The Cover Pebbling Number of Graphs

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Abstract

A pebbling move on a graph consists of taking two pebbles off of one vertex and placing one pebble on an adjacent vertex. In the traditional pebbling problem we try to reach a specified vertex of the graph by a sequence of pebbling moves. In this paper we investigate the case when every vertex of the graph must end up with at least one pebble after a series of pebbling moves. The cover pebbling number of a graph is the minimum number of pebbles such that however the pebbles are initially placed on the vertices of the graph we can eventually put a
pebble on every vertex simultaneously. We find the cover pebbling numbers of trees and some other graphs. We also consider the more general problem where (possibly different) given numbers of pebbles are required for the vertices.

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1 Introduction

The game of pebbling was first suggested by Lagarias and Saks, and introduced to the literature in a paper of Chung [1]. A pebbling move consists of taking two pebbles off of one vertex and placing one pebble on an adjacent vertex. Given a graph $G$, a specified number of pebbles, and a configuration of the pebbles on the vertices of $G$, the goal is to be able to move at least one pebble to any specified target vertex using a sequence of pebbling moves. The pebbling number $\pi(G)$ is the minimum number of pebbles that are sufficient to reach any target vertex regardless of the original configuration of the pebbles. In the present context it is naturally assumed that all graphs considered are connected. Moews [3] found the pebbling number of trees by using a clever path partition of the tree. For a survey of additional results see [2].

In this paper we investigate the following question: How does the pebbling problem change if instead of having a specified target vertex we need to place a pebble simultaneously on every vertex of the graph? In some scenarios this seems to be a more natural question, for example if information needs to be transmitted to several locations of a network, or if army troops need to be deployed simultaneously. We define the cover pebbling number $\gamma(G)$ to be the minimum number of pebbles needed to place a pebble on every vertex of the graph using a sequence of pebbling moves, regardless of the initial configuration. We establish the cover pebbling number for several classes of graphs, including complete graphs, paths, fuses (a fuse is a path with leaves attached at one end), and more generally, trees. We also describe the structure of the largest non-coverable configuration on a tree.

More generally, let a weight function $w$ be given that assigns an integer $w(v)$ to each vertex $v$ of $G$. We say that $w$ is positive if $w(v) > 0$ for all $v$. We define the weighted cover pebbling number $\gamma_w(G)$ to be the minimum number $k$ ensuring that, from any initial configuration with $k$ pebbles there is a sequence of pebbling moves after which all the vertices $v$ simultaneously have $w(v)$ pebbles on them. Our main result on trees in Section 4 determines $\gamma_w(T)$ for every tree $T$ and every positive weight function $w$.

Given a configuration $C$ of pebbles, we will use the following notation. The size $|C|$ of the configuration, denotes the number of pebbles in $C$. The support $\sigma(C)$ of the configuration is the set of support vertices, i.e. those on which there is at least one pebble of $C$. The number of pebbles on $v$ in $C$
is denoted by $C(v)$ (hence, $v \in \sigma(C)$ if and only if $C(v) > 0$). We call a configuration simple if its support consists of a single vertex. We say that a configuration is cover-solvable, or simply coverable (resp. $w$-coverable), if it is possible to transport at least one pebble (resp. $w(v)$ pebbles) to every vertex $v$ of the graph simultaneously (and non-coverable otherwise). As is customary, we denote the vertex set and edge set of $G$ by $V(G)$ and $E(G)$, respectively. If $G$ is of order $n$, we sometimes denote its vertices by $v_1, v_2, \ldots, v_n$.

2 Preliminary Results

We begin with the cover pebbling number of the complete graph $K_n$ on $n$ vertices. Note that the pebbling number for $K_n$, $\pi(K_n)$, is $n$ (see [2]).

**Theorem 1** $\gamma(K_n) = 2n - 1$.

**Proof.** If $2n - 2$ pebbles are placed on vertex $v_n$, then 2 pebbles will be used to cover each of the $n - 1$ other vertices. Thus no pebbles will remain to cover $v_n$. Hence $\gamma(K_n) \geq 2n - 1$.

