ON MODULAR COHOMOTOPY GROUPS

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ABSTRACT

Let \( p \) be a prime and let \( \pi^n(X; \mathbb{Z}/p^r) = [X, M_n(\mathbb{Z}/p^r)] \) be the set of homotopy classes of based maps from CW-complexes \( X \) into the mod \( p^r \) Moore spaces \( M_n(\mathbb{Z}/p^r) \) of degree \( n \), where \( \mathbb{Z}/p^r \) denotes the integers mod \( p^r \). In this paper we firstly determine the modular cohomotopy groups \( \pi^n(X; \mathbb{Z}/p^r) \) up to extensions by classical methods of primary cohomology operations and give conditions for the splitness of the extensions. Secondly we utilize some unstable homotopy theory of Moore spaces to study the modular cohomotopy groups; especially, the group \( \pi^3(X; \mathbb{Z}_{(2)}) \) with \( \text{dim}(X) \leq 6 \) is determined.

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1. Introduction

The cohomotopy groups $\pi^n(X) = [X, S^n]$ of homotopy classes of maps into the $n$-sphere with $\text{dim}(X) \leq 2n - 2$ were firstly introduced by Borsuk [8] and systematically studied by Spanier [29]. Hereafter, Peterson [26] defined the generalized (stable) cohomotopy groups of a CW-pair $(X, A)$ of dimension at most $2n - 2$ by

$$\pi^n(X, A; G) := [X, A; M_n(G), *], \quad \pi^n(X; G) := [X, M_n(G)],$$

where $M_n(G)$ denotes the Moore space with the (reduced) homology group $G$ precisely only in dimension $n$. Clearly

$$\pi^n(X, A; G) \cong \pi^n(X/A; G) \quad \text{and} \quad \pi^n(X; \mathbb{Z}) \cong \pi^n(X).$$

Peterson’s main related results are summarized in Theorem 2.1.

Due to the Pontrjagin–Thom construction (cf. [21, Chapter IX, Theorem 5.5]), the cohomotopy theory of closed smooth manifolds has been extensively studied and is still a hot research direction, for instance [13, 31, 17, 19, 18]. By Bauer and Furuta’s famous work [6], the stable cohomotopy groups of the complex projective space $\mathbb{C}P^{d-1}(d > 1)$ have close relationship with the cohomotopy Seiberg–Witten invariants for a Riemannian four-manifold with the first Betti number $b_1 = 0$. Cohomotopy theory also boosts the development of differential geometry and mathematical physics. In 2005, Auckly and Kapitanski [4] gave an analytic proof of Pontrjagin’s classification of maps from a three-manifold $M$ to $S^2$ [27] by the method of smooth flat connections on $M$. Restricted on those flat connections with finite energy, they further proved the existence of a minimizer of the energy functional $E_\varphi$ introduced by Faddeev [20] of a reference map $\varphi: M \to S^2$.

Let $\mathbb{Z}/p^r$ denote the group of mod $p^r$ integers for some prime $p$ and some positive integer $r$. In this paper we study the cohomotopy groups from the new perspective of the modular cohomotopy groups $\pi^*(X; \mathbb{Z}/p^r)$. In general, modular cohomotopy groups detect the $p$-torsion information of the integral cohomotopy groups, hence to a certain extent, modular cohomotopy groups are useful and advantageous to the computations of the (torsion of) integral ones, which as stated before have close connection with geometry and physics. On the other hand, the following example shows a direct bond between modular cohomotopy and geometric topology. The equivalence classes of oriented real $n$-dimensional vector bundles over a CW-complex $X$ are classified
by \([X, \Omega SO(n)]\), where \(\Omega SO(n)\) is the classifying space of the special orthogonal group \(SO(n)\). Since the Stiefel manifold \(V_{2n+1,2}\) of orthogonal 2-frames in \(\mathbb{R}^{2n+1}\) is a \((4n - 1)\)-dimensional CW-complex with the \((4n - 2)\)-skeleton \(M_{2n-1}(\mathbb{Z}/2)\) (cf. [14, p. 302]), there is an exact sequence

\[
\cdots \to [X, \Omega SO(2n + 1)] \to \pi^{2n-1}(X; \mathbb{Z}/2) \to [X, \Omega SO(2n - 1)]
\]

if \(\text{dim}(X) \leq 4n - 3\). The group \(\pi^{2n-1}(X; \mathbb{Z}/2)\) is closely linked with the classification of vector bundles and tells information that cannot be obtained by the classic tools of characteristic classes and \(K\)-theory, which are appropriate only when \(n\) passes to \(\infty\).

To make more sense of the introduction and formulate the main theorems, we need some global conventions and notations. Unless otherwise stated, complexes mean connected based CW-complexes and maps are base-point-preserving; we don’t notationally distinguish a map with its homotopy class. For an abelian group \(G\) and an integer \(n\), \(H^n(X; G)\) denotes the \(n\)-th reduced cohomology group with coefficients in \(G\), \(H^n(X) = H^n(X; \mathbb{Z})\); similar notations are used for homology. For a prime \(p\), let

\[\mathcal{P}^1 : H^*(-; \mathbb{Z}/p) \to H^{*+2p-2}(-; \mathbb{Z}/p)\]

be Steenrod’s reduced power operations and by convention, \(\mathcal{P}^1\) refers to the Steenrod square \(\text{Sq}^2\) if \(p = 2\). For \(G = \mathbb{Z}, \mathbb{Z}/p\) or \(\mathbb{Z}(p)\), define \(\mathcal{P}^1_G\) by the composition

(1) \[\mathcal{P}^1_G : K_n(G) \to K_n(\mathbb{Z}/p) \xrightarrow{p^1} K_{n+2p-2}(\mathbb{Z}/p),\]

where the first map corresponds to the reduction mod \(p\). If \(f : X \to Y\) is a map inducing a homomorphism

\[f_* : [W, X] \to [W, Y]\]

for a space \(W\), the kernel and cokernel of \(f_*\) are respectively denoted by

\[K_f(W) : = \ker(f_*) = \{\alpha : W \to X : f \circ \alpha = \ast\},\]
\[T_f(W) : = \coker(f_*) = [W, Y]/f_*[W, X].\]

For a complex \(X\) of dimension \(\leq 2n - 2\), the cohomotopy Hurewicz homomorphism \(h^n = h^n_G\) with coefficients in \(G\), or simply the generalized cohomotopy Hurewicz homomorphism is defined by the composition

\[h^n_G : \pi^n(X; G) = [X, M_n(G)] \xrightarrow{(i_{M})_*} [X, K_n(G)] \cong H^n(X; G),\]
where \( i_M : M_n(G) \to K_n(G) \) is the canonical inclusion map into the Eilenberg–MacLane space \( K_n(G) \) of type \((G,n)\) and the isomorphism is due to the Brown’s representability theorem (cf. [14, Theorem 4.57]). Note that \( h_n^G \) is natural with respect to maps \( X \to Y \) and group homomorphisms \( G \to H \). Let \( p \) be a prime and let \( \mathbb{Z}_{(p)} \) be the group of \( p \)-local integers. If \( G = \mathbb{Z}, \mathbb{Z}/p^r \) or \( \mathbb{Z}_{(p)} \), then

\[
H^n(M_n(G); G) \cong G\langle [i_M] \rangle
\]

and we have

\[
h_n^G(f) = f^*([i_M]),
\]

where the cohomology class \([i_M]\) is represented by the inclusion \( i_M \). We call \( h_n^\mathbb{Z} \), \( h_n^{\mathbb{Z}/p^r} \), \( h_n^{\mathbb{Z}_{(p)}} \) the \( n \)-th integral, mod \( p^r \), \( p \)-local cohomotopy Hurewicz homomorphism, respectively.

In [31, Section 6.1], Taylor gave a modern approach to Steenrod’s classification theorem [30]: For a complex \( X \) of dimension at most \( n+1 \), \( n \geq 3 \), there is a short exact sequence of abelian groups

\[
0 \to T_{\Omega Sq^2}(X) \to \pi^n(X) \to H^n(X) \to 0,
\]

where \( Sq^2 : H^n(X; \mathbb{Z}) \to H^{n+2}(X; \mathbb{Z}/2) \). Moreover, using the methods in [23], he showed that the above extension splits if and only if

\[
Sq^2(H^{n-1}(X; \mathbb{Z})) = Sq^2(H^{n-1}(X; \mathbb{Z}/2)).
\]

Our first main theorem is partially motivated by Taylor’s analysis.

**Theorem 1.1:** Let \( p \) be a prime and \( X \) be a complex of dimension \( \leq n + 2p - 3, n \geq 2p - 1 \). For \( G = \mathbb{Z}_{(p)} \) or \( \mathbb{Z}/p^r(r \geq 1) \), there is a short exact sequence of abelian groups:

\[
0 \to T_{\Omega P^1_G}(X) \to \pi^n(X; G) \xrightarrow{h^n} H^n(X; G) \to 0,
\]

where \( T_{\Omega P^1_G}(X) = H^{n+2p-3}(X; \mathbb{Z}/p)/P^1_G(H^{n-1}(X; G)) \); \( P^1_G = Sq^2_G \) if \( p = 2 \).

(1) If \( G = \mathbb{Z}_{(p)}, p \geq 2 \), then the extension (2) splits if and only if

\[
P^1_G(H^{n-1}(X; \mathbb{Z}_{(p)})) = P^1(H^{n-1}(X; \mathbb{Z}/p)).
\]

(2) If \( G = \mathbb{Z}/2^r \), then the extension (2) splits if one of the following conditions holds:
(a) \( r \geq 2 \), the homomorphism \([2]: H^n(X; \mathbb{Z}/2^{r-1}) \to H^n(X; \mathbb{Z}/2^r)\) induced by the coefficient homomorphism \(\mathbb{Z}/2^{r-1} \to \mathbb{Z}/2^r\) is injective and

\[
\text{Sq}^1(H^n(X; \mathbb{Z}/2)) \subseteq \text{Sq}^2_{\mathbb{Z}/2^r}(H^{n-1}(X; \mathbb{Z}/2^r)).
\]

(b) \( r = 1 \), \(\text{Sq}^1: H^n(X; \mathbb{Z}/2) \to H^{n+1}(X; \mathbb{Z}/2)\) is trivial.

(3) If \( G = \mathbb{Z}/p^r, p \geq 3 \), then the extension (2) splits if \( r = 1 \), or \( r \geq 2 \) and the homomorphism \([p]: H^n(X; \mathbb{Z}/p^{r-1}) \to H^n(X; \mathbb{Z}/p^r)\) induced by \(\mathbb{Z}/p^{r-1} \to \mathbb{Z}/p^r\) is injective.

