Quantum Gravitational Collapse of a Charged Dust Shell

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Abstract

A simple self gravitating system—a thin spherical shell of charged pressureless matter—is naively quantized as a test case of quantum gravitational collapse. The model is interpreted in terms of an inner product on the positive energy states. An $S$-matrix is constructed describing scattering between negatively and positively infinite radius.

1 Introduction

In the absence of a successful quantum theory of gravity, it is interesting to study simplified, naively quantized models of self gravitating systems. Although such a model is not backed by any complete theory, it may provide clues as to the properties of a complete theory. Such models may also confront some of the poorly understood issues of quantum gravity in a simplified setting.

One interesting gravitational process predicted by General Relativity is gravitational collapse beyond an event horizon. A quantitatively accurate model of gravitational collapse, taking into account the equation of state of the matter as well as its rotation and other factors is too complicated to solve in general classically; let alone attempt to quantize without a full theory. However, some simple special case of collapse might indicate the qualitative behavior of the more general case. In this paper, the equation of state is dealt with by assuming the collapsing matter to be a pressureless dust. The motion is simplified by assuming spherical symmetry, but this still leaves the system with infinite degrees of freedom. If, instead of a solid ball of matter, the collapsing object is an infinitesimally thick shell, then it has only one degree of freedom, its radius, and it is simple enough to deal with.

Naive quantization of such a dust shell has been accomplished by Hájíček, Kay, and Kuchař [2]. Unfortunately, that model is too simplified to exhibit one of the interesting
properties of general gravitational collapse; in general, if the collapsing object has any angular momentum or charge it may reexpand, possibly into an asymptotically flat region distinct from the one in which it originally collapsed. Allowing the shell to have angular momentum would break its spherical symmetry and complicate matters greatly. A charged shell, however, would exhibit some of the same phenomena, but could still be spherically symmetric and have only one dynamical degree of freedom.

Unfortunately, a major qualitative difference remains between the charged and the rotating cases. In the Kerr space-time there is an accessible, asymptotically flat region where \( r < 0 \); in the Reissner-Nordström metric such a region exists, but it is only connected to \( r > 0 \) at a single point of space, and thus hardly seems like a physical part of the space-time. It seems plausible nonetheless to assume this spherically symmetric model to be continuous with the more general case of nonzero angular momentum and therefore to take the negative radius region seriously.

In this paper I naively quantize the collapse of a charged dust shell using an equation of motion that is a trivial generalization of the Wheeler-DeWitt equation derived in [2]. However, instead of applying a boundary condition at the origin and only allowing positive radii, I apply a matching condition at the origin and allow the shell to attain negative radii.

One issue which this model cannot be expected to resolve is whether singularities occur in quantum gravity. If there is a singularity avoidance mechanism, then it may well involve nonradial degrees of freedom — even in a spherically symmetric situation; but all nonradial degrees of freedom are suppressed in this model.

2 Equations of Motion

Consider a spherically symmetric, infinitesimally thick, dust shell of rest mass \( M \) and charge \( Q \). The interior of the shell is a piece of Minkowski space-time; the exterior is a piece of Reissner-Nordström space-time. The mass appearing in the Reissner-Nordström metric is the total energy \( (E) \) of the system as determined by a distant observer, the charge is that of the shell. Note that Schwarzschild coordinates on the interior and exterior do not join smoothly across the shell \((g_{rr} \text{ is discontinuous})\), but this does not cause any problem in deriving the equation of motion.

The classical equation of motion was derived by Kuchař [1] from the relationship between the surface energy tensor of the shell and the difference between the extrinsic curvatures of its interior and exterior surfaces. The shell’s motion is described by the first integral of motion

\[
E = \frac{M}{\sqrt{1 - \dot{R}^2}} - \frac{k}{2R}
\]

(1)

where \( R \) is the radius of the shell, \( \dot{R} = \frac{dR}{dT} \), \( T \) is the well defined Minkowskian time coordinate on the interior, and \( k = M^2 - Q^2 \). Units are such that \( \hbar = c = G = 1 \). This equation of motion is formally identical to that of a relativistic point particle moving
radially in a Coulomb potential in flat space-time; this analogy is useful because it suggests a ready made interpretation for the quantum model.

