NOTES ABOUT DECIDABILITY OF EXPONENTIAL EQUATIONS

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Abstract. We study relationship among versions of the Knapsack Problem where variables take values in \( \mathbb{Z} \) and the number of them is fixed.

1. Introduction

Let \( G \) be a group given by a recursive presentation \( \langle X \mid R \rangle \). For \( w \in G \) we denote by \(|w|_X\) the length of a shortest word in the alphabet \( X \cup X^{-1} \) representing \( w \). The free group generated by \( X \) is denoted by \( F(X) \). The length of \( u \in F(X) \) with respect to \( X \) will be often written as \(|u|\). By \( WP(G) \) and \( CP(G) \) we denote the word and the conjugacy problems for \( G \), respectively. In formulations of decision problems, we assume that elements of \( G \) are represented by words in the alphabet \( X \cup X^{-1} \).

Let \( PP[n] \) be the set of all tuples \( \bar{g} = (g_0, g_1, \ldots, g_n) \) from \( G^{n+1} \) such that the equation \( g_0 = g_1^{z_1} \cdots g_n^{z_n} \) has a solution which is an \( n \)-tuple of integers. We define the power problem of order \( n \) for \( G \) to be the membership problem for the set \( PP[n] \). For brevity, we denote this decision problem again by \( PP[n] \).

In the case \( n = 1 \) the decision problem \( PP[1] \) is known as the power problem, see [14] and [19]. Our notes are motivated by the following question.

What is the relationship among the decision problems \( PP[n] \) for different \( n \)?

It is easy to see that decidability of the power problem implies decidability of WP. Furthermore, decidability of \( PP[n+1] \) implies decidability of \( PP[n] \). Indeed, in order to verify if \( (g_0, g_1, \ldots, g_n) \in PP[n] \) we just check if \( (g_0, g_1, \ldots, g_n, e) \in PP[n+1] \), where \( e \) is the unit of \( G \). The main result of this paper shows that decidability of \( PP[1] \) does not imply decidability of \( PP[2] \).

Theorem A. There exists a finitely presented group with decidable \( PP[1] \) and undecidable \( PP[2] \). Moreover, this group has decidable conjugacy problem.

The next note concerns estimation of possible solutions of exponential equations arising in \( PP[n] \) by recursive functions on sizes of coefficients. The motivation comes from the fact that this is a usual way to solve such equations. In Proposition 4.5 we show that primitively recursive functions are not sufficient for this aim. More

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information about the complexity of estimating functions for some interesting classes of groups can be found in Remarks 4.6 and 4.7.

In Section 5 we introduce decision problems $PP[g, G^n]$ and $PP[G, g]$ which can be considered as fragments of $PP[n]$ for $G$. We show that these fragments can have diverse r.e. Turing degrees in the same finitely presented group. From a quite general Theorem 5.3 we deduce the following statement.

**Theorem B.** (see Corollary 5.5) There exists a finitely presented torsion-free group $G$ with decidable conjugacy problem and undecidable $PP[1]$ such that any r.e. Turing degree is realised as the Turing degree of the problem $PP[g, G]$ for appropriate $g \in G$.

The methods which we use strongly depend on papers [17] – [19].

In the places where arguments are of computability theory flavor we follow the terminology of [21] (in particular we write “computable” instead of “recursive”). In the remaining parts of the paper we keep the traditional terminology.

**Remark 1.1.** The Knapsack Problem of $G$ is the decidability problem of recognizing of all tuples $\bar{g} \in G^{n+1}$, $n \in \mathbb{N}$, such that $\bar{g} = (g_0, g_1, \ldots, g_n)$ and the equation $g_0 = g_1^{z_1} \cdots g_n^{z_n}$ has a solution which a tuple of natural numbers. It is easy to see that decidability of the Knapsack Problem in $G$ implies decidability of the set $\bigcup_{n \in \mathbb{N}} PP[n]$. The Knapsack Problem was introduced in [16]. It has become a very active area of research where the interplay between group theoretic properties and algorithmic complexity is a typical topic, see [4], [5], [6], [8], [9], [10], [11], [12], and [15]. On the other hand questions similar to ones studied in our paper seem to be open for the Knapsack Problem. We will comment this below.

**Remark 1.2.** Several authors consider exponential equations in the following form:

$$h_1g_1^{z_1}h_2g_2^{z_2} \cdots h_ng_n^{z_n} = 1.$$  

It is worth noting that this equation can be rewritten as

$$f_1^{z_1}f_2^{z_2} \cdots f_n^{z_n} = f_0,$$

where $f_0 = (h_1 \ldots h_n)^{-1}$ and $f_i = (h_1 \ldots h_i)g_i(h_1 \ldots h_i)^{-1}$ for $i = 1, \ldots, n$. In particular, the question about decidability of $PP[n]$ does not depend on the form of exponential equations.

2. **A recursively presented group with decidable $PP[1]$ and undecidable $PP[2]$**

The main purpose of this section is the following weaker version of Theorem A.

**Proposition 2.1.** There exists a recursively presented group $G$ such that $PP[1]$ is decidable, but $PP[2]$ is undecidable.

In the proof of this proposition we use the following lemmas. The first one is obvious.
Lemma 2.2. Let $w$ and $u$ be two nontrivial elements of the free group $F(X)$. If $w = u^z$ for some $z \in \mathbb{Z}$, then $|z| \leq |w|$. 

Lemma 2.3. Let $w(a, b, c)$ be a nonempty reduced cyclic word in $F(a, b, c)$ and let $M = \max\{|z| : w \text{ has a subword of the form } a^z \text{ or } b^z\}$. Suppose that $m > M$. Then $w(a, b, a^mb^m) \neq 1$ in $F(a, b)$. Moreover, if $w(a, b, a^mb^m) = v(a, b)^z$ for some $v(a, b) \in F(a, b)$ and $z \in \mathbb{Z}$, then $|z| \leq |w(a, b, c)|$.

Proof. We assume that $w$ contains at least one $c$ or $c^{-1}$ (otherwise the statement is obvious). For any word $u(a, b, c) \in F(a, b, c)$ let $T(u) \in F(a, b)$ be the reduced word obtained from $u$ by substitution $c \to a^mb^m$ followed by reduction. If $u(a, b, c)$ is considered as a cyclic word, we denote by $CT(u)$ the cyclic word, obtained from $u$ by substitution $c \to a^mb^m$ followed by cyclic reduction. The following claim can be proved by induction on the number of occurrences of letters $c^\pm$ in $u$.

