Lie-algebraic approach to the theory of polynomial solutions.
I. Ordinary differential equations and finite-difference equations in one variable

(submitted to Comm.Math.Phys.)

A. Turbiner

CPT, CNRS-Luminy, Marseille, F-13288, FRANCE

ABSTRACT

A classification of ordinary differential equations and finite-difference equations in one variable having polynomial solutions (the generalized Bochner problem) is given. The method used is based on the spectral problem for a polynomial element of the universal enveloping algebra of \(sl_2(\mathbb{R})\) (for differential equations) or \(sl_2(\mathbb{R})_q\) (for finite-difference equations) in the ”projectivized” representation possessing an invariant subspace. Connection to the recently-discovered quasi-exactly-solvable problems is discussed.

\footnote{On leave of absence from: Institute for Theoretical and Experimental Physics, Moscow 117259, Russia
E-mail: TURBINER@CERNVM or TURBINER@VXCERN.CERN.CH}
S. Bochner (1929) asked about a classification of differential equations

\[ T \varphi = \epsilon \varphi \]  

(0)

where \( T \) is a linear differential operator of \( k \)-th order in one real variable \( x \in \mathbb{R} \) and \( \epsilon \) is the spectral parameter, having an infinite sequence of orthogonal polynomial solutions (see [1]).

**Definition.** Let us give the name of the *generalized Bochner problem* to the problem of classification of the differential equations (0) having \((n + 1)\) eigenfunctions in the form of a polynomial of the order not higher than \( n \).

In Ref. 2 it has been formulated a general method for generating eigenvalue problems for linear differential operators, linear matrix differential operators and linear finite-difference operators in one and several variables possessing the polynomial solutions. The method is based on considering the eigenvalue problem for the representation of a polynomial element of the universal enveloping algebra of the Lie algebra in a finite-dimensional, ’projectivized’ representation of this Lie algebra [2]. Below it is shown that this method provides both necessary and sufficient conditions for the existence of polynomial solutions of linear differential equations and a certain class of finite-difference equations in one variable.

1. **Ordinary differential equations.**

Consider the space of all polynomials of order \( n \)

\[ \mathcal{P}_n = \langle 1, x, x^2, \ldots, x^n \rangle, \]  

(1)

where \( n \) is a non-negative integer and \( x \in \mathbb{R} \).

**Definition.** Let us name a linear differential operator of the \( k \)-th order, \( T_k \) **quasi-exactly-solvable**, if it preserves the space \( \mathcal{P}_n \). Correspondingly, the operator \( E_k \), which preserves the infinite flag \( \mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \ldots \subset \mathcal{P}_n \subset \ldots \) of spaces of all polynomials, is named **exactly-solvable**.
LEMMA 1. (i) Suppose $n > (k - 1)$. Any quasi-exactly-solvable operator of $k$-th order $T_k$, can be represented by a $k$-th degree polynomial of the operators

$$J^+ = x^2 \partial_x - nx,$$

$$J^0 = x \partial_x - \frac{n}{2},$$

$$J^- = \partial_x,$$

(2)

(the operators (2) obey the $sl_2(\mathbb{R})$ commutation relations \[\text{[ ]}\]). If $n \leq (k - 1)$, the part of the quasi-exactly-solvable operator $T_k$ containing derivatives up to the order $n$ can be represented by a $n$-th degree polynomial in the generators (2).

(ii) Inversely, any polynomial in (2) is quasi-exactly solvable.

(iii) Among quasi-exactly-solvable operators there exist exactly-solvable operators $E_k \subset T_k$.

Comment 1. If we define the universal enveloping algebra $U_g$ of a Lie algebra $g$ as the algebra of all polynomials in generators, then $T_k$ at $k < n + 1$ is simply an element of the universal enveloping algebra $U_{sl_2(\mathbb{R})}$ of the algebra $sl_2(\mathbb{R})$ taken in representation (2). If $k \geq n + 1$, then $T_k$ is represented as an element of $U_{sl_2(\mathbb{R})}$ plus $B \frac{d^{n+1}}{dx^{n+1}}$, where $B$ is any linear differential operator of the order not higher than $(k - n - 1)$.

Now let us proceed to the proof of the Lemma 1.

Proof.

Part I. One starts from the statement (i). Suppose $n > (k - 1)$.

