THE THETA DIVISOR OF $SU_C(2,2d)^*$ IS VERY AMPLE IF $C$ IS NOT HYPERELLIPTIC$^1$

by

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0. Introduction.
Let $C$ be an irreducible smooth complex curve of genus $g \geq 2$: in this paper we are dealing with the moduli space

$$X = SU_C(2,2d)$$

of semistable rank 2 vector bundles over $C$ having fixed determinant of even degree. As it is well known

$$X - \text{Sing}X = SU_C(2,2d)^*$$

is the moduli space for stable rank two vector bundles and $\text{Sing}X$ is naturally isomorphic to the Kummer variety of the Jacobian of $C$. Let

$$L \in \text{Pic}X$$

be the generalized theta divisor

$$\theta : X \longrightarrow \mathbb{P}H^0(L)^*$$

the map associated to $L$, the main theorem we want to show is the following

(0.3) THEOREM. Assume $C$ is not hyperelliptic, then

(1) $\theta$ is injective

(2) $d\theta_x$ is injective if $x \in X - \text{Sing}X$.

The proof of the injectivity of $d\theta_x$ at points $x \in \text{Sing}X$ seems a more delicate technical problem which is, at the moment, beyond the capability and perhaps the patience of the authors. Previous results on the embedding properties of $\theta$ were obtained by Beauville [B1]:

(0.4).

(i) $\theta$ is a morphism of degree $\leq 2$ onto its image

(ii) $\theta$ has degree two if and only if $C$ is hyperelliptic and $g \geq 3$

and by Ramananan-Narasimhan for $g \leq 3$ ([NR1], [NR2]). More recently Laszlo showed in [L1] that

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(0.5).
\( \theta \) is an embedding for a general non hyperelliptic curve \( C \)

We recall that \( \text{Pic}(X) \cong \mathbb{Z} \) and that by definition the generalized theta divisor is the ample generator of it; hence, in particular, \( \theta \) is a finite morphism over its image. The fundamental geometric interpretation of \( \theta \) is the following (cf. [B1]): let \( J = \text{Pic}^0(C), \Theta \subset J \) a symmetric theta divisor then, choosing as a fixed determinant for the points of \( X \) the cotangent bundle \( \omega_C \), there exists a canonical isomorphism

\[
H^0(\mathcal{L})^* \cong H^0(O_J(2\Theta))
\]

Under the previous identification the map associated to \( \mathcal{L} \) becomes

\[
\theta : X \longrightarrow |2\Theta |
\]

on the other hand each semistable vector bundle \( \xi \) having determinant \( \omega_C \) defines a divisor

\[
\Theta_\xi \in |2\Theta |
\]

which is set theoretically so defined

\[
\Theta_\xi = \{ e \in J | h^0(\xi(e)) \geq 1 \}
\]

If \( x \in X \) is the moduli point of \( \xi \) it turns out that

\[
\theta(x) = \Theta_\xi
\]

Let us introduce very briefly the method we used to study \( \theta \): we have constructed a projective family \( T \) of rational maps

\[
\eta_t : X \longrightarrow \mathbb{P}^n
\]

\( n = \binom{g+1}{2} \), such that \( \eta_t = \lambda_t \cdot \theta \) where

\[
\lambda_t : |2\Theta | \longrightarrow \mathbb{P}^n
\]

is a suitable linear projection depending on \( t \). Globalizing this construction, we have a rational map

\[
F : X \times T \longrightarrow \mathbb{P}(Q)
\]

where \( Q \) is a vector bundle on \( T \) of rank \( \binom{g+1}{2} + 1 \) and \( F/X \times t = \eta_t \). It turns out that the maps \( \eta_t \) have degree two onto their image as well as the map \( F \). In particular the latter induces a birational involution

\[
j : X \times T \longrightarrow X \times T
\]

Of course \( \theta \) is an embedding if any two distinct points (tangent vectors) of \( X \) are separated by at least one \( \theta_t \) (\( d\theta_t \)). Since \( T \) is projective we introduce in section 6 a weaker version of rigidity lemma which is "ad hoc" for the rational maps \( F \) and \( j \). Applying this rigidity argument we finally deduce in section 7 that, whenever \( \theta \) does not separates two given points of \( X \) or two tangent vectors of \( X - \text{Sing}X \), then the degree of \( \theta \) must be two. Hence \( C \) is hyperelliptic by Beauville’s theorem.
Most of the paper is necessarily devoted to a detailed study of the maps \( \theta_t, F \) and \( j \) (sections 2, 3, 4). This seems interesting in itself in view of the geometry of quadrics behind \( \theta_t \). It is simple to define \( \theta_t \): the parameter space \( T \) is actually \( \text{Pic}^2(C) \) so that \( t \) is a degree 2 line bundle, consider the surface
\[
S_t = \{ t(x - y), \ x, y \in C \} \subset \text{Pic}^0(C)
\]
then \( P^n \) is just the linear system \( |O_{S_t}(2\Theta)| \) and by definition
\[
\lambda_t : 2\Theta \rightarrow P^n
\]
is the restriction map, so that \( \theta_t = \lambda_t \cdot \theta \). We will construct in section 4 a canonical projective isomorphism
\[
|O_{S_t}(2\Theta)| \cong |I_{C_t}(2)|
\]
where
\[
C_t \subset P^{g+2}
\]
is the curve \( C \) embedded by \( \omega_C(2t) \) and \( I_{C_t} \), its ideal sheaf. This relates \( X \) to the geometry of the quadrics containing \( C_t \): the expected dimension for the variety \( Y \) of quadrics of rank \( \leq 6 \) containing \( C_t \) is \( 3g - 3 = \text{dim} X \); let \( Z_t = \theta_t(X) \), we will show that \( Z_t \) is an irreducible component of \( Y \). If \( g \geq 3 \) \( \theta_t \) is the "natural" double covering of \( Z_t \) which parametrizes the two rulings of maximal linear subspaces in a rank 6 quadric, if \( g = 2 \) \( \theta_t = \theta \). The main technical problems to apply the previous method are considered in section 5 where the birational involution \( j \) is suitably extended and it is shown that \( \text{codim} I(x) \geq 2 \) for some special "Brill-Noether" locus \( I(x) \subset T \) depending on \( x \in X \): this follows essentially from Martens' theorem [ACGH] and Lange-Narasimhan results on Maruyama conjecture [LN].

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Let us fix here some of the usual notations:
\[
\begin{align*}
\xi &= \text{semistable rank two vector bundle on } C \text{ with determinant } \omega_C, [\xi] = \text{moduli point of } \xi \\
J &= \text{Jacobian of } C, \Theta = \text{a symmetric theta divisor in } J, C(n) = n-\text{th symmetric product of } C \\
T &= \text{Pic}^2(C), t = \text{an element of } T \\
P^{g+2}_t &= \text{Pic}(\omega_C(2t))^*, C_t = \text{the curve } C \text{ embedded in } P^{g+2}_t \text{ by } \omega_C(2t). \\
E^* &= \text{dual of the vector space (bundle) } E.
\end{align*}
\]

1. Preliminaries: rank 6 quadrics and rank 2 vector bundles.

For our and reader’s convenience we recollect in this section some standard properties which will be used many times. We fix a smooth, irreducible projective curve
\[
C \subset P^n
\]
which is linearly normal and not degenerate. The space of global sections of \( \mathcal{O}_C(1) \) will be
\[
H = H^0(\mathcal{O}_C(1))
\]
so that \( P^n = PH^* \). Then we consider a pair
\[
(\xi, V)
\]
such that:
(1.3).
(i) $\mathcal{E}$ is a rank 2 vector bundle over $C$
(ii) $V$ is a 4-dimensional vector space in $H^0(\mathcal{E})$
(iii) $\text{det}(\mathcal{E}) = O_C(1)$
(iv) $h^0(\mathcal{E}^*) = 0$

from the previous data we construct:

(1.4).
(i) the evaluation map $e_V : V \otimes O_C \to \mathcal{E}$
(ii) the determinant map $d_V : \wedge^2 V \to H$
(iii) the Grassmannian $G^*_V$ of 2-dimensional subspaces of $V^*$

By definition $G^*_V$ is the projectivized set of the irreducible vectors in $\wedge^2 V^*$ hence

$$G^*_V \subset \mathbb{P}^5 = \mathbb{P}(\wedge^2 V^*)$$

as a smooth quadric hypersurface. The dual of of the evaluation map $e_V^* : \mathcal{E}^* \to O_C \otimes V^*$ defines in $V^*$ the family of subspaces

$$\{V^*_x = \text{Im} e^*_V, x \in C\}$$

By definition $e_V$ is *generically surjective* if and only if $\dim V^*_x = 2$ for a general $x$, assuming this one can construct a rational map

$$g_V : C \to G^*_V$$

which associates to $x$ the point

$$g_V(x) = \wedge^2 V^*_x \in G^*_V$$

(1.5) DEFINITION. $g_V$ is the Gauss map of the pair $(\mathcal{E}, V)$

(1.6) PROPOSITION.
(1) $e_V$ is generically surjective if and only if the determinant map $d_V$ is not zero
(2) assume $d_V$ is not zero, let $\delta_V : \mathbb{P}^n \to \mathbb{P} \wedge^2 V^*$ be the projectivization of the dual map $d^*_V$, then

$$\delta_V/C = g_V$$

(3) $\delta_V$ is defined at $x \in C$ iff $e_{V,x}$ is surjective, in particular $\text{SuppCoker}(e_V) \cong C \cdot \mathbb{P} \text{Ker}(d^*_V)$

Proof. Well known.

Let $p$ be an equation for the quadric $G^*_V$ and let

$$q(\mathcal{E}, V)$$

be the pull back of $p$ by the dual map $d^*_V : H^* \to \wedge^2 V^*$. If it is not identically zero $q(\mathcal{E}, V)$ defines a quadric in $\mathbb{P}^n$. This will be denoted by

$$Q(\mathcal{E}, V)$$

and it is the most important object of this paper:
(1.7) **DEFINITION.** \(q(\mathcal{E}, V)\) is a quadratic form defined by \((\mathcal{E}, V)\). \(Q(\mathcal{E}, V)\) is the quadric of \((\mathcal{E}, V)\).

Let \(q = q(\mathcal{E}, V), Q = Q(\mathcal{E}, V)\) obviously \(Q = \delta^{-1}_V(G^*_V)\) so that

\[
\text{rank } q \leq 6 \text{ and } q \text{ vanishes on } C
\]

Let \(K = \text{Ker}(d^*_V), I = \text{Im}d^*_V\) then \(Q\) can be considered as a cone of vertex \(PK\) over the quadric \(PI \cap G^*_V\). In particular \(PK \subseteq \text{Sing}Q\). We want to point out that

(1.9) (i) \(PK = \text{Sing}Q\) if and only if \(PI\) is transversal to \(Q\)
(ii) if \(q\) has rank \(\geq 5\) then \(\text{Sing}Q = PK\)

For any subline bundle \(L \subset \mathcal{E}\) we have in \(H^0(\mathcal{E})\) the natural intersection

\[
V_L = V \cap H^0(L)
\]

(1.11) **PROPOSITION.** Let \(q = q(\mathcal{E}, V), r\) the rank of \(q\). Then:

(1) \(r \leq 4 \iff \mathcal{E}\) contains a subline bundle \(L\) such that \(\dim V_L \geq 2\)
(2) \(r = 0 \iff \mathcal{E}\) contains a subline bundle \(L\) such that \(\dim V_L \geq 3\)
(3) \(e_V\) not generically surjective \(\iff \mathcal{E}\) contains a subline bundle \(L\) such that \(\dim V_L = 4\).

**Proof.** By duality the elements of \(\wedge^2 V\) are linear forms on \(\wedge^2 V^*\). Let \(G_V \subset P(\wedge^2 V)\) be the projectivized set of the irreducible vectors: \(G_V\) is the dual of the quadric \(G^*_V\), hence a non zero irreducible vector \(s_1 \wedge s_2 \in \wedge^2 V\) is a linear form defining a singular hyperplane section of \(G^*_V\). By duality again \(\text{Im}(d^*_V)\) is the zero locus of the linear forms in \(\text{Kerd}_V\). From this it follows that:

(a) \(r \leq 4 \iff \text{Kerd}_V\) contains a non zero irreducible vector \(s_1 \wedge s_2\)
(b) \(r = 0 \iff \text{Kerd}_V\) contains a 3-dimensional vector space \(F\) of irreducible vectors.

On the other hand it is a standard property of two independent sections \(s_1, s_2 \in V\) that

(c) \(d_V(s_1 \wedge s_2) = 0 \iff s_1, s_2\) define a subline bundle \(L \subset \mathcal{E}\) such that \(s_1, s_2 \in V_L\).

The statement follows easily from (a), (b), (c): we omit the details for brevity.

We will say that two pairs \((\mathcal{E}_1, V_1), (\mathcal{E}_2, V_2)\) are isomorphic if there exists an isomorphism \(\sigma : \mathcal{E}_1 \rightarrow \mathcal{E}_2\) such that \(\sigma^*(V_2) = V_1\). We want to construct all the isomorphism classes of pairs \((\mathcal{E}, V)\) as in 1.3 which define the same quadratic form \(q\): we are interested to do this only when \(q\) has rank \(\geq 5\) and \(\text{Sing}Q \cap C = \emptyset, (Q = \{q = 0\}).

(1.12) **DEFINITION.** \(V\) generates \(\mathcal{E}\) iff \(\text{Coker}(e_V) = 0\).

Assume \(V\) generates \(\mathcal{E}\) then in the exact sequence

\[
\begin{array}{c}
0 \longrightarrow \mathcal{E}^* \xrightarrow{e_V^*} V^* \otimes \mathcal{O}_C \longrightarrow \text{Coker}(e_V^*) \longrightarrow 0
\end{array}
\]

\(\text{Coker}(e_V^*)\) is locally free and \(V^*\) is canonically identified to a 4-dimensional space of global sections of it. This defines another pair

\[
(\mathcal{E}, V) = (\text{Coker}(e_V^*), V^*)
\]

satisfying the assumptions in (1.3). Repeating the same construction for \((\mathcal{E}, V)\) one obtains \((\mathcal{E}, V)\) again.
(1.14) DEFINITION. \((\mathcal{E}, V)\) is the dual pair of \((\mathcal{E}, V)\)

The universal and the quotient bundle of \(G^*_V\) will be denoted respectively by

\[ U_V, \quad \overline{U}_V \]

from the definition of the Gauss map \(g_V\) it follows

\[ g^*_V U_V \cong \text{Im}(e^*_V) \cong \text{Im}(e_V)^* \quad \text{and} \quad g^*_V \overline{U}_V \cong \text{Ker}(e^*_V) \cong \text{Ker}(e_V)^* \]

and the exact sequence

\[ 0 \rightarrow g^*_V U'_V \rightarrow E \rightarrow \text{Coker}(e_V) \rightarrow 0 \]

In particular

\[ \mathcal{E} \cong g^*_V U'_V \quad \overline{\mathcal{E}} \cong g^*_V \overline{U}_V \]

if and only if \(V\) generates \(E\). In this case the previous sequence (1.13) is just the pull-back by \(g_V\) of the standard universal/quotient bundle sequence

\[ 0 \rightarrow U_V \rightarrow V^* \otimes \mathcal{O}_{G^*_V} \rightarrow \overline{U}_V \rightarrow 0 \]

(1.18) LEMMA. Let \((\mathcal{E}_1, V_1), (\mathcal{E}_2, V_2)\) be two pairs such that

(i) \(q = q(\mathcal{E}_1, V_1) = q(\mathcal{E}_2, V_2)\)

(ii) \(V_i\) generates \(\mathcal{E}_i, i = 1, 2\)

(iii) \(\text{Ker}(d^*_V) = \text{Ker}(d^*_V)\)

Then either \((\mathcal{E}_1, V_1), (\mathcal{E}_2, V_2)\) are isomorphic or they are dual. Moreover:

let \(h^0(\mathcal{E}_1) = 4\), if the two pairs are dual and \(\mathcal{E}_1 \cong \mathcal{E}_2\) then \(q\) has rank \(\leq 5\).

Proof. For \(i = 1, 2\) consider the Gauss maps \(g_{V_i} : C \rightarrow G^*_V\); since \(V_i\) generates \(\mathcal{E}_i\) it follows \(\mathcal{E}^*_i \cong g^*_V U_{V_i}\).

Since \(\text{Ker}(d^*_V) = \text{Ker}(d^*_V)\) there exists a unique isomorphism \(\phi : \text{Im}(d^*_V) \rightarrow \text{Im}(d^*_V)\) such that \(d^*_V = d^*_V \cdot \phi\). Let \(v \in \wedge^2 V^*_i\) be an irreducible vector, \(v = d^*_V(h)\) where \(h \in H^*\) then \(q(\mathcal{E}_1, V_1)\) is zero on \(h\).

