Optimizing the Age-of-Information for Mobile Users in Adversarial and Stochastic Environments

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Abstract—We study a multi-user downlink scheduling problem for optimizing the freshness of information available to users roaming across multiple cells. We consider both adversarial and stochastic settings and design scheduling policies that optimize two distinct information freshness metrics, namely the average age-of-information and the peak age-of-information. We show that a natural greedy scheduling policy is competitive with the optimal offline policy in the adversarial setting. We also derive fundamental lower bounds to the competitive ratio achievable by any online policy. In the stochastic environment, we show that a Max-Weight scheduling policy that takes into account the channel statistics achieves an approximation factor of 2 for minimizing the average age of information in two extreme mobility scenarios. We conclude the paper by establishing a large deviation optimality result achieved by the greedy policy for minimizing the peak age of information for static users situated at a single cell.

Index Terms—Age-of-Information, Competitive Analysis, Optimal Scheduling, Fundamental Limits.

I. INTRODUCTION

The Quality-of-Service (QoS) offered by a wireless network is traditionally measured along three dimensions, namely, throughput, delay, and energy efficiency. Currently, there exists an extensive body of literature on cross-layer resource allocation algorithms to optimize each of the above metrics [4]–[8]. However, it has been argued that the standard QoS metrics are primarily geared towards quantifying the degree of utilization of the system resources and less towards measuring the actual user experience [9]. With the explosive growth of hand-held mobile devices, the Internet of Things (IoT), real-time AR and VR systems powered by the emerging 5G technology, the Quality of Experience (QoE) for the users is expected to play a major role in the future network design [10]. In order to directly incorporate QoE objective into the system design criteria, a new metric, called Age-of-Information (AoI), has been proposed recently for measuring the freshness of information available to the end-users [11], [12]. Depending on the application, both the average or the peak AoI of a number of users may be of interest [13]. As an example, for non-critical status updates of a number of users, minimizing the average AoI is useful. On the other hand, in applications where the most outdated node is usually the bottleneck (e.g., mission-critical or industrial IoT applications), minimizing the peak-AoI is an appropriate objective. It is well-known that mobility increases the capacity of wireless ad hoc networks [14], [15], [16]. However, to the best of our knowledge, the effect of user mobility on the freshness of information has not been explored before. In this paper, our objective is to design near-optimal scheduling policies in a cellular network for minimizing the average and peak AoI for mobile users under two distinct channel/mobility models, as detailed below.

Most of the existing works on wireless communication assume a stationary channel model for analytical tractability [17]. In highly non-stationary environments, such as high-speed trains and vehicle-to-vehicle communication, the standard stationarity assumption no longer holds in practice. This is particularly true in the emerging 5G mmWave regime, which suffers from severe attenuation loss [18], [19]. On the other hand, designing an accurate and analytically tractable non-stationary wireless channel model remains an overarching challenge to the research community for decades [20], [21]. Furthermore, accurate channel estimation in rapidly-varying environments is often infeasible for applications requiring extremely low latency. Responding to this challenge, we ask the following question in the first half of the paper: Does there exist a scheduling policy that minimizes the overall AoI, irrespective of the channel dynamics and user-mobility patterns? The question is considerably general, as we do not make any assumption on either the channel statistics or the user-mobility, both of which may be dictated by an omniscient adversary in the worst case. The adversarial channel model is also useful for ensuring reliable communication in the presence of tactical jammers, where the interferers, in reality, may behave adversarially [22], [23]. A similar question in the context of the stability of wireless networks was considered in [24] under adversarial arrival and transmission rates.

To formalize the problem, we first introduce an adversarial binary erasure channel model, which may be considered as an adversarial counterpart of the celebrated Binary Erasure Channel (BEC) model. However, unlike a similar adversarial model considered in [25], we neither assume any causality nor impose an upper-bound on the fraction of erasures. Thus, our channel model is considerably more general. In this model, we propose a near-optimal scheduling policy that competitively minimizes both the average and peak-AoI in this model. See Table I for our main results within the adversarial framework.

In contrast with the non-stationary environment, minimizing the AoI in stationary stochastic environments has been considered in many recent papers. In the paper [26], the authors study the average AoI minimization problem for static users associated with a single access point. The authors show that the greedy Max-Age (MA) policy is optimal for minimizing the...
average AoI in a symmetric static network. In the same paper, the authors also propose a $4$-optimal Max-Weight scheduling policy ($MW$) for an arbitrary static network consisting of a single access point. The proposed $MW$ policy has been reported to practically improve the information freshness in WiFi networks by two orders of magnitude [27]. The paper [28] extends the $MW$ policy by taking into account additional throughput constraints. The paper [29] designs optimal stationary scheduling policies for minimizing the AoI in multi-hop networks with static users under general interference constraints. The paper [30] considers the problem of designing an AoI-optimal trajectory for a mobile agent to facilitate the information dissemination from a central station to a set of ground terminals. However, the problem of designing an AoI optimal scheduling policy for mobile users has not been considered before. In the second half of the paper we tackle this question and show that a natural multi-cell extension of the Max-Weight policy, called MMW is 2-optimal under certain conditions. Table III summarizes the main results in the stochastic model.

Our contributions:

1) In this paper, we consider the AoI-optimal scheduling problem for mobile users in both adversarial and stochastic models. The adversarial formulation of the problem is new and, to the best of our knowledge, has not been considered before in the literature. On the other hand, while the AoI-optimal scheduling problem for static users in the stochastic setting is now well-understood [26], [28], the multi-cell extension of the problem with mobile users is new.

2) In the adversarial framework of Section II we show that a greedy online scheduling policy is $2N^2$-competitive for the Average AoI metric. Using Yao’s minimax principle, we also establish a fundamental lower bound by showing that no online policy can have a competitive ratio smaller than $2N – 1$.

3) For the peak-AoI metric, we show that the same greedy scheduling policy achieves a competitive ratio of $2N$ in the adversarial setting. Using Yao’s minimax principle, we show that no online policy can have a competitive ratio better than $\Omega(\frac{N}{\ln(N)})$.

4) In Section III, we consider the AoI-optimal multi-user scheduling problem for mobile users in a stationary environment. For the average-age metric, we design a 2-optimal scheduling policy for mobile users with i.i.d. uniform mobility. As a by-product of our analysis, we improve upon the best-known 4-approximation bound known for the static users [26], [31].

5) For minimizing the peak-AoI metric in the stochastic setting, we show that the greedy policy is optimal for a single-cell static network. We give a short proof of this optimality result by producing a solution to a countable-state average-cost MDP in closed form, which might be of independent interest. This result supplements Theorem 5 of [26], which establishes the optimality of the greedy policy for the average-AoI metric for symmetric static networks using stochastic dominance arguments. We further show that the greedy policy achieves the optimal large-deviation rate.

The rest of the paper is organized as follows. In Section II, we describe the adversarial model and prove upper and lower bounds on the achievable competitive ratios within the adversarial framework. In Section III, we describe the stochastic model and design policies for minimizing the average and peak AoI in the stochastic regime. In Section IV, we compare the performance of the proposed scheduling policies via numerical simulations. Section V concludes the paper with some pointers to a few related open problems.

II. AOI MINIMIZATION IN ADVERSARIAL ENVIRONMENTS

In this Section, we consider the AoI-optimal downlink scheduling problem for $N$ users roaming around in a region having $M$ Base Stations (BS). The environment, which is completely specified by the channel states and the user mobility pattern, is allowed to be non-stationary. Instead of trying to fit a complicated non-stationary probabilistic model with multiple parameters [20], [21], we take a conservative view and consider an adversarial framework for the environment. The adversarial viewpoint can be practically motivated upon considering URLLC-type traffic, which require extremely low latency with very high reliability [32]. Besides being analytically tractable, all achievability results in the adversarial model (Theorem I) carry over to more benign non-stationary stochastic environments. Moreover, as we will see in the sequel, policies having a good competitive ratio in the adversarial setting, sometime translate to optimal policies in the stochastic environment (Theorem 7).

A. Adversarial System Model

The main components of the system are specified below.

a) Network model: The area covered by a Base Station (herein referred to as BS) is referred to as a Cell. The cells are assumed to be spatially disjoint. Time is slotted, and at every slot, a user can either stay in its current cell
or move to any other $M - 1$ cells (need not be adjacent). The movement could be dictated by an omniscient adversary. Our mobility model is considerably general, as it does not make any assumptions (statistical or otherwise) on the speed or user movement patterns. See Figure 1 for a schematic.

b) Traffic Model: We consider a saturated traffic model, where at the beginning of every slot, each of the $M$ BS receives a fresh packet for each user from some external source (e.g., a high-speed optical backbone network). Since our objective is to maximize the freshness of information at the user-end, any stale packet at the BS buffer is replaced by incoming fresh packets at each slot. Each BS can beamform and schedule a downlink packet transmission at each slot to only one user within its coverage area. The saturated traffic model is standard in applications relying on continuous status updates [33], such as monitoring and surveillance with sensor networks [34], velocity and position updates for autonomous vehicles [35], command and control information exchange in mission-critical systems, disseminating stock-index updates and live game scores.

c) Channel states, Control, and Objective: It is the user scheduling decisions by each BS that we can control and optimize. We assume a binary erasure channel model where the channel state for any user at any slot can be either Good or Bad. The schedulers are considered to be oblivious to the current channel state conditions (i.e., no CSIT). An online scheduling policy $\pi$ first selects a user in each cell (if the cell contains at least one user), and then transmits the latest packet from the BS to the selected users over the wireless channel. The set of all admissible scheduling policies is denoted by $\Pi$. If at any slot, a packet transmission is scheduled to a user having a Good channel, the user decodes the packet successfully. Otherwise, the packet is lost. A lost packet is never retransmitted as the scheduler has access to fresh packets at every slot. In the adversarial model, we posit that the channel states are dictated by an omniscient adversary [24]. To be specific, we allow the situation where the adversary knows the scheduling policy in advance and chooses the channel realizations after the scheduling decisions have been made for a slot. On the other hand, the scheduling policy $\pi$ is online and has no information about the channel states in the current or future slots. See Figure 2 for the timeline of events taking place at every slot.

We are concerned with competitively optimizing the information freshness for all users. Formally, our objective is to design a decentralized scheduling policy which minimizes some measure of the aggregate Age-of-Information of the users as defined next. For any slot $t \geq 1$, let $t_i(t) < t$ denote the last time prior to time $t$ at which $i$th user successfully received a packet from some BS. The Age-of-Information (AoI) $h_i(t)$ of the user at time $t$ is defined as

$$h_i(t) \equiv t - t_i(t).$$

In other words, $h_i(t)$ denotes the length of the elapsed time since the $i$th user received its last update packet before time $t$. Thus, $h_i(t)$ quantifies the staleness (or age) of the information available to the $i$th user. Accordingly, we define the $N$-dimensional state-vector $h(t)$, where $h_i(t)$ denotes the AoI of the $i$th user. Clearly, the plot of $h_i(t)$ vs. the time $t$ has a saw-tooth shape that increases linearly with unit-slope until a fresh packet is received. Upon reception of a fresh packet, the AoI $h_i(t)$ instantaneously drops to 1. From that point onwards, $h_i(t)$ again increases linearly, repeating the saw-tooth pattern [26]. See Figure 3 for an illustration. In this paper, we consider optimizing the following two different AoI metrics:

a) Average AoI: The time-averaged cost corresponding to the average AoI for $N$ users up to time $T$ is defined as:

$$\text{Aoi}_{\text{avg}}(T) = \frac{1}{NT} \sum_{t=1}^{T} \left( \sum_{i=1}^{N} h_i(t) \right).$$

b) Peak AoI: The instantaneous peak-AoI at a slot is defined as the maximum age among all users. The time-averaged cost corresponding to the peak-AoI for a time-horizon of length $T$ is defined as:

$$\text{Aoi}_{\text{peak}}(T) = \frac{1}{T} \sum_{t=1}^{T} \max_{i=1}^{N} h_i(t).$$

| Metrics | Cost function | Mobility | Upper Bound (Policy) | Lower Bound | Optimality gap |
|---------|---------------|----------|----------------------|-------------|---------------|
| Average AoI | $N^{-1} \sum_{t=1}^{T} \sum_{i=1}^{N} h_i(t)$ | Yes | $O(N^2)$ (CMA) | $O(N)$ | $O(N)$ |
| Peak AoI   | $\sum_{t=1}^{T} \max_{i=1}^{N} h_i(t)$ | Yes | $O(N)$ (CMA) | $\Omega(\frac{N}{\ln(N)})$ | $O(\ln(N))$ |

![Fig. 2. Timeline of the events at each slot.](image-url)
In the above definition, the supremum is taken over all length admissible sequences σ. Depending on the objective, the cost function in the definition (3) can be taken to be either Eqn. (1) or Eqn. (2). We emphasize that, while the online policy A has only causal information (i.e., knows only the subsequence σ^{t−1}_i at time t), the policy OPT is assumed to be equipped with non-causal knowledge of the entire sequence σ^T_i right at the beginning. Our objective is to design an online scheduling policy A with a small competitive ratio, so that it performs close to the OPT policy.