Claim. Let $u$ be a subword of the cyclic word $w$ from lemma.
1) If $u$ begins with $c$ (resp. with $c^{-1}$) then $T(u)$ begins with $a^m$ (resp. with $b^{-m}$).
2) If $u$ ends with $c$ (resp. with $c^{-1}$), then $T(u)$ ends with $b^m$ (resp. with $a^{-m}$).

Now we show that $CT(w)$ contains at least two syllables. In particular, this will imply $w(a, b, a^mb^m) \neq 1$ in $F(a, b)$. The case where $w$ contains only one occurrence of $c$ or $c^{-1}$ is obvious. We assume that $w$ contains at least 2 occurrences of $c^\pm$.

The case, where the occurrences of $c$ and $c^{-1}$ in $w$ alternate, i.e. if $w = u_1cu_2c^{-1} \ldots cu_nc^{-1}$, where $u_i \in F(a, b)$ for all $i$, is easy. Indeed, each subword $c^{-1}u_ic$ gives at least one $a$-syllable, and each subword $cu_ic$ gives at least one $b$-syllable.

Now suppose that $w$ does not have this form. Then the cyclic word $w$ has the form $cpcq$ or $c^{-1}pc^{-1}q$, where $p \in F(a, b, c)$ and $q \in F(a, b)$. Using inversion, we may assume the first case. Note that $CT(cpcq)$ can be obtained from $T(cpcq)$ by cyclic reduction. By the claim above, the word $T(cpc)$ begins with $a^m$ and ends with $b^m$. By assumption, the absolute values of exponents of all syllables in the word $q$ are less than $m$. Therefore, after cyclic reduction of $T(cpcq)$, some nontrivial parts of the syllables $a^m$ and $b^m$ remain.

Thus, we have proved that the cyclic word $CT(w)$ contains $n$ syllables for some $n \geq 2$. Hence the cyclic word $CT(v)$ also contains at least two syllables, say $m$. Then $z = n/m \leq n/2$. It remains to note that $n \leq 2|w|$. The latter is valid since, after substitution $c \to a^mb^m$ in $w$, the total number of $a$-syllables and $b$-syllables increases by at most $2k$, where $k$ is the number of occurrences of $c^\pm$ in $w$. 

Proof of Proposition 2.7. Our construction resembles McCool’s example from [14]. Let $f : \mathbb{N} \to \mathbb{N}$ be a one-to-one recursive function with non-recursive range. Consider the following infinite presentation:

$$G = \langle \cup_{i \in \mathbb{N}} \{a_i, b_i, c_i\} \mid c_{f(i)} = a_{f(i)}^{i}b_{f(i)}^{i} \ (i \in \mathbb{N}) \rangle.$$  (2.1)
Let $X = \bigcup_{i \in \mathbb{N}} X_i$, where $X_i = \{a_j, b_j, c_i\}$. Let $H_j$ be the subgroup of $G$ generated by $X_i$. Then

$$G = \ast_{j \in \mathbb{N}} H_j$$

(2.2)

where each $H_j$ is free and

$$\text{rk}(H_j) = \begin{cases} 2 & \text{if } j \in \text{im}(f), \\ 3 & \text{if } j \notin \text{im}(f). \end{cases}$$

Claim 1. The word problem is decidable for the presentation (2.1).

Proof. Using the normal form of an element of the free product (2.2), we reduce $WP(G)$ to the following problem. Given $j \in \mathbb{N}$ and a reduced nonempty word $w(a_j, b_j, c_j)$, decide whether the corresponding element of $H_j$ is trivial or not. The difficulty is that we do not know whether $j \in \text{im}(f)$ or not.

From now on we consider $w(a_j, b_j, c_j)$ as a nonempty reduced cyclic word in $F(a_j, b_j, c_j)$. Let $M$ be the maximum of absolute values of exponents of $a_j$ and $b_j$ in the word $w(a_j, b_j, c_j)$.

First we verify whether there exists $m \leq M$ with $j = f(m)$ or not. If such $m$ exists, we substitute $a_j^m b_j^m$ for $c_j$ in $w(a_j, b_j, c_j)$ and verify whether the resulting word is trivial in $F(a_j, b_j)$ or not. This can be done effectively.

We claim that in the remaining cases the word $w$ is nontrivial in $G$. Indeed, if $j \notin \text{im}(f)$, then $H_j \cong F(a_j, b_j, c_j)$, hence $w(a_j, b_j, c_j)$ is nontrivial in $H_j$. If $j = f(m)$ for some $m > M$, then $w(a_j, b_j, c_j) = w(a_j, b_j, a_j^m b_j^m)$ is nontrivial in $F(a_j, b_j)$ by Lemma 2.3.

Claim 2. The group $G$ has undecidable PP[2].

Proof. The equation $c_k = a_k^x b_k^y$ is solvable if and only if $k = f(i)$ for some $i$ (in this case $x = y = i$ is the unique solution). Since the set $\text{im}(f)$ is not recursive, we cannot recognize whether such $i$ exists or not. Therefore we cannot recognize the existence of such $x$ and $y$.

Claim 3. The group $G$ has decidable PP[1].

Proof. Consider an exponential equation

$$u = v^z,$$

(2.3)

where $u$ and $v$ are nontrivial words in the alphabet $X = \bigcup_{i \in \mathbb{N}} X_i$. In order to decide if it is solvable we may assume that $u \neq 1$ and $v \neq 1$ in $G$.

We write $u = u_1 u_2 \ldots u_k$, where $u_i$ is a word in the alphabet $X_{\lambda(i)}$ for some $\lambda(i)$, $i = 1, \ldots, k$, and $\lambda(j) \neq \lambda(j + 1)$ for $j = 1, \ldots, k - 1$. Moreover (using decidability of $WP(G)$), we assume that each $u_i$ represents a nontrivial element of $H_{\lambda(i)}$. Using conjugation, we may additionally assume that $\lambda(1) \neq \lambda(k)$ if $k > 1$. Analogously, we write $v = v_1 v_2 \ldots v_\ell$.

