AN INFEASIBLE FULL NT-STEP INTERIOR POINT METHOD FOR CIRCULAR OPTIMIZATION

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Abstract. In this paper, we design a primal-dual infeasible interior-point method for circular optimization that uses only full Nesterov-Todd steps. Each main iteration of the algorithm consisted of one so-called feasibility step. Furthermore, giving a complexity analysis of the algorithm, we derive the currently best-known iteration bound for infeasible interior-point methods.

1. Introduction. Circular optimization (CO) problems are convex optimization problems in which a linear function is minimized over the intersection of an affine linear manifold with the Cartesian product of a finite number of circular cones. Mathematically, a typical circular cone in $R^{n_j}$ has the form

$$Q_{\theta_j}^{n_j} := \{ (x_0^j, \bar{x}^j) : x_0^j \geq \cot(\theta_j) \| \bar{x}^j \| \},$$

where $\theta \in (0, \frac{\pi}{2})$ is a given angle. Let $Q_\theta^n \subseteq R^n$ be the Cartesian product of several circular cones, i.e.,

$$Q_{\theta_1, \ldots, \theta_N}^{n_1 \ldots n_N} := Q_{\theta_1}^{n_1} \times Q_{\theta_2}^{n_2} \times \cdots \times Q_{\theta_N}^{n_N},$$

where $n = \sum_{j=1}^N n_j$. We consider CO problem in the standard form

$$\min \left\{ \sum_{j=1}^N \langle c_j, x^j \rangle_{\theta_j} : \sum_{j=1}^N (A_j x^j)_{\theta_j} = b, \ x^j \in Q_{\theta_j}^{n_j} \right\}, \quad (P)$$

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where \( b \in \mathbb{R}^m, A_j \in \mathbb{R}^{m \times n_j}, j = 1, 2, \ldots, N \). Moreover, 
\[
\langle c^j, x^j \rangle_{\theta_j} := (c^j)^T I^j_{n_j} x^j, (A_j x^j)_{\theta_j} := A_j I^j_{n_j} x^j
\]
and
\[
I_{\theta_j, n_j} := \begin{bmatrix}
1 & 0 \\
0 & \cot(\theta_j) I_{n_j-1}
\end{bmatrix} \in \mathbb{R}^{n_j \times n_j}, j = 1, \ldots, N.
\]
Defining
\[
Q^n_\theta = Q^n_{\theta_1} \times \cdots \times Q^n_{\theta_N}, A = [A_1, A_2, \ldots, A_N] \in \mathbb{R}^{m \times n}, c = (c^1; c^2; \ldots; c^N) \in \mathbb{R}^n,
\]
\[
I_{\theta, n} = \begin{bmatrix}
I_{\theta_1, n_1} & & \\
& \ddots & \\
& & I_{\theta_N, n_N}
\end{bmatrix}_{n \times n}
\] and 
\[
x = (x^1; x^2; \ldots; x^N) \in Q^n_{\theta},
\]
problem (P) can be written as
\[
\min \{ \langle c, x \rangle_\theta : (Ax)_{\theta} = b, x \in Q^n_{\theta} \}.
\]
Without loss of generality, we assume that \( A \) has full row rank, i.e., \( \text{rank}(A) = m \).

Due to the fact that \( Q^n_{\theta} \) is self-dual [1, Lemma 2], the dual problem of (P) is given by
\[
\max \{ b^T y : A^T y + s = c, y \in \mathbb{R}^m, s \in Q^n_{\theta} \}.
\]

It is well known that CO includes linear optimization (LO) as a special case. Even though CO is less general than symmetric cone optimization (SCO), which includes LO, second-order cone optimization (SOCO) and semidefinite optimization (SDO). The study of primal-dual interior-point methods (IPMs) for SCO was started by Nesterov and Todd [20]. Faybusovich [5] invoked Euclidean Jordan algebra to analyze a variety of search directions for SCO. Schmieta and Alizadeh [25] used the Euclidean Jordan algebraic framework to extend the analysis of the Monteiro-Zhang family [19] to all symmetric cones. Recently, CO and circular cone complementarity problems (CCCP) as the special cases of SCO and symmetric cone complementarity problems (SCCP) have received considerable attention from researchers because of its wide range of applications. Bai et al. [3] developed a primal-dual kernel function-based IPM for CO. Zhou et al. [29] and Chi et al. [4] studied the variational analysis of CO and circular cone eigenvalue complementarity problems (CCECP), respectively. Miao et al. [18] considered the constructions of complementarity functions and merit functions for CCCP.

The first polynomial primal-dual full Newton-step IPM for LO was proposed by Roos et al. [24]. In their work, the small-update path-following methods based on Newton direction is shown to have \( O(\sqrt{n} \log{\frac{1}{\epsilon}}) \) iteration-complexity. In [26, 28], Wang et al. and in [27], Wang and Lesaja extend full Newton step path-following algorithm for LO to SCO, convex quadratic symmetric cone optimization (CQSCO) and the Cartesian \( P_\kappa \)-symmetric cone complementarity problems (SCLCP) using the Nesterov-Todd (NT) direction. Kheirfam and Mahdavi-Amiri [16] and Kheirfam [6] proposed some variants of Roos et al.’s algorithm [24] for SCLCP and the Cartesian \( P_\kappa \)-SCLCP. Recently, Kheirfam [15] analyzed the primal-dual path-following algorithm for CO introduced in [11] by using the NT-directions and established the corresponding complexity bound.

In 2006, Roos [22] presented an infeasible IPM (IIPM) for LO. Later on, this algorithm extended to SCLCP [17] and horizontal LCP (HLCP) [9], and some variants of the algorithm investigated to SDO [7, 8] and SCO [10, 11]
Each main iteration of the aforementioned IIPMs is composed of one so-called feasibility step and a several centering steps to get an ϵ-optimal solution of the underlying problem. Recently, Roos [23] and Kheirfam [12, 13, 14] proposed IIPMs for LO, HLCP, the Cartesian $P_*(κ)$-SCLCP and SCO so that their algorithms do not need centering steps and take only one feasibility step in order to get a new iterate close enough to the central path.

Motivated by Roos [23] and Kheirfam [12, 13, 14], we present a full-NT step IIPM for CO. Each main iteration of the proposed algorithm is consisted of only one feasibility step. Moreover, we analyze the algorithm and derive the iteration-complexity bound which matches the currently best-known iteration bound for IIPMs.

