Abstract

In this paper, the possibility to construct a path integral formalism by using the Hubbard operators as field dynamical variables is investigated. By means of arguments coming from the Faddeev-Jackiw symplectic Lagrangian formalism as well as from the Hamiltonian Dirac method, it can be shown that it is not possible to define a classical dynamics consistent with the full algebra of the Hubbard $X$-operators. Moreover, from the Faddeev-Jackiw symplectic algorithm, and in order to satisfy the Hubbard $X$-operators commutation rules, it is possible to determine the number of constraint that must be included in a classical dynamical model. Following this approach it remains clear how the constraint conditions that must be introduced in the classical Lagrangian formulation, are weaker than the constraint conditions imposed by the full Hubbard operators algebra. The consequence of this fact is analyzed in the context of the path integral formalism. Finally, in the framework of the perturbative theory, the diagrammatic and the Feynman rules of the
model are discussed.

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I. INTRODUCTION

The Hubbard $X$-operators [1] are suitable to give a powerful framework in which the elementary excitations in solids can be explained. The use of $X$-operators is also relevant when electronic correlations are taken into account. This is the scenery in which High-$T_c$ superconductivity occurs, and so the main reason why the Hubbard operators algebra is so interesting at the present time.

The algebra of the Hubbard $\hat{X}$-operators is completely defined by:

a) the commutation rules

$$ [\hat{X}^{\alpha\beta}_i, \hat{X}^{\gamma\delta}_j] = \delta_{ij}(\delta^{\beta\gamma} \hat{X}^{\alpha\delta}_i - \delta^{\alpha\delta} \hat{X}^{\gamma\beta}_i) \tag{1.1} $$

b) the completeness condition

$$ \hat{X}^{++}_i + \hat{X}^{--}_i = \hat{I} \tag{1.2} $$

c) the multiplication rules for a given site

$$ \hat{X}^{\alpha\beta}_i \hat{X}^{\gamma\delta}_i = \delta^{\beta\gamma} \hat{X}^{\alpha\delta}_i \tag{1.3} $$

From now on and for simplicity we consider the case in which the indices $\alpha, \beta$ can only take the values $+$ and $-$, and so the Hubbard $\hat{X}$-operators are boson-like operators of the $SU(2)$ algebra. The spin $s = 1/2$ is naturally contained in this case.

It is easy to show that the equations (1.3) are not all independent, and so the full information contained in the algebra can be recovered from the equations (1.1), (1.2) and the following three independent equations

$$ \hat{X}^{-+} \hat{X}^{++} - \hat{X}^{--} = 0 \tag{1.4a} $$

$$ \hat{X}^{+} \hat{X}^{--} - \hat{X}^{++} = 0 \tag{1.4b} $$

$$ \hat{X}^{+} \hat{X}^{-} - \hat{X}^{++} = 0 \tag{1.4c} $$

Consequently, the full algebra given by equations (1.1)-(1.3) is equivalent to the commutation rules (1.1), the completeness condition (1.2) and the three conditions (1.4).
A many body theory constructed by using the Hubbard operators as field variables, requires the application of techniques used in quantum field theories. From this point of view it is necessary to formulate the Wick theorem for the case in which the field operators are neither usual fermion nor bosons. Progress in this directions where done \cite{2}, but the problem is still open.

Like in quantum field theories, another way to attach the problem is via the path integral formulation. It is important to say that a suitable path integral formulation must be independent of a given representation. On the other hand it must be written in terms of an effective action with a well defined dynamics. This last point of view will be adopted in the present paper.

The paper is organized as follows. In section II and III, by using the Faddeev-Jackiw (FJ) Lagrangian method \cite{3}, a general treatment for first-order Lagrangian systems containing the Hubbard operators as dynamical variables is given. A family of Lagrangian describing these dynamical systems is found. The use of these classical Lagrangians in a path integral quantization formalism is also analyzed. Strong arguments can be given showing that it is not possible to include the full Hubbard algebra (1.1)-(1.3) in a classical dynamical model. In section IV, we confront our results with others previously given in the literature. In section V, the diagrammatic and the Feynman rules for the model are constructed. Finally, conclusions and discussions are given in section VI.

