Constructing Infrared Finite Propagators in Inflating Space-time

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(Dated: February 22, 2010)

The usual (Bunch-Davies) Feynman propagator of a massless field is not well defined in an expanding universe due to the presence of infrared divergences. We propose a new propagator which yields infrared finite answers to any correlation function. The key point is that in a de Sitter spacetime there is an ambiguity in the zero-mode of the propagator. This ambiguity can be used to cancel the apparent divergences which arise in some loop calculations in eternally (or semi-eternally) inflating spacetime. We refer to this process as zero-mode modification. The residual ambiguity is fixed by observational measurement. We also discuss the application of this method to calculations involving the graviton propagator.

PACS numbers: 98.80Cq

I. INTRODUCTION

Improvements in cosmological observations may soon lead to the first detection of non-Gaussianity in the cosmic microwave background [1] or in the large scale structure of the universe [2]. These non-Gaussianities could have a primordial origin in theories where the inflaton has self interactions or interactions with other light fields (see [3] for a review). As such there has been much recent interest in revisiting cosmological perturbation theory beyond leading order. In addition to non-Gaussianities, the inclusion of interactions of the inflaton naturally leads one to wonder about loop corrections to other observables, such as the power spectrum.

From dimensional analysis one might expect these loop corrections to be very small, and many recent calculations have confirmed this expectation in several different settings [4]. Nevertheless, there is some expectation that loop corrections may be bigger than naive dimensional analysis would predict. The reason is that in slow roll inflation, the inflaton is almost massless (i.e. its mass is much lower than the Hubble scale during inflation). If the inflaton is treated as a massless field, loop corrections to correlation functions are afflicted with infrared (IR) divergences due to massless particles propagating in loops, and these make the loop diagrams formally infinite as compared to the tree level contributions. This would lead to the breakdown of perturbation theory, and the tree level calculation would be rendered untrustworthy. A resolution of these IR divergences is essential to extract proper predictions\textsuperscript{1}.

These divergences have been investigated by many authors [5–9]. In principle, one may avoid the issue of these divergences by including deviations from exact masslessness and exact de Sitter geometries. For example, if the inflaton has a mass, the IR divergences will be cut off (indeed, since the mass is perturbatively renormalized, it requires a severe fine tuning to keep the field truly massless). It has been argued that in $\lambda\phi^4$ theory, a mass is dynamically generated [10–12]. The tilt in the potential in slow-roll models can also lead to a cutoff of the divergences. Finally, if inflation only occurs over a finite period, the boundary conditions in the far past also modify the behavior of the low momentum modes and cut off the divergences [13].

For these reasons, it might appear natural to treat the IR divergences of the massless field as unphysical, and to assume that one or more of these mechanisms is responsible for obtaining finite results. This is unsatisfactory for a few reasons. In the first place, it is perfectly possible to consider a massless scalar field that is not the inflaton, and which can have an arbitrary potential. There appears to be no obvious reason that the quantum field theory of such a scalar should break down in the exact de Sitter geometry. Secondly, even if the IR divergences are cut off by these physical corrections, the loop corrections may be large, and we will need to understand their effects. Finally, gravitons in de Sitter space also have propagators similar to those of massless scalar fields. Since their mass is not

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\textsuperscript{1} Note that in single field inflationary models, the curvature perturbation defined on three dimensional hypersurfaces only has derivative interactions and loop corrections are IR finite. We do not expect this to be generally true in multi-field models of inflation or when computing quantities other than correlation functions of the curvature perturbations.
perturbatively renormalized, we are forced to deal with the divergences when we calculate corrections to observables which involve graviton loops.

Here, we shall reexamine these divergences in the context of a massless scalar field theory in a (flat) Friedman-Robertson-Walker (FRW) space-time with constant Hubble rate (this scalar is not the inflaton). This spacetime is a Poincaré patch of exact de Sitter; it is locally identical to de Sitter but has only six global Killing vectors. We will ignore the backreaction of the scalar on the metric (implicitly assuming that $M_{Pl} \to \infty$). We argue that the resolution to the IR divergences involves a correct understanding of what the Feynman propagator is in an inflating spacetime. In fact, just as in [14], we find that a proper treatment of the zero momentum mode removes this divergence. The main difference between the FRW case and the de Sitter case (see [14]) is that the spectrum is continuous, and some care must be taken to correctly change only the zero mode. The end result is IR finite but will in general depend on time (which is fine for a FRW background). We then show that this modification is compatible with other approaches to treating the IR divergences. However, our method is more direct and is easier for computational purposes.

