A MEAN VALUE FORMULA AND A LIOUVILLE THEOREM FOR THE COMPLEX MONGE-AMPÈRE EQUATION

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Abstract. In this paper, we prove a mean value formula for bounded subharmonic Hermitian matrix valued function on a complete Riemannian manifold with nonnegative Ricci curvature. As its application, we obtain a Liouville type theorem for the complex Monge-Ampère equation on product manifolds.

1. Introduction

Understanding various spaces of harmonic functions on complete noncompact Riemannian manifolds is one of the central questions in geometric analysis. During the last 40 years, there have been many significant progress in this question (see e.g. [33, 34, 7, 19, 20, 18, 9, 31, · · ·]). More importantly, the techniques developed in this field are extremely useful when applied to other problems in geometric analysis. In [18], Peter Li proved the following theorem:

Theorem 1.1 (Theorem 2 of [18]). Let $(M^n, \omega)$ be a complete Kähler manifold with nonnegative Ricci curvature and $\mathcal{H}^1(M)$ be the space of linear growth harmonic functions on $(M^n, \omega)$. Then $\dim \mathcal{H}^1(M) \leq 2n + 1$. Moreover, if $\dim \mathcal{H}^1(M) = 2n + 1$ then $M$ must be isometric to $\mathbb{C}^n$ with the standard flat metric.

In Peter Li’s proof of Theorem 1.1 the following mean value theorem for bounded subharmonic functions plays an important role:

Theorem 1.2 (Lemma B of [18]). Let $(M, g)$ be a complete manifold with nonnegative Ricci curvature. Suppose $f$ is a bounded subharmonic function defined on $(M, g)$, then for any $p \in M$

$$\lim_{r \to \infty} \int_{B_r(p)} f dV_g = \sup_M f. \quad (1.1)$$

Despite the application in [18], Theorem 1.2 has some more applications in the study of Riemannian geometry (see e.g. [9]). It is a useful tool in the study of linearly growth harmonic functions on complete Riemannian manifolds with nonnegative Ricci curvature.

In this paper, we study a class of Hermitian matrix valued functions and establish a mean value theorem to them. For convenience, we denote the set of all $m$-order Hermitian matrices by $H_m(m)$, and equip it with the metric induced by the inner product

$$(A, B) = \text{tr} A \overline{B}. \quad (1.2)$$
Definition 1.3. A map $A = (A_{ij})$ from a Riemannian manifold to $\text{Hm}(m)$ is said to be subharmonic, if for any vector $\xi = (\xi_1, \cdots, \xi_m) \in \mathbb{C}^m$, $\xi A^* = A_{ij} \xi_i \xi_j$ is a subharmonic function.

By the definition, it is easy to check that a $C^2$ Hermitian matrix valued function $A = (A_{ij})$ on a Riemannian manifold is subharmonic if and only if $\Delta A = (\Delta A_{ij})$ is semi-positive-defined everywhere. We obtain the following mean value formula to subharmonic Hermitian matrix valued functions.

Theorem 1.4. Let $(M, g)$ be a complete Riemannian manifold with nonnegative Ricci curvature, and $A = (a_{ij})$ be a bounded subharmonic Hermitian matrix valued function on $(M, g)$. Then there exits a Hermitian matrix $A_0$, such that

$$A \leq A_0, \quad (1.3)$$

on $M$, and for any $p \in M$

$$\lim_{r \to \infty} \int_{B_r(p)} A dV_g = \lim_{r \to \infty} \left( \int_{B_r(p)} a_{ij} dV_g \right) = A_0. \quad (1.4)$$

The complex Monge-Ampère equation has significant applications in complex analysis and complex geometry, and many remarkable progresses of complex Monge-Ampère equations were carried out by many people (see e.g. [1, 35, 2, 3, 8, 21, 15, 6, 26, 27, 28, 29, 23, 24, 25, 16, 12, 4, 13, 36, 31, 18, 11, 37, 32, · · · ). In this paper, we concentrate on Liouville theorems for the complex Monge-Ampère equation. In [22], Riebesehl and Schulz proved a Liouville theorem for the complex Monge-Ampère equation on $\mathbb{C}^n$, which can be expressed by Kähler forms as following.

