Wigner functions for fermions in strong magnetic fields

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We compute the covariant Wigner function for spin-1/2 fermions in an arbitrarily strong magnetic field by exactly solving the Dirac equation at non-zero fermion-number and chiral-charge densities. The Landau energy levels as well as a set of orthonormal eigenfunctions are found as solutions of the Dirac equation. With these orthonormal eigenfunctions we construct the fermion field operators and the corresponding Wigner-function operator. The Wigner function is obtained by taking the ensemble average of the Wigner-function operator in global thermodynamical equilibrium, i.e., at constant temperature T and non-zero fermion-number and chiral-charge chemical potentials µ and µ5, respectively. Extracting the vector and axial-vector components of the Wigner function, we reproduce the currents of the chiral magnetic and separation effect in an arbitrarily strong magnetic field.

I. INTRODUCTION

Heavy-ion collisions at ultrarelativistic energies create a new phase of strongly interacting matter, the so-called quark-gluon plasma (QGP) [1–6], for reviews, see, e.g., Refs. [7–13]. In the QGP, quarks and gluons are deconfined and the chiral symmetry of the fundamental theory of the strong interaction, quantum chromodynamics (QCD), is restored. The QGP occurs at temperatures above a deconfinement and chiral symmetry restoring crossover transition at \( T_\chi \sim 150 \text{ MeV} \) (for reviews of lattice-QCD calculations, see, e.g., Refs. [14–16]).

At temperatures above, but not asymptotically far above, \( T_\chi \), the QGP is not a gas of weakly interacting quarks and gluons, but rather a strongly interacting system, with a surprisingly small shear viscosity-to-entropy density ratio \( \eta/s \) (approaching the value estimated from the uncertainty principle [17]). This leads to a strong degree of collectivity of the hot and dense system created in heavy-ion collisions. In fact, the strong collective flow of strongly interacting matter, parametrized in terms of the elliptic flow coefficient \( v_2 \) [1, 2, 18, 19], has become the trademark signature of this system: it has been coined the “most perfect liquid” ever created.

If transport coefficients, like \( \eta/s \), are sufficiently small, the system is close to local thermodynamical equilibrium and a fluid-dynamical description for the dynamical evolution of the system becomes applicable [13–20]. Comparing fluid-dynamical calculations of the collective flow to experimental data, one has attempted to deduce bounds for the \( \eta/s \) ratio [21]. Such studies indicate that \( \eta/s \) could be as small as 0.2, which is not far from the KSS bound 1/(4π) or the quantum limit suggested by the AdS/CFT correspondence [22].

However, the dynamics of a heavy-ion collision is complex and influenced by many effects. The Frankfurt school led by Walter Greiner were pioneers in the study of the physics of strong fields in heavy-ion collisions [23]. It has recently been realized that the magnetic field created by the moving charges in relativistic heavy-ion collisions can be of the order of \( eB \sim m_\pi^2 [24, 32] \). While such fields rapidly decay in the vacuum, they can be sustained for a longer time by an induced current in a conducting medium [24, 30, 32, 33]. Such a magnetic field can then lead to an increase of the elliptic flow \( v_2 \) [24, 34–36]. In turn, this will increase the value of \( \eta/s \) necessary to describe elliptic-flow data.

Due to the fact that QCD has a quantum anomaly which gives rise to an explicit breaking of the \( U(1)_A \) symmetry, strongly interacting matter in a magnetic field can also exhibit other interesting effects. For instance, in case of a local imbalance between right- and left-handed quarks, a magnetic field leads to a current which separates electric charges along the direction of the magnetic field, the so-called Chiral Magnetic Effect (CME) [24, 30–34], for reviews, see, e.g., Refs. [35–37]. The CME is associated with the chiral vortical effect (CVE), where an electric current is induced by the vorticity in a system of charged particles [38–40]. In anomalous hydrodynamics the CME and CVE must coexist in order to guarantee the second law of thermodynamics [38, 50, 51]. In baryon-rich matter, an axial current is generated which separates right- and left-handed quarks along the direction of the magnetic field, the so-called Chiral Separation Effect (CSE) [42, 44–52]. The interplay between CME and CSE leads to so-called Chiral Magnetic Waves (CMW) [53, 54]. The CME has recently been confirmed in materials such as Dirac and Weyl semi-metals [39, 55, 56].

The charged-particle correlations observed in STAR [57–58] and ALICE [59] experiments are consistent with the CME prediction. But there were debates that the observed correlations might arise from other effects such as clustered-particle correlations [60], or local charge conservation [61], so a substantial part of the charged-particle correlation measured in experiments may come from background effects. Recently, the CMS collaboration has measured the
charged-particle correlations in pPb collisions [62] and found a result similar to that of STAR [57, 58] and ALICE [59] in AuAu and PbPb collisions. The CMS result indicates that the measured azimuthal correlation of charged particles at TeV energies may be a background effect. Further theoretical and experimental investigations are needed to separate the signal from the background [63, 65].

The covariant Wigner-function method [66, 72] for spin-1/2 fermions is a useful tool to study the CME, CVE, and other related effects [40, 42, 73–77]. However, previous investigations of these phenomena rely on the assumption that the magnetic field is weak and can be treated as a perturbation. The purpose of the present work is to show that the Wigner-function method can also be applied for magnetic fields of arbitrary strength. To this end, we will derive the exact solution for the fermion Wigner function in a constant, arbitrarily strong magnetic field \( \mathbf{B} \) in an extended system in global thermodynamical equilibrium, i.e., at constant temperature \( T \) and fermion-number chemical potential \( \mu \).

In order to study the CME and CSE, we also allow for a non-zero chiral-charge chemical potential \( \mu_5 \), i.e., we can independently control the number densities of right- and left-handed fermions through their associated chemical potentials \( \mu_L, \mu_R \sim \mu \mp \mu_5 \). We will confirm that the CME and the CSE are natural consequences in such a system.

We also note that the covariant Wigner-function method has also been applied to derive the kinetic equation for gluons in the background fields by one of the authors in collaboration with Walter Greiner [78, 79].

This paper is organized as follows. For determining the Wigner function we need to compute grand canonical ensemble averages of two fermion field operators. The ensemble averages require, in turn, a complete set of basis functions with which one can compute the Gibbs operator-weighted traces. Since these traces become simple if one diagonalizes the Hamilton operator (including fermion-number and chiral-charge chemical potential) of the system, we first derive in Sec. II the exact solution of the one-particle Dirac equation in the presence of a constant magnetic field and chemical potentials \( \mu, \mu_5 \), i.e., we find the corresponding energy eigenvalues and wave functions. It turns out that this can be done in completely analytical form. We will see that the original expression for the energy of the Landau levels is modified in the presence of non-zero \( \mu_5 \). Then, in Sec. III we construct the fermion field operators as an expansion in terms of the exact solutions of the Dirac equation derived in the previous section. The Hamilton operator is diagonal in this basis, i.e., it only contains single-particle creation and annihilation operators. We then compute the Wigner function in Sec. IV. The latter possesses a decomposition in terms of the independent generators of the Clifford algebra, the so-called Dirac-Heisenberg-Wigner functions. From these, we then derive expressions for the fermion-number and chiral-charge currents in Sec. V and recover the results derived previously in the weak-field limit, which give rise to the CME and CSE.

We take fermions to have positive charge \( Q = +e \) and the magnetic field to point in the \( z \)-direction. We use the following notations for four-vectors: \( X = (x^\mu) = (t, \mathbf{r}) = (t, x, y, z) \), \( P = (p^\mu) = (E, \mathbf{p}) = (E, p_x, p_y, p_z) \). We choose the temporal gauge \( A_0 = 0 \) throughout this paper.

