A DECAY ESTIMATE FOR THE FOURIER TRANSFORM OF CERTAIN SINGULAR MEASURES IN $\mathbb{R}^4$ AND APPLICATIONS

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Abstract. We consider, for a class of functions $\varphi : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ satisfying a nonisotropic homogeneity condition, the Fourier transform $\hat{\mu}$ of the Borel measure on $\mathbb{R}^4$ defined by

$$\hat{\mu}(E) = \int_E \chi_E(x, \varphi(x)) \, dx$$

where $E$ is a Borel set of $\mathbb{R}^4$ and $U = \{(t^s, t^u s) : c < s < d, 0 < t < 1\}$. The aim of this article is to give a decay estimate for $\hat{\mu}$, for the case where the set of nonelliptic points of $\varphi$ is a curve in $\mathbb{R} \setminus \{0\}$. From this estimate we obtain a restriction theorem for the usual Fourier transform to the graph of $\varphi_{U} : U \to \mathbb{R}$. We also give $L^p$-improving properties for the convolution operator $T_{\mu} f = \mu * f$.

1. Introduction

Let $U$ be an open bounded set in $\mathbb{R}^n$ and $\varphi : U \to \mathbb{R}^n$ be a continuous function. Let $\mu_U$ be the Borel measure on $\mathbb{R}^{n+m}$ supported on the graph of $\varphi$, defined by

$$\mu_U(E) = \int_U \chi_E(x, \varphi(x)) \, dx,$$

where $dx$ denotes the Lebesgue measure on $\mathbb{R}^n$. Let $\Sigma$ be the graph of $\varphi$, i.e.

$$\Sigma = \{(x, \varphi(x)) : x \in U\}.$$

For $f \in S(\mathbb{R}^{n+m})$, let $T_{\mu} f = \mu_U * f$ and let $R f = \hat{\mu} |_{\Sigma}$, where $\hat{\mu}$ denotes the usual Fourier transform of $\mu$. The $L^p(\mathbb{R}^{n+m}) - L^q(\Sigma)$ boundedness of the restriction operator $\mathcal{R}$ to the submanifold $\Sigma \subset \mathbb{R}^{n+m}$ has been widely studied, in different cases, by many authors. For $S^1 \subset \mathbb{R}^2$ by C. Fefferman in [7], for $S^{n-1} \subset \mathbb{R}^n$, $n \geq 2$, by P. Tomas and E. Stein in [18] and [13] respectively, for quadric surfaces with non vanishing gaussian curvature by R. Strichartz in [17].

The case of compact $n$-dimensional manifolds $\Sigma \subset \mathbb{R}^{2n}$ was studied by E. Prestini in [11] under the assumption that: for each $x_0 \in \Sigma$ there exists a local chart $X(x_1, \ldots, x_n)$ around $x_0$ satisfying that the vectors $\left\{ \frac{\partial X}{\partial x_i} \right\}_{i=1}^n$ and $\left\{ \frac{\partial^2 X}{\partial x_i \partial x_j} \right\}_{i,j=1}^n$ span $\mathbb{R}^{2n}$. The case of two dimensional manifolds $\Sigma = \{(x, \varphi(x)) : x \in U \subset \mathbb{R}^2\} \subset \mathbb{R}^4$ was studied by M. Christ in [1], where a restriction theorem is given under the assumption that: for each $x \in U$ and $\theta \in \mathbb{R}$, if det $H_x \left( (\varphi(x), (\cos(\theta), \sin(\theta))) \right) = 0$ (where $\left( , \right)$ denotes the inner product in $\mathbb{R}^2$ and $H_x$ denotes the Hessian matrix of second partial derivatives with respect to $x$) then $\frac{1}{n} \det H_x \left( (\varphi(x), (\cos(\theta), \sin(\theta))) \right) \neq 0$. More restriction theorems for homogeneous sub-manifolds of $\mathbb{R}^{n+m}$ can be found in [3] and [4]. A very interesting book about the Fourier restriction problem can be found in [5].

On the other hand, the $L^p(\mathbb{R}^{n+m}) - L^q(\mathbb{R}^{n+m})$ boundedness of the convolution operator $T_{\mu} f$ has received also considerable attention in the literature. Define the type set $E_{\mu} \mu$ as the set of the pairs $\left( \frac{1}{p}, \frac{1}{q} \right) \in [0, 1] \times [0, 1]$ such that $\|T_{\mu} f\|_q \leq C_{p,q} \|f\|_p$ for all $f \in S(\mathbb{R}^{n+m})$, where $C_{p,q}$ is a positive constant.

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constant depending only on $p$ and $q$. Explicit descriptions of the set $E_{\mu\nu}$ are known for many cases (see e.g. [12] and the references therein).

In particular, S. W. Drury and K. Guo in [8] studied the case $n = m$, for $\varphi = (\varphi_1, ..., \varphi_n)$ regular enough in an open $V \subset \mathbb{R}^n$, and they proved that if each point in $V$ is elliptic for $\varphi$ and $U$ is a bounded open set such that $\overline{U} \subset V$, then the set $E_{\mu\nu}$ is the closed triangle with vertices $(0,0), (1,1)$ and $(1, \frac{1}{2})$. Elliptic points can be defined as follows: for $x \in V$, consider the function $Q_x$ on $\mathbb{R}^n$ defined by

$$Q_x(\zeta) = \det \left( \sum_{j=1}^n \zeta_j \varphi_j''(x) \right), \quad \zeta = (\zeta_1, ..., \zeta_n) \in \mathbb{R}^n,$$

where $\varphi_j''(x) := (H\varphi_j)(x)$ is the Hessian matrix of the function $\varphi_j$ at $x$. We point out that $Q_x$ is only a quadratic form for $n = 2$. Then, we say that $x \in V$ is elliptic if $\min_{\zeta \in S^{n-1}} |Q_x(\zeta)| > 0$, i.e.: if for all $\zeta \in S^{n-1}$ the surface $\Sigma_\zeta = \{(y, \varphi(y), \zeta) : y \in V\}$ has nonzero curvature at $x$ (although the definition of elliptic point used in [8] is different from the given above, at least for $n = 2$, they are equivalent; see the comments after Lemma 1 below). Similar results about the type set $E_{\mu\nu}$ are given in [8] for the case when $\varphi$ is a non isotropic homogeneous function such that the origin is the unique nonelliptic point.

Our aim in this paper is to study the case where $n = m = 2$ and $\varphi$ is a function with non isotropic homogeneity, whose set of nonelliptic points is a curve in $\mathbb{R}^2$.

Now, we state our precise assumptions and results. Let $\alpha_1, \alpha_2 > 0$ with $\alpha_1 \neq \alpha_2$, and for $t > 0$, $x = (x_1, x_2) \in \mathbb{R}^2$, let

$$t \cdot x = (t^{\alpha_1} x_1, t^{\alpha_2} x_2).$$

For $a < b$, we set

$$V^{a,b} = \{t \cdot (1, s) : s \in (a, b) \text{ and } t > 0\},$$

and for $a < c < d < b$, we put

$$V^{c,d}_1 = \{t \cdot (1, s) : s \in (c, d) \text{ and } 0 < t < 1\},$$

and let $\varphi = (\varphi_1, \varphi_2) : V^{a,b} \to \mathbb{R}^2$ be a function. We assume that the following hypotheses hold:

1. $\varphi$ is a real analytic function on $V^{a,b}$.
2. For some $m \geq 3(\alpha_1 + \alpha_2)$, $\varphi(t \cdot x) = t^m \varphi(x)$ for all $x \in V^{a,b}$, and all $t > 0$.
3. For $a < c < d < b$ and some $\sigma \in [c, d]$, the set of nonelliptic points for $\varphi$ in $V^{c,d}_1 \setminus \{0\}$ is the curve $\{t \cdot (1, \sigma) : t > 0\}$.

Under these assumptions, from Lemma 1 and Lemma 2 (see Preliminaries below), there exist two positive integers $n_1$ and $n_2$, and a positive constant $D$ such that for $\delta$ positive and small enough

$$\min_{\zeta \in S^1} |Q_{\bullet, (1, s)}(\zeta)| \geq D t^\beta |s - \sigma|^{n_1} \quad \text{for all } s \in (\sigma - \delta, \sigma), \text{ and all } t > 0, \text{ if } \sigma \in (c, d),$$

and

$$\min_{\zeta \in S^1} |Q_{\bullet, (1, s)}(\zeta)| \geq D t^\beta |s - \sigma|^{n_2} \quad \text{for all } s \in (\sigma, \sigma + \delta), \text{ and all } t > 0, \text{ if } \sigma \in [c, d).$$

where $\beta = 2(m - \alpha_1 - \alpha_2)$. Taking into account the definition of elliptic points, [8] and [12] suggest to use the number $\max\{n_1, n_2\}$ as a measure of the degeneracy of ellipticity along the curve $\{t \cdot (1, \sigma) : t > 0\}$.
Let \( a < c < d < b \) such that \( \sigma \in [c, d] \) (\( \sigma \) satisfies H3), and let \( \mu \) be the measure, fixed from now on, defined by (11) taking there \( U = V_{1}^{c,d} \), where \( V_{c,d} \) is given by (5). Let \( \tilde{\mu} \) be its Fourier transform given, for \( \xi', \xi'' \in \mathbb{R}^{2} \), by

\[
\tilde{\mu}(\xi', \xi'') = \int_{V_{c,d}^{*}} e^{-\langle (x, \xi') + (\varphi(x), \xi'') \rangle} \, dx.
\]

In Section 3, Theorem 12 we prove that \( \tilde{\mu} \) satisfies, for some constant \( C > 0 \), the following estimate

(8) \[
|\tilde{\mu}(\xi', \xi'')| \leq C |\xi''|^{-\frac{a_1 + a_2}{m}}, \quad \text{for all } \xi' \in \mathbb{R}^{2} \text{ and all } \xi'' \in \mathbb{R}^{2} \setminus \{0\}.
\]

In Section 4, we consider the restriction operator \( \mathcal{R}f = \hat{f} |_{\Sigma} \), with \( \Sigma \) given by (2) (with \( U = V_{1}^{c,d} \)). Following ideas in (11), using (3) and complex interpolation for a suitable analytic family of operators, in Theorem 13 we prove that there exists a constant \( C > 0 \) such that \( \|\mathcal{R}f\|_{L^2(\Sigma)} \leq C \|f\|_{L^2(\mathbb{R}^2)} \) for all \( f \in \mathcal{S}(\mathbb{R}^4) \) if and only if \( \frac{a_1 + a_2 + 4m}{2(a_1 + a_2 + 2m)} \leq \frac{1}{p} \leq 1 \). Finally, using again (3), we study the type set \( E_{\mu} \) for the convolution operator \( T_{\mu} f = \mu * f \), for \( f \in \mathcal{S}(\mathbb{R}^4) \). Theorem 15 states that, for \( p = \frac{a_1 + a_2 + 2m}{a_1 + a_2 + m} \), the closed triangle with vertices \((0,0)\), \((1,1)\) and \((\frac{1}{p}, \frac{1}{p})\) is contained in \( E_{\mu} \); moreover \((\frac{1}{p}, \frac{1}{p}) \in \partial E_{\mu} \). Finally, in Section 5, we give an example of a function \( \varphi = (\varphi_1, \varphi_2) : V_{a,b} \to \mathbb{R}^2 \) satisfying the hypotheses required.

