Contracting Maps and Scalar Curvature
Joachim Lohkamp

Mathematisches Institut, Universität Münster, Einsteinstrasse 62, Germany
e-mail: j.lohkamp@uni-muenster.de

1 Introduction

In these short notes we explain how to derive the following size estimate in scalar curvature geometry for contracting maps. A $C^1$-map $f : V \rightarrow W$ between Riemannian manifolds $V$ and $W$ is $\varepsilon$-contracting when $\|Df(v)\| \leq \varepsilon \cdot \|v\|$ for any tangent vector $v \in TV$. 1-contracting maps are called contracting maps.

Theorem (Contractions and Scalar Curvature) For $n \geq 2$, let $M^n$ be a compact manifold with boundary $\partial M^n$ and $f : (M^n, \partial M^n) \rightarrow (S^n, \{p\})$ a $C^1$-map of non-zero degree mapping a neighborhood of $\partial M^n$ to $p \in S^n$. Then, there is some $\sigma_n \geq 1$, that depends only on $n$, so that for any metric $g$ on $M^n$ making $f$ a contracting map:

\[(1) \quad \text{scal}(g)(z) \leq \sigma_n, \text{ in some point } z \in M^n,\]

where $\text{scal}(g)$ denotes the scalar curvature of the smooth Riemannian metric $g$.

This result was previously known in, what will henceforth be called, the classical cases where $M^n$ either admits a spin structure or it has dimension $n \leq 7$ [GL1], Ch.12. The dimensional constraint arises from the use of minimal hypersurfaces in the non-spin case. These hypersurfaces can carry difficult singularities, occurring in dimensions $\geq 8$, which could not be handled by classical means.

Our extension to the general case is a simple application of scalar curvature splitting theorems. (For the purposes of this paper the basic version derived in [L1], Th.1 and the naturality theory [L2], Th.3 are sufficient.) Roughly speaking, such splitting results shift arguments involving (potentially singular) minimal hypersurfaces back into an entirely smooth scenario.

Applications One may think of (1) as a version of the fundamental Jacobi field estimates along geodesics used in positive Ricci curvature geometry to derive classical results like the Bonnet-Myers theorem. Similarly, (1) implies largeness constraints, now for $\text{scal} > 0$-geometries. We give some basic examples. Recall, that a compact $n$-manifold is called enlargeable if for any $\varepsilon > 0$ there is an orientable covering admitting an $\varepsilon$-contracting map onto the sphere $S^n$, i.e. a $C^1$-map with $\|Df(v)\| \leq \varepsilon \cdot \|v\|$ for all tangent vectors $v$, constant at infinity and of non-zero degree. The Theorem implies extensions of the following result [GL1], Th.12.1, [GL2], Th.A, [LM], Th.5.5 from the classical to the
general case, without changes.

**Corollary 1 (Geometry of Enlargeable manifolds)** An enlargeable manifold cannot carry $\text{scal} > 0$-metrics. If it carries a $\text{scal} \geq 0$-metric, then is covered by a flat torus.

Since the torus $T^n$ is enlargeable, the connected sums $T^n \# N^n$ with any compact manifold $N^n$ are also enlargeable. Hence, we get the following observation one may paraphrase as follows: there is no general mechanism to locally deform a manifold increasing its scalar curvature, even under topological changes. By contrast, we can locally decrease the scalar curvature [L1].

**Corollary 2 (Non-Existence of $S > 0$-Islands)** There exists no complete Riemannian manifold $(M^n, g)$ such that:

- $\text{scal}(g) > 0$ on some non-empty open set $U \subset M^n$, with compact closure.
- $(M^n \setminus U, g)$ is isometric to $(\mathbb{R}^n \setminus B_1(0), g_{\text{Eucl}})$.

It is well-known, from [L3], that this result is equivalent to the Riemannian version of the positive mass theorem.

