0-th quantization or quantum (information) theory in 42 minutes

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Many physicists will tell you that those glorious days of brand new theories are over. But I don’t think so. There are many clues indicating that one should maintain an open mind.

D.M. Greenberger [9]

Abstract

We present a concise introduction of basic concepts in quantum information theory and quantum mechanics prepared as an introduction for a general audience. In our approach the rules of quantum mechanics are presented in a simple form of rules describing the method of constructing quantum objects corresponding to classical objects. As a byproduct of the introduced approach, we present some alternative rules and use them to describe basic ingredients of quantum information theory using different types of objects.

1 Introduction

Quantum mechanics provides a plethora of phenomena which appear to be counterintuitive or counterfactual [13, 12, 18] and which are responsible for speed or security boost offered by quantum information processing. At the same time, these effects, including quantum superposition and quantum entanglement, are hard to understand for newcomers in the field [8].

The main goal of this report is to provide an alternative approach facilitating the introduction of basic concepts used in the description of quantum systems. It is done in terms of simple rules for translating classical concepts into quantum counterparts. We propose to call this approach a zeroth quantization, referring to the standard correspondence between classical and quantum mechanics described as the first quantization [5].

Using the introduced approach we describe some alternative methods of constructing basic ingredients of quantum information theory. We achieve this by substituting common mapping between classical and quantum information theory, with alternative rules operating on higher-dimensional objects.

This report is organized as follows. In Section 2 we introduce the concepts of quantum pure states and the entangled states. In Section 3 we introduce quantum gates required to process information encoded in quantum states. In Section 4 we attempt to provide alternative rules for the zeroth quantization of states and describe the form of resulting operations. Finally, in Section 5 we provide some concluding remarks.
2 Pure states

We start with the basic ingredient needed in any physical theory, namely the states. The path chosen here resembles the one used commonly in the considerations related to the models of computation, where one starts with the deterministic model (e.g., deterministic finite automaton) and subsequently 'upgrades' it to include probabilistic or nondeterministic behavior [16].

In our case we begin with the notion of states used in classical (information) theory and extend it to the quantum realm. We refer to these states as pure in order to distinguish them from the general states (i.e., mixed states), which arise when we deal with statistical mixtures of pure states and are described by density matrices.

Let us start with the requirement for the pure states obtained by using the introduced rules to be elements of a linear space. For a classical bit, the allowed pure states form a set \{0, 1\}, and if one aims to add two elements from this set, the result, i.e., \(a0 + b1\), is not a valid state.

For this reason we introduce pure quantum states by mapping classical bits to vectors. For the sake of simplicity we focus our attention on bits. In this situation the correspondence between classical and quantum systems is described by the following rule.

**Rule 1** (0-th quantization of pure states). *Let us map the states as* \(0 \mapsto (\begin{array}{c}1 \\ 0 \end{array})\) and \(1 \mapsto (\begin{array}{c}0 \\ 1 \end{array})\).

Here \((\begin{array}{c}a \\ b \end{array})\) denotes a two-dimensional vector with complex elements. Dirac notation [4] is commonly used to simplify the notation in quantum mechanics and quantum information theory.

**Fact 1** (Dirac notation). \(|0\rangle \equiv (\begin{array}{c}1 \\ 0 \end{array})\), \(|1\rangle \equiv (\begin{array}{c}0 \\ 1 \end{array})\).

Now \(|0\rangle\) and \(|1\rangle\) are just plain vectors, and thus the state of the quantum bit can be represented by any combination of these two, namely

\[x_0|0\rangle + x_1|1\rangle, \quad x_0, x_1 \in \mathbb{C}.\] (1)

This expresses the basic requirement for the quantum theory to be linear.

One may ask why it is necessary to use complex numbers in the above. The reason for this is that one needs to take into account the states which can be obtained by using the allowed operations. As we will see in Section 3, the quantization of classical operations allows the operations which can introduce complex coefficients.

Rule [1] is also motivated by another assumption, namely simplicity. The space of unnormalized states resulting from this rule is described by \(k = 4\) real parameters. This agrees with Axiom 2 from [11], where quantum mechanics is distinguished from classical one by the number of real parameters required to specify the state. In quantum mechanics this number grows like \(K = N^2\), instead of \(K = N\) in the classical case, where \(N\) denotes the number of distinguishable states.