Remark 1.2:

(1) Condition (2.a) can be strengthened as follows: If \( r \geq 2 \) and the homomorphism \([2]: \) is injective, then the extension (2) splits if and only if the inclusion (4) holds.

(2) If \( G = \mathbb{Z}/p, p \geq 3 \), one can also get the splitness of (2) by applying Theorem 2.1 (2).

When \( n \) is odd, observe that the \( n \)-connected cover \( S^n\langle n \rangle \) of \( S^n \) is a torsion space of finite type. Hence, for a finite suspension \( X \), the group \([X, S^{2n+1}\langle 2n+1 \rangle]\) is determined by its \( p \)-torsion components. Let \( W_{(p)} \) denote the \( p \)-localization of a simply-connected space \( W \). As can be seen from Section 4.1, there is an isomorphism

\[
[X, S^{2n+1}\langle 2n+1 \rangle_{(p)}] \cong \pi^{2n+2p-2}(X; \mathbb{Z}/p)
\]

if \( \text{dim}(X) \leq 2n + 4p - 4, p \geq 5, n \geq 1 \) or \( n = 1, p \geq 2 \). Note that \( p \)-localization loosens the restriction on \( \text{dim}(X) \), especially for greater primes \( p \). Another way is to consider the double suspension

\[ E^2: S^{2n-1} \to \Omega^2 S^{2n+1}, \quad n \geq 1. \]

Let \( C(n) \) be the homotopy fibre of \( E^2 \); as an example, \( C(1) = \Omega^3 S^3\langle 3 \rangle \) [9]. Selick [28] showed that \( C(n)_{(p)} \) is an H-space for each \( n \geq 1, p \geq 2 \) and Gray [11] constructed the classifying space \( BC(n)_{(p)} \) for such \( p, n \). If \( p \) is odd, \( C(n)_{(p)} \) has bottom two cells \( M_{2pn-3}(\mathbb{Z}/p) \) (see Lemma 4.3) and the bond with \( \pi^{2pn-3}(X; \mathbb{Z}/p) \) is set up.
Theorem 1.3: Let $n$ be a positive integer and let $p$ be a prime satisfying: (i) $n \geq 1, p \geq 5$; or (ii) $n = 1, p \geq 2$.

(1) For a complex $X$ of dimension $\leq 2n + 4p - 4$, there is an exact sequence of groups

\[ \pi^{2n+1}(\Sigma^2 X; \mathbb{Z}(p)) \xrightarrow{(\Omega^2 \iota_{(p)})_\sharp} H^{2n+1}(\Sigma^2 X; \mathbb{Z}(p)) \to [X, \Omega S^{2n+1}(2n+1)_{(p)}] \]

\[ \to \pi^{2n+1}(\Sigma X; \mathbb{Z}(p)) \xrightarrow{(\Omega \iota_{(p)})_\sharp} H^{2n+1}(\Sigma X; \mathbb{Z}(p)) \to \pi^{2n+2p-2}(X; \mathbb{Z}/p) \]

\[ \to \pi^{2n+1}(X; \mathbb{Z}(p)) \xrightarrow{(\iota_{(p)})_\sharp} H^{2n+1}(X; \mathbb{Z}(p)), \]

where $\iota_{(p)}: S^{2n+1}_{(p)} \to K_{2n+1}(\mathbb{Z}(p))$ is the canonical map representing a generator of $H^{2n+1}(S^{2n+1}, \mathbb{Z}(p))$. In particular, if $p \geq 3$,

(a) there is a natural isomorphism:

\[ \pi^{2n+2p-2}(X; \mathbb{Z}/p) \cong T_{\Omega \iota_{(p)}}(X) \oplus K_{\iota_{(p)}}(X); \]

(b) there is a natural isomorphism if $\dim(X) \leq 2n + 4p - 5$:

\[ \pi^{2n+2p-3}(X; \mathbb{Z}/p) \cong [X, \Omega S^{2n+1}(2n+1)_{(p)}] \cong T_{\Omega^2 \iota_{(p)}}(X) \oplus K_{\iota_{(p)}}(X). \]

(2) For a complex $X$ of dimension $\leq 4p - 3$, $p \geq 2$, the exact sequence in (1) extends from the right by

\[ \pi^3(X; \mathbb{Z}(p)) \xrightarrow{(\iota_{(p)})_\sharp} H^3(X; \mathbb{Z}(p)) \xrightarrow{\mathcal{P}^1_{\mathbb{Z}(2)}} H^{2p+1}(X; \mathbb{Z}/p), \]

where $\mathcal{P}^1_{\mathbb{Z}(2)} = \text{Sq}^2_{\mathbb{Z}(2)}$.

Note that Theorem 1.3 gives a connection between $\pi^{2n+2p-2}(X; \mathbb{Z}/p)$ and $\pi^{2n+1}(X; \mathbb{Z}(p))$. If $\pi^{2n+2p-2}(X; \mathbb{Z}/p)$ is known, we can apply Theorem 1.3 to determine $\pi^{2n+1}(X; \mathbb{Z}(p))$ (up to extension).

Corollary 1.4: Let $X$ be a complex of dimension $\leq 6$.

(1) There is a short exact sequence of groups:

\[ 0 \to T_{\partial}(X) \to \pi^3(X; \mathbb{Z}(2)) \xrightarrow{h^3} H^3(X; \mathbb{Z}(2)) \to \pi^5(X; \mathbb{Z}/2), \]

where the dashed arrow here means the homomorphism is merely of sets and $T_{\partial}(X)$ is the cokernel of the homomorphism

\[ \partial_{\sharp}: H^2(X; \mathbb{Z}(2)) \to [X, S^3(3)_{(2)}] \cong \pi^4(X; \mathbb{Z}/2) \]

induced by the connecting map $\partial: K_2(\mathbb{Z}(2)) \to S^3(3)_{(2)}$. 

(2) If \( \dim(X) \leq 5 \), then
\[
h^3(\pi^3(X; \mathbb{Z}(2))) = \ker(H^3(X; \mathbb{Z}(2)) \xrightarrow{\text{Sq}^2_{\mathbb{Z}(2)}} H^5(X; \mathbb{Z}/2))
\]
and there is a central group extension
\[
0 \to T_\partial(X) \xrightarrow{\partial} \pi^3(X; \mathbb{Z}(2)) \xrightarrow{h^3} \ker(\text{Sq}^2_{\mathbb{Z}(2)}) \to 0,
\]
which is determined by the following two functions \( \Gamma, \Phi_2 \):
(i) \( \Gamma : \ker(\text{Sq}^2_{\mathbb{Z}(2)}) \times \ker(\text{Sq}^2_{\mathbb{Z}(2)}) \to T_\partial(X) \): if \( x, y \in \ker(\text{Sq}^2_{\mathbb{Z}(2)}) \), let \( g, h \in \pi^3(X; \mathbb{Z}(2)) \) such that \( h^3(g) = x, h^3(h) = y \), then
\[
\Gamma(x, y) = j^{-1}(ghg^{-1}h^{-1}).
\]
(ii) \( \Phi_2 : \ker(\text{Sq}^2_{\mathbb{Z}(2)}) \to T_\partial(X)/2T_\partial(X) \): If \( \bar{\alpha} \in \pi^3(X; \mathbb{Z}(2)) \) satisfies
\[
h^3(\bar{\alpha}) = \alpha = \delta(\alpha') \text{ for some } \alpha' \in H^2(X; \mathbb{Z}/2),
\]
where \( \delta \) is the obvious Bockstein, then
\[
\Phi_2(\alpha) = \bar{\alpha}^2 = [\partial_2 \alpha'] \mod 2T_\partial(X),
\]
where \( \partial_2 \) is given by the decomposition
\[
\partial_1 : H^2(X; \mathbb{Z}(2)) \xrightarrow{\mod 2} H^2(X; \mathbb{Z}/2) \xrightarrow{\partial_2} \pi^4(X; \mathbb{Z}/2).
\]

**Theorem 1.5:** Let \( p \geq 5 \) be a prime and let \( n \geq 1 \). For any complexes \( X \) of dimension \( \leq 2pn + 2n - 5 \), there is an exact sequence of abelian groups
\[
\pi^{2n-1}(\Sigma X; \mathbb{Z}(p)) \xrightarrow{(\Omega E^2_{(p)})_1} \pi^{2n+1}(\Sigma^3 X; \mathbb{Z}(p)) \to [X, C(n)_{(p)}] \\
\to \pi^{2n-1}(X; \mathbb{Z}(p)) \xrightarrow{(E^2_{(p)})_2} \pi^{2n+1}(\Sigma^2 X; \mathbb{Z}(p)) \to \pi^{2pn-2}(X; \mathbb{Z}/p).
\]

In particular, if \( \dim(X) \leq 2pn + 2n - 6 \), there is a natural isomorphism:
\[
\pi^{2pn-3}(X; \mathbb{Z}/p) \cong [X, C(n)_{(p)}] \cong T_{\Omega E^2_{(p)}}(X) \oplus K_{E^2_{(p)}}(X).
\]

Our final result generalizes Peterson’s exact sequence theorem (see Theorem 2.1(3)) for cohomotopy with \( \mathbb{Z}/p^r \)-coefficients.

**Theorem 1.6:** Let \( X \) be a finite complex of dimension \( \leq 4n - 3, n \geq 2 \). Then for each prime \( p \geq 5 \), there is an exact sequence of abelian groups
\[
0 \to \pi^{2n-1}(X; \mathbb{Z}(p)) \otimes \mathbb{Z}/p^r \to \pi^{2n-1}(X; \mathbb{Z}/p^r) \to \pi^{2n+1}(\Sigma X; \mathbb{Z}(p)).
\]

If \( \dim(X) \leq 4n - 4 \), then \( \pi^{2n-1}(X; \mathbb{Z}(p)) \otimes \mathbb{Z}/p^r \) is a direct summand of \( \pi^{2n-1}(X; \mathbb{Z}/p^r) \) and isomorphic to \( \pi^{2n-1}(X) \otimes \mathbb{Z}/p^r \).
Note that if $\dim(X) = 4n - 3$, the group structure of $\pi^{2n-1}(X; \mathbb{Z}/p^r)$ is not obtained by Freudenthal’s suspension isomorphism theorem; the details refer to Section 4.3.

The paper is organized as follows. Section 2 covers Peterson’s main results on the generalized cohomotopy groups and some auxiliary lemmas. In Section 3 we give the proof of Theorem 1.1 and compute partial cohomotopy groups of the projective spaces $\mathbb{F}P^n$ ($\mathbb{F} = \mathbb{C}, \mathbb{H}$) and $M_n(\mathbb{Z}/2^s)$ as examples. In Section 4 we combine the classic unstable homotopy theory of Moore spaces to prove the remaining results.

