Elastoviscoplastic rheology and ageing in a simplified soft glassy constitutive model

Suzanne M. Fielding

1 Department of Physics, Durham University, Science Laboratories, South Road, Durham DH1 3LE, UK

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Yield stress fluids display a rich rheological phenomenology. Beyond the defining existence of a yield stress in the steady state flow curve, this includes in many materials rather flat viscoelastic spectra over many decades of frequency in small amplitude oscillatory shear; slow stress relaxation following the sudden imposition of a small shear strain; stress overshoot in shear startup; logarithmic or sublinear power law creep following the imposition of a shear stress below the yield stress; creep followed by yielding after the imposition of a shear stress above the yield stress; richly featured Lissajous-Bowditch curves in large amplitude oscillatory shear; a Bauschinger effect, in which a material’s effective yield strain is lowered under straining in one direction, following a preceding strain in the opposite direction; hysteresis in up-down shear rate sweeps; and (in some materials) thixotropy and/or rheological ageing. A key challenge is to develop a constitutive model that contains enough underlying mesoscopic physics to have meaningful predictive power for the full gamut of rheological behaviour just described, with only a small number of model parameters, and yet is simple enough for use in computational fluid dynamics to predict flows in complicated geometries, or complicated flows that arise due to spontaneous symmetry breaking instabilities even in simple geometries. Here we introduce such a model, motivated by the widely used soft glassy rheology model, and show that it captures all the above rheological features.

I. INTRODUCTION

Many soft materials behave as so-called “yield stress fluids” [1–8]. Examples include dense colloids, emulsions, foams, star polymers, microgels and lamellar onion phases, as well as low density attractive colloidal gels and clays. At imposed stresses below a critical yield stress, \( \sigma < \sigma_y \), they show solid-like rheological behaviour. In contrast, at larger stresses, \( \sigma > \sigma_y \), they yield and flow like liquids. Their steady state flow curve of shear stress \( \sigma(\dot{\gamma}) \) as a function of shear rate \( \dot{\gamma} \), typically measured in a slow shear rate sweep, is then often fit to a Herschel-Bulkley form [9], \( \sigma(\dot{\gamma}) = \sigma_y + K\dot{\gamma}^n \), with \( 0 < n < 1 \), or Bingham behaviour [10] with \( n = 1 \).

In terms of the physical origin of this behaviour, yield stress fluids can (broadly and loosely) be subdivided into two main categories. In the first category, a material’s constituent mesoscopic substructures, such as colloidal particles, attract to form weakly flocculated aggregates [11]. Even though the volume fraction of the constituent particles can be quite low for the system overall, their aggregates can result in a gel-like response at low loads, but are then pulled apart and reflocculated in shear.

In the second category, a material’s constituent substructures are too densely packed to properly rearrange at low loads, but do rearrange in shear. Examples include colloids, emulsions, foams, and microgels, etc., which respectively comprise densely packed colloidal particles, emulsion droplets, foam bubbles, or microgel beads. Materials in this second category can be further divided into (at least) two idealised limiting subcategories [12, 13]: thermal hard sphere colloids, in which the yield stress has a typical magnitude \( k_B T / R^3 \), where \( R \) is the particle radius, and athermal soft suspensions, in which the yield stress has a typical magnitude set by the modulus of the constituent particles. Materials in the second subcategory have been termed “soft glassy materials” [14–16]. The constitutive model that we shall present in what follows is aimed in particular at dense athermal soft particle suspensions, and motivated by the original, widely used “soft glassy rheology” (SGR) model [14–16]. It is worth noting, however, that the model actually also captures many of the observed rheological features of dense hard sphere colloids, and of low density attractive gels.

Beyond the defining presence of a yield stress in the steady state flow curve, yield stress fluids also display a host of interesting dynamic rheological behaviours. In linear response, the viscoelastic spectra characterising their stress response to a small amplitude oscillatory shear strain typically show a rather flat, power law dependence over many decades of the oscillation frequency, \( \omega \) [17–23]. The storage modulus, \( G'(\omega) \), typically exceeds the loss modulus, \( G''(\omega) \), by about an order of magnitude, consistent with a nearly elastic response overall for these small deformations. The presence of non-trivial dissipation (a non-zero \( G''(\omega) \)) even at the lowest frequencies accessible experimentally however also reveals a broad underlying spectrum of sluggish stress relaxation modes. The stress decay following the sudden imposition of a small shear strain occurs over a similarly wide range of sluggish relaxation timescales [24].

The behaviour of yield stress fluids in time-dependent nonlinear flows is similarly rich, in both strain-imposed and stress-imposed protocols. In shear startup from rest, for example, they typically display an initially elastic solid-like regime in which the stress increases linearly with strain up to a maximum ‘overshoot’ value. Following this stress overshoot, the material yields and the stress declines to its value in the ultimate fluidised flowing state, prescribed by the steady state flow curve. Such behaviour has been observed in foams [19], emulsions [25, 26], carbopol [27, 28], Laponite [29, 30], a fused
silica suspension [31], attractive gels [32, 33], and waxy crude oil [34]. Following the imposition of a step shear stress below the yield stress, a sustained solid-like slow creep response is typically seen, in which the strain increases logarithmically or as a sublinear power of time, with the strain rate accordingly decreasing as a power law. In this way, the material creeps forward at an ever slowing rate, but never attains a steady flow of non-zero rate. Following the imposition of a shear stress just above the yield stress, in contrast, an early time creep regime is followed at later times by a dynamical yielding process in which the shear rate increases up to its final value prescribed by the steady state flow curve. Such behaviour has been observed in carbopol gel [35, 36], carbon black [37–39], polycrystalline hexagonal columnar phases [40], and colloidal glasses [41, 42].

Back-and-forth strain, strain rate or stress ramps or oscillations have also been widely studied. In large amplitude oscillatory shear (LAOS) experiments on yield stress fluids [43–57], parametric Lissajous-Bowditch (LB) plots of stress versus strain typically show a rather complicated progression in shape with increasing amplitude of the applied shear. For example, characteristic diamond shaped LB curves are often seen for intermediate amplitudes. LB curves have recently been modelled within the SGR model in Refs. [58–60]. Yield stress materials often also display a Bauschinger effect [61, 62], in which the apparent yield strain is reduced for straining in one direction, following a preceding plastic strain in the opposite direction. The stress response of yield stress fluids to shear rate sweeps often displays a pronounced hysteresis between the downward and upward sweeps, with the size of the hysteresis loop increasing with increasing sweep rate [63, 64].