The remainder of this paper is organized as follows: In section 2, we provide a brief introduction to the theory of Euclidean Jordan algebra on the circular cones. In section 3, we give the perturbed problems and our new algorithm. The convergence analysis of the algorithm is shown in section 4. We obtain the complexity bound of the algorithm in subsection 4.3. Finally, the paper will end with some concluding remarks follow in section 5.

2. Euclidean Jordan algebra. In this section, we recall associated Jordan algebra with the circular cone, and some results are needed [2, 15]. For $x, s ∈ R^n$, the bilinear operator $o$ with respect to $θ$ is defined as $(x_s)_θ := ((x, s)_θ; x_0s + s_0x)$ where $x = (x_1; \ldots; x_n)$. One can easily verifies that $(R^n, θ, o)$ is a Euclidean Jordan algebra under the circular inner product $⟨·, ·⟩_θ$ with $e = (1; 0) ∈ R^n$ as an identity element; i.e., for all $x, s, y ∈ R^n$, $(x_s)_θ = (s_o x)_θ, (x_o (x^o_o s))_θ = (x^o_o (x_o s))_θ$ where $x^2 = (x_o x)_θ, (x_o s)_θ = (x, s_o y)_θ$ and $(x_o e)_θ = (e_o x)_θ = x$. As a result, it can be easily seen that for each $x ∈ R^n$, $x^2 = 2x_0x + (x_0^2 - cot^2(θ)∥x∥^2)e = 0$, and the corresponding characteristic polynomial: $\lambda_θ^2 - 2x_0\lambda_θ + (x_0^2 - cot^2(θ)∥x∥^2) = 0$, has two roots $\lambda_{θ, min}(x) = x_0 - cot(θ)∥x∥$ and $\lambda_{θ, max}(x) = x_0 + cot(θ)∥x∥$ as the eigenvalues of $x$. Based on the eigenvalues, we define the trace and the determinant as $tr(x) = \lambda_{θ, min}(x) + \lambda_{θ, max}(x) = 2x_0$ and $det(x) = \lambda_{θ, min}(x)\lambda_{θ, max}(x) = x_0^2 - cot^2(θ)∥x∥^2$, respectively. Obviously, we have $c^2_{θ, 1} = (c_{θ, 1} o c_{θ, 1})_θ = c_{θ, 1}, c^2_{θ, 2} = (c_{θ, 2} o c_{θ, 2})_θ = c_{θ, 2}$ and $(c_{θ, 1} o c_{θ, 2})_θ = 0$, i.e., $(c_{θ, 1}, c_{θ, 2})_θ$ is a Jordan frame. Therefore, the circular spectral decomposition of $x$ with respect to $θ$ is given by $x = \lambda_{θ, max}(x)c_{θ, 1} + \lambda_{θ, min}(x)c_{θ, 2}$, where

$$c_{θ, 1} = \frac{1}{2} \left( 1; -\frac{tan(θ)\bar{x}}{∥\bar{x}∥} \right), \quad c_{θ, 2} = \frac{1}{2} \left( 1; -\frac{tan(θ)\bar{x}}{∥\bar{x}∥} \right).$$

Given the circular spectral decomposition $x$, some functions in the eigenvalues can be generated, namely, the inverse $x^{-1} = \lambda_{θ, max}(x)c_{θ, 1} + \lambda_{θ, min}(x)c_{θ, 2}$ and the Frobenius norm $∥x∥_{θ,F} := \sqrt{tr(x^2)} = \sqrt{\lambda_{θ, max}^2(x) + \lambda_{θ, min}^2(x)}$. The cone of squares of the Euclidean Jordan algebra $(R^n, θ, o)$ is the circular cone $Q^o_θ$ [2, Theorem 5]. We say $x ∈ intQ^o_θ$ if $\lambda_{θ, min}(x) > 0$. We say $x$ and $s$ operator commute with respect to $θ$ if they share a Jordan frame, and moreover $x$ and $s$ are similar, denoted as $x ~ s$, if $x$ and $s$ share the same set of eigenvalues. The arrow-shaped matrix $L_θ(x)$ and the quadratic representation $P_θ(x)$ associated with $x$ with respect to $θ$ are respectively defined as $(x_o s)_θ = L_θ(x)s$ and $P_θ(x) = 2L_θ(x)^2 - L_θ(x^2)$. In particular, one has $P_θ(x)e = x^2, P_θ(x)x^{-1} = x$ and $P_θ(x^{-1})P_θ(x) = P_θ(x)P_θ(x^{-1}) = I_n$, i.e., $P_θ(x^{-1}) = P_θ(x)^{-1}$. The following are several useful results which are needed in this paper.
Lemma 2.1. [21 Lemma 2.9] Given $x \in \text{int} Q^n_\theta$, we have
\[
\|x - x^{-1}\|_{\theta,F} \leq \frac{\|x^2 - e\|_{\theta,F}}{\lambda_{\theta, \min}(x)}.
\]

Lemma 2.2. [15 Lemma 4] Let $x, s \in \text{int} Q^n_\theta$, then
\[
\|P_\theta(x^{1/2})s - \mu e\|_{\theta,F} \leq \|(x \circ s)\theta - \mu e\|_{\theta,F}.
\]

Lemma 2.3. [15 Lemma 6] Let $x, s \in \text{int} Q^n_\theta$, then
\[
\lambda_{\theta, \min}(P_\theta(x^{1/2}) s) \geq \lambda_{\theta, \min}((x \circ s)\theta).
\]

In the sequel, we generalize the above definitions to the case where $N > 1$, when the circular cone underlying $Q^n_\theta$ is the Cartesian product of $N$ circular cones $Q^n_{\theta_j}$. For any $x \in \mathbb{R}^n$, the algebra $(\mathbb{R}^n, \theta, \circ)$ is defined as a direct product of the Jordan algebras $(\mathbb{R}^{n_j}, \theta_j, \circ)$ as
\[
(x \circ s)\theta = ((x^1 \circ s^1)\theta_1; \ldots; (x^N \circ s^N)\theta_N).
\]

