Self-dual Chern-Simons Solitons in the Planar Ferromagnet

Yoonbai Kim\(^1\), Phillial Oh\(^1\) and Chaiho Rim\(^2\)

\(^1\)Department of Physics, Sung Kyun Kwan University, Suwon 440-746, Korea
yoonbai@cosmos.skku.ac.kr, ploh@newton.skku.ac.kr

\(^2\)Department of Physics, Chonbuk National University, Chonju 561-756, Korea
rim@phy0.chonbuk.ac.kr

Abstract

We consider a uniaxial planar ferromagnet coupled minimally to an Abelian Chern-Simons gauge field and study self-dual solitons which saturate the Bogomol’nyi bound. We find a rich structure of rotationally symmetric static soliton solutions for various uniform background charge densities. For a given ferromagnet material, the properties of these solitons are controlled only by the external magnetic field and the background charge.

PACS number(s): 03.65.Ge, 11.27.+d, 11.10.Lm
The magnetic solitons of the uniaxial ferromagnetic crystals have attracted much attention recently [1]. It is well known that the macroscopic properties of a ferromagnet material are described by the Landau-Lifshitz equation (LLE) [2] and the order parameters are the spin variables $Q \equiv Q^a(t, x^i) T^a$, $T^a = -i\sigma^a/2$:

$$\partial_t Q + \partial_i [Q, \partial_i Q] - \sum_{a=1}^{3} (J^a Q^a - H^a) [Q, T^a] = 0, \quad Q^a Q^a = 1. \quad (1)$$

The above equation is derived by assuming the potential energy of the form up to a constant

$$W_a = \sum_{a=1}^{3} \left( \frac{1}{2} J^a Q^a Q^a - H^a Q^a \right). \quad (2)$$

The first term of the potential energy accounts for the magnetic anisotropy energy and, for a uniaxial system, $J^a = \lambda \delta_{3a}$. If $\lambda < 0$, it describes a ferromagnet of the easy axis type and, if $\lambda > 0$, the easy plane type. The second term depicts the external magnetic energy and we will assume from here on $H^a = H \delta_{a3}$.

In (1+1) dimensions, LLE supports domain walls. In (2+1) dimensions, prototypical solitons are Belavin-Polyakov type lump solutions, which exist when there is no anisotropy energy [3]. It is obvious by the Derrick’s theorem that these two are the whole spectra of static solitons with finite energy supported by LLE. Furthermore, the Belavin-Polyakov lump solutions are conformally invariant, which makes them less applicable to realistic cases. Phenomenologically, there exist rich spectra of solitons without a scale invariance [1]. To understand these magnetic solitons theoretically, one may include the higher-order spatial derivative terms like Skyrmions [4], or consider charged objects as stationary solutions [5].

Here let us recall the relation between the superfluidity with a global $U(1)$ symmetry and the superconductivity of the Landau-Ginzburg theory of a local $U(1)$ symmetry. In superfluidity, there are global vortices with logarithmically divergent energy, whereas, in superconductivity, local vortices have finite energy. Thus gauging a theory provides a useful mechanism for obtaining the finite energy solitons. Among these local vortices, those which saturate Bogomol’nyi limit are important ones because they give a criterion to distinguish between type-I and type-II superconductivity in Maxwell theory [3]. It turns out that the Bogomol’nyi limit plays also a crucial role in finding a variety of self-dual solitons of intriguing
properties in the Chern-Simons (CS) gauge theories \[7, 8\]. Recent developments along these directions have been in the study of self-dual solitons in the gauged relativistic $O(3)$ nonlinear sigma model with the Maxwell term \[3\], the non-Abelian or the Abelian Chern-Simons term \[10, 11\]. In these models, the $O(3)$ global symmetry is broken down to $U(1)$ without a clear physical motivation. On the other hand, in the nonrelativistic uniaxial ferromagnet system described by Eq.(1), the $O(3)$ global symmetry is broken explicitly to $U(1)$ by the relativistic effect \[2\]. Therefore, it is natural to couple an Abelian gauge field to such $U(1)$ direction in the nonrelativistic theory.