Indeed, the rheological response of many yield stress fluids also shows a pronounced dependence on the ‘waiting time’ since a sample was prepared, before a flow is applied. This phenomenon is often termed rheological ageing and/or thixotropy. For a precise definition of rheological ageing, see Ref. [15]. The definition of thixotropy and its distinction from viscoelasticity and from ageing is a topic of ongoing discussion [11]. Typically, a sample that has waited longer in an undisturbed state before a flow is applied will show a higher viscosity and/or a more solid-like response. The latter can be evidenced, for example, by a slower relaxation of stress following the rapid imposition of a small shear strain, or a larger stress overshoot in shear startup in older samples. Typically, a material is then rejuvenated to a state of lower viscosity and/or lower solidity by an imposed flow.

During the dynamical process whereby a material yields from an initially solid-like to finally fluid-like state, a state of initially homogeneous shear will often become unstable to the formation of heterogeneous shear bands. This has been observed experimentally in yield stress fluids during shear startup [27–31, 34], step stress [35–38, 41], flow curve sweeps [64], and LAOS [53–55, 57]. It has also been studied theoretically and computationally in these same protocols of shear startup [65–76], step stress [72, 77, 78], flow curve sweeps [63], and LAOS [59, 60]. In many materials, the shear bands that form during yielding then gradually heal away to leave a homogeneous steady flowing state. Some yield stress fluids instead support shear bands in the ultimate steady flowing state [79, 80]. We do not consider such materials here: the model that we discuss has a monotonic increasing constitutive relation between stress and strain-rate, precluding steady state shear banding.

From a theoretical viewpoint, a key challenge is to understand how the macroscopic flow properties just described emerge out of the underlying collective dynamics of a material’s constituent mesoscopic substructures, for any given category of yield stress fluids, and to build this understanding into a rheological constitutive model. Ideally, such a model should contain enough of the key mesoscopic physics to have meaningful predictive power for the full gamut of observed rheological phenomena, with just a small number of model parameters. At the same time, it should also be simple enough for use in computational fluid dynamics (CFD) to address flows in complicated geometries, or complicated flows that arise via spontaneous symmetry breaking instabilities even in simple geometries. It should therefore preferably be of time-differential form, which is much easier to implement numerically in a CFD solver than a model of time-integral form. The primary contribution of this work is to introduce a constitutive model that for the first time, to this author’s knowledge, meets all these desirable criteria.

We start in Sec. II by briefly reviewing some of the most widely used models of yield stress rheology in the existing literature. In Sec. III we discuss one such model in more detail: the soft glassy rheology (SGR) model. This does capture all the rheological phenomena described above, but is in its present form far too complicated for use in CFD. Indeed, even computations of homogeneous simple shear flows can be very cumbersome within the SGR model its existing form. Accordingly, in Sec. IV we introduce a simplified SGR model, and in Sec. V demonstrate it to indeed capture all the rheological features discussed above. The potential contributions of this new model are twofold. First, it will allow significantly more straightforward computation of homogeneous shear flows for anyone wishing to fit the SGR model’s predictions to rheometric data. Second, it renders SGR feasible for use in CFD, once suitably tensorialised. We therefore suggest a possible generalisation to tensorial stresses in Sec. VI, before finally setting out our conclusions in Sec. VII.

II. OVERVIEW OF EXISTING CONSTITUTIVE MODELS

Existing elastoviscoplastic constitutive models range from those built from bottom up on the basis of micro-
scopic or mesoscopic physics, to those posed from top down on the basis of macroscopic phenomenology. We now summarise some of the most widely used models in the existing literature, and the extent to which they meet the desirable criteria set out in the penultimate paragraph of Sec. I above.

### A. Microscopically derived models

For dense colloidal suspensions, a rheological constitutive theory has been built on the mode coupling theory (MCT) of the colloidal glass transition [81, 82]. It starts by writing an equation of motion for the microscopic probability distribution in configuration space of the position vectors of a dense ensemble of Brownian particles (ignoring hydrodynamic interactions). This microscopic equation is then projected via a series of approximations onto a time-integral rheological constitutive equation for macroscopic stresses. This takes as its basic input the material’s static and dynamic structure factors for the underlying microscopic density correlation functions.

MCT successfully captures many of the observed rheological features of dense colloidal suspensions, including the existence of a yield stress in the flow curve. Its formalism is heavy to implement in computational practice, however, even in simple homogeneous shear flows. Nonetheless, very recent work has incorporated a simplified schematic – although still time-integral – MCT constitutive equation into a lattice Boltzmann solver for CFD in channel shear flow, assuming translational invariance in the flow direction [83].

### B. Mesoscopic elastoplastic models

Mesoscopic elastoplastic models conceptually divide a macroscopic sample of material into many local mesoscopic regions, each of which is ascribed continuum variables of local strain and stress relative to a locally undisturbed equilibrium. Each such region is represented as an elastoplastic element that loads elastically in flow up to a local threshold, after which it yields plastically, then assumes a new elastic state relative to a new locally undeformed equilibrium.

In lattice-based elastoplastic models [84], the elements just described reside on a lattice, and the stress relaxation involved in any local plastic yielding event results in an explicit redistribution of stress to surrounding elements via an Eshelby propagator, ensuring that force balance is properly maintained [85]. Such models capture many observed features of elastoplastic rheology (even though they often fail properly to account for the advection of an element’s position in flow). In containing detailed spatial information about stress propagation, however, they are too computationally intensive in their present form for use in CFD to predict flows in anything other than small and simply shaped geometries.

Mean field elastoplastic models instead discard any explicit spatial information about the location of elements, and the stress propagation that follows local yielding events. Instead, they model stress propagation in a mean field way. For example, the Hebraud-Lequeux model does so by invoking a diffusive term in the equation of motion for the probability distribution of local strains, with a diffusion constant set by the sample-average yielding rate [86]. Mean field models with broader-tailed stress propagation statistics have also been studied [87]. Most such models, and their lattice-based counterparts described above, assume a flat distribution of local yield energy thresholds.

The soft glassy rheology (SGR) model [14–16] instead assumes a distribution of local yield energy thresholds with an exponential tail. It furthermore models stress propagation by means of an effective temperature that can activate any element out of its local energy minimum and thereby trigger a local yielding event. This activation is taken to model, in a mean field way, stress propagation from other local yielding events elsewhere in the sample. Starting from the initial purely mean field model, SGR was later extended to address flows that develop spatial structuring in one spatial dimension, either via shear banding [88] or extensional necking [89], by allowing stress propagation in the relevant dimension.