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2. Preliminaries

In this section we firstly summarize Peterson’s primary work on the generalized cohomotopy groups and then list some lemmas that will be used in the later sections.

2.1. Peterson’s generalized cohomotopy groups.

THEOREM 2.1 (Peterson, [26]): Let $(X, A)$ be a CW pair of dimension $N \leq 2n - 2$, $n \geq 2$. Let $G$ be an abelian group.

(1) The generalized cohomotopy groups $\pi^n(X, A; G)$ satisfy all the Eilenberg—Steenrod axioms for cohomology theory [10].

(2) If $G$ is a field without 2-torsion, then $\pi^n(X, A; G)$ is a vector space over $G$. 
(3) If $G$ is finitely generated or $(X, A)$ is finite, then there is a universal coefficients exact sequence for each $r > (N + 1)/2$:

\[ 0 \to \pi^r(X, A) \otimes G \to \pi^r(X, A; G) \to \text{Tor}(\pi^{r+1}(X, A), G) \to 0. \]

This sequence is natural with respect to maps $f: (X, A) \to (Y, B)$ and if $G$ has no 2-torsion, then it is natural with respect to homomorphisms $\varphi: G \to H$. Furthermore, the extension (7) splits if $\pi^r(X, A)$ is finitely generated and $G$ has no 2-torsion.

2.2. SOME LEMMAS. Let $n \geq 0$. Recall that a space $X$ is $n$-connected if $\pi_i(X) = 0$ for all $i \leq n$; a map $f: X \to Y$ is $n$-connected if the homotopy fibre of $f$ is $(n - 1)$-connected.

The following lemma is well-known as the universal property of the James space $J(X) \simeq \Omega\Sigma X$ [16]:

**Lemma 2.2:** Let $f: X \to Y$ be a map of spaces where $Y$ is a homotopy associative H-space. Then $f$ extends to an H-map $\bar{f}: \Omega\Sigma X \to Y$ which is unique up to homotopy.

**Lemma 2.3:** Let $f: X \to Y$ be an $n$-connected map, where $X$ is $m$-connected and $Y$ is a homotopy associative H-space, $m, n \geq 1$. Then for any space $W$ of dimension $\leq \min\{n - 1, 2m\}$, the induced map

$$ f^\#: [W, X] \to [W, Y] $$

is an isomorphism, where the group structure of $[W, X]$ is induced by the loop suspension $E: X \to \Omega\Sigma X$ and that of $[W, Y]$ is induced by the H-space structure of $Y$.

**Proof.** By [3, Proposition 2.4.6], the induced map $f^\#: [W, X] \to [W, Y]$ is a bijection. Since $X$ is $m$-connected, the loop suspension $E = E_X: X \to \Omega\Sigma X$ is $(2m + 1)$-connected and hence induces a bijection

$$ E^\#: [W, X] \xrightarrow{1:1} [W, \Omega\Sigma X], $$

which provides $[W, X]$ a group structure. By Lemma 2.2, the map $f$ extends to an H-map $\bar{f}: \Omega\Sigma X \to Y$ and hence the group homomorphism

$$ \bar{f}^\#: [W, \Omega\Sigma X] \to [W, Y]. $$

Since $\bar{f}^\#$ is bijective, it is an isomorphism of groups. ■
Lemma 2.4 (Serre, cf. [3, Theorem 6.4.4]): Let $F \xrightarrow{i} E \xrightarrow{\pi} B$ be a (homotopy) fibration, in which $F$ is $r$-connected, $B$ is $s$-connected, $r \geq 1$, $s \geq 0$. Then for any coefficient group $G$, there is an exact sequence of cohomology groups:

$$H^0(B;G) \xrightarrow{\pi^*} H^0(E;G) \xrightarrow{i^*} H^0(F;G) \xrightarrow{\delta^0} H^1(B;G) \rightarrow \cdots$$

$$\rightarrow H^{N-1}(F) \xrightarrow{\delta^{N-1}} H^N(B;G) \xrightarrow{\pi^*} H^N(E;G) \xrightarrow{i^*} H^N(F;G),$$

where $N = r + s + 1$.

Lemma 2.5 (cf. [24, Theorem 10.3]): Let $p$ be a prime, $1 \leq r \leq \infty$ and by convention, let $\mathbb{Z}/p^{\infty} = \mathbb{Z}$. Let

$$\iota_n \in H^n(K_n(\mathbb{Z}/p^r);\mathbb{Z}/p)$$

be the fundamental class. $H^*(K_n(\mathbb{Z}/p^r);\mathbb{Z}/p)$ is a free, graded-commutative algebra, whose generators $P^I_\iota \iota_n$ with associated excess $e = e(I) \leq 5$ are listed as follows:

| $e$ | $P^I_\iota$ |
|-----|-------------|
| 0   | id          |
| 1   | $\beta_r, P^1_\iota \beta_r, \ldots$ |
| 2   | $P^1_\iota, P^1_\iota \beta_r, \ldots$ |
| 3   | $\beta_1 P^1_\iota, P^2_\iota \beta_r, \ldots$ |
| 4   | $P^2_\iota, \ldots$ |
| 5   | $\beta_1 P^2_\iota, \ldots$ |

where $\beta_r$ is the $r$-th Bockstein associated to the short exact sequence

$$0 \rightarrow \mathbb{Z}/p^r \xrightarrow{p^r} \mathbb{Z}/p^{r+1} \rightarrow \mathbb{Z}/p^r \rightarrow 0$$

for $r < \infty$ and $\beta_\infty = 0$. Note if $p = 2$,

$$\beta_1 \text{Sq}^i = \text{Sq}^1 \text{Sq}^i = (i - 1)\text{Sq}^{i+1}.$$  

Lemma 2.6: Let $p$ be an odd prime, $n \geq 1$. Let

$$S^{2n+2p-2} \xrightarrow{i_p} M_{2n+2p-2}(\mathbb{Z}/p) \xrightarrow{j_p} S^{2n+2p-1}$$

be the cofibration with $i_p = i_p(2n + 2p - 2), j_p = j_p(2n + 2p - 2)$ the canonical inclusion and pinch maps, respectively. The nontrivial $p$-primary components of $\pi_{2n+i}(X)$ for $X = S^{2n+1}, M_{2n+2p-2}(\mathbb{Z}/p)$ in the range $i < 4p - 3$ are listed as follows:
where \( \alpha_k(m) = \Sigma^{m-3}\alpha_k(3) \), \( k = 1, 2 \) and

\[
\alpha_2(2n + 1) \in \langle \alpha_1(2n + 1), p^i2n+2p-2, \alpha_1(2n + 2p - 2) \rangle,
\]

the Toda bracket (or called the secondary composition in [35]); \( \tilde{\alpha}_1 \) satisfies the relation

\[
(8) \quad j_p \tilde{\alpha}_1 = \alpha_1(2n + 2p - 1).
\]

**Proof.** For each \( k \), denote by \( \pi_k(X; p) \) the \( p \)-primary component of \( \pi_k(X) \). The homotopy groups \( \pi_{2n+i}(S^{2n+1}; p) \) refer to [35, Chapter XIII]. Note that for any \( n \geq 1 \), the range \( i \leq 4p - 4 \) is the stable range for the group

\[
\pi_{2n+i}(M_{2n+2p-2}(\mathbb{Z}/p)),
\]

which consequently can be computed by the following short exact sequence for \( i \leq 4p - 4 \):

\[
\pi_{2n+i}(S^{2n+2p-2}; p) \xrightarrow{(i_p)^*} \pi_{2n+i}(M_{2n+2p-2}(\mathbb{Z}/p)) \xrightarrow{(j_p)^*} \pi_{2n+i}(S^{2n+2p-1}; p),
\]

where \( i_p = i_p(2n + 2p - 2), j_p = j_p(2n + 2p - 2) \) are the canonical inclusion and projection maps, respectively.

### 3. Cohomology operations and modular cohomotopy

In this section let \( p \) be a prime and let \( r \) be a positive integer; let \( G = \mathbb{Z}/p^r \) or \( \mathbb{Z}_{(p)} \) and let \( F_n(G) \) be defined by the \( p \)-localized *H-fibration sequence* (homotopy fibration sequence of H-spaces and H-maps):

\[
(9) \quad F_n(G) \to K_n(G) \xrightarrow{\mathcal{P}^1_G} K_{n+2p-2}(\mathbb{Z}/p),
\]

where \( \mathcal{P}^1_G \) is the cohomology operation map defined by (1).
3.1. A CERTAIN GROUP EXTENSION. We shall give a unified expression of a certain extension of abelian groups induced by (9).

**Lemma 3.1:** If $n \geq 2p - 1$, then there exists a canonical map

$$\alpha_G: M_n(G) \to F_n(G),$$

which is $(n + 2p - 2)$-connected.

**Proof.** Consider the defined homotopy fibration (9). There exists a canonical map $\alpha_G: M_n(G) \to F_n(G)$ which is at least $n$-connected. For simplicity, we write

$$H^{n+i}(\mathbb{Z}/p^r, n; \mathbb{Z}/p) = H^{n+i}(K_n(\mathbb{Z}/p^r); \mathbb{Z}/p).$$

(1) $G = \mathbb{Z}/p^r$. If $n \geq 2p - 1$, by Lemma 2.5, the nontrivial cohomology groups $H^{n+i}(K_n(\mathbb{Z}/p^r); \mathbb{Z}/p)$ for $i \leq 2p - 1$ are listed as follows:

| $i$ | 0 | 1 | $2p - 2$ | $2p - 1$ |
|-----|---|---|----------|----------|
| $H^{n+i}(\mathbb{Z}/p^r, n; \mathbb{Z}/p)$ | $\iota_n$ | $\beta_n\iota_n$ | $\mathcal{P}^1\iota_n$ | $\mathcal{P}^1\beta_n\iota_n, \beta_1\mathcal{P}^1\iota_n$ |

where each nonzero generator $x$ represents a direct summand $\mathbb{Z}/p$. Note that when $n = 2p - 1$, $\iota_n^2 = 0$ if $p \geq 3$; otherwise $\beta_n\iota_n = \text{Sq}^3\iota_3 = \iota_3^2$. By Lemma 2.4, the homotopy fibration (9) induces exact sequences of cohomology groups with coefficients in $\mathbb{Z}/p$:

$$0 \to H^{n+i}(\mathbb{Z}/p^r, n) \to H^{n+i}(F_n(\mathbb{Z}/p^r)) \to 0, \quad i = 0, 1, \ldots, 2p - 3;$$

$$0 \to H^{n+2p-2}(\mathbb{Z}/p, n+2p-2) \xrightarrow{(\mathcal{P}^1)^*} H^{n+2p-2}(\mathbb{Z}/p^r, n) \to H^{n+2p-2}(F_n(\mathbb{Z}/p^r))$$

$$\to H^{n+2p-1}(\mathbb{Z}/p, n+2p-2) \xrightarrow{(\mathcal{P}^1)^*} H^{n+2p-1}(\mathbb{Z}/p^r, n) \to H^{n+2p-1}(F_n(\mathbb{Z}/p^r)).$$

