On the Initial–Boundary–Value Problem for the Time–Fractional Diffusion Equation in the Quarter Plane

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Abstract

Taking into account the asymptotic behavior of some Wright functions and the existence of bounds for the Mainardi and the Wright function $W(-x, \frac{\alpha}{2}, 1)$ in $\mathbb{R}^+$, three different initial–boundary–value problems for the time–fractional diffusion equation in the quarter plane, where the time–fractional derivative is taken in the Caputo sense of order $\alpha \in (0, 1)$ are solved. Moreover, the limit when $\alpha \rightarrow 1$ of the respective solutions are analyzed, recovering the respective solutions of the classical boundary–value problems when $\alpha = 1$ and the fractional diffusion equation becomes the heat equation.

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1 Introduction

The one–dimensional Heat Equation has become the paradigm for the all–embracing study of parabolic partial differential equations, linear and nonlinear. A methodical development of a variety of aspects of this paradigm can be seen in [2, 7, 24].

This paper deals with two problems associated to the time–fractional diffusion equation, obtained from the standard diffusion equation by replacing the first order time–derivative by a fractional derivative of order $\alpha > 0$ in the Caputo sense:

\[
\partial_t^\alpha u(x,t) = \lambda^2 u_{xx}(x,t), \quad -\infty < x < \infty, \ t > 0, \ 0 < \alpha < 1,
\]

where the fractional derivative in the Caputo sense of arbitrary order $\alpha > 0$ is given by

\[
\partial_x^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, & n-1 < \alpha < n \\ f^{(n)}(t), & \alpha = n. \end{cases}
\]
where \( n \in \mathbb{N} \) and \( \Gamma \) is the Gamma function defined by \( \Gamma(x) = \int_0^\infty w^{x-1}e^{-w}dw \).

The interest on equation (1) has been in constant increase during the last 30 years. So many authors have studied it \([4, 8, 12, 14, 15, 20, 26]\) and, among the several applications that have been studied, Mainardi \([16]\) focused on the application to the theory of linear viscoelasticity.

A comprehensive analysis of the Cauchy problem associated to this equation can be found in \([6]\) and a physical meaning is discussed in \([23]\).

The two initial–boundary–value problems considered are:

\[
\begin{align*}
&0 \frac{\partial^\alpha c(x,t)}{\partial t^\alpha} = \lambda^2 \frac{\partial^2 c(x,t)}{\partial x^2}, \quad 0 < x < \infty, \ 0 < t < T, \ 0 < \alpha < 1, \\
c(x,0) = f(x) & \quad 0 < x < \infty, \\
c(0,t) = g(t) & \quad 0 < t < T,
\end{align*}
\]

(2)

and

\[
\begin{align*}
&0 \frac{\partial^\alpha c(x,t)}{\partial t^\alpha} = \lambda^2 \frac{\partial^2 c(x,t)}{\partial x^2}, \quad 0 < x < \infty, \ 0 < t < T, \ 0 < \alpha < 1, \\
c(x,0) = f(x) & \quad 0 < x < \infty, \\
\frac{\partial c}{\partial x}(0,t) = g(t) & \quad 0 < t < T.
\end{align*}
\]

(3)

Some variants of this problems have been solved in \([9]\) and \([26]\). The former using scale–invariant techniques and Laplace transform, and the last one in terms of Fox functions using Mellin transform. We propose here a different approach involving convolutions and we prove in each case that the function proposed is a solution of the considered problem.

The paper is presented as follows: Some useful properties about the behavior of Wright functions are given in Section 1. In sections 2, 3 and 4 the two problems enunciated previously will be solved. At the end of sections 2 and 4 the limit when \( \alpha \rightarrow 1 \) of the respective solutions will be done, recovering the respective solutions of the classical boundary–value problems when \( \alpha = 1 \) and equation (1) becomes the heat equation.

2 Preliminaries. Some Results about the special functions involved.

**Definition 1.** For every \( z \in \mathbb{C} \), \( \alpha > -1 \) and \( \beta \in \mathbb{R} \) the Wright function is defined by

\[
\mathcal{W}(z; \alpha; \beta) = \sum_{k=0}^{\infty} \frac{z^k}{k! \Gamma(\alpha k + \beta)}. \tag{4}
\]
Definition 2. For every $z \in \mathbb{C}$, $0 < \nu < 1$ the Mainardi function is defined by

$$M_\nu(z) = W(z; -\nu; 1 - \nu) = \sum_{k=0}^{\infty} \frac{z^k}{k! \Gamma(-\nu k + 1 - \nu)}.$$ (5)

Note 1. This series are absolutely convergent over compact sets and so its derivatives are easy to calculate:

$$\frac{d}{dz} W(z, \alpha, \beta) = \sum_{n=0}^{\infty} \frac{d}{dz} \left( \frac{(z)^n}{n! \Gamma(\alpha n + \beta)} \right) = \sum_{n=0}^{\infty} \frac{(z)^n}{n! \Gamma(\alpha(n+1) + \beta)} = W(z, \alpha, \alpha + \beta).$$

For the special case of the Mainardi function, we have

$$\frac{d}{dz} M_\nu(z) = W(z; -\nu; 1 - 2\nu).$$

2.1 Asymptotic behavior.

The following asymptotic behavior for the Mainardi function was proved in [19].

$$M_\nu \left( \frac{x}{\nu} \right) \sim a(\nu) x^{(\nu-1/2)/(1-\nu)} \exp \left[ -b(\nu) x^{1/(1-\nu)} \right],$$ (6)

where $a(\nu) = \frac{1}{\sqrt{2\pi(1-\nu)}} > 0$ and $b(\nu) = \frac{1 - \nu}{\nu}$.

Theorem 1. If $-1 < \rho < 0$, $y = -z$, $|\arg y| \leq \min \{ \frac{3\pi}{2}, \pi \} - \epsilon$, $\epsilon > 0$, then

$$W(z, \rho, \beta) = I(Y), \quad Y \to \infty,$$

where

$$I(Y) = Y^{1/2-\beta} e^{-Y} \left\{ \sum_{m=0}^{M-1} A_m Y^{-m} + O(Y^{-M}) \right\} \quad \text{and} \quad Y = (1 + \rho) (-(\rho)^{-\rho} y)^{1/\rho}.$$

The coefficients $A_m, m = 0, 1, \ldots$ are defined by the asymptotic expansion

$$\frac{\Gamma(1 - \beta - \rho t)}{2\pi(-\rho)^{-\rho t} (1 + \rho)^{(1+\rho)(t+1)} \Gamma(t+1)} = \sum_{m=0}^{M-1} \frac{(-1)^m A_m}{\Gamma((1+\rho)t + \beta + 1/2 + m)^{t}} + O \left( \frac{1}{\Gamma((1+\rho)t + \beta + 1/2 + M)} \right),$$

valid for $\arg t$, $\arg(-\rho t)$ and $\arg(1 - \beta - \rho t)$ all lying between $-\pi$ and $\pi$ and $t$ tending to infinity.
This theorem was proved in [25]. The next results follows.

**Corollary 1.**

\[
\lim_{x \to \infty} W(-x, -\frac{\alpha}{2}, 1) = 0 \quad \text{and} \quad \lim_{x \to \infty} W(-x, -\frac{\alpha}{2}, 1 + \frac{\alpha}{2}) = 0.
\]

**Corollary 2.**

\[
\lim_{x \to \infty} M_{\alpha/2}(x) = 0.
\]

**Corollary 3.** If \(0 < \alpha < 1\) and \(x \in \mathbb{R}^+\), there exists \(R > 0\) such that,

\[
\left| W(-x, -\frac{\alpha}{2}, 1 - \alpha) \right| < P \left( bx^{\frac{1}{1-\alpha}} \right) \exp \left\{ -bx^{\frac{1}{1-\alpha}} \right\} \quad \forall x > R,
\]

where \(P(x)\) is a polynomial function of degree less or equal than 1 and \(b = (1-\frac{\alpha}{2})(\frac{\alpha}{2})^{\frac{\alpha}{\alpha-\alpha}} > 0\).

