Universal aging properties at a disordered critical point

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We investigate, analytically near the dimension \(d_{uc} = 4\) and numerically in \(d = 3\), the non equilibrium relaxational dynamics of the randomly diluted Ising model at criticality. Using the Exact Renormalization Group Method to one loop, we compute the two times \(t, t_w\) correlation function and Fluctuation Dissipation Ratio (FDR) for any Fourier mode of the order parameter, of finite wave vector \(q\). In the large time separation limit, the FDR is found to reach a non trivial value \(X^{\infty}\) independently of (small) \(q\) and coincide with the FDR associated to the total magnetization obtained previously. Explicit calculations in real space show that the FDR associated to the local magnetization converges, in the asymptotic limit, to this same value \(X^{\infty}\). Through a Monte Carlo simulation, we compute the autocorrelation function in three dimensions, for different values of the dilution fraction \(p\) at \(T_c(p)\). Taking properly into account the corrections to scaling, we find, according to the Renormalization Group predictions, that the autocorrelation exponent \(\lambda_c\) is independent on \(p\). The analysis is complemented by a study of the non equilibrium critical dynamics following a quench from a completely ordered state.

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The study of relaxational dynamics following a quench at a pure critical point has attracted much attention these last few years \cite{1,2,3,4}. Although simpler to study than glasses, critical dynamics display interesting non equilibrium features such as aging, commonly observed in more complex disordered or glassy phases \cite{5}. In this context, the computation of two times \(t, t_w\) response and correlation functions with associated universal exponents has been the subject of numerous analytical as well as numerical studies \cite{4,6,7,8}.

In addition, it has been proposed \cite{6} that a non trivial Fluctuation Dissipation Ratio (FDR) \(X\), originally introduced in the Mean Field approach to glassy systems, which generalizes the Fluctuation Dissipation Theorem (FDT) to non equilibrium situations, is a new universal quantity associated to these critical points. As such, it has been computed using the powerful tools of RG, e.g. for pure \(O(N)\) model at criticality in the vicinity of the upper critical dimension \(d_{uc} = 4\) and for various dynamics \cite{4,6,7,8}.

An important question related to the physical interpretation of \(X\) in terms of an effective temperature \(\tilde{T}\) \(\tilde{T}_{eff} = T/X\) is its dependence on the observables \cite{11,12}. In this respect, a heuristic argument \cite{7} suggests that, for a wide class of critical systems, the local FDR associated to correlation and corresponding response of the local magnetization should be identical, in the large time separation limit, to the FDR for the total magnetization, i.e. for the Fourier mode \(q = 0\). This argument relies strongly on the hypothesis that the time decay of the response function of the Fourier mode \(q\) is characterized by a single time scale \(\tau_q \sim q^{-z}\), with \(z\) the dynamical exponent.

Characterizing the effects of quenched disorder on critical dynamics is a complicated task and indeed the question of how quenched randomness modifies these properties has been much less studied. In particular, in this context of critical disordered systems, the question of universality, i.e. the dependence of the critical exponents on the strength of the disorder, is a controversial issue \cite{12}.

In this paper, we address these questions on the randomly diluted Ising model:

\[ H = \sum_{\langle ij \rangle} \rho_i \rho_j s_i s_j \]

(1)

where \(s_i\) are Ising spins on a \(d\)-dimensional hypercubic lattice and \(\rho_i = 1\) with probability \(p\) and \(0\) with probability \(1 - p\). For the experimentally relevant case of dimension \(d = 3\) \cite{12}, for which the specific heat exponent of the pure model is positive, the disorder is expected, according to Harris criterion \cite{14}, to modify the universality class of the transition. For \(1 - p \ll 1\), the large scale properties of \(\mathbf{u}\) at criticality are then described by the following \(O(1)\) model with a random mass term, the so-called Random Ising Model (RIM):

\[ H^\varphi[\varphi] = \int d^d x \left[ \frac{1}{2} (\nabla \varphi)^2 + \frac{1}{2} \varphi (x) \right] \theta(x) + \frac{\eta}{4!} \varphi^4 \]