Theorem 1.5 ([22]). Let $\omega$ be a Kähler form on $\mathbb{C}^n$ satisfy $C^{-1} \omega_0 \leq \omega \leq C \omega_0$ and $\omega^n = \omega^n_0$, where $\omega_0 = \sqrt{-1} \sum_{i=0}^n dz^i \wedge d\bar{z}^i$ and $C$ is a positive constant. Then $\nabla \omega_0 = 0$, or equivalently

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^n A_{ij} dz^i \wedge d\bar{z}^j \quad (1.5)$$

for some constant Hermitian matrix $(A_{ij})$.

The key of the proof of Theorem 1.5 is a local Calabi $\mathcal{C}^3$ estimate, i.e. an estimate on $|\nabla \omega_0|^2$. However, when considering analogous Liouville type theorems on complete Kähler manifolds with non trivial Riemannian curvature, the Calabi $\mathcal{C}^3$ estimate seems not to work. Recently, Hein ([14]) proved a Liouville theorem for the complex Monge-Ampère equation on product manifolds, which can be expressed in short as:

Theorem 1.6 (Theorem A of [14]). Let $(Y, \omega_Y)$ be a compact Ricci-flat Kähler manifold. Let $\omega$ be a Ricci-flat Kähler form on $\mathbb{C}^m \times Y$. Assume that $C^{-1}(\omega_{C^m} + \omega_Y) \leq \omega \leq C(\omega_{C^m} + \omega_Y)$ for some $C > 1$, where $\omega_{C^m}$ is the standard flat Kähler form on $\mathbb{C}^m$. Then we can find some Kähler form $\omega_Y$ on $Y$, $T_\omega \in \text{Auto}(\mathbb{C}^m \times Y)$ and complex linear map $S \in \text{Auto}(\mathbb{C}^m)$ such that:

$$T^*_\omega \omega = \omega_Y + S^* \omega_{C^m}.$$
Theorem 1.7. Let \((Y^n, \omega_Y)\) be an \(n\) dimensional compact Kähler manifold with nonpositive Ricci curvature, and let \(\omega\) be a Kähler form on \(\mathbb{C}^m \times Y\) with properties

1. \(C^{-1}(\omega_{\mathbb{C}^m} + \omega_Y) \leq \omega \leq C(\omega_{\mathbb{C}^m} + \omega_Y)\), for some positive constant \(C\);
2. \(\omega^{n+m} = (\omega_{\mathbb{C}^m} + \omega_Y)^{m+n}\).

where \(\omega_{\mathbb{C}^m}\) is the standard Kähler form on \(\mathbb{C}^m\). Then there exists a Kähler form \(\omega_Y\) on \(Y\) with \(\text{Ric}(\omega_Y) = \text{Ric}(\omega_Y_0)\) such that \(\nabla_{\omega_{\mathbb{C}^m} + \omega_Y} \omega = 0\). Furthermore, we have the following presentation of \(\omega\)

\[
\omega = \tilde{\omega}_{\mathbb{C}^m} + \omega_Y + \frac{1}{2} \sum_{i=1}^m (dz^i \wedge \eta^i + d\bar{z}^i \wedge \eta^i) \tag{1.6}
\]

where \(\tilde{\omega}_{\mathbb{C}^n} = \frac{1}{2} \sum_{i,j=1}^m u_{ij} dz^i \wedge d\bar{z}^j\) with the constant Hermitian matrix \((u_{ij})\), and every \(\eta^i\) is a \(\omega_Y\)-paralleled \((0,1)\)-form.

Taking the construction of \(\omega_Y\), \(T_l\) and \(S\) in Theorem 1.6 ([14]) in consideration, Theorem 1.7 can be seen as a generalization of Theorem 1.6. Our proof relies on the above mean value formula (i.e. Theorem 1.4) and is very different with Hein’s. Theorem 1.5 also can be seen as an application of the mean value formula (1.4). We hope the mean value formula (1.4) to have more applications in the study of Kähler geometry.