Hence $H^{n+i}(F_n(\mathbb{Z}/p^r); \mathbb{Z}/p)$ is isomorphic to $\mathbb{Z}/p$ for $i = 0, 1$ and 0 for otherwise $i \leq n + 2p - 3$. Since $\iota_{n+2p-2} \circ \mathcal{P}^1 = \mathcal{P}^1\iota_n$, both representing $\mathcal{P}^1_{\mathbb{Z}/p^r}$, the first $(\mathcal{P}^1)^*$ is an isomorphism. The second $(\mathcal{P}^1)^*$ is a monomorphism, whose image is $\mathbb{Z}/p(\beta_r, \mathcal{P}^1\iota_n)$ if $r < \infty$ and otherwise 0. Thus we get

$$H^{n+2p-2}(F_n(\mathbb{Z}/p^r); \mathbb{Z}/p) = 0, \quad H^{n+2p-1}(F_n(\mathbb{Z}/p^r); \mathbb{Z}/p) \neq 0.$$

By the universal coefficient theorem we see that the second nontrivial integral reduced homology group of $F_n(\mathbb{Z}/p^r)$ occurs in dimension $n + 2p - 1$. Thus the map $\alpha_{\mathbb{Z}/p^r}$ is $(n + 2p - 2)$-connected, by the Whitehead’s second theorem (cf. [3, Theorem 6.4.15]). The proof in the case $G = \mathbb{Z}/p^r$ is done.
(2) It is clear that $H^*(K_n(\mathbb{Z}(p)); \mathbb{Z}/p) \cong H^*(K_n(\mathbb{Z}); \mathbb{Z}/p)$; the proof of the case $G = \mathbb{Z}(p)$ is totally similar and omitted here.

PROPOSITION 3.2: Let $X$ be a complex with $\dim(X) \leq n + 2p - 3$, $n \geq 2p - 1$. Then there is a short exact sequence of abelian groups

$$0 \rightarrow T_{\Omega P^1_G}(X) \rightarrow \pi^n(X; G) \xrightarrow{h^n} H^n(X; G) \rightarrow 0,$$

where

$$T_{\Omega P^1_G}(X) = H^{n+2p-3}(X; \mathbb{Z}/p)/P^1_G(H^{n-1}(X; G)).$$

The extension is natural with respect to maps $X \rightarrow Y$ and the reduction homomorphisms $\rho^s_r: \mathbb{Z}/p^r \rightarrow \mathbb{Z}/p^s$ for $1 \leq r \leq s < \infty$; $\rho^s_r = \text{id}$.

Proof. Consider the fibration (9). By Lemma 3.1 and [3, Proposition 2.4.6], the canonical map $\alpha_G: M_n(G) \rightarrow F_n(G)$ induces a bijection

$$[X, M_n(G)] \xrightarrow{\cong} [X, F_n(G)]$$

if $\dim(X) \leq n + 2p - 3$. Note that $F_n(G) = \Omega F_{n+1}(G)$ is a loop space and hence the bijection above is an isomorphism of abelian groups, by Lemma 2.3. The proposition then follows by the induced exact sequence:

$$H^{n-1}(X; G) \xrightarrow{P^1_G} H^{n+2p-3}(X; \mathbb{Z}/p) \rightarrow \pi^n(X; G) \xrightarrow{h^n} H^n(X; G) \rightarrow 0.$$  

The naturality of the extension is clear.

3.2. CONDITIONS FOR THE SPLITNESS OF THE EXTENSION. By [15, Lemma on p. 63], the extension (10) is determined by the multiplication by $p$ on $H^n(X; G)$.

PROPOSITION 3.3: If $G = \mathbb{Z}(p)$, $p \geq 2$, then the extension (10) splits if and only if

$$P^1_G(H^{n-1}(X; G)) = P^1(H^{n-1}(X; \mathbb{Z}/p)) \subseteq H^{n+2p-3}(X; \mathbb{Z}/p).$$

Proof. The proof is totally similar to that of [31, Theorem 6.2] and we omit the details here.

For a prime $p$, let $\pi^n(X; p)$ denote the $p$-primary component of the cohomotopy group $\pi^n(X)$ and by convention, $\pi^n(X; \infty) = \text{the integral direct summand of } \pi^n(X)$.  

Example 3.4: Let $F = \mathbb{C}, \mathbb{H}$ be the fields of the complex numbers and quaternions, respectively. Let $p \geq 2$ be a prime and let $\alpha \in H^d(FP^n; \mathbb{Z}/p)$ be a generator, where $d = \dim_{\mathbb{R}} F$. Recall that

(a) $H^*(FP^n; \mathbb{Z}/p) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$,

(b) $\mathcal{P}^1(\alpha^m) = \frac{dm}{2} \cdot \alpha^{m+\frac{2m-2}{d}}$ for each $m \geq 1$ (cf. [14, 4L]).

By Proposition 3.3 we compute that:

(1) For $p \leq \frac{dn}{4} + 1$,

$$\pi^{dn-2p+3}(FP^n; p) = \pi^{dn-2p+3}(FP^n; \mathbb{Z}/p) = \begin{cases} \mathbb{Z}/p & \text{if } n \equiv \frac{2p-2}{d} \mod p; \\ 0 & \text{otherwise.} \end{cases}$$

(2) For $p \leq \frac{dn+k}{4} + 1, k \geq 1$,

$$\pi^{dn-2p+k+3}(FP^n; \mathbb{Z}/p) \cong H^{dn-2p+k+3}(FP^n; \mathbb{Z}/p);$$

it follows that

$$\pi^{dn-2p+k+3}(FP^n; \infty) = \begin{cases} \mathbb{Z} & \text{if } k \leq 2p - 3 \text{ is odd;} \\ 0 & \text{if } k \leq 2p - 3 \text{ is even.} \end{cases}$$

Taking $p = 3$ and comparing with [36, Theorem 14.2], we have

**Corollary 3.5:** $\pi^{4n-3}(\mathbb{H}P^n) \cong \mathbb{Z}/12$ or $\mathbb{Z}/24$ when $n \equiv 1 \mod 6$.

Now let us focus on the case $G = \mathbb{Z}/p^r$ for some prime $p$. If $r \geq 2$, let

$$K_n(\mathbb{Z}/p^r) \xrightarrow{\rho} K_n(\mathbb{Z}/p^{r-1}) \xrightarrow{[p]} K_n(\mathbb{Z}/p^r)$$

be the maps corresponding to the reduction $\mathbb{Z}/p^r \to \mathbb{Z}/p^{r-1}$ mod $p^{r-1}$ and the multiplication $\cdot p: \mathbb{Z}/p^{r-1} \to \mathbb{Z}/p^r$, respectively. Let $p_K$ be the $p$-th power map on $K_n(\mathbb{Z}/p^r)$, then there holds a homotopy commutative square

$$
\begin{array}{ccc}
K_n(\mathbb{Z}/p^r) & \xrightarrow{\rho} & K_n(\mathbb{Z}/p^{r-1}) \\
\downarrow{[p]} & & \downarrow{[p]} \\
K_n(\mathbb{Z}/p^{r+1}) & \xrightarrow{\rho} & K_n(\mathbb{Z}/p^r)
\end{array}
$$

(11)

The following two propositions respectively give the conditions for the extension (10) to split with $G = \mathbb{Z}/p^r$, $p \geq 2$ and $r \geq 1$. 
Proposition 3.6: Suppose the assumptions in Proposition 3.2 hold and $G = \mathbb{Z}/2^r$.

(1) If $r \geq 2$, the extension (10) splits if
   (a) the homomorphism $[2]: H^n(X; \mathbb{Z}/2^{r-1}) \to H^n(X; \mathbb{Z}/2^r)$ is injective and
   (b) $\text{Sq}^1(H^n(X; \mathbb{Z}/2)) \subseteq \text{Sq}^2_{\mathbb{Z}/2^r}(H^{n-1}(X; \mathbb{Z}/2^r))$.
Moreover, if the condition (a) is true, then the extension (10) splits if and only if (b) holds.

(2) If $r = 1$, the extension (10) splits if $\text{Sq}^1: H^n(X; \mathbb{Z}/2) \to H^{n+1}(X; \mathbb{Z}/2)$ is trivial.

Proof. (1) Consider the following homotopy fibration diagram

\[
\begin{array}{ccccccccc}
K_n(\mathbb{Z}/2) & \overset{[2^{r-1}]}{\longrightarrow} & K_n(\mathbb{Z}/2^r) & \overset{\rho}{\longrightarrow} & K_n(\mathbb{Z}/2^{r-1}) & \overset{\delta}{\longrightarrow} & K_{n+1}(\mathbb{Z}/2) \\
\downarrow \text{Sq}^1 & & \downarrow \psi & & \downarrow 2K & & \downarrow 0 & & \downarrow \text{Sq}^1 \\
K_{n+1}(\mathbb{Z}/2) & \overset{\partial}{\longrightarrow} & F_n(\mathbb{Z}/2^r) & \overset{1}{\longrightarrow} & K_n(\mathbb{Z}/2^r) & \overset{\text{Sq}^2_{\mathbb{Z}/2^r}}{\longrightarrow} & K_{n+2}(\mathbb{Z}/2)
\end{array}
\]

where $\delta = \text{Sq}^1_{\mathbb{Z}/2^{r-1}}$. We claim that after choosing suitably, the map $\psi$ satisfies the relations

$$\iota \psi = 2K, \quad \psi \iota = 2F,$$

where $2F$ is the 2-nd power map on $F_n(\mathbb{Z}/2^r)$. As the dashed arrow indicated, the first equality trivially holds by the commutativity of the middle square. The map $\iota: F_n(\mathbb{Z}/2^r) \to K_n(\mathbb{Z}/2^r)$ induces a commutative exact sequence diagram

\[
\begin{array}{ccccccccc}
H^{n+1}(K_n(\mathbb{Z}/2^r); \mathbb{Z}/2) & \overset{\partial_1}{\longrightarrow} & [K_n(\mathbb{Z}/2^r), F_n(\mathbb{Z}/2^r)] & \overset{\iota_1}{\longrightarrow} & [K_n(\mathbb{Z}/2^r), K_n(\mathbb{Z}/2^r)] & \overset{\partial_2}{\longrightarrow} & H^{n+1}(F_n(\mathbb{Z}/2^r); \mathbb{Z}/2) \\
\downarrow \iota^* & & \downarrow \iota & & \downarrow \iota & & \downarrow \iota^*
\end{array}
\]

