Integrated Differential Geometry. Commutative and Noncommutative.

Hendrik Grundling,
Department of Pure Mathematics, University of New South Wales,
P.O. Box 1, Kensington, NSW 2033, Australia.
email: hendrik@solution.maths.unsw.edu.au

Abstract For a manifold $M$ we define a structure on the group action of $\text{Diff}(M)$ on $C^\infty(M)$ which reduces to the usual differential geometry upon differentiation at zero along the one-parameter groups of $\text{Diff}(M)$. This “integrated differential geometry” generalises to all group actions on associative algebras, including noncommutative ones, and defines an “integrated de Rham cohomology,” which provides a new set of invariants for group actions. We calculate the first few integrated de Rham cohomologies for two examples: a discrete group action on a commutative algebra, and a continuous Lie group action on a noncommutative matrix algebra.

Keywords: group action, cohomology, noncommutative differential geometry.

AMS classification: 46L87, 58B30, 46L55, 18G60, 14F99

Running headline: Integrated Differential Geometry.
1. Introduction.

A problem with blending C*-algebras and differential geometry as in the various approaches to noncommutative differential geometry [1,2,3,4,5], is that C*-algebras deal best with bounded information, whilst differential geometry contains unbounded information expressed infinitesimally. Connected to this is the fact that C*-algebras are appropriate to the category of continuous function spaces with homeomorphisms, whilst differential geometry is appropriate to smooth function spaces with diffeomorphisms. This suggests that one should look for a larger “integrated” structure on a manifold, definable on its continuous functions, which can be “differentiated at zero” on the smooth functions to reproduce the usual differential geometry associated with the manifold. This larger structure can then be generalised to noncommutative C*-algebras with greater ease, thus avoiding derivations of dense *-algebras [4,6].

This paper runs as follows. In Sect. 2 we set up integrated differential forms on a manifold, and generalise this to noncommutative algebras in Sect. 3. On the set of these, we define an “integrated” differential \( \hat{d} \) in Sect. 4, and show that it satisfies \( \hat{d}^2 = 0 \) and reduces to the usual differential in the case of an algebra of smooth functions on a manifold, when we differentiate at zero on one-parameter subgroups of \( \text{Diff}(M) \). This defines then an “integrated de Rham cohomology” of which we calculate the first two for an example in Sect. 5 consisting of the shift automorphism acting on an algebra of sequences. In Sect. 6 we work out the first “integrated de Rham cohomology” for the algebra \( M_2(\mathbb{C}) \) under the action of the identity component of its automorphism group.
2. The Basic Set-Up. Commutative Case.

Let $M$ be an $n$–dimensional manifold, not necessarily compact. Now the vector fields $X \in \mathcal{X}(\mathcal{B})$ (i.e. the derivations of the smooth algebra $\mathcal{B} := C_0^\infty(\mathcal{M}) \subset C_0(\mathcal{M}) := \mathcal{A}$, subscript 0 indicates functions vanishing at infinity) need not be complete (i.e. integrable). However, differential forms are fully defined on the vector fields of compact support $\mathcal{X}_c(\mathcal{B})$, and the latter are indeed complete, and form a Lie ideal of $\mathcal{X}(\mathcal{B})$ which is a $\mathcal{B}$–module. For $X \in \mathcal{X}_c(\mathcal{B})$, denote its flow by $\varphi^X : \mathbb{R} \to \text{Diff } \mathcal{M}$ which in turn defines a one–parameter automorphism group for the $C^*$–algebra $\mathcal{A} = C_0(\mathcal{M})$ by

\[
(\alpha^X_t(f))(m) := f(\varphi^X_t(m)) \quad \forall f \in C_0(\mathcal{M}), \ t \in \mathbb{R}, \ m \in \mathcal{M}
\]

which clearly preserves $\mathcal{B}$.

Consider the one–forms of $\mathcal{M}$, but instead of using the usual definition of $\mathcal{B}$–linear maps from $\mathcal{X}_c(\mathcal{B})$ to $\mathcal{B}$, we use the fact that any smooth one–form $\omega \in \Omega^1(\mathcal{M})$ has a (nonunique) expression as $\omega = \sum_i g_i \, df_i$ ; $g_i, f_i \in \mathcal{B}$. Now for all $X \in \mathcal{X}_c(\mathcal{B})$, $m \in \mathcal{M}$:

\[
df(X)(m) = X(f)(m) = \frac{d}{dt} f(\varphi^X_t(m))\bigg|_{t=0}
\]

and so for $\omega$:

\[
\omega(X)(m) = \left( \sum_i g_i \, df_i \right)(X)(m) = \frac{d}{dt} \sum_i g_i(m) \, f_i(\varphi^X_t(m))\bigg|_{t=0} = \frac{d}{dt} \sum_i g_i(m) \, \alpha^X_t(f_i)(m)\bigg|_{t=0} \quad (2.1)
\]

This suggests the following:

**Definition:** An integrated one–form $\tilde{\omega}$ of $\omega \in \Omega^1(M)$ is a map $\tilde{\omega} : \mathbb{R} \times \mathcal{X}_c(\mathcal{B}) \to \mathcal{B}$ such that

\[
\tau(\tilde{\omega})(X)(m) := \frac{d}{dt} \tilde{\omega}(t, X)(m)\bigg|_{t=0} = \omega(X)(m) \quad \forall X \in \mathcal{X}_c, \ m \in \mathcal{M}
\]

There may be several integrated one–forms for each one–form. In particular, if $\omega = \sum_i g_i \, df_i$ is a representation of $\omega$, then by (2.1), $\tilde{\omega}(t, X) = \sum_i g_i \alpha^X_t(f_i)$
will be an integrated one–form for \( \omega \). We now limit our attention to those integrated one–forms coming from representations \( \omega = \sum g_i df_i \), as above:

**Definition:** Given the set of one–parameter groups \( \alpha : \mathbb{R} \times \mathcal{X}_c(B) \to \text{Aut}A \), define \( \tilde{\Omega}_\alpha^1 \) as the set of all those integrated one–forms of the form

\[
\tilde{\omega}(t, X) = \sum_{i=1}^{k} g_i \alpha_i^X(f_i), \quad f_i, g_i \in B, \quad k < \infty.
\]

Note that \( \tilde{\Omega}_\alpha^1 \) is a \( B \)–linear space, and \( \tau : \tilde{\Omega}_\alpha^1 \to \Omega(M) \) is a surjective \( B \)–linear map, i.e. \( \tau(f\tilde{\omega}) = f\tau(\tilde{\omega}) \) for all \( f \in B \), but since \( \omega = \tau(\tilde{\omega}) \) is \( B \)–linear, we also have that

\[
\tau(\tilde{\omega})(fX)(m) = f(m) \cdot \tau(\tilde{\omega})(X)(m) \quad \forall X \in \mathcal{X}_c(B), \ f \in B,
\]

and this expresses locality w.r.t. \( M \) (which may be lost in the noncommutative case, cf. [5, 1]). First, let us generalise away from the smooth structures on \( M \) to the merely continuous:

**Definition:** Given any set of one–parameter groups \( \beta : \mathbb{R} \times I \to \text{Aut}A \) (\( I \) is an index set), the set \( \hat{\Omega}_\beta^1 \) of total one–forms of \( \beta \) consists of all maps \( \tilde{\omega} : \mathbb{R} \times I \to A \) of the form

\[
\tilde{\omega}(t, X) = \sum_i g_i \beta_i^X(f_i), \quad f_i, g_i \in A; \ X \in I, \ t \in \mathbb{R}.
\]

(Notation: \( \beta_i^X := \beta(t, X) \)).

**Notes:**

1. In the case \( \alpha : \mathbb{R} \times \mathcal{X}_c(B) \to B \) above, \( \hat{\Omega}_\alpha^1 \supset \tilde{\Omega}_\alpha^1 \), where the extra elements come from choices \( g_i, f_i \in A \setminus B \). Clearly the map \( \tau \) will only be definable for those \( \tilde{\omega} \) where \( \alpha_i^X(f_i) \) is differentiable in \( t \) at zero for all \( X \in \mathcal{X}_c(B) \) and \( i \). Denote the set of these by \( D^1(\tau) \).

2. \( \hat{\Omega}_\beta^1 \) is an \( A \)–module, and depends on the choice of \( \beta \).

3. Since \( \tau : D^1(\tau) \to \Omega^1(M) \) maps integrated one–forms of the type \( \tilde{\omega}(t, X) = \alpha_i^X(f) \) to exact one–forms, we will later want to identify the exact integrated one–forms as those of this type.

Next consider smooth two–forms. These also have (nonunique) expressions \( \omega = \sum_i g_i df_i \wedge dh_i \) where \( g_i, f_i, h_i \in B \). Then

\[
\omega(X, Y)(m) = \sum_i g_i(m)(df_i \wedge dh_i)(X, Y)(m)
\]
\[= \frac{1}{2} \sum_i g_i(m) [df_i(X) \cdot dh_i(Y) - dh_i(X) \cdot df_i(Y)](m)\]
\[= \frac{d}{dt} \cdot \frac{d}{ds} \left\{ \frac{1}{2} \sum_i g_i(m) [\alpha_i^X(f_i) \cdot \alpha_s^Y(h_i) - \alpha_s^Y(f_i) \cdot \alpha_i^X(h_i)](m) \right\} \bigg|_{t=0=s}\]
which suggests that an integrated two-form \( \tilde{\omega} : (\mathbb{R} \times \mathcal{C}(\mathcal{B}))^2 \rightarrow \mathcal{B} \) for \( \omega \) is
\[\tilde{\omega}(t, X; s, Y) = \sum_i g_i(\alpha_i^X(f_i) \alpha_s^Y(h_i) - \alpha_s^Y(f_i) \alpha_i^X(h_i)) .\]

In general the full set of integrated \( k \)-forms are:

**Definition 2.2:** Given the set of one-parameter groups \( \alpha : \mathbb{R} \times \mathcal{C}(\mathcal{B}) \rightarrow \text{Aut} \mathcal{A} \), the set \( \tilde{\Omega}_\alpha^k \) of integrated \( k \)-forms consists of all antisymmetric maps
\( \tilde{\omega} : (\mathbb{R} \times \mathcal{C}(\mathcal{B}))^k \rightarrow \mathcal{B} \) such that
\[\frac{d}{dt_1} \cdots \frac{d}{dt_k} \tilde{\omega}(t_1, X_1; \ldots; t_k, X_k) \bigg|_{t_i=0 \forall i} = \omega(X_1, \ldots, X_k)(m)\]
defines a \( k \)-form \( \omega \in \Omega^k(M) \).

**Notes:**
1. Clearly for a \( k \)-form \( \omega = \sum_i g_i \, df_i^1 \wedge \cdots \wedge df_i^k \) one integrated \( k \)-form is
\[\tilde{\omega}(t_1, X_1; \ldots; t_k, X_k) = \sum_i g_i \sum_{\sigma \in S_k} \varepsilon^\sigma \alpha_{t(1)}^X(f_i^1) \cdots \alpha_{t(k)}^X(f_i^k)\]
where \( S_k \) is the permutation group of \( k \) objects and \( \varepsilon^\sigma \) is the parity of \( \sigma \in S_k \).
2. \( \tilde{\Omega}_\alpha^k \) is a \( \mathcal{B} \)-linear space and we have as above, the surjective \( \mathcal{B} \)-linear map \( \tau : D^k(\tau) \rightarrow \Omega^k(M) \) given by
\[\tau(\tilde{\omega})(X_1, \ldots, X_k)(m) := \frac{d}{dt_1} \cdots \frac{d}{dt_k} \tilde{\omega}(t_1, X_1; \ldots; t_k, X_k) \bigg|_{t_i=0 \forall i}\]
hence
\[\tau(\tilde{\omega})(X_1, \ldots, fX_i, \ldots, X_k)(m) = f(m) \cdot \tau(\tilde{\omega})(X_1, \ldots, X_k)(m) \quad \forall f \in \mathcal{B}, \ i,\]
where \( D^k(\tau) \subset \tilde{\Omega}_\alpha^k \) is the domain of \( \tau \).

Next we generalise the previous definitions away from both the smooth structures, and from the one-parameter groups (hence derivations):

**Definition:** Given an indexed set of group actions \( \beta : G \times I \rightarrow \text{Aut} \mathcal{A} \) (\( I \) is an index set), the set \( \hat{\Omega}_\beta^k \) of total \( k \)-forms consists of all maps
\( \tilde{\omega} : (G \times I)^k \rightarrow \mathcal{A} \) of the form:
\[\tilde{\omega}(g_1, X_1; \ldots; g_k, X_k) = \sum_i h_i \sum_{\sigma \in S_k} \varepsilon^\sigma \beta_{g_1(1)}^{X_{\sigma(1)}}(f_i^1) \cdots \beta_{g_k(k)}^{X_{\sigma(k)}}(f_i^k) \quad -(*)\]
for \( h_i, f_i^k \in A \), \( g_i \in G \), \( X_i \in I \), and the sum over \( i \) is finite.

**Notes**

1. \( \hat{\Omega}^k_{\beta} \) is a left and right \( A \)-module, hence an \( A \)-linear space, and depends on \( \beta \). An \( \tilde{\omega} \in \hat{\Omega}^k_{\beta} \) is specified as a map \( \tilde{\omega} : (G \times I)^k \to A \), so may have different representations in the form (*) .

2. Clearly there are canonical choices for \( \beta \), e.g. \( G = \text{Aut} A \) and \( I = \{1\} \). If there is more than one non–diffeomorphic differential structure for \( M \), there are two dense \( C^\infty \)-subalgebras \( B_1 \) and \( B_2 \) of \( A \) such that \( B_1 \) and \( B_2 \) are not in the same automorphism class, then \( G \) can be the diffeomorphism group with respect to either of these.

