INTEGRALS OF PRODUCTS OF HERMITE FUNCTIONS

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ABSTRACT. We compute the integrals of products of Hermite functions using the generating functions. The precise asymptotics of products of 4 Hermite functions are presented below. This estimate is relevant for the corresponding cubic nonlinear equation.

1. Introduction and statement of the theorem

In this note, we compute the integrals of products of 4 (normalized) Hermite functions:

\[ W_{jpk} = \int_{-\infty}^{\infty} h_j(x)h_p(x)h_q(x)h_k(x)dx, \quad j, p, q, k \geq 0 \]  

(1)

using the generating functions. The method applies to arbitrary products. The product of 4 Hermite functions is motivated by the cubic nonlinearity in the equation:

\[ i \frac{\partial}{\partial t} u = \frac{1}{2} \left( -\frac{\partial^2}{\partial x^2} + x^2 \right) u + |u|^2 u. \]

The special case \( p = q = 0 \) was computed in [W], where the author showed stability of the harmonic oscillator under the time dependent perturbation:

\[ V(x, t) = \delta|h_0(x)|^2 \sum_{k=1}^{\nu} \cos(\omega_k t + \phi_k), \]

(2)

for small \( \delta \) and a set of frequencies \( \omega = \{\omega_k\} \) close to full measure. In [W], the precise asymptotics:

\[ W_{j00k} \sim \frac{1}{\sqrt{j+k}} e^{-\frac{(j+k)^2}{2(j+k)}} \quad \text{for } j+k \gg 1, \]

(3)

played an essential role. Using (2)

\[ \|Vh_k\|_2 \leq \frac{1}{k^{1/4}} \to 0, \quad \text{as } k \to \infty. \]

Hence the spatial part of the perturbation diminishes for higher Hermite modes contributing to stability.

In this paper, we compute (1) for arbitrary \( p \) and \( q \). We prove...
Theorem.

\[ |W_{jpqk}| \lesssim C_{p,q} \sum_{\ell=0}^{[\frac{j+k}{2}]} \frac{1}{\ell!} \left( \frac{(j-k)^2}{2(j+k)} \right)^{\ell} \cdot \frac{1}{\sqrt{j+k}} e^{-\frac{(j-k)^2}{2(j+k)}} \]

\[ \lesssim \frac{C_{p,q}}{\sqrt{j+k}} e^{-\frac{(j-k)^2}{3(j+k)}} \quad \text{for } \frac{j-k}{\sqrt{j+k}} \geq \sqrt{p+q}, \]  
\[ \lesssim \frac{C_{p,q}}{\sqrt{j+k}} \quad \text{for all } j, k, \]

\[ W_{jpqk} = 0 \quad j + p + q + k \text{ odd}, \]

where \([]\) denotes the integer part and \(C_{p,q} \leq a^{p+q}\) for some \(a > 1\).

We remark that except for the factor \(C_{p,q}\), the estimate in (4) is essentially the same as in (3). In particular, the polynomial factor in front of the Gaussian is optimal and for fixed \(p\) and \(q\), the Schur norm is of the same order as the operator norm. Using (4), the result in [W] extends immediately to potentials with exponentially decaying Hermite coefficients. The rest of the paper is devoted to the proof of the Theorem.

We first recall some basic facts about the Hermite functions and the proof of (3).

The Hermite functions \(h_j\) are the eigenfunctions of the harmonic oscillator:

\[ H h_j = \left( -\frac{d^2}{dx^2} + x^2 \right) h_j = \lambda_j h_j, \]

with eigenvalues

\[ \lambda_j = 2j + 1, \quad j = 0, 1..., \]

and

\[ h_j(x) = \frac{H_j(x)}{\sqrt{2^j j!}} e^{-x^2/2}, \quad j = 0, 1... \]

(5)

where \(H_j(x)\) is the \(j^{th}\) Hermite polynomial, relative to the weight \(e^{-x^2}\) \((H_0(x) = 1)\) and

\[ \int_{-\infty}^{\infty} e^{-x^2} H_j(x) H_k(x) dx = 2^j j! \sqrt{\pi} \delta_{jk}. \]

(6)

So

\[ W_{jpqk} = \frac{1}{\sqrt{2^j p+q+k + j! q! k!}} \int_{-\infty}^{\infty} H_j(x) H_p(x) H_q(x) H_k(x) dx, \quad j, p, q, k \geq 0 \]

