Faster and simpler approximation of stable matchings.

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Abstract
We give a $\frac{3}{2}$-approximation algorithm for stable matchings that runs in $O(m)$ time. The previously best known algorithm by McDermid has the same approximation ratio but runs in $O(n^{3/2}m)$ time, where $n$ denotes the number of vertices and $m$ the number of edges. Also the algorithm and the analysis are much simpler.

1 Introduction
In the paper we consider a variant of the problem called Stable Matchings, known also in the literature as Stable Marriage. The problem is defined as follows. We are given two sets $W$ and $M$ of women and men and each woman $w$ from $W$ has a linearly ordered preference list $L_w$ of a subset of men $M'_w \subseteq M$ and similarly each man $m$ from $M$ has a linearly ordered preference list $L_m$ of a subset of men $W'_m \subseteq W$. The lists of men and women can contain ties, which are subsets of men (or respectively women), which are equally good for a given woman (resp. man). Thus if $m$ and $m'$ are on list $L_w$ of woman $w$, then either (1) $m < m'$ and then we say that woman $w$ prefers $m$ to $m'$ or that $m$ is better for her than $m'$ or (2) $m = m'$, which means that $m$ and $m'$ are in a tie on $L_w$ and then we say that $w$ is indifferent between $m$ and $m'$ or that $m$ and $m'$ are equally good for her or (3) $m' < m$. If man $m$ does not belong to list $L_w$, then we say that $m$ is unacceptable for $w$. A matching is a set of pairs $(m, w)$ such that $m \in M$, $w \in W$ and $m$ and $w$ are on each other’s preference lists and each man/woman belongs to at most one pair. If $(m_1, w_1)$ belongs to a certain matching $M_1$, then we write $M_1(m_1) = w_1$, which means that in $M_1$ woman $w_1$ is a partner of $m_1$ and analogously that $M_1(w_1) = m_1$. A matching $M_1$ is called stable if it does not contain a blocking pair. A pair $(m, w)$ is blocking for $M$ if (1) $m$ is single or prefers $w$ to $M_1(m)$ and if (2) $w$ is single or prefers $m$ to $M_1(w)$. A stable matching always exists and can be found via Gale/Shapley algorithm ([11]) by breaking ties arbitrarily. However, the problem of finding a stable matching of maximum cardinality is $NP$-hard, which was shown by Manlove et al. in [8]. Therefore it is desirable to devise an approximation algorithm for the problem.
1.1 History of approximation results

In [8] it was shown that an arbitrary stable matching is a $2$-approximation. A first non-trivial $(2 - c \frac{\log n}{n})$-approximation was given by Iwama et al. in [4], where $c$ is a positive constant. Next in [5] the approximation ratio was improved to $2 - \frac{c'}{\sqrt{n}}$, where $c'$ is a positive constant which is at most $\frac{1}{4\sqrt{6}}$ and in [6] to $\frac{15}{8}$. Further on Kiraly ([7]) presented a linear time elegant $\frac{5}{3}$-approximation algorithm. Currently the best approximation algorithm is by McDermid and achieves the approximation guarantee $\frac{3}{2}$. Its running time is $O(n^{3/2}m)$, where $n$ denotes the number of vertices in the graph and $m$ the number of edges in the graph, that is the sum of the lengths of preference lists.

The problem was shown to be APX-complete by Halldorsson et al. in [2] and inapproximable within $\frac{21}{19}$ (unless $P = NP$) by Halldorsson et al. in [3]. Next Yanagisawa ([10]) proved that the problem cannot be approximated within $\frac{33}{29}$. In [10] it is also shown that the problem cannot be approximated within $\frac{7}{5}$ under the assumption that the minimum vertex cover problem cannot be approximated within $2 - \varepsilon$.

**Our results** We give a $3/2$-approximation algorithm that runs in $O(m)$ time and additionally is significantly simpler than that of McDermid.

2 Algorithm

The algorithm consists of two linear (with regard to the number of edges) time phases: one called Algorithm *GS Modified*, the other Algorithm *GS Improve*. The first one is very similar to that of Gale/Shapley but has one modification that handles masculine dangerous paths which are defined later. The second one "improves" the existing matching so that it will not contain any dangerous paths (either masculine or feminine) and as a result will be of size at least $2/3|M_{opt}|$, where $M_{opt}$ denotes a stable matching of largest cardinality.

If man $m$ is matched to woman $w$ and there is at least one free woman $w_1$ such that $w$ and $w_1$ are equally good for $m$, then we say that $m$ is unstable and that $w_1$ is a satellite of $m$. If woman $w$ is matched to an unstable man $m$, then we say that $w$ is unstable. If $e = (m, w)$ is such that both $m$ and $w$ are free and there is at least one free woman $w_1$ such that $w$ and $w_1$ are equally good for $m$, then $e$ is called special.
Algorithm *GS Modified*

while there is a free man $m$ with a nonempty list do

$w \leftarrow$ next woman on $m$’s list

if $m$ is better for $w$ than her current partner $M(w)$ or $w$ is free, then

$M \leftarrow M \cup (m, w) \setminus (w, M(w))$

else

if $w$ is unstable and there is no free woman $w'$ who is equally good for $m$ as $w$, then

$M \leftarrow M \cup \{(m, w), (M(w), w')\} \setminus (w, M(w))$, where $w'$ is a satellite of $M(w)$

While scanning lists $L_m$ we always move forward and do not come back to the edge that was considered previously (or equivalently remove the edge from $L_m$), but there is one exception – if the edge $(m, w)$ is special, then we do not move forward on $L_m$ and the next time we consider $m$ in the algorithm, $w$ will be still on $L_m$.

Before proving that Algorithm *GS Modified* computes a stable matching we give the following fact.

**Fact 1** If woman $w$ becomes matched, then she will stay matched. Woman $w$ can become unstable only at the step of the algorithm, when she is free. If an unstable woman $w$ (matched to some unstable man $m$) becomes matched to some new man $m'$, then she is not unstable any more. If woman $w$ is matched to man $m$ and is not unstable, then she will always be matched to someone at least as good for her as $m$.

**Lemma 1** Algorithm *GS Modified* computes a stable matching.

**Proof.** Suppose to the contrary that a computed matching $M$ is not stable and contains a blocking pair $(m, w)$. Thus $w$ prefers $m$ to her current partner $M(w)$ and $m$ prefers $w$ to his current partner $M(m)$. $m$ must have proposed to $w$ before he proposed to $M(m)$. If she was matched at that moment she would have either accepted $m$ – if she was unstable or matched to someone worse for her than $m$ or rejected him – if she was not unstable and matched to someone at least as good for her as $m$. In either case by Fact[1] she would remain matched to someone at least as good for her as $m$. Therefore $w$ must have been free at the moment $m$ was proposing to her and since she later became matched to someone worse for her than $m$, she was unstable after she became matched to $m$. Thus at the moment $m$ was proposing to $w$ edge $(m, w)$ was special. Then she must have become matched to some man $m'$ and $m$ to one of his satelites. Some time later $m$ must have become free but then at some moment he would propose to $w$ for the second time (because edge $(m, w)$ was special) and then he could get rejected only if $w$ was matched to
someone at least as good for her as \( m \) and by Fact 1 she would have stayed matched to someone at least as good for her as \( m \).

We also prove

**Lemma 2** If \( e \) is special at step \( S \) of Algorithm GS Modified and gets added to \( M \), then it will not become special later in the course of running Algorithm GS Modified.

**Proof.** If at step \( S \) edge \( e = (m, w) \) is special and gets added to \( M \), then it means that \( w \) becomes matched and by Fact 1 she will always stay matched, therefore \( e \) will never become special. \( \square \)

Let us for a moment think of an optimal stable matching \( M_{\text{opt}} \). A path \( P \) or a cycle \( C \) is called alternating if its edges are alternatingly from \( M \) and from \( E \setminus M \). It is well known from matching theory that \( M \oplus M_{\text{opt}} \) can be partitioned into alternating paths and alternating cycles. If \( M \oplus M_{\text{opt}} \) does not contain paths of length 3, then \( |M_{\text{opt}}| \leq \frac{3}{2} |M| \) and \( M \) is a \( \frac{3}{2} \) approximation of \( M_{\text{opt}} \). To achieve a \( \frac{3}{2} \) approximation we will be eliminating paths of length 3 from \( M \oplus M_{\text{opt}} \).

Accordingly we define a dangerous path, which is an alternating path \( P = (w, m, w_1, m_1, m) \) such that \( w \) and \( m \) are unmatched (which means that \( (m_1, w_1) \) is from \( M \) and \( (w, m_1), (w_1, m) \) do not belong to \( M \)) and \( (m_1, w_1) \) is not a blocking pair for matching \( M' = \{(w, m_1), (w_1, m)\} \). Let us notice that if \( P \) is a dangerous path, then either \( m_1 \) is indifferent between \( w \) and \( w_1 \) (and then we say that \( (w, m_1) \) is an equal edge) or \( w_1 \) is indifferent between \( m \) and \( m_1 \) (and then \( (w_1, m) \) is called an equal edge) or both. Thus a dangerous path contains one or two equal edges. If an edge \( e = (w, m_1) \) is equal, then we say that \( P \) is a masculine dangerous path and if \( (w_1, m) \) is equal, then we say that \( P \) is a feminine dangerous path. A path \( P \) can of course be both a masculine and feminine dangerous path.

**Lemma 3** After running algorithm GS Modified there are no masculine dangerous paths in the graph.

**Proof.** Suppose the graph contains a masculine dangerous path \( (w', m, w, m') \). Then \( w' \) is unstable and by Fact 1 she was either free or unstable at the moment \( m' \) was proposing to her. This means that she would have accepted \( m \) and couldn’t ever become unstable. \( \square \)

In Algorithm GS Improve eliminating blocking pairs and masculine dangerous paths has a higher priority than eliminating feminine dangerous paths.

An edge \( e = (m, w) \) is said to be blocking if \( m, w \) are a blocking pair. A man is said to be spoiling if he belongs to a blocking pair or if he is free and belongs to a masculine dangerous path. If in the graph there is a spoiling man than it means that the state of the graph from after running Algorithm GS Modified has been
upset - and now either the matching is unstable or there are masculine dangerous paths in the graph. An edge $e = (m, w)$ is said to be unstable if $m$ is free and $w$ is unstable. Let us notice that if $e = (m, w)$ is unstable, then it is a part of a masculine dangerous path. An edge $e = (m, w)$ that is equal and such that $m$ is free is called $fequal$. An edge $e$ is said to be bad if it is blocking or unstable or $fequal$.

Algorithm \textit{GS Improve}

while there are free spoiling men or feminine dangerous paths do

if there is a free spoiling man $m$, then

\{$
\begin{array}{l}
\text{w} \leftarrow \text{next woman on } m \text{’ s list } L_m
\\
(*) \text{ if } (m, w) \text{ is blocking, then } M \leftarrow M \cup (m, w) \setminus (w, M(w)) \\
\text{else if } w \text{ is unstable and there is no free woman } w' \text{ who is equally good for } m \text{ as } w, \text{ then} \\
M \leftarrow M \cup \{(m, w), (M(w), w')\} \setminus (w, M(w)), \text{ where } w' \text{ is a satellite of } M(w) \\
\text{else if } (m, w) \text{ is } fequal, \text{ then add } w \text{ to the end of list } L'_m.
\end{array}$
\}$

\{ (** \text{ if there is a feminine dangerous path } P \text{ containing free man } m, \text{ then} \\
\begin{array}{l}
(*) \text{ if } (m, w) \text{ is blocking, then } M \leftarrow M \cup (m, w) \setminus (w, M(w)) \\
\text{else if } (m, w) \text{ is } fequal, \text{ then add } w \text{ to the end of list } L'_m.
\end{array}$
\}$

While scanning lists $L_m$ and $L'_m$ we always move forward and do not come back to the edge that was considered previously (or equivalently remove the edge from $L_m$ or $L'_m$), but there is one exception at step (*) – if the edge $(m, w)$ is special and thus also blocking, then we do not move forward on $L_m$ and the next time we consider $m$ in the algorithm, $w$ will be still on $L_m$. In the step (**) we begin scanning from list $L'_m$.