2. Preliminaries

For the sequel, we consider a function \( \varphi = (\varphi_1, \varphi_2) : V_{a,b} \subset \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \) satisfying H1-H3. Given \( x \in V_{a,b} \), let \( Q_x \) be the quadratic form on \( \mathbb{R}^2 \) defined by

(9) \[
Q_x(\zeta) = \text{det} (\zeta_1 \varphi''_1(x) + \zeta_2 \varphi''_2(x)), \quad \zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^2,
\]

where, for each \( j = 1, 2 \), \( \varphi''_j(x) := (\mathbf{H} \varphi_j)(x) \) is the Hessian matrix of the function \( \varphi_j : V_{a,b} \to \mathbb{R} \) at \( x \); and let \( K(x) = (k_{ij}(x)) \) be its associated symmetric matrix, i.e.: satisfying

(10) \[
Q_x(\zeta) = \langle K(x) \zeta, \zeta \rangle, \quad \zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^2.
\]

We say that a point \( x \in V_{a,b} \) is elliptic for \( \varphi \), if \( \min_{\zeta \in \mathbb{S}^1} |Q_x(\zeta)| > 0 \).

**Lemma 1.** If \( x \in V_{a,b} \) is an elliptic point for \( \varphi \), then the eigenvalues of \( K(x) \) are either both positive or both negative and

\[
\min_{\zeta \in \mathbb{S}^1} |Q_x(\zeta)| = \min\{|\Lambda| : \Lambda \text{ is an eigenvalue of } K(x)\}.
\]

**Proof.** Follows immediately from the definition of elliptic point and (10). \(\square\)

Let us quote that a definition of elliptic point, different from the stated at the introduction, is given in (6). Let us recall it: For \( x \in V_{a,b}, h \in \mathbb{R}^2 \), let \( \varphi''(x)h \) be the \( 2 \times 2 \) matrix whose \( j \)-th column is \( \varphi''_j(x)h \), where \( \varphi''_j(x) \) is the Hessian matrix of the function \( \varphi_j : V_{a,b} \to \mathbb{R} \) at \( x \). A point \( x \in V_{a,b} \) is called elliptic in (6), if \( \det (\varphi''(x)h) \neq 0 \) for all \( h \in \mathbb{R}^2 \setminus \{0\} \). This definition is equivalent to ours. Indeed, consider the symmetric matrix \( B(x) \) defined by \( \langle B(x) \zeta, \zeta \rangle = \text{det} (\varphi''(x)\zeta), \zeta \in \mathbb{R}^2 \). An explicit computation shows that \( \text{det} B(x) = \text{det} K(x) \) and so the two definitions agree.

**Lemma 2.** (i) There exist two homogeneous functions \( \Lambda_1 \) and \( \Lambda_2 \) on \( V_{a,b} \) of degree \( \beta = 2(m - a_1 - a_2) \) respect to the dilations given in (11), which give the eigenvalues of \( K(x) \). Moreover, \( \Lambda_1 \) and \( \Lambda_2 \) result real analytic functions on \( V_{a,b} \).

(ii) For \( \delta > 0 \) and small enough, it holds that:
for some $j = 1, 2$, \( \min(\{\Lambda_j(x) : i = 1, 2\}) = |\Lambda_j(x)| \) for all \( x \in V^{\sigma - \delta, \sigma} \), if \( \sigma \in (c, d] \), and for some \( k = 1, 2 \), \( \min(\{\Lambda_k(x) : i = 1, 2\}) = |\Lambda_k(x)| \) for all \( x \in V^{\sigma, \sigma + \delta} \), if \( \sigma \in [c, d) \).

(iii) There exist two positive integers \( n_1 \) and \( n_2 \), and a positive constant \( D \) such that for \( \sigma \) positive and small enough

\[
\min(\{\Lambda_i(t \cdot (1, s)) : i = 1, 2\}) \geq D t^3 |s - \sigma|^{n_1} \quad \text{for all } s \in (\sigma - \delta, \sigma), \text{ and all } t > 0, \text{ if } \sigma \in (c, d],
\]

and

\[
\min(\{\Lambda_i(t \cdot (1, s)) : i = 1, 2\}) \geq D t^3 |s - \sigma|^{n_2} \quad \text{for all } s \in (\sigma, \sigma + \delta), \text{ and all } t > 0, \text{ if } \sigma \in [c, d).
\]

(iv) If \( I \subset [c, d] \) is a closed interval such that \( \sigma \notin I \), then there exists a positive constant \( \tilde{D} \) such that \( \min(\{\Lambda_i(t \cdot (1, s)) : i = 1, 2\}) \geq \tilde{D} t^3 \) for all \( s \in I \) and all \( t > 0 \).

**Proof.** A computation shows that the entries \( k_{ij}(x) \) of \( K(x) \) are given by

\[
k_{11}(x) = \det \varphi_1(x), \quad k_{22}(x) = \det \varphi_2(x),
\]

\[
k_{12}(x) = k_{21}(x) = \frac{1}{2} \left( \frac{\partial^2 \varphi_1}{\partial x_1^2} \frac{\partial^2 \varphi_2}{\partial x_2^2} + \frac{\partial^2 \varphi_1}{\partial x_2^2} \frac{\partial^2 \varphi_2}{\partial x_1^2} - 2 \frac{\partial^2 \varphi_1}{\partial x_1 \partial x_2} \frac{\partial^2 \varphi_2}{\partial x_1 \partial x_2} \right)(x).
\]

From H1 it follows that the functions \( k_{ij}, 1 \leq i, j \leq 2 \), are real analytic on \( V^{a, b} \), and from H2 the \( k_{ij} \)'s result homogeneous of degree \( \beta = 2(m - \alpha_1 - \alpha_2) \), i.e.: for every \( 1 \leq i, j \leq 2 \), \( k_{ij}(t \cdot x) = t^\beta k_{ij}(x) \) for \( x \in V^{a, b} \), and \( t > 0 \). It is easy to check that the eigenvalues \( \Lambda_1(x) \) and \( \Lambda_2(x) \) of \( K(x) \) are given by

\[
\Lambda_1(x) = \frac{1}{2} \left( k_{11}(x) + k_{22}(x) - \sqrt{4k_{12}^2(x) + (k_{11}(x) - k_{22}(x))^2} \right),
\]

and

\[
\Lambda_2(x) = \frac{1}{2} \left( k_{11}(x) + k_{22}(x) + \sqrt{4k_{12}^2(x) + (k_{11}(x) - k_{22}(x))^2} \right).
\]

The homogeneity of \( \Lambda_1 \) and \( \Lambda_2 \) follow from that of the entries \( k_{ij} \). We observe that for every \( s_0 \in (a, b) \) fixed and \( \delta \) positive and small enough, each \( \Lambda_i(1, \cdot) \) is real analytic on \( (s_0 - \delta, s_0 + \delta) \). Indeed, this is clear if either \( k_{12}(1, s_0) \neq 0 \) or \( k_{11}(1, s_0) \neq k_{22}(1, s_0) \). Now, if \( k_{12}(1, s_0) = 0 \) and \( k_{11}(1, s_0) - k_{22}(1, s_0) = 0 \), we consider the function

\[
f(s) = 4k_{12}^2(1, s) + (k_{11}(1, s) - k_{22}(1, s))^2.
\]

If \( f \) is identically zero on \( (s_0 - \delta, s_0 + \delta) \) for some \( \delta > 0 \), then \( \Lambda_1(1, s) = \Lambda_2(1, s) = k_{11}(1, s) \). So that each \( \Lambda_i(1, \cdot) \) is real analytic on \( (s_0 - \delta, s_0 + \delta) \). If \( f \) is not identically zero, since \( f \geq 0 \) and \( f(s_0) = 0 \), from the analyticity of the entries \( k_{ij} \) it follows that there exist \( \delta > 0 \), a positive integer \( q \), and a nonnegative real analytic function \( h \) such that \( h(s_0) > 0 \) and \( f(s) = (s - s_0)^2 q [h(s)]^2 \) for all \( s \in (s_0 - \delta, s_0 + \delta) \). Now, for \( i = 1, 2 \), we set

\[
\tilde{\Lambda}_i(1, s) = \frac{1}{2} (k_{11}(1, s) + k_{22}(1, s) + (-1)^i (s - s_0)^2 h(s)).
\]

It is clear that the functions \( \tilde{\Lambda}_i(1, \cdot) \) are real analytic on \( (s_0 - \delta, s_0 + \delta) \). If \( q \) is even, then \( \tilde{\Lambda}_i(1, s) = \Lambda_i(1, s) \) for \( s \in (s_0 - \delta, s_0 + \delta) \) and \( i = 1, 2 \). If \( q \) is odd, then \( \tilde{\Lambda}_i(1, s) = \Lambda_i(1, s) \) for \( s \in [s_0, s_0 + \delta) \), and \( \tilde{\Lambda}_i(1, s) = \Lambda_i(1, s) \) for \( s \in (s_0 - \delta, s_0) \) (similar identities are obtained for \( \tilde{\Lambda}_i(1, \cdot) \)). Thus, the functions \( x \to \tilde{\Lambda}_i(x), i = 1, 2 \), give the eigenvalues of \( K(x) \). From this analysis and the homogeneity of \( \Lambda_1 \) and \( \Lambda_2 \) we obtain its analyticity on \( V^{a, b} \). Hence, \( (i) \) follows.

To see \( (ii) \), we observe that each point in \( V^{c, \sigma} \) (resp. in \( V^{\sigma, d} \)) is elliptic and so, by Lemma 1, \( \Lambda_1 \) and \( \Lambda_2 \) have the same sign in \( V^{c, \sigma} \) (resp. in \( V^{\sigma, d} \)). Then, \( (ii) \) follows from the fact that \( \Lambda_1 \leq \Lambda_2 \) on \( V^{c, \sigma} \) (resp. on \( V^{\sigma, d} \)).

Finally, Lemma 1 (ii) and (ii) allow us to get \( (iii) \) and \( (iv) \).
Lemma 3. Let \( G : S^1 \times (a,b) \rightarrow \mathbb{R} \) be the function given by \( G(\zeta, s) = \langle \varphi(1,s), \zeta \rangle \). Then,

\[
\alpha_1^2 \det (\zeta_1 \varphi''(1, s) + \zeta_2 \varphi''(1, s)) = m(m - \alpha_1)(G_{ss}(\zeta, s) + \alpha_2(\alpha_1 - \alpha_2)s(G_s G_{ss})(\zeta, s) - (m - \alpha_2)^2 G_s^2(\zeta, s),
\]

for every \( \zeta = (\zeta_1, \zeta_2) \in S^1 \) and every \( s \in (a,b) \).