II. LANDAU LEVELS AND WAVE FUNCTIONS

The momentum spectrum of a free particle is continuous in an infinite volume. In a constant magnetic field, the longitudinal momentum along the field remains continuous while the transverse momentum becomes discrete. The dispersion relation is

\[
E_{p_z}^{(n)} = \sqrt{m^2 + p_z^2 + 2ne B}.
\]  

Here \( n = 0, 1, 2, \cdots \) labels the Landau energy levels [80] [see also the recent review [81]], characterizing the quantization of transverse momentum. In Eq. (II.1) the quantum number \( n \) can also be written as \( n = n' + 1/2 + s \), with \( s = \pm 1/2 \) being the spin of the fermion and \( n' = 0, 1, 2, \cdots \) being the orbital quantum number. The number of states for fixed \( p_z \) is \( |eB|L_xL_y/(2\pi) \) at \( n = 0 \) and \( |eB|L_xL_y/\pi \) at \( n > 0 \), where \( L_xL_y \) is the transverse area of the system. In this section we will solve the Dirac equation including space-time independent chemical potentials for both fermion number and chiral charge and show how the dispersion relation (II.1) is modified. We will also derive the corresponding wave functions.

A. Dirac equation for massive particles and Landau levels

We use the Weyl (or chiral) representation for the Dirac matrices,

\[
\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix},
\]  

(II.2)
with $\sigma^\mu = (1, \sigma)$ and $\sigma^\mu = (1, -\sigma)$. In this representation, $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \text{diag}(-1, 1)$. Including the fermion-number and chiral-charge chemical potentials $\mu$ and $\mu_5$ or equivalently the chemical potentials for left- and right-handed chirality $\mu_{L,R}$, the Dirac Lagrangian in the presence of an external electromagnetic field $A^\sigma$ is

$$\mathcal{L} = \bar{\psi}[i\gamma^\sigma(\partial_\sigma + ieA_\sigma) - m + \mu\gamma^0 + \mu_5\gamma^0\gamma_5]\psi.$$

(II.3)

The corresponding Hamilton density is

$$\mathcal{H} = i\bar{\psi}\partial_t\psi - \mathcal{L} = \bar{\psi}^\dagger [\alpha \cdot (-i\nabla - eA) + m\gamma^0 - \mu - \mu_5\gamma_5]\psi,$$

where $\alpha = \gamma^0\gamma = \text{diag}(-\sigma, \sigma)$. The Dirac equation reads

$$[i\gamma^\sigma(\partial_\sigma + ieA_\sigma) - m + \mu\gamma^0 + \mu_5\gamma^0\gamma_5]\psi(x) = 0.$$

(II.5)

It can be rewritten in the form of a Schrödinger equation,

$$i\frac{\partial\psi}{\partial t} = [\alpha \cdot (-i\nabla - eA) + m\gamma^0 - \mu - \mu_5\gamma_5]\psi,$$

where we can read off the Hamilton operator for a Dirac particle,

$$\hat{H} = \alpha \cdot (-i\nabla - eA) + m\gamma^0 - \mu - \mu_5\gamma_5.$$

(II.7)

In the Landau gauge, a constant and homogeneous external magnetic field pointing in the $z$-direction and the associated vector potential are given by

$$B = Be_z,$$

$$A = -Bye_x.$$  

(II.8)

Without loss of generality, we take $B > 0$. Of course one can also choose a symmetric form for the vector potential, $A = \frac{1}{2}B \times r$. The Wigner function derived in Sec. IV will not depend on the choice of gauge.

Since $\hat{H}$ does not depend on $x$ and $z$, $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial z}$ commute with the Hamilton operator, $[\frac{\partial}{\partial x}, \hat{H}] = [\frac{\partial}{\partial z}, \hat{H}] = 0$. This indicates that $p_x$ and $p_z$ are conserved quantities. Thus, the solution to Eq. (II.6) can be cast into the following form

$$\psi(t, r) = e^{-iEt + ip_xx + ip_zz}\xi(p_x, p_z, y).$$

(II.9)

We can make the decomposition

$$\xi(p_x, p_z, y) = \begin{pmatrix} \chi_L(p_x, p_z, y) \\ \chi_R(p_x, p_z, y) \end{pmatrix},$$

(II.10)

where $\chi_{L,R}$ are Pauli spinors for left- and right-handed chirality, respectively. Then Eq. (II.6) can be simplified as

$$\begin{pmatrix} E + \mu - \mu_5 + \sigma_3p_z + \sigma_1(p_x + eBy) - i\sigma_2\frac{\partial}{\partial y} \end{pmatrix} \chi_L = m\chi_R,$$

$$\begin{pmatrix} E + \mu + \mu_5 - \sigma_3p_z - \sigma_1(p_x + eBy) + i\sigma_2\frac{\partial}{\partial y} \end{pmatrix} \chi_R = m\chi_L.$$  

(II.11)

Using the standard form of the Pauli matrices this becomes

$$\begin{pmatrix} E + \mu - \mu_5 + p_z & \sqrt{2eB}\hat{a}^\dagger \\ \sqrt{2eB}\hat{a} & E + \mu - \mu_5 - p_z \end{pmatrix} \chi_L = m\chi_R,$$

$$\begin{pmatrix} E + \mu + \mu_5 - p_z & -\sqrt{2eB}\hat{a}^\dagger \\ -\sqrt{2eB}\hat{a} & E + \mu + \mu_5 + p_z \end{pmatrix} \chi_R = m\chi_L,$$

(II.12)

where we have introduced the operators,

$$\hat{a} = \frac{1}{\sqrt{2eB}} \left[ \frac{\partial}{\partial y} + eB \left( y + \frac{p_x}{eB} \right) \right],$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2eB}} \left[ -\frac{\partial}{\partial y} + eB \left( y + \frac{p_x}{eB} \right) \right].$$

(II.13)
where the annihilation and creation operators for a harmonic oscillator with mass $m$ and frequency $m\omega = \sqrt{eB}$ and centered at $-p_x/(eB)$. It is straightforward to check that $[\hat{a}, \hat{a}^\dagger] = 1$. Eliminating $\chi_L$ or $\chi_R$ from Eq. (II.12) we can derive equations for $\chi_R$ and $\chi_L$,

$$
\left( \frac{(E + \mu)^2 - \Lambda^-}{2\mu_s\sqrt{eB}} \right) \chi_{R,L}(p_x, p_z, y) = 0,
$$

where we defined the operators

$$
\Lambda^\pm = m^2 + (p_z \pm \mu_s)^2 + 2eB \left( \hat{a} \hat{a}^\dagger + \frac{1}{2} \right) \pm eB.
$$

In order to solve Eq. (II.14), we expand $\chi_{R,L}(p_x, p_z, y)$ in a basis of eigenfunctions of the harmonic oscillator, $\phi_n(p_x, y)$,\[80\],

$$
\chi_{R,L}(p_x, p_z, y) = \sum_{n=0}^{\infty} \left( \frac{c_n(p_x, p_z)}{d_n(p_x, p_z)} \right) \phi_n(p_x, y),
$$

where $c_n$ and $d_n$ depend on $p_x$ and $p_z$, and $\phi_n(p_x, y)$ are given by

$$
\phi_n(p_x, y) = \left( \frac{eB}{\pi} \right)^{1/4} \frac{1}{\sqrt{2^n n!}} \exp \left[ -eB \frac{2}{2} (y + \frac{p_x}{eB})^2 \right] H_n \left[ \sqrt{eB} \left( y + \frac{p_x}{eB} \right) \right],
$$

where $H_n$ is the $n$-th Hermite polynomial. Note that the eigenfunctions $\phi_n(p_x, y)$ do not depend on $p_x$ and $y$ separately, but only on the linear combination $y + p_x/(eB)$, i.e., $\phi_n(p_x, y) \equiv \phi_n(y - y_0)$, where $y_0 \equiv -p_x/(eB)$. The interpretation is that, for given $p_x$, the $y$ coordinate of the center of the Landau orbit is precisely determined by $y_0$. One can show [80] that also its $x$ coordinate is determined by the wave function $\phi_n(p_x, y)$, due to Heisenberg’s uncertainty principle its $x$ coordinate remains completely undetermined, since $p_x$ is a good quantum number. Despite the fact that $\phi_n$ depends on $y - y_0$ only, in the following we keep the notation $\phi_n(p_x, y)$, because when we discuss second quantization in Sec. III we need a label $(p_x)$ to keep track of the Landau orbit in which a particle is created or annihilated.

