A UNIVERSAL NUCLEAR OPERATOR SYSTEM

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Abstract. By means of Fraïssé theory for metric structures developed by Ben Yaacov, we show that there exists a separable 1-exact operator system $G_S$—which we call the Gurarij operator system—of almost universal disposition. This means that whenever $E \subset F$ are finite-dimensional 1-exact operator systems, $\phi : E \to G_S$ is a unital complete isometry, and $\varepsilon > 0$, there is a linear extension $\hat{\phi} : F \to G_S$ of $\phi$ such that $||\hat{\phi}||_o ||\hat{\phi}^{-1}||_o \leq 1 + \varepsilon$. Such an operator system is unique up to complete order isomorphism. Furthermore it is nuclear, homogeneous, and any separable 1-exact operator system admits a complete order embedding into $G_S$. The space $G_S$ can be regarded as the operator system analog of the Gurarij operator space $NG$ introduced by Oikhberg, which is in turn a canonical operator space structure on the Gurarij Banach space. We also show that the canonical $*$-homomorphism from the universal $C^*$-algebra of $G_S$ to the $C^*$-envelope of $G_S$ is a $*$-isomorphism. This implies that $G_S$ does not admit any complete order embedding into a unital exact $C^*$-algebra. In particular $G_S$ is not completely order isomorphic to a unital $C^*$-algebra. With similar methods we show that the Gurarij operator space $NG$ does not admit any completely isometric embedding into an exact $C^*$-algebra, and in particular $NG$ is not completely isometric to a $C^*$-algebra. This answers a question of Timur Oikhberg.

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1. Introduction

A separable Banach space $G$ is of almost universal disposition if, whenever $E \subset F$ are finite-dimensional Banach spaces, $\phi : E \to G$ is an isometric embedding, and $\varepsilon > 0$, there is a linear extension $\hat{\phi} : F \to G$ of $\phi$ such that $||\hat{\phi}||_o ||\hat{\phi}^{-1}||_o \leq 1 + \varepsilon$. It is necessary to allow the extension $\hat{\phi}$ to be only approximately isometric, since there is no separable Banach space satisfying the property above with $\varepsilon = 0$. Such a space—now called the Gurarij space—was shown to exist by Gurarij in [G2], while uniqueness up to linear isometry was later proved by Lusky [L3]. It has been observed in [G1] that such a space must be isometrically universal among separable Banach spaces, as a consequence of the analogous fact for $C[0,1]$. A short proof of uniqueness and universality of $G$ as well as a direct proof of the universality has been recently given in [KS]. Subsequently Ben Yaacov has provided in [BY] an alternative proof of the existence, uniqueness, and

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universality the Gurarij Banach space as an application of the Fraïssé theory for metric structures developed therein.

Fraïssé theory stems from the seminal results of Fraïssé from [F]. Generally speaking, Fraïssé theory studies the common properties shared by homogeneous structures and provides an unified framework for their construction and study. Suppose that \( \mathcal{M} \) is a countable structure which is moreover homogeneous, i.e. any finite partial isomorphism from \( \mathcal{M} \) to itself extends to an automorphism of \( \mathcal{M} \). The age \( \text{Age}(\mathcal{M}) \) of \( \mathcal{M} \) is the collection of all finitely-generated substructures of \( \mathcal{M} \). It satisfies the following properties:

- \( \text{Age}(\mathcal{M}) \) is closed by taking substructures (hereditary property),
- any two elements of \( \text{Age}(\mathcal{M}) \) simultaneously embed into a third one (joint embedding property), and
- if \( \mathcal{A} \subset \mathcal{B}_0 \) and \( \mathcal{B}_1 \) are elements of \( \text{Age}(\mathcal{M}) \) and \( \phi : \mathcal{A} \to \mathcal{B}_1 \) is an embedding then there is \( \mathcal{D} \in \text{Age}(\mathcal{M}) \) and embeddings \( \psi_0 : \mathcal{B}_0 \to \mathcal{D} \) and \( \psi_1 : \mathcal{B}_1 \to \mathcal{D} \) such that \( \phi \circ \psi_1 = (\psi_0)_\mathcal{M} \) (amalgamation property).

The main result of Fraïssé from [F] asserts that, conversely, if \( \mathcal{C} \) is a countable collection of \( \mathcal{L} \)-structures satisfying the three properties above (a Fraïssé class in the modern terminology) then there is a countable homogeneous structure \( \mathcal{M} \) such that \( \text{Age}(\mathcal{M}) = \mathcal{C} \). Moreover such a structure \( \mathcal{M} \) is unique up to isomorphism and contains any separable structure \( \mathcal{X} \) with \( \text{Age}(\mathcal{X}) \subset \mathcal{C} \).

Such a correspondence has been recently generalized to metric structures by Ben Yaacov in [BYBHU]. In this framework structures endowed with a nontrivial metric are considered, and all the embeddings are supposed to be isometric. The hereditary and joint embedding properties are defined analogously, while the amalgamation property is replaced by the near amalgamation property. This is defined analogously as the amalgamation property but the embeddings \( \phi \circ \psi_1 \) and \( (\psi_0)_\mathcal{M} \) are only required to agree up to an arbitrarily small error (measured with respect to the metric in \( \mathcal{D} \)). A separable metric structure \( \mathcal{M} \) is defined to be homogeneous if any isomorphism between finitely generated substructures of \( \mathcal{M} \) can be approximated by an automorphism of \( \mathcal{M} \) up to an arbitrarily small error. The requirement that \( \mathcal{C} \) contains only countably many structures is replaced in this framework by the assumption that a certain metric on \( \mathcal{C} \) be separable and complete. The main results of [BYBHU] completely recover the correspondence between Fraïssé classes and homogeneous structures in the metric setting. Alternative approaches to Fraïssé theory for metric structures have been considered in [S1] and [K2].

It is observed in [BYBHU, Section 3] that the class of finite-dimensional Banach spaces form a Fraïssé class. Moreover [BYBHU, Theorem 3.3] shows that for a Banach space being Gurarij is equivalent to being a limit of the class of finite-dimensional Banach spaces. Thus the general results about Fraïssé classes imply existence, uniqueness, and homogeneity of \( \mathcal{G} \). Analogs of the Gurarij space in the class of \( p \)-Banach space for every \( p \in (0,1] \) have been defined and studied from the point of view of Fraïssé theory in [CSGKW]. Furthermore Fraïssé limits of \( C^* \)-algebras have been considered in [EFH*].

A noncommutative analog of the Gurarij Banach space \( \mathcal{G} \) has been introduced by Oikhberg in [O] in the framework of operator spaces. An operator space \( X \) is, concretely, a closed linear subspace of the space \( B(H) \) of bounded linear operators on a Hilbert space. The inclusion \( X \subset B(H) \) induces on \( X \) a norm, inherited from the operator norm in \( B(H) \). More generally one can consider the inclusion \( M_n(X) \subset M_n(B(H)) \), where \( M_n(X) \) denotes the space of \( n \times n \) matrices with entries in \( X \). The canonical identification of \( M_n(B(H)) \) with the space of bounded linear operators on the Hilbertian \( n \)-fold sum of \( H \) by itself induces a norm on \( M_n(B(H)) \) and, hence, on \( M_n(X) \). This defines a matricial norms structure on \( X \). Operator spaces have been abstractly characterized by Ruan in [R1] as those matricially normed spaces satisfying some natural conditions.
(Ruan’s axioms). If $\phi : X \to Y$ is a linear map between operator spaces, its $n$-th amplification $\phi^{(n)} : M_n (X) \to M_n (Y)$ is defined by $\phi^{(n)} [x_{ij}] = [\phi (x_{ij})]$. The map $\phi$ is completely bounded provided that all its amplifications are bounded and $\sup_n \| \phi^{(n)} \| < +\infty$. Such a supremum is then defined to be the completely bounded norm $\| \phi \|_{cb}$ of $\phi$. The notions of complete contraction and complete isometries are defined similarly. Operator spaces form then a category with completely bounded (or completely contractive) linear maps as morphisms.

Every Banach space $X$ has a canonical operator space structure obtained from the inclusions $X \subset C (\text{Ball } (X^*)) \subset B (H)$. Here $\text{Ball } (X^*)$ is the unit ball of the dual $X^*$ of $X$ endowed with the weak* topology. The space $C (\text{Ball } (X^*))$ of continuous complex-valued functions on $\text{Ball } (X^*)$ is a C*-algebra with respect to the pointwise operations and the sup norm, and hence a subalgebra of $B (H)$. Such an operator space structure on $X$ is called minimal quantization and denoted by $\text{MIN } (E)$. In this case the matricial norms do not provide any new information, being completely coded in the Banach space structure. This is reflected in the fact that any bounded linear map $\phi : X \to Y$ between Banach spaces is automatically completely bounded when regarded as a map between the corresponding minimal quantizations, and moreover $\| \phi : X \to Y \| = \| \phi : \text{MIN } (X) \to \text{MIN } (Y) \|_{cb}$.

The Gurarij operator space is a separable 1-exact operator system of almost universal disposition. This notion is defined analogously as in the Banach space case, where Banach spaces are replaced by 1-exact operator spaces, and the operator norm is replaced by the completely bounded norm [O, Section 1]. Exactness and 1-exactness are regularity notions for operator spaces that have no analog in the commutative setting. An operator space is 1-exact if it can be locally approximated by a subspace of a full matrix algebra with respect to the completely bounded analog of the Banach-Mazur distance. Every Banach space is 1-exact due to the well-known fact that finite-dimensional Banach spaces can be approximated by subspaces of finite-dimensional $\ell^\infty$-spaces. The restriction to 1-exact operator spaces is natural due to an important result of Junge and Pisier [JP, Theorem 2.3], implying that there is no separable operator space that contains completely isometrically every 3-dimensional operator space.

The noncommutative analog of the Gurarij space $\text{NG}$ has been shown to exist by Oikhberg in [O, Theorem 1.1]. Later it has been proved in [L2] that finite-dimensional 1-exact operator spaces form a Fraïssé class. Moreover being a limit of such a class is equivalent to being a noncommutative Gurarij space. This provided a proof of uniqueness up to linear complete isometry of $\text{NG}$. Another corollary is that $\text{NG}$ is homogeneous, universal among separable 1-exact operator spaces, and linearly isometric to the Gurarij Banach space $\mathcal{G}$. Proposition 3.2 of [O] shows that $\text{NG}$ is not isometric to a unital Banach algebra. A similar argument shows that $\text{NG}$ is not completely isometric to a unital operator space; see [GL, Subsection 4.4].

A unital operator space is a closed subspace $X$ of $B (H)$ containing the identity operator. Unital operator spaces can also be defined abstractly as operator spaces with a distinguished unitary element; see [BN, Theorem 2.1]. Particularly important among unital operator spaces are operator systems. These are the unital operator spaces $X \subset B (H)$ that are closed by taking adjoints. Again these can be abstractly characterized as those unital operator spaces that are spanned by their hermitian elements; see [BN, Proposition 3.2]. An operator system $X$ inherits from $B (H)$ a notion positivity for self-adjoint elements of $M_n (X)$. Operator systems can be equivalently characterized in terms of the $*$-vector space structure together with the unit and the matricial positive cones; see [CE2, Section 2] and [P2, Chapter 13]. A linear map between operator systems is positive if it maps positive elements to positive elements, and completely positive if all its amplifications are positive. A unital linear map between operator systems is
completely positive if and only if it is completely contractive [BLM, §1.3.3], and in such a case it is automatically self-adjoint. A unital complete isometry between operator systems is called a complete order embedding, and a complete order isomorphism if it is moreover surjective.

To every unital operator space $X \subset B(H)$ one can canonically assign the operator system $X + X^* = \text{span} \{x, x^*: x \in X\}$. Such an operator system does not depend (up to complete order isomorphism) from the unital completely isometric realization of $X$ as a subspace of $B(H)$; see [BLM, §1.3.7]. Moreover any unital completely contractive (respectively completely isometric) linear map $\phi: X \to Y$ between unital operator systems has a unique extension to a map $\hat{\phi}: X + X^* \to Y + Y^*$ with the same properties. Therefore in some sense there is no real loss of generality in only considering operator systems rather than arbitrary unital operator spaces.

In the present paper we consider the natural operator system analog the notion of operator space of universal disposition. This is obtained by replacing 1-exact operator spaces with 1-exact operator systems, and considering unital linear maps instead of arbitrary linear maps. (An operator system is 1-exact if it is 1-exact as an operator space; see [KPTT, Section 5] for equivalent characterizations.) Therefore we say that a separable 1-exact operator system $GS$ is of almost universal disposition if whenever $E \subset F$ are finite-dimensional 1-exact operator systems, $\phi: E \to GS$ is a unital complete isometry, and $\varepsilon > 0$, there is an extension $\hat{\phi}: F \to GS$ of $\phi$ such that $\|\hat{\phi}\|_{cb} \|\hat{\phi}^{-1}\|_{cb} < 1 + \varepsilon$.

In this paper we prove that the class $E_{1}^{sy}$ of finite-dimensional 1-exact operator systems is a Fraïssé class. Moreover an operator system is of almost universal disposition if and only if it is a limit of $E_{1}^{sy}$. As a consequence we conclude that there exist a unique (up to complete order isomorphism) operator system $GS$ of almost universal disposition, which we call the Gurarij operator system. Furthermore any separable 1-exact operator system admits complete order embedding into $GS$. The homogeneity property of $GS$ asserts that for any finite-dimensional subsystem $E$ of $GS$, complete order embedding $\phi: E \to GS$, and $\varepsilon > 0$, there is a complete order automorphism $\alpha$ of $GS$ such that $\|\alpha|_{E} - \phi\|_{cb} < 1 + \varepsilon$. The Gurarij operator system is nuclear in the sense of [HP, Theorem 3.1], and in fact it is an inductive limit of full matrix algebras with unital completely isometric connecting maps. In particular $GS$ is a C*-system in the sense of [KW], i.e. the second dual $GS^{**}$ of $GS$ is a C*-algebra and the canonical embedding of $GS$ into $GS^{**}$ is unital and completely isometric. Finally show that $GS$ is a universal C*-system as defined in [KW, Section 3]. This means that the canonical *-homomorphism from the universal C*-algebra $C^{*}_{u}(GS)$ to the C*-envelope $C^{*}_{u}(GS)$ is a *-isomorphism. In particular using results of [KW] this implies that $GS$ is not completely order isomorphic to a unital C*-algebra. Using a similar argument involving ternary rings of operators, we prove that the Gurarij operator space $NG$ does not embed completely isometrically into an exact C*-algebra, and in particular it is not completely isometric to a C*-algebra. This answers a question of Timur Oikhberg from [O].

The rest of the paper is organized as follows. In Section 2 we recall some facts about Fraïssé limits for metric structures and about operator systems. Section 3 contains the proof that finite-dimensional 1-exact operator systems form a Fraïssé class. Finally Section 4 contains the proof of the main result, characterizing the Gurarij operator systems $GS$ as the Fraïssé limit of finite-dimensional 1-exact operator systems.

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2. Preliminary notions

2.1. Fraïssé theory. We work in the framework of Fraïssé theory for metric structures developed by Ben Yaacov in [BY]. Similarly as [MT] we adopt the slightly less general approach where all the function and relation symbols are assumed to be Lipschitz with Lipschitz constant that does not depend on the structure.