2. A mean value formula for bounded subharmonic Hermitian matrix valued function

In this section, we first give a proof of Theorem 1.4 and then give a new proof to Theorem 1.5 by using Theorem 1.4 instead of the Calabi \(C^3\) estimate.

A proof of Theorem 1.4. For any vector \(\xi \in \mathbb{C}^m\), define

\[
||\xi||_A^2 = \xi A \xi^* \tag{2.7}
\]

By this definition and the condition on \(A\), for any fixed \(\xi \in \mathbb{C}^m\), \(||\xi||_A^2\) is a bounded subharmonic function, then Theorem 1.2 implies

\[
\lim_{r \to \infty} \int_{B_r(p)} ||\xi||_A^2 = \sup_M ||\xi||_A^2. \tag{2.8}
\]

For \(i = 1, 2, \cdots, n\), let \(e_i\) be the \(i\)-th direction vector in \(\mathbb{C}^n\). We have

\[
A_{ij} = \frac{||e_i + e_j||_A^2 - ||e_i - e_j||_A^2}{4} - \frac{\sqrt{-1}||e_i + \sqrt{-1} e_j||_A^2 - ||e_i - \sqrt{-1} e_j||_A^2}{4}. \tag{2.9}
\]

Together with (2.8), we assert that \(\lim_{r \to \infty} \int_{B_r(p)} A\) exists. Let

\[
A_0 = \lim_{r \to \infty} \int_{B_r(p)} A, \tag{2.10}
\]

then we have for any \(\xi \in \mathbb{C}^m\),

\[
\xi A_0 \xi^* = ||\xi||_A^2 \leq \lim_{r \to \infty} \int_{B_r(p)} ||\xi||_A^2 = \xi A_0 \xi^*. \tag{2.11}
\]

This shows \(A \leq A_0\).

We have the following simple corollary:
Corollary 2.1. Let $A : M \rightarrow \text{Hm}(m)$ satisfy the same condition of Theorem 1.4 and $A_0 = \lim_{r \to \infty} \int_{B_r(p)} A$. Let $F$ be a bounded function on some neighborhood of the closure of $A(M)$ and continuous at $A_0$, then we have

$$\lim_{r \to \infty} \int_{B_r(p)} F(A) = F(A_0).$$  \tag{2.12}$$

Proof. By the condition on $F$ we can find a positive constant $C$, such that

$$F(A) \leq C,$$  \tag{2.13}$$
on $M$. And for any $\varepsilon > 0$, we can find some $\delta > 0$ such that for any $q \in M$ satisfying $|A(q) - A_0| \leq \delta$, there holds

$$|F(A(q)) - F(A_0)| \leq \varepsilon.$$  \tag{2.14}$$

For the mentioned $\varepsilon$ and $\delta$, we have

$$\int_{B_r(p)} |F(A) - F(A_0)| = \int_{B_r(p) \cap \{|A-A_0| \leq \delta\}} |F(A) - F(A_0)| + \int_{B_r(p) \cap \{|A-A_0| > \delta\}} |F(A) - F(A_0)|$$

$$\leq \varepsilon \text{Vol}(B_r(p) \cap \{|A-A_0| \leq \delta\}) + C \text{Vol}(B_r(p) \cap \{|A-A_0| > \delta\})$$

$$\leq \varepsilon \text{Vol}(B_r(p)) + C \text{Vol}(B_r(p) \cap \{|A-A_0| > \delta\}).$$  \tag{2.15}$$

By Theorem 1.4, $A \leq A_0$, so $A_0 = \lim_{r \to \infty} \int_{B_r(p)} A$ implies

$$\lim_{r \to \infty} \int_{B_r(p)} |A - A_0| = 0.$$  \tag{2.16}$$

Together with

$$\text{Vol}(B_r(p) \cap \{|A - A_0| > \delta\}) \leq \delta^{-1} \int_{B_r(p)} |A - A_0|,$$  \tag{2.17}$$
we have

$$\lim_{r \to \infty} \frac{\text{Vol}(B_r(p) \cap \{|A - A_0| > \delta\})}{\text{Vol}(B_r(p))} = 0.$$  \tag{2.18}$$

(2.15) and (2.18) imply

$$\limsup_{r \to \infty} \int_{B_r(p)} |F(A) - F(A_0)| \leq \varepsilon.$$  \tag{2.19}$$

Let $\varepsilon \to 0$, then we get

$$\lim_{r \to \infty} \int_{B_r(p)} |F(A) - F(A_0)| = 0.$$  \tag{2.20}$$

This concludes the proof. \hfill \Box

By Theorem 1.4 and Corollary 2.1 we can give a new proof to Theorem 1.5.

