CLASSIFYING LOCALLY COMPACT SEMITOPOLOGICAL POLYCYCLIC MONOIDS

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ABSTRACT. We present a complete classification of Hausdorff locally compact polycyclic monoids up to a topological isomorphism. A polycyclic monoid is an inverse monoid with zero, generated by a subset $\lambda$ such that $xx^{-1} = 1$ for any $x \in \lambda$ and $xy^{-1} = 0$ for any distinct $x, y \in \lambda$. We prove that any non-discrete Hausdorff locally compact topology with continuous shifts on a polycyclic monoid $M$ coincides with the topology of one-point compactification of the discrete space $M \setminus \{0\}$.

INTRODUCTION

In this paper we present a complete classification of locally compact semitopological polycyclic monoids up to a topological isomorphism.

We shall follow the terminology of [8, 10, 19, 22]. First we recall some information on inverse semigroups and monoids. We identify cardinals with the sets of ordinals of smaller cardinality.

A semigroup is a set $S$ endowed with an associative binary operation $\cdot : S \times S \to S$, $\cdot : (x, y) \mapsto xy$. An element $e \in S$ is called the unit (resp. zero) of $S$ if $xe = x = ex$ (resp. $xe = e = ex$) for all $x \in S$. A semigroup can contains at most one unit (which will be denoted by 1) and at most one zero (denoted by 0). A monoid if a semigroup with a unit.

A semigroup $S$ is called inverse if for every element $a \in S$ there exists a unique element $a^{-1}$ (called the inverse of $a$) such that $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$. An inverse monoid is an inverse semigroup with unit. We say that an inverse monoid $S$ is generated by a subset $\lambda \subset S$ if $S$ coincides with the smallest subsemigroup of $S$ containing the set $\lambda \cup \lambda^{-1}$.

A polycyclic monoid is an inverse monoid $S$ with zero $0 \neq 1$, which is generated by a subset $\lambda \subset S$ such that $xx^{-1} = 1$ for all $x \in \lambda$ and $xy^{-1} = 0$ for any distinct $x, y \in \lambda$. If the generating set $\lambda$ has cardinality $\lambda$, then $S$ is called a $\lambda$-polycyclic monoid. We claim that $|\lambda| \geq 2$. In the opposite case, $\Lambda = \{x\}$ is a singleton and $0 \in S = \{x^{-n}x^m : n, m \in \omega\}$, which implies that $0 = x^{-n}x^m$ for some non-negative numbers $n, m$. Then $0 = x^{n+1} \cdot x^{-m} = x^{n+1}(x^{-n}x^m)x^{-m} = x$ and hence $1 = xx^{-1} = 0x^{-1} = 0$, but this contradicts the definition of a polycyclic monoid.

A canonical example of a $\lambda$-polycyclic monoid can be constructed as follows. Let $M_{\lambda^\pm}$ be the monoid of all words in the alphabet $\{x, x^{-1} : x \in \lambda\}$, endowed with the semigroup operation of concatenation of words. The empty word is the unit 1 of the monoid $M_{\lambda^\pm}$. Let $M_{\lambda^\pm}^0 := M_{\lambda^\pm} \cup \{0\}$ be the monoid $M_{\lambda^\pm}$ with the attached external zero, i.e., an element $0 \notin M_{\lambda^\pm}$ such that $0 \cdot x = 0 = x \cdot 0$ for all $x \in M_{\lambda^\pm}^0$. On the monoid $M_{\lambda^\pm}^0$ consider the smallest congruence $\sim$ containing the pairs $(xx^{-1}, 1)$ and $(xy^{-1}, 0)$ for all distinct elements $x, y \in \lambda$. Then the quotient semigroup $M_{\lambda^\pm}^0/\sim$ is the required canonical example of a $\lambda$-polycyclic monoid, which will be denoted by $P_{\lambda}$ and called the $\lambda$-polycyclic monoid.

