BOGOMOLOV MULTIPLIERS OF GROUPS OF ORDER $p^6$

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Abstract. Let $G$ be a finite group and $B_0(G)$ be its Bogomolov multiplier, i.e., the subgroup of the Schur multiplier $H^2(G, \mathbb{Q}/\mathbb{Z})$ consisting of cohomology classes whose restrictions to all bicyclic subgroups are zero. This invariant of $G$ is of importance in classical Noether’s problem and birational geometry of quotient spaces $V/G$. F. Bogomolov [3] proved that there is some group $G$ of order $p^6$ with $B_0(G) \neq 0$ for any prime number $p$; as a consequence, Noether’s problem has a negative answer for $G$ and $\mathbb{C}$. Recently, all nonabelian groups of order $64$ with $B_0(G) \neq 0$ was classified in [7]. In this paper, we will classify all nonabelian groups of order $p^6$ with $B_0(G) \neq 0$ for any prime $p > 3$. In particular, we also provide an approach to attack Noether’s problem for some $p$-groups with six generators.

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1. Introduction

1.1. Backgrounds. Let $k$ be a field, $G$ a finite group and $V$ be a faithful finite-dimensional representation of $G$ over $k$. Let $k(V)$ be the rational function field which is isomorphic to the field of fractions of the symmetric algebra $S(V^*) \cong k[V]$. Then $G$ can be viewed as a subgroup of the $k$-automorphism group $\text{Aut}_k(k(V))$. We write $k(V)^G = \{f \in k(V) \mid \sigma \cdot f = f \text{ for all } \sigma \in G\}$ for the invariant field. Famous Noether’s problem asks whether $k(V)^G$ is rational (i.e., purely transcendental) over $k$.

This problem has close connection with Lüroth’s problem and inverse Galois problem [13, 24, 27, 25]. We consider the pair $(k, G) = (\mathbb{Q}, C_n)$ with a cyclic action on $\mathbb{Q}(V) = \mathbb{Q}(x_1, \ldots, x_n)$, where $\mathbb{Q}$ is the field of rational numbers and $C_n$ is the cyclic group of order $n$. In 1969, Swan [26] showed that the invariant field $\mathbb{Q}(V)^{C_n}$ is not rational over $\mathbb{Q}$ when $n = 47, 113, 233$. This is the first counterexample to Noether’s problem. But it seems that Swan’s method doesn’t work on the case of an algebraically closed field.

In 1984, Saltman [25] used the unramified cohomology group $H^2_{\text{nr}}(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z})$ as an obstruction to prove that there exists some $p$-group $G$ of order $p^9$ such that $\mathbb{C}(V)^G$ is not rational over the complex field $\mathbb{C}$. In 1988, Bogomolov [3] proved that the unramified cohomology group $H^2_{\text{nr}}(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z})$ is actually isomorphic to

$$B_0(G) = \bigcap_{A \in \mathcal{B}_G} \text{Ker} \left\{ \text{res}_A^G : H^2(G, \mathbb{Q}/\mathbb{Z}) \to H^2(A, \mathbb{Q}/\mathbb{Z}) \right\},$$

where $\mathcal{B}_G$ denotes the set of bicyclic subgroups of $G$ and $\text{res}_A^G$ is the usual cohomological restriction map. The group $B_0(G)$ is a subgroup of the Schur multiplier $H^2(G, \mathbb{Q}/\mathbb{Z})$, so $B_0(G)$ is also called the Bogomolov multiplier of $G$ ([18]). Bogomolov [3] used above description to find some new examples of groups $G$ of order $p^6$ such that $B_0(G) \neq 0$.

On the other hand, one should remark the following result on $p$-groups of small order.

**Theorem 1.1** ([9]). Let $p$ be a prime and $G$ a $p$-group of order $p^4$. Assume that $k$ is a field of char $k \neq p$ and contains a primitive $p^e$th root of unity, where $p^e$ is the exponent of $G$. Then $k(V)^G$ is purely transcendental over $k$ for any linear representation $V$. In particular, $B_0(G) = 0$.

A natural problem is to classify all groups of order $p^5$ and $p^6$ with nontrivial Bogomolov multiplier. But computing the Bogomolov multiplier of a finite group is a complicated task, even for $p$-groups.

Let us recall some developments in computing the Bogomolov multiplier of $p$-groups.

For $p = 2$, a result due to Chu, Hu, Kang and Prokhorov [8] shows that if $G$ is a group of order 32 with exponent $e$, then Noether’s problem for $G$ has positive answer over any field containing a primitive $e$th root of unity, so $B_0(G)$ is trivial. In 2010, Chu, Hu, Kang and Kunyavskii [7] classified all nonabelian groups of order 64 with nontrivial $B_0$. Meanwhile, we notice that for $p \geq 3$, a complete list of all groups of order $p^5$ and $p^6$ is well-known by James’s work [12], in which the nonabelian groups of order $p^5$ and $p^6$ are divided into 9 isoclinism families $\{\Phi_2, \cdots, \Phi_{10}\}$ and 42 isoclinism families $\{\Phi_2, \cdots, \Phi_{43}\}$ respectively.
In [21], Moravec used a notion of nonabelian exterior square $G \wedge G$ of a given group $G$ to obtain a new description of $B_0(G)$ (see Section 2). As an application, it is proved in [21] that there are precisely three groups of order $3^5$ with nontrivial $B_0$. Recently, Hoshi, Kang and Kunyavskii proved the following result, which is also obtained by Moravec [22] using some purely combinatorial methods.

**Theorem 1.2** ([11]). Let $p > 3$ be a prime and $G$ a group of order $p^5$. Then $B_0(G) \neq 0$ if and only if $G$ belongs to the family $\Phi_{10}$.

In [11], an interesting question asks whether $B_0(G_1)$ is isomorphic to $B_0(G_2)$ for two isoclinic $p$-groups $G_1$ and $G_2$. Moravec answered this question affirmatively.

**Theorem 1.3** ([23]). Let $G_1$ and $G_2$ be isoclinic $p$-groups. Then $B_0(G_1)$ is isomorphic to $B_0(G_2)$.

This helpful fact means that if we want to discuss the vanishing of $B_0$ for the groups in some isoclinism family $\Phi_i$, it suffices to pick up one suitable representative $G \in \Phi_i$ and compute its $B_0(G)$.

Furthermore, we notice that there are also some papers addressing on the vanishing question of the Bogomolov multiplier of finite simple groups ([4], [6], [18], [5]) and rigid finite groups ([16]).

**1.2. Main Results.** The purpose of this paper is to compute the Bogomolov multiplier of all nonabelian groups of order $p^6$ for $p > 3$. It follows from the classification of James [12] that each group of order $p^6$ belongs to one of the isoclinism families: $\Phi_2, \Phi_3, \ldots, \Phi_{43}$.

The following is our main result.

**Theorem 1.4.** Let $p > 3$ be a prime number and $G$ be a nonabelian group of order $p^6$. Then $B_0(G) \neq 0$ if and only if $G$ belongs to one of $\Phi_i$, where $i = 10, 18, 20, 21, 36, 38, 39$.

We below will calculate the Bogomolov multiplier of $G$ case by case. Our proof of Theorem 1.4 basically consists of four parts. The most simple part contains those groups for which the vanishing of $B_0$ can be obtained by Theorems 2.5 and 3.1 (see Section 3). We will use the combinatorial method developed by Moravec in [22] to deal with the second part, which contains many isoclinism families. However, we observe that Moravec’s method has its limitation for the situation $G \in \Phi_{15}$. Thus in the third part, we return to investigate Noether’s problem for the group belonging to $\Phi_{15}$. This difficult part also give an approach to solve Noether’s problem for the groups with six generators. Finally, we extend some methods in [10] to discuss these groups with nontrivial $B_0$.

**Remark 1.5.** The result that $B_0(G) = 0$ for $G \in \Phi_2, \Phi_8$ or $\Phi_{14}$ was also proved recently by Michailov [20]. Actually, Noether’s problem for these groups has an affirmative answer if the ground field contains a primitive $e$th root of unity, where $e$ is the group exponent.

**Remark 1.6.** For convenience of the readers, Table 1 gives a summary for the isoclinism families of nonabelian groups of order $p^6$ $(p > 3)$. 
Table 1. Isoclinism families of nonabelian groups of order $p^6$ ($p > 3$)

| Family | Class | $B_0 = 0?$ | Family | Class | $B_0 = 0?$ |
|--------|-------|-----------|--------|-------|-----------|
| $\Phi_2$ | 2 | Yes | $\Phi_{23}$ | 4 | Yes |
| $\Phi_3$ | 3 | Yes | $\Phi_{24}$ | 4 | Yes |
| $\Phi_4$ | 2 | Yes | $\Phi_{25}$ | 4 | Yes |
| $\Phi_5$ | 2 | Yes | $\Phi_{26}$ | 4 | Yes |
| $\Phi_6$ | 3 | Yes | $\Phi_{27}$ | 4 | Yes |
| $\Phi_7$ | 3 | Yes | $\Phi_{28}$ | 4 | Yes |
| $\Phi_8$ | 3 | Yes | $\Phi_{29}$ | 4 | Yes |
| $\Phi_9$ | 4 | Yes | $\Phi_{30}$ | 4 | Yes |
| $\Phi_{10}$ | 4 | No | $\Phi_{31}$ | 3 | Yes |
| $\Phi_{11}$ | 2 | Yes | $\Phi_{32}$ | 3 | Yes |
| $\Phi_{12}$ | 2 | Yes | $\Phi_{33}$ | 3 | Yes |
| $\Phi_{13}$ | 2 | Yes | $\Phi_{34}$ | 3 | Yes |
| $\Phi_{14}$ | 2 | Yes | $\Phi_{35}$ | 5 | Yes |
| $\Phi_{15}$ | 2 | Yes | $\Phi_{36}$ | 5 | No |
| $\Phi_{16}$ | 3 | Yes | $\Phi_{37}$ | 5 | Yes |
| $\Phi_{17}$ | 3 | Yes | $\Phi_{38}$ | 5 | No |
| $\Phi_{18}$ | 3 | No | $\Phi_{39}$ | 5 | No |
| $\Phi_{19}$ | 3 | Yes | $\Phi_{40}$ | 4 | Yes |
| $\Phi_{20}$ | 3 | No | $\Phi_{41}$ | 4 | Yes |
| $\Phi_{21}$ | 3 | No | $\Phi_{42}$ | 4 | Yes |
| $\Phi_{22}$ | 3 | Yes | $\Phi_{43}$ | 4 | Yes |

1.3. Layouts. The present paper is organized as follows. In section 2, we recall some preliminaries and relevant results on the nonabelian exterior square of a finite group. In Section 3, we first use Moravec’s method to compute the Bogomolov multiplier one by one for these groups in Table 1 except for $\Phi_i$, where $i = 15, 18, 20, 21, 28, 29, 36, 38,$ and 39. Afterwards, we provide a method to attack Noether’s problem for the group $\Phi_{15}(21^4)$ in the family $\Phi_{15}$; as a direct consequence, we conclude that $B_0(\Phi_{15}) = 0$. The same techniques can be applied to the cases of $\Phi_{28}$ and $\Phi_{29}$. In Section 4, we will use a nonvanishing criterion for the Bogomolov multiplier due to Hoshi and Kang [10] to prove $B_0(\Phi_i) \neq 0$, where $i = 18, 20, 21, 36, 38,$ and 39.

2. Preliminaries

2.1. Nonabelian Exterior Square. Let $G$ be a group and $x, y \in G$. We define $x^y = y^{-1}xy$ and write $[x, y] = x^{-1}x^y = x^{-1}y^{-1}xy$ for the commutator of $x$ and $y$. We define the commutators of higher weight as $[x_1, x_2, \cdots, x_n] = [[x_1, \cdots, x_{n-1}], x_n]$ for $x_1, x_2, \cdots, x_n \in G$. In particular, we write $[x, y^n]$ for the commutator $[x_1, y, \cdots, y]$ with $n$ copies of $y$.

The nonabelian exterior square of $G$, is a group generated by the symbols $x \wedge y$ ($x, y \in G$), subject to the relations

$$xy \wedge z = (x^y \wedge z^y)(y \wedge z),$$
observe that the commutator map $G \triangleleft G$. Let $[G,G]$ be the commutator subgroup of $G$. We observe that the commutator map $\kappa : G \triangleleft G \to [G,G]$, given by $x \wedge y \mapsto [x,y]$, is a well-defined group homomorphism. Let $M(G)$ denote the kernel of $\kappa$, i.e.,

$$M(G) = \left\{ \prod_{\text{finite}} (x_i \wedge y_i) \in G \triangleleft G \mid \epsilon_i = \pm 1, \prod_{\text{finite}} [x_i, y_i] \epsilon_i = 1 \right\}.$$  

Moreover, we define

$$M_0(G) = \left\{ \langle x \wedge y \in G \triangleleft G \mid [x,y] = 1 \rangle \right\} = \left\{ \prod_{\text{finite}} (x_i \wedge y_i) \in G \triangleleft G \mid \epsilon_i = \pm 1, [x_i, y_i] \epsilon_i = 1 \right\}.$$  

An important result due to Moravec [21] asserts that $B_0(G)$ is exactly isomorphic to $M(G)/M_0(G)$.

Let $G$ be a group. There is also an alternative way to obtain the nonabelian exterior square $G \wedge G$. Let $\phi$ be an automorphism of $G$ and $G^\phi$ be an isomorphic copy of $G$ via $\phi : x \mapsto x^\phi$. We define $\tau(G)$ to be the group generated by $G$ and $G^\phi$, subject to the following relations:

$$[x,y]^\phi = [x^\phi, (y^\phi)^\phi] = [x,y]^\phi \text{ and } [x,x^\phi] = 1$$

for all $x,y,z \in G$. Obviously, the groups $G$ and $G^\phi$ can be viewed as subgroups of $\tau(G)$. Let $[G,G^\phi] = \langle [x,y] \mid x,y \in G \rangle$ be the commutator subgroup. Notice that the map $\phi : G \triangleleft G \to [G,G^\phi]$ given by $x \wedge y \mapsto [x,y]$ is actually an isomorphism of groups (see [21]).

We collect some properties of $\tau(G)$ and $[G,G^\phi]$ that will be used frequently in our proofs.

**Lemma 2.1** ([2]). Let $G$ be a group.

1. $[x,yz] = [x,z][x,y][x,y,z]$ and $[xy,z] = [x,z][x,y][y,z]$ for all $x,y,z \in G$.
2. If $G$ is nilpotent of class $c$, then $\tau(G)$ is nilpotent of class at most $c + 1$.
3. If $G$ is nilpotent of class $\leq 2$, then $[G,G^\phi]$ is abelian.
4. $[x,y'] = [x',y]$ for all $x,y \in G$.
5. $[x,y,z'] = [x,y',z] = [x',y,z] = [x',y',z] = [x,y',z']$ for all $x,y,z \in G$.
6. $[[x,y'],[a,b']] = [[x,y],[a,b']]$ for all $x,y,a,b \in G$.
7. $[x^n,y']^\phi = [x,(y')^\phi]^\phi$ for all integers $n$ and $x,y \in G$ with $[x,y] = 1$.
8. If $[G,G]$ is nilpotent of class $c$, then $[G,G^\phi]$ is nilpotent of class $c$ or $c + 1$.

**Lemma 2.2** ([22], Lemma 3.1). Let $G$ be a nilpotent group of class $\leq 3$. Then

$$[x,y^\phi] = [x,y]^\phi [x,y,y][x,y,y][x,y,y][x,y,y]$$

for all $x,y \in \tau(G)$ and every positive integer $n$. 


Lemma 2.3 ([22], Lemma 3.7). Let $G$ be a nilpotent group of class $\leq 5$. Then
\[ [\alpha^n,\beta] = [\alpha,\beta]^n[\alpha,\beta,\alpha]^{(5)}[\alpha,\beta,\alpha,\alpha]^{(5)}[\alpha,\beta,\alpha,\alpha,\alpha]^{(5)}[\alpha,\beta,\alpha,\alpha,\alpha,\alpha]^{(5)}, \]
where $\sigma(n) = \frac{n(n-1)(2n-1)}{6}$, for all $\alpha,\beta \in \tau(G)$ and every positive integer $n$.

2.2. Polycyclic Groups. We recall several relevant definitions about the polycyclic group. A finite solvable group $G$ is called polycyclic if it has a subnormal series $G = G_1 \triangleright G_2 \triangleright \cdots \triangleright G_{n+1} = 1$ such that every factor $G_i/G_{i+1}$ is cyclic of order $r_i$. A polycyclic generating sequence of a finite solvable group $G$ is a sequence $x_1, \cdots, x_n$ of elements of $G$ such that $G_i = (G_{i+1}, x_i)$ for all $1 \leq i \leq n$. The value $r_i$ is called the relative order of $x_i$. Given a polycyclic generating sequence $x_1, \cdots, x_n$, each element $x$ of $G$ can be expressed uniquely as a product $x = x_1^{e_1} \cdots x_n^{e_n}$ with $e_i \in \{0, \cdots, r_i - 1\}$. An element $x$ of a polycyclic generating sequence of $G$ is absolute if its relative order is equal to the order of $x$.