Now suppose that at least $2n - 1$ pebbles are placed on the vertices. We may suppose that some vertex, say $v_n$, has no pebbles on it, otherwise the graph is already covered. The pigeonhole principle says that some other vertex has at least two pebbles on it; we use those to cover $v_n$. Since there are now at least $2n - 3$ pebbles among the remaining $n - 1$ vertices, induction says we can cover them (of course, $\gamma(K_1) = 1$). Hence $\gamma(K_n) \leq 2n - 1$. $\square$

A similar inductive proof works also for weighted covering, and yields the following result. Denote the total weight by $|w| = \sum_v w(v)$ and define $\min w = \min_v w(v)$.

**Theorem 2** $\gamma_w(K_n) = 2|w| - \min w$ for every positive weight function $w$.

Next we find the cover pebbling number of the path $P_n$ on $n$ vertices $v_1, \ldots, v_n$, with $v_iv_{i+1} \in E$ for $1 \leq i < n$. Note that $\pi(P_n) = 2^{n-1}$ (see [2]).

**Theorem 3** $\gamma(P_n) = 2^n - 1$. 
Proof. If \(2^n - 2\) pebbles are placed at vertex \(v_n\), then covering \(v_1\) will use \(2^{n-1}\) pebbles, covering \(v_2\) will use \(2^{n-2}\) pebbles, \ldots, and covering \(v_{n-1}\) will use 2 pebbles. Then no pebbles will remain to cover \(v_n\). Hence \(\gamma(P_n) \geq 2^n - 1\).

Now suppose that at least \(2^n - 1\) pebbles are placed on the vertices. If there are no pebbles on \(v_n\) then we may use at most \(2^n - 1\) pebbles to cover it, since \(\pi(P_n) = 2^n - 1\). By induction, the remaining \(2^n - 1\) or more pebbles can cover \(P_{n-1}\) (of course, \(\gamma(P_1) = 1\)). If there are pebbles on \(v_n\) then move as many of them as possible to \(v_{n-1}\), leaving 1 or 2 on \(v_n\). Either at least \(2^n - 1\) pebbles have been moved to \(v_{n-1}\), or at most \(2^n - 1 - 2\) moves have been made and at most two pebbles stay on \(v_n\). In any case, at least \(2^n - 1\) pebbles remain on \(P_{n-1}\). Again, induction shows that \(\gamma(P_n) \leq 2^n - 1\). \(\square\)

Among all graphs on \(n\) vertices, the complete graph has the smallest pebbling number \((n)\) and the path has the largest pebbling number \((2^n - 1)\). In both cases, we have \(\gamma(G) = 2\pi(G) - 1\). While this might lead one to guess that such a relation holds for all (connected) graphs, this couldn’t be farther from the truth. As the following theorem shows, the ratio \(\gamma(G)/\pi(G)\) is unbounded, even within the class of trees. The subclass of fuses is defined as follows. The vertices of \(F_l(n)\) \((l \geq 2\) and \(n \geq 3)\) are \(v_1, \ldots, v_n\), so that the first \(l\) vertices form a path from \(v_1\) to \(v_l\), and the remaining vertices are independent and adjacent only to \(v_l\). (The path is sometimes called the wick, while the remaining vertices are sometimes called the sparks.) For example, \(F_2(n)\) is the star \(S_n\) on \(n\) vertices. The fact that \(\gamma(S_n) = 4n - 5\) serves as the base case for the following result.

**Theorem 4** \(\gamma(F_l(n)) = (n - l + 1)2^l - 1\).

Proof. Following the arguments for the path given above, it is easy to see that so many pebbles are required of a simple configuration sitting on \(v_1\).

Likewise, induction on \(n\) shows that so many pebbles suffice to cover the fuse. Indeed, consider the cases whether or not \(v_1\) has pebbles on it and argue as was done for paths, above.

Regarding the base case \(l = 2\), we point out that \(F_2(n)\) is the star on \(n\) vertices, so we can let any leaf play the role of \(v_1\). If all the pebbles are on \(v_2\) then we can cover the star easily. Otherwise, some leaf has at least one pebble on it, and we label that vertex \(v_1\). Now we pebble as many as possible from \(v_1\) to \(v_2\), leaving 1 or 2 on \(v_1\). Induction on the number of leaves finishes the proof. \(\square\)
We define the covering ratio of $G$ to be $\rho(G) = \frac{\gamma(G)}{\pi(G)}$. For a class $\mathcal{F}$ of graphs we define $\rho(\mathcal{F}) = \sup_{G \in \mathcal{F}} \rho(G)$ if it exists, and $\rho(\mathcal{F}) = \infty$ otherwise. Thus, for the families $\mathcal{K}$ of complete graphs and $\mathcal{P}$ of paths, we have $\rho(\mathcal{K}) = \rho(\mathcal{P}) = 2$.

