Average Order of the Euler Phi Function And The Largest Integer Function

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Abstract: Let $x \geq 1$ be a large number, let $[x] = x - \{x\}$ be the largest integer function, and let $\varphi(n)$ be the Euler totient function. The result $\sum_{n \leq x} \varphi([x/n]) = (6/\pi^2)x \log x + O(x(\log x)^2/(\log \log x)^{1/3})$ was proved very recently. This note presents a short elementary proof, and sharpen the error term to $\sum_{n \leq x} \varphi([x/n]) = (6/\pi^2)x \log x + O(x)$. In addition, the first proofs of the asymptotics formulas for the finite sums $\sum_{n \leq x} \psi([x/n]) = (15/\pi^2)x \log x + O(x)$, and $\sum_{n \leq x} \sigma([x/n]) = (\pi^2/6)x \log x + O(x)$ are also evaluated here.

Contents

1 Introduction 1
2 Euler Totient Function 2
3 Auxiliary Results for the Phi Function 3
4 Dedekind Psi Function 5
5 Auxiliary Results for the Psi Function 6
6 Sum of Divisors Function 9
7 Auxiliary Results for the Sigma Function 10
8 Numerical Data 12

1 Introduction

Some new analytic techniques for evaluating the fractional finite sums $\sum_{n \leq x} f([x/n])$ for slow growing functions $f(n) \ll n^\varepsilon$, were recently introduced in [2], and for faster growing functions in the more recent literature as [1]. In this note, the standard analytic techniques originally developed for evaluating the average orders $\sum_{n \leq x} f(n)$ of arithmetic functions are modified to handle the fractional finite sums $\sum_{n \leq x} f([x/n])$ for fast growing multiplicative functions, approximately $f(n) \gg n(\log n)^b$, where $b \in \mathbb{Z}$. The modified standard techniques are simpler, more efficient and produce very short proofs. As demonstrations, the fractional finite sum $\sum_{n \leq x} \varphi([x/n])$ of the Euler phi function $\varphi$ in Theorem 2.1 the
fractional finite sum $\sum_{n \leq x} \psi([x/n])$ of the Dedekind psi function $\psi$ in Theorem 4.1, and the fractional finite sum $\sum_{n \leq x} \sigma([x/n])$ of the sum of divisors function $\sigma$ in Theorem 6.1 are evaluated here. The three functions $\varphi(n) \leq \psi(n) \leq \sigma(n)$ share many similarities such as multiplicative structures, rates of growths, et cetera, and have similar proofs. Theorem 2.1 has a very short proof, and sharpen the error term of a very recent result $\sum_{n \leq x} \varphi([x/n]) = (6/\pi^2)x \log x + O(x(\log x)^{2/3}(\log \log x)^{1/3})$ proved in [4] using a very complicated and lengthy proof. Further, Theorem 4.1, and Theorem 6.1 are new results in the literature.

2 Euler Totient Function

The first result deals with the Euler totient function $\varphi(n) = n \sum_{d|n} \mu(d)/d$. It is multiplicative and satisfies the growth condition $\varphi(n) \gg n/\log \log n$. A very short proof for $\sum_{n \leq x} \varphi([x/n])$ is produced here. It is a modified version of the standard proof for the average order $\sum_{n \leq x} \varphi(n) = (3/\pi^2)x^2 + O(x \log x)$, which appears in [1, Theorem 3.7], and similar references.

**Theorem 2.1.** If $x \geq 1$ is a large number, then,

$$\sum_{n \leq x} \varphi\left(\frac{x}{n}\right) = \frac{6}{\pi^2}x \log x + O(x).$$

**Proof.** Use the identity $\varphi(n) = \sum_{d|n} \mu(d)/d$ to rewrite the finite sum, and switch the order of summation:

$$\sum_{n \leq x} \varphi\left(\left\lfloor\frac{x}{n}\right\rfloor\right) = \sum_{n \leq x} \left\lfloor\frac{x}{n}\right\rfloor \sum_{d|\left\lfloor\frac{x}{n}\right\rfloor} \mu(d)/d$$

$$= \sum_{d \leq x} \mu(d)/d \sum_{n \leq x} \left\lfloor\frac{x}{n}\right\rfloor.$$

Expanding the bracket yields

$$\sum_{n \leq x} \varphi\left(\left\lfloor\frac{x}{n}\right\rfloor\right) = \sum_{d \leq x} \mu(d)/d \sum_{n \leq x} \left(\frac{x}{n} - \left\{\frac{x}{n}\right\}\right)$$

$$= x \sum_{d \leq x} \mu(d)/d \sum_{n \leq x} \left\{\frac{x}{n}\right\}.$$

Therefore, the difference of the subsum $S_1(x)$ computed in Lemma 3.2 and the subsum $S_2(x)$ computed in Lemma 3.3 complete the verification. It is easy to verify that the subsums $S_1(x)$ and $S_2(x)$ imply the omega result

$$\sum_{n \leq x} \varphi\left(\left\lfloor\frac{x}{n}\right\rfloor\right) - \frac{6}{\pi^2}x \log x = \Omega_{\pm}(x)$$

or a better result.
3 Auxiliary Results for the Phi Function

The detailed and elementary proofs of the preliminary results required in the proof of Theorem 2.1 concerning the Euler phi function $\phi(n) = \sum_{d|n} \mu(d) d$ are recorded in this section.