Step 1. It is easy to show that the coefficient functions standing before derivatives in any quasi-exactly-solvable operator $T_k$ should be polynomials.

\[\text{[The representation (2) is one of the 'projectivized' representations (see \[\text{[ ]}\]).}\]
\textbf{Step 2.} Without loss of generality, any operator $T_k$ can be represented as a sum of homogeneous operators:

$$T_k = \sum_{i=0}^{k} a_{k,i}^{(n_i)}(x) \partial_x^i$$

(3)

where $n_i$ indicates the degree of polynomial $a_{k,i}^{(n_i)}(x)$. Suppose, the maximal degree of homogeneity is $M = n_i - i > 0$. Now let us rewrite $T_k$ as sum of operators of fixed degree of homogeneity $m$:

$$T_k = \sum_{m=-k}^{M} T_k^{(m)}, \quad T_k^{(m)} \equiv \sum_{i=0}^{k} A_{k,i}^{(m)} x^{m+i} \partial_x^i$$

(4)

(if $m + i < 0$, the corresponding $A_{k,i}^{(m)} = 0$). Evidently, if $m \leq 0$, then $T_k^{(m)} : \mathcal{P}_n \mapsto \mathcal{P}_n$. Consider the operators $T_k^{(m)}$ at $m > 0$. Their action on monomials is

$$T_k^{(m)} : x^\ell \mapsto x^{\ell + m}$$

(5)

In order to preserve the space (1), the condition

$$\ell + m \leq n, \quad \ell = 0, 1, 2, \ldots n$$

(6)

should be fulfilled. Generically, $T_k^{(m)}$ is characterized by $(k + 1)$ coefficients. The condition (4) leads to the system of $m$ linear homogeneous equations for $(k + 1)$ unknown coefficients $A_{k,i}^{(m)}$. Hence

$$m \leq k$$

(7)

and the maximal degree of homogeneity of a quasi-exactly-solvable operator $T_k$ is equal to $k$.

\textbf{Step 3.} We will prove (i) inductively.

(1). Take $T_0$. Evidently, $T_0 = const$ or, in other words, a zero-order polynomial in generators (2).
(2). Take $T_1$. As it follows from (4),(7)

$$T_1 = \sum_{m=-1}^{1} T_1^{(m)} \equiv (A^{(1)}_{1,0} x + A^{(1)}_{1,1} x^2 \partial_x) + (A^{(0)}_{1,0} + A^{(0)}_{1,1} x \partial_x) + (A^{(-1)}_{1,1} \partial_x). \quad (8)$$

We only need to consider the operator $T_1^{(1)}$, since the others preserve (1) automatically. In this case, the above-mentioned system of linear equations contain only one equation:

$$A^{(1)}_{1,0} + n A^{(1)}_{1,1} = 0, \quad (9)$$

which implies immediately, that

$$T_1^{(1)} = A^{(1)}_{1,1} J^+. \quad (10)$$

Finally,

$$T_1^{(1)} = A^{(1)}_{1,1} J^+ + A^{(0)}_{1,1} J^0 + A^{(-1)}_{1,1} J^- + A^{(0)}_{1,0} + \frac{n}{2} A^{(0)}_{1,1} \quad (11)$$

(3). Now let us assume that a quasi-exactly-solvable operator $T_{k-1}$ is represented through generators (2).

Take $T_k$. In what follows from Step 2, the maximal degree of homogeneity of $T_k$ is equal to $k$. First consider $T^{(k)}_k$. The corresponding system of linear equations on the coefficients $A^{(k)}_{k,i}, i = 0, 1, 2, \ldots k$ contains $k$ equations and hence has a non-trivial solution. This solution can be expressed through one coefficient, e.g. $A^{(k)}_{k,k}$. It is easy to check that obtained coefficients correspond precisely to $(J^+)^k$ (cf.(9),(10)) and hence

$$T^{(k)}_k = A^{(k)}_{k,k} (J^+)^k \quad (12)$$

It is obvious, that (12) preserves the space (1). The remainder of $T_k$ is simply the $k$-th order differential operator characterizing by degrees of homogeneity $-k, -k+1, \ldots, 0, 1, \ldots, k-1$. Now let us consider in the remainder the terms
containing the derivatives of the \( k \)-order. Apparently, they can be rewritten in the following forms:

\[
x^{2k-1} \partial^k = (J^+)^{k-1} J^0 + \alpha_1 x^{2k-2} \partial^{k-1} + \ldots
\]

\[
x^{2k-2} \partial^k = (J^+)^{k-1} J^- + \alpha_2 x^{2k-3} \partial^{k-1} + \ldots
\]

\[
x^{2k-3} \partial^k = (J^+)^{k-2} J^0 J^- + \alpha_3 x^{2k-4} \partial^{k-1} + \ldots
\]

\[
\vdots
\]

\[
\partial^k = (J^-)^k
\]