The eigenfunctions $\phi_n(p_x, y)$ satisfy the orthonormality condition

$$
\int dy \phi_n(p_x, y) \phi_{n'}(p_x, y) = \delta_{nn'}.
$$

Furthermore, applying annihilation and creation operators,

$$
\hat{a} \phi_n(p_x, y) = \sqrt{n} \phi_{n-1}(p_x, y),
$$

$$
\hat{a}^\dagger \phi_n(p_x, y) = \sqrt{n+1} \phi_{n+1}(p_x, y),
$$

where $n \geq 0$ and we assumed $\phi_{-1} = 0$. Inserting the above expansion into Eq. (II.14), we obtain

$$
\sum_{n=0}^{\infty} \left( \frac{(E + \mu)^2 - \lambda_n^-}{2\mu_s\sqrt{2neB}} \phi_n \right) \frac{2\mu_s\sqrt{2(n+1)eB}\phi_{n+1}}{(E + \mu)^2 - \lambda_n^+} \left( \frac{c_n}{d_n} \right) = 0,
$$

where

$$
\lambda_n^\pm = m^2 + (p_z \pm \mu_s)^2 + 2neB.
$$

Using the orthonormality condition II.18 we can derive equations for $c_n$ and $d_n$,

$$
([E + \mu]^2 - \lambda_0^-)c_0 = 0,
$$

$$
([E + \mu]^2 - \lambda_n^-)c_n = -2\mu_s\sqrt{2neB}d_{n-1}, \quad n > 0,
$$

$$
([E + \mu]^2 - \lambda_n^+)d_{n-1} = -2\mu_s\sqrt{2neB}c_n, \quad n > 0.
$$
We observe that \( c_0 \) decouples from the other coefficients while the \( c_n \) \((n > 0)\) always couple to \( d_{n-1} \) \((n > 0)\). The energy eigenvalue for \( n = 0 \) is obtained by demanding a non-zero value for \( c_0 \). Then the first equation \((II.22)\) gives

\[
E^{(0)} = \sqrt{m^2 + (p_z - \mu_5)^2} \tag{II.23}
\]

is the energy of the lowest Landau level. A non-zero \( c_0 \) means that this level is occupied by a fermion with spin up.

The energy eigenvalues \( E \) for \( n > 0 \) are obtained by decoupling the second and third equations in Eq. \((II.22)\). For positive/negative-energy states these eigenvalues are given by 

\[
E^{(n)} = \sqrt{m^2 + \left(\sqrt{p_z^2 + 2neB} - s\mu_5\right)^2} \tag{II.24}
\]

is the energy of a Landau level for \( n > 0 \) and \( s = \pm 1 \) (helicity in the massless case) \cite{ref5}. We note that the energy levels depend on \( p_z, n, \) and \( s \) and are independent of \( p_x \). One also observes that the two-fold degeneracy of the conventional Landau levels \((II.1)\) with respect to spin (or helicity) is now lifted by a non-zero \( \mu_5 \).

In the case of vanishing chiral-charge chemical potential, \( \mu_5 = 0 \), all coefficients \( c_n, d_n \) decouple from each other. Equation \((II.22)\) becomes

\[
\begin{align*}
[(E + \mu)^2 - (m^2 + p_z^2 + 2neB)]c_n &= 0, \\
\{(E + \mu)^2 - [m^2 + p_z^2 + 2(n+1)eB]\}d_n &= 0,
\end{align*}
\tag{II.25}
\]

for \( n \geq 0 \), and we obtain the conventional Landau energy levels \((II.1)\). The lowest Landau level with \( n = 0 \) is occupied by a fermion with spin up. The higher Landau levels are two-fold degenerate, being occupied by fermions with spin up and spin down.

We conclude this subsection with some remarks on the degeneracy of the energy levels in the Landau gauge \((II.8)\). In this gauge a quantum state is labeled by a set of quantum numbers \( \{n, s, p_x, p_z\} \), so the sum over quantum states for a function \( F \) is \( \sim L_xL \sum_{n,s} \frac{dp_x dp_z}{2\pi} F(n, s, p_x, p_z) \). If \( F \) does not depend on \( p_z \), we can trivially perform the integral over \( p_z \). The integral is bounded by the requirement that the Landau orbit labelled by \( p_z \) is still located inside the transverse area \( L_xL_y \), i.e., by the requirement \( 0 \leq y_0 \leq L_y \) (where we neglected the small radius of the orbit with respect to \( L_y \)). Thus, \( 0 \leq p_z/(eB) \leq L_y \), and the degeneracy factor becomes \( L_x \int \frac{dp_x}{2\pi} = \frac{\Phi}{\pi} \) with \( \Phi = BL_xL_y \) being the magnetic flux through the transverse area \( L_xL_y \) \cite{ref6}.

### B. Landau wave functions

We now determine the eigenspinors \( \chi^{(n)}(p_x, p_z, y) \) associated with the Landau energy levels \((II.24)\). We start by assuming that the energy is equal to that of the lowest Landau level \( E^{(0)} \). From Eq. \((II.22)\) we conclude that \( c_0 \) can be non-zero while all other coefficients \( c_n, d_{n-1} \) with \( n > 0 \) have to vanish. The normalized eigenspinor associated with this state is

\[
\chi^{(0)}(p_x, y) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \phi_0(p_x, y). \tag{II.26}
\]

Now assume that the energy is equal to \( E^{(0)}_s \) with \( n > 0 \), cf. Eq. \((II.24)\). Then only the coefficients \( c_n, d_{n-1} \) can be non-zero while all other coefficients \( c_m, d_{m-1} \) with \( m \neq n \) have to vanish. The normalized eigenspinor associated with this Landau level is

\[
\chi^{(n)}(p_x, p_z, y) = \frac{1}{\sqrt{2p_z^2 + 2neB}} \begin{pmatrix} \sqrt{p_z^2 + 2neB + sp_z\phi_n(p_x, y)} \\ \sqrt{p_z^2 + 2neB - sp_z\phi_{n-1}(p_x, y)} \end{pmatrix}. \tag{II.27}
\]

This result is obtained as follows. Since for given energy \( E^{(n)} \) only the coefficients \( c_n \) and \( d_{n-1} \) are non-zero, we conclude from Eq. \((II.14)\) that

\[
\chi^{(n)}(p_x, p_z, y) = \begin{pmatrix} c_n(p_x, p_z)\phi_n(p_x, y) \\ d_{n-1}(p_x, p_z)\phi_{n-1}(p_x, y) \end{pmatrix}. \tag{II.28}
\]
In order to fulfill the second Eq. (II.22) we have to demand that $c_n \sim -2\mu_5 \sqrt{2neB}$ while $d_{n-1} \sim (E_{p_x}^{(n)})^2 - \lambda_n$. Inserting this into Eq. (II.28) and normalizing the eigenspinor yields Eq. (II.27). For later use, we then define $c_n(p_x, p_z)$ and $d_{n-1}(p_x, p_z)$ by the values in Eq. (II.27). The eigenspinors (II.26) and (II.27) fulfill the following orthonormality conditions

$$\int dy \chi^{(0)}(p_x, y) \chi^{(0)*}(p_x, y) = 1,$$

$$\int dy \chi^{(0)}(p_x, y) \chi^{(n)}(p_x, p_z, y) = 0,$$

$$\int dy \chi^{(n)}(p_x, p_z, y) \chi^{(n)*}(p_x, p_z, y) = \delta_{nn'} \delta_{ss'}.$$

