Semisimple Hopf Algebras of Dimension $pq$ are Trivial

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This paper makes a contribution to the problem of classifying finite-dimensional semisimple Hopf algebras $H$ over an algebraically closed field $k$ of characteristic 0. Specifically, we show that if $H$ has dimension $pq$ for primes $p$ and $q$ then $H$ is trivial, that is, $H$ is either a group algebra or the dual of a group algebra.

Previously known cases include dimension $2p$ [M1], dimension $p^2$ [M2], and dimensions $3p$, $5p$ and $7p$ [GW]. Westreich and the second author also obtained the same result for $H$ which is, along with its dual $H^*$, of Frobenius type (i.e. the dimensions of their irreducible representations divide the dimension of $H$) [GW, Theorem 3.5]. They concluded with the conjecture that any semisimple Hopf algebra $H$ of dimension $pq$ over $k$ is trivial.

In this paper we use Theorem 1.4 in [EG] to prove that both $H$ and $H^*$ are of Frobenius type, and hence prove this conjecture.

Throughout $k$ will denote an algebraically closed field of characteristic 0, and $p$ and $q$ will denote two prime numbers satisfying $p < q$. We let $G(H)$ denote the group of grouplike elements of a Hopf algebra $H$, and $D(H)$ denote its Drinfel’d double. Recall that $H$ and $H^{cop}$ are Hopf subalgebras of $D(H)$, and let $i_H : H \hookrightarrow D(H)$ and $i_{H^{cop}} : H^{cop} \hookrightarrow D(H)$ denote the corresponding inclusion maps.

The following result will prove very useful in the sequel.

Theorem 1 [GW, Theorem 2.1] Let $H$ be a semisimple Hopf algebra of dimension $pq$, where $p < q$ are two prime numbers. Then:

1. $|G(H)| \neq q$.

2. If $|G(H)| = p$ then $q = 1(mod p)$.

Suppose there exists a non-trivial semisimple Hopf algebra $H$ of dimension $pq$. Then by Theorem 1, $G(H)$ and $G(H^*)$ have either order 1 or $p$. By [R], $G(D(H)) = G(H^*) \times G(H)$
has therefore either order 1, p or $p^2$. Since by [H], the order of $G(D(H)^*)$ divides the order of $G(D(H))$, it follows that $G(D(H)^*)$ has either order 1, p or $p^2$ too. In the following we show that this leads to a contradiction, and hence conclude that $H$ is trivial.

**Lemma 2** Let $H$ be a semisimple Hopf algebra of dimension $pq$ over $k$, and let $V, U$ be $p$–dimensional irreducible representations of $D(H)$. If $V \otimes U$ contains a 1–dimensional representation $\chi$, then it must contain another 1–dimensional representation $\chi' \neq \chi$.

**Proof:** Without loss of generality we can assume that $\chi$ is trivial and hence that $U = V^*$. Otherwise we can replace $U$ with $U \otimes \chi^{-1}$.

Suppose on the contrary that $V \otimes V^*$ does not contain a non-trivial 1–dimensional representation. By [EG], the dimension of any irreducible representation of $D(H)$ divides $pq$, hence $V \otimes V^*$ is a direct sum of the trivial representation $k$, $p$–dimensional irreducible representations of $D(H)$ and $q$–dimensional irreducible representations of $D(H)$. Therefore we have that $p^2 = 1 + ap + bq$. Clearly $b > 0$. Let $W$ be a $q$–dimensional irreducible representation of $D(H)$ such that $W \subset V \otimes V^*$. Since $0 \neq \text{Hom}_{D(H)}(V \otimes V^*, W) = \text{Hom}_{D(H)}(V, W \otimes V)$ we have that $V \subset W \otimes V$. Since $W \otimes V$ has no linear constituent (because $\text{dim}V \neq \text{dim}W$), $\text{dim}(W \otimes V) = pq$ and $W \otimes V$ contains a $p$–dimensional irreducible representation of $D(H)$ it follows that $W \otimes V = V_1 \oplus \cdots \oplus V_q$ where $V_i$ is a $p$–dimensional irreducible representation of $D(H)$ with $V_1 = V$.