Although the mean field elastoplastic models just described are simpler than their lattice-based counterparts, they still in general involve the time evolution of a full distribution of local strain variables, and remain as yet too complicated to implement in CFD.

An alternative mesoscopic approach, originally intended to model the deformation properties of metallic glasses, is based on the collective statistics of many ‘shear transformation zones’ (STZs), which resemble the yielding local elements of the elastoplastic models just described [74, 75].

### C. Phenomenological macroscopic models

Besides the microscopic and mesoscopic models just described, other constitutive models of yield stress rheology have been built from the top down, on the basis of macroscopic phenomenology. The earliest such models posited a static relation between stress and strain rate [90–95]. When used in CFD, however, these necessitate a cumbersome separate calculation of the ‘yield surface’, or regularisation of sub-yield behaviour [96]. They also miss most of the key physics, including all the dynamical rheological phenomena summarised in Sec. I above.

More recent phenomenological models therefore instead posit an evolution equation for the stress, in order to account for the stress in a material at any time as a functional of the strain rate history it has experienced. This equation may depend on one or more auxiliary variables, for which evolution equations are also posited. Examples include fractional calculus [97, 98], ...
structural evolution, fluidity, and elastoviscoplastic models [34, 56, 99–104]. Although vastly superior to the static models, many involve 10-20+ fitting parameters, limiting their predictive power. Indeed, models benchmarked by fitting to straightforward strain-imposed protocols such as shear startup often then perform poorly in more complicated oscillatory/reversal protocols, and/or in stress-imposed protocols. Such models furthermore often incorporate phenomenological notions such as those of a ‘back stress’ or ‘kinematic hardening’, without always offering a clear understanding of such concepts in terms of the underlying microscopic or mesoscopic physics.

D. Summary of existing models

Among the constitutive models just described, those based on underlying microscopic and mesoscopic physics tend to perform best at predicting the broad gamut of dynamical rheological phenomena described above, but are often prohibitively complicated for use in CFD to predict macroscopic flows in complicated geometries. In contrast, the macroscopic phenomenological models are generally better suited to CFD, but often contain many model parameters, and/or capture only a subset of the desired rheological phenomenology, and/or are limited in the underlying physics they contain.

Indeed, to this author’s knowledge, no currently existing constitutive model of elastoviscoplastic yield stress rheology currently exists that satisfies all the desirable criteria set out above: of containing enough underlying micro/mesoscopic physics to predict the rich dynamical rheology of yield stress fluids with just a small number of model parameters, while also being simple enough – and preferably of time-differential form – for use in CFD to predict flows in complicated geometries.

III. ORIGINAL SOFT GLASSY RHEOLOGY MODEL

The soft glassy rheology (SGR) model [14–16] considers an ensemble of elastoplastic elements, each representing a local mesoscopic region of soft glassy material (a few tens of emulsion droplets, say). Each element is assigned local continuum variables of shear strain $\gamma$ and shear stress $kl$, describing the mesoscopic region’s state of elastic deformation relative to a locally undeformed equilibrium. In between local yielding events, the strain of each element affinely follows the macroscopic shear, $\dot{\gamma}$, giving an elastic buildup of stress.

The stress is intermittently released by local plastic yielding events. In any such event, a mesoscopic region suddenly rearranges into a new configuration locally. This is modelled by its representative element hopping between traps in an energy landscape. An element in a trap of depth $E$ with local shear strain $l$ is assigned a probability per unit time of hopping of $\tau^{-1}(E, l)$, with

$$\tau(E, l) = \tau_0 \exp \left[ (E - \frac{1}{2}kl^2)/x \right]. \quad (1)$$

The stored elastic energy $\frac{1}{2}kl^2$ at any instant therefore offsets the bare trap depth $E$, leading to a reduced local barrier to rearrangement, $E - \frac{1}{2}kl^2$. This confers rheological shear thinning on the sample as a whole. After hopping, an element selects a new trap depth at random from a prior distribution

$$\rho(E) = \frac{1}{x_g} \exp \left( -E/x_g \right), \quad (2)$$

and resets its local strain $l$ to zero, thereby relaxing the local stress.

With the dynamics just described, the probability $P(E, l, t)$ for an element to be in a trap of depth $E$ with local shear strain $l$ evolves according to

$$\dot{P}(E, l, t) + \gamma \frac{\partial P}{\partial l} = -\frac{1}{\tau(E, l)} P + Y(t) \rho(E) \delta(l). \quad (3)$$

The advected derivative on the left hand side captures the affine loading of each element by shear. The first (‘death’) term on the right hand side describes hops out of traps. The second (‘birth’) term describes hops into the bottom of traps, $l = 0$, with the new trap depth chosen at random from the prior distribution, $\rho(E)$, and with an ensemble average hopping rate

$$Y(t) = \int dE \int dl \frac{1}{\tau(E, l)} P(E, l, t). \quad (4)$$

The macroscopic stress of the sample as a whole is defined as the average over the local elemental ones:

$$\sigma(t) = k \int dE \int dl \tau P(E, l, t). \quad (5)$$

Combined with the exponential prior, $\rho(E)$, the exponential activation factor $\tau(E, l)$ confers a glass transition at a noise temperature $x = x_g$. In the absence of any applied flow, the model displays rheological ageing in the glass phase [15], $x < x_g$: following sample preparation at time $t = 0$ by means of a sudden quench from a high initial noise temperature to a value $x < x_g$, the system slowly evolves into ever deeper traps as a function of time $t$. This results in a growing stress relaxation time, $\langle \tau \rangle \sim t$, and therefore in ever more solid-like rheological response as the sample ages. An imposed shear of constant rate $\dot{\gamma}$ will however arrest ageing and rejuvenate the sample to a steady flowing state of effective age $\langle \tau \rangle \sim 1/\dot{\gamma}$. The steady state flow curve has a yield stress $\sigma_y$ that grows linearly with $x_g - x$ in the glass phase $x < x_g$.

For many soft glassy materials, the typical energy barrier for rearrangements greatly exceeds thermal energies. Accordingly, the parameter $x$ is not the true thermodynamic temperature but is taken as an effective noise...
temperature that models in a mean field way coupling with other yielding events elsewhere in the sample.

The full soft glassy rheology model just described captures many features of the rich elastoplastic rheology of yield stress fluids. These include a yield stress in the steady state flow curve; broad and rather flat power-law viscoelastic spectra; ageing in the power-law stress decay following a small amplitude step shear strain; an overshoot in the shear stress following the switch-on of a shear of constant rate $\dot{\gamma}$; slow creep over several time decades followed by yielding and a sudden increase of the shear rate to a steady flowing state after the imposition of a shear stress just above the yield stress; slow creep over several time decades followed by yielding and a sudden increase of the shear rate to a steady flowing state after the imposition of a shear stress just above the yield stress; and characteristic diamond or rhomboidal shaped Lissajous curves in large amplitude oscillatory shear strain.