In section 2, we describe the correct treatment of the zero mode in the propagator, and in section 3 we illustrate with an example of a loop calculation. In section 4, we consider the graviton case. As in the scalar case, the graviton propagator must be modified at tree level. However, this modification applies only to the graviton zero mode, which is a pure gauge mode. By choosing an appropriate gauge, the IR divergences can be gauged away (as argued to be true in [19, 20] (see also [21, 22] for more recent discussions). This provides an IR finite method for computing graviton loop corrections. Our conclusions are presented in section 5.

II. DEFINING THE FEYNMAN PROPAGATOR

We will consider the action for a scalar in a FRW background

$$S = \int dt d^3x \sqrt{g} \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right)$$

(1)

where

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \equiv a^2(\tau)(-d\tau^2 + dx^2)$$

(2)

and we take $a(\tau) = -\frac{1}{H \tau}$. In the non-interacting limit with a pure cosmological constant $V(\phi) = V_0$, the classical equation of motion for the mode functions can be solved to yield

$$\phi(k, \tau) = -i \frac{H}{\sqrt{2}k^{3/2}} (1 - i k \tau)e^{i k \tau}$$

(3)

where we have chosen the usual Bunch-Davies vacuum and $H = \frac{a'}{a} = \frac{1}{3H} \sqrt{V_0 / \pi}$. The Feynman propagator is then

$$G_F(\vec{x}, \tau; \vec{x}', \tau') = \langle 0 | T \phi(\vec{x}, \tau) \phi(\vec{x}', \tau') | 0 \rangle = \int \frac{d^3k}{(2\pi)^3} \frac{H^2}{2k^3} \langle 1 + k^2 \tau \tau' + i k | \tau - \tau' \rangle e^{-ik|\vec{x} - \vec{x}'|} e^{ik(\vec{x} - \vec{x}')}. \quad (4)$$

When interactions are turned on, there will be quantum corrections to this correlation function. Since $\langle \phi(\vec{k}, \tau) \phi(\vec{k}', \tau') \rangle \propto k^{-3}$ for small $k$, one would expect loop corrections to include a contribution from the low-momentum region of the form $\int \frac{d^3k}{k}$, leading to a logarithmic IR divergence. In fact the expression (4) is infinite even for finite scale factor and points which are at finite separation (due to the divergence at $k = 0$). It has been suggested [10–12] that this divergence can be cured by nonperturbative effects, which can generate a mass for the otherwise massless scalar field. However, this nonperturbative contribution can always be canceled by a suitable finite counterterm, again yielding a massless particle. We can therefore assume that our renormalization conditions have been chosen to give us a truly massless field $\phi$, and in this case we still must ask how this IR divergence is to be treated.

Intuitively, though, it does not seem that the divergences in (4) should be physical. For practical purposes, the largest observable scales today exited the inflationary Hubble patch 60 e-folds before the end of inflation. The IR divergence in (4) arises from modes which left the horizon long before that. As these modes themselves are not physically distinguishable from a constant zero-mode, it would seem possible to treat virtual propagation of these modes in a way which removes the divergence.

To address this problem, we note that one can add a homogeneous solution to any inhomogeneous solution to obtain another solution to the inhomogeneous wave-equation; these correspond to propagators with different boundary
conditions. The time-ordering of the Feynman propagator defines the boundary conditions: positive energy modes propagate forward in time, while negative energy modes propagate backward in time. However, in the massless case, there is a zero frequency mode. The boundary condition on this mode is not set by the causality condition, and is an ambiguity in the Feynman propagator. This is perhaps easier to see in position space where the propagator satisfies the inhomogeneous wave equation

$$\partial^\mu \partial_\mu G_F(\vec{x}, \tau; \vec{x}', \tau') = -i\delta^3(\vec{x} - \vec{x}') \delta(\tau - \tau').$$

(5)

This has a constant homogeneous solution, and hence the propagator can be shifted by a constant. This constant piece can be thought of as a shift to the 2-point correlation function, and contributes to the Feynman propagator (we note that since the retarded propagator is $G_R(\vec{x}, \tau; \vec{x}', \tau') = \theta(\tau - \tau')\langle [\phi(\vec{x}, \tau), \phi(\vec{x}', \tau')] \rangle$, the constant shift cancels in the commutator, as required by causality.)

