Correlations between quark mass and flavor mixing hierarchies

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Abstract

We calculate the quark flavor mixing matrix $V$ based on the Hermitian quark mass matrices $M_u$ and $M_d$ with vanishing $(1,1)$, $(1,3)$ and $(3,1)$ entries. The popular leading-order prediction $|V_{ub}/V_{cb}| \simeq \sqrt{m_u/m_c}$ is significantly modified, and the result agrees with the current experimental value. We find that behind the strong mass hierarchy of up- or down-type quarks is the weak texture hierarchy of $M_u$ or $M_d$ characterized by an approximate seesaw-like relation among its $(2,2)$, $(2,3)$ and $(3,3)$ elements.

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1 Introduction

Since the standard model (SM) of particle physics was established in the 1960s and 1970s, its flavor sector has never been fully understood in the sense that all the flavor parameters are theoretically undetermined and their values have to be extracted from various experimental measurements. Such a situation will improve if the flavor structures of leptons and quarks can be well constrained, e.g., with the help of certain flavor symmetries [1, 2]. In this connection the “texture-zero” approach, which was first developed in 1977 to calculate the Cabibbo angle of quark flavor mixing [3–5], is popular and helpful. No matter whether the vanishing entries in a fermion mass matrix originate from a proper choice of the flavor basis or are enforced by a kind of underlying flavor symmetry, they can in practice help establish some testable relations between flavor mixing angles and ratios of fermion masses. The key point is simply that flavor mixing is a measure of the intrinsic mismatch between mass and flavor eigenstates of quarks, and thus the Cabibbo-Kobayashi-Maskawa (CKM) quark mixing matrix $V$ [6, 7] may correlate with the respective mass ratios of quarks via their texture zeros.

Today the values of six quark masses, extracted from quite a number of different measurements [8], have been renormalized to the scale $M_Z = 91.1876$ GeV in the SM framework [9–11]:

\[
\begin{align*}
    m_u &= 1.23 \pm 0.21 \text{ MeV}, \\
    m_c &= 0.620 \pm 0.017 \text{ GeV}, \\
    m_t &= 168.26 \pm 0.75 \text{ GeV}, \\
    m_d &= 2.67 \pm 0.19 \text{ MeV}, \\
    m_s &= 53.16 \pm 4.61 \text{ MeV}, \\
    m_b &= 2.839 \pm 0.026 \text{ GeV}.
\end{align*}
\] (1)

On the other hand, a global fit of current experimental data on quark flavor mixing and CP violation has provided us with the magnitudes of nine CKM matrix elements to a very high degree of accuracy [8]:

\[
\begin{align*}
    |V_{ud}| &= 0.97401 \pm 0.00011, \\
    |V_{us}| &= 0.22650 \pm 0.00048, \\
    |V_{ub}| &= 0.00361^{+0.00011}_{-0.00009}, \\
    |V_{cd}| &= 0.22636 \pm 0.00048, \\
    |V_{cs}| &= 0.97320 \pm 0.00011, \\
    |V_{cb}| &= 0.04053^{+0.00083}_{-0.00061}, \\
    |V_{td}| &= 0.00854^{+0.00023}_{-0.00016}, \\
    |V_{ts}| &= 0.03978^{+0.00082}_{-0.00060}, \\
    |V_{tb}| &= 0.999172^{+0.000024}_{-0.000035}.
\end{align*}
\] (2)

How to link the hierarchical pattern of $V$ to the hierarchical mass spectra of up- and down-type quarks is therefore a burning question.

In this regard one may make a survey of all the phenomenologically allowed zero textures of Hermitian quark mass matrices (see, e.g., Refs. [12–15] for very comprehensive studies) \footnote{Without loss of generality, the up- and down-type quark mass matrices can always be taken to be Hermitian after a proper choice of the flavor basis in the SM or its extensions which have no flavor-changing right-handed currents [16]. A limited number of texture zeros can actually be achieved in this way [17].}. Then it is easy to confirm that the texture [18–20]

\[
M_q = \begin{pmatrix}
0 & C_q & 0 \\
C_q^* & \tilde{B}_q & B_q \\
0 & B_q^* & A_q
\end{pmatrix}
\] (3)

is of particular interest, where $q = u$ (up) or $d$ (down) and $A_q > 0$ can always be arranged. Note that this texture is actually the most natural extension of the original Fritzsch texture [21–23] by allowing for $\tilde{B}_q \neq 0$. Since the structures of $M_u$ and $M_d$ in Eq. (3) are exactly parallel, they should originate from the same underlying flavor dynamics. After making the unitary transformation $V_q^* M_q V_q = \text{diag}\{\lambda_1^q, \lambda_2^q, \lambda_3^q\}$ with $\lambda_i^q$ being the quark mass eigenvalues (for
one may obtain the CKM matrix \( V \equiv V_u^\dagger V_d \). The structural parallelism between \( M_u \) and \( M_d \) implies a structural parallelism between \( V_u \) and \( V_d \), and thus \( V \) is expected to be close to the identity matrix \( I \). This expectation is certainly consistent with the observed pattern of \( V \), which deviates from \( I \) only at the level of \( \mathcal{O}(20\%) \) \[25\]. Another remarkable merit of the zero texture in Eq. (3) is its analytical calculability, which helps us to exactly express the elements of \( V_q \) in terms of \( \lambda^q_i \) and \( A_q \) \[24\] on the one hand and make some reasonable analytical approximations on the other hand.

Although quite a lot of attention has been paid to the above textures of \( M_{u,d} \) and their phenomenological consequences, we realize that in this connection reliable analytical approximations up to a sufficiently good degree of accuracy are still lacking. In particular, the failure of the popular leading-order prediction \(|V_{ub}/V_{cb}| \approx \sqrt{m_u/m_c} \) against current experimental data has not been analytically resolved. Here we are going to fill in this gap and understand why behind the strong mass hierarchy of up- or down-type quarks is the weak texture hierarchy of \( M_u \) or \( M_d \) characterized by an approximate seesaw-like relation \( \tilde{B}_q \sim |B_q|^2/A_q \) (for \( q = u \) or \( d \)). Such insights will be very helpful for a much deeper study of the quark flavor issues.

2 Analytical approximations

One may transform \( M_q \) in Eq. (3) into a real symmetric matrix \( \overline{M}_q \) by a phase redefinition:

\[
\overline{M}_q = P_q M_q P_q^\dagger = \begin{pmatrix}
0 & |C_q| & 0 \\
|C_q| & B_q & |B_q| \\
0 & |B_q| & A_q
\end{pmatrix},
\]

where \( P_q = \text{diag} \{ 1, e^{i\phi_q}, e^{i\phi'_q} \} \) with \( \phi_q \equiv \arg (C_q) \) and \( \phi'_q \equiv \arg (B_q) + \arg (C_q) \). It is obvious that \( \overline{M}_q \) can be diagonalized via an orthogonal transformation \( \tilde{O}_q \overline{M}_q \tilde{O}_q^{-1} = \text{diag} \{ \lambda^q_1, \lambda^q_2, \lambda^q_3 \} \) with \( \lambda^q_i \) denoting the three eigenvalues of \( M_q \). Then \( \tilde{B}_q, |B_q| \) and \( |C_q| \) can be expressed in terms of \( A_q \) and \( \lambda^q_i \) as follows \[25\]:

\[
\begin{align*}
\tilde{B}_q &= \lambda^q_1 + \lambda^q_2 + \lambda^q_3 - A_q, \\
|B_q|^2 &= \frac{(A_q - \lambda^q_1)(A_q - \lambda^q_2)(A_q - \lambda^q_3)}{A_q}, \\
|C_q|^2 &= \frac{-\lambda^q_1\lambda^q_2\lambda^q_3}{A_q}.
\end{align*}
\]