Obviously, if $e^j$ is the identity element in the Jordan algebra for the $j$th circular cone, then
\[
e = (e^1; \ldots; e^N)
\]
is the identity element in $(\mathbb{R}^n, \theta, \circ)$. Moreover, $\text{tr}(e) = 2N$. The arrow-shaped matrix $L_\theta(x)$ and the quadratic representation $P_\theta(x)$ of $(\mathbb{R}^n, \theta, \circ)$ with respect to $\theta$ can be respectively adjusted to
\[
L_\theta(x) := \text{diag}(L_{\theta_1}(x^1), \ldots, L_{\theta_N}(x^N)), \quad P_\theta(x) := \text{diag}(P_{\theta_1}(x^1), \ldots, P_{\theta_N}(x^N)).
\]

Furthermore
\[
\lambda_{\theta, \max}(x) = \max_{1 \leq j \leq N} \{\lambda_{\theta_j, \max}(x^j)\}, \quad \lambda_{\theta, \min}(x) = \min_{1 \leq j \leq N} \{\lambda_{\theta_j, \min}(x^j)\},
\]
and
\[
\|x\|_{\theta,F}^2 = \sum_{j=1}^N \|x^j\|_{\theta_j,F}^2, \quad \text{tr}(x) = \sum_{j=1}^N \text{tr}(x^j).
\]

3. An infeasible algorithm. In this section, we present a full NT-step IIPM for CO. To this end, in next subsection we introduce the perturbed problems.

3.1. The perturbed problems. In accordance with the available results on IIPMs, let $(x^*, y^*, s^*)$ be an optimal solution of (P) and (D) such that $x^* + s^* \preceq_{Q^n} \xi e$, and consider the starting point $(x^0, y^0, s^0) = (\xi e, 0, \xi e)$ which satisfies $(x^0 \circ s^0)\theta = \rho^0 e$ with $\mu^0 = \xi^2$, where $\xi$ is a (positive) number. For any $\nu^0$, with $\nu^0 \in (0,1]$, we consider the perturbed problem pair $(P_{\nu^0})$ and $(D_{\nu^0})$ as follows:
\[
\min \{ (c - \nu^0 r^0_d, x)\theta : b - (Ax)\theta = \nu^0 r^0_p, \ x \in Q^n_\theta \}, \quad (P_{\nu^0})
\]
and
\[
\max \{ (b - \nu^0 r^0_p)^T y : c - ATy - s = \nu^0 r^0_d, y \in \mathbb{R}^{m}, s \in Q^n_\theta \}, \quad (D_{\nu^0})
\]
where $r^0_p = b - (Ax^0)\theta$ and $r^0_d = c - ATy^0 - s^0$ denote the initial values of the primal and dual residual vectors, respectively. The Karush-Kuhn-Tucker (KKT)
optimality conditions for the perturbed problem pair \((P_{\nu_0})\) and \((D_{\nu_0})\) are given as follows:

\[
\begin{align*}
& b - (A x)_{\theta} = \nu y_{\nu,0}^0, \quad x \in Q_{\theta}^n, \\
& c - A^T y - s = \nu y_{\nu,0}^0, \quad s \in Q_{\theta}^n, \\
& (x \circ s)_{\theta} = 0.
\end{align*}
\]

(1)

IPMs replace the complementarity condition \((x \circ s)_{\theta} = 0\) with the perturbed one \((x \circ s)_{\theta} = \mu_\theta e\) to get the system

\[
\begin{align*}
& b - (A x)_{\theta} = \nu y_{\nu,0}^0, \quad x \in Q_{\theta}^n, \\
& c - A^T y - s = \nu y_{\nu,0}^0, \quad s \in Q_{\theta}^n, \\
& (x \circ s)_{\theta} = \mu_\theta e
\end{align*}
\]

(2)

and reduce the parameters \(\mu_\theta\) and \(\nu_0\) to zero, until an \(\epsilon\)-solution of the problem pair \((P)\) and \((D)\) is obtained. Assuming \(P_{\nu}(\cdot)\) as the quadratic representation of \(R^n\) with respect to \(\theta\), considering \(w = P_{\nu}(x^\frac{1}{2})(P_{\nu}(x^\frac{1}{2})s)^{-\frac{1}{2}} [ = P_{\nu}(s^{-\frac{1}{2}})(P_{\nu}(s^\frac{1}{2})x^\frac{1}{2})]\) as the NT-scaling point of \(x\) and \(s\) with respect to \(\theta\), and using \(\delta_{\nu,\theta}\) to get the system \([\ref{system}]\) has a unique solution for each \(\nu_0\) such that \(\nu_0 < \nu_0^{\circ}\) is a strictly feasible solution of \((D_{\nu_0})\).

Note that, if \(\nu_0 = 1\), then \(x = x^0\) yields a strictly feasible solution of \((P_{\nu_0})\), and \((y, s) = (y^0, s^0)\) is a strictly feasible solution of \((D_{\nu_0})\). This means that both perturbed problems \((P_{\nu_0})\) and \((D_{\nu_0})\) satisfy the interior-point condition (IPC) for \(\nu_0 = 1\). Therefore, system \([\ref{system}]\) has a unique solution for each \(\mu_\theta > 0\) (see \([3]\)) and the set of all such solutions form a guide line, so-called central path, to the \(\epsilon\)-solution of the problem pair \((P)\) and \((D)\). In what follows, the parameters \(\mu_\theta\) and \(\nu_0\) always satisfy the relation \(\mu_\theta = \mu_0^0 \nu_0\).

**Lemma 3.1.** Let the original problems \((P)\) and \((D)\) be feasible and \(0 < \nu_0 \leq 1\). Then, the perturbed problems \((P_{\nu_0})\) and \((D_{\nu_0})\) satisfy the IPC.

3.2. **The Algorithm.**

We measure proximity to the central path of the perturbed problem pair \((P_{\nu_0})\) and \((D_{\nu_0})\) by the quantity

\[
\delta(x, s; \mu_\theta) := \delta(v) := \frac{1}{2} \|v - 1 - v\|_{\theta, F}, \quad \text{where } v := \frac{P_{\nu_0}(w^{-\frac{1}{2}})x}{\sqrt{\mu_\theta}} \left[ = \frac{P_{\nu_0}(w^{\frac{1}{2}})s}{\sqrt{\mu_\theta}} \right].
\]

(4)

As an immediate consequence, we have the following lemma.

**Lemma 3.2.** With \(\delta := \delta(v)\), we have

\[
\frac{1}{\rho(\delta)} \leq \lambda_{\nu, \text{min}}(v') \leq \lambda_{\nu, \text{max}}(v') \leq \rho(\delta), \quad j = 1, \ldots, N,
\]

where \(\rho(\delta) = \delta + \sqrt{1 + \delta^2}\).

