ON THE EXACT DIMENSION OF MANDELBROT MEASURE

BY

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Abstract. We develop, in the context of the boundary of a supercritical Galton–Watson tree, a uniform version of the argument used by Kahane (1987) on homogeneous trees to estimate almost surely and simultaneously the Hausdorff and packing dimensions of the Mandelbrot measure over a suitable set \( J \). As an application, we compute, almost surely and simultaneously, the Hausdorff and packing dimensions of the level sets \( E(\alpha) \) of infinite branches of the boundary of the tree along which the averages of the branching random walk have a given limit point.

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1. INTRODUCTION AND MAINS RESULTS

Let \((N, W_1, W_2, \ldots)\) be a random vector taking values in \( \mathbb{N}_+ \times \mathbb{R}_+^{\mathbb{N}_+} \). Then consider \( \{(N_{u0}, W_{u1}, W_{u2}, \ldots)\}_{u \in \bigcup_{n \geq 0} \mathbb{N}_+^n} \), a family of independent copies of this random vector indexed by the finite sequences \( u = u_1 \ldots u_n, n \geq 0, u_i \in \mathbb{N}_+ \) \((n = 0 \) corresponds to the empty sequence denoted by \( \emptyset \)). Let \( T \) be the Galton–Watson tree with defining element \( \{N_u\} \); we have \( \emptyset \in T \), and if \( u \in T \) and \( i \in \mathbb{N}_+ \), then \( u_i \), the concatenation of \( u \) and \( i \), belongs to \( T \) if and only if \( 1 \leq i \leq N_u \). Similarly, for each \( u \in \bigcup_{n \geq 0} \mathbb{N}_+^n \), denote by \( T(u) \) the Galton–Watson tree rooted at \( u \) and defined by \( \{N_{uv}\}, v \in \bigcup_{n \geq 0} \mathbb{N}_+^n \).

For each \( u \in \bigcup_{n \geq 0} \mathbb{N}_+^n \) we denote by \( |u| \) its length, i.e. the number of letters of \( u \), and by \( [u] \) the cylinder \( u \cdot \mathbb{N}_+^{\mathbb{N}_+^*} \), i.e. the set of \( t \in \mathbb{N}_+^{\mathbb{N}_+^*} \) such that \( t_1 t_2 \ldots t_{|u|} = u \). If \( t \in \mathbb{N}_+^{\mathbb{N}_+^*} \), we put \( |t| = \infty \), and the set of prefixes of \( t \) consists of \( \{\emptyset\} \cup \{t_1 t_2 \ldots t_n : n \geq 1\} \cup \{t\} \). Also we set \( t_{|[n]} = t_1 \ldots t_n \) if \( n \geq 1 \) and \( t_{|0} = \emptyset \).

The probability space over which the previous random variables are built is denoted by \((\Omega, \mathcal{A}, \mathbb{P})\), and the expectation with respect to \( \mathbb{P} \) is denoted by \( \mathbb{E} \).

We assume that \( \mathbb{E}(N) > 1 \) so that the Galton–Watson tree is supercritical. Without loss of generality, we also assume that the probability of extinction equals zero, so that \( \mathbb{P}(N \geq 1) = 1 \).
The boundary of $T$ is the subset of $N^N_+$ defined as
\[ \partial T = \bigcap_{n \geq 1} \bigcup_{u \in T_n} [u], \]
where $T_n = T \cap N^n_+$. The set $N^N_+$ is endowed with the standard ultrametric distance
\[ d_1 : (s, t) \mapsto \exp(-|s \wedge t|), \]
where $s \wedge t$ stands for the longest common prefix of $s$ and $t$, and with the convention that $\exp(-\infty) = 0$. The set $\partial T$ endowed with the induced distance is almost surely (a.s.) compact.

For the sake of simplicity we will assume throughout that the logarithmic moment generating function
\[ \tau(q) = \log \mathbb{E}\left( \sum_{i=1}^{N} W_i^q \right) \]
is finite over $\mathbb{R}$. Then, we define, for $u \in \bigcup_{n \geq 0} N^N_+$, the random variable
\[ W_{q,u} = \frac{W_n^q}{\mathbb{E}\left( \sum_{i=1}^{N} W_i^q \right)} = W_n^q e^{-\tau(q)}. \]
Consider the set
\[ J = \{ q \in \mathbb{R} : \tau(q) - q\tau'(q) > 0 \} = \{ q \in \mathbb{R} : \tau^*(\tau'(q)) > 0 \}, \]
where $\tau^*$ is the Legendre transform of the function $\tau$ defined, for all $\alpha \in \mathbb{R}$, as
\[ \tau^*(\alpha) = \inf_{q \in \mathbb{R}} (\tau(q) - q\alpha). \]
Let
\[ \Omega_\gamma^1 = \text{int} \{ q : \mathbb{E}[\left| \sum_{i=1}^{N} W_i^q \right|] < \infty \}, \quad \Omega^1 = \bigcup_{\gamma \in (1,2]} \Omega_\gamma^1 \quad \text{and} \quad J = J \cap \Omega^1. \]
Then, for $n \geq 1$ and $u \in N^n_+$, we define the sequence $(Y_p(q,u))_{p \geq 1}$ as
\[ Y_p(q,u) = \sum_{v \in T_p(u)} \prod_{k=1}^{n} W_{q,u,v_1 \ldots v_k}; \]
when $u = \emptyset$, this quantity will be denoted by $Y_n(q)$, and when $n = 0$, its value equals one.
Since, for all $q \in \mathcal{J}$, we have
\[
\begin{align*}
&\mathbb{E}\left( \sum_{i=1}^{N} W_{q,i} \right) = 1, \\
&\mathbb{E}\left( \sum_{i=1}^{N} W_{q,i} E \log W_{q,i} \right) = q \tau'(q) - \tau(q) < 0, \\
&\mathbb{E}\left( \left( \sum_{i=1}^{N} W_{q,i} \right) \log^+ \left( \sum_{i=1}^{N} W_{q,i} \right) \right) < \infty,
\end{align*}
\]
it follows that $(Y_p(q, u))$ converges to a positive limit $Y(q, u)$ with probability one, while the limit exists and vanishes if the condition is violated. This fact was proven by Kahane in [14] when $N$ is constant and by Biggins in [5] in general. Then, we can associate the Mandelbrot measure defined on the $\sigma$-field $\mathcal{C}$ generated by the cylinders of $\mathbb{N}_+$ as
\[
\mu_q([u]) = \begin{cases} 
W_{q,u_1} W_{q,u_2} \cdots W_{q,u_1 \cdots u_n} Y(q, u) & \text{if } u \in T_n, \\
0 & \text{otherwise},
\end{cases}
\]
and supported on $\partial T$. Moreover, under the property $E(Y(q) \log^+ Y(q)) < \infty$, hence in particular when $E(Y(q)^h) < \infty$ for some $h > 1$, where $Y(q) = Y(q, \emptyset)$, we have, following [14], [16], [4], for all $q \in \mathcal{J}$, a.s., for $\mu_q$-almost every $t \in \partial T$,
\[
\liminf_{n \to \infty} \frac{\log \mu_q([t_n])}{-n} \geq \tau(q) - q \tau'(q).
\]
Hence, for all $q \in \mathcal{J}$, a.s., the lower Hausdorff dimension of $\mu_q$ is
\[
\dim \mu_q \geq \tau(q) - q \tau'(q),
\]
see Section 6 for the definition.

