Instantons and Chern-Simons Flows in 6, 7 and 8 Dimensions

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Abstract—The existence of K-instantons on a cylinder $M^7 = \mathbb{R}^6 \times K/H$ over a homogeneous nearly Kähler 6-manifold $K/H$ requires a conformally parallel or a cocalibrated $G_2$-structure on $M^7$. The generalized anti-self-duality on $M^d$ implies a Chern–Simons flow on $K/H$ which runs between instantons on the coset. For $K$-equivariant connections, the torsionful Yang–Mills equation reduces to a particular quartic dynamics for a Newtonian particle on $C$. When the torsion corresponds to one of the $G_2$-structures, this dynamics follows from a gradient or hamiltonian flow equation, respectively. We present the analytic (kink-type) solutions and plot numerical non-BPS solutions for general torsion values interpolating between the instantonic ones.

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1. INTRODUCTION

Yang–Mills instantons exist dimensions $d$ larger than four only when there is additional geometric structure on the manifold $M^d$ (besides the Riemannian one). In order to formulate generalized first-order anti-self-duality conditions which imply the second-order Yang–Mills equations (possibly with torsion), $M^d$ must be equipped with a so called $G$-structure, which is a globally defined but not necessarily closed $(d-4)$-form $\Sigma$, so that the weak holonomy group of $M^d$ gets reduced.

Instanton solutions in higher dimensions are rare in the literature. In the mid-eighties, Fairlie and Nuyts and also Fubini and Nicolai discovered the Spin(7)-instanton on $\mathbb{R}^8$. Eight years later, a similar $G^2$-instanton on $\mathbb{R}^7$ was found by Ivanova and Popov and also by Günaydin and Nicolai. Our recent work shows that these so called octonionic instantons are not isolated but embedded into a whole family living on a class of conical non-compact manifolds [1].

The string vacua in heterotic flux compactifications contain non-abelian gauge fields which in the supergravity limit are subject to Yang–Mills equations with torsion $\mathcal{H}$ determined by the three-form flux. Prominent cases admitting instantons are $\text{AdS}_{10-d} \times M^d$, where $M^d$ is equipped with a $G$-structure, with $G$ being $\text{SU}(3)$, $G_2$ or Spin(7) for $d = 6, 7$ or $8$, respectively. Homogeneous nearly Kähler 6-manifolds $K/H$ and (iterated) cylinders and (sine-) cones over them provide simple examples, for which all $K$-equivariant Yang–Mills connections can be constructed [2, 3]. Natural choices for the gauge group are $K$ or $G$.

Clearly, the Yang–Mills instantons discussed here serve to construct heterotic string solitons, as was first done in 1990 by Strominger for the gauge five-brane. It is therefore of interest to extend our new instantons to solutions of (string-corrected) heterotic supergravity and obtain novel string/brane vacua [4–6].

In this talk, I present the construction for the simplest case of a cylinder over a compact homogeneous nearly Kähler coset $K/H$, which allows for a conformally parallel or a cocalibrated $G_2$-structure. I display a family of non-BPS Yang–Mills connections, which contain two instantons at distinguished parameter values corresponding to those $G_2$-structures. In these two cases, anti-self-duality implies a Chern–Simons flow on $K/H$.

Finally, I must apologize for the omission—due to page limitation—of all relevant literature besides my own papers on which this talk is based. The reader can find all references therein.

2. SELF-DUALITY IN HIGHER DIMENSIONS

The familiar four-dimensional anti-self-duality condition for Yang–Mills fields $F$ may be generalized to suitable $d$-dimensional Riemannian manifolds $M$,

\[ *F = -\Sigma \wedge F \]
\[ \text{for } F = dA + A \wedge A \]
\[ \Sigma \in \Lambda^{d-4}(M), \]

if there exists a geometrically natural $(d-4)$-form $\Sigma$ on $M$. Applying the gauge-covariant derivative $D = d + [A, \cdot]$ it follows that

\[ D^*F + d\Sigma \wedge F = 0 \]

Yang–Mills with torsion $\mathcal{H} = *d\Sigma \wedge \Lambda^3(M)$. 