**Corollary 3 (Positive Mass Theorem)** Let $(M^n, g)$ be asymptotically flat of order $\tau > \frac{n-2}{2}$. That is, there is a decomposition $M = M_0 \cup M_\infty$, with $M_0$ compact, and $M_\infty$, the end of $M$, diffeomorphic to $\mathbb{R}^n \setminus B_R(0)$, so that this diffeomorphism defines coordinates $\{x^i\}$ on $M_\infty$, with $g_{ij} = \delta_{ij} + O(|x|^{-\tau})$, $\frac{\partial g_{ij}}{\partial x_k} = O(|x|^{-\tau-1})$, $\frac{\partial^2 g_{ij}}{\partial x_k \partial x_l} = O(|x|^{-\tau-2})$. If $\text{scal}(g) \geq 0$, then the total mass

$$E(M, g) := \frac{1}{\text{Vol}(S^{n-1})} \lim_{R \to \infty} \int_{\partial B_R} \sum_{i,j} \left( \frac{\partial g_{ij}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_j} \right) \cdot \nu_j \, dV_{n-1},$$

where $\nu = (\nu_1 \ldots \nu_n)$ is the outer normal vector to $\partial B_R$, is non-negative. Moreover, $E(M, g) = 0$ if and only if $(M, g)$ is isometric to $(\mathbb{R}^n, g_{\text{Eucl}})$.

In a more topological direction, we recall a conjecture related to the Borel conjecture saying that a manifold representing a non-trivial homology class in an aspherical manifold cannot carry any $S > 0$-metric. This has been settled in the classical cases cf.[GL1], Th.13.8. As before, these methods immediately extend to the general case.

**Corollary 4 (Homology of Sec $\leq 0$-Manifolds)** Let $M^n$ be a compact manifold which admits a sec $\leq 0$-metric and $N^k \subset M^n$ be a submanifold with $[N] \neq 0$ in $H_k(M, \mathbb{Q})$. Then $N^k$ does not admit $\text{scal} > 0$-metrics.

**Remark 1.** The present paper gives a more topological argument for the Riemannian positive mass theorem than the one given in [L4]. Moreover, these methods allow us to simplify our proof of the space-time positive mass theorem [L5] without prior reduction to the Riemannian case or, stated differently, we can cover the Riemannian and the space-time case at once.
2. In recent preprints, [CS] and [SY], the authors announce rather different and more classically analytic arguments to address similar results. At any rate, we agree on the statement in [SY] that for such basic results it is valuable to have independent approaches. 3. The present paper is based on the potential theory on singular minimal hypersurfaces and their hyperbolic unfoldings [L2]. We use splitting techniques from [L1] and we assume some basic understanding of these methods and their terminology.

As already noted, the corollaries are well-known consequences of our main theorem discussed, for instance, in [GL1] and [L3] and we refer to these references for further details. The rest of this paper is devoted to the proof of the Theorem.

2 The General Setup

We start with a compact Riemannian manifold \((M^n, g) = (M^n(\sigma), g(\sigma))\) with boundary \(\partial M^n\), so that \(S(g) \geq \sigma\), for some \(\sigma > 0\) and some contracting map \(f : (M^n, \partial M^n) \rightarrow (S^n, \{p\})\) of non-zero degree mapping a neighborhood of \(\partial M^n\) to \(p \in S^n\). We show inductively that this implies the existence of the following structures in all dimensions \(i\), with \(3 \leq i \leq n\).

We get three series of spaces \(M^i\), \(P^i\) and \(\partial M^i\) of dimension \(i\) so that \(\partial M^i = \partial M^i\), \(\partial M^i \subset \partial M^{i+1}\), \(M^{i-1} \supset \ldots M^i \supset \ldots M^3 \) and \(M^{i-1} \supset \ldots M^{i+1} \supset \ldots M^3\). They are defined in a way that resembles exact sequences in homological algebra.

\[ M^n \supset P^n \supset P^{n-1} \supset P^{n-1} \supset \ldots \supset P^i \supset M^i \supset \ldots \supset P^3 \supset M^3 \]

where, in our case, \(A \supset B \supset C\) means that \(C\) is determined from \(A\) and \(B\).

In dimension \(n\) we choose \(M_n^i = M^n\), \(f_n = f\) and \(\varphi_n = 1\). For \(i \leq n - 1\), the \(M^i\) are geometrically regularized subsets of, in general, singular Plateau solutions \(P^i\) defined in \(M^{i+1}\). In turn, the \(M^i\) are smooth extensions of the \(M^i\) homologous to the Plateau solution \(P^i\). They give mapping degrees for maps onto the sphere \(S^n\) a proper sense but there is no control for the curvature on \(M^i \setminus M^i\). In turn, to define the \(M^i\) we apply the splitting theorem, [L1], Th.1.

