COMPLETE CHARACTERIZATION OF THE DENSITY
OF STABLE MAPPINGS

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Abstract. In the middle of the twentieth century, Whitney conjectured that the set of stable mappings would always be dense in the appropriate space of $C^\infty$ mappings, and this conjecture came to be known as the “strong conjecture” (see for example [3]). However, Thom showed that the set is not necessarily dense in all pairs of dimensions of manifolds. Then, around 1970, Mather established a significant theory on the stability of $C^\infty$ mappings and gave a characterization of the density of stable mappings in the case where the source manifold is compact. In this paper, we show that the set of stable mappings is never dense if the source manifold is non-compact. Moreover, by combining Mather’s theorem and the main theorem of this paper, we give a complete characterization of the density of stable mappings.

1. Introduction

In the middle of the twentieth century, Whitney conjectured that the set consisting of all stable mappings would always be dense in the appropriate space of $C^\infty$ mappings, and this conjecture came to be known as the “strong conjecture” (see for example [3]). However, Thom showed that the set is not necessarily dense in all pairs of dimensions of manifolds. Then, around 1970, in a celebrated series [4, 5, 6, 7, 8, 9], Mather established a significant theory on the stability of $C^\infty$ mappings and gave a characterization of the density of stable mappings in the case where the source manifold is compact as in the following theorem. In what follows, unless otherwise stated, all manifolds and mappings belong to class $C^\infty$, and all manifolds are without boundary and assumed to have countable bases. For manifolds $N$ and $P$, we denote the space of all mappings of $N$ into $P$ with the Whitney $C^\infty$ topology by $C^\infty(N, P)$ (for the definition of Whitney $C^\infty$ topology, see for example [2]).

Theorem 1 ([9]). Let $N$ be a compact manifold of dimension $n$, and $P$ be a manifold of dimension $p$. Then, the set consisting of all stable mappings is dense in $C^\infty(N, P)$ if and only if the pair $(n, p)$ satisfies one of the following conditions.

1. $n < \frac{5}{6}p + \frac{8}{7}$ and $p - n \geq 4$
2. $n < \frac{4}{3}p + \frac{9}{7}$ and $3 \geq p - n \geq 0$
3. $p < 8$ and $p - n = -1$
4. $p < 6$ and $p - n = -2$
5. $p < 7$ and $p - n \leq -3$

A dimension pair $(n, p)$ is called a nice dimension if $(n, p)$ satisfies one of the conditions (1)–(5) in Theorem 1.

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After that, the case where a source manifold is non-compact was considered by Dimca, and in 1979, he gave the following result.

**Proposition 1** ([1]). Let $N$ be a non-compact manifold. Then, the set consisting of all (infinitesimally) stable mappings is not dense in $C^\infty(N, \mathbb{R})$.

For the definition of infinitesimal stability, which is defined by Mather in [6], see Section 2 (the definition of stability is also reviewed in this section). The main purpose of this paper is to give a rigorous proof of the following main theorem and a complete characterization of the density of stable mappings.

**Theorem 2.** Let $N$ be a non-compact manifold, and $P$ be a manifold. Then, the set consisting of all (infinitesimally) stable mappings is not dense in $C^\infty(N, P)$.

By Mather’s theorem (Theorem 1 and Theorem 2), we easily obtain the following complete characterization of the density of stable mappings.

**Theorem 3.** Let $N$ and $P$ be manifolds of dimensions $n$ and $p$, respectively. Then, the set consisting of all stable mappings is dense in $C^\infty(N, P)$ if and only if $N$ is compact and $(n, p)$ is a nice dimension.

The remainder of this paper is organized as follows. In Section 2, we prepare some definitions and notations, and give a lemma for the proof of the main theorem (Theorem 2). Finally, Section 3 is devoted to the proof of Theorem 2.

### 2. Preliminaries

In this section, we prepare some definitions and notations, and give a lemma (Lemma 1) for the proof of Theorem 2.