Suppose that $k > 1$. Then the necessary condition for solvability of equation (2.3) is $\ell > 1$ and $v_1$ and $v_\ell$ belong to different subgroups $H_j$ (determined by $u_1$ and $u_k$). If this condition is fulfilled, then any possible solution $z$ of equation (2.3)
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satisfies $|z| = k/\ell$, and the existence of a solution $z$ can be verified using decidability of WP($G$).

Let $k = 1$. Then the necessary condition for solvability of equation (2.3) is $\ell = 1$. Thus, we assume that $u, v$ are words in the alphabet $X_j$ for some $j$. We want to solve the equation

$$u(a_j, b_j, c_j) = v(a_j, b_j, c_j)^z. \tag{2.4}$$

Without loss of generality, we assume that $u(a_j, b_j, c_j)$ is a reduced cyclic word. Let $M$ be the maximum of absolute values of exponents of $a_j$ and $b_j$ in $u(a_j, b_j, c_j)$.

First we check whether some $m \in \{1, \ldots, M\}$ satisfies $f(m) = j$. If such $m$ is found, the equation (2.4) takes the form

$$u(a_j, b_j, a_j^m b_j^m) = v(a_j, b_j, a_j^m b_j^m)^z,$$

and the solvability of this equation can be verified by Lemma 2.2.

If no such $m$ exists, then either $j \notin \text{im } f$, or $j = f(m)$ for some $m > M$. We claim that in these cases the absolute value of a possible solution $z$ of equation (2.4) does not exceed the length of the word $u(a_j, b_j, c_j)$ in $F(a_j, b_j, c_j)$. Indeed, if $j \notin \text{im } f$, then $H_j$ is the free group with basis $\{a_j, b_j, c_j\}$, and the claim follows from Lemma 2.2. If $j \in \text{im } f$, then the claim follows from Lemma 2.3.

Using the estimation for $|z|$ and decidability of WP($G$), we can verify whether equation (2.4) has a solution.

Remark 2.4. One can show that the group $G$ constructed in the proof of Proposition 2.1 has solvable conjugacy problem. However we do not need this for the proof of Theorem A.

3. PROOF OF THEOREM A

Below we deduce Theorem A from Proposition 2.1 and the following result of Ol’shanskii and Sapir.

**Theorem 3.1.** (see [19, Theorem 1]) Every countable group $G = \langle x_1, x_2, \ldots | R \rangle$ with solvable power problem is embeddable into a 2-generated finitely presented group $\overline{G} = \{y_1, y_2 | \overline{R} \}$ with solvable conjugacy and power problems.

**Remark 3.2.** In this remark we recall the main steps of the proof of Theorem 3.1. This task is justified by the additional observation that the embedding $\varphi : G \rightarrow \overline{G}$ constructed in the proof of this theorem is computable. This means that there exists an algorithm, which given $i \in \mathbb{N}$, expresses $x_i$ as a word in $y_1$ and $y_2$. Furthermore, these steps will be also used in arguments of Sections 4 and 5.

Four steps in the construction of Ol’shanskii and Sapir. Before we start, observe that any countable group $G = \langle x_1, x_2, \ldots | R \rangle$ with solvable power problem has solvable word problem, hence it admits a recursive presentation. Thus, we may assume that the given presentation of $G$ is recursive. Moreover, the solvability of power problem implies the solvability of order problem (there exists an algorithm which compute orders of elements).
Step 1. In [3], Collins noticed that if $H$ is a recursively presented group with solvable power problem and $a, b$ are two elements in $H$ of the same order, then the HNN extension $H_{a, b} = \langle H, t \mid t^{-1}at = b \rangle$ has solvable power problem.

Using a sequence of HNN extensions of this type, $G$ can be embedded into a recursively presented group $G_1$ with solvable power problem where every two elements of the same order are conjugate. Thus the conjugacy problem in $G_1$ is decidable. Moreover, the constructed embedding $\varphi_1 : G \to G_1$ is computable.

Step 2. In [17], Olshanskii suggested the following construction for embedding of countable groups into 2-generated groups. Let $H = \langle x_1, x_2, \ldots \mid \mathcal{R} \rangle$ be any countable group. Denote by $\mathcal{R}_1$ the set of words in the alphabet $\{a, b\}$ obtained by substituting the word

$$A_i = a^{100}b^ia^{101}b^i \ldots a^{199}b^i$$

for every $x_i$ in every word from $\mathcal{R}$. It was shown in [17] that the map $x_i \mapsto A_i$, $i \in \mathbb{N}$, extends to an embedding of $H$ into $H_1 = \langle a, b \mid \mathcal{R}_1 \rangle$. Lemmas 10 and 11 from [19] say that if the group $H$ has decidable word or conjugacy problem or power problem, then the same problem is decidable for the group $H_1$.

Applying this construction, we obtain a computable embedding $\varphi_2 : G_1 \to G_2$, where $G_2 = \langle a, b \mid R_2 \rangle$ is 2-generated, recursively presented, and has solvable power and conjugacy problems.

Step 3. Lemma 12 from [19] says that this $G_2$ can be embedded into a finitely presented group $G_3 = \langle a, b, c_1, \ldots, c_n \mid R_3 \rangle$ with solvable power and conjugacy problems. This embedding extends the identity map $a \mapsto a$, $b \mapsto b$.

The corresponding embedding was first described in [18]. We indicate that $G_2$ and $G_3$ play the roles of $K$ and $H$ in [18]. It is worth mentioning that in [18] the set of generators of $H$ consists of $k$-letters, $a$-letters, $\theta$-letters and $x$-letters. The subgroup $K$ in $H$ is generated by a subset $A(P_1)$ of the set of $a$-letters. By Lemma 3.9 of [18] if $K = \langle a_1, \ldots, a_m \mid \mathcal{R} \rangle$, then the map $\phi_H$ defined by $a_i \mapsto a_i(P_1)$ extends to an embedding of $K$ into $H$. This map is obviously computable.

Step 4. Using the construction from Step 2 once more, we embed $G_3$ into a 2-generated finitely presented group $\overline{G} = \langle y_1, y_2 \mid \overline{\mathcal{R}} \rangle$ with solvable conjugacy and power problems.

