Fast estimation method for rank of a high-dimensional sparse matrix

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Abstract

Numerical computing the rank of a matrix is a fundamental problem in scientific computation. The data sets generated by Internet often correspond to the analysis of high-dimensional sparse matrices. Notwithstanding the recent advances in the promotion of traditional singular value decomposition (SVD), an efficient estimation algorithm for rank of a high-dimensional sparse matrix is still lacked. Inspired by the controllability theory of complex networks, we converted the rank of a matrix into max-matching computing. Then we established a fast rank estimation algorithm by using cavity method, a powerful approximate technique for computing the max-matching, to estimate the rank of a sparse matrix. In the merit of its natural low complexity of cavity method, we showed that the rank of a high-dimensional sparse matrix can be estimated in a much faster way than SVD with high accuracy. Our method offers an efficient pathway to fast estimate the rank of the high-dimensional sparse matrix, when the time cost of computing the rank by SVD is unacceptable.

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I. INTRODUCTION

With the development of online social networks, researchers often have to face complex networks composed of huge numbers of individuals and multiple relationships among them. For the analysis of these huge complex networks, we need to convert the network into its corresponding matrix, and get some characteristics of the original network from its matrix based on the traditional matrix theory, such as page-rank method [1], communities detective [2], and some dynamical problem [3, 4]. Rank is one of the most important numerical characteristics of a matrix. In present, a large number of researchers focus on the rank of special matrix [5], low-rank problem [6–9], maximal rank problem [10], and nullity of graph [11, 12]. The most successful method of rank calculation is the traditional singular value decomposition (SVD), which computes the rank through decomposing the original matrix into singular values and computing the statistical properties of the decomposed matrix. However, the complexity of SVD is the cube of the matrix size (denoted as $N^3$), which is numerical difficult for the high-dimensional situation. Thereafter, several methods are developed to tackle the complexity problem based on the novel matrix decompositions [13, 14], Monte Carlo simulation [15], and multi-computing technologies [16–20]. However, all these methods cannot improve the time complexity significantly.

Benefit from the development of control theory of complex network, we know that the rank of coupling matrix reflects the exact controllability of sparse complex networks. On the other hand, the structural controllability can be measured by the maximum matching of sparse complex networks. For those sparse complex networks, the exact controllability is equivalent to the structural controllability [21, 22]. The cavity method, a powerful approximation method developed in statistical mechanics [23, 24], can be designed as to calculate the max-matching of complex networks. Therefore, the controllability of sparse complex networks builds a bridge of cavity method and rank computation.

In other words, for an $N$-dimensional sparse matrix, we can convert it into an $N$-nodes complex network and compute its structural controllability through cavity method. As its sparsity, the structural controllability is equal to the exact controllability, and the rank of the input $N$-dimensional sparse matrix can be estimated approximately. This process, which is a fast estimation method for a sparse matrix rank called FER, can much faster estimate the rank of a high-dimensional sparse matrix than SVD. Then, we apply FER on randomly generalized sparse matrices, and systematically compare FER with SVD from efficiency, accuracy, and applicability, in two typical distributions of non-zero elements of each row (denoted as $\langle k \rangle$). We find the time cost of FER increases little during $N$ growing with a constant $\langle k \rangle$, and the results estimated by FER keep high accuracy, which confirms FER is an efficient tool for estimating the rank of high-dimensional matrices. Then we study the impact of $\langle k \rangle$ on time cost and accuracy of FER, the performance of FER keeps very well. Finally, we applied FER on the matrices with identity of non-zero elements. The efficiency of FER is still very high, and the accuracy of FER performs well. All the results suggest FER is a valid access for estimating the rank of a sparse matrix, especially for estimating the rank of a high-dimensional sparse matrix, which is almost unacceptable to compute the rank by SVD with considering the time cost.
II. MATERIALS AND METHODS

FER is based on the development of controllability theory of complex networks. As we know, there are two existing theoretical frameworks can be used to quantify the controllability of a complex network: structural controllability theory (SCT) and exact controllability theory (ECT) [25]. SCT claims that the structural controllability of any directed network is determined by the maximum matching. The maximum matching can be solved by cavity method when the network is directed with structural matrix. The exact controllability obtained by ECT is determined by the maximum multiplicity of eigenvalues of the coupling matrix. In the sparse situation, ECT has an efficient tool to get the controllability of the networks by calculating the rank of the coupling matrix. When the network is sparse and the weights of links are weakly correlated, the structural controllability and the exact controllability are theoretically equivalent [21]. Therefore, computing the rank of a sparse matrix can be converted into a maximum matching problem, then we can estimate the rank by solving the corresponding coupling equations of cavity method in an efficient way. This is the core of FER.