Let $N$ and $P$ be manifolds. For given mappings $f, g \in C^\infty(N, P)$, we say that $f$ is $A$-equivalent to $g$ if there exist diffeomorphisms $\Phi : N \rightarrow N$ and $\Psi : P \rightarrow P$ such that $f = \Psi \circ g \circ \Phi^{-1}$. Let $f : N \rightarrow P$ be a mapping. We say that $f$ is stable if the $A$-equivalence class of $f$ is open in $C^\infty(N, P)$. A mapping $\xi : N \rightarrow TP$ such that $\Pi \circ \xi = f$ is called a vector field along $f$, where $TP$ is the tangent bundle of $P$ and $\Pi : TP \rightarrow P$ is the canonical projection. Let $\theta(f)$ be the set consisting of all vector fields along $f$. Set $\theta(N) = \theta(id_N)$ and $\theta(P) = \theta(id_P)$, where $id_N : N \rightarrow N$ and $id_P : P \rightarrow P$ are the identity mappings. Following Mather, let $tf : \theta(N) \rightarrow \theta(f)$ (resp., $\omega f : \theta(P) \rightarrow \theta(f)$) be the mapping defined by $tf(\xi) = Tf \circ \xi$ (resp., $\omega f(\eta) = \eta \circ f$), where $Tf : TN \rightarrow TP$ is the derivative mapping of $f$. Then, as in [6], $f$ is said to be infinitesimally stable if

$$tf(\theta(N)) + \omega f(\theta(P)) = \theta(f).$$

Let $N$ and $P$ be manifolds, and let $f : N \rightarrow P$ be a mapping. A point $q \in N$ is called a critical point of $f$ if rank $df_q < \dim P$. We say that a point of $P$ is a critical value if it is the image of a critical point.

In what follows, for a given positive integer $m$, we denote the origin $(0, \ldots, 0)$ of $\mathbb{R}^m$ by $0$, the Euclidean norm of $x \in \mathbb{R}^m$ by $\|x\|$, and the $m$-dimensional open ball with center $x \in \mathbb{R}^m$ and radius $r > 0$ by $B^m(x, r)$, that is,

$$B^m(x, r) = \{ x' \in \mathbb{R}^m \mid \|x - x'\| < r \}.$$  

For a set (resp., a topological space) $X$ and a subset $A$ of $X$, we denote the complement of $A$ (resp., the closure of $A$) by $A^c$ (resp., $\overline{A}$). We denote the set of all positive integers by $\mathbb{N}$.
Lemma 1. Let \( f = (f_1, \ldots, f_p) : B^n(0, r) \to \mathbb{R}^p \) be a mapping such that
\[
f_p(x) = \frac{1}{2} \sum_{i=1}^n x_i^2 + a,
\]
where \( a \) is a real number and \( x = (x_1, \ldots, x_n) \). If \( g = (g_1, \ldots, g_p) : B^n(0, r) \to \mathbb{R}^p \) satisfies that
\[
\left( \sum_{i=1}^n \left( \frac{\partial f_p}{\partial x_i}(x) - \frac{\partial g_p}{\partial x_i}(x) \right)^2 \right)^{\frac{1}{2}} < \frac{r}{2}
\]
for any \( x \in B^n(0, r) \), then there exists a critical point of \( g \) in \( B^n(0, r) \).

Proof of Lemma 4 Since in the case \( n < p \) any point in \( B^n(0, r) \) is a critical point of \( g \), it is sufficient to consider the case \( n \geq p \). Let \( D : B^n(0, r) \to \mathbb{R}^n \) be the mapping given by
\[
D(x) := \left( \frac{\partial f_p}{\partial x_1}(x), \ldots, \frac{\partial f_p}{\partial x_n}(x) \right).
\]
For simplicity, set \( K = B^n(0, \frac{r}{2}) \). Since \( \|D(x)\| < \frac{r}{2} \) for any \( x \in K \), we can define the restriction \( D|_K : K \to K \). Since \( D|_K \) is continuous, there exists a point \( x_0 \in K \) such that \( D|_K(x_0) = x_0 \) by Brouwer’s fixed point theorem. Thus, it follows that \( \frac{\partial g_p}{\partial x_i}(x_0) = 0 \) for any \( i \in \{1, \ldots, n\} \), which implies that \( x_0 \) is a critical point of \( g \). \( \square \)

