From Quantum Mechanics to running $\Lambda$ cosmologies

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Abstract. The cosmological model with running dark energy is considered. We assume that the vacuum of the Universe is in a meta–stable state and decays into a true (bare) vacuum state with increasing time. We use quantum theory of decay processes to find energy of the system in the unstable state and apply obtained results to a description of the running dark energy identified with vacuum energy. We also estimate model parameters using astronomical data. From the astronomical point of view our model is in good agreement with data. Moreover, the framework of this model one can explain naturally, smallness of the cosmological constant parameter $\Lambda_{\text{bare}}$.

1. Introduction

The two most important problems of the contemporary cosmology are: 1) The cosmological constant problem, that is the difference between the measured value of the vacuum energy and the value calculated using quantum field theory methods [1], and 2) The problem why the current densities of matter and dark energy in the Universe have the same order of magnitude [2]. (This question is called the coincidence problem). These mentioned cosmological problems can be considered in the framework of cosmological models where the cosmological constant is running and, in result, its value is changing during the cosmic evolution.

The current observations indicate that the Universe is in an accelerated phase [3] which can be explained by the existence of the dark energy. The analysis of astronomical observations shows that there is a tension between local and primordial measurements of cosmological parameters [3]. A possible explanation of this tension is the dark energy changing in time [4].

In this talk we will assume that the dark energy is metastable and depends on time $t$, $\rho_{\text{de}} = \rho_{\text{de}}(t)$ and decays with the increasing time $t$ to $\rho_{\text{bare}}$ ($\rho_{\text{de}}(t) \to \rho_{\text{bare}} \neq 0$ when $t \to \infty$). Decaying vacuum energy was considered in many papers (see e.g. [5, 6] and also [7, 8]). Shafieloo et al. [9] assumed that $\rho_{\text{de}}(t)$ decays according to the radioactive exponential decay law. Such an assumption seems to be insufficient when one wants to explain the evolution in time of the Universe with decaying dark energy. It is because the creation of the Universe is a quantum process. Therefore the metastable dark energy should be considered as the value of a scalar field.
in the false vacuum state and the process of a decay of the dark energy should be considered as a quantum decay process.

The quantum decay processes consist of the following phases [10, 11]: (i) The early time initial phase, (ii) The canonical or exponential phase, and (iii) The late time non-exponential phase. Consequently, the first phase and the third one are misused in the case of the radioactive decay law only. The theoretical analysis of quantum decaying processes shows that for the late time, the survival probability of the system to be in its initial metastable state (i.e. the decay law), should tend to zero as $t \rightarrow \infty$ much more slowly than any exponential function of time, and that, as a function of time, it has the inverse power–like form at this regime of time [10, 12]. So, all consequences of the decay process of the dark energy as a quantum decay process can be found only if one uses a quantum decay law to describe decaying metastable dark energy. (This idea was considered in [13]). It also appears that energy of the system is different in each phase. In particular at the late time phase it is much smaller than at the canonical phase and this property will be discussed in this talk.

2. Decay of a dark energy as a quantum decay process

Studying quantum unstable systems, one usually uses the survival probability (the decay law)

$$P(t) = |A(t)|^2,$$

(1)

to describe changes in time of the unstable system. This means these properties of the survival amplitudes

$$A(t) = \langle \phi | \phi(t) \rangle$$

(2)

are analyzed in this case. Here, a vector $|\phi\rangle$ represents the unstable state of the considered system and $|\phi(t)\rangle$ is the solution gotten from the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\phi(t)\rangle = \hat{H} |\phi(t)\rangle.$$  

(3)

For the case, the initial condition for Eq. (3) is usually assumed as

$$|\phi(t = t_0 \equiv 0)\rangle \overset{\text{def}}{=} |\phi\rangle,$$  

(4)

or equivalently, $A(0) = 1$.

The $\hat{H}$ symbol denotes the complete (full) self-adjoint Hamiltonian of the system in Eq. (3).

Using a basis in $\mathcal{H}$, which is build from normalized eigenvectors $|E\rangle$ of $\hat{H}$ (where $E \in \sigma_c(\hat{H})$), we can expand $|\phi\rangle$ in this basis, and then we can express the amplitude $A(t)$ as the Fourier integral

$$A(t) \equiv A(t - t_0) = \int_{E_{\text{min}}}^\infty \omega(E) e^{-\frac{i}{\hbar} E(t - t_0)} dE,$$

(5)

where $\omega(E) = \omega(E)^*$ and $\omega(E) > 0$ is the probability to find the energy of the system for the state $|\phi\rangle$ between $E$ and $E + dE$. The above relation (5) denotes that the survival amplitude $A(t)$ is given by a Fourier transform of an absolute integrable function $\omega(E)$. When we apply the Riemann-Lebesgue Lemma to the integral (5) then we can conclude that there must be $A(t) \rightarrow 0$ as $t \rightarrow \infty$. This property and relation (5) are an essence of the Fock–Krylov theory of unstable states [14, 15].

In this approach, the amplitude $A(t)$ and, consequently, the decay law $P(t)$ of the unstable state $|\phi\rangle$, are totally determined by the density of the energy distribution $\omega(E)$ for the system for this state [14, 15] (see also [10, 16], and so on). (This approach is also used in Quantum Field Theory models [17]).