**Theorem 3.1.** ([2, Theorem 2.1]) For $x \geq 3$,

$$ax \log x \leq \sum_{n \leq x} \phi \left( \left\lfloor \frac{x}{n} \right\rfloor \right) \leq bx \log x,$$

where $a > 0$ and $b > 0$ are constants.

**Lemma 3.1.** Let $x \geq 1$ be a large number, and let $1 \leq d, n \leq x$ be integers. Then,

$$\sum_{0 \leq a \leq d-1} e^{i2\pi a [x/n]/d} = \begin{cases} 1 & \text{if } d \mid [x/n], \\ 0 & \text{if } d \nmid [x/n], \end{cases}$$

where $m \neq 0$, and $q \geq 1$ are integers.

**Lemma 3.2.** Let $x \geq 1$ be a large number. Then,

$$x \sum_{d \leq x} \frac{\mu(d)}{d} \sum_{\substack{n \leq x \mid d \mid [x/n]}} \frac{1}{n} = \frac{6}{\pi^2} x \log x + O(x).$$

**Proof.** Apply Lemma 3.1 to remove the congruence on the inner sum index, and break it up into two subsums. Specifically,

$$S_1(x) = x \sum_{d \leq x} \frac{\mu(d)}{d} \sum_{\substack{n \leq x \mid d \mid [x/n]}} \frac{1}{n} \sum_{0 \leq a \leq d-1} e^{i2\pi a [x/n]/d}$$

$$= x \sum_{d \leq x} \frac{\mu(d)}{d^2} \sum_{n \leq x} \frac{1}{n} \sum_{0 \leq a \leq d-1} e^{i2\pi a [x/n]/d}.$$

Equation (3) implies that the integers $n \geq 1$ such that $d \nmid [x/n]$ do not contribute to the triple sum above, but the integers $n \geq 1$ such that $d \mid [x/n]$ do contribute $\sum_{0 \leq a \leq d-1} e^{i2\pi a [x/n]/d} = d - 1$ to the triple sum. Therefore, the last expression reduces to

$$S_1(x) = x \sum_{d \leq x} \frac{\mu(d)}{d^2} \sum_{n \leq x} \frac{1}{n} + x \sum_{d \leq x} \frac{\mu(d)}{d^2} \sum_{n \leq x} \frac{1}{n} (d - 1)$$

$$= x \sum_{d \leq x} \frac{\mu(d)}{d} \sum_{n \leq x} \frac{1}{n}.$$
There are two possible evaluations of (10): Case 1. $d \nmid n$ and Case 2. $d \mid n$.

**Case 1.** If $d \nmid n$, substitute the standard asymptotic formula for the harmonic sum $\sum_{n \leq x} 1/n$, and the asymptotic formula for the finite sum of Mobius function $\sum_{n \leq x} \mu(n)/n = O \left( e^{-c \sqrt{\log x}} \right)$, see [1, Theorem 4.14], then,

$$S_1(x) = x \sum_{d \leq x} \frac{\mu(d)}{d} \sum_{n \leq \frac{x}{d}} \frac{1}{n}$$

$$= x \sum_{d \leq x} \frac{\mu(d)}{d} \left( \log x + \gamma + O \left( \frac{1}{x} \right) \right)$$

$$= O \left( (x \log x) e^{-c \sqrt{\log x}} \right),$$

where $c > 0$ is a constant. But, $S_1(x) = o(x \log x)$ contradicts Theorem 3.1. Hence, $d \mid n$.

**Case 2.** To evaluate this case, let $dm = n \leq x$, and substitute the standard asymptotic for the harmonic sum $\sum_{m \leq \frac{x}{d}} 1/m$ to obtain the following.

$$S_1(x) = x \sum_{d \leq x} \frac{\mu(d)}{d^2} \sum_{m \leq \frac{x}{d}} \frac{1}{m}$$

$$= x \sum_{d \leq x} \frac{\mu(d)}{d^2} \left( \log \left( \frac{x}{d} \right) + \gamma + O \left( \frac{d}{x} \right) \right)$$

$$= x \log x \sum_{d \leq x} \frac{\mu(d)}{d^2} - x \sum_{d \leq x} \frac{\mu(d) \log d}{d^2} + \gamma x \sum_{d \leq x} \frac{\mu(d)}{d^2} + O \left( \sum_{d \leq x} \frac{1}{d} \right)$$

$$= \frac{6}{\pi^2} x \log x + O(x),$$

where $\gamma$ is Euler constant, and the partial sums

$$\sum_{n \leq x} \frac{\mu(n)}{n^2} = \frac{1}{\zeta(2)} + O \left( \frac{1}{x} \right),$$

where $1/\zeta(2) = 6/\pi^2$, see [1, Theorem 3.13], and its ‘derivative’

$$-\sum_{n \leq x} \frac{\mu(n) \log n}{n^2} = \frac{\zeta'(2)}{\zeta(2)^2} + O \left( \frac{\log x}{x} \right),$$

are convergent series, bounded by constants.