Definition of \(M^{i-1}\), \(M^{i-1}\) and \(P^{i-1}\): Let \(D^{i-1} = \{(x_1, ..., x_i, 0) \in S^i : x_i \geq 0\}\) the lower hemisphere \(D^{i-1} \subset S^i\) of a geodesic subsphere for \(S^i \subset \mathbb{R}^{i+1}\) with boundary \(\partial D^{i-1} = S^{i-2}\), where the boundary \(\partial D^{i-1}\) can be assumed to be very close to \(p_i\), but \(p_i \notin D^{i-1}\) and \(f_i\) is transversal to \(D^{i-1}\) and \(\partial D^{i-1}\). We get an area minimizing Plateau solution \(P^{i-1}\) homologous to \(f^{i-1}(D^{i-1})\) in \(M^i\) for the metric \(\varphi_i g|_{M^i}\) and with the boundary \(\partial f^{i-1}(S^{i-2})\) which is disjoint to \(\partial M^i \setminus \partial M^i\). Due to standard boundary regularity results we know that near the smooth boundary \(\partial f^{i-1}(S^{i-2})\) the solution \(P^{i-1}\) is smooth. Then, and this is where we use the conformal splitting theorem, [L1], Th.1, we can modify \(P^{i-1}\) to define \(M^{i-1}\) and, from this, the extension \(M^{i-1}\) homologous to \(P^{i-1}\). These details are explained in the following chapter.
Definition of $f_{i-1}$: We consider the contracting spherical crushing map $c_i : S^i \to D^{i-1}$ given by $c_i((x_1, ..., x_{i+1})) = (x_1, ..., x_{i+1}, \sqrt{x_1^2 + x_{i+1}^2}, 0)$, for $(x_1, ..., x_{i+1}) \in S^i \subset \mathbb{R}^{i+1}$. We compose $c_i$ with a map $d_i : D^{i-1} \to S^{i-1}$, which maps the boundary $\partial D^{i-1}$ and $c(p_i)$ to a point $p_{i-1} \in S^{i-1}$ to define a 4-contracting map $h_i := d_i \circ c_i : S^i \to S^{i-1}$. From the construction of the map $f_{i-1}$ it follows that these new boundary components are all mapped onto the point $p_{i-1} \in S^{i-1}$. We define $f_{i-1} := h_i \circ f_{i-1}|_{M^{i-1}}$. It follows that $f_{i-1}$ is $a_{i-1}$-contracting, where $a_i$ only depends on $i$. We have $\deg(f_{i-1}) \neq 0$ since $\deg(h_i|_{D_{i-1}}) \neq 0$, $\deg(f_i) \neq 0$ and $N$ being homologous to $f_{i-1}(D^{i-1})$.

Definition of $\varphi_{i-1}$: There are smooth functions $\varphi_{i-1} > 0$ on the cores $M^{i-1}_\ominus$ with $\varphi_{i-1} \cdot S(\varphi_{i-1} \cdot g_{M^{i-1}}) \geq \sigma$ and the $M^i_\ominus$ are regular subsets of the generally singular Plateau solution $P^{i-1}$ with $\partial P^{i-1} = f_{i-1}(S^{i-2})$ relative the metric $\varphi_{i+1} \cdot g_{M^{i+1}}$. The definition of the $\varphi_{i-1}$ is related to that of $M_i$ and it will be postponed to chapter 4. The weaker condition $\varphi_i \cdot S(\varphi_i \cdot g_{M^i}) \geq \sigma$, compared to $S(g_{M^n}) \geq \sigma$, has the advantage to survive the dimensional descent we use here.

Final Conclusion and Summary: We run this process until we reach dimension 3. A simple adjustment allows us to continue to dimension 1 where we argue using ordinary differential inequalities to find an upper bound for $\sigma$ depending only on $n$. The overall setup deviates from the classical one in [GL1], Ch. 12 in several ways. We incorporate singular Plateau solutions $P^i$ and regularize $P^i$, in a geometric step, to $M^{i}_\ominus$ and, in a topological one, to $M^i$. The step from $P^i$ to $M^{i}_\ominus$ is not compatible with the symmetrization procedure in [GL1], Ch. 12 and we replace it, even in low dimensions, for a direct conformal deformation.

