On the Problem of Time(s) in Quantum Mechanics and Quantum Gravity: recent integrating developments and outlook

M. Bauer* and C.A. Aguillón**
*Instituto de Física, **Instituto de Ciencias Nucleares,
Universidad Nacional Autónoma de México
Ciudad Universitaria, CP 04510, México, CDMX, MEXICO

Abstract

Canonical quantization applied to closed systems leads to static equations, the Wheeler-deWitt equation in Quantum Gravity and the time independent Schrödinger equation in Quantum Mechanics. How to restore time is the Problem of Time(s). Integrating developments are: a) entanglement of a microscopic system with its classical environment accords it a time evolution description, the time dependent Schrödinger equation, where \( t \) is the laboratory time measured by clocks; b) canonical quantization of Special Relativity yields both the Dirac Hamiltonian and a self adjoint "time" operator, restoring to position and time the equivalent footing accorded to energy and momentum in Relativistic Quantum Mechanics. It introduces an intrinsic time property \( \tau \) associated with the mass of the system, and a basis additional to the usual configuration, momentum and energy basis. As a generator of momentum displacements and consequently of energy, it invalidates Pauli’s objection to the existence of a time operator. It furthermore complies with the requirements to condition the other observables in the conditional interpretation of QG.

As Pauli’s objection figures explicit or implicitly in most current developments of QM and QG, its invalidation opens to research the effect of this new two times perspective on such developments.

"If you are receptive and humble, mathematics will lead you by the hand"

“One must be prepared to follow up the consequences of theory, and feel that one just has to accept the consequences no matter where they lead”

P. A. M. Dirac
1 Introduction

Canonical quantization of the Hamilton-Jacobi formulation applied to closed systems leads to static equations, the Wheeler-deWitt equation in Quantum Gravity (QG)\(^\text{1,2}\) and the time independent Schrödinger equation (TISE) in Quantum Mechanics (QM). The time dependent equation (TDSE), as presented originally by Schrödinger in 1926\(^\text{3}\), is the result of an educated replication of Hamilton’s optical-mechanical analogy rather than a formal derivation.

How to introduce time in QM, an object of research for decades\(^\text{4}\), needed to deal with Pauli’s objection to the existence of a time operator in QM\(^\text{5}\). As a consequence QM fails to treat time and space on an equivalent footing accorded by Special Relativity (SR) to momentum and energy. In QM time appears as a parameter, not as a dynamical variable represented by a self-adjoint operator. It is a c-number, following Dirac’s designation\(^\text{6}\) resulting in the decades discussion of the existence and meaning of a time operator, and of a time energy uncertainty relation\(^\text{4,7,8,10}\). Within the time quantities considered one finds instantaneous values and intervals, e.g., parametric (clock) time, tunneling times, decay times, dwell times, delay times, arrival times or jump times. To quote the introduction in Ref.3: “In fact, the standard recipe to link the observables and the formalism does not seem to apply, at least in an obvious manner, to time observables”. This is the Problem of Time in QM.

Quantum Gravity (QG), the quantization of General Relativity (GR), is still an unsolved problem in physics. One of the difficulties arises from accepting that time in quantum mechanics is a parameter, whereas in general relativity matter determines the structure of spacetime and time and space acquire a dynamical nature. Thus “time” in QM and “time” in GR are seen as mutually incompatible notions. This is the Problem of Time in QG\(^\text{11,12,13,14,15,16,17}\).