(dots in (13) imply terms with lower order derivatives). The representation (13) is unambiguous, since there are only \((2k + 1)\) different monomials of degree \( k \) in the generators \( J^a \) (if we take into account the commutation relations and the elements of the ideal generated by the Casimir operator).

Substituting (12),(13) into \( T_k \) in the form (3), one gets

\[
T_k = \sum_{k_+, k_0, k_- \geq 0} P_{k_+, k_0, k_-} (J_+^0, J^-) + T_{k-1}.
\]

The operator \( T_{k-1} \) can be represented through (2) by assumption. This ends the proof of the first part of (i).

Part II. Suppose \( n \leq k - 1 \). It is clear that the part of \( T_k \), containing derivatives of the orders \( (n + 1), (n + 2), \ldots, k \) annihilates the space (1) and these derivatives can keep any functions as coefficient functions. For the remainder of \( T_k \), containing derivatives up to the \( n \)-th order, the Part I of the proof holds. This concludes the proof of part (i) of the Lemma.

Part III. The part (ii) of the Lemma is evident. The part (iii) of the Lemma is also evident from the proof of part (i) of the Lemma, since a quasi-exactly-solvable operator, containing no operators of positive homogeneity, becomes exactly-solvable. □
Comment 2. If the space (1) is considered not in general position\footnote{not in general position'' means that, it is allowed to have some correlations between coefficients of original polynomial and an operator $T_k$. Otherwords, not any polynomial of degree not higher than $n$ is mapped to $\mathcal{P}_n$.}, there exist linear differential operators others than quasi-exactly-solvable ones, which map a certain polynomial of a degree $n$ to some other polynomial of the same degree.

Since $\mathfrak{sl}_2(\mathbb{R})$ is a graded algebra, let us introduce the grading of generators (2):

\[
\deg(J^+) = +1 \ , \ \deg(J^0) = 0 \ , \ \deg(J^-) = -1,
\]  

hence

\[
\deg[(J^+)^n(J^0)^{n_0}(J^-)^{n_-}] = n_+ - n_-. \tag{16}
\]

The grading allows to classify the operators $T_k$ in Lie-algebraic sense.

\textbf{Lemma 2.} A quasi-exactly-solvable operator $T_k \subset U_{\mathfrak{sl}_2(\mathbb{R})}$ has no terms of positive grading, iff it is an exactly-solvable operator.

Comment 3. It is easy to see that the grading is nothing but the homogeneity, which has been introduced in the proof of the Lemma 1 and the statement of this Lemma becomes obvious.

\textbf{Definition.} Let us call the operator $T(x)$ symmetric, if one can introduce the scalar product

\[
(f, g)_\rho = \int_R f(x)g(x)\rho(x)dx
\]

for positive $\rho > 0, \rho(x) \in S(\mathbb{R})$, such that

\[
(Tf, g)_\rho = (f, Tg)_\rho
\]

where the Schwartz space is defined as

\[
S(\mathbb{R}) = \{ \rho : R \to \mathbb{R}, | \rho^{(k)}(x) | \leq C_{kn}(1+|x|)^{-n}, \forall k, n \}.
\]
THEOREM 1. Let $n$ is non-negative integer. Take the eigenvalue problem for a linear differential operator of the $k$-th order in one variable

$$T_k \phi = \varepsilon \phi ,$$

(17)

where $T_k$ is symmetric. The problem (17) has $(n + 1)$ linear independent eigenfunctions in the form of polynomial in variable $x$ of the order not higher than $n$, iff $T_k$ is quasi-exactly-solvable. The problem (17) has an infinite sequence of polynomial eigenfunctions, iff the operator is exactly-solvable.