(2)

where \(\varphi \equiv \varphi(x)\) and \(\psi(x)\) is a gaussian random variable \(\psi(x) \psi(x') = \delta^d(x - x')\) and \(\theta(x)\), the bare mass, is adjusted so that the renormalized one is zero. The static critical properties of this model have been intensively studied \cite{15} both analytically, mainly using RG, within various schemes, and numerically \cite{16}. The (perturbative) RG calculations below the upper critical dimension in \(d = 4 - \epsilon\), which we will focus on here, confirm the qualitative Harris criterion and predict that the critical properties of these models \cite{12} for different values of \(p\) close to one, are described by a new disordered fixed point, which is independent of \(p\). Therefore, an important statement of this RG analysis is that the critical exponents, which can be computed in an expansion of \(\sqrt{\epsilon}\), e.g. \(\eta = \mathcal{O}(\epsilon)\), are universal, i.e. independent of \(p\).
This was recently confirmed by Monte Carlo simulation in $d = 3$ over a wide range of concentration $p$ above the percolation threshold $p_c = 0.31$. Although quantitative discrepancies were found with perturbative RG calculations [17], universality was demonstrated by taking carefully into account the (strong) corrections to scaling [10]. In the equilibrium dynamics, at variance with the pure case, the perturbative expansion of the dynamical exponent $z$ differs from its high temperature value of 2 already at one loop $z - 2 = \frac{2d}{3} + O(e)$ independently of $p_c$, corrections to two loops have been computed in Ref. [14], and up to three loops in Ref. [20]. After a long debate, a recent numerical simulation [21] where corrections to (dynamical) scaling were taken into account, has confirmed the universality of $z$ in $d = 3$, leading to $z = 2.62(7)$ independently on the spin concentration $p$ above $p_c$.

By contrast, much less is known about the non-equilibrium dynamics of this disordered system at criticality. The critical initial slip exponent $\theta_t$ vanishes to one loop $\theta_t = 0$ and correction to two-loops have been computed [22]. This exponent has been recently computed up to two loops for the case of extended defects in [24]. The two times response in Fourier space, $R_{tt}^q$ and the correlation $C_{tt}^q$, including the associated scaling functions, are known up to one loop. But although the dynamical RG predicts a universal value of the auto-correlation exponent $\lambda_c = 1 - 2d\theta_t$ for $p$ close to 1, this statement remains an open question for a wider range of values of $p$. Furthermore, a non trivial FDR [25], only for the total magnetization, was recently obtained to one loop, and it was argued, using the same aforementioned heuristic argument [1], to coincide with the local FDR. However, it was already noticed in Ref. [22] that, due to the disorder, $R_{tt}^q$ decays as a power law for $q^t \gg 1$. Therefore, the argument of Ref. [1] is challenged for this disordered critical point, and, already at one loop order, the computation of the FDR needs a closer inspection, including an extension of the analysis of Ref. [22] beyond the “diffusive” $q = 0$ mode.

In this paper, using RG to one loop, we obtain, for any finite Fourier mode $q$, the correlation function $C_{tt}^q$ and the FDR $X_{tt}^q$, which are both characterized by scaling functions of the variables $q^t(t - t_w)$ and $t/t_w$. In the asymptotic large time separation regime $t \gg t_w$, the FDR reaches a non trivial value $X^q$, independently of (small) $q$. In addition, we explicitly compute the local FDR, which is a function of $t/t_w$ and reaches the same non-trivial limit $X^q$ when $t \gg t_w$, which thus establishes on firmer ground the heuristic argument of Ref. [1] for the present disordered case. Besides, we perform a Monte Carlo simulation of the non-equilibrium relaxation of $X^q$ following a quench from high temperature with initial magnetization $m_0 = 0$ at $T_c(p)$ and compute the auto-correlation function. In the asymptotic regime, it takes a scaling form compatible with the RG calculations. By taking into account corrections to scaling, we show that the exponent $\lambda_c$ is independent of $p$. Finally, we compute numerically the autocorrelation function for the critical dynamics following a quench for a completely ordered initial condition with $m_0 = 1$. We observe that the system is also aging and show that the decaying exponent $\lambda_c$ is strongly affected by this initial condition.

We study the relaxation dynamics of the Randomly diluted Ising Model in dimension $d = 4 - \epsilon$ described by a Langevin equation:

$$\frac{\partial}{\partial t} \varphi(x, t) = \frac{\delta H^q[\varphi]}{\delta \varphi(x, t)} + \zeta(x, t)$$

where $\zeta(x, t)$ is white noise and $\eta$ the friction coefficient. At initial time $t = 0$, the system is in a random initial configuration with zero magnetization $m_0 = 0$ distributed according to a Gaussian with short range correlations

$$\langle \varphi(x, t = 0) \varphi(x', t = 0) \rangle = \tau_0^{-1} \delta^d(x - x').$$