By Lemma 2.4, the first vertical $\iota^*$ is an isomorphism. The upper exact sequence implies that up to homotopy there are exactly two maps $\psi$ satisfying $\iota \psi = 2K$:

$$\psi_1 = \psi \quad \text{and} \quad \psi_2 = \psi + \partial \varepsilon,$$

where $\varepsilon \in H^{n+1}(K_n(\mathbb{Z}/2^r); \mathbb{Z}/2) \cong \mathbb{Z}/2$ is a generator. The second commutative square and equation then implies that

$$\psi_1 \iota = 2F \quad \text{or} \quad \psi_1 \iota = 2F + \partial \varepsilon',$$

where $\varepsilon' = \iota^*(\varepsilon) \in H^{n+1}(F_n(\mathbb{Z}/p^r); \mathbb{Z}/2) \cong \mathbb{Z}/2$ is a generator.
If \( \psi_1 \iota = 2_F \), the claim is proved; if not, replacing \( \psi_1 \) by \( \psi_2 = \psi + \partial \varepsilon \), then
\[
\psi_2 \iota = 2_F + \partial (\varepsilon' + \varepsilon) = 2_F + 2 \partial \varepsilon' = 2_F.
\]
The proof of the claim is completed.

There is a commutative exact sequence diagram induced by (12):

\[
\begin{array}{ccccccc}
H^{n-1}(X; \mathbb{Z}/2^{r-1}) & \xrightarrow{\delta} & H^n(X; \mathbb{Z}/2) & \xrightarrow{[2^{r-1}]} & H^n(X; \mathbb{Z}/2^r) & \xrightarrow{\rho} & H^n(X; \mathbb{Z}/2^{r-1}) \\
\downarrow[2] & & \downarrow \mathrm{Sq}^1 & & \downarrow \psi \uparrow & & 2 \downarrow \mathrm{Sq}^1 \\
H^n(X; \mathbb{Z}/2^r) & \xrightarrow{\mathrm{Sq}^2_{2^r}} & H^{n+1}(X; \mathbb{Z}/2) & \xrightarrow{\pi^n(X; \mathbb{Z}/2^r)} & H^n(X; \mathbb{Z}/2^{r-1}) \quad \text{and}
\end{array}
\]

Given \( \gamma \in 2H^n(X; \mathbb{Z}/2^r) \), \( 0 = 2\gamma = [2] \rho(\gamma) \) is equivalent to \( \rho(\gamma) = 0 \), by the condition (a). There exists \( \gamma' \in H^n(X; \mathbb{Z}/2) \) satisfying \([2^{r-1}](\gamma') = \gamma \) and hence for any \( \bar{\gamma} \in \pi^n(X; \mathbb{Z}/2^r) \) mapping to \( \gamma \) by \( h^n \), by the relation \( \psi \iota = 2_F \) and the middle commutative square, we obtain
\[
2\bar{\gamma} = \psi(\gamma) = \mathrm{Sq}^1(\gamma') \in \ker(\mathrm{Sq}^2_{2^r}) = T_{\Omega \mathrm{Sq}^2_{2^r}}(X).
\]
Thus by [15, Lemma on p. 63], the extension (10) splits if, in addition,
\[
\mathrm{Sq}^1(H^n(X; \mathbb{Z}/2)) \subseteq \mathrm{Sq}^2_{2^r}(H^{n-1}(X; \mathbb{Z}/2^r)).
\]

(2) Consider the following homotopy fibration diagram

\[
\begin{array}{ccccccc}
K_n(\mathbb{Z}/4) & \xrightarrow{\rho} & K_n(\mathbb{Z}/2) & \xrightarrow{\mathrm{Sq}^1} & K_{n+1}(\mathbb{Z}/2) & \xrightarrow{[2]} & K_{n+1}(\mathbb{Z}/4) \\
\downarrow \mathrm{Sq}^2_{2^r} & & \downarrow \psi' & & \downarrow 0 & & \downarrow \mathrm{Sq}^2_{2^r} \\
K_{n+1}(\mathbb{Z}/2) & \xrightarrow{\partial} & F_n(\mathbb{Z}/2) & \xrightarrow{\iota} & K_n(\mathbb{Z}/2) & \xrightarrow{\mathrm{Sq}^2} & K_{n+2}(\mathbb{Z}/2)
\end{array}
\]

We argue in a similar way to that of the proof of the claim above that the map \( \psi' : K_n(\mathbb{Z}/2) \to F_n(\mathbb{Z}/2) \) satisfies the relations:
\[
\iota \psi' = 2K = 0, \quad \psi' \iota = 2F.
\]

Consider the following induced commutative diagram:

\[
\begin{array}{ccccccc}
H^n(X; \mathbb{Z}/2) & \xrightarrow{[2]} & H^n(X; \mathbb{Z}/4) & \xrightarrow{\rho} & H^n(X; \mathbb{Z}/2) & \xrightarrow{\mathrm{Sq}^1} & H^{n+1}(X; \mathbb{Z}/2) & \xrightarrow{2} & H^n(X; \mathbb{Z}/2) \\
\downarrow 0 & & \downarrow \mathrm{Sq}^2_{2^r} & & \downarrow \psi' & & \downarrow 0 \\
H^{n+1}(X; \mathbb{Z}/2) & \xrightarrow{\mathrm{Sq}^2} & H^{n+1}(X; \mathbb{Z}/2) & \xrightarrow{\pi^n(X; \mathbb{Z}/2)} & \pi^n(X; \mathbb{Z}/2) & \xrightarrow{h^n} & H^n(X; \mathbb{Z}/2)
\end{array}
\]
Given \( \gamma \in 2H^n(X; \mathbb{Z}/2) = H^n(X; \mathbb{Z}/2) \), by assumption and the exactness of the upper row, there exists \( \gamma' \in H^n(X; \mathbb{Z}/4) \) such that \( \rho(\gamma') = \gamma \). For any \( \bar{\gamma} \in \pi^n(X; \mathbb{Z}/2) \) we have

\[
\mathrm{Sq}^1_{\mathbb{Z}/4}(\gamma') = \psi'(\gamma) = 2\bar{\gamma} = 0.
\]

Thus the extension (10) splits, by [15, Lemma on p. 63].

**PROPOSITION 3.7:** Suppose the assumptions in Proposition 3.2 hold and let \( p \geq 3 \), \( 1 \leq r < \infty \). The extension (10) for \( G = \mathbb{Z}/p^r \) splits if one of the following conditions holds:

1. \( r \geq 2 \) and the homomorphism \([p] : H^n(X; \mathbb{Z}/p^{r-1}) \rightarrow H^n(X; \mathbb{Z}/p^r)\) is injective.
2. \( r = 1 \).

**Proof.** (1) There is a homotopy fibration diagram

\[
\begin{array}{ccccccccc}
K_n(\mathbb{Z}/p) & \xrightarrow{[p^{r-1}]} & K_n(\mathbb{Z}/p^r) & \xrightarrow{p} & K_n(\mathbb{Z}/p^{r-1}) & \xrightarrow{\delta} & K_{n+1}(\mathbb{Z}/p) \\
\downarrow 0 & & \downarrow \phi & & \downarrow p_K & & \downarrow [p] & & \downarrow 0 \\
K_{n+2p-3}(\mathbb{Z}/p) & \xrightarrow{\partial} & F_n(\mathbb{Z}/p^r) & \xrightarrow{i} & K_n(\mathbb{Z}/p^r) & \xrightarrow{p^1/P_1} & K_{n+2p-2}(\mathbb{Z}/p) \\
\end{array}
\]

Consider the following commutative exact sequence diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & [K(\mathbb{Z}/p^r, n), F_n(\mathbb{Z}/p^r)] & \xrightarrow{\iota_2} & [K_n(\mathbb{Z}/p^r), K(\mathbb{Z}/p^r, n)] & \downarrow \phi \\
0 & \rightarrow & [F_n(\mathbb{Z}/p^r), F_n(\mathbb{Z}/p^r)] & \xrightarrow{\iota_2} & [F_n(\mathbb{Z}/p^r), K_n(\mathbb{Z}/p^r)] & \downarrow \phi \\
\end{array}
\]

where \( H^{n+2p-3}(F_n(\mathbb{Z}/p^r); \mathbb{Z}/p) = 0 \) is due to Lemma 2.4. It follows that the map \( \phi \) satisfying \( \iota \phi = p_K \) is unique and satisfies the relation \( \phi \iota = p_F \).

The remainder of the proof of (1) is similar to that of Proposition 3.6, by analyzing the induced commutative exact sequence diagram:

\[
\begin{array}{cccccc}
H^{n-1}(X; \mathbb{Z}/p^{r-1}) & \xrightarrow{[p^{r-1}]} & H^n(X; \mathbb{Z}/p^r) \xrightarrow{p} & H^n(X; \mathbb{Z}/p^{r-1}) & \downarrow \phi \\
\downarrow [p] & & \downarrow 0 & & \downarrow \phi \\
H^{n-1}(X; \mathbb{Z}/p^r) & \xrightarrow{p^1/P_1} & H^{n+2p-3}(X; \mathbb{Z}/p) & \xrightarrow{\partial} & \pi^n(X; \mathbb{Z}/p^r) & \xrightarrow{h^n} & H^n(X; \mathbb{Z}/p^r) \\
\end{array}
\]
(2) There is a homotopy fibration diagram
\[
\begin{array}{c}
K_n(\mathbb{Z}/p^2) \xrightarrow{\rho} K_n(\mathbb{Z}/p) \xrightarrow{\delta} K_{n+1}(\mathbb{Z}/p) \xrightarrow{[p]} K_{n+1}(\mathbb{Z}/p^2) \\
\downarrow 0 \downarrow \phi' \downarrow 0 \\
K_{n+2p-3}(\mathbb{Z}/p) \xrightarrow{\partial} F_n(\mathbb{Z}/p) \xrightarrow{\iota} K_n(\mathbb{Z}/p) \xrightarrow{p^i} K_{n+2p-2}(\mathbb{Z}/p)
\end{array}
\]

There hold relations
\[\iota \phi' = p_K = 0, \quad \phi' \iota = p_F = 0\]
and the extension (10) splits. \hfill \blacksquare

**Example 3.8:** Let \( n \geq 3, r, s \geq 1 \).

1. Write \( M_{2s}^n = M_n(\mathbb{Z}/2^s) \). There is a short exact sequence
\[
0 \to H^{n+1}(M_{2s}^n; \mathbb{Z}/2) \to \pi^n(M_{2s}^n; \mathbb{Z}/2^r) \to H^n(M_{2s}^n; \mathbb{Z}/2^r) \to 0,
\]
which splits if and only if \( r \geq 2 \).