We now prove several properties of Algorithm \textit{GS Improve}. First lest us remark that Fact[1] and Lemma[2] are also true of Algorithm \textit{GS Improve}.

\textbf{Lemma 4} Suppose that at step $S$ of Algorithm \textit{GS Improve} edge $e$ is incident on a single man $m$. If at step $S$ edge $e$ is not bad, then it will not become bad later in the course of running Algorithm \textit{GS Improve}. If at step $S$ edge $e$ is $fequal$ and not unstable, then $e$ will not become unstable or blocking.

\textbf{Proof.}
If \( e \) is not bad, then \( w \) is matched to some man \( m' \). Since \( e \) is not unstable, then \( w \) is not unstable and thus by Fact 1 \( w \) will not become unstable, thus \( e \) will not become unstable. If \( w \) prefers \( m' \) to \( m \), then since she is not unstable, she will always be matched to someone she prefers to \( m \), thus \( e \) will not become blocking. If for \( w \) men \( m \) and \( m' \) are equally good, but \( e \) is not an equal edge, then it means that \( m' \) has no free woman incident on him at the moment and thus will never have and if \( w \) gets matched to \( m'' \), then \( m'' \) will be better for her than \( m' \), because \( w \) could become matched to \( m'' \) that is equally good for her as \( m' \) only if \( (w, m') \) belonged to a feminine dangerous path.

If at step \( S \) edge \( e = (m, w) \) is fequal and not unstable, then it means that \( w \) is matched to some man \( m' \) such that \( m \) and \( m' \) are equally good for her and \( w \) is not unstable. Thus by Fact 1 she will not become unstable and she will always be matched to someone at least as good for her as \( m' \).

\[ \square \]

**Lemma 5** At each step of the Algorithm GS Improve, there is at most one spoiling man and if there is one, he is free.

**Proof.** At the beginning of the algorithm or in other words at step 0 matching \( M \) is stable and there are no spoiling men. Suppose that till step \( S - 1 \) the lemma was true. Now we carry out step \( S \). We have two cases.

Case one. If there are no spoiling men, we look for a feminine dangerous path. If there is no such path, the algorithm ends. Suppose we have found a feminine dangerous path with a free man \( m \). \( m \) has no blocking or unstable edges incident on him, but has at least one fequal edge incident on him (otherwise he wouldn’t belong to a feminine dangerous path.) We choose an equal edge \( e = (m, w) \) which is as high on \( m \)'s list as possible. Next we remove \( (w, M(w)) \) from \( M \) and add \( e \) to \( M \). Now \( m \) is not spoiling because he was not spoiling before step \( S \) and by Lemma 4 an edge incident on a single man that is not bad cannot become bad. Similarly a man that was free before step \( S \) is not spoiling. Let us now consider man \( m' \) that before and after step \( S \) is matched. Before step \( S \) he was not spoiling (as there were no spoiling men). The situation of edge \( (m', w'') \) might have changed only if \( w'' \) is now matched to someone different than before step \( S \) and it is true only for \( w \) but \( w \) was not unstable, so she did not become unstable, is now matched to \( m \), who is equally good for her as \( M(w) \) and \( m \) has no free women incident on him, because then \( m \) would have been spoiling before. Thus the only man that can be spoiling is the one that was made single at step \( S \).

Case two. There is one spoiling man \( m \) and \( m \) is free. We choose a bad (but not fequal) edge \( e = (m, w) \) which is as high on \( m \)'s list as possible. \( m \) becomes matched to \( w \) (and if \( w \) was unstable her previous partner \( M(w) \) gets matched to one of his satellites). Now \( m \) is not spoiling because he is matched via a bad edge which is as high on \( m \)'s list as possible. The situation of all men that were free has not changed, because they were free and if they were not spoiling, and they
were not, as only \( m \) was spoiling, by Lemma 4 they are not spoiling now. If \( m' \) is matched, then the situation of an edge \( e' = (m', w') \) might have changed for \( e' \) only if \( w' \) is matched to someone different than she was before that step or if she was free. First suppose that \( e \) was not unstable, therefore the only woman we have to consider is \( w \), but \( w \) is now matched to \( m \), who is better for her than her previous partner (or she was free before), so \( e' \) cannot have become blocking and thus \( m' \) is not spoiling. If \( e \) was unstable, then \( w \) became matched to \( m \) and \( M(w) \) became matched to a satellite \( w_1 \) of \( M(w) \). The situation of edge \( (m', w_1) \) has not changed, because if it was not blocking before, it is not now and \( m' \) cannot be spoiling.

Suppose that some edge \( (m', w) \) is blocking now and was not before. It means that \( m' \) prefers \( w \) to his current partner \( M(m') \), but it means he must have proposed to \( w \) before he proposed to \( M(m') \), but when he was proposing to \( w \), by Fact 1 \( w \) must have been either free or unstable. She could not be unstable because then by Fact 1 she would not be unstable at step \( S \). So she must have been free at that moment but then at step \( S \) \( w \) would be unstable and still matched to \( m' \) and \( m' \) would become matched to one of his satellites, who is equally good for him as \( w \).

\[ \square \]

**Lemma 6** If \( e \) is bad but not special at step \( S \) and gets added to \( M \), then it will not become bad later in the course of running Algorithm GS Modified.

**Proof.** Suppose that at step \( S \) edge \( e = (m, w) \) is bad but not special and \( e \) gets added to \( M \). Then by Fact 1 \( w \) will never become unstable (note that at step \( S \) she might have been unstable) and will always stay matched to \( m \) or will become matched to someone at least as good for her as \( m \). Thus \( e \) will not become unstable or blocking later. If she gets matched to \( m' \), who is equally good for her as \( m \), then it means that at that step, \( (m, w) \) belonged to a feminine dangerous path and thus \( m' \) had not (and thus will never have) a free woman incident on him (because then he would have proposed to her as he would have been blocking because of a free woman incident on him), which means that \( (w, m') \) never becomes a part of a feminine dangerous path, therefore \( e \) will not become fequal.

\[ \square \]

Now we can summarize our observations and state

**Theorem 1** Algorithm computes a stable matching \( M \) that does not contain dangerous alternating paths and thus is a \( \frac{5}{2} \)-approximation algorithm for stable matchings.

**Proof.** Algorithm GS Modified computes a stable matching \( M \). If at certain step \( S \) of Algorithm GS Modified, \( M \) is not stable and contains blocking pairs, then by Lemma 5 all the blocking pairs are incident on one free man \( m \) and by Lemmas 4 and 6 all these blocking edges are still available (on lists \( L_m \)) for the algorithm and
we can carry out an appropriate operation. If at some moment there is a dangerous path, then by Lemmas 4 and 6, appropriate edges – a fequal one in the case of a feminine dangerous path and an unstable one in the case of a masculine dangerous path are still available for the algorithm.

Lemma 7 Algorithms GS Modified and GS Improve run in $O(m)$ time.

Proof. Each edge $e$ is scanned a constant number of times. It may be scanned twice in $L_m$ - once as a special edge, the other time not as a special edge (Lemma 2). It may also be moved to a list $L'_m$ if it is a fequal edge and $m$ is spoiling. Checking whether man $m$ has free women incident on him in the same tie can also be organized so that each edge is checked for that purpose at most once, as woman once matched stays matched.

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We give a $\frac{2}{3}$-approximation algorithm for stable matchings that runs in $O(m)$ time. The previously best known algorithm by McDermid has the same approximation ratio but runs in $O(n^{3/2}m)$ time, where $n$ denotes the number of people and $m$ is the total length of the preference lists in a given instance. Also the algorithm and the analysis are much simpler.

1 Introduction
In the paper we consider a variant of the problem called Stable Matchings, known also in the literature as Stable Marriage. The problem is defined as follows. We are given two sets $W$ and $M$ of women and men and each woman $w$ from $W$ has a linearly ordered preference list $L_w$ of a subset of men $M'_w \subseteq M$ and similarly each man $m$ from $M$ has a linearly ordered preference list $L_m$ of a subset of women $W'_m \subseteq W$. The lists of men and women can contain ties, which are subsets of men (or respectively women), which are equally good for a given woman (resp. man). Thus if $m$ and $m'$ are on list $L_w$ of woman $w$, then either (1) $m < m'$ and then we say that woman $w$ prefers $m$ to $m'$ or that $m$ is better for her than $m'$ or (2) $m \equiv m'$, which means that $m$ and $m'$ are in a tie on $L_w$ and then we say that $w$ is indifferent between $m$ and $m'$ or that $m$ and $m'$ are equally good for her or (3) $m' < m$. If man $m$ does not belong to list $L_w$, then we say that $m$ is unacceptable for $w$. A matching is a set of pairs $(m, w)$ such that $m \in M, w \in W$ and $m$ and $w$ are on each other’s preference lists and each man/woman belongs to at most one pair. If $(m_1, w_1)$ belongs to a certain matching $M_1$, then we write $M_1(m_1) = w_1$, which means that in $M_1$ woman $w_1$ is a partner of $m_1$ and analogously that $M_1(w_1) = m_1$. A matching $M_1$ is called stable if it does not admit a blocking pair. A pair $(m, w)$ is blocking for $M_1$ if (0) $m$ and $w$ are acceptable to each other and (1) $m$ is single or prefers $w$ to $M_1(m)$ and if (2) $w$ is single or prefers $m$ to $M_1(w)$. Each instance of the problem can be represented by a bipartite graph $G = (M \cup W, E)$ with vertices $M$ representing men and vertices $W$ women and edges $E$ connecting all mutually acceptable pairs of men and women. The problem we are interested in in this paper is that of finding a stable
matching that has the largest cardinality. The version in which there are no ties in
the preference lists of men and women has been long known and an algorithm by
Gale and Shapley ([11]) solves it exactly in $O(m)$ time, where $m$ denotes the sum of
the lengths of preference lists. In the version without ties a stable matching always
exists and every stable matching has the same cardinality. If we allow ties, as in the
problem we consider in this paper, then a stable matching also always exists and
can be found via Gale/Shapley algorithm by breaking ties arbitrarily. However, the
sizes of stable matchings can vary considerably and the problem of finding a stable
matching of maximum cardinality is $\text{NP}$-hard, which was shown by Manlove et
al. in [8]. Therefore it is desirable to devise an approximation algorithm for the
problem.

1.1 History of approximation results

In [8] it was shown that an arbitrary stable matching is a 2-approximation. A first
non-trivial $(2 - c \frac{\log n}{n})$-approximation was given by Iwama et al. in [4], where $c$
is a positive constant. Next in [5] the approximation ratio was improved to $2 - \sqrt{c}$,
where $c$ is a positive constant which is at most $1.4\sqrt{6}$ and in [6] to $1.5$. Further on
Kiraly ([7]) presented a linear time elegant $\frac{3}{2}$-approximation algorithm. Currently
the best approximation algorithm is by McDermid [9] and achieves the approximation
guarantee $\frac{3}{2}$. Its running time is $O(n^{3/2}m)$, where $n$ denotes the number of
vertices in the graph and $m$ the number of edges in the graph, that is the sum of the
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The problem was shown to be $\text{APX}$-complete by Halldorsson et al. in [2] and
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Yanagisawa ([10]) proved that the problem cannot be approximated within $\frac{33}{20}$. In
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assumption that the minimum vertex cover problem cannot be approximated within
$2 - \epsilon$.

Our results We give a $3/2$-approximation algorithm that runs in $O(m)$ time and
additionally is significantly simpler than that of McDermid.

2 Algorithm

The algorithm consists of two linear (with regard to the number of edges) time
phases: one called Algorithm $\text{GS Modified}$, the other Algorithm $\text{GS Improve}$. The
first one is very similar to that of Gale/Shapley but has one modification that handles
masculine dangerous paths which are defined later. The second one "improves" the existing
matching so that it will not contain any dangerous paths (either masculine or feminine) and as a result will be of size at least $2/3 |M_{opt}|$, where
$M_{opt}$ denotes a stable matching of largest cardinality.