Note: In the classical sum-throughput maximization problem, the objective is to maximize the total rate of successful packet transmissions to the users [4]. It is interesting to note that finding a scheduling policy with a small competitive ratio for the sum-throughput objective is too strong a requirement, as all deterministic policies have unbounded competitive ratios in this case. This can be understood from the following simple example. Consider a static system consisting of two users in a single cell. If an online policy A schedules the first user at any slot, the adversary sets the channel corresponding to the first user to Bad and the second user’s channel to Good and vice versa. Hence, all transmissions by the policy A are unsuccessful. On the other hand, at any slot, the optimal policy schedules the user having the Good channel state for the current slot. Hence, the OPT policy achieves unit throughput, resulting in unbounded competitive ratio.

### B. Achievability

For our achievability results, we consider the following online scheduling policy:

**Cellular Max-Age (CMA):** At every slot, each BS j schedules a downlink packet transmission to the user with the highest age among all users in BS j’s cell at that slot (ties are broken in an arbitrary but fixed order).

Clearly, the CMA policy is decentralized as the schedulers at each BS need to know the state (AoI) of the users in their local cells only. In the following, we upper bound the competitive ratios of the CMA policy for the Average AoI objective (Eqn. (1)) and the Peak AoI objective (Eqn. (2)). Surprisingly, it turns out that the bounds are independent of the total number of Base Stations M. We now state a few definitions which facilitate the achievability proofs.

**Max-user:** For any slot t, we define the (global) Max-user as the user having the highest age among all N users (ties are broken identically as in the CMA policy). Clearly, the identity of the Max-user changes with time. Observe that, by definition, the CMA policy continues to schedule the current Max-user irrespective of its cell location until the packet transmission to it is successful. In the subsequent slot, a different user assumes the role of the Max-user, and the process continues.

**Super-interval:** The time interval between two consecutive successful transmissions to the current Max-user is called a super-interval. Throughout a super-interval, the identity of the Max-user remains fixed. The Max-user corresponding to the i-th super-interval is denoted by M_i. Note that the super-intervals are contiguous and disjoint. Let T_i be the index of the time slot at which the i-th super-interval ends. Thus, \( \Delta_i = T_i - T_{i-1} \) denotes the length of the i-th super-interval. See Figure 4 for a schematic. Note that, there could be more than one successful transmissions within a super-interval to users other than the Max-user (by different Base Stations). For notational consistency, we define \( T_j = 0 \), and \( \Delta_j = 0, \forall j \leq 0 \). The above definitions and observations lead to the following key result:

**Proposition 1:** At the k-th slot of the i-th super-interval, the age of the Max-user M_i under the CMA policy is upper bounded by \( k + \sum_{j=1}^{N-1} \Delta_{i-j} \).

**Proof:** We claim that the Max-user M_i corresponding to the i-th super-interval must have had a successful transmission within the last N − 1 super-intervals. If not, since there are a total of N users, by the pigeonhole principle, some other user j ≠ M_i must become the Max-user at least twice in the previous N super-intervals. However, this cannot be true as then, the j-th user would have had less age than M_i (which was not scheduled at all in the last N − 1 super-intervals) when the j-th user became the Max-user for the second time. Thus, at the beginning of the i-th super-interval, the age of the new
Max-user is upper bounded by $\sum_{j=1}^{N-1} \Delta_{i-j}$. The proposition follows from this fact.

The previous proposition leads to the following achievability result for the CMA policy for the Average AoI and Peak-AoI objectives.

**Theorem 1 (Upper-bounds):** The competitive ratios of the CMA policy for the Average AoI and Peak-AoI objectives are upper-bounded as

$$\eta_{\text{avg}}^{\text{CMA}} \leq 2N^2,$$

and

$$\eta_{\text{peak}}^{\text{CMA}} \leq 2N.$$

**Proof:** We consider the Average-AoI and Peak-AoI objectives separately.

a) **Average-AoI:** The proof proceeds by upper-bounding the cost incurred by the CMA policy and lower bounding the cost incurred by the OPT policy at each super-intervals. Note that, at any slot of the $i^{th}$ super-interval, the age of every user is upper-bounded by that of the Max-user $M_i$. Hence, using proposition 1, the total cost incurred by the CMA policy during the $i^{th}$ super-interval may be upper-bounded as:

$$C_i^{\text{CMA}} = \sum_{t \in i^{th} \text{ super interval}} \sum_{s=1}^{N} h_i(t)$$

$$\leq \sum_{k=1}^{\Delta_i} \sum_{s=1}^{N} \left( k + \sum_{j=1}^{N-1} \Delta_{i-j} \right)$$

$$= N \left( \frac{\Delta_i(\Delta_i+1)}{2} + \sum_{j=1}^{N-1} \Delta_i \Delta_{i-j} \right)$$

$$\leq N \left( \frac{\Delta_i^2}{2} + \sum_{j=1}^{N-1} \Delta_i \Delta_{i-j} \right),$$

where in the last step, we have used the AM-GM inequality to obtain $\Delta_i \Delta_{i-j} \leq \frac{1}{2} (\Delta_i^2 + \Delta_{i-j}^2), 1 \leq j \leq N-1$.

Let $K$ be the total number of super-intervals in the entire time-horizon of length $T$. The total cost incurred by the CMA policy for the entire time horizon can then be upper bounded as:

$$\text{Cost}^{\text{CMA}}(T) = \sum_{i=1}^{K} C_i^{\text{CMA}}$$

$$\leq \frac{N}{2} \sum_{i=1}^{K} \left( N \Delta_i^2 + \Delta_i + \sum_{j=1}^{N-1} \Delta_{i-j}^2 \right)$$

$$\leq \frac{N}{2} \sum_{i=1}^{K} \left( 2N \Delta_i^2 + \Delta_i \right).$$

On the other hand, observe that the Max-user $M_i$ experiences Bad Channels throughout the $i^{th}$ super-interval. This is true, as otherwise, the Max-user would have successfully received a packet under the CMA policy, thus ending the $i^{th}$ super-interval. Hence, the cost incurred by the OPT policy during the $i^{th}$ super-interval may be lower bounded as:

$$C_i^{\text{OPT}} \geq (N-1) \frac{\Delta_i}{\Delta_i} + \sum_{k=1}^{\Delta_i} (1 + k)$$

$$\geq \frac{1}{2} \Delta_i^2 + N \Delta_i.$$

In the above, the lower-bound to the cost incurred by $M_i$ is obtained by considering that its age increases linearly under any policy throughout the $i^{th}$ super interval, starting from the minimum age of at least 1. Finally, the total cost incurred during the entire horizon of length $T$ is obtained by summing up the cost incurred in the constituent super-intervals. Hence, from Eqns. 1 and 5, the competitive ratio $\eta_{\text{CMA}}$ of the CMA policy may be upper bounded as:

$$\eta_{\text{avg}}^{\text{CMA}} = \frac{1}{NT} \sum_{i=1}^{K} C_i^{\text{CMA}}$$

$$\leq \frac{N}{2} \sum_{i=1}^{K} \left( 2N \Delta_i^2 + \Delta_i \right)$$

$$\leq \frac{N}{2} \sum_{i=1}^{K} \left( \frac{1}{2} \Delta_i^2 + N \Delta_i \right)$$

$$\leq 2N^2.$$

b) **Peak-AoI:** The proof proceeds essentially in the same way as the Average Age case. Using proposition 1, the total cost $C_i^{\text{CMA}}$ incurred by the CMA policy during the $i^{th}$ super-interval may be upper-bounded as:

$$C_i^{\text{CMA}} = \sum_{t \in i^{th} \text{ super interval}} \sum_{s=1}^{N} h_i(t)$$

$$\leq \sum_{k=1}^{\Delta_i} \left( k + \sum_{j=1}^{N-1} \Delta_{i-j} \right)$$

$$\leq \sum_{k=1}^{\Delta_i} \left( \frac{1}{2} \Delta_i^2 + \Delta_i \right) + \sum_{j=1}^{N-1} \Delta_i \Delta_{i-j}$$

$$\leq \frac{1}{2} \Delta_i^2 + \Delta_i + \frac{1}{2} \sum_{j=1}^{N-1} \Delta_{i-j}^2.$$
there are a total of $K$ super-intervals in the time-horizon of length $T$, the total cost incurred by the CMA policy over the entire time horizon is upper bounded as:

$$\text{Cost}^{\text{CMA}}(T) = \sum_{i=1}^{K} C^{\text{CMA}}_i \leq \frac{1}{2} \sum_{i=1}^{K} \left( 2N \Delta_i^2 + \Delta_i \right). \quad (6)$$

On the other hand, the cost incurred by the OPT policy during the $i^{th}$ super-interval is trivially lower bounded by the sum of the ages of the user $M_i$ during the super-interval. Note that, as before, the user $M_i$ consistently experiences Bad channels throughout the $i^{th}$ super-interval. Hence,

$$C^{\text{OPT}}_i \geq \Delta_k (1 + k) = \frac{1}{2} \Delta_i^2 + \frac{3}{2} \Delta_i, \quad (7)$$

Finally, the cost of the entire horizon of length $T$ may be obtained by summing up the cost incurred in each super-intervals. Noting that $\Delta_0 = 0$, using Eqs. (6) and (7), the competitive ratio $\eta^{\text{CMA}}$ of the CMA policy may be upper bounded as follows:

$$\eta^{\text{CMA}}_{\text{avg}} = \frac{1}{T} \sum_{i=1}^{K} \frac{C^{\text{CMA}}_i}{C^{\text{OPT}}_i} \leq \frac{1}{2} \sum_{i=1}^{K} \left( 2N \Delta_i^2 + \Delta_i \right) \leq 2N. \quad (8)$$

Proof: As stated above, to apply Yao’s minimax principle (Eqn. (8)), we need to (a) lower-bound the expected cost incurred by any online policy $\pi$, and (b) upper-bound the expected cost incurred by the optimal offline policy for some suitably chosen channel state distribution $p$. Selecting a channel state distribution $p$, which simultaneously yields a tight lower bound and leads to a tractable analysis, is a non-trivial task. Towards this, we consider the following channel state distribution for static users located in a single cell.

**Channel-State Distribution $p$:** At every slot $t$, a user is chosen independently and uniformly at random, and assigned a Good channel. The rest of the $N-1$ users are assigned Bad channels.