**Theorem 5** Let $\mathcal{T}_n$ be the family of all trees on $n$ vertices. Then $\rho(\mathcal{T}_n) = \infty$.

*Proof.* Since $\pi(F_l(n)) = 2^l + n - l - 1$ (see [3]), we see that, for $n = 2^l + 1$,

$$\rho(F_l(n)) > (n - l)2^l/(n - l + 2^l) > (n - \lg n)/2.$$ 

\square

### 3 The Transition Digraph

The main goal of this section is to prove that any sequence of pebbling moves can be replaced by one which is cycle-free in a well-defined sense. For this, we introduce the following concept.

**Definition.** Given a sequence $S$ of pebbling moves on graph $G$, the transition digraph is a directed multigraph denoted $T(G, S)$ that has $V(G)$ as its vertex set, and each move $s \in S$ along edge $v_i v_j$ (i.e., where two pebbles are removed from $v_i$ and one placed on $v_j$) is represented by one directed edge $v_i v_j$.

**Theorem 6** Let $S$ be a sequence of pebbling moves on $G$, resulting in a configuration $C$. Then there exists a sequence $S^*$ of pebbling moves, terminating with a configuration $C^*$, such that

1. On each vertex $v$, the number of pebbles in $C^*$ is at least as large as that in $C$, and
2. $T(G, S^*)$ does not contain any directed cycles.

*Proof.* We apply induction on the number of directed cycles in $T(G, S)$. The assertion is trivially true for every $S$ where this number is zero.

Let now $S$ be arbitrary, and consider the shortest prefix $S'$ of $S$ that contains a directed cycle. That is, the last move in $S'$ creates a cycle, say $C' = v_1 v_2 \cdots v_k$, in $T(G, S')$. For $i = 1, 2, \ldots, n$, let us denote by $d^-_i$ and $d^+_i$ the in-degree and out-degree, respectively, of vertex $v_i$ in $T(G, S')$. In
the initial configuration, each $v_i$ has to contain at least $2d_i^+ - d_i^-$ pebbles, otherwise some move of $S'$ could not be performed at $v_i$.

Let us consider the edge set $F' = E(T(G, S')) \setminus E(C')$. By the choice of $S'$, this $F'$ does not contain any directed cycles, hence it contains a vertex $v_i$ of in-degree zero. It means $d_i^- = 0$ if $v_i \notin C'$, and $d_i^- = 1$ otherwise. In the former case, $v_i$ initially has at least $2d_i^+$ pebbles and is incident with precisely $d_i^+$ edges; while in the latter, the number of pebbles at $v_i$ is at least $2d_i^+ - 1$ and that of its incident edges is just $d_i^+ - 1$. In either case, $v_i$ has sufficiently many pebbles so that the pebbling moves for all of its incident edges in $F'$ are feasible before any move belonging to $C'$ has been performed.

We now rearrange $S'$ to make all moves of $F'$ involving $v_i$ at the beginning. Analogously, $F' - v_i$ has a vertex $v_j$ of zero in-degree in $F'$, hence after the rearrangement of moves at $v_i$, the moves of edges incident with $v_j$ are feasible completely before $C'$. Eventually we obtain a rearrangement, say $S''$, of $S'$ where the moves of $C'$ are performed at the very end, and of course the concatenation of $S''$ and $S - S'$ terminates in configuration $C$. Now it is immediately seen that the concatenation $S^+$ of $S'' - C'$ and $S - S'$ is a feasible sequence of moves that ends up with a configuration $C^+$ where the vertices $v_1, \ldots, v_k$ have one more pebble than in $C$, and the other vertices have the same number of pebbles in $C$ and $C^+$. Since the number of directed cycles in $T(G, S^+)$ is strictly smaller than that in $T(G, S)$, the assertion follows by induction. \hfill \blacksquare