If man $m$ is matched to woman $w$ and there is at least one free woman $w_1$ such
that $w$ and $w_1$ are equally good for $m$, then we say that $m$ is unstable and that $w_1$
is a satellite of \( m \). If woman \( w \) is matched to an unstable man \( m \), then we say that \( w \) is unstable. If \( e = (m, w) \) is such that both \( m \) and \( w \) are free and there is at least one free woman \( w_1 \) such that \( w \) and \( w_1 \) are equally good for \( m \), then \( e \) is called special.

Algorithm GS Modified

Each man \( m \)'s preference list \( L_m \) is organized in such a way that if \( L_m \) contains tie \( t = (w_1, w_2, \ldots, w_k) \), then free women from \( t \) come before matched women from \( t \). At the beginning all women are free and ties are broken arbitrarily and in the course of running the algorithm whenever woman \( w \) becomes matched for the first time we move her to the end of every tie she belongs to.

while there is a free man \( m \) with a nonempty list do

\[
w \leftarrow \text{next woman on } m \text{'s list}
\]
if \( m \) is better for \( w \) than her current partner \( M(w) \) or \( w \) is free, then
\[
M \leftarrow M \cup (m, w) \setminus (w, M(w))
\]
else if \( w \) is unstable, then
\[
M \leftarrow M \cup \{(m, w), (M(w), w')\} \setminus (w, M(w)), \text{ where } w' \text{ is a satellite of } M(w)
\]

While scanning lists \( L_m \) we always move forward and do not come back to the edge that was considered previously (or equivalently remove the edge from \( L_m \)), but there is one exception – if the edge \( (m, w) \) is special, then we do not move forward on \( L_m \) and the next time we consider \( m \) in the algorithm, \( w \) will be still on \( L_m \) (and thus if \( m \) becomes single he will propose to \( w \) for the second time.)

Before proving that Algorithm GS Modified computes a stable matching we give the following fact.

**Fact 1** If woman \( w \) becomes matched, then she will stay matched. Woman \( w \) can become unstable only at the step of the algorithm, when she is free. If an unstable woman \( w \) (matched to some unstable man \( m \)) becomes matched to some new man \( m' \), then she is not unstable any more. If woman \( w \) is matched to man \( m \) and is not unstable, then she will always be matched to someone at least as good for her as \( m \).

**Proof.** If \( w \) is matched and \( m \) proposes to her, then there is no free woman \( w' \) who is equally good for \( m \) as \( w \) (because then \( m \) would propose to \( w' \) before proposing to \( w \)). As a result if \( w \) becomes matched to \( m \) she will not become unstable and she will cease to be unstable if she was before. Also if \( w \) is matched and not unstable at the moment \( m \) proposes to her, she accepts him only if \( m \) is better for her than her current partner \( M(w) \). \( \square \)
Lemma 1  Algorithm GS Modified computes a stable matching.

Proof. Suppose to the contrary that a computed matching $M$ is not stable and admits a blocking pair $(m, w)$. Thus $w$ prefers $m$ to her current partner $M(w)$ and $m$ prefers $w$ to his current partner $M(m)$. $m$ must have proposed to $w$ before he proposed to $M(m)$. If she was matched at that moment she would have either accepted $m$ – if she was unstable or matched to someone worse for her than $m$ or rejected him – if she was not unstable and matched to someone at least as good for her as $m$. In either case by Fact 1 she would remain matched to someone at least as good for her as $m$. Therefore $w$ must have been free at the moment $m$ was proposing to her and since she later became matched to someone worse for her than $m$, she was unstable after she became matched to $m$. Thus at the moment $m$ was proposing to $w$ edge $(m, w)$ was special. Then she must have become matched to some man $m'$ and $m$ to one of his satellites. Some time later $m$ must have become free but then at some moment he would propose to $w$ for the second time (because edge $(m, w)$ was special) and then he could get rejected only if $w$ was matched to someone at least as good for her as $m$ and by Fact 1 she would have stayed matched to someone at least as good for her as $m$.

We also prove

Lemma 2  If $e$ is special at step $S$ of Algorithm GS Modified and gets added to $M$, then it will not become special later in the course of running Algorithm GS Modified.

Proof. If at step $S$ edge $e = (m, w)$ is special and gets added to $M$, then it means that $w$ becomes matched and by Fact 1 she will always stay matched, therefore $e$ will never become special.

Let us for a moment think of an optimal stable matching $M_{opt}$. A path $P$ or a cycle $C$ is called alternating if its edges are alternatingly from $M$ and from $E \setminus M$. It is well known from matching theory that $M \oplus M_{opt}$ can be partitioned into alternating paths and alternating cycles. (For two sets $X, Y$, the set $X \oplus Y$ denotes $X \setminus Y \cup Y \setminus X$.) If $M \oplus M_{opt}$ does not contain paths of length 3, then $|M_{opt}| \leq \frac{3}{2}|M|$ and $M$ is a $\frac{3}{2}$ approximation of $M_{opt}$. To achieve a $\frac{3}{2}$ approximation we will be eliminating paths of length 3 from $M \oplus M_{opt}$.

Accordingly we define a dangerous path, which is an alternating path $P = (w, m_1, w_1, m)$ such that $w$ and $m$ are unmatched (which means that $(m_1, w_1)$ is from $M$ and $(w, m_1), (w_1, m)$ do not belong to $M$) and $(m_1, w_1)$ is not a blocking pair for matching $M' = \{ (w, m_1), (w_1, m) \}$. Let us notice that if $P$ is a dangerous path, then either $m_1$ is indifferent between $w$ and $w_1$ (and then we say that $(w, m_1)$ is an equal edge) or $w_1$ is indifferent between $m$ and $m_1$ (and then $(w_1, m)$ is called an equal edge) or both. Thus a dangerous path contains one or two equal
edges. If an edge $e = (w, m_1)$ is equal, then we say that $P$ is a **masculine dangerous path** and if $(w_1, m)$ is equal, then we say that $P$ is a **feminine dangerous path**. A path $P$ can of course be both a masculine and feminine dangerous path.

**Lemma 3** After running algorithm GS Modified there are no masculine dangerous paths in the graph.

**Proof.** Suppose the graph contains a masculine dangerous path $(w', m, w, m')$. Then $w$ is unstable and by Fact 1 she was either free or unstable at the moment $m'$ was proposing to her. This means that she would have accepted $m$ and couldn’t ever become unstable. □

In Algorithm GS Improve eliminating blocking pairs and masculine dangerous paths has a higher priority than eliminating feminine dangerous paths.

An edge $e = (m, w)$ is said to be **blocking** if $m, w$ are a blocking pair. A man is said to be **spoiling** if he belongs to a blocking pair or if he is free and belongs to a masculine dangerous path. If in the graph there is a spoiling man, then it means that the state of the graph from after running Algorithm GS Modified has been upset - and now either the matching is unstable or there are masculine dangerous paths in the graph. An edge $e = (m, w)$ is said to be **unstable** if $m$ is free and $w$ is unstable. Let us notice that if $e = (m, w)$ is unstable, then it is a part of a masculine dangerous path. An edge $e = (m, w)$ that is equal and such that $m$ is free is called **fequal**. An edge $e$ is said to be **bad** if it is blocking or unstable or fequal.
Algorithm \textit{GS Improve}

while there are free spoiling men or feminine dangerous paths do

if there is a free spoiling man \( m \), then
\[
\{ \\
  w \leftarrow \text{next woman on } m \text{'s list } L_m \\
  (\star) \text{ if } (m, w) \text{ is blocking, then } M \leftarrow M \cup (m, w) \setminus (w, M(w)) \\
  \text{ else if } w \text{ is unstable and there is no free woman } w'' \text{ who is equally good for } m \text{ as } w, \text{ then} \\
  M \leftarrow M \cup \{(m, w), (M(w), w')\} \setminus (w, M(w)) , \text{ where } w' \text{ is a satellite of } M(w) \\
  \text{ else if } (m, w) \text{ is fequal, then add } w \text{ to the end of list } L'_m. \\
\}
\]

else if there is a feminine dangerous path \( P \) containing free man \( m \), then
\[
\{ \\
  (\star\star) \ w \leftarrow \text{next woman on } m \text{'s lists } L'_m \cup L_m \\
  \text{ if } (m, w) \text{ is fequal, then} \\
  M \leftarrow M \cup (m, w) \setminus (w, M(w)) \\
\}
\]

While scanning lists \( L_m \) and \( L'_m \) we always move forward and do not come back to the edge that was considered previously (or equivalently remove the edge from \( L_m \) or \( L'_m \)), but there is one exception at step (\star) – if the edge \( (m, w) \) is special and thus also blocking, then we do not move forward on \( L_m \) and the next time we consider \( m \) in the algorithm, \( w \) will be still on \( L_m \).

In the step (\star\star) we begin scanning from list \( L'_m \).

We now prove several properties of Algorithm \textit{GS Improve}. First let us remark that Fact[1] and Lemma[2] are also true of Algorithm \textit{GS Improve}.

\textbf{Lemma 4} Suppose that at step \( S \) of Algorithm \textit{GS Improve} edge \( e \) is incident on a single man \( m \). If at step \( S \) edge \( e \) is not bad, then it will not become bad later in the course of running Algorithm \textit{GS Improve}. If at step \( S \) edge \( e \) is fequal and not unstable, then \( e \) will not become unstable or blocking.

\textbf{Proof.}

If \( e = (m, w) \) is not bad, then \( w \) is matched to some man \( m' \). Since \( e \) is not unstable, then \( w \) is not unstable and thus by Fact[1] \( w \) will not become unstable, thus \( e \) will not become unstable. If \( w \) prefers \( m' \) to \( m \), since \( w \) is not unstable, she will always be matched to someone she prefers to \( m \), thus \( e \) will not become blocking. If for \( w \) men \( m \) and \( m' \) are equally good, but \( e \) is not an equal edge, then it means that \( m' \) has no free woman incident on him at the moment and thus will never have and if \( w \) gets matched to \( m'' \), then \( m'' \) will be better for her than \( m' \),
because \( w \) could become matched to \( m'' \) that is equally good for her as \( m' \) only if \( (w, m') \) belonged to a feminine dangerous path.

If at step \( S \) edge \( e = (m, w) \) is fequal and not unstable, then it means that \( w \) is matched to some man \( m' \) such that \( m \) and \( m' \) are equally good for her and \( w \) is not unstable. Thus by Fact 4, she will not become unstable and she will always be matched to someone at least as good for her as \( m' \).

\[ \square \]

**Lemma 5** At each step of the Algorithm GS Improve, there is at most one spoiling man and if there is one, he is free.

**Proof.** At the beginning of the algorithm or in other words at step 0, matching \( M \) is stable and there are no spoiling men. Suppose that till step \( S - 1 \) the lemma was true. Now we carry out step \( S \). We have two cases.

Case one. If there are no spoiling men, we look for a feminine dangerous path. If there is no such path, the algorithm ends. Suppose we have found a feminine dangerous path with a free man \( m \). \( m \) has no blocking or unstable edges incident on him, but has at least one fequal edge incident on him (otherwise he wouldn’t belong to a feminine dangerous path.) We choose an equal edge \( e = (m, w) \) which is as high on \( m \)'s list as possible. Next we remove \( (w, M(w)) \) from \( M \) and add \( e \) to \( M \). Now \( m \) is not spoiling because he was not spoiling before step \( S \) and by Lemma 4, an edge incident on a single man that is not bad cannot become bad. Similarly a man that was free before step \( S \) is not spoiling. Let us now consider man \( m' \) that before and after step \( S \) is matched. Before step \( S \) he was not spoiling (as there were no spoiling men). The situation of edge \( (m', w'') \) might have changed only if \( w'' \) is now matched to someone different than before step \( S \) and it is true only for \( w \) but \( w \) was not unstable, so she did not become unstable, is now matched to \( m \), who is equally good for her as \( M(w) \) and \( m \) has no free women incident on him, because then \( m \) would have been spoiling before. Thus the only man that can be spoiling is the one that was made single at step \( S \).

