Two algorithms in search of a type-system

NORMAN DANNER AND JAMES S. ROYER

ABSTRACT. The authors’ ATR programming formalism is a version of call-by-value PCF under a complexity-theoretically motivated type system. ATR programs run in type-2 polynomial-time and all standard type-2 basic feasible functionals are ATR-definable (ATR types are confined to levels 0, 1, and 2). A limitation of the original version of ATR is that the only directly expressible recursions are tail-recursions. Here we extend ATR so that a broad range of affine recursions are directly expressible. In particular, the revised ATR can fairly naturally express the classic insertion- and selection-sort algorithms, thus overcoming a sticking point of most prior implicit-complexity-based formalisms. The paper’s main work is in refining the original time-complexity semantics for ATR to show that these new recursion schemes do not lead out of the realm of feasibility.

1. Introduction

1.1. Feasible programming and Affine Tiered Recursion. As Hofmann [12] has noted, a problem with implicit characterizations of complexity classes is that they often fail to capture many natural algorithms—usually because the complexity-theoretic types used to control primitive recursion impose draconian restrictions on programming. For example, in Bellantoni and Cook’s [3] and Leivant’s [17] well-known characterizations of the polynomial-time computable functions, a value that is the result of a recursive call cannot itself be used to drive a recursion. But, for instance, the recursion clause of insertion-sort has the form \( \text{ins} \text{sort}(\text{cons}(a, l)) = \text{insert}(a, \text{ins} \text{sort}(l)) \), where \( \text{insert} \) is defined by recursion on its second argument; selection-sort presents analogous problems.

Hofmann [12, 11] addresses this problem by noting that the output of a non-size-increasing program (such as \( \text{ins} \text{sort} \)) can be safely used to drive another recursion, as it cannot cause the sort of complexity blow-up the B-C-L restrictions guard against. To incorporate such recursions, Hofmann defines a higher-order language with typical first-order types and a special type \( \Diamond \) through which functions defined recursively must “pay” for any use of size-increasing constructors, in effect guaranteeing that there is no size increase. Through this scheme Hofmann is able to implement many natural algorithms while still ensuring that any typable program is non-size-increasing polynomial-time computable (Aehlig and Schwichtenberg [1] sketch an extension that captures all of polynomial-time).

Our earlier paper [8, 9], hereafter referred to as ATS, takes a different approach to constructing a usable programming language with guaranteed resource usage. ATS introduces a type-2 programming formalism called ATR, for Affine Tiered Recursion, based on call-by-value PCF for which the underlying model of computation (and complexity) is a standard abstract machine.\(^1\) ATR’s type system comes in two parts: one that is motivated by the tiering and safe/normal notions of [17] and [3] and serves to control the size of objects, and one that is motivated by notions of affine-ness that serves to control time. Instead of restricting to primitive recursion, ATR has an operator for

\(^1\)In our earlier [8] ATR stood for Affine Tail Recursion; we re-christened it in [9].
recursive definitions; affine types and explicit clocking on the operator work together to prevent any complexity blow-up. In ATS we give a denotational semantics to ATR types and terms in which the size restrictions play a key part. This allows us, for example, to give an ATR definition of a primitive-recursion-on-notation combinator (without explicit bounding terms) that preserves feasibility. We also give a time-complexity semantics and use it to prove that each type-2 ATR program has a (second-order) polynomial run-time. Finally, we show that the type-2 basic feasible functionals (an extension of polynomial-time computability to type-2) of Mehlhorn and Cook and Urquhart are ATR definable. However, the version of ATR defined in ATS is still somewhat limited as its only base type is binary words and the only recursions allowed are tail-recursions.

1.2. What is new in this paper. In this paper we extend ATR to encompass a broad class of feasible affine recursions. We demonstrate these extensions by giving fairly direct and natural versions of insertion- and selection-sorts on lists (Section 3) as well as the primitive-recursion-on-notation combinator (in Section 6). As additional evidence of ATR's support for programming we do not add lists as a base type, but instead show how to implement them over ATR's base type of binary words.

The “two algorithms” of the title should not be interpreted as referring to insertion- and selection-sort, but rather the recursion schemes that those two algorithms exemplify. Most implicit characterizations restrict to structural recursion, resulting in somewhat ad-hoc implementations of other kinds of recursion by simulation. We chose insertion- and selection-sort for our prime examples in this paper because they embody key forms non-structural one-use recursion; we capture these key forms in what we call plain affine recursion. We feel that by handling any plain affine recursive program, we have shown that our system can deal with almost all standard feasible linear recursions.

The technical core of this paper is the extension of the Soundness Theorem from ATS (which handled only tail recursions) to the current version of ATR. After defining an evaluation semantics in Section 2 and surveying and simplifying the time-complexity semantics of ATS in Section 4 we introduce and prove the Soundness Theorem for plain affine recursions in Section 5. In Section 6 we use the Soundness Theorem to relate ATR-computable functions to the type-2 basic feasible functions. Since plain affine recursions include those used to implement lists and the sorting algorithms, this significantly extends our original formalism to the point where many standard algorithms can be naturally expressed while ensuring that we do not leave the realm of type-2 feasibility (and in particular, polynomial-time for type 1 programs).

With the exception of the (Shift) typing rule, we provide full definitions of all terms in this paper, and we believe that it can be understood on its own. However, the paper is not entirely self-contained: some of the proofs are adaptations of corresponding proofs in ATS, and in those cases we refer the reader to that paper for details.

1.3. Acknowledgment. Part of the motivation for this paper was a challenge to give natural versions of insertion-, selection-, and quick-sorts within an implicit complexity formalism issued by Harry Mairson in a conversation with the second-author.

2. The ATR formalism

\footnote{These kinds of results may also have applications in the type of static analysis for time-complexity that Frederiksen and Jones investigate.}

\footnote{We discuss quick-sort in Section 7.}
2.1. Types, expressions, and typing. An ATR base type has the form \( N_L \), where labels \( L \) are elements of the set \((\square \top)^* \cup \top (\square \top)^* \) (our use of \( \top \) is unrelated to Hofmann’s); the intended interpretation of \( N_L \) is \( K =_{df} \{0, 1\}^* \). The labels are ordered by \( \varepsilon \leq \top \leq \square \top \leq \square \top \leq \cdots \). We define a subtype relation on the base types by \( N_L \leq N_{L'} \) if \( L \leq L' \) and extend it to function types in the standard way. Roughly, we can think of type-\( N_\top \) values as basic string inputs, type-\( N_\square \top \) values as the result of polynomial-time computations over \( N_\top \)-values, type-\( N_\square \top \cdots \top \)-values as the result applying an oracle (a type-1 input) to \( N_\top \)-values, type-\( N_\square \top \cdots \top \)-values as the result of polynomial-time computations over \( N_\square \top \cdots \top \)-values, etc. To make an analogy with the safe/normal distinction of Bellantoni and Cook [3], oracular types correspond to normal arguments and computational types correspond to safe arguments (once we apply an oracle, we “reset” our notion of what constitutes potentially large data—but we do not “flatten” the notion by having one oracular and one computational type).

ATR’s denotational semantics works to enforce these intuitions. \( N_L \) is called an oracular (respectively, computational) type when \( L \in (\square \top)^* \) (respectively, \( \top (\square \top)^* \)). We let \( b \) (possibly decorated) range over base types. Function types are formed as usual from the base types. We sometimes write \((\sigma_1, \ldots, \sigma_k) \rightarrow \sigma \) or \( \overline{\sigma} \rightarrow \sigma \) for \( \sigma_1 \rightarrow \cdots \rightarrow \sigma_k \rightarrow \sigma \).

**Definition 1.** For any type \( \sigma \) define \( \text{tail}(\sigma) \) by \( \text{tail}(b) = b \) and \( \text{tail}(\sigma \rightarrow \tau) = \text{tail}(\tau) \).

**Definition 2.** A type \( \sigma \) is predicative when \( \sigma \) is a base type or when \( \sigma = \sigma_1 \rightarrow \cdots \rightarrow \sigma_k \rightarrow N_L \) and \( \text{tail}(\sigma_i) \leq N_L \) for all \( i \). A type is impredicative if it is not predicative. A (function) type \( \sigma_1 \rightarrow \cdots \rightarrow \sigma_k \rightarrow N_L \) is flat if \( \text{tail}(\sigma_i) = N_L \) for some \( i \). A type is strict if it is not flat.

The interpretation of the arrow types entails a significant amount of work in the semantics, which we do in \( ATS \). Very briefly, our semantics takes seriously the size information implicit in the labeled base types. In particular, the full type structure is “pruned” to create what we call the well-tempered semantics so that the function spaces of flat and impredicative types consist only of functions with appropriate growth rates. The relevant points are the following:

1. If \( f : (\sigma_1, \ldots, \sigma_k) \rightarrow b \) and \( b \leq: \text{tail}(\sigma_i) \), then \( |f| \) is bounded by a safe polynomial (see Definition 4), where \( |f| \) measures the growth rate of \( f \) and is defined in Definition 14.
2. As a special case of the previous point, if \( f : (\sigma_1, \ldots, \sigma_k) \rightarrow b \) and \( b <: \text{tail}(\sigma_i) \), then \( |f| \) is independent of its \( i \)-th argument.
3. Recursive definitions in ATR typically have flat types; the restriction on growth rates ensures that such recursively-defined functions do not lead us out of the realm of feasibility.

As this paper is concerned primarily with syntactic matters (extending the allowable forms of recursions), we do not go into full details of the denotational semantics here, instead referring the reader to Sections 6–9 of \( ATS \).

The ATR expressions are defined in Figure 1. We use \( v, x, y, z \) for variables, \( a \) for elements of \( K \), \( \alpha, \beta \) for oracles, and \( t \) for expressions (all possibly sub- and super-scripted and with primes). We can think of oracle symbols as external function calls. Formally, they are constant symbols for elements of the ATR-type structure with type-level 1; as such, each oracle symbol is assumed to be labeled with its type, which we write as a superscript when it needs to be indicated. The more-or-less typical expression-forming operations correspond to adding and deleting a left-most bit \((c_0, c_1, \text{ and } d)\), testing whether a word begins with a 0 or a 1 \((t_0 \text{ and } t_1)\), and a conditional. The intended interpretation of \( \text{down } s \ t \) is a length test that evaluates to \( s \) when \( |s| \leq |t| \) and \( \varepsilon \) when \( |s| > |t| \). The recursion operator is \( \text{crec} \), standing for \( \text{clocked recursion} \). In Section 3 we present several sample ATR programs.

\( ^{4} \)As a constant, an oracle symbol is closed, and we will suppress the interpretation of oracle symbols in the semantics.
\[ K ::= \{0, 1\}^* \]
\[ E ::= V | O | K | \lambda V.E | EE \]
\[ | c_0 E | c_1 E | d E | t_0 E | t_1 E | \text{if } E \text{ then } E \text{ else } E \]
\[ | \text{down } E E | \text{crec } K(\lambda_r V.E) \]

**Figure 1.** ATR expressions. \( V \) is a set of variable symbols and \( O \) a set of oracle symbols.

The typing rules are given in Figure 2. Type contexts are split into intuitionistic and affine zones as with Barber and Plotkin’s DILL [2]. When we write \( \Delta_0 \cup \Delta_1 \) we implicitly assume that the environments are consistent (i.e., assign the same type to variables in \( \text{Dom } \Delta_0 \cap \text{Dom } \Delta_1 \)) and when we write \( \Delta_0, \Delta_1 \) we implicitly assume that the environments have disjoint domains. Variables in the intuitionistic zone correspond to the usual \( \rightarrow \) introduction and elimination rules and variables in the affine zone are intended to be recursively defined; variables that occur in the affine zone are said to occur affinely in the judgment. The \text{crec-I} rule serves as both introduction and elimination rule for the implicit \( \rightarrow \) types (in the rule \( \bar{b} = b_1, \ldots, b_k \) and \( \vec{v} : \bar{b} \) stands for \( v_1 : b_1, \ldots, v_k : b_k \)). We use \( \lambda \) as the abstraction operator for variables introduced from the affine zone of the type context to further distinguish them from intuitionistic variables. The typing rules enforce a “one-use” restriction on affine variables that we discuss in Section 5.1. Forbidding affine variables in the conditional test is primarily a convenience and can be easily worked around with let-bindings. Two of the inference rules come with side-conditions:

\( \text{(crec-I) side-condition:} \) If \( b_i \leq b_1 \) then \( b_i \) is oracular (including \( i = 0 \)).

\( \text{(--E) side-condition:} \) At most one of \( \Delta_0 \) and \( \Delta_1 \) is non-empty, and if \( \Delta_1 \) is non-empty then \( \sigma \) is a base type.

Recalling our analogy of oracular types with normal arguments, the \( \text{(crec-I) side-condition} \) says that the clock bound (the first argument in a recursive definition) is normal and its size only depends on normal data. Thus, while the clock bound can be changed during a recursive step, this change is well-controlled. This is the core of the Termination Lemma (Theorem 15), in which we prove a polynomial size-bound on the growth of the arguments to \( f \), which in turn allows us to prove such bounds on all terms. The intuition behind the \( \text{(--E) side-condition} \) is that an affine variable \( f \) may occur in either the operator or argument of an application, but not both. Furthermore, if it occurs in the argument, then it must be a “completed” application in order to prevent the operator from duplicating it (our call-by-value semantics will thus recursively evaluate this complete application once and then plug the result into the operator).

The intuition behind the \( \text{shifts-to relation } \alpha \) between types is as follows. Suppose \( f : N_k \rightarrow N_0 \). We think of \( f \) as being a function that does some polynomial-time computation to its input. If we have an input \( x \) of type \( N_k \) then recalling the intuition behind the base types, we should be able to assign the type \( N_{\square_0} \) to \( f(x) \). The shifts-to relation allows us to shift input types in this way, with a corresponding shift in output type. As a concrete example, the judgment \( f : N_k \rightarrow N_0, x : N_k; \vdash f(x) : N_{\square_0} \) is derivable using (Subsumption) to coerce the type of \( f(x) \) to \( N_{\square_0} \) and (Shift) to shift the type of the outer application of \( f \) to \( N_{\square_0} \rightarrow N_0 \). The definition of \( \alpha \) must take into account multiple arguments and level-2 types, and it must preserve certain relationships between input and output types (for example, shifting must “preserve flatness” in the sense that if \( t : \sigma \rightarrow \tau, \text{tail}(\sigma) = \text{tail}(\tau) \), and \( \sigma \rightarrow \tau \propto \sigma' \rightarrow \tau' \), then \( \text{tail}(\sigma') = \text{tail}(\tau') \)). Our examples in this paper (implementing lists and sorting) do not make use of the (Shift) rule, so in order to not distract the reader from our main theme, we direct him or her to \texttt{ATS} for the full definition.
Changes from ATS. The system we present here differs from the one given in ATS in the following ways:

1. ATS did not restrict (Shift) to have empty affine zone. This restriction is crucial in our discussion of plain affine recursion in Section 5.1. Furthermore, we know of no natural examples in which this constraint is violated. As (Shift) provides a kind of limited polymorphism, this restriction is similar to the restriction in ML that polymorphism is disabled in recursive definitions (see Milner et al. [19] and Pierce [20, Page 338]).