II. CLASSICAL LAGRANGIAN AND DYNAMICAL MODEL

One of the traditional approaches to study the quantization of spin systems or t-J model in which the Hubbard operator algebra takes place, is to consider the constrained systems from the point of view of coherent state phase path integration. Also the usual Dirac's Hamiltonian method for constrained systems by considering slave boson or fermion representation is frequently used.

By writing a family of first-order classical Lagrangian directly in terms of the four Hubbard operators, our main purpose is to obtain information about the kind and the number of
constraints present in these models. In this way it is possible to obtain a response about how many information contained in the algebra (1.1)-(1.3) can be introduced at the classical level. This approach requires the introduction of a suitable set of constraints, a priori unknown, that must be determined later on. To this purpose it is useful to use the FJ Lagrangian method [3–6]. Therefore, we briefly introduce some definitions and key equations.

As is well known, the FJ symplectic quantization method is formulated on actions only containing first-order time-derivatives. The most general first-order Lagrangian is specified in terms of two arbitrary functionals $K_A(\mu^A)$ and $V(0)(\mu)$, and is given by

$$L(\mu_A, \dot{\mu}^A) = \dot{\mu}^A K_A(\mu^A) - V(0)(\mu). \quad (2.1)$$

The functionals, $K_A(\mu^A)$ are the components of the canonical one-form $K(\mu) = K_A(\mu) d\mu^A$ and the functional $V(0)(\mu)$ is the symplectic potential. The general compound index $A$ runs in the different ranges of the complete set of variables that defines the extended configuration space.

The Euler-Lagrange equations of motion obtained from (2.1) are:

$$\sum_B M_{AB} \dot{\mu}^B - \frac{\partial V(0)}{\partial \mu^A} = 0. \quad (2.2)$$

The elements of the symplectic matrix $M_{AB}(\mu)$ are the components of the symplectic two-form $M(\mu) = dK(\mu)$. The exterior derivative of the canonical one-form $K(\mu)$ is written as the generalized curl constructed with partial derivatives and so, the components are given by:

$$M_{AB} = \frac{\partial K_B}{\partial \mu^A} - \frac{\partial K_A}{\partial \mu^B}. \quad (2.3)$$

When the symplectic matrix $M_{AB}$ is non-singular, from the equations of motion (2.2) result

$$\dot{\mu}^A = (M^{AB})^{-1} \frac{\partial V(0)}{\partial \mu^B}. \quad (2.4)$$

As the symplectic potential is just the Hamiltonian of the system, the equation (2.4) is written
\[ \dot{\mu}^A = [\mu^A, V] = [\mu^A, \mu^B] \frac{\partial V^{(0)}}{\partial \mu^B}, \]  
(2.5)  

where

\[ [\mu^A, \mu^B] = (M^{AB})^{-1}, \]  
(2.6)  

are the generalized brackets defined in the FJ symplectic formalism.

It is easy to show that the elements \((M^{AB})^{-1}\) of the inverse of the symplectic matrix \(M_{AB}\) correspond to the Dirac brackets of the theory.

Transition to the quantum theory is realized as usual replacing classical fields by quantum field operators acting on the Hilbert space, where quantum ordering and proper boundary conditions for the quantum field operators must be taken into account. Therefore, the predictions of both FJ and Dirac methods are equivalents.

When the matrix \(M^{AB}\) is singular, the constraints appear as algebraic relations and they are necessary to maintain the consistency of the field equations of motion. In such a case, there exist \(m\) \((m < n)\) left (or right) zero-modes \(v_a\) \((a = 1, \ldots, m, A = 1, \ldots, n)\) of the supermatrix \(M_{AB}\), where each \(v_a\) is a column vector with \(n + m\) entries \(v^A_a\). So the zero-modes verify the following equation

\[ \sum_A v^A_a M_{AB} = 0. \]  
(2.7)  

From the equations of motion (2.2) we see that the quantities \(\Omega_a\) are the true constraints in the FJ symplectic formalism, and they are given by

\[ \Omega_a = v^i_a \frac{\partial}{\partial \varphi_i} V^{(0)} = 0. \]  
(2.8)  

Consequently, in a first iteration the constraints are written in the symplectic part of the Lagrangian by means of Lagrange multipliers as follows

\[ L^{(1)} = \varphi^i a_i(\varphi) + \xi^a \Omega_a - V^{(1)}, \]  
(2.9)  

where the new symplectic potential is by definition \(V^{(1)} = V^{(0)}|_{\Omega=\theta}\). The partition \(\mu^A = (\varphi^i, \xi^a)\) and \(K_A = (a_i, \Omega_a)\) has been made. So, the compound indices \(A, B\) run the set \(A = (i, a)\) and \(B = (j, b)\).
In each iterative procedure the configuration space is enlarged and the symplectic matrix is modified. When no new constraints are found the iterative procedure is finished.

