ZARISKI CANCELLATION PROBLEM FOR
NONCOMMUTATIVE ALGEBRAS

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Abstract. A noncommutative analogue of the Zariski cancellation problem
asks whether $A[x] \cong B[x]$ implies $A \cong B$ when $A$ and $B$ are
noncommutative algebras. We resolve this affirmatively in the case when $A$ is a noncommutative
finitely generated domain over the complex field of Gelfand-Kirillov dimension
two. In addition, we resolve the Zariski cancellation problem for several classes
of Artin-Schelter regular algebras of higher Gelfand-Kirillov dimension.

0. Introduction

Kraft said in his 1995 survey [Kr] that “there is no doubt that complex affine
$n$-space $\mathbb{A}^n = \mathbb{A}^n_\mathbb{C}$ is one of the basic objects in algebraic geometry. It is therefore
surprising how little is known about its geometry and its symmetries. Although
there has been some remarkable progress in the last few years, many basic problems
remain open.” His remark still applies even today—20 years later. Let us start
with one of the famous questions in commutative affine geometry. Throughout the
introduction, we let $k$ be an algebraically closed field of characteristic zero (except
for some results mentioned below).

Question 0.1 (Zariski Cancellation Problem). Does an isomorphism $Y \times \mathbb{A}^1 \cong \mathbb{A}^{n+1}$ imply that $Y$ is isomorphic to $\mathbb{A}^n$? Or equivalently, does an isomorphism $B[t] \cong k[t_1, \ldots, t_n]$ of algebras imply that $B$ is isomorphic to $k[t_1, \ldots, t_n]$?

For simplicity, let ZCP denote the Zariski Cancellation Problem. An algebra $A$ is
called cancellative if $A[t] \cong B[t]$ for some algebra $B$ implies that $A \cong B$. So the
ZCP asks if the commutative polynomial ring $k[x_1, \ldots, x_n]$ is cancellative. Recall
that $k[x_1]$ is cancellative by a result of Abhyankar-Eakin-Heinzer [AEH], while
$k[x_1, x_2]$ is cancellative by Fujita [Fuj] and Miyanishi-Sugie [MS] in characteristic
zero and by Russell [Ru] in positive characteristic. The ZCP was open for many
years. In 2013, a remarkable development was made by Gupta [Gu1, Gu2] who
completely settled the ZCP negatively in positive characteristic for $n \geq 3$. The
ZCP in characteristic zero remains open for $n \geq 3$. We give a list of open questions
and problems that are closely related to the ZCP.

Question 0.2. For the following, let $k^\times$ be $k \setminus \{0\}$.

(ChP:=Characterization Problem) Find an algebro-geometric charac-
terization of $\mathbb{A}^n$.

(EP:=Embedding Problem) Is every closed embedding $\mathbb{A}^n \hookrightarrow \mathbb{A}^{n+n}$
equivalent to the standard embedding?

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(AP:=Automorphism Problem) Describe the group of polynomial automorphisms of \( \mathbb{A}^n \).
(LP:=Linearization Problem) Is every automorphism of \( \mathbb{A}^n \) of finite order linearizable?
(JC:=Jacobian Conjecture) Is every polynomial morphism \( \phi : \mathbb{A}^n \to \mathbb{A}^n \) with \( \det J\phi \in k^\times \) an isomorphism?

There are some known relationships between these problems. For example, a positive solution of the \( \text{LP} \) would imply a positive solution of the \( \text{ZCP} \). When \( n \leq 2 \), most of these questions (except for the \( \text{JC} \)) are resolved and there is a diagram of implications

\[ \text{EP} \implies \text{AP} \implies \text{LP} \implies \text{ZCP} \]

along with a possible “missing link” \( \text{JC} \implies \text{ZCP} \) (see [vdE]). Note that the \( \text{EP} \) (in dimension 2) was solved by Abyhyankar-Moh [AM] and Suzuki [Su]. Gupta’s work [Gu1, Gu2] would suggest a negative solution to the \( \text{ZCP} \), even in characteristic zero. If the “missing link” could be established and if the \( \text{ZCP} \) had a negative solution, then the \( \text{JC} \) could be settled negatively. Many authors have been working on these questions—see the references in [Kr, Lu, vdE].

Some naive and direct translations of these questions into the noncommutative setting are easily seen to have negative solutions. So it is important to carefully formulate noncommutative versions of these questions and to understand for which classes of (commutative or noncommutative) algebras these questions have positive or negative answers. Hopefully new ideas will emerge via the study of the noncommutative versions of these open questions. In this paper we mainly consider the following noncommutative formulation of the \( \text{ZCP} \).

**Question 0.3.** Let \( A \) be a noncommutative noetherian Artin-Schelter regular algebra [AS]. When is \( A \) cancellative?

Since Artin-Schelter regular algebras are considered as a noncommutative generalization of the commutative polynomial rings, the above question can be viewed as a noncommutative version of \( \text{ZCP} \).

In this paper we present two ideas to deal with the \( \text{ZCP} \) for some families of noncommutative algebras. One is to use the **Makar-Limanov invariant** and the other is to use **discriminants**.

Let us first review the Makar-Limanov invariant. Let \( A \) be an algebra and let \( \text{LND}(A) \) be the collection of locally nilpotent \( k \)-derivations of \( A \). The **Makar-Limanov invariant** of \( A \) is defined to be

\[ \text{ML}(A) = \bigcap_{\delta \in \text{LND}(A)} \ker(\delta). \]

The Makar-Limanov invariant was originally introduced by Makar-Limanov [Ma1] and has become a very useful invariant in commutative algebra. We say that \( A \) is **LND-rigid** if \( \text{ML}(A) = A \), or equivalently if \( \text{LND}(A) = \{0\} \). One of our main results (see Theorem 3.6 for the precise statement and proof) is the following, which shows that rigidity controls cancellation.

**Theorem 0.4.** Let \( A \) be a finitely generated domain of finite Gelfand-Kirillov dimension. If \( A \) is LND-rigid, then \( A \) is cancellative.
By the above theorem, we would like to show that various classes of noncommutative algebras are LND-rigid. Here is one of the consequences [Corollary 3.7].

**Theorem 0.5.** Let $k$ be an algebraically closed field of characteristic zero. Let $A$ be a finitely generated domain of Gelfand-Kirillov dimension two. If $A$ is not commutative, then $A$ is cancellative.

By [AEH], every commutative domain of Gelfand-Kirillov dimension (or GK-dimension, for short) one is cancellative. By [Da, Fi] there are commutative domains of GK-dimension two that are not cancellative. Theorem 0.5 ensures that every non-commutative domain of GK-dimension two is cancellative. Crachiola [Cr] showed that commutative UFDs of GK-dimension two are always cancellative.

Next let us talk about the discriminant method. The discriminant method was introduced in [CPWZ1, CPWZ2] to answer the AP for a class of noncommutative algebras. The definition of the discriminant in the noncommutative setting will be reviewed in Section 4. Suppose that $A$ is finitely generated by $\oplus_{i=1}^d kx_i$ as an algebra. An element $f \in A$ is called effective, if for every testing $\mathbb{N}$-filtered $k$-algebra $T$ with $\text{gr} T := \bigoplus F_i T / F_{i-1} T$ being an $\mathbb{N}$-graded domain, and for every testing subset $\{y_1, \ldots, y_d\} \subset T$ satisfying (a) it is linearly independent in the quotient $k$-module $T / k1_T$ and (b) some $y_i$ is not in $F_0 T$, there is a presentation of $f$ of the form $f(x_1, \ldots, x_d)$ when lifted in the free algebra $k\langle x_1, \ldots, x_d \rangle$ such that $f(y_1, \ldots, y_d)$ is either zero or not in $F_0 T$. For example, each monomial $x_1^{a_1} \cdots x_d^{a_d}$, for some positive integers $a_1, \ldots, a_d$, is effective. Note that there are non-monomial effective discriminants (see Examples 5.5 and 5.6). Here is one of our main results by using the discriminant, which provides a uniform way of showing the rigidity for some classes of noncommutative algebras.

**Theorem 0.6.** Suppose that $A$ is a domain which is a finitely generated module over its affine center $C$ and that the discriminant $d(A/C)$ is effective. Then $A$ is cancellative.

The above theorem does not answer the original ZCP as, when $A$ is commutative, the discriminant over its center is trivial and not effective. However, Theorem 0.6 applies to a large family of noncommutative algebras. One can check, for example, that the skew polynomial ring $k_q[x_1, \ldots, x_n]$ where $n$ is even and $1 \neq q$ is a root of unity has effective discriminant. Then, by Theorem 0.6, $k_q[x_1, \ldots, x_n]$ is cancellative. The next result shows a connection between the noncommutative ZCP and the noncommutative AP. Let $C$ denote the center of the algebra $A$ and we refer to Definition 4.5 for the definition of “dominating”. We have the following result (see Theorem 5.7 for an expanded version).

**Theorem 0.7.** Let $A$ be a skew polynomial ring $k_{p_{ij}}[x_1, \cdots, x_n]$ where each $p_{ij}$ is a root of unity. The following are equivalent.

1. The full automorphism group $\text{Aut}(A)$ is affine [CPWZ1 Definition 2.5].
2. The discriminant $d(A/C)$ is dominating.
3. The discriminant $d(A/C)$ is effective.
4. $A$ is LND-rigid.

Consequently, under any of these equivalent conditions, $A$ is cancellative.

In general, by using the Makar-Limanov invariant and Theorem 0.4, we show that if $d(A/C)$ is dominating, then $A$ is cancellative, see Theorem 4.7(2). As an example, we have the following result.
Theorem 0.8. Let $A$ be a finite tensor product of algebras of the form

(a) $k_p[x_1, \cdots, x_n]$ where $1 \neq p \in k^\times$ and $n$ is even;
(b) $k(x,y)/(x^2y-yx^2,y^2x+xy^2)$;
(c) $k(x,y)/(yx-qxy-1)$ where $1 \neq q \in k^\times$.

Then $A$ is LND-rigid. As a consequence, $A$ is cancellative.

Remark 0.9. Suppose that $n$ is odd and that $q \neq 1$ is a root of unity. It is an open question whether $k_q[x_1, \cdots, x_n]$ is cancellative. There are two results related to this.

(1) The following weak cancellative property holds as a consequence of [BZ, Theorem 9]: Let $B$ be a connected graded algebra generated in degree one. If $k_q[x_1, \cdots, x_n][t] \cong B[t]$ as algebras, then $k_q[x_1, \cdots, x_n] \cong B$ as graded algebras.

(2) A result of [CYZ2] says that Veronese subrings of $k_q[x_1, \cdots, x_n](v)$ is cancellative when $m$ and $v$ are not coprime, where $m$ is the order of $q$.

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1. Trivial center vs. cancellation

Throughout the rest of the paper we let $k$ be a base commutative domain. Sometimes we further assume that $k$ is a field. Everything is taken over $k$, for example, $\otimes$ stands for $\otimes_k$. We sometimes consider $k$-flat algebras. If $k$ is a field, then every $k$-module is flat. First we recall the definition of cancellative.