Notice that it has been shown that $\tau_0^{-1}$ is here irrelevant (in the RG sense) in the large time regime studied here [26]. We will focus on the correlation $C_{tt}^q$ in Fourier space and the autocorrelation $C_{tt}^q$ defined by Eq. (3)

$$C_{tt}^q = \langle \varphi(q, t)\varphi(-q, t) \rangle, \quad C_{tt}^q = \langle \varphi(x, t)\varphi(x, t_w) \rangle$$

and the response $R_{tt}^q$ to a small external field $f(-q, t_w)$ as well as on the local response function $R_{tt}^q$ respectively defined, for $t > t_w$

$$R_{tt}^q = \frac{\delta \langle \varphi(q, t) \rangle}{\delta f(q, t_w)} \quad R_{tt}^q = \frac{\delta \langle \varphi(x, t) \rangle}{\delta f(x, t_w)}$$

where $\langle . \rangle$ denote respectively averages w.r.t. disorder and thermal fluctuations. We focus also on the FDR $X_{tt}^q$ associated to the observable $\varphi$.

$$\frac{1}{X_{tt}^q} = \frac{\partial C_{tt}^q}{T R_{tt}^q}$$

defined such that $X_{tt}^q = 1$ at equilibrium. Notice also that for this choice of initial conditions [14], connected and non connected correlations do coincide for large system size.

A convenient way to study the Langevin dynamics defined by Eq. (3) is to use the Martin-Siggia-Rose generating functional. Using the Ito prescription, it can be readily averaged over the disorder. The correlations and response are then obtained from a dynamical (disorder averaged) generating functional or, equivalently, as functional derivatives of the corresponding dynamical effective action $\Gamma$. This functional can be perturbatively computed using the Exact RG equation associated to the multi-local operators expansion introduced in [28, 29]. It allows to handle arbitrary cut off functions $c(q^2/2\Lambda^2)$ and check universality, independence
w.r.t. \( c(x) \) and the ultraviolet scale \( \Lambda_0 \). It describes the evolution of \( \Gamma \) when an additional infrared cut off \( \Lambda_i \) is lowered from \( \Lambda_0 \) to its final value \( \Lambda_i \to 0 \) where a fixed point of order \( O(\sqrt{\epsilon}) \) is reached. In this limit, one obtains \( R_{ttw}^q \) and \( C_{ttw}^q \) (for \( t > t_w \)) from

\[
\partial_t R_{ttw}^q + (q^2 + \mu(t))R_{ttw}^q + \int_{t_i}^t dt_1 \Sigma_{tt}, R_{ttw}^q(t, t_w) = 0 \quad (8)
\]

\[
C_{ttw}^q = 2T \int_{t_i}^t dt_1 R_{tt}^q R_{ttw}^q + \int_{t_i}^t dt_1 \int_{t_i}^t dt_2 R_{ttw}^q D_{ttw}^q \quad (9)
\]

with \( \mu(t) = -\int dt_i \Sigma_{tt}, t \), and where the self energy \( \Sigma_{tt}, t \) and the noise-disorder kernel \( D_{ttw}^q \) are directly obtained from \( \Gamma \) at the fixed point. One finds:

\[
\Sigma_{tt'} = -\frac{1}{2} \frac{6\epsilon}{53} \int_a^\infty (\gamma_a(t-t'))^2 \quad (10)
\]

\[
D_{tt} = \frac{T_c}{2} \frac{6\epsilon}{53} \int_a^\infty (\gamma_a(t-t') - \gamma_a(t+t')) \quad (11)
\]

where \( \gamma_a(x) = (x + a/(2\Lambda_0^2))^{-1} \). For concrete calculations, we have used the decomposition of the cut-off function \( c(x) = \int da \alpha(a) e^{-ax} \equiv \int_a \alpha e^{-ax} \).

The computation of the correlation function \( C_{ttw}^q \) requires the knowledge of the response, which we first focus on. By solving perturbatively to order \( O(\sqrt{\epsilon}) \) the differential equation \( 5 \), similarly to what is done in Ref. \( 27 \), one recovers, in the limit \( q/\Lambda_0 \ll 1 \) keeping the scaling variables \( v = q^2(t-t_w) \) and \( u = t/t_w \) fixed, the solution obtained in Ref. \( 22 \), consistent with the scaling form

\[
R_{ttw}^q = q^{-2+z+\eta} \left( \frac{t}{t_w} \right)^\theta F_R(q^2(t-t_w), t/t_w) \quad (12)
\]

where \( \theta = \frac{1}{2} \sqrt{\frac{6\epsilon}{53}} + O(\epsilon) \) and the universal \( 30 \) scaling function \( F_R(v, u) \) admits also an expansion in powers of \( \sqrt{\epsilon} \) with \( 22 \)

\[
F_R(q^2(t-t_w), t/t_w) = e^{-v} + \frac{6\epsilon}{53} (v-1) \text{Ei}(v) e^{-v} + e^{-u-1}) \quad (13)
\]