By (i) the same holds for \(q(\mathcal{E}_2, V_2)\) so that \(d^*_V(h) = \phi(v)\) is an irreducible vector too. Since it preserves irreducible vectors \(\phi\) is the restriction to \(\text{Im}d^*_V\) of an isomorphism \(\Phi : \wedge^2 V^*_1 \rightarrow \wedge^2 V^*_2\) with the same property. Therefore \(\Phi\) induces a biregular isomorphism

\[ \Phi/G^*_{V_1} = f : G^*_{V_1} \rightarrow G^*_{V_2} \quad \text{such that} \quad g_{V_2} = f \cdot g_{V_1} \]

As it is well known either \(U_{V_1} \cong f^* U_{V_2}\) or \(f^* U_{V_2} \cong U_{V_1}\). Assuming \(f^* U_{V_2} \cong U_{V_1}\) and applying \(f^*\) to the universal/quotient bundle sequence

\[ 0 \rightarrow U_{V_2} \rightarrow V^*_2 \otimes \mathcal{O}_{G^*_{V_2}} \rightarrow \overline{U}_{V_2} \rightarrow 0 \]

one has

\[ 0 \rightarrow \overline{U}_{V_1} \rightarrow V_1 \otimes \mathcal{O}_{G^*_V} \rightarrow U_{V_1} \rightarrow 0 \]

pulling this sequence back by \(g_{V_1}\) and using \(g^*_V \cdot f^* = g^*_V\) we finally obtain

\[ 0 \rightarrow \mathcal{E}^*_1 \rightarrow V_1 \otimes \mathcal{O}_C \rightarrow \mathcal{E}_1 \rightarrow 0 \]
This precisely means that the two pairs are dual. If \( f^*U_{V_2} \cong U_{V_1} \) the same argument implies that the two pairs are isomorphic. To prove the latter statement observe that, by assumption, we can set \( E_1 = E_2 = E \) and \( V_1 = V_2 = V \). Then, tensoring the previous sequence by \( E \) and passing to the long exact sequence, we obtain

\[
0 \to H^0(E \otimes E^*) \to V \otimes V \to H^0(E \otimes E)
\]

where the latter arrow is the natural map \( \mu \). Let \( \mu^- \) be the restriction of \( \mu \) to \( \wedge^2 V \subset V \otimes V \), then \( \text{Im} \mu^- \) is contained in \( H^0(\det E) \subset H^0(E \otimes E) \) and moreover \( \mu^- \) is just the determinant map \( d_V \). Since \( h^0(E \otimes E^*) \geq 1 \) \( \mu \) is not injective: we leave as an exercise to check that then \( \mu^- = d_V \) must be not injective. Since the rank \( r \) of \( q \) is the dimension of \( \text{Im} d_V \) it follows \( r \leq 5 \).

**(1.19) Proposition.** Let \( Q \) be a quadric of rank \( r \leq 6 \) which contains \( C \), then there exists a pair \((E, V)\) such that \( Q = Q(E, V) \). If \( r = 5, 6 \) and \( \text{Sing} Q \cap C = \emptyset \) every pair defining \( Q \) is isomorphic to \((E, V)\) or to its dual \((\overline{E}, \overline{V})\). If the rank is 5 these pairs are isomorphic too.

**Proof.** Consider the linear projection \( \delta : Q \to \mathbb{P}^{r-1} \) of center \( \text{Sing} Q \) and the quadric \( \delta(Q) \); \( \delta(Q) \) can be considered as a linear section of \( G \), where \( G \subset \mathbb{P}^5 \) is the Plücker embedding of a Grassmannian \( G(2, 4) \). Let \( U \) be the universal bundle on \( G \), \( g = \delta/C \), since \( \text{Sing} Q \cap C = \emptyset \) the pull-back \( g^*U^* \) has determinant \( \mathcal{O}_C(1) \). Putting

\[
V = g^*H^0(U^*) \quad , \quad E = g^*U^*
\]

we obtain a pair \((E, V)\) as in (1.3). It is easy to see that this pair defines \( Q \), assume \( r \geq 5 \) then \( \mathbb{P} \ker (d_V^r) = \text{Sing} Q \) (see 1.9) and \( V \) generates \( E \) because \( \text{Sing} Q \cap C = \emptyset \) (see 1.6). Note that these properties hold for every pair defining \( Q \); let \((E_1, V_1), (E_2, V_2)\) be two of them, then, by the previous lemma 1.18, either \((E_1, V_1), (E_2, V_2)\) are isomorphic or they are dual. Finally let \( r = 5 \) then \( \delta(Q) \) is a smooth hyperplane section of \( G \) and hence the universal and the quotient bundle restrict to the same bundle on \( \delta(Q) \) (see e.g. [O]). This easily implies that \((E, V)\) is isomorphic to its dual and completes the proof.

Now we fix a rank two vector bundle \( E \) with \( \text{det} E = \mathcal{O}_C(1) \) and the space of its global sections

\[(1.20) \quad W = H^0(E)\]

we assume \( \dim W \geq 4 \). Let \( \mathcal{I}_C \) be the ideal sheaf of \( C \), the construction of \( q(E, V) \) defines a linear map

\[(1.21) \quad h_E : \wedge^4 W \to H^0(\mathcal{I}_C(2))\]

which is so defined on irreducible vectors: let \( v \in \wedge^4 W \), \( V \) the corresponding 4-dimensional subspace of \( W \) then

\[h_E(v) = q(E, V)\]

We are more interested to the following specialization of the of the previous map: Let us fix a 3-dimensional vector space

\[(1.22) \quad F \subset W\]

and the image

\[(1.23) \quad H_F \subseteq H\]
of $\wedge^2 F$ under the determinant map

$$d : \wedge^2 W \to H = H^0(\mathcal{O}_P^*(1))$$

we restrict $h_\mathcal{E}$ to

$$W_F = (\wedge^3 F) \wedge W \subseteq \wedge^4 W$$

Every non zero $v \in W_F$ represents a 4-dimensional vector space $V$ which contains $F$, in particular $q(\mathcal{E}, V)$ vanishes on the linear space $\Lambda_F \subset \mathbb{P}^n$ which is the zero locus of the elements of $H_F$ and has codimension $\leq 3$ in $\mathbb{P}^n$. Let

$$\mathcal{I}_F = \text{ideal sheaf of } C \cup \Lambda_F$$

then $h_\mathcal{E}$ restricts to a linear map

$$(1.24) \quad h_F : W_F \longrightarrow H^0(\mathcal{I}_F(2))$$

(1.25) Proposition. Let $e_F : F \otimes \mathcal{O}_C \to \mathcal{E}$ be the evaluation map, $d_F : \wedge^2 F \to H$ the determinant map:

(1) if $d_F$ is injective then $h_F$ is injective
(2) if $\text{Coker}(e_F) = 0$ then $h_F$ is an isomorphism.

Proof. (1) Let $F \subset V$, $v = \wedge^4 V$, $q = q(\mathcal{E}, V)$. By proposition 1.11 $q = h_F(v)$ is zero iff $\mathcal{E}$ contains a subline bundle $L$ such that $\dim V_L \geq 3$. Assume such an $L$ exists and consider $F_L = V_L \cap F$: then $\dim F_L \geq 2$; moreover, by remark (c) in the proof of proposition 1.11, the determinant map is zero on $\wedge^2 V_L$ hence on $\wedge^2 F_L$: against the injectivity of $d_F$.

(2) Consider the dual map $e^*_F : \mathcal{E}^* \to \mathcal{O}_V \otimes F^*$ and in $F^*$ the family of subspaces $F^*_x = \text{Im}(e^*_{F,x}), x \in C$. Since $\text{Coker}(e_F) = 0$ $\dim F^*_x = 2$ for all $x \in C$. Then the Gauss map

$$g_F : C \longrightarrow \mathbb{P}^2 = \mathbb{P} \wedge^2 F^*$$

sending $x$ to $\wedge^2 \text{Im}(e^*_{F,x})$ is a morphism and hence

$$\mathcal{E}^* \cong g_F^* \Omega_{\mathbb{P}^2}(1)$$

On the other hand $g_F$ is defined by the subspace $H_F$ of $H = H^0(\mathcal{O}_C(1))$ which means that $g_F$ is the restriction to $C$ of the linear projection of center $\Lambda_F$. To show the surjectivity of $h_F$ consider $q' \in H^0(\mathcal{I}_F(2))$: since it vanishes on $\Lambda_F$ $q'$ has rank $\leq 6$. Hence $q' = q(\mathcal{E}', V')$ for a given pair $(\mathcal{E}', V')$ (prop. 1.19). Up to replacing this pair by its dual we can assume that $V'$ contains a 3-dimensional subspace $F'$ such that the image of $\wedge^2 F'$ by the determinant map is again $H_F$. For this reason, when we construct as above the Gauss map

$$g_F : C \longrightarrow \mathbb{P}^2$$

from $F'$, we obtain $g_F = g_{F'} = \text{linear projection of center } \Lambda_F$. Therefore $\mathcal{E}' \cong \mathcal{E}$ and hence $q' \in \text{Im}(h_F)$. By (1), to show the injectivity of $h_F$ it suffices to show that $d_F$ is injective. Assume $d_F(s_1 \wedge s_2) = 0$ for a given $s_1 \wedge s_2 \in \wedge^2 F$ then, for any $V$ containing $F$, $q(\mathcal{E}, V)$ has rank $\leq 4$ (prop. 1.11). This immediately implies $\Lambda_F \cap C \neq \emptyset$ and $\text{Coker}(e_F) \neq 0$. 
(1.26) **Proposition.** Assume $\Lambda_F \cap C = \emptyset$. Then $h_F$ is an isomorphism and in particular 

$$h^0(\mathcal{E}) = 3 + h^0(\mathcal{I}_F(2))$$

**Proof.** Observe that $\Lambda_F \cap C = \emptyset$ implies that $e_F$ is everywhere surjective. Hence, by the previous proposition, $h_F$ is an isomorphism. In particular $h^0(\mathcal{I}_F((2))) = dim\mathcal{W}_F = dim\mathcal{W} - 3$

2. The fundamental involution and the fundamental map on $SU_C(2,2d) \times Pic^2(C)$.

Let

\[(2.1) \quad U_C(2,2d)\]

be the moduli space of semistable rank 2 vector bundles $\mathcal{E}$ on $C$ of fixed degree $2d$. Since $U_C(2,2d)$ does not depend on the choice of $d$ there is no restriction to assume

$$2d = 2g + 2$$

then the construction of the dual pair we have given in the previous section defines an involution

\[(2.2) \quad i: U_C(2,2d) \longrightarrow U_C(2,2d)\]

Roughly speaking $i$ is so defined: let $u \in U_C(2,2d)$ be the moduli point of $\mathcal{E}$, since $deg(\mathcal{E}) = 2g + 2$ the expected dimension of $V = H^0(\mathcal{E})$ is 4. Assuming this and that $V$ generates $\mathcal{E}$ we can construct the dual $(\mathcal{E},\mathcal{V})$ of the pair $(\mathcal{E},V)$. If semistable $\mathcal{E}$ defines another point $\overline{u} \in U_C(2,2d)$, we define in this case

$$i(u) = \overline{u}$$

Let

\[(2.3) \quad X = SU_C(2,2d) \quad T = Pic^2(C)\]

for reasons which will be evident later we are more interested to the analogous

\[(2.4) \quad j : X \times T \longrightarrow X \times T\]

of the involution $i$: let $(z,t) \in X \times T$, then $z$ is the moduli point of $\xi$ where $det(\xi) = \omega_C$ and $t$ is a line bundle of degree 2. Putting $\mathcal{E} = \xi(t)$, $V = H^0(\xi(t))$ we obtain a pair $(\mathcal{E},V)$ as above. Passing to the dual pair we obtain a rank 2 vector bundle $\overline{\mathcal{E}}$ such that $det(\overline{\mathcal{E}}(−t)) = \omega_C$. If semistable $\overline{\mathcal{E}}(−t)$ defines a point $\overline{z} \in X$, then we define as above

$$j(z,t) = (\overline{z},t)$$

The relation between $i$ and $j$ is very clear: let

\[(2.5) \quad m : X \times T \longrightarrow U_C(2,2d)\]
be the tensor product map sending the moduli of the pair \((\xi, t)\) to the moduli of the vector bundle \(E = \xi(t)\), from the definitions of \(i\) and \(j\) we have the commutative diagram

\[
\begin{array}{ccc}
SU_C(2, 2d) \times T & \longrightarrow & SU_C(2, 2d) \times T \\
\downarrow m & & \downarrow m \\
U_C(2, 2d) & \longrightarrow & U_C(2, 2d)
\end{array}
\]

(2.6)

It is not difficult to see that \(m\) is an étale covering of degree \(2g\), in particular \(j\) can be viewed as the lifting of \(i\) to \(X \times T\). To begin we construct a suitable open subset

\[
U_j \subset X \times T
\]

where \(j\) will be properly defined and biregular. \(U_j\) will be useful later, to define it we consider the exact sequence

\[
0 \to \Omega \to H^0(\omega_C) \otimes O_C \to \omega_C \to 0
\]

(2.8)

\(\Omega\) is a well known semistable vector bundle of rank \(g - 1\) over \(C\). With some abuse we will use sometimes the same notation for a semistable bundle and for its moduli point:

\[
(2.9) \text{DEFINITION. } U_j = \{(\xi, t) \in X \times T/h^0(\Omega \otimes \xi(t)) = 0\}
\]

We want to point out that, tensoring 2.8 with \(\xi(t)\) and passing to the long exact sequence, it follows

\[
h^0(\Omega \otimes \xi(t)) = 0 \iff \text{the multiplication map } \nu : H^0(\omega_C) \otimes H^0(\xi(t)) \to H^0(\omega_C \otimes \xi(t)) \text{ is injective}
\]

furthermore, since \(\chi(\Omega \otimes \xi(t)) = 0\),

\[
h^0(\Omega \otimes \xi(t)) = 0 \iff \nu : H^0(\omega_C) \otimes H^0(\xi(t)) \to H^0(\omega \otimes \xi(t)) \text{ is an isomorphism}
\]

this will be used various times. Of course \(U_j\) is open,

\[
(2.11) \text{PROPOSITION. } \text{Let } (z, t) \in \text{Sing} X \times T, \text{ then}
\]

(i) \(U_j \cap \text{Sing} X \times t \neq \emptyset\) so that \(U_j \cap X \times t\) is not empty

(ii) \(U_j \cap \{z\} \times T \neq \emptyset\).