The rationale behind the above choice of the channel state distribution will become clear when we evaluate OPT’s expected cost. In general, the optimal offline policy’s cost is obtained by solving a Dynamic Program, which is challenging to analyze. However, with the chosen channel distribution $p$, we see that only one user’s channel is in Good state at any slot. This leads to a tractable analysis of OPT’s expected cost for both the Average-AoI and Peak-AoI objectives as shown below.

**Case I: Average AoI metric**

1) **OPT’s expected cost:** Let the random variable $C_i(T)$ denote the total cost incurred by the $i^{th}$ user up to time $T$, i.e.,

$$C_i(T) = \sum_{t=1}^{T} h_i(t).$$

Hence, the limiting time-averaged total expected cost incurred by the OPT policy may be expressed as:

$$\bar{C}(\text{OPT}) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}(C_i(T)) = \sum_{i=1}^{N} \lim_{T \to \infty} \frac{\mathbb{E}(C_i(T))}{T}, \quad (9)$$
In the following, we will show that all of the above limits exist with the assumed choice of the channel state distribution \( p \). We now use the Renewal Reward Theorem \(^{38}\) to evaluate the RHS of Eqn. \((9)\). Since, under the assumed channel state distribution \( p \), only one channel is in \text{Good} state at a slot, the optimal policy \( \text{OPT} \) is easy to characterize - at any slot, \( \text{OPT} \) schedules the user having \text{Good} channel. It can be verified that, under the \( \text{OPT} \) policy, for each user \( i \), the sequence of age random variables \( \{h_i(t)\}_{t=1} \) constitute a renewal process. Clearly, the commencement of scheduling of the \( i \)th user constitutes the renewal instants. A generic renewal interval of length \( \tau \) for the \( i \)th user consists of two parts - (1) a sequence of \text{Good} channels of length \( \tau_G \), and (2) a sequence of \text{Bad} channels of length \( \tau_B \). Hence, the Aol cost \( c_i(\tau) \) incurred by the user \( i \) in any generic renewal cycle may be written as the sum of the costs incurred in two parts:

\[
c_i(\tau) = c_i(\tau_G) + c_i(\tau_B) = \sum_{t=1}^{\tau_G} 1 + \sum_{t=1}^{\tau_B} (1 + t) = \tau_G + \frac{3}{2}\tau_B + \frac{1}{2}\tau_G.
\]

Let \( q \equiv \frac{1}{q} \) be the probability that that the channel is \text{Good} for the \( i \)th user at any slot. Hence, from our construction, the random variables \( \tau_G \) and \( \tau_B \) follows a Geometric distribution having the following p.m.f.

\[
\mathbb{P}(\tau_G = k) = q^{k-1}(1-q), \quad k \geq 1.
\]

\[
\mathbb{P}(\tau_B = k) = (1-q)^{k-1}, \quad k \geq 1.
\]

Thus, the expected cost incurred by the \( i \)th user at any renewal cycle is given by:

\[
\mathbb{E}(c_i(\tau)) = \frac{1}{1-q} + \frac{3}{2q} + \frac{2}{2q^2} - \frac{1}{q^2} = \frac{1}{q^2}(1-q).
\]

Moreover, the expected length of any renewal cycle can be computed to be:

\[
\mathbb{E}(\tau) = \mathbb{E}(\tau_G) + \mathbb{E}(\tau_B) = \frac{1}{q}.
\]

Using Renewal Reward Theorem \(^{38}\), we have

\[
\lim_{T \to \infty} \frac{\mathbb{E}(C_i(T))}{T} = \frac{\mathbb{E}(c_i(\tau))}{\mathbb{E}(\tau)} = \frac{1}{q} = N, \quad \forall i.
\]

Hence, from Eqn. \((9)\), we conclude that the limiting time-averaged total expected cost incurred by \( \text{OPT} \) is given by

\[
\bar{C}(\text{OPT}) = N^2.
\]

2) Lower Bound to the cost incurred by any policy \( \pi \): In order to lower-bound the expected cost incurred by any online policy \( \pi \) under the distribution \( p \), we appeal to a special case of Theorem \(^{5}\) discussed later in Section \[III-A\]. Theorem \(^{5}\) gives a lower bound to the average cost incurred by any scheduling policy in a stochastic setting when the channels are modeled as binary erasure channels (BEC) with fixed probabilities of success. Since the lower bound in Section \[III-A\] does not require the channels of the users to be mutually independent, we see that the bound is applicable with the assumed channel state distribution \( p \) as well. By plugging in \( p_i = \frac{1}{N}, \quad \forall i, \) and \( M = 1 \) in Theorem \(^{5}\) we conclude that under the channel state distribution \( p \), the time-averaged expected cost for any online scheduling policy \( \pi \) is lower bounded as

\[
\bar{C}(\pi) = \lim_{T \to \infty} \frac{1}{T} \sum_{i=1}^{N} \mathbb{E}(C_i(T)) \geq \frac{N^3 + N}{2}.
\]

Finally, using Yao’s minimax principle in conjunction with Eqs. \((12)\) and \((13)\), we conclude that the competitive ratio \( \eta_{avg} \) of any online policy is lower bounded as

\[
\eta_{avg} \geq \frac{C_T(\pi)}{C_T(\text{OPT})} \geq \frac{C_T(\pi)/T}{C_T(\text{OPT})/T} \geq \frac{N}{2} + \frac{1}{2N}.
\]

We point out that the lower bound in Eqn. \((14)\) can be further improved for the case \( N = 2 \). The following Proposition shows that, using a more careful analysis, the lower bound corresponding to the average AoI for \( N = 2 \) users may be improved to 1.5.

**Proposition 2:** For the case of \( N = 2 \) users, we have the following improved bound: \( \eta_{avg}^2 \geq 1.5 \).

For a proof of the above proposition, please refer to Appendix \[VI-B\].

Next, we consider the Peak-AoI objective \((2)\) and derive a minimax lower bound for this cost metric.

Case II- Peak AoI metric

3) Upper bound to \( \text{OPT}'s \) cost: We use the same channel state distribution \( p \) as before. Recall that, under the distribution \( p \), at any slot we have

\[
\mathbb{P}(\text{user } i \text{'s channel is Good}) = \frac{1}{N},
\]

and is \text{Bad} otherwise. The \( \text{OPT} \) policy, with non-causal channel state information, schedules the user having a \text{Good} channel at every slot. Thus, the limiting distribution of the age of any user is Geometric \( \left(\frac{1}{N}\right)\), i.e.,

\[
\lim_{t \to \infty} \mathbb{P}(h_i(t) = k) = \frac{1}{N} \left(1 - \frac{1}{N}\right)^{k-1}, \quad k \geq 1, \forall i.
\]

Hence, for upper-bounding the time-averaged cost of \( \text{OPT} \) using Cesáro’s summation formula, we are required to upper-bound the expected value of maximum of \( N \) dependent and identically Geometrically distributed random variables. The MGF of the Geometric distribution \( G \) is given by:

\[
\mathbb{E}(\exp(\lambda G)) = \begin{cases} \frac{e^{\lambda/N}}{1-e^{-\lambda/N}}, & \lambda < -\log(1-1/N) \\ \infty, & \text{otherwise} \end{cases}
\]

Let the random variable \( H_{\max} \) denote limiting peak-age of the users. We proceed as in the proof of Massart’s lemma for upper bounding \( \mathbb{E}(H_{\max}) \).

For any \( -\log(1-1/N) > \lambda > 0 \), we have:

\[
\exp(\lambda \mathbb{E}(H_{\max})) \leq \sum_{i=1}^{N} \mathbb{E}(\exp(\lambda G_i)) \leq \frac{e^\lambda}{1 - e^\lambda(1 - \frac{1}{N})},
\]

The following Proposition shows that, using a more careful analysis, the upper bound corresponding to the average AoI for \( N = 2 \) users may be improved to 1.5.
where the inequality (a) follows from Jensen’s inequality. Taking natural logarithm of both sides, we get

\[ \mathbb{E}(H_{\text{max}}) \leq 1 - \frac{1}{\lambda} \log \left( 1 - e^{\lambda} \left( 1 - \frac{1}{N} \right) \right). \] (15)

Now, let us choose \( \lambda = \frac{\alpha}{N} \), for some fixed \( \alpha \) \((0 < \alpha < 1)\) that will be fixed later. First, we verify that, with this choice for \( \lambda \), we always have \( \lambda < -\log \left( 1 - \frac{1}{2} \right) \). Using the fact that \( e^{-x} \geq 1 - x, \forall x \), we have

\[ e^{\lambda} \leq \frac{1}{1 - x}, \quad \forall x < 1. \] (16)

As a result,

\[ e^\lambda \leq \frac{1}{1 - \frac{\alpha}{N}} < \frac{1}{1 - \frac{1}{N}} \text{; i.e., } \lambda < -\log \left( 1 - \frac{1}{N} \right). \]

Next, for upper-bounding the RHS of Eqn. (15), we start with the simple analytical fact that for any \( 0 < \alpha < 1 \),

\[ \inf_{0 < x < 1} \frac{1 - (1 - x)e^{\alpha x}}{x} = 1 - \alpha. \] (17)

This result can be verified by using Eqn. (16) to conclude that for any \( 0 < x < 1 \), we have

\[ \frac{1 - (1 - x)e^{\alpha x}}{x} \geq \frac{1}{x} \left( 1 - \frac{1 - x}{1 - \alpha x} \right) = \frac{1 - \alpha}{1 - \alpha x} \geq 1 - \alpha, \]

where the infimum is achieved when \( x \to 0^+ \). Substituting \( x = \frac{1}{N} \) in the inequality (17), we have

\[ 1 - e^{\alpha/N} \left( 1 - \frac{1}{N} \right) \geq 1 - \frac{\alpha}{N}. \]

Hence, using Eqn. (15), we have the following upper bound to the expected Max-age under OPT:

\[ \mathbb{E}(H_{\text{max}}) \leq 1 + \frac{N}{\alpha} \ln \frac{N}{1 - \alpha}, \text{ for some } 0 < \alpha < 1. \]

Setting \( \alpha = 1 - \frac{1}{\ln N} \) yields the following asymptotic bound:

\[ \mathbb{E}(H_{\text{max}}) \leq N \ln N + o(N \ln N). \] (18)

4) Lower Bound to the expected cost of any online policy \( \pi \): To establish a lower bound to the expected cost of any online policy \( \pi \), we use Theorem [7] established in Section III-B. Theorem[7] gives the minimum time-averaged peak-AoI cost in the stationary environment when all channels can be modeled as BEC. As in the average-AoI case, it can be verified that the theorem continues to hold under the channel state distribution \( p \). Taking \( p_i = \frac{1}{N}, \quad \forall i \), and \( M = 1 \) in Theorem [7] we conclude that under the distribution \( p \), the time-averaged expected cost for any online scheduling policy \( \pi \) is lower bounded as:

\[ \liminf_{T \to \infty} \frac{1}{T} \mathbb{E} \text{Cost}^\pi(T) = \liminf_{T \to \infty} \frac{1}{T} \sum_{i=1}^{T} \mathbb{E}(\max h_i^\pi(t)) \geq N^2. \] (19)

Combining Eqns. (18) and (19) with Eqn. (8) and using Cesàro’s summation formula, we have for any online policy \( \pi \):

\[ \eta^\pi_{\text{peak}} \geq \sup_{T \to \infty} \frac{\mathbb{E} \text{Cost}^\pi(T)}{\mathbb{E} \text{Cost}^{\text{OPT}}(T)} \geq \limsup_{T \to \infty} \frac{\mathbb{E} \text{Cost}^\pi(T)/T}{\mathbb{E} \text{Cost}^{\text{OPT}}(T)/T} \geq \Omega\left( \frac{N}{\ln N} \right). \]