where \( \text{Ei}(v) \) is the exponential integral function. At variance with the pure model at one loop \( 4 \), the large \( v \) behavior of \( F_R(v) \) is a power law, \( F_R(v) \propto v^{-2} \), which already indicates that the heuristic argument of Ref.\( 4 \) can not applied here. Besides, when computing the local response \( R_{ttw} \), one is left with an integral over momentum which is logarithmically divergent, indicating that this integral has to be handled with care to obtain the correct result, as the scaling form in \( 12 \) is valid only for \( q/\Lambda_0 \ll 1 \). We thus solve perturbatively the equation \( 5 \) for any fixed \( q \) and obtain an expression for \( R_{ttw}^q \) consistent with the scaling form

\[
R_{ttw}^q = \tilde{g}_1(q) \left( \frac{t}{t_w} \right)^\theta F_R(q^2(t-t_w), t/t_w) \quad (13)
\]

with the universal \( 30 \) small \( q \) behavior:

\[
\tilde{g}_1(q) \sim q^{-2+\eta} \quad , \quad \tilde{g}_2(q) \sim q^{-2} \quad (14)
\]

which actually allows to recover the previous expression in the asymptotic limit \( q/\Lambda_0 \ll 1 \) \( 12 \). By computing the Fourier transform of \( R_{ttw}^q \) as given in Eq. \( 13 \), we explicitly check that the local response \( R_{ttw} \) is consistent with the scaling form

\[
R_{ttw} = \frac{K_d}{2} \frac{A_0^3}{(t-t_w)^{1+(d-2+\eta)/2}} \left( \frac{t}{t_w} \right)^\theta \quad (15)
\]

with \( K_d = S_d/(2\pi)^d \) and where the non-universality is left in the amplitudes \( A_0^d \) and \( A_1^d \):

\[
A_0^d = 1 - \frac{3}{2} \sqrt{\frac{6\epsilon}{53}} + \rho_R \quad , \quad A_1^d = \frac{1}{2} \sqrt{\frac{6\epsilon}{53}} \quad (16)
\]

\[
\rho_R = \sqrt{\frac{6\epsilon}{53}} \int_a^\infty \ln \left( \frac{2A^2}{\alpha} \right) \quad (17)
\]

At the order of our calculations \( O(\sqrt{\epsilon}) \), although \( z \neq 2 \), this scaling form is compatible with local scale invariance arguments \( 21 \). Notice also that, at this order, the scaling form obtained for \( R_{ttw} \) could be written as

\[
R_{ttw} = \frac{K_d}{2} A_0^d \frac{1}{(t-t_w)^{1+(d-2+\eta)/2}} \left( \frac{t}{t_w} \right)^\theta \quad (17)
\]

with \( a \neq (d-2+\eta)/z \). Although this scaling form \( 17 \) can not be ruled out at this stage, which would in principle require a 2-loop calculations, it seems rather unlikely given the scaling form obtained in Fourier space \( 12 \), where instead a logarithmic correction as in \( 14 \) is suggested by the large argument behavior of the function \( F_R^0(v) \).

We now turn to the computation of the correlation function in Fourier space \( C_{ttw}^q \), which was only computed for the \( q = 0 \) mode \( 22 \). Solving \( 9 \), one obtains an explicit expression, which, in the aforementioned scaling limit, is compatible with the scaling form

\[
C_{ttw}^q = T_0 q^{-2+\eta} \left( \frac{t}{t_w} \right)^\theta F_C(q^2(t-t_w), t/t_w) \quad (18)
\]

with the full expression:

\[
F_C(v, u) = F^0_C(v, u) + \sqrt{\epsilon} F^1_C(v, u) + O(\epsilon) \quad (19)
\]

\[
F^0_C(v, u) = e^{-v} - e^{-v(u+1)/(u-1)} \quad (20)
\]

\[
F^1_C(v, u) = F^1_{eq}(v) - F^1_{eq}(\frac{u+1}{u-1}) \quad (20)
\]

\[
+ \sqrt{\frac{6}{53}} e^{-v} - \ln \left( \frac{2v}{u-1} - \gamma_E \right) \quad (21)
\]

where \( \gamma_E \) is the Euler constant. In the limit \( q \to 0 \), our full expression \( 19 \) gives back the result of Ref. \( 23 \). In
the large time separation limit \( u \gg 1 \), keeping \( v \) fixed, one obtains the result:

\[
F_C(v, u) = \frac{1}{u} F_{C,\infty}(v) + O(u^{-2}) \quad (22)
\]

\[
F_{C,\infty}(v) = A_{C,\infty} v F_R^C(v) \quad , \quad A_{C,\infty} = 2 + 2 \sqrt{\frac{6\epsilon}{53}} \quad (23)
\]

This \( O(u^{-1}) \) decay in \( \ref{22} \) is expected from RG arguments, and has been explicitly checked for different pure critical systems \( \ref{3} \). However, this relation \( \ref{23} \) is a priori non trivial and cannot be obtained from general arguments. This identity was also found for the pure \( O(N) \) model at criticality to one loop order \( \ref{7} \) as well as in the glass phase of the Sine Gordon model with random phase shifts \( \ref{32} \) and it would be interesting to investigate whether such a behavior \( \ref{23} \) can be obtained from more general arguments.

