Polynomial splines interpolating prime series

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Abstract: Differentiable real function reproducing primes up to a given number and having a differentiable inverse function is constructed. This inverse function is compared with the Riemann-Von Mangoldt exact expression for the number of primes not exceeding a given value. Software for computation of the direct and inverse functions and their derivatives is developed. Examples of approximate solution of Diophantine equations on the primes are given.

1. Introduction

This article introduces real functions reproducing the values of mutually inverse arithmetic functions \( p(n) : N \to P \) (prime \( p(n) \) at a number \( n \)) and \( p^{-1}(p) : P \to N \) (number \( n \) of the prime \( p(n) \)).

The found functions are employed to create subroutines for computation of \( p(x), \frac{dp(x)}{dx} \) on \( 1 \leq x < \infty \), and \( p^{-1}(x), \frac{dp^{-1}(x)}{dx} \) on \( 2 \leq x < \infty \).

The above-noted programs can be used for a numerical solution of different problems on the set of primes \( P \), including approximate solution of Diophantine equations on \( P \).

The idea consists in establishing a differentiable function which would include the values of primes and which would allow one to construct an inverse function \( p^{-1}(x) \) by a Newton method. More precisely, the sought function \( p(x), 1 \leq x < \infty \) should satisfy the following conditions:

a) \( p(x) \) reproduces primes;

b) there exists a positive derivative \( \frac{dp(x)}{dx} \);

c) there exists an inverse function \( p^{-1}(x) : [2, \infty) \to [1, \infty) \).

As is known, there are no one-variable polynomials which can produce all the primes, or primes only. However, this article shows that there exist polynomial splines reproducing primes in series (along with the continuation of the prime series) and satisfying in addition the conditions b) and c).

A spline formed by polynomials with integer coefficients will be called the arithmetic spline.

This article discusses two candidates for the arithmetic splines, cubic one and parabolic.

These splines do not approximate a prime series. Primes are implanted in the structure of the splines, which ensures that they are exactly reproducible. Such splines lead to explicit soluble systems of linear equations whose coefficients represent arithmetic functions themselves.

The inverse function \( p^{-1}(x) \) constitutes a differentiable analogue of the number–theoretic function \( \pi(x) = \sum_{p \leq x} 1 \) which is comparable with the Riemann exact expression for \( \pi(x) \) through the zeros of the \( \zeta \)–function( [1], page 34).

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2. Cubic spline

Consider the spline

\[
S_{\text{cub}}(x) = \begin{cases} 
  x + 1, & 1 \leq x \leq 1.5, \\
  c_i(x), & i - 0.5 \leq x \leq i + 0.5, \\
  c_i(x) = c_{i+1}(x), & x = i + 0.5, \\
  \frac{d c_i(x)}{d x} = \frac{d c_{i+1}(x)}{d x}, & x = i + 0.5
\end{cases}
\]

with

\[
c_i(x) = 2 \left( a_i(x - i - 0.5)^2 + b_i(x - i - 0.5) + \frac{p(i) + p(i + 1)}{2} \right) (x - i) - 2p(i)(x - i - 0.5).
\]

Exact reproducibility of the primes follows from the identity

\[
c_i(i) \equiv p(i), \quad i = 1, 2, \ldots
\]

At the points of sewing the spline should also obey the identity

\[
c_i(i + 0.5) \equiv \frac{1}{2}(p(i) + p(i + 1)), \quad i = 1, 2, \ldots
\]

which brings into the spline additional information of the prime series behaviour.

There exists an unique cubic spline of the kind \(S_{\text{cub}}(x)\) with the coefficients

\[
a_i = \frac{1}{2}(p(i + 1) - p(i - 1)) - 1, \\
b_i = p(i + 1) - p(i) - 1,
\]

This spline can be considered only as almost–arithmetic. The coefficients \(\gamma_i\) and \(\delta_i\) in \(c_i(x) = \alpha_i x^3 + \beta_i x^2 + \gamma_i x + \delta_i\) appear for some \(i\) in the form \(q + 1/2, q \in \mathbb{N}\).

The negative value of the discriminant

\[
d_i = 4(p(i))^2 - 4(p(i - 1) + p(i + 1))p(i) + \frac{1}{4}(p(i - 1))^2 + \frac{1}{4}(p(i + 1))^2 + \frac{7}{2}p(i - 1)p(i + 1) + 3
\]

gives positivity of the derivative

\[
\frac{d c_i(x)}{d x} = 2(2a_i(x - i - 0.5) + b_i)(x - i) + 2a_i(x - i - 0.5)^2 + 2b_i(x - i - 0.5) - p(i) + p(i + 1).
\]

The inequality \(d_i < 0\) leads to the following condition for prime triplets \(p(i - 1), p(i), p(i + 1)\):

\[
\frac{1}{2}(p(i - 1) + p(i + 1)) - \frac{1}{4}\sqrt{t_i} < p(i) < \frac{1}{2}(p(i - 1) + p(i + 1)) + \frac{1}{4}\sqrt{t_i},
\]
Conditions (3) can be violated in the cases

\[
\begin{align*}
\Delta_1 &= 2, \\
\Delta_2 &= 28.
\end{align*}
\]

Among the first 1000 primes only 5 triplets violate the rule (3):

\[
(2969, 2971, 2999),
(2971, 2999, 3001),
(3271, 3299, 3301),
(6917, 6947, 6949),
(7757, 7759, 7789).
\]

Except these triplets (including twin pairs), condition (3) is violated by triplets of the kind (4) at the following values for \(\Delta_1\) and \(\Delta_2\):

\[
\begin{align*}
\Delta_1 &= 4, \\
\Delta_2 &= 56, \\
\Delta_1 &= 6, \\
\Delta_2 &= 84, \\
\Delta_1 &= 8, \\
\Delta_2 &= 114,
\end{align*}
\]

and all that.
Despite the cases where condition (3) is violated, spline $S_{\text{cub}}(x)$ is convenient for creating subroutines $p(x)$, $\frac{dp(x)}{dx}$ and $p^{-1}(x)$, since the inverse function $p^{-1}(x)$ exists in the neighborhood of each prime number.

3. Parabolic spline

Given the following pairs of parabolas

$$q_i(x) = \begin{cases} 
 q^l_i(x), & i - 0.5 \leq x \leq i, \quad i = 2, 3, \ldots, \\
 q^r_i(x), & i \leq x \leq i + 0.5, \quad i = 2, 3, \ldots,
\end{cases}$$

$$q^l_i(x) = q^r_i(x)$$

$$\frac{dq^l_i(x)}{dx} = \frac{dq^r_i(x)}{dx}$$

with

$$q^l_i(x) = -2a_{i-1}(x - i)^2 + x - i + p(i),$$

$$q^r_i(x) = 2a_i(x - i - 0.5)^2 + (2a_i + 1)(x - i - 0.5) + \frac{p(i) + p(i + 1)}{2},$$

$$a_i = p(i + 1) - p(i) - 1.$$
Figure 3: Inverse functions $p^{-1}(x)$: thick line corresponds to $S_{\text{cub}}$, thin line to $S_{\text{quad}}$.