3. We lose locality information by allowing any group \( G \).

### 3. Noncommutative integrated differential forms.

Maintain the concepts and notation of the last section. In this section we would like to generalise the total \( k \)-forms of the last section to noncommutative algebras \( A \). We remark that in the literature noncommutative differential forms have already appeared, cf. [1,2], but here we follow a different route. We first examine the algebraic context of the integrated differential forms. Following the line of thought above, observe that a reasonable “integrated covariant \( k \)-tensor” will be a map \( \varphi : (G \times I)^k \to A \) of the form:

\[
\varphi(g_1, X_1; \ldots; g_k, X_k) = \sum_i h_i \beta_{g_i}^{X_1}(f_i^1) \cdots \beta_{g_k}^{X_k}(f_i^k)
\]

and such maps form an algebra \( \mathcal{T}^k(A) \) under pointwise multiplication:

\[
(\varphi \cdot \psi)(g_1, X_1; \ldots; g_k, X_k) = \varphi(g_1, X_1; \ldots; g_k, X_k) \cdot \psi(g_1, X_1; \ldots; g_k, X_k).
\]

We also have the usual \( \mathbb{N} \)-graded product \( \star \) given by:

\[
(\varphi \star \psi)(g_1, X_1; \ldots; g_m, X_m) = \varphi(g_1, X_1; \ldots; g_k, X_k) \cdot \psi(g_{k+1}, X_{k+1}; \ldots; g_m, X_m)
\]

where \( \varphi \) is a \( k \)-tensor and \( \psi \) an \((m - k)\)-tensor . However, we will not need the \( \star \)-product much. Note that inside \( \mathcal{T}^k(A) \) the symmetric tensors is a subalgebra whilst the antisymmetric tensors (the \( k \)-forms) is a subspace. Now for the noncommutative generalisation, we henceforth assume \( A \) to be any associative \(*\)-algebra.
Definition: Given the *–algebra $\mathcal{A}$ and a fixed subgroup $G \subset \text{Aut} \mathcal{A}$, let $M^k(G, \mathcal{A})$ be the space of maps $\varphi : G^k \to \mathcal{A}$ and make it into a *–algebra with pointwise operations. For a fixed $A \in \mathcal{A}$, define the elements $\varphi^A_0$ and $\varphi^A_\ell \in M^k(G, \mathcal{A})$ by

$$\varphi^A_0(\alpha_1, \ldots, \alpha_k) := A, \quad \varphi^A_\ell(\alpha_1, \ldots, \alpha_k) := \alpha_\ell(A),$$

for $\alpha_i \in G$, $\ell \in \{1, \ldots, k\}$. Then we define $T^k(\mathcal{A})$ as the *–algebra generated in $M^k(G, \mathcal{A})$ by the set $\{ \varphi^A_\ell \mid A \in \mathcal{A}; \; \ell = 0, \ldots, k \}$.

Notes
1. We think of the elements of $T^k(\mathcal{A})$ as maps $\varphi : G^k \to \mathcal{A}$ of the form

$$\varphi(\alpha_1, \ldots, \alpha_k) = \sum_i A_i^0 \prod_{j=1}^{L_i} \alpha_{s_i(j)}(B_i^j) A_i^j$$

for all $\alpha_\ell \in G$, where $A_i^\ell, B_i^\ell, C_i^\ell \in \tilde{\mathcal{A}}$ are fixed and $s_i : \{1, 2, \ldots, L_i\} \to \{1, 2, \ldots, k\}$ is a surjection, $L_i \geq k$. $\tilde{\mathcal{A}}$ is $\mathcal{A}$ if it has an identity, and it is $\mathcal{A}$ with the identity adjoined otherwise. So $T^k(\mathcal{A})$ excludes maps of the form $\varphi(\alpha_1, \alpha_2) = \alpha_1 \circ \alpha_2(\mathcal{A})$. Clearly $T^0(\mathcal{A}) = \mathcal{A}$. Note that the same tensor $\varphi \in T^k(\mathcal{A})$ may have more than one representation (3.1), given that it is defined as a map.

2. To recover the tensor algebra $T^k(\mathcal{A})$ of above, for $\mathcal{A}$ the continuous functions on a manifold, we let $G = \text{Diff} M \subset \text{Aut} \mathcal{A}$, and identify the one–parameter groups of the compactly supported vector fields in $G$.

3. Note that $T^{k-r}(\mathcal{A}) \subset T^k(\mathcal{A})$ for $0 \leq r \leq k$ where a $\varphi \in T^{k-r}(\mathcal{A})$ is realised as a $k$–tensor $\tilde{\varphi} \in T^k(\mathcal{A})$ which is constant in the last $r$ variables, i.e.

$$\tilde{\varphi}(\alpha_1, \ldots, \alpha_k) = \varphi(\alpha_1, \ldots, \alpha_{k-r}).$$

On $T^k(\mathcal{A})$, identify the symmetrising and antisymmetrising projectors:

$$(P_+ \varphi)(\alpha_1, \ldots, \alpha_k) := \frac{1}{k!} \sum_{\sigma \in S_k} \varphi(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(k)})$$

$$= \frac{1}{k!} \sum_{\sigma \in S_k} \sum_i A_i^0 \prod_{j=1}^{L_i} \alpha_{\sigma(s_i(j))}(B_i^j) A_i^j \quad \text{for } \varphi \text{ as in (3.1)}$$

$$(P_- \varphi)(\alpha_1, \ldots, \alpha_k) := \frac{1}{k!} \sum_{\sigma \in S_k} \epsilon^\sigma \varphi(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(k)})$$
for all \( \varphi \in \mathcal{T}^k(A) \) and \( \alpha_i \in G \). Obviously \( (P_\pm)^2 = P_\pm \). Define the symmetric (resp. antisymmetric) tensors over \( A \) by \( \mathcal{T}^k_\pm(A) := P_\pm \mathcal{T}^k(A) \), then we regard \( \hat{\Omega}^k := \mathcal{T}^k(A) \) as the \textit{integrated k–forms} over \( A \) with respect to \( G \subset \text{Aut}A \). Note that \( \hat{\Omega}^0 = A \).

**Notes**

(1) Under pointwise multiplication \( \mathcal{T}^k_+(A) \cdot \mathcal{T}^k_+(A) \subseteq \mathcal{T}^k_+(A) \supseteq \mathcal{T}^k_-(A) \cdot \mathcal{T}^k_-(A) \) and \( \mathcal{T}^k_+(A) \cdot \mathcal{T}^k_-(A) \subseteq \mathcal{T}^k_-(A) \cdot \mathcal{T}^k_+(A) \), and so \( \mathcal{T}^k_+(A) \cup \mathcal{T}^k_-(A) \) generates a \( \mathbb{Z}_2 \)-graded \(*\)-algebra in \( \mathcal{T}^k(A) \).

(2) When \( A = C_0(M) \), we regain the total k–forms of the last section by replacing in the expression

\[
\omega(\alpha_1, \ldots, \alpha_k) = \frac{1}{k!} \sum_{\sigma \in S_k} e^\sigma \sum_i A_i^0 \prod_{j=1}^{L_i} \alpha_{\sigma(s_i(j))}(B_{ij}^2) A_{ij}^j
\]

\[
\alpha_i = \alpha_{t_i}^{X_i}, \quad \frac{1}{k!} A_i^0 \prod_{j=1}^{L_i} A_{ij}^j = g_i \quad \text{and} \quad \prod_{j \in s_i^{-1}(\ell)} B_{ij}^j = f_{i}^\ell, \quad \text{using commutativity in} \quad A.
\]

(3) When \( G = \text{Aut}A \), on a choice of one–parameter subgroups \( \alpha : \mathbb{R} \times I \to A \), we can define a map \( \tau \) to the infinitesimal k–forms as before (selecting a domain in \( \hat{\Omega}^k \)), but we need to specify in what topology the limits of the differentials should be taken. Possible choices are the C*-topology, weak operator topology of some representation of \( A \), weak *-topology w.r.t. some set of states etc. Note that by the definition of the integrated k-forms, as maps from \( G^k \) to \( A \), there will be some automatic continuity inherited from continuity of the action of \( G \) on \( A \). In the case where we have a C*-dynamical system in which the group is locally compact, this will be useful.
4. The integrated noncommutative de Rham complex.

In this section we want to define a de Rham structure on the integrated k–forms \( \hat{\Omega}^k \) of \( A \) with respect to \( G \subseteq \text{Aut} \ A \). That is, we want a linear map \( \hat{d} : \hat{\Omega}^k \rightarrow \hat{\Omega}^{k+1} \) such that \( \hat{d}^2 = 0 \), making \( \hat{\Omega}^* \) into a differential complex.

We want furthermore for the commutative case \( A = C_0^\infty(M) \) that \( \tau \circ \hat{d} \) be the usual exterior derivative for differential forms. We will not expect \( \hat{d} \) to be a derivation with respect to the \( A \)–action on \( \hat{\Omega}^* \), to enforce that is the work of \( \tau \) when it exists.

Consider the case when \( A = C_0^\infty(M) \). Then a k–form \( \omega = \sum_i g_i df_1^i \wedge \cdots \wedge df_k^i \) has differential \( d\omega = \sum_i dg_i \wedge df_1^i \wedge \cdots \wedge df_k^i \). So if we take as in definition 2.2 an \( \tilde{\omega} \in \tilde{\Omega}^k \) given by

\[
\tilde{\omega}(t_1, X_1; \ldots; t_k, X_k) = \sum_i g_i \sum_{\sigma \in S_k} \epsilon^\sigma \alpha_{l_{\sigma(1)}} X_{\sigma(1)}(f_1^i) \cdots \alpha_{l_{\sigma(k)}} X_{\sigma(k)}(f_i^k)
\]

then it seems reasonable to define \( \tilde{d} \tilde{\omega} \) by analogy

\[
(\tilde{d} \tilde{\omega})(t_1, X_1; \ldots; t_{k+1}, X_{k+1}) := \frac{1}{k+1} \sum_{i=1}^\ell \sum_{\sigma \in S_{k+1}} \epsilon^\sigma \alpha_{l_{\sigma(1)}} X_{\sigma(1)}(g_i) \cdot \alpha_{l_{\sigma(2)}} X_{\sigma(2)}(f_1^i) \cdots \alpha_{l_{\sigma(k+1)}} X_{\sigma(k+1)}(f_i^k)
\]

and then we have that \( \tau \circ \tilde{d} = d \), the usual derivative. However, a quick calculation with small \( k \) shows that \( \tilde{d}^2 \neq 0 \). This will be fixed below, but we first want to generalise \( \tilde{d} \) to \( T^k(A) \) where \( A \) is noncommutative, and also ensure that \( \tilde{d} \) is well–defined. Assuming now that \( A \) is a general *–algebra, let \( \varphi \in T^k(A) \) have the representation

\[
\varphi(\alpha_1, \ldots, \alpha_k) = \sum_i A_i^0 \prod_{j=1}^{L_i} \alpha_{s_i(j)}(B^j_i) A^j_i
\]

where \( G \subseteq \text{Aut} \ A \) is fixed and \( \alpha_{\ell} \in G \). Define:

\[
(\tilde{d}\varphi)(\alpha_1, \ldots, \alpha_{k+1}) := \sum_i \alpha_{k+1}(A^0_i) \prod_{j=1}^{L_i} \alpha_{s_i(j)}(B^j_i) \alpha_{k+1}(A^j_i).
\]

However, due to the possible nonuniqueness of the representation above for the map \( \varphi : G^k \rightarrow A \), it is not clear that \( \tilde{d} \) is well–defined. We rewrite the last
expression to get a more intrinsic expression:

\[(\tilde{d}\varphi)(\alpha_1, \ldots, \alpha_{k+1}) = \alpha_{k+1} \left( \sum_i A_i \prod_{j=1}^{L_i} \alpha_{k+1}^{-1} \circ \alpha_{s_i(j)} (B_j^i) A_i^j \right) \]

\[= \alpha_{k+1} \left( \varphi(\alpha_{k+1}^{-1} \alpha_1, \ldots, \alpha_{k+1}^{-1} \alpha_k) \right) \quad (4.1)\]

and clearly (4.1) is independent of the representation chosen for \( \varphi \), so we henceforth choose (4.1) as the definition of \( \tilde{d} : T^k(\mathcal{A}) \to T^{k+1}(\mathcal{A}) \), which is obviously well-defined. Now \( \tilde{d} \) is linear and we define the odd and even parts of it by

\[\tilde{d}_\pm := P_\pm(\tilde{d}P)\]

which clearly map \( \tilde{d}_\pm : T^k(\mathcal{A}) \to T^{k+1}(\mathcal{A}) \). Explicitly, let \( \psi = P_- \varphi \in \hat{\Omega}^k \), then

\[(\tilde{d}_- P_- \varphi)(\alpha_1, \ldots, \alpha_{k+1}) = (P_- \tilde{d} P_- \varphi)(\alpha_1, \ldots, \alpha_{k+1})
\[
= \frac{1}{(k+1)!} \sum_{\sigma \in S_{k+1}} \varepsilon^\sigma (\tilde{d} P_- \varphi)(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(k+1)})
\[
= \frac{1}{(k+1)!} \sum_{\sigma \in S_{k+1}} \varepsilon^\sigma \alpha_{\sigma(k+1)} \left( (P_- \varphi)(\alpha_{\sigma(k+1)}^{-1} \alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(k+1)}^{-1} \alpha_{\sigma(k)} \right) \quad (4.2)\]