The full expression for \( C^q_{tt_w} \) \( \ref{19} \) also allows to compute the structure factor \( C^q_{tt} \). It is obtained from \( \ref{19} \) in the limit \( v \to 0, u \to 1 \) keeping \( v/(u-1) = q^2 t \) fixed, and we check that one recovers the previous result obtained in Ref. \( \ref{22} \). Thus, one explicitly checks, at order \( O(\sqrt{\epsilon}) \), that the dynamical exponent \( z \) associated to dynamical equilibrium fluctuations is the same as the one associated to nonequilibrium relaxation.

As noticed previously for the response function, the large \( v \) behavior of \( F_C(v, u) \) is a power law \( F_C(v, u) \propto v^{-1} \). Therefore, given the scaling form \( \ref{18} \), the computation of the autocorrelation \( C_{tt} \) has to be done carefully. Just as for the response, we thus compute the correlation function \( C_{tt} \) from \( \ref{19} \) for any fixed \( q \) and then perform the Fourier transformation. One obtains the scaling form

\[
C_{tt} = K_d A_0^0 + A_1^0 \ln \left( \frac{t}{t_w} \right) + A_1^1 \left( \frac{t}{t_w} \right)^\theta F(t/t_w) \quad (24)
\]

with

\[
A_0^0 = \frac{1}{2} \sqrt{\frac{6\epsilon}{53}} + \rho_R \quad , \quad A_1^0 = \frac{1}{2} \sqrt{\frac{6\epsilon}{53}}
\]

\[
F(u) = \frac{1}{1 + u} + O(\epsilon) \quad (25)
\]

The same remarks, concerning the response, made before Eq. \( \ref{17} \) also hold here for the autocorrelation.

We now turn to the FDR, first in Fourier space. Given the scaling forms for the response \( R^q_{tt} \) \( \ref{12} \) and for the correlation \( C^q_{tt} \) \( \ref{18} \) we have explicitly checked here, the FDR \( X^q_{tt_w} \) takes the simple scaling form in the regime \( q/A_0 \ll 1 \):

\[
(X_{tt_w}^q)^{-1} = F_X(q^2 (t-t_w), t/t_w) \quad (26)
\]

We have obtained the complete expression for the scaling function \( F_X(v, u) \), which at variance with the pure \( O(N) \) model at criticality is a function of both \( q^2 t \) and \( q^2 t' \). In the large time separation limit \( u \gg 1 \), keeping \( v \) fixed, one obtains, as a consequence of \( \ref{23} \):

\[
\lim_{u \to \infty} \left( X_{tt_w}^q \right)^{-1} = 2 + \sqrt{\frac{6\epsilon}{53}} + O(\epsilon) \quad (27)
\]

independently of \( v \), i.e. of (small) wave vector \( q \), which coincides of course with the asymptotic value for the \( q = 0 \) mode obtained in Ref. \( \ref{27} \). We can check easily, using the result of Ref. \( \ref{7} \), that this property, independence on \( v \) on the asymptotic limit, holds also for the pure model at one loop, and it was also found in the glass phase of the Sine Gordon model with random phase shifts \( \ref{32} \).

As we saw previously, the large \( v \) power law behavior of the scaling function \( F_R^q(v) \) prevents us to use the argument of Ref. \( \ref{7} \) for the present case. Therefore on computes directly the FDR for the local correlation and associated response \( X_{tt_w}^{q=0} \). It is also characterized by a scaling function of \( t/t_w \), which can be simply written as

\[
(X_{tt_w}^{q=0})^{-1} = F_X(t/t_w) \quad (28)
\]

\[
F_X(u) = \frac{u^2 + 1}{(u + 1)^2} + \sqrt{\frac{6\epsilon}{53}} \left( \frac{u - 1}{u + 1} \right)^2 + O(\epsilon) \quad (29)
\]

where \( F_X(u) \) is a monotonic increasing function of \( u \), it interpolates between 1, in the quasi-equilibrium regime for \( u \to 1 \), and its asymptotic value for \( u \to \infty \) given by

\[
\lim_{t/t_w \to \infty} (X_{tt_w}^{q=0})^{-1} = \lim_{t/t_w \to \infty} (X_{tt_w}^{q=0})^{-1} = 2 + \sqrt{\frac{6\epsilon}{53}} + O(\epsilon) \quad (30)
\]

which shows explicitly, at order \( O(\sqrt{\epsilon}) \) that the asymptotic FDR for both the total and the local magnetization are indeed in the same.