The algorithm works as follows. Suppose for some \(\mu_\theta \in (0, \mu_0^0]\), the algorithm starts from an arbitrary initial point \((x, y, s)\) with \(\delta(x, s; \mu_\theta) < \tau\) where \(\tau\) is a positive threshold value. By solving the search direction system \([\ref{system}]\), the algorithm computes the search directions \(\Delta x, \Delta y\) and \(\Delta s\) to generate the new iterate.
(x⁺, y⁺, s⁺), which is feasible for the new perturbed problem pair (Pν⁺) and (Dν⁺) and δ(x⁺, s⁺; μν⁺) < τ. Updating the parameters μν and νν as μν⁺ = (1 − β)μν and νν⁺ = (1 − β)νν with β ∈ (0, 1), this procedure is repeated until an ϵ-solution of the problem pair (P) and (D) is reached. A formal description of the algorithm is given as below.

Algorithm 1: primal–dual Infeasible IPM

Input: accuracy parameter ε > 0;
barrier update parameter β, 0 < β < 1;
a threshold parameter 0 < τ < 1.

begin
x := ξe; y := 0; s := ξe; μν := ννξ²; νν = 1;
while max (tr((x ◦ s)θ), ∥rν∥θ,F, ∥rν∥θ,F) > ϵ do
begin
(x, y, s) := (x, y, s) + (∆x, ∆y, ∆s);
μν := (1 − β)μν; νν := (1 − β)νν;
end
end

Fig. 1: Algorithm 1

4. Analysis of the algorithm. In this section, we analyze the proposed algorithm in Fig. 1. To this end, let (x, y, s) be a strictly feasible solution of the perturbed pair (Pν) and (Dν) for some μν > 0 and νν > 0, and such that δ(x, s; μν) < τ. We proceed by deriving the search direction (∆x, ∆y, ∆s) in the algorithm to get the new strictly feasible iterate (x⁺, y⁺, s⁺) = (x + ∆x, y + ∆y, s + ∆s) for the new perturbed pair (Pν⁺) and (Dν⁺).

Due to feasibility of the iterate (x, y, s) for the perturbed pair (Pν) and (Dν), the direction (∆x, ∆y, ∆s) is obtained by solving the following system:

\[
\begin{align*}
A^2νPν(w^½)Pν(w^½)Δx &= βννrν², \\
I^2νPν(w^½)ATΔy + I^2νPν(w^½)Δs &= βννI^2νPν(w^½)rν², \\
(Pν(w^½)Δx ◦ Pν(w^½)s)ν + (Pν(w^½)Δs)ν &= (1 − β)μνe − (Pν(w^½)Δx ◦ Pν(w^½)s)ν.
\end{align*}
\]

Introducing the scaled search directions dx and ds as

\[
dx := \frac{Pν(w^½)Δx}{\sqrt{μν}}, \quad ds := \frac{Pν(w^½)Δs}{\sqrt{μν}},
\]

we may easily check that system [5] can be written as follows:

\[
\begin{align*}
A^Tdx &= βννrν², \\
A^T\frac{Δy}{μν} + I^2νPν(w^½)rν² &= βννI^2νPν(w^½)rν², \\
dx + ds &= (1 − β)v − v_n,
\end{align*}
\]

where \( \bar{A} = \sqrt{μν}A^2νPν(w^½) = \sqrt{μν}A(I^2νPν(w^½))^T \) and \( v \) is defined as in [4]. Therefore, by taking a full-NT step, by [4] and [7], the new iterates are given as
follows:

\[ x^+ = \sqrt{\mu_n} P_{\theta}(x + d_x), \quad s^+ = \sqrt{\mu_n} P_{\theta}(w - \frac{1}{2})(v + d_s). \]  

Using the third equation of (7), we have

\[ (v + d_x) \circ (v + d_s) = v^2 + (v \circ (d_x + d_s))_\theta + (d_x \circ d_s)_\theta \]
\[ = (1 - \beta) e + (d_x \circ d_s)_\theta. \]  

**Lemma 4.1.** The iterate \((x^+, y^+, s^+)\) is strictly feasible if \((1 - \beta)e + (d_x \circ d_s)_\theta \succ Q^\theta_0\).

**Proof.** We introduce a step length \(\alpha\) with \(0 \leq \alpha \leq 1\), and define \(v_x(\alpha) = v + \alpha d_x\) and \(v_s(\alpha) = v + \alpha d_s\). From the third equation of (7), it follows that

\[ (v_x(\alpha) \circ v_s(\alpha))_\theta = (v \circ (v + \alpha d_x))_\theta \]
\[ = v^2 + \alpha (v \circ (d_x + d_s))_\theta + \alpha^2 (d_x \circ d_s)_\theta \]
\[ = v^2 + \alpha (v \circ ((1 - \beta)v^{-1} - v))_\theta + \alpha^2 (d_x \circ d_s)_\theta \]
\[ = (1 - \alpha)v^2 + \alpha(1 - \beta)e + \alpha^2 (d_x \circ d_s)_\theta \]
\[ = (1 - \alpha)(v^2 + \alpha(1 - \beta)e) + \alpha^2 ((1 - \beta)e + (d_x \circ d_s)_\theta). \]

Since \(v^2 \succ Q^\theta_0\), we have \(v^2 + \alpha(1 - \beta)e \succ Q^\theta_0\). Hence, if \((1 - \beta)e + (d_x \circ d_s)_\theta \succ Q^\theta_0\), we have \((v_x(\alpha) \circ v_s(\alpha))_\theta \succ Q^\theta_0\). From the last inequality, it follows that

\[ 0 < \det_\theta((v_x(\alpha) \circ v_s(\alpha))_\theta) \leq \det_\theta(v_x(\alpha)) \det_\theta(v_s(\alpha)), \]
and this implies that \(\det_\theta(v_x(\alpha)) \neq 0\) and \(\det_\theta(v_s(\alpha)) \neq 0\), for \(0 \leq \alpha \leq 1\). Since \(\det_\theta(v_x(0)) = \det_\theta(v_s(0)) = \det_\theta(v) > 0\), by continuity, \(\det_\theta(v_x(\alpha))\) and \(\det_\theta(v_s(\alpha))\) stay positive, for all \(0 \leq \alpha \leq 1\). By the circular spectral decomposition, this implies that all the eigenvalues of \(v_x(\alpha)\) and \(v_s(\alpha)\) stay positive, for all \(0 \leq \alpha \leq 1\). Hence, we obtain that \(v_x(1) = v + d_x \succ Q^\theta_0\) and \(v_s(1) = v + d_s \succ Q^\theta_0\). This completes the proof. \(\square\)

