Budgeted Out-Tree Maximization with Submodular Prizes

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Abstract

We consider a variant of the prize collecting Steiner tree problem in which we are given a directed graph $D = (V, A)$, a monotone submodular prize function $p : 2^V \to \mathbb{R}^+ \cup \{0\}$, a cost function $c : V \to \mathbb{Z}^+$, a root vertex $r \in V$, and a budget $B$. The aim is to find an out-subtree $T$ of $D$ rooted at $r$ that costs at most $B$ and maximizes the prize function. We call this problem Directed Rooted Submodular Tree (DRST).

For the case of undirected graphs and additive prize functions, Moss and Rabani [SIAM J. Comput. 2007] gave an algorithm that guarantees an $O(\log |V|)$-approximation factor if a violation by a factor 2 of the budget constraint is allowed. Bateni et al. [SIAM J. Comput. 2018] improved the budget violation factor to $1 + \epsilon$ at the cost of an additional approximation factor of $O(1/\epsilon^2)$, for any $\epsilon \in (0, 1]$. For directed graphs, Ghuge and Nagarajan [SODA 2020] gave an optimal quasi-polynomial time $O\left(\frac{\log n'}{\log \log n'}\right)$-approximation algorithm, where $n'$ is the number of vertices in an optimal solution, for the case in which the costs are associated to the edges.

In this paper, we give a polynomial time algorithm for DRST that guarantees an approximation factor of $O\left(\sqrt{\frac{B}{\epsilon^3}}\right)$ at the cost of a budget violation of a factor $1 + \epsilon$, for any $\epsilon \in (0, 1]$. The same result holds for the edge-cost case, to the best of our knowledge this is the first polynomial time approximation algorithm for this case. We further show that the unrooted version of DRST can be approximated to a factor of $O(\sqrt{B})$ without budget violation, which is an improvement over the factor $O(\Delta \sqrt{B})$ given in [Kuo et al. IEEE/ACM Trans. Netw. 2015] for the undirected and unrooted case, where $\Delta$ is the maximum degree of the graph. Finally, we provide some new/improved approximation bounds for several related problems, including the additive-prize version of DRST, the maximum budgeted connected set cover problem, and the budgeted sensor cover problem.

1 Introduction

Prize collecting Steiner tree problems (PCSTP) have been extensively studied due to their applications in designing computer and telecommunication networks, VLSI design, computational geometry, wireless mesh networks, and cancer genome studies [5, 8, 15, 23, 32]. Very interesting polynomial-time constant/poly-logarithmic approximation algorithms have been proposed for many variants of PCSTP when the graph is undirected [1, 2, 10, 12, 18, 21, 29]. However, these problems are usually much harder on directed graphs. For instance,
there is a simple polynomial-time 2-approximation algorithm for the undirected Steiner tree problem, but no quasi-polynomial-time algorithm for the directed Steiner tree problem achieving an approximation ratio of \( o\left(\frac{\log^2 k}{\log \log k}\right) \) exists, unless \( \text{NP} \subseteq \cap_{0 < c < 1} \text{ZPTIME}(2^{n^c}) \) or the Projection Game Conjecture is false \([13]\), where \( k \) is the number of terminal nodes.

Some of the most relevant variants of \textsc{PCSTP} are represented by prize collecting problems with budget constraints. In such problems, we are usually given a graph with prizes and costs on the nodes and the goal is to find a tree that maximizes the sum of the prize of its nodes, while keeping the total cost bounded by a given budget. Guha et al. \([14]\) introduced the case in which the graph is undirected and the goal is to find a tree that contains a distinguished vertex, called root, respects the budget constraint, and maximizes the prize, we call this problem \textit{Undirected Rooted Additive Tree (URAT)}. They gave an algorithm that achieves an \( \Theta(\log^2 n) \)-approximation factor, where \( n \) is the number of nodes in the graph, but the computed solution requires a factor-2 violation of the budget constraint. Moss and Rabani \([27]\) and Bateni et al. \([2]\) further investigated \textsc{URAT} and improved the results from the approximability point of view. The former paper improved the approximation factor to \( O(\log n) \), with the same budget violation, and the latter one improved the budget violation factor to \( 1 + \varepsilon \) to obtain an approximation factor of \( O\left(\frac{1}{\varepsilon^2} \log n\right) \), for any \( \varepsilon \in (0, 1) \). Kortsarz and Nutov \([22]\) showed that the unrooted version of \textsc{URAT}, so does \textsc{URAT}, admits no \( o(\log \log n) \)-approximation algorithm, unless \( \text{NP} \subseteq \text{DTIME}(n^{\text{polylog}(n)}) \), even if the algorithm is allowed to violate the budget constraint by a factor equal to a universal constant.

In this paper, we consider a generalization of \textsc{URAT} on directed graphs. We are given a directed graph, where each node is associated with a cost, and the prize is defined by a monotone submodular function on the subsets of nodes, and the goal is to find an out-tree (a.k.a. out-arborescence) rooted at a specific vertex \( r \) with the maximum prize such that the total cost of all vertices in the out-tree is no more than a given budget. We term this problem \textit{Directed Rooted Submodular Tree (DRST)}. A closely related problem, called \textit{Submodular Tree Orienteering (STO)}, has been recently introduced by Ghuge and Nagarajan \([11]\). \textsc{STO} is the same problem as \textsc{DRST} except that edges and not nodes have costs. They provided a tight quasi-polynomial-time \( O\left(\frac{\log n'}{\log \log n'}\right) \)-approximation algorithm that requires \( (n \log B)^{O\left(\log^{1+\varepsilon'} n'\right)} \) time, where \( n' \) is the number of vertices in an optimal solution and \( B \) is the budget constraint.

**Contribution.** By extending some ideas of Kuo et al. \([23]\) and Bateni et al. \([2]\), we design a polynomial-time \( O\left(\sqrt{B}/\varepsilon^2\right) \)-approximation algorithm for \textsc{DRST}, violating the budget constraint \( B \) by a factor of at most \( 1 + \varepsilon \), for any \( \varepsilon \in (0, 1) \) (Section 4). Our technique can be used to obtain the same result for \textsc{STO} (Section 6). To our knowledge, this is the first polynomial-time approximation algorithm for \textsc{STO}. We also show that, for any \( 1 + \varepsilon \) budget violation, with \( \varepsilon \in (0, 1) \), our approach provides an \( O\left(\sqrt{B}/\varepsilon^2\right) \)-approximation algorithm for the special cases of \textsc{DRST} and \textsc{STO} where the prize function is additive (Section 7). We also consider the unrooted version of \textsc{DRST} and give an \( O\left(\sqrt{B}\right) \)-approximation algorithm without budget violation (Section 5), which is an improvement over the factor \( O\left(\Delta \sqrt{B}\right) \) \([23]\) for the undirected and unrooted version of \textsc{DRST}, where \( \Delta \) is the maximum node-degree.

Finally, we study some variants of \textsc{DRST} on undirected graphs. We show that, for any \( 1 + \varepsilon \) budget violation, \textsc{URAT} admits an \( O\left(\Delta/\varepsilon^2\right) \)-approximation algorithm, while its quota version admits a \( 2\Delta \)-approximation algorithm. Next, we present some approximation results for some variants of the connected maximum coverage problem, which improve over the bounds given by Ran et al. \([30]\). Finally, we provide two approximation algorithms for the Budgeted Sensor Cover problem, which result in an improvement to the literature \([23, 30, 33, 34]\). We discuss these results in Section 7.
Related Work. Many variants of Prize collecting Steiner Tree problems have been investigated. Here we list those that are more closely related to our study. Further related work is reported in the Appendix.

Kuo et al. [23] studied the unrooted version of DRST on undirected graphs called Maximum Connected Submodular function with Budget constraint (MCSB). They provided an $O(\Delta \sqrt{B})$-approximation algorithm for MCSB, where $\Delta$ is the maximum degree of the graph. Vandin et al. [32] provided a $(\frac{2\Delta - 1}{\Delta - 1})r$-approximation algorithm for a special case of the same problem, where $r$ is the radius of an optimal solution. This problem coincides with the connected maximum coverage problem in which each set has cost one. Ran et al. [30] presented an $O(\Delta \log n)$-approximation algorithm for a special case of the connected maximum coverage problem. Hochbaum and Rao [15] investigated MCSB in which each vertex costs 1 and provided an approximation algorithm with factor $\min\{1/(1 - 1/e)(1/R - 1/B), B\}$, where $R$ is the radius of the graph. Chen et al. [4] investigated the edge-cost version of MCSB. One of the applications of MCSB is a problem in wireless sensor networks called Budgeted Sensor Cover Problem (BSCP), where the goal is to find a set of $B$ connected sensors to maximize the number of covered users, for a given $B$. Kuo et al. [23] provided a $5(\sqrt{B} + 1)/(1 - 1/e)$-approximation algorithm for BSCP, which was improved by Xu et al. [33] to $\lceil \sqrt{B}/(1 - 1/e)\rceil$. Huang et al. [17] proposed a $8((2\sqrt{2}C + 1)^2/(1 - 1/e))$-approximation algorithm for BSCP, where $C = O(1)$.