Definition 1.1. Let $A$ be an algebra.

(a) We call $A$ cancellative if $A[t] \cong B[t]$ for some algebra $B$ implies that $A \cong B$.
(b) We call $A$ strongly cancellative if, for each $d \geq 1$, $A[t_1, \ldots, t_d] \cong B[t_1, \ldots, t_d]$ for some algebra $B$ implies that $A \cong B$.
(c) We call $A$ universally cancellative if, for every $k$-flat finitely generated commutative domain $R$ such that the natural map $k \rightarrow R \rightarrow R/I$ is an isomorphism for some ideal $I \subset R$ and every $k$-algebra $B$, $A \otimes R \cong B \otimes R$ implies that $A \cong B$.

Remark 1.2. By the above definition, it is easy to see that

universally cancellative $\implies$ strongly cancellative $\implies$ cancellative.

But, it is not obvious to us whether any two of them are equivalent.

We have an immediate observation for the noncommutative cancellation problem. Let $C(A)$ denote the center of $A$.

If $A$ is an algebra over a commutative base ring $k$ (which we assume to be a domain but not a field in general), then the Gelfand-Kirillov dimension (or GK-dimension, for short) of $A$ is defined to be

$$\text{GKdim } A = \sup \left( \lim_{n \rightarrow \infty} \log_n \text{rank}_k(V^n) \right)$$

where $V$ varies over all finitely generated $k$-submodules of $A$, and the rank of a finitely generated $k$-module $M$ is defined to be the dimension of $M \otimes_k \text{Frac}(k)$ as
a Frac$(k)$-vector space, where Frac$(k)$ is the field of fractions of $k$. We refer to the book [KL] for basic properties of Gelfand-Kirillov dimension.

**Proposition 1.3.** Let $k$ be a field and $A$ be an algebra with center being $k$. Then $A$ is universally cancellative.

**Proof.** For any algebra $A$, let $C(A)$ denote the center of $A$. Let $R$ be an affine commutative domain such that $R/I = k$ for some ideal $I \subset R$ and suppose that $\phi : A \otimes R \to B \otimes R$ is an algebra isomorphism for some algebra $B$. Since $C(A) = k$, we have $C(A \otimes R) = R$. Since $C(B \otimes R) = C(B) \otimes R$ and since $\phi$ induces an isomorphism between the centers, we have

(E1.3.1) \[ R \cong C(B) \otimes R. \]

Consequently, $C(B)$ is a commutative domain. By considering the GK-dimension of both sides of (E1.3.1), one sees that GKdim $C(B) = 0$, when regarded as a $k$-algebra. (This also follows from Lemma 1.1.2.) Hence $C(B)$ is a field. Since there is an ideal $I$ such that $R/I = k$, $C(B) = k$. Consequently, $C(B \otimes R) = R$. Now the induced map $\phi$ is an isomorphism between $C(A \otimes R) = R$ to $C(B \otimes R) = R$, so we have $R/\phi(I) = k$. Finally, $\phi$ induces an automorphism from $A \cong A \otimes R/(I) \cong B \otimes R/(\phi(I)) \cong B$. \[ \square \]

We list some easy consequences below.

**Example 1.4.** We have the following results.

1. Let $k$ be a field of characteristic zero and $A_n$ the $n$th Weyl algebra. Then $C(A_n) = k$. So $A_n$ is universally cancellative.
2. Let $k$ be a field and $q \in k^\times$. Let $k_q[x_1, \ldots, x_n]$ be the skew polynomial ring generated by $x_1, \ldots, x_n$ subject to the relations $x_i x_j = q x_j x_i$ for all $1 \leq i < j \leq n$. If $n \geq 2$ and $q$ is not a root of unity, then $C(A) = k$. So $A$ is universally cancellative.

2. **Higher derivations and Makar-Limanov invariant**

The Makar-Limanov invariant is a very useful invariant to deal with the cancellation problem. We will also use a modified version of Makar-Limanov invariant to better control the cancellation in positive characteristic. Given a $k$-algebra $A$, let Der$(A)$ denote the collection of $k$-derivations of $A$ and let LND$(A)$ denote the collection of locally nilpotent $k$-derivations of $A$.

For a sequence of $k$-linear endomorphisms $\partial := \{\partial_i\}_{i \geq 0}$ of $A$ with the property that for every $a \in A$ we have $\partial_i(a) = 0$ for $i$ sufficiently large, and for every $c \in k$, we define

(E2.0.1) \[ G_{c, \partial} : A \to A \quad \text{by} \quad a \to \sum_{i=0}^{\infty} c^i \partial_i(a) \]

and

(E2.0.2) \[ G_{\partial, t} : A[t] \to A[t] \quad \text{by} \quad a \to \sum_{i=0}^{\infty} \partial_i(a)t^i, \quad t \to t, \]

for all $a \in A$.

**Definition 2.1.** Let $A$ be an algebra.
(1) [LL, Definition 2.3(1)] A higher derivation (or Hasse-Schmidt derivation [HS]) on $A$ is a sequence of $k$-linear endomorphisms $\partial := \{\partial_i\}_{i=0}^{\infty}$ such that:

$$\partial_0 = id_A, \quad \text{and} \quad \partial_n(ab) = \sum_{i=0}^{n} \partial_i(a)\partial_{n-i}(b)$$

for all $a, b \in A$ and all $n \geq 0$. The collection of higher derivations is denoted by $\text{Der}^H(A)$.

(2) [LL, Definition 2.3(1)] A higher derivation is called iterative if $\partial_i\partial_j = (i+j)\partial_{i+j}$ for all $i, j \geq 0$.

(3) A higher derivation is called locally nilpotent if

(a) for all $a \in A$ there exists $n \geq 0$ such that $\partial_i(a) = 0$ for all $i \geq n$,

(b) the map $G_{\partial, t}$ defined in (E2.1.2) is an algebra automorphism of $A[t]$.

The collection of locally nilpotent higher derivations is denoted by $\text{LND}^H(A)$ and the collection of locally nilpotent iterative higher derivations is denoted by $\text{LND}^I(A)$.

(4) For every $\partial \in \text{Der}^H(A)$, the kernel of $\partial$ is defined to be

$$\ker \partial = \bigcap_{i \geq 1} \ker \partial_i.$$
for all $a, b \in A$, by $k[t]$-linearity. Observe that, for all $a, b \in A$,

$$G_{\partial,t}(ab) = \sum_{i=0}^{\infty} t^i \partial_i(ab)$$

$$= \sum_{i=0}^{\infty} t^i \left( \sum_{j=0}^{i} \partial_j(a) \partial_{i-j}(b) \right)$$

$$= \sum_{j=0}^{\infty} \partial_j(a) t^j \left( \sum_{i=j}^{\infty} t^{i-j} \partial_{i-j}(b) \right)$$

$$= G_{\partial,t}(a) G_{\partial,t}(b),$$

where all interchanging of summations can be justified by the fact that the sums are actually finite. To see that $G_{\partial,t}$ is an automorphism, note that $k[t]$-linearity of $G_{\partial,t}$ and iterativity of $\partial$ give

$$G_{\partial,t} \circ G_{\partial,-t}(a) = G_{\partial,t} \left( \sum_{i=0}^{\infty} (-t)^i \partial_i(a) \right)$$

$$= \sum_{i=0}^{\infty} (-t)^i G_{\partial,t}(\partial_i(a))$$

$$= \sum_{i=0}^{\infty} (-t)^i \left( \sum_{j=0}^{\infty} t^j \partial_j \partial_i(a) \right)$$

$$= \sum_{i=0}^{\infty} (-t)^i \left( \sum_{j=0}^{\infty} t^j \left( \frac{i+j}{j} \right) \partial_{i+j}(a) \right)$$

$$= \sum_{i=0}^{\infty} \sum_{n=0}^{i} \binom{n}{i} (-1)^n \partial_n(a)$$

$$= \sum_{n=0}^{\infty} t^n \partial_n(a) \left( \sum_{i=0}^{n} (-1)^i \binom{n}{i} \right)$$

$$= \partial_0(a)$$

$$= a,$$

which gives that $G_{\partial,t}$ is invertible.

(3) Write $G(a) = \sum_{i \geq 0} \partial_i(a)t^i$ for all $a \in A$. Similar to the first part of the proof of (2), one sees that $\partial := (\partial_i)$ is in $\text{LND}^H(A)$. □

We now recall the definition of the Makar-Limanov invariant.

**Definition 2.3.** Let $A$ be an algebra over $k$. Let $*$ be either blank, or $^H$, or $^I$.

(1) The *Makar-Limanov* $^*$ invariant [Ma1] of $A$ is defined to be

$$\text{ML}^*(A) = \bigcap_{\delta \in \text{LND}^*(A)} \ker(\delta).$$

This means that we have original $\text{ML}(A)$, as well as, $\text{ML}^H(A)$ and $\text{ML}^I(A)$.

(2) We say that $A$ is $\text{LND}^*$-rigid if $\text{ML}^*(A) = A$, or $\text{LND}^*(A) = \{0\}$. 


(3) A is called strongly LND*-rigid if $\text{ML}^*(A[t_1, \ldots, t_d]) = A$, for all $d \geq 1$.

**Example 2.4.** Let $T$ be the polynomial ring $A[t_1, \cdots, t_d]$ over some $k$-algebra $A$. We fix an integer $1 \leq i \leq d$. For each $n \geq 0$, define a divided power version of $A$-linear differential operator

$$\Delta^n_i : t_1^{m_1} \cdots t_d^{m_d} \longrightarrow \begin{cases} \binom{m_i}{n} t_1^{m_1} \cdots t_i^{m_i-n} \cdots t_d^{m_d} & \text{if } m_i \geq n \\ 0 & \text{otherwise} \end{cases}$$

where $\binom{m_i}{n}$ is defined in $\mathbb{Z}$ or in $\mathbb{Z}/(p)$. Then $\{\Delta^n_i\}_{n=0}^\infty$ is a locally nilpotent iterative higher derivation of $T$.

If $k$ contains $\mathbb{Q}$, then $\Delta^n_i$ agrees with $\frac{\partial^n}{\partial t_i^n}$ for all $i$ and $n$.

Using locally nilpotent iterative higher derivations $\partial_i := \{\Delta^n_i\}_{n=0}^\infty$ of $T$, for $i = 1, \cdots, d$, one sees that $\text{ML}^H(T) \subseteq \text{ML}^I(T) \subseteq A$. In particular,

$$\text{ML}^H(k[t_1, \cdots, t_d]) = \text{ML}^I(k[t_1, \cdots, t_d]) = k.$$

**Remark 2.5.** Let $A$ be a $k$-algebra.

1. Suppose $k$ contains $\mathbb{Q}$. By using (E2.1.1), one sees that there is a bijection between $\text{LND}^I(A)$ and $\text{LND}(A)$. As a consequence, $\text{ML}^I(A) = \text{ML}(A)$. Since $\text{LND}^I(A) \subseteq \text{LND}^H(A)$, we obtain that $\text{ML}^H(A) \subseteq \text{ML}(A)$. In particular, if $A$ is $\text{LND}^H$-rigid, then it is $\text{LND}$-rigid.