Recently, a general way of gauging the ferromagnet defined on the Hermitian symmetric space $G/H$ was proposed, in which the maximal torus subgroup of $H$ was gauged and self-dual equations were obtained \[12\]. In this Letter, we will investigate the gauged LLE of the uniaxial ferromagnet system coupled to CS gauge field and study the self-dual solitons which saturate Bogomol’nyi limit.

We start from the action which is a $SU(2)$ version of the generalized planar anisotropic Chern-Simons ferromagnet coupled with the uniform background charge:

$$S = \int dt d^2 x \left\{ \text{Tr} [2Kg^{-1}D_t g + D_i QD_i Q] - \rho_e A_0 - W_a(Q) \right\} + S_g(A_\mu).$$

(3)

Here, $g \in SU(2)$, $K = i \text{ diag}(1/2, -1/2)$, $Q = gKg^{-1}$, and the covariant derivative is given by

$$D_\mu g = \partial_\mu g + A_\mu T^3 g, \quad D_i Q = \partial_i Q + A_i [T^3, Q].$$

(4)

$\rho_e$ is the uniform charge density which will be responsible for a rich structure of self-dual solitons. The magnetic anisotropy energy $W_a$ in Eq.(2) is given by

$$W_a = \frac{\lambda}{2} (Q^3 - v)(Q^3 + \rho_e).$$

(5)

For the gauge dynamics, we consider the Abelian Chern-Simons term

$$S_g(A_\mu) = \frac{\kappa}{2} \epsilon^{\mu\nu\rho} \int dt d^2 x A_\mu \partial_\nu A_\rho.$$  

(6)

The above action is invariant under a left CS gauge transformation:

$$g \rightarrow hg, \ h = \text{ diag}(\exp(-i\Lambda/2), \exp(i\Lambda/2))$$
\[ Q \to hQh^{-1}, \ A_\mu \to A_\mu + \partial_\mu \Lambda, \]  
(7)

Note that it is also invariant under a right local transformation, \( g \to gh \), which corresponds to the \( U(1) \) symmetry of the ungauged \( CP(1) \) model [13].

The Euler-Lagrange equation in terms of spin variables \( Q \) becomes the gauged inhomogeneous LLE:

\[ D_t Q + D_i [Q, D_i Q] - \lambda (Q^3 - \frac{v - \rho_e}{2}) [Q, T^a] = 0. \]  
(8)

The gauge field is governed by the Chern-Simons equation:

\[ \frac{k}{2} \epsilon^{\mu\nu\rho} F_{\nu\rho} = j^\mu. \]  
(9)

Here \( j^\mu \) is a conserved current expressed by

\[ j^\mu \equiv (\rho, j^i) = (Q^3 + \rho_e, 2\text{Tr}[Q, D_i Q]), \]  
(10)

and then \( Q_{U(1)} = \int d^2x Q^3 \) is conserved \( U(1) \) charge.

The gauge invariant topological current is given by

\[ T^\mu = \frac{1}{8\pi} \epsilon^{\mu\nu\rho} \left[ \epsilon^{abc} Q^a D_\nu Q^b D_\rho Q^c + F_{\nu\rho}(w - Q^3) \right] \]  
(11)

\[ = \frac{1}{8\pi} \epsilon^{\mu\nu\rho} \epsilon^{abc} Q^a \partial_\nu Q^b \partial_\rho Q^c + \frac{1}{4\pi} \epsilon^{\mu\nu\rho} \partial_\nu ((w - Q^3) A_\rho), \]  
(12)

where \( w \) is a free parameter. Note that this topological charge is reduced to the winding between a two sphere of compactified 2D space and that of the configuration space of \( Q^a \) when the gauge field vanishes. Eq. (12) tells us that the gauged topological current differs from the ungauged one by the curl of a vector field, and therefore the magnetic flux also contributes to the conserved topological charge \( T \equiv \int d^2x T^0 \).