Note that the amplitude $A(t)$ contains information about the decay law $P(t)$ for the state $|\phi\rangle$: About the decay rate $\gamma_\phi$ for this state and the energy $E_\phi$ of the system for this state. This
information can be extracted from $\mathcal{A}(t)$. It can be done using the rigorous equation governing the time evolution in the subspace of unstable states, $\mathcal{H}_\parallel \ni |\phi_\parallel \equiv |\phi\rangle$. Such an equation results from the Schrödinger equation (3) for the total state space $\mathcal{H}$.

When we use the Schrödinger equation (3), then we can find that within the problem considered

$$i\hbar \frac{\partial}{\partial t} (|\phi\rangle|\phi(t)\rangle) = \langle \phi|\mathcal{S}|\phi(t)\rangle.$$  \hspace{1cm} (6)

From the above relation it follows that the amplitude $\mathcal{A}(t)$ satisfies the following equation

$$i\hbar \frac{\partial \mathcal{A}(t)}{\partial t} = h(t) \mathcal{A}(t),$$  \hspace{1cm} (7)

where

$$h(t) = \frac{\langle \phi|\mathcal{S}|\phi(t)\rangle}{\mathcal{A}(t)} = \frac{i\hbar}{\mathcal{A}(t)} \frac{\partial \mathcal{A}(t)}{\partial t},$$  \hspace{1cm} (8)

$h(t)$ is the effective Hamiltonian which governs the time evolution in the one-dimensional subspace of unstable states $\mathcal{H}_\parallel = \mathbb{P}\mathcal{H}$, where $\mathbb{P} = |\phi\rangle\langle\phi|$ (see [18] and also [19, 20] and references therein). The subspace $\mathcal{H} \subset \mathcal{H}_\parallel = \mathcal{H}_\perp \equiv \mathbb{Q}\mathcal{H}$ is a subspace of decay products. Here, we have $\mathbb{Q} = \mathbb{I} - \mathbb{P}$. One meets the effective Hamiltonian $h(t)$ if one starts with the Schrödinger equation for the total state space $\mathcal{H}$ and looks for the rigorous evolution equation for a distinguished subspace of states $\mathcal{H}_\parallel \subset \mathcal{H}$ [18, 21, 22]. Generally, $h(t)$ is a complex function of time but in the case of $\mathcal{H}_\parallel$ of dimension two or more, the effective Hamiltonian which governs the time evolution in $\mathcal{H}_\parallel$ is a non-hermitian matrix $H_\parallel$ or a non-hermitian operator. There is

$$h(t) = E_\phi(t) - \frac{i}{2} \gamma_\phi(t),$$  \hspace{1cm} (9)

where $E_\phi(t) = \Re [h(t)]$, $\gamma_\phi(t) = -2 \Im [h(t)]$, are respectively instantaneous energy (mass) $E_\phi(t)$ and the instantaneous decay rate, $\gamma_\phi(t)$. (Here $\Re (z)$ and $\Im (z)$ mean the real and imaginary parts of $z$). The quantity $\gamma_\phi(t) = -2 \Im [h(t)]$ can be interpreted as the decay rate since it satisfies the definition of the decay rate which is used in quantum theory: When we use (8) it is easy to check that

$$\frac{\gamma_\phi(t)}{h} \equiv \frac{1}{\mathcal{P}(t)} \frac{\partial \mathcal{P}(t)}{\partial t} = -\frac{1}{|\mathcal{A}(t)|^2} \frac{\partial |\mathcal{A}(t)|^2}{\partial t} \equiv -\frac{2}{\hbar} \Im [h(t)].$$  \hspace{1cm} (10)

We have the relation $|\phi(t)\rangle = \exp \left[-\frac{i}{\hbar} \mathcal{A}(t)\right]|\phi\rangle$. It is easy to see that this property and hermiticity of $\mathcal{S}$ lead to the conclusion that [10]

$$(\mathcal{A}(t))^* = \mathcal{A}(-t).$$  \hspace{1cm} (11)

The conclusion that results from the above property and from the relation (8) is

$$h(-t) = (h(t))^*.$$  \hspace{1cm} (12)

That’s why there should be:

$$E_\phi(-t) = E_\phi(t), \quad \text{and} \quad \gamma_\phi(-t) = -\gamma_\phi(t),$$  \hspace{1cm} (13)

So, the instantaneous energy $E_\phi(t) = \Re [h(t)]$ is an even function of time $t$ and the instantaneous decay rate $\gamma_\phi(t) = -2 \Im [h(t)]$ is an odd function of $t$.

