PARABOLIC OBLIQUE DERIVATIVE PROBLEM IN GENERALIZED MORREY SPACES

LUBOMIRA G. SOFTOVA

Abstract. We study the regularity of the solutions of the oblique derivative problem for linear uniformly parabolic equations with VMO coefficients. We show that if the right-hand side of the parabolic equation belongs to certain generalized Morrey space $M^{p,\phi}(Q)$ than the strong solution belongs to the generalized Sobolev-Morrey space $W^{2,1}_{p,\phi}(Q)$.

1. Introduction

In the present work we consider the regular oblique derivative problem for linear non-divergence form parabolic equations in a cylinder

$$\begin{cases}
    u_t - a^{ij}(x)D_{ij}u = f(x) & \text{a.e. in } Q, \\
    u(x',0) = 0 & \text{on } \Omega, \\
    \partial u/\partial \ell = \ell^i(x)D_iu = 0 & \text{on } S.
\end{cases}$$

The unique strong solvability of this problem was proved in [23]. In [24] we study the regularity of the solution in the Morrey spaces $L^{p,\lambda}$ with $p \in (1, \infty)$, $\lambda \in (0, n+2)$ and also its Hölder regularity. In [26] we extend these studies on generalized Morrey spaces $L^{p,\omega}$ with a weight $\omega$ satisfying the doubling and integral conditions introduced in [18, 20]. The approach associated to the names of Calderón and Zygmund and developed by Chiarenza, Frasca and Longo in [7, 8] consists of obtaining of explicit representation formula for the higher order derivatives of the solution by singular and nonsingular integrals. Further the regularity properties of the solution follows by the continuity properties of these integrals in the corresponding spaces. In [24] and [25] we study the regularity of the corresponding operators in the Morrey and generalized Morrey spaces while in [23] we dispose of the corresponding results obtained in $L^p$ by [9] and [5]. In recent works we study the regularity of the solutions of elliptic and parabolic problems with Dirichlet
data on the boundary in Morrey-type spaces $M^{p,\varphi}$ with a weight $\varphi$ satisfying (2.4) (cf. [12, 13]). Precisely, we obtain boundedness in $M^{p,\varphi}$ for sub-linear integrals generated by singular integrals as the Calderón-Zygmund. More results concerning sub-linear integrals in generalized Morrey spaces can be found in [3, 11, 25], see also the references therein.

Throughout this paper the following notations are to be used, $x = (x', t) = (x', x_n, t) \in \mathbb{R}^{n+1}$, $\mathbb{R}^{n+1}_+ = \{x' \in \mathbb{R}^n, t > 0\}$ and $\mathbb{D}^{n+1}_+ = \{x'' \in \mathbb{R}^{n-1}, x_n > 0, t > 0\}$. $D_i u = \partial u / \partial x_i$, $D_{ij} u = \partial^2 u / \partial x_i \partial x_j$, $D_t u = u_t = \partial u / \partial t$ stand for the corresponding derivatives while $Du = (D_1 u, \ldots, D_\nu u)$ and $D^2 u = \{D_{ij} u\}_{i,j=1}^\nu$ mean the spatial gradient and the Hessian matrix of $u$. For any measurable function $f$ and $A \subset \mathbb{R}^{n+1}$ we write

$$
\|f\|_{p,A} = \left( \int_A |f(y)|^p dy \right)^{1/p}, \quad f_A = \frac{1}{|A|} \int_A f(y) dy
$$

where $|A|$ is the Lebesgue measure of $A$. Through all the paper the standard summation convention on repeated upper and lower indexes is adopted. The letter $C$ is used for various constants and may change from one occurrence to another.

2. DEFINITIONS AND STATEMENT OF THE PROBLEM

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$ be a bounded $C^{1,1}$-domain, $Q = \Omega \times (0, T)$ be a cylinder in $\mathbb{R}^{n+1}$, and $S = \partial \Omega \times (0, T)$ stands for the lateral boundary of $Q$. We consider the problem

$$
\begin{aligned}
\mathfrak{B} u &:= u_t - a^{ij}(x) D_{ij} u = f(x) \quad \text{a.e. in } Q, \\
\mathfrak{I} u &:= u(x', 0) = 0 \quad \text{on } \Omega, \\
\mathfrak{B} u &:= \partial u / \partial \ell = \ell^i(x) D_i u = 0 \quad \text{on } S.
\end{aligned}
$$

under the following conditions:

(i) The operator $\mathfrak{B}$ is supposed to be uniformly parabolic, i.e. there exists a constant $\Lambda > 0$ such that for almost all (a.a.) $x \in Q$

$$
\begin{aligned}
\Lambda^{-1} |\xi|^2 \leq a^{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \\
a^{ij}(x) = a^{ji}(x), \quad i, j = 1, \ldots, n.
\end{aligned}
$$

The symmetry of the coefficient matrix $a = \{a^{ij}\}_{i,j=1}^\nu$ implies essential boundedness of $a^{ij}$’s and we set $\|a\|_{\infty,Q} = \sum_{i,j=1}^\nu \|a^{ij}\|_{\infty,Q}$.

(ii) The boundary operator $\mathfrak{B}$ is prescribed in terms of a directional derivative with respect to the unit vector field $\ell(x) = (\ell^1(x), \ldots, \ell^n(x)) \in S$. We suppose that $\mathfrak{B}$ is a regular oblique
derivative operator, i.e., the field $\ell$ is never tangential to $S$:

\begin{equation}
\langle \ell(x) \cdot n(x) \rangle = \ell^{i}(x)n_{i}(x) > 0 \quad \text{on} \ S, \ \ell^{i} \in \text{Lip}(S).
\end{equation}

Here $\text{Lip}(S)$ is the class of uniformly Lipschitz continuous functions on $S$ and $n(x)$ stands for the unit outward normal to $\partial \Omega$.