Johnson et al. [18] introduced an edge-cost variant of DRST on undirected graphs, where the prize function is additive, called E-URAT. They showed that there exists a $(5 + \epsilon)$-approximation algorithm for the unrooted version of E-URAT using Garg’s 3-approximation algorithm [9] for the $k$-MST problem, and observed that a 2-approximation for $k$-MST would lead to a 3-approximation for E-URAT. This observation along with the Garg’s 2-approximation algorithm [10] for $k$-MST yield a 3-approximation algorithm for the unrooted version of E-URAT. Recently, Paul et al. [29] provided a polynomial-time 2-approximation algorithm for E-URAT.

2 Notation and problem statement

For an integer $k$, let $[k] := \{1, \ldots, k\}$. A directed path is a directed graph made of a sequence of distinct vertices $(v_1, \ldots, v_k)$ and a sequence of directed edges $(v_i, v_{i+1})$, $i \in [k - 1]$. An out-tree (a.k.a. out-arborescence) is a directed graph in which there is exactly one directed path from a specific vertex $r$, called root, to each other vertex. If a subgraph $T$ of a directed graph $D$ is an out-tree, then we say that $T$ is an out-tree of $D$.

Let $D = (V, A)$ be a directed graph with $n$ nodes, $c : V \to \mathbb{Z}^+$ be a cost function on nodes, $p : 2^V \to \mathbb{R}^+ \cup \{0\}$ be a monotone submodular prize function on the subsets of nodes, $r \in V$ be a root vertex, and $B$ be an integer budget. For any subgraph $D'$ of $D$, we denote by $V(D')$ and $A(D')$ the set of nodes and edges in $D'$, respectively. Given $S \subseteq V$, we denote the cost of $S$ by $c(S) = \sum_{v \in S} c(v)$ and we use shortcuts $c(D') = c(V(D'))$ and $p(D') = p(V(D'))$ for a subgraph $D'$ of $D$. In the Directed Rooted Submodular Tree problem (DRST), the goal is to find an out-tree $T$ of $D$ rooted at $r$ such that $c(T) \leq B$ and $p(T)$ is maximum. Throughout the paper, we denote an optimal solution to DRST by $T^*$.

Given two nodes $u$ and $v$ in $V$, a path in $D$ from $u$ to $v$ with the minimum cost is called a shortest path and its cost, denoted by $dist(u, v)$, is called the distance from $u$ to $v$ in $D$.

An algorithm is a bicriteria $(\beta, \alpha)$-approximation algorithm for DRST if, for any instance $I$ of the problem, it returns a solution $Sol_I$ such that $p(Sol_I) \geq \frac{OPT_I}{\alpha}$ and $c(Sol_I) \leq \beta B$, where $OPT_I$ is the optimum for $I$. 

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3 Results and Techniques

Our main result is given in the next theorem.

\textbf{Theorem 1.} DRST admits a polynomial-time bicriteria \((1 + \epsilon, O(\sqrt[3]{B}))\)-approximation algorithm, for any \(\epsilon \in (0, 1)\).

Our approach combines and extends techniques given by Kuo et al. \cite{Kuo} and Bateni et al. \cite{Bateni}. To illustrate our techniques, we now consider the case in which costs are unitary, i.e. \(c(v) = 1\), for each \(v \in V\), and the prize function is additive, i.e. \(p(S) = \sum_{v \in S} p\{v\}\), for any \(S \subseteq V\). In this case, the distance from a node \(u\) to a node \(v\) is equal to the minimum number of nodes in a path from \(u\) to \(v\) and the cost of a tree \(T\) is equal to its size, \(c(T) = |V(T)|\).

W.l.o.g. we also assume that the distance from \(r\) to any node is at most \(B\). We will give the proof for the general case in Section 4.

The algorithm works as follows. For any vertex \(u\), we denote as \(V_u\) the set of all nodes that are at a distance no more than \(\sqrt[3]{B}\) from \(u\), \(V_u := \{v \mid \text{dist}(u, v) \leq \sqrt[3]{B}\}\). We first select a subset \(S_u\) of \(V_u\) of at most \(\sqrt[3]{B}\) nodes with the maximum prize, \(S_u := \arg \max \{p(S) : S \subseteq V_u, |S| \leq \sqrt[3]{B}\}\). We then compute a minimal inclusion-wise out-tree \(T_u\) rooted at \(u\) that spans all nodes in \(S_u\). Note that \(|V(T_u)| \leq B\) since the distance from \(u\) to any node in \(S_u\) is at most \(\sqrt[3]{B}\). Let \(z\) be a node such that \(p(T_z)\) is maximum. If \(z = r\), then we take \(T_z\) as our solution, otherwise we compute a solution by adding to \(T_z\) a shortest path \(P\) from \(r\) to \(z\) and removing the edges in \(A(T_z) \setminus A(P)\) incoming the nodes in \(V(T_z) \cap V(P)\). Let \(T\) be our solution and \(T^*\) be an optimal solution.

We will prove (Lemma 6) that any out-tree \(\hat{T}\) can be covered by at most \(N = O(|\hat{T}|/m)\) out-subtrees \(\{T_i\}_{i=1}^N\) with at most \(m\) nodes each, where \(m\) is any positive integer less than \(|\hat{T}|\). By applying this claim to an optimal solution \(T^*\) and by setting \(m = \lfloor \sqrt[3]{B}\rfloor\), we obtain

\[ p(T^*) = p\left(\bigcup_{i=1}^N V(T_i)\right) \leq Np(T^*), \]

where \(p(T^*) = \max\{p(T_i^*) : i \in [N]\}\), \(|T^*| \leq \sqrt[3]{B}\), and \(N = O(|T^*|/m) = O(\sqrt[3]{B})\). Let \(w\) be the root of \(T^*\). Recall that \(S_w\) is a set of at most \(\sqrt[3]{B}\) nodes that are at a distance no more than \(\sqrt[3]{B}\) from \(w\) and have the maximum prize and \(T_w\) contains all the nodes in \(S_w\). Since \(|T^*| \leq \sqrt[3]{B}\), we have

\[ p(T^*) \leq p(T_w) \leq p(T_z) \leq p(T). \]

Since \(N = O(\sqrt[3]{B})\), we conclude that \(p(T^*) = O(\sqrt[3]{B})p(T)\).

Note that the cost of \(T\) is upper-bounded by \(2B\), as both the cost of \(T_z\) and that of a shortest path from \(r\) to \(z\) are at most \(B\). We can use the trimming procedure introduced by Bateni et al. \cite{Bateni} to obtain an out-subtree of \(T\) with cost at most \((1 + \epsilon)B\) by loosing an approximation factor of \(O(1/\epsilon^2)\), for any \(\epsilon \in (0, 1)\) (see Lemma 2). This shows Theorem 1 for the unit-cost, additive-prize case. In the case in which the prize is a general monotone submodular function, the trimming procedure by Bateni et al. cannot be applied. We show how to generalize this procedure to the case of any monotone submodular prize function by loosing an extra approximation factor of \(O(1/\epsilon)\).

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1. This step can be done in polynomial time since function \(p\) is additive. If \(p\) is monotone and submodular, this step consists in solving the submodular maximization problem. See Section 4 for more details.
We can use the same approach to obtain a polynomial-time bicriteria \( (1 + \varepsilon, O(\frac{\sqrt{B}}{\varepsilon^2})) \)-approximation algorithm for the case of additive prize function and edge-cost. More importantly, we can obtain a polynomial-time bicriteria \( (1 + \varepsilon, O(\frac{\sqrt{B}}{\varepsilon^2})) \)-approximation algorithm for STO, i.e. for the edge-cost case where the prize function is monotone submodular. To the best of our knowledge, this is the first polynomial-time approximation algorithm for STO.

Finally, for the unrooted version the same approach with some minor changes achieves an \( O(\sqrt{B}) \)-approximation with no budget violation.

## 4 Approximation Algorithm for DRST

We now introduce our polynomial-time approximation algorithm for DRST. We start by defining a procedure that takes as input an out-tree of a directed graph \( D \) and returns another out-tree of \( D \) which has a smaller cost but preserves the same prize-to-cost ratio (up to a bounded multiplicative factor).

Bateni et al. [2] introduced a similar procedure for the case of undirected graphs and additive prize function. In their case, we are given an undirected graph \( G = (V, E) \), a distinguished vertex \( r \in V \) and a budget \( B \), where each vertex \( v \in V \) is assigned with a prize \( p'(v) \) and a cost \( c'(v) \). For a tree \( T \), the prize and cost of \( T \) are the sum of the prizes and costs of the nodes of \( T \) and are denoted by \( p'(T) \) and \( c'(T) \), respectively. A graph \( G \) is called \( B \)-proper for the vertex \( r \) if the cost of reaching any vertex from \( r \) is at most \( B \). Bateni et al. proposed a trimming process that leads to the following lemma.

\[ \textbf{Lemma 2 (Lemma 3 in [2]). Let} T \text{ be a tree rooted at } r \text{ with the prize-to-cost ratio } \gamma = \frac{p'(T)}{c'(T)}. \text{ Suppose the underlying graph is } B \text{-proper for } r \text{ and for } \varepsilon \in (0, 1] \text{ the cost of the tree is at least } \frac{\varepsilon B}{2}. \text{ One can find a tree } T' \text{ containing } r \text{ with the prize-to-cost ratio at least } \frac{c'(T')}{\varepsilon} \text{ such that } \varepsilon B/2 \leq c'(T') \leq (1 + \varepsilon)B. \]

We now generalize this trimming process to the case in which the underlying graph is directed and the prize function is monotone and submodular by borrowing ideas from [2].