Note that the strong hierarchy \(|\lambda^q_2| \ll |\lambda^q_3| \ll |\lambda^q_1| \) allows us to choose \( \lambda^q_3 > 0 \) in correspondence to \( A_q > 0 \), and thus \( \lambda^q_1\lambda^q_2 < 0 \) as required by \( \det \overline{M}_q = -A_q |C_q|^2 = \lambda^q_1\lambda^q_2\lambda^q_3 \). In this convention the orthogonal matrix \( \tilde{O}_q \) is exactly given by

\[
\tilde{O}_q = \begin{pmatrix}
\frac{\sqrt{\lambda^q_1\lambda^q_3}(A_q - \lambda^q_1)}{A_q(\lambda^q_2 - \lambda^q_1)(\lambda^q_3 - \lambda^q_1)} & \frac{\sqrt{\lambda^q_1\lambda^q_2}(A_q - \lambda^q_1)}{A_q(\lambda^q_2 - \lambda^q_1)(\lambda^q_3 - \lambda^q_1)} & \frac{\sqrt{\lambda^q_1\lambda^q_3}(A_q - \lambda^q_1)}{A_q(\lambda^q_2 - \lambda^q_1)(\lambda^q_3 - \lambda^q_1)} \\
\frac{-\sqrt{\lambda^q_1\lambda^q_2}(A_q - \lambda^q_2)}{(\lambda^q_2 - \lambda^q_1)(\lambda^q_3 - \lambda^q_2)} & \frac{\sqrt{\lambda^q_2\lambda^q_3}(A_q - \lambda^q_2)}{(\lambda^q_2 - \lambda^q_1)(\lambda^q_3 - \lambda^q_2)} & \frac{-\sqrt{\lambda^q_2\lambda^q_3}(A_q - \lambda^q_2)}{(\lambda^q_2 - \lambda^q_1)(\lambda^q_3 - \lambda^q_2)} \\
\frac{-\sqrt{\lambda^q_1\lambda^q_2}(A_q - \lambda^q_3)}{(\lambda^q_2 - \lambda^q_1)(\lambda^q_3 - \lambda^q_3)} & \frac{-\sqrt{\lambda^q_1\lambda^q_2}(A_q - \lambda^q_3)}{(\lambda^q_2 - \lambda^q_1)(\lambda^q_3 - \lambda^q_3)} & \frac{\sqrt{\lambda^q_1\lambda^q_2}(A_q - \lambda^q_3)}{(\lambda^q_2 - \lambda^q_1)(\lambda^q_3 - \lambda^q_3)}
\end{pmatrix}.
\]
with \( \eta = +1 \) and \( \eta = -1 \) corresponding respectively to \( \lambda_2 > 0 \) and \( \lambda_2 < 0 \). So the explicit relations between \( \lambda_2 \) and real quark masses are \(^2\)

\[
\begin{align*}
(\lambda_1^u, \lambda_2^u, \lambda_3^u) &= (- \eta u_m, \eta u_m, m_u), \\
(\lambda_1^d, \lambda_2^d, \lambda_3^d) &= (- \eta d_m, \eta d_m, m_d),
\end{align*}
\]

(7)

for the up- and down-quark sectors, respectively. As a consequence, nine elements of the CKM matrix \( V = O_u T P_u P_d O_d \) can be expressed as

\[
V_{pq} = (O_u)_{1p} (O_d)_{1q} + (O_u)_{2p} (O_d)_{2q} e^{i \phi_1} + (O_u)_{3p} (O_d)_{3q} e^{i \phi_2},
\]

(8)

where \( p \) and \( q \) run respectively over the flavor indices \((u, c, t)\) and \((d, s, b)\), and the two phase differences are defined as

\[
\phi_1 \equiv \phi_u - \phi_d = \arg(C_u) - \arg(C_d),
\]

\[
\phi_2 \equiv \phi_u' - \phi_d' = \arg(B_u) - \arg(B_d) + \phi_1.
\]

(9)

If the values of six quark masses are input, one may determine the elements of \( V \) and check their consistency with current experimental data on flavor mixing and CP violation by adjusting the four free parameters \( A_u, A_d, \phi_1 \) and \( \phi_2 \). In this regard a careful numerical analysis of the allowed parameter space has been done in Ref. \(^2\). Here we are going to carry out a careful analytical exploration of the salient features of \( V \) by making reliable approximations for \( O_u \) and \( O_d \) in Eq. \( \text{(4)} \) for the first time, so as to truly understand the link between the texture zeros of \( M_q \) and the observed pattern of \( V \) in depth.

To assure that our analytical approximations are reliable, we need to take into account the quark masses and CKM matrix elements in a common energy scale as given in Eqs. \( \text{(1)} \) and \( \text{(2)} \). One can easily see \( m_u/m_c \sim m_c/m_t \sim \lambda_1^2 \) and \( m_d/m_s \sim m_s/m_b \sim \lambda_2^2 \), together with \( |V_{cd}| \approx |V_{us}| = \lambda, |V_{ts}| \approx |V_{cb}| \sim \lambda^2, |V_{td}| \sim \lambda^3 \) and \( |V_{ub}| \approx |V_{td}|/2 \), where \( \lambda \) is defined as a small expansion parameter for the CKM matrix \( V \). The much stronger hierarchy of the up-type quark masses implies that their ratios contribute much less to the elements of \( V \). In other words, the CKM flavor mixing parameters are expected to be dominated by those contributions from the down-quark sector.

### 2.1 Generic approximations

To measure how hierarchical the four entries on the right-bottom corner of \( M_q \) can be in fitting the present experimental data, let us define the following two characteristic parameters:

\[
r_u \equiv 1 - \frac{A_u}{m_t}, \quad r_d \equiv 1 - \frac{A_d}{m_b}.
\]

(10)

The careful numerical analysis made in Ref. \(^2\) indicates that the allowed parameter space of \( r_u \) and \( r_d \) is mainly located in the range of 0.1 to 0.2, although there is also a small possibility

\(^2\)One may consider eliminating the sign uncertainty associated with \( \eta_q \) and thus arrange all the three mass eigenvalues of \( \mathcal{M}_q \) (for \( q = u \) or \( d \)) to be positive by properly redefining the phases of three right-handed quark fields. Such a treatment is equivalent to a new choice of the flavor basis for \( M_q \), and that is why the orthogonal matrix \( O_q \) used to diagonalize \( \mathcal{M}_q \) will be accordingly affected. So different choices of the values of \( \eta_u \) and \( \eta_d \) mean our considerations of somewhat different quark mass matrices (i.e., their corresponding nonzero elements are not exactly equal to one another) — this point will become transparent in our numerical results.
of \( r_u \sim r_d \sim 0.5 \). In this case we typically assume \( r_u \sim r_d \sim \mathcal{O}(\lambda) \) when making our analytical approximations. Up to the accuracy of \( \mathcal{O}(\lambda^3) \), we have

\[
\begin{align*}
(O_u)_{1u} & \simeq 1 - \frac{1}{2} \frac{m_u}{m_c}, \\
(O_u)_{1c} & \simeq \eta_u \sqrt{\frac{m_u}{m_c}}, \\
(O_u)_{1t} & \simeq 0, \\
(O_u)_{2u} & \simeq -\eta_u \sqrt{\frac{m_u}{m_c}} \sqrt{1 - r_u}, \\
(O_u)_{2c} & \simeq \sqrt{1 - r_u} - \frac{1}{2} \frac{m_u}{m_c}, \\
(O_u)_{2t} & \simeq \sqrt{r_u}, \\
(O_u)_{3u} & \simeq \eta_u \sqrt{\frac{m_u}{m_c}}, \\
(O_u)_{3c} & \simeq -\sqrt{r_u}, \\
(O_u)_{3t} & \simeq \sqrt{1 - r_u};
\end{align*}
\]