Proof. For \( \zeta \in S^1 \) fixed, let \( \Psi(x) = \langle \varphi(x), \zeta \rangle \), for \( x \in V_{a,b} \). Clearly, the function \( \Psi \) is homogeneous of degree \( m \) respect to the dilations in \( \mathbb{R} \) and so, from the Euler equations,

\[
m \Psi(x) = \alpha_1 x_1 \Psi_{x_1}(x) + \alpha_2 x_2 \Psi_{x_2}(x)
\]

\[
(m - \alpha_1) \Psi_{x_1_1}(x) + \alpha_2 x_2 \Psi_{x_2_1}(x) + \alpha_1 x_1 \Psi_{x_1_2}(x) + \alpha_2 x_2 \Psi_{x_2_2}(x).
\]

Thus

\[
m G(\zeta, s) = \alpha_1 \Psi_{x_1_1}(1, s) + s \alpha_2 \Psi_{x_2_1}(1, s) = \frac{\alpha_2 s^2}{m - \alpha_2} \Psi_{x_2_1}(1, s) + \frac{\alpha_1 \alpha_2}{m - \alpha_2} \Psi_{x_2_2}(1, s); \]

we also have that

\[
\Psi_{x_2}(1, s) = G_s(\zeta, s) \quad \text{and} \quad \Psi_{x_2x_2}(1, s) = G_{ss}(\zeta, s),
\]

and so

\[
(m - \alpha_2) G_s(\zeta, s) = \alpha_1 \Psi_{x_2_1}(1, s) + \alpha_2 s G_s(\zeta, s).
\]

Then

\[
\Psi_{x_2_2}(1, s) = \frac{m - \alpha_2}{\alpha_1} G_s(\zeta, s) - \frac{\alpha_2}{\alpha_1} s G_{ss}(\zeta, s),
\]

from \((13)\), using \((14)\) and \((15)\) we can express \( \Psi_{x_2_2}(1, s) \) in terms of \( G(\zeta, s), G_s(\zeta, s) \) and \( G_{ss}(\zeta, s) \). Taking into account this expression, and also \((13)\) and \((15)\), a computation of \( \det(\Psi''(1, s)) \) gives \((12)\).

Remark 4. Let \( n_1, n_2, D \) and \( \tilde{D} \) be as in Lemma \( \text{B} \). For \( (\zeta, s) \in S^1 \times (a,b) \), let

\[
H(\zeta, s) = m(m - \alpha_1)(G_{ss}(\zeta, s) + \alpha_2(\alpha_1 - \alpha_2)s(G_s G_{ss})(\zeta, s) - (m - \alpha_2)^2 G_s^2(\zeta, s).
\]

From \((7)\), and Lemmas \( \text{A} \), \( \text{B} \) and \( \text{C} \) we have that

\((i)\) For \( \delta > 0 \) and small enough, \( |H(\zeta, s)| \geq D|s - \sigma|^n_1 \) for all \( s \in (\sigma - \delta, \sigma) \) and all \( \zeta \in S^1 \), if \( \sigma \in (c,d) \).

\((ii)\) For \( \delta > 0 \) and small enough, \( |H(\zeta, s)| \geq D|s - \sigma|^n_2 \) for all \( s \in (\sigma, \sigma + \delta) \) and all \( \zeta \in S^1 \), if \( \sigma \in [c,d) \).

\((iii)\) If \( I \subset [c,d] \setminus \{\sigma\} \) is a closed interval, then \( |H(\zeta, s)| \geq \tilde{D} \) for all \( s \in I \) and all \( \zeta \in S^1 \).

3. An estimate for \( \tilde{\mu} \)

For an open set \( U \subset \mathbb{R}^2 \), we put \( \mu = \mu_U \) for the measure defined by \((1)\). For \((c,d) \subset (a,b) \) such that \( \sigma \in [c,d] \) (see hypothesis \( \text{H3} \)), we take the open set \( U = V_{a,b}^\ast \) given by \((6)\). Then, the change of variable \( x_1 = t^{\alpha_1}, x_2 = st^{\alpha_2} \) gives

\[
\tilde{\mu}(\xi', \xi'') = \alpha_1 \int_c^d \left( \int_0^1 e^{-t(\xi_1 t^{\alpha_1} + \xi_2 st^{\alpha_2} + t^m \langle \varphi(1,s), \xi'' \rangle)} t^{\alpha_1 + \alpha_2 - 1} dt \right) ds,
\]

where \( \xi' = (\xi_1, \xi_2) \) and \( \xi'' = (\xi_3, \xi_4) \).

The following lemmas will allow us to give a decay estimate for \((16)\).
Lemma 5. Let $A_j \in \mathbb{R}$, for $j = 1, 2, 3$. Then there exists a positive constant $C$ such that

$$
\left| \int_0^1 e^{-i(A_1 t^{\alpha_1} + A_2 t^{\alpha_2} + A_3 t^m)} t^{\alpha_1 + \alpha_2 - 1} \, dt \right| \leq C |A_3|^\frac{\alpha_1 + \alpha_2}{m}.
$$

Proof. Without loss of generality we can assume $A_3 > 0$. The change of variable $\tau = A_3^{-\frac{\alpha_1 + \alpha_2}{m}} t^{\alpha_1 + \alpha_2}$ allows us to express the integral in (17) as

$$
\left| \int_0^{A_3^{-\frac{\alpha_1 + \alpha_2}{m}}} e^{-(a \tau^{\gamma_1} + b \tau^{\gamma_2} + c \tau^{\gamma_3})} \, d\tau, \right|
$$

with $a := A_1 A_3^{-\frac{\alpha_1}{m}}$, $b := A_2 A_3^{-\frac{\alpha_2}{m}}$, $\gamma_j := \frac{\alpha_j}{\alpha_1 + \alpha_2}$ for $j = 1, 2$ and $\gamma_3 := \frac{m}{\alpha_1 + \alpha_2}$. Let $\Phi(\tau) = a \tau^{\gamma_1} + b \tau^{\gamma_2} + c \tau^{\gamma_3}$. It is clear that $0 < \gamma_1, \gamma_2 < 1$ and $\gamma_3 \geq 3$ (see hyp. H2), so $\Phi$ do not reduce to zero after taking three derivatives. To prove the lemma it is enough to see that there exists a positive constant $C$ independent of $a$, $b$, and $A_3$ such that for all $B > 1$

$$
\left| \int_1^B e^{-i\Phi(\tau)} \, d\tau \right| \leq C.
$$

For $\gamma \in \mathbb{R}$ and $j \in \mathbb{N}$, we set $\gamma(j) = \gamma(\gamma - 1) \cdots (\gamma - j + 1)$. We observe that $\tau^j \Phi^{(j)}(\tau) = \gamma_1(j) a \tau^{\gamma_1 + 1} + \gamma_2(j) b \tau^{\gamma_2 + 1} + \gamma_3(j) \tau^{\gamma_3 + 1}$, $j = 1, 2, 3$, and that the matrix equation

$$
\begin{bmatrix}
\gamma_1(1) & \gamma_1(2) & \gamma_1(3) \\
\gamma_2(1) & \gamma_2(2) & \gamma_2(3) \\
\gamma_3(1) & \gamma_3(2) & \gamma_3(3)
\end{bmatrix}
\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} =
\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
$$

has unique solution $(c_1, c_2, c_3)$ (since $\det[\gamma(j)] = \gamma_1 \gamma_2 \gamma_3(\gamma_2 - \gamma_1)(\gamma_3 - \gamma_1)(\gamma_3 - \gamma_2) \neq 0$). Thus $\tau^\gamma = \sum_{j=1}^3 c_j \tau^{\Phi^{(j)}(\tau)}$ and so $\tau^\gamma \leq \sum_{j=1}^3 |c_j| \tau^{\Phi^{(1)}(\tau)}$ for all $\tau \geq 1$. Then, $(1, B) \subset \cup_{j=1}^3 I_j$ where $I_j = \{ \tau \in (1, B) : \Phi^{(j)}(\tau) \geq \gamma(j) \}$. Since $\frac{d^3}{d\tau^3} \left( \tau^{\gamma(j) + 1} \frac{d^2}{d\tau^2} \left( \tau^{\gamma(j) + 1} \frac{d}{d\tau} \Phi^{(j)}(\tau) \right) \right) 
eq 0$ for all $\tau \geq 1$ and each $j = 1, 2, 3$, the Rolle’s Theorem applied twice implies that, for each $c \in \mathbb{R}$, the equation $\Phi^{(j)}(\tau) = c$ has at most three solutions. Hence, each $I_j$ has at most six connected components. Being $m \geq 3(\alpha_1 + \alpha_2)$ we have $\gamma_3 \geq 3$, so (18) follows from the Van der Corput lemma (see e.g. [14], p. 332).

Next, we define three functions $f_i : S^1 \times (a, b) \to \mathbb{R}$, $i = 1, 2, 3$, by

$$
f_1(\zeta, s) = (m - \alpha_1) G(\zeta, s) G_{ss}(\zeta, s),
$$

$$
f_2(\zeta, s) = (m - \alpha_2)^2 G_{2s}(\zeta, s),
$$

$$
f_3(\zeta, s) = \alpha_2(\alpha_1 - \alpha_2) s G_{s}(\zeta, s) G_{ss}(\zeta, s),
$$

and for $\delta > 0$ and small, $\zeta \in S^1$ and $C > 0$ let

$$
\begin{align*}
&I_1^\delta, C = \{ s \in (\sigma - \delta, \sigma) : f_1(\zeta, s) > \frac{C}{T} |s - \sigma|^{-n_1} \}, \\
&I_2^\delta, C = \{ s \in (\sigma - \delta, \sigma) : f_2(\zeta, s) < -\frac{C}{T} |s - \sigma|^{-n_1} \}, \\
&I_3^\delta, C = \{ s \in (\sigma - \delta, \sigma) : f_3(\zeta, s) > \frac{C}{T} |s - \sigma|^{-n_1} \}, \\
&I_4^\delta, C = \{ s \in (\sigma - \delta, \sigma) : f_2(\zeta, s) > \frac{C}{T} |s - \sigma|^{-n_1} \}, \\
&I_5^\delta, C = \{ s \in (\sigma - \delta, \sigma) : f_3(\zeta, s) < -\frac{C}{T} |s - \sigma|^{-n_1} \}.
\end{align*}
$$

For $\delta > 0$ and $i = 1, 2, 3, 4, 5$, let $J_i^\delta, C$ be the analogous sets defined again by (20), but replacing there $(\sigma - \delta, \sigma)$ and $n_1$ by $(\sigma, \sigma + \delta)$ and $n_2$ respectively. Finally, for $\tau \in [c, d] \setminus \{\sigma\}$ such that
\(\sigma \not\in (\tau - \delta, \tau + \delta)\) and \(i = 1, 2, 3, 4, 5\), let \(K^{\tau, \delta, \xi, C}\) be the sets defined by (20), but replacing there \((\sigma - \delta, \sigma)\) and \(\frac{C}{4}|s - \sigma|^{n_1}\) by \((\tau - \delta, \tau + \delta) \cap [c, d]\) and \(\frac{C}{4}\) respectively.

**Lemma 6.** (i) For every \(\eta \in S^1\) there exist two positive constants \(\delta_0\) and \(C_0\) and a neighborhood \(W_\eta\) of \(\eta\) in \(S^1\) such that, for all \(\zeta \in W_\eta\) and all \(j = 1, 2, \ldots, 5\), the sets \(J^{\tau, \delta_0, \zeta, C_0}\) and \(J_{\tau, \zeta, C_0}\) have at most \(n_1\) and \(n_2\) connected components respectively.

