ON REPRESENTABILITY OF ALGEBRAIC FUNCTIONS BY RADICALS

ASKOLD KOVANSKII

Department of Mathematics, University of Toronto, Toronto, Canada

Abstract. This preprint is dedicated to a self contained simple proof of the classical criteria for representability of algebraic functions of several complex variables by radicals. It also contains a criteria for representability of algebroidal functions by composition of single-valued analytic functions and radicals, and a result related to the 13-th Hilbert problem. This preprint is an extended version of the author’s 1971 paper. It is written as a part of the comments to a new edition (in preparation) of the classical book “Integration in finite terms” by J.F. Ritt.

Consider an algebraic equation

\[(1) \quad P_n y^n + P_{n-1} y^{n-1} + \cdots + P_0 = 0,\]

whose coefficients \(P_n, \ldots, P_0\) are polynomials of \(N\) complex variables \(x_1, \ldots, x_N\). Camille Jordan discovered that the Galois group of the equation (1) over the field \(\mathbb{R}\) of rational functions of \(x_1, \ldots, x_N\) has a topological meaning (see theorem 3 below): it is isomorphic to the monodromy group of the equation (1).

According to Galois theory, the equation (1) is solvable by radicals over the field \(R\) if and only if its Galois group is solvable. If the equation (1) is irreducible it defines a multivalued algebraic function \(y(x)\). Galois theory and Theorem 3 imply the following criteria for representability of an algebraic function by radicals, which consists of two statements:

1) If the monodromy group of an algebraic function \(y(x)\) is solvable, then \(y(x)\) is representable by radicals.

2) If the monodromy group of an algebraic function \(y(x)\) is not solvable, then \(y(x)\) is not representable by radicals.

We reduce the first statement to linear algebra (see Theorem 10 below) following the book [4].

We prove the second statement topologically without using Galois theory. Vladimir Igorevich Arnold found the first topological proof of this statement [1]. We use another topological approach (see Theorem 15 below) based on the paper [3]. This paper contains the first result of topological Galois theory [4] and it gave a hint for its further development.

Key words and phrases. algebraic function, solvability by radicals, 13-th Hilbert problem.

This work was partially supported by the Canadian Grant No. 156833-17.

Typeset by \(\LaTeX\)
1. Monodromy group and Galois group. Consider the equation (1). Let \( \Sigma \subset \mathbb{C}^n \) be the hypersurface defined by equation \( P_n J = 0 \), where \( P_n \) is the leading coefficient and \( J \) is the discriminant of the equation (1). The monodromy group of the equation (1) is the group of all permutations of its solutions which are induced by motions around the singular set \( \Sigma \) of the equation (1). Below we discuss this definition more precisely.

At a point \( x_0 \in \mathbb{C}^n \setminus \Sigma \) the set \( Y_{x_0} \) of all germs of analytic functions satisfying the equation (1) contains exactly \( n \) elements, i.e. \( Y_{x_0} = \{ y_1, \ldots, y_n \} \). Indeed, if \( P_n(x_0) \neq 0 \) then for \( x = x_0 \) the equation (1) has \( n \) roots counted with multiplicities. If in addition \( J(x_0) \neq 0 \) then all these roots are simple. By the implicit function theorem each simple root can be extended to a germ of a regular function satisfying the equation (1).

Consider a closed curve \( \gamma \) in \( \mathbb{C}^n \setminus \Sigma \) beginning and ending at the point \( x_0 \). Given a germ \( y \in Y_{x_0} \) we can continue it along the loop \( \gamma \) to obtain another germ \( y_\gamma \in Y_{x_0} \). Thus each such loop \( \gamma \) corresponds to a permutation \( S_\gamma : Y_{x_0} \rightarrow Y_{x_0} \) of the set \( Y_{x_0} \) that maps a germ \( y \in Y_{x_0} \) to the germ \( y_\gamma \in Y_{x_0} \). It is easy to see that the map \( \gamma \rightarrow S_\gamma \) defines a homomorphism from the fundamental group \( \pi_1(\mathbb{C}^n \setminus \Sigma, x_0) \) of the domain \( \mathbb{C}^n \setminus \Sigma \) with the base point \( x_0 \) to the group \( S(Y_{x_0}) \) of permutations. The monodromy group of the equation (1) is the image of the fundamental group in the group \( S(Y_{x_0}) \) under this homomorphism.

