Abstract. This paper is devoted to study the box dimension of the orbits of one-dimensional discrete dynamical systems and their bifurcations in nonhyperbolic fixed points. It is already known that there is a connection between some bifurcations in a nonhyperbolic fixed point of one-dimensional maps, and the box dimension of the orbits near that point. The main purpose of this paper is to generalize that result to the one-dimensional maps of class $C^k$ and apply it to one and two-parameter bifurcations of maps with the generalized nondegeneracy conditions. These results show that the value of the box dimension changes at the bifurcation point, and depends only on the order of the nondegeneracy condition. Furthermore, we obtain the reverse result, that is, the order of the nondegeneracy of a map in a nonhyperbolic fixed point can be obtained from the box dimension of a orbit near that point. This reverse result can be applied to the continuous planar dynamical systems by using the Poincaré map, in order to get the multiplicity of a weak focus or nonhyperbolic limit cycle. We also apply the main result to the bifurcations of nonhyperbolic periodic orbits in the plane.

1. Introduction. The main motivation for studying the box dimension and Minkowski content can be found in the recent development of the application of fractal analysis to the solutions of differential equations and dynamical systems. For example, see [7], [10], [11], [16]. Besides that, the fractal dimensions are also being studied in connection to homoclinic bifurcations, for details see [9]. It is particularly interesting that some experiences of fractal analysis in the area of bifurcations of dynamical systems (see [4], [17], [18], [19]) showed that there is a direct connection between the box dimension of trajectories of dynamical systems and the bifurcation of that system. The first article in that direction is [18] in which the authors studied the box dimension of spiral trajectories near weak focus or limit cycle connected to the Hopf bifurcation.

The field of discrete dynamical systems and their one and two-parameter bifurcations is well known (see [1], [6], [14]), but it was not enough studied from the aspect of fractal analysis, which, in our case, includes the box dimension and Minkowski content. The article [4] showed that the above result for box dimension can also be proven for generic saddle-node and period-doubling bifurcations of one-dimensional discrete dynamical systems. In this paper we generalize the results from [4] to the class of finitely nondegenerate maps in $\mathbb{R}$ and apply it to one and two-parameter
bifurcations with generalized sufficient conditions (see [2], [3]). Namely, we show that the box dimension of an orbit near a nonhyperbolic fixed point at the bifurcation value depends only on the order of the nondegeneracy condition of the system in that point. The fact that the box dimension can detect the bifurcation could give an interesting approach in the field of bifurcation analysis. Moreover, in the paper the reverse result is proved and applied to continuous planar dynamical system. Also, the above result for box dimension of bifurcations is applied to the bifurcations of nonhyperbolic periodic orbits in the plane.

Before we go any further, we recall that the fractal dimension such as Hausdorff or box dimension (also known as Minkowski dimension, Minkowski-Bouligand dimension, capacity dimension, limit capacity), can be used to analyze various objects such as various sets, graphs of a function, attractors, trajectories, etc. Of course, during the analysis we are not only interested in calculating the fractal dimension such as various sets, graphs of a function, attractors, trajectories, etc. Of course, but also in connecting it with some other properties of the object which we study. Notice that, for the orbit of one-dimensional discrete dynamical system, the Hausdorff dimension fails to show anything. Namely, because of its property of countable stability, the Hausdorff dimension does not 'see' the countable sets at all. On the other hand, the box dimension is only finitely stable so it can 'see' them clearly. This is the reason why we use the box dimension in studying orbits of discrete dynamical systems.

Now we recall the notions of box dimension and Minkowski content. For further details see e.g. [5], [8], [13], [15]. Let \( A \subseteq \mathbb{R}^N \) be bounded. \textit{Minkowski sausage} of radius \( \varepsilon \) around \( A \) is a \( \varepsilon \)-neighborhood of \( A \), that is \( A_\varepsilon = \{ y \in \mathbb{R}^N | d(y, A) < \varepsilon \} \).

Let \( s \geq 0 \). The lower and upper \( s \)-dimensional Minkowski contents of \( A \) are defined by

\[
\mathcal{M}^s_\varepsilon(A) := \liminf_{\varepsilon \to 0} \frac{|A_\varepsilon|}{\varepsilon^{N-s}} \quad \text{and} \quad \mathcal{M}^{*s}(A) := \limsup_{\varepsilon \to 0} \frac{|A_\varepsilon|}{\varepsilon^{N-s}}.
\]

Then the lower and upper box dimension are defined by

\[
\dim_B A = \inf \{ s > 0 : \mathcal{M}^s_\varepsilon(A) = 0 \} \quad \text{and} \quad \overline{\dim}_B A = \inf \{ s > 0 : \mathcal{M}^{*s}(A) = 0 \}.
\]

If \( \dim_B A = \overline{\dim}_B A \) we denote it by \( \dim_B A \). If there exists \( d \geq 0 \) such that \( 0 < \mathcal{M}^d_\varepsilon(A) \leq \mathcal{M}^{*d}_\varepsilon(A) < \infty \), then we say that set \( A \) is \textit{Minkowski nondegenerate}. Clearly, then \( d = \dim_B A \). If \( |A_\varepsilon| \approx \varepsilon^s \) for \( \varepsilon \) small, then \( A \) is a Minkowski nondegenerate set and \( \dim_B A = N - s \). If \( \mathcal{M}^d_\varepsilon(A) = \mathcal{M}^{*d}_\varepsilon(A) = \mathcal{M}^d(A) \in (0, \infty) \) for some \( d \geq 0 \), then \( A \) is said to be \textit{Minkowski measurable}. Clearly, then \( d = \dim_B A \).

Let \( A \) and \( B \) be two disjoint bounded sets such that \( \dim_B A = \dim_B B \). It is easy to see, using the finite stability of the upper box dimension \( \overline{\dim}_B (A \cup B) = \max \{ \overline{\dim}_B A, \overline{\dim}_B B \} \), the monotonicity of the lower box dimension \( \dim_B (A \cup B) \leq \overline{\dim}_B (A \cup B) \) (for details see [5]), that

\[
\dim_B (A \cup B) = \dim_B A = \dim_B B.
\]

In the paper the following definitions are used. We say that any two sequences \( (a_n)_{n \geq 1} \) and \( (b_n)_{n \geq 1} \) of positive real numbers are \textit{comparable} and write \( a_n \simeq b_n \) as \( n \to \infty \) if \( a \leq a_n/b_n \leq B \) for some \( A, B > 0 \) and \( n \) sufficiently big. Analogously, two positive functions \( f, g : (0, r) \to \mathbb{R} \) are comparable and we write \( f(x) \simeq g(x) \) as \( x \to 0 \) if \( f(x)/g(x) \in [A, B] \) for \( x \) small enough.

Hence we consider a one-dimensional discrete dynamical system

\[
x_{n+1} = F(x_n), \quad x_1 \in \mathbb{R}
\]
generated by a \( C^k \) function \( F : \mathbb{R} \mapsto \mathbb{R} \). The orbit of a system is a sequence \( (x_n)_{n \geq 1} \) such that \( x_{n+1} = F(x_n) \) for some \( x_1 \in \mathbb{R} \). Recall that the fixed point \( x_0 \) \( (F(x_0) = x_0) \) is hyperbolic if \( |F'(x_0)| \neq 1 \) and nonhyperbolic if \( |F'(x_0)| = 1 \). Let \( \text{saddle+/-} \) denote the fixed point \( x_0 \) which is stable/unstable for \( x \in (x_0, x_0 + r) \) and unstable/stable for \( x \in (x_0 - r, x_0) \).

In this paper the main object of our study is a box dimension of orbits of a given system around the fixed point. We will see that the box dimension of an orbit near a nonhyperbolic fixed point is positive, while near a hyperbolic point is zero. Now we introduce the definition of a \( k \)-nondegenerate map in a point.

**Definition 1.1.** Let \( F : (x_0 - r, x_0 + r) \to \mathbb{R}, r > 0, \) be a map of class \( C^k \), and \( x_0 \) is a fixed point of \( F \) such that \( F'(x_0) \neq 0 \). If there is a \( k \geq 3 \) such that \( F''(x_0) = \ldots = F^{(k-1)}(x_0) = 0 \) and \( F^{(k)}(x_0) \neq 0 \), then we say that the map \( F \) is a \( k \)-nondegenerate map in \( x_0 \). Specially, if \( F''(x_0) \neq 0 \), then we say that \( F \) is a 2-nondegenerate map in \( x_0 \). The number \( k \) is called the order of nondegeneracy of \( F \) in \( x_0 \). The map is called a finitely nondegenerate map in \( x_0 \) if \( F \) is a \( k \)-nondegenerate map in \( x_0 \) for some \( k \geq 2 \).

The remainder of this paper is organized as follows. In the next section, we present the main result about the box dimension of the orbit of one-dimensional discrete dynamical systems generated by a \( k \)-nondegenerate map in \( x_0 \). All the possible cases of a nonhyperbolic fixed point are presented. In Section 3 we prove the reverse result for that class of maps, that is, the information about the nonhyperbolicity of a fixed point and the nondegeneracy of a map will follow from the box dimension of some orbit near that fixed point. In Section 4 the results about the box dimension are applied to one and two parameter bifurcations of a nonhyperbolic fixed point in one dimension. We illustrate it by several examples. In the last section we recall the connection between the discrete and continuous dynamical systems by the Poincaré map. Then we apply the reverse result to the continuous planar systems with a weak focus and limit cycle and find their multiplicities. At the end some results for the bifurcations of a nonhyperbolic periodic orbits in the plane are also obtained.

2. **Box dimension of one-dimensional discrete dynamical systems.** We consider one-dimensional maps of class \( C^{k+1}(I) \), \( I \subset \mathbb{R} \) near the nonhyperbolic fixed point, and find the connection between the box dimension of an orbit going to the fixed point and the order \( k \) of the nondegeneracy of a map \( F \) in that nonhyperbolic fixed point.

Since we use Theorem 1 from article [4] in the proof of the main result, we will state it here.