In the large literature, in many cases quantum unstable systems are studied using the Fock–Krylov theory for the Breit–Wigner energy density distribution function $\omega_{BW}(E)$. The use $\omega_{BW}(E)$ is convenient since it describes relatively well a large class of unstable systems and
permits to find an analytical form of the survival amplitude $A(t)$ (see, eg. [23] and other papers). It is because the decay curves which are obtained in this simplest case are very similar in the form to the curves which are calculated for the more general $\omega(E)$, (see [16] and analysis in [10]). Therefore looking for the most typical properties of the decay process it is enough to make the relevant calculations for $\omega(E)$ which is modeled by the Breit–Wigner distribution of the energy density $\omega_{BW}(E)$. For this energy density distribution, we can find relatively easy an analytical form of $A(t)$ for very late times as well as an analytical asymptotic form of $h(t)$, $E(t)$ and $\gamma(t)$ for such times. So, we assume that

$$\omega(E) \equiv \omega_{BW}(E) = \frac{N}{2\pi} \Theta(E - E_{\text{min}}) \frac{\Gamma_0}{(E - E_0)^2 + (\frac{\Gamma_0}{2})^2},$$

where $N$ is a normalization constant. The parameters $E_0$ and $\Gamma_0$ correspond to energy of the system for the unstable state $|\phi\rangle$ and its decay rate at the exponential (or canonical) regime of the decay process. The $E_{\text{min}}$ parameter is the minimal (the lowest) energy of the system. When we insert $\omega_{BW}(E)$ into formula (5) for the amplitude $A(t)$ and assume for simplicity that $t_0 = 0$, after some algebra we can find that

$$A(t) = \frac{N}{2\pi} e^{-\frac{i}{\hbar} E_0 t} I_\beta \left( \frac{\Gamma_0 t}{\hbar} \right),$$

where

$$I_\beta(\tau) = \int_\beta^\infty \frac{1}{\eta^2 + \frac{1}{4}} e^{-i\eta\tau} d\eta.$$  

Here, $\tau = \frac{E_0 t}{\hbar} \equiv \frac{t}{\tau_0}$, $\tau_0 = \frac{\hbar}{E_0}$, and $\beta = \frac{E_0 - E_{\text{min}}}{\Gamma_0} > 0$. The integral $I_\beta(\tau)$ can be decomposed as follows

$$I_\beta(\tau) = I_\beta^{\text{pole}}(\tau) + I_\beta^{\text{L}}(\tau),$$  

where

$$I_\beta^{\text{pole}}(\tau) = \int_{-\infty}^{\infty} \frac{1}{\eta^2 + \frac{1}{4}} e^{-i\eta\tau} d\eta \equiv 2\pi e^{-\frac{\tau}{2}},$$

and

$$I_\beta^{\text{L}}(\tau) = -\int_{+\infty}^{\infty} \frac{1}{\eta^2 + \frac{1}{4}} e^{+i\eta\tau} d\eta.$$  

(The integral $I_\beta^{\text{L}}(\tau)$ can be rewritten in terms of the integral–exponential function [23, 19, 20, 24] (for a definition, see [25, 26])). The above relation (17) implies a natural decomposition of the survival amplitude $A(t)$ into two parts

$$A(t) = A_c(t) + A_L(t),$$

where

$$A_c(t) = \frac{N}{2\pi} e^{-\frac{i}{\hbar} E_0 t} I_\beta^{\text{pole}} \left( \frac{\Gamma_0 t}{\hbar} \right) \equiv N e^{-\frac{i}{\hbar} E_0 t} e^{-\frac{\Gamma_0 t}{\hbar}},$$

and

$$A_L(t) = \frac{N}{2\pi} e^{-\frac{i}{\hbar} E_0 t} I_\beta^{\text{L}} \left( \frac{\Gamma_0 t}{\hbar} \right),$$

$A_c(t)$ is the canonical part of the amplitude $A(t)$ which describes the pole contribution into $A(t)$ and $A_L(t)$ represents the remaining part of $A(t)$.

From the decomposition (20), we have that in the general case within the considered model, the survival probability (1) contains the following parts

$$P(t) = |A(t)|^2 = |A_c(t) + A_L(t)|^2 = |A_c(t)|^2 + 2 |A_c(t) (A_L(t))^*| + |A_L(t)|^2.$$
which means that

\[ \text{where} \]

\[ \gamma \]

\[ \text{the use of the relation (28) gives} \]

\[ \text{curve obtained numerically for} \]

\[ \text{So, when we insert (24) into (22) then we can find a late time for} \]

\[ m \text{ of} \]

\[ \text{The above relation is especially useful when one looks for a contribution of late time properties of the quantum unstable system into the survival amplitude.} \]

\[ \text{The late time form of the integral} \]

\[ I_I^L(\tau) \]

\[ \text{and hence the late time form of the amplitude} \]

\[ A_L(t) \]

\[ \text{is relatively easy to find performing the integration by parts in (19). It appears that for} \]

\[ t \to \infty \text{ (or} \tau \to \infty) \]

\[ \text{the leading term of the late time asymptotic expansion of the integral} \]

\[ I_I^L(\tau) \]

\[ \text{takes the following form} \]

\[ (29) \]

\[ \text{which we are looking for:} \]

\[ (31) \]

\[ \text{So, when we insert (24) into (22) then we can find a late time form of} \]

\[ A_L(t) \]

\[ \text{The late time form of the integral} \]

\[ I_I^L(\tau) \]

\[ \text{for the considered model. These quantities are defined by using the effective Hamiltonian} \]

\[ h(t) \]

\[ \text{Looking for} \]

\[ h(t) \]

\[ \text{we need the quantity} \]

\[ i \hbar \frac{\partial A(t)}{\partial t} \]

\[ (8) \]

\[ \text{From (15) we can find that} \]

\[ \text{If to use (15), (25) and (8) the conclusion follows,} \]

