CANONICAL CONNECTIONS ON SUB-RIEMANNIAN MANIFOLDS WITH CONSTANT SYMBOL

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Abstract. As a new tool in the investigation of the equivalence problem on sub-Riemannian manifolds, we give formulas for a canonical choice of grading and compatible affine connection in this geometry. This grading and affine connection is available on any sub-Riemannian manifold with constant symbol and is based on Morimoto’s normalization condition for a Cartan connection in such a setting. We completely compute these structures in the case of any contact manifold of constant symbol, including those cases where the connections of Tanaka-Webster-Tanno are not defined. We also give a completely original grading and connection on sub-Riemannian (2,3,5)-manifolds, and explain how the grading and connection from Morimoto’s theory compare.

1. Introduction

Since their introduction in 1986 by Strichartz [33], following even earlier work in e.g. [9, 15], there has been relatively few results related to the equivalence problem in sub-Riemannian geometry. The known material can be found in [13, 14, 20, 8, 1, 4]. There are several reasons for why developing tools to determine whether or not two sub-Riemannian manifolds are isometric is a very complicated problem. While any Riemannian manifold at any point will have the Euclidean space as its metric tangent cone, the generic infinitesimal model of a sub-Riemannian manifold belong to a large class of spaces called Carnot groups [24, 6, 17]. As a consequence, in dealing with the equivalence problems, each case will have their own special considerations. Furthermore, the main tool of the equivalence problem in Riemannian geometry is the Levi-Civita connection and its curvature, see e.g. the Cartan-Ambrose-Hicks theorem [10, Chapter 1.12] and result in [32, 29]. With no clear way of defining a canonical connection on sub-Riemannian manifolds, it was difficult to understand how to approach this problem.

A key development related to the lack of a canonical choice of connection was presented by T. Morimoto in his 2008 paper [28], based in his work in [27]. Using Cartan geometry, Morimoto proved that sub-Riemannian manifolds with constant symbol, i.e. with the tangent cone at all points being isometric, have a canonical choice of Cartan connection on its nonholonomic frame bundle. In particular, it is underappreciated how this result gives us a flatness theorem for sub-Riemannian manifolds, which to our knowledge previously only for sub-Riemannian manifolds whose symbol was the standard Heisenberg group. For some result applying Morimoto’s formalism to sub-Riemannian manifolds, see [3, 7].

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We want to make the theory of Morimoto more explicit and available by translating the result from Cartan geometry to using the formalism of a grading of the tangent bundle and a compatible affine connection. In particular, using the theory of selectors developed in [12], we give an explicit description of Morimoto’s connection. We will show the merits of our method of presentation by applying the result to contact manifolds and $(2, 3, 5)$-manifolds. We emphasize that in the case of contact manifolds, we are consider all cases where we have a constant symbol and just the case where the sub-Riemannian metric is defined from the contact form as with connections of Tanaka, Webster and Tanno [34, 37, 35].

The structure of the paper is as follows. In Section 2 we will introduce the basic definitions of sub-Riemannian manifolds, including theory related to compatible affine connections. In Section 3 we introduce Carnot groups and algebras, and their isometry groups. In Section 4 we introduce the symbol of a sub-Riemannian manifold at a point and the extra structure that exists for manifolds with constant symbol. We also introduce the important notion of strongly compatible connections on the nonholonomic tangent bundle and show that these only exist in the case of constant symbols.

In Section 5 we work with sub-Riemannian manifolds that are graded in a way compatible with the growth of the flag $E = E^{-1} \subseteq E^{-2} \subseteq \cdots \subseteq E^{-s}$ produced by including an increasing number of brackets of $E$. We finally use this formalism to write Morimoto’s canonical choice of Cartan connection as a grading and affine connections in Section 6.

In the final two sections, we do two different examples. In Section 7 we work with the case of contact manifold whose constant symbol will be the Heisenberg algebra, but not necessarily with the standard metric. We observe in particular that if we are not in the case of the standard metric, then the grading given from Morimoto’s theory is not necessarily the one given by the Reeb vector field. It is also interesting to observe that even for the case of the standard metric on the Heisenberg algebra, the connection of Morimoto’s theory does not correspond exactly to the connections of Tanaka-Webster-Tanno.

In the final section, Section 8, we introduce a completely original canonical grading and connection for sub-Riemannian manifolds with growth vectors $(2, 3, 5)$. We then write how these structures can be modified to obtain the grading and affine connection corresponding to Morimoto’s normalization condition for Cartan connections.

A central tool for presenting our results will be selectors, introduced in [12]. Some additional helpful results related to these two-vector-valued one-forms and also related to partial connections are found in Appendix A.

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2. Sub-Riemannian manifolds

Let $M$ be a connected manifold. A sub-Riemannian structure is a pair $(E, g)$ where $E$ is a subbundle of the tangent bundle $TM$ and $g = \langle \cdot, \cdot \rangle_g$ is a metric tensor defined only on $E$. We call $E$ the horizontal bundle. For the rest of the paper, we will assume that $E$ is bracket-generating, meaning that the sections of $E$ and their iterated brackets span all of $TM$. With this assumption, we can make $(M, E, g)$
into a metric space with the following construction. An absolutely continuous curve \( \gamma : [a, b] \rightarrow M \) is called horizontal if \( \dot{\gamma}(t) \in E_{\gamma(t)} \) for almost every \( t \). For such a curve, we can define its length as

\[
\text{Length}(\gamma) = \int_a^b \langle \dot{\gamma}, \dot{\gamma} \rangle_g^{1/2}(t) \, dt.
\]

We can then define the Carnot-Carathéodory distance \( d_{cc}(x, y) \) between the points \( x, y \in M \) as the infimum of the lengths of all horizontal curves connecting them. When \( E \) is bracket-generating, this distance is always finite and it induces the same topology as the manifold topology.

In this paper, we will make the additional assumption that \( E \) is equiregular, meaning that for any \( k = 0, 1, \ldots \),

\[
E^{-k} = \text{span}\{[X_1, \cdots, [X_{j-2}, [X_{j-1}, X_j]] \cdots]_x : j = 0, \cdots, k, X_i \in \Gamma(E), x \in M\},
\]

is not only a subset, but a subbundle of \( TM \). In particular, the rank of \( E^{-k} \) does not depend on \( x \). In the above expression, for the cases \( j = 0 \) and \( j = 1 \), we interpret the bracket \([X_1, \cdots, [X_{j-2}, [X_{j-1}, X_j]] \cdots]\) as respectively 0 and \( X_1 \). We have a corresponding flag of subbundles,

\[
E^0 = 0 \subsetneq E^{-1} \subsetneq E^{-2} \subsetneq \cdots \subsetneq E^{-s} = TM.
\]

The minimal integer \( s \) such that \( E^{-s} = TM \) is called the step of \( E \) and the collection of the ranks \( \mathcal{G} = (\text{rank } E^{-1}, \ldots, \text{rank } E^{-s}) \) is called the growth vector of \((M, E, g)\).

For more on sub-Riemannian manifolds, see [26][2].

We make the following definition related to affine connections and compatibility with a sub-Riemannian structure \((E, g)\).

**Definition 2.1.** Let \((E, g)\) be a sub-Riemannian structure on a manifold \( M \).

(i) Let \( \nabla \) be an affine connection on \( TM \). The subbundle \( E \) is called parallel with respect to \( \nabla \) if \( \nabla_Y X |_x \in E_x \) for any \( Y \in \Gamma(TM) \), \( X \in \Gamma(E) \), \( x \in M \). Note that if \( E \) is parallel, we obtain an affine connection on \( E \) by restriction of \( \nabla \).

(ii) An affine connection \( \nabla^E \) on \( E \) is said to be compatible with \( g \) if

\[
Y \langle X_1, X_2 \rangle_g = \langle \nabla^E_Y X_1, X_2 \rangle_g + \langle X_1, \nabla^E_Y X_2 \rangle_g,
\]

for any \( Y \in \Gamma(TM) \), \( X_1, X_2 \in \Gamma(E) \).

(iii) An affine connection \( \nabla \) is compatible with \((E, g)\) if \( E \) is parallel and \( \nabla|E \) is compatible with \( g \).

Without getting into the general theory of length minimizers in sub-Riemannian manifolds, for which we again refer to [26][2], we mention the following relationship between them and compatible connections found in [10][19]. Recall that the torsion \( T \) of a connection \( \nabla \) is defined as

\[
T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].
\]

**Proposition 2.2.** Let \( \nabla \) be any affine connection on \( TM \) compatible with \((E, g)\) and with torsion \( T \). Let \( \gamma : [a, b] \rightarrow M \) be a length minimizer of the distance \( d_{cc} \) between the points \( \gamma(a) \) and \( \gamma(b) \). Then at least one of the following holds.

(i) (Normal geodesic) There is a one-form \( \lambda(t) \) along \( \gamma(t) \) satisfying for any \( u \in E_{\gamma(t)} \) and \( w \in T_{\gamma(t)} M \),

\[
(\nabla_\gamma \lambda(t))(w) = -\lambda(t)(T(\dot{\gamma}(t), w)), \quad \lambda(t)(u) = \langle \dot{\gamma}(t), u \rangle_g.
\]
Remark 2.3. If $E$ is a subbundle of $TM$ with rank greater or equal to 2, then it will generically be bracket-generating in the sense of [25, Proposition 2]. Furthermore, any bracket-generating subbundle $E$ will be equiregular outside of a closed set with empty interior [21, Section 2.1.2, p. 21].

Remark 2.4. Here and in the next section, we use $E^k$ with $k$ negative. Similarly, we will in Section 3 use negative grading for Carnot algebras. This is a convention used here to make the notation coincide with the one used when dealing with Tanaka prolongations, but as the prolongations of Carnot algebras have no positive part [28, Proposition 1], we will not encounter positively graded subspaces.

3. Carnot groups and algebras

3.1. Definitions. A stratified Lie algebra is a graded Lie algebra

$$g_- = g_{-s} \oplus \cdots \oplus g_{-1},$$

such that $[g_{-1}, g_{-k}] = g_{-k-1},$ for $k = 1, \ldots, s - 1$ and with $[g_{-1}, g_{-s}] = 0$. In particular, $g_-$ is nilpotent. If in addition, $g_{-1}$ is furnished with an inner product $(\cdot, \cdot)_{g_{-1}},$ then $g_-$ is called a Carnot algebra. Let $G_-$ be the corresponding simply connected Lie group and define a sub-Riemannian structure $(E, g)$ on $G_-$ by left translation of $g_{-1}$ and its inner product. We say that $(G_-, E, g)$ is a Carnot group which will be a sub-Riemannian manifold with an equiregular horizontal bundle. For examples, see Sections 7.1 and 8.1.

3.2. Isometries. We introduce some notation related to isometry groups of Carnot groups and their Lie algebras. For more details, see [23, 18]. Any isometry of $(G_-, E, g)$ is a composition of a left translation and a group automorphism. More precisely, a group automorphism $\Psi : G_- \to G_-$ is a sub-Riemannian isometry if the corresponding Lie algebra automorphism $\psi = \Psi_{*,1} : g_- \to \tilde{g}_-$ maps $g_{-1}$ onto $\tilde{g}_{-1}$ isometrically. For this reason, we introduce the following definitions.

Definition 3.1. Let $g_- = g_{-s} \oplus \cdots \oplus g_{-1}$ and $\tilde{g}_- = \tilde{g}_{-s} \oplus \cdots \oplus \tilde{g}_{-1}$ be two Carnot algebras.

(i) A linear map $\psi : g_- \to \tilde{g}_-$ will be called an isometry if $\psi$ is a Lie algebra isomorphism that maps $g_{-1}$ onto $\tilde{g}_{-1}$ isometrically. We write $\text{Iso}(g_-, \tilde{g}_-)$ for the space of all such maps.

(ii) We say that $g_-$ and $\tilde{g}_-$ are isometric if there is an isometry connecting them.

Observe that isometries have degree zero from the definition of Carnot algebras. The group $G_0 = \text{Iso}(g_-) = \text{Iso}(g_{-1}, g_-)$ is a Lie group with Lie algebra $g_0 = \text{iso}(g_-)$ consisting of derivations of degree zero such that their restriction to $g_{-1}$ are skew-symmetric. In other word, $g_0$ consists of all maps $D : g_- \to g_-$ such that $D(g_{-k}) \subseteq g_{-k}$ and such that for any $B, C \in g_-$ and $A \in g_{-1},$

$$D[A, B] = [DA, B] + [A, DB], \quad \langle DA, A \rangle_{g_{-1}} = 0.$$
We can define an extended Lie algebra \( g = g_- \oplus g_0 = g_- \oplus \text{iso}(g_-) \) with brackets, 
\[
[D_1 + A, D_2 + B] = [D_1, D_2] + D_1B - D_2A + [A, B], \quad D_1, D_2 \in g_0, A, B \in g_-. 
\]
This will be the Lie algebra of the isometries of \((G-, E, g)\).

3.3. The free nilpotent algebra and induced inner products on Carnot algebras. Recall first the following definition. If \( q : W \to V \) is a surjective linear map between two inner product spaces, we say that \( q \) is a linear submetry if \( q|_{(\ker q)^\perp} \) is an isometry from \((\ker q)^\perp\) onto \( V \). We observe that if \( W \) is an inner product space and \( q : W \to V \) is a surjective linear map to a vector space, there is a unique inner product on \( V \) making \( q \) into a linear submetry.