The parabolic spline

$$S_{\text{quad}}(x) = \begin{cases} x + 1, & 1 \leq x \leq 1.5, \text{ initial polynomial,} \\ q_i(x), & i - 0.5 \leq x \leq i + 0.5, \quad i = 2, 3, \ldots, \\ q_i(x) = q_{i+1}(x) \\ \frac{dq_i(x)}{dx} = \frac{dq_{i+1}(x)}{dx} & x = i + 0.5, \quad i = 1, 2, \ldots; \end{cases}$$

solves the problem better than the spline $S_{\text{cub}}$. It has the following properties:

1) identities analogous to (1) and (2) are applicable

$$q_i(i) \equiv p(i),$$

$$q_i(i + 0.5) \equiv \frac{1}{2}(p(i) + p(i + 1)), \quad i = 1, 2, \ldots;$$

2) the derivatives

$$\frac{dq_i(x)}{dx} = 4a_{i-1}(i - x) + 1,$$

$$\frac{dq_i(x)}{dx} = 4a_i(x - i) + 1$$

take a minimal value +1 at the points of internal sewing (they are the points of interpolation to the spline) and maximal values at the points of external sewing, where they coincide with the derivative of the spline $S_{\text{cub}}$. 
The positive values of the derivatives show that the spline $S_{quad}(x)$ monotonically increases on the semi-axis $[1, \infty)$.

There exists a function $S_{quad}^{-1}(x)$, inverse to the function $S_{quad}(x)$, determined on the axis $[2, \infty)$ and thus the spline $S_{quad}$ fulfills the conditions a), b), c). Moreover, the function $S_{quad}^{-1}(x)$ is differentiable on $(2, \infty)$.

The spline $S_{quad}$ is arithmetic because prime number polynomials

$$q_l^i(x) = \alpha_l^i x^2 + \beta_l^i x + \gamma_l^i \quad \text{and} \quad q_r^i(x) = \alpha_r^i x^2 + \beta_r^i x + \gamma_r^i$$

hold integer coefficients (see Table 1):

$$\alpha_l^i = -2a_{i-1}; \quad \beta_l^i = 4ia_{i-1} + 1; \quad \gamma_l^i = -2i^2a_{i-1} + p(i) - i;$$

$$\alpha_r^i = 2a_i; \quad \beta_r^i = -4ia_i + 1; \quad \gamma_r^i = 2i^2a_i + p(i) - i.$$ 

The joint satisfiability of identities (6), (7) reinforces the hypothesis that the spline $S_{quad}$ is an unique arithmetic spline satisfying the conditions a), b) and c).

Table 1 $(d_l^i = (\beta_l^i)^2 - 4\alpha_l^i\gamma_l^i, \quad d_r^i = (\beta_r^i)^2 - 4\alpha_r^i\gamma_r^i)$

| $i$ | $p(i)$ | $\alpha_l^i$ | $\beta_l^i$ | $\gamma_l^i$ | $d_l^i$ | $\alpha_r^i$ | $\beta_r^i$ | $\gamma_r^i$ | $d_r^i$ |
|-----|--------|------------|------------|------------|--------|------------|------------|------------|--------|
| 2   | 3      | 0          | 1          | 1          | 1      | 2          | -7         | 9          | -23    |
| 3   | 5      | -2         | 13         | -16        | 41     | 2          | -11        | 20         | -39    |
| 4   | 7      | -2         | 17         | -29        | 57     | 6          | -47        | 99         | -167   |
| 5   | 11     | -6         | 61         | -144       | 265    | 2          | -19        | 56         | -87    |
| 6   | 13     | -2         | 25         | -65        | 105    | 6          | -71        | 223        | -311   |
| 7   | 17     | -6         | 85         | -284       | 409    | 2          | -27        | 108        | -135   |
| 8   | 19     | -2         | 33         | -117       | 153    | 6          | -95        | 395        | -455   |
| 9   | 23     | -6         | 109        | -472       | 553    | 10         | -179       | 824        | -919   |
| 10  | 29     | -10        | 201        | -981       | 1161   | 2          | -39        | 219        | -231   |
| 11  | 31     | -2         | 45         | -222       | 249    | 10         | -219       | 1230       | -1239  |
| 12  | 37     | -10        | 241        | -1415      | 1481   | 6          | -143       | 889        | -887   |
| 13  | 41     | -6         | 157        | -986       | 985    | 2          | -51        | 366        | -327   |
| 14  | 43     | -2         | 57         | -363       | 345    | 6          | -167       | 1205       | -1031  |
| 15  | 47     | -6         | 181        | -1318      | 1129   | 10         | -299       | 2282       | -1879  |
| 16  | 53     | -10        | 321        | -2523      | 2121   | 10         | -319       | 2597       | -2119  |
| 17  | 59     | -10        | 341        | -2848      | 2361   | 2          | -67        | 620        | -471   |
| 18  | 61     | -2         | 73         | -605       | 489    | 10         | -359       | 3283       | -2439  |
| 19  | 67     | -10        | 381        | -3562      | 2681   | 6          | -227       | 2214       | -1607  |
| 20  | 71     | -6         | 241        | -2349      | 1705   | 2          | -79        | 851        | -567   |
| 21  | 73     | -2         | 85         | -830       | 585    | 10         | -419       | 4462       | -2919  |

It should be noted here that the coefficient $a_i$ in $q_l^i(x)$ and $q_r^i(x)$ represents a basic arithmetic function – the number of composite numbers in the interval $(p(i), p(i + 1))$.

Figures 1–3 present a comparison of the splines $S_{cub}$ and $S_{quad}$. The interval’s $[428, 432]$ image (argument to the functions $p(x)$ and $dp(x)$) contains the first pair of triplets $[4]$, violative the positivity of the derivative $S_{cub}'(x)$. 

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Figure 3 shows intervals where no inverse function \( p^{-1}(x) \) for the spline \( S_{cub} \) exists.

Figure 2 illustrates the properties of the derivatives \( S'_{cub} \) and \( S'_{quad} \): reaching the minimal and maximal values and the equalities of derivatives at the sewing points.

4. Inverse parabolic spline \( S^{-1}_{quad}(x) \) and its derivative

The pairs of functions

\[
t_i(x) = \begin{cases} 
  t'_i(x), & \frac{p(i) + p(i + 1)}{2} \leq x \leq p(i), \quad i = 2, 3, \ldots, \\
  t''_i(x), & \frac{p(i) + p(i + 1)}{2} \leq x \leq p(i) + p(i + 1), \quad i = 2, 3, \ldots, \\
  t''_i(x) = t'_i(x) \quad & \text{for all } i,
\end{cases}
\]

where

\[
t'_i(x) = i + 1 - \frac{\sqrt{b'_i}}{4a_i - 1}, \quad (t'_2 = x - 1), \quad t''_i(x) = i + \frac{\sqrt{b''_i} - 1}{4a_i},
\]

\[
b'_i = 8a_{i-1}(p(i) - x) + 1, \quad b''_i = 8a_i(x - p(i)) + 1,
\]

\[a_i = p(i+1) - p(i) - 1,\]

determine the inverse spline \( S^{-1}_{quad}(x) \)

\[
S^{-1}_{quad}(x) = \begin{cases} 
  x - 1, & 2 \leq x \leq 2.5, \\
  t_i(x), & \frac{p(i) + p(i + 1)}{2} \leq x \leq \frac{p(i) + p(i + 1)}{2}, \quad i = 2, 3, \ldots, \\
  t_i(x) = t_{i+1}(x) \quad & \text{for all } i,
\end{cases}
\]

The derivative of the inverse spline is as follows

\[
\frac{dS^{-1}_{quad}(x)}{dx} = \begin{cases} 
  1, & 2 \leq x \leq 2.5, \\
  (b'_i)^{-1/2}, & \frac{p(i) + p(i + 1)}{2} \leq x \leq p(i), \quad i = 2, 3, \ldots, \\
  (b''_i)^{-1/2}, & p(i) \leq x \leq \frac{p(i) + p(i + 1)}{2}, \quad i = 2, 3, \ldots.
\end{cases}
\]

5. About subroutines \( p(x), \frac{dp(x)}{dx}, p^{-1}(x) \) and \( \frac{dp^{-1}(x)}{dx} \).

Both splines \( S_{cub} \) and \( S_{quad} \) were employed to create Fortran functions \( p(x) \),

\[dp(x) := \frac{dp(x)}{dx}, \quad p_-(x), \quad p_{-newt}(x) \quad (p_-(x) \text{ and } p_{-newt}(x) \text{ denote } p^{-1}(x)) \quad \text{and} \]

\[dp_-(x) := \frac{dp^{-1}(x)}{dx}.\]
Figure 4: Sewing the spline $S_{\text{quad}}(x)$ with asymptote $\tilde{p}(x)$ at $x=6000$.

