GLOBAL STABILITY OF THE NORMAL STATE OF SUPERCONDUCTORS IN THE PRESENCE OF A STRONG ELECTRIC CURRENT

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Abstract. We consider the time-dependent Ginzburg-Landau model of superconductivity in the presence of an electric current flowing through a two-dimensional wire. We show that when the current is sufficiently strong the solution converges in the long-time limit to the normal state. We provide two types of upper bounds for the critical current where such global stability is achieved: by using the principal eigenvalue of the magnetic Laplacian associated with the normal magnetic field, and through the norm of the resolvent of the linearized steady-state operator. In the latter case we estimate the resolvent norm in large domains by the norms of approximate operators defined on the plane and the half-plane. We also obtain a lower bound, in large domains, for the above critical current by obtaining the current for which the normal state looses its local stability.

1. Introduction

Consider a superconductor placed at a temperature lower than the critical one. It is well-understood from numerous experimental observations, that a sufficiently strong current, applied through the sample, will force the superconductor to arrive at the normal state. To explain this phenomenon mathematically, we use the time-dependent Ginzburg-Landau model which is defined by the following system of equations, and will be referred
to as (TDGL1)

\[\begin{align*}
\frac{\partial \psi}{\partial t} + i\phi \psi &= (\nabla - iA)^2 \psi + \psi \left(1 - |\psi|^2\right), & \text{in } \mathbb{R}^+ \times \Omega, \\
\kappa^2 \text{curl}^2 A + \sigma \left(\frac{\partial A}{\partial t} + \nabla \phi\right) &= \text{Im} (\bar{\psi} \nabla A \psi), & \text{in } \mathbb{R}^+ \times \Omega, \\
\psi &= 0, & \text{on } \mathbb{R}^+ \times \partial \Omega_c, \\
(i\nabla + A)\psi \cdot \nu &= 0, & \text{on } \mathbb{R}^+ \times \partial \Omega_i, \\
\sigma \left(\frac{\partial A}{\partial t} + \nabla \phi\right) \cdot \nu &= J, & \text{on } \mathbb{R}^+ \times \partial \Omega_c, \\
\sigma \left(\frac{\partial A}{\partial t} + \nabla \phi\right) \cdot \nu &= 0, & \text{on } \mathbb{R}^+ \times \partial \Omega_i, \\
\int_{\partial \Omega} \text{curl} A(t, x) \, ds &= h_{ex}, & \text{on } \mathbb{R}^+, \\
\psi(0, x) &= \psi_0(x), & \text{in } \Omega, \\
A(0, x) &= A_0(x), & \text{in } \Omega.
\end{align*}\]

In the above \(\psi\) denotes the order parameter, \(A\) is the magnetic potential, \(\phi\) is the electric potential, \(\kappa\) denotes the Ginzburg-Landau parameter, which is a material property, and the normal conductivity of the sample is denoted by \(\sigma\). We use the notation \(\nabla_A = \nabla - iA\) and \(ds\) for the induced measure on \(\partial \Omega\). In (1.1g) we use the standard notation

\[\int_D \frac{1}{|D|} \int,\]

for any domain \(D \subset \mathbb{R}^2\). The spatial coordinates have been scaled in (1.1) by the coherence length, characterizing variations in \(\psi\). The domain \(\Omega \subset \subset \mathbb{R}^2\), occupied by the superconducting sample, has a smooth interface, denoted by \(\partial \Omega_c\), with a conducting metal which is at the normal state. Thus, we require that \(\psi\) would vanish on \(\partial \Omega_c\) in (1.1c), and allow for a smooth current satisfying

\[\text{(1.2) \quad (J1) \quad } \quad J \in C^2(\partial \Omega_c),\]

to enter the sample in (1.1e). We further require that

\[\text{(1.3) \quad (J2) \quad } \quad \int_{\partial \Omega_c} J \, ds = 0,\]

and, mainly in the last section, that

\[\text{(1.4) \quad (J3) \quad } \quad \text{the sign of } J \text{ is constant on each connected component of } \partial \Omega_c.\]

We allow for \(J \neq 0\) at the corners despite the fact that no current is allowed to enter the sample through the insulator.

The rest of the boundary, denoted by \(\partial \Omega_i\), is adjacent to an insulator. To simplify some of our regularity arguments we introduce the following geometrical assumption (for
We require in the last section that:

\begin{equation}
\psi_0 \in H^1(\Omega, \mathbb{C}) \text{ and } A_0 \in H^1(\Omega, \mathbb{R}^2).
\end{equation}

We further assume everywhere in the sequel that:

\begin{equation}
\|\psi_0\|_{\infty} \leq 1.
\end{equation}

In most of this work we consider Coulomb gauge solutions of (1.1) which satisfy in addition

\begin{equation}
\text{div } A = 0, \quad A \cdot \nu/\partial \Omega = 0.
\end{equation}

To complete the presentation of the problem, we need to make two further assumptions on the normal magnetic and electric potentials which we respectively denote by $A_n$ and $\phi_n$. To this end, we write that $(0, hA_n, h\phi_n)$ is a stationary solution of (1.1), where $h$
is a positive parameter representing the intensity of the applied field. More explicitly, 
\((A_n, \phi_n)\) must satisfy
\[
\begin{cases}
-c \text{curl}^2 A_n + \nabla \phi_n = 0 & \text{in } \Omega, \\
-\sigma \frac{\partial \phi_n}{\partial \nu} = J_r & \text{on } \partial \Omega, \\
\int_{\partial \Omega} \text{curl} A_n \, ds = h_r,
\end{cases}
\]
in which \(c = \kappa^2 / \sigma\) and \(J_r = J / h\), respectively denote some reference electric current and magnetic field. (Obviously, \(J_r \equiv 0\) on \(\partial \Omega_n\).) We fix the Coulomb gauge for \(A_n\), i.e., we require that it satisfies (1.9). In the next section we discuss the existence, uniqueness, and regularity of solutions to the above problem.

The next assumption will be rephrased in the next section. Here we write
\[(1.10) \quad (B) \quad B_n \neq 0 \text{ at the corners},\]
where \(B_n = \text{curl} A_n\).

In the last section we restate that
\[(1.11) \quad (C) \quad \nabla \phi_n \perp \partial \Omega \text{ on } B_n^{-1}(0) \cap \partial \Omega.\]

The reasons for the above assumptions will be clarified in the sections where they are restated.

One possible mechanism which contributes to the breakdown of superconductivity by a strong current is the magnetic field induced by the current. In the absence of electric current, it was proved by Giorgi & Phillips in [17] that, when a sufficiently strong magnetic field is applied on the sample’s boundary (or when \(h\) is sufficiently large), the normal state, for which \(\psi \equiv 0\), becomes the unique solution for the steady-state version of (1.1) (cf. also [16] and the references therein). For the time-dependent Ginzburg-Landau equations it was proved in [14] that every solution must reach an equilibrium in the long-time limit. When combined with the results in [17] it follows that when the applied magnetic field is sufficiently large the normal state becomes globally stable.

No such result is available in the presence of electric currents. The results in [14] are based on the fact that, in the absence of currents, the Ginzburg-Landau energy functional serves as a Lyapunov functional. In the presence of a current one has to take account of the work it produces, which does not necessarily decrease the energy (cf. [35] for instance). Moreover, the magnetic field is not the only mechanism which forces the sample into the normal state when the electric current is sufficiently large. For a reduced model, which neglects the magnetic field (induced and applied) effect it has been proved in [25, 36, 1] that the normal state is at least locally (linearly) stable when the current is sufficiently strong. This reduced model can be easily obtained by setting \(A \equiv 0\) in (1.1), and has received significant attention in the greater context of PT symmetric Schrödinger operators (cf. [34], [5] for instance).

When the magnetic field’s effect is accounted for, we report here the results of three recent contributions [2, 4, 3], we have obtained together with Pan. In all of them we consider the linearization of (1.1a) near the normal state \((0, A_n, \phi_n)\) both in the entire
plane [2], and the half-plane [4, 3]. Thus, we have analyzed the spectrum of two operators associated with the differential operator $-\partial_x^2 + (i\partial_y - \frac{1}{2}x^2)^2 + icy$, where, as above, $c = \kappa^2/\sigma$. We define in [2] $\mathcal{A}$ as the maximal accretive extension in $L^2(\mathbb{R}^2)$ of this differential operator initially defined on $C_0^\infty(\mathbb{R}^2)$ and in [3, 4] $\mathcal{A}_+$ as an unbounded operator in $L^2(\mathbb{R}^2_+)$, where $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : y > 0\}$, using this time Lax-Milgram’s Theorem for the associated sesquilinear form in $H^1_0(\mathbb{R}^2_+)$ (see below (1.16)-(1.19) for more details).

In [2] we show that the spectrum of $\mathcal{A}$ is empty and find estimates on its resolvent norm. In contrast, in [4, 3] we show that $\sigma(\mathcal{A}^+)$ is non-empty and even manage to evaluate the leftmost eigenvalue, in the asymptotic limits $c \to \infty$ [4] and $c \to 0$ [3]. One can easily derive from this leftmost eigenvalue the critical current where the normal state looses its local (linear) stability. In particular, we show that it tends, as $c \to \infty$ (i.e. in the small normal conductivity limit), to the critical value for the reduced model [25]. This result suggests, once again, that stability is being forced not only by the magnetic field that the current induces, but also by the potential term in (1.1a). We conclude the foregoing literature survey by mentioning a few works considering the motion of vortices under the action of an electric current [38, 37, 32].

In the present contribution we prove global, long-time, stability of the normal state, as a solution of (1.1), for sufficiently large currents. In contrast with [2, 4, 3] we consider the fully non-linear problem (1.1) in a bounded domain of the type presented in Fig. 1. While the linear analysis in [2, 4, 3] provides us with some useful insights and tools, employed throughout this work, it cannot be easily applied to obtain long-time stability of the normal state for a wide class of initial conditions, not necessarily close to the normal state in any sense. In particular, it is necessary to bound the effect of the non-linear terms, that are not necessarily small at $t = 0$. The effect of boundaries needs to be taken into account as well.

The rest of this contribution is organized as follows:

We begin by dealing with a few preliminaries in Section 2. In particular, we prove global existence and uniqueness of solutions for (1.1) and obtain their regularity. While these questions have previously been addressed (cf. [7], [15], and [10] to name just a few references) the fact that the boundary is not smooth at the corners requires some additional attention. Some of the results we state in the next section are proved in the appendices.

In Section 3 we prove that if the current is strong enough, the magnetic field it induces forces the semigroup associated with (1.1) to become asymptotically a contraction. Let

$$
\mu(h) = \inf_{\substack{u \in H^1(\Omega, \mathbb{C}) \\
u_{|\alpha_n} = 0; \|u\|_2 = 1}} \|\nabla h_{A_n} u\|_2^2.
$$

The main result of Section 3 is the following
Theorem 1.1. Let $(\psi, A, \phi)$ denote a solution of (1.1) and (1.9) satisfying (1.8). Then, there exists $\gamma > 0$ for which whenever

$$
\mu(h) > 1 + \frac{\gamma}{\kappa^2} + \frac{\gamma^2}{2 \kappa^4},
$$

there exist $C = C(\Omega, \kappa, c, \|\psi_0\|_2, \|A_0\|_2, h) > 0$ and $\lambda_m = \lambda_m(c, \kappa, \mu(h), \Omega) > 0$, where $c = \kappa^2/\sigma$, such that, for all $t > 0$, we have:

$$
\|\psi\|_2 + \|A - hA_n\|_2 + \|\phi - h\phi_n\|_2 \leq Ce^{-\lambda_m t}.
$$

Furthermore, there exists $t^*(\kappa, c, \|A_0\|_2, \Omega)$ such that $[t^* + 1, +\infty) \ni t \mapsto \|\psi(t, \cdot)\|_2$ is monotone decreasing.

Note that, as is explained at the beginning of Section 3, (1.12) means that the semigroup associated with the linearized version of (1.1) is a contraction. The reader is referred to Theorem 3.1 and to Proposition 3.5 for the precise values of $\gamma$, $\lambda_m$, and $t^*$ in the large $\kappa$ limit.

Let

$$
L_h = -\nabla^2 h A_n + ih\phi_n,
$$

be defined over the domain

$$
D(L_h) = \{u \in H^2(\Omega) \mid u|_{\partial\Omega_c} = 0 ; \nabla u \cdot \nu|_{\partial\Omega_o} = 0\}.
$$

In Section 4 we prove that a proper bound on the resolvent of $L_h$, which is the elliptic operator in (1.1a) linearized near $(0, hA_n, h\phi_n)$, obtained over a vertical line in the complex plane, guarantees global stability of the normal state. In particular we show:

Theorem 1.2. Let $\nu \geq 0$. There exists $\kappa_0 > 0$ and $C_1 > 0$ such that, if for some $\kappa > \kappa_0$ we have

$$
\sup_{\gamma \in \mathbb{R}} \|(L_h - i\gamma - \nu)^{-1}\| < 1 - \frac{C_1}{\kappa^2},
$$

then, any solution of (1.1) must satisfy

$$
\int_0^\infty e^{2\nu t} \|\psi(t, \cdot)\|^2_2 dt < \infty.
$$

Unlike (1.12), (1.14) does not guarantee that the semigroup necessarily becomes a contraction in the long-time limit. The above stability is proved in the large $\kappa$ limit both for (1.1) and, in Section 5, for the same system, scaled with respect to the penetration depth, which is obtained by applying the transformation $x \to x/\kappa$ in (1.1), (cf. Proposition 5.6).

As the resolvent of $L_h$ in an arbitrary domain is difficult to control, we provide, in Section 6, an estimate of its norm for large values of $h$, which can be applied for either large domains (with respect to the coherence length), or large $\kappa$ values for penetration depth scaling. We show that its norm can be controlled using bounds derived from two approximate problems, with constant current defined either in $\mathbb{R}^2$ or in $\mathbb{R}^2_+$ with Dirichlet boundary conditions. From the resolvent estimates, together with the results in [2, 4, 3] we deduce that the critical current, for which the normal state looses its local stability, can be approximated by the same critical current obtained for the above $\mathbb{R}^2_+$ problem.
For a more precise description of the results in Section 6, we recall from [2] and [4] the definitions of these model operators in \( \mathbb{R}^2 \) and \( \mathbb{R}^2_+ \). Let

\[(1.16) \quad A(j, c) = -\left( \nabla - ij \frac{2}{3} i_y \right)^2 + icjy , \]

(where \( i_y \) is a unit vector in the \( y \) direction) defined on \( D(A) \) where

\[(1.17) \quad D(A) = \{ u \in L^2(\mathbb{R}^2) \mid Au \in L^2(\mathbb{R}^2) \} . \]

Let \( A_+(j, c) \) be defined by the same differential operator defining \( A \) but on the domain

\[(1.18) \quad D(A_+) = \{ u \in \tilde{V} : A_+u \in L^2(\mathbb{R}^2_+, \mathbb{C}) \}, \]

where

\[(1.19) \quad \tilde{V} = H_0^{1, \text{max}}(\mathbb{R}^2_+, \mathbb{C}) \cap L^2(\mathbb{R}^2_+, \mathbb{C}; t \, ds \\ dt) . \]

Set

\[(1.20) \quad h|\nabla B_n(z_0)| = j(z_0) , \]

and then define

\[(1.21) \quad A(z_0) = A(j(z_0), c) ; \quad A_+(z_0) = A_+(j(z_0), c) . \]

We show in Section 2 that under all of the above assumptions: (J), (R), (B), and (C), \( B_{n}^{-1}(0) \) is either empty, or else consists of a single curve \( \Gamma \) connecting the two connected components of \( \partial \Omega_c \). We denote the two points of intersection by \( z_1 \) and \( z_2 \) and then set

\[(1.22) \quad j_+ = \inf_{i=1,2} j(z_i) . \]

We then let

\[(1.23) \quad \nu_m(j, c) = \inf_{\lambda \in \sigma(A_+(j, c))} \Re \lambda . \]

A straightforward dilatation argument, which we detail in Section 6, shows that

\[(1.24a) \quad \nu_m(j, c) = j^{2/3} \nu_m(1, c) \]

\[(1.24b) \quad \|A^{-1}(j, c)\| = j^{-2/3} \|A^{-1}(1, c)\| \]

\[(1.24c) \quad \sup_{\gamma \in \mathbb{R}} \|(A_+(j, c) - i\gamma)^{-1}\| = j^{-2/3} \sup_{\gamma \in \mathbb{R}} \|(A_+(1, c) - i\gamma)^{-1}\| \]

We can now state

**Theorem 1.3.** Let \( \mu_R \) and \( \mu_\infty \) be respectively defined by

\[(1.25) \quad \mu_R = R^2 \inf_{\lambda \in \sigma(L^2_{\mu_R})} \Re \lambda \quad \text{and} \quad \mu_\infty = \liminf_{R \to \infty} \mu_R . \]

Then

\[(1.26) \quad \mu_\infty = \lim_{R \to \infty} \mu_R = \nu_m , \]

with

\[(1.27) \quad \nu_m = \nu_m(j_+, c) . \]
Furthermore, let $\nu < \mu_\infty$. Then, there exist positive $R_0$ and $C$, depending only on $\Omega$, $\nu$, and $h$ such that, for $R \geq R_0$, we have

\begin{equation}
2 \sup_{\gamma \in \mathbb{R}} \| (L^R h - R^2 \nu - i R^3 \gamma)^{-1} \| \leq \max \left( \sup_{z_0 \in \Gamma} \| (A(z_0) - \nu)^{-1} \|, \sup_{\gamma \in \mathbb{R}} \| (A_+(z_i) - \nu - i \gamma)^{-1} \| \right) \left( 1 + \frac{C}{R^{1/4}} \right) + \frac{C}{R^{1/4}}.
\end{equation}

**Remark 1.4.** One can deduce from (1.28) an upper bound for the critical current where the normal state $(0, hA_n, h\phi_n)$ becomes globally stable. Let

\begin{equation}
j_m = \inf_{z \in \Gamma} j(z),
\end{equation}

and let $j_+$ be defined by (1.22). As is proved in Section 6, and in particular in (6.4), when the domain size is multiplied by $R$, the resolvent norm of $L_h$ is given by the left-hand-side of (1.28). By (1.14) it then follows that if both the domain and $\kappa$ are sufficiently large, and if

\begin{equation}
j_m > \| A^{-1}(1, c) \|^3/2
\end{equation}

and

\begin{equation}
j_+ > \sup_{\gamma \in \mathbb{R}} \| (A_+(1, c) - i \gamma)^{-1} \|^3/2,
\end{equation}

then the normal state must be globally stable. The above conditions serve as an upper bound for the critical current where the normal state becomes globally stable.

An obvious lower bound for this global stability current, is the critical current for which the normal state becomes linearly unstable. For large domains such instability is granted when $\mu_R < 1$. By (1.26), for sufficiently large $R$, it follows that the loss of stability would take place when $\nu_m < 1$. Using (1.24) it then follows that whenever

\begin{equation}
j_+ < \nu_m(1, c)^{-3/2},
\end{equation}

local stability is lost for sufficiently large $R$. The optimality of the above bound and of (1.30) is left for future research.

We conclude this work by providing some well-known elliptic-regularity results for domains with corners in Appendices A and B. Then in Appendix C we show how to use these results for parabolic problems. Finally, in Appendix D we use the results of the previous appendices to prove global existence, uniqueness, and regularity for solutions of (1.1).

2. Preliminaries

2.1. Equivalent boundary conditions.
Instead of considering the boundary conditions (1.1e,f,g), it is possible to use an equivalent
boundary condition where we prescribe instead the magnetic field. As in [38] we note that by (1.1b,e,f), on each point on ∂Ω, except for the corners, we have

\begin{equation}
\frac{\partial}{\partial \tau} \text{curl} A(t, \cdot) = \frac{1}{\kappa^2} J(\cdot),
\end{equation}

where \( \partial/\partial \tau \) denotes the tangential derivative along ∂Ω in the positive trigonometric direction. For convenience we set

\begin{equation}
J(x) \equiv 0 \text{ on } \partial \Omega_i.
\end{equation}

Thus, if we introduce on the boundary the function \( B \) via

\begin{equation}
\text{curl} A(t, x) = h B(t, x) \text{ on } \partial \Omega,
\end{equation}

where \( h \) denotes a parameter measuring the intensity of the magnetic field, we first observe that it satisfies

\begin{equation}
B(t, x) = B(t, x_0) + \frac{1}{h \kappa^2} \int_{\Gamma(x, x_0)} J(\tilde{x}) \, ds(\tilde{x}),
\end{equation}

where \( (x, x_0) \in \partial \Omega \times \partial \Omega, \Gamma(x, x_0) \) is the portion on the boundary connecting \( x_0 \) and \( x \) in the positive trigonometric direction, and \( ds \) is a length element. For later reference, we define the reference current \( J_r \)

\begin{equation}
h J_r = J.
\end{equation}

Clearly, \( J_r(x) \) is as smooth as \( J \), i.e. at least \( C^2(\partial \Omega_c) \). Note that by (1.3) we have

\begin{equation}
\int_{\partial \Omega} J_r(x) \, ds = 0.
\end{equation}

One can recover the magnetic field \( B(t, \cdot) \) at \( x_0 \) by integrating of (2.4) over \( \partial \Omega \) (\( x_0 \) remaining fixed). This gives, with the aid of (1.1g),

\begin{equation*}
B(t, x_0) = h_r - \frac{1}{\kappa^2} \int_{\partial \Omega} \left( \int_{\Gamma(\tilde{x}, x_0)} J_r(\tilde{x}) \, ds(\tilde{x}) \right) \, ds(\tilde{x}) \text{ for } x_0 \in \partial \Omega,
\end{equation*}

where \( h_r = h_{ex}/h \).

We can thus conclude that \( B(t, x) \) does not depend on \( t \), that is: \( B(t, x) = B(x) \). Switching the order of integration then yields for \( B \):

\begin{equation}
B(x) = h_r - \frac{1}{\kappa^2} \int_{\partial \Omega} |\Gamma(\tilde{x}, x)| J_r(\tilde{x}) \, ds(\tilde{x}) \text{ for } x \in \partial \Omega.
\end{equation}

Note that by (2.4) and (1.2), \( B \) must be continuous along \( \partial \Omega \) and, moreover, we have the property:

\begin{equation}
The magnetic field \( B \) is constant along each component of \( \partial \Omega_i \).
\end{equation}
Hence (pending on the verification of the spaces in which we should consider the solutions) the system (TDGL1) is equivalent to the system (TDGL2)

\[
\begin{align*}
\frac{\partial \psi}{\partial t} + i\phi \psi &= (\nabla - iA)^2 \psi + \psi \left(1 - |\psi|^2\right), & \text{in } \mathbb{R}_+ \times \Omega, \\
\kappa^2 \text{curl}^2 A + \sigma \left(\frac{\partial A}{\partial t} + \nabla \phi\right) &= \text{Im} (\overline{\psi} \nabla \psi), & \text{in } \mathbb{R}_+ \times \Omega, \\
\psi &= 0, & \text{on } \mathbb{R}_+ \times \partial \Omega_c, \\
(i\nabla + A) \psi \cdot \nu &= 0, & \text{on } \mathbb{R}_+ \times \partial \Omega, \\
\text{curl } A(t, x) &= hB(x), & \text{on } \mathbb{R}_+ \times \partial \Omega, \\
\psi(0, x) &= \psi_0(x), & \text{in } \Omega, \\
A(0, x) &= A_0(x), & \text{in } \Omega.
\end{align*}
\]

where $B$ is given by (2.7).

Conversely, a solution of (TDGL2) must satisfy (TDGL1) with

\[
J_r = \kappa^2 \frac{\partial B}{\partial \tau}, \quad \text{and } h_r = \int_{\partial \Omega} B(x) ds,
\]

having in mind that $J = hJ_r$ and $h_{ex} = hh_r$.

**Remark 2.1.** Note the above equivalence has only been established formally, as the regularity of the solutions has not been addressed yet. We return to this point in Subsection 2.5 where we provide a precise definition of the spaces where the solutions reside.

### 2.2. Stationary states.

For a normal state we have $\psi \equiv 0$ by definition. Furthermore, denoting the corresponding stationary magnetic and electric potentials respectively by $hA_n$ and $h\phi_n$ we obtain, after dividing by $h$, that $(A_n, \phi_n)$ weakly satisfy

\[
\begin{align*}
-\Delta \phi_n + \nabla^2 A_n &= 0 & \text{in } \Omega, \\
\text{curl } A_n &= B & \text{on } \partial \Omega,
\end{align*}
\]

where

\[
c = \kappa^2 / \sigma
\]

is a positive parameter and $B$ is defined by (2.7).

As we later discuss, we choose the Coulomb gauge and assume that $A_n$ satisfies:

\[
A_n \in H^1(\Omega), \quad \text{div } A_n = 0, \quad A_n \cdot \nu/\partial \Omega = 0.
\]

We now show that (2.12) combined with (2.10) is uniquely solvable. We begin by constructing $\phi_n$ as a solution of the following problem, which can formally be obtained by taking the divergence of (2.10),

\[
\begin{align*}
-\Delta \phi_n &= 0 & \text{in } \Omega, \\
\frac{\partial \phi_n}{\partial \nu} &= -\frac{J_r(x)}{\sigma} & \text{on } \partial \Omega.
\end{align*}
\]
Remark 2.2. Let $v$ denote a function in $H^2(\Omega)$. Then the trace of its normal derivative is well defined in $H^{1/2}(\Gamma)$, where $\Gamma$ is any regular component of $\partial \Omega$. For convenience of notation we write $v \in H^{1/2}(\partial \Omega)$ in the sequel. The reader should thus be careful not to adopt the conventional interpretation of this notation which may not apply in some cases (consider, for instance, the case where $J$ is discontinuous at the corners).

We seek a solution to the problem (2.13) in $H^2(\Omega)$ such that

$$\int_{\Omega} \phi_n \, dx = 0. \tag{2.14}$$

Since $J_r \in C^2(\partial \Omega)$, and

$$\int_{\partial \Omega} J_r(x) \, ds = 0. \tag{2.16}$$

We can now use Proposition A.2 and property (R1) of $\Omega$ to obtain that $\phi_n$ uniquely exists and that

$$\phi_n \in W^{2,p}(\Omega), \tag{2.15}$$

for all $p \geq 2$.

Similarly, we construct $B_n = \text{curl} A_n$ as the solution of a problem which can be formally derived by taking the curl of (2.10),

$$\begin{cases} \Delta B_n = 0 & \text{in } \Omega, \\ B_n = B & \text{on } \partial \Omega. \end{cases} \tag{2.17}$$

Although one can obtain an explicit formula for $B_n$ in $\Omega$ (which amounts to extending (2.7) into $\Omega$) using the strong regularity of $J$, we prefer to use (2.16) which allows us to rely on the regularity results in Appendix A. In particular, by Proposition A.1, we have, as $\Omega$ satisfies condition (R1), a unique $B_n \in H^2(\Omega)$ solution of (2.16). Moreover

$$B_n \in W^{2,p}(\Omega), \forall p \geq 2. \tag{2.18}$$

We can now proceed to determine $A_n$. To ensure that $\text{div} A_n = 0$, we look for $A_n$ in the form:

$$A_n = \nabla_\perp \theta_n. \tag{2.19}$$

Since $B_n = \text{curl} A_n$, we have

$$\begin{cases} -\Delta \theta_n = B_n & \text{in } \Omega, \\ \theta_n = 0 & \text{on } \partial \Omega. \end{cases} \tag{2.19}$$

Hence, we set $\theta_n$ to be the variational solution of the above Dirichlet problem. The Dirichlet condition $\theta_n$ satisfies ascertains that $A_n$ meets the condition $A_n \cdot \nu = 0$ on $\partial \Omega$. From Proposition A.1 (see (A.4)) it then follows that $\theta_n \in W^{3,p}(\Omega)$ for all $p < 2$, and that $\theta_n \not\in W^{3,p}(\Omega)$ for $p > 2$ unless $B$ vanishes at every corner (a case which certainly doesn’t fall into the (J3) category in (1.4)). Hence,

$$A_n \in W^{2,p}(\Omega, \mathbb{R}^2), \text{ for all } p < 2. \tag{2.19}$$
It remains to show that \((\phi_n, A_n)\) is indeed a solution of \((2.10)\), as we have only established, so far, that \(V_n := -c \text{curl}^2 A_n + \nabla \phi_n\) satisfies:

\[
V_n \in L^2(\Omega, \mathbb{R}^2), \quad \text{div} \, V_n = 0 \quad \text{and} \quad \text{curl} \, V_n = 0 \quad \text{in} \quad \Omega, \quad V_n \cdot \nu = 0 \quad \text{on} \quad \partial \Omega.
\]

To obtain the last property of the previous line, we used \((2.1)\) and the boundary condition for \(\phi_n\) in \((2.13)\). We can now use the decomposition Proposition B.1 to conclude that \(V_n = 0\), and hence \((\phi_n, A_n)\) satisfy \((2.10)\). Finally, note that \(\phi_n\) is unique, by \((2.13)\) and \((2.14)\), and that the uniqueness of \(A_n\) follows from \((2.10)\).

We summarize the above discussion by the following proposition:

**Proposition 2.3.**

Suppose that \(\partial \Omega\) has the property \((R1)\). There exists a unique solution \((\phi_n, A_n) \in H^2(\Omega) \times H^1(\Omega)\) satisfying \((2.10)\), \((2.12)\), and \((2.14)\). This solution belongs to \(W^{2,p}(\Omega) \times W^{2,q}(\Omega, \mathbb{R}^2)\), for all \(p \geq 2\) and \(q < 2\). Moreover \((0, h\phi_n, hA_n)\) is a stationary solution of \((1.1)\), and \(\text{curl} \, A_n \in W^{2,p}(\Omega, \mathbb{R}^2)\) for all \(p \geq 2\).

Using Sobolev embeddings, we deduce in particular that:

\[
\phi_n, B_n, \quad \text{and} \quad \text{curl} \, B_n \quad \text{belong to} \quad C^1(\Omega).
\]

2.3. **A magnetic Laplacian.** Next, we define

\[
(2.21) \quad \mu(h) = \min_{u \in H^{1,0}_{\partial \Omega} \mid \|u\|_2 = 1} \|\nabla h A_n u\|_2^2,
\]

where

\[
H^{1,0}_{\partial \Omega} = \{ u \in H^1(\Omega, \mathbb{C}) \mid u|_{\partial \Omega} = 0 \},
\]

in which the boundary data appear in a trace sense. Using the diamagnetic inequality \([16]\), it is easy to show that

\[
(2.22) \quad \mu(h) \geq \mu(0) > 0,
\]

since \(\Omega\) is relatively compact.

Under relatively weak assumptions one can obtain that

\[
(2.23) \quad \lim_{h \to +\infty} \mu(h) = +\infty.
\]

One can estimate the rate of divergence of \(\mu\), in the large \(h\) limit, by assuming first that \((R2), (J2)\) and \((J3)\) hold true. In that case, \(B\) is strictly monotone on each component of \(\partial \Omega_i\). We now argue as in [1] (proof of Proposition 4.1 there). Observing that \(B_n\) is continuous on \(\overline{\Omega}\) and harmonic in \(\Omega\), the maximum principle shows that the minimum \(B_{\text{min}}\) of \(B_n\) in \(\Omega\) is attained on one component of \(\partial \Omega_i\) and that the maximum \(B_{\text{max}}\) is attained at the other component.

Assume further that

\[
(2.24) \quad (B) \quad B_{\text{min}}^{-1}(0) = \emptyset \quad \text{or} \quad B_{\text{min}} < 0 < B_{\text{max}},
\]

which implies that \(B_{\text{min}}^{-1}(0)\) lies away from the corners.
Remark 2.4. This condition can be expressed in terms of the boundary data (1.1). With the aid of (2.7) we obtain that \(0 \in (B_{\min}, B_{\max})\) is equivalent to

\[
\frac{1}{\kappa^2} \int_{\partial \Omega} \Gamma(\tilde{x}, c_{\min}) |J_r(\tilde{x})| ds(\tilde{x}) < h_r < \frac{1}{\kappa^2} \int_{\partial \Omega} \Gamma(\tilde{x}, c_{\max}) |J_r(\tilde{x})| ds(\tilde{x})
\]

where \(c_{\min}\) and \(c_{\max}\) lie both on \(\partial \Omega\) and satisfy \(B(c_{\min}) = B_{\min}\) and \(B(c_{\max}) = B_{\max}\).

