Stochastic domination for the last passage percolation model

DAVID COUPIER∗
PHILIPPE HEINRICH†

Laboratoire Paul Painlevé, UMR 8524
UFR de Mathématiques, USTL, Bât. M2
59655 Villeneuve d’Ascq Cedex France
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Abstract: A competition model on $\mathbb{Z}^2_+$ governed by directed last passage percolation is considered. A stochastic domination argument between subtrees of the last passage percolation is put forward.

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1 Introduction

The directed last passage percolation model goes back to the original work of Rost [8] in the case of i.i.d. exponential weights. In this paper, Rost proved a shape theorem for the infected region and exhibited for the first time a link with the one-dimensional totally asymmetric simple exclusion process (TASEP). A background on exclusion processes can be found in the book [5] of Liggett. Since then, this link has been done into details by Ferrari and its coauthors [1, 2, 3] to obtain asymptotic directions and related results for competition interfaces. Other results have been obtained in the case of i.i.d. geometric weights: see Johansson [4]. For i.i.d. weights but with general weight distribution, Martin [6] proved a shape theorem and described the behavior of the shape function close to the boundary. See also the survey [7].

Let us consider $\Omega = [0, \infty)^{\mathbb{Z}^2}$ referred as the configuration space and endowed with a Borel probability measure $\mathbb{P}$. All throughout this paper, $\mathbb{P}$ is assumed translation-invariant: for all $a \in \mathbb{Z}^2$,

$$\mathbb{P}' = \mathbb{P} \circ \tau_a^{-1},$$

∗david.coupier@math.univ-lille1.fr
†philippe.heinrich@math.univ-lille1.fr
where \( \tau_a \) denotes the translation operator on \( \Omega \) defined by \( \tau_a(\omega) = \omega(a + \cdot) \). This is the only assumption about the probability measure \( \mathbb{P} \). We are interested in the behavior of optimal paths from the origin to a site \( z \in \mathbb{Z}_2^2 \). The collection of optimal paths forms the last passage percolation tree \( T \). In this paper, a special attention is paid to the subtree of \( T \) rooted at \( (1,1) \): see Figure 1.

![Figure 1: An example of the last passage percolation tree on the set \([0; 15]^2\). The subtree rooted at \((1,1)\) is surrounded by dotted lines. Here, the upper dotted line corresponds to the competition interface studied by Ferrari and Pimentel in \( [3] \).](image)

Our goal is to stochastically dominate subtrees of the last passage percolation tree by the one rooted at \((1,1)\). Our results (Theorems 2 and 3) essentially rely on elementary properties of the last passage percolation model; its directed nature and the positivity of weights.

The paper is organized as follows. In the rest of this section, optimal paths and the last passage percolation tree \( T \) are precisely defined. The growth property which allows us to compare subtrees of \( T \) is introduced. Theorems 2 and 3 are stated and commented in Section 2. They are proved in Section 3.

### 1.1 Paths, low-optimality, percolation tree

We will focus on (up-right oriented only) paths which can be defined as sequences (finite or not) \( \gamma = (z_0, z_1, \ldots) \) of sites \( z_i \in \mathbb{Z}^2 \) such that \( z_{i+1} - z_i = (1,0) \) or \((0,1)\).

For a given configuration \( \omega \), we define the length of a path \( \gamma \) as

\[
\omega(\gamma) = \sum_{z \in \gamma} \omega(z).
\]

If \( \Gamma_z \) is the (finite) set of paths from \((0,0)\) to \( z \), a path \( \gamma \in \Gamma_z \) is \( \omega \)-optimal if its length \( \omega(\gamma) \) is maximal on \( \Gamma_z \). The quantity \( \max_{\gamma \in \Gamma_z} \omega(\gamma) \) is known
as the last passage time at $z$. To avoid questions on uniqueness of optimal paths, it is convenient to call low-optimal the optimal path below all the others.

**Proposition 1.** Given $\omega \in \Omega$, each $\Gamma_z$ contains a (unique) low-optimal path denoted by $\gamma^\omega_z$.

