Modulation instability and rogue waves for the sixth-order nonlinear Schrödinger equation with variable coefficients on a periodic background

Wei Shi · Zhaqilao

Received: 24 February 2022 / Accepted: 16 May 2022 / Published online: 10 June 2022
© The Author(s), under exclusive licence to Springer Nature B.V. 2022

Abstract In this paper, rogue wave solutions of a sixth-order focusing nonlinear Schrödinger (NLS) equation with variable coefficients are investigated on a periodic background. To get the results, we take advantage of Darboux transformation approach and the nonlinearization of spectral problem and we firstly find one kind of rogue wave solution that evolves periodically with time on a periodically spatial background. Besides, we also find this kind of rogue wave solution dissipates over time. Modulation instability (MI) of the sixth-order focusing NLS equation with variable coefficients is also studied.

Keywords Rogue wave on a periodic background · Sixth-order focusing nonlinear Schrödinger equation · Variable coefficient · Modulation instability

1 Introduction

In nonlinear science, modulation instability is a basic physical phenomenon, which refers to the self-modulation of amplitude and frequency for continuous waves or quasi-continuous waves when coming through nonlinear dispersive medium. It is a phenomenon that leads to many interesting physical problems, for example, it results in the fracture of water-gravity waves in the deep ocean, the appearance of a periodic train of localized waves or pulses, the formation of envelope solitons in electrical transmission lines and so on. As a result, MI has been studied in many fields, such as hydrodynamics [1], biology [2], plasmas [3] and optical fibers [4–6]. MI can be also known as the Benjamin–Feir instability [1]. In addition, for the reason that MI gives rise to the exponentially growth of random disturbance, MI has something to do with the formation of rogue waves in some degree [7] and the phenomenon of MI is quite worthy studying.

Damping, which is a physical phenomenon with a close connection with MI, also needs to be considered here. Even when damping is weak, it can have a strong effect on the formation of rogue waves. As in [8], damping is critical to prevent the further development of rogue waves. When permanent downshifting occurs, rogue waves will stop developing. Recently, there are many papers about damping. In [9], the damped NLS equation was transformed into the standard NLS equation with constant coefficients by using a suitable transformation. The effects of the wind and the nonlinear damping of the MI were also considered. In [10], the effects of wind and nonlinear
damping on permanent downshift and the formation of rogue waves were investigated for a higher-order NLS model.

Rogue waves are considered to be a kind of weird waves with extremely large amplitude. The phenomena of rogue waves exist in many fields, such as the ocean [11], the optical fibers [12] and the Bose–Einstein condensates [13, 14]. It usually comes out from nowhere and disappears without trace [15], which may lead to some fatal catastrophe in ocean. Therefore, the study of rogue waves is of considerable necessity and practicality and conforms to people’s universal values. If we reveal the formation mechanism and nature of rogue waves, we can forecast it more accurate in advance and give forewarnings, which may safeguard lives and property.

The analytical expression of the first-order rogue wave was proposed by Peregrine [16] in 1983. In the current soliton theory, the problem of rogue wave on a periodic background is a subject with considerable attention. By means of the nonlinearization of spectral problem [17] and Darboux transformation approach [18–22], periodic standing waves of various equations have been investigated, such as mKdV equation [23], the NLS equation [24–29], the Hirota equation [30], the sine-Gordon equation [31] and the fifth-order Ito equation [32]. However, all the above mentioned are equations with constant coefficients, and in this paper, we focus on a variable-coefficient equation.

Mathematically speaking, rogue waves can be well simulated by NLS equation [15], which plays a significant role in the field of nonlinear physics. It was proposed firstly in 1926 by Erwin Schrödinger. It is a partial differential equation used to describe nonlinear waves and can be applied to many nonlinear physics problems, such as nonlinear optics, quantum mechanics, and ion acoustic waves in plasma. Recently, there are many studies for the higher-order rogue wave solutions of the NLS equation. For example, Yue et al. [28] studied MI, spectral analysis and rogue waves for the sixth-order NLS equation. Zhang et al. [29] investigated rogue wave solutions on the periodic background of the fourth-order NLS equation. Sun et al. [33] considered the existence and properties of vector rogue waves for the higher-order matrix NLS equation. In addition, there have been abundant works with profound significance and advance on the higher-order NLS hierarchy in the past few years. For example, Kedziora et al. [34] proposed an infinite NLS equation hierarchy of integrable equations in 2015. In 2016, Ankiewicz et al. [35] studied the infinite integrable NLS equation hierarchy and put forward the generalized Lax pair and various solutions. In 2019, Anastasia et al. [36] considered the generalized matrix NLS hierarchy and identified recursion relations that yield the Lax pairs for the whole matrix NLS-type hierarchy. The infinite integrable NLS equation has the following form