Let us next present results from our Monte Carlo simulations of the relaxational dynamics of the randomly diluted Ising model \( \ref{11} \) in dimension \( d = 3 \), which were done on \( L \times L \times L \) cubic lattices with periodic boundary conditions. We first focus on the following situation where the system is initially prepared in a random initial configuration with zero magnetization \( m_0 = 0 \). At each time step, the \( L^3 \) sites are then sequentially updated : for each site \( i \), the move \( s_i \to -s_i \) is accepted or rejected according to Metropolis rule. If one gradually decreases \( p \) the fraction of magnetic sites will be reached below which the system no longer exhibits a transition to ferromagnetic order at any finite temperature. This happens at the percolation threshold, for which \( \rho_c(p_c) = 0 \) \( \ref{12} \) \( \ref{11} \). For different values of \( p > p_c \), we compute the spin-spin auto-correlation function defined as

\[
C_{tt} = \frac{1}{L^3} \sum_i \langle s_i(t) s_i(t_w) \rangle \quad (31)
\]

In the following we will also be interested in the connected correlation function \( \tilde{C}(t, t_w) \) defined as

\[
\tilde{C}(t, t_w) = \frac{1}{L^3} \sum_i \left( \langle s_i(t) s_i(t_w) \rangle - \langle s_i(t) \rangle \langle s_i(t_w) \rangle \right) \quad (32)
\]
In order to obtain better statistics, $C_{tt_w}$ (or $\tilde{C}(t, t_w)$) is averaged over a suitably chosen time window $\Delta t$ around $t$, with $\Delta t \ll t$. All our data are obtained for a lattice linear size $L = 100$, as an average over 500 independent initial conditions and disorder configurations. We also produced data (not shown here) for the spatial correlation function, for the same system size, to ensure that our results are not influenced by finite size effects.

Fig. 1 shows the auto-correlation function $C_{tt_w}$ as a function of $t - t_w$ for different values of the waiting time $t_w = 2^4, 2^5, 2^6, 2^7$ and $2^8$ at $p = 0.8$. One observes a clear dependence on $t_w$, which indicates a non-equilibrium dynamical regime. We have also checked that for this choice of initial conditions, $C(t, t_w)$ and $\tilde{C}(t, t_w)$ do coincide. The scaling form obtained from the RG analysis \cite{21} suggests, discarding the logarithmic correction, to plot $t_w^{(1+\eta)/z} C_{tt_w}$ as a function of $t/t_w$. Taking the values $\eta = 0.0374$ from Ref. \cite{16} and $z = 2.62$ from Ref. \cite{21}, we see in the inset of Fig. 1 that, for $p = 0.8$, one obtains a good collapse of the curves for different $t_w$. Notice that such scaling forms are also obtained in more complicated disordered systems like 3-dimensional spin-glasses \cite{33}.

However, for different values of $p$, the best collapse, under this form \cite{21}, would be obtained for a $p$-dependent exponent $(1 + \eta)/z$. Thus one would conclude that this exponent is non universal \cite{12}. Nevertheless, it is known \cite{21} that such $p$-dependence occur due to corrections to scaling. Therefore, to include them, we extend the scaling form \cite{21} as

$$C_{tt_w} \propto \frac{1}{(t - t_w)^{(1+\eta)/z}} \left( \tilde{F}_p(t/t_w) - \frac{D(p)}{(t - t_w)^b} \tilde{G}_p(t/t_w) \right)$$

\text{(33)}

with $b = \omega_d/z$, where $\omega_d$ corresponds to the biggest irrelevant eigenvalue of the RG in the dynamics, which is a priori different from the leading corrections in the statics.\cite{21}. Unfortunately, we do not have any information on the function $\tilde{G}_p(x)$. We will thus propose the simplest hypothesis, $\tilde{G}_p(x) = \tilde{F}_p(x)$. In Fig. 2 we show a plot of $t_w^{(1+\eta)/z} C_{tt_w}/f(t - t_w)$, with $f(x) = 1 - D(p)x^{-b}$ : this results in a reasonably good data collapse of the curves for different $t_w$, for $p = 0.5, 0.6, 0.65$ and $0.8$. For each value of $p$, this data collapse is obtained via the fitting of 3 parameters : the exponents $b, z$ and the amplitude $D(p)$. We found quite stable value of the exponents $z = 2.6 \pm 0.1$ and $b = 0.23 \pm 0.02$, which are \textit{both independent} of $p$. Our value of $z$, together with $\omega_d = 0.61 \pm 0.06$ are consistent with the value obtained by Parisi \textit{et. al.} \cite{21}. All the $p$-dependence is thus contained in the non universal amplitude $D(p)$, as shown in the inset of Fig. 2. According to our data, the corrections to scaling in the quasi-equilibrium regime vanish for $p = 0.8$, \textit{i.e.} $D(p = 0.8) \simeq 0$, in agreement with a previous numerical computation of the equilibrium auto-correlation function \cite{12}. Notice that this value $p = 0.8$ is also known \cite{10}, in the statics, to minimize the corrections to scaling.