2. Suppose $k$ contains $\mathbb{Q}$. It is not obvious to us whether $\text{ML}^H(A) = \text{ML}(A)$ in general. In particular, we don’t know if $\text{LND}$-rigidity is equivalent to $\text{LND}^H$-rigidity.

3. Suppose the prime field of $k$ is finite, but not $\mathbb{F}_2$. Let $A$ be the skew polynomial ring $k_{-1}[x_1, x_2]$ and $\partial$ be the nonzero locally nilpotent derivation of $A$ given in [CPWZ1] Example 3.9. Then, by definition, $\text{ML}(A) \subseteq A$. On the other hand, by Theorem 3.7(1) and Example 1.8(1) in Section 1, $A$ is $\text{LND}^H$-rigid, namely, $\text{ML}^H(A) = A$. Therefore

$$\text{ML}^I(A) = \text{ML}^H(A) = A \supseteq \text{ML}(A).$$

In particular, $\text{LND}^H$-rigidity is not equivalent to $\text{LND}$-rigidity. In this example, $A$ is (strongly) cancellative, see Theorem 4.7(2).

4. It follows from part (c) that the locally nilpotent derivation $\partial$ given in [CPWZ1] Example 3.9 (when char $k = p > 2$) can not be extended to a locally nilpotent higher derivation, but it is standard that $\partial$ can be extended to an iterative higher derivation by using an idea similar to (E2.4.1).

**Remark 2.6.** Suppose $A$ contains $\mathbb{Z}$. Let * be either blank, $H$ or $I$.

1. It is clear that $\text{ML}^*(A[t_1, \ldots, t_d]) \subseteq \text{ML}^*(A)$, but, it is not obvious to us whether $\text{ML}^*(A[t_1, \ldots, t_d]) = \text{ML}^*(A)$.

2. Makar-Limanov made the following conjecture in [Ma2]. If $A$ is a commutative domain over a field of characteristic zero, then $\text{ML}(A[t_1, \ldots, t_d]) = \text{ML}(A)$. And he proved that the conjecture holds when $\text{GKdim } A = 1$ [Ma2].

3. **Rigidity controls cancellation**

We shall investigate the relationship between LND-rigidity (respectively, strong LND-rigidity) and cancellation (respectively, strong cancellation).

We need the following lemma which is [KL, Proposition 3.11 and Lemma 6.5] when $k$ is a field. See the definition of Gelfand-Kirillov dimension, denoted by
Lemma 3.1. Let $A$ be a $k$-algebra and $R$ be an affine commutative $k$-algebra.

1. $\GKdim A = \GKdim Q(A \otimes Q)$ where $Q$ is the field of fractions of $k$. In particular, if $A$ is affine and commutative, $\GKdim A$ is an integer.

2. [KL, Proposition 3.11] $\GKdim A \otimes R = \GKdim A + \GKdim R$.

3. [KL, Lemma 6.5] Let $\{F_i A\}_{i \in \mathbb{Z}}$ be a filtration of $A$ in the sense of [KL, p. 73]. Let $M$ be a filtered right $A$-module with filtration $\{F_i M\}_{i \in \mathbb{Z}}$ in the sense of [KL, p. 74]. Then $\GKdim \text{gr}(M) \leq \GKdim M$.

Proof. (1) This follows from the definition of $\GKdim$ and the equation \[ \text{rank}_k(V^n) = \dim_Q(V \otimes_k Q)^n. \]

(2) This follows from part (1) and [KL, Proposition 3.11].

(3) This follows from part (1) and [KL, Lemma 6.5]. \qed

Lemma 3.2. Let $Y := \bigoplus_{i=0}^{\infty} Y_i$ be an $\mathbb{N}$-graded domain. If $Z$ is a subalgebra of $Y$ containing $Y_0$ such that $\GKdim Z = \GKdim Y_0 < \infty$, then $Z = Y_0$.

Proof. Let $X$ denote the subalgebra $Y_0$. Suppose $Z$ strictly contains $X$ as a subalgebra. Since $Y$ is a graded algebra, $Z$ is an $\mathbb{N}$-filtered algebra with $F_0 Z = X$. By Lemma 3.1(3), $\GKdim Z \geq \GKdim \text{gr} Z$. Since $\text{gr} Z$ is an $\mathbb{N}$-graded sub-domain of $Y$ that strictly contains $X$ as the degree zero part of $\text{gr} Z$, one can easily see that $\GKdim \text{gr} Z \geq \GKdim(\text{gr} Z)_0 + 1 = \GKdim X + 1$. Combining these inequalities, one obtains that $\GKdim Z \geq \GKdim X + 1$. This contradicts the hypothesis that $\GKdim Z = \GKdim X$. Therefore $Z = X$. \qed

It is well-known that a domain of finite $\GKdim$ is an Ore domain. Here is the main result of this section.

Theorem 3.3. Let $A$ be a finitely generated domain of finite $\GKdim$. Let $\ast$ be either blank, or $H$, or $I$. When $\ast$ is blank we further assume $A$ contains $Z$.

1. If $A$ is strongly LND$^\ast$-rigid, then $A$ is strongly cancellative.

2. If $\text{ML}^\ast(A[t]) = A$, then $A$ is cancellative.

Proof. We prove (1) and note that the proof of (2) is similar.

Let $\phi : A[t_1,\ldots,t_d] \to B[t_1,\ldots,t_d]$ be an isomorphism for some algebra $B$. By Lemma 3.1(2), $\GKdim B = \GKdim A < \infty$. For each $i$, let $\partial_i := \frac{\partial}{\partial t_i}$ when $\ast$ is blank and $\partial_i := \{\Delta^p_i\}_{n=0}$ as in Example 2.4 when $\ast$ is either $H$ or $I$. We have that $\text{ML}^\ast(B[t_1,\ldots,t_d])$ is contained in $B$ since $\partial_i$, for $i = 1,\ldots,d$, are locally nilpotent (higher) derivations of $B[t_1,\ldots,t_d]$ and the intersection of the kernels of these maps is exactly $B$ (see Example 2.4). On the other hand, we have that $\text{ML}^\ast(A[t_1,\ldots,t_d]) = A$ by hypothesis.

If $\partial$ is a locally nilpotent derivation of $B[t_1,\ldots,t_d]$ then $\phi^{-1} \circ \partial \circ \phi$ is a locally nilpotent derivation of $A[t_1,\ldots,t_d]$. Similarly if $\partial'$ is a locally nilpotent derivation of $A[t_1,\ldots,t_d]$ then $\phi \circ \partial' \circ \phi^{-1}$ is a locally nilpotent derivation of $B[t_1,\ldots,t_d]$. Similarly, the higher derivations of $A[t_1,\ldots,t_d]$ and $B[t_1,\ldots,t_d]$ correspond. Thus $\phi$ induces an algebra isomorphism

\[ \text{ML}^\ast(A[t_1,\ldots,t_d]) \cong \text{ML}^\ast(B[t_1,\ldots,t_d]). \]
In particular $\phi$ maps $A$ into $B$. Let $Y = A[t_1,\ldots,t_d]$ with $\deg t_i = 1$ and $Y_0 = A$ and $Z = \phi^{-1}(B)$. Then Lemma 3.2 implies that $\phi^{-1}(B) = A$. So $A$ and $B$ are isomorphic. The result follows. \hfill \Box

For the rest of this section we give some corollaries. We begin with a well-known result (see [BS] Lemma 3.2 or [Ba] Lemma 2.1 for related results). If $A$ is an Ore domain, let $Q(A)$ denote the fraction division ring of $A$.

**Lemma 3.4.** Let $A$ be an Ore domain containing $\mathbb{Z}$. Suppose that $A$ is endowed with a nonzero locally nilpotent derivation $\delta$. Then the following hold.

1. $A$ is embedded in the Ore extension $E[x;\delta_0]$ and $E[x;\delta_0]$ is embedded in $Q(A)$, where $E = \{a \in Q(A) \mid \delta(a) = 0\}$ and $\delta_0$ is a derivation of $E$.
2. $Q(A) = Q(E[x;\delta_0])$.
3. $\delta$ can be extended to a locally nilpotent derivation of $E[x;\delta_0]$ by declaring that $\delta(E) = 0$ and $\delta(x) = 1$.

**Proof.** (1) Let $E$ denote the kernel of the unique extension of $\delta$ to $Q(A)$. Then $E$ is a division subalgebra of $Q(A)$. Since $\delta$ is nonzero and locally nilpotent, we can find $x \in Q(A) \setminus E$ such that $\delta(x) \in E$. By replacing $x$ by $\alpha x$ for some $\alpha \in E$ we may assume that $\delta(x) = 1$. Now for every $a \in E$ we have $\delta([x,a]) = [\delta(x),a] = [1,a] = 0$. Thus $[x,a] \in E$ for all $a \in E$. In particular, $[x,-]$ induces a derivation $\delta_0$ of $E$.

Let $W = \{a \in Q(A) \mid \delta^n(a) = 0, \text{ for some } n \geq 0\}$. We claim that $W$ is a subset of the subalgebra of $Q(A)$ generated by $E$ and $x$. Since $[x,E] \subseteq E$, we have that this subalgebra is just

$$\sum_{i \geq 0} Ex^i.$$ 

To see the claim, we let $a \in W$. Then there is some smallest $n$ for which $\delta^n(a) = 0$. We prove the claim by induction on $n$. When $n = 0$ we have $a \in E$ and so the result follows. Now suppose that the claim holds whenever $\delta^j(a) = 0$ for some $j < n$ and consider the case where $\delta^n(a) = 0$ but $\delta^j(a) \neq 0$ for $j < n$. Then $\delta^{n-1}(a) = \alpha \in E$ with $\alpha \neq 0$. Since $\delta^{n-1}(\alpha x^{n-1}/(n-1)!)$, we see that $\delta^{n-1}(a - \alpha x^{n-1}/(n-1)!)$ = 0 and so by the induction hypothesis $a \in \sum Ex^i$. The claim follows.

It is clear that $\sum Ex^i \subseteq W$. So $W = \sum Ex^i$. Since $\delta$ is in $\text{LND}(A)$, $A \subseteq W$. Thus $A$ embeds in the subalgebra $W$ generated by $E$ and $x$. Since $[x,\alpha] = \delta_0(\alpha)$ for $\alpha \in E$, we see that $W$ is isomorphic to a homomorphic image of $E[t;\delta_0]$. We claim that $W$ cannot be isomorphic to a proper homomorphic image of $E[t;\delta_0]$. To see this, note that if it were $x$ would satisfy a non-trivial equation

$$x^d + \beta_{d-1}x^{d-1} + \cdots + \beta_0 = 0$$

for some $d \geq 1$ and $\beta_{d-1},\ldots,\beta_0 \in E$. We may assume without loss of generality that $d$ is minimal. Then applying $\delta$ and using the fact that $\delta(x) = 1$ and that $\delta$ is zero on $E$ gives

$$x^{d-1} + \sum_{j=1}^{d-1} jd^{-1}\beta_jx^{j-1} = 0,$$

contradicting the minimality of $d$. Thus we see that $A$ embeds in $W$ which is isomorphic to $E[x;\delta_0]$ as required.