The Mandelbrot measure $\mu_q$ is naturally considered when studying the multifractal analysis of some random sets (see [10], [19], [1]–[3], [7]). By exploiting the simultaneous construction of the Mandelbrot measure $\mu_q$, $q \in \mathcal{J}$, and using a uniform version of the argument applied by Kahane in [13] on homogeneous trees, we get the following result.

**Theorem 1.1.** With probability one, for all $q \in \mathcal{J}$, $\dim \mu_q \geq \tau(q) - q \tau'(q)$.

As an application we study, for $q \in \mathcal{J}$, the set $E(\tau'(q))$ associated with the branching random walk with $(X_i = \log(W_i))_{1 \leq i \leq N}$ (see Section 4). Since, with probability one, for all $q \in \mathcal{J}$, the set $E(\tau'(q))$ is supported by $\mu_q$ and its packing dimension is smaller than $\tau^*(\tau'(q))$ (see Proposition 2.7 in [2]), we get
\[
as, \forall q \in \mathcal{J}, \quad \overline{\dim} \mu_q \leq \tau(q) - q \tau'(q),
\]
where $\overline{\dim} \mu_q$ is the upper packing dimension of $\mu_q$ (see Section 6 for the definition). As a consequence, we infer that the measures are exact dimensional.
COROLLARY 1.1. With probability one, for all \( q \in J \),
\[
\dim \mu_q = \text{Dim} \mu_q = \tau(q) - q\tau'(q),
\]
where \( \dim \mu_q \) and \( \text{Dim} \mu_q \) denote the Hausdorff and packing dimensions of \( \mu_q \), respectively.

REMARK 1.1. These results are known (see [1], [3]). Using a uniform version of a percolation argument, we will give a new proof of the sharp lower bounds for the lower Hausdorff dimension of these measures.

2. PRELIMINARIES

Given an increasing sequence \( \{A_n\}_{n \geq 1} \) of sub-\( \sigma \)-fields of \( \mathcal{A} \) and a sequence of random functions \( \{P_n(t, \omega)\}_{n \geq 1} \) such that
\[
1. P_n(t) = P_n(t, \omega) \text{ are non-negative and independent processes; } P_n(\cdot, \omega) \text{ is }
\]
Borelian for almost all \( \omega \);
\[
2. \mathbb{E}(P_n(t)) = 1 \text{ for all } t \in \partial T.
\]
Such a sequence \( \{P_n\} \) is called a sequence of weights adopted to \( \{A_n\} \). Let
\[
Q_n(t) = Q_n(t, \omega) = \prod_{k=1}^{n} P_k(t, \omega).
\]

For any \( n \geq 1 \) and any positive Radon measure \( \sigma \) on \( \partial T \) (we write \( \sigma \in \mathcal{M}^+(\partial T) \)), we consider the random measures \( Q_n\sigma \) defined as
\[
Q_n\sigma(A) = \int_A Q_n(t) d\sigma(t) \quad (A \in \mathcal{B}(\partial T)),
\]
where \( \mathcal{B}(\partial T) \) is the Borel field on \( \partial T \). For all \( A \in \mathcal{B}(\partial T) \), \( Q_n\sigma(A) \) is a positive martingale so it converges almost surely. Also, for all \( \sigma \in \mathcal{M}_+^+(\partial T) \), the random measure \( Q_n\sigma \) converges weakly, almost surely, to the random measure \( Q\sigma \).

There are two possible extreme cases. The first one is that \( Q_n\sigma(\partial T) \) converges almost surely to zero, i.e. \( Q\sigma = 0 \) a.s. In this case, we say that \( Q \) degenerates on \( \sigma \) or \( \sigma \) is said to be \( Q \)-singular. The second one is that \( Q_n\sigma(\partial T) \) converges in \( L^1 \) so that \( \mathbb{E}(Q_n(\sigma)(\partial T)) = \sigma(\partial T) \). In this case we say that \( Q \) fully acts on \( \sigma \) or \( \sigma \) is said to be \( Q \)-regular.

THEOREM 2.1. Let \( \alpha \) be a positive number such that \( \mathcal{H}^\alpha(\partial T) < \infty \), where \( \mathcal{H}^\alpha \) denotes the \( \alpha \)-dimensional Hausdorff measure. Let \( 0 < h < 1 \) and \( C > 0 \). Suppose
\[
(2.1) \quad \sup_{t \in \bar{B}} \left( Q_n(t)^h \right) \leq C |B|^\left(1-h\right)\alpha
\]
for all balls \( B \) and some \( n = n(B) \) depending on \( B \). Then \( Q \) is completely degenerate, that is, \( Q\sigma = 0 \) a.s. for all \( \sigma \in \mathcal{M}^+(\partial T) \).
This provides a good tool to verify the $Q$-singularity of $\sigma$. Indeed, if a measure is not killed, it means that it has a lower Hausdorff dimension at least $\alpha$.