Lemma 2.4 ([2], Proposition 20). Let $G$ be a finite solvable group with a polycyclic generating sequence $x_1, \cdots, x_n$. Then the group $[G, G^p]$ is generated by $\{[x_i, x'_j] | i, j = 1, \cdots, n, i > j \}$.

Theorem 2.5 ([22], Proposition 3.2). Let $p > 3$ be a prime and $G$ be a $p$-group of class $\leq 3$. Let $x_1, \cdots, x_n$ be a polycyclic generating sequence of $G$. Suppose that all nontrivial commutators $[x_i, x_j](i > j)$ are different absolute elements of the polycyclic generating sequence. Then $B_0(G) = 0$.

2.3. Moravec’s Strategy. Here we state the main idea in [22] as follows. Let $\kappa^* = \kappa \cdot \phi^{-1}$ be the composite map from $[G, G^p]$ to $[G, G]$, $M^*(G) = \ker \kappa^*$ and $M_0^*(G) = \phi(M_0(G))$. More precisely,
\[
M^*(G) = \left\{ \prod_{\text{finite}} [x_i, y_i]^{e_i} \in [G, G^p] \mid e_i = \pm 1, \prod_{\text{finite}} [x_i, y_i]^{e_i} = 1 \right\},
\]
and
\[
M_0^*(G) = \left\{ \prod_{\text{finite}} [x_i, y_i]^{e_i} \in [G, G^p] \mid e_i = \pm 1, [x_i, y_i] = 1 \right\}.
\]
It is immediate that $B_0(G)$ is isomorphic to $M^*(G)/M_0^*(G)$. To prove $B_0(G) = 0$, it suffices to show that $M^*(G) \subseteq M_0^*(G)$.

3. Trivial Bogomolov Multipliers

3.1. Kang’s Theorem. The following result is very useful in our discussions.

Theorem 3.1 (Kang [15]). Let $G$ and $H$ be finite groups. Then $B_0(G \times H)$ is isomorphic to $B_0(G) \times B_0(H)$. As a corollary, if $B_0(G)$ and $B_0(H)$ are both trivial, then also is $B_0(G \times H)$.

Theorems 2.5, 3.1, 1.1, 1.2 and 1.3 can be applied to obtain the vanishing result for the groups of order $p^6$. For example, in the classification of James ([12], page 621), the group $\Phi_2(411)_a = \Phi_2(41) \times (1)$ is the direct product of $\Phi_2(41)$ and a cyclic group of order $p$. It follows from Theorem 1.2 that $B_0(\Phi_2(41)) = 0$. 

By Theorem 1.1, we see that $B_0((1)) = 0$. Thus $B_0(\Phi_2(411)_a)$ is trivial by Theorem 3.1. Another example is the group $\Phi_2(51)$, which has a polycyclic presentation $(\alpha, \alpha_1, \alpha_2 | [\alpha_1, \alpha] = \alpha_2 = \alpha_1^p, \alpha_1^p = \alpha_2^p = 1)$ that satisfies the assumption of Theorem 2.5. Thus $B_0(\Phi_2(51)) = 0$. Of course, it follows from Theorem 1.3 that any group belonging to $\Phi_2$ has trivial Bogomolov multiplier.

We proceed in this way to check James’s classification, and we will obtain

**Corollary 3.2.** Let $p > 3$ be a prime number and $G$ belong to one of the families $\Phi_i$, where

$$i = 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 14, 16, 17, 32.$$  

Then $B_0(G) = 0$.

**Corollary 3.3.** Let $p > 3$ be a prime number and $G \in \Phi_{10}$. Then $B_0(G) \neq 0$.

**Proof.** We choose the group $\Phi_{10}(1^6) = \Phi_{10}(1^5) \times (1)$. Since $B_0(\Phi_{10}(1^5)) \neq 0$, it follows from Theorem 3.1 that $B_0(\Phi_{10}(1^6)) \neq 0$. Theorem 1.3 implies that $B_0(G) \neq 0$ for any $G \in \Phi_{10}$. □

3.2. **Extending Moravč’s Methods.** In what follows, we always assume that $G$ is a group of order $p^6(p > 3)$, and we will omit all trivial commutator relations among the generators in a polycyclic presentation of $G$.

In this subsection, we use Moravč’s strategy to show that $B_0(G) = 0$ for any $G$ belonging to one of the families $\Phi_i$, where $i = 13, 19, 22, \cdots, 27, 30, 31, 33, 34, 35, 37, 40, \cdots, 43$. We always choose $G = \Phi_i(1^6)$ as a representative in these families $\Phi_i$ except for $i = 25, 26, 34, 42, 43$.

**Proposition 3.4.** If $G \in \Phi_{13}$, then $B_0(G) = 0$.

**Proof.** The group $G = \Phi_{13}(1^6)$ has a polycyclic presentation

$$\langle \alpha_1, \cdots, \alpha_4, \beta_1, \beta_2 | [\alpha_1, \alpha_2] = \beta_1, [\alpha_1, \alpha_3] = [\alpha_2, \alpha_4] = \beta_2, \alpha_i^p = \alpha_i^p = \beta_i^p = 1, (i = 1, 2) \rangle.$$  

By Lemma 2.4, the group $[G, G^{\tau}]$ is generated by $[\alpha_1, \alpha_2^\prime]$, $[\alpha_1, \alpha_3^\prime]$ and $[\alpha_2, \alpha_4^\prime]$ modulo $M_0^r(G)$. Since $[G, G^{\tau}]$ is abelian by (3) of Lemma 2.1, each element in $[G, G^{\tau}]$ can be expressed as

$$[\alpha_1, \alpha_2^\prime] [\alpha_1, \alpha_3^\prime] [\alpha_2, \alpha_4^\prime] \bar{w},$$

where $\bar{w} \in M_0^r(G)$. Let $w = [\alpha_1, \alpha_2^\prime] [\alpha_1, \alpha_3^\prime] [\alpha_2, \alpha_4^\prime] \bar{w} \in M^r(G)$. Then $1 = \kappa^r(w) = \beta_1^{r^2t}$. Since $\beta_1$ and $\beta_2$ are in the polycyclic generating sequence, $\beta_1^{r^2t} = 1$. Thus $r$ divides $r$ and $s + t$ respectively.

We claim that $[\alpha_1, \alpha_2^\prime]^p = 1$. By (4) of Lemma 2.1, it is sufficient to show that $[\alpha_1^\prime, \alpha_2]^p = 1$. By Lemma 2.2 we have

$$1 = [\alpha_1^\prime, 1] = [\alpha_1^\prime, \alpha_2^\prime] = [\alpha_1^\prime, \alpha_2]^p[\alpha_1^\prime, \alpha_2, \alpha_2]^{\ell_1}[\alpha_1^\prime, \alpha_2, \alpha_2]^{\ell_2}.$$  

Thus it suffices to show that $[\alpha_1^\prime, \alpha_2, \alpha_2]^{\ell_1}$ and $[\alpha_1^\prime, \alpha_2, \alpha_2]^{\ell_2}$ are both 1. Actually, it follows from (2) of Lemma 2.1 that $\tau(G)$ is nilpotent of class $\leq 3$. So $[\alpha_1^\prime, \alpha_2, \alpha_2] = 1$. By (5) of Lemma 2.1 we have

$$1 = [\alpha_1^\prime, \alpha_2, \alpha_2]^{\ell_1}.$$
\[ [\alpha_1', \alpha_2, \alpha_2]^{(3)} = [\alpha_1, \alpha_2, \alpha_2']^{(3)} = [\beta_1, \alpha_2']^{(3)}. \] Since \([\beta_1, \alpha_2] = 1\), it follows from (7) of Lemma 2.1 that \([\beta_1, \alpha_2']^{(3)} = [\beta_1^{(3)}, \alpha_2'] = [1, \alpha_2] = 1\). Thus the claim follows.

Similar arguments will imply that \([\alpha_1, \alpha_2']^p = 1\) and \([\alpha_2, \alpha_2']^p = 1\), so we have

\[ w = (([\alpha_1, \alpha_2']^p)^{-1} \overline{w}) \]

for any \(w \in M^*(G)\). If \([\alpha_1, \alpha_2']([\alpha_2, \alpha_2']^{-1}) \in M^0_0(G)\), then we are done. We use the formulae in (1) of Lemma 2.1 to check that \([\alpha_1 \alpha_2 \alpha_4, \alpha_1 \alpha_2 \alpha_3] = 1\). Indeed,

\[
[\alpha_1 \alpha_2 \alpha_4, \alpha_1 \alpha_2 \alpha_3] = [\alpha_1 \alpha_2 \alpha_4, \alpha_3][\alpha_1 \alpha_2 \alpha_4, \alpha_1 \alpha_2][\alpha_1 \alpha_2 \alpha_4, \alpha_1 \alpha_2, \alpha_3] = \beta_{2} \cdot [\beta_2, \alpha_4] \cdot \beta_{2}^{-1} \cdot [\beta_2^{-1}, \alpha_3] = 1.
\]

Thus \([\alpha_1 \alpha_2 \alpha_4, (\alpha_1 \alpha_2 \alpha_3')] \in M^0_0(G)\). Expanding it, we obtain

\[
[\alpha_1 \alpha_2 \alpha_4, (\alpha_1 \alpha_2 \alpha_3')] = [\alpha_1 \alpha_2 \alpha_4, \alpha_3'] [\alpha_1 \alpha_2 \alpha_4, \alpha_4] [\alpha_1 \alpha_2 \alpha_4, \alpha_2, \alpha_3']
\]

\[
= [\alpha_1 \alpha_2, \alpha_3'] [\alpha_1 \alpha_2, \alpha_4] [\alpha_4, \alpha_4] [\alpha_4, \alpha_2, \alpha_3']
\]

\[
= [\alpha_1, \alpha_3'] [\alpha_1, \alpha_4] [\alpha_4, \alpha_2, \alpha_3'] [\beta_2, \alpha_4] [\alpha_4, \alpha_4] [\alpha_4, \alpha_2] [\alpha_4, \alpha_1] [\beta_2, \alpha_2, \alpha_3'] [\alpha_4, \alpha_2, \alpha_3'].
\]

We can see that, except \([\alpha_1, \alpha_3']\) and \([\alpha_4, \alpha_2']\), the others belong to \(M^0_0(G)\). So

\[
[\alpha_1, \alpha_3'] [\alpha_4, \alpha_2'] = [\alpha_1, \alpha_3'] [\alpha_2, \alpha_4']^{-1} \in M^0_0(G),
\]

as required. Hence \(B_0(G) = 0\).

\[ \square \]

**Proposition 3.5.** If \(G \in \Phi_{19}\), then \(B_0(G) = 0\).

**Proof.** The group \(G = \Phi_{19}(1^6)\) has a polycyclic presentation

\[
\langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1] = \beta_2, [\alpha_1, \alpha_2] = \alpha_1^p = \beta_2^p = \beta_2^{-1} = 1, (i = 1, 2) \rangle.
\]

By Lemma 2.4, the group \([G, G]^p\) is generated by the set \(\mathbb{A} = \{[\alpha_1, \alpha_2'], [\beta, \alpha_2'], [\beta, \alpha_2'], [\alpha_2, \alpha_1']\}\) modulo \(M^0_0(G)\).

We claim that any two elements in the set \(\mathbb{A}\) are commutating modulo \(M^0_0(G)\). First of all, by (6) of Lemma 2.1, we have \([[\alpha_1, \alpha_2'], [\beta, \alpha_2']] = [[\alpha_1, \alpha_2], [\beta, \alpha_2]] = [\beta, \beta_1'] \in M^0_0(G)\), where \(i = 1, 2\). Secondly, \([[\alpha_1, \alpha_2'], [\alpha, \alpha_1']] = [[\alpha_1, \alpha_2], [\alpha, \alpha_1]] = [\beta, \beta_1'] \in M^0_0(G)\). The remaining cases will be checked easily.

Thus each element in \([G, G]^p\) can be expressed as

\[
[\alpha_1, \alpha_2']^m [\beta, \alpha_2']^n [\beta, \alpha_2']^t [\alpha, \alpha_2']^s \overline{w},
\]

where \(\overline{w} \in M^0_0(G)\). Let \(w = [\alpha_1, \alpha_2']^m [\beta, \alpha_2']^n [\beta, \alpha_2']^t [\alpha, \alpha_2']^s \overline{w} \in M^*(G)\), then \(1 = \kappa(w) = \beta^m \beta_2^t \beta_2^{-1} \). So \(p\) divides \(m, n + t\) and \(s\) respectively. By Lemma 2.2 we have

\[
1 = [\beta', \alpha_2'^p] = [\beta', \alpha_2'^p] [\beta', \alpha_2] [\beta', \alpha_2] [\beta', \alpha_2] [\beta', \alpha_2] [\beta', \alpha_2] [\beta', \alpha_2].
\]
Notice that $[\beta', \alpha, \alpha_2] = [\beta, \alpha_2, \alpha'] = [\beta_2, \alpha'] = [\beta, \alpha_3] = [\beta_2, \alpha_2] = 1$ and $[\beta, \alpha_2'] = [\beta_2, \alpha_2'] = [1, \alpha_2'] = 1$. Thus $[\beta, \alpha_2'] = [\beta, \alpha_2] = 1$. Similarly, one can prove that $[\beta, \alpha_1'] = 1, [\alpha, \alpha_2'] = 1$ and $[\alpha, \alpha_1'] = 1$. Hence we have $w = ([\beta, \alpha_1'][\alpha, \alpha_1']^{-1})\bar{w}$.

Now we need to prove that $[\beta, \alpha_1'][\alpha, \alpha_1']^{-1} \in M_0^*(G)$. Observe that

\[
[\beta \alpha_1, \alpha_1] = [\beta \alpha_1, \alpha][\beta \alpha_1, \alpha_1] = [\beta, \alpha][\beta, \alpha_1][\alpha, \alpha][\beta, \alpha_1][\alpha, \alpha] = 1.
\]

Thus $[\beta \alpha_1, (\alpha_1 \alpha')] \in M_0^*(G)$. On the other hand,

\[
[\beta \alpha_1, (\alpha_1 \alpha')] = [\beta \alpha_1, \alpha'][\beta \alpha_1, \alpha_1'] = [\beta, \alpha'][\beta, \alpha_1][\alpha_1, \alpha'][\beta, \alpha_1][\alpha_1, \alpha] = 1.
\]

We can see that $[\beta, \alpha'], [\beta_1, \alpha'], [\beta, \alpha', 1]$, and $[\beta, \alpha', 1]$ all belong to $M_0^*(G)$. So

\[
[\beta, \alpha_1'][\alpha, \alpha'] = [\beta, \alpha_1'][\alpha, \alpha_1']^{-1} \in M_0^*(G).
\]

Hence $B_0(G) = 0$. \qed

**Proposition 3.6.** If $G \in \Phi_{22}$, then $B_0(G) = 0$.

**Proof.** The group $G = \Phi_{22}(1^6)$ has a polycyclic presentation

\[
\langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2 | [\alpha_i, \alpha] = \alpha_{i+1}, [\beta_1, \beta_2] = \alpha_3, \alpha_1^{(p)} = \alpha_2^p = \alpha_{i+1}^p = \beta_1^p = 1, (i = 1, 2) \rangle,
\]

where $\alpha_i^{(p)} = \alpha_{i}^{p} \alpha_2^{(p/2)} \alpha_3^{(p)}$. The group $[G, G^p]$ is generated by $[\alpha, \alpha'], [\beta_1, \beta_2', \alpha_1, \alpha']$ modulo $M_0^*(G)$. It is not hard to check that any two of these three generators are commutating modulo $M_0^*(G)$. Thus each element in $[G, G^p]$ can be expressed as

\[
[\alpha_1, \alpha'] \alpha_1^{p} \beta_1^{p} \beta_2^{p},
\]

where $w \in M_0^*(G)$. Let $w = [\alpha_i, \alpha'] \alpha_1^{p} \beta_1^{p} \beta_2^{p} w \in M^*(G)$. Then $1 = \kappa^*(w) = \alpha_2^p \alpha_2^{p} \beta_1^{p} \beta_2^{p} \alpha^{p}$ and so $p$ divides $r$ and $s + t$ respectively. Notice that

\[
1 = [\alpha_i^{p}] = [\alpha_1, \alpha']^{p} [\alpha_1, \alpha, \alpha']^{(p)} [\alpha_1, \alpha]^{(p)} = [\alpha_i, \alpha]^{p}, (i = 1, 2).
\]

Similarly, $[\beta_1, \beta_2^{p}]^{p} = 1$, so we have $w = ([\alpha_1, \alpha'] [\beta_1, \beta_2])^{p} \bar{w}$. Now we want to prove that

\[
[\alpha_2, \alpha'][\beta_1, \beta_2]^{-1} \in M_0^*(G).
\]

Observe that $1 = [\alpha_1, \alpha][\beta_2, \beta_1] = [\alpha_2 \beta_2, \alpha \beta_1], \text{ so } [\alpha_2 \beta_2, (\alpha \beta_1)] \in M_0^*(G)$. On the other hand,

\[
[\alpha_2 \beta_2, (\alpha \beta_1)] = [\alpha_2 \beta_2, \beta_1'] \alpha_2 \beta_2, \alpha'][\alpha_2 \beta_2, \alpha', \beta_1']
\]

\[
= [\alpha_2, \beta_1'][\alpha_2, \beta_2'][\beta_2, \beta_1'][\alpha_2, \alpha'][\alpha_2, \beta_2][\beta_2, \alpha'][\alpha_3, \beta_1']
\]

\[
= [\alpha_2, \beta_1'][\beta_2, \beta_1'][\alpha_2, \alpha'][\alpha_3, \beta_1'][\beta_2, \alpha'][\alpha_3, \beta_1']
\]
and \([\alpha_2, \beta'_1], [\alpha_3, \beta'_2], [\alpha_3, \beta'_1], [\beta_2, \alpha']\) are in \(M'_0(G)\). Thus \([\alpha_2, \alpha'][\beta_2, \beta'_1] = [\alpha_2, \alpha'] [\beta_1, \beta'_2]^{-1} \in M'_0(G)\) and we are done.