Now, we are going to apply the FJ quantization formalism to a dynamical model for the Hubbard operators.

As it is well known in all the examples in which the field variables are the components of the spin operators, the starting point is to consider first-order Lagrangians. This also happens in the t-J model when it is written in slave boson or fermion representation. The FJ quantization algorithm is suitable to study this kind of dynamical systems described by constrained first-order Lagrangians in which the constraints play a crucial role.

Therefore, in the case under consideration we assume that the first-order classical Lagrangian as functional of the Hubbard operators is written as follows

\[ L = a_{\alpha\beta}(X) \dot{X}^{\alpha\beta} - H(X) - \lambda_a \Omega^a, \]  

where \( H(X) \) is for instance the Hamiltonian of the Heisenberg model written in terms of the Hubbard operators. The site indices were dropped since they are irrelevant in the analysis we will develop. Without any difficulty the site indices can be opportune included.

In the equation (2.10) \( \lambda_a \) is an adequate set of Lagrange multipliers which allows the introduction of the constraints in the Lagrangian formalism. \( \Omega^a(X) \) is a set of suitable unknown constraints, initially considered ad hoc in the Lagrangian. Both the constraints \( \Omega^a(X) \) as well as the range of the index \( a \) must be determined by consistency. The coefficients \( a_{\alpha\beta}(X) = a^*_\beta\alpha(X) \) are found in such a way that the algebra (1.1)-(1.3) for the Hubbard operators must be verified.

Looking at equation (2.10) we see that the initial set of dynamical symplectic variables is defined by \( (X^{\alpha\beta}, \lambda_a) \) and the symplectic potential \( V^{(0)} \) is given by

\[ V^{(0)} = H(X) + \lambda_a \Omega^a. \]  

So, the symplectic matrix (2.3) obtained from the Lagrangian (2.10) is singular, therefore the constraints are obtained by using the equation (2.8) and they read

\[ \frac{\partial V^{(0)}}{\partial \lambda_a} = \Omega^a, \]
and the first-iterated Lagrangian writes

\[ L^{(1)} = a_{\alpha\beta}(X) \dot{X}^{\alpha\beta} + \dot{\xi}_a \Omega^a - H(X). \]  

(2.13)

The modified symplectic matrix associated to the Lagrangian (2.13) is

\[ M_{AB} = \begin{pmatrix} \frac{\partial a_{\gamma\delta}}{\partial X^{\alpha\beta}} - \frac{\partial a_{\alpha\beta}}{\partial X^{\gamma\delta}} \frac{\partial \Omega_b}{\partial X^{\alpha\beta}} & \frac{\partial \Omega_b}{\partial X^{\alpha\beta}} \\ - \frac{\partial \Omega_b}{\partial X^{\gamma\delta}} & 0 \end{pmatrix}, \]  

(2.14)

where the indices \( A = \{(\alpha\beta), a\} \), \( B = \{\gamma\delta, b\} \).

At this stage the problem is to determine which, and how many constraints can be deduced from the algorithm of the method in such a way to obtain a non-singular symplectic matrix.

In this way, from the Lagrangian (2.13) the symplectic matrix (2.14) is constructed and its inverse can be computed. By equating each elements \((M^{AB})^{-1}\) of the inverse of the symplectic matrix \(M_{AB}\) to each one of the commutations rules (1.1), differential equations on the coefficients \( a_{\alpha\beta}(X) \) and on the constraints \( \Omega^a \) are obtained.