A language $\mathcal{L}$ is a collection of function and relation symbols, each one with a given arity and Lipschitz constant. We will assume $\mathcal{L}$ to contain a distinguished binary relation symbol for the metric. An $\mathcal{L}$-structure $\mathfrak{A}$ is given by a complete metric spaces $A$ together with interpretations $f^\mathfrak{A}$ and $R^\mathfrak{A}$ of appropriate arity for any function symbol $f$ and relation symbol $R$. An embedding between $\mathcal{L}$-structure is a (necessarily isometric) map that commutes with the interpretations of all the function and relation symbols. A finite partial embedding is an embedding defined on a finitely generated substructure. A isomorphism is a surjective embedding, while an automorphism is an isomorphism from a structure to itself.

If $\overline{a}$ is a tuple of elements of $\mathfrak{A}$ then $\langle \overline{a} \rangle$ denotes the smallest substructure of $\mathfrak{A}$ containing the tuple $\overline{a}$. If $\overline{a}, \overline{b}$ are $n$-tuples of elements of $\mathfrak{A}$ and $\phi : \mathfrak{A} \to \mathfrak{B}$ is an embedding we denote by $d(\overline{a}, \overline{b})$ the maximum of $d(a_i, b_i)$ for $i \leq n$, and by $d(\overline{a})$ the $n$-tuple $(\phi(a_i))_{i \leq n}$. An $\mathcal{L}$-structure $\mathfrak{A}$ is homogeneous if for every finite partial embedding $\phi : \langle \overline{a} \rangle \subset \mathfrak{A} \to \mathfrak{A}$ and every $\varepsilon > 0$ there is an automorphism $\psi$ of $\mathfrak{A}$ such that $d(\phi(\overline{a}), \psi(\overline{a})) < \varepsilon$.

Suppose that $\mathcal{C}$ is a collection of finitely-generated $\mathcal{L}$-structures. A separable $\mathcal{L}$-structure $\mathfrak{A}$ is called a $\mathcal{C}$-structure provided that every finitely-generated substructure of $\mathfrak{A}$ belongs to $\mathcal{C}$.

Definition 2.1. The class $\mathcal{C}$ satisfies:

1. the hereditary property (HP) if every finitely generated substructure of an element of $\mathcal{C}$ belongs to $\mathcal{C}$;
2. the joint embedding property (JEP) if every two elements of $\mathcal{C}$ embed simultaneously in a third one;
3. the near amalgamation property (NAP) if for any $\langle \overline{a} \rangle \subset \mathfrak{B}_0$ and $\mathfrak{B}_1$ in $\mathcal{C}$, embedding $\phi : \mathfrak{A} \to \mathfrak{B}_1$, and $\varepsilon > 0$, there are $\mathfrak{D} \in \mathcal{C}$ and embeddings $\psi_0 : \mathfrak{B}_0 \to \mathfrak{D}$ and $\psi_1 : \mathfrak{B}_1 \to \mathfrak{D}$ such that $d(\psi_0(\overline{a}), (\psi_1 \circ \phi)(\overline{a})) < \varepsilon$.

Suppose now that $\mathcal{C}$ is a class of finitely-generated $\mathcal{L}$-structure satisfying (HP), (JEP), and (NAP). For $k \in \mathbb{N}$ let $\mathcal{C}(k)$ denote the space of pairs $(\overline{a}, \mathfrak{A})$ where $\mathfrak{A}$ is a structure in $\mathcal{C}$ and $\overline{a}$ is a distinguished tuple of generators of $\mathfrak{A}$. We identify two such pairs $(\overline{a}, \mathfrak{A})$ and $(\overline{b}, \mathfrak{B})$ provided that there is an isomorphism $\phi : \mathfrak{A} \to \mathfrak{B}$ such that $\phi(\overline{a}) = \overline{b}$. For brevity we denote the pair $(\overline{a}, \mathfrak{A})$ simply by $\overline{a}$, and denote $\mathfrak{A}$ by $\langle \overline{a} \rangle$.

Definition 2.2. The Fraïssé metric $d_\mathcal{C}$ on $\mathcal{C}(k)$ is defined by

$$d_\mathcal{C}(\overline{a}, \overline{b}) = \inf_{\phi, \psi} d(\phi(\overline{a}), \psi(\overline{b}))$$

where $\phi$ and $\psi$ range over all simultaneous embeddings of $\langle \overline{a} \rangle$ and $\langle \overline{b} \rangle$ into a third element $\mathfrak{D}$ of $\mathcal{C}$.

Requiring that the metric spaces $\mathcal{C}(k)$ be separable for every $k \in \mathbb{N}$ can be seen as a “smallness” condition on the class $\mathcal{C}$, analogous to the requirement that there be countably many isomorphism classes in the discrete case.

Definition 2.3. A Fraïssé class $\mathcal{C}$ in the language $\mathcal{L}$ is a class of finitely-generated $\mathcal{L}$-structures satisfying (HP), (JEP), and (NAP), such that moreover the space $\mathcal{C}(k)$ with the Fraïssé metric $d_\mathcal{C}$ is complete and separable for every $k \in \mathbb{N}$. A subclass $\mathcal{C}_0$ of $\mathcal{C}$ is dense if $\mathcal{C}_0(k)$ is dense in $\mathcal{C}(k)$ with respect to the metric $d_\mathcal{C}$ for every $k \in \mathbb{N}$.
Definition 2.4. A separable $C$-structure $\mathcal{M}$ is a limit of $C$ provided that for every structure $\mathcal{A}$ in $C$, for every finite tuple $\vec{a}$ in $\mathcal{A}$, for every embedding $\phi : \langle \vec{a} \rangle \rightarrow \mathcal{M}$, and for every $\varepsilon > 0$, there is an embedding $\psi : \mathcal{A} \rightarrow \mathcal{M}$ such that $d(\phi(\vec{a}), \psi(\vec{a})) < \varepsilon$.

Definition 2.4 is equivalent to Definition 2.15 in [BY] in view of [BY, Corollary 2.20]. The following statement combines the main results from [BY].

Theorem 2.5 (Ben Yaacov). Suppose that $C$ is a Fraïssé class. Then $C$ has a limit $\mathcal{M}$. Any two limits of $C$ are isomorphic as $C$-structures. Moreover $\mathcal{M}$ is homogenous, and every $C$-structure admits an embedding into $\mathcal{M}$.

2.2. Approximate morphisms. Even though this is not strictly necessary to state the results, it is useful in Fraïssé theory for metric structures to consider more general morphisms than embeddings. These morphisms, which in general are not functions in the usual sense, can be described in terms of approximate isometries as defined in [BY, Section 1].

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are $C$-structures as in Subsection 2.1. An approximate isometry $\phi$ from $\mathcal{A}$ to $\mathcal{B}$ is a function $\phi : A \times B \rightarrow [0, +\infty]$ that is 1-Lipschitz and Katětov in each variable. We write $\phi : \mathcal{A} \sto \mathcal{B}$ to indicate that $\phi$ is an approximate isometry from $\mathcal{A}$ to $\mathcal{B}$. Intuitively an approximate isometry $\phi : A \sto \mathcal{B}$ can be thought as prescribing that the point $a$ of $A$ be mapped at distance at most $\phi(a, b)$ from the point $b$ of $B$ for every $a \in A$ and $b \in B$. If $\phi, \psi : A \sto \mathcal{B}$ we write $\phi \leq \psi$ if the comparison holds pointwise. Every (partial) embedding from $\mathcal{A}$ to $\mathcal{B}$ can be regarded as an approximate isometry, by identifying it with the distance function from its graph. Approximate isometries can be naturally composed, extended, restricted, and inverted similarly as usual functions; see [BY, Section 1] for more details. For example the composition $\psi \phi : \mathcal{A} \sto \mathcal{D}$ of $\phi : \mathcal{A} \sto \mathcal{B}$ and $\psi : \mathcal{B} \sto \mathcal{D}$ is defined by

$$
A \times D \rightarrow [0, +\infty], \quad (a, d) \mapsto \inf_{b \in B} (\phi(a, b) + \psi(b, d)).
$$

It is easy to check that this definition is compatible with the usual definition of composition of functions.

Suppose now that $C$ is a Fraïssé class and $\mathcal{A}, \mathcal{B}$ are two $C$-structures. Let $\text{Apx}_{2,C}(\mathcal{A}, \mathcal{B})$ be the set of approximate isometries from $\mathcal{A}$ to $\mathcal{B}$ of the form $\psi \phi$, where $\mathcal{D}$ is a $C$-structure and $\phi : \mathcal{A} \sto \mathcal{D}$, $\psi : \mathcal{D} \sto \mathcal{B}$ are finite partial embeddings. Define then $\text{Apx}_C(\mathcal{A}, \mathcal{B})$ to be the set of $\phi : \mathcal{A} \sto \mathcal{B}$ such that for every finite subset $A_0$ of $A$ and $B_0$ of $B$ and every $\varepsilon > 0$ there are approximate isometries $\psi \in \text{Apx}_{2,C}(\mathcal{A}, \mathcal{B})$ and $\phi_0 : \mathcal{A} \sto \mathcal{B}$ such that $\psi \leq \phi_0$ and $|\phi_0(a, b) - \phi(a, b)| \leq \varepsilon$ for every $(a, b) \in A_0 \times B_0$. The elements of $\text{Apx}(\mathcal{A}, \mathcal{B})$ are called $(C$-intrinsic) approximate morphisms from $\mathcal{A}$ to $\mathcal{B}$. A (C-intrinsic) strictly approximate morphism from $\mathcal{A}$ to $\mathcal{B}$ is an element $\phi$ of $\text{Apx}_C(\mathcal{A}, \mathcal{B})$ such that $\phi \geq \psi|_{A_0 \times B_0} + \varepsilon$ for some element $\psi$ of $\text{Apx}_{2,C}(\mathcal{A}, \mathcal{B})$, $\varepsilon > 0$, and finite subsets $A_0$ of $A$ and $B_0$ of $B$.

The Fraïssé metric $d_C$ as well as the notion of limit can be rephrased in terms of approximate morphisms. For example if $\vec{a}, \vec{b} \in C(k)$ then

$$
d_C(\vec{a}, \vec{b}) = \inf_{\phi} \max_{i} \phi(a_i, b_i)
$$

where $\phi$ varies among all the (strictly) approximate morphisms from $\langle \vec{a} \rangle$ to $\langle \vec{b} \rangle$; see [BY, Definition 2.11]. Similarly one can reformulate the property of being limit. A $C$-structure $\mathcal{M}$ is a limit of $C$ if and only if for every structure $\mathcal{A}$ in $C$, for every finite tuple $\vec{a}$ in $\mathcal{A}$, for every strictly approximate morphism $\psi : \mathcal{A} \sto \mathcal{M}$, and for every $\varepsilon > 0$ there exist a finite tuple $\vec{b}$ in $\mathcal{M}$ and an approximate morphism $\phi : \mathcal{A} \sto \mathcal{M}$ such that $\phi \leq \psi$ and $\phi(a_i, b_i) \leq \varepsilon$ for every $i$; see [BY, Lemma 2.16 and Corollary 2.20]. If
moreover $C_0 \subset C$ is a dense subclass in the sense of Definition 2.3, then in the above characterization of limits of $C$ one can just consider structures in $C_0$.

2.3. Operator spaces. A vector space $X$ is matricially normed provided that, for every $k \in \mathbb{N}$, the space $M_k(X)$ of $k \times k$ matrices with entries in $X$ is endowed with a complete norm. An operator space is a matricially normed vector space satisfying the following:

- for every $x \in M_k(X)$ and $y \in M_n(X)$
  $$\left\| \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \right\|_{M_{k+n}(X)} = \max \left\{ \|x\|_{M_k(X)}, \|y\|_{M_n(X)} \right\},$$

- for every $\alpha \in M_{k,n}$, $x \in M_n(X)$, and $\beta \in M_{n,k}$
  $$\|\alpha x \beta\|_{M_k(X)} \leq \|\alpha\| \|x\|_{M_n(X)} \|\beta\|.$$  

Here $M_{k,n}$ denotes the ring of $k \times n$ matrices with entries from $C$, and $\|\alpha\|$ denotes the norm of $\alpha$ regarded as a linear operator from $\ell^2(k)$ to $\ell^2(n)$. The notation $\alpha x \beta$ stands for the usual product of matrices.

Suppose that $H$ is a Hilbert space. Denote by $B(H)$ the algebra of bounded linear operators on $H$ endowed with the usual vector space structure and the operator norm. Every closed subspace of $X$ is canonically endowed with an operator space structure obtained by the canonical identification of $M_k(X)$ with a closed subset of $B(H^{\otimes k})$, where $H^{\otimes k}$ denotes the $k$-fold Hilbertian sum of $H$ by itself. Conversely every operator space can be concretely represented as a closed linear subspace of $B(H)$ [R1, Theorem 3.1].

Suppose that $X, Y$ are operator spaces, and that $\phi : X \to Y$ is a linear map. The $n$-th amplification $\phi^{(n)} : M_n(X) \to M_n(Y)$ is defined by

$$\phi^{(n)}[x_{ij}] = [\phi(x_{ij})]$$

for $[x_{ij}] \in M_n(X)$. A map $\phi : X \to Y$ is completely bounded provided that all the amplifications $\phi^{(n)}$ of $\phi$ are bounded and

$$\sup_n \|\phi^{(n)}\| < +\infty.$$  

In such a case the supremum above is called the completely bounded norm (or cb-norm) of $\phi$ and denoted by $\|\phi\|_{cb}$. A map is completely contractive if all its amplifications are contractive, and completely isometric if all its amplifications are isometric.

Denote by $K_0$ the increasing union of $M_n$ for $n \in \mathbb{N}$, where $M_n$ is identified with the upper left corner of $M_{n+1}$. If $X$ is an operator space, then the matricial norms of $X$ can be regarded as a norm on the (algebraic) tensor product $K_0 \otimes X$. The corresponding completion is then a complete normed space with a natural $K_0$-bimodule structure. As observed in [L2, Section 2.4], this allows one to regard operator spaces as (complete) structures in a (single-sorted) language $\mathcal{L}_{OS}$. Alternative descriptions of operator spaces as metric structures are considered in [GS, Appendix B] and [EFP+, Section 3.3].

In the terminology of [BN] an element $u$ of an operator space $X$ is a unitary if there is a complete isometry $u$ from $X$ to $B(H)$ mapping $u$ to the identity operator. By [BN, Theorem 1.1] an element $u$ of $X$ is a unitary if and only if

$$\left\| \begin{bmatrix} u_n & x \\ 0 & 0 \end{bmatrix} \right\|_{M_{2n}(X)} = \left\| \begin{bmatrix} u_n & 0 \\ 0 & 0 \end{bmatrix} \right\|_{M_{2n}(X)} = \|x\|^2_{M_n(X)}$$

for every $n \in \mathbb{N}$ and $x \in M_n(X)$, where $u_n$ denotes the element of $M_n(X)$ with $u$ in the diagonal and zeros elsewhere. A unital operator space is an operator space with a distinguished unitary. Unital operator spaces can be naturally regarded as structures.
2.4. Operator systems. An operator system is a unital operator space $X$ such that there exists a complete isometry $\phi : X \to B(H)$ mapping the distinguished unitary to the identity operator and mapping $X$ onto a self-adjoint subspace of $B(H)$. Operator systems are precisely the unital operator space that are spanned by their hermitian elements; see [BN, Definition 3.1 and Proposition 3.2]. This provides an intrinsic characterization of operator systems among unital operator spaces.