We wish to show that for any $i = 1, \ldots, q$ there exists a 1–dimensional representation $\chi_i$ such that $V_i = V \otimes \chi_i$. Suppose on the contrary that this is not true for some $i$. Then $V \otimes V_i^*$ has no linear constituent, hence it must be a direct sum of the $p$–dimensional irreducible representations of $D(H)$. Therefore, $W \otimes (V \otimes V_i^*)$ has no linear constituent. But, $(W \otimes V) \otimes V_i^* = (V_1 \oplus \cdots) \otimes V_i^* = k \oplus \cdots$ which is a contradiction. Therefore, $W \otimes V = V \otimes (\chi_1 \oplus \cdots \oplus \chi_q)$. Note that for all $i$, $\chi_i$ is uniquely determined. Indeed, if there exists $\chi'_i \neq \chi_i$ such that $V \otimes \chi_i = V \otimes \chi'_i$, then $\chi'_i \chi_i^{-1}$ is a non-trivial linear constituent of $V \otimes V^*$ which is a contradiction.

We now consider two possible cases. First, suppose that any 1–dimensional representation $\chi$ such that $\chi \otimes W = W$, is trivial. Then for any $q$–dimensional irreducible representation $W'$ of $D(H)$, $W \otimes W'$ contains no more than one 1–dimensional representation. Therefore, since $V \otimes V^*$ contains $b \geq 1$ $q$–dimensional irreducible representations of $D(H)$, it follows that $W \otimes V \otimes V^*$ contains at most $b$ 1–dimensional representations. Therefore, $b \geq q$, and $p^2 = \text{dim}(V \otimes V^*) \geq bq \geq q^2$, which is a contradiction.

Second, suppose there exists a non-trivial 1–dimensional representation $\chi$ such that $\chi \otimes W = W$. Then $\chi \otimes W \otimes V = W \otimes V$ which is equivalent to $\chi \otimes V \otimes (\chi_1 \oplus \cdots \oplus \chi_q) = V \otimes (\chi_1 \oplus \cdots \oplus \chi_q)$. Write $\chi_1 \oplus \cdots \oplus \chi_q = \sum_{\alpha \in G(D(H)^*)} m_{\alpha} \alpha$. Then $V \otimes \sum_{\alpha \in G(D(H)^*)} m_{\alpha} \chi^{-1} \alpha = V \otimes \sum_{\alpha \in G(D(H)^*)} m_{\alpha} \chi \alpha = V \otimes \sum_{\alpha \in G(D(H)^*)} m_{\alpha} \alpha$. But, $V \otimes \alpha_1 = V \otimes \alpha_2$ implies $\alpha_1 = \alpha_2$, so $m_{\alpha} = m_{\alpha} \chi$. Since the order of $\chi$ in $G(D(H)^*)$ is divisible by $p$, this implies that $q = \sum_{\alpha} m_{\alpha}$ is divisible by $p$ which is a contradiction. This concludes the proof of the lemma.

**Lemma 3** Let $H$ be a semisimple Hopf algebra of dimension $pq$ over $k$. Then $G(D(H)^*)$ is non-trivial.
Proof: Suppose on the contrary that \(|G(D(H)^\ast)| = 1\). By \([EG]\), the dimension of any irreducible representation of \(D(H)\) divides \(pq\), hence \(p^2q^2 = 1 + ap^2 + bq^2\) for some integers \(a, b > 0\). In particular \(D(H)\) has a \(p\)–dimensional irreducible representation \(V\). But then by Lemma \(2\), \(D(H)\) must have a non-trivial 1–dimensional representation which is a contradiction. ■

**Lemma 4** Let \(H\) be a non-trivial semisimple Hopf algebra of dimension \(pq\) over \(k\). Then the order of \(G(D(H)^\ast)\) is not \(p^2\).