\[ \begin{align*}
\mathcal{M}_2(q) &= q_{xx} + 2q|q|^2, \\
\mathcal{M}_5(q) &= q_{xxxxx} + 6|q|^2 q_x, \\
\mathcal{M}_4(q) &= q_{xxxx} + 6q^* q_x^2 + 4q|q|^2 + 8|q|^2 q_x \\
&+ 2q^2 q_x^2 + 6|q|^4 q_x, \\
\mathcal{M}_5(q) &= q_{xxxxx} + 10|q|^2 q_{xxx} + 30|q|^4 q_x + 10q_{xx} q_{xxx}^* \\
&+ 10q q_x q_{xxx} + 20q^* q_x q_{xx} + 10q^2 q_{xx}^*, \\
\mathcal{M}_6(q) &= q_{xxxxxx} + q^3 [60|q_x|^2 q^* - 50q_{xx}(q^*)^2 \\
&+ 2q_{xxxx}^2] + q^2 [12q^2 q_{xxx} + 18q^2 q_{xx} + 8q^3 q_{xxx} \\
&+ 60q^2 q_{xx} + 30q^3 q_{xxx} + 5q^2 q_x \\
&+ 2q q_{xxx} + 10q^2 q_{xx}^* + (q^*)^2 + 20q^2 q_{xx}] \\
&+ 20q|q|^6, \\
\mathcal{M}_7(q) &= q_{xxxxxxx} + 70q_{xx} q_{xxx}^* + 112q_{xxx} q_{xx}^2 \\
&+ 98q_{xxx}^2 q_{xx} + 28q_{xxx} q_{xx}^* + 70q^2 [2q^2 q_{xx} + (q^*)^3] \\
&+ q^*(2q_{xx} q_{xxx}^* + q_{xxx} q^*) + 14q^2 (20q_{xx} q_{xxx}^* q_{xx} \\
&+ q_{xxxxx} + 3q_{xxx} q_{xx}^* + 2q_{xxx} q_{xx}^* + 2q_{xxx} q_{xx}^*) \\
&+ q_{xxx} + q_{xxx}^* + 20q_{xxx} q_{xx}^*(q^*)^2] + 140q^6 q_x \\
&+ 70q_{xx}^2 (q^*)^2 + 14q^2 (5q_{xxx} q_{xx} + 3q_{xxxxx}),
\end{align*} \]

where \( q = q(x, t) \) is the spatial variable, \( t \) is the time variable, \( |q| = |q(x, t)| \) means the envelope of the wave, the asterisk * denotes the complex conjugate and \( q_{j}(j = 2, 3, 4, \cdots, \infty) \) represents the \( j \)-order real

\( \circ \) Springer
coefficients. It is also necessary to talk about \( M_j(q)(j = 2, 3, \ldots, 7) \). \( M_5(q) \) is the NLS operator. If we take \( x_j \neq 0 \) and \( x_j = 0(j = 3, 4, \ldots, 7) \), we obtain the fundamental NLS equation, which has been specifically mentioned above. \( M_3(q) \) is the Hirota operator. On the condition of the fundamental NLS equation, if we add \( x_3 \neq 0 \), we obtain the Hirota equation. It was firstly proposed in 1973 by Ryogo Hirota [37], which describes the propagation of the fundamental NLS equation, if we add \( x_4 \neq 0 \), we obtain the LPD equation. It was firstly proposed in 1988 by Lakshmanan et al [38], which is used to describe the propagation of the fundamental NLS equation, if we add \( x_5 \neq 0 \), we obtain the LPD equation. It was firstly proposed in 2014 by Adrian Ankiewicz et al [35]. In addition, we take \( x_6 \neq 0 \) and \( x_7 \neq 0 \). Thereby, we also take \( x_2 = \frac{1}{2} \). Therefore, we also take \( x_2 = \frac{1}{2} \).

In this paper, we mainly discuss rogue waves on a periodic background. Based on Eq. (1), we consider a sixth-order focusing NLS equation with variable coefficients as

\[-q + \frac{1}{2} M'_5(q) - i \varepsilon_5(t)M'_5(q) + \varepsilon_5(t)M'_6(q)\]

(3)

where \( \varepsilon_j(t) (j = 3, 4, 5, 6) \) are the time-dependent variable parameters and \( M'_j(j = 2, 3, 4, 5, 6) \) have the following form

\[ M'_5(q) = iq_{xx}^* + 2iq^*|q|^2, \]

\[ M'_6(q) = iq_{xxx}^* + 6qi^2|q|^2q^*, \]

\[ M'_7(q) = iq_{xxxx}^* + 6iq(q^*)^2 + 4iq^*|q|^2 + 8iq^*q_{xx}^* + 2i(q^*)^2q_{xx} + 6i|q|^4q^*. \]

\[ M'_8(q) = iq_{xxxxx}^* + 10i|q|^2q_{xx}^* + 30i|q|^4q^* \]

+ 10iq^*q_{xx}^* + 10iq^*q_{xx}^* + 20iqq_{xx}^* \]

+ 10i(q^*)^2q^*.

\[ M'_9(q) = iq_{xxxxx}^* + 1 + (q^*)^2[60|q|^2q + 50q^2q_{xx}] \]

+ 2iq_{xxxx}^* + q^*[12iq^*q_{xx} + 18iq^*q_{xx} \]

+ 8iq^*q_{xxx} + 70q^2(q^*)^2 + 22|q_{xx}|^2 \]

+ 10iq^*[3qq_{xxx} + 5qq_{xxx} + 2q_{xx}^*] \]

+ 10iq^*[32qq_{xx} + (q^*)^2 + 20i(q^*)_{xx}^* \]

+ 20iq^*|q|^6. \]

Equation (3) has the Lax pair

\[ \Phi_x = U\Phi, U' = \begin{pmatrix} \lambda & q \\ -q & -\lambda \end{pmatrix}, \Phi = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \]

(5)

\[ \Phi_y = V\Phi, V' = \sum_{j=0}^{6} \lambda^j \begin{pmatrix} A_j^s & B_j^s \\ B_j^s & -A_j^s \end{pmatrix}, \]

