Lower bound for entanglement cost of antisymmetric states

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Abstract

This report gives a lower bound of entanglement cost for antisymmetric states of bipartite $d$-level systems to be $\log_2 \frac{d}{d-1}$ ebit (for $d = 3, E_c \geq 0.585\ldots$). The paper [1] claims that the value is equal to one ebit for $d = 3$, since all of the eigenvalues of reduced matrix of any pure states affiliating to $\mathcal{H}^\otimes N$ is not greater than $2^{-N}$, thus the von Neumann entropy is not less than $N$, but the proof is not true. Hence whether the value is equal to or less than one ebit is not clear at this moment.

1 Introduction

Entanglement cost is determined by asymptotic behavior of entanglement formation [3], but it is regarded to be very difficult to calculate. The paper [1] claims that the entanglement cost of antisymmetric states of bipartite three-level system is one ebit. However, the proof in that paper is not correct (for the version of January 11, 2002) as explained as follows. The essential point of the proof of that paper is, all of the eigenvalues of reduced matrix of any pure states affiliating to $\mathcal{H}^\otimes N$ for $d = 3$ is not greater than $2^{-N}$. Thus the von Neumann entropy $-\sum \lambda_i \log_2 \lambda_i$ is not less than $N$ bit. Hence any mixed states supported on antisymmetric states, whose decomposition is always on antisymmetric states, have the entanglement formation not less than $N$ bit. Therefore entanglement cost is not less than one ebit. However, there exist counterexamples, i.e., the largest eigenvalue of the reduced matrix of $\frac{1}{\sqrt{3}} \sum_{1 \leq i < j \leq 3} (\langle ii | \langle jj | - \langle jj | \langle ii | )^\otimes 2 \in \mathcal{H}^\otimes 2$ is $\frac{\sqrt{2}}{3} (\frac{1}{2})$. Hence at this moment, it is not clear whether the entanglement cost of antisymmetric states for bipartite three-level system is one ebit or not.

This report furnish a lower bound of the entanglement cost of antisymmetric states for bipartite $d$-level systems. It is proved that all of the eigenvalues of reduced matrix of any pure states affiliating to $\mathcal{H}^\otimes N$ for general $d$ is not greater than $\left( \frac{d-1}{d} \right)^N$. This is proved by investigating a certain map $\tilde{\Lambda}$, which is defined at expression (2) later, whether it is CP or not.

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2 Results

2.1 Problem Setup

Let us assume each of $\mathcal{H}_A$ and $\mathcal{H}_B$ is a $d$-dimentional Hilbert space with basis $D := \{|i\rangle\}_{i=1,...,d}$ and $\mathcal{H}_{AB} := \mathcal{H}_A \otimes \mathcal{H}_B$. For $1 \leq i \leq j \leq d$, 

$$|(i,j)) := \frac{|iangle \otimes |j\rangle_B - |j\rangle \otimes |i\rangle_B}{\sqrt{2}} \in \mathcal{H}_{AB}$$

and $D' := \{|(i,j)|\}_{1 \leq i \leq j \leq d}$, the antisymmetric space $\mathcal{H}_- := \text{span} D' \subset \mathcal{H}_{AB}$.

**Notation 1 (matrices)** For a positive integer $m$, $\mathcal{M}_m$ is a set of $m \times m$-dimentional matrices with each entry a complex number $\mathbb{C}$. For a set $\mathcal{X}$, $[a_{ij}]_{i,j \in \mathcal{X}}$ is a matrix with $(i,j)$-component specified $a_{ij}$, and $\mathcal{M}(\mathcal{X}) := \{[a_{ij}]_{i,j \in \mathcal{X}} | a_{ij} \in \mathbb{C}\}$ is a collection of matrices each rows and columns are labelled with elements of $\mathcal{X}$.

**Notation 2 (partial order between matrices)** The partial order $\leq$ in $\mathcal{M}(\mathcal{X})$ is introduced as follows. For $X_1, X_2 \in \mathcal{M}(\mathcal{X})$, $X_1 \geq X_2 \Leftrightarrow X_1 - X_2 \geq 0 \Leftrightarrow X_1 - X_2$ is a positive matrix. Note that a positive matrix is always hermitian.