As it can be seen, the dimension of the symplectic matrix (2.14) is \(4 + a\), where \( a \) enumerates the constraints. Because of the antisymmetric property of \( M_{AB} \) the index \( a \) has even range. From the properties of this matrix we can conclude:

i) If \( a > 4 \) or odd, the symplectic matrix is singular.

ii) For \( a = 4 \) the symplectic matrix can be invertible, but it is not possible to obtain the commutation rules (1.1). The commutators obtained by using equation (2.6) vanish, independently of the value of the coefficients \( a_{\alpha\beta}(X) \). On the other hand, when the number of constraints equals the number of fields there is no dynamics. So, it is not possible by means of Lagrange multipliers to enforce the constraint (1.2) together with the other three conditions (1.4).

Consequently, we must resign the introduction in a classical first-order Lagrangian of the complete information contained in the algebra (1.1)-(1.3).

Then, the unique possibility is to have only two constraints. The equation (1.2) or completeness condition must be imposed accounting their physical meaning. It avoids at quantum level the configuration with doubly occupied sites. The remaining constraint can not be any
one of that given in (1.4), because the commutators (1.1) can not be recovered. Therefore, we can expect that the remaining constraint can be provided naturally by consistency, when the symplectic method is used.

Consequently, we assume an arbitrary constraint \( \Omega = \Omega (X^+, X^-, u) \), where \( u = X^{++} - X^{--} \). This assumption is not a restriction because, by the completeness condition, the sum \( (X^{++} + X^{--}) \) is equal to one. From the requirement that the matrix elements of the inverse of the symplectic matrix (2.14) must be equal to each one of the Hubbard commutation rules (1.1), and by solving the differential equation on this constraint the solution we find is

\[
\Omega = X^+X^- + \frac{1}{4} u^2 - \beta = 0 ,
\] (2.15)

where \( \beta \) is an arbitrary constant.

We emphasize that the constraint (2.15) is not an imposition but appears naturally from our method. This is the unique possible constraint in order to satisfy the commutation rules and the completeness condition. Of course in equation (2.15) there is less dynamical information than the contained in the three equations (1.4).

We will discuss this point connected to the fact that the path integral for this kind of fields represents the system in some limit of the operatorial approach.

III. DETERMINATION OF THE LAGRANGIAN COEFFICIENTS.

The next step is to determine the functions \( a_{\alpha\beta}(X) \) written in the Lagrangian (2.13). The two constraints \( \Omega_a \) we must consider are given in equations (1.2) and (2.15). Once the symplectic matrix (2.14) is constructed its inverse can be computed. Taking into account the equation (2.6) and the commutation rules (1.1), by consistency the following differential equation is found,

\[
2 \left[ \frac{\partial a_{++}}{\partial u} - \frac{\partial a_{u}}{\partial X^{+-}} \right] X^{+-} - 2 \left[ \frac{\partial a_{+-}}{\partial u} - \frac{\partial a_{u}}{\partial X^{-+}} \right] X^{-+} + \left[ \frac{\partial a_{-+}}{\partial X^{+-}} - \frac{\partial a_{u}}{\partial X^{-+}} \right] u = i ,
\] (3.1)

where \( a_u = \frac{1}{2}(a^{++} - a^{--}) \).
We assume that the coefficients $a_{+-}$, $a_{-+}$ and $a_u$ can be written as products of arbitrary functions of the $u$ variable by polynomials in the $X^{+-}$ and $X^{-+}$ variables. For simplicity we try to look for a particular family of solutions by taking first-order polynomials in the $X^{+-}$ and $X^{-+}$ variables, i.e.

$$a_{+-} = f(u)[e + bX^{+-} + cX^{-+}] , \quad (3.2a)$$

$$a_{-+} = a_{+-}^* = f^*(u)[e^* + c^*X^{+-} + b^*X^{-+}] , \quad (3.2b)$$

$$a_u = h(u)[p + qX^{+-} + rX^{-+}] , \quad (3.2c)$$

where the constant coefficients $p$, $q$, $r$, $e$, $b$ and $c$ are arbitrary ones.

Once the expressions (3.2) are introduced in the equation (3.1) by straightforward computation we find

$$ph(u) = (ph(u))^* \quad (3.3a)$$

$$qh(u) = (rh(u))^* \quad (3.3b)$$

$$qh(u) = e \frac{df}{du} \quad (3.3c)$$

$$cf(u) - c^*f^*(u) = 2i\text{Im} cf = 2i \frac{u + \alpha}{4\beta - u^2} \quad (3.3d)$$

with the conditions $b = 0$, and being $\alpha$ an arbitrary integration constant.