Henceforth, we denote the collection of the $i$ dimensional data $M^{i}_\ominus$, $M^i$ and $P^i$ with the maps $f_i$ and $\varphi_i$ by $C_i$ and the transition from $C_i$ to $C_{i-1}$ by $\triangleright$. 

3 Conformal Splittings and $C_i \triangleright C_{i-1}$

We start with the basic step $C_n \triangleright C_{n-1}$. For the given compact Riemannian manifold $(M^n, g)$ with $S(g) \geq \sigma$ and the contracting map $f : (M^n, \partial M^n) \to (S^n, \{p\})$ we choose a lower hemisphere $D^{n-1} \subset S^n$ with boundary $\partial D^{n-1} = S^{n-2}$, so that $\partial D^{n-1}$ is close to $p$, but $p \notin D^{n-1}$ and we may assume that $f$ is transversal to $D^n$ and $\partial D^n$.

We get an area minimizing Plateau solution $P^{n-1}$ homologous to $f^{-1}(D^{n-1})$ in $M^n$ with the boundary $f^{-1}(S^{n-2})$. It may happen that $P^{n-1}$ hits $\partial M$, but this does not cause problems since $f_{n-1}$ maps also a neighborhood of $P^{n-1} \cap \partial M$ the new basepoint $p_{n-1}$. Thus, in our present argument we can assume that $P^{n-1} \cap \partial M \subset \partial P^{n-1}$. (In these places the regularity of $P$ is not needed.) Since $P^{n-1}$ is area minimizing we have $A''(f) := A_{\nu}(f \cdot \nu) \geq 0$ for any smooth infinitesimal variation supported away from the singular set $\Sigma P \subset P$ and the boundary $f \cdot \nu$ of $P \setminus (\Sigma P \cup \partial P)$, where $\nu$ is a unit normal vector field, (we may assume $M$ is orientable). A direct computation shows:

$$A''(f) = \int_P |\nabla_f f|^2 - f^2(|A|^2 + Ric_M(\nu, \nu))dA \geq 0 \iff$$
\[
\int_P |\nabla f|^2 + \frac{n - 2}{4(n - 1)}(\text{scal}_P - \text{scal}_M)f^2 dA \geq \int_P \frac{n}{2(n - 1)}|\nabla f|^2 + \frac{n - 2}{4(n - 1)}f^2 |A|^2 dA.
\]

Thus for a Hardy $S$-transform $\langle A \rangle$ we get

\[
\int_P |\nabla f|^2 + \frac{n - 2}{4(n - 1)}(\text{scal}_P - \text{scal}_M)f^2 dA \geq \lambda_{\langle A \rangle} \cdot \int_P \langle A \rangle^2 \cdot f^2 dA,
\]

for some (largest) constant $\lambda_{\langle A \rangle} > 0$. In particular we get on $P \setminus \Sigma_P$ the \textit{ground state} for the $S$-adapted Schrödinger operator $L(P) := -\frac{4(n - 1)}{n - 2} \Delta + \text{scal}(g|_P) - \text{scal}_M$, that is, a smooth function $\psi > 0$ with $\psi \equiv 0$ on $\partial P$ and with

\[
- \frac{4(n - 1)}{n - 2} \Delta \psi + \text{scal}(g|_P)\psi - \text{scal}_M\psi = \lambda_{\langle A \rangle} \langle A \rangle^2 \psi
\]

This observation and growth estimates for such functions $\psi$ from [L2], Th.6 are the basis for the following result which is a version of [L1], Th.1 in the case $i = n - 1$. The cases $3 \leq i < n - 1$, where we first need to define appropriate Plateau solutions will be proved inductively.