Both (2) and (3) are clear. \hfill \Box
The following result was proved in [Ma2] in the commutative case.

**Lemma 3.5.** Let $A$ be a finitely generated Ore domain over $k$ that contains $\mathbb{Z}$. If $A$ is LND-rigid, then $\text{ML}(A[x]) = A$.

**Proof.** Let $C = \text{ML}(A[x])$. Note that $C \subseteq A$ since differentiation with respect to $x$ gives a locally nilpotent derivation of $A[x]$ and the kernel of this map is exactly $A$. It suffices to show that $C \supseteq A$. Suppose that there is a locally nilpotent derivation $\delta$ of $A[x]$ that does not send $A$ to zero. Suppose $a_1, \ldots, a_s$ generate $A$ as a $k$-algebra. Then $\delta(A) \subseteq A\delta(a_1)A + \cdots + A\delta(a_s)A$ and so there exists some smallest $m \geq 0$ such that $\delta(A) \subseteq A + Ax + \cdots + Ax^m$. If $m = 0$, then $\delta(A) \subseteq A$. Since $A$ is LND-rigid, $\delta(A) = 0$. This yields a contradiction and therefore $m \geq 1$. We write

$$\delta(a) = \mu(a)x^m + \text{lower degree terms}$$

for some derivation $\mu$ of $A$. We now consider the following three cases.

**Case I:** $\delta(x) \in A + Ax + \cdots + Ax^m$.

In this case we have $\delta(x^i) \subseteq \sum_{n=0}^{i+m-1} Ax^n$ and $\delta(Ax^i) \subseteq \sum_{n=0}^{i+m} Ax^n$ for all $i$. Thus for every $a \in A$ we have

$$\delta^2(a) = \mu^2(a)x^{2m} + \text{lower degree terms}.$$ More generally, we see that

$$\delta^i(a) = \mu^i(a)x^{mj} + \text{lower degree terms}.$$ Thus $\mu$ is a locally nilpotent derivation and so $\mu(A) = 0$, contradicting the minimality of $m$. Thus $\delta(A) = 0$ in this case.

**Case II:** $\delta(x) = bx^{m+1} + \text{lower degree terms for some } b \neq 0 \text{ in } A$.

Applying $\delta$ to the equation $[x, a] = 0$, one sees that $b$ commutes with every $a$ in $A$, and so $b$ is in the center of $A$. Now we define a new derivation $\delta'$ of $A[x]$ by declaring that $\delta'(a) = \mu(a)x^m$ for $a \in A$ and $\delta'(x) = bx^{m+1}$. Then we see that $\delta'$ sends $Ax^i$ to $Ax^{i+m}$ for every $i \geq 0$. We can view $\delta'$ as an associated graded derivation of $\delta$. Since $\delta$ is locally nilpotent, $\delta'$ is a locally nilpotent derivation of $A[x]$. Applying Lemma 3.3 to the algebra $A[x], A[x]$ embeds in $E[y; \delta_0]$ where $\delta_0$ is a derivation of $E$. Moreover, $\delta'$ extends to a locally nilpotent derivation of $E[y; \delta_0]$ by declaring that $\delta'(E) = 0$ and $\delta'(y) = 1$. Under this embedding $x = p(y)$ for some nonzero polynomial $p$. Let $d$ denote the degree of this polynomial. Then $bx^{m+1}$ gets sent to $q(y)p(y)^{m+1}$ for some nonzero polynomial $q(y)$. But since $\delta'(x)$ is nonzero, it has degree exactly $d - 1$ and so we have $(m + 1)d + \deg q(y) = d - 1$, which is impossible.

**Case III:** $\delta(x) = bx^i + \text{lower degree terms for some } b \neq 0 \text{ in } A \text{ and some } i > m+1$.

In this case we see that, for each $n \geq 2$,

$$\delta^n(x) = \left\{ \prod_{s=1}^{n-1} ((i - 1)s + 1) \right\} b^n x^{(i-1)n+1} + \text{lower degree terms},$$

so $\delta$ cannot be locally nilpotent, which contradicts the hypothesis.

Combining these cases, we see that $\delta(A) = 0$. The result follows. \hfill $\square$

We next give the proof of Theorem 0.4.
Theorem 3.6. Let $A$ be a finitely generated domain containing $\mathbb{Z}$ and suppose that $A$ has finite GK-dimension. If $A$ is LND-rigid, then $A$ is cancellative.

Proof. Since $A$ is a domain of finite GK-dimension, it is an Ore domain. By Lemma 3.5, $\text{ML}(A[x]) = A$. The assertion follows from Theorem 3.3(2). □

We now prove Theorem 0.5. We say an algebra $A$ is PI if it satisfies a polynomial identity.

Corollary 3.7. Let $A$ be a domain of GK-dimension two over an algebraically closed field $k$ of characteristic zero.

1. If $A$ is PI and not commutative, then $A$ is LND-rigid. As a consequence, if we assume in addition that $A$ is finitely generated over $k$, then $A$ is cancellative.

2. If $A$ is not PI, then $A$ is universally cancellative.

Proof. (1) If $A$ is not LND-rigid, then there is a nonzero locally nilpotent derivation $\delta$ of $A$. So the kernel of $\delta$ is not equal to $A$. As in Lemma 3.1, let $E$ denote the set of elements $a \in Q(A)$ such that $\delta(a) = 0$. By Lemma 3.4, $A$ embeds in $W := E[x; \delta]$ for some derivation $\delta_0$ of $E$. Since $W$ is a subalgebra of $Q(A)$, $Q(A)$ is infinite-dimensional as a left and right $E$-vector space. Hence $E$ has GK-dimension one [Be, Theorem 1.3]. Since $E$ is a subalgebra of $Q(A)$, it is PI and so by Tsen’s theorem, $E$ is commutative, whence $E$ is a field. By Lemma 3.3(3), $W := E[x; \delta_0]$ is a subring of $Q(A)$. Since $A$ is PI, $Q(A)$ is also PI and hence $E[x; \delta_0]$ is PI. We observe that this gives $\delta_0 = 0$. To see this, suppose that there is some $\alpha \in E$ such that $\beta := \delta_0(\alpha) \neq 0$. Then $[\beta^{-1}x, \alpha] = \beta^{-1}\delta_0(\alpha) = 1$ and so in this case we would have that $E[x; \delta_0]$ contains a copy of the Weyl algebra over $\mathbb{Q}$, which contradicts the fact that $E[x; \delta_0]$ is PI. Thus $\delta_0 = 0$ and $W$ is commutative. So $A$ is commutative, yielding a contradiction. The result follows.

The consequence follows from the main assertion and Theorem 3.6.

(2) If $A$ is not PI and has GK-dimension two, then, by [SZ, Corollary 2], $C(A) = k$. The assertion now follows from Proposition 1.7. □

Definition 3.8. An Ore domain $A$ is called birationally affine-ruled if $Q(A) = D(x)$ for some division algebra $D$ and birationally Weyl-ruled if $Q(A) = Q(E[x; \delta_0])$ for some division algebra $E$ and some nonzero derivation $\delta_0$ of $E$.

By Lemma 3.4, if $A$ has a nonzero locally nilpotent derivation, then $A$ is either birationally affine-ruled or birationally Weyl-ruled.

Corollary 3.9. Let $A$ be a finitely generated PI domain containing $\mathbb{Z}$ with finite GK-dimension. If $A$ is not birationally affine-ruled, then $A$ is LND-rigid and cancellative.

Proof. By Theorem 3.6, it suffices to show that $A$ is LND-rigid.

If $A$ is not LND-rigid, then $A$ is endowed with a nonzero locally nilpotent derivation. By Lemma 3.4, $A \subset E[x; \delta_0] \subset Q(A)$, where $E$ is a division subring of $Q(A)$. Since $A$ is PI, so are $Q(A)$ and $E[x; \delta_0]$. Then the center of $E[x; \delta_0]$ is not a subring of $E$. Let $f = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1$ be a central element in $E[x; \delta_0]$ for some $n \geq 1$ and $a_0 \neq 0$. Since $f$ is central,

$$0 = [x, f] = \sum_{i=0}^{n} [x, a_i]x^i = \sum_{i=0}^{n} \delta_0(a_i)x^i,$$
implying that \( \delta_0(a_i) = 0 \) for all \( i \). For every \( e \in E \),
\[
0 = [e, f] = [e, a_n]x^n + \text{lower degree terms},
\]
which implies that \([e, a_n] = 0\). Hence, \( a_n \) is in the center of \( E[x; \delta_0] \). By replacing \( f \) by \( a_n^{-1}f \), we may assume that \( a_n = 1 \). A straightforward calculation gives
\[
0 = [e, f] = ex^n - (ex^n + n\delta_0(e)x^{n-1} + \text{lower degree terms})
+ [e, a_{n-1}]x^{n-1} + \text{lower degree terms}
= (-n\delta_0(e) + [e, a_{n-1}])x^{n-1} + \text{lower degree terms}.
\]
Hence \(-n\delta_0(e) + [e, a_{n-1}] = 0\) or \( \delta_0(e) = [e, b] \) where \( b = \frac{1}{x}a_{n-1} \). Then \( E[x, \delta_0] = E[x'] \) where \( x' = x + b \). So \( A \) is birationally affine-ruled, a contradiction. \( \square \)

4. Discriminant

We recall the definition of the discriminant in the noncommutative setting and everything in this section is taken from [CPWZ1, CPWZ2]. Let \( R \) be a commutative algebra and let \( B \) and \( F \) be algebras both of which contain \( R \) as a subalgebra. In our applications, \( F \) will either be \( R \) or a ring of fractions of \( R \). An \( R \)-linear map \( \text{tr} : B \to F \) is called a trace map if \( \text{tr}(ab) = \text{tr}(ba) \) for all \( a, b \in B \).

If \( B \) is the \( w \times w \)-matrix algebra \( M_w(R) \) over \( R \), the internal trace \( \text{tr}_{\text{int}} : B \to R \) is defined to be the usual matrix trace, namely, \( \text{tr}_{\text{int}}((r_{ij})) = \sum_{i=1}^w r_{ii} \). Let \( B \) be an \( R \)-algebra, and suppose that \( B_F := B \otimes_R F \) is finitely generated and free over \( F \), where \( F \) is a localization of \( R \). Then left multiplication defines a natural embedding of \( R \)-algebras \( \text{lm} : B \to B_F \to \text{End}_F(B_F) \cong M_w(F) \), where \( w \) is the rank \( \text{rk}(B_F/F) \). Then we have a regular trace, by composing:
\[
\text{tr}_{\text{reg}} : B \overset{\text{lm}}{\longrightarrow} M_w(F) \overset{\text{tr}_{\text{int}}}{\longrightarrow} F.
\]
Usually we use the regular trace even if other trace functions exist. The following definition is well-known, see Reiner’s book [Re]. For an algebra \( A \), let \( A^\times \) denote the set of invertible elements in \( A \). If \( f, g \in A \) and \( f = cg \) for some \( c \in A^\times \), then we write \( f =_A^g \).