**Proof** We can assume that $\text{Card}(\Gamma_z) \geq 2$. Given $\omega \in \Omega$, consider two arbitrary optimal paths $\gamma, \gamma'$ of $\Gamma_z$. If they have no common point (except endpoints $(0,0)$ and $z$), then one path is below the other. If $\gamma$ and $\gamma'$ meet in sites, say $u_1, \ldots, u_k$, it’s easy to see that the path which consists in concatenation of lowest subpaths of $\gamma, \gamma'$ between consecutive $u_i, u_{i+1}$ is also an optimal path of $\Gamma_z$. This procedure can be (finitely) repeated to reach the low-optimal path of $\Gamma_z$ for the configuration $\omega$. ■

In literature, optimal paths are generally unique and called geodesics. This is the case when $\text{IP}$ is a product measure over $\mathbb{Z}^2$ of non-atomic laws. Here, low-optimality ensures uniqueness without particular restriction and Proposition 1 allows then to define the (last passage) percolation tree $T_\omega$ as the collection of low-optimal paths $\gamma^\omega_z$ for all $z \in \mathbb{Z}^2_+$. Moreover, the subtree of $T_\omega$ rooted at $z$ is denoted by $T^\omega_z$.

### 1.2 Growth property

Let us introduce the set $T$ of all substrees of $T$:

$$T = \{T^\omega_z : z \in \mathbb{Z}^2_+, \omega \in \Omega\}.$$  

For a tree $T \in T$, $r(T)$ and $V(T)$ denote respectively its root and its vertex set.

**Definition 1.** A subset $A$ of $T$ satisfies the growth property if

$$(T \in A, T' \in T, V(T) - r(T) \subset V(T') - r(T')) \implies T' \in A. \quad (1)$$

For example, if $k \in \mathbb{Z}_+ \cup \{\infty\}$, the set $\{T \in T : \text{Card}V(T) \geq k\}$ satisfies the growth property. But so does not the set

$$\{T \in T : T \text{ have at least two infinite branches}\}.$$  

Indeed, the partial ordering on the set $T$ induced by Definition 1 does not take into account the graph structure of trees.
2 Stochastic domination

The following results compare subtrees of the last passage percolation tree through subsets of $T$ satisfying the growth property.

**Theorem 2.** Let $a \in \mathbb{Z}_+^2$ and a subset $A$ of $T$ satisfying the growth property (1). Set also $\Omega^a = \{ \omega \in \Omega : a \text{ belongs to } \gamma_a^{(1,0)} \text{ and } \gamma_a^{(0,1)} \}$. Then,

$$\mathbb{P}(T_{a+(1,1)} \in A, \Omega^a) \leq \mathbb{P}(T_{(1,1)} \in A, \tau_a(\Omega^a)).$$

In particular, if $\mathbb{P}$ is in addition a product measure, we have

$$\mathbb{P}(T_{a+(1,1)} \in A \mid \Omega^a) \leq \mathbb{P}(T_{(1,1)} \in A).$$

To illustrate the meaning of this result, assume that the vertices of $T_{(1,1)}$ are painted in blue and those of $T_{(2,0)}$ and $T_{(0,2)}$ in red. This random coloration leads to a competition of colors. The red area is necessarily unbounded since the model forces every vertex $(x,0)$ or $(0,x)$ with $x \in \{2,3,\ldots\}$ to be red. But the blue area can be bounded. Now, consider $a \in \mathbb{Z}_+^2$ and the same way to color but only in the quadrant $a + \mathbb{Z}_+^2$: this time, the blue area consists of the vertices of $T_{a+(1,1)}$ and the red one of $T_{a+(x,0)}$ and $T_{a+(0,x)}$, for $x \geq 2$.

Roughly speaking, Theorem 2 says that, conditionally to $\Omega^a$, the competition is harder for the latter blue area.

The proof of Theorem 2 can be summed up as follows. From a configuration $\omega$, a new one which is a perturbed translation of $\omega$, namely $\omega^a = \tau_a(\omega) + \varepsilon$ is built in order to satisfy

$$T_{a+(1,1)}^\omega = a + T_{(1,1)}^\omega.$$

But $\varepsilon$ is chosen such that for $\omega \in \Omega^a$, we have $V(T_{(1,1)}^\omega) \subset V(T_{(1,1)}^{\tau_a(\omega)})$ and the growth property leads to

$$T_{a+(1,1)}^{\omega} \in A \implies T_{(1,1)}^{\tau_a(\omega)} \in A.$$

It remains then to use the translation invariance of $\mathbb{P}$ to get the result.