**Proof.**

Let us consider the function \(W(-x, -\frac{\alpha}{2}, 1 - \alpha)\).

\[ z = -x \Rightarrow y = x \quad \text{and} \quad \arg y = 0. \]

By Theorem 1, taking

\[
Y = \left( 1 - \frac{\alpha}{2} \right) \left( \frac{\alpha}{2} x \right)^{\frac{1}{1-\alpha}} = bx^{\frac{1}{1-\alpha}} \quad \text{and} \quad b = \left( 1 - \frac{\alpha}{2} \right) \left( \frac{\alpha}{2} \right)^{\frac{\alpha}{\alpha-\alpha}} > 0,
\]

\[
W \left( -x, -\frac{\alpha}{2}, 1 - \alpha \right) = \left( bx^{\frac{1}{1-\alpha}} \right)^{\alpha-1/2} \exp \left\{ -bx^{\frac{1}{1-\alpha}} \right\} \left\{ \sum_{m=0}^{M-1} A_m (bx^{\frac{1}{1-\alpha}})^{-m} + O \left( (bx^{\frac{1}{1-\alpha}})^{-M} \right) \right\}.
\]

Or equivalently,

\[
\frac{W \left( -x, -\frac{\alpha}{2}, 1 - \alpha \right)}{\left( bx^{\frac{1}{1-\alpha}} \right)^{\alpha-1/2} \exp \left\{ -bx^{\frac{1}{1-\alpha}} \right\}} - \sum_{m=0}^{M-1} A_m (bx^{\frac{1}{1-\alpha}})^{-m} = O \left( (bx^{\frac{1}{1-\alpha}})^{-M} \right).
\]

Taking \(M = 1\), there exists \(R > 0\) such that

\[
\left| \frac{W \left( -x, -\frac{\alpha}{2}, 1 - \alpha \right)}{\left( bx^{\frac{1}{1-\alpha}} \right)^{\alpha-1/2} \exp \left\{ -bx^{\frac{1}{1-\alpha}} \right\}} - A_0 \right| \leq C \quad \text{if} \ x > R.
\]
Then
\[ |\mathcal{W}\left(-x, -\frac{\alpha}{2}, 1 - \alpha\right)| \leq (bx^{1/\alpha})^{a-1/2} \left( C(bx^{1/\alpha})^{-1} + |A_0| \right) \exp\{-bx^{1/\alpha}\}, \ \text{if} \ x > R. \]

(7)

If \( 0 < \alpha < \frac{1}{2} \), \( (bx^{1/\alpha})^{a-1/2} < \frac{1}{(bR^{1/\alpha})^{1/2-a}} \).

If \( \alpha = \frac{1}{2} \), \( (bx^{1/\alpha})^{a-1/2} = 1 \).

If \( \frac{1}{2} < \alpha < 1 \), taking \( R \) large enough so that \( bx^{1/\alpha} > 1 \) if \( x > R \), it follows that
\[ (bx^{1/\alpha})^{a-1/2} < (bx^{1/\alpha}). \]

Hence there exists two constants \( B_0 \) and \( B_1 \) depending on \( \alpha \) such that
\[ (bx^{1/\alpha})^{a-1/2} < B_0 + B_1(bx^{1/\alpha}). \]

Finally,
\[ (bx^{1/\alpha})^{a-1/2} \left( C(bx^{1/\alpha})^{-1} + |A_0| \right) < \left( B_0 + B_1(bx^{1/\alpha}) \right) \left( C(bx^{1/\alpha})^{-1} + |A_0| \right) = \]
\[ = P_1 \left( (bx^{1/\alpha})^{-1} \right) + P_2 \left( bx^{1/\alpha} \right) \leq \tilde{C} + P_2 \left( bx^{1/\alpha} \right) = P \left( bx^{1/\alpha} \right), \]

where \( P \) is a polynomial function of degree less or equal than 1.

Therefore
\[ |\mathcal{W}(-x, -\frac{\alpha}{2}, 1 - \alpha)| < P \left( bx^{1/\alpha} \right) \exp\{-bx^{1/\alpha}\}, \ \text{if} \ x > R. \]

\[ \square \]

**Corollary 4.** If \( 0 < \alpha < 1 \) and \( x \in \mathbb{R}^+ \), there exists \( R > 0 \) such that,
\[ |\mathcal{W}(-x, -\frac{\alpha}{2}, 1)| < Ke^{-bx}, \ \forall \ x > R \quad \text{where} \ b = \left( 1 - \frac{\alpha}{2} \right) \left( \frac{\alpha}{2} \right)^{2-\alpha}. \]
2.2 Some bounds and convergence.

The assertions in this subsection were proved in [21].

**Lemma 1.** If \(0 < \alpha < 1\), \(M_{\alpha/2}(x)\) is a strictly decreasing positive function in \(\mathbb{R}^+\).

**Corollary 5.** If \(x > 0\), \(M_{\alpha/2}(x) < \frac{1}{\Gamma(1-\frac{\alpha}{2})}\).

**Corollary 6.** If \(0 < \alpha < 1\), \(W(-x, -\frac{\alpha}{2}, 1)\) is a positive and decreasing function in \(\mathbb{R}^+\) such that \(0 < W(-x, -\frac{\alpha}{2}, 1) \leq 1\), \(\forall x \in \mathbb{R}^+_0\).

**Note 2.** Note that

\[
W\left(-x, \frac{1}{2}, 1\right) = \int_{\infty}^{x} \left( \frac{\partial}{\partial x} W\left(-\xi, -\frac{1}{2}, 1\right) \right) d\xi = \int_{\infty}^{x} -W\left(-\xi, -\frac{1}{2}, \frac{1}{2}\right) d\xi = \\
= \int_{x}^{\infty} W\left(-\xi, -\frac{1}{2}, \frac{1}{2}\right) d\xi = \int_{x}^{\infty} \frac{1}{\sqrt{\pi}} e^{-\xi^2/4} d\xi = \\
= \frac{2}{\sqrt{\pi}} \int_{x/2}^{\infty} \frac{1}{\sqrt{\pi}} e^{-\xi^2} d\xi = erf\left(\frac{x}{\sqrt{2}}\right).
\]

Hence

\[1 - W\left(-x, -\frac{1}{2}, 1\right) = erf\left(\frac{x}{\sqrt{2}}\right).\]

**Lemma 2.** If \(x \in \mathbb{R}^+_0\) and \(\alpha \in (0, 1)\),

1. \(\lim_{\alpha \to 1} M_{\alpha/2}(x) = M_{1/2}(x) = \frac{e^{-x^2}}{\sqrt{\pi}}\)

2. \(\lim_{\alpha \to 1} \left[1 - W\left(-x, -\frac{\alpha}{2}, 1\right)\right] = \frac{1}{\sqrt{\pi}} erf\left(\frac{x}{\sqrt{2}}\right)\).

3 Solving the Initial–Boundary–Value Problem for the Time–Fractional Diffusion Equation in the Quarter Plane with Temperature–Boundary Condition.

Let us consider problem (2). The principle of superposition is valid due to the linearity of the Caputo derivative. Then, solve problem (2) is equivalent to solve the two auxiliary
problems:

\[
\begin{aligned}
0 D^\alpha_t c_1(x, t) &= \lambda^2 \frac{\partial^2 c_1}{\partial x^2}(x, t) \quad 0 < x < \infty, 0 < t < T, 0 < \alpha < 1, \\
c_1(x, 0) &= f(x) \quad 0 < x < \infty, \\
c_1(0, t) &= 0 \quad 0 < t < T.
\end{aligned}
\]  

(8)

and

\[
\begin{aligned}
0 D^\alpha_t c_2(x, t) &= \lambda^2 \frac{\partial^2 c_2}{\partial x^2}(x, t) \quad 0 < x < \infty, 0 < t < T, 0 < \alpha < 1, \\
c_2(x, 0) &= 0 \quad 0 < x < \infty, \\
c_2(0, t) &= g(t) \quad 0 < t < T.
\end{aligned}
\]  

(9)

Problem (8) was solved in [13] and its solution is given by

\[
c_1(x, t) = \int_0^\infty \frac{1}{2\lambda t^{\alpha/2}} \left[ M_{\alpha/2} \left( \frac{|x - \xi|}{t^{\alpha/2}} \right) - M_{\alpha/2} \left( \frac{|x + \xi|}{t^{\alpha/2}} \right) \right] f(\xi) d\xi
\]  

(10)

where the function \( M_{\alpha/2}(\cdot) \) is the Mainardi function defined in (5) and \( f \) is a continuous bounded function in \( \mathbb{R}_0^+ \) (which guarantees that \( c_1 \) is a solution, see the Cauchy problem in [5]).

In [21] it was proved that

\[
z(x, t) = A + (B - A) W \left( \frac{-x}{\lambda t^{\alpha/2}}, -\frac{\alpha}{2}, 1 \right),
\]  

(11)

where \( W(\cdot, -\frac{\alpha}{2}, 1) \) is the Wright function of parameters \(-\frac{\alpha}{2}\) and 1, defined in [4], is a solution to problem

\[
\begin{aligned}
0 D^\alpha_t z(x, t) &= \lambda^2 \frac{\partial^2 z}{\partial x^2}(x, t) \quad 0 < x < \infty, 0 < t < T, 0 < \alpha < 1, \\
z(x, 0) &= A \quad 0 < x < \infty, \\
z(0, t) &= B \quad 0 < t < T.
\end{aligned}
\]  

(12)

Then, we can assure that

\[
v(x, t) = W \left( \frac{-x}{\lambda t^{\alpha/2}}, -\frac{\alpha}{2}, 1 \right)
\]  

(13)

is a solution to problem

\[
\begin{aligned}
0 D^\alpha_t v(x, t) &= \lambda^2 \frac{\partial^2 v}{\partial x^2}(x, t) \quad 0 < x < \infty, 0 < t < T, 0 < \alpha < 1, \\
v(x, 0) &= 0 \quad 0 < x < \infty, \\
v(0, t) &= 1 \quad 0 < t < T.
\end{aligned}
\]  