Proof: Suppose on the contrary that \(|G(D(H)^\ast)| = p^2\). By \([R]\), any \(\chi \in G(D(H)^\ast)\) is of the form \(\chi = g \otimes \alpha\), where \(g \in G(H)\) and \(\alpha \in G(H^\ast)\). In particular, either \(G(H)\) or \(G(H^\ast)\) is non-trivial, and hence by Theorem \([I]\), either \(G(H)\) or \(G(H^\ast)\) has order \(p\). Therefore we must have that both \(G(H)\) and \(G(H^\ast)\) are of order \(p\), and hence that \(G(D(H)^\ast) = G(H) \times G(H^\ast)\). This implies that there exists a non-trivial \(g \in G(H)\) so that \(g \otimes 1 \in G(D(H)^\ast)\). By \([R]\), \(1 \otimes g \in G(D(H))\) is central in \(D(H)\). In particular \(g\) is central in \(H\), and hence \(H\) possesses a normal Hopf subalgebra isomorphic to \(kC_p\), which implies that \(H\) is commutative (see the proof of Theorem 2.1 in \([GW]\)). This is a contradiction and the result follows. ■

**Lemma 5** Let \(H\) be a non-trivial semisimple Hopf algebra of dimension \(pq\) over \(k\). Then there exists a quotient Hopf algebra \(A\) of \(D(H)\) of dimension \(pq^2\) which contains \(H\) and \(H^{*\cop}\) as Hopf subalgebras, and \(H \cap H^{*\cop} = kG(A) = kC_p\).

Proof: By Lemmas \(3\) and \(4\), the order of \(G(D(H)^\ast)\) is \(p\). If either \(G(H)\) or \(G(H^\ast)\) is trivial then either \(H^\ast\) or \(H\) respectively contains a central grouplike element. Hence, as in the proof of Lemma \(4\), \(H\) is trivial. Therefore, \(G(H)\) and \(G(H^\ast)\) have both order \(p\).

Since \(D(H)\) is of Frobenius type \([EG]\) and since by Theorem \([I]\), \(q = 1(\text{mod } p)\), we must have that \(p^2q^2 = p + ap^2 + bq^2\), where \(a = \frac{q}{p} - 1\) and \(b = p(p - 1)\). In particular \(D(H)\) has irreducible representations of dimension \(p\) and \(q\). We wish to show that if \(V\) and \(U\) are two irreducible representations of \(D(H)\) of dimension \(p\) then \(V \otimes U\) is a direct sum of \(1\)–dimensional irreducible representations of \(D(H)\) and \(p\)–dimensional irreducible representations of \(D(H)\) only. Indeed, by Lemma \(4\) either \(V \otimes U\) does not contain any \(1\)–dimensional representation or it must contain at least two different \(1\)–dimensional representations. But if it contains two different \(1\)–dimensional representations, then since the \(1\)–dimensional representations of \(D(H)\) form a cyclic group of order \(p\), it follows that \(V \otimes U\) contains all the \(p\) \(1\)–dimensional representations of \(D(H)\). We conclude that either \(p^2 = ap + bq\) or \(p^2 = p + ap + bq\). At any rate \(b = 0\), and the result follows. Therefore, the subcategory \(C\) of \(\text{Rep}(H)\) generated by the \(1\) and \(p\)–dimensional representations of \(D(H)\) gives rise to a quotient Hopf algebra \(A\) of \(D(H)\) of dimension \(pq^2\) whose irreducible representations have either dimension \(1\) or \(p\) \((A\) is the quotient of \(D(H)\) over the Hopf ideal \(I = \sum_{V \in C} \text{End}_k(V))\). Let \(\phi\) denote the corresponding surjective homomorphism, and consider the sequences:

\[
\begin{align*}
H & \xrightarrow{i_H} D(H) \xrightarrow{\phi} A & & H^{*\cop} & \xrightarrow{i_{H^{*\cop}}} D(H) \xrightarrow{\phi} A.
\end{align*}
\]
Since \( \phi(H) \), \( \phi(H^{\text{cop}}) \) are Hopf subalgebras of \( A \) it follows that they have dimension 1, \( p \), \( q \) or \( pq \) \([\text{NZ}]\). If, say, \( \dim(\phi(H)) = q \) then \( \phi(H) = kC_q \) is a group algebra \([\text{Z}]\). Since \( (kC_q)^* \cong kC_q \) the surjection of Hopf algebras \( \phi : H \to kC_q \) gives rise to an inclusion of Hopf algebras \( \phi^* : kC_q \to H^* \). Since \( \phi^* (kC_q) \subset kG(H^*) \) it follows that \( q \) divides \( |G(H^*)| = p \) which is impossible. Similarly, \( \dim(\phi(H^{\text{cop}})) \neq q \), and we conclude that \( \phi(H) \), \( \phi(H^{\text{cop}}) \) have dimension 1, \( p \) or \( pq \). Since \( A = \phi(H) \phi(H^{\text{cop}}) \) is of dimension \( pq^2 \) it follows that the dimensions must equal \( pq \) and hence that \( H \) and \( H^{\text{cop}} \) can be considered as Hopf subalgebras of \( A \). Furthermore, \( G(H) \) and \( G(H^{\text{cop}}) \) are mapped onto \( G(A) \), otherwise \( |G(A)| \) is divisible by \( p^2 \) which is impossible. Therefore \( G(H) \cap G(H^{\text{cop}}) = G(A) \) in \( A \). Since \( H \neq H^{\text{cop}} \) in \( A \), this implies that \( H \cap H^{\text{cop}} = kG(A) = kC_p \). \( \blacksquare \)

We are ready to prove our main result.

**Theorem 6** Let \( H \) be a semisimple Hopf algebra of dimension \( pq \) over \( k \), where \( p \) and \( q \) are distinct prime numbers. Then \( H \) is trivial.

**Proof:** Suppose on the contrary that \( H \) is non-trivial. We wish to prove that \( H \) and \( H^* \) are of Frobenius type. Indeed, consider the Hopf algebra quotient \( A \) which exists by Lemma \([\text{Z}]\). For any 1-dimensional representation \( \chi \) of \( A \) we define the induced representations

\[
V_+^\chi = A \otimes_H \chi \quad \text{and} \quad V_-^\chi = A \otimes_{H^*} \chi.
\]

Note that since finite-dimensional Hopf algebras are free over any of their Hopf subalgebras \([\text{NZ}]\), it follows that \( V_+^\chi = H^* \otimes_{H \cap H^*} \chi = H^* \otimes_{kC_p} \chi \), and \( V_-^\chi = H \otimes_{H \cap H^*} \chi = H \otimes_{kC_p} \chi \).

We first show that if \( \chi \) is non-trivial then \( \chi \) is non-trivial on \( H \) and \( H^* \) as well. Indeed, for any non-trivial 1-dimensional representation \( \tilde{\chi} \) we have by Frobenius reciprocity that \( \text{Hom}_A(V_+^\chi, \tilde{\chi}) = \text{Hom}_H(\chi, \tilde{\chi}) \). Therefore, if \( \chi \) is trivial on \( H \) then \( \tilde{\chi} \) is also trivial on \( H \) (since \( \tilde{\chi} = \chi^l \) as \( \chi \) generates the group of 1-dimensional representations of \( A \)), and \( \text{Hom}_A(V_+^\chi, \tilde{\chi}) = k \). But then \( V_+^\chi \) is a sum of \( p \) 1-dimensional representations and \( p \)-dimensional irreducible representations which contradicts the fact that \( p \) does not divide \( q \). Similarly, \( \chi \) is non-trivial on \( H^* \).