**Definition 3 ($\Lambda : \mathcal{M}(D') \to \mathcal{M}(D)$)** The map $\Lambda : \mathcal{M}(D') \to \mathcal{M}(D)$ is defined as follows. First $X \in \mathcal{M}(D')$, is regarded as an antisymmetric state $\rho_1 := \sum_{I,J \in D} X_{IJ} |I\rangle \langle J| \in \mathcal{H}_-$. Then $\rho_1$ is reduced into $\mathcal{H}_A$ by the operation $\rho_2 := \text{Tr}_B \rho_1 \in \mathcal{H}_A$, and is converted into the matrix representation $Y \in \mathcal{M}(D)$ with basis $D$ satisfying $\rho_2 = \sum_{i,j \in D} Y_{ij} |i\rangle \langle j|$. This transformation $X \mapsto Y$ is the map $\Lambda$.

The derivations of this map $\Lambda$ are investigated in the section 2.2.

**Notation 4 ($e_{ij}^X$)** For a set $\mathcal{X}$ and $i,j \in \mathcal{X}$, $e_{ij}^X \in \mathcal{M}(\mathcal{X})$ is a matrix with entry 1 only at $(i,j)$-component and 0 elsewhere. For example, for $\mathcal{X} = \{1,2,3\}$, $e_{1,2}^X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

The notations such as $[e_{ij}^X]_{i,j \in \mathcal{X}}$, which is equal to

$$
\begin{pmatrix}
    e_{11}^X & e_{12}^X & e_{13}^X \\
    e_{21}^X & e_{22}^X & e_{23}^X \\
    e_{31}^X & e_{32}^X & e_{33}^X \\
\end{pmatrix}
= 
\begin{pmatrix}
    1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
$$

when $\mathcal{X} = \{1,2,3\}$, for example, will be used in this report. This example indicates a $3 \times 3$ block matrix with $3 \times 3$ matrices, and can be treated as a $9 \times 9$.

**Notation 5 ($\Lambda^{(1)}$)** $\Lambda^{(1)}$ is defined as a mapping $X \mapsto \Lambda(X^\dagger)$ that is a compound map $\Lambda$ preceded by matrix adjoint (Hermitan transpose). Note that $\Lambda^{(1)}$ operates on hermitian matrices as same as $\Lambda$ operate on, i.e. for a hermitian matrix $X$, $\Lambda^{(1)}(X) = \Lambda(X)$ since $X^\dagger = X$. 

2
In this report "map" is a mapping between matrices.

**Notation 6 (identities)** Let us assume each of \( M, M' \) is either of \( M_m \) or \( M(\mathbb{F}) \). Then id, Id, \( Id^{\#} \), are denoted as follows: id is an identity matrix of \( M, \) Id is an identity map on \( M \), \( Id^{\#} \) is a linear map \( M \ni X \mapsto (\text{Tr} X) \cdot \text{id} \in M' \). \( M, M' \) will be dropped sometimes, such as, id, Id and \( Id^{\#} \).

### 2.2 Propositions and theorems

**Lemma 7** For scalars \( x, y \), eigenvalues of \( \left( \frac{x}{y} - 1 \right) \left( x + y \Lambda^{(1)} \right) \left[ E_{ij}^{D'} \right]_{i,J \in D'} \) are 

\[-y, \frac{1}{y}, \frac{x-1}{x} x + \frac{1}{x} y.\]

**proof** The considering matrix is equal to \( \Xi := \left( \frac{x}{y} - 1 \right) \left( x + y \Lambda^{(1)} \right) \left[ E_{ij}^{D'} \right]_{i,J \in D'} \). For \( (i, j), (k, l) \in D' \), \( \Lambda \left( E_{(i,j)(k,l)}^{D'} \right) = \text{Tr}_B |(i, j)\rangle \langle (k, l)| = \frac{1}{2} \left( \begin{array}{cc} \delta_{i,k} & -\delta_{i,l} \\ -\delta_{i,k} & \delta_{i,l} \end{array} \right) \). Observing the whole matrix \( \Xi \), it is decomposed into the form of direct product \( \Xi = \frac{x}{y} \Xi_1 + \left( \frac{x}{y} \Xi_2 + \frac{x}{y} \Xi_3 \right) \) where