Case two. There is one spoiling man \( m \) and \( m \) is free. We choose a bad (but not fequal) edge \( e = (m, w) \) which is as high on \( m \)'s list as possible. \( m \) becomes matched to \( w \) (and if \( w \) was unstable her previous partner \( M(w) \) gets matched to one of his satellites). Now \( m \) is not spoiling because he is matched via a bad edge which is as high on \( m \)'s list as possible. The situation of all men that were free has not changed, because they were free and if they were not spoiling, and they were not, as only \( m \) was spoiling, by Lemma 4, they are not spoiling now. If \( m' \) is matched, then the situation of an edge \( e' = (m', w') \) might have changed for \( e' \) only if \( w' \) is matched to someone different than she was before that step or if she was free. First suppose that \( e \) was not unstable, therefore the only woman we have to consider is \( w \), but \( w \) is now matched to \( m \), who is better for her than her previous partner (or she was free before), so \( e' \) cannot have become blocking and thus \( m' \) is not spoiling. If \( e \) was unstable, then \( w \) became matched to \( m \) and \( M(w) \)
became matched to a satellite $w_1$ of $M(w)$. The situation of edge $(m', w_1)$ has not changed, because if it was not blocking before, it is not now and $m'$ cannot be spoiling.

Suppose that some edge $(m', w)$ is blocking now and was not before. It means that $m'$ prefers $w$ to his current partner $M(m')$, but it means he must have proposed to $w$ before he proposed to $M(m')$, but when he was proposing to $w$, by Fact 1 $w$ must have been either free or unstable. She could not be unstable because then by Fact 1 she would not be unstable at step $S$. So she must have been free at that moment but then at step $S$ $w$ would be unstable and still matched to $m'$ and $m'$ would become matched to one of his satellites, who is equally good for him as $w$. Contradiction.

\[\text{Lemma 6} \quad \text{If } e \text{ is bad but not special at step } S \text{ and gets added to } M, \text{ then it will not become bad later in the course of running Algorithm GS Improve.} \]

\[\text{Proof.} \quad \text{Suppose that at step } S \text{ edge } e = (m, w) \text{ is bad but not special and } e \text{ gets added to } M. \text{ Then by Fact 1 } w \text{ will never become unstable (note that at step } S \text{ she might have been unstable) and will always stay matched to } m \text{ or will become matched to someone at least as good for her as } m. \text{ Thus } e \text{ will not become unstable or blocking later. If she gets matched to } m', \text{ who is equally good for her as } m, \text{ then it means that at that step, } (m, w) \text{ belonged to a feminine dangerous path and thus } m' \text{ had not (and thus will never have) a free woman incident on him (because then he would have proposed to her as he would have been blocking because of a free woman incident on him), which means that } (w, m') \text{ never becomes a part of a feminine dangerous path, therefore } e \text{ will not become fequal.} \]

Now we can summarize our observations and state

\[\text{Theorem 1} \quad \text{Algorithm computes a stable matching } M \text{ that does not contain dangerous alternating paths and thus is a } \frac{3}{2} \text{-approximation algorithm for stable matchings.} \]

\[\text{Proof.} \quad \text{Algorithm GS Modified computes a stable matching } M. \text{ If at certain step } S \text{ of Algorithm GS Modified, } M \text{ is not stable and contains blocking pairs, then by Lemma 5 all the blocking pairs are incident on one free man } m \text{ and by Lemmas 4 and 6 all these blocking edges are still available (on lists } L_m) \text{ for the algorithm and we can carry out an appropriate operation. If at some moment there is a dangerous path, then by Lemmas 4 and 6 appropriate edges – a fequal one in the case of a feminine dangerous path and an unstable one in the case of a masculine dangerous path are still available for the algorithm.} \]

\[\text{Lemma 7} \quad \text{Algorithms GS Modified and GS Improve run in } O(m) \text{ time.} \]
Proof. Each edge $e$ is scanned a constant number of times. It may be scanned twice in $L_m$ - once as a special edge, the other time not as a special edge (Lemma[3]). It may also be moved to a list $L'_m$ if it is a fequal edge and $m$ is spoiling. Checking whether man $m$ has free women incident on him in the same tie can also be organized so that each edge is checked for that purpose at most once, as woman once matched stays matched.

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Faster and simpler approximation of stable matchings

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Abstract

We give a $\frac{3}{2}$-approximation algorithm for stable matchings that runs in $O(m)$ time. The previously best known algorithm by McDermid has the same approximation ratio but runs in $O(n^{3/2}m)$ time, where $n$ denotes the number of people and $m$ is the total length of the preference lists in a given instance. Also the algorithm and the analysis are much simpler. We also give the extension of the algorithm for computing stable many-to-many matchings.

1 Introduction

In the paper we consider a variant of the problem called Stable Matchings, known also in the literature as the Stable Marriage problem. The problem is defined as follows. We are given two sets $W$ and $U$ of women and men. Each woman $w$ of $W$ has a preference list $L_w$ of a subset of men and similarly each man $m$ of $U$ has a preference list $L_m$ of a subset of women. The preference lists are linearly ordered lists of ties, which are subsets of men (or resp. women), who are equally good for a given woman (resp. man). Ties are disjoint and can contain also one person, appropriately a man or a woman. Thus if $m$ and $m'$ are on list $L_w$ of woman $w$, then either (1) $w$ prefers $m$ to $m'$ or in other words $m$ is better for $w$ than $m'$ or (2) $m$ and $m'$ are in a tie on $L_w$ and then we say that $w$ is indifferent between $m$ and $m'$ or that $m$ and $m'$ are equally good for her or (3) $w$ prefers $m'$ to $m$. Man $m$ and woman $w$ are said to be mutually acceptable to each other if they belong to each other’s preference lists. The most preferred person(s) is(are) at the top the preference lists. A matching is a set of pairs $(m, w)$ such that $m \in U$, $w \in W$ and $m$ and $w$ are mutually acceptable and each man/woman belongs to at most one pair. If $(m, w)$ belongs to a certain matching $M$, then we write $M(m) = w$, which means that in $M$ woman $w$ is a partner of $m$ and analogously that $M(w) = m$. If man $m$ (or woman $w$) is not contained in any pair of a matching $M$, then we say that $m$ (or $w$) is unmatched or free in $M$. A matching $M$ is called stable if it does not admit a blocking pair. A pair $(m, w)$ is blocking for $M$ if (0) $m$ and $w$ are mutually acceptable and (1) $m$ is unmatched or prefers $w$ to $M(m)$ and (2) $w$ is unmatched or prefers $m$ to $M(w)$. Each instance of the problem can be represented by a bipartite graph $G = (U \cup W, E)$ with vertices $U$ representing men, vertices $W$ representing women and edges $E$ connecting all mutually acceptable pairs of men and women. The problem we are interested in is that of finding a stable matching that has the largest cardinality. The version in which there are no ties in the preference lists of men and women has been long known and an algorithm by Gale and Shapley [4] solves it exactly in $O(m)$ time, where $m$ denotes the number of edges in the underlying graph. In the version without ties a stable matching always exists and every stable matching has the same cardinality. If we allow ties, as in the problem we consider in this paper, then a stable matching always exists and can be found via the Gale/Shapley algorithm by breaking ties arbitrarily. However, the

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sizes of stable matchings can vary considerably and the problem of finding a stable matching of maximum cardinality is \( NP \)-hard, which was shown by Manlove et al. in [13]. Therefore it is desirable to devise an approximation algorithm for the problem.

**Previous results** Previous approximation algorithms were presented in [13], [8], [9], [10], [11]. Currently the best approximation algorithm is by McDermid [14] and achieves the approximation guarantee \( \frac{3}{2} \). Its running time is \( O(n^{3/2} m) \), where \( n \) denotes the number of vertices and \( m \) the number of edges. Inapproximability results were shown in [9], [6], [10].

**Our results** While constructing approximation algorithms the goal is not only to achieve a good approximation guarantee but also good running time (to name just two examples, see [1], [15]). We give a \( \frac{3}{2} \)-approximation algorithm that runs in \( O(m) \) time and additionally is significantly simpler than that of McDermid. In devising the algorithm we were led by the observation that it suffices to find a stable matching that will not create a dangerous path, which is defined later. We also give the extension of the algorithm for computing stable many-to-many matchings, which runs in \( O(mlogc) \) time, where \( c \) denotes the minimum of the maximal capacities in each side of the bipartition. In particular it means we give an \( O(m) \)-time algorithm for the Hospitals-Residents problem, improving on an \( O(d^{5/2}n^{3/2}m) \) time algorithm given by McDermid, where \( d \) denotes the maximal capacity of a hospital. McDermid's algorithm follows from the reduction of the Hospitals-Residents problem to the Stable Matchings problem by "cloning" hospitals. The approach by cloning does not work if the vertices on both sides of the bipartition are allowed to have capacities larger than 1. Since the problems have many practical applications (see [2], [3], [7] for example), we believe our algorithms will be of help.

## 2 Algorithm

For a given instance of the problem let \( M_{\text{opt}} \) denote an optimal (i.e. largest) stable matching and let \( M, M' \) be any two matchings. We say that \( e \) is an \( M \)-edge if \( e \in M \). A path \( P \) or a cycle \( C \) is called alternating (wrt \( M \)) if its edges alternate between \( M \)-edges and edges of \( E \setminus M \). It is well known from matching theory (see [12] for example) that \( M \oplus M' \) can be partitioned into a set of alternating paths and alternating cycles. (For two sets \( X, Y \), the set \( X \oplus Y \) denotes \( (X \setminus Y) \cup (Y \setminus X) \).) Let \( S \) denote a set of alternating paths and cycles of \( M \oplus M_{\text{opt}} \). Consider any alternating cycle \( c \) of \( S \) or any alternating path \( p \) of even length of \( S \). Then both \( c \) and \( p \) contain the same number of \( M \)-edges and \( M_{\text{opt}} \)-edges. Consider an alternating path \( p \) of length \( 2k + 1 \) of \( S \). Then either
\[
|\frac{|M_{\text{opt}} \cap P|}{|M' \cap P|}| = \frac{k+1}{k} \quad \text{or} \quad |\frac{|M \cap P|}{|M_{\text{opt}} \cap P|}| = \frac{k+1}{k}.
\]

Therefore if \( M \) is stable and \( M \oplus M_{\text{opt}} \) does not contain a path of length 3 with the middle edge being an \( M \)-edge, then \( |M_{\text{opt}}| \leq \frac{3}{2} |M| \) and \( M \) is a \( \frac{3}{2} \)-approximation of \( M_{\text{opt}} \). To achieve a \( \frac{5}{4} \)-approximation we will eliminate such potential paths of length 3 of \( M \oplus M_{\text{opt}} \).

Accordingly we define a dangerous path wrt to a stable \( M \) to be an alternating path \( P = (w, m_1, w_1, m) \) such that \( w \) and \( m \) are unmatched (which means that \((m_1, w_1)\) is in \( M \) and \((w, m_1)\), \((w_1, m)\) do not belong to \( M \)) and \((m_1, w_1)\) is not a blocking pair for matching \( M' = \{(w, m_1), (w_1, m)\} \). Let us notice that if \( P \) is a dangerous path, then either \( m_1 \) is indifferent between \( w \) and \( w_1 \) and then we say that \( P \) is a masculine dangerous path or \( w_1 \) is indifferent between \( m \) and \( m_1 \) and then we say that \( P \) is a feminine dangerous path. A path \( P \) can of course be both a masculine and feminine dangerous path.

We also introduce the following terminology. If man \( m \) is matched to woman \( w \) and there is at least one free woman \( w_1 \) such that \( w \) and \( w_1 \) are equally good for \( m \), then we say that \( w_1 \) is a satellite of \( m \) and \( m \) is satellitic. If woman \( w \) is matched to a satellite man \( m \), then we say that \( w \) is insecure. If \( e = (m, w) \) is such that \( w \) is free and there is at least one free woman \( w_1 \) such that \( w \) and \( w_1 \) are equally good for \( m \), then \( e \) is called special. If man \( m \)
has at least one free woman incident with him, then he is said to be subsatellite. Woman \( w \) matched to a subsatellite man \( m \) and not insecure is said to be uneasy wrt to \( m' \) if \( m \) and \( m' \) are equally good for her.