Note that the propagator for a massless field in Minkowski space suffers from a precisely analogous ambiguity; there, however, the requirement from cluster decomposition that the propagator fall off at large distance eliminates the ambiguity. In de Sitter space (or FRW) such a requirement cannot be imposed; the propagator necessarily grows at large distances.

We can therefore add a (possibly divergent) constant to the propagator to make it finite. We modify the propagator by putting an IR cutoff at $k = \mu$ (and taking $\mu \to 0$). The finite propagator is defined to be

$$G_F^{\text{fin}}(\vec{x}, \tau; \vec{x}', \tau') = \lim_{\mu \to 0} \int \frac{d^3k}{(2\pi)^3} \frac{H^2}{2k^3} (1 + k^2 \tau \tau' + i k|\tau - \tau'|) e^{-ik|\tau - \tau'|} e^{i\vec{k}(\vec{x} - \vec{x}')} + f\left(\frac{k_{IR}}{\mu}\right)$$

(6)

where we have added a constant $f\left(\frac{k_{IR}}{\mu}\right)$. To cancel the leading divergence as $\mu \to 0$ we take $f\left(\frac{k_{IR}}{\mu}\right) = -\left(\frac{H}{2\pi}\right)^2 \ln \frac{k_{IR}}{\mu}$ (here we have introduced a new comoving scale $k_{IR}$.) The propagator in momentum space then can be written as

$$G_F^{\text{fin}}(k, \tau; \tau') = \lim_{\mu \to 0} \int \frac{d^3k}{(2\pi)^3} \frac{H^2}{2k^3} (1 + k^2 \tau \tau' + i k|\tau - \tau'|) e^{i\vec{k}(\vec{x} - \vec{x}')} e^{i\vec{k}(\tau - \tau')} \theta(k - \mu) + f\left(\frac{k_{IR}}{\mu}\right) (2\pi)^3 \delta^3(\vec{k}).$$

(7)

It is easy to see that the Feynman propagator is finite. Indeed, all calculations in perturbation theory will now yield results free of IR divergences. We refer to this procedure as zero-mode modification. Note that while the divergences have canceled, we have introduced a scale $k_{IR}$ which is a finite ambiguity in the propagator; this ambiguity will need to be fixed by an experimental measurement. A similar analysis applies for the two point correlation function. The usual formula for this quantity is

$$G_0(\vec{x}, \tau; \vec{x}', \tau') = \langle \phi(\vec{x}, \tau) \phi(\vec{x}', \tau') \rangle = \int \frac{d^3k}{(2\pi)^3} \frac{H^2}{2k^3} (1 + i k \tau) (1 - i k \tau') e^{-i\vec{k}(\vec{x} - \vec{x}')} e^{i\vec{k}(\tau - \tau')}.$$

(8)

The zero-mode modified correlation function is defined analogously to be

$$G_0^{\text{fin}}(\vec{x}, \tau; \vec{x}', \tau') = \lim_{\mu \to 0} \int \frac{d^3k}{(2\pi)^3} \frac{H^2}{2k^3} (1 + i k \tau) (1 - i k \tau') e^{-i\vec{k}(\vec{x} - \vec{x}')} e^{i\vec{k}(\tau - \tau')} + f\left(\frac{k_{IR}}{\mu}\right)$$

(9)

and is also free of divergences.

It may appear that this method of treating the zero-mode is discontinuous; one adds a constant to the propagator of the zero-mode, while leaving undisturbed the propagator of modes with arbitrarily small wavenumber. However, this discontinuity cannot be observed, as any physical experiment can only probe wavelengths smaller than the size of the observable universe. One cannot distinguish the manner in which small wavenumber modes are treated from the manner of treatment of the zero-mode.