and

\[
\begin{align*}
(O_d)_{1d} & \simeq 1 - \frac{1}{2} \frac{m_d}{m_s} \left(1 - \frac{3 m_d}{4 m_s}\right), \\
(O_d)_{1s} & \simeq \eta_d \sqrt{\frac{m_d}{m_s}} \left(1 - \frac{1}{2} r_d - \frac{1}{2} \eta_d m_d m_b\right), \\
(O_d)_{1b} & \simeq \sqrt{r_d} \frac{m_s}{m_b} \sqrt{\frac{m_d}{m_s}}, \\
(O_d)_{2d} & \simeq -\eta_d \sqrt{\frac{m_d}{m_s}} \left[\sqrt{1 - r_d} - \frac{1}{2} r_d \left(1 - \frac{1}{2} r_d\right)\right], \\
(O_d)_{2s} & \simeq \sqrt{1 - r_d} - \frac{1}{2} \frac{m_d}{m_s} \left(1 - \frac{3 m_d}{4 m_s}\right) + \frac{1}{4} r_d m_d m_s \left(1 + \frac{1}{4} r_d\right) - \frac{1}{2} \eta_d m_d m_b \left(1 + \frac{1}{2} r_d\right), \\
(O_d)_{2b} & \simeq \sqrt{r_d} \left(1 + \frac{1}{2} \eta_d \frac{m_s}{m_b}\right), \\
(O_d)_{3d} & \simeq \eta_d \sqrt{\frac{r_d m_d}{m_s}} \left[1 - \frac{1}{2} \left(m_d m_a + \eta_d m_a m_b\right)\right], \\
(O_d)_{3s} & \simeq -\sqrt{r_d} \left[1 + \frac{1}{2} \left(\eta_d m_b - m_d m_s\right)\right], \\
(O_d)_{3b} & \simeq \sqrt{1 - r_d} - \frac{1}{2} \eta_d r_d m_s m_b \left(1 + \frac{1}{2} r_d\right);
\end{align*}
\]

where \( \sqrt{1 - r_q} \) is only written for short and it should be expanded as \( \sqrt{1 - r_q} \simeq 1 - r_q/2 - r_q^2/8 + \cdots \) in order to assure that the relevant terms can reach the precision of \( \mathcal{O}(\lambda^4) \). Since the three up-type quarks have a much stronger mass hierarchy, the analytical approximations made for the elements of \( O_u \) are much simpler at the \( \mathcal{O}(\lambda^3) \) level as compared with the corresponding results of \( O_d \) to the same degree of accuracy.

\(^3\)The accuracy of our analytical approximations is found to be only slightly worse in the case of \( r_u \sim r_d \sim 0.5 \).
With the help of Eqs. (8), (11) and (12), we obtain the following approximate results for nine elements of the CKM matrix $V$:

$$
V_{ud} \simeq 1 - \frac{1}{2} \frac{m_u}{m_s} + \eta_d \eta_u e^{i \phi_1} \sqrt{\frac{m_u}{m_c}} \sqrt{\frac{m_u}{m_s}}
$$

$$
V_{us} \simeq \eta_d \sqrt{\frac{m_u}{m_c}} \left( 1 - \frac{1}{2} \frac{m_u}{m_s} \right) + \frac{1}{2} \eta_u e^{i \phi_1} \sqrt{\frac{m_u}{m_c}} \left( r_u - 2 \sqrt{r_u r_d} e^{-i \delta} + r_d - 2 \right),
$$

$$
V_{ab} \simeq \frac{1}{2} \eta_u \sqrt{\frac{m_u}{m_c}} \left[ (2 - r_d) \sqrt{r_u} e^{i \phi_2} - (2 - r_u) \sqrt{r_d} e^{i \phi_1} \right] + \sqrt{r_d} \frac{m_u}{m_b} \sqrt{\frac{m_d}{m_s}},
$$

$$
V_{cd} \simeq \frac{1}{2} \eta_d e^{i \phi_1} \sqrt{\frac{m_d}{m_s}} \left[ \frac{m_d}{m_s} + \frac{1}{4} (r_u - r_d)^2 + r_u - 2 \sqrt{r_u r_d} e^{-i \delta} + r_d - 2 \right] + \eta_u \sqrt{\frac{m_u}{m_c}},
$$

$$
V_{cs} \simeq e^{i \phi_1} \left[ 1 - \frac{1}{4} \left( 2 - \frac{m_u}{m_s} \right) (r_u - 2 \sqrt{r_u r_d} e^{-i \delta} + r_d) - \frac{1}{16} (r_u + r_d + 2) (r_u - r_d)^2

- \frac{1}{2} \frac{m_u}{m_s} - \frac{1}{2} \eta_d \frac{m_u}{m_b} \left( r_d - \sqrt{r_u} r_d e^{-i \delta} \right) + \eta_u \eta_u e^{-i \phi_1} \sqrt{\frac{m_u}{m_c}} \sqrt{\frac{m_d}{m_s}} \right],
$$

$$
V_{cb} \simeq e^{i \phi_1} \left\{ \sqrt{r_d} \left[ 1 - \frac{1}{2} r_u - \frac{1}{8} r_u^2 - \frac{1}{16} r_u^3 + \frac{1}{4} \eta_d (2 - r_u) \frac{m_u}{m_s} \right]

- \sqrt{r_u} e^{-i \delta} \left[ 1 - \frac{1}{2} r_d - \frac{1}{8} r_d^2 - \frac{1}{16} r_d^3 - \frac{1}{2} \eta_d \frac{m_u}{m_b} \right] \right\},
$$

$$
V_{td} \simeq \eta_d e^{i \phi_1} \sqrt{\frac{m_d}{m_s}} \left[ \sqrt{r_d} e^{-i \delta} \left( 1 - \frac{1}{2} r_u - \frac{1}{8} r_u^2 - \frac{1}{16} r_u^3 - \frac{1}{2} \eta_d \frac{m_u}{m_b} \right)

- \sqrt{r_u} \left( 1 - \frac{1}{2} r_d - \frac{1}{8} r_d^2 - \frac{1}{2} \frac{m_u}{m_s} \right) \right],
$$

$$
V_{ts} \simeq e^{i \phi_1} \left\{ \sqrt{r_u} \left[ 1 - \frac{1}{2} r_d - \frac{1}{8} r_d^2 - \frac{1}{16} r_d^3 - \frac{1}{4} (2 - r_d) \frac{m_u}{m_s} - \frac{1}{2} \eta_d \frac{m_u}{m_b} \right]

- \sqrt{r_d} e^{-i \delta} \left[ 1 - \frac{1}{2} r_u - \frac{1}{8} r_u^2 - \frac{1}{16} r_u^3 + \frac{1}{4} (2 - r_u) \left( \eta_d \frac{m_u}{m_b} - \frac{m_d}{m_s} \right) \right] \right\},
$$

$$
V_{tb} \simeq e^{i \phi_2} \left[ 1 - \frac{1}{2} (r_u - 2 \sqrt{r_u} r_d e^{-i \delta} + r_d) - \frac{1}{16} (r_u + r_d + 2) (r_u - r_d)^2

- \frac{1}{2} \eta_u \frac{m_u}{m_b} \left( r_d - \sqrt{r_u} r_d e^{-i \delta} \right) \right],
$$

(13)

where $\delta \equiv \phi_1 - \phi_2 = \arg (B_d) - \arg (B_u)$ is defined. Then the moduli of nine CKM matrix elements $|V_{pq}| = \sqrt{V_{pq} V_{qp}^*}$ (for $p = u, c, t$ and $q = d, s, b$) are found to be

$$
|V_{ud}| \simeq 1 - \frac{1}{2} \frac{m_u}{m_s} + \eta_d \eta_u e^{i \phi_1} \sqrt{\frac{m_u}{m_c}} \sqrt{\frac{m_d}{m_s}} \cos \phi_1,
$$

$$
|V_{us}| \simeq \left( 1 - \frac{1}{2} \frac{m_d}{m_s} \right) \sqrt{\frac{m_d}{m_c}} + \frac{1}{2} \eta_u \eta_d \frac{m_u}{m_b} \sqrt{\frac{m_u}{m_c}} \left[ (r_u + r_d - 2) \cos \phi_1