In the following, besides the parabolic metric $\rho(x) = \max(|x'|, |t|^{1/2})$ and the defined by it parabolic cylinders

$$I_{r}(x) = \{ y \in \mathbb{R}^{n+1} : |x' - y'| < r, |t - \tau| < r^{2} \} \quad |I_{r}| = Cr^{n+2}. $$

we use the equivalent one $\rho(x) = \left( \frac{|x'|^{2} + \sqrt{|x'|^{4} + 4t^{2}}}{2} \right)^{1/2}$ (see [9]). The balls with respect to this metric are ellipsoids

$$E_{r}(x) = \{ y \in \mathbb{R}^{n+1} : \frac{|x' - y'|^{2}}{r^{2}} + \frac{|t - \tau|^{2}}{r^{4}} < 1 \} \quad |E_{r}| = Cr^{n+2}. $$

Because of the equivalence of the metrics all estimates obtained over ellipsoids hold true also over parabolic cylinders and in the following we shall use this without explicit references.

**Definition 2.1.** ([14, 22]) Let $a \in L^{1}_{\text{loc}}(\mathbb{R}^{n+1})$, denote by

$$\eta_{a}(R) = \sup_{E_{r}, r \leq R} \frac{1}{|E_{r}|} \int_{E_{r}} |f(y) - f_{E_{r}}| dy \quad \text{for every} \ R > 0$$

where $E_{r}$ ranges over all ellipsoids in $\mathbb{R}^{n+1}$. The Banach space $BMO$ (bounded mean oscillation) consists of functions for which the following norm is finite

$$\|a\|_{*} = \sup_{R > 0} \eta_{a}(R) < \infty.$$ 

A function $a$ belongs to $VMO$ (vanishing mean oscillation) with $VMO$-modulus $\eta_{a}(R)$ provided

$$\lim_{R \to 0} \eta_{a}(R) = 0.$$ 

For any bounded cylinder $Q$ we define $BMO(Q)$ and $VMO(Q)$ taking $a \in L^{1}(Q)$ and $Q_{r} = Q \cap I_{r}$ instead of $E_{r}$ in the definition above.

According to [1, 15] having a function $a \in BMO/VMO(Q)$ it is possible to extend it in the whole $\mathbb{R}^{n+1}$ preserving its $BMO$-norm or $VMO$-modulus, respectively. In the following we use this property without explicit references.

**Definition 2.2.** Let $\varphi : \mathbb{R}^{n+1} \times \mathbb{R}_{+} \to \mathbb{R}_{+}$ be a measurable function and $p \in (1, \infty)$. A function $f \in L^{p}_{\text{loc}}(\mathbb{R}^{n+1})$ belongs to the generalized
parabolic Morrey space $M^{p,\varphi}(\mathbb{R}^{n+1})$ provided

$$
\|f\|_{p,\varphi;\mathbb{R}^{n+1}} = \sup_{(x,r)\in\mathbb{R}^{n+1} \times \mathbb{R}^{+}} \varphi(x,r)^{-1} \left( r^{-(n+2)} \int_{\mathcal{E}_r(x)} |f(y)|^p dy \right)^{1/p} < \infty.
$$

The space $M^{p,\varphi}(Q)$ consists of $L^p(Q)$ functions provided the following norm is finite

$$
\|f\|_{p,\varphi;Q} = \sup_{(x,r)\in Q \times \mathbb{R}^{+}} \varphi(x,r)^{-1} \left( r^{-(n+2)} \int_{Q_r(x)} |f(y)|^p dy \right)^{1/p}.
$$

The generalized Sobolev-Morrey space $W^{p,\varphi}_{2,1}(Q)$, $p \in (1, \infty)$ consist of all Sobolev functions $u \in W^{p}_{2,1}(Q)$ with distributional derivatives $D^l D^s u \in M^{p,\varphi}(Q)$, $0 \leq 2l + |s| \leq 2$, endowed by the norm

$$
\|u\|_{W^{p,\varphi}_{2,1}(Q)} = \|u\|_{p,\varphi;Q} + \sum_{|s| \leq 2} \|D^s u\|_{p,\varphi;Q}.
$$

$W^{p,\varphi}_{2,1}(Q) = \{ u \in W^{p,\varphi}_{2,1}(Q) : u(x) = 0, x \in \partial Q \}$,

$$
\|u\|_{W^{p,\varphi}_{2,1}(Q)} = \|u\|_{p,\varphi;Q}
$$

where $\partial Q$ means the parabolic boundary $\Omega \cup \{ \partial \Omega \times (0, T) \}$.

**Theorem 2.3. (Main result)** Let (i) and (ii) hold, $a \in VMO(Q)$ and $u \in W^{p}_{2,1}(Q)$, $p \in (1, \infty)$ be a strong solution of (2.1). If $f \in M^{p,\varphi}(Q)$ with $\varphi(x,r)$ being measurable positive function satisfying

$$
(2.4) \quad \int_{r}^{\infty} \left( 1 + \ln \frac{s}{r} \right)^{\frac{\varphi(x,\zeta)}{s^{\zeta+\frac{n+2}{p}}}} ds \leq C
$$

for each $(x,r) \in Q \times \mathbb{R}^{+}$ then $u \in W^{p,\varphi}_{2,1}(Q)$ and

$$
(2.5) \quad \|u\|_{W^{p,\varphi}_{2,1}(Q)} \leq C \|f\|_{p,\varphi;Q}
$$

with $C = C(n, p, \Lambda, \partial \Omega, T, \|a\|_{\infty;Q}, \eta_{a})$ and $\eta_{a} = \sum_{i,j=1}^{n} \eta_{a^{ij}}$.