2. ATS imposed no constraint on b₀ in (crec-I). Again, we know of no natural programs in which this constraint is violated.

3. ATS restricted (d-I) and (tᵦ-I) to computational types. There was no real need for this, as these term constructors represent operations that are not size-increasing.

4. ATS restricted (crec-I) to tail-recursion. Of course, this is the major improvement of the current work.

5. ATS did not allow affine variables in the argument of (→-E). This is another non-trivial improvement of the current work.

2.2. Operational semantics. Motivated by the approach of Jones [14], we define the cost of computing a program to be the cost of a call-by-value evaluation derivation. The evaluation relation ↓ relates closures to values, which are inductively defined as follows:

1. A closure \((\Gamma; \Delta \vdash t : \tau)\) consists of a term \(\Gamma; \Delta \vdash t : \tau\) and a \((\Gamma, \Delta, t)\)-environment \(\rho\). We shall always drop reference to the explicit typing and talk of closures \(tp\).

5 In ATS we give an abstract machine semantics based on defunctionalized continuations; see Appendix A for a proof of the equivalence between that semantics and the one we present here.

6 If one is only interested in computing, then the typing information in the following definitions can be dropped. However, we will address properties of closures that arise from terms (specifically, bounds on the cost of evaluation) and will need to make use of that typing information, so we include it here.
(2) A \((\Gamma, \Delta, t)\)-environment \(\rho\) is a finite map from variables to extended values such that \(\text{fv}(t) \subseteq \text{Dom} (\Gamma, \Delta)\), \(\text{fv}(t) \subseteq \text{Dom} \rho\) and if \(x \in \text{fv}(t)\) and \((x : \sigma) \in (\Gamma, \Delta)\) then \(\rho(x)\) is of type \(\sigma\). The empty environment is denoted \([],\).  

(3) A value \(z\theta\) is a closure in which \(z\) is either a string constant, oracle, or abstraction. 

(4) An extended value \(z\theta\) is a closure that is a value or has \(z = \text{crec} \ a (\lambda r f. \lambda \vec{v}. t)\) for some string constant \(a\), variables \(f\) and \(\vec{v}\), and term \(t\).  

For an environment \(\rho\), \(\rho[x \mapsto z\theta]\) is the environment that is the same as \(\rho\) on variables other than \(x\), and maps \(x\) to \(z\theta\). We write \(\rho[x_1, \ldots, x_n \mapsto z_1\theta_1, \ldots, z_n\theta_n]\) for the obvious simultaneous extension, and often abbreviate this by \(\rho[\bar{x} \mapsto z\theta]\) or \(\rho[x_i \mapsto z_i\theta_i]\), where in the latter \(i\) has a range that should be clear from context. We will also occasionally write \(\rho[x_i \mapsto z_i\theta_i]_{i..j}\) for \(\rho[x_i, \ldots, x_j \mapsto z_i\theta_i, \ldots, z_j\theta_j]\). 

The evaluation relation \(t\rho \downarrow z\theta\) is defined in Figure 3. It is a fairly standard call-by-value operational semantics; we just make a few points about some of the rules:

- Because environments may assign \text{crec} terms to variables, we cannot assume that \(\rho(x)\) is a value in \((\text{Env})\). However, we note that \(\rho(x) \downarrow z\theta\) is an instance of either the \((\text{Val})\) or \((\text{crec})\) axioms.
- In the \((\text{crec})\) rule, \(\text{"a} \leq |v_1|\) is shorthand for \(\text{down}(c_0 a)(c_0 v_1)\).
- In the \((\text{down}_1)\) rules, \(a_s\) and \(a_t\) are string constants, so the length comparison makes sense. Our cost model will take into account the actual cost of the length comparison.
- Recalling that oracles name type-1 functions and that the only type-0 values are string constants, the evaluation rules \(O_0\) and \(O_1\) say to treat multiple-argument oracles as though they are in curried form, returning the curried oracle result until all arguments have been provided. 

The cost of a derivation is the sum of the costs of the rules. All rules have cost 1 except:

- \((\text{Env})\): if \(z\) is a string constant this rule has cost \(1 \lor |z|\); otherwise if \(z\) is an abstraction or oracle, this rule has cost 1. This reflects a length-cost model of accessing the environment, where string constants are copied into memory bit-by-bit, but higher-type values are simply stored in memory as references.
- \((\text{down}_1)\): the cost of this rule is \(2|a_t| + 1\). This reflects the cost of comparing \(a_s\) and \(a_t\) bit-by-bit to determine which is longer.
- \((O_0)\): the cost of this rule is \(1 \lor |a'|\), similar to accessing a base-type value in the environment.
- \((O_1)\): the cost of this rule is 1, similar to accessing a higher-type value in the environment.

Definition 3. \(\text{cost}(t\rho)\) is defined to be the cost of the evaluation derivation of \(t\rho\). We write \(t\rho \downarrow_n z\theta\) to indicate that \(t\rho \downarrow z\theta\) and \(\text{cost}(t\rho) \leq n\).

\(A\ pri\ori\ \text{cost}(t\rho)\) may be infinite, as there may not be an evaluation derivation of \(t\rho\). Intuitively the problem may be that the “clock” \(|v_1|\) in the \((\text{crec})\) rule may be increased during the recursive call, thus leading to a non-terminating recursion. The main work of this paper to show that \(\text{cost}(t\rho)\) is always finite and in fact second-order polynomially bounded.

3. Programming in ATR

To illustrate ATR programming we give a data-type implementation of lists of binary strings and then present versions of insertion- and selection-sort using this implementation. These programs are fairly close to straightforward ML for these algorithms, with a few crucial differences discussed...
Figure 3. ATR evaluation. Note that in the $O_1$ rules $a$ is necessarily a string constant, hence $\theta$ is irrelevant.

below. Also, lists and both sorts nicely highlight various forms of affine recursion that we will need to treat in our analysis of the complexity properties of ATR programs.

In these programs we use the ML notation $fn\ x\ \Rightarrow\ \ldots$ for $\lambda$-abstraction. Also let $\text{val}\ x\ =\ s\ in\ t\ \text{end}$ abbreviates $(fn\ x\ \Rightarrow\ t)s$ and let $\text{letrec}\ f\ =\ s\ in\ t\ \text{end}$ abbreviates $t[f\ \mapsto\ \text{crec}\ \varepsilon(\lambda_r.f.s)]$.

We implement lists of binary words as concatenated self-delimiting strings. Specifically, we code the word $w = b_0 \ldots b_{k-1}$ as $s(w) = 1b_01b_1 \ldots 1b_{k-1}0$ and the list $\langle w_0, \ldots, w_{k-1} \rangle$ as $s(w_0) \oplus \cdots \oplus s(w_{k-1})$, where $\oplus$ is the concatenation operation. Code for the basic list operations is given in Figure 4. Note that the $\text{cons}$, $\text{head}$, and $\text{tail}$ programs all use cons-tail recursion—that is, the application of the recursively-defined function is followed by some number of basic operations. Insertion-sort is expressed in essentially its standard form, as in Figure 5. This implementation requires another form of recursion, in which the complete application of the recursively-defined function appears in an argument to some operator. Selection-sort (Figure 6) requires yet another form of recursion in which the complete application of the recursively-defined function appears in the body of a $\lambda$-expression. All of these recursion schemes are special cases of what we call plain affine recursion, which we discuss in Section 5.1.

Our $\text{head}$ and $\text{ins}\_\text{sort}$ programs use the $\text{down}$ operator to coerce the type $N_0$ to $N_e$. Roughly, $\text{down}$ is used in places where our type-system is not clever enough to prove that the result of a
val \texttt{nil} = \varepsilon : N_\varepsilon

val \texttt{cons} : N_\varepsilon \to N_\Diamond \to N_\Diamond =
\texttt{fn } x \texttt{ } xs \Rightarrow \texttt{letrec } \texttt{enc} : N_\varepsilon \to N_\Diamond \to N_\Diamond =
\texttt{fn } b \texttt{ } y \Rightarrow \texttt{if } y \texttt{ } \texttt{then } \texttt{if } t_0(y) \texttt{ } \texttt{then } c_1(c_0(\texttt{enc } b \texttt{ (d } y)))
\texttt{else } c_1(c_0(\texttt{enc } b \texttt{ (d } y)))
\texttt{else } c_0(xs)
in \texttt{enc } x \texttt{ } x \texttt{ end}

val \texttt{head} : N_\varepsilon \to N_\varepsilon =
\texttt{fn } xs \Rightarrow \texttt{letrec } \texttt{dec} : N_\varepsilon \to N_\Diamond \to N_\Diamond =
\texttt{fn } b \texttt{ } ys \Rightarrow \texttt{if } t_1(ys) \texttt{ } \texttt{then}
\texttt{if } t_0(d \texttt{ (d } ys)) \texttt{ } \texttt{then } c_0(\texttt{dec } b \texttt{ (d(d(y)s))}) \texttt{else } c_1(\texttt{dec } b \texttt{ (d(d(y)s))})
\texttt{else } \varepsilon
\texttt{in } \texttt{down } (\texttt{dec } xs \texttt{ } xs)(\texttt{xs}) \texttt{ end}

val \texttt{tail} : N_\varepsilon \to N_\varepsilon =
\texttt{fn } xs \Rightarrow \texttt{letrec } \texttt{strip} : N_\varepsilon \to N_\varepsilon \to N_\varepsilon =
\texttt{fn } b \texttt{ } ys \Rightarrow \texttt{if } t_1(ys) \texttt{ } \texttt{then } \texttt{strip } b \texttt{ (d} (d(ys))) \texttt{else } d(ys)
in \texttt{strip } xs \texttt{ } xs \texttt{ end}

val \texttt{insert} : N_\varepsilon \to N_\varepsilon \to N_\Diamond =
\texttt{fn } x \texttt{ } xs \Rightarrow \texttt{letrec } \texttt{ins} : N_\varepsilon \to N_\varepsilon \to N_\Diamond =
\texttt{fn } b \texttt{ } ys \Rightarrow \texttt{if } ys \texttt{ } \texttt{then}
\texttt{if } \texttt{leq } x \texttt{ } \texttt{head} (ys) \texttt{then } \texttt{cons } x \texttt{ } ys
\texttt{else } \texttt{cons } (\texttt{head } ys) \texttt{ (ins } b \texttt{ (tail } ys))
\texttt{else } \texttt{cons } x \texttt{ nil}
in \texttt{ins } xs \texttt{ } xs \texttt{ end}

val \texttt{ins\_sort} : N_\varepsilon \to N_\Diamond =
\texttt{fn } xs \Rightarrow \texttt{letrec } \texttt{isort} : N_\varepsilon \to N_\varepsilon \to N_\Diamond =
\texttt{fn } b \texttt{ } ys = \texttt{if } ys \texttt{ } \texttt{then } \texttt{insert } (\texttt{head } ys) \texttt{ (down } (\texttt{isort } b \texttt{ (tail } ys)) \texttt{ys}) \texttt{else } \varepsilon
\texttt{in } \texttt{isort } xs \texttt{ } xs \texttt{ end}

\textbf{Figure 4.} The basic list operations in ATR.

\textbf{Figure 5.} Insertion-sort in ATR. The function \texttt{leq} tests two integers written in binary for inequality; we leave its full definition as an exercise for the reader.

recursion is of size no larger than one of the recursion’s initial arguments; the burden of supplying these proofs is shifted off to the correctness argument for the recursion. A cleverer type system (say, along the lines of Hofmann’s [11]) could obviate many of these \texttt{down’s}, but at the price of more complex syntax (i.e., typing), semantics (of values and of time-complexities), and, perhaps, pragmatics (i.e., programming). Our use of \texttt{down} gives us a more primitive (and intensional) system
Figure 6. Selection-sort in ATR.

than found in pure implicit complexity,\textsuperscript{8} but it also gives us a less cluttered setting to work out the basics of complexity-theoretic compositional semantics—the focus of the rest of the paper. Also, in practice the proofs that the uses of \texttt{down} forces into the correctness argument are for the most part obvious, and thus not a large burden on the programmer.

4. Time-complexity semantics and soundness for non-recursive terms

The key fact we want to establish about ATR and its operational semantics is that the cost of evaluating a term to a value is, in an appropriate sense, polynomially bounded. This section sets up the framework for proving this and establishes the result for non-recursive terms.

The key technical notion is that of bounding a closure $t\rho$ by a time-complexity, which provides upper bounds on both the cost of evaluating $t\rho$ to a value $z\theta$ as well as the potential cost of using $z\theta$. The potential of a base-type closure is just its (denotation’s) length, whereas the potential of a function $f$ is itself a function that maps potentials $p$ to the time complexity of evaluating $f$ on arguments of potential $p$ (more on this later—we give precise definitions in Section 4.1). The bounding relation gives a time-complexity semantics for ATR-terms; a soundness theorem asserts the existence of a bounding time-complexity for every ATR term. In this paper, our soundness theorems also assert that the bounding time-complexities are safe (Definition 6), which in particular implies type-2 polynomial size and cost bounds for the closure. We thereby encapsulate the Soundness, polynomial-size-boundedness, and polynomial-time-boundedness theorems of ATS (the value semantics for the meaning of ATR terms and corresponding soundness theorem are essentially unchanged).

4.1. Time-complexity semantics. Our prior discussion of ATR types and terms situated their semantics in the realm of values—i.e., $\textbf{0}$-$\textbf{1}$-strings, functions over strings, functionals over functions over strings, etc. To work with time-complexities and potentials we introduce a new type system and new semantic realm for bounds. We will connect the realms of values and bounds in Definition 4 where we introduce bounding relations.

\textsuperscript{8}Leivant’s recursion under a high-tier bound \cite[§3.1]{Leivant93} implements a similar idea.
We start by defining cost, potential, and time-complexity types, all of which are elements of the simple product type structure over the time-complexity base types \( \{T\} \cup \{T_L : L \text{ is a label}\} \). The intended interpretation of these base types is the unary numerals and of product types the usual cartesian product. The arrow types are interpreted as the pointwise monotone non-decreasing functions and are further “pruned” analogously to the well-tempered semantics for ATR (see the discussion following Definition 2)—for more details see Section 12 of ATS and in particular Definition 49.