Comment 4. The ”if” part of the first statement is obvious. The ”only if” part is a direct corollary of Lemma 1.

This theorem gives a general classification of differential equations

$$\sum_{j=0}^{k} a_j(x) \phi^{(j)}(x) = \varepsilon \phi(x)$$

(18)

having at least one polynomial solution in $x$, thus resolving the generalized Bochner problem. The coefficient functions $a_j(x)$ must have the form

$$a_j(x) = \sum_{i=0}^{k+j} a_{j,i} x^i$$

(19)

The explicit expressions (19) for coefficient function in (18) are obtained by the substitution (2) into a general, $k$-th degree polynomial element of the universal enveloping algebra $U_{sl_2(R)}$. Thus the coefficients $a_{j,i}$ can be expressed through the coefficients of the $k$-th degree polynomial element of the universal enveloping algebra $U_{sl_2(R)}$. The number of free parameters of the polynomial solutions is defined by the number of parameters characterizing a general $k$-th degree polynomial element of the universal enveloping algebra $U_{sl_2(R)}$. Rather straightforward calculation leads to the following formula

$$\text{par}(T_k) = (k + 1)^2$$

(20)

4 Counting free parameters, one should introduce a certain ordering of generators to avoid double counting because of commutation relations. Also the quadratic Casimir operator and the double-sided ideal generated by it should not be taken into account.
where we denoted the number of free parameters of operator $T_k$ by the symbol $\text{par}(T_k)$. For the case of an infinite sequence of polynomial solutions the expression (19) simplifies to

$$a_j(x) = \sum_{i=0}^{j} a_{j,i} x^i$$

(21)

in agreement to the results of H.L.Krall’s classification theorem \cite{3}(see also \cite{1}). In this case the number of free parameters is equal to

$$\text{par}(E_k) = \frac{(k+1)(k+2)}{2}$$

(22)

In present approach Krall’s theorem is simply a description of differential operators of $k-$th order in one variable preserving a finite flag $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \ldots \subset \mathcal{P}_k$ of spaces of polynomials. One can easily show that the preservation of such a set of polynomial spaces implies the preservation of an infinite flag of such spaces.

One may ask a more general question: which non-degenerate linear differential operators have finite-dimensional invariant sub-space of the form

$$\langle \alpha(x), \alpha(x)\beta(x), \ldots, \alpha(x)\beta(x)^n \rangle,$$

(23)

where $\alpha(x)$ is a function and $\beta(x)$ is a diffeomorphism of the line. Such operators are obtained from the ones of the Theorem 1 by the change of variable $x \mapsto \beta(x)$ and the ”gauge” transformation $\varphi(x) \mapsto \alpha(x)\varphi(x)$).

Let us consider the set of second order differential equations (17), which can possesses polynomial solutions. From Theorem 1 it follows that the corresponding differential operator must be quasi-exactly-solvable and can be represented as

$$T_2 = c_{++} J^+ J^+ + c_{+0} J^+ J^0 + c_{+-} J^+ J^- + c_0 J^0 J^- + c_{--} J^- J^- +$$

$$c_+ J^+ + c_0 J^0 + c_- J^- + c,$$

(24)
where $c_{\alpha\beta}, c_\alpha, c \in \mathbb{R}$. The number of free parameters is $\text{par}(T_2) = 9$. Under the condition $c_{++} = c_{+0} = c_+ = 0$, the operator $T_2$ becomes exactly-solvable (see Lemma 2) and the number of free parameters is $\text{par}(E_2) = 6$.

**Lemma 3.** If the operator (21) is such that

$$c_{++} = 0 \quad \text{and} \quad c_+ = (\frac{n}{2} - m)c_{+0}, \text{ at } m = 0, 1, 2, \ldots$$

then the operator $T_2$ preserves both $\mathcal{P}_n$ and $\mathcal{P}_m$. The number of free parameters is $\text{par}(T_2) = 7$.

**Proof.** By straightforward analysis.

In fact, the Lemma 3 means that $T_2(J^\alpha(n), c_{\alpha\beta}, c_\alpha)$ can be rewritten as $T_2(J^\alpha(m), c'_{\alpha\beta}, c'_\alpha)$. As a consequence of Lemma 3 and Theorem 1, among polynomial solutions of (17) there are polynomials of order $n$ and order $m$.