By the maximum principle, we first deduce that \(B^{-1}(\mu)\) cannot contain a loop for any \(\mu \in (B_{\min}, B_{\max})\). Then, we use the above-stated monotonicity to conclude that \(B^{-1}(\mu) \cap \partial \Omega\) consists of precisely two points: one on each connected component of \(\partial \Omega_{\epsilon}\). Hence \(B^{-1}(\mu)\) must be a simple smooth curve joining the two components of \(\partial \Omega_{\epsilon}\). By Hopf’s lemma for harmonic functions (cf. §6.4.2 in [13] for instance), it thus follows that \(\nabla B_n \neq 0\) on \(\Omega \cap B^{-1}(\mu)\). We have thus proved that

\[
\nabla B_n \neq 0 \text{ in } B^{-1}([B_{\min}, B_{\max}]) = \bar{\Omega},
\]

and in particular on \(B^{-1}_n(0)\). It is now possible to use the same methods as in [31] to obtain the existence of some \(\mu_0 > 0\) such that

\[
\mu(h) \geq \inf_{u \in H^1(\Omega, \mathbb{C})} \frac{\|
abla h A_n u\|_2^2}{\|u\|_2} \geq \mu_0 h^{2/3}, \forall h \geq 1.
\]

2.4. Another spectral entity.

To be able to state the main result in the next section we need to define yet another entity. Let then

\[
\lambda = \inf_{V \in \mathcal{H}_d} \frac{\|
abla V\|_2^2}{\|V\|_2},
\]

where

\[
\mathcal{H}_d = \{ V \in H^1(\Omega, \mathbb{R}^2) | \text{div} V = 0, V|_{\partial \Omega} \cdot \nu = 0 \}.
\]

We next provide an alternative characterization of \(\lambda\).

Proposition 2.5. Under condition (R1),

\[
\lambda = \lambda^D,
\]

where \(\lambda^D\) is the ground state energy of the Dirichlet Laplacian \(-\Delta^D\).

Proof. We have seen in Proposition A.1 that the domain of \(\Delta^D\) is \(\mathcal{H}_p := H^2(\Omega) \cap H_0^1(\Omega)\). Let \(u\) denote an \(L^2\)-normalized ground state of \(-\Delta^D\). Then \(\nabla^\perp u\) belongs to \(\mathcal{H}_d\), and \(\text{curl} \nabla^\perp u = -\Delta u = \lambda^D u\). Hence

\[
\|	ext{curl} \nabla^\perp u\|_2 = \lambda^D \langle u, \text{curl} \nabla^\perp u \rangle = \lambda^D \|
abla^\perp u\|_2^2.
\]

From the above we deduce that

\[
\lambda \leq \lambda^D.
\]
Conversely, let \( V \in \mathcal{H}_d \). Under assumption (R1) there exists \( \Phi \in \mathcal{H}_p \) such that \( V = -\nabla_\perp \Phi \) (cf. Proposition B.1). Moreover \( \nabla_\perp \) is a bijection from \( \mathcal{H}_p \) onto \( \mathcal{H}^d \). Hence, we can rewrite (2.27), in terms of \( \Phi \), in the form

\[
(2.30) \quad \lambda = \inf_{\Phi \in \mathcal{H}_p} \| \Delta \Phi \|_2^2/p \| \nabla \Phi \|_2^2 = 1.
\]

It can be readily verified that the functional in (2.30) is lower semicontinuous. Furthermore, it is also coercive in view of Proposition A.1. We can thus conclude the existence of a minimizer which we denote by \( \Phi_{\text{min}} \). Evaluating the first variation we can conclude that

\[
\int_\Omega \Delta \Phi_{\text{min}} (\Delta \eta + \lambda \eta) \, dx = 0, \quad \forall \eta \in \mathcal{H}_p.
\]

Clearly, if \( \Delta + \lambda : \mathcal{H}_p \to L^2(\Omega) \) is invertible, then we must have \( \Delta \Phi_{\text{min}} = 0 \) and since \( \Phi_{\text{min}} \in H^1_0(\Omega) \), it follows that \( \Phi_{\text{min}} \equiv 0 \), contradicting the requirement that \( \| \nabla \Phi_{\text{min}} \|_2 = 1 \). Consequently, \( \lambda \) is an eigenvalue of the Dirichlet Laplacian \( -\Delta^D \), hence satisfying \( \lambda \geq \lambda^D \).

2.5. Gauge equivalence and weak solutions.
We assume that \( \Omega \) has property (R1) (see (1.5)). Let \( A \in L^2_{\text{loc}}([0, \infty); H^1(\Omega, \mathbb{R}^2)) \), \( \psi \in L^2_{\text{loc}}([0, \infty); H^1(\Omega, \mathbb{C})) \), and \( \phi \in L^2_{\text{loc}}([0, \infty); L^2(\Omega, \mathbb{R}^2)) \). Following [7], we say that \( (\psi', A', \phi') \) is gauge equivalent to \( (\psi, A, \phi) \) if there exists \( \omega \in L^2_{\text{loc}}([0, \infty); H^2(\Omega)) \cap H^1_{\text{loc}}([0, \infty); H^1(\Omega)) \) such that

\[
(2.31) \quad A' = A + \nabla \omega, \quad \phi' = \phi - \frac{\partial \omega}{\partial t}, \quad \psi' = \psi e^{i\omega}.
\]

We say that \( (\psi', A', \phi') = G_\omega(\psi, A, \phi) \) in that case. It is easy to show that (2.31) is an equivalence relation. We begin by defining the gauge (cf. [7])

\[
(2.32) \quad \mathbb{H} = \{(u, v) \in H^1(\Omega, \mathbb{R}^2) \times L^2(\Omega) \mid c \text{ div } u + v = 0; \ u \cdot \nu|_{\partial \Omega} = 0 \}.
\]

Let \( (\psi, A, \phi) \in L^2_{\text{loc}}([0, \infty); H^1(\Omega, \mathbb{C})) \times L^2_{\text{loc}}([0, \infty); H^1(\Omega, \mathbb{R}^2)) \times L^2_{\text{loc}}([0, \infty); L^2(\Omega, \mathbb{R}^2)) \), such that \( A_0|_{\partial \Omega} \cdot \nu = 0 \) (where \( A_0 = A(0, \cdot) \)). We first show that there exists a unique gauge equivalent \( (\psi_d, A_d, \phi_d) \in L^2_{\text{loc}}([0, \infty); H^1(\Omega, \mathbb{C})) \times L^2_{\text{loc}}([0, \infty); \mathbb{H}) \). To prove this, following [7], we first solve

\[
(2.33) \begin{cases}
\frac{\partial \chi}{\partial t} - c \Delta \chi = c \text{ div } A + \phi & \text{in } (0, \infty) \times \Omega \\
\frac{\partial \nu}{\partial t} = -A \cdot \nu & \text{on } (0, \infty) \times \partial \Omega \\
\chi|_{t=0} = 0 & \text{in } \Omega.
\end{cases}
\]

In Appendix C we prove that there exists a solution \( \chi \in L^2_{\text{loc}}([0, \infty); H^2(\Omega)) \cap H^1_{\text{loc}}([0, \infty); L^2(\Omega)) \) to the above problem. It can be readily verified that \( (\psi_d, A_d, \phi_d) = G_\chi(\psi, A, \phi) \). In the case where \( A_0|_{\partial \Omega} \cdot \nu \neq 0 \) we define first the gauge function

\[
\begin{cases}
\Delta \chi_0 = 0 & \text{in } \Omega \\
\frac{\partial \chi_0}{\partial t} = -A_0 \cdot \nu & \text{on } \partial \Omega.
\end{cases}
\]

Then, we replace \( (\psi, A, \phi) \) by \( G_{\chi_0}(\psi, A, \phi) \) and proceed as before.
Let
\[ W_1 = \{ V \in H^1(\Omega; \mathbb{R}^2), V \cdot \nu = 0 \text{ on } \partial \Omega \} , \]
and
\[ W_2 = \{ \psi \in H^1(\Omega), \psi = 0 \text{ on } \partial \Omega_c \} . \]

Let \( A \in L^2_{\text{loc}}([0, \infty); H^1_1(\Omega_0)) \cap H^1_{\text{loc}}([0, \infty); W_2^1) \), and \( \phi \in L^2_{\text{loc}}([0, \infty); L^2(\Omega, \mathbb{R}^2)) \) denote a weak solution of (1.1). (The reader is referred to [7] for the definition of a weak solution.) It is easy to show that \( G_\omega(\psi, A, \phi) \) is also a weak solution of (1.1) for any \( \omega \in L^2_{\text{loc}}([0, \infty); H^2(\Omega)) \cap H^1_{\text{loc}}([0, \infty); L^2(\Omega)) \). In Theorem 2.6 and Appendix D we prove (relying on [15]) that the solution of (2.9) in \( L^2_{\text{loc}}([0, \infty); H^1(\Omega, \mathbb{C})) \times L^2_{\text{loc}}([0, \infty); L^2(\Omega, \mathbb{R}^2)) \) is unique. From the foregoing discussion we can thus conclude that all the weak solutions of (1.1) are gauge equivalent. In particular, this would mean that all the results established in the next sections for the decay of \(|\psi|\) are valid for all possible solutions of (2.9). In the next subsections, we concentrate on strong solutions of (2.9).

2.6. The strong solution in the Coulomb gauge. In view of the discussion in the previous subsection we fix the Coulomb gauge, i.e., we look for global solutions in \( L^2_{\text{loc}}([0, +\infty), H^1(\Omega, \mathbb{R}^2)) \) of (1.1) satisfying
\[ \text{(2.34)} \]
\[ \text{div} \, A(t, \cdot) = 0 \text{ in } L^2_{\text{loc}}([0, +\infty), L^2(\Omega)), \quad A(t, \cdot) \cdot \nu|_{\partial \Omega} = 0 \text{ in } L^2_{\text{loc}}([0, +\infty), H^1(\partial \Omega)), \]
and we also assume:
\[ \text{(2.35)} \]
\[ \int_{\Omega} \phi(t, x) \, dx = 0 \text{ in } L^2_{\text{loc}}([0, +\infty)). \]

Suppose first that the initial condition \( A_0 \) satisfies
\[ \text{(2.36)} \]
\[ \text{div} \, A_0 = 0 \text{ in } \Omega, \quad A_0 \cdot \nu = 0 \text{ on } \partial \Omega, \]
where
\[ A_0 \in H^2(\Omega, \mathbb{R}^2). \]
We further assume that
\[ \text{(2.37)} \]
\[ \psi_0 \in H^2(\Omega, \mathbb{C}), \]
and (1.8).

We show that the solution \((\psi_d, A_d, \phi_d)\) established in Theorem D.2 with \( \hat{A}_0 = A_0 \) and \( \hat{\psi}_0 = \psi_0 \) is gauge-equivalent to the solution of (1.1) and (2.34).
To this end we define the gauge function \( \omega \) as the solution of
\[ \text{(2.39)} \]
\[ \begin{cases} -\Delta \omega = \text{div} \, A_d & \text{in } (0, +\infty) \times \Omega, \\ \frac{\partial \omega}{\partial \nu} = 0 & \text{on } (0, +\infty) \times \partial \Omega, \\ \int_{\Omega} \omega(t, x) \, dx = 0 & \text{in } (0, +\infty). \end{cases} \]
As \( A_d \in C([0, +\infty); W^{1+\alpha,2}(\Omega, \mathbb{R}^2)) \) for any \( 0 < \alpha < 1 \), it follows by Sobolev embeddings and Proposition A.2 that \( \omega \in C([0, +\infty); W^{1,p}(\Omega)) \) for all \( p \geq 2 \). Furthermore, since
\[ \text{div} \, A_d \in L^2_{\text{loc}}([0, +\infty), H^1(\Omega)) \] we get by (A.7) \begin{equation}
abla \omega \in L^2_{\text{loc}}([0, +\infty), H^3(\Omega)). \tag{2.40} \end{equation}

Next, we observe that the projector \( \pi_1 \) introduced in Proposition B.1 extends (by tensor product) to a projector \( \Pi_1 \) in \( H^1_{\text{loc}}([0, +\infty); L^2(\Omega, \mathbb{R}^2)) \) and that by the uniqueness of the decomposition established in the proposition and (2.39):

\begin{equation}
-\nabla \omega = \Pi_1 A_d, \tag{2.41} \end{equation}

in \( \mathcal{D}'((0, +\infty); L^2(\Omega, \mathbb{R}^2)) \), where \( \mathcal{D}'((0, +\infty); L^2(\Omega, \mathbb{R}^2)) \) denotes the space of distributions on \((0, +\infty)\) with value in \( L^2(\Omega, \mathbb{R}^2) \).

Note that (2.41) simply reads

\begin{equation}
-\nabla \left( \int \omega(t, \cdot) \phi(t) dt \right) = \pi_1 \left( \int A_d(t, \cdot) \phi(t) dt \right), \tag{2.42} \end{equation}

for all \( \phi \in C^\infty_0(0, +\infty) \).

The right hand side of (2.41) being in \( H^1_{\text{loc}}((0, +\infty); L^2(\Omega, \mathbb{R}^2)) \), this implies that \( \nabla \omega \in H^1_{\text{loc}}((0, +\infty); L^2(\Omega, \mathbb{R}^2)) \), and hence

\begin{equation}
\partial_t \omega \in L^2_{\text{loc}}((0, +\infty); H^1(\Omega, \mathbb{R}^2)). \tag{2.43} \end{equation}

It can now be readily verified from (2.40) and (2.43) that the Coulomb gauge solution \( (\psi_c, A_c, \phi_c) = G_\omega(\psi_d, A_d, \phi_d) \) satisfies:

\begin{equation}
\psi_c \in C([0, +\infty); W^{1+\alpha/2}(\Omega, \mathbb{C})) \cap H^1_{\text{loc}}([0, +\infty); L^2(\Omega, \mathbb{C})), \forall \alpha < 1, \tag{2.44} \end{equation}

\begin{equation}
A_c \in C([0, +\infty); W^{1,p}(\Omega, \mathbb{R}^2)) \cap H^1_{\text{loc}}([0, +\infty); L^2(\Omega, \mathbb{R}^2)), \forall p \geq 1, \tag{2.45} \end{equation}

which follows from the fact that by (2.41) \( \nabla \omega \in C((0, +\infty); W^{1,p}(\Omega, \mathbb{R}^2)) \), and

\begin{equation}
\phi_c \in L^2_{\text{loc}}([0, +\infty); H^1(\Omega)). \tag{2.46} \end{equation}

Relying on Theorem D.2 and the above discussion we can now state:

**Theorem 2.6.** Suppose that \( \Omega \) satisfies condition (R1) and that \( B \) is in \( H^1(\partial \Omega) \) (see Remark 2.2). Suppose further that \( (\psi_0, A_0) \) satisfies (2.37), (2.36), (2.38) and (1.8).

Then, there exists a unique weak solution \( (\psi_c, A_c, \phi_c) \) of (TDGL2) in the Coulomb gauge. Moreover, this solution is strong in the sense that it satisfies (2.44)-(2.46) and

\begin{equation}
\|\psi_c(t, \cdot)\|_\infty \leq 1, \forall t > 0. \tag{2.47} \end{equation}

Finally, let \( A_1 = A_c - h A_n \) where \( A_n \) satisfies (2.10) and (2.12). Then

\begin{equation}
A_1 \in L^2_{\text{loc}}([0, +\infty); H^2(\Omega, \mathbb{R}^2)). \tag{2.48} \end{equation}

The last statement of the theorem is a consequence of (D.4), (2.40) and Theorem D.2.

We can indeed write:

\[ A_1 = \tilde{A}_1 - \nabla \omega - h(A_n - A_{n,d}), \]

where \( \tilde{A}_1 \) is defined in (D.6).
Remark 2.7. If $A_0$ does not satisfy (2.36), we let $\omega_0 \in H^2(\Omega)$ be solution of
\[
\begin{cases}
-\Delta \omega_0 = \text{div} A_0 & \text{in } \Omega, \\
\frac{\partial \omega_0}{\partial \nu} = -A_0 \cdot \nu & \text{on } \partial \Omega, \\
\int_{\Omega} \omega_0(x) \, dx = 0.
\end{cases}
\]
It follows by Proposition A.2, that, for $\Omega$ satisfying property (R), $\omega_0 \in H^2(\Omega)$. We then consider $G_{\omega_0}(\psi, A, \phi)$ which is a solution of (1.1) with initial conditions satisfying (2.36).

We can now return to the solution of (TDGL1).

Theorem 2.8. Under the assumptions of the previous theorem, assuming that $J$ satisfies (1.2)-(1.3), and $B$ by (2.4), the solution of (TDGL2) has the additional property that $\phi_c \in C([0, +\infty); W^{1,p}(\Omega))$ for all finite $p$, and is a solution of (TDGL1).

Proof. Let $(\psi_c, A_c, \phi_c)$ denote a solution of (TDGL2) and (2.34). One has to clarify first the sense in which the trace condition (1.1e)-(1.1f) is satisfied. By Theorem 2.6 we have that $\partial_t A_c + \nabla \phi_c$ belongs to $L^2_{\text{loc}}([0, +\infty), L^2(\Omega, \mathbb{R}^2))$. Hence, we can use the fact (see for example Theorem 2.2 in [19]) that for a vector field $V$ in $L^2_{\text{loc}}([0, +\infty); L^2(\Omega; \mathbb{R}^2))$ with $\text{div} V \in L^2_{\text{loc}}([0, +\infty); L^2(\Omega))$, the normal component of its trace, $V \cdot \nu|_{\partial \Omega}$, belongs to $L^2_{\text{loc}}([0, +\infty); H^{-\frac{1}{2}}(\partial \Omega))$.

Consider then $V = \partial_t A_c + \nabla \phi_c$. By (2.9b) and (2.34) we obtain:
\[
(2.49) \quad \sigma \text{div} V = \sigma \text{div} \nabla \phi_c = \text{Im} \text{div} (\bar{\psi}_c \cdot \nabla A_c \psi_c).
\]

It is easy to show that the left hand side is in $L^2_{\text{loc}}([0, +\infty); L^2(\Omega))$. As $\Delta A_c \psi_c \in L^2_{\text{loc}}([0, +\infty); L^2(\Omega))$ we can use (2.47) to conclude that $\psi_c \Delta A_c \psi_c \in L^2_{\text{loc}}([0, +\infty); L^2(\Omega))$. Furthermore, $\nabla \psi_c \in C([0, +\infty); L^4(\Omega, \mathbb{R}^2))$ and $A_c \in C([0, +\infty) \times \Omega)$ in view of (2.44) and (2.45), hence $\nabla \psi_c \cdot \nabla A_c \psi_c \in L^2_{\text{loc}}([0, +\infty); L^2(\Omega))$. Consequently, $V \cdot \nu$ is well defined in $L^2_{\text{loc}}([0, +\infty); H^{-\frac{1}{2}}(\partial \Omega))$, and we can discern that
\[
V \cdot \nu|_{\partial \Omega} = \partial_t \phi_c,
\]
due to the fact that $\partial_t A_c \cdot \nu = 0$ in $D'(0, +\infty; H^\frac{1}{2}(\partial \Omega))$ by (2.34).

Consider again (2.9b). Each term of the equality has a meaningful normal component for its trace and hence, as the right hand side has a zero ”normal” trace,
\[
(2.50) \quad V \cdot \nu = \kappa^2 \partial_c \text{curl} A_c = \kappa^2 h \partial_r B = J,
\]
in $L^2_{\text{loc}}([0, +\infty); H^{-\frac{1}{2}}(\partial \Omega))$, as expected. \qed

Remark 2.9. Although the main focus of this work is on Coulomb gauge solutions, it seems worthwhile to note that any weak solution for which $A \cdot \nu = 0$ on $\partial \Omega$, div $A \in L^2_{\text{loc}}([0, +\infty); H^1(\Omega))$, $\phi \in L^2_{\text{loc}}([0, +\infty); H^1(\Omega))$, and the initial data are regular, is also a strong solution. To prove this, we return to Equation (2.33) to determine the regularity of $\chi$. Then, one has to show that $\chi$ satisfies (2.40) and (2.43), which we prove by considering
the equation satisfied by \( V = \nabla \chi \). Differentiating (2.33) yields,

\[
\begin{aligned}
\frac{\partial V}{\partial t} - (\nabla \text{div} + \nabla^\perp \text{curl})V &= \varepsilon \nabla \text{div} A + \nabla \phi & \text{in } (0, +\infty) \times \Omega \\
\text{curl}V &= 0 & \text{on } (0, +\infty) \times \partial \Omega \\
V \cdot \nu &= 0 & \text{on } (0, +\infty) \times \partial \Omega \\
V_{t=0} &= 0 & \text{in } \Omega.
\end{aligned}
\]

(2.51)

The above problem possesses a unique solution \( V \in L^2_{\text{loc}}([0, +\infty) ; D(L(1))) \) and \( \partial_t V \in L^2_{\text{loc}}([0, +\infty) ; L^2(\Omega, \mathbb{R}^2)) \) (cf. Proposition A.3). Since by Proposition B.3, \( D(L(1)) \subset H^2(\Omega, \mathbb{R}^2) \), the desired regularity of \( \chi \) readily follows.

3. Asymptotic contraction properties of the semi-group

In this section we obtain our simplest estimate for the critical current, for which the normal state \((0, A_n, \phi_n)\), given by (2.10), becomes globally stable. We concentrate here on currents for which the (non-linear) semi-group associated with (1.1) and (2.34) becomes a contraction for sufficiently long times.

3.1. Analysis of the linearized problem.

Consider first the linearized version of (1.1):

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= L u, & \text{in } \mathbb{R}_+ \times \Omega, \\
(i \nabla + h A_n) u \cdot \nu &= 0, & \text{on } \mathbb{R}_+ \times \partial \Omega_i, \\
u &= 0, & \text{on } \mathbb{R}_+ \times \partial \Omega_c, \\
u(0, \cdot) &= u_0(\cdot), & \text{in } \Omega.
\end{aligned}
\]

(3.1)

In the above

\[ L = (\nabla - ih A_n)^2 + ih \phi_n + 1. \]

It is easy to show using integration by parts that for any \( v \in D(L) \) we have

\[ \langle v, L v \rangle = -\| \nabla h A_n v \|_2^2 + \| v \|_2^2. \]

By (2.21) we have that

\[ \langle v, L v \rangle \leq -(\mu - 1)\| v \|_2^2. \]

Note that if \( v \) is a ground state of \( L \) the above inequality becomes an identity. Hence, it follows that the operator \( L \) is dissipative if and only if \( \mu \geq 1 \). Consequently, it is easy to show from the Lumer-Phillips Theorem (Theorem 8.3.5 in [8]) that the semigroup associated with (3.1) is a contraction semigroup if and only if \( \mu \geq 1 \). If \( \mu > 1 \) one can easily show that any solution of (3.1) decays exponentially fast (with a decay like \( \exp \left\{- (\mu - 1)t\right\} \)) and hence, that \( u \equiv 0 \) is asymptotically stable.

If we now consider the linearized part of (1.1b), (after taking its curl), we get the equation for the first variation \( w \) of \( \text{curl} A \)

\[
\begin{aligned}
\sigma \partial_t w - \kappa^2 \Delta w &= 0 & \text{in } \mathbb{R}_+ \times \Omega, \\
w(t, \cdot) &= 0 & \text{on } \mathbb{R}_+ \times \partial \Omega, \\
w(0, \cdot) &= w_0(\cdot) & \text{on } \Omega.
\end{aligned}
\]
From the above we can conclude an $O(e^{-\lambda_D t})$-decay for $w(t, \cdot)$, where $\lambda_D$ is defined below (2.29) and $c = \frac{\sigma^2}{\kappa}$ (cf (2.11)). We recall from Proposition 2.5 that $\lambda = \lambda_D$.

We say that $f : \mathbb{R} \to \mathbb{R}_+$ has a monotone $O(e^{-\alpha t})$ decay, if $e^{\alpha t}f$ is monotone. From the foregoing discussion it therefore follows that we cannot hope for a better monotone decay than $e^{-\min(\mu - 1, \lambda c)t}$ for the asymptotic behavior of the nonlinear problem that we consider in the next subsection. (See Formula (3.3).)

We note that if, instead of imposing that both $e^{\alpha t}\|\psi(t, \cdot)\|_2$ and $e^{\alpha t}\|A(\cdot, t) - hA_n\|_2$ become monotone for sufficiently large $t$, we impose the weaker requirement that they are both bounded, we can obtain greater values of $\alpha$. This is precisely the focus of Sections 4 and 5.

3.2. Asymptotic analysis. In light of the above discussion we expect that asymptotic stability of $\psi \equiv 0$ could be achieved for $\mu > 1$ (at least in some asymptotic regimes). We also expect that the semigroup associated with (1.1) and (2.34) would turn asymptotically into a contraction semigroup, when $\|\psi(t, \cdot)\|_2$ becomes very small. The statement and the proof of these intuitions are made in part, in the following theorem for fixed values of $\sigma$ and $\kappa$. A particular attention is devoted to the limit $\kappa \to \infty$ with fixed $c = \frac{\kappa^2}{\sigma}$.

Theorem 3.1. Let $(\psi, A, \phi) \in \mathcal{H}$ denote a solution of (1.1) and (2.34) satisfying (1.8). Then, whenever

\[
\mu(h) > 1 + \frac{\gamma}{\kappa^2} + \frac{\gamma^2}{\kappa^4},
\]

where

\[
\gamma = (\lambda c + 2) \left[ \frac{2}{c\lambda^3} \right]^{1/2},
\]

there exist $C = C(\lambda, \kappa, \|\psi_0\|_2, \|A_0\|_2, h) > 0$ and $\lambda_m = \lambda_m(c, \kappa, \mu(h), \lambda) > 0$ such that, for all $t > 0$, we have:

\[
\|\psi\|_2 + \|A - hA_n\|_2 \leq Ce^{-\lambda_m t},
\]

where $A_n$ is the stationary normal solution introduced in (2.10)-(2.13) and satisfying (2.34).

Moreover, when $\mu > 1$, there exists $C = C(\mu, \lambda, c) > 0$ and $\kappa_0(\mu, \lambda, c)$ such that, for $\kappa \geq \kappa_0(\mu, \lambda, c),$

\[
\lambda_m \geq \min((\mu - 1), \lambda c) - \frac{C}{\kappa}
\]

More precisely,

(1) If $0 < \mu - 1 < \lambda c$ then

\[
\lambda_m \geq (\mu - 1) - 4c\mu(\lambda c - \mu + 1)^{-1} \kappa^{-2} + \mathcal{O}(\kappa^{-4}).
\]

(2) If $\mu - 1 = \lambda c$

\[
\lambda_m \geq \lambda c - \frac{2c^2}{\kappa} \left[ 3^{-4}(\lambda c + 1)^{\frac{3}{2}} \right] + \mathcal{O}(\kappa^{-2}).
\]
If $\mu - 1 > \lambda c$

$$\lambda_m \geq \lambda c - \frac{cd}{\kappa^2} + O(\kappa^{-4}).$$

where

$$d = 4\mu^{\frac{1}{2}}(\lambda c + 1)^{\frac{1}{2}}(\mu - 1 - \lambda c)^{-1}.$$ 

Proof:

Set

$$A_1 = A - hA_n ; \quad \phi_1 = \phi - h\phi_n.$$ 

Note that by (2.35) and (2.14) we have

$$\int_{\Omega} \phi_1(t,x) \, dx = 0.$$ 

Substituting into (1.1b,e,f) yields with the aid of (2.34),

$$\sigma \frac{\partial A_1}{\partial t} + \kappa^2 \text{curl}^2 A_1 + \sigma \nabla \phi_1 = \text{Im} (\bar{\psi} \nabla A \psi) \quad \text{in } \mathbb{R}_+ \times \Omega,$$

$$\frac{\partial \phi_1}{\partial \nu} = 0 \quad \text{on } \mathbb{R}_+ \times \partial \Omega.$$ 

Taking the scalar product in $L^2(\Omega, \mathbb{R}^2)$ of (3.11a) with $\nabla \phi_1$ yields after integration by parts, with the aid of (2.34),

$$\|\nabla \phi_1(t, \cdot)\|_2 \leq \frac{1}{\sigma} \|\text{Im} (\bar{\psi} \nabla A \psi)(t, \cdot)\|_2.$$ 

We next multiply (3.11a) by $A_1$ and integrate over $\Omega$. Observing that by (2.9e), (2.10), and (3.9) we have

$$\text{curl} A_1(t,x) = 0 \quad \text{for any } (t,x) \in \mathbb{R}_+ \times \partial \Omega,$$

we obtain

$$\frac{1}{2} \sigma \frac{d}{dt} \|A_1(t, \cdot)\|_2^2 + \kappa^2 \|\text{curl} A_1(t, \cdot)\|_2^2 = \langle A_1(t, \cdot), \text{Im} (\bar{\psi} \nabla A \psi)(t, \cdot) - \sigma \nabla \phi_1(t, \cdot) \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the $L^2(\Omega, \mathbb{R}^2)$ inner product. In view of (3.12) and (2.27) we have, as $\|\psi\|_\infty \leq 1$,

$$\frac{1}{2} \sigma \frac{d}{dt} \|A_1(t, \cdot)\|_2^2 + \lambda c \|A_1(t, \cdot)\|_2^2 \leq \frac{2c}{\kappa^2} \|A_1(t, \cdot)\|_2 \|\nabla A \psi(t, \cdot)\|_2.$$ 

With the aid of Cauchy’s inequality, we then obtain for any $\alpha > 0$

$$\frac{1}{2} \frac{d}{dt} \|A_1(t, \cdot)\|_2^2 + (\lambda c - \frac{c\alpha}{\kappa^2}) \|A_1(t, \cdot)\|_2^2 \leq \frac{c}{\alpha\kappa^2} \|\nabla A \psi(t, \cdot)\|_2^2,$$

where $\lambda$ has been introduced in (2.27).