In the present paper a clarifying and integrating picture is shown to arise from recent developments, namely:

a) the time independent Schrödinger describing the (approximately) closed system of joint microscopic quantum element and classical environment, gives rise to the corresponding time dependent Schrödinger equation (TDSE) for the quantum system in a disentangling approximation where “the motion of the environment provides a time derivative which monitors the development of the quantum system”\(^\text{2}\). Consequently “time enters quantum mechanics only when

\(^{1^}\) In the older literature on quantum mechanics, we often find the operator equation [\(H, T\)] = -\(i\)\(\mathbf{I}\) ...... It is generally not possible, however, to construct a Hermitian operator (e.g. as function of \(p\) and \(q\)) which satisfies this equation. This is so because, from the C.R. written above, it follows that \(H\) possesses continuously all eigenvalues from \(-\infty\) to \(+\infty\) , whereas on the other hand, discrete eigenvalues of \(H\) can be present. We, therefore, conclude that the introduction of an operator \(t\) is basically forbidden and the time \(t\) must necessarily be considered as an ordinary number (“c-number”) in Quantum Mechanics”. As opposed to this, operators are usually called q-numbers.

\(^{2}\)In this respect it is important to avoid, following Hilgevoord\(^\text{18}\), the confusion between the space coordinates \((x, y, z)\) of a system of reference and the quantum mechanical position operator \(\hat{\mathbf{r}} = (\hat{x}, \hat{y}, \hat{z})\) whose expectation value gives the time evolution of the position of a system described by a certain state vector \(|\Psi(t)\rangle\). In the same way, a distinction should
an external force on the quantum system is considered classically \[19, 20\].

b) the canonical quantization of Special Relativity, together with Born’s reciprocity principle\[21, 22, 23\], yields both the Dirac Hamiltonian and a self adjoint “time” operator that restores to position and time the equivalent footing accorded to energy and momentum in Relativistic Quantum Mechanics (RQM). It introduces an intrinsic time property \(\tau_0\) of the system, provides a basis (and a representation) additional to the usual configuration, momentum and energy basis and invalidates or circumvents Pauli’s objection \[24, 25, 26, 27\]. It furthermore complies with the requirements to condition the other observables in the conditional interpretation of QG \[28\].

After an abbreviated presentation of such developments, it is finally noted the need to introduce this new perspective in the extensive developments to date of both QM and QG where the non existence of a time operator seems to have been taken for granted, explicit or implicitly.

2 The emergence of the TDSE \[19, 20\]

The Hamilton-Jacobi formulation of classical mechanics is based on assuming that there is a (sufficient) isolation of the system. Canonical quantization results in the constraint equation:

\[ \hat{H} |\Psi\rangle = 0 \] (1)

Introducing the free motion Hamiltonian, Briggs and Rost have shown that the TDSE emerges from the TISE as a classical-quantum equation describing the relational evolution of a quantum system \(S\) entangled with an environment \(E\) that is large enough to be treated classically. Quantum system and environment together are assumed to constitute a (sufficiently) closed system (the only one completely closed is the universe) to apply a Hamilton-Jacobi formulation. The wave function satisfying the TISE is written as:

\[ \Psi(x, R) = \chi_E(R)\psi_S(x, R). \] (2)

where \(\{x\}\) and \(\{R\}\) refer to the quantum system and environment coordinates respectively. Projection with the wave functions \(\psi_S(x, R)\) and \(\chi_E(R)\) leads to

3\(^\text{rd}\) There is strong formal evidence for the hypothesis, which I have called the principle of reciprocity, that the laws of nature are symmetrical with regard to space-time and momentum-energy, or more precisely, that they are invariant under the transformation \(\hat{x}_k \Rightarrow \hat{p}_k\); \(\hat{p}_k \Rightarrow -\hat{x}_k\). The most obvious indications are these: The canonical equations of classical mechanics \(\dot{x}^k = \partial H/\partial p^k\); \(\dot{p}^k = -\partial H/\partial x^k\) are indeed invariant under the transformation, if only the first 3 components of the 4-vectors \(x^k\) and \(p^k\) are considered. These equations hold also in the matrix or operator form of quantum mechanics. The commutation rules \([\hat{x}^k, \hat{p}_l] = i\hbar \delta^k_l\) and the components of the angular momentum, \(m_{kl} = x_l p_k - p_l x_k\), show the same invariance, for all 4 components \[21\]. Although Born did not succeed in his intended application, the reciprocity principle is currently subject of renewed interest \[22, 23\].
coupled TISE for both system and environment. Disentangling approximations are based on assuming the influence of the environment on the large system is negligible so that the environment wave function can be taken as

