QUOTIENTS OF CUBIC SURFACES

ANDREY TREPALIN

Abstract. Let \( k \) be any field of characteristic zero, \( X \) be a cubic surface in \( \mathbb{P}^3_k \) and \( G \) be a group acting on \( X \). We show that if \( X(k) \neq \emptyset \) and \( G \) is not trivial and not a group of order 3 acting in a special way then the quotient surface \( X/G \) is rational over \( k \). For the group \( G \) of order 3 we construct examples of both rational and nonrational quotients of both rational and nonrational \( G \)-minimal cubic surfaces over \( k \).

1. Introduction

Let \( k \) be any field of characteristic zero. If \( k \) is algebraically closed then any quotient of a rational surface by an action of a finite group is rational by Castelnuovo criterion. For del Pezzo surfaces of degree 4 and higher the following theorem holds.

**Theorem 1.1** ([Tr13, Theorem 1.1]). Let \( k \) be a field of characteristic zero, \( X \) be a del Pezzo surface over \( k \) such that \( X(k) \neq \emptyset \) and \( G \) be a finite subgroup of automorphisms of \( X \). If \( K^2_X \geq 5 \) then the quotient variety \( X/G \) is \( k \)-rational. If \( K^2_X = 4 \) and the order of \( G \) is not equal to 1, 2 or 4 then \( X/G \) is \( k \)-rational.

In this paper we find for which finite groups a quotient of cubic surface is \( k \)-rational and for which is not. The main result of this paper is the following.

**Theorem 1.2.** Let \( k \) be a field of characteristic zero, \( X \) be a del Pezzo surface over \( k \) of degree 3 such that \( X(k) \neq \emptyset \) and \( G \) be a subgroup of \( \text{Aut}_k(X) \). Suppose that \( G \) is not trivial and \( G \) is not a group of order 3 having no curves of fixed points. Then \( X/G \) is \( k \)-rational.

Note that if \( G \) is trivial and \( X \) is minimal then \( X \) is not \( k \)-rational (see [Man74, Theorem V.1.1]). This gives us an example of a del Pezzo surface of degree 3 such that its quotient by the trivial group is not \( k \)-rational. For a group \( G \) of order 3 acting without curves of fixed points on \( X \) we construct examples of quotients of \( G \)-minimal cubic surface \( X \) such that \( X \) is \( k \)-rational and \( X/G \) is \( k \)-rational, \( X \) is \( k \)-rational and \( X/G \) is not \( k \)-rational, \( X \) is not \( k \)-rational and \( X/G \) is \( k \)-rational, and \( X \) is not \( k \)-rational and \( X/G \) is not \( k \)-rational.

To prove Theorem 1.2 we consider possibilities for groups \( G \) acting on \( X \). Our main method is to find a normal subgroup \( N \) in \( G \) such that the quotient \( X/N \) is \( G/N \)-birationally equivalent to a del Pezzo surface of degree 5 or more. Therefore \( k \)-rationality of \( X/G \) is equivalent to \( k \)-rationality of the quotient of the obtained del Pezzo surface by the group \( G/N \) and we can use Theorem 1.1.

The plan of this paper is as follows. In Section 2 we recall some facts about minimal rational surfaces, groups, singularities and quotients. In Section 3 we consider quotients
of cubic surfaces by nontrivial groups of automorphisms and show that all them except a case are always $k$-rational. In Section 4 for non-$k$-rational quotients of a $k$-rational cubic surface $X$ by a group of order 3 we find all possibilities of the image of the Galois group $\text{Gal}(\overline{k}/k)$ in the Weyl group $W(E_6)$ acting on the Picard group of $X$. In Section 5 we find an explicit geometric interpretation of the obtained actions of the Galois group in terms of equations of $X$. In Section 6 for a group $G$ of order 3 acting on a $G$-minimal cubic surface $X$ without curves of fixed points we construct examples of $k$-rational and non-$k$-rational quotients.

The author is grateful to his adviser Yu. G. Prokhorov and to C. A. Shramov for useful discussions.

**Notation.** Throughout this paper $k$ is any field of characteristic zero, $\overline{k}$ is its algebraic closure. For a surface $X$ we denote $X \otimes k$ by $\overline{X}$. For a surface $X$ we denote the Picard group (resp. $G$-invariant Picard group) by $\text{Pic}(X)$ (resp. $\text{Pic}(X)^G$). The number $\rho(X) = \text{rk} \text{Pic}(X)$ (resp. $\rho(X)^G = \text{rk} \text{Pic}(X)^G$) is the Picard number (resp. $G$-invariant Picard number) of $X$. If two surfaces $X$ and $Y$ are $k$-birationally equivalent then we write $X \approx Y$. If two divisors $A$ and $B$ are linearly equivalent then we write $A \sim B$.

## 2. Preliminaries

### 2.1. $G$-minimal rational surfaces

In this subsection we review main notions and results of $G$-equivariant minimal model program following the papers [Man67], [Isk79], [DI09a]. Throughout this subsection $G$ is a finite group.

**Definition 2.1.** A rational variety $X$ is a variety over $k$ such that $\overline{X} = X \otimes k$ is birationally equivalent to $\mathbb{P}^n_k$.

A $k$-rational variety $\overline{X}$ is a variety over $k$ such that $X$ is birationally equivalent to $\mathbb{P}^n_k$.

A variety $X$ over $k$ is a $k$-unirational variety if there exists a $k$-rational variety $Y$ and a dominant rational map $\varphi : Y \dashrightarrow X$.

**Definition 2.2.** A $G$-surface is a pair $(X, G)$ where $X$ is a projective surface over $k$ and $G$ is a finite subgroup of $\text{Aut}_k(X)$. A morphism of $G$-surfaces $f : X \to X'$ is called a $G$-morphism if for each $g \in G$ one has $fg = gf$.

A smooth $G$-surface $(X, G)$ is called $G$-minimal if any birational morphism of smooth $G$-surfaces $(X, G) \to (X', G)$ is an isomorphism.

Let $(X, G)$ be a smooth $G$-surface. A $G$-minimal surface $(Y, G)$ is called a minimal model of $(X, G)$ or $G$-minimal model of $X$ if there exists a birational $G$-morphism $X \to Y$.

The following theorem is a classical result about $G$-equivariant minimal model program.

**Theorem 2.3.** Any $G$-morphism $f : X \to Y$ can be factorized in the following way:

$$X = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \ldots \xrightarrow{f_{n-2}} X_{n-1} \xrightarrow{f_{n-1}} X_n = Y,$$

where each $f_i$ is a contraction of a set $\Sigma_i$ of disjoint $(-1)$-curves on $X_i$, such that $\Sigma_i$ is defined over $k$ and $G$-invariant. In particular,

$$K_Y^2 - K_X^2 \geq \rho(Y)^G - \rho(Y)^G.$$

The classification of $G$-minimal rational surfaces is well-known due to V. Iskovskikh and Yu. Manin (see [Isk79] and [Man67]). We introduce some important notions before surveying it.
Definition 2.4. A smooth rational $G$-surface $(X, G)$ admits a conic bundle structure if there exists a $G$-morphism $\varphi : X \to B$ such that any scheme fibre is isomorphic to a reduced conic in $\mathbb{P}^2_k$ and $B$ is a smooth curve.

Definition 2.5. A del Pezzo surface is a smooth projective surface $X$ such that the anticanonical divisor $-K_X$ is ample. A singular del Pezzo surface is a normal projective surface $X$ such that the anticanonical divisor $-K_X$ is ample and all singularities of $X$ are Du Val singularities. A weak del Pezzo surface is a smooth projective surface $X$ such that the anticanonical divisor $-K_X$ is nef and big.

The number $d = K^2_X$ is called the degree of a (singular) del Pezzo surface $X$.

A del Pezzo surface $X$ of degree 3 is isomorphic to a smooth cubic surface in $\mathbb{P}^3_k$.

The following theorem classifies $G$-minimal rational surfaces.

Theorem 2.6 ([Isk79, Theorem 1]). Let $X$ be a $G$-minimal rational $G$-surface. Then either $X$ admits a $G$-equivariant conic bundle structure with $\text{Pic}(X)^G \cong \mathbb{Z}$, or $X$ is a del Pezzo surface with $\text{Pic}(X)^G \cong \mathbb{Z}$.

Theorem 2.7 (cf. [Isk79, Theorem 4]). Let $X$ admit a $G$-equivariant conic bundle structure. Suppose that $K^2_X = 3, 5, 6$ or $X$ is a blowup of $\mathbb{P}^2_k$ at a point. Then $X$ is not $G$-minimal.

The following theorem is an important criterion of $k$-rationality over an arbitrary perfect field $k$.

Theorem 2.8 ([Isk96, Chapter 4]). A minimal rational surface $X$ over a perfect field $k$ is $k$-rational if and only if the following two conditions are satisfied:

(i) $X(k) \neq \emptyset$;
(ii) $K^2_X \geq 5$.

Corollary 2.9. Let $X$ be a rational $G$-surface such that $X(k) \neq \emptyset$ and $\rho(X)^G + K^2_X \geq 7$. Then there exists a $G$-minimal model $Y$ of $X$ such that $K^2_Y \geq 6$. In particular, $X$ is $k$-rational.

Proof. By Theorem 2.6 there exists a birational $G$-morphism $f : X \to Z$ such that $\rho(Z)^G \leq 2$. By Theorem 2.3 one has

$$K^2_Z \geq K^2_X + \rho(X)^G - \rho(Z)^G \geq 7 - \rho(Z)^G.$$ 

If $\rho(Z)^G = 1$ then $K^2_Z \geq 6$. If $\rho(Z)^G = 2$ and $K^2_Z = 5$ then $Z$ is not $G$-minimal by Theorem 2.7. Therefore there exists a $G$-minimal model $Y$ of $Z$ such that $K^2_Y \geq 6$.

The set $X(k)$ is not empty. Thus $Y(k) \neq \emptyset$ and $X \approx Y$ is $k$-rational by Theorem 2.8.

In this paper we use the notation of the following remark.

Remark 2.10. Let $X$ be a cubic surface in $\mathbb{P}^3_k$. Then $\overline{X}$ can be realized as a blowup $f : \overline{X} \to \mathbb{P}^2_k$ at 6 points $p_1$, $\ldots$, $p_6$ in general position. Put $E_i = f^{-1}(p_i)$ and $L = f^*(l)$, where $l$ is the class of a line on $\mathbb{P}^2_k$. One has

$$-K_{\overline{X}} \sim 3L - \sum_{i=1}^{6} E_i.$$
The $(-1)$-curves on $X$ are $E_i$, the proper transforms $L_{ij} \sim L - E_i - E_j$ of the lines passing through a pair of points $p_i$ and $p_j$, and the proper transforms

$$Q_j \sim 2L + E_j - \sum_{i=1}^{6} E_i$$

of the conics passing through five points from the set $\{p_1, p_2, p_3, p_4, p_5, p_6\}$.

In this notation one has:

$$E_i \cdot E_j = 0; \quad E_i \cdot L_{ij} = 1; \quad E_i \cdot L_{jk} = 0;$$

$$L_{ij} \cdot L_{ik} = 0; \quad L_{ij} \cdot L_{kl} = 1; \quad E_i \cdot Q_i = 0; \quad E_i \cdot Q_j = 1;$$

$$Q_i \cdot Q_j = 0; \quad Q_i \cdot L_{ij} = 1; \quad Q_i \cdot L_{jk} = 0.$$

where $i$, $j$ and $k$ are different numbers from the set $\{1, 2, 3, 4, 5, 6\}$.