A kinematical quantity specifying the CS solitons as the anyons is the angular momentum and its gauge field contribution is defined by

\[ J = \int d^2x \varepsilon_{ij} x_i \left\{ 2\text{Tr}[-T^3 (g^{-1} \partial_j g - (g^{-1} \partial_j g)|_{|\vec{x}| \to \infty}) - A_j Q^3] \right\}, \]  
(13)

where \( |\vec{x}| \to \infty \) denotes the boundary value of the soliton \( g(\infty) \) for \( \rho_e = 0 \), and the momentum density \( p_i \) is given by

\[ p_i = 2\text{Tr}(-T^3 g^{-1} D_i g). \]  
(14)
Here we count only the unambiguous contribution of the angular momentum from the CS
gauge field, which is finite in the limit of zero background CS charge [14].

Let us derive the Bogomol’nyi limit of the Chern-Simons ferromagnet system in Eq. (3)
under the specific anisotropic energy in Eq. (5). When \( \lambda = \mp \frac{2}{\kappa} \), the Chern-Simons system
achieves the Bogomol’nyi bound:

\[
E = \int d^2 x \{ \partial_i \left( -\frac{\kappa}{2} \epsilon_{ij} A_0 A_j \right) + A_0 (\kappa B - Q^3 - \rho_e) + \frac{1}{2} (D_i Q^a)^2 + W_a \} \\
= \int d^2 x \frac{1}{4} |D_i Q^a \pm \epsilon^{abc} \epsilon_{ij} Q^b D_j Q^c|^2 \pm 4\pi T,
\]

where \( T \) is the topological charge in which the free parameter \( w \) is fixed to be the parameter \( v \) in Eq. (5), and we used the Gauss’ law \( (B \equiv \frac{1}{2} \epsilon_{ij} F_{ij}) \)

\[
\kappa B - Q^3 - \rho_e = 0.
\]

Note that the spatial integration of the above Gauss’ law neglecting the uniform charge
density term \( \rho_e \) tells us that any flux-carrying Chern-Simons solitons are charged in this
model; \( \Phi \equiv \int d^2 x B = \frac{Q_{U(1)}}{\kappa} \).

The self-dual solitons which saturate the Bogomol’nyi bound satisfy the self-dual equations:

\[
D_i Q^a = \mp \epsilon^{abc} \epsilon_{ij} Q^b D_j Q^c.
\]

Introducing a parameterization of the spherical coordinates \( Q^a = (\sin F \cos \Theta, \sin F \sin \Theta, 
\cos F) \), we express the gauge field \( A_i \) in terms of the scalar fields by solving Eq. (17):

\[
A_i = -\partial_i \Theta \mp \epsilon_{ij} \partial_j \ln \tan \frac{F}{2}.
\]

Substituting Eq. (18) into the Gauss law in Eq. (16), we obtain a scalar equation for the
soliton configurations:

\[
\nabla^2 \phi \mp \epsilon^{ij} \partial_i \partial_j \Theta = -\frac{dV}{d\phi},
\]

where \( \phi = \ln \tan \frac{F}{2} \). The shapes of “effective” potential for scalar field \( \phi \) are (See Fig. 1)

\[
V(\phi) = \pm \frac{1}{\kappa} (\ln \cosh \phi - \rho_e \phi).
\]
One can easily notice that the Bogomol’nyi equations in Eq. (19) and Eq. (20) are independent of the external magnetic field $H$ in contrast to the fact that the vacuum configuration of the spin variable and the dispersion relation of the magnon are independent of the background CS charge.

Let us concentrate on the upper sign (self-dual) in Eq. (17). The anti self-dual case can be reached by replacing $\kappa$ with $-\kappa$. For the rotationally symmetric solutions, the ansatz in the cylindrical coordinate $(r, \theta)$ is given by

$$\phi = \phi(r), \quad \Theta = n\theta, \quad A_i = \frac{\epsilon_{ij}x_j}{r^2}a(r).$$ (21)

Then the equation of motion in Eq. (19) becomes an analogue of the one dimensional Newton’s equation for $r > 0$, if we regard $r$ as “time” and $\phi$ as the position of the hypothetical particle with unit mass:

$$\frac{d^2\phi}{dr^2} = -\frac{dV(\phi)}{d\phi} - \frac{1}{r} \frac{d\phi}{dr}.$$ (22)

The exerting forces are the conservative force from the effective potential $V(\phi)$ in Eq. (20).
and time-dependent friction $-\frac{1}{r} \frac{d\phi}{dr}$. When $n \neq 0$, there is an impact term at $r = 0$ due to $\varepsilon^{ij} \partial_i \partial_j n \theta = \frac{\alpha}{r} \delta(r)$ in Eq. (19).