3. PROOF OF THEOREM 1.1

For each $\beta \in (0, 1]$, let $W_\beta$ be a random variable taking the value $1/\beta$ with probability $\beta$ and the value 0 with probability $1-\beta$. Then, let $\{W_{\beta,u}\}_{u \in \bigcup_{n \geq 0} \mathbb{N}_+^n}$ be a family of independent copies of $W_\beta$. Denote by $(\Omega_\beta, \mathcal{A}_\beta, \mathbb{P}_\beta)$ the probability space on which this family is defined.

We naturally extend to $(\Omega_\beta \times \Omega, \mathcal{A}_\beta \otimes \mathcal{A}, \mathbb{P}_\beta \otimes \mathbb{P})$ the random variables $W_{\beta,u}$ and the random vectors $(N_{u0}, W_{u1}, \ldots)$ as

$$W_{\beta,u}(\omega_\beta, \omega) = W_{\beta,u}(\omega)$$

and

$$(N_{u0}(\omega_\beta, \omega), W_{u1}(\omega_\beta, \omega), \ldots) = (N_{u0}(\omega), W_{u1}(\omega), \ldots),$$

so that the families $\{W_{\beta,u}\}_{u \in \bigcup_{n \geq 0} \mathbb{N}_+^n}$ and $\{(N_{u0}, W_{u1}, \ldots)\}_{u \in \bigcup_{n \geq 0} \mathbb{N}_+^n}$ are independent.

The expectation with respect to $\mathbb{P}_\beta \otimes \mathbb{P}$ will also be denoted by $\mathbb{E}$. For $n \geq 1$ and $\beta \in (0, 1]$, we set $F_n = \sigma((N_u, W_{u1}, W_{u2}, \ldots) : u \in \bigcup_{k=0}^n \mathbb{N}_+^{k-1})$ and $F_{\beta,n} = \sigma((W_{\beta,u_1}, W_{\beta,u_2}, \ldots) : u \in \bigcup_{k=0}^n \mathbb{N}_+^{k-1})$. We denote by $F_0$ and $F_{\beta,0}$ the trivial $\sigma$-field.

If $\beta \mathbb{E}(N) > 1$, the random variables

$$N_{\beta,u}(\omega_\beta, \omega) = \sum_{i=1}^{N_u(\omega)} \mathbf{1}_{\{\beta-1\}}(W_{\beta,ui}(\omega_\beta))$$

define a new supercritical Galton–Watson process with which the trees $T_{\beta,n} \subset T_n$ and $T_{\beta,n}(u) \subset T_n(u)$, $u \in \bigcup_{n \geq 0} \mathbb{N}_+^n$, $n \geq 1$, are associated, as well as the infinite tree $T_{\beta} \subset T$ and the boundary $\partial T_{\beta} \subset \partial T$ conditional on non-extinction.

For $u \in \bigcup_{n \geq 0} \mathbb{N}_+^n$, $1 \leq i \leq N(u)$, and $q \in J$ we define

$$W_{\beta,q,ui} = W_{\beta,ui}W_{q,ui}.$$

For $q \in J$, $\beta \mathbb{E}(N) > 1$, $n \geq 0$ and $u \in \bigcup_{n \geq 0} \mathbb{N}_+^n$, we define

$$Y_n(\beta, q, u) = \sum_{v_1 \ldots v_n \in T_n(u)} \prod_{k=1}^n W_{\beta,q,u,v_1 \ldots v_n}.$$

When $u = 0$, this quantity will be denoted by $Y_n(\beta, q)$, and when $n = 0$, its value equals one.
We set
\[ J_{\beta,\epsilon} = \{ q \in J : \tau^*(\tau'(q)) > -\log \beta + \epsilon \}. \]
Notice that \( \tau^*(\tau'(q)) \) takes values between zero and \( \tau(0) = \log (E(N)) \) over \( J \).

Then
\[ J = \bigcup_{\beta \in (E(N)^{-1}, 1], \epsilon > 0} J_{\beta,\epsilon}. \]

The following propositions will be established in Section 3.

**Proposition 3.1.** (1) For all \( u \in \bigcup_{n \geq 0} \mathbb{N}_+^n \), the sequence of continuous functions \( Y_n(\cdot, u) \) converges uniformly, almost surely and in \( L^1 \) norm, to a positive limit \( Y(\cdot, u) \) on \( J \).

(2) With probability one, for all \( q \in J \), the mapping
\[ (3.2) \quad \mu_q([u]) = \left( \prod_{k=1}^n W_{q,u_1...u_k} \right) Y(q,u) \]
defines a positive measure on \( \partial T \).

**Proposition 3.2.** Let \( \beta \in (0, 1] \) such that \( \beta E(N) > 1 \). Then, for all \( \epsilon \in \mathbb{Q}_+^* \):

(1) the sequence of continuous functions \( Y_n(\beta, \cdot) \) converges uniformly, almost surely and in \( L^1 \) norm, to a positive limit \( Y(\beta, \cdot) \) on \( J_{\beta,\epsilon} \);

(2) the sequence of continuous functions
\[ q \mapsto \tilde{Y}_n(\beta, q) = \sum_{u \in T_n} \left( \prod_{k=1}^n W_{\beta,u_1...u_k} \right) \mu_q([u]) \]
converges uniformly, almost surely and in \( L^1 \) norm, to\( Y(\beta, \cdot) \) on \( J_{\beta,\epsilon} \).

**3.2. Proof of Theorem 1.1.** Let \( \epsilon \in \mathbb{Q}_+^* \) and \( \beta \in (0, 1] \) such that \( \beta E(N) > 1 \). For every \( t \in \partial T \) and \( \omega_\beta \in \Omega_\beta \) set
\[ Q_{\beta,n}(t, \omega_\beta) = \prod_{k=1}^n W_{\beta,t_k}, \]
so that for \( q \in J_{\beta,\epsilon} \), \( \tilde{Y}_n(\beta, q) \) is the total mass of the measure \( Q_{\beta,n}(t, \omega_\beta) \cdot d\mu_q^\epsilon(t) \).