(6)

with

\[ A_0 = -\frac{1}{2}|q|^2 - 3\varepsilon_4(t)|q|^4 - 10\varepsilon_6(t)|q|^6 \]

+ \varepsilon_4(t)|q|^2 - \varepsilon_6(t)|q_{xx}|^2 - 5\varepsilon_6(t)q^2(q^*)^2 \]

+ (q^*)^2q_{xx}^* - (10\varepsilon_6(t)|q|^2 + \varepsilon_4(t))(qq_{xx}^* \]

+ q^*q_{xx} - \varepsilon_6(t)(q_{xx}^* + q_{xxx}^* - q_{xxx}) \]

- q^*_{xxx} - (i\varepsilon_5(t) + 6i\varepsilon_5(t)|q|^2)(q^*q_{xx} - q_{xx}^*q) \]

- i\varepsilon_5(t)(q_{xx}^* + q_{xxx}^* - q_{xxx}). \]

\[ A'_1 = 2\varepsilon_5(t)|q|^2 + 6\varepsilon_5(t)|q|^4 - 2\varepsilon_5(t)|q_{xx}|^2 \]

+ 2\varepsilon_5(t)(qq_{xx}^* + q^*q_{xx}) - (2i\varepsilon_4(t) \]

+ 12i\varepsilon_6(t)|q|^2)(q^*q_{xx} - q_{xx}^* \]

- 2i\varepsilon_6(t)(q_{xx}^* + q_{xxx}^* - q_{xxx} - q_{xxx}), \]

\[ A'_2 = 1 + 4\varepsilon_4(t)|q|^2 + 12\varepsilon_6(t)|q|^4 \]

+ 4\varepsilon_6(t)(q^*q_{xx}^* + q^*q_{xx} - |q|^2) - 4i\varepsilon_5(t)(qq_{xx}^* \]

- q^*q_{xx}), \]

\[ A'_3 = -4i\varepsilon_3(t) - 8\varepsilon_5(t)|q|^2 - 8i\varepsilon_6(t)(qq_{xx}^* - q^*q_{xx}) \]

\[ A'_4 = -8\varepsilon_4(t) - 16\varepsilon_6(t)|q|^2, \]

\[ A'_5 = 16\varepsilon_5(t), A'_6 = 32\varepsilon_6(t), \]

\[ \Theta \] Springer
$B'_0 = 2i\varepsilon_3(t) q^* |q|^2 + 6i\varepsilon_5(t) q^* |q|^2 \frac{3}{2} + 4i\varepsilon_5(t) q^* |q|^2$

$+ 6i\varepsilon_5(t) q^* |q|^2 + 2i\varepsilon_5(t) q^* |q|^2 q_{xx} + i\varepsilon_3(t) q_{xx}^*$

$+ 8i\varepsilon_5(t) q^* q_{xx}^* + i\varepsilon_5(t) q_{xxx}^* - \frac{1}{2} q_x^*$

$- i\varepsilon_4(t) q_{xxx}^* - i\varepsilon_6(t) q_{xxxx}^* - 6i\varepsilon_6(t) q|^2 q_x^*$

$- 10i\varepsilon_6(t) q^* q_{xx}^* + q^* q_{xx}^* + |q|^2 q_{xxx}^*$

$+ |q|^2 q_{xxxx}^*$

$B'_1 = i q^* + 4i\varepsilon_4(t) q^* |q|^2 + 12i\varepsilon_6(t) q^* |q|^4$

$+ 8i\varepsilon_6(t) q^* q_{xx}^* + 2i\varepsilon_6(t) q_{xx}^* + 16i\varepsilon_6(t) q|^2 q_{xx}^*$

$+ 2i\varepsilon_6(t) q_{xxxx}^* + 2i\varepsilon_6(t) q_x^* + 12i\varepsilon_6(t) q|^2 q_x^*$

$+ 2i\varepsilon_6(t) q_{xxxx}^*$

$B'_2 = -4i\varepsilon_3(t) q^* - 8i\varepsilon_3(t) q^* |q|^2 - 4i\varepsilon_3(t) q_{xx}^*$

$+ 4i\varepsilon_4(t) q_x^* + 24i\varepsilon_6(t) q|^2 q_x^* + 4i\varepsilon_6(t) q_{xx}^*$

$B'_3 = -8i\varepsilon_4(t) q^* - 16i\varepsilon_6(t) q^* |q|^2 - 8i\varepsilon_6(t) q_{xx}^*$

$- 8i\varepsilon_3(t) q_x^*, B'_4 = 16i\varepsilon_3(t) q^* - 16i\varepsilon_6(t) q_x^*$

$B'_5 = 32i\varepsilon_3(t) q^*, B'_6 = 0$, \hspace{1cm} (7)

where $\lambda$ is a spectral parameter independent of variable $x$, $t$ and Lax pair (5)–(6) satisfies the compatibility condition $U'_x - V'_t + [U', V'] = 0$.

As we know, rogue waves on a periodic background of Eq. (3) have not been constructed. In the rest of the paper, our purpose is to construct the rogue waves on a periodic background of Eq. (3) and the structure of this paper is given as follows. In Sect. 2, we analyze the MI of Eq. (3) to ensure the existence of rogue waves. In Sect. 3, we deduce two families of periodic solutions for Eq. (3), which are called Jacobian elliptic functions $dn$ and $cn$. In Sect. 4, the Lax pair of Eq. (3) is nonlinearized. In Sect. 5, we obtain periodic and non-periodic wave solutions of Eq. (3). In Sect. 6, we construct rogue wave and rogue wave solutions of Eq. (3) on the $dn$-periodic background and $cn$-periodic background via Darboux transformation. In Sect. 7, some conclusions are given.