(ii) For every \(\tau \in [c, d] \setminus \{\sigma\}\) and \(\eta \in S^1\) there exist two positive constants \(\delta_0, \tau\) and \(C_{\eta, \tau}\) and a neighborhood \(W_{\eta, \tau}\) of \(\eta\) in \(S^1\) such that, for all \(\zeta \in W_{\eta, \tau}\) and all \(j = 1, 2, \ldots, 5\), either \(K^{\tau, \delta_0, \zeta, C_{\eta, \tau}} = \emptyset\) or \(K^{\tau, \delta_0, \zeta, C_{\eta, \tau}} = (\tau - \delta_0, \tau + \delta_0) \cap [c, d]\).

**Proof.** To see (i), we will show, for an appropriate positive constant \(C\), that the sets where the \(f_i\)'s (see (19)) are equal to \(\pm \frac{C}{4}|s - \sigma|^{n_j}\) \((j = 1, 2)\) are boundedly finite, so the boundedness of the number of connected components will follow from this. For them, we proceed as follows. Given 

\[f \in C^\infty(a, b),\]

let \(d_f\) be defined by

\[d_f = \begin{cases} \min \{l \geq 0 : f^{(l)}(\sigma) \neq 0\} & \text{if } f^{(k)}(\sigma) \neq 0 \text{ for some } k \geq 0, \\ +\infty & \text{otherwise}. \end{cases} \]

For \(i = 1, 2, 3\), let \(f_i : S^1 \times (a, b) \to \mathbb{R}\) be the functions given by (19) and, for \(\eta \in S^1\) fixed, let \(k_i = d_{f_i(\eta, \tau)}\). By Remark 4 (i), we have that \(d_{f_{i, j - 2}(\eta, \tau)} \leq n_1\) and thus \(k_i \leq n_1\) for some \(i = 1, 2, 3\). Let \(A = \{i \in \{1, 2, 3\} : k_i \leq n_1\}\) and for \(\eta \in S^1\) let \(C_0 := \min \left\{ \left| \frac{C}{4}\right|^n f_i(\eta, \sigma) : i \in A \right\}\), where \(\partial_s\) denotes the partial derivative respect to \(s\). For \((\zeta, s) \in S^1 \times (a, b)\) let

\[F_i(\zeta, s) = f_i(\zeta, s) - \frac{C^2}{4}(s - \sigma)^{n_1}, \quad i = 1, 2, 3.\]

If \(i \in A\), we have that \(\partial_s^n F_i(\eta, \sigma) \neq 0\) and so there exists \(\delta_{i, \eta} > 0\) and a neighborhood \(W_{\eta, i}\) of \(\eta\) in \(S^1\) such that \(\partial_s^n F_i(\zeta, s) \neq 0\) for all \((\zeta, s) \in W_{\eta, i} \times (\sigma - \delta_{i, \eta}, \sigma)\). Then, by the Rolle’s Theorem for \(\zeta \in W_{\eta, i}\) and \(i \in A\), the equation \(f_i(\zeta, s) = \frac{C^2}{4}(s - \sigma)^{n_1}\) has at most \(k_i\) roots in \((\sigma - \delta_{i, \eta}, \sigma)\) with \(k_i \leq n_1\). Now, if \(i \not\in A\), then \(\partial_s^n F_i(\eta, \sigma) \neq 0\) and so, proceeding as above, we find also in this case \(\delta_{i, \eta} > 0\) and \(W_{\eta, i}\) such that for \(\zeta \in W_{\eta, i}\), \(f_i(\zeta, s) = \frac{C^2}{4}(s - \sigma)^{n_1}\) has at most \(n_1\) roots in \((\sigma - \delta_{i, \eta}, \sigma)\). Similarly, by considering \(\tilde{F}_i(\zeta, s) = f_i(\zeta, s) + \frac{C^2}{4}(s - \sigma)^{n_1}\) instead of \(F_i\), and diminishing \(W_{\eta, i}\) and \(\delta_{i, \eta}\) if necessary, we obtain that for \(\zeta \in W_{\eta, i}\), \(f_i(\zeta, s) = \frac{C^2}{4}(s - \sigma)^{n_1}\) has at most \(n_1\) roots in \((\sigma - \delta_{i, \eta}, \sigma)\). From these facts it is clear that the assertion in (i) holds for the sets \(J^{\tau, \delta_0, \zeta, C_{\eta, \tau}}\) if we take \(\tilde{\delta}_0 = \min \{\delta_{i, \eta} : i = 1, 2, 3\}\) and \(W_{\eta, i} = \bigcap_{i=1}^{3} W_{\eta, i}\). The proof of the statement for the sets \(J_{\tau, \zeta, C_0}\) is similar.

To see (ii), let \(\tau \in [c, d] \setminus \{\sigma\}\) and let \(f_i\) be the functions defined above. For a constant \(\tilde{D} > 0\) as in Lemma 2 (iv), from Remark 4 (iii), we have that

\[|f_1(\zeta, \tau) - f_2(\zeta, \tau) + f_3(\zeta, \tau)| \geq \tilde{D},\]

for all \(\zeta \in S^1\). For \(\eta \in S^1\), we define \(A = \{i \in \{1, 2, 3\} : f_i(\eta, \tau) \neq 0\}\) and \(C_{\eta, \tau} := \min \{|\tilde{D}\} \cup \{|f_i(\eta, \tau)| : i \in A\}\}\). For \((\zeta, s) \in S^1 \times (a, b)\) and \(i = 1, 2, 3\), let \(F_i(\zeta, s) = f_i(\zeta, s) - \frac{C^2}{4}\) and \(\tilde{F}_i(\zeta, s) = f_i(\zeta, s) + \frac{C^2}{4}\). Thus \(F_i(\eta, \tau) \neq 0\) and \(\tilde{F}_i(\eta, \tau) \neq 0\) for \(i = 1, 2, 3\) and so there exist \(\delta_{i, \tau, i, \tau}\) and a neighborhood \(W_{\eta, \tau, i}\) of \(\eta\) in \(S^1\) such that \(F_i(\zeta, s) \neq 0\) and \(\tilde{F}_i(\zeta, s) \neq 0\) for all \(\zeta \in W_{\eta, \tau, i}\) and all \(s \in (\tau - \delta_{i, \tau, i, \tau}, \tau + \delta_{i, \tau, i, \tau}) \cap [c, d]\). We set \(W_{\eta, \tau} = \bigcap_{i=1}^{3} W_{\eta, \tau, i}\) and \(\delta_{\tau, i} = \min \{\delta_{i, \tau, i, \tau} : i = 1, 2, 3\}\). Since for \(\zeta \in W_{\eta, \tau}\) and \(i = 1, 2, 3\), the equations \(f_i(\zeta, s) = \pm \frac{C^2}{4}\) have no roots in \((\tau - \delta_{i, \tau, i, \tau}, \tau + \delta_{i, \tau, i, \tau}) \cap [c, d]\) it follows, for every \(j\), that \(K^{\tau, \delta_{\tau, i}, \zeta, C_{\eta, \tau}} = \emptyset\) or \(K^{\tau, \delta_{\tau, i}, \zeta, C_{\eta, \tau}} = (\tau - \delta_{\tau, \eta, \tau}, \tau + \delta_{\tau, \eta, \tau}) \cap [c, d]\). This completes the proof. □

**Remark 7.** We observe that if \(D\) is a positive constant as in Lemma 2 (iii), then for each \(\eta \in S^1\) we can find a positive constant \(C_0 \leq D\) for which Lemma 6 (i), still holds.
Let $f \in C^1([\theta_1, \theta_2]), \gamma > 1$ and let $I$ be a closed interval such that $I \subseteq [\theta_1, \theta_2]$. If one of the following two conditions holds

\begin{equation}
(f'(s))^2 \geq A(s - \theta_1)^\alpha \quad \text{for all } s \in I,
\end{equation}

\begin{equation}
(f'(s))^2 \geq A(\theta_2 - s)^\alpha \quad \text{for all } s \in I,
\end{equation}

for positive constants $A$ and $\alpha$ with $\alpha < 2\gamma - 2$, then there exists a constant $C > 0$ depending only on $A$, $\alpha$, $\theta_1$, $\theta_2$ and $\gamma$ such that

\begin{equation}
\int_I |f(s)|^{-\frac{\gamma}{\alpha}} ds \leq C.
\end{equation}

\textbf{Proof.} Let $I = [a, b]$. Assume that (24) holds. Since $f'$ is continuous we have that either (i) $f'(s) \geq 0$ for all $s \in I$ or (ii) $f'(s) \leq 0$ for all $s \in I$. Consider the case (i). It is clear that $f'(s) > 0$ for all $s \in (a, b)$, so $f$ is strictly increasing on $[a, b]$. Thus only one of the following three cases occurs

a) $f$ is positive everywhere in $(a, b]$ with $f(a) \geq 0,$

b) $f$ is negative everywhere in $[a, b)$ with $f(b) \leq 0,$

c) $f$ vanishes at exactly a point $s_0 \in (a, b)$.

If a) holds, then for every $s \in I$

\[ |f(s)| \geq \int_a^s f'(t) \, dt \geq \frac{A^{1/2}}{1 + \frac{\alpha}{2}} ((s - \theta_1)^{1+\frac{\alpha}{2}} - (a - \theta_1)^{1+\frac{\alpha}{2}}) \geq \frac{A^{1/2}}{1 + \frac{\alpha}{2}} (s - a)^{1+\frac{\alpha}{2}}, \]

which, taking into account that $\alpha < 2\gamma - 2$, implies (23) with the constant $C > 0$ required.

If b) holds, then for every $s \in I$

\[ |f(s)| \geq \int_s^b f'(t) \, dt \geq \frac{A^{1/2}}{1 + \frac{\alpha}{2}} ((b - \theta_1)^{1+\frac{\alpha}{2}} - (s - \theta_1)^{1+\frac{\alpha}{2}}) \geq \frac{A^{1/2}}{1 + \frac{\alpha}{2}} (b - s)^{1+\frac{\alpha}{2}}, \]

which implies (23) with the constant $C > 0$ required since $\alpha < 2\gamma - 2$.

If c) holds, we have for $a \leq s \leq s_0$

\[ |f(s)| \geq \int_s^{s_0} f'(t) \, dt \geq \frac{A}{1 + \frac{\alpha}{2}} ((s_0 - \theta_1)^{1+\frac{\alpha}{2}} - (s - \theta_1)^{1+\frac{\alpha}{2}}) \geq \frac{A}{1 + \frac{\alpha}{2}} (s_0 - s)^{1+\frac{\alpha}{2}}, \]

and thus $\int_a^{s_0} |f(s)|^{-\frac{\gamma}{\alpha}} \, ds \leq C$, where $C$ is the constant required. Similarly, one can estimate $\int_s^{s_0} |f(s)|^{-\frac{\gamma}{\alpha}} \, ds$. So (23) holds in the case c). The case (ii) reduces to (i) by considering there $-f$ instead of $f$. Finally, the proof of that $\text{(22)} \implies \text{(23)}$ is similar. \hfill $\square$

\textbf{Lemma 9.} Let $f \in C^2([\theta_1, \theta_2]), \gamma > 1$ and let $I$ be a closed interval such that $I \subseteq [\theta_1, \theta_2]$. If one of the following two conditions holds

\begin{equation}
|f'(s)f''(s)| \geq A(s - \theta_1)^\alpha \quad \text{for all } s \in I,
\end{equation}

\begin{equation}
|f'(s)f''(s)| \geq A(\theta_2 - s)^\alpha \quad \text{for all } s \in I,
\end{equation}

for positive constants $A$ and $\alpha$ with $\alpha < 2\gamma - 3$, then (23) holds for some $C > 0$ depending only on $A$, $\alpha$, $\theta_1$, $\theta_2$ and $\gamma$.