Consequently, the equations (3.2) for the Lagrangian coefficients and (3.3), determine a family of Lagrangians compatible with the commutation rules (1.1), the completeness condition (1.2) and the constraint (2.15).

Not losing generality, in equations (3.2b) and (3.3d) we can choose $c = i$ and the function $f(u)$ results

$$f(u) = \frac{u + \alpha}{4\beta - u^2} ,$$
and so two different families of solutions are obtained:

i) If \( e = 0 \) the solution reads

\[
a_{+-} = \frac{(u + \alpha)}{(4\beta - u^2)} X^{--}, \tag{3.4a}
\]

\[
a_{-+} = -i \frac{(u + \alpha)}{(4\beta - u^2)} X^{+-}, \tag{3.4b}
\]

\[
a_u = \frac{1}{2} (a_{++} - a_{--}) = h(u), \tag{3.4c}
\]

where \( h(u) \) is an arbitrary real function which also can be taken equal to zero.

ii) If \( e \neq 0 \) the solution reads

\[
a_{+-} = \frac{(u + \alpha)}{(4\beta - u^2)} (1 + iX^{--}) , \tag{3.5a}
\]

\[
a_{-+} = \frac{(u + \alpha)}{(4\beta - u^2)} (1 - iX^{+-}) , \tag{3.5b}
\]

\[
a_u = \frac{1}{2} (a_{++} - a_{--}) = h(u)[1 + X^{+-} + X^{--}] , \tag{3.5c}
\]

where in this second case \( h(u) \) verifies the equation (3.3c).

Really, the two different families of solutions (3.4) and (3.5) take into account the majority of the significant cases.

Finally, we note that making the following linear transformation to real variables \((S_1, S_2, S_3)\)

\[
X^{++} = S_1 + iS_2 , \tag{3.6a}
\]

\[
X^{--} = S_1 - iS_2 , \tag{3.6b}
\]

\[
X^{+-} - X^{--} = 2S_3 , \tag{3.6c}
\]

and by defining the vectors \( \mathbf{a} = (a_{S_1}, a_{S_2}, a_{S_3}) \), \( \nabla = (\partial_{S_1}, \partial_{S_2}, \partial_{S_3}) \) and \( \mathbf{S} = (S_1, S_2, S_3) \) where
\[ a_{S_1} = a_{+-} + a_{-+} , \quad (3.7a) \]

\[ a_{S_2} = i(a_{+-} - a_{-+}) , \quad (3.7b) \]

\[ a_{S_3} = a_{++} - a_{--} , \quad (3.7c) \]

the equation (3.1) can be written in a more simple way and it reads

\[
(\nabla \times \mathbf{a}) \cdot \mathbf{S} = 1 . \quad (3.8)
\]

The form of the differential equation (3.8) is equal to that obtained in Refs.[9,10]. Then, the fact that the kinetic term can be written as a function of a vector field \( \mathbf{a} \) which satisfies the equation (3.8) is recovered. It must be noted that the equation (3.8) is a good definition for a curl on a \( S^2 \) manifold. Then, the equation (3.8) together with (2.15) written in terms of the new variables \( S_1 \), \( S_2 \) and \( S_3 \), allows us to write the kinetic term in the Lagrangian as the area of a sphere with \( \beta^{1/2} \) radius. This is the principal argument to say that \( \beta^{1/2} \) must be integer or half-integer. For a complete discussion about this argument see Refs.[9,10].