2.2. Groups. In this subsection we collect some results and notation concerning groups used in this paper.

We use the following notation:

- $\mathfrak{C}_n$ denotes the cyclic group of order $n$;
- $\mathfrak{D}_{2n}$ denotes the dihedral group of order $2n$;
- $\mathfrak{S}_n$ denotes the symmetric group of degree $n$;
- $\mathfrak{A}_n$ denotes the alternating group of degree $n$;
- $(i_1 \ldots i_j)$ denotes a cyclic permutation of $i_1, \ldots, i_j$;
- $\mathfrak{V}_4$ denotes the Klein group isomorphic to $\mathfrak{C}_2^2$;
- $\langle g_1, \ldots, g_n \rangle$ denotes a group generated by $g_1, \ldots, g_n$;
- $\text{diag}(a_1, \ldots, a_n)$ denotes the diagonal $n \times n$ matrix with entries $a_1, \ldots, a_n$;
- $i = \sqrt{-1}$;
- $\xi_n = e^{2\pi i/n}$;
- $\omega = \xi_3 = e^{2\pi i/3}$.

The group $\mathfrak{S}_5$ can act on a cubic surface. Therefore it is important to know some facts about subgroups of this group. The following lemma is an easy exercise.

**Lemma 2.11.** Any nontrivial subgroup $G \subset \mathfrak{S}_5$ contains a normal subgroup $N$ conjugate in $\mathfrak{S}_5$ to one of the following groups:

- $\mathfrak{C}_2 \cong \langle (12) \rangle$,
- $\mathfrak{C}_2 \cong \langle (12)(34) \rangle$,
- $\mathfrak{C}_3 \cong \langle (123) \rangle$,
- $\mathfrak{V}_4 \cong \langle (12)(34), (13)(24) \rangle$,
- $\mathfrak{C}_5 \cong \langle (12345) \rangle$,
- $\mathfrak{A}_5$.

2.3. Singularities. In this subsection we review some results about quotient singularities and their resolutions.

All singularities appearing in this paper are toric singularities. These singularities are locally isomorphic to the quotient of $\mathbb{A}^2$ by a cyclic group generated by $\text{diag}(\xi_m, \xi_m^q)$. Such a singularity is denoted by $\frac{1}{m}(1,q)$. If $\gcd(m,q) > 1$ then the group

$$\mathfrak{C}_m \cong \langle \text{diag}(\xi_m, \xi_m^q) \rangle$$
contains a reflection and the quotient singularity is isomorphic to a quotient singularity with smaller \( m \).

A toric singularity can be resolved by a sequence of weighted blowups. Therefore it is easy to describe numerical properties of a quotient singularity. We list here these properties for singularities appearing in our paper.

**Remark 2.12.** Let the group \( C_m \) act on a smooth surface \( X \) and \( f : X \to S \) be a quotient map. Let \( p \) be a singular point on \( S \) of type \( \frac{1}{m}(1, q) \). Let \( C \) and \( D \) be curves passing through \( p \) such that \( f^{-1}(C) \) and \( f^{-1}(D) \) are \( C_m \)-invariant and tangent vectors of these curves at the point \( f^{-1}(p) \) are eigenvectors of the natural action of \( C_m \) on \( T_{f^{-1}(p)}X \) (the curve \( C \) corresponds to the eigenvalue \( \xi_m \) and the curve \( D \) corresponds to the eigenvalue \( \xi_q \)).

Let \( \pi : \tilde{S} \to S \) be the minimal resolution of the singular point \( p \). Table 1 presents some numerical properties of \( \tilde{S} \) and \( S \) for the singularities appearing in this paper.

The exceptional divisor of \( \pi \) is a chain of transversally intersecting exceptional curves \( E_i \) whose selfintersection numbers are listed in the last column of Table 1. The curves \( \pi^{-1}(C) \) and \( \pi^{-1}(D) \) transversally intersect at a point only the first and the last of these curves respectively and do not intersect other components of the exceptional divisor of \( \pi \).

### 2.4. Quotients

In this subsection we collect some additional information about quotients of rational surfaces.

We use the following definition for convenience.

**Definition 2.13.** Let \( X \) be a \( G \)-surface (resp. surface), \( \tilde{X} \to X \) be its minimal resolution of singularities, and \( Y \) be a \( G \)-minimal model (resp. minimal model) of \( \tilde{X} \). We call the surface \( Y \) a \( G \)-MMP-reduction (resp. MMP-reduction) of \( X \).

We need some results about quotients of del Pezzo surfaces of degree 4.

**Lemma 2.14** ([Tr13, Remark 6.2]). Let a finite group \( G \) act on a del Pezzo surface \( X \) of degree 4 and \( N \cong \mathbb{Z}_2 \) be a normal subgroup in \( G \) such that \( N \) has no curves of fixed points. Then the surface \( X/N \) is \( G/N \)-birationally equivalent to a conic bundle \( \tilde{Y} \) with \( K_{\tilde{Y}}^2 = 2 \). If there exists a \( G \times \text{Gal}(\overline{k}/k) \)-fixed point then \( Y \) is not \( G/N \)-minimal and there exists a \( G/N \)-MMP-reduction \( Z \) of \( \tilde{Y} \) such that \( K_Z^2 = 8 \).
Table 2.

| Type | Order | Equation | Action |
|------|-------|----------|--------|
| 1    | 2     | $x^3 + y^3 + z^3 + \alpha xyz + t^2(ux + vy + wz) = 0$ | $(x : y : z : -t)$ |
| 2    | 2     | $x^3 + y^3 + xz(z + \alpha t) + yt(z + \beta t) = 0$ | $(x : y : -z : -t)$ |
| 3    | 3     | $x^3 + y^3 + z^3 + \alpha xyz + t^3 = 0$ | $(x : y : z : \omega t)$ |
| 4    | 3     | $x^3 + y^3 + z^3 + t^3 = 0$ | $(x : y : \omega z : \omega t)$ |
| 5    | 3     | $x^3 + y^3 + zt(ux + vy) + z^3 + t^3 = 0$ | $(x : y : \omega z : \omega^2 t)$ |
| 6    | 5     | $x^2 y + y^2 z + z^2 t + t^3 x = 0$ | $(x : \xi_5 y : \xi_5^2 z : \xi_5^3 t)$ |

3. Del Pezzo surface of degree 3

In this Section we prove Theorem 1.2. We start from cyclic groups of prime order. The following theorem classifies actions of cyclic groups of prime order on smooth cubics.

**Theorem 3.1** (cf. [DI09a, Theorem 6.10]). Let a group $\mathfrak{C}_p$ of prime order $p$ act on a del Pezzo surface of degree 3. Then one can choose homogeneous coordinates $x, y, z, t$ in $\mathbb{P}^3_\mathbb{K}$ such that the equation of $\overline{X}$ and the action of $\mathfrak{C}_p$ are presented in Table 2, where $u, v, w, \alpha$ and $\beta$ are coefficients. These actions have different sets of fixed points on $\overline{X}$ and correspond to different conjugacy classes of cyclic subgroups in the Weyl group $W(E_6)$ acting on $\text{Pic}(\overline{X})$.

In this section we prove Theorem 1.2. Note that elements of type 3 and 4 of Table 2 have curves of fixed points $t = 0$ and $x = y = 0$ respectively. Therefore an element of order 3 having no curves of fixed points has type 5 of Table 2.

In the latter case the following lemma holds.

**Lemma 3.2.** Let a finite group $G$ act on a del Pezzo surface $X$ of degree 3 and $N \cong \mathfrak{C}_3$ be a normal subgroup in $G$ such that $N$ acts as in type 5 of Table 2. Then the surface $X/N$ is $G/N$-birationally equivalent to a del Pezzo surface of degree 3.

**Proof.** Let $\overline{X}$ be given by equation

$$x^3 + y^3 + zt(ux + vy) + z^3 + t^3 = 0$$

in $\mathbb{P}^3_\mathbb{K}$ and $N$ act as

$$(x : y : z : t) \mapsto (x : y : \omega z : \omega^2 t).$$

The fixed points of $N$ lie on the line $z = t = 0$. Thus $N$ has three fixed points $q_1, q_2$ and $q_3$. One can easily check that on the tangent spaces of $\overline{X}$ at these points $N$ acts as $\langle \text{diag}(\omega, \omega^2) \rangle$. Denote by $C_1$ and $C_2$ invariant curves $z = 0$ and $t = 0$ each passing through the three points $q_i$.

Let $f : X \to X/N$ be the quotient morphism and

$$\pi : \overline{X}/N \to X/N$$

be the minimal resolution of singularities. The curves $f(C_1)$ and $f(C_2)$ meet each other at the three singular points of $X/N$ and $f(C_1) \cdot f(C_2) = 1$. Thus two curves $\pi_*^{-1} f(C_j)$ are disjoint. Moreover (see Table 1), one has

$$\pi_*^{-1} f(C_j)^2 = f(C_j)^2 - 3 \cdot \frac{2}{3} = \frac{1}{3} C_j^2 - 2 = -1.$$
Therefore we can \(G/N\)-equivariantly contract the two \((-1)\)-curves \(\pi_j^{-1}f(C_j)\) and get a surface \(Y\) with \(K_Y^2 = 3\).

The surface \(X/N\) has only Du Val singularities. Therefore \(X/N\) is a singular del Pezzo surface and \(\tilde{X}/N\) is a weak del Pezzo surface containing exactly six curves \(\pi^{-1}(q_i)\) whose selfintersection is less than \(-1\). Thus \(Y\) does not contain curves with selfintersection less than \(-1\). So \(Y\) is a del Pezzo surface of degree 3. □

**Remark 3.3.** Note that in the notation of Lemma 3.2 there are two points on the surface \(Y\) where three \((-1)\)-curves meet each other. These points are images of \(\pi^{-1}f(C_j)\). Such a point is called an **Eckardt point** (see Definition 5.3 below).

**Remark 3.4.** Note that in the notation of Lemma 3.2 if \(\rho(X) > 1\) then \(X\) is not \(G\)-minimal by Theorem 2.7. Therefore the quotient of \(X/N\) is equivalent to a quotient of a del Pezzo surface with degree greater than 3 by a group of order 3. By Theorem 1.1 such a quotient is \(k\)-rational.

In Section 4 for non-\(k\)-rational quotient \(X/C_3\) of \(k\)-rational surface \(X\) we find restrictions on the image of the Galois group \(\text{Gal}(\overline{k}/k)\) in the Weyl group \(W(E_6)\) which acts on \(\text{Pic}(X)\).

Now we show that in all other cases of Theorem 1.2 the quotient of \(X\) is \(k\)-rational.

**Lemma 3.5.** Let a finite group \(G\) act on a del Pezzo surface \(X\) of degree 3 and \(N \cong C_p\) be a normal cyclic subgroup of prime order in \(G\) such that \(N\) acts not as in type 5 of Table 2. Then there exists a \(G/N\)-MMP-reduction \(Y\) of \(X/N\) such that \(K_Y^2 \geq 5\).

**Proof.** Let us consider the possibilities case by case.

In types 1 and 3 of Table 2 the group \(N\) has a pointwisely fixed the hyperplane section \(t = 0\). In type 3 there are no other fixed points and in type 1 there is only one other fixed point \((0:0:0:1)\). Therefore by the Hurwitz formula

\[
K_{X/\epsilon_2}^2 = \frac{1}{2} (2K_X)^2 = 6, \quad K_{X/\epsilon_3}^2 = \frac{1}{3} (3K_X)^2 = 9
\]

in types 1 and 3 respectively. The surface \(X/N\) has at most du Val singularities. Therefore for the minimal resolution of singularities \(\tilde{X}/N \to X/N\) one has \(K_{X/N}^2 = K_{\tilde{X}/N}^2\). Thus for any \(G/N\)-MMP-reduction \(Y\) of \(X/N\) one has \(K_Y^2 \geq 6\).