Now observe that for any \( \varphi \in T^k(\mathcal{A}) \) we have:

\[(\tilde{d} \tilde{d} \varphi)(\alpha_1, \ldots, \alpha_{k+2}) = \alpha_{k+2} \left[ (\tilde{d} \varphi)(\alpha_{k+2}^{-1} \alpha_1, \ldots, \alpha_{k+2}^{-1} \alpha_{k+1}) \right]
\[
= \alpha_{k+2} \left[ \alpha_{k+2}^{-1} \alpha_{k+1} \varphi(\alpha_{k+1}^{-1} \alpha_{k+2}^{-1} \alpha_1, \ldots, \alpha_{k+1}^{-1} \alpha_{k+2}^{-1} \alpha_{k+1} \alpha_k) \right]
\[
= \alpha_{k+1} \left[ \varphi(\alpha_{k+1}^{-1} \alpha_1, \ldots, \alpha_{k+1}^{-1} \alpha_k) \right]
\[
= (\tilde{d} \varphi)(\alpha_1, \ldots, \alpha_{k+1}) . \]

Thus it is independent of \( \alpha_{k+2} \). Now to evaluate \( (\tilde{d}_-)^2 \), let \( \omega \in \hat{\Omega}^k = T^k_-(\mathcal{A}) \), then using (4.2):

\[(\tilde{d}_- \tilde{d}_- \omega)(\alpha_1, \ldots, \alpha_{k+2}) = \frac{1}{(k+2)!} \sum_{\sigma \in S_{k+2}} \varepsilon^\sigma \alpha_{\sigma(k+2)} \left( (\tilde{d}_- \omega)(\alpha_{\sigma(k+2)}^{-1} \alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(k+2)}^{-1} \alpha_{\sigma(k+1)} \right) \]

\[= \frac{1}{(k+1)!(k+2)!} \sum_{\sigma \in S_{k+2}} \varepsilon^\sigma \alpha_{\sigma(k+2)} \left[ \sum_{\bar{\sigma} \in S_{k+1}} \varepsilon^\bar{\sigma} \alpha_{\bar{\sigma}(k+2)} \left[ \omega(\alpha_{\bar{\sigma}(k+1)}^{-1} \alpha_{\bar{\sigma}(1)}, \ldots, \alpha_{\bar{\sigma}(k+1)}^{-1} \alpha_{\bar{\sigma}(k)} \right] \right] \]

\[= \frac{1}{(k+1)!(k+2)!} \sum_{\sigma \in S_{k+2}} \sum_{\bar{\sigma} \in S_{k+1}} \varepsilon^\sigma \alpha_{\sigma(k+1)} \left[ \omega(\alpha_{\sigma(k+1)}^{-1} \alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(k+1)}^{-1} \alpha_{\sigma(k)} \right] \]

\[= \frac{1}{(k+1)!(k+2)!} \sum_{\sigma \in S_{k+2}} \sum_{\bar{\sigma} \in S_{k+1}} \varepsilon^{\sigma \bar{\sigma}} (\tilde{d} \omega)(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(k+1)}) \quad (4.3)\]
Note that (4.3) is a linear combination of terms, each dependent on only \( k + 1 \) of the \( k + 2 \) variables \( \alpha_1, \ldots, \alpha_{k+2} \).

**Definition:** Define the \( \mathcal{A} \)-linear maps \( \Delta_\ell : \mathcal{T}^k(\mathcal{A}) \to \mathcal{T}^k(\mathcal{A}) \), \( 1 \leq \ell \leq k \) by

\[
(\Delta_\ell \varphi)(\alpha_1, \ldots, \alpha_k) := \varphi(\alpha_1, \ldots, \alpha_k) - \varphi(\alpha_1, \ldots, \alpha_{\ell-1}, e, \alpha_{\ell+1}, \ldots, \alpha_k)
\]

where \( e \in G \subseteq \text{Aut} \mathcal{A} \) is the identity automorphism. Then

\[
\Delta^{(k)} := \Delta_1 \Delta_2 \cdots \Delta_k.
\]

Note that a tensor \( \varphi \in \mathcal{T}^k(\mathcal{A}) \) is independent of the \( \ell \)-th entry, \( \alpha_\ell \), iff \( \Delta_\ell \varphi = 0 \). If \( \varphi \) is independent of any one of its arguments, then \( \Delta^{(k)} \varphi = 0 \). When it is obvious what degree tensor we are dealing with, we will omit the superscript on \( \Delta \).

Using the linearity of \( \Delta \), we see from (4.3) that

\[
\Delta(\tilde{d} \tilde{d} \omega) = 0 \quad \forall \omega \in \hat{\Omega}^k.
\]

Note that for \( \omega \in \hat{\Omega}^k \) we have by antisymmetry:

\[
(\Delta \omega)(\alpha_1, \ldots, \alpha_k) = \omega(\alpha_1, \ldots, \alpha_k) - \omega(e, \alpha_2, \ldots, \alpha_k) - \cdots - \omega(\alpha_1, \ldots, \alpha_{k-1}, e)
\]

**Theorem 4.4.** Assume the hypotheses and notation above. Then

\[
(i) \quad \Delta^2 = \Delta,
\]

\[
(ii) \quad P_- \Delta = \Delta P_-, \n\]

\[
(iii) \quad \text{define } \hat{d} : \hat{\Omega}^k \to \hat{\Omega}^{k+1} \text{ by } \hat{d} := \Delta \tilde{d} \Delta, \text{ then } \hat{d}^2 = 0.
\]

**Proof:**

\( (i) \) By definition \( (\Delta \psi)(\alpha_1, \ldots, \alpha_k) = \psi(\alpha_1, \ldots, \alpha_k) + \) terms in which some of the \( \alpha_i \)'s have been replaced by \( e \). The latter terms are in \( \text{Ker} \Delta \), so clearly

\[
(\Delta \Delta \psi)(\alpha_1, \ldots, \alpha_k) = (\Delta \psi)(\alpha_1, \ldots, \alpha_k) \quad \forall \psi \in \mathcal{T}^k(\mathcal{A}).
\]

\( (ii) \) Let \( \psi \in \mathcal{T}^k(\mathcal{A}) \), then

\[
(\Delta \psi)(\alpha_1, \ldots, \alpha_k) = \psi(\alpha_1, \ldots, \alpha_k) - \left[ \psi(e, \alpha_2, \ldots, \alpha_k) + \cdots + \psi(\alpha_1, \ldots, \alpha_{k-1}, e) \right]
\]

\[
\quad + \sum_{i<j} \psi(\alpha_1, \ldots, \alpha_i-1, e, \alpha_{i+1}, \ldots, \alpha_{j-1}, e, \alpha_{j+1}, \ldots, \alpha_k) - \cdots
\]

\[
=: \left( 1 - \sum_{\alpha_i \to e} + \sum_{\alpha_i, \alpha_j \to e} - \cdots \right) \psi(\alpha_1, \ldots, \alpha_k)
\]

\[\text{Note: } k \text{ is a fixed integer.}\]
in self-evident notation. So

\[
(P_\Delta \psi)(\alpha_1, \ldots, \alpha_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \epsilon^\sigma(\Delta \psi)(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(k)})
\]

\[
= \frac{1}{k!} \sum_{\sigma \in S_k} \epsilon^\sigma \left( 1 - \sum_{\alpha_{\sigma(i)} \rightarrow e}^k + \sum_{\alpha_{\sigma(i)}, \alpha_{\sigma(j)} \rightarrow e}^k - \cdots \right) \psi(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(k)}).
\]

Observe that for each \( \sigma \in S_k \) we have

\[
\sum_{\alpha_{\sigma(i)} \rightarrow e}^k \psi(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(k)}) = \sum_{\alpha_i \rightarrow e}^k \psi(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(k)})
\]

and

\[
\sum_{\alpha_{\sigma(i)}, \alpha_{\sigma(j)} \rightarrow e}^k \psi(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(k)}) = \sum_{\alpha_i, \alpha_j \rightarrow e}^k \psi(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(k)})
\]

since the sums are over all possible single replacements or pairs of replacements by \( e \). Similar statements are true for the higher terms.

Thus

\[
(P_\Delta \psi)(\alpha_1, \ldots, \alpha_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \epsilon^\sigma \left( 1 - \sum_{\alpha_i \rightarrow e}^k + \sum_{\alpha_i, \alpha_j \rightarrow e}^k - \cdots \right) \psi(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(k)})
\]

\[
= \left( 1 - \sum_{\alpha_i \rightarrow e}^k + \sum_{\alpha_i, \alpha_j \rightarrow e}^k - \cdots \right) \frac{1}{k!} \sum_{\sigma \in S_k} \epsilon^\sigma \psi(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(k)})
\]

\[
= (\Delta P_\psi)(\alpha_1, \ldots, \alpha_k).
\]

\( (iii) \) To show that \( \tilde{d}^2 = \Delta \tilde{d}_- \Delta \tilde{d}_- \Delta = 0 \) on \( \hat{\Omega}^k \), recall the result \( \Delta \tilde{d}_- \Delta \tilde{d}_- = 0 \) from above. Then we will have that \( \tilde{d}^2 = 0 \) if we can show that \( \Delta \tilde{d}_-(1 - \Delta)\tilde{d}_- \Delta = 0 \) on \( \hat{\Omega}^k \). Now recall that \( \tilde{d}_- = P_\Delta \tilde{d}_0 P_\Delta \) maps into \( \hat{\Omega}^{k+1} \) and that \( 1 - \Delta \) on \( \omega \in \hat{\Omega}^{k+1} \) has the form

\[
\sum_{\alpha_1 \rightarrow e}^{k+1} \omega(\alpha_1, \ldots, \alpha_{k+1}),
\]

so it produces a sum with terms, each depending on only \( k \) of the \( k+1 \) original variables. Thus, since \( \tilde{d}_- \) can only add one dependence, we find that \( \tilde{d}_-(1 - \Delta)\tilde{d}_- \Delta \) is a sum of terms, each depending on only \( k+1 \) of the \( k+2 \) variables. Hence \( \Delta \tilde{d}_-(1 - \Delta)\tilde{d}_- \Delta = 0 \), i.e. \( \tilde{d}^2 = 0 \). This can also be done by explicit calculation.
Remarks: (1) By this theorem we have obtained a new chain complex, hence cohomology theory which can be associated to any group action on an associative algebra \( A \). Since \( \text{Aut} A \) is an intrinsic group action for \( A \), the cohomology of \( A \) w.r.t the group \( \text{Aut} A \) is an invariant of \( A \).

(2) In the case where \( A = C_0(M) \), and \( G = \text{Aut} A = \text{Homeo}(M) \), we thus have obtained a cohomology theory for any topological space \( M \), independent of differential structures. In particular, let \( M \) be a space having more than one differential structure, e.g. \( \mathbb{R}^d \), so we have two dense \( C^\infty \) subalgebras \( B_1, B_2 \subset A \) corresponding to nondiffeomorphic differential structures with automorphism groups \( \text{Diff}_1M \) and \( \text{Diff}_2M \) resp. Then the integrated de Rham cohomologies for \( M \) are simply the restrictions of the cohomology for \( A \) with \( \text{Aut} A \) to the algebras \( B_i \) with groups \( \text{Diff}_iM \), \( i = 1, 2 \) and under the corresponding \( \tau \)-maps these map to the de Rham cohomologies. So the integrated de Rham cohomology of \( A \) with \( \text{Aut} A \) is some sort of “universal receptacle” for all the de Rham cohomologies associated with \( M \).

(3) Observe that the range of \( \Delta \) consists of those \( \omega \in \mathcal{T}^k \) for which \( \omega(\alpha_1, \ldots, \alpha_k) = 0 \) if any \( \alpha_i \) is equal to \( e \). That is:

\[
\Delta \mathcal{T}^k(A) = \{ \omega \in \mathcal{T}^k(A) \mid \text{Ker} \omega \supset \{ e \} \times G \times \cdots \times G \cup \cdots \cup G \times \cdots \times G \times \{ e \} \}
\]

**Theorem 4.5.** Let \( M \) be a finite dimensional manifold, \( A = C_0^\infty(M) \), \( G = \text{Diff}_0 M \). Then

(i) there is a linear map \( \tau : \hat{\Omega}^k \rightarrow \Omega^k(M) \) for each \( k \), such that \( \tau \circ \hat{d} = d \circ \tau \), where \( d \) is the usual exterior derivative.

(ii) Denote the (integrated de Rham) cohomology produced by the differential complex \( (\hat{\Omega}^*, \hat{d}) \) by \( \hat{H}^*(A) \), then there is a surjective homomorphism to the usual de Rham cohomology groups \( \hat{\tau} : \hat{H}^k(A) \rightarrow H^k(M) \) for each \( k \).

**Proof:** (i) Let \( \omega \in \hat{\Omega}^k \) and \( X_1, \ldots, X_k \in \mathcal{X}_c(A) \), and define

\[
(\tau \omega)(X_1, \ldots, X_k)(m) := \left. \frac{d}{dt_1} \cdots \frac{d}{dt_k} \omega(\alpha_{t_1}^{X_1}, \ldots, \alpha_{t_k}^{X_k})(m) \right|_{t_i = 0 \forall i}
\]
where $\alpha^X_t$ is as before. Clearly $\tau$ is linear. Now $\hat{d}\omega = \Delta \tilde{d} - \Delta \omega = \tilde{d} - \omega + $ terms depending on fewer than $k$ variables. Then since $\tau$ is zero on the latter terms,

$$(\tau(\hat{d}\omega))(X_1, \ldots, X_{k+1})(m) = \frac{d}{dt_1} \cdots \frac{d}{dt_{k+1}}(\hat{d}\omega)(\alpha^X_{t_1}, \ldots, \alpha^X_{t_{k+1}})\bigg|_{t_i=0 \forall i}$$

$$= \tau(\tilde{d}\omega)(X_1, \ldots, X_{k+1})(m)$$

$$= (d\omega)(X_1, \ldots, X_{k+1})(m)$$

where the last equality is already known. Thus $\tau \circ \hat{d} = d \circ \tau$.