$$\chi_E(R) = A(R)e^{iW/\hbar}$$  \hspace{1cm} (3)

where $W(R, E)$ is the (time-independent) action of the classical free motion Hamiltonian $H_E$. The leading coupling term in the system equation is of the form:

$$\frac{1}{M} \frac{\partial W}{\partial R} \frac{\partial}{\partial R} = P \frac{\partial}{\partial \mu} \frac{\partial}{\partial R} \frac{\partial}{\partial t} = \frac{\partial}{\partial t}$$  \hspace{1cm} (4)

This introduces the classical time dependence and transforms the TISE for the quantum system into a TDSE, where the time dependence is assigned to the state vector - the Schrödinger picture. The alternate Heisenberg picture arises from a unitary transformation that translates the time dependence to the operators describing the dynamical variables.

The extensive experimental confirmation of the TDSE clearly identifies this time as the laboratory time, thus part of the space–time frame of reference associated to an observer that uses clocks and measuring rods\cite{29}. The classical environment monitors a time evolution of the atomic system within the full static system. This mechanism has already been demonstrated experimentally\cite{30}.

As this is an extremely schematic presentation, the reader is referred to the original papers to follow the details of the approximations and their interesting discussion of the relation to QG.

3 \hspace{0.5cm} Time operator in RQM: the demise of Pauli’s objection\cite{24}

The constancy of the speed of light in vacuum for all inertial observers, denoted by $c$, introduces scalar invariant constraints in time and space, and in energy and momentum as follows:

$$p_\mu p^\mu = \eta^{\mu\nu} p_\mu p_\nu = p_0^2 - \mathbf{p}^2 = \pi^2 \hspace{0.5cm} ; \hspace{0.5cm} x_\mu x^\mu = \eta^{\mu\nu} x_\mu x_\nu = x_0^2 - \mathbf{r}^2 = \sigma^2$$  \hspace{1cm} (5)

where $\eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$. Using $c$, the invariants are defined as $\pi := m_0 c$, and similarly $\sigma := \tau_0 c$. As $m_0$ is identified with the rest mass in any rest frame independent of location, $\tau_0$ would be also an intrinsic property with dimension of time evident when the particle is found at the origin $\mathbf{r} = 0$ of any inertial frame independent of its velocity.

The canonical quantization consists in substituting the Hamilton-Jacobi dynamical variables by self adjoint operators acting on normalized state vectors in a Hilbert space representing the system and subject to constraints as follows:

$$\{\hat{p}_\mu \hat{p}^\mu - (m_0 c)^2\} |\Psi\rangle = 0 \hspace{0.5cm} ; \hspace{0.5cm} \{x_\mu x^\mu - (\tau_0 c)^2\} |\Psi\rangle = 0$$  \hspace{1cm} (6)

Factorization and ordering results in the linear equations:

$$[\rho^\nu \hat{p}_\nu - m_0 c] |\Psi\rangle = 0 \hspace{0.5cm} ; \hspace{0.5cm} [\rho^\nu \hat{x}_\nu - \tau_0 c] |\Psi\rangle = 0$$  \hspace{1cm} (7)
provided that:

\[ [p_\mu, p_\nu] = 0 \ ; \ \{ \rho^\mu \rho^\nu + \rho^\nu \rho^\mu \} = 2\eta^{\mu\nu}I_4 \ ; \ [x_\mu, x_\nu] = 0 \] (8)

where \( I_4 \) is the 4 × 4 identity matrix. Thus the coefficients \( \rho^\mu \) obey a Clifford algebra and are represented by matrices[6, 31, 32, 33].