Let \( W \) be a vector space and define \( T(W, s) = \bigoplus_{j=1}^s W^{\otimes j} \) as the truncated tensor algebra up to step \( s \). We consider \( T(W, s) \) as the algebra with product \((A, B) \mapsto A \otimes B\) with the convention that tensor products of more than \( s \) elements in \( W \) vanish. We define a subalgebra \( K(W, s) \) of elements 
\[
K(W, s) = \text{span}\{A \otimes B + B \otimes A, A \otimes B \otimes C + B \otimes C \otimes A + C \otimes A \otimes B : A, B, C \in T(W, s)\}. 
\]
and a submersion
\[
(k : T(W, s) \to \text{free}(W, s) := T(W, s)/K(W, s),
\]
We furnish \( \text{free}(W, s) \) with a Lie algebra structure given by 
\[
[k(A), k(B)] = k(A \otimes B - B \otimes A).
\]
This will be a stratified Lie algebra 
\[
f_- = \text{free}(W, s) = f_-s \oplus \cdots \oplus f_-1, \quad \text{with} \quad f_-j = k(W^{\otimes j});
\]
so in particular \( f_-1 = W \). The Lie algebra \( f_- = \text{free}(W, s) \) is called the free nilpotent algebra of step \( s \) generated by \( W \). If \( W \) is an inner product space, then \( f = \text{free}(W, s) \) has the structure of a Carnot algebra. If we define \( F_0 = \text{Iso}(f) \), then there is a bijection between linear isometries \( q : W \to W = f_-1 \) and elements \( \psi_q \in F_0 \) given by \( \psi_q|_{f_-1} = q \).

Let \( g_- = g_-s \oplus \cdots \oplus g_-1 \) be any stratified Lie algebra. Then there is a surjective Lie algebra homomorphism
\[
P : \text{free}(g_-1, s) \to g_-
\]
given by dividing out the additional relations. It follows that \( g_- \) is uniquely determined by the ideal \( a = \ker P \). Furthermore, the Lie group \( G_0 = \text{Iso}(g_-) \) is isomorphic to the Lie group 
\[
\{\psi \in F_0 : \psi(a) \subseteq a\}.
\]
Assume that \( g_- = g_-s \oplus \cdots \oplus g_-1 \) has the structure of a Carnot algebra. The inner product on \( g_-1 \) gives us an induced inner product on \( T(g_-1, s) \). To obtain simplified formulas in later sections, we make the following conventions when regarding to induced inner products on tensor products. If \( A_1, \ldots, A_n \) is an orthonormal basis \( g_-1 \), we give \( g_-^\otimes n \) an inner product such that
\[
\{2^{i_1/2}A_{i_1} \otimes \cdots \otimes A_{i_j} : 1 \leq i_1, \ldots, i_j \leq n\},
\]
is an orthonormal basis of \( g_-^\otimes n \). We make the decomposition \( T(g_-1, s) = \bigoplus_{j=1}^s g_-^\otimes j \) orthogonal to get a full inner product. Taking the next step, there is then an induced inner product on \( \text{free}(g_-1, s) \) such that \( k \) as in (3.1) is a submetry, and consequently there is an inner product on \( g_- \) such that \( P \) in (3.2) is a submetry.
3.2 Remark. We note the following about the induced metric on \( g_- \).

(i) If \( A_1, A_2, \ldots, A_n \) is an orthonormal basis of \( g_{-1} \) and \( B \in g_{-j}, \) then

\[
\|B\|^2 = \sum_{k_1, \ldots, k_j=1}^n \langle B, [A_{k_1}, [A_{k_2}, \cdots, [A_{k_j-1}, A_{k_j}]]] \cdots \rangle^2.
\]

Furthermore, \( j > 1 \) and if \( B_1, \ldots, B_m \) is an orthonormal basis of \( g_{-j+1}, \) then

\[
\|B\|^2 = \sum_{j=1}^n \sum_{k=1}^m \langle B, [A_j, B_k] \rangle^2.
\]

(ii) If \( D \in g_0 = \text{iso}(g_-) \), then it is also a skew-symmetric map on \( g_- \) with respect to the induced inner product. To show this fact, write \( D_{-1} = D|_{g_{-1}} \) and also use \( D_{-1} \) to denote the induced map on \( T(g_-, s) \) by the rule

\[
D_{-1}(A \otimes B) = D_{-1}A \otimes B + A \otimes D_{-1}B, \quad A, B \in T(g_{-1}, s).
\]

This map will then also be skew-symmetric. Finally, from the relationship \( D(k \circ P) = (k \circ P)D_{-1} \) it follows that \( D \) is a skew-symmetric map on all of \( g_- \).

3.3 Remark. The above construction also make sense when \( W \) is a vector bundle rather than a vector space. We will take advantage of this in Section 4.2.

4. Nonholonomic tangent bundle and constant symbol

4.1 Sub-Riemannian manifolds with constant symbols. Let \((M, E, g)\) be a sub-Riemannian manifold. Assume that \( E \) is bracket-generating and equiregular. We define the nonholonomic tangent bundle of \((M, E, g)\) by

\[
gr = \text{gr}(E) = E^{-s}/E^{-s+1} \oplus \cdots \oplus E^{-2}/E + E.
\]

For each \( x \in M \), \( \text{gr}_x \) can be given the the structure of a Carnot algebra, with grading \( \text{gr}_{x,-k} := (\text{gr}_x)^{-k} = E^{-k}/E^{-k+1} \) and with Lie brackets \([\cdot, \cdot]\) defined by

\[
[X_x \mod E^{-j}, Y_x \mod E^{-j}] = [X, Y]_x \mod E^{-j}.
\]

Here, \([X, Y]_x\) denotes the commutator bracket of the vector fields \( X \) and \( Y \) extending the vectors \( X_x \) and \( Y_x \). We observe that this bracket is well defined as the right hand side does not depend on the chosen extensions. Since \( \text{gr}_{x,-1} = E_x \) has an inner product, we have a Cartan algebra structure on \( \text{gr}_x \). We say that \( \text{gr}_x \) with its Cartan algebra structure is the symbol of \((M, E, g)\) at \( x \in M \).

Definition 4.1. Let \((M, E, g)\) be a sub-Riemannian manifold with \( E \) bracket-generating and equiregular and let \( g_- \) be a Cartan algebra. We say that \((M, E, g)\) has constant symbol \( g_- \) if \( \text{gr}_x \) is isometric to \( g_- \) for any \( x \in M \).

If \((M, E, g)\) has constant symbol \( g_- \), we can then define nonholonomic frame bundle as the principal bundle

\[
G_0 \to \mathcal{F} \to M,
\]

where \( \mathcal{F}_x = \text{Iso}(g_-|_{\text{gr}_x}) \) for any \( x \in M \), \( G_0 = \text{Iso}(g_-) \) and \( G_0 \) acts on \( \mathcal{F}_x \) by composition on the right.
Remark 4.2. Let $\text{Gr}_x$ be the Carnot group corresponding to $\text{gr}_x$ with its sub-Riemannian structure. Recall the definition of the Carnot Caratheodory metric $d_{cc}$ in Section 2. The space $\text{Gr}_x$ is then the tangent cone of the metric space $(M, d_{cc})$ at $x \in M$, see [23] [6] [17]. Hence, assuming constant symbol is equivalent to assuming that all of the tangent cones of $(M, d_{cc})$ are isometric.

4.2. Compatible connections on the nonholonomic tangent bundle. Assume that $E$ is bracket generating and define $\text{gr} = \text{gr}(E)$ as the nonholonomic tangent bundle. Let $\nabla^{gr}$ be an affine connection on $\text{gr}$. We make the following definition regarding such connections.

Definition 4.3. (i) We say that $\nabla^{gr}$ is a compatible connection if $\text{gr}_{-j} = E^{-j}/E^{-j+1}$ is a parallel subbundle for $j = 1, \ldots, s$ and, furthermore, its restriction to $\text{gr}_{-1} = E$ is compatible with the sub-Riemannian metric $g$.

(ii) We say that $\nabla^{gr}$ is strongly compatible if it is compatible and furthermore, for any $Y \in \Gamma(TM)$, $A, B \in \Gamma(\text{gr})$, we have

$$
\nabla^{gr}_Y [A, B] = [\nabla^{gr}_Y A, B] + [A, \nabla^{gr}_Y B].
$$

(4.1)

Compatible connections will always exist by taking a direct sum of connections on each $\text{gr}_{-j}$, $j = 1, \ldots, s$, with the restriction that the one used on $\text{gr}_{-1}$ has to be metric. By contrast, strongly compatible only exists in the following special case.

Proposition 4.4. There exists a strongly compatible connection on $\text{gr}$ if and only if $(M, E, g)$ has constant symbol.

Proof. Assume first that $\nabla^{gr}$ is strongly compatible. For a curve $\gamma : [0, 1] \to M$ connecting two arbitrary points $x = \gamma(0)$ and $y = \gamma(1)$, let $/\gamma : \text{gr}_x \to \text{gr}_{\gamma(t)}$ denote the parallel transport along the curve. Since it is compatible, the map $/\gamma : \text{gr}_x \to \text{gr}_y$ preserves the grading and is a linear isometry from $E_x$ to $E_y$. Furthermore, since it is strongly compatible, $/\gamma$ will also be a Lie algebra homomorphism. Since it is invertible, it follows that $/\gamma$ is an isometry of Cartan algebras. As $x$ and $y$ was arbitrary, $(M, E, g)$ has constant symbol.

Conversely, if $(M, E, g)$ has constant symbol $\mathfrak{g}_-$ we can define the nonholonomic frame bundle $G_0 \to \mathcal{F} \to M$. By [22] Theorem II.2.1, any such principal bundle has a choice of a principal connection $\omega$. Consider the representation $\text{Ad}$ of $G_0$ on $\mathfrak{g}_-$, given by

$$
\text{Ad}(\psi)(A) = \psi A, \quad \psi \in G_0 = \text{Iso}(\mathfrak{g}_-), A \in \mathfrak{g}_-.
$$

(4.2)

We can then identify the associated bundle $\text{Ad}(\mathcal{F})$ with $\text{gr}(E)$, through the vector bundle isomorphism

$$
f \times \text{Ad} A \mapsto f(A), \quad f \in \mathcal{F}, A \in \mathfrak{g}_-.
$$

Define $\nabla^{gr}$ as the induced connection on $\text{gr}$ by $\omega$. By definition, parallel transport $/\gamma : \text{gr}_x \to \text{gr}_{\gamma(t)}$ along any curve $\gamma$ will be a Cartan algebra isometry, which is equivalent to being strongly compatible.

If $(M, E, g)$ is a manifold with constant symbol, we have the following description of strongly compatible connections. Assume that $\nabla^{gr}$ is a strongly compatible connection on $\text{gr} = \text{gr}(E)$. Let $\nabla^E = \nabla^{gr}|E$ denote the restriction of $\nabla^{gr}$ to $\text{gr}_{-1} = E$. By (4.1) it follows that $\nabla^E$ iteratively determine $\nabla^{gr}$, but such a connection cannot be arbitrary as the next result shows.
Proposition 4.5. Let \((M, E, g)\) be a sub-Riemannian manifold of constant symbol. Let \(\nabla^E\) be a connection on \(E\) compatible with the sub-Riemannian metric. Let \(\text{Hol}_x(\nabla^E)\) denote the holonomy group of \(\nabla^E\) at \(x\). Then the following are equivalent.

(i) \(\nabla^E\) is the restriction of a strongly compatible connection \(\nabla^\text{gr}\) on \(\text{gr}\).

(ii) For any \(x \in M\), if \(S_x = \text{Iso}(\text{gr}_x)\),

then \(\text{Hol}_x(\nabla^E) \subseteq S_x|_E\).

Proof. Let \(\nabla^E\) be a compatible connection on \(E\). Using the notation from Section 3.3, we note that \(\nabla^E\) induces a connection on the tensor space \(T(E, s)\). We furthermore observe that \(K(E, s)\) is parallel under this connection, giving us an induced connection on free\((E, s)\) which we denote by \(\nabla^\text{free}\). By definition, the Lie brackets of free\((E, s)\) will be parallel with respect to this connection. If we define \(P\) as in (3.2), then we have a strongly compatible connection \(\nabla^\text{gr}\) on \(\text{gr}\) whose restriction to \(E\) is \(\nabla^E\) if and only if \(\nabla^\text{gr} P = P \nabla^\text{free}\). We can use such a relation to define a connection \(\nabla^\text{gr}\) if and only if ker\(P\) is parallel with respect to \(\nabla^\text{free}\). By our comment on self-isometries of Carnot algebras in Section 3.3, this is the case if and only if \(\text{Hol}(\nabla^\text{free})|_E = \text{Hol}(\nabla^E)|_E \subseteq \text{Iso}(\text{gr}_x)|_E\).

We have the following relationship in the simply connected case, which is always the case locally. Recall the definition of curvature of a connection \(\nabla^E\) on \(E\) being given by

\[
R^E(Y_1, Y_2)X = \nabla^E_{Y_1} \nabla^E_{Y_2} X - \nabla^E_{Y_2} \nabla^E_{Y_1} X - \nabla^E_{[Y_1, Y_2]} X, \quad Y_1, Y_2 \in \Gamma(TM), X \in \Gamma(E).
\]

The next result then follow from the Ambrose-Singer theorem [5].

Corollary 4.6. Let \((M, E, g)\) be a sub-Riemannian manifold of constant symbol, with \(M\) simply connected. Let \(\nabla^E\) be a connection on \(E\) compatible with the sub-Riemannian metric and with curvature \(R^E\). Then the following are equivalent.

(i') \(\nabla^E\) is the restriction of a strongly compatible connection \(\nabla^\text{gr}\) on \(\text{gr}\).

(ii') For any \(x \in M\), if \(s_x = \text{iso}(\text{gr}_x)\),

then \(R^E(u, v) \in s_x|_E\) for any \(u, v \in T_x M\).

5. Graded sub-Riemannian structures

5.1. Equiregular subbundles and grading. Let \((M, E, g)\) be the sub-Riemannian manifold with \(E\) bracket-generating and equiregular of step \(s\). Related to such a subbundle, we introduce the following important definition.

Definition 5.1. Let

\[
TM = \bigoplus_{k=1}^s (TM)_{-k},
\]

be a grading of \(TM\). We will call it an \(E\)-grading if

\[
E^{-k} = (TM)_{-k} \oplus \cdots \oplus (TM)_{-1}.
\]

We will use \(\text{pr}_{-k}\) to denote the associated projection \(TM \rightarrow (TM)_{-k}, k = 1, \ldots, s\). We then have a corresponding vector bundle isomorphism \(I : TM \rightarrow \text{gr}\) given by

\[
I : v \mapsto \bigoplus_{k=1}^s I_{-k} v = \bigoplus_{k=1}^s \text{pr}_{-k} v \bmod E^{-k+1}.
\]
Then for every $k = 1, \ldots, s$,
\begin{equation}
I_{-k} w = w \mod E^{-k+1}
\end{equation}
for every $w \in E^{-k}$.
In particular, the restriction of $I_{-k}$ to $E^{-k}$ does not depend on the choice of $E$-grading.