(II.29)

From Eq. (II.9) we obtain the Dirac wave functions corresponding to the various Landau levels,

$$\psi^{(0)}(t, r) = \exp[-irE_{p_x}^{(0)} t + i\mu t + ip_x x + ip_z z] \xi_r^{(0)}(p_x, p_z, y),$$

$$\psi^{(n)}(t, r) = \exp[-irE_{p_x}^{(n)} t + i\mu t + ip_x x + ip_z z] \xi_r^{(n)}(p_x, p_z, y),$$

(II.30)

where $r = \pm$ denotes positive- or negative-energy states and $s$ denotes the helicity of the state. Here the Dirac spinors $\xi_r^{(0)}$ and $\xi_r^{(n)}$, which depend on momentum $p_x$, $p_z$, and $y$, are defined by

$$\xi_r^{(0)}(p_x, p_z, y) = \frac{1}{\sqrt{2E_{p_x}^{(0)}}} \left( \frac{r \sqrt{E_{p_x}^{(0)} - r(p_z - \mu_5)}}{E_{p_x}^{(0)} + r(p_z - \mu_5)} \right) \chi^{(0)}(p_x, y),$$

$$\xi_r^{(n)}(p_x, p_z, y) = \frac{1}{\sqrt{2E_{p_x}^{(n)}}} \left( \frac{r \sqrt{E_{p_x}^{(n)} + r\mu_5 - rs\sqrt{p_x^2 + 2neB}}}{E_{p_x}^{(n)} - r\mu_5 + rs\sqrt{p_x^2 + 2neB}} \right) \chi^{(n)}(p_x, p_z, y).$$

(II.31)

We can easily check that the quantities under the roots in Eq. (II.31) have non-negative values because $E_{p_x}^{(0)} \geq |p_z - \mu_5|$ and $E_{p_x}^{(n)} \geq \sqrt{p_x^2 + 2neB - s\mu_5}$. The first equation is obtained by using Eq. (II.12) to express $\chi_{L,R}$ in terms of $\chi_{R,L}$, assuming $\chi_{R,L} \sim \chi^{(0)}$, and normalizing the resulting Dirac spinor. The second equation is obtained analogously, assuming $\chi_{R,L} \sim \chi^{(n)}$. The Dirac spinors (II.31) satisfy the following orthonormality relations,

$$\int dy \xi_r^{(0)*}(p_x, p_z, y) \xi_r^{(0)}(p_x, p_z, y) = \delta_{rr'},$$

$$\int dy \xi_r^{(0)*}(p_x, p_z, y) \xi_r^{(n)}(p_x, p_z, y) = 0,$$

$$\int dy \xi_r^{(n)*}(p_x, p_z, y) \xi_r^{(n)}(p_x, p_z, y) = \delta_{nn'} \delta_{rr'} \delta_{ss'}.$$

(II.32)

III. FIELD OPERATORS, HAMILTON OPERATOR, AND DISTRIBUTION FUNCTION

A. Field operators

As shown in the last section, the Dirac eigenspinors (II.30) form an orthonormal basis and can thus be used in an expansion of the fermion field operator,

$$\psi(t, r) = e^{i\mu t} \sum_{n, s} \int_{p_x, p_z} \left\{ a_{p_x, p_z}^{(n)} \xi_{r+, s}(p_x, p_z, y) \exp \left[-iE_{p_x}^{(n)} t + ip_x x + ip_z z \right] \right. \left. + b_{-p_x, -p_z}^{(n)^\dagger} \xi_{r-, s}(p_x, p_z, y) \exp \left[iE_{p_x}^{(n)} t + ip_x x + ip_z z \right] \right\}.$$\n
(III.1)

Here, $a_{p_x, p_z}^{(n)}$ is the annihilation operator for fermions with momentum $p_x, p_z$ in the Landau level $E_{p_x}^{(n)}$ and $b_{-p_x, -p_z}^{(n)^\dagger}$ is the creation operator for anti-fermions with momentum $-p_x, -p_z$ in the same Landau level. We also defined $\int_{p_x, p_z} \equiv \int dp_x dp_z/(2\pi)^2$ and

$$\sum_{n, s} f_s^{(n)} \equiv f_s^{(0)} + \sum_{n > 0, s = \pm} f_s^{(n)}.$$\n
(III.2)
for any function \( f_s^{(n)} \) which depends on the Landau level \( n \) and the helicity \( s \). The first term is for the lowest Landau level with \( n = 0 \), which is always occupied by a fermion/anti-fermion with spin up/down. We see that the chemical potential \( \mu \) contributes only a global phase factor \( e^{i \mu t} \) to the field.

Note again that the momentum variable \( p_x \) serves as a label for the individual Landau orbits \( \{ \Phi_p \} \) with center located at \( y_0 = -p_x/(eB) \). The integral over \( p_x \) can thus also be interpreted as a (continuous) summation over these Landau levels. However, we cannot trivially perform the \( p_x \) integral (giving rise to the well-known degeneracy factor \( 2\pi/eB \)), because the integrand in Eq. (III.1) depends on \( p_x \); each Landau orbit has its own Fock space on which the annihilation and creation \( a_{p_x p_z s}^{(n)} \), \( \hat{a}_{p_x p_z s}^{(n)} \) act, and each wavefunction of the respective particle needs to be associated to the particular Landau orbit (labelled by \( p_x \)) where it was annihilated or created.

We assume that all operators satisfy the following anti-commutation relations:

\[
\begin{align*}
\{ a_{p_x p_z s}^{(n)}, a_{p_x p_z s'}^{(n')} \} &= (2\pi)^2 \delta(p_x - q_x) \delta(p_z - q_z) \delta_{nn'} \delta_{ss'}, \\
\{ b_{p_x p_z s}^{(n)}, b_{p_x p_z s'}^{(n')} \} &= (2\pi)^2 \delta(p_x - q_x) \delta(p_z - q_z) \delta_{nn'} \delta_{ss'},
\end{align*}
\]  

(III.3)

while all other anti-commutators vanish. Then one can verify the following equal-time anticommutation relations for the field operators,

\[
\begin{align*}
\{ \psi_\alpha(t, \mathbf{r}), \psi_\beta^\dagger(t, \mathbf{r}') \} &= \delta_{\alpha\beta} \delta^{(3)}(\mathbf{r} - \mathbf{r}') , \\
\{ \psi_\alpha(t, \mathbf{r}), \psi_\beta(t, \mathbf{r}') \} &= \{ \psi_\alpha^\dagger(t, \mathbf{r}), \psi_\beta(t, \mathbf{r}') \} = 0.
\end{align*}
\]  

(III.4)

B. Hamilton operator for Dirac fields and distribution functions

Integrating the Hamilton density \( \hat{H} \) over space, using Eq. (III.1) as well as the orthonormality relations (III.32) and anticommutation relations (III.3), (III.4), we obtain the Hamilton operator in the presence of fermion-number and anticommutation relations (III.3), (III.4), (III.5), (III.6), and (III.7), the integral over \( p_x \) can be performed trivially and, as already mentioned above, gives the well-known degeneracy factor \( e^{i \mu t}/(2\pi) \).

