Pricing Asian Options For Jump Diffusions

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Jump Diffusions

Introduce discontinuity into the stock price dynamics

- Capture excess kurtosis, better describe catastrophic events,
- Generate implied volatility skew / smile.

\[
dS_t = (r - \mu) S_t \, dt + \sigma S_t \, dB_t + S_t \int_\mathbb{R} (y - 1) N(dt, dy),
\]

where \( N(dt, dy) \) is a Poisson random measure with the mean measure \( \lambda \, dt \, \nu(dy) \). Here \( \nu(dy) \) is a finite measure on \( \mathbb{R}^+ \).

\[
\int_\mathbb{R} (y - 1) N(dt, dy) = \sum N_t (Y - 1).
\]

At the time of jump, \( S_t \rightarrow S_t - Y \), \( Y \) has distribution \( \nu(dy) \), (jump up for \( Y > 1 \), down for \( Y < 1 \)).

\( \mu \equiv \lambda (\xi - 1) \) and assume \( \xi \equiv \mathbb{E}[Y] < +\infty \).

Example:

- Merton's model: log \( Y \) is normal;
- Kou's model: log \( Y \) is double exponential.
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Assume the stock price $S$ follows jump diffusions (under the risk neutral measure $\mathbb{P}$ calibrated from the market)

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\]

- $N(dt, dy)$ is a Poisson random measure with the mean measure $\lambda dt \nu(dy)$. Here $\nu(dy)$ is a finite measure on $\mathbb{R}_+$. $\int_{\mathbb{R}} (y - 1) N(dt, dy) = d \sum_{t} N_t (Y - 1)$.
- At the time of jump, $S_t \rightarrow S_t Y$, $Y$ has distribution $\nu(dy)$, (jump up for $Y > 1$, down for $Y < 1$).
- $\mu \triangleq \lambda (\xi - 1)$ and assume $\xi \triangleq \mathbb{E}[Y] < +\infty$. 

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$$ dS_t = (r - \mu)S_t dt + \sigma S_t dB_t + S_t \int_{\mathbb{R}} (y - 1)N(dt, dy), $$

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- At the time of jump, $S_{t-} \rightarrow S_{t-} Y$, $Y$ has distribution $\nu(dy)$, (jump up for $Y > 1$, down for $Y < 1$).
- $\mu \equiv \lambda (\xi - 1)$ and assume $\xi \equiv \mathbb{E}[Y] < +\infty$.

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Asian Options

The value of European style continuous averaging Asian option is

\[ V(S_0) \triangleq \mathbb{E}^P \left\{ e^{-rT} \left( \zeta \cdot \left( \frac{1}{T} \int_0^T S_t dt - K_1 S_T - K_2 \right) \right)^+ \right\}, \]

- \( K_1 \): Floating Strike, \( K_2 \): Fixed Strike,
- \( \zeta \in \{-1, 1\} \) indicates put/call option,
- Maturity \( T \) is fixed.
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$V$ satisfies a degenerate PDE with two space dimensions. Numerical solution of this PDE is difficult.
Define a process

$$Z^J_t \triangleq \frac{X_t}{S_t}, \quad t \in [0, T], \quad Z^J_0 = z = q_0 - e^{-rT} \frac{K_2}{S_0}.$$ 

$X = \{X_t, t \in [0, T]\}$ is a self-financing portfolio with dynamics

$$dX_t = q_t dS_t + r (X_{t-} - q_t S_{t-}) dt, \quad X_0 = x = q_0 S_0 - e^{-rT} K_2.$$ 

$q_t, t \in [0, T]$, is the number of shares invested in stock at time $t$,

$$q_t \triangleq \frac{1}{rT} \left( 1 - e^{-r(T-t)} \right).$$ 

Then

$$X_T = \frac{1}{T} \int_0^T S_t dt - K_2.$$
Define a process
\[ Z_t^J \triangleq \frac{X_t}{S_t}, \quad t \in [0, T], \quad Z_0^J = z = q_0 - e^{-rT} K_2. \]

\( X = \{X_t, t \in [0, T]\} \) is a self-financing portfolio with dynamics
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Introduce a new measure \( \mathbb{Q} \) by the Randon-Nykodym derivative
\[ \frac{d\mathbb{Q}}{d\mathbb{P}} \bigg|_{\mathcal{F}_t} = e^{-rt} \frac{S_t}{S_0}, \quad t \in [0, T], \]
Proposition (Večeř and Xu (04))