**Proposition 3.7.** If \(G \in \Phi_{23}\), then \(B_0(G) = 0\).

**Proof.** The group \(G = \Phi_{23}(1^6)\) has a polycyclic presentation

\[
\langle \alpha, \alpha_1, \cdots, \alpha_4, \gamma \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \gamma, \alpha^p = \alpha_1^{(p)} = \alpha_1^{(p)} = \gamma^{p} = 1, (i = 1, 2, 3) \rangle.
\]

where \(\alpha_j = \alpha_j^{(p)} \alpha_{j+1}^{(q)} \cdots \alpha_4^{(r)}\), \(j, i = 2, 3, 4\). By Lemma 2.4, the group \([G, G^\varphi]\) is generated by

\[
\mathcal{B} = \{[\alpha_1, \alpha'], [\alpha_2, \alpha'], [\alpha_3, \alpha'], [\alpha_1, \alpha'_2] \}
\]

modulo \(M'_0(G)\). Notice that any two elements in the set \(\mathcal{B}\) are commuting modulo \(M'_0(G)\); for instance, \([\alpha_1, \alpha'], [\alpha_2, \alpha']\) = \([\alpha_1, \alpha], [\alpha_2, \alpha']\) = \([\alpha_2, \alpha'_2] \in M'_0(G)\), and the similar arguments can used to check the other cases. Thus each element in \([G, G^\varphi]\) can be expressed as

\[
\prod_{i=1}^{3} [\alpha_i, \alpha']^{m_i} [\alpha_1, \alpha'_2]^{m_w} \bar{w},
\]

where \(\bar{w} \in M'_0(G)\). Let \(w = \prod_{i=1}^{3} [\alpha_i, \alpha']^{m_i} [\alpha_1, \alpha'_2]^{m_w} \in M'(G)\), then \(1 = \kappa^*(w) = \alpha_2^{m_1} \alpha_3^{m_2} \gamma^{m_3}\). Therefore \(p\) divides \(m_i\) and \(n\) respectively.

It follows from (2) of Lemma 2.1 that \(\tau(G)\) is nilpotent of class \(\leq 5\), so Lemma 2.3 implies that

\[
1 = [\alpha^p_1, \alpha'_2] = [\alpha_1, \alpha'_2]^{p} [\alpha_1, \alpha'_2, \alpha_1]^{(p)} [\alpha_1, \alpha'_2, \alpha_1, \alpha_1]^{(p)} [\alpha_1, \alpha'_2, \alpha_1, \alpha_1, \alpha_1]^{\sigma(p)},
\]

where \(\sigma(p) = \frac{p(p-1)(2p-1)}{6}\). Notice that \([\alpha_1, \alpha'_2, \alpha_1]^{(p)} = [\alpha_1, \alpha_2, \alpha'_1]^{(p)} = [\gamma, \alpha'_1]^{(p)}\). Since \([\gamma, \alpha_1] = 1\), it follows from (7) of Lemma 2.1 that \([\gamma, \alpha'_1]^{(p)} = [\gamma]^{(p)} = [1, \alpha'_2] = 1\). By the same way, we observe that \([\alpha_1, \alpha'_2, \alpha_1]^{(p)} = [\alpha_1, \alpha'_2, \alpha_1, \alpha_1]^{(p)} = [\alpha_1, \alpha'_2, \alpha_1, \alpha_1, \alpha_1]^{\sigma(p)} = 1\). Thus

\[
1 = [\alpha^p_1, \alpha'_2] = [\alpha_1, \alpha'_2]^{p}.
\]

Similarly, \([\alpha_i, \alpha']^{p} = 1 (i = 1, 2, 3)\), so we have

\[
w = \prod_{i=1}^{3} [\alpha_i, \alpha']^{m_i} [\alpha_1, \alpha'_2]^{m_w} \bar{w} = \bar{w} \in M'_0(G).
\]

Hence \(B_0(G) = 0\). \(\square\)

**Proposition 3.8.** If \(G \in \Phi_{24}\), then \(B_0(G) = 0\).

**Proof.** Since the group \(G = \Phi_{24}(1^6)\) has a polycyclic presentation

\[
\langle \alpha, \alpha_1, \cdots, \alpha_4, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_4, \alpha_1^{(p)} = \beta^{p} = \alpha_1^{(p)} = 1, (i = 1, 2, 3) \rangle,
\]

the group \([G, G^\varphi]\) is generated by \([\alpha_i, \alpha'](i = 1, 2, 3)\) and \([\alpha_1, \beta']\) modulo \(M'_0(G)\). It is not hard to check that any two elements among this four generators are commuting modulo \(M'_0(G)\). Thus each element in
$[G, G^c]$ can be expressed as

$$\prod_{i=1}^{3} [a_i, a']^{m_i} [\alpha_1, \beta']^{n_i} \bar{w},$$

where $\bar{w} \in M_0^*(G)$. Let $w = \prod_{i=1}^{3} [a_i, a']^{m_i} [\alpha_1, \beta']^{n_i} \bar{w} \in M^*(G)$. Then $1 = \kappa^*(w) = \alpha_1^{m_1} \alpha_2^{m_2} \alpha_3^{m_3+n} + p$ divides $m_1, m_2$ and $m_3 + n$ respectively. By (7) of Lemma 2.1 and Lemma 2.3 we have

$$1 = [a_i^p, a']^p = [a_i, a']^p [a_i, a'] [a_i, a']^p [a_i, a'] [a_i, a']^p [a_i, a'] [a_i, a']^p [a_i, a'] [a_i, a']^p [a_i, a']^p$$

$$= [a_i, a']^p [a_i, a'] [a_i, a']^p [a_i, a'] [a_i, a'] [a_i, a']^p [a_i, a'] [a_i, a']^p [a_i, a']^p$$

$$= [a_i, a']^p [a_i, a']^p [a_i, a'] [a_i, a'] [a_i, a']^p [a_i, a'] [a_i, a']^p [a_i, a']^p [a_i, a'] [a_i, a']^p$$

$$= [a_i, a']^p,$$

where $i = 1, 2, 3$. Thus $[a_i, a']^p = 1$ for $i = 1, 2, 3$. Similarly, one can prove that $[\alpha_1, \beta']^p = 1$, so

$$w = ([a_3, a'] [\alpha_1, \beta']^{-1})^{m_3} \bar{w}.$$

Now we need to prove that $[a_3, a'] [\alpha_1, \beta']^{-1} \in M_0^*(G)$. Notice that

$$[a_3 \beta, a a_1] = [a_3, a a_1] [\alpha_3, \alpha_1, \beta] [\beta, \alpha a_1]$$

$$= [a_3, a_1] [\alpha_3, \alpha] [\alpha_3, \alpha_1] [\alpha_4, \beta] [\beta, \alpha_1] [\beta, \alpha] [\beta, \alpha, a_1]$$

$$= 1.$$ 

Thus $[a_3 \beta, (aa_1)^{-1}] \in M_0^*(G)$. Expanding it, we obtain

$$[a_3 \beta, (aa_1)^{-1}] = [a_3, a_1] [\alpha_3, a_1, \beta] [\beta, a_1 a_1]$$

$$= [a_3, a_1] [\alpha_3, a] [\alpha_3, a_1] [\alpha_4, \beta] [\beta, a_1] [\beta, a_1 a_1]$$

$$= [a_3, a_1] [\alpha_3, a] [\alpha_4, a_1] [\alpha_4, \beta] [\beta, a_1] [\beta, a_1 a_1].$$

Since $[a_3, a_1], [\alpha_4, a_1], [\alpha_4, \beta]$ and $[\beta, a_1]$ are in $M_0^*(G)$, $[a_3, a_1] [\beta, a_1] = [a_3, a_1] [a_1, \beta']^{-1}$ belongs to $M_0^*(G)$, as desired. 

**Proposition 3.9.** If $G \in \Phi_{25}$ or $\Phi_{26}$, then $B_0(G) = 0$.

**Proof.** Here we only give the proof for the case $G \in \Phi_{25}$ because their proofs are almost same.

We choose $G = \Phi_{25}(222)$ as a representative, it has a polycyclic presentation

$$\langle \alpha, \alpha_1, \ldots, \alpha_4 | [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_3, \alpha] = \alpha_4, \alpha_i^{(p)} = \alpha_i^{p} = \alpha_{i+2} = 1, (i = 1, 2) \rangle.$$

Lemma 2.4 implies that the group $[G, G^c]$ is generated by $[\alpha_i, a'] (i = 1, 2, 3)$ modulo $M_0^*(G)$. Notice that for any $i, j \in \{1, 2, 3\}$ and $i < j$,

$$[\alpha_i, \alpha'] [\alpha_j, \alpha'] = [[\alpha_i, \alpha], [\alpha_j, a']'] = [\alpha_{i+1}, a_{i+1}'] \in M_0^*(G).$$
Thus $[\alpha_i, \alpha']$ and $[\alpha_j, \alpha']$ commutates modulo $M^*(G)$ and each element in $[G, G^\varphi]$ can be expressed as

$$\prod_{i=1}^{3}[\alpha_i, \alpha']^{m_i}\bar{w},$$

where $\bar{w} \in M^*_0(G)$. Let $w = \prod_{i=1}^{3}[\alpha_i, \alpha']^{m_i}\bar{w} \in M^*(G)$, then $1 = \kappa^*(w) = \alpha_2^{m_1}\alpha_3^{m_2}\alpha_4^{m_3}$. Thus $p$ divides $m_2$ and $m_3$, and $p^2$ divides $m_1$. Note that $\tau(G)$ is nilpotent of class $\leq 5$, we have

$$1 = [\alpha^p, \alpha'] = [\alpha_3, \alpha']^p[\alpha_3, \alpha', \alpha_3]^{\varphi}(\alpha_3, \alpha_3)\alpha_3, \alpha', [\alpha_3, \alpha']^{\varphi(p)} = [\alpha_3, \alpha']^p[\alpha_4, \alpha_3]^{\varphi}(\alpha_4, \alpha_3)\alpha_4, \alpha_3, [\alpha_3, \alpha']^{\varphi(p)} = [\alpha_3, \alpha']^p.$$  

Similarly, one can prove that $[\alpha_1, \alpha']^p = 1$ and $[\alpha_2, \alpha']^p = [\alpha_3, \alpha'] = [\alpha_4, \alpha'] \in M^*_0(G)$. Hence

$$w = \prod_{i=1}^{3}[\alpha_i, \alpha']^{m_i}\bar{w} \in M^*_0(G).$$

and $B_0(G) = 0$. \hfill \square

**Proposition 3.10.** If $G \in \Phi_{27}$, then $B_0(G) = 0$.

**Proof.** The group $G = \Phi_{27}(1^6)$ has a polycyclic presentation

$$\langle \alpha, \alpha_1, \cdots, \alpha_4, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = [\alpha_1, \alpha_2] = \alpha_4, \alpha^p = \alpha_1^{(p)} = \beta^p = 1, (i = 1, 2, 3) \rangle.$$

Then the group $[G, G^\varphi]$ is generated by the set

$$\mathcal{C} = \{[\alpha_1, \alpha'], [\alpha_2, \alpha'], [\alpha_3, \alpha'], [\alpha_1, \beta'], [\alpha_1, \alpha_2']\}$$

modulo $M^*_0(G)$. As before, it is easy to check that any two elements in the set $\mathcal{C}$ are commutating modulo $M^*_0(G)$. Thus each element in $[G, G^\varphi]$ can be expressed as

$$\prod_{i=1}^{3}[\alpha_i, \alpha']^{m_i}[\alpha_1, \beta']^{n_1}[\alpha_2, \alpha_2']^{m_1}\bar{w},$$

where $\bar{w} \in M^*_0(G)$. Let $w = \prod_{i=1}^{3}[\alpha_i, \alpha']^{m_i}[\alpha_1, \beta']^{n_1}[\alpha_2, \alpha_2']^{m_1}\bar{w} \in M^*(G)$, then $1 = \kappa^*(w) = \alpha_2^{m_1}\alpha_3^{m_2}\alpha_4^{m_3+s+t}$. So $p$ divides $m_1$, $m_2$ and $m_3 + s + t$ respectively.

On the other hand, we have

$$1 = [\alpha^p, \alpha_2'] = [\alpha_1, \alpha_2']^p[\alpha_1, \alpha_2', \alpha_1]^{\varphi}(\alpha_1, \alpha_2', \alpha_1)\alpha_1, \alpha_2', \alpha_1, [\alpha_1, \alpha_2']^{\varphi(p)} = [\alpha_1, \alpha_2']^p[\alpha_4, \alpha_1]^{\varphi}(\alpha_4, \alpha_1)\alpha_4, \alpha_1, [\alpha_1, \alpha_2']^{\varphi(p)} = [\alpha_1, \alpha_2']^p[\alpha_4, \alpha_1]' = \alpha_1'.\]
Similarly, $[\alpha_1, \beta']^p = 1$ and $[\alpha_i, \alpha']^p = 1 (i = 1, 2, 3)$, so

$$w = [\alpha_3, \alpha']^m[\alpha_1, \beta']^{-m-1}[\alpha_1, \alpha']^{-1}w = ([\alpha_3, \alpha']^{-1}[\alpha_1, \beta']^{-1}[\alpha_1, \alpha'])^{-1}w.$$  

We need to prove that $[\alpha_3, \alpha'][\alpha_1, \beta']^{-1}$ and $[\alpha_1, \beta']^{-1}[\alpha_1, \alpha']$ are both in $M_0^r(G)$. Observe that

$$[\alpha_3, \alpha][\alpha_1, \beta][\alpha_1] = [\alpha_3, \alpha][\alpha_1, \beta][\alpha_1] = [\alpha_3, \alpha][\alpha_1, \beta][\alpha_1] = [\alpha_3, \alpha][\beta, \alpha_1] = 1.$$  

Thus $[\alpha_3, \alpha' \alpha_1'] \in M_0^r(G)$. Since

$$[\alpha_3, \alpha' \alpha_1'] = [\alpha_3, \alpha'][\alpha_3, \alpha_1'] = [\alpha_3, \alpha'][\alpha_3, \alpha_1'] = [\alpha_3, \alpha'][\alpha_3, \alpha_1'] = [\alpha_3, \alpha'][\alpha_3, \alpha_1'].$$

and $[\alpha_3, \alpha'], [\alpha_4, \alpha'], [\alpha_4, \beta'], [\beta, \alpha']$ are in $M_0^r(G)$, $[\alpha_3, \alpha'][\beta, \alpha_1'] = [\alpha_3, \alpha'][\beta, \alpha_1'] \in M_0^r(G)$.