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1 The article is published in the original.
This torsionful Yang–Mills equation extremizes the action
\[ S_{YM} + S_{CS} = \int_M \text{tr} \{ F \wedge * F + (-)^{d-3} \Sigma \wedge F \wedge F \} \]
\[ = \int_M \text{tr} \left\{ F \wedge * F + \frac{1}{2} d\Sigma \wedge (\text{Ad}A + \frac{2}{3} A^3) \right\}. \tag{3} \]
Related to this generalized anti-self-duality is the gradient Chern–Simons flow on \( M \),
\[ \frac{dA}{d\tau} = \delta S_{CS} = * (d\Sigma \wedge F) - * \Sigma J F \tag{4} \]
In fact, this equation follows from generalized anti-self-duality on the cylinder \( M = \mathbb{R}_\tau \times M \) over \( M \) (in the \( A_\tau = 0 \) gauge).

The question is therefore: Which manifolds admit a global \((d-4)\)-form? And the answer is: \( G \)-structure manifolds, i.e. manifolds with a weak special holonomy. Let me give the key cases we shall encounter in this talk.

Some of those cases are related via the following scheme, with examples in square brackets.

For this talk I shall consider (reductive non-symmetric) coset spaces \( M = K/H \) in \( d = 6 \) as well as cylinders and cones over them. In all these cases, the gauge group is chosen to be \( K \).

### 3. Six Dimensions: Nearly Kähler Coset Spaces

All known compact nearly Kähler 6-manifolds \( M^6 \) are nonsymmetric coset spaces \( K/H \):
\[
S^6 = \frac{G_2}{SU(3)}, \quad \frac{Sp(2)}{Sp(1) \times U(1)}, \quad \frac{SU(3)}{U(1) \times U(1)},
\]
\[
S^3 \times S^3 = \frac{SU(2) \times SU(2) \times SU(2)}{SU(2)}.
\tag{5}
\]
The coset structure \( H \preceq K \) implies the decomposition
\[ \text{Lie}(K) = \mathfrak{f} = \mathfrak{h} \oplus \mathfrak{m} \]
with \( \mathfrak{h} = \text{Lie}(H) \) and \( \{ \mathfrak{h}, \mathfrak{m} \} \subset \mathfrak{m} \).

Interestingly, the reflection automorphism of symmetric spaces gets generalized to a so called tri-symmetry automorphism \( S : K \to K \) with \( S^3 = \text{id} \) implying
\[ s : \mathfrak{f} \to \mathfrak{f} \text{ with } s|_{\mathfrak{h}} = 1 \]
and \( s|_{\mathfrak{m}} = -\frac{1}{2} + \frac{\sqrt{3}}{2} J = \exp \left\{ \frac{2\pi i}{3} J \right\}, \tag{7} \]
effecting a \( 2\pi/3 \) rotation on \( TM^6 \). I pick a Lie-algebra basis
\[ \{ I_a = 1, \ldots, 6, I_i = 7, \ldots, \dim K \} \]
with \( [I_a, I_b] = f_{ab}^c I_c \), involving the structure constants \( f_{ab}^c \). The Cartan-Killing form then reads
\[ \langle \cdot, \cdot \rangle_{\mathfrak{h}} = -\text{tr}_\mathfrak{h}(\text{ad}(\cdot) \circ \text{ad}(\cdot)) \]
\[ = 3 \langle \cdot, \cdot \rangle_{\mathfrak{m}} = 3 \langle \cdot, \cdot \rangle_{\mathfrak{m}} = 1. \tag{9} \]
Expanding all structures in a basis of canonical one-forms \( e^a \) framing \( T^0(K/H) \),
\[ g = \delta_{ab} e^a \wedge e^b, \quad \omega = \frac{1}{2} f_{ab}^c e^a \wedge e^b \wedge e^c, \quad \Omega = \frac{1}{\sqrt{3}} (f + iJ)_{abc} e^a \wedge e^b \wedge e^c, \tag{10} \]
we see that the almost complex structure \( (J_{ab}) \) and the structure constants \( f_{abc} \) rule everything.