**Definition 4.1.** [CPWZ1, Definition 1.3] Let \( \text{tr} : B \to F \) be a trace map and \( v \) be a fixed integer. Let \( Z := \{z_i\}_{i=1}^v \) be a subset of \( B \).

1. The **discriminant** of \( Z \) is defined to be
\[
d_v(Z : \text{tr}) = \det(\text{tr}(z_i z_j))_{i,j=1}^v \in F.
\]
2. [Re, Section 10, p. 126]. The **\( v \)-discriminant ideal** (or **\( v \)-discriminant \( R \)-module**) \( D_v(B, \text{tr}) \) is the \( R \)-submodule of \( F \) generated by the set of elements \( d_v(Z : \text{tr}) \) for all \( Z = \{z_i\}_{i=1}^v \subset B \).
3. Suppose \( B \) is an \( R \)-algebra which is finitely generated free over \( R \) of rank \( w \). If \( Z \) is an \( R \)-basis of \( B \), the **discriminant of \( B \) over \( R \)** is defined to be
\[
d(B/R) = R^\times \cdot d_v(Z : \text{tr}).
\]
Note that \( d(B/R) \) is well-defined up to a scalar in \( R^\times \) [Re, p.66, Exer 4.13].

We refer to the books [AW, Re, St] for the classical definition of the discriminant and its connection with the above definition.

To cover a larger class of algebras that are not free over their centers, we need a modified version of the discriminant. Let \( B \) be a domain and let \( \mathcal{D} := \{d_i\}_{i \in I} \)
be a set of elements in \( B \). A normal element \( x \in B \) is called a common divisor if \( d_i = d'_i x \) for some \( d'_i \) for all \( i \in I \). We say a normal element \( x \in B \) is the greatest common divisor or gcd of \( D \), denoted by \( \text{gcd} D \), if

(a) \( x \) is a common divisor of \( D \), and

(b) for every common divisor \( y \) of \( D \), \( x = cy \) for some \( c \in B \).

It follows from part (b) that the gcd of any subset \( D \subseteq B \) (if it exists) is unique up to a scalar in \( B^\times \).

**Definition 4.2.** [CPWZ2 Definition 1.2] Let \( \text{tr} : B \to R \) be a trace map and let \( v \) be a positive integer. Let \( Z \) (respectively, \( Z' \)) denote a \( v \)-element subset \( \{z_i\}_{i=1}^v \) (respectively, \( \{z'_i\}_{i=1}^v \)) of \( B \).

1. The **discriminant** of the pair \( (Z, Z') \) is defined to be
   \[
d_v(Z, Z' : \text{tr}) = \det(\text{tr}(z_i z'_j))_{i,j=1}^v \in R.
\]

2. The **modified \( v \)-discriminant ideal** \( MD_v(B, \text{tr}) \) is the ideal of \( R \) generated by the set of elements \( d_v(Z, Z' : \text{tr}) \) for all \( Z, Z' \subset B \).

3. The **\( v \)-discriminant** \( d_v(B/R) \) is defined to be the gcd in \( B \) (possibly not in \( R \)) of elements \( d_v(Z, Z' : \text{tr}) \) for all \( Z, Z' \subset B \). Equivalently, the \( v \)-discriminant \( d_v(B/R) \) is the gcd in \( B \) of all elements in \( MD_v(B, \text{tr}) \).

In Definition 4.2(3), we are taking the gcd in \( B \), not in \( R \). If \( d_v(B/R) \) exists, then the ideal \( (d_v(B/R)) \) of \( B \) generated by \( d_v(B/R) \) is the smallest principal ideal of \( B \) that contains \( MD_v(B, \text{tr})B \). If \( B \) is an \( R \)-algebra which is finitely generated free over \( R \) and if \( w = \text{rk}(B/R) \), then \( MD_w(B : \text{tr}) \) equals \( D_w(B : \text{tr}) \), both of which are generated by a single element \( d(B/R) \). In this case it is also true that \( d(B/R) =_{B^\times} d_w(B/R) \). If \( v > \text{rk}(B/R) \), then \( d_v(B/R) = 0 \) [CPWZ2 Lemma 1.9(2)].

Some explicit examples of discriminants are given in [CPWZ1 CPWZ2 CYZ1 CYZ2].

The next lemma is straightforward by using commutative algebra argument.

**Lemma 4.3.** Let \( A \) and \( B \) be PI domains with centers \( C_A \) and \( C_B \) respectively. Let \( \text{tr} : A \to Q(C_A) \) be a \( C_A \)-linear trace function. Let \( R \) be a \( k \)-flat commutative algebra such that \( A \otimes R \) is a domain.

1. If \( \text{tr} \) is the regular trace, then \( \text{tr} \otimes R : A \otimes R \to Q(C_A) \otimes R \) is the regular trace.

2. Suppose that \( R \) is \( k \)-free. Then the image of \( \text{tr} \otimes R \) is in \( C_A \) if and only if the image of \( \text{tr} \otimes R \) is in \( C_A \otimes R \).

3. Suppose that \( \phi : A \to B \) is an algebra isomorphism. Then \( \phi \circ \text{tr} \circ \phi^{-1} : B \to Q(C_B) \) is the regular trace if and only if \( \text{tr} \) is the regular.

One of our key lemmas is the following, which suggests that the discriminant controls the group of automorphisms.

**Lemma 4.4.** Let \( \phi : A \to B \) be an isomorphism of algebras. Let \( C_A \) and \( C_B \) be the center of \( A \) and \( B \) respectively. Suppose that \( \text{tr}_A \) (respectively, \( \text{tr}_B \)) is the regular trace \( A \to C_A \) (resp. \( B \to C_B \)) and that the image of \( \text{tr}_A \) is in \( C_A \) (resp. the image of \( \text{tr}_B \) is in \( C_B \)). Let \( w \) be a positive integer. Then the following hold:

1. \( \phi \) maps the discriminant ideal \( D_w(A, \text{tr}_A) \) to \( D_w(B, \text{tr}_B) \);

2. if \( A \) is a finitely generated free module over \( C_A \), then \( \phi(d(A/C_A)) = c_B^\phi \cdot d(B/C_B) \);
(3) \( \phi \) maps the modified discriminant ideal \( MD_w(A, tr_A) \) to \( MD_w(B, tr_B) \);
(4) \( \phi \) maps the \( w \)-discriminant \( d_w(A/C_A) \) to \( d_w(B/C_B) \).

Proof. By Lemma 4.6, \( \phi(tr_A(x)) = tr_B(\phi(x)) \) for all \( x \in A \). The rest follows from this observation.

The concept of a dominating element was introduced in [CPWZ1, CPWZ2] to handle the noncommutative AP. We now recall this notion.

Let \( R \) be an algebra over \( k \). We say \( R \) is connected graded if \( R = k \oplus R_1 \oplus R_2 \oplus \cdots \) and \( R \) is locally finite if each \( R_i \) is finitely generated over \( k \). We now consider filtered rings \( A \). Let \( Y \) be a finitely generated free \( k \)-submodule of \( A \) such that \( k1_A \cap Y = \{0\} \). Consider the standard filtration defined by \( F_n A := (k1_A + Y)^n \) for all \( n \geq 0 \). Assume that this filtration is exhaustive and that the associated graded ring \( gr A \) is connected graded (or the map \( k \to A \) is injective). For each element \( f \in F_n A \setminus F_{n-1} A \), the associated element in \( gr A \) is defined to be \( gr f = f + F_{n-1} A \in (gr F) A_n \). The degree of an element \( f \in A \), denoted by \( \deg f \), is defined to be the degree of \( gr f \).

If \( gr A \) is a domain, then, for any elements \( f_1, f_2 \in A \),

\[
\deg(f_1 f_2) = \deg f_1 + \deg f_2.
\]

If \( gr A \) is a connected graded domain, it is easy to see that \( A^x = k^x \). As usual, we assume that \( k \subseteq A \). In this case \( gr A \) is connected graded. If \( R \) is a subalgebra of \( A \), then \( R^x \subseteq A^x = k^x \).

**Definition 4.5.** [CPWZ1, Definition 2.1(2)] Retain the above notation. Suppose that \( Y = \bigoplus_{i=1}^n kx_i \) generates \( A \) as an algebra. Assume that \( gr A \) is a connected graded domain. An element \( f \in A \) is called dominating if, for every testing \( \mathbb{N} \)-filtered PI algebra \( T \) with \( gr T \) being a connected graded domain, and for every testing subset \( \{ y_1, \ldots, y_n \} \subset T \) that is linearly independent in the quotient \( k \)-module \( T/F_0 T \), there is a presentation of \( f \) of the form \( f(x_1, \ldots, x_n) \) in the free algebra \( k[x_1, \ldots, x_n] \), such that the following hold: either \( f(y_1, \ldots, y_n) = 0 \), or

- (a) \( \deg f(y_1, \ldots, y_n) \geq \deg f \), and
- (b) \( \deg f(y_1, \ldots, y_n) > \deg f \) if, further, \( \deg y_{i_0} > 1 \) for some \( i_0 \).

Suppose now \( A \) is generated by elements in \( Y = \bigoplus_{i=1}^n kx_i \) of degree 1. A monomial \( x_1^{b_1} \cdots x_n^{b_n} \) is said to have degree component-wise less than \( (b_1, \ldots, b_n) \) (or, \( cwlt \), for short) \( x_1^{a_1} \cdots x_n^{a_n} \) if \( b_k \leq a_k \) for all \( i \) and \( b_{i_0} < a_{i_0} \) for some \( i_0 \). We write \( f = cx_1^{b_1} \cdots x_n^{b_n} + (cwlt) \) if \( f = cx_1^{b_1} \cdots x_n^{b_n} \) is a linear combination of monomials with degree component-wise less than \( x_1^{b_1} \cdots x_n^{b_n} \). If \( f = x_1^{b_1} \cdots x_n^{b_n} + (cwlt) \) for some \( b_1, \ldots, b_n \geq 1 \), then \( f \) is dominating (see the proof of [CPWZ1] Lemma 2.2). In the next section we will introduce a notion of effectiveness to deal with noncommutative ZCP.

The next result is a key lemma. Let \( R \) be a commutative algebra. We say that \( A \otimes R \) is \( A \)-closed if, for every \( 0 \neq f \in A \) and \( x, y \in A \otimes R \), the equation \( xy = f \) implies that \( x, y \in A \) up to units of \( A \otimes R \). For example, if \( R \) is connected graded and \( A \otimes R \) is a domain, then \( A \otimes R \) is \( A \)-closed.

**Lemma 4.6.** [CPWZ2, Lemma 1.12] Let \( tr : A \to C \) be a \( C \)-linear trace function where \( C \) is a central subalgebra of \( A \). Let \( R \) be a \( k \)-flat commutative algebra such that \( A \otimes R \) is a domain and \( v \) be a positive integer.