2. Generally, for any an \((n - 1)\)-connected \((n + 1)\)-dimensional finite complex \( X \), the short exact sequence
\[
0 \to H^{n+1}(X; \mathbb{Z}/2) \to \pi^n(X; \mathbb{Z}/2^r) \xrightarrow{h^n} H^n(X; \mathbb{Z}/2^r) \to 0
\]
splits if and only if \( H_n(X) \) has no direct summand \( \mathbb{Z}/2 \).

**Proof.** (1) Since \( M_{2s}^n \) is \((n - 1)\)-connected, \( T_{\Omega Sq_{2s}^2}(M_{2s}^n) \cong H^{n+1}(M_{2s}^n; \mathbb{Z}/2) \). The condition (a) in Proposition 3.6(1) is true. Thus the extension (13) splits if and only if
\[0 = Sq^1: H^n(M_{2s}^n; \mathbb{Z}/2) \to H^{n+1}(M_{2s}^n; \mathbb{Z}/2),\]
which holds if and only if \( s \geq 2 \), by the Bockstein exact sequence associated to the short exact sequence \( 0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0 \). The statement for the mod \( 2^s \) Moore space follows by the Spanier–Whitehead duality:
\[[M_{2s}^n, M_{2r}^n] \cong [M_{2r}^n, M_{2s}^n].\]

(2) It is well-known that an \((n - 1)\)-connected \((n + 1)\)-dimensional finite complex \( X \) has the homotopy type of a wedge of spheres and mod \( p^r \) Moore spaces for different primes \( p \geq 2 \) and positive integers \( r \); that is, there is a homotopy equivalence
\[X \simeq \bigvee_i S_i^n \vee \bigvee_j S_j^{n+1} \vee \bigvee_{p,r} M_n(\mathbb{Z}/p^r),\]
where $S_i^n = S^n$ and $S_j^{n+1} = S^{n+1}$. By Theorem 2.1(1), it suffices to prove (2) for the indecomposable spaces $X = S^n, S^{n+1}$ and $M_n(\mathbb{Z}/p^r)$. The $X = S^n, S^{n+1}$ or $M_n(\mathbb{Z}/p^r)$ with $p \geq 3$ cases are trivial and the equivalent condition in (1) is obviously characterized by the condition that $H_n(X)$ has no direct summand $\mathbb{Z}/2$. The proof of (2) is completed.

**Proof of Theorem 1.1.** Theorem 1.1 consists of Propositions 3.2, 3.3, 3.6 and 3.7.

**4. Unstable homotopy theory and modular cohomotopy**

In this section we combine some classic unstable homotopy theory of Moore space to study the modular cohomotopy groups. Let $p$ be a prime. It is well-known that $S^{2n-1}_{(p)}$ is an H-space for $p \geq 3, n \geq 1$ and $S^{n-1}_{(2)}$ is an H-space if and only if $n = 1, 2, 4, 8$ [1]. If $p \geq 5, n \geq 1$ (or $p \geq 2, n = 1, 2$), $S^{2n-1}_{(p)}$ is homotopy associative; if $p \geq 5$ (or $p \geq 3, n \geq 2$), $S^{2n-1}_{(p)}$ is homotopy commutative; see [25, p. 465, Exercises].

**4.1. $(2n+1)$-connected cover of $S^{2n+1}$**

Consider the following $p$-localized homotopy fibration sequence

$$K_{2n}(\mathbb{Z}_{(p)}) \xrightarrow{\partial_{(p)}} S^{2n+1}(2n+1)_{(p)} \xrightarrow{\pi_{(p)}} S^{2n+1}_{(p)} \xrightarrow{\iota_{(p)}} K_{2n+1}(\mathbb{Z}_{(p)})$$

where $\iota = \iota_{2n+1}$ represents a generator of $H^{2n+1}(S^{2n+1})$. It is clear that for each $p \geq 3$, the map $\iota_{(p)}$ is an H-map and hence the above sequence is an H-fibration sequence.

**Lemma 4.1:** For each prime $p \geq 2$, $\iota_{(p)} = \Omega \alpha$ for some $\alpha \in H^4(BS^3_{(p)}; \mathbb{Z}_{(p)})$, hence $S^3_{(3)}_{(p)}$ inherits the H-space structure of $S^3_{(p)}$.

**Proof.** Since $S^3$ has the homotopy type of a loop space, so does $S^3_{(3)}$; hence the classifying space $BS^3_{(p)}$ exists. The evaluation $e : \Sigma \Omega BS^3_{(p)} \to BS^3_{(p)}$ is 7-connected and hence the loop map

$$[BS^3_{(p)}, K_4(\mathbb{Z}_{(p)})] \xrightarrow{e^*} [\Sigma S^3_{(p)}, K_4(\mathbb{Z}_{(p)})] \cong [S^3_{(p)}, K_3(\mathbb{Z}_{(p)})]$$

is surjective, by [3, Proposition 2.4.13].
PROPOSITION 4.2: Let \( p \) be a prime and let \( n \geq 1 \).

1. There exists a map \( \alpha: M_{2n+2p-2}(\mathbb{Z}/p) \to S^{2n+1}\langle 2n+1 \rangle_{(p)} \) which is \( N \)-connected with

\[
N = \begin{cases} 
2n + 4p - 3 & \text{for } p \geq 3 \text{ or } p = 2, n = 1; \\
2n + 3 & \text{for } p = 2, n \geq 2.
\end{cases}
\]

2. If \( p \geq 3 \) or \( p = 2, n = 1 \), then there exists a map

\[
M_{2n+2p-3}(\mathbb{Z}/p) \to \Omega S^{2n+1}\langle 2n+1 \rangle_{(p)}
\]

which is \( (2n+4p-4) \)-connected.

3. For each \( p \geq 2 \), there exists a map \( M_{2p+1}(\mathbb{Z}/p) \to BS^3\langle 3 \rangle_{(p)} \) which is \( 4p \)-connected.

Proof. For simplicity we write \( M_k = M_k(\mathbb{Z}/p) \) for \( k \geq 1 \) in the proof. The \( n = 1 \) cases in (1) and (2) refer to [25, Corollary 4.6.2 and Proposition 4.6.4], respectively. Suppose \( n > 1 \).

1. By Lemma 2.6 we have an isomorphism

\[
[M_{2n+2p-2}, S^{2n+1}] \xrightarrow{i_p^*} [S^{2n+2p-2}, S^{2n+1}] \cong \mathbb{Z}/p\langle \alpha_1(2n+1) \rangle.
\]

Let \( \bar{\alpha}_1 \) be the inverse image of \( \alpha_1(2n+1) \) under \( i_p^* \). There exists a canonical map \( \alpha: M_{2n+2p-2} \to S^{2n+1}\langle 2n+1 \rangle_{(p)} \) satisfying \( \bar{\alpha}_1 = \pi(p) \circ \alpha \). Since \( \pi(p) \) induces an isomorphism of homotopy groups in dimensions greater than \( 2n+1 \), \( \alpha \) is \( N \)-connected if and only if \( \bar{\alpha}_1 \) is \( N \)-connected.

(i) Let \( p \geq 3 \). Using the elements defined in Lemma 2.6, the following relation equalities holds:

\[
\begin{align*}
\bar{\alpha}_1 \circ i_p &= \alpha_1(2n+1), \text{ by definition.} \\
\bar{\alpha}_1 \circ (i_p \alpha_1(2n+2p-2)) &= \alpha_1(2n+1) \circ \alpha_1(2n+2p-2). \\
\bar{\alpha}_1 \circ \bar{\alpha}_1 &= -\alpha_2, \text{ due to [34, Proposition 4.2] by taking} \\
\langle \alpha, \beta, \gamma \rangle &= \langle \alpha_1(2n+1), \nu_{2n+2p-2}, \alpha_1(2n+2p-2) \rangle.
\end{align*}
\]

It follows that \( \bar{\alpha}_1 \) induces an isomorphism of homotopy groups in dimensions \( \leq 2n+4p-4 \). Observe that

\[
\pi_{2n+4p-3}(M_{2n+2p-2}) \neq 0,
\]

while \( \pi_{2n+4p-3}(S^{2n+1}; p) = 0 \) by [35, Theorem 13.4]. Thus \( \bar{\alpha}_1 \), or equivalently \( \alpha \), is \( (2n+4p-3) \)-connected if \( p \geq 3 \).
(ii) If \( p = 2 \), the map \( \tilde{\alpha}_1 = \tilde{\eta}_1 : M_{2n+2} \to S^{2n+1} \) satisfies the relation \( \tilde{\eta}_1 i_2 = \eta \).
By [7] we have \( \pi_{2n+3}(M_{2n+2}) \cong \mathbb{Z}/2 \langle i_2 \eta \rangle \) and \( \pi_{2n+4}(M_{2n+2}) \cong \mathbb{Z}/4 \langle \tilde{\eta}_1 \rangle \), where \( \tilde{\eta}_1 \) satisfies \( j_2 \tilde{\eta}_1 = \eta \). Recall that \( \pi_{2n+4}(S^{2n+1}; 2) \cong \mathbb{Z}/8 \langle \nu \rangle \) if \( n > 1 \). Hence \( \tilde{\alpha}_1 \) induces an isomorphism of homotopy groups in dimensions \( \leq 2n+3 \) and therefore \( \tilde{\alpha}_1 \) is \((2n+3)-\)connected. This completes the proof of (1).

(2) The loop suspension map \( E : M_{2n+2p-3} \to \Omega M_{2n+2p-2} \) is \((4n+4p-7)-\)connected. Consider the adjoint map \( \alpha^\circ = (\Omega \alpha) \circ E : 
M_{2n+2p-3} \xrightarrow{E_1} \Omega M_{2n+2p-2} \xrightarrow{\Omega S^{2n+1}} \Omega S^{2n+1}(2n+1)_p \).
Since \( (\Omega \alpha) \) is \((2n+4p-4)-\)connected and \( 2n+4p-4 \leq 4n+4p-7 \) for \( n > 1 \), we see that \( \alpha^\circ \) is \((2n+4p-4)-\)connected.