1. \( V(S_0) = S_0 \cdot \mathbb{E}^Q_z[(\zeta \cdot (Z^J_t - K_1))^+] \), where
   \[
dZ^J_t = (q_t - Z^J_t) \left\{ \sigma dW_t + \int_{\mathbb{R}^+} \frac{y-1}{y} \left[ N(dt, dy) - y\nu(dy)dt \right] \right\}.
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Proposition (Večeř and Xu (04))

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\]

2. Let \( v(z, t) \) be the solution of
\[
\frac{\partial}{\partial t} v + A(t)v - \lambda \xi v + \lambda \cdot (Pv)(z, t) = 0, \quad (z, t) \in \mathbb{R} \times [0, T),
\]
\[
v(z, T) = (\zeta \cdot (z - K_1))^+,
\]
where \( A(t) := -\mu(q_t - z) \frac{\partial}{\partial z} + \frac{1}{2} \sigma^2(q_t - z)^2 \frac{\partial^2}{\partial z^2} \) and
\[
Pv(z, t) = \int_{\mathbb{R}^+} v \left( \frac{z}{y} + q_t \frac{y - 1}{y}, t \right) y \nu(dy).
\]

If \( v_t, v_z \) and \( v_{zz} \) are continuous, then \( V(S_0) = S_0 \cdot v(z, 0). \)
Proposition (Večer and Xu (04))

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If \( v_t, v_z \) and \( v_{zz} \) are continuous, then \( V(S_0) = S_0 \cdot v(z, 0) \).

Our goals: Show the assumptions are satisfied, use a sequence of diffusion problems to approximate the jump diffusion problem.
**Functional Operator \( J \)**

Let us introduce the functional operator \( J \) through its action on a test function \( f : \mathbb{R} \times [0, T] \to \mathbb{R}_+ \):

\[
Jf(z, t) = \mathbb{E}_{t, z}^Q \left\{ e^{-\lambda \xi (T-t)} (\zeta \cdot (Z_T - K_1))^+ 
+ \int_t^T e^{-\lambda \xi (s-t)} \lambda \cdot Pf(Z_s, s) \, ds \right\},
\]

in which \( \mathbb{E}_{t, z}^Q \) is the conditional expectation and the process \( Z = \{Z_t; s \geq 0\} \) is a diffusion process with the dynamics

\[
dZ_s = -\mu(q_s - Z_s) \, ds + \sigma(q_s - Z_s) \, dW_s.
\]

Recall \( Pf(Z_s, s) = \int_{\mathbb{R}_+} f \left( \frac{Z_s}{y} + q_s \frac{y-1}{y}, t \right) y \nu(dy) \).
Functional Operator $J$

Let us introduce the functional operator $J$ through its action on a test function $f : \mathbb{R} \times [0, T] \to \mathbb{R}_+$:

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**Remark:** Intuitively, let $\tau_1$ be the first jump time of a Poisson process with the parameter $\lambda \xi$,

$$Jf(x, t) = \mathbb{E}_{t,z}^Q \left\{ \left( \zeta \cdot (Z_T - K_1) \right)^+ 1_{\{\tau_1 > T-t\}} + f(S_{\tau_1}, \tau_1) 1_{\{\tau_1 \leq T-t\}} \right\}.$$
The Approximating Sequence

Using the operator $J$, let us introduce

$$v_0(z, t) \triangleq (\zeta \cdot (z - K_1)^+),$$

$$v_{n+1}(z, t) \triangleq Jv_n(z, t), \quad n \geq 0, \text{ for } (z, t) \in \mathbb{R} \times [0, T].$$
The Approximating Sequence

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$$v_{n+1}(z, t) \triangleq Jv_n(z, t), \quad n \geq 0, \text{ for } (z, t) \in \mathbb{R} \times [0, T].$$

We want to show:

1. $\{v_n\}_{n \geq 0}$ converges to a limit $v_{\infty}$,
2. $v_{\infty}$ is the unique classical solution, i.e. $v_{\infty} \in C^{2,1}$, of the following equation

$$\frac{\partial}{\partial t}v_{\infty} + A(t)v_{\infty} - \lambda \xi v_{\infty} + \lambda \cdot (Pv_{\infty})(z, t) = 0,$$
$$v_{\infty}(z, T) = (\zeta \cdot (z - K_1))^+. $$

Therefore, $V(S_0) = S_0 \cdot v_{\infty}(z, 0)$. 
Properties of $J$

The operator $J$ can be rewritten as

$$Jf(z, t) = \mathbb{E}_{Q}\{e^{-\lambda \xi (T-t)}(\xi \cdot (zH_0T-t+bT-t) + 1) + \int_{T-t}^0 e^{-\lambda \xi s} \lambda \cdot Pf(zH_0s+b, t+s) ds\},$$

where $H_0 s \equiv \exp((\mu - \frac{1}{2} \sigma^2)s - \sigma W_s)$ and $b_s$ is represented by $H$ and $q$. 