Nearly Kähler 6-manifolds are special in that the torsion term in (2) vanishes by itself! What is more, this

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**Table 1.** Examples of \( G \)-structure manifolds in \( d = 6, 7, 8 \)

| \( d \) | \( G \) | \( \Sigma \) | Cases | Example | Structure |
|---|---|---|---|---|---|
| 6 | \( \text{SU}(3) \) | \( \omega \) | Kähler | \( \mathbb{C} P^3 \) | \( \text{do} = 0 \) |
| 6 | \( \text{SU}(3) \) | \( \omega \) | nearly Kähler | \( S^6 = \frac{G_2}{SU(3)} \) | \( \text{do} - \text{Im} \Omega, \text{dRe} \Omega - \omega^3 \) |
| 7 | \( G_2 \) | \( \psi \) | conf. parallel \( G_2 \) | \( \mathbb{R}_\tau \times \text{nearly Kähler} \) | \( \text{d} \psi - \psi \text{Ad} \tau, \text{d} \psi - * \psi \text{Ad} \tau \) |
| 7 | \( G_2 \) | \( \psi \) | nearly parallel \( G_2 \) | \( \mathbb{R}_\tau / \text{parallel } G_2 \) | \( \text{d} \psi - * \psi \Rightarrow \text{d} \psi = 0 \) |
| 7 | \( G_2 \) | \( \psi \) | conf. parallel \( G_2 \) | cone (nearly Kähler) | \( \text{d} \psi = 0 = \text{d} \psi \) |
| 8 | \( \text{Spin}(7) \) | \( \Sigma \) | parallel \( \text{Spin}(7) \) | \( \mathbb{R}_\tau \times \text{parallel } G_2 \) | \( \text{d} \Sigma = 0, * \Sigma = \Sigma \) |
property is actually equivalent to the generalized anti-self-duality condition (1):

\[ *F = -\omega \land F \Rightarrow 0 = d\omega \land F \sim \text{Im}\Omega \land F \]  

(11)

\[ \Leftrightarrow \text{DUY equations}, \]

where the Donaldson–Uhlenbeck–Yau (DUY) equations\(^2\) state that

\[ F^{0,2} = F^{0,2} = 0 \quad \text{and} \quad \omega \land F = 0. \]  

(12)

Another interpretation of this anti-self-duality condition is that it projects \( F \) to the 8-dimensional eigenspace of the endomorphism \( *\omega \land \cdot \) with eigenvalue \(-1\), which contains the part of \( F^{3,1} \) orthogonal to \( \omega \). The equations (11) imply also \( \text{Re}\Omega \land F = 0 \) and the (torsion-free) Yang–Mills equations \( D^*F = 0 \). Clearly, they separately extremize both \( S_{\gamma_M} \) and \( S_{CS} \) in (3), but of course yield only BPS-type classical solutions. In components the above relations take the form

\[ \frac{1}{2} \varepsilon_{abcdef} F_{ef} = -J_{\{ab} F_{c\}} \Leftrightarrow 0 = f_{abc} F_{bc}, \]  

(13)

\[ \Rightarrow \omega_{ab} F_{ab} = 0, \quad (J_f)_{abc} F_{bc} = 0, \quad D_a F_{ab} = 0. \]  

(14)

I notice that each Chern–Simons flow \( A_a \rightarrow f_{abc} F_{bc} \) on \( M^6 \) ends in an instanton.