**Lemma 3.3.** Let $x \geq 1$ be a large number. Then,

$$\sum_{d \leq x} \frac{\mu(d)}{d} \sum_{n \leq \frac{x}{d}} \left\{ \frac{x}{n} \right\} = O(x).$$

**Proof.** Apply Lemma 3.1 to remove the congruence on the inner sum index, and break it
up into two subsums. Specifically,
\[
S_2(x) = \sum_{d \leq x} \frac{\mu(d)}{d} \sum_{n \leq x \atop d \mid \lfloor x/n \rfloor} \left\{ \frac{x}{n} \right\}
\]
(16)
\[
= \sum_{d \leq x} \frac{\mu(d)}{d^2} \sum_{n \leq x} \left\{ \frac{x}{n} \right\} - \sum_{d \leq x} \frac{\mu(d)}{d^2} \sum_{n \leq x} \left\{ \frac{x}{n} \right\} \sum_{0 < a \leq d - 1} e^{i2\pi a \lfloor x/n \rfloor}/d.
\]

Equation (12) implies that the integers \( n \geq 1 \) such that \( d \nmid \lfloor x/n \rfloor \) do not contribute to the triple sum above, but the integers \( n \geq 1 \) such that \( d \mid \lfloor x/n \rfloor \) do contribute \( \sum_{0 < a \leq d - 1} e^{i2\pi a \lfloor x/n \rfloor}/d = d - 1 \) to the triple sum. Therefore, the last expression reduces to
\[
S_2(x) = \sum_{d \leq x} \frac{\mu(d)}{d} \sum_{n \leq x \atop d \mid \lfloor x/n \rfloor} \left\{ \frac{x}{n} \right\} + \sum_{d \leq x} \frac{\mu(d)}{d^2} \sum_{n \leq x} \left\{ \frac{x}{n} \right\} (d - 1)
\]
(17)
\[
= \sum_{d \leq x} \frac{\mu(d)}{d} \sum_{n \leq x \atop d \mid \lfloor x/n \rfloor} \left\{ \frac{x}{n} \right\}.
\]

Since \( d \mid n \), see (11), and (12), substitute \( dm = n \leq x \), and complete the estimate:
\[
S_2(x) = \sum_{d \leq x} \frac{\mu(d)}{d} \sum_{m \leq x/d \atop d \mid \lfloor x/m \rfloor} \left\{ \frac{x}{dm} \right\}
\]
(18)
\[
\leq x \sum_{d \leq x} \frac{1}{d^2}
\]
\[
= O(x),
\]
where the partial sum \( \sum_{n \leq x} 1/n^2 \) is bounded by a constant, see [1, Theorem 3.2].

4 Dedekind Psi Function

The second result deals with the Dedekind function \( \psi(n) = n \sum_{d \mid n} \mu(d)^2/d \). It is multiplicative and satisfies the growth condition \( \psi(n) \gg n \). The first asymptotic formula for the fractional finite sum of the Dedekind function is given below.

**Theorem 4.1.** If \( x \geq 1 \) is a large number, then,
\[
\sum_{n \leq x} \psi \left( \left\lfloor \frac{x}{n} \right\rfloor \right) = \frac{15}{\pi^2} x \log x + O(x).
\]
(19)

**Proof.** Use the identity \( \psi(n) = n \sum_{d \mid n} \mu^2(d)/d \) to rewrite the finite sum, and switch the order of summation:
\[
\sum_{n \leq x} \psi \left( \left\lfloor \frac{x}{n} \right\rfloor \right) = \sum_{n \leq x} \left\lfloor \frac{x}{n} \right\rfloor \sum_{d \mid \lfloor x/n \rfloor} \frac{\mu^2(d)}{d}
\]
(20)
\[
= \sum_{d \leq x} \frac{\mu^2(d)}{d} \sum_{n \leq x \atop d \mid \lfloor x/n \rfloor} \left\lfloor \frac{x}{n} \right\rfloor.
Expanding the bracket yields

\[
\sum_{n \leq x} \phi \left( \left\lfloor \frac{x}{n} \right\rfloor \right) = \sum_{d \leq x} \mu^2(d) \sum_{n \leq x, d \mid \lfloor x/n \rfloor} \left( \frac{x}{n} - \left\{ \frac{x}{n} \right\} \right) = x \sum_{d \leq x} \mu^2(d) \frac{1}{d} \sum_{n \leq x, d \mid \lfloor x/n \rfloor} \left\{ \frac{x}{n} \right\} = S_3(x) - S_4(x).
\]