(14)
Taking into account Note 1 and Corollary 2, function $v$ can be expressed as

$$v(x, t) = \int_0^t \frac{\partial}{\partial \tau} \mathcal{W} \left( \frac{-x}{\lambda^{\alpha/2}}, -\frac{\alpha}{2}, 1 \right) d\tau = \int_0^t \mathcal{M}_{\alpha/2} \left( \frac{x}{\lambda^{\alpha/2}} \right) \frac{\alpha x}{2\lambda^{\alpha/2+1}} d\tau.$$ 

Let be

$$K(x, t) = \mathcal{M}_{\alpha/2} \left( \frac{x}{\lambda^{\alpha/2}} \right) \frac{\alpha x}{2\lambda^{\alpha/2+1}}$$

and

$$1_{[0,t_0]}(t) = \begin{cases} 1 & \text{if } 0 < t < t_0 \\ 0 & \text{else} \end{cases}.$$ 

Then function $v$ can be written as a convolution in the $t$–variable

$$v(x, t) = K(x, t) * 1_{[0,t]}.$$ 

This new way of expressing $v$ lead us to propose the following function

$$c_2(x, t) = K(x, t) * g(t)1_{[0,t]} = \int_0^t \mathcal{M}_{\alpha/2} \left( \frac{x}{\lambda(t - \tau)^{\alpha/2}} \right) \frac{\alpha x}{2\lambda(t - \tau)^{\alpha/2+1}} g(\tau) d\tau \quad (15)$$

as a solution to problem (9).

In order to prove this assertion, let us enunciate the following lemma.

**Lemma 3.** Let $K(t - \tau)f(\tau)$ be a function that verifies the following conditions:

$$\frac{\partial}{\partial \tau} K(t - \tau)f(\tau) \leq g(\tau) \in L^1[0, t],$$

$$\left| \frac{\partial}{\partial \eta} K(\eta - \tau)f(\tau) \right| \in L^1(\Omega), \text{ where } \Omega = \{(\eta, \tau) \in \mathbb{R}^2 : \eta \in (0, t), 0 \leq \tau \leq \eta\}$$

$$\lim_{\tau \uparrow \eta} K(\eta - \tau)f(\tau) = h(\eta) \in L^1(0, t).$$

Then

$$0D_t^\alpha \left( \int_0^t K(t - \tau)f(\tau) d\tau \right) = \int_0^t (0D_t^\alpha K(t - \tau)) f(\tau) d\tau + 0I_t^{1-\alpha} (h(t)),$$

where $0I_t^{1-\alpha}$ is the fractional integral of Riemann–Liouville of order $1 - \alpha$ defined by

$$0I_t^{1-\alpha} h(t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - \tau)^{(1-\alpha)-1} h(\tau) d\tau.$$
Proof. Due to (17) and (19)
\[ \frac{d}{dt} \int_0^t K(t-\tau) f(\tau) d\tau = \int_0^t \frac{\partial}{\partial t} K(t-\tau) f(\tau) d\tau + \lim_{\tau \to t} K(t-\tau) f(\tau). \]

Now
\[ 0D_t^\alpha \left( \int_0^t K(t-\tau) f(\tau) d\tau \right) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial}{\partial \eta} \int_0^\eta K(\eta-\tau) f(\tau) d\tau d\eta = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial}{\partial \eta} \int_0^\eta K(\eta-\tau) f(\tau) d\tau + \lim_{\tau \to \eta} K(\eta-\tau) f(\tau) \left[ \int_0^\eta \frac{\partial}{\partial \eta} K(\eta-\tau) f(\tau) d\tau + \lim_{\tau \to \eta} K(\eta-\tau) f(\tau) \right] d\eta. \]

Since (18) holds, (20) is equal to
\[ \int_0^t \frac{\partial}{\partial \eta} K(\eta-\tau) f(\tau) d\tau + \frac{1}{\Gamma(1-\alpha)} \int_0^t \lim_{\tau \to \eta} K(\eta-\tau) f(\tau) (t-\eta)^{\alpha} d\eta. \]

Substituting \( s = \eta - \tau \),
\[ \frac{1}{\Gamma(1-\alpha)} \int_\tau^t \frac{\partial}{\partial \eta} K(\eta-\tau) (t-\eta)^{\alpha} d\eta = \frac{1}{\Gamma(1-\alpha)} \int_0^{t-\tau} \frac{\partial}{\partial s} K(s) (t-s)^{\alpha} ds = 0D_t^\alpha K(t-\tau). \]

On the other hand,
\[ \frac{1}{\Gamma(1-\alpha)} \int_0^t \lim_{\tau \to \eta} K(\eta-\tau) f(\tau) \frac{(t-\eta)^{\alpha}}{(t-\eta)^{\alpha}} d\eta = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{h(\eta)}{(t-\eta)^{\alpha}} d\eta = 0D_t^\alpha \left( h(t) \right). \]

Hence
\[ 0D_t^\alpha \left( \int_0^t K(t-\tau) f(\tau) d\tau \right) = \int_0^t \left( 0D_t^\alpha K(t-\tau) \right) f(\tau) d\tau + 0D_t^\alpha \left( h(t) \right). \]

Now the purpose is to prove that the kernel
\[ \mathcal{M}_{\alpha/2} \left( \frac{x}{\lambda(t-\tau)^{\alpha/2}} \right) \frac{\alpha x}{2\lambda(t-\tau)^{\alpha/2+1}} g(\tau) \]
verifies the hypothesis of Lemma 3.

\[ \bullet \text{Hypothesis (16)}:\]
\[
\int_{0}^{t} \frac{M_{\alpha/2} \left( \frac{x}{\lambda(t-\tau)^{\alpha/2}} \right)}{2\lambda(t-\tau)^{\alpha/2+1}} \frac{\alpha x}{2\lambda(t-\tau)^{\alpha/2+1}} g(\tau) \, d\tau = \int_{x/\lambda^{\alpha/2}}^{\infty} M_{\alpha/2} (y) \left| g \left( t - \left( \frac{x}{\lambda y} \right)^{2/\alpha} \right) \right| \, dy
\]

We know that:

\[
\int_{0}^{\infty} y^{\alpha} M_{\alpha/2}(y) \, dy = \frac{\Gamma(n + 1)}{\Gamma \left( \frac{\alpha}{2} n + 1 \right)} \quad \text{for all } \alpha \in (0, 2) \quad \text{(see [10]).} \tag{23}
\]

The Mainardi function is a positive and decreasing function in \( \mathbb{R}^+ \), (see [21]). \( g \) is a bounded function in \( [0, t] \), that is
\[
|g(\tau)| \leq M \quad \forall \tau \in [0, t]. \tag{25}
\]

Then (22) is convergent and (21) is \( \tau - \text{integrable} \) in \( [0, t] \).

- Hypothesis (17):
  \[
  \frac{\partial}{\partial t} \left[ M_{\alpha/2} \left( \frac{x}{\lambda(t-\tau)^{\alpha/2}} \right) \frac{\alpha x}{2\lambda(t-\tau)^{\alpha/2+1}} \right] g(\tau) =
  \]
  \[
  = W \left( -\frac{x}{\lambda(t-\tau)^{\alpha/2}}, -\frac{\alpha}{2}, 1 - \alpha \right) \left( \frac{\alpha x}{2\lambda(t-\tau)^{\alpha/2+1}} \right)^2 g(\tau) -
  \]
  \[
  - M_{\alpha/2} \left( \frac{x}{\lambda(t-\tau)^{\alpha/2}} \right) \frac{(\alpha/2 + 1)\alpha x}{2\lambda(t-\tau)^{\alpha/2+2}} g(\tau) \tag{26}
  \]

Applying Corollary 3 there exists \( \delta > 0 \) such that, for all \( \tau \in (t - \delta, t) \),
\[
\left| W \left( -\frac{x}{\lambda(t-\tau)^{\alpha/2}}, -\frac{\alpha}{2}, 1 - \alpha \right) \left( \frac{\alpha x}{2\lambda(t-\tau)^{\alpha/2+1}} \right)^2 g(\tau) \right| \leq
\]
\[
\leq \left( c + d \frac{x}{\lambda(t-\tau)^{\alpha/2}} \right) \exp \left\{ -b \left( \frac{x}{\lambda(t-\tau)^{\alpha/2}} \right)^{2-\alpha} \right\} \left( \frac{\alpha x}{2\lambda(t-\tau)^{\alpha/2+1}} \right)^2 |g(\tau)|,
\]

\( b > 0, \ c, d \) constants.