\[\Xi_1 = \bigoplus_{1 \leq i < j < k \leq d} (i,j) \otimes (i,k) \otimes (j,k) \otimes i \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix},\]

\[\Xi_2 = \bigoplus_{1 \leq i \leq d} i \begin{pmatrix} 1 & \ldots & 1 & -1 & \ldots & -1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -1 & \ldots & -1 & 1 & \ldots & 1 \end{pmatrix},\]

\[\Xi_3 = \bigoplus_{1 \leq i \leq d} i \begin{pmatrix} 1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 1 \end{pmatrix} \Xi_1 \text{ has eigenvalues } -y, \frac{y}{x}, \text{ and } \left( \frac{x}{y} \Xi_2 + \frac{x}{y} \Xi_3 \right) \text{ has eigenvalues } \frac{x}{y}, \frac{x-1}{x} x + \frac{1}{x} y.\]
Lemma 8 Let $\lambda(x, y) := \max\{|-y|, |\frac{d-1}{2}x + \frac{1}{2}y|\}$. Then

$$\begin{align*}
\arg \min_{x+y=1} \lambda(x, y) &= \left(\frac{1}{d}, \frac{d-1}{d}\right), \\
\min_{x+y=1} \lambda(x, y) &= \lambda\left(\frac{1}{d}, \frac{d-1}{d}\right) = \frac{d-1}{d}.
\end{align*}$$

Notation 9 ($\tilde{\lambda}, \tilde{\Lambda}$) Let

$$\begin{align*}
\tilde{\lambda} &:= \frac{d-1}{d}, \\
\tilde{\Lambda} &:= \frac{1}{d}\Lambda + \frac{d-1}{d}\Lambda^{(i)}.
\end{align*}$$

Note that due to the last two lemmata,

$$-\tilde{\lambda}\text{id} \leq \left(\text{Id} \otimes \tilde{\Lambda}\right) \left[\mathcal{E}_{IJ}^{D'}\right]_{I, J \in D'} \leq \tilde{\lambda}\text{id}$$

i.e. the absolute values of every eigenvalues of the central side are not larger than $\tilde{\lambda}$. Note that (2) operates on hermitian matrices as same as $\Lambda$ operates on.

Notation 10 If it is written $D^{N'}$ where $D'$ is the set of basis of $\mathcal{H}_{-}$, it associates the direct product of the set $D'$, indicates the basis of $\mathcal{H}_{-}^{\otimes N}$.

Lemma 11

$$\left(\text{Id} \otimes \tilde{\Lambda}^{\otimes N}\right) \left[\mathcal{E}_{IJ}^{D'}\right]_{I, J \in D'} \geq 0$$

This is equivalent to that the left hand side is positive matrix. Here, $\text{Id}^{\#}$ is a map from $\mathfrak{M}(D')^{\otimes N}$ into $\mathfrak{M}(D)^{\otimes N}$.

proof The inequality (4) is equivalent to

$$\left(\text{Id} \otimes \tilde{\lambda}^{N}\text{id}^{\#}\right) \left[\mathcal{E}_{IJ}^{D'}\right]_{I, J \in D'} \geq \left(\text{Id} \otimes \tilde{\Lambda}^{\otimes N}\right) \left[\mathcal{E}_{IJ}^{D'}\right]_{I, J \in D'}$$

To verify this inequality, the following is enough.

$$\begin{align*}
\text{LHS.} & = \tilde{\lambda}^{N} \left[\text{Id}^{\#} \left(\mathcal{E}_{IJ}^{D'}\right)\right]_{I, J} = \tilde{\lambda}^{N} [\delta_{IJ} \text{id}]_{I, J} = \tilde{\lambda}^{N} \text{id} \\
\text{RHS.} & = \left(\text{Id}^{\otimes N} \otimes \tilde{\Lambda}^{\otimes N}\right) \left[\mathcal{E}_{IJ}^{D'}\right] \otimes N = \left(\text{Id}^{\otimes N} \otimes \tilde{\Lambda}\right) \left[\mathcal{E}_{IJ}^{D'}\right] \otimes N \\
& \leq \left(\tilde{\lambda} \text{id}\right)^{\otimes N} = \tilde{\lambda}^{N} \text{id} \quad (\leq \text{ due to (3) })
\end{align*}$$

The last lemma successively induces next two propositions.