Algebraic properties of the $\lambda$-polycyclic monoid were deeply investigated in [5]. According to [5, Theorem 2.5], the semigroup $P_{\lambda}$ is congruence-free, which implies that each $\lambda$-polycyclic monoid is algebraically isomorphic to $P_{\lambda}$.

The aim of this paper is to describe Hausdorff locally compact topologies on $P_{\lambda}$, compatible with the algebraic structure of the semigroup $P_{\lambda}$. A suitable compatibility condition is given by the notion of a semitopological semigroup.
A semitopological semigroup is a semigroup $S$ endowed with a Hausdorff topology making the binary operation $S \times S \to S$, $(x, y) \mapsto xy$, separately continuous. If this operation is jointly continuous, then $S$ is called a topological semigroup.

For a cardinal $\lambda \geq 2$ by $\mathcal{P}_\lambda^d$ we shall denote the $\lambda$-polycyclic monoid $\mathcal{P}_\lambda$ endowed with the discrete topology and by $\mathcal{P}_\lambda^c$ the monoid $\mathcal{P}_\lambda$ endowed with the compact topology $\tau = \{ U \subset \mathcal{P}_\lambda : 0 \in U \Rightarrow (\mathcal{P}_\lambda \setminus U \text{ is finite}) \}$ of one-point compactification of the discrete space $\mathcal{P}_\lambda \setminus \{0\}$. It is clear that $\mathcal{P}_\lambda^d$ is a topological monoid. On the other hand, $\mathcal{P}_\lambda^c$ is a compact semitopological monoid, which is not a topological semigroup.

By [5], each locally compact topological $\lambda$-polycyclic monoid is discrete and hence is topologically isomorphic to $\mathcal{P}_\lambda^d$. In the semitopological case we have the following dichotomy, which is the main result of this paper.

**Main Theorem.** Any locally compact semitopological polycyclic monoid $S$ is either discrete or compact. More precisely, $S$ is topologically isomorphic either to $\mathcal{P}_\lambda^d$ or to $\mathcal{P}_\lambda^c$ for a unique cardinal $\lambda \geq 2$.

Since the compact semitopological $\lambda$-polycyclic monoid $\mathcal{P}_\lambda^c$ fails to be a topological semigroup, Main Theorem implies the mentioned result of [5]:

**Corollary.** Any locally compact topological polycyclic monoid $S$ is discrete. More precisely, $S$ is topologically isomorphic to the topological $\lambda$-polycyclic monoid $\mathcal{P}_\lambda^d$ for a unique cardinal $\lambda \geq 2$.

Some other topologizability results of the same flavor can be found in [24, 23, 17, 2, 20, 13, 4, 5, 6].

**Proof of Main Theorem**

The proof of Main Theorem is divided into a series of 12 lemmas.

Let $S$ be a non-discrete locally compact semitopological polycyclic monoid and let $\Lambda$ be its generating set. By [5, Proposition 2.2], $S$ is algebraically isomorphic to the $\lambda$-polycyclic monoid $\mathcal{P}_\lambda$ for a unique cardinal $\lambda \geq 2$. So, we can identify $S$ with $\mathcal{P}_\lambda$ and the cardinal $\lambda$ with the generating set $\Lambda$ of the inverse monoid $S$.

Let $S^+$ be the submonoid of $S$, generated by the set $\Lambda$ (i.e., $S^+$ is the smallest submonoid of $S$ containing the generating set $\Lambda$). Elements of $S^+$ can be identified with words in the alphabet $\Lambda$. Such words will be called positive. The relations between the generators of $S$ guarantee that each non-zero element $a$ of $S$ can be uniquely written as $u^{-1}v$ for some positive words $u, v \in S^+$. Then by $\downarrow a$ we denote the set of all prefixes of the word $u^{-1}v$. For a subset $C \subset S$ we put $\downarrow C = \bigcup_{a \in C} \downarrow a$.

The following algebraic property of a polycyclic monoid is proved in [5, Proposition 2.7].