Therefore, the difference of the subsum \( S_3(x) \) computed in Lemma 5.1 and the subsum \( S_4(x) \) computed in Lemma 5.2 complete the verification. ■

It is easy to verify that the subsums \( S_3(x) \) and \( S_4(x) \) imply the omega result

\[
\sum_{n \leq x} \psi \left( \left\lfloor \frac{x}{n} \right\rfloor \right) - \frac{15}{\pi^2} x \log x = \Omega(x)
\]

or a better result. A sketch of the standard proof for the average order

\[
\sum_{n \leq x} \psi(n) = \frac{15}{\pi^2} x^2 + O(x \log x),
\]

appears in [1, Exercise 13, p. 71].

## 5 Auxiliary Results for the Psi Function

The detailed and elementary proofs of the preliminary results required in the proof of Theorem 4.1 concerning the Dedekind psi function \( \psi(n) = \sum_{d \mid n} \mu^2(d)d \) are recorded in this section.

**Theorem 5.1.** For a large number \( x \geq 1 \),

\[
a x \log x \leq \sum_{n \leq x} \psi \left( \left\lfloor \frac{x}{n} \right\rfloor \right) \leq b x \log x,
\]

where \( a > 0 \) and \( b > 0 \) are constants.

**Proof.** Since \( \varphi(n) \leq \psi(n) \), the lower bound \( a x \log x \leq \sum_{n \leq x} \psi([x/n]) \) follows from Theorem 3.1. The upper bound is computed via the identity

\[
\sum_{n \leq x} \psi \left( \left\lfloor \frac{x}{n} \right\rfloor \right) = \sum_{n \leq x} \psi(n) \left( \left\lfloor \frac{x}{n} \right\rfloor - \left\lfloor \frac{x}{n + 1} \right\rfloor \right) = x \sum_{n \leq x} \frac{\psi(n)}{n(n + 1)} + \sum_{n \leq x} \psi(n) \left( -\left\{ \frac{x}{n} \right\} + \left\{ \frac{x}{n + 1} \right\} \right).
\]

The difference of consecutive fractional parts has the upper bound

\[
\left| -\left\{ \frac{x}{n} \right\} + \left\{ \frac{x}{n + 1} \right\} \right| = \left| -\left\{ \frac{x}{n} \right\} + \left( \frac{x}{n} - \frac{x}{n(n + 1)} \right) \right| \leq \left\{ \frac{1}{n(n+1)} \right\} \text{ if } n \mid x, \text{ if } n \nmid x.
\]


Moreover, if \( x \geq 1 \) is an integer as specified in (26), there is a correction term

\[
\sum_{n|x} \psi(n) \leq 2 \sum_{n|x} n \log \log n \leq 2 \log \log x \sum_{n|x} n \leq 4x(\log \log x)^2, \tag{27}
\]

(recall that \( \psi(n) \leq 2n \log \log n \)). Hence, partial summation, and the result in (23) lead to

\[
\sum_{n \leq x} \psi \left( \left\lfloor \frac{x}{n} \right\rfloor \right) \leq b_0 x \sum_{n \leq x} \frac{\psi(n)}{n(n + 1)} + \sum_{n|x} \psi(n) \leq bx \log x,
\]

where \( b_0, b, c > 0 \) are nonnegative constants.

**Lemma 5.1.** Let \( x \geq 1 \) be a large number. Then,

\[
x \sum_{d \leq x} \frac{\mu^2(d)}{d} \sum_{n \leq x \atop d \mid [x/n]} \frac{1}{n} = \frac{15}{\pi^2} x \log x + O(x). \tag{29}
\]

**Proof.** Apply Lemma 5.1 to remove the congruence on the inner sum index, and break it up into two subsums. Specifically,

\[
S_3(x) = x \sum_{d \leq x} \frac{\mu^2(d)}{d} \sum_{n \leq x \atop d \mid [x/n]} \frac{1}{n}
= x \sum_{d \leq x} \frac{\mu^2(d)}{d} \sum_{n \leq x} \frac{1}{n} \cdot \frac{1}{d} \sum_{0 \leq a \leq d-1} e^{2\pi i a[x/n]/d}
= x \sum_{d \leq x} \frac{\mu^2(d)}{d^2} \sum_{n \leq x} \frac{1}{n} + x \sum_{d \leq x} \frac{\mu^2(d)}{d^2} \sum_{n \leq x} \frac{1}{n} \sum_{0 \leq a \leq d-1} e^{2\pi i a[x/n]/d}.
\]