Conversely, let $I = \oplus_{k=1}^s I_{-k} : TM \to \text{gr}$ be a vector bundle map satisfying (5.1). Define subbundles
\[ V^{-k} = \{ w \in TM : I_{-j} w = 0 \mod E^{-j+1}, k \neq j \}. \]
Then $E^{-k} = E^{-k+1} \oplus V^{-k}$ and we can define $\text{pr}_{-k}$ as the projection to $V^{-k} = (TM)_{-k}$ according to the decomposition $TM = V^{-s} \oplus \cdots \oplus V^{-1}$. We can hence equivalently define an $E$-grading as a vector bundle map $I : TM \to \text{gr}$ satisfying (5.1). In what follows, we will use the two points of view interchangeably.

We call $(M, E, g, I)$ a graded sub-Riemannian manifold.

### 5.2. Strongly compatible connections on graded sub-Riemannian manifolds

Let $(M, E, g)$ be a sub-Riemannian manifold with a bracket generating and equiregular horizontal bundle $E$. Consider an affine connection $\nabla$ on $TM$. We introduce the following definitions.

If each $E^{-k}$ is parallel with respect to $\nabla$, we get an induced connection $\nabla^{\text{gr}}$ on $\text{gr} = \text{gr}(E)$ by
\begin{equation}
\nabla^{\text{gr}} Y \mod E^{-k} = \nabla_X Y \mod E^{-k}.
\end{equation}
An $E$-grading $I = \oplus_{k=1}^s I_{-k} : TM \to \text{gr}$ is parallel if the kernel of each $I_{-k}$ is parallel. Equivalently, it is parallel if each $(TM)_{-k}$ is parallel for $k = 1, \ldots, s$. In particular, each $E^{-k}$ is parallel. Conversely, if $\nabla$ is any connection such that the growth flag of $E$ is parallel and $\nabla^{\text{gr}}$ is defined as in (5.2), then $I$ is parallel if and only if $\nabla^{\text{gr}} I = I \nabla$.

**Definition 5.2.** Let $(M, E, g, I)$ be a graded sub-Riemannian manifold and $\nabla$ a connection on $TM$.

(I) We say that $\nabla$ is compatible with $(E, g, I)$ if it is compatible with $(E, g)$ and $I$ is $\nabla$-parallel.

(II) We say that $\nabla$ is strongly compatible with $(E, g, I)$ if $\nabla$ is compatible with $(E, g, I)$ and the induced connection $\nabla^{\text{gr}}$ on $\text{gr} = \text{gr}(E)$ is strongly compatible.

We note the following about the torsion $T$ of a connection $\nabla$ compatible $(E, g, I)$. Recall that if $\mathcal{A} \to M$ is a vector bundle and $\nabla'$ is a connection on $\mathcal{A}$, then for $\mathcal{A}$-valued forms $\Gamma(\wedge^k T^*M \otimes \mathcal{A})$, the covariant exterior differential $d\nabla'$ is defined according to the following rules,
\[ d\nabla' A = \nabla' A, \quad d\nabla' (\zeta \wedge \alpha) = (d\nabla' \zeta) \wedge \alpha + (-1)^k \zeta \wedge d\alpha, \]
where $A \in \Gamma(\mathcal{A})$, $\zeta$ is any $\mathcal{A}$-valued $k$-form and $\alpha$ is any real-valued form.

**Lemma 5.3.** If $\nabla$ is a compatible connection of $(M, E, g, I)$ with induced connection $\nabla^{\text{gr}}$ on $\text{gr}$, then its torsion $T$ is given by
\[ IT = d\nabla^{\text{gr}} I. \]

**Proof.** By definition, we have the identity
\[ IT(X, Y) = I \nabla_X Y - I \nabla_Y X - I[X, Y] = \nabla^{\text{gr}}_X Y - \nabla^{\text{gr}}_Y X - I[X, Y] = d\nabla^{\text{gr}} I(X, Y). \]
As a consequence of Lemma 5.3 we have that for \( I^{-k}X = 0 \mod E^{-k+1} \) and \( I^{-k}Y = 0 \mod E^{-k+1} \), then
\[
I^{-k}T(X,Y) = -I^{-k}[X,Y].
\]
In particular, if \( X \in \Gamma(E^{-i}) \), \( Y \in \Gamma(E^{-j}) \) and \( k > i + j \), then
\[
(I^{-i-j}T(X,Y)) = -[I^{-i}X, I^{-j}Y], \quad I^{-k}T(X,Y) = 0. \tag{5.3}
\]
We therefore introduce the notation.
\[
\mathcal{T}(X,Y) = (T)_0(X,Y) = -I^{-1}\{[IX, IY]\},
\]
for the homogeneous part of \( T \) of degree 0. Note the following simple observation.

**Proposition 5.4.** A connection \( \nabla \) on \( TM \) is strongly compatible with \((E, g, I)\) if and only if it is compatible with the same graded sub-Riemannian structure and also satisfies
\[
\nabla \mathcal{T} = 0.
\]

### 5.3. Gradings and selectors

We include the following definition from [12]. Let \( E \) be a bracket-generating and equiregular subbundle with canonical flag as in (2.1).

**Definition 5.5.** A map \( \chi : TM \to \wedge^2 TM \) is a selector of \( E \) if
(a) \( \chi(E^{-k}) \subseteq \wedge^2 E^{-k+1} \),
(b) If \( \alpha \) is any one-from vanishing on \( E^{-k} \) for any \( k \geq 1 \), we have that
\[
v \mapsto (\text{id} + \chi^*d)\alpha(v) = \alpha(v) + d\alpha(\chi(v)),
\]
vanishes on \( E^{-k-1} \).

Any bracket-generating and equiregular subbundle \( E \) has at least one selector. If we have a graded sub-Riemannian manifold \((M, E, g, I)\), then we have a canonically associated selector of \( E \). By Section 5.3 we have a canonical extension of \( g \) to a metric tensor \( \bar{g} \) on \( \text{gr} = \text{gr}(E) \) from its fiber-wise Carnot algebra structure. We can induce an inner product on \( \wedge^2 \text{gr} \) such that if \( A_1, \ldots, A_n \) is a local orthonormal basis of \( \text{gr} \), then \( \{A_i \wedge A_j : i < j\} \) is a local orthonormal basis for \( \wedge^2 \text{gr} \). We then define a map \( \chi_0 : \text{gr} \to \wedge^2 \text{gr} \) by
\[
([A, B], C)_{\bar{g}} = \langle A \wedge B, \chi_0(C) \rangle_{\bar{g}}, \quad A, B, C \in \text{gr}.
\]
In particular, if \( \chi_0(C) = \sum_{i=1}^k A_i \wedge B_i \), then \( \sum_{i=1}^k [A_i, B_i] = C \) for any \( C \in [\text{gr}, \text{gr}] \). We finally have a selector \( \chi_I \) corresponding to the \( E \)-grading \( I \) given by
\[
\chi_I = (\wedge^2 I^{-1})\chi_0 I. \tag{5.5}
\]

For more on selectors, see Appendix A

**Example 5.6 (Step 2).** Consider the special case when we have step \( s = 2 \). Then \( I^{-2}v = v \mod E \), while \( I^{-1} = \text{pr}_{-1} : TM \to E \) is a projection. The inverse is given by
\[
I^{-1}(u \oplus w \mod E) = u+w-\text{pr}_{-1}w, \quad u \in E = \text{gr}_{-1}, w \mod E \in \text{gr}_{-2}.
\]
Since \( \chi_0 \) vanishes on \( E \) and has image in \( \text{gr}_{-1} \), it follows from (5.5) that the selector \( \chi_I \) is actually independent of the \( E \)-grading \( I \) chosen.
6. Morimoto’s Cartan connection as a grading and an affine connection

6.1. Cartan connections. We recall the definition of Cartan connections, see e.g. [31, 36] for more details. Let \( \mathfrak{h} \) be a subalgebra of the Lie algebra \( \mathfrak{g} \) of codimension \( n \). Let \[
H \to P \xrightarrow{\pi} M,
\]
be a principal bundle with the group acting on the right such that \( M \) has dimension \( n \) and such that \( H \) has Lie algebra \( \mathfrak{h} \). Let \( \text{Ad} \) be a representation of \( H \) on \( \mathfrak{g} \) extending the usual adjoint action of \( H \) on \( \mathfrak{h} \). A Cartan connection \( \varpi \) on \( P \) modeled on \( (\mathfrak{g}, \mathfrak{h}) \) is a \( \mathfrak{g} \)-valued one form
\[
\varpi : TP \to \mathfrak{g},
\]
such that

(i) For each \( p \in P \), \( \varpi|_p \) is a linear isomorphism from \( T_pP \) to \( \mathfrak{g} \).
(ii) For each \( a \in H \), \( v \in TP \),
\[
\varpi(v \cdot a) = \text{Ad}(a^{-1})\varpi(v).
\]
(iii) For every \( D \in \mathfrak{h} \), \( p \in P \), we have \( \varpi\left( \frac{d}{dt}p \cdot \exp_H(tD)|_{t=0} \right) = D \).

The curvature of a Cartan connection is the \( \mathfrak{g} \)-valued two-form defined by
\[
K = d\varpi + \frac{1}{2}[\varpi, \varpi].
\]
We verify that if \( v \in TP \) satisfies \( \varpi(v) \in \mathfrak{h} \) then \( K(v, \cdot) = 0 \). Hence, the curvature can represented by a smooth function \( \kappa : P \to \wedge^2(\mathfrak{g}/\mathfrak{h})^* \otimes \mathfrak{g} \), given by
\[
K(v, w) = \kappa(p)(\varpi(v), \varpi(w)), \quad v, w \in T_pM.
\]
Assume that \( \mathfrak{g} \) is an reductive \( \mathfrak{h} \)-module, i.e. assume that \( \mathfrak{g} \) admits an \( \text{Ad}(H) \)-invariant splitting \( \mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h} \). We can then identify \( \mathfrak{g}/\mathfrak{h} \) with \( \mathfrak{m} \). Furthermore, if we decompose the Cartan connection as \( \varpi = \theta + \omega \) with \( \theta \) and \( \omega \) taking values if respectively \( \mathfrak{m} \) and \( \mathfrak{h} \), then \( \omega \) is a principal connection on \( P \).

6.2. Canonical sub-Riemannian Cartan connections. Let \((M, E, g)\) be a sub-Riemannian manifold with constant symbol \( g_\cdot \). Consider \( \mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \) as in Section 3.2 with the inner product induced from \( g_\cdot \). Define a representation \( \text{Ad} \) of \( G_0 = \text{Isom}(g_\cdot) \) on \( \mathfrak{g}_- \) as the usual adjoint action on \( \mathfrak{g}_0 \) and defined as in (4.2) on \( \mathfrak{g}_- \). Introduce the Spencer differential \( \partial : \wedge^k \mathfrak{g}_-^* \otimes \mathfrak{g} \to \wedge^{k+1} \mathfrak{g}_-^* \otimes \mathfrak{g}, k = 1, \ldots, n \), by
\[
(\partial \alpha)(A_0, \ldots, A_k) = \sum_{i=0}^{n} (-1)^i[A_i, \alpha(A_0, \ldots, \hat{A}_i, \ldots, A_k)]
+ \sum_{i<j} (-1)^{i+j}\alpha([A_i, A_j], A_0, \ldots, \hat{A}_i, \ldots, \hat{A}_j, \ldots, A_k),
\]
where \( A_0, \ldots, A_k \in \mathfrak{g}_- \) and where the hat denotes terms that should be omitted. We get an induced inner product on \( \wedge^k \mathfrak{g}_-^* \otimes \mathfrak{g} \) from the inner products of \( \mathfrak{g}_- \) and \( \mathfrak{g} \). More specifically, if \( A_1, \ldots, A_n \) is an orthonormal basis of \( \mathfrak{g}_- \), with dual \( A_1^*, \ldots, A_n^* \) and \( D_1, \ldots, D_m \) is an orthonormal basis of \( \mathfrak{g}_0 \), then we define an inner product on \( \wedge^k \mathfrak{g}_-^* \otimes \mathfrak{g} \) such that,
\[
\begin{aligned}
\left\{ A_{i_1}^* \wedge \cdots \wedge A_{i_k}^* \otimes A_{r}, i_1 < \cdots < i_k, j_1 < \cdots < j_k, \\
A_{j_1}^* \wedge \cdots \wedge A_{j_k}^* \otimes D_{s} : 1 \leq r \leq n, 1 \leq s \leq m \right\},
\end{aligned}
\]
is an orthonormal basis. Write $\partial^*$ for the dual of $\partial$ with respect to our mentioned inner product. We then have the following result by Morimoto \cite{28}.

**Theorem 6.1.** Let $(M, E, g)$ be a sub-Riemannian manifold with constant symbol $g_-$. Let $G_0 \to \mathcal{F} \to M$ be its nonholonomic frame bundle. Then there is a unique $(\mathfrak{g}, g_0)$-Cartan connection $\varpi : T\mathcal{F} \to \mathfrak{g}$ such that its curvature $\kappa : \mathcal{F} \to \wedge^2 g^* \otimes \mathfrak{g}$ satisfies

$$\partial^* \kappa = 0.$$ 

We note the following important point related to isometries. Let $(M, E, g)$ and $(M', E', g')$ be two sub-Riemannian manifolds with constant symbol $g_-$. Assume that there is an isometry $\Phi : M \to M'$, i.e. a diffeomorphism such that $\Phi_*$ maps $E$ to $E'$ isometrically on each fiber. Write $\text{gr} = \text{gr}(E)$ and $\text{gr}' = \text{gr}(E')$ and let $\mathcal{F}$ and $\mathcal{F}'$ be their respective non-holonomic frame bundles, and consider the induced map

$$\Phi_* : \mathcal{F} \to \mathcal{F}', \quad \Phi_* : f \in \text{Iso}(\mathfrak{g}_-, \text{gr}_x) \mapsto (\Phi_* \circ f) \in \text{Iso}(\mathfrak{g}_-, \text{gr}'(\Phi(x))).$$

We can then verify that if $\varpi'$ is the unique Cartan connection on $\mathcal{F}'$ satisfying Theorem 6.1, then $\varpi = \Phi^* \varpi'$ will also satisfy the same condition on $\mathcal{F}$ and their curvatures are related by $\kappa'(\Phi(p)) = \kappa(p)$.