3. PROOF OF THE MAIN THEOREM

Set \( n = \dim N \) and \( p = \dim P \), respectively. By Whitney’s embedding theorem, there exist a positive integer \( \ell \) and an embedding \( F : N \to \mathbb{R}^\ell \) such that \( F(N) \) is a closed set of \( \mathbb{R}^\ell \). By taking \( \ell \) larger if necessary we can assume that \( F(N) \neq \mathbb{R}^\ell \). Then, there exists a point \( z_0 \in \mathbb{R}^\ell \setminus F(N) \). Since \( N \) is non-compact, \( F(N) \) is also non-compact. Thus, \( F(N) \) is not bounded, which implies that there exists a sequence \( \{ R_\alpha \}_{\alpha \in \mathbb{N}} \) of positive real numbers and a sequence \( \{ z_\alpha \}_{\alpha \in \mathbb{N}} \) of points in \( \mathbb{R}^\ell \) such that
\[
\begin{align*}
R_\alpha &< R_{\alpha+1} \text{ for any } \alpha \in \mathbb{N} \text{ and } \lim_{\alpha \to \infty} R_\alpha = \infty, \\
z_\alpha &\in F(N) \cap (B^\ell(z_0, R_{\alpha+1}) \setminus \overline{B^\ell(z_0, R_\alpha)}), \text{ for any } \alpha \in \mathbb{N}.
\end{align*}
\]
Let \( \alpha \) be any positive integer. Set \( q_\alpha = F^{-1}(z_\alpha) \). Here, note that \( F^{-1}(B^\ell(z_0, R_{\alpha+1}) \setminus \overline{B^\ell(z_0, R_\alpha)}) \) is an open neighborhood of \( q_\alpha \). Then, there exists a coordinate neighborhood \( (U_\alpha, \varphi_\alpha) \) of \( N \) with the following properties:
\[
\begin{align*}
&U_\alpha \text{ is compact,} \\
&q_\alpha \in U_\alpha \subset F^{-1}(B^\ell(z_0, R_{\alpha+1}) \setminus \overline{B^\ell(z_0, R_\alpha)}), \\
&\varphi_\alpha(q_\alpha) = 0 \in \mathbb{R}^n.
\end{align*}
\]
Moreover, there exist an open neighborhood \( U_\alpha' \) of \( q_\alpha \), a positive real number \( r_\alpha \), and \( \rho_\alpha : N \to \mathbb{R} \) such that
\[
\begin{align*}
&\overline{U_\alpha'} \subset U_\alpha, \\
&\rho_\alpha(q) = 1 \text{ for any } q \in \overline{U_\alpha'}, \\
&\text{supp } \rho_\alpha \subset U_\alpha,
\end{align*}
\]
where supp \( \rho_\alpha = \{ q \in \mathbb{N} \mid \rho_\alpha(q) \neq 0 \} \). Note that supp \( \rho_\alpha \) is compact since \( U_\alpha \) is compact. Here, by choosing \( U'_\alpha \) smaller for each \( \alpha \in \mathbb{N} \) we can assume that

- \( \varphi_\alpha(U'_\alpha) = B^n(0, r_\alpha) \),
- \( \lim_{\alpha \to \infty} r_\alpha = 0 \),

where each \( r_\alpha \) is a positive real number.

Let \( \gamma = (\gamma_1, \ldots, \gamma_p) : \mathbb{N} \to \mathbb{R}^p \) be a bijection, and let \( \eta_\alpha : \varphi_\alpha(U_\alpha) \to \mathbb{R}^p \) be the mapping defined by

\[
\eta_\alpha(x) = \left( \gamma_1(\alpha), \ldots, \gamma_{p-1}(\alpha), \frac{1}{2} \sum_{i=1}^{n} x_i^2 + \gamma_p(\alpha) \right)
\]

for each \( \alpha \in \mathbb{N} \), where \( x = (x_1, \ldots, x_n) \). Let \((V, \psi)\) be a coordinate neighborhood of \( P \) satisfying \( \psi(V) = \mathbb{R}^p \). Since \( U_\alpha \cap U_\beta = \emptyset \) if \( \alpha \neq \beta \), we can define \( f : N \to P \) as follows:

\[
f(q) = \begin{cases} 
\psi^{-1}(\rho_\alpha(q)(\eta_\alpha \circ \varphi_\alpha)(q)) & \text{if } q \in U_\alpha, \\
\psi^{-1}(0) & \text{if } q \notin \bigcup_{\alpha \in \mathbb{N}} U_\alpha.
\end{cases}
\]

We show that \( f \) is of class \( C^\infty \). Let \( q \in N \) be any point. If \( q \in \bigcup_{\alpha \in \mathbb{N}} U_\alpha \), then by the definition of \( f \) it is clearly seen that \( f \) is of class \( C^\infty \) at \( q \). Thus, we consider the case \( q \in (\bigcup_{\alpha \in \mathbb{N}} U_\alpha)^c \). Since \( \lim_{\alpha \to \infty} R_\alpha = \infty \), there exists \( \beta \in \mathbb{N} \) such that \( q \in F^{-1}(B^\ell(z_0, R_\beta)) \). For simplicity, set

\[
A = F^{-1}(B^\ell(z_0, R_\beta)) \cap \left( \bigcup_{\alpha \in \mathbb{N}} \text{supp } \rho_\alpha \right)^c.
\]