Now, Proposition [5,2] claims that there exists a measurable subset \( A \) of \( \Omega \times \Omega_\beta \) of full probability in the set of those \( (\omega, \omega_\beta) \) such that \( (T_{\beta,n})_{n \geq 1} \) survives and for all \( (\omega, \omega_\beta) \in A \), for all \( q \in J_{\beta,\epsilon} \), \( \tilde{Y}_n(\beta, q) \) does not converge to zero. Moreover, since the branching number of the tree \( T \) is \( \mathbb{P} \)-almost surely equal to the constant \( \mathbb{E}(N) \) and \( \beta \mathbb{E}(N) > 1 \), conditional on \( T \), the \( \mathbb{P}_\beta \)-probability of non-extinction of \( (T_{\beta,n})_{n \geq 1} \) is positive ([12], Theorem 6.2). Thus, the projection of \( A \) to \( \Omega \) has
The probability one and there exists a measurable subset $\Omega(\beta, \epsilon)$ of $\Omega$ such that $\mathbb{P}(\Omega(\beta, \epsilon)) = 1$ and for all $\omega \in \Omega(\beta, \epsilon)$, there exists $\Omega^\beta_{\omega} \subset \Omega^\beta_{\beta}$ of positive probability such that for all $\omega \in \Omega(\beta, \epsilon)$, for all $q \in \mathcal{J}_{\beta, \epsilon}$, for all $\omega_\beta \in \Omega^\beta_{\beta}$, $\hat{Y}_n(\beta, q)$ does not converge to zero. In terms of the multiplicative chaos theory developed in [12], this means that for all $\omega \in \Omega(\beta, \epsilon)$ and $q \in \mathcal{J}_{\beta, \epsilon}$, the set of those $\omega_\beta$ such that the multiplicative chaos $(Q_{\beta, n}(\cdot, \omega))_{n \geq 1}$ has not killed $\mu_q$ on the compact set $\partial T$ has a positive $\mathbb{P}_\beta$-probability. Now, the good property of $(Q_{\beta, n}(\cdot, \omega))_{n \geq 1}$ is

$$
\mathbb{E}_\beta \left( \sup_{t \in B} (Q_{\beta, n}(t))^h \right) = e^{n(1-h)\log(\beta)} = (|B|)^{(1-h)\log(\beta)}
$$

for any $h \in (0, 1)$ and any ball $B$ of generation $n$ in $\partial T$, where $|B|$ stands for the diameter of $B$ and $\mathbb{E}_\beta$ stands for the expectation with respect to $\mathbb{P}_\beta$. Thus, we can apply Theorem 3 of [12] and claim that for all $\omega \in \Omega(\beta, \epsilon)$ and all $q \in \mathcal{J}_{\beta, \epsilon}$, no piece of $\mu_q$ is carried by a Borel set of Hausdorff dimension less than $-\log(\beta)$.

Let $\Omega' = \bigcap_{\beta \in \mathbb{E}(\mathbb{N})^{-1}, 1} |Q_{\beta, \epsilon}|_{\epsilon \in Q_{\beta, \epsilon}^*} \Omega(\beta, \epsilon)$. This set is of $\mathbb{P}$-probability one. Let $q \in \mathcal{J}$. By (3.1), there exists a sequence of points $(\beta_n, \epsilon_n) \in (\mathbb{E}(\mathbb{N})^{-1}, 1) \times Q_{\beta, \epsilon}^*$ such that $\tau(q) - q\tau'(q) > -\log(\beta_n) + \epsilon_n/2$ for all $n \geq 1$, $\lim_{n \to \infty} -\log(\beta_n) = \tau(q) - q\tau'(q)$, $\lim_{n \to \infty} \epsilon_n = 0$ and $q \in \bigcap_{n \geq 1} \mathcal{J}_{\beta_n, \epsilon_n}$. Consequently, the previous paragraph implies that for all $\omega \in \Omega'$,

$$\dim(\mu_n^\omega) \geq \limsup_{n \to \infty} -\log(\beta_n) = \tau(q) - q\tau'(q).$$

4. APPLICATION

Let $(N, X_1, X_2, \ldots)$ be a random vector taking values in $\mathbb{N}_+ \times (\mathbb{R})^{\mathbb{N}_+}$. Then consider $\{((N_u, X_{u_1}, X_{u_2}, \ldots))_{u \in \mathbb{N}_+}\}$ a family of independent copies of the vector $(N, X_1, X_2, \ldots)$ indexed by the set of finite words over the alphabet $\mathbb{N}_+$. We assume that $\mathbb{E}(N) > 1$ and $\mathbb{P}(N \geq 1) = 1$. Suppose that, for all $u \in T$, $X_u$ is integrable and the sequences $(X_u)_{u \in \mathbb{N}_+}$ are i.i.d. Given $t \in \partial T$, by the strong law of large numbers, we have $\lim_{n \to \infty} n^{-1} S_n(t) = \mathbb{E}(X_1)$ almost surely, where $S_n(t) = \sum_{k=1}^n X_{u_{1}\ldots u_k}$. Since $\partial T$ is not countable, the following question naturally arises: are there some $t \in \partial T$ so that $\lim_{n \to \infty} n^{-1} S_n(t) = \alpha \neq \mathbb{E}(X_1)$? Multifractal analysis is a framework adapted to answer this question. Consider the set $\mathcal{I}$ of those $\alpha \in \mathbb{R}$ such that

$$E(\alpha) = \left\{ t \in \partial T : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n X_{u_{1}\ldots u_k} = \alpha \right\} \neq \emptyset.$$

These level sets can be described geometrically through their Hausdorff dimensions. They have been studied by many authors, see [11], [12], [10]–[3], [17]; all these papers also deal with the multifractal analysis of associated Mandelbrot measure (see also [14], [21], [16] for the study of Mandelbrot measures dimension).
Take, for \( u \in \bigcup_{n \geq 0} \mathbb{N}_+ \), the random variable \( W_u = e^{X_u} \) and set
\[
I = \{ \tau'(q); \ q \in \mathcal{J} \}.
\]

**Theorem 4.1.** With probability one, for all \( \alpha \in I \), the multifractal formalism holds at \( \alpha \), i.e.,
\[
\dim E(\alpha) = \text{Dim} E(\alpha) = \tau^*(\alpha);
\]
in particular, \( E(\alpha) \neq \emptyset \).

**Proof.** A simple covering argument yields, with probability one, for all \( \alpha \in I \), \( \text{Dim} E(\alpha) \leq \tau^*(\alpha) \) (see, for example, Proposition 2.7 in [2]). In addition, consider the Mandelbrot measure \( \mu_q, q \in \mathcal{J} \), defined by (1.1). It is known (see, for example, Corollary 2.5 in [1]) that with probability one, \( \mu_q(\tau'(q)) = 1 \). In addition, according to Theorem 1.1, we have, with probability one, for all \( q \in \mathcal{J}, \dim \mu_q \geq \tau(q) - q\tau'(q) \). We deduce the result from the mass distribution principle (Theorem 1.2 below).