**Remark 6.2.** In this paper, we are considering a slightly different normalization condition than what is considered in \cite{28}, with our convention of how to extend the metric from an inner product space to its tensors in \cite{13}. We make this convention to simplify our presentation. In particular, it our formula for the selector $\chi_{I}$ in Section 5.3 and our result in Theorem 6.4 would have been more complicated without this convention.

**Remark 6.3.** An advantage of using a normalization condition as in Theorem 6.1 is that along with the Bianchi identity $\partial \kappa = 0$, we have that the connection is uniquely determined by the harmonic part of its curvature. For more details, see \cite{3}.

### 6.3. Unique affine connection and grading

We can rewrite Morimoto’s connection using the concepts of $E$-gradings and affine connections. Let $(M, E, g)$ be a sub-Riemannian manifold with constant symbol $g$ and with non-holonomic frame bundle $\text{gr} = \text{gr}(E)$.

- Recall that $\text{gr}(E)$ has a metric $\bar{g}$ induced from $g$ and the Cartan algebra structure on each fiber.
- Define a subbundle $\mathfrak{s}$ of $\text{End} \text{gr}$ consisting of isometry algebras $\mathfrak{s}_x = \text{iso}(\text{gr}_x)$ on each fiber.
- Let $I$ be a chosen $E$-grading and define a Riemannian metric $g_I$ on $M$ by

$$\langle v, w \rangle_{g_I} = \langle v, w \rangle_{\bar{g}}, \quad v, w \in T\mathcal{M}.$$ 

Observe that $g_I$ *tames* the sub-Riemannian metric $g$, i.e. $g_I|E = g$. Also define a subbundle $\mathfrak{s}_I \subseteq \text{End} T\mathcal{M}$ by $\mathfrak{s}_I = I^{-1} \circ \mathfrak{s} \circ I$.

- Let $\nabla$ be a strongly compatible connection with $(E, g, I)$, with respectively torsion $T$ and curvature $R$. Also let $\nabla = (T)_0$ be the part of the torsion of degree zero as in \cite{5,4}. Write $T_v = T(v, \cdot)$ and $T_v = T(v, \cdot)$.
- We note that by Corollary 4.10, any strongly compatible connection $\nabla$ with $(E, g, I)$ will have $R$ with property $R(u, v) \in \mathfrak{s}_I$ for any $u, v \in T\mathcal{M}$.
- Let $\chi_{I}$ be the selector defined as in Section 5.3 relative to $I$. 

Theorem 6.4. Let \((M, E, g)\) be a sub-Riemannian manifold with constant symbol \(\mathfrak{g}_-\). Then there is a unique \(E\)-grading \(I\) and affine connection \(\nabla\) on \(TM\) satisfying the following.

(i) \(\nabla\) is strongly compatible with \((E, g, I)\);
(ii) The torsion \(T\) and curvature \(R\) of \(\nabla\) satisfies for any \(v \in TM\),

\[
R(\chi_I(v)) = \text{pr}_{\mathfrak{g}_I} T_v = \frac{1}{2} \text{pr}_{\mathfrak{g}_I} (T_v - T_v^*),
\]

\[
T(\chi_I(v)) = \text{tr}_{\mathfrak{g}_I} T^*_X v + I_{-1} v - v - \text{tr}_{\mathfrak{g}_I} T^*_X T^*_X v,
\]

where the trace, duals and the orthogonal projection \(\text{pr}_{\mathfrak{g}_I}\) : \(\text{End} \, TM \to \mathfrak{g}_I\) are all defined relative to \(\mathfrak{g}_I\).

We will refer to the grading and connection satisfying the above theorem as respectively the Morimoto grading and the Morimoto connection.

Proof. Consider the canonical Cartan connection \(\varpi = \theta \oplus \omega : T \mathcal{F} \to \mathfrak{g}\) as in Theorem 6.1 with \(\theta\) taking values in \(\mathfrak{g}_-\) and \(\omega\) taking values in \(\mathfrak{g}_0\). Then \(\omega\) is a principal connection on \(\pi\). Define \(\mathcal{H} = \ker \omega\) and write the corresponding horizontal lift by \(h_f : T_{\pi(f)} \to \mathcal{H}_f \subseteq T_f \mathcal{F}\), that is, \(h_f \in \mathcal{H}_f\) is the unique element satisfying \(\pi_* h_f v = v\). Define a grading \(I : TM \to \text{gr}\) by

\[
I v = f(\theta(h_f v), \quad v \in T_x M, f \in \mathcal{F}_x, x \in M.
\]

This is well defined since \(\mathcal{H}\) is a principal Ehresmann connection. Any other element in \(\mathcal{F}_x\) which will be on the form \(f \cdot a, a \in G_0\), we observe that

\[
(f \cdot a)(\theta(h_f a v)) = (f \cdot a)(\theta(h_f v \cdot a)) = (f \cdot a)(\text{Ad}(a^{-1}) \theta(h_f v) = (f \cdot a)a^{-1}(\theta(h_f v) = f \theta(h_f v).
\]

We now turn to the curvature \(\kappa\) of \(\varphi\). We first observe that for \(X, Y \in \Gamma(TM)\), \(\bar{D} \in \mathfrak{g}_0, \bar{B} \in \mathfrak{g}_-\),

\[
\langle \kappa(\theta(hX), \theta(hY)) \rangle_f, \bar{D} = \langle -\omega([hX, hY]) \rangle_f, \bar{D}
\]

\[
= \langle R^{\pi f}(X, Y), f \bar{D} f^{-1} \rangle_{\bar{g}} = \langle R(X, Y), I^{-1} f \bar{D} f^{-1} I \rangle_{g_I},
\]

\[
\langle \kappa(\theta(hX), \theta(hY)) \rangle_f, \bar{B} = \langle h_f X f^{-1} I(Y) - h_f Y f^{-1} I(X) - I([X, Y]), \bar{B} \rangle
\]

\[
+ \langle f^{-1} [I(X), I(Y)], \bar{B} \rangle
\]

\[
= \langle T(X, Y) - T(X, Y), I^{-1} (f \bar{B}) \rangle_{g_I}.
\]

It follows that \(\kappa\) is only non-zero in positive degrees. Hence, we only need to show that \(\kappa\) is orthogonal to the image of positive elements of \(\mathfrak{g}_-^+ \otimes \mathfrak{g}\) under \(\partial\).

If \(A \in (\text{gr}_x)^{-1}, B \in (\text{gr}_x)^{-1}\) and \(f \in \mathcal{F}_x\) with \(i > j \geq 1\), then

\[
0 = \langle \kappa, \partial((f^{-1} A)^* \otimes (f^{-1} B)) \rangle
\]

\[
= \langle f \kappa^{-1}, A^* \wedge [B, \cdot] - A^*[\cdot, \cdot] \otimes B \rangle_{\bar{g}}
\]

\[
= \text{tr}_{TM} (IT(I^{-1} A, x), [B, I \times]) - (IT(\chi_I(I^{-1} A)), B)_{\bar{g}}
\]

\[
= - (\text{tr}_{TM} T^*_X I^{-1} A + T(\chi_I(I^{-1} A)), I^{-1} B)
\]
It follows that for any \( v \in (TM)_{-i} \),
\[
\sum_{j=1}^{i-1} \text{pr}_{-j} T(\chi_{1}(v)) = T(\chi_{1}(v)) + \sum_{j=2}^{s} \text{pr}_{j} v = T(\chi_{1}(v)) + v - I_{-1}v
\]
\[
= - \sum_{j=1}^{i-1} \text{pr}_{-j} \text{tr}_{TM} T_{x}^{*} T_{x}v = - \text{tr}_{TM} T_{x}^{*} T_{x}v + \text{tr}_{TM} T_{x}^{*} T_{x}v.
\]

Next, if \( A \in (\text{gr}(E))_{-i}, i \geq 1, D \in \mathfrak{s}_{x}, f \in \mathcal{F}_{x} \), then
\[
0 = (\kappa, \partial((f^{-1}A)^{\ast} \otimes f^{-1}Df)) = \langle f_{\kappa} f^{-1}, A^{\ast} \wedge D - A^{\ast}[\cdot, \cdot] \otimes D \rangle_{g}
\]
\[
= \text{tr}_{TM}\langle IT(I^{-1}A, x), D \chi \rangle_{g} - (R^{\sigma}(\chi(I^{-1}A)), D)_{g}
\]
It follows that for any \( v \in TM \),
\[
R(\chi(v)) = \text{pr}_{\ast} T(v, \cdot). \quad \square
\]

Remark 6.5. As one can also observe from the proof, if we consider
\[
\langle T(\chi(v)), w \rangle_{g_{\ast}}, \quad v \in (TM)_{-i}, w \in (TM)_{-j},
\]
and then apply (6.2), then we only get non-trivial information when \( i > j \).

In the following sections, we will consider examples showing how the Morimoto grading and curvature can be computed.

6.4. Strongly compatible connections and flatness. We emphasize the following important consequence of our previous proof.

Corollary 6.6 (Sub-Riemannian flatness theorem). Let \((M, E, g)\) be a sub-Riemannian manifold with constant symbol \( g_{-} \). Let \((G_{-}, E', g')\) be the Carnot group corresponding to the symbol \( g_{-} \) with its sub-Riemannian structure.
(a) If \( I \) is an \( E \)-grading and there exist \( \nabla \) is a strongly compatible connection with \((E, g, I)\) satisfying
\[
T(X, Y) = T(X, Y), \quad R(X, Y) = 0,
\]
for any \( X, Y \in \Gamma(TM) \), then \((M, E, g)\) is locally isometric to \((G_{-}, E', g')\). Furthermore, if \( I \) and \( \nabla \) is then respectively the Morimoto grading and connection of \((M, E, g)\).
(b) Conversely, if \((M, E, g)\) is locally isometric to \((G_{-}, E', g')\), and \( I \) and \( \nabla \) are respectively the Morimoto grading and connection, then they satisfy (6.3).

Proof. Reversing the steps in the proof of Theorem 6.4 we see that any grading and strongly compatible connection can be seen as a Cartan connection \( \varpi \) on the nonholonomic frame bundle \( \mathcal{F} \), with the conditions (6.3) being equivalent to the curvature vanishing. Connections with curvature \( \kappa \equiv 0 \) will in particular satisfy Theorem 6.1. Let \( G \) be the simply connected Lie group corresponding to \( G \). If the curvature vanishes, then by [31] Chapter 3, Theorem 6.1, for every \( f \in \mathcal{F} \), there is a neighborhood \( U \) and a map \( \Psi : U \to G \) such that
\[
\varpi(w) = \Phi(f)^{-1} \cdot \Psi_{\ast}w, \quad w \in T_{f} \mathcal{F}, f \in U.
\]
Following similar steps as in [31] Chapter 5.5, we use invariance under \( G_{0} \) to reduce to a map \( \Phi : \pi(U) \to G_{1} \) mapping \( E|_{U} \) into \( E' \) isometrically on each fiber. This proves (a). The converse follows from observing that the Morimoto grading on a Carnot group is given by the left translation of the stratification and the Morimoto
connection is the connection determined by making all left invariant vector fields parallel.

7. Sub-Riemannian manifolds with contact horizontal bundles

7.1. The Heisenberg algebra. The $n$-th Heisenberg algebra is the step 2 nilpotent algebra $\mathfrak{h}_n = \mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ where

$$\mathfrak{g}_{-2} = \text{span}\{C\}, \quad \mathfrak{g}_{-1} = \text{span}\{A_1, \ldots, A_n, B_1, \ldots, B_n\},$$

and with the only non-zero brackets being

$$[A_j, B_j] = C, \quad j = 1, 2, \ldots, n.$$ We consider Carnot algebras structures on $\mathfrak{h}_n$. For any vector $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ such that

$$(7.1) \quad 1 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n,$$

we define an inner product $\mathfrak{g}_-$ by

$$\langle A_i, B_j \rangle_{\mathfrak{g}_-} = 0, \quad \langle A_j, A_j \rangle_{\mathfrak{g}_-} = \langle B_j, B_j \rangle_{\mathfrak{g}_-} = \lambda_j.$$ It is simple to verify that any Carnot algebra structure on $\mathfrak{h}_n$ is isometric to exactly one choice of vector $\lambda$ as in (7.1). We write this Carnot algebra $\mathfrak{h}_n(\lambda)$ Let us define maps $D_{ij} : \mathfrak{h}_n(\lambda) \to \mathfrak{h}_n(\lambda)$, $i, j = 1, 2, \ldots, n$ by $D_{ii} = 0$ and for $i \neq j,$

$$D_{ij}(A_k) = \delta_{ki}A_j - \delta_{kj}A_i, \quad D_{ij}(B_k) = \delta_{ki}B_j - \delta_{kj}B_i, \quad D_{ij}C = 0.$$ Furthermore, define $J : \mathfrak{h}_n(\lambda) \to \mathfrak{h}_n(\lambda)$ by

$$J(A_k) = B_k, \quad J(B_k) = -A_k, \quad JC = 0.$$ Then the isometry algebra $\mathfrak{g}_0 = \text{iso}(\mathfrak{h}_n(\lambda))$ is given by

$$\mathfrak{g}_0 = \text{span}\{J, D_{ij} : i < j, \lambda_i = \lambda_j\}.$$ 7.2. Contact manifolds with constant symbols. A subbundle $E$ on $M$ is called a contact distribution if it has codimension 1 and if the bilinear map

$$\wedge^2 E \to \text{gr}_{-2} = TM/E^{-2}, \quad X \wedge Y \mapsto [X, Y] \mod E^{-2},$$

is non-degenerate. It follows that $E$ is necessarily bracket generating, equiregular and of even rank. We write this rank as $2n$, meaning that the manifold $M$ has dimension $2n+1$. If $(M, E, g)$ is a sub-Riemannian manifold with $E$ being a contact distribution, then $\text{gr}_x$ is isometric to the Heisenberg algebra with a given inner product, but this inner product will depend on $x \in M$ in general.