**Proposition 3.1 (Conformal Splitting in Dimension $i$)** For any $\varepsilon > 0$, there are smooth functions $\lambda_\varepsilon, \phi_{\varepsilon i} > 0$ on $P^i \setminus \Sigma_{P^i}$ with $\phi_{\varepsilon i} \equiv 0$ on $\partial P^i$, with $c_{P^i} \leq \lambda_\varepsilon \leq C_{P^i}$, for some constants $c_{P^i}, C_{P^i} > 0$ and a neighborhood $U_\varepsilon$ of $\Sigma_{P^i}$, within an $\varepsilon$-neighborhood of $\Sigma_{P^i}$ so that

\[
L_\varepsilon := -\frac{4(i - 1)}{i - 2} \Delta + \text{scal}(g|_{P^i}) - \text{scal}_M - \lambda_\varepsilon \cdot \langle A \rangle^2 \text{ is } S\text{-adapted and } L_\varepsilon \phi_{\varepsilon i} = 0
\]

and $M^i_\varepsilon = M^i_{\varepsilon}((P^i \setminus U_\varepsilon, \phi_{\varepsilon i}^{4/i - 2}, g_{P^i})$ is a scal $> 0$-manifold with (locally) area minimizing boundary $\partial M^i_\varepsilon$. Moreover, there is a smooth manifold $M^i$ with boundary $\partial P^i$ with $M^i_\varepsilon \subset M^i$ homologous to $P^i$.

**Remark 3.2** In the following argument $\varepsilon > 0$ is a parameter we gradually shrink to a tiny value. For $i \leq 7$, we choose $U_\varepsilon = \emptyset$ and $M^i = M^i_\varepsilon = P^i$. For $i \geq 8$, the extension from $M^i_\varepsilon$ to $M^i$ is the relative version of the general observation that for any closed manifold $X^n$ one has $H_{n - 1}(X, \mathbb{Z}) \cong H^1(X, \mathbb{Z}) \cong [X, S^1]$. Namely, for any $\alpha \in H_{n - 1}(X, \mathbb{Z})$ and any smooth map $f : X \to S^1$ that represents the homotopy class in $[X, S^1]$ associated to $\alpha$, the smooth preimage $f^{-1}(c)$ of some generic $c \in S^1$ represents $\alpha$. \hfill \Box

For the induction step $C_\varepsilon \supset C_{\varepsilon i - 1}$, $i < n$, we start from $M^i$. We observe that $[3.1]$ does not control for the curvatures on $M^i \setminus M^i_{\varepsilon}$. However, the area minimizing Plateau solution $P^{i - 1}$ homologous to $f_{i - 1}(D^{i - 1})$ in $M^i$ with the boundary $f_{i - 1}^{-1}(S^{i - 2})$ may intersect this subset. We use the minimality of $\partial M^i_{\varepsilon}$ to rebuild $P^{i - 1}$ to bypass $M^i \setminus M^i_{\varepsilon}$ and to define the next two spaces $M^{i - 1}_{\varepsilon}$ and $M^{i - 1}$.

To this end, we may assume, after selecting one of its components, that $\partial M^i_{\varepsilon}$ is connected. For small $\varepsilon > 0$, we may assume that $\partial P^{i - 1} \cap \partial M^i_{\varepsilon} = \emptyset$. If $P^{i - 1} \cap \partial M^i_{\varepsilon} \neq \emptyset$ the
intersection is transversal in all regular points due to the strict maximum principle and there are two cases where $\partial M_i^i$ plays the role of an either repelling or attracting end:

**Repelling Ends** $\partial P^{i-1}_i \cap \partial M_i^i = \partial U$ for some open subset $U \subset \partial M_i^i$. Then we can replace $(P^{i-1}_i \setminus (M^i_i \setminus M^i_i)) \cup U$ for some other Plateau solution $Q^{i-1}_i \cap \partial M_i^i = \emptyset$ again using the strict maximum principle.

**Attracting Ends** Otherwise, we may assume that $\partial P^{i-1}_i \cap \partial M_i^i$ locally separates $\partial M_i^i$ but $\partial M_i^i \setminus (P^{i-1}_i \cap \partial M_i^i)$ remains connected. Then, for small $\varepsilon > 0$ and any $m \in \mathbb{Z}^\geq 1$, we get, using the strict maximum principle another time:

- an at least locally area minimizing $Q^{i-1}_m$ with two (not necessarily connected) disjoint boundary components $\partial Q^{i-1}_m[A] := \partial P^{i-1}_i$ and the far end $\partial Q^{i-1}_m[B] := P^{i-1}_i \cap \partial M_i^i$.
- $[Q^{i-1}_m]$ is homologous to $[P^{i-1}_i \cap M_i^i] + m \cdot [(\partial M_i^i \setminus (P^{i-1}_i \cap \partial M_i^i))']$ in $H_{n-1}(M^i_i, \partial M_i^i; \mathbb{Z})$, where $\sim$ means the non-embedded one-sided closure of $\partial M_i^i \setminus (P^{i-1}_i \cap \partial M_i^i)$.