For convenience in applications the functions $p(x)$ and $dp(x)$ have been extended to $+\infty$ by the asymptote (2, page 140)

$$
\tilde{p}(x) = x \left( \ln x + \ln \ln x + \frac{\ln \ln x - 2}{\ln x} - \frac{(\ln \ln x)^2/2 - 3 \ln \ln x + 5.5}{(\ln x)^2} - 1 \right). \tag{8}
$$

To obtain primes, two alternative programs have been employed – subroutine eratosthenes(n) and subroutine primes(n). The former accomplishes this by generating primes up to a given value $n$, whereas the latter achieves the result by reading $n$ primes from given 6 column file named primes. In these programs, the spline and its derivative are automatically sewed with asymptote (8) and its derivative.

For purposes of building the inverse function $p^{-1}(x)$ in the programs $p_{\text{-newt}}(x)$ and $p_{\text{-}}(x)$, two different approaches have been used.

The program $p_{\text{-newt}}(x)$ is based on an autoregularized variant of the Newton method (3, page 43).

\[
\begin{align*}
\begin{cases}
y_0, \quad \varepsilon_0 > 0, \quad y_{k+1} = y_k - \frac{p(y_k) - x}{dp(y_k) + \varepsilon_k}, \quad k = 0, 1, 2, \ldots, \\
\varepsilon_k = \frac{1}{2} \left( \sqrt{(dp(y_k))^2 + 4N|p(y_k) - x|} - dp(y_k) \right), \\
N = (\varepsilon_0^2 + \varepsilon_0 dp(y_0))/|p(y_0) - x|,
\end{cases}
\end{align*}
\tag{9}
\]

where several combinations of the initial value $y_0 = li(x)$ and initial regularizator $\varepsilon_0$ (these
combinations can be seen at the beginning of the $p_{\text{newt}}(x)$ program’s body) ensure a construction of the inverse function $p^{-1}(x)$ on the interval $[2, 10^8]$.

The way of setting the initial approximation $y_0$ within the values

$$
\frac{x}{\ln x}, \quad li(x) = \int_2^x \frac{dt}{\ln t},
$$

$$
R(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} f(x^{1/n}) \quad (\text{see } [1], \text{ page } 35), \quad (10)
$$

where $f(x) = li(x)$, $\mu(n)$ is 0 if $n$ is divisible by a prime square, 1 if $n$ is a product of an even number of distinct primes, and $-1$ if $n$ is a product of an odd number of distinct primes was checked: the conclusion is that the method $[1]$ and $y_0 = li(x)$ are an acceptable combination.

Figure 4 shows the sewing of the function $p(x)$ with the asymptote $[8]$.

The programs $p_-(x)$ and $dp_-(x)$ are based on the application of the inverse spline $S_{\text{quad}}^{-1}$ and its derivative. In these programs, an algorithm has been applied by virtue of which the needed pairs $t_i(x)$ and $b_i(x)$ are established by approximating $ix = \lfloor li(x) \rfloor$ (see the beginning of the programmes $p_-(x)$ and $dp_-(x)$). It is worth mentioning that this algorithm works in the case where $\pi(x) < li(x)$, as well as in the case where $\pi(x) > li(x)$, i.e., the algorithm does not depend on the knowledge the minimal value of $x$ for which the difference $li(x) - \pi(x)$ changes the sign.

The programs $p(x)$, $dp(x)$, $p_{\text{newt}}(x)$, $p_-(x)$ and $dp_-(x)$ have been realized in the Fortran90. These programs form the package named $pp_{\text{-}90}[1, \infty)$, available in Appendix 1. They employ the natural extension of the function to $-\infty$, based on the values of the initial polynomial $x + 1$.

The package $pp_{\text{-}90}[1, \infty)$ is immediately applicable to Compaque– and MS– Fortran and is facile transportable to other Fortran versions.

Fortran functions $p(x)$ and $p^{-1}(x)$ are as easily applicable as the intrinsic functions $\sin(x)$ and $\exp(x)$.

The end of Appendix 1 contains a program called $test_{\text{-}pp}$ which has been used to compute all the tables supporting the graphics in this article and which serves as an illustration for application of the functions $p(x)$ and $p^{-1}(x)$.

6. Possible applications of the functions $p(x)$ and $p^{-1}(x)$.

6.1. Functions $p(x)$ and $p^{-1}(x)$ can be used to introduce new functions $\sin p(x)$, $\cos p(x)$, $\tan p(x)$, $e^{-p(x)}$ and $\ln p(x)$, applicable when the specific character of nonasymptotic prime number distribution is necessary to be accounted for (see, e.g., [1]). In particular, the functions $\sin p(x)$ and $\cos p(x)$ can be used for creation of a prime number harmonic analysis.

6.2. Diophantine equations can be solved within the following approximate method: to the Diophantine equation

$$
f_1(x_1, \ldots, x_n) = 0 \quad (11)
$$

one adds a new equation, from reals-to-integers equation

$$
f_2^h(x_1, \ldots, x_n) := \sin^2(\pi x_1) + \sin^2(\pi x_2) + \cdots + \sin^2(\pi x_n) = 0, \quad (12)
$$
or one adds from reals-to-primes equation
\[ f_2(x_1, \ldots, x_n) := \sin^2(\pi p^{-1}(x_1)) + \sin^2(\pi p^{-1}(x_2)) + \cdots + \sin^2(\pi p^{-1}(x_n)) = 0. \] (13)

By solving either system (11), (12) or (11), (13) one can find solutions to (11) as real approximations to the natural or the prime numbers.

The above systems can be solved by methods working in the case of degeneration of the derivative at the solution (see, e.g., [5]).

Here two examples of solving such systems by means of autoregularized iterative processes \((rgn)\) [6] are presented; \(rgn\)-processes are combined with both the \(svd\)-method [7] and the adaptive scaling [8].

In this case the program \(afxy\) [9] is used to find all solutions of the systems (11), (12) and (11), (13) in a given definition domain.

Let the problem (11), (13) be written as
\[ Fx = 0, \] (14)

where
\[ Fx = f^T(x)(fx - \overline{y}), \quad f = [f_1(x), f_2(x)]^T, \quad f : D_f \subset R^n \to R^m, \quad x \in R^n, \quad \overline{y} \in R^m, \]

and \(D_f\) is an open convex domain in \(R^n\).