1. \( MD_v(A \otimes R : tr \otimes R) = MD_v(A : tr) \otimes R. \)
(2) Suppose $A \otimes R$ is $A$-closed. If $d_{w}(A/C)$ exists, then $d_{w}(A \otimes R/C \otimes R)$ exists and is equal to $d_{w}(A/C)$.

Now we are ready to state the main result of this section, which is basically [CPWZ1, Lemma 3.3(3)]. The proof is given in the next section.

**Theorem 4.7.** Let $A$ be a PI algebra. Suppose that the $w$-discriminant $d_{w}(A/C)$ is dominating for some $w$. Then the following hold.

1. $A$ is strongly LND$^H$-rigid.
2. If $A$ has finite GK-dimension, then $A$ is strongly cancellative.

The above theorem applies to many algebras including ones listed below.

**Example 4.8.** It is known that the following algebras have dominating discriminants [CPWZ1].

1. $k[x_1, \ldots, x_n]$ where $n$ is an even number and $1 \neq q$ is a root of unity.
2. $k[x, y]/(x^2y - yx^2, y^2x + xy^2)$.
3. $k[x, y]/(yx - qx - 1)$ where $1 \neq q$ is a root of unity.
4. finite tensor products of algebras of the form (1),(2),(3) above [CPWZ1, Lemma 5.4].

By Theorem 4.7(2), these algebras are strongly cancellative.

5. Effectiveness controls cancellation

First we introduce the notion of effectiveness that plays an important role in the resolution of the noncommutative ZCP.

**Definition 5.1.** Let $A$ be a domain and suppose that $Y = \bigoplus_{i=1}^{n} kx_{i}$ generates $A$ as an algebra. An element $f \in A$ is called effective if the following conditions hold.

1. There is an $N$-filtration $\{F_{i}A\}_{i \geq 1}$ on $A$ such that the associated graded ring $\text{gr} A$ is a domain (one possible filtration is the trivial filtration $F_{0}A = A$).
   - With this filtration we define the degree of elements in $A$, denoted by $\text{deg}_{A}$.
2. For every testing $N$-filtered PI algebra $T$ with $\text{gr} T$ being an $N$-graded domain and for every testing subset $\{y_{1}, \ldots, y_{n}\} \subset T$ satisfying
   - (a) it is linearly independent in the quotient $k$-module $T/k1_{T}$, and
   - (b) $\deg y_{i} \geq \deg x_{i}$ for all $i$ and $\deg y_{i_{0}} > \deg x_{i_{0}}$ for some $i_{0}$,
   - there is a presentation of $f$ of the form $f(x_{1}, \ldots, x_{n})$ in the free algebra $k\langle x_{1}, \ldots, x_{n} \rangle$, such that either $f(y_{1}, \ldots, y_{n})$ is zero or $\deg_{T} f(y_{1}, \ldots, y_{n}) > \deg_{A} f$.

Note that the definition of a dominating element [Definition 4.3] is slightly different from the definition of effectiveness. For example, we do not require Definition 4.3(b) in the definition of effectiveness. On the other hand, one only needs to test those $T$ such that $\text{gr} T$ is connected graded in the definition of a dominating element. It is easy to check that elements $f := x_{1}^{a_{1}} \cdots x_{n}^{a_{n}}$ (cwlt) is effective. We have already seen that there are many examples (those example given in CPWZ1, CPWZ2) of noncommutative algebras whose discriminant is dominating and of the form $x_{1}^{a_{1}} \cdots x_{n}^{a_{n}}$ (cwlt), whence effective. Here is the main result in this section, which is a slight generalization of Theorem 4.7.

**Theorem 5.2.** Let $A$ be a PI domain such that the $w$-discriminant over its center is effective (or dominating in part (2)) for some $w$. 

(1) Suppose $A$ has finite GK-dimension. Let $R$ be an affine $k$-free connected graded commutative domain such that $A \otimes R$ is a domain. If $A \otimes R \cong B \otimes R$ for some algebra $B$, then $A \cong B$. As a consequence, $A$ is strongly cancellative.

(2) $A$ is strongly $\text{LND}^H$-rigid.

Proof. (1) Let $\phi$ be the isomorphism from $A \otimes R$ to $B \otimes R$. By Lemma 3.1(2), $\text{GKdim} B = \text{GKdim}(B \otimes R) - \dim R = \text{GKdim}(A \otimes R) - \text{GKdim} R = \text{GKdim} A < \infty$.

Let $C_A$ and $C_B$ be the center of $A$ and $B$ respectively. Since $R$ is $k$-free, the center of $A \otimes R$ and $B \otimes R$ are $C_A \otimes R$ and $C_B \otimes R$ respectively. By Lemma 4.3, the regular trace functions for different algebras are all compatible. By Lemma 4.4(4), $\phi$ maps $d_w(A \otimes R/C_A \otimes R)$ to $d_w(B \otimes R/C_B \otimes R)$. By Lemma 4.0(2),

$$d_w(A \otimes R/C_A \otimes R) = d_w(A/C_A), \quad \text{and} \quad d_w(B \otimes R/C_B \otimes R) = d_w(B/C_B).$$

This implies that $\phi(d_w(A/C_A)) = d_w(B/C_B) \in B$.

Let $f$ denote $d_w(A/C_A)$. By hypothesis, $f$ is effective. Suppose $A$ is generated by $Y = \bigoplus_{i=1}^b kx_i$ as a $k$-algebra, and write $f$ as $f(x_1, \cdots, x_d)$ as in Definition 5.1. We take the testing algebra to be $T = B \otimes R$. Since $T$ is a domain (since $T \cong A \otimes R$), and $R$ is connected graded, $T$ is an $\mathbb{N}$-graded domain by setting $\deg b = 0$ for all $b \in B$ and $\deg r = \deg R r$ for all homogeneous element $r \in R$. In particular, $T$ is an $\mathbb{N}$-filtered algebra with $F_0 T = B$ such that $\text{gr} T$ is a domain. Now take a testing subset $\{y_1, \cdots, y_d\} \subset T$ by setting $y_i = \phi(x_i) \in T$ for $i = 1, \cdots, d$. We claim that $y_i \in B$ for all $i$. If not, there is some $i_0$ such that $y_{i_0}$ is not in $B = F_0 T$. By the effectiveness of of $f$, $f(y_1, \cdots, y_d)$ is either zero or not in $B := F_0 T$. However,

$$f(y_1, \cdots, y_d) = f(\phi(x_1), \cdots, \phi(x_d)) = f(\phi(x_1), \cdots, x_d) = \phi(f).$$

By the last statement in the previous paragraph, $\phi(f) = d_w(B/C_B)$ is a nonzero element in $B$, a contradiction. Therefore each $y_i \in B$. This means that $\phi$ maps $x_i$ to $y_i$ in $B$. Since $A$ is generated by $x_i$, the image of $A$ under $\phi$ is a subalgebra of $B$. So $\phi^{-1}(B)$ is a subalgebra $A \otimes R$ that contains $A$ as a subalgebra. Note that $\text{GKdim} \phi^{-1}(B) = \text{GKdim} B = \text{GKdim} A$, by the first paragraph of the proof. By Lemma 5.2 $\phi^{-1}(B) = A$. Therefore the image of $A$ under $\phi$ is exactly $B$, which implies that $\phi : A \cong B$.

(2) Since the proofs for the “effective” case and the “dominating” case are very similar, we combine two proofs together.

Suppose $A$ is generated by $\{x_1, \cdots, x_n\}$ as in Definition 5.1 (or Definition 5.5). Let $R = k[t_1, \cdots, t_d]/[t]$. By Lemma 4.0(2),

$$d_w(A \otimes R/C_A \otimes R) = d_w(A/C_A) =: f,$$

which is effective (or dominating) by hypothesis. Let $\partial \in \text{LND}^H(A[t_1, \cdots, t_d])$. By definition, $G := G_{\partial, t} \in \text{Aut}_{k[t]}(A[t_1, \cdots, t_d]/[t])$. For each $j$,

$$G(x_j) = x_j + \sum_{i \geq 1} t^i \partial_i(x_j).$$

We take the test algebra $T$ to be $A[t_1, \cdots, t_d]/[t]$ where the filtration on $T$ is induced by the filtration on $A$ together with $\deg t_s = 1$ for all $s = 1, \ldots, d$ and $\deg t = \alpha$ where $\alpha$ is larger than $\deg \partial_i(x_j)$ for all $j = 1, \ldots, n$ and all $i \geq 1$. (In the
dominating case, \( \text{gr} \, T \) is a connected graded domain.) Now set \( y_j = G(x_j) \in T \).

By the choice of \( \alpha \), we have that

(a) \( \deg y_j \geq \deg x_j \), and that

(b) \( \deg y_j = \deg x_j \) if and only if \( y_j = x_j \).

If \( G(x_j) \neq x_j \) for some \( j \), by effectiveness as in Definition 5.1 (or dominating as in Definition 5.3), \( \deg f(y_1, \ldots, y_n) > \deg f \). So \( f(y_1, \ldots, y_n) \neq f \times f \). But \( f(y_1, \ldots, y_n) = G(f) = A \times f \) by Lemma 4.4, a contradiction. Therefore \( G(x_j) = x_j \) for all \( j \). As a consequence, \( \partial_i(x_j) = 0 \) for all \( i \), or \( x_j \in \ker \partial \). Since \( A \) is generated by \( x_j \)'s, \( A \subseteq \ker \partial \). Thus \( A \subseteq \text{ML}^H(A[t_1, \ldots, t_d]) \). It is clear that \( A \subseteq \text{ML}^H(A[t_1, \ldots, t_d]) \) [Example 2.4], so the assertion follows.

Part (1) of the above theorem shows that \( A \) is close to be universally cancellative. Part (2) is Theorem 0.6 is a special case of Theorem 5.2. We are now ready to show Theorem 0.6.

Proof of Theorem 0.6. Since \( A \) is finitely generated over its affine center, \( A \) has finite GK-dimension. The assertion follows immediately from Theorem 5.2.

Next we consider some examples studied in [CPWZ1, CPWZ2]. Effectiveness of an element is easy to check sometimes. The following lemma is easy.

Lemma 5.3. Suppose \( A \) is generated by \( \{x_1, \ldots, x_n\} \) as in Definition 5.1

1. \( f = g_1 x_1 g_2 x_2 \cdots x_{n-1} g_{n-1} x_n g_n \) is effective if \( g_i \in A \) are nonzero.
2. \( f = x_1^{b_1} \cdots x_n^{b_n} \) then \( f \) is effective if and only if \( b_i \geq 1 \) for all \( i \).
3. \( f \) is effective, and \( g \neq 0 \), then \( fg \) and \( gf \) are effective.
4. Suppose that \( A \) is generated by two subalgebras \( A_1 \) and \( A_2 \). If \( f_1 \) and \( f_2 \) are effective elements in \( A_1 \) and \( A_2 \) respectively, then \( f_1 g f_2 \) is an effective element in \( A \) for every nonzero \( g \in A \).
5. If \( A \) is generated by \( \{x_1, x_2\} \) and \( g, h \in A \) and \( f = g(x_1 x_2 + a x_2 x_1) = h(x_1 x_2 + b x_2 x_1) \) for some scalars \( a \neq b \). Then \( f \) is effective.
6. If \( b_1, \ldots, b_n \) are positive integers and \( f = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n} + (\text{cwt}), \) then \( f \) is effective and dominating.