**IV. A SIMPLE CASE AND ITS RELATION WITH PREVIOUS WORKS**

From section III, we can assert that a big family of Lagrangian really exists and any one of them can be considered as a good candidate for describing the dynamics contained in the commutation rules of the X-operators. The aim of this section is to discuss some important points by using explicitly one of the possible Lagrangians found in the previous section. Thus, by taking \( a_{++} = a_{--} = 0 \) in equation (3.4c) and calling \( \alpha = -2s \) and \( \beta = s^2 \), the Lagrangian (2.10 can be written

\[
L(X, \dot{X}) = \frac{i}{2} \left( \frac{X^{-+}X^{+-} - X^{++}X^{-+}}{s + \frac{1}{2}(X^{++} - X^{--})} \right) - H(X) , \quad (4.1)
\]

with the two constraints

\[
X^{+-}X^{-+} + \frac{1}{4}(X^{++} - X^{--})^2 = s^2 , \quad (4.2a)
\]
\[ X^{++} + X^{--} = 1 \tag{4.2b} \]

The equations (4.1) and (4.2) describe the classical dynamics of a system in which the commutation rules (1.1) are verified.

We note that the same result also can be found by using the Dirac theory for constrained systems [7]. From this approach it is easy to show that the constraints given in equations (4.2) together with the constraints coming from the definition of the momentum of the \( X \)'s variables, is a set of second class constraints. The Dirac brackets associated to this set of constraints are exactly the correct commutation rules for the Hubbard operators.

Now, we are able to write the partition function by using the path integral Faddeev-Senjanovic approach [11] and it reads

\[
Z = \int DX \delta \left[ X^{+-} X^{-+} + \frac{1}{4}(X^{++} - X^{--})^2 - s^2 \right] \delta(X^{++} + X^{--} - 1) \\
\times \exp i \int dt \ L(X, \dot{X}) , \tag{4.3}
\]

where \( L(X, \dot{X}) \) is given by (4.1).

By integrating in the \( X^{--} \) variable we obtain for the partition function \( Z \) the following expression

\[
Z = \int DX^{++} DX^{+-} DX^{-+} \delta \left[ X^{+-} X^{-+} + \frac{1}{4}(2X^{++} - 1)^2 \right] \\
\times \exp i \int dt \ L^{(1)}(X, \dot{X}) , \tag{4.4}
\]

where

\[
L^{(1)}(X, \dot{X}) = -i \frac{X^{+-} X^{-+} - X^{-+} X^{+-}}{2s + \frac{1}{4}(2X^{++} - 1)} - H(X) . \tag{4.5}
\]

Making in equation (4.5) the following change of variables,

\[
S_1 = \frac{X^{++} + X^{--}}{2} , \tag{4.6a}
\]

\[
S_2 = \frac{X^{+-} - X^{-+}}{2i} , \tag{4.6b}
\]

\[
S_3 = \frac{1}{2}(2X^{++} - 1) , \tag{4.6c}
\]
the functional integral (4.4) can be written as

\[ Z = \int DS \, \delta(S_1^2 + S_2^2 + S_3^2 - s^2) \exp i \int dt \, L^{(2)}(S, \dot{S}), \quad (4.7) \]

where

\[ L^{(2)}(S, \dot{S}) = \frac{S_2 \dot{S}_1 - S_1 \dot{S}_2}{s + S_3} - H(S) \quad (4.8) \]

where the constant Jacobian of the transformation (4.6) was absorbed in the functional integral measure. Therefore, the equation (4.7) for the partition function agrees with the expression (3.17) of Ref.[12], obtained by means of different arguments.

Now, it is easy to show that this expression is consistent with the quantization of a spin system in the limit of large \( s \). Applying again the Dirac theory, but now to the Lagrangian (4.8) with the constraints

\[ |S|^2 = s^2, \quad (4.9) \]

we find that the second class nature of the constraint defining Dirac brackets are again exactly the commutation rules (1.1) for the spin components. It is interesting to note that in the quantization procedure, the second class constraint (4.9) must be considered as a strong equation among operators. Then,

\[ \hat{S}^2 = s^2 \hat{I}. \quad (4.10) \]

From the comment given at the end of section III, it is known that the number \( s \) must be integer or half-integer. Consequently, in the equation (4.10) it is not possible to write \( s^2 \) as \( s'(s' + 1) \) with \( s' \) integer or half-integer.