The linear problem at the \(k\)th iteration of \(rgn\)-process is of the kind
\[ (f^T(x^k)f'(x^k) + \varepsilon_k I)(x^{k+1} - x^k) = -Fx^k, \] (15)

where
\[ \varepsilon_k = \frac{1}{2} \left( \sqrt{\tau_k^2 + 4N\rho_k - \tau_k} \right), \]
\[ \tau_k = \|f^T(x^k)f'(x^k)\|_\infty, \quad \rho_k = \|Fx^k\|_\infty, \quad N = (\varepsilon_0 + \varepsilon_0\tau_0)/\rho_0. \]

For simplicity in equality (15) and in the expressions for \(\tau_k, \rho_k\) and \(\varepsilon_k I\), the scaling operators are neglected. Just for the problem (15) the \(svd\)-method is in use.

For the purpose of finding all solutions of equation (14) in \(D_f\), the program \(afxy\) realizes an algorithm in which the vector \(Fx_k\) is multiple factored by local extractors of the kind
\[ e_r(x, \overline{x}(r)) = \left( 1 - e^{-\|x - \overline{x}(r)\|_2} \right)^{-1}, \]

where \(\overline{x}(r) \in D_f\) is the \(r\)th solution of equation (14). The transformed problem
\[ F_{r^*}x := \left( \prod_{r=1}^{r^*} e_r(x, \overline{x}(r)) \right) Fx = 0, \quad r^* \geq 1 \] (16)

is multiple solvable by the program \(afxy\).

For each new problem (16) \(afxy\) realizes different \(rgn\) iterative processes: with different guesses \(x_0\) (randomly formed by the initially given \(x_0\) and by peculiarities of the domain \(D_f\)) and with different initial regularizators \(\varepsilon_0\) (from the preset table of regularizators).

Example 1. Solution of the equation \(x_1^2 + x_2^2 = x_3^2 + 1\) over primes.
Figure 5: The differentiable inverse function $p^{-1}(x)$ (thick line) versus the Riemann–Von Mangoldt step-function $\pi_R(x)$. 
Consider problem \((\text{I}4)\) with
\[
f x = \begin{cases} 
  f_1(x) := x_1^2 + x_2^2 - x_3^2 - 1 = 0, \\
  f_2(x) := \sin^2(\pi p^{-1}(x_1)) + \sin^2(\pi p^{-1}(x_2)) + \cdots + \sin^2(\pi p^{-1}(x_n)) = 0,
\end{cases}
\]
\[
n = 3, m = 2, D_f = [2, 100] \times [2, 100] \times [2, 100], x = (x_1, x_2, x_3)^T, \gamma = [0, 0]^T.
\]

Run-time section of Application 2 contains a subroutine \(f xy(m, n, np, neq, f, x, pp, d f, yr)\), where the equation \(f x = 0\) and derivatives \(\frac{df_x(x)}{dx_i} (i = 1, 2, 3, j = 1, 2)\) are coded. Here, in the pre-exe section of the program \(af xy\), the main controls \(m, n, np, lsmh, mgq, nsolh, x_1^{(0)}, x_2^{(0)} \) and \(D_f\) are given as well.

If in the pre-exe section some main control is not prescribed, then \(af xy\) switches to an interactive mode and demands from display an adjustment to the value of this control.

The solution \((x_1, x_2, x_3)\) of the equation \(f_1(x) = 0\) (\(\text{I}0\), page 35) is thought to be a quasi-Pythagorean prime triplet (the equation \(x_1^2 + x_2^2 = x_3^2\) has no solutions on primes). There exist two series of natural numbers satisfying equation \(f_1(x) = 0\):
\[
(2n + 1, n^2 + n - 1, n^2 + n + 1), \quad (2n(4n + 1), 16n^3 - 1, 16n^3 + 2n), \quad (10), \quad \text{page 35,}
\]
\[
(2n(4n + 1), 16n^3 - 1, 16n^3 + 2n), \quad (10), \quad \text{page 36.}
\]

The present state of the program \(af xy\) can only produce up to 20 (i.e. \(r^* \leq 20\)) extractions and can find only 20 quasi-Pythagorean prime triplets in \(D_f\)(see FOUND SOLUTIONS in Application 2).

**Example 2.** Solution of the equation \(x_1^2 + x_2^2 = x_3^2 + 1\) over twins.

Consider problem \((\text{I}4)\) with
\[
f x = \begin{cases} 
  f_1(x) := x_1^2 + x_2^2 - x_3^2 - 1 = 0, \\
  f_2(x) := \sin^2(\pi p^{-1}(x_1)) + \sin^2(\pi p^{-1}(x_2)) + \cdots + \sin^2(\pi p^{-1}(x_n)) = 0, \\
  f_3(x) := x_3 - x_1 - 2 = 0,
\end{cases}
\]
\[
n = 3, m = 3, D_f = [2, 100] \times [2, 100] \times [2, 100], \quad \text{and} \quad \gamma = [0, 0, 0]^T.
\]

The needed subroutine \(f xy\) for this example is presented in Application 3.

Application 3 shows that the program \(af xy\) finds all 5 quasi-Pythagorean prime triplets containing twin pairs in the domain \(D_f\).

**6.3.** In Figure 5, the inverse function \(p^{-1}(x)\) is compared with the step–function \(\pi(x)\) and with the Riemann–Von Mangoldt continuous step–function \(\pi_R(x)\) which results from \((10)\) by means of the substitution
\[
f(x) = \text{li}(x) - \sum_{n=1}^{\infty} \text{li}(e^{\rho_n \ln x}) + \int_{x}^{\infty} \frac{da}{(a^2 - 1)a \ln a} - \ln 2,
\]
where \( \{\rho_n\} \) are the complex zeros of the equation

\[ \zeta(s) = 0 \]  \hspace{1cm} (17)

in the form \( \rho_n = \frac{1}{2} \pm it_n, n = 1, 2, \ldots \), and \( \zeta(s) \) is Riemann’s \( \zeta \)-function \([\text{I}]\).

In our case, counting function \( \pi(x) \) is expressed by the inverse function \( p^{-1}(x) \) as the formula

\[ \pi(x) = \lfloor p^{-1}(x) \rfloor. \]  \hspace{1cm} (18)

Comparing the two ways, the function \( p^{-1}(x) \) proves more convenient for application than the Riemann–Von Mangoldt function \( \pi_R(x) \); moreover, \( p(x) \) and \( p^{-1}(x) \) are applicable just now, not waiting for the final representation of the function \( \pi_R(x) \) through the zeros \( \rho_n \).

6.4. For purposes of investigation the nonasymptotic behaviour of primes, instead of the functions \( p(x) \) and \( p^{-1}(x) \), one can resort the functions:

\[ A(x) = p(x) - \tilde{p}(x) - (p(x_0) - \tilde{p}(x_0)), \quad x \in [x_0, x_0 + \varepsilon(x_0)] \] – a local variance of the function \( p(x) \), where \( \varepsilon(x_0) \) is relatively small in comparison with \( x_0 \), and

\[ B(x) = p^{-1}(x) - R(x) - (p^{-1}(x_0) - R(x_0)), \quad x \in [x_0, x_0 + \varepsilon(x_0)] \] – a local variance of the function \( p^{-1}(x) \).

Here \( \tilde{p}(x) \) is the asymptote \([\text{S}]\), and \( R(x) \) is the Riemann simplified formula \([\text{I0}]\) for \( \pi(x) \). Figures 7-9 serve as examples to the behaviour of \( A(x) \) and \( B(x) \).
Figure 6: Function $p(x)$. The interval $[154.78, 168.2]$ is chosen so as the definition interval $[900, 1000]$ of the function $p^{-1}(x)$ coincides with the definition interval of the function number variance of the zeros $\rho_n$ from Figure 4 [11], p. 406.