We introduce some additional definitions. Let \( T \) be an out-tree rooted at \( r \). A full out-subtree of \( T \) rooted at some node \( v \) is an out-subtree of \( T \) containing all the vertices that are reachable from \( r \) through \( v \) in \( T \). The set of strict out-subtrees of \( T \) is the set of all full out-subtrees of \( T \) other than \( T \) itself. The set of immediate out-subtrees of \( T \) is the set of all full out-subtrees rooted at the children of \( r \) in \( T \). A directed graph \( D = (V, A) \) is \( B \)-appropriate for a node \( r \) if \( \text{dist}(r, v) \leq B \) for any node \( v \in V \).

\[ \textbf{Lemma 3. Let} D = (V, A) \text{ be a } B \text{-appropriate graph for a node } r. \text{ Let } T \text{ be an out-tree of } D \text{ rooted at } r \text{ with the prize-to-cost ratio } \gamma = \frac{p(T)}{c(T)}, \text{ where } p \text{ is a monotone submodular function. Suppose that } \frac{\gamma B}{2} \leq c(T) \leq hB, \text{ where } h \in (1, n] \text{ and } \varepsilon \in (0, 1]. \text{ One can find an out-subtree } \hat{T} \text{ rooted at } r \text{ with the prize-to-cost ratio at least } \frac{c'(\hat{T})}{\varepsilon} \text{ such that } \varepsilon B/2 \leq c'(\hat{T}) \leq (1 + \varepsilon)B. \]

\[ \textbf{Proof.} \text{ We run the following initial trimming procedure. We iteratively remove a strict out-subtree } T' \text{ from } T \text{ that satisfies two conditions: (i) the prize-to-cost ratio of } T \setminus T' \text{ is at least } \gamma, \text{ and (ii) } c(T \setminus T') \geq \frac{\gamma}{2} B. \text{ We repeat this process until no such strict out-subtree exists. Let } T' \text{ be the remaining out-tree after applying this process on } T. \]

\text{Now if } c(T') \leq (1 + \varepsilon)B, \text{ the desired out-subtree is obtained. Suppose it is not the case. A full out-subtree } T' \text{ is called } \text{rich if } c(T') \geq \frac{\gamma}{2} B \text{ and the prize-to-cost ratio of } T' \text{ and all its strict out-subtrees are at least } \gamma. \text{ We claim that if there exists a rich out-subtree, then we can find the desired out-subtree } \hat{T}. \]
Claim 4. Given a rich out-subtree $T'$, the desired out-subtree $\hat{T}$ can be found.

Proof. We first find a rich out-subtree $T''$ of $T'$ such that the strict out-subtrees of $T''$ are not rich, i.e., $c(T'') \geq \frac{2}{3}B$ while the cost of any strict out-subtree of $T''$ (if any exist) is less than $\frac{2}{3}B$. Let $C$ be the total cost of the immediate out-subtrees of $T''$. We distinguish between two cases:

1. If $C < \frac{2}{3}B$, then let $\hat{T}$ be the union of $T''$ and a shortest path $P$ from $r$ to the root $r''$ of $T''$. $\hat{T}$ has cost at most $C + B \leq (1 + \epsilon)B$ and prize at least $\gamma(\frac{2}{3}B)$. This implies that $\hat{T}$ has ratio at least $\frac{\frac{4}{3}B}{\frac{4}{3}B} \geq \frac{\gamma}{\gamma} \geq \frac{1}{2}$.

2. If $C \geq \frac{2}{3}B$, we proceed as follows. Since each immediate out-subtree of $T''$ has a cost strictly smaller than $\frac{2}{3}B$, we can partition all the immediate out-subtrees of $T''$ into $M$ groups $S_1, \ldots, S_M$ in such a way that for each $i \in [M - 1]$ the total cost of immediate out-subtrees in $S_i$ is at least $\frac{2}{3}B$, and for each $i \in [M]$ it is at most $\epsilon B$. We can always group in this way since the cost of each immediate out-subtree of $T''$ is less than $\frac{2}{3}B$ while $C \geq \frac{2}{3}B$. Since the total cost of all the immediate out-subtrees of $T''$ is upper bounded by $hB$, then the number of selected groups $M$ is at most

$$M \leq \left\lceil \frac{hB}{\frac{2}{3}B} \right\rceil \leq \left\lceil \frac{2h}{\epsilon} \right\rceil + 1 \leq \frac{4h}{\epsilon} \leq \frac{4h}{\epsilon}.$$

We now add the root $r''$ of $T''$ to each group $S_i$ and denote the new group by $S'_i$, i.e., $S'_i = S_i \cup \{r''\}$, for any $i \in [M]$. By the monotonicity and submodularity of $p$, we have $\sum_{i=1}^M p(S'_i) \geq p(S'_i) + \sum_{i=2}^M p(S_i) \geq p(S'_i \cup \bigcup_{i=2}^M S_i) = p(T'')$. Now among $S'_1, \ldots, S'_M$, we select the group $S'_2$ that maximizes the prize, i.e., $z = \arg \max_{i \in [M]} p(S'_i)$. We know that

$$p(S'_2) \geq \frac{1}{M} \sum_{i=1}^M p(S'_i) \geq \frac{p(T'')}{M} \geq \frac{\epsilon}{4h} p(T'') \geq \frac{\epsilon}{4h} \cdot \frac{y\epsilon}{2} B = \frac{y\epsilon^2}{8h} B.$$

In case $z = M$ and $c(S'_M) < \frac{2}{3}B$, we select a subset of immediate out-subtrees from $\bigcup_{i=1}^{M-1} S_i$ with the total cost of at least $\frac{2}{3}B$ and at most $\epsilon B - c(S'_M)$, and add it to $S'_2$.

Finally, let $\hat{T}$ be the union of a shortest path $P$ from $r$ to $r''$, $S'_2$, and the edges from $r''$ to the roots of the out-subtrees in $S_2$ (see Figure 1). By monotonicity, $\hat{T}$ has the total prize at least $p(\hat{T}) \geq p(S'_2) \geq \frac{y\epsilon^2}{6h} B$. Note that $c(\hat{T}) \leq (1 + \epsilon)B$ as $c(S'_2 \setminus \{r''\}) = c(S_2) \leq \epsilon B$ and the shortest path from $r$ to $r''$ costs at most $B$ (since the graph is $B$-appropriate). This implies that the prize-to-cost ratio of $\hat{T}$ is at least $\frac{y\epsilon^2}{6h(1+\epsilon)} \geq \frac{y\epsilon^2}{2\epsilon B}$. <

It only remains to consider the case when there is no rich out-subtree. Since $T_-$ is not rich and $c(T_-) \geq \frac{2}{3}B$, the ratio of at least one strict out-subtree of $T_-$ is less than $\gamma$. Now we find a strict out-subtree $T'$ with ratio less than $\gamma$ such that the ratio of all of its strict out-subtrees (if any exist) is at least $\gamma$. We first need to show that $c(T_- \setminus T') < \frac{2}{3}B$.

Claim 5. $c(T_- \setminus T') < \frac{2}{3}B$.

Proof. By the submodularity of $p$, we know that $p(T_- \setminus T') + p(T') \geq p(T_-)$. This implies that

$$\frac{p(T_-. T')} {c(T_-. T')} \geq \frac{p(T_-) - p(T')} {c(T_-) - c(T')} \geq \frac{\epsilon}{c(T_-)} - \frac{c(T')} {c(T_-) - c(T')}.$$

Let $y' = \frac{\epsilon}{c(T_-)}$ be the prize-to-cost ratio of $T'$. We know that

$$p(T_-) - p(T') = c(T_-) y - c(T') y' > c(T_-) y - c(T') y,$$
As the ratio and cost of the prize-to-cost ratio of $T'$ is more than $\gamma$ but $T'$ has not been removed from $T$ during the initial phase, then $c(T- \setminus T') < \frac{\gamma}{2} B$. This concludes the proof of the claim. 

