Mean parity of a single quantum excitation of some optical fields in thermal environments

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Abstract

The mean parity (the Wigner function at the origin) of excited binomial states, excited coherent states and excited thermal states in thermal environments is investigated in detail. It is found that the single-photon excited binomial state and the single-photon excited coherent state exhibit certain similarity in the aspect of their mean parity in the thermal environment. We show the negative mean parity can be regarded as an indicator of non-classicality of single-photon excitation of optical fields with a little coherence, especially for the single-photon excited thermal states.

Among various kinds of indicators of non-classicality of optical fields [1–6], the partial negativity of the Wigner function (PNWF) indicates the highly non-classical character of the optical fields. Quantum excitation of general optical fields can always exhibit PNWF which will be destroyed and eventually completely disappear at the same decay time $\gamma t_c = \ln 2 + n_{\text{ph}}$ in a thermal environment [7]. The Wigner function $W(q, p)$ is proportional to the expectation value of the parity operator that performs reflections about the phase-space point $(q, p)$ [8]. For an optical field in the state $\rho$, the value of the Wigner function at the origin of the state can be derived by [9, 10]

$$W(0, 0) = \frac{2}{\pi} \text{Tr}[(\hat{O}_e - \hat{O}_o)\rho],$$

where $\hat{O}_e = \sum_{k=0}^{\infty} |2k\rangle\langle 2k|$ and $\hat{O}_o = \sum_{k=0}^{\infty} |2k+1\rangle\langle 2k+1|$ are the even and odd parity operators, respectively. Therefore, the value of the Wigner function at the origin equals the constant $\frac{2}{\pi}$ times the expectation value of the parity operator for the state $\rho$. Hereafter the expectation value of the parity operator is named ‘mean parity’.

There have been several kinds of experimental schemes for reconstructing or measuring the Wigner distribution of the optical fields or the motional states of trapped atoms [11, 12]. Here, we briefly outline an operational approach for measuring the Wigner function which is based on the Fresnel transformation of measured Rabi oscillations [13]. Assume a two-level atom (qubit) initially in the excited state $|e\rangle$ is resonantly coupled with the field prepared in the quantum state of interest whose dynamical behaviour is governed by the Jaynes–Cummings model in the interaction picture $\lambda(a^\dagger \sigma_+ + a\sigma_-)$. Via observing Rabi oscillations of the two-level atom, the mean parity of the initial state of interest can be derived as follows:

$$\frac{\pi}{2} W(0, 0) = \frac{4}{\sqrt{1}} \int_0^{\infty} e^{i\tau^2/\pi} \left[ P_g(\tau) - \frac{1}{2} \right] d\tau = \frac{2}{\sqrt{1}} \int_0^{\infty} e^{i\tau^2/\pi} \left[ P_e(\tau) - P_x(\tau) \right] d\tau,$$

where $P_g(\tau)$ ($P_e(\tau)$) is the probability of finding the atom in the ground state $|g\rangle$ (the excited state $|e\rangle$) as a function of dimensionless interaction time $\tau = \lambda t$. Obviously, $P_e(\tau) - P_x(\tau)$ can be regarded as the mean parity of the two-level atom. Thus, the initial mean parity of an optical field is directly related to the Fresnel integral of the mean parity of a resonant coupled two-level atom. Though the mean parity of a quantum field contains less information than its full Wigner function, it can partly determine the dynamical behaviour of Rabi oscillation of the resonant coupled qubit, which is important in those quantum information processes containing the resonant coupling between non-classical optical fields and qubit. Therefore, it is desirable to investigate the dynamical behaviour of mean parity of some kinds of non-classical states in thermal environments.
When the state $\rho$ evolves in the thermal environment, its evolution can be described by [14]

$$\frac{d\rho}{dt} = \gamma(n + 1) \left( 2\rho a^\dagger a - a^\dagger a\rho - \rho a^\dagger a \right) + \frac{\gamma n}{2} \left( 2a^\dagger \rho a - a a^\dagger \rho - \rho a a^\dagger \right),$$

(3)

where $\gamma$ represents dissipative coefficient and $n$ denotes the mean thermal photon number of the thermal environment. In the thermal environment described by the master equation (2), the time evolution Wigner function satisfies the following Fokker–Planck equation [15]

$$\frac{\partial}{\partial t} W(q, p, t) = -\frac{\gamma}{2} \frac{\partial}{\partial q} \left( \frac{\partial W}{\partial q} + \frac{\partial W}{\partial p} \right) - \frac{\gamma}{2} \frac{\partial^2 W}{\partial p^2} + \frac{\gamma}{8} \left( \frac{\partial^2 W}{\partial q^2} + \frac{\partial^2 W}{\partial p^2} \right),$$