**Lemma 1.** For any non-zero elements $a, b, c \in S$, the set $\{ x \in S : axb = c \}$ is finite.

This lemma will be applied in the proof of the following useful fact, proved in [5, Proposition 3.1].

**Lemma 2.** All non-zero elements of $S$ are isolated points in the space $S$.

**Proof.** For convenience of the reader we present a short proof of this important lemma. First we show that the unit 1 is an isolated point of the semitopological monoid $S$. Take any generator $g \in \Lambda$ and consider the idempotent $e = g^{-1}g$ of $S$. Since the map $S \to eS$, $x \mapsto ex$, is a retraction of the Hausdorff space $eS$ onto $eS$, the principal right ideal $eS = g^{-1}S$ is closed in $S$. By the same reason, the principal left ideal $Se = Sg$ is closed in $S$. The separate continuity of the semigroup operation yields a neighborhood $U_1 \subset S \setminus (g^{-1}S \cup Sg)$ of 1 such that $0 \notin (e \cdot U_1) \cap (U_1 \cdot e)$. We claim that $U_1 = \{1\}$. In the opposite case, $U_1$ contains some element $a \neq 1$, which can be written as $u^{-1}v$ for some positive words $u, v \in S^+$. Since $a \neq 1$ one of the words $u, v$ is not empty. If $u$ is not empty, then $a \in U_1 \subset S \setminus g^{-1}S$ implies that the word $u^{-1}$ does not start with $g^{-1}$. In this case $ea = g^{-1}gu^{-1}v = g^{-1} \cdot 0 = 0$, which contradicts the choice of the neighborhood $U_1 \ni a$. If the word $v$ is not empty, then $a \in U_1 \subset S \setminus Sg$ implies that $v$ does not end with $g$. In this case $ae = u^{-1}vg^{-1}g = 0$, again contradicting the choice of $U_1$. This contradiction shows that the unit 1 is an isolated point of $S$. 


Now we can prove that each non-zero point \( a \in S \) is isolated. Write \( a \) as \( u^{-1}v \) for some positive words \( u, v \in S^+ \). Since \( uv^{-1} = 1 \), the separate continuity of the semigroup operation on \( S \), yields an open neighborhood \( O_a \subset S \) of \( a \) such that \( uO_av^{-1} \subset U_1 = \{1\} \). By Lemma 4, the neighborhood \( O_a \) is finite and hence the singleton \( \{a\} = O_a \setminus (O_a \setminus \{a\}) \) is open, which means that the point \( a \) is isolated in \( S \).

Lemma 2 implies that the locally compact space \( S \) has a neighborhood base at zero, consisting of compact sets. It also implies the following useful lemma.

**Lemma 3.** For any compact neighborhoods \( U_0, V_0 \subset S \) of zero the set \( U_0 \setminus V_0 \) is finite.

For an element \( u \in S \) by \( \mathcal{R}_u := \{x \in S : xS = uS\} \) we denote its Green \( \mathcal{R} \)-class in \( S \). Here \( uS = \{us : s \in S\} \) is the right principal ideal generated by the element \( u \).

**Lemma 4.** Every non-zero \( \mathcal{R} \)-class in \( S \) coincides with the \( \mathcal{R} \)-class \( \mathcal{R}_{u^{-1}} = \mathcal{R}_{u^{-1}u} \) for some positive word \( u \in S^+ \).

**Proof.** Each non-zero element of the semigroup \( \mathcal{P}_\Lambda \) can be written as \( u^{-1}v \) for some positive words \( u, v \in S^+ \). Taking into account that \( u^{-1}v \cdot v^{-1} = u^{-1} \), we conclude that \( \mathcal{R}_{u^{-1}} = \mathcal{R}_{u^{-1}u} \).

In the following Lemmas 5–12 we assume that \( U_0 \) is any fixed compact neighborhood of zero in the semitopological monoid \( S \). Since zero is a unique non-isolated point in \( S \), the neighborhood \( U_0 \) is infinite.