(7)

As in [W], the idea is to view the above integral as an \(L^2\) product with the new measure \(e^{-2x^2}\) and reexpress the products of Hermite polynomials in \(x\) as new Hermite polynomials in \(\sqrt{2x}\):

\[ H_j(x) H_k(x) = \sum_{r=0}^{j+k} a_r H_r(\sqrt{2x}). \]

(8)
and similarly

\[ H_p(x)H_q(x) = \sum_{\ell=0}^{p+q} b_{\ell} H_{\ell}(\sqrt{2}x). \quad (9) \]

Using (8) and (9) in the integral in (7), and assuming (without loss of generality), \( p + q \leq j + k \), we then have

\[ I = \int_{-\infty}^{\infty} H_j(x)H_p(x)H_q(x)H_k(x)dx = \sum_{\ell=0}^{p+q} a_{\ell} b_{\ell} c_{\ell}, \quad (10) \]

where

\[ c_{\ell} = \int_{-\infty}^{\infty} [H_{\ell}(\sqrt{2}x)]^2 e^{-2x^2} dx = 2^{\ell-\frac{1}{2}} \ell! \sqrt{\pi}. \quad (11) \]

To find the coefficients \( a_r \) and \( b_\ell \), we use the generating functions as follows. Since

\[ e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x), \quad (12) \]

\[ e^{2sx-s^2} = \sum_{m=0}^{\infty} \frac{s^m}{m!} H_m(x), \quad (13) \]

which can be found in any mathematics handbook (cf. [T] for connections with the Mehler formula), multiplying (12, 13), we obtain

\[ e^{2(t+s)x-(t^2+s^2)} = \sum_{n,m} \frac{t^n s^m}{n!m!} H_n(x)H_m(x) \]

\[ = e^{2\left(\frac{t+s}{\sqrt{2}}\right)\sqrt{2}x-(\frac{t+s}{\sqrt{2}})^2} \cdot e^{-\frac{1}{2}(t-s)^2} \]

\[ = \sum_{\ell=0}^{\infty} H_{\ell}(\sqrt{2}x) \cdot \frac{(t+s)^\ell}{\ell!} \cdot \sum_{r=0}^{\infty} (-1)^r \frac{(t-s)^{2r}}{2^r r!}. \quad (14) \]

Using (14) in (8), we note that

\[ a_0 = \frac{(-1)^{\frac{j+k}{2}}}{2^{\frac{j+k}{2}}(j+k)!} \cdot (j+k)!, \quad j + k \text{ even} \]

\[ = 0 \quad \text{otherwise}, \quad (15) \]

by taking \( 2r = j + k \), which is the only contributing term. In [W], we computed the case \( p = q = 0 \): \( W_{j00k} = a_0 b_0 c_0 \), which we recall below.

Lemma.

\[ W_{jk} \overset{\text{def}}{=} W_{j00k} = \int_{-\infty}^{\infty} h_0^2(x)h_j(x)h_k(x)dx \]

\[ = \frac{(-1)^{\frac{j+k}{2}}}{2^{j+k} j! k!} \cdot \frac{(j+k)!}{(\frac{j+k}{2})!} \sqrt{\frac{\pi}{2}} j + k \text{ even}, \]

\[ = 0 \quad \text{otherwise}. \quad (16) \]
Let
\[ J = \frac{j + k}{2}, \quad K = \frac{j - k}{2}, \]  
assuming \( j \geq k \), without loss. When \( J \gg 1 \),
\[ W_{jk} = \left[ 1 + O\left( \frac{1}{J} \right) \right] \frac{(-1)^{\frac{j-k}{2}} J!}{\sqrt{2J(J+K)!(J-K)!}} \]  
(18)
\[ |W_{jk}| \leq \frac{1}{\sqrt{J}} e^{-K^2/2J}. \]  
(19)