\[ h(t) = i \hbar \frac{1}{A(t)} \frac{\partial A(t)}{\partial t} = E_0 + \Gamma_0 \frac{J_\beta(\tau(t))}{I_\beta(\tau(t))}, \]

\[ \text{which means that} \]

\[ E(t) = \Re [h(t)] = E_0 + \Gamma_0 \Re \left[ \frac{J_\beta(\tau(t))}{I_\beta(\tau(t))} \right]. \]

\[ \text{For a visualization of properties of} \]

\[ E(t) \]

\[ \text{it is convenient to use the following function:} \]

\[ \kappa(t) \]

\[ \text{The use of the relation (28) gives} \]

\[ E(t) - E_{min} = E_0 - E_{min} + \Gamma_0 \Re \left[ \frac{J_\beta(\tau(t))}{I_\beta(\tau(t))} \right], \]

\[ \text{When to divide two sides of the above equation by} \]

\[ E_0 - E_{min} \]

\[ \text{then we obtain the function} \]

\[ \kappa(t) \]

\[ \text{(see (29)) which we are looking for:} \]
The typical behavior of the instantaneous energy $E(t)$ changing in time $t$ is shown in Fig. (2).

To find the asymptotic form of $h(t)$ we need to know the late time form for $J_\beta(\tau)$. Using similar methods as in the case of $I_\beta(\tau)$ we find for $\tau \to \infty$ [24]

$$J_\beta(\tau) \simeq \frac{i}{\tau} \frac{e^{i\beta \tau}}{\beta^2 + \frac{1}{4}} \left\{ \beta + \left[ 1 - \frac{2\beta^2}{\beta^2 + \frac{1}{4}} \right] \frac{i}{\tau} + \frac{2\beta}{\beta^2 + \frac{1}{4}} \left[ \frac{4\beta^2}{\beta^2 + \frac{1}{4}} - 3 \right] \left( \frac{i}{\tau} \right)^2 \right. \right. \right.$$

$$\left. \left. \left. \left. + \frac{6}{\beta^2 + \frac{1}{4}} \left[ - \frac{8\beta^4}{(\beta^2 + \frac{1}{4})^2} + \frac{8\beta^2}{\beta^2 + \frac{1}{4}} - 1 \right] \left( \frac{i}{\tau} \right)^3 \right. \right. \right.$$

$$\left. \left. \left. \left. + \frac{24\beta}{(\beta^2 + \frac{1}{4})^2} \left[ \frac{16\beta^4}{(\beta^2 + \frac{1}{4})^2} - \frac{20\beta^2}{\beta^2 + \frac{1}{4}} + 5 \left( \frac{i}{\tau} \right)^4 \right] \right. \right. \right.$$  

$$(32)$$

Starting from the asymptotic expression (32) and using formula (27) we can find the late time asymptotic form of $h(t)$ and hence of $E(t)$ and $\gamma_\phi(t)$ for the considered model [24],

$$E(t) \big|_{t \to \infty} = \Re \left[ h(t) \right] \big|_{t \to \infty} \simeq E_{\min} - 2 \frac{E_0 - E_{\min}}{\Gamma_0} \left( \frac{h}{t} \right)^2 + \frac{1}{4} \frac{6 - 21\beta + 48\beta^2 - 64\beta^3 - 288\beta^4 + 464\beta^5}{\Gamma_0 (\beta^2 + \frac{1}{4})^4} \left( \frac{h}{t} \right)^4 + \ldots.$$  

This last relation is valid for $t > T_C$, where $T_C$ is the cross-over time, i.e. the time when the canonical exponential and late time inverse power law contributions to the survival amplitude start to be comparable.

In the general case we get for $t \to \infty$,

$$E(t) \big|_{t \to \infty} \simeq E_{\min} + \sum_{k \geq 1} f_{2k} \left( \frac{h}{t} \right)^{2k},$$  

$$(34)$$

where $f_{2k} = (f_{2k})^*$. This expansion is consistent with the property (13), which says that $E(t) \big|_{t \to \infty}$ must be an even function of time $t$. 
All the above described properties of the quantum unstable system are general one and are independent on the specific form of interactions which force the decay process or a mechanism responsible for such a process. General properties of the decay law are the same in all quantum decay processes. The decay law always has three phases: an initial non–exponential phase, the exponential (canonical) phase and the late time non-exponential phase. This concerns also such a quantum process like the decaying dark energy.

When we analyze the temporal behavior of the decaying dark energy as a function of time \( t \) then the following strategies can be used: we can analyze the decay process as the quantum decay process which is governed by the quantum mechanical decay law \( \mathcal{P}(t) \), or we can consider the dark energy as the energy density in the meta–stable false vacuum state and analyze temporal properties of this energy density similarly to the properties of the instantaneous energy of the system for the unstable state \( |\phi\rangle \).

In the case of the first strategy using

\[
\rho_{\text{de}}(t) \equiv \rho_{\text{de}}(t) - \rho_{\text{bare}},
\]

we have \( \lim_{t \to \infty} (\rho_{\text{de}}(t) - \rho_{\text{bare}}) = 0 \) and we can describe such a process assuming that (see [13]), \( \rho_{\text{de}}(t) = \rho_{\text{de}}(t_0) \mathcal{P}(t) \) where \( \mathcal{P}(t) \) is given by the relation (23). Such an approach can be deliberated as a generalization of the idea studied by Shafieloo et al. in [9]. (This was done in the paper [13]).