Note that \( q \in A \). Since \( R_\alpha < R_{\alpha+1} \) for any \( \alpha \in \mathbb{N} \), we see that \( F^{-1}(B^\ell(z_0, R_\beta)) \subset (\text{supp } \rho_\alpha)^c \) for any \( \alpha \in \mathbb{N} \) satisfying \( \alpha > \beta \). Thus, we have

\[
A = F^{-1}(B^\ell(z_0, R_\beta)) \cap \left( \bigcap_{\alpha \in \mathbb{N}} (\text{supp } \rho_\alpha)^c \right) = F^{-1}(B^\ell(z_0, R_\beta)) \cap \left( \bigcap_{\alpha \leq \beta} (\text{supp } \rho_\alpha)^c \right),
\]

which implies that \( A \) is an open set of \( N \). Since \( \rho_\alpha|_A \) is a constant function with a constant value 0 for each \( \alpha \in \mathbb{N} \), the mapping \( f|_A \) is also constant. Therefore, \( f \) is of class \( C^\infty \) at \( q \).

Since \( z_0 \in \mathbb{R}^\ell \setminus F(N) \), we can define the following continuous function \( \delta : N \to \mathbb{R} \):

\[
\delta(q) = \frac{1}{\|F(q) - z_0\|}.
\]

Let \( \pi : J^1(N, P) \to N \times P \) be the natural projection defined by \( \pi(j^1 g(q)) = (q, g(q)) \). Then, for any \( \alpha \in \mathbb{N} \), set

\[
O_\alpha = \{ j^1 g(q) \in \pi^{-1}(U_\alpha \times V) \mid j^1 g(q) \text{ satisfies (3.1) and (3.2)} \}.
\]
By showing that $\| (\psi \circ f)(q) - (\psi \circ g)(q) \| < \delta(q)$,\)
\begin{equation}
(3.2) \quad \sqrt{\sum_{i=1}^{n} \left( \frac{\partial (\psi_p \circ f \circ \varphi^{-1}) (\varphi(q))}{\partial x_i} - \frac{\partial (\psi_p \circ g \circ \varphi^{-1}) (\varphi(q))}{\partial x_i} \right)^2} < \frac{r_\alpha}{2}.
\end{equation}

In (3.2), $\psi_p$ is the $p$-th component of $\psi$. From (3.1) and (3.2), it is not hard to see that $O_\alpha$ is an open set of $J^1(N, P)$.

We show that $\cap_{\alpha \in \mathbb{N}} (\overline{U_\alpha})^c$ is an open set of $N$. Let $q \in \cap_{\alpha \in \mathbb{N}} (\overline{U_\alpha})^c$ be any point. Since $\lim_{\alpha \to \infty} R_\alpha = \infty$, there exists $\beta \in \mathbb{N}$ such that $q \in F^{-1}(B^\ell(z_0, R_\beta))$.

Since $R_\alpha < R_{\alpha+1}$ for any $\alpha \in \mathbb{N}$, we have $F^{-1}(B^\ell(z_0, R_\beta)) \subset (\overline{U_\alpha})^c$ for any $\alpha \in \mathbb{N}$ satisfying $\alpha > \beta$, which implies that
\[
F^{-1}(B^\ell(z_0, R_\beta)) \cap \left( \bigcap_{\alpha \leq \beta} (\overline{U_\alpha})^c \right) \subset \bigcap_{\alpha \in \mathbb{N}} (\overline{U_\alpha})^c.
\]

Since the left side of the above expression is an open neighborhood of $q$, it follows that $\cap_{\alpha \in \mathbb{N}} (\overline{U_\alpha})^c$ is open. Thus, since $\pi$ is continuous,
\[
O := \left( \bigcup_{\alpha \in \mathbb{N}} O_\alpha \right) \cup \pi^{-1} \left( \left( \bigcap_{\alpha \in \mathbb{N}} (\overline{U_\alpha})^c \right) \times V \right)
\]
is open in $J^1(N, P)$. Hence, we can construct the following open set of $C^\infty(N, P)$:
\[
U := \{ g \in C^\infty(N, P) \mid j^1 g(N) \subset O \}.
\]