**Proof.** (16) follows directly from (15, 7). We only need to obtain the asymptotics in (18, 19). This is an exercise in Stirling’s formula:
\[ j! = \left( \frac{J}{e} \right)^j \sqrt{2\pi j} (1 + \frac{1}{12j} + \frac{1}{288j^2} + ...) \]
or its log version
\[ \log j! = (j + \frac{1}{2}) \log j - j + \log \sqrt{2\pi} + ... \]  
(20)
Here it is more convenient to use the latter. Using (17, 20),
\[ \log \frac{(j + k)!}{2^{j+k}(\frac{j+k}{2})!} = \log \frac{(2J)!}{J!} - \log 2^{2J} \]
\[ = J \log J - J + \frac{1}{2} \log J + O(J^{-1}). \]  
(21)
So
\[ \frac{(j + k)!}{2^{j+k}(\frac{j+k}{2})!} = \left[ 1 + O\left( \frac{1}{J} \right) \right] \frac{J!}{\sqrt{\pi J}}, \]
using (21). Hence
\[ W_{jk} = \left[ 1 + O\left( \frac{1}{J} \right) \right] \frac{(-1)^{\frac{j-k}{2}} J!}{\sqrt{2J(J+K)!(J-K)!}}, \quad J \gg 1 \]
which is (18). Using the fact that
\[ j! = \sqrt{2\pi j} \left( \frac{j}{e} \right)^j e^{\lambda_j} \]
with
\[ \frac{1}{12j} + 1 < \lambda_j < \frac{1}{12j}, \quad \text{for all } j \geq 1, \]
and applying the inequalities (with \( x = K/J \)):
\[ \phi(x) \overset{\text{def}}{=} (1 + x) \log(1 + x) + (1 - x) \log(1 - x) \geq x^2 \]
for all \( x \in [0, 1) \) and \( \phi(x) \geq ax^2 \) with \( a > 1 \) for \( x \in [7/10, 1) \), we obtain (19). (When \( x = K/J = 1 \), (19) follows by a direct computation using Stirling’s formula.) \( \square \)
2. Proof of the theorem

From (14), the \( t^j s^k \) term in the RHS is among the terms:

\[
\sum_{\ell=0, \ell \sim j+k}^{j+k} H_\ell (\sqrt{2x}) \cdot \frac{(t+s)\ell}{2^\ell \ell!} \left[ (-1)^r \frac{(t-s)^{2r}}{2r!} \right]_{2r+\ell=j+k}
\]

\[
= \sum_{\ell=0, \ell \sim j+k}^{j+k} H_\ell (\sqrt{2x}) \cdot \frac{(t+s)\ell}{2^\ell \ell!} (-1)^{\frac{j+k-\ell}{2}} \frac{(t-s)^{j+k-\ell}}{2^{\frac{j+k-\ell}{2}}(\frac{j+k-\ell}{2})!},
\]

where \( \ell \sim j + k \) means that \( \ell \) has the same parity as \( j + k \). Equating the coefficients in front of the \( t^j s^k \) term in the second line of (14) and (22), we obtain

\[
H_j(x)H_k(x) = j!k! \sum_{\ell=0, \ell \sim j+k}^{j+k} H_\ell (\sqrt{2x}) \cdot \frac{(-1)^{\frac{j+k-\ell}{2}}}{2^{\frac{j+k-\ell}{2}}(\frac{j+k-\ell}{2})!} \sum_{r=0}^{\ell} (-1)^{k-r} C_{\ell}^r C_{j+k-\ell}^{k-r} \]

(23)

from (8). Similarly

\[
H_p(x)H_q(x) = p!q! \sum_{i=0, i \sim p+q}^{p+q} H_i (\sqrt{2x}) \cdot \frac{(-1)^{\frac{p+q-i}{2}}}{2^{\frac{p+q-i}{2}}(\frac{p+q-i}{2})!} \sum_{m=0}^{i} (-1)^{q-m} C_{i}^m C_{p+q-i}^{q-m} \]

(24)

from (9).

So

\[
a_{\ell} = \frac{(-1)^{\frac{j+k-\ell}{2}}j!k!}{2^{\frac{j+k-\ell}{2}}(\frac{j+k-\ell}{2})!} \sum_{r=0}^{\ell} (-1)^{k-r} C_{\ell}^r C_{j+k-\ell}^{k-r},
\]

(25)

for \( \ell = j + k, j + k - 2, ... (\ell \geq 0) \),

\[
= 0 \quad \text{otherwise},
\]

and

\[
b_{\ell} = \frac{(-1)^{\frac{p+q-\ell}{2}}p!q!}{2^{\frac{p+q-\ell}{2}}(\frac{p+q-\ell}{2})!} \sum_{m=0}^{\ell} (-1)^{q-m} C_{\ell}^m C_{p+q-\ell}^{q-m},
\]