Figure 7: Function $A(x)$; note a qualitative intimacy with the function number variance of the zeros $\rho_n$ from Figure 4 [11], p. 406, as well as the same number of peaks equal to 14 in the given interval; the function $A(x)$ covers 14 successive primes: 907, 911, 919, 929, 937, 941, 947, 953, 967, 971, 977, 983, 991 and 997.
Figure 8: Function $B(x)$. Inverse function to the function $A(x)$ from Figure 7.

Figure 9: Function $B(x)$ on the enlarged interval [900,1150]. How many peaks does the function number variance of the zeros $\rho_n$ have on the interval [1000, 1150]? Do we really have no 17 peaks?
Application 1. Program package *pp_f90*[1, ∞)

```fortran
function p(x) ! 1 <= x < infinity
implicit real*8(a-h,o-z)
common/wn/wn
if(x <= 2.d0) then
  p = x+1.d0; return
endif
if(x > 2.d0 .and. x <= wn) then
  ix=floor(x+0.5d0); xx=dfloat(ix); p = sq(ix,x,xx); return
endif
if(x > wn) then
  p = r(x); return
endif
contains
function sq(ix,x,xx) ! left-right quadric pair
common/protiarithmi/q(10000000)
if(x <= xx) then
  sq = -2.d0*(q(ix)-q(ix-1)-1.d0)*(x-xx)**2+x-xx+q(ix)
else
  sq = 2.d0*(q(ix+1)-q(ix)-1.d0)*(x-xx-0.5d0)**2+(2.d0*(q(ix+1)-q(ix))-1.d0)*(x-xx-0.5d0)+(q(ix)+q(ix+1))/2.d0
endif
end function sq
end function p

function dp(x) ! 1 <= x < infinity ; derivative of p(x)
implicit real*8(a-h,o-z)
common/wn/wn
if(x <= 2.d0) then
  dp = 1.d0; return
endif
if(x > 2.d0 .and. x <= wn) then
  ix=floor(x+0.5d0); xx=dfloat(ix); dp = dsq(ix,x,xx); return
endif
if(x > wn) then
  dp = dr(x); return
endif
contains
function dsq(ix,x,xx) ! derivative of left-right quadrics
common/protiarithmi/q(10000000)
if(x <= xx) then
  dsq = -4.d0*(q(ix)-q(ix-1)-1.d0)*(x-xx)+1.d0
else
  dsq = 4.d0*(q(ix+1)-q(ix)-1.d0)*(x-xx-0.5d0)+2.d0*(q(ix+1)-q(ix))-1.d0
endif
end function dsq
end function dp
```

function p_newt(u) ! 2 <= x < infinity
implicit real*8(a-h,o-z)
parameter(xy0=1.d3,xeps1=58.d3,eps1=3.d0,xeps2=17.d5, &
eps2=300,eps3=700.d0,ytol=1.d-11,dytol=1.d-11,ktol=1000)
common/rcorr/rc,drc/kk/k/rw/r

drcw=drc; drc=0.d0
x=u; xw=x
if(x >= 3.d0) then
  if(x < xy0) then
    y=x/dlog(x)
  else
    y=dlogintegral(x)
  endif
else
  p_newt=x-1.d0
return
endif
if( x < xeps1) eps0=eps1 ! initial value of
if(xeps1 <= x < xeps2) eps0=eps2 ! initial value of
if(xeps2 <= x) eps0=eps3 ! autoregularizator
20 dt0=dp(y); r0=dabs(p(y)-x) ! constant for
en=(eps0**2+eps0*dabs(dt0))/r0 ! autoregularizator formula
r=0.d0; k=0; small=1.d200
1 k=k+1
  yy=y; rr=r; t=p(yy)-x; dt=dp(yy); r=dabs(t) ! currant value of
eps=0.5d0*(dsgrt(dt**2+4.d0*en*r)-dabs(dt)) !autoregularizator
  y=yy-t/(dt+eps) ! autoregularized Newtonian iterator
  dif=dabs(y-yy)
  dr=dabs(r-rr)
if(r >= ytol .and. k <= ktol .and. dr > dytol) then
  if(dif <= small) then
    small=dif; ybest=y; rr=r
  endif
  goto 1
else
  p_newt= ybest
endif
drc=drcw
return
end
function p_(x)  ! 2 <= x < infinity
implicit real*8(a-h,o-z)
common/protiarithmetic/q(10000000)/wn/wn/ck/vkoch/nn
1 if(x >= 3.d0 .and. x <= q(nn)) then
   if(x <= 1.d3) ixx=floor(x/dlog(x))
   if(x > 1.d3) ixx=floor(dlogintegral(x)-vkoch*dsqrt(x)*dlog(x))
   if(q(ixx) == 0.d0) qq=r(dfloat(ixx))
   if(q(xx) /= 0.d0) qq=q(ixx)
   if(qq <= x) then
      is=-1; si=-1.d0
   else
      is= 1; si= 1.d0
   endif
else
   if(x < 3.d0) then
      p_=x-1.d0; return
   endif
endif
do i=1, nn
   ii=ixx-is*i
   if(si*(q(ii)-x) <= 0.d0) then
      ix=ii; goto 3; endif
endo
3 if(dabs(q(ix)-x) >= dabs(q(ix+is)-x)) ix=ix+is
xx=q(ix); xi=dfloat(ix); p_= y(ix,xi,x,xx)
if(vkoch /= 0.d0) goto 1
contains
function y(ix,xi,x,xx)  ! left-right inverse quadric pair
implicit real*8(a-h,o-z)
common/protiarithmetic/q(10000000)/ck/vkoch
vkoch=0.d0
if(x <= xx) then
   a=q(ix)-q(ix-1)-1.d0; b=8.d0*a*(q(ix)-x)+1.d0
   if(b <= 0.d0) then
      vkoch=1.d0
   else
      if(a == 0.d0) then
         y=x-1.d0; else; y=xi+(1.d0-dsqrt(b))/(4.d0*a); endif
   endif
else
   a=q(ix+1)-q(ix)-1.d0; b=8.d0*a*(x-q(ix))+1.d0
   if(b < 0.d0) then
      vkoch=1.d0
   else
      y=xi+(dsqrt(b)-1.d0)/(4.d0*a)
   endif
endif
end function y
end function p_
function dp_(x)        ! 2 <= x < infinity; inverse of dp(x)
implicit real*8(a-h,o-z)
common/protiarithmi/q(10000000)/wn/wn/ck/vkoch/nn/nn
1 if(x >= 3.d0 .and. x <= q(nn)) then
  if(x <= 1.d3) ixx=floor(x/dlog(x))
  if(x > 1.d3) ixx=floor(dlogintegral(x)-vkoch*dsqrt(x)*dlog(x))
  if(q(ixx) == 0.d0) qq=r(dfloat(ixx))
  else
    is=-1; si=-1.d0
  endif
else
  if(x < 3.d0) then
    dp_=1.d0
    return
  endif
endif
do i=1, nn
  ii=ixx-is*i
  if(si*(q(ii)-x) <= 0.d0) then
    ix=ii; goto 3; endif
endo
3 if(dabs(q(ix)-x) >= dabs(q(ix+is)-x)) ix=ix+is
xx=q(ix); xi=dfloat(ix); dp_= dy(ix,x,xx)
if(vkoch /= 0.d0) goto 1
contains
function dy(ix,x,xx)        ! left-right inverse quadric pair
common/protiarithmi/q(10000000)/ck/vkoch
vkoch=0.d0
if(x <= xx) then
  a=q(ix)-q(ix-1)-1.d0; b=1.d0+8.d0*a*(q(ix)-x)
  if(b <= 0.d0) then
    vkoch=1.d0
  else
    dy=1.d0/dsqrt(b)
  endif
else
  a=q(ix+1)-q(ix)-1.d0; b=1.d0+8.d0*a*(x-q(ix))
  if(b <= 0.d0) then
    vkoch=1.d0
  else
    dy=1.d0/dsqrt(b)
  endif
endif
end function dy
end function dp_