We define a subtype relation on base types by \( T_L \leq T \) if \( L \leq L' \) and \( T_L \leq T \) for all \( L \), and extend it to product and function types in the standard way. The only cost type is \( T \). For each ATR-type \( \sigma \) we define the time-complexity type \( \|\sigma\| \) and potential type \( \langle\langle \sigma\rangle\rangle \) by

\[
\|\tau\| = T \times \langle\langle \tau\rangle\rangle \quad \langle\langle N_L \rangle\rangle = T_L \quad \langle\langle \sigma \rightarrow \tau\rangle\rangle = \langle\langle \sigma\rangle\rangle \rightarrow \langle\langle \tau\rangle\rangle.
\]

We denote the left- and right-projections on \( \|\tau\| \) by \( \text{cost}(\cdot) \) and \( \text{pot}(\cdot) \), respectively. Define \( \text{tail}(\|\tau\|) = \langle\langle \text{tail}(\tau)\rangle\rangle \). Extend the notions of predicative, impredicative, etc. from Definition 2 to time-complexity and potential types in the obvious way. We note that \( \|\sigma\| \leq \|\tau\| \) iff \( \sigma \leq \tau \).

We need to describe objects in the time-complexity types and introduce a small formalism to do so. We will only consider terms of cost, potential, and time-complexity type. We use a fresh set of variables that we call time-complexity variables and for each ATR oracle symbol \( \alpha^\sigma \) we have a time-complexity oracle symbol \( \alpha^\|\sigma\| \). Define a time-complexity context to be a finite map from t.c. variables to cost and potential types. For a t.c. context \( \Sigma \), a \( \Sigma \) environment \( \Sigma \text{-Env} \) is a finite map from \( \text{Dom} \Sigma \) to the interpretation of the time-complexity types that respects the type \( \Sigma \) assigns to each variable; we denote the set of \( \Sigma \)-environments by \( \Sigma\text{-Env} \). We use the same extension notation for t.c. environments as for term environments. We extend \( \|\cdot\| \) to ATR-type contexts by introducing t.c. variables \( x \) and \( x_p \) for each ATR-variable \( x \) and setting \( \|\Gamma; \Delta\| = \bigcup (x, \sigma) \in (\Gamma; \Delta) \{ x_c : T, x_p : \langle\langle \sigma\rangle\rangle \} \). A time-complexity denotation of t.c. type \( \gamma \) w.r.t. a t.c. environment \( \Sigma \) is a function \( X : \Sigma\text{-Env} \rightarrow \gamma \). The projections cost and pot extend to t.c. denotations as \( \text{cost}(X) = q \mapsto \text{cost}(Xq) \) and \( \text{pot}(X) = q \mapsto \text{pot}(Xq) \). We now come to the main technical notion, that of bounding a term by a t.c. denotation.

**Definition 4.**

1. Suppose \( tp \) is a closure and \( z\theta \) a value, both of type \( \tau; \chi \) a time-complexity of type \( \|\tau\| \); and \( q \) a potential of type \( \langle\langle \tau\rangle\rangle \). Define the bounding relations \( tp \sqsubseteq^\tau \chi \) and \( z\theta \sqsubseteq^\text{pot} q \) as follows:

   a. \( tp \sqsubseteq^\tau \chi \) if \( tp \downarrow_{\text{cost}(\chi)} z\theta \) and \( z\theta \sqsubseteq^\text{pot} \text{pot}(\chi) \) (recall that the subscript on \( \downarrow \) indicates an upper bound on the cost of the evaluation derivation).
   
   b. \( z\theta \sqsubseteq^b \text{pot} q \) if \( |z| \leq q \).
   
   c. \( (\lambda v. t)\theta \sqsubseteq^\sigma_{\text{pot}^\tau} q \) when for all values \( z\eta \) and all potentials \( p \), if \( z\eta \sqsubseteq^\sigma_{\text{pot}} p \), then \( t\theta[v \mapsto z\eta] \sqsubseteq^\tau q(p) \).
   
   d. \( a\theta \sqsubseteq^\sigma_{\text{pot}^\tau} q \) when for all values \( z\eta \), if \( z\eta \sqsubseteq^\sigma_{\text{pot}} p \), then \( (a(z\eta))[\mapsto^\tau q(p)] \).

2. For \( \rho \in (\Gamma; \Delta)\text{-Env} \) and \( q \in \|\Gamma; \Delta\|\text{-Env} \), we write \( \rho \sqsubseteq q \) if for all \( v \in \text{Dom} \rho \) we have that \( \nu\rho \sqsubseteq (g(v_c), g(v_p)) \).

3. For an ATR-term \( \Gamma; \Delta \vdash t : \tau \) and a time-complexity denotation \( X \) of type \( \|\tau\| \) w.r.t. \( \|\Gamma; \Delta\| \), we say \( t \sqsubseteq X \) if for all \( \rho \in (\Gamma; \Delta)\text{-Env} \) and \( q \in \|\Gamma; \Delta\|\text{-Env} \) such that \( \rho \sqsubseteq q \) we have that \( tp \sqsubseteq Xq \).

---

9For obvious reasons, we shall start abbreviating “time-complexity” as “t.c.”

10We will drop the superscript when it is clear from context.
and suppose \( \Sigma \). Suppose \( \Sigma \parallel \vdash \) the obvious way. We shall frequently write \( p \). We define second-order polynomial expressions of cost, potential, and time-complexity types

\[
\begin{align*}
\Sigma \vdash e : T_e & \quad \Sigma \vdash 0^n : T_\emptyset & \quad \Sigma \vdash \alpha^{||\sigma||} : ||\sigma|| \\
\Sigma, x : \gamma \vdash x : \gamma & \\
\Sigma \vdash p : \gamma & \quad \Sigma \vdash p : \gamma' \quad (\gamma \propto \gamma') & \quad \Sigma \vdash p : \gamma' \quad (\gamma \leq \gamma') \\
\Sigma \vdash p : b & \quad \Sigma \vdash q : b & \quad \Sigma \vdash p \cdot q : b \\
\Sigma, x : \langle \langle \sigma \rangle \rangle \vdash p : ||\sigma|| & \quad \Sigma \vdash q : \langle \sigma \rangle & \quad \Sigma \vdash p : \langle \sigma \rangle \\
\Sigma \vdash \lambda x.p : \langle \sigma \rightarrow \tau \rangle & \quad \Sigma \vdash pq : ||\tau|| & \quad \Sigma \vdash \langle \langle \sigma \rightarrow \tau \rangle \rangle : \langle \sigma \rangle \\
\Sigma \vdash (p, q) : ||\tau|| & \quad \Sigma \vdash \langle \sigma \rangle \\
\Sigma \vdash p : T & \quad \Sigma \vdash q : ||\tau|| & \quad \Sigma \vdash p : ||\tau|| & \quad \Sigma \vdash \langle \sigma \rangle \\
\Sigma \vdash \langle \sigma \rangle \\
\end{align*}
\]

**Figure 7.** Typing rules for time-complexity polynomials. The type \( b \) is a t.c. base type, \( \gamma \) and \( \gamma' \) are any t.c. or potential types, and \( \sigma \) and \( \tau \) are any ATR-types. The operation \( \bullet \) is \( + \) or \( \ast \) and in this rule \( b \) is either \( T \) or \( T_{O_k} \) for some \( k \).

We define second-order polynomial expressions of cost, potential, and time-complexity types using the operations \(+\), \(*\), and \(\vee\) (plus, times, and binary maximum); the typing rules are given in Figure 7. Of course, a polynomial \( \Sigma \vdash p : \gamma \) corresponds to a t.c. denotation of type \( \gamma \) w.r.t. \( \Sigma \) in the obvious way. We shall frequently write \( p_\rho \) for \( \text{pot}(p) \). Our primary interest is in constructing a bounding t.c. polynomial \( ||\Gamma; \Delta|| \vdash p : ||\tau|| \) for each term \( \Gamma; \Delta \vdash t : \tau \). Rather than writing \( p = \cdots (x_c, x_p) \cdots \) each \( x \in \text{Dom}(\Gamma \cup \Delta) \), we shall just write \( p = \cdots x \cdots \).

**Definition 5.** Suppose \( \Sigma \vdash p : \gamma \) is a t.c. polynomial and \( s \) is a subterm occurrence of \( p \). We say that \( s \) is shadowed if (1) \( s \) occurs in a context \( ts \) where the occurrence of \( t \) has impredicative type \( \sigma \rightarrow \tau \) with \( \text{tail}(\tau) \leq \text{tail}(\sigma) \), or (2) the occurrence of \( s \) appears properly within another shadowed subterm occurrence.

**Definition 6.** Let \( \gamma \) be a potential type, \( b \) a time-complexity base type, \( p \) a potential polynomial, and suppose \( \Sigma \vdash p : \gamma \).

1. \( p \) is \( b\)-strict w.r.t. \( \Sigma \) when \( \text{tail}(\gamma) \leq b \) and every unshadowed free-variable occurrence in \( p \) has a type with tail \( \leq b \).
2. \( p \) is \( b\)-chary w.r.t. \( \Sigma \) when \( \gamma = b \) and \( p = p_1 \vee \cdots \vee p_m \) with \( m \geq 0 \) where \( p_i = (vq_1 \cdots q_k) \) with \( v \) a variable or oracle symbol and each \( q_j \) \( b\)-strict w.r.t. \( \Sigma \). As special cases we get \( p = 0 \) (\( m = 0 \)) and \( p = v \) for \( v \) a base-type potential variable (\( m = 1 \) and \( k = 0 \)).
3. \( p \) is \( b\)-safe w.r.t. \( \Sigma \) if:
   1. \( \gamma \) is a base type and \( p = q \odot b r \) where \( q \) is \( b\)-strict and \( r \) is \( b\)-chary, \( \odot b = \vee \) if \( b \) is oracular, and \( \odot b = + \) if \( b \) is computational.
   2. \( \gamma = \langle \sigma \rightarrow \tau \rangle \) and \( \text{pot}(p\nu) \) is \( b\)-safe w.r.t. \( \Sigma, v : \langle \sigma \rangle \).
4. A t.c. polynomial \( \Sigma \vdash q : \langle \sigma \rangle \) is \( b\)-safe if \( \text{pot}(q) \) is.
5. A t.c. denotation \( X \) of type \( ||\tau|| \) w.r.t. \( \Sigma \) is \( b\)-safe if there is a \( b\)-safe t.c. polynomial \( \Sigma \vdash p : ||\tau|| \) such that \( X \leq p^{[11]} \) \( X \) is \( \text{safe} \) if \( X \) is \( \text{tail}(||\tau||)\)-safe.

\[11\text{Remember that this inequality is with respect to the well-tempered semantics discussed at the beginning of this section.}\]
For full details and basic properties of safety, see ATS Section 8. Here we just give a couple of example propositions to get a feel for how to manipulate safe polynomials.

**Proposition 1.** If \( \Sigma, x : T \vdash p : T_L \) is a \( T_L \)-safe polynomial, then every occurrence of \( x \) in \( p \) is shadowed.

**Proof.** Set \( b = T_L \). We have that \( p = q \circ_b r \) where \( q \) is \( b \)-strict and \( r \) is \( b \)-chary. Since \( q \) is \( b \)-strict and \( T_L \leq: T \), any occurrence of \( x \) must be shadowed in \( q \). The polynomial \( r \) cannot have the form \( \cdots \lor x \lor \cdots \) because this latter expression can only have type \( T \). Thus any occurrence of \( x \) in \( r \) must occur in some \( b \)-strict polynomial, and the argument just given tells us that any such occurrence must be shadowed. \( \square \)

Under the well-tempered semantics, shadowed subterms do not contribute to the value of a polynomial. Thus we can w.l.o.g. assume that any safe potential polynomial contains only variables of potential type by replacing every occurrence of every variable of type \( T \) with ~\( \varepsilon \).

**Proposition 2.** If \( p \) and \( p' \) are \( b \)-safe potential polynomials, then there is a \( b \)-safe potential polynomial \( p^* \) such that \( p \lor p' \leq p^* \).

**Proof.** If \( b \) is computational, then \( p = q + r \) and \( p' = q' + r' \) where \( q \) and \( q' \) are \( b \)-strict and \( r \) and \( r' \) are \( b \)-chary. Thus \( p + p' = (q + r) \lor (q' + r') \leq q + q' + (r \lor r') \) is \( b \)-safe. Similarly, if \( b \) is oracular, then \( p + p' = (q \lor r) \lor (q' \lor r') = (q \lor q') \lor (r \lor r') \). \( \square \)

### 4.2. Soundness for non-recursive terms.

The Soundness Theorem asserts that every term is bounded by a safe t.c. denotation; in particular, the potential component is bounded by a safe type-2 polynomial (we shall also be able to conclude that the cost component is bounded by a type-2 polynomial in the lengths of \( t \)'s free variables). At base type, the statement about the potential corresponds to the “poly-max” bounds that can be computed for Bellantoni-Cook and Leivant-style tiered functions (e.g., [3, Lemma 4.1]). The bulk of the work is in handling \( \text{crec} \) terms. To ease the presentation, we first extract out the main claim for \( \text{ATR}^- \), the sub-system of \( \text{ATR} \) that does not include \( \text{crec} \). Although we could prove a version of the Soundness Theorem directly for \( \text{ATR}^- \) by structural induction on terms, we state instead a slightly more general proposition from which the Soundness Theorem follows directly. The reason is that when analyzing \( \text{crec} \) terms we will frequently need to construct bounding t.c. denotations for terms \( t \) given assumptions about bounding t.c. denotations for the subterms of \( t \). Thus we need to extract out what is really just the induction step of the proof of the \( \text{ATR}^- \) Soundness Theorem into its own lemma (Lemma [3]).

Figure [8] gives a number of operations on time complexity denotations that correspond to the \( \text{ATR}^- \) term-forming operations other than application and abstraction. In that figure and the following, we use the notation \( \lambda x, \cdots \) to denote the (semantic) map \( x \mapsto \cdots \). For application and abstraction, we make the following definitions:

**Definition 7.**

(1) For a potential \( p \), if \( p \) is of base type, \( \text{val} \) \( p = (1 \lor p, p) \); if \( p \) is of higher type, then

\[ \text{val} p = (1, p) \]  

For a t.c. environment \( \theta \) and \( \text{ATR} \) variable \( v \) we write \( \theta [v \mapsto \chi] \) for

\[ \theta [v_c, v_p \mapsto \text{cost}(\chi), \text{pot}(\chi)] \].

(2) If \( Y \) is a t.c. denotation of type \( ||\tau|| \) w.r.t. \( \Sigma, ||v : \sigma|| \), then

\[ \lambda v.Y =_{df} \lambda \theta (1, \lambda v_p.Y (\theta [v \mapsto \text{val} v_p])) \]

is a t.c. denotation of type \( ||\sigma \rightarrow \tau|| \) w.r.t. \( \Sigma \).

\[ ^{12} \text{Notice that} \ \text{val}(p) \ \text{is a time-complexity that bounds a value with potential} \ p. \]
conclude that 

\[ X \ast Y \]

Lemma 3.

would normally be an induction hypothesis into the statement of the lemma itself.