Equation (20) implies that the integers \( n \geq 1 \) such that \( d \mid [x/n] \) do not contribute to the triple sum above, but the integers \( n \geq 1 \) such that \( d \mid [x/n] \) do contribute \( \sum_{0 \leq a \leq d-1} e^{2\pi i a[x/n]/d} = d - 1 \) to the triple sum. Therefore, the last expression reduces to

\[
S_3(x) = x \sum_{d \leq x} \frac{\mu^2(d)}{d^2} \sum_{n \leq x} \frac{1}{n} + x \sum_{d \leq x} \frac{\mu^2(d)}{d^2} \sum_{n \leq x} \frac{1}{n} (d - 1) \tag{31}
= x \sum_{d \leq x} \frac{\mu^2(d)}{d} \sum_{n \leq x} \frac{1}{n}.
\]

There are two possible evaluations of (31): Case 1. \( d \nmid n \) and Case 2. \( d \mid n \).

**Case 1.** If \( d \nmid n \), substitute the standard asymptotic formula for the harmonic sum \( \sum_{n \leq x} 1/n \), then,

\[
S_3(x) = x \sum_{d \leq x} \frac{\mu^2(d)}{d} \sum_{n \leq x} \frac{1}{n}
= x \sum_{d \leq x} \frac{\mu^2(d)}{d} \left( \log x + \gamma + O\left(\frac{1}{x}\right) \right)
= \frac{15}{\pi^2} x \log^2 x + O(x).
\]
But, \( S_3(x) \gg x \log^2 x \) contradicts Theorem 5.1. Hence, \( d \mid n \).

**Case 2.** To evaluate this case, let \( dm = n \leq x \), and substitute the standard asymptotic for the harmonic sum \( \sum_{m \leq x/d} 1/m \) to obtain the following.

\[
S_3(x) = x \sum_{d \leq x} \frac{\mu^2(d)}{d^2} \sum_{m \leq x/d} \frac{1}{m}
\]

\[
= x \sum_{d \leq x} \frac{\mu^2(d)}{d^2} \left( \log \left( \frac{x}{d} \right) + \gamma + O \left( \frac{d}{x} \right) \right)
\]

\[
= x \log x \sum_{d \leq x} \frac{\mu^2(d)}{d^2} - x \sum_{d \leq x} \frac{\mu^2(d) \log d}{d^2} + \gamma x \sum_{d \leq x} \frac{\mu^2(d)}{d^2} + O \left( \sum_{d \leq x} \frac{1}{d} \right)
\]

\[
= \frac{15}{\pi^2} x \log x + O(x),
\]

where \( \gamma \) is Euler constant, and the partial sums

\[
\sum_{n \leq x} \frac{\mu^2(n)}{n^2} = \frac{\zeta(2)}{\zeta(4)} + O \left( \frac{1}{x} \right),
\]

where \( \zeta(2)/\zeta(4) = 15/\pi^2 \), and its ‘derivative’

\[
- \sum_{n \leq x} \frac{\mu^2(n) \log n}{n^2} = -c_1 + O \left( \frac{\log x}{x} \right),
\]

where \( c_1 > 0 \) is a constant, are convergent series, bounded by constants.

**Lemma 5.2.** Let \( x \geq 1 \) be a large number. Then,

\[
\sum_{d \leq x} \frac{\mu^2(d)}{d} \sum_{\substack{n \leq x \atop d \mid [x/n]}} \left\{ \frac{x}{n} \right\} = O(x).
\]

**Proof.** Apply Lemma 5.1 to remove the congruence on the inner sum index, and break it up into two subsums. Specifically,

\[
S_4(x) = \sum_{d \leq x} \frac{\mu^2(d)}{d} \sum_{\substack{n \leq x \atop d \mid [x/n]}} \left\{ \frac{x}{n} \right\}
\]

\[
= \sum_{d \leq x} \frac{\mu^2(d)}{d^2} \sum_{n \leq x} \left\{ \frac{x}{n} \right\} \cdot \frac{1}{d} \sum_{0 \leq a \leq d-1} e^{i2\pi a[x/n]/d}
\]

\[
= \sum_{d \leq x} \frac{\mu^2(d)}{d^2} \sum_{n \leq x} \left\{ \frac{x}{n} \right\} + \sum_{d \leq x} \frac{\mu^2(d)}{d^2} \sum_{n \leq x} \left\{ \frac{x}{n} \right\} \sum_{0 \leq a \leq d-1} e^{i2\pi a[x/n]/d}.
\]

Equation \( \ref{20} \) implies that the integers \( n \geq 1 \) such that \( d \mid [x/n] \) do not contribute to the triple sum above, but the integers \( n \geq 1 \) such that \( d \mid [x/n] \) do contribute \( \sum_{0 \leq a \leq d-1} e^{i2\pi a[x/n]/d} = d - 1 \) to the triple sum. Therefore, the last expression reduces to

\[
S_4(x) = \sum_{d \leq x} \frac{\mu^2(d)}{d^2} \sum_{n \leq x} \left\{ \frac{x}{n} \right\} + \sum_{d \leq x} \frac{\mu^2(d)}{d^2} \sum_{n \leq x} \left\{ \frac{x}{n} \right\} (d - 1)
\]

\[
= \sum_{d \leq x} \frac{\mu^2(d)}{d} \sum_{n \leq x} \left\{ \frac{x}{n} \right\}.
\]
Euler Phi Function and Largest Integer Function