\text{Proof.} \text{ Let } z \in \text{Sing} X, \text{ it is well known that } z \text{ is the moduli point of all semistable extensions}

\[
0 \to \alpha \to \xi \to \omega_C \otimes \alpha^{-1} \to 0 \quad \text{or} \quad 0 \to \omega_C \otimes \alpha^{-1} \to \xi \to \alpha \to 0
\]

for a given \(\alpha \in \text{Pic}g^{-1}(C)\). Tensoring the previous sequences by \(\Omega(t)\) it follows that \(h^0(\Omega \otimes \xi(t)) = 0\) iff \(h^0(\Omega \otimes \alpha(t)) = h^0(\Omega \otimes \omega \otimes \alpha^{-1}) = 0\). Tensoring the sequence 2.8 by \(\alpha(t)\) the latter condition is equivalent to the injectivity of the multiplication \(H^0(\omega_C) \otimes H^0(\alpha(t)) \longrightarrow H^0(\omega_C \otimes \alpha(t))\) and of the analogous multiplication for \(\omega_C \otimes \alpha^{-1}\). Fixing \(t\) the injectivity of these maps for a general \(\alpha\) is well known and it follows from the base-point-free pencil trick [ACGH]. Fixing \(\alpha\) (hence \(z\)) the same argument gives the injectivity for a general \(t\). Hence \(U_j\) intersects both \(z \times T\) and \(\text{Sing} X \times t\).
(2.12) **Proposition.** Let \( (\xi, t) \) define a point of \( U_j \) if and only if the following conditions are satisfied:

1. \( h^0(\xi(t)) = 4 \)
2. \( \xi(t) \) is globally generated
3. \( h^0(\xi(t)) = 4 \), where \( \xi \) is defined by the fundamental exact sequence

\[
0 \rightarrow (\xi(t))^* \rightarrow H^0(\xi(t))^* \otimes O_C \rightarrow \xi(t) \rightarrow 0
\]

**Proof.** If (1),(2),(3) hold the sheaf \( \xi(t) \) defined by the previous sequence is a rank 2 vector bundle such that \( h^1(\xi(t)) = 0 \). The semistability of \( \xi(t)^* \) implies \( h^0(\xi(t)^*) = 0 \) so that, taking the long exact sequence of 2.13, we obtain the isomorphisms

\[
H^0(\xi(t))^* \cong H^0(\xi(t)) \quad \text{and} \quad H^1(\xi(t))^* \cong H^0(\xi(t))^* \otimes H^1(O_C)
\]

By Serre duality the latter one is the dual of the multiplication map

\[
\nu : H^0(\omega_C) \otimes H^0(\xi(t)) \to H^0(\omega \otimes \xi(t))
\]

since \( \nu \) is an isomorphism \( h^0(\Omega \otimes \xi(t)) = 0 \) and \( (\xi, t) \in U_j \). Conversely let \( \nu \) be an isomorphism: note that \( h^1(\omega_C \otimes \xi(t)) = h^0(\xi(t)^*) \); then \( h^0(\omega_C \otimes \xi(t)) = 4g \) and \( h^0(\xi(t)) \leq 4 \). Hence, by Riemann-Roch, \( h^0(\xi(t)) = 4 \) and (1) holds. From the semistability of \( \xi(t-x)^* \) we have again \( 0 = h^0(\xi(t-x))^* = h^1(\omega_C \otimes (\xi(t-x))) \), \( \forall x \in C \). In particular it follows

\[
h^0(\omega_C \otimes (\xi(t-x))) = 4g - 2 \quad \forall x \in C
\]

so that \( \omega_C \otimes \xi(t) \) globally generated. Let \( x \in C \), consider in \( H^0(\omega_C) \otimes H^0(\xi(t)) \) the vector space \( S_x = \nu^{-1}(H^0(\omega_C \otimes (\xi(t-x))) \): this is just the sum

\[
S_x = H^0(\omega_C(-x)) \otimes H^0(\xi(t)) + H^0(\omega_C) \otimes H^0(\xi(t-x))
\]

moreover the intersection of the latter two vector spaces is \( H^0(\omega_C(-x) \otimes H^0(\xi(t-x))) \). If \( \xi(t) \) is not globally generated at \( x \) on sees by computing dimensions that \( \dim S_x > 4g - 2 = \dim \nu(S_x) \): against the injectivity of \( \nu \). Hence (2) holds. Since (1) and (2) hold it is very easy to show (3) using the exact sequence 2.13.

(2.14) **Proposition.** Assume that \( (\xi, t) \) defines a point \( U_j \). Let \( \xi \) be constructed from \( (\xi, t) \) using the exact sequence 2.13. Then \( \xi \) is semistable and \( (\xi, t) \) defines a point of \( U_j \).

**Proof.** Assume \( \xi \) not semistable the there exists an exact sequence of vector bundles

\[
0 \rightarrow L \rightarrow (\xi(t)) \rightarrow M \rightarrow 0
\]

such that \( \deg(L) \geq g + 2 \). By the previous proposition we have \( h^0(\xi(t)) = 4 \), therefore, passing to the long exact sequence, we obtain \( h^1(\xi(t)) = h^1(M) = 0 \). Then only two cases are possible: (1) \( h^0(L) = 3 \) so that \( \deg(M) = g \), \( h^0(M) = 1 \) and (2) \( h^0(L) = 4 \). In case (1) \( \xi(t) \) is not globally generated at points of \( \text{Supp}M \), in case (2) at all points of \( C \). On the other hand \( \xi(t) \) is defined by the exact sequence 2.13 hence it is globally generated. Therefore \( \xi \) must be semistable. The rest of the proof is an immediate consequence of prop. 2.12.
(2.15) **PROPOSITION.** Assume that \((\xi, t)\) defines a point of \(U_j\), then there is no subline bundle \(L\) of \(\xi(t)\) having degree \(\leq g\) and \(h^0(L) \geq 2\).

**Proof.** As above we construct from \((\xi, t)\) the pair \((\xi, t)\). We know from the previous proposition that \(\xi\) is semistable. Assume such a line bundle \(L\) exists, there is no restriction to assume that \(L\) is saturated in \(\xi(t)\). Let \(h^0(L) = 2\), \(L\) globally generated, \(\deg(L) \leq g\). Then we have the exact diagram

\[
\begin{array}{ccccccc}
0 & 0 & 0 \\
\downarrow & & & & & & \\
0 & L^* & \longrightarrow & H^0(L) \otimes \mathcal{O}_C & \longrightarrow & L & \longrightarrow & 0 \\
\downarrow & & & & & & \\
0 & \xi(t)^* & \longrightarrow & H^0(\xi(t)) \otimes \mathcal{O}_C & \longrightarrow & \xi(t) & \longrightarrow & 0 \\
\downarrow & & & & & & \\
0 & M^* & \longrightarrow & H^0(M) \otimes \mathcal{O}_C & \longrightarrow & M & \longrightarrow & 0 \\
\downarrow & & & & & & \\
0 & 0 & 0 & 0 \\
\end{array}
\]

where the middle horizontal row is just the dual of 2.13. In particular \(L^*\) is a subline bundle of \(\xi(t)^*\) and \(\deg(L^*) > \frac{1}{2} \deg(\xi(t)^*)\): against the semistability of \(\xi(t)\). Assume \(h^0(L) \geq 3\) or \(h^0(L) = 2\) and \(L\) not globally generated then it follows \(h^0(\xi(t-x)) \geq 3\) for at least one \(x \in C\). This implies \(h^0(\xi(t)) > 4\) or \(\xi(t)\) is not globally generated: a contradiction.

Using the exact sequence 2.13 we define a bijective involution

\[ j : U_j \longrightarrow U_j \]

such that

\[ j(\xi, t) = (\xi, t) \]

it is standard to check that \(j\) is a birational involution on \(X \times T\) which is biregular on \(U_j\).

(2.16) **DEFINITION.** Let \(X = SU_C(2), T = Pic^2(C)\). We will say that

\[ j : X \times T \longrightarrow X \times T \]

is the fundamental involution of \(X \times T\).

(2.17) **REMARKS**

(1) Let \(K\) be the kernel of the evaluation map \(e : H^0(\xi(t)) \otimes \mathcal{O}_C \rightarrow \xi(t)\): it is standard to deduce that \(j(\xi, t) = (K^*(-t), t)\) as soon as \(K\) is a rank two vector bundle which is semistable and of degree \(2g + 2\). It is not difficult to check for which \(t\) \(K\) is a rank two vector bundle and it is possible to suitably extend \(j\) in case \(\xi(t)\) is not globally generated (see section 5). The main problem is the stability of \(K\), (cfr. [Bu]).
Let $p_1 : X \times T \to T$ be the first projection: knowing that $X = p_1(U_j)$ would be very useful in this paper. This property means that, for each semistable rank 2 vector bundle $\xi$ with canonical determinant, the locus
$$D_\xi = \{ t \in T / h^0(\Omega \otimes \xi(t)) > 0 \}$$
is a divisor and not all $T$. In any case let $Y = X - p_1(U_j)$; $Y$ is clearly closed and contained in $X - Sing(X)$ by prop. 2.11. Since $Pic(X) \cong \mathbb{Z}$ $Y$ cannot be a divisor:
\begin{equation}
(2.18) \quad \text{codim}_XY \geq 2
\end{equation}
This remark will be important in the proof of the main theorem (see section 7 and thm. 6.13). We continue with the previous notations and introduce in addition the following ones:
\begin{equation}
(2.19) \quad \text{let } t \in T
\end{equation}
$P^{g+2}_t = PH^0(\omega_C(2t))^*$
$C_t = \text{the curve } C \text{ embedded in } P^{g+2}_t \text{ by } \omega_C(2t)$
\begin{equation}
(2.20) \quad \mathcal{I}_t = \text{ideal sheaf of } C_t
\end{equation}
We want to construct a vector bundle $Q \to T$ of fibre $H^0(\mathcal{I}_t(2))$ over $t$ and a rational map $F : X \times T \to \mathbb{P}(Q)$
To construct $Q$ we fix a Poincaré bundle $\mathcal{P} \to Pic^{2g+2}(C) \times C$
and consider the vector bundles
\begin{equation}
(2.21) \quad \mathcal{F} = p_\ast \mathcal{P} \quad \text{and} \quad \mathcal{G} = p_\ast (\mathcal{P} \otimes 2)
\end{equation}
where $p$ is the first projection. Then $\mathbb{P}(F) = C^{(2g+2)}$, the fibre of $\mathcal{F}$ is $H^0(L)$ and the fibre of $\mathcal{G}$ is $H^0(L^{\otimes 2})$ (at the point $L \in Pic^{2g+2}(C)$). Moreover there exists a uniquely defined morphism
\begin{equation}
(2.22) \quad \nu : Sym^2 \mathcal{F} \to \mathcal{G}
\end{equation}
such that $\nu_L : (Sym^2 \mathcal{F})_L \to \mathcal{G}_L$ is the usual multiplication map $\nu_L : Sym^2 H^0(L) \to H^0(L^{\otimes 2})$
$\nu$ Since $deg(L) \geq 2g + 1$ $\nu_L$ is always surjective, hence
\begin{equation}
(2.23) \quad V = Kerv\nu
\end{equation}
is a vector bundle and one computes
$$\text{rank of } V = \left( \frac{g + 1}{2} \right) + 1$$
We define now a squaring map $\gamma : T \to Pic^{2g+2}(C)$
by setting $\gamma(t) = \omega_C(2t)$
then we take the pull-back of $V$ by $\gamma$ and give the following
**Definition.** $Q = \gamma^* \text{Ker} \nu$ is the fundamental bundle.

Considering the multiplication map

$$\mu_t : \text{Sym}^2 H^0(\omega_C(2t)) \to H^0(\omega_C^\otimes 4(4t))$$

it is clear that $\text{Ker} \mu_t$ is the fibre of $Q$ at $t$, therefore

$$Q_t = \text{Ker} \mu_t = H^0(I_t(2))$$

Now we want to construct $F$: we fix the open set

$$U = U_j \cap (X - \text{Sing} X)$$

where $U_j$ is defined as in the previous section. Let $(z, t) \in U$ then $z$ is the moduli point of a stable $\xi$ such that $h^0(\xi(t)) = 4$ (prop.2.12). Putting as usual $E = \xi(t)$, $V = H^0(\xi(t))$ we construct from $(E, V)$ the quadratic form $q(E, V) \in H^0(I_t(2))$. Since $\xi$ is stable and $(z, t) \in U_j$ there is no subline bundle $L$ of $E$ with $h^0(L) \geq 2$ (prop. 2.15), in particular $q(E, V)$ is not zero (prop.1.11). Therefore its zero locus $Q(E, V)$ is a quadric containing $C_t$ i.e. a point of $PQ_t$. $Q(E, V)$ depends only on $(z, t)$, we set

$$F(z, t) = Q(E, V)$$

which is regular on $U$.

**Definition.** We will say that $F : X \times T \to PQ$ is the fundamental map. The restriction of $F$ to $X \times t$ will be denoted by

$$F_t : X \to PQ_t$$

### 3. Geometry of the fundamental map: rank 6 quadrics

The family of maps $F_t : X \to PQ_t$ has the following important feature: let

$$\theta : X \to PH^0(\mathcal{L})^*$$

be the morphism associated to the generalized theta divisor $\mathcal{L}$ then $F_t$ is defined by a linear subsystem of $| \mathcal{L} |$. In other words $\{ F_t(X), \ t \in T \}$ is a complete family of linear projections of $\theta(X)$. This is shown in section 4 and it will be used to discuss the very ampleness of $\mathcal{L}$. Now we want to study the map $F_t$; this provides a natural description of $X$ in terms of rank 6 quadrics. Let

$$U = U_j \cap X - \text{Sing} X \quad U_t = U \cap X \times t$$

by proposition 2.11(i) $U_t$ is not empty for all $t \in T$. $F_t$ is regular on $U_t$, we consider the closure

$$Z_t \subset PQ_t$$

of $F_t(X)$ and the scheme theoretic intersection

$$W_t^{(r)} = p_t^{(r)} \cdot PQ_t$$

where $P_t^{(r)} \subset PH^0(O_{P_t^{r+2}}(2))$ is the variety of all quadrics in $P_t^{r+2}$ having rank $\leq r$. It is clear that

$$Z_t \subset W_t^{(6)}$$

moreover the expected dimension of $W_t^{(6)}$ is $3g - 3$ and this is also the dimension of $X$: the situation is described by the next
THEOREM. Assume $g \geq 3$, then

1. the general point of $Z_t$ is a quadric of rank 6
2. the rational map $F_t : X \to Z_t$ has degree two
3. $Z_t$ is a reduced irreducible component of $W_t$.

Proof. Preliminarily we show that the map $F_t : X \to Z_t$ has degree $\leq 2$: Let $z \in U_t$, then $z$ is the moduli point of a stable bundle $\xi$; since $z \in U_t$, $\xi(t)$ is globally generated, there is no subline bundle $L$ of $\xi(t)$ with $h^0(\xi(t))$ and $h^0(\xi(t)) = 4$ (prop. 2.12, 2.15). Therefore (by prop. 1.11) $5 \leq \text{rank}F_t(z) \leq 6$.

Since these conditions hold we can apply to $F_t(z)$ proposition 1.19 which says that there exist at most two isomorphism classes of pairs $(\mathcal{E}, V)$ such that $Q(\mathcal{E}, V) = F_t(z)$. Hence $\deg F_t \leq 2$ and moreover $\dim Z_t = 3g - 3$ because $F_t$ is of finite degree. We can now prove our statements: assume (1) does not hold, then every point of $Z_t$ is a quadric of rank $\leq 5$ and $Z_t \subseteq W_t^{(5)}$.

Fix $Q = F_t(z)$ with $z \in U_t$ then $Q$ has rank exactly 5. Consider the projectivized tangent space $T_Q^{(5)}$ to $W_t^{(5)}$ at the point $Q$, it is standard that $T_Q^{(5)} = \mathbb{P}H^0(I_{\text{Sing}Q \cup C(2)})$ where $I_{\text{Sing}Q \cup C(2)}$ is the ideal sheaf of $\text{Sing}Q \cup C$. Since $Z_t$ is a closed subset of $W_t^{(5)}$ it follows $\dim T_Q^{(5)} \geq 3g - 3 = \dim Z_t$. Let us show that this is impossible for $g \geq 3$ since $\text{Sing}Q \cap C$ is empty the general maximal linear subspace $\Lambda$ of $Q$ does not intersect $C$. Then, by proposition 1.26, $h^0(I_{\Lambda \cup C}(2)) = h^0(\mathcal{E}) - 3$

where $I_{\Lambda \cup C}$ is the ideal sheaf of $\Lambda \cup C$ and $(\mathcal{E}, V)$ is a pair defining $Q$. Since $Q = F(z, t)$ with $z \in U_t$ we can assume $\mathcal{E} = \xi(t)$ where $\xi$ is the bundle considered at the beginning of this proof. But then $h^0(\mathcal{E}) = 4$ so that $h^0(I_{\Lambda \cup C}(2)) = 1$.

On the other hand we can consider the exact sequence $0 \to H^0(I_{\Lambda \cup C}(2)) \to H^0(I_{\text{Sing}Q \cup C}(2)) \to H^0(J(2))$ where $J$ is the ideal of $\text{Sing}Q$ in $\Lambda$. We have $h^0(I_{\text{Sing}Q \cup C}(2)) \geq 3g - 2 = \dim T_Q^{(5)} + 1$, from $\dim \text{Sing}Q = g - 3$ and $\dim \Lambda = g - 1$ we compute immediately $h^0(J(2)) = 2g - 1$. Thus we finally obtain $h^0(I_{\Lambda \cup C}(2)) \geq g - 1$. 

which is impossible for \( g \geq 3 \). This shows (1), to show (2) observe that, if \( F_t(z) \) has rank 6, there exist exactly two isomorphism classes of pairs \((E, V)\) defining \( Q \). To show (3) let \( Q = F_t(z) \) with \( z \in U_t \) and rank of \( Q = 6 \). We consider the projectivized tangent space to \( W_t^{(6)} \) at \( Q \) that is

\[
T_Q^{(6)} = \mathbb{P} H^0(\mathcal{I}_{SingQUC}(2))
\]

Then we use the previous sequence 3.14: this time we have \( dimSingQ = g - 4, dim\Lambda = g - 1 \) so that \( h^0(\mathcal{J}(2)) = 3g - 1 \). On the other hand, choosing \( \Lambda \) general in at least one of the two rulings of maximal linear subspaces, we have \( h^0(\mathcal{I}_{AUC}(2)) = 1 \) as above. Therefore \( h^0(\mathcal{I}_{SingQUC}(2)) \leq 3g - 2 \) and \( dimT_Q^{(6)} \leq 3g - 3 \). Since \( dimT_Q^{(6)} \geq dimZ_t = 3g - 3 \) it follows \( dimT_Q^{(6)} = 3g - 3 \). Hence \( Z_t \) is an irreducible component of \( W_t^{(6)} \) which is smooth at \( Q \).