We have described here the SGR model in its original form, which considers only scalarised shear stresses, and which contains no spatial information about the location of any element. As such, it addresses only homogeneous simple shear flows. Extensions of the model have since been put forward to address tensorial stresses [105], and flows that become heterogeneous in one spatial dimension [17].

We shall undertake this simplification first in the context of a scalarised approach that considers only shear strains and stresses, before returning in Sec. VI below to suggest a tensorial generalisation, as needed for CFD.

IV. SIMPLIFIED SOFT GLASSY RHEOLOGY MODEL

A. Motivation for a simplified model

The fact that the SGR model captures the rich phenomenology just described within a relatively simple and generic set of physical assumptions, and with a small number of model parameters, is a remarkable achievement. Indeed, once suitable units have been chosen, the model’s only free parameter is the noise temperature, $x$.

Set against this appeal is the considerably cumbersome task of computing these rheological behaviours in practice. Even for homogeneous simple shear flows, this requires the solution either of the full partial differential equation (PDE) given above, $\partial_t P(E,E,t) = \cdots$, or the solution of two coupled nonlinear integral constitutive equations with power-law memory kernels (derived from the original PDE) [16], or the direct simulation of hopping SGR elements [60], typically with $10^5$ elements required for reliable predictions. (This direct simulation is generally easier to implement computationally than a numerical solution of the differential or integral equations, but is still costly in terms of computer time.)

To utilise the (tensorially extended) full SGR model in computational fluid dynamics (CFD) to address heterogeneous flows in complicated geometries would require comparably involved computation at each lattice site: a task that is likely to prove prohibitively formidable in both practical implementation and computational cost. Indeed, to this author’s knowledge, such a task has not been attempted to date.

The present manuscript therefore aims to develop a simplified SGR model that captures the same rich phenomenology as the original model, but with greatly reduced computational demand. The contributions of this will be twofold. First, the calculation of homogeneous flows will be made simpler for anyone wishing to compare rheometric experimental data with SGR. Second, the SGR model will be rendered simple enough for practical use in CFD for predicting elastoplastic flows in complicated geometries, or complicated flow patterns that arise via spontaneous symmetry breaking instabilities even in simple geometries.

B. Simplified model

We start by exactly rewriting the full SGR model equation, Eqn. 3, as follows:

$$\dot{P}(E,l,t) + \dot{\gamma} \frac{\partial P}{\partial l} = -\frac{1}{\tau(E)} f(l) P + Y(t) \rho(E) \delta(l),$$

in which we have written the ‘bare’ hopping time

$$\tau(E) = \tau_0 \exp \left( \frac{E}{x} \right),$$

and the ‘boost factor’ to the hopping rate:

$$f(l) = \exp \left( \frac{k l^2}{2x} \right).$$

We now exactly rewrite the full joint probability distribution $P(E,l,t)$ of an element being in a trap of depth $E$ with a local strain $l$ as the probability $G(E,t)$ of an element being in a trap of depth $E$ multiplied by the conditional probability $P_l(l|E,t)$ of an element having a local strain $l$, given that it is in a trap of depth $E$:

$$P(E,l,t) = G(E,t) P_l(l|E,t).$$

We have suggestively used the notation $G(E,t)$ rather than $P(E,t)$ in the first term on the right hand side, because this quantity will below assume the behaviour of a modulus-like quantity (once re-dimensionalised by $k$).

We can then exactly rewrite Eqn. 6 as

$$\dot{G}(E,t) P_l + \dot{\gamma} G \frac{\partial P_l}{\partial l} = -\frac{G}{\tau(E)} f(l) P_l + Y(t) \rho(E) \delta(l).$$

Averaging this equation over $l$ at fixed $E$ then gives

$$\frac{d}{dt} G(E,t) = -\frac{G(E,t)}{\tau(E)} f(l)(E,t) + Y(t) \rho(E).$$
Instead pre-multiplying Eqn. 10 by \( l \) before averaging over \( l \) at fixed \( E \) gives the additional equation:

\[
\frac{d}{dt} \left[ G(E,t)\bar{l}(E,t) \right] = G(E,t)\dot{\gamma} - \frac{G(E,t)}{\tau(E,l(E,t))} l P_\gamma(l) (E,t). \tag{12}
\]

In these equations, we have used the notation

\[
\bar{a}(l)(E,t) = \int dl P_\gamma(l) E(t)a(l). \tag{13}
\]

for any function \( a(l) \).

Eqs. 11 and 12 together constitute an exact rewriting of the full SGR model. We now make an approximation by rewriting

\[
\bar{a}(l) = a(\bar{l}). \tag{14}
\]

This amounts to assuming that the distribution \( P_\gamma(l) \) has the form of a delta function located at \( \bar{l}(E,t) \): i.e., that traps of depth \( E \) have a single slaved local strain, \( \bar{l}(E,t) \). For simplicity we further now drop the overbar notation from \( l \). Eqs. 11 and 12 can then be drop the overbar notation from \( l \). Eqs. 11 and 12 can then be written:

\[
\dot{G}(E,t) = \frac{G(E,t)}{\tau(E,l(E,t))} + Y(t)\rho(E), \tag{15}
\]

and

\[
\dot{\sigma}(E,t) = kG(E,t)\dot{\gamma} - \frac{\sigma(E,t)}{\tau(E,l(E,t))}. \tag{16}
\]

Here we premultiplied Eqn. 12 by \( k \) and have defined the stress in traps of depth \( E \),

\[
\sigma(E,t) = kG(E,t)l(E,t). \tag{17}
\]

The average hopping rate,

\[
Y(t) = \int dE \frac{G(E,t)}{\tau(E,l(E,t))}, \tag{18}
\]

with an \( E \)-dependent relaxation timescale

\[
\tau(E,l(E,t)) = \tau_0 \exp \left[ (E - \frac{1}{2}k l(E,t))^2/x_\gamma \right]. \tag{19}
\]

Normalisation of overall element numbers demands that

\[
\int dE G(E,t) = 1. \tag{20}
\]

As above, the prior distribution is

\[
\rho(E) = \frac{1}{x_\gamma} \exp(-E/x_\gamma). \tag{21}
\]

Throughout what follows, in both the original and simplified SGR models, we shall choose units of time in which \( \tau_0 = 1 \) and of energy in which \( x_\gamma = 1 \). We further rescale strains such that \( k = 1 \), making the typical yield strain of order unity.