Since \( d \mid n \), see (32), and (33), substitute \( dm = n \leq x \), and complete the estimate:

\[
S_4(x) = \sum_{d \leq x} \frac{x}{d} \sum_{m \leq x/d} \left\{ \frac{x}{dm} \right\} \leq x \sum_{d \leq x} \frac{1}{d^2} = O(x),
\]

where the partial sum \( \sum_{n \leq x} 1/n^2 \) is bounded by a constant, see [1, Theorem 3.2].

6 Sum of Divisors Function

The third result deals with the sum of divisors function \( \sigma(n) = n \sum_{d|n} 1/d \). It is multiplicative and satisfies the growth condition \( \sigma(n) \gg n \). The first asymptotic formula for the fractional sum of divisor function is given below.

**Theorem 6.1.** If \( x \geq 1 \) is a large number, then,

\[
\sum_{n \leq x} \sigma \left( \left\lfloor \frac{x}{n} \right\rfloor \right) = \frac{\pi^2}{6} x \log x + O(x).
\]

**Proof.** Use the identity \( \sigma(n) = n \sum_{d|n} 1/d \) to rewrite the finite sum, and switch the order of summation:

\[
\sum_{n \leq x} \sigma \left( \left\lfloor \frac{x}{n} \right\rfloor \right) = \sum_{n \leq x} \left\lfloor \frac{x}{n} \right\rfloor \sum_{d \mid \left\lfloor x/n \right\rfloor} \frac{1}{d} = \sum_{d \leq x} \frac{1}{d} \sum_{n \leq x} \left\lfloor \frac{x}{n} \right\rfloor.
\]

Expanding the bracket yields

\[
\sum_{n \leq x} \sigma \left( \left\lfloor \frac{x}{n} \right\rfloor \right) = \sum_{d \leq x} \frac{1}{d} \sum_{n \leq x \atop d \mid \left\lfloor x/n \right\rfloor} \left( \frac{x}{n} - \left( \frac{x}{n} \right) \right) = x \sum_{d \leq x} \frac{1}{d} \sum_{n \leq x \atop d \mid \left\lfloor x/n \right\rfloor} \left\lfloor \frac{x}{n} \right\rfloor - \sum_{d \leq x} \frac{1}{d} \sum_{n \leq x \atop d \mid \left\lfloor x/n \right\rfloor} \left\{ \frac{x}{n} \right\} = S_5(x) - S_6(x).
\]

Therefore, the difference of the subsum \( S_5(x) \) computed in Lemma 7.1 and the subsum \( S_6(x) \) computed in Lemma 7.2 complete the verification.

It is easy to verify that the subsums \( S_5(x) \) and \( S_6(x) \) imply the omega result

\[
\sum_{n \leq x} \sigma \left( \left\lfloor \frac{x}{n} \right\rfloor \right) - \frac{\pi^2}{6} x \log x = \Omega(x)
\]

or a better result. The standard proof for the average order

\[
\sum_{n \leq x} \sigma(n) = \frac{\pi^2}{12} x^2 + O(x \log x),
\]

appears in [1, Theorem 3.4].
7 Auxiliary Results for the Sigma Function

The detailed and elementary proofs of the preliminary results required in the proof of Theorem 6.1 concerning the sum of divisor function $\sigma(n) = \sum_{d|n} d$ are recorded in this section.

**Theorem 7.1.** For a large number $x \geq 1$, 

$$ax \log x \leq \sum_{n \leq x} \sigma\left(\left\lfloor \frac{x}{n} \right\rfloor\right) \leq bx \log x,$$  

where $a > 0$ and $b > 0$ are constants.

**Proof.** Since $\varphi(n) \leq \sigma(n)$, the lower bound $ax \log x \leq \sum_{n \leq x} \sigma([x/n])$ follows from Theorem 3.1. The upper bound is computed via the identity

$$\sum_{n \leq x} \sigma\left(\left\lfloor \frac{x}{n} \right\rfloor\right) = \sum_{n \leq x} \sigma(n) \left(\left\lfloor \frac{x}{n} \right\rfloor - \frac{x}{n+1}\right).$$  

The difference of consecutive fractional parts has the upper bound

$$\left|\{\frac{x}{n}\} + \frac{x}{n+1}\right| = \left|\{\frac{x}{n}\} + \frac{x}{n} - \frac{x}{n(n+1)}\right| \leq \left\{\frac{1}{n(n+1)}\right\} \text{ if } n \mid x, $$  

$$\left\{\frac{x}{n}\right\} \text{ if } n \nmid x.$$  

Moreover, if $x \geq 1$ is an integer as specified in (47), there is a correction term

$$\sum_{n \mid x} \sigma(n) \leq 2 \sum_{n \mid x} n \log n \leq 2 \log \log x \sum_{n \mid x} n \leq 4x(\log \log x)^2,$$  