Lemma 70(b) in ATS to which we alluded earlier.

This and other similar computations of the full proof rely on simple properties of the well-tempered semantics of ATS which we alluded earlier.

\[ \text{pot}(\sigma) = \langle \langle \sigma \rangle \rangle \rightarrow \| \tau \| \] for some \( \sigma \)-safe polynomial \( p \). Since \( \text{pot}(X \ast Y) \leq \text{pot}(p_X p_Y) \) we conclude that \( X \ast Y \) is \( b \)-safe.\(^{13}\)
Proof. The proof is by induction on the typing inference. The cases of the induction step corresponding to the syntax-directed rules are given by Lemma 5 and if the last line of the typing inference is either (Shift) or (Subsumption), then the corresponding typing rule for t.c. polynomials applies. So we are just left with establishing the base cases. The constants are easy and inference is either (Shift) or (Subsumption), then the corresponding typing rule for t.c. polynomials responding to the syntax-directed rules are given by Lemma 3 and if the last line of the typing definitions until Section 6, when we show how to extract second-order polynomial bounds on the A TR of α there by proving Soundness for A TR. First we define plain affine recursion in Section 5.1 which captures (up to η-equivalence) how a recursively-defined function can function in its definition. In Section 5.2 we prove the Decomposition Lemma (Theorem 11), which characterizes the t.c. denotations that bound plain affine recursive definitions. Specifically, we give an algebraic characterization in which the cost of the application of the affine variable occurs as a linear term with coefficient 1 (hence our terminology). In Section 5.3, we use the Decomposition Lemma to prove the Unfolding Lemma (Theorem 12 and Corollary 13), which gives polynomial bounds on recursively-defined functions in terms of their recursion depth (Definition 12). We also prove the Termination Lemma (Theorem 15) which gives polynomial bounds on recursively-defined functions in terms of their recursion depth. This provides the last step needed to prove Soundness for A TR (Theorem 16 and Corollary 17).

5. Soundness for ATR

Our goal in this section is to extend the Soundness argument for ATR− to handle crec terms, thereby proving Soundness for ATR. First we define plain affine recursion in Section 5.1 which captures (up to η-equivalence) how a recursively-defined function can function in its definition. In Section 5.2 we prove the Decomposition Lemma (Theorem 11), which characterizes the t.c. denotations that bound plain affine recursive definitions. Specifically, we give an algebraic characterization in which the cost of the application of the affine variable occurs as a linear term with coefficient 1 (hence our terminology). In Section 5.3, we use the Decomposition Lemma to prove the Unfolding Lemma (Theorem 12 and Corollary 13), which gives polynomial bounds on recursively-defined functions in terms of their recursion depth (Definition 12). We also prove the Termination Lemma (Theorem 15) which gives polynomial bounds on the recursion depth. This provides the last step needed to prove Soundness for ATR (Theorem 16 and Corollary 17).

5.1. Plain affine recursion. As already noted, our list-operation and sorting programs use several forms of recursion that go beyond tail recursion. However, they all boil down to (essentially) filling in the argument positions of the recursively-defined function, then using the result in basic operations or as an argument to an application. In fact, they are all instances of the scheme of plain affine recursion:

Definition 9. Suppose that Γ; f : b₁ → · · · → bₖ → b₀ ⊢ t : b. t is a plain affine recursive definition of f, or f is in plain affine position in t, if:

1. f ∈ fv(t); or
2. t = f t₁ . . . tₖ where f ∈ fv(tᵢ) for any i (we call this a complete application of f); or
3. t = if s then s₀ else s₁ where f ∈ fv(s) and each sᵢ is a plain affine recursive definition of f; or
4. t = op s where op is any of cₐ, d, or tₐ and s is a plain affine recursive definition of f; or
5. t = down s₀ s₁ where s₀ is a plain affine recursive definition of f and f ∈ fv(s₁); or
6. t = st₁ . . . tₙ where f ∈ fv(s) and there is i such that tᵢ is a plain affine recursive definition of f and f ∈ fv(tⱼ) for j ≠ i; or

Formally, of course, we should write ||Γ; Δ ⊢ t : τ||, but the typing should always be clear from context.
(7) \( t = (\lambda x_1 \ldots x_m.s)t_1 \ldots t_m \) where \( s \) is a plain affine recursive definition of \( f \) and \( f \not\in \text{fv}(t_i) \) for any \( i \) (we call this a let-binding).

Whereas in ATS we enforced a side condition on (crec-I) that the recursively-defined function be in tail position, it would be much nicer to be able to say that if \( \Gamma; f : \gamma \vdash t : b \), then \( f \) occurs in plain affine position in \( t \). As stated, this does not quite hold. An exception is \( (\lambda x.f s)t_1t_2 \), which is typeable with \( f : b_1 \rightarrow b_2 \rightarrow b \) from appropriate typings of \( s, t_1, \text{and} t_2 \); but \( f \) is not in plain affine position in this expression. A trivial syntactic change “fixes” this expression without changing the meaning: simply replace \( \lambda x.f s \) with \( \lambda y.f s y \) where \( y \) is a fresh variable. In fact, it is not hard to show that this exception illustrates essentially the only way in which \( f \) can occur affinely in a term without being in plain affine position.

More precisely, we define a recursive operation on base-type terms \( t \mapsto t^\dagger \) as follows. If \( t = c_0 \, s \) then \( t^\dagger = c_0 \, s^\dagger \), and the operation “pushes through” \( c_1, d, t_0, \text{if, and down} \) similarly. Assume we have a term \( t \) such that \( \Gamma; f : \gamma \vdash t : b \) where \( \gamma = b_1 \rightarrow \cdots \rightarrow b_k \rightarrow b_0 \). Consider any base-type subterm of the form \( s s_1 \ldots s_m \) that is not an immediate subterm of an application and for which \( s \) is not an application. If \( f \in \text{fv}(s_i) \) then necessarily \( s_i \) is of base type, so \( s s_1 \ldots s_i \ldots s_{i+1}^\dagger s_{i+2} \ldots s_m \) is a plain affine definition of \( f \). If \( f \in \text{fv}(s) \), then \( f \not\in \text{fv}(s_i) \) for any \( i \) and \( s \) cannot be a crec-term, so \( s \) has the form \( (\lambda x_1 \ldots x_i.s') \) for some \( i \) where \( s' \) is not an abstraction. Replace \( s \) with \( (\lambda x_1 \ldots x_m.(s x_{i+1} \ldots x_m))^\dagger \); note that we have “filled out” the arguments of \( s \) so that \( s x_{i+1} \ldots x_m \) is of base type. Of course, a formal definition would impose an appropriate measure on terms and define \( t^\dagger \) recursively in terms of that measure; we leave the details to the interested reader. The relevant properties are as follows, all of which are easily verified by unwinding the definitions:

**Proposition 7.** Suppose that \( \Gamma; f : \gamma \vdash t : b \). Then:

1. \( \Gamma; f : \gamma \vdash t^\dagger : b \).
2. \( f \) is in plain affine recursive position in \( t^\dagger \).
3. For any environment \( \rho \), \( t \rho \downarrow z \theta \) iff \( t^\dagger \rho \downarrow z \theta \).
4. If \( t^\dagger \sqsubseteq X \) then \( t \sqsubseteq X \).

In particular, we can w.l.o.g. assume that the body of every crec expression is a plain affine recursive definition.

The next proposition shows that typing derivations of plain affine recursive definitions can placed in a normal form. We will use this normal form in our proof of the Decomposition Lemma (Theorem 11), which characterizes the t.c. denotations that bound plain affine recursive definitions. We call the premis of \( \rightarrow \text{-E} \) that types the operator the major premis of the rule.

**Proposition 8.** Suppose \( \mathcal{D} \) is a derivation of \( \Gamma; \Delta, f : \gamma \vdash t : b \) where \( t \) is a plain affine definition of \( f \), \( f \in \text{fv}(t) \), and \( \gamma = (b_1, \ldots, b_k) \rightarrow b_0 \). Then:

1. No (Subsumption) inference is the last line of the major premis of an \( (\rightarrow \text{-E}) \) inference in which \( f \) occurs free.
2. No (Subsumption) inference immediately follows an \( (\rightarrow \text{-I}) \) inference in which \( f \) occurs free.

**Proof.** The proof is by induction on the shape of \( t \) and we consider the possible typings of each shape in turn. The cases in which the induction hypothesis does not immediately apply are \( t = ft_1 \ldots t_k \) and \( t = (\lambda x_1 \ldots x_m.s)t_1 \ldots t_m \).
Suppose \( t = f t_1 \ldots t_k \); for concreteness we take \( k = 2 \) and we write \( \Sigma \) for \( \Gamma; \Delta, f : \gamma \). Then \( \mathcal{D} \) has the following general form:\(^{15}\)

\[
\text{Subsumption} \quad \Sigma \vdash f : b_1 \to b_2 \to b_0 \\
\Sigma \vdash f : b'_1 \to b'_0 \\
\Sigma \vdash f t_1 : b'_1 \to b'_0 \\
\Sigma \vdash f t_1 t_2 : b'_0 \\
\Gamma; \_ \vdash t_1 : b'_1 \\
\Gamma; \_ \vdash t_2 : b'_2 \\
\Sigma \vdash f t_1 : b'_2 \to b'_0 \\
\Sigma \vdash f t_2 : b'_{0}
\]

Since \( b'_1 \leq b_1, b'_2 \leq b_2, \) and \( b_0 \leq b'_0 \leq b'_{0}, \) we can rewrite this derivation as

\[
\text{Subsumption} \quad \Sigma \vdash f : \gamma \\
\Gamma; \_ \vdash t_1 : b'_1 \\
\Sigma \vdash f t_1 : b_2 \to b_0 \\
\Sigma \vdash f t_1 t_2 : b_0 \\
\Gamma; \_ \vdash t_2 : b'_2 \\
\Gamma; \_ \vdash t_2 : b_2 \\
\Sigma \vdash f t_1 t_2 : b'_2 \to b'_0 \\
\Sigma \vdash f t_1 t_2 : b'_0
\]

If \( t = (\lambda x_1 \ldots x_m.s)\vec{t} \) then first apply the induction hypothesis to the typing of \( s \). Any (Subsumption) inferences that follow one of the \((\to-I)\) inferences can be moved to the end of all those inferences. Thus as in the previous case, we can move any (Subsumption) inferences that occur as the last line of a major premis in one of the \((\to-E)\) inferences \((\lambda \vec{x}.s)t_1 \ldots t_i\) to the minor premise, concluding with a possible last (Subsumption) inference. \( \square \)

The \( \text{let} \)-binding clause of plain affine recursion leads us to consider t.c. denotations of the form \((\lambda \vec{x}.X) \star \vec{Y}\), so we characterize them here. First we define a function on t.c. denotations that allows us to neatly express the “overhead cost” of combining t.c. denotations:

**Definition 10.** For any t.c. denotation \( X \),

\[ \text{dally}(m, X) = \lambda \varrho (m + \text{cost}(X \varrho), \text{pot}(X \varrho)). \]

**Proposition 9.** If \( X \) is a safe t.c. denotation, then so is \( \text{dally}(m, X) \).

**Proposition 10.** Let \( X \) be a t.c. denotation w.r.t. \( \Sigma, \| x_1 : \sigma_1, \ldots, x_m : \sigma_m \| \) and \( Y_1, \ldots, Y_m \) be t.c. denotations w.r.t. \( \Sigma \). Then

\[ (\lambda \vec{x}.X) \star \vec{Y} = \lambda \varrho. \text{dally}(2m + \sum_{i=1}^{m} \text{cost}(Y_i \varrho), X \varrho[x_i \mapsto \text{val}(\text{pot}(Y_i \varrho))]). \]

**Proof.** The proof is by induction on \( m \); the base case is immediate. For the induction step we apply the induction hypothesis and unwind definitions. In the following calculation we write \( Y_i' \varrho \)

\(^{15}\)It is here that we use the restriction that (Shift) cannot be applied if the affine zone is non-empty; without this restriction, we could have a sequence of (Shift) and (Subsumption) inferences interleaved with the \((\to-E)\) inferences, and this proof would not carry through.
for \(\text{cost}(Y_{i}\phi), \quad \phi_{m} \) for \(\phi[x_{i} \mapsto \text{val}(\text{pot}(Y_{i}\phi))]\) where \(i = 1, \ldots, m\), and similarly for \(\phi_{m+1}\):

\[
(\lambda x_{1} \ldots x_{m+1}.X) \cdot Y_{1} \cdot \cdots \cdot Y_{m+1}
\]

\[
= (\lambda \phi. \text{dally}(2m + \sum_{i=1}^{m} Y_{i}\phi, (\lambda x_{m+1}.X)\phi_{m})) \cdot Y_{m+1}
\]

\[
= (\lambda \phi. \text{dally}(2m + \sum_{i=1}^{m} Y_{i}\phi, \langle \langle 1, \lambda x_{m+1}.p, X\phi'[x_{m+1} \mapsto \text{val}(x_{m+1}.p)]\rangle \phi_{m} \rangle)) \cdot Y_{m+1}
\]

\[
= \lambda \phi. (1 + 2m + \sum_{i=1}^{m} Y_{i}\phi + 1 + Y_{m+1}e\phi + \text{cost}(X\phi_{m+1}), \text{pot}(X\phi_{m+1}))
\]

\[
= \lambda \phi. \text{dally}(2(m + 1) + \sum_{i=1}^{m+1} Y_{i}\phi, X\phi_{m+1}).
\]

\[
\square
\]

5.2. Bounds for recursive definitions: the Decomposition Lemma. We now state and prove the Decomposition Lemma. Throughout this section and the next we will need to assume that induction hypothesis of the Soundness Theorem holds, because the Decomposition Lemma will be used in its induction step. So to shorten the statements of the coming claims, we name the induction hypothesis:

**Inductive Soundness Assumption (ISA):** A term \(\Gamma; f: \gamma \vdash t: b\) (where \(\gamma = (b_{1}, \ldots, b_{k}) \rightarrow b_{0}\)) satisfies the *inductive Soundness assumption* if \(t\) is a plain affine recursive definition of \(f\) and whenever \(\Gamma'; \_ \vdash s: \tau\) is a subterm of \(t\), there is a \(\text{tail}(|\tau|)\)-safe t.c. polynomial \((P_{s}, p_{s})\) w.r.t. \(|\Gamma'|\) such that \(s \sqsubseteq (P_{s}, p_{s})\).

For the statement of the Decomposition Lemma, recall our convention that in writing a t.c. polynomial \(p\) w.r.t. \(|\Gamma; \Delta|\), if \(x \in \text{Dom}(\Gamma \cup \Delta)\) we write \(p(\ldots, x, \ldots)\) to abbreviate \(p(\ldots, x, c, x, p, \ldots)\).