### 2.1 Description of Algorithm GS Modified

Algorithm GS Modified given further on is to some extent modeled on the Gale-Shapley algorithm in which men propose to women on their lists and women dispose. In the course of running the algorithm preference lists \( L_m \) will diminish and some additional lists \( L'_m \) will be built. If at some point a free man \( m \) has a nonempty list \( L_m \), it means that he has not yet proposed to all women on his list \( L_m \) and potentially belongs to a blocking pair or a masculine dangerous path. If a free man \( m \) has a nonempty list \( L'_m \), it means that he potentially belongs to a feminine dangerous path.

Whenever it is man \( m \)'s turn to propose and \( L_m \neq \emptyset \), he would like to get matched to the best possible woman on his list without creating a blocking pair (as in GS algorithm) but also ensure that he does not belong to any masculine dangerous path. To this end \( m \) proposes to the woman \( w \) to whom he has not yet proposed and who is as high on \( L_m \) as possible. If \( w \) is free or matched to someone worse for her than \( m \), she accepts \( m \) and rejects her current partner if she had one. If \( w \) is insecure, which means that she is matched to some man \( m' \) such that there is a free woman \( w' \) who is equally good for \( m' \) as \( w \), then it means that \( m \) currently belongs to a masculine dangerous path \((m, w, m', w')\). In this case \( w \) does not care whether \( m \) is better for her than \( m' \) and accepts him while rejecting \( m' \) and immediately afterwards \( m' \) proposes to \( w' \), who accepts him. This operation can be very well viewed as though \( m' \) proposed to \( w' \) without having proposed to \( w \) first and some time later \( m \) proposed to \( w \) (here edge \((m', w)\) was special at the moment \( m' \) proposed to \( w \) for the first time and that’s why if it is \( m' \)'s next turn to propose, he will propose to \( w \) again, because in this case \( w \) was not removed from \( L_m \)). To avoid multiple operations of this kind concerning one woman we will assume that given a tie a man proposes to unmatched women before proposing to matched ones. If a woman \( w \), to whom \( m \) proposes is matched to man \( m' \) equally good for her as \( m \) and \( w \) is uneasy wrt to \( m \), meaning that \( m' \) has some free women on his list, then at the current moment \( m \) belongs to a feminine dangerous path. What happens now is that \( w \) rejects \( m \) but \( m \) adds \( w \) to his list \( L'_m \). In every other case \( w \) rejects \( m \).

If man \( m \) has proposed to all women on his list \( L_m \) and remained free but his list \( L'_m \) is nonempty, he will propose to women on \( L'_m \) starting from the top. If he proposes to \( w \) and \( w \) is matched to some man \( m' \) who is equally good for her as \( m \) and additionally \( m' \) is subsatellite, then \( w \) accepts \( m \) and rejects \( m' \). Otherwise \( w \) rejects him. (Notice that if \( m \) proposes to \( w \in L'_m \) (this means also that \( L_m = \emptyset \)), then it cannot be the case that \( m \) is better for \( w \) than \( M(w) \).)
### Algorithm GS Modified

Each man $m$’s preference list $L_m$ is organized in such a way that if $L_m$ contains a tie $t$, then free women in $t$ come before matched women in $t$. At the beginning all women are free and ties on men’s lists are broken arbitrarily and in the course of running the algorithm whenever woman $w$ becomes matched for the first time, say to $m$, we move her to the end of every tie she belongs to but the one on list $L_m$.

```
while there exists a free man $m$ with a nonempty list $L_m$ or a nonempty list $L'_m$
    if $L_m \neq \emptyset$, then
        $w \leftarrow$ woman at the top of $m$’s list $L_m$
        if $(m, w)$ is not special, remove $w$ from $L_m$
        if $w$ is free, then $M \leftarrow M \cup \{m, w\}$
        else if $w$ is insecure, then
            let $w'$ be a satellite of $M(w)$
            if $(M(w), w')$ is not special, remove $w'$ from $L_{M(w)}$
            $M \leftarrow M \cup \{(m, w), (M(w), w')\} \setminus \{w, M(w)\}$
        else if $w$ prefers $m$ to $M(w)$, then $M \leftarrow M \cup \{(m, w) \setminus \{w, M(w)\}\}$
        else if $w$ is uneasy wrt to $m$, then add $w$ to the end of list $L'_m$
    else
        $w \leftarrow$ woman at the top of $m$’s list $L'_m$
        remove $w$ from $L'_m$
        if $w$ is uneasy wrt to $m$, then $M \leftarrow M \cup \{m, w\} \setminus \{w, M(w)\}$
```

First we show how Algorithm GS Modified runs on the following example. Suppose the preference lists of men $m_1, m_2, m_3, m_4$ and women $w_1, w_2, w_3, w_4$ are as follows. The brackets indicate ties.

| $m_1$ | $m_2$ | $m_3$ | $m_4$ |
|-------|-------|-------|-------|
| $(w_1, w_2)$ | $w_3$ | $w_1$ | $w_3$ |
| $w_3$ | $w_4$ | $w_1$ | $w_4$ |
| $w_2$ | $w_3$ | $(m_2, m_4)$ | $m_2$ |
| $w_3$ | $w_4$ | $m_2$ | |

Suppose that $m_1$ starts. $m_1$ proposes to $w_1$ and gets accepted ($(m_1, w_1)$ is a special edge and $w_2$ is a satellite of $m_1$). Now suppose that it is $m_2$’s turn to propose. (It might also be $m_3$ or $m_4$.) $m_2$ proposes to $w_1$ and gets accepted because $w_1$ is insecure. $m_1$ gets matched with $w_2$, $m_3$ proposes to $w_2$ and gets accepted. $m_1$ proposes to $w_1$ (as $(m_1, w_1)$ was a special edge) and gets accepted. $m_2$ proposes to $w_3$ and gets accepted. $m_4$ proposes to $w_3$ and gets rejected but $w_3$ is uneasy wrt to $m_4$ and $m_4$ adds $w_3$ to his list $L'_{m_4}$. Afterwards $m_4$ proposes to $w_3$ again, this time from $L'_{m_4}$, and gets accepted. $m_2$ proposes to $w_4$ and gets accepted.

### 3 Correctness of Algorithm GS Modified

In this section we prove the correctness of Algorithm GS Modified.

If $w \in L_m$ and $m$ proposes to $w$, then we will sometimes say that $m$ proposes from $L_m$ (to $w$). If $L_m = \emptyset$, $w \in L'_m$ and $m$ proposes to $w$, then we will sometimes say that $m$ proposes from $L'_{m}$.  

#### Lemma 1
1) If woman $w$ becomes matched, she will stay matched. 2) Woman $w$ can become insecure only the first time someone, say $m$, proposes to her and only if at the time of proposal edge $(m, w)$ is special. If an insecure woman $w$ receives a proposal, she always accepts it and is no longer insecure. 3) If woman $w$ is matched to man $m$ and not insecure, she can accept man $m'$ only if $m'$ is at least as good for her as $m$. Moreover, if $m'$ is better for her than $m$, she always accepts him. If $m'$ is equally good for her as $m$, then she accepts him, only if she is
uneasy wrt to $m'$ and $m'$ proposes from $L_m'$. 4) If woman $w$ matched to man $m$ is not insecure and changes $m$ for $m'$, who is equally good for her as $m$, then $m$ is subsatellite and $m'$ is not.

**Proof.** Statements 1) and 3) follow directly from the description of Algorithm GS Modified. 2) If $w$ is matched and $m$ proposes to her, then there is no free woman $w'$ incident with $m$ who is equally good for $m$ as $w$ (because then $m$ would propose to $w'$ before proposing to $w$). As a result if $w$ becomes matched to $m$ she will not become insecure and she will cease to be insecure if she was before. 4) If $w$ changes $m$ for $m'$ who is equally good for her as $m$ (and $w$ is not insecure), then by the above statement $m'$ proposes from $L_m'$ and $m$ is subsatellite. Man $m'$ proposing from $L_m'$ does not have any free women incident with him. \(\square\)

**Lemma 2** Let $M$ denote a matching computed by Algorithm GS Modified. Then the graph does not contain blocking pairs and dangerous paths.

**Proof.** Suppose that $(m, w)$ are a blocking pair. $m$ is either free or $M(m)$ is worse for him than $w$. It means that at some point $m$ proposed to $w$ from $L_m$ when edge $(m, w)$ was not special. (Clearly at some point $m$ proposed to $w$ from $L_m$. Assume that at that point edge $(m, w)$ was special. Then $w$ was free and accepted $m$. However $m$ got rejected later and therefore proposed to $w$ from $L_m$ again, when edge $(m, w)$ was no longer special.) If $w$ rejected him then, then by Lemma 1 $w$ was not insecure and matched to someone at least as good for her as $m$ and thus would have stayed matched to someone as good for her as $m$. If $w$ accepted $m$, then after getting matched to $m$ she was not insecure and by Lemma 1 would have stayed matched to someone at least as good for her as $m$. Either way we get a contradiction.

Suppose now that the graph contains a masculine dangerous path $(m', w, m, w')$ such that $m = M(w)$. Thus $m$ is satellitic and $w$ is insecure. Since she is insecure, it means that the only proposal she ever got was from $m$, but $m'$ must have proposed to her too, a contradiction.

Finally suppose that the graph contains a feminine dangerous path $(m', w, m, w')$ such that $m = M(w)$. Thus $m$ is subsatellite and $w$ is uneasy wrt to $m'$, also $m'$ is not subsatellite. At some point $m'$ proposed to $w$ from $L_m'$ while $(m, w)$ was not special. If he got accepted at that moment, then later on he could not become rejected, because by Lemma 1 after accepting $m'$ woman $w$ was not insecure and could not accept a subsatellite man equally good for her as her current partner. Therefore he was rejected then and $w$ was already matched with $m$ (by Lemma 1 3) and 4)). Hence $w$ was uneasy wrt to $m'$ (because $m$ was subsatellite) and $m'$ added $w$ to the end of list $L_m'$. Thus later $m'$ proposed to $w$ from $L_m'$. According to the algorithm $w$ would have accepted him and could not later on become matched to someone equally good for her as $m'$ and subsatellite. Contradiction. \(\square\)

**Theorem 1** Algorithm GS Modified computes a stable matching $M$ which is a $\frac{3}{2}$-approximation of the optimal solution. Algorithm GS Modified runs in $O(m)$ time.

**Proof.** By Lemma 2 matching $M$ computed by Algorithm GS Modified is stable and does not contain dangerous paths. Therefore $M$ is a $\frac{3}{2}$-approximation of the optimal solution.

The running time of the algorithm is proportional to the total length of lists $L_m$ and $L_m'$. Each edge of $L_m$ is scanned at most twice - twice, only if the first time it was scanned, it was special and each edge of $L_m'$ is scanned at most once. \(\square\)

Let us finally make the following remark.

If we break ties and run the classic Gale/Shapley algorithm, then the cardinality of the computed matching depends on the order in which we break ties. Algorithm GS Modified
outputs a matching that would have been output by the GS algorithm if ties were broken as follows. Men’s lists would be identical to those at the end of Algorithm GS Modified but for one thing: if at some point of running Algorithm GS Modified man \( m \) proposes to an insecure woman \( w \) and as a result \( m \) gets matched to \( w \) and \( w \)'s partner \( M(w) \) gets matched to his satellite \( w' \), then a tie on \( L_{M(w)} \) would be broken in such a way that \( w' \) comes before \( w \). Every tie \( t \) on a woman \( w \)'s list would be first broken into \( (m_1, m_2, \ldots, m_n) \) in such a way that \( m_1 \) denotes the first man of \( t \) to whom \( w \) got matched without becoming insecure and assuming that it happened at some step \( S \), \( m_2 \) denotes the first man of \( t \) who proposed to \( w \) after step \( S \), \( m_3 \) denotes the second man of \( t \), who proposed to \( w \) after step \( S \) and so on. Next we would make the following alterations on women’s lists: if at some point man \( m \in F \) proposes to an uneasy woman \( w \) matched to \( M(w) \), then a tie on \( L_w \) would be broken in such a way that \( m \) comes before \( M(w) \).