### 2 Modulation instability analysis

To analyze modulation instability, we firstly consider a plane wave solution of Eq. (3) in the following form

$$q = Q_0 e^{i(\kappa x + \omega t)}, \hspace{1cm} (8)$$

where $Q_0$ is a real constant representing the amplitude of the wave, $\kappa$ means wave number of the plane wave and $\omega$ means background frequency.

Substituting Eq. (8) into Eq. (3) yields

$$\kappa = -\frac{1}{2} w^2 + Q_0^2 + (w^3 + 6wQ_0^2) \varepsilon_3(t)$$

$$+ (w^4 - 12w^2Q_0^2 + 6Q_0^4) \varepsilon_4(t)$$

$$+ (w^5 - 20w^3Q_0^2 + 30wQ_0^4) \varepsilon_5(t)$$

$$+ (-w^6 + 30w^4Q_0^2 - 90w^2Q_0^4 + 20Q_0^6) \varepsilon_6(t). \hspace{1cm} (9)$$

In order to analyze MI, we need a carrier that contains it, so we introduce a small random perturbation function $r(x,t)$ in Eq. (3) to get a perturbation wave solution as

$$q = (Q_0 + \epsilon r(x,t)) e^{i(\kappa x + \omega t)}, \hspace{1cm} (10)$$

$$r(x,t) = r_1 e^{i(Kx + \Omega + t)} + r_2 e^{-i(Kx + \Omega + t)},$$

where $Q_0$ is a real constant defined in (8), $\epsilon$ is a small arbitrary constant, $K$ means perturbation wave numbers in $x$ direction, $\Omega$ means frequency of perturbation wave and $r_1, r_2$ are constants.

If we find $K$ or $\Omega$ exists the imaginary part, there must exists a exponentially increase in the perturbation function $r(x,t)$, which destroys the stability and leads to MI. As a result, we concentrate on $K$ here in this paper.

Substituting Eqs. (10) into Eq. (3) and separating the coefficients of $e^{i((\omega - \Omega)x + (\kappa - K)t)}$ and $e^{-i((\omega - \Omega)x + (\kappa - K)t)}$, we obtain the following time-dependent linear homogeneous equations for $r_1$ and $r_2$. 
where \( \epsilon_j(t) (j = 3, \ldots, 6) \) have been defined in (3).

To ensure the existence of the random perturbation, that is \( r_1 \) and \( r_2 \) are not equal to zero at the same time, we make the determinant of the coefficient matrix for the system of \( r_1 \) and \( r_2 \) equal to zero, which yields a univariate quadratic equation about \( K \). After solving this equation, we obtain the MI gain as

\[
G = |\text{Im}(K)| = \frac{1}{2} \text{Im} \left( \Omega \sqrt{(\Omega^2 - 4Q_0^2)g^2} \right),
\]

\[
g = 1 + 6\Omega \epsilon_3(t) - 2(\Omega^2 + 6\Omega^2 - 6Q_0^2)\epsilon_4(t) - 10(\Omega^2w + 2w^3 - 6wQ_0^2)\epsilon_5(t) + (2\Omega^4 + 30\Omega^2w^2 + 30w^4 - 20Q_0^2(\Omega^2 + 9w^2) - 3Q_0^2)\epsilon_6(t).
\]

According to (12), when \( \Omega^2 - 4Q_0^2 < 0 \), there exists imaginary part for \( K \), which makes the small random perturbation function \( r(x,t) \) grow exponentially and lose its stability. In addition, the MI band can also be further obtained, that is \( |\Omega| < 2Q_0 \). It is easy to find that there exists a positive correlation between the width of the MI band and \( Q_0 \).

For convenience, we make \( F = (\Omega^2 - 4Q_0^2)g^2 \). From Figs. 1, 2 and 3, we plot pictures about \( F \) to analyze MI of Eq. (3). As in Fig. 1, we take \( \epsilon_3(t) = \sin(t), \epsilon_4(t) = \sin(t), \epsilon_5(t) = 2\sin(t), \epsilon_6(t) = 3\sin(t) \), \( Q_0 = 0.2, w = 1 \). In Fig. 2, we take \( \epsilon_3(t) = -\exp(t), \epsilon_4(t) = \exp(t), \epsilon_5(t) = 2\exp(t), \epsilon_6(t) = 3\exp(t), Q_0 = 0.2, w = 1 \). In Fig. 3, we take \( \epsilon_3(t) = -t, \epsilon_4(t) = t, \epsilon_5(t) = 2t, \epsilon_6(t) = 3t, Q_0 = 0.2, w = 1 \).

As we can see in these pictures, for Fig. 1, the most instable regions evolve periodically with time. For Fig. 2, the MI region gets more and more instable over time. For Fig. 3, the MI region gets more and more instable when time tends to positive and negative infinity. These conditions are consistent with the characteristics of the functions \( \epsilon_j(t) (j = 3, \ldots, 6) \) take. In other words, there exists a relationship between the variable coefficient of the nonlinear term with highest order, because the higher-order nonlinear term has more influence, and the distribution of MI band.

Based on Figs. 1, 2 and 3 above, it is clear to find the areas less than zero, in which MI arises.