We know that $c(T_\ast) > (1 + \varepsilon) B$ and the cost from $r$ to the root of $T'$ is at most $B$. Then by Claim 5, the total cost of immediate out-subtrees of $T'$ is at least $\frac{\gamma}{2} B$. Also, the cost of an immediate out-subtree of $T'$ is less than $\frac{\gamma}{2} B$, otherwise, we have a rich out-subtree. As the ratio and cost of $T_\ast$ are at least $\gamma$ and $\frac{\gamma}{2} B$, respectively, then $p(T_\ast) \geq \frac{\gamma}{2} B$. Now we distinguish between two cases:

1. If $p(T') \geq \frac{\gamma}{4} B$, by similar reasoning as above, we group the immediate out-subtrees of $T'$ into $M$ groups $S_1, \ldots, S_M$ in such a way that for each $i \in [M - 1]$ the total cost of immediate out-subtrees in $S_i$ is at least $\frac{\gamma}{2} B$, and for each $i \in [M]$ it is at most $\varepsilon B$. Now define a new group $S'_i = S_i \cup \{r\}$, for any $i \in [M]$. Let $z = \text{arg max}_{i \in [M]} p(S'_i)$. Then the group $S'_z$, which maximizes the prize is selected. We know that $M \leq \frac{4B}{\varepsilon}$. Hence, $p(S'_z) \geq \frac{\varepsilon}{4B} p(T') \geq \frac{\varepsilon}{4B} \cdot \frac{\gamma}{2} B = \frac{\varepsilon^2}{10B} B$.

Note that in case $z = M$ and $c(S'_M) < \frac{\gamma}{2} B$, we select a subset of immediate out-subtrees from $\bigcup_{i=1}^{M-1} S_i$ with the total cost of at least $\frac{\gamma}{2} B$ and at most $\varepsilon B - c(S'_M)$, and add it to $S'_z$. Let $\tilde{T}$ be the union of a shortest path $P$ from $r$ to $r'$, $S'_z$, and the edges from $r'$ to the roots of the out-subtrees in $S_z$. The cost of $\tilde{T}$ is at most $(1 + \varepsilon) B$ and the prize-to-cost ratio is at least $\frac{\varepsilon^2}{10B(1 + \varepsilon)} \geq \frac{\varepsilon^2}{32\varepsilon}$.

2. If $p(T') < \frac{\gamma}{4} B$, we proceed as follows. Consider the out-subtree $T'' = T_\ast \setminus T'$, which is rooted at $r$. Recall that by Claim 5, we have $c(T'') < \frac{\gamma}{2} B$. We connect a subset of immediate out-subtrees $T'_1, \ldots, T'_q$ of $T'$ with cost $\frac{\gamma}{2} B - c(T'') \leq c(\bigcup_{i=1}^{q} T'_i) \leq \varepsilon B - c(T'')$ to the root of $T''$ through the root of $T'$. Since the cost of each immediate out-subtree of $T'$ is less than $\frac{\gamma}{2} B$ (otherwise, we have a rich out-subtree) and $c(T') > (1 + \frac{\gamma}{2}) B$, a subset of immediate out-subtrees $T'_1, \ldots, T'_q$ of $T'$ with such a cost can be found. We call the resulting out-subtree $\tilde{T}$ and observe that $c(\tilde{T}) \geq \frac{\gamma}{2} B$ (see Figure 2). We now bound the prize-to-cost ratio of $\tilde{T}$. First note that by the submodularity of $p$, $p(T'') + p(T') \geq p(T_\ast)$. Thus by the subcase assumption and the monotonicity of $p$, we have $p(\tilde{T}) \geq p(T'') \geq \frac{\gamma}{4} B$. Since $\frac{\gamma}{2} B - c(T'') \leq c(\bigcup_{i=1}^{q} T'_i) \leq \varepsilon B - c(T'')$ and the graph is $B$-appropriate, $c(\tilde{T}) \leq (1 + \varepsilon) B$. 

![Figure 1](image-url)
Algorithm 1. In words, Algorithm 1 first computes the maximal inclusion-wise
The proof is complete.
To propose our algorithm, we need a last element. Let
The blue dashed circle represents a subset of immediate out-subtrees \( T'_1, \ldots, T'_q \) of \( T' \) with cost
\( \frac{\gamma \epsilon}{2} B - c(T'') \leq c(\cup_{i=1}^q T'_i) \leq \epsilon B - c(T''). \) The red out-subtree represents \( \hat{T} \), which is the union of \( T'' \),
the edge from \( T'' \) to \( r' \), \( \cup_{i=1}^q T'_i \) and the edges from \( r' \) to \( T'_1, \ldots, T'_q \).

Therefore, the prize-to-cost ratio of the resulting out-subtree \( \hat{T} \) is
\[
\frac{V_{\hat{T}}}{2(1+\epsilon^2)} \geq \frac{V'_{\hat{T}}}{8} \geq \frac{V'}{32 \epsilon}.
\]
The proof is complete.
To propose our algorithm, we need a last element. Let \( U = \{x_1, \ldots, x_n\} \) be a ground set, 
\( c: U \to \mathbb{Z}^+ \) be a cost function, \( f: 2^U \to \mathbb{R}^+ \cup \{0\} \) be a monotone submodular function, and \( K \) be an integer budget. In the Submodular Maximization problem (SM), we are looking for a subset \( S \subseteq U \) such that \( |S| \leq K \) and \( f(S) \) is maximum. Nemhauser et al. [28] provided a greedy algorithm that starts from \( S := \emptyset \) and runs \( K \) iterations in which, at each iteration, it adds to \( S \) the element \( x \) which maximizes \( f(S \cup \{x\}) - f(S) \). This algorithm guarantees a 
\((1 - \epsilon^{-1})\)-approximation for SM. We denote by RSM the rooted variant of SM in which, additionally, a specific element \( v \in U \) is required to be included in the solution, that is we are looking for a subset \( S \subseteq U \) such that \( |S| \leq K, v \in S \) and \( f(S) \) is maximum. We can run Nemhauser et al. [28]'s approach for RSM with a minor change: we initialize \( S := \{v\} \) and run \( K - 1 \) greedy iterations. We call this approach Greedy. It can be shown that Greedy
guarantees a \((1 - \epsilon^{-1})\)-approximation algorithm for RSM (see e.g. [23]).

Now we can propose our approximation algorithm for DRST, which is reported in Algorithm 1. In words, Algorithm 1 first computes the maximal inclusion-wise \( B \)-appropriate subgraph for \( r \) of a given graph \( D \) by removing all the nodes at a distance larger than \( B \) from \( r \). Let \( D = (V, A) \) be the resulting directed graph. For each node \( u \), it computes the set \( V_u \) of all nodes that are at a distance no more than \( c(u) + \lfloor \sqrt{B} \rfloor \) from \( u \). Let \( S'_u \) be a subset of \( V_u \) such that \( |S'_u| \leq \lfloor \sqrt{B} \rfloor + 1, u \in S'_u \), and \( p(S'_u) \) is maximum. Finding \( S'_u \) requires to solve an instance \( RSM \) of RSM where the elements are \( V_u \), the budget is \( \lfloor \sqrt{B} \rfloor + 1 \), the specific element is \( u \), and profits are defined by function \( p(\cdot) \). Using Greedy, Algorithm 1 computes in polynomial time an approximate solution \( S_u \) to \( RSM \) with \( u \in S_u, |S_u| \leq \lfloor \sqrt{B} \rfloor + 1 \) and \( p(S_u) \geq (1 - \epsilon^{-1})p(S'_u) \). Finally, for each \( u \in V \), Algorithm 1 computes a spanning out-tree \( T_u \) rooted in \( u \) that spans all the nodes in \( S_u \). Let \( z \) be a node such that \( p(T_z) \) is maximum, i.e., \( z = \arg \max_{x \in V} p(T_x) \). Then, we have \( c(T_z \setminus \{z\}) \leq B \) as \( |S_z \setminus \{z\}| \leq \lfloor \sqrt{B} \rfloor \) and 
\( dist(z, v) \leq c(z) + \lfloor \sqrt{B} \rfloor \) for any \( v \in S_z \). If \( z = r \), then Algorithm 1 defines \( T = T_z \). Otherwise, it computes a shortest path \( P \) from \( r \) to \( z \) and defines \( T \) as the union of \( T_z \) and \( P \). Since the obtained graph might not be an out-tree, Algorithm 1 removes the possible edges incoming the nodes in \( V(T_z) \cap V(P) \) that belong only to \( T_z \). The obtained out-tree \( T \) has a cost of at
An out-subtree $I_i$ is the root of $V_i$, and let $I_i$ be the full out-subtree of $I_i$ rooted at $u$, for each child $u$ of $v_i$. Each out-subtree $I_i$ is further divided into out-subtrees as follows:
Budgeted Out-Tree Maximization with Submodular Prizes

for all children \( u \) of \( v_i \) such that \( c(I_u) \geq m/2 \), we generate an out-subtree \( I_u \), observe that \( c(I_u) \leq m + c(u) \) because \( I_u \) is feasible. If all the children of \( v_i \) are in this category, we generate a further out-subtree made of only node \( u_i \).

All children \( u \) of \( v_i \) such that \( c(I_u) < m/2 \) are partitioned into groups of cost between \( m/2 \) and \( m \), plus a possible group of cost smaller than \( m/2 \). It is always possible to partition the nodes in this way since \( c(I_u) < m/2 \) for all such nodes. Then, for each of these groups, we generate an out-subtree by connecting \( v_i \) to the roots of the out-subtrees in the group.