(4)

First, we will investigate the influence of the thermal noise on the mean parity of the single-photon excited coherent state (ECS) described by unnormalized wave vector $a^\dagger |\alpha\rangle$ [16]. Let us recall the results in [17, 18], and the time-evolving Wigner function at the origin of the single-photon ECS in the thermal environment can be derived as

$$W^S(0, 0, \gamma t) = \frac{2e^{\gamma t} |\alpha|^2 (1 - c^2)^2 + c^2 - 1}{\pi (1 + |\alpha|^2)^2 (1 + c^2)^2} \exp \left[ -\frac{2|\alpha|^2}{1 + c^2} \right],$$

$$c = (\exp(\gamma t) - 1)(1 + 2n)^{1/2}.$$  

(5)

It is easy to check that

$$W^S(0, 0, \gamma t) < 0 \quad \text{for } \gamma t < \gamma t_c \quad \text{and} \quad |\alpha| < 1$$

$$W^S(0, 0, \gamma t) \geq 0 \quad \text{for } \gamma t \leq \gamma t_{c1} \quad \text{and} \quad |\alpha| \geq 1$$

$$W^S(0, 0, \gamma t) < 0 \quad \text{for } \gamma t_{c1} < \gamma t < \gamma t_c \quad \text{and} \quad |\alpha| \geq 1,$$

(6)

where

$$\gamma t_c = \ln \frac{2 + n}{1 + n},$$

(7)

and

$$\gamma t_{c1} = \ln \left[ \frac{2|\alpha|^2 (1 + n) + 2n}{(1 + 2n)(1 + |\alpha|^2)} \right].$$

(8)

The above results imply the mean parity of the initial pure single-photon ECS exhibits transition-like behaviour with a critical point at $|\alpha| = 1$. When $|\alpha| < 1$, the pure single-photon ECS has negative mean parity. Otherwise, it has positive definite mean parity. The negativity of mean parity of the single-photon ECS with $|\alpha| < 1$ will be maintained until the threshold decay time $\gamma t_c$, beyond which the Wigner function of the thermal dissipative single-photon ECS become positive definite over the whole phase space. The mean parity of the single-photon ECS with $|\alpha| > 1$ is positive before the threshold decay time $\gamma t_{c1}$, and then becomes negative until $\gamma t_c$. In this sense, the mean parity may be suitable to be regarded as one of the indicators of the quantum-classical transition of the single-photon ECS.

In what follows, excited binomial states (EBSs) are chosen as the example to investigate then dynamical behaviour of the Wigner distribution and mean parity in the thermal environment. The EBSs have been introduced [19], which represent the intermediate non-Gaussian state between quantum Fock state and ECS. The EBS of the radiation field can be generated by repeated application of the photon creation operator on binomial states [19]. The binomial states of optical fields can be generated in some nonlinear processes [20–22]. Then, by a scheme similar to that for preparing the ECS [16], one can produce the EBS in a cavity initially prepared in a binomial state. If a travelling optical field in the binomial state has been produced, one can also adopt the experimental scheme of Zavatta et al [23] to generate the single-photon-excited binomial state.
Let us briefly recall the definition of the EBS [19]. The EBS is defined by
\[ |k, \eta, M\rangle = N(k, \eta, M)a^{|k}\rangle |\eta, M\rangle , \]  
(9)
where
\[ |\eta, M\rangle = \sum_{l=0}^{M} \left( \frac{C_{M}^{l}}{M!} \right)^{1/2} \eta^{|l/2|} \left( 1 - \eta^{|l/2|} \right)^{(M-l)/2} |l\rangle \]  
(10)
is the binomial state [24]. Here \( k, l \) and \( M \) are integers. \( C_{M}^{l} = \frac{M!}{l!(M-l)!} \) is the binomial coefficient, \( \eta \) is real number with \( 0 \leq \eta \leq 1 \). \(|l\rangle\) is a Fock state. \( a^{|i|} (a) \) is the creation operator (annihilation operator) of the optical mode. \( N(k, \eta, M) \) is normalization constant of the EBS, which is given by
\[ N(k, \eta, M) = \frac{1}{\sqrt{\eta^{|k(M+k)|} M!}} 2F_{1}(\frac{-M, -M - k; \eta_{1}^{k} - 1}{\eta}) . \]  
(11)
where \( 2F_{1}(\varepsilon, \zeta; k; x) \) is the hypergeometric function. The statistics properties of pure EBSs have been investigated in the past few years [19]. In [25], a scheme for preparing a two-photon binomial state in a single-mode high-Q cavity has been proposed. In fact, by slightly changing that scheme, one can generate the EBS with \( k = 1 \) and \( M = 2 \). Let an excited atom pass through the cavity initially in the two-photon binomial state, one can produce the EBS with \( k = 1 \) and \( M = 2 \). In [26], the analytical expressions of the Wigner function of the EBSs have been derived. Substituting them into equation (4) as the initial conditions and numerically solving this partial differential equation, one can obtain the time-evolving Wigner function. As an illustration, in figure 1, we have plotted the evolution of the cross-section of the Wigner distribution of \( |1, \eta, 2\rangle \) in the thermal environment labelled with \( n = 0.5 \) for three different values of \( \eta \). The partial negativity of the cross-section of the Wigner function indicates the non-classical nature of the single-photon EBS. The thermal noise destroys the partial negativity of the Wigner function. This figure explicitly exhibits the role of the single-photon-occurring probability \( \eta^{2} \) in its corresponding Wigner distribution.