4 Trees

In this section we determine the (weighted) cover pebbling number for an arbitrary tree $T$. For $v \in V(T)$ define

$$s(v) = \sum_{u \in V(T)} 2^{d(u,v)},$$

where $d(u,v)$ denotes the distance from $u$ to $v$, and let

$$s(T) = \max_{v \in V(T)} s(v).$$

Analogously, if a positive weight function $w$ is given, we define

$$s_w(v) = \sum_{u \in V(T)} w(u) \cdot 2^{d(u,v)}$$

7
and

\[ s_w(T) = \max_{v \in V(T)} s_w(v). \]

Clearly, for a simple configuration sitting on \( v \), \( s_w(v) \) pebbles are necessary and sufficient to cover \( T \). Thus \( \gamma_w(T) \geq s_w(T) \) for every \( T \) and every positive \( w \). We are going to prove that this obvious lower bound is in fact tight.

**Theorem 7** For positive weight functions \( w \) we have \( \gamma_w(T) = s_w(T) \).

**Proof.** The theorem can be reformulated in the following equivalent form:

For every non-coverable configuration \( C \) there exists a simple non-coverable configuration \( C^* \) such that \( |C^*| = |C| \).

The proof of this latter assertion is essentially induction, where we either reduce the tree to another tree with fewer vertices or keep \( T \) unchanged but decrease the support \( \sigma(C) \) of \( C \) without making its size \( |C| \) decrease.

We shall use the following terminology concerning a configuration \( C \). We say that a vertex \( v \) is a

- D-vertex with demand \( D(v) = w(v) - C(v) \) if \( w(v) - C(v) > 0 \).
- N-vertex (neutral) if \( C(v) = w(v) \). Then we define \( D(v) = 0 \).
- S-vertex with supply \( S(v) = C(v) - w(v) \) if \( C(v) - w(v) > 0 \).

It is immediate by definition that every non-coverable configuration contains at least one D-vertex.

**Case 1.** \( T = K_1 \) or \( T = K_2 \).

These are trivial initial cases, handled already in the more general context of Theorem 2.

**Case 2. Some leaf of \( T \) is not an S-vertex.**

Let \( v \) be such a leaf, and let \( u \) be its neighbor in \( T \). We now delete \( v \) from \( T \) (with all its pebbles), and increase \( w \) at \( u \) to the value \( w'(u) = w(u) + 2D(v) \). Keeping \( w'(x) = w(x) \) unchanged for all \( x \notin \{u, v\} \), the configuration \( C' = C - v \) on the tree \( T' = T - v \) with the weight function \( w' \) is coverable if and only if so is \( C \) on \( T \) with \( w \). This follows from Theorem 6 which implies that if \( T \) is coverable, then there is a sequence of pebbling moves where no pebble gets moved from \( v \) to \( u \). (To make \( v \) properly covered, we need to place at
least $D(v)$ additional pebbles on it; and this requires taking $2D(v)$ pebbles off of $u$.)

**Case 3.** Every leaf of $T$ is an S-vertex.

For a given leaf $v = v_1$, define the path $v_1v_2\cdots v_m$ so that $v_m$ is the other leaf if $T$ is a path and is the only vertex of the path having degree at least 3 in $T$ otherwise. In the latter case we call $v_m$ the *split* vertex of $v_1$. If there is a support vertex other than $v_1$ on this path, we call the one having minimum subscript the *nearest support* vertex of $v_1$.

Since $v_1$ is an S-vertex we can move $s_1 = \lfloor S(v_1)/2 \rfloor$ pebbles to $v_2$. Moreover, if $s_1 > w(v_2) - C(v_2)$ then we can further transmit $s_2 = \lfloor (s_1 + C(v_2) - w(v_2))/2 \rfloor$ pebbles to $v_3$, and so on. For a vertex $v_k$ on this path we say that $v_1$ supplies $v_k$ if at least one of the pebbles from $v_1$ can reach $v_k$ in this way. There are three possibilities for $v_1$, namely, $v_1$ supplies its split vertex, $v_1$ supplies its nearest support vertex, or $v_1$ supplies neither of these. We consider these possibilities in reverse order.