**Theorem 11** (Decomposition Lemma). Suppose \(\Gamma; f: \gamma \vdash t: b\) satisfies the ISA and that \(\text{Dom} \Gamma = \bar{y}\). Then \(t \sqsubseteq (P(\bar{y}, \text{pot}(f \ast \bar{p})) + \text{cost}(f \ast \bar{p}), \quad p(\bar{y}, \text{pot}(f \ast \bar{p}))))\)

where \(P(\bar{y}, w^{(\bar{b})}); \: T\) is a cost polynomial, \(p(\bar{y}, w^{(\bar{b})}); \: \langle \langle \bar{b} \rangle \rangle\) is a \(\langle \langle \bar{b} \rangle \rangle\)-safe potential polynomial, and \(\bar{p} = p_{1}, \ldots, p_{k}\) where for each \(i, \quad p_{i} = p_{i}(\bar{y}): \|b_{i}\|\) is a \(\text{tail}(|\|b_{i}\||)\)-safe t.c. polynomial\(^{16}\).\)

**Proof.** The proof is by induction on the typing of \(t\). For clarity we drop mention of the parameters \(\bar{y}\) everywhere. If \(f \notin \text{fv}(t)\), then the claim follows from the ISA. Also notice that if the last line of the typing of \(t\) is (Subsumption) then the claim follows immediately from the induction hypothesis, because if \(b' \leq b\), then any \(\langle \langle b' \rangle \rangle\)-safe polynomial is \(\langle \langle b \rangle \rangle\)-safe. The last line cannot be (Shift) because this rule cannot be applied to a judgment with non-empty affine zone.

If the last line of the typing is (op-I), (if-1), or (down-I) then the claim follows from the induction hypothesis by using the appropriate operation from Figure 8; we present the (if-I) case as an example. Suppose the last line of the typing is (if-I), so that \(t = \text{if } s \text{ then } t_{0} \text{ else } t_{1}\). By the ISA

\(^{16}\)Recall from Proposition \(\|\) that since \(p\) is a potential polynomial, we can in fact assume that \(p(\bar{y}, w) = p(\ldots, y_{q}, \ldots, w)\).
we have that $s \sqsubseteq (P_s, p_s)$, and by the induction hypothesis that $t_i \sqsubseteq (P^i, p^i, p_i^\circ)$ for appropriate polynomials $P^i, p^i$, and $p_i^\circ = p_{i_1} \ldots p_{i_k}$. By Lemma 3 we have that

$$t \sqsubseteq (1 + P_s + (P^1(pot(f \star \mathbf{p})) + cost(f \star \mathbf{p})))$$

and the ISA tells us that $1 + P_s$ is a safe t.c. polynomial greater than $P^1 \cup P^2$, and $p_i$ is a safe t.c. polynomial greater than $p_i^1 \cup p_i^2$. The only other possibility is that the last line is $(\rightarrow E)$, and for that we break into cases depending on the exact form of $t$.

Case 1: $t = ft_1 \ldots t_k$. By Proposition 5 we can assume that we have typings $\Gamma; t_i \vdash b_i$. Since $f \notin \text{fv}(t_i)$ we have $\|b_i\|$-safe t.c. polynomials $p_i$ such that $t_i \sqsubseteq p_i$ and it follows from Lemma 3 that

$$t \sqsubseteq (1 + P_s + (P^1(pot(f \star \mathbf{p})) + cost(f \star \mathbf{p}))), (pot(pot(f \star \mathbf{p}))), (pot(pot(f \star \mathbf{p})))$$

Case 2: $t = st_1 \ldots t_m$ where w.l.o.g. $t_m$ is a plain affine definition of $f$ and $f \in \text{fv}(t_m)$. We can assume that $\Gamma; b' \vdash b \rightarrow b$ and $\Gamma; f : \gamma \vdash t_m : b'$ for some $b'$. Since $f \notin \text{fv}(st_1 \ldots t_{m-1})$ the ISA tells us that $st_1 \ldots t_{m-1} \sqsubseteq (P_s, p_s) : \|b'\| \rightarrow b$ where $(P_s, p_s)$ is $(\|b\|)$-safe. The induction hypothesis tells us that $t_m \sqsubseteq (P(pot(f \star \mathbf{p})) + cost(f \star \mathbf{p})), (pot(pot(f \star \mathbf{p})))$ so by Lemma 3 we conclude that

$$t \sqsubseteq (1 + P_s + (P^1(pot(f \star \mathbf{p})) + cost(f \star \mathbf{p})))$$

Since $p_s : \|b'\| \rightarrow \|b\|$ is $(\|b\|)$-safe and $p(w^{\langle b_0 \rangle}) : \|b'\| \rightarrow \|b'\|$-safe, we have that $p_s(pot(f \star \mathbf{p})) : \|b\|$ is $(\|b\|)$-safe, and hence that $pot(p_s(pot(f \star \mathbf{p})))$ is $(\|b\|)$-safe, completing the proof for this case.

Case 3: $t = (\lambda x_1 \ldots x_m) t_1 \ldots t_m$ where $s$ is a plain affine definition of $f$. By Proposition 8 we may assume that we have typings $\Gamma, \vec{\sigma} : \vec{b}; f : \gamma \vdash s : b$ and $\Gamma; t_i : \sigma_i$. The induction hypothesis tells us that $s \sqsubseteq (P_s(x, pot(f \star \mathbf{p})) + cost(f \star \mathbf{p})), p_s(x, pot(f \star \mathbf{p}))$ where $p_i = p_i(x)$ and the ISA tells us that $t_i \sqsubseteq (P^i, p_i^\circ)$. Using Lemma 3 and Proposition 10 we conclude that

$$t \sqsubseteq (2m + \sum_{i=1}^m P^i + P_s(val(p^1), \ldots, val(p^m), pot(f \star \mathbf{p}))) + cost(f \star \mathbf{p})))$$

where $p^i_s = p_i(val(p^1), \ldots, val(p^m))$. Since each $p^i_s : \|\sigma_i\|$ is tail$(\|\sigma_i\|)$-safe, $p^i_s$ is $(\|b_i\|)$-safe, and substituting safe polynomials into safe polynomials yields a t.c. denotation that is bounded by a safe polynomial (ATS Lemma 32), the claim is established.

5.3. Polynomial bounds for recursive terms.

5.3.1. Bounds in terms of recursion depth: the Unfolding Lemma. From the Decomposition Lemma we know that if $\Gamma, \vec{\sigma} : \vec{b}; f : \gamma \vdash t : b$ satisfies the ISA, then

$$t \sqsubseteq (P(\vec{v}, pot(f \star \mathbf{p}))) + cost(f \star \mathbf{p}), q \circ (r \lor pot(f \star \mathbf{p}))$$

\[17\text{Actually, bounded by a } \langle b \rangle\text{-safe polynomial; from now on we shall assume that the reader can insert the \"bounded by\" qualification as needed.} \]
where \( q = q(\vec{v}) \) is \( \langle \langle \mathbf{b} \rangle \rangle \)-strict and \( r = r(\vec{v}) \) is \( \langle \langle \mathbf{b} \rangle \rangle \)-chary (we have suppressed mention of the variables other than \( \vec{v} \) and \( f \)). Let \( X_t \) denote this t.c. denotation. Also define the (syntactic) substitution function
\[
\xi_t = \left[ \text{cost}(\text{val}(p_{\ell p})), \text{pot}(\text{val}(p_{\ell p}))/v_{ic}, v_{\ell p} \right]
\]
and set \( \xi_0^t = \text{id} \) and \( \xi^{n+1}_t = \xi^n_t \circ \xi_t \) (we write the syntactic substitution of the polynomial \( p \) for the variable \( x \) in the t.c. denotation \( X \) by \( X[p/x] \)). The point behind these functions is that if \( p(v_1, \ldots, v_k) \) is a polynomial, then
\[
(\mathbf{A}, v_1 \ldots v_k, p) \ast p_1 \cdots p_k = \text{dally} \left( 2k + \sum_{i=1}^{k} p_{ic}, p_\xi_t \right)
\]
by Proposition 10 and expressions of this form arise frequently in our analysis.

To analyze the of closures of the form \( t_\rho[f \mapsto (\text{rec}(\mathbf{0}^\ell)(\lambda r.f.\lambda \vec{v}.t))p] \) where \( t \) is a plain affine recursive definition of \( f \), we will actually need to analyze subterms of \( t \) under extensions of the environment indicated here. To that end, we make some definitions in order to simplify the statements of the coming claims.

**Definition 11.** Suppose \( \Gamma, v_1 : \mathbf{b}_1, \ldots, v_k : \mathbf{b}_k; f : \gamma \vdash t : \mathbf{b} \) satisfies the ISA. Define
1. \( \Gamma_\vec{v} = \Gamma, v_1 : \mathbf{b}_1, \ldots, v_k : \mathbf{b}_k; \)
2. \( C_{t,\ell} = \mathbf{a} \text{f rec}(\mathbf{0}^\ell)(\lambda r.f.\lambda \vec{v}.t); \)
3. \( T_{t,\ell} = \mathbf{a} \text{f } \lambda \vec{v}.\text{if } |\mathbf{0}^\ell| < |v_1| \text{ then } t \text{ else } \varepsilon; \)
4. \( \text{For } \rho \in \Gamma_\vec{v}-\text{Env}, \rho_{t,\ell} = \mathbf{a} \text{f } [f \mapsto C_{t,\ell}] \).

Notice that \( C_{t,\ell} \rho \downarrow T_{t,\ell} \rho_{t,\ell}+1 \) is an axiom of the evaluation relation. We write \( t_\rho \) for \( t_{\rho_{t,\ell}} \).

**Definition 12.** Suppose \( \Gamma_\vec{v}; f : \gamma \vdash t : \mathbf{b} \) satisfies the ISA, \( \Gamma^* = \Gamma_\vec{v}; f : \gamma \vdash t^* : \mathbf{b}^* \) is a subterm of \( t \), \( \rho^* \in \Gamma^*\text{-Env} \) is an extension of \( \rho \). The recursion-depth of \( t^* \rho^*_{t,\ell} \), \( \text{rdp}(t^* \rho^*_{t,\ell}) \) is defined to be the number of \( \text{rec} \) axioms \( C_{t,m} \rho \downarrow T_{t,m} \rho_{t,m}+1 \) in the evaluation derivation of \( t^* \rho^*_{t,\ell} \) when \( t^* \rho^*_{t,\ell} \downarrow z_\theta \) for some \( z_\theta \), and \( \text{rdp}(t^* \rho^*_{t,\ell}) = \infty \) otherwise.

The Unfolding Lemma establishes bounds on evaluating closures in terms of recursion depth. The proof is a nested induction: first on the recursion depth, and then on the shape of the plain affine definition. Because of the many cases its length may hide the simplicity of what is going on, so we make that explicit here: a careful calculation of the cost of one recursive call in the evaluation.

**Theorem 12 (Unfolding Lemma).** Suppose \( \Gamma_\vec{v}; f : \gamma \vdash t : \mathbf{b} \) satisfies the ISA. Let \( \xi = \xi_t \) be given as above. Suppose \( \rho \in \Gamma_\vec{v}-\text{Env}, \rho \in \parallel \Gamma_\vec{v} \parallel -\text{Env}, \rho \subseteq \bar{\rho}, \text{and that } \text{rdp}(t_\rho) = d < \infty. \text{ Then:} \)
1. If \( \mathbf{b} \) is computational,
\[
t_\rho \subseteq (d(10 + 3p_{1p}) + (d + 1)(2k + \sum_{i=1}^{k} p_{ic} + P(dq + r)), (d + 1)q + r) \xi^d \rho.
\]
2. If \( \mathbf{b} \) is oracular,
\[
t_\rho \subseteq (d(10 + 3p_{1p}) + (d + 1)(2k + \sum_{i=1}^{k} p_{ic} + P(q \vee r)), q \vee r) \xi^d \rho.
\]

**Proof.** The proof is by induction on \( d \). For the base case \( (d = 0) \) we prove the following claim:
Suppose $\Gamma^*_{\theta} ; f : \gamma \vdash t^* : b^*$ is a subterm of $t$ and take $X^*$ so that $t^* \subseteq X^*$ by the Decomposition Lemma. Suppose $\rho^* \in \Gamma^*_{\theta} \text{Env}$ is an extension of $\rho$, $\theta^* \in \| \Gamma^*_{\theta} \| \text{Env}$ is an extension of $\theta$, and $\rho^* \subseteq \theta^*$. If $\text{rdp}(t^* \rho^*_{t,\ell}) = 0$ then $t^* \rho^*_{t,\ell} \subseteq X^*[\varepsilon/f] \theta^*$ where $\varepsilon = \lambda^* \varepsilon.(0,0)$.

First let us see that this claim yields the desired bound when $d = 0$. It tells us that $t \rho_{t,\ell} \subseteq X_t[\varepsilon/f] \theta$.

Thus if $b$ is computational

$$t \rho_{t,\ell} \subseteq (P(\text{pot}(f * \overline{p})) + \text{cost}(f * \overline{p}), q + (r \lor \text{pot}(f * \overline{p})))[\varepsilon/f] \theta$$

$$= (P(0) + (2k + \sum p_{ic}), q + (r \lor 0)) \theta$$

$$\leq (2k + \sum p_{ic} + P(r), q + r) \xi^0 \theta.$$

The calculation is similar when $b$ is oracular.

We prove the claim by induction on the shape of $t^*$ (a plain affine definition of $f$ that satisfies the ISA). For each case of the induction, we import the notation from the corresponding case in the proof of the Decomposition Lemma. We give the details for a few cases, leaving the rest to the reader. The case in which $t^* = f t_1 \ldots t_k$ is not possible, because necessarily $\text{rdp}((f t_1 \ldots t_k) \rho^*_{t,\ell}) > 0$.

Case 1: $t^* = s$ then $t_0$ else $t_1$. Consider the subcase in which $s \rho^*_{t,\ell} \downarrow \varepsilon \theta$ (the other subcase is analogous). An analysis of the evaluation of $t^* \rho^*_{t,\ell}$ yields

$$\text{cost}(t^* \rho^*_{t,\ell}) = 1 + \text{cost}(s \rho^*_{t,\ell}) + \text{cost}(t_0 \rho^*_{t,\ell})$$

(by applying the ISA to $s$ and secondary induction hypothesis to $t_0$)

$$\leq (1 + P_s + \text{cost}(X_t[\varepsilon/f] \lor \text{cost}(X_{t_1}[\varepsilon/f]))) \theta^*$$

$$= \text{cost}(X^*[\varepsilon/f] \theta^*).$$

Furthermore, if $t^* \rho^*_{t,\ell} \downarrow z \theta$ then $t_0 \rho^*_{t,\ell} \downarrow z \theta$, so again by the secondary induction hypothesis we have that

$$z \theta \subseteq \text{pot}(X_{t_0}[\varepsilon/f]) \theta^* \leq \text{pot}(X_{t_0}[\varepsilon/f] \lor X_{t_1}[\varepsilon/f]) \theta^* = \text{pot}(X^*[\varepsilon/f] \theta^*).$$

The two facts together tell us that $t^* \rho^*_{t,\ell} \subseteq X^*[\varepsilon/f] \theta^*$.