(ii) This part follows from (i), since clearly if $\hat{d}\omega = 0$ then $d(\tau\omega) = 0$, hence $\tau$ maps closed forms to closed forms, and by the explicit formulae above for forms, we see that any closed de Rham form is an image under $\tau$ of a closed total form. Moreover if $\tau\omega$ is exact, i.e. $\tau\omega = d\varphi = d\tau\tilde{\varphi} = \tau \cdot \hat{d}\tilde{\varphi}$, then $\tau\omega$ is the image under $\tau$ of an exact form in $\hat{\Omega}^*$. Thus $\tau$ respects cohomology classes, so $\tau$ lifts to a linear surjective map by $\hat{\tau}[\omega] := [\tau\omega]$ where $[\omega]$ is the class of $\omega \in \hat{\Omega}^k$, as claimed.

Notes

(1) These two theorems establish the claim that the current constructions produce a cohomology theory which generalises de Rham cohomology. It is also clear that the cohomology classes $\hat{H}^*(\mathcal{A})$ are nontrivial for some $\mathcal{A}$, or else $\hat{\tau}$ will not map onto de Rham cohomology.

(2) Observe that the kernel of $\tau$ contains all forms which are invariant in some entry. The invariant forms in $\hat{\Omega}^1$, as an algebra under pointwise operations, is isomorphic to $\mathcal{A}$. For the forms in $\hat{\Omega}^k$, $k \geq 2$, invariance in one slot automatically implies that such forms vanish by antisymmetry.

(3) This cohomology is relevant for actions, hence it can be used to study a single automorphism by letting $G \subset \text{Aut}\mathcal{A}$ be the group generated by that automorphism. Thus it can be used to study operators on linear spaces and transformations on topological spaces (with the choice $\mathcal{A} = C_b(X)$ or $C_0(X)$ if $X$ is locally compact) without recourse to measure theory.

(4) Once we have equipped $\hat{\Omega}^*$ with a wedge product in the obvious way, we do not expect it to be a differential algebra with respect to $\hat{d}$. That can only
be expected at the infinitesimal level, once we have a map \( \tau \) as in theorem 4.5. Some homomorphic property of \( \hat{d} \) remains, cf. (6.0) below.

(5) Whilst in general it may be hard to compute the integrated cohomology \( \hat{H}^* \), in the case of a Lie group action \( G \subset \text{Aut} \mathcal{A} \) it may be easier to calculate the infinitesimal cohomology, for which we would need a \( \tau \)-map:

\[
\frac{d}{dt_i} \cdots \frac{d}{dt_k} \omega(\alpha^{(1)}_{t_i}, \ldots, \alpha^{(k)}_{t_k}) \bigg|_{t_i=0 \forall i} : = (\tau \omega)(X_i, \ldots, X_k)
\]

where \( \alpha^{(n)}_t = \exp tX_n \) are one–parameter groups in \( G \). To make sense of this, some topology on \( \mathcal{A} \) must have been given. The \( \tau \)-map then produces the infinitesimal (possibly noncommutative!) de Rham cohomology from \( \hat{H}^* \). This will be done in the second example below in Sect.6.

5. Examples: A Discrete Action on a Commutative Algebra.

Next we wish to work out \( \hat{H}^1 \) and \( \hat{H}^2 \) for some concrete examples, but before doing so, first need the general formulii for closed and exact forms.

**Lemma 5.1.** Let \( \mathcal{A} \) be an associative algebra, and \( G \subset \text{Aut} \mathcal{A} \) given. Then a form \( \omega \in \hat{\Omega}^1 \) is

(i) exact iff \( \omega(\alpha) = \alpha(A) - A \) for all \( \alpha \in G \) and some fixed \( A \in \mathcal{A} \) (depending on \( \omega \))

(ii) closed iff for all \( \alpha, \alpha_0 \in G \):

\[
\alpha_0(\omega(\alpha^{-1} \cdot \alpha)) - \alpha_0(\omega(\alpha_0^{-1})) + \omega(\alpha_0) = \alpha(\omega(\alpha^{-1} \cdot \alpha)) - \alpha(\omega(\alpha^{-1})) + \omega(\alpha).
\]

**Proof:** (ii) We first prove the second part. Now \( \omega \in \hat{\Omega}^1 \) is closed if \( (\hat{d}\omega)(\alpha, \alpha_0) = 0 \) for all \( \alpha, \alpha_0 \in G \). Expand this equation:

\[
0 = (\hat{d}\omega)(\alpha, \alpha_0) = (\Delta P_- \tilde{d} P_\Delta \omega)(\alpha, \alpha_0)
\]

\[
= (P_- \tilde{d} P_\Delta \omega)(\alpha, \alpha_0) - (P_- \tilde{d} P_\Delta \omega)(e, \alpha_0) - (P_- \tilde{d} P_\Delta \omega)(\alpha, e)
\]

\[
= \frac{1}{2} \left[ (\tilde{d} P_\Delta \omega)(\alpha, \alpha_0) - (\tilde{d} P_\Delta \omega)(\alpha, \alpha) - (\tilde{d} P_\Delta \omega)(e, \alpha_0) + (\tilde{d} P_\Delta \omega)(\alpha_0, e)
\right.

\[
- (\tilde{d} P_\Delta \omega)(\alpha, e) + (\tilde{d} P_\Delta \omega)(e, \alpha)
\]

\[
= \frac{1}{2} \left[ \alpha_0((P_- \Delta \omega)(\alpha^{-1} \alpha)) - \alpha((P_- \Delta \omega)(\alpha^{-1} \alpha_0)) - \alpha_0((P_- \Delta \omega)(\alpha_0^{-1}))
\right.

\[
+ (P_- \Delta \omega)(\alpha) - (P_- \Delta \omega)(\alpha) + \alpha((P_- \Delta \omega)(\alpha^{-1})) \right]
\]
\[-16-\]

\[= \frac{1}{2}[\alpha_0(\omega(\alpha_0^{-1} \cdot \alpha) - \omega(e)) - \alpha(\omega(\alpha^{-1} \cdot \alpha_0) - \omega(e)) - \alpha_0(\omega(\alpha_0^{-1}) - \omega(e))
+ \omega(\alpha_0) - \omega(e) - \omega(\alpha) + \omega(\alpha) + \alpha(\omega(\alpha^{-1}) - \omega(e))]
\]

\[= \frac{1}{2}[\alpha_0(\omega(\alpha_0^{-1} \cdot \alpha)) - \alpha(\omega(\alpha^{-1} \cdot \alpha_0)) - \alpha_0(\omega(\alpha_0^{-1})) + \alpha(\omega(\alpha^{-1})) + \omega(\alpha_0) - \omega(\alpha)]
\]

which proves \( (ii) \).

\( (i) \quad \omega \in \hat{\Omega}^1 \) is exact if \( \omega = \hat{d}\varphi \) for some \( \varphi \in \hat{\Omega}^0 = A \). Say \( \varphi = A \), then

\[\omega(\alpha) = (\hat{d}\varphi)(\alpha) = (\Delta P_\alpha \hat{d} P_\alpha \Delta \varphi)(\alpha)
= (\hat{d} P_\alpha \Delta \varphi)(\alpha) = (\hat{d} P_\alpha \Delta \varphi)(e) = \alpha(A) - A\]

**Notes:**
(1) Observe that if a one–form is invariant (i.e. \( \omega(\alpha) = \omega(e) \)), then it is closed, but obviously the only exact invariant one–form is zero. So the algebra \( A \) itself will always constitute part of \( \tilde{H}^1(A) \), corresponding to the invariant one–forms. Forms of the type \( \omega(\alpha) = \sum_i [B_i \alpha(B_i) + \alpha(B_i) B_i] \), \( B_i \in A \) are always closed, as we can check from (5.1ii).

(2) A zero–form \( \varphi = A \in \hat{\Omega}^0 = A \) is closed iff \( (\hat{d}\varphi)(\alpha) = \alpha(A) - A = 0 \), i.e. \( A \) is \( G \)–invariant. Hence \( \tilde{H}^0(A) = A^G \).

**Lemma 5.2.** Given an associative algebra \( A \) and group \( G \subseteq \text{Aut} A \), a two–form \( \omega \in \hat{\Omega}^2 \) is

\( (i) \) exact whenever there is a \( \varphi \in \hat{\Omega}^1 \) such that for all \( \alpha_i \in G \):

\[\omega(\alpha_1, \alpha_2) = \alpha_2(\varphi(\alpha_2^{-1} \alpha_1) - \varphi(\alpha_2^{-1})) - \alpha_1(\varphi(\alpha_1^{-1} \alpha_2) - \varphi(\alpha_1^{-1})) + \varphi(\alpha_2) - \varphi(\alpha_1)
= (2(\hat{d}\varphi)(\alpha_1, \alpha_2))\]

\( (ii) \) closed whenever for all \( \alpha_i \in G \):

\[0 = \omega(\alpha_3, \alpha_2) + \omega(\alpha_1, \alpha_3) + \omega(\alpha_2, \alpha_1) + \alpha_1[\omega(\alpha_1^{-1} \alpha_2, \alpha_1^{-1} \alpha_3) + \omega(\alpha_1^{-1} \alpha_3, \alpha_1^{-1})
+ \omega(\alpha_1^{-1}, \alpha_1^{-1} \alpha_2)] + \alpha_2[\omega(\alpha_2^{-1} \alpha_3, \alpha_2^{-1} \alpha_1) + \omega(\alpha_2^{-1} \alpha_1, \alpha_2^{-1} \alpha_3) + \omega(\alpha_2^{-1} \alpha_1, \alpha_2^{-1})
+ \alpha_3[\omega(\alpha_3^{-1} \alpha_1, \alpha_3^{-1} \alpha_2) + \omega(\alpha_3^{-1} \alpha_2, \alpha_3^{-1}) + \omega(\alpha_3^{-1} \alpha_1, \alpha_3^{-1})] \]

**Proof:** \( (i) \) Now \( \omega \) is exact iff \( \omega = \hat{d}\varphi \) for some \( \varphi \in \hat{\Omega}^1 \). Expand this equation, using (4.4ii) and the fact that \( P_- \) on \( \hat{\Omega}^k \) is just the identity:

\[\omega(\alpha_1, \alpha_2) = (\hat{d}\varphi)(\alpha_1, \alpha_2) = (\Delta P_\alpha \hat{d} P_\alpha \Delta \varphi)(\alpha_1, \alpha_2) = (\Delta P_\alpha \Delta \varphi)(\alpha_1, \alpha_2)\]
\[-17-\]
\[= (P_0 \tilde{d} \Delta \varphi)(\alpha_1, \alpha_2) - (P_0 \tilde{d} \Delta \varphi)(\alpha_1, e) - (P_0 \tilde{d} \Delta \varphi)(e, \alpha_2)\]
\[= \frac{1}{2} \left[ (\tilde{d} \Delta \varphi)(\alpha_1, \alpha_2) - (\tilde{d} \Delta \varphi)(\alpha_2, \alpha_1) - (\tilde{d} \Delta \varphi)(\alpha_1, e) + (\tilde{d} \Delta \varphi)(e, \alpha_1)\right.\]
\[= \frac{1}{2} \left[ \alpha_2((\Delta \varphi)(\alpha_1^{-1} \alpha_2)) - \alpha_1((\Delta \varphi)(\alpha_1^{-1} \alpha_2)) - (\Delta \varphi)(\alpha_1) + \alpha_1((\Delta \varphi)(\alpha_1^{-1})) \right.\]
\[= \frac{1}{2} \left[ \alpha_2(\varphi(\alpha_1^{-1} \alpha_2)) - \alpha_1(\varphi(\alpha_1^{-1} \alpha_2)) - \alpha_1(\varphi(\alpha_1^{-1})) - \alpha_1(\varphi(\alpha_1^{-1})) + \alpha_1(\varphi(\alpha_1^{-1})) + \alpha_1(\varphi(\alpha_1^{-1}))\right.\]
\[= \frac{1}{2} \left[ \alpha_2(\varphi(\alpha_1^{-1} \alpha_2) - \varphi(\alpha_1^{-1})) - \alpha_1(\varphi(\alpha_1^{-1} \alpha_2) - \varphi(\alpha_1^{-1})) + \varphi(\alpha_2) - \varphi(\alpha_1)\right.\]
\[which
s\proves\ (i) .\]

\[(ii)\ This\ is\ proven\ by\ (a\ lengthy)\ expansion\ of\]
\[0 = (\tilde{d} \omega)(\alpha_1, \alpha_2, \alpha_3) = (\Delta P_0 \tilde{d} \Delta \omega)(\alpha_1, \alpha_2, \alpha_3)\]

which\ we\ omit\ as\ straightforward\ algebra.

Notes: Observe\ that\ \(\omega \in \hat{\Omega}^2\) is\ closed\ if\ it\ satisfies\]
\[\omega(\alpha_3, \alpha_2) + \omega(\alpha_1, \alpha_3) + \omega(\alpha_2, \alpha_1) = 0 .\]
\[(5.3)\]

Another\ kind\ of\ closed\ form\ (of\ all\ orders)\ can\ be\ deduced\ from\ the\ representation\ (3.1)\ of\ forms,\ when\ the\ \(A_i^\ell\)\ are\ all\ \(G\)-invariant.\ At\ the\ level\ of\ maps,\ this\ condition\ will\ read\ for\ such\ an\ \(\omega \in \hat{\Omega}^k :\]
\[\alpha_1(\omega(\alpha_1^{-1} \alpha_2, \ldots, \alpha_1^{-1} \alpha_{k+1})) = \omega(\alpha_2, \ldots, \alpha_{k+1})\]

for\ all\ \(\alpha_i \in G\) , in\ which\ case\ \(\tilde{d} \omega = 0\)\ and\ so\ \(\tilde{d} \omega = 0\) .\ For\ the\ case\ in\ theorem\ 4.5,\ these\ closed\ forms\ map\ under\ \(\tau\)\ to\ the\ exact\ de\ Rham\ forms.