With the Hamilton operator (III.5), the grand partition function of the system at temperature \( T = \beta^{-1} \) reads

\[
\Xi = \text{Tr} \left[ \exp \left( -\beta \hat{H} \right) \right] = \text{Tr} \left\{ \exp \left[ -\beta \sum_{n,s} \int \left( E_{p_x p_z}^{(n)} - \mu \right) \hat{n}_{p_x p_z}^{(n)} - \beta \sum_{n,s} \int \left( E_{-p_x p_z}^{(n)} + \mu \right) \left( \hat{n}_{p_x p_z}^{(n)} - 1 \right) \right] \right\}.
\]  

(III.6)

The grand canonical ensemble average of an operator \( \hat{O} \) is given by

\[
\langle \hat{O} \rangle = \frac{1}{\Xi} \text{Tr} \left[ \hat{O} \exp \left( -\beta \hat{H} \right) \right].
\]  

(III.7)

In a similar way we can calculate the average particle number in the state with quantum numbers \( (n, s, p_x, p_z) \),

\[
\langle \hat{n}_{p_x p_z}^{(n)} \rangle = \frac{\text{Tr} \left\{ \hat{a}_{p_x p_z s}^{(n)} \exp \left[ -\beta \left( E_{p_x p_z}^{(n)} - \mu \right) \hat{n}_{p_x p_z}^{(n)} \right] \right\} \text{Tr} \left\{ \exp \left[ -\beta \left( E_{p_x p_z}^{(n)} - \mu \right) \hat{n}_{p_x p_z}^{(n)} \right] \right\}}{\text{Tr} \left( \exp \left[ -\beta \left( E_{p_x p_z}^{(n)} - \mu \right) \hat{n}_{p_x p_z}^{(n)} \right] \right)^2}.
\]  

(III.8)

where the reduced trace is defined as

\[
\text{Tr}_R(\hat{O}) = \langle 0 | \hat{O} | 0 \rangle + \langle 0 | a_{p_x p_z s}^{(n)} \hat{O} a_{p_x p_z s}^{(n)\dagger} | 0 \rangle.
\]  

(III.9)
Here \(|0\rangle\) is the vacuum state and \(a_{p_\mu p_\nu s}^{(n)\dagger}|0\rangle\) is the one-particle state. States with more than one particle with the same quantum numbers do not exist due to the Pauli principle. Then it is straightforward to obtain

\[
\langle \hat{p}_{p_\mu p_\nu s}^{(n)} \rangle = \frac{1}{\exp \left[ \beta(E^{(n)}_{p_\mu p_\nu s} - \mu) \right] + 1} = f_{FD}(E^{(n)}_{p_\mu p_\nu s} - \mu). \tag{III.10}
\]

In the same way we obtain

\[
\langle \hat{p}_{p_\mu p_\nu s}^{(n)} \rangle = f_{FD}(E^{(n)}_{-p_\mu p_\nu s} + \mu). \tag{III.11}
\]

We see that the number distributions of particles and anti-particles follow Fermi-Dirac statistics, where the energies are given in Eqs. (II.23), (II.24). We note that the energy for \(n > 0\) is an even function of \(p_\mu\) while that for the lowest level is not if \(\mu_5\) is non-zero.

As a final remark we would like to point out that, if the magnetic field is strong enough so that only the lowest Landau level is occupied, due to the large energy gap between the lowest and the higher Landau levels the transport coefficients have very special properties \[84, 85\]. Nevertheless, one can construct an effective theory, and even fluid dynamics, for a system where only the lowest Landau level is occupied \[86\].

### IV. WIGNER FUNCTIONS

The gauge-invariant Wigner function for fermions is defined by \[67 \, 70\]

\[
W_{\alpha\beta}(X, P) = \int \frac{d^4X'}{(2\pi)^4} \exp(-ip_\mu x'^\mu) \left\langle \bar{\psi}_\beta \left( X + \frac{1}{2}X' \right) U \left( A, X + \frac{1}{2}X', X - \frac{1}{2}X' \right) \psi_\alpha \left( X - \frac{1}{2}X' \right) \right\rangle, \tag{IV.1}
\]

where \(U \left( A, X + \frac{1}{2}X', X - \frac{1}{2}X' \right)\) is the gauge link between \(X - \frac{1}{2}X'\) and \(X + \frac{1}{2}X'\). Since we consider a constant and homogeneous external magnetic field along the \(z\) direction, for which the electromagnetic gauge potential can be chosen as \(A^\mu(X) = (0, -By, 0, 0)\), cf. Eq. (III.5), the gauge link is just a phase, \(U \left( A, X + \frac{1}{2}X', X - \frac{1}{2}X' \right) = \exp(-ieByx')\). Thus the Wigner function is given by

\[
W(X, P) = \int \frac{d^4X'}{(2\pi)^4} \exp(-ip_\mu x'^\mu - ieByx') \left\langle \bar{\psi} \left( X + \frac{1}{2}X' \right) \otimes \psi \left( X - \frac{1}{2}X' \right) \right\rangle. \tag{IV.2}
\]

The Wigner function can be decomposed in terms of the 16 independent generators of the Clifford algebra \[69\],

\[
W(X, P) = \frac{1}{4} \left( \mathcal{F} + i\gamma^5\mathcal{P} + \gamma^\mu \mathcal{V}_\mu + \gamma^5\gamma^\mu \mathcal{A}_\mu + \frac{1}{2}\sigma^{\mu\nu}S_{\mu\nu} \right), \tag{IV.3}
\]

where the coefficients \(\mathcal{F}, \mathcal{P}, \mathcal{V}_\mu, \mathcal{A}_\mu,\) and \(S_{\mu\nu}\) are the scalar, pseudo-scalar, vector, axial-vector, and tensor components of the Wigner function, respectively. The tensor component is anti-symmetric so we can equivalently introduce two vector functions

\[
T = \frac{1}{2} \epsilon_{ijk}e_i(S^{0ij} - S^{i0}) , \quad S = \frac{1}{2} \epsilon_{ijk}e_iS_{ijk}. \tag{IV.4}
\]

The functions \(\mathcal{F}, \mathcal{P}, \mathcal{V}_\mu, \mathcal{A}_\mu, T,\) and \(S\) are called Dirac-Heisenberg-Wigner (DHW) functions. All of them are real functions over phase space and some of them have an obvious physical meaning \[57\]. For example, \(\mathcal{V}_\mu(X, P)\) is the fermion-current density.

In order to determine the DHW functions in a constant magnetic field, we insert the field operator \[III.1\] into the definition \(\mathcal{W}^{(n)}\) of the Wigner function. The only combinations of creation and annihilation operators which survive when ensemble-averaging are \(a_{p_\mu p_\nu s}^{(n)\dagger} a_{p_\mu p_\nu s}^{(n)} = \delta_{p_\mu, p_\nu} \delta_{p_\mu, p_\nu} \delta_{p_\mu, p_\nu} \delta_{p_\mu, p_\nu} = \delta_{p_\mu, p_\nu} \). These have been calculated in the previous section, see Eqs. (III.10), (III.11). Since we assume constant chemical potentials and temperature, the Wigner function does not depend on space-time,

\[
W(P) = \sum_{n, s} \left\{ f_{FD}(E^{(n)}_{p_\mu p_\nu s} - \mu) \delta(p_0 + \mu - E^{(n)}_{p_\mu p_\nu s}) W^{(n)}_{+\mu, s}(p) + [1 - f_{FD}(E^{(n)}_{p_\mu p_\nu s} + \mu)] \delta(p_0 + \mu + E^{(n)}_{p_\mu p_\nu s}) W^{(n)}_{-\mu, s}(p) \right\}. \tag{IV.5}
\]
Here the 1 in the square brackets is the vacuum contribution arising from the anti-commutation relation for \( b_{p,s}^{(n)} \), \( b_{p,s}^{(n)^\dagger} \). We will show in Sec. [V] that this vacuum term contributes to the chiral magnetic effect. The matrix-valued functions \( W_{rs}^{(n)}(p) \) denote the contributions of fermions/anti-fermions in the \( n \)-th Landau level with \( E_{p,s}^{(n)} \). They are straightforwardly computed as

\[
W_{rs}^{(n)}(p) = \frac{1}{(2\pi)^3} \int dy' \exp \left( i p y' \right) \xi_{rs}^{(n)^\dagger} \left( p_x, p_z, -\frac{y'}{2}, \gamma^0 \otimes \xi_{rs}^{(n)} \left( p_x, p_z, -\frac{y'}{2} \right) \right),
\]

where we used the property \( \phi_n \left( p_x - eB y, y - \frac{1}{2} y' \right) = \phi_n \left( p_x, -\frac{1}{2} y' \right) \) and the fact that the dependence of \( \xi_{rs}^{(n)} \) on \( p_x \) and \( y \) only appear in the eigenfunctions \( \phi_n \) and \( \phi_{n-1} \) of the harmonic oscillator, see Eqs. (II.17), (II.26), (II.27), and (II.31).