4 Extension to stable \( b \)-matchings

Suppose we have a bipartite graph \( G = (V, E) \), where \( V = U \cup W \) and \( U, W \) are disjoint sets, and a function \( b : V \rightarrow N \). Then a subset \( M \subseteq E \) is called a \( b \)-matching if for each \( v \in V \) it is \( deg_M(v) \leq b(v) \), where \( deg_M(v) \) denotes the degree of vertex \( v \) in a graph \( G_M = (U \cup W, M) \). We will call vertices of \( U \) - \( U \)-agents and vertices of \( W \) - \( W \)-agents and vertices of \( U \cup W \) -agents. Each \( U \)-agent \( u \) of \( U \) has a preference list \( L_u \) of a subset of \( W \)-agents and analogously each \( W \)-agent \( w \) has a preference list \( L_w \) of a subset of \( U \)-agents. The preference lists are linearly ordered lists of ties. The majority of the terminology for stable matchings goes through for stable \( b \)-matchings. Instead of saying that some agent or vertex is free we will use the term unsaturated: agent \( v \) is unsaturated in a \( b \)-matching \( M \) if \( deg_M(v) < b(v) \) and if \( deg_M(v) = b(v) \), then we will say that \( v \) is saturated. For any agent \( v \) by \( M(v) \) we will denote the set \( \{ w \in U \cup W : (v, w) \in M \} \). A pair \( (u, w) \) is blocking for a \( b \)-matching \( M \) if (0) \( u \) and \( w \) are mutually acceptable and (1) \( u \) is unsaturated or prefers \( w \) to one of \( W \)-agents of \( M(u) \) and (2) \( w \) is unsaturated or prefers \( u \) to one of \( U \)-agents of \( M(w) \). A \( b \)-matching \( M \) is said to be stable if it does not admit a blocking pair. As previously we are interested in finding a stable \( b \)-matching of largest size. Let us also note that if for each \( u \in U \) we have \( b(u) = 1 \), then the problem is known under the name of the Hospitals-Residents problem or one-to-many stable matching problem.

Alternating paths and cycles are defined for \( b \)-matchings in an analogous way as for matchings but we do not require paths and cycles to be simple, i.e. an alternating path \( P \) wrt a \( b \)-matching \( M \) is defined as any sequence of edges \( \{(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k)\} \) such that the edges alternate between \( M \)-edges and edges of \( E \setminus M \) and an alternating cycle \( C \) wrt \( M \) is defined as an alternating path (wrt \( M \)) that ends and begins with the same vertex, i.e. the sequence of edges has the form \( \{(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_1)\} \). As before for any two \( b \)-matchings \( M, M' \), a symmetric difference \( M \oplus M' \) can be partitioned into alternating paths and cycles. A given stable \( b \)-matching \( M \) might be not a 3/2-approximation of \( M_{\text{opt}} \), where \( M_{\text{opt}} \) denotes a stable \( b \)-matching of maximum size, if the graph contains a dangerous path defined as follows. If \( M \) is a stable \( b \)-matching, then a path \( P = (w, u_1, w_1, u) \) is called dangerous if \( (u_1, w_1) \) is in \( M \), \( (w, u_1) \), \( (w_1, u) \) are not in \( M \), \( w \) and \( u \) are unsaturated, \( u_1, w_1 \) are saturated and \( (u_1, w_1) \) is not a blocking pair for a \( b \)-matching \( M' = (M \setminus (u_1, w_1)) \cup (w, u_1) \cup (w_1, u) \). Since \( (w, u_1) \) is not blocking for \( M \), \( w \) is not better for \( u_1 \) than any of the \( W \)-agents he is currently matched with and analogously \( u \) is not better for \( w_1 \) than any of the \( U \)-agents he is currently matched with. Thus if \( P \) is dangerous, then either \( w, w_1 \) are equally good for \( u_1 \) and then \( P \) is called a masculine dangerous path, or \( u, u_1 \) are equally good for \( w_1 \) and then \( P \) is called a feminine dangerous path.

An approximation algorithm for stable \( b \)-matchings is constructed analogously to the al-
algorithm for stable matchings. $U$-agents play the role of men and $W$-agents play the role of women. For convenience we shall refer to a $U$-agent as "he" and to a $W$-agent as "she". We adapt the terminology from the one-to-one setting to the current one as follows. If a $U$-agent $u$ is matched with a $W$-agent $w$ and there is at least one unsaturated $W$-agent $w_1$ such that $w$ and $w_1$ are equally good for $u$, then we say that $w_1$ is a satellite of $u$ wrt $w$ and $u$ is satellitic wrt $w$. $W$-agent $w$ matched to a $U$-agent $u$ satellitic wrt $w$ is said to be insecure. If $e = (u, w)$ is such that $w$ is unsaturated and there is at least one unsaturated $W$-agent $w_1$ such that $w$ and $w_1$ are equally good for $u$, then $e$ is called special. If $U$-agent $u$ has at least one unsaturated $W$-agent incident with him, then he is called subsatellitic. A saturated $W$-agent $w$ matched to a subsatellitic man $u$ and not insecure is said to be uneasy wrt $u'$ if $u$ and $u'$ are equally good for her. By the worst $U$-agent matched with a $W$-agent $w$ we will mean any $U$-agent in $u \in M(w)$ such that there is no other $U$-agent $u' \in M(w)$ who is worse for $w$ than $u$.

**Algorithm ASBM** (short for Approximate Stable b-Matching)

Each $U$-agent $u$'s preference list $L_u$ is organized in such a way that if $L_u$ contains a tie $t$, then unsaturated $W$-agents in $t$ come before saturated $W$-agents in $t$. At the beginning all $W$-agents are unsaturated and ties on $U$-agents’s lists are broken arbitrarily and in the course of running the algorithm whenever $W$-agent $w$ becomes matched for the first time, say to $U$-agent $u$, we move her to the end of every tie she belongs to but the one on list $L_u$.

\[\text{while} \text{ there exists an unsaturated } U\text{-agent } u \text{ with a nonempty list } L_u \text{ or a nonempty list } L'_u \]

\[\text{if } L_u \neq \emptyset, \text{ then} \]
\[w \leftarrow W\text{-agent at the top of } u' \text{'s list } L_u \]
\[\text{if } (u, w) \text{ is not special, then remove } w \text{ from } L_u \]
\[\text{if } w \text{ is unsaturated, then } M \leftarrow M \cup (u, w) \]
\[\text{else if } w \text{ is insecure, then} \]
\[\text{let } w' \text{ be a satellite of a } U\text{-agent } u' \in M(w) \text{ wrt to } w \]
\[\text{if } (u', w') \text{ is not special, remove } w' \text{ from } L_{u'} \]
\[M \leftarrow M \cup \{(u, w), (u', w')\} \setminus (w, u') \]
\[\text{else if } w \text{ prefers } u \text{ to the worst } U\text{-agent in } M(w), \text{ then} \]
\[\text{let } u' \text{ denote the worst } U\text{-agent matched with } w \text{ who is subsatellitic, if such one exists;} \]
\[\text{otherwise let } u' \text{ denote any worst } U\text{-agent matched with } w \]
\[M \leftarrow M \cup (u, w) \setminus (w, u') \]
\[\text{if } w \text{ is uneasy wrt } u', \text{ then add } w \text{ to the end of list } L'_{u'} \]
\[\text{else if } w \text{ is uneasy wrt } u, \text{ then add } w \text{ to the end of list } L'_u. \]
\[\text{else} \]
\[w \leftarrow W\text{-agent at the top of } u' \text{'s list } L'_u \]
\[\text{remove } w \text{ from } L'_u \]
\[\text{if } w \text{ is uneasy wrt to } u, \text{ then} \]
\[\text{let } u' \text{ denote a subsatellitic } U\text{-agent in } M(w) \text{ equally good for } w \text{ as } u \]
\[\text{if } w \text{ is uneasy wrt to } u', \text{ add } w \text{ to the end of } L'_{u'} \]
\[M \leftarrow M \cup (u, w) \setminus (w, u') \]

**4.1 Data structures and running time**

Each agent $a$ (either a $U$-agent or $W$-agent) has a preference list $L_a$, which is a list of lists i.e. we have a list for each tie. For each list we have the access to both its first and last element.

Each agent has a pointer to their position in every tie (1-element list is here also considered a tie) they belong to. Whenever $W$-agent $w$ gets saturated for the first time, $w$ goes over her whole list $L_w$ and moves herself to the end of every tie she belongs to but the one, as explained in the algorithm ASBM. This operation takes $O(|L_w|)$ time.
Every $W$-agent $w$ stores information about $U$-agents currently matched with $w$ in a priority queue. $U$-agents matched with $w$ who are equally good for $w$ are kept in one list, thus the priority queue contains lists. This way checking by $w$ if there exists a $U$-agent $u' \in M(w)$ such that $w$ prefers some given $u$ to $u'$ takes $O(\log b(w))$ time.

Each $U$-agent $u$ has the counter of the number of unsaturated $W$-agents incident with him and whenever a saturated $W$-agent moves herself to the end of the ties, $U$-agents also decrease respective counters. Therefore checking if $u$ is subsatellitic takes constant time.

Each $W$-agent $w$ has a separate list $S_w$ of satellitic $U$-agents wrt $w$ matched with $w$. Every time $w$ gets matched to some new $U$-agent $u$, who is satellitic wrt $w$, $w$ adds $u$ to $S_w$. When we want to check if $w$ is insecure, we go over $S_w$ and for each $u \in S_w$ check if $u$ is still satellitic wrt to $w$. If $u$ is not satellitic wrt to $w$, we remove $u$ from $S_w$, otherwise we do an appropriate exchange. Once $u$ is removed from $S_w$, he will not be added to $S_w$ again. It is so since once $u$ has no unsaturated $W$-agents equally good for him as $w$ on his list, it will stay so. Hence the overall time Algorithm ASBM spends on $S_w$ is $O(|U|)$.

Every list in the priority queue of $U$-agents matched with $w$ is organized in such a way that subsatellitic $U$-agents proceed $U$-agents that are not subsatellitic. Whenever a $U$-agent $u$ ceases to be subsatellitic we move him to the end of every list in every priority queue he is in. Moving $u$ to the end of every such list takes $O(\sum_{u \in M(w)} \log (b(w)))$ time. Every $u$ ceases to be subsatellitic at most once in the course of running the algorithm. This way to see if $w$ is insecure, we go over $S_w$ and for each $u \in S_w$, denote the number of the edges. Every $U$-agent $u$ makes a proposal to every $W$-agent on $L_u$ at most twice and to every $W$-agent on $L_u'$ at most once.

Summing all the arguments together, we get that the running time of Algorithm ASBM is $O(m \min \{1, \log \max \{b(w) : w \in W\}\})$, where $m$ denotes the number of edges in $G$. If $\max \{b(v) : u \in U\} > \max \{b(w) : w \in W\}$ then we can swap the roles of $U$-agents and $W$-agents. Therefore we can state

**Theorem 2** The running time of Algorithm ASBM is $O(m \min \{1, \log c\})$, where $c = \min \{\max \{b(v) : v \in U\}, \max \{b(v) : v \in W\}\}$ and $m$ denotes the number of the edges.

## 5 Correctness of Algorithm ASBM

The correctness of Algorithm ASBM is proved in a very similar way as the correctness of Algorithm GS Modified.

**Lemma 3** 1) If $W$-agent becomes saturated, she will stay saturated. 2) An insecure $W$-agent $w$ accepts every proposal. Once a saturated $W$-agent is not insecure, she cannot become insecure later. 3) A $W$-agent $w$ matched with $u$ can reject $u$ only if $w$ is saturated and a) $u$ is satellitic wrt to $w$ ($w$ is insecure) or b) $w$ is not insecure and $u$ is the worst $U$-agents currently matched with $w$ and $w$ receives a proposal from $u'$, who is better for $w$ than $u$ or c) $w$ is not insecure and $u$ is (one of) the worst $U$-agents currently matched with $w$ and $u$ is subsatellitic and $w$ is uneasy wrt to $u'$ who proposes from $L_u'$ 4) A saturated $W$-agent $w$ and not insecure can accept a $U$-agent $u$ only if $u'$ is at least as good for $w$ as the worst $U$-agent $u \in M(w)$; moreover if $u'$ is equally good for $w$ as $u$, then $w$ accepts $u'$ only if $w$ is uneasy wrt to $u'$ and $u'$ proposes from $L_u'$.