Let me look for \( K \)-equivariant connections \( A \) on \( M^6 \). If I restrict their value to 0, the answer is unique: the only '\( H \)-instanton' is the so-called canonical connection

\[ A^{\text{can}} = e^i I_i \rightarrow F^{\text{can}} = \frac{1}{2} f_{abc} e^a \wedge e^b I_i, \]  

(15)

where \( e^i = e^i_a e^a \). Generalizing to '\( K \)-instantons', I extend to

\[ A = e^i I_i + e^a \Phi_{ab} I_b \text{ with ansatz} \]

\[ (\Phi_{ab}) := \Phi = \phi_1 I + \phi_2 J, \]  

(16)

which is in fact general for \( G_2 \) invariance on \( S^6 \). Its curvature is readily computed to

\[ F_{ab} = F_{ab}^{1,1} + F_{ab}^{0,1,0,2} \]

(17)

\[ = (|\Phi|^2 - 1) I_{ab} I_c + [(\overline{\Phi}^2 - \Phi) I_{abc} I_c, \]

and displays the tri-symmetry invariance under \( \Phi \rightarrow \exp(-2\pi/3J)\Phi \). The solutions to the BPS conditions (11) are

\[ \Phi^2 = \Phi \Rightarrow \Phi = 0 \text{ or } \Phi = \exp\left\{ \frac{2\pi k}{3} J \right\} \]  

(18)

for \( k = 0, 1, 2, \)

which yields three flat \( K \)-instanton connections besides the canonical curved one,

\[ A^{(k)} = e^i I_i + e^a (\delta_k I)_{a} \]  

\[ \text{and } A^{\text{can}} = e^i I_i. \]  

(19)

4. SEVEN DIMENSIONS: CYLINDER OVER NEARLY KÄHLER COSETS

Let me step up one dimension and consider 7-manifolds \( M^7 \) with weak \( G_2 \) holonomy associated with a \( G_2 \)-structure three-form \( \psi \). Here, the 7 generalized anti-self-duality equations project \( F \) onto the \(-1\) eigenspace of \(*\psi \land \cdot \), which is 14-dimensional and isomorphic to the Lie algebra of \( G_2 \),

\[ *F = -\psi \land F \Rightarrow *\psi \land F = 0 \Rightarrow \psi \land F = 0, \]  

(20)

providing an alternative form of the condition. In components, it reads

\[ \frac{1}{2} \varepsilon_{abcdfe} F_{fe} = -\psi_{[abc} F_{def]} \Leftrightarrow 0 = \psi_{abc} F_{bc}. \]  

(21)

For the parallel and nearly parallel \( G_2 \) cases, the previous accident (11) recurs,

\[ d\psi \sim *\psi \Rightarrow d\psi \land F = 0 \Rightarrow D^*F = 0, \]  

(22)

and the torsion decouples. Note that on a general weak \( G_2 \)-manifold there are two different flows,

\[ \frac{dA(\sigma)}{d\sigma} = *d\psi \land F(\sigma) \quad \text{and} \quad \frac{dA(\sigma)}{d\sigma} = \psi \land F(\sigma) \]  

(23)

for \( \sigma \in \mathbb{R} \),

which coincide in the nearly parallel case. The second flow ends in an instanton on \( M^7 \).

In this talk I focus on cylinders \( M^7 = \mathbb{R}_r \times K/H \) over nearly Kähler cosets, with a metric \( g = (dz)^2 + \delta_{ab} e^a e^b \), on which I study the Yang–Mills equation with a torsion given by

\[ *\mathcal{H} = \frac{1}{3} \kappa d\omega \land d\tau \Leftrightarrow T_{abc} = \kappa f_{abc} \]  

(24)

with a real parameter \( \kappa \). We shall see that for special values of \( \kappa \) my torsionful Yang–Mills equation

\[ D^*F + \frac{1}{3} \kappa d\omega \land d\tau \land F = 0, \]  

(25)

descends from an anti-self-duality condition (20).