(3) By [5, (3.2)], the evaluation map \( e : \Sigma S^3(3)_p \to BS^3(3)_p \) is \((4p+1)-\)connected. Then (1) implies that the adjoint map
\[
\alpha^\circ : M_{2p+1} \xrightarrow{\Sigma \alpha} \Sigma S^3(3)_p \xrightarrow{e} BS^3(3)_p
\]
is \(4p\)-connected.

Proof of Theorem 1.3. By Proposition 4.2 and Lemma 2.3, if \( p \geq 3, n \geq 1 \) or \( p = 2, n = 1 \), there are induced bijections
\[
\pi^{2n+2p-2}(X; \mathbb{Z}/p) \xrightarrow{1:1} [X, S^{2n+1}(2n+1)_p]
\]
for \( \dim(X) \leq 2n+4p-4 \) and
\[
\pi^{2n+2p-3}(X; \mathbb{Z}/p) \xrightarrow{1:1} [X, \Omega S^{2n+1}(2n+1)_p]
\]
for \( \dim(X) \leq 2n+4p-5 \).

(1) If \( p \geq 5 \) or \( n = 1, p \geq 2 \), \( S^{2n+1}(p)_p \) and \( S^{2n+1}(2n+1)_p \) are homotopy associative H-spaces. By Lemma 2.3, the bijection (15) is an isomorphism. Similarly the bijection (16) is an isomorphism of abelian groups if \( n \geq 2 \). For \( n = 1 \), this bijection yields an abelian group structure on \( \pi^{2p-1}(X; \mathbb{Z}/p) \), and hence the exact sequence in (1) follows.

If \( p \geq 3 \), \( \pi^{2n+2p-2}(X; \mathbb{Z}/p) \) \( (n \geq 1) \) and \( \pi^{2n+2p-3}(X; \mathbb{Z}/p) \) \( (n \geq 2) \) are both vector spaces over \( \mathbb{Z}/p \), by Theorem 2.1 (2). If \( n = 1 \), by [33, Proposition 8.4], the Anick’s space \( T_{2p-1}(p) \) is homotopy equivalent to \( \Omega S^3(3)_p \).
By [33, Theorem 1.3] and its comments, for each \( p \geq 3 \), \( \Omega S^3(3)_p \) has H-space exponent \( p \), which means the order of the identity map is \( p \). Hence the group \( \pi^{2p-1}(X; \mathbb{Z}/p) \cong [X, \Omega S^3(3)_p] \) is annihilated by \( p \); that is, \( \pi^{2p-1}(X; \mathbb{Z}/p) \) is a vector space over \( \mathbb{Z}/p \).
(2) By Lemma 2.6, the map $\mathcal{B}S^3\langle 3 \rangle_{(p)} \to K_{2p+1}(\mathbb{Z}/p)$ representing a generator of

$$H^{2p+1}(\mathcal{B}S^3\langle 3 \rangle_{(p)}; \mathbb{Z}/p) \cong \mathbb{Z}/p$$

is $(4p - 2)$-connected. It follows that the induced homomorphism

$$[X, \mathcal{B}S^3\langle 3 \rangle_{(p)}] \to H^{2p+1}(X; \mathbb{Z}/p)$$

is an isomorphism if $\dim(X) \leq 4p - 3$.

There is a homotopy fibration diagram induced by the third homotopy commutative square:

$$
\begin{array}{cccccc}
K_2(\mathbb{Z}(p)) & \xrightarrow{\partial(p)} & S^3\langle 3 \rangle_{(p)} & \xrightarrow{\kappa(p)} & K_3(\mathbb{Z}(p)) \\
\downarrow & & \downarrow \beta & & \downarrow \alpha \\
K_2(\mathbb{Z}(p)) & \xrightarrow{p^1_{z(p)}} & K_{2p}(\mathbb{Z}/p) & \xrightarrow{F_1(\mathbb{Z}(p))} & K_3(\mathbb{Z}(p))
\end{array}
$$

By Lemma 4.1(1), the above fibration diagram extends from the right by

$$
\begin{array}{ccc}
K_3(\mathbb{Z}(p)) & \xrightarrow{\mathcal{B}\partial(p)} & \mathcal{B}S^3\langle 3 \rangle_{(p)} \\
\downarrow & & \downarrow \mathcal{B}\beta \\
K_3(\mathbb{Z}(p)) & \xrightarrow{p^1_{z(p)}} & K_{2p+1}(\mathbb{Z}/p)
\end{array}
$$

Then the extended exact sequence in (2) follows at once.

**Proof of Corollary 1.4.** (1) If $\dim(X) \leq 6$, the canonical map

$$M_5(\mathbb{Z}/2) \to \mathcal{B}S^3\langle 3 \rangle_{(2)}$$

in Proposition 4.2 (3) induces an isomorphism

$$\pi^5(X; \mathbb{Z}/2) \cong [X, \mathcal{B}S^3\langle 3 \rangle_{(2)}].$$

Then (1) follows by applying Theorem 1.3.

(2) The map $\mathcal{B}\beta : \mathcal{B}S^3\langle 3 \rangle_{(2)} \to K_5(\mathbb{Z}/2)$ defined by the commutative square (17) with $(n, p) = (1, 2)$ gives a generator of

$$H^5(\mathcal{B}S^3\langle 3 \rangle_{(2)}; \mathbb{Z}/2) \cong \mathbb{Z}/2$$

and is 6-connected. Thus $[X, \mathcal{B}S^3\langle 3 \rangle_{(2)}] \to H^5(X; \mathbb{Z}/2)$ is an isomorphism for $\dim(X) \leq 5$. By Lemma 4.1 and [22, Theorem 1.1], the central extension (5) then follows by (1) and (17).
Since \( \ker(\text{Sq}^2_{\mathbb{Z}/2}) \) has order 2, the central extension (5) is determined by the two functions \( \Gamma, \Phi_2 \) given in the statement (2), by [22, Theorem 3.1]. We prove the formula for \( \Phi_2 \) as follows. There is a homotopy fibration diagram:

\[
\begin{array}{ccccccc}
K_2(\mathbb{Z}/2) & \xrightarrow{\rho} & K_2(\mathbb{Z}/2) & \rightarrow & K_3(\mathbb{Z}/2) & \xrightarrow{2} & K_3(\mathbb{Z}/2) & \xrightarrow{\rho} & K_3(\mathbb{Z}/2) \\
\downarrow & & \downarrow \phi & & \downarrow \phi & & \downarrow \phi & & \downarrow \phi \\
K_2(\mathbb{Z}/2) & \xrightarrow{\partial_2} & S^3(\mathbb{Z}/2) & \rightarrow & S^3(\mathbb{Z}/2) & \xrightarrow{\iota(2)} & K_3(\mathbb{Z}/2) & \xrightarrow{B\partial(2)} & BS^3(\mathbb{Z}/2) \\
\end{array}
\]

where \( \rho \) is the mod 2 reduction. By a similar argument to that of the claim of the proof of Proposition 3.6 (1) we deduce that there is an appropriate \( \phi_k \) satisfying the relations

\[ \iota(2) \circ \phi = 2, \quad \phi \circ \iota(2) = 2. \]

Consider the induced commutative diagram of exact sequences:

\[
\begin{array}{ccccccc}
H^2(X;\mathbb{Z}/(2)) & \xrightarrow{\delta} & H^3(X;\mathbb{Z}/(2)) & \xrightarrow{[2]} & H^3(X;\mathbb{Z}/(2)) \\
\downarrow & & \downarrow \phi & & \downarrow \phi \\
H^2(X;\mathbb{Z}/(2)) & \xrightarrow{\partial_2} & \pi^4(X;\mathbb{Z}/(2)) & \rightarrow & \pi^3(X;\mathbb{Z}/(2)) & \xrightarrow{h^2} & H^3(X;\mathbb{Z}/(2)) \\
\end{array}
\]

Then given any

\[ \alpha = h^3(\tilde{\alpha}) = \ker(\text{Sq}^2_{\mathbb{Z}/(2)}) \]

with \( \tilde{\alpha} \in \pi^3(X;\mathbb{Z}/(2)) \), \( 2 \cdot \alpha = 0 \) implies that there exists \( \alpha' \in H^2(X;\mathbb{Z}/(2)) \) such that \( \delta'(\alpha') = \alpha \) and we obtain

\[ \tilde{\alpha}^2 = \phi(\alpha) = \partial_2'(\alpha') \in \pi^4(X;\mathbb{Z}/(2))/\partial_2(H^2(X;\mathbb{Z}/(2))). \]

4.2. The double suspension. Consider the \( p \)-localized H-fibration sequence constructed by Gray [11, Theorem 9] for each prime \( p \geq 5 \):

\[ C(n) \xrightarrow{\Omega_n^m} S^{2n-1} \xrightarrow{E_2^m} \Omega^2 S^{2n+1} \xrightarrow{\nu_n} BC(n). \]

**Lemma 4.3:** Let \( p \geq 3 \) be a prime and let \( n \geq 1 \).

1. There exists a canonical map \( \beta : M_{2pn-3}(\mathbb{Z}/p) \rightarrow C(n) \) which is \( (2pn + 2n - 5) \)-connected.
2. There exists a canonical map \( M_{2pn-2}(\mathbb{Z}/p) \rightarrow BC(n) \) which is \( (2pn + 2n - 4) \)-connected.
Proof. (1) From [25, p. 309], for each \( p \geq 3, n \geq 1 \), we have
\[
H_k(C(n)(p)) \cong \begin{cases} 
\mathbb{Z}/p & \text{for } k = 2pn - 3; \\
0 & \text{otherwise for } k \leq 2pn + 2n - 5.
\end{cases}
\]
It follows that the canonical inclusion map \( \beta: M_{2pn-3}(\mathbb{Z}/p) \to C(n)(p) \) is \((2pn + 2n - 5)\)-connected.