The next result suggests a second stochastic domination argument in the spirit of Theorem 2.

**Theorem 3.** Let $m \in \mathbb{N}$ and a subset $A$ satisfying the growth property (1). Set $\Omega_m = \{ \omega : \gamma_m^{(m,1)} = ((0,0),(1,0),\ldots,(m,0),(m,1)) \}$. Then

$$\mathbb{P}(T_{(m,1)} \in A, \Omega_m) \leq \mathbb{P}(T_{(1,1)} \in A, \Omega_1).$$

(2)

Now some comments are needed.
Figure 2: Are represented the subtrees of the last passage percolation tree rooted at sites \((1, 1)\) and \((m, 1)\), for a configuration \(\omega \in \Omega^1 \cap \Omega^m\).

- Note that \(\Omega_1 = \{\omega : \omega(1, 0) > \omega(0, 1)\}\).

- It is worth pointing out here that Theorem 3 is, up to a certain extend, better than Theorem 2. If \(a = (m, 0)\) then the events \(\Omega^a\) and \(\Omega_m\) are equal and the probability \(\mathbb{P}(T_{a + (1,1)} \in A, \Omega^a)\) can be splitted into

\[
\mathbb{P}(T_{(m+1,1)} \in A, \Omega_m, \omega(m+1,0) < \omega(m,1)) \quad (3)
\]

and

\[
\mathbb{P}(T_{(m+1,1)} \in A, \Omega_m, \omega(m+1,0) \geq \omega(m,1)). \quad (4)
\]

On the event \(\{\omega(m+1,0) < \omega(m,1)\}\), \(T_{(m+1,1)}\) is as a subtree of \(T_{(m,1)}\). Hence, if \(A\) satisfies the growth property (1) then \(T_{(m+1,1)} \in A\) forces \(T_{(m,1)} \in A\). It follows that (3) is bounded by \(\mathbb{P}(T_{(1,1)} \in A, \Omega_1)\) which is at most \(\mathbb{P}(T_{(1,1)} \in A, \Omega_1)\) by Theorem 3.

On the other hand, \(\{\Omega_m, \omega(m+1,0) \geq \omega(m,1)\}\) is included in \(\Omega_{m+1}\). Consequently, (4) is bounded by \(\mathbb{P}(T_{(m+1,1)} \in A, \Omega_{m+1})\), and also by \(\mathbb{P}(T_{(1,1)} \in A, \Omega_1)\) by Theorem 3 again.

Combining these bounds, we get

\[
\mathbb{P}(T_{a + (1,1)} \in A, \Omega^a) \leq 2 \mathbb{P}(T_{(1,1)} \in A, \Omega_1) .
\]

To sum up, whenever \(2 \mathbb{P}(T_{(1,1)} \in A, \Omega_1)\) is smaller than \(\mathbb{P}(T_{(1,1)} \in A)\) (this is the case when \(\mathbb{P}\) and \(A\) are invariant by the symmetry with respect to the diagonal \(x = y\)), Theorem 2 with \(a = (m,0)\) can be obtained as a consequence of Theorem 3.

- Let us remark that further work seems to lead to the following improvement of Theorem 3: the application

\[
m \mapsto \mathbb{P}(T_{(m,1)} \in A, \Omega^m)
\]

should be non increasing.
Finally, by symmetry, Theorem 3 obviously admits an analogous version on the other axis. Roughly speaking, the subtree of the last passage percolation tree rooted at the site \((1, m)\) is stochastically dominated by the one rooted at \((1, 1)\).

Here are two situations in which Theorem 3 can be used. An infinite low-optimal path is said non trivial if it does not coincide with one of the two axes \(\mathbb{Z}(1, 0)\) and \(\mathbb{Z}(0, 1)\). If the set \(V(T_{(1,1)})\) is unbounded (which can be referred as “coexistence”) then, since each vertex in a subtree has a bounded number of children (in fact, at most 2), the tree \(T_{(1,1)}\) contains an infinite low-optimal path. So, if we set

\[
Coex = \{\text{Card}(V(T_{(1,1)})) = \infty\},
\]

then

\[
\mathbb{P}(Coex) > 0 \implies \mathbb{P}\left(\text{there exists a non trivial, infinite low-optimal path}\right) > 0. \quad (5)
\]