Now\ we\ are\ ready\ to\ do\ examples.

Example 0:

Let\ \(\beta : H \to \text{Aut} A\)\ be\ a\ trivial\ action\ on\ an\ associative\ algebra\ \(A\) ,\ i.e. \(\beta(H) = \iota = G\) .\ Then\ we\ have\ \(\hat{\Omega}^0 = A = \hat{\Omega}^1\)\ and\ \(\hat{\Omega}^k = 0\)\ for\ all\ \(k \geq 2\) .\ Thus\ \(\hat{H}^0(A) = A = \hat{H}^1(A)\)\ and\ \(\hat{H}^k(A) = 0\)\ for\ all\ \(k \geq 2\) .

Example 1:
Here we want to study a single homeomorphism $T : X \to X$ of a locally compact space $X$. Let $\mathcal{A} = C_0(X)$ and define the automorphism $\alpha(f)(x) := f(Tx)$ for all $f \in \mathcal{A}$. Let $G$ be the group generated by $\alpha$ in Aut$\mathcal{A}$, which is obviously a factor group of $\mathbb{Z}$. Then a general one–form $\omega \in \hat{\Omega}^1$ has an expression

$$\omega(\alpha^n)(x) = \sum_{\ell=1}^{L} \{ f^\ell(x) \alpha^n(h^\ell)(x) + \alpha^n(g^\ell)(x) + k^\ell(x) \}$$

for all $n \in \mathbb{Z}$ and fixed $f^\ell, h^\ell, g^\ell, k^\ell \in \mathcal{A}$. Closed forms must satisfy 5.1ii, i.e. for all $n, m \in \mathbb{Z}$:

$$\alpha^n(\omega(\alpha^{m-n})) - \alpha^n(\omega(\alpha^{-n})) + \omega(\alpha^n) = \alpha^m(\omega(\alpha^{n-m})) - \alpha^m(\omega(\alpha^{-m})) + \omega(\alpha^m)$$

so

$$\sum_{\ell=1}^{L} \{ \alpha^n(f^\ell)\alpha^m(h^\ell) - \alpha^n(f^\ell)h^\ell + f^\ell\alpha^n(h^\ell) \} = \sum_{\ell=1}^{L} \{ \alpha^m(f^\ell)\alpha^n(h^\ell) - \alpha^m(f^\ell)h^\ell + f^\ell\alpha^m(h^\ell) \}$$

Exact one–forms are of the type $\omega(\alpha^n) = \alpha^n(f) - f$.

In particular, let us work out the first and second cohomology classes for the shift operator on $\mathbb{Z}$. That is, we set $X = \mathbb{Z}$, $T : \mathbb{Z} \to \mathbb{Z}$ by $Tn = n + 1$, so $\mathcal{A} = C_0(\mathbb{Z})$ consists of sequences $f = \{ f_i \}_{i \in \mathbb{Z}}$ which go to zero at both ends, and pointwise multiplication is $f \cdot g = \{ f_i \cdot g_j \} = \{ f_ig_i \}_{i \in \mathbb{Z}}$ in $\mathcal{A}$. Note that $\mathcal{A}$ has no nonzero elements invariant under $\alpha$, and a general one–form $\omega$ has now an expression

$$\omega(\alpha^n) = \sum_{\ell=1}^{L} \{ f^\ell_i h^\ell_{i-n} + g^\ell_{i-n} + k^\ell_{i} \}_{i \in \mathbb{Z}} \quad \forall n$$

and hence $\omega$ is closed iff

$$\sum_{\ell=1}^{L} \{ f^\ell_{i-n}h^\ell_{i-m} - f^\ell_{i-n}h^\ell_{i-n} + f^\ell_{i}h^\ell_{i-n} \}_{i \in \mathbb{Z}} = \sum_{\ell=1}^{L} \{ f^\ell_{i-m}h^\ell_{i-n} - f^\ell_{i-m}h^\ell_{i-n} + f^\ell_{i}h^\ell_{i-m} \}_{i \in \mathbb{Z}}$$

and on equating each entry separately, we find:

$$\sum_{\ell=1}^{L} \{ f^\ell_{i-n}(h^\ell_{i-m} - h^\ell_{i}) + f^\ell_{i}(h^\ell_{i-n} - h^\ell_{i-m}) + f^\ell_{i-m}(h^\ell_{i} - h^\ell_{i-n}) \} = 0$$
for all \( n, m, i \in \mathbb{Z} \) . Taking the limit \( n \to \infty \):

\[
\sum_{\ell}^{L} \left\{ -f_i^\ell h_{i-m}^\ell + f_{i-m}^\ell h_i^\ell \right\} = 0 \quad \forall i, m \in \mathbb{Z}
\]

i.e.

\[
\sum_{\ell}^{L} f_i^\ell h_j^\ell = \sum_{\ell}^{L} f_j^\ell h_i^\ell \quad \forall i, j \in \mathbb{Z}
\]

Note that this condition is also sufficient for \( \omega \) to be closed. Thus the closed one–forms \( \omega \) , are precisely those which can be written:

\[
\omega(\alpha^n) = \sum_{\ell=0}^{L} \left\{ f_i^\ell h_{i-n}^\ell + f_{i-n}^\ell h_i^\ell + g_{i-n}^\ell + k_{i}^\ell \right\}_{i \in \mathbb{Z}} \quad (5.3)
\]

for all \( n \in \mathbb{Z} \) . Denote the space of these by \( \hat{Z}^1 \) . The exact one–forms are of the type \( \omega(\alpha^n) = \{ f_i^n - f_i \}_{i \in \mathbb{Z}} \) and the space of these is

\[
\hat{B}^1 = (\alpha^Z - i)(\mathcal{A}) := \{ \alpha^n(A) - A \mid A \in \mathcal{A}, n \in \mathbb{Z} \} . \quad (5.4)
\]

Claim 5.5. There is a linear bijection from \( \hat{H}^1(\mathcal{A}) \) to the linear spaces

(i) \( \mathcal{L} \subset \hat{Z}^1 \) consisting of those \( \omega \) of the form

\[
\omega(\alpha^n) = \left\{ \sum_{\ell=1}^{L} (f_i^\ell h_{i-n}^\ell + f_{i-n}^\ell h_i^\ell + k_{i}) \right\}_{i \in \mathbb{Z}} \quad (5.6) \quad \text{and}
\]

(ii) \( L[(\hat{B}^1) \cdot (\hat{B}^1)] + \mathcal{A} \subset \hat{Z}^1 \) consisting of \( \omega \) of the form

\[
\omega(\alpha^n) = \left\{ \sum_{\ell=1}^{L} (f_{i-n}^\ell - f_i^\ell)(h_{i-n}^\ell - h_i^\ell) + k_{i} \right\}_{i \in \mathbb{Z}} , \quad (5.7)
\]

\( \{ f_i^\ell \}, \{ h_i^\ell \}, \{ k_{i} \} \in \mathcal{A} \).

Proof: (i) From (5.3) and (5.4) we see that every \( \omega \in \hat{Z}^1 \) is cohomologous to an element of \( \mathcal{L} \) , in fact \( \mathcal{L} \) is in \( \hat{Z}^1 \) and has nonempty intersection with each cohomology class in \( \hat{Z}^1 \) . We only need to show that an \( \omega \in \mathcal{L} \) is exact iff it is zero. Let \( \omega \in \mathcal{L} \cap \hat{B}^1 \) , so there is a \( \varphi = \{ g_{i} \}_{i \in \mathbb{Z}} \) such that \( \omega = \hat{d}\varphi \) and \( \omega \) is of the form (5.6), i.e.

\[
\sum_{\ell=1}^{L} (f_i^\ell h_{i-n}^\ell + f_{i-n}^\ell h_i^\ell) + k_{i} = g_{i-n} - g_i \quad (5.8)
\]
for all \( i, n \in \mathbb{Z} \). Let \( n \to \infty \) to find \( k_i = -g_i \), so on substitution into (5.8) and replacing \( i \) with \( i + n \):

\[
\sum_{\ell}^L\left( f_{i+n}^\ell h_i^\ell + f_i^\ell h_{i+n}^\ell \right) = g_i \quad \forall i, n
\]

so that in the limit \( n \to \infty \) we find \( g_i = 0 = k_i \) for all \( i \), hence

\[
\omega(\alpha^n) = \hat{d}\{g_i\} = \{g_{i-n} - g_i\}_{i \in \mathbb{Z}} = 0 .
\]

(ii) Now every element of \( L \) is cohomologous to an element of \( L[(\hat{B}^1) \cdot (\hat{\mathring{B}}^1)] + A \) because

\[
f_i h_{i-n} + h_i f_{i-n} + k_i = (f_{i-n} - f_i)(h_i - h_{i-n}) + (k_i + 2f_i h_i) + (f_{i-n} h_{i-n} - f_i h_i)
\]

and the last term is clearly exact. Thus \( L[(\hat{B}^1) \cdot (\hat{\mathring{B}}^1)] + A \subset \hat{\mathring{B}}^1 \) has nonempty intersection with each cohomology class. We show that

\[
\left( L[(\hat{B}^1) \cdot (\hat{\mathring{B}}^1)] + A \right) \cap \hat{\mathring{B}}^1 = \{0\} .
\]

Choose an \( \omega \) of the form (5.7) which is exact: \( \omega = \hat{d}\varphi \), for \( \varphi = \{g_i\}_{i \in \mathbb{Z}} \), so

\[
\sum_{\ell}^L\left( f_{i-n}^\ell - f_i^\ell \right)(h_{i-n}^\ell - h_i^\ell) + k_i = g_{i-n} - g_i \quad (5.9)
\]

for all \( i, n \in \mathbb{Z} \). For \( n = 0 \) we see \( k_i = 0 \). Let \( n \to \infty \) to find

\[
\sum_{\ell}^L f_i^\ell h_i^\ell = -g_i ,
\]

so (5.9) simplifies to

\[
\sum_{\ell}^L\left( 2f_{i-n}^\ell h_{i-n}^\ell - f_i^\ell h_{i-n}^\ell - f_{i-n}^\ell h_i^\ell \right) = 0 \quad \forall i, n \in \mathbb{Z} .
\]

Replace \( i \) with \( i + n \) and take the limit \( n \to \infty \) to find:

\[
2 \sum_{\ell}^L f_i^\ell h_i^\ell = 0 , \quad \text{hence} \quad g_i = 0 , \quad \text{so} \quad \omega = 0 .
\]

To compute \( \hat{\mathring{H}}^2(A) \), we start with the exact two–forms (5.2i). Since a general one–form \( \varphi \) has the form

\[
\varphi(\alpha^n) = \left\{ \sum_{\ell}^L f_i^\ell h_{i-n}^\ell + g_{i-n} + k_i \right\}_{i \in \mathbb{Z}}
\]

substitution of this into (5.2i) produces

\[
(\hat{d}\varphi)(\alpha^n, \alpha^m) = \frac{1}{2} \sum_{\ell}^L \left\{ (f_{i-m}^\ell - f_i^\ell)(h_{i-n}^\ell - h_i^\ell) - (f_{i-n}^\ell - f_i^\ell)(h_{i-m}^\ell - h_i^\ell) \right\}_{i \in \mathbb{Z}} . \quad (5.10)
\]
Denote the space of these by $\hat{B}^2$. Now a general two-form has an expression

$$\omega(\alpha^n, \alpha^m) = \sum_{\ell} \left\{ f_{i}^{\ell}(h_{i-n}^{\ell}g_{i-m}^{\ell} - h_{i-m}^{\ell}g_{i-n}^{\ell}) + (t_{i-n}^{\ell} - t_{i-m}^{\ell}) ight.$$ 

$$\left. + (u_{i-n}^{\ell}v_{i-m}^{\ell} - v_{i-n}^{\ell}u_{i-m}^{\ell}) + r_{i}^{\ell}(s_{i-n}^{\ell} - s_{i-m}^{\ell}) \right\}_{i \in \mathbb{Z}}$$

where $f^{\ell}, h^{\ell}, g^{\ell}, u^{\ell}, v^{\ell}, r^{\ell}, t^{\ell} \in \mathcal{A}$. Observe that by regrouping we can readjust the r–s part to make the u–v part exact, and that the r–s and t–parts are only need to consider the f–h–g part. So now the closure equation for $\omega$ reads:

$$0 = (\hat{d}\omega)(\alpha^n, \alpha^m, \alpha^k)$$

$$= \sum_{\ell} \left\{ f_{i}^{\ell}(h_{i-k}^{\ell}g_{i-m}^{\ell} - h_{i-m}^{\ell}g_{i-k}^{\ell} + h_{i-n}^{\ell}g_{i-k}^{\ell} + h_{i-m}^{\ell}g_{i-n}^{\ell} - h_{i-n}^{\ell}g_{i-m}^{\ell}) ight.$$ 

$$+ f_{i-n}^{\ell}(h_{i-m}^{\ell}g_{i-k}^{\ell} - h_{i-k}^{\ell}g_{i-m}^{\ell} + h_{i-k}^{\ell}g_{i-n}^{\ell} + h_{i-m}^{\ell}g_{i-n}^{\ell} - h_{i-n}^{\ell}g_{i-m}^{\ell})$$ 

$$+ f_{i-m}^{\ell}(h_{i-k}^{\ell}g_{i-n}^{\ell} - h_{i-n}^{\ell}g_{i-k}^{\ell} + h_{i-k}^{\ell}g_{i-n}^{\ell} + h_{i-n}^{\ell}g_{i-k}^{\ell} - h_{i-k}^{\ell}g_{i-n}^{\ell})$$ 