**Lemma 5.** The neighborhood \( U_0 \) has infinite intersection with some \( \mathcal{R} \)-class of \( S \).

**Proof.** To derive a contradiction, assume \( U_0 \) has finite intersection with each \( \mathcal{R} \)-class of the semigroup \( S \). Taking into account that \( U_0 \) is infinite and applying Lemma 4, we can see that the set \( B = \{u \in S^+ : \mathcal{R}_{u^{-1}} \cap U_0 \neq \emptyset\} \) is infinite. For every \( u \in B \) denote by \( v_u \) a longest positive word in \( S^+ \) such that \( u^{-1}v_u \in \mathcal{R}_{u^{-1}} \cap U_0 \) (such word \( v_u \) exists as the set \( \mathcal{R}_{u^{-1}} \cap U_0 \) is finite). It follows that \( \Lambda = \{u^{-1}v_u : u \in B\} \) is an infinite subset of \( U_0 \). Fix any element \( g \) of the generating set \( \Lambda \) of \( S \). Since \( 0 \cdot g = 0 \), we can use the separate continuity of the semigroup operation of \( S \) and find a compact neighborhood \( V_0 \subset U_0 \) of zero such that \( V_0 \cdot g \subset U_0 \). But then \( V_0 \subset U_0 \setminus A \) which contradicts Lemma 3.

**Lemma 6.** The neighborhood \( U_0 \) has infinite intersection with each non-zero \( \mathcal{R} \)-class of the semigroup \( S \).

**Proof.** By Lemma 4 any non-zero \( \mathcal{R} \)-class of the semigroup \( S = \mathcal{P}_\Lambda \) is of the form \( \mathcal{R}_{v^{-1}} \) for some positive word \( v \in S^+ \). By Lemmas 4 and 6 for some element \( u \in S^+ \) the intersection \( U_0 \cap \mathcal{R}_{u^{-1}} \) is infinite. Observe that \( v^{-1}u \cdot \mathcal{R}_{u^{-1}} \subset \mathcal{R}_{v^{-1}} \). By the separate continuity of the semigroup operation at \( 0 = v^{-1}u \cdot 0 \), there exists a neighborhood \( V_0 \subset S \) of zero such that \( v^{-1}u \cdot V_0 \subset U_0 \). By Lemma 5 the difference \( U_0 \setminus V_0 \) is finite, which implies that the intersection \( V_0 \cap \mathcal{R}_{u^{-1}} \) is infinite. Then the set \( v^{-1}u \cdot (V_0 \cap \mathcal{R}_{u^{-1}}) \subset U_0 \cap \mathcal{R}_{v^{-1}} \) is infinite, too.

**Lemma 7.** If the generating set \( \Lambda \) is finite, then the neighborhood \( U_0 \) contains all but finitely many elements of the \( \mathcal{R} \)-class \( \mathcal{R}_1 = \{x \in S : xS = S\} \).

**Proof.** To derive a contradiction, assume that the set \( A := \mathcal{R}_1 \setminus U_0 \) is infinite. We claim that for every \( g \in \Lambda \) the set \( A_g = \{a \in A : ag \in U_0\} \) is finite. Indeed, suppose that \( A_g \) is infinite. By Proposition 11 \( A_g \cdot g \) is an infinite subset of \( U_0 \). Since \( 0 \cdot g^{-1} = 0 \), the separate continuity of the semigroup operation on \( S \) yields a compact neighborhood \( V_0 \subset U_0 \) of zero such that \( V_0 \cdot g^{-1} \subset U_0 \). Then \( V_0 \subset U_0 \setminus (A_g \cdot g) \) which contradicts Lemma 5.

