ON ZARISKI’S MULTIPLICITY CONJECTURE

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1. Introduction

Since the early 1960’s, O. Zariski developed a comprehensive theory of equisingularity in codimension one. He initiated an equisingularity program with topological, differential geometrical and purely algebraical point of view and proposed a problem list in [22] as an extraction of many possible conjectures in singularity theory [23]. In this part we will be concerned with topological aspects of this program and more specifically with the so-called Zariski’s multiplicity conjecture. We first recall some definitions. Let \( f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) be two germs of holomorphic functions and \( V_f \) and \( V_g \) be two germs at the origin of the hypersurfaces defined by \( f^{-1}(0) \) and \( g^{-1}(0) \) respectively. We suppose \( 0 \in \mathbb{C}^n \) is an isolated singularity of the functions. The algebraic multiplicity \( m_f \) of the germs of \( V_f \) or \( f \) is the order of vanishing of function \( f \) at \( 0 \in \mathbb{C}^n \) or equivalently is the order of the first nonzero leading term in the Taylor expansion of \( f \)

\[
f = f_\nu + f_{\nu+1} + \cdots
\]

where \( f_i \) is homogeneous polynomial of degree \( i \).

Definition 1. We say \( V_f \) and \( V_g \) are topologically equisingular or topologically V-equivalent if there is a germ of homeomorphism \( \phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0) \) sending \( V_f \) onto \( V_g \). More precisely, there are neighborhoods \( U \) and \( U' \) of \( 0 \in \mathbb{C}^n \) such that \( f \) and \( g \) are defined and a homeomorphism \( \phi : U \to U' \) such that \( \phi(f^{-1}(0) \cap U) = g^{-1}(0) \cap U' \) and \( \phi(0) = 0 \).

Zariski conjecture. Topological equisingularity of germs of hypersurfaces implies equimultiplicity.

A well known result by Burau [4] and Zariski [23] states an affirmative answer in the case of curves (\( n = 2 \)). In higher dimension the conjecture is still open despite more than three decades effort to prove it.

Here we discuss some features of the problem, especially the relations of the work of A’Campo on the zeta function of a monodromy and the Zariski’s multiplicity conjecture. Also some previous results are sharpened;
the results of [6] and [18] in theorem (3.2) and the one in [7] in theorem (5.2). In an analogy with hypersurfaces, J.F. Mattei asked the same question about multiplicity for holomorphic foliations. In section (6) we recall some remarkable results for foliations.

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2. Preliminaries

In [15], Milnor has opened a beautiful account on the complex hypersurfaces. The main achievement of it, is the Milnor fibration which we mention here. Also we recall briefly some generalities about complex hypersurfaces.

Let \( f : U \subset \mathbb{C}^n \to \mathbb{C} \) be a holomorphic function on an open neighborhood of 0 in \( \mathbb{C}^n \) and \( f(0) = 0 \). We denote \( D_\epsilon = \{ z \in \mathbb{C}^n : \|z\| \leq \epsilon \} \), \( S_\epsilon = \partial D_\epsilon \), \( H_0 = \{ z \in \mathbb{C}^n | f(z) = 0 \} \) and \( d_z f = (\frac{\partial f}{\partial z_1}(z), \cdots, \frac{\partial f}{\partial z_n}(z)) \).

We say the origin is an isolated singularity of \( f \) if \( d_0 f = 0 \) and \( d_z f \neq 0 \) for a neighborhood of 0 in \( \mathbb{C}^n \) except 0.

Let \( \mathcal{O}_n \) be the ring of germs of holomorphic functions defined in some neighborhood of 0 in \( \mathbb{C}^n \) and let \( \langle \frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n} \rangle \) be the ideal generated by the germs at 0 of derivative components of \( f \). We define Milnor number \( \mu \) of the holomorphic function \( f \) at 0 as

\[
\mu = \text{dim}_{\mathbb{C}} \mathcal{O}_n / \langle \frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n} \rangle
\]

This number is finite and nonzero if and only if 0 is an isolated singularity of \( f \), a hypothesis which we will assume from now on. In this case \( \mu \) coincides with the topological degree of the Gauss mapping induced by \( d_z f \) on \( S_\epsilon \) for \( \epsilon \) small enough. The following lemma is useful to deal with the Milnor number.