Multiplying by \( c \rho_0 \) and defining \( \rho_0 := \beta, \rho_i := \alpha_i \) one obtains :

\[ c\hat{p}_0 |\Psi\rangle = \{ c\alpha \cdot \hat{p} + \beta m_0 c^2 \} |\Psi\rangle \ ; \ (\hat{x}_0/c) |\Psi\rangle = \{ \alpha \cdot \hat{r} / c + \beta \tau_0 \} |\Psi\rangle \] (9)

that exhibits the Dirac Hamiltonian \( \hat{H}_D := c\alpha \cdot \hat{p} + \beta m_0 c^2 \) (one recognizes here the procedure followed by Dirac to obtain a first order linear equation in energy and momentum that agrees with the second order one resulting from the energy momentum constraint) and in the same way the time operator \( \hat{T} := \alpha \cdot \hat{r} / c + \beta \tau_0 \), introduced earlier by analogy to the Dirac Hamiltonian[25].

Finally, the antisymmetrized scalar product of the time space and energy momentum quadrivectors \( \hat{O}^- = \eta_{\mu\nu}[x_\mu, p_\nu] \) is the Lorentz invariant that satisfies Born’s reciprocity principle. Upon quantization it introduces the constraint:

\[ \hat{O}^- |\Psi\rangle = \eta_{\mu\nu}[\hat{x}_\mu, \hat{p}_\nu] |\Psi\rangle = \{ \{ \hat{x}_0, \hat{p}_0 \} - \delta_{ij} [x_i, p_j] \} |\Psi\rangle = 0 \] (10)

As \( (\hat{O}^-)^\dagger = -\hat{O}^- \), it is a purely imaginary constant. The constraint can then be satisfied by the Planck constant \( \hbar \) multiplied by any imaginary constant. One can choose:

\[ [\hat{x}_0, \hat{p}_0] = i3\hbar \quad [x_i, p_j] = i\hbar \delta_{ij} \] (11)

where \( \hbar \) is the reduced Planck constant. This completes the commutation relations between space and momentum operators from which one derives: a) the \{x_i\} and \{p_i\} basis are continuous from \(-\infty\) to \(+\infty\); b) the representations of the space and momentum operators in these basis; c) the reciprocal Fourier transform relation between the state vector representations; and d) the existence of the Heisenberg uncertainty relation between space and momentum expectation values. This unified derivation is unfortunately absent in most QM textbooks where some of the elements are presented as independent anzats.

Most remarkable is that the over the years much debated question of the introduction of complex numbers in QM, is here a derived consequence of Born’s reciprocity principle[34, 35].

What about Pauli’s objection? The time operator being self adjoint is the generator of a unitary transformation (Stone’s theorem[37]):

\[ U_T = \exp(-i\delta \varepsilon \hat{T} / \hbar) \approx \exp(-i\delta \varepsilon \alpha \cdot \hat{r} / \hbar) \exp(-i\delta \varepsilon \beta \tau_0 / \hbar) \] (12)

where the infinitesimal \( \delta \varepsilon \) is real. In the momentum representation where \( \hat{r} \Rightarrow i\hbar \nabla_p \), this yields a momentum displacement \( \delta p = (\delta \varepsilon / c^2)\alpha = (\delta \varepsilon / c^2)d\hat{r} / dt \) equal to a relativistic mass increase multiplied by the relativistic velocity (for a wave packet \( \langle d\hat{r} / dt \rangle = v_{gp} \), the group velocity[31]), and a phase shift \( \delta \phi = \beta \delta \varepsilon \tau_0 / \hbar \). Thus \( \hat{T} \) acts on the momentum space where there is no discontinuity, and not on the energy space as assumed in Pauli’s argument, but the change
in momentum entails a change in energy in both energy branches, as \( E(p) \Rightarrow E(p + mv_{gp}) \), circumventing Pauli’s objection.