If $\varphi(x,r) = r^{(\lambda-n-2)/p}$ then $M^{p,\varphi} \equiv L^{p,\lambda}$ and the condition (2.4) holds with a constant depending on $n$, $p$ and $\lambda$. If $\varphi(x,r) = \omega(x,r)^{1/p} r^{-(n+2)/p}$ with $\omega : \mathbb{R}^{n+1} \times \mathbb{R}^{+} \to \mathbb{R}^{+}$ satisfying the conditions

$$
k_1 \leq \frac{\omega(x_0,s)}{\omega(x_0,r)} \leq k_2 \quad \forall x_0 \in \mathbb{R}^{n+1}, \ r \leq s \leq 2r
$$

$$
\int_{r}^{\infty} \frac{\omega(x_0,s)}{s} ds \leq k_3 \omega(x_0,r) \quad k_i > 0, \ i = 1, 2, 3
$$
than we obtain the spaces $L^{p,\omega}$ studied in [18, 20]. The following results are obtained in [13] and treat continuity in $M^{p,\psi}$ of certain singular and nonsingular integrals.

**Definition 2.4.** A measurable function $\mathcal{R}(x; \xi) : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}$ is called variable parabolic Calderón-Zygmund kernel (PCZK) if:

i) $\mathcal{R}(x; \cdot)$ is a PCZK for a.a. $x \in \mathbb{R}^{n+1}$:
   a) $\mathcal{R}(x; \cdot) \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$,
   b) $\mathcal{R}(x; \mu \xi) = \mu^{-(n+2)} \mathcal{R}(x; \xi)$ for all $\mu > 0$,
   c) $\int_{\mathbb{R}^n} \mathcal{R}(x; \xi) d\sigma_\xi = 0$, $\int_{\mathbb{R}^n} |\mathcal{R}(x; \xi)| d\sigma_\xi < +\infty$.

ii) $\|D_\xi^\beta \mathcal{R}\|_{\mathbb{R}^{n+1} \times \mathbb{S}^n} \leq M(\beta) < \infty$ for each multi-index $\beta$.

Consider the singular integrals

$$Kf(x) = P.V. \int_{\mathbb{R}^{n+1}} \mathcal{R}(x; x - y) f(y) dy$$

(2.6)

$$\mathcal{C}[a, f](x) = P.V. \int_{\mathbb{R}^{n+1}} \mathcal{R}(x; x - y) [a(y) - a(x)] f(y) dy.$$  

**Theorem 2.5.** For any $f \in M^{p,\psi}(\mathbb{R}^{n+1})$ with $(p, \varphi)$ as in Theorem 2.3 and $a \in BMO$ there exist constants depending on $n, p$ and the kernel such that

$$\|Kf\|_{p,\psi; \mathbb{R}^{n+1}} \leq C\|f\|_{p,\psi; \mathbb{R}^{n+1}},$$

(2.7)

$$\|\mathcal{C}[a, f]\|_{p,\psi; \mathbb{R}^{n+1}} \leq C\|a\|_\infty \|f\|_{p,\psi; \mathbb{R}^{n+1}}.$$  

**Corollary 2.6.** Let $Q$ be a cylinder in $\mathbb{R}^{n+1}_+$, $f \in M^{p,\psi}(Q)$, $a \in BMO(Q)$ and $\mathcal{R}(x, \xi) : Q \times \mathbb{R}^{n+1}_+ \setminus \{0\} \to \mathbb{R}$. Then the operators (2.6) are bounded in $M^{p,\psi}(Q)$ and

$$\|Kf\|_{p,\psi; Q} \leq C\|f\|_{p,\psi; Q},$$

(2.8)

$$\|\mathcal{C}[a, f]\|_{p,\psi; Q} \leq C\|a\|_\infty \|f\|_{p,\psi; Q}$$

with $C$ independent of $a$ and $f$.

**Corollary 2.7.** Let $a \in VMO$ and $(p, \varphi)$ be as in Theorem 2.3. Then for any $\varepsilon > 0$ there exists a positive number $r_0 = r_0(\varepsilon, \eta_0)$ such that for any $E_r(x_0)$ with a radius $r \in (0, r_0)$ and all $f \in M^{p,\psi}(E_r(x_0))$

$$\|\mathcal{C}[a, f]\|_{p,\psi; E_r(x_0)} \leq C\varepsilon \|f\|_{p,\psi; E_r(x_0)}$$

(2.9)

where $C$ is independent of $\varepsilon$, $f$, $r$ and $x_0$.

For any $x' \in \mathbb{R}^n_+$ and any fixed $t > 0$ define the *generalized reflection*

$$\mathcal{T}(x) = (\mathcal{T}'(x), t), \quad \mathcal{T}'(x) = x' - 2x^n \frac{a^n(x', t)}{a^{nn}(x', t)}$$

(2.10)
where $a^n(x)$ is the last row of the coefficients matrix $a(x)$ of (2.1).
The function $T'(x)$ maps $\mathbb{R}_+^n$ into $\mathbb{R}_+^n$ and the kernel $\mathcal{K}(x; T(x) - y) = \mathcal{K}(x; T'(x) - y', t - \tau)$ is a nonsingular one for any $x, y \in \mathbb{D}_+^{n+1}$. Taking $\tilde{x} = (x'', -x_n, t)$ there exist positive constants $\kappa_1$ and $\kappa_2$ such that
\begin{equation}
\kappa_1 \rho(\tilde{x} - y) \leq \rho(T(x) - y) \leq \kappa_2 \rho(\tilde{x} - y).
\end{equation}
For any $f \in M^{p,\varphi}(\mathbb{D}_+^{n+1})$ with a norm
\[ \|f\|_{p,\varphi; \mathbb{D}_+^{n+1}} = \sup_{(x,r) \in \mathbb{D}_+^{n+1} \times \mathbb{R}_+} \varphi(x,r)^{-1} \left( r^{-(n+2)} \int_{E_r(x)} |f(y)|^p dy \right)^{1/p} \]
and $a \in BMO(\mathbb{D}_+^{n+1})$ define the nonsingular integral operators
\begin{align*}
\tilde{\mathcal{K}}f(x) &= \int_{\mathbb{D}_+^{n+1}} \mathcal{K}(x; T(x) - y) f(y) dy \\
\tilde{\mathcal{C}}[a, f](x) &= \int_{\mathbb{D}_+^{n+1}} \mathcal{K}(x; T(x) - y) [a(y) - a(x)] f(y) dy.
\end{align*}