In type 2 of Table 2 the group \(C_2\) fixes pointwisely the lines \(x = y = 0\) and \(z = t = 0\). If one of these lines is tangent to \(X\) at a point \(t\) then in the neighbourhood of this point the group \(C_2\) acts as a reflection. Therefore there is a curve of \(C_2\)-fixed points passing through \(t\) contained in \(X\). Thus one of the pointwisely fixed lines is contained in \(X\). Therefore this line is defined over \(k\) and can be \(G\)-equivariantly contracted. If both lines \(x = y = 0\) and \(z = t = 0\) intersect the surface \(X\) transversally then there are six \(C_2\)-fixed points on \(X\) but by the Lefschetz fixed-point formula there are exactly four \(C_2\)-fixed points if \(C_2\) does not have pointwisely fixed curves.

So the quotient \(X/N\) is \(G/N\)-birationally equivalent to the quotient of del Pezzo surface of degree 4 by a group of order 2 having 4 fixed points one of which is \(G \times \text{Gal}(\overline{k}/k)\)-fixed. By Lemma 2.14 there exists a \(G/N\)-MMP-reduction \(Y\) of the latter quotient such that \(K_Y^2 = 8\).
In type 4 of Table 2 the group $C_3$ fixes pointwisely the lines $x = y = 0$ and $z = t = 0$. These lines intersect $X$ given by 

$$x^3 + y^3 + z^3 + t^3 = 0$$

at points $p_1$, $p_2$, $p_3$ and $q_1$, $q_2$, $q_3$ respectively. Let $C_{ij}$ be a line in $\mathbb{P}^3_k$ passing through $p_i$ and $q_j$.

Assume that $C_{ij}$ does not lie in $X$. For some integer $a$ the involution 

$$(x : y : z : t) \mapsto (\omega^a z : t : \omega^a x : y)$$

permutes points $p_i$ and $q_j$, thus the line $C_{ij}$ is invariant under the action of this involution. Therefore the line $C_{ij}$ cannot be tangent to $X$ at any of the points $p_i$ and $q_j$. Then the third point of intersection of $C_{ij}$ with $X$ is $C_3$-fixed. Thus there are three $C_3$-fixed points on $C_{ij}$ but this contradicts the fact that the action of $C_3$ is faithful on $C_{ij}$. So $C_{ij}$ lies in $X$ and $C_{ij}^2 = -1$.

Let $f : X \to X/N$ be the quotient morphism and 

$$\pi : \tilde{X}/N \to X/N$$

be the minimal resolution of singularities. Then $f(p_i)$ and $f(q_j)$ are singularities of type $\frac{1}{3}(1, 1)$. Thus $\pi^{-1}(C_{ij})$ are 9 disjoint $(-1)$-curves (see Table 1). We can contract these curves and get a surface $Y$. One has 

$$K_Y^2 = K_{\tilde{X}/N}^2 + 9 = K_{X/N}^2 + 9 - 6 \cdot \frac{1}{3} = \frac{1}{3} K_X^2 + 7 = 8.$$ 

In type 6 of Table 2 the group $C_5$ has two invariant lines $x = z = 0$ and $y = t = 0$ lying in $\tilde{X}$ given by the equation 

$$x^2 y + y^2 z + z^2 t + t^2 x = 0.$$ 

One can $G$-equivariantly contract this pair and get a del Pezzo surface of degree 5.

So the quotient $X/N$ is $G/N$-birationally equivalent to the quotient of del Pezzo surface of degree 5 by a group of order 5. By Theorem 1.1 this quotient is $k$-rational so it is $G/N$-birationally equivalent to a surface $Y$ such that $K_Y^2 \geq 5$.

**Corollary 3.6.** Let a finite group $G$ of order 6 act on a del Pezzo surface $X$ of degree 3. Then the surface $X/G$ is birationally equivalent to a surface $Y$ such that $K_Y^2 \geq 5$.

**Proof.** Let $N \subset G$ be the subgroup of order 3. Then by Lemmas 3.5 and 3.2 the quotient $X/N$ is $G/N$-birationally equivalent to a surface $\tilde{Z}$ such that either $K_{\tilde{Z}}^2 \geq 5$ or $\tilde{Z}$ is a del Pezzo surface of degree 3. There exists an MMP-reduction $Y$ of $X/G \approx Z/(G/N)$ such that $K_Y^2 \geq 5$ by Theorem 1.1 and Lemma 3.5 respectively. \hfill $\Box$

**Remark 3.7.** Note that for an element $g$ of type 3 of Table 2 the quotient $X/\langle g \rangle$ is isomorphic to $\mathbb{P}^2_k$. Therefore one has 

$$\rho \left( \tilde{X}/\langle g \rangle \right) = \rho \left( \mathbb{P}^2_k \right) = 1.$$ 

To prove Theorem 1.2 we need to list all possible automorphism groups of cubic surfaces.

**Theorem 3.8 (cf. [DI09a, Subsection 6.5, Table 4]).** Let $\tilde{X}$ be a del Pezzo surface of degree 3. Then one can choose homogeneous coordinates $x, y, z, t$ in $\mathbb{P}^3_k$ such that the
Table 3.

| Type | Group | Equation |
|------|-------|----------|
| I    | $\mathbb{C}_3 \rtimes \mathbb{S}_4$ | $x^3 + y^3 + z^3 + t^3 = 0$ |
| II   | $\mathbb{S}_5$ | $x^2y + y^2z + z^2t + t^2x = 0$ |
| III  | $H_3(3) \rtimes \mathbb{C}_4$ | $x^3 + y^3 + z^3 + \alpha xyz + t^3 = 0$ |
| IV   | $H_3(3) \rtimes \mathbb{C}_2$ | $x^3 + y^3 + z^3 + \alpha xyz + t^3 = 0$ |
| V    | $\mathbb{S}_4$ | $t(x^2 + y^2 + z^2) + \alpha xyz + t^4 = 0$ |
| VI   | $\mathbb{S}_3 \times \mathbb{C}_2$ | $x^3 + y^3 + \alpha zt(x + y) + z^3 + t^3 = 0$ |
| VII  | $\mathbb{S}_8$ | $x^3 + xy^2 + yz^2 + zt^2 = 0$ |
| VIII | $\mathbb{S}_3$ | $x^3 + y^3 + z^3 + \alpha xyz + t^3 + t^4 = 0$ |
| IX   | $\mathbb{C}_4$ | $x^3 + \alpha y^3 + xy^2 + yz^2 + zt^2 = 0$ |
| X    | $\mathbb{C}_2^3$ | $x^3 + y^3 + z^3 + \alpha xyz + t^2(x + y + uz) = 0$ |
| XI   | $\mathbb{C}_2$ | $x^3 + y^3 + z^3 + \alpha xyz + t^2(x + uy + vz) = 0$ |

equation of $\overline{X}$ and the full automorphism group $\text{Aut}(\overline{X})$ are presented in Table 3, where $u$, $v$ and $\alpha$ are coefficients, and $H_3(3)$ is a group generated by the transformation $$(x : y : z : t) \mapsto (x : \omega y : \omega^2 z : t)$$ and a cyclic permutation of $x$, $y$ and $z$.

In the paper [DI09a] there is one more column in this table which contains conditions on the parameters. But we are interested only in the structure of the group and its action on $\mathbb{P}^3_k$ so we omit this column.

**Lemma 3.9.** Let a finite group $G$ act on a del Pezzo surface $X$ of degree 3 and $N \cong \mathfrak{V}_4$ be a normal subgroup in $G$ such that nontrivial elements of $N$ act as in type 2 of Table 2. Then there exists a $G/N$-MMP-reduction $Y$ of $X/N$ such that $K_Y^2 = 6$.

**Proof.** One can choose coordinates in $\mathbb{P}^3_k$ in which $\overline{X}$ is given by the equation $$t(x^2 + y^2 + z^2) + \alpha xyz + t^3 = 0$$ and nontrivial elements of $\mathfrak{V}_4$ switch signs of $t$ and one other variable. The set of fixed by nontrivial elements of group $\mathfrak{V}_4$ points consists of three pointwisely fixed lines $$x = t = 0, \quad y = t = 0, \quad z = t = 0$$ lying in $\overline{X}$ and six isolated fixed points $(1 : 0 : 0 : \pm i), (0 : 1 : 0 : \pm i)$ and $(0 : 0 : 1 : \pm i)$. Thus the quotient $X/N$ is a singular del Pezzo surface with three $A_1$ singularities. By the Hurwitz formula $$K_{X/N}^2 = \frac{1}{4} (2K_X)^2 = 3.$$ Let $q_1$, $q_2$ and $q_3$ be singular points of $X/N$. Consider the anticanonical embedding $$X/N \hookrightarrow \mathbb{P}^3_k.$$ Denote by $C_{ij}$ a line in $\mathbb{P}^3_k$ passing through $q_i$ and $q_j$. Such a line contains two singular points on the surface $X/N$ of degree 3, therefore all lines $C_{ij}$ lie in $X/N$. Moreover, one has $$K_{X/N} : C_{ij} = -1.$$
Thus we can resolve the singularities of $X/N$, then $G/N$-equivariantly contract the proper transforms of $C_{ij}$ and get a surface $Y$ with $K_Y^2 = 6$.

\[\square\]

**Lemma 3.10.** Let a finite group $G$ act on a del Pezzo surface $X$ of degree 3 and $N \cong \mathfrak{A}_5$ be a normal subgroup in $G$. Then there exists a $G/N$-MMP-reduction $Y$ of $X/N$ such that $K_Y^2 \geq 6$.

**Proof.** Consider the group $\mathfrak{A}_5$ acting on $\mathbb{P}^2_\mathbb{k}$. By a direct computation one can show that each element $g$ of $\mathfrak{A}_5$ is a composition of two elements $h_1$ and $h_2$ of order two. Any element of order 2 in $\text{PGL}_3(\mathbb{k})$ has a line of fixed points. Thus each element $g$ of $\mathfrak{A}_5$ has an isolated fixed point $p$ which is the intersection point of the lines fixed by elements $h_1$ and $h_2$. The stabilizer group $N_p$ of $p$ acts on the tangent space of $\mathbb{P}^2_\mathbb{k}$ at $p$. Therefore $N_p$ is a subgroup in $\mathfrak{A}_5$ and $\text{GL}_2(\mathbb{k})$. Thus this is isomorphic to $\mathfrak{C}_5$, $\mathfrak{S}_3$ or $\mathfrak{D}_{10}$ if ord $g$ is 2, 3 or 5 respectively. The images of these group in $\text{GL}_2(\mathbb{k})$ are generated by reflections.

Consider such a point $p$ for an element $g$ of order 5. The stabilizer $N_p$ of $p$ is isomorphic to $\mathfrak{D}_{10}$ and $g$ acts in the tangent space of $\mathbb{P}^2_\mathbb{k}$ at $p$ as $\text{diag}(\xi_5, \xi_5)$. One can easily check that the action of $g$ in the tangent spaces of the two other fixed points is conjugate to $\text{diag}(\xi_5^2, \xi_5^2)$.

The group $\mathfrak{A}_5$ contains six subgroups isomorphic to $\mathfrak{C}_5$. Let $p_1, \ldots, p_6$ be fixed points of these subgroups whose stabilizers are isomorphic to $\mathfrak{D}_{10}$. Consider an $\mathfrak{A}_5$-equivariant blowup $\sigma : \overline{X} \to \mathbb{P}^2_\mathbb{k}$ of the points $p_1, \ldots, p_6$. The surface $\overline{X}$ is a del Pezzo surface of degree 3. From Theorem 3.8 one can see that there is a unique cubic surface with $\mathfrak{A}_5$ action on it. This cubic surface is called Clebsch cubic. We show that the action of $\mathfrak{A}_5$ on the Clebsch cubic is conjugate to the action of $\mathfrak{A}_5$ on the blowup $\overline{X} \to \mathbb{P}^2_\mathbb{k}$ at the six points $p_i$.