A unital complete isometry $\phi : X \to B(H)$ induces an involution $x \mapsto x^*$ on $X$, as well as a notion of positivity in $M_n(X)$ for every $n \in \mathbb{N}$. We will regard operator systems as structures in the language $\mathcal{L}_{uOS}$ of unital operator spaces with a unitary function symbol for the involution. The intrinsic characterization [BN, Proposition 3.2] shows that an $\mathcal{L}_{uOS}$-structure is an operator system if and only if all its finitely generated substructures are operator systems.

An earlier and more commonly used abstract description of operator spaces is due to Choi and Effros [CE2]. Such a description only involves the unit, the involution, and the positive cones in all the matrix amplifications. The operator space structure is completely recovered by the relation

$$\|x\|_{M_n(X)} \leq 1 \iff \begin{bmatrix} I_n & x \\ x^* & I_n \end{bmatrix} \in M_{2n}(X)$$ is positive.

Suppose that $X$ and $Y$ are operator systems. A map $\phi : X \to Y$ is unital provided that it maps the unit to the unit, and self-adjoint if $\phi(x^*) = \phi(x)^*$ for every $x \in X$. The positive cones in $X$ and $Y$ also define a notion of positivity for maps. Thus $\phi : X \to Y$ is positive if it maps positive elements to positive elements, and completely positive if its amplifications $\phi^{(n)}$ are positive for every $n \in \mathbb{N}$. Every positive map is automatically self-adjoint. It is well known that a unital linear map between operator systems is completely positive if and only if it is completely contractive; see [BLM, 1.3.3]. In the following we will abbreviate “unital completely positive” with $ucp$, as it is customary. We will often tacitly use Arveson’s extension theorem, asserting that $B(H)$ is an injective element in the category of operator systems and with $ucp$ maps as morphisms; see [P2, Theorem 7.5].

**Definition 2.6.** An operator space is matricial if it admits a completely isometric embedding into $M_n$ for some $n \in \mathbb{N}$. An operator system is matricial if it admits a complete order embedding into $M_n$ for some $n \in \mathbb{N}$.

The theory of injective envelopes of operator spaces has been developed independently by Hamana [H2,H1] and Ruan [R2]. It follows from the main theorem of [S3] that an operator space is matricial if and only if it its injective envelope is finite-dimensional. Moreover by [BLM, Corollary 4.2.8] an operator system is matricial if and only if it is matricial as an operator space.

2.5. Exactness. If $X \subset B(H)$ and $Y \subset B(H)$ are operator spaces, then their minimal tensor product $X \otimes_{min} Y$ is the completion of the algebraic tensor product with respect to the norm induced by the inclusion $X \otimes Y \subset B(H \otimes H)$ where $H \otimes H$ denotes the Hilbertian tensor product. The minimal tensor product is canonically endowed with an operator space structure induced by the inclusion $X \otimes_{min} Y \subset B(H \otimes H)$.

An operator space $X$ 1-exact whenever for every $C^*$-algebra $B$ and ideal $I$ of $B$ the sequence

$$0 \to I \otimes_{min} X \to B \otimes_{min} X \to B/I \otimes_{min} X \to 0$$
is exact, and moreover the induced isomorphisms
\[
\frac{B \otimes_{\min} X}{I \otimes_{\min} X} \to B/I \otimes_{\min} X
\]
is a complete isometry.

If \(E\) and \(F\) are operator spaces, the completely bounded distance \(d_{cb}(E,F)\) of \(E\) and \(F\) is the infimum of \(\|\phi\|_{cb}\|\phi^{-1}\|_{cb}\) when \(\phi\) ranges among all isomorphisms from \(E\) to \(F\). Theorem 1 of [P3] provides the following equivalent reformulation of 1-exactness in terms of the completely bounded distance. An operator space \(X\) is 1-exact if and only if for every finite dimensional subspace \(E\) of \(X\) and every \(\varepsilon > 0\) there is \(n \in \mathbb{N}\) and a subspace \(F\) of \(M_n\) such that \(d_{cb}(E,F) \leq 1 + \varepsilon\).

Suppose now that \(X,Y\) are operator systems, \(B\) is a unital C*-algebra, and \(I\) is an ideal of \(B\). The minimal tensor product \(X \otimes_{\min} Y\) is an operator system obtained as before as a completion of the algebraic tensor product via the inclusion \(X \otimes Y \subset B(H \otimes H)\). It is shown in [KPTT] that the quotient \(\frac{B \otimes_{\min} X}{I \otimes_{\min} X}\) is canonically endowed with an operator system structure. A operator system \(X\) is 1-exact [KPTT, Definition 5.4] if for every unital C*-algebra \(B\) and every ideal \(I\) of \(B\) the sequence
\[
0 \to I \otimes_{\min} X \to B \otimes_{\min} X \to B/I \otimes_{\min} X \to 0
\]
is exact, and moreover the induced (unital) isomorphism
\[
\frac{B \otimes_{\min} X}{I \otimes_{\min} X} \to B/I \otimes_{\min} X
\]
is a complete isometry. It is then shown in [KPTT, Proposition 5.5] that an operator system \(X\) is 1-exact if and only if it is 1-exact as an operator space. Moreover the same proof as [P3, Theorem 1] shows that if \(X\) is a 1-exact operator system then for every finite subset \(E\) of \(X\) and for every \(\varepsilon > 0\) there exist \(n \in \mathbb{N}\), a subspace \(F\) of \(M_n\), and a ucp invertible map \(\phi : E \to F\) such that \(\|\phi^{-1}\|_{cb} < 1 + \varepsilon\). Summarizing we can list the following equivalent characterizations of 1-exactness for operator spaces.

**Proposition 2.7** (Kavruk-Paulsen-Todorov-Tomforde, Pisier). Suppose that \(E\) is a finite-dimensional operator system. The following statements are equivalent:

1. \(E\) is 1-exact;
2. for every \(\varepsilon > 0\) there is a matricial operator space \(F\) such that \(d_{cb}(E,F) < 1 + \varepsilon\);
3. for every \(\varepsilon > 0\) there is \(n \in \mathbb{N}\) and an injective ucp map \(\phi : E \to M_n\) such that \(\|\phi^{-1}\|_{cb} < 1 + \varepsilon\).

Moreover an operator system \(X\) is 1-exact if and only if all its finite-dimensional subspaces are 1-exact.

The implications 3 \(\Rightarrow\) 2 in Proposition 2.7 is obvious. The implication 2 \(\Rightarrow\) 1 follows from [P3, Theorem 1] and [KPTT, Proposition 5.5]. Finally the implication 1 \(\Rightarrow\) 3 follows from the proof of [P3, Theorem 1].

In Condition (3) of Proposition 2.7 we denote as customary by \(\|\phi^{-1}\|_{cb}\) the norm of \(\phi^{-1}\) regarded as a map from \(\phi[E]\) to \(E\). Such a convention will be also used in the rest of this paper.

2.6. **Pushouts of operator systems.** A \(*\)-vector space is a complex vector space \(V\) endowed with a conjugate linear involutive map \(v \mapsto v^*\) from \(V\) to \(V\). A unital \(*\)-vector space is a \(*\)-vector space \(V\) endowed with a distinguished element \(1_V\) such that \((1_V)^* = 1_V\). Clearly any operator system is in particular a unital \(*\)-vector space. A
map $\phi : V \to B (H)$ is unital and self-adjoint if $\phi (1_V)$ is the identity operator and $\phi (v^*) = \phi (v)^*$ for every $v \in V$.

Suppose that $P$ is a collection of unital self-adjoint maps $\phi : V \to B (H_\phi)$ such that $\sup_{\phi} \| \phi (x) \| < +\infty$ for every $x \in V$. Let $J_P$ be the subspace

$$\bigcap_{\phi \in P} \text{Ker} (\phi)$$

of $V$. Define on $V/J_P$ the norm

$$\| x + J_P \| = \sup_{\phi \in P} \| \phi (x) \|_{B (H_\phi)} .$$

Then the completion of $V_P$ of $V/J_P$ is an operator system with unit $1_V + J_P$ and matrix norms

$$\| [x_{ij} + J_P] \| = \sup_{\phi \in P} \| \phi (x_{ij}) \|_{B (H_\phi)} .$$

Equivalently $V_P$ can be defined as the closure inside $\oplus_\phi B (H_\phi) \subset B (\oplus_\phi H_\phi)$ of the image of $V$ under the map

$$v \mapsto (\phi (v))_{\phi \in P} .$$

By definition any element $\phi$ of $P$ induces a ucp map from $V_P$ to $B (H_\phi)$.

We use this construction to define pushouts in the category of operator systems with ucp maps. Suppose that $Z, X, Y$ are operator systems and $\alpha_X : Z \to X$ and $\alpha_Y : Z \to Y$ are ucp maps. Let $V$ be the quotient of the algebraic direct sum $X \oplus Y$ by the subspace $N = \text{span} \{ (-1_X, 1_Y) \}$. Consider the collection $P$ of unital self-adjoint maps $\phi : V \to B (H)$ of the form

$$(x, y) + N \mapsto \phi_X (x) + \phi_Y (y)$$

where $\phi_X$ and $\phi_Y$ are ucp maps on $X$ and $Y$ such that $\phi_X \circ \alpha_X = \phi_Y \circ \alpha_Y$. Let $W$ be the operator system associated with the collection $P$ as in the paragraph above. The canonical morphisms $\psi_X : X \to W$ and $\psi_Y : Y \to W$ are obtained from the first and second coordinate inclusions of $X$ and $Y$ into $(X \oplus Y)/N$. When moreover the maps $\alpha_X : Z \to X$ and $\alpha_Y : Z \to Y$ are unital complete isometries, then $\psi_X : X \to W$ and $\psi_Y : Y \to W$ are unital complete isometries. It follows easily from the definition that this construction is indeed a pushout in the category of operator systems with ucp maps as morphisms.

As shown in [GS, Appendix B] operator systems can be regarded as structures in a suitable language in the logic for metric structures as defined in [FHS, Section 2]; see also [BYBHU]. In particular if $(X_n)_{n \in \mathbb{N}}$ is a sequence of operator systems and $U$ is an ultrafilter on $\mathbb{N}$, then one can define the ultraproduct $\prod_U X_n$ of the $X_n$'s in such a language with respect to the ultrafilter $U$. The ultraproduct can also be explicitly constructed as follows. Define

$$\ell^\infty (X_n) := \left\{ (x_n) \in \prod_n X_n : \sup_n \| x_n \| < +\infty \right\} .$$

Such a space has a natural unital $*$-vector space structure with unit $(1_{X_n})_{n \in \mathbb{N}}$ and involution $(x_n)^* = (x_n^*)^*$. Let then $\mathcal{P}_U$ be the collection of ucp maps of $\ell^\infty (X_n)$ of the form

$$\ell^\infty (X_n) \to \prod_U B (H_{\phi_n})$$

$$(x_n) \mapsto [\phi_n (x_n)] ,$$
where \( \phi_n : X_n \to B(H_{\phi_n}) \) is a ucp map and \( \prod_{\mathcal{U}} B(H_{\phi_n}) \) denote the C*-algebraic ultraproduct; see [B2, II.8.1.7]. The ultraproduct \( \prod_{\mathcal{U}} X_n \) is the operator system obtained from the collection ucp maps \( \mathcal{P}_M \) on the unital \( \ast \)-vector space \( \ell^\infty(X_n) \) as at the beginning of this subsection.

We recall that the category of operator systems also admits inductive limits, which can be defined as in [K1, Section 2]; see also the proof of Proposition 16 in [KW].

2.7. \( M_n \)-systems. Here we recall the definition of \( M_n \)-spaces and the functor \( \text{MIN}_n \) defined and studied in [L1]; see also [OR, Section 2] and [O, Subsection 2.1]. Let \( X \) be an operator space and \( n \in \mathbb{N} \). The operator space \( \text{MIN}_n(X) \) has the same underlying vector space as \( X \) with matrix norms

\[
\|x\|_{\text{MIN}_n(X)} = \sup_{\phi} \|\phi^{(k)}(x)\|_{\text{MIN}_n(M_n)}
\]

where \( \phi \) ranges over all completely contractive maps from \( X \) to \( M_n \). This defines an operator space structure on \( X \) such that the identity map \( \text{id}_X : X \to \text{MIN}_n(X) \) is completely contractive. Such a space is characterized by the following property. If \( Z \) is an operator space and \( \phi : Z \to X \) is a linear map, then \( \phi : Z \to \text{MIN}_n(X) \) is completely bounded if and only if \( \phi^{(n)} : M_n(Z) \to M_n(X) \) is bounded, and in such a case

\[
\|\phi : Z \to \text{MIN}_n(X)\|_{cb} = \|\phi^{(n)} : M_n(Z) \to M_n(X)\|.
\]

This is a consequence of Smith’s lemma [S2, Theorem 2.10].

An operator space \( X \) is called an \( M_n \)-space provided that \( \text{id}_X : \text{MIN}_n(X) \to X \) is a complete isometry. It follows easily from the above mentioned property that \( M_n \)-spaces form a full subcategory of the category of operator spaces. Moreover the inclusion functor implements an equivalence of categories with inverse the functor \( \text{MIN}_n \). Assume now that \( E \) is a finite-dimensional operator space. Then \( E \) is 1-exact if and only if for every \( \varepsilon > 0 \) there is \( n \in \mathbb{N} \) such that

\[
\|\text{id}_X : \text{MIN}_n(E) \to E\|_{cb} \leq 1 + \varepsilon;
\]

see [OR, Lemma 2.2]. In particular all \( M_n \)-spaces are 1-exact.

The natural analogs of the above notions in the category of operator systems have been defined in [X]. Suppose that \( X \) is an operator system. Then \( \text{OMIN}_n(X) \) is the operator system having the same unital \( \ast \)-vector space structure as \( X \) and matrix norms

\[
\|x\|_{\text{OMIN}_n(X)} = \sup_{\phi} \|\phi^{(k)}(x)\|_{\text{OMIN}_n(M_n)}
\]

where \( \phi \) ranges over all ucp maps from \( X \) to \( M_n \). This is a particular case of the construction of operator systems presented in Subsection 2.6 where \( \mathcal{P} \) is the collection of ucp maps from \( X \) to \( M_n \). It is shown in [X] that \( \text{OMIN}_n \) has analogous properties as \( \text{MIN}_n \) where one replaces operator spaces with operator systems and (completely) contractive maps with unital (completely) positive maps. In the proofs the use of Smith’s lemma for operator spaces [ER, Proposition 2.2.2] is replaced by [P2, Theorem 13.1] that \( X \) and \( \text{OMIN}_n(X) \) have the same matricial positive cones up to the \( n \)-th amplification. Moreover since a unital 2-positive map is contractive, \( \text{OMIN}_n(X) \) and \( X \) are \( n \)-isometric.