Remark. Instead of the point \( x_0 \) one can choose any other point \( x_1 \in \mathbb{C}^n \setminus \Sigma \). Such a choice will not change the monodromy group up to an isomorphism. To fix this isomorphism one can choose any curve \( \gamma : I \rightarrow \mathbb{C}^n \setminus \Sigma \) where \( I \) is the segment \( 0 \leq t \leq 1 \) and \( \gamma(0) = x_0 \), \( \gamma(1) = x_1 \) and identify each germ \( y_{x_0} \) of solution of (1) with its continuation \( y_{x_1} \) along \( \gamma \).

Instead of the hypersurface \( \Sigma \) one can choose any bigger algebraic hypersurface \( D, \Sigma \subset D \subset \mathbb{C}^n \). Such a choice will not change the monodromy group: one can slightly move a curve \( \gamma \in \pi_1(\mathbb{C}^n \setminus \Sigma, x_0) \) without changing the map \( S_\gamma \) in such a way that \( \gamma \) will not intersect \( D \).

The field of rational functions of \( x_1, \ldots, x_N \) is isomorphic to the field \( \mathcal{R} \) of germs of rational functions at the point \( x_0 \in \mathbb{C}^n \setminus \Sigma \). Consider the field extension \( \mathcal{R}\{y_1, \ldots, y_n\} \) of \( \mathcal{R} \) by the germs \( y_1, \ldots, y_n \) at \( x_0 \) satisfying the equation (1).

Lemma 1. Every permutation \( S_\gamma \) from the monodromy group can be uniquely extended to an automorphism of the field \( \mathcal{R}\{y_1, \ldots, y_n\} \) over the field \( \mathcal{R} \).

Proof. Every element \( f \in \mathcal{R}\{y_1, \ldots, y_n\} \) is a rational function of \( x, y_1, \ldots, y_n \). It can be continued meromorphically along the curve \( \gamma \in \pi_1(\mathbb{C}^m \setminus \Sigma, x_0) \) together with \( y_1, \ldots, y_n \). This continuation gives the required automorphism, because the continuation preserves the arithmetical operations and every rational function returns back to its original values (since it is a single-valued valued function). The automorphism is unique because the extension is generated by \( y_1, \ldots, y_n \).

By definition the Galois group of the equation (1) is the group of all automorphisms of the field \( \mathcal{R}\{y_1, \ldots, y_n\} \) over the field \( \mathcal{R} \). According to Lemma 1 the monodromy group of the equation (1) can be considered as a subgroup of its Galois group. Recall that by definition a multivalued function \( y(x) \) is algebraic if all its meromorphic germs satisfy the same algebraic equation over the field of rational functions.
Theorem 2. A germ $f \in \mathcal{R}\{y_1, \ldots, y_n\}$ is fixed under the monodromy action if and only if $f \in \mathcal{R}$.

Proof. A germ $f \in \mathcal{R}\{y_1, \ldots, y_n\}$ is fixed under the monodromy action if and only if $f$ is a germ of a single valued function. The field $\mathcal{R}\{y_1, \ldots, y_n\}$ contains only germs of algebraic functions. Any single valued algebraic function is a rational function.

According to Galois theory Theorem 2 can be formulated in the following way.

Theorem 3. The monodromy group of the equation (1) is isomorphic to the Galois group of the equation (1) over the field $\mathcal{R}$.

Below we will not rely on Galois theory. Instead we will use Theorem 2 directly.

Lemma 4. The monodromy group acts on the set $Y_{x_0}$ transitively if and only if the equation (1) is irreducible over the field of rational functions.

Proof. Assume that there is a proper subset $\{y_1, y_2, \ldots y_k\}$ of $Y_{x_0}$ invariant under the monodromy action. Then the basic symmetric functions $r_1 = y_1 + \cdots + y_k$, $r_2 = \sum_{i<j} y_i y_j$, $\ldots$, $r_k = y_1 \cdot \cdots \cdot y_k$ belong to the field $\mathcal{R}$. Thus $y_1, y_2, \ldots y_k$ are solutions of the degree $k < n$ equation $y^k - r_1 y^{k-1} + \ldots + (-1)^k r_k = 0$. So the equation (1) is reducible. On the other hand if the equation (1) can be represented as a product of two equations over $\mathcal{R}$ then their roots belong to two complementary subsets of $Y_{x_0}$ which are invariant under the monodromy action.