The same techniques can be applied to the fact that $[\beta \alpha_1, \alpha_1 \alpha_2] = 1$. Expanding $[\beta \alpha_1, \alpha_1' \alpha_2']$ by the same way as above, we will deduce that $[\alpha_1, \beta']^{-1}[\alpha_1, \alpha'] \in M_0^r(G)$. Therefore $B_0(G) = 0$. \qed

**Proposition 3.11.** If $G \in \Phi_{3d}$, then $B_0(G) = 0$.

**Proof.** The group $G = \Phi_{3d}(1^6)$ has a polycyclic presentation

$$\langle \alpha, \alpha_1, \cdots, \alpha_4, \beta_1 \mid [\alpha, \alpha] = \alpha_{i+1}, [\alpha, \beta] = \alpha_{i+2}, [\alpha_3, \alpha] = \alpha_4 \alpha_i^{(p)} = \alpha_3 \beta = \alpha_4 \beta = 1, (i = 1, 2), \rangle.$$  

By Lemma 2.4, the group $[G, G^p]$ is generated by $[\alpha_i, \alpha']$, $[\alpha_i, \beta'] (i = 1, 2)$ and $[\alpha_3, \alpha']$ modulo $M_0^r(G)$.

Notice that any two elements among these generators are commuting modulo $M_0^r(G)$, so each element in $[G, G^p]$ can be expressed as

$$[\alpha_1, \alpha']^m[\alpha_2, \alpha']^n[\alpha_1, \beta']^p[\alpha_2, \beta']^q[\alpha_3, \alpha']^r[w],$$

where $w \in M_0^r(G)$.

Let $w = [\alpha_1, \alpha']^m[\alpha_2, \alpha']^n[\alpha_1, \beta']^p[\alpha_2, \beta']^q[\alpha_3, \alpha']^r[w] \in M^*(G)$. Then $1 = \kappa^*(w) = \alpha_3^{m+\tau} \alpha_4^{n+\tau}$ and $p$ divides $m, n + r$ and $s + t$ respectively. By (7) of Lemma 2.1 and Lemma 2.3 we have

$$1 = [\alpha_i, \alpha'] = [\alpha_i, \alpha']^p[\alpha_1, \alpha', \alpha_i]^{(p)}[\alpha_1, \alpha', \alpha_i]^{(p)}[\alpha_1, \alpha', \alpha_i]^{(p)}[\alpha_1, \alpha', \alpha_i]^{(p)}[\alpha_1, \alpha', \alpha_i]^{(p)}[\alpha_1, \alpha', \alpha_i]^{(p)}[\alpha_1, \alpha', \alpha_i]^{(p)}[\alpha_1, \alpha', \alpha_i]^{(p)} = [\alpha_i, \alpha']^p[\alpha_1, \alpha', \alpha_i]^{(p)}[\alpha_1, \alpha', \alpha_i]^{(p)}[\alpha_1, \alpha', \alpha_i]^{(p)}[\alpha_1, \alpha', \alpha_i]^{(p)}[\alpha_1, \alpha', \alpha_i]^{(p)}[\alpha_1, \alpha', \alpha_i]^{(p)}[\alpha_1, \alpha', \alpha_i]^{(p)}[\alpha_1, \alpha', \alpha_i]^{(p)} = [\alpha_i, \alpha']^p, (i = 1, 2).$$

Similarly, $[\alpha_i, \beta']^p = 1 (i = 1, 2)$ and $[\alpha_3, \alpha']^p = 1$. Therefore

$$w = ([\alpha_2, \alpha']^p[\alpha_3, \beta']^{-1})^p([\alpha_2, \beta']^{-1}[\alpha_3, \alpha']^p)^{\tau(w)}.$$
To complete the proof, it suffices to prove that $[\alpha_2, \alpha'][\alpha_1, \beta']^{-1}$ and $[\alpha_2, \beta']^q[\alpha_3, \alpha']^{-1} \in M_0^*(G)$. Actually,

$$[\alpha_2 \beta, \alpha \alpha_1] = [\alpha_2, \alpha \alpha_1][\alpha_2, \alpha \alpha_1, \beta][\beta, \alpha \alpha_1]$$
$$= [\alpha_2, \alpha_1][\alpha_2, \alpha][\alpha_2, \alpha_1][\alpha_3, \beta][\beta, \alpha_1][\beta, \alpha, \alpha_1]$$
$$= [\alpha_2, \alpha][\beta, \alpha_1] = 1.$$

Thus $[\alpha_2 \beta, (\alpha \alpha_1)'] \in M_0^*(G)$. Expanding it, we obtain

$$[\alpha_2 \beta, \alpha' \alpha_1'] = [\alpha_2, \alpha' \alpha_1'][\alpha_2, \alpha' \alpha_1'][\beta, \alpha' \alpha_1']$$
$$= [\alpha_2, \alpha'][\alpha_2, \alpha'][\alpha_3, \beta'][\beta, \alpha'][\beta, \alpha', \alpha_1']$$
$$= [\alpha_2, \alpha'][\alpha_2, \alpha'][\alpha_3, \beta'][\beta, \alpha'][\beta, \alpha']$$

Notice that $[\alpha_2, \alpha_1'], [\alpha_3, \alpha_1'], [\alpha_3, \beta']$ and $[\beta, \alpha']$ are all in $M_0^*(G)$, so $[\alpha_2, \alpha'][\alpha_1, \beta']^{-1} \in M_0^*(G)$. Similar arguments from the fact that $[\alpha_2, \beta, \alpha_3] = 1$, will imply that $[\alpha_2, \beta'][\alpha_3, \alpha']^{-1} \in M_0^*(G)$, as desired. \qed

**Proposition 3.12.** If $G \in \Phi_{31}$ or $\Phi_{35}$, then $B_0(G) = 0$.

**Proof.** The group $G = \Phi_{31}(1^6)$ has a polycyclic presentation

$$\langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma \mid [\alpha_i, \alpha] = \beta_i, [\alpha_i, \beta_j] = \gamma, \alpha_i^p = \alpha^p = \beta_i^p = \gamma^p = 1, (i = 1, 2) \rangle.$$

Then the group $[G, G^p]$ is generated by $[\alpha_i, \alpha']$, $[\alpha_i, \beta'_i](i = 1, 2)$ modulo $M_0^*(G)$. One can check that each element in $[G, G^p]$ can be expressed as

$$[\alpha_1, \alpha']^\mu[\alpha_2, \alpha']^\nu[\alpha_1, \beta'_1]^i[\alpha_2, \beta'_2]^i\bar{w},$$

where $\bar{w} \in M_0^*(G)$. Let $w = [\alpha_1, \alpha']^\mu[\alpha_2, \alpha']^\nu[\alpha_1, \beta'_1]^i[\alpha_2, \beta'_2]^i\bar{w} \in M^*(G)$. Then $1 = \kappa^*(w) = \beta_1^m \beta_2^n \gamma^{s+t}$ and $p$ divides $m$, $n$ and $s + t$. By (5) of Lemma 2.1 and Lemma 2.2 we have

$$1 = [\alpha_i', \alpha^p] = [\alpha_i', \alpha^p][\alpha_i', \alpha]([\alpha_i', 3 \alpha]([\alpha_i', 3 \alpha]([\alpha_i', \alpha])^\nu[\alpha_1, \alpha])^\nu = [\alpha_i', \alpha]^p, (i = 1, 2).$$

Similarly, $[\alpha_i, \beta'_i]^p = 1(i = 1, 2)$, so we have $w = ([\alpha_1, \beta'_1][\alpha_2, \beta'_2]^{-1})^\nu \bar{w}$. To prove $B_0(G) = 0$, it suffices to prove that $[\alpha_1, \beta'_1][\alpha_2, \beta'_2]^{-1}$ belongs to $M_0^*(G)$. Notice that

$$[\alpha_1 \beta_2, \beta_1 \alpha_2] = [\alpha_1 \beta_2, \alpha_2][\alpha_1 \beta_2, \beta_1][\alpha_1 \beta_2, \beta_1, \alpha_2]$$
$$= [\alpha_1, \alpha_2][\alpha_1, \alpha_2, \beta_2][\beta_2, \alpha_2][\alpha_1, \beta_1][\alpha_1, \beta_1, \beta_2][\beta_2, \beta_1][\gamma, \alpha_2]$$
$$= [\beta_2, \alpha_2][\alpha_1, \beta_1] = 1.$$

Thus $[\alpha_2 \beta_2, (\beta_1 \alpha_2)'] \in M_0^*(G)$. On the other hand,

$$[\alpha_1 \beta_2, (\beta_1 \alpha_2)'] = [\alpha_1 \beta_2, \alpha'_2][\alpha_1 \beta_2, \beta'_1][\alpha_1 \beta_2, \beta'_1, \alpha'_2]$$
$$= [\alpha_1, \alpha'_2][\alpha_1, \alpha'_2, \beta_2][\beta_2, \alpha'_2][\alpha_1, \beta'_1][\alpha_1, \beta'_1, \beta_2][\beta_2, \beta'_1][\gamma, \alpha'_2]$$
$$= [\alpha_1, \alpha'_2][\beta_2, \alpha'_2][\alpha_1, \beta'_1][\gamma, \beta'_2][\beta_2, \beta'_1][\gamma, \alpha'_2].$$
But \([\alpha_1, \alpha'_2], [\gamma, \beta'_2], [\gamma, \alpha'_2] \text{ and } [\beta_2, \beta'_1]\) belong to \(M_*(G)\). Therefore

\[ [\alpha_1, \beta'_1] [\beta_2, \alpha'_2] = [\alpha_1, \beta'_1] [\alpha_2, \beta'_2]^{-1} \in M_*(G). \]

Similar arguments can be applied to the case \(G \in \Phi_3\). We here only outline the proof. First of all, the group \(G = \Phi_3(1^s)\) has a polycyclic presentation

\[ \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma \mid [\alpha, \alpha] = \beta_1, [\alpha_1, \alpha_1] = [\alpha_2, \alpha] = \gamma, \alpha^p = \alpha_1^p = \alpha_2^p = \beta_1^p = \beta_2^p = \gamma^p = 1, (i = 1, 2) \rangle, \]

where \(\alpha_2^p = \alpha_2^p \gamma^p\). Every element \(w\) in \([G, G^p]\) can be expressed as \([\alpha_1, \alpha']^m [\alpha_2, \alpha']^n [\alpha_1, \beta'_1] [\beta_2, \alpha']^t \tilde{w}\), where \(\tilde{w} \in M_0^*(G)\). Moreover, we can prove that \(w = (\alpha_1, \beta_1^p, \gamma, \gamma, \alpha_1^p) \in M_0^*(G)\) and \([\alpha_1, \alpha_1, (\beta_1^p, \beta_2)] = [\alpha_1, \beta_2] [\alpha_1, \beta_2] [\gamma, \alpha'] [\beta_1^p, \gamma, \alpha'_1] [\gamma, \beta_2] \). Finally, since \([\alpha_1, \beta'_2], [\gamma, \alpha'], [\gamma, \beta'_2]\) and \([\alpha, \beta'_1]\) are in \(M_0^*(G)\), \([\alpha_1, \beta'_1] [\beta_2, \alpha']^{-1} \in M_0^*(G)\), i.e., \(M^*(G) \subseteq M_0^*(G)\) and \(B_0(G) = 0\). \(\Box\)

**Proposition 3.13.** If \(G \in \Phi_3\), then \(B_0(G) = 0\).

**Proof.** We choose \(G = \Phi_3(321)\) as a representative. The group \(G\) has a polycyclic presentation

\[ \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma \mid [\alpha, \alpha] = \beta_1, [\alpha_1, \alpha_1] = [\alpha_2, \alpha] = \gamma, \alpha^p = \alpha_1^p = \alpha_2^p = \beta_1^p = \beta_2^p = \gamma^p = 1(i = 1, 2) \rangle. \]

We notice that the group \(([G, G^p]\)) is generated by \([\alpha_1, \alpha'], [\alpha_2, \alpha'], [\beta_2, \alpha']\) and \([\alpha_1, \beta'_1]\) modulo \(M_0^*(G)\). It is easy to check that each element in \(([G, G^p]\)) can be expressed as

\[ [\alpha_1, \alpha']^m [\alpha_2, \alpha']^n [\beta_2, \alpha']^t [\alpha_1, \beta'_1]^s \tilde{w}, \]

where \(\tilde{w} \in M_0^*(G)\). Let \(w = [\alpha_1, \alpha']^m [\alpha_2, \alpha']^n [\beta_2, \alpha']^t [\alpha_1, \beta'_1]^s \tilde{w} \in M^*(G)\). Then \(1 = \kappa'(w) = \beta_1^p \beta_2^p \gamma^{s+t}\) and \(p^n\) divides \(n\), \(s + t\), and \(p^2\) divides \(m\). Notice that

\[ 1 = [\alpha', \beta'_2] = [\alpha', \beta_2^p] [\alpha_1, \beta_2^p] [\alpha_1, \beta'_2]^s [\alpha', \beta'_2]^p = [\alpha', \beta_2]^p. \]

Similarly, \([\alpha_1, \alpha']^p = 1, [\alpha_2, \alpha']^p = 1\) and \([\alpha_1, \beta'_1]^p \in M_0^*(G)\), so we have \(w = ([\beta_2, \alpha'] [\alpha_1, \beta_1]^{-1}) \tilde{w}\). We observe that

\[ [\beta_2, \beta_1, \alpha_1] = [\beta_2, \beta_1, \alpha_1] [\beta_2, \beta_1, \alpha_1] \]

Thus \([\beta_2, \beta_1, (\alpha_1, \alpha_1)] \in M_0^*(G)\). Expanding it, we obtain

\[ [\beta_2, \beta_1, (\alpha_1, \alpha_1)] = [\beta_2, \beta_1, \alpha_1] [\beta_2, \beta_1, \alpha_1], \]

Since \([\beta_2, \alpha'_1], [\gamma, \beta'_1], [\gamma, \alpha'_1], [\beta_1, \alpha']\) are all in \(M_0^*(G)\),

\[ [\beta_2, \alpha'_1] [\beta_1, \alpha'_1] = [\beta_2, \alpha'_1] [\alpha_1, \beta'_1]^{-1} \in M_0^*(G). \]
Hence $B_0(G) = 0$ and we are done. \hfill \Box

**Proposition 3.14.** If $G \in \Phi_{35}$, then $B_0(G) = 0$.

**Proof.** The group $G = \Phi_{35}(1^6)$ has a polycyclic presentation

$$\langle \alpha, \alpha_1, \cdots, \alpha_5 | [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_{i+1}^p = 1, (i = 1, 2, 3, 4) \rangle.$$ 

By Lemma 2.4, the group $[G, G^\alpha]$ is generated by $[\alpha_i, \alpha'](i = 1, 2, 3, 4)$ modulo $M_0^*(G)$. Since

$$[[\alpha_i, \alpha'], [\alpha_j, \alpha']] = [[\alpha_i, \alpha], [\alpha_j, \alpha']'] = [\alpha_{i+1}, \alpha_{j+1}'],$$

and $[\alpha_{i+1}, \alpha_{j+1}'] \in M_0^*(G)$ ($1 \leq i < j \leq 4$), any two generators of $[G, G^\alpha]$ commutes each other modulo $M_0^*(G)$. Thus each element in $[G, G^\alpha]$ can be expressed as

$$w = \prod_{i=1}^3 [\alpha_i, \alpha'^{m_i} w],$$

where $w \in M_0^*(G)$. Let $w = \prod_{i=1}^4 [\alpha_i, \alpha'^{m_i} w] \in M^*(G)$, then $1 = \kappa(w) = \alpha_2^m \alpha_3^m \alpha_4^m \alpha_5^m$, so $p$ divides $m_i(i = 1, 2, 3, 4)$. Note that $\tau(G)$ is nilpotent of class $\leq 5$, we have

$$1 = \alpha_4^{p} \alpha' = [\alpha_4, \alpha']^p [\alpha_4, \alpha']^\sigma(p)$$

where $\sigma(p) = [\alpha_4, \alpha']^p [\alpha_4, \alpha']^\sigma(p) = [\alpha_4, \alpha']^{2p}$. Similarly, one can prove that $[\alpha_i, \alpha']^p = 1(i = 1, 2, 3)$. Hence

$$w = \prod_{i=1}^4 [\alpha_i, \alpha']^{m_i} \bar{w} = \bar{w} \in M_0^*(G)$$

and $B_0(G) = 0$. \hfill \Box

**Proposition 3.15.** If $G \in \Phi_{37}$, then $B_0(G) = 0$.

**Proof.** The group $G = \Phi_{37}(1^6)$ has a polycyclic presentation

$$\langle \alpha, \alpha_1, \cdots, \alpha_5 | [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_2, \alpha_3] = [\alpha_3, \alpha_1] = [\alpha_4, \alpha_1] = \alpha_5,$$

$$\alpha^p = \alpha_1^p = \alpha_{i+1}^p = \alpha_5^p = 1(i = 1, 2, 3) \rangle.$$ 

Notice that the group $[G, G^\alpha]$ is generated by $[\alpha_i, \alpha'](i = 1, 2, 3), [\alpha_2, \alpha'_3], [\alpha_3, \alpha'_1]$ and $[\alpha_4, \alpha'_1]$ modulo $M_0^*(G)$. Except for $[\alpha_1, \alpha']$ and $[\alpha_2, \alpha']$, any two elements of these generators commutes modulo $M_0^*(G)$. Indeed, $[[\alpha_1, \alpha'], [\alpha_2, \alpha']] = [\alpha_2, \alpha'_3]$. Thus every element $w$ in $[G, G^\alpha]$ can be expressed as

$$w = \prod_{i=1}^3 [\alpha_i, \alpha']^{m_i} [\alpha_2, \alpha'_3]^{m_i} [\alpha_3, \alpha'_1]^{m_i} [\alpha_4, \alpha'_1]^{m_i} \bar{w},$$

where $ar{w} \in M_0^*(G)$. Let $w = \prod_{i=1}^4 [\alpha_i, \alpha']^{m_i} \bar{w} \in M^*(G)$, then $1 = \kappa(w) = \alpha_2^m \alpha_3^m \alpha_4^m \alpha_5^m$, so $p$ divides $m_i(i = 1, 2, 3, 4)$. Note that $\tau(G)$ is nilpotent of class $\leq 5$, we have

$$1 = \alpha_4^{p} \alpha' = [\alpha_4, \alpha']^p [\alpha_4, \alpha']^\sigma(p)$$

where $\sigma(p) = [\alpha_4, \alpha']^p [\alpha_4, \alpha']^\sigma(p) = [\alpha_4, \alpha']^{2p}$. Similarly, one can prove that $[\alpha_i, \alpha']^p = 1(i = 1, 2, 3)$. Hence

$$w = \prod_{i=1}^4 [\alpha_i, \alpha']^{m_i} \bar{w} = \bar{w} \in M_0^*(G)$$

and $B_0(G) = 0$. \hfill \Box
Similarly, \( \overline{w} \in M_0'(G) \). If \( w \in M'(G) \), then \( 1 = \kappa^*(w) = \alpha_2^{m_1} \alpha_3^{m_2} \alpha_4^{m_3} \alpha_5^{s+t} \). Thus \( p \) divides \( m_i \) and \( r + s + t \).