Working locally, we may assume that Ann($E$) is a trivial line bundle. Choose any orientation. Let $\theta$ be a positively oriented non-vanishing section of Ann($E$). We introduce a corresponding map $J^\theta : E \to E$ defined such that

$$d\theta(u, v) = \langle u, J^\theta v \rangle_g.$$ Observe that $J^\theta$ is a skew-symmetric map with a trivial kernel and hence $-(J^\theta)^2$ is positive definite. We normalize $\theta$ by requiring it to be the unique one-form such that the maximal eigenvalue of $-(J^\theta)^2$ at every point is 1.

Notice that every eigenvalue of $-(J^\theta)^2$ has at least multiplicity 2, since if $u$ is an eigenvector, then $J^\theta u$ will also be an eigenvector with the same eigenvalue. Hence, we can describe all of the eigenvalues of $J^\theta$ at every point by a function

$$M \to \mathbb{R}^n, \quad x \mapsto \lambda_x = (\lambda_{x,1}, \ldots, \lambda_{x,n}), \quad 1 = \lambda_{x,1} \leq \cdots \leq \lambda_{x,n},$$
such that the eigenvalues, with multiplicities are

\[ 1 = 1 \geq \frac{1}{\lambda_{x,2}} \geq \cdots \geq \frac{1}{\lambda_{x,n}} = 1. \]

It is then simple to verify that the symbol of \((M, E, g)\) at \(x\) is \(h_n(\lambda_x)\), and so the manifold has constant symbol if and only if \(\lambda_x = \lambda\) is independent of \(x \in M\).

We will continue the discussion assuming that \((M, E, g)\) has constant symbol.

Let \(1 = \lambda[1] < \lambda[2] < \cdots < \lambda[k]\) be the values of \(\lambda = (\lambda_1, \ldots, \lambda_n)\) written without repetitions, and write the corresponding eigenspace decomposition of \(E\) by

\[ E = E[1] \oplus \cdots \oplus E[k], \]

where \(E[j]\) is the eigenspace of \(\lambda[j]\). Define \(\text{pr}[j] : E \to E[j]\) as the corresponding projections. Remark that this is an orthogonal decomposition since \(-J^2\) is symmetric. Define a vector bundle map \(\Lambda : E \to E\) by \(\Lambda(E[j]) \subseteq E[j]\) and with

\[ \Lambda|_{E[j]} = \lambda[j] \, \text{id}_{E[j]} . \]

Write

\[ J^0 = \Lambda^{-1} J \]

and notice that we now have \(J^2 = -\text{id}_E\).

We define the Reeb vector field \(Z^0\) such that \(\theta(Z^0) = 1\) and \(d\theta(Z^0, \cdot) = 0\). Write \(V^0 = \text{span}\{Z^0\}\). We then observe that any \(E\)-grading \(I\) is uniquely determined by the vector field \(W\) with values in \(E\) and defined by

\[ W = J I_{-1} Z^0 . \]

Write \(\ker I_{-1} = V^W = \text{span}\{Z^W\}\) where \(Z^W = Z^0 - JW\). We observe that for any \(X \in \Gamma(E)\),

\[ d\theta(Z^W, X) = -\theta([Z^W, X]) = -\sum_{i=1}^k \frac{1}{\lambda[i]} \langle W, \text{pr}[i] X \rangle_{g_I} . \]

Define \(g_I\) as the taming Riemannian metric of \(g\) such that that \(Z = Z^W\) is a unit vector orthogonal to \(E\). We also extend the definition of \(\Lambda, \Lambda^{-1}, J\) and the maps \(\text{pr}[j]\) to the whole tangent bundle by requiring them to vanish on \(V = V^W\).

We observe that the decomposition (7.2), the subbundle \(V\) and the metric \(g_I\) does not depend on the choice of orientation \(\theta\), and can hence be defined globally. The same is true for the vector field \(W \in \Gamma(E)\). Since we are in step 2, we have a canonical selector \(\chi : \wedge^2 TM \to TM\). Locally, this is given by

\[ \chi(Z) = \frac{1}{\sqrt{\text{tr} \Lambda^2}} \sum_{j=1}^n \Lambda X_j \wedge J X_j , \]

where \(X_1, \ldots, X_n\) is a choice of local unit vector fields such that such that

\[ \{X_1, \ldots, X_n, JX_1, \ldots, JX_n\} , \]

is a local orthonormal basis.
7.3. Connections on contact manifolds. For the case when Λ is a projection to E, i.e. the case when 1 is the only eigenvalue of $J^0$, there exists several known canonical choices of connections defined by Tanaka and Webster for the integrable case [34, 37] and by Tanno [35] when $J = J^0$ is not integrable almost complex structure on E. We will introduce the following generalization.

Let $I$ be any $E$-grading with $\ker I_{-1} = V^W = V$. Introduce tensors $\tau(X)Y = \tau_XY$ defined by linearity and the following relations.

(1) If $\tilde{Z}$ is a section of $V$, then $\tau_Y\tilde{Z} = 0$ for any $Y \in \Gamma(TM)$.

(2) For any $Y \in \Gamma(TM)$, we have that $\tau_Y(E[j]) \subseteq E[j]$ for any $j = 1, \ldots, k$.

(3) If $X_1, X_2 \in \Gamma(E[j])$, then

$$\tau_{X_1}X_2 = 0.$$ 

Furthermore, for any $X \in E[i], i \neq j$ and any $\tilde{Z} \in \Gamma(V)$, we have

$$\langle \tau_XX_1, X_2 \rangle = \frac{1}{2}(\mathcal{L}_X g_I)(X_1, X_2), \quad \langle \tau_{\tilde{Z}}X_1, X_2 \rangle = \frac{1}{2}(\mathcal{L}_{\tilde{Z}} g_I)(X_1, X_2).$$

Let $\nabla'$ be the Levi-Civita connection of $g_I$. Define first a connection $\nabla' = \nabla'^W$ such that it is compatible with $g_I$ and such that each of the subbundles in the decomposition $TM = E[1] \oplus \cdots \oplus E[k] \oplus V$ are parallel. In particular, any such connection is compatible with $(E, g)$ and any local constant length section of $V$ will be parallel. For $X \in \Gamma(E[i]), X_1 \in \Gamma(E[j])$ and any section $\tilde{Z} \in \Gamma(V)$, we define

$$\nabla'_X X_1 = \begin{cases} 
\text{pr}[j]\tilde{Z},X_1 & \text{if } i = j, \\
\text{pr}[j][X, X_1] + \tau_X X_1 & \text{if } i \neq j,
\end{cases}$$

$$\nabla'_2 X_1 = \text{pr}[j][\tilde{Z}, X_1] + \tau_{\tilde{Z}} X_1.$$ 

We observe that for any $X_1, X_2 \in \Gamma(E[j])$ and any $Y \in \Gamma(TM)$, we have

$$\langle T'(Y, X_1), X_2 \rangle = \langle \tau_Y X_1, X_2 \rangle,$$

where $T'$ is the torsion $\nabla'$.

We next define a connection $\nabla'' = \nabla'^W$ by $\nabla'' \tilde{Z} = \nabla' \tilde{Z}$ for any $\tilde{Z} \in \Gamma(V)$ and for any $Y \in \Gamma(TM), X \in \Gamma(E)$

$$\nabla''_Y X = \nabla'_Y X + \frac{1}{2}(\nabla'_Y J)X.$$

We observe the following

**Lemma 7.1.** (a) The connection $\nabla'$ is compatible with $(E, g)$ and $\nabla''$ is strongly compatible with $(E, g, I)$.

(b) For any $X \in \Gamma(E)$,

$$\text{tr}_E(\langle \nabla'_X J, X \rangle)_{g_I} = -\text{tr}_E(\langle \nabla'_X J \times, X \rangle)_{g_I} = 0.$$

(c) If $T''$ is the torsion of $\nabla''$ and $Y \in \Gamma(TM)$ is any vector field, then for any $D \in \mathfrak{s}_I$,

$$\langle T''_Y, D \rangle_{g_I} = 0.$$

**Proof.** We will be working locally thought the proof. We will always let $Z = Z^W = Z^0 - JW$ denote the modification of the Reeb vector field with respect to a chosen orientation on $\text{Ann}(E)$. To simplify notation in the proof, we will simply write $\langle \cdot, \cdot \rangle_{g_I}$ as just $\langle \cdot, \cdot \rangle$. 


(a) We first note that $\nabla'$ is compatible with $(E,g)$ by definition. Since for any $X_1, X_2 \in \Gamma(E)$, $Y \in \Gamma(TM)$, we have
\[
\langle X_1, J(\nabla_YJ)X_2 \rangle_g = \langle (\nabla_YJ)JX_1, X_2 \rangle = -\langle J(\nabla_YJ)X_1, X_2 \rangle,
\]
so $\nabla''$ is also compatible with $(E,g)$. Here we have used that $J^2 = -\text{pr}_E$ is $\nabla'$-parallel. Finally, for any $Y_1, Y_2, Y_3 \in \Gamma(TM)$,
\[
\langle \nabla''_{Y_3}T \rangle(Y_1, Y_2) = \langle Y_1, \Lambda^{-1}(\nabla''_{Y_3}J)Y_2 \rangle Z
\]
\[
= \langle Y_1, \Lambda^{-1}(\nabla'_{Y_3}J)Y_2 \rangle Z
\]
\[
+ \frac{1}{2}(Y_1, \Lambda^{-1}(\nabla'_{Y_3}J)J^2Y_2)Z - \frac{1}{2}(W_1, \Lambda^{-1}(\nabla'_{Y_3}J)JW_2)Z = 0.
\]

(b) For $X_1, X_2 \in \Gamma(E[j]), X \in \Gamma(E[i])$, we use the first Bianchi identity for connections with torsion, with $\odot$ denoting the cyclic sum
\[
\odot \langle \Lambda^{-1}(\nabla_XJ)X_1, X_2 \rangle = -\odot \langle \nabla_XT' \rangle(X_1, X_2, Z)
\]
\[
= -\odot \langle R'(X, X_1)X_2, Z \rangle + \odot \langle T'(X, X_1), X_2, Z \rangle = \odot \langle T'(X, X_1), \Lambda^{-1}JX_2 \rangle.
\]
If $i = j$, we have
\[
\odot \langle (\nabla'_{X}J)X_1, X_2 \rangle = \odot \langle (\nabla'_{X}J)X_1, X_2 \rangle = 0.
\]
If we insert $X = JX_1$, we obtain
\[
\langle (\nabla'_{JX_1}J)X_1, X_2 \rangle = -\langle (\nabla'_{X_2}J)JX_1, X_1 \rangle - \langle (\nabla'_{X_1}J)X_2, JX_1 \rangle = -\langle J(\nabla'_{X_1}J)X_1, X_2 \rangle.
\]
It follows that if $X \in \Gamma(E[j])$, then
\[
2 \text{tr}_E \langle (\nabla'_XJ)X, X \rangle = \text{tr}_E \langle (\nabla'_XJ)X, X \rangle + \text{tr}_E \langle (\nabla'_XJ)X, X \rangle
\]
\[
= \text{tr}_E \langle (\nabla'_XJ)X, X \rangle + \text{tr}_E \langle X, J(\nabla'_XJ)X \rangle
\]
\[
= \text{tr}_E \langle (\nabla'_XJ)X, X \rangle + \text{tr}_E \langle X, (\nabla'_XJ)X \rangle = 0.
\]

(c) For some $X_1, X_2 \in \Gamma(E[j])$, $j = 1, \ldots, k$, define
\[
\eta(X_1, X_2)(Y) = \langle X_1, Y \rangle \bar{g}X_2 - \langle X_2, Y \rangle \bar{g}X_1.
\]
Then elements of $\mathfrak{s}_I$ is spanned by $J$ and elements on the form $\eta(X_1, X_2) - J\eta(X_1, X_2)J$ for $\langle X_1, X_2 \rangle = \langle X_1, JX_2 \rangle = 0$ by Section 7.1. We first observe that
\[
\langle T''_Z, J \rangle = \langle T''_Z, J \rangle + \frac{1}{2} \langle J(\nabla''_ZJ), J \rangle = \langle \tau_Z, J \rangle = 0.
\]
and
\[
(7.4) \quad \langle T''_Z, \eta(X_1, X_2) - J \eta(X_1, X_2)J \rangle
\]
\[
= \langle \tau_Z, \eta(X_1, X_2) - J \eta(X_1, X_2)J \rangle + \frac{1}{2} \langle J(\nabla''_ZJ), \eta(X_1, X_2) - J \eta(X_1, X_2)J \rangle
\]
\[
= \frac{1}{2} \langle J(\nabla''_ZJ)X_1, X_2 \rangle - \frac{1}{2} \langle J(\nabla''_ZJ)X_2, X_1 \rangle
\]
\[
+ \frac{1}{2} \langle J(\nabla''_ZJ)JX_1, JX_2 \rangle - \frac{1}{2} \langle J(\nabla''_ZJ)JX_2, JX_1 \rangle = 0.
\]
For $X \in \Gamma(E[i])$ and with $i \neq j$, then by a similar calculation as in (7.4), we have
\[
\langle T''_X, \eta(X_1, X_2) - J \eta(X_1, X_2)J \rangle = 0.
If $i = j$, then

\[
\langle T''_X, \eta(X_1, X_2) - J\eta(X_1, X_2)J \rangle \\
= \frac{1}{2} \langle J(\nabla'_X J)X_1, X_2 \rangle - \frac{1}{2} \langle J(\nabla'_X J)X_2, X_1 \rangle \\
+ \frac{1}{2} \langle J(\nabla'_X J)JX_1, JX_2 \rangle - \frac{1}{2} \langle J(\nabla'_X J)JX_2, JX_1 \rangle \\
- \frac{1}{2} \langle J(\nabla'_X J)X_1, X_2 \rangle - \frac{1}{2} \langle J(\nabla'_X J)X_2, X_1 \rangle \\
- \frac{1}{2} \langle J(\nabla'_X J)X_1, JX_2 \rangle - \frac{1}{2} \langle J(\nabla'_X J)X_2, JX_1 \rangle \\
= -\frac{1}{2} \langle (\nabla'_X J)X_1, X_2 \rangle + \frac{1}{2} \langle (\nabla'_X J)X_2, X \rangle \\
+ \frac{1}{2} \langle (\nabla'_X J)X_1, JX_2 \rangle - \frac{1}{2} \langle (\nabla'_X J)X_2, JX_1 \rangle = 0.
\]

Finally,

\[
\langle T''_X, J \rangle = \langle \tau_X, J \rangle + \frac{1}{2} \langle J(\nabla'_X J), J \rangle - \frac{1}{2} \langle J(\nabla'_X J)X, J \rangle \\
= \frac{1}{2} \langle (\nabla'_X J), id_E \rangle - \frac{1}{2} \langle (\nabla'_X J)X, id_E \rangle = 0,
\]

by (b). \hfill \Box

7.4. The Morimoto grading and connection for the contact case. We will now present the Morimoto connection for a sub-Riemannian manifold $(M, E, g)$ of constant symbol $h_\nu(\lambda)$.