Equivalently the Dirac Hamiltonian is the generator of a unitary transformation

\[
U_{H_D} = \exp(i\delta t \hat{H}_D/\hbar) \approx \exp(i\delta t \alpha \hat{\mathbf{p}}/\hbar) \exp(i\delta t \beta m_0 c^2/\hbar)
\]

for \( \delta t \) an infinitesimal time displacement. In the configuration representation where \( \hat{\mathbf{p}} \Rightarrow -i\hbar \nabla_r \), this yields a position displacement \( \delta \mathbf{r} = (\delta t)c\alpha \) and a phase shift \( \delta \phi = \beta (\delta t)m_0 c^2/\hbar \). For a wave packet, \( \Psi(r) \Rightarrow \Psi(r + v_{gp}\delta t) \).

The eigenvalues \( \tau(r) = \pm \sqrt{(r/c)^2 + \tau_0} \) of the time operator \( \hat{T} := \alpha \hat{\mathbf{r}}/c + \beta \tau_0 \) exhibit a positive and a negative time branches separated by a \( 2\tau_0 \) gap in the same way that the energy spectrum \( e(p) = \pm \sqrt{(cp)^2 + m_0 c^2} \) of \( \hat{H}_D := c\alpha \hat{\mathbf{p}} + \beta m_0 c^2 \) is composed by a positive and a negative energy branches separated by a \( 2m_0 c^2 \) gap. As \( \tau_0 = h/m_0 c^2 \), the de Broglie period, these gaps are complementary: to a small energy gap corresponds a large time gap, and vice versa. The Heisenberg picture relates the time operator eigenvalues to the laboratory time as:

\[
\frac{d\hat{T}}{dt} = \frac{1}{i\hbar} [\hat{T}, \hat{H}_D] = \{I + 2\beta K\} + \frac{2\beta}{\hbar} \{\tau_0 \hat{H}_D - m_0 c^2 \hat{T}\}
\]

where \( K = \beta (2s/\hbar^2 + 1) \) is a constant of motion. Integration yields an oscillation (Zitterbewegung) about a linear dependence, as occurs with \( \hat{\mathbf{r}}(t) \).

To be noted finally is that the association of an intrinsic time property with mass has been already exhibited experimentally.

### 4 The time operator and the conditional interpretation of time in QG

The Wheeler-deWitt (WdW) equation results from the canonical quantization of Einstein’s GR equations for the Universe, the only truly closed system. Notwithstanding that the procedure involves a foliation of spacetime into three-dimension spacelike hypersurfaces and a one-dimension timelike vector that may characterize the foliation, no time variable appears in the equation. The problem of time is that there is no time, leaving indefinite the choice of foliation among other problems. The WdW equation predicts a static universe.

Page and Wooters advanced the idea that evolution from the point of view of an internal observer can be introduced through conditional probabilities between two of the system observables, the continuum spectrum of one of them serving as the ”internal time parameter” for the other. The conditional probability between projection operators \( \hat{B} \) and \( \hat{C} \) that the observation of \( \hat{B} \) is subject to the observation of \( \hat{C} \) is given by:

\[
P(B \mid C) = \frac{\left\langle \Psi \mid \hat{C} \hat{B} \hat{C} \mid \Psi \right\rangle}{\left\langle \Psi \mid \hat{C} \mid \Psi \right\rangle}
\]
(where $|\Psi\rangle$ is a solution of the constraint $\hat{H} |\Psi\rangle = 0$). The operator $\hat{C}$ must represent an observable with a continuum spectrum, and must not commute with $\hat{H}$ as otherwise it would be a constant of motion \cite{13}. It is clear that the intrinsic time operator $\hat{T}$ introduced satisfies all the necessary requirements, namely:

i) it is self adjoint and can represent an observable;

ii) it is a timelike operator, as it is given in terms of the position operator $\hat{r}(t)$ whose expectation value represents the worldline of the system;

iii) it does not commute with the Hamiltonian (see Eq.14), so it evolves with time. In the Heisenberg picture one obtains:

$$\hat{T}(t) = \int_0^t (1/\hbar) [\hat{T}, \hat{H}_D] = \hat{T}(0) + \{1 + 2\beta \hat{K}_D\}t + \text{oscillating terms} \quad (16)$$