All the generated out-subtrees have the same root \( v_i \) and cost at most \( m + c(v_i) \).

The generated out-subtrees cover all the nodes in \( I_i \). We add \( I_{i+1} \) to the set of generated out-subtrees, if it exists. Let \( T^1, \ldots, T^N \) be the set of generated out-subtrees. Since \( I_1, \ldots, I_{i+1} \) cover all the nodes of \( \hat{T} \), then so do \( T^1, \ldots, T^N \). Moreover, each generated out-subtree \( T^j \) costs at most \( m + c(r_j) \), where \( r_j \) is the root of \( T^j \).

We now bound the number \( N \) of generated out-subtrees. Given an infeasible out-subtree \( I_i \), for some \( i \in [s] \), each out-subtree generated from \( I_i \) costs at least \( m/2 \), except for the possible out-subtree made of only the root node of \( I_i \) and a possible out-subtree of cost smaller than \( m/2 \). Note that, by construction, at most one of these two additional out-subtrees can be generated. Hence, for each \( i \in [s] \), the number \( s_i \) of out-subtrees generated from \( I_i \) is

\[
s_i \leq \left\lfloor \frac{c(I_i)}{m/2} \right\rfloor + 1 \leq 2 \left\lfloor \frac{c(I_i)}{m} \right\rfloor + 2 \leq 4 \left\lfloor \frac{c(I_i)}{m} \right\rfloor.
\]

Since \( I_1, \ldots, I_{i+1} \) are disjoint, then the overall number of generated out-subtrees is at most

\[
N \leq 1 + \sum_{i \in [s]} s_i \leq 1 + \sum_{i \in [s]} 4 \left\lfloor \frac{c(I_i)}{m} \right\rfloor \leq 5 \left\lfloor \frac{c(I_i)}{m} \right\rfloor.
\]

Now we are ready to prove Theorem 1.

**Proof of Theorem 1.** By applying Lemma 6 to an optimal solution \( T^* \) and by setting \( m = \lfloor \sqrt{B} \rfloor \), we obtain \( N \leq 5 \lfloor \sqrt{B} \rfloor \) out-subtrees \( T_i = (V_i, A_i) \) of \( T^* \), for \( i \in [N] \), where \( V_i \subseteq V(T^*) \), \( A_i = (V_i \times V_i) \cap A(T^*) \), \( c(V_i) \leq c(r_i) + \lfloor \sqrt{B} \rfloor \), \( r_i \) is the root of \( T_i \), and \( \bigcup_{i=1}^{N} V_i = T^* \).

Let \( p(T^*) = \max\{p(T_i) : i \in [N]\} \) and \( w \) be the root of \( T_i \). The submodularity of \( p \) implies

\[
p(T^*) = p \left( \bigcup_{i=1}^{N} V(T_i) \right) \leq Np(T_i),
\]

which implies

\[
p(T) \geq p(T_1) \geq p(S_w) \geq (1 - e^{-1})p(S_w) \geq (1 - e^{-1})p(T^*) \geq \frac{1 - e^{-1}}{N} p(T^*) \geq \frac{1 - e^{-1}}{5 \lfloor \sqrt{B} \rfloor} p(T^*), \tag{3}
\]

where the first two inequalities hold by the definitions of \( z \) and \( S_w \) and by the monotonicity of function \( p \); The third inequality holds because \( S_w \) is a \((1 - e^{-1})\)-approximate solution for instance \( IRSM \); The fourth inequality holds as \((i) \) \( T^* \) contains nodes at a distance no more than \( c(w) + \lfloor \sqrt{B} \rfloor \) from \( w \) and contains at most \( 1 + \lfloor \sqrt{B} \rfloor \) nodes (since the minimum cost of a node is at least 1) and \((ii) \) \( p(S_w) = \max\{p(S) : |S| \leq 1 + \lfloor \sqrt{B} \rfloor \} \) and \( \text{dist}(w, v) \leq c(w) + \lfloor \sqrt{B} \rfloor \), for all \( v \in S \).

Before the trimming process in Lemma 3, the ratio between the prize and the cost of \( T \) is at least \( \gamma = \frac{1 - e^{-1}}{5 \lfloor \sqrt{B} \rfloor} p(T^*) \) as \( c(T) \leq 2B \). After applying the trimming process in Lemma 3 (with \( h = 2 \)) to \( T \), the cost of \( T \) is at most \((1 + \epsilon)B \) and its prize-to-cost ratio is:

\[
p(T) \geq c(T) \geq \frac{e^2 \gamma}{64} = a \frac{e^2}{\sqrt{B}} p(T^*),
\]

where \( a = \frac{1 - e^{-1}}{64} \). As \( c(T) \geq \epsilon B/2 \), we have \( p(T) \geq \frac{a \epsilon^2}{2 \sqrt{B}} p(T^*) \), which concludes the proof. \( \blacksquare \)
The unrooted version of DRST

Here we consider the unrooted version of DRST, denoted by DUST, in which the goal is to find an out-tree $T$ of $D$ such that $c(T) \leq B$ and $p(T)$ is maximum. Note that $T$ can be rooted at any vertex. By guessing the root of an optimal solution, we can apply the algorithm in the previous section to obtain a bicriteria $(1+\varepsilon, O(\sqrt{\frac{\beta}{\varepsilon}}))$ approximation. We now show that DUST admits an $O(\sqrt{B})$-approximation algorithm with no budget violations. To do this, we first provide an unrooted version of Lemma 3 in which it is not necessary to violate the budget constraint when each vertex costs at most half of the budget. This trimming process follows the same procedure as that of Lemma 3, but we include it for the sake of completeness.

Lemma 7. Let $T$ be an out-tree with the prize-to-cost ratio $\gamma = \frac{p(T)}{c(T)}$, where $p$ is a monotone submodular function. Suppose $\frac{B}{2} \leq c(T) \leq hB$, where $h \in (1, \eta]$ and the cost of each vertex is at most $\frac{B}{2}$. One can find an out-subtree $\tilde{T} \subseteq T$ with the prize-to-cost ratio at least $\frac{\gamma}{32\sqrt{8}}$ such that $B/4 \leq c(\tilde{T}) \leq B$.

Proof. In the initial step, we remove a strict out-subtree $T'$ of $T$ if (i) the prize-to-cost ratio of $T \setminus T'$ is at least $\gamma$, and (ii) $c(T \setminus T') \geq \frac{B}{4}$. This process is performed iteratively, until no such out-subtree exists. Let $T_-$ be the remaining out-subtree after applying this iterative process on $T$.

If $c(T_-) \leq B$, the desired out-subtree is obtained and we are done. Suppose it is not the case. A full out-subtree $T'$ is called rich if $c(T') \geq \frac{B}{4}$ and the prize-to-cost ratio of $T'$ and all its strict out-subtrees are at least $\gamma$. As in Lemma 3, we claim that the lemma follows from the existence of a rich out-subtree.

Claim 8. Given a rich out-subtree $T'$, the desired out-subtree $\tilde{T}$ can be found.

Proof. Let $T''$ be the lowest rich out-subtree of $T'$ such that the strict out-subtrees of $T''$ are not rich, i.e., $c(T'') \geq \frac{B}{4}$ while the cost of strict out-subtrees of $T''$ (if any exist) is less than $\frac{B}{4}$. Let $C$ be the total cost of the immediate out-subtrees of $T''$. We distinguish between two cases:

1. If $C < \frac{B}{4}$, then $c(T'') \leq \frac{3B}{4}$ as the root of $T''$ costs at most $\frac{B}{4}$. Since $T''$ has the prize-to-cost ratio at least $\gamma$ and cost at least $\frac{B}{4}$ (as it is rich), $\tilde{T} = T''$ is the desired out-subtree.
2. If $C \geq \frac{B}{4}$, we first group the immediate out-subtrees of $T''$ into $M$ groups $S_1, \ldots, S_M$ in such a way that for each $i \in [M-1]$ the total cost of immediate out-subtrees in $S_i$ is at least $\frac{B}{2}$, and for each $i \in [M]$ it is at most $\frac{B}{2}$. As $c(T_-) \leq hB$, we have

\[ M \leq \left\lceil \frac{hB}{B/4} \right\rceil = \lceil 4h \rceil \leq 4h + 1. \]

For each $i \in [M]$, let $S'_i = S_i \cup \{r''\}$, where $r''$ is the root of $T''$. Let $z = \arg\max_{i \in [M]} p(S'_i)$. Hence by the submodularity and monotonicity of $p$, we have

\[ p(S'_i) \geq \frac{\sum_{j=1}^{i-1} p(S'_j)}{4h+1} \geq \frac{p(S'_i) + \sum_{j=2}^{M} p(S_j)}{4h+1} \geq \frac{p(S'_i) \cup \bigcup_{j=2}^{M} S_j}{4h+1} = \frac{p(T'')}{4h+1} \geq \frac{\gamma}{16h+4} B, \]

where the last inequality holds as $p(T'') \geq \gamma \frac{B}{4}$ (since $T''$ is rich).