Let \( A^* = A \setminus \bigcup_{g \in \Lambda} \downarrow A_g \) (we recall that \( \downarrow A_g = \bigcup_{a \in A_g} \downarrow a \) where \( \downarrow a \) is the set of all prefixes of the word \( a \)). It follows that \( A^* \) is a cofinite (and hence infinite) subset of \( A \). Now we are going to show that \( A^* \) is a right ideal of \( \mathcal{R}_1 \). In the opposite case we could find elements \( c \in \mathcal{R}_1 \) and \( v \in A^* \) such that \( vc \notin A^* \). Let \( c^* \) be the longest prefix of \( c \) such that \( vc^* \in A^* \) (the word \( c^* \) can be empty, in which case it is the unit of \( S \)). Then \( vc^*g \notin A^* \) for some \( g \in \Lambda \). Observe that \( vc^* \in A^* \subset A \cap \mathcal{R}_1 \) implies \( vc^*g \in \mathcal{R}_1 \).
Assuming that $ve^g \in U_0$, we conclude that $ve^g \in A_g \subset \downarrow A_g$, which contradicts the inclusion $ve^g \in A^*$. So, $ve^g \notin U_0$ and hence $ve^g \in A$. Then $ve^g \notin A^*$ implies that $ve^g \notin \downarrow A_f$ for some $f \in \Lambda$ and thus $ve^g \notin \downarrow A_f$, too. But this contradicts the inclusion $ve^g \in A^*$. The obtained contradiction implies that $A^*$ is a right ideal of $R_1$.

Let $u \in A^*$ be an arbitrary element. Since $u \cdot 0 = 0$, the separate continuity of the semigroup operation yields a compact neighborhood $V_0 \subset U_0$ of zero such that $u \cdot V_0 \subset U_0$. Proposition $1$ and Lemma $6$ imply that $u \cdot (V_0 \cap R_1)$ is an infinite subset of $A^* \cap U_0 \subset A \cap U_0$. In particular, $A \cap U_0$ is not empty, which contradicts the definition of the set $A := R_1 \setminus U_0$.

**Lemma 8.** If the cardinal $\lambda = |\Lambda|$ is finite, then the neighborhood $U_0$ contains all but finitely many elements of any $R$-class $R_x$, $x \in S$.

**Proof.** The lemma is trivial if $x = 0$. So we assume that $x \neq 0$. By Lemma $4$, $R_x = R_{u^{-1}}$ for some positive word $u \in S^+$. Since $u^{-1} \cdot 0 = 0$, the separate continuity of the semigroup operation yields an neighborhood $V_0 \subset U_0$ of zero such that $u^{-1} \cdot V_0 \subset U_0$. By Lemmas $8$ and $7$, $R_1 \subset * V_0$ (which means that $R_1 \setminus V_0$ is finite). Then $R_x = R_{u^{-1}} = u^{-1} \cdot R_1 \subset u^{-1} \cdot V_0 \subset U_0$, which means that $U_0$ contains all but finitely many points of the $R$-class $R_x$.

The following lemma proves Main Theorem in case of finite cardinal $\lambda = |\Lambda|$.

**Lemma 9.** If the cardinal $\lambda$ is finite, then the set $S \setminus U_0$ is finite.

**Proof.** To derive a contradiction, assume that $S \setminus U_0$ is infinite. By Lemma $8$ for each $u \in S^+$ the set $R_{u^{-1}} \setminus U_0$ is finite. Since the complement $S \setminus U_0 = \bigcup_{u \in S^+} R_{u^{-1}} \setminus U_0$ is infinite, the set $B = \{u \in S^+ : R_{u^{-1}} \setminus U_0 \neq \emptyset\}$ is infinite, too. For every $u \in B$ denote by $v_u$ the longest word in $S^+$ such that $u^{-1} v_u \in R_{u^{-1}} \setminus U_0$. Then $C = \{u^{-1} v_u : u \in B\} \subset R_{u^{-1}} \setminus U_0$ is infinite and by Proposition $1$, for every $g \in \Lambda$ the set $C \cdot g$ is an infinite subset of $U_0$. Since $0 \cdot g^{-1} = 0$, the separate continuity of the semigroup operation yields a neighborhood $V_0 \subset U_0$ of zero such that $V_0 \cdot g^{-1} \subset U_0$. By Lemma $8$ the set $U_0 \setminus V_0$ is finite. Since the set $C g \subset U_0$ is infinite, there is an element $c \in C$ with $c g \in V_0$. Then $c = c g g^{-1} \in V_0 g^{-1} \subset U_0$, which contradicts the inclusion $C \subset R_1 \setminus U_0$.