III. AOI MINIMIZATION IN STATIONARY ENVIRONMENTS

In this section, we study the problem of AoI-optimal multi-user scheduling in a cellular wireless network when the channel and user mobility can be modeled as a stationary stochastic process. In the following, we highlight the major differences between the adversarial model in section II and the stochastic model in this section. Our main results for the stochastic model is summarized in Table II

Stochastic System Model

a) Channel model: As in the adversarial case, we consider a cellular system where \( N \) users roam in an area hosting \( M \) Base Stations. Contrary to the adversarial model, the wireless link to the \( i^{th} \) user from the BS in its current cell is modeled by a stationary binary erasure channel (BEC) with the probability of successful reception of a transmitted packet being \( p_i, 0 < p_i \leq 1, \forall i \). Hence, when the associated BS schedules a downlink packet transmission to the \( i^{th} \) user, the packet is either successfully received with probability \( p_i \) (if the channel is in Good state) or is lost otherwise (if the channel is in Bad state). Due to possible power control mechanisms employed by the Base Stations, the success probabilities (i.e., the parameter \( p_i \)’s) may vary among the users [39]. The channels are i.i.d. with respect to the time, but need not be independent across the users.

b) Mobility model: Contrary to the adversarial case, where we allow arbitrary mobility patterns, here we assume that the user mobility is given by a stationary ergodic process. Formally, let the random variable \( C_i(t) \in \{1, 2, \ldots, M\} \) denote the index of the cell to which the \( i^{th} \) user is associated with at time \( t \). We assume that the stochastic process \( \{C_i(t)\}_{t=1}^\infty \) is a stationary ergodic process such that \( \mathbb{P}(C_i(t) = j) = \psi_{ij}, \forall i \in [N], j \in [M], t \geq 1 \). The probability measure \( \psi \) denotes the time-invariant occupancy distribution of the cells by the users. The mobility of the users may be correlated or independent of each other. Many different stochastic mobility models proposed in the literature fall under the above general scheme, including the i.i.d. mobility model, random walk model, and the random waypoint model [40–43].

c) Packet arrival model and Policy Space: We use the same saturated traffic model for packet arrivals that we used for the adversarial environment. The policy space \( \Pi \) is identical to that of the adversarial setup.

d) Performance metric: Our goal in this section is to design scheduling policies which minimizes the long-term expected average AoI as well as the expected peak AoI of all users. In view of this, we define the following long-term Average AoI objective:

\[ \text{Aol}^\pi_{\text{avg}} = \inf_{\pi \in \Pi} \limsup_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}(\max h_i(t)). \] (20)

The long-term Peak AoI objective is defined as:

\[ \text{Aol}^\pi_{\text{peak}} = \inf_{\pi \in \Pi} \limsup_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}(\max h_i(t)). \] (21)
TABLE II
SUMMARY OF THE RESULTS FOR THE STATIONARY ENVIRONMENT

| Metrics          | Cost function | Mobility                        | Upper Bound (Policy)      | Lower Bound                                  | Approximation Factor |
|------------------|---------------|---------------------------------|---------------------------|----------------------------------------------|---------------------|
| Average AoI      | \(\limsup_{T \to \infty} \frac{1}{NT} \sum_{t=1}^{T} \left( \sum_{i=1}^{N} \mathbb{E}^{\pi}(h_i(t)) \right)\) | Yes                | \(\frac{N}{M_p(1-(1-M_p)^N)}\) (MMW) | \(\frac{1}{2Ng(\psi)} \left( \sum_{i=1}^{N} \mathbb{E}(\max_i h_i(t)) \right)^2 + \frac{1}{2}\) | 2                   |
| Peak AoI         | \(\limsup_{T \to \infty} \frac{1}{T} \mathbb{E}(\max_i h_i(t))\) | No                 | \(\sum_{i=1}^{N} \frac{1}{p_i}\) (CMA) | \(\sum_{i=1}^{N} \frac{1}{p_i}\) | 1 (Optimal)         |

A. Minimizing the Average AoI in the Stochastic Framework

1) Converse: The Average AoI minimization problem given by (20) is an example of an infinite horizon average-cost MDP with countably infinite state-space [44]. Apart from a few problems with special structures, infinite-state MDPs are notoriously difficult to solve exactly [45]. Moreover, the standard numerical approximation schemes for infinite-state MDPs do not typically provide theoretical performance guarantees [46]. In this section, we take a different approach to approximately solve the problem (20). We first obtain a fundamental lower bound to the objective achievable by any policy \(\pi \in \Pi\). Then, in Theorem 6 we show that a simple online scheduling policy \(\pi_{MMW}\) achieves the within a factor of 2 under certain assumptions on the mobility and channel statistics.

**Theorem 5 (Converse):** In the stationary setup, the objective in (20) is lower bounded as:

\[
\text{Aol}_{\text{avg}}^* \geq \frac{1}{2Ng(\psi)} \left( \sum_{i=1}^{N} \frac{1}{1/p_i} \right)^2 + \frac{1}{2},
\]

where the quantity \(g(\psi)\) denotes the expected number of cells with at least one user under the stationary occupancy distribution \(\psi\). In particular, since \(g(\psi) \leq \min\{M, N\}\), we also have the following sub-optimal lower bound which is agnostic of the user mobility statistics:

\[
\text{Aol}_{\text{avg}}^* \geq \frac{1}{2N \min\{M, N\}} \left( \sum_{i=1}^{N} \frac{1}{1/p_i} \right)^2 + \frac{1}{2}.
\]

**Proof:** Similar to our analysis of the static users in a single cell [26], we give a sample-path-based argument to obtain an almost sure lower bound to the objective (20). Finally, we use Fatou’s lemma [47] to convert the almost sure bound to a bound in expectation.

Consider a sample path under the action of any arbitrary scheduling policy \(\pi \in \Pi\). Let the r.v. \(N_i(T)\) denote the number of packets received by the \(i\)th user up to time \(T\). Also, let the r.v. \(T_{ij}\) denote the time interval between receiving the \((j-1)\)th packet and the \(j\)th packet, and the r.v. \(D_i\) denote the time interval between receiving the last \((N_i(T))\)th packet and the time-horizon \(T\) for the \(i\)th user. Hence, we can write

\[
T = \sum_{j=1}^{N_i(T)} T_{ij} + D_i.
\]

Since the AoI of any user increases by one at each slot until a new packet is received and then it drops to one again (please refer to Figure [5]), the average AoI up to time \(T\) may be lower bounded as:

\[
\text{AoI}_T \equiv \frac{1}{NT} \sum_{i=1}^{N} \sum_{j=1}^{T} h_i(t)
\]

\[
= \frac{1}{NT} \sum_{i=1}^{N} \left( \sum_{j=1}^{T} \frac{1}{2} T_{ij} T_{ij} + 1 \right) + \frac{1}{2} D_i (D_i + 1)
\]

\[
= \frac{1}{2NT} \sum_{i=1}^{N} \left( N_i(T) \left( \frac{1}{N_i(T)} \sum_{j=1}^{N_i(T)} T_{ij}^2 \right) + D_i^2 \right) + \frac{1}{2} \leq \frac{1}{2NT} \sum_{i=1}^{N} \left( N_i(T) \bar{T}_i^2 + D_i^2 \right) + \frac{1}{2},
\]

where in (a) we have used Eqn. (23), and in (b) we have defined \(\bar{T}_i \equiv \frac{1}{N_i(T)} \sum_{j=1}^{N_i(T)} T_{ij}\) and used Jensen’s inequality. Rearranging the Eqn. (23), we can express the random variable \(\bar{T}_i\) as:

\[
\bar{T}_i = \frac{T - D_i}{N_i(T)}.
\]

With this substitution, the term within the bracket in Equation (24) evaluates to

\[
N_i(T) \bar{T}_i^2 + D_i^2 \geq \frac{(T - D_i)^2}{N_i(T)} + D_i^2 \geq \frac{T^2}{N_i(T) + 1},
\]

where the last inequality is obtained by minimizing the resulting expression by viewing it as a quadratic in the variable \(D_i\).

Hence, from Eqns. (24) and (25), we obtain the following lower bound to the average AoI under the action of any admissible scheduling policy:

\[
\text{Aol}_T \geq \frac{T}{2N} \sum_{i=1}^{N} \frac{1}{N_i(T) + 1} + \frac{1}{2}.
\]

Next, we incorporate the scheduling constraints to lower bound the RHS of the inequality (26). Let the r.v. \(A_i(T)\) denote the total number of transmission attempts made to the \(i\)th user by all BS up to time \(T\). Also, let the r.v. \(g_j(T)\) denote the fraction of time that BS\(_j\) contained at least one user in its coverage area. Since, a BS can attempt a downlink transmission only when there is at least one user in its coverage area, the total number of transmission attempts to all users by the Base Stations is upper bounded by the following global balance condition:

\[
\sum_{i=1}^{N} A_i(T) \equiv T \sum_{j=1}^{M} g_j(T) \equiv T g(T),
\]

where \(g(T) \equiv \sum_j g_j(T)\). Using the inequality (27), we can further lower bound the inequality (26) as:

\[
\text{Aol}_T \geq \frac{1}{2Ng(T)} \left( \sum_{i=1}^{N} A_i(T) \right) \left( \sum_{i=1}^{N} \frac{1}{N_i(T) + 1} \right) + \frac{1}{2}.
\]
An application of the Cauchy-Schwartz inequality yields:

$$\text{AoI}_T \geq \frac{1}{2Ng(T)} \left( \sum_{i=1}^{N} \sqrt{\frac{A_i(T)}{N_i(T) + 1}} \right)^2 + \frac{1}{2}. \tag{28}$$

Note that, the $i$th user successfully received $N_i(T)$ packets out of a total of $A_i(T)$ packet transmission-attempts made by the Base Stations via the erasure channels with success probability $p_i$. Without any loss of generality, we may fix our attention on those scheduling policies only for which $\lim_{T \to \infty} A_i(T) = \infty, \forall i$. Otherwise, at least one of the users receive a finite number of packets, resulting in infinite average AoI. Hence, using the Strong law of large numbers [47], we obtain:

$$\lim_{T \to \infty} N_i(T) = A_i(T), \forall i \text{ w.p. 1.} \tag{29}$$

Moreover, using the ergodicity property of the user mobility, we conclude that almost surely:

$$\lim_{T \to \infty} g_j(T) = \mathbb{P}_\psi(\text{BS}_j \text{ contains at least one user}),$$

where we recall that $\psi$ denotes the stationary cell occupancy distribution. Thus, we have almost surely

$$\lim_{T \to \infty} g(T) = \lim_{T \to \infty} \sum_j g_j(T) = \sum_{j=1}^{M} \mathbb{P}_\psi(\text{BS}_j \text{ contains at least one user}) = g(\psi), \tag{30}$$

where the function $g(\psi)$ denotes the expected number of non-empty cells where the expectation is evaluated w.r.t. the stationary occupancy distribution $\psi$. Hence, combining equations (29) and (30) together with the lower bound in (28), we have almost surely:

$$\liminf_{T \to \infty} \text{AoI}_T \geq \frac{1}{2Ng(\psi)} \left( \sum_{i} \sqrt{\frac{1}{p_i}} \right)^2 + \frac{1}{2}. \tag{31}$$

Finally,

$$\text{AoI}^* \geq \liminf_{T \to \infty} \mathbb{E}(\text{AoI}_T) \overset{(a)}{=} \mathbb{E}(\liminf_{T \to \infty} \text{AoI}_T) \geq \frac{1}{2Ng(\psi)} \left( \sum_{i} \sqrt{\frac{1}{p_i}} \right)^2 + \frac{1}{2},$$

where the inequality (a) follows from Fatou’s lemma. This concludes the proof of Theorem 5. Note that the proof continues to hold even when the mobility of the users are not independent of each other.