**Theorem 2.8.** Let $a \in BMO(\mathbb{D}_+^{n+1})$ and $f \in M^{p,\varphi}(\mathbb{D}_+^{n+1})$ with $(p, \varphi)$ as in Theorem 2.3. Then the operators $\tilde{\mathcal{K}}f$ and $\tilde{\mathcal{C}}[a, f]$ are continuous in $M^{p,\varphi}(\mathbb{D}_+^{n+1})$ and
\begin{equation}
\|\tilde{\mathcal{K}}f\|_{p,\varphi; \mathbb{D}_+^{n+1}} \leq C \|f\|_{p,\varphi; \mathbb{D}_+^{n+1}},
\end{equation}
\begin{equation}
\|\tilde{\mathcal{C}}[a, f]\|_{p,\varphi; \mathbb{D}_+^{n+1}} \leq C \|a\|_\ast \|f\|_{p,\varphi; \mathbb{D}_+^{n+1}}
\end{equation}
with a constant independent of $a$ and $f$.

**Corollary 2.9.** Let $a \in VMO$, then for any $\varepsilon > 0$ there exists a positive number $r_0 = r_0(\varepsilon, \|a\|_\ast)$ such that for any $E^+_r(x^0) = E_r(x^0) \cap \mathbb{D}_+^{n+1}$ with a radius $r \in (0, r_0)$ and center $x^0 = (x'', 0, 0)$ and for all $f \in M^{p,\varphi}(E^+_r(x^0))$ holds
\begin{equation}
\|\tilde{\mathcal{C}}[a, f]\|_{p,\varphi; E^+_r(x^0)} \leq C \varepsilon \|f\|_{p,\varphi; E^+_r(x^0)},
\end{equation}
where $C$ is independent of $\varepsilon$, $f$, $r$ and $x^0$.

3. Proof of the main result

As it follows by [24], the problem (2.1) is uniquely solvable in $\hat{W}^{p,\varphi}_{2,1}(Q)$.

We are going to show that $f \in M^{p,\varphi}(Q)$ implies $u \in \hat{W}^{p,\varphi}_{2,1}(Q)$. For this goal we obtain an a priori estimate of $u$. The proof is divided in two steps.

**Interior estimate.** For any $x_0 \in \mathbb{R}_+^{n+1}$ consider the parabolic semi-cylinders $C_r(x_0) = B_r(x_0') \times (t_0 - r^2, t_0)$. Let $v \in C_0^\infty(C_r)$ and suppose
that \(v(x,t) = 0\) for \(t \leq 0\). According to [5, Theorem 1.4] for any \(x \in \text{supp} \ v\) the following representation formula for the second derivatives of \(v\) holds true

\[
D_{ij}v(x) = P.V. \int_{\mathbb{R}^{n+1}} \Gamma_{ij}(x; x-y)[a^{hk}(y) - a^{hk}(x)]D_{hk}v(y)dy \\
+ P.V. \int_{\mathbb{R}^{n+1}} \Gamma_{ij}(x; x-y)\mathcal{P}v(y)dy + \mathcal{P}v(x) \int_{S^n} \Gamma_j(x; y)\nu_id\sigma_y,
\]

(3.15)

where \(\nu(\nu_1, \ldots, \nu_{n+1})\) is the outward normal to \(S^n\). Here \(\Gamma(x; \xi)\) is the fundamental solution of the operator \(\mathcal{P}\) and \(\Gamma_{ij}(x; \xi) = \partial^2 \Gamma(x; \xi)/\partial \xi_i \partial \xi_j\).

Because of density arguments the representation formula (3.15) still holds for any \(v \in W^0_{2,1}(C_r(x_0))\). The properties of the fundamental solution (cf. [5, 16, 23]) imply \(\Gamma_{ij}\) are Calderón-Zygmund kernels in the sense of Definition 2.4. We denote by \(\mathcal{K}_{ij}\) and \(\mathcal{C}_{ij}\) the singular integrals defined in (2.6) with kernels \(\mathcal{K}(x; x-y) = \Gamma_{ij}(x; x-y)\). Then we can write that

\[
D_{ij}v(x) = \mathcal{C}_{ij}[a^{hk}, D_{hk}v](x) \\
+ \mathcal{K}_{ij}(\mathcal{P}v)(x) + \mathcal{P}v(x) \int_{S^n} \Gamma_j(x; y)\nu_id\sigma_y.
\]

(3.16)

Because of Corollaries 2.6 and 2.7 and the equivalence of the metrics we get

\[
\|D^2v\|_{p,\phi; C_r(x_0)} \leq C(\|D^2v\|_{p,\phi; C_r(x_0)} + \|\mathcal{P}u\|_{p,\phi; C_r(x_0)})
\]

for some \(r\) small enough. Moving the norm of \(D^2v\) on the left-hand side we get

\[
\|D^2v\|_{p,\phi; C_r(x_0)} \leq C\|\mathcal{P}v\|_{p,\phi; C_r(x_0)}
\]

with a constant depending on \(n, p, \eta_\alpha(r), \|a\|_{\infty,Q}\) and \(\|D\Gamma\|_{\infty,Q}\). Define a cut-off function \(\phi(x) = \phi_1(x')\phi_2(t)\), with \(\phi_1 \in C_c^\infty(B_r(x_0'))\), \(\phi_2 \in C_c^\infty(\mathbb{R})\) such that

\[
\phi_1(x') = \begin{cases} 
1 & x' \in B_{\theta r}(x_0') \\
0 & x' \notin B_{\theta r}(x_0')
\end{cases}, \quad \phi_2(t) = \begin{cases} 
1 & t \in (t_0 - (\theta r)^2, t_0] \\
0 & t < t_0 - (\theta r)^2
\end{cases}
\]