In figure 2, \( W(0, 0) \) of the Wigner function at the origin of the single-photon EBS with \( M = 3 \) in the thermal environment with \( n = 0.5 \) is plotted as the function of decay time \( \gamma t \) and \( \eta \). With varying of \( \eta \) from 0 to 1, \( W(0, 0) \) exhibits different dynamical behaviour in the thermal environment: for small values of \( \eta < \eta^{3}_{1} \simeq 0.4 \), \( W(0, 0) \) is always negative before the threshold decay time \( \gamma t_{c} \). For \( \eta^{3}_{1} \leq \eta < 1 \), \( W(0, 0) \) is initially non-negative and then becomes negative before the threshold decay time \( \gamma t_{c} \). As \( \eta = 1 \), \( W(0, 0) \) is always non-negative. In figure 3, \( W(0, 0) \) of the Wigner function at the origin of the single-photon EBS with \( M = 4 \) in the thermal environment with \( n = 0.5 \) is plotted as the function of decay time \( \gamma t \) and \( \eta \). In this case, for small values of \( \eta < \eta^{4}_{1} \simeq 0.4 \), \( W(0, 0) \) is always negative before the threshold decay time \( \gamma t_{c} \). For \( \eta^{4}_{1} \leq \eta < \eta^{4}_{2} \simeq \sqrt{2}/2 \), \( W(0, 0) \) is initially non-negative, then becomes negative before the threshold decay time \( \gamma t_{c} \). For \( \eta^{4}_{2} \leq \eta < 1 \), \( W(0, 0) \) is initially negative, then becomes non-negative, and becomes negative again until the threshold decay time \( \gamma t_{c} \). We have also calculated the cases with larger values of \( M \), and found similar behaviour. Now for single-photon EBSs, the dependence of thermal-noise-induced behaviour of \( W(0, 0) \) on \( \eta \) can be classified: for the cases with \( M = 2m + 1, (m = 1, 2, \ldots) \), \( 0 \leq \eta < \eta^{1}_{1} \), \( W(0, 0) < 0 \) negative before the threshold decay time \( \gamma t_{c} \). For the cases with \( M = 2m + 1, (m = 1, 2, \ldots) \), \( \eta^{1}_{1} \leq \eta < 1 \), \( W(0, 0) \) is initially positive and then becomes negative until the threshold decay time \( \gamma t_{c} \). The critical point \( \eta^{1}_{1} \) decreases with the increase of \( M \). For the cases with \( M = 2m, (m = 1, 2, \ldots) \), \( 0 \leq \eta < \eta^{2}_{2} \), \( W(0, 0) \) is initially negative and then becomes negative until the threshold decay time \( \gamma t_{c} \). For the cases with \( M = 2m, (m = 1, 2, \ldots) \), \( \eta^{2}_{2} \leq \eta < 1 \), \( W(0, 0) \) is initially negative and then becomes positive and then becomes negative again until the threshold decay time \( \gamma t_{c} \), \( \eta^{2}_{2} \) increases with \( M \).