As shown on the log-log plot in Fig. 2 and consistently with the RG prediction \cite{21}, $\tilde{F}_p(t/t_w)$ \textit{decays} as a power law for $t \gg t_w$. However, this plot in Fig. 2 would suggest that the decaying exponent depends, namely decreases, with $p$. We expect instead that this $p$-dependence is again due to corrections to scaling \cite{21}. Consistently with the corrections we introduced in the quasi-equilibrium part of $C_{tt_w}$ in Eq. \textit{(33)}, we propose the form

$$\tilde{F}_p(x) = A(p)x^{(1+\eta - \lambda_c)/z} (1 + B(p)x^{-b})$$

\text{(34)}

where we (reasonably) assume that the dynamical cor-
reactions to scaling are characterized by the same, \( p \)-independent, exponent \( b = 0.23 \pm 0.02 \) as obtained previously \[23\]. Therefore, for each value of \( p \) one has three parameters to fit: the exponent \( \lambda_c/z \) and the amplitudes \( A(p), B(p) \). We obtain a quite stable fit for the different values of \( p \), with the \( p \)-independent value of the decaying exponent \( \lambda_c/z: \)

\[
\frac{\lambda_c}{z} = 1.05 \pm 0.03
\]

all the \( p \)-dependence being contained in the non-universal amplitudes \( A(p), B(p) \) (see the inset in Fig. 3). As shown in Fig. 3, the curves for different values of \( p \) (and different \( t_w \)) in Fig. 2 collapse on a master curve when we plot \( t_w^{(1+\eta)/z} C_{tt_w}/(f(t-t_w)g(t/t_w)) \), with \( g(x) = A(p)(1 + B(p)x^{-b}) \), as a function of \( t/t_w \). This fact supports universality of the long-time non-equilibrium relaxation in this model. Our value for the exponent \( \lambda_c/z \), together with \( z = 2.6 \pm 0.1 \) gives for the initial slip exponent \( \theta' = 0.1 \pm 0.035 \), which is in rather good agreement with the two-loops RG result \( \theta'_{\text{loops}} = 0.0868 \) \[22\]. Alternatively, this exponent could be measured by studying the initial stage of the relaxational dynamics starting from a non-zero magnetization: this is left for future investigations \[24\].

Here also, one obtains that the corrections to scaling in Eq. (34) vanish for \( p = 0.8 \). We notice that this result is in apparent contradiction with the previous analysis of the non-equilibrium relaxation in this model performed in Ref. [21], where the focus was on the non-connected susceptibility, a one-time quantity, which instead claimed a “perfect Hamiltonian” for \( p \approx 0.63 \). However, the statistical precision of our data does not allow us to make a strong statement about this point, which certainly deserves further investigations.

FIG. 3: Universality of \( C_{tt_w} \) for \( p = 0.6, 0.65, 0.8 \). The function \( g(x) \) is defined in the text. Inset: Non-universal amplitudes \( A(p), B(p) \) as functions of concentration \( p \). Here, the system is initially prepared in a random initial configuration with zero magnetization.

So far, we have focused on the relaxational dynamics occurring after a quench from a completely disordered initial condition, with zero initial magnetization \( m_0 = 0 \) to \( T_c(p) \). But it is also interesting to study how these aging properties depend on the initial conditions \[25, 36, 37\]. We have therefore performed numerical simulations where the system is initially prepared in a completely ordered state:

\[
S_i(t=0) = +1 \quad \forall \quad \text{occupied site } i
\]

such that the initial magnetization is \( m_0 = 1 \). The system is then quenched at \( t = 0 \) to \( T_c(p) \) and evolves according to the same aforementioned dynamical rules. We also compute the autocorrelation function \( C(t, t_w) \) as defined in Eq. 32. The result of this computation for \( p = 0.65 \) is shown on Fig. 4, where we plot \( C(t, t_w) \) as a function of \( t - t_w \), for different \( t_w = 2^5, 2^7, 2^9 \). Here also, one observes a clear dependence on the waiting time \( t_w \), which indicates that the system is aging. Notice however that, at variance with the previous situation (Fig. 1), the correlation for a given \( t - t_w \), decreases as \( t_w \) increases. In addition, at variance with the previous case \( m_0 = 0 \), the behaviors of the connected \( C(t, t_w) \) and the non-connected \( C(t, t_w) \) correlations are qualitatively different: this is shown in the inset of Fig. 4 when one observes that \( C(t, t_w) \) decays indeed much faster \[35\]. This property could be relevant for the computation of the FDR in this situation. The quantitative analysis of the correlation function \( C(t, t_w) \) is shown on Fig. 4. Indeed, the curves for different \( t_w \) can be plotted on a master curve if one plots, for different \( t_w, t_w^{(1+\eta)/z} C_{tt_w}/f(t - t_w) \) as a function of \( t/t_w \), which suggests that also in that case the correlation function can be written under the scaling form as in Eq. (18) with \( F_p(x) = G_p(x) \). However, as illustrated on Fig. 4 the behavior of \( C(t, t_w) \) is strongly affected by the initial condition, the decay be-

