemma 2.3 and 2.12, we have
\[
\| j \mapsto 2^{js} \| R^j_L \|_{L^p} \|_{L^p} \lesssim \| \nabla v \|_{B_{p_1,q_1}^{s_1}} \| f \|_{B_{p_2,q_2}^{s_2}} \quad \text{if } s > -1,
\]
\[
\| j \mapsto 2^{js} \| R^j_L \|_{L^p} \|_{L^p} \lesssim \| \nabla v \|_{B_{p_1,q_1}^{s_1}} \| f \|_{B_{p_2,q_2}^{s_2}} \quad \text{if } s > 0.
\]
4. Proof of blowup rates

We now give the

Proof of Theorem 4.4. Note first that the regularity assumptions on \( u \) are strong enough to justify the calculations used in this proof.

Fix \( \epsilon \in [1, 2] \) and \( p, q \in [1, \frac{\infty}{\epsilon}] \). By virtue of the inequality (4.1), it suffices to prove the estimate

\[
\|u(t)\|_{\dot{B}^s_{p,r}} \geq \epsilon^{-1/2}(T-t)^{-\epsilon/2}
\]

for any fixed \( r \in [p \vee q \vee 2, \frac{\infty}{\epsilon}] \) (e.g., \( r = \max\{p \vee q \vee 2, \frac{\infty}{\epsilon}\} \)), where \( \bar{r} = r \) in the case \( \epsilon = 2 \), and \( \bar{r} = 1 \) in the case \( \epsilon = 2 \).

Applying \( \Delta _t \) to \( \omega = \nabla \times u \), we see that \( \omega \) satisfies

\[
\partial_t \omega_{ij} - \Delta \omega_{ij} + (u \cdot \nabla) \omega_{ij} + \omega_{ik} \nabla_k u_j = \omega_{jk} \nabla_k u_i.
\]

By Lemma 4.1 we deduce that (4.1) is equivalent to

\[
\|\omega(t)\|_{\dot{B}^s_{p,r}} \geq \epsilon^{-1/2}(T-t)^{-\epsilon/2},
\]

and that (4.3) is equivalent to

\[
\lim_{t \to T} \|\omega(t)\|_{\dot{B}^s_{p,r}} = \infty.
\]

Applying the operator \( X \mapsto \nabla_i X_j - \nabla_j X_i \) to the Navier-Stokes equations (1.1), we see that \( \omega \) satisfies

\[
\partial_t \omega_{ij} - \Delta \omega_{ij} + (u \cdot \nabla) \omega_{ij} + \omega_{ik} \nabla_k u_j = \omega_{jk} \nabla_k u_i.
\]

Applying \( \Delta _t \) to the equation (4.1), multiplying by \( |\Delta _t \omega|^{-2} \Delta _t \omega_{ij} \), summing over \( i, j \) and integrating over \( \mathbb{R}^n \), we obtain

\[
\frac{1}{r} \frac{\partial}{\partial t} \left( \|\Delta _t \omega\|^2_{L^r} \right) - \left( \Delta _t \omega_{ij}, |\Delta _t \omega|^{-2} \Delta _t j_\omega \right) = - \left( \Delta _t j_((u \cdot \nabla) \omega), |\Delta _t \omega|^{-2} \Delta _t j_\omega \right) - \left( \Delta _t (\omega_{ik} \nabla_k u_j - \omega_{jk} \nabla_k u_i), |\Delta _t \omega|^{-2} \Delta _t \omega_{ij} \right).
\]

