As a direct source of information on chiral symmetry breaking within QCD, the sigma commutator is of considerable importance. Since hadron structure is a non-perturbative problem, numerical calculations on a space-time lattice are currently the only rigorous approach. With recent advances in the calculation of hadron masses within full QCD, it is of interest to see whether the sigma commutator can be calculated directly from the dependence of the nucleon mass on the input quark mass. We show that, provided the correct chiral behaviour of QCD is respected in the extrapolation to realistic quark masses, one can indeed obtain a fairly reliable determination of the sigma commutator using present lattice data. For two-flavour dynamical fermion QCD the sigma commutator lies between 45 and 55 MeV based on recent data from CP-PACS and UKQCD.

1. WHAT IS THE SIGMA COMMUTATOR?

In the quest to understand hadron structure within QCD, small violations of fundamental symmetries play a vital role. The sigma commutator, \( \sigma_N \):

\[
\sigma_N = \frac{1}{3} \langle N | [Q_{i5}, [Q_{i5}, H]] | N \rangle = \bar{m} \langle N | \bar{u}u + \bar{d}d | N \rangle = \bar{m} \frac{\partial M_N}{\partial \bar{m}}
\]  

(with \( Q_{i5} \) the two-flavour \((i=1, 2, 3)\) axial charge) is an extremely important example of such a symmetry.

2. PREVIOUS ATTEMPTS

\( \sigma_N \) cannot be accessed directly by experimental measurements. However, one can infer from world data a value of 45 ± 8 MeV [1]. This result has been under some scrutiny recently due to the progress in new determinations of the pion-nucleon scattering lengths [2, 3] and new phase shift analyses [4, 5]. The full lattice QCD calculations upon which our work is based involve only two active flavours, the heavier third flavour is approximated by a renormalisation of the strong coupling constant. As a guide, recent work suggests that the best value of \( \sigma_N \) may be 8 to 26 MeV larger than the value quoted above [6].

One can notionally use QCD to directly calculate the value of \( \sigma_N \), but in practice the calculation has proven to be difficult. Early attempts [7] to extract \( \sigma_N \) from the quark mass dependence of the nucleon mass (using Eq.(1)) in quenched QCD with naive
extrapolations produced values in the range 15 to 25 MeV. Attention subsequently turned to determining $\sigma_N$ by calculating the scalar matrix element of the nucleon $\langle N|\bar{u}u + \bar{d}d|N\rangle$. There it was discovered that the sea-quark loops make a dominant contribution to $\sigma_N$ [8,9]. These works, based on quenched QCD simulations found values in the 40 to 60 MeV range, which are more compatible with the experimental values quoted earlier.

On the other hand, the most recent estimate of $\sigma_N$, and the only one based on a two-flavour, dynamical-fermion lattice QCD calculation, comes from the SESAM collaboration. They obtain a value of $18 \pm 5$ MeV [10], through a direct calculation of the scalar matrix element $\langle N|\bar{u}u + \bar{d}d|N\rangle$ and the quark mass $\bar{m}$.

The fact that neither $\langle N|\bar{u}u + \bar{d}d|N\rangle$, nor $\bar{m}$ is renormalisation group invariant introduces a major difficulty in calculating the sigma commutator in this approach. One must reconstruct the scale invariant result from the product of the scale dependent matrix element and the scale dependent quark masses. The latter are extremely difficult to determine precisely and are the chief sources of uncertainty. Furthermore, since lattice calculations are made at quite large pion masses, typically above 500 or 600 MeV, one needs to extrapolate, in the pion mass down to the physical value at 140 MeV. An important innovation adopted by Dong et al. was to extrapolate $\langle N|\bar{u}u + \bar{d}d|N\rangle$ using a form motivated by chiral symmetry, namely $a + b\bar{m}^{1/2}$. Regrettably, the value of $b$ used was not constrained by chiral symmetry and higher order terms of the chiral expansion were not considered. Furthermore, since the work was based on a quenched calculation, the chiral behaviour implicit in the lattice results involves incorrect chiral physics [11].

3. THE CURRENT CALCULATION

Our recent work [12] was motivated by the improvements in computing power, together with the development of improved actions [13], which have led to accurate calculations of the mass of the nucleon within full QCD (for two flavours) as a function of $\bar{m}$ down to $m_{\pi} \sim 500$ MeV. (Since $m_{\pi}^2$ is proportional to $\bar{m}$ over the range studied we choose to display all results as a function of $m_{\pi}^2$.) We showed that provided that one has control over the extrapolation of this lattice data to the physical pion mass, one can calculate $\sigma_N$ from $\sigma_N = m_{\pi}^2 \partial M_N/\partial m_{\pi}^2$ (which is equivalent to Eq. (1) where $\bar{m}$ was used) at $m_{\pi} = 140$ MeV. This approach has the important advantage that one only needs to work with renormalization group invariant quantities.