By assuming the externally measured energy $E$ to be the energy conjugate to $T$, numerically equal to the Hamiltonian that generates motion in $T$, the momentum conjugate to $R$ is found to be

$$P^R = \frac{M \dot{R}}{\sqrt{1 - \dot{R}^2}}$$

(2)

and from this a super-Hamiltonian

$$h = - \left( P^T + \frac{k}{2R} \right)^2 + P^R^2 + M^2$$

(3)

is found which is constrained to vanish and which generates correct equations of motion. See [2].

Promoting the momenta in $h$ to operators makes the constraint $h = 0$ into a Wheeler-DeWitt equation, namely

$$0 = \left( i \frac{\partial}{\partial T} + \frac{k}{2R} \right)^2 \Psi + \frac{\partial^2 \Psi}{\partial R^2} - M^2 \Psi.$$

(4)

Consistent with the above analogy, (4) is equivalent to the Klein-Gordon equation for a scalar particle in a Coulomb potential under the replacement $\Psi = R \Phi$ (equivalent to a change of measure). Equation (4) is the basis of this naive quantization.

### 3 Probability Interpretation

This quantum model is most appropriate to an observer located in the interior of the shell. $T$ and $R$ are well defined in terms of an observer at $r = 0$ who measures the shell’s motion by timing the delays of light pulses reflected off the inside of the shell. This model may not, however, be able to describe observations made by an external observer outside the shell. An important effect detectible only by an external observer would be the shell passing beyond an event horizon. In this model there is no qualitative distinction between the motion of the shell outside and inside an event horizon, as there would be in a model based upon an exterior observer who only sees $R > 2E$. This is emphasized by the invariance of both the classical and quantum equations of motion under the rescaling:

$$(T, R, M, E, Q) \rightarrow \left( aT, aR, aM, aE, \sqrt{Q^2 - (1 - a^2)M^2} \right)$$

(5)

for any $a > 0$. This equates points inside the event horizon in one case with points outside the event horizon in another.
Following the scalar particle analogy Hájíček et al \cite{2} construct “charge” (really number), and energy currents which are generalized as locally conserved bilinear currents:

\[
J_T(\Psi_1, \Psi_2) = \frac{i}{2} \left[ \bar{\Psi}_1 \dot{\Psi}_2 - \dot{\bar{\Psi}}_1 \Psi_2 \right] + \frac{k}{2R} \bar{\Psi}_1 \Psi_2 \tag{6}
\]

\[
J_R(\Psi_1, \Psi_2) = \frac{i}{2} \left[ \bar{\Psi}_1 \Psi'_2 - \bar{\Psi}'_1 \Psi_2 \right] \tag{7}
\]

\[
E_T(\Psi_1, \Psi_2) = \frac{1}{2} \left[ \dot{\bar{\Psi}}_1 \dot{\Psi}_2 + \bar{\Psi}'_1 \Psi'_2 + \left( M^2 - \left[ \frac{k}{2R} \right]^2 \right) \bar{\Psi}_1 \Psi_2 \right] \tag{8}
\]

\[
E_R(\Psi_1, \Psi_2) = \frac{-1}{2} \left[ \bar{\Psi}'_1 \dot{\Psi}_2 + \bar{\Psi}_1 \dot{\Psi}'_2 \right] \tag{9}
\]

where \(’\) denotes \(\frac{\partial}{\partial R}\).

Assuming \(J(\Psi, \Psi)\) to be a probability current gives a tentative interpretation of the wavefunction for \(R > 0\). In the defining experiment for this model an observer at \(r = 0\) sends a light pulse at time \(U\) which reflects off the shell at \((T, R)\) and is seen again at time \(V\). \(T\) and \(R\) are given by

\[
T = \frac{1}{2} (V + U) \\
R = \frac{1}{2} (V - U). \tag{10}
\]

In the quantum model, if the light pulse is sent at the observer’s time \(U\) then the probability that it will return at time \(V\) is \(\frac{1}{2} J^U dV\) where \(J^U = J^T - J^R\), evaluated at \((T, R)\) given by (10). (I’m assuming an appropriately normalized wavefunction.)