Considering the second strategy: As it was mentioned earlier the general properties of the quantum decay process shortly described above are independent on the mechanism responsible for a decay. The inherent mechanism of quantum decay processes presented earlier in Fig. (2) is a reduction of the energy of the system for the meta–stable state \( |\phi\rangle \) from the large values measured at exponential decays time region to much smaller values for the asymptotically late times region. Our hypothesis is that this pure quantum effect occurs also in the case of the decaying dark energy.

Now we will consider the cosmological scenario for which the false vacuum decays at the inflationary stage of the Universe. This corresponds with the hypothesis which was analyzed by Krauss and Dent in [5, 6]. (The hypothesis in which some false vacuum regions do survive well up to the time \( T_C \) or later was formulated in these papers). Let \( |\phi\rangle = |0\rangle_{\text{false}}, \) be a false state, \( |0\rangle_{\text{true}} \) — a true state of vacuum and \( E_0 = E_{0, \text{false}} \) be energy of a state corresponding to the false vacuum measured at the canonical decay time, which leads to the vacuum energy density calculated by means of quantum field theory methods. Similarly, let \( E_{0, \text{true}} \) be the energy of the true vacuum (i.e. the true ground state of the system).

From the results presented earlier we get that the energy of those false vacuum regions which survived up to \( T_C \) and much later differs from \( E_{0, \text{false}} \). Now, when we assume that \( E_{0, \text{true}} = E_{\text{min}} \) and \( E_{0, \text{false}} = E_0 \) and take into account results described above then we can conclude that energy of the system in the false vacuum state has the following form at asymptotically late times \( t \gg T_C \).

\[
E^{\text{false}}(t) \simeq E_{0, \text{true}} + f_2 \frac{\hbar^2}{t^2} + f_4 \frac{\hbar^4}{t^4} \ldots \neq E_{0, \text{false}}.
\]

Next we will identify \( \rho_{\text{de}}(t_0) \) with energy \( E_0 \) of the unstable system divided by the volume \( V_0 \) (where \( V_0 \) is the volume of the system at \( t = t_0 \)): \( \rho_{\text{de}}(t_0) \equiv \rho_{\text{de}}^{\text{qft}} \equiv \rho_{\text{de}} = \frac{E_{0, \text{false}}}{V_0} \) and \( \rho_{\text{bare}} = \frac{E_{0, \text{true}}}{V_0} \). (\( \rho_{\text{de}}^{\text{qft}} \) is the vacuum energy density calculated by the use of quantum field theory methods).

Hence at times \( t \gg T_C \),

\[
\rho_{\text{de}}(t) = \rho_{0, \text{false}}^{\text{qft}}(t) \simeq \rho_{\text{bare}} + \frac{d_2}{t^2} + \frac{d_4}{t^4} + \ldots, \quad (t \gg T_C),
\]
where \( d_{2k} = d_{2k}^* \). Analogous relations are for \( \Lambda(t) = \frac{8\pi G}{c^2} \rho(t) \), or \( \Lambda(t) = 8\pi G \rho(t) \) where we assume \( \hbar = c = 1 \) units:

\[
\Lambda(t) \simeq \Lambda_{bare} + \frac{\alpha_2}{t^2} + \frac{\alpha_4}{t^4} + \ldots, \quad (t \gg T_C).
\]  

(38)

The coefficients \( d_{2k} \) and \( \alpha_{2k} \) are functions of \( \beta \) and \( \Gamma_0 \) (see (33)) and in this case we get that

\[
\beta = \frac{E_0 - E_{\text{min}}}{\Gamma_0} \equiv \frac{\rho_{\text{de}} - \rho_{\text{bare}}}{\gamma_0} > 0,
\]

(39)

(where \( \gamma_0 = \Gamma_0/V_0 \)), or equivalently, \( \Gamma_0/V_0 \equiv \frac{\rho_{\text{de}} - \rho_{\text{bare}}}{\beta} \).

We get the good approximation of Eq. (38) replacing the cosmological time \( t \) with the Hubble cosmological scale time \( t_H = \frac{1}{H} \). As the result, instead of (38) we get

\[
\Lambda(t) = \Lambda(H(t)) \simeq \Lambda_{bare} + \alpha_2 (H(t))^2 + \alpha_4 (H(t))^4 + \ldots,
\]

(40)

(where \( H(t) \) is the Hubble function), which is exactly the parametrization considered in [27, 28] and in many papers of these and other authors.

Note that the form of \( \kappa(t) \) is unchanged when one passes from energies \( E(t), E_0, E_{\text{min}} \) to the above defined energy density \( \rho \) and \( \Lambda \):

\[
\kappa(t) = \frac{E(t) - E_{\text{min}}}{E_0 - E_{\text{min}}} \equiv \frac{E(t)}{E_0} - \frac{E_{\text{min}}}{E_0} = \frac{\rho_{\text{de}}(t) - \rho_{\text{bare}}}{\rho_{\text{de}}^0 - \rho_{\text{bare}}^0} = \frac{\Lambda(t) - \Lambda_{bare}}{\Lambda_{\text{qft}} - \Lambda_{bare}},
\]

(41)

This means that time evolution of \( \rho_{\text{de}}(t) \) and \( \Lambda(t) \) should the same form as it is presented earlier in Fig. (2).