Proposition 12 $\tilde{\lambda}^{N} \text{id}^{\#} - \tilde{\Lambda}^{\otimes N}$ is a CP map.
This is due to (4) and [2]. Namely, for a map \( \Gamma : \mathcal{M}_m \to \mathcal{M}_n \), it is CP iff \((\text{Id}_{\mathcal{M}_m} \otimes \Gamma)(E_{ij})_{i,j=1...m} = [\Gamma(E_{ij})]_{i,j=1...m}\) is a positive matrix.

**Proposition 13** \( \Lambda^\otimes N(X) \leq \tilde{\lambda}^N \text{id} \) for \( X \in \mathcal{M}(D')^\otimes N \) if \( X \geq 0 \), \( \text{Tr} \; X = 1 \).

Let us denote entanglement measures \( E, E_f \) and \( E_c \) as von Neumann entropy, entanglement formation and entanglement cost, respectively. Proposition 13 is applied to calculate (a lower bound of) these values. In this report, the base of entropy is always fixed to two, regardless the dimension \( d \) of either \( \mathcal{H}_A \) or \( \mathcal{H}_B \). That is to say, for density matrix \( \rho \), the von Neumann entropy is \( E(\rho) = -\text{Tr} \rho \log_2 \rho \), not \( -\text{Tr} \rho \log_d \rho \).

**Theorem 14** \( E(|\Psi\rangle) \geq N \log_2 \frac{d}{d-1} \) for any pure state \( |\Psi\rangle \in \mathcal{H}_-^\otimes N \).

This is because the last proposition indicates that all of the eigenvalues of the reduced matrix from arbitrary antisymmetric states are less than or equal to \((d/d - 1)^{-N}\).

**Lemma 15** \( E_f(\sigma) \geq N \log_2 \frac{d}{d-1} \) for any density matrix \( \sigma \) supported on \( \mathcal{H}_-^\otimes N \).

**proof** Entanglement formation is defined as

\[
E_f(\rho) := \min_{(p_i,|\Phi_i\rangle)} \sum_i p_i E(\Phi_i) \tag{5}
\]

where

\[
\Delta(\rho) := \left\{ (p_i,|\Phi_i\rangle) \left| (p_i > 0, \|\Phi_i\| = 1) \forall i, \sum_i p_i = 1, \sum_i p_i |\Phi_i\rangle\langle\Phi_i| = \rho \right\} \tag{6}
\]

is the collection of all possible decompositions of \( \rho \). It is known that all of \( |\Phi_i\rangle \) induced from \( \Delta(\rho) \) satisfy \( |\Phi_i\rangle \in \text{Range}(\rho) \), where \( \text{Range}(\rho) \) is sometimes called image space of a matrix \( \rho \) which is a collection of \( \rho|\psi\rangle \) with \( |\psi\rangle \) running over the domain of \( \rho \). Hence

\[
E_f(\rho) \geq \min\{ E(\Phi)|\Phi \in \text{Range}(\rho), \|\Phi\| = 1 \}. \tag{7}
\]

The condition of the lemma above implies \( \text{Range}(\rho) \subseteq H_-^\otimes N \), therefore the last theorem implies \( E_f(\sigma) \geq N \).

The paper [3] claims that \( E_c(\rho) = \lim_{N \to \infty} \frac{E_f(\rho^\otimes N)}{N} \), therefore the value of entanglement cost is given as follows.

**Theorem 16** \( E_c(\sigma) \geq N \log_2 \frac{d}{d-1} \) for any density matrix \( \sigma \) supported on \( \mathcal{H}_-^\otimes N \).

**Corollary 17** (The lower bound of entanglement cost for \( \mathcal{H}_- \))

\[
E_c(\sigma) \geq \log_2 \frac{d}{d-1} \tag{8}
\]

for any density matrix \( \sigma \) supported on \( \mathcal{H}_- \).
3 Conclusion and Discussion

This report gave a lower bound of entanglement cost of antisymmetric states for $d$-dimentional antisymmetric states as inequality (8). However, it is still open probem whether the entanglement cost for $d = 3$ is one ebit or not is not clear.

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