ONE REMARK ON THE B-SEMIAMPLENESS OF THE MODULI PART

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Abstract. In this short note we reduce the b-semiampleness conjecture for lc-trivial fibrations to the b-semiampleness conjecture for klt-trivial fibrations.

1. Introduction

An important problem in birational geometry is the following: given a morphism \( f: X \to Z \) and an lc pair \((X, B)\) such that \( K_X + B \sim_{Qf} 0 \), find a \( \mathbb{Q} \)-divisor \( \Delta_Z \) on \( Z \) such that \((Z, \Delta_Z)\) is lc and

\[
K_X + B \sim_{Q} f^*(K_Z + \Delta_Z).
\]

The problem has been solved by Ambro when \((X, B)\) is klt and \( B \geq 0 \) by using the canonical bundle formula [3, Theorem 0.2] and by Fujino and Gongyo when \( f \) is generically finite and \((X, B)\) is lc by [8, Lemma 1.1].

In general, we write canonical bundle formulas for lc-trivial (resp. klt-trivial fibrations), that are fibrations \( f: (X, B) \to Z \) such that \((X, B)\) is lc (resp. klt) on the generic point of \( Z \) and \( K_X + B \sim_{Qf} 0 \) (see Definition 2.2 for a precise definition). The canonical bundle formula consists in writing

\[
K_X + B \sim_{Q} f^*(K_Z + B_Z + M_Z)
\]

where \( B_Z \) is called discriminant and contains information on the singular fibers of \( f \) and \( M_Z \) is called the moduli part and, conjecturally, is related to the birational variation of the fibers. The main steps in the proof of [3, Theorem 0.2] are the following:

- the pair \((Z, B_Z)\) has the same singularities as \((X, B)\) [1, Proposition 3.4], namely \((Z, B_Z)\) is klt (resp. lc) \( \iff \) \((X, B)\) is klt (resp. lc);

- if \((X, B)\) is klt on the generic point of \( Z \), then \( M_Z \) is the pullback of a big divisor [3, Theorem 3.3] or, more precisely, there exist a generically finite morphism \( \tau: \tilde{Z} \to Z \), a morphism \( \rho: \tilde{Z} \to Z' \) and a big divisor \( M_{Z'} \), such that

\[
\tau^* M_Z = \rho^* M_{Z'}.
\]

Since klt is an open condition and the pair \((Z, B_Z)\) is klt, it is sufficient to chose

\[
\Delta_Z = B_Z + \frac{1}{m} \tau_* \rho^* D
\]

where \( D \in |mM_{Z'}| \) and \( m \) is big enough.
If we want to follow the path of this proof, in order to solve the problem for \((X, B)\) is lc, we need a stronger condition on \(M_Z\), namely

**Conjecture 1.1.** Let \(f: (X, B) \to Z\) be a lc-trivial fibration. Then there exists a birational morphism \(\mu: Z' \to Z\) such that \(M_{Z'}\) is semiample.

We can also state the conjecture for klt-trivial fibrations (cf. [12, Conjecture 7.13.3]).

**Conjecture 1.2.** Let \(f: (X, B) \to Z\) be a klt-trivial fibration. Then there exists a birational morphism \(\mu: Z' \to Z\) such that \(M_{Z'}\) is semiample.

Conjecture 1.2 is true if the fiber of the klt-trivial fibration is a curve by [12, Theorem 8.1] and if the fiber is isomorphic to a K3 surface or to an abelian variety by [6, Theorem 1.2]. Moreover Conjecture 1.1 is true if \(\dim Z = 1\) by [3, Theorem 3.5] in the klt case and by [5, Theorem 1.4].

In [9] the authors prove that, for an lc-trivial fibration, the moduli part is the pull-back of a big divisor. They consider a center \(W\) of \((X, B)\) that is minimal among the centers such that \(f(W) = Z\) and they find a quasi-finite morphism \(Z'' \to Z\) such that

1. the restriction \(f'|_{W'}\) defines a klt-trivial fibration;
2. if we set \(M_{Z''}^{\text{min}}\) the moduli part of \(f'|_{W'}\) then we have

\[ M_{Z''}^{\text{min}} \sim_{\mathbb{Q}} M_{Z'}.