By the identities \( \nabla \cdot u = 0 \), \( \nabla \left( |\Delta _t \omega|^2 \right) = r(\nabla_k \Delta _t \omega_{ij}) |\Delta _t \omega|^{-2} \Delta _t \omega_{ij} \) and \( \omega_{ij} = -\omega_{ji} \), we see that the right hand side of (4.3) is equal to

\[
\left( |u \cdot \nabla, \Delta _t |\omega, |\Delta _t \omega|^{-2} \Delta _t j_\omega \right) - 2 \left( \Delta _t (\omega \cdot \nabla u), |\Delta _t \omega|^{-2} \Delta _t j_\omega \right),
\]

where we define \( (\omega \cdot \nabla)_{ij} := \omega_{ik} \nabla_k u_j \). Writing \( \Omega_j := [u \cdot \nabla, \Delta _t |\omega - 2\Delta _t (\omega \cdot \nabla u), \) and noting the inequality\(^{12}\)

\[
- \left( \Delta v, |v|^{-2} v \right) \geq \epsilon^{-1/2}(T-t)^{-\epsilon/2},
\]

\[
\epsilon^{-1/2}(T-t)^{-\epsilon/2} \|v\|_{L^r}^{r-1},
\]

In the case \( n = 3 \), this is related to the vorticity vector \( \omega := \nabla \times u \) by \( \omega := (\omega_{31}, \omega_{12}, \omega_{23})^T \). One could view \( \omega \) and \( \Delta _t \omega \) as being different ways of representing the exterior derivative of the 1-form \( \sum_{i=1}^n u_i \, dr^i \).

\(^{13}\) Valid for \( n \geq 3 \) and \( t \in [2, \infty) \), proved in [5] Lemmas 1-2.
we deduce that
\[
\frac{\partial}{\partial t} \left( \|\xi_J \omega\|_{L^r}^r \right) + \|\Delta_J \omega\|_{L^\infty}^r \lesssim_{n,r} \langle \Omega_J, |\Delta_J \omega|^{-2} \Delta_J \omega \rangle. 
\]
(4.10)

The case $\epsilon < 2$. Suppose first that $\epsilon \in [1, 2)$. From (4.10) we have
\[
\frac{\partial}{\partial t} \left( \|\omega\|_{B^r_{\frac{r}{r-1}}}^{r} \right) + \|\Delta_J \omega\|_{L^\infty}^r \lesssim_{n,r} \sum_{J \in \mathcal{Z}} 2^{r \epsilon (s+1)} \langle \Omega_J, |\Delta_J \omega|^{-2} \Delta_J \omega \rangle. 
\]
(4.11)

Since $\epsilon \in (0, 2)$ and $r \in (1, \infty)$, the interval $I_{n,r;\epsilon} := (\frac{2n}{r} - \frac{\epsilon}{r}, \frac{2n}{r}) \cap (\frac{2n}{r} - 2 + \epsilon, \frac{2n}{r} - \frac{\epsilon}{r})$ is non-empty, so we are free to choose $r_1$ satisfying $\frac{2n}{r} \in I_{n,r;\epsilon}$. Let $r_2$ and $r_3$ be given by

\[
\frac{n}{r_2} = s_r + \epsilon - 1 - \frac{n}{r_1}, \quad r_3 = \frac{r_1 r_2}{r_2 + r}.
\]

Then $\frac{2n}{r_1} > \frac{2n}{r_2} - \frac{\epsilon}{r_2}$ is equivalent to $r < r_1 < \frac{2n}{r_2}$, while $\frac{2n}{r_1} - \frac{\epsilon}{r_1} > \frac{2n}{r_2} - 2 + \epsilon$ is equivalent to $r_3 < r < r_4$. Therefore
\[
\left( \frac{r_1 r_2}{r_2 + r} \right) = \frac{r_1}{r_2} < r_3 < r < r_1 < \frac{r_2}{r_2 + r}.
\]

Writing $r_4 = r_3' (r_1 - 1)$, by Hölder’s inequality we have
\[
2^{r \epsilon (s+1)} \langle \Omega_J, |\Delta_J \omega|^{-2} \Delta_J \omega \rangle \leq \sum_{J \in \mathcal{Z}} 2^{r \epsilon (s+1)} \|\Omega_J\|_{L^3} \|\Delta_J \omega\|_{L^4}^{-2} \|
\]
(4.12)