Next we recall some examples given in [CPWZ1, CPWZ2, CYZ1] that have dominating (and effective) discriminant. In the following examples, we assume that \( k \) is a field (and could be a finite field).

Example 5.4. [CYZ1] Theorem 0.1] Let

\[ A = k(x, y)/(xy - qyx - 1) \]

where \( 1 \neq q \) is an \( n \)th root of unity. Its center is \( C = k[x^n, y^n] \) and \( A \) is free over \( C \) of rank \( n^2 \). By [CYZ1] Theorem 0.1 the discriminant of \( d := d_{n^2}(A/C) \) is of the form

\[ d = k^\times x^{n^2(n-1)y^{n^2(n-1)} + \sum_{j<n^2(n-1)} a_j(xy)^j} \]

for some \( a_j \in k \). This \( d \) is dominating and effective by Lemma 5.3(6).

Example 5.5. [CPWZ1] Example 5.1] Consider the algebra

\[ S(p) := k(x, y)/(y^2x - pxy^2, yx^2 + px^2y) \]

where \( p \in k^\times \). By [AS] (8.11), \( S(p) \) is a noetherian Artin-Schelter regular domain of global dimension 3, which is of type \( S_2 \) in the classification given in [AS].
If, further, $k$ contains $\mathbb{Q}$, then the above are also equivalent to

8. $A$ is LND-rigid.
9. $A$ is strongly LND-rigid.
Proof. By [CPWZ2, Theorem 3.1], (1), (2) and (3) are equivalent. By [CPWZ2, Theorem 2.11], (2) and (5) are equivalent. By the proof of [CPWZ2, Theorem 2.11], (4) and (5) are equivalent.

(2) ⇒ (7): This is Theorem 4.7(1).

(7) ⇒ (6): Clear.

(6) ⇒ (5): If (5) fails, by [CPWZ2, Theorem 2.11], there is a homogeneous element \( f \) of degree at least 2 such that \( x_if = px_ifx_i \) for all \( s \) (such an element corresponds to an element in \( T_\star \) [E6.1.1] in the next section). Then

\[
g : x_i \mapsto \begin{cases} x_i & \text{if } i \neq s, \\ x_s + tf & \text{if } s = i, \\ t \to t \end{cases}
\]
defines an algebra automorphism of \( A[t] \) over \( k[t] \). By Lemma 2.2(3), \( g = G_{\partial, t} \) for some nonzero \( \partial \in \text{LND}^H(A) \), yielding a contradiction.

Next assume that \( k \) contains \( \mathbb{Q} \).

(7) ⇒ (9): This follows from the fact the map \( \text{LND}^H(A) \to \text{LND}(A) \) is surjective.

(9) ⇒ (8): Obvious.

(8) ⇒ (2): [CPWZ2, Theorem 3.1].

\[\square\]

6. The Makar-Limanov Invariant for Skew Polynomial Rings

In this section we study \( \text{ML}^\star(k_{p_{ij}}[x_1, \ldots, x_n]) \). We start with the following example.

Example 6.1. Suppose \( n \geq 3 \) is odd and \( 1 \neq q \) is a root of unity. Let \( A = k_q[x_1, \ldots, x_n] \) where \( k \) contains \( q^{\pm 1} \) and \( \mathbb{Z} \). Then \( \text{ML}(A) = k \). To see this, we first construct some locally nilpotent derivations. Suppose the order of \( q \) is \( \ell \). If \( w \) is odd, let \( \partial_w \) be the locally nilpotent derivation of \( A \) determined by

\[
x_i \mapsto \begin{cases} 0 & \text{if } i \neq w, \\ x_{\ell^{-1}}x_2x_{3}^{-1} \cdots x_{n-2}x_{n-1}^{-1}x_n^{-1} & \text{if } i = w. 
\end{cases}
\]

If \( w \) is even, let \( \partial_w \) be the locally nilpotent derivation of \( A \) determined by

\[
x_i \mapsto \begin{cases} 0 & \text{if } i \neq w, \\ x_1x_{\ell^{-1}}x_3^{-1}x_5^{-1} \cdots x_{n-2}x_{n-1}^{-1}x_n^{-1} & \text{if } i = w.
\end{cases}
\]

For any \( f \in A \setminus k \), there is a \( w \) and polynomials \( f_i \) of \( x_1, \ldots, x_w, \ldots, x_n \) such that \( f = \sum_{i=0}^n f_i x_w^i \) with \( f_n \neq 0 \) and \( n > 0 \). Then since \( \partial_w(x_w) \) commutes with \( x_w \) we have that \( \partial_w(x_w^i) = ix_{w}^{i-1} \partial_w(x_i) \) and so \( \partial_w(f) = \sum_{i=1}^n f_i i\partial_w(x_w)x_w^{i-1} \neq 0 \). Therefore \( f \notin \text{ML}(A) \) and the assertion follows.

Fix a parameter set \( \{p_{ij} \mid 1 \leq i < j \leq n \} \subseteq k^\ast \) and for \( i > j \) define \( p_{ij} = p_{ji}^{-1} \) and define \( p_{ii} = 1 \) for all \( i \). For \( s \in \{1, \ldots, n\} \), recall from [CPWZ2, p.757] that

\[
T_s := \{(d_1, \ldots, d_s, \ldots, d_n) \in \mathbb{N}^{n-1} \mid \prod_{j=1, j \neq s}^n p_{ij}^{d_j} = p_{is}, \forall i \neq s\}.
\]

By [CPWZ2, Lemma 2.9], if \( (d_1, \ldots, d_s, \ldots, d_n) \in T_s \), then the equation

\[
\prod_{j=1, j \neq s}^n p_{ij}^{d_j} = 1.
\]

Similar to the argument in Example 6.1 we have the following result.
Theorem 6.2. Let $A = k_{p_{ij}}[x_1, \ldots, x_n]$ where all $p_{ij}$ are roots of unity.

(1) Suppose that $k$ contain $\mathbb{Z}$. Then $ML^I(A)$ is the subalgebra of $A$ generated by $\{x_s \mid T_s = \emptyset\}$. As a consequence, $ML^I(A)$ is a skew polynomial ring.

(2) $ML^H(A)$ is the subalgebra of $A$ generated by $\{x_s \mid T_s = \emptyset\}$. As a consequence, $ML^H(A)$ is a skew polynomial ring.

Proof. (1) By replacing $k$ by its fraction field, we may assume that $k$ is a field containing $\mathbb{Q}$. In this case, $ML = ML^I$.

Let $B$ be the subalgebra generated by $\{x_s \mid T_s = \emptyset\}$. First we show that $ML(A) \subset B$. Pick $f \in A \setminus B$. Then there is a $w$ such that $T_w \neq \emptyset$ and polynomials $f_i$ of $x_1, \ldots, x_n$ such that $f = \sum_{i=0}^{n} f_i x_i^w$ with $f_n \neq 0$ and $n > 0$. Since $T_w \neq \emptyset$, we can pick $(d_1, \ldots, d_n) \in T_w$. Using this element, we define a locally nilpotent derivation $\partial_w$ as follows:

\[
\partial_w : x_i \mapsto \begin{cases} 
0 & \text{if } i \neq w \\
\sum_{i=1}^{n} f_i d_i w_{x_i^w} x_i^{w_{x_i^w}} & \text{if } i = w.
\end{cases}
\]

It follows from (E6.1.2) that $x_w$ commutes with $\partial_w(x_w)$. Then

\[
\partial_w(f) = \sum_{i=1}^{n} f_i \partial_w(x_w) x_i^{w_{x_i^w}} \neq 0.
\]

So $f$ is not in $ML(A)$. This implies that $ML(A) \subset B$.

To finish the argument, one needs to show that $\partial(B) = 0$ for every $\partial \in LND(A)$. Or it suffices to show that $\partial(x_s) = 0$ for all $s$ satisfying $T_s = \emptyset$.

So we need to prove the following claim: if $\partial \in LND(A)$ and $T_s = \emptyset$, then $\partial(x_s) = 0$. Let $g$ be the automorphism of the $k[t]$-algebra $A[t]$ of the form $exp(t\partial)$. Let $C$ be the center of $A[t]$. Since $A[t]$ is also a skew polynomial ring over the base ring $k[t]$, by the proof of [CPWZ2, Theorem 2.11(1)], $d_v(A[t]/C)$ is a monomial (where $v$ is taken to be the rank $A[t]$ over $C$). By [CPWZ2, Theorem 2.11(2)], $d_v(A[t]/C) = \prod_{s \mid T_s = \emptyset} x_s^a_s$ for some $a_s > 0$. Since every automorphism preserves $d_v(A[t]/C)$, $g(x_s)$ has $t$-degree 0. This implies that $\partial(x_s) = 0$, as required.

(2) The proof of (2) is similar to the proof of part (1). The main difference is to replace locally nilpotent derivations by locally nilpotent higher derivations. We only provide a sketch here.

Let $w$ be the integer such that $T_w \neq \emptyset$. We claim that there is a locally nilpotent higher derivation $\{\Delta_{w}^n\}_{n=0}^{\infty}$ such that $\Delta_{w}^1$ is the derivation $\partial_w$ defined in (E6.2.1) (and $\Delta_{w}^0$ is the identity by default). For each $n \geq 2$, the $k$-linear map $\Delta_{w}^n$ is determined by

\[
\Delta_{w}^n : f x_w^m \mapsto \begin{cases} 
\binom{m}{n} f (\partial_w(x_w))^n x_w^{m-n} & \text{if } m \geq n \\
0 & \text{otherwise},
\end{cases}
\]

for any $f$ being a polynomial of $x_1, \ldots, x_w, \ldots, x_n$. By an easy combinatorial computation, $\{\Delta_{w}^n\}_{n=0}^{\infty}$ is a locally nilpotent higher derivation. Replacing $\partial_w$ by $\{\Delta_{w}^n\}_{n=0}^{\infty}$, the first half of the proof of part (1) can be recycled. The second half of the proof can be copied when $\exp(t\partial)$ is replaced by $G_{\partial,t}$. Details are omitted. □

7. Mod-$p$ reduction

In this section we introduce a method that deals with the ZCP for certain non-PI algebras. We start with a temporary definition.
**Definition 7.1.** Let $A$ be a $k$-algebra that is free over $k$. Fix a $k$-basis $\{x_i\}_{i \in I}$ of $A$. Let $K$ be a subring of $k$. We say a subring $B \subset A$ is a $K$-order of $A$ if $\{x_i\}_{i \in I}$ is a $K$-basis of $B$.

**Lemma 7.2.** Let $K$ be a commutative domain and $A$ be a $K$-algebra with $K$-basis $\{x_i\}_{i \in I}$. Assume that $K$ is affine over $\mathbb{Z}$. Suppose that, for every quotient field $F := K/m$, $A \otimes_K F$ is LND$^H$-rigid (respectively, strongly LND$^H$-rigid). Then $A$ is LND$^H$-rigid (respectively, strongly LND$^H$-rigid).