By following the same approach as in [9] we can prove the following result

**Theorem 1.3.** Conjecture 1.2 implies Conjecture 1.1.

2. Definitions and known results

We will work over \(\mathbb{C}\). In the following \(\equiv, \sim\) and \(\sim_{\mathbb{Q}}\) will respectively indicate numerical, linear and \(\mathbb{Q}\)-linear equivalence of divisors.

**Definition 2.1.** Let \((X, B)\) be a pair and \(\nu: X' \to X\) a log resolution of the pair. We set

\[ A(X, B) = K_{X'} - \nu^*(K_X + B) \]

and

\[ A^*(X, B) = A(X, B) + \sum_{a(E, X, B) = 1} E. \]

**Definition 2.2.** A klt-trivial (resp. lc-trivial) fibration \(f: (X, B) \to Z\) consists of a surjective morphism with connected fibers of normal varieties \(f: X \to Z\) and of a log pair \((X, B)\) satisfying the following properties:

1. \((X, B)\) has klt (resp. lc) singularities over the generic point of \(Z\);
2. \(\text{rank } f^!_X \mathcal{O}_X([A(X, B)]) = 1\) (resp. \(\text{rank } f^*_X \mathcal{O}_X([A^*(X, B)]) = 1\)) where \(f' = f \circ \nu\) and \(\nu\) is a given log resolution of the pair \((X, B)\);
(3) there exists a positive integer $r$, a rational function $\varphi \in \mathbb{C}(X)$ and a $\mathbb{Q}$-Cartier divisor $D$ on $Z$ such that

$$K_X + B + \frac{1}{r}(\varphi) = f^*D.$$  

**Remark 2.3.** The smallest possible $r$ that can appear in Definition 2.2 is the minimum of the set

$$\{ m \in \mathbb{N} | m(K_X + B)|_F \sim 0 \}$$

that is the Cartier index of the fiber. We will always assume that the $r$ that appears in the formula is the smallest one.

**Definition 2.4.** Let $P \subseteq Z$ be a prime Weil divisor. The log canonical threshold $\gamma_P$ of $f^*(P)$ with respect to the pair $(X, B)$ is defined as follows. Let $\bar{Z} \to Z$ be a resolution of $Z$. Let $\mu: \bar{X} \to X$ be the birational morphism obtained as a desingularisation of the main component of $X \times_Z \bar{Z}$. Let $\bar{f}: \bar{X} \to \bar{Z}$. Let $\bar{B}$ be the divisor defined by the relation

$$K_{\bar{X}} + \bar{B} = \mu^*(K_X + B).$$

Let $\bar{P}$ be the strict transform of $P$ in $\bar{Z}$. Set

$$\gamma_P = \sup\{ t \in \mathbb{Q} | (\bar{X}, \bar{B} + t\bar{f}^*(\bar{P})) \text{ is lc over } \bar{P} \}.$$  

We define the discriminant of $f: (X, B) \to Z$ as

$$B_Z = \sum_P (1 - \gamma_P)P.$$  

**Remark 2.5** ([10], p.14 [1]). The log canonical threshold $\gamma_P$ is a rational number and the sum above is finite, thus the discriminant is a $\mathbb{Q}$-Weil divisor

**Definition 2.6.** Fix $\varphi \in \mathbb{C}(X)$ such that $K_X + B + \frac{1}{r}(\varphi) = f^*D$. Then there exists a unique divisor $M_Z$ such that we have

$$K_X + B + \frac{1}{r}(\varphi) = f^*(K_Z + B_Z + M_Z)$$

where $B_Z$ is as in (2.1). The $\mathbb{Q}$-Weil divisor $M_Z$ is called the moduli part.

We have the two following results.

**Theorem 2.7** (Theorem 0.2 [2], [4]). Let $f: (X, B) \to Z$ be an lc-trivial fibration. Then there exists a proper birational morphism $Z' \to Z$ with the following properties:

(i): $K_{Z'} + B_{Z'}$ is a $\mathbb{Q}$-Cartier divisor, and for every proper birational morphism $\nu: Z'' \to Z'$

$$\nu^*(K_{Z'} + B_{Z'}) = K_{Z''} + B_{Z''}.$$  

(ii): $M_{Z'}$ is a nef $\mathbb{Q}$-Cartier divisor and for every proper birational morphism $\nu: Z'' \to Z'$

$$\nu^*(M_{Z'}) = M_{Z''}.$$
Proposition 2.8 (Proposition 5.5 [2]). Let \( f : (X, B) \to Z \) be an lc-trivial fibration. Let \( \tau : Z' \to Z \) be a generically finite projective morphism from a non-singular variety \( Z' \). Assume there exists a simple normal crossing divisor \( \Sigma_{Z'} \) on \( Z' \) which contains \( \tau^{-1}\Sigma_Z \) and the locus where \( \tau \) is not étale. Let \( M_{Z'} \) be the moduli part of the induced lc-trivial fibration \( f' : (X', B') \to Z' \). Then \( M_{Z'} = \tau^*M_Z \).

The formula (2.2) is called the canonical bundle formula.

The following result is a well known result. We recall the proof here for the reader’s convenience.

Lemma 2.9. Let \( \tau : W \to Y \) be a generically finite morphism. Let \( D \) be a divisor on \( Y \) and assume that \( \tau^*D \) is base point free. Let \( d = \deg \tau \). Then

\[
\text{Codim}(Bs|dD|) \geq 2.
\]

Proof. Consider the Stein factorization of \( \tau \)

\[
\begin{array}{c}
W \xrightarrow{\tau} Y \\
\downarrow g \quad \quad \quad \downarrow h \\
Y' \quad \quad \quad \quad Y
\end{array}
\]

The morphism \( g \) is birational. Let \( z \in Y \setminus \tau(\text{Exc}(g)) \). Then the set \( \tau^{-1}(z) \) is finite. A general element \( E \in |\tau^*D| \) is such that \( \text{Supp}E \cap \tau^{-1}(z) = \emptyset \), hence \( z \not\in \text{Supp}\tau_*E \). Since \( \tau_*E \in |dD| \), we proved that

\[
Bs|dD| \subseteq \tau(\text{Exc}(g))
\]

which is a subset of codimension at least 2. \( \square \)

3. Proof of the main result

This section is devoted to the proof of Theorem 1.3. We will be dealing with lc-trivial fibrations that are not klt-trivial fibrations. The main tool is the subadjunction formula for centers that are minimal over the generic point of \( Z \). The next remark explains the geometric construction used in [9] and the notation that will be used.

Remark 3.1 (cf. [9]). Let \( f : (X, B) \to Z \) be an lc-trivial fibration such that \( (X, B) \) is not klt over the generic point of \( Z \). Then there exists a center \( W \subseteq X \) that is minimal among the centers \( C \) of \( (X, B) \) such that \( f(C) = Z \). Let

\[
\begin{array}{c}
W \xrightarrow{f|W} Z \\
\downarrow \alpha \quad \quad \downarrow \beta \\
Z_1
\end{array}
\]
be the Stein factorization of \( f|_{W} \). Let \( Z_{1}' \) be a desingularisation of \( Z_{1} \). Consider the lc-trivial fibration induced by the base change

\[
\begin{array}{ccc}
(X_{1}', B_{1}') & \longrightarrow & (X, B) \\
\downarrow f' & & \downarrow f \\
Z_{1}' & \longrightarrow & Z.
\end{array}
\]

Let \( W_{1}' \) be the preimage of \( W \) in \( X_{1}' \) and let \( n: W_{1}'' \to W_{1}' \) be its normalization. The variety \( W_{1}'' \) is birational to \( W \) and, by adjunction (see [1 Proposition 3.9.2]), there exists \( B^{W_{i}} \) such that

- the pair \( (W_{1}''', B^{W_{i}}) \) is klt;
- we have \( K_{W_{1}'''} + B^{W_{i}} = n^*(K_{X_{1}'} + B) \).

Then the restriction of \( f' \) induces a klt-trivial fibration

\[
g_{1} = f' \circ n: (W_{1}', B^{W_{i}}) \to Z_{1}.
\]

to whom we can associate a discriminant \( B_{Z_{1}'}^{\min} \) and a moduli part \( M_{Z_{1}'}^{\min} \). If \( B_{Z_{1}'} \) and \( M_{Z_{1}'} \) are discriminant and moduli part of \( f'_{1} \), we have

\[
K_{Z_{1}'} + B_{Z_{1}'} + M_{Z_{1}'} \sim_{\mathbb{Q}} K_{Z_{1}'} + B_{Z_{1}'}^{\min} + M_{Z_{1}'}^{\min}.
\]

Though not explicitly stated, the following proposition is the key result in [1].