**Theorem 2.1.** ([4], Theorem 1)

Let \( \alpha > 1 \) and let \( f : (0, r) \to (0, \infty) \) be a monotonically nondecreasing function such that \( f(x) \simeq x^{\alpha} \) as \( x \to 0 \), and \( f(x) < x \) for all \( x \in (0, r) \). Consider the sequence \( S(x) := (x_n)_{n \geq 1} \) defined by \( x_{n+1} = x_n - f(x_n), x_1 \in (0, r) \). Then

\[
x_n \simeq n^{-\frac{1}{\alpha - 1}}, \text{ as } n \to \infty.
\]

Furthermore,

\[
\text{dim}_B S(x) = 1 - \frac{1}{\alpha},
\]

and the set \( S(x) \) is Minkowski nondegenerate.
Next Theorem 2.2 is a generalization of Theorem 3 from [4] where the result is shown for $\alpha = 2$.

**Theorem 2.2. Box dimension in the stable case with $F'(x_0) = 1$**

Let $F : (x_0 - r, x_0 + r) \to \mathbb{R}$ be a map of class $C^{\alpha+1}$, where $\alpha \in \mathbb{N}$, $\alpha \geq 3$ such that $F(x_0) = x_0$, $F'(x_0) = 1$, and

$$F^{(k)}(x_0) = 0, k = 2, \ldots, \alpha - 1, \quad F^{(\alpha)}(x_0) < 0.$$ 

Then there is $r_1 > 0$ such that for the sequence $S(x_1) = (x_n)_{n \geq 1}$ defined by $x_{n+1} = F(x_n)$, $x_1 \in (x_0, x_0 + r_1)$ we have $|x_n - x_0| \approx n^{-\frac{\alpha - 1}{\alpha}}$ as $n \to \infty$ and

$$\dim_B S(x_1) = 1 - \frac{1}{\alpha}.$$ 

Moreover, set $S(x_1)$ is Minkowski nondegenerate.

**Proof.** Without loss of generality we may assume that $x_0 = 0$ and $x > 0$. Then Taylor formula for the map $F$ around the point $x_0 = 0$ is:

$$F(x) = F(0) + F'(0)x + \frac{1}{2} F''(0)x^2 + \ldots + \frac{F^{(\alpha)}(0)}{\alpha!} x^\alpha + \frac{F^{(\alpha+1)}(x_*)}{(\alpha+1)!} x^{\alpha+1}$$

where $x_* \in (0, x)$. By using $F(0) = 0$, $F'(0) = 1$, $F^{(k)}(0) = 0, k = 2, \ldots, \alpha - 1$ and $F^{(\alpha)}(0) < 0$ we obtain

$$F(x) = x + \frac{F^{(\alpha)}(0)}{\alpha!} x^\alpha + \frac{F^{(\alpha+1)}(x_*)}{(\alpha+1)!} x^{\alpha+1}.$$ 

Let

$$f(x) = x - F(x) = - \frac{F^{(\alpha)}(0)}{\alpha!} x^\alpha - \frac{F^{(\alpha+1)}(x_*)}{(\alpha+1)!} x^{\alpha+1}. \quad (2)$$

Now we need to verify that there exists some $r_1 > 0$ such that the function $f$ satisfies the conditions of Theorem 2.1 for $x \in (0, r_1)$.

**Condition 1.** $f(x) \simeq x^\alpha$, when $x \to 0$ and $\alpha > 1$.

Using $F^{(\alpha)}(0) < 0$, we get $f(x) = \frac{|F^{(\alpha)}(0)|}{\alpha!} x^\alpha - \frac{F^{(\alpha+1)}(x_*)}{(\alpha+1)!} x^{\alpha+1}$. So the condition is satisfied.

**Condition 2.** $f$ is a nondecreasing function, that is $f'(x) \geq 0$

By deriving (2) and solving the inequality

$$\left| \frac{F^{(\alpha)}(0)}{(\alpha - 1)!} x^{\alpha-1} - \frac{F^{(\alpha+1)}(x_*)}{\alpha!} x^\alpha \right| \geq 0$$

for positive $x$, we get

$$F^{(\alpha+1)}(x_*) x \leq \alpha \left| \frac{F^{(\alpha)}(0)}{(\alpha - 1)!} \right|.$$ 

(3)

The solution of (3) depends on the sign of $F^{(\alpha+1)}(x_*)$, so we see that if it is negative then $f$ is nondecreasing for all positive $x$, and if it is positive, then there exists $r_2 > 0$ such that $f'(x) \geq 0$ for all $x \in (0, r_2)$, where is $r_2 = \min \left\{ r_3, \frac{\alpha |F^{(\alpha)}(0)|}{M} \right\}$, and $r_3 > 0$ such that $|F^{(\alpha+1)}(x)| < M$ for $|x| < r_3$.

**Condition 3. $f(x) < x$ for $x \in (0, r_4)$**

Notice that $f(x) < x$, in fact, means $F(x) > 0$. We know that $F(0) = 0$ and $F$ is increasing in $0$ ( $F'(0) = 1 > 0$). It follows that there exists $r_4 > 0$ such that $F(x) > 0$ for $x \in (0, r_4)$. 

Finally, if we take \( r_1 = \min\{r_2, r_4\} \), \( f \) satisfies the conditions of Theorem 2.1 for \( x \in (0, r_1) \).

**Remark 1.** The Condition 2 and 3, which are satisfied by the function \( f \) in the proof of Theorem 2.2, show that the sequence \( S(x_1) = (x_n)_{n \geq 1}, x_1 \in (x_0, x_0 + r) \) is a decreasing sequence of positive numbers which tends to \( x_0 \). In fact, it means that nonhyperbolic fixed point \( x_0 \) is (asymptotically) stable for \( x > x_0 \), and for every \( \alpha \).

**Remark 2.** If \( \alpha \) is odd, then Theorem 2.2 holds for \( F^{(\alpha)}(x_0) < 0 \) and \( x_1 \in (x_0 - r, x_0) \). Notice that if \( F^{(\alpha)}(x_0) < 0 \) and \( \alpha \) odd, then the fixed point \( x_0 \) is stable. If \( \alpha \) is even, then Theorem 2.2 holds for \( F^{(\alpha)}(x_0) > 0 \) and \( x_1 \in (x_0 - r, x_0) \). Notice that for \( F^{(\alpha)}(x_0) > 0 \) and \( \alpha \) even, the fixed point \( x_0 \) is stable from the left. Afterwards we will see that, in that case, the point \( x_0 \) is a saddle-

The following lemma is needed in order to prove the unstable case.

**Lemma 2.3.** If \( f : (−r, r) \to \mathbb{R} \) is the function of a form \( f(x) = c_1x + c_kx^k + c_{k+1}x^{k+1} + \ldots \), where \( c_1 = \pm 1 \), then \( f^{-1}(y) = d_1y + d_ky^k + O(y^{k+1}) \), where \( d_1 = c_1 \) and \( d_k = -\frac{c_k}{c_1} \).

**Corollary 1. (Unstable case with \( F'(x_0) = 1 \))** Let \( F : (x_0 - r, x_0 + r) \to \mathbb{R} \) be a map of class \( C^{\alpha+1} \), \( \alpha \in \mathbb{N}, \alpha \geq 3 \) such that
\[
F(x_0) = x_0, \quad F'(x_0) = 1, \quad F^{(k)}(x_0) = 0, \quad k = 2, \ldots, \alpha - 1, \quad F^{(\alpha)}(x_0) > 0.
\]
Then there exists \( r_1 > 0 \) such that for the sequence \( S(x_1) = (x_n)_{n \geq 1} \) defined by
\[
x_{n+1} = F^{-1}(x_n), \quad x_1 \in (x_0, x_0 + r_1) \text{ we have } |x_n - x_0| \asymp n^{-\frac{1}{\alpha+1}} \text{ as } n \to \infty \text{ and dim}_B S(x_1) = 1 - \frac{1}{\alpha}.
\]
Moreover, set \( S(x_1) \) is Minkowski nondegenerate.

**Proof.** \( F \) is a map of class \( C^{\alpha+1} \) on \( (x_0 - r, x_0 + r) \) and \( F'(x_0) \neq 0 \). By using the Inverse Function Theorem we get that there exists \( F^{-1} \) of class \( C^{\alpha+1} \) on some neighborhood of \( x_0 = F(x_0) \), that is, there exist \( r_1, r_2 > 0 \) such that \( F^{-1} \) is of class \( C^{\alpha+1} \) on \( (x_0 - r_2, x_0 + r_2) = F(x_0 - r_1, x_0 + r_1) \) where \( r_1 < r \).

Without loss of generality we may assume that \( x_0 = 0 \) and \( x > 0 \). Then from Lemma 2.3 we obtain:
\[
(F^{-1})'(0) = 0, \quad (F^{-1})'(0) = 1, \quad (F^{-1})'(k)(0) = 0, \quad k = 2, \ldots, \alpha - 1;
\]
\[
(F^{-1})'(\alpha)(0) = -F^{(\alpha)}(x_0) < 0.
\]
Now the conclusion that the inverse function \( F^{-1} \) satisfies the conditions of Theorem 2.2 implies the claim of the theorem.

**Remark 3.** Corollary 1 gives a sufficient condition under which the fixed point \( x_0 \) is unstable from the right (\( \lim_{n \to -\infty} x_n = x_0 \), where \( x_{n+1} = F(x_n), x_1 > 0 \)) and it holds for every \( \alpha \).

**Remark 4.** If \( \alpha \) is odd, then Corollary 1 holds for \( F^{(\alpha)}(x_0) > 0 \) and \( x_1 \in (x_0 - r, x_0) \). Notice that if \( F^{(\alpha)}(x_0) > 0 \) and \( \alpha \) odd, then the fixed point \( x_0 \) is unstable. If \( \alpha \) is even, then Corollary 1 holds if \( F^{(\alpha)}(x_0) < 0 \) and \( x_1 \in (x_0 - r, x_0) \). In that case the fixed point \( x_0 \) is a saddle-

**Remark 5.** Notice that the nonhyperbolic fixed point of the finite nondegenerate function of even order is always a saddle. That is the reason why the sufficient nondegeneracy condition for a saddle-node bifurcation is connected to the even order of nondegeneracy.