where the oscillation occurs about a linear dependence with the laboratory time, and is present when both positive and negative intrinsic time components are found in the wave packet. This is in entire analogy with the evolution of the position operator \cite{6, 31, 32}, namely:

$$\hat{r}(t) = \hat{r}(0) + \{c^2 \hat{p}/\hbar D\}t + \text{oscillating terms}(\text{Zitterbewegung}) \quad (17)$$

iv) its spectrum is a single valued continuum as a function of $r$ in either positive or negative time branch;

v) it follows that one can construct a free wave packet $|\varphi_{t_0}\rangle = \int dt_\nu c(t_\nu) |t_\nu\rangle$ such that $\langle \varphi_{t_0} | \hat{T} | \varphi_{t_0}\rangle = T(0)$. One can then set $\hat{C}_0 = |\varphi_0\rangle \langle \varphi_0|$, and $\hat{B} = |b\rangle \langle b|$ for any operator that represents an observable and commutes with $\hat{C}$. Then Eq.10 yields:

$$P\{t_\nu, b\} = \frac{\sum_{\nu} |\Psi(t_\nu, b)|^2}{\sum_{\nu} \int dt'_\nu \int db |\Psi(t'_\nu, b)|^2} \quad (18)$$

Consequently $P(t_\nu, b)$ is interpreted as the joint probability of finding the system with spin $\nu$, eigenvalue $b$ and intrinsic time $t_\nu = (r/c)^2 + \tau_0^2$. Contrary to the reservations raised by Unruh and Wald \cite{39}, it can be asserted that this time operator constitutes an additional observable that invalidates Pauli’s objection and provides a time reference by conditionning the other dynamical variables.

## 5 Conclusion and Outlook

Two times are involved in the problem of time in QM and QG, the time present in the TDSE and an intrinsic time property associated with the system’s mass, which is an eigenvalue of a self adjoint time operator that restores to time and
space that equal footing of energy and momentum accorded by SR. They are not unrelated, as shown by Eq.14, providing an integrated perspective. The continued experimental confirmation of the predictions and applications of the TDSE support the identification of the first as the time measured by laboratory clocks[29]. As for the existence of the intrinsic time property and the representation associated to the dynamical time operator, some support is found in the experimental association of time with mass[34], in electron tunneling and channeling[42, 43, 44] and in simulations of the Dirac equation[45, 46, 47, 48], but additional evidence should become a research subject. To quote: "Physics does not depend on the choice of basis, but which is the most convenient choice depends on the physics"[49].

Indeed, although the presentation in this paper is at a basic level, the fundamental point resides in the invalidation of Pauli’s objection to the existence of a time operator, which figures explicit or implicitly in most relevant work on QM and QG. Taking into account that agreement of QG with relativistic quantum field theory is to be expected locally[50], to be investigated is its inclusion in Quantum Field Theory[51] and in the extensive developments of QG such as Loop Quantum Gravity and String Theory[15]; and whether it contributes to the solution of the diverse facets of the problem of time in QG[11], e.g., it may have relevance in the reduction of the multiple foliation problem of the canonical formulation of QG, providing a specific choice for the time variable[13], with its one to one correspondence with the timelike worldline r(t).

References

[1] DeWitt, B. S., "Quantum Theory of Gravity I: The Canonical Theory". Phys. Rev. 160 (5): 1113–1148 (1957)
[2] J. A. Wheeler, in Battelle Rencontres: 1967 Lectures on Mathematical Physics (Benjamin, New York, 1968).
[3] E. Schrödinger, Ann. Phys. (Leipzig) 79, 361 (1926); Ann. Phys. (Leipzig) 81, 109 (1926).
[4] J.G. Muga, R. Sala Mayato, I.L. Egusquiza (eds.), "Time in Quantum Mechanics -Vol. 1", Lect.Notes.Phys. 734, Berlin Springer (2008); J.G. Muga, A. Ruschaupt, A. del Campo (eds.), "Time in Quantum Mechanics -Vol. 2", Springer, Berlin Heidelberg (2009)
[5] Pauli, W., "The general principles of quantum mechanics", Springer-Verlag, Berlin Heidelberg New York (1980) p.63
[6] Dirac, P.A.M., “The principles of quantum mechanics” (4th ed.), Oxford, Clarendon Press (1958)
[7] Busch, P., "The time-energy uncertainty relation", in J.G. Muga, R. Sala-Mayato, I.L. Egusquiza (eds.) "Time in Quantum Mechanics-Vol.1", Berlin Springer (2008)