Since $\left( \frac{r_2}{r_2 + r} \right)$, $r < r_3$, it follows that $r' > r_3' \leq \frac{2n}{r_2}$ and hence $r < r_4 < \frac{2n}{r_2}$. By (2.7), we deduce that
\[
\|\omega\|_{B^r_{\frac{r}{r-1}}} \leq \|\omega\|_{B^r_{\frac{r}{r-1}}} \|\omega\|_{B^r_{\frac{r}{r-1}}} \quad \text{for } \mu = \frac{rn}{2} \left( \frac{1}{r_4} - \frac{n - 2}{r} \right). 
\]
(4.13)

We now need to estimate $\|J \rightarrow 2^{r \epsilon (s+1)} \|\Omega_J\|_{L^3}$. By the Bony estimates (2.10) and (2.12), the inequality $\|v\|_{B^r_{\frac{r}{r-1}}} \lesssim |v|_{L^2}$, and the assumption $r < \frac{2n}{r}$ (which is equivalent to $s_r + \epsilon > 1$), we have
\[
\|\omega \cdot \nabla u\|_{B^r_{\frac{r}{r-1}}} \lesssim_{s_r, r} \|\omega\|_{B^r_{\frac{r}{r-1}}} \|\nabla u\|_{L^2} \|\omega\|_{L^2} \|\nabla u\|_{B^r_{\frac{r}{r-1}}} \quad \text{for } I = 2, 3, 4, 5. 
\]
(4.14)

On the other hand, by Proposition 3.11 we have $[u \cdot \nabla, \Delta_J] \omega = \sum_{I=1}^5 R^I_J$, where
\[
\|J \rightarrow 2^{r \epsilon (s+1)} \|R^I_J\|_{L^3} \leq \|\nabla u\|_{L^2} \|\omega\|_{B^r_{\frac{r}{r-1}}} \quad \text{for } I = 1, 2, 3, 4, 5. 
\]
(4.15)

Combining (4.14) and (4.15), and noting the relations (4.4), (4.5) and Lemma 2.3 we therefore have
\[
\|J \rightarrow 2^{r \epsilon (s+1)} \|\Omega_J\|_{L^3} \lesssim_{s_r, r} \|\omega\|_{B^r_{\frac{r}{r-1}}} \quad \text{for } \mu = \frac{rn}{2} \left( \frac{1}{r_4} - \frac{n - 2}{r} \right). 
\]
(4.16)

The indices $r_1, r_2$ we chosen to ensure that $r < r_1 < \frac{2n}{r_2}$ and $s_r + \epsilon = 1 = \frac{n}{r_2} - \frac{n - 2}{r_2}$ and that the spaces $B^r_{\frac{r}{r-1}}$ and $L^2$ have the same scaling; from these conditions we deduce the interpolation inequality
\[
\|\omega\|_{B^r_{\frac{r}{r-1}}} \lesssim_{s_r, r} \|\omega\|_{B^r_{\frac{r}{r-1}}} \|\omega\|_{B^r_{\frac{r}{r-1}}} \quad \text{for } \mu = \frac{rn}{2} \left( \frac{1}{r_4} - \frac{n - 2}{r} \right), 
\]
(4.17)

where the estimate on $\|\omega\|_{B^r_{\frac{r}{r-1}}}$ follows from (2.7), while the estimate on $\|\omega\|_{L^2}$ is justified (writing $r_5 = \frac{r_1 r_2}{r_2 + r}$ and $s_r + \epsilon = 1 = \frac{n}{r_2} = \frac{n - 2}{r_2}$) by the embedding $\|\omega\|_{L^2} \leq \|\omega\|_{B^r_{\frac{r}{r-1}}}$, and the calculations
\[
\text{If } r_2 \geq r_5 : \quad \|\omega\|_{B^r_{\frac{r}{r-1}}} \lesssim_{s_r, r} \|\omega\|_{B^r_{\frac{r}{r-1}}} \|\omega\|_{B^r_{\frac{r}{r-1}}} \|\omega\|_{B^r_{\frac{r}{r-1}}} \|\omega\|_{B^r_{\frac{r}{r-1}}} \|\omega\|_{B^r_{\frac{r}{r-1}}} 
\]
\[
\text{If } r_2 < r_5 : \quad \|\omega\|_{B^r_{\frac{r}{r-1}}} \lesssim_{s_r, r} \|\omega\|_{B^r_{\frac{r}{r-1}}} \|\omega\|_{B^r_{\frac{r}{r-1}}} \|\omega\|_{B^r_{\frac{r}{r-1}}} \|\omega\|_{B^r_{\frac{r}{r-1}}} \|\omega\|_{B^r_{\frac{r}{r-1}}} \|\omega\|_{B^r_{\frac{r}{r-1}}} \|\omega\|_{B^r_{\frac{r}{r-1}}} \|\omega\|_{B^r_{\frac{r}{r-1}}} 
\]