$$+ f_{i-k}^{\ell}(h_{i-n}^{\ell}g_{i-m}^{\ell} - h_{i-m}^{\ell}g_{i-n}^{\ell} + h_{i-n}^{\ell}g_{i-m}^{\ell} + h_{i-m}^{\ell}g_{i-n}^{\ell} - h_{i-n}^{\ell}g_{i-m}^{\ell}) \right\}_{i \in \mathbb{Z}}$$

for all $n, m, k \in \mathbb{Z}$. Now let $k \to \infty$ and regroup to find:

$$0 = \sum_{\ell} \left\{ f_{i}^{\ell}(h_{i-m}^{\ell}g_{i-n}^{\ell} - h_{i-n}^{\ell}g_{i-m}^{\ell}) + h_{i}^{\ell}(f_{i-n}^{\ell}g_{i-m}^{\ell} - f_{i-m}^{\ell}g_{i-n}^{\ell}) ight.$$ 

$$+ g_{i}^{\ell}(f_{i-m}^{\ell}h_{i-n}^{\ell} - f_{i-n}^{\ell}h_{i-m}^{\ell}) \right\}_{i \in \mathbb{Z}}$$

for all $n, m$. More compactly, it says that for all $i, j, k \in \mathbb{Z}$:

$$0 = \sum_{\ell} \left\{ f_{i}^{\ell}(h_{j-k}^{\ell}g_{j}^{\ell} - h_{k}^{\ell}g_{j}^{\ell}) + h_{i}^{\ell}(f_{j-k}^{\ell}g_{j}^{\ell} - f_{j}^{\ell}g_{k}^{\ell}) + g_{i}^{\ell}(f_{j}^{\ell}h_{k}^{\ell} - f_{k}^{\ell}h_{j}^{\ell}) \right\}_{i \in \mathbb{Z}}$$

$$= \sum_{\ell} \sum_{\sigma \in \mathcal{S}_3} \epsilon^{\sigma} f_{\sigma(i)}^{\ell} h_{\sigma(j)}^{\ell} g_{\sigma(k)}^{\ell}$$

$$= P_- S_{ijk}, \quad (5.11)$$

where we think of $\mathcal{S}_3$ as permutations acting on the set $\{i, j, k\}$, and we used the notation $S_{ijk} := \sum_{\ell} f_{i}^{\ell} h_{j-k}^{\ell} g_{j}^{\ell}$ and $P_- = \sum_{\sigma \in \mathcal{S}_3} \epsilon^{\sigma}$ which is idempotent. On comparing equation (5.11) with the closure equation above for $\omega$, we see that
(5.11) is also sufficient for $\omega$ to be closed. (Note that if one of $f, h$ or $g$ is a linear combination of the others for each $\ell$, e.g. $f_i^\ell = \alpha^i h_i^\ell + \beta^i g_i^\ell$ for each $i$ and $\ell$, then the closure equation will be satisfied.) Thus $S_{ijk} \in (P_-)^\perp$, i.e. $S_{ijk}$ is symmetric with respect to some pair of indices in $\{i, j, k\}$, so a closed form can always be written in the form:

$$\omega(\alpha^n, \alpha^m) = \sum_{\ell} \left\{ f_i^\ell (h_{i-n}^\ell g_{i-m}^\ell - h_{i-m}^\ell g_{i-n}^\ell) - h_i^\ell (g_{i-n}^\ell f_{i-m}^\ell - g_{i-m}^\ell h_{i-n}^\ell) \\
+ (t_{i-n}^\ell - t_{i-m}^\ell) + r_i^\ell (s_{i-n}^\ell - s_{i-m}^\ell) \right\}_i + (\hat{d} \varphi)(\alpha^n, \alpha^m)$$

where the added exact form takes care of the u–v part in the original expression. Conversely, such a form is always closed. To now examine the factor space $\hat{H}^2$, we want to write an $\omega$ in terms of products of exact one–forms (having (5.7) in mind). A small calculation shows that by absorbing the cross–terms into the $r$–s and $t$–parts, we can write any closed two–form, up to an exact form as:

$$\omega(\alpha^n, \alpha^m) = \sum_{\ell} \left\{ f_i^\ell \left[ (h_{i-n}^\ell - h_i^\ell) (g_{i-m}^\ell - g_i^\ell) - (g_{i-n}^\ell - g_i^\ell) (h_{i-m}^\ell - h_i^\ell) \right] \\
- h_i^\ell \left[ (g_{i-n}^\ell - g_i^\ell) (f_{i-m}^\ell - f_i^\ell) - (f_{i-n}^\ell - f_i^\ell) (g_{i-m}^\ell - g_i^\ell) \right] \\
+ (t_{i-n}^\ell - t_{i-m}^\ell) + r_i^\ell (s_{i-n}^\ell - s_{i-m}^\ell) \right\}_i \in \mathbb{Z} \\
(5.12)$$

Notice that each of the square brackets is an exact form. Denote the space of two–forms having an expression as in (5.12) by $Q$. Then we show there is a linear bijection between $Q$ and $\hat{H}^2$, which is done by proving $Q \cap \hat{B}^2 = \{0\}$. Let $\omega \in Q$, so it has an expression (5.12), and observe that due to the coefficients $f_i^\ell, h_i^\ell, r_i^\ell$, we have

$$\lim_{k \to \infty} \alpha^{-k} \omega(\alpha^{n+k}, \alpha^{m+k}) = \sum_{\ell} (t_{i-n}^\ell - t_{i-m}^\ell) \quad \forall n, m \in \mathbb{Z}.$$ 

However, for an exact form as in (5.10), say

$$(\hat{d} \varphi)(\alpha^n, \alpha^m) = \sum_{\ell} \left\{ (p_{i-m}^\ell - p_i^\ell) (q_{i-n}^\ell - q_i^\ell) - (q_{i-m}^\ell - q_i^\ell) (p_{i-n}^\ell - p_i^\ell) \right\}_i ,$$
we have that
\[
\lim_{k \to \infty} \alpha^{-k}(\hat{d}\varphi)(\alpha^{n+k}, \alpha^{m+k}) = \lim_{k \to \infty} \sum_{\ell} \left\{ (p_{i-m} - p_{i+k}) (q_{i-n} - q_{i+k}) \\
- (q_{i-m} - q_{i+k}) (p_{i-n} - p_{i+k}) \right\}_i \\
= \sum_{\ell} \{ p_{i-m} q_{i-n} - q_{i-m} p_{i-n} \}_i
\]
and this clearly goes to zero when either \( n \) or \( m \to \infty \), whereas
\[
\lim_{m \to \infty} (t_{i-n}^\ell - t_{i-m}^\ell) = t_{i-n}^\ell .
\]
Thus if \( \omega \) is exact, \( t_{i}^\ell = 0 \) for all \( \ell \) and \( i \), and so \( \lim_{k \to \infty} \alpha^{-k}\omega(\alpha^{n+k}, \alpha^{m+k}) = 0 \), and (5.13) can only be zero when \( \hat{d}\varphi = 0 \).
So we have proven:

**Claim 5.14:** There is a linear bijection between \( Q \) and \( \hat{H}^2(A) \).

6. Examples: (II) A Lie Group Action on a Noncommutative Algebra.

Next we wish to do a simple noncommutative example, but since the exact one–forms \( \hat{d}A \) played such an important role in (5.7) and (5.12), want to exploit these explicitly. Recall that in differential geometry an n–form has an expression
\[
\omega = \sum_i f_i \, dg_i^1 \wedge \cdots \wedge dg_i^n
\]
and in Connes’ differential envelope over an algebra \( A \), an n–form is a linear combination of formal expressions
\[
\omega = (a_0 + \lambda 1) \, da_1 \, da_2 \ldots da_n
\]
where \( a_i \in A, \lambda \in \mathbb{C} \), and given any monomial made up from \( a_i \in A \) and \( db_i \), \( b_i \in A \), we can convert it to this form using the assumption that \( d \) is a graded derivation on the differential envelope. In the present integrated differential geometry, we wish to get as close as possible to such an expression of a general n–form. Recall that a zero–form \( \varphi \) is just an element \( A \in A \). Then \( (\hat{d}\varphi)(\alpha) = \alpha(A) - A =: \hat{d}_\alpha A \) for \( \alpha \in G \). Now
\[
\hat{d}_\alpha(AB) = \alpha(A) \alpha(B) - AB = \hat{d}_\alpha(A) \cdot \hat{d}_\alpha(B) + \hat{d}_\alpha(A) \cdot B + A \hat{d}_\alpha(B)
\]

which can be thought of as an integrated form of the Leibniz rule. This rule has already been used in Cuntz’s algebra of formal differences [3]. So by

\[ \hat{d}_\alpha(A) \cdot B = \hat{d}_\alpha(AB) - \hat{d}_\alpha(A) \cdot \hat{d}_\alpha(B) - A \hat{d}_\alpha(B), \]

(6.0)

we can convert any expression of the form

\[ \omega(\alpha) = \sum_i A_i^0 \prod_{j=1}^{L_i} \hat{d}_\alpha(B^j_i) A_i^j \]

(6.1)

to the form

\[ \omega(\alpha) = \sum_i C_i^0 \prod_{j=1}^{K_i} \hat{d}_\alpha(E^j_i). \]

(6.2)

Since furthermore any general one–form

\[ \omega(\alpha) = \sum_i A_i^0 \prod_{j=1}^{L_i} \alpha(B^j_i) A_i^j \]

(cf. (3.1)) can be written in the form (6.1), we conclude that every one–form has an expression (6.2), which comes close to the expression of a one–form for differential geometry. For n–forms \( \omega \in \hat{\Omega}^n \) we have likewise that they can be expressed in the form:

\[ \omega(\alpha_1, \ldots, \alpha_n) = \sum_{\sigma \in S_n} \epsilon^\sigma \sum_i A_i^0 \prod_{j=1}^{L_i} (\hat{d}B^j_i)(\alpha_{\sigma(s_i(j))}) \]

where \( s_i : \{1, 2, \ldots, L_i\} \rightarrow \{1, 2, \ldots, n\} \) is a map and \( L_i \geq n \).

Exact one–forms are of the type \( \omega(\alpha) = \alpha(A) - A \), \( A \in \mathcal{A} \), so if we use the expression for a one–form

\[ \omega(\alpha) = \sum_{n=0}^{N} \sum_{i=0}^{K_n} A_i^n \prod_{j=1}^{n} \hat{d}_\alpha(B^j_i) \]

(6.3)

then the closure equation (5.1ii) is

\[ 0 = \sum_{n=0}^{K_n} \sum_{i=0}^{n} \left( \alpha'(A_i^n) \prod_{j=1}^{n} (\alpha(B^j_i) - \alpha'(B^j_i)) - \alpha(A_i^n) \prod_{j=1}^{n} (\alpha'(B^j_i) - B^j_i) \right) \]
for all \( \alpha, \alpha' \in G \). This equation for closure of one–forms has an interesting resemblance to the closure condition for ordinary one–forms in differential geometry. Next consider two–forms which we know by (5.2) to be exact when there is a \( \varphi \in \hat{\Omega}^1 \) such that

\[
\omega(\alpha_1, \alpha_2) = \alpha_2(\varphi(\alpha_2^{-1} \cdot \alpha_1) - \varphi(\alpha_2^{-1})) - \alpha_1(\varphi(\alpha_1^{-1} \cdot \alpha_2) - \varphi(\alpha_1^{-1})) + \varphi(\alpha_2) - \varphi(\alpha_1)
\]

so on substituting in the canonical form:

\[
\varphi(\alpha) = \sum_n \sum_{i=0}^{K_n} E_{n_i} \prod_j \hat{d}_{\alpha}(D_{i_j})
\]

we find that for all \( \alpha_i \in G \):

\[
\omega(\alpha_1, \alpha_2) = \sum_n \sum_{i=0}^{K_n} \left\{ \hat{d}_{\alpha_2}(E_{n_i}) \prod_j \hat{d}_{\alpha_1}(D_{i_j}) - \hat{d}_{\alpha_1}(E_{n_i}) \prod_j \hat{d}_{\alpha_2}(D_{i_j}) \right\}
\]  \hspace{1cm} (6.5)

The closure equation for two–forms in canonical form is very messy, and we omit it.