Taking the \( A_0 = 0 \) gauge and borrowing the ansatz (16) from the nearly Kähler base, I write

\[ A_a = e^i_a I_i + [\Phi(\tau) I]_a \Rightarrow F_{0a} = [\Phi I]_a, \]

\[ F_{ab} = (|\Phi|^2 - 1) I_{ab} I_c + [(\overline{\Phi}^2 - \Phi) I_{abc} I_c, \]

(26)

which depends on a complex function \( \Phi(\tau) \) (values in the \((\bar{z}, J)\) plane). Sticking this into (25) and computing for a while, one arrives at
Nice enough, I have obtained a model with an action

$$\mathcal{S}[\Phi] \sim \int_{\mathbb{R}} \{3|\Phi|^2 + V(\Phi)\}$$

for $V(\Phi) = (3-\kappa) + 3(\kappa - 1)|\Phi|^2 - (3 + k)(\Phi^3 + \Phi^5) + 6|\Phi|^4$,

devoid of rotational symmetry (for $\kappa \neq -3$) but enjoying tri-symmetry in the complex plane. It leads me to a mechanical analog problem of a Newtonian particle on $\mathcal{M}$ in a potential $-V$. I obtain the same action by plugging (26) directly into (3) with $d\Sigma = \hat{\mathcal{H}}$ from (24).

For the case of $K/H = S^6 = G_2/SU(3)$ equation (27) produces in fact all $G$-equivariant Yang–Mills connections on $\mathbb{C} \times K/H$. On $Sp(2)/Sp(1) \times U(1)$ and $SU(3)/U(1) \times U(1)$, however, the most general $G$-equivariant connections involves two respective three complex functions of $\tau$. The corresponding Newtonian dynamics on $\mathbb{C}^2$ respective $\mathbb{C}^3$ is of similar type but constrained by the conservation of Noether charges related to relative phase rotations of the complex functions.

5. SEVEN DIMENSIONS: SOLUTIONS

Finite-action solutions require Newtonian trajectories between zero-potential critical points $\hat{\Phi}$. With two exotic exceptions, $dV(\hat{\Phi}) = 0 = V(\hat{\Phi})$ yields precisely the BPS configurations on $K/H$:

- $\hat{\Phi} = e^{2\pi k/3}$ with $V(\hat{\Phi}) = 0$ for all values of $\kappa$ and $k = 0, 1, 2$
- $\hat{\Phi} = 0$ with $V(\hat{\Phi}) = 3 - \kappa$ vanishing only at $\kappa = 3$

Kink solutions will interpolate between two different critical points, while bounces will return to the critical starting point. Thus for generic $\kappa$ values one may have kinks of 'transversal' type, connecting two third roots of unity, as well as bounces. For $\kappa = 3$ 'radial' kinks, reaching such a root from the origin, may occur as well. Numerical analysis reveals the domains of existence in $\kappa$:

| $\kappa$ interval of trajectory | Radial & Bounce | Transversal & Kink | Radial & Kink | Radial & Bounce |
|---------------------------------|--------------|-----------------|-------------|--------------|
| $(-\infty, -3)$                | Radial       | Transversal     | Radial      | Radial       |
| $(-3, +3)$                     |               |                 |             |              |
| $(-3, +5)$                     |               |                 |             |              |
| $(+3, +5)$                     |               |                 |             |              |
| $(+5, +\infty)$               |               |                 |             |              |

In Fig. 2 I display contour plots of the potential and finite-action trajectories for eight choices of $\kappa$. They reveal three special values of $\kappa$: At $\kappa = -3$ rotational symmetry emerges; this is a degenerate situation. At
\( \kappa = -1 \) and at \( \kappa = +3 \), the trajectories are straight, indicating integrability. Indeed, behind each of these two cases lurks a first-order flow equation, which originates from anti-self-duality and hence a particular \( G_2 \)-structure \( \psi \).