Next, since \( \dim(\text{Hom}_A(V_+^\chi, \tilde{\chi})) = \dim(\text{Hom}_H(\chi, \tilde{\chi})) = \delta_{\chi, \tilde{\chi}} \) for any 1-dimensional representation \( \tilde{\chi} \) and \( \dim V_+^\chi = q \), it follows that

\[
V_+^\chi = \chi \oplus V_1 \oplus \cdots \oplus V_{q-1}^p
\]

where \( V_i \) is a \( p \)-dimensional irreducible representation of \( A \) for all \( 1 \leq i \leq \frac{q-1}{p} \). We wish to show that the \( V_i \)'s do not depend on \( \chi \). Indeed, let \( \chi \) be a non-trivial 1-dimensional representation of \( A \). Then any 1-dimensional representation of \( A \) is of the form \( \chi^l \) for some \( 0 \leq l \leq p-1 \), and \( V_+^\chi = V_+^\chi \otimes \chi^{l-1} = \chi^l \oplus (V_1 \otimes \chi^{l-1}) \oplus \cdots \oplus (V_{\frac{q-1}{p}} \otimes \chi^{l-1}) \). But by Lemma \([\text{Z}]\) \( \chi^{l-1} \subset V_i \otimes V^*_i \) for all \( i \), and hence \( V_i \otimes \chi^{l-1} \cong V_i \) which proves the claim. Similarly, \( V_-^\chi = \chi \oplus W_1 \oplus \cdots \oplus W_{\frac{q-1}{p}} \) where \( W_i \) is a \( p \)-dimensional irreducible representation of \( A \) for all \( 1 \leq i \leq \frac{q-1}{p} \).
Finally, we wish to show that the set
\[ C = \{ \chi_{|H^*}| \chi \text{ is a } 1 - \text{dimensional representation of } A \} \cup \{ V_{i|H^*}|1 \leq i \leq (q - 1)/p \} \]
is a full set of representatives of the irreducible representations of \( H^* \). Indeed, consider the regular representation of \( H^* \):
\[
H^* = H^* \otimes_{kC_p} kC_p = \bigoplus_{\chi} H^* \otimes_{kC_p} \chi = \bigoplus_{\chi} V_{\chi|H^*} = \bigoplus_{\chi} \chi \oplus \bigoplus_{i} pV_{i|H^*}.
\]
Since in the regular representation any representation can occur no more times than its dimension we get that \( V_{i|H^*} \) are irreducible. By summation of degrees, \( C \) is a full set of representatives, and since \( \text{dim}(V_{i|H^*}) = p \) we have that \( H^* \) is of Frobenius type. Similarly \( H \) is of Frobenius type. But Theorem 3.5 in [GW] states that if this is the case then \( H \) is trivial. This concludes the proof of the theorem.

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References

[EG] P. Etingof and S. Gelaki, Some Properties of Finite-Dimensional Semisimple Hopf Algebras, Mathematical Research Letters 5 (1998), 191-197.

[GW] S. Gelaki and S. Westreich, On Semisimple Hopf Algebras of Dimension \( pq \), Proceedings of the AMS, to appear, q-alg/9801128.

[M1] A. Masuoka, Semisimple Hopf Algebras of Dimension \( 2p \), Communication in Algebra 23 No.5 (1995), 1931-1940.

[M2] A. Masuoka, The \( p^n \) theorem for semisimple Hopf Algebras, Proceedings of the AMS 124 (1996), 735-737.

[NZ] W. D. Nichols and M. B. Zoeller, A Hopf algebra freeness theorem, Amer. J. of Math. 111 (1989), 381-385.

[R] D. E. Radford, Minimal quasitriangular Hopf algebras, J. of Algebra 157 (1993), 281-315.

[Z] Y. Zhu, Hopf algebras of prime dimension, International Mathematical Research Notices No.1 (1994), 53-59.