The proof is very similar to that of Lemma 4.4 and follows directly from the description of Algorithm ASBM.

**Theorem 3** Let $M$ denote a b-matching computed by Algorithm ASBM. Then $M$ is a $3/2$-approximation of an optimal stable b-matching.
**Proof.** We will show that the graph does not contain blocking pairs and dangerous paths.

Suppose that \((u, w)\) are a blocking pair. \(u\) is either unsaturated or there exists \(w' \in M(u)\) worse for \(u\) than \(w\). It means that at some point \(u\) proposed to \(w\) from \(L_u\) when edge \((u, w)\) was not special. If \(u\)'s proposal to \(w\) was rejected, then at that point \(w\) was saturated and not insecure and the worst \(w' \in M(w)\) was at least as good as \(u\) for \(w\) (by Lemma 3) and thus (also by Lemma 3) \(w\) could not later become matched to some \(u''\) who is worse for \(w\) than \(u\). Therefore \(u\) got accepted then and later got rejected. Since at the moment of that proposal edge \((u, w)\) was not special, \(u\) was not satellitic wrt to \(w\) (and clearly could not become satellitic later.) By Lemma 3\(\) \(3)\) at the moment of rejecting \(u\) W-agent \(w\) was not insecure and the worst \(U\)-agent matched with \(w\) was \(u\). Therefore by Lemma 3\(\) \(4)\) \(w\) could not later become matched to some \(u''\) who is worse for her than \(u\). A contradiction.

Suppose now that the graph contains a masculine dangerous path \((u', w, u, w')\) such that \(u \in M(w)\). Thus \(u\) is satellitic wrt to \(w\) and \(w\) is insecure. It means that at some point \(u\) proposed to \(w\) from \(L_u\) when edge \((u, w)\) was not special. Then \(w\) was either insecure, because she is insecure now, or unsaturated. Therefore \(u\) got accepted. Later on he was clearly rejected. However by Lemma 3\(\) \(3)\) and the description of the algorithm ABSM, it is impossible because an insecure \(w\) reject only satellitic wrt to \(w\) \(U\)-agents.

Finally suppose that the graph contains a feminine dangerous path \((u', w, u, w')\) such that \(u \in M(w)\). Thus \(w\) is uneasy wrt to \(u'\) and \(u\) is subsatellitic. At some point \(u'\) proposed to \(w\) from \(L_{u'}\) when edge \((u, w)\) was not special. If he got rejected then, then \(w\) was not insecure and the worst \(U\)-agent \(u'\) she was matched with was equally good for her as \(u\). By Lemma 3\(\) \(4)\) at that point \(w\) was uneasy wrt to \(u'\) and \(u'\) added \(w\) to the end of list \(L'_{u'}\). If he got accepted at that point, then later he was rejected and also had to add \(w\) to the end of list \(L'_{u'}\). When \(u'\) proposed to \(w\) from \(L'_{u'}\), \(w\) was still uneasy wrt to \(u'\) (because \(w\) is uneasy wrt to \(u'\) now), hence \(u'\) was accepted (because \(u'\) proposing from \(L'_{u'}\) is subsatellitic) and could not get rejected later if there were still subsatellitic \(U\)-agents matched with \(w\), who were equally good as \(u'\) for \(w\). A contradiction. \[\square\]

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Faster and simpler approximation of stable matchings

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Abstract

We give a $\frac{3}{2}$-approximation algorithm for stable matchings that runs in $O(m)$ time. The previously best known algorithm by McDermid has the same approximation ratio but runs in $O(n^{3/2}m)$ time, where $n$ denotes the number of people and $m$ is the total length of the preference lists in a given instance. Also the algorithm and the analysis are much simpler. We sketch the extension of the algorithm for computing stable many-to-many matchings.

1 Introduction

In the paper we consider a variant of the problem called Stable Matchings, known also in the literature as Stable Marriage. The problem is defined as follows. We are given two sets $W$ and $U$ of women and men and each woman $w$ of $W$ has a linearly ordered preference list $L_w$ of a subset of men $U' \subseteq U$ and similarly each man $m$ of $U$ has a linearly ordered preference list $L_m$ of a subset of women $W' \subseteq W$. The lists of men and women can contain ties, which are subsets of men (or respectively women), which are equally good for a given woman (resp. man). Thus if $m$ and $m'$ are on list $L_w$ of woman $w$, then either (1) $m <_w m'$ and then we say that woman $w$ prefers $m$ to $m'$ or that $m$ is better for her than $m'$ or (2) $m \equiv_w m'$, which means that $m$ and $m'$ are in a tie on $L_w$ and then we say that $w$ is indifferent between $m$ and $m'$ or that $m$ and $m'$ are equally good for her or (3) $m' <_w m$. If man $m$ does not belong to list $L_w$, then we say that $m$ is unacceptable for $w$. A matching is a set of pairs $(m, w)$ such that $m \in U$, $w \in W$ and $m$ and $w$ are on each other’s preference lists and each man/woman belongs to at most one pair. If $(m_1, w_1)$ belongs to a certain matching $M_1$, then we write $M_1(m_1) = w_1$, which means that in $M_1$ woman $w_1$ is a partner of $m_1$ and analogously that $M_1(w_1) = m_1$. If man $m$ (or woman $w$) is not contained in any pair of matching $M$, then we say that $m$ ($w$) is unmatched or single or free in $M$. A matching $M_1$ is called stable if it does not admit a blocking pair. A pair $(m, w)$ is blocking for $M_1$ if (0) $m$ and $w$ are acceptable to each other and (1) $m$ is single or prefers $w$ to $M_1(m)$ and if (2) $w$ is single or prefers $m$ to $M_1(w)$. Each instance of the problem can be represented by a bipartite graph $G = (U \cup W, E)$ with vertices $U$ representing men and vertices $W$ women and edges $E$ connecting all mutually acceptable pairs of men and women. The problem we are interested in in this paper is that of finding a stable matching that has the largest cardinality. The version in which there are no ties in the preference lists of men and women has been long known and an algorithm by Gale and Shapley (1) solves it exactly in $O(m)$ time, where $m$ denotes the sum of the lengths of preference lists. In the version without ties a stable matching always exists and every stable matching has the same cardinality. If we allow ties, as in the problem we consider in this paper, then a stable matching also always exists and can be found via the Gale/Shapley algorithm by breaking ties arbitrarily. However, the sizes of stable matchings

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can vary considerably and the problem of finding a stable matching of maximum cardinality is \( NP \)-hard, which was shown by Manlove et al. in [9]. Therefore it is desirable to devise an approximation algorithm for the problem.

**Previous results** Previous approximation algorithm were presented in [9], [4], [5], [6], [7]. Currently the best approximation algorithm is by McDermid [10] and achieves the approximation guarantee \( \frac{3}{2} \). Its running time is \( O(n^{3/2}m) \), where \( n \) denotes the number of vertices and \( m \) the number of edges. Inapproximability results were shown in [2], [3], [11].

**Our results** We give a \( 3/2 \)-approximation algorithm that runs in \( O(m) \) time and additionally is significantly simpler than that of McDermid. We sketch the extension of the algorithm for computing stable many-to-many matchings, which runs in \( O(m\log c) \) time, where \( c \) denotes the minimum of the maximal capacities in each side of the bipartition. In particular it means we give an \( O(m) \)-time algorithm for the Hospitals-Residents problem, improving on an \( O(d^{3/2}n^{3/2}m) \) time algorithm given by McDermid, where \( d \) denotes the maximal capacity of a hospital. Since the problems are said to have many practical applications, we believe our algorithms will be of help.

## 2 Algorithm

Let \( M_{opt} \) denote an optimal (i.e. largest) stable matching and \( M, M' \) any two matchings. We say that \( e \) is an \( M \)-edge if \( e \in M \). A path \( P \) or a cycle \( C \) is called **alternating (with respect to** \( M \)) if its edges alternate between \( M \)-edges and edges of \( E \setminus M \). It is well known from matching theory (see [8] for example) that \( M \oplus M' \) can be partitioned into a set of alternating paths and alternating cycles. (For two sets \( X, Y \), the set \( X \oplus Y \) denotes \( X \setminus Y \cup Y \setminus X \.) Let \( S \) denote a set of alternating paths and cycles of \( M \oplus M_{opt} \). Consider any alternating cycle \( c \) of \( S \) or any alternating path \( p \) of even length of \( S \). Then both \( c \) and \( p \) contain the same number of \( M \)-edges and \( M_{opt} \)-edges. Consider an alternating path \( p \) of length \( 2k + 1 \) of \( S \). Then either \( |M_{opt}\cap p| = \frac{k+1}{k} \) or \( |M\cap p| = \frac{k+1}{k} \). Therefore if \( M \oplus M_{opt} \) does not contain paths of length 3 with the middle edge of a path being an \( M \)-edge, then \( |M_{opt}| \leq \frac{3}{2} |M| \) and \( M \) is a \( \frac{3}{2} \)-approximation of \( M_{opt} \). To achieve a \( \frac{3}{2} \)-approximation we will be eliminating such potential paths of length 3 of \( M \oplus M_{opt} \).

Accordingly we define a **dangerous path**, which is an alternating path \( P = (w, m_1, w_1, m) \) such that \( w \) and \( m \) are unmatched (which means that \( (m_1, w_1) \) is in \( M \) and \( (w, m_1), (w_1, m) \) do not belong to \( M \) and \( (m_1, w_1) \) is not a blocking pair for matching \( M' = \{(w, m_1), (w_1, m)\} \). Let us notice that if \( P \) is a dangerous path, then either \( m_1 \) is indifferent between \( w \) and \( w_1 \) (and then we say that \( (w, m_1) \) is an **equal edge** or \( w_1 \) is indifferent between \( m \) and \( m_1 \) (and then \( (w_1, m) \) is called an equal edge) or both. Thus a dangerous path contains one or two equal edges. If an edge \( e = (w, m_1) \) is equal, then we say that \( P \) is a **masculine dangerous path** and if \( (w_1, m) \) is equal, then we say that \( P \) is a **feminine dangerous path**. A path \( P \) can of course be both a masculine and feminine dangerous path.

We also introduce the following terminology. If man \( m \) is matched to woman \( w \) and there is at least one free woman \( w_1 \) such that \( w \) and \( w_1 \) are equally good for \( m \), then we say that \( m \) is **unstable** and \( w_1 \) is a **satellite** of \( m \). If woman \( w \) is matched to an unstable man \( m \), then we say that \( w \) is **unstable**. If \( e = (m, w) \) is such that \( w \) is free and \( m \) is either free or matched to \( w' \), who is equally good for him as \( w \) and there is at least one free woman \( w_1 \) such that \( w \) and \( w_1 \) are equally good for \( m \), then \( e \) is called **special**. An edge \( (m, w) \) is said to be **blocking** if \( m, w \) are a blocking pair. An edge \( (m, w) \) is said to be **unstable** if \( m \) is free and \( w \) is unstable. Let us notice that if \( (m, w) \) is unstable, then it is a part of a masculine dangerous path. An edge \( (m, w) \) that is equal and such that \( m \) is free is called **f-equal**. An edge \( e \) is said to be **bad** if it is blocking or unstable or f-equal.
In the algorithm given below set $X$ contains single men that have not yet proposed to all women on their lists (and potentially belong to blocking pairs or feminine dangerous paths) and set $F$ contains single men, who have already proposed to all women on their lists and potentially belong to masculine dangerous paths.

| Algorithm GS Modified |
|------------------------|
| Each man $m$’s preference list $L_m$ is organized in such a way that if $L_m$ contains a tie $t = (w_1, w_2, \ldots, w_k)$, then free women in $t$ come before matched women in $t$. At the beginning all women are free and ties are broken arbitrarily and in the course of running the algorithm whenever woman $w$ becomes matched for the first time we move her to the end of every tie she belongs to. |
| $X := U$ (all men) |
| $F := \emptyset$ |
| while $X$ or $F$ is nonempty |
| if there is a man $m \in X$, then |
| if $L_m = \emptyset$, then remove $m$ from $X$ and if $L_m' \neq \emptyset$, add $m$ to $F$ |
| else |
| $w \leftarrow$ next woman on $m’$’s list $L_m$ |
| if $(m, w)$ is not special, then remove $w$ from $L_m$ |
| if $(m, w)$ is blocking, then |
| add $M(w)$ to $X$ |
| $M \leftarrow M \cup (m, w) \setminus (w, M(w))$ |
| remove $m$ from $X$ |
| else if $w$ is unstable, then |
| let $w'$ be a satellite of $M(w)$ |
| if $(M(w), w')$ is special, remove $w'$ from $L_M(w)$ |
| $M \leftarrow M \cup \{(m, w), (M(w), w')\} \setminus (w, M(w))$ |
| remove $m$ from $X$ |
| else if $(m, w)$ is $f$-equal, then add $w$ to the end of list $L_m'$. |
| else |
| there is a man $m \in F$ |
| if $L_m' = \emptyset$, remove $m$ from $F$ |
| else |
| $w \leftarrow$ next woman on $m’$’s list $L_m'$ |
| remove $w$ from $L_m'$ |
| if $(m, w)$ is $f$-equal, then |
| add $M(w)$ to $X$ |
| $M \leftarrow M \cup (m, w) \setminus (w, M(w))$ |

First we show how Algorithm GS Modified runs on the following example. Suppose the preference lists of men $m_1, m_2, m_3$ and women $w_1, w_2, w_3$ are as follows. The brackets indicate ties.