Using Rule [1] we can easily obtain states which have no direct counterparts in the classical theory. For example, the system can be described by a vector

\[|0\rangle + i|1\rangle,\] (2)

or if we want for our states to be normalized, by a vector

\[\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle).\] (3)

Such states – superpositions of base states – are crucial for quantum algorithms as they allow processing multiply values in a single computational step. This is exploited in the most prominent way in Grover’s algorithm [10].


2.1 Compound systems

In quantum mechanics the most interesting things happen when we use more than one system or a system with two subsystems. In the case of classical systems the state of the compound system is described by the state space constructed as a Cartesian product of spaces used to describe subsystems. In quantum case the situation is more involved. The state space of the compound system is constructed as a tensor product of the spaces used to describe the subsystems.

2.1.1 Two subsystems

Let us assume for a start that we deal with the system composed of two subsystems. In this situation we use the following rule.

**Rule 2** (0-th quantization of a bipartite system). If \((b_0, b_1) \in \{0, 1\} \times \{0, 1\}\) we put \((b_0, b_1) \mapsto |b_0\rangle \otimes |b_1\rangle\), where \(\otimes\) denotes the tensor product.

The first question appearing here is what exactly the tensor product is.

For the physicists this term is related mostly to the theory of General Relativity (see eg. [23]), but in this context tensor is used as a multi-linear form (map) on \(V^p \times (V^*)^q\) into \(F\), where \(V\) is some vector space over \(F\). For example, if we get \(p = 1\) and \(q = 1\), the tensors are just matrices. For \(p > 1\) or \(q > 1\) the tensors are represented by \(n\)-way arrays, which are called tensors as well.

Tensors (multilinear maps) form a linear space and one can define a tensor product of such spaces. The concept of a tensor product, however, can be defined for any vector space. The crucial part of this extension is that the tensor product transforms multilinear maps into linear ones.

**Fact 2** (Universal property of tensor product). If \(V\) and \(W\) are finite-dimensional vector spaces, then a tensor product of \(V\) and \(W\) is a vector space \(T\) with a bilinear map \(\tau : V \times W \mapsto T\), such that

\[
\forall f : V \times W \mapsto X \exists ! g : T \mapsto X \quad f = \tau \circ g,
\]

ie. for any bilinear map \(f : V \times W \mapsto X\) into some vector space \(X\), there exists a unique \(g : T \mapsto X\), such that \(f = \tau \circ g\).

The above property defines a tensor product in a unique way. In the case of finite-dimensional spaces the tensor product is known as a Kronecker product and can be found in various areas of applied mathematics [24].

One should note that the use of a tensor product for representing the state of compound system can be justified by using a different approach [2]. It is also worth noting that by using a Kronecker product for describing compound systems, we obtain a theory which agrees with Axiom 4 from [11], which requires from the system composed of two subsystems with dimensions \(N_A\) and \(N_B\) to be described by a space of dimension \(N_A N_B\).

In the case of two quantum bits, the base states (ie. states obtained directly by the application of Rule 2) are

\[
|00\rangle \equiv |0\rangle \otimes |0\rangle, \quad |01\rangle \equiv |0\rangle \otimes |1\rangle, \quad |10\rangle \equiv |1\rangle \otimes |0\rangle, \quad |11\rangle \equiv |1\rangle \otimes |1\rangle,
\]

and the system can be also in any state obtained as a linear combination of the above states

\[
|x\rangle = x_{00}|00\rangle + x_{01}|01\rangle + x_{10}|10\rangle + x_{11}|11\rangle.
\]

In the compound systems we can distinguish between separable and non-separable (ie. entangled) states. This distinction is based on the following fact, known as the Schmidt decomposition [22].
Fact 3 (Schmidt decomposition). The matrix of coefficients

\[ M(x) = \begin{pmatrix} x_{00} & x_{01} \\ x_{10} & x_{11} \end{pmatrix} \]  

(7)
can be represented in a diagonal form by local changes of bases on both subsystems.

The representation mentioned above is obtained by calculating a Singular Value Decomposition of the coefficient matrix and can be used for any n-ary space [15].

Using the Schmidt decomposition we can introduce two different classes of states, namely

- separable states – vectors such that the diagonalized matrix has only one entry,
- entangled states – vectors such that the diagonalized matrix has two entries.

An example of the entangled state is given by so-called Bell state, one of which reads

\[ |\psi^+\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle). \]  

(8)

In this case the matrix of coefficients reads

\[ M_{\psi^+} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]  

(9)

and it has two singular values.

2.1.2 Three or more subsystems

The construction of the states describing systems composed of more than two subsystems relies on a general construction of a tensor product.