In case $z = M$ and $c(S'_M) < \frac{B}{4}$, we select a subset of immediate out-subtrees from $\bigcup_{i=1}^{M-1} S_i$ with the total cost of at least $\frac{B}{4}$ and at most $\frac{B}{2} - c(S_M)$, and add it to $S_z$. Let $\tilde{T}$ be the union of $r''$, the edges from $r''$ to the roots of the out-subtrees in $S_z$, and $S_z$. The cost of $\tilde{T}$ is at most $B$. Hence the prize-to-cost ratio of $\tilde{T}$ is at least $\frac{\gamma}{16h+4} \geq \frac{\gamma}{32\sqrt{8}}$. ▶
It only remains to consider the case when there is no rich out-subtree. Since $T_-$ is not rich and $c(T_-) \geq \frac{R}{4}$, the ratio of at least one of the strict out-subtrees of $T_-$ is less than $\gamma$. Now we find an out-subtree $T'$ with ratio less than $\gamma$ such that the ratio of all of its strict out-subtrees (if any exist) is at least $\gamma$. Since the ratio of $T'$ is less than $\gamma$ and $T'$ is not removed in the initial process, $c(T_--T') < \frac{R}{4}$ (this can be shown by the same argument as that of Claim 5). As $c(T_-) > B$ and the cost of the root of $T'$ is at most $\frac{R}{4}$, the total cost of the immediate out-subtrees of $T'$ is at least $\frac{R}{4}$. Also, the cost of an immediate out-subtree of $T'$ is less than $\frac{R}{4}$, otherwise we have a rich out-subtree. As the ratio and cost of $T_-$ are at least $\gamma$ and $\frac{R}{4}$ respectively, then $p(T_-) \geq \frac{R}{4}$. We distinguish between two cases.

1. If $p(T') \geq \frac{R}{4}$, by the similar reasoning as above, we partition the immediate out-subtrees of $T'$ into $M$ groups $S_1, \ldots, S_M$ in such a way that for each $i \in [M-1]$ the total cost of immediate out-subtrees in $S_i$ is at least $\frac{R}{4}$, and for each $i \in [M]$ it is at most $\frac{R}{4}$. For each $i \in [M]$, let $S'_i = S_i \cup \{r'\}$ where $r'$ is the root of $T'$. Let $z = \arg\max_{i \in [M]} p(S'_i)$.

As $M \leq 4h+1$, by the submodularity and monotonicity of $p$ we have:

$$p(S'_z) \geq \frac{\sum_{i=1}^M p(S'_i)}{4h+1} \geq \frac{p(S'_1) + \sum_{i=2}^M p(S_i)}{4h+1} \geq \frac{p(S'_1 \cup \bigcup_{i=2}^M S_i)}{4h+1} \geq \frac{p(T')} {4h+1} \geq \frac{\gamma}{32h+8},$$

where the last inequality holds as $p(T') \geq \frac{R}{4}$.

Note that in case $z = M$ and $c(S'_M) < \frac{R}{4}$, we select a subset of immediate out-subtrees from $\bigcup_{i=1}^{M-1} S_i$ with the total cost of at least $\frac{R}{4}$ and at most $\frac{R}{4} - c(S_M)$, and add it to $S_z$. Let $\hat{T}$ be the union of $r'$, the edges from $r'$ to the roots of the out-subtrees in $S_z$ and $S_z$. The cost of $\hat{T}$ is at most $B$ and its prize-to-cost ratio is at least $\frac{\gamma}{32h+8}$.

2. If $p(T') < \frac{R}{4}$, we proceed as follows. Consider the out-subtree $T'' = T_--T'$. Recall that by the above discussion we have $c(T'') < \frac{R}{4}$. Thus we find a subset $S$ of the immediate out-subtrees of $T''$ with cost between $\frac{R}{4} - c(T'') \leq c(S) \leq \frac{R}{4} - c(T'')$. Note that such set $S$ can be found as each immediate out-subtree of $T''$ costs less than $\frac{R}{4}$ (otherwise we have a rich subtree) and $c(T'') > \frac{R}{4}$ (as $c(T_-) > B$ and $c(T'') < \frac{R}{4}$). Then let $\hat{T}$ be the union of $T''$ to $r'$ in $T_-, S$, and the edges from $r'$ to the roots of the out-subtrees in $S$, where $r'$ is the root of $T'$. We now bound the prize-to-cost ratio of $\hat{T}$. Recall that $T'' = T_--T'$. First note that by the submodularity' properties $p(T'') + p(T') \geq f(T_-)$. Thus by the case assumption and monotonicity, we have $f(\hat{T}) \geq p(T'') \geq \frac{R}{4}$. Since $\frac{R}{4} - c(T'') \leq c(S) \leq \frac{R}{4} - c(T'')$ and $c(r') \leq \frac{R}{4}$, $c(\hat{T}) \leq B$. Therefore, the prize-to-cost ratio of $\hat{T}$ is at least $\frac{\gamma}{32h+8}$.

The proof is complete.

**Theorem 9.** $\textbf{DUST}$ admits a polynomial-time $O(\sqrt{B})$-approximation algorithm.

**Proof.** We follow arguments similar to those in Theorem 4 from Bateni et al. [2], but for the sake of completeness the proof is provided here.

An out-tree is called flat if each vertex of the out-tree costs no more than $\frac{R}{4}$. Let $x$ be a vertex of an out-tree with the largest cost. An out-tree is called saddled if $c(x) > \frac{R}{4}$ and the cost of every other vertex of the out-tree is no more than $\frac{R-c(x)}{2}$. Let $T^*_f$ (resp. $T^*_s$) be the optimal flat (resp. saddled) out-tree, i.e., a flat (resp. saddled) out-tree with cost at most $B$ maximizing the prize. We first show that given an optimal solution $T^*$ to $\textbf{DUST}$, then either $p(T^*_f) \geq \frac{p(T^*)}{2}$, or $p(T^*_s) \geq \frac{p(T^*)}{2}$.

**Claim 10.** Either $p(T^*_f) \geq \frac{p(T^*)}{2}$ or $p(T^*_s) \geq \frac{p(T^*)}{2}$, where $T^*$ is an optimal solution to $\textbf{DUST}$. ▷
Proof. If $T^*$ has only one vertex, then it is either flat or saddled and we are done. If $T^*$ has more than one vertex and it is neither flat nor saddled, then we proceed as follows. Let $x$ and $y$ be two vertices in $T^*$ with the maximum cost and the second maximum cost, respectively. Since $T^*$ is not flat then $c(x) > \frac{B}{2}$ and $c(y) \leq \frac{B}{2}$. Also as $T^*$ is not saddled, $c(y) > \frac{B-c(x)}{2}$. By removing the edge $e$ adjacent to $y$ on the path between $x$ and $y$, we can partition $T^*$ into two out-subtrees $T^*_x$ and $T^*_y$ that contain $x$ and $y$, respectively. Clearly, each vertex in $T^*_y$ costs no more than $\frac{B}{2}$, then $T^*_y$ is flat. Also, each vertex in $T^*_x$ except $x$ costs at most $\frac{B-c(x)}{2}$, implying that $T^*_x$ is saddled. By the submodularity of $p$, $p(T^*_x) + p(T^*_y) \geq p(T^*)$, meaning that one of $T^*_x$ and $T^*_y$ has at least half of the optimum prize $p(T^*)$, which concludes the claim. □

Now we restrict Algorithm 1 to only flat and saddled out-trees. Indeed, we can reduce the case of saddled out-trees to flat out-trees as follows. We first find a vertex $x$ with the maximum cost. We then set the cost of $x$ to zero and define a new budget $B' = B - c(x)$. Note that the cost of any other vertex in the optimal saddled out-tree $T^*_x$ is at most half of the remaining budget. This means that we only need to find an approximation solution when restricted to flat out-trees.

