Geometric model of the pursuit problem on a plane for the case of sets of targets and pursuers

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Abstract. This article examines a kinematic model of a group pursuing several targets by the method of parallel approach. The model is based on the fact that pursuers try to adhere to pre-designed trajectories. The pursuers' trajectories have curvature constraints. The initial directions of the pursuers' velocities are arbitrary, which changes the well-known method of parallel approach. In our model, targets are chased by the pursuers simultaneously. This is due to the change in the lengths of the predicted trajectories in such a way as to synchronize the time to reach the target. The change in the lengths of the predicted trajectories occurs due to an increase in the radius of curvature in the initial segment of the trajectory.

1. Introduction
One of the peculiarities of the method of parallel approach on the plane [1], [2], [3] is that the speed of the pursuer $P_i$ at some moment in time is directed to a point on the circle of Apollonius. In Figure 1, this is point $K_i$, and point $T_i$ is the position of the target at a given time.

![Figure 1. Method of parallel pursuit](image)

The iterative scheme of the parallel approach method is shown in Figure 1. The pursuer coordinates $P_i$ will be calculated as follows:

$$P_{i+1} = P_i + V_P \cdot \frac{P_i K_i}{|P_i K_i|} \cdot \Delta T, T_{i+1} = T_i + V_T \cdot \frac{T_i K_i}{|T_i K_i|} \cdot \Delta T.$$
Where $\Delta T$ is the discrete time interval.
Radius $R_i$ and the center of the circle of Apollonius $Q_i$ are calculated as follows:

$$R_i = \frac{V^2_f}{V^2_p - V^2_T} |T_i - P_i|, Q_i = T_i + \frac{V^2_f}{V^2_p - V^2_T} (T_i - P_i).$$

$K_i$ point coordinates are the result of solving a system of equations for a continuous parameter $t$:

$$\begin{cases}
(K_i - Q_i)^2 = R_i^2 \\
K_i = T_i + V_T \frac{T_{i+1} - T_i}{|T_{i+1} - T_i|} \cdot t
\end{cases}$$

2. Problem statement

It can be seen from the description of the parallel approach method that the initial velocity of the pursuer cannot have an arbitrary direction.

In this article, we want to implement a method close to the parallel approach method. At the initial stage of the solution, we consider two pursuers $P_1$, $P_2$, the speeds of which $V_1$, $V_2$ are directed arbitrarily (Figure: 2). Target $T$ moves in a straight line and evenly.

The radius of curvature of the trajectories of the pursuers cannot be less than a certain value. Therefore, we form one-parameter sets of compound lines, which are analogous to the line of sight $(P_i T_i)$ (Figure 1).

In our case, these will be compound lines connecting points $P_1$, $P_2$ with the point $T$ (Figure 2), consisting of an arc segment and a straight line segment.

Let's say the pursuer $P_2$ when calculating the trajectory to achieve the target $T$ has less time. We can change the radius of curvature of the predicted trajectory of the pursuer $P_2$ upward (Figure 2), to achieve the simultaneous achievement of the target together with $P_1$.

If we add one more goal and one more pursuer (Fig. 3), then in this case, the standard is chosen the pursuer who has the longest time to reach its' target in the preliminary calculation.

Figure 2. Pursuit of one target by two pursuers
Figure 3 shows the pursuers $P_1$, $P_2$ pursue the target $T_1$, and the pursuer $P_3$ pursues the target $T_2$. We have written a test program where the pursuers $P_2$ and $P_3$ change the radius of curvature of the predicted trajectories, adjusting to the time of reaching the target $T_1$ by the pursuer $P_1$.

3. Theory

3.1. Composite curve modeling

To solve this problem, we have to simulate a compound curve for each of the pursuers (Fig. 4). Since in our model there are restrictions on the curvature of the trajectory of all participants in the pursuit problem, then our pursuer $P$ in the predicted trajectory (Fig. 4) will pass in an arc $P_1P_t$, then it will come out to the straight section $[P_tT]$ to the target $T$.

Radius of curvature $r$ of a circle $(C, r)$ in our model is considered to be given and can only change upward.

Center $C$ of circle $(C, r)$ satisfies the system of equations:

\[ |C - P| = r \]
\[ V \cdot (C - P) = 0. \]

In the local coordinate system $(H_1, H_2)$ centered at point $C$ are equation $PP_t$ will be:

\[ L_{\text{circle}}(\alpha) = r \cdot \begin{bmatrix} \cos \left( \frac{\pi}{2} - \alpha \right) \\ \sin \left( \frac{\pi}{2} - \alpha \right) \end{bmatrix}. \]

Where $\alpha$ takes values from 0 to $\arccos \left( \frac{(P-C) \cdot (P_t-C)}{|P-C| \cdot |P_t-C|} \right)$. Basic vectors $(H_1, H_2)$ are equal to:

\[ H_1 = \frac{V}{|V|}, H_2 = \frac{P - C}{|P - C|}. \]
Figure 4. Modeling of the set of parallel lines

Translation to world coordinate system of a line $L_{\text{circle}}(\alpha)$ is:

$$L_{\text{circle}}(\alpha) = \begin{bmatrix} L_{\text{circle}}(\alpha)^* \cdot E_1^* \\ L_{\text{circle}}(\alpha)^* \cdot E_2^* \end{bmatrix} + C$$

$$E_1^* = \begin{bmatrix} E_1 \cdot H_1 \\ E_1 \cdot H_2 \end{bmatrix}, E_2^* = \begin{bmatrix} E_2 \cdot H_1 \\ E_2 \cdot H_2 \end{bmatrix}$$

The equation for the straight section $[P, T]$ is represented as: $L_{\text{line}}(\varepsilon) = (1 - \varepsilon) \cdot P + \varepsilon \cdot T$.