[8] Busch, P., T. Heinonen and P. Lahti, "Heisenberg’s uncertainty principle", Phys.Rep. 452, 155 (2007)

[9] M. Bauer, P.A. Mello, “On the lifetime-width relation for a decaying state and the uncertainty principle”, Proc.Nat.Acad.Sci.USA 73, 283-285 - sponsored by E. Wigner- (1976)

[10] Bauer, M. and P.A. Mello, "The time-energy uncertainty relation", Ann.Phys. 111, 38-60 (1978)

[11] Anderson E., "The Problem of Time - Quantum Mechanics Versus General Relativity", Springer International Publishing (2017)

[12] Isham, C.J., "Canonical Quantum Gravity and the Problem of Time", arXiv:gr-qc/9210011v1, (1992); "Prima Facie Questions in Quantum Gravity", arXiv:gr-qc/9310031v1 (1993)

[13] Kuchar, K.V., "Canonical quantum gravity", arXiv:gr-qc/9304012v1 (1993); "Time and interpretations of quantum gravity", Int.J.Mod.Phys. D 20, (2011)

[14] Butterfield, J. and C.J. Isham, "On the Emergence of Time in Quantum Gravity", in The Arguments of Time, ed. J. Butterfield, Oxford University Press (1999)

[15] C. Kiefer, "Concept of Time in Canonical Quantum Theory and String Theory", J.Phys. Conference Series 174, 012021 (2009)

[16] A. Ashtekar, "Gravity and the quantum", New J.Phys. 7, 198 (2005); "The winding road to quantum gravity", Curr.Sci. 89, No.12 (2005)

[17] Smolin, L., "How far are we from the quantum theory of relativity", arXiv:0303185v2 (2003)

[18] J. Hilgevoord, "Time in quantum mechanics: a story of confusion", Studies in History and Philosophy of Modern Physics 36, 29-60 (2005)

[19] J.S. Briggs and J.M. Rost, "Time dependence in quantum mechanics", Eur.Phys.J. 10, 311 (2000); "On the Derivation of the Time-dependent Schrödinger Equation", Found.Phys. 31, 694 (2001)

[20] J.S. Briggs, "Equivalent emergence of time dependence in classical and quantum mechanics", Phys.Rev. A 052119 (2015)

[21] Born, M., "A suggestion for unifying quantum theory and relativity", Proceedings of the Royal Society London A , 165, 291 (1938); "Reciprocity theory of elementary particles", Rev.Mod.Phys. 21, 463 (1949)
[22] Freidel, L., R.G. Leigh and D. Minic, "Born reciprocity in string theory and the nature of spacetime", Phys.Lett. B 730, 302-304 (2014)

[23] Morgan, S., "A Modern Approach to Born Reciprocity", Thesis, University of Tasmania, (2010); Govaerts, J., P.D. Jarvis, S. Morgan and S.G. Low, "World-line Quantization of a Reciprocally Invariant System", Journal of Physics A: Mathematical and Theoretical 40, 12095–12111 (2007)

[24] C.A. Aguillón, M Bauer and G.E. García, "Time and energy operators in the canonical quantization of special relativity", Eur.J.Phys., 41, No.3, 035601 (2020)

[25] M. Bauer, "A dynamical time operator in Dirac's relativistic quantum mechanics", Int.J.Mod.Phys. 29, 14500365 (2014)

[26] M. Bauer, "On the problem of time in quantum mechanics", Eur.J.Phys. 38, 035402 (2017)