Notice that

\[
1 = [\alpha_3' \alpha_3] = [\alpha_2, \alpha_3, \alpha_3'] [\alpha_3, \alpha_2, \alpha_3] [\alpha_3, \alpha_3', \alpha_3] [\alpha_3, \alpha', \alpha_3, [\alpha_3', \alpha']^p] \\
= [\alpha_2, \alpha_3, \alpha_3'][\alpha_2, \alpha_3', \alpha_3] [\alpha_2, \alpha_3, [\alpha_3', \alpha']^p] \\
= [\alpha_2, \alpha_3', \alpha_3] = [\alpha_2, \alpha_3', \alpha_3] \\

Similarly, \( [\alpha_i, \alpha'_j]^p = 1(i = 1, 2) \), \( [\alpha_2, \alpha'_3]^p = 1 \), \( [\alpha_3, \alpha'_1]^p = 1 \) and \( [\alpha_4, \alpha'_1]^p = 1 \). Thus we have

\[
w = ([\alpha_2, \alpha'_3][\alpha_3, \alpha'_1]^{-1} [\alpha_4, \alpha'_1])^p \overline{w}.
\]

Observe that \( [\alpha_2, \alpha'_3, \alpha_1] = [\alpha_2, \alpha_3][\alpha_2, \alpha_3, \alpha_1] [\alpha_1, \alpha_3] = 1 \). Thus \( [\alpha_2, \alpha'_3] \in M_0'(G) \). On the other hand,

\[
[\alpha_2, \alpha'_3, \alpha_1] = [\alpha_2, \alpha'_3][\alpha_2, \alpha'_3, \alpha_1][\alpha_1, \alpha'_3].
\]

Since \( [\alpha_2, \alpha'_3, \alpha_1] = [\alpha_5, \alpha'_1] \) belongs to \( M_0'(G) \), \( [\alpha_2, \alpha'_3][\alpha_1, \alpha'_3] = [\alpha_2, \alpha'_3][\alpha_3, \alpha'_1]^{-1} \in M_0'(G) \). By the same way, we have

\[
[\alpha_2, \alpha'_3, \alpha_1] = [\alpha_2, \alpha'_3][\alpha_2, \alpha'_3, \alpha_1][\alpha_1, \alpha'_3] \\
= [\alpha_2, \alpha'_3][\alpha_2, \alpha'_3, \alpha_1][\alpha_1, \alpha'_3] \\
= [\alpha_2, \alpha'_3][\alpha_2, \alpha'_3, \alpha_1][\alpha_1, \alpha'_3] = 1.
\]

So \( [\alpha_2, \alpha'_3, \alpha_1] \in M_0'(G) \). We expand this commutator and eventually deduce that \( [\alpha_3, \alpha'_1]^{-1} \) \( \in M_0'(G) \). Thus \( B_0(G) = 0 \).

**Proposition 3.16.** If \( G \in \Phi_{40} \), then \( B_0(G) = 0 \).

**Proof.** The group \( G = \Phi_{40}(1^6) \) has a polycyclic presentation

\[
\langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2, \gamma \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\beta_1, \alpha_2] = \beta_2, \alpha_1 \gamma = \gamma \rangle
\]

where \( n \) denotes the smallest positive integer which is a non-quadratic residue (mod \( p \)). Then the group \( [G, G^p] \) is generated modulo \( M_0'(G) \) by \( [\alpha_1, \alpha'_2], [\beta, \alpha'_1(i = 1, 2)], [\beta_1, \alpha'_2] \) and \( [\beta_2, \alpha'_1] \). The direct computation shows that any two elements of these generators are commuting modulo \( M_0'(G) \). Thus every element \( w \) in \( [G, G^p] \) can be expressed as

\[
w = [\alpha_1, \alpha'_2]^m [\beta, \alpha'_1]^p [\beta, \alpha'_1]'^p [\beta_1, \alpha'_2]^p [\beta_2, \alpha'_1]^p \overline{w},
\]

where \( \overline{w} \in M_0'(G) \). If \( w \in M'(G) \), then \( 1 = \kappa^*(w) = \beta^{\alpha_1} \beta_1^{\alpha_1} \gamma^{s+t} \) and \( p \) divides \( m, n, r \) and \( s + t \). Notice that

\[
1 = [\beta^p, \alpha_1'] = [\beta, \alpha_1']^p [\beta, \alpha_1', \beta]^{(p)} [\beta, \alpha_1', \beta_1]^{(p)} [\beta, \alpha_1', \beta_1]^{(p)} \beta, \alpha_1', \beta, [\beta, \alpha_1']^{(p)} \\
= [\beta, \alpha_1']^p [\beta, \beta_1']^{(p)} [\beta_1', \beta_1']^{(p)} [\beta_1', \beta_1']^{(p)} [\beta, \alpha_1']^{(p)} \\
= [\beta, \alpha_1']^p [\beta_1^{(p)}, \beta_1'] = [\beta, \alpha_1']^p (i = 1, 2).
\]
Thus $[\beta, \alpha'_1]^p = 1$. Similarly, $[\beta_1, \alpha'_2]^p = 1$, $[\beta_2, \alpha'_1]^p = 1$ and $[\alpha_1, \alpha'_2]^p = 1$. Thus we have

$$w = ([\beta_1, \alpha'_2][\beta_2, \alpha'_1]^{-1})^{\overline{w}}.$$  

Notice that

$$[\alpha_1\alpha_2\beta_1, \alpha_1\alpha_2\beta_2] = [\alpha_1\alpha_2\beta_1, \beta_2][\alpha_1\alpha_2\beta_1, \alpha_1\alpha_2][\alpha_1\alpha_2\beta_1, \alpha_1\alpha_2, \beta_2]$$

$$= [\alpha_1\alpha_2, \beta_2][\alpha_1\alpha_2, \beta_1][\beta_1, \alpha_1\alpha_2, \beta_2][\beta_1, \alpha_1\alpha_2, \beta_2]$$

$$= [\alpha_1, \beta_2][\alpha_1, \beta_2][\alpha_1, \beta_2][\gamma^{-1}, \beta_1][\beta_1, \alpha_1][\gamma, \beta_2]$$

$$= [\alpha_1, \beta_1][\alpha_1, \beta_2] = 1.$$  

Thus $[\alpha_1\alpha_2\beta_1, \alpha'_1\alpha'_2\beta_2] \in M_0^*(G)$. On the other hand,

$$[\alpha_1\alpha_2\beta_1, \alpha'_1\alpha'_2\beta_2] = [\alpha_1\alpha_2\beta_1, \beta_2][\alpha_1\alpha_2\beta_1, \alpha'_1\alpha'_2][\alpha_1\alpha_2\beta_1, \alpha'_1\alpha'_2, \beta_2]$$

$$= [\alpha_1\alpha_2, \beta_2][\alpha_1, \alpha_2, \beta_2][\beta_1, \alpha_1\alpha_2, \beta_2][\beta_1, \alpha_1\alpha_2, \beta_2]$$

$$= [\alpha_1, \beta_2][\alpha_1, \beta_2][\alpha_1, \beta_2][\gamma^{-1}, \beta'_1][\beta_1, \beta_2][\beta_1, \alpha'_1][\beta_1, \alpha'_2][\beta_1, \alpha'_1][\gamma, \beta'_2]$$

Except $[\beta_1, \alpha'_2]$ and $[\alpha_1, \beta'_1]$, the other commutators are in $M_0^*(G)$. Hence $[\beta_1, \alpha'_2][\beta_2, \alpha'_1]^{-1}$ belongs to $M_0^*(G)$, and $B_0(G) = 0$.  

\textbf{Proposition 3.17.} If $G \in \Phi_{41}$, then $B_0(G) = 0$.

\textbf{Proof.} The group $G = \Phi_{41}(16)$ has a polycyclic presentation

$$\langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2, \gamma \mid [\alpha_1, \alpha_2] = \beta, \gamma, [\alpha_1, \beta_1]^{-\nu} = [\alpha_2, \beta_1] = \gamma^{-\nu}, \alpha_i^p = \beta_i^p = \gamma^p = 1(i = 1, 2) \rangle.$$  

where $\nu$ denotes the smallest positive integer which is a non-quadratic residue (mod $p$). Notice that the group $[G, G^\nu]$ is generated by $[\alpha_1, \alpha'_2], [\beta, \alpha'_1]$ and $[\alpha_i, \beta'_j](i = 1, 2)$ modulo $M^*_0(G)$. As before, the commutativity among these generators implies that every element $w$ in $[G, G^\nu]$ can be expressed as

$$w = [\alpha_1, \alpha_2]^m[\beta, \alpha'_1]^n[\beta, \alpha'_2]^n[\alpha_1, \beta_1]^n[\alpha_2, \beta_2]^{n},$$

where $\overline{w} \in M_0^*(G)$. If $w \in M^*(G)$, then $1 = \kappa^*(w) = \beta_1^m\beta_2^n\gamma^{s-vt}$, so $p$ divides $m$, $n$, $r$ and $s - vt$. Notice that

$$1 = [\beta_i, \alpha'_j] = \beta_i[\beta, \alpha'_j][\beta, \alpha'_j] = \beta_i[\beta, \alpha'_j][\beta, \alpha'_j]$$

$$= [\beta, \alpha'_j][\beta, \alpha'_j][\beta, \alpha'_j][\beta, \alpha'_j] = [\beta, \alpha'_j][\beta, \alpha'_j][\beta, \alpha'_j] = [\beta, \alpha'_j][\beta, \alpha'_j] = \beta_i[\beta, \alpha'_j] = \beta_i[\beta, \alpha'_j].$$

Thus $[\beta, \alpha'_1]^p = 1$. Similarly, $[\beta_i, \alpha'_j]^p = 1(i = 1, 2)$, and $[\alpha_1, \alpha'_2]^p = 1$. Hence

$$w = ([\alpha_1, \beta'_1]^p[\alpha_2, \beta'_2])^{\overline{w}}.$$
On the other hand,

\[
[a_1a_2,\beta_i^\prime\beta_2] = [a_1,\beta_i^\prime\beta_2][a_1,\beta_i^\prime\beta_2,\alpha_2][\alpha_2,\beta_i^\prime\beta_2]
\]

\[
= [a_1,\beta_2][a_1,\beta_i^\prime][a_1,\beta_i^\prime,\beta_2][\gamma,\alpha_2][a_2,\beta_2,\beta_i^\prime][a_2,\beta_i^\prime,\beta_2]
\]

\[
= [a_1,\beta_1^\prime][\alpha_2,\beta_2] = 1.
\]

Thus \([\alpha_1a_2, (\beta_i^\prime\beta_2)^\prime] \in M_0(G)\). We expand this commutator as follows,

\[
[a_1a_2, (\beta_i^\prime\beta_2)^\prime] = [a_1,\beta_i^\prime\beta_2][a_1,\beta_i^\prime\beta_2,\alpha_2][\alpha_2,\beta_i^\prime\beta_2]
\]

\[
= [a_1,\beta_2][a_1,\beta_i^\prime][a_1,\beta_i^\prime,\beta_2][\gamma,\alpha_2][a_2,\beta_2,\beta_i^\prime][a_2,\beta_i^\prime,\beta_2]
\]

\[
= [a_1,\beta_2^\prime][a_1,\beta_i^\prime][\gamma,\beta_i^\prime,\beta_2][\gamma,\alpha_2][\alpha_2,\beta_2][\alpha_2,\beta_i^\prime].
\]

We notice that \([\alpha_1,\beta_2^\prime], [\gamma,\beta_i^\prime,\beta_2], [\gamma,\alpha_2^\prime] \) and \([\alpha_2,\beta_i^\prime] \) are in \(M_0(G)\), so \([\alpha_1,\beta_i^\prime][\alpha_2,\beta_i^\prime] \in M_0(G)\). Hence \(B_0(G) = 0\). □

**Proposition 3.18.** If \(G \in \Phi_{42}\), then \(B_0(G) = 0\).

**Proof.** Let \(G = \Phi_{42}(222)_{\omega_3}\) be the group in the family \(\Phi_{42}\), then it has a polycyclic presentation

\[
\langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2, \gamma \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_i] = \beta_i, [\alpha_1, \beta_2] = [\alpha_2, \beta_1] = \beta^p = \gamma, \alpha_i^p = \beta_i^{-1} \gamma^{-1/2}, \alpha_2^p = \beta_2 \gamma^{s+1/2}, \beta_i^p = \gamma^p = 1(i = 1, 2) \rangle.
\]

where \(v\) denotes the smallest positive integer which is a non-quadratic residue (mod \(p\)). The group \([G, G^\varphi]\) is generated by \([\alpha_1, \alpha_2^\prime], [\beta, \alpha_i^\prime], [\beta, \alpha_2^\prime], [\alpha_1, \beta_2^\prime] \) and \([\alpha_2, \beta_1^\prime] \) modulo \(M_0(G)\). Since any two elements of these generators are commuting modulo \(M_0(G)\), every element \(w\) in \([G, G^\varphi]\) can be expressed as

\[
w = [\alpha_1, \alpha_2^\prime]^m[\beta, \alpha_1^\prime]^p[\beta, \alpha_2^\prime]^p[\alpha_2, \beta_1^\prime]^p \bar{w},
\]

where \(\bar{w} \in M_0(G)\). If \(w \in M^\varphi(G)\), then \(1 = \varphi^\varphi(w) = \beta^m \beta_1 \beta_2 \gamma^{s+t}\), so \(p^2\) divides \(m\), \(p\) divides \(n\), \(r\) and \(s + t\). Notice that

\[
1 = [\beta^p, \alpha_1^\prime] = [\beta, \alpha_1^\prime]^p[\beta, \alpha_1^\prime]^p[\beta, \alpha_1^\prime, \beta]^p[\beta, \alpha_1^\prime, \beta, [\beta, \alpha_1^\prime]]^{\varphi(p)}
\]

\[
= [\beta, \alpha_1^\prime]^p[\beta, \alpha_1^\prime, \beta]^p[\beta, \alpha_1^\prime, \beta, [\beta, \alpha_1^\prime]]^{\varphi(p)}
\]

\[
= [\beta, \alpha_1^\prime]^p[\beta, \alpha_1^\prime]^{\varphi(p)}(i = 1, 2).
\]

Similarly, \([\beta_1, \alpha_2^\prime]^p = 1, [\beta_2, \alpha_1^\prime]^p = 1\) and \([\alpha_1, \alpha_2^\prime]^p = 1\). Thus we have

\[
w = (\alpha_1, \alpha_2^\prime)[\alpha_2, \beta_1^\prime]^{-1} \bar{w}.
\]

The direct computation shows that

\[
[a_1a_2\beta_1, a_1a_2\beta_2] = [a_1a_2\beta_1, \beta_1][a_1a_2\beta_1, \alpha_1a_2][\alpha_1a_2\beta_1, \alpha_1a_2, \beta_2]
\]

\[
= [a_1a_2, \beta_1][a_1a_2, \beta_2, \beta_1][\beta_1, \alpha_1a_2][\beta_1, \alpha_1a_2, \beta_2]
\]

\[
= [a_1, \beta_1][a_1, \beta_2, \beta_1][\beta_1, \alpha_1a_2][\beta_1, \alpha_1a_2, \beta_2]
\]

\[
= [a_1, \beta_2][a_1, \beta_2, \alpha_1][\alpha_1, \beta_2, \alpha_1a_2, \beta_2][\gamma, \beta_1, \beta_1a_2][\beta_1, \alpha_1a_2, \beta_2][\beta_1, \alpha_1a_2, \beta_2, \alpha_1] \]
This implies that $[\alpha_1 \alpha_2 (\beta_1, \alpha'_2 \beta_2')] \in M'_0(G)$. On the other hand,

$$
[\alpha_1 \alpha_2 \beta_1, \alpha'_2 \beta'_2] = [\alpha_1 \alpha_2 \beta_1, \alpha_1 \beta_2, \alpha'_2 \beta'_2] = [\alpha_1 \alpha_2, \beta_2, \beta_1 | \beta_1, \beta'_2 | \beta_1, \alpha'_2 | \beta_1, \alpha_2, \beta_2]
$$

Thus $[\alpha_1, \alpha'_2 \beta'_2] = [\alpha_2, \beta_2], [\alpha_1, \beta_2], [\alpha'_2, \beta'_2].$

ExCEPT $[\beta_1, \alpha'_2]$ and $[\alpha_1, \beta'_2]$, the other commutators all belong to $M'_0(G)$. Thus $[\alpha_1, \beta'_2][\alpha_2, \beta'_2]^{-1}$ is in $M'_0(G)$, and we are done.  