And this is an integrable function, in fact, making the substitution
\[
\frac{x}{\lambda(t-\tau)^{\alpha/2}} = r
\]

and considering the inequality
\[
\exp \{-x\} \leq \frac{n!}{x^n}, \quad \forall \ n \in \mathbb{N}, \ \forall \ x > 0, \tag{28}
\]

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it results that
\[
\int_{t-\delta}^{t} \left( c + d \frac{x}{\lambda(t-\tau)^{\alpha/2}} \right) \exp \left\{ -b \frac{x}{\lambda(t-\tau)^{\alpha/2}} \right\} \left( \frac{\alpha x}{2\lambda(t-\tau)^{\alpha/2+1}} \right)^2 |g(\tau)| \, d\tau =
\]
\[
= \int_{\lambda^{\alpha/2}}^{\infty} (c + dr) \exp \left\{ -br^{2/\alpha} \right\} \frac{\alpha r}{2} \left( \frac{r^{\alpha/2}}{x} \right)^{2/\alpha} |g \left( t - \left( \frac{x}{\lambda r} \right)^{2/\alpha} \right)| \, dr \leq
\]
\[
\leq C_{x,\lambda,\alpha} M \int_{\lambda^{\alpha/2}}^{\infty} (c + dr) r^{1+2/\alpha} \exp \left\{ -br^{2/\alpha} \right\} \, dr \leq C_{x,\lambda,\alpha} M \int_{\lambda^{\alpha/2}}^{\infty} \left( \frac{c + dr}{b r^{2n/2-\alpha}} \right) n! \, dr. \tag{29}
\]
It is easy to see that for any \( \alpha \in (0, 1) \), there exists \( n \in \mathbb{N} \) such that (29) is convergent. For example, if \( \alpha = 1/4 \) we can take \( n = 10 \).

Then, the first term of the sum (26) is bounded by an integrable function.

Let us consider the second term of sum (26). Making the substitution (27) and taking into account that the Mainardi function is a positive function, we have
\[
\int_{0}^{t} \left| M_{\alpha/2} \left( \frac{x}{\lambda(t-\tau)^{\alpha/2}} \right) \frac{(\alpha/2 + 1)\alpha x}{2\lambda(t-\tau)^{\alpha/2+1}} g(\tau) \right| \, d\tau \leq
\]
\[
\leq M \int_{\lambda^{\alpha/2}}^{\infty} (\alpha/2 + 1)M_{\alpha/2} \left( r \right) \left( \frac{\lambda r}{x} \right)^{2/\alpha} \, dr \leq MC_{x,\lambda,\alpha} \int_{0}^{\infty} M_{\alpha/2}(r) r^{2/\alpha} \, dr. \tag{30}
\]
Now, for any \( \alpha \in (0, 1) \), there exists \( k \in \mathbb{N} \) such that \( \frac{1}{k} < \frac{\alpha}{2} < 1 \). Then, using (23), it yields
\[
MC_{x,\lambda,\alpha} \int_{0}^{\infty} M_{\alpha/2}(r) r^{2/\alpha} \, dr \leq MC_{x,\lambda,\alpha} \int_{0}^{\infty} M_{\alpha/2}(r) r^{k} \, dr = MC_{x,\lambda,\alpha} \frac{\Gamma(k + 1)}{\Gamma \left( \frac{\alpha}{2} k + 1 \right)}. \tag{31}
\]
-

**Hypothesis (18):**
We have to prove that
\[
\frac{\partial}{\partial \eta} \left[ M_{\alpha/2} \left( \frac{x}{\lambda(\eta-\tau)^{\alpha/2}} \right) \frac{\alpha x}{2\lambda(\eta-\tau)^{\alpha/2+1}} \right] \frac{g(\tau)}{(t-\eta)^{\alpha}} \in L^1(\Omega),
\]
where \( \Omega = \{ (\eta, \tau) \in \mathbb{R}^2 : \eta \in (0, t), 0 \leq \tau \leq \eta \} \). Or equivalently,
\[
\frac{\partial}{\partial \eta} \left[ M_{\alpha/2} \left( \frac{x}{\lambda(\eta-\tau)^{\alpha/2}} \right) \right] \left( \frac{\alpha x}{2\lambda(\eta-\tau)^{\alpha/2+1}} \right) \frac{g(\tau)}{(t-\eta)^{\alpha}} -
\]
\[
- M_{\alpha/2} \left( \frac{x}{\lambda(\eta-\tau)^{\alpha/2}} \right) \frac{(\alpha/2 + 1)\alpha x}{2\lambda(\eta-\tau)^{\alpha/2+1}} \frac{g(\tau)}{(t-\eta)^{\alpha}} = I + II \in L^1(\Omega).
\]
Reasoning like in the previous item, using Corollary 3, inequality (28), Corollary 5 and Tonelli's theorem (see [1], p. 55), the following assertions are true:

\[ \int_0^{\eta} |I + II| \, d\tau < \infty \quad \forall \eta \in (0, t). \]  

(32)

Taking \( \delta \) small according to Corollary 3,

\[ \int_0^t \int_0^{\eta-\delta} |I + II| \, d\tau \, d\eta = \int_0^t \int_0^{\eta-\delta} |I + II| \, d\tau \, d\eta + \int_0^t \int_{\eta-\delta}^\eta |I + II| \, d\tau \, d\eta. \]  

(33)

Now, noting that

\[ \frac{\partial}{\partial \eta} \left[ M_{\alpha/2} \left( \frac{x}{\lambda(\eta - \tau)^{\alpha/2}} \right) \right] = -\frac{\partial}{\partial \tau} \left[ M_{\alpha/2} \left( \frac{x}{\lambda(\eta - \tau)^{\alpha/2}} \right) \right] \]  

(34)

and that

\[ -\frac{\partial}{\partial \tau} M_{\alpha/2} \left( \frac{x}{\lambda(\eta - \tau)^{\alpha/2}} \right) > 0 \quad \text{(it is a consequence of Lemma 4.2 from [21]).} \]  

(35)

Let be \( M \) defined in (25) and \( C \) any constant depending on \( \delta, \alpha, x \) or \( n \). Then

\[ \int_0^t \int_0^{\eta-\delta} |I| \, d\tau \, d\eta \leq \int_0^t \frac{MC}{(t-\eta)^{\alpha \delta^{\alpha/2+1}}} \int_0^{\eta-\delta} -\frac{\partial}{\partial \tau} M_{\alpha/2} \left( \frac{x}{\lambda(\eta - \tau)^{\alpha/2}} \right) \, d\tau \, d\eta = \]

\[ = \int_0^t \frac{MC}{(t-\eta)^{\alpha \delta^{\alpha/2+1}}} \left[ M_{\alpha/2} \left( \frac{x}{\lambda\eta^{\alpha/2}} \right) - M_{\alpha/2} \left( \frac{x}{\lambda\delta^{\alpha/2}} \right) \right] \, d\eta \leq \]

\[ \leq \frac{2MC}{\Gamma \left(1 - \frac{\alpha}{2}\right) \delta^{\alpha/2+1}} \int_0^t \frac{1}{(t-\eta)^{\alpha}} \, d\eta < \infty \]

due to Lemma 4.2 [21] and that \( \alpha \in (0, 1) \). Then,

\[ \int_0^t \int_0^{\eta-\delta} |I| \, d\tau \, d\eta < \infty. \]  

(36)

On the other hand,

\[ \int_0^t \int_0^{\eta-\delta} |II| \, d\tau \, d\eta = \int_0^t \int_0^{\eta-\delta} \left| M_{\alpha/2} \left( \frac{x}{\lambda(\eta - \tau)^{\alpha/2}} \right) \frac{(\alpha/2 + 1)\alpha x}{2\lambda(t - \tau)^{\alpha/2+1}} \, g(\tau) \right| \, d\tau \, d\eta = \]

\[ = \int_0^t \frac{1}{(t-\eta)^{\alpha}} \int_0^{\eta-\delta} \left| M_{\alpha/2} \left( \frac{x}{\lambda(\eta - \tau)^{\alpha/2}} \right) \frac{(\alpha/2 + 1)\alpha x}{2\lambda(t - \tau)^{\alpha/2+1}} g(\tau) \right| \, d\tau \, d\eta. \]
We proved in the previous item that
\[ \int_0^\eta \left| M_{\alpha/2} \left( \frac{x}{\lambda(\eta - \tau)^{\alpha/2}} \right) \frac{(\alpha/2 + 1)\alpha x}{2\lambda(\eta - \tau)^{\alpha/2+1}} g(\tau) \right| d\tau < \infty. \]

Recalling that \( \alpha \in (0, 1) \), it yields that
\[ \int_0^t \int_0^{\eta-\delta} |II| d\tau d\eta < \infty. \quad (37) \]

Then
\[ \int_0^t \int_0^{\eta-\delta} |I + II| d\tau d\eta < \infty. \quad (38) \]

Proceeding like in item 2, it can be proved that
\[ \int_0^t \int_{\eta-\delta}^{\eta} |II| d\tau d\eta < \infty. \quad (39) \]

Finally, (38) and (39) yield
\[ \int_0^t \int_0^{\eta} |I + II| d\tau d\eta < \infty. \quad (40) \]

From (36) and (40), Tonelli’s theorem holds and
\[ \frac{\partial}{\partial \eta} \left[ M_{\alpha/2} \left( \frac{x}{\lambda(\eta - \tau)^{\alpha/2}} \right) \frac{x}{\lambda(\eta - \tau)^{\alpha/2+1}} \frac{\alpha}{2} \left( \frac{2\lambda(\eta - \tau)^{\alpha/2+1}}{x} \right) g(\tau) \right] \in L^1(\Omega). \]