function r(t) ! asymptote of function p(t)
! M. Cipolla, "La determinazione assintotica dell nimo numero primo",
! Rend. Acad. Sci. Fis. Mat. Napoli, Ser. 3, 8 (1902), 132-166
implicit real*8(a-h,o-z)
common/rcorr/rc,drc
r=t*(dlog(t)+dlog(dlog(t))+(dlog(dlog(t))-2.d0)/dlog(t)-((dlog(dlog(t)))**2-6.d0*dlog(dlog(t))+11.d0)/(2.d0*dlog(t)**2)-1.d0)+rc
end function r

function dr(x) ! derivative of asymptote r(t)
implicit real*8(a-h,o-z)
common/rcorr/rc,drc
t1 = dlog(x); t2 = dlog(t1); t3 = t1**2; t4 = t3*t1
t7 = t1-2.d0; t8 = dlog(t7); t18 = t2**2; t21 = t3**2
t35 = -56.d0-4.d0*t2*t4+4.d0*t8*t1+50.d0*t1-13.d0*t3+2.d0*t4-6.d0*t8*t3-26.d0*t2*t1+50.d0*t1+2.d0*t2*t21+2.d0*t8*t4-1.d0*t18*t3+6.d0*t2*t3+28.d0*t2-4.d0*t18-4.d0*t21+2.d0*t21*t1
dr = 0.5d0*t35/t4/t7+drc
end function dr

subroutine eratosthenes(k)
implicit real*8(a-h,o-z)
common/protiarithmi/q(10000000)/rcorr/rc,drc/wn/wn/nn/nn
color/ck/vkoch
rc=0.d0; drc=0.d0; np=0; vkoch=0.d0
do n=2,k
id=1
isqrtn=dsqrt(dfloat(n))
1 id=id+1
if(n == 2) goto 2
if(mod(n,id) == 0) goto 3
if(id >= isqrtn) goto 2
goto 1
2 np=np+1; q(np)=n
3 continue
enddo
wn=dfloat(np); nn=np; rc=p(wn)-r(wn); drc=dp(wn)-dr(wn)
end subroutine eratosthenes

subroutine primes(n)
implicit real*8(a-h,o-z)
common/protiarithmi/q(10000000)/rcorr/rc,drc/wn/wn/nn/nn
color/ck/vkoch
rc=0.d0; drc=0.d0; nn=n; wn=dfloat(n); vkoch=0.d0
open(222, file='primes')
do i=1, n/6+1
i1=6*i-5; i2=6*i-4; i3=6*i-3; i4=6*i-2; i5=6*i-1; i6=6*i;
read(222, *) q(i1),q(i2),q(i3),q(i4),q(i5),q(i6)
enddo
nn=n; rc=p(wn)-r(wn); drc=dp(wn)-dr(wn)
end subroutine primes

! function dlogintegral(x)
implicit real*8(a-h,o-z)
parameter(itol=100, small=1.d-10)
dli=0.d0; i=0
1 i=i+1; dliw=dli; dm=(dlog(x))**i
if(dm > small) then
    dli=dli+dfloat(factorial(i-1))/dm
    if(dabs(dliw-dli) > small.and.i <= itol) goto 1
endif

end function dlogintegral
contains
integer recursive function factorial(l) result(lf)
integer (4) l
lf=1
if(l > 0) then
    lf=l*factorial(l-1); return
endif
end function factorial
end function dlogintegral

program test_pp_f
implicit real*8(a-h,o-z)
open(21, file='p.txt'); open(22, file='p_.txt')
open(23, file='dp.txt'); open(24, file='dp_.txt')
n=6000

call eratosthenes(n) ! call primes(n)
!
s=5970.d0; sm=0.05d0; ns=1000
! uses function primes(n) at n=6000; returns tab for fig 4
! s=428.d0; sm=0.01d0; ns=600 ! returns tabs for figs 1, 2 and 3
! s=154.78d0; sm=0.01d0; ns=1350 ! returns tabs for figs 6, 7, 8 and 9
s=1.d0; sm=0.1d0; ns=250; ! returns tab for fig 5
ss=s
do i=1, ns
    s=s+sm
    write(21,*) s, p(s)
    write(23,*) s, dp(s)
enddo

sm=(p(s)-p(ss))/dfloat(ns); s=p(ss)
do i=1, ns
    s=s+sm
    write(22,*) s, p_(s) ! p_newt(s)
    write(24,*) s, dp_(s)
enddo
end
Application 2.
Solution of the equation $x(1)^2 + x(2)^2 = x(3)^2 + 1$ over primes

```fortran
!---user---module---to---the---main---program---afxy---------------------
subroutine FXY(m,n,np,neq,f,x,pp,df,yr)
  implicit real*8(a-h,o-z)
  DIMENSION X(1),pp(1),DF(1),YR(1)
  COMMON/LSMH/LSMH,MQH,NSOLH/BXH/D1,D2,BL(600),BR(600)
  COMMON/RETFH/LF1,LF2,LF3,NDAT/FR/FR/SLMH/SSVH,S3H
  go to (1,2), np
!-----run-time--section-------------------------------------------------
2 pi=dacos(-1.D0)
goto(21,22), neq
!first..equation.....................................................
21 f=x(1)**2+x(2)**2-x(3)**2-1.d0 ! diophantine equation
   df(1)= 2.d0*x(1); df(2)= 2.d0*x(2); df(3)=-2.d0*x(3) ! derivatives
   return
!second..equation....................................................
22 f=0.d0
   do i=1,3
     df(i)=pi*dsin(2.d0*pi*p_(x(i)))*dp_(x(i)) ! derivatives
     f=f+(dsin(pi*p_(x(i))))**2 ! from-reals-to-primes equation
   enddo
   return
!-------pre-execution--section--set--the--controls--to--afxy-program---
1 nn=10000; call eratosthenes(nn) ! prime number table creation
   n=3 ! number of unknowns
   m=2 ! number of equations
   np=-3 ! produce autoregularized Gauss-Newton process
   f=1.D-34 ! accuracy level for the residual f(x)-y
   lsmh=1 ! Gene Golub's SVD-method
   ssvh=1.d-16 ! minimal characteristic numb in SVD-method
   mqh=1 ! Jorge More's adaptive scaling
   nsolh=20 ! limit of the sought solutions
   do i=1, n
     x(i)=20.d0 ! guesses
     bl(i)=2.d0; br(i)=100.d0 ! constraints (definition domain Df)
   enddo
   lf1=1; lf2=2
   return
!
FOUND SOLUTIONS:
Solution # 1 (Plan 2; x0 # 2; eps0=2.88D-08):
K= 32---------------------------------------------------------------------
(Df)'(fx-y) fx-y chi.sqr. (Df)'Df l.p.prec. eps en/r
2.66223D-21 6.39687D-15 4.09199D-29 1.3172D+04 3.5976D-23 0.000D+00 4.8D-06
Unknowns:
X( 1)= 2.8999999979D+01 X( 2)= 2.3000000010D+01 X( 3)= 3.6999999989D+01
```
Solution # 2 (Plan 2; x0 # 2; eps0=2.88D-03) :
K= 26