Since the embeddings at all steps are computable, their composition $\varphi : G \to \overline{G}$ is computable as well.

Proof of Theorem A. By Proposition 2.1, there is a recursively presented group $G$ with decidable PP[1] and undecidable PP[2]. Using Theorem 3.1 and Remark 3.2 we obtain a computable embedding $\varphi : G \to \overline{G}$, where $\overline{G}$ is finitely presented and has decidable PP[1]. Since $\varphi$ is computable, the undecidability of PP[2] for $G$ implies the undecidability of PP[2] for $\overline{G}$.

Indeed, consider an arbitrary equation $g_0 = g_1g_2^y$ with $g_0, g_1, g_2 \in G$ written as words in the generators of $G$. Using computability of $\varphi$, we can write $\varphi(g_0), \varphi(g_1), \varphi(g_2)$ as words in the generators of $\overline{G}$. The equation $\varphi(g_0) = \varphi(g_1)^x\varphi(g_2)^y$ has the same solutions as the original one. If we could decide whether this equation is
solvable, we could decide whether the original equation is solvable. However PP[2] is undecidable for $G$. Hence it is undecidable for $\overline{G}$.

Remark 3.3. The Knapsack counterpart of PP[2] is also undecidable in $\overline{G}$.

4. Estimation functions for solutions of exponential equations

Let $G$ be a group generated by a set $X$. For any finite tuple $\bar{g} = (g_0, \ldots, g_n)$ of elements of $G$, the $\infty$-norm of this tuple is the number

$$\|\bar{g}\|_X = \max\{|g_0|_X, \ldots, |g_n|_X\}.$$ 

In the case where $G = \mathbb{Z}$ and $X = \{1\}$ we omit $X$ and write $\|\bar{g}\|$.

Definition 4.1. Let $G$ be a group generated by a set $X$. A function $f : \mathbb{N} \to \mathbb{N}$ is called a PP$[n]$-bound for $G$ (with respect to $X$) if for any exponential equation $g_0 = g_1^{x_1} \cdots g_n^{x_n}$ over $G$ with nonempty set of solutions, there exists a solution $k = (k_1, \ldots, k_n)$ with

$$\|k\| \leq f(\|\bar{g}\|_X).$$

Remark 4.2. Let $G$ be a group and let $X$ and $Y$ be two generating sets of $G$. Suppose that $\sup_{y \in Y} |y|_X < \infty$.

If there exists a (recursive) PP$[n]$-bound for $G$ with respect to $X$, then there exists a (recursive) PP$[n]$-bound for $G$ with respect to $Y$.

The following lemma relates decidability of PP$[n]$ in $G$ and existence of a total recursive PP$[n]$-bound. It is a counterpart of the fact that a group $G$ with a finite generating set $X$ has solvable WP if and only if the Dehn function of $G$ with respect to $X$ is computable.

Lemma 4.3. Let $G$ be a group generated by a finite set $X$. For any $n \in \mathbb{N}$ the following two conditions are equivalent.

1. PP$[n]$ is decidable in $G$.
2. WP$(G)$ is decidable and there exists a total recursive PP$[n]$-bound for $G$ with respect to $X$.

Proof. (1) $\Rightarrow$ (2). Suppose that PP$[n]$ is decidable in $G$. Then, clearly WP$(G)$ is decidable. Now we define the desired function $f : \mathbb{N} \to \mathbb{N}$ at arbitrary point $m \in \mathbb{N}$ in four steps.

1) Let $B(m)$ be the set of all tuples $\bar{g} = (g_0, g_1, \ldots, g_n)$ of words in the alphabet $X$ satisfying $\|\bar{g}\|_X \leq m$. Since $X$ is finite, the set $B(m)$ is finite and we can compute it.

2) Let $B(m)'$ be the subset of $B(m)$ consisting of the tuples $\bar{g} = (g_0, g_1, \ldots, g_n)$ for which that the equation $g_0 = g_1^{x_1} \cdots g_n^{x_n}$ has a solution. We can compute $B(m)'$ using decidability of PP$[n]$ in $G$.

3) For each tuple $\bar{g} \in B(m)'$ we can find some solution $\bar{k} = (k_1, \ldots, k_n)$ of the equation $g_0 = g_1^{x_1} \cdots g_n^{x_n}$ using an effective enumeration of $n$-tuples of integers and the decidability of WP$(G)$. We denote this solution by $\bar{k}(\bar{g})$. 
4) Finally we set $f(m)$ to be the maximum of $\|k(\bar{g})\|$ over all $g \in B(m)'$ if $B(m)' \neq \emptyset$, and we set $f(m) = 1$ if $B(m)' = \emptyset$.

The function $f$ is total and recursive, and satisfies Definition 4.1

$(2) \Rightarrow (1)$. Consider an exponential equation $g_0 = g_1^{a_1} \cdots g_n^{a_n}$ over $G$. To decide, whether this equation has a solution, we verify whether the equality $g_0 = g_1^{k_1} \cdots g_n^{k_n}$ holds for at least one tuple $\bar{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ with $\|\bar{k}\| \leq f(\|\bar{g}\|)_X$. The verification for a concrete tuple $\bar{k}$ can be done using $WP(G)$.\hfill $\square$

Remark 4.4. In [7], Kharlampovich constructed a group $G$ which is finitely presented in the variety $x^m = 1$ and has undecidable word problem. By Lemma 4.3 $PP[n]$ is undecidable for each $n$. On the other hand the constant function $f(k) = m$, $k \in \mathbb{N}$, is a total recursive $PP[1]$-bound for $G$.

The following proposition shows that there is a finitely presented group with decidable $PP[1]$ which does not have a primitively recursive $PP[1]$-bound.

Proposition 4.5. There exists a finitely presented group $G = \langle X \mid R \rangle$ with decidable $PP[1]$ and there exists a collection of elements $(c_n)_{n \in \mathbb{N}}$ of $G$ such that the following holds.

1) For any $n$ the equation $c_1 = c_n^2$ has a unique solution, say $k_n$; this solution is positive.

(2) There is no primitively recursive function $f$ such that $k_n \leq f(\max\{c_n \mid X, c_1 \mid X\})$.