3. Cosmological implications of decaying vacuum

Now we will consider cosmological equations with \( \rho_{\text{de}} = \Lambda \). It describes the density of dark energy. Here, the form of \( \Lambda \) has been derived in the previous section and is given by

\[
\Lambda \equiv \Lambda_{\text{eff}}(t) = \Lambda_{bare} + \delta \Lambda(t) = \rho_{\text{de}},
\]

(42)

where \( \delta \Lambda(t) \) describes quantum corrections given by a series with respect to \( \frac{1}{t} \), i.e.

\[
\delta \Lambda(t) = \sum_{n=1}^{\infty} \alpha_{2n} \left( \frac{1}{t} \right)^{2n},
\]

(43)

and \( t \) is the cosmological scale time. The function \( \delta \Lambda(t) \) has a reflection symmetry with respect to the cosmological time \( \delta \Lambda(-t) = \delta \Lambda(t) \).

The cosmological equations has the following form:

\[
3H(t)^2 = \rho_m(t) + \rho_{\text{de}}(t),
\]

(44)

\[
\dot{\rho}_m = -3H(t)\rho_m(t) - \dot{\rho}_{\text{de}}(t),
\]

(45)

where \( H(t) \equiv \frac{\dot{a}}{a} \) is the Hubble function, the dot denotes the derivative with respect to time: \( \dot{a} \equiv \frac{d}{dt} \), and \( a \) is the scale factor. The equation (44) is called the Friedmann equation and the equation (45) is called the continuous equation.
The equations of state for matter and dark energy are assumed to have the form $p_m = 0$ and $p_{de} = -\rho_{de}$ respectively, where $p_m$ is a pressure of matter and $p_{de}$ is a pressure of the dark energy. The Friedmann equation for the FRW metric is given by

$$\frac{dH(t)}{dt} = -\frac{1}{2}(\rho_{\text{eff}}(t) + p_{\text{eff}}(t)) = -\frac{1}{2}\rho_m(t) = -\frac{1}{2}\left(3H(t)^2 - \Lambda_{\text{bare}} - \delta\Lambda(t)\right), \quad (46)$$

where $\rho_m$ is the density of the matter, $\rho_{\text{eff}} = \rho_m + \rho_{de}$ and $p_{\text{eff}} = \rho_m + p_{de}$.

Now we cut off the series (43) on the second term. As a result, we get

$$\delta\Lambda(t) = \frac{\alpha_2}{t^2} + \frac{\alpha_4}{t^4}. \quad (47)$$

In this case, the Friedmann equation has the following form

$$3H^2 = \rho_m + \Lambda_{\text{bare}} + \frac{\alpha_2}{t^2} + \frac{\alpha_4}{t^4}. \quad (48)$$

For the case $\delta\Lambda(t) = \frac{\alpha_2}{t^2} \ (\alpha_4 = 0)$, the solution of Eq. (46) looks as follows

$$H(t) = \frac{1 - 2n}{3t} + \sqrt{\frac{\Lambda_{\text{bare}}}{3}} \frac{I_{n-1}\left(\sqrt{\frac{3\Lambda_{\text{bare}}}{2}} t\right)}{I_n\left(\sqrt{\frac{3\Lambda_{\text{bare}}}{2}} t\right)}. \quad (49)$$

where $I_n$ denotes the Bessel function of the first kind. The parameter $n$ is equal to $\frac{1}{2}\sqrt{1 + 3\alpha_2}$.

Now we can obtain from the Friedmann equation the formula for $\rho_m(t)$ for the case $\alpha_4 = 0$. The result is:

$$\rho_m(t) = \left(1 - \frac{2n}{\sqrt{3t}} + \sqrt{\Lambda_{\text{bare}}} \frac{I_{n-1}\left(\sqrt{\frac{3\Lambda_{\text{bare}}}{2}} t\right)}{I_n\left(\sqrt{\frac{3\Lambda_{\text{bare}}}{2}} t\right)}\right)^2 - \Lambda_{\text{bare}} - \frac{\alpha_2}{t^2}. \quad (50)$$

If $\alpha_4 \neq 0$ then $\delta\Lambda(t)$ has the form given by (47) and the Friedmann equation rewritten in dimensionless terms looks as follows:

$$\frac{H(t)^2}{H_0^2} = \Omega_{m,0} f(t) + \Omega_{\Lambda_{\text{bare}},0} + \Omega_{\alpha_2,0} \frac{T_0^2}{t^2} + \Omega_{\alpha_4,0} \frac{T_0^4}{t^4}, \quad (51)$$

where $\Omega_{m,0} = \frac{\rho_{m,0}}{3H_0^2}$, $\Omega_{\Lambda_{\text{bare}},0} = \frac{\Lambda_{\text{bare}}}{3H_0^2}$, $\Omega_{\alpha_2,0} = \frac{\alpha_2}{3H_0^4}$, $\Omega_{\alpha_4,0} = \frac{\alpha_4}{3H_0^6}$, $H_0$ is the today’s value of the Hubble constant, $f(t) = \frac{\rho_m(t)}{\rho_{m,0}}$, and $T_0$ is the age of the Universe. Here, $\rho_{m,0}$ parameter denotes the present day value of the energy density of matter.