Without loss of generality, we consider an arbitrary sparse input matrix $A$ with weakly correlated non-zero elements as shown in Fig. 1a, where only the white grids represent the zero elements and the darker color means the larger value of non-zero element. Then we apply FER on the input matrix $A$, and the procedure of FER can be described as the following five steps:

**Step 1.** Transfer the input matrix $A$ into a structural matrix $B$, in which the elements can only be 0 or 1. The 0s represent the zero elements donated as white grids, and the 1s represent the non-zero elements donated as black grids as shown in Fig. 1a-b;

**Step 2.** Consider the structural matrix $B$ as a coupling matrix of a complex network, and construct a directed network as shown in Fig. 1b-c;

**Step 3.** Obtain the in-degree ($P_{\text{in}}(k)$) and out-degree ($P_{\text{out}}(k)$) distributions of the directed network, where $P_{\text{in(out)}}(k) = n_{\text{in(out)}}(k)/N$. The $n_{\text{in(out)}}(k)$ is the number of nodes with the in(out)-degree value $k$ in the whole network as illustrated in Fig. 1d-e;

**Step 4.** Calculate the structural controllability ($N_C$) of the directed network according to the degree distribution by cavity method [22], which illustrated as Fig. 1b-e,

$$
N_C = N - \frac{1}{2} [G(\bar{\omega}_2) + G(1 - \bar{\omega}_1) - 1] + [\hat{G}(\bar{\omega}_2) + \hat{G}(1 - \omega_1) - 1] \\
+ \frac{\langle k \rangle}{2} [\bar{\omega}_1 (1 - \omega_2) + \omega_1 (1 - \bar{\omega}_2)]
$$

where $G(x)$, $\hat{G}(x)$ are ordered by the following equations,

$$
G(x) = \sum_{k=0}^{\infty} P_{\text{out}}(x) x^k,
$$

$$
\hat{G}(x) = \sum_{k=0}^{\infty} P_{\text{in}}(x) x^k,
$$
FIG. 1: Illustration of the fast estimation algorithm for matrix rank. (a) Matrix $A$ represents a general sparse matrix as the input matrix, and each grid represents an element in the matrix, in which white grids denote zeros, and darker grids stand for the non-zero elements. (b) shows the structural matrix of the input matrix $A$. (c) transfers the structural matrix $B$ to a directed network. (d) statistics the in-degree and out-degree distributions of the complex network, where the horizontal axis represents the degrees, and the vertical axis represents the relative frequency of the corresponding degree. (e) inputs the degree distributions of the network into the coupling equations of cavity method, and solves the values of four coupling parameters. Following eq. (1), the structural controllability $N_C$ can be calculated. (f) according to $N_C$, obtains the rank approximation of the input matrix $A$.

and $\omega_1, \omega_2, \hat{\omega}_1, \hat{\omega}_2$ are the solutions of following coupling equations:

\[
\begin{align*}
\omega_1 &= H(\hat{\omega}_2), \\
\omega_2 &= 1 - H(1 - \hat{\omega}_1), \\
\hat{\omega}_1 &= \hat{H}(\omega_2), \\
\hat{\omega}_2 &= 1 - \hat{H}(1 - \omega_1),
\end{align*}
\]

in the above equations, the functions of $H(\ast)$ and $\hat{H}(\ast)$ are shown as:

\[
\begin{align*}
H(x) &= \sum_{k=0}^{\infty} \frac{(k+1)P_{out}(k+1)x^k}{\sum_{k=0}^{\infty} kP_{out}(k)}, \\
\hat{H}(x) &= \sum_{k=0}^{\infty} \frac{(k+1)P_{in}(k+1)x^k}{\sum_{k=0}^{\infty} kP_{in}(k)}.
\end{align*}
\]
as the result, we can obtain $\omega_1, \omega_2, \hat{\omega}_1, \hat{\omega}_2$ by numerically solving eq. (2) by using $P_{in}(*)$ and $P_{out}(*)$ obtained in step 3. Finally, we can obtain the structural controllability $N_C$ from eq. (1).