The boundary conditions for the regular soliton configurations are the followings. (i) At the origin, $n F(0) = 0$ for the singlevaluedness of $Q$, and $a(0) = 0$ for the gauge field $A_i$ to be non-singular. (ii) At the spatial infinity, $F(\infty)$ is determined by the condition $\frac{dV(a)}{d\phi} \big|_{\phi(\infty)} = 0$, and the finiteness of energy requires $a(\infty) = n$ when $F(\infty)$ is neither zero nor $\pi$.

Under the ansatz in Eq. (21), the topological charge $T$ is expressed by

$$T = \frac{n}{2} \left[ \cos F(0) - \cos F(\infty) \right] + \frac{a(\infty)}{2} \left( \cos F(\infty) - \rho_e + \kappa H \right),$$

and the magnetic flux $\Phi$ (or equivalently the $U(1)$ charge) is given by

$$\Phi = -2\pi a(\infty).$$

A straightforward computation of Eq. (13) in terms of Eq. (21) yields the angular momentum when $\rho_e = 0$:

$$J \big|_{\rho_e = 0} = -\pi \kappa a(\infty)(a(\infty) - 2n).$$

| $(\kappa, \rho_e)$ | $(-, 1)$ | $(-, -1 < \rho_e < 1)$ | $(+, +1), (1, -1)$ | $(+, -1)$ |
|-----------------|--------|----------------|-----------------|--------|
| $F(0)$          | 0      | 0              | $0 < F(0) < \infty$ | 0      |
| $F(\infty)$     | $\pi$  | $2 \tan^{-1} \sqrt{\frac{1-\rho_e}{1+\rho_e}}$ | 0      | 0      |
| $a(\infty)$     | $\alpha$, $(0 < \alpha < n - 1)$ | $n$ | $\alpha$, $(\rho_e \alpha < -1)$ | $\alpha$, $(\alpha > n + 1)$ |
| $T$             | $n - \alpha(1 - \frac{\kappa H}{2})$ | $\frac{n}{2}(1 - \rho_e + \kappa H)$ | $\frac{\alpha}{2}(1 - \rho_e + \kappa H)$ | $\frac{\alpha}{2}(2 + \kappa H)$ |
| $\Phi$          | $-2\pi \alpha$ | $-2\pi n$ | $-2\pi \alpha$ | $-2\pi \alpha$ |
| $J \big|_{\rho_e = 0}$ | $-\pi \kappa \alpha(\alpha - 2n)$ | $\pi \kappa n^2$ | $-\pi \kappa \alpha^2$ | $-\pi \kappa \alpha(\alpha - 2n)$ |
| Species         | topological lump ($n \geq 2$) | topological vortex | nontopological soliton | nontopological vortex |

Table 2. Self-dual CS solitons for the sign of $\kappa$ and various values of $\rho_e$ (n means a positive integer and $\alpha$ means an arbitrary real number).
Figure 2. (a) Scalar functions $F(r)$ of various solitons and (b) Gauge fields $a(r)$ of various solitons: (i) Topological lump for $(-1, 1)$, (ii) Topological vortices for $(-1, 0)$ and $(-1, -0.5)$, (iii) Nontopological solitons for $(1, +1)$ and $(1, -1)$, (iv) Nontopological vortex for $(1, -1)$. 
Therefore, one can easily find all the possible rotationally symmetric soliton solutions by the shooting argument \cite{15} and they are classified in Table 1. Note that in Table 1 we do not include trivial solutions and the solutions approaching their boundary values with oscillations. Numerical solutions of the self-dual solitons are given in Figure 2.