Let \( j : X \times T \to X \times T \) be the fundamental involution, it follows from the previous proof that generically

\[
F^{-1}(F(z,t)) = \{(z,t), j(z,t)\}
\]

hence we have also obtained:

\[\text{(3.5)THEOREM. Assume } g \geq 3, \text{ then:} \]

1. the fundamental map \( F \) has degree two;
2. the birational involution induced by \( F \) is the fundamental involution \( j \).

Assume \( g \geq 3 \), then \( Z_t \) is a family of quadrics of even rank 6: a well known construction defines a finite double covering

\[\tilde{F}_t : \tilde{Z}_t \to Z_t\]

where, as a set, \( \tilde{Z}_t \) is the family of pairs \((Q, \Lambda)\) such that \( Q \in Z_t, \Lambda \) is a connected component of the variety of linear subspaces of \( Q \) having codimension 2 and \( \tilde{F}_t(Q, \Lambda) = Q \). In particular \( \tilde{F}_t \) is étale at each \( Q \in Z_t \) having rank 6.

\[\text{(3.7)THEOREM. } F_t = \tilde{F}_t \cdot \pi_t \text{ where } \pi_t : X \to \tilde{Z}_t \text{ is a birational morphism} \]

\textbf{Proof.} \( F_t(z) = Q(E, V) \) for a given pair \((E, V) \). As in section 1 we consider the grassmannian \( G_t^* \) and the universal bundle \( \mathcal{U}_V \): one of the two families of planes in \( G_t^* \) is the family of the zero sets of the non zero global sections of \( \mathcal{U}_V \). Recall that \( Q(E, V) = \delta_V^{-1}(G_t^*) \) where \( \delta_V \) is the linear projection defined in 1.5. Taking the inverse images by \( \delta_V \) of the planes of this family we obtain a connected component \( \Lambda \) of the variety of codimension 2 linear subspaces of \( Q(E, V) \). One defines \( \pi_t \) by setting \( \pi_t(z,t) = (Q, \Lambda) \).

We finish this section by an auxiliary result to be used later:

\[\text{(3.8)PROPOSITION. Let } \xi \text{ be a stable bundle, } z \in X \text{ its moduli point, } t \in T. \text{ Assume } h^0(\xi(t)) = 4, \xi(t) \text{ globally generated and } F(z,t) \text{ of rank 6. Then } F(t,z) \text{ is a smooth point of } Z_t \text{ and the tangent map } dF \text{ is injective at } (z, t) \text{ as well as } (dF_t) \text{ at } t. \]

\textbf{Proof.} Let \( Q = F(z,t) \): in theorem 3.3 we proved \( dimZ_t = 3g - 3 \) exactly by showing that the tangent space \( T_{Z_t, Q} \) has dimension \( 3g - 3 \) (under the assumptions we have). Hence \( Q \) is non singular for \( Z_t \). Since \( \xi(t) \) is globally generated and \( Q \) has rank \( \geq 5 \) it follows from our usual argument (prop. 1.19) that \( z \) is a connected component of \( F^{-1}(z) \). \( z \) is smooth because \( \xi \) is stable, since \( F_t = \tilde{F}_t \cdot \pi_t \) and \( \tilde{F}_t \) is étale over \( Q \) because the rank of \( Q \) is 6 we conclude that \( dF_t(z) \) is injective. Finally let \( p : PQ \to T \) be the natural map, since \( p \cdot F = id_P \) then \( dP_Q \cdot dF_t(z) : T_{X,z} \oplus T_{T,t} \to T_{T,t} \) is the obvious projection, on the other hand \( dF_{(z,t)}(z) : T_{X,z} \to T_{T,t} \). This implies that \( dF_{(z,t)}(z) \) is injective.
4. Geometry of the fundamental map: the generalized theta divisor.

In this section, and throughout all the paper, we fix the following notations

\[(4.1)\]

- \(C^{(n)}\) = \(n\)-th symmetric product of the curve \(C\),
- \(J = \text{Pic}^0(C)\),
- \(\Theta = \text{a symmetric theta divisor in} \ J,\)
- \(X = SU(2,C)\),
- \(\mathcal{L} = \text{generalized theta divisor of} \ SU(2,C)\)

Moreover we will set

\[S = C^{(2)}\]

and

\[T = \text{Pic}^2(C)\]

The vector space \(H^0(\mathcal{L})^*\) will be canonically identified to \(H^0(\mathcal{O}_J(2\Theta))\) as in [B1]. Though it is actually not needed we will assume in this section that \(C\) is not hyperelliptic: this simplifies somehow the exposition. Our first purpose is to construct a suitable map of vector bundles over \(T\)

\[\lambda : H^0(\mathcal{L})^* \otimes \mathcal{O}_T \longrightarrow \mathcal{Q}\]

where \(\mathcal{Q}\) is the fundamental bundle defined in section 3. In order to do this fix \(t \in T\) and consider the morphism

\[\alpha_t : S \longrightarrow J\]

such that

\[\alpha_t(x + y) = t - x - y\]

\(\alpha_t\) is just the Abel map multiplied by -1, its image in \(J\) is \(t - S\); we denote it by

\[(4.2) \quad S_t\]

This defines the correspondence

\[(4.3) \quad S = \{(t,e) \in T \times J | e \in S_t\}\]

together with its two natural projections

\[T \leftarrow \pi_1 \quad S \quad \pi_2 \rightarrow J\]

Applying \(\pi_1 \pi_2^*\) to the evaluation map \(e : \mathcal{O}_J \otimes H^0(\mathcal{O}_J(2\Theta)) \longrightarrow \mathcal{O}_J(2\Theta)\) we obtain the morphism of sheaves

\[\pi_1 \pi_2^*(e) : \pi_1 \pi_2^* \mathcal{O}_J \otimes H^0(\mathcal{O}_J(2\Theta)) \longrightarrow \pi_1 \pi_2^* \mathcal{O}_J(2\Theta)\]

let us set

\[(4.4) \quad \mathcal{Q} = \pi_1 \pi_2^* \mathcal{O}_J(2\Theta)\]
and
\[(4.5) \hat{\lambda} = \pi_1, \pi_2(e)\]
since \(\pi_1, \pi_2 \circ H^0(O_J(2\Theta)) = O_T \otimes H^0(O_J(2\Theta))\) we have defined a morphism
\[\hat{\lambda} = \pi_1, \pi_2(e) : H^0(O_J(2\Theta)) \otimes O_T \to \hat{Q}\]
Note that \(\pi_1 t = S_i\) so that the stalk of \(\hat{Q}\) at \(t\) is
\[\hat{Q}_t = H^0(O_{S_i}(2\Theta))\]
and the map
\[\hat{\lambda}_t : H^0(O_J(2\Theta)) \to \hat{Q}_t\]
is just the restriction
\[H^0(O_J(2\Theta)) \to H^0(O_{S_i}(2\Theta))\]
\[(4.6)\textsc{Definition.} \text{ We will say that } \hat{\lambda} \text{ is the restriction map.}\]
If \(\dim(\hat{Q})\) is a constant function \(\hat{Q}\) is a vector bundle: this is ensured by the next proposition 4.9.
\[(4.7)\textsc{Lemma.} \text{ Let } d = \Sigma z_i \in \text{Div}(C), D = \Sigma (z_i + C) \text{ be the corresponding divisor in } S, \Delta \text{ the diagonal in } S. \text{ Then:}\]
\[(1) O_{\Delta}(D) \cong O_C(2d)\]
\[(2) \text{there exists a canonical isomorphism } \psi : \text{Sym}^2 H^0(O_C(d)) \to H^0(O_S(D)) \text{ such that}\]
\[(3) \text{composing the restriction } \rho : H^0(O_S(D)) \to H^0(O_C(2d)) \text{ with } \psi \text{ we obtain the multiplication map}\]
\[\mu : \text{Sym}^2 H^0(O_C(d)) \to H^0(O_C(2d))\]
In particular \(\rho\) is surjective if \(\deg(d) \geq 2g + 1\).
\[\text{Proof.} \text{ It is elementary to check that } (z + C) \text{ and } \Delta \text{ intersect at the unique point } 2z \in S \text{ with multiplicity 2: this implies (1). To show (2) consider the natural involution } \iota : C \times C \to C \times C \text{ together with the quotient map } \pi : C \times C \to S \text{ and the projections } p_i : C \times C \to C. \text{ From } p_i^* O_C(d) \otimes p_2 O_C(d) \cong \pi^* O_S(D) \text{ we obtain the isomorphism}\]
\[\tau : H^0(O_C(d)) \otimes H^0(O_C(d)) \to H^0(\pi^* O_S(D))\]
such that \(\tau(s_i \otimes s_j) = p_i^* s_i \otimes p_2 s_j\). On the other hand we have the involution \(\iota^*\) on \(H^0(\pi^* O_S(D))\) defined by \(\iota\). Let \(j = (\tau^{-1} \cdot \iota^* \cdot \tau)\) observe that \(j\) is the natural involution on \(H^0(O_C(d)) \otimes H^0(O_C(d))\) sending \(s_i \otimes s_j\) to \(s_j \otimes s_i\).
Considering the +1 eigenspaces we have \(\text{Sym}^2 H^0(O_C(d))\) for \(j\) and \(\pi^* H^0(O_S(D))\) for \(\iota^*\). Restricting \(\tau\) to \(\text{Sym}^2 H^0(\omega_C(2t))\) we obtain the diagram
\[\text{Sym}^2 H^0(O_C(2)) \begin{array}{c} \tau \end{array} \pi^* H^0(O_S(D)) \begin{array}{c} \iota^* \end{array} H^0(O_S(D))\]
from this (composing properly the maps) we obtain the isomorphism \(\psi\) required in (2). The equality \(\mu = \rho \cdot \psi\) can be checked on vectors \(s_i \otimes s_j + s_j \otimes s_i\). Finally \(\mu\) is surjective if \(\deg d \geq 2g + 1\) so that the same holds for \(\rho\).
We denote by
\[\Theta_t \in \text{Div}(S)\]
a divisor such that
\[O_S(\Theta_t) \cong \alpha_t^* O_J(\Theta)\]
for brevity we omit the proof of the following
\textbf{(4.8) Lemma.} \( \mathcal{O}_S(2\Theta_t) \cong \mathcal{O}_S(\Sigma(z_i + C) - \Delta) \) where \( \mathcal{O}_C(\sum z_i) \cong \omega_C(2t) \).

\textbf{(4.9) Proposition.} \( \forall t \in T \) we have

\begin{enumerate}[(1)]
\item \( h^0(\mathcal{O}_S(2\Theta_t)) = \left(\frac{g+1}{2}\right) + 1 \)
\item \( h^1(\mathcal{O}_S(2\Theta_t)) = 0 \)
\end{enumerate}

\textit{Proof.} (1) By the previous two lemmas we have the exact sequence

\[ 0 \to \mathcal{O}_S(2\Theta_t) \to \mathcal{O}_S(\Sigma(z_i + C)) \to \mathcal{O}_C(2\Sigma z_i) \to 0 \]

with \( \mathcal{O}_C(\Sigma z_i) \cong \omega_C(2t) \). Consider the restriction map

\[ \rho : H^0(\mathcal{O}_S(\Sigma(z_i + C)) \to H^0(\mathcal{O}_C(2\Sigma z_i)) \]

up to the isomorphism \( \psi \) given in lemma 4.7 \( \rho \) is just the multiplication map

\[ \mu_t : \text{Sym}^2 H^0(\omega_C(2t)) \to H^0(\omega_C^{\otimes 2}(4t)) \]

since \( \text{deg}(\Sigma z_i) \geq 2g + 1 \) \( \mu_t \) is surjective and \( \text{Ker} \mu_t \cong H^0(\mathcal{O}_S(2\Theta_t)) \). Then we can compute \( h^0(\mathcal{O}_S(2\Theta_t)) = \left(\frac{g+1}{2}\right) + 1 \).

(2) By Riemann–Roch \( h^0(\mathcal{O}_S(2\Theta_t)) = \left(\frac{g+1}{2}\right) + 1 + h^1(\mathcal{O}_S(2\Theta_t)) - h^0(\mathcal{O}_S(2\Theta_t)) \). By Serre duality \( h^2(\mathcal{O}_S(2\Theta_t)) = h^0(\mathcal{O}_S(K_S - 2\Theta_t)) \). It is easy to see that \( h^0(\mathcal{O}_S(K_S - 2\Theta_t)) = 0 \); indeed \( z + C \) is nef and \( (z + C) \cdot (K_S - 2\Theta_t) = -3 \). Hence, by (1), \( h^2(\mathcal{O}_S(2\Theta_t)) = h^1(\mathcal{O}_S(2\Theta_t)) = 0 \).

\textbf{(4.12) Corollary.} \( \hat{Q} \) is a vector bundle of rank \( \left(\frac{g+1}{2}\right) + 1 \) over \( \text{Pic}^2(C) \)

Note that this is also the rank of the fundamental bundle \( \mathcal{Q} \) constructed in section 3. Moreover we have the exact sequence

\[ 0 \to H^0(\mathcal{O}_S(2\Theta_t)) \to \text{Sym}^2 H^0(\omega_C(2t)) \to H^0(\omega_C^{\otimes 2}(4t)) \to 0 \]

where \( \mu_t \) is the multiplication map: this is obtained from the sequence 4.10 applying lemma 4.7. Since the fibre of \( \mathcal{Q} \) on \( t \) is \( \mathcal{Q}_t = \text{Ker} \mu_t \) we have a natural isomorphism

\[ \sigma_t : \hat{Q}_t \to Q_t \]

\textbf{(4.14) Proposition.} Up to tensoring by a line bundle there exists an isomorphism \( \sigma : \hat{Q} \to Q \) having as its fibrewise maps the previous isomorphisms \( \sigma_t (\forall t \in T) \).

Since this is essentially not needed we omit the proof. Composing the restriction map \( \lambda \) with \( \sigma \) we obtain the morphism of vector bundles

\[ \lambda = \sigma \circ \hat{\lambda} : H^0(\mathcal{O}_j(2\Theta)) \otimes \mathcal{T} \to \mathcal{Q} \]

From \( \lambda \) we construct the diagram

\[ \begin{array}{ccc}
X \times T & \xrightarrow{\theta \times \text{id}} & |2\Theta| \times T \\
\downarrow \theta \times \text{id} & & \downarrow \theta \\
|2\Theta| \times T & \xrightarrow{\Phi} & \mathcal{PQ}
\end{array} \]

where \( \theta : X \to |2\Theta| \) is the morphism associated to the generalized theta divisor and \( \theta \) is the projectivization of \( \lambda \).

This defines the rational map

\[ \Phi = \lambda \circ (\theta \times \text{id}_T) : X \times T \to \mathcal{PQ} \]

on the other hand we have from section 2 the fundamental map

\[ F : X \times T \to \mathcal{PQ} \]

The main step is now the following
(4.17) THEOREM. $F$ and $\Phi$ are the same map.

In order to show the theorem we need some preparation.

(4.18) LEMMA. Let $C \subseteq \mathbb{P}^r$ be a not degenerate smooth curve, $\text{Sec}(C)$ the variety of its bisecant lines. Then there is no quadric containing $\text{Sec}(C)$.

Proof. Assume there exists a quadric $Q$ containing $\text{Sec}(C)$ then $Q$ contains the linear span of $<d>$ for every $d \in C^{(3)}$ (indeed $<d>$ is a 3-secant plane or a 3-secant line to $C$: in the former case $Q$ contains three lines of $<d>$ hence $<d>$). Using the same argument one can show by induction on $k \geq 3$ that $Q$ contains the linear span of $<d>$ for every $d \in C^{(k)}$. Since $C$ spans $\mathbb{P}^r$ this implies $Q = \mathbb{P}^r$: obviously a contradiction.

Let

(4.19) $\text{Sec}(C_t) \subset \mathbb{P}^{g+2}_t$

be the variety of bisecant lines to $C_t$: $\forall x, y \in C$ we denote by $\overline{xy}_t$ the bisecant line to $C_t$ joining $x$ to $y$. We consider the incidence correspondence

$$\Sigma_t = \{(x + y, z) \in S \times \mathbb{P}^{g+2}_t/ z \in \overline{xy}_t\}$$

together with its two projections

(4.20) $S \leftarrow \Sigma_t \rightarrow \text{Sec}(C_t)$

Let

(4.21) $E = \{(x + y, z) \in \Sigma_t/ x = z \text{ or } y = z\}$

$E$ is a copy of $C \times C$ and it is contracted to $C_t$ by $\beta$.