By showing that $j^1 f(N) \subset O$, we will prove that $U \neq \emptyset$. Let $j^1 f(q) \in N$ be any element of $j^1 f(N)$. If there exists $\alpha \in \mathbb{N}$ such that $q \in U_\alpha$, we have $j^1 f(q) \in O_\alpha$ ($\subset O$) since $f(q) \in V$ and $j^1 f(q)$ clearly satisfies (3.1) and (3.2). In the case where $q \notin \bigcup_{\alpha \in \mathbb{N}} U_\alpha$, since
\begin{equation}
(3.3) \quad N = \left( \bigcup_{\alpha \in \mathbb{N}} U_\alpha \right) \cup \left( \bigcap_{\alpha \in \mathbb{N}} (\overline{U_\alpha})^c \right),
\end{equation}

it must follow that $q \in \bigcap_{\alpha \in \mathbb{N}} (\overline{U_\alpha})^c$. Therefore, since $f(q) \in V$, we obtain
\[
j^1 f(q) \in \pi^{-1} \left( \left( \bigcap_{\alpha \in \mathbb{N}} (\overline{U_\alpha})^c \right) \times V \right) \subset O).
\]

Hence, we have $U \neq \emptyset$. We give the following lemma on properties of a mapping in $U$.

**Lemma 2.** For any mapping $g \in U$, we have $g(N) \subset V$ and there exists a sequence $\{ q'_\alpha \}_{\alpha \in \mathbb{N}}$ of points in $N$ with the following properties.

1. For each $\alpha \in \mathbb{N}$, $q'_\alpha$ is a critical point of $g$ in $U'_\alpha$.
2. The set $\{ g(q'_\alpha) \mid \alpha \in \mathbb{N} \}$ is dense in $V$.

**Proof of Lemma 2.** By the definition of $U$, we have $g(N) \subset V$.

Let $\alpha$ be any positive integer. Then, we have
\[
(\psi_p \circ f \circ \varphi^{-1})(x) = \frac{1}{2} \sum_{i=1}^{n} x_i^2 + \gamma_p(\alpha)
\]
for any \( x = (x_1, \ldots, x_n) \in \varphi(a(U'_a) = B^n(0, r_a)) \). For any \( q \in U'_a \), we obtain
\[ j^1 g(q) \in O_a \text{ since we have } (\ref{eq:j1g}) \text{ and } U'_a \text{ is contained in } U_a \] which does not intersect with \( U_\beta (\beta \neq \alpha) \). Hence, it follows that \((\psi_p \circ g \circ \varphi_\alpha^{-1})|_{B^n(0, r_a)} \) satisfies (\ref{eq:3.2}), which implies that there exists a critical point of \((\psi_p \circ g \circ \varphi_\alpha^{-1})|_{B^n(0, r_a)} \) in \( B^n(0, r_a) \) by Lemma \ref{lemma:3.2}. Namely, there exists a critical point of \( g \) in \( U'_a \). We denote its point by \( q'_a \).

Since \( \{ q'_a \}_{a \in \mathbb{N}} \) satisfies (\ref{eq:3.1}) by the above argument, it is sufficient to show that the sequence of points also satisfies (\ref{eq:3.3}). Let \( V' \) be any open set of \( V \). We show that \( \{ g(q'_a) \mid a \in \mathbb{N} \} \cap V' \neq \emptyset \). Then, by choosing \( V' \) smaller, we can assume that \( \psi(V') = B^p(y_0, \epsilon) \), where \( y_0 \) is a point of \( \mathbb{R}^p \) and \( \epsilon \) is a positive real number. Note that for any \( a \in \mathbb{N} \), we have
\begin{equation}
(3.4) \quad \left\| (\psi \circ g)(q'_a) - y_0 \right\| \leq \left\| (\psi \circ g)(q'_a) - (\psi \circ f)(q_a) \right\| + \left\| (\psi \circ f)(q'_a) - (\psi \circ f)(q_a) \right\| + \left\| (\psi \circ f)(q_a) - y_0 \right\|. 
\end{equation}

Thus, it follows that \( \{ q'_a \}_{a \in \mathbb{N}} \) satisfies (\ref{eq:3.3}) and \( \lim_{\alpha \to \infty} R_\alpha = \infty \), there exists \( \alpha_1 \in \mathbb{N} \) such that \( \delta(q'_a) < \frac{\epsilon}{3} \) for any \( a \in \mathbb{N} \) satisfying \( \alpha \geq \alpha_1 \). Here, note that for any \( a \in \mathbb{N} \), we have
\[ \| (\psi \circ g)(q'_a) - (\psi \circ f)(q'_a) \| < \delta(q'_a) \]
by (\ref{eq:3.1}) since \( j^1 g(q'_a) \in O_a \). Thus, it follows that for any \( a \in \mathbb{N} \),
\begin{equation}
(3.5) \quad \alpha \geq \alpha_1 \implies \| (\psi \circ g)(q'_a) - (\psi \circ f)(q'_a) \| < \frac{\epsilon}{3}.
\end{equation}