A new proof to Theorem 1.5 We can write $\omega$ as

$$\omega = \sqrt{-1} \frac{1}{2} \sum_{i,j=1}^{n} u_{ij} dz^{i} \wedge d\bar{z}^{j},$$  \tag{2.21}$$

where $(u_{ij})$ is a function valued in $\text{Hm}(n)$. Denote $(u^{ij}) = (u_{ij})^{-1}$, $u_{ijk} = \frac{\partial}{\partial z^i} u_{jk}$, $u_{ijkl} = \frac{\partial}{\partial z^i} u_{jk}$, etc. Since $\omega$ is closed, we have

$$u_{ijk} = u_{kji}, \quad u_{ijk} = u_{ikj}.$$  \tag{2.22}$$

By the equation $\omega$ satisfied, we have

$$\det(u_{ij}) = 1.$$  \tag{2.23}$$
Direct computation shows
\[
\Delta \omega u_{ij} = u_{k\bar{l}} u_{\bar{p}q} u_{kj} u_{i\bar{p}j}. \tag{2.24}
\]
For any \(\xi = (\xi^1, \xi^2, \cdots, \xi^n) \in \mathbb{C}^n\), consider the Hermitian quadratic form \(F : \mathbb{C}^{3n} \times \mathbb{C}^{3n} \to \mathbb{R}\) defined by
\[
(A, B) \mapsto u^{\alpha\bar{\beta}} u^{\delta\bar{\gamma}} \xi^k \xi^l A_{ijk} B_{\alpha\beta\gamma}. \tag{2.25}
\]
By choosing a proper frame on \(\mathbb{C}^n\), one can easily check that \(F\) is semi-positive-defined. So
\[
\xi^i (\Delta \omega u_{ij}) \xi^j = u^{\alpha\bar{\beta}} u^{\delta\bar{\gamma}} \xi^k \xi^l u_{ijk} u_{\alpha\beta\gamma} \geq 0. \tag{2.26}
\]
This implies that \((u_{ij})\) is subharmonic. The condition on \(\omega\) implies that \((u_{ij})\) is bounded and \((\mathbb{C}^n, \omega)\) is a complete Ricci flat Kähler manifold. By Theorem 1.4 and Corollary 2.1, we can find a constant Hermitian matrix \(A\) such that
\[
(u_{ij}) \leq A, \tag{2.27}
\]
on \(M\) and
\[
\lim_{r \to \infty} \int_{B_{\bar{r}}(O)} \det(u_{ij}) \omega^n = \det A. \tag{2.28}
\]
By (2.28) and the previous equality, we have
\[
\det A = \det(u_{ij}) = 1. \tag{2.29}
\]
Since \((u_{ij})\) is positive-defined, the previous equality and (2.27) imply \((u_{ij})\) is the constant function \(A\). This concludes the proof.

**Remark:**

1. To prove \((u_{ij})\) is subharmonic, besides direct computation, we can also use the following argument: For any \(\xi \in \mathbb{C}^m\), let \(X_\xi = \xi \frac{\partial}{\partial z}\), then
\[
\xi_i u_{ij} \xi_j^i = 2 |X_\xi|^2. \tag{2.30}
\]
Using the Bochner formula for holomorphic fields and the fact that \(\text{Ric}(\omega) = 0\), one can easily check that \(\xi_i u_{ij} \xi_j^i\) is subharmonic.