In our model we observe a small deviation from the canonical scaling of the energy density of matter ($\rho_m \propto a^{-3}$) [27, 28]. This deviation is defined as

$$\lambda(t) = \ln \frac{\rho_m(t)}{\rho_{m,0}} \ln a(t) + 3. \quad (52)$$

As result we get

$$\rho_m = \rho_{m,0} a^{-3 + \lambda}. \quad (53)$$

Our model has five parameters: $H_0, \Omega_{m,0}, \Omega_{\Lambda_{\text{bare}},0}, \Omega_{\alpha_2,0}, \Omega_{\alpha_4,0}$. But, for the present age of the Universe, from the above equation, we obtain the following condition

$$1 = \Omega_{m,0} + \Omega_{\Lambda_{\text{bare}},0} + \Omega_{\alpha_2,0} + \Omega_{\alpha_4,0}. \quad (54)$$
which reduces the number of independent parameters from five to four. Now we need to determine all these four parameters using observational data. Having all these parameters we can use the above presented equations to test our model. In order to do this we used parameter obtained from the astronomical data: supernovae of type Ia, Baryon Acoustic Oscillations (Sloan Digital Sky Survey Release 7, 6dF Galaxy Redshift Survey, WiggleZ), measurements of the Hubble parameter $H(z)$ of galaxies, the Alcock-Paczynski test and measurements of Cosmic Microwave Background (CMB) and lensing by Planck satellite and low-$\ell$ polarization from the WMAP (WP) (the acoustic scale, the shift parameter).

We took the data of supernovae of type Ia from the Union 2.1 dataset [29]. In this case, the likelihood function is given by the following equation

$$\ln L_{\text{SN}} = -\frac{1}{2} \left[ A - B^2/C + \log(C/(2\pi)) \right],$$

(55)

where $A = (\mu_{\text{obs}} - \mu_{\text{th}})C^{-1}(\mu_{\text{obs}} - \mu_{\text{th}})$, $B = C^{-1}(\mu_{\text{obs}} - \mu_{\text{th}})$, $C = \text{Tr}C^{-1}$ and $C$ is a covariance matrix for SNIa. The observer distance modulus $\mu_{\text{obs}}$ is given by equation $\mu_{\text{obs}} = m - M$ (where $m$ is the apparent magnitude and $M$ is the absolute magnitude of SNIa). The theoretical distance modulus is defined by equation $\mu_{\text{th}} = 5\log_{10} D_L + 25$ (where the luminosity distance has the form $D_L = c(1+z)\int_0^z \frac{dz'}{H(z')}$.)

For our estimation, we got the BAO data: Sloan Digital Sky Survey Release 7 (SDSS DR7) dataset at $z = 0.275$ [30], 6dF Galaxy Redshift Survey measurements at redshift $z = 0.1$ [31], and WiggleZ measurements at redshift $z = 0.44, 0.60, 0.73$ [32]. In this case the likelihood function is given by the equation

$$\ln L_{\text{BAO}} = -\frac{1}{2} \left( D_{\text{obs}} - D_{\text{th}} \right)^2,$$

(56)

where $r_s(z_d)$ is the sound horizon for the drag epoch [33, 34].

We took measurements of the Hubble parameter $H(z)$ of galaxies from [35, 36, 37]. In this case, the likelihood function is defined by the following formula

$$\ln L_{H(z)} = -\frac{1}{2} \sum_{i=1}^{N} \left( H(z_i)_{\text{obs}} - H(z_i)_{\text{th}} \right)^2.$$

(57)

The likelihood function for the Alcock-Paczynski test [38, 39] looks as follows

$$\ln L_{\text{AP}} = -\frac{1}{2} \sum_i \left( A P_{\text{th}}(z_i) - A P_{\text{obs}}(z_i) \right)^2.$$

(58)

where $A P(z)_i = H(z) f z^2 \frac{dz'}{H(z')}$ and $A P(z)_i$ are observational data [40, 41, 42, 43, 44, 45, 46, 47, 48].

We use the likelihood function for the measurements of CMB [3] and lensing by Planck, and low-$\ell$ polarization from the WMAP (WP) which can be written in the following form

$$\ln L_{\text{CMB+lensing}} = -\frac{1}{2} (x^{\text{th}} - x^{\text{obs}})^{(x^{\text{th}} - x^{\text{obs}})} - x^{\text{obs}}),$$

(59)

where $C$ is the covariance matrix describing the errors and $x$ is a vector having the following components: the acoustic scale $l_A$, the shift parameter $R$ and $\Omega_m h^2$.

$$l_A = \frac{\pi}{r_s(z^*)} \int_0^{z^*} \frac{dz'}{H(z')}$$

(60)

$$R = \sqrt{\Omega_m h^2} \int_0^{z^*} \frac{dz'}{H(z')}$$

(61)
where $z^*$ is the redshift at the epoch of the recombination [33].

The final formula for likelihood function has the following form

$$L_{\text{tot}} = L_{\text{SNIa}}L_{\text{BAO}}L_{\text{AP}}L_{\text{H}(z)}L_{\text{CMB+lensing}}.$$  (62)

Performing the statistical analysis we used our own code CosmoDarkBox, where Metropolis-Hastings algorithm [49, 50] is used.