For later reference we note that by setting $\alpha = \kappa^2 \lambda$, we obtain a weaker estimate:

$$\frac{1}{2} \frac{d}{dt} \|A_1(t, \cdot)\|_2^2 \leq \frac{c}{\lambda\kappa^2} \|\nabla A \psi(t, \cdot)\|_2^2.$$
Multiplying (1.1a) by $\bar{\psi}$ and integrating by parts we obtain for the real part

\begin{equation}
\frac{1}{2} \frac{d}{dt} \|\psi(t, \cdot)\|^2_2 + \|\nabla A \psi(t, \cdot)\|^2_2 \leq \|\psi(t, \cdot)\|^2_2.
\end{equation}

From this, with the aid of Cauchy's inequality, we obtain that, for all $\epsilon > 0$,

\begin{equation}
\frac{1}{2} \frac{d}{dt} \|\psi\|^2_2 + (1 - \epsilon) \|\nabla hA_n \psi\|^2_2 \leq \|\psi\|^2_2 + \frac{1}{\epsilon} \|A_1 \psi\|^2_2.
\end{equation}

From (2.21) we then conclude

\begin{equation}
\frac{1}{2} \frac{d}{dt} \|\psi\|^2_2 + [\mu(h)(1 - \epsilon) - 1] \|\psi\|^2_2 \leq \frac{1}{\epsilon} \|A_1 \psi\|^2_2.
\end{equation}

Using (2.47), we infer from the above that

\begin{equation}
\frac{1}{2} \frac{d}{dt} \|\psi\|^2_2 + [\mu(h)(1 - \epsilon) - 1] \|\psi\|^2_2 \leq \frac{1}{\epsilon} \|A_1 \psi\|^2_2.
\end{equation}

We next combine (3.15) and (3.17), so that $\|\nabla A \psi\|^2_2$ is eliminated, to obtain

\begin{equation}
\frac{d}{dt} \left[ \|A_1\|^2_2 + \frac{c}{\alpha \kappa^2} \|\psi\|^2_2 \right] + 2(\lambda c - \frac{c\alpha}{\kappa^2}) \|A_1\|^2_2 \leq \frac{2c}{\alpha \kappa^2} \|\psi\|^2_2.
\end{equation}

Then, substituting into the above the variable transformation

\begin{equation}
u(t) = \|A_1(t, \cdot)\|^2_2 + \frac{c}{\alpha \kappa^2} \|\psi(t, \cdot)\|^2_2, \quad v(t) = \|\psi(t, \cdot)\|^2_2,
\end{equation}

we obtain the following vector inequality

\begin{equation}
\frac{d}{dt} \begin{bmatrix} u \\ v \end{bmatrix} \leq \begin{bmatrix} -2\lambda c + \frac{2c\alpha}{\kappa^2} & \frac{2c(\lambda c + 1)}{\alpha \kappa^2} - \frac{2c^2}{\kappa^4} \\ \frac{2\alpha}{\kappa^2} -2[\mu(h)(1 - \epsilon) - 1] - \frac{2c}{\alpha \kappa^2} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix},
\end{equation}

which is merely an alternative representation of (3.19) and (3.20). Denote the matrix on the right-hand-side by $M(\epsilon, \alpha)$. Next, we consider for $a > 0$ the quantity $\langle (au, v) | M(\epsilon, \alpha) | (u, v) \rangle$ which can be represented in the form $\langle (a^2 u, v) | M_a(\epsilon, \alpha) | (a^2 u, v) \rangle$, where

$$M_a(\epsilon, \alpha) := \begin{bmatrix} -2\lambda c + \frac{2c\alpha}{\kappa^2} & \frac{2c(\lambda c + 1)}{\alpha \kappa^2} - \frac{2c^2}{\kappa^4} \\ \frac{2\alpha}{\kappa^2} -2[\mu(h)(1 - \epsilon) - 1] - \frac{2c}{\alpha \kappa^2} \end{bmatrix}.$$ 

We then choose such $a$, for which $M_a$ is symmetric. Suppose that

\begin{equation}
\alpha c < (\lambda c + 1)\kappa^2.
\end{equation}

We then set

\begin{equation}
\alpha^{-1} = \epsilon \left( \frac{c(\lambda c + 1)}{\alpha \kappa^2} - \frac{c^2}{\kappa^4} \right).
\end{equation}

Let $\hat{\lambda}_1(\epsilon, \alpha)$ and $\hat{\lambda}_2(\epsilon, \alpha)$ denote the eigenvalues of $M_a$, which are identical with the eigenvalues of $M$. Without loss of generality we may assume that

$$\hat{\lambda}_1 \leq \hat{\lambda}_2.$$ 

We obtain that

\begin{equation}
\frac{1}{2} \frac{d}{dt} (a|u|^2 + |v|^2) \leq \hat{\lambda}_2 (a|u|^2 + |v|^2).
\end{equation}
This proves that

\[ a|u(t)|^2 + |v(t)|^2 \leq e^{2\lambda_2 t}(a|u(0)|^2 + |v(0)|^2) , \]

which implies (3.3) with

(3.25) \quad \lambda_m = -\hat{\lambda}_2 .

One can easily obtain from the above an explicit formula for \( C(\lambda, \kappa, \|\psi_0\|_2, \|A_0\|_2, h) \).

We now determine the conditions under which \( \hat{\lambda}_2 < 0 \). We search for the optimal values of \( \alpha \) and \( \epsilon \) which achieve that goal. We separately obtain conditions for which \( \hat{\lambda}_2 \) is negative for arbitrary values of \( \kappa \) and in the large \( \kappa \) limit. We first obtain results of the former type that are, naturally, non-optimal. Then, we consider the regime \( \kappa \to +\infty \) where we obtain results that are asymptotically close to those of the linearized problem discussed in the previous subsection.

**Arbitrary \( \kappa \)**

We set

(3.26) \quad \epsilon = \frac{2b}{\kappa^2}, \quad \alpha = \frac{\lambda}{2\kappa^2},

where

\[ b = \left[ \frac{c}{2\mu\lambda} \right]^{1/2} . \]

(Note that (3.23) is satisfied for the above value of \( \alpha \).) From (3.24) we get

\[ a = \frac{\lambda \kappa^6}{2b(\lambda c^2 + 2c)} , \]

and hence,

\[ M_a = \begin{bmatrix} -\lambda c & \frac{\beta}{\kappa} \\ \frac{\beta}{\kappa} & -2(\mu - 1) \end{bmatrix} , \]

where

\[ \beta^2 = \frac{2\lambda c^2 + 2c}{\lambda b} \]

Clearly, under Assumption (3.2), \( \mu > 1 \) and \( \text{Tr}(M_a) < 0 \). Hence for \( M_a \) to be strictly negative we must require that \( \det M_a > 0 \). As

\[ \det M_a = 2\lambda c(\mu - 1) - \frac{\beta^2}{\kappa^2} = \det M . \]

A straightforward computation shows that (3.2) guarantees that \( \det M_a \) is positive. The rate of decay \( \lambda_m \) can now be obtain from (3.25) and

\[ \hat{\lambda}_2 = \frac{\text{Tr}(M)}{2} + \sqrt{\frac{\text{Tr}(M)^2}{4} - \det M} . \]

**Large \( \kappa \)**

We now compute the asymptotics of \( \hat{\lambda}_2 \) as \( \kappa \to +\infty \). In this limit we can distinguish between three zones:

(1) \( 0 < \mu - 1 < \lambda c \);

(2) \( \mu - 1 = \lambda c \);
In cases 1 and 3, \(|\mu - 1 - \lambda c|\) (as well as \(c\) and \(\lambda\)) must be bounded away from 0 as \(\kappa \to \infty\). In all cases we search for the values of \(\epsilon\) and \(\alpha\) for which the minimal value of \(\overline{\lambda}_2\) is obtained and then prove that it is negative.

The case \(\lambda c < \mu - 1\)

We first note that in view of the discussion in Subsection 3.1 one cannot have \(\lambda_2 < \lambda c\). For convenience we divide all elements of \(M_a\) by 2 to obtain the matrix

\[
\begin{bmatrix}
-\lambda c + \frac{c \epsilon}{\kappa^2} & \frac{c(\lambda c + 1)}{\alpha \kappa^2} - \frac{c^2}{\kappa^4} \\
\frac{1}{\epsilon} & -[\mu(h)(1 - \epsilon) - 1] - \frac{c}{\alpha \kappa^2}
\end{bmatrix} = \begin{bmatrix}
p_1 & \frac{c(\lambda c + 1)}{\alpha \kappa^2} - \frac{c^2}{\kappa^4} \\
\frac{1}{\epsilon} & p_2
\end{bmatrix}
\]

whose characteristic polynomial in \(\nu\) is given by

\[
(\rho_1 - \nu)(\rho_2 - \nu) = \frac{1}{\epsilon} \left( \frac{c(\lambda c + 1)}{\alpha \kappa^2} - \frac{c^2}{\kappa^4} \right).
\]

Suppose that \(\epsilon\) and \(\alpha\) are chosen so that the right-hand-side vanishes as \(\kappa \to \infty\). Then \(\nu\) is confined to a close neighborhood of \(\{p_1, p_2\}\). We seek an estimate of \(\nu\) in the neighborhood of \(p_1\). Thus, we neglect the terms \((\rho_1 - \nu)^2\) and \(\frac{c^2}{\kappa^4}\) to obtain that

\[
(\rho_1 - \nu)(\rho_2 - \rho_1) \sim \frac{c(\lambda c + 1)}{\epsilon \alpha \kappa^2}.
\]

We now write

\[
\epsilon(\rho_1 - \rho_2) = (\mu - 1 - \lambda c)\epsilon + \frac{c \alpha}{\kappa^2} - \mu \epsilon^2 + \frac{c}{\alpha \kappa^2},
\]

and maximize it with respect to \(\epsilon\), for fixed \(\alpha\), to obtain

\[
2\mu \epsilon = (\mu - 1 - \lambda c) + \frac{c \alpha}{\kappa^2} \sim (\mu - 1 - \lambda c).
\]

Setting \(\epsilon = \frac{\mu - 1 - \lambda c}{2\mu}\), we obtain that

\[
\epsilon(\rho_1 - \rho_2) = \frac{(\mu - 1 - \lambda c)^2}{4\mu} + \frac{c \alpha (\mu - 1 - \lambda c)}{2\mu \kappa^2} + \frac{c}{\alpha \kappa^2},
\]

and hence

\[
\nu \sim -\lambda c + \frac{c \alpha}{\kappa^2} + \frac{c(\lambda c + 1)}{\alpha \kappa^2} \frac{4\mu}{(\mu - 1 - \lambda c)^2}.
\]

Minimizing \(\nu\) over \(\alpha\) then yields

\[
\nu = -\lambda c + \frac{cd}{\kappa^2} + O(\kappa^{-4}),
\]

where

\[
d = \inf_{\alpha} \left( \alpha + \frac{4\mu(\lambda c + 1)}{(\mu - 1 - \lambda c)^2 \alpha} \frac{1}{\alpha} \right).
\]

This leads to

\[
\alpha = \sqrt{\frac{4\mu(\lambda c + 1)}{(\mu - 1 - \lambda c)^2}}
\]

and to the value of \(d\) given in (3.8).
Once $\alpha$ and $\epsilon$ had been set, it can be easily established that
\[ \hat{\lambda}_2 = -2\lambda c + \frac{2cd}{\kappa^2} + O(\kappa^{-4}), \]
which verifies (3.7). Moreover, by (3.24) we obtain that
\[ a \sim \frac{4\mu}{c(\mu - 1 - \lambda c)^2(\lambda c + 1)^2} \kappa^2. \]

The case: $\lambda c > \mu - 1$

In this case, since $\rho_2 > \rho_1$, we look for the solution of (3.27) which lies in the close vicinity of $\rho_2$. Neglecting the term $(\rho_2 - \nu)^2$ leads to
\[ (3.29) \quad (\rho_2 - \nu)(\rho_1 - \rho_2) = \frac{c(\lambda c + 1)}{\epsilon \hat{\alpha}} \kappa^2 - c^2 \epsilon^{-1} \kappa^{-2}. \]

We next set
\[ \alpha = \hat{\alpha} \kappa^2 \quad \text{and} \quad \epsilon = \hat{\epsilon} \kappa^{-2}, \]
which is in line with the choice we made for arbitrary values of $\kappa$. Then we can rewrite (3.29) in the form
\[ (3.30) \quad (\rho_2 - \nu)(\rho_1 - \rho_2) \sim \frac{c(\lambda c + 1)}{\hat{\epsilon} \hat{\alpha}} \kappa^2 - c^2 \hat{\epsilon}^{-1} \kappa^{-2}. \]

It follows that
\[ \nu \sim -[\mu(h) - 1] + \hat{\epsilon} \mu \kappa^2 - \frac{c}{c \hat{\alpha}} \kappa^2 + \frac{c(\lambda c + 1)}{(\hat{\epsilon} \hat{\alpha})(\lambda c - \mu + 1 - c \hat{\alpha})} \kappa^{-2}. \]

In view of the above we have to minimize the coefficient of $\kappa^{-2}$ which is given by
\[ q(\hat{\alpha}, \hat{\epsilon}) := \hat{\epsilon} \mu + \frac{c \mu}{(\hat{\epsilon} \hat{\alpha})(\lambda c - \mu + 1 - c \hat{\alpha})}. \]

To minimize $q$, we first minimize over $\alpha$ to obtain
\[ \hat{\alpha} = \frac{1}{2c} (\lambda c - \mu + 1) \]
(note that (3.23) is satisfied), which leads to the minimization, this time over $\epsilon$, of
\[ \hat{\epsilon} \mu + \frac{4c^2 \mu}{\hat{\epsilon}(\lambda c - \mu + 1)^2} \]
and finally to
\[ \inf_{\hat{\epsilon}, \hat{\alpha}} q(\hat{\alpha}, \hat{\epsilon}) = 4c \mu (\lambda c - \mu + 1)^{-1}, \]
where the minimum is obtained for
\[ \hat{\epsilon} = \frac{2c}{\lambda c - \mu + 1}. \]

Hence, we have shown that when $\mu - 1 < \lambda c$ then for $\kappa$ large enough, the largest eigenvalue $\hat{\lambda}_2$ satisfies
\[ (3.31) \quad \hat{\lambda}_2 = -2(\mu - 1) + 8c \mu (\lambda c - \mu + 1)^{-1} \kappa^{-2} + O(\kappa^{-4}). \]
A simple computation shows that there exists a constant $C(\lambda, \mu, c)$ and $\kappa_0(\lambda, \mu, c)$ such that if $\kappa \geq \kappa_0$ and

$$\mu > 1 + \frac{4}{\lambda \kappa^2} + C \kappa^{-4},$$

then $\hat{\lambda}_2 < 0$.

Note that the above condition is weaker than (3.2) in the sense that $4/\lambda \leq \gamma$, but unlike (3.2) its validity is limited for large values $\kappa$.

The asymptotic behavior of $a$ in this case is given by

$$a^{-1} \sim 2c^3 (\lambda c + \mu + 1)^{-1} \kappa^{-6},$$

which is of the same magnitude as in the estimate for arbitrary values of $\kappa$.

The case $\mu - 1 = \lambda c$

In this case we search for the optimal eigenvalues of

$$\begin{bmatrix}
-\lambda c + \frac{c\alpha}{\kappa} & \frac{c(\lambda c + 1)}{\alpha \kappa^2} - \frac{c^2}{\kappa^4} \\
\frac{1}{\epsilon} & -[(\lambda c + 1)(1 - \epsilon) - 1] - \frac{c}{\epsilon \alpha \kappa^2}
\end{bmatrix}
= \begin{bmatrix}
\rho_1 & \frac{c(\lambda c + 1)}{\alpha \kappa^2} - \frac{c^2}{\kappa^4} \\
\frac{1}{\epsilon} & \rho_2
\end{bmatrix}$$

whose characteristic equation is given by

$$(\rho_1 - \nu) (\rho_2 - \nu) = \frac{1}{\epsilon} \left( \frac{c(\lambda c + 1)}{\alpha \kappa^2} - \frac{c^2}{\kappa^4} \right).$$

Here we set $\alpha = \tilde{\alpha} \kappa$ and $\epsilon = \tilde{\epsilon} \kappa^{-1}$. Then, neglecting lower order terms the characteristic equation becomes

$$(\nu + \lambda c - \frac{c\tilde{\alpha}}{\kappa})(\nu + \lambda c - \frac{\tilde{\epsilon}(\lambda c + 1)}{\kappa}) \sim \frac{c(\lambda c + 1)}{\tilde{\alpha} \tilde{\epsilon} \kappa^2}.$$ 

Hence,

$$\nu \sim -\lambda c + \frac{1}{\kappa} \left( \frac{1}{2} (c\tilde{\alpha} + \tilde{\epsilon}(\lambda c + 1)) + \sqrt{\frac{c(\lambda c + 1)}{\tilde{\alpha} \tilde{\epsilon}}} + \frac{1}{4} (c\tilde{\alpha} + \tilde{\epsilon}(\lambda c + 1))^2 \right).$$

The minimum of the coefficient of $\kappa^{-1}$ is achieved for

$$c\tilde{\alpha} = \tilde{\epsilon}(\lambda c + 1) = (c(\lambda c + 1))^\frac{1}{4} 3^{-\frac{1}{4}},$$

leading to

$$\nu = -\lambda c + \frac{2 c^{\frac{3}{4}} 3^{-\frac{1}{4}} (\lambda c + 1)^\frac{1}{2}}{\kappa} + \mathcal{O}(\kappa^{-2}).$$

The asymptotic behavior of $a$ is given by

$$a \sim c^{-2} (\lambda c + \mu + 1) (\lambda c + 1)^{\kappa^4}.$$ 

This completes the proof of the theorem.
3.3. \( L^2 \) estimate of \( A_1(t, \cdot) \).
We continue this section by the following simple bound for \( \|A_1(t, \cdot)\|_2 \).

Lemma 3.2. Let \( A_1 \) be defined by (3.9). Then, for any positive values of \( h, \kappa, c, \) and \( \lambda \), we have
\[
\|A_1(t, \cdot)\|_2^2 \leq \|A_1(0, \cdot)\|_2^2 e^{-\lambda c t} + \left( \frac{2c}{\lambda c^4} + \frac{4}{\lambda^2 c^2} \right) |\Omega|.
\]

Proof. By (3.22) with the choice (3.26) for \( (\alpha, c) \) and Gronwall’s inequality we have, with \( v \) and \( u \) introduced in (3.21),
\[
u(t) \leq u(0)e^{-\lambda c t} + \left( \frac{2c}{\kappa^4} + \frac{4}{\lambda^2 \kappa^4} \right) \int_0^t e^{-\lambda c (t-\tau)} v(\tau) d\tau.
\]

By (2.47) we have \( v(t) \leq |\Omega| \) and hence
\[
u(t) \leq u(0)e^{-\lambda c t} + \left( \frac{2c}{\kappa^4} + \frac{4}{\lambda^2 \kappa^4} \right) |\Omega|.
\]

The lemma readily follows. \( \blacksquare \)

3.4. \( L^\infty \) decay of \( A_1(t, \cdot) \).
We now show how to deduce from an \( L^2 \)-estimate of \( A_1 \), an \( L^\infty \) estimate for it.

Proposition 3.3. Let \( M > 0, (\psi, A, \phi) \) denote a solution of (1.1) and (2.34), and \( A_1 \) be defined by (3.9). We further require that
\[
\|A_1(0, \cdot)\|_2 \leq M.
\]

Let further, for any \( \kappa \in (0, +\infty), \)
\[
t^*(\kappa, M) = \max \left( \frac{2}{\lambda c} \ln((\kappa^2 M)^{1/2}), 1 \right).
\]

For any \( \kappa_0 > 0, \) and any \( \delta \in (0, 1), \) there exists \( C = C(\Omega, c, \delta, \kappa_0) > 0, \) such that, for all \( \kappa \in [\kappa_0, +\infty), \) and \( t \geq t^*(\kappa, M) + 2\delta, \) we have
\[
\|A_1(t, \cdot)\|_\infty \leq C \left( \|A_1(t - \delta, \cdot)\|_2 + \frac{1}{\kappa^2} \|\psi(t - \delta, \cdot)\|_2 \right).
\]

Remark 3.4. Note that in view of (3.37), \( A_1(t, \cdot) \) satisfies for \( t \geq t^*(\kappa, M) \)
\[
\|A_1(t, \cdot)\|_2 \leq \frac{C(\Omega, c)}{\kappa^2}.
\]

Proof. Let \( 0 < \tilde{\delta} < 1/3, \) and \( t_0 \geq t^*(\kappa, M). \) Let further \( T = t_0 + 4\tilde{\delta}, t_1 = t_0 + \delta, t_2 = t_0 + 2\tilde{\delta}, \) and \( t_3 = t_0 + 3\tilde{\delta}. \) Hence we have \( 0 < t_0 < t_1 < t_2 < T. \) We begin by applying (C.4) in the case of \( L^{11}, \) with \( X = A_1 \) on the interval \((t_0, T)\) and \( X_0 = A_1(t_0, \cdot) \) (recalling that \( A_1 \) satisfies (2.34)), Sobolev embedding then yields
\[
\|A_1\|_{L^2(\Omega, L^\infty(\Omega, \mathbb{R}^2))} + \|A_1\|_{L^\infty(\Omega, H^1(\Omega, \mathbb{R}^2))} \leq C \left[ \frac{1}{\kappa^2} \|\text{Im} (\bar{\psi} \nabla A\psi)\|_{L^2(\Omega, T; L^2(\Omega, \mathbb{C}))} + \|A_1(t_0, \cdot)\|_2^2 \right].
\]
for some $C > 0$ depending on $\delta$, $c$, and $\Omega$. Integrating (3.17) on $(t_0, T)$ yields, in view of (2.47), that

\begin{equation}
(3.41) \quad \|\Im (\tilde{\psi} \nabla A \psi)\|_{L^2(t_0, T; L^2(\Omega, C))}^2 \leq \|\nabla_A \psi\|_{L^2(t_0, T; L^2(\Omega, C))}^2 \leq \int_{t_0}^T \left[ \|\psi(\tau, \cdot)\|_2^2 - \frac{1}{2} \frac{d}{d\tau} \|\psi(\tau, \cdot)\|_2^2 \right] d\tau \leq \|\psi\|_{L^2(t_0, T; L^2(\Omega, C))}^2 + \frac{1}{2} \|\psi(0, \cdot)\|_2^2.
\end{equation}

We then obtain that

\[ \|A_1\|_{L^2(t_0, T; L^2(\Omega, C))}^2 + \|A_1\|_{L^2(t_0, T; H^1(\Omega))}^2 \leq C \left[ \|A_1(t_0, \cdot)\|_2^2 + \frac{1}{\kappa^2} \|\psi(0, \cdot)\|_2^2 \right]. \]

Using (3.17) again yields for all $t_0 \leq \tilde{t} < t \leq T$

\begin{equation}
(3.42) \quad \|\psi(t, \cdot)\|_2 \leq e^{(t-\tilde{t})} \|\psi(\tilde{t}, \cdot)\|_2 \leq e^{\delta t} \|\psi(0, \cdot)\|_2.
\end{equation}

Hence,

\begin{equation}
(3.43) \quad \|\psi\|_{L^2(t_0, T; L^2(\Omega, C))}^2 \leq 4 \delta e^{\delta t} \|\psi(0, \cdot)\|_2^2.
\end{equation}

Consequently,

\begin{equation}
(3.44) \quad \|A_1\|_{L^2(t_0, T; L^2(\Omega, C))}^2 + \|A_1\|_{L^2(t_0, T; H^1(\Omega))}^2 \leq C(\delta, c, \Omega) \left[ \|A_1(t_0, \cdot)\|_2^2 + \frac{1}{\kappa^2} \|\psi(0, \cdot)\|_2^2 \right].
\end{equation}

For later reference we mention that by (3.16) and (3.41) we have, for all $t_0 < t \leq T$,

\begin{equation}
(3.45) \quad \|A_1(t, \cdot)\|_2^2 \leq \|A_1(t_0, \cdot)\|_2^2 + \frac{C(\delta, c, \Omega)}{\kappa^2} \|\psi(0, \cdot)\|_2^2.
\end{equation}

Let $t_1 \leq \tilde{t} \leq t_2$. We next apply (C.3) in the case of the Dirichlet-Neumann problem to (1.1a) on the interval $(\tilde{t}, s)$, for any $\tilde{t} < s \leq T$, to obtain that

\begin{equation}
(3.46) \quad \|\psi\|_{L^2(\tilde{t}, s; H^1(\Omega, C))} \leq C(\Omega) \left[ \|\psi(\tilde{t}, \cdot)\|_{L^2(\tilde{t}, s; L^2(\Omega, C))} + \|A \cdot \nabla \psi\|_{L^2(\tilde{t}, s; L^2(\Omega, C))} + \|A^2 \psi\|_{L^2(\tilde{t}, s; L^2(\Omega, C))} + \|\psi(1 - |\psi|^2)\|_{L^2(\tilde{t}, s; L^2(\Omega, C))} \right].
\end{equation}

For the last term inside the brackets on the right-hand-side we obtain with the aid of (2.47) and (4.32) that

\begin{equation}
(3.47) \quad \|\psi(1 - |\psi|^2)\|_{L^2(\tilde{t}, s; L^2(\Omega, C))} \leq 3 \delta \|\psi(\tilde{t}, \cdot)\|_2^2.
\end{equation}

For the second term inside the brackets on the right-hand-side we have

\begin{equation}
(3.48) \quad \|A \cdot \nabla \psi\|_{L^2(\tilde{t}, s; L^2(\Omega, C))} \leq 2 \int_{\tilde{t}}^s (\|A_n\|_\infty^2 + \|A_1(\tau, \cdot)\|_2^2) \|\nabla \psi(\tau, \cdot)\|_2^2 d\tau.
\end{equation}

For the third term inside the brackets we have, using Sobolev embedding and the fact that $A_n$ is bounded in $L^8(\Omega; \mathbb{R}^2)$ by a constant depending only on $c$ and $\Omega$,

\[ \|A^2 \psi\|_{L^2(\tilde{t}, s; L^2(\Omega, C))} \leq C(\Omega) \int_{\tilde{t}}^s (\|A_n\|_8^4 + \|A_1(\tau, \cdot)\|_4^4) \|\psi(\tau, \cdot)\|_2^2 d\tau \leq \tilde{C}(c, \Omega) \left[ \|\psi\|_{L^2(\tilde{t}, s; H^1(\Omega, C))}^2 \right] \int_{\tilde{t}}^s \|A_1(\tau, \cdot)\|_{L^2}^4 \|\psi(\tau, \cdot)\|_{L^2}^2 d\tau. \]
With the aid of (3.44) we then obtain that
\[
\|A^2\psi\|^2_{L^2(\Omega, C)} \leq C(\delta, c, \Omega) \left\{ \|\psi\|^2_{L^2(\Omega, C)} + \left[ \|A_1(\tau, \cdot)\|^2_2 + \frac{1}{\kappa^2} \|\psi(\tau, \cdot)\|^2_2 \right]^2 \right\}^{1/2} \int_{t_1}^T \|\psi(\tau, \cdot)\|^2_{1,2} d\tau
\]

From (3.39), we then get
\[
(3.49) \quad \|A^2\psi\|^2_{L^2(\Omega, C)} \leq C(\delta, c, \Omega, \kappa_0) \|\psi\|^2_{L^2(\Omega, C)}.
\]

For the fourth term on the right-hand-side of (3.46) we obtain, using (2.47) and Sobolev embedding
\[
(3.50) \quad \|\phi\|_{L^2(\Omega, C)} \leq 2 \|\phi\|_{L^2(\Omega, C)} + 2 \int_{t_1}^T \|\phi_1(\tau, \cdot)\|^2_2 \|\psi(\tau, \cdot)\|^2_2 d\tau \leq C(\delta, c, \Omega, \kappa_0) \left( \|\psi\|^2_{L^2(\Omega, C)} + \int_{t_1}^T \|\phi_1(\tau, \cdot)\|^2_{1,2} d\tau \right).
\]

To obtain the last inequality we had to use (2.13), to conclude that
\[
\|A_n\|_{\infty} \leq C(\Omega)/\sigma \leq C(c, \Omega, \kappa_0),
\]
for all \(\kappa \geq \kappa_0\). Using (3.12) and Poincaré’s inequality for \(\phi_1\) then yields,
\[
\|\phi\|_{L^2(\Omega, C)} \leq C(\delta, c, \Omega, \kappa_0)^2 \left[ \|\psi\|^2_{L^2(\Omega, C)} + \frac{1}{\kappa^2} \int_{t_1}^T \|\nabla A\psi(\tau, \cdot)\|^2_2 d\tau \right].
\]

By (3.17) (or, more precisely, its integrated version over \((t_1, s)\)), we then obtain
\[
\|\phi\|_{L^2(\Omega, C)} \leq C(\delta, c, \Omega, \kappa_0) \left[ \|\psi\|^2_{L^2(\Omega, C)} + \|\psi(\tau, \cdot)\|^2_2 \right],
\]
which together with (3.42) gives way to
\[
(3.51) \quad \|\phi\|_{L^2(\Omega, C)} \leq C(\delta, c, \Omega, \kappa_0) \|\psi(\tau, \cdot)\|^2_2.
\]

Substituting (3.48), (3.49), and (3.51) into (3.46) then yields
\[
\|\psi\|^2_{L^2(\Omega, C)} \leq C(\delta, c, \Omega, \kappa_0) \left[ \int_{t_1}^T (1 + \|A_1(\tau, \cdot)\|^2_\infty) \|\psi(\tau, \cdot)\|^2_{1,2} d\tau + \|\psi(\tau, \cdot)\|^2_{1,2} \right].
\]

We now apply a variant of Gronwall’s inequality.

Let \(f(t) = \|\psi(t, \cdot)\|^2_{1,2} \) and \(g(t) = C(\delta, c, \Omega, \kappa_0)(1 + \|A_1(t, \cdot)\|^2_\infty) \) and \(C_0 = C(\delta, c, \Omega, \kappa_0)\|\psi(\tau, \cdot)\|^2_{1,2} \).

By the inequality below (3.51) we obtain that
\[
f(s) \leq C_0 + \int_{t_1}^s f(\tau)g(\tau) d\tau \quad \forall t_1 < s \leq T.
\]

One can now apply Gronwall’s inequality (Theorem III.1.1 in [22]) to obtain that
\[
f(s) \leq C_0 \exp \left\{ \int_{t_1}^T g(\tau) d\tau \right\} \leq C_0 \exp \left\{ \int_{t_1}^T g(\tau) d\tau \right\},
\]
for all \( \bar{t}_1 < s \leq T \). By taking the supremum over \( s \in (\bar{t}_1, T] \), we obtain:

\[
\|\psi\|_{L^\infty(\bar{t}_1, T; H^1(\Omega, \mathbb{C}))} \leq C \|\psi(\bar{t}_1, \cdot)\|^2_{L^2(\Omega, \mathbb{C})} \exp \left\{ C \int_{\bar{t}_1}^{T} (1 + \|A_1(t, \cdot)\|^2_\infty) \, dt \right\},
\]

which together with (3.44) and (3.39) yields (recall that \( t_0 \geq t^*(\kappa, M) \)),

\[(3.52) \quad \|\psi\|_{L^\infty(\bar{t}_1, T; H^1(\Omega, \mathbb{C}))} \leq C(\delta_0, \Omega, \kappa_0) \|\psi(\bar{t}_1, \cdot)\|_{L^2(\Omega, \mathbb{C})}.
\]

To find \( \bar{t}_1 \) for which we can estimate \( \|\psi(\bar{t}_1, \cdot)\|_{L^2(\Omega, \mathbb{C})} \), we first observe that

\[
\|\nabla \psi(\cdot, t)\|_{L^2(\Omega, \mathbb{C})} \leq \|\nabla A \psi(t, \cdot)\|_{L^2(\Omega, \mathbb{C})} + \|A \psi(t, \cdot)\|_{L^2(\Omega, \mathbb{C})} \leq \|\nabla A \psi(t, \cdot)\|_{L^2(\Omega, \mathbb{C})} + \|A_0(t)\|_{L^\infty(\Omega, \mathbb{C})} \|\nabla \psi(t, \cdot)\|_{L^2(\Omega, \mathbb{C})} + \|A_1(t, \cdot)\|_{L^2(\Omega, \mathbb{C})}.
\]

Integrating the above between \( t_1 \) and \( t_2 \) yields, with the aid of (3.17) that

\[
\|\psi\|_{L^2(t_1, t_2; H^1(\Omega, \mathbb{C}))} \leq C(1) \|\psi\|_{L^2(t_1, t_2; L^2(\Omega, \mathbb{C}))} + \|A_1(t_1, \cdot)\|_{L^2(\Omega, \mathbb{C})} + \|\psi(t_1, \cdot)\|_{L^2(\Omega, \mathbb{C})}.
\]

With the aid of (3.45) and (3.42) we then obtain that

\[
\|\psi\|_{L^2(t_1, t_2; H^1(\Omega, \mathbb{C}))} \leq C(1) \|\psi(t_1, \cdot)\|_{L^2(\Omega, \mathbb{C})} + \|\psi(t_1, \cdot)\|_{L^2(\Omega, \mathbb{C})}.
\]

We can, thus, conclude that there exists \( \bar{t}_1 \in [t_1, t_2] \) such that

\[
\|\psi(\bar{t}_1, \cdot)\|_{L^2(\Omega, \mathbb{C})} \leq C(1) \|A_1(\bar{t}_1, \cdot)\|_{L^2(\Omega, \mathbb{C})} + \|\psi(t_1, \cdot)\|_{L^2(\Omega, \mathbb{C})}.
\]

In conjunction with (3.52), (3.42), and (3.45) the above inequality yields the existence of some constant \( C \) such that:

\[(3.53) \quad \|\psi\|_{L^\infty(\bar{t}_1, t_2; H^1(\Omega, \mathbb{C}))} \leq C \|A_1(\bar{t}_1, \cdot)\|_{L^2(\Omega, \mathbb{C})} + \|\psi(t_1, \cdot)\|_{L^2(\Omega, \mathbb{C})}.
\]

Let \( t_3 = t_2 + \delta \). We continue by applying (C.5) (in the case of the operator \( \mathcal{L}(1) \)) to the first line of (3.11) (recalling (2.34) and (3.13)) in \( (t_2, T) \) to obtain, with the aid of (3.12)

\[(3.54) \quad \|A_1\|^2_{L^\infty(t_3, T; L^\infty(\Omega, \mathbb{R}^2))} \leq C \left( \frac{1}{\kappa^2} \|\Im(\nabla A \psi)\|^2_{L^2(t_3, T; H^1(\Omega, \mathbb{R}^2))} + \|A_1(t_2, \cdot)\|^2_{L^2(\Omega, \mathbb{C})} \right).
\]

By (2.47) we have

\[(3.55) \quad \|\nabla \Im(\nabla A \psi)(t, \cdot)\|_{L^2(\Omega, \mathbb{C})} \leq \|\nabla^2 \psi(t, \cdot)\|_{L^2(\Omega, \mathbb{C})} + \|D^2 \psi(t, \cdot)\|_{L^2(\Omega, \mathbb{C})} + \|A \nabla \psi(t, \cdot)\|_{L^2(\Omega, \mathbb{C})} + \|A \psi(t, \cdot)\|_{L^2(\Omega, \mathbb{C})},
\]

for all \( t \geq t_2 \), where \( D^2 \psi \) denotes the Hessian matrix of \( \psi \).