Lemma: For any $t \in [0, T]$, 

$$|f(z, t) - f(\tilde{z}, t)| \leq D |z - \tilde{z}|, z, \tilde{z} \in \mathbb{R},$$

Then $Jf$ satisfies 

$$|Jf(z, t) - Jf(\tilde{z}, t)| \leq E |z - \tilde{z}|, z, \tilde{z} \in \mathbb{R},$$

with $E = \max\{1, D\}$. 
Properties of $J$

$J$ maps “nice” functions to “nicer” functions.

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$$\left. + \int_0^{T-t} e^{-\lambda \xi s} \lambda \cdot Pf(zH^0_s + b_s, t + s) \, ds \right\},$$

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Properties of $J$ cont.

Defining $M_f \triangleq \sup_{t \in [0, T]} f(0, t)$ and $M_{Jf} \triangleq \sup_{t \in [0, T]} Jf(0, t)$, we have $f$ and $Jf$ both satisfy linear growth conditions. Moreover,

**Lemma**

\[ M_{Jf} \leq U + \alpha \left( M_f + \frac{B}{\xi} \right), \]

in which $\alpha = 1 - e^{-\lambda \xi T} < 1$, and $U$, $B$ are positive constants depending on $T$. 
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**Lemma**

Assume $|f(z, t) - f(\tilde{z}, t)| \leq D|z - \tilde{z}|$, for $z, \tilde{z} \in \mathbb{R}$, then

\[ |Jf(z, t) - Jf(z, s)| \leq F \left( 1 + |z| \right) (s - t)^{\frac{1}{2}}, \quad 0 \leq t < s \leq T, \]

in which $F$ is a positive constant that only depends on $\lambda$, $\xi$, $T$ and $M_f$. 
Properties of $J$ cont.

**Theorem**

Assume function $f$ satisfies

$$|f(z, t) - f(\tilde{z}, s)| \leq D|z - \tilde{z}| + F(1 + |z|)|s - t|^\frac{1}{2},$$

then the function $Jf$ is the unique classical solution, i.e. $Jf \in C^{2,1}$, of

$$\mathcal{L}_D Jf(z, t) \triangleq \frac{\partial}{\partial t} Jf + A(t) Jf - \lambda \xi Jf = -\lambda \cdot Pf(z, t)$$

$$Jf(z, T) = (\zeta \cdot (z - K_1))^+. $$
Properties of $J$ cont.

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**Proof:** For any point $(z, t) \in D = [z_1, z_2] \times [0, T]$.

\[
\mathcal{L}_D u(z, t) = -\lambda \cdot Pf(z, t), \quad u(z, t) = Jf(z, t), \quad (z, t) \in \partial_0 D.
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Properties of $J$ cont.

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Pervious Lemmas $\implies$ 1. $Jf$ is joint continuous, 2. $Pf(z, t)$ is Lipschitz in $z$ and Hölder continuous in $t$ uniformly in $D.$
Properties of $J$ cont.

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\mathcal{L}_D Jf(z, t) \equiv \frac{\partial}{\partial t} Jf + A(t) Jf - \lambda \xi Jf = -\lambda \cdot Pf(z, t)
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Pervious Lemmas $\implies$ 1. $Jf$ is joint continuous, 2. $Pf(z, t)$ is Lipschitz in $z$ and Hölder continuous in $t$ uniformly in $D$. Theory of parabolic PDE $\implies$ there is an unique classical solution.
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**Theorem**

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\mathcal{L}_D Jf(z, t) \triangleq \frac{\partial}{\partial t} Jf + A(t) Jf - \lambda \xi Jf = -\lambda \cdot Pf(z, t),
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Pervious Lemmas $\implies$ 1. $Jf$ is joint continuous, 2. $Pf(z, t)$ is Lipschitz in $z$ and Hölder continuous in $t$ uniformly in $D$. Theory of parabolic PDE $\implies$ there is an unique classical solution. Its representation is exactly $Jf$. 