This interpretation fails, however, because the probability given is not necessarily positive. Following the scalar particle analogy further would suggest that the negative densities might represent pair creation, but this model is too crude to make a meaningful prediction of that sort. It would be like modeling a car crashing into a wall as a potential barrier, and concluding that automobiles would be pair produced.

3.1 Boundary Condition

If only \(R \geq 0\) is allowed then \(J\) and \(E\) will only be globally conserved if a boundary condition is imposed. Namely,

\[
J^R(\Psi_1, \Psi_2) = E^R(\Psi_1, \Psi_2) = 0 \tag{11}
\]

at \(R = 0\) for all \(T\).

This boundary condition rules out \textit{ab initio} a truly irreversible collapse. When the shell reaches 0 radius it is required to pass through itself and reexpand. However, the collapse may appear irreversible; if the shell reaches \(R = 0\) it is within an event horizon and an external observer would never see it reexpand.
Negative radius might seem unphysical, but as I have mentioned there are plausible reasons for considering it. Among the stable, approximately spherically symmetric vacuum space-times (i.e., black holes) the Reissner-Nordström space-time is exceptional. For any such space-time with nonzero angular momentum the causal structure is that of the Kerr space-time, and there is an accessible region where $r < 0$. In the quantized case, classical states are smeared and the Reissner-Nordström causal structure might well give in to its generic neighbor.

In view of the interpretation in terms of an observer at $r = 0$, allowing $R < 0$ seems especially meaningless—an observer could not possibly continue to exist inside the shell after it has completely collapsed. However, if the shell reexpands then observations could be recommenced and their correlation with previous observations would depend upon the $R < 0$ behavior.

### 3.2 Revised Interpretation

When all values $-\infty < R < +\infty$ are allowed, $J$ gives a conserved inner product equal to the flux of $J$ across any Cauchy surface; that is:

$$\langle \Psi_1 | \Psi_2 \rangle = \int_{-\infty}^{+\infty} J^T(\Psi_1, \Psi_2) \, dR. \quad (12)$$

Because $J^T(\Psi, \Psi)$ is not always positive, it is not clear that this inner product is positive definite, as is necessary for normalizable wavefunctions. In fact it is positive definite if and only if the wavefunctions are restricted to the subspace of positive energy solutions (see appendix). This restriction seems acceptable and excuses the failure of the initial interpretation. The local currents were not meaningful because the position operators $R$ or $V$ do not leave the positive energy subspace invariant and are thus not good observables.

### 4 Classical Motion

First, consider the motion through $R > 0$. In [3] Boulware analysed the motion of a charged shell through the possible regions of the extended Reissner-Nordström space-time that its exterior is a piece of. The Schwarzschild coordinates do not completely specify the position of the shell. The key tool which helps to distinguish between regions with identical coordinates is the $r$ component of the unit outward normal to the outer surface of the shell, which points from the interior (Minkowski) to the exterior (Reissner-Nordström) region. This is given simply by:

$$n^r = \frac{E}{M} - \frac{M^2 + Q^2}{2MR}. \quad (13)$$

When $n^r > 0$ the exterior of the shell is the $r > R$ region of Reissner-Nordström space-time. When $n^r < 0$ the “exterior” of the shell is the $r < R$ region of Reissner-Nordström.
For sufficiently small $R > 0$, $n^r$ is necessarily negative. This means that the shell cannot collapse to form a singularity of charge $Q$. If it collapses to $R = 0$ it in fact collides with a singularity on its exterior with charge $-Q$.

Setting $\dot{R} = 0$ in (1) and solving for $R$ gives a turning radius

$$R_+ = \frac{k}{2(M - E)} = \frac{M^2 - Q^2}{2(M - E)}.$$  \hspace{1cm} (14)

Consideration of $n^r$ and $R_+$ gives the principal distinct classical trajectories through $R > 0$ (along with trivial cases such as $k = 0$).

1: For $E > M$ and $R_+ > 0$ (hence $|Q| > M$), the shell starts from $R = \infty$, falls to a finite radius and bounces back to $\infty$. A further distinction is

   a) $|Q| > E$ and no event horizon occurs, or

   b) $|Q| < E$, an event horizon is formed and the shell reexpands in a different asymptotically flat region than it collapsed from.

2: For $E > M$ and $R_+ < 0$ (hence $|Q| < M$), The shell either falls from $R = \infty$ to 0, or the time reverse, exploding from 0 to $\infty$.