By (C.3) (in the case of the Dirichlet-Neumann Laplacian) applied to (1.1a) in \( (t_2, T) \) we have:

\[(3.56) \quad \|\psi\|_{L^2(t_2, T; H^2(\Omega))} \leq C(1) \left( \|A \cdot \nabla \psi\|^2_{L^2(t_2, T; L^2(\Omega, \mathbb{C}))} + \|\psi\|^2_{L^2(t_2, T; L^2(\Omega, \mathbb{C}))} + \|A \psi\|^2_{L^2(t_2, T; L^2(\Omega, \mathbb{C}))} + \|\psi(t_2, \cdot)\|^2_{L^2(\Omega, \mathbb{C})} \right).
\]

Using (3.53) yields

\[
\|A \cdot \nabla \psi\|^2_{L^2(t_2, T; L^2(\Omega, \mathbb{C}))} \leq \|A\| \|\nabla \psi\|^2_{L^2(t_2, T; L^2(\Omega, \mathbb{C}))} \leq \int_{t_2}^{T} \|A(\tau, \cdot)\|^2_{L^\infty(\Omega, \mathbb{C})} \|\nabla \psi(\tau, \cdot)\|^2_{L^2(\Omega, \mathbb{C})} \, d\tau \leq \int_{t_2}^{T} \|A(\tau, \cdot)\|^2_{L^\infty(\Omega, \mathbb{C})} \|\nabla \psi(\tau, \cdot)\|^2_{L^2(\Omega, \mathbb{C})} \, d\tau \leq C(1) \|\psi(t_0, \cdot)\|^2_{L^2(\Omega, \mathbb{C})} + \|A_1(t_0, \cdot)\|^2_{L^2(\Omega, \mathbb{C})} \int_{t_2}^{T} \|A(\tau, \cdot)\|^2_{L^\infty(\Omega, \mathbb{C})} \, d\tau,
\]

which together with (3.44) and (3.42) yields

\[
\|A\| \|\nabla \psi\|^2_{L^2(t_2, T; L^2(\Omega, \mathbb{C}))} \leq C(1) \|\psi(t_0, \cdot)\|^2_{L^2(\Omega, \mathbb{C})} + \|A_1(t_0, \cdot)\|^2_{L^2(\Omega, \mathbb{C})} + \|A_1(t_0, \cdot)\|^2_{L^2(\Omega, \mathbb{C})} \int_{t_2}^{T} \|A(\tau, \cdot)\|^2_{L^\infty(\Omega, \mathbb{C})} \, d\tau.
\]
Hence, by (3.39) and (2.47) we get
\begin{equation}
(3.57) \quad \|A|\nabla \psi\|_{L^2(t;L^2(\Omega,\mathbb{C}))} \leq C \left( \|A_1(t_0,\cdot)\|_2 + \|\psi(t_0,\cdot)\|_2 \right).
\end{equation}

By (3.49) and (3.53) we have that
\begin{equation}
\|A^2 \psi\|_{L^2(t;L^2(\Omega,\mathbb{C}))} \leq C \|\psi\|_{L^2(t;H^1(\Omega,\mathbb{C}))} \leq \hat{C} \|\psi(t_0,\cdot)\|_2.
\end{equation}

Combining the above with (3.57), (3.56), (3.42), (3.45), (3.50), and (3.33) yields
\begin{equation}
(3.58) \quad \|\psi\|_{L^2(t;H^2(\Omega))} \leq C \left( \|A_1(t_0,\cdot)\|_2 + \|\psi(t_0,\cdot)\|_2 \right).
\end{equation}

For the first term on the right-hand-side of (3.55) we have by (3.58) and Sobolev embeddings that
\begin{equation}
(3.59) \quad \|\nabla \psi\|_{L^2(t;L^1(\Omega))} \leq C \|\psi\|_{L^2(t;H^2(\Omega))} \leq \hat{C} \left( \|A_1(t_0,\cdot)\|_2 + \|\psi(t_0,\cdot)\|_2 \right).
\end{equation}

The second term on the right-hand-side of (3.55) can similarly be estimated
\begin{equation}
\|D^2 \psi\|_{L^2(t;L^2(\Omega))} \leq C \left( \|A_1(t_0,\cdot)\|_2 + \|\psi(t_0,\cdot)\|_2 \right).
\end{equation}

Combining the above, (3.59), (3.57), (3.44), (3.39), and (3.55) yields
\begin{equation}
\|\text{Im} (\psi \nabla A \psi)\|_{L^2(t;T;H^1(\Omega,\mathbb{R}^2))} \leq C \left( \|A_1(t_0,\cdot)\|_2 + \|\psi(t_0,\cdot)\|_2 \right).
\end{equation}

Substituting the above into (3.54) yields (3.38) for \(\delta = 3\tilde{\delta}\). \(\blacksquare\)

3.5. Decay estimate for \(\phi_1\) and asymptotic contraction.

We conclude this section by establishing an exponential rate of decay for \(\phi_1\).

**Proposition 3.5.** Under the same assumptions of Theorem 3.1 and Proposition 3.3, there exist \(\kappa_0\) and \(C(\kappa)\) such that, for \(\kappa \geq \kappa_0\) and \(t \geq t^*(\kappa, M) + 1\), where \(t^*(\kappa, M)\) is given by (3.37), we have
\begin{equation}
(3.60) \quad \|\phi(t,\cdot) - h\phi_n(\cdot)\|_2 \leq C(\kappa) e^{-\lambda_m t},
\end{equation}
and \([t^*(\kappa, M) + 1, +\infty) \ni t \mapsto \|\phi(t,\cdot)\|_2\) is monotone decreasing.

**Proof.** Recalling (3.12), we use (2.47), (3.53), and (3.38) to obtain that, for any \(0 < \delta < 1\), there exists \(C(\Omega, \delta) > 0\) such that
\begin{equation}
(3.61) \quad \|\nabla \phi_1(t,\cdot)\|_2 \leq C(\Omega, \delta) \left[ \left( \|A_n\|_\infty + \|A_1(t - \delta,\cdot)\|_2 + \|\psi(t - \delta,\cdot)\|_2 \right) \|\psi(t,\cdot)\|_2 + \|A_1(t - \delta,\cdot)\|_2 + \|\psi(t - \delta,\cdot)\|_2 \right].
\end{equation}

Using (3.3) for \(\psi\) and \(A_1\) together with Poincaré’s inequality (recall that \(\int_{\Omega} \phi_1(t, x) \, dx = 0\)) completes the proof of (3.60).

We now show that \(\psi(t,\cdot)\) becomes decreasing for sufficiently large \(\kappa\) and \(t \geq t^*(\kappa, M) + 1\). To this end we combine (3.3), and (3.18) to obtain that for all \(\epsilon \in (0, 1)\)
\[\frac{1}{2} \frac{d}{dt} \|\psi\|_2^2 + [\mu(h)(1 - \epsilon) - 1] \|\psi(t,\cdot)\|_2^2 \leq \frac{1}{\epsilon} \|A_1(t,\cdot)\|_\infty^2 \|\psi\|_2^2.\]
Using the choice of $\epsilon$

$$\epsilon = \frac{1}{2} \frac{\mu(h) - 1}{\mu(h)},$$

then yields

$$\frac{1}{2} \frac{d}{dt} \|\psi(t, \cdot)\|^2_2 + \left[ \frac{\mu - 1}{2} - \frac{2\mu}{\mu - 1} \|A_1(t, \cdot)\|^2_\infty \right] \|\psi\|^2_2 \leq 0.$$ 

Hence, for sufficiently large $\kappa_0$, we get from (3.38) and (3.3) that

$$\frac{d}{dt} \|\psi(t, \cdot)\|^2_2 < 0.$$ 

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4. Stable semigroup - coherence length scale

Let

$$\mathcal{L}_h = -\nabla^2_{hA_n} + ih\phi_n,$$

where $(A_n, \phi_n)$ are defined by (2.10), and $h$ denotes the external current’s intensity. We define $D(\mathcal{L}_h)$ as

$$D(\mathcal{L}_h) = \{ u \in H^2(\Omega) | u|_{\partial\Omega_\text{c}} = 0 ; \nabla u \cdot \nu|_{\partial\Omega_i} = 0 \}.$$ 

Let $Q_L$ denote the domain of the sesquilinear form $q_L$ associated with $\mathcal{L}_h$, that is

$$Q_L := \{ u \in H^1(\Omega) , u|_{\partial\Omega_\text{c}} = 0 \}.$$ 

Then the domain of the operator $\hat{\mathcal{L}}_h$, which can be obtained using the Lax-Milgram’s procedure in [4], is the space:

$$D(\hat{\mathcal{L}}_h) := \{ u \in Q_L , \nabla u \cdot \nu|_{\partial\Omega_i} = 0 , (-\nabla^2_{hA_n} + ih\phi_n)u \in L^2 \}.$$ 

To show that $D(\hat{\mathcal{L}}_h) = D(\mathcal{L}_h)$, we use (A.9) to prove $H^2$ regularity for every $u \in D(\hat{\mathcal{L}}_h)$. Note that since $A_n \cdot \nu = 0$ on $\partial\Omega_i$, $u$ satisfies a Neumann boundary condition on $\partial\Omega_i$.

We seek a lower bound for the critical value $J^d$, so that the normal state is globally stable whenever $\|J\|_{L^\infty(\partial\Omega_\text{c})} > J^d$. Let $c$ be defined by (2.11),

$$J_r(x) = \kappa^2 J_0(x),$$

and $h_r$ be fixed. We assume that $c$ is fixed and that $J_0$ is independent of $\kappa$. Hence (see (4.2e)) $\mathcal{L}_h$ is independent of $\kappa$ as well. We begin by the following statement on the steady-state version of (1.1).
Proposition 4.1. Let $(\psi, A, \phi)$ denote a steady-state solution of (1.1), i.e.

\begin{equation}
-\nabla^2 A \psi + i\phi \psi = \psi (1 - |\psi|^2), \quad \text{in } \Omega,
\end{equation}

\begin{equation}
-\text{curl}^2 A + \frac{1}{c} \nabla \phi = \frac{1}{\kappa^2} \text{Im}(\bar{\psi} \nabla A \psi), \quad \text{in } \Omega,
\end{equation}

\begin{equation}
\psi = 0, \quad \text{on } \partial \Omega_c,
\end{equation}

\begin{equation}
\nabla A \psi \cdot \nu = 0, \quad \text{on } \partial \Omega_i,
\end{equation}

\begin{equation}
\frac{\partial \phi}{\partial \nu} = -hcJ_0(x), \quad \text{on } \partial \Omega_c,
\end{equation}

\begin{equation}
\frac{\partial \phi}{\partial \nu} = 0, \quad \text{on } \partial \Omega_i,
\end{equation}

\begin{equation}
\int_{\partial \Omega} \text{curl} A(x) ds = h \cdot h_r.
\end{equation}

There exists $\kappa_0 > 0$ and $C_1 > 0$ such that if

\begin{equation}
\|L_h^{-1}\| < 1 - \frac{C_1}{\kappa^2},
\end{equation}

for some $\kappa > \kappa_0$, $(0, hA_n, h\phi_n)$ is the unique solution of (4.2) satisfying (2.34).

Proof.

Set $A_1$ and $\phi_1$ to be as in (3.9). Then,

\begin{equation}
-\text{curl}^2 A_1 = \frac{1}{\kappa^2} \text{Im}(\bar{\psi} \nabla A \psi) - \frac{1}{c} \nabla \phi_1 \quad \text{in } \Omega,
\end{equation}

\begin{equation}
\frac{\partial \phi_1}{\partial \nu} = 0 \quad \text{on } \partial \Omega,
\end{equation}

\begin{equation}
\text{curl} A_1 = 0 \quad \text{on } \partial \Omega.
\end{equation}

The last boundary condition is a consequence of (4.2c-f) and (2.10). Since $\phi_1$ satisfies (3.12), and since $\|\psi\|_{\infty} \leq 1$ by (2.47), it follows that

\begin{equation}
\|\text{curl}^2 A_1\|_2 \leq \frac{2}{\kappa^2} \|\nabla A \psi\|_2.
\end{equation}

Observing that curl $A_1$ vanishes on the boundary, we obtain that

\begin{equation}
\|\text{curl} A_1\|_{H^1(\Omega)} \leq \frac{C_1}{\kappa^2} \|\nabla A \psi\|_2,
\end{equation}

which leads, via Sobolev’s injection, to

\begin{equation}
\|\text{curl} A_1\|_{L^p(\Omega)} \leq \frac{C_2}{\kappa^2} \|\nabla A \psi\|_2,
\end{equation}

for $p \in [2, +\infty)$.

Let $\Phi$ denote the solution in $H^1(\Omega)$ of

\begin{equation}
\begin{cases}
\Delta \Phi = \text{curl} A_1 & \text{in } \Omega, \\
\Phi = 0 & \text{on } \partial \Omega.
\end{cases}
\end{equation}
We now use (A.3) and (4.5) to obtain that \( \Phi \in W^{2,p}(\Omega) \) and

\[
(4.7) \quad \| \Phi \|_{2,p} \leq \frac{C(p, \Omega)}{\kappa^2} \| \nabla_A \psi \|_2,
\]

for all \( 2 \leq p < \infty \).

We next show that \( A_1 = -\nabla_\perp \Phi \). We set first \( u = A_1 + \nabla_\perp \Phi \). Since \( A_1 \cdot \nu = 0 \) on \( \partial \Omega \), and since by the Dirichlet condition \( \Phi \) satisfies we additionally have \( \nabla_\perp \Phi \cdot \nu = 0 \) on \( \partial \Omega \), it follows that \( u \in H^1(\Omega, \mathbb{R}^2) \) satisfies

\[
\begin{cases}
\text{curl } u = 0 & \text{in } \Omega \\
\text{div } u = 0 & \text{in } \Omega \\
u \cdot u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Since \( u \equiv 0 \) is the unique solution to the above problem (see Proposition B.1), we obtain that \( A_1 = -\nabla_\perp \Phi \).

From the embedding of \( W^{1,p}(\Omega) \) in \( L^\infty(\Omega) \) for any \( p \in (2, +\infty) \), we then get

\[
(4.8) \quad \| A_1 \|_\infty \leq C(\Omega, p) \| A_1 \|_{1,p} \leq \tilde{C}(\Omega, p) \| \Phi \|_{2,p} \leq \tilde{C} \frac{\kappa^2}{\kappa^2} \| \nabla_A \psi \|_2.
\]

Multiplying (4.2a) by \( \bar{\psi} \) and integrating by parts we obtain

\[
(4.9) \quad \| \nabla_A \psi \|_2^2 = \| \psi \|_2^2 - \| \psi \|_4^2 \leq \| \psi \|_2^2.
\]

Substituting the above into (4.8) yields

\[
(4.10) \quad \| A_1 \|_\infty \leq \frac{C}{\kappa^2} \| \psi \|_2.
\]

We next write (4.2a) in the form

\[
\mathcal{L}_h \psi = 2iA_1 \cdot \nabla_A \psi - |A_1|^2 \psi - i\phi_1 \psi + \psi (1 - |\psi|^2).
\]

By (2.47), (3.12), and (4.9), we have

\[
\| \phi_1 \psi \|_2 \leq \| \phi_1 \|_2 \leq \frac{C}{\kappa^2} \| \nabla_A \psi \|_2 \leq \frac{C}{\kappa^2} \| \psi \|_2.
\]

Thus, we obtain that

\[
\| \mathcal{L}_h \psi \|_2 \leq \frac{(C_0 \kappa^2 + 1)}{\kappa^2} \| \psi \|_2.
\]

The proposition now easily follows since, if we choose \( C_1 = C_0 \) in (4.3), we have that

\[
\| \psi \|_2 \leq \| \mathcal{L}_h^{-1} \| \| \mathcal{L}_h \psi \|_2 \leq \left( 1 - \frac{C_0^2}{\kappa^4} \right) \| \psi \|_2.
\]

From the above we readily conclude \( \psi \equiv 0 \). One can then show that \( A = hA_n \) and \( \phi = h\phi_n \) from (2.10) and the discussion which follows.}

We now move to consider the time-dependent problem (1.1).
Proof of Theorem 1.2.
We begin the proof by defining some useful entities. We first rewrite (1.1a) in the form
\[ \frac{\partial \psi}{\partial t} + L_{\text{h}} \psi = 2iA_1 \cdot \nabla_A \psi - |A_1|^2 \psi + i \phi_1 \psi + (1 - |\psi|^2) \psi. \]
Set then
\[ F = 2iA_1 \cdot \nabla_A \psi - |A_1|^2 \psi + i \phi_1 \psi + (1 - |\psi|^2) \psi. \]
We next define the Laplace transform. Let \( u \in e^{\omega} L^2(\mathbb{R}_+; L^2(\Omega, \mathbb{C})) \). Then
\[ \hat{u}(s, x) = \int_0^\infty e^{st} u(t, x) \, dt \]
denotes the Laplace transform of \( u \), which is well defined whenever \( \Re s \leq \omega \). Denote then
\[ \Gamma_\omega = \{ s \in \mathbb{C} \mid \Re s = \omega \}. \]
It is well known that \( \hat{u} \in L^2(\Gamma_\omega; L^2(\Omega, \mathbb{C})) \). Finally we define the cutoff function
\[ \chi_{T, \epsilon} = \begin{cases} 0 & t < \frac{1}{\epsilon} \\ 1 & \frac{1}{\epsilon} < t < T \\ e^{-\epsilon(t-T)} & t \geq T, \end{cases} \]
for some \( T > 1/\epsilon \).

Step 1: Let \( t^*(\kappa, M) \) be given by (3.37). We prove that there exists \( C > 0 \) such that for sufficiently large \( \kappa \) and \( \epsilon \) satisfying
\[ \frac{1}{\epsilon} > t^*(\kappa, M), \]
we have
\[ \| \chi_{T, \epsilon} F \|_{L^2(\Gamma_\omega; L^2(\Omega, \mathbb{C}))} \leq \left( 1 + \frac{C}{\kappa^2} \right) \left[ \| \chi_{T, \epsilon} \psi \|_{L^2(\Gamma_\omega; L^2(\Omega, \mathbb{C}))}^2 + \| \psi(\epsilon^{-1}, \cdot) \|_2^2 \right]. \]
By Parseval’s identity we have
\[ \frac{1}{2\pi} \| \chi_{T, \epsilon} F \|_{L^2(\Gamma_\omega; L^2(\Omega, \mathbb{C}))}^2 = \| \chi_{T, \epsilon} F \|_{L^2(\mathbb{R}_+; L^2(\Omega, \mathbb{C}))}^2. \]
We, therefore, attempt to estimate the norm of the right hand side. Let then
\[ F = F_1 + F_2, \]
where
\[ F_1 = (1 - |\psi|^2 - |A_1|^2) \psi \]
and
\[ F_2 = i \phi_1 \psi + 2iA_1 \cdot \nabla_A \psi. \]
To bound the norm of \( \chi_{T, \epsilon} F_1 \), we recall (3.39) which holds for every \( t \geq t^*(\kappa, M) \). Hence, for sufficiently large \( \kappa \), we have that \( \| A_1(t, \cdot) \|_\infty \leq 1 \) for all \( t > t^* \), from which we conclude that
\[ |F_1(t, x)| \leq |\psi(t, x)|. \]
We thus conclude that there exists $C(\Omega, c)$ such that, for sufficiently large $\kappa$ large and for any $\epsilon$ satisfying (4.14), we have

\begin{equation}
(4.16) \quad \int_0^\infty \chi_{T, \epsilon}^2(t) \|F(t, \cdot)\|^2_2 \, dt \leq \left( 1 + \frac{C(\Omega, c)}{\kappa^2} \right) \int_0^\infty \chi_{T, \epsilon}^2(t) \|\psi(t, \cdot)\|^2_2 \, dt
\end{equation}

To bound the norm of $\chi_{T, \epsilon} F$, we use (3.39), (3.12), and Poincaré inequality, applied to $\phi_1$, to obtain that for sufficiently large $t$

$$\|F_2(t, \cdot)\|_2 \leq \|\phi_1\|_2 + \frac{C}{\kappa^2} \|\nabla_A \psi(t, \cdot)\|_2 \leq \frac{\widehat{C}}{\kappa^2} \|\nabla_A \psi(t, \cdot)\|_2.$$ 

Hence, for $\epsilon$ satisfying (4.14), we have

$$\int_0^\infty \chi_{T, \epsilon}^2(t) \|F_2(t, \cdot)\|^2_2 \, dt \leq \frac{\widehat{C}^2}{\kappa^4} \int_0^\infty \chi_{T, \epsilon}^2(t) \|\nabla_A \psi(t, \cdot)\|^2_2 \, dt.$$ 

We next use (3.17) to obtain that

$$\|\chi_{T, \epsilon} F\|_{L^2([0, T); L^2(\Omega, \mathbb{C})]} \leq \frac{C(\Omega, c)}{\kappa^4} \int_0^\infty \chi_{T, \epsilon}^2(t) \left[ \|\psi(t, \cdot)\|_2^2 - \frac{1}{2} \frac{d\|\psi(t, \cdot)\|^2_2}{dt} \right] \, dt.$$ 

Integration by parts then yields, using the fact that $\chi_{T, \epsilon}'(t) \leq 0$ for all $t > \epsilon^{-1}$,

$$\|\chi_{T, \epsilon} F\|_{L^2([0, T); L^2(\Omega, \mathbb{C})]} \leq \frac{C(\Omega, c)}{\kappa^4} \left[ \int_0^\infty \chi_{T, \epsilon}^2(t) \|\psi(t, \cdot)\|_2^2 \, dt + \frac{1}{2} \|\psi(\epsilon^{-1}, \cdot)\|^2_2 \right].$$ 

In conjunction with (4.16), the above inequality readily yields (4.15).

**Step 2**: We now prove (1.15). Multiplying (4.11) by $\chi_{T, \epsilon}$ and then applying the Laplace transform with $s = \nu + i\gamma$ yields

$$(\mathcal{L}_h - \nu - i\gamma) \chi_{T, \epsilon} \widehat{\psi} - \chi_{T, \epsilon} \widehat{\psi} - e^{(\nu + i\gamma)/\epsilon} \psi(\epsilon^{-1}, \cdot) = \chi_{T, \epsilon} \mathcal{F}.$$ \hspace{1cm} (4.17)

\[ \chi_{T, \epsilon} \] denotes here the extension to $[0, \infty)$ of the derivative of $\chi_{T, \epsilon}$ in $(\frac{1}{\epsilon}, +\infty)$, which is obtained by setting $\chi_{T, \epsilon}'(0) = 0$ on $[0, \frac{1}{\epsilon}]$.

By (1.14) it follows that, for any $\gamma \in \mathbb{R}$,

$$\|\chi_{T, \epsilon} \widehat{\psi}(\nu + i\gamma, \cdot)\|_2 \leq \|\mathcal{L}_h - \nu - i\gamma\|^{-1} \left[ \|\chi_{T, \epsilon} \mathcal{F}(\nu + i\gamma, \cdot)\|_2 + \|\chi_{T, \epsilon} \widehat{\psi}(\nu + i\gamma, \cdot)\|_2 + e^{\nu/\epsilon} \|\psi(\epsilon^{-1}, \cdot)\|_2 \right] \leq \left( 1 - \frac{C_1}{\kappa^2} \right) \left[ \|\chi_{T, \epsilon} \mathcal{F}(\nu + i\gamma, \cdot)\|_2 + \|\chi_{T, \epsilon} \widehat{\psi}(\nu + i\gamma, \cdot)\|_2 + e^{\nu/\epsilon} \|\mathcal{L}_h - \nu - i\gamma\|^{-1} \|\psi(\epsilon^{-1}, \cdot)\|_2 \right],$$

where the precise value of $C_1$ will be determined later. Hence, by Cauchy’s inequality, we have for every $\delta > 0$,

$$\|\chi_{T, \epsilon} \widehat{\psi}(\nu + i\gamma, \cdot)\|_2^2 \leq \left( 1 - \frac{C_1}{\kappa^2} \right) \left( 1 + 2\delta \|\mathcal{L}_h - \nu - i\gamma\|^{-1} \|\psi(\epsilon^{-1}, \cdot)\|_2^2 \right) + \frac{\varepsilon^{\nu/\epsilon}}{\delta} \|\mathcal{L}_h - \nu - i\gamma\|^{-1} \|\psi(\epsilon^{-1}, \cdot)\|_2^2.$$ 

Since $\phi_n \in H^2(\Omega) \subset L^\infty(\Omega)$, it is easy to show from the identity

$$\text{Im} \langle u, \mathcal{L}_h u \rangle = h\langle u, \phi_n u \rangle,$$
that, for some positive \(C\),

\[
\| (\mathcal{L}_h - \nu - i\gamma)^{-1} \| \leq \frac{C}{1 + |\gamma|}.
\]

Consequently, we may integrate (4.17) over \(\Gamma\) to obtain

\[
\|
\hat{\chi}_{T,\epsilon} \psi \|_{L^2(\mathbb{R}^+; L^2(\Omega, C))}^2 \\
\leq \left( 1 - \frac{C_1}{\kappa^2} \right) (1 + 2\delta) \| \chi_{T,\epsilon} F \|_{L^2(\mathbb{R}^+; L^2(\Omega, C))}^2 \\
+ \frac{1}{\delta} \| \chi_{T,\epsilon} \psi \|_{L^2(\mathbb{R}^+; L^2(\Omega, C))}^2 + \frac{C e^{\nu/\epsilon}}{\delta} \| \psi(\epsilon^{-1}, \cdot) \|_2^2.
\]

We next use Parseval’s identity together with (4.15) and the fact that

\[-\chi_{T,\epsilon}' \leq \epsilon \chi_{T,\epsilon},\]

to obtain that, for any \(\delta > 0,\)

\[
\| e^{\nu t} \chi_{T,\epsilon} \psi \|_{L^2(\mathbb{R}^+; L^2(\Omega, C))}^2 \\
\leq \left( 1 - \frac{C_1}{\kappa^2} \right) (1 + 2\delta) \left( 1 + \frac{C \epsilon^2}{\delta} + C e^{\nu/\epsilon} \right) \| e^{\nu t} \chi_{T,\epsilon} \psi \|_{L^2(\mathbb{R}^+; L^2(\Omega, C))}^2 \\
+ \frac{C e^{\nu/\epsilon}}{\delta} \| \psi(\epsilon^{-1}, \cdot) \|_2^2.
\]

Finally, we choose \(C_1 = 8(C + 1), \delta = C/\kappa^2,\) and \(\epsilon = \min(1/\kappa^2, 1/t)\), to obtain that

\[
\| e^{\nu t} \chi_{T,\epsilon} \psi \|_{L^2(\mathbb{R}^+; L^2(\Omega, C))}^2 \leq C e^{\nu/\epsilon} \kappa^4 \| \psi(\epsilon^{-1}, \cdot) \|_2^2.
\]

The theorem now easily follows by taking the limit \(T \to \infty.\)

**Remark 4.2.** It can be easily verified that (4.3), implies that any \(\mu \in B(0, (1 - C_1/\kappa^2)^{-1})\) is in the resolvent set of \(\mathcal{L}_h.\) As \(\langle u, \mathcal{L}_h u \rangle \geq 0\) for all \(u \in D(\mathcal{L}_h),\) we can thus conclude that \((-\infty, (1 - C_1/\kappa^2)^{-1}) \cap \sigma(\mathcal{L}_h) = \emptyset.\) A similar conclusion can be reached from the lower bound (3.2) on \(\mu(h)\) in Section 3. Indeed, it can be easily verified that \(\| \mathcal{L}_h^{-1} \| \leq 1/\mu(h).\)

The formulation of (1.14) is reminiscent of conditions appearing in the statement of the Gearhart-Pruss Theorem (see Theorem 1.11 (p. 302-304) in [12]) or more quantitatively in Helffer-Sjöstrand [23] (see Remarks 1.3 and 1.4 there).

### 5. Penetration depth scaling

We begin by considering a steady-state solution for domains scaled with respect to the penetration depth. If one applies the transformation

\[
x \to \frac{x}{\kappa} ; \quad \phi \to \kappa \phi,
\]

\[\text{(5.1)}\]
the system (4.2) becomes
\begin{align}
(5.2a) & \quad -\nabla^2_{\kappa A} \psi + i\kappa \phi \psi = \kappa^2 (1 - |\psi|^2) \psi \quad \text{in } \Omega, \\
(5.2b) & \quad -\kappa^2 \text{curl}^2 A + \sigma \nabla \phi = \kappa \text{Im}(\bar{\psi} \nabla_{\kappa A} \psi) \quad \text{in } \Omega, \\
(5.2c) & \quad \psi = 0 \quad \text{on } \partial \Omega_c, \\
(5.2d) & \quad \nabla_{\kappa A} \psi \cdot \nu = 0 \quad \text{on } \partial \Omega_i, \\
(5.2e) & \quad \frac{\partial \phi}{\partial \nu} = -h\frac{\kappa^4 J_0(x)}{\sigma} \quad \text{on } \partial \Omega_c, \\
(5.2f) & \quad \frac{\partial \phi}{\partial \nu} = 0 \quad \text{on } \partial \Omega_i, \\
(5.2g) & \quad \int_{\partial \Omega} \text{curl} A \, ds = h \cdot h_r.
\end{align}

We consider again fixed \( c = \kappa^2/\sigma \) and \( J_0 \). The normal state for this scaling is defined once again by (2.10). We define the linear operator (with Dirichlet-Neumann conditions)
\[
L_{\kappa} = L_{\kappa h} := -\nabla^2_{\kappa h A} + i\kappa^3 h \phi_n.
\]

Note that unlike \( L_h \), \( L_{\kappa} \) does depend on \( \kappa \).

**Lemma 5.1.** For any \( \beta < 1 \) there exists \( C_{\beta}(\Omega, h) > 0 \) and \( \kappa_{\beta}(\Omega, h) \) such that for \( \kappa \geq \kappa_{\beta} \) we have
\[
\| \psi \|_2 \leq C_{\beta} \kappa^{-\beta/2}.
\]

**Proof.**
From Assumption (B) (implying (2.25)) and Assumption (R1) on \( \Omega \) we get that \( \partial \Omega \cup \{ \text{curl} A_n = 0 \} \) is a union of piecewise \( C^1 \) curves. For \( \alpha \in (0, 1) \), whose value is to be determined later, we define
\[
S_{\kappa, \alpha} = \{ x \in \Omega, | \text{curl} A_n(x) | \geq \kappa^{-\alpha} \} \cap \{ d(x, \partial \Omega) \geq \kappa^{-\alpha} \}.
\]

Note that \( |\Omega \setminus S_{\kappa, \alpha}| \leq C \kappa^{-\alpha} \) for some fixed \( C > 0 \). We can now consider (for \( \kappa \) large enough) a covering of \( S_{\kappa, \alpha} \) by balls of size \( \kappa^{-\alpha'} \) with \( 1 > \alpha' > \alpha \) with support in \( \Omega \).