It follows from Proposition 2.7 and the properties of \( \text{OMIN}_n \) that a finite-dimensional operator system \( E \) is 1-exact if and only if for every \( \varepsilon > 0 \) there is \( n \in \mathbb{N} \) such that

\[
\|\text{id}_E : \text{OMIN}_n(E) \to E\|_{cb} \leq 1 + \varepsilon.
\]

**Definition 2.8.** An \( M_n \)-system is an operator system \( X \) such that \( \text{id}_X : \text{OMIN}_n(X) \to X \) is a complete isometry.
Equivalently $X$ is an $M_n$-system if and only if it admits a complete order embedding into $C(K, M_n)$ for some compact Hausdorff space $K$. As before $M_n$-systems form a full subcategory of the category of operator systems with ucp maps as morphisms, and the inclusion functor is an equivalence of categories with inverse $\text{OMIN}_n$. This readily implies in view of Subsection 2.6 that the category of $M_n$-systems has pushouts. Moreover such a pushout can be obtained by applying $\text{OMIN}_n$ to the pushout in the category of operator systems. In the following we will tacitly use the fact that every $M_n$-system can be approximated by a matricial $M_n$-system. In fact if $X$ is an $M_n$-system and $\varepsilon > 0$ then there is a ucp map $\phi : X \to M_n \oplus^\infty \cdots \oplus^\infty M_n$ such that $\|\phi^{-1}\|_{cb} < 1 + \varepsilon$.

It is a consequence of [P2, Theorem 6.1] that a unital linear map between operator systems is completely contractive if and only if it is $n$-contractive if and only if its $n$-th amplification is positive. Finally we observe that the ultraproduct $\prod M_n$ can be obtained with $C(T, M_n)$ where $T$ is the compact Hausdorff space $\sum K_i$ defined as in [B1]. It follows that the class of $M_n$-systems is closed under ultraproducts.

3. Finite-dimensional 1-exact operator systems

We regard operator systems as structures in the language $\mathcal{L}_{OSy}$ defined in Subsection 2.4. We denote by $\mathcal{E}^{sy}_1$ the class of finite-dimensional 1-exact operator systems. It follows easily from the characterization of 1-exactness provided by Proposition 2.7 together with the abstract characterization of operator systems among unital operator spaces [BN, Proposition 3.2] that the $\mathcal{E}^{sy}_1$-structures in the sense of Subsection 2.1 are precisely the 1-exact operator systems. We define also $\mathcal{M}_{\infty}^0 \subset \mathcal{E}^{sy}_1$ to be the class of matricial operator systems in the sense of Definition 2.6. In this section we will show that $\mathcal{E}^{sy}_1$ is a Fraïssé class, and $\mathcal{M}_{\infty}^0$ is a dense subclass.

3.1. Amalgamation of 1-exact operator systems. In this subsection we will show that the class $\mathcal{E}^{sy}_1$ of finite-dimensional 1-exact operator systems satisfies (NAP) from Definition 2.3. We will start by considering matricial operator systems. If $X, Y$ are operator systems and $f : X \to Y$ is a linear map, define $f^* : X \to Y$ by setting

$$f^* (x) = f (x^*)^* .$$

Define then $\text{Re} (f) = \frac{1}{2} (f + f^*)$ and $\text{Im} (f) = \frac{1}{2i} (f - f^*)$.

The following lemma is an approximate version of [P2, Lemma 2.10].

**Lemma 3.1.** Suppose that $X$ is an operator system, $\delta \in (0, 1]$, and $\phi$ is a unital linear functional on $X$ such that $\|\phi\| \leq 1 + \delta$ for $\delta \in [0, 1]$. Then $|\text{Im} (\phi (x))| \leq 2\sqrt{\delta} \|x\|$ whenever $x \in X$ is self-adjoint.

**Proof.** Suppose that $x \in X$ is self-adjoint and $\|x\| \leq 1$. Denote by $\sigma (x)$ the spectrum of $x$. If $M > 0$ then $\sigma (x)$ is contained in the disc of center $iM$ and radius $(M^2 + 1)^{\frac{1}{2}}$. Therefore $\sigma (x - iM)$ is contained in the disc of center $0$ and radius $(M^2 + 1)^{\frac{1}{2}}$. Thus

$$\|\phi (x) - iM\| \leq (1 + \delta) \|x - iM\| \leq (1 + \delta) (M^2 + 1)^{\frac{1}{2}} .$$

Therefore

$$\text{Im} (\phi (x)) \geq M - (1 + \delta) (M^2 + 1)^{\frac{1}{2}} .$$
Consider

\[ \psi \triangleright (1 + \delta) (M^2 + 1)^{\frac{1}{2}} - x = (1 + \delta) M \left( 1 + \frac{1}{M^2} \right)^{\frac{1}{2}} - M \]

\[ \leq (1 + \delta) M \left( 1 + \frac{1}{2M^2} \right) - M \]

\[ \leq \frac{1}{M^2} + \delta M. \]

Setting \( M = \frac{1}{\sqrt{\delta}} \), this yields \( \text{Im} (\phi (x)) \geq -2\sqrt{\delta} \). A similar argument shows that \( \text{Im} (\phi (x)) \leq 2\sqrt{\delta} \).

**Lemma 3.2.** Suppose that \( X, Y \) are operator systems, \( \delta \in (0, 1] \), and \( f : X \to Y \) is a unital linear map such that \( \| f \|_{cb} \leq 1 + \delta \). Then \( \| \text{Im} (f) \|_{cb} \leq 4\sqrt{\delta} \).

**Proof.** Suppose that \( n \in \mathbb{N} \) and \( \rho \) is a state of \( M_n (Y) \). Applying Lemma 3.1 to \( \phi = \rho \circ f^{(n)} \) one obtains that, whenever \( x \in M_n (X) \) is self-adjoint,

\[ \left| \left( \rho \circ \text{Im} (f)^{(n)} \right) (x) \right| = \text{Im} \left( \left( \rho \circ f^{(n)} \right) (x) \right) \leq 2\sqrt{\delta}. \]

Therefore \( \| \text{Im} (f)^{(n)} \| \leq 4\sqrt{\delta} \), and \( \| \text{Im} (f) \|_{cb} \leq 4\sqrt{\delta} \).

We will need the following perturbation lemma, which is a minor variation of [EH, Theorem 2.5].

**Lemma 3.3.** Suppose that \( E \) is a finite-dimensional operator system, \( Y \) is an operator system, and \( \delta \in [0, 1] \). Denote by \( \dim (E) \) the dimension of \( E \) as a vector space. If \( f : E \to Y \) is unital linear map such that \( \| f \|_{cb} \leq 1 + \delta \), then there is a ucp map \( \psi : E \to Y \) such that \( \| f - \psi \|_{cb} \leq 20 (\dim (E) + 1) \sqrt{\delta} \).

**Proof.** For \( \delta = 0 \) this follows from [P2, Lemma 2.10]. Suppose now that \( \delta \) is nonzero. Consider \( Y \) as a subsystem of \( B (H) \), and let \( n \) be \( \dim (E) \). By Lemma 3.2 we have that \( \| \text{Im} (f) \|_{cb} \leq 4\sqrt{\delta} \). By Wittstock’s decomposition theorem [W]—see also [P1, Corollary 2.6]—there are completely positive maps \( \phi_1, \phi_2 : E \to B (H) \) such that \( \text{Re} (f) = \phi_1 - \phi_2 \) and \( \| \phi_1 + \phi_2 \|_{cb} \leq \| \text{Re} (f) \|_{cb} \). Let \( a_i = \phi_i (1) \) for \( i = 1, 2 \). By [P2, Proposition 3.6] we have that

\[ \| a_1 \| \leq \| a_1 + a_2 \| = \| \phi_1 + \phi_2 \|_{cb} \leq \| \text{Re} (f) \|_{cb} \leq 1 + \delta + 4\sqrt{\delta}. \]

If \( \xi \in H \) has norm 1, then

\[ \langle a_2 \xi, \xi \rangle = \langle (a_1 - 1) \xi, \xi \rangle \leq \delta + 4\sqrt{\delta} \leq 5\sqrt{\delta}. \]

Therefore \( \| \phi_2 \|_{cb} = \| a_2 \| \leq 5\sqrt{\delta} \). By [EH, Lemma 2.4] there is a linear functional \( \theta \) on \( E \)—which we will regard as a function from \( E \) to \( Y \)—such that \( \| \theta \| \leq 5n\sqrt{\theta} \) and \( \theta - \phi_2 \) is completely positive. Define \( \psi' = \text{Re} (f) + \theta = \phi_1 - \phi_2 + \theta \) and observe that \( \psi' \) is completely positive, being sum of completely positive maps. Moreover \( \| \psi' - f \|_{cb} \leq \| \text{Im} (f) \|_{cb} + \| \theta \| \leq 4\sqrt{\delta} + 5n\sqrt{\delta} \) and \( \| \psi' \|_{cb} \leq 1 + 5 (n + 1) \sqrt{\delta} \). Set \( \psi'' = \frac{1}{1 + 5 (n + 1) \sqrt{\delta}} \psi' \) and observe that \( \psi'' \) is completely positive and completely contractive. Moreover

\[ \| \psi'' (1) - 1 \| \leq \| \psi'' (1) - \psi' (1) \| + \| \psi' (1) - f (1) \| \leq 10 (n + 1) \sqrt{\delta}. \]

Let \( \rho \) be a state on \( E \) and set \( \psi (x) = \psi'' (x) + \rho (x) (1 - \psi'' (1)) \). Observe that \( \psi \) is completely positive being sum of completely positive maps. Moreover \( \psi \) is unital and

\[ \| \psi - f \|_{cb} \leq \| 1 - \psi'' (1) \| + \| \psi' - f \|_{cb} + \| \psi' - \psi'' \|_{cb} \leq 20 (n + 1) \sqrt{\delta}. \]
Lemma 3.4. Suppose that \( E \subset F_0 \) and \( F_1 \) are matricial operator systems, and \( \delta \in [0, 1] \). If \( f : E \to F_1 \) is an invertible unital map such that \( \|f\|_{cb} \leq 1 + \delta \) and \( \|f^{-1}\|_{cb} \leq 1 + \delta \), then there exist \( d \in \mathbb{N} \) and unital completely isometric embeddings \( i : F_0 \to M_d \) and \( j : F_1 \to M_d \) such that \( \|j \circ f - i\|_{cb} \leq 100 \dim (E) \delta^\frac{1}{2} \).

Proof. Without loss of generality we can assume that \( F_0 = M_n \) and \( F_1 = M_k \) for some \( n, k \in \mathbb{N} \). By Lemma 3.3 and injectivity of \( M_k \) there is a ucp map \( \phi : M_n \to M_k \) such that \( \|\phi - f\|_{cb} \leq 50 \dim (E) \delta^\frac{1}{2} \). Similarly there is a ucp map \( \psi : M_k \to M_n \) such that \( \|\psi \circ f - id\|_{cb} \leq 50 \dim (E) \delta^\frac{1}{2} \) and hence \( \|\psi \circ f - \phi\|_{cb} \leq 100 \dim (E) \delta^\frac{1}{2} \). Set \( d = n + k \) and define maps \( i : M_n \to M_d \) and \( j : M_k \to M_d \) by \( i(x) = \begin{bmatrix} x & 0 \\ 0 & \phi(x) \end{bmatrix} \) and \( j(y) = \begin{bmatrix} \psi(y) & 0 \\ 0 & y \end{bmatrix} \) for \( x \in M_n \) and \( y \in M_k \). Observe that \( i \) and \( j \) are unital complete isometries such that

\[
\|j \circ f - i\|_{cb} = \max \left\{ \|\psi \circ f - id\|_{cb}, \|f - \phi\|_{cb} \right\} \leq 100 \dim (E) \delta^\frac{1}{2}.
\]

\[\square\]

We are now ready to bootstrap Lemma 3.4 to arbitrary finite-dimensional 1-exact operator systems. The proof is analogous to the proof of [L2, Proposition 4.2].

Proposition 3.5. Suppose that \( E \subset X \) and \( Y \) are finite-dimensional 1-exact operator systems, and \( \delta \in [0, 1] \). If \( f : E \to Y \) is a unitar map such that \( \|f\|_{cb} < 1 + \delta \) and \( \|f^{-1}\|_{cb} < 1 + \delta \), then there exist a separable 1-exact operator system \( Z \) and unital completely isometric embeddings \( i : X \to Z \) and \( j : Y \to Z \) such that \( \|j \circ f - i\|_{cb} < 100 \dim (E) \delta^\frac{1}{2} \).

Proof. Fix \( \varepsilon, \delta' \in (0, 1) \) such that \( \|f\|_{cb} < 1 + \delta', \|f^{-1}\|_{cb} < 1 + \delta' \), and \( \delta' + 2\varepsilon < \delta \).

Set

\[
\eta = \left( \frac{\varepsilon}{100 \dim (E)} \right)^2.
\]

We will define by recursion on \( k \) sequences \((n_k)_{k \in \mathbb{N}}, (Z_k)_{k \in \mathbb{N}}, (i_k)_{k \in \mathbb{N}}, (j_k)_{k \in \mathbb{N}}\) and \((\phi_k)_{k \in \mathbb{N}}\) such that, for every \( k \in \mathbb{N} \),

1. \( Z_k \) is an \( M_{n_k} \)-system,
2. \( i_k : X \to Z_k \) is a ucp map such that \( \|i_k^{-1}\|_{cb} \leq 1 + \eta 2^{-2k} \),
3. \( j_k : Y \to Z_k \) is a ucp map such that \( \|j_k^{-1}\|_{cb} \leq 1 + \eta 2^{-2k} \) and

\[
\left\| j_k \circ f - (i_k)_{cb} \right\|_{cb} \leq 100 \dim (E) \delta^\frac{1}{2} + 2\varepsilon \left( \sum_{i<k} 2^{-i} \right),
\]

4. \( \phi_k : Z_k \to Z_{k+1} \) is a ucp map such that \( \|\phi_k^{-1}\| \leq 1 + \eta 2^{-4k} \),

\[
\left\| \phi_k \circ i_k - i_{k+1} \right\|_{cb} \leq \varepsilon 2^{-k} \quad \text{and} \quad \left\| \phi_k \circ j_k - j_{k+1} \right\|_{cb} \leq \varepsilon 2^{-k}.
\]

Granted the construction we can let \( Z \) be the inductive limit \( \lim_{k} Z_k \) and define the maps

\[
i = \lim_{k} i_k : X \to Z \quad \text{and} \quad j = \lim_{k} j_k : Y \to Z.
\]