We use the notation of Remark 2.10 Let $g_2$, $g_3$ and $g_5$ be elements in $\mathfrak{A}_5$ of order 2, 3 and 5 respectively.

The stabilizer of any point $p_i$ in $\mathfrak{A}_5$ is isomorphic to $\mathfrak{D}_{10}$. Therefore there are 5 lines passing through the point $p_i$ that are pointwisely fixed by an element of order 2 in $\mathfrak{A}_5$. But in $\mathfrak{A}_5$ there are only 15 elements of order 2. Thus each element of order 2 fixes pointwisely a line passing through a pair of points $p_i$ and $p_j$ on $\mathbb{P}^2_\mathbb{k}$ and fixes pointwisely a $(-1)$-curve $L_{ij}$ on $\overline{X}$. By the Lefschetz fixed-point formula the element $g_2$ has three isolated fixed points. Two of them are $E_i \cap Q_j$ and $Q_i \cap E_j$ and the third is the preimage of the isolated fixed point $p$ of $g_2$ on $\mathbb{P}^2_\mathbb{k}$. In the tangent space of $\overline{X}$ at $p$ the stabilizer group $N_p$ of $p$ acts as $\mathfrak{C}_5^2$ generated by reflections.

An element $g_3$ does not have any invariant $(-1)$-curve. Therefore it cannot have curves of fixed points. Thus the action of $g_3$ on $\mathbb{P}^2_\mathbb{k}$ is conjugate to $\text{diag}(1, \omega, \omega^2)$. Hence on the surface $\overline{X}$ the element $g_3$ has three isolated points and acts in the tangent spaces of $\mathbb{P}^2_\mathbb{k}$ at these points as $\text{diag}(\omega, \omega^2)$. These points cannot be points of the blowup since the stabilizer of any point $p_i$ in $\mathfrak{A}_5$ is $\mathfrak{D}_{10}$. Therefore there are three $g_3$-fixed points on $\overline{X}$ and in the tangent spaces of $\overline{X}$ at these points as $\text{diag}(\omega, \omega^2)$. Two of these points do not lie on $(-1)$-curves and the third one is a point of intersection of three $(-1)$-curves.

An element $g_5$ has three fixed points on $\mathbb{P}^2_\mathbb{k}$, namely $p_k$ for some $k \in \{1, 2, 3, 4, 5, 6\}$ and two points in whose tangent space the action of $g_5$ is conjugate to $\text{diag}(\xi_5^2, \xi_5^2)$. The $(-1)$-curve $Q_k$ is $g_5$-invariant thus the quadric $\sigma(Q_k)$ passes through two $g_5$-fixed points different from $p_k$. Therefore the element $g_5$ has four fixed points on $\overline{X}$, two of them lie...
on $E_k$ and two on $Q_k$. The element $g_5$ acts in the tangent spaces of $X$ at $g_5$-fixed points as diag($\xi_5, \xi_5^2$).

Let $f : X \to X/N$ be the quotient morphism,

$$\pi : \tilde{X}/N \to X/N$$

be the minimal resolution of singularities, and put $E = f(E_i), Q = f(Q_j)$. There are four singular points on $X/N$: two singular points of type $\frac{1}{5}(1, 2)$ lie on the curves $E$ and $Q$ respectively, one singular point of type $A_1$ is the intersection point $E \cap Q$ and one singular point of type $A_2$ lies neither on $E$ nor on $Q$. We have (see Table 1):

$$K^2_{\tilde{X}/N} = K^2_{X/N} - \frac{4}{5} = \frac{1}{60}(6K_X)^2 - \frac{4}{5} = 1,$$

$$\rho(\tilde{X}/N)^{G/N} \geq \rho(X/N)^{G/N} + 4 = \rho(X)^G + 4 \geq 5,$$

$$\pi_*^{-1}(E)^2 = E^2 - \frac{1}{2} - \frac{2}{5} = \frac{1}{60}\left(\sum_{i=1}^{6} E_i\right)^2 \geq \frac{9}{10} = -1,$$

$$\pi_*^{-1}(Q)^2 = Q^2 - \frac{1}{2} - \frac{2}{5} = \frac{1}{60}\left(\sum_{i=1}^{6} Q_i\right)^2 \geq \frac{9}{10} = -1.$$ 

Thus we can $G/N$-equivariantly contract curves $\pi_*^{-1}(E)$ and $\pi_*^{-1}(Q)$. We obtain a surface $Z$ such that $K^2_Z = 3$ and $\rho(Z)^{G/N} \geq 4$. By Corollary 2.9 there exists a $G/N$-minimal model $Y$ of $Z$ such that $K^2_Y \geq 6$.

Now we prove Theorem 1.2.

**Proof of Theorem 1.2.** We consider each case of Theorem 3.8 and show that if the group $G$ is not trivial and is not conjugate to $C_3$ acting as in type 5 of Table 2 then there exists a $G/N$-MMP-reduction $Y$ of $X/N$ such that $K^2_Y \geq 5$.

In case I the group $\text{Aut}(X)$ is $C_3 \rtimes S_4$. The group $C_3$ is a diagonal subgroup of $\text{PGL}_4(k)$ and $S_4$ permutes coordinates. We consider a normal subgroup $H = G \cap C_3^3$. If there is an element of type 3 in this subgroup then we consider a group $N \subset H$ generated by elements of type 3. Then the group $N$ is normal and one of the following possibilities holds:

- If $N$ is generated by one element of type 3 then $N \cong C_3$, the quotient $X/N$ is smooth and by the Hurwitz formula one has

$$K^2_{X/N} = \frac{1}{3}(3K_X)^2 = 9.$$ 

- If $N$ is generated by two elements of type 3 then $N \cong C_3^2$, the quotient $X/N$ has only one singular point of type $\frac{1}{5}(1, 1)$ and by the Hurwitz formula one has

$$K^2_{X/N} = \frac{1}{9}(5K_X)^2 = \frac{25}{3}.$$ 

- If $N$ is generated by three elements of type 3 then $N \cong C_3^3$, the quotient $X/N$ is smooth and by the Hurwitz formula one has

$$K^2_{X/N} = \frac{1}{27}(9K_X)^2 = 9.$$
For any $G/N$-MMP-reduction $Y$ of $X/N$ one has $K_Y^2 \geq 8$.

If the group $H$ does not contain elements of type 3 then either $H$ is trivial or $H$ is isomorphic to $\mathfrak{c}_3$ generated by an element of type 4 or 5 or $\mathfrak{c}_3^2$ generated by the elements $\text{diag}(1,1,\omega,\omega)$ and $\text{diag}(1,\omega,1,\omega)$.

In the last case the group $G$ is a subgroup of $\mathfrak{c}_3^2 \times \mathfrak{d}_8$ where $\mathfrak{d}_8 = \langle (1243), (23) \rangle$. If $G$ contains the element $(14)(23)$ then the group $N = \langle (14)(23) \rangle$ is normal in $G$ and there exists a $G/N$-MMP-reduction $Y$ of $X/N$ such that $K_Y^2 \geq 5$ by Lemma 3.5. Otherwise the group $G$ is isomorphic to $\mathfrak{c}_3^2 \times \mathfrak{c}_2$ or $\mathfrak{c}_3^2$.

If $G$ contains a normal subgroup $N \cong \mathfrak{c}_3$ generated by element of type 4 then there exists a $G/N$-MMP-reduction $Y$ of $X/N$ such that $K_Y^2 \geq 5$ by Lemma 3.5. Otherwise $G$ is conjugate to $\mathfrak{c}_3^2 \times \mathfrak{c}_2$ where $\mathfrak{c}_2$ is either $\langle (23) \rangle$ or $\langle (14) \rangle$. In this case $G$ contains a normal subgroup $N \cong \mathfrak{c}_3$ generated by an element of type 5. By Lemma 3.2 the quotient $X/N$ is $G/N$-birationally equivalent to a del Pezzo surface $Z$ of degree 3 and the group $G/N \cong \mathfrak{g}_3$. The quotient $Z/\mathfrak{g}_3$ is $k$-birationally equivalent to a surface $Y$ with $K_Y^2 \geq 5$ by Corollary 3.6.

If the group $H \cong \mathfrak{c}_3$ is generated by an element of type 4 then there exists a $G/H$-MMP-reduction $Y$ of $X/H$ such that $K_Y^2 \geq 5$ by Lemma 3.5.

If the group $H \cong \mathfrak{c}_3$ is generated by an element of type 5 then $G \subset \mathfrak{g}_3 \times \mathfrak{c}_2$ and the quotient $X/H$ is $G/H$-birationally equivalent to a del Pezzo surface $Z$ of degree 3 by Lemma 3.2. Thus if $G/H$ is nontrivial then it contains a subgroup $N$ of order 2. There exists a $G/N$-MMP-reduction $Y$ of $X/N$ such that $K_Y^2 \geq 5$ by Lemma 3.5.

If the group $H$ is trivial then $G$ is a subgroup of $\mathfrak{g}_4$. Then the group $G$ contains a normal subgroup $N$ isomorphic to $\mathfrak{c}_2 \times \mathfrak{c}_3 \times \mathfrak{c}_4 \times \mathfrak{c}_5$ or $\mathfrak{e}_5$ by Lemma 2.11. If $N \cong \mathfrak{c}_2 \times \mathfrak{c}_3$ or $N \cong \mathfrak{c}_4$ then there exists a $G/N$-MMP-reduction $Y$ of $X/N$ such that $K_Y^2 \geq 5$ by Lemma 3.5 or Lemma 3.9 respectively. If $N \cong \mathfrak{c}_3$ then either $G$ is generated by an element of type 5 or $G \cong \mathfrak{g}_3$ and there exists an MMP-reduction $Y$ of $X/G$ such that $K_Y^2 \geq 5$ by Corollary 3.6.

In case II the group $G$ contains a normal subgroup $N$ isomorphic to $\mathfrak{c}_2 \times \mathfrak{c}_3 \times \mathfrak{c}_4 \times \mathfrak{c}_5$ or $\mathfrak{e}_5$ by Lemma 2.11. If $N$ is not isomorphic to $\mathfrak{c}_3$ then there exists a $G/N$-MMP-reduction $Y$ of $X/N$ such that $K_Y^2 \geq 5$ by Lemma 3.5, Lemma 3.9 or Lemma 3.10. Otherwise $N \cong \mathfrak{c}_3$ and $G$ is a subgroup of $\mathfrak{g}_3 \times \mathfrak{c}_2$. Subgroups of this group are considered in case I.

In cases III and IV let us consider the group $H = G \cap H_3(3)$. If $\text{ord} \ H > 3$ then $H$ contains a normal subgroup $N$ generated by an element of order 3 acting as in type 3 of Table 2. The group $N$ is normal in $G$ and there exists a $G/N$-MMP-reduction $Y$ of $X/N$ such that $K_Y^2 \geq 5$ by Lemma 3.5.

If $H$ is generated by an element of type 5 and $G/H$ is not trivial then $G \cong \mathfrak{g}_3$ and there exists an MMP-reduction $Y$ of $X/G$ such that $K_Y^2 \geq 5$ by Corollary 3.6.

If $H$ is trivial and $G$ is not trivial then $G \cong \mathfrak{g}_4$ contains a normal subgroup $N$ of order two and there exists an $G/N$-MMP-reduction $Y$ of $X/N$ such that $K_Y^2 \geq 5$ by Lemma 3.5.

In case VII if $G$ is not trivial then $G$ contains a normal subgroup $N$ of order two and there exists an $G/N$-MMP-reduction $Y$ of $X/N$ such that $K_Y^2 \geq 5$ by Lemma 3.5.