3: For $E < M$, the shell cannot reach $R = \infty$. Either,

   a) $E < \frac{Q^2 + M^2}{2M}$ and the shell never emerges into an asymptotically flat region, or

   b) $E > \frac{Q^2 + M^2}{2M}$, the shell emerges from an event horizon, expands to a maximum radius of $R_+$, then recollapses, hitting singularities in its past and future.

Because of the rescaling symmetry (5), there is no qualitative distinction between the solutions for the $a$ and $b$ cases. An interior observer can only guess if the shell has passed within an event horizon.

4.1 Extension to $R < 0$

Transforming $r \rightarrow -r$ in the Reissner-Nordström metric shows that the $r < 0$ domain of the complete Reissner-Nordström space-time is an asymptotically flat region containing a singularity with mass $-E$, and charge either $\pm Q$; by following the electric field lines through $r = 0$ the charge is seen to be $-Q$. From (13) $n^r$ must be positive when $R < 0$; therefore, if the shell enters $R < 0$, this singularity is on the exterior of the shell within a compact region. The $r$ component of the normal to the inner surface of the shell also changes sign; the interior thus continues to be a ball of Minkowski space.

If the classical solution is followed past $R = 0$, the radical in (14) must change sign; this analytically continued equation of motion is presumably the only one continuous with that for nonzero angular momentum. There is now a second turning point at

$$R_- = \frac{-k}{2(M + E)} = -\frac{M^2 - Q^2}{2(M + E)}.$$  \hspace{1cm} (15)
For case 2, the shell passes in a finite proper time to $R = R_- < 0$ and then bounces back out to $R = +\infty$. For case 3, the shell oscillates between $R_+ > 0$ and $R_- < 0$ with a finite period. So, classically, if the shell is allowed to pass into $R < 0$, it will always return to $R > 0$ and potentially be observed again, except in the special case $k = 0$ in which the equation of motion reduces to $\dot{R} = \text{Const.}$

5 Solution of the Wave Equation

Now find a complete set of solutions to the Wheeler-DeWitt equation (14) by considering definite energy states ($P^T$ eigenstates) $\Psi(T, R) = e^{-iET} \psi(R)$. With this ansatz, (14) becomes

$$0 = \psi'' + \left( E + \frac{k^2}{2R} \right)^2 - M^2 \psi = \psi'' + \left( E^2 - M^2 \right) + \frac{Ek}{R} + \frac{k^2}{4R^2} \psi.$$  \hfill (16)

Now assume $E > M$. Defining

$$\chi = \sqrt{E^2 - M^2}; \quad x = \chi R$$ \hfill (17)

(14) becomes

$$0 = \frac{d^2 \psi}{dx^2} + \left[ 1 + \frac{Ek}{\chi x} + \frac{k^2}{4x^2} \right] \psi.$$ \hfill (18)

The characteristic equation about the regular singularity at $x = 0$ is

$$0 = \lambda(\lambda - 1) + \frac{k^2}{4}$$ \hfill (19)

with roots $\lambda_\pm = \frac{1 \pm \alpha}{2}$ where $\alpha = \sqrt{1 - k^2}$. As noted in [2], this model breaks down for $|k| > 1$, corresponding to an overcritical Coulomb potential which can only be described by a second quantized theory. It is far from clear what an appropriate analogue of second quantization would be here, and it also seems implausible that an analysis of higher order effects could be meaningful in a model with virtually all the system’s degrees of freedom suppressed. I therefore only consider $|k| < 1$.

First, only consider the solution of (18) in $x > 0$; there must exist independent solutions such that

$$\psi = x^{\lambda\pm} (1 + O(x)).$$ \hfill (20)

Because the coefficients in (18) are real, the solutions (20) are real for all $x > 0$. Adopting one more new constant $\beta = \frac{Ek}{2\chi}$, (18) may be written

$$0 = \frac{d^2 \psi}{d(2ix)^2} + \left[ \frac{1}{4} - \frac{i\beta}{2ix} + \frac{1}{4} - \frac{1}{4} \alpha^2 \right] \psi.$$ \hfill (21)
which is just Whittaker’s equation (see [4] p505). Define two functions, equal to the solutions (20):

\[ \phi_\pm(\alpha, \beta; x > 0) = e^{\pm ix} x^{\lambda_\pm} M(\lambda_\pm + i\beta, 2\lambda_\pm, 2ix) \] (22)

where M is the regular Kummer confluent hypergeometric function. These solutions are indeed real (see [4] eq. 13.1.27).