It is immediate to verify that conditions (1)–(4) above guarantee that these maps have the desired properties. We will now present the recursive construction. By Lemma 3.4 and Proposition 2.7 one can find \( n_1 \in \mathbb{N} \) and ucp maps \( i_1 : X \to M_{n_1} \) and \( j_1 : Y \to M_{n_1} \) that satisfy conditions (1)–(3) above for \( k = 1 \) and \( Z_1 = M_{n_1} \). Suppose that \( n_k, Z_k, i_k, j_k, \) and \( \phi_k \) have been defined for \( k \leq m \). By Proposition 2.7 there exist \( d \in \mathbb{N} \) and ucp maps \( \theta_X : X \to M_d \) and \( \theta_Y : Y \to M_d \) such that

\[
\max \left\{ \|\theta_X^{-1}\|_{cb}, \|\theta_Y^{-1}\|_{cb} \right\} \leq 1 + \eta 2^{-2(m+1)}.
\]
Moreover by Lemma 3.3 there are ucp maps $\alpha_X : i_m [X] \subset Z_m \to M_d$ and $\alpha_Y : i_m [Y] \subset Z_m \to M_d$ such that
\[
\max \left\{ \left\| \alpha_X - \theta_X \circ i_m^{-1} \right\|_{cb}, \left\| \alpha_Y - \theta_Y \circ j_m^{-1} \right\|_{cb} \right\} \leq \varepsilon 2^{-m}.
\]
Injectivity of $M_d$ ensures that we can extend $\alpha_X$ and $\alpha_Y$ to ucp maps defined on all $Z_m$. Set now $n_{m+1} = n_m + d$. Define maps
\[
\begin{align*}
\hat{\theta}_X : X &\to M_{n_{m+1}} \\
x &\mapsto \begin{bmatrix} i_m(x) & 0 \\
0 & \theta_X(x) \end{bmatrix}, \\
\hat{\theta}_Y : Y &\to M_{n_{m+1}} \\
y &\mapsto \begin{bmatrix} j_m(y) & 0 \\
0 & \theta_Y(y) \end{bmatrix},
\end{align*}
\]
and
\[
\begin{align*}
\hat{\alpha}_X : Z_m &\to M_{n_{m+1}} \\
z &\mapsto \begin{bmatrix} z & 0 \\
0 & \alpha_X(z) \end{bmatrix}, \\
\hat{\alpha}_Y : Z_m &\to M_{n_{m+1}} \\
z &\mapsto \begin{bmatrix} z & 0 \\
0 & \alpha_Y(z) \end{bmatrix}.
\end{align*}
\]
Observe that $\hat{\alpha}_X$ and $\hat{\alpha}_Y$ are unital complete isometries, while $\hat{\theta}_X$ and $\hat{\theta}_Y$ are ucp maps such that
\[
\max \left\{ \left\| \hat{\theta}_X^{-1} \right\|_{cb}, \left\| \hat{\theta}_Y^{-1} \right\|_{cb} \right\} \leq \max \left\{ \left\| \theta_X^{-1} \right\|_{cb}, \left\| \theta_Y^{-1} \right\|_{cb} \right\} \leq 1 + 2 \varepsilon 2^{\frac{4}{m+1}}.
\]
Moreover
\[
\max \left\{ \left\| \hat{\theta}_X - \hat{\alpha}_X \circ i_m \right\|_{cb}, \left\| \hat{\theta}_Y - \hat{\alpha}_Y \circ j_m \right\|_{cb} \right\} \\
\leq \max \left\{ \left\| \alpha_X - \theta_X \circ i_m^{-1} \right\|_{cb}, \left\| \alpha_Y - \theta_Y \circ j_m^{-1} \right\|_{cb} \right\} \leq \varepsilon 2^{-m}.
\]
Now let $Z_{m+1}$ be the pushout of $\hat{\alpha}_X : Z_m \to M_{n_{m+1}}$ and $\hat{\alpha}_Y : Z_m \to M_{n_{m+1}}$ in the category of $M_{n_{m+1}}$-systems with canonical unital complete isometries $\psi_X : M_{n_{m+1}} \to Z_{m+1}$ and $\psi_Y : M_{n_{m+1}} \to Z_{m+1}$; see Subsection 2.6 and Subsection 2.7. Set $\phi_m := \psi_X \circ \hat{\alpha}_X = \psi_Y \circ \hat{\alpha}_Y$, $i_{m+1} := \psi_X \circ \hat{\theta}_X$, and $j_{m+1} := \psi_Y \circ \hat{\theta}_Y$. Observe that
\[
\max \left\{ \left\| i_{m+1}^{-1} \right\|_{cb}, \left\| j_{m+1}^{-1} \right\|_{cb} \right\} \leq \max \left\{ \left\| \hat{\alpha}_X^{-1} \right\|_{cb}, \left\| \hat{\alpha}_Y^{-1} \right\|_{cb} \right\} \leq 1 + 2 \varepsilon 2^{\frac{4}{m+1}}.
\]
Moreover
\[
\left\| \phi \circ i_m - i_{m+1} \right\|_{cb} = \left\| \phi \circ i_m - \psi_X \circ \hat{\theta}_X \right\|_{cb} \\
\leq \left\| \phi \circ i_m - \psi_X \circ \hat{\alpha}_X \circ i_m \right\|_{cb} + 2 \varepsilon 2^{-m} \\
\leq 2 \varepsilon 2^{-m}
\]
and similarly $\left\| \phi \circ j_m - j_{m+1} \right\|_{cb} \leq 2 \varepsilon 2^{-m}$. Furthermore
\[
\left\| (i_{m+1})_E - j_{m+1} \circ f \right\|_{cb} = \left\| \left( \psi_X \circ \hat{\alpha}_X \right)_E - \psi_Y \circ \hat{\theta}_Y \circ f \right\|_{cb} \\
\leq \left\| \psi_X \circ \hat{\alpha}_X \circ i_m - \psi_Y \circ \hat{\alpha}_Y \circ j_m \circ f \right\|_E + 2 \varepsilon 2^{-m+1} \\
\leq \left\| \phi_m \circ i_m - \phi_m \circ j_m \circ f \right\|_E + 2 \varepsilon 2^{-m+1} \\
\leq 100 \dim(E) \delta \frac{1}{\varepsilon^2} + 2 \varepsilon \sum_{k \leq m} 2^{-k}.
\]
This concludes the recursive construction. \qed
It is not difficult to modify the proof of Proposition 3.5 to cover the case when $X$ and $Y$ are not necessarily finite-dimensional.

3.2. Embeddings of 1-exact operator systems. The goal of this section is to show that every 1-exact separable operator system admits a unit completely isometric embedding into an inductive limit of full matrix algebras with unital completely isometric connective maps. The construction of the inductive limit of an inductive sequence of operator systems can be found in [KW, Section 2]; see also the proof of Proposition 16 in [K1]. The proof of the following proposition is similar to the one of [EOR, Theorem 4.7].

Proposition 3.6. Suppose that $X$ is a separable 1-exact operator system. Then there are a sequence natural numbers $n_k$ and unital complete isometries $\phi_k : M_{n_k} \to M_{n_{k+1}}$ such that $X$ embeds unitally completely isometrically into the inductive limit $\lim\nolimits_{(\phi_k)} M_{n_k}$.

Proof. Let $(E_k)_{k \in \mathbb{N}}$ be an increasing sequence of finite-dimensional subsystems of $X$ with dense union. Set $\varepsilon_k = (50 \dim (E_k))^{-2}$, where $\dim (E_k)$ denotes the dimension of $E_k$ as a vector space. We will define the sequences $(n_k)$ and $(\phi_k)$ by recursion on $k \in \mathbb{N}$ together with ucp maps $i_k : E_k \to M_{n_k}$ such that $\|i_k^{-1}\|_{cb} \leq 1 + \varepsilon_k 2^{-2k}$ and $\|\phi_k \circ i_k - i_{k+1}\|_{cb} \leq 2^{-k}$. Granted this construction we let $i : X \to \lim\nolimits_{(\phi_k)} M_{n_k}$.

The map $i_1 : E_1 \to M_{n_1}$ can be defined applying the characterization of 1-exactness provided by Proposition 2.7. Suppose that $n_k$, $i_k$, and $\phi_k$ satisfying the conditions above have been defined for $k \leq m$. Again by Proposition 2.7 one can find $d \in \mathbb{N}$ together with a ucp map $\theta : E_{m+1} \to M_d$ such that $\|\theta^{-1}\|_{cb} \leq 1 + \varepsilon_{m+1} 2^{-2(m+1)}$. By Lemma 3.3 there is a ucp map $\alpha : i_m [E_m] \subset M_{n_m} \to M_d$ such that $\|\alpha - \theta \circ i_m^{-1}\|_{cb} \leq 2^{-m}$. By injectivity of $M_n$ we can extend $\alpha$ to a ucp map from $M_{n_m}$ to $M_d$. Define now $n_{m+1} = n_m + d$, $\phi_m : M_{n_m} \to M_{n_{m+1}}$ and $i_{m+1} : E_{m+1} \to M_{n_{m+1}}$ by setting $\phi_m(z) = \begin{bmatrix} z & 0 \\ 0 & \alpha(z) \end{bmatrix}$ and $i_{m+1}(x) = \begin{bmatrix} i_m(x) & 0 \\ 0 & \theta(x) \end{bmatrix}$ for $z \in M_{n_m}$ and $x \in E_{m+1}$. Observe that $\phi_m$ is a unital complete isometry, and $i_{m+1}$ is a ucp map such that $\|i_{m+1}^{-1}\|_{cb} \leq \|\theta^{-1}\|_{cb} \leq 1 + \varepsilon_{m+1} 2^{-2(m+1)}$.

Moreover
\[
\|\phi_m \circ i_m - i_{m+1}\|_{cb} = \|\theta - \alpha \circ i_m\|_{cb} \leq \|\alpha - \theta \circ i_m^{-1}\|_{cb} \leq 2^{-m}.
\]

This concludes the recursive construction. \hfill $\square$

3.3. The Fraïssé metric space. Recall that we denote by $\mathcal{E}_1^{sy}$ the class of finite-dimensional 1-exact operator systems. It is clear that $\mathcal{E}_1^{sy}$ satisfies (HP) from Definition 2.1. Proposition 3.5 shows that $\mathcal{E}_1^{sy}$ has (NAP). Finally (JEP) is a consequence of (NAP) and the observation that the operator system $\mathbb{C}$ is an initial object in the category of operator systems. In order to conclude that $\mathcal{E}_1^{sy}$ is a Fraïssé class it is enough to show that for every $k \in \mathbb{N}$ the space $\mathcal{E}_1^{sy}(k)$ endowed with the Fraïssé metric $d_{\mathcal{E}_1^{sy}}$ from Definition 2.2 is complete and separable. Without loss of generality we can assume that $\mathcal{E}_1^{sy}(k)$ only contains the pairs $(\mathfrak{a}, E)$ where $E$ is a finite-dimensional 1-exact operator
system and $\pi$ is a linear basis of $E$. Any two such pairs $(\pi, E)$ and $(\overline{b}, F)$ are identified if there is a complete order isomorphism from $E$ to $F$ mapping $\pi$ to $\overline{b}$. For brevity we denote an element $(\pi, E)$ of $\mathcal{E}^{sy}_1(k)$ simply by $\pi$ and set $E = (\overline{\pi})$.

Recall that a basis $\{x_1, \ldots, x_n\}$ of a Banach space $X$ with dual basis $\{x_1', \ldots, x_n'\}$ is an Auerbach basis provided that $\|x_i\| = \|x_i'\| = 1$ for $i \leq n$; see [B3, Theorem 13]. In analogy we call an element $\pi$ of $\mathcal{E}^{sy}_1$ with dual basis $\pi'$ $N$-Auerbach for $N \in \mathbb{N}$ if $\|a_i\| \leq N$ and $\|a_i'\| \leq N$ for $i \leq n$. The following is a standard lemma, known as the standard perturbation argument; see [BO, Lemma 12.3.15].

**Lemma 3.7.** Suppose that $\pi, \overline{b} \in \mathcal{E}^{sy}_1(k)$ are $N$-Auerbach. If $d_{\mathcal{E}^{sy}_1}(\pi, \overline{b}) \leq \frac{1}{4kN\pi}$, then there is a unital isomorphism $\rho$ of $\hat{\mathcal{E}}^{sy}_1(k)$ provided that $\|f\|_{cb} \leq 1 + kNd_{\mathcal{E}^{sy}_1}(\pi, \overline{b})$ and $\|f^{-1}\|_{cb} \leq 1 + kNd_{\mathcal{E}^{sy}_1}(\pi, \overline{b})$.

**Proof.** Set $\delta = d_{\mathcal{E}^{sy}_1}(\pi, \overline{b})$. We can assume that $\langle \pi \rangle$ and $\langle \overline{b} \rangle$ are unitally contained in a unital $C^*$-algebra $A$ in such a way that $\max_i |a_i - b_i| \leq \delta$. Define the map

$$\theta : A \to A,$$

$$z \mapsto z + \sum_{i \leq n} b_i'(z) (b_i - a_i).$$

Observe that $\theta(a_i) = b_i$ for $i \leq k$ and

$$\|id_A - \theta\|_{cb} \leq \sum_{i \leq n} \|b_i'\| \leq kN\delta.$$

Reversing the roles of $\pi$ and $\overline{b}$ allows one to define a map $\tilde{\theta} : A \to A$ such that $\tilde{\theta}(b_i) = a_i$ and $\|id_A - \tilde{\theta}\|_{cb} \leq kN\delta$. Let now $f : \langle \pi \rangle \to \langle \overline{b} \rangle$ be $\theta|_{\langle \pi \rangle}$, and observe that $f$ satisfies the desired conclusions.

The following lemma is an immediate consequence of Proposition 3.5.

**Lemma 3.8.** Suppose that $\delta \in [0, \frac{1}{20}]$ and $\pi, \overline{b} \in \mathcal{E}^{sy}_1(k)$. If there is an isomorphism $f : \langle \pi \rangle \to \langle \overline{b} \rangle$ such that $\|f(1) - 1\| \leq \delta$, $\|f\|_{cb} \leq 1 + \delta$ and $\|f^{-1}\|_{cb} \leq 1 + \delta$, then $d_{\mathcal{E}^{sy}_1}(\pi, \overline{b}) \leq 100k\delta^2$.

**Proof.** Let $\rho$ be a state on $\langle \pi \rangle$. Define $g(x) = f(x) + \rho(x)(1 - f(1))$. Observe that $\|g - f\|_{cb} \leq \delta$ and hence $\|g\|_{cb} \leq 1 + 2\delta$. Furthermore $g$ is unital, and an easy calculation shows that $g$ is invertible with $\|g^{-1}\|_{cb} \leq 1 + 10\delta$. By Proposition 3.5 we can now conclude that $d_{\mathcal{E}^{sy}_1}(\pi, \overline{b}) \leq 100k(10\delta)^2 \leq 1000k\delta^2$.

We are now ready to show that $(\mathcal{E}^{sy}_1(k), d_{\mathcal{E}^{sy}_1})$ is a complete metric space. The proof of this fact involves a standard argument; see also [P3, Proposition 12].

**Proposition 3.9.** Suppose that $k \in \mathbb{N}$. Then $\mathcal{E}^{sy}_1(k)$ is a complete metric space with respect to the metric $d_{\mathcal{E}^{sy}_1}$ and $\mathcal{M}^0_{\mathcal{E}^{sy}_1}(k)$ is a dense subspace.

**Proof.** Let us first show that $\mathcal{E}^{sy}_1(k)$ is complete. Suppose that $(\pi^{(m)})_{m \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{E}^{sy}_1(k)$. It is not difficult to verify that there is $N \in \mathbb{N}$ such that $\pi^{(m)}$ is $N$-Auerbach for every $m \in \mathbb{N}$. Fix a nonprincipal ultrafilter $\mathcal{U}$ over $\mathbb{N}$ and let $E := \prod_{\mathcal{U}} \langle \pi^{(m)} \rangle$ be the corresponding ultraproduct. Let $a_i$ for $i \leq k$ be the element of $X$ with representative sequence $(a_i^{(m)})_{m \in \mathbb{N}}$. Let $\phi_m : \langle \pi^{(m)} \rangle \to E$ be the linear map such that $\phi_m(a_i^{(m)}) = a_i$. 