One may compare this case with that of a geodesic ray on a surface approaching a closed geodesic in a gradually narrowing spiral.

We observe, for $\varepsilon > 0$ small and $m$ large enough, starting and counting backwards from $\partial Q^{i-1}_m[B]$, the hypersurface $Q^{i-1}_m$ can be chopped into $m$ pieces, we choose $m$ even, $V_{m,k} \subset Q^{i-1}_m$, $k = 1, \ldots, m$, so that viewed as currents

$$V_{m,k} \to \partial M_i^i \setminus (P^{i-1}_i \cap \partial M_i^i) \text{ for } k \leq m/2, m \to \infty, \text{ in flat norm.} \tag{5}$$

We apply [3,1] to the possibly singular $\partial M_i^i$ in its $scal > 0$-ambient space $M_i^i$ and get, as an intermediate step, another conformal $scal > 0$-geometry written $N_i^i(\partial M_i^i)_i^{i-1}$ with area minimizing boundary $\partial N_i^{i-2}$ to some intermediate smooth manifold $N_i^{i-1}$.

Next we also apply the splitting result to $Q^{i-1}_m$ and notice from the use of the naturality of the underlying Martin theory for the family of Schrödinger operators $L(\cdot)$ form [L2], Th.3 to periodically define the same $scal > 0$-geometry and extensions for the $V_{m,k}$. We call the $scal > 0$-geometry $W^{i-1}_{m,k}$ with minimal border $\partial W^{i-1}_{m,k}$ and the smooth extension geometry $W^{i-1}_m$, we can define, from [3], to become asymptotically periodical for large $m$ and $k \leq m/2$. The limit for $m \to \infty$ is written $W^{i-1}_m$ and asymptotically periodical extension pieces we associate to the $V_{m,k}$ are labelled $WV_{m,k}$.

Next, for sufficiently large $m \gg 1$ we may assume that we have an area minimizing hypersurface $X^{i-2} \subset WV_{m,k-1} \cup \ldots \cup WV_{m,k-\ell} \subset Q^{i-1}_m$, for $k = 1/2 \cdot m, \ell = 1/4 \cdot m$, separating $WV_{m,k}$ from $WV_{m,k-\ell}$. This can easily be accomplished, e.g. applying a slight scaling of the pieces of $W^{i-1}_{m,k}$ towards infinity, which keeps the scalar curvature condition [7] below up to a factor $\in (1/2, 2)$.

We write $W^{i-1}_{m,k}$ for the component that contains $WV_{m,k}$. Again we are in the situation that $X^{i-2}$ may intersect $\partial W^{i-1}_{m,k}$ and reach a subset without controlled curvatures. However, in this case we are in an appended end piece of $Q^{i-1}_m$. Hence, for $\varepsilon > 0$ small
enough and large $m$, the boundary $\partial W^{i-1}_{m,X}$ is a repelling end. Thus we get other area minimizing boundaries $Y^{i-2} = Y^{i-2}(m, \varepsilon) \subset W^{i-1}_{m,X}$ with $Y^{i-2}(m, \varepsilon) \to X^{i-2}$ for $\varepsilon \to 0$ and $Y^{i-2}(m, \varepsilon) \to W^{i-1}_\circ$ for $m \to \infty$, in flat norm. We first choose a large $m$ and then shrink $\varepsilon$ to define the new $M^{i-1}_\circ$ as the component of $W^{i-1}_\circ$ that does not contain $X^{i-2}$. This reproduces $\text{(3)}$ in dimension $i - 1$ and from this we readily get the other components of $C_{i-1}$.