For any \( a \in \mathbb{N} \), since \( q_a, q'_a \in U'_a \), we have
\[ \| (\psi \circ f)(q'_a) - (\psi \circ f)(q_a) \| \leq \| \psi_\alpha(\varphi_\alpha(q'_a)) - \gamma(\alpha) \| = \frac{\| \varphi_\alpha(q'_a) \|^2}{2} < \frac{\epsilon^2}{2}. \]
Since \( \lim_{a \to \infty} r_\alpha = 0 \), there exists \( \alpha_2 \in \mathbb{N} \) such that for any \( \alpha \in \mathbb{N} \),
\begin{equation}
(3.6) \quad \| (\psi \circ f)(q_a) - (\psi \circ f)(q'_a) \| < \frac{\epsilon}{3}.
\end{equation}

Since \( (\psi \circ f)(q_a) = \gamma(\alpha) \) for each \( \alpha \in \mathbb{N} \), we have
\[ \{ (\psi \circ f)(q_a) \mid a \in \mathbb{N} \} = \mathbb{Q}^p. \]
Hence, there exists \( \alpha_3 \in \mathbb{N} \) such that \( \alpha_3 > \max \{ \alpha_1, \alpha_2 \} \) and
\begin{equation}
(3.7) \quad \| (\psi \circ f)(q_{\alpha_3}) - y_0 \| < \frac{\epsilon}{3}.
\end{equation}
Thus, we have \( \| (\psi \circ g)(q'_{\alpha_3}) - y_0 \| < \epsilon \) by (\ref{eq:3.6}) to (\ref{eq:3.7}), which implies that \( g(q'_{\alpha_3}) \in V' \). \qed

We show that any mapping in \( \mathcal{U} \) is not infinitesimally stable. Let \( g \in \mathcal{U} \) be any mapping, and let \( \Sigma \) be the set consisting of all critical points of \( g \). Set \( K = \psi^{-1}(B^p(0, r)) \), where \( r \) is a positive real number. Note that \( K \) is a compact set in \( P \). Then, from Lemma \ref{lemma:2.1} (2), \( (g|\Sigma)^{-1}(K) \) contains a countable subset of \( \{ q'_a \mid a \in \mathbb{N} \} \). Since \( F(g(q'_a)) \notin B^p(z_0, R_\alpha) \) for each \( a \in \mathbb{N} \) and \( \lim_{\alpha \to \infty} R_\alpha = \infty \), the set \( F((g|\Sigma)^{-1}(K)) \) is not compact, which implies that \( (g|\Sigma)^{-1}(K) \) is not compact.
Since \( g|_{\Sigma}: \Sigma \to P \) is not proper, \( g \) is not infinitesimally stable (note that this fact follows from [8, Proposition 5.1]).

We show that any mapping in \( \mathcal{U} \) is not stable. Let \( g \in \mathcal{U} \) be an arbitrary mapping, and let \( U_g \) be any open neighborhood of \( g \). Then, there exist a non-negative integer \( k \) and an open set \( O' \) of \( J^k(N, P) \) such that

\[
g \in \{ h \in C^\infty(N, P) \mid j^kh(N) \subset O' \} \subset \mathcal{U}_g.
\]

In order to prove that \( g \) is not stable, it is sufficient to show that there exists a mapping \( h \in C^\infty(N, P) \) satisfying the following properties.

- We have \( j^kh(N) \subset O' \).
- There exist \((p+1)\)-critical points of \( h \) which share the same critical value.

Note that the second property implies that \( h \) is not stable.

For any \( \alpha \in \mathbb{N} \) and \( c \in \mathbb{R}^p \), let \( G_{\alpha,c}: N \to P \) be the mapping defined by

\[
G_{\alpha,c} = \psi^{-1} \circ (\psi \circ g + \rho_\alpha c).
\]

**Lemma 3.** Let \( \alpha \) be any positive integer. Then, there exists a positive real number \( r'_{\alpha} \) such that \( j^kG_{\alpha,c}(N) \subset O' \) for any \( c \in B^p(0, r'_\alpha) \).