\section*{Proposition 3.19.} If $G \in \Phi_{43}$, then $B_0(G) = 0.$

\begin{proof}
Let $G = \Phi_{43}(222)_{la}$, then it has a polycyclic presentation

$\langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2, \gamma \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_i] = \beta_i, [\alpha_i, \beta_1]^{-v} = [\alpha_2, \beta_2] = \gamma^{-v},$

$$\alpha'_1 = \beta_2 \gamma^k, \alpha'_2 = \beta'_1 \gamma^l, \beta'_2 = \gamma^n, \beta''_2 = \gamma^p = 1 (i = 1, 2) \rangle.$$

where the $n = v + \left(\frac{p}{3}\right)$, and $k, l$ are the smallest positive integers satisfying $(k - v)^2 - v(l + v)^2 \equiv r (mod p)$, for $r = 0, 1, \cdots, p - 1$.

Notice that the group $[G, G^p]$ is generated by $[\alpha_1, \alpha'_2], [\beta, \alpha_i']$ and $[\alpha_i, \beta_i'](i = 1, 2)$ modulo $M'_0(G)$. The direct computation shows that any two elements of these generators are commutating modulo $M'_0(G)$, so every element $w$ in $[G, G^p]$ can be expressed as

$$w = [\alpha_1, \alpha'_2]^m [\beta, \alpha_1']^n [\beta, \alpha'_2] \gamma [\alpha_1, \beta_1']^i [\alpha_2, \beta_2']^i \bar{w},$$

where $\bar{w} \in M'_0(G)$. We assume that $w \in M^*(G)$. Then $1 = \kappa^*(w) = \beta''_2 [\beta', \alpha'_2] \gamma^{-v}$, so $p^2$ divides $m$, $p$ divides $n$, $r$ and $s - v$. Notice that

$$[\gamma^n, \alpha'_i] = [\beta''_2, \alpha'_i] = [\beta, \alpha'_i]^p [\beta, \alpha'_i, \beta] [\beta, \alpha'_i, \beta] [\beta, \alpha'_i, \beta, [\beta, \alpha'_i]]^{\sigma(p)}$$

$$= [\beta, \alpha'_i]^p [\beta', \alpha'_i] [\beta', \alpha'_i]^{\sigma(p)}$$

Thus $\bar{w} \in M'_0(G)$. Similarly, we deduce that $[\beta_i, \alpha'_i]^p = 1 (i = 1, 2)$, and $[\alpha_1, \alpha'_2]^p = 1.$ Thus

$$w = ([\alpha_1, \beta_1'][\alpha_2, \beta_2']^i\bar{w}.$$ 

We notice that

$$[\alpha_1 \alpha_2, \beta'_i \beta_2] = [\alpha_1, \beta_1'][\alpha_1, \beta'_i \beta_2, \alpha_2][\alpha_2, \beta'_i \beta_2]$$

$$= [\alpha_1, \beta_2][\alpha_1, \beta_1'][\alpha_1, \beta'_i \beta_2][\gamma^n, \alpha_2][\alpha_2, \beta_2][\alpha_2, \beta'_i \beta_2]$$

$$= [\alpha_1, \beta_1'][\alpha_2, \beta_2] = 1.$$
Thus \([\alpha_1 \alpha_2, (\beta'_i \beta_2)'] \in M^*_0(G)\). On the other hand,
\[
[\alpha_1 \alpha_2, (\beta'_i \beta_2)'] = [\alpha_1, \beta'_i \beta_2] [\alpha_1, \beta'_i \beta_2, \alpha_2] [\alpha_2, \beta'_i \beta_2] \\
= [\alpha_1, \beta'_i \beta_2] [\alpha_1, \beta'_i \beta_2, \gamma, \alpha'_2] [\alpha_2, \beta'_2] [\alpha_2, \beta'_i \beta_2] [\alpha_2, \beta'_1 \beta_2] \\
= [\alpha_1, \beta'_i \beta_2] [\alpha_1, \beta'_i \beta_2, \gamma, \alpha'_2] [\alpha_2, \beta'_2] [\alpha_2, \beta'_1 \beta_2].
\]

The commutators \([\alpha_1, \beta'_i \beta_2], [\gamma, \beta'_2], [\gamma, \alpha'_2] \) and \([\alpha_2, \beta'_1 \beta_2] \) are in \(M^*_0(G)\), thus \([\alpha_1, \beta'_i \beta_2] \) belongs to \(M^*_0(G)\) and hence \(B_0(G) = 0\).

\(\square\)

### 3.3. Noether’s Problem

Let \(G\) be a finite \(p\)-group of exponent \(e\) and \(k\) be any field of characteristic prime to \(e\). Let \(G\) act on the rational function field \(k(x_h : h \in G)\) by \(g \cdot x_h = x_{gh}\) for all \(g, h \in G\). We write \(k(G)\) for the fixed field \(k(x_h : h \in G)^G\). The main purpose of this subsection is to prove that if \(G = \Phi_{15}(2^{14})\) and \(k\) contains a primitive \(p^2\)-th root of unity, then \(k(G)\) is rational over \(k\). As a direct consequence, we have \(B_0(G) = 0\). Similar arguments can be applied to the cases where \(G \in \Phi_{28}\) or \(\Phi_{29}\), so we omit the detailed proofs.

To do this, we need some results which will be used frequently in our proof.

**Theorem 3.20** (Fischer ([27], Theorem 6.1)). Let \(G\) be a finite abelian group of exponent \(e\). Let \(k\) be a field of characteristic prime to \(e\) and containing the \(e\)th roots of unity. Let \(V\) be a finite dimensional representation of \(G\) over \(k\). Then the fixed field \(k(V)^G\) is rational over \(k\).

**Lemma 3.21** (No-name Lemma ([13], page 22)). Let \(G\) be a finite group acting faithfully on a finite-dimensional \(k\)-vector space \(V\), and let \(W\) be a faithful \(k[G]\)-submodule of \(V\). Then the extension of the fixed fields \(k(V)^G / k(W)^G\) is rational.

**Theorem 3.22** ([1]). Let \(L\) be a field and \(G\) be a finite group acting on the rational function field \(L(x)\). Assume that for any \(g \in G\), \(g(L) \subseteq L\) and \(g(x) = a_g \cdot x + b_g\), where \(a_g, b_g \in L\) and \(a_g \neq 0\). Then \(L(x)^G = L^G(f)\) for some polynomial \(f \in L[x]\).

Recall that a \(k\)-automorphism \(\beta \in \text{Aut}_k(k(x_1, \cdots, x_m))\) is said to be linearized if there exists an injection from the cyclic group \(\langle \beta \rangle\) to \(GL_m(k)\). Equivalently, \(\beta\) is linearized if and only if there are \(m\) elements \(z_1, \cdots, z_m \in k(x_1, \cdots, x_m)\) such that \(k(x_1, \cdots, x_m) = k(z_1, \cdots, z_m)\) and \(\beta \cdot (z_i) = \sum_{j=1}^{m} b_{ij} z_j\), where \((b_{ij})\) is an \(m \times m\) invertible matrix over \(k\).

**Lemma 3.23** ([14], page 226). Let \(p\) be a prime number and \(k\) be a field of characteristic \(\neq p\). Let \(\beta\) be a \(k\)-automorphism of \(k(x_1, \cdots, x_{p-1})\) with the action \(\beta : x_1 \mapsto x_2 \mapsto \cdots \mapsto x_{p-1} \mapsto (x_1 x_2 \cdots x_{p-1})^{-1} \mapsto x_1\). Then \(\beta\) can be linearized.
Corollary 3.24. Let $p$ be a prime number and $k$ be a field of characteristic $\neq p$. Let $\beta$ be a $k$-automorphism of $k(x_1, \cdots, x_{p-1})$ with the action
\[
\beta : x_1 \mapsto x_1x_2^0 \\
x_2 \mapsto x_3 \mapsto \cdots \mapsto x_{p-1} \mapsto \frac{1}{x_1x_2^{p-1}x_3^{p-2} \cdots x_{p-1}^{p-2}x_{p-2}x_{p-1}} \mapsto x_2.
\]
Then $\beta$ can be linearized.

Proof. Define $z_1 = x_2$, $z_i = \beta^{i-1} \cdot x_2$ for $2 \leq i \leq p - 1$. Then $k(z_i : 1 \leq i \leq p - 1) = k(x_1, \cdots, x_{p-1})$ with the action $\beta : z_1 \mapsto z_2 \mapsto \cdots \mapsto z_{p-1} \mapsto (z_1z_2 \cdots z_{p-1})^{-1} \mapsto z_1$. It follows from Lemma 3.23 that $\beta$ is linearized. \qed

Lemma 3.25. Let $p$ be a prime number and $k$ be a field of characteristic prime to $p$ and containing the $p$th roots of unity. Let $G = \langle \alpha, \beta \rangle$ be a group of order $p^2$ and as the $k$-automorphism group act on $k(x_{ij} : 1 \leq i, j \leq p - 1)$ by
\[
\alpha : x_{i1} \mapsto x_{i2} \mapsto \cdots \mapsto x_{ip-1} \mapsto (x_{i1}x_{i2} \cdots x_{ip-1})^{-1},
\]
\[
\beta : x_{ij} \mapsto x_{ij} \mapsto \cdots \mapsto x_{p-1,j} \mapsto (x_{1j}x_{2j} \cdots x_{p-1,j})^{-1},
\]
where $1 \leq i, j \leq p - 1$. Then the fixed field $k(x_{ij} : 1 \leq i, j \leq p - 1)^G$ is rational over $k$.

Proof. By Lemma 3.23, for each $1 \leq i \leq p - 1$, the restriction of $\alpha$ on the subfield $k(x_{i1}, x_{i2}, \cdots, x_{i,p-1})$ can be linearized. It follows from Fischer’s Theorem 3.20 that the fixed field $k(x_{ij} : 1 \leq i, j \leq p - 1)^{(\alpha)}$ is rational over $k$. We assume that $k(x_{ij} : 1 \leq i, j \leq p - 1)^{(\alpha)} = k(z_{ij} : 1 \leq i, j \leq p - 1)$ with $z_{ij} = f_j(x_{i1}, x_{i2}, \cdots, x_{i,p-1})$ for some rational functions $f_j$. Then $k(x_{ij} : 1 \leq i, j \leq p - 1)^G = (k(x_{ij} : 1 \leq i, j \leq p - 1)^{(\alpha)})^{(\beta)} = k(z_{ij} : 1 \leq i, j \leq p - 1)^{(\beta)}$, where the action of $\beta$ is given as follows:
\[
\beta : z_{1j} \mapsto z_{2j} \mapsto \cdots \mapsto z_{p-1,j} \mapsto (z_{1j}z_{2j} \cdots z_{p-1,j})^{-1}.
\]
Apply Lemma 3.23 and Fischer’s Theorem 3.20 again. Thus $k(z_{ij} : 1 \leq i, j \leq p - 1)^{(\beta)}$ is rational over $k$. \qed

The following is our main result of this subsection.

Theorem 3.26. Let $G = \Phi_{15}(21^4) = \langle \alpha_1, \cdots, \alpha_4, \beta_1, \beta_2 | \alpha_2 = \alpha_3 = \alpha_4 = \beta_1 = \beta_2 = 1, [\alpha_1, \alpha_2] = [\alpha_3, \alpha_4] = \beta_1 = \alpha_1^\theta, [\alpha_1, \alpha_3] = \beta_2, [\alpha_2, \alpha_4] = \beta_2^\theta \rangle$ be a nonabelian group of order $p^6$, where $\theta$ is the smallest positive integer which is a primitive root (mod $p$). Assume that the base field $k$ is of characteristic prime to $p$ and contains a primitive $p^2$-th root of unity. Then the fixed field $k(G)$ is rational over $k$. In particular, $B_0(G) = 0$.

Proof. Our proof consists of the following three steps:
STEP 1 We will construct a faithful subrepresentation $W$ of $V^*$, where $V^* = \oplus_{g \in G} k \cdot x_g$ be the dual of the regular representation $V$ of $G$.

Let $\eta$ be a primitive $p^2$-th root of unity. Then $\omega = \eta^p$ is a primitive $p$-th root of unity. Define

$$X_1 = \sum_{0 \leq j \leq p^2-1} x_{\alpha_j^1}, \quad X_2 = \sum_{0 \leq j \leq p^2-1} x_{\alpha_j^2}.$$  

Then $\alpha_1 \cdot X_1 = X_1$ and $\alpha_4 \cdot X_2 = X_2$. Define

$$Y_1 = \sum_{0 \leq j \leq p^2-1} \omega^{-j} \alpha_j^1 \cdot X_1 = X_1 + \omega^{-1} \alpha_4 \cdot X_1 + \cdots + \omega^{-(p^2-1)} \alpha_1^{p^2-1} \cdot X_1,$$

$$Y_2 = \sum_{0 \leq j \leq p^2-1} \eta^{-j} \alpha_j^1 \cdot X_2 = X_2 + \eta^{-1} \alpha_1 \cdot X_2 + \cdots + \eta^{-(p^2-1)} \alpha_1^{p^2-1} \cdot X_2.$$  

Since $[\alpha_1, \alpha_4] = 1$, it follows that

$$\alpha_1 : Y_1 \mapsto Y_1, \quad Y_2 \mapsto \eta \cdot Y_2$$

$$\alpha_4 : Y_1 \mapsto \omega \cdot Y_1, \quad Y_2 \mapsto Y_2.$$  

Notice that $\beta_1 = \alpha_1^p$. We define

$$\bar{Y}_1 = \sum_{0 \leq j \leq p^2-1} \beta_j^1 \cdot Y_1, \quad \bar{Y}_2 = \sum_{0 \leq j \leq p^2-1} \omega^{-j} \beta_j^1 \cdot Y_2.$$  

Since $\beta_1$ belongs to the center of $G$, we have

$$\alpha_1 : \bar{Y}_1 \mapsto \bar{Y}_1, \quad \bar{Y}_2 \mapsto \eta \cdot \bar{Y}_2$$

$$\alpha_4 : \bar{Y}_1 \mapsto \omega \cdot \bar{Y}_1, \quad \bar{Y}_2 \mapsto \bar{Y}_2$$

$$\beta_1 : \bar{Y}_1 \mapsto \bar{Y}_1, \quad \bar{Y}_2 \mapsto \omega \cdot \bar{Y}_2.$$  

Define

$$\bar{X}_1 = \sum_{0 \leq j \leq p^2-1} \omega^{-j} \beta_j^2 \cdot \bar{Y}_1, \quad \bar{X}_2 = \sum_{0 \leq j \leq p^2-1} \omega^{-j} \beta_j^2 \cdot \bar{Y}_2.$$  

Since $\beta_2$ is also an element in the center of $G$, it follows that

$$\alpha_1 : \bar{X}_1 \mapsto \bar{X}_1, \quad \bar{X}_2 \mapsto \eta \cdot \bar{X}_2$$

$$\alpha_4 : \bar{X}_1 \mapsto \omega \cdot \bar{X}_1, \quad \bar{X}_2 \mapsto \bar{X}_2$$

$$\beta_1 : \bar{X}_1 \mapsto \bar{X}_1, \quad \bar{X}_2 \mapsto \omega \cdot \bar{X}_2$$

$$\beta_2 : \bar{X}_1 \mapsto \omega \cdot \bar{X}_1, \quad \bar{X}_2 \mapsto \omega \cdot \bar{X}_2.$$  

Now to realize $\alpha_2$ and $\alpha_3$, for $0 \leq i, j \leq p - 1$, we define

$$x_{ij} = \alpha_2^i \alpha_3^j \cdot \bar{X}_1, \quad y_{ij} = \alpha_2^i \alpha_3^j \cdot \bar{X}_2.$$  

Applying the commutator relation among the generators in the presentation of $G$, we have

$$\alpha_2 : x_{0j} \mapsto x_{1j} \mapsto \cdots \mapsto x_{p-1,j} \mapsto x_{0j}, \quad y_{0j} \mapsto y_{1j} \mapsto \cdots \mapsto y_{p-1,j} \mapsto y_{0j}.$$
It is clear that $W = (\oplus_{0 \leq j \leq p - 1} k \cdot x_j) \oplus (\oplus_{0 \leq i, j \leq p - 1} k \cdot y_{ij})$ is a faithful representation of $G$.