This is one of the important reasons which allows to ensure that in the path integral formalism for the spin systems, the information of large \( s \) approximation is contained from the beginning. This fact is connected with our results making impossible the inclusion of the full \( X \)-operators algebra in a classical Lagrangian formalism, or equivalently in a path integral formulation.
V. DIAGRAMMATIC AND FEYNMAN RULES

Now, in order to obtain the diagrammatic and the Feynman rules for the model the perturbative treatment is analyzed. The starting point is to consider the following partition function

\[ Z = \int DX^+ DX^- Du \delta(X^+X^- + \frac{1}{4}u^2 - \beta) \times \exp i \int dt L(X, \dot{X}) , \]  

(5.1)

where the integration on the \((X^+ + X^-)\) variable has been made by using the function \(\delta(X^+ + X^- - 1)\).

Consequently, the Lagrangian \(L(X, \dot{X})\) can be written:

\[ L = a_{++} \dot{X}^+ + a_{+\cdot} \dot{X}^- + a_u \dot{u} - H(X^+, X^-, u) . \]  

(5.2)

Taking into account the equations (3.4) for the coefficients, we consider the perturbative development for large value of the parameter \(\beta\). Therefore the non-polynomial Lagrangian (5.2), up to first order in \(\beta^{-1}\) reads:

\[ L(X, \dot{X}) = i\frac{\alpha}{4\beta} (X^-X^+ - X^+X^-) + a_u \dot{u} + \frac{i}{4\beta} u(X^+X^- - X^-X^+) - H(X) . \]  

(5.3)

In equation (5.3), we consider for the Hamiltonian \(H(X)\) the Heisenberg ferromagnetic form:

\[ H(X) = -\frac{1}{2} J (X^+X^- + X^-X^+ + \frac{1}{2}uu) . \]  

(5.4)

where \(J > 0\).

By using in the path integral (5.1) the Gaussian representation

\[ \delta(x) = \lim_{\sigma \to 0} \frac{1}{\pi \sqrt{\sigma}} \exp(-\frac{1}{\sigma} x^2) \]

for the delta function, the partition function can be written in terms of an effective Lagrangian as follows.
\[ Z = \int DV \exp i \int_0^T dt \, L^{eff}(V) . \]  

(5.5)

In the equation (5.5), the effective Lagrangian \( L^{eff}(V) \) is written in terms of an extended complex vector field \( V \) whose components are given by

\[ V = (X^+ - , X^- + , u) , \]

and it can be partitioned as follows

\[ L^{eff} = L^{(2)}(V) + L^{(3)}(V) + L^{(4)}(V) . \]  

(5.6)

As it is usual the quadratic part \( L^{(2)}(V) \) of the effective Lagrangian defines the free propagator of the model, and the remaining parts \( L^{(3)}(V) \) and \( L^{(4)}(V) \) represent the interaction vertices, i.e the three and four legs vertices of the model respectively. So, from equation (5.5) it can be seen that the quantum problem remains defined in terms of a path integral which contains the three independent fields \( X^+ - , X^- + \) and \( u \).

In equation (5.6) the quadratic part \( L^{(2)}(V) \) is given by,

\[ L^{(2)}(V) = \frac{1}{2} V^\alpha D_{\alpha\beta} V^\beta , \]  

(5.7)

where

\[ D_{\alpha\beta} = \begin{pmatrix} 0 & \frac{i\alpha}{4\beta} \partial_t + \frac{\beta}{\sigma} + J & 0 \\ -\frac{i\alpha}{4\beta} \partial_t + \frac{\beta}{\sigma} + J & 0 & 0 \\ 0 & 0 & a\partial_t + \frac{\beta}{2\sigma} + \frac{J}{2} \end{pmatrix} . \]  

(5.8)

The simplest case in which \( a_u = h(u) = au \) (where \( a \) is an arbitrary constant) has been considered when the matrix (5.8) was computed.

The matrix \( D_{\alpha\beta} \) is Hermitian and non-degenerate, and so the propagator \( (D_{\alpha\beta})^{-1} \) in the \([q, \omega]\) - space can be evaluated and it results,

\[ (D_{\alpha\beta})^{-1}(\omega, \omega') = \begin{pmatrix} 0 & \frac{4\beta}{a(\omega^2 + \frac{4\beta^2}{a\sigma} - \frac{4\beta}{a} J_q)} & 0 \\ \frac{4\beta}{a(-\omega^2 + \frac{4\beta^2}{a\sigma} + \frac{4\beta}{a} J_q)} & 0 & 0 \\ 0 & 0 & \frac{1}{ia\omega + \frac{\beta}{2\sigma} - J_q} \end{pmatrix} \delta(\omega, \omega') . \]  