- 2 \sqrt{r_u} r_d \cos \phi_2 \right],
$$

$$
|V_{ab}| \simeq \frac{1}{2} \sqrt{R} \frac{m_u}{m_c} \left[ 2 R - 2 r_u r_d + (r_u + r_d) \sqrt{r_u} r_d \cos \delta \right] - \frac{1}{2} \sqrt{R} \eta_u \frac{m_u}{m_b} \sqrt{\frac{m_d}{m_s}} \times \left( \frac{r_d \cos \phi_1 - \sqrt{r_u} r_d \cos \phi_2} {r_d \cos \phi_1 - \sqrt{r_u} r_d \cos \phi_2} \right),
$$

$$
|V_{cd}| \simeq \frac{1}{2} \sqrt{R} \left[ 1 - \frac{1}{2} R - \frac{1}{2} \frac{m_d}{m_s} + \frac{1}{2} r_u r_d \sin^2 \delta - \frac{1}{8} (r_u - r_d)^2 \right] + \frac{1}{2} \eta_d \frac{m_u}{m_b} \sqrt{\frac{m_u}{m_c}} \sin^2 \phi_1
$$

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\[-\eta_d \eta_\lambda \sqrt{\frac{m_u}{m_c}} (\cos \phi_1 + \sqrt{r_u r_d} \sin \phi_1 \sin \delta),\]

\[|V_{cs}| \simeq 1 - \frac{1}{2} R - \frac{1}{4} (2 - R) \frac{m_d}{m_s} - \frac{1}{16} \left\{ \left( r_u^3 + r_d^3 + 2 (r_u - r_d)^2 - r_u r_d \right) \left(3 - 2 \cos 2\delta\right) \times (r_u + r_d + 4 \sin \delta (2 \sin \delta - \sqrt{r_u r_d} \sin 2\delta) \right\} + \eta_u \eta_d \sqrt{\frac{m_u}{m_c}} \sqrt{\frac{m_d}{m_s}} \cos \phi_1 \]

\[-\frac{1}{2} \eta_d \frac{m_u}{m_b} (r_d - \sqrt{r_u r_d} \cos \delta),\]

\[|V_{cb}| \simeq \sqrt{R} + \frac{1}{2 \sqrt{R}} \sqrt{r_u r_d} \left[ (r_u + r_d) \cos \delta - 2 \sqrt{r_u r_d} \right] + \frac{1}{2 R \sqrt{R}} \left\{ \frac{1}{4} (\cos 2\delta - 3) r_u^2 r_d^2 + \frac{1}{4} (r_u + r_d)^3 \sqrt{r_u r_d} \cos \delta - \frac{3}{4} r_u r_d (r_u^2 + r_d^2) \cos^2 \delta + \eta_d \frac{m_s}{m_b} \left( r_d^2 - (r_u + 3 r_d) \times \sqrt{r_u r_d} \cos \delta + r_u r_d (2 + \cos 2\delta) \right) \right\},\]

\[|V_{td}| \simeq \frac{1}{2} \sqrt{R} \left(2 - \frac{m_d}{m_s}\right) R + \frac{1}{\sqrt{r_u r_d}} \left[ (r_u + r_d) \cos \delta - 2 \sqrt{r_u r_d} \right] \]

\[-\frac{1}{4 R} \sqrt{r_u r_d} \left[ (r_u^2 + r_d^2)^2 \cos \delta - r_u r_d \left( 5 + \cos 2\delta \right) - 4 \sqrt{r_u r_d} (r_u + r_d) \cos \delta \right] + \frac{1}{4} (r_u - r_d)^2 \sqrt{r_u r_d} \cos \delta - \eta_d \frac{m_s}{m_b} (r_d - \sqrt{r_u r_d} \cos \delta) \right\},\]

\[|V_{ts}| \simeq \frac{1}{2} \sqrt{R} \left(2 - \frac{m_d}{m_s}\right) + \frac{1}{2 R \sqrt{R}} \sqrt{r_u r_d} \left[ (r_u + r_d) \cos \delta - 2 \sqrt{r_u r_d} \right] + \frac{1}{8 R \sqrt{R}} \]

\[\times \left\{ \sqrt{r_u r_d} \left( r_u + r_d \right)^3 \cos \delta - r_u r_d \left[ 3 (r_u - r_d)^2 \cos^2 \delta + 4 r_u r_d (1 + \cos^2 \delta) \right] \right\} + \frac{1}{4} \eta_d \frac{m_s}{m_b} \left[ r_d^2 + r_u r_d (2 + \cos 2\delta) - \sqrt{r_u r_d} (r_u + 3 r_d) \cos \delta \right]\right\},\]

\[|V_{tb}| \simeq 1 - \frac{1}{2} R - \frac{1}{16} (r_u + r_d + 2) (r_u - r_d)^2 + \frac{1}{4} (2 + R) r_u r_d \sin^2 \delta \frac{1}{2} \eta_d \frac{m_s}{m_b} \]

\[\times (r_d - \sqrt{r_u r_d} \cos \delta), \quad (14)\]

where \(R \equiv r_u + r_d - 2 \sqrt{r_u r_d} \cos \delta\) has been defined. Note that in Eq. (14) the expansion terms of \(|V_{cb}|\) and \(|V_{ts}|\) are only kept up to \(\mathcal{O}(\lambda^{2.5})\) for the sake of simplicity, and the accuracy of the other CKM matrix elements is at the level of \(\mathcal{O}(\lambda^{3.5})\). Note also that \(R \lesssim \mathcal{O}(\lambda)\) is required to assure the analytical approximations made above to be valid. If the experimental values of \(|V_{ub}|, |V_{cb}|, |V_{td}|\) and \(|V_{ts}|\) are taken into consideration, we actually find that \(R\) should be of \(\mathcal{O}(\lambda^4)\). One will see later on that the smallness of \(R\) mainly results from \(r_u \simeq r_d\) and \(\delta \sim 0\).

With the help of Eq. (14), we immediately arrive at the following three typical ratios of the CKM matrix elements:

\[\frac{|V_{ub}|}{|V_{cb}|} \simeq \sqrt{\frac{m_u}{m_c} + \eta_d \frac{m_s}{m_b} \sqrt{r_d m_d \sqrt{r_u} r_u r_d + r_d - 2 \sqrt{r_u r_d} \cos \delta}},\]

\[\frac{|V_{td}|}{|V_{ts}|} \simeq \sqrt{\frac{m_d}{m_s} \left[ 1 - \eta_d \frac{m_s}{m_b} \frac{r_d^2 + (2 + \cos 2\delta) r_u r_d - (3 r_d + r_u) \sqrt{r_u r_d} \cos \delta}{(r_u + r_d - 2 \sqrt{r_u r_d} \cos \delta)^2} \right]}, \quad (15)\]

and \n
\[\frac{|V_{us}|}{|V_{ud}|} \simeq \sqrt{\frac{m_d}{m_s} + \frac{1}{2} \eta_d \eta_u \sqrt{\frac{m_u}{m_c} \left( (r_u + r_d - 2) \cos \phi_1 + 2 \sqrt{r_u r_d} \cos \phi_2 \right) + \frac{1}{2} \frac{m_u}{m_c} \frac{m_s}{m_d} \sin^2 \phi_1}}. \quad (16)\]
One can see that $|V_{td}/V_{ts}| \simeq \sqrt{m_d/m_s}$ and $|V_{us}/V_{ud}| \simeq \sqrt{m_d/m_s}$ hold at the leading order. These two simple and instructive relations are well in accordance with current experimental data. But a similar leading-order relation $|V_{ub}/V_{cb}| \simeq \sqrt{m_u/m_c}$ is obviously in conflict with the values of $m_u, m_c$, $|V_{ub}|$ and $|V_{cb}|$ given in Eqs. (14) and (2), although it is reasonably expected in the $m_b \to \infty$ limit [27,28]. That is why the second term on the right-hand side of $|V_{ub}/V_{cb}|$ in Eq. (15) is crucial to fill in the gap and resolve the issue.