(4.22) LEMMA.
(1) $\beta^*\mathcal{O}_{\text{Sec}(C_t)}(2) \otimes \mathcal{O}_\Sigma(-E) \cong \alpha^*\mathcal{O}_S(2\Theta_t)$
(2) Let $s \in H^0(\mathcal{O}_S(2\Theta_t)), s \neq 0$ then

$$\alpha^*\text{div}(s) - E = \beta^*Q$$

where $Q$ is the quadric defined by $\sigma_t(s)$.

Proof. Let $s \in \hat{Q}_t = H^0(\mathcal{O}_S(2\Theta_t)), s \neq 0$: at first we construct

(4.23) $q = \sigma_t(s)$

(up to a non zero constant factor). For this it suffices to use the proof of lemma 4.7 and to recall the definition of $\sigma_t$ as it has been given in 4.13: the sequence 4.10 defines an injection $H^0(\mathcal{O}_S(2\Theta_t)) \rightarrow H^0(\mathcal{O}_S(D))$. Let 

\[ r \]
be the image of $s$ in $H^0(O_S(D))$: using the quotient map $\pi : C \times C \to S$ we lift $r$ to a global section

$$b = \pi^* r \in H^0(\pi^* O_S(D))$$

As in the proof of lemma 4.7 we have the standard identifications

$$H^0(\omega_C(2t)) \otimes H^0(\omega_C(dt)) = H^0(\pi^* O_S(D))$$

and

$$\text{Sym}^2 H^0(\omega_C(2t)) = \pi^* H^0(O_S(D))$$

Hence we can view $b$ as a symmetric bilinear form on $H^0(\omega_C(2t))$: let $q$ be its associated quadratic form

$$q = \sigma_t(s)$$

To prove (1) and (2), it suffices to show that

$$\text{div}(\beta^* q_s) - E = \alpha^* \text{div}(s)$$

Let us check this equality set theoretically: by definition of $b$ we have

$$x + y \in \text{div}(r) \iff \langle x, y \rangle \text{ is an isotropic space for } b$$

(4.24)

where $\langle x, y \rangle \subset H^0(\omega_C(2t))$ denotes the vector space having as its projectivization the linear span of $x, y$ in $\mathbb{P}^{g+2}_t$. Furthermore $\text{div}(r) = \text{div}(s) + \Delta$ so that $b(x, x) = 0 \forall x \in C$ and, as it must be, $q$ vanishes on $C_t$; let $Q$ be the quadric defined by $q$ and

$$R = Q \cap \text{Sec}(C_t)$$

since $Q$ contains $C_t$ one can easily show that $R$ is union of bisecant lines $\mathfrak{f}_t$. Assume $x \neq y$ then, using the previous equivalence and $\text{div}(r) = \text{div}(s) + \Delta$, it follows

$$\mathfrak{f}_t \subset Q \iff b_s(\langle x \rangle, \langle y \rangle) \iff x + y \in \text{div}(s)$$

Hence

$$\beta^{-1}(R) = \alpha^{-1}(\text{div}(s)) \cup E$$

(4.25)

For a general $s$ $\text{div}(s)$ is reduced, hence $\alpha^* \text{div}(s) = \alpha^{-1}(\text{div}(s))$; moreover a standard computation in $\text{Num}_{\Sigma_t}$ shows that $\alpha^* \text{div}(s)$ is numerically equivalent to $\beta^* q - E$. From this and the set theoretical equality the proof follows.

Now we can give a
(4.26) PROOF OF THEOREM 5.20.

Proof. Let us recall that:

(1) $F$ is the composition of the following maps

$$X \times T \xrightarrow{\theta \times \text{id}_T} 2\Theta \vert \times \times T \xrightarrow{\lambda} \mathbb{P} \hat{Q} \xrightarrow{\sigma} \mathbb{P}^Q$$

(2) given $z \in X$ we have

$$\theta(z) = \Theta_\xi = \{e \in J/h^0(\xi(e)) \geq 1\}$$

where $\xi$ is any bundle with moduli point $z$.

Assume $(z,t)$ is in the domain of $F$, since $\hat{\lambda}_t$ is the restriction map $H^0(O_J(2\Theta)) \to H^0(O_{S_t}(2\Theta))$ it follows that

$$\hat{\lambda}(\theta(z),t)$$

is the curve

$$\Theta_{t,z} = \theta(z) \cdot S_t$$

let $\Theta_{z,t} = \text{div}(s)$, $q = \sigma_t(s)$, then

$$Q = F(z,t) \in \mathbb{P}Q_t$$

is the quadric defined by $q$. In particular it follows that:

$(z,t)$ is in the domain of $F$ $\iff$ $\Theta_{t,z}$ is well defined as a divisor in $S_t$ $\iff$ $\theta(z)$ does not contain $S_t$

Actually it is possible to show that these conditions are equivalent to $h^0(E) = 4$ and $q(E,V)$ not identically zero, where $E = \xi(t)$, $V = H^0(E)$; in any case, for a general $(z,t)$ in the domain of $F$, the latter conditions are satisfied by $\xi$. This gives to us a second quadric

$$Q' = Q(E,V)$$

which is defined defined by the pair $(E,V)$ as in section 1. We must show $Q = Q'$. Since no quadric contains $\text{Sec}(C_t)$ it suffices to show $Q \cdot \text{Sec}(C_t) = Q' \cdot \text{Sec}(C_t)$ that is

$$\beta^* Q - E = \beta^* Q' - E$$

At first we check this equality set theoretically: since $Q$ is defined by $\sigma_t(s)$ with $\text{div}(s) = \Theta_{t,z}$ we have from the previous lemma

$$\alpha^* \Theta_{t,z} - E = \beta^* Q$$

which means

$$x + y \in \Theta_{t,z} \iff \overline{xy} \subset Q$$

on the other hand

$$x + y \in \Theta_{z,t} \iff x + y \in \Theta_\xi \cap S_t \iff h^0(\xi(t - x - y)) \geq 1$$

by definition of $\Theta_\xi$. Now recall that $Q' = Q(E,V)$ so that, using the same notations of section 1, we have the linear projection

$$\delta_V : Q_1 \to G^*_V$$

such that $\delta_V/C_t$ is the Gauss map $g_V$. It is a standard exercise to check that

$$h^0(\xi(t - x - y)) \geq 1 \iff \delta_V(\overline{xy}) \subset G^*_V \iff xy \subset Q'$$

Therefore $Q \cap \text{Sec}(C_t) = Q' \cap \text{Sec}(C_t)$. To complete the proof it suffices to show that $Q \cap \text{Sec}(C_t) = Q \cdot \text{Sec}(C_t)$ which is true if $\Theta_{z,t}$ is reduced. Now, for a general $z$, $\Theta_\xi$ is reduced; moreover, by transversality of general translate, $\Theta_{\xi} \cap S_t$ is reduced too for a general $t$. 

5. Geometry of the fundamental involution: technical lemmas.

Let \( j : X \times T \to X \times T \) be the fundamental involution: from now on we will denote respectively by

\[
I_j \quad \text{and} \quad B_j
\]

the indeterminacy locus of \( j \) and the maximal (with respect to inclusion) open subset along which \( j \) is biregular. The proof of our main theorem relies on a weaker version of rigidity lemma which is given in the next section. This rigidity argument can be certainly applied if: for every \( o \in X - \Sing X \)

1. \( o \times T \cap I_j \) has codimension \( \geq 2 \) in \( T \)
2. \( o \times T \cap B_j \) is not empty.

Since we did not come to a complete proof of this statement, we found more convenient showing a weaker result which is sufficient for our purposes: this is the content of the next theorems 5.6, 5.7, 5.8. Let \( \xi \) be a semistable rank 2 vector bundle, \( \det \xi = \omega_C \), we define from \( \xi \) the following subsets of \( T \)

\[
\begin{align*}
V_{00}(\xi) &= \{ t \in T / \ h^0(\xi(t - x - y)) \geq 1 \ \text{for all} \ x, y \in C \}, \\
V_0(\xi) &= \{ t \in T / \ h^0(\xi(t - 2x)) \geq 1 \ \text{for all} \ x, \in C \} \quad (2) \\
V_m(\xi) &= \{ t \in T / \ h^0(\xi(t - mx)) \geq 4 - m \ \text{for some} \ x \in C \}, \ m = 1, 2, 3 \\
W(\xi) &= \{ t \in T / \ h^0(\xi(t - L)) \geq 1 \ \text{for some} \ L \in W_d^1, d \leq g \}
\end{align*}
\]

where \( W_d^1 \subset \Pic^d(C) \) is the Brill-Noether locus of line bundles \( L \) having degree \( d \) and \( h^0(L) \geq 2 \). In addition, considering the following subset of \( V_1(\xi) \cap V_3(\xi) \) will be quite important:

\[
V_{13}(\xi) = \{ t \in T / \ h^0(\xi(t - x)) \geq 3, \ h^0(\xi(t - 3x)) \geq 1 \ \text{for the same} \ x \in C \} \subset V_1(\xi) \cap V_3(\xi)
\]

By semicontinuity these sets are closed in \( T \), it is easy to see that they depend only on the moduli point \( [\xi] \) of \( \xi \); let

\[
I(o) = V_0(\xi) \cup V_{13}(\xi) \cup W(\xi)
\]

if \( [\xi] = o \) and let

\[
I(o_1 \ldots o_r) = I(o_1) \cup \cdots \cup I(o_r)
\]

for points \( o_1 \ldots o_r \in X \)

(5.5) **DEFINITION.** \( I(o_1, \ldots, o_r) \) is the special set of \( \{o_1 \ldots o_r\} \).

(5.6) **THEOREM.** Let \( o, \overline{r} \in X \), assume: (1) \( o \times T \cap B_j \) is not empty; (2) \( j(o \times T) \subset \overline{r} \times T \). Then

\[
I_j \cap o \times T \subseteq o \times I(o, \overline{r})
\]

and the same holds for \( \overline{r} \).
(5.7) **THEOREM.** Let \( o = [\xi] \in X \) then \( \text{codim} I(o) \geq 2 \) unless the pair \((C, \xi)\) satisfies one of the following exceptional conditions:

1. \( o \in \text{Sing} X \) i.e. \( \xi \) is not stable
2. \( C \) is hyperelliptic
3. there exists a double covering \( \pi : C \to Y \) of an elliptic curve and \( \xi = \pi^* \eta \otimes L \) where \( L^2 = \omega_C \otimes \text{det}(\pi^* \eta^*) \) and \( \eta \) is the unique (up to twisting by a degree zero line bundle) irreducible rank two vector bundle of degree 1.
4. \( C \) is a smooth quartic curve in \( \mathbb{P}^2 \), \( \xi = T_{\mathbb{P}^2}(-1) \otimes \mathcal{O}_C(e) \), \((2e \sim 0)\).

The next result is what we need in the proof of the main theorem: it is an immediate consequence of the previous statements 5.6 and 5.7.

(5.8) **THEOREM.** Let \( C \) be not hyperelliptic of genus \( \geq 4 \), \( o \) a point of \( X - \text{Sing} X \), \( o \) not the moduli point of the previous bundle \( \pi^* \eta \otimes L \) if \( C \) is bielliptic. Assume \( j \) is generically defined on \( o \times T \) and moreover that

\[ p_1 \cdot j : o \times T \to X \]

extends to a constant map, where \( p_1 : X \times T \to T \) is the first projection. Then

\[ I_j \cap o \times T \]

has codimension \( \geq 2 \) in \( T \).

The first theorem we want to show is 5.7. Let \( \xi \) be as above, we need to consider one closed set more:

(5.9)

\[ H(\xi) = \{ l \in \text{Pic}^1(C)/h^0(\xi(l)) \geq 3 \} \]

it is clear that

\[ l \in H(\xi) \iff -C_l \cup C_l \subset \Theta_\xi \]

where \( C_l = \{ l - x, ~ x \in C \} \)

(5.10) **LEMMA.** \( g - 3 \leq \text{dim} H(\xi) \leq g - 2 \)

**Proof.** Let \( \alpha : \text{Pic}^1(C) \times C \to \text{Pic}^0(C) \) be the difference map, consider the natural projection \( p : \alpha^* \Theta_\xi \to \text{Pic}^1(C) \) and notice that \( p^*(l) = C_l \cdot \Theta_\xi \) for each \( l \in \text{Pic}^1(C) \). Then \( p \) is generically finite on each irreducible component of \( \alpha^* \Theta_\xi \) and the locus of points \( l \) where the fibre of \( p \) is not finite is \( H(\xi) \); therefore \( \text{dim} H(\xi) \leq g - 2 \). On the other hand, if \( H(\xi) \cap C(3) - t \) is not empty for any \( t \in T \) it follows from transversality of general translate and the ampleness of the cycle \( C(3) - t \) that \( g - 3 \leq \text{dim} H(\xi) \). It is known ([G]) that, for at least one effective divisor \( d \) of degree 3, one has \( h^0(\xi(t - d)) \geq 1 \). Putting \( l = d - t \) and applying Riemann-Roch it follows \( h^1(\xi(-l)) = h^0(\xi(l)) \geq 3 \). Hence \( d - t \in H(\xi) \cap C(3) - t \neq \emptyset \).

(5.11) **LEMMA.** \( V_{00}(\xi) \) and \( V_m(\xi), m = 0 \ldots 3 \), are proper closed subsets of \( T \), moreover:

1. \( V_{00}(\xi) \subset V_0(\xi), \text{codim} V_0(\xi) \geq 2, \text{codim} V_{00}(\xi) \geq 3 \)
2. \( \text{codim} V_1(\xi) \geq 1 \)
3. \( \text{codim} V_2(\xi) \geq 2 \) if \( \Theta_\xi \) is reduced
4. \( \text{codim} V_3(\xi) \geq 1 \)

**Proof.** (1) observe that \( t \in V_{00}(\xi) \) if and only if the surface \( S_t = \{ t - x - y, ~ x, y \in C \} \) is in \( \Theta_\xi \). Fix a point \( e \) not in \( \Theta_\xi \) and consider in \( T \) the surface \( S_e = \{ e + x + y, ~ x, y \in C \} \). Since \( S_e \) is ample and
$S_e \cap V_{00}(\xi) = \emptyset$ it follows $\text{codim} V_{00}(\xi) \geq 3$. Replacing the surface $S_e$ by the curve $\Delta_e = \{e + 2x, \ x \in C\}$ and repeating word by word the same argument one shows $\text{codim} V_6(\xi) \geq 2$.

(2): by definition $t \in V_1(\xi)$ if and only if $t = l + x$ where $x \in C$ and $l \in H(\xi)$. Considering the sum map $\beta : H(\xi) \times C \rightarrow T$, we have $V_1(\xi) = \beta(H(\xi) \times C)$. Since $\text{codim} H(\xi) \geq 2$ it follows $\text{codim} V_1(\xi) \geq 1$.

(3): by a theorem of Laszlo [L3] $h^0(\xi(t - 2x)) \geq 2$ implies $t - 2x \in \text{Sing} \Theta_\xi$. Therefore $V_2(\xi) \subseteq \beta(\text{Sing} \Theta_\xi \times \Delta)$ where $\beta$ is the sum map and $\Delta \subset C(2)$ the diagonal. The result follows immediately.

(4): observe that $t \in V_3(\xi)$ if and only if $t = 3x - l$ where $x \in C$ and $h^0(\xi(-l)) \geq 1$. The latter condition is equivalent to $h^0(\xi(l)) \geq 3$ so that $l \in H(\xi)$. Consider the map $\gamma : H(\xi) \times C \rightarrow T$ which is so defined: $\gamma(l, x) = 3x - l$; then $V_3(\xi) = \gamma((H(\xi) \times C)$. Since $\text{codim} H(\xi) \geq 2$ it follows $\text{codim} V_3(\xi) \geq 1$.