**Remark.** From the Lie-algebraic point of view Lemma 3 means the existence of representations of second-degree polynomials in the generators (2) possessing two invariant sub-spaces. In general, if $n$ is not a non-negative integer in (2) (correspondingly, (2) becomes infinite-dimensional), then among representations of $k$-th degree polynomials in the generators (2), lying in the universal enveloping algebra, there exist representations possessing $0, 1, 2, \ldots, k$ invariant sub-spaces. Also this properly implies existence of representations of the polynomial elements of the universal enveloping algebra, which can be obtained starting from different representations of the original algebra. Even starting from an infinite-dimensional representation of the original algebra, one can construct the elements of the universal enveloping algebra having finite-dimensional representation (e.g. the parameter $n$ in (25) is non-integer, however $T_2$ has the invariant sub-space of the dimension $(m + 1)$).

Substituting (2) into (24) and then into (17), we obtain

$$P_4(x)\partial_x^2\varphi(x) + P_3(x)\partial_x\varphi(x) + P_2(x)\varphi(x) = \varepsilon\varphi(x),$$

(26)
where the $P_j(x)$ are polynomials of $j$-th order with coefficients related to $c_{\alpha \beta}, c_\alpha$ and $n$ (see (9)). In general, problem (26) has $(n + 1)$ polynomial solutions. If $n = 1$, as a consequence of Lemma 1, a more general spectral problem than (26) arises

$$F_3(x)\partial_x^2 \varphi(x) + Q_2(x)\partial_x \varphi(x) + Q_1(x)\varphi(x) = \varepsilon \varphi(x), \quad (27)$$

possessing only two polynomial solutions of the form $(ax + b)$, where $F_3$ is an arbitrary real function of $x$ and $Q_j(x), j = 1, 2$ are polynomials of order $j$. For the case $n = 0$ (one polynomial solution) the spectral problem (11) becomes

$$F_2(x)\partial_x^2 \varphi(x) + F_1(x)\partial_x \varphi(x) + Q_0\varphi(x) = \varepsilon \varphi(x), \quad (28)$$

where $F_{2,1}(x)$ are arbitrary real functions of real $x$ and $Q_0$ is a real constant. After the transformation

$$t : \varphi \mapsto \varphi(x(z))e^{A(z)}, \quad (29)$$

where $z \mapsto x(z)$ is a diffeomorphism of the line and $A(z)$ is a certain real function, one can reduce (26)–(28) to the Sturm–Liouville problem

$$(\partial_z^2 + V(z))\varphi = \varepsilon \varphi, \quad (30)$$

with the potential

$$V(z) = (A')^2 + A'' + P_2(x(z))$$

where $A = \int (\frac{\alpha}{2}) dx - log z'$ for (23). If the functions (29), obtained after transformation, belong to the $L_2(D)$-space, we reproduce the recently discovered quasi-exactly-solvable problems[4], where a finite number of eigenstates was found algebraically. For example,

$$T_2 = -4J^0 J^- + 4aJ^+ + 4bJ^0 - 2(n + 1)J^- \quad (31)$$

In dependence on a diffeomorphism $z \mapsto x(z)$, the space $D$ can be the infinite real line, semi-infinite real line and a finite real interval.
leads to the spectral problem (30) with the potential 
\[ V(z) = a^2 z^6 + 2ab z^4 + (b^2 - (4n + 3)a)z^2, \] (32)
for which the first \((n+1)\) eigenfunctions, even in \(x\) can be found algebraically.

It is worth noting that the use of (27) as input leads to one-functional family of the Schrödinger operators with two explicitly known eigenstates. One of such operators has been described at [5], where the authors confusingly stated about non-existence of \(sl_2(R)\) algebra behind.

Taking different exactly-solvable operators \(E_2\) for the eigenvalue problem (17) one can reproduce the equations having the Hermite, Laguerre, Legendre and Jacobi polynomials as solutions [2] [4]. Also under special choices of general element \(E^4\), one can reproduce all known fourth order differential equations giving rise infinite sequences of orthogonal polynomials (see [1] and other papers in this volume).