Rule 3 (0-th quantization of multipartite states). If \((b_0, b_1, \ldots, b_{n-1}) \in \{0,1\}^n\) we map this state as \((b_0, b_1, \ldots, b_{n-1}) \rightarrow |b_0\rangle \otimes |b_1\rangle \otimes \cdots \otimes |b_{n-1}\rangle\).

In this situation the construction of the tensor product assures that any \(n\)-linear map is translated into a linear map. However, in this situation one cannot use Schmidt decomposition to distinguish between separable and entangled states [21].

Even for the case of three subsystem the situation is far more complicated than in the case of two subsystems. A classification of states with respect of their degree of entanglement in three-partite systems is given in [1].

3 Quantum gates

Having introduced the states we wish to operate on, it is easy to provide a quantization of allowed operations. Once again, the main rule is to use linearity of quantum theory. Moreover, it is required from the resulting operation to be reversible.

3.1 One-qubit gates

We start with operations acting on one bit. In the classical case we have only two allowed reversible operations: identity and negation.

Rule 4 (0-th quantization of one-bit circuits). For a give reversible classical operation, its quantum counterpart is defined by its application on the base states \(|0\rangle\) and \(|1\rangle\).
For operations acting on one bit, the simplest examples is the binary negation (NOT) operation. Its quantum counterpart is given by a matrix which simply interchanges the base states,

\[
\text{NOT} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\] (10)

Now, as we are able to use linear combinations of states, we would like to use gates which can result in such superpositions. The simplest examples is the $\sqrt{\text{NOT}}$ gate, defined as

\[
\sqrt{\text{NOT}} = \frac{1}{2} \begin{pmatrix} i + 1 & i - 1 \\ i - 1 & i + 1 \end{pmatrix},
\] (11)

which results in states half way between the input state and the negated input state. Gate $\sqrt{\text{NOT}}$ acts on the base states $|0\rangle$ and $|1\rangle$ as

\[
\sqrt{\text{NOT}} |0\rangle = \frac{1 + i}{2} |0\rangle + \frac{1 - i}{2} |1\rangle,
\]
\[
\sqrt{\text{NOT}} |0\rangle = \frac{1 - i}{2} |0\rangle + \frac{1 + i}{2} |1\rangle.
\] (12)

The use of gates such as $\sqrt{\text{NOT}}$ justifies to some degree the need for using complex numbers in quantum mechanics.

Although we require any quantum circuit to be reversible, we are able to introduce the operations which do not have classical counterparts. The most prominent example is the Hadamard matrix,

\[
H = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.
\] (13)

Another example, this time based on the ability to use complex numbers, is

\[
R(\phi) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}.
\] (14)

The ability of using continuous transformations between the states is the requirement which distinguishes quantum mechanics from the classical theory [11]. Without this requirement, one would be able to enumerate allowed states of the system and the obtained space of states would be equivalent to the space of classical states.

### 3.2 Two-qubit gates

The natural extension of Rule [11] provides a method for constructing quantum gates for reversible 2-bit operations.

**Rule 5** (0-th quantization of reversible two-bit circuits). For a given reversible classical operation acting on two bits, $f_2 : \{0, 1\} \times \{0, 1\} \mapsto \{0, 1\} \times \{0, 1\}$, its quantum counterpart is defined by the application on the base states,

\[
G_{f_2} |x_1\rangle |x_2\rangle = |f_2(x_1, x_2)\rangle.
\] (15)

Moreover, the ability to operate on two bits allows introducing a wider class of operations. In particular, one can introduce irreversible one-bit operations using the following rule.

**Rule 6** (0-th quantization of irreversible one-bit circuits). For a given irreversible classical operation acting on one bit, $f_1 : \{0, 1\} \mapsto \{0, 1\}$, its quantum counterpart is defined as

\[
G_{f_1} |x_1\rangle |x_2\rangle = |x_1\rangle |f_1(x_1) + x_2 \pmod{2}\rangle.
\] (16)
For example, a quantum gate for binary function \( f_1^{(0)}(x) = 0 \) (i.e., reset operation) defined using this rules is given by identity matrix, while quantum counterpart of binary function \( f_1^{(1)}(x) = 1 \) is given by
\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}.
\] (17)

The ability to operate on two bits allows the introduction of the basic ingredient used in all programming languages, namely conditional statements. The simplest gate we can consider here is \( \text{NOT} \), and this gives the classical pseudocode illustrating conditional statements presented in Listing 1

```
if \( x_1 \) == 1 then
  \( x_2 := \text{not} \ (x_2) \)
end
```

Listing 1: Pseudocode for the simplest conditional statement with ‘==’ denoting comparison and ‘:=’ denoting assignment.