Properties of $v_n$

Lemma

1. Define $M_n = \sup_{t \in [0, T]} \{v_n(0, t)\}$, then
   
   $M_n < M_\infty \triangleq \frac{U}{1-\alpha} + \frac{\alpha B}{1-\alpha} \xi + K_1 < \infty$ for $n \geq 0$.

2. For $n \geq 0$, $|v_n(z, t) - v_n(\tilde{z}, t)| \leq |z - \tilde{z}|$, $z, \tilde{z} \in \mathbb{R}$.

3. $|v_n(z, t) - v_n(z, s)| \leq F_n(1 + |z|)(s - t)^{\frac{1}{2}}$, $0 \leq t < s \leq T$, in which $F_n$ are finite constants depends on $T$.

4. $\{v_n(z, t)\}_{n \geq 0}$ is a Cauchy sequence.
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Remark:

- $1+2 \implies v_n(z, t) \leq M_\infty + |z| \triangleq L(z)$. 
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   $$M_n < M_\infty \triangleq \frac{U}{1-\alpha} + \frac{\alpha}{1-\alpha} \frac{B}{\xi} + K_1 < \infty \text{ for } n \geq 0.$$

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4. $\{v_n(z, t)\}_{n \geq 0}$ is a Cauchy sequence.

Remark:

- 1+2 $\implies$ $v_n(z, t) \leq M_\infty + |z| \triangleq L(z)$.
- 4 $\implies$ the pointwise limit for $\{v_n\}_{n \geq 0}$ exists, we call it $v_\infty$. Moreover, $v_\infty \leq L(z)$ and for any compact domain $\mathcal{D} \in \mathbb{R}$,

$$|v_\infty(z, t) - v_n(z, t)| \leq M_\mathcal{D} \left(1 - e^{-\lambda \eta (T-t)}\right)^n,$$

where $M_\mathcal{D}$ is a constant depending on $\mathcal{D}$ and $\eta = \max\{\xi, 1\}$. 
Combining 2, 3 and the Theorem for $Jf$, we have that $v_{n+1}$ is the unique classical solution, i.e. $v_{n+1} \in C^{2,1}$, of

$$\frac{\partial}{\partial t} v_{n+1} + A(t)v_{n+1} - \lambda \xi v_{n+1} + \lambda \cdot (Pv_{n})(z, t) = 0,$$

$$v_{n+1}(z, T) = (\zeta \cdot (z - K_1))^+. $$

This is a parabolic PDE with an integral term as the driving term.
Properties of $v_\infty$

Lemma

1. $v_\infty$ is a fixed point of the operator $J$.
2. $|v_\infty(z, t) - v_\infty(\tilde{z}, t)| \leq |z - \tilde{z}|$.
3. $|v_\infty(z, t) - v_\infty(z, s)| \leq F_\infty (1 + |z|) |t - s|^{\frac{1}{2}}$. 
Properties of $v_\infty$

Lemma

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2. $|v_\infty(z, t) - v_\infty(\tilde{z}, t)| \leq |z - \tilde{z}|$.
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Apply the Theorem on $Jf$ to $v_\infty$, we obtain

Theorem (Main Theorem)

The function $v_\infty$ is the unique classical solution, i.e. $v_\infty \in C^{2,1}$, of

$$\frac{\partial}{\partial t} v_\infty + A(t)v_\infty - \lambda \xi v_\infty + \lambda \cdot (Pv_\infty)(z, t) = 0,$$
$$v_\infty(z, T) = (\zeta \cdot (z - K_1))^+.$$

Proof: Combining 2, 3 and Theorem on $Jf$, we have that $Jv_\infty$ is the unique classical solution of the parabolic PDE with the integral term $\lambda \cdot (Pv_\infty)$. The theorem follows since $Jv_\infty = v_\infty$. 
Numerical Algorithm

We solve the sequence of PDEs satisfied by $v_n$ iteratively, using the finite difference method.

- Crank-Nicolson discretization + SOR,
- trapezoidal rule to evaluate the integral $Pv_n$.

Let $\tilde{v}_n$ and $\tilde{v}_\infty$ be the numerical solutions for the discretized PDEs satisfied by $v_n$ and $v_\infty$ respectively, we have also shown

- The algorithm is stable, $\tilde{v}_n$ converges to $\tilde{v}_\infty$ uniformly and at an exponential rate.
- $\tilde{v}_\infty \rightarrow v_\infty$ as discretizations go to 0.
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Numerical Results

Table: The approximated price for continuously averaged European type Asian options for a double exponential jump model.