In each of the balls (for \( \kappa \) large enough) we can assume that the sign of \( \text{curl} A_n \) is constant. Associated with this (finite) covering, we can associate a partition of unity \( \eta_j \) such that
\[
\sum_j \eta_j^2 = 1 \quad \text{on } S_{\kappa, \alpha}, \quad \sum_j |\nabla \eta_j|^2 \leq C \kappa^{2\alpha'} \quad \text{and } \text{Supp } \eta_j \subset \Omega.
\]

Multiplying (5.2a) by \( \eta_j \bar{\psi} \) and integrating by parts, taking into account the boundary conditions (5.2c,d) yields for the real part
\[
\| \nabla_{\kappa A} (\eta_j \psi) \|_2^2 = \| \psi \eta_j \|_2^2 + \kappa^2 (\| \eta_j \psi \|_2^2 - \| \eta_j^{1/2} \psi \|_4^4).
\]

With the aid of Theorem 4 in [30] we then obtain
\[
\kappa \int \text{curl} A |\eta_j \psi|^2 \, dx \leq C (\| \nabla \eta_j \|_2^2 + \kappa^2 \| \eta_j \psi \|_2^2).
\]
Using the fact that \(\text{curl} \, A_n\) does not change sign in each of the connected components of \(S_{\kappa,\alpha}\) we obtain that for \(\kappa\) large enough

\[
\kappa^{3-\alpha} h \int_{S_{\kappa,\alpha}} |\psi|^2 \, dx \leq \kappa \int_\Omega |\text{curl} \, A_1| \, |\psi|^2 \, dx + C \kappa^2 \int_\Omega |\psi|^2 \, dx.
\]

From (5.15) below we get that

\[
\|\text{curl} \, A_1\|_2 \leq C(\Omega) \|\psi\|_2,
\]

which, when substituted into (5.8) yields, with the aid of (2.47) and Cauchy-Schwarz inequality, the existence of \(C(\hbar,\Omega) > 0\) such that, for sufficiently large \(\kappa\),

\[
\|\psi\|_{L^2(S_{\kappa,\alpha})} \leq C \kappa^{1-\alpha} \|\psi\|_2^2.
\]

Choosing \(\alpha < 1\), keeping in mind the control of the measure of \(\Omega \setminus S_{\kappa,\alpha}\), yields

\[
\int_{S_{\kappa,\alpha}} |\psi|^2 \, dx \leq C \hbar \kappa^{-1},
\]

and completing by the integral over the complementary of \(S_{\kappa,\alpha}\), we obtain

\[
\int_\Omega |\psi|^2 \, dx \leq C \hbar \kappa^{-\alpha}.
\]

To complete the proof, we can take \(\alpha = \beta\) and then \(\alpha' = (1 + \beta)/2\). Consequently, we obtain (5.4). This proves the lemma.

We can now state our steady-state estimate for the critical current

**Proposition 5.2.** Suppose that there exist \(\gamma < 1/2\), \(C > 0\) and \(\kappa_0 > 0\) such that

\[
\|Q^{-1}\|_{2} < \frac{1}{\kappa^2} \left[ 1 - \frac{C}{\kappa^\gamma} \right],
\]

for all \(\kappa > \kappa_0\). Then, there exists \(\kappa_1 \geq \kappa_0\) such that \(\psi \equiv 0\) is the unique solution of (5.2) for all \(\kappa \geq \kappa_1\).

**Proof.** As in the proof of Proposition 4.1 we set

\[
A_1 = A - h\kappa^2 A_n \quad ; \quad \phi_1 = \phi - h\kappa^2 \phi_n.
\]

It is easy to show that \(A_1\) and \(\phi_1\) satisfy

\[
-\kappa^2 \text{curl} \, A_1 = \kappa \text{Im} \left( \overline{\psi} \nabla_{\kappa A} \psi \right) - \sigma \nabla \phi_1 \quad \text{in} \ \Omega,
\]

\[
\frac{\partial \phi_1}{\partial \nu} = 0 \quad \text{on} \ \partial \Omega,
\]

\[
\text{curl} \, A_1 = 0 \quad \text{on} \ \partial \Omega.
\]

Taking the scalar product with \(\nabla \phi_1\) of the first line of (5.13) yields

\[
\|\nabla \phi_1\|_2 \leq \frac{\kappa}{\sigma} \|\text{Im} \left( \overline{\psi} \nabla_{\kappa A} \psi \right)\|_2,
\]

and hence,

\[
\|\text{curl} \, A_1\|_2 \leq \frac{2}{\kappa} \|\nabla_{\kappa A} \psi\|_2.
\]
By (5.2a) we then have (see the proof of (4.9))

\[(5.15) \quad \| \text{curl}^2 A_1 \|_2 \leq 2 \| \psi \|_2.\]

We can now set \( A_1 = -\nabla \perp \Phi \), and follow the same route as in the proof of (4.8) (see (4.10)), to obtain, with the aid of Sobolev embedding that

\[(5.16) \quad \| A_1 \|_\infty \leq C \| \psi \|_2.\]

By (5.14), (4.8) and (5.4) we obtain that for all \( \tilde{\gamma} > 1/2 \) there exists \( C_\gamma(h, \Omega) \) such that

\[
\text{for sufficiently large } \kappa \text{ we have }
\| 2i\kappa A_1 \cdot \nabla \kappa \psi - \kappa^2 |A_1|^2 \psi + i\kappa \phi_1 \psi \|_2 \leq C_\tilde{\gamma} \kappa^{1+\tilde{\gamma}} \| \psi \|_2.
\]

Consequently, we obtain that

\[
\| \psi \|_2 \leq \left( \kappa^2 + C_\gamma \kappa^{1+\tilde{\gamma}} \right) \| \mathcal{L}_\kappa^{-1} \| \| \psi \|_2,
\]

from which the proposition readily follows by choosing \( 1/2 < \tilde{\gamma} < 1 - \gamma \).

We next prove the decay, in the long time limit, of solutions of the time-dependent version of (5.2), i.e.,

\[
(5.17a) \quad \frac{\partial \psi}{\partial t} - \nabla^2 \kappa \psi + i\kappa \Phi \psi = \kappa^2 (1 - |\psi|^2) \psi \quad \text{in } \Omega,
\]

\[
(5.17b) \quad -\kappa^2 \text{curl}^2 A + \sigma \left( \nabla \Phi + \frac{\partial A}{\partial t} \right) = \kappa \text{Im} (\bar{\psi} \nabla \kappa \psi) \quad \text{in } \Omega,
\]

\[
(5.17c) \quad \psi = 0 \quad \text{on } \partial \Omega_c,
\]

\[
(5.17d) \quad \nabla \kappa \psi \cdot \nu = 0 \quad \text{on } \partial \Omega_i,
\]

\[
(5.17e) \quad \nabla \Phi \cdot \nu = -\frac{h}{\sigma} J_0(x) \quad \text{on } \partial \Omega_c,
\]

\[
(5.17f) \quad \nabla \Phi \cdot \nu = 0 \quad \text{on } \partial \Omega_i,
\]

\[
(5.17g) \quad \int_{\partial \Omega} \text{curl} A \, ds = h \kappa \cdot h_r.
\]

We begin by proving a few auxiliary estimates.

**Lemma 5.3.** Let \( A_1 \) be defined by (5.12). There exists \( C(c, \Omega) \) and \( \kappa_0 \) such that, for \( \kappa \geq \kappa_0 \),

\[
(5.18) \quad \| A_1(t, \cdot) \|_2^2 \leq \left( \| A_1(0, \cdot) \|_2^2 + \frac{C(c, \Omega)}{\kappa^2} \right) e^{-\lambda t} + C(c, \Omega) \int_0^t e^{-\lambda(t-\tau)} \| \psi(\tau, \cdot) \|_2^2 d\tau,
\]

and a constant \( C_1(c, \Omega) \) such that:

\[
(5.19) \quad \int_0^t e^{-2\lambda(t-\tau)} \| \text{curl} A_1(\tau, \cdot) \|_2^2 d\tau \leq C_1(c, \Omega) \left\{ \left( \| A_1(0, \cdot) \|_2^2 + \frac{1}{\kappa^2} \right) e^{-\lambda t} + \int_0^t e^{-\lambda(t-\tau)} \| \psi(\tau, \cdot) \|_2^2 d\tau \right\}.
\]
Proof. In a similar manner to the one used to derive (3.14) we obtain that
\[ \frac{1}{2} d\|A_1(t, \cdot)\|^2 dt + \frac{\lambda \kappa^2}{\sigma} \|A_1(t, \cdot)\|^2 \leq \frac{C}{\kappa} \|A_1(t, \cdot)\|_2 \|\nabla A \psi(t, \cdot)\|_2, \]
from which we readily obtain that
\[ \frac{d\|A_1(t, \cdot)\|^2}{dt} + \lambda c \|A_1(t, \cdot)\|^2 \leq \frac{C}{\kappa} \|\nabla A \psi(t, \cdot)\|_2. \]
Hence,
\[ \|A_1(t, \cdot)\|_2 \leq \|A_1(0, \cdot)\|_2 e^{-\lambda c t} + \frac{C}{\kappa} \int_0^t e^{-\lambda c (t-\tau)} \|\nabla A \psi(\tau, \cdot)\|_2 d\tau \]
\[ \leq \|A_1(0, \cdot)\|_2 e^{-\lambda c t} + \frac{C}{\kappa} \left[ \int_0^t e^{-\lambda c (t-\tau)} \|\nabla A \psi(\tau, \cdot)\|_2^2 d\tau \right]^{1/2}. \]
Note that $C$ depends on $\Omega$ and $c$. Since, as in (3.17)
\[ \|\nabla A \psi\|^2 \leq \kappa^2 \|\psi\|^2 - \frac{1}{2} \frac{d\|\psi\|^2}{dt}, \]
we obtain after an integration by parts
\[ \|A_1(t, \cdot)\|_2 \leq \|A_1(0, \cdot)\|_2 e^{-\lambda c t} + C \left[ \int_0^t e^{-\lambda c (t-\tau)} \|\psi\|^2 d\tau + \frac{1}{2\kappa^2} \|\psi(0, \cdot)\|^2 e^{-\lambda c t} \right]^{1/2}, \]
from which (5.18) readily follows.

To prove (5.19) we use the fact that, by straightforward integration by parts we have,
\[ \frac{1}{2} \frac{d\|A_1(\tau, \cdot)\|^2}{d\tau} + c \|\text{curl} A_1(\tau, \cdot)\|^2 \leq \frac{C}{\kappa} \|A_1(\tau, \cdot)\|_2 \|\nabla A \psi(\tau, \cdot)\|_2. \]
Multiplying by $e^{-2\lambda c (t-\tau)}$ and integrating with respect to $\tau$ yields after integration by parts
\[ \int_0^t e^{-2\lambda c (t-\tau)} \|\text{curl} A_1(\tau, \cdot)\|^2 d\tau \leq C \int_0^t e^{-2\lambda c (t-\tau)} \left[ \|A_1(\tau, \cdot)\|^2 + \frac{1}{\kappa^2} \|\nabla A \psi(\tau, \cdot)\|^2 \right] d\tau \]
\[ + \frac{1}{2} \|\nabla A_1(0, \cdot)\|^2 + \lambda c \int_0^t e^{-2\lambda c (t-\tau)} \|A_1(\tau, \cdot)\|^2_2 d\tau. \]
The above, together with (5.18) and (5.20), leads to (5.19).

\[ \text{Lemma 5.4. Let } A_1 \text{ be defined by (5.12) and } 0 < \beta < 1/2. \text{ Suppose that } \|A_1(\cdot, 0)\| \leq M. \text{ Then, there exist } \kappa_\beta(h, c, \Omega) \text{ and } C_\beta(h, c, \Omega), \text{ and, for } \kappa \geq \kappa_\beta, t_\beta(M, \kappa) \text{ such that, for all } t \geq t_\beta(\kappa), \]
\[ \|A_1(t, \cdot)\|_2 + \|\psi(t, \cdot)\|_2 \leq \frac{C_\beta}{\kappa^\beta}. \]

\[ \text{Proof. Let } S_{\kappa, \alpha} \text{ be defined by (5.5) and } \eta_j \text{ by (5.6). We multiply (5.17a) by } \eta_j^2 \bar{\psi} \text{ and integrate over } \Omega \text{ to obtain} \]
\[ \frac{1}{2} \frac{d\|\eta_j \psi\|^2}{dt} + \|\nabla A(\eta_j \psi)\|^2 = \|\psi \nabla \eta_j\|^2 + \kappa^2 (\|\eta_j \psi\|^2 - \|\eta_j^{1/2} \psi\|^2). \]
In the same manner used to prove (5.8) we can show that
\[ \|\nabla A(\eta_j \psi)\|^2 \geq \kappa^{3-\alpha} h \|\eta_j \psi\|^2, \]
Summing over $j$, we obtain with the aid of the Cauchy-Schwarz inequality and (2.47):

\[
\frac{1}{2} \frac{d}{dt} \sum_j \|\eta_j \psi\|^2_2 + \kappa^3 \|h \|_{\psi} \leq \kappa \int_\Omega |\text{curl} A_1| \|\eta_j \psi\|^2_2 + \int |\psi \nabla \eta_j|^2 dx + C\kappa^2 \int_\Omega |\eta_j \psi|^2 dx.
\]

We rewrite this inequality in the following way, for $\kappa$ large enough,

\[
\frac{1}{2} \frac{d}{dt} \sum_j \|\eta_j \psi\|^2_2 + \frac{1}{2} \kappa^3 \|h \|_{\psi} \leq \kappa \|\text{curl} A_1\|_2 \left( \sum_j \|\eta_j \psi\|_2 \right)^\frac{1}{2} + C\kappa^2 \|\psi\|^2_2.
\]

Let

\[
\Theta(t) := \Theta_{k,\alpha}(t) = \sum_j \|\eta_j(\cdot) \psi(t, \cdot)\|^2_2.
\]

From (5.23) we deduce that

\[
\Theta(t) \leq \Theta(0) e^{-\kappa^3 \|h\|_{\psi} t} + 2 \int_0^t e^{-\kappa^3 \|h\|_{\psi} (t-\tau)} \left[ \kappa \|\text{curl} A_1(\tau, \cdot)\|_2 \Theta(\tau) \right]^{\frac{1}{2}} + C\kappa^{2-\alpha} \, d\tau,
\]

from which we obtain

\[
\Theta(t) \leq \Theta(0) e^{-\kappa^3 \|h\|_{\psi} t} + 2\kappa \left[ \int_0^t e^{-2\lambda c(t-\tau)} \|\text{curl} A_1(\tau, \cdot)\|^2_2 \, d\tau \right]^{1/2} + C\kappa^{-1}.
\]

Let $t^* = t^*(\kappa, M)$ be given by (3.37). Suppose now that $t \geq t^*$. It readily follows from (5.19) and (3.37) that

\[
\int_0^t e^{-2\lambda c(t-\tau)} \|\text{curl} A_1(\tau, \cdot)\|^2_2 \, d\tau \leq C \left( \frac{1}{\kappa^2} + \|\psi\|^2_2 \right).
\]

Substituting the above into (5.24) yields, with the aid of (2.47), for sufficiently large $\kappa$,

\[
\Theta(t) \leq \Theta(0) e^{-\kappa^3 t} + C \left( \frac{1}{\kappa} + \frac{1}{\kappa(1-\alpha/2)} \|\psi\|^2_2 \right) \leq \hat{C} \left( \frac{1}{\kappa(1-\alpha/2)} + \frac{1}{\kappa^2} \right).
\]

Hence, for all $t \geq t^*$ we have

\[
\|\psi(t, \cdot)\|^2_2 \leq \Theta(t) + |\Omega \setminus S_{k,\alpha}| \leq \Theta(0) \exp(-\kappa^3 t) + C \left( \frac{1}{\kappa(1-\alpha/2)} \|\psi\|^2_2 \right) \leq \hat{C} \left( \frac{1}{\kappa(1-\alpha/2)} + \frac{1}{\kappa^2} \right).
\]

leading to:

\[
\|\psi(t, \cdot)\|^2_2 \leq e^{-\kappa^3 t} + C \left( \frac{1}{\kappa(1-\alpha/2)} \|\psi\|^2_2 \right) \leq C\kappa^{-\frac{1}{3}}.
\]

Optimizing over $\alpha$ gives, for $\alpha = \frac{1}{3}$, the existence of $\kappa_0$ and $t^*$ such that, for $\kappa \geq \kappa_0$ and $t \geq t^*$, we have:

\[
\|\psi(t, \cdot)\|^2_2 \leq C\kappa^{-\frac{1}{3}}.
\]
Here we have chosen $\kappa_0$ such that:
\[
e^{-t^* h\kappa^2} \leq \kappa^{-2}, \forall \kappa \geq \kappa_0.
\]

We can now apply the above procedure for $t \geq nt^*$ with $n \geq 1$ to obtain a generalization of (5.25). Suppose that we have found for some $n$ and for any $p \leq n$, two increasing sequences $\kappa_p$ and $\alpha_p$, and constants $C_p$ such that $\kappa \geq \kappa_p$ and for $t \geq pt^*(\kappa, M)$, we have
\[
\|\psi(t, \cdot)\|_2^2 \leq C_p \kappa^{-\alpha_p}.
\]

Then, we show that the above is true for $p = n + 1$.

We rewrite (5.25) with initial point $nt^*$ in the form:
\[
\|\psi(t, \cdot)\|_2^2 \leq e^{-t^* h\kappa^2} + C \left( \frac{1}{\kappa^{(1-\alpha)/2}} \|\psi\|_{L^\infty(nt^*, t; L^2(\Omega, C))} + \frac{1}{\kappa^\alpha} \right), \forall t \geq (n + 1)t^*.
\]

Using the recursion argument, we obtain, for any $\alpha \in (0, 1)$ and $\kappa \geq \kappa_n$,
\[
\|\psi(t, \cdot)\|_2^2 \leq \kappa^{-2} + C(\alpha) \max(C_n, 1) \left( \frac{1}{\kappa^{(1-\alpha)/2} + \alpha} + \frac{1}{\kappa^\alpha} \right), \forall t \geq (n + 1)t^*.
\]

Optimizing over $\alpha$ leads to the choice:
\[
\alpha_{n+1} = \frac{1}{3} + \frac{2}{3} \alpha_n.
\]

and to (5.27) for $p = (n + 1)$.

It can be easily shown that
\[
\alpha_n = 1 - \left(\frac{2}{3}\right)^n.
\]

Hence, for any $\beta < 1/2$ there exists $n \in \mathbb{N}$ such that $\alpha_n \geq 2\beta$. It follows that for all $t \geq n(\beta)t^*$,
\[
\|\psi(t, \cdot)\|_2^2 \leq \frac{C_\beta}{\kappa^{2\beta}}.
\]

We next substitute (5.28), in conjunction with (2.47), into (5.18) to obtain that for all $t \geq n_\beta t^* + \frac{2}{\kappa\alpha} \ln \kappa$
\[
\|A_1(t, \cdot)\|_2^2 \leq \frac{C}{\kappa^2} + C \int_0^{n_\beta t^*} e^{-\lambda_c(t-\tau)} d\tau + \int_{n_\beta t^*}^t e^{-\lambda_c(t-\tau)} \frac{C_\beta^2}{\kappa^{2\beta}} d\tau.
\]

Hence, for new constants $C, \tilde{C}, C_\beta$, we get:
\[
\|A_1(t, \cdot)\|_2^2 \leq \frac{C}{\kappa^2} + \frac{C_\beta}{\kappa^{2\beta}} \leq \frac{\tilde{C}_\beta}{\kappa^{2\beta}}.
\]

The above, together with (5.28), readily verifies (5.21) with $t_\beta(\kappa) = n_\beta t^*(\kappa) + \frac{2}{\kappa\alpha} \ln \kappa$.

**Corollary 5.5.** Let $A_1$ be defined by (5.12) and satisfy $\|A_1(\cdot, 0)\|_2 \leq M$ for some $M > 0$. Let further $0 < \gamma < 1/2$, and $t_\beta(\kappa, M)$ be defined as in Lemma 5.4. For sufficiently large $\kappa$, there exists $C_\gamma(h, c, \Omega) > 0$ such that
\[
\|A_1(t, \cdot)\|_{\infty} \leq \frac{C_\gamma}{\kappa^{\gamma}},
\]

(5.29)
We next use (5.17a) to obtain from the above that
\begin{equation}
\frac{\partial A_1}{\partial t} + c \text{curl}^2 A_1 = \frac{1}{\kappa} \text{Im} (\bar{\psi} \nabla_{\kappa A} \psi) - \nabla \phi_1 \quad \text{in } \mathbb{R}_+ \times \Omega, \tag{5.30a}
\end{equation}
\begin{equation}
\text{div} A_1 = 0 \quad \text{in } \mathbb{R}_+ \times \Omega, \tag{5.30b}
\end{equation}
\begin{equation}
A_1 \cdot \nu = 0 \quad \text{on } \mathbb{R}_+ \times \partial \Omega, \tag{5.30c}
\end{equation}
\begin{equation}
\int_{\partial \Omega} \text{curl} A_1(t, x) \, ds = 0 \quad \text{for all } t > 0. \tag{5.30d}
\end{equation}

As in (5.14) we obtain here that
\begin{equation}
\| \nabla \phi_1 \|_2 \leq \frac{c}{\kappa} \| \nabla_{\kappa A} \psi \|_2. \tag{5.31}
\end{equation}

We can now apply (C.4) and (C.3) to (5.30) (in the case of \( L^{(1)} \)) with \( X = A_1 \) on the interval \([t_0, T]\) for some \( t_0 > t_\beta, t_1 = t_0 + 1, \) and \( T = t_0 + 2, \) to obtain
\begin{equation}
\left\| \frac{\partial A_1}{\partial t} \right\|_{L^2(t_1, T; L^2(\Omega))} + \| A_1 \|_{L^2(t_1, T; H^2(\Omega))} + \| A_1 \|_{L^\infty(t_1, T; H^1(\Omega))}
 \leq C \left[ \| A_1(t_0, \cdot) \|_2 + \| \nabla \phi_1 \|_{L^2(t_0, T; L^2(\Omega))} + \frac{1}{\kappa^2} \| \text{Im} (\bar{\psi} \nabla_{\kappa A} \psi) \|_2 \right]. \tag{5.32}
\end{equation}

With the aid of (5.31), (5.20), and (2.47) we then conclude that there exists \( C_\beta(c, \Omega, \kappa) \) such that for all \( 0 < \beta < 1/2, \)
\begin{equation}
\left\| \frac{\partial A_1}{\partial t} \right\|_{L^2(t_1, T; L^2(\Omega))} + \| A_1 \|_{L^2(t_1, T; H^2(\Omega))} + \| A_1 \|_{L^\infty(t_1, T; H^1(\Omega))}
 \leq C \left[ \| A_1(t_0, \cdot) \|_2 + \| \psi \|_{L^2(t_0, T; L^2(\Omega))} + \frac{1}{\kappa^2} \| \psi(t_0, \cdot) \|_2 \right] \leq \frac{C_\beta}{\kappa^\beta}. \tag{5.33}
\end{equation}

We next obtain a bound on \( \| \nabla_{\kappa A} \psi \|_{L^\infty(t_0, T; L^2(\Omega))}. \) By (5.20) we have that
\begin{equation}
\int_{t_0}^{T} \| \nabla_{\kappa A} \psi(t, \cdot) \|_2^2 \, dt \leq \kappa^2 \| \psi \|_{L^2(t_0, T; L^2(\Omega))}^2 + \frac{1}{2\kappa^2} \| \psi(t_0, \cdot) \|_2^2 \leq C \kappa^{2(1-\beta)}. \tag{5.33}
\end{equation}

It is easy to show that
\begin{equation}
\frac{d}{dt} \| \nabla_{\kappa A} \psi(t, \cdot) \|_2^2 = -\left\langle \frac{\partial \psi}{\partial t}, \nabla_{\kappa A} \psi \right\rangle - \left\langle i\kappa \frac{\partial A}{\partial t}, \nabla_{\kappa A} \psi \right\rangle. \tag{5.33}
\end{equation}

We next use (5.17a) to obtain from the above that
\begin{equation}
\frac{d}{dt} \| \nabla_{\kappa A} \psi(t, \cdot) \|_2^2 = \frac{1}{2} \| \psi(t, \cdot) \|_2^2 - \frac{1}{4} \| \psi(t, \cdot) \|_4^4 - \frac{1}{4} \| \psi(t, \cdot) \|_4^4.
\end{equation}

Hence,
\begin{equation}
\frac{d}{dt} \| \nabla_{\kappa A} \psi(t, \cdot) \|_2^2 \leq 2 \frac{d}{dt} \left( \frac{1}{2} \| \psi(t, \cdot) \|_2^2 - \frac{1}{4} \| \psi(t, \cdot) \|_4^4 + \frac{\kappa^2 \| \frac{\partial A_1}{\partial t} \|_2^2 + \| \nabla_{\kappa A} \psi \|_2^2 + \frac{\kappa^2}{2} \| \phi \|_2^2.
\end{equation}
By (2.13) and (5.31) we have:

\[ \|\phi\|_{1,2}^2 \leq C(\kappa^4 + \|\nabla_{\kappa A}\psi\|^2_2). \]  

(5.34)

Let \( t_0 < \tilde{t}_0 < t_1 \). Integrating the above on \((\tilde{t}_0, t)\) yields, with the aid of (5.33) and (5.32), that for every \( t \in [t_1, T] \) it holds:

\[ \|\nabla_{\kappa A}\psi(t, \cdot)\|^2_2 \leq \|\nabla_{\kappa A}\psi(\tilde{t}_0, \cdot)\|^2_2 + C\kappa^6. \]

(5.35)

By (5.33), there exists \( \tilde{t}_0 \in (t_0, t_1) \) such that

\[ \|\nabla_{\kappa A}\psi(\tilde{t}_0, \cdot)\|^2_2 \leq C\kappa^{2(1-\beta)}, \]

which readily yields

\[ \|\nabla_{\kappa A}\psi\|_{L^\infty(t_1, T; L^2(\Omega))} \leq C\kappa^3. \]

We next apply (C.4) to (1.1a) on the interval \((t_1, T)\) to obtain that

\[ \|\psi\|_{L^\infty(t_1, T; H^1(\Omega, \mathbb{C}))} + \|\psi\|_{L^2(t_1, T; H^2(\Omega, \mathbb{C}))} \leq C(\Omega) \left[ \|\psi(t_1, \cdot)\|_{1,2} + \kappa\|A \cdot \nabla_{\kappa A}\psi\|_{L^2(t_1, T; L^2(\Omega, \mathbb{C}))} + \kappa^2\|A^2\psi\|_{L^2(t_1, T; L^2(\Omega, \mathbb{C}))} + \kappa\|\nabla\psi\|_{L^2(t_1, T; L^2(\Omega, \mathbb{C}))} + \kappa^2\|\psi(1 - |\psi|^2)\|_{L^2(t_1, T; L^2(\Omega, \mathbb{C}))} \right]. \]

(5.36)

We now estimate the various terms on the right-hand-side of (5.36). For the first term we have, in view of (5.32) and (5.35), that

\[ \|\psi(t_1, \cdot)\|_{1,2} = \|\psi(t_1, \cdot)\|_2 + \|\nabla_{\kappa A}\psi(t_1, \cdot)\|_2 + \kappa\|A\psi(\cdot, t_1)\|_2 \leq C\left[\kappa^{-\beta} + \kappa^{1-\beta} + \kappa\left(\|A_1\|_{L^\infty(t_0, T; L^2(\Omega))} + \kappa^2\|A_n\|_\infty\right)\right] \leq C\kappa^3. \]

(5.37)

For the second term we have by (5.35) and (5.33)

\[ \kappa\|A \cdot \nabla_{\kappa A}\psi\|_{L^2(t_1, T; L^2(\Omega, \mathbb{C}))} \leq \kappa\|\nabla_{\kappa A}\psi\|_{L^\infty(t_1, T; L^2(\Omega))} \left(\|A_1\|_{L^2(t_0, T; L^\infty(\Omega))} + \kappa^2\|A_n\|_\infty\right) \leq C\kappa^6. \]

(5.38)

For the third term we have, with the aid of (2.47), (5.32), and Sobolev embedding

\[ \kappa^2\|A^2\psi\|_{L^2(t_1, T; L^2(\Omega, \mathbb{C}))} \leq \kappa^2\left(\|A_1\|^2_{L^\infty(t_0, T; H^1(\Omega))} + \|A_n\|^2_\infty\right) \leq C\kappa^6. \]

(5.39)

For the fourth term we have, in view of (5.34)

\[ \kappa\|\nabla\psi\|_{L^2(t_1, T; L^2(\Omega, \mathbb{C}))} \leq C\kappa^3. \]

Finally, for the last term on the right hand-side we have

\[ \kappa^2\|\psi(1 - |\psi|^2)\|_{L^2(t_1, T; L^2(\Omega, \mathbb{C}))} \leq C\kappa^{2-\beta}. \]

Combining the above with (5.37), (5.38), and (5.39) yields

\[ \|\psi\|_{L^\infty(t_1, T; H^1(\Omega, \mathbb{C}))} + \|\psi\|_{L^2(t_1, T; H^2(\Omega, \mathbb{C}))} \leq C\kappa^6. \]

(5.40)

We continue from here in precisely the same manner as in the proof of Lemma 3.3 to obtain that,

\[ \|A_1\|_{L^\infty(t_1, T; H^2(\Omega, \mathbb{R}^2))} \leq C\left[\frac{1}{\kappa}\|\text{Im}(\tilde{\psi}\nabla_{\kappa A}\psi)\|_{L^2(t_2, T; H^1(\Omega, \mathbb{R}^2))} + \|A_1(t_2, \cdot)\|_2^2\right]. \]

(5.41)

With the aid of (5.40) it can then be proved that

\[ \|A_1\|_{L^\infty(t_1, T; H^2(\Omega, \mathbb{R}^2))} \leq C\kappa^9, \]
and hence
\[ \| \nabla A_1 \|_{L^\infty((t_1,T;H^1(\Omega,\mathbb{R}^2)))} \leq C \kappa^9. \]

Note that by (5.32)
\[ \| \nabla A_1 \|_{L^\infty((t_1,T;L^2(\Omega)))} \leq C \beta \kappa^\beta. \]

We next apply a standard interpolation inequality (cf. Section 7.1 in [18]) together with the above to obtain
\[ (5.41) \quad \| \nabla A_1(t, \cdot) \|_{2+\delta} \leq \| \nabla A_1(t, \cdot) \|_p \leq C \kappa^{-s\beta+9(1-s)}, \]
where
\[ s = \frac{\frac{1}{2} - \frac{1}{p}}{\frac{1}{2} - \frac{1}{p}} \geq 1 - \delta \frac{p}{2p-4}. \]

For \( p \geq 4 \) we get
\[ s \geq 1 - \delta, \]
which combined with (5.41) yields
\[ \| \nabla A_1(t, \cdot) \|_{2+\delta} \leq C \kappa^{-\beta+\delta(9-\beta)}. \]

Since we may choose \( \delta \) to be arbitrarily small the corollary easily follows with the aid of Sobolev embeddings. \( \blacksquare \)

We can now conclude long-time decay of solutions of (5.17).

**Proposition 5.6.** Suppose that for some \( 0 < \alpha < 1 \), there exists some \( C > 0 \) and \( \kappa_0 > 0 \) we have
\[ (5.42) \quad \sup_{\gamma \in \mathbb{R}} \|(\mathcal{L}_\kappa - i\gamma)^{-1}\| < \frac{1}{\kappa^2} \left[ 1 - \frac{C}{\kappa^{1-\alpha}} \right] \]
for all \( \kappa > \kappa_0 \). Then, there exists \( \kappa_1 \geq \kappa_0 \) such that (1.15) with \( \nu = 0 \) holds true.