Recently, A.Gonzalez-Lopez, N.Kamran and P.Olver [3] gave the complete description of second-order polynomial elements of \(U_{sl_2(R)}\) leading to the square-integrable eigenfunctions of the Sturm-Liouville problem (30) after transformation (29). Consequently, for second-order ordinary differential equations (26) the combination of this result and Theorem 1 gives a general solution of the Bochner problem as well as the more general problem of classification of equations possessing finite number of orthogonal polynomial solutions.

2. Finite-difference equations in one variable.

The generalized Bochner problem is defined in the same way, as it has been done for differential equations. The only difference is to consider the operator \(T\) in the problem (0) as a linear finite-difference operator of a finite

\[ \text{For instance, putting the parameter } a = 0 \text{ in (31), the equation (26) converts to the Hermite equation (after some substitution.)} \]
order. For the case of one real variable a solution of the classification problem is very similar to the case of ordinary differential equations described above.

Let us introduce the finite-difference analogue of the operators (2) \[7\]

\[
\tilde{J}^+ = x^2 D - \{n\} x
\]

\[
\tilde{J}^0 = x D - \hat{n}
\]

\[
\tilde{J}^- = D,
\]

where \(\hat{n} \equiv \frac{\{n\}(n+1)}{\{2n+2\}}, \{n\} = \frac{1-q^n}{1-q}\) is the quantum symbol, \(q\) is a number characterizing the deformation, \(Dz = 1 + qzD\) and \(Df(z) = \frac{D(f(z)-f(qz))}{1-qz} + f(qz)D\) is a shift or a finite-difference operator (or so called the Jackson symbol (see \[7\])). The operators (33) after multiplication by some factors as

\[
\tilde{J}^0 = \frac{q^{-n}}{p+1} \frac{\{2n+2\}}{\{n+1\}} \tilde{J}^0
\]

\[
\tilde{J}^\pm = q^{-n/2} \tilde{J}^\pm
\]

(see \[7\]) form a quantum \(sl_2(R)_q\) algebra with the following commutation relations

\[
q^0 \tilde{J}^- - \tilde{J}^- \tilde{J}^0 = -\tilde{J}^-
\]

\[
q^2 \tilde{J}^+ \tilde{J}^- - \tilde{J}^- \tilde{J}^+ = -(q + 1) \tilde{J}^0
\]

\[
\tilde{J}^0 \tilde{J}^+ - q \tilde{J}^+ \tilde{J}^0 = \tilde{J}^+
\]

(this algebra corresponds to the second Witten’s quantum deformation of \(sl_2\) in the classification of C.Zachos \[8\]). If \(q \to 1\), the commutation relations (34) reduce to the standard \(sl_2(R)\) ones. A remarkable property of generators (33) is such that, if \(n\) is a non-negative integer, they form the finite-dimensional representation.

Similarly as for differential operators one can introduce quasi-exactly-solvable and exactly-solvable operators.
**Definition.** Let us name a linear difference operator of the $k$-th order, $\tilde{T}_k$ quasi-exactly-solvable, if it preserves the space $\mathcal{P}_n$. Correspondingly, the operator $\tilde{E}_k$, which preserves the infinite flag $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \ldots \subset \mathcal{P}_n \subset \ldots$ of spaces of all polynomials, is named exactly-solvable.

The analogue of the Lemma 1 holds.

**LEMMA 4.** (i) Suppose $n > (k - 1)$. Any quasi-exactly-solvable operator of $k$-th order $\tilde{T}_k$, can be represented by a $k$-th degree polynomial of the operators (33). If $n \leq (k - 1)$, the part of the quasi-exactly-solvable operator $\tilde{T}_k$ containing derivatives up to the order $n$ can be represented by a $n$-th degree polynomial in the generators (33).

(ii) Inversely, any polynomial in (33) is quasi-exactly solvable.

(iii) Among quasi-exactly-solvable operators there exist exactly-solvable operators $\tilde{E}_k \subset \tilde{T}_k$.

**PROOF.** Straightforward analogue of the proof of the Lemma 1.

**Comment 5.** If we define an analogue of the universal enveloping algebra $U_g$ of a Lie algebra $g$ as an algebra of all polynomials in generators $sl_2(\mathbb{R})_q$. Then a quasi-exactly-solvable operator $\tilde{T}_k$ at $k < n + 1$ is simply an element of the ‘universal enveloping algebra’ $U_{sl_2(\mathbb{R})_q}$ of the algebra $sl_2(\mathbb{R})_q$ taken in representation (33). If $k \geq n + 1$, then $\tilde{T}_k$ is represented as an element of $U_{sl_2(\mathbb{R})_q}$ plus $BD^{n+1}$, where $B$ is any linear difference operator of the order not higher than $(k - n - 1)$.