(5.12) **Lemma.** Let $Z$ be an irreducible component of $V_{13}(\xi)$. Assume $Z$ is not contained in $W(\xi)$, then

$$\text{codim} Z \geq 2$$

**Proof.** Let $U = Z - (W(\xi) \cap Z)$, we consider in $U \times C$ the closed set

$$\tilde{U} = \{(t, x) \in U \times C/ h^0(\xi(t - x)) \geq 3 \text{ and } h^0(\xi(t - 3x)) \geq 1\}$$

Under the difference map $\alpha : \tilde{U} \longrightarrow \text{Pic}^1(C)$, $(\alpha(t, x) = t - x)$, we have $\alpha(\tilde{U}) \subseteq H(\xi)$ hence $\alpha(\tilde{U})$ is at most $g - 2$-dimensional by lemma 5.10. On the other hand the fibre of $\alpha$ is either the curve $C$ or a finite set. Let $Y$ be any irreducible component of $\alpha(\tilde{U})$ having maximal dimension $g - 2$: if, for $l$ general in $Y$, $\alpha^{-1}(l)$ is finite the result follows. Let

$$Y_n = \{l \in Y/ h^0(\xi(l)) \geq n\}$$

it is obvious that $Y_3 = Y$, moreover $Y_n$ is closed in $Y$. We first assume that $Y_4$ is a proper subset so that $h^0(\xi(l)) = 3$ for $l$ general. Consider the determinant map

$$d : \wedge^2 H^0(\xi(l)) \longrightarrow H^0(\omega_C(2l))$$

and assume $\alpha^{-1}(l)$ is not finite, then $\alpha^{-1}(l) = \{(l + x, x) \in C\}$ so that

$$h^0(\xi(l + x - 3x)) = h^0(\xi(l - 2x)) \geq 1 \forall x \in C$$

This is impossible if $d$ is injective because then, in the 2-dimensional linear system defined by $\text{Im} d$ we would have a pencil of divisors containing $2x$ for each $x$ in $C$. Hence there exist two independent vectors $s_1, s_2 \in H^0(\xi(l))$ such that $d(s_1 \wedge s_2) = 0$ and they define a subline bundle $L$ of $\xi(l)$ with $h^0(L) \geq 2$, $\text{deg} L \leq g$. Since $L$ is also a subline bundle of $\xi(l + x)$ it follows $l + x \in W(\xi), \forall x \in C$: hence $l$ cannot be in $\alpha(\tilde{U})$, a contradiction.

Secondly we assume $Y = Y_4$. Note that, fixing a point $x \in C$, $x + Y_3$ is contained in $V_{00}(\xi)$ which is at most $g - 3$-dimensional by the previous lemma. Hence, generically on $Y$, $h^0(\xi(l)) = 4$. Assume $\alpha^{-1}(l)$ is not finite for a general $l \in Y$ then

$$h^0(\xi(l - 2x)) \geq 1 \text{ for all } x \in C \text{ and moreover } h^0(\xi(l)) = 4$$

to complete the proof we consider the surface $R = P(\xi(l)^*)$ together with its tautological bundle $H$ (i.e. $p_* H = \xi(l)$) and the rational map

$$f_H : R \longrightarrow P^3 = PH^0(\xi(l))^*$$
associates to \( H, R_x = p^{-1}(x) \). Let \( p : R \to C \) be the obvious projection, \( R_x = p^{-1}(x) \), we recall that 
\[
h^0(\xi(l-mx)) = h^0(\mathcal{O}_R(H-mR_x)).
\]
There are two cases to be considered:

CASE (1) \( h^0(\mathcal{O}_R(H-mR_x)) \geq 2 \) for all \( x \in C \) and a fixed \( m \geq 2 \).

Observe that the tangent map \((df)_l\) for \( \dim f \) CASE (2) with \( h \) 
Assuming this and keeping \( h \) associated to \( z \in R_x \): therefore \( f_H(R) \) cannot be 2-dimensional in this case. Since \( \dim f_H(R) \leq 1 \) there exists a subline bundle \( L \) of \( \xi(l) \) with \( h^0(L) = 4 \) \( f_H(R) \). Hence \( l + x \in W(\xi), \forall x \in C \) and \( l \) cannot belong to \( \alpha(\tilde{U}) \).

CASE (2) \( \dim f_H(R) = 2 \) and \( h^0(\mathcal{O}_R(H-2R_x)) = 1 \) for a general \( x \in C \).

Assuming this and keeping \( x \) general we observe that there exists a unique element \( H_x \in | H-2R_x |, \) moreover \( | H_x - R_x | \) is (generically) empty: otherwise, fixing the point \( x \), we would have \( 3x - l \in V_{00}(\xi) \) for \( l \) general, hence \( \dim Y_4 = g - 3 \). From these two remarks it follows that the pencil \( | H-R_x | \) has a unique base point \( b(x) \) on the fibre \( R_x \). This defines the holomorphic section 
\[
b : C \to R
\]
sending \( x \) to \( b(x) \) and the curve \( B = b(C) \). Let \( i(x) \) be the intersection index of \( B \) and \( H \) at \( b(x) \): assume 
| \( H \) | has no base points on \( R_x \) then, by the construction of \( B \), 
\[
i(x) \geq 2 \quad \text{if and only if } H \text{ contains } R_x
\]
This implies that either \( f_H(R_x) \) is a tangent line to the curve \( f_H(B) \) or \( f_H(R) \) is a cone of vertex the point \( f_H(B) \). In any case fix a general pencil \( P \subset | H | \) and consider its restriction \( P_B \) to \( B \): if \( d \) is the degree of the moving part of \( P_B \) one computes from the previous equivalence and Hurwitz formula: \( 2d - H^2 \leq 2 - 2g \), hence \( d \leq 1 \) because \( H^2 = \deg(\xi(l)) = 2g \). Since this is impossible \( P_B \) has no moving part and an element of \( P \) must contains \( B \); moreover \( \dim | H-B | \geq 3 \) and \( f_H(R) \) is a cone. Therefore there exists a subline bundle \( L \) of \( \xi(l) \) with \( h^0(L) \geq 3 \) and one completes the proof as in case (1).

The next lemma is standard, we omit for brevity its proof

(5.13) LEMMA: (1) Let \( Y \subset \text{Pic}^n(C) \) be a closed subset, \( \alpha : C^{(n)} \times Y \to J = \text{Pic}^0(C) \) the difference map, \( Z = \alpha(C^{(n)} \times Y) \). If \( \dim Z < g \) then \( \alpha \) is generically finite onto \( Z \). In particular:
(2) Let \( Y = \{ l \in \text{Pic}^n(C)/C^{(n)} - l \subseteq Z \} \) where \( Z \) is a proper closed subset of \( \text{Pic}^0(C) \). Then \( \dim Y \leq \dim(Z) - n \).

The next result is an application of Martens’ theorem:

(5.14) THEOREM. For a semistable rank two vector bundle \( \xi \) of determinant \( \omega_\xi \)-let 

\[
(5.15) W^1_n(\xi, i) = \{ l \in \text{Pic}^i(C)/ \text{there exists } L \in \text{Pic}^n(C) \text{ such that } h^0(\xi(l) \otimes L^{-1}) \geq 1, h^0(L) \geq 2 \}
\]

assume \( 0 \leq i \leq n < g \) then:
(1) if \( C \) is not hyperelliptic \( \dim W^1_n(\xi, i) \leq g + i - 4 \)
(2) if \( C \) is hyperelliptic \( \dim W^1_n(\xi, i) = g + i - 3 \) for \( i \leq 3 \)
Assume \( 0 \leq i \leq n = g \) then:
(3) \( \dim W^1_n(\xi, i) \leq g + i - 3 \)

Proof. Let \( W^1_n \subset \text{Pic}^n(C) \) be the Brill-Noether locus of line bundles \( L \) with \( h^0(L) \geq 2 \). We can construct in \( W^1_n \times \text{Pic}^i(C) \) the closed subset 
\[
B = \{ (L, l) \in W^1_n \times \text{Pic}^i(C)/h^0(\xi(l) \otimes L^{-1}) \geq 1 \}
\]
considering the two projections
\[
W_n^1 \xleftarrow{p_1} B \xrightarrow{p_2} \text{Pic}^i(C)
\]
it is clear that
\[
W_n^1(\xi, i) = p_2(B)
\]
On the other hand, fix \( L \in p_1(B) \) and consider the fibre
\[
F_L = p_1^{-1}(L)
\]
Let \( l \in F_L, m = L(-l) \) then \( h^0(\xi(-m)) \geq 1 \) so that \( h^0(\xi(d - m)) \geq 1 \) for all \( d \in C^{(n-i)} \) and
\[
C^{(n-i)} - m \subseteq \Theta
\]
In particular \( F_L \subseteq \{ m \in \text{Pic}^{(n-i)}/C^{n-i} - m \subseteq \Theta \} \), hence \( \text{dim}(F_L) \leq (g - 1) - (n - i) \) by the previous corollary; on the other hand, by Martens theorem, \( \text{dim}W_n^1 \leq n - 3 \) if \( C \) is not hyperelliptic. Therefore
\[
\text{dim}W_n^1(\xi, i) \leq \text{dim}B \leq \text{dim}W_n^1 + \text{dim}F_L \leq g + i - 4
\]
We leave as an exercise the hyperelliptic case. The proof of the case \( n = g \) is exactly the same: of course in this case \( \text{dim}W_g^1 = g - 2 \) so that \( \text{dim}W_g^1(\xi, i) \leq g + i - 3 \).

Finally, from a theorem of Lange and Narasimhan [LN], we obtain

\[5.16\] \textbf{PROPOSITION.} \( \text{Codim} \text{W}(\xi) \geq 2 \) unless:

(1) \( \xi \) is not stable

(2) \( C \) is hyperelliptic

(3) there exists a double covering \( \pi : C \to Y \) of an elliptic curve and \( \xi = \pi^*\eta \otimes L \) where \( L^2 = \omega_C \otimes \det \pi^*\eta \) and \( \eta \) is the unique irreducible rank two vector bundle of degree 1, (up to twisting by a degree zero line bundle).

(3) \( C \) is a smooth quartic curve in \( \mathbb{P}^2 \), \( \xi = T_{\mathbb{P}^2}(-1) \otimes \mathcal{O}_C(e), 2e \sim 0 \).

\textbf{Proof.} Using the previous notations we have \( \text{W}(\xi) = \bigcup W_n^1(\xi, 2), \ 2 \leq n \leq g \)
By the previous theorem 5.14 \( \text{codim}W_n^1(\xi, 2) \geq 2 \) if \( C \) is not hyperelliptic and \( n \leq g - 1 \). So we have only to consider the case \( n = g \): by definition \( t \in W_g^1(\xi, 2) \) iff there exists \( L_t \in W_g^1 \) such that \( L_t(-t) \) is a subbundle of \( \xi \). A simple dimension count shows that if \( \text{codim}W_g^1(\xi, 2) = 1 \) then \( \xi \) has (at least) a 1 dimensional family of such subline bundles of degree \( g - 2 \). If \( \xi \) is stable they are of maximal degree \( g - 2 \), hence by a theorem of Lange-Narasimhan \( \xi \) ([LN] thm. 5.1), \( \xi \) is as in cases (3) or (4). If \( \xi \) is semistable not stable it is easy to see that \( \text{codim}W(\xi) = 1 \).

\textbf{PROOF OF THEOREM 5.7:} this is now an immediate consequence of the previous results: let \( o = [\xi] \in X \), by lemma 5.12 and 5.13 the only component of \( I(o) \) having possibly codimension 1 is the "Brill-Noether locus" \( W(\xi) \). On the other hand, by the previous theorem and proposition 5.16, one has, with the prescribed exceptions, \( \text{codim}W(\xi) = 2 \). Hence the same holds for \( I(o) \).

In order to show theorem 5.6 we need some preparation. In particular we want to give the following

\textbf{ALTERNATIVE DEFINITION OF j:}
Let \((o, t) \in X \times T, \quad o = [\xi]\)

we denote by
\[ e : H^0(\xi(t)) \otimes O_C \rightarrow \xi(t) \]

the evaluation map and by
\begin{equation}
U^*
\end{equation}

the image sheaf \(Im(e)\). Then we consider the two projections
\begin{equation}
C \leftarrow^{p_1} C \times C \rightarrow^{p_2} C
\end{equation}

and the diagonal
\[ \Delta \subset C \times C \]

Tensoring
\[ 0 \rightarrow \mathcal{O}_{C \times C}(-2\Delta) \rightarrow \mathcal{O}_{C \times C}(-\Delta) \rightarrow \mathcal{O}_{\Delta}(-\Delta) \rightarrow 0 \]

by
\begin{equation}
\mathcal{F} = p_1^*U^*
\end{equation}

and applying the \(p_{2*}\) functor, we obtain the long exact sequence
\begin{equation}
0 \rightarrow p_{2*}\mathcal{F}(-2\Delta) \rightarrow p_{2*}\mathcal{F}(-\Delta) \rightarrow R^1p_{2*}\mathcal{F}(-2\Delta) \rightarrow R^1p_{2*}\mathcal{F}(-\Delta) \rightarrow 0
\end{equation}

Before of using it we fix the following open set in \(X \times T\)
\textbf{(5.21) DEFINITION.}
\[ U_j' = \{(o, t) \in X \times T / \quad t \text{ is not in } V_0(\xi) \cup V_{13}(\xi), \quad o = [\xi]\} \]

let \((o, t) \in U_j', \) since \(t\) is not in \(V_0(\xi)\) it follows:

(1) \(h^0(\xi(t)) = 4\)

(2) the evaluation map \(e : V \otimes O_C \rightarrow \xi(t)\) is generically surjective, where \(V = H^0(\xi(t))\).

This defines the exact sequence
\begin{equation}
0 \rightarrow \mathcal{U} \xrightarrow{e^*} V^* \otimes O_C \rightarrow \overline{\mathcal{U}} \rightarrow 0
\end{equation}

where both
\[ \mathcal{U} = Im(e^*) = Im(e)^* \quad \text{and} \quad \overline{\mathcal{U}} \]

are rank two vector bundles. Considering the Grassmanian \(G\) of 2-spaces in \(V^*\) and the Gauss map 1.5
\[ g : C \rightarrow G \]

of the pair \((V, \xi(t))\) it turns out that the previous sequence is the pull back by \(g\) of the universal/quotient bundle sequence on \(G\) (see section 1)
\[ 0 \rightarrow \mathcal{U}_V \rightarrow V^* \otimes O_C \rightarrow \overline{\mathcal{U}}_V \rightarrow 0 \]
(5.23) **Lemma.** If \((o, t) \in U_j^r\) the sequence (5.20) becomes

\[
0 \longrightarrow \overline{U}^r \longrightarrow U^r \otimes \omega_C \longrightarrow R^1 p_{2*} \mathcal{F}(-2\Delta) \longrightarrow R^1 p_{2*} \mathcal{F}(-\Delta) \longrightarrow 0
\]

In particular this induces the exact sequence

\[
(5.24) \quad 0 \longrightarrow \overline{U}^r \longrightarrow U^r \otimes \omega_C \longrightarrow D \longrightarrow 0
\]

where \(D = \text{Coker}u\) is an \(O_C\)-module of finite length. \(D\) is supported exactly on points \(x\) such that \(h^0(U^r(-2z)) \geq 1\).

**Proof.** Let \(z \in C\): by definition, tensoring (5.20) by \(O_{C,z}/m_z\), we obtain the long exact sequence

\[
0 \longrightarrow H^0(U^r(-2z)) \longrightarrow H^0(U^r(-z)) \longrightarrow u_z^* U^r_x \otimes \omega_C \longrightarrow \]

\[
\longrightarrow H^1(U^r(-2z)) \longrightarrow H^1(U^r(-z)) \longrightarrow 0
\]

Since \(U^r\) is globally generated and \(H^0(U^r) = V\) it follows that \(H^0(U^r(-z))\) is of constant dimension 2. Since \(H^0(U^r(-z))\) is the fibre of \(p_{2*} \mathcal{F}(-\Delta)\) at \(z\) this latter sheaf is a rank two vector bundle. Furthermore the universal/quotient bundle sequence

\[
0 \rightarrow \overline{U}^r \rightarrow V \otimes O_C \rightarrow U^r \rightarrow
\]

yields the canonical isomorphims \(\sigma_z : \overline{U}^r \rightarrow H^0(U^r(-z)), z \in C\). This implies that \(p_{2*} \mathcal{F}(-\Delta)\) is isomorphic to \(\overline{U}^r\).

Note that \(O_{\Delta(-\Delta)} \cong \omega_C \cong p^*_2 \omega_C\). Therefore, by projection formula,

\[
p_{2*} \mathcal{F} \otimes O_{\Delta(-\Delta)} = p_{2*}(p_1^* U^* \otimes O_{\Delta}) \otimes p^*_2 \omega_C = U^* \otimes \omega_C
\]

Since \(t\) is not in \(V_0(\xi)\) we have \(h^0(U^r(-2z)) = h^0(\xi(t - 2z)) = 0\) for a general \(z\). This implies that \(p_{2*} \mathcal{F}(-2\Delta) = 0\) and that \(\text{Coker}u\) is supported on points \(z\) such that \(h^0(U^r(-2z)) > 0\).