Proof. We enumerate all primitively recursive functions $g_1, g_2, \ldots$ and, for any $n \in \mathbb{N}$, we define a function $f_n : \mathbb{N} \rightarrow \mathbb{N}$ by the rule

$$f_n(x) = \sum_{i=1}^{n} \sum_{j=1}^{x} g_i(j), \quad x \in \mathbb{N}.$$ 

Clearly, $f_n$ is primitively recursive, nondecreasing, $g_n \leq f_n$, and $f_n \leq f_{n+1}$. Finally, we define a function $F : \mathbb{N} \rightarrow \mathbb{N}$ by the rule

$$F(n) = n!(f_n(100n + 14500) + 1).$$

Clearly, $F$ is recursive. We also define rational numbers $c_1 = 1$ and $c_n = \frac{1}{F(n)}$ for $n \geq 2$. Then the following recursive presentation (written multiplicatively) defines the group $Q$:

$$\langle c_1, c_2, \ldots \mid c_n^{F(n)} = c_1, [c_n, c_m] = 1 (n, m \in \mathbb{N}) \rangle.$$

It is obvious that $PP[1]$ is decidable for this presentation. We embed $Q$ into a finitely presented group $G_3$ by the Ol’shanskii - Sapir construction which we described in Steps 1-4 in Section 3. Note that since $Q$ has decidable conjugacy problem, we do not need to do Step 1. Thus, we start with Step 2, where we use the following map.

- Let $\varphi_2$ maps each $c_i$ to the word $a^{100b^ia^{101b^i}} \cdots a^{199b^i}$ (of length $100i + 14500$), $i \in \mathbb{N}$.

By Step 2, $\varphi_2$ extends to an embedding $\varphi_2 : Q \rightarrow G_2$, where the group $G_2 = \langle a, b \mid R_2 \rangle$ is 2-generated, recursively presented, and has solvable power and conjugacy problems. Then we only apply Step 3. By this step the map $a \mapsto a$, $b \mapsto b$
extends to an embedding $\varphi_3 : G_2 \to G_3$, where $G_3 = \langle X \mid R_3 \rangle$ is a finite presentation with solvable power and conjugacy problems, and $\{a, b\} \subseteq X$. We set $G = G_3$.

The statement (1) is valid: for any $n$, the equation $c_n^x = c_1$ has a unique solution, namely $k_n = F(n)$. To prove statement (2), we first observe that

$$\max\{|c_n|_X, |c_1|_X\} \leq \max\{|c_n|_{\{a,b\}}, |c_1|_{\{a,b\}}\} \leq 100n + 14500. \quad (4.1)$$

Suppose that statement (2) is not valid, i.e. there exists a primitively recursive function $g_m$ such that

$$k_n \leq g_m(\max\{|c_n|_X, |c_1|_X\}) \quad (4.2)$$

for any $n$. Using that $g_m \leq f_m$ and that $f_m$ is nondecreasing, we deduce from (4.1) and (4.2) that

$$F(n) \leq f_m(100n + 14500)$$

for any $n$. In particular, $F(m) \leq f_m(100m + 14500)$. This contradicts the definition of $F$. \hfill \Box

**Remark 4.6.** Theorem 4.2 in [12] states that the Knapsack Problem in the Baumslag–Solitar group $BS(1,2)$ is NP-complete but the function estimating nonnegative solutions of exponential equations with 3 variables cannot be essentially smaller than the exponential function (note that in [12], the authors use the binary representation of natural numbers). This statement can be considered as a counterpart of Proposition 4.5 at the level of polynomial computability.

**Remark 4.7.** However for hyperbolic groups such functions for any $n$ can be chosen to be linear (a polynomial estimation was known earlier, see [16]). This follows from a forthcoming paper [1] of the first named author and A. Bier. It is proved in [1] that similar linearity result holds for acylindrically hyperbolic groups in the case of loxodromic coefficients. Another result in [1] states that there is a linear reduction to peripheral subgroups in the case of relatively hyperbolic groups.

5. **Restricted versions of PP$[n]$**

We introduce two new algorithmic problems which can be considered as fragments of PP$[n]$. Informally we call them the the left and the right fragments of PP$[n]$. We show that these fragments can take diverse computational complexities for the same finitely presented group, see Corollary 5.5.

5.1. **Definitions and observations.** Below we assume that $G$ is given by a recursive presentation and $X$ is the corresponding set of generators.

**Definition 5.1.** (1) Let $g_1, \ldots, g_n \in G$. By PP$[G, g_1, \ldots, g_n]$ we denote the set of all $g \in G$ such that the equation $g = g_1^{x_1} \cdots g_n^{x_n}$ has a solution which is a tuple of integers.

(2) For a fixed $g \in G$ let PP$[g, G^n]$ consist of all tuples $\bar{g} = (g_1, \ldots, g_n)$ such that the equation $g = g_1^{x_1} \cdots g_n^{x_n}$ has a solution which is a tuple of integers.
Note that for a tuple of units $\bar{e}$ the membership problem for $PP[G, \bar{e}]$ is equivalent to the word problem. Decidability of the problem $PP[n]$ is a uniform form of decidability of all $PP[G, \bar{g}]$ (resp. $PP[g, G^n]$). Indeed, if for each $g \in G$ there is an algorithm (depending on the word $g$) which decides the membership problem for $PP[g, G^n]$, then $PP[n]$ is decidable. The similar statement holds for problems $PP[G, g_1, \ldots, g_n]$.

**Remark 5.2.** Suppose that $G$ is a group given by a recursive presentation. Let $g$ be an nontrivial element of $G$. Suppose that $PP[G, g]$ is decidable in $G$ and the order of $g$ is known. Then WP is decidable in $G$.

Indeed, in order to determine whether a given $h$ is trivial in $G$, we first verify whether $h$ is a power of $g$. If $h$ is not a power of $g$ then $h \neq 1$. If we know that $h$ is a power of $g$ let us start a diagonal computation for verification of the following equalities: $h = 1$, $h = g$, $h = g^2$, $\ldots$. Here we use the recursive presentation of $G$. At some stage we will find a number $k$ with $h = g^k$. Since the order of $g$ is known, we can check, whether $h = 1$ or not.