(26)

for \( \ell = p + q, p + q - 2, ... (\ell \geq 0) \),

\[
= 0 \quad \text{otherwise}.
\]
Using (25, 26) in (10, 7), assuming \( p + q + j + k \) even, otherwise \( I = 0 \), we then have

\[
W_{jpqk} = \frac{\sqrt{j!k!q!} \sqrt{\pi}}{2^{j+p+q+k}} \sum_{\ell=0, \ell \sim p+q}^{p+q} \frac{(-1)^{\frac{j+k+p+q}{2}}-\ell}{\ell!} \frac{\ell}{p+q-\ell} \sum_{r=0}^{\ell} (-1)^{k-r} C_{\ell}^r C_{j+k-\ell}^{k-r}
\]

\[
\left( \sum_{m=0}^{\ell} (-1)^{q-m} C_{\ell}^m C_{p+q-\ell}^{q-m}, \quad j + p + q + k \text{ even.} \right)
\]

(27)

For each \( \ell = p + q, p + q - 2, \ldots, (\ell \geq 0) \), we then need to estimate

\[
I_{jk}^{(\ell)} = \frac{\sqrt{j!k!}}{2^{j+k}} \frac{1}{(\ell - j-k)^!} \sum_{r=0}^{\ell} (-1)^{k-r} C_{\ell}^r C_{j+k-\ell}^{k-r}.
\]

(28)

and

\[
I_{pq}^{(\ell)} = \frac{\sqrt{p!q!}}{2^{p+q}} \frac{1}{(\ell - p+q)^!} \sum_{m=0}^{\ell} (-1)^{q-m} C_{\ell}^m C_{p+q-\ell}^{q-m}.
\]

(29)

Then

\[
W_{jpqk} = \sqrt{\frac{n}{2}} \sum_{\ell=0, \ell \sim p+q}^{p+q} (-1)^{\frac{j+k+p+q}{2}}-\ell \frac{2^{\ell}}{\ell!} I_{jk}^{(\ell)} I_{pq}^{(\ell)}, \quad p + q \leq j + k.
\]

(30)

We check that when \( p + q = 0 \), the sum in (30) reduces to the term \( \ell = 0 \) and

\[
W_{j00k} = \frac{(-1)^{\frac{j+k}{2}}}{2^{j+k}} \frac{(j+k)!}{\sqrt{j!k!}} \frac{\sqrt{\pi}}{2}, \quad j + k \text{ even,}
\]

same as in (16).

**Estimates on** \( I_{jk}^{(\ell)} \).

We first look at (28). When \( \ell > 0 \), we need to perform the sum over \( r \). Rewrite

\[
I_{jk}^{(\ell)} = \frac{\sqrt{j!k!}}{2^{j+k}} \frac{1}{(\ell - j-k)^!} \sum_{r=0}^{\ell} (-1)^{k-r} \frac{\ell!}{r!(\ell-r)!(k-r)!(j-\ell+r)!}
\]

\[
= (-1)^{\frac{k}{2}} \frac{(j+k-\ell)!}{(\ell - j-k)^!} \sqrt{j!k!} \sum_{r=0}^{\ell} (-1)^{r} \frac{k!}{r!(k-r)!(\ell-r)!(j-\ell+r)!}
\]

\[
= (-1)^{\frac{k}{2}} F_1 F_2, \quad \ell = p + q, p + q - 2, \ldots (\ell \geq 0),
\]

where \( F_2 \) denotes the sum.

We note that

\[
I_{jk}^{(0)} = \frac{1}{2^{j+k}} \frac{(j+k)!}{\sqrt{j!k!} (\ell - j-k)^!} \sqrt{\frac{n}{2}} \sim \frac{1}{\sqrt{j+k}} e^{-\frac{(j-k)^2}{2(j+k)}}
\]
from the Lemma. So we write

\[ F_1 = I_{jk}^{(0)} \cdot \frac{\ell!}{2^{\frac{\ell}{2}}} \cdot \frac{(j + k - \ell - 1)!!}{(j + k - 1)!!} \]

\[ |F_1| \lesssim |I_{jk}^{(0)}| \cdot \frac{\ell!}{2^{\frac{\ell}{2}}} \cdot \frac{e^{\frac{\ell}{2}}}{(j + k)!!} \]