(2) The evaluation map
\[
e: \Sigma C(n)(p) = \Sigma \Omega BC(n)(p) \to BC(n)(p)
\]
is \((4pn - 5)\)-connected, hence the adjoint map \( \beta^o = e \circ \Sigma(\beta) \) is \((2pn + 2n - 4)\)-connected and the proof is completed. ■

Proof of Theorem 1.5. The H-fibration sequence (18) induces an exact sequence
\[
[X, \Omega S^{2n-1}(p)] \xrightarrow{(\Omega E^2_p)_*} [X, \Omega^3 S^{2n+1}(p)] \to [X, C(n)(p)] \xrightarrow{(E^2_p)_*} [X, \Omega^2 S^{2n+1}(p)] \xrightarrow{(v_n)_*} [X, BC(n)(p)].
\]
By Lemmas 4.3 and 2.3, there are natural isomorphisms of abelian groups
\[
\pi^{2pn-3}(X; \mathbb{Z}/p) \xrightarrow{\cong} [X, C(n)(p)], \text{ if dim}(X) \leq 2pn + 2n - 6;
\]
\[
\pi^{2pn-2}(X; \mathbb{Z}/p) \xrightarrow{\cong} [X, BC(n)(p)], \text{ if dim}(X) \leq 2pn + 2n - 5.
\]
Since \( C(n)(p) \) has H-space exponent \( p \) cf. [9, Corollary 1.5] or by Theorem 2.1(2), \( \pi^{2pn-3}(X; \mathbb{Z}/p) \) is a vector space over \( \mathbb{Z}/p \) and the proof is completed. ■

4.3. Anick’s fibration. Let \( p \) be an odd prime in this subsection. When \( p \geq 5, n, r \geq 1 \), Anick [2] constructed the following homotopy fibration (19), which was extended to \( p = 3 \) by Gray and Theriault [12]:

Lemma 4.4 (cf. [12]): For integers \( n, r \geq 1 \), there exists a space \( T^{2n-1}_{\infty}(p^r) \) and an H-fibration sequence
\[
\Omega^2 S^{2n+1}(p) \xrightarrow{\phi_r} S^{2n-1}(p) \xrightarrow{i_r} T^{2n-1}_{\infty}(p^r) \xrightarrow{\pi_{\infty}} \Omega S^{2n+1}(p)
\]
with the following two properties:
\[
\phi_r \circ E^2_{(p)} \simeq p^r, \quad E^2_{(p)} \circ \phi_r \simeq \Omega^2 p^r,
\]
where \( E^2_{(p)} \) is the double suspension localized at \( p \).
Moreover, there is a coalgebra isomorphism

\[ H_\ast(T^{2n-1}_\infty(p^r); \mathbb{Z}/p) \cong \mathbb{Z}/p[v_{2n}] \otimes \Lambda(u_{2n-1}), \]

where subscripts indicate dimensions and \( \beta^r(v_{2n}) = u_{2n-1} \) under the \( r \)-th Bockstein.

The space \( T^{2n-1}_\infty(p^r) \) is called Anick’s space and the fibration (19) is the called Anick’s fibration.

**Lemma 4.5:** Let \( p \) be an odd prime and let \( n, r \geq 1 \).

1. There exists a canonical map \( \iota_M : M_{2n-1}(\mathbb{Z}/p^r) \to T^{2n-1}_\infty(p^r) \) which is \( (4n-2) \)-connected.
2. If \( n \geq 2 \), there is a canonical map \( M_{2n-2}(\mathbb{Z}/p^r) \to \Omega T^{2n-1}_\infty(p^r) \) which is \( (4n-5) \)-connected.

*Proof.* (1) is a direct result of the coalgebra structure of \( H_\ast(T^{2n-1}_\infty(p^r); \mathbb{Z}/p) \).

(2) Write \( M = M_{2n-2}(\mathbb{Z}/p^r) \) for short. The adjoint map \( \iota_M^0 \) is the composition \((\Omega \iota_M)|E_M\), where \( E_M : M \to \Omega \Sigma M \) is the loop suspension. By (1) the loop map \((\Omega \iota_M)\) is \( (4n-3) \)-connected. Since \( E_M \) is \( (4n-5) \)-connected, \( \iota_M^0 \) is \( (4n-5) \)-connected and the proof is completed. ■

**Remark 4.6:**

1. Let \( i_{(p)} : S^{2n-1}_{(p)} \to M_{2n-1}(\mathbb{Z}/p^r) \) be the canonical inclusion map localized at \( p \). Then we have

\[ \iota_r = \iota_M \circ i_{(p)}. \]

2. Let \( S^{2n-1}_{(p)} \) be the homotopy fibre of the \( p^r \)-th power map of \( S^{2n-1}_{(p)} \).

By [25, p. 472], for each \( n \geq 2 \), the canonical map

\[ M_{2n-2}(\mathbb{Z}/p^r) \to S^{2n-1}_{(p)} \]

is \( (4n-5) \)-connected. Together with Proposition 4.5 (2), we obtain isomorphisms for \( \dim(X) \leq 4n-6 \):

\[ \pi^{2n-2}(X; \mathbb{Z}/p^r) \cong [X, S^{2n-1}_{(p)}] \cong [X, \Omega T^{2n-1}_\infty(p^r)]. \]

Since

\[ T^1_\infty(p^r) \cong K_1(\mathbb{Z}/p^r) \times \Omega S^3(3)_{(p)} \]

([32, Lemma 5.1]), the group \([X, \Omega S^3(3)_{(p)}]\) has been discussed; we may assume that \( n \geq 2 \).
Proof of Theorem 1.6. For any prime \( p \geq 5 \) and any complex \( X \), \([X, T_{2n-1}^\infty(p^r)]\) is an abelian group since \( T_{2n-1}^\infty(p^r) \) is a homotopy commutative H-space [33, Theorem 1.2]. The Anick’s fibration (19) induces an exact sequence of abelian groups
\[
[X, \Omega^2 S_{2n+1}^{(p)}] \xrightarrow{(\phi_r)_{\sharp}} [X, S_{2n-1}^{(p)}] \xrightarrow{(\iota_r)_\sharp} [X, T_{2n-1}^\infty(p^r)] \xrightarrow{(\pi_\infty)_{\sharp}} [X, \Omega S_{2n+1}].
\]

Consider the following bijection for complexes \( X \) of dimension \( \leq 4n-3 \):
\[
(20) \quad \pi_{2n-1}(X; \mathbb{Z}/p^r) \to [X, T_{2n-1}^\infty(p^r)].
\]
If \( \dim(X) \leq 4n-4 \), by Lemma 2.3 this bijection is an isomorphism of abelian groups. If \( \dim(X) = 4n-3 \), the bijection (20) provides \( \pi_{2n-1}(X; \mathbb{Z}/p^r) \) an abelian group structure.

Consider the commutative square of abelian groups:
\[
\begin{array}{ccc}
[X, S_{2n-1}^{(p)}] & \xrightarrow{p^r} & [X, S_{2n-1}^{(p)}] \\
\downarrow (E_{(p)}^2)_{\sharp} & & \downarrow \\
[X, \Omega^2 S_{2n+1}^{(p)}] & \xrightarrow{(\phi_r)_{\sharp}} & \pi_{2n-1}(X; \mathbb{Z}(p))
\end{array}
\]
Since \( E_{(p)}^2 \) is \((2pn-3)\)-connected, \((E_{(p)}^2)_{\sharp}\) is an isomorphism and hence
\[
T_{\phi_r}(X) \cong \text{coker} \left( \pi_{2n-1}(X; \mathbb{Z}(p)) \xrightarrow{p^r} \pi_{2n-1}(X; \mathbb{Z}(p)) \right)
\cong \pi_{2n-1}(X; \mathbb{Z}(p)) \otimes \mathbb{Z}/p^r
\]
and the exact sequence in Theorem 1.6 is proved.

By [33, Theorem 1.3], \( T_{2n-1}^\infty(p^r) \) has H-space exponent \( p^r \) for each \( p \geq 5 \). Hence
\[
\pi_{2n-1}(X; \mathbb{Z}/p^r) \cong [X, T_{2n-1}^\infty(p^r)]
\]
is annihilated by \( p^r \); especially, \( \pi_{2n-1}(X; \mathbb{Z}/p) \) is a vector space over \( \mathbb{Z}/p \), \( T_{\phi_1}(X) \) is a direct summand. The proof that \( T_{\phi_r}(X) \) is a direct summand for \( r \geq 2 \) is similar to that of [26, Lemma 7.3]; the details are given as follows. Consider the commutative diagram induced by coefficient homomorphism \( \rho: \mathbb{Z}/p^r \to \mathbb{Z}/p \):
\[
\begin{array}{ccc}
0 & \longrightarrow & \pi_{2n-1}(X; \mathbb{Z}(p)) \otimes \mathbb{Z}/p^r & \xrightarrow{i_r} & \pi_{2n-1}(X; \mathbb{Z}/p^r) & \longrightarrow & 0 \\
& & \downarrow 1 \otimes \rho & & \downarrow \rho_1 & & \\
0 & \longrightarrow & \pi_{2n-1}(X; \mathbb{Z}(p)) \otimes \mathbb{Z}/p & \xrightarrow{i_1} & \pi_{2n-1}(X; \mathbb{Z}/p) & \longrightarrow & 0
\end{array}
\]
Since \( \pi^{2n-1}(X; \mathbb{Z}(p)) \) is finitely generated, we can write

\[
\pi^{2n-1}(X; \mathbb{Z}(p)) = \mathbb{Z}\langle \alpha_1, \ldots, \alpha_k \rangle \oplus \mathbb{Z}/p^u \langle \beta_1 \rangle \oplus \cdots \oplus \mathbb{Z}/p^u \langle \beta_l \rangle.
\]

Then \( \{\bar{\alpha}_i, \bar{\beta}_j\} \) form a basis of \( \pi^{2n-1}(X; \mathbb{Z}(p)) \oplus \mathbb{Z}/p^r \). Obviously \((1 \otimes \rho)(\gamma_k) \neq 0\) for arbitrary \( \gamma_k \in \{\bar{\alpha}_i, \bar{\beta}_j\} \), hence

\[
i_1(1 \otimes \rho)(\gamma_k) \neq 0
\]

for all \( k \). Assume that \( i_r(\pi^{2n-1}(X; \mathbb{Z}(p)) \oplus \mathbb{Z}/p^r) \subseteq \pi^{2n-1}(X; \mathbb{Z}/p^r) \) is not a direct summand. Then there exists \( \gamma_{k_0} \) satisfying \( i_r(\gamma_{k_0}) = p\delta \) for some \( \delta \in \pi^{2n-1}(X; \mathbb{Z}/p^r) \). Hence

\[
p\rho^*_r(\delta) = \rho^*_r \circ i_r(\gamma_{k_0}) = i_1(1 \otimes \rho)(\gamma_{k_0}) = 0,
\]

which is a contradiction.

If \( \dim(X) \leq 4n-4 \), \( T_{\rho^*}(X) \cong \pi^{2n-1}(X) \otimes \mathbb{Z}/p^r \) follows by Theorem 2.1(3).

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