**Proof.** The proof is almost identical with the proof of Theorem 1.2, and hence we bring therefore only a brief summary of it. We first set
\[ F = 2i\kappa A_1 \cdot \nabla \kappa A_1 \psi - |\kappa A_1|^2 \psi + i\kappa \phi_1 \psi + \kappa^2(1 - |\psi|^2) \psi. \]

We then recall the definition of \( \chi_{T,\xi} \) in (4.13), of \( \Gamma_\omega \) in (4.12), and of the Laplace transform, and then, in the same manner we prove (4.15) above, we show that for any \( 0 < \gamma < 1 \) there exists \( C_\gamma > 0 \) such that
\[ (5.43) \quad \| \hat{\chi}_{T,\xi} F \|_{L^2(\Gamma_\omega;L^2(\Omega,\mathbb{C}))} \leq \kappa^2 \left( 1 + \frac{C_\gamma}{\kappa^{1-\gamma}} \right) \left[ \| \hat{\chi}_{T,\xi} \psi \|_{L^2(\Gamma_\omega;L^2(\Omega,\mathbb{C}))}^2 + \| \psi^{-1} \cdot \|_{L^2(\Omega,\mathbb{C}}^2 \right]. \]

We next write (5.17a) in the form
\[ \frac{\partial \psi}{\partial t} + \mathcal{L}_\kappa \psi = F, \]
and then take its Laplace transform. Then, we make use of (5.43) in conjunction with (5.42) and Parseval’s identity to obtain

\[ \|\chi_{T,\epsilon}\psi\|_{L^2(\mathbb{R}_+; L^2(\Omega,\mathcal{C}))}^2 \leq \left[ 1 - \frac{C}{\kappa^{1-\alpha}} \right] (1 + 2\delta) \left( 1 + \frac{C\gamma}{\kappa^{1-\gamma}} + C\epsilon^2 \frac{\delta}{\delta} \right) \|\chi_{T,\epsilon}\psi\|_{L^2(\mathbb{R}_+; L^2(\Omega,\mathcal{C}))}^2 \\
+ \frac{C}{\delta} \|\psi(\epsilon^{-1}, \cdot)\|_{L^2}^2. \]

Finally, we choose \( \delta = \epsilon = 1/\kappa \) and \( \gamma < \alpha \) to obtain that

\[ \|\chi_{T,\epsilon}\psi\|_{L^2(\mathbb{R}_+; L^2(\Omega,\mathcal{C}))}^2 \leq C\kappa^4 \|\psi(\epsilon^{-1}, \cdot)\|_{L^2}^2, \]

and take the limit \( T \to \infty \) to complete the proof. \( \blacksquare \)
6. Resolvent estimates in the large domain limit

6.1. Presentation of the problem. In the previous sections we have obtained sufficient conditions for the stability of the semi-group associated with (1.1). These conditions were phrased in terms of the resolvent norm of the linear operator $L_h$, which is defined in (4.1). As the operator is defined on a general class of domains in $\mathbb{R}^2$, we attempt to estimate the resolvent $(L_h - \lambda)^{-1}$, in the large domain limit, by approximate operators defined on $\mathbb{R}^2$ and $\mathbb{R}_+^2$, $h$ being fixed and strictly positive.

Let then $R > 0$. We denote by $\Omega_R$ the image of $\Omega$ under the dilation
\[(6.1) \quad x \rightarrow Rx.\]

We assume that the domain $\Omega$ has the property (R1)-(R2) and that assumptions (J1)-(J3), (B) and (C) are met.

Denote the transformed electric field by $\phi_R$. It satisfies the problem
\[
\begin{cases}
\Delta \phi_R = 0 & \text{in } \Omega_R, \\
\frac{\partial \phi_R}{\partial \nu} = -\frac{J_R(x)}{\sigma} & \text{on } \partial \Omega_R,
\end{cases}
\]
where the current density $J_R$ remains fixed except for the dilation $J_R(x) = J_r(x/R)$, in which $J_r(x)$ is the reference current density defined in (2.5).

Note that $\phi_R(x) = R \phi_n(x/R)$.

The transformed magnetic potential, which we denote by $A_R$ then satisfies
\[
(6.2) \quad \begin{cases}
-\text{curl}^2 A_R + \frac{1}{\epsilon} \nabla \phi_R = 0 & \text{in } \Omega_R, \\
\text{curl} A_R = B_R(x) & \text{on } \partial \Omega_R,
\end{cases}
\]
where $B_R(x) = R B(x/R)$.

It can be easily proved that $A_R(x) = R^2 A_n(x/R)$.

Let then
\[
(6.3) \quad L^R_h = -\nabla^2 h A_R + ih\phi_R.
\]

The form domain associated with $L^R_h$ is given by
\[
H^{1,\partial \Omega_R,e}_0 = \left\{ u \in H^1(\Omega_R, \mathbb{C}) \left| u|_{\partial \Omega_R,e} = 0 \right. \right\},
\]
and the domain of $L^R_h$ is
\[
D_R = \left\{ u \in H^2(\Omega_R, \mathbb{C}) \left| u|_{\partial \Omega_R,e} = 0, \right. \frac{\partial u}{\partial \nu}|_{\partial \Omega_R,i} = 0 \right\}.
\]

We attempt to estimate $\sup_{\gamma \in \mathbb{R}} \| (L^R_h - \mu - i\gamma)^{-1} \|$ as $R \rightarrow \infty$. Once the problem has been defined, we apply the inverse transformation of (6.1) to (6.3) to obtain that
\[
(6.4) \quad \sup_{\gamma \in \mathbb{R}} \| (L^R_h - \mu - i\gamma)^{-1} \| = R^2 \sup_{\gamma \in \mathbb{R}} \| (L_{R\Omega_R} - \mu R^2 - i\gamma)^{-1} \|.\]
We attempt to estimate the right-hand side in the sequel, as it is more in line with the standard practice in semi-classical analysis than the estimate of the left-hand-side. Furthermore, estimating the right-hand-side would be valuable also for the analysis of the penetration-scale problem presented in the previous section. In particular, (5.42), which is a sufficient condition for global stability of the normal state can be written in the form

\[(6.5) \quad \kappa^2 \sup_{\gamma \in \mathbb{R}} \| (\mathcal{L}_{\kappa^3 h} - i\gamma)^{-1} \| \leq 1 - \frac{C}{\kappa^{1-a}}. \]

Thus, an estimate of the right-hand-side is valuable for this problem as well as for verifying that (1.14) is satisfied. Additionally, we can use the estimate of (6.4) with \(\mu = 1\) to find the critical current where the normal state looses its stability, which amounts to determining the values of \(h\) for which (6.4) becomes infinite.

Let \(\gamma \in \mathbb{R}\), \(B_n = \text{curl} A_n\) and

\[(6.6) \quad F = \phi_n + icB_n. \]

From (2.10), which is nothing else as the Cauchy-Riemann equations for \(F\), we see that \(F\) is holomorphic in \(\Omega\) as a function of \(x_1 + ix_2\).

**Lemma 6.1.** Under assumptions (B), (J1)-(J3) and (R1)-(R2), for any \(\gamma\), \(F - \gamma/h\) has at most one simple zero in \(\Omega\).

**Proof.** Since \(\gamma\) is real, any zero of \(F - \gamma/h\) must lie in \(B_n^{-1}(0)\). It has been established in (2.25) that \(B_n^{-1}(0)\) is either empty or that it is a regular curve \(\Gamma\) joining the two components of \(\partial \Omega\), on which \(\nabla B_n \neq 0\). By (2.10), \(\nabla \phi_n \neq 0\) and is tangent to \(\Gamma\). Hence \(\phi_n\) is strictly monotone on \(\Gamma\), which completes the proof of the lemma. 

To obtain the supremum with respect to \(\gamma\) of the resolvent as is clear from (6.4), we allow for dependence of \(\gamma\) on \(R\).

**Suppose first** that a zero of \(F - \gamma(R)/h\) exists in \(\Omega\), and let \(z_0(R) = (x_0, y_0)\) denote it. We distinguish between two different cases:

\[(6.7a) \quad d(z_0, \partial \Omega) \geq 2R^\alpha^{-1}, \]
\[(6.7b) \quad d(z_0, \partial \Omega) < 2R^\alpha^{-1}, \]

where \(\alpha \in (0, 1)\) will be determined later.

In the case (6.7a) we approximate \(\| (\mathcal{L}_{R^3 h} - \lambda)^{-1} \|\) by the norm of a similar operator on \(L^2(\mathbb{R}^2)\), whereas in case (6.7b), we approximate it by the norm of an operator on \(L^2(\mathbb{R}^2_+)\).

**Suppose next** that \(F \neq \gamma(R)/h\) for all \(x \in \bar{\Omega}\). Let \(\Gamma\) denote the set in \(\Omega\) where \(B_n = 0\) (\(\Gamma = B_n^{-1}(0)\)). By Assumption (B) together with (R2), this set is a single curve joining the two components of \(\partial \Omega\), away from the corners. Denote the points of intersection of \(\Gamma\) with \(\partial \Omega\) by \(z_1\) and \(z_2\). Without loss of generality we assume that \(\phi_n(z_1) < \phi_n(z_2)\) and that \(\gamma/h < \phi_n(z_1)\) (the case \(\gamma/h > \phi_n(z_2)\) can be treated similarly). We distinguish then
between two cases

(6.7c) \[-\gamma/h + \phi_n(z_1) \geq C^* R^{\alpha-1},\]

(6.7d) \[-\gamma/h + \phi_n(z_1) < C^* R^{\alpha-1}\]

where \(C^*\) is determined below.

In the case (6.7d) we set

\[ z_0 = z_1 + \frac{\gamma h - F(z_1)}{F'(z_1)}, \]

which is clearly well defined, since \(|F'(z_j)| = |\nabla \phi_n(z_j)|\) which is strictly positive, as has already been stated in the proof of Lemma 6.1. Let

\[ d_0 = d(z_0, \Omega). \]

By Assumption (C) \(\nabla \phi_n(z_1)\) is perpendicular to \(\partial \Omega\) at \(z_1\), and so is, by the Cauchy-Riemann relations, \(F'(z_1)\). For sufficiently large \(R\), we thus have \(z_0 \notin \Omega\),

\[ d_0 = |z_0 - z_1|. \]

We thus choose

\[ C^* = 2|\nabla \phi_n(z_1)| \]

which guarantees that

\[ d_0 < 2R^{\alpha-1}, \]

for \(R\) large enough, when (6.7d) is met.

Hence in Cases (6.7a,b,d), we have constructed a point \(z_0(R)\) which for \(R\) large enough is in a fixed neighborhood \(\mathcal{V}(\Omega)\) of \(\Omega\). We shall consolidate in the sequel the treatment the cases (6.7d) and (6.7b). In the case (6.7c) we shall prove that \(\lambda_R\) is not in the spectrum and that \(\| (L_{R^3 h} - \lambda_R)^{-1} \| \rightarrow 0 \) as \(R \rightarrow \infty\), where

\[ \lambda_R = \nu(R)R^2 + i\gamma(R)R^3. \]

6.2. A resolvent estimate. We seek an estimate for \(\| (L_{R^3 h} - \lambda_R)^{-1} \| \) for the cases (6.7a,b,d). Let then \(\chi, \tilde{\chi} \in C^\infty(\mathbb{R}_+, [0, 1])\) form a partition of unity satisfying

\[ \chi(t) = \begin{cases} 1 & \text{for } t \leq 1, \\ 0 & \text{for } t \geq 2, \end{cases} \]

and

\[ \chi^2 + \tilde{\chi}^2 = 1 \text{ on } \mathbb{R}^+. \]

We then introduce

\[ \chi_1(x) = \chi(R^{1-\alpha}|x - z_0|) \text{ and } \chi_2(x) = \tilde{\chi}(R^{1-\alpha}|x - z_0|), \]

where

\[ 0 < \alpha < 1/3 \]

is kept fixed throughout the sequel.

We establish the following auxiliary result:
Lemma 6.2. Suppose that for some \( \ell \) and \( R_0 \), \( \nu(R) \leq \ell \) in (6.10) for all \( R \geq R_0 \) and let \( z_0(R) \) as defined in the previous subsection for the cases (6.7a,b,d)). Then, there exist \( R_1 \geq R_0 \) and \( C > 0 \), such that, if \( R \geq R_1 \) and \( \lambda_R \) belongs to the resolvent set \( \rho(\mathcal{L}_{R^3h}) \) of \( \mathcal{L}_{R^3h} \), then
\[
\| \chi_2(\mathcal{L}_{R^3h} - \lambda_R)^{-1} \| \leq C R^{-\alpha} (R^{-2} + \| (\mathcal{L}_{R^3h} - \lambda_R)^{-1} \|).
\]

Remark 6.3. In case (6.7c), we set \( \chi_2 = 1 \) on \( \Omega \) to obtain \( \lambda_R \) is not in the spectrum of \( \mathcal{L}_{R^3h} \) and that:
\[
\| (\mathcal{L}_{R^3h} - \lambda_R)^{-1} \| \leq C R^{-\alpha-2}.
\]

Proof.

It can be readily verified that there exists \( C_0 > 0 \) and \( R_1 \) such that for \( R \geq R_1 \) and \( x \in \Omega \setminus B(z_0, R^{\alpha-1}) \),
\[
|h\phi_n(x) - \gamma| + |B_n(x)| \geq C_0 R^{\alpha-1}.
\]

Note that in Case (6.7c), this inequality is satisfied for \( \forall x \in \Omega \) as \( B(z_0, R^{\alpha-1}) \) is not in \( \Omega \) in this case. We then define the following subdomains of \( \Omega \setminus B(z_0, R^{\alpha-1}) \):
\[
\mathcal{D}_1^+ = \left\{ x \in \Omega \setminus B(z_0, R^{\alpha-1}) \mid B_n \geq \frac{C_0}{2} R^{\alpha-1} \right\};
\]
\[
\mathcal{D}_1^- = \left\{ x \in \Omega \setminus B(z_0, R^{\alpha-1}) \mid B_n \leq \frac{C_0}{2} R^{\alpha-1} \right\};
\]
\[
\mathcal{D}_2^+ = \left\{ x \in \Omega \setminus B(z_0, R^{\alpha-1}) \mid h\phi_n - \gamma \geq \frac{C_0}{2} R^{\alpha-1} \right\};
\]
\[
\mathcal{D}_2^- = \left\{ x \in \Omega \setminus B(z_0, R^{\alpha-1}) \mid h\phi_n - \gamma \leq \frac{C_0}{2} R^{\alpha-1} \right\}.
\]

It readily follows from (6.15) that
\[
\Omega \setminus B(z_0, R^{\alpha-1}) \subseteq \mathcal{D}_1^+ \cup \mathcal{D}_1^- \cup \mathcal{D}_2^+ \cup \mathcal{D}_2^-.
\]

Denote then by \( \chi_3, \chi_4, \chi_5, \chi_6 \in C^\infty(\Omega, [0, 1]) \) cutoff \( R \)-dependent functions satisfying
\[
\chi_3(x) = 1 \text{ for } x \in \mathcal{D}_1^+, \text{ Supp } \chi_3 \subset \mathcal{D}_1^+, \text{ and } |\nabla \chi_3| \leq CR^{1-\alpha},
\]
\[
\chi_4(x) = 1 \text{ for } x \in \mathcal{D}_1^-, \text{ Supp } \chi_4 \subset \mathcal{D}_1^-, \text{ and } |\nabla \chi_4| \leq CR^{1-\alpha},
\]
\[
\chi_5(x) = 1 \text{ for } x \in \mathcal{D}_2^+, \text{ Supp } \chi_5 \subset \mathcal{D}_2^+, \text{ and } |\nabla \chi_5| \leq CR^{1-\alpha},
\]
and
\[
\chi_6(x) = 1 \text{ for } x \in \mathcal{D}_2^-, \text{ Supp } \chi_6 \subset \mathcal{D}_2^-, \text{ and } |\nabla \chi_6| \leq CR^{1-\alpha}.
\]
Having introduced these cut-off functions, we consider
\[ f \in L^2(\Omega) \text{ and } u = (\mathcal{L}_{R^3} - \lambda_R)^{-1} f. \]
In order to control \( \chi_2 u \) in \( L^2 \) we successively control \( \chi_j \chi_2 u \) for \( j = 3, \ldots, 6 \), observing that the support of these \( \chi_j \) (\( j = 3, \ldots, 6 \)) cover by (6.16) the support of \( \chi_2 \). Let \( \eta_1 = \chi_2 \chi_3 \).

An integration by parts readily yields the support of \( \chi_3 \) and Assumption (R1):
\[ \| \nabla R^{hA_n} (\eta_1 u) \|_2^2 \geq \frac{\Theta_2 C_0}{4} R^{2+\alpha} \| \eta_1 u \|_2^2, \]
where \( \Theta_2 \) is the lowest eigenvalue of the magnetic Laplacian with constant magnetic field equal to \( 1 \) in an infinite sector.

Hence, since by assumption \( \nu(R) \) is bounded, we obtain from (6.17) and (6.18)
\[ \| \chi_2 u \|_{L^2(D_1^+)}^2 \leq C R^{-2\alpha} (R^{-4} \| f \|_2^2 + R^{-\alpha} \| u \|_2^2). \]

We next introduce \( \eta_2 = \chi_2 \chi_3 \). An integration by parts yields again
\[ \Im \langle \eta_2^2 u, \mathcal{L}_{R^3} u \rangle = -2 \Im \langle \eta_2 u \nabla \eta_2, \nabla R^{hA_n} u \rangle + \langle R^3 (h\phi_n - \gamma) \eta_2 u, \eta_2 u \rangle = \Im \langle \eta_2 u, \eta_2 f \rangle . \]

Since
\[ \frac{C_0}{4} R^{2+\alpha} \| \eta_2 u \|_2^2 \leq \| \eta_2 u \|_2 \| f \|_2 + CR^{1-\alpha} \| \eta_2 u \| \| \nabla h^{R^3A_n} u \|_2, \]
we obtain that
\[ \frac{C_0}{8} R^{2+\alpha} \| \eta_2 u \|_2^2 \leq \frac{16}{C_0} R^{-2-\alpha} \| f \|_2^2 + \hat{C} R^{-3\alpha} \| \nabla h^{R^3A_n} u \|_2^2. \]

Observing that
\[ \| \nabla h^{R^3A_n} u \|_2^2 \leq \nu R^2 \| u \|_2^2 + \| u \|_2 \| f \|_2, \]
we finally get
\[ \| \chi_2 u \|_{L^2(D_2^+)}^2 \leq C R^{-2\alpha} (R^{-4} \| f \|_2^2 + \| u \|_2^2). \]

In a similar manner, we obtain that
\[ \| \chi_2 u \|_{L^2(D_1^+)}^2 + \| \chi_2 u \|_{L^2(D_2^+)}^2 \leq C R^{-2\alpha} (R^{-4} \| f \|_2^2 + \| u \|_2^2), \]
which, together with (6.19) and (6.20), proves
\[ \| \chi_2 u \|_2^2 \leq C R^{-2\alpha} (R^{-4} \| f \|_2^2 + \| u \|_2^2), \]
and the claim of the lemma.
6.3. **The entire plane limit case.** Consider first the case (6.7a). We choose a frame of reference whose origin is located at $z_0$, and with the $x$ and $y$ axes respectively directed tangentially at $z_0$ to the level curves of $\phi_n$ and $B_n$. We recall that $j$ was introduced in (1.20). It follows then from (6.2) that

$$j = h|\nabla \phi_n(z_0)|/c \neq 0.$$

We next define a potential function $V : \mathbb{R}^2 \to \mathbb{R}$ via

$$\nabla V(x) = \frac{1}{2}x \cdot D^2A_n(z_0)x + x \cdot \nabla A_n(z_0) + A_n(z_0) - \frac{1}{2}jx^2i_y \quad \text{and} \quad V(0) = 0,$$

where $x = (x, y)$. It can be readily verified that $V$ is properly defined, since the curl of the right-hand-side identically vanishes in $\mathbb{R}^2$ (recall that $B_n \sim jx$ near $z_0$). We can thus define

$$\hat{\mathcal{L}}_h = -\nabla_h^2 + i\hbar \phi_n.$$  

By the gauge transformation $u \mapsto e^{iV}u$, it readily follows that

$$\|(\mathcal{L}_{R\hbar} - \lambda_{R})^{-1}\| = \|(\mathcal{L}_{R\hbar} - \lambda_{R})^{-1}\|.$$ 

For convenience we drop the accent from $\mathcal{L}_{R\hbar}$ in the sequel and refer to $A_n - \nabla V$ as $A_n$. We have introduced $\mathcal{A}(j, c)$ and its domain in (1.16) and (1.17). Define the dilation operator $T_ju(x) = u(j^{1/3}x)$. It can be readily verified that

$$T_j^{-1}\mathcal{A}(j, c)T_j = j^{1/3}\mathcal{A}(1, c),$$

and hence $j^{1/3}\mathcal{A}(1, c)$ is unitarily equivalent to $\mathcal{A}(j, c)$. As in (1.21) that we set $\mathcal{A}(z_0) = \mathcal{A}(j(z_0), c)$ or simply $\mathcal{A}$. Recall from [2], that $D(\mathcal{A})$ is the closure of $C_0^\infty(\mathbb{R}^2)$ under the graph norm $\|u\|_2 + \|Au\|_2$, that the spectrum of $\mathcal{A}$ is empty, and

$$\|(\mathcal{A} - \lambda)^{-1}\| = \|(\mathcal{A} - \text{Re} \lambda)^{-1}\|$$

for all $\lambda \in \mathbb{C}$. Moreover (see Lemma 4.4 in [2]), for any $\lambda_0 \in \mathbb{R}$, there exists $C(\lambda_0)$ such that for all $\lambda \in \mathbb{C}$ such that $\text{Re} \lambda \leq \lambda_0$, we have:

$$\|(\mathcal{A} - \lambda)^{-1}\| \leq C(\lambda_0).$$

We can now state the following:

**Lemma 6.4.** Let $\lambda_R = R^2\nu(R) + iR^3\gamma(R)$. Suppose that there exist positive $R_0$ and $\ell$ such that for all $R \geq R_0$ we have:

1. $\nu(R) \leq \ell$;
2. (6.7a) holds true;
3. $\lambda_R \in \rho(\mathcal{L}_{R\hbar})$.

Let

$$M(R) = R^2\|(\mathcal{L}_{R\hbar} - \lambda_{R})^{-1}\|.$$ 

Then, there exist $C > 0$ and $R_1 \geq R_0$ depending only on $\ell$, $\Omega$, $j$, and $\alpha$, such that, for all $R \geq R_1$ we have

$$M(R) \leq \|(\mathcal{A} - \nu(R))^{-1}\| + C(R^{-\alpha} + R^{-(1-3\alpha)})\|(\mathcal{A} - \nu(R))^{-1}\| + 1 + M(R)^2.$$.  


We can, thus, readily conclude and that
\[ R = R^{-2}U^{-1}_R(A - \nu(R))^{-1}U_R. \]
Note that
\[ R \in (A_R - \nu(R)R^2)^{-1}. \]
where \( A_R(z_0) : D(A_R) \rightarrow L^2(\mathbb{R}^2) \) is given by
\[ A_R = \left( \nabla - iR^3 \frac{x^2}{2} \right)^2 + icR^3y. \]

Proof.
Let \( U_R : L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2) \) denote the unitary dilation operator
\[ u \mapsto U_R u(x) = R^{-1}u(x/R), \]
and then set
\[ R = R^{-2}U^{-1}_R(A - \nu(R))^{-1}U_R. \]

We attempt to approximate \( (L_{R^\gamma} - \lambda_R)^{-1} \) by the following operator
\[ (6.28) \quad R = \chi_1 R \infty_1 + \chi_2 (L_{R^\gamma} - \lambda_R)^{-1} \chi_2. \]
Clearly,
\[ (6.29) \quad (L_{R^\gamma} - \lambda_R)R = I + [L_{R^\gamma}, \chi_1]R \infty_1 + \chi_1 (L_{R^\gamma} - iR^3\gamma - A_R)R \infty_1 \]
\[ + [L_{R^\gamma}, \chi_2] (L_{R^\gamma} - \lambda_R)^{-1} \chi_2, \]
We now estimate the norms of the three operators appearing on the right-hand-side of
(6.29) after the identity.

For the first operator, we evaluate the commutator as follows:
\[ (6.30) \quad [L_{R^\gamma}, \chi_1] = -\Delta \chi_1 + 2\nabla \chi_1 \cdot \nabla R^\gamma A_n = \]
\[ \Delta \chi_1 + 2\nabla \chi_1 \cdot \left( \nabla - iR^3 \frac{x^2}{2} \right)^2 \] \[ + 2i\nabla \chi_1 \cdot R^3 \left( hA_n - j \frac{x^2}{2} \right), \]
and then successively estimate in \( L(L^2) \) the three resulting terms on the right-hand-side, i.e., \( -\Delta \chi_1 \infty_1, 2\nabla \chi_1 \cdot \left( \nabla - iR^3 \frac{x^2}{2} \right) \infty_1, \) and \( 2i\nabla \chi_1 \cdot R^3 \left( hA_n - j \frac{x^2}{2} \right) \infty_1. \)

For the first term, since by assumption \( \nu(R) \leq \ell \), we observe, by using (6.25), that
\[ (6.31) \quad \| \infty_1 \| = \frac{\| (A - \nu(R))^{-1} \|}{R^2} \leq C(\ell) \frac{1}{R^2}, \]
and that
\[ \| - \Delta \chi_1 \| \leq CR^{-2a}. \]

We can, thus, readily conclude
\[ (6.32) \quad \| - \Delta \chi_1 \infty_1 \| \leq CR^{-2a}. \]

To estimate the second term, let \( u = \infty_1 f \) for some \( f \in L^2(\Omega) \). It follows that
\[ \left\| \left( \nabla - iR^3 \frac{x^2}{2} \right) u \right\| L^2_2 = \nu(R)R^2 \| u \| L^2_2 + \text{Re} \langle u, f \rangle \leq \nu(R)R^2 \| u \| L^2_2 + \| u \| L^2_2 \| f \| L^2_2. \]
Thus, by (6.31),
\[ (6.33) \quad \left\| \left( \nabla - iR^3 \frac{x^2}{2} \right) \infty_1 f \right\| L^2_2 \leq \frac{C}{R} \| f \| L^2_2, \]
and hence,
\[(6.34) \quad \| \nabla \chi_1 \cdot \left( \nabla - i R^3 \frac{x^2}{2} \hat{y} \right) \mathcal{R}_\infty \chi_1 \| \leq C R^{-\alpha} .
\]

For the third sub-term, as in view of (6.23),
\[(6.35) \quad \| h A_n - \frac{j x^2}{2} \hat{y} \|_{L^\infty(B(z_0, 2 R^{\alpha-1}))} \leq C R^{-3(1-\alpha)} ,
\]

we obtain
\[(6.36) \quad \| 2 i R^3 \nabla \chi_1 \cdot \left( h A_n - \frac{j x^2}{2} \hat{y} \right) \|_\infty \leq C R^{1+2\alpha} ,
\]

and then, using (6.31) yields
\[(6.37) \quad \| \mathcal{L}_{R^3 h} \chi_1 \mathcal{R}_\infty \chi_1 \| \leq C(R^{-\alpha} + R^{2\alpha-1}) \leq CR^{-\alpha} ,
\]

for any choice of \( \alpha \in (0, \frac{1}{3}) \).

Consider next the second operator after the identity on the right hand side of (6.29). We use the decomposition:
\[(6.38) \quad (\mathcal{L}_{R^3 h} - i R^3 \gamma(R) - \mathcal{A}_R) = 2 i R^3 \left( h A_n - \frac{j x^2}{2} \hat{y} \right) \cdot \left( \nabla - i R^3 \frac{x^2}{2} \hat{y} \right) + R^2 \| h A_n - \frac{j x^2}{2} \hat{y} \|^2 + i R^3 (\phi_n - \gamma(R) - c \hat{y}) .
\]

By (6.36) we have that
\[(6.39) \quad \| \chi_1 R^2 h A_n - \frac{j x^2}{2} \hat{y} \|_{\mathcal{R}_\infty \chi_1} \leq C R^{-2(1-3\alpha)} .
\]

Similarly, by (6.35) and (6.33) we obtain that
\[(6.40) \quad \| \chi_1 R^2 \left( h A_n - \frac{j x^2}{2} \hat{y} \right) \cdot \left( \nabla - i R^3 \frac{x^2}{2} \hat{y} \right) \mathcal{R}_\infty \chi_1 \| \leq C R^{-(1-3\alpha)} .
\]

Finally, as
\[(6.41) \quad \| \phi_n - \gamma(R) - c \hat{y} \|_{L^\infty(B(0, 2 R^{\alpha-1}))} \leq C R^{-2(1-\alpha)} ,
\]

we obtain,
\[(6.42) \quad \| \chi_1 (\mathcal{L}_{R^3 h} - i R^3 \gamma(R) - \mathcal{A}_R) \mathcal{R}_\infty \chi_1 \| \leq C R^{-(1-3\alpha)} .
\]

For the last operator on the right hand side of (6.29) we use the decomposition:
\[|\mathcal{L}_{R^3 h} \chi_2 (\mathcal{L}_{R^3 h} - \lambda R)^{-1} \chi_2 = (-\Delta \chi_2 + 2 \nabla \chi_2 \cdot \nabla R^3 h A_n) (\mathcal{L}_{R^3 h} - \lambda R)^{-1} \chi_2 .
\]

It can be easily verified, as in the derivation of (6.34), that
\[\| \nabla R^3 h A_n (\mathcal{L}_{R^3 h} - \lambda R)^{-1} \| \leq C \frac{R}{R} (1 + M(R)) .
\]
Hence,

\[
\|(L^{R^3}_h - \lambda)^{-1}_2\| \leq C\frac{M(R) + 1}{R^a}.
\]

We next rewrite (6.29) in the following form

\[
(L^{R^3}_h - \lambda)\left(\mathcal{R} - (L^{R^3}_h - \lambda)^{-1}_2(L^{R^3}_h - \lambda)^{-1}_2\right)
= I + [L^{R^3}_h, \chi_1](A_R - \nu(R)^{R^2})^{-1}_1 + \chi_1(L^{R^3}_h - iR^3\gamma - A_R)(A_R - \nu(R)^{R^2})^{-1}_1.
\]

In view of (6.37) and (6.42) we obtain that for sufficiently large \( R \) the right-hand-side of the above identity becomes invertible. Hence,

\[
(L^{R^3}_h - \lambda)^{-1} = \left(\mathcal{R} - (L^{R^3}_h - \lambda)^{-1}_2(L^{R^3}_h - \lambda)^{-1}_2\right) \times
\]

\[
\times \left( I + [L^{R^3}_h, \chi_1]\mathcal{R}_\infty \chi_1 + \chi_1(L^{R^3}_h - iR^3\gamma(R) - A_R)\mathcal{R}_\infty \chi_1 \right)^{-1}_1.
\]

By (6.13) there exists \( C > 0 \) such that

\[
\|(L^{R^3}_h - \lambda)^{-1}_2\| \leq \frac{\|(A - \nu(R)^{R^2})^{-1}_1\|}{R^2} + CR^{-(2+\alpha)}(1 + M(R)).
\]

By (6.37), (6.42), (6.43), (6.44), and (6.45), we easily obtain (6.27). □

6.4. The half-plane limit case. We next consider together the cases (6.7b) and (6.7d). Let \( \hat{z}_0(R) \) denote the projection of \( z_0(R) \) on \( \partial\Omega \), \( (d(z_0, \partial\Omega) = d(z_0(R), \hat{z}_0)) \). Note that by Assumption (B) (see (2.24)) and (R2), since the curve \( \Gamma = \{B^{-1}_r(0)\} \) intersects \( \partial\Omega \) on the interior of \( \partial\Omega \), it follows that \( \partial\Omega \) is smooth near \( \hat{z}_0(R) \). Hence, since \( d(z_0, \partial\Omega) \leq 2R^{a-1} \), \( \hat{z}_0(R) \) must exist. In case (6.7d) where \( z_0 \) lies outside \( \Omega \) we have \( \hat{z}_0(R) = z_1 \). We define a curvilinear coordinate system \( (s, t) \) such that \( t = d(x, \partial\Omega) \), and \( s \) denotes the arc length of \( \partial\Omega \) from \( \hat{z}_0(R) \) in the positive trigonometric direction to \( z(s) \), which denotes the projection of \( x(s, t) \) on \( \partial\Omega \). Thus (cf. [28]),

\[
x = \mathcal{F}(s, t) = z(s) - t\nu(s),
\]

where \( \nu(s) \) denotes the outward normal on \( \partial\Omega \) at \( z(s) \). We further set

\[
g = |\det \mathcal{D}\mathcal{F}| = 1 - t\kappa_r(s),
\]

where \( \kappa_r(s) \) denotes the relative curvature of \( \partial\Omega \) at \( z(s) \). Note that, outside a fixed neighborhood of the corners, there exists \( C > 0 \) depending only on \( \Omega \) such that

\[
|\kappa_r| + |\kappa'_r| \leq C.
\]

In case (6.7b) we then set

\[
\nabla V_+(x) = \frac{1}{2}x \cdot D^2A_n(z_0)x + x \cdot \nabla A_n(z_0) + A_n(z_0) - \frac{1}{2}j(x \cdot \hat{n}_r(z_0))^2\hat{n}_r(z_0),
\]

where \( V_+(0) = 0 \). In the case (6.7d) we set

\[
\hat{A}_n(x) = \frac{1}{2}x \cdot D^2A_n(z_1)x + x \cdot \nabla A_n(z_1) + A_n(z_1),
\]
and then define $V_+$ in the following manner
\[
\nabla V_+(\mathbf{x}) = \frac{1}{2} \mathbf{x} \cdot D^2 \tilde{A}_n(z_0) \mathbf{x} + \mathbf{x} \cdot \nabla \tilde{A}_n(z_0) + \tilde{A}_n(z_0) + \frac{1}{2} j(\mathbf{x} \cdot \mathbf{i}_s(z_0))^2 \mathbf{i}_t(z_0).
\]

Then we let
\[
\hat{L}_h = -\nabla^2_{h[A_n - \nabla V_+]} + ih\phi_n,
\]
and attempt to estimate $\|(\hat{L}_{R^2} - \lambda_R)^{-1}\|$. As before, we drop the accent from $L_{R^2}$ in the sequel and refer to $A_n - \nabla V_+$ as $A_n$.