**Proof of Lemma 3.** Let \( \Gamma_\alpha: N \times \mathbb{R}^p \to J^k(N, P) \) be the mapping defined by \( \Gamma_\alpha(q,c) = j^kG_{\alpha,c}(q) \). For any \( q \in \text{supp } \rho_\alpha \), since \( \Gamma_\alpha(q,0) = j^kg(q) \in O' \) and \( \Gamma_\alpha \) is continuous at \((q,0)\), there exist an open neighborhood \( U_q \) of \( q \in N \) and an open neighborhood \( W_q \) of \( 0 \in \mathbb{R}^p \) such that \( \Gamma_\alpha(U_q \times W_q) \subset O' \). Since \( \{ U_q \}_{q \in \text{supp } \rho_\alpha} \) is an open covering of the compact set \( \text{supp } \rho_\alpha \), there exists a finite subset \( S \) of \( \text{supp } \rho_\alpha \) such that \( \text{supp } \rho_\alpha \subset \bigcup_{q \in S} U_q \). Since \( \bigcap_{q \in S} W_q \) is an open neighborhood of \( 0 \in \mathbb{R}^p \), there exists a positive real number \( r'_{\alpha} \) such that \( B^p(0, r'_\alpha) \subset \bigcap_{q \in S} W_q \).

Let \( c \in B^p(0, r'_\alpha) \) and \( q \in N \) be any points. If \( q \notin \text{supp } \rho_\alpha \), then we have \( j^kG_{\alpha,c}(q) \in O' \) since \( G_{\alpha,c} = g \) on the open neighborhood \( (\text{supp } \rho_\alpha)c \) of \( q \). If \( q \in \text{supp } \rho_\alpha \), then there exists a point \( q_0 \in S \) such that \( q \in U_{q_0} \). Since \( c \in W_{q_0} \), we obtain \( j^kG_{\alpha,c}(q) = \Gamma_\alpha(q,c) \in O' \). \( \square \)

Since \( g \in \mathcal{U} \), note that there exists a sequence \( \{ q'_\alpha \}_{\alpha \in \mathbb{N}} \) of points in \( N \) satisfying (1) and (2) of Lemma 2.

**Lemma 4.** Let \( m \) be any positive integer. Then, there exist \( (m+1) \) distinct positive integers \( \alpha_1, \ldots, \alpha_{m+1} \) and \( m \) positive real numbers \( r'_{\alpha_1}, \ldots, r'_{\alpha_m} \) \((r'_{\alpha_1} > \cdots > r'_{\alpha_m})\) such that for any \( j \in \{ 1, \ldots, m \} \),

1. \( j^kG_{\alpha_j,c}(N) \subset O' \) for any \( c \in B^p(0, r'_{\alpha_j}) \),
2. \( \left\| (\psi \circ g)(q'_{\alpha_j+1}) - (\psi \circ g)(q'_{\alpha_j}) \right\| < \frac{r'_{\alpha_j}}{p} \).

**Proof of Lemma 4.** We prove the lemma by induction on \( m \).

Let \( \alpha_1 \) be any positive integer. By Lemma 3 there exists a positive real number \( r'_{\alpha_1} \) such that \( j^kG_{\alpha_1,c}(N) \subset O' \) for any \( c \in B^p(0, r'_{\alpha_1}) \). By Lemma 2 (2), there exists \( \alpha_2 \in \mathbb{N} \setminus \{ \alpha_1 \} \) such that \( \left\| (\psi \circ g)(q'_{\alpha_2}) - (\psi \circ g)(q'_{\alpha_1}) \right\| < \frac{r'_{\alpha_1}}{p} \). Hence, the case \( m = 1 \) holds.

We assume that the lemma is true for \( m = i \), where \( i \) is a positive integer. By Lemma 3 there exists a positive real number \( r'_{\alpha_{i+1}} \) \((r'_{\alpha_i} > r'_{\alpha_{i+1}})\) such that \( j^kG_{\alpha_{i+1},c}(N) \subset O' \) for any \( c \in B^p(0, r'_{\alpha_{i+1}}) \). By Lemma 2 (2), there exists \( \alpha_{i+2} \in \mathbb{N} \setminus \{ \alpha_{i+1} \} \) such that \( \left\| (\psi \circ g)(q'_{\alpha_{i+2}}) - (\psi \circ g)(q'_{\alpha_{i+1}}) \right\| < \frac{r'_{\alpha_{i+1}}}{p} \). Hence, the case \( m = i + 1 \) holds.
For simplicity, set $I = \{1, \ldots, p\}$. By Lemma 4 in the case $m = p$, there exist $(p+1)$ distinct positive integers $\alpha_1, \ldots, \alpha_{p+1}$ and $p$ positive real numbers $r'_{\alpha_1}, \ldots, r'_{\alpha_p}$ $(r'_{\alpha_1} > \cdots > r'_{\alpha_p})$ such that for any $j \in I$,