\textbf{Proof.} Let $I = [a, b]$. If (24) holds, we have that either (i) $(f')^2(t) \geq 2A(t - \theta_1)^\alpha$ for all $t \in [a, b]$ or (ii) $(f')^2(t) \leq -2A(t - \theta_1)^\alpha$ for all $t \in [a, b]$. If (i) holds, by integrating on $[a, s]$, we obtain

\[ (f')^2(s) \geq (f')^2(a) + c_1 ((s - \theta_1)^{1+\alpha} - (a - \theta_1)^{1+\alpha}) \geq c_1 (s - a)^{1+\alpha} \quad \text{for all } s \in I, \]

where $c_1$ is a positive constant depending only on $A$ and $\alpha$. If (ii) holds, by integrating on $[s, b]$, we get similarly as above that $(f')^2(s) \geq c_2(b - s)^{1+\alpha}$ for all $s \in I$, with $c_2 = c_2(A, \alpha)$. Since $1+\alpha < 2\gamma - 2$, the lemma follows, in both cases (i) and (ii), from Lemma 8. Finally, the case when (25) holds is similar. \hfill $\square$
Lemma 10. Let \( f \in C^2([\theta_1, \theta_2]), \gamma > 1 \) and let \( I \) be a closed interval such that \( I \subseteq [\theta_1, \theta_2] \). If one of the following two conditions holds

\[
\begin{align*}
(26) & \quad f(s) f''(s) \geq A(s - \theta_1)^\alpha & \text{for all } s \in I, \\
(27) & \quad f(s) f''(s) \geq A(\theta_2 - s)^\alpha & \text{for all } s \in I,
\end{align*}
\]

for positive constants \( A \) and \( \alpha \) with \( \alpha < 2\gamma - 2 \), then \((26)\) holds for some \( C > 0 \) depending only on \( A, \alpha, \theta_1, \theta_2 \) and \( \gamma \).

Proof. Let \( I = [a, b] \). We give the proof only for the case when \((26)\) holds, since the proof when \((27)\) holds is similar. We assume that \((26)\) occurs, in this case either

(i) \( f \geq 0 \) and \( f'' \geq 0 \) on \( I \),

or

(ii) \( f \leq 0 \) and \( f'' \leq 0 \) on \( I \).

If (i) holds, let \( s_1 \) be the point where the minimum of \( f \) on \( I \) is achieved. If \( s_1 = a \), then \( f(a) \geq 0 \) and \( f'(a) \geq 0 \). From \((26)\), to integrate by parts, we get

\[
\frac{1}{2} (f^2)'(t) \geq f(t) f'(t) - f(s) f'(a) - \int_a^t (f')^2(r) dr = \int_a^t f(r) f''(r) dr,
\]

for a positive constant \( c_1 = c_1(A, \alpha) \) and for all \( t \in I \). A new integration on \([a, s]\) gives, for some positive constant \( c_2 = c_2(A, \alpha) \),

\[
f^2(s) \geq c_2 s(a - s)^{2+\alpha}, \text{ for all } s \in I.
\]

Since \( \alpha < 2\gamma - 2 \), \((28)\) gives \((29)\) with a constant \( C \) as required. Now, if \( s_1 = b \) we have that \( f(b) > 0 \) and \( f''(r) \leq f'(b) \leq 0 \) for every \( r \in I \). Using \((26)\), to integrate by parts on \([t, b]\), we get

\[
-(f^2)'(t) \geq c_3 (b - t)^{1+\alpha} \text{ for a positive constant } c_3 = c_3(A, \alpha) \text{ and for all } t \in I.
\]

Then, the proof in this case follows as above, but integrating on \([s, b]\). Now, we consider the case \( a < s_1 < b \). This case follows to apply the two cases above on the intervals \([s_1, b]\) and \([a, s_1]\) respectively.

Finally, the case (ii) reduces to (i) by considering there \(-f\) instead of \( f \).

Lemma 11. Let \( f \in C^2([\theta_1, \theta_2]), \gamma > 1 \) and let \( I \) be a closed interval such that \( I \subseteq [\theta_1, \theta_2] \). If one of the following two conditions holds

\[
\begin{align*}
(29) & \quad f(s) f''(s) \leq -A(s - \theta_1)^\alpha & \text{for all } s \in I, \\
(30) & \quad f(s) f''(s) \leq -A(\theta_2 - s)^\alpha & \text{for all } s \in I,
\end{align*}
\]

for positive constants \( A \) and \( \alpha \) with \( \alpha < 2\gamma - 2 \), then \((29)\) holds for some \( C > 0 \) depending only on \( A, \alpha, \theta_1, \theta_2 \) and \( \gamma \).

Proof. The proof of the case when \((29)\) holds is analogous to that of when \((26)\) occurs, so we will prove only that \((29) \implies (28)\). Let \( I = [a, b] \). From \((29)\) we have either (i) \( f(s) > 0 \) and \( f''(s) < 0 \) for all \( s \in I \setminus \{\theta_1\} \) or (ii) \( f(s) < 0 \) and \( f''(s) > 0 \) for all \( s \in I \setminus \{\theta_1\} \). Consider the case (i). Let \( u \) and \( \Phi \) be the functions defined on \([a, b]\) by

\[
u(s) = \sin \left( \frac{(s-a)\pi}{b-a} \right),
\]

and

\[
\Phi(s) = A(b-a)^\alpha \pi^{-\alpha} u^\alpha(s).
\]

From \((20)\) we have that \( f(a) \geq 0 \), \( f(b) > 0 \) and \( f''(s) \geq \Phi(s)/f(s) \) for all \( s \in (a, b) \). Let \( \tilde{u} \) and \( \tilde{\Phi} \) be the functions defined on \([a, b]\) by

\[
\tilde{u}(s) = u^{1+\tilde{\Phi}}(s),
\]

and

\[
\tilde{\Phi}(s) = \frac{\pi^2 (1 + \tilde{\Phi})}{(b-a)^2} \left[ u^{2+\alpha}(s) - \frac{\alpha}{2} \cos \left( \frac{(s-a)\pi}{b-a} \right) u^\alpha(s) \right].
\]
A computation shows that \( \tilde{u} \) satisfies \(-\tilde{u}'' = \Phi/\tilde{u} \) on \((a, b)\) and \( \tilde{u}(a) = \tilde{u}(b) = 0 \). Also, \( \Phi \leq \beta \Phi \) on \((a, b)\), with \( \beta = A^{-1}(b-a)^{-2+\alpha} \pi^{2+\alpha}(1 + \frac{2}{\pi})^2 \). We claim that
\[
(31) \quad \tilde{u} \leq \beta^{1/2} f \quad \text{on} \quad (a, b).
\]
Indeed, we have \(-\tilde{u}'' \leq \beta \Phi/\tilde{u} \) and \(-\beta^{1/2} f' \geq \beta \Phi (\beta^{1/2} f)^{-1} \) on \((a, b)\). Then
\[
(32) \quad -\beta^{1/2} f - \tilde{u}'' \geq -\beta \Phi \beta^{1/2} f - \tilde{u} \quad \text{on} \quad (a, b).
\]
Let \( I^* = \{ s \in (a, b) : \beta^{1/2} f < \tilde{u} \} \) and suppose that \( I^* \neq \emptyset \). Let \( J \) be a connected component of \( I^* \). Since \( \beta^{1/2} f - \tilde{u} \) is continuous on \( I^* \) and \( \beta^{1/2} f - \tilde{u} \) is nonnegative at \( a \) and at \( b \), it follows that \( \beta^{1/2} f - \tilde{u} \) is nonnegative on \( J \). Being \( \beta^{1/2} f - \tilde{u} \) negative on \( J \), we get a contradiction. Thus \( 31 \) holds. Then
\[
\int_{I} |f(s)|^{-\frac{1}{d}} ds \leq \beta^{\frac{1}{d}} \int_{I} |\tilde{u}(s)|^{-\frac{1}{d}} ds = \beta^{\frac{1}{d}} (b-a)^{-1} \int_{0}^{\pi} (\sin(\theta))^{-\frac{2+\alpha}{\pi}} d\theta < \infty
\]
since \( \alpha < 2\gamma - 2 \). Finally, the case (ii) follows from the previous one applied to \(-f\). \( \square \)

We recall the definition of our singular measure \( \mu \) in \( \mathbb{R}^4 \) which is supported on the graph of a real analytic function \( \varphi : V^{a, b} \to \mathbb{R}^2 \) satisfying H1 - H3. We assume, in addition to H1 - H3, that

\[ H4 \text{ max}\{n_1, n_2\} < \frac{2m}{\alpha_1 + \alpha_2} - 3, \]

where \( n_1 \) and \( n_2 \) are the two positive integers appearing in Lemma 2 (iii). The number \( \text{max}\{n_1, n_2\} \) is a kind of measure of the degeneracy of ellipticity along the curve \( \{ \bullet (1, \sigma) : t > 0 \} \) (see hypothesis H3).

Now, we consider the set \( V_{c, d}^{a, b} \subset V^{a, b} \) given by \( 5 \), where \( \sigma \in [c, d] \). Then, for every Borel set \( E \) of \( \mathbb{R}^4 \), we define the singular measure \( \mu \) by
\[
(33) \quad \mu(E) = \int_{V_{c, d}^{a, b}} \chi_{E}(x, \varphi(x)) \, dx.
\]

In the following theorem, assuming H1 - H4, we give a decay estimate for the Fourier transform of \( \mu \).