In the other cases the group $G$ is conjugate in $\text{PGL}_4(\kappa)$ to a subgroup of $\mathfrak{g}_5$. All these possibilities were considered in case II.
In all cases one has $Y(k) \neq \emptyset$ since $X(k) \neq \emptyset$. Thus
$$Y/(G/N) \approx X/G$$
is $k$-rational by Theorem 1.1.

4. Minimality conditions

Let $X$ be a cubic surface in $\mathbb{P}^3_k$ and let a group $G \cong C_3$ act on $X$ as in type 5 of Table 2. In this section we find some conditions for the action of the Galois group $\text{Gal}(k/k)$ on the set of $(-1)$-curves under which the surface $X$ is $k$-rational and $X/G$ is not $k$-rational.

Throughout this section we use the notation of Remark 2.10. Let $\Gamma$ be the image of the Galois group $\text{Gal}(k/k)$ in the Weyl group $W(E_6)$ acting on $\text{Pic}(X)$ (see [Man74, §IV.3]). The group $\Gamma$ effectively acts on the set of $(-1)$-curves on $X$. The group $W(E_6)$ contains a subgroup $S_6$ acting in the following way: for $\sigma \in S_6$ one has
$$\sigma E_i = E_{\sigma(i)}, \quad \sigma L_{ij} = L_{\sigma(i)\sigma(j)}, \quad \sigma Q_i = Q_{\sigma(i)}.$$

**Lemma 4.1.** The image of the group $G$ in the Weyl group $W(E_6)$ is conjugate to $\langle (123)(456) \rangle$.

**Proof.** The order of the Weyl group $W(E_6)$ is equal to $51840 = 2^7 \cdot 3^4 \cdot 5$. By Sylow theorem all groups of order 81 are conjugate in $W(E_6)$. The group of order 81 acts on the Fermat cubic (see Table 3)
$$x^3 + y^3 + z^3 + t^3 = 0.$$ Thus any element of order 3 in $W(E_6)$ has type 3, 4 or 5 of Table 2.

For any element $g$ of type 3 of Table 2 one has $\rho(X)^{(g)} = 1$ by Remark 3.7. In the subgroup $S_6 \subset W(E_6)$ there is an element $(123)(456)$ which has type 4 or 5 of Table 2 since
$$\rho(X)^{((123)(456))} > 1.$$ But an element of type 4 has invariant $(-1)$-curves (see the proof of Lemma 3.5) and the element $(123)(456)$ does not have invariant $(-1)$-curves. Therefore the action of the group $G$ is conjugate to $\langle (123)(456) \rangle$ in the group $W(E_6)$.

**Remark 4.2.** An alternative proof of Lemma 4.1 is the following. One can look at Table 1 in [Man74, Chapter IV, §5] and see that conjugacy classes of elements of order 3 in the group $W(E_6)$ correspond to the rows 3, 18 and 22 of this table. But in the eighth column of the table one can see that the element $g$ correspond to the 18-th row. In the ninth column one can see that $g$ is conjugate to $(123)(456)$ in $W(E_6)$. Also one can see that the order of the centralizer of $G$ in $W(E_6)$ is 108.

From now on we can assume that the group $G$ acts on the set of $(-1)$-curves on $X$ as $\langle (123)(456) \rangle$. The Galois group $\text{Gal}(k/k)$ commutes with the group $G$. Therefore to describe possibilities for the group $\Gamma$ we should find the centralizer of $G$ in $W(E_6)$.

**Lemma 4.3.** The centralizer of $G = \langle (123)(456) \rangle$ in $S_6$ is a subgroup $C_3^2 \rtimes C_2$ generated by $a = (123), b = (456)$ and $c = (14)(25)(36)$. 

13
Proof. Note that in the group \( \mathfrak{S}_6 \) there are
\[
\frac{6!}{3! \cdot 3! \cdot 2} = 40
\]
elements conjugate to \((123)(456)\). Therefore the order of the centralizer of \( G = \langle (123)(456) \rangle \) is equal to 18.

The elements \( a, b \) and \( c \) obviously commute with the element \((123)(456)\) and the group \( \mathfrak{C}_3^2 \times \mathfrak{C}_2 = \langle a, b, c \rangle \) has order 18. Thus the centralizer of \( G = \langle (123)(456) \rangle \) in \( \mathfrak{S}_6 \) is a subgroup \( \mathfrak{C}_3^2 \times \mathfrak{C}_2 = \langle a, b, c \rangle \).

\( \square \)

Note that the group \( G \) has exactly three orbits that consist of \((-1\)-curves meeting each other: \( \{L_{14}, L_{25}, L_{36}\} \), \( \{L_{15}, L_{26}, L_{34}\} \) and \( \{L_{16}, L_{24}, L_{35}\} \). The other orbits consist of disjoint \((-1\)-curves. Therefore the set of nine \((-1\)-curves
\[ L_{ij}, \text{ where } i \in \{1, 2, 3\} \text{ and } j \in \{4, 5, 6\} \]
is invariant under the action of centralizer of \( G \) in \( W(E_6) \). The group
\[ \mathfrak{C}_3^2 \times \mathfrak{C}_2 \cong \langle a, b, c \rangle \]
can realize any permutation of this set of \((-1\)-curves that preserves the intersection form. Therefore to find the centralizer of \( G \) in \( W(E_6) \) we should find a subgroup in \( W(E_6) \) acting trivially on the set of nine \((-1\)-curves \( L_{ij} \), where \( i \in \{1, 2, 3\} \) and \( j \in \{4, 5, 6\} \).

Lemma 4.4. The subgroup of \( W(E_6) \) fixing each of the nine \((-1\)-curves \( L_{ij} \), where \( i \in \{1, 2, 3\} \) and \( j \in \{4, 5, 6\} \), is a group \( \mathfrak{S}_3 \) generated by elements \( r \) and \( s \) of order 3 and 2 respectively acting on the set of \((-1\)-curves in the following way:
\[
 s (E_i) = Q_i, \quad s (Q_i) = E_i, \quad s (L_{ij}) = L_{ij},
 r (E_i) = Q_i \text{ if } i \in \{1, 2, 3\}, \quad r^2 (E_i) = Q_i \text{ if } i \in \{4, 5, 6\},
 r (E_i) = L_{jk} \text{ if } i \in \{4, 5, 6\} \text{ and } j, k \in \{4, 5, 6\} \text{ differ from } i,
 r^2 (E_i) = L_{jk} \text{ if } i \in \{1, 2, 3\} \text{ and } j, k \in \{1, 2, 3\} \text{ differ from } i.
\]

Proof. Let us consider the \((-1\)-curve \( E_1 \). Since \( L_{14}, L_{15} \) and \( L_{16} \) are invariant, the image of \( E_1 \) can be only \( E_1, L_{23} \) or \( Q_1 \). The action of the group on these three \((-1\)-curves defines the whole action of the group \( \langle r, s \rangle \) on the set of \((-1\)-curves. The group \( \mathfrak{S}_3 = \langle r, s \rangle \) fixes all the nine \((-1\)-curves \( L_{ij} \), where \( i \in \{1, 2, 3\} \) and \( j \in \{4, 5, 6\} \), and permutes \( E_1, L_{23} \) and \( Q_1 \) in all possible ways.

\( \square \)

Proposition 4.5. The centralizer of \( G = \langle (123)(456) \rangle \) in \( W(E_6) \) is a subgroup
\[ H \cong \left( \mathfrak{C}_3^2 \times \mathfrak{C}_2 \right) \times \mathfrak{S}_3 \]
where the first factor is generated by \( a, b \) and \( cs \), and the second factor is generated by \( r \) and \( s \).

Proof. By Lemmas 4.3 and 4.4 the centralizer of \( G \) in \( W(E_6) \) is generated by the subgroups \( \mathfrak{C}_3^2 \times \mathfrak{C}_2 = \langle a, b, c \rangle \) and \( \mathfrak{S}_3 = \langle r, s \rangle \). Obviously, the elements \( a, b, cs \) and \( r, s \) generate this group. One can easily check that \( a, b, cs \) and \( r, s \) pairwisely commute.

\( \square \)
By Remark 3.4 if the quotient of $X$ is not $k$-rational then $\rho(X)^G = 1$. Moreover, if $\rho(X) = 1$ then $X$ is not $k$-rational by Theorem 2.8. Thus to construct non-$k$-rational quotient of a $k$-rational cubic surface we should find all possibilities of the Galois group $\Gamma$ such that $\rho(X) > 1$ and $\rho(X)^G = 1$.

The group $\Gamma$ is a subgroup of the group $H \cong (\mathcal{C}_3^2 \rtimes \mathcal{C}_2) \rtimes \mathcal{S}_3$ where the first factor is generated by $a$, $b$ and $cs$, and the second factor is generated by $r$ and $s$. We denote the projection on the first factor
\[ H \cong (\mathcal{C}_3^2 \rtimes \mathcal{C}_2) \times \mathcal{S}_3 \rightarrow \mathcal{C}_3^2 \rtimes \mathcal{C}_2 \]
by $\pi_1$, and the projection on the second factor
\[ H \cong (\mathcal{C}_3^2 \rtimes \mathcal{C}_2) \times \mathcal{S}_3 \rightarrow \mathcal{S}_3 \]
by $\pi_2$.

**Lemma 4.6.** If $\pi_2(\Gamma)$ is trivial and $\pi_1(\Gamma) \subset \mathcal{C}_3^2$ or $\pi_2(\Gamma) = \langle s \rangle$ but $s \notin \Gamma$ then $\rho(X)^G > 1$.

**Proof.** In these cases the group $\Gamma$ is a subgroup of the group $\langle a, b, c \rangle$. Therefore the groups $\Gamma$ and $G \cong \mathcal{C}_3$ preserve $\sum_{i=1}^{6} E_i$ and $\rho(X)^G > 1$. \hfill $\square$

**Lemma 4.7.** If $\pi_2(\Gamma) = \langle s \rangle$ and $\pi_1(\Gamma) \subset \mathcal{C}_3^2$ then $\rho(X)^G > 1$.

**Proof.** In this case the groups $\Gamma$ and $G$ preserve the divisor $\sum_{i=1}^{3} E_i - \sum_{i=4}^{6} E_i$. Therefore one has $\rho(X)^G > 1$. \hfill $\square$

**Lemma 4.8.** One has $\rho(X)^{\langle abcs \rangle} = \rho(X)^{\langle a^2b, cs \rangle} = 1$.

**Proof.** One has $(abcs)^4 = ab$ and $(abcs)^3 = cs$.

Note that the groups Pic$(X)^{\langle ab \rangle}$ and Pic$(X)^{\langle a^2b \rangle}$ are generated by $K_X$, $\sum_{i=1}^{3} E_i$ and $\sum_{i=4}^{6} E_i$. One can check that $cs$-invariants in Pic$(X)^{\langle ab \rangle}$ and Pic$(X)^{\langle a^2b \rangle}$ are generated by $K_X$. \hfill $\square$

**Corollary 4.9.** Suppose that $\pi_1(\Gamma)$ contains the element $cs$ and at least one element of order 3 and $cs \in \Gamma$ then $\rho(X) = 1$.

**Lemma 4.10.** One has $\rho(X)^{\langle abr \rangle} = \rho(X)^{\langle a^2br \rangle} = 1$.

**Proof.** Note that any $abr$ and $a^2br$ orbit of a $(-1)$-curve consists of three $(-1)$-curves meeting each other. Therefore these elements has type 3 of Table 2. Thus by Remark 3.7 one has
\[ \rho(X)^{\langle abr \rangle} = \rho(X)^{\langle a^2br \rangle} = 1. \]
\hfill $\square$

**Corollary 4.11.** If $r \in \pi_2(\Gamma)$, and $\pi_1(\Gamma)$ contains $ab$ or $a^2b$ then $\rho(X) = 1$.

As a result of all the previous lemmas we have the following proposition.