Working locally, let $J$ be defined relative to a choice of orientation on $\text{Ann}(E)$. For any $i = 1, \ldots, k$, define vector fields,

\[
\Upsilon_i = J \text{tr}_E \text{pr}[i] \left[ (\text{id} - \text{pr}[i]) \times, (\text{id} - \text{pr}[i]) \times \right].
\]

and write

\[
\Upsilon = \frac{1}{\sqrt{\text{tr} A^2}} \sum_{i=1}^{k} \lambda[i]^2 \Upsilon_i.
\]

We observe that these vector fields are independent of orientation and are hence well defined globally. Recall the definition of the subbundles $V^W$ from Section 7.2 and the connection $\nabla''^W$ from Section 7.3.

**Theorem 7.2.** The Morimoto grading $I$ is the unique grading with

\[
\ker I_{-1} = \mathcal{V}^\Upsilon.
\]

Furthermore if we consider the connection $\nabla'' = \nabla''^W$ with curvature $R''$, then the Morimoto connection is given by,

\[
\nabla Y_1 Y_2 = \nabla'' Y_1 Y_2 + \frac{1}{2} R''(\chi(Y_1))Y_2, \quad Y_1, Y_2 \in \Gamma(TM).
\]

**Proof.** We will work locally, considering $Z^0$ as the Reeb vector field with corresponding map $J$. We write the vector field $Z = Z'' = Z^0 - J\Upsilon$.

We already know that $\nabla''$ is strongly compatible with $(E, g, I)$. By compatibility with $g_I$ and strong compatibility, it follows that $R''(\chi(Y_1))$ is $g_I$-skew-symmetric map that preserves the grading and commutes with $J$. This gives us that $\nabla$ is also strongly compatible with $(E, g, I)$.
We first see that it satisfies (6.2). Observe first that

$$-T''(\chi(Z)) = -T'(\chi(Z)) = Z - \frac{1}{\sqrt{\text{tr} \Lambda}} \sum_{i=1}^{k} \lambda[i] JY_i.$$

Here we have used that $\text{tr}_E(\nabla^* J) \times = 0$. It follow that the only nontrivial relation of (6.2) is satisfied from the fact that for any $X \in \Gamma(E)$,

$$\langle T(\chi(Z)), X \rangle_{g_1} = \langle T''(\chi(Z)), X \rangle_{g_1} = \frac{1}{\sqrt{\text{tr} \Lambda^2}} \sum_{i=1}^{k} \lambda[i] \langle JY_i, \text{pr}[i]X \rangle_{g_1}$$

$$= -\text{tr}_{TM} \langle T(x, Z), T(x, X) \rangle_{g_1} = -\langle T(JX, Z), Z \rangle_{g_1} =$$

$$= d\theta(Z, JX) = -d\theta(JY, JX) = \sum_{i=1}^{k} \frac{1}{\lambda[i]} \langle JY_i, \text{pr}[i]X \rangle = \frac{1}{\sqrt{\text{tr} \Lambda^2}} \sum_{i=1}^{k} \lambda[i] \langle JY_i, \text{pr}[i]X \rangle_{g_1}.$$

To see that (6.1) is also satisfied, we see that for any $X \in \Gamma(E)$,

$$0 = R(\chi(X)) = \text{pr}_{s_j} T''_X = 0,$$

by Lemma 7.1 Finally,

$$R(\chi(Z)) = R''(\chi(Z)) - \frac{1}{2} R''(\chi(Z))$$

$$= \frac{1}{2} R''(\chi(Z)) = \text{pr}_{s_j} T_Z = \text{pr}_{s_j} T''_Z + \frac{1}{2} R''(\chi(Z)) = \frac{1}{2} R''(\chi(Z)). \quad \Box$$

8. The Cartan nilpotent algebra and (2, 3, 5)-sub-Riemannian manifolds

8.1. On the Carnot nilpotent algebra. We want to consider sub-Riemannian manifolds with growth vector $\Theta = (2, 3, 5)$. These are all of constant symbol, since, up to isometry, there only exists one Carnot algebra with this growth vector; the Cartan nilpotent algebra. It can be written as $g = g_{-3} \oplus g_{-2} \oplus g_{-1} = \text{span}\{C_1, C_1\} \oplus \text{span}\{B\} \oplus \text{span}\{A_1, A_2\}$ with $A_1, A_2$ being an orthonormal basis of $g_{-1}$ and with the only non-zero brackets given by

$$[A_1, A_2] = B, \quad [A_j, B] = C_j.$$

The corresponding isometry algebra is given by $g_0 = \text{span}\{J\}$ where

$$J : A_1 \mapsto A_2, \quad A_2 \mapsto -A_1, \quad B \mapsto 0, \quad C_1 \mapsto C_2, \quad C_2 \mapsto -C_1.$$

8.2. Structure on (2, 3, 5) manifolds. Let $(M, E, g)$ be a sub-Riemannian manifold with growth vector field $(2, 3, 5)$. We will begin by working locally, so we may assume that $E$ is trivializable and and equipped with an orientation. Define an endomorphism $J : E \to E$ such that for any unit vector $u \in E_x, x \in M$, we have that $u, Ju$ is a positively oriented orthonormal basis of $E_x$. Define an $E$-valued one-form $\varphi : TM \to E$ such that $\ker \varphi = E^{-2}$ and for $X \in \Gamma(E)$,

$$\varphi([X, [X, JX]]) = \|X\|^2 JX. \quad (8.1)$$
Here we use that the map \(-JX \xrightarrow{\cong} \|X\|^{-2}[X, [X, JX]]\) mod \(E^{-2}\) is an invertible linear map from \(E\) to \(TM/E^{-2}\) and hence the map \(\varphi\) is well defined by

\[
\begin{array}{ccc}
TM & \xrightarrow{\varphi} & E \\
& \cong & \\
TM/E^{-2} & \xrightarrow{\cong} & 
\end{array}
\]

Introduce the following two-form on \(M\),

\[
\Psi(v, w) := \langle J\varphi(v), \varphi(w) \rangle_g.
\]

Let \(I' : TM \to \text{gr}(E)\) be an \(E\)-grading, also written as \(TM = (TM)_{-3}^{'} \oplus (TM)_{-2}^{'} \oplus (TM)_{-1}^{'}\). Since we are working locally, we may again assume that \((TM)_{-2}^{'}\) is a trivial line bundle. Let \(\theta\) be the unique one-form with \(\ker \theta = (TM)_{-3}^{'} \oplus (TM)_{-1}^{'}\) and with normalization condition

\[
d\theta(u, v) = \langle u, Jv \rangle_g, \quad u, v \in E.
\]

We will use the forms \(\Psi\) and \(\theta\) to determine a grading. We emphasize that \(\Psi\) is independent of \(I'\) while \(\theta\) is not.

**Lemma 8.1.** There is a unique grading \(I'\) such that

(i) for any \(u \in E = (TM)_{-1}^{'}\) and \(w_1, w_2 \in (TM)_{-3}^{'}\),

\[
d\Psi(u, w_1, w_2) = 0.
\]

(ii) for any \(u \in E\), \(w \in (TM)_{-2}^{'} \oplus (TM)_{-3}^{'}\)

\[
d\theta(u, w) = 0.
\]

**Proof.** Working locally, let \(X_1\) and \(X_2\) be a positively oriented orthonormal basis of \(E\). Define one-forms \(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\) as the dual basis of

\[
X_1, X_2, [X_1, X_2], [X_1, [X_1, X_2]], [X_2, [X_1, X_2]].
\]

Then

\[
\Psi = \alpha_4 \wedge \alpha_5.
\]

We note that \(\Psi\) and \(\text{span}\{\alpha_4, \alpha_5\}\) = \(\text{Ann} E^{-2}\) is independent of of choice of basis. Observe also that for some functions \(a_1, a_2, a_3 \in C^\infty(M)\),

\[
d\Psi = a_3 \wedge \alpha_1 \wedge \alpha_5 - a_3 \wedge \alpha_2 \wedge \alpha_4 + a_1 \alpha_1 \wedge \Psi + a_2 \alpha_2 \wedge \Psi + a_3 \alpha_3 \wedge \Psi
\]

\[= \theta \wedge (\alpha_1 \wedge \alpha_5 - \alpha_4 \wedge \alpha_2 + a_3 \Psi)
\]

where

\[
\theta = a_3 - a_1 \alpha_4 - a_2 \alpha_5.
\]

Hence, in order for (i), to be satisfied, we must have \((TM)_{-1}^{'} \oplus (TM)_{-3}^{'} = \ker \theta\) with \(\theta\) given as above. Furthermore, since we know that \(d\theta(X_2, X_1) = 1\), we can define

\[
\beta_1 = d\theta(X_2, \cdot), \quad \beta_2 = -d\theta(X_1, \cdot).
\]

Then \(\beta_1\) and \(\beta_2\) will depend on the choice of \(X, Y\), but \(\text{span}\{\beta_1, \beta_2\}\) will be independent of this basis. Define

\[
(TM)_{-2}^{'} \oplus (TM)_{-3}^{'} = \ker \beta_1 \cap \ker \beta_2,
\]

which satisfy (ii). □
Changing the orientation will only change the sign of Ψ and θ, so the grading $I'$ will not depend on choice of orientation and can hence be defined globally. Furthermore, we see that $\varphi$ does not depend on choice of orientation and will be defined globally as well. We hence have a vector bundle isomorphism $\ell' : E = (TM)'_{-1} \to (TM)'_{-3}$ defined by $\varphi(\ell' v) = v$.

**Corollary 8.2.** Let $I'$ be as in Lemma \[8.1\] with corresponding selector $\chi'$. Then there exists a unique strongly compatible connection $\nabla'$ of $(E, g, I)$ such that for any $X_1, X_2 \in \Gamma(E)$, $Y \in \Gamma(E)$,

$$\begin{align*}
\pr_{-1}' T'(X_1, X_2) &= 0, \quad \text{and} \quad R'(\chi'(Y))X_1 = 0
\end{align*}$$

**Proof.** From Lemmas \[A.3\] Appendix it follows that there is a unique partial connection on $E$ satisfying the requirement on torsion. This can extended uniquely to a connection on $E$ with curvature vanishing along $\chi'$ by Lemma \[A.2\] Appendix. Finally, since $G_0 = \Iso(g_-) \cong O(2)$, any compatible connection defined on $E$ induces a strongly compatible connection on $TM$ by Proposition \[1.3\]. \hfill \Box

We can rewrite requirement (i) for $I'$ in Lemma \[8.1\] to say that for any $X, \tilde{X} \in \Gamma(E)$,

$$0 = d\Psi(X, \ell' \tilde{X}, \ell' J\tilde{X}) = -\Psi([X, \ell' \tilde{X}], \ell' J\tilde{X}) + \Psi([X, \ell' J\tilde{X}], \ell' \tilde{X})$$

$$= -\langle \varphi([X, \ell' \tilde{X}]), \tilde{X} \rangle_g - \langle \varphi([X, \ell' J\tilde{X}]), J\tilde{X} \rangle_g.$$

If $\nabla'$ is as in Corollary \[8.2\], we can write this as

$$\tr_E (d\nabla' \varphi(X, \ell' \times), \times) = 0.$$

For any $X, \tilde{X}, \tilde{X}_2 \in \Gamma(E)$, we define a symmetric map $\tau_X : E \to E$,

$$\langle \tau_X \tilde{X}, \tilde{X}_2 \rangle = \frac{1}{2} \langle \mathcal{L}_X g_{\ell'} \rangle (\ell' \tilde{X}, \ell' \tilde{X}_2) = \frac{1}{2} (\langle d\nabla' \varphi(X, \ell' \tilde{X}), \tilde{X}_2 \rangle + \langle d\nabla' \varphi(X, \ell' \tilde{X}_2), \tilde{X} \rangle).$$

Then it follows that $\tr_E (\tau_X \times, \times) = 0$. Furthermore, if we define a vector field $\Upsilon$ by

$$\langle \Upsilon, X \rangle = \frac{1}{2} \tr_E (d\nabla' \varphi(X, \ell' \times), J\times)_g,$$

then

$$d\nabla' \varphi(X, \ell' \tilde{X}_2) = \langle X, \Upsilon \rangle J\tilde{X}_2 + \tau_X \tilde{X}_2.$$

From the Jacobi identity, we note that for any $\tilde{X}_1, \tilde{X}_2 \in \Gamma(E)$,

$$d\nabla' \varphi(\tilde{X}_1, \ell' J\tilde{X}_2) - d\nabla' \varphi(\tilde{X}_2, \ell' J\tilde{X}_1)$$

$$= J\nabla'_{\tilde{X}_1} \tilde{X}_2 - J\nabla'_{\tilde{X}_2} \tilde{X}_1 - \varphi([\tilde{X}_1, [\tilde{X}_2, Z]] - [\tilde{X}_2, [\tilde{X}_1, Z]])$$

$$= J\nabla'_{\tilde{X}_1} \tilde{X}_2 - J\nabla'_{\tilde{X}_2} \tilde{X}_1 - \varphi([[\tilde{X}_1, \tilde{X}_2], Z]) = J \pr_{-1}' T'(\tilde{X}_1, \tilde{X}_2) = 0.$$

Inserting $\tilde{X}_1 = JX$, we have

$$d\nabla' \varphi(JX, \ell' J\tilde{X}_2) + d\nabla' \varphi(\tilde{X}_2, \ell' X) = 0,$$

or

$$\tau_{\tilde{X}_2} X = -\tau_{JX} JX_2 - \langle X_2, \Upsilon \rangle JX + \langle JX, \Upsilon \rangle X_2.$$

We will use these identities for the proof of the Morimoto grading and connection in Section \[8.4\].
Remark 8.3. There is a different approach to gradings of $(2, 4, 5)$ manifolds in [30], Chapter 3, using generalized contact forms. We leave it to the reader to verify that by choosing a generalized contact form normalized with respect to the metric, requiring a generalized Reeb vector field as in [30, Proposition 3.5] spans $(TM)_{-2}$ and then adding requirement (ii) of Lemma 8.1, we can obtain the same grading $I'$ as in Lemma 8.1.