**Step 5.** As the SCT and the ECT is equivalent when the input matrix $A$ is sparse, the structural controllability $N_C$ is equal to the exact controllability $N - \text{Rank}(A)$. Thus, we can estimate the rank of the input sparse matrix $A$ as illustrated as Fig. 1b-f:

$$\text{Rank}(A) \approx N - N_C$$

(3)

It is worth noting that, if a matrix contains totally irrelevant element values (every non-zero element is a real random number), $\text{Rank}(A)$ is theoretically equal to $\text{Rank}(B)$ based on the SCT and ECT, and the process of FER is rigorous. However, the assumption is too strict for the general cases, which means the result of FER is just an estimation of the rank of the input matrix. And the correlation strength of non-zero elements in input matrix indeed affects the accuracy of FER.

### III. RESULTS

Some comparisons between FER and SVD are exhibited from the efficiency and accuracy aspects in some typical situations. For analyzing the impact of the matrix size ($N$), we generate some matrices randomly with a fixed sparsity, i.e., the average number of non-zero elements in each row ($\langle k \rangle$). The non-zero elements are generated following two typical distributions: random distribution and power-law distribution. Then we apply FER and SVD to the generated matrices, and the results of comparing the efficiency and the accuracy are shown in Fig. 2. The efficiency of the algorithm is defined by time cost of solving the task, denoted as $T_{cost}$. As Fig. 2a and Fig. 2c shown, if $N$ increases, $T_{cost}$ of SVD increases following its theoretical computational complexity $O(N^3)$. Although we can use GPU for accelerating, $T_{cost}$ of SVD increases beyond $O(N^2)$ as $N$ increasing. In contrast, $T_{cost}$ of FER increases very little as $N$ increasing, which suggests that its computational complexity is determined by the size of the matrix and the average number of non-zero elements in each row together. the estimated rank by FER (denoted as $r_{FER}^M$) and SVD (denoted as $r_{SVD}^M$) almost overlap together as shown in both Fig. 2b and Fig. 2h, which means these two methods obtain a similar result no matter with how $N$ increasing. In order to explain the high accuracy of FER in more detail, we treat the rank computed by SVD as the ground truth, and define the relative error as:

$$\Delta r_M = \left| \frac{r_{FER}^M - r_{SVD}^M}{r_{SVD}^M} \right|,$$

(4)

On the other hand, for comparing the accuracy of FER as $N$ increasing, we define a normalized rank (denoted as $r_M$) as the following equation:

$$r_M \equiv \frac{\text{Rank}(A)}{N},$$

(5)

the inset figures in Fig. 2b and Fig. 2i show the relative error $\Delta r_M$ in random distribution and power-law situations respectively. $\Delta r_M$ are quite small with fluctuations as $N$ increasing, which keeps below 0.003 in random situation and keeps below 0.001 in power-law situation. The results indicate that FER has well performance both in the two typical. When $N$ grows larger, $\Delta r_M$ has a downward trend in both two distributions, which is implied the relative error between FER and SVD should be very small when $N$ is enough large. In summary, for a high-dimensioned
FIG. 2: The impact of $N$ on the efficiency and accuracy of FER. The changes of $T_{\text{cost}}$ to compute the rank of the input matrix using SVD and FER, when input matrix size $N$ increases. The $\langle k \rangle$ follows two typical distributions: random distribution (a) and power-law distribution ($\gamma = 3$) (c). The results of the rank calculated by SVD and the rank estimated by FER in random distribution (b) and power-law distribution (d) follow the eq. (5). The inset figures in (b) and (d) show the relative errors of FER calculated by eq. (4). The $\langle k \rangle$ is kept 2, and all the non-zero elements are random in the generated matrices. The results of $N \leq 5000$ are averaged over 50 independent calculations, and the results of $5000 < N \leq 10000$ are averaged over 20 independent calculations.

sparse matrix, we can use FER to obtain an accurate estimation of rank efficiently with the similar accuracy as obtained by SVD, no matter in random or power-law distribution.

As shown in Fig. 3, we checked how the sparsity of input matrix, measured by $\langle k \rangle$, affects the efficiency and accuracy of FER, when the matrix size is fixed as $N = 3000$. It is shown that $T_{\text{cost}}^{\text{SVD}}$ and $T_{\text{cost}}^{\text{FER}}$ are the functions of $\langle k \rangle$, in random situation (Fig. 3a) and power-law situation (Fig. 3c). And $T_{\text{cost}}^{\text{FER}}$ is much smaller than $T_{\text{cost}}^{\text{SVD}}$ in each situation. In Fig. 3b and Fig. 3d, we analyzed the accuracy of FER as $\langle k \rangle$ increasing. There is almost no differences between the result of FER and SVD, and the scatters in the main figures almost overlap. Then, we consider the relative error of FER, as shown in the inset figures of Fig. 3b and Fig. 3d. If $\langle k \rangle$ increases,