2. To prove Theorem 1.7, one can also consider the \((u^{ij}) = (u_{ij})^{-1}\). For any \(\xi \in \mathbb{C}^m\), let \(f_\xi = \text{Re}(\xi z^i)\), then
\[
\xi_i u^{ij} \xi_j^i = |df_\xi|^2. \tag{2.31}
\]
Clearly \(f_\xi\) is a pluri-harmonic function and hence a harmonic function with respect to \(\omega\). Using the Bochner formula and the fact that \(\text{Ric}(\omega) = 0\) one can easily check that \(\xi_i u^{ij} \xi_j^i\) is subharmonic.

3. A Liouville theorem for the complex Monge-Ampère equation

In this section, we obtain a Liouville theorem for the complex Monge-Ampère equation as an application of the mean value formula (1.4), i.e. we give a proof of Theorem 1.7. First we introduce the following lemma concerning the computation of determine of a blocked Hermitian matrix.
Lemma 3.1. Let $M$ be an invertible Hermitian matrix. If
\[ M = \begin{pmatrix} A & C \\ C^* & B \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} \bar{A} & \bar{C} \\ \bar{C}^* & \bar{B} \end{pmatrix}, \]
where $A$ is invertible. Then
\[ \det M = \det A \det \bar{B}^{-1}. \]

Proof. Since $A$ is invertible, we have
\[
\begin{pmatrix} I & 0 \\ -C^*A^{-1} & I \end{pmatrix} \begin{pmatrix} A & C \\ C^* & B \end{pmatrix} \begin{pmatrix} I & -A^{-1}C \\ O & I \end{pmatrix} = \begin{pmatrix} A & O \\ O & B - C^*A^{-1}C \end{pmatrix},
\]
this implies
\[ \det M = \det A \det(B - C^*A^{-1}C). \tag{3.32} \]

$M$ is invertible, so $B - C^*A^{-1}C$ is also invertible. At the same time, we have
\[ M^{-1} = \begin{pmatrix} I & -A^{-1}C \\ O & I \end{pmatrix} \begin{pmatrix} A & O \\ O & B - C^*A^{-1}C \end{pmatrix}^{-1} \begin{pmatrix} I & O \\ -C^*A^{-1} & I \end{pmatrix}, \]
which implies
\[ \bar{B} = (B - C^*A^{-1}C)^{-1}. \tag{3.33} \]
The required equality is a combination of (3.32) and (3.33). \qed

A proof to Theorem 1.7. Let $\pi_Y$ and $\pi_{C^m}$ be the two projections:
\[ \pi_Y : \mathbb{C}^m \times Y \rightarrow Y, \quad \pi_Y(z, y) = y, \tag{3.34} \]
\[ \pi_{C^m} : \mathbb{C}^m \times Y \rightarrow \mathbb{C}^m, \quad \pi_{C^m}(z, y) = z. \tag{3.35} \]
By Künneth’s formula (see e.g. Section 5 of \[5\]) and the result on the de Rahm cohomology groups of $\mathbb{C}^m$
\[ H^k_{dR}(\mathbb{C}^m) = \begin{cases} \mathbb{R}, & k = 0, \\ 0, & k \geq 1, \end{cases} \tag{3.36} \]
there exists a closed real 2-form $\Theta$ on $Y$, such that
\[ [\omega] = [\pi_Y^* \Theta], \tag{3.37} \]
in the sense of de Rahm cohomology classes. For any $z \in \mathbb{C}^m$, denote the embedding from $Y$ to $\mathbb{C}^m \times Y$
\[ y \mapsto (z, y), \tag{3.38} \]
by $i_z$. For distinct $z_1, z_2 \in \mathbb{C}^m$, since $\pi_Y \circ i_{z_1} = id_Y = \pi_Y \circ i_{z_2}$, we have
\[ [i_{z_1}^* \omega] = [\Theta] = [i_{z_2}^* \omega]. \tag{3.39} \]
Obviously all $i_z^* \omega$ are Kähler forms on $Y$, so (3.39) shows that all $i_z^* \omega$ are in the same Kähler class.

By Calabi-Yau theorem (see \[35\]), there is a unique Kähler form $\omega_Y$ in this Kähler class satisfying $Ric(\omega_Y) = Ric(\omega_{Y_0})$ and consequently
\[ \omega^n_Y = c \omega^n_{Y_0} \tag{3.40} \]
for some positive constant $c$.