8.3. Morimoto’s connection on $(2, 3, 5)$-manifolds. We will now present the Morimoto grading and connection locally, but noting from the formulas that it follows that it is well defined globally. Let $(M, E, g)$ be a sub-Riemannian manifold with growth vector $(2, 3, 5)$ and with $E$-valued one-form $\varphi$ as in (8.2). Let $I'$ and $\nabla'$ be as in respectively Lemma 8.1 and Corollary 8.2. Let $\ell' : (TM)'_{-1} \to (TM)'_{-3}$ be defined by $\varphi(\ell' v) = v$ and use this to define the tensor $\tau$ as in (8.2). Choose locally an orientation $J : E \to E$ and let $Z'$ and $\Upsilon$ be as in the previous section. Both of the latter vector fields will change signs if we change orientation. Finally, introduce a vector bundle endomorphism $\mathcal{A} : E \to E$ by

$$2\langle \mathcal{A} X, \tilde{X} \rangle = \langle (L_{Z'} J) X + J(L_{Z'} \varphi) \ell' X, \tilde{X} \rangle - \frac{3}{4} \langle \nabla' \Upsilon, X \rangle + \langle \nabla' X, \tilde{X} \rangle + \langle \tau_{\Upsilon} X, JX \rangle + \|\Upsilon\|^2 \langle X, \tilde{X} \rangle$$

which we observe is actually independent of orientation.

Theorem 8.4. The Morimoto grading is given by $TM = (TM)_{-3} \oplus (TM)_{-2} \oplus (TM)_{-1}$, with $(TM)_{-1} = E$, and with $(TM)_{-2}$ and $(TM)_{-3}$ spanned by respectively $Z$ and $\ell X$, $X \in \Gamma(E)$, where

$$Z := Z' - \Upsilon,$$

$$\ell X := \ell' X - \frac{3}{4} \langle \Upsilon, X \rangle Z' + \mathcal{A} X$$

Furthermore, if we extend the the map $J$ by defining $JZ = 0$ and $\ell X = \ell JX$, then Morimoto connection is determined by the properties that for any $X \in \Gamma(E)$, $Y \in \Gamma(TM)$

$$\nabla Z = 0, \quad \nabla Y \ell X = \ell \nabla Y X, \quad \nabla Y X = \nabla_\Upsilon X + \mu(Y) JX,$$

where the one-form $\mu$ is iteratively determined by

$$\mu(X) = -\frac{5}{4} \langle \Upsilon, X \rangle$$

$$\mu(Z) = d\mu(\chi(Z)) - \frac{1}{4} \langle T'_{Z'}, J \rangle_{gI},$$

$$\mu(\ell X) = d\mu(\ell X) + \frac{1}{2} \langle \Upsilon, X \rangle_{gI} tr(E' R(\chi(Z)) \times, JX)_{gI} - \frac{1}{4} \langle T'_{\ell X}, J \rangle_{gI}.$$
and define $Z$, $\ell X_1$ and $\ell X_2$ similarly. If we extend $\ell$ and $\ell'$ by linearity to maps from $E$ to respectively $(TM)_{-3}$ and $(TM)_{-3}'$, then $\varphi(\ell X) = \varphi(\ell' X) = X$ and $\varphi([X, Z]) = \varphi([X, Z']) = JX$, for any $X \in \Gamma(E)$. By our definition of $g$, we have $\|Z\| = 1$ and $\langle \ell X, \ell \tilde{X} \rangle = \langle X, \tilde{X} \rangle$.

Define vector fields $W_1, W_2 \in \Gamma(E)$ and an endomorphism $\mathcal{A} : E \to E$ such that for any $X \in \Gamma(E)$,

$$Z - Z' = W_1,$$

$$\ell X - \ell' X = -\langle W_2, X \rangle g Z' + \mathcal{A} X,$$

Extend $J$ by linearity and the identities $JZ = 0$ and $J\ell X = \ell JX$. Then $\mathfrak{s}_I = \text{span}\{J\}$. We let $\theta$ be as in Section 5.2, i.e. the one-form satisfying $\theta(Z') = 1$ and $\theta(X) = \theta(\ell' X) = 0$ for any $X \in \Gamma(E)$. Finally, we observe that if $\chi$ and $\chi'$ are the selectors of respectively $I$ and $I'$, then

$$\chi(Z) = \chi'(Z) = \chi(Z') = X_1 \wedge X_2, \quad \chi(\ell X) = -JX \wedge Z.$$

$$\chi'(\ell X) = \chi(\ell' X) - \langle W_2, X \rangle \chi(Z') = -JX \wedge Z' - \langle W_2, X \rangle X_1 \wedge X_2$$

$$= -JX \wedge Z - \langle W_1 + W_2, X \rangle X_1 \wedge X_2.$$

Let $\nabla$ be the Morimoto connection. By strong compatibility, we will have $\nabla Z = 0$ and $\nabla Y \ell X = \ell \nabla Y X$ for any $Y \in \Gamma(TM)$, $X \in \Gamma(E)$. We hence only need to determine covariant derivatives of sections of $E$. By compatibility of the metric, we know that for any $Y \in \Gamma(TM)$, $X \in \Gamma(E)$,

$$\nabla Y X = \nabla_Y X + \mu(Y) JX,$$

for some one-form $\mu$. Let $T$ be the torsion of $\nabla$ and write

$$\text{pr}_{-1}' T(X_1, X_2) = -W_3.$$

We then have the following identities.

**Lemma 8.5.** Let $X, \tilde{X} \in \Gamma(E)$ be arbitrary horizontal vector fields. Then

$$\mu(X) = \langle X, W_1 + W_3 \rangle,$$

$$\varphi(T(X, \ell \tilde{X})) = \tau_X \tilde{X} + \langle X, W_1 + W_3 + \Upsilon \rangle J\tilde{X} + \langle W_2, \tilde{X} \rangle JX,$$

$$\langle T(X, Z), Z \rangle = \langle X, J(W_1 + W_2) \rangle,$$

$$\langle T(\ell X, \tilde{X}), Z \rangle = \langle J\tilde{X}, \mathcal{A} X \rangle - \langle \nabla_{\tilde{X}} W_2, X \rangle g - \langle W_2, X \rangle \langle W_2, J\tilde{X} \rangle$$

$$- \langle W_2, JX \rangle \langle \Upsilon, \tilde{X} \rangle - \langle W_2, \tau_X X \rangle.$$

**Proof.** Throughout this proof, $X, \tilde{X}$ and $\tilde{X}$ will be sections of $E$.

Eq. (8.5): Since

$$\text{pr}_{-1}' T(X_1, X_2) = -\text{pr}_{-1}'(Z + W_3) = -W_1 - W_3,$$

then by Lemma A.3 Appendix,

$$\langle \nabla_X X_1 - \nabla_X' X_1, X_2 \rangle g = \mu(X)$$

$$= \frac{1}{2} \langle T(X, X_1), X_2 \rangle - \frac{1}{2} \langle T(X, X_2), X \rangle - \frac{1}{2} \langle T(X, X_2), X_1 \rangle$$

$$= \frac{1}{2} \langle W_1 + W_3, X_2 \rangle \langle X, X_2 \rangle + \frac{1}{2} \langle W_1 + W_3, X \rangle + \frac{1}{2} \langle W_1 + W_3, X_1 \rangle \langle X_1, X \rangle$$

$$= \langle X, W_1 + W_3 \rangle.$$
Eq. (8.8):
\[ \langle T(X, t\tilde{X}), t\tilde{X}\rangle = \langle \nabla_X \tilde{X}, \tilde{X}\rangle - \langle [X, t\tilde{X}], t\tilde{X}\rangle_g \]
\[ = \langle \nabla'_X \tilde{X}, \tilde{X}\rangle + \langle X, W_1 + W_3\rangle \langle J\tilde{X}, \tilde{X}\rangle - \langle \varphi([X, t\tilde{X}]), \tilde{X}\rangle_g + \langle W_2, \tilde{X} \rangle \langle \varphi([X, Z']), \tilde{X}\rangle \]
\[ = \langle d^\nu(X, t\tilde{X}), \tilde{X}\rangle + \langle X, W_1 + W_3\rangle \langle J\tilde{X}, \tilde{X}\rangle + \langle W_2, \tilde{X} \rangle \langle JX, \tilde{X}\rangle \]
\[ = \langle \tau_X \tilde{X}, \tilde{X}\rangle + \langle X, W_1 + W_3 + \Upsilon \rangle \langle J\tilde{X}, \tilde{X}\rangle + \langle W_2, \tilde{X} \rangle JX \]

Eq. (8.7): Observe that for any vector field \( Y \in \Gamma(TM) \)
\[ \langle Y, Z \rangle = \theta(Y) + \langle W_2, \varphi(Y) \rangle. \]
Using this identity, we have that for any \( X \in \Gamma(E) \),
\[ - \langle T(X, Z), Z \rangle = \langle [X, Z'], Z \rangle + \langle [X, Z'], Z \rangle + \langle W_1, JX \rangle \]
\[ = \theta([X, Z']) + \langle W_2, \varphi([X, Z']) \rangle + \langle W_1, JX \rangle = \langle W_1 + W_2, JX \rangle. \]

Eq. (8.8): Using \( \theta \) again, we deduce that for \( X, \tilde{X} \in \Gamma(E) \),
\[ \langle T(t\tilde{X}, X), Z \rangle = \langle [\tilde{X}, t\tilde{X}], Z \rangle = \theta([\tilde{X}, t\tilde{X}]) + \langle W_2, \varphi([\tilde{X}, t\tilde{X}]) \rangle \]
\[ = \langle J\tilde{X}, \mathscr{A}X \rangle - \tilde{X} \langle W_2, X \rangle_g + \langle W_2, \nabla'_X X \rangle \]
\[ - \langle W_2, X \rangle \langle W_2, J\tilde{X} \rangle - \langle W_2, d^\nu \varphi(\tilde{X}, t\tilde{X}) \rangle \]
\[ = \langle J\tilde{X}, \mathscr{A}X \rangle - \langle \nabla'_X W_2, X \rangle_g - \langle W_2, X \rangle \langle W_2, J\tilde{X} \rangle \]
\[ - \langle W_2, JX \rangle \langle \Upsilon, \tilde{X} \rangle - \langle W_2, \tau_X X \rangle. \]

We are now ready to go through the steps of finding the Morimoto grading and connection.

**Step 1**: \((6.1)\) with horizontal vector fields. If \( X \) is any horizontal vector field then
\[ 0 = \langle T_X, J \rangle = - \langle T(X_1, X_2), X \rangle + \langle T(X, \ell X_1), \ell X_2 \rangle - \langle T(X, \ell X_2), \ell X_1 \rangle \]
\[ \overset{(6.8)}{=} \langle W_3, X \rangle + 2 \langle X, W_1 + W_3 + \Upsilon \rangle + \langle W_2, X_1 \rangle \langle JX, X_2 \rangle - \langle W_2, X_2 \rangle \langle JX, X_1 \rangle \]
\[ = \langle 2W_1 + W_2 + 3W_3 + 2\Upsilon, X \rangle \]
In conclusion
\[ 2W_1 + W_2 + 3W_3 = -2\Upsilon. \]
We observe also that by definition, \( 0 = \langle T_X, J \rangle = 4\mu(X) + \langle T'_X, J \rangle \), where we have used that \( \nabla'_X \ell X = \ell \nabla'_X X \mod E^{-2} \).