Chiral perturbation theory ($\chi$PT) predicts that the leading non-analytic (LNA) correction to the self energy contribution to the nucleon mass is proportional to $m_{\pi}^3$ (or $\bar{m}^{3/2}$). It can be seen in Fig. 2 that the preliminary point from CP-PACS [14] at $m_{\pi}^2 \sim 0.1$ GeV$^2$ does indeed suggest some curvature in this low mass region. These observations led the CP-PACS group to extrapolate their data with the simple, phenomenological form:

$$M_N = \alpha + \beta m_{\pi}^2 + \gamma m_{\pi}^3,$$

rather than a naive linear form ($\gamma \equiv 0$), as shown in Fig. 4. The corresponding fit to the combined data set, using Eq. (2), is shown as the short-dashed curve in Figs. 1 and 4. We found that this fit gives $\sigma_N = 29.7$ MeV. The difficulty with this purely phenomenological analysis was discussed in Ref. [14]. The problem is that a derivative is required when evaluating $\sigma_N$ and the value of $\gamma$ found in the fit ($-0.761$ GeV$^{-2}$) is almost an order of
Figure 1. Nucleon mass versus $m_{\pi}^2$. The solid data points are CP-PACS results [14], whilst the open points are UKQCD data [15]. Both curves are fits using Eq. (2). The solid curve has $\tilde{\gamma} \equiv 0$, whilst the short-dashed curve has $\tilde{\gamma}$ unconstrained. The vertical line indicates the physical pion mass.

Figure 2. Data as labelled in Fig. 1. The solid curve is a fit to Eq. (3) with a dipole form factor, the dashed curve is the same fit using a sharp cut-off form factor. The long-dash curve is a fit to Eq. (3) excluding the lowest data point.

Magnitude smaller than the model independent LNA coefficient, $\gamma^{\text{LNA}} = -5.60 \text{ GeV}^{-2}$, indicated by $\chi^2$. Recently, an alternative approach was suggested in Ref. [16]. There it was realised that the pion loop diagrams shown in Fig. 3 yield not only the most important non-analytic structure, but also give rise to the most significant variation in the nucleon mass as $m_\pi \to 0$. This leads to the following extrapolation function for $M_N$:

$$M_N = \alpha + \beta m_\pi^2 + \sigma_{NN}(m_\pi, \Lambda) + \sigma_{N\Delta}(m_\pi, \Lambda),$$

where $\sigma_{NN}$ and $\sigma_{N\Delta}$ are the self-energy contributions of Figs. 3(a) and 3(b), respectively, using a cut-off in momentum controlled by $\Lambda$. The full analytic expressions for $\sigma_{NN}$ and $\sigma_{N\Delta}$ are given in Ref. [16]. For our purposes it suffices that they have precisely the correct LNA and next-to-leading non-analytic behaviour required by chiral perturbation theory as $m_\pi \to 0$. In addition, $\sigma_{N\Delta}$ contains the correct, square root branch point

Figure 3. One-loop pion induced self energy of the nucleon.
\((\sim [m_\pi^2 - (M_\Delta - M_N)^2]^{1/2})\) at the \(\Delta - N\) mass difference, which is essential for extrapolations from above the \(\Delta - N\pi\) threshold.

Fitting Eq. (3) to the data, including the point near 0.1 GeV\(^2\), gives the dot-dash curve in Fig. 2. The corresponding value of \(\sigma_N\) is 54.6 MeV and the physical nucleon mass is 870 MeV. Omitting the lowest data point from the fit yields the long-dash curve in Fig. 2 with \(\sigma_N = 65.8\) MeV, demonstrating the need for lattice simulations of QCD at light quark masses.

4. CONCLUSION

The importance of the inclusion of the correct chiral behaviour is clearly seen by the fact that it increases the value of the sigma commutator from the 30 MeV of the unconstrained cubic fit to around 50 MeV. Nevertheless, it is a remarkable result that the present lattice data for two-flavour dynamical-fermion QCD, yields a stable \cite{12} and accurate answer for the sigma commutator, an answer which is already within the range of the experimental values.

This work was supported by the Australian Research Council.

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