Lemma 2.1. Let \( 0 < \mu < \infty \). Given \( \epsilon > 0 \) there exists \( \delta > 0 \) such that for any \( c \in \mathbb{C}^n \) with \( \|c\| < \delta \) the number of solutions of the equation \( d_z f = c \) in the ball \( D_\epsilon \) is at most \( \mu \). Moreover, if \( p_1, \cdots, p_m, m \leq \mu \), are such solutions, then \( \sum_{i=1}^{m} \mu(f - \sum_{i=1}^{n} z_i c_i, p_i) = \mu \).

The following theorem is called the Milnor fibration theorem:

Theorem 2.2. For \( \epsilon \) small enough the mapping \( \psi_\epsilon : S_\epsilon \setminus H_0 \to S^1 \) defined by \( \psi_\epsilon(z) = f(z)/\|f(z)\| \) is a smooth fibration which is called the Milnor fibration. Moreover the fibers of \( \psi_\epsilon \) have the homotopy type of a bouquet of \( \mu \) (the Milnor number of the holomorphic function \( f \) at 0) spheres of dimension \( n - 1 \).

Also we call the number of spheres, the number of vanishing cycles of \( f \) at 0. The following theorem is due to Lê [14]:
Theorem 2.3. If $V_f$ and $V_g$ are topologically equisingular then the number of vanishing cycles at 0 of $f$ and $g$ are the same.

Now we recall some definitions and facts about deformations of functions. A deformation of a holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is a family $(f_t)_{t \in [0,1]}$ of germs of holomorphic functions with isolated singularities at $0 \in \mathbb{C}^n$. The jump of the family $(f_t)$ is $\mu(f_0) - \mu(f_t)$, where $\mu$ is the Milnor number at the origin. It is independent of $t$ for $t$ small enough, moreover by the upper semi-continuity of $\mu$ this number is a non-negative integer.

We use frequently the following theorem proved by Lê and Ramagujam [12]:

Theorem 2.4. Let $(f_s)_{s \in [0,1]}$ be a $C^\infty$ family of hypersurfaces having an isolated singularity at the origin. If the Milnor number of singularity does not change then the topological type of singularity does not change too provided that $n \neq 3$.

In [11], theorem (2.4) is generalized which we recall in section (5). Finally we recall an interesting result of P. Samuel [19]:

Theorem 2.5. Every germ $V_f$ is analytically equivalent with $V_g$ in which $g$ is a polynomial.

Moreover we may choose a polynomial $g$ with cutting the Taylor expansion of $f$ in somewhere. By the theorem of (2.5) it is enough to consider polynomials to prove the conjecture.

3. The topological right equivalent complex hypersurfaces

In this section we recall several ways to define a topological type of a holomorphic function and relations between them according to [10], [17] and [20].

Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two germs of holomorphic functions with an isolated singularity at the origin.

Definition 2. $f$ and $g$ are topologically right equivalent if there is a germ of homeomorphism $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ satisfying $f = g \circ \varphi$

Definition 3. $f$ and $g$ are topologically right-left equivalent if there are germs of homeomorphism $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ and $\psi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ satisfying $f = \psi \circ g \circ \varphi$

Put $V_f := f^{-1}(0)$. By [15], $S^2n-1_\varepsilon \cap V_f$ is a smooth $(2n - 3)$-dimensional manifold for $\varepsilon > 0$ sufficiently small. The pair $(S^2n-1_\varepsilon, S^2n-1_\varepsilon \cap V_f)$ is called the link of the singularity of $f$.

Definition 4. $f$ and $g$ are link equivalent if $(S^2n-1_\varepsilon, S^2n-1_\varepsilon \cap V_f)$ is homeomorphic to $(S^2n-1_\varepsilon, S^2n-1_\varepsilon \cap V_g)$ for all sufficiently small $\varepsilon$.

By the definitions, the right equivalence implies the right-left equivalence, which in turn implies the V-equivalence. The outstanding result, obtained by King [10] in $n \neq 3$ and by Perron [17] in $n = 3$, is the following:
Theorem 3.1. The topological $V$-equivalence implies topologically right-left equivalence. Moreover if $f$ and $g$ are topologically right-left equivalent then $g$ is topologically right equivalent either to $f$ or to $\overline{f}$, the complex conjugate of $f$.

Using theorem (3.1), Risler and Trotman in [18] proved that right-left bilipschitz equivalence implies equimultiplicity.