**Lemma 10.** The set $R_1 \setminus U_0$ is finite.

**Proof.** To derive a contradiction, assume that the complement $A := R_1 \setminus U_0$ is infinite. By Lemma $6$ the set $U_0 \cap R_1$ is finite.

For a finite subset $F \subset \Lambda$, let $S_F$ be the smallest subsemigroup of $S$ containing the set $F \cup F^{-1} \cup \{0, 1\}$. If $|F| \geq 2$, then $S_F$ is a polycyclic monoid. Separately, we shall consider two cases.

1. First assume that for every finite subset $F \subset \Lambda$ the set $U_0 \cap S_F$ is finite. In this case for every point $g \in \Lambda$, consider the set $W_g = \{a \in U_0 \cap R_1 : ag \notin U_0\}$. The separate continuity of the semigroup operation yields a neighborhood $V_0 \subset U_0$ of zero such that $V_0 \cdot g \subset U_0$. Lemma $8$ implies that the set $W_g \subset U_0 \setminus V_0$ is finite and hence for every non-empty finite subset $F \subset \Lambda$ the set $U_F := (U_0 \cap R_1) \setminus \bigcup_{g \in F} W_g$ is finite. We claim that $U_F \cdot y \subset U_F$ for every $y \in S_F \cap R_1$. In the opposite case, there exist elements $y \in S_F \cap R_1$ and $x \in U_F$ such that $xy \notin U_F$. Let $y^*$ be the longest prefix of $y$ such that $xy^* \in U_F$ (note that $y^*$ could be equal to $1$). Then $xy^* g \notin U_F$ for some $g \in F$. Hence $xy^* \in W_g$ which contradicts the definition of $U_F \ni xy^*$. Hence $U_F \cdot y \subset U_F$ for each element $y \in S_F \cap R_1$.

Fix any element $v \in U_F$ and find a finite subset $D \subset \Lambda$ such that $v \in S_D$, $F \subset D$ and $|D| \geq 2$. Proposition $1$ implies that $v \cdot (S_F \cap R_1)$ is an infinite subset of $U_F \cap S_D$, which contradicts our assumption.

2. Next, assume that for some finite subset $F \subset \Lambda$ the intersection $U_0 \cap S_F$ is infinite. For every $g \in F$ consider the subset $A_g := \{a \in A : ag \in U_0\}$ of the infinite set $A = R_1 \setminus U_0$. The separate continuity of the semigroup operation yields a neighborhood $V_0 \subset S \setminus U_0$ of zero such that $V_0 \cdot g^{-1} \subset U_0$. We claim that for every $a \in A_g$ we get $ag \notin V_0$. In the opposite case we would get $a = agg^{-1} \in V_0 \cdot g^{-1} \subset U_0$, which contradicts the inclusion $a \in A$. Then $A_g = \{a \in A : ag \in U_0 \setminus V_0\}$ and this set is finite by Lemmas $8$ and $1$. It follows that $A_F = A \setminus \bigcup_{g \in F} A_g$ is a cofinite (and hence infinite) subset of $A$. 