C. Discussion of the simplified model

The simplified SGR model just described has a rather appealing physical structure. Indeed, for any fixed value of energy depth \( E \), Eqn. 16 takes the form of a Maxwell model in which an elastic loading term with an effective modulus \( G(E,l) \) (in our units) competes with a plastic stress relaxation term. The relaxation time \( \tau(E,l(E)) \) is however a strongly nonlinear function of the energy depth \( E \) and local strain \( l \). The modulus, \( G(E,t) \), which is prescribed by the fraction of elements in traps of depth \( E \), furthermore has its own dynamics given by Eqn. 15.

In assuming a single value of the local strain \( l(E,t) \) for all elements in traps of a given energy depth \( E \), we have reduced the partial differential equation of the full SGR model, \( \partial_t P(E,l,t) = \cdots \), with two dynamical variables, to two equations, \( \partial_t G(E,t) = \cdots \) and \( \partial_t \sigma(E,t) = \cdots \), each with just one dynamical variable. Neither equation now contains any derivatives with respect to \( E \), further simplifying any numerics.

Nonetheless, one must still in principle evolve the two full functions \( G(E,t) \) and \( l(E,t) \) over time \( t \). A second, pragmatic simplification however arises in recognising that the continuous spectrum of energy values can be discretised on a grid of \( N \) values linearly distributed in a suitably chosen range \( 0 \leq E \leq E_{\text{max}} \). This leaves \( 2N \) differential equations, which, apart from the integral couplings of Eqn. 18 and 20, are ‘ordinary’ in form.

Numerical results obtained within this simplified model then in principle need converging to the limit \( E_{\text{max}} \to \infty, N \to \infty \). Once converged, the results show excellent agreement with all the qualitative predictions of the full SGR model, in every rheological protocol studied. Indeed, they show exact quantitative agreement in the regime of linear rheology. As we shall see below, however, the quantitative numbers can differ from full SGR typically by a worst-case factor of about 2 in the most nonlinear regime of protocols such as those in Figs. 3 and 4.

Given this quantitative discrepancy from full SGR, together with the fact that even full SGR, with its minimal – and therefore powerfully generic – set of physical assumptions, is anyway not expected to model any particular experimental sample in a fully quantitative way, we follow in our numerical computations below the pragmatic philosophy of taking the minimal value of \( N \) required to give convergence to the \( N \to \infty \) limit of the simplified model to within 1%. Typically, we find \( E_{\text{max}} = 12.0 \) and \( N = 32 \) sufficient for most protocols. Indeed, even \( N = 16 \) gives convergence to 5%, but we take \( N = 32 \) minimally in what follows.

This brings the number of degrees of freedom required to predict homogeneous simple shear flows into the numerically trivial and represents a considerable simplification compared with the full SGR model, which required the solution of the full partial differential equation \( \partial_t P(E,l,t) = \cdots \), or the solution of two coupled nonlinear integral constitutive equations with power-law
V. RHEOLOGICAL PREDICTIONS OF THE SIMPLIFIED SGR MODEL

We now present our numerical results for the predictions of the simplified SGR model in homogeneous simple shear flow. We start with linear rheology in Sec. V A, before turning to address nonlinear flows. For any protocol in which the sample age explicitly features, we model sample preparation at time \( t = 0 \) via a sudden quench from an infinite initial noise temperature to a final noise temperature, usually (as just noted) in the glass phase, \( x < 1 \). This gives an initial distribution of trap depths \( G(E, t = 0) = \rho(E) \). We further assume all local strains and stresses to be zero in this initial state, corresponding to an initially well relaxed sample. We then age the sample undisturbed for a time \( t_w \), before imposing a strain or stress according to the protocol in question.

A. Linear rheology

1. Stress relaxation after linear step shear strain

A standard rheological test consists of suddenly straining a previously undeformed material by an amount \( \gamma_0 \) at a time \( t_w \). The shear strain is accordingly \( \gamma(t) = \gamma_0 \Theta(t - t_w) \), where \( \Theta \) is the Heaviside function. The shear stress response can be written generally as

\[
\sigma(t) = \gamma_0 G_{\text{step}}(t - t_w, t_w; \gamma_0).
\] (22)

In the limit of linear response, \( \gamma_0 \to 0 \), the \( \gamma_0 \) dependence disappears from the stress relaxation function, \( G_{\text{step}} \).

The left panel of Fig. 1 shows results for \( G_{\text{step}}(t - t_w, t_w) \) in this linear regime, computed within the full SGR model, for several samples ages \( t_w \) at a fixed noise temperature in the glass phase. As can be seen, the model predicts slow power law stress relaxation. It furthermore captures rheological ageing, in which this stress relaxation takes place on a typical timescale that grows as the age of the sample \( t_w \). This gives the observed collapse of the data for different values of \( t_w \), as a function of the rescaled time interval, \( (t - t_w)/t_w \).

Corresponding results for the simplified SGR model are shown in the right panel of Fig. 1, for matched parameter values. Excellent agreement is obtained between the full and simplified models, consistent with our above statement that the assumption made in moving from the full to the simplified model is exactly correct in the linear rheological regime.

2. Viscoelastic spectra

We now consider the viscoelastic spectra that characterise a material’s stress response to a small amplitude oscillatory shear strain. As discussed in Ref. [15], the definition of viscoelastic spectra in an ageing material
needs some care, because the time-translational invariance (TTI) that is usually implicitly assumed in defining these spectra breaks down as a consequence of ageing.

Consider an experiment in which a sample is freshly prepared at time \( t = 0 \) then allowed to age undisturbed to a time \( t_s \). A small amplitude oscillatory shear strain of amplitude \( \gamma_0 \) is started at this time \( t_s \), and maintained up to a later time \( t \). For such a protocol, one can unambiguously define a time-dependent viscoelastic spectrum:

\[
G^*(\omega, t, t_s) = i\omega \int_{t_s}^{t} dt' e^{-i\omega(t-t')} G_{\text{step}}(t - t', t') \\
+ e^{-i\omega(t-t_s)} G_{\text{step}}(t - t_s, t_s)
\]

where \( G_{\text{step}}(t - t', t') \) is the (non-TTI) stress relaxation function defined in Eqn. 22 above, in the limit \( \gamma_0 \rightarrow 0 \). In Ref. [15], it was shown that the dependence of \( G^* \) on \( t_s \) becomes negligible in full SGR once many cycles have been performed, \( \omega(t - t_w) \gg 1 \), giving \( G^*(\omega, t, t_s) \rightarrow G^*(\omega, t) \). Although intuitively reasonable, this simplification is not in fact guaranteed upfront in a glassy material with long term memory. Nonetheless, we now adopt \( G^*(\omega, t) \) as a working definition of the time-dependent viscoelastic spectrum for an ageing material.