Since $(D_{z}^{m}ER_{f}^{2} \cap V_{f})$ is homeomorphic to the cone cover over the link $(S_{z}^{m}ER_{f}^{2} \cap V_{f})$ ([15]), the link equivalence implies the $V$-equivalence. Conversely Saeki [20] showed that the topological $V$-equivalence implies the link equivalence. This means that there is a homeomorphism $\varphi_{1} : (\mathbb{C}^{n}, 0) \rightarrow (\mathbb{C}^{n}, 0)$, sending $V_{f}$ onto $V_{g}$ and such that $|\varphi_{1}(z)| = |z|$. By theorem (3.1) there is a homeomorphism $\varphi_{2} : (\mathbb{C}^{n}, 0) \rightarrow (\mathbb{C}^{n}, 0)$ such that $|f(z)| = |g \circ \varphi_{2}(z)|$. Comte, Milman and Trotman [6] showed that if there is a germ of homeomorphism $\varphi : (\mathbb{C}^{n}, 0) \rightarrow (\mathbb{C}^{n}, 0)$ having simultaneously the properties of $\varphi_{1}$ and of $\varphi_{2}$ then the multiplicity conjecture is true. In fact they proved that it suffices to assume that there are positive constants $A$, $B$, $C$ and $D$ such that:

1. $A|z| \leq |\varphi(z)| \leq B|z|$, for all $z$ near 0, and
2. $C|f(z)| \leq |g \circ \varphi(z)| \leq D|f(z)|$, for all $z$ near 0.

Now we prove that it is enough to assume the conditions (1) and (2) are valid for some special sequences converge to the origin. Given two holomorphic function germs $f, g : (\mathbb{C}^{n}, 0) \rightarrow (\mathbb{C}, 0)$, by an analytic change of coordinates one may assume that the 1-axis is not contained in the tangent cones $C(V_{f}), C(V_{g})$ (respectively the zero set of first non zero jet of $f$ and $g$), so that $f(z_{1}, 0, \cdots, 0) \neq 0$ and $g(z_{1}, 0, \cdots, 0) \neq 0$ for a neighborhood of 0 in the 1-axis, and by theorem (2.5) one may assume $f$ and $g$ are polynomials.

In this situation we have the following:

Theorem 3.2. Suppose there are a germ of homeomorphism $\varphi : (\mathbb{C}^{n}, 0) \rightarrow (\mathbb{C}^{n}, 0)$ with inverse $\psi$ and positive constants $A$, $B$, $C$ and $D$ and two sequences $w_{m}$ and $w'_{m}$ in the 1-axis which converge to the origin with the following properties:

(i) $|\psi(w'_{m})| \leq A|w'_{m}|$, $|\varphi(w_{m})| \leq B|w_{m}|$ and
(ii) $C|f(w_{m})| \leq |g \circ \varphi(w_{m})|$, $D|g(w'_{m})| \leq |f \circ \psi(w'_{m})|

then $m_{f} = m_{g}$.

The conditions (i) and (ii) are slightly weaker than conditions (1) and (2) above.

Proof. Write $f(z) = f_{k}(z) + f_{k+1}(z) + \cdots + f_{k+r}(z),$ $g(z) = g_{l}(z) + g_{l+1}(z) + \cdots + g_{l+s}(z).$ $f_{i}$ and $g_{j}$ are homogeneous parts of degree $i$ and $j$ of $f$ and $g$ respectively. $f_{k}$ and $g_{l}$ are not identically zero. We want to prove $k = l$. By contrary suppose $l > k$. The other case is similar. Let $w_{1} = (z_{1}, 0, \cdots, 0)$ and
$w_m = (t_m z_1, 0, \cdots , 0)$, $t_m \neq 0$ and converges to the origin. Also write $g$ in the following form:

$$g(z) = \sum_{j=1}^{l+s} \sum_{|\beta|=j} C_{\beta}^j z_1^\beta,$$

where $z = (z_1, \cdots , z_n)$ and $\beta = (\beta_1, \cdots , \beta_n)$, $\beta_i \in \mathbb{N} \cup \{0\}$. Now we have

$$f(w_m) = f_k(w_m) + f_{k+1}(w_m) + \cdots + f_{k+r}(w_m) \text{ or}$$

$$f(w_m) = t_m^k f_k(w_1) + t_m f_{k+1}(w_1) + \cdots + t_{m}^r f_{k+r}(w_1)$$

and

$$|(g \circ \varphi)(w_m)| \leq \sum_{j=1}^{l+s} \sum_{|\beta|=j} |C_{\beta}^j|B_j|t_m z_1|^j$$

by (i). Now we use condition (ii). It is:

$$C|f(w_m)| \leq |g \circ \varphi(w_m)| \text{ or}$$

$$C|t_m^k f_k(w_1) + t_m f_{k+1}(w_1) + \cdots + t_{m}^r f_{k+r}(w_1)| \leq \sum_{j=1}^{l+s} \sum_{|\beta|=j} |C_{\beta}^j|B_j|t_m z_1|^j.$$