4 Recursive Conformal Deformations

Now we discuss the propagation of scalar curvature estimates while we run through the steps $C_i \triangleright C_{i-1}$. As before we start from be a compact manifold $M^n$ with boundary $\partial M^n$ and a contracting $C^1$-map $f : (M^n, \partial M^n) \to (S^n, \{p\})$ of non-zero degree, which maps a neighborhood of $\partial M$ to a point $p \in S^n$. The compactness gives some $\kappa \in \mathbb{R}$ so that $\text{scal}(g_M) \geq \kappa$ and our goal is to show that there is upper bound on the possible values of $\kappa$ for any such given $M$, $f$ depending only on the dimension $n$.

We recall the scalar curvature transformation law for an $k$-dimensional Riemannian manifold $(N^k, g_N)$, $k \geq 3$, under conformal deformations $u^{4/k-2} \cdot g_N$ for smooth $u > 0$:

$$\text{scal}(u^{4/k-2} \cdot g_N) \cdot u^{\frac{k-2}{4}} = -\Delta_N u + \frac{k-2}{4(k-1)} \cdot \text{scal}_N \cdot u$$

(6)

From this equality and $\text{scal}(g_M) \geq \kappa$ we get for a positive solutions $\psi_{M^{n-1}_\circ}$ of $\text{(3)}$:

$$\text{scal}(\psi^{4/n-3}_{M^{n-1}_\circ} \cdot g_{M^{n-1}_\circ}) \cdot \psi^{4/n-3}_{M^{n-1}_\circ} \geq \kappa.$$  

(7)

Now we argue inductively in two steps. We choose the area minimizer of the last chapter and find for some positive functions $\lambda_{[i]} > 0$ from $\text{(3)}$ for some small $\varepsilon > 0$, so that

$$(M^{n-2}_\circ, \psi^{4/n-3}_{M^{n-1}_\circ} \cdot g_{M^{n-2}_\circ}) \subset (M^{n-1}_\circ, \psi^{4/n-3}_{M^{n-1}_\circ} \cdot g_{M^{n-1}_\circ})$$

$$\text{scal} \left( \psi^{4/n-4}_{M^{n-2}_\circ} \cdot \psi^{4/n-3}_{M^{n-1}_\circ} \cdot g_{M^{n-2}_\circ} \right) \cdot \psi^{4/n-4}_{M^{n-2}_\circ} \geq \lambda_{[n-2]} \cdot (A)^2_{M^{n-2}_\circ} + \text{scal}(\psi^{4/n-3}_{M^{n-1}_\circ} \cdot g_{M^{n-1}_\circ})$$

(8)

We multiply by $\psi^{4/n-3}_{M^{n-1}_\circ}$ to apply the inductive assumption

$$\lambda_{[n-2]} \cdot (A)^2_{M^{n-2}_\circ} \cdot \psi^{4/n-3}_{M^{n-1}_\circ} + \text{scal}(\psi^{4/n-3}_{M^{n-1}_\circ} \cdot g_{M^{n-1}_\circ}) \cdot \psi^{4/n-3}_{M^{n-1}_\circ} \geq \lambda_{[n-2]} \cdot (A)^2_{M^{n-2}_\circ} \cdot \psi^{4/n-3}_{M^{n-1}_\circ} + \kappa \geq \kappa.$$  

Next we choose

$$(M^{n-3}_\circ, \psi^{4/n-4}_{M^{n-2}_\circ} \cdot g_{M^{n-3}_\circ}) \subset (M^{n-2}_\circ, \psi^{4/n-4}_{M^{n-2}_\circ} \cdot (\psi^{4/n-3}_{M^{n-2}_\circ} \cdot g_{M^{n-2}_\circ}))$$

and repeat the previous arguments iteratively until we reach

$$\text{scal} \left( \psi^{4/n-3}_{M^3_\circ} \cdot \ldots \cdot \psi^{4/n-3}_{M^{n-2}_\circ} \cdot \psi^{4/n-3}_{M^{n-1}_\circ} \cdot g_{M^3_\circ} \right) \cdot \psi^{4/n-3}_{M^3_\circ} \cdot \ldots \cdot \psi^{4/n-3}_{M^{n-2}_\circ} \cdot \psi^{4/n-3}_{M^{n-1}_\circ} \geq$$