3 Two families of periodic solutions for Eq. (3)

We consider a solution for Eq. (3) as follows:

\[
q(x,t) = Q(x)e^{iK(t)},
\]
where \( Q(x) \) is a real periodic function and \( c(t) \) is a time-dependent function presenting the speed of waves. It is also easy to find that \(|q|^2 = q'^2 = Q^2\).

Substituting (13) into Eq. (3) yields a sixth-order nonlinear ordinary differential equation, which is difficult to obtain exact solutions. However, the sixth-order nonlinear ordinary differential equation can be simplified to a first-order nonlinear ordinary differential equation by means of Jacobi elliptic function expansion approach. Then, we finally obtain two families of periodic solutions for Eq. (3), which are expressed by Jacobi elliptic functions \( \text{dn} \) and \( \text{cn} \) as

\[
Q(x) = \text{dn}(x; k), (2 - k^2)e_3(t) + (6 - 6k^2 + k^4)e_5(t) = 0,
\]

\[
c(t) = \int \left[ -(6 - 6k^2 + k^4)e_4(t) + \left( -1 + \frac{1}{2}k^2 \right) \right] dt + c_0,
\]

\[
(1 + 2(10 - 10k^2 + k^4)e_6(t))|dt + c_0,
\]

\[
(14)
\]

\[
\text{and}
\]

\[
Q(x) = k\text{cn}(x; k), (-1 + 2k^2)e_3(t) + (1 - 6k^2 + k^4)e_5(t) = 0,
\]

\[
c(t) = \int \left[ (-1 + 6k^2 - 6k^4)e_4(t) - \left( -\frac{1}{2} + k^2 \right) \right] dt + c_1,
\]

\[
(1 + (2 - 20k^2 + 20k^4)e_6(t))|dt + c_1,
\]

\[
(15)
\]

where \( k \in (0, 1) \) is elliptic modulus and \( c_0, c_1 \) are two arbitrary integral constants. For convenience, we take \( c_0 = c_1 = 0 \). When \( e_4(t) \) and \( e_6(t) \) take different types of time-dependent functions, the pictures of the two periodic solutions (14)-(15) almost have no change, so we just take \( e_4(t) = t, e_6(t) = 2t \) and \( k = \frac{1}{2} \) in (14)-(15) and display the pictures in Fig. 4.

Clearly, (14)-(15) satisfy the following two elliptic equations

\[
Q_{xx} = -2Q^3 + a_0Q, Q_x^2 = -Q^4 + a_0Q^2 + a_1,
\]

\[
(16)
\]

where \( a_0 \) and \( a_1 \) are two real constants. As for \( \text{dn} \)-function solution, we take \( a_0 = 2 - k^2 \) and \( a_1 = k^2 - 1 \). As for \( \text{cn} \)-function solution, we take \( a_0 = 2k^2 - 1 \) and \( a_1 = k^2(1 - k^2) \) on the other hand.

### 4 Nonlinearization of the Lax pair

In this section, we introduce the Bargmann constraint [42, 43] to make Lax pair (5)-(6) nonlinear. Consider the following Bargmann constraint:

\[
q(x, t) = \varphi_1^2 + \varphi_2^2,
\]

\[
(17)
\]

where \( \Phi = (\varphi_1, \varphi_2)^T \) is a nonzero solution of the Lax pair (5)-(6) with \( \lambda = \lambda_1 \).

Substituting (17) into (5), we obtain a finite-dimensional Hamiltonian system as

\[
\frac{d\varphi_1}{dx} = \frac{\partial H}{\partial \varphi_2}, \quad \frac{d\varphi_2}{dx} = -\frac{\partial H}{\partial \varphi_1},
\]

\[
(18)
\]

where

\[
\begin{align*}
\text{Fig. 1} & \quad \text{The MI analysis of Eq. (3) with } e_3(t) = -\sin(t), e_4(t) = \sin(t), e_5(t) = 2\sin(t), e_6(t) = 3\sin(t), Q_0 = 0.2 \text{ and } w = 1
\end{align*}
\]
\[
H = \lambda_1 \varphi_1 \varphi_2 + \lambda_1^* \varphi_1^* \varphi_2^* + \frac{1}{2} (|\varphi_1|^2 + |\varphi_2|^2) (\varphi_1^{2*} + \varphi_2^{2*}) .
\]

Then, we introduce two conserved integrals of (18) as

\[
G_0 = i (\varphi_1 \varphi_2 - \varphi_1^* \varphi_2^*) ,
\]

(20)

\[
G_1 = \lambda_1 \varphi_1 \varphi_2 + \lambda_1^* \varphi_1^* \varphi_2^* + \frac{1}{2} (|\varphi_1|^2 + |\varphi_2|^2)^2 ,
\]

(21)

it is easy to find a relationship between \( H, G_0 \) and \( G_1 \) as

\[
H = G_1 - \frac{1}{2} G_0^2 .
\]

Considering Eqs. (17) and (20) together, we have

\[
\lambda_1 \varphi_1^2 - \lambda_1^* \varphi_1^{2*} = \frac{1}{2} q_1 + iqG_0 .
\]

(22)

According to [24], we obtain three constraints with \( \lambda_1 = \rho + i\mu \) in the following form

\[
q_1 q^* - q q^* = 2i(q^{1/2}(2\mu - G_0) + 2iG_0(G_0^2 + 2G_0\mu - 2G_1)) ,
\]