Results for the real and imaginary parts of \( G^*(\omega, t) \), \( G'(\omega, t) \) and \( G''(\omega, t) \), are shown for the full SGR model by the solid and dashed lines respectively in Fig. 2. The left panel shows results for a fixed sample age \( t_s \) at a noise temperature in the glass phase. The right panel shows results above the glass transition temperature. The spectra show a broad power-law dependence on frequency, consistent with the model’s underlying spectrum of relaxation timescales. Corresponding results for the simplified SGR model are shown by symbols in the same figure. Excellent agreement with the full SGR model again substantiates our claim that the simplified model exactly agrees with the full model in the linear rheological regime.

B. Nonlinear steady state flow curves

Having discussed the linear rheological regime, in which the full and simplified models exactly coincide, we now address nonlinear flows. We start by considering the steady state relationship between the shear stress and shear rate, as encoded in the flow curve, \( \sigma(\dot{\gamma}) \). Results for this quantity computed in the full SGR model are shown in the left panel of Fig. 3. For high noise temperatures, \( x > 2 \), the model displays Newtonian flow response in which \( \sigma \sim \dot{\gamma} \). For intermediate noise temperatures, \( 1 < x < 2 \), it shows power-law fluid behaviour in which \( \sigma \sim \dot{\gamma}^{\frac{1}{x}-1} \). For noise temperature in the glass phase, \( x < 1 \), the flow curve displays a yield stress \( \sigma_y(x) \), such that \( \sigma(\dot{\gamma}) - \sigma_y(x) \sim \dot{\gamma}^{\frac{1}{x}-1} \). The yield stress \( \sigma_y(x) \) shows a linear onset with \( x_y - x \) below the glass point.

Corresponding results computed within the simplified SGR model are shown in the right panel of the same figure. All the same quantitative features as in the full SGR model are preserved, but with qualitative differences of about a factor 2 between the full and simplified models in the most strongly linear flows, \( i.e., \) in the glass phase.

C. Dynamical nonlinear rheology: imposed strain

1. Shear startup from rest

Consider now a startup experiment in which a shear of rate \( \dot{\gamma}_0 \) is suddenly switched on at time \( t_w \), with the shear rate held constant thereafter. We thus have \( \dot{\gamma}(t) = \dot{\gamma}_0 \Theta(t - t_w) \), where \( \Theta \) is the Heaviside function.
The left panel of Fig. 4 shows results for the stress response as a function of accumulating strain, $\gamma(t) = \gamma_0(t - t_w)$, computed in the glass phase of the full SGR model. At early time intervals, for which the accumulated strain is modest, the model shows an elastic solid-like response in which the stress increases linearly with strain, $\sigma(t) = \gamma(t)$, consistent with elements being in deep enough traps that their plastic relaxation is initially negligible. By contrast, in the limit of long times $t \to \infty$ and large strains $\gamma \to \infty$, the sample flows in a liquid-like way, with the stress assuming a steady state value prescribed by the flow curve $\sigma(\gamma_0)$ described in the previous subsection. These early-time solid-like and late-time liquid-like responses accordingly show no dependence on the age of the sample before the shearing commenced. In contrast, at intermediate strains the stress overshoots its final steady state value, and the size of this overshoot shows a strong dependence on the sample age, $t_w$.

The right panel of Fig. 4 shows corresponding results computed within the simplified SGR model, for matched parameter values. All the qualitative features are preserved in moving from the full to simplified SGR model, and with only modest quantitative differences.

So far, then, we have seen that the simplified SGR model exactly reproduces the predictions of the full model in the regime of linear rheology. We have further shown that it reproduces the full model’s qualitative behaviour in the nonlinear steady state flow curve, and in nonlinear shear startup, with modest quantitative differences. Having thus developed some confidence in the simplified SGR model, we now proceed to present some new rheological predictions within the simplified model that have not, to this author’s knowledge, been previously computed within full SGR.

The right panel of Fig. 4 shows corresponding results computed within the simplified SGR model, for matched parameter values. All the qualitative features are preserved in moving from the full to simplified SGR model, and with only modest quantitative differences.

Having discussed in the previous subsection shear startup from an initial rest state, $\dot{\gamma} = 0$, to a shear rate $\gamma_0$, we turn now to consider a protocol in which a sample is sheared at some initial rate $\dot{\gamma}_1$ until it attains a steady flowing state, and is then subject at some time $t = 0$ to a shear rate jump to a final value $\gamma_2$. To this author’s knowledge, such a protocol has not previously been studied in the full SGR model. All the results presented here are computed within the simplified model.

In the left panel of Fig. 5, we show results for the time-dependent stress response $\sigma(t)$ in several different upward strain rate jumps, $\gamma_2 > \gamma_1$. In each case, the stress starts at early times at its value as prescribed by the steady state flow curve, $\sigma(\gamma_1)$, and tends at late times to a different value that is also prescribed by the steady state flow curve, $\sigma(\gamma_2)$. (Because the flow curve is a rather flat function of shear rate for $\dot{\gamma} \ll 1$ in the glass phase, $x < 1$, these initial and final values, $\sigma(\gamma_1)$ and $\sigma(\gamma_2)$, are rather similar.)

Between these short- and long-time asymptotes, the stress displays an overshoot that depends on both $\gamma_1$ and $\gamma_2$. The time at which the overshoot occurs appears to scale roughly as $t \sim \gamma_2^{-1}$, to within logarithmic corrections set by $\gamma_1$. The stress overshoot accordingly happens when the accumulated strain approaches a value $O(1)$ (to within logarithmic corrections), consistent with elements then being pulled out of their traps by the imposed strain.
The height of the overshoot is set by \( \gamma_2/\gamma_1 \).

The right panel of Fig. 5 shows counterpart results for downward strain rate jumps, \( \gamma_2 < \gamma_1 \). In this case the stress signal shows an undershoot in between its initial and final steady state values. The time at which this undershoot occurs increases with decreasing \( \gamma_2 \), and its height is set by \( \gamma_2/\gamma_1 \).