[27] M. Bauer, “A Time Operator in Quantum Mechanics”, Ann. Phys. (N.Y.) 150 pp.1-21 (1983)

[28] Bauer, M., C.A. Aguillón and G.E. García, "Conditional interpretation of time in quantum gravity and a time operator in relativistic quantum mechanics", Int.J.Mod.Phys. A 35, 2050114-1 (2020)

[29] A. Einstein, “Autobiographical notes”, p.31, in Albert Einstein: Philosopher-Scientist, Ed. Paul Arthur Schilpp, Cambridge University Press (1949)

[30] E. Moreva et al., "Time from quantum entanglement", Phys.Rev. A 89, 052122 (2014); "The time as an emergent property of quantum mechanics, a synthetic description of a first experimental approach", J.Phys.: Conference Series 626, 012019 (2015)

[31] B. Thaller, "The Dirac Equation", Springer-Verlag (1956)

[32] Greiner, W., “Relativistic Quantum Mechanics - Wave equations”, (3rd ed.) Springer, Berlin Heidelberg New York (2000)

[33] Messiah, A., "Quantum Mechanics", Vol.I, p. 442, North-Holland Publishing Company, Amsterdam, and John Wiley&Sons, New York London Sidney, 4th printing (1966)

[34] S.Y. Lan et al., "A clock directly linking time to a particle mass", Science 339 554–7 (2013)

[35] W.E. Baylis, "Why "i"?", Am.J.Phys. 60 (9), 788 (1992)

[36] R. Karam, "Why are complex numbers needed in quantum mechanics? Some answers for the introductory level", Am.J.Phys. 88, 39 (2020)
[37] Stone, M. H., "On one-parameter unitary groups in Hilbert Space", Annals of Mathematics 33 (3): 643–648 (1932)

[38] D.N. Page and W.K. Wotters, "Evolution without evolution: Dynamics described by stationary observables", Phys Rev D 27, 2885-2892 (1983)

[39] W.G. Unruh and R.M. Wald, “Time and the interpretation of canonical quantum gravity”, Phys.Rev. D 40, pp.2598-2614 (1989)

[40] Baylis, W.E., "De Broglie waves as an effect of clock desynchronization", Can.J.Phys. 85, 1317-1323 (2007)

[41] R. Gerritsma et al, "Quantum Simulation of the Dirac Equation", Nature 463, 68-71, (2010)

[42] M. Bauer, "de Broglie Clock, Electron Channeling and Time in Quantum Mechanics", Can.J.Phys. 97, 37-41 (2018)

[43] P. Catillon et al., “A Search for de Broglie Particle Internal Clock by means of Electron Channeling”, Found.Phys. 38, pp.659-664 (2008)

[44] G.R. Osche, “Electron channeling resonance and de Broglie’s internal clock”, Annales de la Fondation Louis de Broglie 36, pp.61-80 (2011)

[45] Bauer, M., "A time operator in the simulations of the Dirac equation", Int.J.Mod.Phys. A 34, 1950114 (2019)

[46] S. Kling et al.,"Atomic Bloch-Zener Oscillations and Stuckelberg Interferometry in Optical Lattices", Phylykhyovsk. Rev. Lett. 105, 215301 (2010)

[47] J. LeBlanc et al, "Direct observation of zitterbewegung in a Bose-Einstein condensate", New J. Phys. 15, 073011 (2013)

[48] F. Dreisow et al., "Classical Simulation of Relativistic Zitterbewegung in Photonic Lattices", Phys.Rev.Lett. 105, 143902 (2010)

[49] A. Zee, "Quantum Field Theory in a Nutshell", Princeton University Press (2010)

[50] Bonder, Y., Ch. Ghryssmalakos and D. Sudarsky, "Extracting Geometry from Quantum Spacetime: Obstacles Down the Road", Found.Phys. 48, 1038 (2018)

[51] M. Bauer and C.A. Aguillón, "Second quantization of time and energy in Relativistic Quantum Mechanics", arXiv:submit/3584811 [quant-ph] 1 Feb 2021