We can now write $L_{R^2}$ in terms of the curvilinear coordinates (cf. [28])
\[
L_{R^2} = -\left(\frac{1}{g} \left[ \frac{\partial}{\partial s} - ihR^3 a_s \right] \right)^2 - \frac{1}{g} \left( \frac{\partial}{\partial t} - ihR^3 a_t \right) \left( \frac{\partial}{\partial t} - ihR^3 a_t \right) + ihR^3 \phi_n(s, t),
\]
where
\[
a_s = g A \cdot \mathbf{i}_s \quad ; \quad a_t = A \cdot \mathbf{i}_t.
\]

We have defined the operator $\mathcal{A}_+(j, c)$ in (1.18). Since (6.24) is valid for $\mathcal{A}_+$ as well, it readily follows that $\mathcal{A}_+(j, c)$ is unitarily equivalent to $j^2 \mathcal{A}_+(1, c)$. As before (see (1.21)), we may set $\mathcal{A}_+(z_0) = \mathcal{A}_+(j(z_0), c)$ or more simply $\mathcal{A}_+$. The domain of $\mathcal{A}_+$ is given by (1.17) (cf. [4]).

Note that, by Assumption (C), $\nabla \phi_n$ (which is always orthogonal to $\nabla B_n$ by (2.10)) is perpendicular to $\partial \Omega$ at the intersection with the curve $\Gamma$, which explains why $\nabla \phi_n$ is parallel to $\mathbf{i}_t$ in (1.18).

We begin by stating a rather standard estimate.

**Lemma 6.5.** Let $\nu \in \mathbb{R}$, $j \neq 0$ and $c \neq 0$. Then, there exists $C > 0$, such that, for any $f \in L^2(\mathbb{R}^2_+)$, with compact support in $\mathbb{R}^2_+$, and for all $\gamma \in \mathbb{R}$ such that $\lambda = \nu + i\gamma \in \rho(\mathcal{A}_+(j, c))$, we have:

\[
\|u_{ss}\|_2 \leq C(\|u\|_2 + \|(1 + |s|)f\|_2),
\]

where $u = (\mathcal{A}_+(j, c) - \lambda)^{-1} f$.

*Proof.*

We first establish local estimates before assembling them together by a covering argument. Let $s_0 \in \mathbb{R}$. We set $v = e^{i\nu t/2} u$. We then have

\[
\begin{align*}
-\Delta v = -ij(s^2 - s_0^2) \frac{\partial v}{\partial t} - \left[ j^2 \frac{(s^2 - s_0^2)^2}{4} + icjt - \lambda \right] v + f & \quad \text{in } \mathbb{R}^2_+ \\
v = 0 & \quad \text{on } \partial \mathbb{R}^2_+.
\end{align*}
\]

Let $\tau_0 \geq 1$, and let $x_0 = (s_0, \tau_0)$. It is easy to show that

\[
-\Delta v = -ij(s^2 - s_0^2) \left( \frac{\partial v}{\partial t} - ij \frac{s^2 - s_0^2}{2} \right) + \left[ j^2 \frac{(s^2 - s_0^2)^2}{4} - icjt + \lambda \right] v + f.
\]
As \(|s^2 - s_0^2| \leq 1 + 2|s_0|\) in \(B(x_0, 1)\), standard elliptic estimates show that
\begin{equation}
(6.50)
\|u_{ss}\|_{L^2(B(x_0,1/2))} = \|u_{ss}\|_{L^2(B(x_0,1/2))} \leq C(\nu, j) \left[ (1 + |s_0|) \left\| \left( \frac{\partial}{\partial t} - i\frac{s^2 - s_0^2}{2} \right) v \right\|_{L^2(B(x_0,1))} + (1 + s_0^2) \| v \|_{L^2(B(x_0,1))} + \|(t - \gamma)v\|_{L^2(B(x_0,1))} + \| f \|_{L^2(B(x_0,1))} \right].
\end{equation}

Recall the cutoff function \(\chi\) defined by (6.11). Multiplying (6.49) by \(\chi^2 \bar{v}\) and integrating yield for the real part
\[
\left\| \left( \nabla - i\frac{s^2 - s_0^2}{2} \right) \left( \chi v \right) \right\|_2^2 = \| v \nabla \chi \|_2^2 + \nu \| \chi v \|_2^2 + \text{Re} \langle \chi f, \chi v \rangle,
\]
which holds also for \(\tau_0 > 0\). Consequently,
\[
\left\| \left( \nabla - i\frac{s^2 - s_0^2}{2} \right) v \right\|_{L^2(B(x_0,1))} \leq C(\nu, j) \left[ \| (1 + |s|^2) v \|_{L^2(B(x_0,2) \cap \mathbb{R}_+^2)} + \| f \|_{L^2(B(x_0,2) \cap \mathbb{R}_+^2)} \right],
\]
where \(B_+(x_0, r) = B(x_0, r) \cap \mathbb{R}_+^2\). Substituting the above into (6.50) yields, using the fact that \(|s_0| \leq |s| + 1\) in \(B(x_0, 1)\),
\begin{equation}
(6.51)
\|u_{ss}\|_{L^2(B(x_0,1/2))} \leq C(\nu, j) \left[ \| (1 + |s|^2) v \|_{L^2(B(x_0,2) \cap \mathbb{R}_+^2)} + \| f \|_{L^2(B(x_0,2) \cap \mathbb{R}_+^2)} \right].
\end{equation}

In a similar manner we obtain that, when \(\tau_0 = 0\), we have
\begin{equation}
(6.52)
\|u_{ss}\|_{L^2(B_+(x_0,1))} \leq C(\nu, j) \left[ \| (1 + |s|^2) v \|_{L^2(B_+(x_0,2))} + \| f \|_{L^2(B_+(x_0,2))} \right].
\end{equation}

We can now define a covering of \(\mathbb{R}_+^2\) by discs of radius 1/2 and semi-discs of radius 1 centered on the boundary, such that each point in \(\mathbb{R}_+^2\) is covered a finite number of times. Summing up (6.51) and (6.52) over these discs and semi-discs yields
\begin{equation}
(6.53)
\|u_{ss}\|_{2} \leq C(\nu, j) \left[ \| (1 + |s|^2) u \|_{2} + \| (t - \gamma) u \|_{2} + \| (1 + |s|) f \|_{2} \right].
\end{equation}

Note that in [4] it has been established that \(u\) has finite moments of any order, when \(f\) is compactly supported in \(\mathbb{R}_+^2\).

We next estimate \(\|s^2 u\|_{2}\). It has been proved in [4] (cf. (5.20) therein) that for \(k \geq 0\) we have
\[\|s^{(k+1)/2} u\|_{2} \leq C(\nu, j) \left[ \| (1 + |s|^{k/2}) u \|_{2} + \| (s^{(k+1)/2} u | s^{(k-1)/2} f) \right].\]
From this it readily follows that for \(k \geq 1\)
\begin{equation}
(6.54)
\|s^{(k+1)/2} u\|_{2} \leq C \left[ \| (1 + |s|^{k/2}) u \|_{2} + \| s^{(k-1)/2} f \|_{2} \right].
\end{equation}
For \(k = 0\), we have
\[\|s^{1/2} u\|_{2} \leq C \left[ \| u \|_{2} + \| f \|_{2} \right].\]
Applying the above together with (6.54) recursively for \(k = 1, 2, 3\) yields
\begin{equation}
(6.55)
\|s^2 u\|_{2} \leq C(\nu, j) \left[ \| u \|_{2} + \| (1 + |s|) f \|_{2} \right].
\end{equation}
Finally, we obtain an estimate for \( \| (t - \gamma) u \|_2 \). An integration by parts yields

\[
\text{Im} \langle (t - \gamma) u, (A_+ (j, c) - \lambda) u \rangle = \text{Im} \left( \langle u, \left( \frac{\partial}{\partial t} - i \frac{s^2}{2} \right) u \rangle \right) + \text{je}\|t - \gamma| u\|_2^2 = \text{Im} \langle (t - \gamma) u, f \rangle,
\]

from which it readily follows that

\[
\| t - \gamma \| u \|_2^2 \leq C \left[ \| t - \gamma \| u \|_2 f \|_2 + \| u \|_2 \left\| \frac{\partial}{\partial t} - i \frac{s^2}{2} \right\| u \|_2 \right].
\]

As

\[
\text{Re} \langle u, (A_+ (j, c) - \lambda) u \rangle = \left\| \left( \nabla - i \frac{s^2}{2} \right) u \right\|_2^2 - \nu \| u \|_2^2 = \text{Re} \langle u, f \rangle,
\]

we immediately conclude that

\[
\left\| \frac{\partial}{\partial t} - i \frac{s^2}{2} \right\| u \|_2 \leq C \left[ \| u \|_2 + \| f \|_2 \right].
\]

Substituting into (6.56) then yields

\[
\| t - \gamma \| u \|_2 \leq C \left[ \| u \|_2 + \| f \|_2 \right],
\]

which, in conjunction with (6.55) and (6.53) yields (6.48). \( \blacksquare \)

We can now provide another estimate for \( \| (\mathcal{L}_{R^3 h} - \lambda_R)^{-1} \| \).

**Lemma 6.6.** Let \( \lambda_R = \nu(R) R^2 + i R^3 \gamma(R) \). Suppose that there exist positive \( R_0 \) and \( \ell \) such that for all \( R \geq R_0 \) we have

1. \( \nu(R) \leq \ell \);
2. (6.7b) or (6.7d) hold true;
3. \( \lambda_R \in \rho(\mathcal{L}_{R^3 h}) \).

Let \( M(R) \) be defined by (6.26). Then, there exist \( C > 0 \) and \( R_1 \geq R_0 \) depending only on \( \ell, \Omega, j, \) and \( \alpha \), such that for all \( R \geq R_1 \) we have

\[
M(R) \leq \| (A_+ - \nu - icj R_0)^{-1} \| + C (R^{-\alpha} + R^{-(1-3\alpha)}) \left[ \| (A_+ - \nu - icj R_0)^{-1} \| + 1 + M(R)^2 \right],
\]

where \( t_0 = t_0 \) in case (6.7b) and \( t_0 = -t_0 \) in case (6.7d).

**Proof.**

We prove (6.57) in a similar manner to (6.27). Recall the definition of \( \chi \) from (6.11). We let \( \eta_1 = \chi(4R^{a_1}|x - z_0|) 1_{\Omega} \) and \( \eta_2 = \tilde{\chi}(4R^{b_1}|x - z_0|) 1_{\Omega} \). Let further

\[
\mathcal{R}^\infty_+ = U_R^{-1} (A_+ - \nu - icj R_0)^{-1} U_R = (A^R_+ - R^2 [\nu - icj R_0])^{-1},
\]

where

\[
A^R_+ = \left( \nabla - i R^3 \frac{s^2}{2} \right)^2 + ic R^3 j t.
\]

Then we set

\[
\mathcal{R}_+ = \eta_1 \mathcal{R}^\infty_+ \eta_1 + \eta_2 (\mathcal{L}_{R^3 h} - \lambda_R)^{-1} \eta_2.
\]

Clearly,

\[
(\mathcal{L}_{R^3 h} - \lambda_R) \mathcal{R}_+ = I + [\mathcal{L}_{R^3 h}, \eta_1] \mathcal{R}^\infty_+ \eta_1 + \eta_1 (\mathcal{L}_{R^3 h} - i R^2 (\gamma - icj R_0) - A^R_+ \mathcal{R}^\infty_+ \eta_1 + [\mathcal{L}_{R^3 h}, \eta_2] (\mathcal{L}_{R^3 h} - \lambda_R)^{-1} \eta_2.
\]
The estimate for the various terms on the right-hand-side of (6.58) proceeds in exactly the same manner as in the proof of (6.27). We note that as in the case (6.27) we have

\begin{equation}
\|hA_n - j\frac{s^2}{2}i\|_{L^\infty(B(z_0, 8R^{\alpha-1}))} \leq CR^{-3(1-\alpha)},
\end{equation}

from which we obtain, as in the proof of (6.37) that

\begin{equation}
\|\|LR^h, \eta\|R^\infty_+\| \leq C (R^{-\alpha} + R^{\alpha-1}) (1 + \| (A_+ - \nu - i \eta R t_0)^{-1} \|).
\end{equation}

Consider next the third term on the right hand side of (6.58). We first note that for every $x \in \Omega \cap B(z_0, 8R^{\alpha-1})$

\begin{equation*}
(L_{Rh} - i R^3(\gamma - cjt_0) - A_+) = -(g^{-2} - 1) \frac{\partial^2}{\partial s^2} + g^{-3}(g_s + 2i h R^3 g a_s) \frac{\partial}{\partial s} + g^{-2} h^2 R^6|a_s|^2 + g^{-1}[g_s + 2i g R^3 \left(h a_t - j\frac{s^2}{2}\right)] \left(\frac{\partial}{\partial t} - i j R^3 \frac{s^2}{2}\right) + R^6 \left(h a_t - j\frac{s^2}{2}\right)^2 + i R^3[h\phi_n - \gamma - cjt(t - t_0)].
\end{equation*}

By (6.66) we have

\begin{equation*}
\|\eta_1(g^{-2} - 1)\| \leq \frac{C}{R^{1-\alpha}}.
\end{equation*}

In view of (6.48) we thus obtain

\begin{equation}
\|\eta_1(g^{-2} - 1)\| \frac{\partial^2}{\partial s^2} R^\infty_+ \| \leq \frac{C}{R^{1-2\alpha}} (1 + \| (A_+ - \nu - i \eta R t_0)^{-1} \|).
\end{equation}

As

\begin{equation*}
\|g_s\| \leq \frac{C}{R^{1-\alpha}},
\end{equation*}

and since

\begin{equation*}
\left\| \frac{\partial}{\partial s} R^\infty_+ \right\| \leq \left\| \left(\nabla - i R^3 \frac{s^2}{2}i\right) (A_+^R - R^2 \nu - i \eta R^3 t_0)^{-1} \right\| \leq \frac{C}{R} (1 + \| (A_+^R - R^2 \nu - i \eta R^3 t_0)^{-1} \|),
\end{equation*}

we readily obtain, in view of (6.59), that

\begin{equation}
\|\eta_1 \left[g^{-3}(g_s + 2i h R^3 g a_s) \frac{\partial}{\partial s} + g^{-2} h^2 R^6|a_s|^2 \right] (A_+ - \nu - i \eta t_0)^{-1} \| \leq \frac{C}{R^{1-3\alpha}} (1 + \| (A_+ - \nu - i \eta t_0)^{-1} \|).
\end{equation}

In a similar manner we show that

\begin{equation}
\|\eta_1 g^{-1}\left\{[g_s + 2i g R^3 \left(h a_t - j\frac{s^2}{2}\right)] \left(\frac{\partial}{\partial t} - i R^3 \frac{s^2}{2}\right) + R^6 \left(h a_t - j\frac{s^2}{2}\right)^2 \right\} (A_+^R - R^2 \nu - i \eta R^3 t_0)^{-1} \| \leq \frac{C}{R^{1-3\alpha}} (1 + \| (A_+ - \nu - i \eta R t_0)^{-1} \|).
\end{equation}

Finally, as

\begin{equation*}
\|\eta_1[h\phi_n - \gamma - cjt(t - t_0)]\| \leq CR^{-2(1-\alpha)},
\end{equation*}
we obtain, by combining the above together with (6.61), (6.62), and (6.63) that

\[(6.64) \| \eta_1 (L_{R^3} - iR^3(\gamma - i\eta t_0) - A_+)(A_+^R - R^2\nu - i\gamma R t_0)^{-1}\eta_1 \| \leq C(R^{-\alpha} + R^{-(1-3\alpha)}) \left( 1 + \| (A_+ - \nu - i\gamma R t_0)^{-1} \| \right). \]

It is easy to show that (6.43) holds true when \( \chi_2 \) is replace by \( \eta_2 \), and hence in view of (6.60) and (6.64) we obtain we can complete the proof of (6.57) in the same manner of (6.27).

**Remark 6.7.** In the previous proofs, we have substantially relied on Assumption (C) (formulated in (1.11)). If we do not make this assumption, we could still attempt to approximate \( L_{h,R} - \lambda_R \) by the Dirichlet realization in \( \mathbb{R}^2_+ \) of the following operator

\[ A_+^0(z_0) = -\left( \nabla - i \left[ j \cos \theta \frac{s^2}{2} i_z + j \sin \theta \frac{t^2}{2} i_z \right] \right)^2 + ic[j(t - t_0) \cos \theta + (s - s_0) \sin \theta]. \]

In the above \( \theta \) represents the angle between the curve \( \Gamma \) and the boundary at the point of intersection. However, since we are unaware of any study addressing the above operator, and even the definition of this operator in the half-plane appears to be non-trivial whenever \( \theta \neq \frac{\pi}{2} \), we defer the discussion of the more general case to a later stage.

### 6.5. Proof of Theorem 1.3.

The theorem follows from Lemmas 6.2, 6.4, and 6.6. Let \( \gamma \in \mathbb{R} \) and \( \nu < \nu_m \) where \( \nu_m \) is given by (1.27). Then, for sufficiently large \( R \), we must be in one of the cases listed in (6.7). Therefore, it is easy to show that a path \( \gamma = \gamma(R) \), \( \nu = \nu_0 \), exists in \( [R_0, \infty) \) such that we remain in the same (6.7) case with \( \nu R^2 + i\gamma R^3 \in \rho(L_{R^3}) \) for all \( R \geq R_0 \). In case (6.7a) we can thus apply (6.27), if either (6.7b) or (6.7d) is satisfied, we can apply (6.57). Finally, in cases where (6.7c) is satisfied, we can employ (6.14). Let

\[ C_0(\nu) = \max \left( \sup_{z_0 \in \Gamma} \| (A(z_0) - \nu)^{-1} \|, \sup_{\gamma \in \mathbb{R}} \| (A_+(z_i) - \nu - i\gamma)^{-1} \| \right). \]

Choosing \( \alpha = 1/4 \) in all of the estimates of the two previous subsections yields that whenever \( \nu < \nu_m \) and \( \lambda_R \in \rho(L_{R^3}) \) we have

\[ M(R) \leq C_0 + \frac{C}{R^{1/4}} \left[ C_0 + 1 + M(R)^2 \right], \]

where \( M(R) \) is given by (6.26). It follows that there exist \( C_1 \) and \( C_2 \) such that either

\[(6.65) M(R) \leq C_0 \left[ 1 + \frac{C_1}{R^{1/4}} \right], \]

or

\[(6.66) M(R) \geq C_2 R^{1/4}, \]

for all sufficiently large \( R \). Note that \( [C_2 R^{1/4}, \infty) \) and \( (-\infty, C_0 \left[ 1 + \frac{C_1}{R^{1/4}} \right] \) are disjoint for sufficiently large \( R \) and \( \nu < \nu_m \).

Let \( \nu \leq -1 \). It is easy to show that \( M(R) \leq 1 \) independently of \( \gamma \) and \( R \) in that case. Consequently, for sufficiently large \( R \), (6.65) must be satisfied. We next increase \( \nu \) gradually, keeping \( \gamma \) and \( R \) fixed, thereby changing \( M \) continuously. It follows that
as long as \( \nu \leq \nu_m - \delta \) for any fixed positive \( \delta \), then for sufficiently large \( R \), \( M(R) \) must satisfy (6.65) since \( \nu \mapsto M(\nu, R) \) maps \((-1, \nu_m - \delta)\) onto an interval in \( \mathbb{R} \).

From the above discussion we can conclude that

\[
M(R, \nu, \gamma) \leq C_0(\nu) \left[ 1 + \frac{2C_1(\nu)}{R^{1/4}} \right],
\]

for all \( \nu < \nu_m \) and \( R > R_0 \). Let

\[
M_0(R, \nu) = \sup_{\gamma \in \mathbb{R}} M(R, \gamma, \nu).
\]

Since \( C_1 \) and \( C_0 \) are both independent of \( \gamma \), it follows that

\[
M_0(R) \leq C_0 \left[ 1 + \frac{2C_1}{R^{1/4}} \right],
\]

for all \( R > R_0 \). The above inequality readily provides us with both (1.28) and a lower bound for \( \mu_\infty \), i.e.,

\[
\mu_\infty \geq \nu_m.
\]

To complete the proof of the theorem we need to show that

\[
\lim_{R \to \infty} \sup \mu_R \leq \nu_m.
\]

If \( \nu_m \) is infinite, then (1.26) is readily proved. If \( \nu_m \) is finite, we now prove that for any sequence \( \{R_k\}_{k=1}^\infty \), such that \( R_k \uparrow \infty \), there exists a corresponding sequence of eigenvalues of \( \mathcal{L}_{R_k^3h} \), which we denote by \( \{\lambda_{R_k}\}_{k=1}^\infty \), satisfying

\[
R_k^2 \Re \lambda_{R_k} \to \nu_m.
\]

For convenience of notation we denote \( \mathcal{A}_+(z_i) \), at the point \( z_i \) where \( \inf_{\lambda \in \sigma(\mathcal{A}_+(z_i))} \Re \lambda = \nu_m \), by \( \mathcal{A}_+ \) in the sequel.

We first claim that there exists \( \lambda_{\min} \in \sigma(\mathcal{A}_+) \) which lies on the left margin of \( \sigma(\mathcal{A}_+) \), i.e., \( \Re \lambda_{\min} = \nu_m \). We prove this claim by using the same techniques as in the proof of Lemma 7.2 in [4], to show that there exists \( C > 0 \), such that \( \nu + i\gamma \in \rho(\mathcal{A}_+) \) for \( \nu \in [0, \nu_m + 1] \) and \( |\gamma| \geq C \). Hence the infimum defining \( \nu_m \) is attained by some \( \lambda_{\min} \) whose real part is \( \nu_m \).

Let then \( \lambda_{\min} \in \sigma(\mathcal{A}_+) \) satisfy \( \Re \lambda_{\min} = \nu_m \). We choose \( \gamma \) such that \( z_0 \in \partial \Omega \), i.e., so that \( F(z_1) - \gamma/h = 0 \), and hence \( d_0 = 0 \) in (6.9). Since (6.58) holds for any \( \lambda_R \in \rho(\mathcal{L}_{R^3h}) \) for which \( \nu < \nu_m \), we choose \( \nu = 0 \), and hence \( \lambda_R = i\gamma R^3 \).

Hence, we have

\[
(\mathcal{L}_{R^3h} - \lambda_R)\mathcal{R}_+ - \mathcal{I} = [\mathcal{L}_{R^3h}, \eta_1](A^R_+)^{-1} \eta_1 + [\mathcal{L}_{R^3h} - i\gamma R^3 - \lambda_R]^2 \eta_1 + [\mathcal{L}_{R^3h}, \eta_2](\mathcal{L}_{R^3h} - \lambda_R)^{-1} \eta_2.
\]

By (6.60) (recalling that \( \mathcal{R}_+^\infty = (A^R_+)^{-1} \) since \( t_0 = \nu = 0 \), (6.64), (6.13), and (1.28) we obtain that

\[
R^2 \|\mathcal{R}_+ - (\mathcal{L}_{R^3h} - \lambda_R)^{-1}\| \leq \frac{C}{R^{1/4}},
\]

and hence, using (6.13) once again yields

\[
R^2 \|\eta_1 (A^R_+)^{-1} \eta_1 - R^2 (\mathcal{L}_{R^3h} - \lambda_R)^{-1}\| \leq \frac{C}{R^{1/4}}.
\]
Equivalently we may state that
\[
\|\eta^R_1 A_+^{-1} \eta^R_1 - R^2 U(R(L R_h - \lambda R)^{-1} U_R^{-1})\| = \|\eta^R_1 A_+^{-1} \eta^R_1 - (L^R - R^{-2} \lambda R)^{-1}\| \leq \frac{C}{R^{1/4}},
\]
where \(\eta^R_1 = U R \eta_1\).

We can now define \(R_+^R : L^2(\mathbb{R}_+^2) \to H_{mag}(\mathbb{R}_+^2)\) by
\[
R_+^R = \eta^R_1 A_+^{-1} \eta^R_1.
\]
It can be easily verified that \(R_+^R \to A_+^{-1}\) in \(\mathcal{L}(L^2(\mathbb{R}_+^2))\). Here we use that the form domain of the operator \(A_+\) is contained in a suitable weighted space \(L^2(\mathbb{R}_+^2, \rho)\) with \(\rho \to +\infty\) as \(|x| \to \infty\) (see the proof of Proposition 2.4 in [2]). Hence, by Section IV, §3.5 in [26], it follows that, for any sequence \(R_k \uparrow \infty\), there exists \(\Lambda R_k \in \sigma((L^R R_h - R^{-2} \lambda R)^{-1})\) such that
\[
\hat{\lambda}_R = \frac{1}{R^2_k \Lambda R_k} + \frac{1}{R^2_k \lambda R_k}.
\]

**Remark 6.8.** By the proof of Lemma 7.2 in [4] it follows that
\[
(6.68) \quad \|(A(1, c) - \nu)^{-1}\| = \lim_{\gamma \to +\infty} \|(A_{+}(1, c) - \nu - i\gamma)^{-1}\|.
\]
Let \(j_m\) and \(j_+\) be given by (1.29). It follows from (6.68) that, if \(j_m = j_+\) we can drop \(\sup_{z_0 \in \Gamma} \|(A(z_0) - \nu)^{-1}\|\) from the right-hand-side of (1.28).
APPENDIX A. SCALAR REGULARITY IN NON-SMOOTH DOMAINS

In this section we provide some elliptic estimates, valid for domains with right-angled corners. For simplicity we assume property (R1) for the domain. We note that one can extend these estimates to domains composed of \( C^2 \) curves (or curved polygons). For the case of homogeneous boundary condition we prove, for the so-called variational (or weak) solution obtained by the Lax-Milgram Theorem, regularity in \( W^{2,p}(\Omega) \) for any \( p \geq 2 \). Inhomogeneous boundary conditions are converted into homogeneous ones by subtracting an appropriate function from the variational solution.

While most of the estimates we present here were first obtained in the pioneering works of Kondratiev [27] and Mazya-Plamenevskii [29], we prefer to refer the reader to [20], [21] or Chapter 1 in [19], where the results we use here can be obtained more easily. We finally note that while many of the results we cite are stated for polygons, they are equally valid for domains with property (R1). The regularity away from the corners follows from standard elliptic estimates, and the regularity near the corners can be obtained using the arguments applied in the case of polygons.

**Proposition A.1.**

Let \( \Omega \) satisfy property (R1) and \( p \geq 2 \), let \( f \in L^p(\Omega) \), \( g_i \in W^{2-\frac{1}{p},p}(\partial \Omega_i) \) \( g_c \in W^{2-\frac{1}{p},p}(\partial \Omega_c) \) such that

\[
\int_{\Omega} f(x) \, dx + \int_{\partial \Omega_c} g_c \, ds + \int_{\partial \Omega_i} g_i \, ds = 0.
\]

Then, \( u \in W^{2,p}(\Omega) \) and there exists \( C(p,\Omega) \) such that

\[
\|u\|_{2,p} \leq C(p,\Omega) \left( \|f\|_p + \|g_i\|_{W^{2-\frac{1}{p},p}(\partial \Omega_i)} + \|g_c\|_{W^{2-\frac{1}{p},p}(\partial \Omega_c)} \right).
\]

Furthermore, if \( f \in W^{1,q}(\Omega) \) for some \( 1 < q < 2 \) and \( g_i \equiv g_c \equiv 0 \), then \( u \in W^{3,q}(\Omega) \)

\[
\|u\|_{3,q} \leq C \|f\|_{1,q}.
\]

**Proof.**

The proof of (A.3) follows immediately from Corollary 4.4.3.8 [20]. The proof of (A.4) follows from Theorem 5.1.2.3 there. \( \blacksquare \)

**Proposition A.2.**

Let \( \Omega \) satisfy property (R1) and \( p \geq 2 \), let \( f \in L^p(\Omega) \), \( g_i \in W^{1-\frac{1}{p},p}(\partial \Omega_i) \) \( g_c \in W^{1-\frac{1}{p},p}(\partial \Omega_c) \) such that

\[
\int_{\Omega} f(x) \, dx + \int_{\partial \Omega_c} g_c \, ds + \int_{\partial \Omega_i} g_i \, ds = 0.
\]
Let $u \in H^1(\Omega)$ denote the weak solution for the following problem

(A.5) \[
\begin{cases}
-\Delta u = f & \text{in } \Omega \\
\int_{\Omega} u \, dx = 0 \\
\frac{\partial u}{\partial \nu} = g_i & \text{in } \partial \Omega_c \\
\frac{\partial u}{\partial \nu} = g_c & \text{in } \partial \Omega_i.
\end{cases}
\]

Then $u \in W^{2,p}(\Omega)$ and there exists $C(p, \Omega)$ such that

(A.6) \[\|u\|_{2,p} \leq C \left( \|f\|_p + \|g_i\|_{W^{1-\frac{1}{p}}(\partial \Omega_i)} + \|g_c\|_{W^{1-\frac{1}{p}}(\partial \Omega_c)} \right).\]

Furthermore, if $f \in W^{1,q}$ for some $1 < q < \infty$ and $g_i \equiv g_c \equiv 0$, then $u \in W^{3,q}(\Omega)$ and

(A.7) \[\|u\|_{3,q} \leq C \|f\|_{1,q}.
\]

Proof. The proof of (A.6) follows immediately from Corollary 4.4.3.8 in [20]. The proof of (A.7) follows from Theorem 5.1.2.3 there. \hfill \bl

Proposition A.3.
Let $\Omega$ satisfy property (R1) and $p \geq 2$. Let $f \in L^p(\Omega)$ and $u \in H^1(\Omega)$ denote the variational solution for the following problem

(A.8) \[
\begin{cases}
-\Delta u = f & \text{in } \Omega \\
u = 0 & \text{in } \partial \Omega_c \\
\frac{\partial u}{\partial \nu} = 0 & \text{in } \partial \Omega_i.
\end{cases}
\]

Then $u \in W^{2,p}(\Omega)$ and there exists $C(p, \Omega)$ such that, for any $f \in L^p(\Omega)$,

(A.9) \[\|u\|_{2,p} \leq C\|f\|_p.
\]

Proof. This is an immediate consequence of Theorem 2.4.3 in [21] (in the case $p = 2$) and Theorem 4.4.3.7 in [20] for $p > 2$. For $p = 2$, the reader is referred to p. 55 in [21] to verify that indeed mixed Dirichlet-Neumann conditions near a right-angled corners do not produce any singularities. \hfill \bl

Remark A.4.
It is necessary to address, in addition, (A.8) with non-homogeneous boundary conditions. In particular we consider:

\[
\frac{\partial u}{\partial \nu} = B \text{ on } \partial \Omega_i.
\]

Assuming that $B \in H^\frac{1}{2}(\partial \Omega)$, an $H^2(\Omega)$ regularity of $u$ does not follow as in the homogeneous case unless $B$ vanishes at the corners. The best regularity we can obtain in this case is:

\[u \in \cup_{q<2} W^{2,q}(\Omega).
\]

We refer the reader this to Corollary 4.4.3.8 in [20] for the proof of this result.
Appendix B. Vector Regularity in Non-Smooth Domains

B.1. Decomposition of vector fields. We begin by presenting a well-known decomposition of vector fields in $L^2(\Omega; \mathbb{R}^2)$ using their curl and their div.