What is the relationship among possible algorithmic complexities of $PP[n]$, $PP[g, G^n]$ and $PP[G, g_1, \ldots, g_n]$ for tuples $g, g_1, \ldots, g_n \in G$? Some additional issue of this problem is nicely illustrated by McCool’s example from [14]. Having a computable function $F$ with a non-computable $\text{im}F$ the group

$$\left\{ \bigcup_{n \in \mathbb{N}} \{a_n, b_n\} \big| \{[a_n, b_n] = 1 \mid n \in \mathbb{N}\}, \{a_{F(n)} = b_n^k \mid n \in \mathbb{N}\} \right\}$$

has decidable WP but undecidable $PP[1]$. On the other hand each $PP[g_0, G]$ (resp. $PP[G, g_0]$) is a decidable problem whenever the set

$$\{n \in \text{im}F \mid a_n \text{ or } b_n \text{ occurs in } g_0\}$$

is provided. This claim can be easily verified (for example see arguments in the proof of Theorem 5.3 below).

5.2. **Example.** Let $p_n$ denote the $n$-th prime number. For any function $F : \mathbb{N} \to \mathbb{N}^2$ and any $n \in \mathbb{N}$ we denote $(\text{im}F)_n = \{m \in \mathbb{N} \mid (n, m) \in \text{im}(F)\}$ and write $F = (F_1, F_2)$.

Let $F : \mathbb{N} \to \mathbb{N}^2$ be a total, injective, computable function such that for any $n \in \mathbb{N}$ we have either $(\text{im}F)_n = \emptyset$ or

$$p_n \in (\text{im}F)_n \subseteq \{p^k \mid k \in \mathbb{N} \setminus \{0\}\}.$$ 

Thus all sets $(\text{im}F)_n$, $n \in \mathbb{N}$, are pairwise disjoint. We put

$$X = \{a_n \mid n \in \mathbb{N}\} \cup \{b_m \mid m \text{ is a power of a prime number}, m \neq 1\}$$

and consider the group with the following recursive presentation

$$G = \langle X \mid \{[a_n, b_m] = 1 \mid \exists k (m = p^k_n)\}, a_{F_1(n)} = b_{F_2(n)}^n \mid n \in \mathbb{N}\rangle. \quad (5.1)$$

**Theorem 5.3.** For the above defined group $G$ the following statements are valid.
(1) CP($G$) is decidable. In particular, PP[$e, G$] and PP[$G, e$] are decidable.

(2) PP[1] is undecidable for $G$ if the set $\text{im} (F_1)$ is not computable.

(3) For any fixed $g_0 \in G$ the problem PP[$g_0, G$] (resp. PP[$G, g_0$]) is decidable or there is a number $n$ such that PP[$g_0, G$] (resp. PP[$G, g_0$]) is Turing reducible to $\text{im} (F)_n$. Each of these possibilities can be effectively recognized and the corresponding number $n$ can be computed.

(4) If $n \in \text{im} F_1$ then the problem PP[$a_n, G$] is computably equivalent to the membership problem for $\text{im} (F)_n$.

Proof. Before we start to prove these statements, we establish the structure of $G$. We decompose $X = \bigcup_{i \in \mathbb{N}} X_i$, where

$$X_i = \{a_i\} \cup \{b_j \mid j \text{ is a power of } p_i\}.$$ 

Let $H_i$ be the subgroup of $G$ generated by $X_i$. Then

$$G = \ast_{i \in \mathbb{N}} H_i.$$ 

(5.2)

To describe the structure of $H_i$, we first introduce the following subgroups of $H_i$:

$$H_i^- = \langle b_j \mid j \text{ is a power of } p_i, j \notin (\text{im} F)_i \rangle,$n

$$H_i^+ = \langle b_j \mid j \text{ is a power of } p_i, j \in (\text{im} F)_i \rangle.$$ 

Then $H_i^-$ is the free product of all its subgroups $\langle b_j \rangle$, and $H_i^+$ is the amalgamated product over $\langle a_i \rangle$ of all its subgroups $\langle b_j \rangle$. Moreover, we have

$$H_i = (\langle a_i \rangle \times H_i^-) \ast_{\langle a_i \rangle} H_i^+.$$ 

(5.3)

Note that $\langle a_i \rangle$ is the center of $H_i$.

Before we start the proof of statement (1), we make the following important observation.

**Observation.** Let $i, j \in \mathbb{N}$ and $k \in \mathbb{Z}$. Then $b_j^k$ is a power of $a_i$ if and only if $j$ is a power of $p_i$ and there exists a positive divisor $d$ of $k$ such that $F(d) = (i, j)$. We can recognize the existence of such $d$ since $F$ is computable. If such $d$ exists, then $a_i = b_j^d$ and hence $a_i^{k/d} = b_j^k$.

**Proof of statement (1).** First we prove that WP($G$) is decidable. Using the normal form of an element of the free product (5.2), we reduce this problem to the following one. Given $i \in \mathbb{N}$ and a cyclically reduced nonempty word $a_i^s w(\bar{b})$, where $w(\bar{b})$ is over $X_i \setminus \{a_i\}$, decide whether the corresponding element of $H_i$ is trivial or not.

We may assume that the word $w(\bar{b})$ is nonempty. Indeed, otherwise $a_i^s w(\bar{b})$ lies in the cyclic subgroup $\langle a_i \rangle$ of $H_i$ and therefore is trivial exactly when $s = 0$.

Using the above observation, we verify whether some subword $b_j^k$ of $w(\bar{b})$ is a power of $a_i$ or not. If no one such subword is a power of $a_i$, then the element $a_i^s w(\bar{b})$ is nontrivial in the amalgamated product (5.3). Suppose that some subword $b_j^k$ of $w(\bar{b})$ is a power of $a_i$, say $a_i^k = b_j^k$. Since $a_i$ lies in the center of $H_i$, we can move this
subword to the left and adjoin to \(a^s\). After this operation \(|w(b)|_{X \setminus \{a_i\}}\) decreases and we can proceed by induction.

Now we show that the conjugacy problem in \(G\) is decidable. Using (5.2), we reduce this problem to the conjugacy problem in the groups \(H_i, i \in \mathbb{N}\). By (5.3), each \(H_i\) is an amalgamated product over the center of \(H_i\). This fact, the decidability of WP(\(G\)) and a criterium for conjugacy of elements in amalgamated products (see [13, Chapter IV, Theorem 2.9]), imply that there is a universal algorithm deciding the conjugacy problem in each \(H_i\) and hence in \(G\).