Received line segments $L_{\text{circle}}(\alpha)$ and $L_{\text{line}}(\varepsilon)$ must be combined into one composite line and parametrized in relation to the arc length.

In the test program written on the basis of the materials of the article, we received the combined arrays of coordinates $\{X_i, Y_i\}, i \in 0..N$ of the composite curve. We introduce the formal parameter $\tau$, which continuously runs through values from 0 to $N$. After the cubic spline interpolation procedure, we will have continuous coordinate functions $X(\tau)$ and $Y(\tau)$ from the formal parameter $\tau$.

3.2. Calculation of the input process

From the equation for the total arc length differential $ds^2 = dX^2 + dY^2$ we get a first order differential equation for further transmission to the built-in solvers of the Cauchy problem:

$$D(\tau, s) = \frac{ds}{d\tau} = \frac{1}{\sqrt{\frac{dX^2}{d\tau} + \frac{dY^2}{d\tau}}}, \tau(0) = 0.$$

Thus, we obtained the dependencies $X(s) \equiv Y(s)$ from the arc length parameter. If the length parameter satisfies the relation $s = V \cdot t$, where $t$ is real time, then we will get dependencies $X(t) \equiv Y(t)$, which are the coordinate functions of the baseline $l(t)$.

The compound line that connects the pursuer and the target at the moment of the beginning of the pursuit will be called the baseline.

To highlight the line corresponding to the target position $T(t)$, it is necessary to add vector $T(t) - T(0)$ (Figure 2) to the basic line equation $l(t)$.

If we need to increase the length of the baseline, then we increase the radius of the minimum curvature. In general, the baseline length depends on the following parameters: $T$ target coordinates, coordinates $P$ of the pursuer, velocity vector $V$ pursuer and radius $r$ of the minimum radius of curvature (Figure 2, 3).
Let our pursuer $P$ have a speed module $V_P$. In our task at the moment $t_i$ the coordinates of the points of the pursuer are calculated $P_i$ and target $T_i$. The equation of the predicted motion line calculated from its arc length $l_i(s)$.

![Figure 5. Calculating the next step of the pursuer](image)

At time $t_{i+1}$, the coordinates of the target are known. Then the parallel shear line $l_{i+1}(s)$ is calculated like this:

$$l_{i+1}(s) = l_i(s) + (T_{i+1} - T_i).$$

Pursuer's next move point $P_{i+1}$ is a line reversal point $l_{i+1}(s)$ and the circle radius $V_P \cdot (t_{i+1} - t_i)$ centered at point $P_i$ (Figure 5).

The test program first calculates the approximate time intervals for the pursuers to achieve their targets. Then, the largest one is selected as the reference. Then, in the cycle, small increments of the radius of the admissible curvature of the base trajectories are made $\delta r$ (Fig. 2, 3) until the alignment of the values of the time intervals occurs.

4. Experimental results

Based on the materials of the article, a test program was developed [16], in which two pursuers with initial arbitrary directions and velocities begin to pursue a target moving in a straight line at a constant speed.

![Figure 6. Pursuit of one target by two pursuers](image)
Figure 6 shows the first frame of the program. Figure 6 is supplemented with a link to an animated image [8].

Let’s note that the text of the program can be found on the author's website [9]. In our work, we relied on the results obtained in [4-7]. Attention was also paid to works [10-13]. Also, the authors wrote a program of pursuit by a group of three pursuers of a group of two targets. Targets are achieved simultaneously.

Figure 7. Pursuit of two targets by a group of three pursuers

Figure 7 shows the first frame of the animated image [14] of the model of simultaneous target achievement. An animated image is posted on the resource [15] where the iterative pursuit process is shown without predicted lines of movement of the pursuers' trajectories.

5. The discussion of the results

The mathematical model of the pursuit problem presented in the article assumes that the trajectories of the pursuers at a certain point in time are calculated as if the targets are moving in a straight line and uniform. But nothing prevents us from making calculations of predicted trajectories for other directions of target movement, with different speeds.

We decided not to make direction changes in the test program, so as not to add an additional nested computational cycle to the iterative process.

The main thing in the proposed model is the imposition of restrictions on the curvature of the pursuers' trajectories. This is typical for objects that do not have the ability to change the direction of speed instantly.

6. Conclusion

An important issue in the presented model is the distribution of pursuers by targets. In the test program, distribution was done manually. We would like to have an automated distribution to targets, without operator involvement. The main thing in the developed model is the calculation and modification of baselines for synchronization with the maximum time to reach one of the pursuers of its target.

If it is possible to simulate the process of simultaneously achieving targets, then we can change the model, where the achievement of targets will occur on a timer.
7. References

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