**Subcase A.** Some leaf supplies neither its split nor its nearest support vertices.

We follow a similar argument as in Case 2. Let $v_1$ be such a leaf and let $k$ be the largest subscript so that $v_1$ supplies $v_k$ (then $k < m$ and $v_i$ is not a support vertex for any $2 \leq i \leq k$). Let $C'$ and $w'$ be the restrictions of $C$ and $w$ to $T' = T - \{v_1, \ldots, v_k\}$, except that $w'(v_{k+1}) = w(v_{k+1}) + 2D'$, where $D' = w(v_k) - s_{k-1}$ is the resulting demand on $v_k$ after being supplied by $v_1$. Then $C'$ is non-$w'$-coverable on $T'$, and since $|T'| < |T|$ there is a simple non-$w'$-coverable configuration of size $|C'|$ on $T'$. This yields a non-$w$-coverable configuration $C''$ of size $|C|$ on $T$ that sits on two vertices. If $T$ has at least three leaves then some leaf is not an S-vertex and we are done by Case 2. Otherwise $T$ is a path and $\sigma(C'') = \{v_1, v_n\}$. Non-$w$-coverability now means that $v_n$ can supply $v_k$ with strictly fewer than $D'$ pebbles. Finally we test if $k - 1 \geq n - k$. If so, then for every $j$ in the range $k \leq j \leq n$, $d(v_1, v_j) \geq d(v_j, v_k)$. Thus, defining $C^*(v_n) = 0$ and $C^*(v_1) = C'(v_1) + C''(v_n) = |C|$, we obtain a simple non-coverable configuration, as required. If $k - 1 < n - k$ we do the opposite.

**Subcase B.** Some leaf supplies its nearest support vertex.

Let $v_1$ be such a vertex and $v_k$ its nearest support vertex (then $v_i \notin \sigma(C)$ for $1 < i < k$). We define $C'(v_k) = 0$ and $C'(v_1) = C'(v_1) + C(v_k)$, keeping $C'$ identical to $C$ on every other vertex. Then $|C'| = |C|$, $|\sigma(C')| < |\sigma(C)|$, and
C′ is non-coverable whenever C is, because the supply from \(v_1\) yields fewer pebbles on \(v_k\) in \(C′\) than in \(C\).

**Subcase C. Every leaf supplies its split vertex.**

By Subcase B we may assume that no leaf supplies its nearest support vertex. There must be some vertex \(v\) that is the split vertex for two different leaves (indeed, choose any leaf and let \(v\) be any vertex of degree at least 3 at farthest distance from it – the two leaves past \(v\) witness this). Label these leaves \(v_1\) and \(v_\ell\) so that \(P = v_1 \cdots v_m \cdots v_\ell\) is the unique path between them and \(v = v_m\). Recall that \(v_i\) is not a support vertex for any \(1 < i < \ell\) and that both \(v_1\) and \(v_\ell\) supply \(v_m\). Let us denote by \(s_m\) their total supply for \(v_m\).

If \(s_m > w(v_m)\), then \(P\) can supply \(T - P\) with \(s' = \lfloor \frac{1}{2}(s_m - w(v_m)) \rfloor\) pebbles (at most); and otherwise it needs to receive at least \(s'' = w(v_m) - s_m\) pebbles from \(T - P\). In both cases we consider the problem restricted to \(P\), where \(w(v_i)\) is kept unchanged for all \(i \neq m\), and \(w(v_m)\) is modified to \(s_m + 1\). This configuration on \(P\) is non-coverable. Thus, according to Subcase A, the \(C(v_1) + C(v_\ell)\) pebbles can be placed on one vertex \((v_1\) or \(v_\ell)\), keeping \(P\) non-coverable. It follows that the modified configuration, too, either supplies \(T - P\) with at most \(s'\) pebbles or needs to receive at least \(s''\) pebbles from \(T - P\). In either case, the new configuration on \(T\) is non-coverable and has at least one D-vertex leaf, thus we are done by Case 2.

\[\square\]

From this proof we see that a non-coverable configuration of maximum size can be assumed to be simple. The next result shows that the single support vertex must be an end of a longest path. (This is the case even for weight functions \(w\) where the longest paths are not of maximum weight.)

**Theorem 8** Given a tree \(T\) and a positive weight function \(w\), let \(C\) be a non-coverable simple configuration of maximum size, with \(\sigma(C) = \{v\}\). Then \(v\) is a leaf of a longest path in \(T\).