FIG. 4: Log-log plot of the correlation \( C(t, t_w) \) as a function of \( t - t_w \) for \( p = 0.65 \). Inset: Log-log plot of the connected correlation \( C(t - t_w) \) as a function of \( t - t_w \). In the inset, we use the same symbols as in the main figure. Here \( m_0 = 1 \).
completely determined by the equilibrium exponents: 

\[ C^\beta/\nu \]

in that case of a fully ordered initial condition \((\lambda = 0)\) and where we have used the hyperscaling relation \(\beta/\nu\). This relation (38) can be understood by considering \(t\) for the present problem, we believe that one has 

\[ C(t) = \left( \frac{t}{t_0} \right)^{-\beta/\nu} \]

for \(t \gg t_0\) where \(t_0\) is the characteristic relaxation time of the initial condition \(\lambda = 0\). Indeed, for this particular initial condition (36), \(f(x)\) defined in the text. The straight line is a guide line for the eyes.

ing much faster when the system is initially in a random configuration with \(m_0 = 0\). More precisely, as suggested on Fig. 4 our data for \(m_0 = 1\) are compatible with the following scaling form

\[ C_{ttw} \sim \frac{1}{(t - t_w)^{1 + \eta \beta/\nu}} \left( 1 - \frac{D(p)}{(t - t_w)\nu} \right) \left( \frac{t}{t_w} \right)^{1 + \eta \beta/\nu} \]

\[ \sim t^{-\frac{\beta}{\nu}}, \quad t \gg t_w \] (37)

where \(\beta, \nu\) are the standard equilibrium critical exponents and where we have used the hyperscaling relation \(\beta/\nu = (d - 2 + \eta)/2\). Thus, although we can not show it analytically for the present problem, we believe that in that case of a fully ordered initial condition \((m_0 = 1)\), although the system displays aging, the exponent \(\lambda_c\) is completely determined by the equilibrium exponents:

\[ \lambda_c = \frac{\beta}{\nu} \] (38)

This relation (38) can be understood by considering \(C(t, 0)\). Indeed, for this particular initial condition (38), one has \(C(t, 0) = M(t)\), where \(M(t)\) is the global magnetization at time \(t\). Therefore, at large time, from standard scaling argument \(C(t, 0) \sim t^{-\frac{\beta}{\nu}}, \) which thus gives

the relation (38). Notice that this relation (38) is also found in the context of pure critical point [30, 31].

To sum up, we have performed a rather detailed analysis of the relaxational dynamics up to one loop of the randomly diluted Ising model in dimension \(d = 4 - \epsilon\). The computation of the correlation function \(C_q\), including its associated scaling function, allows us to show that the Fluctuation Dissipation Ratio reaches, in the large time separation limit, a non trivial value \(X^\infty\), independently of small wave vector \(q\). Although, due to the broad relaxation time spectrum induced by the disorder, the standard argument of Ref. [7] can not be applied here, we have performed an explicit computation in real space which shows explicitly that the limiting FDR associated to the total magnetization, on the one hand, and the local one, on the other hand, do coincide. And in this respect, it would be interesting to further investigate the FDR associated to other observables, like the energy for instance [10, 11]. These properties could also be tested in numerical simulations.

In addition, we have computed numerically, in \(d = 3\) the autocorrelation function. It is characterized by a scaling form fully compatible with our one loop RG calculation in real space. We have however shown that this two times quantity is strongly affected by corrections to scaling, which remain to be understood more deeply from an analytical point of view. By taking them properly into account, our data suggest a universal, \(i.e., p\)-independent autocorrelation exponent \(\lambda_c\), which provides an “indirect” measurement of the initial slip exponent \(\theta'\), which is in reasonably good agreement with two-loops RG prediction. Finally, we have shown that the critical dynamics following a quench from a completely ordered state \((m_0 = 1)\) displays also aging, but with a quantitative different behavior, the decaying exponent \(\lambda_c\) being in that case completely determined by the equilibrium exponents.

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