Besicovitch Covering Property for homogeneous distances on the Heisenberg groups

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The **Besicovitch Covering Property (BCP)** holds on \((M, d)\) if there exists \(N \geq 1\) such that if \(A \subset M\) is bounded and \(\mathcal{B}\) is a family of balls such that each point of \(A\) is the center of some ball of \(\mathcal{B}\), then there is a subfamily \(\mathcal{F} \subset \mathcal{B}\) such that
\[
11_A \leq \sum_{B \in \mathcal{F}} 11_B \leq N.
\]

A family \(\mathcal{B} = \{B_d(x_B, r_B)\}\) of balls in \((M, d)\) is a family of **Besicovitch balls** if
- \(\text{Card } \mathcal{B} < +\infty\),
- \(x_B \notin B'\) for all \(B, B' \in \mathcal{B}, B \neq B'\),
- \(\bigcap_{B \in \mathcal{B}} B \neq \emptyset\).

The **Weak Besicovitch Covering Property (w-BCP)** holds on \((M, d)\) if there exists \(N \geq 1\) such that \(\text{Card } \mathcal{B} \leq N\) for every family \(\mathcal{B}\) of Besicovitch balls.
BCP $\Rightarrow$ w-BCP

If $(M, d)$ is doubling then BCP $\Leftrightarrow$ w-BCP
We say that the differentiation theorem holds for the locally finite Borel measure $\mu$ on $(M, d)$ if

$$\lim_{r \to 0^+} \frac{1}{\mu(B_d(p, r))} \int_{B_d(p, r)} f(q) \, d\mu(q) = f(p)$$

for $\mu$-almost every $p \in M$ and for each $f \in L^1(\mu)$.

**Theorem (Besicovitch - Preiss)**

Let $(M, d)$ be a complete separable metric space. Then the differentiation theorem holds for each locally finite Borel measure $\mu$ on $(M, d)$ iff $M = \bigcup_{n \in \mathbb{N}} M_n$ where, for each $n \in \mathbb{N}$, $w$-BCP holds for family of balls centered on $M_n$ with radii less than $r_n$ for some $r_n > 0$. 
Examples of metric spaces satisfying BCP:
- The Euclidean space $\mathbb{R}^n$
- Finite dimensional normed vector spaces
- Riemannian manifolds of class $C^2$

Examples of metric spaces that do not satisfy BCP:
- Infinite dimensional Hilbert spaces
- The Heisenberg group equipped with the (Cygan-)Korányi distance [Sawyer-Wheeden, Korányi-Reimann]
- The Heisenberg group equipped with the Carnot-Carathéodory distance [R]
The validity of BCP depends strongly (and only) on the shape of the balls in the metric space \((M, d)\).

It is in particular not stable under a bi-Lipschitz change of distance.

**Theorem (Preiss, Le Donne-R)**

Let \((M, d)\) be a metric space. Assume that there exists an accumulation point in \((M, d)\). Let \(0 < c < 1\). Then there exists a distance \(\overline{d}\) on \(M\) such that \(c \, d \leq \overline{d} \leq d\) and for which \(w\)-BCP, and hence BCP, does not hold.
The Heisenberg group $\mathbb{H}^1 \cong \mathbb{R}^3$

- **Group law:**
  \[(x, y, z) \cdot (x', y', z') := (x + x', y + y', z + z' + \frac{1}{2}(x y' - y x'))\]

- **Dilations:** $\delta_\lambda(x, y, z) = (\lambda x, \lambda y, \lambda^2 z)$

**Definition**

We say that a distance $d$ on $\mathbb{H}^1$ is homogeneous if

\[
d(p \cdot q, p \cdot q') = d(q, q')
\]

\[
d(\delta_\lambda(p), \delta_\lambda(q)) = \lambda \ d(p, q)
\]

for all $p, q, q' \in \mathbb{H}^1$ and all $\lambda > 0$. 
Let $A \subset \mathbb{H}^1$ be compact. Assume that

- $p \in A, q \in A, t \in [0, 1] \Rightarrow \delta_t(p) \cdot \delta_{1-t}(q) \in A$
- $p \in A \Rightarrow p^{-1} \in A$.