**Remark 4.1.** This result has been proved in [3] when \( N \) is not random, and in the weaker form, for each fixed \( \alpha \in I \), almost surely \( \dim E(\alpha) = \tau^*(\alpha) \) in [10], [19], [7], when \( N \) is random.

**Remark 4.2.** Using the Cauchy formula, we can prove Theorem 1.1 (see [1]). Then our result gives a new approach to estimate, almost surely and simultaneously, the lower Hausdorff dimension of the Mandelbrot measure over \( \mathcal{J} \).

5. **Proof of Propositions 3.1 and 3.2**

Define, for \( (q, p, \beta) \in \mathcal{J} \times [1, \infty) \times (0, 1] \), the function
\[
\varphi_{\beta}(p, q) = \exp \left( \tau(pq) - p\tau(q) + (1 - p) \log \beta \right).
\]

**Lemma 5.1.** For all nontrivial compact \( K \subset \mathcal{J}_{\beta, \epsilon} \) there exists a real number \( 1 < p_K < 2 \) such that for all \( 1 < p \leq p_K \) we have
\[
\sup_{q \in K} \varphi_{\beta}(p_K, q) < 1.
\]

**Proof.** Let \( q \in \mathcal{J}_{\beta, \epsilon}; \) we have \( \frac{\partial \varphi_{\beta}}{\partial p}(1^+, q) < 0 \) and there exists \( p_q > 1 \) such that \( \varphi_{\beta}(p_q, q) < 1 \). Therefore, in a neighborhood \( V_q \) of \( q \), we have \( \varphi_{\beta}(p_q, q') < 1 \) for all \( q' \in V_q \). If \( K \) is a nontrivial compact of \( \mathcal{J}_{\beta, \epsilon} \), it is covered by a finite number of such \( V_q \). Let \( p_K = \inf_{q \in K} p_q \). If \( 1 < p \leq p_K \) and \( \sup_{q \in K} \varphi_{\beta}(p, q) \geq 1 \), there exists \( q \in K \) such that \( \varphi_{\beta}(p, q) \geq 1 \), and \( q \in V_{q_i} \) for some \( i \). By log-convexity of the mapping \( p \mapsto \varphi_{\beta}(p, q) \) and the fact that \( \varphi_{\beta}(1, q) = 1 \), since \( 1 < p \leq p_{q_i} \), we have \( \varphi_{\beta}(p, q) < 1 \), which is a contradiction.  \( \blacksquare \)
LEMMA 5.2. For all compact $K \subset \mathcal{J}$, there exists $\tilde{p}_K > 1$ such that
\[
\sup_{q \in K} \mathbb{E}\left( (\sum_{i=1}^{N} W_i^q)^{\tilde{p}_K} \right) < \infty.
\]

Proof. Since $K$ is compact and the family of open sets $J \cap \Omega_\gamma$ increases to $\mathcal{J}$ as $\gamma$ decreases to one, there exists $\gamma \in (1, 2]$ such that $K \subset \Omega_\gamma$. Take $\tilde{p}_K = \gamma$. The conclusion comes from the fact that the function $q \mapsto \mathbb{E}\left( (\sum_{i=1}^{N} W_i^q)^{\tilde{p}_K} \right)$ is continuous over $\Omega_\gamma$.

LEMMA 5.3 (Biggins [6]). If $\{X_i\}$ is a family of integrable and independent complex random variables with $\mathbb{E}(X_i) = 0$, then $\mathbb{E}\left| \sum_{i=1}^{N} X_i \right|^p \leq 2^p \sum_{i=1}^{N} \mathbb{E}|X_i|^p$ for $1 \leq p \leq 2$.

The same lines as in Lemma 2.11 in [1], we get the following lemma.

LEMMA 5.4. Let $(N, V_1, V_2, \ldots)$ be a random vector taking values in $\mathbb{N}^+ \times \mathbb{C}^{N}$ and such that $\sum_{i=1}^{N} V_i$ is integrable and $\mathbb{E}\left( \sum_{i=1}^{N} V_i \right) = 1$. Consider a sequence $\{(N_u, V_{u1}, V_{u2}, \ldots)\}_{u \in \mathbb{N}_0}$ of independent copies of $(N, V_1, \ldots, V_N)$. We define the sequence $(Z_n)_{n \geq 0}$ by $Z_0 = 1$ and for $n \geq 1$
\[
Z_n = \sum_{u \in T_n} \left( \prod_{k=1}^{n} V_{u|_k} \right).
\]

Let $p \in (1, 2]$. There exists a constant $C_p$ depending on $p$ only such that for all $n \geq 1$,
\[
\mathbb{E}(|Z_n - Z_{n-1}|^p) \leq C_p \left( \mathbb{E}\left( \sum_{i=1}^{N} |V_i|^p \right) \right)^{n-1} \left( \mathbb{E}\left( \sum_{i=1}^{N} |V_i|^p \right) + 1 \right).
\]

Proof of Proposition 5.3. (1) Recall that the uniform convergence result uses an argument developed in [6]. Fix a compact $K \subset \mathcal{J}_{\beta, \epsilon}$. By Lemma 5.2 we can fix a compact neighborhood $K'$ of $K$ and $\tilde{p}_{K'} > 1$ such that
\[
\sup_{q \in K'} \mathbb{E}\left( (\sum_{i=1}^{N} W_i^q)^{\tilde{p}_{K'}} \right) < \infty.
\]