Divided two sides of above inequality by $|t_m^k|$ we obtain the following:

$$C|f_k(w_1) + t_m f_{k+1}(w_1) + \cdots + t_{m}^r f_{k+r}(w_1)| \leq \sum_{j=1}^{l+s} \sum_{|\beta|=j} |C_{\beta}^j|B_j|t_m z_1|^j-k,$$

or

$$C|f_k(w_1)| \leq \sum_{j=1}^{l+s} \sum_{|\beta|=j} |C_{\beta}^j|B_j|t_m z_1|^j-k + C|t_m f_{k+1}(w_1) + \cdots + t_{m}^r f_{k+r}(w_1)|.$$

When $t_m$ goes to zero, the right hand of the last inequality goes to zero but the left hand is a positive constant. This contradiction shows $l = k$. \qed

4. The zeta function of a monodromy

Now we recall some features from [1] and [2]. Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial so that $f(0) = 0$ and consider the hypersurface defined by it, $V_f = f^{-1}(0)$. The map

$$\pi : z \in S^{2n-1} \setminus V_f \longmapsto \arg(f(z)) \in S^1,$$

defines a Milnor fibration of the hypersurface $V_f$ at the origin. The fibre $F_\theta = \pi^{-1}(\theta)$, $\theta \in S^1$, is a $2(n-1)$-dimensional differential manifold and the characteristic homeomorphism of this fibration

$$h : F_\theta \to F_\theta$$
is the geometric monodromy of $V_f$ at the origin. By definition the zeta function of $h$ is the following:

$$Z(t) = \prod_{q \geq 0} \left\{ \det(I - th^*; H^q(F_\theta, \mathbb{C})) \right\}^{(-1)^{q+1}}.$$

When the origin of $\mathbb{C}^n$ is an isolated singular point of $V_f$, one has

$$H^q(F_\theta, \mathbb{C}) = \begin{cases} \mathbb{C} & q = 0 \\ 0 & q \neq 0, q \neq n \\ \mathbb{C}^\mu & q = n, \end{cases}$$

where $\mu$ is the Milnor number of $f$ and therefore the characteristic polynomial $\Delta(t)$ of the monodromy at degree $n$ is deduced from the zeta function $Z(t)$ by the formula

$$\Delta(t) = t^\mu \left( 1 - \frac{1}{t} \right)^n Z\left( \frac{1}{t} \right)^{(-1)^{n+1}}.$$

For an integer $k \geq 1$, let the integer number

$$\Lambda(h^k) = \sum_{q \geq 0} (-1)^q \text{Trace}[(h^*)^k; H^q(F_\theta, \mathbb{C})]$$

the Lefschetz number of the $k$-th power of $h$. Let $s_1, s_2, \cdots$ be the integers defined by the following recurrence relations:

$$\Lambda(h^k) = \sum_{i | k} s_i,$$

$k \geq 1$, then the zeta function of $h$ is given by

$$Z(t) = \prod_{q \geq 0} (1 - t^q)^{-s_i}.\$$

The Lefschetz numbers $\Lambda(h^k)$ are topological invariants of the singularity of $V_f$, therefore the integers $s_1, s_2, \cdots$ are topological invariants.

Remark 4.1. In [2], A’Campo has calculated $\Lambda(h)$ as following:

$$\Lambda(h) = \begin{cases} 0 & \text{if } d_0 f = 0 \\ 1 & \text{if } d_0 f \neq 0. \end{cases}$$

This tells us that if $f$ is regular and $g$ is singular at the origin there is no topological equivalence between germs of $V_f$ and $V_g$ at the origin.

Remark 4.2. More generally Deligne has explained in a letter to A’Campo (see [1], [9]) that

$$\Lambda(h^k) = 0, \text{ if } 0 < k < \text{ multiplicity of } V_f \text{ at the origin.}$$

The Lefschetz numbers $\Lambda(h^k)$ are topological invariants of the singularity of $V_f$, therefore the integers $s_1, s_2, \cdots$ are topological invariants. A’Campo discovered the meaning of the topological invariants $s_1, s_2, \cdots$ as following:
Let $\pi : X \to \mathbb{C}^n$ be a proper modification such that in all points of $S := \pi^{-1}(0)$, the divisor $V'_f := \pi^{-1}(V_f)$ has normal crossings. Such a local resolution of $(\mathbb{C}^n, V_f)$ at the origin exists by the theorem of resolution of singularities due to Hironaka [8]. For every $m \in \mathbb{N}$, let $S_m$ be all points $s \in S$ such that the equation of $V'_f$ at $s$ is of the form $z_1^m = 0$ for a local coordinate $z$ of $X$ at $s$ and denote by $\chi(S_m)$ the Euler-Poincaré characteristic of $S_m$. A’Campo proved that $s_m = m\chi(S_m)$. More precisely:

**Theorem 4.3.** One has

1. $\Lambda(h^k) = \sum_{m \geq 1} m\chi(S_m)$,
2. $\Lambda(h^0) = \chi(F_0) = \sum_{m \geq 1} m\chi(S_m)$,
3. $\mu = \dim H^{n-1}(F_0, \mathbb{C}) = (-1)^{n-1}[-1 + \sum_{m \geq 1} m\chi(S_m)]$.

Therefore the numbers $\chi(S_m)$ don’t depend on the chosen resolution and are topological invariants of the singularity. As a consequence we have the following result that may be useful for resolving the multiplicity conjecture.

**Proposition 4.4.** If $f(z) = f_k(z) + f_{k+1}(z) + \cdots + f_l(z)$, $g(z) = g_l(z) + g_{l+1}(z) + \cdots + g_l(z)$ and $k + r < l$ then there is no topological equivalence between germs of $V_f$ and $V_g$ at the origin.

**Proof.** Let $h_1$ and $h_2$ be the monodromies associated to $f$ and $g$ and $s_1, s_2, \cdots$ and $s'_1, s'_2, \cdots$ the two related sequences of $f$ and $g$ respectively as above. If there exists such an equivalence then $\Lambda(h_i) = \Lambda(h'_i) = 0$ and $s_j = s'_j = 0$ for every $j$. Hence $\mu_f = \mu_g = (-1)^{n-1}[-1 + \sum_{j \geq 1} s_j] = (-1)^n$. If $n$ is odd this is impossible and if $n$ is even, then $\mu_f = \mu_g = 1$. In this case $k = l = 2$. Contradiction! \hfill \Box

The second result is the following [1], [9]:

**Theorem 4.5.** Given two germs of hypersurfaces $V_f$ and $V_g$. Let $\mathbb{P}C(V_f)$, respectively $\mathbb{P}C(V_g)$, denote the projectivized tangent cone which is a subvariety of $\mathbb{C}P^{n-1}$. If $\chi(\mathbb{C}P^{n-1} \backslash \mathbb{P}C(V_f)) \neq 0$ and $\chi(\mathbb{C}P^{n-1} \backslash \mathbb{P}C(V_g)) \neq 0$, then topological equisingularity of $V_f$ and $V_g$ implies $m_f = m_g$.

The key point of the proof is that: if $\chi(\mathbb{C}P^{n-1} \backslash \mathbb{P}C(V_f)) \neq 0$ then by theorem (4.3), $m_f = \inf \{s \in \mathbb{N} | \Lambda(h^s) \neq 0\}$.

It is unknown whether $\chi(\mathbb{C}P^{n-1} \backslash \mathbb{P}C(V_f))$ is a topological invariant or not.

**Example 4.6.** Let $g = z_1^l + z_2^l + \cdots + z_n^l$, $V_g = g^{-1}(0) \subset \mathbb{C}^n$ and $C(V_g) \subset \mathbb{C}P^{n-1}$ and $F_0$ be the fibre of the Milnor fibration of $g$ at the origin. By (6.1) in the appendix we have

$$\mu_g = (l - 1)^n.$$

With one blowing up at the origin, the singularity of $g$ may be resolved and then we may apply the theorem (4.3): $S = \mathbb{C}P^{n-1}$ and

$$S_m = \begin{cases} \phi & \text{if } m \neq l \\ \mathbb{C}P^{n-1} \backslash V_g & \text{if } m = l. \end{cases}$$
By theorem (4.3),

$$\mu_g = (-1)^{n-1}[1 + l\chi(S_l)].$$

The numbers $$\chi(S_l)$$ and $$\chi(C(V_g))$$ are related by

$$\chi(S_l) + \chi(C(V_g)) = \chi(\mathbb{C}P^{n-1}) = n.$$ 

Therefore we obtain the following well known formula

$$\chi(C(V_g)) = n - \frac{1 - (1 - l)^n}{l}.$$

**Example 4.7.** Let $$\mathcal{A}$$ be the set of all holomorphic functions $$g$$ such that $$0 \in \mathbb{C}^n$$ is an isolated singularity for the first nonzero homogeneous part of the Taylor expansion of $$g$$. Then by an argument (see (6.3) in the appendix) the origin is an isolated singularity of $$g$$. Let $$g \in \mathcal{A}$$ with algebraic multiplicity $$l$$ and the leading term $$g_1$$. Since $$g$$ and $$g_1$$ have the same projetivized tangent cones and $$V_{g_1}$$ is topologically equivalent to $$V_{z_1^l + z_2^l + \cdots + z_n^l}$$ then by the previous example