Now we can write the conditions on $\omega$ as follow
1. $C^{-1}(\omega_{C^m} + \omega_Y) \leq \omega \leq C(\omega_{C^m} + \omega_Y)$, for some positive constant $C$;
2. $\omega^{n+m} = e^{-1}(\omega_{C^m} + \omega_Y)^{m+n}$;
3. for any $z \in \mathbb{C}^m$, $i_z^* \omega$ and $\omega_Y$ are in the same Kähler class.
Denote $g_0$ and $g$ to be the Riemann metric associated with $\omega_{\mathbb{C}^m} + \omega_Y$ and $\omega$ respectively and let $g^{-1}$ be the metric on $T^*(\mathbb{C}^m \times Y)$ induced by $g$. Let \( \{ z^a \}_{a=1}^m \) be the standard complex coordinate system on $\mathbb{C}^m$, for $i,j = 1,2,\ldots, m$, define
\[
 u^{ij} = \frac{1}{2} g^{-1}(dz^i, dz^j), \quad u_{ij} = 2g \left( \frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j} \right). \tag{3.41}
\]
For any point $(z,y)$, we choose a complex normal coordinates system \( \{ z^a \}_{a=1}^m \) around $y$ with respect to $\omega_Y$. Computing under the coordinate system \( \{ z^a \}_{a=1}^m \) mentioned above and applying Theorem 1.4 and Corollary 2.1, we can find some constant Hermitian matrix $A$ such that
\[
 (u^{ij}) \leq A, \tag{3.44}
\]
everywhere, and
\[
 \lim_{r \to \infty} \int_{B^r_c(x_0, z_0)} \det(u^{ij})dV_g = \det A. \tag{3.45}
\]
We can find some sufficiently large constant $C$ such that for $r \gg \text{diam} Y$,
\[
 B_r(z_0) \times Y \subset B^r_c, (z_0, y_0) \subset B_{C^2r}(z_0) \times Y,
\]
so we have
\[
 0 \leq \int_{B_r(z_0) \times Y} (\det A - \det(u^{ij}))dV_{g_0} \leq C^{-1} \int_{B^r_c(x_0, z_0)} (\det A - \det(u^{ij}))dV_g,
\]
and
\[
 \text{Vol}_{g_0}(B_r(z_0) \times Y) = C^{-4n} \text{Vol}(B_{C^2r}(z_0) \times Y) \geq C^{-4n} c^{-1} \text{Vol}_{\omega}(B^r_c(z_0, y_0)).
\]
Then we can obtain
\[
 0 \leq \int_{B_r(z_0) \times Y} (\det A - \det(u^{ij}))dV_{g_0} \leq C^{-4n} \left( \int_{B^r_c(x_0, y_0)} (\det A - \det(u^{ij}))dV_g \right),
\]
consequently
\[
 \lim_{r \to \infty} \int_{B_r(z_0) \times Y} \det(u^{ij})dV_{g_0} = \det A. \tag{3.46}
\]
By computing under the local coordinate system \( \{ z^a \}_{a=1}^m \) mentioned above and applying Lemma 3.1, we can check that
\[
 (\det(u^{ij}))^{-1} (\iota^* \omega)^n \frac{\omega^n}{\omega_Y^n} = \frac{\omega^{n+m}}{\omega_Y + \omega_{\mathbb{C}^m}} = c^{-1}, \tag{3.47}
\]
at any point \((z, y)\). Then we have

\[
\int_{\{z\} \times Y} \det(u^i) \omega^n_Y = \int_{\{z\} \times Y} c(i^* \omega)^n = c.
\]  

(3.48)

This tells

\[
\int_{B_r(z_0) \times Y} \det(u^i) dV_{g_0} = c,
\]

(3.49)

for any \(z_0\) and \(r > 0\). Together with (3.44) and (3.45), we have

\[
\det(u^i) \leq \det A = c,
\]  

(3.50)

and then

\[
\det(u^i) = \det A = c.
\]  

(3.51)

Since \((u^i)\) is positive-defined, using (3.44) again, we obtain

\[
(u^i) \equiv A.
\]  