The following lemma gives the connection between the \( n \)-derivation of \( F^2 \) \( ((F^2)^{(\alpha)}) \) and the \( r \)-derivations of \( F \) \( (F^{(r)}, \ r \leq n) \).
Lemma 2.4. Let $F : I \to \mathbb{R}$ be a map of class $C^n$. Then for $n > 2$ and $x \in I$ is
\[ (F^2)^{(n)}(x) = F^{(n)}(F(x)) \cdot (F'(x))^n + F^{(n)}(x)F'(F(x)) + \sum_{i=2}^{n-1} F^{(i)}(F(x))G_{n,i}(F'(x), F''(x), \ldots, F^{(n-1)}(x)) \]  
where $G_{n,i}$ are the polynomials in the variables $F', F'', \ldots, F^{(n-1)}$ of order less and equal to $i$ such that $G_{n,i}(F'(x_0), 0, \ldots, 0) = 0$ for $i = 2, \ldots, n-1$ and the polynomial $G_{n,n-1}$ does not depend on a variable $F^{(n-1)}(x)$ for $n > 3$.

Lemma 2.4 can easily be proven by using the method of mathematical induction.

We apply this lemma to the case of a $n$-nondegenerate function in a nonhyperbolic fixed point, and get the following corollary.

Corollary 2. Let $F : (x_0 - r, x_0 + r) \to \mathbb{R}$ be a map of class $C^{n+1}$, and let $F(x_0) = x_0$ and $F'(x_0) = -1$. If $F''(x_0) = F'''(x_0) = \ldots = F^{(n-1)}(x_0) = 0$, $F^{(n)}(x_0) \neq 0$, then
\[ (F^2)''(x_0) = \ldots = (F^2)^{(n-1)}(x_0) = 0, \quad (F^2)^{(n)}(x_0) = F^{(n)}(x_0)[(-1)^n - 1]. \]
Moreover, if $n$ is odd, then $(F^2)^{(n)}(x_0) = -2F^{(n)}(x_0)$, and if $n$ is even, then $(F^2)^{(n)}(x_0) = 0$ and $(F^2)^{(n+1)}(x_0) = -2F^{(n+1)}(x_0)$.

Remark 6. Notice that under the conditions of Corollary 2 the map $F^2$ is, at least, a $n$-nondegenerate map in $x_0$. Moreover, if $F^2$ is a $n$-nondegenerate map in $x_0$, then it follows that the map $F$ can be at most $n$-nondegenerate in $x_0$.

The next theorem is a generalization of Theorem 4 from [4], but the proof presented here is much simpler. We do not observe the alternating sequence as such, but just study its positive subsequence and apply certain properties of box dimension in order to obtain the result.

Theorem 2.5. (Stable case with $F'(x_0) = -1, \alpha$ odd)
Let $F : (x_0 - r, x_0 + r) \to \mathbb{R}$ be a map of class $C^{n+1}$, for $\alpha \in \mathbb{N}$, $\alpha \geq 3$, $\alpha$ odd, such that
\[ F(x_0) = x_0, \quad F'(x_0) = -1, \quad F^{(k)}(x_0) = 0, \quad k = 2, \ldots, \alpha - 1, \quad F^{(\alpha)}(x_0) > 0. \]
Then there exists $r_1 \in (0, r)$ such that for the sequence $S(x_1) = (x_n)_{n \geq 1}$ defined by $x_{n+1} = F(x_n)$, $x_1 \in (x_0, x_0 + r_1)$ we have $|x_n - x_0| \approx \frac{n^{-\alpha}}{\alpha!}$ as $n \to \infty$ and
\[ \dim_B S(x_1) = 1 - \frac{1}{\alpha}. \]
Moreover, the set $S(x_1)$ is Minkowski nondegenerate.

Proof. Without loss of generality we may assume that $x_0 = 0$. Taylor’s formula for the function $F$ around the point $x_0 = 0$ is
\[ F(x) = -x + \frac{F^{(\alpha)}(0)}{\alpha!} x^\alpha + \frac{F^{(\alpha+1)}(x_0)}{\alpha + 1!} x^{\alpha+1} \]
where is $x_0 \in (0, x)$. Now, we would like to show that the sequence $x_{n+1} = F(x_n)$ is an alternating sequence on some neighborhood of $x_0 = 0$, that is, there exists $r_* \in (0, r)$ such that $F(x) < 0$ for $x \in (0, r_*)$ and $F(x) > 0$ for $x \in (-r_*, 0)$. This problem is equivalent to finding the neighborhood on which the following inequality
Proof. We find \( r \). Furthermore, the set 

In the case of two subsequences \((a_n)\) and \((b_n)\) where \((a_n)\) is defined by \( a_{n+1} = F^2(a_n) \), with \( a_1 = x_1 > 0 \), and \( b_{n+1} = F^2(b_n) \), with \( b_1 = F(x_1) < 0 \). We notice that the both subsequences \( A(a_1) = (a_n)_{n \geq 1} \) and \( B(b_1) = (b_n)_{n \geq 1} \) are defined with the same map \( F^2 \), but with the different initial point. Afterwards we will show that \( F^2 \) satisfies Theorem 2.2. Then by Remark 2 for \( \alpha \) odd, the result for box dimension also holds for \( x_1 \in (x_0 - r, x_0) \), that is, \( \dim_B B(b_1) = \dim_B A(a_1) \). Then by \( S(x_1) = A(a_1) \cup B(b_1) \) and (1) we obtain \( \dim_B S(x_1) = \dim_B A(a_1) \). So it suffices to prove the theorem only for the positive subsequence \( A(a_1) \).

Now let us consider the function \( F^2 \) and see that \( F^2 \) is a map of class \( C^{\alpha + 1} \), with \( F^2(0) = 0 \), \( F^2(1) = 1 \), \( (F^2)'(0) = 0 \). The assumption of \( \alpha \)-nondegeneracy of map \( F \) and Corollary 2 imply

\[
(F^2)'(0) = \ldots = (F^2)'(0) = 0, \quad (F^2)'(0) = -2F'(0) < 0.
\]

Now \( F^2 \) satisfies the conditions of Theorem 2.2, so there exists \( r_1 \in (0, r) \) such that for the positive subsequence \( A(x_1) = (x_n)_{n \geq 1} \) defined by \( a_{n+1} = F^2(a_n) \), \( a_1 = x_1 > 0 \), it holds \( |a_n| \approx n^{-\frac{1}{\alpha - 1}} \) and \( \dim_B A(x_1) = 1 - \frac{1}{\alpha} \). According to Remark 2 and 4, we can get the same result for the negative subsequence \((b_n)_{n \geq 1}\). Now (5) follows directly from the equality: \( \dim_B S(x_1) = \dim_B A(x_1) = 1 - \frac{1}{\alpha} \).

Finally, we show the result for the asymptotic behavior. The form of the sequence \( S(x_1) = (x_n)_{n \geq 1} \) of class \( C^{\alpha + 1} \), and \( \alpha \geq 3 \), \( \alpha \) odd such that

\[
F(x_0) = x_0, \quad F'((x_0) = -1, \quad F(\alpha) = 0, \quad k = 2, \ldots, \alpha - 1, \quad F^\alpha(x_0) < 0.
\]

Then there exists \( r \in (0, r) \) such that for the sequence \( S(x_1) = (x_n)_{n \geq 1} \) defined by \( x_{n+1} = F^{-1}(x_n), x_1 \in (x_0, x_0 + r) \) we have \( |x_n - x_0| \approx n^{-\frac{1}{\alpha - 1}} \) as \( n \to \infty \) and \( \dim_B S(x_1) = 1 - \frac{1}{\alpha} \). Furthermore, the set \( S(x_1) \) is Minkowski nondegenerate.

Remark 3. (Unstable case with \( F'(x_0) = -1, \alpha \) odd)

Let \( F : (x_0 - r, x_0 + r) \to \mathbb{R} \) be a map of class \( C^{\alpha + 1} \), and \( \alpha \in \mathbb{N}, \alpha \geq 3, \alpha \) odd such that

\[
F(x_0) = x_0, \quad F'(x_0) = -1, \quad F^{\alpha}(x_0) < 0.
\]

Then there exists \( r \in (0, r) \) such that for the sequence \( S(x_1) = (x_n)_{n \geq 1} \) defined by \( x_{n+1} = F^{-1}(x_n), x_1 \in (x_0, x_0 + r) \) we have \( |x_n - x_0| \approx n^{-\frac{1}{\alpha - 1}} \), as \( n \to \infty \) and \( \dim_B S(x_1) = 1 - \frac{1}{\alpha} \). Furthermore, the set \( S(x_1) \) is Minkowski nondegenerate.

Proof. If \( F \) is a map of class \( C^{\alpha + 1} \) on \((x_0 - r, x_0 + r)\) and \( F'(x_0) \neq 0 \), then, by the Inverse Function Theorem, map \( F^{-1} \) is also of class \( C^{\alpha + 1} \) on some neighborhood of \( x_0 = F(x_0) \), that is, there exists \( \epsilon_1, \epsilon_2 > 0 \) such that \( F^{-1} \) of class \( C^{\alpha + 1} \) on \((x_0 - \epsilon_2, x_0 + \epsilon_2) = F(x_0 - \epsilon_1, x_0 + \epsilon_1) \) where \( \epsilon_1 < r \). Without loss of generality we may assume that \( x_0 = 0 \) and \( x > 0 \). Now Lemma 2.3 implies that: \( F^{-1}(0) = 0, (F^{-1})'(0) = 0, (F^{-1})^{(k)}(0) = 0, k = 2, \ldots, \alpha - 1; (F^{-1})^{(\alpha)}(0) = -F^\alpha(x_0) > 0 \). Since now the inverse function \( F^{-1} \) satisfies the conditions of Theorem 2.5, we obtain the claim of the corollary.
Remark 8. In the case with $F^\alpha(x_0) < 0$ and $\alpha$ odd, the fixed point $x_0$ is unstable.