Then

$$d(p, q) := \inf(t > 0; \delta_{\frac{1}{t}}(p^{-1} \cdot q) \in A)$$

defines a homogeneous distance on $\mathbb{H}^1$. It is the homogeneous distance such that $B_d(0, 1) = A$.

Let $B_\alpha := \{(x, y, z) \in \mathbb{H}^1; x^2 + y^2 + z^2 \leq \alpha^2\}$. Then

$$d_\alpha(p, q) := \inf(t > 0; \delta_{\frac{1}{t}}(p^{-1} \cdot q) \in B_\alpha)$$

defines a homogeneous distance on $\mathbb{H}^1$ for all $\alpha < \alpha^*$ for some $\alpha^* > 0$ [Hebisch-Sikora].

\[1\]This holds true more generally for any Carnot group.
Theorem (Le Donne-R)

Let $\alpha > 0$ be such that $d_\alpha$ defines a homogeneous distance on $\mathbb{H}^1$. Then BCP holds for the homogeneous distance $d_\alpha$ on $\mathbb{H}^1$.

More generally, BCP holds in $\mathbb{H}^n$ equipped with a homogeneous distance whose unit ball centered at the origin is an Euclidean ball centered at the origin.
The distance \( d_\alpha \)

- **Analytic expression of \( d_\alpha \)**
  
  Let \( p = (x_p, y_p, z_p) \in \mathbb{H}^1 \) and set \( \rho_p := \sqrt{x_p^2 + y_p^2} \).

  \[
  d_\alpha(0, p) = r \iff d_\alpha(0, \delta_1(p)) = 1 
  \iff \frac{\rho_p^2}{r^2} + \frac{z_p^2}{r^4} = \alpha^2 
  \]

  \[
  d_\alpha(0, p) = \sqrt{\frac{\rho_p^2 + \sqrt{\rho_p^4 + 4\alpha^2 z_p^2}}{2\alpha^2}}. 
  \]
The distance \( d_\alpha \)

\[
d_\alpha(p, q) = \sqrt{\frac{\rho(p, q)^2 + d_{K,\alpha}(p, q)^2}{2\alpha^2}}
\]

\( \rho(p, q) := \sqrt{(x_q - x_p)^2 + (y_q - y_p)^2} \) is a homogeneous pseudo-distance.

\( d_{K,\alpha}(p, q) := 4\sqrt{\rho(p, q)^4 + 4\alpha^2(z_q - z_p)^2} \).

For \( \alpha = 2 \), \( d_{K,2} \) is the (Cygan-)Korányi distance.

For \( 0 < \alpha \leq 2 \), \( d_{K,\alpha} \) is a homogeneous distance on \( \mathbb{H}^1 \).

For \( 0 < \alpha \leq 2 \), \( d_\alpha \) defines a homogeneous distance on \( \mathbb{H}^1 \).
The distance $d_\alpha$ already appeared in other problems.

- Lee and Naor proved that the distances $d_\alpha$ are of negative type on $\mathbb{H}^1$, i.e., $(\mathbb{H}^1, \sqrt{d_\alpha})$ is isometric to a subset of a Hilbert space.

- Combined with the fact that $\mathbb{H}^1$ equipped with a homogeneous distance does not admit a bi-Lipschitz embedding into $L^1$ [Cheeger-Kleiner], this allowed them to provide a counterexample to the Goemans-Linial conjecture of theoretical computer science.

NB: The (Cygan-)Korányi distance $d_{K,2}$ is not of negative type on $\mathbb{H}^1$. 
Let $d$ be a homogeneous distance on $\mathbb{H}^1$.

Then BCP does not hold in $(\mathbb{H}^1, d)$ in the following cases:

- One can find $p \in \partial B_d(0, 1)$ and $\lambda > 0$ such that $d(p, \delta_\lambda(p)) > 1$ for all $0 < \lambda < \lambda$.

Examples:

- The Carnot-Carathéodory distance [R].
- The unit ball centered at the origin has an inward cone-like singularity in the Euclidean sense at the poles [Le Donne-R].
One can find a sequence of points \((q_n)\) in \(\partial B_d(0, 1)\) and a positive sequence \((\lambda_n)\) such that

\[
d(q_k, \delta_{\lambda}(q_n)) > 1 \quad \text{for all } 0 \leq k < n \text{ and all } 0 < \lambda \leq \lambda_n.
\]

Example:

- The unit ball centered at the origin is given near the north pole by the subgraph \(\{z \leq \varphi(x, y)\}\) of a \(C^2\) function \(\varphi\) whose first and second order partial derivatives vanish at the origin [Le Donne-R].
  This applies to the \(d_\infty\) and the (Cygan-)Korányi distances.
The unit ball centered at the origin has an outward cone-like singularity in the Euclidean sense at the poles [Le Donne-R].