for $\nu, \sigma, \rho \in (0, 1)$ determined by (2.20) and (2.22). By the bounds (4.12) and (4.13), and the interpolation inequalities (4.14) and (4.15), we obtain
\[
\sum_{J \in \mathcal{Z}} 2^{r \epsilon (s+1)} \langle \Omega_J, |\Delta_J \omega|^{-2} \Delta_J \omega \rangle \lesssim_{\sigma, r} \|\omega\|_{B^r_{\frac{r}{r-1}}} \|\omega\|_{B^r_{\frac{r}{r-1}}} \|\omega\|_{B^r_{\frac{r}{r-1}}} \|\omega\|_{B^r_{\frac{r}{r-1}}} 
\]
where $\mu$ and $\nu$ are given by (4.13) and (4.17). Noting that
\[
\frac{r-1}{r_4} \cdot \frac{2}{r_1} = 1 - \frac{1}{r_3} + \frac{2}{r_1} = 1 + \frac{1}{r_1} - \frac{1}{r_2} = 1 + \frac{1}{r} + \frac{\epsilon - 2}{n},
\]
we see that
\[
(r-1)\mu + 2\nu = \frac{rn}{2} \left( (r-1) \left( \frac{1}{r_4} - \frac{n-2}{rn} \right) + 2 \left( \frac{1}{r_1} - \frac{n-2}{rn} \right) \right) \frac{r-1}{r} = \frac{rn}{2} \left( (r-1) \left( \frac{2}{r_1} + \frac{2}{r_1} \right) - (r+1)(n-2) \right) = 1 + \frac{r}{2}
\]
and hence
\[
\sum_{J \in \mathbb{Z}} 2^{Jr(s_\epsilon+\epsilon-1)} \left( \Omega_J, |\Delta_J \omega|^{-2} \Delta_J \omega \right) \lesssim_{\omega,x} \|\omega\|_{B^{s_\epsilon+1}_r}^{1+\frac{rn}{2}} \|\omega\|_{B^{(s_\epsilon+1)-1}_r}^{\frac{rn}{2}}. \tag{4.18}
\]
By (4.11), (4.18) and Young’s product inequality, we deduce that
\[
\frac{\partial}{\partial t} \left( \|\omega\|_{B^{s_\epsilon+1}_r} \right) \lesssim_{n,r} \|\omega\|_{L^\infty}.
\]
Applying Lemma 4.1 with $X(t) = \|\omega(t)\|_{B^{s_\epsilon+1}_r}$ and $\gamma = 2 - \frac{2}{r}$, and noting (4.11), we conclude that (4.10) holds, which (as noted above) implies (4.10).