Since \(U^r = \text{Im}(e)\) we have the exact sequence

\[
(5.25) \quad 0 \longrightarrow U^r \longrightarrow \xi(t) \longrightarrow C \longrightarrow 0
\]

with

\[
(5.26) \quad C = \text{Coker}(e)
\]

Both

\[
D = \text{Supp} D \quad \text{and} \quad Z = \text{Supp} C
\]

are divisors in \(C\), in particular

\[
C = O_Z, \quad D = O_D
\]

We want to point out that

\[
(5.27) \quad \text{Supp} Z \cap \text{Supp} D = \emptyset \quad \text{so that} \quad D \otimes C = 0
\]
this is due to our choice of \((o, t)\) in \(U_j\) because then \(t\) is not in \(V_{13}(\xi)\). Another remark is that, whenever \(\xi(t)\) is globally generated and \(U_j\) semistable, then \(\det U_j(-t) \cong \omega_C\) and moreover
\[
j(o, t) = (\sigma, t)
\]
for the moduli point \(\sigma\) of \(U_j(-t)\): this just follows from the definition of \(j\). We want to use the previous constructions to give a partial extension of \(j\) to points \((o, t)\) such that \(\xi(t)\) is not globally generated. With this purpose we consider the exact sequence
\[
0 \longrightarrow \xi(t)^* \longrightarrow U \xrightarrow{\epsilon} C \longrightarrow 0
\]
which is the dual of 5.25 and note that the map \(\epsilon\) is an element of
\[
U^* \otimes C = Hom(U, C)
\]
Then we tensor by \(C\) the sequence
\[
0 \longrightarrow \xi(t)^* \longrightarrow U \xrightarrow{\epsilon} C \longrightarrow 0
\]
constructed in 5.24. Since \(C \otimes D = 0\) we obtain
\[
U^* \otimes C \otimes \omega_C \longrightarrow U^* \otimes C \longrightarrow C \otimes D \otimes \omega_C = 0
\]
and hence a natural isomorphism
\[
\sigma : \xi(t)^* \otimes \omega_C \longrightarrow U^* \otimes C
\]
Let
\[
\tau = \sigma^{-1}(\epsilon)
\]
then \(\tau \in Hom(U \otimes C, C)\). Since \(\epsilon\) is an epimorphism it is easy to see that \(\tau\) is an epimorphism too. This defines the rank two vector bundle
\[
\text{Ker} \tau
\]
such that \(\det(\text{Ker} \tau) = \det \xi(t) \otimes \omega_C^2\). Then we fix the following
\[(5.28)\text{DEFINITION.}\] \(\text{Ker} \tau \otimes \omega_C\) will be always denoted by \(\phi_t\)
and briefly sketch the conclusion: the construction of \(\text{Ker} \tau\) easily extends from one pair to a family of pairs \((\xi, t)\) such that \((|\xi|, t) \in U_j\). Then it is standard to deduce that there exists a rational map
\[(5.29)\]
\[
j' : X \times T \longrightarrow X \times T
\]
which is so defined on \(U_{j'}\):
\[
\forall (|\xi|, t) \in U_{j'} \quad j'(|\xi|, t) = (|\phi_t|, t)
\]
provided \(\phi_t\) is semistable. From the definitions of \(j\) and \(j'\) it follows
\[
j(o, t) = j'(o, t) \quad \text{on } U_{j'} \cap U_j
\]
where \(U_j\) is the open set defined in section 2. Therefore \(j'\) and \(j\) define the same birational involution.
Let us recall two elementary facts:

(5.29) REMARKS
(1) If $\xi(t)$ is globally generated $C = \text{Coker}(e) = 0$. Hence $\epsilon = \tau = 0$ and $\overline{\phi}_t(t) = \overline{U} = \text{Ker}(e)^*$. 
(2) We have obvious canonical inclusions

$$V^* \subset H^0(\overline{U}) \subseteq H^0(\overline{\phi}_t(t))^*$$

using the standard arguments of section 1 and the exact sequence

$$0 \rightarrow \mathcal{U} \rightarrow V^* \otimes \mathcal{O}_C \rightarrow \overline{U} \rightarrow 0$$

it is easy to see that the pairs $(V, \xi(t))$ and $(V^*, \overline{\phi}_t(t))$ define the same quadric $Q$ containing the embedded curve $C_t \subset P_{g+2}$:

$$Q = Q(V, \xi(t)) = Q(V^*, \overline{\phi}_t(t))$$

**PROOF OF THEOREM 5.6.**

*Proof.* Let $o = [\xi]$, at first we assume that $\xi$ is *stable*. The assumption of the theorem we want to prove is that

$$j(o, t) = (\overline{\xi}, t)$$

for a general $t$ and a fixed point

$$\overline{\xi} = [\overline{\xi}]$$

As it is easy to show (see next lemma), for each point $u$ corresponding to a semistable not stable bundle and $t$ general $j(u, t) = (u, t)$; therefore $\overline{\xi}$ is stable too. Let $I(o, \overline{\xi}) = I(o) \cup I(\overline{\xi})$ be the special set of $\{o, \overline{\xi}\}$ defined in 5.5, we must show that the open subset

$$U = T - I(o, \overline{\xi})$$

is in the domain of $j$. Let $t \in U$, from the previous construction of $j'$ we already know that

$$j'(o, t) = j(o, t) = ([\overline{\phi}_t], t)$$

as soon as the rank two vector bundle $\overline{\phi}_t$ is semistable. To show that $\overline{\phi}_t$ is semistable we fix a Poincaré bundle on $C \times T$ and construct in a standard way a sheaf

$$\mathcal{R} \rightarrow C \times T$$

such that

$$\mathcal{R}_t = \mathcal{R} \otimes \mathcal{O}_{C \times t} = \overline{\phi}_t(t)$$

for all $t \in U$. On the other hand, for a *general* $t \in U$, we have

$$\overline{\xi}(t) \cong \overline{\phi}_t(t)$$
To see this take \( t \) in the complement of \( I(o,\overline{\pi}) \cup V_1(\xi) \cup V_1(\overline{\xi}) \) and such that \( (o,t) \) is in the domain of \( f \). Under this choice both \( \xi(t) \) and \( \overline{\xi}(t) \) are globally generated and moreover the sequence

\[ 0 \longrightarrow \overline{\xi}(t)^* \longrightarrow H^0(\xi(t)) \otimes \mathcal{O}_C \longrightarrow \xi(t) \longrightarrow 0 \]

is exact. Since \( \xi(t) \) is globally generated it is clear that the construction of \( \overline{\phi}_t(t) \) gives exactly \( \overline{\xi}(t) \), (cfr. remark 5.29(1)). By semicontinuity the condition \( \overline{\xi}(t) \cong \mathcal{R}_t \) for a general \( t \) implies

\[ h^0(\mathcal{R}_t \otimes \overline{\xi}(t)^*) \geq 1 \]

for all \( t \in T \). Equivalently, there exists a non zero morphism

\[ \rho_t : \mathcal{R}_t \longrightarrow \overline{\xi}(t) \]

for each \( t \in T \). Assume \( t \in U \) and \( \mathcal{R}_t = \phi_t(t) \) unstable, then \( \rho_t \) cannot be an isomorphism; since \( \phi_t, \overline{\xi} \) have the same slope and the latter is stable we have an exact sequence

\[ 0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{O}_t(t) \longrightarrow \mathcal{B} \longrightarrow 0 \]

where

\[ \mathcal{A} = Ker(\rho_t) \text{ and } \mathcal{B} = Im(\rho_t) \]

are line bundles and \( \mathcal{B} \) is a subline bundle of \( \overline{\xi}(t) \) of degree \( \leq g \). To obtain a contradiction we show that

\[ h^0(\mathcal{B}) \geq 2 \]

because then \( h^0(\mathcal{B}) \geq 2 \Rightarrow t \in W(\overline{\xi}) \subset I(o,\overline{\pi}) \Rightarrow t \) is not in \( U \). By construction (see section 1 and remark 5.29) we have

\[ q(V,\xi(t)) = q = cq(V^*,\phi_t(t)) \]

\( (c \in \mathbb{C}^*) \) for the pairs \((V,\xi(t))\) and \((V^*,\overline{\phi}_t(t))\); the quadratic form \( q \) is vanishing on the embedded curve \( C_t \). If \( h^0(\mathcal{B}) \leq 1 \) then \( H^0(\mathcal{A}) \) cuts on \( V^* \) a subspace of codimension \( \leq 1 \) and hence \( q \) is identically zero by prop. 1.11. Since \( q \) is also defined by \((V,\xi(t))\) the same proposition implies that \( \xi(t) \) has a subline bundle \( \mathcal{C} \) with \( h^0(\mathcal{C}) \geq 3 \): this is impossible because \( t \) is in \( U \), hence not in \( W(\xi) \). Therefore \( h^0(\mathcal{B}) \geq 2 \).

It remains to consider the semistable case: this is done in the next lemma.

\textbf{(5.30) LEMMA.} Let \( o \in Sing(X) \) then:

(1) \( I_j \cap o \times T \subset I(o) \)

(2) \( j(o,t) = (o,t) \) for \( t \in T - I(o) \)

\textbf{Proof.} Since \( o \in SingX \) \( o \) is the moduli point of a family of \( (S\text{-equivalent, see [B1]}) \) semistable not stable vector bundles. Exactly one of them is split: let

\[ \xi = M \oplus N \]

such a bundle: if \( t \in V_1(\xi) \) then \( |M(t)| \) \( (\text{or } |N(t)|) \) has a base point \( x \) and \( L = M(t-x) \) is a subline bundle of \( \xi(t) \) with \( h^0(L) \geq 2, \deg L \leq g \). Hence \( t \in W(\xi) \). Conversely, if \( t \in W(\xi) \), it is easy to see that \( \xi(t) \) is not globally generated so that \( t \in V_1(\xi) \). Therefore \( V_1(\xi) = W(\xi) \) so that

\[ I(o) = W(\xi) \cup V_0(\xi) \]

If \( t \in T - I(o) \), one has immediately the exact sequence

\[ 0 \longrightarrow M(t)^* \oplus N(t)^* \longrightarrow H^0(M(t)) \otimes \mathcal{O}_C \oplus H^0(N(t)) \otimes \mathcal{O}_C \longrightarrow M(t) \oplus N(t) \rightarrow \]

which implies \( j(o,t) = (o,t) \). This completes the proof.

\textbf{(5.31) REMARK} Using the previous lemma and our main theorem one could deduce that \( I_j \cap o \times T = W(\xi) \) if \([\xi] \in SingX \) and \( C \) is not hyperelliptic. Note also that \( W(\xi) \) is a divisor if \([\xi] \in SingX \).
6. An application of Rigidity lemma.

In this section we fix

(6.1) \( X = \) integral quasi projective variety

and

(6.2) \( \mathcal{L} = \) ample, globally generated line bundle on \( X \)

so that the map associated to \( \mathcal{L} \)

(6.3) \( \theta : X \to \mathbb{P}^N = \mathbb{P}(\mathcal{H}^0\mathcal{L})^* \)

is a finite morphism. We want to check whether \( \mathcal{L} \) is very ample by composing \( \theta \) with the elements of a \textit{complete} family of linear projections of \( \mathbb{P}^N \). More precisely this means that we fix

\[ T = \text{integral projective variety} \]

and a non zero map of vector bundles over \( T \)

(6.4) \( \lambda : \mathcal{H}^0(\mathcal{L})^* \otimes \mathcal{O}_T \to \mathcal{Q} \)

Denoting by

(6.5) \( \mathcal{X} : \mathbb{P}^N \times T \to \mathbb{P}(\mathcal{Q}) \)

the induced map of projective bundles and composing it with

(6.6) \( \theta \times id_T : X \times T \to \mathbb{P}^N \times T \)

we obtain a rational map

(6.7) \( \Phi = \mathcal{X} \cdot (\theta \times id_T) : X \times T \to \mathbb{P}(\mathcal{Q}) \)

Clearly, for each \( t \in T \), the restriction

\[ \Phi_t : X \times t \to \mathbb{P}(\mathcal{Q}_t) \]

of \( \Phi \) to \( X \times t \) is just the composition of \( \theta \) with the linear projection

\[ \mathcal{X}_t : \mathbb{P}^N \to \mathbb{P}(\mathcal{Q}_t). \]

In principle, when \( X \) is projective, one can try to show the very ampleness of \( \mathcal{L} \) by checking if any pair of points (of tangent vectors) can be separated by at least one \( \Phi_t \). Under some suitable assumptions such a family of rational maps \( \Phi_t \) can also be used to show that

\[ \deg(\theta) = 1 \implies \mathcal{L} \text{ very ample.} \]

In this section we show this result under some assumptions which are "ad hoc" constructed for the case \( X = SU_2(C), \mathcal{L} = \) generalized theta divisor. Essentially, every result here is a consequences of the well known
RIGIDITY LEMMA. Let $\Phi : X \times T \to Z$ be a morphism. Assume that there exists $o \in X$ such that

$$\Phi/(o \times T)$$

is constant

then $\Phi$ factors through the first projection $p_1 : X \times T \to T$ i.e.

$$\Phi = f \cdot p_1$$

where $f : X \to Z$ is a morphism.

As a slight extension of the lemma we have

(6.8)PROPOSITION. Let $\Phi : X \times T \to Z$ be only a rational map, $I_\Phi$ its indeterminacy locus. Assume there exists $o \in X$ such that

(1) $\text{codim}_T(I_\Phi \cap o \times T) \geq 2$

(2) $\Phi$ restricted to $o \times T$ extends to a constant map.

Then $\Phi$ factors through the first projection $p_1 : X \times T \to X$ i.e.: 

$$\Phi = f \cdot p_1$$

where $f : X \to Z$ is a rational map.

Proof. Let

$$A = \{x \in X | \text{codim}_T(I_\Phi \cap x \times T) \geq 2\}$$

by assumption (1) and dimension theory $A$ is a non empty open set. Let us show that $\Phi$ restricted to $x \times T$ extends to a constant map for all $x \in A$: take any two points $t_1, t_2 \in T$

such that $(x, t_1), (x, t_2)$ are not in $I_\Phi$. Since $T$ is projective and $\text{codim}_T(I_\Phi \cap x \times T) \geq 2$, $\text{codim}_T(I_\Phi \cap o \times T) \geq 2$ there exists a projective curve $\Gamma \subset T$ containing $t_1, t_2$ and such that $I_\Phi \cap (x \times \Gamma) = I_\Phi \cap (o \times \Gamma) = \emptyset$. Now consider the (non empty) open set $A_\Gamma = \{x \in X | I_\Phi \cap x \times \Gamma = \emptyset\}$: by assumption (2) and the Rigidity Lemma applied to $\Phi$ restricted to $A_\Gamma \times B$, we obtain

$$\Phi(x, t_1) = \Phi(x, t_2)$$

Since $t_1, t_2$ are choosen on an open set of $T$ it follows that $\Phi/x \times T$ is a constant map. Then we can consider the function

$$f : A \to Z$$

sending $x \in A$ to the point $\Phi(x \times T)$: clearly $f$ is a rational map from $X$ to $Z$ and $\Phi = f \cdot p_1$.

Let us point out a special corollary of this:

(6.9)PROPOSITION. Let $j : X \times T \to X \times T$ be a birational involution, $I_j$ its indeterminacy locus, $p_1 : X \times T \to X$ the natural projection. Assume there exists $o \in X$ such that:

(1) $\text{codim}_T(I_j \cap o \times T) \geq 2$

(2) $p_1 \cdot j$ restricted to $o \times T$ extends to a constant map
then there exists a birational involution \( f : X \to X \) which makes commutative the diagram

\[
\begin{array}{ccc}
X \times T & \xrightarrow{j} & X \times T \\
\downarrow p_1 & & \downarrow p_1 \\
X & \xrightarrow{f} & X
\end{array}
\]

**Proof.** Consider the rational map \( \Phi = p_1 \cdot j : X \times T \to X \) and observe that \( I_j \) contains the indeterminacy locus of \( \Phi \); then apply to \( \Phi \) proposition 6.8: by assumptions (1) and (2) \( \Phi \) factors through \( p_1 \) and a rational map \( f : X \to X \); therefore \( f \cdot p_1 = p_1 \cdot j \) this gives the required commutative diagram. In particular \( f \cdot f \cdot p_1 = f \cdot p_1 \cdot j = p_1 \cdot j = p_1 \) so that \( f \cdot f = \text{id}_X \) and \( f \) is a birational involution.