The effect of zero-mode modification is to effectively remove the contributions from modes below $k_{IR}$; it therefore has some similarities to imposing a hard cutoff at that scale (it is important to note, however, that our scale $k_{IR}$ does not correspond to an actual cutoff on the mode integration. In our formalism, all modes are integrated over, and the effective scale $k_{IR}$ is independent of this limiting process.) However, when we consider higher point functions, the integrand of the momentum integration will have additional terms, i.e. it will be of the form $\frac{1}{2}(1 + a_1 k + a_2 k^2 + ...).$ The IR singularity from the leading term is still canceled by zero-mode modification, but subleading terms will yield a finite contribution from the modes $k < k_{IR}$. This contribution would not be present in the case of a hard-cutoff at $k_{IR}$, and this feature could potentially distinguish these two approaches.

Note that in order for eq. (7) to be a solution of eq. (5), $k_{IR}$ and $\mu$ must be comoving scales and thus independent of time. UV regularization on the other hand needs to be done with a UV cutoff fixed in physical scale ($\Lambda_{UV} = k_{UV} a(\tau)$) and therefore the coincidence limit of the two-point function depends on time (as in [13])

$$G_F^{\text{fin}}(\vec{x}, \tau; \vec{x}, \tau) \sim \ln \frac{a(\tau)}{a_0}$$

(10)
with an arbitrary constant $a_0$.

This formula also offers an alternative way of understanding our modified propagator. Equation (10) implies that the variance of a massless field grows with time. This fact is most easily understood in stochastic inflation. In this approach, one treats all modes of wavelength larger than some cutoff (typically the scale of the observable universe) as part of a zero mode. As each mode at the scale of the horizon freezes out, it shifts the value the stochastic field by a fixed magnitude, but with a random phase. The value of the field thus executes a random walk, which leads to a variance which grows linearly with time. If inflation is taken to begin at time $t = 0$, then for any finite time, we get an infinite variance. Alternatively, one can quantize the zero mode of the field separately, such that vacuum fluctuations cause the variance to increase with time, and this can only be de Sitter-invariant if the variance is finite at $t = 0$.

The result is naturally consistent with the stochastic interpretation; in the stochastic interpretation, the random walk of the scalar field causes the variance to increase with time, and this can only be de Sitter-invariant if the variance is finite at $t = 0$.

In the zero-mode modified propagator, this divergence is not present. Our propagator should be therefore interpreted as having boundary conditions where the variance is finite at finite times, but is not finite as $t = -\infty$. Alternatively, we are considering a state where the wavefunction of the zero mode is not infinitely delocalized. Our choice breaks de Sitter invariance; by taking a limit where the constant $k_{IR}$ in (4) becomes zero, we can restore de Sitter invariance at the cost of reintroducing the IR divergences.

It is also possible to think of our zero mode modification as a limiting case of the more standard scheme of modifying the Bunch-Davies vacua. Indeed, as considered in [18], one way to obtain IR finite answers in a FRW spacetime is to modify the wave functions to

$$\phi(\vec{k}, \tau) = c_1(k)\tau^{1/2}H^{(1)}_{3/2}(-k\tau) + c_2(k)\tau^{1/2}H^{(2)}_{3/2}(-k\tau)$$

where $H^{(1,2)}_{3/2}$ are the Hankel functions. For appropriate choices of $c_1, c_2$, e.g.

$$c_1 = k^{-p}, \quad c_2 = \left(k^{-2p} + \frac{3\pi}{4}\right)^{1/2}$$

the divergences can be removed. The Bunch-Davies vacuum (3) corresponds to $c_1 = 0$ and $c_2 = \sqrt{\frac{3\pi}{4}}$. Zero-mode modification corresponds to the limit $p \to \infty$, which goes over to the Bunch Davies vacuum for all nonzero $k$. For $k = 0$, the limit is singular; this corresponds again to the ambiguity in the zero mode.