Now we observe what happens in the case for $\alpha$ even, and notice that the result is more complicated.

Theorem 2.6. (Case $F'(x_0) = -1, \alpha$ even)

Let $F : (x_0 - r, x_0 + r) \to \mathbb{R}$ be a map of class $C^\infty$, and let $\alpha \geq 3, \alpha$ even, such that

$$F(x_0) = x_0, \quad F'(x_0) = -1, \quad F^{(k)}(x_0) = 0, k = 2, \ldots, \alpha - 1, \quad F^{(\alpha)}(x_0) > 0.$$ 

Then there exists $r_1 \in (0, r)$ such that for the sequence $S(x_1) = (x_n)_{n \geq 1}$ defined by $x_{n+1} = F(x_n), x_1 \in (x_0, x_0 + r_1)$ we have the following:

a) If there exist some odd numbers $\beta_i, i \in \{1, \ldots, j\}$ such that $\alpha < \beta_1 < 2\alpha - 1$ and $F^{(\beta)}(x_0) \neq 0$, then for $\beta = \min \{\beta_i \mid \beta_i \} \in (1, \ldots, j)$ we have $|x_n - x_0| \simeq n^{-\frac{1}{\alpha - 1}}$ as $n \to \infty$ and $\dim_B S(x_1) = 1 - \frac{1}{\beta'}$.

b) If $L \geq 2\alpha - 1$, then $a = \frac{F^{(\alpha)}(x_0)}{\alpha!}$ and $b = \frac{F^{(\beta)}(x_0)}{\beta!}$.

- If $2b \neq -\alpha^2$, then $|x_n - x_0| \simeq n^{-\frac{1}{\alpha - 1}}$ as $n \to \infty$ and $\dim_B S(x_1) = 1 - \frac{1}{2\alpha - 1}$;
- If $2b = -\alpha^2$, then $|x_n - x_0| \simeq n^{-\frac{1}{\alpha - 1}}$ as $n \to \infty$ and $\dim_B S(x_1) = 1 - \frac{1}{2\alpha - 1}$.

Moreover, the set $S(x_1)$ is Minkowski nondegenerate.

Proof. Without loss of generality we may assume that $x_0 = 0$. Then the Taylor formula for the function $F$ around the point $x_0 = 0$ is

$$F(x) = -x + \frac{F^{(\alpha)}(0)}{\alpha!} x^\alpha + \frac{F^{(\alpha+1)}(x_*)}{(\alpha+1)!} x^{\alpha+1},$$

where $x_* \in (-x, x)$. In this case, we show that the sequence $x_{n+1} = F(x_n)$ is an alternating sequence on some neighborhood of $x_0 = 0$, i.e. there exists $r_1 \in (0, r)$ such that $F(x) < 0$ for $x \in (0, r_1)$ and $F(x) > 0$ for $x \in (-r_1, 0)$. Finding that neighborhood can be reduced to find the neighborhood on which the inequality $x^{\alpha-1}(A + Bx) < 1$ holds, with $A = \frac{F^{(\alpha)}(0)}{\alpha!} > 0$ and $B = \frac{F^{(\alpha+1)}(x_*)}{(\alpha+1)!}$. Now we easily get $r_* = (\frac{1}{A+Bx})^{\frac{1}{\alpha-1}}$.

Notice that the alternating sequence $x_{n+1} = F(x_n), x_1 > 0$ consists of two subsequences $(a_n)$ and $(b_n)$, where the subsequence $(a_n)$ is defined by $a_{n+1} = F^2(a_n)$, with $a_1 = x_1 > 0$, and the other one is defined by $b_{n+1} = F^2(b_n)$, with $b_1 = F(x_1) > 0$. Note that both subsequences $A(a_1) = (a_n)_{n \geq 1}$ and $B(b_1) = (b_n)_{n \geq 1}$ are defined by the same map $F^2$, but with the different initial points. Afterwards we will show that $F^2$ satisfies Theorem 2.2. Then by Remark 2 for $\alpha$ even, the result for box dimension is also true for $x_1 \in (x_0 - r, x_0)$, that is, $\dim_B B(b_1) = \dim_B A(a_1)$. Then by $S(x_1) = A(a_1) \cup B(b_1)$ and (1) we have $\dim_B S(x_1) = \dim_B A(a_1)$. This means that it suffices to prove the statement of the theorem for the positive subsequence $A(a_1)$, and that is exactly what is done in the rest of the proof.

Now let us consider the map $F^2$, and we know that it is a map of class $C^{\alpha+1}$ on some neighborhood of $x_0 = 0$, and that $F^2(0) = 0, (F^2)'(0) = 1, (F^2)''(0) = 0$. 


The map can be written in a form $F(x) = -x + ax^\alpha + bx^\beta + cx^\gamma$, where $a = \frac{F^{(\alpha)}(0)}{\alpha!}, b = \frac{F^{(\beta)}(0)}{\beta!}$ and $c = \frac{F^{(\gamma)}(x_*)}{\gamma!}$, $\alpha < \beta < \gamma$, $\alpha$ even. Now we calculate

$$F^2(x) = x - ax^\alpha - bx^\beta - cx^\gamma + ax^\alpha(1 - ax^{\alpha-1} - bx^{\beta-1} - cx^{\gamma-1})^\alpha +$$
$$+(-1)^\alpha bx^\beta(1 - ax^{\alpha-1} - bx^{\beta-1} - cx^{\gamma-1})^\beta +$$
$$+(-1)^\gamma bx^\gamma(1 - ax^{\alpha-1} - bx^{\beta-1} - cx^{\gamma-1})^\gamma =$$

$$= x + ((-1)^\alpha b - b)x^\beta + ((-1)^\gamma c - c)x^\gamma - \alpha a^2 x^{2\alpha-1} -$$
$$-( -1)^\beta \frac{\partial}{\partial x} \frac{\partial^2}{\partial x^2} (x^{2\beta-1} - (-1)^\gamma c^2 x^{2\gamma-1} + O(x^{\alpha + \beta - 1}),$$

for $x$ such that $|x^{\alpha-1}(a + bx^{\beta-a} + cx^{\gamma-o})| < 1$. It can be easily shown that this inequality holds for $x \in (-r_2, r_2)$, with $r_2 = \left(\frac{a + \beta - 1}{\alpha + |\beta-a| + |\gamma-o|}\right)^{\frac{1}{\gamma-1}}$ and $r_2 < r_*$.

Let $A(x_1) = (x_n)_{n \geq 1}$ be defined by $x_{n+1} = F^2(x_n), x_1 \in (x_0, x_0 + r_1)$. From (6) follow three possibilities:

**Case 1.** If $\beta$ and $\gamma$ are even numbers, then

$$F^2(x) = x - \alpha a^2 x^{2\alpha-1} + O(x^{2\beta-1}).$$

The map $F^2$ satisfies the conditions:

$(F^2)^{(\alpha)}(0) = \ldots = (F^2)^{(2\alpha-2)} = 0$, $(F^2)^{(2\alpha-1)}(0) = -\alpha a^2 < 0$, so by Theorem 2.2 we obtain $\dim_B A(x_1) = 1 - \frac{1}{2\alpha-1}$.

**Case 2.** If $\beta$ is an odd number, then

$$F^2(x) = x - 2bx^\beta - \alpha a^2 x^{2\alpha-1} + O(x^\delta),$$

where $\delta = \min\{\gamma, 2\beta - 1\}$.

(a) If $\beta > 2\alpha - 1$, then we have again **Case 1**.

(b) If $\beta < 2\alpha - 1$, then map $F^2$ is a $\beta$-nondegenerate in $x_0 = 0$, so it follows from the Theorem 2.2 that $\dim_B A(x_1) = 1 - \frac{1}{2\alpha-1}$.

(c) If $\beta = 2\alpha - 1$, then

$$F^2(x) = x - (2b + \alpha a^2)x^{2\alpha-1} + O(x^\delta).$$

We see that the map $F^2$ is a $(2\alpha - 1)$-nondegenerate in $x_0 = 0$ if $2b + \alpha a^2 \neq 0$, so by Theorem 2.2 we get $\dim_B A(x_1) = 1 - \frac{1}{2\alpha-1}$.

But if $2b + \alpha a^2 = 0$, then the order of the nondegeneracy of the map $F^2$ in $x_0 = 0$ is surely greater than $2\alpha - 1$.

**Case 3.** If $\beta$ is even and $\gamma$ is odd, then we have again **Case 2** but for $\beta = \gamma$. □

**Remark 9.** Notice that in the case of $F^{(\alpha)}(x_0) < 0$, the proof is analogous because the series of $F^2$ is equal (the coefficient $a$ appears only in the form $a^2$ so its sign does not have any influence). The only difference is that the fixed point $x_0$ is unstable so we have to consider the sequence defined by $x_{n+1} = F^{-1}(x_n)$.

**Remark 10.** Notice that from Theorem 2.6 we obtain the following: if the map $F$ is $\alpha$-nondegenerate in $x_0$ where $\alpha$ is even, then the map $F^2$ is nondegenerate in $x_0$ with the order greater than $\alpha$. Consequently, the result in that case of $\alpha$ even is the following: if the map $F^2$ is $\alpha$-nondegenerate in $x_0$, then the map $F$ is $\gamma$-nondegenerate in $x_0$, where $\gamma < \alpha$. Recall that for $\alpha$ odd, Theorem 2.5 and Corollary 3 say that the orders of the nondegeneracy of $F$ and $F^2$ are equal. Now we can state the entire reverse result for the case $F^2(x_0) = -1$: if a map $F^2$ is $\beta$-nondegenerate in $x_0$, then the map $F$ has the order of nondegeneracy less or equal
than $\beta$. We will further discuss this reverse result in Theorem 3.3 in the following section.