Let me first discuss \( \kappa = +3 \). For this value I find that

\[
3 \Phi = \frac{\partial V}{\partial \Phi} \iff \sqrt{2} \Phi = \pm \frac{\partial W}{\partial \Phi},
\]

with \( W = \frac{1}{3} (\Phi^3 + \bar{\Phi}^3) - |\Phi|^2 \),

which is a gradient flow with a real superpotential \( W \), as

\[
V = 6 \left| \frac{\partial W}{\partial \Phi} \right|^2 \quad \text{for} \quad \kappa = \pm 3.
\]

It admits the obvious analytic radial kink solution,

\[
\Phi(\tau) = e^{\frac{2\pi i k}{3} \left( \frac{1}{2} \pm \frac{1}{2} \tan \frac{\tau}{2\sqrt{3}} \right)}.
\]

What is the interpretation of this gradient flow in terms of the original Yang–Mills theory? Demanding that the torsion in (24) comes from a \( G_2 \)-structure, \( \ast \mathcal{H} = d\psi \), I am led to

\[
\psi = \frac{1}{3} \kappa \omega \wedge d\tau + \alpha \text{Im}\Omega
\]

where \( \alpha \) is undetermined. This is a conformally parallel \( G_2 \)-structure, and (20) quantizes the coefficients to \( \alpha = 1 \) and \( \kappa = 3 \), fixing

\[
\psi = \omega \wedge d\tau + \text{Im}\Omega = r^{-3} (r^2 \omega \wedge dr + r^3 \text{Im}\Omega)
\]

\[
= r^{-3} \psi_{\text{cone}} \quad \text{with} \quad e^\tau = r,
\]

where I displayed the conformal relation to the parallel \( G_2 \)-structure on the cone over \( K/H \).

Alternatively, with this \( G_2 \)-structure the 7 anti-self-duality equations (20) turn into

\[
\omega \cdot J_{ab} F_{ab} = 0 \quad \text{and} \quad \dot{A} \sim d\omega \cdot F \sim e^\tau f_{abc} F_{bc}.
\]

With the ansatz (26), the first relation is automatic, and the second one indeed reduces to (29). As a consistency check, one may verify that

\[
\begin{align*}
3 \Phi &= \frac{\partial V}{\partial \Phi} \iff \sqrt{2} \Phi = \pm \frac{\partial W}{\partial \Phi} \\
\text{with} \quad W &= \frac{1}{3} (\Phi^3 + \bar{\Phi}^3) - |\Phi|^2,
\end{align*}
\]
I now come to the other instance of straight trajectories, \( \kappa = -1 \). For this value I find that
\[
\left( \Phi ^{3} + \Phi ^{3} \right) - |\Phi |^{2},
\]
which is a hamiltonian flow (note the imaginary multiplier!), running along the level curves of the function \( H \), that is identical to \( W \). It has the obvious analytic transverse kink solution,
\[
\Phi (\tau) = \frac{1}{2} \pm \frac{\sqrt{2}}{2} i \left( \tan \frac{\tau}{2} \right)
\]
and its images under the tri-symmetry action.

Have I discovered another hidden \( G_2 \)-structure here? Let me try the other obvious choice,
\[
3 \Phi = \frac{\partial V}{\partial \Phi} \leftarrow \sqrt{2} \Phi = \pm i \frac{2W}{\partial \Phi}
\]
with \( H = \frac{1}{3} (\Phi^3 + \Phi^3) - |\Phi|^2 \),
which is a hamiltonian flow (note the imaginary multiplier!), running along the level curves of the function \( H \), that is identical to \( W \). It has the obvious analytic transverse kink solution,
\[
\Phi (\tau) = \frac{1}{2} \pm \frac{\sqrt{2}}{2} i \left( \tan \frac{\tau}{2} \right)
\]

Employing the anti-self-duality with respect to \( \tilde{\psi} \),
\[
\tilde{\psi} \wedge F = 0 \Rightarrow \omega \wedge \omega \wedge F = 2 \mbox{Im} \Omega \wedge d \tau \wedge F,
\]
it works out, adjusting the coefficients to \( \tilde{\kappa} = 3 \) and \( \tilde{\alpha} = -1 \). Hence, my cocalibrated \( G_2 \)-structure
\[
\tilde{\psi} = \omega \wedge d \tau - \mbox{Re} \Omega
\]
is responsible for the hamiltonian flow. To see this directly, I import (41) into (20) and get
\[
J_{ab} F_{ab} = 0 \quad \text{and} \quad \dot{A}_a \sim [J f]_{abc} F_{bc}.
\]
Again, the ansatz (26) fulfills the first relation, but the second one nicely turns into (36).