**Corollary 1.** After the full-NT step the new iterate \((x^+, y^+, s^+)\) is strictly feasible if \(1 - \beta + \lambda_i((d_x \circ d_s)_\theta) > 0, \ i = 1, \ldots, 2N.\)

In what follows, we set \(\bar{\omega}(v) := \frac{1}{2}(\|d_x\|^2_\theta,F + \|d_s\|^2_\theta,F)\) and assume that \(\bar{\omega}(v) < 1 - \beta, \ \text{for all} \ \beta \in [0, 1].\) Then, we get

\[ |\lambda_i((d_x \circ d_s)_\theta)| \leq \|\theta \circ d_s\|^2_\theta,F = \|L_\theta(d_x)\|d_s,\theta,F \leq \|L_\theta(d_x)\|\|d_s\|_\theta,F \leq \|d_x\|_\theta,F \|d_s\|_\theta,F \leq \frac{1}{2}(\|d_x\|^2_\theta,F + \|d_s\|^2_\theta,F) = \bar{\omega}(v). \]

It follows that \(|\lambda_i((d_x \circ d_s)_\theta)| < 1 - \beta, \ \text{for} \ i = 1, \ldots, 2N.\) Hence, \(\bar{\omega}(v) < 1 - \beta\) implies that the iterate \((x^+, y^+, s^+)\) is strictly feasible, by Corollary 3.2. In the sequel, we proceed by driving an upper bound for \(\delta(x^+, s^+; \mu_0^+)\), and we denote simply by

\[ \delta(v^+) = \frac{1}{2}\|v^+ - (v^+)^-\|_\theta,F, \]

where \(v^+ = \frac{P_{\theta}(w - \frac{1}{2}) x^+}{\sqrt{\mu_0^+}}\) and \(w^+\) is the NT-scaling point of \(x^+\) and \(s^+\).

**Lemma 4.2.** ([15, Lemma 11]) One has

\[ (1 - \beta)(v^+)^2 \sim P_{\theta}((v + d_x)^2)(v + d_s). \]
Lemma 4.3. We have
\[ \lambda_{\theta, \min}(v^+) \geq \sqrt{1 - \bar{\omega}(v)} \Big/ (1 - \beta). \]

**Proof.** Using Lemmas 4.2, 2.3 and Equation (9), we obtain
\[
\lambda_{\theta, \min}(v^+)^2 = \lambda_{\theta, \min}\left( P_\theta \left( \left( \frac{v + d_x}{\sqrt{1 - \beta}} \right)^2 \left( \frac{v + d_s}{\sqrt{1 - \beta}} - e \right) \right) \right)
\]
\[
\geq \lambda_{\theta, \min}\left( \left( \frac{v + d_x}{\sqrt{1 - \beta}} \circ \left( \frac{v + d_s}{\sqrt{1 - \beta}} \right) \right) \theta \right)
\]
\[
= \frac{1}{1 - \beta} \lambda_{\theta, \min}\left( (1 - \beta)e + (d_x \circ d_s) \right)
\]
\[
\geq \frac{1}{1 - \beta} \left( 1 - \bar{\omega}(v) \right) \lambda_{\theta, \min}\left( (d_x \circ d_s) \circ \theta \right)
\]
\[
\geq 1 - \frac{\bar{\omega}(v)}{1 - \beta}.
\]
This implies the lemma. \(\square\)

Lemma 4.4. If \(\bar{\omega}(v) < 1 - \beta\), then
\[
\delta(v^+) = \frac{\bar{\omega}(v)}{2 \sqrt{(1 - \beta)(1 - \beta - \bar{\omega}(v))}}.
\]

**Proof.** Using Lemmas 2.1, 4.2, 4.3, 2.2 and Equation (9), we get
\[
2\delta(v^+) = \|v^+ - (v^+)^{-1}\|_{\theta,F} = \frac{\|v^+\|^2 - e\|_{\theta,F}}{\lambda_{\theta, \min}(v^+)}
\]
\[
= \frac{1}{\lambda_{\theta, \min}(v^+)} \left\| P_\theta \left( \left( \frac{v + d_x}{\sqrt{1 - \beta}} \right)^2 \left( \frac{v + d_s}{\sqrt{1 - \beta}} - e \right) \right) \right\|_{\theta,F}
\]
\[
\leq \frac{\sqrt{1 - \beta}}{1 - \beta - \bar{\omega}(v)} \left\| \left( \frac{v + d_x}{\sqrt{1 - \beta}} \circ \left( \frac{v + d_s}{\sqrt{1 - \beta}} \right) \right) \theta - e \right\|_{\theta,F}
\]
\[
= \frac{1}{\sqrt{(1 - \beta)(1 - \beta - \bar{\omega}(v))}} \left\| (d_x \circ d_s) \theta \right\|_{\theta,F}
\]
\[
\leq \frac{\bar{\omega}(v)}{\sqrt{(1 - \beta)(1 - \beta - \bar{\omega}(v))}}.
\]
This completes the proof. \(\square\)

4.1. Upper bound for \(\bar{\omega}(v)\).

**Lemma 4.5.** The solution \((v_1, z, v_2)\) of the linear system
\[
\begin{align*}
Av_1 &= 0, \\
\bar{A}^T z + \bar{I}_{\theta, n}^2 v_2 &= 0, \\
v_1 + v_2 &= \bar{a}
\end{align*}
\]

satisfies the following equality
\[
\|v_1\|_{\theta,F}^2 + \|v_2\|_{\theta,F}^2 = \|\bar{a}\|_{\theta,F}^2.
\]
Proof. From the first two equations of (10), it follows that $(v_1, v_2)_\theta = 0$. Taking norm-squared with respect to \( \theta \) of the third equation, we get
\[
\|\bar{a}\|_{\beta,F}^2 = \|v_1 + v_2\|_{\beta,F}^2 = \|v_1\|_{\beta,F}^2 + \|v_2\|_{\beta,F}^2 + 2(v_1, v_2)_\theta = \|v_1\|_{\beta,F}^2 + \|v_2\|_{\beta,F}^2.
\]
This proves the lemma.