Corollary 5. An irreducible equation (1) defines a multivalued algebraic function $y(x)$ whose set of germs at $x_0 \in \mathbb{C}^N \setminus \Sigma$ is the set $Y_{x_0}$ and whose monodromy group coincides with the monodromy group of the equation (1).

Theorem 3, Corollary 5 and Galois theory immediately imply the following result.

Theorem 6. An algebraic function whose monodromy group is solvable can be represented by rational functions using the arithmetic operations and radicals.

A stronger version of Theorem 6 can be proven using linear algebra (see Theorem 10 in the next section).

2. Action of solvable groups and representability by radicals. In this section, we prove that if a finite solvable group $G$ acts on a $\mathbb{C}$-algebra $V$ by automorphisms, then all elements of $V$ can be expressed by the elements of the invariant subalgebra $V_0$ of $G$ by taking radicals and adding.

This construction of a representation by radicals is based on linear algebra. More precisely we use the following well known statement: any finite abelian group of linear transformations of a finite-dimensional vector space over $\mathbb{C}$ can be diagonalized in a suitable basis.

Lemma 7. Let $G$ be a finite abelian group of order $n$ acting by automorphisms on $\mathbb{C}$-algebra $V$. Then every element of the algebra $V$ is representable as a sum of elements $x_i \in V$, such that $x_i^n$ lies in the invariant subalgebra $V_0$ of $G$, i.e., in the fixed-point set of the group $G$.

Proof. Consider a finite-dimensional vector subspace $L$ in the algebra $V$ spanned by the $G$-orbit of an element $x$. The space $L$ splits into a direct sum $L = L_1 + \cdots + L_k$ of eigenspaces for all operators from $G$. Therefore, the vector $x$ can be represented
Theorem 8. Let \( x \) representable as a sum of eigenvectors for all the operators from the group. The corresponding eigenvalues are \( n \)-th roots of unity. Therefore, the elements \( x_1^1, \ldots, x_k^k \) belong to the invariant subalgebra \( V_0 \).

**Definition.** We say that an element \( x \) of algebra \( V \) is an \( n \)-th root of an element \( a \) if \( x^n = a \).

We can now restate Lemma 7 as follows: every element \( x \) of the algebra \( V \) is representable as a sum of \( n \)-th roots of some elements of the invariant subalgebra.

**Theorem 8.** Let \( G \) be a finite solvable group of order \( n \) acting by automorphisms on \( \mathbb{C}\text{-algebra } V \). Then every element \( x \) of the algebra \( V \) can be obtained from the elements of the invariant subalgebra \( V_0 \) by takings \( n \)-th roots and summing.

We first prove the following simple statement about an action of a group on a set. Suppose that a group \( G \) acts on a set \( X \), let \( H \) be a normal subgroup of \( G \), and denote by \( X_0 \) the subset of \( X \) consisting of all points fixed under the action of \( G \).

**Lemma 9.** The subset \( X_H \) of the set \( X \) consisting of the fixed points under the action of the normal subgroup \( H \) is invariant under the action of \( G \). There is a natural action of the quotient group \( G/H \) on the set \( X_H \) with the fixed-point set \( X_0 \).

**Proof.** Suppose that \( g \in G \), \( h \in H \). Then the element \( g^{-1}hg \) belongs to the normal subgroup \( H \). Let \( x \in X_H \). Then \( g^{-1}hg(x) = x \), or \( h(g(x)) = g(x) \), which means that the element \( g(x) \in X \) is fixed under the action of the normal subgroup \( H \). Thus the set \( X_H \) is invariant under the action of the group \( G \). Under the action of \( G \) on \( X_H \), all elements of \( H \) correspond to the identity transformation. Hence the action of \( G \) on \( X_H \) reduces to an action of the quotient group \( G/H \).

We now proceed with the proof of Theorem 8.

**Proof (of Theorem 8).** Since the group \( G \) is solvable, it has a chain of nested subgroups \( G = G_0 \supset \cdots \supset G_m = e \) in which the group \( G_m \) consists of the identity element \( e \) only, and every group \( G_i \) is a normal subgroup of the group \( G_{i-1} \). Moreover, the quotient group \( G_{i-1}/G_i \) is abelian. Let \( V_0 \subset \cdots \subset V_m = V \) denote the chain of invariant subalgebras of the algebra \( V \) with respect to the action of the groups \( G_0, \ldots, G_m \). By Lemma 9 the abelian group \( G_{i-1}/G_i \) acts naturally on the invariant subalgebra \( V_i \), leaving the subalgebra \( V_{i-1} \) pointwise fixed. The order \( m_i \) of the quotient group \( G_{i-1}/G_i \) divides the order of the group \( G \). Therefore, Lemma 7 is applicable to this action. We conclude that every element of the algebra \( V_i \) can be expressed with the help of summation and \( n \)-th root extraction by the elements of the algebra \( V_{i-1} \). Repeating the same argument, we will be able to express every element of the algebra \( V \) by the elements of the algebra \( V_0 \) using a chain of summations and \( n \)-th root extractions.