The story has an eight-dimensional twist, which can be inferred from the diagram in Section 2. There it is indicated that my cylinder is embedded into an 8-manifold \( M^8 \) equipped with a parallel Spin(7)-structure \( \Sigma \). It can be regarded as the cylinder over the cone over \( K/H \). The four-form \( \Sigma \) descends to the cocalibrated \( G_2 \)-structure \( \psi \), while \( \psi \) is obtained by reducing to the cone and applying a conformal transformation.

The anti-self-duality condition on \( M^8 \) represents 7 relations, which project \( F_8 \) to the 21-dimensional \(-1\) eigenspace of \( \ast (\Sigma \wedge \cdot) \). Contrary to the \( G_2 \) situation (34), where 7 anti-self-duality equations split to 6 flow equations and the supplementary condition \( \omega \wedge F = 0 \), for
Spin(7) the count precisely matches, as I have also 7 flow equa-
tions. Indeed, there is equivalence:
\[ s_8 F_8 = \Sigma \wedge F_8 \Leftrightarrow \frac{\partial A_7(\sigma)}{\partial \alpha} = \gamma(\text{d} \psi \wedge F_7(\sigma)). \] (43)

6. PARTIAL SUMMARY
Let me schematically sum up the construction.

\[ \psi \wedge F = -s_7 F_7 \]

\[ \dot{A}_a \sim f_{abc} F_{bc} \]

on \( \mathbb{R} \times \frac{K}{H} \)

\[ \dot{\psi} \wedge F = -s_7 F \]

\[ \dot{A}_a \sim [J \, f]_{abc} F_{bc} \]

\[ \text{ansatz } A = e^i I_i + e^a [\Phi I]_a \]

\[ \sqrt{2} \Phi = \pm \frac{\partial W}{\partial \Phi} \]

\[ \sqrt{2} \Phi = \pm i \frac{\partial H}{\partial \Phi} \]

\[ W = \frac{1}{3} (\Phi^3 + \bar{\Phi}^3) - |\Phi|^2 = H \]

\[ F(\tau) = \text{d} \tau \wedge e^a [\Phi I]_a + \frac{1}{2} e^a \wedge e^b \{(|\Phi|^2 - 1)f_{ab} I_i + [I_a I_a I_c] \} \]

are \( G_2 \)-instantons for Yang–Mills with torsion \( D^* \) + \( F(\ast \partial \xi) \wedge F = 0 \) from

\[ S[A] = \int_{\mathbb{R} \times \frac{K}{H}} \text{tr} \left\{ F \wedge * F + \frac{1}{3} \kappa \omega \wedge \text{d} \tau \wedge F \wedge F \right\} \]

with \( \kappa = +3 \) or \(-1 \) and obey gradient/hamiltonian flow equations for \( \int_{\frac{K}{H}} \text{tr} \{\omega \wedge F \wedge F \} \propto W(\Phi) + 1/3. \)

REFERENCES

1. K.-P. Gemmer and O. Lechtenfeld, C. Nölle, and A. D. Popov, J. High Energy Phys. 09, 103 (2011). arXiv:1108.3951
2. D. Harland, T. A. Ivanova, O. Lechtenfeld, and A. D. Popov, Commun. Math. Phys. 300, 185 (2010). arXiv:0909.2730
3. I. Bauer, T. A. Ivanova, O. Lechtenfeld, and F. Lubbe, J. High Energy Phys. 10, 44 (2010). arXiv:1006.2388
4. O. Lechtenfeld, C. Nölle, and A. D. Popov, J. High Energy Phys. 09, 074 (2010). arXiv:1007.0236
5. A. Chatzistavrakidis, O. Lechtenfeld, and A. D. Popov, High Energy Phys. 4, 114 (2012). arXiv:1202.1278.
6. K.-P. Gemmer, A. Haupt, O. Lechtenfeld, C. Nölle, and A. D. Popov, arXiv:1202.5046.