Let us proceed to make a ball-park estimate of the allowed ranges of those free parameters appearing in Eq. (15). If $|V_{ub}/V_{cb}| \simeq 0.089$, $\sqrt{m_u/m_c} \simeq 0.045$ and $m_s/m_b \sqrt{m_d/m_s} \simeq 0.004$ are typically taken into account, we find that

$$\sqrt{r_d} \frac{\sqrt{r_u} \cos \phi_2 - \sqrt{r_d} \cos \phi_1}{r_u + r_d - 2 \sqrt{r_u r_d} \cos \delta} \sim 10$$

should hold so as to make the first formula in Eq. (15) valid. On the other hand, we keep in mind that the relation $|V_{us}| \simeq \sqrt{m_d/m_s}$ coincides with the experimental data very well. Thus the term proportional to $\sqrt{m_u/m_c}$ in the expression of $|V_{us}|$ as shown in Eq. (14) must be very small. This observation implies that $\cos \phi_1 \sim 0$ is naturally expected. Substituting $\cos \phi_1 \sim 0$ into Eq. (17), one has

$$\frac{\xi \cos \phi_2}{1 + \xi^2 \pm 2 \xi \sin \phi_2} \sim 10,$$

where $\xi \equiv \sqrt{r_u/r_d}$ is defined, and “±” corresponds to $\phi_1 \sim \pm \pi/2$. The left-hand side of Eq. (18) is a function of $\phi_2$ and takes its maximum value $|\xi/ (1 - \xi^2)|$ at $\sin \phi_2 = \pm 2\xi/(1 + \xi^2)$. As a consequence, the condition in Eq. (18) leads us to the constraint $|\xi/ (1 - \xi^2)| \gtrsim 10$, or equivalently $(\sqrt{401 - 1})/20 \lesssim \xi \lesssim (\sqrt{401 + 1})/20$. This result means that the ratio $r_u/r_d$ lies in the range of 0.9 to 1.1. In other words, $r_u \simeq r_d$ is expected to hold as a good approximation. This interesting point has been seen from a careful numerical analysis of the parameter space of $r_u$ and $r_d$ in Ref. [26], but here we reach the same observation based mainly on our reliable analytical approximations made above.

### 2.2 A special case with $r_u = r_d$

Now that $r_u \simeq r_d$ is favored by current experimental data (especially by today’s experimental result for $|V_{ub}/V_{cb}|$), we are well motivated to consider the special case $r_u = r_d \equiv r$ as another natural consequence of the structural parallelism between the up- and down-quark sectors. In this case Eq. (17) is simplified to $|\sin \phi/ \sin (\delta/2)| \sim 20$, where $\phi \equiv (\phi_1 + \phi_2)/2$ has been defined. Then we are left with $\phi_2 \sim \phi_1 \sim \pm \pi/2$, an estimate consistent with Eq. (18) for $\xi = 1$. Moreover, the expression of $R$ is simplified to $R = 4r \sin^2 (\delta/2)$, which can easily reach the level of $\mathcal{O}(\lambda^2)$ for $r \sim \mathcal{O}(\lambda)$ due to the smallness of $\delta = \phi_1 - \phi_2$ in magnitude. As a result, the approximate formulas for the CKM matrix elements in Eq. (14) are now simplified to

$$|V_{ud}| = 1 - \frac{1}{2} \frac{m_d}{m_s} + \eta_t \eta_d \sqrt{m_d/m_s} \cos \phi_1,$$

$$|V_{us}| = \sqrt{m_d/m_s} \left(1 - \frac{1}{2} \frac{m_d}{m_s}\right) + \eta_t \eta_d \sqrt{m_u/m_c} \left((r - 1) \cos \phi_1 - r \cos \phi_2 \right) + \frac{1}{2} \frac{m_u}{m_c} \sqrt{m_s/m_d} \sin \phi_1,$$

$$|V_{ub}| = \sqrt{r} \left(2 - r \right) \sqrt{m_u/m_c} + \eta_t \eta_d \sqrt{m_d/m_s} \sin \phi \left[\sin \frac{\delta}{2}\right].$$
Accordingly, Eqs. (15) and (16) are simplified to

\[ |V_{cd}| = \sqrt{\frac{m_d}{m_s}} \left( 1 - 2r \sin^2 \frac{\delta}{2} + \frac{1}{2} r^2 \sin^2 \delta - \frac{1}{2} m_d \right) + \frac{1}{2} m_u \frac{m_d}{m_c} \sqrt{\frac{m_s}{m_d}} \sin^2 \phi_1 \]

\[-n_u n_d \left( \frac{m_u}{m_c} - r \sin \phi_1 \sin \delta + \cos \phi_1 \right),\]

\[ |V_{cs}| = 1 - \frac{1}{2} \frac{m_d}{m_s} - r \left( 2 - \frac{m_d}{m_s} + n_u \left( \frac{m_u}{m_b} \right) \right) \frac{\sin^2 \frac{\delta}{2}}{2} + \frac{1}{2} r^2 \left( 1 + 2r \sin^2 \frac{\delta}{2} \right) \sin^2 \delta \]

\[ + n_u n_d \sqrt{\frac{m_u}{m_c} - \frac{m_d}{m_s} \cos \phi_1},\]

\[ |V_{cb}| = \sqrt{r} \left( 2 - r - \frac{1}{4} r^2 + \frac{1}{2} n_u \frac{m_u}{m_b} \right) \left| \sin \frac{\delta}{2} \right|,\]

\[ |V_{td}| = \sqrt{r} \left( 2 - r - \frac{1}{4} r^2 + \frac{1}{2} n_u \frac{m_u}{m_b} - \frac{m_d}{m_s} \right) \left| \sin \frac{\delta}{2} \right|,\]

\[ |V_{ts}| = \sqrt{r} \left( 2 - r - \frac{1}{4} r^2 + \frac{1}{2} n_u \frac{m_u}{m_b} - \frac{m_d}{m_s} \right) \left| \sin \frac{\delta}{2} \right|,\]

\[ |V_{tb}| = 1 - r \left( 2 + n_d \frac{m_d}{m_b} \right) \frac{\sin^2 \frac{\delta}{2}}{2} + \frac{1}{2} r^2 \left( 1 + 2r \sin^2 \frac{\delta}{2} \right) \sin^2 \delta. \quad (19)\]

Accordingly, Eqs. (15) and (16) are simplified to

\[ \frac{|V_{ub}|}{|V_{cb}|} = \sqrt{\frac{m_u}{m_c}} + \frac{1}{2} n_u \frac{m_u}{m_b} \sqrt{\frac{m_d}{m_s} \sin \frac{\delta}{2}},\]

\[ \frac{|V_{td}|}{|V_{ts}|} = \sqrt{\frac{m_d}{m_s}} \left( 1 - \frac{1}{2} n_d \frac{m_d}{m_b} \right), \quad (20)\]

and

\[ \frac{|V_{us}|}{|V_{ud}|} = \sqrt{\frac{m_d}{m_s}} + n_u n_d \sqrt{\frac{m_u}{m_c}} \left( r - 1 \right) \cos \phi_1 - r \cos \phi_2 \right] + \frac{1}{2} m_u \frac{m_d}{m_c} \sqrt{\frac{m_s}{m_d}} \sin^2 \phi_1, \quad (21)\]

respectively. Some brief comments are in order.

- One can see that the expressions of $|V_{ub}|$, $|V_{cb}|$, $|V_{td}|$ and $|V_{ts}|$ are all approximately proportional to $| \sin (\delta/2) |$ in this special case. This result is exactly a consequence of the up-down structural parallelism, implying some significant cancellations between the two sectors so as to arrive at sufficiently small values for the four off-diagonal CKM matrix elements associated with the third-family quarks. Note, however, that $\sin (\delta/2)$ should not be too small, in order to assure $|V_{cb}|$ and $|V_{ts}|$ to reach the level of $O(\lambda^2)$.

- Eq. (19) shows that $|V_{ub}| > |V_{cs}|$, $|V_{ud}| > |V_{cs}|$ and $|V_{cb}| > |V_{ts}|$ hold as a natural result of the smallness of $r$ and $\delta$. These fine inequalities reveal some details of the obtained pattern of the unitary CKM matrix $V$, and they are fully consistent with the experimental data listed in Eq. (2) [8].