\( r = 0.15, \ S_0 = 100, \ T = 1, \ p = 0.6 \) and \( \eta_1 = \eta_2 = 25 \). Monte Carlo method uses \( 10^6 \) simulations and \( 10^3 \) time steps. "C - P" is the difference between our approximated call and put option prices. "Parity" is the difference predicted by the put-call parity. All our computations are performed on a Pentium IV 3.0 GHz machine with C++ implementation. Run times are in seconds.

| \( \sigma \) | \( K_2 \) | \( \lambda \) | Call Option (C) | Put Option (P) | C - P | Parity | Monte Carlo (Call Option) |
|-------------|--------|-------------|----------------|----------------|-------|--------|----------------|----------------|
|             |        |             | Value | Time | Value | Time | Value | Stan. Err. | Time |
| 0.1         | 90     | 1           | 15.419 | 1.0  | 0.012 | 1.0  | 15.407 | 15.412 | 15.398 |
|             |        | 3           | 15.457 | 1.5  | 0.045 | 1.5  | 6.794  | 6.800  | 6.791  |
|             | 100    | 1           | 7.170  | 1.0  | 0.376 | 1.0  | -1.818 | -1.820 | -1.817 |
|             |        | 3           | 7.456  | 1.5  | 0.656 | 1.6  | 1.697  | 2.207  | 1.697  |
|             | 110    | 1           | 1.702  | 1.0  | 3.520 | 1.0  | 15.686 | 15.699 | 15.686 |
|             |        | 3           | 2.220  | 1.5  | 4.040 | 1.6  | 15.407 | 15.412 | 15.407 |
| 0.2         | 90     | 1           | 15.699 | 1.0  | 0.292 | 1.0  | 15.407 | 15.412 | 15.398 |
|             |        | 3           | 15.802 | 1.5  | 0.390 | 1.6  | 6.795  | 6.796  | 6.791  |
|             | 100    | 1           | 8.540  | 1.0  | 1.745 | 1.0  | 8.540  | 8.784  | 8.540  |
|             |        | 3           | 8.790  | 1.5  | 1.994 | 1.6  | 8.784  | 8.784  | 8.784  |
|             | 110    | 1           | 3.723  | 1.0  | 5.541 | 1.0  | -1.819 | -1.819 | -1.817 |
|             |        | 3           | 4.045  | 1.6  | 5.864 | 1.6  | 3.721  | 4.038  | 3.721  |
Table: The approximated price for continuously averaged European type Asian options for normal jump diffusion model.

$r = 0.15$, $S_0 = 100$, $T = 1$, $\lambda = 1$, $\bar{\mu} = -0.1$ and $\bar{\sigma} = 0.3$. Monte Carlo method uses $10^6$ simulations and $10^3$ time steps. "C - P" is the difference between our approximated call and put option prices. "Parity" is the difference predicted by the put-call parity. All our computations are performed on a Pentium IV 3.0 GHz machine with C++ implementation. Run times are in seconds.

| $\sigma$ | $K_2$ | Call Option | Put Option | C - P | Parity |
|----------|-------|-------------|------------|-------|--------|
|          |       | Value | Time | Value | Time | Value | Time | Value | Time |
| 0.1      | 90    | 16.997 | 0.5  | 1.601 | 0.5  | 15.396 | 15.398 |
|          | 100   | 10.062 | 0.5  | 3.272 | 0.5  | 6.789  | 6.791 |
|          | 110   | 4.836  | 0.5  | 6.653 | 0.5  | -1.817 | -1.816 |
| 0.2      | 90    | 17.346 | 0.5  | 1.950 | 0.5  | 15.396 | 15.398 |
|          | 100   | 10.959 | 0.5  | 4.170 | 0.5  | 6.789  | 6.791 |
|          | 110   | 6.303  | 0.5  | 8.120 | 0.5  | -1.817 | -1.816 |
|          |       |        |      |       |       | 16.991 | 0.014 | 913 |
|          |       |        |      |       |       | 10.046 | 0.013 | 910 |
|          |       |        |      |       |       | 4.834  | 0.011 | 915 |
|          |       |        |      |       |       | 17.339 | 0.017 | 919 |
|          |       |        |      |       |       | 10.968 | 0.015 | 917 |
|          |       |        |      |       |       | 6.310  | 0.012 | 913 |
Thanks for your attention!