**Remark 11.** In Theorems 2.2, 2.5 and 2.6 and Corollaries 1 and 3, we have the result of Minkowski nondegeneracy of the sequence $S(x_1) = (x_n)_{n \geq 1}$, $x_1 \in (x_0, x_0 + r)$, that is

$$0 < M^d_1(S(x_1)) \leq M^d(S(x_1)) < \infty.$$ 

Furthermore, by Theorem 3.1 from [12] we can obtain the lower and upper estimates for Minkowski contents of $S(x_1)$. It is done in the following way for the case in which $S(x_1)$ is a decreasing sequence of positive numbers which tend to 0 ($x_0 = 0$ is stable). Let $\Omega$ be a union of open intervals in $\mathbb{R}$

$$\Omega = \bigcup_{j=1}^{\infty} I_k, \quad I_k = (x_{k+1}, x_k).$$

Then the sequence $S(x_1)$ is, in fact, a boundary of $\Omega$: $S(x_1) = \partial \Omega$. We denote by $(l_k)_{k \geq 1}$ the sequence of the lengths of the intervals $I_k$:

$$l_k = x_k - x_{k+1}.$$ 

Notice that $l_k = f(x_k)$. Now we see that $(l_k)_{k \geq 1}$ is a positive non-increasing sequence such that $l_k \sim k^{-\frac{1}{d}}$, where $d = \dim_B S(x_1) \in (0, 1)$. Now by [12](Theorem 3.1, Part a) we obtain

$$2^{1-d} (u^d + \frac{d}{1-d} uv^{d-1}) \leq M^d_1(S(x_1)) \leq M^d(S(x_1)) \leq 2^{1-d} (v^d + \frac{d}{1-d} vu^{d-1})$$

where $u = \lim \inf_{j \to \infty} l_j^{\frac{1}{d}}$ and $v = \lim \sup_{j \to \infty} l_j^{\frac{1}{d}}$.

3. **Non-hyperbolic fixed point and box dimension.** The article [12] showed that the trajectory in the sufficiently small environment of a stable hyperbolic fixed point has the box dimension equal to zero. It is easy to see that the same is true for unstable hyperbolic fixed point. Following results give us the connection between the nonhyperbolicity of a fixed point and box dimension.

**Theorem 3.1. Stable hyperbolic fixed point ([4], Theorem 5)**

Let $F : (x_0 - r, x_0 + r) \to \mathbb{R}$ be a map of class $C^1$, $F(x_0) = x_0$, and $|F'(x_0)| < 1$. Then there exists $r_1 \in (0, r)$ such that for any sequence $S(x_1) = (x_n)_{n \geq 1}$ defined by $x_{n+1} = F(x_n), \; x_1 \in (x_0 - r_1, x_0 + r_1)$ we have $\dim_B S(x_1) = 0$.

**Corollary 4. Unstable hyperbolic fixed point**

Let $F : (x_0 - r, x_0 + r) \to \mathbb{R}$ be a map of class $C^1$, $F(x_0) = x_0$, and $|F'(x_0)| > 1$. Then there exists $r_1 \in (0, r)$ such that for any sequence $S(x_1) = (x_n)_{n \geq 1}$ defined by $x_{n+1} = F^{-1}(x_n), \; x_1 \in (x_0 - r_1, x_0 + r_1)$ we have $\dim_B S(x_1) = 0$.

**Proof.** By the Inverse Function Theorem, we have that the map $F$ of class $C^1$ on $(x_0 - r, x_0 + r)$ has an inverse map $F^{-1}$ on some neighborhood of $x_0 = F(x_0)$ of the same class $C^1$, and it holds that $(F^{-1})'(y) = \frac{1}{F'(x)}$. Now the statement follows from the previous theorem because the map $F^{-1}$ satisfies its conditions: class $C^1$ and $|(F^{-1})'(x_0)| < 1$.

Before the reverse result, we prove a lemma which shows that all the sequences with the same limit have equal box dimensions. In the following results we consider the sequence $S(x_1) = (x_n)_{n \geq 1}$ defined by

$$x_{n+1} = \begin{cases} F(x_n), & x_0 \text{ stable} \\ F^{-1}(x_n), & x_0 \text{ unstable} \end{cases}$$

(7)
Lemma 3.2. (Excision lemma) Let $x_0$ be a fixed point of a map $F : (x_0 - r, x_0 + r) \rightarrow \mathbb{R}$ of class $C^m, \ m \geq 1$, $F'(x_0) \neq 0$. Assume that there is $r_1 \in (0, r)$ such that for any decreasing sequence $S(x_1) = (x_n)_{n \geq 1}$ defined by (7), $x_1 \in (x_0, x_0 + r_1)$ is $\dim_B S(x_1) = A \geq 0$. Then any sequence $S(x_1) = (x_n)_{n \geq 1}$ defined by (7) with the initial point $x_1^* > x_0$ such that $\lim_{n \rightarrow \infty} x_n = x_0$ has the box dimension $\dim_B S(x_1^*) = A \geq 0$.

Proof. Without loss of generality we may assume that $x_0 > 0$ is a stable fixed point for $x \in (x_0, x_0 + r)$. Then the decreasing sequence $S(x_1) = (x_n)_{n \geq 1}$ defined by $x_{n+1} = F(x_n), x_1 \in (x_0, x_0 + r_1)$ has $\dim_B S(x_1) = A \geq 0$.

Let $S(x_1^*) = (x_n^*)_{n \geq 1}$ be defined by $x_{n+1} = F(x_n), x_1^* > x_0 + r_1$. Then there exists $k \in \mathbb{N}, k > 1$ such that $S(x_1^*) = \{x_1^*\} \cup \{x_k^*\} \cup \ldots \cup S(x_k^*)$, where $x_k^* \in (x_0, x_0 + r_1)$. Thus, $\dim_B S(x_k^*) = A$. Because of the monotonicity of lower box dimension we get:

$$\dim_B S(x_1^*) \geq \dim_B S(x_k^*) = A.$$ 

The finite stability of upper box dimension gives

$$\overline{\dim}_B S(x_1^*) = \max\{\overline{\dim}_B S(x_k^*), 0, \ldots, 0\} = A.$$ 

So

$$A \leq \dim_B S(x_1^*) \leq \overline{\dim}_B S(x_k^*) = A,$$

implies $\dim_B S(x_1^*) = A$. \hfill \Box

Remark 12. This result can be analogously shown for $x_1 \in (x_0 - r, x_0)$ and for the alternating sequence on the interval $(x_0 - r, x_0 + r)$.

The next theorem shows how can the nonhyperbolicity and the order of nondegeneracy in the fixed point follow from the box dimension.

Theorem 3.3. Let $F : (x_0 - r, x_0 + r) \rightarrow \mathbb{R}$ be a map of class $C^{\alpha+1}, F(x_0) = x_0$. Assume that $F$ is a finitely nondegenerate map in $x_0$. Assume there is a sequence $S(x_1) = (x_n)_{n \geq 1}$ defined by (7), $x_1 \in (x_0, x_0 + r)$ with the box dimension in a form $\dim_B S(x_1) = 1 - \frac{1}{\alpha}$, for some $\alpha \geq 2$. Then the fixed point $x_0$ is nonhyperbolic. Moreover, the following is true:

(i) if $F'(x_0) = 1$, then $F$ is an $\alpha$-nondegenerate map in $x_0$.

(ii) if $F'(x_0) = -1$, then $F$ is a $k$-nondegenerate map in $x_0$ with $k \leq \alpha$.

Proof. The nonhyperbolicity of the fixed point $x_0$ follows from the contrapositive statements of Theorem 3.1 and Corollary 4. Namely, if the point $x_0$ is a hyperbolic fixed point, then by Lemma 3.2 the box dimensions of all the sequences which tend to $x_0$ will be equal to 0, what is in the contradiction with the assumption of the theorem. So, $x_0$ is nonhyperbolic.

The second claim of the theorem about the nondegeneracy follows from the form of box dimension and from Theorems 2.2, 2.5 i 2.6.

(i) $F'(x_0) = 1$

Let us assume that the sequence $S(x_1) = (x_n)_{n \geq 1}$ defined by $x_{n+1} = F(x_n), x_1 \in (x_0, x_0 + r)$ is decreasing, that is, $x_0$ is stable from the right (stable+). The assumption of the theorem is that there is a sequence $S(x_1) = (x_n)_{n \geq 1}$ defined by $x_{n+1} = F(x_n), x_1 \in (x_0, x_0 + r)$ with $\dim_B S(x_1) = 1 - \frac{1}{\alpha}$. Assume that the order of nondegeneracy of the map $F$ in $x_0$ is $\alpha^* \neq \alpha$. Then Theorem 2.2 implies that there exists $r^* \in (0, r)$ such that any sequence $S(x_1) = (x_n)_{n \geq 1}$ defined by $x_{n+1} = F(x_n), x_1^* \in (x_0, x_0 + r^*)$ has $\dim_B S(x_1) = 1 - \frac{1}{\alpha^*}$. But, for $x_1 < x_0 + r^*$
that is in contradiction with the assumption of this theorem, and for $x_1 > x_0 + r^*$ is in contradiction with Lemma 3.2. Consequently, $F$ must be an $\alpha$-nondegenerate function in $x_0$.

(ii) $F'(x_0) = -1$

In this case, we consider the map $F^2$. Namely, note that $(F^2)'(x_0) = 1$, and $\dim_B A(x_1) = \dim_B S(x_1)$ (see the proof of Theorems 2.5 and 2.6). Now we can apply the case (i) on $F^2$, and conclude that the map $F^2$ is an $\alpha$-nondegenerate map in $x_0$. Now Remark 10 of Theorem 2.6 implies that $F$ is a finitely nondegenerate map of order at most $\alpha$ in $x_0$. \hfill \Box

Remark 13. Theorem 3.3 is applied to continuous planar dynamical systems in Section 5. So we will see that from the box dimension of the sequence defined by the Poincaré map we can get the information about the multiplicity of a weak focus or limit cycle. Hopefully, this result could be used as useful numerical tool for getting the ‘good’ estimation of multiplicity of focus or limit cycle for a given continuous planar system.