Lemma 4.6. Let \((d_x, \Delta y, d_s)\) be a solution of (7). Then, we have
\[
\|d_x\|_{\beta,F}^2 + \|d_s\|_{\beta,F}^2 \leq \left[2(1 - \beta)\delta + \beta \sqrt{2\bar{N}} \rho(\delta) + \frac{3\beta \nu_0\|\beta, F\|_{\theta}^2}{\sqrt{\mu_\theta}} \sqrt{\text{tr}(w^2 + w^{-2})}\right]^2,
\]
where \(w\) is the NT-scaling point of \(x\) and \(s\) with respect to \(\theta\) and \(\rho(\delta)\) is defined as in Lemma 3.3.

Proof. Let \((\bar{u}, \bar{y}, s)\) satisfy \(\bar{A}(\bar{d}_x - \bar{u}) = 0\), and \(\bar{A}^T \frac{\bar{u}}{\mu_\theta} + I_{\theta,F}(\bar{d}_s - \bar{v}) = \frac{\beta \nu_0}{\sqrt{\mu_\theta}} I_{\theta,F} P_\theta(w^2) r_0^0\).

Then, we get
\[
\bar{A}(d_x - \bar{u}) = 0,
\]
\[
\bar{A}^T \left(\frac{\Delta y}{\mu_\theta} - \frac{\bar{y}}{\mu_\theta}\right) + I_{\theta,F}(d_s - \bar{v}) = 0,
\]
\[
(d_x - \bar{u}) + (d_s - \bar{v}) = (1 - \beta)u^{-1} - v - (\bar{u} + \bar{v}).
\]
For any \((u, v) \in \mathbb{R}^n \times \mathbb{R}^n\), let \(||(u, v)||_\pi := \|u\|_{\theta,F}^2 + \|v\|_{\theta,F}^2\). It is trivial to show that \(||(\cdot, \cdot)||_\pi\) is a norm on \(\mathbb{R}^n \times \mathbb{R}^n\). Then, applying Lemma 4.5 to the system (11), we obtain
\[
\|(d_x, d_s)\|_\pi \leq \|(d_x - \bar{u}, d_s - \bar{v})\|_\pi + \|(\bar{u}, \bar{v})\|_\pi
\]
\[
= \|(1 - \beta)v^{-1} - v - (\bar{u} + \bar{v})\|_{\theta,F} + \|(\bar{u}, \bar{v})\|_\pi
\]
\[
\leq \|(1 - \beta)v^{-1} - v\|_{\theta,F} + \||u||_{\theta,F} + \||v||_{\theta,F} + \sqrt{||u||_{\theta,F}^2 + ||v||_{\theta,F}^2}
\]
\[
\leq \|(1 - \beta)v^{-1} - v\|_{\theta,F} + \sqrt{3||u||_{\theta,F}^2 + ||v||_{\theta,F}^2},
\]
where the last inequality follows from \(a + b \leq \sqrt{2(a^2 + b^2)} \leq 2\sqrt{a^2 + b^2}\). Let \((x^0, y^0, s^0)\) and \((x^*, y^*, s^*)\) be as defined in Subsection 3.1. Then, we have
\[
\beta \nu_0 r_0^0 = \beta \nu_0 \mu_\theta (b - (Ax^0)_\theta) = \beta \nu_0 \mu_\theta A\bar{l}_{\theta,F}(x^* - x^0)
\]
\[
= \beta \nu_0 \mu_\theta A\bar{l}_{\theta,F} P_\theta(w^\frac{1}{2}) P_\theta(w^{-\frac{1}{2}})(x^* - x^0)
\]
\[
= \frac{\beta \nu_0}{\sqrt{\mu_\theta}} (\sqrt{\mu_\theta} A\bar{l}_{\theta,F} P_\theta(w^\frac{1}{2})) P_\theta(w^{-\frac{1}{2}})(x^* - x^0)
\]
\[
= \frac{\beta \nu_0}{\sqrt{\mu_\theta}} \bar{A} P_\theta(w^{-\frac{1}{2}})(x^* - x^0).
\]
Similarly, we get
\[
\frac{\beta \nu_0}{\sqrt{\mu_\theta}} I_{\theta,F} P_\theta(w^\frac{1}{2}) r_0^0 = \bar{A}^T \frac{\beta \nu_0 (y^* - y^0)}{\mu_\theta} + \frac{\beta \nu_0}{\sqrt{\mu_\theta}} I_{\theta,F} P_\theta(w^\frac{1}{2})(s^* - s^0) - (\bar{u} + \bar{v}).
\]
Therefore, for \(\bar{u} = \frac{\beta \nu_0}{\sqrt{\mu_\theta}} P_\theta(w^{-\frac{1}{2}})(x^* - x^0)\) and \(\bar{v} = \frac{\beta \nu_0}{\sqrt{\mu_\theta}} P_\theta(w^\frac{1}{2})(s^* - s^0)\), we obtain
\[
\|\bar{u}\|_{\theta,F}^2 + \|\bar{v}\|_{\theta,F}^2 = \frac{\beta^2 \nu_0^2}{\mu_\theta} (\|P_\theta(w^{-\frac{1}{2}})(x^* - x^0)\|_{\theta,F}^2 + \|P_\theta(w^\frac{1}{2})(s^* - s^0)\|_{\theta,F}^2).
\]
Since $x^* \text{ and } (y^*, s^*)$ are feasible for (P) and (D) respectively, we have $x^* \succeq Q^n_0$ and $s^* \succeq Q^n_0$. Hence, $0 \preceq Q^n_0 x^* \preceq Q^n_0 x^* + s^* \preceq Q^n_0 \xi e$ and similarly $0 \preceq Q^n_0 s^* \preceq Q^n_0 \xi e$. Therefore, it follows that

$$0 \preceq Q^n_0 x^0 - x^* \preceq Q^n_0 \xi e, \quad 0 \preceq Q^n_0 s^0 - s^* \preceq Q^n_0 \xi e. \quad (14)$$