The functions \( W_{rs}^{(0)}(p) \) and \( W_{rs}^{(n)}(p) \) are evaluated in Appendix A. The results are given in Eqs. (A.1), (A.2) and Eqs. (A.9), (A.10). We can extract all DHW functions from the Wigner function (IV.5) with \( W_{rs}^{(0)}(p) \) and \( W_{rs}^{(n)}(p) \) given by Eqs. (A.9), (A.10). In order to write these functions in compact form, we divide the 16 DHW functions into four groups, each group forming a four-dimensional vector,

\[
G_1(P) = \left( \mathcal{F}(P), \mathcal{S}(P) \right), \quad G_2(P) = \left( V_0(P), A(P) \right), \\
G_3(P) = \left( A_0(P), V(P) \right), \quad G_4(P) = \left( T(P), P(P) \right).
\]

All of these are functions of four-momentum \( P \). In order to separate the \( p_T \) dependence we define four-dimensional vectors for \( n \geq 0 \)

\[
e_1^{(n)}(p_T) = \left( \Lambda_+^{(n)}(p_T), 0, \Lambda_-^{(n)}(p_T) \right), \quad e_2^{(n)}(p) = \left( \frac{p_z \Lambda_-^{(n)}(p_T)}{p_x} \right) = \left( \frac{p_z \Lambda_-^{(n)}(p_T)}{p_x} \right).
\]

Here \( 0_T = (0, 0)^T \) is a two-dimensional null vector and \( \Lambda_\pm^{(n)} \) is defined in Eqs. (A.6-A.7). Then, the DHW functions read

\[
G_1(P) = \left( \begin{array}{c} \sum_{n=0}^\infty V_n(p_0, p_z) e_1^{(n)}(p_T) + \sum_{n=1}^\infty \frac{1}{\sqrt{p_x^2 + 2neB}} A_n(p_0, p_z) e_2^{(n)}(p) \end{array} \right) \left( \begin{array}{c} m \\ p_0 + \mu \end{array} \right), \\
G_2(P) = \left( \begin{array}{c} \sum_{n=0}^\infty V_n(p_0, p_z) e_1^{(n)}(p_T) \\
+ \sum_{n=1}^\infty \left[ \frac{\mu_5}{\sqrt{p_x^2 + 2neB}} A_n(p_0, p_z) - \frac{\mu_5 V_n(p_0, p_z)}{\sqrt{p_x^2 + 2neB}} \right] e_1^{(n)}(p_T) \\
+ \sum_{n=1}^\infty \left[ V_n(p_0, p_z) - \frac{\mu_5}{\sqrt{p_x^2 + 2neB}} A_n(p_0, p_z) \right] e_2^{(n)}(p) \end{array} \right), \\
G_3(P) = 0.
\]

Here \( V_n \) and \( A_n \) for \( n > 0 \) are given by

\[
V_n(p_0, p_z) = \frac{2}{(2\pi)^3} \sum_s \delta \left( (p_0 + \mu)^2 \right) \left\{ \theta(p_0 + \mu) f_{FD}(p_0) + \theta(-p_0 - \mu) f_{FD}(-p_0) + \theta(-p_0 - \mu) f_{FD}(-p_0) \right\}, \\
A_n(p_0, p_z) = \frac{2}{(2\pi)^3} \sum_s s \delta \left( (p_0 + \mu)^2 \right) \left\{ \theta(p_0 + \mu) f_{FD}(p_0) + \theta(-p_0 - \mu) f_{FD}(-p_0) + \theta(-p_0 - \mu) f_{FD}(-p_0) \right\}.
\]

The lowest Landau level does not depend on helicity \( s \), thus

\[
V_0(p_0, p_z) = \frac{2}{(2\pi)^3} \delta \left( (p_0 + \mu)^2 \right) \left\{ \theta(p_0 + \mu) f_{FD}(p_0) + \theta(-p_0 - \mu) f_{FD}(-p_0) + \theta(-p_0 - \mu) f_{FD}(-p_0) \right\}.
\]

In this paper we choose the Landau gauge when solving the Dirac equation. If we choose a different gauge, the single-particle wave functions will be different, but the gauge link in the definition of the Wigner function will change at the same time, so that the latter is gauge invariant.
V. FERMION-NUMBER CURRENT AND CHIRAL-CHARGE CURRENT

In this section, we will derive the fermion-number current $j^{\mu}$ and chiral-charge current $j^{\mu}_5$ from the DHW functions $\Psi^{\mu}$ and $A^{\mu}$, respectively, by integrating over four-momentum $P$,

$$j^{\mu} = \int d^4P \Psi^{\mu}(P),$$

$$j^{\mu}_5 = \int d^4P A^{\mu}(P).$$

(V.1)

The analytical formulas for these DHW functions are given by Eq. (IV.9). The terms corresponding to $e_2^{(n)}(p)$ vanish when integrating over $P$ because they are odd under $p \to -p$. This means that the $x$ and $y$ components of the functions $G_i(P)$, $i = 1, 2, 3, 4$, vanish. Thus, also the $x$ and $y$ components of the fermion-number and chiral-charge currents are zero, which is a consequence of fact that the motion of the particles is confined to Landau orbits in the transverse plane.

A. Fermion-number and current density

The $t$ and $z$ component of the fermion-number current, which denote the fermion-number density and the current density pointing along the magnetic field, respectively, are non-zero. From Eqs. (IV.7), (IV.8), and (IV.9) we get

$$\rho = 2\pi eB \int dp_0 dp_z \sum_{n=0}^{\infty} (p_0 + \mu) V_n(p_0, p_z),$$

$$j_z = 2\pi eB \int dp_0 dp_z (p_z - \mu_5) V_0(p_0, p_z),$$

(V.2)

where the functions $V_n(p_0, p_z)$ are given in Eqs. (IV.10), (IV.11) and we have used Eq. (A.8). Note that only the lowest Landau level contributes to $j_z$ [37]. In order to perform the $p_0$ integration we use the following properties of the Dirac delta function,

$$\delta \left\{ (p_0 + \mu)^2 - [E_{p,s}^{(n)}]^2 \right\} \theta[r(p_0 + \mu)] = \frac{1}{2E_{p,s}^{(n)}} \delta(p_0 + \mu - rE_{p,s}^{(n)}),$$

$$\delta \left\{ (p_0 + \mu)^2 - [E_{p,s}^{(n)}]^2 \right\} \theta[r(p_0 + \mu)] = \frac{r}{2E_{p,s}^{(n)}} \delta(p_0 + \mu - rE_{p,s}^{(n)}).$$

(V.3)