(9)
\[ 
\lambda[3] \cdot \langle A \rangle^2_{M_3^0} \cdot \psi^{A/2}_{M_4^0} \cdot \psi^{4/n-4}_{M_{n-2}^0} \cdot \psi^{4/n-3}_{M_{n-1}^0} + \text{scal}(\psi^{4/2}_{M_4^0} \cdot \psi^{4/n-4}_{M_{n-2}^0} \cdot \psi^{4/n-3}_{M_{n-1}^0} \cdot g_{M_4^0}) \cdot \psi^{A/2}_{M_4^0} \cdot \psi^{4/n-4}_{M_{n-2}^0} \cdot \psi^{4/n-3}_{M_{n-1}^0} \\
\geq \lambda[3] \cdot \langle A \rangle^2_{M_3^0} \cdot \psi^{A/2}_{M_4^0} \cdot \psi^{4/n-4}_{M_{n-2}^0} \cdot \psi^{4/n-3}_{M_{n-1}^0} + \kappa \geq \kappa.
\]

To extend this dimensional descent to the two and finally the one dimensional case we observe that the argument for [L1], Th.1 and the estimates [L2], Th.6 (and specifically Prop.4.6) show that we can apply the previous arguments not only to \( M \) but also to the Riemannian product \( M \times S^1 \times S^1 \) for conformal deformations of \( M \times S^1 \times S^1 \) depending only on the \( M \). This makes the various deformation and construction steps we have seen before invariant under the \( S^1 \times S^1 \)-action and we finally reach a 3-manifold with connected components of the form \( I \times S^1 \times S^1 \), where \( I \subset \mathbb{R} \) is an interval.

**Remark 4.1** We note in passing that the bending effect towards singularities, needed to find an area minimizing border, would not be strong enough to make the symmetrization argument of [GL], Ch.12 work in high dimension.

For such a flat three manifold \( I \times S^1 \times S^1 \) we have from (1), (6) and (9)

\[ 
-8 \frac{\Delta \psi}{\psi} = \text{scal}(\varphi \cdot (g|_I + g_{S^1 \times S^1})) \cdot \varphi \geq \kappa \text{ with } \psi^4 := \varphi.
\]

For at least one component \( I \times S^1 \times S^1 \) we have \( f_1 : I \rightarrow S^1 \) is of nonzero degree with \( f_1(I) = S^1 \). After reparametrization of \( I \) by arc-length we have that \( f_1 \) is \( 4^n \)-contracting and this can be reduced to the condition \( \frac{\varphi''}{\varphi} \geq \kappa/8 \) on the interval \([0, 4^{-n}]\), where \( \varphi(x) = \int_{[x]} \psi dV \), since \( \int_{[x]} \Delta_{[x]} \psi \big|_{[x]} \psi \big|_{[x]} dV \right) = 0 \). An elementary computation, very similar to that in [GL], Ch.12, then gives the desired upper bound on \( \kappa \).

**References**

[CS] Cecchini, S. and Schick, T.: Enlargeable metrics on nonspin manifolds, [arXiv:1810.02116v2]

[GL1] Gromov, M. and Lawson, B.: Positive scalar curvature and the Dirac operator on complete Riemannian manifolds, Publ. Math. IHES 58 (1983), 295-408

[GL2] Gromov, M. and Lawson, B.: Spin and scalar curvature in the presence of a fundamental group, Ann. Math. 111 (1980), 209-230

[L1] Lohkamp, J.: Skin Structures in Scalar Curvature Geometry, arXiv: 1512.08252

[L2] Lohkamp, J.: Potential Theory on Minimal Hypersurfaces II: Hardy Structures and Schrödinger Operators, [arXiv:1810.03154]

[L3] Lohkamp, J.: Scalar Curvature and Hammocks, Math. Ann. 313 (1999), 385-407

[L4] Lohkamp, J.: The Higher Dimensional Positive Mass Theorem I, [arXiv:math/0608795v2]

[L5] Lohkamp, J.: The Higher Dimensional Positive Mass Theorem II, [arXiv:1612.07505v2]

[LM] Lawson, B. and Michelsohn, M.L.: Spin geometry, Princeton Univ. Press, Princeton (1989)