Corollary 5. Let $F : (x_0 - r, x_0 + r) \rightarrow \mathbb{R}$ be a map of class $C^r$, $F(x_0) = x_0$. Assume that $F$ is a finitely nondegenerate function in $x_0$. Let us assume there is a sequence $S(x_1) = (x_n)_{n \geq 1}$ defined by (7), $x_1 \in (x_0, x_0 + r)$ with the box dimension $\dim_B S(x_1) = 0$. Then the fixed point $x_0$ is a hyperbolic fixed point of $F$.

This reverse result for the hyperbolic fixed point follows from Theorem 3.3 and Lemma 3.2.

4. Box dimension of bifurcations of nonhyperbolic fixed points.

4.1. One-parameter bifurcations. In a one-parameter family $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ two types of local bifurcations can occur, corresponding to the eigenvalue 1 or $-1$. The bifurcations of nonhyperbolic fixed point with an eigenvalue equal to 1 are: a saddle-node, transcritical and pitchfork bifurcation, and with an eigenvalue $-1$ is a period doubling bifurcation. For more details about the one-parameter bifurcations of fixed point see for example [6] and [14]. In the article [2] they generalize the sufficient conditions for this bifurcation to higher degree. We apply our box dimension results to the one-parameter bifurcations with nondegeneracy conditions of higher degree.

In [2] we have the following generalized sufficient conditions for bifurcations in the family with a nonhyperbolic fixed point:

- **ND1.** $f_{xx}(0,0) = \ldots = f_{x^{2n}}(0,0) = 0, f_{x^{2n+1}}(0,0) \neq 0$
- **ND2.** $f_{xxx}(0,0) = \ldots = f_{x^{2n-1}}(0,0) = 0, f_{x^{2n+1}}(0,0) \neq 0$
- **PD1.** $f_{xxx}(0,0) = f_{xx}(0,0) = \ldots = f_{x^{2n}}(0,0) = 0, f_{x^{2n+1}}(0,0) \neq 0$
- **ND3.** $f_{\mu}(0,0) = f_{\mu\mu}(0,0) = \ldots = f_{\mu^{2n}}(0,0) = 0, f_{\mu^{2n+1}}(0,0) \neq 0$
- **ND4.** $f_{x\mu}(0,0) = f_{x\mu\mu}(0,0) = \ldots = f_{x^{2n}\mu}(0,0) = 0, f_{x^{2n+1}\mu}(0,0) \neq 0$

For each bifurcation we need only two conditions, the first is a nondegeneracy condition (ND1, ND2 or PD1) and the second is a transversality condition (ND3 or ND4). The bifurcation part (i)-(iv) of Theorem 4.1 is proven in [2], but the part about box dimension is a new result.

**Theorem 4.1. Box dimension of one-parameter bifurcations**

Let $f : (-r, r) \times I \rightarrow \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an open interval, be a uniparameter family of $C^{r+1}$ maps.

(i) Suppose that the above family has at $\mu_0 = 0$ the fixed point $x_0 = 0, f_x(0,0) = 1$
and \( s = \max\{2n, 2m+1\} \). If the family satisfies the nondegeneracy conditions \( ND1 \) and \( ND3 \), then it undergoes a saddle-node bifurcation.

(iii) Suppose that the above family has the fixed point \( x_0 = 0 \) for all \( \mu \), \( f_\mu(0,0) = 1 \) and \( s = \max\{2n, 2m+2\} \). If the family satisfies the nondegeneracy conditions \( ND1 \) and \( ND4 \), then it undergoes a transcritical bifurcation.

(iv) Suppose that the above family has at \( \mu_0 = 0 \) the fixed point \( x_0 = 0 \), \( f_\mu(0,0) = -1 \) and \( s = \max\{2n+1, 2m+2\} \). If the family satisfies the nondegeneracy conditions \( PD1 \) and \( ND4 \), then it undergoes a period doubling bifurcation.

Then there is \( r_1 > 0 \) such that for the sequence \( S(x_1) = (x_k)_{k \geq 1} \) defined by \( x_{k+1} = f(x_k,0) \) (\( x_0 \) stable+) or \( x_{k+1} = f^{-1}(x_k,0) \) (\( x_0 \) unstable+), \( x_1 \in (0,r_1) \), we have

- for (i) and (ii): \(|x_k| \simeq k^{-\frac{1}{2n}}, \text{ as } k \to \infty \text{ and } \dim_B S(x_1) = 1 - \frac{1}{2n}\).

- for (iii) and (iv): \(|x_k| \simeq k^{-\frac{1}{2n}}, \text{ as } k \to \infty \text{ and } \dim_B S(x_1) = 1 - \frac{1}{2n+2}\).

Moreover, set \( S(x_1) \) is Minkowski nondegenerate.

Proof. The proofs of claims (i)-(iv) can be found as Theorems 1-4 in [2].

Now we look at the box dimension for the cases (i) and (ii). Notice that the conditions \( f(0,0) = 0 \), \( f_\mu(0,0) = 1 \) and \( ND1 \) are, in fact, the assumptions of Theorem 2.2 and Corollary 1 with \( F(x) = f(x,0) \) i \( \alpha = 2n \), so the box dimensions for a saddle-node and transcritical bifurcation follow from that theorem, its corollary and their remarks.

Concerning a pitchfork bifurcation (case (iii)), the function \( f \) is odd in a variable \( x \), so we get that \( f(0,0) = 0 \) and that all even derivatives in \((0,0)\) are 0. Now together with the conditions \( f_\mu(0,0) = 1 \) and \( ND2 \) the assumptions of Theorem 2.2 and Corollary 1 are satisfied with \( F(x) = f(x,0) \) and \( \alpha = 2n+1 \). So we obtain the claim for the box dimension of a pitchfork bifurcation.

Concerning the period doubling bifurcation (case (iv)), from the conditions \( f(0,0) = 0 \), \( f_\mu(0,0) = -1 \) and \( PD1 \) it follows that the function \( F(x) = f^2(x,0) \) satisfies the conditions of Theorem 2.2 and Corollary 1 with \( \alpha = 2n+1 \), so the box dimension of the sequence \( x_{k+1} = F(x_k), x_1 \in (0,r_1) \) is \( \dim_B S_2(x_1) = 1 - \frac{1}{2n+2} \). But we know that the sequence \( x_{k+1} = f(x_k,0), x_1 \in (0,r_1) \) is, in fact, the union of \( x_{k+1} = F(x_k), x_1 \in (0,r_1) \) and \( x_{k+1} = F(x_k), x_1 = f(x_1) \in (-r_1,0) \). Analogously as in the proof of Theorem 2.5 we get the needed result.

Remark 14. Notice that the box dimension depends only on the first nondegeneracy condition.

Remark 15. Notice that if the nondegeneracy condition has the lowest possible degree for which we have the bifurcation \((n = 1)\), then that degree represents the maximum possible fixed or periodic points which can occur by bifurcation. Theorem 4.1 shows that nondegeneracy conditions with the higher order degeneracies give the larger box dimension, but the bifurcation itself does not change. We know that in order to increase the number of possible fixed or periodic points, we must increase the number of parameters and, as a consequence, change the transversality condition.

Example 1. Saddle-node bifurcation \((m = 0, n = 1)\)

The family \( f(x,\mu) = \mu + x - x^2 \) undergoes a saddle-node bifurcation at the bifurcation value \( \mu = 0 \) in the fixed point \( x_0 = 0 \). For \( \mu < 0 \) there is no fixed point (Figure
1a), but at the bifurcation point \( \mu_0 = 0 \), the map is \( f(x,0) = x - x^2 \) and the fixed point \( x_0 = 0 \) is a nonhyperbolic point with \( f'(0,0) = 1 \). Then by Theorem 4.1, we have \( \dim_B S(x_1) = \frac{1}{2} \), where \( S(x_1) \) is a sequence defined by \( x_{n+1} = f(x_n, 0) \), with \( x_1 \) is near \( x_0 = 0 \) (Figure 1b).

**Example 2. Pitchfork bifurcation** \((m = 0, n = 1)\)

The family \( f(x, \mu) = \mu x + x - x^3 \) undergoes a pitchfork bifurcation in \( x_0 = 0 \) at \( \mu = 0 \). For \( \mu < 0 x_0 = 0 \) is a hyperbolic fixed point and \( \dim_B S(x_1) = 0 \) (Figure 2a). But at the bifurcation point \( \mu_0 = 0 \), the map \( f(x,0) = x - x^3 \) has a nonhyperbolic fixed point \( x_0 = 0 \) with \( f'(0,0) = 1 \), and by Theorem 4.1 we have \( \dim_B S(x_1) = \frac{2}{3} \) (Figure 2b).

**Example 3. Period doubling bifurcation** \((m = 0, n = 1)\)

The family \( f(x, \mu) = -(1 + \mu)x + x^3 \) undergoes a period doubling bifurcation in \( x_0 = 0 \) at \( \mu = 0 \). For \( \mu < 0 x_0 = 0 \) is a hyperbolic fixed point and \( \dim_B S(x_1) = 0 \) (Figure 3a). At the bifurcation point \( \mu_0 = 0 \), the map \( f(x,0) = -x + x^3 \) has a nonhyperbolic fixed point \( x_0 = 0 \) with \( f'(0,0) = -1 \). Now by Theorem 4.1 we have \( \dim_B S(x_1) = \frac{2}{3} \) (Figure 3b).

4.2. **Two-parameter bifurcations.** Let us consider two-parameter families of one-dimensional maps \( f : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}, \)

\[ f_{(\mu)}(x) = f(x, \mu), \quad \mu = (\mu_1, \mu_2). \]

For details see [6] and [14]. The two-parameter bifurcations which occur for one-dimensional discrete dynamical systems are the cusp bifurcation and the period
doubling of codimension two. In [3] there are results for these bifurcations with the higher degree conditions which we put in the next theorem as A and B.