3. Flow curve sweeps

Consider now a protocol in which a sample is presheared to a steady flowing state by executing \( \gamma_{\text{preshear}} \) strain units at a high shear rate \( \dot{\gamma}_{\text{max}} \). The strain rate is then stepped downwards in \( N_{\text{sweep}} \) logarithmic increments to a low strain rate \( \dot{\gamma}_{\text{min}} \), waiting a time \( \Delta t \) at each strain rate value before further reducing the strain rate by a constant factor \( (\dot{\gamma}_{\text{max}}/\dot{\gamma}_{\text{min}})^{1/N_{\text{sweep}}} \). Once \( \dot{\gamma}_{\text{min}} \) is attained the sweep is reversed, with the strain rate stepped upwards through the same \( N_{\text{sweep}} \) values of strain rate, spending the same time \( \Delta t \) at each strain rate.

Fig. 6 shows results for the stress obtained by the final time for each strain rate value, plotted as a function of that strain rate, for two different values of \( \Delta t \). In each case, the black symbols denote the initial down-sweep and the red symbols the later up-sweep. These curves show all the same features as in the full SGR model [63], which can be summarised as follows.

Notable hysteresis is clearly evident between the down- and up-sweeps. Consider first the down-sweep. For strain rates higher than about \( 10^{-2} \), the stress lies on the steady state flow curve, which has a slight upward curvature as a function of \( \dot{\gamma} \). At lower strain rates, the stress falls away from the steady state flow curve. This occurs because the system cannot age into deeper traps quickly enough to keep pace with the ever decreasing strain rate. Accordingly, the sample remains in a more fluid-like state than it would be at a true steady state for any imposed strain rate. The viscosity and shear stress therefore remain low compared with the values they would assume on the true steady state flow curve.

Once the strain rate reaches \( \dot{\gamma}_{\text{min}} = 10^{-6} \), the up-sweep is commenced. During this upsweep, the stress initially (i.e., at low strain rates) lies below that seen during the down-sweep. Indeed, it even decreases with increasing \( \dot{\gamma} \). This is because the stress response of the SGR model is intrinsically viscoelastic. Accordingly, the sample retains some memory of the stress it had accumulated at the earlier high values of strain rate during the down-sweep, which is still slowly relaxing even as the shear rate increases again during the up-sweep.

This regime of declining stress ends with a steep upturn in the stress as the strain rate increases yet further. The up-sweep stress then rises above its down-sweep counterpart and indeed overshoots its flow-curve value, before finally declining to meet the steady state flow curve at the highest strain rates. This overshoot is the counterpart to that seen in shear startup in Fig. 4, and other upward shear rate jumps in Fig. 5. The shear rate at which the overshoot occurs is seen to scale as \( 1/\Delta t \). Indeed, all the features just described shift a decade to the left between the curves for \( \Delta t = 25.0 \) and \( \Delta t = 250.0 \).

4. Large amplitude oscillatory shear (LAOS)

Let us consider now a protocol in which a sample is freshly prepared at an initial time \( t = 0 \), then left to age undisturbed to a time \( t_w \) before an oscillatory shear strain is commenced, \( \gamma(t) = \gamma_0 \sin(\omega t) \). We present here results obtained within the simplified SGR model. LAOS has been previously studied in the full SGR model (extended to allow shear banding) in Refs. [59, 60].

For values of \( \gamma_0 \) in the nonlinear regime, we find that the system attains after many strain cycles a state in which the stress response is invariant under cycle-to-cycle translations, \( t \rightarrow t + 2\pi/\omega \). For small \( \gamma_0 \), the sample instead continues to age slightly from cycle to cycle, as seen in the viscoelastic spectra of Fig. 2, left. Fig. 7 shows parametric so-called Lissajous-Bowditch (LB) plots of the stress \( \sigma(t) \) as a function of strain \( \gamma(t) \) for the 99th and 100th cycles, indeed with no discernible difference in the stress response between these two cycles. Curves are shown for several values of the imposed strain amplitude,
γ₀, for a fixed frequency ω. The value of γ₀ in each case can be simply read off from the maximum value of γ(t) attained during the cycle.

For low values of γ₀, each LB curve takes the form of a highly elongated, almost needle-like ellipse, oriented so as to have a slope σ'(γ) ≈ 1 over much of the cycle (except, obviously, at the turning points of maximum and minimum strain). This is consistent with the SGR model showing rather elastic behaviour at stresses below the yield stress, with modulus k = 1 in our units. For the intermediate strain amplitudes explored, γ₀ = 3.0 and γ₀ = 3.5, the LB curves instead adopt a characteristic diamond shape, again with a slope ≈ 1 for stresses below the yield stress, but now with a reduced slope for higher stresses. At the highest strain amplitudes γ₀, the LB curves likewise show a slope ≈ 1 for stresses below the yield stress, but are much flatter above yield.

The same progression in the shapes of the LB curves with increasing strain amplitude γ₀ is also seen in the full SGR model. Curves such as these have been discussed in detail in the LAOS literature for yield stress fluids in terms of a sequence of physical processes, with elastic caging at low stresses and yielding at higher stresses [106].

In Ref. [56], experimental data for a carbopol gel in large amplitude oscillatory shear (LAOS) (albeit in that work for LAOSTress rather than LAOSTrain) was compared with the so-called elastic Herschel-Bulkley (EHB) model. That model is constructed to give a stress linear in strain below the yield stress, and a Herschel-Bulkley relationship between stress and strain rate above yield (as in the steady state flow curve of the SGR model). The EHB model fails to capture the diamond shaped LB curves seen experimentally in Ref. [56], and reproduced here for intermediate values of strain amplitudes in SGR. In particular, the part of these diamond-shaped LB curves above yield shows a hardening compared with EHB. (Care is warranted with nomenclature here, because this part of the curve represents a softening relative to the elastic regime below yield.)

Motivated by this observed hardening relative to EHB, many recent attempts to build constitutive models of elastoplastic rheology have incorporated the concept of ‘kinematic hardening’. This is typically discussed as modelling the movement of the centre of a material’s yield surface, and is captured by including in the constitutive model equations an additional variable termed the ‘back-stress’. Attempts to justify this back-stress in terms of underlying mesoscopic physics however remain largely unsatisfactory to date.