By Lemma 3.7 and Los’ theorem for ultraproducts [FHS, Proposition 4.3] we have that
\[ \lim_{m \to +\infty} \| \phi_m^{\text{cb}} \phi_m^{-1} \|_{\text{cb}} \to 1. \]
Therefore \( X \) is 1-exact by Proposition 2.7. Moreover \( d_{\xi_1} (\pi, \pi(m)) \to 0 \) by Lemma 3.8. The fact that \( M_0^0 (k) \) is dense in \( \mathcal{E}_{1}^{sy} (k) \) is an immediate consequence of Proposition 3.6. To prove separability it is enough to show that \( M_0^0 (k) \) is separable. For every \( n \in \mathbb{N} \) let \( D_n \subseteq M_n \) be a self-adjoint countable dense subset containing the identity. Then the set of \( \pi \in M_0^0 \) such that \( \pi \subseteq D_n \) for some \( n \in \mathbb{N} \) is a countable dense subset of \( M_0^0 (k) \). □

This concludes the proof that \( \mathcal{E}_{1}^{sy} \) is a Fraïssé class, and \( M_0^0 \) is a dense subclass as in Definition 2.3. In view of Theorem 2.5 we can consider the corresponding limit. Equivalent characterizations of such a limit will be obtained in Subsection 4.1.

4. The Gurarij operator system

4.1. The limit of the class \( \mathcal{E}_{1}^{sy} \).

**Definition 4.1.** A separable 1-exact operator system \( \mathbb{G} \) is Gurarij if whenever \( E \subset F \) are finite-dimensional 1-exact operator systems, \( \phi : E \to \mathbb{G} \) is a unital complete isometry, and \( \varepsilon > 0 \), there is a linear map \( \hat{\phi} : F \to \mathbb{G} \) extending \( \phi \) such that
\[ \left\| \phi \left\|_{\text{cb}} \right\| \hat{\phi}^{-1} \right\|_{\text{cb}} \leq 1 + \varepsilon. \]

The following characterization of the limit of \( \mathcal{E}_{1}^{sy} \) follows easily from our results above.

**Proposition 4.2.** Suppose that \( Z \) is a separable 1-exact operator system. The following statements are equivalent:

1. \( Z \) is the Fraïssé limit of the class \( \mathcal{E}_{1}^{sy} \) in the sense of Definition 2.4;
2. Whenever \( E \subset F \) are finite-dimensional 1-exact operator systems, \( \phi : E \to Z \) is a unital complete isometry, and \( \varepsilon > 0 \), there is a unital complete isometry \( \psi : F \to Z \) such that \( \left\| \psi \|_{E} - \phi \right\| \leq \varepsilon; \)
3. \( Z \) satisfies the same condition as (2) when \( F = M_n \) for some \( n \in \mathbb{N} \);
4. \( Z \) is Gurarij;
5. \( Z \) satisfies the same conditions as in Definition 4.1 when \( F = M_n \) for some \( n \in \mathbb{N} \);
6. For every matricial operator space \( E \), finite subset \( B \) of \( E \), \( \psi \in \text{Stx}_{\mathcal{E}_{1}^{sy}} (E, Z) \), and \( \varepsilon > 0 \), there is \( \phi \in \text{Apx}_{\mathcal{E}_{1}^{sy}} (E, Z) \) such that \( \phi \leq \psi \) and for every \( b \in B \) there is \( z \in Z \) such that \( \phi (b, z) \leq \varepsilon. \)

**Proof.** We present a proof of the nontrivial implications below.

(1) \( \Rightarrow \) (2): By definition of Fraïssé limit \( Z \) is a separable \( \mathcal{E}_{1}^{sy} \)-structure, and hence a separable 1-exact operator system by [BN, Proposition 3.2]. Observe that the embeddings between structures in the language \( L_{OSy} \) for operator systems are precisely the unital complete isometries. Therefore (2) is a reformulation of being limit as in Definition 2.4;

(3) \( \Rightarrow \) (5): Suppose that \( E \subseteq M_n \) and \( \phi : E \to Z \) is a unital complete isometry. Fix a strictly positive real number \( \delta \). By hypothesis there is a unital complete isometry \( \psi : M_n \to Z \) such that \( \left\| \psi \|_{E} - \phi \right\| < 1 + \delta \). If \( \delta \) is small enough, then by Lemma 3.7 one can perturb \( \psi \) to a unital linear map \( \phi : M_n \to Z \) such that \( \left\| \phi \|_{cb} \right\| \phi^{-1} \right\|_{cb} \leq 1 + \varepsilon \) and \( \phi = \phi. \)

(2) \( \Rightarrow \) (4): Analogous to the implication (3) \( \Rightarrow \) (5).
and operator system embeds unitally and completely isometrically into operator system is unique up to complete order isomorphism. Any separable 1-exact operator system has the following homogeneity property: if $GS$ exists a $100 \dim (E) \geq 1 + \delta$, then there exists a complete order automorphism $\alpha$ of $GS$ such that $\|\alpha(E)\|_{cb} < 1 + \delta$. This implies that $d_{E_1^v}(\tau, f(\tau)) \leq 1000k\delta^\frac{1}{2}$ by Lemma 3.8. Therefore there are a finite-dimensional 1-exact operator system $F$ and unitar complete isometries $f_0 : (\tau) \rightarrow F$ and $f_1 : (f(\tau)) \rightarrow F$ such that $\|f_0(c_i) - (f_1 \circ f)(c_i)\| \leq 1000k\delta^\frac{1}{2}$ for every $i$. Let $\phi : (\tau) \rightarrow \langle f(\tau) \rangle$ be the composition of $f^{-1}_1$ and $f_0$ as approximate morphisms. It is clear that by choosing $\delta$ small enough we can ensure that $\phi(x, f(x)) \leq \varepsilon$ for every $x \in A \cup B$. Observe finally that

\[
\phi \in \operatorname{Apx}(\langle \tau \rangle, \langle f(\tau) \rangle) \subset \operatorname{Apx}(E, Z)
\]

and

\[
\psi \geq f|_A + \varepsilon \geq \phi.
\]

This concludes the proof.

(6)⇒(1): Since by Proposition 3.9 the class $\mathcal{M}_\infty^0$ of matricial operator systems is dense in $\mathcal{E}_1^{sy}$ as in Definition 2.3, the conclusion follows from the characterization of the Fraïssé limit provided by [BY, Lemma 2.16].

\[\square\]

4.2. Existence and uniqueness of the Gurarij operator system. Recall that we have shown in Subsection 3.1 and Subsection 3.3 that $\mathcal{E}_1^{sy}$ is a Fraïssé class. We can therefore consider its limit. Proposition 4.2 shows that being limit of $\mathcal{E}_1^{sy}$ is equivalent to being a Gurarij operator system.

The following theorem is now an immediate consequence of Ben Yaacov’s main results on Fraïssé limits of metric structures; see Theorem 2.5. The following lemma can be obtained from Proposition 3.5, similarly as Lemma 2.2 is obtained from Lemma 2.1 in [KS].

**Lemma 4.3.** Suppose that $E \subset GS$ is a finite-dimensional subsystem, $Y$ is a finite-dimensional 1-exact operator system, $\delta \in (0, 1]$, and $f : E \rightarrow Y$ is a unital invertible linear map such that $\|f\|_{cb} < 1 + \delta$ and $\|f^{-1}\|_{cb} < 1 + \delta$. Then for every $\eta > 0$ there exists a $g : Y \rightarrow NG$ such that $\|g\|_{cb} < 1 + \eta$, $\|g^{-1}\|_{cb} < 1 + \eta$, and $\|g \circ f - id_X\|_{cb} < 100 \dim (E) \delta^\frac{1}{2}$.

**Theorem 4.4.** There is a Gurarij operator system $GS$ as in Definition 4.1. Such an operator system is unique up to complete order isomorphism. Any separable 1-exact operator system embeds unitally and completely isometrically into $GS$. Furthermore $GS$ has the following homogeneity property: if $E$ is a finite-dimensional subsystem of $GS$, $\delta \in (0, 1]$, and $\phi : X \rightarrow GS$ is a unital invertible linear map such that $\|\phi\|_{cb} < 1 + \delta$ and $\|\phi^{-1}\|_{cb} < 1 + \delta$, then there exists a complete order automorphism $\alpha$ of $GS$ such that $\|\phi - \alpha(E)\|_{cb} < 100 \dim (X) \delta^\frac{1}{2}$. 
Proof. The existence, uniqueness, and universality statements follow from Theorem 2.5. The proof of the homogeneity property is analogous to the proof of [KS, Theorem 1.1] where [KS, Lemma 2.2] is replaced by Lemma 4.3. □

4.3. An explicit construction of GS. Lemma 3.4 shows that the class of matricial operator systems is Fraïssé. It follows from this observation and the proof of the existence statement for Fraïssé limits [BY, Lemma 2.7] that GS can be written as a direct limit \( \lim_{\longrightarrow} X_k \), where \( X_k \) is an operator system completely order isomorphic to \( M_{n_k} \) for some \( n_k \in \mathbb{N} \) and \( \phi_k : X_k \to X_{k+1} \) is a complete order embedding. We will present in this subsection an explicit construction of GS that makes this fact apparent.

Let us say that a subset \( D \) of a metric space \( X \) is \( \varepsilon \)-dense for some \( \varepsilon > 0 \) if every element of \( X \) is at distance at most \( \varepsilon \) from some element of \( D \). For \( m, k \in \mathbb{N} \), let \( D_{m,k} \) be a finite \( 2^{-k} \)-dense subset of the ball of radius 2 of \( X_k \); \( (\varphi_{m,k,i}) \) be an enumeration of the finite tuples from \( D_{m,k} \). Set \( E_{m,k,i} = \langle \varphi_{m,k,i} \rangle \subset M_m \) for \( m, i \in \mathbb{N} \). We define by recursion on \( k \) sequences \( (n_k) \), \( (X_k) \), \( (D_k^X) \), \( (\phi_k) \), \( (f_{m,k,i,j}) \) such that:

1. \( X_k \) is an operator system completely order isomorphic to \( M_{n_k} \);
2. \( D_k^X \) is a \( 2^{-k} \)-dense subset of the ball of radius 2 of \( X_k \);
3. \( \phi_k : X_k \to X_{k+1} \) is a complete order embedding;
4. \( (f_{m,k,i,j}) \) enumerates all the unital linear maps \( f \) from \( E_{m,k,i} \) to \( X_k \) such that \( f(\varphi_{m,k,i}) \in D_k^X \);
5. for every \( m, k, i, j \) such that \( m \leq k \) there exists a complete order embedding \( g_{m,k,i,j} : M_m \to X_{k+1} \) such that

\[
\|g_{m,k,i,j} - \phi_k \circ f_{m,k,i,j}\|_{cb} \leq 100 \dim(E_{m,k,i}) \left( \max \left\{ \|f_{m,k,i,j}\|_{cb}, \|f_{m,k,i,j}^{-1}\|_{cb} \right\} - 1 \right)^{1/2}.
\]

Granted the construction define \( X = \lim_k X_k \) and identify \( X_k \) with its image inside \( X \). Suppose that \( m \in \mathbb{N} \), \( E \subset M_m \) is a subsystem, \( \varepsilon > 0 \), and \( f : E \to X \) is a complete order embedding. Fix \( \eta \in (0, 1) \) small enough. By the small perturbation argument [BO, Lemma 12.3.15] there exist \( k, i, j \in \mathbb{N} \) such that \( m \leq k \), \( \|f_{m,k,i,j} - f\|_{cb} \leq \eta \), \( \|f_{m,k,i,j}\|_{cb} \leq 1 + \eta \), and \( \|f_{m,k,i,j}^{-1}\|_{cb} \leq 1 + \eta \). Therefore

\[
\|g_{m,k,i,j} - f\|_{cb} \leq \|g_{m,k,i,j} - f_{m,k,i,j}\|_{cb} + \|f_{m,k,i,j} - f\|_{cb} \leq 100m\eta^{1/2} + \eta \leq \varepsilon
\]

for \( \eta \) small enough. This proves that \( X \) satisfies Condition (3) of Proposition 4.2. Since \( X \) is 1-exact, it follows that \( X \) is completely order isomorphic to GS. We now show how to recursively define sequences as above. Let \( n_1 = 1 \), \( X_1 = \mathbb{C} \). Suppose that the sequences \( (n_k) \), \( (X_k) \), \( (D_k^X) \), \( (\phi_k) \), \( (f_{m,k,i,j}) \) have been defined for \( k \leq l \). Observe that condition (5) only concerns finitely many functions \( f_{m,k,i,j} \). Therefore one can obtain \( X_{k+1} \), \( \phi_k : X_k \to X_{k+1} \), and \( (g_{m,k,i,j}) \) by repeatedly applying Lemma 3.4. This concludes the construction.

4.4. Unital operator spaces. In this subsection we want to observe that GS is of universal disposition for 1-exact unital operator spaces. Recall that a unital operator space is an operator space with a distinguished unitary element in the sense of [BN]. A unital operator space can be concretely represented as a unital subspace of \( B(H) \) with the inherited matrix norms and the identity operator as distinguished unitary. If \( X \subset B(H) \) is a unital operator space, define the operator system \( X + X^* = \{ x_1 + x_2^* : x_1, x_2 \in X \} \). Recall that if \( \varphi : X \to Y \) is a unital completely contractive (resp. completely isometric) map from \( X \) into an operator system \( Y \), then \( x_1 + x_2^* \mapsto \varphi(x_1) + \varphi(x_2)^* \) is a unital completely contractive (resp. completely isometric) map; see [BLM, Lemma 1.3.6]. This implies that the complete order isomorphism class of \( X + X^* \) does not depend from the concrete representation of \( X \) as a unital subspace of \( B(H) \).
A unital operator space is 1-exact if it is 1-exact as an operator space. We now want to observe that a unital operator space $X$ is 1-exact if and only if the operator system $X + X^*$ is 1-exact. The proof of [P3] shows that a unital operator space $X$ is 1-exact if and only if for every finite-dimensional unital operator space $E \subset X$ and for every $\delta > 0$ there exist $n \in \mathbb{N}$ and a unital completely contractive map $\phi : E \to M_n$ such that $\|\phi^{-1}\|_{cb} \leq 1 + \delta$. The proof of the following lemma is analogous to the proof of Proposition 3.6 with the additional ingredient of [BLM, Lemma 1.3.6].

**Lemma 4.5.** Suppose that $X$ is a 1-exact unital operator space. Then $X$ admits a unital completely isometric embedding into an inductive limit of full matrix algebras with unital completely isometric connective maps.

Suppose now that $E$ is a finite-dimensional unital 1-exact operator space. By Lemma 4.5 we can assume that $E \subset Z$, where $Z$ is an inductive limit of full matrix algebras with unital completely isometric connective maps. Hence

$$E + E^* = \{x_1 + x_2^*: x_1, x_2 \in E\} \subset Z.$$

Let $\pi$ be a basis for $E$ and $\varepsilon > 0$. Pick $\delta > 0$ small enough. There exist a subsystem $Z_0 \subset Z$ completely order isomorphic to a full matrix algebra and a tuple $\tilde{\psi}$ in $Z_0$ such that $\max_i |a_i - b_i| \leq \delta$ and, hence, $\max_i |a_i^* - b_i^*| \leq \delta$. The proof of Lemma 3.7 shows that for $\delta$ small enough one has that $d_{cb}(E + E^*, F) \leq \varepsilon$ for some subsystem $F$ of $Z_0$. This shows that $E + E^*$ is a 1-exact operator system. The fact that a unital operator space $X$ is 1-exact if and only if $X + X^*$ is 1-exact now follows immediately.