(3.52)

By (3.47), we have

\[
(i^*_z \omega)^n = \omega^n_Y
\]

(3.53)

for any \(z \in \mathbb{C}^m\). We already know that \(i^*_z \omega\) and \(\omega_Y\) are in the same Kähler class, so

\[
i^*_z \omega = \omega_Y.
\]  

(3.54)

For any \(i = 1, 2, \ldots, m\), by (3.42) and (3.52) we have

\[
0 = \Delta \omega u^{ii} \geq C^{-1} \sum_{a_1, a_2=1}^{m+n} |g_{a_1i a_2}|^2.
\]  

(3.55)

(3.55) implies that \(u^{ii}\) is a constant function, then consequently

\[
0 = \Delta \omega u^{ii} \geq g^{a_1i} g^{a_2i} g_{a_1 a_2} g_{a_1 a_2} \geq C^{-1} \sum_{a,b=1}^{m+n} |g_{ab}|^2.
\]  

(3.56)

(3.56) implies \((u^{ij})\) is a constant matrix. At the same time, (3.54) implies

\[
\sum_{\alpha=1}^{m+n} \sum_{\beta=m+1}^{m+n} |g_{\alpha \beta a}|^2 = 0.
\]  

(3.57)

(3.55), (3.56) and (3.57) together show \(\nabla_{g_0} g = 0\).

We define

\[
\eta^i = i^*_z \left( \frac{\partial}{\partial z^i} \omega \right),
\]

(3.58)

where \(z \in \mathbb{C}^m, i = 1, 2, \ldots, m\). Since \(\nabla_{g_0} g = 0\), this definition doesn’t depend on the choice of \(z\) and every \(\eta^i\) is an \(\omega_Y\)-paralleled \((0,1)\)-form. The expression (1.1) can be easily checked under a local coordinate system. This concludes the proof of Theorem 1.7.

\[\square\]
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REFERENCES

[1] T. Aubin, *Equations du type de Monge-Ampère sur les varietes Kähleriennes compactes*, C. R. Acad. Sci. Paris, 283 (1976), 119-121.

[2] E. Bedford and B.A. Taylor, *The Dirichlet problem for a complex Monge-Ampère operator*, Invent. Math. 37 (1976), 1-44.

[3] E. Bedford and B.A. Taylor, *Variational properties of the complex Monge-Ampère equation, II. Intrinsic norms*, Amer. J. Math., 101 (1979), 1131-1166.

[4] Z. Blocki, *Interior regularity of the complex Monge-Ampère equation in convex domains*, Duke Math. J. 105 (2000), no. 1, 167-181.

[5] R. Bott, L. W. Tu, *Differential Forms in Algebraic Topology*, Graduate Texts in Mathematics, 82. Springer-Verlag, New York-Berlin, 1982. xiv+331 pp. ISBN: 0-387-90613-4.

[6] L. Caffarelli, J. J. Kohn, L. Nirenberg, J. Spruck, *The Dirichlet problem for nonlinear second-order elliptic equations. II. Complex Monge-Ampère and uniformly elliptic equations*, Comm. Pure Appl. Math. 38 (1985) (2), 209-252.

[7] S. Y. Cheng, S. T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math. 28 (1975), 333-354.

[8] S. Y. Cheng, S. T. Yau, *On the existence of a complete Kähler-Einstein metric on non-compact complex manifolds and the regularity of Fefferman’s equation*, Comm. Pure Appl. Math. 33 (1980) 507-544.

[9] T. H. Colding, W. P. II Minicozzi, *Linear growth harmonic functions on complete manifolds with nonnegative Ricci curvature*, Geom. Funct. Anal. 5 (1995), no. 6, 948-954.

[10] J. P. Demailly, N. Pali, *Degenerate complex Monge-Ampère equations over compact Kähler manifolds*, Internat. J. Math., 21 (2010), no. 3, 357-405.

[11] S. Dinew, X. Zhang, X. W. Zhang, *The $C^{2,\alpha}$ estimate of complex Monge-Ampère equation*, Indiana Univ. Math. J., 60 (2011), (5), 1713-1722.