- It is worth pointing out that the analytical approximations given in Eqs. (19)–(21) keep unchanged under the replacements $n_u \rightarrow -n_u$, $n_d \rightarrow n_d$ and $\phi_{1,2} \rightarrow \phi_{1,2} \pm \pi$. This observation means that once $n_d$ is fixed (i.e., either $n_d = +1$ or $-1$), the $n_u = \pm 1$ cases will lead to the same results for the moduli of nine CKM matrix elements with the help of Eqs. (19)–(21). The two independent phases in these two cases differ from each other
by $\pi$. One may easily check that such a conclusion is also valid for the generic analytical approximations made in the $r_u \neq r_d$ case, as shown in Eq. (14). But one should keep in mind that the validity of the above observations is essentially a consequence of $(O_u)_{1u} \simeq 0$ up to $O(\lambda^4)$ in Eq. (11), which leads us to $|V_{tq}| \simeq |(O_u)_{2t} (O_d)_{2q} e^{i\phi_1} + (O_u)_{3t} (O_d)_{3q} e^{i\phi_2}|$ (for $q = d, s, b$) as can be seen from Eq. (8). In view of the structural parallelism between $M_u$ and $M_d$, one might naively expect that a similar conclusion could be drawn for the $\eta_d = \pm 1$ cases against a fixed choice of $\eta_u$. However, it is indeed not the case because the much weaker hierarchy of three down-type quark masses results in $(O_d)_{1b} \sim O(\lambda^3)$ as shown in Eq. (12). Hence $|V_{pb}|$ is sensitive to the contribution from $(O_u)_{1p} (O_d)_{1b}$ (for $p = u, c, t$) to some extent in our analytical approximations.

In short, the Ansatz based on the assumption of $r_u = r_d$ might be interesting from the point of view of model building with the help of a kind of flavor symmetry, but an apparent asymmetry between the mass hierarchies of up- and down-type quarks implies that $r_u \simeq r_d$ should be more reasonable and thus more likely. Of course, why $m_t/m_b \sim 60, m_c/m_s \sim 12$ but $m_u/m_d \sim 0.5$ hold at a given energy scale remains a big puzzle in particle physics, especially in view of the fact that the masses of up- and down-type quarks originate from the similar Yukawa interactions in the SM and its natural extensions.

### 2.3 Textures of $M_{u,d}$ and CP violation

Once the free parameters $r_u$ and $r_d$ are well constrained by current experimental data on the CKM matrix elements and quark masses based on the zero textures of $M_u$ and $M_d$, one will be able to determine or constrain those nonzero elements of $M_u$ and $M_d$ to gain a deeper understanding of their hierarchical textures. So let us go back to Eq. (5) and reexpress those real matrix elements as follows:

\[
\frac{|C_u|}{m_t} = \frac{m_c}{m_t} \sqrt{\frac{1}{1 - r_u m_c}},
\]

\[
\frac{\tilde{B}_u}{m_t} = r_u - \eta_u \frac{m_u}{m_t} + \eta_u \frac{m_c}{m_t},
\]

\[
\frac{|B_u|}{m_t} = \sqrt{r_u \left( 1 - r_u + \frac{m_u}{m_t} \right)} - \frac{1}{1 - r_u} \frac{m_u m_c}{m_t^2},
\]

and

\[
\frac{|C_d|}{m_b} = \frac{m_s}{m_b} \sqrt{\frac{1}{1 - r_d m_s}},
\]

\[
\frac{\tilde{B}_d}{m_b} = r_d - \eta_d \frac{m_d}{m_b} + \eta_d \frac{m_s}{m_b},
\]

\[
\frac{|B_d|}{m_b} = \sqrt{r_d \left( 1 - r_d + \frac{m_d}{m_b} \right)} - \frac{1}{1 - r_d} \frac{m_d m_s}{m_b^2}.
\]

It is then straightforward to obtain the approximate analytical results of these parameters by making appropriate expansions of the right-hand sides of Eqs. (22) and (23) in terms of the quark mass ratios and $r_q$ (for $q = u, d$). Here we focus on the ratios $\tilde{B}_u/|B_u|$ and $\tilde{B}_d/|B_d|$, in
order to understand why fitting the experimental result of $|V_{ub}/V_{cb}|$ requires the zero textures of $M_u$ and $M_d$ to have a somewhat weak hierarchy on their right-bottom corners.

To be explicit, we have

$$\frac{\hat{B}_u}{|B_u|} \simeq \sqrt{r_u} \left(1 + \frac{1}{2} r_u + \frac{3}{8} r_u^2\right),$$

$$\frac{\hat{B}_d}{|B_d|} \simeq \sqrt{r_d} \left[1 + \left(\frac{1}{2} r_d + \eta_d \frac{m_a}{r_d} m_b\right) + \left(\frac{3}{8} r_d^2 + \eta_d \frac{m_a}{m_b}\right)\right],$$

(24)

up to the accuracy of $O(\lambda^3)$ for $r_u \sim r_d \sim O(\lambda)$. Given the fact that $r_u$ and $r_d$ measure the deviations of $A_u$ and $A_d$ respectively from $m_i$ and $m_b$, we arrive at

$$\frac{\hat{B}_u}{|B_u|} \sim \frac{|B_u|}{A_u} \sim \sqrt{r_u}, \quad \frac{\hat{B}_d}{|B_d|} \sim \frac{|B_d|}{A_d} \sim \sqrt{r_d}.$$  

(25)

This approximate observation implies that the four nonzero elements on the right-bottom corner of $M_q$ (for $q = u$ or $d$) should not have a very strong hierarchy in magnitude. Instead, their values satisfy an approximate “seesaw” relation $\tilde{B}_q \sim |B_q|^2/A_q$. Although such a relation has been observed before [25,26], our present analytical approximations provide a much better understanding of why this is the case. If the $\textit{Ansatz}$ with $r_u = r_d \equiv r$ is taken into account, one will be similarly left with $\hat{B}_u/|B_u| \sim \hat{B}_d/|B_d| \sim \sqrt{r}$.

Let us highlight that the seesaw-like relation $\tilde{B}_q \sim |B_q|^2/A_q$ associated with the structure of $M_q$ is actually a phenomenological compromise between resolving the failure of the popular relation $|V_{ub}/V_{cb}| \simeq \sqrt{m_u/m_c}$ and generating a sufficiently strong quark mass hierarchy (i.e., $m_c \ll m_u$ and $m_s \ll m_b$). In this way we are left with a relatively weak texture hierarchies on the right-bottom corners of $M_u$ and $M_d$, and achieve a sufficiently large correction to $|V_{ub}/V_{cb}|$ from the down-type quark sector as shown in Eq. (15) or Eq. (20).

It is well known that the strength of CP violation in the quark sector can be described by the rephasing-invariant Jarlskog parameter $\mathcal{J}$ [29], which is defined by

$$\mathcal{J} \sum_\gamma \varepsilon_{\alpha \beta \gamma} \sum_k \varepsilon_{ijk} = \text{Im} \left(V_{a_\alpha} V_{b_\beta} V_{c_\gamma} V_{d_k}^* \varepsilon_{ijk}\right),$$

(26)

where the Greek and Latin subscripts run over the up- and down-type quark flavors, respectively. Taking advantage of the analytical approximations made for the CKM matrix elements in Eq. (13), we obtain the approximate expression of the Jarlskog invariant as follows:

$$\mathcal{J} \simeq \eta_u \sqrt{r_u r_d} \frac{m_d}{m_b} \sin \delta - \frac{1}{2} \eta_u \eta_d \sqrt{\frac{m_u}{m_c}} \sqrt{\frac{m_d}{m_s}} \left\{ \left[r_u + r_d - 2 \right] \left[r_u + r_d - 4 \sqrt{r_u r_d} \cos \delta\right]
+ 4 \sqrt{r_u r_d} \left(\sqrt{r_u r_d} - \cos \delta\right) \right\} \sin \phi_1 - 2 \sqrt{r_u r_d} \left[r_u + r_d - 2 \sqrt{r_u r_d} \cos \delta\right] \sin \phi_2,$$