The case $\epsilon = 2$. By (4.10) and Hölder’s inequality, for all $J,t$ satisfying $\Delta_J \omega(t) \neq 0$ in $S'$ we have
\[
\frac{\partial}{\partial t} \left( \|\omega\|_{B^{s_\epsilon+1}_r} \right) \lesssim_{n,r} \|\Omega_J\|_{L^r}.
\]
If $\Delta_J \omega(t_0) = 0$ in $S'$, then either $\frac{\partial}{\partial t} \left( \|\Delta_J \omega\|_{L^r} \right) \big|_{t=t_0} = 0$ (in which case (4.19) is true for $t = t_0$) or $\frac{\partial}{\partial t} \left( \|\Delta_J \omega\|_{L^r} \right) \big|_{t=t_0} \neq 0$ (in which case (4.19) is true for $t$ close to $t_0$, so by continuity it is true for $t = t_0$). Therefore (4.19) holds for all $J \in \mathbb{Z}$ and $t \in (0,T)$, so we can estimate
\[
\frac{\partial}{\partial t} \left( \|\omega\|_{B^{s_\epsilon+1}_r} \right) \lesssim_{n,r} \left\| J \mapsto 2^{Jr(s_\epsilon+1)} \|\Omega_J\|_{L^r(t)} \right\|_{L^1}.
\]
We now need to estimate $\left\| J \mapsto 2^{Jr(s_\epsilon+1)} \|\Omega_J\|_{L^r(t)} \right\|_{L^1}$. By the Bony estimates (2.10) and (2.12), the inequality $\|v\|_{B^s_{\infty,\infty}} \lesssim_v \|v\|_{L^\infty}$, and the assumption $r < \infty$ (which is equivalent to $s_\epsilon + 1 > 0$), we have
\[
\|\omega \cdot \nabla u\|_{B^{s_\epsilon+1}_r} \lesssim_{\omega,r} \|\omega\|_{B^{s_\epsilon+1}_r} \|\nabla u\|_{L^\infty} + \|\omega\|_{B^{s_\epsilon+1}_r} \|\nabla u\|_{B^{s_\epsilon+1}_r}.
\]
On the other hand, by Proposition 3.1 we have $[u \cdot \nabla, \Delta] \omega = \sum_{I=1}^5 R_I \omega$, where
\[
\left\| J \mapsto 2^{Jr(s_\epsilon+1)} \|R_I\|_{L^r(t)} \right\|_{L^1} \lesssim_{\omega,r} \|\nabla u\|_{L^\infty} \|\omega\|_{B^{s_\epsilon+1}_r},
\]
\[
\left\| J \mapsto 2^{Jr(s_\epsilon+1)} \|R_I\|_{L^r(t)} \right\|_{L^1} \lesssim_{\omega,r} \|\nabla u\|_{B^{s_\epsilon+1}_r} \|\omega\|_{B^{s_\epsilon+1}_r} \text{ for } I = 2,3,4,5.
\]
Combining (4.21) and (4.22), and noting the relations (4.4)–(4.5) and Lemma 2.3 we therefore have
\[
\left\| J \mapsto 2^{Jr(s_\epsilon+1)} \|\Omega_J\|_{L^r(t)} \right\|_{L^1} \lesssim_{\omega,r} \|\omega\|_{B^{s_\epsilon+1}_r}^2.
\]
By (4.20) and (4.23) we have
\[
\frac{\partial}{\partial t} \left( \|\omega\|_{B^{s_\epsilon+1}_r} \right) \lesssim_{n,r} \|\omega\|_{B^{s_\epsilon+1}_r}^2.
\]
Hence if $\epsilon = 2$, applying Lemma 4.1 with $X(t) = \|\omega\|_{B^{s_\epsilon+1}_r}$ and $\gamma = 1 + \frac{2}{r}$, and noting (4.10), we conclude that (4.10) holds, which (as noted above) implies (4.10). □

As a side remark, we observe that if an estimate of the form
\[
\sum_{J \in \mathbb{Z}} 2^{Jr(s_\epsilon+1-\alpha)} \left( \Omega_J, |\Delta_J \omega|^{-2} \Delta_J \omega \right) \lesssim_{\omega,x,r,\alpha,\beta} \|\omega\|_{B^{s_\epsilon+1}_r} \|\omega\|_{B^{(s_\epsilon+1)-1}_r}^\beta \text{ holds for all antisymmetric } \omega, \text{ then necessarily } \alpha = 1 + \frac{r}{2} \text{ and } \beta = \frac{r}{2}(2 - \epsilon). \]
This observation can be justified by considering the effect of the rescaling $\omega \mapsto \kappa \omega$ and $x \mapsto 2^N x$ for $\kappa > 0$ and $N \in \mathbb{Z}$.
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