• Hypothesis (19):

Let us prove that
\[ \lim_{\tau \rightarrow \eta} M_{\alpha/2} \left( \frac{x}{\lambda(\eta - \tau)^{\alpha/2}} \right) \frac{\alpha x}{2\lambda(\eta - \tau)^{\alpha/2+1}} g(\tau) = 0. \quad (41) \]

Note that, due to (23) and (25),
\[ \lim_{\tau \rightarrow \eta} M_{\alpha/2} \left( \frac{x}{\lambda(\eta - \tau)^{\alpha/2}} \right) \frac{\alpha x}{2\lambda(\eta - \tau)^{\alpha/2+1}} g(\tau) = \lim_{s \rightarrow +0} M_{\alpha/2} \left( \frac{x}{\lambda s^{\alpha/2}} \right) \frac{\alpha x}{2\lambda s^{\alpha/2+1}} g(\eta - s) = \lim_{y \rightarrow \infty} M_{\alpha/2} \left( \frac{x}{\lambda y^{\alpha/2}} \right) \frac{\alpha x y^{\alpha/2+1}}{2\lambda} g \left( \eta - \frac{1}{y^{\alpha/2+1}} \right) = 0. \]

Finally, we can apply Lemma \[ \text{Lemma } \] to kernel (15). Then,
Proposition 1. The following limits holds:

1. \( \lim_{x \to 0^+} \int_0^t \mathcal{M}_{\alpha/2} \left( \frac{x}{\lambda(t-\tau)^{\alpha/2}} \right) \frac{\alpha x}{2\lambda(t-\tau)^{\alpha/2+1}} g(\tau) d\tau = 1. \)

2. \( \lim_{t \to 0^+} \int_0^t \mathcal{M}_{\alpha/2} \left( \frac{x}{\lambda(t-\tau)^{\alpha/2}} \right) \frac{\alpha x}{2\lambda(t-\tau)^{\alpha/2+1}} d\tau = 0. \)
Proof.
1. Taking \( n = 0 \) in (23), it yields \( \int_0^\infty \mathcal{M}_{\alpha/2}(u) du = 1 \).
Making the substitution (27) and applying Corollary 5 we have
\[
\left| \int_0^t \mathcal{M}_{\alpha/2} \left( \frac{x}{\lambda(t-\tau)^{\alpha/2}} \right) \frac{\alpha x}{2\lambda(t-\tau)^{\alpha/2+1}} d\tau - 1 \right| \leq \left| \int_{x/\lambda^{\alpha/2}}^\infty \mathcal{M}_{\alpha/2}(u) du - \int_0^\infty \mathcal{M}_{\alpha/2}(u) du \right| \leq \int_0^{x/\lambda^{\alpha/2}} \mathcal{M}_{\alpha/2}(u) du \leq \frac{1}{\Gamma(1 - \frac{\alpha}{2})} \frac{x}{\lambda^{\alpha/2}} \to 0 \quad \text{if} \quad x \searrow 0.
\]
2. It is a consequence of applying substitution (27).

Let us check now the border conditions.

• \( c_2(x, 0) = \lim_{t \searrow 0} c_2(x, t) \).

From Proposition [1] and [25],
\[
0 \leq \lim_{t \searrow 0} \left| \int_0^t \mathcal{M}_{\alpha/2} \left( \frac{x}{\lambda(t-\tau)^{\alpha/2}} \right) \frac{\alpha x}{2\lambda(t-\tau)^{\alpha/2+1}} g(\tau) d\tau \right| \leq \lim_{t \searrow 0} \int_0^t \mathcal{M}_{\alpha/2} \left( \frac{x}{\lambda(t-\tau)^{\alpha/2}} \right) \frac{\alpha x}{2\lambda(t-\tau)^{\alpha/2+1}} \frac{\alpha}{2} |g(\tau)| d\tau \leq \lim_{t \searrow 0} M \int_0^t \mathcal{M}_{\alpha/2} \left( \frac{x}{\lambda(t-\tau)^{\alpha/2}} \right) \frac{\alpha x}{2\lambda(t-\tau)^{\alpha/2+1}} \frac{\alpha}{2} d\tau = 0 \\
\therefore \quad c_2(x, 0) = 0.
\]

• \( c_2(0, t) = \lim_{x \searrow 0} c_2(x, t) \).

Note that
\[
\int_0^t \mathcal{M}_{\alpha/2} \left( \frac{\alpha x}{2\lambda(t-\tau)^{\alpha/2}} \right) \frac{x}{\lambda(t-\tau)^{\alpha/2+1}} g(\tau) d\tau = \frac{1}{\Gamma(1 - \frac{\alpha}{2})} \frac{x}{\lambda^{\alpha/2}} \to 0 \quad \text{if} \quad x \searrow 0.
\]

Applying Proposition [1] to the second member of the sum,
\[
\lim_{x \searrow 0} \int_{t}^{\infty} M_{\alpha/2} \left( \frac{x}{\lambda(t - \tau)^{\alpha/2}} \right) \frac{\alpha x}{2\lambda(t - \tau)^{\alpha/2+1}} g(t)d\tau =
\]
\[
= g(t) \lim_{x \searrow 0} \int_{t}^{\infty} M_{\alpha/2} \left( \frac{x}{\lambda(t - \tau)^{\alpha/2}} \right) \frac{\alpha x}{2\lambda(t - \tau)^{\alpha/2+1}} d\tau = g(t).
\]

Let be
\[
I = \int_{0}^{t} M_{\alpha/2} \left( \frac{x}{\lambda(t - \tau)^{\alpha/2}} \right) \frac{x}{\lambda(t - \tau)^{\alpha/2+1}} \frac{\alpha}{2} [g(\tau) - g(t)] d\tau.
\]

The next goal is to prove that \(\lim_{x \searrow 0} I = 0\).

Let \(\delta > 0\) to be determined.

\[
I = \int_{0}^{t-\delta} M_{\alpha/2} \left( \frac{x}{\lambda(t - \tau)^{\alpha/2}} \right) \frac{\alpha x}{2\lambda(t - \tau)^{\alpha/2+1}} [g(\tau) - g(t)] d\tau +
\]
\[
+ \int_{t-\delta}^{t} M_{\alpha/2} \left( \frac{x}{\lambda(t - \tau)^{\alpha/2}} \right) \frac{\alpha x}{2\lambda(t - \tau)^{\alpha/2+1}} \frac{\alpha}{2} [g(\tau) - g(t)] d\tau = I_1 + I_2.
\]

Applying Corollary 5

\[
|I_1| = \left| \int_{0}^{t-\delta} M_{\alpha/2} \left( \frac{x}{\lambda(t - \tau)^{\alpha/2}} \right) \frac{\alpha x}{2\lambda(t - \tau)^{\alpha/2+1}} [g(\tau) - g(t)] d\tau \right| \leq
\]
\[
\leq \int_{0}^{t-\delta} \frac{1}{\Gamma \left(1 - \frac{\alpha}{2}\right)} \frac{\alpha x}{2\lambda(t - \tau)^{\alpha/2+1}} \frac{\alpha}{2} |g(\tau) - g(t)| d\tau \leq \frac{1}{\Gamma \left(1 - \frac{\alpha}{2}\right)} \frac{\alpha x}{2\lambda \delta^{\alpha/2+1}} \int_{0}^{t-\delta} |g(\tau) - g(t)| d\tau
\]
\[
\leq \left( \frac{1}{\Gamma \left(1 - \frac{\alpha}{2}\right)} \frac{\alpha}{2\lambda \delta^{\alpha/2+1}} \right) \int_{0}^{t} |g(\tau) - g(t)| d\tau = C_{t,\delta,\alpha} x < \frac{\epsilon}{2}, \text{ if } x < \frac{\epsilon}{2C_{t,\delta,\alpha}}. \tag{48}
\]

Now, \(g\) is continuous in \(t\). Then, for \(\epsilon > 0\) given, there exists \(\delta > 0\) such that \(|g(\tau) - g(t)| < \frac{\epsilon}{2}\) if \(|t - \tau| < \delta\). Using this fact and making the substitution \((27)\),

\[
|I_2| \leq \frac{\epsilon}{2} \int_{t-\delta}^{t} M_{\alpha/2} \left( \frac{x}{\lambda(t - \tau)^{\alpha/2}} \right) \frac{x}{\lambda(t - \tau)^{\alpha/2+1}} \frac{\alpha}{2} d\tau = \frac{\epsilon}{2} \int_{\lambda \delta^{\alpha/2}}^{\infty} M_{\alpha/2} \left( \frac{x}{\lambda \delta^{\alpha/2}} \right) d\lambda < \frac{\epsilon}{2} \tag{49}
\]