Conversely, assume that \(\mathbb{P}(Coex)\) is zero. Since the set

\[
\{T \in \mathcal{T} : \text{Card}(V(T)) = \infty\}
\]

satisfies the growth property, Theorem 3 implies that for all \(m \in \mathbb{N}\)

\[
\mathbb{P}\left(\text{Card}(V(T_{(m,1)})) = \infty, \Omega_m\right) = 0.
\]

Hence, \(\mathbb{P} - \text{a.s.},\) each subtree coming from the axis \(\mathbb{Z}(1, 0)\) is finite. This result can be generalized to the two axes \(\mathbb{Z}(1, 0)\) and \(\mathbb{Z}(0, 1)\) by symmetry. Then, \(\mathbb{P} - \text{a.s.},\) there is no non trivial, infinite low-optimal path and (5) becomes an equivalence.

Now, Set

\[
\Delta_n = \{(x, y) \in \mathbb{Z}^2_+ : x + y = n\},
\]

and let us denote by \(\alpha_n\) the (random) number of vertices of \(T_{(1,1)}\) meeting \(\Delta_n\):

\[
\alpha_n = \text{Card}\left(V(T_{(1,1)}) \cap \Delta_n\right).
\]

The event \(Coex\) can be written \(\bigcap_{n \in \mathbb{N}} \{\alpha_n > 0\}\). We say there is “strong coexistence” if

\[
\limsup_{n \to \infty} \frac{\alpha_n}{n} > 0.
\]

In a future work, Theorem 3 is used so as to give sufficient conditions ensuring strong coexistence with positive probability.
3 Proofs

3.1 Proof of Theorem 2

Recall that \( \gamma_\omega \) denotes the low-optimal path from 0 to \( z \) for \( \omega \).

I/ Let \( \omega \) and \( \omega + \varepsilon \) be configurations where \( \varepsilon \) is a vanishing configuration except on the axes \( \mathbb{Z}_+(1,0) \) and \( \mathbb{Z}_+(0,1) \) i.e \( \varepsilon(x,y) = 0 \) if \( xy \neq 0 \). We shall show that if \( \varepsilon \) satisfies

\[
\varepsilon(0,1) + \varepsilon(0,2) \geq \varepsilon(1,0) \quad \text{and} \quad \varepsilon(1,0) + \varepsilon(2,0) \geq \varepsilon(0,1),
\]

then

\[
V(T_{\omega+\varepsilon}^{(1,1)}) \subset V(T_{\omega}^{(1,1)}).
\]

Let \( z \in V(T_{\omega}^{(1,1)}) \). By definition, \( z \) is a vertex such that the low-optimal path \( \gamma_\omega \) contains \( (1,1) \); we have to show that \( z \in V(T_{\omega+\varepsilon}^{(1,1)}) \) i.e. the low-optimal path \( \gamma_{\omega+\varepsilon} \) also contains \( (1,1) \).

There is nothing to prove if \( \gamma_{\omega+\varepsilon} = \gamma_\omega \), so we assume that \( \gamma_{\omega+\varepsilon} \neq \gamma_\omega \). By additivity of \( \omega \mapsto \omega(\gamma) \) and optimality of \( \gamma_{\omega+\varepsilon} \) and \( \gamma_\omega \), we have

\[
\varepsilon(\gamma_\omega) = (\omega + \varepsilon)(\gamma_{\omega+\varepsilon}) - \omega(\gamma_\omega) \\
\leq (\omega + \varepsilon)(\gamma_{\omega+\varepsilon}) - \omega(\gamma_{\omega+\varepsilon}) \\
= \omega(\gamma_{\omega+\varepsilon}) + \varepsilon(\gamma_{\omega+\varepsilon}) - \omega(\gamma_\omega) \\
\leq \varepsilon(\gamma_{\omega+\varepsilon}).
\]

Note also that by low-optimality of \( \gamma_{\omega+\varepsilon} \) and \( \gamma_\omega \), we have

\[
(\omega + \varepsilon)(\gamma_{\omega+\varepsilon}) = (\omega + \varepsilon)(\gamma_\omega) \quad \text{and} \quad \omega(\gamma_{\omega+\varepsilon}) = \omega(\gamma_\omega) \implies \gamma_{\omega+\varepsilon} = \gamma_\omega.
\]