**Theorem 4.2. Box dimension of two-parameter bifurcations**  
Suppose that a two-parameter family $f : (-r, r) \times \mathbb{R}^2 \to \mathbb{R}$, of $C^{2n+2}$ maps, has at $\mu_0 = (0, 0)$ the fixed point $x_0 = 0$.

**A.** Let $f_x(0, 0, 0) = 1$. Assume that the family satisfies the following nondegeneracy conditions:

- $A_1$: $f_{xx}(0, 0, 0) = \ldots = f_{x^{2n}}(0, 0, 0) = 0, f_{x^{2n+1}}(0, 0, 0) \neq 0$
- $A_2$: $(f_{\mu_1} f_{x^{\mu_2}} - f_{x^{2n}}(0, 0, 0) \neq 0$

Then the family undergoes a period doubling bifurcation of codimension two. Furthermore, then there exists $r_1 > 0$ such that for the sequence $S(x_1) = (x_k)_{k \geq 1}$ defined by $x_{k+1} = f(x_k, 0, 0) \ (x_0 \text{ stable})$ or $x_{k+1} = f^{-1}(x_k, 0, 0) \ (x_0 \text{ unstable})$, $x_1 \in (0, r_1)$, we have $|x_k - x_0| \sim k^{-\frac{1}{2n}}$ as $k \to \infty$ and $\dim B S(x_1) = 1 - \frac{1}{2n+1}$.

Moreover, set $S(x_1)$ is Minkowski nondegenerate.

**Proof.** For the proofs of cases A. and B. see Theorems 1 and 3 in [3].

**Part A.** We see that the conditions $f(0, 0, 0) = 0, f_x(0, 0, 0) = 1$ and A1 are the same as the assumptions of Theorem 2.2 and Corollary 1 with $F(x) = f(x, 0, 0)$ and $\alpha = 2n+1$. Hence, the claims about the box dimension and asymptotics of a cusp bifurcation follow from Theorem 2.2, Corollary 1 and Remarks 1-4.

**Part B.** The conditions $f(0, 0, 0) = 0, f_x(0, 0, 0) = -1$ and B1 mean that the function $f^2(x, 0, 0)$ satisfies the conditions of Theorem 2.2 and Corollary 1 with $F(x) = f^2(x, 0, 0)$ and $\alpha = 2n+1$. So the box dimension of sequence $x_{k+1} = f(x_n), x_1 \in (0, r_1)$ is equal to $\dim B S_2(x_1) = 1 - \frac{1}{2n+1}$. But we know that the sequence $x_{k+1} = f(x_n, 0, 0), x_1 \in (0, r_1)$ is the union of two subsequences $x_{k+1} = f(x_k), x_1 \in (0, r_1)$ and $x_{k+1} = F(x_n), x_1 = f(x_1) \in (-r_1, 0)$. Now analogously as in the proof of Theorem 2.5 we get the result for the generalized period doubling bifurcation. \hfill \Box

**Example 4 Cusp bifurcation** ($n = 2$):  
The family $f(x, \mu_1, \mu_2) = x + \mu_1 + \mu_2 x - x^5$ undergoes a cusp bifurcation at the bifurcation value $\mu_0 = (0, 0)$ in the fixed point $x_0 = 0$. At the bifurcation point $\mu_0 =$
(0,0), the map is \( f(x,0,0) = x - x^5 \) and the fixed point \( x_0 = 0 \) is a nonhyperbolic point with \( f'(0,0,0) = 1 \). Then by Theorem 4.2, we have \( \dim_B S(x_1) = \frac{4}{5} \), where \( S(x_1) \) is a sequence defined by \( x_{n+1} = f(x_n,0,0) \) and \( x_1 \) is near \( x_0 = 0 \). At Figures 4a and 4b, you can ‘see’ the change of box dimension.

\[
\begin{align*}
(\alpha) \ & \mu_1 = 0, \mu_2 > 0, \quad \dim_B S(x_1) = 0 \\
(\beta) \ & (\mu_1,\mu_2) = (0,0), \quad \dim_B S(x_1) = \frac{4}{5}
\end{align*}
\]

**Figure 4. Cusp bifurcation**

**Example 5 Generalized period doubling bifurcation** (\( n = 3 \)):

The family \( f(x,\mu_1,\mu_2) = -(1 + \mu_1)x + \mu_2x^3 + x^7 \) undergoes a period doubling bifurcation in \( x_0 = 0 \) at \( \mu_0 = (0,0) \). At the bifurcation point \( \mu_0 = (0,0) \), the map \( f(x,0,0) = -x + x^7 \) has a nonhyperbolic fixed point \( x_0 = 0 \) with \( f'(0,0) = -1 \). Now by Theorem 4.2 we have \( \dim_B S(x_1) = \frac{6}{7} \). See Figures 5a and 5b. Notice that the difference between the dimensions \( \frac{4}{5} \) (Figure 4b) and \( \frac{6}{7} \) (Figure 5b) is too small to be visible.

\[
\begin{align*}
(\alpha) \ & \mu_1 > 0, \mu_2 = 0, \quad \dim_B S(x_1) = 0 \\
(\beta) \ & (\mu_1,\mu_2) = (0,0), \quad \dim_B S(x_1) = \frac{6}{7}
\end{align*}
\]

**Figure 5. Generalized period doubling bifurcation**

**Remark 16.** All the results for bifurcations of fixed point can be applied to the periodic points by considering the relevant iterate of the map. The result for box dimensions will also be the same because of the finite stability of box dimension.

5. **Application to continuous planar dynamical systems.** Our main motivation for studying the one-dimensional discrete dynamical systems is their connection with continuous planar dynamical systems. Namely, we know that the Poincaré map of planar continuous systems generates one-dimensional discrete dynamical system.
In this section we apply the results from the previous sections to planar continuous systems with weak focus or limit cycle. This approach is also taken in the article [17], but here we give the reverse result. For details about the continuous dynamical systems, weak focus, limit cycles and their bifurcations see e.g. [12].

5.1. **Box dimension and multiplicity of weak focus.** First, we should remember the result from the article [17] in which they study the connection between the box dimension of a spiral trajectory and the Poincaré map.

We consider analytic planar dynamical systems with weak focus in the origin:

\[
\begin{align*}
\dot{x} &= -y + p(x, y) \\
\dot{y} &= x + q(x, y)
\end{align*}
\]  

where \(p(x, y)\) and \(q(x, y)\) are analytic functions such that

\[
\begin{align*}
p(x, y) &= \sum_{k=2}^{\infty} p_k(x, y) \\
q(x, y) &= \sum_{k=2}^{\infty} q_k(x, y)
\end{align*}
\]

with \(p_k\) and \(q_k\) homogeneous polynomials of order \(k\). The Poincaré map for (8) around focus is given by

\[
P(r_0) = r(2\pi, r_0) = r_0 + \sum_{k=2}^{\infty} u_k(2\pi)r_0^k.
\]

Coefficients \(u_k(2\pi), k \geq 2\) are called the Lyapunov coefficients for the weak focus. We denote by \(V_k\) the first Lyapunov coefficient different from zero. It is known that \(k\) is always odd, and the integer \(m = \frac{k-1}{2}\) is called the multiplicity of a focus. If \(m > 0\) this focus is a weak focus. Notice that \(P^{(k)}(0) = (k)!V_k\). So we see that the first Lyapunov coefficient different from 0 is \(V_k\) if and only if Poincaré map is a \(k\)-nondegenerate map in \(x_0\).

**Theorem 5.1.** ([17], Theorem 7) Let \(\Gamma\) be a spiral trajectory near the origin of planar system (8). If \(V_{2k+1}\) is the first Lyapunov coefficient different from 0, then

\[
\dim_B \Gamma = 2\left(1 - \frac{1}{2k+1}\right).
\]

Moreover, \(\Gamma\) is Minkowski nondegenerate.

Notice that in Theorem 5.1 the number \(k\) is a multiplicity of a weak focus of (8). The next theorem is reverse of Theorem 5.1 since the multiplicity of a weak focus follows from the box dimension.

**Theorem 5.2. (Multiplicity of a weak focus)** Let \(\Gamma\) be a spiral trajectory near the origin of analytic systems (8). Let \(P : [0, r) \to \mathbb{R}\) be the Poincaré map associated to \(\Gamma\) near the origin. Let the sequence \(S(x_1) = (x_n)_{n \geq 1}\) defined by \(x_{n+1} = P(x_n)\) (stable focus) or \(x_{n+1} = P^{-1}(x_n)\) (unstable focus), \(x_1 \in (0, r)\) has box dimension of the form \(\dim_B S(x_1) = 1 - \frac{1}{\alpha}\), with \(\alpha \geq 3\) odd. Then

\[
P''(0) = \ldots = P^{(\alpha-1)}(0) = 0, P^{(\alpha)}(0) \neq 0,
\]

that is, the origin is a weak focus with multiplicity \(m = \frac{\alpha-1}{2}\).

**Proof.** We know that the Poincaré map of an analytic system is also analytic near a focus. Since \(\dim_B S(x_1) > 0\), then \(x_0\) is a nonhyperbolic fixed point of \(P\) and \(P'(x_0) = 1\). Because of the positive box dimension and analyticity, \(P\) must be a finitely nondegenerate function in \(x_0\). Then from the box dimension and Theorem
Remark 17. If \( x_0 \) is a strong or hyperbolic focus, then \( x_0 \) is also hyperbolic fixed point of the Poincaré map, so \( \dim B \{ P(x_n) \} = 0 \). If \( x_0 \) is a center, then the sequence \( \{ P(x_n) \} \) has only one point and the box dimension is also 0.

Let \( \Gamma \) be a spiral trajectory near weak focus of analytic planar system (8), and let \( P \) be the associated Poincaré map. We denote by \( \{ x \} \) the sequence defined by \( x_{n+1} = P(x_n) \) (for stable focus) or \( x_{n+1} = P^{-1}(x_n) \) (for unstable focus) with \( x_1 \) near weak focus. Then
\[
\dim B \Gamma = 2 \dim B Po(x_1).
\] (9)

This connection was presented in the article [17] but in different form. We use it in the proof of Theorem 5.5. Of course, it does not hold for strong (hyperbolic) focus or center, since then \( \dim B \Gamma = 1 \) and \( \dim B \{ P(x_n) \} = 0 \).