(27)

in which $\delta$ is correlated with $\phi_1$ and $\phi_2$ through $\delta = \phi_1 - \phi_2$. If the $\textit{Ansatz}$ with $r_u = r_d$ is taken into consideration, then Eq. (27) can be simplified to

$$\mathcal{J} \left(r_u = r_d \equiv r\right) \simeq 4 \eta_u \eta_d \sqrt{\frac{m_u}{m_c}} \sqrt{\frac{m_d}{m_s}} \left[\left(1 - 2r\right) \sin \phi_1 + r \sin \phi_2\right] \sin^2 \frac{\delta}{2} \eta_d r \frac{m_d}{m_b} \sin \delta.$$

(28)

The two terms on the right-hand side of Eq. (28) should contribute comparably to the Jarlskog invariant, because both $\delta$ and $r$ are small. In this case the observed positivity of $\mathcal{J}$ [8] requires
\[ \eta_u \eta_d \sin \phi_1 > 0 \] and \[ \eta_d \sin \delta > 0. \] To make the result of \( |V_{ub}|/|V_{cb}| \) in Eq. (20) coincide with the experimental data, \( \eta_u \sin \phi / \sin (\delta/2) > 0 \) is demanded. Similarly, the result of \( |V_{us}| \) given in Eq. (19) requires \( \eta_u \eta_d \cos \phi_1 < 0 \). For \( \phi_4 \sim \phi_2 \sim \pm \pi/2 \), we find that \( \sin \phi_1, \sin \phi_2 \) and \( \sin \phi \) should all have the same sign. Thus we have

\[ \phi_1 \sim \phi_2 \sim \frac{\pi}{2} \text{ or } \frac{3\pi}{2}, \quad \phi_1 \geq \frac{\pi}{2} \text{ or } \frac{3\pi}{2}, \quad \eta_u \eta_d \sin \phi_1 > 0, \quad \eta_d \delta > 0, \quad (29) \]

and \( \eta_u \sin \phi / \sin (\delta/2) > 0 \) is accordingly satisfied. The constraints obtained in Eq. (29) are fully consistent with those obtained in Ref. [26], where different phase conventions were used.

### 3 Numerical results

With the help of the approximate expressions of \( |V_{us}|, |V_{ub}| \) and \( |V_{cb}| \) given in Eq. (19) and the experimental data shown in Eq. (2), one may easily determine the values of \( r, \phi_1 \) and \( \phi_2 \) for the Ansatz considered in section 2.2. In this regard the positivity of \( J \) in Eq. (28) can be used to resolve possible ambiguities associated with the two phase parameters. For the sake of simplicity, here we only adopt the central values of \( |V_{us}|, |V_{ub}|, |V_{cb}| \) and six quark masses listed in Eqs. (1) and (2) in our numerical calculations.

#### 3.1 \( \eta_u = +1 \) and \( \eta_d = +1 \)

In this case we have \( (r, \phi_1, \phi_2) = (0.18556, 0.53216\pi, 0.49914\pi) \). Then we arrive at

\[ |V| \simeq \begin{pmatrix} 0.97388 & 0.22650 & 0.00361 \\ 0.22632 & 0.97308 & 0.04053 \\ 0.00874 & 0.03941 & 0.99918 \end{pmatrix}, \quad (30) \]

and

\[ \frac{|V_{ub}|}{|V_{cb}|} \simeq 0.08497, \quad \frac{|V_{td}|}{|V_{ts}|} \simeq 0.22201, \quad \frac{|V_{us}|}{|V_{ud}|} \simeq 0.23213, \quad (31) \]

where Eqs. (19), (20) and (21) have been used. If the above three ratios are directly calculated from the results in Eq. (30), their values will deviate from those obtained in Eq. (31) by 4.6%, 0.13% and 0.19%, respectively. Using Eq. (28) to calculate the Jarlskog invariant, we obtain a reasonable value \( J \simeq 3.42 \times 10^{-5} \).

To check the accuracy of our analytical approximations, we recalculate the moduli of nine CKM matrix elements by means of Eqs. (4) and (8) in the case of \( r_u = r_d = r \). We find that the numerical results obtained in this way (named as “exact”) deviate slightly from those achieved from the analytical approximations in Eq. (30):

\[ \frac{|V|_{\text{approx}} - |V|_{\text{exact}}}{|V|_{\text{exact}}} \simeq \begin{pmatrix} -0.03149 & +0.35053 & -5.70473 \\ +0.33591 & -0.03112 & +0.11371 \\ -0.21264 & -0.66504 & +0.0054 \end{pmatrix} \% . \quad (32) \]

It is obvious that the two smallest CKM matrix elements (namely, \( |V_{ub}| \) and \( |V_{td}| \)) involve the largest uncertainties, but the relative errors are at most at the level of \( O(6\%) \). Moreover,
the corresponding value of the Jarlskog invariant is found to be $J_{\text{exact}} \simeq 3.29 \times 10^{-5}$, and the corresponding textures of quark mass matrices read as follows:

$$\overline{M}_u \simeq m_t \begin{pmatrix} 0 & 0.00118 & 0 \\ 0.00118 & 0.18924 & 0.38787 \\ 0 & 0.38787 & 0.81444 \end{pmatrix},$$

$$\overline{M}_d \simeq m_b \begin{pmatrix} 0 & 0.00465 & 0 \\ 0.00465 & 0.20335 & 0.38787 \\ 0 & 0.38787 & 0.81444 \end{pmatrix},$$

which lead us to $\tilde{B}_u/|B_u| \simeq 0.488$ and $\tilde{B}_d/|B_d| \simeq 0.529$. The values of these two ratios are quite close to $1/2$, an interesting possibility which has been discussed by one of us in Ref. [30].

Eq. (33) clearly shows that the texture hierarchy associated with the right-bottom corner of $M_q$ is somewhat weaker than naively expected, and an approximate seesaw-like relation $\tilde{B}_q \sim |B_q|^2/A_q$ holds (for $q = u$ or $d$). One can therefore understand why those previous studies in the seemingly “natural” assumption of $A_q \gg |B_q| \gg \tilde{B}_q$ (see, e.g., Refs. [18–20,30–39]) are invalid today to explain the experimental result for $|V_{ub}/V_{cb}|$.

3.2 $\eta_u = +1$ and $\eta_d = -1$

In this case we have $(r, \phi_1, \phi_2) = (0.18371, 1.51969\pi, 1.55317\pi)$. Then we obtain

$$|V| = \begin{pmatrix} 0.97427 & 0.22650 & 0.00361 \\ 0.22632 & 0.97348 & 0.04053 \\ 0.00892 & 0.03940 & 0.99918 \end{pmatrix},$$

and

$$\frac{|V_{ub}|}{|V_{cb}|} \simeq 0.08419, \quad \frac{|V_{td}|}{|V_{ts}|} \simeq 0.22621, \quad \frac{|V_{us}|}{|V_{ud}|} \simeq 0.23213,$$

where Eqs. (19), (20) and (21) have been used. If the above three ratios are simply calculated from the results in Eq. (34), their values will deviate from those give in Eq. (35) by $5.5\%$, $0.13\%$ and $0.15\%$, respectively. As for the Jarlskog invariant, we arrive at $J \simeq 3.46 \times 10^{-5}$ by means of the approximate formula given in Eq. (28).