We claim that $A_F \cdot y \subseteq A_F$ for every $y \in S_F \cap \mathcal{R}_1$. In the opposite case, we can find elements $y \in S_F \cap \mathcal{R}_1$ and $x \in A_F$ such that $xy \notin A_F$. Let $y^*$ be the longest prefix of $y$ such that $xy^* \in A_F$ (note that $y^*$ could be equal to 1). Then $xy^* g \notin A_F$ for some $g \in F$. It follows from $xy^* \in A_F \subseteq A = \mathcal{R}_1 \setminus U_0$ and $gg^{-1} = 1$ that $xy^* g \in \mathcal{R}_1$. Assuming that $xy^* g \in U_0$, we conclude that $xy^* \in A_g$, which contradicts the inclusion $xy^* \in A_F$. So, $xy^* g \in \mathcal{R}_1 \setminus U_0 = A$ and then $xy^* g \notin A_F$ implies that $xy^* g \in \downarrow A_h$ for some $h \in F$ and finally $xy^* \in \downarrow A_h$, which contradicts the inclusion $xy^* \in A_F$. This contradiction completes the proof of the inclusion $A_F \cdot y \subseteq A_F$ for each $y \in S_F \cap \mathcal{R}_1$.

Fix any element $v \in A_F$ and find a finite subset $D \subseteq \Lambda$ such that $v \in S_D$, $F \subseteq D$ and $|D| \geq 2$. The subset $S_D$ contains the unique non-isolated point of the space $S$ and hence is closed in $S$. The local compactness of $S$ implies the local compactness of the polycyclic monoid $S_D$ endowed with the subspace topology. Lemma 3 and our assumption guarantee that the semitopological polycyclic monoid $S_D$ is not discrete. By Proposition 1, $v \cdot (S_F \cap \mathcal{R}_1)$ is an infinite subset of $A_F \cap S_D \subseteq S_D \setminus U_0$. But this contradicts Lemma 9 (applied to the locally compact polycyclic monoid $S_D$ and the neighborhood $U_0 \cap S_D$ of zero in $S_D$).

Lemma 11. The neighborhood $U_0$ contains all but finitely many points of each $\mathcal{R}$-class in $S$.

Proof. By Lemma 4, it suffices to check that for any $u \in S^+$ the set $\mathcal{R}_{u^{-1}} \setminus U_0$ is finite. The separate continuity of the semigroup operation yields a compact neighborhood $V_0 \subseteq U_0$ of zero such that $u^{-1} V_0 \subseteq U_0$. By Lemmas 11 and 3 we get $\mathcal{R}_1 \cap S^* V_0$. Then $\mathcal{R}_{u^{-1}} = u^{-1} \cdot \mathcal{R}_1 \cap S^* u^{-1} \cdot V_0 \subseteq U_0$, which means that the set $\mathcal{R}_{u^{-1}} \setminus U_0$ is finite.

Our final lemma combined with Lemma 2 proves Main Theorem and shows that the semitopological polycyclic monoid $S$ carries the topology of one-point compactification of the discrete space $S \setminus \{0\}$.

Lemma 12. The complement $S \setminus U_0$ is finite and hence $S$ is compact.

Proof. To derive a contradiction, assume that the set $S \setminus U_0$ is infinite. By Lemma 11, for each $u \in S^+$ the set $\mathcal{R}_{u^{-1}} \setminus U_0$ is finite. Since $S = \bigcup_{u \in S^+} \mathcal{R}_{u^{-1}}$, the set $B = \{u \in S^+ : \mathcal{R}_{u^{-1}} \setminus U_0 \neq \emptyset\}$ is infinite. For every $u \in B$ denote by $v_u$ the longest word in $S^+$ such that $u^{-1} v_u \in \mathcal{R}_{u^{-1}} \setminus U_0$. Then $C = \{u^{-1} v_u : u \in B\}$ is an infinite subset of $S \setminus U_0$. By Lemma 11, for any $g \in \Lambda$ the set $C \cdot g$ is infinite. The separate continuity of the semigroup operation yields a neighborhood $V_0 \subseteq U_0$ of zero such that $V_0 \cdot g^{-1} \subseteq U_0$. Then $V_0 \subseteq U_0 \setminus (C \cdot g)$ which contradicts Lemma 3.

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