Due to $\|x\|_{\theta,F}^2 = \text{tr}((x \circ x)_\theta) = 2x^T I^2_{\theta,n} x = 2\langle x, x \rangle_\theta$, we have

$$\|P_\theta(w^\frac{1}{2})(s^0 - s^*)\|_{\theta,F}^2 = 2(P_\theta(w^\frac{1}{2})(s^0 - s^*))^T I^2_{\theta,n} P_\theta(w^\frac{1}{2})(s^0 - s^*)$$

$$= 2(s^0 - s^*)^T P_\theta(w^\frac{1}{2}) I^2_{\theta,n} P_\theta(w^\frac{1}{2})(s^0 - s^*)$$

$$= 2(s^0 - s^*)^T I^2_{\theta,n} P_\theta(w^\frac{1}{2})(s^0 - s^*)$$

$$= 2(s^0 - s^*)^T I^2_{\theta,n} P_\theta(w)(s^0 - s^*)$$

$$= 2(s^0 - s^*, P_\theta(w)(s^0 - s^*))_\theta$$

$$= 2\langle \xi e, P_\theta(w)(s^0 - s^*) \rangle_\theta - 2\langle s^* - s^0 + \xi e, P_\theta(w)(s^0 - s^*) \rangle_\theta$$

$$\leq 2\langle \xi e, P_\theta(w)(s^0 - s^*) \rangle_\theta - 2\langle s^* - s^0 + \xi e, P_\theta(w)(s^0 - s^*) \rangle_\theta$$

$$= 2\xi e^T P_\theta(w) I^2_{\theta,n}(s^0 - s^*) = 2\xi \langle P_\theta(w)s^0 - s^* \rangle_\theta$$

$$= 2\xi \langle w^2, s^0 - s^* \rangle_\theta = 2\xi \langle w^2, \xi e \rangle_\theta - 2\xi \langle w^2, \xi e - s^0 + s^* \rangle_\theta$$

$$\leq 2\xi \langle w^2, \xi e \rangle_\theta = 2\xi^2 \langle w^2, e \rangle_\theta = \xi^2 \text{tr}((w^2 \circ e)_\theta) = \xi^2 \text{tr}(w^2).$$

In the same way, it follows that

$$\|P_\theta(w^{-\frac{1}{2}})(x^0 - x^*)\|_{\theta,F}^2 \leq \xi^2 \text{tr}(w^{-2}).$$

Substitution of the last two inequalities in $\|d_x, d_s\| \leq \|(1 - \beta)v^{-1} - v\|_{\theta,F} + \frac{3\beta \nu_0 \xi}{\sqrt{\mu_\theta}} \sqrt{\text{tr}(w^2 + w^{-2})}$ gives

$$\|d_x, d_s\| \leq \|(1 - \beta)v^{-1} - v\|_{\theta,F} + \frac{3\beta \nu_0 \xi}{\sqrt{\mu_\theta}} \sqrt{\text{tr}(w^2 + w^{-2})}$$

$$\leq 2(1 - \beta)\delta + \beta\|v\|_{\theta,F} + \frac{3\beta \nu_0 \xi}{\sqrt{\mu_\theta}} \sqrt{\text{tr}(w^2 + w^{-2})}$$

$$\leq 2(1 - \beta)\delta + \beta \sqrt{2N \rho(\delta)} + \frac{3\beta \nu_0 \xi}{\sqrt{\mu_\theta}} \sqrt{\text{tr}(w^2 + w^{-2})},$$

where the second inequality follows from the triangular inequality and $\|v\|_{\theta,F}$ and the last inequality is due to Lemma 3.2. This implies the inequality in lemma. \hfill \Box

Since $\lambda_{\theta, \min}(\lambda_{\theta, \max}(x)e - x) \geq \lambda_{\theta, \max}(x) - \lambda_{\theta, \max}(x) = 0$. This means that $\lambda_{\theta, \max}(x)e - x \in Q^n_0$ and also we have $s \in Q^n_0$. From these last two inequalities, we
conclude that \( \langle x, s \rangle \leq \lambda_{\theta, \text{max}}(x, e, s) \). Therefore, by \( u = (P_\theta(x^\frac{1}{2}))s^{-\frac{1}{2}} \), we obtain

\[
\text{tr}(w^2) = 2\langle w, w \rangle_{\theta} = 2\langle P_\theta(x^\frac{1}{2})u, P_\theta(x^\frac{1}{2})u \rangle_{\theta}
= 2\langle P_\theta(x^\frac{1}{2})u^T I_{\theta, n}^2 P_\theta(x^\frac{1}{2})u = 2u^T P_\theta(x^\frac{1}{2})I_{\theta, n}^2 P_\theta(x^\frac{1}{2})u
\leq 2\lambda_{\theta, \text{max}}(u)e^T I_{\theta, n}^2 P_\theta(u) = 2\lambda_{\theta, \text{max}}(u)(P_\theta(x)e)^T I_{\theta, n}^2 u
= 2\lambda_{\theta, \text{max}}(u)(x^T v)^T I_{\theta, n}^2 u = 2\lambda_{\theta, \text{max}}(u)(x^2, u)_{\theta}
\leq 2\lambda_{\theta, \text{max}}(u)(x^2, e)_{\theta} = \lambda_{\theta, \text{max}}(u)^2 \text{tr}((x^2 \circ e)_{\theta})
= \lambda_{\theta, \text{max}}(u)^2 \text{tr}(x^2) = \frac{\text{tr}(x^2)}{\lambda_{\theta, \text{min}}(P_\theta(x^\frac{1}{2}))s} = \frac{\text{tr}(x^2)}{\mu_\theta \lambda_{\theta, \text{min}}(v)^2}.
\]

Similarly, we have

\[
\text{tr}(w^{-2}) \leq \frac{\text{tr}(s^2)}{\mu_\theta \lambda_{\theta, \text{min}}(v)^2}.
\]

Therefore, we get

\[
\sqrt{\text{tr}(w^2 + w^{-2})} \leq \sqrt{\frac{\text{tr}(x^2 + s^2)}{\mu_\theta \lambda_{\theta, \text{min}}(v)^2}} \leq \frac{\rho(\delta)\text{tr}(x + s)}{\xi \sqrt{v_\theta}}, \tag{15}
\]

where the last inequality follows from \( \mu_\theta = \xi^2 v_\theta \) and Lemma 3.2

**Lemma 4.7.** Let \( x \) and \( (y, s) \) be feasible for the perturbed problem \((P_{\nu_\theta})\) and \((D_{\nu_\theta})\), respectively, and \((x^0, y^0, s^0)\) and \((x^*, y^*, s^*)\) as defined in Section 3.1. Then, we have

\[
\text{tr}(x + s) \leq 2(1 + \rho(\delta)^2)\eta_\xi.
\]