**Proof of statement (2).** This statement easily follows from the equivalence

\[ n \in \text{im}F_1 \iff a_n \text{ is a power of } b_{p_n}. \]

Indeed, if \(\text{im}F_1\) is not computable, we cannot decide, given \(n \in \mathbb{N}\), whether the equation \(a_n = b_{p_n}^s\) has a solution or not.

**Proof of statement (3).** For a fixed element \(g_0 \in G\) we study the problem \(\text{PP}[g_0, G]\). Given another element \(g_1 \in G\), we shall consider the exponential equation

\[ g_0 = g_1^z. \]

We may assume that \(g_0 \neq 1\), otherwise \(\text{PP}[g_0, G]\) is decidable since \(G\) is torsion-free and WP(\(G\)) is decidable. Having \(g_0 \neq 1\), we may assume that \(g_1 \neq 1\). Standardly, we assume that \(g_0\) and \(g_1\) are represented by some words \(u\) and \(v\) in the alphabet \(X = \bigcup_{i \in \mathbb{N}} X_i\). Thus, we consider the following exponential equation in \(G\):

\[ u = v^z. \quad (5.4) \]

We write \(u = u_1u_2 \ldots u_k\), where \(u_i\) is a word in the alphabet \(X_{\lambda(i)}\) for some \(\lambda(i)\), \(i = 1, \ldots, k\), and \(\lambda(j) \neq \lambda(j + 1)\) for \(j = 1, \ldots, k - 1\). Moreover, we assume that each \(u_i\) represents a nontrivial element of \(H_{\lambda(i)}\). This can be recognized by decidability of WP(\(G\)). Using conjugation, we may additionally assume that \(\lambda(1) \neq \lambda(k)\) if \(k > 1\). Analogously, we write \(v = v_1v_2 \ldots v_t\).

Suppose that \(k > 1\). Then the necessary condition for solvability of equation (5.4) is \(\ell > 1\) and \(v_1\) and \(v_t\) belong to different subgroups \(H_i\) (determined by \(u_1\) and \(u_k\)). If this condition is fulfilled, then any possible solution \(z\) of equation (5.4) satisfies \(|z| = k/\ell\), and the existence of a solution \(z\) can be verified using decidability of WP(\(G\)).

Note that until this moment the corresponding algorithm is uniform on \(u\) and \(v\). In the following case we will call some oracle depending on \(u\).

Suppose that \(k = 1\). Then \(u \in H_n\) for \(n = \lambda(1)\), and this \(n\) can be determined using the definition of \(X_n\). Using the procedure described in the proof of WP(\(G\)), we write \(u\) in the normal form with respect to the amalgamated product (5.3), i.e., we write \(u = a_{n}^{s_1}b_{i_1}^{s_1} \ldots b_{i_p}^{s_p}\) where, in particular, each \(b_{i_1}^{s_1}, \ldots, b_{i_p}^{s_p}\) does not have a subword which is a power of \(a_n\). Conjugating, we may additionally assume that \(u\) is cyclically reduced in the sense that \(i_1 \neq i_p\) if \(p > 1\).

Furthermore, we may now assume that \(v\) belongs to \(H_n\) too. We also write \(v\) in the normal form with respect to the amalgamated product (5.3), \(v = a_{n}^{t_1}b_{j_1}^{t_1} \ldots b_{j_q}^{t_q}\)
If $p > 1$, then the necessary condition for solvability of equation (5.4) is $q > 1$. In this case any possible solution $z$ of (5.4) satisfies $|z| = p/q$, and the existence of the corresponding $z$ can be verified using decidability of $\text{WP}(G)$.

Suppose that $p = 1$. Then the necessary condition for solvability of (5.4) is $q = 1$ and $i_1 = j_1$. In this case we only need to verify the existence of $z$ satisfying (5.4) in the group $\langle a_n, b_i \rangle$. This is the only place where we need the oracle for $(\text{im } F)_{1}$. Substituting the appropriate power of $b_i$ instead of $a_n$ both in $u$ and $v$ we obtain an equation in the cyclic group $\langle b_i \rangle$. Now the number $z$ can be computed. This gives an appropriate algorithm which is computable with respect to $(\text{im } F)_{n}$.

The case $p = 0$ is trivial and we leave it to the reader.

This completes the proof of statement (3) for $\text{PP}[g_0, G]$. The argument for $\text{PP}[G, v]$ is analogous.

Proof of statement (4). Let $n \in (\text{im } F)_1$. The following equivalence recognizes $j \in (\text{im } F)_n$ under the oracle for $\text{PP}[a_n, G]$:

$$j \in (\text{im } F)_n \iff j \text{ is of the form } p^n_k \text{ and } a_n \text{ is a power of } b_j.$$ 

\[\square\]

Remark 5.4. Statements (1) and (4) also hold for the corresponding versions of the Knapsack Problem.

The following corollary is Theorem B from Introduction.

Corollary 5.5. There exists a finitely presented torsion-free group $G$ with decidable conjugacy problem and undecidable $\text{PP}[1]$ such that any r.e. Turing degree is realised as the Turing degree of the problem $\text{PP}[g, G]$ for appropriate $g \in G$.

Proof. First we construct a recursively presented group $G$ with these properties. Let $\varphi(x, y)$ be Kleene’s universal computable function. Let

$$W = \{(x, z) \mid \exists y (z = \varphi(x, y))\}$$

and let $\Phi : \mathbb{N} \to \mathbb{N}^2$ be a total, injective, computable function with $\text{im } \Phi = W$. Below we use notations introduced at the beginning of this subsection. Obviously, the sets $(\text{im } \Phi)_n, n \in \mathbb{N}$, have all possible r.e. Turing degrees. Now we extend the set $W$ as follows:

$$\hat{W} = W \cup \{(x, 1) \mid \exists z : (x, z) \in W\}.$$ 

Let $\hat{\Phi} : \mathbb{N} \to \mathbb{N}^2$ be a total, injective, computable function with $\text{im } \hat{\Phi} = \hat{W}$. We have $(\text{im } \hat{\Phi})_n = (\text{im } \Phi)_n \cup \{1\}$ for any $n \in \mathbb{N}$. Therefore the sets $(\text{im } \hat{\Phi})_n, n \in \mathbb{N}$ have all possible r.e. Turing degrees as well.