**Theorem 10.** An algebraic function whose monodromy is solvable can be represented by rational functions by root extractions and summations.

**Proof.** One can prove Theorem 10 by applying Theorem 8 to the monodromy action by automorphisms on the extension \( \mathcal{R}\{y_1, \ldots, y_n\} \) with the field of invariants \( \mathcal{R} \).

3. **Topological obstruction to representation by radicals.** Let us introduce some notation.

By \( G^m \) we denote the \( m \)-th commutator of the group \( G \). For any \( m \geq \) the group \( G^m \) is a normal subgroup in \( G \).
By \(F(D, x_0)\) we denote the fundamental group of the domain \(U = \mathbb{C}^n \setminus D\) with the base point \(x_0 \in U\), where \(D\) is an algebraic hypersurface in \(\mathbb{C}^n\).

Let \(H(D, m)\) be the covering space of the domain \(\mathbb{C}^n \setminus D\) corresponding to the subgroup \(F^m(D, x_0)\) of the fundamental group \(F(D, x_0)\).

We will say that an algebraic function is an \(R\)-function if it becomes a single-valued function on some covering \(H(D, m)\).

**Lemma 11.** If \(m_1 \geq m_2\) and \(D_1 \supset D_2\) then there is a natural projection \(\rho : H(D_1, m_1) \to H(D_2, m_2)\). Thus if a function \(y\) becomes a single-valued function on \(H(D_2, m_2)\) then it certainly becomes a single-valued function on \(H(D_1, m_1)\).

**Proof.** Let \(p_* : F(D_1, x_0) \to F(D_2, x_0)\) be the homomorphism induced by the embedding \(p : \mathbb{C}^n \setminus D_1 \to \mathbb{C}^n \setminus D_2\). Lemma 11 follows from the following obvious inclusions: \(p_*^{-1}[F^m_2(D_2, x_0)] \subset F^m_2(D_1, x_0)\) and \(F^m_2(D_1, x_0) \subset F^m_1(D_1, x_0)\).

**Lemma 12.** If \(y_1\) and \(y_2\) are \(R\)-functions then \(y_1 + y_2\), \(y_1 - y_2\), \(y_1 \cdot y_2\) and \(y_1/y_2\) also are \(R\)-functions.

**Proof.** Assume that \(R\)-functions \(y_1\) and \(y_2\) become single-valued functions on the coverings \(H(D_1, m_1)\) and \(H(D_2, m_2)\). By Lemma 11 the functions \(y_1, y_2\) become single-valued on the covering \(H(D, m)\) where \(D = D_1 \cup D_2\) and \(m = \max(m_1, m_2)\).

Thus the functions \(y_1 + y_2\), \(y_1 - y_2\), \(y_1 \cdot y_2\) and \(y_1/y_2\) also become single-valued on the covering \(H(D, m)\). The proof is completed since \(y_1 + y_2\), \(y_1 - y_2\), \(y_1 \cdot y_2\) and \(y_1/y_2\) are algebraic functions.

**Lemma 13.** Composition of an \(R\)-function with the degree \(q\) radical is an \(R\)-function.

**Proof.** Assume that the function \(y\) defined by (1) is \(R\)-function which becomes a single-valued function on the covering \(H(D, m)\). Let \(D_2 \subset \mathbb{C}^n\) be the hypersurface defined by the equation \(P_nP_0 = 0\), where \(P_n\) and \(P_0\) are the leading coefficient and the constant term of the equation (1). According to Lemma 11 the function \(y\) becomes a single-valued function on the covering \(H(D, m)\) where \(D = D_1 \cup D_2\). Let \(h_0 \in H(D, m)\) be a point whose image under the natural projection \(\rho : H(D, m) \to \mathbb{C}^n \setminus D\) is the point \(x_0\). One can identify the fundamental groups \(\pi_1(H(D, m), h_0)\) and \(F^m(D, x_0)\).