Discussion: Theorem 5 gives a universal lower bound to the optimal average AoI achievable by any admissible scheduling policy $\pi \in \Pi$. Interestingly, it reveals that user mobility enters in the lower bound only through the stationary cell-occupancy distribution $\psi$. Hence, given the stationary distribution $\psi$, the lower bound (22) is agnostic of the details of the mobility model. A similar phenomenon has been observed earlier for the capacity region of wireless networks (see [48], Corollary 5, p. 88). The appearance of the quantity $g(\psi)$ in the lower bound should not be surprising as it denotes the typical number of non-empty cells at a slot in the long run. Since a BS can transmit a packet only if at least one user is present in its coverage area, the quantity $g(\psi)$, in some sense, represents the multi-user diversity of the system.

Closed-form expression for $g(\psi)$: To get a sense of the bound (22), we now work out a closed-form expression for $g(\psi)$. Using the linearity of expectation, we have

$$g(\psi) = \mathbb{E}_\psi \sum_{j=1}^{M} \mathbbm{1}(\text{BS}_j \text{ contains at least one user}) = \sum_{j=1}^{M} \mathbb{P}_\psi(\text{BS}_j \text{ contains at least one user}). \tag{32}$$

Since the cells are disjoint, we readily conclude from (32) that $g(\psi) \leq \min\{M, N\}$. Recall that $\psi_{ij}$ denotes the marginal probability that the $i$th user is in BS$_j$. If the mobility of the users are independent of each other, the expected number of non-empty cells $g(\psi)$ in Eqn. (32) simplifies to:

$$g(\psi) = \sum_{j=1}^{M} (1 - \mathbb{P}_\psi(\text{BS}_j \text{ contains at least one user})). \tag{33}$$

We now evaluate the above expression for the case when the limiting occupancy distribution of each user is uniform across all BSs, i.e., $\psi_{ij} = \frac{1}{M}, \forall i, j$. The uniform stationary distribution arises, for example, when the user mobility can be modelled as a random walk on a regular graph [49]. In this case, Eqn. (33) simplifies to:

$$g(\psi^{\text{unif}}) = M \left( 1 - \left( 1 - \frac{1}{M} \right)^N \right). \tag{34}$$

For $M = 1$, we have $g(\psi) = 1$. We now derive the bounds for $M \geq 2$. For $M \geq 2$, we have the following bounds:

$$e^{-\beta} \overset{(a)}{\leq} \left( 1 - \frac{1}{M} \right)^{\frac{1}{N}} \overset{(b)}{\leq} e^{-\frac{1}{M}}, \tag{35}$$

where $\beta \equiv \log(4) \leq 1.387$. The inequality (b) is standard. To prove the inequality (a), consider the concave function

$$f(x) = 1 - x - e^{-\beta x}, 0 \leq x \leq \frac{1}{2}$$

for some $\beta > 0$. Since a concave function of a real variable defined on an interval attains its minima at one of the end points of the closed interval, and since $f(0) = 0$, we have $f(x) \geq 0, \forall x \in [0, \frac{1}{2}]$, if $f(1/2) \geq 0, i.e., e^{\beta/2} \geq 2$, i.e., $\beta \geq \ln(4)$. Thus, the inequality (a) holds for $M \geq 2$ with $\beta = \ln(4)$. The inequality (35) directly leads to the following bounds for $M \geq 2$:

$$M \left( 1 - e^{-\frac{N}{M}} \right) \leq g(\psi^{\text{unif}}) \leq M \left( 1 - e^{-1.387 \frac{N}{M}} \right). \tag{36}$$

2) Achievability: We now propose an online scheduling policy $\pi^{\text{MMW}}$ which approximately minimizes the average AoI for mobile users (the abbreviation MMW stands for “Multi-cell Max-Weight”).
The policy $\pi^{\text{MMW}}$. At every slot, each BS schedules a user under its coverage that has the highest index among all other users. The index $I_i(t)$ of the $i^{th}$ user is defined as $I_i(t) = p_i h_{i}^2(t)$.

Note that the throughput-optimal policy, which always schedules the user having the highest success probability (i.e., $\arg\max_i p_i$), does not work and may result in unbounded AoI. Our proposed policy is a multi cell generalization of the 4-approximate single cell scheduling policy $\text{MW}$ proposed in [26]. In Theorem 6 we show that $\pi^{\text{MMW}}$ is a 2-approximation policy for identical users with i.i.d. uniform mobility, where $p_i = p$, and $\psi_{ij} = \frac{1}{M}, \forall i \in [N], j \in [M]$ as well as for arbitrary stationary users with different success probabilities $p_i$’s located in a single cell. Hence, our result in this section strictly improves upon the 4-approximation guarantee of the $\text{MW}$ policy for stationary non-identical users [26]. Our result also complements Theorem 5 of [26], where the authors showed that the $\text{MMW}$ policy is exactly optimal for the case of identical stationary users at a single cell.

**Theorem 6 (Achievability):** $\pi^{\text{MMW}}$ is a 2-approximation scheduling policy for

1) stationary users in a single cell with arbitrary success probabilities $\{p_i, 1 \leq i \leq N\}$ and
2) statistically identical users with i.i.d. uniform mobility (i.e., when $p_i = p_j, \psi_{ij} = \frac{1}{M}, \forall i, j$ and the r.v.s $C_i(t)$’s are independent across time).

Note that scenarios (1) and (2) represent two extreme ends of the user mobility landscape. Scenario (1) corresponds to zero mobility, and scenario (2) corresponds to infinite mobility, which has been considered earlier in [16].

**Proof:** Let the scheduling decisions at slot $t$ be denoted by the binary control vector $\mu(t) \in \{0, 1\}^M$, where $\mu_i(t) = 1$ if and only if the following two conditions hold simultaneously: (1) $C_i(t) = j$, i.e., the $i^{th}$ user is within the coverage area of the $j^{th}$ BS at slot $t$, for some $1 \leq j \leq M$, and (2) BS $j$ schedules a packet transmission to the $i^{th}$ user at time $t$. Since a BS can schedule only one transmission per slot to a user in its coverage area, the control vector must satisfy the following constraint:

$$\sum_{i \in C_t(t)=j} \mu_i(t) \leq 1, \quad \forall j, t.$$

For performance analysis, we consider the following Lyapunov function, which is linear in the ages of the users:

$$L(h(t)) = \sum_{i=1}^{N} \frac{h_i(t)}{\sqrt{p_i}}. \quad (37)$$

The above linear Lyapunov function should be compared with the quadratic Lyapunov function used in [26]. The conditional transition probabilities for the age of the $i^{th}$ user may be written as follows:

$$P(h_i(t+1) = 1|h(t), \mu(t+1), C(t+1)) = \mu_i(t+1)$$

$$P(h_i(t+1) = h_i(t) + 1|h(t), \mu(t+1), C(t+1)) = 1 - \mu_i(t+1).$$

where the first conditional probability corresponds to the event $S_i$ when the $i^{th}$ user was scheduled and the packet transmission was successful, and the second equation gives the conditional probability of the complement event $S_i^c$. Hence, for each user $i \in [N]$, we have:

$$\mathbb{E}(h_i(t+1)|h(t), \mu(t+1), C(t+1)) = h_i(t) - \mu_i(t+1)p_i h_i(t) + 1. \quad (38)$$

From the equation above, we can evaluate the one-step conditional drift as follows:

$$\mathbb{E}(L(h(t+1)) - L(h(t))|h(t), \mu(t+1), C(t+1)) = -\sum_{i=1}^{N} \mu_i(t+1) \frac{1}{\sqrt{p_i}} + \frac{1}{\sqrt{p_i}}. \quad (39)$$

Finally, consider the policy Multi-Cell MW ($\pi^{\text{MMW}}$), under which, each Base Station $BS_j$ schedules the user $i$ having the highest weight $\sqrt{p_i}h_i(t)$ among all users in $BS_j$’s cell. Next, define a stationary randomized scheduling policy $\text{RAND}$, under which every BS randomly schedules a user in its cell with probability $\mu_i^{\text{RAND}}(t+1) \propto 1/\sqrt{p_i}$. Using the fact that maximum of a set of real numbers is at least as large as their any convex combination, we have

$$\mathbb{E}\left(\sum_{i=1}^{N} \mu_i^{\text{MMW}}(t+1) \sqrt{p_i}h_i(t)|h(t), \mu(t+1), C(t+1)\right) \geq \sum_{j=1}^{M} \frac{\sum_{i \in C_t(t+1)=j} h_i(t)}{\sum_{i \in C_t(t+1)=j} \frac{1}{\sqrt{p_i}}}. \quad (40)$$

This leads to the following upper-bound of the drift $\mathbb{E}\mathbb{E}^{\text{MMW}}$ under the MMW policy:

$$\mathbb{E}^{\text{MMW}}(L(h(t+1)) - L(h(t))|h(t), C(t+1)) \leq -\sum_{j=1}^{M} \frac{\sum_{i \in C_t(t+1)=j} h_i(t)}{\sum_{i \in C_t(t+1)=j} \frac{1}{\sqrt{p_i}}} + \frac{1}{\sqrt{p_i}}. \quad (40)$$

Taking expectation of the above drift-inequality w.r.t. the random cell-occupancy vector $C(t+1)$, we have

$$\mathbb{E}^{\text{MMW}}(L(h(t+1)) - L(h(t))|h(t)) \leq -\sum_{j=1}^{M} \mathbb{E}(Z_j(t)|h(t)) + \frac{1}{\sqrt{p_i}}. \quad (41)$$

where $Z_j(t) \equiv \frac{\sum_{i \in C_t(t+1)=j} h_i(t)}{\sum_{i \in C_t(t+1)=j} \frac{1}{\sqrt{p_i}}}$, where we define $0/0 \equiv 0$. This is consistent, because if a cell is empty, the corresponding term is not present in the summation [41]. We start with the simplest case.

\(^2\)We use the usual convention that summation over an empty set is zero.
CASE I: STATIONARY USERS IN A SINGLE CELL. In this case, we have \( M = 1 \), and hence,

\[
\mathbb{E}_{\text{MMW}}^{\text{MMW}}(L(h(t+1)) - L(h(t))) 
\leq - \frac{1}{N} \sum_{i=1}^{N} \frac{h_i(t)}{\sqrt{P_i}} + \frac{1}{P} \sum_{i=1}^{N} \frac{1}{\sqrt{P_i}}.
\]

Taking expectations of both sides w.r.t \( h(t) \), we obtain

\[
\mathbb{E}(L(h(t+1)) - L(h(t))) 
\leq - \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}(h_i(t)) + \frac{1}{N} \left( \sum_{i=1}^{N} \frac{1}{\sqrt{P_i}} \right)^2.
\]

Comparing the above with the lower bound in Eqn. (22) and realizing that \( g(\psi) = 1 \) in this case, we have

\[
\text{AoI}_{\text{avg}}^{\text{MMW}} \leq 2\text{AoI}_{\text{avg}}^{*}.
\]

CASE II: IDENTICAL USERS WITH I.I.D. UNIFORM MOBILITY. In the case of i.i.d. uniform mobility, the users move to any one of the \( M \) Base Stations independently and uniformly at random. This model is appropriate for high-speed users. We start from the drift upper bound given in equation (41). Note that, given the age vector \( h(t) \), with i.i.d. uniform mobility, the r.v.s \( Z_j(t) \)'s are identically distributed. Hence,

\[
\sum_{j=1}^{M} \mathbb{E}(Z_j(t)|h(t)) = M \mathbb{E}(Z_1(t)|h(t)).
\]

The RHS can be expressed as the expectation of the ratio of two correlated r.v.s

\[
\mathbb{E}(Z_1(t)|h(t)) = \mathbb{E}_h(t) \left[ \frac{\sum_{i=1}^{N} h_i(t) W_i}{\sum_{i=1}^{N} \frac{1}{\sqrt{P_i}} W_i} \right],
\]

where the indicator r.v.s \( \{W_i\}_{i=1}^{N} \) are i.i.d. such that

\[
P(W_i = 1) = \frac{1}{M} = 1 - P(W_i = 0).
\]

In general, it is non-trivial to obtain a tight lower bound to the expression (42), which is the expectation of the ratio of two correlated random variables. In the following, we consider the special case of statistically identical users with \( p_i = p, \forall i \), and evaluate the expectation exactly.