[12] B. Guan, *The Dirichlet problem for complex Monge-Ampère equations and regularity of the pluricomplex Green’s function*, Comm. Anal. Geom., 8(2000), 213-218.

[13] P. F. Guan. *The extremal function associated to intrinsic norms*, Ann. of Math. (2) 156 (2002), no. 1, 197-211.

[14] H.-J. Hein, *A Liouville theorem for the complex Monge-Ampère equation on product manifolds*, preprint, arXiv:1701.06147.

[15] R. Kobayashi, *Kähler-Einstein metrics on an open algebraic manifold*, Osaka. J. Math. 21 (1984) 399-418.

[16] S. K. Kolodziej, *The complex Monge-Ampère equation*, Acta Math. 180 (1998), 69-117.

[17] P. Li, *Large time behavior of the heat equation on complete manifolds with nonnegative Ricci curvature*, Comm. Pure Appl. Math. 33 (1980) 507-544.

[18] P. Li and R. Schoen, *$L^p$ and mean value properties of subharmonic functions on Riemannian manifolds*, Acta Math. 153 (1984), 279-301.

[19] P. Li and L. F. Tam, *Linear growth harmonic functions on a complete manifold*, J. Diff. Geom. 29 (1989), 421-425.

[20] N. Mok, S. T. Yau, *Completeness of the Kähler-Einstein metric on bounded domains and the characterization of domains of holomorphy by curvature conditions*, Proc. Symp. in Pure Math. 39 (1983) 41-59.

[21] D. H. Hein, *A Priori Estimates and a Liouville Theorem for Complex Monge-Ampère Equations*, Math. Z. 186 (1984), 57-66.

[22] G. Tian, *On the existence of solutions of a class of Monge-Ampère equations*, Acta Math. Sin. 4(3), 250-265 (1988)

[23] G. Tian, *On Calabi’s conjecture for complex surfaces with positive first Chern class*, Invent. Math. 101 (1990), no. 1, 101-172.

[24] G. Tian, *Kähler-Einstein metrics with positive scalar curvature*, Invent. Math. 130 (1997), no. 1, 1-37.

[25] G. Tian, S. T. Yau, *Existence of Kähler-Einstein metrics on complete Kähler manifolds and their applications to algebraic geometry*, in Mathematical aspects of string theory, ed. by S. T. Yau (1987) 575-627, World Scientific.

[26] G. Tian, S. T. Yau, *Kähler-Einstein metrics on complex surfaces with $c_1(M)$ positive*, Comm. Math. Phys. 112 (1987) 175-203.

[27] G. Tian, S. T. Yau, *Complete Kähler manifolds with zero Ricci curvature I*, J. Amer. Math. Soc. 3 (1990) 579-609.

[28] G. Tian, S. T. Yau, *Complete Kähler manifolds with zero Ricci curvature II*, Inventiones Math. 106 (1991) 27-60.
[30] V. Tosatti, B. Wenkove, S. T. Yau, *Taming symplectic forms and the Calabi-Yau equation*, Proc. Lond. Math. Soc. (3), 97 (2008), no. 2, 401–424.

[31] J. P. Wang, *Linear growth harmonic functions on complete manifolds*, Comm. Anal. Geom. 3 (1995), no. 3-4, 683–698.

[32] Y. Wang, *On the $C^{2,\alpha}$-regularity of the complex Monge-Ampère equation*, Math. Res. Lett. 19 (2012), no. 4, 939–946.

[33] S. T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. 28 (1975), 201C228.

[34] S. T. Yau, *Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry*, Indiana Math. J. 25 (1976), 659C670.

[35] S. T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation*, Comm. Pure Appl. Math. 31 (1978), 339–411.

[36] Z. Zhang, *On degenerate Monge-Ampère equations over closed Kähler manifolds*, Int. Math. Res. Not. 2006, Art. ID 63640, 18 pp.

[37] X. Zhang, X. W. Zhang, *Regularity estimates of solutions to complex Monge-Ampère equations on Hermitian manifolds*, J. Funct. Anal. 260 (2011), no. 7, 2004–2026.

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