From (48) and (49), it results that \(|I| < \epsilon\), for every \(\epsilon > 0\). Then,

\[
\lim_{x \searrow 0} \int_{0}^{t} M_{\alpha/2} \left( \frac{x}{\lambda(t - \tau)^{\alpha/2}} \right) \frac{x}{\lambda(t - \tau)^{\alpha/2+1}} \frac{\alpha}{2} [g(\tau) - g(t)] d\tau = 0,
\]

and we can assure that

\[
c_2(0, t) = g(t). \tag{50}
\]
Theorem 2. Let be $f$ a continuous bounded function in $\mathbb{R}_0^+$ and $g$ a continuous function in $[0,T)$. Then
\[
c(x, t) = \int_0^\infty \frac{1}{2\lambda t^{\alpha/2}} \left[ M_{\alpha/2} \left( \frac{|x - \xi|}{\lambda t^{\alpha/2}} \right) - M_{\alpha/2} \left( \frac{|x + \xi|}{\lambda t^{\alpha/2}} \right) \right] f(\xi) d\xi + \int_0^t M_{\alpha/2} \left( \frac{x}{\lambda(t - \tau)^{\alpha/2}} \right) \frac{\alpha x}{2\lambda(t - \tau)^{\alpha/2 + 1}} g(\tau) d\tau
\]
is a solution to problem
\[
\begin{cases}
D^\alpha c(x, t) = \lambda^2 \frac{\partial^2 c}{\partial x^2}(x, t) & 0 < x < \infty, 0 < t < T, 0 < \alpha < 1, \\
c(x, 0) = f(x) & 0 < x < \infty, \\
c(0, t) = g(t) & 0 < t < T.
\end{cases}
\]

Theorem 3. The limit when $\alpha \uparrow 1$ of the solution to problem
\[
\begin{cases}
D^\alpha c_2(x, t) = \lambda^2 \frac{\partial^2 c_2}{\partial x^2}(x, t) & 0 < x < \infty, 0 < t < T, 0 < \alpha < 1, \\
c_2(x, 0) = 0 & 0 < x < \infty, \\
c_2(0, t) = g(t) & 0 < t < T,
\end{cases}
\]
is the classical solution to the analogous problem when $\alpha = 1$ and we recover the heat equation
\[
\begin{cases}
\frac{\partial}{\partial t} w(x, t) = \lambda^2 \frac{\partial^2 c}{\partial x^2}(x, t) & 0 < x < \infty, 0 < t < T, \\
w(x, 0) = 0 & 0 < x < \infty, \\
w(0, t) = g(t) & 0 < t < T.
\end{cases}
\]

Proof. Let be
\[
w(x, t) = \int_0^t e^{-\frac{x^2}{4\lambda^2(t-\tau)^{3/2}}} \frac{x}{\lambda(t - \tau)^{3/2}} g(\tau) d\tau
\]
the solution of problem (54) (see [2]) and let be $c_2^\alpha$ the solution of problem (53) given by Theorem 2
\[
c_2^\alpha(x, t) = \int_0^t M_{\alpha/2} \left( \frac{x}{\lambda(t - \tau)^{\alpha/2}} \right) \frac{x}{\lambda(t - \tau)^{\alpha/2 + 1}} g(\tau) d\tau.
\]
\[
\lim_{\alpha \to 1} c_2^\alpha (x, t) = \lim_{\alpha \to 1} \left\{ \int_0^t M_{\alpha/2} \left( \frac{x}{\lambda (t-\tau)^{\alpha/2}} \right) \frac{\alpha x}{2\lambda (t-\tau)^{\alpha/2+1}} g(\tau) d\tau \right\} = \\
= \int_0^t \lim_{\alpha \to 1} M_{\alpha/2} \left( \frac{x}{\lambda (t-\tau)^{\alpha/2}} \right) \frac{\alpha x}{2\lambda (t-\tau)^{\alpha/2+1}} g(\tau) d\tau = \\
= \int_0^t e^{-\frac{x^2}{4\left( t-\tau \right)}} \frac{x}{2\sqrt{\pi} \lambda (t-\tau)^{3/2}} g(\tau) d\tau = w(x, t).
\]

4 Solving the Initial-Boundary-Value Problem for the Time-Fractional Diffusion Equation in the Quarter Plane with null Flux-Boundary Condition.

In this section, problem
\[
\begin{aligned}
0D^\alpha c_3(x, t) &= \lambda^2 \frac{\partial^2 c_3}{\partial x^2}(x, t) \quad 0 < x < \infty, 0 < t < T, 0 < \alpha < 1 \\
\frac{\partial c_3}{\partial x}(0, t) &= 0 \quad 0 < t < T \\
c_3(x, 0) &= f(x) \quad 0 < x < \infty
\end{aligned}
\]

(57)

Let us consider the following auxiliary problem
\[
\begin{aligned}
0D^\alpha z(x, t) &= \lambda^2 \frac{\partial^2 c_3}{\partial x^2}(x, t) \quad -\infty < x < \infty, 0 < t < T, 0 < \alpha < 1 \\
z(x, 0) &= \tilde{f}(x) \quad -\infty < x < \infty
\end{aligned}
\]

where \(\tilde{f}\) is an even extension of \(f\).

This problem was solved in [11] and its solution is given by the following function
\[
c_3(x, t) = \int_{-\infty}^{\infty} \frac{t^{-\frac{\alpha}{2}}}{2\lambda} M_{\alpha/2} \left( \frac{|x - \xi|}{\lambda t^{\frac{\alpha}{2}}} \right) \tilde{f}(\xi) d\xi = \\
= \frac{1}{2\lambda t^{\frac{\alpha}{2}}} \int_0^{\infty} \left[ M_{\alpha/2} \left( \frac{x + \xi}{\lambda t^{\frac{\alpha}{2}}} \right) + M_{\alpha/2} \left( \frac{|x - \xi|}{\lambda t^{\frac{\alpha}{2}}} \right) \right] f(\xi) d\xi. \quad (58)
\]

The next goal is to prove that
\[
\lim_{x \to 0} \frac{\partial}{\partial x} c_3(x, t) = 0. \quad (59)
\]

\[
\frac{\partial c_3}{\partial x}(0, t) = \lim_{x \to 0} \frac{1}{2\lambda t^{\frac{\alpha}{2}}} \frac{\partial}{\partial x} \int_0^{\infty} \left[ M_{\alpha/2} \left( \frac{x + \xi}{\lambda t^{\frac{\alpha}{2}}} \right) + M_{\alpha/2} \left( \frac{|x - \xi|}{\lambda t^{\frac{\alpha}{2}}} \right) \right] f(\xi) d\xi =
\]

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Due to the continuity of the Mainardi function, Corollary 3 and \( f \in C(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+) \), the next equalities are true:

\[
\frac{\partial}{\partial x} \int_0^x \left[ M_{\alpha/2} \left( \frac{x + \xi}{\lambda t^{\alpha/2}} \right) + M_{\alpha/2} \left( \frac{x - \xi}{\lambda t^{\alpha/2}} \right) \right] f(\xi) d\xi + \int_x^\infty \left[ M_{\alpha/2} \left( \frac{x + \xi}{\lambda t^{\alpha/2}} \right) + M_{\alpha/2} \left( \frac{\xi - x}{\lambda t^{\alpha/2}} \right) \right] f(\xi) d\xi = \lim_{\xi \to x} \left[ M_{\alpha/2} \left( \frac{x + \xi}{\lambda t^{\alpha/2}} \right) + M_{\alpha/2} \left( \frac{\xi - x}{\lambda t^{\alpha/2}} \right) \right] f(\xi).
\]

(60)

and

\[
\frac{\partial}{\partial x} \int_x^\infty \left[ M_{\alpha/2} \left( \frac{x + \xi}{\lambda t^{\alpha/2}} \right) + M_{\alpha/2} \left( \frac{\xi - x}{\lambda t^{\alpha/2}} \right) \right] f(\xi) d\xi =
\]

\[
= - \int_0^x \frac{\partial}{\partial x} \left[ M_{\alpha/2} \left( \frac{x + \xi}{\lambda t^{\alpha/2}} \right) + M_{\alpha/2} \left( \frac{\xi - x}{\lambda t^{\alpha/2}} \right) \right] f(\xi) d\xi - \lim_{\xi \to x} \left[ M_{\alpha/2} \left( \frac{x + \xi}{\lambda t^{\alpha/2}} \right) + M_{\alpha/2} \left( \frac{\xi - x}{\lambda t^{\alpha/2}} \right) \right] f(\xi).
\]

(61)

Note that

\[
\int_0^x \frac{\partial}{\partial x} \left[ M_{\alpha/2} \left( \frac{x + \xi}{\lambda t^{\alpha/2}} \right) + M_{\alpha/2} \left( \frac{\xi - x}{\lambda t^{\alpha/2}} \right) \right] f(\xi) d\xi =
\]

\[
= - \frac{1}{\lambda t^{\alpha/2}} \int_0^x \left[ W \left( - \frac{x + \xi}{\lambda t^{\alpha/2}}, - \frac{\alpha}{2}, 1 - \alpha \right) + W \left( - \frac{x - \xi}{\lambda t^{\alpha/2}}, - \frac{\alpha}{2}, 1 - \alpha \right) \right] f(\xi) d\xi.
\]