**Ex:** The $l^1$-sum of the pseudo-distance $\rho(p, q)$ with the (Cygan-)Korányi distance. More generally, any distance $d$ of the form $d(p, q) := \beta \rho(p, q) + d_{K,\alpha}(p, q)$.

The distance $d_\alpha$ lies in between these two cases. Its unit ball centered at the origin is smooth with positive curvature in the Euclidean sense. Up to a multiplicative constant, it is the $l^2$-sum of the pseudo-distance $\rho(p, q)$ with the distance $d_{K,\alpha}$. 
Proof. Recall that $\mathcal{F} = \{B_d(x_B, r_B)\}$ is a family of Besicovitch balls if

- $\text{Card } \mathcal{F} < +\infty$,
- $x_B \not\in B'$ for all $B, B' \in \mathcal{F}$, $B \neq B'$,
- $\bigcap_{B \in \mathcal{F}} B \neq \emptyset$.

We want to find some $N \geq 1$ such that $\text{Card } \mathcal{F} \leq N$ for every family $\mathcal{F}$ of Besicovitch balls.
Proof of the validity of BCP for $d_\alpha$

**Step 1.** One can find $R > 0$ large enough, $\theta > 0$ small enough, $a > 1$ large enough and $b < 1$ small enough, such that, if $\mathcal{F}$ is a family of Besicovitch balls, one can find a family $\mathcal{B} = \{B(p_j, r_j)\}_{j=1}^{k}$ of Besicovitch balls such that:

- $\text{Card } \mathcal{F} \leq 2 \left( \frac{\pi}{\theta} + 1 \right) \text{Card } \mathcal{B} + 2$,
- $r_j = d(0, p_j)$, i.e., $0 \in \cap_{j=1}^{k} \partial B(p_j, r_j)$,
- $\rho_{p_1} \leq \rho_{p_2} \leq \cdots \leq \rho_{p_k}$,
- $R = \min_{\{j=1, \ldots, k\}} (r_j)$,
- $p_j \in C^- := \{p \in \mathbb{H}^1; |y_p| < x_p \tan \theta, z_p \leq 0\} \setminus U(0, R)$ for all $j = 1, \ldots, k$,
- $C^- \cap \{-b < z_p \leq 0\} \subset \mathcal{P}(a, b, \theta)$,
- $C^- \cap \{\rho_p < b\} \subset \mathcal{T}(a, b)$.

Here $U(0, R)$ denotes the open ball with center $0$ and radius $R > 0$. 
Proof of the validity of BCP for $d_{\alpha}$
Proof of the validity of BCP for $d_{\alpha}$

**Step 2.** We have

$$\text{Card}(\{p_j\}_{j=1}^k \cap \mathcal{P}(a, b, \theta)) \leq 1,$$

$$\text{Card}(\{p_j\}_{j=1}^k \cap \mathcal{T}(a, b)) \leq 1.$$ 

**Step 3.** We have

$$z_j < 2z_{j+1} \quad \text{and} \quad \rho_j < \cos(2\theta) \rho_{j+1}. $$
Theorem (Le Donne-R)

Let $G$ be a Carnot group of step $\geq 3$. Let $d$ be a homogeneous distance on $G$ whose unit ball centered at the origin is an Euclidean ball centered at the origin. Then BCP does not hold in $(G, d)$.

More generally, assume that

$$B_d(0, 1) = \{ c_1 |x_1|^{\gamma_1} + \cdots + c_n |x_n|^{\gamma_n} \leq 1 \}.$$

Then BCP does not hold in $(G, d)$.
Open questions

- For Carnot groups of step $\geq 3$, does there exist homogeneous distances for which BCP holds?

- For Carnot groups of step 2 other than the Heisenberg groups,
  - does BCP hold for homogeneous distances whose unit ball centered at the origin is an Euclidean ball?
  - does there exist homogeneous distances for which BCP holds?