(Df)'(fx-y) fx-y chi.sqr. (Df)' l.p.precision. eps en/r
5.32001D-18 9.98342D-13 9.96687D-25 1.6260D+04 1.2170D-13 0.000D+00 4.8D-02
Unknowns:
X( 1)= 2.9000000270D+01 X( 2)= 2.8999999622D+01 X( 3)= 4.1000000164D+01

Solution # 3 (Plan 2; x0 # 2; eps0=2.88D-01) :
K= 50

(Df)'(fx-y) fx-y chi.sqr. (Df)' l.p.precision. eps en/r
7.48702D-21 1.30119D-14 1.69311D-28 2.6924D+04 7.0632D-23 0.000D+00 5.3D-01
Unknowns:
X( 1)= 4.299999971D+01 X( 2)= 3.100000004D+01 X( 3)= 5.299999979D+01

Solution # 4 (Plan 2; x0 # 3; eps0=2.42D-08) :
K= 35

(Df)'(fx-y) fx-y chi.sqr. (Df)' l.p.precision. eps en/r
1.74942D-20 2.45302D-14 6.01732D-28 1.3197D+04 1.3640D-13 0.000D+00 3.1D-05
Unknowns:
X( 1)= 2.299999992D+01 X( 2)= 2.899999964D+01 X( 3)= 3.69999997D+01

Solution # 5 (Plan 2; x0 # 3; eps0=2.42D-07) :
K= 52

(Df)'(fx-y) fx-y chi.sqr. (Df)' l.p.precision. eps en/r
9.75966D-22 3.70459D-15 1.37240D-29 5.0600D+03 1.7760D-14 0.000D+00 3.1D-04
Unknowns:
X( 1)= 1.900000009D+01 X( 2)= 1.300000011D+01 X( 3)= 2.300000013D+01

Solution # 6 (Plan 2; x0 # 3; eps0=2.42D-04) :
K= 32

(Df)'(fx-y) fx-y chi.sqr. (Df)' l.p.precision. eps en/r
6.32452D-21 1.18087D-14 1.39445D-28 2.1246D+04 6.7279D-23 0.000D+00 3.1D+00
Unknowns:
X( 1)= 2.900000000D+01 X( 2)= 3.699999973D+01 X( 3)= 4.699999979D+01

Solution # 7 (Plan 2; x0 # 4; eps0=2.01D-06) :
K= 52

(Df)'(fx-y) fx-y chi.sqr. (Df)' l.p.precision. eps en/r
9.97306D-21 1.59910D-14 2.55714D-28 2.0899D+04 1.0602D-22 0.000D+00 3.1D-05
Unknowns:
X( 1)= 2.300000008D+01 X( 2)= 4.099999968D+01 X( 3)= 4.699999976D+01

Solution # 8 (Plan 2; x0 # 4; eps0=2.01D-05) :
K= 56

(Df)'(fx-y) fx-y chi.sqr. (Df)' l.p.precision. eps en/r
2.27957D-20 3.09817D-14 9.59867D-28 1.6685D+04 2.5783D-13 0.000D+00 3.1D-03
Unknowns:
X( 1)= 1.300000029D+01 X( 2)= 4.1000000030D+01 X( 3)= 4.300000037D+01

!
Solution # 9 (Plan 2; x0 # 5; eps0=1.74D-08) :
K= 40

(Df)'(fx-y) fx-y chi.sqr. (Df)'Df l.p.prec. eps en/r
2.16010D-16 1.23358D-11 1.52172D-22 2.6935D+04 1.5541D-13 0.000D+00 1.1D-06
Unknowns:
X( 1)= 3.0999999926D+01 X( 2)= 4.3000000887D+01 X( 3)= 5.3000000676D+01

Solution # 10 (Plan 2; x0 # 5; eps0=1.74D-04) :
K= 37

(Df)'(fx-y) fx-y chi.sqr. (Df)'Df l.p.prec. eps en/r
4.52579D-21 1.02193D-14 1.04433D-28 8.8041D+03 2.8456D-14 0.000D+00 1.1D-02
Unknowns:
X( 1)= 2.9000000019D+01 X( 2)= 1.100000013D+01 X( 3)= 3.100000022D+01

Solution # 11 (Plan 2; x0 # 5; eps0=1.74D+00) :
K= 29

(Df)'(fx-y) fx-y chi.sqr. (Df)'Df l.p.prec. eps en/r
4.96913D-21 1.02836D-14 1.05752D-28 2.0492D+04 2.3436D-13 0.000D+00 2.2D+01
Unknowns:
X( 1)= 4.300000024D+01 X( 2)= 1.899999989D+01 X( 3)= 4.700000018D+01

Solution # 12 (Plan 2; x0 # 6; eps0=2.43D-03) :
K= 44

(Df)'(fx-y) fx-y chi.sqr. (Df)'Df l.p.prec. eps en/r
1.26391D-23 1.87462D-16 3.51422D-32 5.0621D+03 3.5534D-14 0.000D+00 1.6D+00
Unknowns:
X( 1)= 1.300000003D+01 X( 2)= 1.900000003D+01 X( 3)= 2.300000003D+01

Solution # 13 (Plan 2; x0 # 7; eps0=2.70D-08) :
K= 58

(Df)'(fx-y) fx-y chi.sqr. (Df)'Df l.p.prec. eps en/r
7.20035D-21 1.26460D-14 1.59223D-28 4.3148D+04 4.7940D-16 0.000D+00 6.7D-03
Unknowns:
X( 1)= 4.100000029D+01 X( 2)= 5.299999979D+01 X( 3)= 6.700000001D+01

Solution # 14 (Plan 2; x0 # 7; eps0=2.70D-02) :
K= 44

(Df)'(fx-y) fx-y chi.sqr. (Df)'Df l.p.prec. eps en/r
7.26781D-17 6.74038D-12 4.54328D-23 6.5404D+04 4.7940D-13 0.000D+00 6.7D-01
Unknowns:
X( 1)= 7.1000000315D+01 X( 2)= 4.300000054D+01 X( 3)= 8.3000000546D+01

Solution # 15 (Plan 2; x0 # 9; eps0=2.73D-03) :
K= 28

(Df)'(fx-y) fx-y chi.sqr. (Df)'Df l.p.prec. eps en/r
2.31418D-21 6.77216D-15 4.58621D-29 8.8041D+03 3.1949D-14 0.000D+00 1.6D+00
Unknowns:
X( 1)= 1.100000015D+01 X( 2)= 2.900000013D+01 X( 3)= 3.100000017D+01

! 24
Solution # 16 (Plan 2; x0 # 10; eps0=2.78D-05):
K= 71

(Df)'(fx-y) fx-y chi.sqr. (Df)'Df l.p.prec. eps en/r
1.32835D-20 2.01058D-14 4.04243D-28 4.3148D+04 9.9131D-23 0.000D+00 1.7D-04
Unknowns:
X( 1)= 5.2999999967D+01 X( 2)= 4.099999995D+01 X( 3)= 6.699999970D+01