(32)

\[ F_2 \text{ can be written as} \]

\[ F_2 = \sum_{r=0}^{\ell} (-1)^r \frac{k!}{r!(k-r)!(\ell-r)!(j-r)!} \]

\[ = \sum_{r=0}^{[\ell/2]} (-1)^r C_k^r C_j^{\ell-r} \]

\[ = \sum_{r=0}^{[\ell/2]} (-1)^r C_k^r C_j^{\ell-2r}, \quad j \geq k, j + k \geq \ell, \]

(33)

since \((t - s)^k(t + s)^j = (t^2 - s^2)^k(t + s)^{j-k}\) and both expressions in (33) give the coefficients in front of the \(s^\ell t^{j+k-\ell}\) term \((\ell \leq j + k)\). So

\[ |F_2| \leq \sum_{r=0}^{[\ell/2]} \frac{k^r (j-k)^{\ell-2r}}{r!(\ell-2r)!}. \]

We also need to estimate \(I_{pq}^{(\ell)}, \ell = p + q, p + q - 2, \ldots (\ell > 0)\). Since \(p + q \leq j + k\), we use norm estimates. Comparing (29) with (26), we have

\[ I_{pq}^{(\ell)} = \frac{(-1)^{p+q-\ell+1} \ell!}{2^{\frac{\ell}{2}} \sqrt{p!q!}} b_\ell, \]

(34)

where \(b_\ell\) is as defined in (24). From (24)

\[ b_\ell \int H_\ell^2(\sqrt{2}x)e^{-2x^2}dx = \int H_p(x)H_q(x)H_\ell(\sqrt{2}x)e^{-2x^2}dx \]

\[ \leq \left[ \int H_p^2(x)H_q^2(x)e^{-2x^2}dx \right]^{1/2} \left[ \int H_\ell^2(\sqrt{2}x)e^{-2x^2}dx \right]^{1/2} \]

So

\[ |b_\ell| \leq 2^{\frac{p+q}{2}} \sqrt{p!q!}, \]

(35)

where we used the normalization conditions in (5, 6) and the \(L^\infty\) estimate \([T]\)

\[ \|h_p\|_{L^\infty} \leq \frac{1}{p^{1/2}} < C. \]
Using (35) in (34), we then have

$$|I_{pq}^{(\ell)}| \leq \sqrt{\frac{\ell!}{2\ell}}.$$ 

So the terms in the sum in (30) can be estimated as follows:

$$\frac{2\ell}{\ell!} |I_{jk}^{(\ell)} I_{pq}^{(\ell)}| \leq |I_{jk}^{(0)}| e^{\frac{\ell}{2}} \sqrt{\ell!} \sum_{r=0}^{[\ell/2]} \left(\frac{j-k}{\sqrt{j+k}}\right)^{\ell-2r} \cdot \left(\frac{k}{j+k}\right)^r \cdot \frac{1}{r!(\ell-2r)!} \leq |I_{jk}^{(0)}| e^{\frac{\ell}{2}} \sqrt{\ell!} \sum_{r=0}^{[\ell/2]} \frac{1}{2r} X^{\ell-2r} \cdot \frac{1}{r!(\ell-2r)!} \leq \frac{1}{\sqrt{j+k}} e^{-\frac{x^2}{2}} e^{\frac{\ell}{2}} X^{\ell} \sum_{r=0}^{[\ell/2]} \frac{\sqrt{\ell!}}{(2X^2)^r} \cdot \frac{1}{r!(\ell-2r)!},$$

where $X = \frac{j-k}{\sqrt{j+k}}$. Using Stirling’s formula to relate $\sqrt{\ell!}$ and $(\ell-2r)!$ to $\frac{\ell!}{2\ell}$ and $\left(\frac{\ell}{2}-r\right)!$, we have

$$\text{(36)} \leq \frac{C\ell}{\sqrt{j+k}} e^{-\frac{x^2}{2}} \sum_{r=0}^{[\ell/2]} \frac{\left(\frac{x^2}{2}\right)^{\frac{\ell}{2}-r}}{(\ell-r)!} \cdot \frac{(\ell/2)!}{r!(\ell/2-r)!},$$

for some $C > 1$. Summing over $r$ and then $\ell$, we obtain the Theorem.

\[ \square \]

**References**

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