(23)

\[
q_{1t} + 2|q|^2 q - 4q(\rho^2 + \mu^2 + G_1 - \frac{1}{2} G_0^2 - 2\mu G_0) = 2iq_1(2\mu - G_0) ,
\]

(24)

\[
|q_1|^2 = -|q|^4 + 4|q|^2(\rho^2 + \mu^2 + G_1 - \frac{1}{2} G_0^2 - 2\mu G_0) + 4\rho^2 G_0^2 - (G_0^2 + 2\mu G_0 - 2G_1)(5G_0^2 - 2\mu G_0 - 2G_1) .
\]

(25)

Substituting (13) into (23), it is easy to notice that the left-hand side of (23) is zero, which yields

\[\text{Fig. 2} \text{ The MI analysis of Eq. (3) with } e_3(t) = -\exp(t), e_4(t) = \exp(t), e_5(t) = 2\exp(t), e_6(t) = 3\exp(t), Q_0 = 0.2 \text{ and } w = 1\]

\[\text{Fig. 3} \text{ The MI analysis of Eq. (3) with } e_3(t) = -t, e_4(t) = t, e_5(t) = 2t, e_6(t) = 3t, Q_0 = 0.2, \text{ and } w = 1\]
There exist two cases for the second equation in (26), one is \( \mu = 0 \) and the other is \( \mu \neq 0 \), \( G_1 = 4 \mu^2 \).

**Case 1** When \( \mu = 0 \), relations in (26) yield \( G_0 = 0 \) and relations in (27) yield

\[
a_0 = 4(\rho^2 + G_1), a_1 = -4G_1^2. \tag{28}
\]

As a result of \( a_1 = -4G_1^2 < 0 \), we choose the cn-function solution in (14), which yields

\[
G_1 = \pm \frac{1}{2} \sqrt{1 - k^2}, \lambda_1^2 = \frac{1}{4} \left( 2 - k^2 \pm 2 \sqrt{1 - k^2} \right). \tag{29}
\]

The expression for eigenvalue \( \lambda_1 \) with two real eigenvalues in the right half-plane is now available as

\[
\lambda_1 = \rho = \lambda_\pm = \frac{1}{2} \left( 1 \pm \sqrt{1 - k^2} \right), \tag{30}
\]

as well as other two \(-\lambda_\pm\) in the left half-plane.

**Case 2** When \( \mu \neq 0 \), relations in (26) yield \( G_0 = 2\mu \), \( G_1 = 4 \mu^2 \) and relations in (27) yield

\[
a_0 = 4(\rho^2 - \mu^2), a_1 = 16 \mu^2 \rho^2. \tag{31}
\]

As a result of \( a_1 = 16 \mu^2 \rho^2 > 0 \), we choose the cn-function solution in (15), which yields

\[
\lambda_1^2 = \frac{1}{4} \left( 2k^2 - 1 \pm 2ik \sqrt{1 - k^2} \right). \tag{32}
\]

There also exist two pairs of eigenvalues with \( \lambda_1 = \pm \lambda_\pm \) and \( \lambda_1 = -\lambda_\pm \) in the following form

\[
\lambda_\pm = \frac{1}{2} \left( k \pm i \sqrt{1 - k^2} \right). \tag{33}
\]

## 5 Periodic and non-periodic solutions of the Lax pair

In this section, we introduce a function \( \theta(x, t) \) to establish a connection between periodic solutions and non-periodic solutions for the Lax pair (5)–(6). Therefore, the chief aim of this section is to find out the expression of \( \theta(x, t) \).

Based on (17), (22) and (26), we have

\[
\varphi_1^2 = \frac{2\lambda_1 q + q_x}{2(\lambda_1 + \lambda_1^*)}, \varphi_2^2 = \frac{2\lambda_1^* q - q_x}{2(\lambda_1 + \lambda_1^*)}. \tag{34}
\]

Due to \( q(x, t) = Q(x)e^{ik(t)} \), we take

\[
\varphi_1(x, t) = \Phi_1(x)e^{i\iota(x)}, \varphi_2(x, t) = \Phi_2(x)e^{-i\iota(x)}. \tag{35}
\]

Substituting (13) and (35) into (34) yields

\[
\Phi_1^2(x) = \frac{2\lambda_1 Q + Q_x}{2(\lambda_1 + \lambda_1^*)}, \Phi_2^2(x) = \frac{2\lambda_1^* Q - Q_x}{2(\lambda_1 + \lambda_1^*)}. \tag{36}
\]

According to (36), we have

\[
\Phi_1^2 + \Phi_2^2 = \frac{2\lambda_1 Q + Q_x}{(\lambda_1 + \lambda_1^*)}, \Phi_1^2 - \Phi_2^2 = \frac{Q_x}{(\lambda_1 + \lambda_1^*)}. \tag{37}
\]

Substituting (35) into (19) and (20) yields \( \lambda_1 \Phi_1 \Phi_2 + \lambda_1^* \Phi_1^* \Phi_2 - H + \frac{1}{2} Q^2 = 0 \) and \( \Phi_1^* \Phi_2 = \Phi_1 \Phi_2 + iG_0 \).