Since for the new instance no other vertex except $x$ with cost more than $\frac{B}{2}$ can be contained in the final solution, we remove all vertices with cost more than $\frac{B}{2}$ and run Lines 1-8 of Algorithm 1 on the new resulting graph to achieve an out-tree $T$ with cost $c(T) \leq 2B$ (as $c(T \setminus \{x\}) = B$ and $c(x) \leq B$) and prize $p(T) \geq \frac{1-\epsilon^{-1}}{5\sqrt{B}}p(T^*_x) \geq \frac{1-\epsilon^{-1}}{10\sqrt{B}}p(T^*)$. So, the prize-to-cost ratio of $T$ is $\gamma \geq \frac{p(T)}{c(T)}$. As $T$ is flat, we can apply Lemma 7 to achieve an out-subtree $\hat{T}$ of $T$ with the cost $B/4 \leq c(\hat{T}) \leq B$ and the prize-to-cost ratio $\frac{p(\hat{T})}{c(\hat{T})} \geq \frac{\gamma}{32h+8} = \frac{\gamma}{72}$ as $h \leq 2$. This implies that

$$p(\hat{T}) \geq \frac{\gamma}{72} \cdot \frac{B}{4} = \frac{\gamma}{288} B, \quad \frac{p(T)}{c(T)} \geq \frac{1-\epsilon^{-1}}{5760\sqrt{B}} p(T^*).$$

6 Submodular Tree Orienteering

Recently, Ghuge and Nagarajan [11] studied the Submodular Tree Orienteering problem (STO), which is similar to DRST with the only difference that the costs are associated to the edges of a directed graph instead of the nodes. In particular, in STO, we are given a directed graph $D = (V, A)$, a vertex $r \in V$, a budget $B$, a monotone submodular function $p : 2^V \to \mathbb{R}^+$, and a cost $c : A \to \mathbb{Z}^+$, and the goal is to find an out-subtree $T$ of $D$ rooted at $r$ such that $\sum_{e \in AT} c(e) \leq B$ and $p(T) = p(V(T))$ is maximum. Ghuge and Nagarajan [11] proposed an $O\left(\frac{\log k}{\log \log k}\right)$-approximation algorithm for STO that runs in $(n \log B)^{O(1/\epsilon^2 \log k)}$ time for any constant $\epsilon > 0$, where $k \leq |V|$ is the number of vertices in an optimal solution.

Here we first show that DRST can be reduced to STO, preserving the approximation factor, by assigning the cost of each node $v$ to all edges entering $v$.

**Theorem 11.** There is an $O\left(\frac{\log k}{\log \log k}\right)$-approximation algorithm for DRST that runs in $(n \log B)^{O(1/\epsilon^2 \log k)}$ time for any constant $\epsilon > 0$, where $k \leq n = |V|$ is the number of vertices in an optimal solution.

**Proof.** To prove the theorem, we show that one can transform an instance $J = (D' = (V', A'), p', c', r', B')$ of DRST to an instance $I = (D = (V, A), p, c, r, B)$ of STO as follows. We set $V = V'$, $r = r'$, $A = A' \setminus \{(a, r') | (a, r') \in A'\}$, $B = B' - c'(r')$ and for any subset...
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\[ S \subseteq V, \ p(S) = p'(S). \] For any \( e = (i, j) \in A \) in \( I \), we set \( c(e) = c'(j) \). The theorem follows by observing that any out-subtree \( T \) of \( D \) is an out-subtree for \( D' \), \( c'(T) = \sum_{e \in V(T)} c'(e) = \sum_{e=(u,v) \in A(T)} c(e) + c'(j) = c(T) + c'(j) \), and \( p'(T) = p(T) \).

Moreover, we show that Algorithm 1 can be used to approximate STO. To our knowledge, this is the first polynomial-time bicriteria approximation algorithm for STO.

**Theorem 12.** There exists a polynomial-time bicriteria \((1 + \epsilon, O(\sqrt{\frac{\epsilon}{\epsilon'}}))\)-approximation algorithm for STO, for any \( \epsilon \in (0, 1] \).

**Proof.** We first transform an instance \( I_S = (D_S = (V_S, A_S), p_S, c_S, r, B) \) of STO to an instance \( I_D = (D_D = (V_D, A_D), p_D, c_D, r, B) \) of DRST, where \( V_D = V_S \cup V_A, V_A = \{ v_e : e \in A_S \} \), \( A_D = \{(i, v_e), (v_e, j) : e = (i, j) \in A_S \} \), \( p_D(S) = p_S(S \cap V_S) \), for each \( S \subseteq V_D \), \( c_D(v) = 0 \) for each \( v \in V_S \), and \( c_D(v_e) = c_S(e) \) for each \( v_e \in V_A \).

Let \( T^*_D \) be an optimal solution for \( I_D \) and let \( T^*_D \) be the out-subtree of \( D' \) corresponding to \( T^*_S \) (i.e. \( V(T^*_D) = V(T^*_S) \cup \{ v_e : e \in A(T^*_S) \} \), \( A(T^*_D) = \{ (i, v_e), (v_e, j) : e = (i, j) \in A(T^*_S) \} \)). We observe that \( p_D(T^*_D) = p_S(T^*_S), c_D(T^*_D) = c_S(T^*_S), \) and \( T^*_D \) is an optimal solution for \( I_D \), since if there exists an out-subtree \( T_D \) of \( D_D \) with \( p_D(T_D) > p_D(T^*_D) \), then we can construct an out-subtree \( T_S = (V(T_D) \cap V_S, (e \in A_S : v_e \in V(T_D) \cap V_A)) \) of \( D_S \) such that \( p_S(T_S) > p_S(T^*_S) \).

We decompose \( T^*_D \) as in Lemma 6, with \( m = \sqrt{B} \); let \( T^*_D \) be the out-subtree that maximizes the prize among those returned by the lemma, and let \( w \) be the root of \( T^*_D \). We have that \( p_D(T^*_D) \geq \frac{1}{5\sqrt{B}}p_D(T_D) \) and \( c(T^*_D) \leq c(w) + 5\sqrt{B} \). It follows that the distance from \( w \) to any other node in \( T^*_D \) is at most \( c(w) + 5\sqrt{B} \).

We now show that \( |V(T^*_D) \cap V_S| \leq \sqrt{B} + 1 \). Since the cost of nodes in \( V_S \) is equal to 0, then \( c(V(T^*_D) \cap V_A \setminus \{ w \}) = c(V(T^*_D) \cap V_A) \leq \sqrt{B} \). Therefore, as the cost of each edge in \( A_S \) is at least 1, \( |V(T^*_D) \cap V_A \setminus \{ w \}| \leq \sqrt{B} \). For every node in \( (V(T^*_D) \cap V_S) \setminus \{ w \} \), there exists a distinct node in \( V(T^*_D) \cap V_A \), which means that \( |V(T^*_D) \cap V_A| = |V(T^*_D) \cap V_A \cap \{ w \}| \leq \sqrt{B} \). If \( w \in V_S \), then \( |V(T^*_D) \cap V_A| = |V(T^*_D) \cap V_A \cap \{ w \}| \leq \sqrt{B} \). If \( w \in V_A \), then \( |(V(T^*_D) \cap V_A)| \leq \sqrt{B} + 1 \).

In both cases \( |V(T^*_D) \cap V_S| \leq \sqrt{B} + 1 \).

Let \( T_D \) be the output of lines 1-11 of Algorithm 1 for instance \( I_D \). We have that

\[ p_D(T_D) \geq (1 - \epsilon^{-1})p_D(S^*_w) \geq (1 - \epsilon^{-1})p_D(T^*_D) \geq \frac{1 - \epsilon^{-1}}{5\sqrt{B}} p_D(T^*_D) = \frac{1 - \epsilon^{-1}}{5\sqrt{B}} p_S(T^*_S), \]

where the second inequality is due to the fact that (i) \( T^*_D \) contains nodes at a distance no more than \( c(w) + 5\sqrt{B} \) from \( w \) and contains at most \( \sqrt{B} + 1 \) nodes in \( V_S \), and (ii) \( p_D(S) = p_D(S \cap V_S) \), for each \( S \subseteq V_D \), and therefore \( p_D(S^*_w) = \max_{S \subseteq V_S} \{ p_D(S) : |S \cap V_S| \leq \sqrt{B} + 1 \text{ and } \text{dist}(w, v) \leq c(w) + 5\sqrt{B} \} \), for all \( v \in S \). The other inequalities are analogous to those in (3).

The cost of \( T_D \) is at most \( 2B \), as in Theorem 1 we can trim \( T_D \) to reduce its cost to \( (1 + \epsilon)B \) and maintaining a prize of \( p_D(T_D) = \frac{2\epsilon}{\sqrt{B}} p_D(T^*_D) \), for some constant \( \alpha \) and any arbitrary \( \epsilon > 0 \).

Let us consider the out-subtree \( T_S \) of \( D_S \) corresponding to \( T_D \), \( T_S = (V(T_D) \cap V_S, (e \in A_S : v_e \in V(T_D) \cap V_A)) \), then \( p_S(T_S) = p_D(T_D) \) and \( c_S(T_S) = c_D(T_D) \), which concludes the proof.

### 7 Further Results on Some Variants of DRST

In this section, we provide approximation results on some variants of DRST. Due to space constraints, here we only state our results, all the details are given in a long version of the paper [6].

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2 Note that the Lemma 3 and 6 hold even if node costs are allowed to be equal to 0.
Additive prize function and Directed Tree Orienteering (DTO). We consider the special case of DRST in which the prize function is additive, i.e., for any $S \subseteq V$, $p(S) = \sum_{v \in S} p(v)$, called DRAT. We show that there exists a polynomial-time bicriteria $(1 + \epsilon, O(\sqrt{B}/\epsilon^2))$-approximation algorithm for DRAT (Theorem B.1 in [6]). By using the reduction in Theorem 12, it follows that this result also holds for DTO, which is the special case of STO in which the prize function is additive [11].