with \(\theta \in (0,1), \theta' = \theta(3-\theta)/2 > \theta\) and \(|D^s\phi| \leq C(\theta(1-\theta)r)^{-s}\), \(s = 0, 1, 2, \ |\phi_t| \sim |D^2\phi|\). For any solution \(u \in W^0_{2,1}(Q)\) of (2.1) define \(v(x) = \phi(x)u(x) \in W^0_{2,1}(C_r)\). Then we get

\[
\|D^2u\|_{p,\phi; C_{\theta r}(x_0)} \leq C\|D^2v\|_{p,\phi; C_{\theta r}(x_0)} \leq C\|\mathcal{P}v\|_{p,\phi; C_{\theta r}(x_0)} \leq C\left(\|f\|_{p,\phi; C_{\theta r}(x_0)} + \frac{\|Du\|_{p,\phi; C_{\theta r}(x_0)}}{\theta(1-\theta)r^2} + \frac{\|u\|_{p,\phi; C_{\theta r}(x_0)}}{[\theta(1-\theta)r^2]}\right).
\]
By the choice of $\theta'$ it holds $\theta(1-\theta) \leq 2\theta'(1-\theta')$ which leads to
\[ \left[ \theta(1-\theta)r \right]^2 \| D^2 u \|_{p,\varphi;C_r(x_0)} \leq C \left( r^2 \| f \|_{p,\varphi;Q} + \theta'(1-\theta')r \right) \| D u \|_{p,\varphi;C_{r',}(x_0)} + \| u \|_{p,\varphi;C_{r'}(x_0)} \right) \). 

Introducing the semi-norms
\[ \Theta_s = \sup_{0 < \theta < 1} \left[ \theta(1-\theta)r \right]^s \| D^s u \|_{p,\varphi;C_{r'/2}(x_0)} \quad s = 0, 1, 2 \]
and taking the supremo with respect to $\theta$ and $\theta'$ we get
\[ (3.17) \quad \Theta_2 \leq C \left( r^2 \| f \|_{p,\varphi;Q} + \Theta_1 + \Theta_0 \right) . \]

The interpolation inequality [26, Lemma 4.2] gives that there exists a positive constant $C$ independent of $r$ such that
\[ \Theta_1 \leq \varepsilon \Theta_2 + \frac{C}{\varepsilon} \Theta_0 \quad \text{for any } \varepsilon \in (0, 2) . \]

Thus (3.17) becomes
\[ \left[ \theta(1-\theta)r \right]^2 \| D^2 u \|_{p,\varphi;C_r(x_0)} \leq \Theta_2 \leq C \left( r^2 \| f \|_{p,\varphi;Q} + \Theta_0 \right) \]
for each $\theta \in (0, 1)$. Taking $\theta = 1/2$ we get the Caccioppoli-type estimate
\[ \| D^2 u \|_{p,\varphi;C_{r/2}(x_0)} \leq C \left( \| f \|_{p,\varphi;Q} + \frac{1}{r^2} \| u \|_{p,\varphi;C_r(x_0)} \right) . \]

To estimate $u_t$ we exploit the parabolic structure of the equation and the boundedness of the coefficients
\[ \| u_t \|_{p,\varphi;C_{r/2}(x_0)} \leq \| a \|_{\infty;Q} \| D^2 u \|_{p,\varphi;C_{r/2}(x_0)} + \| f \|_{p,\varphi;C_{r/2}(x_0)} \leq C \left( \| f \|_{p,\varphi;Q} + \frac{1}{r^2} \| u \|_{p,\varphi;C_r(x_0)} \right) . \]

Consider cylinders $Q' = \Omega' \times (0, T)$ and $Q'' = \Omega'' \times (0, T)$ with $\Omega' \Subset \Omega'' \Subset \Omega$, by standard covering procedure and partition of the unity we get
\[ (3.18) \| u \|_{W^{2,\infty}_{1,1}(Q)} \leq C \left( \| f \|_{p,\varphi;Q} + \| u \|_{p,\varphi;Q''} \right) \]
where $C$ depends on $n, p, \Lambda, T, \| D\Gamma \|_{\infty;Q}, \| a \|_{\infty;Q}$ and $\text{dist}(\Omega', \partial \Omega'')$.

**Boundary estimates.** For any fixed $R > 0$ and $x^0 = (x'', 0, 0)$ define the semi-cylinders
\[ \mathcal{C}_R^+(x^0) = \mathcal{C}_R(x^0) \cap \mathbb{B}^{n+1}_+ . \]

Without lost of generality we can take $x^0 = (0, 0, 0)$. Define
\[ \mathcal{B}_R^+ = \{ |x'| < R, x_n > 0 \}, \quad S_R^+ = \{ |x''| < R, x_n = 0, t \in (0, R^2) \} \text{ and} \]
\[ \mathcal{B}_R^- = \{ |x'| < R, x_n > 0 \}, \quad S_R^- = \{ |x''| < R, x_n > 0, t \in (-R^2, 0) \}. \]
consider the problem
\[
\begin{align*}
\mathfrak{B}u & := u - a^{ij}(x)D_{ij}u = f(x) \quad \text{a.e. in } \mathcal{C}^+_R, \\
\mathfrak{G}u & := u(x', 0) = 0 \quad \text{on } \mathcal{B}^+_R, \\
\mathfrak{M}u & := \ell(x)D_i u = 0 \quad \text{on } S^+_R.
\end{align*}
\]
(3.19)