The quantum counterpart of this operation can be obtained by writing the truth table for this operation. As a result we get the quantum version of the code from Listing 1

\[
|0\rangle|0\rangle \mapsto |0\rangle|0\rangle, \\
|0\rangle|1\rangle \mapsto |0\rangle|1\rangle, \\
|1\rangle|0\rangle \mapsto |1\rangle|1\rangle, \\
|1\rangle|1\rangle \mapsto |1\rangle|0\rangle,
\] (18)

and one can easily see that this operation is realized by the gate
\[
\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
\] (19)

The conditional application of a general gate
\[
U = \begin{pmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{pmatrix},
\] (20)
results in a conditional operation give as
\[
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
\] (21)

However, the syntax similar to the pseudo-code in Listing 1 is available in some quantum programming languages [19][16].

Rule 6 suggests the general method for obtaining quantum version of general binary functions \( f_{m,n} : \{0,1\}^m \mapsto \{0,1\}^n \).

**Rule 7** (0-th quantization of irreversible circuits). For a given irreversible classical operation acting on \( m \) bits, \( f : \{0,1\}^m \mapsto \{0,1\}^n \), its quantum counterpart is defined as
\[
G_f |x\rangle|y\rangle = |x\rangle f(x_1) + y_1 \pmod{2},
\] (22)
where \( |x\rangle \) and \( |y\rangle \) denote \( m \)-qubit and \( n \)-qubit states.

### 4 Alternative rules

The rules presented in Sections 2.3 allows the reproduction of a standard presentation of quantum information theory [17]. There is no reason, however, why we cannot try to describe the quantization of classical information theory using a different set of rules.
4.1 Zeroth quantization with qudits and ququarts

The simplest example of generalizing the standard zeroth quantization is to use larger space in the place of $\mathbb{C}^2$.

To do this we start by substituting Rule 1, with the following one.

**Alternative Rule 1** (0th quantization with qudits). Let us map the states 0 and 1 as $0 \rightarrow (1,0,0)^T$ and $1 \rightarrow (0,0,1)^T$.

Using the nomenclature of quantum information theory the above rule maps bit to qudit (instead of qubit). However, as we do not use full $\mathbb{C}^3$, the third of the vectors has to be fixed, i.e. set to $(0,1,0)^T$.

It is easy to see that in this case NOT operation (i.e. operation exchanging the base vectors encoding our bit) reads

\[
\left( \begin{array}{ccc}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{array} \right).
\]

(23)

One can also easily obtain a $\sqrt{\text{NOT}}$ operation, which, for the above mapping reads

\[
\frac{1}{2} \left( \begin{array}{cccc}
1+i & 0 & 1-i & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 1+i & 1+i \\
1-i & 1+i & 0 & 0
\end{array} \right).
\]

(24)

Clearly the above mapping is equivalent to operating on a fixed subspace of the $\mathbb{C}^3$ – the vectors from the subspace $\text{span}\{(0,1,0)^T\}$ are simply ignored. Instead, one could try to use a different mapping for 0, e.g.

\[
0 \rightarrow \text{span}\{(1,0,0)^T, (0,1,0)^T\},
\]

(25)

but this could not be used to obtain a unitary matrix in place of $\text{NOT}$ operation.

The above considerations suggest to use the mapping onto $\mathbb{C}^4$ given in the following rule.

**Alternative Rule 2.** Let us map the states 0 and 1 as

\[
0 \rightarrow \text{span}\{(1,0,0,0)^T, (0,0,0,1)^T\}
\]

and

\[
1 \rightarrow \text{span}\{(0,1,0,0)^T, (0,0,1,0)^T\}.
\]

(26)

(27)

This basically encodes bits into two-dimensional subspaces of $\mathbb{C}^4$. Gate $\text{NOT}$ obtained by using this mapping can be expressed as

\[
\text{NOT}_4 = \left( \begin{array}{cccc}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{array} \right),
\]

(28)

which simply flips $(1,0,0,0)^T \Rightarrow (0,1,0,0)^T$ and $(0,0,0,1)^T \Rightarrow (0,0,1,0)^T$, and results in $\sqrt{\text{NOT}}$ operation given as

\[
\frac{1}{2} \left( \begin{array}{cccc}
1+i & 1-i & 0 & 0 \\
1-i & 1+i & 0 & 0 \\
0 & 0 & 1+i & 1+i \\
0 & 0 & 1-i & 1+i
\end{array} \right).
\]