Since \( \gamma_{\omega+\varepsilon} \) and \( \gamma_\omega \) are different, this allows us to strengthen (8):

\[
\varepsilon(\gamma_\omega) < \varepsilon(\gamma_{\omega+\varepsilon}).
\]

Now, it’s got to be one thing or the other:

- Either the site \( (1,0) \) belongs to \( \gamma_{\omega+\varepsilon} \). In this case, the right hand side of (8) which becomes \( \varepsilon(0,0) + \varepsilon(1,0) \) and the strict inequality imply that \( \gamma_\omega \) can not run through \( (1,0) \). Moreover, if \( \gamma_\omega \) ran through \( (0,2) \), we would have

\[
\varepsilon(0,0) + \varepsilon(0,1) + \varepsilon(0,2) \leq \varepsilon(\gamma_\omega) < \varepsilon(0,0) + \varepsilon(1,0),
\]

but this would be in contradiction with (6). We conclude that \( \gamma_\omega \) must run through \( (0,1) \) and \( (1,1) \).
• Or the site \((0, 1)\) belongs to \(\gamma^\omega_{z+\varepsilon}\), and symmetrically we conclude that, if \((6)\) hold, \(\gamma^\omega_z\) runs through \((1, 0)\) and \((1, 1)\).

To sum up, if \(\varepsilon\) satisfies \((6)\) then \((7)\) holds. The conditions \((6)\) can be understood as follows; the second one (for example) prevents the set \(V(T_{(2, 0)})\) to drop vertices in favour of \(V(T_{(1, 1)})\) passing from \(\omega\) to \(\omega + \varepsilon\).

II/ For given configuration \(\omega \in \Omega\) and site \(a \in \mathbb{Z}^2_+\), we construct a new configuration \(\omega^a\) such that

\[
\forall z \in \mathbb{Z}^2_+, \quad \omega^a(\gamma^\omega_{a+z}) = \omega(\gamma^\omega_{a+z}).
\]  

(10)

The idea of the construction is to translate \(\omega\) from \(a\) to the origin and to modify then weights on the axes: more precisely, set

\[
\omega^a(z) = \begin{cases} 
\omega(\gamma^\omega_a) & \text{if } z = (0, 0); \\
\omega(\gamma^\omega_{a+z}) - \omega(\gamma^\omega_{a+z-(1,0)}) & \text{if } z = (x,0) \text{ with } x \in \mathbb{N}; \\
\omega(\gamma^\omega_{a+z}) - \omega(\gamma^\omega_{a+z-(0,1)}) & \text{if } z = (0,y) \text{ with } y \in \mathbb{N}; \\
\omega(a+z) & \text{otherwise}.
\end{cases}
\]

Let \(\bar{z}\) be the latest site of \(a+(\mathbb{Z}_+(1,0) \cup \mathbb{Z}_+(0,1))\) whereby \(\gamma^\omega_{a+z}\) passes. The configuration \(\omega^a\) is defined so as to the last passage time to \(\bar{z}\) for \(\omega\) is equal to the last passage time to \(\bar{z}-a\) for \(\omega^a\), i.e. \(\omega(\gamma^\omega_{\bar{z}}) = \omega^a(\gamma^\omega_{\bar{z}-a})\). Combining with \(\omega^a(\cdot) = \omega(a+\cdot)\) on \(a+\mathbb{N}^2\), the identity \((10)\) follows.

Finally, by low-optimality, the translated path \(a+\gamma^\omega_{\bar{z}}\) coincides with the restriction of \(\gamma^\omega_{a+z}\) to the quadrant \(a+\mathbb{N}^2\). See Figure 3.

![Figure 3](image-url)

Figure 3: To the left, the low-optimal path to \(a+z\) for a given configuration \(\omega\) is represented. Let us denote by \(\bar{z}\) the latest site of \(a+(\mathbb{Z}_+(1,0) \cup \mathbb{Z}_+(0,1))\) whereby \(\gamma^\omega_{a+z}\) passes. To the right, the low-optimal path to \(z\) for the corresponding configuration \(\omega^a\) is represented.