**Step 2**: Using \((6.2)\) with \( Z \). If \( X \in \Gamma(E) \) is any horizontal vector field, then
\[ \langle T(\chi(Z)), X \rangle = \langle T(X_1, X_2), X \rangle = -\langle W_3, X \rangle = -\text{tr}_{TM} \langle T(\times, Z), \Upsilon(\times, X) \rangle \]
\[ = -\langle T(JX, Z), X \rangle \overset{(6.8)}{=} -\langle X, W_1 + W_2 \rangle. \]
In conclusion
\[ W_3 = W_1 + W_2. \]
Step 3: \((6.2)\) with elements of degree \(-3\). Let \(X \in \Gamma(E)\) be arbitrary. We use \((6.2)\) with \(\ell X\) to obtain

\[
\langle T(\ell X), Z \rangle = -\langle T(JX, Z), Z \rangle = -\langle X, W_1 + W_2 \rangle
\]

\[
= -\text{tr}_{TM} \langle T(x, \ell X), T(x, Z) \rangle = \text{tr}_E \langle T(x, \ell X), \ell Jx \rangle = \text{tr}_E \langle \varphi(T(x, \ell X)), Jx \rangle
\]

\[
\lesssim \text{tr}_E \langle \tau_x X, Jx \rangle + \text{tr}_E \langle x, W_1 + W_3 + \Upsilon \rangle \langle Jx, JX \rangle + \langle W_2, X \rangle \text{tr}_E \langle Jx, Jx \rangle
\]

\[
= -\text{tr}_E \langle \tau_J XJx, Jx \rangle - \text{tr}_E \langle x, \Upsilon \rangle \langle JX, Jx \rangle + \langle X, W_1 + W_3 + \Upsilon \rangle + 2\langle W_2, X \rangle
\]

\[
= \langle X, W_1 + 2W_2 + W_3 \rangle
\]

Hence

\[
2W_1 + 3W_2 + W_3 = 0.
\]

From these expressions, one determines

\[
W_1 = -\Upsilon, \quad W_2 = \frac{3}{4} \Upsilon, \quad W_3 = -\frac{1}{4} \Upsilon.
\]

We continue with \(X, \tilde{X} \in \Gamma(E)\). Again we apply \((6.2)\) to \(\ell X\) and obtain

\[
\langle T(\chi(\ell X)), \tilde{X} \rangle = -\langle T(JX, Z), \tilde{X} \rangle = -\text{tr}_{TM} \langle T(x, \ell X), T(x, \tilde{X}) \rangle
\]

\[
= -\langle T(J\tilde{X}, \ell X), Z \rangle = -\langle T(Z, \ell X), \ell J\tilde{X} \rangle.
\]

Rearranging the terms, we observe that

\[
- \langle T(JX, Z), \tilde{X} \rangle + \langle T(Z, \ell X), \ell J\tilde{X} \rangle = \langle T(Z, JX) - J\varphi(T(Z, \ell X)), \tilde{X} \rangle
\]

\[
= \langle J\nabla_Z X - [Z, JX] - J\varphi(\ell \nabla_Z X - [Z, \ell X]), \tilde{X} \rangle
\]

\[
= \langle -[Z, JX] + J\varphi([Z, \ell X]), \tilde{X} \rangle
\]

\[
= \langle \mathcal{A} X, \tilde{X} \rangle + \langle -[Z', JX] + J\varphi([Z', \ell' X]), \tilde{X} \rangle
\]

\[
+ \langle -[W_1, JX] + J\varphi([W_1, \ell' X]), \tilde{X} \rangle
\]

\[
= \langle \mathcal{A} X, \tilde{X} \rangle + \langle -(\mathcal{L}_{Z'} JX - J(\mathcal{L}_{Z'} \varphi)\ell' X), \tilde{X} \rangle
\]

\[
+ \langle T(W_1, JX) - Jd\varphi(W_1, \ell' X) + \nabla'_{JX} W_1, \tilde{X} \rangle
\]

\[
= \langle \mathcal{A} X, \tilde{X} \rangle + \langle -(\mathcal{L}_{Z'} JX - J(\mathcal{L}_{Z'} \varphi)\ell' X), \tilde{X} \rangle
\]

\[
- \langle X, W_1 \rangle \langle W_3, \tilde{X} \rangle + \langle \tau_{W_1} X, JX \rangle + \langle W_1, \Upsilon \rangle \langle X, \tilde{X} \rangle + \langle \nabla'_{JX} W_1, \tilde{X} \rangle
\]

\[
= \langle T(\ell X, \tilde{X}), Z \rangle
\]

\[
\lesssim -\langle \tilde{X}, \mathcal{A} X \rangle - \langle \nabla'_{JX} W_2, X \rangle + \langle W_2, X \rangle \langle W_2, \tilde{X} \rangle
\]

\[
\lesssim -\langle W_2, JX \rangle \langle \Upsilon, J\tilde{X} \rangle - \langle W_2, \tau_{JX} X \rangle
\]

From this, we get a formula for \(\mathcal{A}\). This completes the proof for \(I\).
Step 4: \([6.1]\) with \(Z\). We use that \(\mathfrak{g}\) is spanned by \(J\) to obtain
\[
\langle R(\chi(Z)), J \rangle = \langle R(X_1, X_2), J \rangle = 2\langle R(X_1, X_2)X_1, X_2 \rangle + 2\langle R(X_1, X_2)\ell X_1, \ell X_2 \rangle
\]
\[
= 4\langle R(X_1, X_2)X_1, X_2 \rangle = \langle T_Z, J \rangle = 4\mu(Z) + \langle T'_Z, J \rangle,
\]
where we have used that \(\ell\) is parallel with respect to \(\nabla\). Using the properties of \(\nabla'\), we observe that
\[
\langle R(X_1, X_2)X_1, X_2 \rangle = \langle R'(X_1, X_2)X_1, X_2 \rangle + (\nabla_{X_1}\mu)(X_2)
\]
\[
- (\nabla_{X_2}\mu)(X_1) + \mu(T(X_1, X_2))
\]
\[
= \langle d\mu(\chi(Z)) \rangle
\]
Finally, this leads to
\[
\langle (\id - \chi^*d)\eta \rangle(Z) = -\frac{1}{4}\langle T'_Z, J \rangle.
\]
Step 4: \([6.1]\) with vector fields of degree \(-3\). For the final computation, we have that for any \(X \in \Gamma(E)\), we have
\[
\langle R(\chi(\ell X)), J \rangle = -\langle R(JX, Z), J \rangle = -4\langle R(JX, Z)X_1, X_2 \rangle
\]
\[
= -4\langle R'(JX, Z)X_1, X_2 \rangle + 4d\mu(\chi(\ell X))
\]
\[
= -\langle \gamma, \chi \rangle \langle R'(X_1, X_2)X_1, X_2 \rangle + 4d\mu(\chi(\ell X))
\]
\[
= \langle T_{\ell X}, J \rangle = \langle T'_{\ell X}, J \rangle + 4\mu(\ell X)
\]
This completes the proof.

Appendix A. Selectors and partial connection

A.1. Selectors and partial one-forms. Let \(\chi\) be the selector of an equiregular subbundle \(E\) of step \(s\). Then given information about a one-form along just the subbundle \(E\) and knowing the value of the differential along the selector will give us a unique way of extending the partial data to a full one-form. The statement can be found below and can be deduced from [12].

Lemma A.1. Let \(\beta\) be any one-form and let \(\eta\) be any one-form vanishing on \(E\). Then the unique solution \(\alpha\) of the equations
\[(A.1) \quad \alpha|_{E} = \beta|_{E}, \quad \chi^*d\alpha = \eta,\]
is given by
\[
\alpha = (\id + \chi^*d)^{s-1}\beta - \sum_{j=0}^{s-2} \binom{s-1}{j+1} (\chi^*d)^j \eta
\]
\[
= \beta^\chi - \frac{(\id - \chi^*d)^{s-1} - \id}{\chi^*d} \eta.
\]
We include a proof, since the formulation of the result is a bit different than in our reference.

Proof. If \(\eta = 0\): We note that \(\chi^*d\beta\) is a one-form vanishing on \(E\) so \((\id + \chi^*d)^{s-1}\chi^*d\beta = 0\). Hence
\[
\chi^*d(\id + \chi^*d)^{s-1}\beta = (\id + \chi^*d)^{s-1}\chi^*d\beta = 0,
\]
and furthermore, we have \((\id + \chi^*d)^{s-1}\beta|_{E} = \beta|_{E}\).
If $\beta = 0$: We note that then $\alpha$ vanishes on $E$, then $\alpha + \eta = (\text{id} + \chi^* d)\alpha$ vanishes on $E^{-2}$. It follows that

$$
0 = (\text{id} + \chi^* d)^{s-2}(\alpha + \eta) = \sum_{j=0}^{s-2} \binom{s-2}{j} (\chi^* d)^j (\alpha + \eta)
$$

$$
= \alpha + \sum_{j=0}^{s-3} \binom{s-2}{j} (\chi^* d)^j \eta + \sum_{j=0}^{s-2} \binom{s-2}{j} (\chi^* d)^j \eta
$$

$$
= \alpha + \sum_{j=0}^{s-2} \binom{s-1}{j} (\chi^* d)^j \eta
$$

The general case: The general solution can be found by adding two previous equations. □

A.2. Selectors and partial connections. In Lemma A.1, we can replace $d$ with any operator $L$ from one-forms to two-forms such that $(\text{id} + \chi^* L)\alpha$ vanishes on $E^{-k-1}$ if $\alpha$ vanishes $E^{-k}$. This holds true also when the forms in question have values in a vector bundle. We will use this result for partial connections.

Let $\mathcal{A}$ be any vector bundle and let $E$ be a subbundle of the tangent bundle. A partial connection $\mathcal{D}$ on $\mathcal{A}$ in the direction of $E$ is a map $\Gamma(E) \times \Gamma(\mathcal{A}) \to \Gamma(\mathcal{A})$

$$(X,A) \mapsto \mathcal{D}_X A$$

which is tensorial in the first argument and satisfies $\mathcal{D}_X \phi A = (X\phi) A + \phi \mathcal{D}_X A$ for any $\phi \in \mathcal{C}^\infty(M)$. We note the following result, which can also be deduced from [12].

Lemma A.2. Let $\mathcal{D}$ be a partial connection be a connection on $\mathcal{A}$ in the direction of $E$.

(a) Then there exists a unique affine connection $\nabla = \mathcal{D}^\chi$ on $\mathcal{A}$, such that $\nabla|_E = \mathcal{D}$ and such that the curvature $R$ of $\nabla$ satisfies

$$R(\chi(\cdot)) = 0.$$ 

We also have $\text{Hol}(\nabla) = \text{Hol}(\mathcal{D})$.

(b) Let $\eta$ be a one-form vanishing on $E$ with values in $\text{End} \mathcal{A}$. Then there is a unique affine connection $\nabla$ such that $\nabla|_E = \mathcal{D}$ and such that its curvature $R$ satisfies

$$R(\chi(\cdot)) = \eta.$$ 

This connection is given by

$$\nabla_X Y = \mathcal{D}_X Y - \sum_{j=0}^{s-2} \binom{s-1}{j+1} ((\chi^* d^{\mathcal{D}^\chi})^j \eta)(X) Y.$$ 

Proof. Let $k$ be the rank of $\mathcal{A}$. Consider the principal bundle

$$\text{GL}(k) \to \text{GL}(\mathcal{A}) \to M.$$ 

Then a partial connection correspond to a partial form $\omega : \pi^{-1}_* E \to \mathfrak{gl}(k)$ satisfying $\omega(v \cdot a) = \text{Ad}(a^{-1}) \omega(v)$. Let $L : \Gamma(T^* \text{GL}(\mathcal{A}) \otimes \mathfrak{gl}(k)) \to \Gamma(\wedge^2 T^* \text{GL}(\mathcal{A}) \otimes \mathfrak{gl}(k))$ be the operator

$$L\alpha = d\alpha + \frac{1}{2} [\alpha, \alpha].$$
Let \( \hat{\omega} : T GL(A) \to gl(A) \) be any principal connection with \( \hat{\omega}|_{\pi^{-1}E} = \omega \). Write \( \mathcal{H} = \ker \omega \) and let \( v \in T_{\pi(f)}M \mapsto h_f v \in \mathcal{H} \) be the corresponding horizontal lift. Define a selector \( \hat{\chi} \) of \( \pi^{-1}E \) by

\[
\hat{\chi} : h_f v \mapsto h_f \chi(v).
\]

(a) We define \( \omega^\chi \) as the solution of

\[
\omega^\chi = \omega, \quad \hat{\chi}^* L \omega^\chi = 0.
\]

This has the the same holonomy by [12].

(b) Introduce \( \hat{\eta} \in \Gamma(T^* GL(A) \otimes gl(k)) \),

\[
\hat{\eta}(w) = f^{-1} \eta(\pi_* w) f, \quad w \in T_f P.
\]

We define \( \omega \),

\[
\omega = \omega, \quad \hat{\chi}^* L \omega = \hat{\eta}.
\]

The solution is

\[
\omega = \omega^\chi - \sum_{j=0}^{s-2} \left( s - 1 \right) \left( \hat{\chi}^* d \right)^j \hat{\eta}.
\]

We finally observe that

\[
\hat{\chi}^* d \hat{\eta}(w) = f^{-1} (\hat{\chi}^* d \hat{\eta})(\pi_* w) f. \quad \square
\]

### A.3. Partial connections on \( E \) and complements

If we have a sub-Riemannian manifold \((M, E, g)\) with a chosen complement to \( E \) or equivalently a chosen projection \( \text{pr}_E : TM \to E \), then it was observed in [11, Lemma 2.9] that we have the following unique way of choosing a partial connection on \( E \). For any partial connection \( \mathcal{D} \), define its partial torsion by

\[
t(X,Y) = \mathcal{D}_X Y - \mathcal{D}_Y X - [X,Y], \quad X,Y \in \Gamma(E),
\]

which takes values in \( TM \).

**Lemma A.3.** Let \( \text{pr}_E : TM \to E \) be any projection. Then there is a unique partial connection \( \mathcal{D}^0 \) such that its partial torsion \( t^0 \) satisfies \( \text{pr}_E t^0(E, E) = 0 \). Furthermore, let \( \mathcal{D} \) be any compatible partial connection on \( E \) in the direction of \( E \) with partial torsion \( t \). Define \( t_E(X, Y) = \text{pr}_E t(X, Y) \). Then for any \( X, Y, Z \in \Gamma(E) \),

\[
\langle (\mathcal{D} - \mathcal{D}^0)_Y Z \rangle = \frac{1}{2} \langle t_E(X, Y), Z \rangle_g - \frac{1}{2} \langle t_E(Y, Z), X \rangle_g - \frac{1}{2} \langle t_E(X, Z), Y \rangle_g.
\]

Note that if we take any taming Riemannian metric \( \tilde{g} \) of \( g \) such that \( \text{pr}_E \) becomes the orthogonal projection, then we can write \( \mathcal{D}^0 \) as \( \mathcal{D}^0_X Y = \text{pr}_E \nabla^g_X Y, \quad X, Y \in \Gamma(E) \) where \( \nabla^g \) is the Levi-Civita connection of \( \tilde{g} \). Using the Koszul formula for the Levi-Civita connection, we conclude that

\[
\langle \mathcal{D}^0_X Y, Z \rangle = \frac{1}{2} \langle X(Y, Z)_g + Y(X, Z)_g - Z(X, Y)_g \rangle_g
\]

\[
\quad \quad \quad \quad \quad \frac{1}{2} \left( \langle \text{pr}_E[X, Y], Z \rangle_g - \langle \text{pr}_E[X, Z], Y \rangle_g - \langle \text{pr}_E[Y, Z], X \rangle_g \right).
\]
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