The value of $\Delta r_M$ are both much smaller in two situations, which fluctuates obviously in random situation. While $\Delta r_M$ almost keeps constant, which is smaller than $5 \times 10^{-5}$, in power-law
FIG. 3: The impact of $\langle k \rangle$ on FER when $N$ fixed. The impact of $\langle k \rangle$ on $T_{\text{cost}}^{SVD}$ and $T_{\text{cost}}^{FER}$ in random distribution (a) and power-law distribution ($\gamma = 3$) (c). The comparison between the rank calculated by SVD and FER on generated matrices with random distribution (b) or power-law distributions (d) for different $\langle k \rangle$. The inset figures in (b,d) show the relative errors of FER versus $\langle k \rangle$. The fixed size of all the simulated networks is $N = 3000$, and all the results are averaged over 100 independent calculations.

situation. In summary, we find that FER is much more efficient than SVD, no matter when $\langle k \rangle$ increasing, and the impact of $\langle k \rangle$ on the efficiency and accuracy of FER is quite small in both two situations.

As we know, FER works strictly only if all the non-zero elements in the sparse matrix are uncorrelated. However, there are lots of relevant elements in the real data, which means that errors are unavoidable if the non-zero elements are correlated. Thus, we should discuss whether the result obtained approximately by FER is acceptable when the non-zero elements of the input matrix are correlated. In Fig. 4, we consider an extreme case where all the non-zero elements in the input matrix are identity (set as 1), which means all the non-zero elements are correlated strongly. The strongly correlation has tiny effect on $T_{\text{cost}}$ in both random situation (Fig. 4a) and power-law situation (Fig. 4c). And $T_{\text{cost}}^{FER}$ is still much smaller than $T_{\text{cost}}^{SVD}$. However the results shown in Fig. 4b and Fig. 4d indicate that the accuracy of FER has a significant decline comparing
FIG. 4: The efficiency and accuracy of FER when the non-zero elements of input matrix are all strongly correlated. The $T_{cost}^{SVD}$ and the $T_{cost}^{FER}$ on input matrices with random (a) or power-law (c) distributions for different $\langle k \rangle$. b,d, the ranks obtained by SVD and FER on generated matrices with random distribution and power-law distributions (d) for different average degrees. All the non-zero elements in generated matrices are set as 1. The size of all the simulated networks is $N = 3000$, and all the results are averaged over 100 independent calculations.

with Fig. 5b and Fig. 5d. It means the correlation of the input matrix does affect the accuracy of FER, which is agree with the limitation of structural controllability as well as cavity method. Although the accuracy of FER has decreased, we can also learn from the inset figures that the relative error of FER is still very small in both two situation. Especially when $\langle k \rangle$ increases over 4, $\Delta r_M$ has an obvious descent. In a word, even the non-zero elements in the sparse matrix are strongly correlated, the performance of FER is also acceptable in terms of efficiency and accuracy. The robustness of FER suggests the effectiveness in estimating the rank of more general matrix extracted from the empirical data set.
IV. DISCUSSION

In summary, we established an efficient rank estimation algorithm for a high-dimensional sparse matrix, which called FER. We introduce the cavity method to estimate the rank of a sparse matrix based on SCT and ECT. Then, we discussed the impact of the input matrix size ($N$), the sparsity of matrix (measured by $\langle k \rangle$), and the correlation of non-zero element on the efficiency (measured by $T_{\text{cost}}$) and accuracy (measured by $\Delta r_M$) of FER, in random situation and power-law situation. We uncovered FER has remarkable performances both in efficiency and accuracy in random distribution and power-law distribution. Although the characteristics of non-zero elements affect the results, FER can still be applied to most sparse matrix to estimate its rank with fast speed and high accuracy. It can significantly outperform SVD on time cost, and have the similar accuracy with SVD. Therefore, FER provides an efficient and accurate method for estimating the rank of a sparse matrix. Especially when we need to deal with a large real networks by some algorithms with its matrix rank, FER can do a good job to estimate the rank from its high-dimensional sparse matrix, although SVD is inapplicable as its excessive time cost. Furthermore, in some special situations, where only the structural information of a social network can be detected, FER is still applicable to estimate the rank of its corresponding matrix. It means FER potentially can be used in some algorithms, which designed for incomplete data or data polluted by interference noise.

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