**Proof.** Let $d$ be a non-negative integer. We only need to show that

$$A \subseteq \text{ML}^H(A[t_1, \ldots, t_d])$$

if the above holds when replacing $A$ by $A_F := A \otimes_K F$ for all quotient field $F = K/m$. We proceed by contradiction. If $\partial := \{\partial_j\}_{j \geq 0} \in \text{LND}^H(A[t_1, \ldots, t_d])$ and $\partial(f) \neq 0$ for some $f \in A$. Then there is some $j \geq 0$ such that $\partial_j(f) = \sum_{i \in I} c_i x_i$ where some $c_{i_0}$ is nonzero. Consider $K_1 = K[c_{i_0}^{-1}]$ and take a quotient field $F$ of $K_1$. Then $F$ is finite and $F$ is a quotient field of $K$ as well. Remember that $c_{i_0}$ is invertible in $K_1$, whence invertible in $F$. Write $\partial_F = (\partial \otimes_K F)$. It is clear that $\partial_F$ is in $\text{LND}^H(A_F[t_1, \ldots, t_d])$ since $G_{\partial_F,t}$ is the automorphism $G_{\partial,F} \otimes_K F \in \text{Aut}(A_F[t_1, \ldots, t_d][t])$. Let $f'$ be the image of $f$ in $A_F$. Then $(\partial_F)_j(f') = \sum_{i \in I} c'_i x_i \neq 0$ where $c'_i$ are image of $c_i$ in $F$. This contradicts the fact that $\text{ML}^H(A_F[t_1, \ldots, t_d]) = A_F$. Therefore the assertion follows. \qed

**Definition 7.3.** Let $A$ be a $K$-algebra with a $K$-basis $\{x_i\}_{i \in I}$. We call $\{x_i\}$ manageable if for each $w \geq 0$ and each $K$-algebra automorphism $G \in \text{Aut}(A[y_1, \ldots, y_w])$ there is an affine $\mathbb{Z}$-subalgebra $K_1 \subset K$ such that $B := \oplus_{i \in I} K_1 x_i$ is a $K_1$-order and $G$ is induced from an automorphism of $B[y_1, \ldots, y_w]$.

**Lemma 7.4.** Let $K$ be a commutative domain and $A$ be a finitely generated $K$-algebra with a manageable $K$-basis $\{x_i\}_{i \in I}$. If, for every affine $\mathbb{Z}$-subalgebra $K_1 \subset K$, there is an affine $\mathbb{Z}$-subalgebra $K_2 \subset K$ containing $K_1$ such that $B := \oplus_{i \in I} K_2 x_i$ is a $K_2$-order of $A$ and that $B$ is LND$^H$-rigid (respectively, strongly LND$^H$-rigid), then $A$ is LND$^H$-rigid (respectively, strongly LND$^H$-rigid).

**Proof.** Again, let $d$ be a non-negative integer. We will show that

$$A \subseteq \text{ML}^H(A[t_1, \ldots, t_d]).$$

If not, pick $\partial \in \text{LND}^H(A[t_1, \ldots, t_d])$ and $f \in A$ such that $\partial(f) \neq 0$. Write, for every $j \geq 0$, $\partial_j(f) = \sum_{i \in I} c_{ji} x_i$ where some $c_{j0}$ is nonzero. Consider $G = G_{\partial,t} \in \text{Aut}(A[t_1, \ldots, t_d][t])$. Since $\{x_i\}_{i \in I}$ is manageable, $G = H \otimes_K K$ where $K_2$ is a finitely generated $\mathbb{Z}$-subalgebra of $K$ and $H \in \text{Aut}(B[t_1, \ldots, t_d][t])$. By the hypothesis, we may assume that $K_2$ contains all $c_{ji}$ and $\text{ML}^H(B[t_1, \ldots, t_d]) = B$. Since $H(b) \equiv b \pmod{t}$ for all $b \in B[t_1, \ldots, t_d]$, by Lemma 7.2, $H = G_{\partial,t}$. Since $\text{ML}^H(B[t_1, \ldots, t_d]) = B$, $\partial'(f) = 0$, which implies that $\partial(f) = 0$, a contradiction. \qed

Here is the main result of this section.

**Theorem 7.5.** Let $\Phi := \{A\}$ be a collection of finitely generated algebras over various base commutative domains $K$. Assume that the following hold.

1. Each $A$ is finitely generated over $K$ and has a manageable $K$-basis.
(2) If $A \in \Phi$ where $A$ is a $K$-algebra and $K$ is an affine $\mathbb{Z}$-algebra, then $A \otimes_K F$ is in $\Phi$ where $F$ is a finite quotient field of $K$.

(3) If $A \in \Phi$ where $A$ is a $K$-algebra, then for every affine $\mathbb{Z}$-subalgebra $K_1 \subset K$, there is an affine $\mathbb{Z}$-subalgebra $K_2 \subset K$ containing $K_1$ such that $A = B \otimes_{K_2} K$ for $B \in \Phi$ that is a $K_2$-order of $A$.

(4) If $K$ is a finite field, then $A$ is LND$^H$-rigid (respectively, strongly LND$^H$-rigid).

Then every $A \in \Phi$ is LND$^H$-rigid (respectively, strongly LND$^H$-rigid).

Proof. By Lemma 7.2 and hypothesis (4), $A$ is LND$^H$-rigid (respectively, strongly LND$^H$-rigid) if $K$ is affine over $\mathbb{Z}$. Then by Lemma 7.3 and hypothesis (3), every $A \in \Phi$ is LND$^H$-rigid (respectively, strongly LND$^H$-rigid).

Now we are ready to prove Theorem 0.8.

Proof of Theorem 0.8. We now construct the collection $\Phi$ as follows: a $K$-algebra $A$ is in $\Phi$ if $A$ is a finite tensor product (over $K$) of different copies $K[y_1, \ldots, y_n]$ when $n$ is even (for different values of $p \neq 1$), copies of $K[x, y]/(x^2y - yx^2, y^2x + xy^2)$, and different copies of $K[x,y]/(yx - qxy - 1)$ (for different values of $q \neq 1$). We require that the base commutative rings $K$ are domains containing the following elements

$$(E7.5.1) \quad 2^{-1}, p^{\pm 1}, q^{\pm 1}, (p - 1)^{-1}, (q - 1)^{-1}$$

for different $p$ and $q$ occurring in $A$. In particular,

$$(E7.5.2) \quad p \neq 0, 1 \quad and \quad q \neq 0, 1 \quad and \quad 2 \neq 0.$$  

The conditions in (E7.5.2) will survive when passing $K$ to a finite quotient field $K/I$ due to (E7.5.1). If $K$ is a finite field, Example 4.8(4) says that $A$ has dominating discriminant. By Theorems 4.7(1) and 5.2(2), $A$ is strongly LND$^H$-rigid. This verified hypothesis (4) of Theorem 7.5. Hypotheses (1,2,3) of Theorem 7.5 are easy to verify, see Lemma 7.6 below. Therefore every member in $\Phi$ is strongly LND$^H$-rigid.

Since every $A$ is a domain of finite GK-dimension, by Theorem 6.3 $A$ is strongly cancellative.

Lemma 7.6. Retain notation as in the proof of Theorem 0.8 above.

(1) Each $A$ is finitely generated over $K$ and has a manageable $K$-basis.

(2) If $A \in \Phi$ where $A$ is a $K$-algebra and $K$ is an affine $\mathbb{Z}$-algebra, then $A \otimes_K F$ is in $\Phi$ where $F$ is a finite quotient field of $K$.

(3) If $A \in \Phi$ where $A$ is a $K$-algebra, then for every affine $\mathbb{Z}$-subalgebra $K_1 \subset K$, there is an affine $\mathbb{Z}$-subalgebra $K_2 \subset K$ containing $K_1$ such that $A = B \otimes_{K_2} K$ for $B \in \Phi$ that is a $K_2$-order of $A$.

Proof. Part (2) is clear by the definition of $\Phi$. We only prove parts (1) and (3).

For each $K$-algebra $A$ in $\Phi$, it is well-known that there is a $K$-linear basis $\mathcal{A}$ of $A$ that consists of a family of noncommutative monomials. One property of this basis is that it is independent of the parameters $p, q$ and independent of the base ring $K$. The multiplication constant with respect to this basis are in the $\mathbb{Z}$-subalgebra of $K$ generated by elements in (E7.5.1). Then (3) follows by taking $K_2$ to be the subalgebra generated by $K_1$ and elements in (E7.5.1).
For part (1), we show that the basis $\mathcal{A}$ used in the last paragraph is manageable. Let $E := A[y_1, \cdots, y_w]$ where $A$ is in $\Phi$. It is clear that $E$ has a $K$-basis, denoted by $E_1$, of the form

$$\{b_n y_1^{d_1} \cdots y_w^{d_w} \mid b_n \in A, d_s \geq 0\}.$$ 

Since $A$ is finitely generated over $K$, so is $E$. Let $G \in \text{Aut}(E)$. Then there is an affine $\mathbb{Z}$-subalgebra $K_1$ of $K$ such that $G(f_s)$ and $G^{-1}(f_s)$, for $s = 1, \cdots, z$, are all in the $K_1$-span of $E_1$. Without loss of generality we assume that $K_1$ contains elements in $E_7.5.1$. Then $K_1A$ is an algebra, which is denoted by $B$. (By part (3), we might further assume that $B$ is in $\Phi$.) Since $A$ and $B$ has the “same” basis (over different commutative rings), $B$ is a $K_1$-order of $A$. By the choice of $K_1$, $G$ restricts to an algebra automorphism $G'$ of $B[y_1, \cdots, y_w]$. Since $E$ and $B[y_1, \cdots, y_w]$ has the “same” basis, $G$ is induced form $G'$. Therefore part (1) holds. $\square$

We summarize the key steps of solving $\text{ZCP}$ for noncommutative algebras similar to those in Theorem 0.8 as follows. For an algebra $A$ over a base commutative ring $k$ satisfying certain finiteness conditions, one uses reduction modulo $p$ to reduce the problem in the special case when $k$ is a finite field. When $k$ is a finite field, $A$ ends up being PI (which is true for a large class of quantized algebras). Then one can compute the discriminant of $A$ over its center, say $d := d(A/C)$. If one can verify that $d$ is effective (or dominating), then $A$ is strongly $LND_H$-rigid by Theorems 4.7(1) and 5.2(2). Finally, by Theorem 3.3(1), $A$ is strongly cancellative. So we have the following diagram.

\[
\begin{array}{ccc}
A \text{ with finiteness conditions} & \xrightarrow{\text{reduction mod } p} & A \text{ over a finite field } k \\
& \xrightarrow{\text{computing discriminant } d} & \\
& \xrightarrow{\text{Theorem 5.2(2)}} & A \text{ is strongly } LND_H \text{-rigid} \\
& \xrightarrow{\text{Theorem 5.3(1)}} & A \text{ is strongly cancellative.}
\end{array}
\]

For algebras of GK-dimension two we should apply Theorem 0.5 directly.

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