M is a single valued function, but because of the \( x^{\lambda_\pm} \) factor \( \phi_\pm \) are generally multi-valued functions in the complex plane with a branch point at \( x = 0 \). In order to analyse the \( R < 0 \) behavior it must be determined which branch which \( x \) axis lies on. (This analysis also applies when \( E < M \).) As a criterion, note that (4) is invariant under the time reversal transformation \( \Psi(T, R) \rightarrow \Psi(-T, R) \) or equivalently \( \psi(R) \rightarrow \bar{\psi}(R), E \rightarrow E \). Because of this we will demand that the complex conjugate of any physical solution of (16) must also be a physical solution. This means that \( \phi_\pm \) must be real for \( x < 0 \) as well as \( x > 0 \), otherwise \( \phi_\pm \) and \( \bar{\phi}_\pm \) would constitute more than 2 independent solutions to (16).

A second criterion is provided by the current form \( J^R \). For two definite energy states of the same energy, conservation implies that \( J^R \) must be constant with respect to \( R \). So

\[ \phi_+(\alpha, \beta; x)\phi'_-(\alpha, \beta; x) - \phi'_+(\alpha, \beta; x)\phi_-(\alpha, \beta; x) = \text{Const.}(x). \] (23)

This implies that one of the \( \phi_\pm \) is approximately even, and the other approximately odd near \( x = 0 \); it doesn’t matter which. Arbitrarily choose \( \phi_+ \) as approximately even\(^2\).

Now note that (21) is invariant under \((x, \beta) \rightarrow (-x, -\beta)\) so

\[ \phi_\pm(\alpha, \beta; x) = \pm \phi_\pm(\alpha, -\beta; -x). \] (24)

### 6 Transmission and Reflection coefficients

In a definite energy state for \( R \rightarrow \pm\infty \), \( J^T(\Psi, \Psi) \) becomes simply \( E \cdot |\Psi|^2 \); the interpretation simplifies. For a fixed energy, the wavefunction near infinity may be interpreted as a relative probability amplitude, and an S-matrix may be constructed. Time reversal invariance and unitarity restrict the S-matrix to the form:

\[ S = \begin{pmatrix} S_{++} & S_{+-} \\ S_{-+} & S_{--} \end{pmatrix} \] (25)

where

\[ S_{++} = e^{i\delta} \cos \gamma \]
\[ S_{+-} = i e^{i\delta} \sin \gamma. \] (26)

\(^2\)Strictly speaking, this sign combination (or its alternative) cannot be attained as a branch of the multi-valued solution for certain rational values of \( \alpha \), but the complement of that set of values is dense, and this is physics.
If (16) is viewed as a time independent Schrödinger equation, the effective potential is long ranged, and so the wavefunctions cannot asymptote to definite momentum waves with stable phase. As a result the phase $\delta$ is not well defined; only the reflection and transmission coefficients $\cos^2 \gamma$ and $\sin^2 \gamma$ are meaningful. The reflection coefficient is interpreted as the probability that a shell collapsing from $+\infty$ will return to $+\infty$; assuming this model to have any validity, that probability should be independent of how the position of the shell is measured.

These reflection and transmission coefficients for the scattering of wave packets between $+\infty$ and $-\infty$ are determined from the asymptotic behavior of the two basis solutions $\psi(R) = \varphi_{\pm}(\alpha, \beta; \chi R)$. Equation 13.5.1 of [4] gives the asymptotic behavior of $\varphi_{\pm}$:

$$\varphi_{\pm}(\alpha, \beta; x \to +\infty) = (\pm \alpha)! \left[ \left( i \right)^{\lambda_{\pm}} \frac{\Gamma(\lambda_{\pm} - i\beta)}{\Gamma(\lambda_{\pm} + i\beta)} e^{ix(\pm 2x) - i\beta} + \text{c.c.} \right] \left( 1 + O(1) \right)$$

and by (24),

$$\varphi_{\pm}(\alpha, \beta; x \to -\infty) = \pm(\pm \alpha)! \left[ \left( i \right)^{\lambda_{\pm}} \frac{\Gamma(\lambda_{\pm} + i\beta)}{\Gamma(\lambda_{\pm} - i\beta)} e^{ix(\pm 2x) - i\beta} + \text{c.c.} \right] \left( 1 + O(1) \right)$$