Let \( u \in W^{p, 1}_K(\mathcal{C}^+_R) \) with \( u = 0 \) for \( t \leq 0 \) and \( x_n \leq 0 \), then the following representation formula holds (see [17, 23])
\[
D_{ij}u(x) = I_{ij}(x) - J_{ij}(x) + H_{ij}(x)
\]
where
\[
I_{ij}(x) = \text{P.V.} \int_{\mathcal{C}^+_R} \Gamma_{ij}(x; x - y)F(x; y)dy
\]
\[
+ f(x) \int_{\mathbb{S}^n} \Gamma_{j}(x; y)\nu_id\sigma_y, \quad i, j = 1, \ldots, n;
\]
\[
J_{ij}(x) = \int_{\mathcal{C}^+_R} \Gamma_{ij}(x; T(x) - y)F(x; y)dy;
\]
\[
J_{in}(x) = J_{ni}(x) = \int_{\mathcal{C}^+_R} \Gamma_{il}(x; T(x) - y)\left(\frac{\partial T(x)}{\partial x_n}\right)^{i} F(x; y)dy,
\]
\[
i, j = 1, \ldots, n - 1
\]
\[
J_{nn}(x) = \int_{\mathcal{C}^+_R} \Gamma_{ls}(x; T(x) - y)\left(\frac{\partial T(x)}{\partial x_n}\right)^{i} \left(\frac{\partial T(x)}{\partial x_n}\right)^{s} F(x; y)dy;
\]
\[
F(x; y) = f(y) + [a^{hk}(y) - a^{hk}(x)]D_{hk}u(y)
\]
\[
H_{ij}(x) = (G_{ij} * \mathcal{Q})(x) + g(x'', t) \int_{\mathbb{S}^n} G_{j}(x; y'', x_n, \tau)n_id\sigma_{(y'', \tau)},
\]
\[
i, j = 1, \ldots, n,
\]
\[
\frac{\partial T(x)}{\partial x_n} = \left(-2\frac{a^{n1}(x)}{a^{nn}(x)}, \ldots, -2\frac{a^{nn-1}(x)}{a^{nn}(x)}, -1\right).
\]

Here the kernel \( G = \Gamma \mathcal{Q} \), is a byproduct of the fundamental solution and a bounded regular function \( \mathcal{Q} \). Hence its derivatives \( G_{ij} \) behave as \( \Gamma_{ij} \) and the convolution that appears in \( H_{ij} \) is defined as follows
\[
(G_{ij} * \mathcal{Q})(x) = \text{P.V.} \int_{\mathcal{S}^+_R} G_{ij}(x; x'' - y'', x_n, t - \tau)g(y'', 0, \tau)dy''d\tau,
\]
\[
g(x'', 0, t) = \left[\left(\ell^k(0) - \ell^k(x'', 0, t)\right)D_ku - \ell^k(0)(\Gamma_k * F)\right]_{x_n=0}(x'', 0, t),
\]
\[
(\Gamma_k * F)(x) = \int_{\mathcal{C}^+_R} \Gamma_k(x; x - y)F(x; y)dy.
\]
Here $I_{ij}$ are a sum of singular integrals and bounded surface integrals hence the estimates obtained in Corollaries 2.6 and 2.7 hold true. On the nonsingular integrals $J_{ij}$ we apply the estimates obtained in Theorem 2.8 and Corollary 2.9 that give

\begin{equation}
\|I_{ij}\|_{p,\varphi C^+_R} + \|J_{ij}\|_{p,\varphi C^+_R} \leq C \left( \|f\|_{p,\varphi C^+_R} + \eta_k(R) \|D^2 u\|_{p,\varphi C^+_R} \right)
\end{equation}

for all $i, j = 1, \ldots, n$. To estimate the norm of $H_{ij}$ we suppose that the vector field $\ell$ is extended in $C^+_R$ preserving its Lipschitz regularity and the norm. This automatically leads to extension of the function $g$ in $C^+_R$ that is

\begin{equation}
g(x) = \left( \ell^k(0) - \ell^k(x) \right) D_k u(x) - \ell^k(0)(\Gamma_k * F)(x).
\end{equation}

Applying the estimates for the heat potentials [16, Chapter 4] and the trace theorems in $L^p$ [2, Theorems 7.48, 7.53] (see also [23, Theorem 1]) we get

\[
\int_{\mathcal{C}_R} |(G_{ij} * g)(y)|^p dy \leq C \left( \int_{\mathcal{C}_R} |g(y)|^p dy + \int_{\mathcal{C}_R} |D g(y)|^p dy \right).
\]

Taking a parabolic cylinder $\mathcal{I}_r(x)$ centered in some point $x \in \mathcal{C}_R$ we have

\[
\int_{\mathcal{C}_R \cap \mathcal{I}_r(x)} |(G_{ij} * g)(y)|^p dy \leq C_{r,n+2} \left( \frac{\varphi(x, r)^{-p}}{r^{n+2}} \int_{\mathcal{C}_R \cap \mathcal{I}_r(x)} |g(y)|^p dy \right.
\]

\[
+ \frac{\varphi(x, r)^{-p}}{r^{n+2}} \int_{\mathcal{C}_R \cap \mathcal{I}_r(x)} |D g(y)|^p dy \right)
\]

\[
\leq C_{r,n+2} \left( \|g\|_{p,\varphi C^+_R}^p + \|D g\|_{p,\varphi C^+_R}^p \right).
\]

Moving $\frac{\varphi(x, r)^{-p}}{r^{n+2}}$ on the left-hand side and taking the supremo with respect to $(x, r) \in \mathcal{C}_R \times \mathbb{R}_+$ we get

\[
\|G_{ij} * g\|_{p,\varphi C^+_R}^p \leq C \left( \|g\|_{p,\varphi C^+_R}^p + \|D g\|_{p,\varphi C^+_R}^p \right).
\]