Undirected graphs. All our results hold also in the case in which the input graph is undirected and the output graph is a tree. In particular, our $O(\sqrt{B})$-approximation algorithm for the unrooted case improves over the factors $O((\Delta + 1)\sqrt{B})$ [23] and $\min\{1/((1 − 1/e)(1/R − 1/B)), B\}$ [15], where $R$ is the radius of the input graph $G$. For the case in which the prize function is additive, we show that there exists a polynomial-time bicriteria approximation algorithm whose approximation factor only depends on the the maximum degree $\Delta$ of the given graph. In particular, it guarantees a bicriteria $(1 + \epsilon, 16\Delta/\epsilon^2)$-approximation (Theorem B.2 in [6]).

Quota problem. We consider the problem in which we are given an undirected graph $G = (V, E)$, a cost function $c : V \rightarrow \mathbb{R}^+$, a prize function $p : V \rightarrow \mathbb{R}^+$, a quota $Q \in \mathbb{R}^+$, and a vertex $r$, and the goal is to find a tree $T$ such that $p(T) \geq Q$, $r \in V(T)$ and $c(T)$ is minimum. We prove that this problem admits a 2$\Delta$-approximation algorithm (Theorem B.3 in [6]).

Maximum Weighted Budgeted Connected Set Cover (MWBCSC). Let $X$ be a set of elements, $S \subseteq 2^X$ be a collection of sets, $p : X \rightarrow \mathbb{R}^+$ be a prize function, $c : S \rightarrow \mathbb{R}^+$ be a cost function, $G_S$ be a graph on vertex set $S$, and $B$ be a budget. In MWBCSC, the goal is to find a subcollection $S' \subseteq S$ such that $c(S') = \sum_{S \in S'} c(S) \leq B$, the subgraph induced by $S'$ is connected and $p(S') = \sum_{x \in X_{S'}} p(x)$ is maximum, where $X_{S'} = \bigcup_{S \in S'} S$. We show that MWBCSC admits a polynomial-time $af$-approximation algorithm, where $f$ is the maximum frequency of an element and $a$ is the performance ratio of an algorithm for the unrooted version of MCSB with additive prize function (Theorem B.4 in [6]). Moreover, one can have a polynomial-time $O(\log n)$-approximation algorithm for MWBCSC under the assumption that if two sets have an element in common, then they are adjacent in $G_S$ (Corollary B.2 in [6]). This last result is an improvement over the factor $2(\Delta + 1)a/(1 − e^{-1})$ by Ran et al. [30].

Budgeted Sensor Cover Problem (BSCP). In BSCP, we are given a set $S$ of sensors, a set $\mathcal{P}$ of target points in a metric space, a sensing range $R_s$, a communication range $R_c$, and a budget $B$. A target point is covered by a sensor if it is within distance $R_s$ from it. Two sensors are connected if they are at a distance at most $R_c$. The goal is to find a subset $S' \subseteq S$ such that $|S'| \leq B$, the number of covered target points by $S'$ is maximized and $S'$ induces a connected subgraph. We give a 2$f$-approximation algorithm for BSCP (Theorem B.5 in [6]), where $f$ is the maximum number of sensors that cover a target point. We also show that, under the assumption that $R_c \leq R_c/2$, BSCP admits a polynomial-time $8/(1 − e^{-1})$-approximation algorithm (Theorem B.8 in [6]), which improves the factors $8(2\sqrt{2C} + 1)^2/(1 − 1/e)$ [17] and $8(4C/\sqrt{3})^2/(1 − 1/e)$ [34], where $C = R_s/R_c$. Note that Huang et al. [17] do not assume that $R_c/R_c \leq R_c/2$, however, our technique improves over their result if $R_c \leq R_c/2$. 


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A Further Related Work

Zelikovsky [35] provided the first approximation algorithm for Directed Steiner Tree Problem (DSTP) with factor $O(k(\log^{1/2}k))$ that runs in $O(n^{1/2})$, where $|V| = k$. Charikar et al. [3] proposed a better approximation algorithm for DSTP with a factor $O(\log^2 k)$ in quasi-polynomial time. Grandoni et al. [13] improved this factor and provided a randomized $O(\frac{\log^2 k}{\log \log k})$-approximation algorithm $n^{O(\log^2 k)}$ time. Also, they showed that, unless $NP \subseteq \cap_{0<\epsilon<1} \text{ZPTIME}(2^{\epsilon n})$ or the Projection Game Conjecture is false, there is no quasi-polynomial time algorithm for DSTP that achieves an approximation ratio of $o(\frac{\log^2 k}{\log \log k})$. Ghuge and Nagarajan [11] showed that their approximation algorithm results in a deterministic $O(\frac{\log^2 k}{\log \log k})$-approximation algorithm for DSTP in $n^{O(\log^{1+\epsilon} k)}$ time. Very recently, Li and Laekhanukit [26] showed that the lower bound on the integrality gap of the flow LP is polynomial in the number of vertices.

Danilchenko et al. [7] investigated a closely related problem to BSCP, where the goal is to place a set of connected disks (or squares) such that the total weight of target points in the plane is maximized. They provided a polynomial-time $O(1)$-approximation algorithm for this problem. MCSV is also closely related to the budgeted connected dominating set problem, where the goal is to select at most $B$ connected vertices in a given undirected graph to maximize the profit function on the set of selected vertices. Khuller et al. [19] investigated this problem in which the profit function is a special submodular function. Khuller et al. [19] designed a $\frac{11K}{10K-7}$-approximation algorithm. By generalizing the analysis of Khuller et al. [19], Lamprou et al. [24] showed that there is a $\frac{11K}{10K-e^{-1/17}}$-approximation algorithm for the budgeted connected dominating set problem. They also showed that for this problem we cannot achieve in polynomial time an approximation factor better than $\left(\frac{1}{1-e^{-1/17}}\right)$, unless $P = NP$.

Lee and Dooley [25] provided a $(B - 2)$-approximation algorithm for URAT, where each vertex costs 1. Zhou et al. [36] studied a variant of E-URAT in the wireless sensor networks and provided a 10-approximation algorithm. Seufert et al. [31] investigated a special case of the unrooted version of URAT, where each vertex has cost 1 and we aim to find a tree with at most $B$ nodes maximizing the accumulated prize. This coincides with the unrooted version of E-URAT when the cost of each edge is 1 and we are looking for a tree containing at most $B - 1$ edges to maximize the accumulated prize. Seufert et al. [31] provided a $(5 + \epsilon)$-approximation algorithm for this problem. Similarly, Huang et al. [16] investigated this variant of E-URAT (or URAT) in the plane and proposed a 2-approximation algorithm.

The quota variant of URAT also has been studied, which is called Q-URAT. Here we wish to find a tree including a vertex $r$ in a way that the total cost of the tree is minimized and its prize is no less than some quota $r$. By using Moss and Rabbani [27] ‘black box and the ideas of Könenmann et al. [21], and Bateni et al. [2], we have an $O(\log n)$-approximation algorithm for Q-URAT. This bound is tight [27]. The edge cost variant of Q-URAT, called EQ-URAT, has been investigated by Johnson et al. [18]. They showed that by adapting an $\alpha$-approximation algorithm for the $k$-MST problem, one can have an $\alpha$-approximation algorithm for EQ-URAT. Hence, the 2-approximation algorithm of Garg [10] for the $k$-MST problem results in a 2-approximation algorithm for EQ-URAT.

The prize collecting variants of URAT have also been studied. Könenmann et al. [21] provided a Lagrangian multiplier preserving $O(\ln n)$-approximation algorithm for NW-PCST, where the goal is to minimize the cost of the nodes in the resulting tree plus the penalties of vertices not in the tree. Bateni et al. [2] considered a more general case of NW-PCST and provided an $O(\log n)$-approximation algorithm. There exists no $o(\ln n)$-approximation algorithm for NW-PCST, unless $NP \subseteq \text{DTIME}(n^{\text{Polylog}(n)})$ [20]. The edge cost variant
of NW-PCST has been investigated by Goemans and Williamson [12]. They provided a 2-approximation algorithm for EW-PCST. Later, Archer et al. [1] proposed a \((2 - \varepsilon)\)-approximation algorithm for EW-PCST which was an improvement upon the long standing bound of 2.

Table 1 A summary of the best bounds on some variants of prize collecting problems.

| Problem   | Best Bound                                      |
|-----------|-------------------------------------------------|
| STO       | \(O\left(\frac{\log n}{\log \log n}\right)\) [11] (tight) |
| DTO       | \(O\left(\frac{\log n}{\log \log n}\right)\) [11] (tight) |
| DSTP      | \(O\left(\frac{\log k}{\log \log k}\right)\) [11, 13] (tight) |
| NW-PCST   | \(O(\log n)\) [2, 21] (tight)                  |
| EW-PCST   | \(2 - \varepsilon\) [1]                       |
| URAT      | \((1 + \varepsilon, O\left(\frac{\log n}{\log \log n}\right)\)) [2, 21, 27] |
| E-URAT    | \(2\) [29]                                      |
| Q-URAT    | \(O(\log n)\) [2, 21, 27] (tight)              |
| EQ-URAT   | \(2\) [10, 18]                                  |