By Lemma 5.1 we can fix $1 < p_K \leq \min(2, \tilde{p}_{K'})$ such that $\sup_{q \in K} \varphi_{\beta}(p_K, q) < 1$. Then for each $q \in K$, there exists a neighborhood $V_q \subset \mathbb{C}$ of $q$ whose projection to $\mathbb{R}$ is contained in $K'$ and such that for all $u \in T$ and $z \in V_q$, the random variable
\[
W_{\beta, z, u} = W_{\beta, u} \frac{e^{z \log W_u}}{\mathbb{E}\left( \sum_{i=1}^{N} e^{z \log W_i} \right)}
\]
is well defined, and we have
\[ \sup_{z \in V} \varphi_{\beta}(p_K, z) < 1, \]
where for all \( z \in \mathbb{C} \)
\[ \varphi_{\beta}(p_K, z) = \beta^{1-p_K} E \left( \sum_{i=1}^{N} e^{z \log W_i} |p_K| \right) \left| E \left( \sum_{i=1}^{N} e^{z \log W_i} \right)^{-p_K} \right. \]

By extracting a finite covering of \( K \) from \( \bigcup_{q \in K} V_q \), we find a neighborhood \( V \subset \mathbb{C} \) of \( K \) such that \( \sup_{z \in V} \varphi_{\beta}(p_K, z) < 1 \). Since the projection of \( V \) to \( \mathbb{R} \) is included in \( K' \) and the mapping \( z \mapsto E \left( \sum_{i=1}^{N} e^{z \log W_i} \right) \) is continuous and does not vanish on \( V \), by considering a smaller neighborhood of \( K \) included in \( V \) if necessary, we can assume that
\[ A_V = \sup_{z \in V} E \left( \left| \sum_{i=1}^{N} e^{z \log W_i} |p_K| \right| \left| \sum_{i=1}^{N} e^{z \log W_i} \right|^{-p_K} \right) + 1 < \infty. \]

Now, for \( u \in T \), we define the analytic extension of \( Y_n(\beta, q, u) \) to \( V \) given by
\[ Y_n(\beta, z, u) = \sum_{v \in T_n(u)} \prod_{k=1}^{n} W_{\beta, z, u v_1 \ldots v_k}. \]
We denote also \( Y_n(\beta, z, \emptyset) \) by \( Y_n(\beta, z) \). Now, applying Lemma 5.4 with \( V_i = W_{\beta, z, i} \), we obtain
\[ E \left( \left| Y_n(\beta, z) - Y_{n-1}(\beta, z) \right|^{|p_K|} \right) \leq C_{p_K} \left( E \left( \left| \sum_{i=1}^{N} V_i \right|^{|p_K|} \right) + 1 \right)^{n-1}. \]
Notice that \( E \left( \sum_{i=1}^{N} |V_i|^{p_K} \right) = \varphi_{\beta}(p_K, z) \). Then,
\[ E \left( \left| Y_n(\beta, z) - Y_{n-1}(\beta, z) \right|^{|p_K|} \right) \leq C_{p_K} A_V \sup_{z \in V} \varphi(p_K, z)^{n-1}. \]
With probability one, the functions \( z \in V \mapsto Y_n(\beta, z), n \geq 0, \) are analytic. Fix a closed disc \( D(z_0, 2\rho) \subset V \). Theorem 6.1 below implies
\[ \sup_{z \in D(z_0, \rho)} |Y_n(\beta, z) - Y_{n-1}(\beta, z)| \leq 2 \int_{[0,1]} |Y_n(\beta, \zeta(\theta)) - Y_{n-1}(\beta, \zeta(\theta))| d\theta, \]
where, for \( \theta \in [0, 1], \ \zeta(\theta) = z_0 + 2\rho e^{i2\pi \theta}. \) Furthermore, Jensen’s inequality and
Fubini’s theorem give

$$\mathbb{E} \left( \sup_{z \in D(0, A_0, \rho)} |Y_n(\beta, z) - Y_{n-1}(\beta, z)|^{P_K} \right)$$

$$\leq \mathbb{E} \left( \int_{[0,1]} |Y_n(\beta, \zeta(\theta)) - Y_{n-1}(\beta, \zeta(\theta))|^{P_K} \, d\theta \right)$$

$$\leq 2^{P_K} \int_{[0,1]} \mathbb{E} \left( |Y_n(\beta, \zeta(\theta)) - Y_{n-1}(\beta, \zeta(\theta))|^{P_K} \, d\theta \right)$$

$$\leq 2^{P_K} \int_{[0,1]} \mathbb{E} |Y_n(\beta, \zeta(\theta)) - Y_{n-1}(\beta, \zeta(\theta))|^{P_K} \, d\theta$$

$$\leq 2^{P_K} \sup_{z \in V \setminus \{0\}} \mathbb{E} \left( \varphi_{\beta}(p_K, z)^{n-1} \right).$$

Since \( \sup_{z \in V} \varphi_{\beta}(p_K, z) < 1 \), it follows that

$$\sum_{n \geq 1} \mathbb{E} \left( \sup_{z \in D(0, A_0, \rho)} |Y_n(\beta, z) - Y_{n-1}(\beta, z)|^{P_K} \right) < \infty.$$

This implies that \( z \mapsto Y_n(\beta, z) \) converge uniformly, almost surely and in \( L^{P_K} \) norm over the compact \( D(0, A_0, \rho) \), to a limit \( z \mapsto Y(\beta, z) \). This also implies that

$$\mathbb{E} \left( \sup_{z \in D(0, A_0, \rho)} |Y(\beta, z)|^{P_K} \right) < \infty.$$

Since \( K \) can be covered by finitely many such discs \( D(0, A_0, \rho) \), we get the uniform convergence, almost surely and in \( L^{P_K} \) norm, of the sequence \( \{q \in K \mapsto Y_n(\beta, q)\}_{n \geq 1} \) to \( q \in K \mapsto Y(\beta, q) \). Moreover, since \( \mathcal{J}_{\beta, \sigma} \) can be covered by a countable union of such compact \( K \), we get the simultaneous convergence for all \( q \in \mathcal{J}_{\beta} \). The same holds simultaneously for all the functions \( q \in \mathcal{J}_{\beta} \mapsto Y_n(\beta, q, u) \), \( u \in \bigcup_{n \geq 0} \mathbb{N}^n \), because \( \bigcup_{n \geq 0} \mathbb{N}^n \) is countable.