Applying Mean Value Theorem, it yields that

\[
\left| \int_0^x \left[ W \left( - \frac{x + \xi}{\lambda t^{\alpha/2}}, - \frac{\alpha}{2}, 1 - \alpha \right) + W \left( - \frac{x - \xi}{\lambda t^{\alpha/2}}, - \frac{\alpha}{2}, 1 - \alpha \right) \right] f(\xi) d\xi \right| =
\]

\[
= \left| \left[ W \left( - \frac{x + \theta}{\lambda t^{\alpha/2}}, - \frac{\alpha}{2}, 1 - \alpha \right) + W \left( - \frac{x - \theta}{\lambda t^{\alpha/2}}, - \frac{\alpha}{2}, 1 - \alpha \right) \right] f(\theta) x \right| \quad \theta \in [0, x].
\]

Then,

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Applying Lebesgue Convergence Theorem to the first integral,
\[
\lim_{x \to \infty} \int_0^x \frac{\partial}{\partial x} \left[ \mathcal{M}_{\alpha/2} \left( \frac{x + \xi}{\lambda t^2} \right) + \mathcal{M}_{\alpha/2} \left( \frac{x - \xi}{\lambda t^2} \right) \right] f(\xi) d\xi = 0. \tag{63}
\]

On the other side,
\[
\int_x^\infty \frac{\partial}{\partial x} \left[ \mathcal{M}_{\alpha/2} \left( \frac{x + \xi}{\lambda t^2} \right) + \mathcal{M}_{\alpha/2} \left( \frac{x - \xi}{\lambda t^2} \right) \right] f(\xi) d\xi =
\]
\[
= \int_x^\infty -\frac{1}{\lambda t^{\alpha/2}} \left[ \mathcal{W} \left( -\frac{x + \xi}{\lambda t^2}, -\frac{\alpha}{2}, 1 - \alpha \right) - \mathcal{W} \left( -\frac{x - \xi}{\lambda t^2}, -\frac{\alpha}{2}, 1 - \alpha \right) \right] f(\xi) d\xi =
\]
\[
= \int_0^\infty -\frac{1}{\lambda t^{\alpha/2}} \left[ \mathcal{W} \left( -\frac{x + \xi}{\lambda t^2}, -\frac{\alpha}{2}, 1 - \alpha \right) - \mathcal{W} \left( -\frac{x - \xi}{\lambda t^2}, -\frac{\alpha}{2}, 1 - \alpha \right) \right] f(\xi) d\xi
\]
\[
- \int_0^x -\frac{1}{\lambda t^{\alpha/2}} \left[ \mathcal{W} \left( -\frac{x + \xi}{\lambda t^2}, -\frac{\alpha}{2}, 1 - \alpha \right) - \mathcal{W} \left( -\frac{x - \xi}{\lambda t^2}, -\frac{\alpha}{2}, 1 - \alpha \right) \right] f(\xi) d\xi.
\]
Applying Lebesgue Convergence Theorem to the first integral,
\[
\lim_{x \to \infty} \int_0^x -\frac{1}{\lambda t^{\alpha/2}} \left[ \mathcal{W} \left( -\frac{x + \xi}{\lambda t^2}, -\frac{\alpha}{2}, 1 - \alpha \right) - \mathcal{W} \left( -\frac{x - \xi}{\lambda t^2}, -\frac{\alpha}{2}, 1 - \alpha \right) \right] f(\xi) d\xi =
\]
\[
= \int_0^\infty \lim_{x \to \infty} -\frac{1}{\lambda t^{\alpha/2}} \left[ \mathcal{W} \left( -\frac{x + \xi}{\lambda t^2}, -\frac{\alpha}{2}, 1 - \alpha \right) - \mathcal{W} \left( -\frac{x - \xi}{\lambda t^2}, -\frac{\alpha}{2}, 1 - \alpha \right) \right] f(\xi) d\xi =
\]
\[
= \int_0^\infty -\frac{1}{\lambda t^{\alpha/2}} \left[ \mathcal{W} \left( -\frac{x + \xi}{\lambda t^2}, -\frac{\alpha}{2}, 1 - \alpha \right) - \mathcal{W} \left( -\frac{x - \xi}{\lambda t^2}, -\frac{\alpha}{2}, 1 - \alpha \right) \right] f(\xi) d\xi = 0.
\]

For the second integral we apply Mean Value Theorem as before. Then,
\[
\lim_{x \to \infty} \int_x^\infty \frac{\partial}{\partial x} \left[ \mathcal{M}_{\alpha/2} \left( \frac{x + \xi}{\lambda t^2} \right) + \mathcal{M}_{\alpha/2} \left( \frac{x - \xi}{\lambda t^2} \right) \right] f(\xi) d\xi = 0. \tag{64}
\]

From (63) and (64),
\[
\frac{\partial c_3}{\partial x}(0, t) = 0.
\]

**Theorem 4.** Let be \( f \) a continuous bounded function in \( \mathbb{R}^+_0 \), then
\[
c(x, t) = \frac{1}{2\lambda t^2} \int_0^\infty \left[ \mathcal{M}_{\alpha/2} \left( \frac{x + \xi}{\lambda t^2} \right) + \mathcal{M}_{\alpha/2} \left( \frac{|x - \xi|}{\lambda t^2} \right) \right] f(\xi) d\xi
\]
is a solution to problem
\[
\left\{
\begin{array}{ll}
D^\alpha c(x, t) = \lambda^2 \frac{\partial^2 c}{\partial x^2}(x, t), & 0 < x < \infty, 0 < t < T, 0 < \alpha < 1, \\
\frac{\partial c}{\partial x}(0, t) = 0, & 0 < t < T, \\
c(x, 0) = f(x), & 0 < x < \infty.
\end{array}
\right.
\]
5 The Initial–Boundary–Value Problem for the Time–Fractional Diffusion Equation in the Quarter Plane with Flux–Boundary Condition.

In this last section, the following problem will be solved:

\[
\begin{align*}
0 & D_t^\alpha c(x, t) = \lambda^2 \frac{\partial^2 c}{\partial x^2}(x, t) \quad & 0 < x < \infty, \ 0 < t < T, \ 0 < \alpha < 1 \\
c(x, 0) & = f(x) \quad & 0 < x < \infty \\
\frac{\partial}{\partial x} c(0, t) & = g(t) \quad & 0 < t < T
\end{align*}
\]

As in the previous sections two auxiliary problems are considered:

\[
\begin{align*}
0 & D_t^\alpha c_4(x, t) = \lambda^2 \frac{\partial^2 c_1}{\partial x^2}(x, t) \quad & 0 < x < \infty, \ 0 < t < T, \ 0 < \alpha < 1 \\
c_4(x, 0) & = f(x) \quad & 0 < x < \infty \\
\frac{\partial}{\partial x} c_4(0, t) & = 0 \quad & 0 < t < T
\end{align*}
\]

and

\[
\begin{align*}
0 & D_t^\alpha c_5(x, t) = \lambda^2 \frac{\partial^2 c_2}{\partial x^2}(x, t) \quad & 0 < x < \infty, \ 0 < t < T, \ 0 < \alpha < 1 \\
c_5(x, 0) & = 0 \quad & 0 < x < \infty \\
\frac{\partial}{\partial x} c_5(0, t) & = g(t) \quad & 0 < t < T
\end{align*}
\]

From Theorem 4, \(c_4(x, t)\) is a solution to (66).

In view of the results obtained in Section 3,

\[
c_5(x, t) = \int_0^\infty \left[ M_{\frac{\alpha}{2}} \left( \frac{x + \xi}{\lambda t^{\frac{\alpha}{2}}} \right) + M_{\frac{\alpha}{2}} \left( \frac{|x - \xi|}{\lambda t^{\frac{\alpha}{2}}} \right) \right] f(\xi) d\xi
\]

is proposed as a solution to problem (67).

Lemma 4. Let \(c(x, t)\) be a solution of the time–fractional diffusion equation \(0 D_t^\alpha c(x, t) = \lambda^2 \frac{\partial^2 c}{\partial x^2}(x, t)\) such that:

\[
\begin{align*}
(i) & \quad \text{For every } (x, t), \ \text{the function } F(x, t) = \int_x^\infty c(\xi, t) d\xi \text{ is well defined}, \\
(ii) & \quad \lim_{x \to \infty} \frac{\partial}{\partial x} c(x, t) = 0,
\end{align*}
\]
(iii) \( \left| \frac{\partial}{\partial \tau} c(\xi, \tau) \right| \leq g(\xi) \in L^1(x, \infty) , \)  

(71)

(iv) \( \frac{\partial}{(t-\tau)^\alpha} c(\xi, \tau) \in L^1((x, \infty) \times (0, t)) . \)  

(72)

Then \( \int_x^\infty c(\xi, t)d\xi \) is a solution to the time fractional diffusion equation.

Proof.