**Proposition 4.12.** Let $X$ be a del Pezzo surface of degree 3 and $G \cong \mathcal{C}_3$ be a group acting as in type 5 of Table 2. Let $\Gamma$ be the image of the Galois group Gal$(\overline{k}/k)$ in the Weyl group $W(E_6)$. If $\rho(X) > 1$ and $\rho(X)^G = 1$ then we have the following possibilities for $\Gamma$ up to conjugation:
(1) $\Gamma = \langle cs \rangle \cong \mathbb{C}_2$;
(2) $\Gamma = \langle c, s \rangle \cong \mathbb{C}_2^2$;
(3) $\Gamma = \langle r \rangle \cong \mathbb{C}_3$;
(4) $\Gamma = \langle ar \rangle \cong \mathbb{C}_3$;
(5) $\Gamma = \langle a, r \rangle \cong \mathbb{C}_3^2$;
(6) $\Gamma = \langle cs, r \rangle \cong \mathbb{C}_6$;
(7) $\Gamma = \langle r, s \rangle \cong \mathbb{S}_3$;
(8) $\Gamma = \langle a, r, s \rangle \cong \mathbb{C}_3 \times \mathbb{S}_3$;
(9) $\Gamma = \langle r, c \rangle \cong \mathbb{S}_3$;
(10) $\Gamma = \langle r, c, s \rangle \cong \mathbb{S}_3 \times \mathbb{C}_2$.

Proof. At first we show that in all other cases one has either $\rho(X) = 1$ or $\rho(X)^G > 1$.

If $r \notin \pi_2(\Gamma)$ then if $\pi_1(\Gamma)$ contains an element $ab$ or $a^2b$ then $\rho(X) = 1$ by Corollary 4.11. Therefore in this case $\pi_1(\Gamma)$ should be trivial or conjugate to $\langle a \rangle$ or $\langle cs \rangle$. These possibilities correspond to cases (3)–(10) of the proposition.

Now we can assume that $r \notin \pi_2(\Gamma)$. If $\pi_1(\Gamma)$ contains the element $cs$ and at least one element of order 3 then by Corollary 4.9 one has $\rho(X) = 1$ in all cases except $\Gamma = \langle ab, c \rangle$, $\Gamma = \langle a^2b, c \rangle$ and $\Gamma = \langle a, b, c \rangle$. In the last three cases we have $\rho(X)^G > 1$ by Lemma 4.6. If $cs \notin \pi_1(\Gamma)$ then $\rho(X)^G > 1$ by Lemmas 4.6 and 4.7. Therefore $\pi_1(\Gamma) = \langle cs \rangle$. This possibility corresponds to cases (1) and (2) of the proposition.

Now we show that in all these cases one has $\rho(X) > 1$ and $\rho(X)^G = 1$.

In cases (1), (2), (6), (9) and (10) the $(-1)$-curves $L_{14}$, $L_{25}$ and $L_{36}$ are $\Gamma$-invariant. Therefore $X$ is not minimal and $\rho(X) > 1$. In cases (3), (4), (5), (7) and (8) the triple of disjoint $(-1)$-curves $L_{14}$, $L_{24}$ and $L_{34}$ is $\Gamma$-invariant. Therefore $X$ is not minimal and $\rho(X) > 1$.

In cases (1) and (2) one has $\rho(X)^G = 1$ by Lemma 4.8. In the other cases one has $\rho(X)^G = 1$ by Lemma 4.10.

Note that if one can contract a $(-1)$-curve defined over $k$ and $\rho(X) = 2$ or $\rho(X) = 3$ then the obtained del Pezzo surface either is not minimal or can be minimal del Pezzo surface of degree 4. So for each case of Proposition 4.12 we should check whether the surface $X$ is $k$-rational.

Lemma 4.13. If the Galois group $\Gamma$ contains the element $cs$ then $X$ is not $k$-rational.

Proof. Note that the $(-1)$-curves $L_{14}$, $L_{25}$ and $L_{36}$ are $cs$-invariant and the other $(-1)$-curves form $cs$-invariant pairs of $(-1)$-curves which are not disjoint. The curves $L_{14}$, $L_{25}$ and $L_{36}$ meet each other. Therefore we can contract no more than one $(-1)$-curve and $X$ is not $k$-rational by Theorem 2.8.

Lemma 4.14. If the Galois group $\Gamma$ contains the elements $c$ and $r$ then $X$ is not $k$-rational.

Proof. Note that if a $\langle c, r \rangle$-orbit contains $E_i$ then it contains $Q_j$ with $i \neq j$. Therefore we cannot contract any of these orbits. Also we cannot contract $\langle c, r \rangle$-invariant pairs $L_{15}$ and $L_{24}$, $L_{16}$ and $L_{34}$, $L_{26}$ and $L_{35}$. The $(-1)$-curves $L_{14}$, $L_{25}$ and $L_{36}$ are $\langle c, r \rangle$-invariant and meet each other. Therefore we cannot contract more than one $(-1)$-curve and $X$ is not $k$-rational by Theorem 2.8.

Lemma 4.15. If the Galois group $\Gamma$ is contained in $\langle a, r, s \rangle$ then $X$ is $k$-rational.
Proof. We can contract \((-1)\)-curves $E_4$, $L_{56}$ and $Q_4$, and get a del Pezzo surface of degree 6 which is $\mathbb{k}$-rational by Theorem 2.8. □

Now we can prove the following theorem.

**Theorem 4.16.** Let $X$ be a $\mathbb{k}$-rational del Pezzo surface of degree 3 and $G \cong \mathfrak{S}_3$ be a group acting as in type 5 of Table 2. Let $\Gamma$ be the image of the Galois group $\text{Gal}(\mathbb{K}/\mathbb{k})$ in the Weyl group $W(E_6)$. If $X/G$ is not $\mathbb{k}$-rational then we have the following possibilities for $\Gamma$ up to conjugation:

1. $\Gamma = \langle r \rangle \cong \mathfrak{C}_3$;
2. $\Gamma = \langle ar \rangle \cong \mathfrak{C}_3$;
3. $\Gamma = \langle a, r \rangle \cong \mathfrak{C}_2 \times \mathfrak{C}_3$;
4. $\Gamma = \langle r, s \rangle \cong \mathfrak{S}_3$;
5. $\Gamma = \langle a, r, s \rangle \cong \mathfrak{C}_3 \times \mathfrak{S}_3$.

Proof. For cases (1), (2), (6), (10) of Proposition 4.12 the Galois group $\Gamma$ contains the element $cs$. Therefore $X$ is not $\mathbb{k}$-rational by Lemma 4.13. For case (9) of Proposition 4.12 the surface $X$ is not $\mathbb{k}$-rational by Lemma 4.14.

For cases (3), (4), (5), (7), (8) of Proposition 4.12 the Galois group $\Gamma$ is contained in $\langle a, r, s \rangle \cong \mathfrak{C}_3 \times \mathfrak{S}_3$. Therefore $X$ is $\mathbb{k}$-rational by Lemma 4.15. □

5. Geometric interpretation

In this section we give geometric interpretation of the actions of elements in the Galois group $\Gamma$ considered in Section 4.

For convenience we assume that the field $\mathbb{k}$ contains $\omega$. Therefore we can choose homogeneous coordinates in $\mathbb{P}^3_{\mathbb{k}}$ such that the group $G$ acts as

$$(x : y : z : t) \mapsto (x : y : \omega z : \omega^2 t)$$

on the cubic surface $X$ given by the equation

$$P(x : y) + zt(ux + vy) + z^3 + \alpha t^3 = 0 \quad (5.1)$$

where $P(x : y)$ is a homogeneous polynomial of degree 3.

Let us consider a line $x = y = 0$. This line intersects $X$ in three points $e_1$, $e_2$ and $e_3$ given by the equation

$$z^3 + \alpha t^3 = 0. \quad (5.2)$$

**Definition 5.3.** A point $p$ on a cubic surface is called an *Eckardt point* if there are three \((-1)\)-curves passing through $p$.

**Lemma 5.4.** The points $e_1$, $e_2$ and $e_3$ are Eckardt points.

Proof. The surface $X$ is given by equation (5.1) in $\mathbb{P}^3_{\mathbb{k}}$. In coordinates $x$, $y$, $z$, $t$ the points $e_1$, $e_2$ and $e_3$ are $(0 : 0 : -\sqrt[3]{\alpha} : 1)$, $(0 : 0 : -\omega \sqrt[3]{\alpha} : 1)$ and $(0 : 0 : -\omega^2 \sqrt[3]{\alpha} : 1)$. Consider the tangent plane at the point $e_1$. Its equation is

$$u \sqrt[3]{\alpha} x + v \sqrt[3]{\alpha} y = 3(\sqrt[3]{\alpha} z + \alpha t).$$

We have

$$zt(ux + vy) + z^3 + \alpha t^3 = 3zt(\sqrt[3]{\alpha} z + \sqrt[3]{\alpha^2} t) + z^3 + \alpha t^3 =$$
\[(z + \sqrt[3]{\alpha} t)^3 = \left(\frac{\sqrt[3]{\alpha^{-1}} u x + \sqrt[3]{\alpha^{-1}} vy}{3}\right)^3.\]

So that equation (5.1) can be rewritten as

\[P(x : y) + \frac{(ux + vy)^3}{27\alpha} = 0.\] (5.5)

The last equation has three roots \((\lambda_1 : \mu_1), (\lambda_2 : \mu_2)\) and \((\lambda_3 : \mu_3)\). The three \((-1)\)-curves passing through the point \(e_1\) are given by

\[u\sqrt[3]{\alpha} x + v\sqrt[3]{\alpha} y = 3(\sqrt[3]{\alpha^2} z + \alpha t)\]

and \(u, x = \lambda_i y\). Similarly we can show that the three \((-1)\)-curves passing through \(e_2\) are given by

\[u\sqrt[3]{\alpha} x + v\sqrt[3]{\alpha} y = 3(\omega\sqrt[3]{\alpha^2} z + \omega^2 t)\]

and \(u, x = \lambda_i y\), and the three \((-1)\)-curves passing through \(e_3\) are given by

\[u\sqrt[3]{\alpha} x + v\sqrt[3]{\alpha} y = 3(\omega^2\sqrt[3]{\alpha^2} z + \omega t)\]

and \(u, x = \lambda_i y\).

\[\square\]

Remark 5.6. Applying explicit equations given in the proof of Lemma 5.4 one can see that the \(G\)-orbit of any \((-1)\)-curve passing through a point \(e_i\) consists of three \((-1)\)-curves meeting each other at a point. The image of \(G\) in the Weyl group \(W(E_6)\) is conjugate to \(((123)(456))\) by Lemma 4.1. Therefore nine curves passing through the Eckardt points \(e_i\) are \(L_{ij}\) where \(i \in \{1, 2, 3\}\) and \(j \in \{4, 5, 6\}\). We can assume that the curves \(L_{14}, L_{26}\) and \(L_{35}\) pass through \(e_1\), the curves \(L_{16}, L_{25}\) and \(L_{34}\) pass through \(e_2\) and the curves \(L_{15}, L_{24}\) and \(L_{36}\) pass through \(e_3\).

Now we give explicit geometric interpretation of the action of the group \(\pi_1(\Gamma)\).