(recall that $\sigma(n) \leq 2n \log \log n$). Hence, partial summation, and the result in (23) lead to

$$\sum_{n \leq x} \sigma\left(\left\lfloor \frac{x}{n} \right\rfloor\right) \leq b_0x \sum_{n \leq x} \frac{\sigma(n)}{n(n+1)} + \sum_{n \mid x} \sigma(n) \leq bx \log x,$$  

where $b_0, b, c > 0$ are nonnegative constants.

**Lemma 7.1.** Let $x \geq 1$ be a large number. Then,

$$x \sum_{d \leq x} \frac{1}{d} \sum_{n \leq x \atop d \mid [x/n]} \frac{1}{n} = \frac{\pi^2}{6} x \log x + O(x).$$  

**Proof.** Apply Lemma 5.1 to remove the congruence on the inner sum index, and break it up into two subsums. Specifically,

$$S_5(x) = x \sum_{d \leq x} \frac{1}{d} \sum_{n \leq x \atop d \mid [x/n]} \frac{1}{n} = x \sum_{d \leq x} \frac{1}{d} \sum_{n \leq x \atop d \mid [x/n]} \frac{1}{n} \sum_{0 \leq a \leq d-1} e^{2\pi i n [x/n]/d} = x \sum_{d \leq x} \frac{1}{d^2} \sum_{n \leq x} \frac{1}{n} + x \sum_{d \leq x} \frac{1}{d^2} \sum_{n \leq x \atop 0 \leq a \leq d-1} e^{2\pi i n [x/n]/d}.$$
Equation (41) implies that the integers \( n \geq 1 \) such that \( d \nmid \lfloor x/n \rfloor \) do not contribute to the triple sum above, but the integers \( n \geq 1 \) such that \( d \mid \lfloor x/n \rfloor \) do contribute to the triple sum. Therefore, the last expression reduces to

\[
S_3(x) = x \sum_{d \leq x} \sum_{n \leq x} \frac{1}{d} \sum_{n \leq x} \frac{1}{n} (d - 1)
\]

(52)

There are two possible evaluations of (52): Case 1. \( d \nmid n \) and Case 2. \( d \mid n \).

**Case 1.** If \( d \nmid n \), substitute the standard asymptotic formula for the harmonic sum \( \sum_{n \leq x} 1/n \), then,

\[
S_3(x) = x \sum_{d \leq x} \frac{1}{d} \sum_{n \leq x} \frac{1}{n}
\]

(53)

\[
= x \sum_{d \leq x} \frac{1}{d} \left( \log x + \gamma + O \left( \frac{1}{x} \right) \right)
\]

\[
= \frac{\pi^2}{6} x \log^2 x + O(x).
\]

But, \( S_3(x) \gg x \log^2 x \) contradicts Theorem 7.1. Hence, \( d \mid n \).

**Case 2.** To evaluate this case, let \( dm = n \leq x \), and substitute the standard asymptotic for the harmonic sum \( \sum_{m \leq x/d} 1/m \) to obtain the following.

\[
S_5(x) = x \sum_{d \leq x} \frac{1}{d^2} \sum_{m \leq x/d} \frac{1}{m}
\]

(54)

\[
= x \sum_{d \leq x} \frac{1}{d^2} \left( \log \left( \frac{x}{d} \right) + \gamma + O \left( \frac{1}{d^2} \right) \right)
\]

\[
= x \log x \sum_{d \leq x} \frac{1}{d^2} - x \sum_{d \leq x} \frac{\log d}{d^2} + \gamma x \sum_{d \leq x} \frac{1}{d^2} + O \left( \sum_{d \leq x} \frac{1}{d^2} \right)
\]

\[
= \frac{\pi^2}{6} x \log x + O(x),
\]

where \( \gamma \) is Euler constant, and the partial sums

\[
\sum_{n \leq x} \frac{1}{n^2} = \frac{\pi^2}{6} + O \left( \frac{1}{x} \right),
\]

(55)

and its ‘derivative’

\[
- \sum_{n \leq x} \frac{\log n}{n^2} = -\zeta'(2) + O \left( \frac{\log x}{x} \right),
\]

(56)

are convergent series, bounded by constants. The constant \( -\zeta'(2) = 0.937548 \ldots \), archived in OEIS A073002, has a rather complicated expression, see [3, Eq. 25.6.15].

\[\square\]
Lemma 7.2. Let $x \geq 1$ be a large number. Then,

$$
\sum_{d \leq x} \frac{1}{d} \sum_{n \leq x \atop d \mid \lfloor x/n \rfloor} \left\{ \frac{x}{n} \right\} = O(x).
$$