From (69), \( F(x, t) = \int_x^\infty c(\xi, t)d\xi \) is well defined. The next equalities are valid due to (70), (71) and (72):

\[
0 D_t^\alpha F(x, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial/\partial \tau F(x, \tau)}{(t-\tau)^\alpha} d\tau = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-\tau)^\alpha} \left( \frac{\partial}{\partial \tau} \int_x^\infty c(\xi, \tau)d\xi \right) d\tau =
\]

\[
= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-\tau)^\alpha} \int_x^\infty \frac{\partial}{\partial \tau} c(\xi, \tau) d\xi d\tau = \int_x^\infty \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial/\partial \tau c(\xi, \tau)}{(t-\tau)^\alpha} d\tau =
\]

\[
= \int_x^\infty 0 D_t^\alpha c(\xi, t)d\xi = \int_x^\infty \lambda^2 \frac{\partial^2 c}{\partial x^2} (\xi, t)d\xi = -\lambda^2 \frac{\partial c}{\partial x}(\xi, t) \bigg|_x^\infty =
\]

\[
= -\lambda^2 \frac{\partial c}{\partial x}(x, t) + \lambda^2 \lim_{x \to \infty} \frac{\partial c}{\partial x}(x, t) = \frac{\partial^2}{\partial x^2} \left( \int_x^\infty c(\xi, t)d\xi \right) = \frac{\partial^2}{\partial x^2} F(x, t).
\]

It can be proved that (68) is under the hypothesis of Lemma 4.

Respect on the border conditions:

- Observing that

\[
\int_0^t \mathcal{M}_{t/2} \left( \frac{\xi}{\lambda(t-\tau)^{t/2}} \right) \frac{\xi}{\lambda(t-\tau)^{t/2+1}} \frac{\alpha}{2} g(\tau)d\tau \leq MW \left( -\frac{\xi}{\lambda^{t/2}}, \frac{\alpha}{2}, 1 \right),
\]

Lebesgue Convergence Theorem can be applied and

\[
c_5(x, 0) = \lim_{t \to 0} -\int_x^\infty \int_0^t \mathcal{M}_{t/2} \left( \frac{x}{\lambda(t-\tau)^{t/2}} \right) \frac{x}{\lambda(t-\tau)^{t/2+1}} \frac{\alpha}{2} g(\tau)d\tau d\xi.
\]

\[
= \lim_{t \to 0} \left| \int_x^\infty \int_0^t \mathcal{M}_{t/2} \left( \frac{\xi}{\lambda(t-\tau)^{t/2}} \right) \frac{\xi}{\lambda(t-\tau)^{t/2+1}} \frac{\alpha}{2} g(\tau)d\tau d\xi \right| =
\]

\[
= \left| \int_x^\infty \lim_{t \to 0} \int_0^t \mathcal{M}_{t/2} \left( \frac{\xi}{\lambda(t-\tau)^{t/2}} \right) \frac{\xi}{\lambda(t-\tau)^{t/2+1}} \frac{\alpha}{2} g(\tau)d\tau d\xi \right| \leq
\]

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\[
\leq \int_x^\infty \left| \lim_{t \searrow 0} M_W \left( -\frac{\xi}{\lambda t^{\alpha/2}}, -\frac{\alpha}{2}, 1 \right) \right| d\xi = 0
\]

- From [50],

\[
\frac{\partial}{\partial x} c_5(0, t) = \lim_{x \searrow 0} \frac{\partial}{\partial x} \left( -\int_x^\infty \int_0^t M_{\alpha/2} \left( \frac{\xi}{\lambda(t-\tau)^{\alpha/2}} \right) \frac{\xi}{\lambda(t-\tau)^{\alpha/2+1}} \frac{\alpha}{2} g(\tau) d\tau d\xi \right) = \\
\lim_{x \searrow 0} \int_0^t M_{\alpha/2} \left( \frac{x}{\lambda(t-\tau)^{\alpha/2}} \right) \frac{x}{\lambda(t-\tau)^{\alpha/2+1}} \frac{\alpha}{2} g(\tau) d\tau = g(t).
\]

**Theorem 5.** Let \( f \) be a continuous bounded function in \( \mathbb{R}_0^+ \) and \( g \) a continuous function in \([0, T]\). Then

\[
c(x, t) = \frac{1}{2\lambda t^{\alpha/2}} \int_0^\infty \left[ M_{\alpha/2} \left( \frac{x + \xi}{\lambda t^{\alpha/2}} \right) + M_{\alpha/2} \left( \frac{|x - \xi|}{\lambda t^{\alpha/2}} \right) \right] f(\xi) - \\
- \int_x^\infty \int_0^t M_{\alpha/2} \left( \frac{x}{\lambda(t-\tau)^{\alpha/2}} \right) \frac{x}{\lambda(t-\tau)^{\alpha/2+1}} \frac{\alpha}{2} g(\tau) d\tau d\xi
\]

is a solution to problem

\[
\left\{
\begin{array}{ll}
D^\alpha c(x, t) = \lambda^2 \frac{\partial^2 c}{\partial x^2}(x, t), & 0 < x < \infty, 0 < t < T, 0 < \alpha < 1, \\
c(x, 0) = f(x), & 0 < x < \infty, \\
\frac{\partial}{\partial x} c(0, t) = g(t), & 0 < t < T.
\end{array}
\right.
\]  
\tag{74}

**Theorem 6.** The limit when \( \alpha \searrow 1 \) of the solution to problem

\[
\left\{
\begin{array}{ll}
D^\alpha c_5(x, t) = \lambda^2 \frac{\partial^2 c_2}{\partial x^2}(x, t), & 0 < x < \infty, 0 < t < T, 0 < \alpha < 1, \\
c_5(x, 0) = 0, & 0 < x < \infty, \\
\frac{\partial}{\partial x} c_5(0, t) = g(t), & 0 < t < T,
\end{array}
\right.
\]  
\tag{75}

is the classical solution to the analogous problem when \( \alpha = 1 \) and we recover the heat equation

\[
\left\{
\begin{array}{ll}
\frac{\partial}{\partial t} w(x, t) = \lambda^2 \frac{\partial^2 w}{\partial x^2}(x, t), & 0 < x < \infty, 0 < t < T, \\
w(x, 0) = 0, & 0 < x < \infty, \\
\frac{\partial}{\partial x} w(0, t) = g(t), & 0 < t < T.
\end{array}
\right.
\]  
\tag{76}

**Proof.** It can be seen in [2] that

\[
w(x, t) = -\int_0^t \frac{e^{-\frac{4\lambda^2}{\pi(t-\tau)}g(\tau)}}{\sqrt{\pi(t-\tau)}} d\tau
\]  
\tag{77}

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is a solution to problem \((76)\).

Let be \(c^\alpha\) a solution to problem \((75)\) given by Theorem \(5\),

\[
c^\alpha(x, t) = -\int_x^\infty \int_0^t M_{\alpha/2} \left( \frac{x}{\lambda(t-\tau)^{\alpha/2}} \right) \frac{x}{\lambda(t-\tau)^{\alpha/2+1}} \frac{\alpha}{2} g(\tau) d\tau d\xi.
\]

Applying Lebesgue Convergence Theorem, Lemma \(2\) and Fubini’s Theorem,

\[
\lim_{\alpha \to 1} c^\alpha(x, t) = \lim_{\alpha \to 1} \left\{ -\int_x^\infty \int_0^t M_{\alpha/2} \left( \frac{x}{\lambda(t-\tau)^{\alpha/2}} \right) \frac{x}{\lambda(t-\tau)^{\alpha/2+1}} \frac{\alpha}{2} g(\tau) d\tau d\xi \right\} =
\]

\[
= -\int_x^\infty \int_0^t \lim_{\alpha \to 1} M_{\alpha/2} \left( \frac{x}{\lambda(t-\tau)^{\alpha/2}} \right) \frac{x}{\lambda(t-\tau)^{\alpha/2+1}} \frac{\alpha}{2} g(\tau) d\tau d\xi =
\]

\[
= -\int_x^\infty \int_0^t e^{-\frac{x^2}{4(t-\tau)}} \frac{x}{2\sqrt{\pi} \lambda(t-\tau)^{3/2}} g(\tau) d\tau =
\]

\[
= -\int_0^t \int_x^\infty e^{-\frac{4x^2}{4(t-\tau)}} \frac{x}{2\sqrt{\pi} \lambda(t-\tau)^{3/2}} g(\tau) d\tau = -\int_0^t \frac{e^{-\frac{x^2}{4(t-\tau)}}}{\sqrt{\pi}(t-\tau)} g(\tau) d\tau = w(x,t).
\]

### 6 Conclusions

On the basis of the asymptotic behavior of some Wright functions and the existence of bounds for the Mainardi and the Wright function \(W(\alpha, \frac{\alpha}{2}, 1)\) in \(\mathbb{R}^+\), three different initial-boundary value problems for the time-fractional diffusion equation in the quarter plane were solved (considering temperature boundary condition, null flux boundary condition and flux boundary condition in the fixed face \(x = 0\)). In each case, certain conditions must be verified for the data to obtain the solution and the convergence of this solution when \(\alpha \to 1\) was analyzed, recovering the classical solutions of the respective boundary-value problems corresponding to the heat equation in the quarter plane.

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