(29)

However, there are also three other matrices which can be used in place of $\text{NOT}$ gate, namely the matrices

\[
\left( \begin{array}{cccc}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{array} \right), \quad \left( \begin{array}{cccc}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{array} \right), \quad \left( \begin{array}{cccc}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{array} \right).
\]

(30)

Using Alternative Rule 2 one can also easily construct $\frac{\text{NOT}_4}{2}$ gate, which negates only some bits. Let us assume that this gate flips $(0,0,0,1)^T \Rightarrow (0,0,1,0)^T$ only. Then it is given as

\[
\frac{\text{NOT}_4}{2} = \left( \begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{array} \right),
\]

(31)
which is simply CNOT gate on qubits with control on the first and target on the second qubit. Gate flipping $(1,0,0,0)^T \rightarrow (0,1,0,0)^T$ only is equivalent to the controlled NOT operation of qubits with the first qubit being the and target.

4.2 Zeroth quantization with matrices

Taking into account our requirement for the obtained structure to be linear, one can use any mapping which turns bits into elements of some vector space.

The most straightforward generalization of the above examples is obtained by using matrices in the place of vectors.

Alternative Rule 3. Let us map the states 0 and 1 as

\[
0 \mapsto \text{span}\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}, \\
1 \mapsto \text{span}\{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\},
\]

In order to construct quantum gates acting on such states, one can note that the matrix-vector multiplication is composed of dot (ie. scalar) products between vectors forming a matrix and the vector the matrix acts on. The resulting vector is subsequently transposed. This suggests that in the case of matrices one should use Hilbert-Schmidt scalar product to define quantum gates. This appears to be more tricky, but one can easily obtain the form of the basic operations by using isomorphism between $\mathbb{M}_n(\mathbb{C})$ and $\mathbb{C}^{n^2}$.

To achieve this we use res operation [15], which maps elements of $\mathbb{M}_n(\mathbb{C})$ into vectors using row-order. For examples, if

\[
A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \]

we have

\[
\text{res} A = (a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2})^T.
\]

Using this notation, NOT gate resulting from Alternative Rule 3 is given as

\[
\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},
\]

As expected, the resulting gate is equivalent to the SWAP operation defined for qubits.

Without using res mapping, this can be represented as a tensor (ie. n-way array),

\[
\text{NOT}_{\mathbb{M}_2(\mathbb{C})} = (\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix})
\]

of dimension $(2, 2, 4)$. Given a pure state obtained using Alternative Rule 3

\[
X = (x_{1,1}, x_{2,1}, x_{1,2}, x_{2,2}),
\]

tensor $\text{NOT}_{\mathbb{M}_2(\mathbb{C})}$ acts on it as

\[
\text{NOT}_{\mathbb{M}_4(\mathbb{C})} X = (\text{tr}(\text{NOT}_{\mathbb{M}_2(\mathbb{C})} ; 1 X), \ldots, \text{tr}(\text{NOT}_{\mathbb{M}_2(\mathbb{C})} ; 4 X))^{T(1,2);3},
\]

where $(\text{NOT}_{\mathbb{M}_2(\mathbb{C})} ; i)$ denotes $i$-th element in third way (dimension) of the tensor and $T_{(1,2);3}$ is a general transposition operation, which exchanges between indices $(1, 2)$ and $3$.

As a matter of fact, any mapping from bits to $\mathbb{M}_2(\mathbb{C})$, eg. given as

\[
0 \mapsto \text{span}\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}, 1 \mapsto \text{span}\{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}\},
\]

will be isomorphic to quantization obtained using qutrits. The only difference we get in this case is the representation of quantum operations in the form of tensors.
This situation shows the importance of the simplicity axiom stated in [11]. This axiom basically states, that the number of degrees of freedom should take the minimal value which still enables it to be consistent with the formulation of the theory. From the above considerations, one can see the introduced rules, which are based on objects like matrices, do not introduce any qualitative changes into the quantum theory obtained with them.

5 Summary

The main goal of this report was to provide a clear and consistent introduction to the basic concepts of quantum information theory. We aimed at show that the basic rule of linearity allows an easy quantization of classical information theory and introduction of the concepts of superposition and entanglement.