$m_1 : (w_1, w_2) \; w_3 \; w_1 : m_1 \; m_2 \; m_3$

$m_2 : w_1 \; w_3 \; w_2 \; w_2 : m_3 \; m_1 \; m_2$

$m_3 : w_2 \; w_1 \; w_3 \; w_3 : m_1 \; m_2 \; m_3$

Suppose that $m_1$ starts. $m_1$ proposes to $w_1$ and gets accepted ($(m_1, w_1)$ is a special edge and $w_2$ is a satellite of $m_1$). Now suppose that it is $m_2$’s turn to propose. (It might also be $m_3$.) $m_2$ proposes to $w_1$ and gets accepted because $w_1$ is unstable. $m_1$ gets matched with $w_2$. $m_3$ proposes to $w_2$ and gets accepted. $m_1$ proposes to $w_1$ (as $(m_1, w_1)$ was a special edge) and gets accepted. $m_2$ proposes to $w_3$ and gets accepted.

If we break ties arbitrarily and run the Gale/Shapley algorithm, then the cardinality of the computed matching depends on the order in which we break ties and the order in which men propose. Algorithm GS Modified outputs a matching that would have been output by the GS algorithm if the order of ties and proposals of men were as follows. It would be identical to that in Algorithm GS Modified but for two things: (1) if $m$ proposes to $w$, $(m, w)$ is not blocking, $w$ is unstable, $w'$ is a satellite of $M(w)$, then a tie in the list of $L_M(w)$ would be broken so that $w'$ would come before $w$ (whereas in Algorithm GS Modified $w$ comes before $w'$) and thus $M(w)$ would propose first to $w'$ and not $w$, (2) if $m \in F$ and $(m, w)$ is $f$-equal, then $m$ would propose to $w$ before $M(w)$.
Now we prove the correctness of Algorithm GS Modified.

**Fact 1** If woman \( w \) becomes matched, then she will stay matched. Woman \( w \) can become unstable only the first time someone proposes to her. If an unstable woman \( w \) (matched to some unstable man \( m \)) becomes matched to some new man \( m' \), then she is not unstable any more. If woman \( w \) is matched to man \( m \) and is not unstable, then she will always be matched to someone at least as good for her as \( m \).

**Proof.** If \( w \) is matched and \( m \) proposes to her, then there is no free woman \( w' \) who is equally good for \( m \) as \( w \) (because then \( m \) would propose to \( w' \) before proposing to \( w \)). As a result if \( w \) becomes matched to \( m \) she will not become unstable and she will cease to be unstable if she was before. Also if \( w \) is matched and not unstable at the moment \( m \) proposes to her, she accepts him only if \( m \) is not worse for her than her current partner \( M(w) \).

**Lemma 1** If \( e \) is special at some step of Algorithm GS Modified and gets added to \( M \), then it will not become special later. Suppose that at some step \( S \) of Algorithm GS Modified edge \( e \) is incident with a single man \( m \). If at step \( S \) edge \( e \) is not bad, then it will not become bad later. If at step \( S \) edge \( e \) is f-equal and not unstable, then \( e \) will not become unstable or blocking. If \( e \) is bad but not special at step \( S \) and gets added to \( M \), then it will not become bad later in the course of running the algorithm.

**Proof.** If at step \( S \) edge \( e = (m, w) \) is special and gets added to \( M \), then it means that \( w \) becomes matched and by Fact 1 she will always stay matched, therefore \( e \) will never become special.

If \( e = (m, w) \) is not bad, then \( w \) is matched to some man \( m' \). Since \( e \) is not unstable, then \( w \) is not unstable and thus by Fact 1 \( w \) will not become unstable, thus \( e \) will not become unstable. If \( w \) prefers \( m' \) to \( m \), since she is not unstable, she will always be matched to someone she prefers to \( m \), thus \( e \) will not become blocking. If for \( w \) men \( m \) and \( m' \) are equally good, but \( e \) is not an equal edge, then it means that \( m' \) has no free woman incident on him at the moment and thus will never have and if \( w \) gets matched to \( m'' \), then \( m'' \) will be better for her than \( m' \), because \( w \) could become matched to \( m'' \) that is equally good for her as \( m' \) only if \((w, m') \) belonged to a feminine dangerous path.

If at step \( S \) edge \( e = (m, w) \) is f-equal and not unstable, then it means that \( w \) is matched to some man \( m' \) such that \( m \) and \( m' \) are equally good for her and \( w \) is not unstable. Thus by Fact 1 she will not become unstable and she will always be matched to someone at least as good for her as \( m' \).

Suppose that at step \( S \) edge \( e = (m, w) \) is bad but not special and \( e \) gets added to \( M \). Then by Fact 1 \( w \) will never become unstable (note that at step \( S \) she might have been unstable) and will always stay matched to \( m \) or will become matched to someone at least as good for her as \( m \). Thus \( e \) will not become unstable or blocking later. If she gets matched to \( m' \), who is equally good for her as \( m \), then it means that at that step, \((m, w)\) belonged to a feminine dangerous path and thus \( m' \) had not (and thus will never have) a free woman incident on him (because then he would have proposed to her as he would have been blocking because of a free woman incident on him), which means that \((w, m') \) never becomes a part of a feminine dangerous path, therefore \( e \) will not become f-equal.

**Theorem 1** Algorithm GS Modified computes a stable matching \( M \) that does not contain dangerous alternating paths and thus is a \( \frac{2}{3} \)-approximation algorithm. Algorithm GS Modified runs in \( O(m) \) time.
Proof. Suppose that matching $M$ computed by the algorithm is not stable. Then it contains a blocking edge $(m, w)$. Thus $m$ is either single or matched to woman $w'$, who is worse for him than $w$. Therefore at some step of the algorithm $m$ must have proposed to $w$. If at that step $(m, w)$ was blocking, it got added to $M$ and by Lemma 1, $(m, w)$ could not become blocking later and if it was not blocking, it also could not become blocking later.

For $\frac{2}{3}$-approximation, it suffices to show that the graph does not contain dangerous paths. If the graph contains a masculine or feminine dangerous path, then it contains an edge $e = (m, w)$, that is unstable or f-equal. But then at some step of the algorithm $m$ proposed to $w$ and if $e$ was not unstable or f-equal then, by Lemma 1 it could not become unstable or f-equal later. If it was unstable at that step it got added to $M$ and also could not become unstable or f-equal later. If it was f-equal, $w$ was added to $L'_m$ and $e$ was considered again at some later step and either it got added to $M$ because it still was f-equal or not because it was not and by Lemma 1 could not become f-equal later.

The running time of the algorithm is proportional to the total length of lists $L_m$ and $L'_m$. Each edge of $L_m$ is scanned at most twice - twice, only if the first time it was scanned, it was special and each edge of $L'_m$ is scanned at most once. $$\square$$

3 Extension to stable b-matchings

Suppose we have a bipartite graph $G = (V, E)$, where $V = U \cup W$ and $U, W$ are disjoint sets, and a function $b : V \to N$. Then a subset $M \subseteq E$ is called a b-matching if for each $v \in V$ it is $\deg_M(v) \leq b(v)$, where $\deg_M(v)$ denotes the degree of vertex $v$ in a graph $G_M = (U \cup W, M)$. We will call vertices of $U$ - U-agents and vertices of $W$ - W-agents. Each U-agent $u$ of $U$ has a linearly ordered preference list $L_u$ of a subset of W-agents $W'_u \subseteq W$ possibly containing ties and analogously each W-agent $w$ has a linearly ordered preference list $L_w$ of a subset of $U$-agents. The majority of the terminology for stable matchings goes through for stable b-matchings. $v_1$ is acceptable for $v_2$ if $v_1$ is on $L_{v_2}$. Instead of saying that some agent or vertex is single or free we will use the term unsaturated: agent $v$ is unsaturated by a b-matching $M$ if $\deg_M(v) < b(v)$. A pair $(u, w)$ is blocking for a b-matching $M$ if (0) $u$ and $w$ are acceptable to each other and (1) $u$ is unsaturated or prefers $w$ to one of $W$-agents of $M(u)$ and if (2) $w$ is unsaturated or prefers $u$ to one of $U$-agents of $M(w)$. A b-matching $M$ is said to be stable if it does not admit a blocking pair. As previously we are interested in finding a stable b-matching of largest size. Let us also note that if for each $u$ in $U$ we have $b(u) = 1$, then the problem is known under the name Hospitals-Residents problem or one-to-many stable matching problem.

An approximation algorithm for stable b-matchings is constructed analogously to the algorithm from the previous section. U-agents play the role of men and W-agents play the role of women. Each U-agent $u$ makes a proposal to each W-agent on $L_u$. Each W-agent $w$ stores information about U-agents currently matched with $w$ in a priority queue. We translate the notions from the previous section to the current setting as follows. If a saturated U-agent $u$ is matched with a W-agent $w$ and there is at least one unsaturated W-agent $w_1$ such that $w$ and $w_1$ are equally good for $u$, then $u$ and $w$ are said to be unstable and $w_1$ is said to be a satellite of $u$. If $e = (u, w)$ is such a non-M-edge that $u, w$ are unsaturated and there is at least one unsaturated W-agent $w_1$ such that $w$ and $w_1$ are equally good for $u$, then $e$ is called special. We also have dangerous paths. Suppose we have a stable matching $M$, then a path $P = (w, u_1, w_1, u)$ is called dangerous if $(u_1, w_1)$ is in $M$, $(w, u_1), (w_1, u)$ are not in $M$, $w$ and $u$ are unsaturated, $u_1, w_1$ are saturated and it is not true that $u_1$ prefers $w_1$ to $w$ and $w_1$ prefers $u_1$ to $u$. Since $(w, u_1)$ is not blocking for $M$, $w$ is not better for $u_1$ than any of the W-agents he is currently matched with and analogously $u$ is not better for $w_1$ than any of the
$U$-agents he is currently matched with. Thus if $P$ is dangerous, then either $w, w_1$ are equally good for $u_1$ and $(w, u_1)$ is called an equal edge and $P$ a masculine dangerous path, or $u, u_1$ are equally good for $w_1$ and $(u, w_1)$ is called an equal edge and $P$ a feminine dangerous path. Analogously we define blocking, unstable, f-equal and bad edges.

Whenever $W$-agent $w$ receives a proposal from $u$, $w$ accepts $u$ if it is unsaturated or unstable (then a proper exchange also takes place) or compares $u$ to the worst $u'$ that is currently matched with $w$. Finding the worst $u'$ that is currently matched with $w$ takes $O(\log(b(w)))$ time.

The running time of the algorithm is $O(m \log c)$, where $c = \min\{\max\{b(v) : v \in U\}, \max\{b(v) : v \in W\}\}$ and $m$ denotes the number of the edges and the approximation factor is $\frac{3}{2}$.

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