We are now ready to attempt to find \( \hat{H}^1(A) \) in a simple noncommutative case. Let \( A = M_2(\mathbf{C}) \), and \( G = \text{Aut}_0A \cong (U(2)/T)_0 \), the connected component of the identity of the automorphism group. Because \( G \) is a Lie group, we will be able to use differentiation at zero on the one–parameter groups, to obtain a map \( \tau \) from \( \hat{H}^n(A) \) to infinitesimal cohomology for \( A \). The action of \( G \) on \( A \) in terms of one–parameter groups is

\[
\alpha_t(A) = e^{iBt}Ae^{-iBt} \quad \forall t \in \mathbb{R}, \ A \in A, \ B = B^* \in A.
\]
Now for a selfadjoint matrix $B$ we have either

$$B = \begin{pmatrix} γ & 0 \\ 0 & β \end{pmatrix} \quad \text{or} \quad B = r \begin{pmatrix} γ & e^{iθ} \\ e^{-iθ} & β \end{pmatrix}$$

where $γ, β, r, θ \in \mathbb{R}$. Since $r$ can be absorbed into the $t$, we will set $r = 1$ henceforth. So for Case 1 where $B = \begin{pmatrix} γ & 0 \\ 0 & β \end{pmatrix}$, we have obviously

$$U_t = \exp(itB) = \begin{pmatrix} e^{itγ} & 0 \\ 0 & e^{itβ} \end{pmatrix}. \quad (6.6)$$

Now $B = \begin{pmatrix} γ & \exp(iθ) \\ \exp(-iθ) & β \end{pmatrix}$ has eigenvalues $E_1 = λ + μ$, $E_2 = λ − μ$ where $λ := \frac{1}{2}(γ + β) \in \mathbb{R}$, $μ := \frac{1}{2}\sqrt{(γ − β)^2 + 4} \in [1, +∞)$. So on exponentiation we find for Case 2 where $γ ≥ β$ that

$$U_t = e^{itB} = \frac{i\exp(itλ)}{μ} \begin{pmatrix} √(μ^2 − 1)\sin tμ − iμ\cos tμ & e^{iθ}\sin tμ \\ e^{-iθ}\sin tμ & −√(μ^2 − 1)\sin tμ − iμ\cos tμ \end{pmatrix} \quad (6.7)$$

and for Case 3 where $γ < β$ we have

$$U_t = e^{itB} = \frac{i\exp(itλ)}{μ} \begin{pmatrix} −√(μ^2 − 1)\sin tμ − iμ\cos tμ & e^{iθ}\sin tμ \\ e^{-iθ}\sin tμ & √(μ^2 − 1)\sin tμ − iμ\cos tμ \end{pmatrix} \quad (6.8)$$

Thus $(\hat{d}A)(α_t) = U_tAU_{−t} − A = \exp(it\text{ad }B)(A) − A$, and hence for an $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we get in case 1:

$$(\hat{d}A)(α_t) = \begin{pmatrix} 0 & b(e^{iθ(γ − β)} − 1) \\ c(e^{iθ(β − γ)} − 1) & 0 \end{pmatrix} \quad (6.9)$$

and when $α_t$ is case 2 we have that $(\hat{d}A)(α_t)$ is

$$\sin tμ \begin{pmatrix} q\sin tμ + iy\cos tμ & e^{iθ}[−(μy + q√(μ^2 − 1))\sin tμ + i(μq + y√(μ^2 − 1))\cos tμ] \\ e^{-iθ}[−(μy − q√(μ^2 − 1))\sin tμ − i(μq − y√(μ^2 − 1))\cos tμ] & −q\sin tμ + iy\cos tμ \end{pmatrix} \quad (6.10)$$

where $y = μ^2(b(-e^{iθ} − ce^{iθ}))$ and $q = μ^2[dy + (e^{iθ}b + e^{iθ}c)√(μ^2 − 1)].$

When $α_t$ is case 3 we find that $(\hat{d}A)(α_t)$ is

$$\sin tμ \begin{pmatrix} q’\sin tμ + iy\cos tμ & e^{iθ}[−(μy + q’√(μ^2 − 1))\sin tμ + i(μq’ − y√(μ^2 − 1))\cos tμ] \\ e^{-iθ}[−(μy − q’√(μ^2 − 1))\sin tμ − i(μq’ + y√(μ^2 − 1))\cos tμ] & −q’\sin tμ + iy\cos tμ \end{pmatrix} \quad (6.11)$$
where $y$ is as above, and $q' := \mu^{-2}[d - a - (e^{-i\theta}b + e^{i\theta}c)\sqrt{\mu^2 - 1}]$. Now observe that the power series of $(\hat{dA})(\alpha_t)$ in powers of $t$ has lowest order one. On substitution of an $\omega$ of the form (6.3) into the closure relation (6.4), we find that for all $\tilde{\alpha}_t, \alpha_s \in G$ we have

$$0 = (\hat{d}\omega)(\alpha_s, \tilde{\alpha}_t)$$

which can be expressed as a polynomial in $s$ and $t$ with constant matrix coefficients, and we see for each order of $s^k t^\ell$ that the coefficient must vanish.

The coefficient of the lowest order of (6.12) (using the fact that each $(\hat{dA})(\alpha_t)$ starts with order one in $t$) is found from

$$0 = \lim_{t \to 0} \frac{1}{t} \sum_{i=0}^{K_0} \left( (\hat{dA}_i^0)(\tilde{\alpha}_t) \prod_{j=1}^{n} (\hat{dB}_j^i)(\alpha_s) - (\hat{dA}_i^0)(\alpha_s) \prod_{j=1}^{n} (\hat{dB}_j^i)(\tilde{\alpha}_t) \right)$$

for all $\alpha_t \in G$. On setting $A^0 := \sum_{i=0}^{K_0} A_i^0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we find from (6.9) that $b = c = y = 0$ and $q = (d - a)/\mu^2$, so (6.10) becomes for $\lim_{t \to 0} \frac{1}{t}(\hat{dA}^0)(\alpha_t)$:

$$\lim_{t \to 0} \frac{\sin t\mu}{t} \begin{pmatrix} q \sin t\mu & e^{i\theta}q[-\sqrt{\mu^2 - 1}\sin t\mu + i\mu \cos t\mu] \\ -e^{-i\theta}q[\sqrt{\mu^2 - 1}\sin t\mu + i\mu \cos t\mu] & -q \sin t\mu \end{pmatrix} = 0$$

so $q = 0$, i.e. $d = a$, hence $A^0 = aI$ and $\hat{d}A^0 = 0$. Similarly we obtain from (6.11) the same conclusion. Now for the coefficient of the $st$–term in (6.12), using $A^0 = aI$, we find

$$0 = \lim_{s \to 0} \lim_{t \to 0} \frac{1}{st} \sum_{i=0}^{K_1} \left( (\hat{dA}_i^1)(\tilde{\alpha}_t)(\hat{dB}_i^1)(\alpha_s) - (\hat{dA}_i^1)(\alpha_s)(\hat{dB}_i^1)(\tilde{\alpha}_t) \right)$$

for all $\alpha_s$ and $\tilde{\alpha}_t \in G$. Now from (6.9) we see for $\alpha_s$ type 1:

$$\lim_{s \to 0} \frac{1}{s}(\hat{dA})(\alpha_s) = i \begin{pmatrix} 0 & b(\gamma - \beta) \\ c(\beta - \gamma) & 0 \end{pmatrix}$$

and from (6.8) for type 2:

$$\lim_{s \to 0} \frac{1}{s}(\hat{dA})(\alpha_s) = i \begin{pmatrix} y & e^{i\theta}(\mu q + y\sqrt{\mu^2 - 1}) \\ -e^{-i\theta}(\mu q - y\sqrt{\mu^2 - 1}) & y \end{pmatrix}$$
 whilst (6.11) produces for type 3:

\[
\lim_{s \to 0} s^{-1}(\hat{d}A)(\alpha_s) = i \left( -e^{-i\theta}(\mu q' + y\sqrt{\mu^2 - 1}) e^{i\theta}(\mu q' - y\sqrt{\mu^2 - 1}) \right)
\]

(6.14)

So there are six possible substitutions to make into (6.13). Observe that when both \( \alpha_s, \tilde{\alpha}_t \) are type 1, (6.13) only produces an identity. For the rest, we collect the results in the following claim, where we assume that

\[
A_i^1 = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \quad \text{and} \quad B_i^1 = \begin{pmatrix} e_i & f_i \\ g_i & h_i \end{pmatrix}
\]

Claim 6.15. With notation above we have that

(i) if an \( \omega \) of the form (6.3) is closed, then

\[
\sum_i g_i b_i = \sum_i c_i f_i \quad (6.15i)
\]

\[
\sum_i b_i(h_i - e_i) = 0 = \sum_i g_i(d_i - a_i) \quad (6.15ii)
\]

\[
\sum_i b_if_i = 0 = \sum_i g_ic_i \quad (6.15iii)
\]

\[
\sum_i f_i(d_i - a_i) = 0 = \sum_i c_i(h_i - e_i) \quad (6.15iv)
\]

\[
0 = \sum_i (d_i - a_i)(h_i - e_i) \quad (6.15v)
\]

\[
\sum_i b_ig_i = 0 = \sum_i c_if_i, \quad (6.15vi)
\]

and there are no further conditions on \( A_i^1 \) and \( B_i^1 \).

(ii) if \( \omega \in \hat{\Omega}^1 \) is closed, its first order term in \( \hat{d} \), i.e. \( \sum_i A_i^1\hat{d}_\alpha(B_i^1) \), is exact.

(iii) There is a linear bijection from \( \hat{H}^1 \) to the closed forms of the type

\[
\omega(\alpha) = aI + \sum_{n=2}^{N} \sum_{i=0}^{K_n} A_i^n \prod_{j=1}^{n} \hat{d}_\alpha(B_i^j) .
\]

(6.16)

Proof: Let \( \tilde{\alpha}_t \) be type 1 and \( \alpha_s \) type 2, then we obtain for the st–coefficient (6.13) that

\[
\sum_{i=0}^{K_1} \begin{pmatrix} 0 & b_i(\tilde{\gamma} - \tilde{\beta}) \\ c_i(\tilde{\beta} - \tilde{\gamma}) & 0 \end{pmatrix} \begin{pmatrix} y_i^B & e^{i\theta}(\mu q_i^B + y_i^B\sqrt{\mu^2 - 1}) \\ -e^{-i\theta}(\mu q_i^B - y_i^B\sqrt{\mu^2 - 1}) & y_i^B \end{pmatrix}
\]

\[
= \sum_{i=0}^{K_1} \begin{pmatrix} y_i^B & e^{i\theta}(\mu q_i^A + y_i^A\sqrt{\mu^2 - 1}) \\ -e^{-i\theta}(\mu q_i^A - y_i^A\sqrt{\mu^2 - 1}) & y_i^B \end{pmatrix} \begin{pmatrix} 0 & f_i(\tilde{\gamma} - \tilde{\beta}) \\ g_i(\tilde{\beta} - \tilde{\gamma}) & 0 \end{pmatrix}
\]
where

\[ q^A_i := \mu^{-2}(d_i - a_i + (e^{-i\theta}b_i + e^{i\theta}c_i)\sqrt{\mu^2 - 1}) \]

\[ q^B_i := \mu^{-2}(h_i - e_i + (e^{-i\theta}f_i + e^{i\theta}g_i)\sqrt{\mu^2 - 1}) \]

\[ y^A_i := \mu^{-2}(b_i e^{-i\theta} - c_i e^{i\theta}) \]

\[ y^B_i := \mu^{-2}(f_i e^{-i\theta} - g_i e^{i\theta}) \].

Multiplying out and equating matrix entries we find:

\[
\sum_i b_i(\mu q^B_i - y^B_i \sqrt{\mu^2 - 1}) = e^{i2\theta} \sum_i g_i(\mu q^A_i + y^A_i \sqrt{\mu^2 - 1}) \quad (i)
\]

\[
\sum_i y^B_i b_i = \sum_i y^A_i f_i \quad (ii)
\]

\[
\sum_i c_i y^B_i = \sum_i y^A_i g_i \quad (iii)
\]

\[
\sum_i f_i(\mu q^A_i - y^A_i \sqrt{\mu^2 - 1}) = e^{i2\theta} \sum_i c_i(\mu q^B_i + y^B_i \sqrt{\mu^2 - 1}) \quad (iv)
\]

Expand (ii) to find for all \( \theta \) :

\[
\sum_i (f_i e^{-i\theta} - g_i e^{i\theta})b_i = \sum_i (b_ie^{-i\theta} - c_ie^{i\theta})f_i
\]

so on cancelling we obtain (6.15i), which would also have followed from (iii). Next expand (i) and cancel to find

\[
\sum_i b_i \left( h_i - e_i + e^{-i\theta}f_i(1 - \mu^{-1})\sqrt{\mu^2 - 1} \right) = e^{i2\theta} \sum_i g_i \left( d_i - a_i + e^{i\theta}c_i(1 - \mu^{-1})\sqrt{\mu^2 - 1} \right) \quad (*)
\]

for all \( \mu \) and \( \theta \). In the case when \( \mu = 1 \) we obtain for all \( \theta \) that

\[
\sum_i b_i(h_i - e_i) = e^{i2\theta} \sum_i g_i(d_i - a_i) \quad \text{from which we deduce (6.15ii).}
\]

When we substitute this back into (*) when \( \mu \neq 1 \) and cancel, we find

\[
\sum_i b_i f_i e^{-i\theta} = \sum_i g_i c_i e^{i2\theta} \quad \text{for all \( \theta \), from which we deduce (6.15iii).}
\]

Similarly, by expanding (iv) we obtain (6.15iv).