To complete the proof of (1), we must show that a.s., \( q \in K \mapsto Y(\beta, q) \) does not vanish. Without loss of generality we suppose that \( K = [0, 1] \). If \( I \) is a dyadic closed subinterval of \([0, 1]\), we denote by \( E_I \) the event \( \{3 q \in I : Y(\beta, q) = 0\} \). Let \( I_0, I_1 \) stand for two dyadic subintervals of \( I \) in the next generation. The event \( E_I \) being a tail event of probability zero or one, if we suppose that \( P(E_I) = 1 \), there exists \( j \in \{0, 1\} \) such that \( P(E_{I_j}) = 1 \). Suppose now that \( P(E_{I_0}) = 1 \). The previous remark allows us to construct a decreasing sequence \( \{I(n)\}_{n \geq 0} \) of dyadic subintervals of \( K \) such that \( P(E_{I(n)}) = 1 \). Let \( q_0 \) be the unique element of \( \bigcap_{n \geq 0} I(n) \). Since \( q \mapsto Y(\beta, q) \) is continuous, we have \( P(Y(\beta, q_0) = 0) = 1 \), which contradicts the fact that \( (Y_n(\beta, q_0))_{n \geq 1} \) converges to \( Y(\beta, q_0) \) in \( L^1 \).

(2) Here we develop, in the context of the boundary of a supercritical Galton–Watson tree, a uniform version of the argument used by Kahane in [13] on homogeneous trees, and written in complete rigor in [24]. Fix \( \epsilon > 0 \) and a compact set...
\( K \) in \( \mathcal{F}_{\beta, e} \). Denote by \( E \) the separable Banach space of the real-valued continuous functions over \( K \) endowed with the supremum norm.

For \( n \geq m \geq 1 \) and \( q \in K \) let

\[
Z_{m,n}(\beta, q) = \sum_{u \in Y_m} Y_{n-m}(q, u) \prod_{k=1}^{m} W_{\beta,q,u_1 \cdots u_k}.
\]

Notice \( Z_{n,n}(\beta, q) = Y_n(\beta, q) \). Moreover, since \( Y_n(\beta, \cdot) \) converges almost surely and in \( L^1 \) norm to \( Y(\beta, \cdot) \) as \( n \to \infty \), \( Y_n(\beta, \cdot) \) belongs to \( L^1_E = L^1_E(\Omega \times \Omega, \mathcal{A}_\beta \times \mathcal{A}, \mathbb{P}_{\beta} \times \mathbb{P}) \) (where we use the notation of Section V-2 in \([20]\)), so that the continuous random function \( \mathbb{E}(Z_{n,n}(\beta, q)|\mathcal{F}_{\beta,m} \otimes \mathcal{F}_n) \) is well defined by Proposition V-2-5 in \([20]\); also, for any fixed \( q \in K \), we can deduce from the definitions and the independence assumptions that

\[
Z_{m,n}(\beta, q) = \mathbb{E}(Z_{n,n}(\beta, q)|\mathcal{F}_{\beta,m} \otimes \mathcal{F}_n)
\]

almost surely. By Proposition V-2-5 in \([20]\) again, since \( g \in E \mapsto q(g) \) is a continuous linear form over \( E \), we thus have

\[
Z_{m,n}(\beta, q) = \mathbb{E}(Z_{n,n}(\beta, \cdot)|\mathcal{F}_{\beta,m} \otimes \mathcal{F}_n)(q)
\]

almost surely. By considering a dense countable set of \( q \) in \( K \), we can conclude that the random continuous functions \( Z_{m,n}(\beta, \cdot) \) and \( \mathbb{E}(Z_{n,n}(\beta, \cdot)|\mathcal{F}_{\beta,m} \otimes \mathcal{F}_n) \) are equal almost surely.

Similarly, since for each \( q \in K \) the martingale \( (Y_n(\beta, q), \mathcal{F}_{\beta,n} \otimes \mathcal{F}_n) \) converges to \( Y(\beta, q) \) almost surely and in \( L^1 \), and \( Y(\beta, \cdot) \in L^1_E \), by using Proposition V-2-5 in \([20]\) again we can get almost surely

(5.1)

\[
Z_{n,n}(\beta, \cdot) = \mathbb{E}(Y(\beta, \cdot)|\mathcal{F}_{\beta,n} \otimes \mathcal{F}_n), \text{ hence } Z_{m,n}(\beta, \cdot) = \mathbb{E}(Y(\beta, \cdot)|\mathcal{F}_{\beta,m} \otimes \mathcal{F}_n).
\]

Moreover, it follows from Proposition 3.2(1) and the definition of \( \mu_q([u]) \) that \( Z_{m,n}(\beta, \cdot) \) converges almost surely uniformly and in \( L^1 \) norm, as \( n \to \infty \), to \( \widetilde{Y}_m(\beta, \cdot) \). This and (5.1) yield, by Proposition V-2-6 in \([20]\),

\[
\widetilde{Y}_m(\beta, \cdot) = \lim_{n \to \infty} Z_{m,n}(\beta, \cdot) = \mathbb{E}(Y(\beta, \cdot)|\mathcal{F}_{\beta,m} \otimes \sigma(\bigcup_{n \geq 1} \mathcal{F}_n))
\]

and finally

\[
\lim_{m \to \infty} \widetilde{Y}_m(\beta, \cdot) = \mathbb{E}(Y(\beta, \cdot)|\sigma(\bigcup_{m \geq 1} \mathcal{F}_{\beta,m}) \otimes \sigma(\bigcup_{n \geq 1} \mathcal{F}_n)) = Y(\beta, \cdot)
\]

almost surely (since, by construction, \( Y(\beta, \cdot) \) is \( \sigma(\bigcup_{m \geq 1} \mathcal{F}_{\beta,m}) \otimes \sigma(\bigcup_{n \geq 1} \mathcal{F}_n) \)-measurable), where the convergences hold in the uniform norm. Moreover, since \( \mathcal{F}_{\beta,e} \) can be covered by a countable union of such compact \( K \), we get the simultaneous convergence for all \( q \in \mathcal{F}_{\beta,e} \). \( \blacksquare \)
Proof of Proposition 3.1. The proof of the first point is similar to the proof of Proposition 3.2(1) \((\beta = 1)\). The second point is a consequence of the branching property:

\[ Y_{n+1}(q, u) = \sum_{i=1}^{N} W_{q, ui} Y_n(q, ui). \]

6. APPENDICES

APPENDIX 1 — CAUCHY FORMULA

Definition 6.1. Let \(D(\zeta, r)\) be a disc in \(\mathbb{C}\) with center \(\zeta\) and radius \(r\). The set \(\partial D\) is the boundary of \(D\). Let \(g \in C(\partial D)\) be a continuous function on \(\partial D\). We define the integral of \(g\) on \(\partial D\) as

\[ \int_{\partial D} g(\zeta) d\zeta = 2i\pi r \int_{[0,1]} g(\zeta(t)) e^{i2\pi t} dt, \]

where \(\zeta(t) = \zeta + re^{i2\pi t}\).