From (33), we conclude that $\alpha_2$ parameter is negative so we can assume for the statistical analysis that $\Omega_{\alpha_2,0} < 0$. Results of our estimations are presented below in Tab. 1. Figs 3, 4, 5, and 6 show the predictions of our model based on the values of parameters resulting from the fitting process.

**Table 1.** The best fit and errors for the estimated model with $H_0$ assumed as 67.74 $\frac{\text{km}}{\text{s Mpc}}$, $\Omega_{m,0}$, $\Omega_{\alpha_2,0}$, $\Omega_{\alpha_4,0}$ and $\Omega_{b,0}$ assumed as 0.048468. The value of $\chi^2$ per degree of freedom is equal to 0.094 for our model. For $\Lambda$CDM model, $\chi^2$ per degree of freedom is equal to 0.099.

| parameter | best fit | 68% CL | 95% CL |
|-----------|----------|--------|--------|
| $\Omega_{m,0}$ | 0.3045 | +0.0045 | +0.0089 |
| $\Omega_{m,0}$ | -0.0043 | -0.0087 |
| $\Omega_{\alpha_2,0}$ | -0.0056 | +0.0056 |
| $\Omega_{\alpha_2,0}$ | -0.0257 | -0.0391 |
| $\Omega_{\alpha_4,0}$ | $4.1 \times 10^{-6}$ | +0.000032 | +0.000059 |
| $\Omega_{\alpha_4,0}$ | $-6.2 \times 10^{-6}$ | $-6.8 \times 10^{-6}$ |

**Figure 3.** The diagram presents the evolution of $\frac{\rho_m(t)}{3H_0^2}$, for the best fit values of model parameters. The cosmological time $t$ is expressed in $\frac{\text{s Mpc}}{\text{km}}$. Here, the present age of the Universe is equal to 0.014 $\frac{\text{s Mpc}}{\text{km}}$.

**Figure 4.** The diagram presents the evolution of $H(t)$, for the best fit values of model parameters. The cosmological time $t$ is expressed in $\frac{\text{s Mpc}}{\text{km}}$. Here, the present age of the Universe is equal to 0.014 $\frac{\text{s Mpc}}{\text{km}}$. 
4. Summary

From statistical analysis, we get that the best fit of values of dimensionless density parameters $\Omega_{\alpha_2,0}$ and $\Omega_{\alpha_4,0}$

$$
\Omega_{\alpha_2,0} = -0.0056, \quad \Omega_{\alpha_4,0} = 4 \times 10^{-6}, \quad \Omega_{m,0} = 0.3045,
\Omega_{\text{bare}} = 1 - \Omega_{\alpha_2,0} - \Omega_{\alpha_4,0} - \Omega_{m,0} = 0.701096.
$$

These parameters characterize the phenomenological properties of decaying quantum vacuum. While there exists theoretical prior for negative value $\Omega_{\alpha_2}$ there is no prior for $\Omega_{\alpha_4}$. Having values of these parameters one can calculate another characteristic of the process of the decaying dark energy.

The considered model is in good agreement with the astronomical data. This shows that $\frac{\alpha_2}{t^2}$ term dominates in formula for $\delta \Lambda(t)$ and $\frac{\alpha_4}{t^4}$ term is negligible comparing to the term $\frac{\alpha_2}{t^2}$. This means that the function $\delta \Lambda(t)$ is negative in the present epoch. As the result, we get that the value of the energy density of dark matter increases for the our epoch and for the late times. In consequence, we get the quantum process of the annihilation of particles of dark matter for the late Universe.

We found a deviation from the standard scaling of the energy density of matter. We can explain the source of this deviation showing that the energy density of matter decreases more rapidly or slowly due to the energy transfer from dark matter to dark energy sector or in the opposite direction. The direction of the energy transfer crucially depends on the sign of $\delta \Lambda(t)$ in our model. For the best fit value, we get that the deviation speeds up decreasing of the energy density of matter.

We found that for the best fit value of the model parameters, the evolution of the Universe in our model and in the standard cosmological model $\Lambda$CDM are similar in the present epoch and for late epochs. In this model $\Lambda$ decays into $\Lambda_{\text{bare}}$. As a result, we can treated the $\Lambda$ term as a constant for our epochs. The process of decaying of dark energy has not a big influence for the evolution of the current Universe. Therefore, this model can be considered as a simple extension of the $\Lambda$CDM model. The deviation of this model from the $\Lambda$CDM model is less than $1\sigma$ today. Note that our model explains naturally the smallness of the cosmological constant parameter $\Lambda_{\text{bare}}$: In the model the running $\Lambda = \Lambda(t)$ term, which is very, very large near the initial singularity (i.e., for very small times $t$), tends to much, much smaller $\Lambda_{\text{bare}}$ when $t \to \infty$. 

Figure 5. The diagram presents the evolution of $a(t)$, for the best fit values of model parameters. The cosmological time $t$ is expressed in $\text{s Mpc}^{-1}$. Here, the present age of the Universe is equal to 0.014 $\text{s Mpc}^{-1}$.

Figure 6. The diagram presents the evolution of $\lambda(t)$, for the best fit values of model parameters. The cosmological time $t$ is expressed in $\text{s Mpc}^{-1}$. Here, the present age of the Universe is equal to 0.014 $\text{s Mpc}^{-1}$.
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