Note that, we can express the random variable \( \sum_{j=1}^{M} Z_j(t) \) as

\[
\sum_{j=1}^{M} Z_j(t) = \sum_{i=1}^{N} h_i(t) Y_i(t),
\]

where \( Y_i(t) = \left( \frac{1}{\sqrt{P_i}} + \sum_{k+i} \frac{1}{\sqrt{P_k}} I(C_i(t) = C_k(t)) \right)^{-1} \). Given our i.i.d. mobility assumption, the r.v. \( C(t+1) \) is independent of \( h(t) \). Hence,

\[
\mathbb{E}(Y_i(t)|h(t)) = \sum_{n=0}^{N-1} \sum_{S_i \neq S_j} \left( \frac{1}{\sqrt{P_i}} + \sum_{k+i} \frac{1}{\sqrt{P_k}} \right)^{-1} \times \frac{1}{M^n} \left( 1 - \frac{1}{M} \right)^{N-n-1}.
\]

Since the users are identical, i.e., \( p_i = p, \forall i \), the summation (43) has a closed-form expression. Clearly, for all \( 0 \leq n \leq N - 1 \), we have:

\[
Y_i(t) = \sqrt{p} \frac{1}{n+1}, \quad \text{w.p.} \left( \frac{N-1}{n} \left( 1 - \frac{1}{M} \right)^{N-n-1} \right).
\]

To evaluate the expectation of \( Y_i(t) \), we integrate the binomial expansion of \( (1 + x)^N \) in the range \([0, \beta]\) to obtain the identity:

\[
\frac{1}{N} \left( 1 + \beta \right)^N - 1 = \beta \sum_{n=0}^{N-1} \frac{1}{n+1} \left( \frac{N-1}{n} \right) \beta^n.
\]

Substituting \( \beta = \frac{1}{\sqrt{p}} \) in the above, we obtain

\[
\mathbb{E}(Y_i(t)|h(t)) = \sqrt{p} \frac{1}{n+1} \left( 1 - \left( 1 - \frac{1}{M} \right)^N \right) \equiv Y^*(\text{say}).
\]

From Eqns. (41) and (44), we have

\[
\mathbb{E}_{\text{MMW}}\left( L(h(t+1)) - L(h(t)) \right) \leq -Y^* \sum_i h_i(t) + \frac{N}{\sqrt{p}}.
\]

Taking expectation of both sides, we have

\[
\mathbb{E}_{\text{MMW}}\left( L(h(t+1)) - L(h(t)) \right) \leq - Y^* \mathbb{E}_h(t) + \frac{N}{\sqrt{p}}.
\]

Summing up the above inequalities for \( t = 1, 2, \ldots, T \), dividing both sides by \( T \) and then taking limit as \( T \to \infty \), we obtain

\[
\text{AoI}_{\text{MMW}} = \limsup_{T \to \infty} \frac{1}{NT} \sum_{t=1}^{T} \sum_i h_i(t) \leq \frac{N}{Y^* \sqrt{p}} \equiv \frac{N}{Mp \left( 1 - \left( 1 - \frac{1}{M} \right)^N \right)}.
\]

On the other hand, the lower bound from Theorem 3 specialized to the case of identical users, yields:

\[
\text{AoI}^* \geq \frac{N}{2Mp \left( 1 - \left( 1 - \frac{1}{M} \right)^N \right)}.
\]

Eqs. (45) and (46), we have

\[
\text{AoI}_{\text{MMW}} \leq 2\text{AoI}^*.
\]

The above result shows that the policy \text{MMW} is 2–optimal in the case of statistically identical users with uniform i.i.d. mobility. ■
B. Minimizing the Peak-AoI in the Stochastic Framework

In this section, we consider the problem of minimizing the long-term peak-AoI metric \(\rho_i\), for static users in a single cell. By directly solving the associated countable state Bellman equation, we prove that the greedy CMA scheduling policy is optimal for minimizing the Peak-AoI. Furthermore, we also establish the large-deviation optimality of the CMA policy.

**Theorem 7 (Optimal Policy for Peak-Age):** The greedy CMA policy is optimal for the problem \((\rho_i)\). Moreover, the optimal long-term peak-AoI is given by \(\lambda^* = \sum_{i=1}^{N} \frac{1}{p_i}\).

We prove this theorem by proposing a closed-form candidate solution of the Bellman’s equation of the associated average-cost MDP and then verifying that the candidate solution indeed satisfies the Bellman’s equation.

**Proof:** The stochastic control problem under investigation is an instance of a countable-state average-cost MDP with a finite action space. The state \(h(t)\) of the system at a slot \(t\) given by the current AoI vector of all users, i.e., \(h(t) \equiv (h_1(t), h_2(t), \ldots, h_N(t))\). The per-stage cost at time \(t\) is \(\max_{i=1}^{N} h_i(t)\), which is unbounded, in general. Finally, the finite action space \(A = \{1, 2, \ldots, N\}\) corresponds to the index of the user scheduled at a given slot.

Let the optimal cost for the problem \(P_{sched}\) be denoted by \(\lambda^*\) and the differential cost-to-go from the state \(h\) be denoted by \(V(h)\). Then, following the standard theory of average cost countable state MDP (Proposition 4.6.1 of \cite{50}), we set up the following Bellman Eqn.

\[
\begin{align*}
\lambda^* + V(h) &= \min_i \{p_i V(h^{-1}_i + 1) + (1 - p_i) V(h + 1)\} \\
&\quad + \max_i h_i,
\end{align*}
\]

where using the standard notation, the vector \(h^{-1}_i\) denotes the \(N - 1\) dimensional vector of all coordinates excepting the \(i^{th}\) coordinate and \(1\) is a all-one vector of appropriate dimension.

**Discussion:** The Bellman Equation (47) may be derived as follows. Suppose that the current age of the users is given by the vector \(h\). If the policy schedules a transmission to the \(i^{th}\) user, the transmission is successful with probability \(p_i\) and is unsuccessful with probability \(1 - p_i\). If the transmission is successful, the AoI of all users, excepting the \(i^{th}\) user, is incremented by 1, and the AoI of the \(i^{th}\) user is reduced to 1. This explains the first term. On the other hand, if the transmission to the \(i^{th}\) user is unsuccessful, the AoI of all users are incremented by 1. This explains the second term within the bracket. Finally, the term \(\max_i h_i\) denotes the stage cost.

**Solution to the Bellman Equation (47):** We verify that the following constitutes a solution to the Bellman Equation (47):

\[
V(h) = \sum_j h_j p_j, \quad \lambda^* = \sum_j \frac{1}{p_j}.
\]

To verify the above solution, we start with the RHS of (47). Upon substitution from Eqn. (48), the expression corresponding to the \(i^{th}\) user inside the min operator of Eqn. (47) is simplified to:

\[
\begin{align*}
p_i V(h^{-1}_i + 1) &= p_i \sum_j h_j + 1 + (1 - p_i) V(h + 1) \\
&= p_i \sum_j h_j + 1 + p_i - 1 \sum_j h_j + 1 + (1 - p_i) \sum_j h_j + 1 \\
&= \sum_j h_j - h_i + \sum_j \frac{1}{p_j},
\end{align*}
\]

Hence,

\[
\begin{align*}
\text{RHS} &= \min_i \{p_i V(h^{-1}_i + 1) + (1 - p_i) V(h + 1)\} + \max_i h_i \\
&= \sum_j h_j + 1 - \max_i h_i + \max_i h_i \\
&= \lambda^* + V(h) \\
&= \text{LHS}.
\end{align*}
\]

Finally, to verify the regularity condition (Eqn. 4.122 of \cite{50}), note that

\[
\frac{\mathbb{E}(V(h(t))|h(1))}{t} = \frac{1}{t} \mathbb{E} \left[ \sum_{j=1}^{N} h_j(t) / p_j | h(1) \right] \\
\leq \mathbb{E} \left[ \max_i h_i(t) / t | h(1) \right] \left( \sum_i 1 / p_i \right).
\]

Hence, for any scheduling policy \(\pi\) for which the regularity condition is violated, i.e., \(\lim_{t \to \infty} \frac{\mathbb{E}(V(h(t))|h(1))}{t} > 0\), for some \(h(1)\) with positive probability, the above bound implies that

\[
\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}^\pi[ \max_i h_i(t) ] = \infty.
\]

Hence, without any loss of optimality, we can confine our attention to those policies for which the regularity condition holds. The optimality result now follows from Proposition 4.6.1 of \cite{50}.

**Large-Deviation Optimality:** Theorem 7 establishes that the greedy CMA scheduling policy is optimal for minimizing the long-term expected peak-AoI. However, for mission-critical URLLC applications, the scheduling policy is additionally required to ensure that the peak-AoI metric stays within limit with a high probability after sufficiently long time. Making use of the notion of large-deviations \cite{51}, this problem can be succinctly formulated as follows: Design a scheduling policy \(\pi\) which maximizes the large deviation exponent, i.e.,

\[
\max_{\pi \in \Pi} \left\{ -\lim_{k \to \infty} \lim_{t \to \infty} \frac{1}{k} \log \mathbb{P}^\pi(\max_{i=1}^{N} h_i(t) \geq k) \right\}.
\]

The following theorem shows that the CMA policy is also optimal in the large-deviation sense.
This paper investigates the fundamental limits of Age-of-Information for mobile users in adversarial and stochastic environments and proposes provably near-optimal scheduling policies. We showed that the greedy scheduling policy works reasonably well in the adversarial setting. Using Yao’s principle, we derived fundamental lower bounds for the competitive ratios in the adversarial model. We showed that a Max-Weight policy is 2-optimal for two extreme mobility
scenarios in the stochastic setting. The problem of designing a peak-AoI optimal policy for mobile users in the stochastic environment is open and maybe investigated in the future. Designing optimal algorithms in the adversarial setting when some approximate channel state information is known (due to channel estimations) is an interesting open problem.

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VI. APPENDIX

A. Proof of Theorem 2

Proof: We prove the theorem by exhibiting a channel state sequence for which the CMA policy achieves a competitive ratio of at least $N^2$ for the average AoI objective and at least $2N - 1$ for the peak AoI objective.