By definition of \(D_2\) the function \(y\) never equals to zero or to infinity on \(H(D, m)\). Hence \(y\) defines a map \(y : H(D, m) \to \mathbb{C} \setminus \{0\}\). Let \(y_* : \pi_1(H(D, m), h_0) \to \pi_1(\mathbb{C} \setminus \{0\}, y(h_0))\) be the induced homomorphism of the fundamental groups. The group \(\pi_1(H(D, m), h_0)\) is identified with the group \(F^m(D, x_0)\) and the group \(\pi_1(\mathbb{C} \setminus \{0\}, y(h_0))\) is isomorphic to \(\mathbb{Z}\). So \(\ker y_* \subset F^{m+1}(D, x_0)\). Thus all loops from the group \(y_* \big( F^{m+1}(D, x_0) \big)\) do not wind around the origin \(0 \in \mathbb{C}\). Hence any germ of \(y^{1/q}\) does not change its value after continuation along a loop from the group \(F^{m+1}(D, x_0)\). So \(y^{1/q}\) is a single-valued function on \(H(D, m + 1)\). The proof is completed since \(y^{1/q}\) is an algebraic function.

**Lemma 14.** An algebraic function \(y\) is an \(R\)-function if and only if its monodromy group is solvable.

**Proof.** Assume that \(y\) is defined by (1). Let \(D\) be the hypersurface \(P_nJ = 0\) where \(P_n\) is the leading coefficient and \(J\) is the discriminant of (1). Let \(M\) be the monodromy group of \(y\). Consider the natural homomorphism \(p : F(D, x_0) \to M\). If \(M\) is solvable then for some natural number \(m\) the \(m\)-th commutator of \(M\) is the
identity element $e$. The function $y$ becomes single-valued on the covering $H(D, m)$ since $F^m(D, x_0) \subset p^{-1}(M^m) = p^{-1}(e)$. Conversely, if $y$ is a single-valued function on some covering $H(D, m)$ then $p(F^m(D, x_0)) = e$. But $p(F^m(D, x_0)) = M^m$. Thus the monodrogy group $M$ is solvable.

**Theorem 15.** If an algebraic function has unsolvable monodromy group then it cannot be represented by compositions of rational functions and radicals.

**Proof.** Lemma 12 and Lemma 13 show that the class of $R$-functions is closed under arithmetic operations and compositions with radicals. Lemma 14 shows that the monodromy group of any $R$-function is solvable.

4. **Compositions of analytic functions and radials.** In this section we describe a class of multivalued functions in a domain $U \subset \mathbb{C}^N$ representable by composition of single-valued analytic functions and radicals.

A multivalued function $y$ in $U$ is called an *algebroidal function* in $U$ if it satisfies an irreducible equation

$$y^n + f_{n-1}y^{n-1} + \cdots + f_0 = 0$$

whose coefficients $f_{n-1}, \ldots, f_0$ are analytic functions in $U$. An algebroidal function could be considered as a continuous multivalued function in $U$ which has finitely many values.

**Theorem 16 ([2], [3]).** A multivalued function $y$ in the domain $U$ can be represented by composition of radicals and single valued analytic functions if and only if $y$ is an algebroidal function in $U$ with solvable monodromy group.

To prove the “only if” part one can repeat the proof of Theorem 15 replacing coverings over domains $\mathbb{C}^N \setminus D$ by coverings over domains $U \setminus \tilde{D}$ where $D$ is an analytic hypersurface in $U$.

To prove Theorem 16 in the opposite direction one can use Theorem 8 in the same way as it was used in the proof of Theorem 10.

5. **Local representability.** In this section we describe a a local version of Theorem 16.

Let $y$ be an algebroidal function in $U$ defined by (2). One can localize the equation (2) at any point $p \in U$, i.e. one can replaced the coefficients $f_i$ of the equation (2) by their germs at $p$. After such a localization the equation (2) can became reducible, i.e. it can became representable as a product of irreducible equations. Thus an algebroidal functions $y$ in arbitrary small neighborhood of a point $p$ defines several algebroidal functions, which we will call *ramified germs of $y$ at $p$*. For a ramified germ of $y$ at $p$ the monodromy group is defined (as the monodromy group of an algebroidal function in an arbitrary small neighborhood of the point $p$).