Theorem 6.1. Let \(D = D(a, r)\) be a disc in \(\mathbb{C}\) with radius \(r > 0\), and \(f\) be a holomorphic function in a neighborhood of \(D\). Then, for all \(z \in D\)

\[ f(z) = \frac{1}{2i\pi} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta. \]

It follows that

\[ \sup_{z \in D(a, r/2)} |f(z)| \leq 2 \int_{[0,1]} |f(\zeta(t))| dt. \]

APPENDIX 2 — MASS DISTRIBUTION PRINCIPLE

Theorem 6.2 (Falconer [9]). Let \(\nu\) be a positive and finite Borel probability measure on a compact metric space \((X, d)\). Assume that \(M \subseteq X\) is a Borel set such that \(\nu(M) > 0\) and

\[ M \subseteq \left\{ t \in X, \lim_{r \to 0^+} \frac{\log \nu(B(t, r))}{\log r} \geq \delta \right\}. \]

Then the Hausdorff dimension of \(M\) is bounded from below by \(\delta\).

APPENDIX 3 — HAUSDORFF AND PACKING MEASURES AND DIMENSIONS

Given a subset \(K\) of \(\mathbb{N}_+^N\) endowed with a metric \(d\) making it \(\sigma\)-compact, \(g : \mathbb{R}_+ \to \mathbb{R}_+\) a continuous non-decreasing function near zero and such that \(g(0) = 0\),
and $E$ a subset of $K$, the Hausdorff measure of $E$ with respect to the gauge function $g$ is defined as
\[
\mathcal{H}^g(E) = \lim_{\delta \to 0^+} \inf \left\{ \sum_{i \in \mathbb{N}} g(\operatorname{diam}(U_i)) \right\},
\]
the infimum being taken over all the countable coverings $(U_i)_{i \in \mathbb{N}}$ of $E$ by subsets of $K$ of diameters less than or equal to $\delta$.

If $s \in \mathbb{R}_+^*$ and $g(u) = u^s$, then $\mathcal{H}^s(E)$ is also denoted by $\mathcal{H}^E$ and called the $s$-dimensional Hausdorff measure of $E$. Then, the Hausdorff dimension of $E$ is defined as
\[
\dim E = \sup \{ s > 0 : \mathcal{H}^s(E) = \infty \} = \inf \{ s > 0 : \mathcal{H}^s(E) = 0 \},
\]
with the convention $\sup \emptyset = 0$ and $\inf \emptyset = \infty$.

Packing measures and dimensions are defined as follows. Given $g$ and $E \subset K$ as above, one first defines
\[
\overline{P}^g(E) = \lim_{\delta \to 0^+} \sup \left\{ \sum_{i \in \mathbb{N}} g(\operatorname{diam}(B_i)) \right\},
\]
the supremum being taken over all the packings $(B_i)_{i \in \mathbb{N}}$ of $E$ by balls centered on $E$ and with diameter smaller than or equal to $\delta$. Then, the packing measure of $E$ with respect to the gauge $g$ is defined as
\[
\mathcal{P}^g(E) = \lim_{\delta \to 0^+} \inf \left\{ \sum_{i \in \mathbb{N}} \mathcal{P}^g(E_i) \right\},
\]
the infimum being taken over all the countable coverings $(E_i)_{i \in \mathbb{N}}$ of $E$ by subsets of $K$ of diameters less than or equal to $\delta$. If $s \in \mathbb{R}_+^*$ and $g(u) = u^s$, then $\mathcal{P}^s(E)$ is also denoted by $\mathcal{P}^s(E)$ and called the $s$-dimensional measure of $E$. Then, the packing dimension of $E$ is defined as
\[
\operatorname{Dim} E = \sup \{ s > 0 : \mathcal{P}^s(E) = \infty \} = \inf \{ s > 0 : \mathcal{P}^s(E) = 0 \},
\]
with the convention $\sup \emptyset = 0$ and $\inf \emptyset = \infty$. For more details the reader is referred to [9].

If $\mu$ is a positive and finite Borel measure supported on $K$, then its lower Hausdorff and packing dimensions are defined as
\[
\dim(\mu) = \inf \{ \dim F : F \text{ Borel, } \mu(F) > 0 \}, \quad \overline{\dim}(\mu) = \inf \{ \dim F : F \text{ Borel, } \mu(F) > 0 \},
\]
and its upper Hausdorff and packing dimensions are defined as
\[
\underline{\dim}(\mu) = \inf \{ \dim F : F \text{ Borel, } \mu(F) = \|\mu\| \}, \quad \underline{\dim}(\mu) = \inf \{ \dim F : F \text{ Borel, } \mu(F) = \|\mu\| \}.
\]
We have (see [8], [11])

\[
\dim(\mu) = \text{ess inf}_\mu \liminf_{r \to 0^+} \frac{\log(\mu(B(t, r)))}{\log(r)},
\]

\[
\text{Dim}(\mu) = \text{ess inf}_\mu \limsup_{r \to 0^+} \frac{\log(\mu(B(t, r)))}{\log(r)},
\]

and

\[
\overline{\dim}(\mu) = \text{ess sup}_\mu \liminf_{r \to 0^+} \frac{\log(\mu(B(t, r)))}{\log(r)},
\]

\[
\overline{\text{Dim}}(\mu) = \text{ess sup}_\mu \limsup_{r \to 0^+} \frac{\log(\mu(B(t, r)))}{\log(r)},
\]

where \(B(t, r)\) stands for the closed ball of radius \(r\) centered at \(t\). If \(\dim(\mu) = \overline{\dim}(\mu)\) (resp. \(\text{Dim}(\mu) = \overline{\text{Dim}}(\mu)\)), this common value is denoted by \(\dim \mu\) (resp. \(\text{Dim} \mu\)), and if \(\dim \mu = \text{Dim} \mu\), one says that \(\mu\) is exact dimensional.

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