By No-name Lemma 3.21, it suffices to show that the invariant field $k(W)^G$ is rational over $k$. Let $K$ be the fixed field $k(x_{ij} : 0 \leq i, j \leq p - 1)^G$. Then $k(W)^G = K(y_{ij} : 0 \leq i, j \leq p - 1)^G$.

**Step 2** We will prove that $K = k(x_{ij} : 0 \leq i, j \leq p - 1)^G$ is rational over $k$. In what follows, we write $I$ for the set $\{0, 1, 2, \cdots, p - 1\}$, $I^2$ for the Cartesian set $I \times I$, and $J$ for $I^2 - \{(0, 0)\}$.

Let $u_{ij} = \frac{u_{ij}}{u_{j,1}}$, $u_j = \frac{u_j}{u_{j,1}}$ for all $1 \leq i, j \leq p - 1$. Then $k(x_{ij} : 0 \leq i, j \leq p - 1) = k(x_{00}, u_{ij} : (i, j) \in J)$. We write $L$ for the subfield $k(u_{ij} : (i, j) \in J)$. For every $\rho \in G, \rho \cdot x_{00} \in L \cdot x_{00}$, while $L$ is invariant by the action of $G$, i.e.,

$$\begin{align*}
\alpha_2 : & \quad u_{ij} \mapsto u_{ij} \cdot u_{0j}^{-1} \cdot u_{0j} \quad (1 \leq j \leq p - 1), \\
 & \quad u_{1j} \mapsto u_{2j} \mapsto \cdots \mapsto u_{p-1,j} \mapsto (u_{1j}u_{2j} \cdots u_{p-1,j})^{-1} \mapsto u_{1j} \quad (0 \leq j \leq p - 1), \\
\alpha_3 : & \quad u_{01} \mapsto u_{02} \mapsto \cdots \mapsto u_{0,p-1} \mapsto (u_{01}u_{02} \cdots u_{0,p-1})^{-1} \mapsto u_{01}, \\
 & \quad u_{0} \mapsto u_{01} \mapsto \cdots \mapsto u_{i,p-1} \mapsto u_{0} \quad (1 \leq i \leq p - 1), \\
\alpha_4 : & \quad u_{0j} \mapsto u_{0j} \quad (1 \leq j \leq p - 1), \\
 & \quad u_{ij} \mapsto \omega \cdot u_{ij} \quad (0 \leq j \leq p - 1, 1 \leq i \leq p - 1), \\
\alpha_1 : & \quad u_{0j} \mapsto \omega \cdot u_{0j} \quad (1 \leq j \leq p - 1), \\
 & \quad u_{ij} \mapsto u_{ij} \quad (0 \leq j \leq p - 1, 1 \leq i \leq p - 1), \\
\beta_1, \beta_2 : & \quad u_{ij} \mapsto u_{ij} \quad ((i, j) \in J).
\end{align*}$$

Apply Theorem 3.22, it suffices to show that $L^G$ is rational over $k$. Since the action of $\langle \beta_1, \beta_2 \rangle$ is trivial, $L^G = L^{\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle}$.

Let $v_{10} = u_{10}^p$, $v_{00} = \frac{u_{00}}{u_{0,1}}$ for $2 \leq i \leq p - 1$. Let $v_{0j} = u_{0j}$ for $1 \leq i \leq p - 1$. Let $v_{ij} = \frac{u_{ij}}{u_{i,p-1}}$ for all $1 \leq i, j \leq p - 1$. Then $L^{\langle \alpha_4 \rangle} = k(v_{ij} : (i, j) \in J)$. Note that the action of $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ on $k(v_{ij} : (i, j) \in J)$ are given by:

$$\begin{align*}
\alpha_2 : & \quad v_{0j} \mapsto v_{1j} \cdot v_{0j} \quad (1 \leq j \leq p - 1), \\
 & \quad v_{10} \mapsto v_{10} \cdot v_{20}^p,
\end{align*}$$
\[
\begin{align*}
\alpha_3 : & v_{01} \mapsto v_{02} \mapsto \cdots \mapsto v_{0,p-1} \mapsto (v_{01}v_{02} \cdots v_{0,p-1})^{-1} \mapsto v_{01}, \\
v_{10} & \mapsto v_{11} \cdot v_{10}, \quad v_{0} \mapsto v_{1i}v_{-1,i}^{-1} \cdot v_{0} \quad (2 \leq i \leq p-1), \\
v_{11} & \mapsto v_{12} \mapsto \cdots \mapsto v_{i,p-1} \mapsto (v_{11}v_{12} \cdots v_{i,p-1})^{-1} \mapsto v_{i1} \quad (1 \leq i \leq p-1), \\
\alpha_1 : & v_{0j} \mapsto \omega \cdot v_{0j} \quad (1 \leq j \leq p-1), \\
v_{ij} & \mapsto v_{ij} \quad (0 \leq j \leq p-1, 1 \leq i \leq p-1).
\end{align*}
\]

Let \( w_{01} = v_{01}^p, w_{0j} = \frac{v_{0j}}{v_{0,j-1}} \) for \( 2 \leq j \leq p-1 \), and \( w_{ij} = v_{ij} \), otherwise. Obviously, \( k(v_{ij} : (i, j) \in J) = k(v_{ij} : (i, j) \in J) \) with the action of \( \langle \alpha_2, \alpha_3 \rangle \):

\[
\begin{align*}
\alpha_2 : & w_{01} \mapsto w_{01} \cdot w_{10}^p, \quad w_{0j} \mapsto w_{1j} \cdot w_{1,j-1}^{-1} \cdot w_{0j} \quad (2 \leq j \leq p-1), \\
w_{10} & \mapsto w_{10} \cdot w_{20}^p, \\
w_{20} \mapsto w_{30} \mapsto \cdots \mapsto w_{p-1,0} \mapsto \frac{1}{w_{10}w_{20}^p \cdots w_{p-1,0}^{-1}} \mapsto w_{01}w_{02}^p \cdots w_{0,p-2}w_{0,p-1} \mapsto w_{02}, \\
w_{1j} & \mapsto w_{2j} \mapsto \cdots \mapsto w_{p-1,j} \mapsto (w_{1j}w_{2j} \cdots w_{p-1,j})^{-1} \mapsto w_{1j} \quad (1 \leq j \leq p-1), \\
\alpha_3 : & w_{01} \mapsto w_{01} \cdot w_{02}^p, \\
w_{02} \mapsto w_{03} \mapsto \cdots \mapsto w_{0,p-1} \mapsto \frac{1}{w_{01}w_{02}^p \cdots w_{0,p-2}^{-1}} \mapsto w_{01}w_{02}^p \cdots w_{0,p-2}w_{0,p-1} \mapsto w_{02}, \\
w_{10} & \mapsto w_{11} \cdot w_{10}, \quad w_{0} \mapsto w_{1i}w_{-1,i}^{-1} \cdot w_{0} \quad (2 \leq i \leq p-1), \\
w_{1i} & \mapsto w_{12} \mapsto \cdots \mapsto w_{i,p-1} \mapsto (w_{1i}w_{12} \cdots w_{i,p-1})^{-1} \mapsto w_{i1} \quad (1 \leq i \leq p-1).
\end{align*}
\]

Since \( L^G = L^{(\alpha_1, \alpha_2, \alpha_3, \alpha_4)} = k(v_{ij} : (i, j) \in J) \), we need to prove that \( k(v_{ij} : (i, j) \in J) \) is rational over \( k \).

Let \( r_i = w_{0i} \) and \( s_i = w_{i0} \) for \( 1 \leq i \leq p-1 \). Let \( M = k(v_{ij} : 1 \leq i, j \leq p-1) \). Then

\[
k(v_{ij} : (i, j) \in J) = \left( M^{(\alpha_2, \alpha_3)}(r_i, s_i : 1 \leq i \leq p-1) \right)^{(\alpha_2, \alpha_3)}. \]

It follows from Lemma 3.25 that \( M^{(\alpha_2, \alpha_3)} \) is rational over \( k \). We assume that \( F = M^{(\alpha_2, \alpha_3)} \). Then \( k(v_{ij} : (i, j) \in J) = F(r_i, s_i : 1 \leq i \leq p-1) \), and we need to show that \( F(r_i, s_i : 1 \leq i \leq p-1) \) is rational over \( F \).

We rewrite the action of \( \langle \alpha_2, \alpha_3 \rangle \) on \( F(r_i, s_i : 1 \leq i \leq p-1) \):

\[
\begin{align*}
\alpha_2 : & r_1 \mapsto s_1^p \cdot r_1, \quad r_i \mapsto a_i^{-1} \cdot r_i \quad (2 \leq i \leq p-1), \\
&s_1 \mapsto s_1 \cdot s_2^p, \\
&s_2 \mapsto s_3 \mapsto \cdots \mapsto s_{p-1} \mapsto \frac{1}{s_1s_2^p \cdots s_{p-2}^p} \mapsto s_1s_2^{p-2} \cdots s_{p-2}^2s_{p-1} \mapsto s_2,
\end{align*}
\]

\[
\begin{align*}
\alpha_3 : & v_{01} \mapsto v_{02} \mapsto \cdots \mapsto v_{0,p-1} \mapsto (v_{01}v_{02} \cdots v_{0,p-1})^{-1} \mapsto v_{01}, \\
v_{10} & \mapsto v_{11} \cdot v_{10}, \quad v_{0} \mapsto v_{1i}v_{-1,i}^{-1} \cdot v_{0} \quad (2 \leq i \leq p-1), \\
v_{11} & \mapsto v_{12} \mapsto \cdots \mapsto v_{i,p-1} \mapsto (v_{11}v_{12} \cdots v_{i,p-1})^{-1} \mapsto v_{i1} \quad (1 \leq i \leq p-1), \\
\alpha_1 : & v_{0j} \mapsto \omega \cdot v_{0j} \quad (1 \leq j \leq p-1), \\
v_{ij} & \mapsto v_{ij} \quad (0 \leq j \leq p-1, 1 \leq i \leq p-1).
\end{align*}
\]
\[ \alpha_3 : r_1 \mapsto r_1 \cdot r_2^p, \]
\[ r_2 \mapsto r_3 \mapsto \cdots \mapsto r_{p-1} \mapsto \frac{1}{r_1 r_2^{p-1} r_3^{p-2} \cdots r_{p-1}^2} \mapsto r_1 r_2^{p-2} r_3^{p-3} \cdots r_{p-2}^2 r_{p-1} \mapsto r_2, \]
\[ s_1 \mapsto b_1^p \cdot s_1, \quad s_i \mapsto b_i b_{i-1} \cdot s_i \quad (2 \leq i \leq p-1), \]

where all \(a_i, b_i \in F\).

Let \(s'_1 = s_2, r'_1 = r_2, s'_i = a_2^{-1} \cdot s_2\) and \(r'_i = a_3^{-1} \cdot r_2\) for \(2 \leq i \leq p-1\). Then \(F(r_i, s_i) = F(r'_i, s'_i)\). Note that \([a_2, a_3] = 1\). We have

\[ \alpha_2 : r'_i \mapsto a_2 a_1^{-1} \cdot r'_i \quad (1 \leq i \leq p-1), \]
\[ s'_1 \mapsto s'_2 \mapsto \cdots \mapsto s'_{p-1} \mapsto (s'_1 s'_2 \cdots s'_{p-1})^{-1} \mapsto s'_1, \]
\[ \alpha_3 : r'_1 \mapsto r'_2 \mapsto \cdots \mapsto r'_p \mapsto (r'_1 r'_2 \cdots r'_{p-1})^{-1} \mapsto r'_1, \]
\[ s'_1 \mapsto b_2 b_1^{-1} \cdot s'_i \quad (1 \leq i \leq p-1). \]

Define \(t_1 = r_1^{p^p}\) and \(t_i = \frac{r'_i}{r_{i-1}}\) for \(2 \leq i \leq p-1\). Then \(F(r'_i : 1 \leq i \leq p-1)^{(a_2, a_3)} = F(t_1, \ldots, t_{p-1})^{(a_3)}\)
and

\[ \alpha_3 : t_1 \mapsto t_1 \cdot t_2^p, \]
\[ t_2 \mapsto t_3 \mapsto \cdots \mapsto t_{p-1} \mapsto \frac{1}{t_1^p t_2^{p-2} \cdots t_{p-1}^2} \mapsto t_1 t_2^{p-2} t_3^{p-3} \cdots t_{p-2}^2 t_{p-1} \mapsto t_2. \]

By Corollary 3.24 and Theorem 3.20, we have \(F(r'_i : 1 \leq i \leq p-1)^{(a_2, a_3)}\) is rational over \(F\). Let \(F' = F(r'_i : 1 \leq i \leq p-1)^{(a_2, a_3)}\). Notice that \(F(r'_i, s'_i : 1 \leq i \leq p-1)^{(a_2, a_3)} = F'(s'_1, \ldots, s'_{p-1})^{(a_2, a_3)}\), which is rational over \(F'\) by the same reason, so is also rational over \(F\).

**Step 3** Finally, we will use the method developed in step 2 to prove that \(K(y_{ij} : 0 \leq i, j \leq p-1)^G\) is rational over \(K\).

Let \(u_{0j} = \frac{y_{ij}}{y_{i,0}^{j+1}}, u_{00} = \frac{y_{ij}}{y_{i,0}^{j+1}}\) and \(u_{ij} = \frac{y_{ij}}{y_{i,0}^{j+1}}\) for all \(1 \leq i, j \leq p-1\). Then \(K(y_{ij} : 0 \leq i, j \leq p-1) = K(y_{00}, u_{ij} : (i, j) \in J)\). We write \(L\) for the subfield \(K(u_{ij} : (i, j) \in J)\). For every \(\rho \in G, \rho \cdot y_{00} \in L \cdot y_{00}\), while \(L\) is invariant by the action of \(G\), i.e.,

\[ \alpha_2 : u_{0j} \mapsto u_{1j} u_{1,j-1}^{-1} \cdot u_{0j} \quad (1 \leq j \leq p-1), \]
\[ u_{1j} \mapsto u_{2j} \mapsto \cdots \mapsto u_{p-1,j} \mapsto (u_{1j} u_{2j} \cdots u_{p-1,j})^{-1} \mapsto u_{1j} \quad (0 \leq j \leq p-1), \]
\[ \alpha_3 : u_{01} \mapsto u_{02} \mapsto \cdots \mapsto u_{0,p-1} \mapsto (u_{01} u_{02} \cdots u_{0,p-1})^{-1} \mapsto u_{01}, \]
\[ u_{0} \mapsto u_{1} \mapsto \cdots \mapsto u_{i,p-1} \mapsto u_{0} \quad (1 \leq i \leq p-1), \]
\[ \alpha_4 : u_{0j} \mapsto \omega^{-1} \cdot u_{0j} \quad (1 \leq j \leq p-1), \]
\[ u_{ij} \mapsto \omega^{-\theta} \cdot u_{ij} \quad (0 \leq j \leq p-1, 1 \leq i \leq p-1), \]
\[ \alpha_1 : u_{0j} \mapsto \omega \cdot u_{0j} \quad (1 \leq j \leq p-1), \]
\[ u_{ij} \mapsto \omega \cdot u_{ij} \quad (0 \leq j \leq p-1, 1 \leq i \leq p-1), \]
Let $L^G$ is rational over $K$. Since the action of $\langle \beta_1, \beta_2 \rangle$ is trivial, $L^G = L^{(\alpha_1, \alpha_2, \alpha_3, \alpha_4)}$.