III. APPLICATION TO A LOOP CALCULATION

We now apply this modified propagator to a simple loop calculation. Consider a function $N(\phi) = N_0 + N_1\phi + N_2\phi^2 + \ldots$. As our notation suggest, this function could be related to the number of e-folds which in turn determines the scalar curvature perturbation. The two-point correlation function is

$$\langle N(x, \tau)N(y, \tau') \rangle = N_0^2 + N_1^2\langle\phi(x, \tau)\phi(y, \tau')\rangle + N_2^2\langle\phi^2(x, \tau)\phi^2(y, \tau')\rangle + \cdots$$

$$= N_0^2 + N_1^2G_0(x, t; y, t') + N_2^2(G_0(x, t; y, t'))^2 + \cdots$$

where $G_0$ is the 2-point correlator. When this is evaluated in momentum space, we find

$$\langle N(\vec{k}, \tau)N(\vec{k}, \tau') \rangle = N_0^2 + N_1^2\langle\phi(\vec{k}, \tau)\phi(\vec{k}, \tau')\rangle + N_2^2\int \frac{d^3k'}{(2\pi)^3}\langle\phi(\vec{k}', \tau)\phi(\vec{k}' - \vec{k}, \tau')\phi(\vec{k}' - \vec{k}, \tau)\rangle + \cdots$$

$$= N_0^2 + N_1^2G_0(\vec{k}, \tau, \tau') + N_2^2\int \frac{d^3k'}{(2\pi)^3}G_0(\vec{k}', \tau, \tau')G_0(\vec{k}' - \vec{k}, \tau, \tau') + \cdots$$

The last term can be represented as a loop diagram. If one uses the expression (8) for the correlation function, the loop integral will exhibit an IR divergence. In particular, if the lower limit of integration is some cutoff scale $\mu$, then the divergence is manifested through the appearance of $\ln \mu$, which diverges in the $\mu \to 0$ limit.

We now show that by using the zero-mode modified correlation function (9), there is no IR divergence. An infrared divergence in the integral in (14) arises if $k' \sim 0$ or $k' \sim k$, in which case one factor $G_0$ is potentially divergent. The integral converges very quickly for $k' \gg k$ and so the incoming momenta serve as a UV cutoff for this loop diagram.
Without loss of generality, consider the integrand in the limit \( k' \sim 0 \), and integrate over the modes between zero and \( k \). We then find that the relevant integral is (after zero-mode modification)

\[
N_2^2 \int_0^k \frac{d^3k'}{(2\pi)^3} G_0^{\text{fin}}(\vec{k'}, \tau, \tau') G_0^{\text{fin}}(\vec{k} - \vec{k'}, \tau, \tau') \sim N_2^2 G_0^{\text{fin}}(\vec{k}, \tau, \tau') \int_0^k \frac{d^3k'}{(2\pi)^3} G_0^{\text{fin}}(\vec{k'}, \tau, \tau') .
\]  

(15)

After zero-mode modification, the integral over all modes with momentum \( < k_{1R} \) cancels against the zero mode, and we get the contribution

\[
N_2^2 G_0^{\text{fin}}(\vec{k}, \tau, \tau') \int_{k_{1R}}^k \frac{d^3k'}{(2\pi)^3} G_0^{\text{fin}}(\vec{k'}, \tau, \tau') \sim N_2^2 G_0^{\text{fin}}(\vec{k}, \tau, \tau') \left( \frac{H}{2\pi} \right)^2 \ln \frac{k}{k_{1R}} .
\]  

(16)

We see that the contribution to the integrand from the putatively divergent regions is canceled. The integral over the remaining modes yields a dependence on \( \ln(k/k_{1R}) \), but there is no divergence.

Note also that a shift in the quantity \( k_{1R} \), shifts the correlation function by a term proportional to \( G_0^{\text{fin}} \), and can be absorbed by a change in \( N_1 \). This is analogous to a renormalization group flow, where the shift in the scale leads to a shift in the coupling constants. In any experimental measurement, we can choose \( k_{1R} \) to be the scale of observation; for example, in [15], one was interested in temperature correlation function in the CMB on large scales as measured by WMAP and Planck and hence \( k_{1R} \) was chosen at a scale of order of \( H_0 \), the Hubble radius today. The ambiguity in the propagator is thus absorbed into coefficients like \( N_1 \), which must be measured. More generally, a smoothing (window) function that depends on some experimental scale must be used when comparing to any experiments; this smoothing procedure ends up setting the arbitrary scale \( k_{1R} \) to be the observational smoothing scale \( 1/H_0 \) in many calculations. Not every physical quantity needs to be smoothed; gravitational waves, for example, are not subtracted against any background zero mode and, as we will discuss below, our modified propagator is quite useful in this case.