**Theorem 12.** Let \( \mu \) be the singular measure given by (33). Then there exist a positive constant \( C \) such that
\[
(34) \quad |\hat{\mu}(\xi', \xi'')| \leq C |\xi''|^{-\alpha_1 + \alpha_2} m, \quad \text{for all} \ \xi' \in \mathbb{R}^2 \ \text{and all} \ \xi'' \in \mathbb{R}^2 \setminus \{0\}.
\]

**Proof.** We consider the case \( \sigma \in (c, d) \); if \( c = \sigma \) we work with the second inclusion in (35) below, now if \( d = \sigma \), then we work with the first. From (10) and Lemma 5 we have for universal constant \( C > 0 \) that
\[
|\hat{\mu}(\xi', \xi'')| \leq C |\xi''|^{-\alpha_1 + \alpha_2} \int_{c}^{d} |G(\zeta, s)|^{-\alpha_1 + \alpha_2} ds, \quad \text{for all} \ \xi' \in \mathbb{R}^2 \times \mathbb{R}^2 \setminus \{0\},
\]

where \( \zeta = \frac{\xi''}{m} \). Given \( D > 0 \) as in Lemma 2 (iii), from Remark 1 (i) and (ii), for \( \delta > 0 \) and small enough we have
\[
(35) \quad (\sigma - \delta, \sigma) \subset \bigcup_{j=1}^{5} I_{j}^{\delta, \zeta, D} \quad \text{and} \quad (\sigma, \sigma + \delta) \subset \bigcup_{j=1}^{5} J_{j}^{\delta, \zeta, D} \quad \text{for all} \ \zeta \in S^1,
\]

where the sets \( I_{j}^{\delta, \zeta, D} \) and \( J_{j}^{\delta, \zeta, D} \) are defined by (29). For \( \eta \in S^1 \) fixed, let \( \delta_{n}, C_{n}, \) and \( W_{n} \) be as in Lemma 3 (i). By Remark 7 we take \( C_{n} \leq D \). From the compactness of \( S^1 \) there exists a finite set of points \{ \( n_{k} \)b_{k} \subset S^1 \) such that \( S^1 = \bigcup_{k=1}^{N} W_{n_{k}} \). Now, we pick \( \epsilon \) such that \( 0 < \epsilon < \delta \) and \( 0 < \epsilon < \min\{\delta_{n_{k}} : 1 \leq k \leq N\} \), so \( (\sigma - \epsilon, \sigma) \cup (\sigma, \sigma + \epsilon) \subset \bigcup_{j=1}^{5} \left( I_{j}^{\delta, \zeta, D} \cup J_{j}^{\delta, \zeta, D} \right) \) for all \( \zeta \in S^1 \).
Given ζ ∈ S¹, we have that ζ ∈ Wₖ for some k = k(ζ). Since δₖ(ζ) ≥ ϵ and Cₖ(ζ) ≤ D, it follows that I'_j(ζ, D) ⊂ I'_ₖ(ζ, Cₖ(ζ)) and J'_j(ζ, D) ⊂ J'ₖ(ζ, Cₖ(ζ)) for every j = 1, 2, ..., 5. Then

\[(σ - ϵ, σ) \cup (σ, σ + ϵ) \subset \bigcup_{j=1}^{5} \left( I'_ₖ(ζ, Cₖ(ζ)) \cup J'ₖ(ζ, Cₖ(ζ)) \right).\]

Hence

\[\int_{σ-ϵ}^{σ+ϵ} |G(ζ, s)|^{-a_{1+α} + n_2} ds \leq \int_{I'_1(ζ, Cₖ(ζ))} |G(ζ, s)|^{-a_{1+α} + n_2} ds + \int_{J'_1(ζ, Cₖ(ζ))} |G(ζ, s)|^{-a_{1+α} + n_2} ds.\]

By Lemma 3 (ii), the sets I'_j(ζ, Cₖ(ζ)) and J'_j(ζ, Cₖ(ζ)) have at most n₁ and n₂ connected components respectively, so we can estimate, uniformly on ζ, the integrals on the right-hand side of this inequality by applying Lemmas 8, 9, 10, and 11 to the function f(s) = G(ζ, s) with γ = \(\frac{m}{n₁ + n₂}\) and α = n₁ or α = n₂ according to the case and taking into account the hypothesis H4.

Let D > 0 be as in Lemma 2 (ii), and let F = [c, d] \ (σ - ϵ, σ + ϵ), where ϵ is as above. It is clear that F is compact. For τ ∈ [c, d] \ (σ - ϵ, σ + ϵ) and η ∈ S¹, let δₙ, τ, Cₙ, τ ≤ D and Wₙ, τ be as in Lemma 2 (ii). From the compactness of S¹ there exists a finite set of points \(\{ηₖ\}_{k=1}^{M} \subset S¹\) such that S¹ = \(∪_{k=1}^{M} Wₙ, τ\). Now, from Remark 3 (iii), for δ > 0 and small enough we have that

\[(τ - δ, τ + δ) \cap [c, d] ⊂ \bigcup_{j=1}^{5} K'_j(τ, Cₖ, D), \text{ for all } ζ ∈ S¹.\]

Let δₖ = min{δₖ, τ : 1 ≤ k ≤ M}. We take c = min{δₖ, τ}. Now, given ζ ∈ S¹ we have that ζ ∈ Wₖ for some k = k(ζ). Since δₖ(ζ) = δₖ, τ ≥ c and Cₖ(ζ) = Cₖ, τ ≤ D, it follows that Iₙ := (τ - c, τ + c) \ [c, d] ⊂ \bigcup_{j=1}^{5} K'_j(τ, Cₖ, D). Thus

\[\int_{Iₙ} |G(ζ, s)|^{-a_{1+α} + n_2} ds ≤ \sum_{j=1}^{5} \int_{K'_j(τ, Cₖ, D)} |G(ζ, s)|^{-a_{1+α} + n_2} ds.\]

By Lemma 3 (ii), each K'_j(τ, Cₖ, D) is either empty or connected and so the integrals on the right-hand side of this last inequality can be estimated, uniformly on ζ, by using Lemmas 8, 9, 10, and 11 and the hypothesis H4. Since F is covered by a finite number of these intervals (τ - c, τ + c), we obtain an estimate, uniformly on ζ, for the integral \(\int_{F} |G(ζ, s)|^{-a_{1+α} + n_2} ds\). This concludes the proof of the theorem.

4. A restriction theorem for the Fourier transform and Lᵖ-improving properties of μ

Let Σ = \(\{x, Φ(x) : x ∈ V_{1,q}^d\}\), with σ ∈ [c, d] (see hyp. H3). For \(f ∈ L¹(ℝ⁴)\), let \(Rf = \widehat{f}|ξ\): where \(\hat{f}\) denotes the usual Fourier transform. The following lemma gives a necessary condition for the \(Lᵖ(ℝ⁴) - Lᵖ(Σ)\) boundedness of the restriction operator R and it is an adaptation, to our setting, of the well known Knapp’s homogeneity argument.

Lemma 13. If there exists a positive constant C such that

\[||Rf||_{Lᵖ(Σ, dμ)} ≤ C||f||_{Lᵖ(ℝ⁴)}, \text{ } f ∈ S(ℝ⁴).\]

then \(\frac{1}{q} ≥ \frac{(α₁ + α₂ + 2m)}{(α₁ + α₂)} \frac{1}{p}\).

Proof. We observe that if (36) holds for \(f ∈ S(ℝ¹)\), then it holds also for \(f ∈ L¹(ℝ¹) \cap Lᵖ(ℝ⁴)\). Let \(Q = \{t₁(1, s) : s ∈ [s₁, s₂] \text{ and } \frac{1}{2} ≤ t ≤ 1\}\), where \(c < s₁ < s₂ < d\). We fix \(0 < η < 1\) such that \(|Φ(x)| < 1\) for all \(x ∈ ηQ\). For ζ', ζ'' ∈ ℝ² and \(t > 0\) let \(t ∩ (ζ', ζ'') = (tζ', tζ''\)", and for \(0 < η < 1\) we set \(E₁ = η^{-1} \cap [0, 1]^2\) and \(f_η = ζE₁\).
For \( x = (x_1, x_2) \in Q \), a change of variable gives
\[
\hat{f}_e(x, \varphi(x)) = \int_{F_\epsilon} e^{-i(x_1 \xi_1 + x_2 \xi_2 + \xi_3 \varphi_1(x) + \xi_4 \varphi_2(x))} d\xi
\]
\[
= e^{-(\alpha_1 + \alpha_2 + 2m)} \int_{[0,1]^4} e^{-i(\xi_1 \varphi_1(x) + \xi_2 \varphi_1(x) + \epsilon^{-m} \xi_4 \varphi_2(x))} d\xi,
\]
where \( d\xi = d\xi_1 \cdots d\xi_4 \). We define \( F_\epsilon = (\eta) \cdot Q \) and \( \Sigma_\epsilon = \{ (x, \varphi(x)) : x \in F_\epsilon \} \). Then,
\[
\| Rf_\epsilon \|_{L^q(\Sigma_\epsilon, d\mu)} \geq \| Rf_\epsilon \|_{L^q(\Sigma_\epsilon, d\mu)} \geq e^{-q(\alpha_1 + \alpha_2 + 2m)} \int_{F_\epsilon} \left| \int_{[0,1]^4} e^{-i(\xi_1 \varphi_1(x) + \xi_2 \varphi_1(x) + \epsilon^{-m} \xi_4 \varphi_2(x))} d\xi \right|^q dx.
\]
Also
\[
\int_{[0,1]^4} e^{-i(\xi_1 \varphi_1(x) + \xi_2 \varphi_1(x) + \epsilon^{-m} \xi_4 \varphi_2(x))} d\xi \leq \int_{[0,1]^4} e^{-i(\xi_1 \varphi_1(x) + \xi_2 \varphi_1(x) + \epsilon^{-m} \xi_4 \varphi_2(x))} d\xi_1 \int_{[0,1]^4} e^{-i(\xi_1 \varphi_1(x) + \xi_2 \varphi_1(x) + \epsilon^{-m} \xi_4 \varphi_2(x))} d\xi_2 | e^{-i(\xi_1 \varphi_1(x) + \xi_2 \varphi_1(x) + \epsilon^{-m} \xi_4 \varphi_2(x))} d\xi_3 | e^{-i(\xi_1 \varphi_1(x) + \xi_2 \varphi_1(x) + \epsilon^{-m} \xi_4 \varphi_2(x))} d\xi_4 |.
\]
(37)

For every \( x \in F_\epsilon \), fixed, we have that \( x = (x_1, x_2) = ((\eta t)^{\alpha_1}, (\eta t)^{\alpha_2}s) \) with \( \frac{1}{2} \leq t \leq 1 \) and \( s_1 \leq s \leq s_2 \). To estimate the two integrals in (37), from below we choose \( \eta \in (0, 1) \) such that \([\eta^{\alpha_1}, \eta^{\alpha_2}] \subset [-1, 1]\), so \( e^{-1} \cdot x = (e^{-\alpha_1}x_1, e^{-\alpha_2}x_2) = ((\eta t)^{\alpha_1}, (\eta t)^{\alpha_2}s) \in [-1, 1] \times [-1, 1] \). Thus
\[
\left| \int_{0}^{1} e^{-i\epsilon^{-\alpha} \xi_j x_j} d\xi_j \right| = \left| \frac{e^{-i\epsilon^{-\alpha} \xi_j x_j} - 1}{\epsilon^{-\alpha} x_j} \frac{\sin(\epsilon^{-\alpha} x_j)}{\epsilon^{-\alpha} x_j} \right| \geq \sin(\varphi_1(\epsilon^{-1} \cdot x)) \geq \sin(1), \text{ for every } j = 1, 2.
\]

For \( x \in F_\epsilon \), we also have that \( |\varphi_1(\epsilon^{-1} \cdot x)| < 1 \), so the third integral in (37) can be lower bounded as
\[
|\int_{0}^{1} e^{-i\epsilon^{-\alpha_1} \xi_j x_j} d\xi_j | = \left| \frac{e^{-i\epsilon^{-\alpha_1} \xi_j x_j} - 1}{\varphi_1(\epsilon^{-1} \cdot x)} \right| \geq \sin(\varphi_1(\epsilon^{-1} \cdot x)) \geq \sin(1).
\]
The last integral in (37) can be estimated similarly. Thus
\[
\| Rf_\epsilon \|_{L^q(\Sigma_\epsilon, d\mu)} \geq Ca^{-(\alpha_1 + \alpha_2 + 2m) + \frac{\alpha_1 + \alpha_2}{q}},
\]
with \( C > 0 \) independent of \( \epsilon \). On the other hand \( \| f_\epsilon \|_{L^p(\mathbb{R}^4)} = e^{-a^{-(\alpha_1 + \alpha_2 + 2m) + \frac{\alpha_1 + \alpha_2}{q}}} \), so (36) and (38) lead to
\[
Ce^{-(\alpha_1 + \alpha_2 + 2m) + \frac{\alpha_1 + \alpha_2}{q}} \leq \epsilon^{a^{-(\alpha_1 + \alpha_2 + 2m) + \frac{\alpha_1 + \alpha_2}{p}}},
\]
this implies, taking \( \epsilon \) small enough, that \( \frac{1}{q} > \frac{(\alpha_1 + \alpha_2 + 2m)}{(\alpha_1 + \alpha_2 + 4m)} \cdot \frac{1}{p} \). \( \square \)

P. Tomas in [13] pointed out the following result (see also [17], Lemma 1).

Lemma 14. If for some \( 1 \leq p < 2 \), the inequality \( \| \hat{\mu} * f \|_{L^p(\mathbb{R}^4)} \leq C_p \| f \|_{L^p(\mathbb{R}^4)} \) holds for all \( f \in \mathcal{S}(\mathbb{R}^4) \), then for that \( p \)
\[
\| Rf \|_{L^2(\Sigma_\epsilon, d\mu)} \leq C_p \| f \|_{L^p(\mathbb{R}^4)},
\]
for all \( f \in \mathcal{S}(\mathbb{R}^4) \).