5.2. Box dimension and multiplicity of limit cycle. We consider continuous planar system
\[
\dot{x} = f(x),
\] (10)
with \( f \) analytic. Let \( \gamma \) be a limit cycle of (10). Let \( P \) be the Poincaré map near the limit cycle \( \Gamma \), and \( d \) is a displacement map \( d(s) = P(s) - s \). If \( P'(0) \neq 1 \), then \( \gamma \) is a hyperbolic limit cycle. The stability of a hyperbolic limit cycle is determined by the sign of \( d'(0) \), in the following way: if \( d'(0) < 0 \), then the cycle is stable, and if \( d'(0) > 0 \), it is unstable.

If \( d'(0) = 0 \), that means that \( P'(0) = 1 \). Then \( \gamma \) is a nonhyperbolic limit cycle, and \( x = 0 \) is a nonhyperbolic fixed point of a map \( P \). We see that the nonhyperbolic periodic orbit has an order of multiplicity at least 2.

Definition 5.3. Let \( P(s) \) be the Poincaré map for a cycle \( \gamma \) of a planar analytic system (10), and let \( d(s) = P(s) - s \) be the displacement function. Then if
\[
d(0) = d'(0) = \ldots = d^{(k-1)}(0) = 0, \quad d^{(k)}(0) \neq 0,
\]
then \( \gamma \) is called a multiple limit cycle of multiplicity \( k \). If \( k = 1 \), then \( \gamma \) is called a simple limit cycle.

Notice that \( d(0) = d'(0) = \ldots = d^{(k-1)}(0) = 0, \quad d^{(k)}(0) \neq 0 \), is equivalent to \( P'(0) = 1, \quad P''(0) = \ldots = P^{(k-1)}(0) = 0, \quad P^{(k)}(0) \neq 0 \). So if \( P \) is a \( k \)-nondegenerate map in \( x_0 = 0 \), then \( \gamma \) is a limit cycle with the multiplicity \( k \).

Let \( \Gamma \) be a spiral trajectory near limit cycle of planar vector field of class \( C^1 \). Let \( P \) be the associated Poincaré map. We denote by \( Po(x_1) = \{ x \} \) the sequence defined by
\[
x_{n+1} = \begin{cases} P(x_n), & \gamma \text{ stable} \\ P^{-1}(x_n), & \gamma \text{ unstable} \end{cases}
\] (11)
with \( x_1 \) near \( \gamma \). If \( \Gamma \) is a limit cycle spiral trajectory of power \( \beta > 0 \), then
\[
\dim B \Gamma = 1 + \dim B Po(x_1).
\] (12)
This connection is proved in the article [17], Theorem 4 but it is presented in a different form. We need this form for proving Theorem 5.5. In this case, the equation does not hold for hyperbolic limit cycle, because then \( \dim B \Gamma = 1 \) and \( \dim B \{ P(x_n) \} = 0 \).
The following theorem gives the reverse connection between the box dimension of spiral trajectory near limit cycle and the multiplicity of limit cycle from the one stated in the article [17].

**Theorem 5.4. (Multiplicity of limit cycle)**

Let $\Gamma$ be a spiral trajectory near limit cycle $\gamma$ of analytic planar vector field, and $P$ is the Poincaré map near limit cycle $\gamma$. Let us assume that there exist the decreasing sequence $S(x_1) = (x_n)_{n \geq 1}$ defined by (11) with $x_1 \in (0, r)$ which has box dimension of the form $\dim_B S(x_1) = 1 - \frac{1}{\alpha}, \alpha > 1$.

Then

$$P''(0) = \ldots = P^{(\alpha-1)}(0) = 0, \ P(\alpha)(0) \neq 0$$

that is, the multiplicity of limit cycle is equal to $\alpha$.

**Proof.** We know that the Poincaré map of analytic system is also analytic near limit cycle. It holds $\dim_B S(x_1) > 0$, so $x_0$ is a nonhyperbolic fixed point of $P$, and $P'(x_0) = 1$. The positive dimension and analyticity provide that $P$ is a finitely nondegenerate function in $x_0$. From the box dimension and Theorem 2.2, we obtain that $P$ is a $\alpha$-nondegenerate in $x_0 = 0$ and the limit cycle has multiplicity $\alpha$. $\square$

### 5.3. Bifurcation of nonhyperbolic periodic orbits.

We consider the one-parameter family of planar systems

$$\dot{x} = f(x, \mu) \tag{13}$$

where $f \in C^4(E \times J)$, where $E$ is an open subset of $\mathbb{R}^2$ and $J \subset \mathbb{R}$ is an interval. Let $\gamma_0$ be a periodic orbit of (13) at the bifurcation value $\mu_0 = 0$, and let $P(x, \mu)$ be the Poincaré map $\gamma_0$ of the system (13). By $DP$ we denote the partial derivative of $P$ with respect to the variable $x$. The periodic orbit $\gamma_0$ is nonhyperbolic at $\mu = \mu_0$ if $|DP(x_0, \mu_0)| = 1$, for $x_0 \in \gamma_0$, that is, $x_0$ is a nonhyperbolic fixed point of $P$. Because of this connection, the bifurcations of nonhyperbolic periodic orbit are determined by the bifurcations of nonhyperbolic fixed point of associated Poincaré map. Since the Poincaré map is a one-dimensional map, it generates a one-dimensional discrete dynamical systems. So the study of bifurcation of periodic orbit in the plane is reduced to study the bifurcation of one-dimensional discrete dynamical system. Therefore, this is a perfect example for the application of box dimension result from Section 4.

Hence, there are two possibilities: $DP(x_0, \mu_0) = 1$ and $DP(x_0, \mu_0) = -1$. In the case $DP(x_0, \mu_0) = 1$, we have a saddle-node, transcritical and pitchfork bifurcation. We know that the case $DP(x_0, \mu_0) = -1$ does not occur in the plane, but only in the higher dimensions. From the geometrical point of view it is impossible to have a period-doubling bifurcation of periodic orbit for a planar system since the trajectories do not cross in the plane. In the higher dimensions, there also exists another possibility, that is, $DP(x_0, \mu_0)$ has a pair of complex conjugate eigenvalues on the unit circle. Then generically $\gamma_0$ bifurcates to an invariant two-dimensional torus which corresponds to the Neimark Sacker bifurcation of Poincaré map.

So in the planar case there are only bifurcations with $DP(x_0, \mu_0) = 1$ which are presented as parts (i)-(iii) in Theorem 5.5. In fact, this parts are taken from [12], p.331, but the dimension part is a new result.

**Theorem 5.5.** Suppose that $f \in C^3(E \times J)$, where $E$ is an open subset of $\mathbb{R}^2$ and $J \subset \mathbb{R}$ is an interval. Assume that for $\mu = \mu_0$ the system (13) has a periodic orbit $\gamma_0 \in E$ and that $P(s, \mu)$ is the Poincaré map for $\gamma_0$ defined in a neighborhood
$N_\delta(0, \mu_0)$. Let $P(0, \mu_0) = 0$ and $DP(0, \mu_0) = 1$.

(i) If

$$D^2P(0, \mu_0) \neq 0, \quad P_\mu(0, \mu_0) \neq 0,$$

it follows that a saddle-node bifurcation occurs at the nonhyperbolic periodic orbit $\gamma_0$ at the bifurcation value $\mu = \mu_0$, i.e. depending on the signs of the expressions (14), there are no periodic orbit of (13) near $\gamma_0$ for $\mu < \mu_0$ (or $\mu > \mu_0$) and there are two periodic orbits of (13) near $\gamma_0$ for $\mu > \mu_0$ (or $\mu < \mu_0$). The two critical orbits are hyperbolic and of the opposite stability.

(ii) If $D^2P(0, \mu_0) = 0, \quad P_\mu(0, \mu_0) = 0$ and $DP_\mu(0, \mu_0) \neq 0$, then a transcritical bifurcation occurs at the nonhyperbolic periodic orbit $\gamma_0$ at the bifurcation value $\mu = \mu_0$.

(iii) If $D^2P(0, \mu_0) = 0$, $D^3P(0, \mu_0) \neq 0$, $P_\mu(0, \mu_0) = 0$ and $DP_\mu(0, \mu_0) \neq 0$ then a pitchfork bifurcation occurs at the nonhyperbolic periodic orbit $\gamma_0$ at the bifurcation value $\mu = \mu_0$.

Let $\Gamma$ be a spiral trajectory near the limit cycle $\gamma_0$ at the bifurcation point $\mu = \mu_0$. Then

- for (i) or (ii): $\dim_B \Gamma = \frac{3}{2}$.
- for (iii): $\dim_B \Gamma = \frac{5}{3}$.

Moreover, $\Gamma$ is Minkowski nondegenerate.

Proof. Denote by $Po(x_1) = (x_n)_{n \geq 1}$ the sequence defined by $x_{n+1} = P(x_n, \mu_0)$ for $\gamma$ stable, or $x_{n+1} = P^{-1}(x_n, \mu_0)$ for $\gamma$ unstable, with $x_1 \in (0, r)$ and $r$ sufficiently small. Then Theorem 4.1 for $n = 1$ and for cases (i) and (ii) gives $\dim_B Po(x_1) = \frac{1}{2}$, and for case (iii) $\dim_B Po(x_1) = \frac{5}{3}$. Notice that in the cases (i) and (ii), $\Gamma$ is a limit cycle spiral of power $\beta = 1$, and in the case (iii) of power $\beta = \frac{1}{2}$. Now using (12) we obtain the box dimensions: for the cases (i) and (ii) $\dim_B \Gamma = \frac{3}{2}$ and for (iii) $\dim_B \Gamma = \frac{5}{3}$. \hfill \Box

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