Our intension was not to derive quantum theory from a fixed, minimalistic, set of axioms. Such attempts were made in several papers and the reader interested in this matter is advised to consult [11, 7, 3, 6] for the recent developments.

Standard quantum mechanics provides one of possible theories constructed as a generalization of the classical theory. The study of different approaches for construction of such theories was undertaken by many authors [14, 25, 20]. The rules introduced in this report allowed the introduction of alternative methods for obtaining quantization for classical information theory. The presented alternative rules enable a better understanding of the choices made during the representation used in quantum information theory, especially in for the sake the simplicity criterion.

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References

[1] A. Acín, A. Andrianov, L. Costa, E. Jané, J. I. Latorre, and R. Tarrach. Generalized Schmidt decomposition and classification of three-quantum-bit states. Phys. Rev. Lett., 85(7):1560–1563, 2000.

[2] D. Aerts and I. Daubechies. Physical justification for using the tensor product to describe two quantum systems as one joint system. Helv. Phys. Acta, 51:661–675, 1978.

[3] G.M. D’Ariano. Operational axioms for quantum mechanics. 2006. arXiv:quant-ph/0611094.

[4] P.A.M. Dirac. A new notation for quantum mechanics. Math. Proc. Camb. Philos. Soc., 35(3):416–418, 1939.

[5] P.A.M. Dirac. The principles of quantum mechanics, volume 27 of International Series of Monographs on Physics. Oxford University Press, Oxford, UK, 4 edition, 1981.
[6] D. Fivel. Derivation of the rules of quantum mechanics from information-theoretic axioms. *Found. Phys.*, 42(2):291–318, 2012.

[7] Ch. A. Fuchs. Quantum mechanics as quantum information (and only a little more). 2002. arXiv:quant-ph/0205039.

[8] A. Goff. Quantum tic-tac-toe: A teaching metaphor for superposition in quantum mechanics. *Am. J. Phys.*, 74(11):962–973, 2006.

[9] D.M. Greenberger. The tic-tac-toe theory of gravity. *Found. Phys.*, 42:46–52, 2012.

[10] Lov K. Grover. Quantum mechanics helps in searching for a needle in a haystack. *Phys. Rev. Lett.*, 79:325–328, 1997.

[11] L. Hardy. Quantum theory from five reasonable axioms. 2001. arXiv:quant-ph/0101012.

[12] O. Hosten, M.T. Rakher, J.T. Barreiro, N.A. Peters, and P.G. Kwiat. Counterfactual quantum computation through quantum interrogation. *Nature*, 439(7079):949–952, 2006.

[13] Richard Jozsa. Quantum effects in algorithms. In ColinP. Williams, editor, *Quantum Computing and Quantum Communications*, volume 1509 of *Lecture Notes in Computer Science*, pages 103–112. Springer Berlin Heidelberg, 1999.

[14] B. Mielnik. Generalized quantum mechanics. *Commun. Math. Phys.*, 37(3):221–256, 1974.

[15] J.A. Miszczak. Singular value decomposition and matrix reorderings in quantum information theory. *Int. J. Mod. Phys. C*, 22(9):897–918, 2011.

[16] J.A. Miszczak. *High-level Structures for Quantum Computing*, volume #6 of *Synthesis Lectures on Quantum Computing*. Morgan & Claypool Publishers, May 2012.

[17] M. A. Nielsen and I. L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, Cambridge, U.K., 2000.

[18] T.-G. Noh. Counterfactual quantum cryptography. *Phys. Rev. Lett.*, 103(23):230501, 2009.

[19] B. Ömer. *Structured Quantum Programming*. PhD thesis, Vienna University of Technology, 2003.

[20] M. Pawłowski and A. Winter. “Hyperbits”: The information quasiparticles. *Phys. Rev. A*, 85:022331, 2012. arXiv:1106.2409.

[21] A. Peres. Higher order schmidt decompositions. *Phys. Lett. A*, 202(1):16 – 17, 1995.

[22] E. Schmidt. Zur theorie der linearen und nichtlinearen integralgleichungen. *Mathematische Annalen*, 63:161–174, 1907.

[23] B. F. Schutz. *A first course in general relativity*. Cambridge University Press, Cambridge, U.K., 1985.

[24] C.F. Van Loan. The ubiquitous Kronecker product. *J. Comput. Appl. Math.*, 123(1-2):85–100, 2000.

[25] K. Życzkowski. Quartic quantum theory: an extension of the standard quantum mechanics. *J. Phys. A: Math. Theor.*, 41(35):355302, 2008.