Case 2: $t^* = s t_1 \ldots t_m$ where w.l.o.g. $t_m$ is a plain affine definition of $f$ and $f \in \text{fv}(t_m)$. By the secondary induction hypothesis we may assume that $t_m \rho^*_{t,\ell} \subseteq X_{t_m}[\varepsilon/f] \theta^*$ and following the notation of the Decomposition Lemma $s t_1 \ldots t_{m-1} \subseteq (P_s, p_s)$.

Suppose $(s t_1 \ldots t_{m-1}) \rho^*_{t,\ell} \downarrow (\lambda x.s') \theta'$ (the case of evaluating to an oracle is similar), $t_m \rho^*_{t,\ell} \downarrow z'' \theta''$, and $s' \theta'[x \mapsto z'' \theta''] \downarrow z \theta$ (these evaluations are all defined because they are subevaluations of that of $t^* \rho^*_{t,\ell}$). By definition of $\subseteq$ we have that $s' \theta'[x \mapsto z'' \theta''] \subseteq p_s(\text{pot}(X_{t_m}[\varepsilon/f] \theta^*))$. An analysis of the evaluation of $t^* \rho^*_{t,\ell}$ yields

$$\text{cost}(t^* \rho^*_{t,\ell}) = 1 + \text{cost}((s t_1 \ldots t_{m-1}) \rho^*_{t,\ell}) + \text{cost}(t_m \rho^*_{t,\ell}) + \text{cost}(s' \theta'[x \mapsto z'' \theta''])$$

$$\leq 1 + P_s + \text{cost}(X_{t_m}[\varepsilon/f]) + \text{cost}(p_s(\text{pot}(X_{t_m}[\varepsilon/f] \theta^*)))$$

$$= \text{cost}(X^*[\varepsilon/f] \theta^*).$$

And if $t^* \rho^*_{t,\ell} \downarrow z \theta$ then $s' \theta'[x \mapsto z'' \theta''] \downarrow z \theta$ so we conclude that

$$z \theta \subseteq \text{pot}(p_s(\text{pot}(X_{t_m}[\varepsilon/f] \theta^*))) = \text{pot}(X^*[\varepsilon/f] \theta^*).$$
CASE 3: \( t^* = (\lambda x_1 \ldots x_m.s)^f \) where \( s \) is a plain affine definition of \( f \). Say that \( t_i\rho_{t,\ell}^* \downarrow z_i[\theta_i'] \) and \( sp_{t,\ell}^*[x_i \mapsto z_i[\theta_i']] \downarrow z\theta \) (the evaluations of the subterms and body are all defined because they are sub-evaluations of \( t^*\rho_{t,\ell}^* \)). Following the notation of the Decomposition Lemma we have \( t_i \subseteq (P^i, p^i) \), so \( z_i[\theta_i'] \subseteq \text{pot} \ p^i \). By the secondary induction hypothesis we have that \( sp_{t,\ell}^*[x_i \mapsto z_i[\theta_i']] \subseteq X_s[\varepsilon/f]g^*[x_i \mapsto val(p^i \ g^*)] \). An analysis of the evaluation derivation of \( t^*\rho_{t,\ell}^* \) yields
\[
\begin{align*}
\text{cost}(t^*\rho_{t,\ell}^*) &= 2m + \sum \text{cost}(t_i\rho_{t,\ell}^*) + \text{cost}(sp_{t,\ell}^*[x_i \mapsto z_i[\theta_i']]) \\
&\leq 2m + \sum P^i g^* + \text{cost}(X_s[\varepsilon/f]g^*[x_i \mapsto val(p^i \ g^*)]) \\
&= \text{cost}(X^*[\varepsilon/f]g^*)
\end{align*}
\]
where \( p'_j = p_j(\ldots, val(p^i), \ldots) \) and
\[
\begin{align*}
z\theta &\subseteq \text{pot} (X_s[\varepsilon/f]g^*[x_i \mapsto val(p^i \ g^*)]) \\
&= p_s(\bar{x}, \text{pot}(f \star \bar{p}))[\varepsilon/f]g^*[x_i \mapsto val(p^i \ g^*)] \\
&= p_s(\ldots, p^i, \ldots, \text{pot}(f \star \bar{p}))[\varepsilon/f]g^* \\
&= \text{pot}(X^*[\varepsilon/f]g^*).
\end{align*}
\]
Thus \( t^*\rho_{t,\ell}^* \subseteq X^*[\varepsilon/f]g^* \). This completes the proof of the Unfolding Lemma.

For the induction step, suppose that \( \text{rdp}(t\rho_{t,\ell}) = d + 1 \). We show just the case when \( b \) is computational; the oracular case is similar. Set
\[
Y = (d(10 + 3p_{1p}) + (d + 1)(2k + \sum p_{ic} + P(dq + r)), (d + 1)q + r)\xi^d.
\]
We will prove the following claim:

Suppose \( t^*, X^*, \rho^*, \) and \( g^* \) are as in the claim for the base case \( d = 0 \) and suppose
\[
X^* = (P^*(\text{pot}(f \star \bar{p}^*) + \text{cost}(f \star \bar{p}^*)), \text{cost}(\text{pot}(f \star \bar{p}^*))).
\]
If \( \text{rdp}(t^*\rho_{t,\ell}^*) = d + 1 \) then
\[
t^*\rho_{t,\ell}^* \subseteq \text{dally}(10 + 3p_{1p}, X^*[\lambda_s \bar{v}, Y/f])[\xi^*].
\]
Again we first show that this claim is sufficient for establishing desired bound for the induction step. From it we calculate
\[
t_{\rho_{t,\ell}} \subseteq \text{dally}(10 + 3p_{1p}, X_t[\lambda_s \bar{v}, Y/f])[\xi]
\]
\[
= (10 + 3p_{1p} + P(\text{pot}(f \star \bar{p})), q + (r \lor \text{pot}(f \star \bar{p}))[\lambda_s \bar{v}, Y/f])[\xi]
\]
\[
\leq (10 + 3p_{1p} + P(\text{pot}(Y\xi)), (2k + \sum p_{ic} + \text{cost}(Y\xi)), q + (r \lor \text{pot}(Y\xi)))[\xi]
\]
\[
\leq \left(10 + 3p_{1p} + P(((d + 1)q + r)\xi^{d+1}) + 2k + \sum p_{ic} +
\right.
\]
\[
n (d(10 + 3p_{1p}) + (d + 1)(2k + \sum p_{ic} + P(dq + r)))\xi^{d+1},
\]
\[
q + (r \lor ((d + 1)q + r)\xi^{d+1})[\xi]
\]
\[
\leq ((d + 1)(10 + 3p_{1p}) + (d + 2)(2k + \sum p_{ic} + P((d + 1)q + r))(d + 2)q + r)\xi^{d+1}[\xi]
\]
5.3.2. Bounds on recursion depth: the Termination Lemma.

Establishing the claim is very similar to the $d = 0$ claim; we present just one key case here. Suppose that $t^* = ft_1 \ldots t_k$ and $t_i \rho_{t,\ell} \downarrow z_i \theta_i$. Also take $p_i^*$ so that $t_i \sqsubseteq p_i^*$ so that $X^* = f \star \overline{p^*}$. Then analysing the evaluation derivation we have that

$$\text{cost}(t^* \rho_{t,\ell}^*) = (2 + k + \sum p_i^{*} \theta^* + (8 + 3p_{1p})\theta^* + \text{cost}(t \rho_{t,\ell+1}[v_i \mapsto z_i \theta_i]))$$

(the $8 + 3p_{1p}$ term is from the clock test when evaluating $T_i.t \rho_{t,\ell+1}[v_i \mapsto z_i \theta_i]$). Since $\text{rdp}(t^* \rho_{t,\ell}^*) = d + 1$ and the evaluation of $t \rho_{t,\ell+1}[v_i \mapsto z_i \theta_i]$ is a subevaluation we have that $\text{rdp}(t \rho_{t,\ell+1}[v_i \mapsto z_i \theta_i]) = d$ and so the main induction hypothesis applies to let us conclude that $t \rho_{t,\ell+1}[v_i \mapsto z_i \theta_i] \sqsubseteq Y \rho[v_i \mapsto \text{val}(p_{1p} \theta^*)]$. Thus

$$\text{cost}(t^* \rho_{t,\ell}^*) \leq (2 + k + \sum p_i^{*} + (8 + 3p_{1p}) + \text{cost}(Y \rho[v_i \mapsto \text{val}(p_{1p} \theta^*)]) \theta^*$$

$$= (10 + 3p_{1p} + \text{cost}((\mathbf{A}_s \bar{v}.Y) \star \overline{p^*}) \theta^*)$$

$$= (10 + 3p_{1p} + \text{cost}(f \star \overline{p^*}))[\mathbf{A}_s \bar{v}.Y/f] \theta^*.$$

Furthermore, if $t^* \rho_{t,\ell} \downarrow z \theta$ then $t \rho_{t,\ell+1}[v_i \mapsto z_i \theta_i] \downarrow z \theta$ and so

$$z \theta \sqsubseteq_{\text{pot}} \text{pot}(Y \rho[v_i \mapsto \text{val}(p_{1p} \theta^*)]) = \text{pot}((\mathbf{A}_s \bar{v}.Y) \star \overline{p^*}) \theta^* = \text{pot}(f \star \overline{p^*}[\mathbf{A}_s \bar{v}.Y/f]) \theta^*.$$

We conclude that $t^* \rho_{t,\ell} \sqsubseteq \text{dally}(10 + 3p_{1p}, X^*[\mathbf{A}_s \bar{v}.Y/f]) \theta^*$. \hfill \Box

**Corollary 13** (Polynomial Unfolding Lemma). Suppose $\Gamma; v : t : b$ satisfies the ISA, $\rho \in \Gamma_{\bar{v}}\text{-Env}$, $\rho \sqsubseteq \varrho$. Then there is a $\langle \varrho \rangle$-safe time-complexity polynomial $\varphi(\bar{v}, d(\langle b \rangle))$ such that for all $\ell$ such that $\text{rdp}(t \rho_{\ell}) < \infty$, $t \rho_{\ell} \sqsubseteq \varphi(\bar{v}, \text{rdp}(t \rho_{\ell})).$

**Proof.** Using the Unfolding Lemma, it suffices to show that the map $v_{\varrho} : \langle b \rangle$ is a safe polynomial w.r.t. $v_{\varrho} : \langle b \rangle, d : \langle b \rangle$. This is precisely the content of the One-step and n-step Lemmas of ATS (Lemmas 44 and 45). \hfill \Box

5.3.2. Bounds on recursion depth: the Termination Lemma. Next we prove the Termination Lemma, which establishes a polynomial bound on $\text{rdp}(t \rho_{\ell})$; this will allow us to apply the Unfolding Lemma. Since we cannot \textit{a priori} assume that we have an evaluation of $t \rho_{\ell}$, we need a formalism that allows us to refer to “non-terminating evaluations.” We sketch the idea here. Introduce an additional axiom $[\text{?}].$ Define the truncated evaluation relation $s \rho \downarrow z \theta$ just like the usual evaluation relation $\downarrow$, but with an additional axiom:

$$([\text{crec}(0^d)(\lambda v.f.\lambda \bar{v}.t)] \rho \sqsubseteq [\text{?}])$$

Furthermore, for each inference rule of $\downarrow$ we add additional rules that say that if one of the hypotheses evaluates to $[\text{?}]$, then the remaining hypotheses (to the right) are ignored and the conclusion evaluates to $[\text{?}]$. For example, we have the additional inferences

$$\frac{rs \rho \sqsubseteq [\text{?}]}{(rs) \rho \sqsubseteq [\text{?}]}, \quad \frac{(rs) \rho \sqsubseteq [\text{?}]}{s \rho \downarrow (\lambda x.r') \theta' \sqsubseteq [\text{?}].}$$

We will use these truncated evaluations to establish a bound on the recursion depth of ordinary evaluations. The idea is to establish a uniform bound on the size of any “clock test” in any truncated evaluation of $t \rho_{\ell}$. Once we do that, we can consider a truncated evaluation with recursion depth greater than this bound. In such a evaluation, either the recursion terminates normally or the clock test will fail before any truncation axiom can be evaluated. Either way, there are no truncation axioms, so in fact we have an ordinary evaluation with the given bound on its recursion depth. Thus we will be able to apply the Unfolding Lemma.
First we make an observation about evaluating \( \text{crec} \) terms. The case of interest is a closure of the form \((\text{crec}(0^N)(\lambda_{y_1}f.\lambda_{y_2}t_1 \ldots t_k)\rho)\) of base type. The first (lowest) evaluation of \( t \) evaluates the closure \( tp_{t+1}[v_1 \mapsto z_{\ell,i}] \) where \( t,\rho \downarrow z_{\ell,i} \). Furthermore, if \( m \geq \ell \) then the evaluation of \( tp_{t,m+1}[v_1 \mapsto z_{m,i}^{\ell+1}] \) has the form

\[
\begin{align*}
\vdots \\
\frac{``m + 1 < |z_{m+1,i}|'' \quad tp_{t,m+2}[v_1 \mapsto z_{m+1,i}^{\ell+1}] \mid z\theta}{T_{m+1}p_{t,m+2}[v_1 \mapsto z_{m+1,i}^{\ell+1}] \mid z\theta} \\
\end{align*}
\]

\[
(f s_1 \ldots s_k)p_{t,m+1}[v_1 \mapsto z_{m,i}^{\ell+1}] [\tilde{y} \mapsto z'^{\ell+1}] \mid z\theta
\]

\[
\vdots
\]

\[
tp_{t,m+1}[v_1 \mapsto z_{m,i}^{\ell+1}] \mid z\theta
\]

where:

- The \( y \)'s are the \( \xi \)-bound variables in \( t \);
- \( f\tilde{s} \) is one of the complete applications of \( f \) in \( t \);
- \( s_i p_{t,m+1}[v_1 \mapsto z_{m,i}^{\ell+1}] [\tilde{y} \mapsto z'^{\ell+1}] \downarrow z_{m+1,i}^{\ell+1} \) (\( f \notin \text{fvs}(s_i) \), so the evaluation of \( s_i \) cannot involve a truncation axiom);
- We assume that the evaluation of \( f\tilde{s} \) hidden by the \( \ldots \) does not use a truncation axiom to evaluate the \( \text{crec} \) term to which \( f \) evaluates.