An immediate consequence of (3.21) is the estimate

\[
\|g\|_{p,\varphi C^+_R} \leq \|\ell^k(0) - \ell^k(\cdot)\|_{p,\varphi C^+_R} + \|\ell^k(0)(\Gamma_k * F)\|_{p,\varphi C^+_R}
\]

\[
\leq C R \|\ell\|_{\text{Lip}(\mathcal{S})} \|D u\|_{p,\varphi C^+_R} + \|\Gamma_k * f\|_{p,\varphi C^+_R}
\]

\[
+ \|\Gamma_k * (a^{hk}(\cdot) - a^{hk}(x))D_{hk} u\|_{p,\varphi C^+_R}.
\]
The convolution $\Gamma_k \ast f$ is a Riesz potential. On the other hand

$$
| (\Gamma_k \ast f)(x) | \leq C \int_{\mathcal{C}_R^+} \frac{|f(y)|}{\rho(x-y)^{n+1}} \, dy 
\leq CR \int_{\mathcal{C}_R^+} \frac{|f(y)|}{\rho(x-y)^{n+2}} \, dy \leq C \int_{\mathcal{C}_R^+} \frac{|f(y)|}{\rho(x-y)^{n+2}} \, dy
$$

with a constant depending on $T$ and diam $\Omega$. Because of [10, Lemma 7.12]

$$
\| \Gamma_k \ast f \|_{p,\mathcal{C}_R^+} \leq C \| f \|_{p,\mathcal{C}_R^+}
$$

which allows to apply [13, Theorem 3.3] that gives

$$
\| \Gamma_k \ast f \|_{p,\varphi,\mathcal{C}_R^+} \leq C \| f \|_{p,\varphi,\mathcal{C}_R^+}.
$$

Analogously

$$
| (\Gamma_k \ast [a^{hk}(\cdot) - a^{hk}(x)]D_{hk}u(\cdot)) | \leq C \int_{\mathcal{C}_R^+} \frac{|a^{hk}(y) - a^{hk}(x)||D_{hk}u(y)|}{\rho(x-y)^{n+2}} \, dy
$$

with a constant depending on diam $\Omega$ and $T$. The kernel $\rho(x-y)^{-(n+2)}$ is a nonnegative singular one and the [4, Theorem 0.1] gives

$$
\| \Gamma_k \ast [a^{hk}(\cdot) - a^{hk}]D_{hk}u \|_{p,\mathcal{C}_R^+} \leq C \| a \|_{\ast} \| D^2 u \|_{p,\mathcal{C}_R^+}.
$$

Applying again the results for sub-linear integrals [13, Theorem 3.7] we get

$$
\| \Gamma_k \ast [a^{hk}(\cdot) - a^{hk}]D_{hk}u \|_{p,\varphi,\mathcal{C}_R^+} \leq C \| a \|_{\ast} \| D^2 u \|_{p,\varphi,\mathcal{C}_R^+}.
$$

Hence

$$
(3.22) \quad \| g \|_{p,\varphi,\mathcal{C}_R^+} \leq C \left( R \| \ell \|_{L^p(S)} \| Du \|_{p,\varphi,\mathcal{C}_R^+} + \| f \|_{p,\varphi,\mathcal{C}_R^+} + R \eta_a(R) \| D^2 u \|_{p,\varphi,\mathcal{C}_R^+} \right).
$$

Further, the Rademacher theorem asserts existence almost everywhere of the derivatives $D_h \ell^k \in L^\infty$, thus

$$
D_h g(x) = -D_h \ell^k(x) D_k u(x) + [\ell^k(0) - \ell^k(x)]D_{hk} u - \ell^k(0)(\Gamma_{hk} \ast F)(x).
$$

The $M^{p,\varphi}$ norm of the last term is estimated as above and

$$
(3.23) \quad \| Dg \|_{p,\varphi,\mathcal{C}_R^+} \leq C \left( \| D\ell \|_{\infty;S} \| Du \|_{p,\varphi,\mathcal{C}_R^+} + R \| \ell \|_{L_p(S)} \| D^2 u \|_{p,\varphi,\mathcal{C}_R^+} + \| f \|_{p,\varphi,\mathcal{C}_R^+} + \eta_a(R) \| D^2 u \|_{p,\varphi,\mathcal{C}_R^+} \right).
$$

Finally unifying (3.20), (3.22) and (3.23) we get

$$
\| D^2 u \|_{p,\varphi,\mathcal{C}_R^+} \leq C \left( \| f \|_{p,\varphi,\mathcal{C}_Q} + (1 + R) \| Du \|_{p,\varphi,\mathcal{C}_R^+} + (R + \eta_a(R) + R \eta_a(R)) \| D^2 u \|_{p,\varphi,\mathcal{C}_R^+} \right).
$$
with a constant depending on known quantities and \( \| \ell \|_{\text{Lip}(S)} \) and \( \| D\ell \|_{\infty,S} \). Direct calculations lead to an interpolation inequality in \( M^{p,\varphi} \) analogous to [16, Lemma 3.3] (cf. [26])

\[
\| Du \|_{p,\varphi;C_R^+} \leq C \left( \| f \|_{p,\varphi;Q} + R \| D^2 u \|_{p,\varphi;C_R^+} + \frac{C}{R} \| u \|_{p,\varphi;C_R^+} \right) + (R + \eta_a(R) + R\eta_a(R)) \| D^2 u \|_{p,\varphi;C_R^+}.
\]

Choosing \( R \) small enough and moving the terms containing the norm of \( D^2 u \) on the left-hand side we get

\[
\| D^2 u \|_{p,\varphi;C_R^+} \leq C \left( \| f \|_{p,\varphi;C_R^+} + \frac{1}{R} \| u \|_{p,\varphi;C_R^+} \right).
\]

Because of the parabolic structure of the equation analogous estimate holds also for \( u_t \). Further the Jensen inequality applied to \( u(x) = \int_0^t u_s(x',s)ds \) gives

\[
\| u \|_{p,\varphi;C_R^+} \leq CR^2 \| u_t \|_{p,\varphi;C_R^+} \leq C \left( R^2 \| f \|_{p,\varphi;C_R^+} + R \| u \|_{p,\varphi;C_R^+} \right).
\]