**Theorem 3.2** (Theorem 1.1 [9]). Let \( f: (X, B) \to Z \) be an lc-trivial fibration. Assume that \( (X, B) \) is not klt over the generic point of \( Z \) and let \( W \) be center of \( (X, B) \) that is minimal over the generic point of \( Z \) and such that \( f(W) = Z \).

Assume that \( M_{Z} \) is such that for any birational morphism \( \mu: Z_{1} \to Z \) we have \( M_{Z_{1}} = \mu^*M_{Z} \).

Then there exists a generically finite mophism \( \tau: Z' \to Z \) with the following properties. Let \( f': (X', B') \to Z' \) be the lc-trivial fibration obtained by base change, let \( W' \) be the preimage of \( W \) in \( X' \) and let \( n: W'' \to W' \) be its normalization. Then

1. \( g = f'|_{W'} \circ n: (W'', B^W) \to Z' \) is a klt-trivial fibration;
2. for any birational morphism \( \mu: Z'' \to Z' \) we have \( M_{Z''}^{\min} = \mu^*M_{Z'}^{\min} \);
3. \( B_{Z''}^{\min} = B_{Z'} \).

Let us prove Theorem 1.3.

**proof of Theorem 1.3**. Let \( f: (X, B) \to Z \) be an lc-trivial fibration. If \( f \) is a klt-trivial fibration, we are done. Then we can assume that \( (X, B) \) is not klt over the generic point of \( Z \). Let \( W \) be an arbitrary log canonical center of \((X, B)\) that is dominant onto \( Z \) and is minimal over the generic point of \( Z \). By Theorem 2.7, we can assume that \( M_{Z} \) is nef and that for any birational morphism \( \mu: Z_{1} \to Z \) we have \( M_{Z_{1}} = \mu^*M_{Z} \).

Let \( \tau: Z' \to Z \) be a generically finite base change as in Proposition 3.2. Then, by using the same notation as in Remark 3.1, we have \( M_{Z'}^{\min} \sim_{\mathbb{Q}} M_{Z} \). By hypothesis there exists a birational morphism \( \nu: Z'' \to Z' \) such that \( M_{Z''}^{\min} \) is semiample. Let \( m \) be an integer such that

- \( mM_{Z''}^{\min} \) is base point free;
\( \bullet \) \( mM^\min_Z \sim mM_{Z'} \).

Set \( \tilde{\tau} = \tau \circ \nu \). The morphism \( \tilde{\tau} \) is generically finite and let \( d = \deg \tilde{\tau} \). By Lemma 2.9
\[
\text{Codim}(\text{Bs}|mdM_Z|) \geq 2.
\]
Let \( \mu : \tilde{Z} \to Z \) be a resolution of the base locus of \( |mdM_Z| \). Then
\[
|\mu^*(mdM_Z)| = |M| + F,
\]
the linear system \( |M| \) is base point free and \( F \) is the base locus of \( |\mu^*(mdM_Z)| \). Let \( \tilde{Z}' \) be a desingularisation of the main component of the fiber product \( Z'' \times_Z \tilde{Z} \). We have the following commutative diagram:

\[
\begin{array}{ccc}
Z'' & \xrightarrow{\tilde{\tau}} & Z \\
\mu'' \downarrow & & \downarrow \mu \\
Z'' & \xrightarrow{\sigma} & \tilde{Z}
\end{array}
\]

Since the morphism \( \mu'' \) is birational, we have \( M_{Z''} = \mu''^* M_{Z''} \) and \( M_{Z''} \) is semiample. Moreover the degree of \( \sigma \) is the same as the degree of \( \tilde{\tau} \) and \( M_{Z''} = \sigma^* M_Z \). Then, by Lemma 2.9 we have
\[
\text{Codim}(\text{Bs}|mdM_{\tilde{Z}}|) \geq 2.
\]
On the other hand, since
\[
|mdM_Z| = \mu^* |mdM_Z|,
\]
the base locus of \( mdM_Z \) has pure codimension 1. This implies that \( F = 0 \) and that the linear system \( |mdM_Z| \) is base point free. \( \square \)

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