Taking $r_u = r_d \equiv r$ and using Eqs. (6) and (8), we recalculate the moduli of nine CKM matrix elements and compare the results with those approximate ones obtained in Eq. (34):

$$\frac{|V|_{\text{approx}} - |V|_{\text{exact}}}{|V|_{\text{exact}}} \simeq \begin{pmatrix} +0.02942 & -0.03841 & -4.50882 \\ -0.05558 & +0.01314 & -0.64582 \\ -0.01523 & -1.46565 & +0.00196 \end{pmatrix} \%. \quad (36)$$

In addition, we obtain $J_{\text{exact}} \simeq 3.26 \times 10^{-5}$ for the Jarlskog invariant. The textures of quark mass matrices are found to be

$$\overline{M}_u \simeq m_t \begin{pmatrix} 0 & 0.00118 & 0 \\ 0.00118 & 0.18739 & 0.38638 \\ 0 & 0.38638 & 0.81629 \end{pmatrix},$$

$$\overline{M}_d \simeq m_b \begin{pmatrix} 0 & 0.00464 & 0 \\ 0.00464 & 0.16593 & 0.39144 \\ 0 & 0.39144 & 0.81629 \end{pmatrix},$$

from which we get $\tilde{B}_u/|B_u| \simeq 0.485$ and $\tilde{B}_d/|B_d| \simeq 0.424$. 

13
3.3 \( \eta_u = -1 \) and \( \eta_d = +1 \)

In this case we have \((r, \phi_1, \phi_2) = (0.18556, 1.53216\pi, 1.49914\pi)\). Then we are left with

\[
|V| \simeq \begin{pmatrix} 0.97388 & 0.22650 & 0.00361 \\ 0.22632 & 0.97308 & 0.04053 \\ 0.00874 & 0.03941 & 0.99918 \end{pmatrix},
\]

(38)

and

\[
\left| \frac{V_{ub}}{V_{cb}} \right| \simeq 0.08497, \quad \left| \frac{V_{td}}{V_{ts}} \right| \simeq 0.22201, \quad \left| \frac{V_{us}}{V_{ud}} \right| \simeq 0.23213,
\]

(39)

where Eqs. (19), (20) and (21) have been used. If the above three ratios are directly calculated by using the results obtained in Eq. (38), their values will deviate from those given in Eq. (39) by 4.6\%, 0.13\% and 0.19\%, respectively. Moreover, we arrive at \( J \simeq 3.42 \times 10^{-5} \).

Taking \( r_u = r_d \equiv r \) and using Eqs. (6) and (8), we recalculate the moduli of nine CKM matrix elements and compare the results with those approximate ones obtained in Eq. (38):

\[
\left| \frac{|V|_{\text{approx}} - |V|_{\text{exact}}}{|V|_{\text{exact}}} \right| \simeq \begin{pmatrix} -0.03134 +0.34760 -5.65788 \\ +0.32934 -0.03021 -2.0508 \\ +0.22741 -1.10930 +0.00107 \end{pmatrix} \%
\]

(40)

In addition, we obtain \( J_{\text{exact}} \simeq 3.23 \times 10^{-5} \) and

\[
\overline{M}_u \simeq m_t \begin{pmatrix} 0 & 0.00018 & 0 \\ 0.00018 & 0.18189 & 0.38963 \\ 0 & 0.38963 & 0.81444 \end{pmatrix},
\]

\[
\overline{M}_d \simeq m_b \begin{pmatrix} 0 & 0.00465 & 0 \\ 0.00465 & 0.20335 & 0.38448 \\ 0 & 0.38448 & 0.81444 \end{pmatrix},
\]

(41)

from which \( \tilde{B}_u/|B_u| \simeq 0.467 \) and \( \tilde{B}_d/|B_d| \simeq 0.529 \) can be extracted.

3.4 \( \eta_u = -1 \) and \( \eta_d = -1 \)

In this case we have \((r, \phi_1, \phi_2) = (0.18371, 0.51969\pi, 0.55318\pi)\) and obtain

\[
|V| \simeq \begin{pmatrix} 0.97427 & 0.22650 & 0.00361 \\ 0.22632 & 0.97348 & 0.04053 \\ 0.00892 & 0.03940 & 0.99918 \end{pmatrix},
\]

(42)

and

\[
\left| \frac{V_{ub}}{V_{cb}} \right| \simeq 0.08419, \quad \left| \frac{V_{td}}{V_{ts}} \right| \simeq 0.22621, \quad \left| \frac{V_{us}}{V_{ud}} \right| \simeq 0.23213,
\]

(43)

where Eqs. (19), (20) and (21) have been used. If the above three ratios are simply calculated by means of the results given in Eq. (42), they will deviate from those obtained in Eq. (43) by 5.5\%, 0.13\% and 0.15\%, respectively. As for the Jarlskog invariant of CP violation, we get \( J \simeq 3.46 \times 10^{-5} \) with the help of Eq. (28).
Taking \( r_u = r_d \equiv r \) and using Eqs. (6) and (8), we recalculate the moduli of nine CKM matrix elements and compare the results with those approximate ones obtained in Eq. (42):

\[
\frac{|V|_{\text{approx}} - |V|_{\text{exact}}}{|V|_{\text{exact}}} \simeq \begin{pmatrix}
+0.02972 & -0.04378 & -4.82382 \\
-0.05785 & +0.03054 & -0.06763 \\
-2.01585 & -0.76827 & +0.00101
\end{pmatrix} \%
\]

Furthermore, we obtain \( J_{\text{exact}} \simeq 3.31 \times 10^{-5} \) and

\[
\begin{align*}
\bar{M}_u & \simeq m_t \begin{pmatrix}
0 & 0.00018 & 0 \\
0.00018 & 0.18003 & 0.38812 \\
0 & 0.38812 & 0.81629
\end{pmatrix}, \\
\bar{M}_d & \simeq m_b \begin{pmatrix}
0 & 0.00464 & 0 \\
0.00464 & 0.16593 & 0.39144 \\
0 & 0.39144 & 0.81629
\end{pmatrix},
\end{align*}
\]

from which \( \tilde{B}_u/|B_u| \simeq 0.464 \) and \( \tilde{B}_d/|B_d| \simeq 0.424 \) can be achieved.

Our numerical results confirm that the moduli of nine CKM matrix elements and the Jarlskog invariant of CP violation obtained from our analytical approximations are the same in the cases with \( \eta_d = +1 \) (or \( \eta_d = -1 \)) and \( \eta_u = \pm 1 \), as pointed out in section 2.2. As mentioned in section 2.2, however, this conclusion only approximately holds. By comparing between Eqs. (32) and (40) or between Eqs. (36) and (44), one can see that the moduli of nine CKM matrix elements calculated exactly by means of Eqs. (6) and (8) are slightly different from each other in the cases of \( \eta_d = +1 \) (or \( \eta_d = -1 \)) and \( \eta_u = \pm 1 \). Moreover, the nonzero elements of up- and down-type quark mass matrices are not exactly the same in the above four cases. That is why in the beginning of section 2 we have pointed out that different choices of the values of \( \eta_u \) and \( \eta_d \) correspond to different flavor bases and thus to slightly different models of quark mass matrices.

4 Conclusions

Since 1977, many attempts have been made towards understanding the zero textures of quark mass matrices and their connections with the CKM matrix elements. Imposing the Hermiticity and parallelism on the up- and down-type quark sectors, we are now left with the zero textures of \( M_u \) and \( M_d \) shown in Eq. (3) as the simplest viable possibility. To fit current experimental data on quark flavor mixing and CP violation (especially the observed value of \( |V_{ub}/V_{cb}| \)), however, we have analytically demonstrated that the (2, 2), (2, 3) and (3, 3) elements of \( M_q \) should have a relatively weak texture hierarchy, much weaker than the corresponding quark mass hierarchy. That is why the three relevant nonzero elements of \( M_q \) satisfy an approximate seesaw-like relation \( \tilde{B}_q \sim |B_q|^2/A_q \) (for \( q = u \) or \( d \)), and \( |V_{ub}/V_{cb}| \) receives a quite large correction from the down-type quark sector.

We highlight that our reliable analytical approximations have helped quite a lot in resolving the phenomenological failure of the popular relation \( |V_{ub}/V_{cb}| \simeq \sqrt{m_u/m_c} \) and achieving a better understanding of the correlation between quark mass and flavor mixing hierarchies. In particular, we have clarified a somewhat misleading point of view that a strong quark mass hierarchy “naturally” corresponds to a comparably strong texture hierarchy of the quark mass.
matrix. We emphasize that the same caution should be exercised when studying the flavor issues of charged leptons and massive neutrinos.

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