Proposition B.1.
Any vector $U \in L^2(\Omega; \mathbb{R}^2)$ can be uniquely written as the sum

\[(B.1)\quad U = V + W,\]

where

\[V \in H^0_0(\text{curl}, \Omega) = \{\hat{V} \in L^2(\Omega, \mathbb{R}^2), \text{curl} \hat{V} = 0\}\]

and

\[W \in H^0_0(\text{div}) := \{\hat{W} \in L^2(\Omega, \mathbb{R}^2), \text{div} \hat{W} = 0 \text{ and } \hat{W} \cdot \nu = 0 \text{ on } \partial \Omega\}.\]

The maps $U \mapsto \pi_1 U = V$ and $U \mapsto \pi_2 U = W$ in (B.1) are mutually orthogonal projections in $L^2(\Omega; \mathbb{R}^2)$. Moreover $\pi_1$ and $\pi_2$ are continuous from $H^1(\Omega, \mathbb{R}^2)$ into $H^1(\Omega, \mathbb{R}^2)$ and there exist $q$ and $r$ such that

\[(B.2)\quad V = \nabla q, \quad W = \nabla^\perp r,\]

where $q \in H^2(\Omega)$ is the variational solution (orthogonal to the constant) of the Neumann problem

\[(B.3)\quad \Delta q = \text{div} U, \quad \partial_n q = U \cdot \nu, \quad \int_\Omega q \, dx = 0,\]

and $r \in H^1_0(\Omega) \cap H^2(\Omega)$ is the variational solution of the Dirichlet problem

\[(B.4)\quad \begin{cases} \Delta r = \text{curl} U & \text{in } \Omega \\ r = 0 & \text{on } \partial \Omega \end{cases}\]

Proof. This is an immediate consequence of Remarks 3.3 and Theorem 3.4 in [19].

Remark B.2. Note that by applying Proposition A.1 to (B.4) and Proposition A.2 to (B.3) we obtain that, if $U \cdot \nu = 0$ on $\partial \Omega$, for all $p \geq 2$ there exist $C(p, \Omega)$ such that

\[(B.5)\quad \|r\|_{2,p} \leq C \|\text{curl} U\|_p; \quad \|q\|_{2,p} \leq C \|\text{div} U\|_p.\]

By (B.2) we then obtain that, if $U \cdot \nu = 0$ on $\partial \Omega$,

\[(B.6)\quad \|U\|_{1,p} \leq C(p, \Omega) \left(\|V\|_{1,p} + \|W\|_{1,p}\right) \leq C_1(p, \Omega) \left(\|\text{curl} U\|_p + \|\text{div} U\|_p\right),\]

holds for any $p \geq 2$. 

B.2. A new Laplacian. Set

\[ \mathcal{L}^{(1)} = \nabla^\perp \text{curl} - \text{grad div} \]

which is defined as the linear operator associated with the form

\[ (V, W) \mapsto \int_{\Omega} \text{curl} V \cdot \text{curl} W \, dx + \int_{\Omega} \text{div} V \cdot \text{div} W \, dx, \]

where \( V \in H^1(\Omega; \mathbb{R}^2) \), such that \( V \cdot \nu = 0 \) on \( \partial \Omega \). It can be easily verified that the domain of \( \mathcal{L}^{(1)} \) is given by

\[ D(\mathcal{L}^{(1)}) = \{ V \in H^2(\Omega; \mathbb{R}^2), V \cdot \nu = 0, \text{curl} V = 0 \} \]

Note, that, for any smooth function \( V = (u, v) \), we have \( \mathcal{L}^{(1)} V = (-\Delta u, -\Delta v) \) at every point in \( \Omega \).

Proposition B.3.
Let \( \Omega \) satisfy property (R1) and \( p \geq 2 \). Let \( F \in L^p(\Omega, \mathbb{R}^2) \), \( V \in H^1(\Omega, \mathbb{R}^2) \) denote the variational solution for the above problem, i.e.

\[ \begin{align*}
\nabla^\perp \text{curl} V - \text{grad div} V &= F \quad \text{in } \Omega, \\
V \cdot \nu &= 0 \quad \text{on } \partial \Omega, \\
\text{curl} V &= 0 \quad \text{on } \partial \Omega.
\end{align*} \]

Then \( V \in W^{2,p}(\Omega) \) and there exists \( C(p, \Omega) \) such that, for any \( F \in L^p(\Omega) \),

\[ \| V \|_{W^{2,p}(\Omega)} \leq C \| F \|_p. \]

Proof. Due to Assumption (R1), it is sufficient to estimate \( \| V \|_{2,p} \) in a neighborhood of one of the right angle corners. Set then a coordinate system whose origin is located at the corner and its axes coincide with the boundary in some neighborhoods of the corner. We set \( V = (u, v) \) and \( F = (f, g) \). Without loss of generality we can represent the problem satisfied by \( u \) near the corner by

\[ \begin{align*}
-\Delta u &= f \quad \text{in } B(0, r) \cap \Omega \\
u(x_1, 0) &= 0 \quad \text{for } 0 < x_1 < r \\
\frac{\partial u}{\partial x_1}(0, x_2) &= 0 \quad \text{for } 0 < x_2 < r,
\end{align*} \]

and

\[ \begin{align*}
-\Delta v &= g \quad \text{in } B(0, r) \cap \Omega \\
\frac{\partial v}{\partial x_2}(x_1, 0) &= 0 \quad \text{for } 0 < x_1 < r \\
v(0, x_2) &= 0 \quad \text{for } 0 < x_2 < r.
\end{align*} \]

Hence, the system is decoupled near each corner into a Dirichlet-Neumann problem, for \( u \) (and a Neumann-Dirichlet problem for \( v \)). Multiplying \( u \) and \( v \) by a cutoff function supported in \( B(0, r) \), we can transform the problem in \( \Omega \) to a problem in a polygon. We can then apply Proposition A.3 to obtain that

\[ \| V \|_{W^{2,p}(\Omega \cap B(0,r/2))} \leq C(\| F \|_p + \| V \|_{L^p(\Omega \cap B(0,r/2))}). \]

With the aid of standard elliptic estimates away from the corners we can glue together the estimates and establish that

\[ \| V \|_{2,p} \leq C(\| F \|_p + \| V \|_p). \]
We complete the proof by substituting (B.6) into the above, in conjunction with the inequality
\[ \| \text{curl} U \|_p + \| \text{div} U \|_p \leq \| F \|_p, \]
which easily follows from (B.7) and (B.9).

**Remark B.4.**
In Appendix D we meet a non-homogeneous version of (B.9). In particular we consider the boundary condition
\[ \text{curl} V = B, \text{ on } \partial \Omega. \]
Assuming that \( B \in H^1(\partial \Omega) \), an \( H^2(\Omega) \) regularity of \( u \) does not follow as in the homogeneous case unless \( B \) vanishes at the corners. The best regularity we can obtain in this case is:
\[ u \in \cup_{q<2} W^{2,q}(\Omega). \]
Once we have observed the above decoupling near each corner, the proof is a direct consequence of Remark A.4.

**Appendix C. Time Dependent Regularity**

We derive here some regularity results for solutions of parabolic equations in domains satisfying property (R1), relying heavily on the elliptic estimates recalled in the previous subsections. These estimates are exploited throughout the work and in the next appendix, where global existence and uniqueness of solutions for (TDGL2) is established. We derive all estimates for a general abstract setting. When we refer to them throughout the work we cite the specific example in which they are applied.

Let \((\mathcal{V}, \mathcal{H}, \mathcal{V}')\) denote a triplet of Hilbert spaces such that \((\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}')\) (where \( \mathcal{V}' \) denotes the dual space of \( \mathcal{V} \)) with continuous injection such that \( \mathcal{V} \) is dense in \( \mathcal{H} \) with compact injection. Let \( a : \mathcal{V} \times \mathcal{V} \to \mathbb{R} \) denote a coercive, continuous, and symmetric bilinear map. For some \( \alpha > 0 \) we thus have
\[ \alpha \| X \|_\mathcal{V}^2 \leq a(X, X), \forall X \in \mathcal{V}. \]
By Lax-Milgram theorem we can associate two bijective operators \( A_\mathcal{V} \) and \( A_{\mathcal{V}'} \) from \( \mathcal{V} \) onto \( \mathcal{V}' \) satisfying
\[ a(u, v) = \langle A_\mathcal{V} u, v \rangle_\mathcal{V} = \langle A_{\mathcal{V}'} u, v \rangle_{\mathcal{V}'}, \forall u \in \mathcal{V}, \forall v \in \mathcal{V}, \]
and \( A \) which is a self-adjoint strictly positive unbounded operator in \( \mathcal{H} \), with compact resolvent, satisfying
\[ a(u, v) = \langle Au, v \rangle_\mathcal{H}, \forall u \in D(A), \forall v \in \mathcal{V}. \]

Use of the above general setting is being made throughout this work in the following particular cases
1. The Dirichlet Laplacian \(-\Delta^D\) with \( \mathcal{H} = L^2(\Omega), \mathcal{V} = H^1_0(\Omega), \) and \( D(A) = H^2(\Omega) \cap H^1_0(\Omega); \)
2. The Neumann Laplacian \(-\Delta^N\) with \( \mathcal{H} = \{ u \in L^2(\Omega), \int_\Omega u \, dx = 0 \}, \mathcal{V} = H^1(\Omega) \cap \mathcal{H}, \) and \( D(A) = \{ u \in H^2(\Omega), \int_\Omega u \, dx = 0, \partial_n u = 0 \) on each regular piece of \( \partial \Omega \}; \)
(3) The Dirichlet-Neumann Laplacian $-\Delta_{DN}^{\mathcal{H}}$ with $\mathcal{H} = L^2(\Omega)$, $\mathcal{V} = H_0^{1,\partial\Omega}(\Omega)$, and $D(\mathcal{A}) = \{ u \in H^2(\Omega), u_{/\partial\Omega} = 0, \partial_n u_{/\partial\Omega} = 0 \}$;

(4) The operator $\mathcal{L}^{(1)}$ with $\mathcal{H} = L^2(\Omega; \mathbb{R}^2)$, $\mathcal{V} = \{ V \in H^1(\Omega; \mathbb{R}^2), V \cdot \nu_{/\partial\Omega} = 0 \}$, and $D(\mathcal{A}) = \{ V \in H^2(\Omega; \mathbb{R}^2), V \cdot \nu_{/\partial\Omega} = 0, \operatorname{curl} V_{/\partial\Omega} = 0 \}$;

In each of the above cases the domain is deduced with the aid of the relevant regularity result in the previous appendices. Furthermore, there exists in each case $C(\Omega, \mathcal{A})$ such that

\begin{equation}
\|X\|_{D(\mathcal{A})} \leq C \|AX\|_{\mathcal{H}}.
\end{equation}

We now state and prove the general statement.

**Theorem C.1.** Let $T > 0$, $F \in L^2(0, T; \mathcal{H})$, and $X_0 \in \mathcal{H}$. Then, there exists a unique $X(t, \cdot) \in L^2(0, T; \mathcal{V})$, which serves as weak solution for

\begin{equation}
\begin{cases}
X_t - AX = F \quad &\text{in } L^2(0, T; \mathcal{V}'), \\
X(0, \cdot) = X_0 \quad &\text{in } \mathcal{H}.
\end{cases}
\end{equation}

Additionally, $X' \in L^2(0, T; \mathcal{V}')$ and $X \in L^\infty(0, T; \mathcal{H})$. Furthermore, for any $0 < t_0 < T$ we have:

\begin{equation}
\|X\|_{L^\infty(t_0, T; \mathcal{V})} \leq C(t_0) \left[ \|F\|_{L^2(0, T; \mathcal{H})} + \|X_0\|_{\mathcal{H}} \right].
\end{equation}

Moreover, if in addition $X_0 \in \mathcal{V}$, then $X \in L^2(0, T; D(\mathcal{A})) \cap H^1(0, T; \mathcal{H})$ and there exists $C > 0$, independent of $T$, such that

\begin{equation}
\|X\|_{L^2(0, T; D(\mathcal{A}))} + \|X'\|_{L^2(0, T; \mathcal{H})} + \|X\|_{L^\infty(0, T; \mathcal{V})} \leq C \left[ \|F\|_{L^2(0, T; \mathcal{H})} + \|X_0\|_{\mathcal{V}} \right].
\end{equation}

Finally, if $F \in L^2(0, T; \mathcal{V})$ and $X_0 \in \mathcal{H}$, then, for any $0 < t_0 < T$ we have

\begin{equation}
\|X\|_{L^\infty(t_0, T; D(\mathcal{A}))} \leq C(t_0) \left[ \|F\|_{L^2(0, T; \mathcal{V})} + \|X_0\|_{\mathcal{H}} \right].
\end{equation}

**Proof.**

We begin by concluding the existence and uniqueness of a weak solution for (C.2) from Theorem 3.2.2 in [24]. The remainder is devoted to the proof of the above estimates.

Suppose first that $F \in L^2(0, T; \mathcal{H})$. Denote by $\{\lambda_n\}_{n=1}^\infty$ the increasing sequence of the eigenvalues of $A$ counted with multiplicity and let $\{u_n\}_{n=1}^\infty$ denote the corresponding orthonormal basis of eigenfunctions. Let further $a_n = \langle X, u_n \rangle_\mathcal{H}$ and $b_n = \langle F, u_n \rangle_\mathcal{H}$. Taking the product of (C.2) by $u_n$ in $\mathcal{H}$ yields

\begin{equation}
\frac{da_n}{dt} + \lambda_n a_n = b_n.
\end{equation}

Multiplying (C.6) by $\frac{da_n}{dt}$, using Cauchy-Schwarz, and integrating between $0$ and $t_1$ (with $t_1 \in (0, T)$) yields

\begin{equation}
\left\| \frac{da_n}{dt} \right\|_{L^2(0, t_1)}^2 + \lambda_n \left[ |a_n(t_1)|^2 - |a_n(0)|^2 \right] \leq \|b_n\|_{L^2(0, t_1)}^2.
\end{equation}

Summing up $n$ gives

\begin{equation}
\|X'\|_{L^2(0, T; \mathcal{H})}^2 + \|X\|_{L^\infty(0, T; \mathcal{V})}^2 \leq \|F\|_{L^2(0, T; \mathcal{H})}^2 + \|X_0\|_{\mathcal{V}}^2.
\end{equation}
By (C.2) and (C.1) we have
\[ \|X\|_{L^2(0,T,D(A))} \leq C\|AX\|_{L^2(0,T;H)} \leq C\left[\|F\|_{L^2(0,T;H)} + \|X'\|_{L^2(0,T;H)}\right]. \]

Combining the above with (C.8) yields (C.4).

Suppose now that \( F \in L^2(0, T; V) \). Multiplying by \( \lambda_n + 1 \) in (C.2) and summing up over \( n \) yields for another constant \( C > 0 \):
\[ \|X_t\|^2_{L^2(0,T;V)} + \|AX\|^2_{L^\infty(0,T;H)} \leq C\left[\|F\|^2_{L^2(0,T;V)} + \|X_0\|^2_{D(A)}\right]. \]

With the aid of (C.1) we then obtain (C.5).

We next return to (C.7) to obtain
\[ a_n(t) = a_n(0)e^{-\lambda_n t} + \int_0^t e^{-\lambda_n(t-\tau)} b_n(\tau) \, d\tau, \]
from which we readily obtain that
\[ |a_n(t)|^2 \leq 2|a_n(0)|^2e^{-2\lambda_n t} + \frac{2}{\lambda_n}\|b_n\|_2^2. \]

Let
\[ C_0(t, \Omega) = \frac{1}{\alpha t} \geq \sup_{n \in \mathbb{N}} \lambda_n e^{-\lambda_n t} \]
and
\[ C_1(t, \Omega) = \frac{4}{t^2 e^2} \geq \sup_{n \in \mathbb{N}} \lambda_n^2 e^{-\lambda_n t}. \]

Obviously, \( C_i(t, \Omega) \) are decreasing functions of \( t \), and hence
\[ C_i(t, \Omega) \leq C_i(t_0, \Omega), \quad \text{for } i = 0, 1, \]
for all \( t \geq t_0 \).

Upon multiplying (C.11) by \( \lambda_n \), we sum over \( n \) to obtain, for \( t \in [t_0, T) \),
\[ \|u(t, \cdot)\|_V^2 \leq C\|F(t, \cdot)\|^2_{L^2(0,T;H)} + 2C_0(t_0)^2\|u_0\|^2_H, \]
yielding (C.3). Multiplying (C.11) by \( \lambda_n^2 \) and summing over \( n \) yields, for \( t \in [t_0, T) \),
\[ \|Au(t, \cdot)\|_2^2 \leq C\|F(t, \cdot)\|^2_{L^2(0,T;V)} + 2C_1(t_0)^2\|u_0\|^2_H, \]
which together with (C.1) yields (C.5).

**Appendix D. Global existence and uniqueness**

We adopt here the gauge
\[ \phi + c \text{ div } A = 0 \quad ; \quad A \cdot \nu|_{\partial \Omega} = 0, \]
and assume that \( B \in H^{\frac{3}{2}}(\partial \Omega) \) and that condition (R1) is satisfied. The normal field \( A_{n,d} \) is defined by
\[ \begin{cases} 
\text{curl}^2 A_{n,d} - \nabla \text{ div } A_{n,d} = 0 & \text{in } \Omega, \\
\text{curl } A_{n,d} = B & \text{on } \partial \Omega, \\
A_{n,d} \cdot \nu = 0 & \text{on } \partial \Omega.
\end{cases} \]
Using Remark B.4, we note that unless $B$ vanishes at the corners (an atypical case) $A_{n,d} \notin H^2(\Omega; \mathbb{R}^2)$. One can guarantee, however, that

\[(D.3)\quad A_{n,d} \in \bigcup_{q<2} W^{2,q}(\Omega, \mathbb{R}^2).\]

**Remark D.1.**
From (2.10) and (2.12), we get that $(A_n - A_{n,d})$ is in the domain of $L^{(1)}$ and satisfies

\[L^{(1)}(A_n - A_{n,d}) = -\frac{1}{c} \nabla \phi_n.\]

But $\nabla \phi_n \in L^p(\Omega, \mathbb{R}^2)$ for all $p \geq 2$ by (2.15), and hence, by Proposition B.3, we obtain

\[(D.4)\quad A_n - A_{n,d} \in W^{2,p}(\Omega), \forall p \geq 2.\]

In particular, we have that

\[(D.5)\quad \text{div} A_{n,d} \in W^{1,p}(\Omega), \forall p \geq 2.\]

Next we set,

\[(D.6)\quad \hat{A}_1 = A - hA_{n,d}\]

to obtain, in the above gauge, the system

\[(D.7a)\quad \frac{\partial \psi}{\partial t} - \Delta \psi = F_1 \quad \text{in } \mathbb{R}_+ \times \Omega,\]

\[(D.7b)\quad \frac{\partial \hat{A}_1}{\partial t} + c L^{(1)} \hat{A}_1 = F_2 \quad \text{in } \mathbb{R}_+ \times \Omega,\]

\[(D.7c)\quad \psi = 0, \quad \text{on } \mathbb{R}_+ \times \partial\Omega_c,\]

\[(D.7d)\quad \frac{\partial \psi}{\partial \nu} = 0, \quad \text{on } \mathbb{R}_+ \times \partial\Omega_i,\]

\[(D.7e)\quad \text{curl} \hat{A}_1(t, x) = 0, \quad \text{on } \mathbb{R}_+,\]

\[(D.7f)\quad \psi(0, x) = \psi_0(x), \quad \text{in } \Omega,\]

\[(D.7g)\quad \hat{A}_1(0, x) = A_0(x) - hA_{n,d}(x), \quad \text{in } \Omega.\]

In the above,

\[(D.8)\quad F_1 = i \left( \frac{1}{c} - 2 \right) \psi \text{div} A - 2iA \cdot \nabla A \psi + |A|^2 \psi + \psi \left( 1 - |\psi|^2 \right)\]

\[(D.9)\quad F_2 = \frac{1}{\sigma} \text{Im} \left( \bar{\psi} \nabla A \psi \right).\]

We can now make the following statement:

**Theorem D.2.**
Suppose that for some $1/2 < \alpha < 1$, $(\psi_0, \hat{A}_0) \in W^{1+\alpha,2}(\Omega, \mathbb{C}) \times W^{1+\alpha,2}(\Omega, \mathbb{R}^2)$ satisfy $\hat{A}_0 \cdot \nu = 0$ on $\partial\Omega$ and $\|\psi_0\|_{\infty} \leq 1$.

Then, there exists a weak solution $(\psi_d, A_d$) of (D.7) and (D.1), such that:

\[\psi_d \in C([0, +\infty); W^{1+\alpha,2}(\Omega, \mathbb{C})) \cap H^1_{\text{loc}}([0, +\infty); L^2(\Omega, \mathbb{C}))\]

and

\[A_d \in C([0, +\infty); W^{1+\alpha,2}(\Omega, \mathbb{C})) \cap H^1_{\text{loc}}([0, +\infty); L^2(\Omega, \mathbb{R}^2)).\]
Moreover $\psi$ and $A$ satisfy:

$$
\|\psi_d(t, \cdot)\|_\infty \leq 1, \quad \forall t > 0,
$$

$$
\psi_d \in L^2_{loc}([0, +\infty), H^2(\Omega, \mathbb{C})),
$$

$$
\hat{A}_1 \in L^2_{loc}([0, +\infty), H^2(\Omega, \mathbb{R}^2)),
$$

and

$$
\text{div} \ A_d \in L^2_{loc}([0, +\infty); H^1(\Omega)).
$$

**Proof.** We follow the same steps as in the proof of Theorem 1 in [15].

**Local existence and uniqueness:**

Set

$$
\mathcal{A} = \begin{pmatrix} -\Delta_{DN} & c \mathcal{L}^{(1)} \end{pmatrix},
$$

where $\Delta_{DN}$ denotes the Dirichlet-Neumann Laplacian whose domain is given by

$$
D(\Delta_{DN}) = \{v \in H^2(\Omega, \mathbb{C}) \mid v|_{\partial\Omega_c} = 0 \; ; \; \partial v/\partial \nu|_{\partial\Omega_i} = 0\}.
$$

Set further $u = (\psi, A_1)$ and $\mathcal{F} = (F_1, F_2)$.

The system (D.7) can be represented in the form

$$
\frac{\partial u}{\partial t} + \mathcal{A}u = \mathcal{F}.
$$

Since both $-\Delta_{DN}$, by Proposition A.3, and $\mathcal{L}^{(1)}$, by Proposition B.3 are positive self-adjoint operators, and hence sectorial, it follows that $\mathcal{A} = D(\Delta_{DN}) \times D(\mathcal{L}^{(1)})$ is sectorial as well. Furthermore it can be easily verified (cf. [15]) that $\mathcal{F} : W^{1+\alpha,2}(\Omega, \mathbb{C}) \times W^{1+\alpha,2}(\Omega, \mathbb{R}^2) \rightarrow L^2(\Omega, \mathbb{C}) \times L^2(\Omega, \mathbb{R}^2)$ is Lipschitz continuous. We can therefore apply Theorem 3.3.3 in [24] to establish the stated existence in $(0, T)$ for some $T > 0$.

**Global existence:**

We begin by establishing (2.47) for all $t > 0$. We use here the fact that

$$
\Delta(|\psi|^2 - 1) - \frac{\partial(|\psi|^2 - 1)}{\partial t} - 2|\psi|^2(|\psi|^2 - 1) = 2|\nabla \psi|^2 \geq 0,
$$

By (1.1c,d), (1.8), and the maximum principle (cf. Theorem 3.7 in [33]) a non-negative maximum cannot exist for all $t > 0$.

Next we take the inner product of (D.7a) with $\psi$ to obtain that

$$
\frac{1}{2} \frac{d}{dt} \|\psi(t, \cdot)\|_2^2 + \|\nabla \psi(t, \cdot)\|_2^2 \leq \|\psi(t, \cdot)\|_2^2.
$$

With the aid of the above and (2.47) we then obtain

$$
\|F_2\|_{L^2(0, T; L^2(\Omega, \mathbb{R}^2))} \leq C \left[ \frac{1}{L^2} \|\psi\|_{L^2(0, T; L^2(\Omega, \mathbb{C})))} + \|\psi(0, \cdot)\|_2 \right].
$$

We can now use Proposition C.1 with $\mathcal{A} = -c\mathcal{L}^{(1)}$, on the interval $(0, T)$, to obtain that

$$
(D.10) \quad \|\frac{\partial \hat{A}_1}{\partial t}\|_{L^2(0, T; L^2(\Omega, \mathbb{R}^2))} + \|\hat{A}_1\|_{L^2(0, T; H^2(\Omega, \mathbb{R}^2))} + \|\hat{A}_1\|_{L^\infty(0, T; H^1(\Omega, \mathbb{R}^2))} \leq C \left[ \frac{1}{L^2} \|\psi\|_{L^2(0, T; L^2(\Omega, \mathbb{C})))} + \|\hat{A}_1(0, \cdot)\|_{L_2^1} + \frac{1}{K^2} \|\psi(0, \cdot)\|_2 \right].
$$
With the aid of (2.47) we then obtain that
\[
\left\| \frac{\partial \hat{A}_1}{\partial t} \right\|_{L^2(0,T;L^2(\Omega))} + \left\| \hat{A}_1 \right\|_{L^\infty(0,T;H^1(\Omega,\mathbb{R}^2))} \leq C(1 + T).
\]

It can be easily verified that
\[
\frac{d}{dt} \left\| \nabla \psi(t,\cdot) \right\|^2_2 + \left\| \frac{\partial \psi}{\partial t} \right\|^2_2 = \frac{d}{dt} \left( \frac{1}{2} \left\| \psi(t,\cdot) \right\|^2_2 - \frac{1}{4} \left\| \psi(t,\cdot) \right\|^4_4 \right)
\]
\[
- \left\langle i \frac{\partial \hat{A}_1}{\partial t} \psi, \nabla \psi \right\rangle + \left\langle \frac{\partial \psi}{\partial t}, ic\psi \text{div } \nabla \psi \right\rangle.
\]

From the above we deduce that
\[
(D.11) \quad \left\| \nabla \psi(t,\cdot) \right\|^2_2 + \left\| \frac{\partial \psi}{\partial t} \right\|^2_2 \leq C(1 + T).
\]

We have thus obtained uniform boundedness in [0, T] of the $H^1$ norm of $\psi$ and $\hat{A}_1$, and an $L^2(0,T)$-bound for $\left\| \psi(t,\cdot) \right\|_2$ and $\left\| \psi(t,\cdot) \right\|_2$.

We need yet to obtain boundedness in $C([0,T];W^{1+\alpha,2}(\Omega))$ of both objects. To this end we use the smoothing property of the semigroup $e^{-tA}$. Since $\sigma(A)$ is discrete and lies on the positive real axis, there exists $\delta > 0$ such that for any $\lambda \in \sigma(A)$ we have $\lambda > \delta$. We may thus use Theorem 1.4.3 in [24], or more straightforwardly the spectral theorem for self-adjoint operators, to establish that for any $\alpha > 0$ there exists $C_\alpha \in \mathbb{R}_+$ such that
\[
\left\| A^\alpha e^{-tA} \right\| \leq C_\alpha t^{-\alpha} e^{-\delta t}, \quad \forall t > 0.
\]

It follows that there exists $C(\Omega,\alpha)$ s.t. for all $0 < \alpha < 1$,
\[
\left\| \hat{A}_1(t,\cdot) \right\|_{1+\alpha,2} \leq C \left\| A^{(1+\alpha)/2} \hat{A}_1 \right\|_2
\]
\[
\leq C t^{-\alpha} e^{-\delta t} \left\| \hat{A}_0 - A_{n,d} \right\|_{1+\alpha,2} + C \left\| \int_0^t A^{(1+\alpha)/2} e^{-(t-\tau)A} F_1(\tau,\cdot) d\tau \right\|_2.
\]

Here we observe from (D.3) and the Sobolev injection theorem that
\[
(D.12) \quad A_{n,d} \in W^{1+\alpha,2}(\Omega,\mathbb{R}^2), \quad \forall \alpha \in (\frac{1}{2}, 1).
\]

For the last term on the right-hand-side we have
\[
\left\| \int_0^t A^{(1+\alpha)/2} e^{-(t-\tau)A} F_2(\tau,\cdot) d\tau \right\|_2 = \left\| \int_0^t A^{(1+\alpha)/2} e^{-\tau A} F_2(t-\tau) d\tau \right\|
\]
\[
\leq C \int_0^t \tau^{-(1+\alpha)/2} e^{-\delta \tau} \left\| F_2(t-\tau,\cdot) \right\|_2 d\tau \leq C \int_0^t \tau^{-(1+\alpha)/2} \left\| F_2(t-\tau,\cdot) \right\|_2 d\tau
\]
\[
\leq C t^{(1-\alpha)/2} \left\| F_2 \right\|_{L^\infty(0,T;L^2(\Omega,\mathbb{R}^2))},
\]

where we have used the fact that $\alpha < 1$.

By (D.11) and (2.47) we have that
\[
\left\| F_2 \right\|_{L^\infty(0,T;L^2(\Omega,\mathbb{R}^2))} \leq C(1 + T).
\]

Hence,
\[
\left\| \hat{A}_1(T,\cdot) \right\|_{1+\alpha,2} \leq C(1 + T^2).
\]
In a similar manner we establish first that
\[ \| \psi(t, \cdot) \|_{1+\alpha, 2} \leq C \left[ t^{-\alpha} e^{-\delta t} \| \psi_0 \|_{1+\alpha, 2} + t^{(1-\alpha)/2} \| F_1 \|_{L^\infty(0,T;L^2(\Omega, \mathbb{C}))} \right]. \]

By the continuous embedding of \( W^{1+\alpha, 2} \) into \( L^\infty \) and (D.11) we have that
\[ \| F_1 \|_{L^\infty(0,T;L^2(\Omega, \mathbb{C}))} \leq C (1 + T^5), \]
and hence
\[ \| \psi(T, \cdot) \|_{1+\alpha, 2} \leq C (1 + T^6). \]

This completes the proof of global existence. Note that by (D.11) and (D.10) we obtain that both \( \psi \) and \( \hat{A}_1 \) are in \( H^1(0, T; L^2(\Omega)) \) for any \( T > 0 \).

Finally, as \( \hat{A}_1 \in L^2_{loc}([0, +\infty), H^2(\Omega, \mathbb{R}^2)) \) then \( \text{div} \hat{A}_1 \in L^2_{loc}([0, +\infty); H^1(\Omega)) \). Furthermore, by (D.5) \( \text{div} \hat{A}_{n,d} \) is in \( H^1(\Omega) \). Consequently, we obtain that \( \text{div} \hat{A} \in L^2_{loc}([0, +\infty); H^1(\Omega)) \).

Furthermore, since by (2.47), we have for any \( T > 0 \),
\[ \| F_1 \|_{L^2((0,T) \times \Omega)} = C (\| \text{div} A \|_{L^2((0,T) \times \Omega)} + \| \nabla A \psi \|_{L^\infty(0,T;L^2(\Omega))} \| A \|_{L^2(0,T,L^\infty(\Omega))} + \| A \|^2_{L^2(0,T,L^\infty(\Omega))} + 1), \]

it follows by (C.4) that \( \psi \in L^2_{loc}([0, +\infty), H^2(\Omega, \mathbb{C})). \)

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