Now we apply the previous propositions to the special situation which is relevant for this paper; therefore let \( \Phi : X \times T \to \mathbb{P}(Q) \) be the rational map constructed in 6.7 we make the following

**6.10 Assumption.** \( \Phi \) has degree two onto its image.

Moreover, given such a \( \Phi \), we will always use the following

**6.11 Notations.**
1. \( j : X \times T \to X \times T := \) the involution induced by \( \Phi \)
2. \( I_j := \) the indeterminacy locus of \( j \)
3. \( B_j := \) the maximal \( j \)-invariant open set such that \( j/B_j \) is a biregular isomorphism.

As usual we say that \( \theta \) is not an embedding at \( x \in X \) if \( \theta \) is not injective at \( x \) or if the tangent map \( d\theta_x : T_{X,x} \to T_{\theta(X),\theta(x)} \) is not an isomorphism. Let

\[
N(\mathcal{L}) = \{ x \in X/ \theta \text{ is not an embedding at } x \}
\]

then

**6.13 Theorem.** Let \( \theta : X \to \mathbb{P}^n \) be the morphism associated to \( \mathcal{L} \). Assume:
1. \( \forall x \in N(\mathcal{L}) \) \( \text{codim}_T(I_j \cap x \times T) \geq 2 \)
2. \( \forall x \in N(\mathcal{L}), p_1 \cdot j \) restricted to \( x \times T \) extends to a constant map

Then, if \( \mathcal{L} \) is not very ample, there exists a non identical birational involution \( f : X \to X \) such that \( j = f \times \text{id}_T \)

Furthermore, assume in addition that:
1. \( X \) is normal, \( \text{Pic}(X) \cong \mathbb{Z} \) and the restriction \( \text{Pic}(X) \to \text{Pic}(X - \text{Sing}X) \) is an isomorphism
2. \( \text{codim}_X Y \geq 2 \), where \( Y = X - p_1(B_j) \)

Then, if \( \mathcal{L} \) is not very ample, there exists a projective involution \( \overline{f} \) of \( \mathbb{P}^n \) such that \( \theta \cdot f = \overline{f} \cdot \theta \)

**Proof.** If \( \mathcal{L} \) is not very ample \( N(\mathcal{L}) \) is not empty. Then, by proposition 6.9, there exists a birational involution \( f \) on \( X \) such that \( p_1 \cdot j = f \cdot p_1 \) where \( p_1 \) is the first projection of \( X \times T \). Since \( j(x,t) \) is the identity in \( t \) it follows \( j = f \times \text{id}_T \) and in particular \( f \) is non identical.
Since \( f = j \times \text{id}_T \) the open set \( X - Y \) is \( f \)-invariant and \( f \) is biregular on it. Let \( Z = Y \cup \text{Sing}(X) \): from assumption (4) and the normality of \( X \) it follows that \( Z \) has codimension \( \geq 2 \) in \( X \). Consider the open set \( U = X - Z \) from assumption (3) it follows that the restriction \( \text{Pic}(X) \to \text{Pic}(U) \) is an isomorphism. Clearly \( U \) is \( f \)-invariant and \( f/U \) is a biregular involution of \( U \). Hence, in particular, \((f/U)^*L/U \cong L/U \). Since \( X - U \) has codimension \( \geq 2 \) the restriction \( H^0(L) \to H^0(L/U) \) is an isomorphism too. This implies that \( f/U \) induces on \( \mathbb{P}^N = H^0(L)^* \) a projective involution \( f \) such that \( \theta \cdot f = f' \cdot \theta \).

(6.14) REMARK The previous theorem is only a corollary to proposition 6.9, nevertheless it plays an essential role for showing the very ampleness of the generalized theta divisor. Let us see why: of course if we produce a subset \( S \subset X \) such that \( \theta(S) \) spans \( \mathbb{P}^N \) and \( f/S = \text{id}_S \), then \( \overline{f} \) is the identity and \( \deg(\theta) \geq 2 \). In the case \( X = SU_C(2), L = \text{generalized theta divisor} \) we will see that such an \( S \) is \( \text{Sing}X \); therefore we will be able to conclude \( L \) not very ample \( \implies \deg(\theta) \geq 2 \implies C \) hyperelliptic, where the second implication is Beauville's result (0.4) (iii).

7. Proof of the main theorem.

Let \( \theta : X \to |2\Theta| \) be the morphism associated to the generalized theta divisor, we want to show our main theorem (0.3) that is:

\( \begin{align*}
(1) & \, \theta \text{ is injective} \\
(2) & \, d\theta_x \text{ is injective } \forall x \in X - \text{Sing}X.
\end{align*} \)

Since everything is known for genus \( g \leq 3 \) we assume \( C \) not hyperelliptic of genus \( g \geq 4 \), then we consider the fundamental map

\[ F : X \times T \to \mathbb{P}(Q) \]

and its associated fundamental involution

\[ j : X \times T \to X \times T \]

which have been studied in detail in the previous sections. We want to apply theorem 6.13 to \( F \) and \( j \), therefore we must check wether the assumptions (1),(2),(3),(4) of this theorem are satisfied:

Assumption (3) is satisfied because \( X \) is normal and \( \text{Pic}(X) - \text{Sing}X \cong \text{Pic}X \) (see [B1]3.1). By remark 2.18 assumption (4) is satisfied too. Assumptions (1) and (2) concern the locus

\[ N(L) = \{ o \in X/\theta \text{is not an embedding at } o \} \]

a point \( o \in N(L) \) satisfies them iff:

\( \begin{align*}
(1) & \, o \times T \cap I_j \text{ has codimension } \geq 2 \text{ in } T, \text{ where } I_j \text{ is the indeterminacy locus of } j \\
(2) & \, \text{let } p_1 : X \times T \to X \text{ be the first projection, then } p_1 \cdot j \text{ is generically defined and constant when restricted to } o \times T.
\end{align*} \)

(7.1) LEMMA.

(i) Let \( o \in N(L) \cap X - \text{Sing}X \) then \( o \) satisfies assumptions (1) and (2)

(ii) let \( o \in \text{Sing}X \) then \( \theta^{-1}(o) = \{ o \} \).

Proof. (i) At first we show that condition (2) is satisfied by \( o \). We distinguish two cases:
CASE (A) \( \theta \) is not injective at \( o \). Then there exists \( \tau \neq o \) such that \( \theta(o) = \theta(\tau) \), we will assume \( o = [\xi] \), \( \tau = \xi \). Consider in \( T \) the special closed set \( I(o, \tau) \) defined in 5.5 and the complement

\[ U_{o, \tau} = T - I(o, \tau) \]

of the special closed set \( I(o, \tau) = I(o) \cup I(\tau) \) defined in 5.5: we already know that, putting

\[ \mathcal{E} = \xi(t), V = H^0(\xi(t)) \quad \overline{\mathcal{E}} = \overline{\xi(t)}, \overline{V} = H^0(\xi(t)) \]

and choosing \( t \) in \( U_{o, \tau} \), the dimension of \( V, \overline{V} \) is 4 and moreover the pairs \((\mathcal{E}, V), (\overline{\mathcal{E}}, \overline{V})\) define two quadrics

\[ Q_t = Q(\mathcal{E}, V), \quad \overline{Q}_t = Q(\overline{\mathcal{E}}, \overline{V}) \]

having rank \( \geq 4 \). Actually, since \( o \) is not in \( \text{Sing} X \) and \( t \) is not in \( I(o) \), there is no subline bundle \( L \) of \( \mathcal{E} \) with \( h^0(L) \geq 2 \) so that

\[ \text{rank} Q_t \geq 5 \]

by prop. 1.11. Choosing \( t \) general in \( U_{o, \tau} \) (that is: \( t \in U_0 - (U_0 \cap V_1(\xi)) \)) \( \xi(t) \) is globally generated, hence

\[ \text{Sing} Q_t \cap C_t = \emptyset \]

by prop. 1.9. From the definition of the fundamental map \( F \) we have

\[ Q_t = F(o, t) \quad \text{and} \quad \overline{Q}_t = F(\tau, t) \]

Since \( F = \lambda \cdot (\theta \times id_T) \) and \( \theta(o) = \theta(\tau) \) it follows \( F(o, t) = F(\tau, t) \) that is

\[ Q_t = \overline{Q}_t \]

Since \( Q_t \) has rank \( \geq 5 \) and \( \text{Sing} Q_t \cap C_t = \emptyset \) there exist at most two isomorphism classes of pairs defining \( Q_t \), moreover they are either isomorphic or dual (prop. 1.19, lemma 1.18). In particular this holds for \((\mathcal{E}, V), (\overline{\mathcal{E}}, \overline{V})\); since \( o \neq \tau \) \( \xi \) is not isomorphic to \( \overline{\xi} \) hence these two pairs cannot be isomorphic and they are dual. Then the rank of \( Q_t \) is 6 and

\[ j(o, t) = (\tau, t) \]

Since this holds for a general \( t \in T \) \( j \) is generically defined on \( o \times T \) and \( p_1 \cdot j : o \times T \to X \) is the constant map onto the point \( \tau \). Hence condition (2) of theorem 6.13 is satisfied by \( o \). The proof of the next case is similar:

CASE (B) the tangent map \( d\theta_o : T_{X,o} \to T_{\theta(X), \theta(o)} \) is not an isomorphism. Since we have already shown case (A) we can assume \( \theta^{-1}(o) = \{ o \} \): this clearly implies \( d\theta_o \) not injective. We consider as above the quadric \( Q_t = F(o, t) \): we know that its rank is 5 or 6 for general \( t \) (precisely for \( t \in T - W(\xi) \)). Let us show that in this case the rank is 5: since \( F = \overline{\lambda} \cdot (\theta \times id_T) \) and \( d\theta \) is not injective at \( o \) \( dF \) is not injective at \( o, t \). We have shown in proposition 3.18 that this is impossible if the rank of \( Q_t \) is 6. Since the rank is 5 \((\mathcal{E}, V)\) is isomorphic to its dual so that \( j(o, t) = (o, t) \) for a general \( t \). Hence \( o \) satisfies conditions (2) of theorem 6.13. To complete the proof of (i) we have now to show that \( o \) satisfies also condition (1). This is an immediate consequence of theorem 5.8 which says that, with the prescribed exceptions, \( \text{codimo} \times T \cap I_j \geq 2 \) if \( p_1 \cdot j : o \times T \to X \) is constant. Indeed, since we are assuming \( C \) not hyperelliptic, \( g \geq 4 \), \( o \) non singular, only the following exception is possible:
there exists a double covering \( \pi : C \to Y \) of an elliptic curve and \( o \) is the moduli point of the bundle constructed from \( \pi \) as in the statement of theorem 5.8. Our claim is that \( \theta \) is an embedding at such a special point \( o \) so that \( o \) is not in \( N(\mathcal{L}) \): we will show this claim at the end of the paper.

(ii) Let \( u \in X - SingX \) we have already remarked that \( F(u, t) \) is a quadric of rank \( \geq 5 \) if \( t \) is general. On the other hand the rank of \( F(o, t) \) is always \( \leq 4 \) if \( o = [q] \in SingX \): this is due to the fact that \( \xi(t) \) has a subline bundle \( L \) with \( h^0(\mathcal{L}) \geq 2 \) for every \( t \), \( (\deg L = g + 1) \). Therefore \( F(u, t) \neq F(o, t) \) at least for one \( t \) and hence \( \theta(u) \neq \theta(o) \). In particular \( \theta^{-1}(o) \subset SingX \), but it is well known that \( \theta/SingX \) is injective hence \( \theta^{-1}(o) = o \).

Finally we can complete the

**PROOF OF THE MAIN THEOREM.**

Assume there exists a point \( o \in X \) such that

(1) \( \theta \) is not injective at \( o \) in \( X \) or

(2) \( o \) is non singular and \( d\theta_o \) is not an isomorphism.

In case (1), by statement (ii) of the previous lemma 7.1, \( o \in X - SingX \cap N(\mathcal{L}) \) hence, by statement (i) of the same lemma, \( o \) satisfies assumptions (1), (2) of theorem 6.13. In case (2) \( o \in X - SingX \cap N(\mathcal{L}) \) again hence the same holds by the same lemma 7.1. Therefore we can apply to \( o \) the rigidity argument used in theorem 6.13: since \( o \) exists and satisfies assumptions (1) and (2) \( j \) is induced by a non identical birational involution \( f \) on \( X \). That is

\[
j = f \times id_T
\]

Once we have \( f \) we conclude by showing that

\[
\theta \cdot f = \theta
\]

so that \( \deg(\theta) \geq 2 \). For this it suffices to produce a subset \( S \) of \( X \) such that \( \theta(S) \) spans the ambient space \( |2\Theta| \) of \( \theta(X) \) and \( f/S = id_S \) (see remark 6.14). Let \( S = SingX \) then \( \theta(S) \) is the Kummer variety of \( J \) naturally embedded in \( |2\Theta| \) and of course it spans this space. Let \( s \in S \): since, for a general \( t \), \( j(s, t) = (s, t) \) it follows \( f(s) = s \) and \( f/S = id_S \). Therefore \( \deg(\theta) \geq 2 \): since \( C \) is not hyperelliptic this is impossible ([B1]), hence the point \( o \) cannot exists. This implies the theorem.

**COMPLETION OF THE PROOF OF LEMMA 7.1 (i) (the bielliptic case)** Let \( \pi : C \to Y \) be a double cover of an elliptic curve, \( \eta \) an irreducible rank two vector bundle on \( Y \) of degree 1: as claimed in the proof of lemma 7.1 (i) we must show that \( \theta \) is an embedding at the moduli point \( o \) of

\[
\xi \pi^* \eta \otimes L
\]

where \( L^2 \cong \omega_C \otimes det \pi^* \eta^* \). We recall that \( \xi \) is stable (cfr.[LN]) and that, fixing its determinant, \( \eta \) is unique. Hence for every \( \pi \) there are \( 2^g \) exceptional points like \( o \): one for each \( L \). We will just sketch proof: by [LN] \( \xi \) has a family of subline bundles of maximal degree \( g - 2 \) which is exactly the following

\[
Y_L = \{ L(f), f \in \pi^* Pic^0(Y) \}
\]

Let \( C_t \subset P_t^{g+2} \) be the curve \( C \) embedded by \( \omega_C(2t) \), as usual we can choose \( t \in T \) so that in the quadric

\[
Q_t = F(o, t)
\]
has rank $\geq 5$ and $C_t \cap Q_t = \emptyset$. We can also assume $h^0(L(f + t)) = 1$ and $h^1(L(f - t)) = 3$ for each $L(f) \in Y_L$. Then, denoting by $< D_f >$ the linear span in $\mathbb{P}^{r+2}_t$ of the unique element $D_f \in |L(f + t)|$, it follows

$$\text{codim} < D_f > = 3$$

and moreover

$$< D_f > \subset Q_t$$

because $h^0(\xi(t - D_f)) \geq 1$. Obviously $< D_f > \neq < D_g >$ if $f \neq g$; moreover, since they are maximal subspaces of $Q_t$ in the same ruling we have

$$\dim < D_f > \cap < D_g > = g - 5$$

Now observe that, if the rank of $Q_t$ is 5, then $dim \text{Sing} Q_t = g - 5$ so that $< D_f > \cap < D_g > = \text{Sing} Q_t$. Assuming this and considering the variety $R = \bigcup < D_f >$, $f \in \pi^* \text{Pic}_0(Y)$ it is not difficult to deduce that $R$ would be a cone of vertex $\text{Sing} Q_t$ over $\overline{R} \subset \overline{Q} \subset \mathbb{P}^4$, where $\overline{R}$ is a smooth scroll over $Y$ and $\overline{Q}$ a smooth quadric. Since this is impossible $Q_t$ must have rank 6. Then all the assumptions of proposition 3.18 are satisfied by the pair $(o, t)$ and hence $(df)_{o} = (d\lambda)_{o} \cdot (d\theta)_{o}$ is injective. Therefore $d\theta_{o}$ is injective. It remains to show that $\theta^{-1}(o) = \{o\}$: essentially this follows from the remark that $\Theta_{\xi}$ contains the irreducible components

$$W - Y_L, \quad Y_L - W \quad (\text{where } W = C(g - 2))$$

Assume $\theta(\overline{\xi}) = \theta(o)$: by lemma 7.1 (ii) $\overline{\xi}$ is stable, the main point is to deduce from $W - Y_L \subset \Theta_{\xi}$ ($= \Theta_{\xi}$) that $L(f)$ is a subline bundle of $\overline{\xi}$ for every $L(f) \in Y_L$: we omit this. Since, by the quoted result of Lange-Narasimhan ([LN]thm.5.1), the latter property characterizes $\xi$ it follows $\overline{\xi} \cong \xi$.

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