Solution # 17 (Plan 2; x0 # 13; eps0=2.58D+00):
K= 36

(Df)'(fx-y) fx-y chi.sqr. (Df)'Df l.p.prec. eps en/r
7.87327D-20 5.71405D-14 3.26504D-27 7.4410D+04 2.8416D-13 0.000D+00 8.1D+01
Unknowns:
X( 1)= 7.9000000002D+01 X( 2)= 4.099999930D+01 X( 3)= 8.89999970D+01

Solution # 18 (Plan 2; x0 # 15; eps0=2.47D-03):
K= 55

(Df)'(fx-y) fx-y chi.sqr. (Df)'Df l.p.prec. eps en/r
1.70828D-20 2.50831D-14 6.29162D-28 4.7012D+04 5.1160D-13 0.000D+00 3.4D+00
Unknowns:
X( 1)= 7.1000000031D+01 X( 2)= 1.7000000020D+01 X( 3)= 7.300000035D+01

Solution # 19 (Plan 2; x0 # 16; eps0=2.29D-09):
K= 47

(Df)'(fx-y) fx-y chi.sqr. (Df)'Df l.p.prec. eps en/r
6.22186D-21 1.25141D-14 1.56603D-28 1.6703D+04 1.2445D-13 0.000D+00 1.4D-05
Unknowns:
X( 1)= 4.100000024D+01 X( 2)= 1.30000008D+01 X( 3)= 4.30000025D+01

Solution # 20 (Plan 2; x0 # 21; eps0=1.97D-08):
K= 34

(Df)'(fx-y) fx-y chi.sqr. (Df)'Df l.p.prec. eps en/r
1.02797D-20 1.75525D-14 3.08090D-28 2.1356D+04 1.5072D-13 0.000D+00 2.1D-05
Unknowns:
X( 1)= 4.099999971D+01 X( 2)= 2.29999991D+01 X( 3)= 4.69999971D+01

Singular values of the matrix Z=(Df)'Df at the best approximation X( 34)
(errSVD= 0):
1) 0.0000000000000D+00  2) 2.0194839173658D-28  3) 1.8088976985834D+04

Cond(Z(x( 34)))=104 ; GMcond(Z(x( 34)))= 45 ; nullity(Z(x( 34)))= 2 .

Residual_tab of the vector r=y-f(x( 34)) (M= 2):
0.0D+00,  1;  1.8D-14,  1;

Final report : 202 iterative processes were tried,
9093 times subroutine FXY was called,
8911 iterations were produced, and
20 solutions were found.
Application 3.
Solution of the system \( x(1)^2 + x(2)^2 = x(3)^2 + 1, x(3) - x(1) = 2 \) over twins

```fortran
!-----user's---module--to--the--main--program--afxy-----------------------------
subroutine FXY(m,n,np,neq,f,x,pp,df,yr)
  implicit real*8(a-h,o-z)
  DIMENSION X(1),pp(1),DF(1),YR(1)
  COMMON/LSMH/LSMH,MQH,NSOLH/BXH/D1,D2,BL(600),BR(600)
  COMMON/RETFH/LF1,LF2,LF3,NDAT/SLMH/SSVH,S3H
  go to (1,2), np
!-----run-time--section------------------------------------------------------
2 continue; pi=dacos(-1.D0)
  goto(21,22,23), neq
!...first..equation....................................................
21 f=x(1)**2+x(2)**2-x(3)**2-1.d0 ! diophantine equation
   df(1)= 2.d0*x(1); df(2)= 2.d0*x(2); df(3)=-2.d0*x(3) ! derivatives
   return
!...second..equation..................................................
22 f=x(3)-x(1)-2.d0 ! twin-pair equation
   df(1)=-1.d0; df(2)= 0.d0; df(3)= 1.d0 ! derivatives
   return
!...third..equation..................................................
23 f=0.d0
   do i=1,3
     df(i)=pi*dsin(2.d0*pi*p_(x(i)))*dp_(x(i)) ! derivatives
     f=f+(dsin(pi*p_(x(i))))**2 ! from-reals-to-primes equation
   enddo
   return
!-------pre-execution--section--set--the--controls--to--afxy-program-----
1 continue
  nn=10000; call eratosthenes(nn)! prime number table creation
  n=3 ! number of unknowns
  m=3 ! number of equations
  np=-3 ! produce autoregularized Gauss-Newton process
  f=1.D-34 ! accuracy level for the residual f(x)-y
  lsmh=1 ! Gene Golub’s SVD-method
  ssvh=1.d-16 ! minimal characterist numb in SVD-method
  mqh=1 ! Jorge More’s addaptive scaling
  nsolh=10 ! limit of the sought solutions
  do i=1, n
    x(i)=50.d0 ! guesses
    bl(i)=2.d0; br(i)=101.d0 ! constraints (definition domain Df)
  enddo
  lf1=1; lf2=2
  return
end
```
FOUND SOLUTIONS:

Solution # 1 (Plan 2; x0 # 1; eps0=3.00D-02):
K= 31-----------------------------------------------------------------------
(Df)'(fx-y) fx-y chi.sqr. (Df)'Df l.p.prec. eps en/r
9.94498D-20 7.96082D-14 6.33746D-27 4.7014D+04 6.3947D-14 0.000D+00 3.6D-01
Unknowns:
X( 1)= 7.0999999937D+01 X( 2)= 1.6999999993D+01 X( 3)= 7.2999999937D+01

Solution # 2 (Plan 2; x0 # 1; eps0=3.00D+00):
K= 36-----------------------------------------------------------------------
(Df)'(fx-y) fx-y chi.sqr. (Df)'Df l.p.prec. eps en/r
1.84559D-21 5.58920D-15 3.12392D-29 1.6686D+04 7.1257D-15 0.000D+00 7.2D+00
Unknowns:
X( 1)= 4.1000000017D+01 X( 2)= 1.3000000003D+01 X( 3)= 4.3000000017D+01

Solution # 3 (Plan 2; x0 # 2; eps0=2.88D-08):
K= 34-----------------------------------------------------------------------
(Df)'(fx-y) fx-y chi.sqr. (Df)'Df l.p.prec. eps en/r
9.06718D-24 1.65074D-16 2.72495D-32 4.7800D+02 1.7750D-15 0.000D+00 3.1D-04
Unknowns:
X( 1)= 5.0000000028D+00 X( 2)= 5.0000000011D+00 X( 3)= 7.0000000028D+00

Solution # 4 (Plan 2; x0 # 2; eps0=2.88D-07):
K= 34-----------------------------------------------------------------------
(Df)'(fx-y) fx-y chi.sqr. (Df)'Df l.p.prec. eps en/r
1.14803D-22 8.85711D-16 7.84485D-31 1.6145D+03 3.5523D-15 0.000D+00 3.1D-04
Unknowns:
X( 1)= 1.0999999993D+01 X( 2)= 6.9999999981D+00 X( 3)= 1.2999999993D+01

Solution # 5 (Plan 2; x0 # 2; eps0=2.88D-03):
K= 33-----------------------------------------------------------------------
(Df)'(fx-y) fx-y chi.sqr. (Df)'Df l.p.prec. eps en/r
4.85228D-22 2.29735D-15 5.27781D-30 8.8060D+03 3.5523D-15 0.000D+00 3.1D-01
Unknowns:
X( 1)= 2.9000000011D+01 X( 2)= 1.1000000002D+01 X( 3)= 3.1000000011D+01

Final report: 300 iterative processes were tried,
6108 times subroutine FXY was called,
6027 iterations were produced, and
5 solutions were found.
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