A ramified germ of an algebroidal function $y$ of one variable $x$ in a neighborhood of a point $p \in \mathbb{C}^1$ has a simple structure: its monodromy group is a cyclic group $\mathbb{Z}/m\mathbb{Z}$ and it can be represented as a composition of a radical and an analytic single-valued function: $y(x) = f((x - p)^{1/m})$ where $m$ is the ramification order of $y$. The following corollary follows from Theorem 16.
Corollary 17 ([2], [3]). 1) If a multivalued function \( y \) in the domain \( U \) can be represented by composition of an algebroidal functions of one variable and single valued analytic functions then the monodromy group of any ramified germ of \( y \) is solvable.

2) If the monodromy group of a ramification germ of \( y \) at \( p \) is solvable then in a small neighborhood of \( p \) it can be represented by composition of radicals and single valued analytic functions.

The local monodromy group of an algebroidal function \( y \) at a point \( p \in U \) is the monodromy group of the equation (2) in an arbitrary small neighborhood of the point \( p \). The ramified germs of \( y \) at the point \( p \) correspond to the orbits of the local monodromy group actions. This statement can be proven in the same way as Lemma 4 was proved.

6. Application to the 13-th Hilbert problem. In 1957 A.N. Kolmogorov and V.I. Arnold proved the following totally unexpected theorem which gave a negative solution to the 13-th Hilbert problem.

Theorem (Kolmogorov–Arnold). Any continuous function of \( n \) variables can be represented as the composition of functions of a single variable with the help of addition.

The 13-th Hilbert problem has the following algebraic version which still remains open: Is it possible to represent any algebraic function of \( n > 1 \) variables by algebraic functions of a smaller number of variables with the help of composition and arithmetic operations?

An entire algebraic function \( y \) in \( \mathbb{C}^N \) is an algebraic function defined in \( U = \mathbb{C}^N \) by an equation (2) whose coefficient \( f_i \) are polynomials. An entire algebraic function could be considered as a continuous algebraic function.

It turns out that in Kolmogorov–Arnold Theorem one can not replace continuous functions by entire algebraic functions.

Theorem 18([2], [3]). If an entire algebraic function can be represented as a composition of polynomials and entire algebraic functions of one variable, then its local monodromy group at each point is solvable.

Proof. Theorem 18 follows from from Corollary 17.

Corollary 19. A function \( y(a, b) \), defined by equation \( y^5 + ay + b = 0 \), cannot be expressed in terms of entire algebraic functions of a single variable by means of composition, addition and multiplication.

Proof. Indeed, it is easy to check that the local monodromy group of \( y \) at the origin is the unsolvable permutation group \( S_5 \) (see [2], [3]).

Division is not a continuous operation and it destroys the locality. One cannot add division to the operations used in Theorem 18. It is easy to see that the function \( y(a, b) \) from Corollary 18 can be expressed in terms of entire algebraic functions of a single variable by means of composition and arithmetic operations: \( y(a, b) = g(b/\sqrt[5]{a^5}) \sqrt[5]{a} \), where \( g(u) \) is defined by equation \( u^5 + u + a = 0 \).

The following particular case of the algebraic version of the 13-th Hilbert problem still remains open.
Problem. Show that there is an algebraic function of two variables which cannot be expressed in terms of algebraic functions of a single variable by means of composition and arithmetic operations.

7. Acknowledgement. I would like to thank Michael Singer who invited me to write comments for a new edition of the classical J.F. Ritt’s book “Integration in finite terms” [5]. This preprint was written as a part of these comments. I also am grateful to Fedor Kogan who edited my English.

REFERENCES

[1] V.B. Alekseev, Abel’s Theorem in Problems and Solutions. Based on the lectures of Professor V.I. Arnold, Kluwer Academic Publishers, 2004.

[2] A.G. Khovanskii, The representability of algebroidal functions by superpositions of analytic functions and of algebroidal functions of one variable. Functional Analysis and its applications, V. 4, N 2, 74–79, 1970; translation in Funct. Anal. Appl. 4 (1970), no. 2, 152–156.

[3] A.G. Khovanskii, On compositions of analytical functions with radicals, UMN, 26:2 (1971), 213–214.

[4] A. Khovanskii, Topological Galois theory. Solvability and unsolvability of equations in finite terms. Translated by Valentina Kiritchenko and Vladlen Timorin. Series: Springer Monographs in Mathematics. Springer Berlin Heidelberg. 2014, XVIII, 305 pp. 6 illus.

[5] J. Ritt, Integration in finite terms. Liouville’s theory of elementary methods, N.Y. Columbia Univ. Press. 1948.