In lieu of definite momentum waves, the appropriate ingoing and outgoing asymptotic waves are $e^{\pm i\phi}$ where

$$\phi = \chi R + \text{sgn}(R) \beta \ln |2\chi R|$$

In terms of $\phi$, the asymptotic forms of the wavefunctions are

$$\psi(R \to +\infty) = (\pm \alpha)! \left[ \left( i \right)^{\lambda_{\pm}} \frac{\Gamma(\lambda_{\pm} - i\beta)}{\Gamma(\lambda_{\pm} + i\beta)} e^{i\phi} + \text{c.c.} \right] \left( 1 + O(1) \right)$$

$$\psi(R \to -\infty) = \pm(\pm \alpha)! \left[ \left( i \right)^{\lambda_{\pm}} \frac{\Gamma(\lambda_{\pm} + i\beta)}{\Gamma(\lambda_{\pm} - i\beta)} e^{-i\phi} + \text{c.c.} \right] \left( 1 + O(1) \right)$$

which are of the form

$$\psi(r \to +\infty) \rightarrow c_{\text{in}+} e^{i\phi} + c_{\text{out}+} e^{-i\phi}$$

$$\psi(r \to -\infty) \rightarrow c_{\text{in}+} e^{-i\phi} + c_{\text{out}+} e^{i\phi}$$

By applying the definition of the $S$-matrix:

$$\begin{pmatrix} c_{\text{out}+} \\ c_{\text{out}-} \end{pmatrix} = S \begin{pmatrix} c_{\text{in}+} \\ c_{\text{in}-} \end{pmatrix}$$

one obtains,

$$i e^{\pm \frac{x}{2} \beta} = \pm S_{++} e^{i\beta} + S_{+-} \frac{\Gamma(\lambda_{\pm} - i\beta)}{\Gamma(\lambda_{\pm} + i\beta)}$$
Combining the $+$ and $-$ equations and solving gives

$$
\delta = \arg [\Gamma (\lambda_+ - i\beta) \Gamma (\lambda_+ + i\beta)]
$$

(35)

$$
\cos^2 \gamma = \sin^2 \left(\frac{\pi}{2} \alpha \right) \sech^2 \pi \beta.
$$

(36)

Classically, the shell cannot reach $R = -\infty$, unless $k = 0$ in which case it must. Correspondingly, the transmission coefficient, $\cos^2 \gamma$, is necessarily $< 1$ unless $k = 0$, in which case it is necessarily $= 1$. The transmission coefficient goes to 0 when $k \to \pm 1$ or $E \to M$.

There are poles in the transmission coefficient when $\cosh \pi \beta = 0$; accepting only normalizable (positive energy) solutions, this implies the existence of bound states with energies

$$
E_n = \frac{M(2n - 1)}{\sqrt{(2n - 1)^2 + k^2}}
$$

(37)

with $n = 1, 2, 3, \ldots$. This is a different energy spectrum than results if the boundary condition is imposed. For $k < 0$, the bound states correspond to states with no asymptotically flat region. For $k > 0$ the bound states all correspond to classical states with maximum radius $(R_+)$ outside the event horizon (classical case 3b). Because this model is unable to deal with an observer outside the shell, it cannot decide questions of when the shell does or does not pass beyond an event horizon any better than this.

The bound state solutions can be found explicitly in the same way as the scattering solutions, but in contrast to the bound states found in [2] with the boundary condition imposed, they are not algebraically special.

7 Conclusions

By necessity, this Quantum model is based upon several tentative assumptions. The standard Schwarzschild time $t$ is the proper time of a distant, stationary observer; such an observer will measure the total mass of the system to be $E$; therefore $E$ is the energy conjugate to $t$. The first assumption made in this model was that $E$ is conjugate to $T$. This works if there is a time slicing such that $T$ and $t$ differ only by a constant; that may be accomplished locally but not necessarily globally in this system. This issue will be further addressed for the uncharged case in [5]. If the correct energy for $T$ is not $E$, then it must be some constant of motion invariant under time translations (as $P_T$ commutes with itself); this restricts it to be some homogeneous function of $E$, $M$, and $Q$.