Similar to $sl_2(\mathbb{R})$, one can introduce the grading of generators (31) of $sl_2(\mathbb{R})_q$ (see (15)) and, hence, of monomials of the universal enveloping $U_{sl_2(\mathbb{R})_q}$ (see (16)). The analogue of Lemma 2 holds.

**LEMMA 5.** A quasi-exactly-solvable operator $\tilde{T}_k \subset U_{sl_2(\mathbb{R})_q}$ has no terms of positive grading, iff it is an exactly-solvable operator.
PROOF. Straightforward analogue of the proof of the Lemma 2.

THEOREM 2. Let \( n \) is non-negative integer. Take the eigenvalue problem for a linear difference operator of the \( k \)-th order in one variable

\[
\tilde{T}^k \varphi(x) = \varepsilon \varphi(x),
\]

where \( \tilde{T}_k \) is symmetric. The problem (35) has \( (n + 1) \) linear independent eigenfunctions in the form of polynomial in variable \( x \) of the order not higher than \( n \), iff \( T_k \) is quasi-exactly-solvable. The problem (35) has an infinite sequence of polynomial eigenfunctions, iff the operator is exactly-solvable \( \tilde{E}_k \).

Comment 6. Saying the operator \( \tilde{T}_k \) is symmetric, we imply, that considering action of this operator on a space of polynomials of degree not higher than \( n \), one can introduce a positively-defined scalar product and the operator \( \tilde{T}_k \) is symmetric with respect to it.

Comment 7. The ”if” part of the first statement is obvious. The ”only if” part is a direct corollary of Lemma 4.

This theorem gives a general classification of finite-difference equations

\[
\sum_{j=0}^{k} \tilde{a}_j(x)D^j \varphi(x) = \varepsilon \varphi(x)
\]

(36)

having polynomial solutions in \( x \). The coefficient functions must have the form

\[
\tilde{a}_j(x) = \sum_{i=0}^{k+j} \tilde{a}_{j,i}x^i.
\]

(37)

Particularly, this form occurs after substitution (33) into a general \( k \)-th degree polynomial element of the universal enveloping algebra \( U_{sl_2(R)} \). It guarantees the existence of at least a finite number of polynomial solutions. The coefficients \( \tilde{a}_{j,i} \) are related to the coefficients of the \( k \)-th degree polynomial.
element of the universal enveloping algebra $U_{\mathfrak{sl}_2(\mathbb{R})}_q$. The number of free parameters of the polynomial solutions is defined by the number of free parameters of a general $k$-th order polynomial element of the universal enveloping algebra $U_{\mathfrak{sl}_2(\mathbb{R})}_q$. A rather straightforward calculation leads to the following formula

$$\text{par}(\tilde{T}_k) = (k + 1)^2 + 1$$

(for the second order finite-difference equation $\text{par}(\tilde{T}^2) = 10$). For the case of an infinite sequence of polynomial solutions the formula (37) simplifies

$$\tilde{a}_j(x) = \sum_{i=0}^{j} \tilde{a}_{j,i} x^i \quad (38)$$

and the number of free parameters is given by

$$\text{par}(\tilde{E}_k) = \frac{(k + 1)(k + 2)}{2} + 1$$

(for $k = 2$, $\text{par}(\tilde{E}^2) = 7$). The increase in the number of free parameters compared to ordinary differential equations is due to the presence of the deformation parameter $q$. In [2] one can find the description in present approach of the $q$-deformed Hermite, Laguerre, Legendre and Jacobi polynomials (for definition these polynomials see [3]).