Up to now, we have been looking at a free field and modifying the propagators at tree-level. Interactions can introduce further infrared divergences, which will need to be canceled by modifying \( f\left( \frac{k_{1R}}{H} \right) \) at higher orders. For example, the two point function \( \langle \phi(\vec{k}, \tau_1)\phi(-\vec{k}, \tau_2) \rangle \) receives a series of corrections in an interacting theory of the form

\[
\int d\tau_1 d\tau_2 G_0^{\text{fin}}(\vec{k}, \tau_1, \tau') \Sigma(k, \tau_1, \tau_2) G_0^{\text{fin}}(\vec{k}, \tau', \tau_2)
\]  

(17)

where \( \Sigma \) is a loop diagram. In general, \( \Sigma \) will contain terms of the form \( \ln(k_{1R}\tau_1) \) or \( \ln(k_{1R}\tau_2) \), which will lead to apparent divergences when integrating over \( \tau_1, \tau_2 \).

However, the integrand is cut off for \( |k\tau| \gg 1 \) by the oscillatory terms in \( G_0(\vec{k}, \tau_1, \tau_2) \). This removes the divergence for small \( \tau_1, \tau_2 \), but leads to terms of the form \( \ln(k_{1R}k) \). Such terms are finite if \( k > k_{1R} \) but significantly modify the propagator for \( k \ll k_{1R} \). To cancel the contribution from these modes, \( f\left( \frac{k_{1R}}{H} \right) \) will have to be adjusted order by order in perturbation theory.

**IV. THE GRAVITON PROPAGATOR**

The quantization of gravitons is in many respects similar to the massless scalar case. Indeed the two physical polarizations of the graviton satisfy the same wave equation as the massless scalar. The issues with IR divergences are therefore similar.

To preserve the gauge invariance, it is preferable to work with the Lagrangian formalism and impose a gauge fixing condition. A suitable gauge fixing condition was found in [16]. Following them, we define

\[
\gamma_{\mu\nu} dx^\mu dx^\nu = \frac{1}{H^2\tau^2}(-dt^2 + d\vec{x}^2)
\]

\[
g_{\mu\nu} = \gamma_{\mu\nu} + h_{\mu\nu}
\]

\[
\psi^\mu = h^\mu - \frac{1}{2} \delta^\mu_\mu h^\lambda
\]  

(18)

where \( \gamma_{\mu\nu} \) is the unperturbed metric of de Sitter space and \( h_{\mu\nu} \) is the metric perturbation. Indices are raised and lowered with the metric \( \gamma_{\mu\nu} \), at lowest order. We then impose the gauge-fixing condition

\[
D_\nu \psi^\nu_\mu = 2H^2\tau \psi^0_\mu
\]  

(19)
and find that the equations of motion for the quadratic part of the action reduce to
\[
\begin{align*}
\left(\Box + \frac{2}{\tau} \partial_{\tau}\right) \chi_{ij} &= 0 \\
\left(\Box + \frac{1}{\tau} \partial_{\tau}\right) \chi_{0i} &= 0 \\
\left(\Box + \frac{1}{\tau}\right) \chi &= 0
\end{align*}
\tag{20}
\]
where \(\chi_{00} = -\psi_{0}^{0}, \chi_{0i} = -\psi_{i}^{0}, \chi_{ij} = \psi_{i}^{j}\) and \(\chi = \chi_{11} + \chi_{22} + \chi_{33} + \chi_{00}\).