(57)

Proof. Apply Lemma 3.1 to remove the congruence on the inner sum index, and break it up into two subsums. Specifically,

$$
S_6(x) = \sum_{d \leq x} \frac{1}{d} \sum_{n \leq x \atop d \mid \lfloor x/n \rfloor} \left\{ \frac{x}{n} \right\}
$$

(58)

$$
= \sum_{d \leq x} \frac{1}{d} \sum_{n \leq x} \left\{ \frac{x}{n} \right\} \cdot \frac{1}{d} \sum_{0 \leq a \leq d-1} e^{i2\pi a \lfloor x/n \rfloor /d}
$$

$$
= \sum_{d \leq x} \frac{1}{d^2} \sum_{n \leq x} \left\{ \frac{x}{n} \right\} + \sum_{d \leq x} \frac{1}{d^2} \sum_{n \leq x} \left\{ \frac{x}{n} \right\} \sum_{0 < a \leq d-1} e^{i2\pi a \lfloor x/n \rfloor /d}.
$$

Equation (51) implies that the integers $n \geq 1$ such that $d \nmid \lfloor x/n \rfloor$ do not contribute to the triple sum above, but the integers $n \geq 1$ such that $d \mid \lfloor x/n \rfloor$ do contribute

$$
\sum_{0 < a \leq d-1} e^{i2\pi a \lfloor x/n \rfloor /d} = d - 1
$$

to the triple sum. Therefore, the last expression reduces to

$$
S_6(x) = \sum_{d \leq x} \frac{1}{d} \sum_{n \leq x} \left\{ \frac{x}{n} \right\} + \sum_{d \leq x} \frac{1}{d^2} \sum_{n \leq x} \left\{ \frac{x}{n} \right\} (d - 1)
$$

(59)

$$
= \sum_{d \leq x} \frac{1}{d} \sum_{n \leq x} \left\{ \frac{x}{n} \right\}.
$$

Since $d \mid n$, see (50), and (51), substitute $d m = n \leq x$, and complete the estimate:

$$
S_6(x) = \sum_{d \leq x} \frac{1}{d} \sum_{m \leq x/d} \left\{ \frac{x}{dm} \right\}
$$

(60)

$$
\leq x \sum_{d \leq x} \frac{1}{d^2}
$$

$$
= O(x),
$$

where the partial sum $\sum_{n \leq x} 1/n^2$ is bounded by a constant, see [1, Theorem 3.2].

8 Numerical Data

Small numerical tables were generated by an online computer algebra system, the range of numbers $x \leq 10^5$ is limited by the wi-fi bandwidth. The error terms are defined by

$$
E_1(x) = \sum_{n \leq x} \varphi \left( \left\lfloor \frac{x}{n} \right\rfloor \right) - \frac{6}{\pi^2} x \log x,
$$

(61)

and

$$
E_2(x) = \sum_{n \leq x} \sigma \left( \left\lfloor \frac{x}{n} \right\rfloor \right) - \frac{\pi^2}{6} x \log x,
$$

(62)

respectively. All the calculations are within the predicted ranges $E_i(x) = O(x)$. 

Table 1: Numerical Data For $\sum_{n \leq x} \varphi([x/n])$.

| $x$  | $\sum_{n \leq x} \varphi([x/n])$ | $6\pi^{-2}x \log x$ | Error $E_1(x)$ |
|------|----------------------------------|----------------------|----------------|
| 10   | 17                               | 14.00                | 3.00           |
| 100  | 275                              | 279.96               | -4.96          |
| 1000 | 4053                             | 4199.41              | 146.41         |
| 10000| 52201                            | 55992.16             | -3791.16       |
| 100000| 673929                           | 699901.94            | -25972.94      |

Table 2: Numerical Data For $\sum_{n \leq x} \sigma([x/n])$.

| $x$  | $\sum_{n \leq x} \sigma([x/n])$ | $6^{-1}\pi^2x \log x$ | Error $E_2(x)$ |
|------|----------------------------------|------------------------|----------------|
| 10   | 39                               | 37.88                  | 1.12           |
| 100  | 804                              | 757.52                 | 46.48          |
| 1000 | 12077                            | 11362.80               | 714.20         |
| 10000| 167617                           | 151504.03              | 16112.97       |
| 100000| 2033577                          | 1893800.33             | 139776.67      |

References

[1] Apostol, Tom M. Introduction to analytic number theory. Undergraduate Texts in Mathematics. Springer-Verlag, New York-Heidelberg, 1976.

[2] Olivier Bordelles, Randell Heyman, Igor E. Shparlinski. On a sum involving the Euler function, arXiv:1808.00188.

[3] F. Olver, M. McClain, et al, Editors. Digital Library of mathematical Functions. http://dlmf.nist.gov/5.4.el

[4] Zhai, Wenguang. On a sum involving the Euler function. J. Number Theory 211 (2020), 199-219.