With the help of Cramer’s Rule, we obtain...
\[
\Phi_1 \Phi_2 = -\frac{Q^2 - 2H + 2iG_0 \lambda_1^2}{2(\lambda_1^2 + \lambda_1'^2)}. \tag{38}
\]

As for the \(dn\)-function solution \(Q(x) = dn(x; k)\) in (14), we already know that \(G_0 = 0\), \(H = G_1 = \pm \frac{1}{2} \sqrt{1 - k^2}\) and choose \(\lambda_1 = \rho = \lambda_\pm = \frac{1}{2} \left(1 + \sqrt{1 - k^2}\right)\). Therefore, (37) and (38) can be rewritten as
\[
\Phi_1 \Phi_2 = -\frac{1}{4\rho} \left( Q^2 + \sqrt{1 - k^2} \right) = -\frac{dn^2(x; k) + \sqrt{1 - k^2}}{2 \left(1 + \sqrt{1 - k^2}\right)}, \tag{39}
\]
\[
\Phi_1^2 + \Phi_2^2 = Q = dn(x; k),
\]
\[
\Phi_1^2 - \Phi_2^2 = \frac{1}{2\rho} Q_x = -\frac{k \text{sn}(x; k) \text{cn}(x; k)}{1 + \sqrt{1 - k^2}}.
\]

According to (36) with \(\lambda_1 = \lambda_1^* = \rho\), we find \(\Phi_1^2\) and \(\Phi_2^2\) are real. And as a result of \(\Phi_1^2 + \Phi_2^2 = Q = dn(x; k) > 0\), we finally determine that \(\Phi_1\) and \(\Phi_2\) are real so that the first equation of (39) can be rewritten as
\[
\Phi_1 \Phi_2 = -\frac{1}{4\rho} \left( Q^2 + \sqrt{1 - k^2} \right) = -\frac{dn^2(x; k) + \sqrt{1 - k^2}}{2 \left(1 + \sqrt{1 - k^2}\right)}. \tag{40}
\]

As for the \(cn\)-function solution \(Q(x) = k \text{cn}(x; k)\) in (15), we already know that \(G_0 = 2\mu, G_1 = 4\mu^2, H = 2\mu^2\) and choose \(\lambda_1 = \lambda_\pm = \frac{1}{2} \left(k \pm i \sqrt{1 - k^2}\right)\). Therefore, (38) can be rewritten as
\[
\Phi_1 \Phi_2 = -\frac{1}{2k} \left( Q^2 + ik \sqrt{1 - k^2} \right). \tag{41}
\]

Eqs. (21) and (38) yield
\[
(|\Phi_1|^2 + |\Phi_2|^2)^2 = 1 - k^2 + |u|^2.
\]
If we consider the positive square root, we have
\[
|\Phi_1|^2 + |\Phi_2|^2 = \text{dn}(x; k). \tag{42}
\]

Since \(G_1 = G_0^2\), we have
\[
\dot{\lambda}_1 \Phi_1 \Phi_2 + \dot{\lambda}_1^* \Phi_1^* \Phi_2^* + \Phi_1^2 \Phi_2^2
\]
\[
+ \Phi_1^{2*} \Phi_2^{2*} + \frac{1}{2} (|\Phi_1|^2 - |\Phi_2|^2)^2 = 0. \tag{43}
\]

Eqs. (33), (41) and (43) yield
\[
(|\Phi_1|^2 - |\Phi_2|^2)^2 = |u|^2 - \frac{1}{2} |u|^4. \quad \text{If we consider the negative square root, we have}
\]
\[
|\Phi_1|^2 - |\Phi_2|^2 = -k \text{sn}(x; k) \text{cn}(x; k). \tag{44}
\]

According to Eqs. (42) and (44), we have
\[
|\Phi_1|^2 = \frac{1}{2} [\text{dn}(x; k) - k \text{sn}(x; k) \text{cn}(x; k)],
\]
\[
|\Phi_2|^2 = \frac{1}{2} [\text{dn}(x; k) + k \text{sn}(x; k) \text{cn}(x; k)].
\]

According to (33) and (36), we have
\[
\Phi_1^2 \Phi_2^2 = \frac{1}{4} [k^2 \text{cn}^2(x; k) - \text{sn}^2(x; k) \text{dn}^2(x; k)
\]
\[
+ 2i \sqrt{1 - k^2} \text{sn}(x; k) \text{cn}(x; k) \text{dn}(x; k)],
\]
which yields
\[
\Phi_1 \Phi_2 = -\frac{1}{2} [\text{cn}(x; k) \text{dn}(x; k) + i \sqrt{1 - k^2} \text{sn}(x; k)]. \tag{47}
\]

Here, we introduce a function \(\theta(x, t)\). Let us make an assumption that \((\varphi_1, \varphi_2)^T\) is the periodic solution of the Lax pair (5)–(6) with \(\lambda = \lambda_1\), and \((\psi_1, \psi_2)^T\) is the second linearly independent solution of the Lax pair (5)–(6) with the same \(\lambda = \lambda_1\), where \(\psi_1\) and \(\psi_2\) are non-periodic solutions and have the following forms
\[
\psi_1 = \frac{\theta - 1}{\varphi_2}, \quad \psi_2 = \frac{\theta + 1}{\varphi_1}, \tag{48}
\]
where \(\theta = \theta(x, t)\) is a function to be determined.