A pleasing feature of SGR is that it naturally captures these diamond shaped LB curves (and many other features of elastoplastic rheology besides) without recourse to the notion of a back-stress. How any effective back-stress emerges from the SGR model remains an interesting question. Feasibly, it could represent one of the next higher moments of the local strain distribution, besides the average strain encoded in the first moment.
5. Strain cycling: Bauschinger effect

In 1886, Bauschinger reported an effective reduction in the tensile yield stress of a polycrystalline metal following a tensile pre-strain in the opposite direction [107]. The effect has since also been discussed in the context of shear deformations. We shall now explore this Bauschinger effect within the simplified SGR model. To do so, let us consider a protocol in which a sample is freshly prepared at time \( t = 0 \), then left to age undisturbed before a forward shear rate \( \dot{\gamma} \) is applied up to a forward strain \( \gamma_0 \). The strain is then reversed at an equal and opposite applied shear rate, \(-\dot{\gamma},\) up to a final strain of \(-10\) units.

Figure 8 shows the predictions of the simplified SGR model in this protocol. The initial sample age \( t_w \) increases by a factor 10 between each successive panel from left to right across the figure. In each panel, results are shown for several values of the total forward strain \( \gamma \) applied before the strain direction is reversed.

During the initial forward straining phase SGR predicts a stress overshoot. Indeed, we have already discussed this in the context of a simple forward shear startup experiment. Recall Fig. 4. The associated yield stress and strain increase with increasing sample age \( t_w \) across the panels of Fig. 8 from left to right. The degree to which this stress overshoot and subsequent stress decline are explored, for any \( t_w \), increases with increasing \( \gamma_0 \). For values of \( \gamma_0 \) large enough to give significant yielding in this forward direction, a reduced yield stress is then observed during the subsequent backward straining. This corresponds to a Bauschinger effect (asymmetry between the initial forward and subsequent backward yield stress), the size of which increases with increasing age of the sample \( t_w \) before the first, forward shear commenced.

D. Dynamical nonlinear rheology: imposed stress

So far, we have discussed the predictions of the simplified SGR model for rheological protocols in which the shear strain is imposed as a function of time. We turn finally to a common stress-imposed protocol. In particular, we consider a sample that is freshly prepared at time \( t = 0 \) and left to age undisturbed up to a time \( t_w \), when a step shear stress of amplitude \( \sigma_0 \) is suddenly applied. The imposed stress is accordingly \( \sigma(t) = \sigma_0 \Theta(t-t_w) \), where \( \Theta \) is the Heaviside function.

The strain rate response as a function of time \( t-t_w \) is shown in Fig. 9 for a fixed value of the sample age \( t_w \) and several values of the imposed stress \( \sigma_0 \) from below to above the yield stress \( \sigma_y \) (defined as the stress attained in the limit \( \dot{\gamma} \rightarrow 0 \) of the steady state flow curve).

For the smallest imposed stress values shown, the shear rate decays as a function of time, and the corresponding strain response increases sublinearly. In this way, the material creeps ever forward, but at an ever declining shear rate, never attaining a flowing state of constant non-zero \( \dot{\gamma} \). For the imposed stresses just above \( \sigma_y \), the sample initially displays a window of sublinear creep in which the shear rate progressively decreases, much as it would for an imposed stress below yield. In marked contrast, however, at later times the sample yields and the shear rate suddenly increases to attain its value as prescribed by the steady state flow curve \( \sigma(\dot{\gamma}) \). The same behaviour was reported (in the format of strain versus time) in the full SGR model in Ref. [15].

This ability of the SGR model to capture power creep followed by fluidisation and yielding following the imposition of a step shear stress just above the yield stress should be particularly noted. Reports of such behaviour in other constitutive models of elastoplastic fluids (at least in those that have monotonic underlying constitutive curves \( \sigma(\dot{\gamma}) \), precluding steady state shear banding) are rare [108]: it appears difficult to capture this complicated behaviour in a simple constitutive model with just a small number of dynamical variables.

VI. POSSIBLE TENSORIAL GENERALISATION

So far, we have presented a simplified SGR model with only a scalarised shear stress. In order to perform CFD, one needs a constitutive model with a fully tensorial stress. We offer finally one possible choice for such a model, following here the simplest path to tensorialising the scalar model discussed above, and leaving other more sophisticated generalisations for future work.

We define the stress carried in wells of depth \( E \) as

\[
\Sigma(E, t) = G(E, t) I(E, t),
\]

where we have defined a new tensorial strain variable \( I \). We then write the evolution equations...
here $K$ is the velocity gradient tensor, and $D$ its symmetric part.

We then have the usual definition of the hopping rate:

$$Y(t) = \int dE \frac{G(E,t)}{\tau(E,I(E,t))}$$  \hspace{1cm} (26)

and the prior distribution

$$\rho(E) = \exp(-E).$$  \hspace{1cm} (27)

The time constant is now defined as

$$\tau(E,l) = \tau_0 \exp \left( \frac{E - I(l)}{x} \right),$$  \hspace{1cm} (28)

in which $I$ is a suitable invariant of the local strain tensor $l$, for which we suggest $I = \frac{1}{2}l : l$.

To compute the response of this tensorial model in homogeneous flow would require the evolution of $7N$ time-differential equations: one at each $N$ for $G$ and 6 for each independent component of the symmetric tensor $\Sigma$, having again discretised $E$ on a grid of $N$ values. To perform CFD would require $7N$ such variables at each lattice site, with three additional variables for the flow velocity vector.

VII. CONCLUSIONS

In this work, we have introduced a simplified constitutive model for the elastoviscoplastic rheology of yield stress fluids, motivated by the widely used soft glassy rheology model. We have demonstrated this simplified model to capture a wide array of observed rheological behaviours, in both strain-imposed and stress-imposed flow protocols, in both the linear and nonlinear rheological regimes. Once suitable units of modulus, length and time are chosen, the model has only one dimensionless parameter: the effective noise temperature, $x$.

The original soft glassy rheology model on which this simplified model is based has been widely used in the literature. However, the computation within it of even homogeneous simple shear flows is considerably cumbersome, involving the solution of a partial differential equation, $\partial_t P(E,l,t) = \cdot \cdot \cdot$, or the solution of two coupled nonlinear integral equations, or the direct simulation of typically $10^5$ hopping SGR elements.

In contrast, the computation of homogeneous simple shear flows within the simplified model requires the time-evolution of only $2N$ relatively simple differential equations, with values of $N$ as low as 16 giving good results. This renders SGR much more readily accessible to anyone wishing to fit its predictions to rheometric data. And whereas the original model was prohibitively costly for use in CFD to address flows in complicated geometries, or complicated flows arising due to spontaneous symmetry breaking instabilities even in simple geometries, the simplified model is now sufficiently simple for use in CFD, once suitably tensorialised. Indeed, work is currently in progress to benchmark its behaviour in the canonical CFD geometries of 2D flow past a cylinder and 3D flow past a sphere.

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