In particular, we can write with some abuse of notation

\[
a + T_{(1,1)}^\omega = T_{a+(1,1)}^\omega.
\]  

(11)
The induction formula’s
\[ \omega(\gamma_u^\omega) = \max(\omega(\gamma_{u-(1,0)}^\omega), \omega(\gamma_{u-(0,1)}^\omega)) + \omega(u), \] (12)
allows to rewrite the configuration \(\omega^a\):
\[ \omega^a = \tau_a(\omega) + \varepsilon, \] (13)
where \(\varepsilon\) is defined on the axes by
\[ \varepsilon(0,0) = \max(\omega(\gamma_{a-(1,0)}^\omega), \omega(\gamma_{a-(0,1)}^\omega)) \] (14)
\[ \varepsilon(x,0) = \max(0, \omega(\gamma_{a+(x,-1)}^\omega) - \omega(\gamma_{a+(x-1,0)}^\omega)) \quad (x \in \mathbb{N}) \] (15)
\[ \varepsilon(0,y) = \max(0, \omega(\gamma_{a+(1,y)}^\omega) - \omega(\gamma_{a+(0,y-1)}^\omega)) \quad (y \in \mathbb{N}) \] (15)
\[ \varepsilon(x,y) = 0 \quad \text{otherwise.} \] (16)

III/ Consider \(a\) and \(\Omega^a\) as in the statement of Theorem 2. Let \(\omega \in \Omega^a\) so that the length \(\omega(\gamma_u^\omega)\) is bigger than \(\omega(\gamma_{a+(1,-1)}^\omega)\) and \(\omega(\gamma_{a+(-1,1)}^\omega)\). It follows from (14) and (15) that \(\varepsilon(1,0) = \varepsilon(0,1) = 0\). Conditions (6) are then trivially satisfied so that (7) holds for \(\omega\) and also for \(\tau_a(\omega)\):
\[ V(\mathcal{T}_a(\omega)+\varepsilon) \subset V(\mathcal{T}_a(\omega)). \] (17)
Combined with (11) and (13), this leads to
\[ V(\mathcal{T}_a(\omega)) - a \subset V(\mathcal{T}_a(\omega)). \] (18)

Now, if \(A\) satisfies the growth property (11) then
\[ \mathcal{T}_a(\omega) \in A \implies \mathcal{T}_a(\omega) \in A. \] (19)
To summarize, we have \(\mathcal{T}_a(\omega) \subset \tau_a^{-1}\{\mathcal{T}_a(\omega) \in A\} \) on \(\Omega^a\), and since \(\mathbb{P}\) is translation-invariant and \(\Omega^a = \tau_a^{-1}(\tau_a(\Omega^a))\), we conclude that
\[ \mathbb{P}(\mathcal{T}_a(\omega) \in A, \Omega^a) \leq \mathbb{P}(\tau_a^{-1}\{\mathcal{T}_a(\omega) \in A\}, \Omega^a) = \mathbb{P}(\mathcal{T}_a(\omega) \in A, \tau_a(\Omega^a)). \] (20)

The first part of Theorem 2 is proved. In order to prove the second part, let us assume \(\mathbb{P}\) is a product measure. It suffices to remark the events \(\tau_a(\Omega^a)\) which means both low-optimal paths from \(-a\) to \((1,0)\) and \((0,1)\) run through the origin, and \(\mathcal{T}_a(\omega) \subset \tau_a(\Omega^a)\) are independent. Actually, the random variable \(\omega(0,0)\) is the only weight of \(\mathbb{Z}_2^4\) of which \(\tau_a(\Omega^a)\) depends on, and it is involved in all optimal paths coming from the origin. So, it does not affect the event \(\mathcal{T}_a(\omega) \subset \tau_a(\Omega^a)\).
3.2 Proof of Theorem 3

I/ Let \( \omega \in \Omega_1 \) and \( \omega + \varepsilon \) be two configurations where \( \varepsilon \) is a vanishing configuration except on the axis \( \mathbb{Z}_+ (0, 1) \): \( \varepsilon (x, y) = 0 \) whenever \( x > 0 \).
We also assume that \( \omega \) and \( \varepsilon \) verify \( \omega (1, 0) > \omega (0, 1) + \varepsilon (0, 1) \) (i.e. \( \omega + \varepsilon \in \Omega_1 \)). The goal of the first step consists in stating:

\[
V (T_{\omega} (1, 1)) \subset V (T_{\omega + \varepsilon} (1, 1)) .
\]