(a) $j^k G_{\alpha_j, c}(N) \subset O'$ for any $c \in B^p(0, r'_{\alpha_j})$,

(b) $\|(\psi \circ g)(q'_{\alpha_{i+1}}) - (\psi \circ g)(q_{\alpha_i})\| < \frac{r'_{\alpha_j}}{p}$.

Let $h : N \rightarrow P$ be the mapping defined by

$$h = \psi^{-1} \circ \left( \psi \circ g + \sum_{i=1}^{p} \rho_{\alpha_i} c_i \right),$$

where $c_i = (\psi \circ g)(q'_{\alpha_{i+1}}) - (\psi \circ g)(q_{\alpha_i}) \in \mathbb{R}^p$.

First, we show that $j^k h(N) \subset O'$. Let $q \in N$ be an arbitrary point. In the case where $q$ is an element of $(\{\bigcup_{i=1}^{p} \text{supp } \rho_{\alpha_i}\})^c$, since $h = g$ on the open neighborhood $(\{\bigcup_{i=1}^{p} \text{supp } \rho_{\alpha_i}\})^c$ of $q$, we have $j^k h(q) = j^k g(q) \in O'$. We consider the case where there exists $j \in I$ such that $q \in \text{supp } \rho_{\alpha_j}$. Since $\text{supp } \rho_{\alpha_j} \subset \bigcap_{i \in I \setminus \{j\}} (\text{supp } \rho_{\alpha_i})^c$ and $h = G_{\alpha_j, c_j}$ on the open neighborhood $\bigcap_{i \in I \setminus \{j\}} (\text{supp } \rho_{\alpha_i})^c$ of $q$, we have $j^k h(q) = j^k G_{\alpha_j, c_j}(q)$. Moreover, since

$$\|c_j\| = \|(\psi \circ g)(q'_{\alpha_{j+1}}) - (\psi \circ g)(q_{\alpha_j})\|$$

$$\leq \sum_{i=j}^{p} \|(\psi \circ g)(q'_{\alpha_{i+1}}) - (\psi \circ g)(q_{\alpha_i})\|$$

$$< \sum_{i=j}^{p} \frac{r'_{\alpha_i}}{p}$$

$$\leq r'_{\alpha_j},$$

we have $c_j \in B^p(0, r'_{\alpha_j})$. Note that the last two inequality in (3.8) follow from (4) and the fact that $r'_{\alpha_1} > \cdots > r'_{\alpha_p}$, respectively. Thus, we obtain $j^k G_{\alpha_j, c_j}(q) \in O'$ by (3), which implies that $j^k h(q) \in O'$.

Finally, we show that there exist $(p+1)$-critical points of $h$ which share the same critical value. For any $i, j \in I$, since $\rho_{\alpha_i}(q'_{\alpha_{i+1}}) = 0$ and $\rho_{\alpha_i}(q'_{\alpha_{j+1}}) = 0$, we obtain

$$\left( \psi \circ g + \sum_{i=1}^{p} \rho_{\alpha_i} c_i \right)(q'_{\alpha_j}) = (\psi \circ g)(q'_{\alpha_j}) + c_j = (\psi \circ g)(q'_{\alpha_{j+1}}) = (\psi \circ h)(q'_{\alpha_{j+1}}),$$

where $\delta_{ij}$ is the Kronecker delta. Thus, we have $h(q'_{\alpha_i}) = \cdots = h(q'_{\alpha_{j+1}})$. Moreover, for any $j \in I$, the point $q'_{\alpha_j}$ (resp., $q'_{\alpha_{j+1}}$) is a critical point of $h$ since $h = \psi^{-1} \circ (\psi \circ g + c_j)$ on an open neighborhood of $q'_{\alpha_j}$ (resp., $h = g$ on an open neighborhood of $q'_{\alpha_{j+1}}$). Namely, $q'_{\alpha_1}, \ldots, q'_{\alpha_{p+1}}$ share the same critical value of $h$. \hfill \square

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