$$\chi(\mathbb{C}P^{n-1} \setminus \mathbb{P}C(V_g)) = \frac{1 - (1 - l)^n}{l} \neq 0.$$ 

Hence by theorem (4.5) any topological equivalence between two elements of $$\mathcal{A}$$ preserves multiplicities.

Still it is unknown whether there is any topological equivalence between $$g \in \mathcal{A}$$ and $$f \notin \mathcal{A}$$. By contrary if there exists such an equivalence then $$k < l$$, where $$k$$ and $$l$$ are the multiplicities of $$f$$ and $$g$$ respectively. The reason is that by (6.2) the Milnor number $$\mu_f > (k - 1)^n$$ and $$\mu_g = (l - 1)^n$$ and by theorem (4.3), Milnor number is a topological invariant. Therefore it remains to show that: Let $$g = z_1^l + z_2^l + \cdots + z_n^l$$ and $$f = f_k + \cdots + f_{k+r}$$ with $$k < l$$ and $$k + r \geq l$$ then the germs $$V_f$$ and $$V_g$$ at the origin are not topologically equisingular.

5. On the deformation of complex hypersurfaces

Let us, instead of dealing with a pair of hypersurfaces, consider families of hypersurfaces, $$V_{f_t}$$, all having an isolated singular point at the origin and depending continuously in $$t \in [0, 1]$$ and $$f_0 = f$$ and $$f_1 = g$$. We denote by $$C(V_{f_t})$$, the tangent cone at 0 of $$V_{f_t}$$, that is, the zero set of the initial polynomial of $$f_t$$. H. King generalized theorem (2.4) as follows [11]:

**Theorem 5.1.** Suppose $$f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$$, $$t \in [0, 1]$$ is a continuous family of holomorphic germs with the same Milnor number and $$n \neq 3$$. Then there is a continuous family of germs of homeomorphisms $$h_t : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$$ so that $$f_0 = f_t \circ h_t$$

Now we have the following result:
Theorem 5.2. If for every $t_0 \in [0,1]$ there exist a neighborhood $I_{t_0}$ of $t_0$ in $[0,1]$ and a line $L_{t_0}$ through 0 in $\mathbb{C}^n$ such that $L_{t_0} \cap C(V_{f_s}) = \{0\}$ for $s \in I_{t_0}$, then topological equisingularity of the family implies equimultiplicity provided that $n \neq 3$.

Proof. By theorem (5.1) there exists a continuous family of homeomorphisms $\varphi_t$ such that $f_t = f \circ \varphi_t$. Therefore for every $t_0 \in [0,1]$ we may write

$$f_s = f_{t_0} \circ \varphi_{t_0}s,$$

where $\varphi_{t_0}s = \varphi_t^{-1} \circ \varphi_s$. Since $f_{t_0}$ is uniformly continuous on a compact small ball $B_r \subset \mathbb{C}^n$ around 0, there exists $\eta > 0$ such that, for any $z, w \in B_r$,

$$|z - w| < \eta \implies |f_{t_0}(z) - f_{t_0}(w)| < \min_{u \in S_\delta}|f_{t_0}(u)|,$$

where $S_\delta$ is the boundary of $\overline{D_\delta}$, the closed disc with radius $\delta < r/2$ in $L_{t_0}$ around 0. Let $\varepsilon := \min\{\eta, \delta \}$. By continuity of $\varphi_s$, if $I_{t_0}$ is sufficiently small then $|\varphi_{t_0}s(z) - z| < \varepsilon$ for $s \in I_{t_0}$. Then for all $z$ in the closed ball $B_\delta \subset \mathbb{C}^n$, $\varphi_{t_0}s(z) \in B_r$ and

$$|f_{t_0}(z) - f_s \circ \varphi_{t_0}s(z)| < \min_{u \in S_\delta}|f_{t_0}(u)|.$$

In particular for all $z \in S_\delta$ we have

$$|f_{t_0}(z) - f_s(z)| < |f_{t_0}(z)|,$$

for $s \in I_{t_0}$.