(5.10)

We note that \( J_q \) is the Fourier transforms of \( J_{ij} = J \) only if \( i, j \) are nearest neighbor sites.
The three and four legs vertices are respectively given by the parts

\[ L^{(3)}(V) = \frac{1}{3!} \lambda_{\alpha \beta \gamma} V^\alpha V^\beta V^\gamma, \]  

\[ L^{(4)}(V) = \frac{1}{4!} \lambda_{\alpha \beta \gamma \delta} V^\alpha V^\beta V^\gamma V^\delta, \]

where

\[ \lambda_{\alpha \beta \gamma} = \frac{1}{4\beta}(\omega' - \omega)\delta(-\omega + \omega' + \omega'')\delta(-q + q' + q'') \quad \alpha \neq \beta \neq \gamma, \]  

\[ \lambda_{\alpha \beta \gamma \delta} = -\frac{3}{2\sigma}\delta(\omega + \omega' + \omega'' + \omega''')\delta(q + q' + q'' + q''') \quad \alpha = \beta = \gamma = \delta = 3, \]  

\[ \lambda_{\alpha \beta \gamma \delta} = -\frac{1}{\sigma}\delta(-\omega + \omega' + \omega'' + \omega''')\delta(-q + q' + q'' + q''') \quad \alpha = 1, \beta = 2, \gamma = \delta = 3 \]

and all the permutations,  

\[ \lambda_{\alpha \beta \gamma \delta} = -\frac{4}{\sigma}\delta(-\omega + \omega' - \omega'' + \omega''')\delta(-q + q' - q'' + q''') \quad \alpha = 1, \beta = 2, \gamma = 1 \]

and all the permutations.

From the above results we can see that for \( \alpha = -2\sqrt{\beta} = -2s \) and by choosing for the parameter \( \sigma = \frac{\beta}{Jz} \), where \( z \) is the number of nearest neighbor sites, the matrix element reads

\[ (D_{12})^{-1} \equiv \langle T[X_q^{+}(-\omega)X_q^{-}(\omega')] \rangle = \frac{2s}{\omega - 2sz(J - J_q)} \delta(\omega - \omega')\delta(q - q'). \]

The above equation gives precisely the magnon propagator of the usual spin-wave theory.

From (5.10), it can be seen that the longitudinal mode \( \langle T[u u] \rangle \) has a pole on the imaginary axis. This non-physical mode is related with the fact that there is no longitudinal dynamics in the lowest order of the spin-wave theory of the Heisenberg ferromagnetism. So, not losing physical information, also can be taken \( a_\alpha = 0 \).

By computing the propagator and vertices for the solution (3.5) at the same perturbative order, it is easy to show that the same results are obtained. In particular the free propagator takes the form (5.8) with \( a = 0 \).

In a further work under preparation, we will apply our perturbative approach to the renormalization and dumping of magnon energies.
VI. CONCLUSIONS AND DISCUSSIONS

In this paper a new discussion about the path integral formalism for dynamical systems written in terms of Hubbard operators is done.

If it could have been possible, this path integral must have contained the full algebra (1.1)-(1.3) for the $X$-operators. Using the Faddeev-Jackiw symplectic formalism we have shown that this proposal is not possible, and in order to satisfy the commutation relations rules (1.1) we have resigned to include the complete information contained in the $X$-operator algebra (1.1)-(1.3).

By consistency of the formalism and in order to satisfy the Hubbard commutation rules we have found the number of constraint conditions. From our point of view and in a total independent way we arrive to a path integral which is consistent with those obtained by means of coherent states method.

We have also shown that this path integral for the spin system case, is valid in the large spin $s$ limit. Then, our conclusion assert that this limit is closely related with the impossibility to include the full algebra of the Hubbard $X$-operators in a classical dynamics.

On the basis of our path integral formulation we present the diagrammatic and the Feynman rules for the perturbation theory. We have shown that our free theory is consistent with the results provided by the lowest order of the spin-wave theory.

Finally, we can emphasize that from our approach a large family of kinetic terms of effective Lagrangians can be found. Some of them can be related with previous Lagrangians obtained by different methods.

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