**Proposition 4.6.** If $E \subset F$ are unital 1-exact operator spaces, $\phi : E \to GS$ is a unital complete isometry, and $\varepsilon > 0$, then there exists a unital complete isometry $\psi : F \to GS$ such that $\|\psi|_E - \phi\|_{cb} \leq 1 + \varepsilon$.

**Proof.** Since $F$ is 1-exact, also $F + F^*$ is 1-exact. Moreover by [BLM, Lemma 1.3.6] the map $\tilde{\phi} : E + E^* \to GS$ defined by $x_1 + x_2^* \mapsto \phi(x_1) + \phi(x_2)^*$ is a unital complete isometry. Therefore there is a unital complete isometry $\tilde{\psi} : F + F^* \to GS$ such that $\|\tilde{\psi}|_{E + E^*} - \tilde{\phi}\|_{cb} \leq 1 + \varepsilon$. Setting $\psi := \tilde{\psi}|_E$ yields a unital complete isometry from $F$ to $GS$ such that $\|\psi - \phi\|_{cb} \leq 1 + \varepsilon$. \qed

Let us now consider $GS$ as a unital operator space, i.e. a structure in the language $L_{uOS}$ of unital operator spaces. Proposition 4.6 shows that $GS$ is a homogeneous $L_{uOS}$-structure. Therefore the age of $GS$ in the language $L_{uOS}$, which is the collection of finite-dimensional 1-exact unital operator spaces, is a Fraïssé class with limit $GS$.

### 4.5. A unital function system of almost universal disposition

We will now consider the class of $M_n$-systems defined in Subsection 2.7. Let us say that a unital $M_n$-space is a subspace of an $M_n$-system containing the unit. Observe that a unital $M_n$-space is in particular an $M_n$-space in the sense of [BLM, 4.1.1]. We can regard unital $M_n$-spaces as structures in a language $L_{uM_n}$, which is the language of $M_n$-spaces described in [L2, Subsection 2.5] with the addition of a constant symbol for the unit. Similarly $M_n$-systems can be regarded as structures in the language $L_{uM_n^*}$ obtained from $L_{uM_n}$ adding a symbol for the involution.

The proofs of Lemma 4.7 and Proposition 4.8 is analogous to the proofs of Lemma 3.4 and Lemma 3.5. We denote the $\infty$-sum of $k$ copies of $M_n$ by $\ell_\infty^k(M_n)$.

**Lemma 4.7.** Suppose that $F_0, F_1 \subset \ell_\infty^k(M_n)$, $E \subset F_0$, and $\delta \in [0, 1]$. If $f : E \to F_1$ is an invertible unital map such that $\|f\|_{cb} \leq 1 + \delta$ and $\|f^{-1}\|_{cb} \leq 1 + \delta$, then there exist $d \in \mathbb{N}$ and unital completely isometric embeddings $i : F_0 \to \ell_\infty^d(M_n)$ and $j : F_1 \to \ell_\infty^d(M_n)$ such that $\|j \circ f - i|_E\|_{cb} \leq 100 \dim(E) \delta^2$. 

Proposition 4.8. Suppose that \( X \) and \( Y \) are \( M_q \)-systems, \( E \subset X \) is a finite-dimensional subsystem, \( \varepsilon > 0 \), and \( \delta \in [0, 1] \). If \( f : E \to Y \) is a unital map such that \( \|f\|_{cb} \leq 1 + \delta \) and \( \|f^{-1}\|_{cb} \leq 1 + \delta \), then there exist a separable \( M_q \)-system \( Z \) and unital completely isometric embeddings \( i : X \to Z \) and \( j : Y \to Z \) such that \( \|j \circ f - i\|_{cb} \leq 100 \dim (E) \delta^2 + \varepsilon \).

Proposition 4.8 shows that \( M_n \)-systems form a Fraïssé class. Using the properties of \( M_n \)-systems and the functor \( \text{OMIN}_n \) defined in Subsection 2.7, one obtain the following characterization for the limit \( G^n_u \) of the Fraïssé class of \( M_n \)-systems.

Proposition 4.9. Suppose that \( Z \) is a separable \( M_n \)-system. The following conditions are equivalent:

1. \( Z \) is completely order isomorphic to \( G^n_u \);
2. \( Z \) is completely order isomorphic to \( \text{OMIN}_n (GS) \);
3. whenever \( E \subset F \) are finite-dimensional \( M_n \)-systems, \( \phi : E \to Z \) is a unital complete isometry, and \( \varepsilon > 0 \), there is a unital complete isometry \( \psi : F \to Z \) such that \( \|\psi_{|E} - \phi\|_{cb} \leq \varepsilon \);
4. whenever \( E \subset F \) are finite-dimensional \( M_n \)-systems, \( \phi : E \to Z \) is a unital complete isometry, and \( \varepsilon > 0 \), there is a unital linear map \( \psi : F \to Z \) such that \( \|\widehat{\psi_{|E}}\|_{cb} \leq 1 + \varepsilon \).

Similar characterizations hold when \( E \) and \( F \) in (3) and (4) are only assumed to be finite-dimensional unital \( M_n \)-spaces. Observe that unital \( M_1 \)-spaces are precisely \emph{unital function spaces}, i.e. subspaces of \( C (K) \) for some compact Hausdorff space \( K \) containing the function 1\(_K\) constantly equal to 1. Similarly \( M_1 \)-systems are \emph{function systems}, i.e. subspaces of \( C (K) \) for some compact Hausdorff space \( K \) containing 1\(_K\) and closed under taking adjoints. We can therefore conclude from Proposition 4.9 that \( G^n_u \) is a separable unital function system of almost universal disposition for unital function spaces. In other words whenever \( E \subset F \) are unital function spaces, \( \phi : E \to G^n_u \) is a unital isometry, and \( \varepsilon \geq 0 \), there is a unital isometry \( \psi : F \to G^n_u \) such that \( \|\psi_{|E} - \phi\| \leq 1 + \varepsilon \).

4.6. The C*-envelope of the Gurarij operator system. We now want to show that the canonical \(*\)-homomorphism from the universal C*-algebra of \( GS \) to the C*-envelope of \( GS \) is a \(*\)-isomorphism.

Suppose that \( X \) is an operator system. A C*-cover of \( X \) is a unital completely isometric embedding \( \phi \) of \( X \) into a C*-algebra \( A \) such that the image of \( X \) under \( \phi \) generates \( A \) as a C*-algebra. It was shown by Hamana [H2, H1] and, independently, Ruan [R2] that there always exists a (projectively) minimal C*-cover \( i_X : X \to C^* (X) \), called the \emph{C*-envelope} of \( X \) (or the regular C*-algebra in [KW]). The C*-envelope has the property that whenever \( \phi : X \to A \) is a C*-cover, there is a (necessarily unique and surjective) \(*\)-homomorphism \( \pi : A \to C^*_e (X) \) such that \( \pi \circ \phi = i_X \).

Similarly it is shown in [KW, Section 3] that there always exists a (projectively) maximal C*-cover \( u_X : X \to C^*_u (X) \), called the \emph{universal C*-algebra} of \( X \). This has the property that whenever \( \phi : X \to A \) is a C*-cover, there is a (necessarily unique and surjective) \(*\)-homomorphism \( \pi : C^*_u (X) \to A \) such that \( \pi \circ u_X = \phi \). In particular there is a \(*\)-homomorphism \( \sigma_X : C^*_u (X) \to C^*_e (X) \).

An operator system \( X \) for which \( \sigma_X \) is injective (or, equivalently, a \(*\)-isomorphism) is called \emph{universal} in [KW]. In particular this property implies that if \( \phi : X \to A \) is any C*-cover of \( X \), then \( A \cong C^*_u (X) \cong C^*_e (X) \). It is proved in [KW, Theorem 15 and Corollary 18] that if \( X \) is a universal operator system of dimension at least 2, then \( X \) does not embed unitally completely isometrically into any exact C*-algebra. An example of a nuclear separable universal operator system is constructed in [KW, Theorem 17].
In the rest of this subsection we will show that the Gurarij operator system is universal in the sense of [KW]. The argument is similar to the one in the proof of [KW, Theorem 17]. In the following we identify an operator system \( X \) with its image inside the C*-envelope \( C^*_e(X) \), and we denote by \( u_X : X \to C^*_u(X) \) the canonical embedding of \( X \) into its universal C*-algebra. It is shown in [KW, Proposition 9] that a unital complete isometry \( \phi : X \to Y \) between operator systems has a unique “lift” to an injective \(*\)-homomorphism \( \overline{\phi} : C^*_u(X) \to C^*_u(Y) \) such that \( \overline{\phi} \circ u_X = u_Y \circ \phi \). In particular this allows one to identify the universal C*-algebra of an inductive limit with the inductive limit of the universal C*-algebras of the building blocks.

**Theorem 4.10.** The canonical \(*\)-homomorphism \( \sigma_{GS} : C^*_u(GS) \to C^*_e(GS) \) is a \(*\)-isomorphism.

**Proof.** As observed in Subsection 4.3, GS is the limit of an inductive sequence \( (X_k)_{k \in \mathbb{N}} \) with unital completely isometric connective maps \( \phi_k : X_k \to X_{k+1} \), where \( X_k \) is completely order isomorphic to a full matrix algebra. Denote by \( \iota_k : X_k \to GS \) the canonical inclusion, and observe that the set of elements of the form \( \| \pi (z) \| \geq (1 - \epsilon) \| z \| \). Set \( \theta := \pi \circ u_{X_k} : X_k \to M_d \), and observe that \( \pi \) is the unique \(*\)-homomorphism from \( C^*_u(X_k) \) to \( M_d \) such that \( \pi \circ u_{X_k} = \theta \). By the approximate injectivity property of \( GS \)—see Proposition 4.2—there is a unital complete isometry \( \eta : M_d \to GS \) such that \( \| \eta \circ \theta - u_k \|_{\text{op}} \leq \delta \). Observe that, for \( \delta \) small enough, we have

\[
\| (\eta \circ \theta) (z) - \pi_k (z) \| \leq \varepsilon.
\]

By [CE1, Theorem 4.1]—see also [KW, Lemma 6]—there is a \(*\)-homomorphism \( \mu : C^* (\eta [M_d]) \subset C^*_e(GS) \to M_d \) such that \( \mu \circ \eta = id_{M_d} \). Observe now that

\[
\mu \circ \sigma_{GS} \circ \eta \circ \theta : C^*_u(X_k) \to Y
\]

is a \(*\)-homomorphism such that

\[
\mu \circ \sigma_{GS} \circ \eta \circ \theta \circ u_{X_k} = \mu \circ \sigma_{GS} \circ u_{GS} \circ \eta \circ \theta
= \mu \circ \eta \circ \theta
= \theta.
\]

Therefore \( \mu \circ \sigma_{GS} \circ \eta \circ \theta = \pi \). Hence we have that

\[
\| \sigma_{GS} (\pi_k (z)) \| \geq \| (\sigma_{GS} \circ \eta \circ \theta) (z) \| - \varepsilon
\geq \| (\mu \circ \sigma_{GS} \circ \eta \circ \theta) (z) \| - \varepsilon
= \| \pi (z) \| - \varepsilon
\geq \| \pi_k (z) \| - 2\varepsilon.
\]

This concludes the proof that \( \sigma \) is injective. \( \square \)

**Corollary 4.11.** The Gurarij operator system GS does not admit any complete order embedding into an exact C*-algebra. Moreover if GS \( \subset B (H) \) is a unital completely isometric representation of GS, then the C*-algebra generated by GS inside \( B (H) \) is \(*\)-isomorphic to \( C^*_e(GS) \).

In particular it follows from Corollary 4.11 that GS is not completely order isomorphic to a unital C*-algebra. (Recall that a C*-algebra is exact if and only if it is 1-exact as an operator system.) In fact by [BN, Proposition 4.2] one can also conclude that GS is not completely isometric to a unital operator algebra (even without assuming that the complete isometry preserves the unit).
4.7. The triple envelope of the Gurarij operator space. The (noncommutative) Gurarij operator space \( NG \) has been defined by Oikhberg in [O] and proven to be unique and universal among separable 1-exact operator spaces in [L2]. A 1-exact operator space \( Z \) is completely isometric to \( NG \) if and only if for every \( n \in \mathbb{N}, E \subset M_n \), complete isometry \( \phi : E \to NG \), there exists a complete isometry \( \psi : F \to NG \) such that \( \| \psi|E - \phi \| \leq \varepsilon \). A similar argument as the one in Subsection 4.3 shows that \( NG \) is completely isometric to a direct limit \( \lim_{\phi_k} X_k \) where \( X_k \) is an operator space completely isometric to \( M_{n_k} \) for some \( n_k \in \mathbb{N} \) and \( \phi_k : X_k \to X_{k+1} \) is a complete isometry. Here one needs to replace Lemma 3.4 with the following lemma, which can be proved in the same way using injectivity of \( M_d \) in the category of operator spaces with completely contractive maps.

**Lemma 4.12.** Suppose that \( k, m \in \mathbb{N}, F_0, F_1 \) are matricial operator spaces, \( E \subset F_0 \), and \( \delta > 0 \). If \( f : E \to F_1 \) is an invertible linear map such that \( \| f \|_{cb} \leq 1 + \delta \) and \( \| f^{-1} \|_{cb} \leq 1 + \delta \), then there exist \( d \in \mathbb{N} \) and unital completely isometric embeddings \( i : F_0 \to M_d \) and \( j : F_1 \to M_d \) such that \( \| j \circ f - i \|_{cb} \leq 2\delta \).

Here we want to point out that the methods employed in Subsection 4.6 can used to show that \( NG \) is not completely isometric to a C*-algebra. This gives an answer to [O, Remark 3.3]. The key idea is to replace the C*-envelope with the triple envelope, and the universal C*-algebras with the universal TRO. Recall that a ternary ring of operators (TRO) is a subspace \( V \) of \( B(H, K) \) for some Hilbert spaces \( H, K \) such that \( x y^* z \in V \) for any \( x, y, z \in V \). The operation \( (x, y, z) \mapsto x y^* z \) on \( V \) is called triple product. A triple morphism between TROs is a linear map that preserves the triple product. Observe that (the restriction of) a \( \ast \)-homomorphism is in particular a triple product. A TRO has a canonical operator space structure, where the matrix norms are uniquely determined by the triple product [KR, Proposition 2.1]. A triple morphism between TROs is automatically completely contractive, and it is injective if and only if is completely isometric [BLM, Lemma 8.3.2].

Suppose that \( X \) is an operator space. A triple cover of \( X \) is a pair \( (\phi, V) \) where \( V \) is a TRO and \( \phi : X \to V \) is a linear complete isometry. Triple covers naturally form a category, where a morphism from \( (\phi, V) \) to \( (\phi', V') \) is a triple morphism \( \psi : V \to V' \) such that \( \psi \circ \phi = \phi' \). The (unique up to isomorphism) terminal object in such a category is called the triple envelope \( \cal{T}_u (X) \) of \( X \). The existence of such an object has been proved independently by Ruan [R2] and Hamana [H3]. The (unique up to isomorphism) initial object in the category of triple covers is the universal TRO \( \cal{T}_u (X) \) of \( X \). We could not find the existence of such an object explicitly stated in the literature. However this can be easily established with a minor modification of the proof of the existence of the universal C*-algebra of an operator system; see [KW, Proposition 8]. Moreover the same proof as [KW, Proposition 9]—where one replaces Arveson’s extension theorem with Wittstock’s extension theorem [P2, Theorem 8.2]—shows that a complete isometry between TROs “lifts” to an injective triple morphism between the corresponding universal TROs.