This description of the evaluation is easy to prove by induction on the shape of \( t \). What we must do to prove the Termination Lemma is to get a handle on the sizes of the values \( z_{m,i} \) for \( m \geq \ell \).

**Lemma 14.** Suppose that \( \Gamma_G; f : \gamma \vdash t : \mathbb{b} \) satisfies the ISA, \( \rho \in \Gamma_G\mathbb{v}\text{-Env} \), \( \theta \in \|\Gamma_G\|\text{-Env} \), \( \rho[v_i \mapsto z_{\ell,i}] \| \theta_i \). Consider any truncated evaluation of \( tp_{t,\ell+1}[v_i \mapsto z_{\ell,i}] \). Referring to the notation just introduced, for any \( m \geq \ell \), \( |z_{m,i}| \leq v_{ip\ell}^m \cdot |\theta| \).

**Proof sketch.** The proof is by induction on \( m - \ell \) with the base case given by assumption. For the induction step, we first bound \( |z_{\ell+1,i}| \). Here we need another claim about subterms of \( t \) as in the proof of the Unfolding Lemma:

Suppose that \( \Gamma_G; f : \gamma \vdash t^* : \mathbb{b}^* \) is a subterm of \( t \) and take \( X^* = (P(\text{pot}(f \ast \tilde{p})) \ast \text{cost}(f \ast \tilde{p})) \ast (P(\text{pot}(f \ast \tilde{p})) \ast \text{cost}(f \ast \tilde{p})) \) by the Decomposition Lemma so that \( t^* \subseteq X^* \). Suppose \( \rho^* \in \Gamma_G\mathbb{v}\text{-Env} \) is an extension of \( \rho \), \( \theta^* \in \|\Gamma_G\|\text{-Env} \) is an extension of \( \theta \), and \( \rho^*[v_i \mapsto z_{\ell,i}] \| \theta^* \). Then using notation analogous to that just introduced, to just introduced, in the evaluation of \( t^* p_{t,\ell+1}^* : X_{\ell+1,i}^* \leq \rho_{ip\ell}^m \theta^* \).

The proof of the claim is by induction on the shape of \( t^* \) and is by now routine. Applying the claim to \( t \) we conclude that \( |z_{\ell+1,i}| \leq \rho_{ip\ell} \theta \) and so \( \rho[v_i \mapsto z_{\ell+1,i}] \| \theta_i \| v_{ip\ell} \| \text{val}(p_{ip\ell}) \). So for \( m \geq \ell + 1 \) the induction hypothesis tells us that \( |z_{m,i}| \leq v_{ip\ell}^m \cdot |\theta| \| v_{ip\ell} \| \text{val}(p_{ip\ell}) \) \( = v_{ip\ell}^m \cdot |\theta| \). \( \square \)

**Theorem 15** (Termination Lemma). Under the assumptions of Lemma 14, \( \text{rdp}(tp_{t,\ell+1}[v_i \mapsto z_{\ell,i}]) \leq (2 + |p_{ip\ell}|)\theta \).

**Proof.** A key component of the One-step and \( n \)-step Lemmas of ATS (Lemmas 44 and 45) is that we can take \( p_{ip\ell} \) such that \( p_{ip\ell}(\xi) = p_{ip} \) (this makes critical use of the restriction that if \( b_i \leq b_j \) then \( b_i \) is oracular). Hence \( v_{ip\ell}^d \theta = p_{ip\ell} \theta \) for any \( d \geq 2 \).

Suppose we choose \( d \geq 2 \) such that \( \ell + d - 1 \geq p_{ip\ell} \theta \). Consider any truncated evaluation of \( tp_{t,\ell+1}[v_i \mapsto z_{\ell,i}] \) of recursion depth \( d \). Such an evaluation recursively evaluates \( tp_{t,m+1}[v_i \mapsto \theta_{m,i}] \) for \( m = \ell, \ldots, \ell + d - 1 \). By Lemma 14 we have that \( |z_{\ell+d-1,i}| \leq v_{ip\ell}^{d+1-\ell} \theta = p_{ip\ell}^{d+2-\ell} \theta = p_{ip\ell} \leq \ell + 2 \).
d − 1. Thus either the evaluation terminates normally (i.e., the evaluation of \( t p_{\ell, m+1}[v_i \mapsto z_{m,i} \theta_{m,i}] \) does not recursively evaluate \( f \) at all for some \( \ell \leq m < \ell + d - 1 \)) or one of the clock tests fails, thereby terminating the evaluation. Either way we have a standard evaluation of \( t p_{\ell, \ell+1}[v_i \mapsto z_{\ell,i} \theta_{\ell,i}] \) of recursion depth \( \leq d \). Taking \( d = (2 + p_{1p})q \) yields the theorem. □

5.3.3. The Soundness Theorem.

**Theorem 16.** For every ATR term \( \Gamma; \Delta \vdash t : \tau \) there is a tail(\( |\tau|\))-safe t.c. denotation \( X \) of type \( |\tau| \) w.r.t. \( \|\Gamma; \Delta\| \) such that \( t \sqsubseteq X \).

**Proof.** The proof is by induction on terms; for non-crec terms use Lemma [3]. Let \( s \) be the term \( \Gamma; \_ \vdash \text{crec}(\ell)(\lambda \gamma \cdot f. \lambda \vec{v} . t) : b \rightarrow b \). Suppose \( \rho \in \Gamma\cdot\text{-Env}, \varrho \in \|\|\cdot\|-\text{Env}, \rho \subseteq \varrho \). Since \( \text{sdp}(\ell \vec{v} . t) \rho_{\ell+1} \) in one step, if \( (\lambda \vec{v} . t) \rho_{\ell+1} \sqsubseteq \chi \) then \( \text{sdp} \subseteq \text{dally}(1, \chi) \), so we focus on characterizing such time-complexities \( \chi \). Unwinding the definition of \( \sqsubseteq \) we have \( (\lambda \vec{v} . t) \rho_{\ell+1} \sqsubseteq \chi \) if whenever \( z_i \theta_i \sqsubseteq \text{pot} \) \( p_i \) (\( p_i \) is an arbitrary potential here, not necessarily a polynomial), we have that:

1. \( 1 \leq \text{cost}(\chi), \text{cost}(\text{pot}(\chi)p_1), \ldots, \text{cost}(\text{pot}(\ldots(\text{pot}(\chi)p_1)p_2) \ldots) p_{k-1} \).
2. \( T \rho_{\ell+1}[v_i \mapsto z_i \theta_i] \sqsubseteq \text{pot}(\ldots(\text{pot}(\text{pot}(\ldots(\text{pot}(\chi)p_1)p_2) \ldots) p_{k-1}) \).

Since \( z_i \theta_i \sqsubseteq \text{pot} \) \( p_i \) we have that \( \rho[v_i \mapsto z_i \theta_i] \sqsubseteq \varrho[v_i \mapsto \text{val}(\rho)] \). Let \( \rho' \) and \( \varrho' \) denote these extended environments. By the Termination Lemma (Theorem [15]) we have that \( \text{rdp}(\rho'_{\ell+1}, \varrho'_{\ell+1}) \leq (2 + p_{1p})q \), where \( p_{1p} \) is the \( \langle b_1 \rangle \)-safe polynomial given by the Decomposition Lemma (Theorem [11] for \( t \). By the Polynomial Unfolding Lemma (Corollary [13]) there is a \( b \)-safe polynomial \( \varphi(\vec{v}, d'^{(b_1)}) \) such that

\[
\rho'_{\ell+1} \sqsubseteq \varphi(\vec{v}, p_{1p} + 2) \varrho' \text{ and hence }
\]

\[
T \rho'_{\ell+1} \sqsubseteq \text{pot}(\ldots(\text{pot}(\text{pot}(\ldots(\text{pot}(\chi_2 \vec{v} . \varphi(\vec{v}, p_{1p} + 2)) \varrho))p_1)p_2) \ldots) p_{k-1} \).
\]

Since \( \text{cost}(\chi_2, X) = 1 \) for any \( x \) and \( \chi_2 \) and \( p_i \) were chosen arbitrarily, we conclude that

\[
(\lambda \vec{v} . t) \rho_{\ell+1} \sqsubseteq (\chi_2 \vec{v} . \text{dally}(8 + v_{1p}, \varphi(\vec{v}, p_{1p} + 2))) \varrho.
\]

Since \( \rho \) and \( \varrho \) were chosen arbitrarily, we can therefore conclude that

\[
\text{crec}(\ell)(\lambda \gamma \cdot f. \lambda \vec{v} . t) \sqsubseteq \text{dally}(1, \chi_2 \vec{v} . \text{dally}(8 + v_{1p}, \varphi(\vec{v}, p_{1p} + 2))),
\]

and by Propositions [3] and [9] this is a safe t.c. polynomial. □

**Definition 13.** For an ATR term \( \Gamma; \Delta \vdash t : \tau \) we define the time-complexity interpretation of \( t \), \( ||t|| \), to be the t.c. denotation of Theorem [16].

**Corollary 17** (Soundness for ATR). For every ATR term \( \Gamma; \Delta \vdash t : \tau \), \( ||t|| \) is tail(\( |\tau|\))-safe w.r.t. \( \|\Gamma; \Delta\| \) and \( t \sqsubseteq ||t|| \).

6. Second-order polynomial bounds

Our last goal is to connect time-complexity polynomials to the usual second-order polynomials of Kapron and Cook [15] and show that any ATR program is computable in type-2 polynomial time. The polynomial here will be in the lengths of the program’s arguments, and hence we need a semantics of lengths, which lives inside the simple type structure over the time-complexity base types. We give a brief outline here, referring the reader to Section 2 of ATS for full details.

For each ATR-type \( \sigma \) we define \( |\sigma| \) by

\[
|N_L| = T_L \quad |\sigma \rightarrow \tau| = |\sigma| \rightarrow |\tau|.
\]
We are concerned primarily with two kinds of objects in these length-types: the lengths of the meanings of ATR programs and the meanings of second-order polynomials. For the former, recall that the interpretation of the ATR base types is $K = \{0, 1\}$; for any $a \in K$, $|a|$ is defined as expected and the length of a function is defined as follows:

**Definition 14.** If $f$ is a type-1 $k$-ary function, set

$$|f| = \lambda n_1 \ldots n_k. \max \{|f(v_1, \ldots, v_k)| : \forall i(|v_i| \leq n_i)\}.$$  

The notion of length for objects of type-level $\geq 2$ is much more difficult to pin down; as we do not need it here, we omit any discussion of it.

With the notion of length in hand, we can give the definition of $\|\alpha\|$ promised in Theorem 5.

**Definition 15.** If $\alpha^{(b_1, \ldots, b_k)}$ is an oracle symbol, then

$$\|\alpha^{(b_1, \ldots, b_k)}\| = (1, \lambda n_1^{(b_1)}(1, \lambda n_2^{(b_2)}(\ldots (1, \lambda n_k^{(b_k)}(1 \vee |\alpha|(|\vec{n}|), |\alpha|(|\vec{n}|)) \ldots)))).)$$  

The second-order length polynomials are defined by the typing rules in Figure 9. There is nothing surprising here, and the intended interpretation is just as expected. As with the time-complexity types, we define $|\sigma| \propto |\tau|$ iff $|\sigma| \propto \sigma$ and $|\sigma| \leq |\tau|$ iff $|\sigma| \leq \tau$. In these rules, a type-context $\Sigma$ is an assignment of length-types to variables. For an ATR type-context $\Gamma; \Delta$ set $|\Gamma; \Delta| = \bigcup_{(x; \sigma) \in \Gamma; \Delta} \{|x| : |\sigma|\}$ where for each ATR variable $x$, $|x|$ is a new variable symbol.

Our real concern is with closed ATR programs of the form $\lambda \vec{x}.t$ where $t$ is of base type. We know that $\lambda \vec{x}.t \subseteq \lambda x_{\vec{x}}.\|t\| = (1, \lambda x_{1p}(\ldots (1, \lambda x_{kp}(P, P)) \ldots))$ where $P$ and $p$ are base-type polynomials over the potential variables $\vec{x}$. Since the time-complexity polynomial calculus is just a simple applied $\lambda$-calculus, it is strongly normalizing, and so we can assume that the polynomials are in normal form. Thus we start with an analysis of time-complexity polynomials in normal form:

**Lemma 18.** Suppose $x_1 : \langle |\sigma_1| \rangle, \ldots, x_k : \langle |\sigma_k| \rangle \vdash p : \gamma$ is a t.c. polynomial in normal form. Then $p$ has one of the following forms:

1. $0^n$ for some $n \geq 0$;
2. $\text{pot}(\ldots \text{pot}(\text{pot}(vq_1)q_2)\ldots)q_i$ where $v$ is either an oracle symbol or one of the $x_j$’s and each $q_i$ is in normal form and of potential type (this term is of potential type);
3. $\text{cost}(\ldots \text{pot}(\ldots \text{pot}(vq_1)q_2)\ldots)q_i$ where $v$ is either an oracle symbol or one of the $x_j$’s and each $q_i$ is in normal form and of potential type (this term is of cost type);
(4) \( q_1 \ast q_2, q_1 + q_2, q_1 \lor q_2 \) where each \( q_i \) is in normal form and of base type (this term is of base type);
(5) \( \alpha^{|p|} \);
(6) \( (q_0, q_1) \) where \( q_0 \) is in normal form and of cost type and \( q_1 \) is in normal form and of potential type (this term is of time-complexity type);
(7) \( \text{pot}(\ldots \text{pot}(\text{pot}(vq_1)q_2)\ldots)q_f \) where \( v \) is either an oracle symbol or one of the \( x_j \)'s and each \( q_i \) is in normal form and of potential type (this term is of time-complexity type).

Note that (2) includes the special case \( x_j \) when \( \sigma_j \) is a base type and in (2), (3), and (7), \( \ell \) may be strictly less than the arity of \( v \).

Proof. By induction on the typing derivation. \qed

Proposition 19. Suppose \( p(x_1, \ldots, x_k) \) is as in Lemma 18; \( \alpha_i^{q_i} \) an oracle symbol for \( i = 1, \ldots, k \). Then \( p(\text{pot}(|\alpha_1|), \ldots, \text{pot}(|\alpha_k|)) \) is a second-order polynomial in \( |\alpha_1|, \ldots, |\alpha_k| \).

Combining the Soundness Theorem (Corollary 17) with Proposition 19 yields:

Theorem 20. If \( \Gamma \vdash t : \tau \), then \( t \) is computable in type-2 polynomial time.