The second assumption is that a Wheeler-DeWitt equation based on the super-Hamiltonian is the correct method of quantization; this is probably not completely true and is the naive aspect of this model.
The third assumption is that, based on the equivalence of the equations of motion with those of a scalar particle, the inner product (12) gives the correct probability interpretation of the wavefunctions.

The restriction of my interpretation to measurements made by an observer on the interior is necessary. One might attempt to apply this model to observations made from $\infty$; if a distant observer reflects light off of the shell (to be precise a pulse is sent from some $I^-$ and received at some $I^+$, possibly in a different asymptotically flat region), then the appropriate coordinates are the Kruskal coordinates of the shell in the exterior region. Unfortunately, the transformation to these coordinates from $T$ and $R$ depends upon the whole trajectory of the shell; there is no obvious quantum equivalent. A model analogous to this one, but based upon an outside observer appears intractable. The first step to constructing it would be to write $E$ as a function of $R$ and $\frac{dR}{dt}$, but this is the solution of a quartic equation and is rather unwieldy.

This restriction to interior observers means that this model cannot directly deal with the interesting questions of when (or rather with what amplitude) the shell will pass beyond an event horizon and remerge in a different asymptotically flat region. This can only be addressed crudely by comparing quantum states with classical trajectories. A typical wavepacket will be a combination of definite energy states whose classical equivalents do and do not contain event horizons. This combination can be crudely interpreted to give the probability of an event horizon occurring.

This model has in some sense addressed the issue of whether singularities occur. When $R < 0$, the corresponding classical space-time will include a singularity at $r = 0$. However, the dynamics of this model are only mildly singular at $R = 0$, mildly enough that the solution could be simply continued across that singularity. This might be taken as a hopeful sign that the dynamics of a full quantum theory of gravitation will be singularity free.

8 Appendix

Proposition: The restriction to positive energies makes the inner product (12) positive definite.

To see this consider a state $\Psi = e^{-iET}\psi(R)$ with definite energy $E > 0$ (it is straightforward to show that different energy states are orthogonal). We have

$$\langle \Psi | \Psi \rangle = \int_{-\infty}^{+\infty} \left( E + \frac{k^2}{2R} \right) |\psi|^2 dR.$$  \hspace{1cm} (38)

For scattering states this must be positive as $|R| \approx \infty$ contributions dominate. For bound states $\psi(R \rightarrow \pm \infty) \longrightarrow 0$, hence for $a > 0$,

$$0 = \left[ (\bar{\psi}(aR))'\psi(R) - \bar{\psi}(aR)\psi'(R) \right]_{-\infty}^{+\infty}$$
\[ \int_{-\infty}^{+\infty} \left[ (\bar{\psi}(aR))^\prime \psi(R) - \bar{\psi}(aR) \psi''(R) \right] dR = \int_{-\infty}^{+\infty} \left[ (E + \frac{k}{2R})^2 - (aE + \frac{k}{2R})^2 - (1 - a^2) M^2 \right] \bar{\psi}(aR) \psi(R) dR \]

so for \( a \neq 1 \)
\[ 0 = \int_{-\infty}^{+\infty} \left[ E \left[ 1 + a E \frac{k}{R} \right] - (1 + a) M^2 \right] \bar{\psi}(aR) \psi(R) dR, \]

This integrand diverges at \( R \to 0 \) as \( R^{-\alpha} \) (recall \( |\alpha| < 1 \)) and approaches 0 exponentially as \( R \to \pm \infty \). As a result, this improper integral may be uniformly approximated to arbitrary accuracy by a proper integral and is therefore a continuous function of \( a \) and so
\[ \langle \Psi | \Psi \rangle = \frac{M^2}{E} \int_{-\infty}^{+\infty} |\psi(R)|^2 dR > 0 \]
which is manifestly positive. Similarly this is clearly negative for \( E < 0 \); hence restriction to positive energy is both necessary and sufficient for the inner product to be positive definite.

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