Lemma 5.7. Let \(X\) be a cubic surface given by equation (5.1) and \(\Gamma\) be the image of the Galois group \(\text{Gal}(\mathbb{K}/k)\) in the Weyl group \(W(E_6)\). Let \(\Gamma_1\) and \(\Gamma_2\) be the Galois groups of equations (5.2) and (5.5) respectively. Then in the notation of Section 4 one has the following:

- if \(\pi_1(\Gamma)\) is trivial then \(\Gamma_1\) and \(\Gamma_2\) are trivial;
- if \(\pi_1(\Gamma) = \langle cs \rangle \cong \mathcal{C}_3\) then \(\Gamma_1\) is trivial and \(\Gamma_2 \cong \mathcal{C}_2\);
- if \(\pi_1(\Gamma) = \langle ab \rangle \cong \mathcal{C}_3\) then \(\Gamma_1 \cong \mathcal{C}_3\) and \(\Gamma_2\) is trivial;
- if \(\pi_1(\Gamma) = \langle a \rangle \cong \mathcal{C}_3\) then \(\Gamma_1 \cong \mathcal{C}_3\), \(\Gamma_2 \cong \mathcal{C}_3\) and equations (5.2) and (5.5) have the same splitting field;
- if \(\pi_1(\Gamma) = \langle a^2b, cs \rangle \cong \mathcal{G}_3\) then \(\Gamma_1\) is trivial and \(\Gamma_2 \cong \mathcal{G}_3\);
- if \(\pi_1(\Gamma) = \langle ab, cs \rangle \cong \mathcal{C}_6\) then \(\Gamma_1 \cong \mathcal{C}_3\) and \(\Gamma_2 \cong \mathcal{C}_2\);
- if \(\pi_1(\Gamma) = \langle a, b \rangle \cong \mathcal{C}_3^2\) then \(\Gamma_1 \cong \mathcal{C}_3\), \(\Gamma_2 \cong \mathcal{C}_3\) and equations (5.2) and (5.5) have different splitting fields;
- if \(\pi_1(\Gamma) = \langle a, b, cs \rangle \cong \mathcal{C}_3^2 \times \mathcal{C}_2\) then \(\Gamma_1 \cong \mathcal{C}_3\) and \(\Gamma_2 \cong \mathcal{G}_3\).

Proof. Note that the group \(\Gamma_1\) permutes the Eckardt points \(e_1, e_2\) and \(e_3\), and the group \(\Gamma_2\) permutes three \((-1)\)-curves passing through an Eckardt point. In the notation of Remark 5.6 one can see that the elements \(a^2b\) and \(cs\) of \(W(E_6)\) preserve the Eckardt points \(e_1, e_2\) and \(e_3\), and permute the \((-1)\)-curves passing through each of them. Thus
the availability of elements conjugate to \(a^2b\) and \(cs\) in \(\pi_1(\Gamma)\) is equivalent to the availability of elements of order 3 and 2 in \(\Gamma_2\) respectively.

The element \(ab\) permutes three \((-1)\)-curves \(E_{14}, E_{25}\) and \(E_{36}\). These curves lie in a plane given by \(\mu_iz = \lambda_iy\), where \((\lambda_i : \mu_i)\) is a root of equation (5.5). Similarly, the element \(ab\) preserves the other roots of equation (5.5). Thus the availability of the element \(ab\) in \(\pi_1(\Gamma)\) is equivalent to the availability of an element of order 3 in \(\Gamma_1\).

The group \((a, b, cs) \cong C_3 \times C_2\) is generated by the elements \(ab, a^2b, cs\). So for any subgroup of \((a, b, cs)\) one can obtain the result of this lemma.

\(\square\)

Now we want to find geometric interpretation of the actions of the elements \(r\) and \(s\).

Consider the class \(L\) in \(\text{Pic}(X)\). We have

\[
\begin{align*}
3L &= 4L - \sum_{i=1}^{3} E_i - 2 \sum_{i=4}^{6} E_i, \\
2L &= 4L - \sum_{i=1}^{3} E_i - 6 \sum_{i=4}^{6} E_i, \\
L &= 5L - \sum_{i=1}^{6} E_i, \\
2L &= 5L - 2 \sum_{i=1}^{6} E_i, \\
L &= 6L - \sum_{i=1}^{6} E_i, \\
L &= 6L - 3 \sum_{i=1}^{6} E_i.
\end{align*}
\]

The three fixed points of \(G\) on \(X\) lie on the line \(z = t = 0\). We denote these points by \(q_1, q_2\) and \(q_3\) given by the equation

\[
P(x : y) = 0.
\]

There are two \(G\)-invariant hyperplane sections \(z = 0\) and \(t = 0\) passing through the fixed points of \(G\). We denote these sections by \(C_1\) and \(C_2\).

Let \(h : X \to \mathbb{P}^2\) be a \(G\)-equivariant blowup of \(\mathbb{P}^2\) at six points \(p_1, p_2, p_3, p_4, p_5\) and \(p_6\) and \(l\) be a class of line on \(\mathbb{P}^2\). Then \(G\) has three fixed points on \(\mathbb{P}^2\). For each two of these fixed points there is exactly one \(G\)-invariant curve passing through these two points that belongs to one of the following six classes: a line, a quadric passing through \(p_1, p_2\) and \(p_3\), a quadric passing through \(p_4, p_5\) and \(p_6\), a quartic passing through \(p_1, p_2, p_3, p_4, p_5\) and \(p_6\) and having nodes at \(p_1, p_2\) and \(p_3\), a quartic passing through \(p_1, p_2, p_3\) and \(p_4, p_5, p_6\) or a quintic having nodes at \(p_1, p_2, p_3, p_4, p_5, p_6\). Proper transforms of these curves on \(X\) are \(G\)-invariant and can be permuted by the group \(\Gamma\). Denote these curves by \(R_{ij}^K\), where \(K\) is a class of curve in \(\text{Pic}(X)\) and \(i\) and \(j\) are indices of points \(q_i\) and \(q_j\), which \(R_{ij}^K\) is passing through.

**Lemma 5.9.** We can choose notation in such way that the following conditions hold:

- the curve \(C_1\) is tangent to the curves \(R_{12}^L, R_{12}^L, R_{13}^L, R_{13}^L, R_{13}^L, R_{13}^L\) at the point \(q_1\), tangent to the curves \(R_{12}^L, R_{12}^L, R_{12}^L, R_{12}^L, R_{12}^L, R_{12}^L\) at the point \(q_2\) and tangent to the curves \(R_{13}^L, R_{13}^L, R_{13}^L, R_{13}^L, R_{13}^L, R_{13}^L\) at the point \(q_3\);
- the curve \(C_2\) is tangent to the curves \(R_{12}^L, R_{12}^L, R_{12}^L, R_{12}^L, R_{12}^L, R_{12}^L\) at the point \(q_1\), tangent to the curves \(R_{12}^L, R_{12}^L, R_{12}^L, R_{12}^L, R_{12}^L, R_{12}^L\) at the point \(q_2\) and tangent to the curves \(R_{13}^L, R_{13}^L, R_{13}^L, R_{13}^L, R_{13}^L, R_{13}^L\) at the point \(q_3\).

**Proof.** One has \(C_1 \cdot R_{12}^L = C_2 \cdot R_{12}^L = 3\). Note that the curves \(C_1, C_2\) and \(R_{12}^K\) pass through the \(G\)-fixed points \(q_1\) and \(q_2\), therefore \(R_{12}^L\) can not meet \(C_1\) and \(C_2\) at any other point since that point should be \(G\)-invariant. Therefore \(R_{12}^L\) is tangent to \(C_1\) and \(C_2\). The curves \(C_1\) and \(C_2\) have different tangents at the points \(q_i\). Thus we can assume that \(R_{12}^L\) is tangent to \(C_1\) at \(q_1\) and tangent to \(C_2\) at \(q_2\).
In the same way we can show that for any class
\[ K \in \{ L, rL, r^2L, sL, srL, sr^2L \} \]
the curve \( R_{ij}^K \) is tangent to \( C_1 \) and \( C_2 \) at points \( q_i \) and \( q_j \). One has \( R_{ij}^K \cdot R_{ik}^{srs} = 2 \) therefore these curves meet each other transversally and have different tangents at points \( q_i \) and \( q_j \). Moreover, one has \( R_{ij}^K \cdot R_{jk}^K = 1 \) therefore these curves meet each other transversally and have different tangents at point \( q_j \). Lemma 5.9 follows from these two facts.

Now we give explicit geometric interpretation of the action of the group \( \pi_2(\Gamma) \).

**Lemma 5.10.** In the notation of Section 4 the group \( \pi_2(\Gamma) \) contains an element conjugate to \( s \) if and only if the Galois group \( \Gamma_3 \) of equation (5.8) is of even order.

**Proof.** Let the group \( \Gamma_3 \) contain an element \( h \) such that \( h(q_1) = q_3 \) and \( h(q_3) = q_2 \). By Lemma 5.9 the curve \( R_{i2}^L \) is tangent to \( C_2 \) at \( q_2 \). Thus the curve \( h(R_{i2}^L) \) is tangent to \( h(C_2) = C_2 \) at \( q_3 \) and passes through \( q_1 \). Therefore by Lemma 5.9 the curve \( h(R_{i2}^L) \) is \( R_{i3}^{sL}, R_{i3}^{srL} \) or \( R_{i3}^{sr^2L} \). Hence the group \( \pi_2(\Gamma) \) contains an element conjugate to \( s \).

Now assume that the group \( \pi_2(\Gamma) \) contains an element conjugate to \( s \). If the Galois group \( \Gamma_3 \) is of odd order then this element fixes the points \( q_1, q_2 \) and \( q_3 \). Therefore the curve \( R_{i2}^L \) is mapped by \( s \) to \( R_{i2}^{sL} \). But \( R_{i2}^L \) is tangent to \( C_1 \) at \( q_1 \) and \( R_{i2}^{sL} \) is tangent to \( C_2 \) at \( q_2 \). This contradiction finishes the proof.

**Lemma 5.11.** Let \( X \) be a \( G \)-minimal cubic surface given by equation (5.11) and the Galois group \( \Gamma_3 \) of equation (5.8) is isomorphic to \( \mathfrak{C}_2 \). Then the quotient \( X/G \) is birationally equivalent to a minimal del Pezzo surface \( Z \) of degree 4. In particular \( X/G \) is not \( k \)-rational.

**Proof.** The Galois group \( \Gamma_3 \) of equation (5.8) is isomorphic to \( \mathfrak{C}_2 \). Therefore we can assume that the \( G \)-fixed point \( q_1 \) is defined over \( k \) and two other \( G \)-fixed points \( q_2 \) and \( q_3 \) are permuted by \( \Gamma_3 \).

Let \( f : X \to X/G \) be the quotient morphism and
\[ \pi : \widetilde{X/G} \to X/G \]
be the minimal resolution of singularities.

There are three singular points of type \( A_2 \) on \( X/G \), namely \( f(q_1), f(q_2) \) and \( f(q_3) \). The curves \( C_1, C_2 \) and the point \( q_1 \) are defined over \( k \). Thus the irreducible components of \( \pi^{-1}f(q_1) \) are defined over \( k \). The group \( \Gamma_3 \cong \mathfrak{C}_2 \) maps the irreducible components of \( \pi^{-1}f(q_2) \) to the irreducible components of \( \pi^{-1}f(q_3) \). Therefore, one has
\[ \rho(\widetilde{X/G}) = \rho(X/G) + 4 = \rho(X)^G + 4 = 5. \]
As in the proof of Lemma 5.2 two curves \( \pi^{-1}\sigma f(C_i) \) are \((-1)\)-curves defined over \( k \). We can contract this pair and get a del Pezzo surface \( Y \) such that \( K_Y^2 = 3 \) and \( \rho(Y) = 3 \).

The Galois group \( \Gamma_3 \) acts on the set of 27 \((-1)\)-curves on \( Y \). One cannot contract more than four \((-1)\)-curves on \( Y \) since \( \rho(Y) = 3 \). But in Table 1 in [Man74, Chapter IV, §5] there is only one class of elements of order 2 satisfying this property. This class corresponds to the 11-th row of the table. For this class one cannot contract more than one \((-1)\)-curve on \( Y \) (see the second column of the table). Therefore one can contract this
curve on $Y$ and get a minimal del Pezzo surface $Z$ of degree 4 with $\rho(Z) = 2$ admitting a structure of conic bundle. The surface $Z \approx X/G$ is not $k$-rational by Theorem 2.8.