Consider a single-cell scenario where a BS serves $N$ stationary users. In this case, it is easy to see that the CMA policy reduces to a persistent round-robin policy - the users are scheduled in a round-robin fashion such that each user is scheduled continuously until its transmission is successful. Consider the following channel-state sequence: all super-intervals are of constant length $\Delta \gg N$, where we take $\Delta \equiv 1 \mod (N - 1)$. During each super-interval, all users, apart from the Max-user, have Good channels at every slot. Hence, under the CMA policy, the time interval between two consecutive successful packet transmissions to any user is $N\Delta$ slots. Thus, at the beginning of any super-interval (apart from the first $N - 1$ super-intervals) the ages of the users under the CMA policy in ascending order is given by $1, \Delta + 1, 2\Delta + 1, \ldots, (N - 1)\Delta + 1$. Next, we consider the following two objectives.

a) Avg-AoI: The total cost incurred by the CMA policy in any interval (apart from the first $N - 1$ super-intervals) is given by:

$$C_i^{\text{CMA}} = \sum_{j=1}^{N} \sum_{k=1}^{\Delta} ((j - 1)\Delta + k) = \frac{1}{2}(N^2\Delta^2 + N\Delta).$$

Next, we upper-bound the cost incurred by the offline optimal policy $\text{OPT}$ by comparing it to another (potentially sub-optimal) offline policy $\mathcal{P}$. The policy $\mathcal{P}$ serves each of the $N - 1$ users other than the Max-user in a round-robin fashion in each super-interval and finally, it serves the Max-user at the last time slot of each super-interval (See Figure 4). Clearly, under the action of the policy $\mathcal{P}$, the set of ages of the users at the beginning of every super-interval is given by $\{1, 2, \ldots, N\}$. Thus the total cost incurred by the policy $\mathcal{P}$ during the $i$th super-interval is bounded as:

$$C_i^{\mathcal{P}} \leq \sum_{k=1}^{\Delta} k + (N - 1)\left(\left(\frac{\Delta - 1}{N - 1} - 1\right)\frac{N(N - 1)}{2} + \frac{3N(N - 1)}{2}\right) + \frac{\Delta^2}{2} + \frac{\Delta N^2}{2} + N^3. \quad (51)$$

Let $K$ be the number of super-intervals in the time-horizon $T$. We have

$$\text{Cost}^{\text{OPT}}(T) \leq \sum_{i=1}^{K} C_i^{\mathcal{P}}.$$

Hence, for large enough $K$, the competitive ratio of the CMA policy is lower bounded as:

$$\eta_{\text{avg}}^{\text{CMA}} \geq \frac{\sum_{i=1}^{K} C_i^{\text{CMA}}}{\sum_{i=1}^{K} C_i^{\text{OPT}}} \geq \frac{1}{2}(N^2\Delta^2 + N\Delta).$$

By taking $\Delta$ to be arbitrarily large, it follows from the above expression that

$$\eta_{\text{avg}}^{\text{CMA}} \geq N^2.$$

b) Peak-AoI: We compute the competitive ratio for the same channel state sequence as before for the peak-AoI objective. The cost incurred by the CMA policy during any super-interval is:

$$C_i^{\text{CMA}} = \sum_{k=1}^{\Delta} ((N - 1)\Delta + k) = (N - 1)\Delta^2 + \frac{\Delta(\Delta + 1)}{2}. \quad (52)$$

Similar to the Average-AoI case, we use the policy $\mathcal{P}$ to upper-bound the cost incurred by the OPT policy for the given channel state sequence. As before, the total cost incurred by the policy $\mathcal{P}$ during the $i$th super-interval is bounded as:

$$C_i^{\mathcal{P}} \leq \sum_{k=1}^{\Delta} (N + k) = N\Delta + \frac{\Delta(\Delta + 1)}{2}.$$

Thus, summing over all super-intervals, we obtain

$$\eta_{\text{peak}}^{\text{CMA}} \geq \frac{\sum_{i=1}^{K} C_i^{\text{CMA}}}{\sum_{i=1}^{K} C_i^{\text{OPT}}} \geq \frac{(N - 1)\Delta^2 + \frac{\Delta(\Delta + 1)}{2}}{N\Delta + \frac{\Delta(\Delta + 1)}{2}}.$$

The result now follows by taking the length of the sub-intervals $\Delta$ to be arbitrarily large.

\[\qed\]

B. Proof of Proposition 2

Proof: Define $\mathcal{F}_{t-1} = \sigma(\hat{h}(k), \mu(k), 1 \leq k \leq t - 1)$ to be the sigma-algebra generated by the r.v.s of age and control vectors observed up to time $t - 1$. Since the policy is online, the scheduling decision $\mu(t)$ at time $t$ must be measurable in $\mathcal{F}_{t-1}$ for all $t \geq 1$. Let $H_{\text{sum}}(t) = \mathbb{E}[(h_1(t) + h_2(t))]$ be the expected sum of the ages of the users at time $t$. Let
$B_t \in \mathcal{F}_t$ be the event for which the user 1 is scheduled under the policy $\pi$. Then, we can write

$$\mathbb{E}^\pi(h_1(t+1)|\mathcal{F}_t)$$

\begin{align}
&= (1 + \frac{1}{2}h_1(t))\mathbb{1}(B_t) + (1 + h_1(t))\mathbb{1}(B^c_t) \\
&= 1 + \frac{1}{2}h_1(t) + \frac{1}{2}h_1(t)\mathbb{1}(B^c_t) \\
&\geq 1 + \frac{1}{2}h_1(t) + \frac{1}{2}\min\{h_1(t), h_2(t)\}\mathbb{1}(B^c_t),
\end{align}

(54)

Similarly, we can also write

$$\mathbb{E}^\pi(h_2(t+1)|\mathcal{F}_t) \geq 1 + \frac{1}{2}h_2(t) + \frac{1}{2}\min\{h_1(t), h_2(t)\}\mathbb{1}(B^c_t).$$

(55)

Since $\mathbb{1}(B_t) + \mathbb{1}(B^c_t) = 1$, from the equations (53) and (55), we have

$$\mathbb{E}^\pi(h_1(t+1) + h_2(t+1)|\mathcal{F}_t) \geq 2 + \frac{1}{2}(h_1(t) + h_2(t)) + \frac{1}{2}\min\{h_1(t), h_2(t)\}.$$ 

Taking expectations of both sides of the above equation, we get

$$H_{\text{sum}}(t+1) \geq 2 + \frac{1}{2}H_{\text{sum}}(t) + \frac{1}{2}\mathbb{E}(S(t)).$$

(56)

Let the random variable $S(t)$ denote the time elapsed since the last successful transmission (by any user) before time $t$. Clearly,

$$\min\{h_1(t), h_2(t)\} \geq S(t)$$

(the above inequality holds with equality for the two user case). Hence, the above inequality implies

$$H_{\text{sum}}(t+1) \geq 2 + \frac{1}{2}H_{\text{sum}}(t) + \frac{1}{2}\mathbb{E}(S(t)).$$

Summing up the above inequalities for $t = 1, 2, \ldots, T$, and dividing both sides by $T$, we obtain

$$2\frac{H_{\text{sum}}(T+1)}{T} + \frac{1}{T}\sum_{t=1}^TH_{\text{sum}}(t) \geq 4 + \frac{1}{T}\sum_{t=1}^T\mathbb{E}(S(t)).$$

(57)

It is to be noted that $\{S(t)\}_{t \geq 1}$ is a renewal process with the time-stamp of successful transmissions constituting the renewal instants. Let the random variable $\tau$ denote the length of any generic renewal cycle. Hence, using the renewal reward theorem $^{38}^{52}$, it follows that

$$\lim_{T \to \infty} \frac{1}{T}\sum_{t=1}^T\mathbb{E}(S(t)) = \frac{\mathbb{E}\left(\int_0^{\tau} S(t) dt\right)}{\mathbb{E}(\tau)} = \frac{\mathbb{E}(1 + 2 + \ldots + \tau)}{\mathbb{E}(\tau)} = \frac{\mathbb{E}(\tau^2) + \mathbb{E}(\tau)}{2\mathbb{E}(\tau)} = 2,$$

where the last inequality follows from the fact that the renewal cycle lengths $T$ are distributed geometrically with the parameter $p = 1/2$. Thus, the limit of the RHS of Eqn. (57) exists and the limiting value is equal to 6. Next, we consider two possible cases.

Case I: $\liminf_{T \to \infty} \frac{H_{\text{sum}}(T+1)}{T} = 0$: In this case, consider a subsequence $\{T_k\}_{k \geq 1}$ along which $\lim_{k \to \infty} \frac{H_{\text{sum}}(T_k+1)}{T_k} = 0$.

For this subsequence, we have from Eqn. (57):

$$2\frac{H_{\text{sum}}(T_k+1)}{T_k} + \frac{1}{T_k}\sum_{t=1}^{T_k}H_{\text{sum}}(t) \geq 4 + \frac{1}{T_k}\sum_{t=1}^{T_k}\mathbb{E}(S(t)).$$

Taking $k \to \infty$, we conclude that

$$\limsup_{T \to \infty} \frac{1}{T}\sum_{t=1}^T H_{\text{sum}}(t) \geq 6.$$ 

(58)

Case II: $\liminf_{T \to \infty} \frac{H_{\text{sum}}(T+1)}{T} = \alpha > 0$: From the definition of $\liminf$, it follows that there exists a finite $T_0$ such that, for all $T \geq T_0$, we have

$$\frac{H_{\text{sum}}(T+1)}{T} \geq \frac{\alpha}{2}. $$

(59)

Thus, for any $T \geq T_0$, we can write

$$\frac{1}{T}\sum_{t=1}^T H_{\text{sum}}(t) \geq \frac{\alpha}{2T}\sum_{t=T_0+1}^T H_{\text{sum}}(t) \geq \frac{\alpha}{2}\sum_{t=T_0}^{T-1} t = \Omega(T).$$

Hence, in this case, we have

$$\limsup_{T \to \infty} \frac{1}{T}\sum_{t=1}^T H_{\text{sum}}(t) = \infty.$$ 

Hence, from Eqns. (58) and (60), we conclude that, in either case, we have

$$\limsup_{T \to \infty} \frac{1}{T}\sum_{t=1}^T H_{\text{sum}}(t) \geq 6.$$ 

(60)

Thus, In the case when $N = 2$, using the result of Proposition $^{2}$ the competitive ratio is lower bounded by

$$\eta(2) \geq \frac{6}{2^2} = 1.5.$$ 

Nevertheless, the achievability result in Theorem $^{1}$ shows that the bound in Eqn. (13) is tight within a factor of 2. In particular, Eqn. (13) has the order-optimal dependence on $N$.

\section{C. Proof of Theorem $^3$}

\textbf{Proof:} Let $i^* = \arg \min_i p_i$. Now, under the action of any arbitrary policy $\pi$, at any slot $t \geq k$ and for all $k \geq 1$, we have

$$\mathbb{P}^\pi_i(\max_{i} h_i(t) \geq k) \geq \mathbb{P}^\pi_i(h_{i^*}(t) \geq k) \geq (1 - p_{\text{min}})^k,$$ 

(61)

where the inequality (a) follows from the fact that consecutive $k$ erasures just prior to time $t$ for the $i^*$ th user (which occurs with probability $(1 - p_{\text{min}})^k$) ensures that the age of the user at time $t$ is at least $k$.

Next, we analyze the large-deviation exponent under the action of the CMA policy. Using the union bound, we have

$$\mathbb{P}(\max_i h_i(t) \geq k) \leq \sum_{i=1}^N \mathbb{P}(h_i(t) \geq k).$$ 

(62)

Now, for any user $i$, the event $h_i(t) \geq k$ occurs if and only if it has been at least $k$ slots since the $i$ th user received a packet successfully before time $t$. Define $p_{\text{max}} \equiv \max_i p_i$ and $p_{\text{min}} \equiv \min_i p_i$. Since the CMA policy transmits other users
follows that, during the last $k$ slots prior to time $t$, at most $N - 1$ users have successfully received a packet. Thus, we have the following bound:

$$\Pr(h_i(t) \geq k) \leq \left( \frac{k}{N-1} \right) (1 - p_{\text{min}})^k \times \left( \frac{1 - p_{\text{min}}}{p_{\text{max}} + p_{\text{min}}} \right) \left( \frac{p_{\text{max}}}{1 - p_{\text{min}}} \right)^N - 1 \right)$$

where we have used the bound\(^\text{(1)}\) and defined

\begin{equation}
\epsilon'(N, p) = \left( \frac{k}{N-1} \right) (1 - p_{\text{min}})^k \times \left( \frac{1 - p_{\text{min}}}{p_{\text{max}} + p_{\text{min}}} \right) \left( \frac{p_{\text{max}}}{1 - p_{\text{min}}} \right)^N - 1 \right) \end{equation}

Combining the above bound with Eqs. (62) and (61), we conclude that the CMA policy is optimal for the problem (50) and

$$- \lim_{k \to \infty} \frac{1}{k} \log \Pr[i_{\text{CMA}}(\max_i h_i(t) \geq k) = - \log(1 - \min_{i=1}^N p_i)] .$$

\section*{References}

[1] A. Srivastava, A. Sinha, and K. Jagannathan, “On minimizing the maximum age-of-information for wireless erasure channels,” in 2019 International Symposium on Modeling and Optimization in Mobile Ad Hoc, and Wireless Networks (WiOPT), 2019, pp. 1–6.