The inherent physics picture may be explained as follows: when \( \eta \) equals zero or one, the EBS reduce to the Fock state. The parity property of Fock states tell us that only those Fock states with odd photon numbers have negative value of the Wigner function at the origin. For very small values of \( \eta \), the
single-photon EBS looks like the single-photon Fock state. For intermediate value of $\eta^{\alpha} < \eta < \eta^{\beta}$, the single-photon EBS is similar to the single-photon ECS with $|\alpha| > 1$ in the aspect of thermal noise induced evolution of mean parity.

Up to now, we have understood the dynamical characteristics of mean parity of both the single-photon ECS and EBS in the thermal environment. These states have no population in the vacuum state. We also investigate the states with population in the vacuum state such as the binomial state and thermal states in the thermal environment. The single photon excitation of classical or thermal states in the thermal environment labelled by the same effective temperature, and the results show that they more rapidly completely lose the PNWF than those states with zero population in the vacuum state. In other words, the initial possible negative mean parity of the binomial state is more fragile against the thermal noise than the single-photon EBS. The negative mean parity is a sufficient condition for non-classicality of quantum optical fields, thus the single-photon EBS may have advantage in potential applications in quantum optical information processes.

While for some two-photon EBSs with initial negative mean parity, our numerical result indicates that the negative mean parity of the two-photon EBS is more fragile against the thermal noise than the single-photon EBS. In figure 4, $W(0, 0)$ of the two-photon EBS with $M = 2$ in photon loss channel with $n = 0$ is plotted as the function of decay time $\gamma t$ and $\eta$. Initially, the mean parity is negative only for those states with intermediate $\eta$. For small value of $\eta$, the mean parity is always positive. For intermediate value of $\eta$, the mean parity maintains its negative value in a period shorter than $\gamma t_c$, and then perpetually becomes positive.

For the single-photon excited thermal state [27], it is also easy to obtain its time-evolving Wigner function at the origin in the thermal environment as follows [7]:

$$W_T (0, 0, \gamma t) = \frac{\kappa}{\pi^3} e^{\kappa \gamma t},$$

$$\kappa = -8(\bar{n} - n)(1 + n) + 2(1 + 2n)^2 e^{\gamma t}$$

$$+ 4(\bar{n}(1 + 2n) - (1 + 2n)^2) e^{\gamma t},$$

where $\bar{n}$ denotes the mean photon number of the initial thermal state. Its mean parity is always negative before the threshold decay time $\gamma t_c = \ln \frac{2n + 1}{\ln 2}$. In this case, the negative mean parity can be regarded as a suitable indicator of non-classicality of a single-photon excited thermal state in the thermal environment for its monotonic corresponding to the volume of negative part of the Wigner function.

The above several examples show that the negativity of mean parity, as defined, is a useful indicator of the non-classicality of some specific class of non-classical fields. In particular, it exhibits the advantage if used as an indicator of convenient experimental analysis whether the thermal noise has completely destroyed the negativity of the Wigner distribution of any photon-added optical fields or not. The results in [7] have demonstrated that arbitrary photon-added optical fields completely lose their negativity of the Wigner distribution at the same threshold decay time in the thermal environment labelled by the same effective temperature, and their mean parity is zero at the threshold decay time.

In summary, we have investigated the mean parity of single quantum excitations of coherent states, binomial states and thermal states in the thermal environment. The single quantum excitation usually strengthens the negative mean parity. When the single quantum excitation of classical or non-classical optical fields lacks enough coherence, such as the single-photon excited thermal state, ECS and EBS which have high overlap with the single-photon Fock state, the initial negative mean parity monotonically increases and becomes positive after the decay time $\gamma t_c$. In these cases, the negative mean parity can be regarded as a good indicator of non-classicality of optical fields. For those single-photon ECS or EBS with sufficient large coherence, their mean parity will exhibit transient negativity even if initially the mean parity is positive.

For the single-photon ECS, its mean parity can be also regarded as the indicator for the transition from the Fock-like state to the coherent-like state.

To experimentally investigate whether the thermal noise has completely destroyed the PNWF, one has to reconstruct the whole phase-space Wigner distribution of real-time states, though reconstruction of the whole phase-space Wigner distribution is still very complicated. Thus, for some specific single-photon excitation of optical fields with a little coherence, to measure the mean parity may be a feasible method.

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