**Proof.** Let \( (x, y, s) \) be a feasible solution for the perturbed problem pair \((P_{\nu_\theta})\) and \((D_{\nu_\theta})\), and \((x^0, y^0, s^0)\) and \((x^*, y^*, s^*)\) be as defined in Section 3.1. Then, one easily verifies that

\[
\langle A(x - \nu_\theta \xi e - (1 - \nu_\theta)x^*) \rangle_{\theta} = 0,
A^T(y - (1 - \nu_\theta)y^*) + (s - \nu_\theta \xi e - (1 - \nu_\theta)s^*) = 0,
\]
or equivalently

\[
(x - \nu_\theta \xi e - (1 - \nu_\theta)x^*)^T I_{\theta}^2 A^T = 0,
I_{\theta}^2 A^T (y - (1 - \nu_\theta)y^*) + I_{\theta}^2 (s - \nu_\theta \xi e - (1 - \nu_\theta)s^*) = 0.
\]

The last two equations imply that

\[
(x - \nu_\theta \xi e - (1 - \nu_\theta)x^*, s - \nu_\theta \xi e - (1 - \nu_\theta)s^*)_{\theta} = 0.
\]

By expanding the above equation, we get

\[
\xi \nu_\theta \langle x, e \rangle_{\theta} + \langle s, e \rangle_{\theta} + \nu_\theta^2 \xi^2 \langle e, e \rangle_{\theta} + \nu_\theta (1 - \nu_\theta) \xi \langle (x^*, e)_{\theta} + (s^*, e)_{\theta} \rangle
\leq 2\langle x, s \rangle_{\theta} + \nu_\theta^2 \xi^2 N + \nu_\theta (1 - \nu_\theta) \xi \text{tr}(x^* + s^*)
\leq N \mu_\theta \rho(\delta)^2 + \nu_\theta^2 \xi^2 N + \nu_\theta (1 - \nu_\theta) \xi^2 N
= N \mu_\theta \rho(\delta)^2 + \nu_\theta \xi^2 N = N \nu_\theta \xi^2 \rho(\delta)^2 + \nu_\theta \xi^2 N.
\]

This implies the inequality in lemma. \(\square\)
Using Lemma 4.7 and Lemma 4.6, we obtain the following upper bound for $\bar{\omega}(v)$:

$$\bar{\omega}(v) \leq \left[ \sqrt{2}(1 - \beta)\delta + \beta\sqrt{N}\rho(\delta) + 3\sqrt{2N}\beta\rho(\delta)(1 + \rho(\delta)^2) \right]^2. \tag{16}$$

### 4.2. Values for $\beta$ and $\tau$

The main goal of this section is to devote some positive values for the parameters $\beta$ and $\tau$ such that the proposed algorithm in Fig. 1 is well-defined. That is, after a full NT-step, the iterate $(x^+, y^+, s^+)$ is strictly feasible and $\delta(x^+, s^+; \mu^*) \leq \tau$. Due to Lemma 4.4, this will hold if

$$\bar{\omega}(v) < 1 - \beta, \tag{17}$$

$$\frac{\bar{\omega}(v)}{2\sqrt{(1 - \beta)(1 - \beta - \bar{\omega}(v))} \leq \tau. \tag{18}$$

Using (16), the inequality (17) holds if

$$\sqrt{2}(1 - \beta)\delta + \beta\sqrt{N}\rho(\delta) + 3\sqrt{2N}\beta\rho(\delta)(1 + \rho(\delta)^2) < \sqrt{1 - \beta}. \tag{19}$$

One easily verifies that the left-hand side expression in (19) is increasing with respect to $\delta$. Hence, assuming $\delta \leq \tau$, it suffices to have

$$\sqrt{2}(1 - \beta)\tau + \beta\sqrt{N}\rho(\tau) + 3\sqrt{2N}\beta\rho(\tau)(1 + \rho(\tau)^2) < \sqrt{1 - \beta}. \tag{20}$$

Choosing $\tau = \frac{1}{20N}$, one may easily check that the inequality (20) is satisfied if $\beta = \frac{1}{20N}$. This means that the iterates $(x^+, y^+, s^+)$ are strictly feasible. Noting the left-hand side of (20) provides an upper bound for $\bar{\omega}(v)$, we then have

$$\delta(v^+) = \frac{\bar{\omega}(v)}{2\sqrt{(1 - \beta)(1 - \beta - \bar{\omega}(v))} \leq 0.0613 < \tau. \tag{20}$$

Therefore, the algorithm is well-defined in the sense that $\delta(v) \leq \tau$ is maintained in all iterations.

### 4.3. Complexity of the algorithm

Note that after each iteration of the proposed algorithm, we have

$$b - (Ax^+_\theta) = b - AI_{\theta,n}^2(x + \Delta x) = b - AI_{\theta,n}^2x - AI_{\theta,n}^2\Delta x = \nu_{\theta}r_\theta^0 - \beta\nu_{\theta}r_\theta^0 = (1 - \beta)\nu_{\theta}r_\theta^0,$$

and similarly

$$c - ATy^+ - s^+ = (1 - \beta)\nu_{\theta}r_\theta^0.$$

From the above relations it follows that the norm of the residual vectors, and similarly the duality gap, are reduced by a factor $1 - \beta$. Therefore, the total number of iterations is bounded above by

$$\frac{1}{\beta \log \frac{\max \left(\text{tr}((x^0 \circ s^0)_{\theta}), \|r_{\theta}^0\|_{\theta,F}, \|r_{\theta}^0\|_{\theta,F} \right)}{\epsilon}}.$$

Thus, we may state without further proof the main result of the paper as follows:

**Theorem 4.8.** Let $(P)$ and $(D)$ be feasible and $\xi > 0$ such that $x^* + s^* \preceq_Q^\theta \xi e$ for some optimal solutions $x^*$ of $(P)$ and $(y^*, s^*)$ of $(D)$. Then, after at most

$$20N \log \frac{\max \left(\text{tr}((x^0 \circ s^0)_{\theta}), \|r_{\theta}^0\|_{\theta,F}, \|r_{\theta}^0\|_{\theta,F} \right)}{\epsilon}$$

iterations the algorithm finds an $\epsilon$-solution of $(P)$ and $(D)$.
5. **Conclusion.** In this paper, we presented a full NT-step IIPM for CO. In the proposed algorithm, after one feasibility step, the new generated iterate is strictly feasible and close enough to the central path. The polynomial iteration complexity of the algorithm is established.

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