The analogue of Lemma 3 holds as well

\textbf{LEMMA 6.} If the operator $\tilde{T}_2$ (see (24)) is such that

$$\tilde{c}_{++} = 0 \quad \text{and} \quad \tilde{c}_+ = (\hat{n} - \{m\}) \tilde{c}_{+0}, \quad \text{at} \ m = 0, 1, 2, \ldots \quad (39)$$

\footnote{For quantum $\mathfrak{sl}_2(\mathbb{R})_q$ algebra there are no polynomial Casimir operators [3]. However, in the representation (33) the relationship between generators analogous to the quadratic Casimir operator

$$q\tilde{J}^+ \tilde{J}^- - \tilde{J}^0 \tilde{J}^0 + (\{n + 1\} - \hat{n}) \tilde{J}^0 = \hat{n}(\hat{n} - \{n + 1\})$$

appears. It reduces the number of independent parameters of the second-order polynomial element of $U_{\mathfrak{sl}_2(\mathbb{R})}_q$. It becomes the standard Casimir operator at $p \to 1.$}
then the operator $\tilde{T}_2$ preserves both $\mathcal{P}_n$ and $\mathcal{P}_m$ and polynomial solutions in $x$ with 8 free parameters occur.

**PROOF.** By straightforward analysis.

Rather interesting situation occurs, if the parameter of deformation $q$ equals to the root of unity.

**LEMMA 7.** If a quasi-exactly-solvable operator $\tilde{T}_k$ preserves the polynomial space $\mathcal{P}_n$ and the parameter $q$ is satisfied to the equation

$$q^n = 1,$$  \hfill (40)

then the operator $\tilde{T}_k$ preserves an infinite flag of polynomial spaces $\mathcal{P}_0 \subset \mathcal{P}_n \subset \mathcal{P}_{2n} \subset \ldots \subset \mathcal{P}_{kn} \subset \ldots$.

It is worth emphasizing, that in the limit $q$ stresses to one, Lemmas 4,5,6 and Theorem 2 coincide to the Lemmas 1,2,3 and Theorem 1,respectively. Thus the case of differential equations in one variable can be considered as particular case of finite-difference ones. Evidently, one can consider the finite-difference operators, which are a mixture of generators (33) with the same value of $n$ and different $q$’s.

In closing, I would like to thank to L.Michel, V.Ovsienko, S.Tabachnikov and, especially, V.Arnold and M.Shubin for numerous useful discussions. Also I am very grateful to the Centre de Physique Theorique for hospitality extended to me, where this work has been completed.
References

[1] "Orthogonal polynomial solutions to ordinary and partial differential equations"
L.L.Littlejohn, Proceedings of an International Symposium on Orthogonal Polynomials and their Applications, Segovia, Spain, Sept.22-27, 1986
Lecture Notes in Mathematics No.1329, M.Alfaro et al. (Eds.), Springer-Verlag (1988), pp. 98-124

[2] "On polynomial solutions of differential equations"
A.V.Turbiner, Preprint CPT-91/P.2628 (November 1991)

[3] "Certain differential equations for Chebyshev polynomials"
H.L.Krall, Duke Math.J., 4, 705-718 (1938)

[4] "Spectral Riemannian surfaces of the Sturm-Liouville operators and quasi-exactly-solvable problems"
A.V.Turbiner, Sov.Math.–Funk.Analysis i ego Prilogenia, 22 (1988) 92-94;
"Quasi-exactly-solvable problems and sl(2, R) algebra"
A.V. Turbiner, Comm. Math. Phys. 118 (1988) 467-474

[5] "A quasi-exactly-solvable problem without sl(2) symmetry"
D.P.Jatkar, C.Nagaraja Kumar, A.Khare, Phys.Lett.A142 (1989) 200-202

[6] "Normalizability properties of one-dimensional quasi-exactly-solvable Schroedinger operators"
A. Gonzales-Lopez, N. Kamran and P.J. Olver. Preprint of the School of Math., Univ. Minnesota (Invited talk at the Session of AMS No.873, March 20-21, Springfield, USA)
[7] "$sl(2, \mathbb{R})_q$ and quasi-exactly-solvable problems"
   O. Ogievetski and A. Turbiner, Preprint CERN-TH: 6212/91 (1991)

[8] "Elementary paradigms of quantum algebras"
   C. Zachos, Proceedings of the Conference on Deformation Theory of
   Algebras and Quantization with Applications to Physics, Contemporary
   Mathematics, J. Stasheff and M. Gerstenhaber (eds.), AMS, in
   press (1991)

[9] "$q$-Hypergeometrical functions and applications"
   H. Exton, Horwood Publishers, 1983