By hypothesis $L_{t_0} \cap C(V_{f_s}) = \{0\}$ for $s \in I_{t_0}$, then $m_{f_s}$ is the order at 0 of $f_s|L_{t_0}$ for $s \in I_{t_0}$. By theorem (1.6) in [12, Ch.VI], $f_{t_0}|L_{t_0}$ and $f_s|L_{t_0}$ have the same number of zeros, counted with their multiplicities in the interior of $\overline{D_\delta}$. As $f_{t_0}|L_{t_0}$ and $f_s|L_{t_0}$ vanish only at 0 on $\overline{D_\delta}$, the orders at 0 of $f_{t_0}|L_{t_0}$ and $f_s|L_{t_0}$ are equal. So $m_{f_{t_0}} = m_{f_s}$ for $s \in I_{t_0}$. This tells us that the multiplicity of the deformation is constant. $\square$

6. Topological invariants for holomorphic vector fields

Let $U \subset \mathbb{C}^n$ be an open neighborhood of $0 \in \mathbb{C}^n$ and $X : U \to \mathbb{C}^n$, $X(0) = 0$, a holomorphic vector field with a singularity at $0 \in \mathbb{C}^n$. The integral curves of $X$ are complex curves i.e. Riemann surfaces parametrized locally as the solutions of the differential equation

$$\frac{dx}{dt} = X(x), \quad x \in U, \quad t \in \mathbb{C}.$$

These curves define a complex one dimensional foliation $\mathcal{F} = \mathcal{F}_X$ with singularity at $0 \in \mathbb{C}^n$. We define the algebraic multiplicity of $X$ as the degree of its first nonzero jet, i.e. $m = m_X$ where

$$X = X_m + X_{m+1} + \cdots$$

is the Taylor series of $X$ and $X_m$ is not identically zero.

In analogy with the case of hypersurfaces we define the Milnor number of the vector field $X$ at $0 \in \mathbb{C}^n$ as

$$\mu = \dim \mathbb{C} \left\langle \frac{O_n}{\langle X_1, \cdots, X_n \rangle} \right.$$
where \( \mathcal{O}_n \) is the ring of germs of holomorphic functions at \( 0 \in \mathbb{C}^n \) and 
\(< X_1, \cdots, X_n >\) is the ideal generated by the germs at \( 0 \in \mathbb{C}^n \) of the coordinate functions of \( X \). This number is finite if and only if \( 0 \in \mathbb{C}^n \) is an isolated singularity of \( X \), a hypothesis which we will assume from now on.

We say \( \mathcal{F}_X \) is topologically equivalent with \( \mathcal{F}_{X'} \), \( X' \) is a holomorphic vector field defined in a neighborhood \( U' \) of \( 0 \in \mathbb{C}^n \), if there is a homeomorphism \( \varphi : U \to U' \) fixing the origin (singularity) and sending every leaf of the foliation \( \mathcal{F}_X \) into a leaf of the foliation \( \mathcal{F}_{X'} \).

A similar question is the following: is \( m_X \) a topological invariant of the foliation \( \mathcal{F}_X \)?

In a remarkable work [5], C. Camacho, A. Lins Neto and P. Sad deal with this problem. First of all they prove the following result:

**Theorem 6.1.** The Milnor number of a holomorphic vector field \( X \) as above is a topological invariant provided that \( n \geq 2 \).

Now we restrict ourselves to \( n = 2 \) and recall some of the features coming from [5]. A germ of a vector field \( X \) with an isolated singularity may be given by \( X = a(x,y) \frac{\partial}{\partial x} + b(x,y) \frac{\partial}{\partial y} \) where \( a \) and \( b \) are holomorphic functions with isolated zero in a neighborhood \( U \subset \mathbb{C}^2 \). Denote by \( \mathcal{F} \) the foliation induced by \( X \). The main tool in the local study is the resolution theorem of Seidenberg [21] that establishes a canonical reduction. More precisely, there is a holomorphic map \( \pi : M \to \mathbb{C}^2 \) obtained as a composition of a finite number of blowing ups at points over \{0\}, such that at each singular point \( m \in M \) of \( \tilde{\mathcal{F}} \) the foliation with isolated singularity constructed from \( \pi^*(\omega) \) is reduced: there are coordinate charts \((z,w)\) such that \( z(m) = w(m) = 0 \), \( \tilde{\mathcal{F}} \) is given locally by the expression \( A(z,w) \frac{\partial}{\partial x} + B(z,w) \frac{\partial}{\partial y} = 0 \) and the Jacobian \( \frac{\partial(A,B)}{\partial(z,w)}(0,0) \) has at least one nonzero eigenvalue. Moreover if \( \lambda_1 \neq 0 \neq \lambda_2 \) are eigenvalues of the matrix then \( \lambda_1/\lambda_2 \notin \mathbb{Q}_+ \). If one of the eigenvalues of the above matrix of a singularity is zero and another different from zero we call it saddle-node. By definition a generalized curve is a germ of a vector field \( X \) at \( 0 \in (\mathbb{C}^2,0) \) and singular at the origin such that its desingularization does not admit any saddle-node. In [5] Camacho, Lins Neto and Sad proved that this property is invariant under topological equivalences and finally they deduced the following:

**Theorem 6.2.** The algebraic multiplicity of a generalized curve is a topological invariant.

Also we may say the same as theorem (3.2) for polynomial foliations.

7. **Appendix**

Let \( \mathcal{A} \) be the set of all holomorphic functions \( f \) such that \( 0 \in \mathbb{C}^n \) is an isolated singularity not only for \( f \) but also for the first nonzero homogeneous polynomial of the Taylor expansion of the \( f \). Actually if the origin is an isolated singularity of the leading term of \( f \) then the same holds for \( f \).
Remark 7.1. We have the following relation between multiplicity and Milnor number of $f$ (see page 194 in [3]):

$$\mu_f = (m_f - 1)^n.$$ 

Remark 7.2. If $0 \in \mathbb{C}^n$ is not an isolated singularity of the first nonzero homogeneous polynomial then $\mu_f > (m_f - 1)^n$.

The following proposition is true in any dimension. But the following proof is based on the theorem of Lé and Ramanujam which is valid for $n \neq 3$.

**Proposition 7.3.** The germ at the origin of the hypersurface defined by an element $f \in \mathcal{A}$ with the algebraic multiplicity $k$ is topologically equivalent with the germ at the origin of the hypersurfaces defined by $z_1^k + \cdots + z_n^k$.

**Proof.** By a symbol $f \sim g$ between two germs of holomorphic functions at the origin we mean $V_f$ and $V_g$, two germs of hypersurfaces defined by $f$ and $g$ respectively, are topological equivalent. Let

$$f = f_k + f_{k+1} + \cdots$$

be the Taylor expansion of $f$, where $f_i$ is homogeneous polynomial of degree $i$. The family $(H_t)_{t \in [0,1]} \in \mathcal{A}$:

$$H_t = f_k + tf_{k+1} + t^2 f_{k+2} + \cdots$$

defines a $\mu$-constant family between $f$ and $f_k$. So $f \sim f_k$.

Now our task is to show $P(z) \sim (z_1^k + \cdots + z_n^k)$ where $P(z)$ is a homogeneous polynomial of degree $k$.

**Claim:** There is a non zero complex number $\alpha$ such that $0$ is an isolated singularity of $F_t(z) := (1 - t)(z_1^k + \cdots + z_n^k) + t\alpha P(z)$ for $t \in [0,1]$.

**The proof of claim:** The partial derivatives of $F_t(z)$ form a system of bihomogeneous polynomials of bidegree $(1, k - 1)$:

$$\frac{\partial F_t}{\partial z_1} = k(1-t)z_1^{k-1} + t\alpha \frac{\partial P}{\partial z_1}$$

$$\vdots$$

$$\frac{\partial F_t}{\partial z_n} = k(1-t)z_n^{k-1} + t\alpha \frac{\partial P}{\partial z_n}$$

and $V := \text{Zero}(\frac{\partial F_t}{\partial z_1}, \cdots, \frac{\partial F_t}{\partial z_n})$ is an algebraic subset of $\mathbb{C}P(1) \times \mathbb{C}P(n-1)$.

Now consider the projection $\pi : \mathbb{C}P(1) \times \mathbb{C}P(n-1) \to \mathbb{C}P(1)$. Image of $V$, $\pi(V)$, is a Zariski-closed subset of $\mathbb{C}P(1)$ (see for instance [16] Pg. 33).

Since $F_t(z)$ for $t = 0$ has the isolated singularity, $(1 : 0)$ is not in the $\pi(V)$. Therefore $\pi(V)$ is finite and there are infinitely many lines in the complement of $\pi(V)$ in $\mathbb{C}P(1)$. Since $P(z)$ has an isolated singularity at $0 \in \mathbb{C}^n$ we may choose lines passing through the origin. This means that there is $\alpha$ such that the claim is true for every $t \in \mathbb{R}$.

$\square$
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