**Proof.** Since \(\gamma_w(T) = s_w(v)\) for some \(v\), we need to show that the maximum value of \(s_w(v)\) is attained only on some endpoints of the longest path(s) of \(T\). We are going to prove something stronger: every longest path has at least one endpoint \(x\) whose \(s_w(x)\) is larger than \(s_w(u)\) for every \(u\) which is not the endpoint of some longest path.

Suppose first that \(T\) is just a path \(v_1 v_2 \cdots v_n\). Consider any internal vertex \(v_k\) \((1 < k < n)\). We compare the partial sums \(s^- = \sum_{1 \leq i < k} w(v_i) \cdot 2^{d(v_i, v_k)}\)
and \( s^+ = \sum_{k<i \leq n} w(v_i) \cdot 2^{d(v_i, v_k)}. \) If \( s^- \leq s^+, \) then \( s_w(v_{k-1}) > s_w(v_k) \); and if \( s^- \geq s^+, \) then \( s_w(v_{k+1}) > s_w(v_k) \). Thus, \( s_w(k) \) can never be largest.

Suppose next that \( T \) is a tree with precisely three leaves. Applying the previous idea, from any non-leaf vertex we can move to one of its neighbors and find there a larger value of \( s_w \). Hence, let \( v, v', v'' \) be the three leaves, and suppose that the longest path \( P \) in \( T \) is the one connecting \( v' \) with \( v'' \). We need to show \( s_w(v') < \max \{ s_w(v'), s_w(v'') \} \). Let \( u \) be the unique degree-3 vertex of \( T \). We have \( d(u, v) < d(u, v') \) and \( d(u, v) < d(u, v'') \) (for otherwise the \( v-v' \) path or the \( v-v'' \) path were at least as long as the \( v'-v'' \) path, contrary to the assumption on \( v \)). From this it is easily seen that for every vertex \( x \), at least one of \( d(v', x) \) and \( d(v'', x) \) is at least \( d(v, x) + 1 \). Consequently, \( s_w(v') + s_w(v'') > 2s_w(v) \), i.e. \( s_w(v) \) cannot be largest.

Finally, let \( T \) be a tree with more than three leaves. Let \( P \) be one of its longest paths, \( v^* \) a leaf that does not belong to any longest path of \( T \), and \( v \neq v^* \) a leaf not on \( P \) (but maybe on some other longest path of \( T \)). We apply the transformation on \( v \) as described in Case 2 of the proof of Theorem 7. This modification keeps the function \( s_w \) unchanged on all vertices of \( T - v \), moreover \( P \) remains a longest path and \( v^* \) does not become the endpoint of any longest path in \( T - v \). Thus, by induction on the number of vertices, \( s_w \) is larger on some endpoint of \( P \) than on \( v^* \). This completes the proof. \( \square \)

## 5 Open Problems

There are several natural problems and questions to ask.

**Problem 9** Find \( \gamma(G) \) for other graphs \( G \), for example cubes, complete \( r \)-partite graphs, etc.

**Question 10** Is it true for all graphs \( G \) that at least one of the largest non-coverable configurations on \( G \) is simple?

**Problem 11** Find classes of graphs \( \mathcal{F} \) whose covering ratio \( \rho(\mathcal{F}) \) is bounded.

**Question 12** Can the question, “Is \( \rho(G) \leq k? \)” be answered efficiently?

These questions extend to positive weight functions in a natural way. Let us note, however, that the situation drastically changes when “positive” is
replaced by “nonnegative” for $w$. This fact is already shown by the complete graph $K_n$ ($n \geq 3$) where only one vertex is required to be covered, which corresponds to the weights $1, 0, 0, \ldots, 0$. Here the unique maximal non-coverable configuration has the pebble distribution $0, 1, 1, \ldots, 1$, in striking contrast to the case where $w > 0$ and all pebbles may be concentrated on a suitably chosen single vertex. Such considerations must be tackled in order to pursue the *weighted pebbling number* of a graph $G$, defined as $\pi_w(G) = \max_w \gamma_w(G)$, where the maximum is taken over all nonnegative weight functions $w$ of size $|w| = w$. The pebbling number $\pi(G)$ is the case $w = 1$.

**Problem 13** Find $\pi_w(T)$ for any tree $T$ and weight $w$.

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