Here $r = \pm$ for fermions or anti-fermions. Then the fermion-number and current densities can be expressed as

$$\rho = \frac{eB}{(2\pi)^2} \sum_{n,s} \int dp_z \left[ f_{FD}(E_{p,s}^{(n)} - \mu) - f_{FD}(E_{p,s}^{(n)} + \mu) + 1 \right],$$

$$j_z = \frac{eB}{(2\pi)^2} \int dp_z \frac{p_z - \mu_5}{E_{p_z}^{(0)}} \left[ f_{FD}(E_{p_z}^{(0)} - \mu) + f_{FD}(E_{p_z}^{(0)} + \mu) - 1 \right].$$

(V.4)

In Fig. (V.1) we show the ratio of the renormalized fermion-number density, i.e., the expression without the vacuum term, to the fermion-number density for $B = 0$,

$$\rho_0 = \frac{1}{(2\pi)^3} \int d^3p \sum_{r,s=\pm} r f_{FD} \left(\sqrt{m^2 + p^2} - r\mu + s\mu_5\right),$$

(V.5)

as a function of $\beta^2 eB$. We choose four different configurations: (i) $\beta m = 1$, $\beta \mu = 2$, $\beta \mu_5 = 0$, which represents a chirally symmetric system, (ii) $\beta m = 1$, $\beta \mu = 2$, $\beta \mu_5 = 0.5$, representing a system with an imbalance in the number of right- and left-handed fermions, (iii) $\beta m = 1$, $\beta \mu = 3$, $\beta \mu_5 = 0.5$, representing the same case but at a larger fermion-number chemical potential. The last one, (iv) $\beta m = 0$, $\beta \mu = 3$, $\beta \mu_5 = 0.5$, corresponds to the massless case. In all cases the fermion-number density increases with $B$ and approaches Eq. (V.5) in the weak-field limit.

We now turn to the computation of the current density. The integration can be done analytically by first limiting the integration to the region $\pm \Lambda$ and then taking the limit $\Lambda \to +\infty$. The result is

$$j_z = -\frac{eB}{4\pi^2 \beta} \lim_{\Lambda \to +\infty} \frac{\ln \left\{ 1 + \exp(-\beta(E^{(0)}_{\Lambda} - \mu)) \right\} \left\{ 1 + \exp(-\beta(E^{(0)}_{-\Lambda} + \mu)) \right\}}{\left\{ 1 + \exp(-\beta(E^{(0)}_{-\Lambda} - \mu)) \right\} \left\{ 1 + \exp(-\beta(E^{(0)}_{\Lambda} + \mu)) \right\}} - \frac{eB}{4\pi^2 \beta} \lim_{\Lambda \to +\infty} \left( E^{(0)}_{\Lambda} - E^{(0)}_{-\Lambda} \right).$$

(V.6)
The first term is zero while a careful calculation of the second term gives
\[ j_z = \frac{e \mu_5}{2\pi^2} B. \]  
(V.7)
We have thus reproduced the previous, well-known result for the CME [24, 36, 37].

B. Chiral-charge and current density

Analogously we can derive the chiral-charge and current densities from Eqs. (IV.7), (IV.8), and (IV.9),
\[ \rho_5 = 2\pi eB \int dp_0 dp_z \left\{ (p_z - \mu_5)V_0(p_0, p_z) + \sum_{n=1}^{\infty} \left[ \sqrt{p_z^2 + 2neBA_n(p_0, p_z)} - \mu_5 V_n(p_0, p_z) \right] \right\}, \]
\[ j_{5z} = 2\pi eB \int dp_0 dp_z (p_0 + \mu)V_0(p_0, p_z). \]  
(V.8)
Using the property (V.3) of the delta function it is straightforward to perform the \( p_0 \) integration,
\[ \rho_5 = \frac{eB}{(2\pi)^2} \int dp_z \left\{ \frac{p_z - \mu_5}{E_{p_z}^{(0)}} \left[ f_{FD}(E_{p_z}^{(0)} - \mu) + f_{FD}(E_{p_z}^{(0)} + \mu) - 1 \right] \right\} + \sum_{n=1}^{\infty} \sum_s \frac{\sqrt{p_z^2 + 2neBA} - s\mu_5}{E_{p_z,s}^{(n)}} \left[ f_{FD}(E_{p_z,s}^{(n)} - \mu) + f_{FD}(E_{p_z,s}^{(n)} + \mu) - 1 \right] \right\}, \]
\[ j_{5z} = \frac{eB}{(2\pi)^2} \int dp_z \left[ f_{FD}(E_{p_z}^{(0)} - \mu) - f_{FD}(E_{p_z}^{(0)} + \mu) + 1 \right]. \]  
(V.9)
Focusing on the chiral-charge density, we numerically perform the \( p_z \) integration and compare with the \( B = 0 \) limit,
\[ \rho_{5,0} = \frac{1}{(2\pi)^3} \int d^3p \sum_{r,s=\pm} rs f_{FD} \left( \sqrt{m^2 + p^2} - r\mu + s\mu_5 \right). \]  
(V.10)
The ratio \( \rho_5(B)/\rho_{5,0} \) is shown in Fig. V.2. We observe that the ratio is 1 in the weak-field limit (up to numerical errors in our algorithm).

We now turn to the chiral-charge current density, cf. the last line of Eq. (V.9). In general, the result cannot be given in a closed analytic form. As a general remark, however, note that \( j_{5z} \) does not depend on \( \mu_5 \), by virtue of a shift of the integration variable \( p_z \rightarrow p_z + \mu_5 \). In the limit \( m \ll T \), we may expand the integrand in a power series in \( \beta m \). The leading term, corresponding to \( m = 0 \), can be analytically calculated. The first two terms in this expansion read
\[ j_{5z} = \frac{eB\mu}{2\pi^2} - \frac{eBT(\beta m)^2}{(2\pi)^2} \int_0^\infty dp e^{\beta(p-\mu)}(e^{2\beta p} - 1)(e^{2\beta p} - 1) \left[ \frac{1}{p[1 + e^{\beta(p+\mu)}]} - \frac{1}{[1 + e^{\beta(p-\mu)}]^2} \right] + O[(\beta m)^4]. \]  
(V.11)
The leading-order term was first calculated by Metlitski and Zhitnitsky [88] and later reproduced by many groups in different approaches [42, 44].
VI. SUMMARY

We have computed the covariant Wigner function for spin-1/2 fermions in an arbitrarily strong magnetic field $B$ by exactly solving the Dirac equation at non-zero fermion-number and chiral-charge densities (or equivalently non-zero chemical potentials $\mu$ and $\mu_5$ for the fermion number and chiral charge, respectively). The Landau energy levels and the corresponding orthonormal eigenfunctions were obtained. With these orthonormal eigenfunctions we have constructed the fermion field operators in canonical quantization and, consequently, the Wigner function operator. The Wigner function was then obtained by taking the ensemble average of the Wigner function operator in global thermodynamical equilibrium, i.e., at constant temperature $T$ and non-zero $\mu$ and $\mu_5$. By extracting the vector and axial-vector components of the Wigner function and carrying out four-momentum integrals, we obtain the fermion-number and chiral-charge currents, which agree with the standard results for the CME and CSE, respectively, in an arbitrarily strong magnetic field.

Acknowledgments

QW thanks I. Shovkovy for helpful discussions. DHR acknowledges support by the High-End Visiting Expert project GDW20167100136 of the State Administration of Foreign Experts Affairs (SAFEA) of China and by the Deutsche Forschungsgemeinschaft (DFG) through the grant CRC-TR 211 "Strong-interaction matter under extreme conditions". QW is supported in part by the Major State Basic Research Development Program in China (973 program) under the Grant No. 2015CB856902 and 2014CB845402 and by the National Natural Science Foundation of China (NSFC) under the Grant No. 11535012. This work was first presented at the Frankfurt Institute of Advanced Studies International Symposium on Discoveries at the Frontiers of Science held in memory of Walter Greiner (1935-2016). We dedicate this work to Walter Greiner, who was teacher, mentor, and friend of DHR, DV, and QW.