Choosing \( R \) smaller, if necessary, we get \( \| u \|_{p,\varphi;C_R^+} \leq C \| f \|_{p,\varphi;C_R^+} \) and therefore

\[
(3.24) \quad \| u \|_{W^{2,1}_{p,\varphi}(C_R^+)} \leq C \| f \|_{p,\varphi;C_R^+} \leq C \| f \|_{p,\varphi;C_R^+}.
\]

Making a covering \( \{C^+_\alpha\}, \alpha \in \mathcal{A} \) such that \( Q \setminus Q' \subset \bigcup_{\alpha \in \mathcal{A}} C^+_\alpha \), considering a partition of unity subordinated to that covering and applying (3.24) for each \( C^+_\alpha \) we get

\[
(3.25) \quad \| u \|_{W^{2,1}_{p,\varphi}(Q,Q')} \leq C \| f \|_{p,\varphi;Q}
\]

with a constant depending on \( n, p, \Lambda, T, \text{diam } \Omega, \| D\Gamma \|_{\infty,Q}, \eta_a, \| a \|_{\infty,Q}, \| \ell \|_{\text{Lip}(S)} \), and \( \| D\ell \|_{\infty,S} \).

The estimate (2.5) follows from (3.18) and (3.25).

**References**

[1] P. Acquistapace, *On BMO regularity for linear elliptic systems*, Ann. Mat. Pura Appl., 161, 231–270, 1992.

[2] R. Adams, *Sobolev Spaces*, Academic Press, New York 1975.

[3] A. Akbulut, V.S. Guliyev, R. Mustafayev, *On the boundedness of the maximal operator and singular integral operators in generalized Morrey spaces*, Math. Bohem., 137 (1), 27–43, 2012.

[4] M. Bramanti, *Commutators of integral operators with positive kernels*, Le Matematiche, 49, 149–168, 1994.
[5] M. Bramanti, M.C. Cerutti, $W^{1,2}_p$ solvability for the Cauchy–Dirichlet problem for parabolic equations with VMO coefficients, Comm. Partial Diff. Eq., 18, 1735–1763, 1993.

[6] F. Chiarenza, M. Frasca, Morrey spaces and Hardy-Littlewood maximal function, Rend. Mat., 7, 273–279, 1987.

[7] F. Chiarenza, M. Frasca, P. Longo, Interior $W^{2,p}$-estimates for nondivergence elliptic equations with discontinuous coefficients, Ricerche Mat., 40, 149–168, 1991.

[8] F. Chiarenza, M. Frasca, P. Longo, $W^{2,p}$-solvability of the Dirichlet problem for non divergence elliptic equations with VMO coefficients, Trans. Amer. Math. Soc., 336, 841–853, 1993.

[9] E.B. Fabes, N. Rivière, Singular integrals with mixed homogeneity, Studia Math., 27, 19–38, 1996.

[10] D. Gilberg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, 2nd edn. Springer-Verlag, Berlin 1983.

[11] V.S. Guliyev, S.S. Aliyev, T. Karaman, P. Shukurov, Boundedness of sublinear operators and commutators on generalized Morrey spaces, Integral Equ. Oper. Theory, 71 (3), 327–355, 2011.

[12] V.S. Guliyev, L.G. Softova, Global regularity in generalized Morrey spaces of solutions to non-divergence elliptic equations with VMO coefficients, Potential Anal., (on line first), DOI 10.1007/s11118-012-9299-4.

[13] V.S. Guliyev, L.G. Softova, Generalized Morrey regularity for parabolic equations with discontinuity data, arXiv:1210.6566.

[14] F. John, L. Nirenberg, On functions of bounded mean oscillation, Commun. Pure Appl. Math., 14, 415–426, 1961.

[15] P.W. Jones, Extension theorems for BMO, Indiana Univ. Math. J., 29, 41–66, 1980.

[16] O.A. Ladyzhenskaya, V.A. Solonnikov, N.N. Ural’tseva, Linear and Quasilinear Equations of Parabolic Type, Transl. Math. Monographs 23, Amer. Math. Soc., Providence, R.I., 1968.

[17] A. Maugeri, D.K. Palagachev, L.G. Softova, Elliptic and Parabolic Equations with Discontinuous Coefficients, Wiley-VCH, Berlin 2000.

[18] T. Mizuhara, Boundedness of some classical operators on generalized Morrey spaces, Harmonic Anal., Proc. Conf., Sendai/Jap. 1990, ICM-90 Satell. Conf. Proc., 183–189, 1991.

[19] C.B. Morrey, On the solutions of quasi-linear elliptic partial differential equations, Trans. Amer. Math. Soc., 43, 126–166, 1938.

[20] E. Nakai, Hardy-Littlewood maximal operator, singular integral operators and the Riesz potentials on generalized Morrey spaces, Math. Nachr., 166, 95–103, 1994.

[21] D.K. Palagachev, L.G. Softova, Singular integral operators, Morrey spaces and fine regularity of solutions to PDE’s, Potential Anal., 20, 237–263, 2004.

[22] D. Sarason, On functions of vanishes mean oscillation, Trans. Amer. Math. Soc., 207, 391–405, 1975.

[23] L.G. Softova, Oblique derivative problem for parabolic operators with VMO coefficients, Manuscr. Math., 103, No.2, 203-220, 2000.

[24] L.G. Softova, Parabolic equations with VMO coefficients in Morrey spaces, Electron. J. Differ. Equ. 2001, No. 51, 1–25, 2001.
[25] L.G. Softova, *Singular integrals and commutators in generalized Morrey spaces*, Acta Math. Sin., Engl. Ser., **22**, 757–766, 2006.
[26] L.G. Softova, *Morrey-type regularity of solutions to parabolic problems with discontinuous data*, Manuscr. Math., **136** (3–4), 365–382, 2011.

Department of Civil Engineering, Second University of Naples, Via Roma 29, 81031 Aversa, Italy

*E-mail address*: luba.softova@unina2.it