Now we define a function $F : \mathbb{N} \to \mathbb{N}^2$ by the formula

$$F = f \circ \hat{\Phi},$$
where \( f : \mathbb{N}^2 \to \mathbb{N}^2 \) is the function sending each \((n, m)\) to \((n, p^m_n)\). The function \( F \) satisfies the conditions formulated at the beginning of this subsection, since it is total, injective, computable, and for any \( n \in \mathbb{N} \) we have
\[
p_n \in (\text{im} F)_n \subseteq \{p^k_n \mid k \in \mathbb{N} \setminus \{0\}\}.
\]

Finally, we define a recursively presented group \( G \) by formula (5.1) and apply Theorem 5.3. By statements (1) and (2) of this theorem, \( \text{CP}(G) \) is decidable and \( \text{PP}[1] \) is undecidable for \( G \).

The statement of the corollary about Turing degrees follows from statement (4) of Theorem 5.3 which says that, for any \( n \in \mathbb{N} \), the problem \( \text{PP}[a_n, G] \) is computably equivalent to the membership problem for \((\text{im} F)_n\). It remains to note that
\[
(\text{im} F)_n = \{p^m_n \mid m \in (\text{im} \hat{\Phi})_n\},
\]
therefore these sets have all possible r.e. Turing degrees.

Now we embed the group \( G \) into a finitely presented group \( \overline{G} \) using Ol’shanskii–Sapir construction explained in Remark 3.2. Note that we do not need to do Step 1 there since \( G \) already has solvable conjugacy problem, and we do not need to do Step 4 since we do not specially want \( \overline{G} \) to be 2-generated.

Thus, using notations of this remark, we may assume that \( G = G_1 \) and that we have embeddings \( G_1 \to G_2 \to G_3 \), where \( G_3 = \overline{G} \). Simplifying notation, we assume \( G_1 \subseteq G_2 \subseteq G_3 \). By this construction, \( G_3 \) has solvable conjugacy problem if \( G_1 \) has solvable conjugacy problem. The latter is valid, hence \( \overline{G} \) has solvable conjugacy problem. It remains to prove the following claim.

**Claim.** For any \( g \in G \) the problems \( \text{PP}[g, G_1] \) and \( \text{PP}[g, G_3] \) are computationally equivalent.

**Proof.** The computational equivalence of \( \text{PP}[g, G_1] \) and \( \text{PP}[g, G_2] \) follows from the proof of Lemma 11 in [19]. (We stress that we use the proof and not the formulation of this lemma which requires solvability of power problem in \( G_1 \).) Indeed, given an exponential equation \( g = u^z \) with \( u \in G_2 \), the proof (depending on \( u \)) either recursively reduces this equation to \( \text{PP}[g, G_1] \) or gives a linear upper bound for \( |z| \) in terms of \( g \) and \( u \).

The computational equivalence of \( \text{PP}[g, G_2] \) and \( \text{PP}[g, G_3] \) analogously follows from the proof of Lemma 12 in [19].

\[\square\]

6. Further observations. Complexity

Applying the approach of [2] we obtain the following proposition.

**Proposition 6.1.**  
(1) Detecting a group with decidable \( \text{PP}[n] \) is \( \Sigma^0_3 \) in the class of recursively presented groups.

(2) The same conclusion holds both for the problem \( \bigcup_{n \in \mathbb{N}} \text{PP}[n] \) and the Knapsack Problem.
Proof. We will use standard terminology from [21]. The universal computable function $\varphi(x, y)$ will be applied to several families of objects. As usual these objects are coded by natural numbers.

Take a computable indexation $G_i = \langle X | R_i \rangle$, $i \in \omega$, of all recursively presented groups with respect to generators $X = \{x_1, x_2, \ldots \}$. Fix an algorithm which for the input $(i, s)$ outputs the $s$-th equality of the form $w = 1$ satisfied in $G_i$. We see that the set of pairs $(G_i, w)$, where $G_i \models w = 1$ with $w \in F(X)$ is computably enumerable. There also exists a computable enumeration of the set of pair $(G_i, \bar{w})$ where $\bar{w} = (w_0, w_1, \ldots , w_n) \in PP[n](G_i)$. Thus the set $I_{\bar{w}} = \{G_i | (w_0, w_1, \ldots , w_n) \in PP[n](G_i) \}$ belongs to $\Sigma_0^1$. On the other hand the set $\bar{I}_{\bar{w}} = \{G_i | (w_0, w_1, \ldots , w_n) \notin PP[n](G_i) \}$ belongs to $\Pi_0^1$. The property $(w_0, w_1, \ldots , w_n) \notin PP[n](G_i)$ exactly means that for any $(s_0, s_1, \ldots , s_n)$ the equality $w_0 = w_1^{s_1} \cdot \ldots \cdot w_n^{s_n}$ is not recognized in $G_i$ at step $|s_0|$. Developing these observations we formulate decidability of $PP[n]$ for $G_i$ as follows.

There is a number $m \in \mathbb{N}$ such that for any tuple $w_0, w_1, \ldots , w_n \in F(X)$ and any $(s_0, s_1, \ldots , s_n) \in \mathbb{Z}^{n+1}$ there exist numbers $s, t \in \mathbb{N}$ such that the following properties hold:

• the algorithm $\varphi(m, \cdot)$ applied to the code of $\bar{w}$ gives the value 0 or 1 at step $s$;
• the algorithm $\varphi(m, \cdot)$ applied to the code of $\bar{w}$ gives the value 0 at step $s$ or the membership $G_i \in I_{\bar{w}}$ is confirmed at step $t$ of computation;
• the algorithm $\varphi(m, \cdot)$ applied to the code of $\bar{w}$ gives the value 1 at step $s$ or the equality $w_0 = w_1^{s_1} \cdot \ldots \cdot w_n^{s_n}$ is not recognized at step $|s_0|$.

The second statement of the proposition is similar. □

Question. Are $PP[n]$ and the Knapsack Problem $\Sigma_3^0$-complete in the class of recursively presented groups?

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