Let \( z \) be a vertex such that the low-optimal path \( \gamma_{\omega + \varepsilon} \) contains \( (1, 1) \).
If the low-optimal paths \( \gamma_{\omega} \) and \( \gamma_{\omega + \varepsilon} \) are different then it follows as for (9):

\[
\varepsilon \left( \gamma_{\omega} \right) < \varepsilon \left( \gamma_{\omega + \varepsilon} \right) .
\]

Henceforth, the condition \( \omega + \varepsilon \in \Omega_1 \) implies that \( \gamma_{\omega + \varepsilon} \) runs through \( (1, 0) \) and leads to a contradiction:

\[
\varepsilon (0, 0) \leq \varepsilon \left( \gamma_{\omega} \right) < \varepsilon \left( \gamma_{\omega + \varepsilon} \right) = \varepsilon (0, 0) .
\]

So, \( \gamma_{\omega} \) and \( \gamma_{\omega + \varepsilon} \) are equal, which implies \( z \) is a vertex of \( T_{\omega} (1, 1) \). Relation (16) is proved. It is worth to note that condition \( \omega + \varepsilon \in \Omega_1 \) ensures that the random interface between sets \( V (T_{\omega} (1, 1)) \) and \( V (T_{\omega} (2, 0)) \) remains unchanged if we add \( \varepsilon \) to \( \omega \). Hence, the set \( V (T_{\omega} (1, 1)) \) can only decrease.

II/ Let \( \omega \) be a configuration and \( b = (m - 1, 0) \). In the spirit of the proof of Theorem 2, a configuration \( \omega^b \) is built by translating \( \omega \) by vector \( -b \) and preserving the last passage percolation tree structure. The right construction is the following:

\[
\omega^b (z) = \begin{cases} 
\omega (\gamma_{\omega}^b) & \text{if } z = (0, 0); \\
\omega (\gamma_{b+z}^\omega) - \omega (\gamma_{b+z-1,0}^\omega) & \text{if } z \in \{0\} \times \mathbb{N}; \\
\omega (b + z) & \text{otherwise.}
\end{cases}
\]

By construction, the configuration \( \omega^b \) satisfies \( \omega^b (\gamma_{\omega}^b) = \omega (\gamma_{b+z}^\omega) \), for all \( z \in \mathbb{Z}_+^2 \). Thus, we can deduce from low-optimality:

\[
b + T_{\omega} (1, 1) = T_{\omega + \varepsilon} (1, 1) .
\]

The induction formula’s (12) allows to write for all \( z \in \mathbb{Z}_+^2 \),

\[
\omega^b (z) = \tau_b (\omega) (z) + \varepsilon (z) ,
\]

with

\[
\varepsilon (z) = \begin{cases} 
\omega (\gamma_{b-1,0}^\omega) & \text{if } z = (0, 0); \\
\max \left( \omega (\gamma_{b+z-1,0}^\omega) - \omega (\gamma_{b+z-1,0}^\omega), 0 \right) & \text{if } z \in \{0\} \times \mathbb{N}; \\
0 & \text{otherwise.}
\end{cases}
\]
Besides,
\[
\omega \in \Omega_m \iff \omega(\gamma^\omega_{b^+(0,1)}) < \omega(\gamma^\omega_{b^+(1,0)})
\]
\[
\iff \omega^b(0,1) + \omega(\gamma^\omega_{b^+}) < \omega^b(0,1) + \omega(b + (1,0))
\]
\[
\iff \omega^b(0,1) < \omega^b(1,0)
\]
\[
\iff \omega^b \in \Omega_1.
\] (18)

III/ Given \(\omega \in \Omega_m\), equivalence (18) implies \(\omega^b = \tau^b(\omega) + \varepsilon \in \Omega_1\). As a by-product, we have \(\tau^b(\omega) \in \Omega_1\) and from (16) and (17), we deduce
\[
V(T_{b+(1,1)}^\omega) - b \subset V(T_{(1,1)}^\omega) \subset V(T_{(1,1)}^{\tau^b(\omega)}).
\]

If \(A \subset \mathbb{T}\) satisfies the growth property (11) then
\[
\left( \omega \in \Omega_m \text{ and } T_{(m,1)}^\omega \in A \right) \implies \left( \tau^b(\omega) \in \Omega_1 \text{ and } T_{(1,1)}^{\tau^b(\omega)} \in A \right).
\]

Finally, (2) easily follows from the translation invariance of the probability measure \(\mathbb{P}\).

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