We are now in position to prove a sharp estimate for the restriction operator \( R \). For them, we embed the operator \( T_\bar{\mu} \) defined by \( T_\bar{\mu} f = \hat{\mu} * f \) in an analytic family \( \{ T_\bar{\mu} \} \) of operators on the strip \(-\alpha_1 - \alpha_2 \frac{m}{4} \leq \Re(z) \leq 2\), and then we apply the complex interpolation theorem followed by Lemma 14 with \( p = \frac{2(\alpha_1 + \alpha_2 + 2m)}{\alpha_1 + \alpha_2 + 4m} \). 

Theorem 15. There exists a positive constant \( C \) such that
\[
\| Rf \|_{L^2(\Sigma_\epsilon, d\mu)} \leq C \| f \|_{L^p(\mathbb{R}^4)}, \quad f \in \mathcal{S}(\mathbb{R}^4)
\]
if and only if \( \frac{\alpha_1 + \alpha_2 + 4m}{p(\alpha_1 + \alpha_2 + 2m)} \leq \frac{1}{p} \leq 1 \).
Proof. To prove the statement of the theorem we consider the family \( \{ |s|^{-2} \} \) of functions initially defined when \( \Re(z) > 0 \) and \( s \in \mathbb{R}^2 \setminus \{0\} \). This family of functions can be extended, in the \( z \) variable, to an analytic family of distributions on \( \mathbb{C} \setminus \{-2k : k \in \mathbb{N} \cup \{0\}\} \). By abuse of notation, we denote this extension by \( |s|^{-2} \). The family \( \{ |s|^{-2} \} \) have simple poles in \( z = -2k \) for \( k \in \mathbb{N} \cup \{0\} \). Since the meromorphic continuation of the function \( \Gamma(\frac{\alpha}{2}) \) (we keep the notation for his continuation) has simple poles at the same points (i.e. \( z = -2k \)), the family \( \{ I_z \} \) of distributions defined by

\[
I_z(s) = \frac{2^{-\frac{\alpha}{2}}}{\Gamma \left( \frac{\alpha}{2} \right)} |s|^{-2}
\]

results in an entire family of distributions (see pp. 71-74 in [9]).

From this construction, and by taking the ratios of the corresponding residues at \( z = 0 \), we have \( I_0 = \delta \), where \( \delta \) is the Dirac distribution at the origin on \( \mathbb{R}^2 \) (see equation (9), pp. 74 in [9]), also \( \tilde{I}_z = 4\pi I_{2-z} \) (see equation (2'), pp. 194 in [9]).

For \( z \in \mathbb{C} \), we also define \( J_z \) as the distribution on \( \mathbb{R}^2 \) given by the tensor product

\[
J_z = \delta \otimes I_z,
\]

where \( \delta \) is the Dirac distribution at the origin on \( \mathbb{R}^2 \) and \( I_z \) is given by (40).

Let \( \{ T_z \} \) be the analytic family of operators on the strip \( -\frac{\alpha + \alpha_2}{m} \leq \Re(z) \leq 2 \), given by

\[
T_z f = (J_z * \mu) * f, \quad \text{for} \quad f \in S(\mathbb{R}^4).
\]

It is clear that \( T_0 = T_\mu \). For \( \Re(z) = -\frac{\alpha + \alpha_2}{m} \), we have

\[
\left| T_z \right| (\xi) \leq \frac{2^{\frac{\alpha}{2} + 1}}{\Gamma \left( 1 - \frac{\alpha}{2} \right)} \left| \xi'' \right|^{-\frac{\alpha + \alpha_2}{m}}, \quad \text{for all} \quad \xi = (\xi', \xi'') \in \mathbb{R}^2 \times \mathbb{R}^2.
\]

Thus, Young’s inequality and Theorem 12 give

\[
\| T_z f \|_{L^\infty(\mathbb{R}^4)} = \| (J_z \mu) * f \|_{L^\infty(\mathbb{R}^4)} \leq C \frac{2^{\frac{\alpha}{2} + 1}}{\Gamma \left( 1 - \frac{\alpha}{2} \right)} \| f \|_{L^1(\mathbb{R}^4)}, \quad \text{for} \quad \Re(z) = -\frac{\alpha + \alpha_2}{m},
\]

where \( C \) does not depend on \( z \). On the other hand, one also can see that \( (J_z * \mu)(x', x'') = \frac{2^{\frac{\alpha}{2}}}{\pi(1 - \frac{\alpha}{2})} |x''| - |x'| \) for \( x', x'' \in \mathbb{R}^2 \). Then, for \( \Re(z) = 2 \) we have

\[
\| J_z \mu \|_{L^\infty(\mathbb{R}^4)} \leq \frac{2^{-\frac{\alpha}{2}}}{\pi \Gamma \left( \frac{\alpha}{2} \right)}.
\]

So

\[
\| T_z f \|_{L^2(\mathbb{R}^4)} = \| T_z \|_{L^2(\mathbb{R}^4)} = \frac{2^{-\frac{\alpha}{2}}}{\pi \Gamma \left( \frac{\alpha}{2} \right)} \| f \|_{L^2(\mathbb{R}^4)} = \frac{2^{-\frac{\alpha}{2}}}{\pi \Gamma \left( \frac{\alpha}{2} \right)} \| f \|_{L^2(\mathbb{R}^4)}, \quad \text{for} \quad \Re(z) = 2.
\]

Now, we check that the family \( \{ T_z \} \) satisfies the hypotheses of the Stein’s complex interpolation theorem (see [15], pp. 205) on the strip \( -\frac{\alpha + \alpha_2}{m} \leq \Re(z) \leq 2 \). For them, let

\[
M_{\frac{\alpha + \alpha_2}{m}}(y) := \frac{2^{-\frac{\alpha + \alpha_2}{m} - \frac{\alpha}{2} + 1}}{\Gamma \left( 1 + \frac{\alpha + \alpha_2}{2m} - \frac{\alpha}{2} \right)} \quad \text{and} \quad M_2(y) := \frac{2^{-\frac{2m}{\alpha}}}{\pi \Gamma \left( \frac{2m}{\alpha} \right)}.
\]

We recall the Stirling’s formula (see e.g. [16]). This states that

\[
\Gamma(z) \sim \sqrt{2\pi} z^{-\frac{1}{2}} e^{-z}, \quad \text{as} \quad |z| \to \infty \quad \text{and} \quad |\arg(z)| \leq \frac{\pi}{2}.
\]

Then, for every \( a > 0 \) fixed, with the aid of the Stirling’s formula, it is easy to check that

\[
\sup_{-\infty < y < +\infty} e^{-a|y|} \log M_{\frac{\alpha + \alpha_2}{m}}(y) < +\infty \quad \text{and} \quad \sup_{-\infty < y < +\infty} e^{-a|y|} \log M_2(y) < +\infty.
\]

From (12), (13) and (44) taking there \( 0 < a < \frac{\pi}{2 + \alpha + \alpha_2} \) follow that the family \( \{ T_z \} \) satisfies the hypotheses of the complex interpolation theorem on the strip \( -\frac{\alpha + \alpha_2}{m} \leq \Re(z) \leq 2 \) and then, since \( \frac{2m}{\alpha + \alpha_2} - 2(1-t) = 0 \) for \( t = \frac{2m}{\alpha + \alpha_2} - 2m \), \( T_0 = T_\mu \) is a bounded operator from \( L^p(\mathbb{R}^4) \) into \( L^p(\mathbb{R}^4) \).
for $\frac{1}{p} = t + \frac{1-t}{2} = \frac{\alpha_1+\alpha_2+4m}{2(\alpha_1+\alpha_2+2m)}$. This gives (39) for that $p$ (see Lemma 14). Finally, the global estimate $\|\hat{f}\|_{L^\infty(\mathbb{R}^4)} \leq \|f\|_{L^1(\mathbb{R}^4)}$ implies that $\mathcal{R} : L^1(\mathbb{R}^4) \to L^2(\Sigma, \mu)$ is bounded, so the theorem follows from the Riesz-Thorin Convexity Theorem and Lemma 13.

Corollary 16. If $(\frac{1}{p}, \frac{1}{q})$ belongs to the closed quadrilateral with vertices $(1, 0)$, $(1, 1)$, $(\frac{\alpha_1+\alpha_2+4m}{2(\alpha_1+\alpha_2+2m)}, 1)$ and $(\frac{\alpha_1+\alpha_2+4m}{2(\alpha_1+\alpha_2+2m)}, \frac{1}{2})$, then the restriction operator $\mathcal{R}$ is bounded from $L^p(\mathbb{R}^4)$ into $L^q(\Sigma, \mu)$.

Proof. The global estimate $\|\hat{f}\|_{L^\infty(\mathbb{R}^4)} \leq \|f\|_{L^1(\mathbb{R}^4)}$ gives, for every $1 \leq q < \infty$, the $L^1(\mathbb{R}^4) \to L^q(\Sigma, \mu)$ bound for $\mathcal{R}$. To apply Holder’s inequality with $p = 2$ followed by Theorem 15 we obtain the $L^{\frac{2(\alpha_1+\alpha_2+2m)}{\alpha_1+\alpha_2+4m}}(\mathbb{R}^4) \to L^1(\Sigma, \mu)$ bound for $\mathcal{R}$. Finally, the corollary follows from the Riesz-Thorin Convexity Theorem.