A word of caution in interpreting this result is in order. The basic feasible functionals of Mehlhorn [18] and Cook and Urquhart [3] are an extension of polynomial-time functions to higher type. They live in the full (set-theoretic) type structure and for type-level \( \leq 2 \) are defined as follows. The basic model is an oracle Turing machine with function oracles, and the cost of an oracle query is the length of the answer. A functional \( F(f, x) \) of type-level \( \leq 2 \) is basic feasible if it is computed by such an oracle Turing machine with oracle \( f \) in time \( p(|f|, |x|) \), where \( p \) is a second-order polynomial (this is the characterization of Kapron and Cook [15]; Ignjatovic and Sharma [13] give a similar characterization for unit-cost oracle queries). Now, ATR is not interpreted in the full type structure but rather in the well-tempered semantics discussed in Section 4.1. Thus, we have not quite yet characterized the basic feasible functionals. However, on ATR-types that are both strict and predicative (see Definition 2), the well-tempered semantics agrees with the full type structure (recalling the discussion after Definition 2 the relevant point here is that no restrictions are made on function spaces of strict and predicative type). Thus we conclude:

Theorem 21. If \( \Gamma \vdash t : \tau \), all variables of \( t \) are of strict and predicative type, and \( t \) contains no oracle symbols, then \( t \) defines a basic feasible functional.

In fact, some ATR programs compute function(al)s that are not basic feasible but are nonetheless second-order polynomial-time computable according to Theorem 20. For example, consider the following ATR program for the primitive recursion on notation combinator (roughly, \texttt{foldr} for binary strings):

\[
\text{val } \text{prn} : ((\mathbb{N}, \rightarrow \mathbb{N} \rightarrow \mathbb{N}, \rightarrow \mathbb{N}, \rightarrow \mathbb{N}, \rightarrow \mathbb{N})^2, \mathbb{N}) \rightarrow (\mathbb{N}, \rightarrow \mathbb{N} \rightarrow \mathbb{N}, \rightarrow \mathbb{N})
\]

\[
\text{fn } f 0, f 1, x \Rightarrow
g \text{letrec } F : \mathbb{N}, \rightarrow \mathbb{N} \rightarrow \mathbb{N} \rightarrow \mathbb{N} =
\text{fn } b x' \Rightarrow
g \text{if } x' \text{ then } (t_0 x') \text{ if } t_0 x' \text{ then } f 0 (d x') \mathbb{y} (F b (d x'))
\text{else } f 1 (d x') \mathbb{y} (F b (d x'))
\text{else } a
\text{in } F c_0 x x \text{ end}
\]

This combinator is not basic feasible, because in the full type structure it could be applied to arguments with non-trivial growth rates, and this would lead out of the realm of feasibility. However,
in ATR the types of the arguments control the growth rates of the functions to which it is applied (specifically, the type of the function argument ensures that it has a “small” growth rate in terms of the size of the recursive call). Thus we can have our cake and eat it too: we can define natural programming combinators (like prn), but the type system will keep us from using them in a way that results in infeasible computations.

7. Concluding remarks

In ATS we introduced the formalism ATR which captures the basic feasible functionals at type-levels \( \leq 2 \). In the current paper we have extended the formalism to include a broad range of affine recursion schemes (plain affine recursive definitions) that allow for more natural programming and demonstrated the new formalism by implementing lists of binary strings and insertion- and selection-sort. We have extended the original time-complexity semantics of ATS to handle the more involved programs expressible via plain affine recursion and shown that these new programs do not take use out of the realm of feasibility. We conclude by indicating some possible extensions and future research directions:

**Branching recursion.** This paper has focused on affine (one-use) recursions, and of course there are feasible algorithms that do not fit this mold. Especially germane to the examples of this paper are sorting algorithms such as merge-sort and quick-sort that are based on branching recursions. Let us consider the latter to see some of what would be involved in adding branching recursion to an ATR-like language. Here is a functional version of quick-sort over lists:

```plaintext
val quicksort = fn xs ⇒ letrec qsort = fn ys ⇒ if (length ys) ≤ 1 then ys else let val (pivot, small, big) = partition ys
    in append (qsort small) (cons pivot (qsort big)) end
    in qsort xs end
```

We assume that \( small \) is the list of items in \( ys \) with values \( \leq pivot \) (excluding the pivot item itself), and \( big \) is the list of items in \( ys \) with values \( > pivot \).

The tightest upper bounds on the sizes of the individual arguments are \( |small| < |ys| \) and \( |big| < |ys| \), and this only allows us to extract exponential upper bounds on the run-time of this definition. In order to establish a polynomial run-time bound one also needs to know that that the arguments of the two branches of the recursion satisfy the joint size restriction \( |small| + |big| < |ys| \). It is hard to see how to gracefully assert this sort of joint size bound using ATR-style types and combinators. Another problem is that in a recursive definition, it may be difficult to know which of the various recursive calls can together form a set of branching calls, and hence it may be difficult to know what sets of joint size constraints one needs to satisfy to guarantee a polynomial run-time.

Rather than attempting to handle general feasible branching recursions, we propose investigating combinators that express particular flavors of branching recursions that work well with ATR-style types and deal with the problems noted above. Here is a reworked version of quick-sort using a possible such combinator, inspired by Blelloch and colleagues’ work on the parallel programming language NESL [5, 4]:

```plaintext
val quicksort = fn xs ⇒ letrec qsort = fn ys ⇒ if (length ys) ≤ 1 then ys else let val (zs, part_idx) = partition ys
    in qsort zs end
```

If Proposition 23.

is quite plausible that a combinator like map occurs in one place where typing has a chance of constraining its size. Based on this, we claim it recursions. In fact, NESL uses a parallel map on NESL we know that such a combinator can express a great many useful divide-and-conquer

28 TWO ALGORITHMS IN SEARCH OF A TYPE-SYSTEM

Lazy ATR. A version of ATR with lazy evaluation would be very interesting, regardless of whether the constructors are strict or lazy (yielding streams). There are many technical challenges in analyzing such a system but we expect that the general outline will be the approach we have used in this paper. Of course one can implement streams in the current call-by-value setting in standard ways (raising the type-level), but a direct lazy implementation of streams is likely to be more informative. We expect the analysis of such a lazy-ATR to require an extensive reworking of the various semantic models we have discussed here and in ATS.

Real-number algorithms. ATR is a type-2 language, but here we have focused on type-1 algorithms. We are interested in type-2 algorithms, specifically in real-number algorithms as discussed in, e.g., Ko [16], where real numbers are represented by type-1 oracles. This can be done in either a call-by-value setting in which algorithms take a string of length \( n \) as input and return something like an \( n \)-bit approximation of the result, or a lazy setting in which the algorithm returns bits of the result on demand. Combined with lazy constructors, the latter would allow us to view real numbers themselves as streams; in particular, since real numbers would be base-type objects, we could look at operators on real functions.

APPENDIX A. EQUIVALENCE OF THE OPERATIONAL SEMANTICS AND THE ABSTRACT MACHINE SEMANTICS OF ATS

Here we sketch the proof of equivalence between the abstract-machine semantics for ATR in ATS and the evaluation-derivation semantics we have used here. We refer the reader to ATS for a detailed definition of the abstract machine. The abstract machine semantics works with configurations of the form \( \langle t, \rho, \kappa \rangle \), where \( t \) is an expression, \( \rho \) an environment, and \( \kappa \) a (defunctionalized) continuation, and defines a transition relation \( c \rightsquigarrow c' \) between configurations. Continuations are defined as a sequence of keywords, expressions, and environments, always ending in the keyword halt. If \( \kappa \) and \( \kappa' \) are two continuations, we define \( \kappa \kappa' \) to be the continuation obtained by deleting the keyword halt from \( \kappa \) and then concatenating \( \kappa' \) to the result. For configurations \( c \) and \( c' \) we write \( c \rightsquigarrow^n c' \) if \( c = c_0 \rightsquigarrow c_1 \rightsquigarrow \cdots \rightsquigarrow c_n = c' \) and \( c \rightsquigarrow^n c' \) if \( c \rightsquigarrow^n c' \) for some \( n \). In the following, \( z \) denotes a value.

Lemma 22. If \( \langle t, \rho, \kappa \rangle \rightsquigarrow^n \langle z, \theta, \kappa_1 \rangle \), and \( \kappa' \) is any continuation, \( \langle t, \rho, \kappa \kappa' \rangle \rightsquigarrow^n \langle z, \theta, \kappa_1 \kappa' \rangle \). In particular, if \( \langle t, \rho, \langle \text{halt} \rangle \rangle \rightsquigarrow^n \langle z, \theta, \langle \text{halt} \rangle \rangle \), then for any continuation \( \kappa \), \( \langle t, \rho, \kappa \rangle \rightsquigarrow^n \langle z, \theta, \kappa \rangle \).

Proposition 23. If \( t \rho \vdash_n z \theta \) then \( \langle t, \rho, \langle \text{halt} \rangle \rangle \rightsquigarrow^m \langle z, \theta, \langle \text{halt} \rangle \rangle \) for some \( m \leq 3n \).
Proof. By induction on the height of the derivation. Lemma 22 allows us to make use of the induction hypothesis.

**Lemma 24.** If $\langle t, \rho, \kappa_0 \rangle \rightsquigarrow^* \langle z, \theta, \kappa_1 \rangle$, then the transition sequence has an initial segment of the form $\langle t, \rho, \kappa_0 \rangle \rightsquigarrow^n \langle z', \theta', \kappa_0 \rangle$ for some value $z'\theta'$ such that $t\rho \downarrow^n z'\theta'$.

**Proof.** By induction on the length of the transition sequence.

**Proposition 25.** If $\langle t, \rho, \langle \text{halt} \rangle \rangle \rightsquigarrow^n \langle z, \theta, \langle \text{halt} \rangle \rangle$, then $t\rho \downarrow^n z\theta$.

**Proof.** By Lemma 24, there are $m$ and $\ell$ such that the given transition sequence has the form $\langle t, \rho, \langle \text{halt} \rangle \rangle \rightsquigarrow^m \langle z', \theta', \langle \text{halt} \rangle \rangle \rightsquigarrow^\ell \langle z, \theta, \langle \text{halt} \rangle \rangle$. Since $z'$ is a value, there are no transitions that start from $\langle z', \theta', \langle \text{halt} \rangle \rangle$, and so we conclude that $\ell = 0$ and hence that $z'\theta' = z\theta$. And by Lemma 24, $t\rho \downarrow^n z'\theta' = z\theta$.

**References**

[1] K. Aehlig and H. Schwichtenberg. A syntactical analysis of non-size-increasing polynomial time computation. *ACM Transactions on Computational Logic*, 3(3):383–401, 2002. URL http://doi.acm.org/10.1145/507382.507386

[2] A. Barber. Dual intuitionistic linear logic. Technical Report ECS-LFCS-96-347, Laboratory for Foundations of Computer Science, 1996. URL http://www.lfcs.inf.ed.ac.uk/reports/96/ECS-LFCS-96-347/index.html

[3] S. Bellantoni and S. Cook. A new recursion-theoretic characterization of the polynome functions. *Computational Complexity*, 2(2):97–110, 1992. URL http://dx.doi.org/10.1007/BF01201998

[4] G. E. Blelloch. Programming parallel algorithms. *Communications of the Association for Computing Machinery*, 39(3):85–97, 1996. URL http://doi.acm.org/10.1145/227234.227246

[5] G. E. Blelloch, S. Chatterjee, J. C. Hardwick, J. Sipelstein, and M. Zagha. Implementation of a portable nested data-parallel language. *Journal of Parallel and Distributed Computing*, 21(1):4–14, Apr. 1994. URL http://dx.doi.org/10.1006/jpdc.1994.1038

[6] M. M. T. Chakravarty, R. Leshchinskiy, S. P. Jones, G. Keller, and S. Marlow. Data parallel haskell: a status report. In *DAMP ’07: Proceedings of the 2007 Workshop on Declarative Aspects of Multicores Programming* (Nice, France, 2007), pages 10–18, New York, NY, USA, 2007. ACM Press. URL http://doi.acm.org/10.1145/1248648.1248652

[7] S. Cook and A. Urquhart. Functional interpretations of feasibly constructive arithmetic. *Annals of Pure and Applied Logic*, 63(2):103–200, 1993. URL http://dx.doi.org/10.1016/0168-0072(93)90044-E

[8] N. Danner and J. S. Royer. Adventures in time and space. In *POPL ’06: Conference Record of the 33rd ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages* (Charleston, SC, 2006), pages 168–179, New York, 2006. Association for Computing Machinery. URL http://doi.acm.org/10.1145/1111037.1111053

[9] N. Danner and J. S. Royer. Adventures in time and space. *Logical Methods in Computer Science*, 3(9), 2007. URL http://arxiv.org/abs/cs/0612116

[10] C. C. Frederiksen and N. D. Jones. Recognition of polynomial-time programs. Technical Report TOPPS/D-501, DIKU, University of Copenhagen, 2004. URL http://www.diku.dk/topps/bibliography/2004.html

[11] M. Hofmann. Linear types and non-size-increasing polynomial time computation. *Information and Computation*, 183(1):57–85, 2003. URL http://dx.doi.org/10.1016/S0890-5401(03)00009-9
[12] M. Hofmann. The strength of non-size increasing computation. In POPL ’02: Proceedings of the 29th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (Portland, OR, 2002), pages 260–269, New York, 2002. ACM Press. URL http://doi.acm.org/10.1145/503272.503297

[13] A. Ignjatovic and A. Sharma. Some applications of logic to feasibility in higher types. ACM Transactions on Computational Logic, 5(2):332–350, 2004. URL http://xxx.lanl.gov/abs/cs.LO/0204045.

[14] N. D. Jones. The expressive power of higher-order types or, life without cons. Journal of Functional Programming, 11(1):55–94, 2001. URL http://dx.doi.org/10.1017/S0956796800003889.

[15] B. M. Kapron and S. A. Cook. A new characterization of type-2 feasibility. SIAM Journal on Computing, 25(1):117–132, 1996. URL http://dx.doi.org/10.1137/S0097539794263452.

[16] K.-I. Ko. Computational Complexity of Real Functions. Birkhäuser Boston, 1991.

[17] D. Leivant. Ramified recurrence and computational complexity I: Word recurrence and polytime. In Feasible Mathematics II (Ithaca, NY, 1992), pages 320–343. Birkhäuser Boston, Boston, MA, 1995.

[18] K. Mehlhorn. Polynomial and abstract subrecursive classes. In Proceedings of the Sixth Annual ACM Symposium on Theory of Computing (Seattle, WA, 1974), pages 96–109, New York, NY, USA, 1974. ACM Press. URL http://doi.acm.org/10.1145/800119.803890.

[19] R. Milner, M. Tofte, R. Harper, and D. MacQueen. The Definition of Standard ML (Revised). The MIT Press, Cambridge, MA, 1997.

[20] B. C. Pierce. Types and Programming Languages. The MIT Press, Cambridge, MA, 2002.

E-mail address: ndanner@wesleyan.edu

Department of Electrical Engineering and Computer Science, Syracuse University, Syracuse, NY 13210, USA
E-mail address: royer@ecs.syr.edu