The \(\chi_{ij}\) therefore obey the same wave equation as massless scalars, and their propagators will be of the form (7) with a different constant shift to the zero-mode for each of the \(\chi_{ij}\). These shifts are related to the gauge symmetry; the gauge-fixing condition (19) imposes the constraints
\[
\begin{align*}
-\partial_{\tau} \chi_{00} + \partial_{i} \chi_{0i} + \tau^{-1} \chi &= 0 \\
-\partial_{\tau} \chi_{0i} + \partial_{j} \chi_{ji} + 2\tau^{-1} \chi_{0i} &= 0
\end{align*}
\tag{21}
\]
which can be used to fix \(\chi_{00}\) and \(\chi_{0i}\). However, there is a residual gauge symmetry which is not fixed by the above constraints. The residual gauge transformations can be used to generate a constant shift of the graviton perturbations (in fact, the IR divergence occurs because the gauge fixing term (19) does not completely fix the gauge). The correlator can be made finite by an appropriate gauge choice which fixes the shifts, yielding
\[
G_{\chi,ijmn}^{\text{fin}}(k, \tau', \tau) = \langle \chi_{ij}(k) \chi_{mn}(k) \rangle
= (g_{im}g_{jn} + g_{in}g_{jm}) \left( \lim_{\mu \to 0} \frac{H^{2}}{k^{2}} (1 + ik\tau)(1 - ik\tau') e^{ik(\tau - \tau')} \delta(k - \mu) - 4\pi H^{2} \ln \frac{k_{IR}}{\mu} \delta^{3}(k) \right)
\tag{22}
\]
which has no IR divergence. The propagators for \(\chi_{ij}, \chi_{0i},\) and \(\chi_{00}\) are thus free of IR divergences\(^2\). Furthermore, the ambiguity arising from the choice of \(k_{IR}\) will not be present in any gauge invariant expression, since the value of \(k_{IR}\) can be shifted by a gauge transformation.

V. CONCLUSION

We have constructed infrared finite propagators in inflating spacetimes by taking the usual Bunch-Davies propagators and modifying the zero mode; a procedure we dubbed zero mode modification. We do not implicitly assume that all scalar fields are massive, or that inflation is not eternal. Instead, this method utilizes the fact that in de Sitter spacetime, there is an inherent ambiguity in boundary conditions for the zero mode of the Feynman propagator. This ambiguity can be chosen to cancel the divergences which usually arise from the integration of low-lying modes in a loop diagram. This leaves over a finite ambiguity in the propagator, which can only be set by observation.

Physically, this modification corresponds to the choice of a state in which the variance of scalar field is finite at finite conformal time. In practice, the only observable effect of this choice is to change the variance (and in general, higher zero-mode moments) of a stochastic field, and the result is to divorce the moments of the field from the history of inflation before the era when observable modes froze out. The result is a coherent justification of the practical approach which has often been taking when dealing with loop diagrams in the quantum expansion of inflationary theories. At leading order, one can see that calculations performed using zero mode modification yield results similar to those obtained from a hard IR cutoff at some momentum \(k_{IR}\), but these approaches will differ by terms which are subleading in \(k_{IR}\). Note that at a fundamental level, our message is that the description of a massless scalar field in de Sitter space or its Poincaré patch depends on a scale \(H\) and an extra arbitrary parameter (in the form of \(k_{IR}\) here).

We have moreover argued that this approach can also be used to calculate graviton loop diagrams, in which case the ambiguity in the propagator is in fact a gauge ambiguity. In a more detailed calculation in a theory where gravity is dynamical, one should be able to see that, unlike the case of the scalar propagator, observables are independent of

\(^2\) It has been suggested [23, 24] that Gupta-Bleuler quantization of the graviton also yields unambiguous finite results for observables in de Sitter space.
this ambiguity in graviton propagator. It would be interesting to address this issue with a concrete calculation of a gravitational observable.

Acknowledgments

We are grateful to Niayesh Afshordi, Cliff Burgess, Richard Holman, David Seery, Sarah Shandera and Andrew Tolley for useful conversations. J.K. is grateful to the Perimeter Institute for its hospitality.

This research has been supported in part by funds from the Natural Sciences and Engineering Research Council (NSERC) of Canada, and Perimeter Institute. Research at the Perimeter Institute is supported in part by the Government of Canada through NSERC and by the Province of Ontario through the Ministry of Research and Information (MRI). A.R. is supported in part by NSF Grant No. PHY-0653656.

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