Let $\mu$ be a Borel measure and let $T_\mu$ be the convolution operator by $\mu$, defined by $T_\mu f = \mu * f, f \in \mathcal{S}(\mathbb{R}^4)$, and let $E_\mu$ be the set of all pairs $(\frac{1}{p}, \frac{1}{q}) \in [0, 1] \times [0, 1]$ such that $\|T_\mu f\|_{L^p(\mathbb{R}^4)} \leq C\|f\|_{L^q(\mathbb{R}^4)}$ for all $f \in \mathcal{S}(\mathbb{R}^4)$ and for some positive constant $C$ depending only on $p$ and $q$. We say that the measure $\mu$ is $L^p$-improving if $E_\mu$ does not reduce to the diagonal $\frac{1}{p} = \frac{1}{q}$.

The following proposition gives a necessary condition for the $L^p(\mathbb{R}^4) - L^q(\mathbb{R}^4)$ boundedness of the operator $T_\mu$. This result is a particular case of Propozizione 2.2 in [12] applied to the measure $\mu$ given by (39). For readers’ convenience, we include a proof.

Proposition 17. If $(\frac{1}{p}, \frac{1}{q}) \in E_\mu$, then $\frac{1}{q} \geq \frac{1}{p} - \frac{\alpha_1+\alpha_2}{\alpha_1+\alpha_2+2m}.$

Proof. For $c < s_1 < s_2 < d$, let $Q = \{(t^{\alpha_1}, t^{\alpha_2} s) : \frac{1}{2} \leq t \leq 1, s_1 \leq s \leq s_2\}$ and $M = \sup \{|\varphi(x)| : x \in Q\}$. For $\delta > 0$ and small, we define

$$R_\delta = (-2\delta^{\alpha_1}, 2\delta^{\alpha_1}) \times (-2\delta^{\alpha_2}, 2\delta^{\alpha_2}) \times (-2(2M+1)\delta^{\alpha_1} + (2M+1)\delta^{\alpha_2})^2,$$

$$f_\delta = \chi_{R_\delta}, E_\delta = \delta \cdot Q$$

and

$$A_\delta = \{(x', x'') \in E_\delta \times \mathbb{R}^2 : \|x'' - \varphi(x')\|_{\mathbb{R}^2} \leq \delta\}.$$

To prove the proposition it suffices to show that there exists a positive constant $C$ independent of $\delta$ such that $|\mu * f_\delta(x', x'')| \geq C\delta^{\alpha_1+\alpha_2}$. Indeed,

$$\mu * f_\delta(x', x'') = \int_{\mathbb{R}^2} \chi_{R_\delta}(x' - y, x'' - \varphi(y)) dy.$$

Then, for every $(x', x'') \in A_\delta$ fixed, from the homogeneity of $\varphi$, it follows that $(x' - y, x'' - \varphi(y)) \in R_\delta$ for all $y \in E_\delta$. So,

$$|\mu * f_\delta(x', x'')| \geq |E_\delta| = \delta^{\alpha_1+\alpha_2}|Q|,$$

and therefore

$$\|\mu * f_\delta\|_q \geq \left(\int_{A_\delta} |\mu * f_\delta|^q\right)^{1/q} \geq C\delta^{\alpha_1+\alpha_2}|A_\delta|^{1/q} = C\delta^{\alpha_1+\alpha_2+\frac{\alpha_1+\alpha_2+2m}{2}},$$

where $C = |Q|$. On the other hand, $(\frac{1}{p}, \frac{1}{q}) \in E_\mu$ implies that $\|\mu * f_\delta\|_q \leq C_{p,q}\|f_\delta\|_p \leq C_{p,q}\delta^{\frac{\alpha_1+\alpha_2+2m}{2}}$. Thus $\delta^{\alpha_1+\alpha_2+\frac{\alpha_1+\alpha_2+2m}{2}} \leq C\delta^{\frac{\alpha_1+\alpha_2+2m}{2}}$ for all $\delta > 0$ and small. This implies, taking $\delta$ small enough, that $\frac{1}{q} \geq \frac{1}{p} - \frac{\alpha_1+\alpha_2}{\alpha_1+\alpha_2+2m}$.

The following theorem states that the measure $\mu$ given by (39) is $L^p$-improving.

Theorem 18. For $p = \frac{\alpha_1+\alpha_2+2m}{\alpha_1+\alpha_2+m}$, the closed triangle with vertices $(0, 0), (1, 1)$ and $(\frac{1}{p}, \frac{1}{p})$ is contained in $E_\mu$ and $(\frac{1}{p}, \frac{1}{p}) \in \partial E_\mu$. 

Proof. It is known that if \( \left( \frac{1}{p}, \frac{1}{q} \right) \in \mathcal{E}_\mu \), then \( \frac{1}{q} \leq \frac{1}{p} \) and \( \frac{1}{q} \geq \frac{2}{p} - 1 \) (cf. [15], pp. 33; and [10], Theorem 1); from Proposition 12 we also have that \( \frac{1}{q} \geq \frac{1}{p} - \frac{\alpha_1 + \alpha_2 + m}{\alpha_1 + \alpha_2 + 2m} \). In particular, these facts imply that if \( \left( \frac{1}{p}, \frac{1}{p} \right) \in \mathcal{E}_\mu \), then \( \frac{1}{q} \leq \frac{1}{p} \leq \frac{\alpha_1 + \alpha_2 + m}{\alpha_1 + \alpha_2 + 2m} \). On the other hand, by Riesz-Thorin Theorem, \( \mathcal{E}_\mu \) is a convex set and, since \( \mu \) is finite, the diagonal \( \frac{1}{q} = \frac{1}{p} \) is contained in \( \mathcal{E}_\mu \). Thus, to prove the theorem it is enough to see that \( \left( \frac{1}{p}, \frac{1}{p} \right) \in \mathcal{E}_\mu \) for \( p = \frac{\alpha_1 + \alpha_2 + m}{\alpha_1 + \alpha_2 + 2m} \). Consider now, for \( z \in \mathbb{C} \), the analytic family of distributions \( J_z \) defined by \( \psi \). For \( z \) in the strip \( \frac{\alpha_1 + \alpha_2}{m} \leq \Re(\zeta) \leq 2 \), let \( S_z \) be the analytic family of operators defined by

\[
S_z f = \mu \ast J_z \ast f, \quad f \in \mathcal{S}(\mathbb{R}^4).
\]

From Theorem 14 we have \( \| \tilde{\mu}(\xi', \xi'') \| \leq C(\xi'')^{-\frac{\alpha_1 + \alpha_2}{m}} \), and since \( \tilde{J}_z = 1 \otimes 4\pi I_{2-z} \) we obtain that

\[
\| S_z \|_{L^2(\mathbb{R}^4) \rightarrow L^2(\mathbb{R}^4)} \leq \left\| \frac{\Gamma(1 - \frac{3}{2})}{\Gamma(1 - \frac{3}{2})} \right\|_{C(\xi'')^{-\frac{\alpha_1 + \alpha_2}{m}}} \leq \frac{2^{\frac{3}{2} + 1}}{\Gamma(1 - \frac{3}{2})} \text{ for } \Re(\zeta) = -\frac{\alpha_1 + \alpha_2}{m},
\]

where \( C \) does not depend on \( z \). Since \( (\mu \ast J_z)(x', x'') = \frac{2^{\frac{3}{2} + 1}}{\Gamma(1 - \frac{3}{2})} |x'' - \psi'(x')|^2 \) for \( x', x'' \in \mathbb{R}^2 \), we obtain

\[
\| S_z \|_{L^1(\mathbb{R}^4) \rightarrow L^\infty(\mathbb{R}^4)} = \| \mu \ast J_z \|_\infty \leq \frac{2^{\frac{3}{2} + 1}}{\Gamma(1 - \frac{3}{2})} \text{ for } \Re(\zeta) = 2.
\]

As in Theorem 15 the family \( \{ S_z : -\frac{\alpha_1 + \alpha_2}{m} \leq \Re(\zeta) \leq 2 \} \) satisfies the hypotheses of the complex interpolation theorem and then, since \( -\frac{\alpha_1 + \alpha_2}{m} + 2(1-t) = 0 \) for \( t = \frac{2m}{\alpha_1 + \alpha_2 + 2m} \), \( S_0 = T_\mu \) is a bounded operator from \( L^p(\mathbb{R}^4) \) into \( L^q(\mathbb{R}^4) \) for \( \frac{1}{p} = \frac{3}{2} + (1-t) = \frac{\alpha_1 + \alpha_2 + m}{\alpha_1 + \alpha_2 + 2m} \). Finally, being \( S_0 = T_\mu \) the theorem follows.

Remark 19. If one consider \( \varphi \in C^\infty(V^{a,b}) \) with additional hypotheses in the lemma 2, then the theorems 17 and 18 are still true. We observe that the lemma 2 still holds if we replace the word analytic by smooth, the proof depends on certain "well-controlled" monotonicity conditions given in some derivatives of fixed order being non-vanishing. The remain of the lemmas hold considering only \( \varphi \in C^\infty(V^{a,b}) \). We leave the details to the interested reader.

5. An example

Let \( \varphi : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by

\[
\varphi(x, y) = (\varphi_1(x, y), \varphi_2(x, y)) = \left( \frac{-4}{132} x^{12} + \frac{1}{132} y^{12} + \frac{3}{132} y^{12} + \frac{1}{30} y^{12} + \frac{5}{90} y^{12} \right), \quad (x, y) \in \mathbb{R}^2.
\]

In this case we have that \( \varphi(t \bullet (x, y)) = t^m \varphi(x, y) \) with \( \alpha_1 = \frac{1}{2}, \alpha_2 = 1 \) and \( m = 6 \), so H1 and H2 hold. A computation shows that the discriminant of the quadratic form

\[
(\zeta_1, \zeta_2) \to \det (\zeta_1 \varphi_1'(1, y) + \zeta_2 \varphi_2'(1, y)) = (K(1, y) \zeta_1, \zeta_1)
\]

is

\[
\text{det} K(1, y) = -4y^4 \left( -3 + \frac{5}{3} y^2 \right) \left( y^4 + \frac{y^2}{3} \right) + 4y^8 - \frac{1}{4} \left( 4 \left( y^4 + \frac{y^2}{3} \right) + y^8 \right) \left( 3 - \frac{5}{3} y^2 \right)^2.
\]

We observe that \( K_{11}(1, y) = -4y^4 \), and the discriminant is positive for every \( y \in \left[ \frac{17}{99}, 1 \right) \) and it has a simple zero for \( y = 1 \). By Lemma 2 we have that the eigenvalues \( \Lambda_1(1, \cdot) \) and \( \Lambda_2(1, \cdot) \) of \( K(1, \cdot) \) are real analytic on \( \mathcal{V} \). Since \( \Lambda_1(1, 1) \Lambda_2(1, 1) = \det K(1, 1) = 0 \) and \( \frac{dy}{dy} \text{det} K(1, y) \bigg|_{y=1} \neq 0 \), it follows that \( \Lambda_1(1, 1) \neq 0 \) and \( \frac{dy}{dy} \Lambda_2(1, y) \bigg|_{y=1} \neq 0 \) or \( \Lambda_2(1, 1) \neq 0 \) and \( \frac{dy}{dy} \Lambda_1(1, y) \bigg|_{y=1} \neq 0 \). Thus, H3 holds with \( \sigma = 1 \). It is clear that H4 also holds. Finally H1 - H4 hold for \( \varphi \) and \( V = \{(1/2, y) : \frac{17}{99} < y < 1 \text{ and } 0 < t < 1 \} \).

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