When finalizing this work, we became aware that the authors of [89] were performing a related study which reaches similar conclusions as our work.
Appendix A: Derivation of $W_r^{(0)}(p)$ and $W_{rs}^{(n)}(p)$

In this appendix, we will give the detailed derivation of $W_r^{(0)}(p)$ and $W_{rs}^{(n)}(p)$. Substituting $\xi_r^{(0)}$ and $\xi_{rs}^{(n)}$ from Eqs. (II.26), (II.31) into Eq. (IV.6), we obtain $W_r^{(0)}(p)$ for the lowest Landau level,

$$ W_r^{(0)}(p) = \frac{1}{(2\pi)^3} \int dy' \exp{(ip y')} \xi_r^{(0)} \left( p_x, p_z, -\frac{1}{2} y' \right) \xi_r^{(0)*} \left( p_x, p_z, \frac{1}{2} y' \right) \gamma^0 $$

$$ = \frac{1}{(2\pi)^3} \frac{1}{2E_{p_z}^{(0)}} \int dy' \exp{(ip y')} \phi_0 \left( p_x, -\frac{1}{2} y' \right) \phi_0 \left( p_x, \frac{1}{2} y' \right) $$

$$ \times \left( \frac{rm}{E_{p_z}^{(0)} + r(p_z - \mu_5)} \right) \times \left( 1 \begin{array}{cc} 0 \\ 0 \end{array} \right). \quad (A.1) $$

We obtain $W_{rs}^{(n)}(p)$ for the higher Landau levels,

$$ W_{rs}^{(n)}(p) = \frac{1}{(2\pi)^3} \int dy' \exp{(ip y')} \xi_{rs}^{(n)} \left( p_x, p_z, -\frac{1}{2} y' \right) \xi_{rs}^{(n)*} \left( p_x, p_z, \frac{1}{2} y' \right) \gamma^0 $$

$$ = \frac{1}{(2\pi)^3} \frac{1}{2E_{p_z}^{(n)}} \int dy' \exp{(ip y')} \phi_0 \left( p_x, -\frac{1}{2} y' \right) \phi_0 \left( p_x, \frac{1}{2} y' \right) $$

$$ \times \left( \frac{rm}{E_{p_z}^{(n)} + rs \left( \sqrt{p_z^2 + 2neB} - s\mu_5 \right)} \right) \times \left( \begin{array}{c} c_n^2 I_{nn} \\ c_n d_{n-1} d_{n-1,n-1} \end{array} \right), \quad (A.2) $$

where the integrals are defined by

$$ I_{ij} = \int dy' \exp{(ip y')} \phi_i \left( p_x, -\frac{1}{2} y' \right) \phi_j \left( p_x, \frac{1}{2} y' \right) \quad (A.3) $$

for $i, j = n$ or $n - 1$. The coefficients are evaluated from Eqs. (II.27), (II.28) as

$$ c_n^2 = \frac{1}{2} \left( 1 + \frac{sp_z}{\sqrt{p_z^2 + 2neB}} \right), $$

$$ c_n d_{n-1} = \frac{s}{2} \sqrt{2neB} \left( \frac{sp_z}{\sqrt{p_z^2 + 2neB}} \right), $$

$$ d_{n-1}^2 = \frac{1}{2} \left( 1 - \frac{sp_z}{\sqrt{p_z^2 + 2neB}} \right). \quad (A.4) $$

For $n > 0$, the integrals in Eq. (A.3) can be computed analytically as

$$ \frac{1}{2} (I_{n,n} \pm I_{n-1,n-1}) = \Lambda^{(n)}_{\pm}(p_T), $$

$$ \frac{1}{2} (I_{n,n} + I_{n-1,n-1}) = \frac{p_x \sqrt{2neB}}{p_T^2} \Lambda^{(n)}_+(p_T), $$

$$ \frac{1}{2} (I_{n,n} - I_{n-1,n-1}) = \frac{ip_x \sqrt{2neB}}{p_T^2} \Lambda^{(n)}_-(p_T), \quad (A.5) $$

where $p_T = \sqrt{p_x^2 + p_y^2}$ is the modulus of the transverse momentum and $\Lambda^{(n)}_\pm(p_T)$ ($n > 0$) are defined as

$$ \Lambda^{(n)}_\pm(p_T) = (-1)^n \left[ L_n \left( \frac{2p_T^2}{eB} \right) \mp L_{n-1} \left( \frac{2p_T^2}{eB} \right) \right] \exp \left( -\frac{p_T^2}{eB} \right), \quad (A.6) $$

where $L_n(x)$ are the Laguerre polynomials with $L_{-1}(x) = 0$. For the lowest Landau level, $n = 0$, we have

$$ I_{00} = \Lambda^{(0)}_+(p_T) = \Lambda^{(0)}_-(p_T) = 2 \exp \left( -\frac{p_T^2}{eB} \right). \quad (A.7) $$
One can check that when integrating over $p_T = (p_x, p_y)^T$, the functions $\Lambda_+^{(n)}(p_T)$ ($n > 0$) and $\Lambda^{(0)}(p_T)$ will give the density of states and $\Lambda_-(p_T)$ ($n > 0$) will give zero,

$$
\int \frac{d^2 p_T}{(2\pi)^2} \Lambda_+^{(n)}(p_T) = \int \frac{d^2 p_T}{(2\pi)^2} \Lambda^{(0)}(p_T) = \frac{eB}{2\pi}.
$$

$$
\int \frac{d^2 p_T}{(2\pi)^2} \Lambda_-^{(n)}(p_T) = 0.
$$

(A.8)

We can expand Eq. (A.1) as in Eq. (IV.3) and obtain

$$
W_r^{(0)}(p) = \frac{r}{4(2\pi)^3 E_{p_z}^{(0)}} \Lambda^{(0)}(p_T) \left[ m(1 + \sigma^{12}) + r E_{p_z}^{(0)}(\sigma^0 - \gamma^3) - (p_z - \mu_5)(\gamma^3 - \gamma^5\gamma^0) \right].
$$

(A.9)

Similarly we can also expand Eq. (A.2) as

$$
W_{rs}^{(n)}(p) = \frac{1}{4(2\pi)^3 E_{p_z}^{(n)}} \left\{ \left[ \Lambda_+^{(n)}(p_T) + \frac{p_z}{\sqrt{p_z^2 + 2neB}} \Lambda_-^{(n)}(p_T) \right] \left[ m + r E_{p_z}^{(n)}(\sigma^0 + (s\sqrt{p_z^2 + 2neB} - \mu_5)\gamma^5\gamma^0) \right] \\
- \left[ \Lambda_-^{(n)}(p_T) + \frac{p_z}{\sqrt{p_z^2 + 2neB}} \Lambda_+^{(n)}(p_T) \right] \left[ \left( s\sqrt{p_z^2 + 2neB} - \mu_5 \right) \gamma^3 + r E_{p_z}^{(n)}(\sigma^0)\gamma^3 - m\sigma^{12} \right] \\
- \frac{2neB}{p_z^2 + 2neB} \Lambda_+^{(n)}(p_T) \left[ \left( \sqrt{p_z^2 + 2neB} - s\mu_5 \right) (p_z\gamma^1 + p_y\gamma^2) \\
+ rs E_{p_z}^{(n)}(p_z\gamma^5\gamma^1 + p_y\gamma^5\gamma^2) - sm (p_z\sigma^{23} - p_y\sigma^{13}) \right] \right\}.
$$

(A.10)

The global factors $r/E_{p_z}^{(0)}$ and $r/E_{p_z}^{(n)}$ in front of the brackets will be used later to write the delta functions in Eq. (IV.5) in a covariant form.

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