The universal property of \( \cal{T}_u (X) \) yields a (necessarily surjective) canonical triple morphism \( \sigma_X : \cal{T}_u (X) \to \cal{T}_e (X) \). Our goal is to show that such a triple morphism is injective (and hence a triple isomorphism) in the case of the Gurarij operator space \( NG \). First we need an adaptation to TROs of a well known result of Choi and Effros; see [CE1, Theorem 4.1].

**Proposition 4.13.** Suppose that \( V, W \) are TROs and \( \theta : V \to W \) is a linear complete isometry. Then there is a triple morphism \( \eta : \cal{T} (\theta [V]) \to V \) such that \( \eta \circ \theta = id_V \), where \( \cal{T} (\theta [V]) \) is the sub-TRO of \( W \) generated by the image of \( V \) under \( \theta \).
Proof. The proof consists in reducing the problem to the unital case via the Paulsen trick; see [BLM, 1.3.14]. Represent $V$ as a subspace of $B(H_1, K_1)$ and define the Paulsen system

$$S(V) = \begin{bmatrix} CH_1 & V \\ V^* & CI_{K_1} \end{bmatrix} = \left\{ \begin{pmatrix} \lambda & x \\ y^* & \mu \end{pmatrix} : x, y \in V, \lambda, \mu \in \mathbb{C} \right\} \subset B(H_1 \oplus K_1).$$

One can similarly define the Paulsen system $S(W) \subset B(H_2 \oplus K_2)$ of $W$. The linear complete isometry $\theta : V \to W$ yields a linear map $\hat{\theta} : S(V) \to S(W)$ defined by

$$\begin{pmatrix} \lambda & x \\ y^* & \mu \end{pmatrix} \mapsto \begin{pmatrix} \lambda & \theta(x) \\ \theta(y)^* & \mu \end{pmatrix}.$$ 

By [BLM, Lemma 1.3.15] $\hat{\theta}$ is a unital complete isometry. Let $\tilde{\theta}^{-1} : \tilde{\theta}[S(V)] \to S(V) \subset B(H_1 \oplus K_1)$ be the inverse of $\tilde{\theta}$. By the Arveson extension theorem [P2, Theorem 7.5] there is a unital completely positive map $\eta : C^*(\tilde{\theta}[S(V)]) \to B(H_1 \oplus K_1)$ extending $\tilde{\theta}$. Here $C^*(\tilde{\theta}[S(V)])$ is the $C^*$-algebra generated by $\tilde{\theta}[S(V)]$ inside $B(H_1 \oplus K_1)$.

We claim that $\eta$ is a $*$-homomorphism. By Choi’s multiplicative domain theorem [P2, Theorem 3.18] it is enough to show that for every self-adjoint element $a$ of $S(V)$, one has that $\eta(\theta(a)^2) = (\eta(\theta(a))^2$. This easily follow from the Kadison-Schwartz inequality [BLM, Proposition 1.3.9] applied to $\eta$ and $\theta$. Therefore $\eta$ is a $*$-homomorphism, and its restriction to the subTRO $\mathcal{T}(\theta[V]) \subset B(H_2, K_2)$ of $W$ generated by the image of $V$ under $\theta$ is a triple morphism. Since moreover $\eta \circ \theta$ is the identity of $V$ and $V$ is a TRO, it follows that $\eta$ maps $\mathcal{T}(\theta[V])$ into $V$. This concludes the proof.

Theorem 4.14. The canonical triple morphism $\sigma_{NG} : \mathcal{T}_u(NG) \to \mathcal{T}_u(NG)$ is a triple isomorphism.

Proof. The proof is entirely similar to the one of Theorem 4.10. Here one needs to use the fact that $NG$ is the inductive limit of an inductive sequence $(X_k)$ of operator spaces with completely isometric connective maps $\phi_k : X_k \to X_{k+1}$, as observed above. Moreover by [L2, Theorem 4.12] $NG$ satisfies the following approximate injectivity property: whenever $E \subset F$ are 1-exact finite-dimensional operator spaces, $\phi : E \to NG$ is a linear complete isometry, and $\varepsilon > 0$, then there is a linear complete isometry $\psi : F \to NG$ such that $\|\psi_E - \phi\|_b < \varepsilon$. Finally one needs to replace the use of [CE1, Theorem 4.1] with Proposition 4.13.

We now discuss how Theorem 4.14 implies that $NG$ does not admit any completely isometric embedding into an exact TRO. One can canonically assign to a TRO a $C^*$-algebra called its linking algebra. The local properties of a TRO are closely reflected by the local properties of its linking algebra. In particular a TRO is exact if and only if it is 1-exact if and only if its linking algebra is exact [KR, Theorem 4.4]. Moreover a (surjective) triple morphism between TROs induces a (surjective) $*$-homomorphism between the corresponding linking algebras. Since the class of exact C*-algebras is closed under quotients [BO, Corollary 9.4.3], it follows that the image of an exact TRO under a triple morphism is exact.

Observe now that if $X$ is an operator system, $C^*_u(X)$ is its universal C*-algebra, and $\mathcal{T}_u(X)$ is the triple envelope of $X$, then the universal property of $\mathcal{T}_u(X)$ implies the existence of a surjective triple morphism from $\mathcal{T}_u(X)$ to $C^*_u(X)$. If $C^*_u(X)$ is not exact, then $\mathcal{T}_u(X)$ is not exact as well. Since the universal C*-algebra of $M_2(\mathbb{C})$ is not exact [KW, Section 5], it follows that the universal TRO of $M_2(\mathbb{C})$ is not exact. Recall that, as observed above, completely isometric embeddings between operator spaces "lift" to injective ternary morphisms between the corresponding universal TROs. Since $M_2(\mathbb{C})$ embeds completely isometrically into $NG$, it follows that $\mathcal{T}_u(M_2(\mathbb{C}))$ embeds as
a subTRO of $\mathcal{T}_u(\mathcal{NG}) \cong \mathcal{T}_e(\mathcal{NG})$. Therefore $\mathcal{T}_e(\mathcal{NG})$ is not exact. Now, if $\mathcal{NG}$ embeds completely isometrically into a TRO $V$, then the universal property of $\mathcal{T}_e(\mathcal{NG})$ implies that $\mathcal{T}_e(\mathcal{NG})$ is the image of under a triple morphism of the subTRO of $V$ generated by (the image of) $\mathcal{NG}$. Hence $V$ is not exact as well.

**Corollary 4.15.** The Gurarij operator space $\mathcal{NG}$ does not admit any unital completely isometric embedding into an exact C*-algebra. Moreover if $\mathcal{NG} \subset B(H)$ is a unital completely isometric representation of $\mathcal{NG}$ then the TRO generated by $\mathcal{NG}$ inside $B(H)$ is $*$-isomorphic to $\mathcal{T}_e(\mathcal{NG})$.

Since $\mathcal{NG}$ is 1-exact, it follows in particular that $\mathcal{NG}$ is not completely isometric to a TRO or a C*-algebra.

**References**

[B1] P. Bankston, *Reduced coproducts of compact hausdorff spaces*, The Journal of Symbolic Logic 52 (1987), no. 2, 404–424.

[B2] B. Blackadar, *Operator algebras*, Encyclopaedia of Mathematical Sciences, vol. 122, Springer-Verlag, Berlin, 2006.

[B3] B. Bollobás, *Linear analysis*, Second, Cambridge University Press, Cambridge, 1999.

[BLM] D. P. Blecher and C. Le Merdy, *Operator algebras and their modules—an operator space approach*, London Mathematical Society Monographs. New Series, vol. 30, Oxford University Press, Oxford, 2004.

[BN] D. P. Blecher and M. Neal, *Metric characterizations of isometries and of unital operator spaces and systems*, Proceedings of the American Mathematical Society 139 (2011), no. 3, 985–998.

[BO] N. P. Brown and N. Ozawa, *C*-algebras and finite-dimensional approximations, Graduate Studies in Mathematics, vol. 88, American Mathematical Society, Providence, RI, 2008.

[BY] I. Ben Yaacov, *Fraïssé limits of metric structures*, Journal of Symbolic Logic, to appear.

[BYBH] I. Ben Yaacov, A. Berenstein, C. W. Henson, and A. Usvyatsov, *Model theory for metric structures*, Model theory with applications to algebra and analysis. vol. 2, 2008, pp. 315–427.

[CE1] M.-D. Choi and E. G. Effros, *The completely positive lifting problem for C*-algebras*, Annals of Mathematics 104 (1976), no. 3, 585–609.

[CE2] __________, *Injectivity and operator spaces*, Journal of Functional Analysis 24 (1977), no. 2, 156–209.

[CSGWK] F. Cabello Sánchez, J. Garbulińska-Wgrzyn, and W. Kubiś, *Quasi-Banach spaces of almost universal disposition*, Journal of Functional Analysis 267 (2014), no. 3, 744–771.

[EFH+] C. J. Eagle, I. Farah, B. Hart, B. Kadets, V. Kalashnyk, and M. Lupini, *Fraïssé limits of C*-algebras*, arXiv:1411.0466 (2014).

[EFP+] G. A. Elliott, I. Farah, V. Paulsen, C. Rosendal, A. S. Toms, and A. Törnquist, *The isomorphism relation for separable C*-algebras*, Mathematical Research Letters 20 (2013), no. 6, 1071–1080.

[EH] E. G. Effros and U. Haagerup, *Lifting problems and local reflexivity for C*-algebras*, Duke Mathematical Journal 52 (1985), no. 1, 103–128.

[EOR] E. G. Effros, N. Ozawa, and Z.-J. Ruan, *On injectivity and nuclearity for operator spaces*, Duke Mathematical Journal 110 (2001), no. 3, 489–521.

[ER] E. G. Effros and Z.-J. Ruan, *Operator spaces*, London Mathematical Society Monographs. New Series, vol. 23, Oxford University Press, 2000.

[F] R. Fraïssé, *Sur l’extension aux relations de quelques propriétés des ordres*, Annales Scientifiques de l’École Normale Supérieure. Troisième Série 71 (1954), 363–388.

[FHS] I. Farah, B. Hart, and D. Sherman, *Model theory of operator algebras II: Model theory*, Israel Journal of Mathematics. to appear.

[G1] Yu. L. Gevorkjan, *Universality of the spaces of almost universal placement*, Functional Analysis and its Applications 8 (1974), no. 2, 72.

[G2] V. I. Gurariĭ, *Spaces of universal placement, isotropic spaces and a problem of Mazur on rotations of Banach spaces*, Siberian Mathematical Journal 7 (1966), 1002–1013.

[GL] I. Goldbring and M. Lupini, *Model-theoretic aspects of the Gurarij operator space*, arXiv:1501.04332 (2015).

[GS] I. Goldbring and T. Sinclair, *On Kirchberg’s embedding problem*, arXiv:1404.1861 (2014).
M. Hamana, *Injective envelopes of C*-algebras*, Journal of the Mathematical Society of Japan 31 (1979), no. 1, 181–197.

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**H2** M. Hamana, *Injective envelopes of operator systems*, Publications of the Research Institute for Mathematical Sciences 15 (1979), no. 3, 773–785.

**H3** M. Hamana, *Triple envelopes and Šilov boundaries of operator spaces*, Mathematics Journal of Toyama University 22 (1999), 77–93.

**HP** K. H. Han and V. I. Paulsen, *An approximation theorem for nuclear operator systems*, Journal of Functional Analysis 261 (2011), no. 4, 999–1009.

**JP** M. Junge and G. Pisier, *Bilinear forms on exact operator spaces and B(H)⊗B(H)*, Geometric and Functional Analysis 5 (1995), no. 2, 329–363.

**K1** E. Kirchberg, *On subalgebras of the CAR-algebra*, Journal of Functional Analysis 129 (1995), no. 1, 35–63.

**K2** W. Kubiš, *Fraïssé sequences: category-theoretic approach to universal homogeneous structures*, Annals of Pure and Applied Logic 165 (2014), no. 11, 1755–1811.

**KPTT** A. S. Kavruk, V. I. Paulsen, I. G. Todorov, and M. Tomforde, *Quotients, exactness, and nuclearity in the operator system category*, Advances in Mathematics 325 (2013), 321–360.

**KR** M. Kaur and Z.-J. Ruan, *Local properties of ternary rings of operators and their linking C*-algebras*, Journal of Functional Analysis 195 (2002), no. 2, 262–305.

**KS** W. Kubiš and S. Solecki, *A proof of uniqueness of the Gurarij space*, Israel Journal of Mathematics 195 (2013), no. 1, 449–456.

**KW** E. Kirchberg and S. Wassermann, *C*-algebras generated by operator systems, Journal of Functional Analysis 155 (1998), no. 2, 324–351.

**L1** F. Lehner, *M_n-spaces, sommes d’unitaires et analyse harmonique sur le groupe libre*, Ph.D. Thesis, 1997.

**L2** M. Lupini, *Uniqueness, universality, and homogeneity of the noncommutative Gurarij space*, arXiv:1410.3345 (2014).

**L3** W. Lusky, *The Gurarij spaces are unique*, Archiv der Mathematik 27 (1976), no. 6, 627–635.

**MT** J. Melleray and T. Tsankov, *Extremely amenable groups via continuous logic*, arXiv:1404.4590 (2014).

**O** T. Oikhberg, *The non-commutative Gurarij space*, Archiv der Mathematik 86 (2006), no. 4, 350–364.

**OR** T. Oikhberg and É. Ricard, *Operator spaces with few completely bounded maps*, Mathematische Annalen 328 (2004), no. 1-2, 229–259.

**P1** V. I. Paulsen, *Completely bounded maps on C*-algebras and invariant operator ranges*, Proceedings of the American Mathematical Society 86 (1982), no. 1, 91–96.

**P2** V. I. Paulsen, *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Mathematics, vol. 78, Cambridge University Press, Cambridge, 2002.

**P3** G. Pisier, *Exact operator spaces*, Astérisque 232 (1995), 159–186. Recent advances in operator algebras (Orléans, 1992).

**R1** Z.-J. Ruan, *Subspaces of C*-algebras*, Journal of Functional Analysis 76 (1988), no. 1, 217–230.

**R2** Z.-J. Ruan, *Injectivity of operator spaces*, Transactions of the American Mathematical Society 315 (1989), no. 1, 89–104.

**S1** K. Schoretsanitis, *Fraïssé theory for metric structures*, Ph.D. Thesis, 2007.

**S2** R. R. Smith, *Completely bounded module maps and the haagerup tensor product*, Journal of Functional Analysis 102 (1991), no. 1, 156–175.

**S3** R. R. Smith, *Finite dimensional injective operator spaces*, Proceedings of the American Mathematical Society 128 (2000), no. 11, 3461–3462.

**W** G. Wittstock, *Ein operatorwertiger Hahn-Banach Satz*, Journal of Functional Analysis 40 (1981), no. 2, 127–150.

**X** B. Xhabli, *The super operator system structures and their applications in quantum entanglement theory*, Journal of Functional Analysis 262 (2012), no. 4, 1466–1497.

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