[2] S. Banerjee, R. Bhattacharjee, and A. Sinha, “Fundamental limits of age-of-information in stationary and non-stationary environments,” in 2020 IEEE International Symposium on Information Theory (ISIT), 2020, pp. 1741–1746.

[3] R. Bhattacharjee and A. Sinha, “Competitive algorithms for minimizing the maximum age-of-information,” arXiv preprint arXiv:2005.05873 (appeared on the proceedings of Mathematical performance Modeling and Analysis (MAMA)), 2020.

[4] L. Tassiulas and A. Ephremides, “Stability properties of constrained queueing systems and scheduling policies for maximum throughput in multihop radio networks,” Automatic Control, IEEE Transactions on, vol. 37, no. 12, pp. 1936–1948, 1992.

[5] A. Mandelbaum and A. L. Stolyar, “Scheduling flexible servers with convex delay costs: Heavy-traffic optimality of the generalized cμt-rule,” Operations Research, vol. 52, no. 6, pp. 836–855, 2004.

[6] A. Sinha and E. Modiano, “Optimal control for generalized network-flow problems,” IEEE/ACM Transactions on Networking, vol. 26, no. 1, pp. 506–519, Feb 2018.

[7] M. J. Neely, “Stochastic network optimization with application to communication and queuing systems,” Synthesis Lectures on Communication Networks, vol. 3, no. 1, pp. 1–211, 2010.

[8] U. C. Kocat, I. Koutsopoulos, and L. Tassiulas, “A framework for cross-layer design of energy-efficient communication with qos provisioning in multi-hop wireless networks,” in IEEE INFOCOM 2004, vol. 2, IEEE, 2004, pp. 1446–1456.

[9] A. Gurijala and C. Molina, “Defining and monitoring qos metrics in the next generation wireless networks,” in 2004 IEEE Telecommunications Quality of Services: The Business of Success QoS 2004, March 2004, pp. 37–42.

[10] N. Banović-Čurguz and D. Liščević, “Mapping of qos/qoe in 5g networks,” in 2019 42nd International Convention on Information and Communication Technology, Electronics and Microelectronics (MIPRO), May 2019, pp. 404–408.

[11] S. Kaul, R. Yates, and M. Gruteser, “Real-time status: How often should one update?” in INFOCOM, 2012 Proceedings IEEE, IEEE, 2012, pp. 2731–2735.

[12] A. Kosta, N. Pappas, V. Angelakis et al., “Age of information: A new concept, metric, and tool,” Foundations and Trends® in Networking, vol. 12, no. 3, pp. 162–259, 2017.

[13] S. Farazi, A. G. Klein, and D. R. Brown III, “Fundamental bounds on the age of information in multi-hop global status update networks,” Journal of Communications and Networks, vol. 21, no. 3, pp. 268–279, 2019.

[14] M. Grossglauser and D. N. Tse, “Mobility increases the capacity of ad hoc wireless networks,” IEEE/ACM Transactions on Networking (ToN), vol. 10, no. 4, pp. 477–486, 2002.

[15] A. E. Gamal, J. Mammen, B. Prabhakar, and D. Shah, “Throughput-delay tradeoff in wireless networks,” in IEEE INFOCOM 2004, vol. 1, IEEE, 2004.

[16] M. J. Neely and E. Modiano, “Capacity and delay tradeoffs for ad hoc mobile networks,” IEEE Transactions on Information Theory, vol. 51, no. 6, pp. 1917–1937, 2005.

[17] W.-J. Hsu, T. Spyropoulos, K. Psounis, and A. Helmy, “Modeling time-variant user mobility in wireless mobile networks,” in IEEE INFOCOM 2007-26th IEEE International Conference on Computer Communications, IEEE, 2007, pp. 758–766.

[18] A. Ghazal, Y. Yuan, C. Wang, Y. Zhang, Q. Yao, H. Zhou, and W. Duan, “A non-stationary int-advanced mimo channel model for high-mobility wireless communication systems,” IEEE Transactions on Wireless Communications, vol. 16, no. 4, pp. 2057–2068, April 2017.

[19] S. Wu, C.-X. Wang, M. M. Alwakeel, X. You et al., “A general 3-d non-stationary 5g wireless channel model,” IEEE Transactions on Communications, vol. 66, no. 7, pp. 3065–3078, 2017.

[20] J. Bian, J. Sun, C. Wang, R. Feng, J. Huang, Y. Yang, and M. Zhang, “A winner+ based 3-d non-stationary wideband mimo channel model,” IEEE Transactions on Wireless Communications, vol. 17, no. 3, pp. 1755–1767, March 2018.

[21] A. Ghazal, C. Wang, Y. Liu, P. Fan, and M. K. Chahine, “A generic non-stationary mimo channel model for different high-speed train scenarios,” in 2015 IEEE/CIC International Conference on Communications in China (ICCC), Nov 2015, pp. 1–6.

[22] R. Poisel, Modern communications jamming principles and techniques. Artech House, 2011.

[23] A. Mpitziopoulos, D. Gavalas, C. Konstantopoulos, and G. Pantziou, “A survey on jamming attacks and countermeasures in wsns,” IEEE Communications Surveys & Tutorials, vol. 11, no. 4, pp. 42–56, 2009.

[24] M. Andrews and L. Zhang, “Routing and scheduling in multihop wireless networks with time-varying channels,” ACM Transactions on Algorithms (TALG), vol. 3, no. 3, pp. 33–es, 2007.

[25] R. Bassily and A. Smith, “Causal erasure channels,” in Proceedings of the twenty-fifth annual ACM-SIAM symposium on Discrete algorithms. SIAM, 2014, pp. 1844–1857.

[26] I. Kadota, A. Sinha, E. Uysal-Biyikoglu, R. Singh, and E. Modiano, “Scheduling policies for minimizing age of information in broadcast wireless networks,” IEEE/ACM Transactions on Networking (TON), 2018, pp. 2637–2650, 2018.

[27] I. Kadota, M. S. Rahaman, and E. Modiano, “Age of information in wireless networks: From theory to implementation,” in Proceedings of the 26th Annual International Conference on Mobile Computing and Networking, ser. MobiCom ’20, New York, NY, USA: Association for Computing Machinery, 2020. [Online]. Available: https://doi.org/10.1145/3372224.3418171

[28] I. Kadota, A. Sinha, and E. Modiano, “Optimizing age of information in wireless networks with throughput constraints,” IEEE INFOCOM 2018-IEEE Conference on Computer Communications. IEEE, 2018, pp. 1844–1852.

[29] R. Talak, S. Karaman, and E. Modiano, “Minimizing age-of-information in multi-hop wireless networks,” in 2017 55th Annual Allerton Conference on Communication, Control, and Computing (Allerton). IEEE, 2017, pp. 486–493.

[30] V. Tripathi, R. Talak, and E. Modiano, “Age optimal information gathering and dissemination on graphs,” in IEEE INFOCOM 2019-IEEE Conference on Computer Communications. IEEE, 2019, pp. 2422–2430.

[31] I. Kadota, A. Sinha, and E. Modiano, “Scheduling algorithms for optimizing age of information in wireless networks with throughput constraints,” IEEE/ACM Transactions on Networking, 2019.

[32] P. Popovski, C. Stefanovic, J. J. Nielsen, E. de Carvalho, M. Angelichowski, K. F. Trillingsgaard, and A. Bana, “Wireless access in ultra-reliable low-latency communication (urllc),” in IEEE INFOCOM 2019-IEEE Conference on Computer Communications, IEEE, 2019, pp. 2422–2430.
[34] A. Javani and Z. Wang, “Age of information in multiple sensing of a single source,” arXiv preprint arXiv:1902.01975, 2019.
[35] S. Kaul, M. Gruteser, V. Rai, and J. Kenney, “Minimizing age of information in vehicular networks,” in 2011 8th Annual IEEE Communications Society Conference on Sensor, Mesh and Ad Hoc Communications and Networks. IEEE, 2011, pp. 350–358.
[36] A. Fiat and G. J. Woeginger, Online algorithms: The state of the art. Springer, 1998, vol. 1442.
[37] S. Albers, Competitive online algorithms. Citeseer, 1996.
[38] R. G. Gallager, Discrete stochastic processes. Springer Science & Business Media, 2012, vol. 321.
[39] F. de Sousa Chaves, M. Abbas-Turki, H. Abou-Kandil, and J. M. T. Romano, “Transmission power control for opportunistic qos provision in wireless networks,” IEEE Transactions on Control Systems Technology, vol. 21, no. 2, pp. 315–331, 2013.
[40] X. Ge, J. Ye, Y. Yang, and Q. Li, “User mobility evaluation for 5g small cell networks based on individual mobility model,” IEEE Journal on Selected Areas in Communications, vol. 34, no. 3, pp. 528–541, 2016.
[41] I. F. Akyildiz, Y.-B. Lin, W.-R. Lai, and R.-J. Chen, “A new random walk model for pcs networks,” IEEE Journal on Selected Areas in Communications, vol. 18, no. 7, pp. 1254–1260, 2000.
[42] D. B. Johnson and D. A. Maltz, “Dynamic source routing in ad hoc wireless networks,” in Mobile computing. Springer, 1996, pp. 153–181.
[43] F. Bai and A. Helmy, “A survey of mobility models,” Wireless Adhoc Networks. University of Southern California, USA, vol. 206, p. 147, 2004.
[44] D. P. Bertsekas, Dynamic programming and optimal control. Athena scientific Belmont, MA, 1995, vol. 2, no. 2.
[45] I. Lee, M. A. Epelman, H. E. Romeijn, and R. L. Smith, “Simplex algorithm for countable-state discounted markov decision processes,” Operations Research, vol. 65, no. 4, pp. 1029–1042, 2017.
[46] N. Ferns, P. Panangaden, and D. Precup, “Metrics for markov decision processes with infinite state spaces,” arXiv preprint arXiv:1207.1386, 2012.
[47] D. Williams, Probability with martingales. Cambridge university press, 1991.
[48] M. J. Neely, “Dynamic power allocation and routing for satellite and wireless networks with time varying channels,” Ph.D. dissertation, Massachusetts Institute of Technology, 2003.
[49] L. Lovász et al., “Random walks on graphs: A survey,” Combinatorics, Paul erdos is eighty, vol. 2, no. 1, pp. 1–46, 1993.
[50] D. P. Bertsekas, Dynamic programming and optimal control. Athena scientific Belmont, MA, 2005, vol. 2, no. 3.
[51] A. Shwartz and A. Weiss, Large deviations for performance analysis: queues, communication and computing. CRC Press, 1995, vol. 5.
[52] R. G. Gallager, Stochastic processes: theory for applications. Cambridge University Press, 2013.