Next, we consider the case where \( \tilde{\alpha}_t \) is type 1 and \( \alpha_s \) is type 3 in (6.13). Note first from (6.14) that type 2 is converted to type 3 by the substitutions \( \theta \rightarrow \theta + \pi \) and \( q \rightarrow -q' \). On application of these to (i – iv) we find

\[
\sum_i (-\mu q^B_i - y^B_i \sqrt{\mu^2 - 1}) = e^{i2\theta} \sum_i g_i(-\mu q^A_i + y^A_i \sqrt{\mu^2 - 1}) \quad (i')
\]
\[
\sum_i y_i^B b_i = \sum_i y_i^A f_i \quad (ii)
\]
\[
\sum_i c_i y_i^B = \sum_i y_i^A g_i \quad (iii)
\]
\[
\sum_i f_i (-\mu q_i^A - y_i^A \sqrt{\mu^2 - 1}) = e^{i2\theta} \sum_i c_i (-\mu q_i^B + y_i^B \sqrt{\mu^2 - 1}) \quad (iv')
\]
where:
\[
q_i^A := \mu^{-2} [d_i - a_i - (e^{-i\theta} b_i + e^{i\theta} c_i) \sqrt{\mu^2 - 1}]
\]
\[
q_i^B := \mu^{-2} [h_i - e_i - (e^{-i\theta} f_i + e^{i\theta} g_i) \sqrt{\mu^2 - 1}].
\]

On expanding (i') and (iv') we find they are already satisfied by virtue of (6.15i–iv). Next we let both \(\alpha_s\) and \(\tilde{\alpha}_t\) be type 2 in (6.13):
\[
\sum_{i=1}^{K_1} \begin{pmatrix}
\bar{y}_i^A & e^{i\theta} (\mu q_i^A + \bar{y}_i^A \sqrt{\mu^2 - 1}) & y_i^B \\
-e^{-i\theta} (\mu q_i^A - y_i^A \sqrt{\mu^2 - 1}) & \bar{y}_i^A & e^{i\theta} (\mu q_i^B + y_i^B \sqrt{\mu^2 - 1}) \\
-e^{-i\theta} (\mu q_i^A - y_i^A \sqrt{\mu^2 - 1}) & e^{i\theta} (\mu q_i^A + y_i^A \sqrt{\mu^2 - 1}) & y_i^B
\end{pmatrix}
\]
\[
= \sum_{i=1}^{K_1} \begin{pmatrix}
y_i^A & e^{i\theta} (\mu q_i^A + y_i^A \sqrt{\mu^2 - 1}) & \bar{y}_i^B \\
-e^{-i\theta} (\mu q_i^A - y_i^A \sqrt{\mu^2 - 1}) & y_i^A & e^{i\theta} (\mu q_i^B + \bar{y}_i^B \sqrt{\mu^2 - 1}) \\
-e^{-i\theta} (\mu q_i^A - y_i^A \sqrt{\mu^2 - 1}) & y_i^B & \bar{y}_i^B
\end{pmatrix} \quad (\oplus)
\]

Multiplying out and equating matrix entries, we find for the upper diagonal entry:
\[
\sum_i \left[ \bar{y}_i^A y_i^B e^{i(\theta - \tilde{\theta})} (\mu q_i^A + \bar{y}_i^A \sqrt{\mu^2 - 1}) (\mu q_i^B - y_i^B \sqrt{\mu^2 - 1}) \right]
\]
\[
= \sum_i \left[ y_i^A \bar{y}_i^B e^{i(\theta - \tilde{\theta})} (\mu q_i^A + y_i^A \sqrt{\mu^2 - 1}) (\mu q_i^B - \bar{y}_i^B \sqrt{\mu^2 - 1}) \right]
\]

On expansion of this, and eliminating terms via equations (6.15i–iv), we get
\[
\sum_i \left[ b_i g_i (\mu - 1)(\tilde{\mu} - 1) \sqrt{(\mu^2 - 1)(\tilde{\mu}^2 - 1)} \sin(2(\theta - \tilde{\theta})) + (d_i - a_i)(h_i - e_i) \mu \tilde{\mu} \sin(\theta - \tilde{\theta}) \right] = 0
\]
for all \(\mu, \tilde{\mu}, \theta, \tilde{\theta}\). On setting \(\mu = \tilde{\mu} = 1\) we obtain (6.15v), and on substituting it back and using (6.15i) we obtain (6.15vi). Now it is a straightforward verification to check that the set of equations (6.15i–vi) guarantee that the matrix equation \((\oplus)\) is satisfied for all its entries, and moreover for the two remaining choices \(\alpha_s\) and \(\tilde{\alpha}_t\) being either both type 3 or one type 2 and the other type 3; we find that the set
of equations (6.15) are also sufficient for (6.13) to hold. We omit the calculations.

(ii) From part (i) we know that a closed one-form has expression

\[ \omega(\alpha) = aI + \sum_{i=0}^{K_1} A_1^i \hat{d}_{\alpha}(B_1^i) + \sum_{n=2}^{N} \sum_{i=0}^{K_n} A_n^i \prod_{j=1}^{n} \hat{d}_{\alpha}(B_1^j) \]  

(6.17)

where \( A_1^i \) and \( B_1^i \) satisfy equations (6.15). Now observe from (6.10) and (6.11) that when \( \alpha_t \) is either type 2 or 3, the entries of \( (\hat{d}B_1^1)(\alpha_t) \) consist of linear combinations of \( yB_1^i = \mu^{-2}(f_i e^{-i\theta} - g_i e^{i\theta}) \), \( qB_1^i = \mu^{-2}(h_i e_i + (f_i e^{-i\theta} + g_i e^{i\theta})\sqrt{\mu^2 - 1}) \) and \( q'\!\!B_1^i = \mu^{-2}(h_i e_i - (f_i e^{-i\theta} + g_i e^{i\theta})\sqrt{\mu^2 - 1}) \) and so in the expression \( \sum_{i=0}^{K_1} A_1^i \hat{d}_{\alpha_t}(B_1^i) \) we see by eqs (6.15) that only combinations involving \( a_i \) and \( d_i \) are nonzero, i.e. \( \sum_{i} A_1^i \hat{d}_{\alpha_t}(B_1^i) = \sum_{i} \left( \begin{array}{c} a_i \\ 0 \\ d_i \end{array} \right) \hat{d}_{\alpha_t}(B_1^i) \). In fact, using 6.15ii, iv and v we have \( a_i = d_i \), so \( \sum_{i} A_1^i \hat{d}_{\alpha_t}(B_1^i) = \sum_{i} a_i \hat{d}_{\alpha_t}(B_1^i) = \hat{d}_{\alpha_t} \left( \sum_{i} a_i B_1^i \right) \). When \( \alpha_t \) is type 1, we have from (6.9), by (6.15vi) that

\[
\sum_{i} \left( \begin{array}{ccc} a_i & b_i & c_i \\ 0 & f_i(e^{it(\gamma-\beta)} - 1) & 0 \\ g_i(e^{it(\beta-\gamma)} - 1) & 0 & 0 \end{array} \right) = \sum_{i} \left( \begin{array}{ccc} 0 & a_i f_i(e^{it(\gamma-\beta)} - 1) & 0 \\ d_i g_i(e^{it(\beta-\gamma)} - 1) & 0 & 0 \end{array} \right)
\]

and so by (6.15iv) in this case too, we have that \( \sum_{i} A_1^i \hat{d}_{\alpha_t}(B_1^i) = \hat{d}_{\alpha} \left( \sum_{i} a_i B_1^i \right) \), which is therefore true for all \( \alpha \in G \). Thus the first order term in \( \hat{d} \) for a closed one-form \( \omega \) is exact.

(iii) We already know that by the preceding parts of the claim, there is a closed form of the type (6.16):

\[ \omega(\alpha) = aI + \sum_{n=2}^{N} \sum_{i=0}^{K_n} A_n^i \prod_{j=1}^{n} \hat{d}_{\alpha}(B_1^j) \]

in each cohomology class. We only need to show that such a closed one-form is exact iff it is zero. Observe from (6.9), (6.10) and (6.11) that for all types of \( \alpha_t \), the power series of \( \omega(\alpha_t) \) in \( t \) has no first order term. Now any exact one-form \( \hat{d}(A)(\alpha_t) \) must necessarily have a nonzero first order term, since otherwise we see from the expression

\[ \hat{d}(A)(\alpha_t) = \exp(\text{ad}itB)(A) - A \]
that its higher order terms are also zero. Thus a one–form of the type (6.16) is exact iff it is zero.

**Remarks:**

1. So by this claim we can visualize \( \hat{H}^1 \) as the linear space of closed one–forms of the type (6.16). We will not here work out the conditions on the entries of \( A^n_i \) and \( B^j_i \), \( n \geq 2 \) to ensure that such an \( \omega \) is closed.

2. Clearly for a closed \( \omega \) of type (6.16), we have \( \left. \frac{d}{dt} \omega(\alpha_t) \right|_{t=0} = 0 \), hence the *infinitesimal* first cohomology for \( G \) acting on \( \mathcal{A} = M_2(\mathbb{C}) \) is zero.

7. **Further Developments.**

First some comments on relating integrated de Rham cohomology to existing cohomologies for algebras and groups.

(i) Hochschild cohomology for algebras is constructed from cochains consisting of \( n \)–linear maps from an algebra \( \mathcal{A} \) to an \( \mathcal{A} \)–module \( X \). This is quite different from the cochains of integrated de Rham cohomology, consisting of maps \( \omega : G^n \to \mathcal{A} \) where \( G \) acts on \( \mathcal{A} \), so there seems to be little connection. Moreover, Hochschild cohomology is intrinsic to algebras, regardless of any group actions.

(ii) Group cohomology starts from cochains which are maps \( \varphi : G^n \to Y \) where \( G \) is a group and \( Y \) is a coefficient group on which \( G \) acts. So given an action \( G \subset \text{Aut} \mathcal{A} \) on an algebra, we can regard \( \mathcal{A} \) with its additive structure as such a coefficient group. In this case we can regard the cochains of integrated de Rham cohomology \( \omega : G^n \to \mathcal{A} \) as a subset of the cochains of group cohomology with coefficient group \( \mathcal{A} \). However which particular subset it will be, depends on the algebraic structure of \( \mathcal{A} \). Moreover, the group coboundary operator is relatively insensitive to the action of \( G \) on \( \mathcal{A} \), whilst \( \hat{d} \) is extremely sensitive to the action. So again, there seems to be little connection.

(iii) In Connes’ differential envelope over an algebra \( \mathcal{A} \), there is no reference to a group action. One may try to take care of this, using tensor products and homomorphisms, but this is unlikely to succeed for the following reasons:
a) in Connes’ differential envelope, formal expressions of the type:

\[ \omega = (a_0 + \lambda_1) \, da_1 \, da_2 \ldots da_n \]  

are the basic objects, whilst in our case, the maps \( \omega : G^n \to \mathcal{A} \) are basic, and the same map may have different expressions

\[ \omega(\alpha_1, \ldots, \alpha_n) = (a_0 + \lambda_1) \hat{d}_{\alpha_i} a_1 \ldots \hat{d}_{\alpha_k} a_m \]  

which we identify.

b) The expression (*) in the differential envelope for an n–form has precisely n factors \( da_i \), whilst in our case, in (+) there is no upper bound on the (finite) number of \( \hat{d}_{\alpha_i} a \) factors which can occur for an n–form.

c) \( d \), as an operator on the differential envelope satisfies the graded Leibniz rule, whereas \( \hat{d} \) satisfies (6.0).

(iv) Cuntz in [3] defines an algebra of formal differences in which the basic objects do satisfy (6.0) and the algebra consist of formal products of these. There is no reference to a group action, and it also appears difficult to connect to integrated differential geometry for reason (a) above.

The rest of the machinery of differential geometry is quite easy to define in integrated differential geometry, for instance it has been done for push–forwards, pull–backs, Lie derivatives, principal fibre bundles and connections on them. In each case, the integrated object is defined in such a way that under the \( \tau \) map on \( \mathcal{C}^\infty(M) \) it reduces to the usual object. The main application for such an extension of differential geometry would be to Hamiltonian mechanics and classical gauge theories. Whilst we can easily imitate the formal structure of Hamiltonian mechanics in integrated differential geometry, what is really needed is a way of doing actual Hamiltonian mechanics using only structures of integrated differential geometry (without reference to the infinitesimal level, i.e. the map \( \tau \)). That is, from the Hamiltonian function and symplectic form (integrated), we should obtain the same time evolution groups on \( \mathcal{C}^\infty(M) \) by such an integrated method, as that obtained by the usual Hamiltonian mechanics. Thus far, such a method has been eluding the author, and so we leave the further development of integrated differential geometry for a future project.
8. Discussion.

Above, we have shown that for a manifold $M$, there is a larger “integrated differential geometry” structure defined on the action of $\text{Diff}(M)$ on $C^\infty(M)$ such that when we differentiate at zero along the one-parameter groups we obtain ordinary differential geometry. This structure generalised readily to all group actions on associative algebras, and provided a chain complex from which we could define “integrated de Rham cohomology,” thus establishing a set of new invariants for group actions. This was applied to two examples: calculating $\hat{H}^1$ and $\hat{H}^2$ for the shift operator acting on an algebra of sequences, and finding $\hat{H}^1$ for the algebra $M_2(\mathbb{C})$ under its automorphism group.

Of the many possible directions for developing this structure further, we note a few: first, examining topological questions when $G$ and $A$ are endowed with topologies; second, how this structure intertwines with the covariant representation theory of the group action, and thirdly, do a more difficult example, e.g. $\text{Diff}(S^1)$ acting on $C(S^1)$. For comparison with Connes’ approach in an example, a good example would be the action of the permutation group on the algebra of functions on a discrete set, cf. [7]. Apart from this, there is the development of integrated Hamiltonian mechanics and gauge theory, as mentioned above.

Acknowledgements.

The idea for this paper occurred to me whilst working on a joint project with Mark Gotay and Angas Hurst on geometric quantisation. It was developed purely for my own pleasure, and is intended as a minor footnote to that large and deep theory [1,2] developed by Alain Connes and his co-workers. I am also grateful to Norman Wildberger for his incredulous but patient ear over many coffees, to Keith Hannabuss for harboring me in Oxford where I tried getting the examples straight, to prof. Cuntz for his remarks and preprints, and to Prof. R. Coquereaux for his hospitality at the CNRS, Luminy.
Bibliography.

[1] Connes, A.: Non-commutative differential geometry, Publ. Math. IHES \textbf{62}, 257–360 (1985)
Connes, A: Non–commutative differential geometry, Academic Press (in press).

[2] Cuntz, J.: Representations of quantized differential forms in non–commutative geometry. Preprint, Heidelberg 1994.

[3] Cuntz, J.: A survey of some aspects of non–commutative differential geometry, Jber. d. Dt. Mat.–Verein. \textbf{95}, 60–84 (1993)

[4] Bratteli, O.: Derivations, dissipations and group actions on C∗–algebras, Springer Lect. Notes Math. 1229 (1986)

[5] Dubois–Violette, M., Kerner, R., Madore, J.: Noncommutative differential geometry of matrix algebras, J. Math. Phys. \textbf{31}, 316–322 (1990)

[6] Blackadar, B., Cuntz, J.: Differential Banach algebra norms and smooth sub–algebras of C∗–algebras, J. Op. Theory \textbf{26}, 255–282 (1991)

[7] Dimakis, A., Müller–Hoissen, F.: Discrete differential calculus, graphs, topologies and gauge theory, J. Math. Phys. to appear.