Let $v_{10} = u_{10}^{\alpha_{i_0}}$, $v_0 = u_{i_0}^{\alpha_{j_0}}$, $v_{01} = u_{i_0}^{\alpha_{i_1}}$ and $v_{0j} = u_{i_0}^{\alpha_{j_1}}$ for $2 \leq i, j \leq p - 1$. Let $v_{ij} = u_{i_0}^{\alpha_{j_1}}$ for all $1 \leq i, j \leq p - 1$. Then $L^{(\alpha_{i_1})} = K(v_{ij} : (i, j) \in J)$. Note that the action of $\langle \alpha_2, \alpha_3, \alpha_4 \rangle$ on $K(v_{ij} : (i, j) \in J)$ are given by:

\[ \alpha_2 : \quad v_{01} \mapsto v_{11}v_{20}^{-1} \cdot v_{01}, \quad v_{0j} \mapsto v_{1j}v_{1,j-1}^{-1} \cdot v_{0j} \quad (2 \leq j \leq p - 1), \]
\[ v_{10} \mapsto v_{10} \cdot v_{20}^p, \]
\[ v_{20} \mapsto v_{30} \mapsto \cdots \mapsto v_{p-1,0} \mapsto \frac{1}{v_{10}v_{20}^{-1}v_{30}^{-1} \cdots v_{p-2,0}v_{p-1,0}} \mapsto v_{10}v_{20}^{-1}v_{30}^{-1} \cdots v_{p-2,0}v_{p-1,0} \mapsto v_{20}, \]
\[ v_{1j} \mapsto v_{2j} \mapsto \cdots \mapsto v_{p-1,j} \mapsto (v_{1j}v_{2j} \cdots v_{p-1,j})^{-1} \mapsto v_{1j} \quad (1 \leq j \leq p - 1), \]

\[ \alpha_3 : \quad v_{01} \mapsto v_{02}v_{11}^{-1} \cdot v_{01}, \]
\[ v_{02} \mapsto v_{03} \mapsto \cdots \mapsto v_{0,p-1} \mapsto \frac{1}{v_{01}v_{02}^{-1}v_{03}^{-1} \cdots v_{0,p-2}v_{0,p-1}} \mapsto v_{01}v_{02}^{-1}v_{03}^{-1} \cdots v_{0,p-2}v_{0,p-1} \mapsto v_{02}, \]
\[ v_{10} \mapsto v_{11}^{p_1} \cdot v_{10}, \quad v_{0} \mapsto v_{11}^{-1} \cdot v_{0} \quad (2 \leq i \leq p - 1), \]
\[ v_{1i} \mapsto v_{12} \mapsto \cdots \mapsto v_{i,p-1} \mapsto (v_{1i}v_{12} \cdots v_{i,p-1})^{-1} \mapsto v_{1i} \quad (1 \leq i \leq p - 1), \]

\[ \alpha_4 : \quad v_{01} \mapsto \omega^{p_1-1} \cdot v_{01}, \quad v_{ij} \mapsto v_{ij} \quad \text{(otherwise)}. \]

Let $w_{01} = v_{01}^{\beta_{i_0}}$ and $w_{ij} = v_{ij}$, otherwise. Obviously, $K(v_{ij} : (i, j) \in J)^{(\alpha_4)} = K(w_{ij} : (i, j) \in J)$. Thus $L^G = L^{(\alpha_1, \alpha_2, \alpha_3, \alpha_4)} = K(w_{ij} : (i, j) \in J)^{(\alpha_2, \alpha_3)}$. The rationality of $K(w_{ij} : (i, j) \in J)^{(\alpha_2, \alpha_3)}$ over $K$ can be proved as same as in step 2. Hence $K(v_{ij} : 0 \leq i, j \leq p - 1)^G$ is rational over $K$.

This completes the proof. \( \square \)

Applying the same techniques, we will obtain

**Theorem 3.27.** Let $G = \Phi_{28}(222)$ (or $\Phi_{29}(222)$) be a nonabelian group of order $p^6$. Assume that the base field $k$ is of characteristic prime to $p$ and contains a primitive $p^2$-th root of unity. Then the fixed field $k(G)$ is rational over $k$. In particular, $B_0(G) = 0$.

**Remark 3.28.** The method above can be applied to discuss Noether’s problem for these groups in the family $\Phi_{15}$ or other $p$-groups with six generators.

### 4. Nontrivial Bogomolov Multipliers

In this section, we use the following nonvanishing criterion for the Bogomolov multiplier to complete the proof of Theorem 1.4. Throughout this section, $\binom{x}{y}$ denotes the binomial coefficient when $x \geq y \geq 1$ and we adopt the convention $\binom{x}{y} = 0$ if $1 \leq x < y$. 
**Lemma 4.1** (Hoshi-Kang [10]). Let $G$ be a finite group, $N$ be a normal subgroup of $G$. Assume that

1. the transgression map $\text{tr}: H^1(N, \mathbb{Q}/\mathbb{Z})^G \to H^2(G/N, \mathbb{Q}/\mathbb{Z})$ is not surjective, and

2. for any bicyclic subgroup $A$ of $G$, the group $AN/N$ is a cyclic subgroup of $G/N$.

Then $B_0(G) \neq 0$.

**Proposition 4.2.** If $G$ belongs to one of $\Phi_i (i = 18, 20, 21, 36, 38, 39)$, then $B_0(G) \neq 0$.

**Remark 4.3.** Here we only give the detailed proof for the case $G \in \Phi_{18}$. The similar arguments can be applied to the remaining cases. However, we need to remark that the normal subgroup $N$ is taken to be $\langle \beta, \beta_1, \beta_2 \rangle$ in the cases $\Phi_{20}$ and $\Phi_{21}$; to be $\langle \alpha_3, \alpha_4, \alpha_5 \rangle$ in the cases $\Phi_{36}$, $\Phi_{38}$ and $\Phi_{39}$.

**Proof.** We choose $G = \Phi_{18}(1^6)$ as a representative, it has a polycyclic presentation

$$\langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta, \gamma \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, [\alpha, \beta] = \gamma, \alpha^p = \beta^p = \alpha_1^p = \gamma^p = 1(i = 1, 2) \rangle.$$

Let $N = \langle \alpha_3, \beta, \gamma \rangle$ be the normal subgroup of $G$. We will prove that $N$ satisfies the two conditions in Lemma 4.1, thus $B_0(G) \neq 0$.

Since $N \cong C_p \times C_p \times C_p$, it follows that $H^1(N, \mathbb{Q}/\mathbb{Z}) = \text{Hom}(N, \mathbb{Q}/\mathbb{Z}) \cong C_p \times C_p \times C_p$. Define $\varphi_1, \varphi_2, \varphi_3 \in H^1(N, \mathbb{Q}/\mathbb{Z})$ by

$$\varphi_1(\alpha_3) = \frac{1}{p}, \varphi_1(\beta) = 0, \varphi_1(\gamma) = 0;$$

$$\varphi_2(\alpha_3) = 0, \varphi_2(\beta) = \frac{1}{p}, \varphi_2(\gamma) = 0;$$

$$\varphi_3(\alpha_3) = 0, \varphi_3(\beta) = 0, \varphi_3(\gamma) = \frac{1}{p}.$$

We have $H^1(N, \mathbb{Q}/\mathbb{Z}) = \langle \varphi_1, \varphi_2, \varphi_3 \rangle$. The action of $G$ on $\varphi_1, \varphi_2, \varphi_3$ are given by

$$\alpha \cdot \varphi_1(\alpha_3) = \varphi_1(\alpha^{-1} \alpha_3 \alpha) = \varphi_1(\alpha_3) = \frac{1}{p};$$

$$\alpha \cdot \varphi_1(\beta) = \varphi_1(\alpha^{-1} \beta \alpha) = \varphi_1(\beta \gamma^{-1}) = \varphi_1(\beta) + \varphi_1(\gamma^{-1}) = 0;$$

$$\alpha \cdot \varphi_1(\gamma) = \varphi_1(\alpha^{-1} \gamma \alpha) = \varphi_1(\gamma) = 0.$$

Thus $\alpha$ fixes $\varphi_1$. Similarly,

$$\alpha \cdot \varphi_2(\alpha_3) = 0, \alpha \cdot \varphi_2(\beta) = \frac{1}{p}, \alpha \cdot \varphi_2(\gamma) = 0,$$

$$\alpha \cdot \varphi_3(\alpha_3) = 0, \alpha \cdot \varphi_3(\beta) = \frac{1}{p}, \alpha \cdot \varphi_3(\gamma) = \frac{1}{p}.$$

Hence, $\alpha \cdot \varphi_3 = -\varphi_2 + \varphi_3$ and $\alpha$ fixes $\varphi_2$. With an analogous argument, we eventually obtain

$\alpha_1 : \varphi_1 \mapsto \varphi_1 - \varphi_2, \varphi_2 \mapsto \varphi_2, \varphi_3 \mapsto \varphi_3$;

$\alpha_2 : \varphi_1 \mapsto \varphi_1, \varphi_2 \mapsto \varphi_2, \varphi_3 \mapsto \varphi_3$. 
For any $\varphi \in H^1(N, \mathbb{Q}/\mathbb{Z})$, we write $\varphi = a_1\varphi_1 + a_2\varphi_2 + a_3\varphi_3$ for some integers $a_1, a_2, a_3 \in \mathbb{Z}$ (modulo $p$). It is easy to check that $\varphi \in H^1(N, \mathbb{Q}/\mathbb{Z})^G$ if and only if $a_1 = a_2 = a_3 = 0$. Obviously, $\varphi_2 \in H^1(N, \mathbb{Q}/\mathbb{Z})^G$. Thus $H^1(N, \mathbb{Q}/\mathbb{Z})^G = \langle \varphi_2 \rangle \cong \mathbb{C}_p$. Notice that $G/N$ is a nonabelian group of order $p^3$ and of exponent $p$, it follows from Proposition 6.3 in [19] (or see [17], Theorem 3.3.6) that $H^2(G/N, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{C}_p \times \mathbb{C}_p$. Thus the transgression map $\text{tr}: H^1(N, \mathbb{Q}/\mathbb{Z})^G \to H^2(G/N, \mathbb{Q}/\mathbb{Z})$ is not surjective.

The second step is to prove that the group $AN/N$ is a cyclic subgroup of $G/N$ for any bicyclic subgroup $A$ of $G$. Recall that a group $A$ is said to be bicyclic if $A$ is either cyclic or a direct product of two cyclic groups. The following formulae follows from the commutator relations of $G$:

\begin{align}
(4.1) & \quad a_i^j a^j = a^i \alpha_i \alpha_j \alpha_j^i, \\
(4.2) & \quad a_i^j \beta^j = \beta^i \alpha_i \alpha_j \alpha_j^i, \\
(4.3) & \quad a_i^j \alpha^j = \alpha^i \alpha_i \alpha_j \alpha_j^i, \\
(4.4) & \quad a_i^j \beta^j = \beta^i \alpha_i \alpha_j \alpha_j^i,
\end{align}

where $1 \leq i, j \leq p - 1$. Let $A = \langle y_1, y_2 \rangle$ be a bicyclic subgroup of $G$. We observe that $AN/N$ is abelian and $G/N$ is nonabelian, so $AN/N$ is a proper subgroup of $G/N$. Thus the order of $AN/N$ is either $p$ or $p^2$. If the order of $AN/N$ is $p$, then it is cyclic, we are done.

Assume that the order of $AN/N$ is $p^2$, we will prove that this is impossible. In $G/N$, we write $y_1N = a_1a_1^a_1^2$, and $y_2N = a_2a_2^b_2$. The almost same proof as in Lemma 2.2 of Hoshi-Kang [10] implies that there are only three possibilities:

\[ (y_1N, y_2N) = (a_1N, a_2N), (aa^a_2, a_1a_2^b_2), (aa^a_2, a_2N) \]

if it is necessary to change some suitable generators $y_1, y_2$ and integers $a_2, a_3, b_2$. Finally we will show that all three possibilities will lead to contradiction.

For the first case, we write $y_1 = a_1a_1^a_1^2$ and $y_2 = a_2a_2^b_2$. Since $y_1$ and $y_2$ are commutating, $a_1a_1^a_1^2 = a_2a_2^b_2$. It follows from (4.2) that $[a_1, a_2] \neq 1$, which is a contradiction. Second, suppose $y_1N = aa^a_2N$ and $y_2N = a_1a_2^b_2N$. Using (4.1) and (4.3), we obtain that $y_1N$ and $y_2N$ do not commute. This is a contradiction again. The last case is similar. We write $y_1 = aa^a_1a_1^a_1^2$ and $y_2 = a_2a_2^b_2$. Notice that $y_1y_2 = y_2y_1$, but we use (4.1)-(4.4) to get a contradictory fact that $y_1$ and $y_2$ do not commute.

The proof is completed.

\[ \square \]

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References

[1] H. Ahmad, M. Hajja and M. Kang, Rationality of some projective linear actions. J. Algebra 228 (2000) 643-658.
[2] R. Blyth and R. Morse, Computing the nonabelian tensor square of polycyclic groups. J. Algebra 321 (2009) 2139-2148.
[3] F. Bogomolov, The Brauer group of quotient spaces by linear group actions. Math. USSR Izv. 30 (1988) 455-485.
[4] F. Bogomolov, J. Maciel and T. Petrov, Unramified Brauer groups of finite simple groups of Lie type A\_\ell. Amer. J. Math. 126 (2004) 935-949.
[5] F. Bogomolov and T. Petrov, Unramified cohomology of alternating groups. Cent. Eur. J. Math. 9 (2011) 936-948.
[6] F. Bogomolov, T. Petrov and Y. Tschinkel, Unramified cohomology of finite groups of Lie type. In “Cohomological and Geometric Approaches to Rationality Problems” (F. Bogomolov and Y. Tschinkel, eds.), Progress in Math. vol. 282, Birkhäuser, Boston (2010), pp. 55-73.
[7] H. Chu, S. Hu, M. Kang and B. Kunyavskii, Noether’s problem and the unramified Brauer groups for groups of order 64. Intern. Math. Res. Notices 12 (2010) 2329-2366.
[8] H. Chu, S. Hu, M. Kang and Y. Prokhorov, Noether’s problem for groups of order 32. J. Algebra 320 (2008) 3022-3035.
[9] H. Chu and M. Kang, Rationality of p-group actions. J. Algebra 237 (2001) 673-690.
[10] A. Hoshi and M. Kang, Unramified Brauer groups for groups of order p^5. arXiv:1109.2966.
[11] A. Hoshi, M. Kang and B. Kunyavskii, Noether’s problem and unramified Brauer groups. To appear in Asian J. Math. arXiv:1202.5812.
[12] R. James, The groups of order p^6 (p an odd prime). Math. Comp. 34 (1980) 613-637.
[13] C. Jensen, A. Ledet and N. Yui, Generic polynomials: constructive aspects of the inverse Galois problem. Cambridge University Press, Cambridge (2003).
[14] M. Kang, Noether’s problem for p-groups with a cyclic subgroup of index p^2. Adv. Math. 226 (2011) 218-234.
[15] M. Kang, Bogomolov multipliers and retract rationality for semi-direct products. arXiv:1207.5467.
[16] M. Kang and B. Kunyavskii, The Bogomolov multiplier of rigid finite groups. arXiv:1304.2691.
[17] G. Karpilovsky, The Schur multiplier. Lond. Math. Soc. Monographs 2, Oxford University Press, New York (1987).
[18] B. Kunyavskii, The Bogomolov multiplier of finite simple groups. In “Cohomological and Geometric Approaches to Rationality Problems” (F. Bogomolov and Y. Tschinkel, eds.), Progress in Math. vol. 282, Birkhäuser, Boston (2010), pp. 209-217.
[19] G. Lewis, The integral cohomology rings of groups of order p^3. Trans. Amer. Math. Soc. 132 (1968) 501-529.
[20] I. Michailov, Noether’s problem for abelian extensions of cyclic p-groups. arXiv:1301.7284.
[21] P. Moravec, Unramified Brauer groups of finite and infinite groups. Amer. J. Math. 134 (2012) 1679-1704.
[22] P. Moravec, Groups of order p^4 and their unramified Brauer groups. J. Algebra 372 (2012) 420-427.
[23] P. Moravec, Unramified Brauer groups and isoclinism. ARS Math. Contemp. 7 (2014) 337-340.
[24] D. Saltman, Generic Galois extensions and problems in field theory. Adv. Math. 43 (1982) 250-283.
[25] D. Saltman, Noether’s problem over an algebraically closed field. Invent. Math. 77 (1984) 71-84.
[26] R. Swan, Invariant rational functions and a problem of Steenrod. Invent. Math. 7 (1969) 148-158.
[27] R. Swan, Noether’s problem in Galois theory. In: Emmy Noether in Bryn Mawr (B. Srinivasan and J. Sally eds.), Springer-Verlag, Berlin (1983).

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