Assume that the $G$-fixed point $q_1$ on the cubic surface $X$ is defined over $k$. Then after the change of coordinates this cubic is given by the equation

$$wx(x^2 - \lambda y^2) + zt(ux + vy) + z^3 + \alpha t^3 = 0.$$ 

For this cubic surface the following lemma holds.

**Lemma 5.12.** In the notation of Section 4 the group $\pi_2(\Gamma)$ contains $r$ if and only if the Galois group $\Gamma_4$ of the equation

$$4\alpha \mu^3 - \left(u^2 - \frac{u^2}{\lambda}\right) \mu^2 - 2u\mu v - w^2 = 0.$$ 

contains an element of order 3.

**Proof.** Note that the divisors $R^L_{23} + R^{st}_L, R^L_{23} + R^{st2}_L$ and $R^{x2}_L + R^{sr}_L$ are linearly equivalent to $-2K_X$. Therefore these $G$-invariant pairs of curves passing through $q_2$ and $q_3$ are cut from $X$ by the quadric surfaces of the following form

$$\mu(x^2 - \lambda y^2) = zt.$$ 

Let us find reducible members in these family of curves. One has

$$wx(x^2 - \lambda y^2)t^3 + \mu(ux + vy)(x^2 - \lambda y^2)t^3 + \mu^3(x^2 - \lambda y^2)^3 + \alpha t^6 = 0.$$ 

If the polynomial in the left hand side fo the latter equation is reducible over $\mathbb{k}(x, y, t)$ then it factorizes in the following way

$$\left(A(x - y\sqrt{\lambda})(x^2 - \lambda y^2) + \sqrt{\alpha}t^3\right) \left(B(x + y\sqrt{\lambda})(x^2 - \lambda y^2) + \sqrt{\alpha}t^3\right) = 0$$

and therefore we have $AB = \mu^3$ and

$$A(x - y\sqrt{\lambda})\sqrt{\alpha} + B(x + y\sqrt{\lambda})\sqrt{\alpha} = \mu(ux + vy) + wx.$$ 

Therefore the following system of equations holds

$$\begin{cases} 
(A + B)\sqrt{\alpha} = \mu u + w, \\
(B - A)\sqrt{\lambda \alpha} = \mu v.
\end{cases}$$

Solving this system one has

$$A = \frac{(\mu u + w)\sqrt{\lambda} - \mu v}{2\sqrt{\lambda \alpha}}, \quad B = \frac{(\mu u + w)\sqrt{\lambda} + \mu v}{2\sqrt{\lambda \alpha}}.$$ 

Since $AB = \mu^3$, the reducible members of the linear system $| - 2K_X|$ passing through $q_2$ and $q_3$ are given by equation (5.13).

The roots of this equation correspond to the pairs of curves $R^L_{23}$ and $R^{st}_L, R^L_{23}$ and $R^{st2}_L, R^{x2}_L$ and $R^{sr}_L$ which are cyclically permuted by $\Gamma$ if and only if the group $\pi_2(\Gamma)$ contains $r$. 

□
Remark 5.14. At the beginning of this section we assume that the field \( k \) contains \( \omega \). For any field \( k \) the action of a generator of \( G \) can be written as

\[(x : y : z : t) \mapsto (y : z : x : t).\]

One can remake the computations (which is much more complicated) for this action. Then Lemmas 5.4, 5.7, 5.9 and 5.12 hold. But Lemma 5.10 does not hold since the curves \( C_1 \) and \( C_2 \) are not defined over \( k \).

6. Examples

In this section we construct explicit examples of quotients of del Pezzo surfaces of degree 3 by a group \( G \cong \mathbb{C}_3 \) acting as in type 5 of Table 2. We use the notation of Section 5.

Lemma 6.1. Let \( X \) be a cubic surface given by equation \((5.1)\). Suppose that the Galois groups \( \Gamma_1, \Gamma_2, \Gamma_3 \) of equations \((5.2), (5.5), (5.8)\) are trivial and the Galois group \( \Gamma_4 \) of equation \((5.13)\) contains an element of order 3. Then the surface \( X \) is \( G \)-minimal and \( k \)-rational, and the quotient \( X/G \) is also \( k \)-rational.

Proof. The group \( \Gamma_1 \) is trivial. Therefore \( X(k) \) contains the points \( e_1, e_2 \) and \( e_3 \).

By Lemmas 5.7, 5.10 and 5.12 the group \( \Gamma \) is conjugate to \( \langle r \rangle \). Therefore one can Galois equivariantly contract the curves \( E_1, L_{23} \) and \( Q_1 \) and get a del Pezzo surface of degree 6 which is \( k \)-rational by Theorem 2.8.

The image of the group \( G \) in the Weyl group \( W(E_6) \) is \( \langle ab \rangle \) thus \( X \) is \( G \)-minimal by Corollary 4.11.

Let \( f: X \to X/G \) be the quotient morphism and

\[\pi: \tilde{X}/G \to X/G\]

be the minimal resolution of singularities. The group \( \Gamma_3 \) is trivial. Therefore the points \( q_1, q_2 \) and \( q_3 \) are defined over \( k \). Thus \( \rho(\tilde{X}/G) = 7 \), and \( \tilde{X}/G \) and \( X/G \) are \( k \)-rational by Corollary 2.9.

Example 6.2. If the field \( k \) contains \( \omega \) and an element \( \nu \) such that \( \sqrt[3]{\nu} \notin k \) then the cubic surface given by the equation

\[2\nu x(x^2 - y^2) + z^3 + t^3 = 0\]

satisfies the conditions of Lemma 6.1.

Lemma 6.3. Let \( X \) be a cubic surface given by equation \((5.1)\). Suppose that the Galois groups \( \Gamma_1, \Gamma_2 \) of equations \((5.2), (5.5)\) are trivial, the Galois group \( \Gamma_3 \) of equation \((5.8)\) is isomorphic to \( \mathbb{C}_2 \) and the Galois group \( \Gamma_4 \) of equation \((5.13)\) contains an element of order 3. Then the surface \( X \) is \( G \)-minimal and \( k \)-rational, and the quotient \( X/G \) is not \( k \)-rational.

Proof. The group \( \Gamma_1 \) is trivial. Therefore \( X(k) \) contains the points \( e_1, e_2 \) and \( e_3 \).

By Lemmas 5.7, 5.10 and 5.12 the group \( \Gamma \) is conjugate to \( \langle r, s \rangle \). Therefore one can Galois equivariantly contract the curves \( E_1, L_{23} \) and \( Q_1 \) and get a del Pezzo surface of degree 6 which is \( k \)-rational by Theorem 2.8.

The image of the group \( G \) in the Weyl group \( W(E_6) \) is \( \langle ab \rangle \) thus \( X \) is \( G \)-minimal by Corollary 4.11.
The quotient $X/G$ is not $k$-rational by Lemma 5.11.

**Example 6.4.** Suppose that the field $k$ contains $\omega$, and does not contain $\sqrt{2}$ and any root of the equation
\[4\mu^3 - 9\mu^2 - 6\mu - 1 = 0.\]
Then the cubic surface given by the equation
\[x(x^2 - 2y^2) + 3xzt + z^3 + t^3 = 0\]
satisfies the conditions of Lemma 6.3.

**Lemma 6.5.** Let $X$ be a cubic surface given by equation (5.1). Suppose that the Galois groups $\Gamma_1$ and $\Gamma_3$ of equations (5.2) and (5.8) are trivial, the Galois group $\Gamma_2$ of equation (5.5) is isomorphic to $C_2$ and the Galois group $\Gamma_4$ of equation (5.13) contains an element of order 3. Then the surface $X$ is $G$-minimal and not $k$-rational, and the quotient $X/G$ is $k$-rational.

**Proof.** The group $\Gamma_1$ is trivial. Therefore $X(k)$ contains the points $e_1, e_2$ and $e_3$.

By Lemmas 5.7 and 5.10 the group $\Gamma$ is conjugate to $\langle c, r \rangle$. Therefore $X$ is not $k$-rational by Lemma 4.14.

The image of the group $G$ in the Weyl group $W(E_6)$ is $\langle ab \rangle$ thus $X$ is $G$-minimal by Corollary 4.11.

Let $f : X \to X/G$ be the quotient morphism and
\[\pi : \widetilde{X/G} \to X/G\]
be the minimal resolution of singularities. The group $\Gamma_3$ is trivial. Therefore the points $q_1, q_2$ and $q_3$ are defined over $k$. Thus $\rho(\widetilde{X/G}) = 7$, and $X/G$ and $X/G$ are $k$-rational by Corollary 2.9.

**Example 6.6.** In the assumptions of Example 6.4 the cubic surface given by the equation
\[x(x^2 - y^2) + 3xzt + z^3 + t^3 = 0\]
satisfies the conditions of Lemma 6.5.

**Lemma 6.7.** Let $X$ be a cubic surface given by equation (5.1). Suppose that the Galois group $\Gamma_1$ of equation (5.2) is trivial, the Galois group $\Gamma_2$ of equation (5.5) is isomorphic to $C_2$ and the Galois group $\Gamma_3$ of equation (5.8) is isomorphic to $C_2$. Then the surface $X$ is $G$-minimal and not $k$-rational, and the quotient $X/G$ is also not $k$-rational.

**Proof.** The group $\Gamma_1$ is trivial. Therefore $X(k)$ contains the points $e_1, e_2$ and $e_3$.

By Lemmas 5.7 and 5.10 the group $\Gamma$ contains a subgroup $\langle cs \rangle \cong C_2$. Therefore $X$ is not $k$-rational by Lemma 4.14.

The image of the group $G$ in the Weyl group $W(E_6)$ is $\langle ab \rangle$ thus $X$ is $G$-minimal by Corollary 4.9.

The quotient $X/G$ is not $k$-rational by Lemma 5.11.

**Example 6.8.** If the field $k$ contains $\omega$ and an element $\lambda$ such that $\sqrt[3]{\lambda} \notin k$ then the cubic surface given by the equation
\[ux(x^2 - \lambda y^2) + z^3 + t^3 = 0\]
satisfies the conditions of Lemma 6.7.

Remark 6.9. Note that the conditions of Examples 6.2, 6.4, 6.6, 6.8 hold for $k = \mathbb{Q}(\omega)$.

References

[DI09a] I. V. Dolgachev, V. A. Iskovskikh, Finite subgroups of the plane Cremona group, In: Algebra, arithmetic, and geometry, vol. I: In Honor of Yu. I. Manin, Progr. Math., 269, 443–548, Birkhäuser, Basel, 2009

[Isk79] V. A. Iskovskikh, Minimal models of rational surfaces over arbitrary field, Math. USSR Izv., 1979, 43, 19–43 (in Russian)

[Isk96] V. A. Iskovskikh, Factorization of birational mappings of rational surfaces from the point of view of Mori theory, Uspekhi Mat. Nauk, 1996, 51, 3–72 (in Russian); translation in Russian Math. Surveys, 1996, 51, 585–652

[Man67] Yu. I. Manin, Rational surfaces over perfect fields. II, Math. USSR-Sbornik, 1967, 1, 141–168 (in Russian)

[Man74] Yu. I. Manin. Cubic forms: algebra, geometry, arithmetic, In: North-Holland Mathematical Library, Vol. 4, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., New York, 1974

[Tr13] A.S. Trepalin, Quotients of del Pezzo surfaces of high degree, preprint, see http://arxiv.org/abs/1312.6904.

Institute for Information Transmission Problems, 19 Bolshoy Karetnyi side-str., Moscow 127994, Russia

Laboratory of Algebraic Geometry, National Research University Higher School of Economics, 7 Vavilova str., Moscow 117312, Russia

E-mail address: trepalin@mccme.ru