In summary, we studied the Bogomol’nyi limit of a uniaxial ferromagnet system coupled to the Abelian CS gauge field. We found various solitons: the nontopological solitons and vortices, the topological vortices, and the topological lumps. When there is no background CS charge, there is uniquely the topological vortex of half winding. We make the following remarks: (i) For $\rho_e = +1(-1)$, the Bogomol’nyi equation for the nontopological solitons and vortices reduces to the Liouville equation when $Q_3$ is near $-1(+1)$, and these solitons are the same as those in the nonrelativistic Abelian self-dual CS scalar model \cite{8}. (ii) The unboundedness of the effective potential $V(\phi)$ in Eq. (20) at $\phi = \pm \infty$ disappears in the relativistic counterpart of our model \cite{11} as is the case between the relativistic \cite{7} and nonrelativistic \cite{8} Abelian self-dual CS scalar models. (iii) The vacuum value of $Q_3$ determined by the minimization of the anisotropy energy depends on the external magnetic field $H$. On the other hand, the boundary values of solitons $Q_3(\infty)$ are fixed by the background CS charge. Therefore, the vacuum value of $Q_3$ need not coincide with the boundary value of solitons even when the background CS charge vanishes.

We conclude with a final comment. The Bogomol’nyi limit of our model leaves no room for free parameters. For example, the original free parameter $w$ of the topological charge is decided by the external magnetic field $H$ and the background charge $\rho_e$. Our model, therefore, possesses the merit of predicting the observables if the limit could be indeed realized in a ferromagnet sample with a given $\lambda$.

We thank Q-H. Park for useful discussions. This work is supported by the KOSEF through the CTP at Seoul National University and the project number(96-0702-04-01-3, 96-1400-04-01-3), and by the Ministry of Education through the Research Institute for Basic Science(BSRI/96-1419, 96-2434).
References

[1] For a review, see A.M. Kosevich, B.A. Ivanov and A.S. Kovalev, Phys. Rep. 194, 117 (1990).

[2] For a review, see E.M. Lifshitz and L.P.P. Pitaevskii, *Statistical Physics Part 2: Landau and Lifshitz Course of Theoretical Physics Vol. 9*, (Pergamon Press, Oxford, 1980).

[3] A.A. Belavin and A.M. Polyakov, JETP Lett. 22, 245 (1975).

[4] B.A. Ivanov and V.A. Stephanovich, Phys. Lett. A 141, 89 (1989); B.A. Ivanov, V.A. Stephanovich and A.A. Zhmudskii, J. Magn. Magn. Mat. 88, 116 (1990).

[5] R.A. Leese, Nucl. Phys. B 344, 33 (1990); B 366, 283 (1991).

[6] E.B. Bogomol’nyi, Yad. Fiz. 24, 861 (1976) [Sov. J. Nucl. Phys. 24, 449 (1976)].

[7] J. Hong, Y. Kim and P. Y. Pac, Phys. Rev. Lett. 64, 2230 (1990); R. Jackiw and E. J. Weinberg, *ibid* 64, 2234 (1990); R. Jackiw, K. Lee and E. J. Weinberg, Phys. Rev. D 42, 3488 (1990).

[8] R. Jackiw and S.-Y. Pi, Phys. Rev. Lett. 64, 2969 (1990); Phys. Rev. D 42, 3500 (1990); 48, 3929(E) (1993).

[9] B. J. Schroers, Phys. Lett. B 356, 291 (1995); J. Gladikowski, B. M. A. G. Piette and B. J. Schroers, Phys. Rev. D 53, 844 (1996).

[10] G. Nardelli, Phys. Rev. Lett. 73, 2524 (1994).

[11] P. K. Ghosh and S. K. Ghosh, Phys. Rev. B 366, 199 (1996); K. Kimm, K. Lee and T. Lee, Phys. Rev. D 53, 4436 (1996). K. Arthur, D. H. Tchrakian and Y. Yang, Phys. Rev. D 54, 5245 (1996).

[12] P. Oh and Q-H. Park, [hep-th/9612063](http://arxiv.org/abs/hep-th/9612063), Phys. Lett. B (in press).
[13] P. Oh and Q-H. Park, Phys. Lett. B 383, 333 (1996).

[14] For other definition, see, e.g., N. Papanicolaou and T.N. Tomaras, Nucl. Phys. B 360, 425 (1991); E.G. Floratos, Phys. Lett. B 279, 117 (1992).

[15] C. Kim, S. Kim and Y. Kim, Phys. Rev. D 147, 5434 (1993); C. Kim and Y. Kim, Phys. Rev. D 50, 1040 (1994).