Using (48) and (5) yields
\[
\theta_t = \theta \frac{q \varphi_2^2 - q' \varphi_2^2}{\varphi_1 \varphi_2} + \frac{q \varphi_2^2 + q' \varphi_1^2}{\varphi_1 \varphi_2}. \tag{49}
\]

Using (13) and (35), we rewrite (49) as
\[
\theta_t = \theta Q \frac{\Phi_2^2 - \Phi_1^2}{\Phi_1 \Phi_2} + Q \frac{\Phi_2^2 + \Phi_1^2}{\Phi_1 \Phi_2}. \tag{50}
\]

Substituting (37) and (38) into (50) yields
\[
\theta = (Q^2 - 2H + 2iG_0\lambda_1^2) \Gamma_0 + \Gamma_1,
\]
where \( \Gamma_0 = i[1 + 4i\lambda_1\varepsilon_3(t) + 2(a_0 + 4\lambda_1^2)\varepsilon_4(t)] \)
\[
\theta_0(t) = i[1 + 4i\lambda_1\varepsilon_3(t) + 2(a_0 + 4\lambda_1^2)\varepsilon_4(t) + 4i(a_0\lambda_1 + 4\lambda_1^2)\varepsilon_5(t) + (2a_0^2 - 4a_1 + 8a_0\lambda_1^2 + 32\lambda_1^4)\varepsilon_6(t)]t + \gamma,
\]
where \( \gamma \) is an arbitrary constant.

Substituting (57) into (52), we finally arrive at the expression of \( \theta(x, t) \)
\[
\theta(x, t) = (Q^2 - 2H + 2iG_0\lambda_1^2) \int_{0}^{x} \frac{-4\lambda_1Q^2(y)}{(Q^2(y) - 2H + 2iG_0\lambda_1^2)^2} dy + \Gamma_0t + \gamma,
\]
(58)

6 Rogue waves on the periodic background

6.1 Darboux transformation

According to \([44]\), we redefined the elementary Darboux transformation of Eq. (3) in the following form
\[
q[1] = q + \frac{2(\lambda_1 + \lambda_1^2)\phi_{11}\phi_{21}^*}{|\phi_{11}|^2 + |\phi_{21}|^2},
\]
(60)
where \( (\phi_{11}, \phi_{21})^T \) is a nonzero solution of the Lax pair (5)–(6) with \( \lambda = \lambda_1 \).

6.2 Rogue waves on the dn-periodic background

In order to construct the rogue waves of Eq. (3) on the dn-periodic background, we apply onefold Darboux transformation (60) to the Jacobian elliptic function dn, take the seed solution as \( q = Q(x)e^{\kappa(x(t))} \) and choose
the real eigenvalue $\lambda_1 = \rho = \lambda_+ = \frac{1}{2} \left(1 + \sqrt{1 - k^2}\right)$ in (30). Substituting $(\psi_{11}, \psi_{21})^T = (\psi_1, \psi_2)^T$ defined by (48) into (58) and using (39)–(40), we construct the rogue wave solution of Eq. (3) on the dn-periodic background as

$$q_{\text{dn-wave}} = e^{\rho(t)} \left[ \text{dn}(x;k) + \frac{\left(1 - 2i\text{Im}\theta_{\text{dn}} - |\theta_{\text{dn}}|^2\right) \left(\text{dn}^2(x;k) + \sqrt{1 - k^2}\right)}{\left(1 + |\theta_{\text{dn}}|^2\right) \text{dn}(x;k) + 2\left(1 - \sqrt{1 - k^2}\right) \text{Re}\theta_{\text{dn}} \text{sn}(x;k) \text{cn}(x;k)} \right],$$

(61)

with

$$\theta_{\text{dn}} = \left(\text{dn}^2(x;k) + \sqrt{1 - k^2}\right)^{-1/2} \left[-2\left(1 + \sqrt{1 - k^2}\right) \int_0^x \frac{\text{dn}^2(y)}{\left(\text{dn}^2(y) + \sqrt{1 - k^2}\right)^{3/2}} dy + \Gamma_0 t + \gamma\right],$$

(62)

where $\Gamma_0$ is defined in (59) and $\gamma$ is a constant.

If we make $\varepsilon_3(t) \neq 0$ and $\varepsilon_j(t), (j = 4, 5, 6)$ arbitrary, the rogue wave solution on the dn-periodic background (61) we have obtained is for the sixth-order NLS equation with variable coefficients. If we make $\varepsilon_3(t) = 0$, $\varepsilon_j(t) \neq 0$ and $\varepsilon_4(t)$ arbitrary, (61) is for the fifth-order NLS equation with variable coefficients. Similarly, we can obtain rogue wave solutions on the dn-periodic background for remaining lower-order NLS equations with variable coefficients. In addition, if we make $\varepsilon_j(t), (j = 4, 5, 6)$ degenerate to arbitrary constants, we can obtain rogue wave solutions on the dn-periodic background for higher-order NLS equation with constant coefficients.

For convenience and the beauty of figures, we make $\varepsilon_3(t) = 0$. In Fig. 5, we take $\varepsilon_4(t) = \sin(t), \varepsilon_5(t) = \sin(t)$. In Fig. 6, we take $\varepsilon_4(t) = \sin(t), \varepsilon_5(t) = \exp(t)$. In Fig. 7, we take $\varepsilon_4(t) = \sin(t), \varepsilon_5(t) = t$. The reason for taking all the $\varepsilon_4(t)$ is to ensure periodicity. After that, we take the variable coefficient of the highest-order nonlinear term for different types of functions so that we can pave the way for the next analysis. When $\varepsilon_j(t), (j = 4, 5, 6)$ are arbitrary constants [27, 29], the corresponding figures are rather similar to Fig. 7. Therefore, it does not make so much sense to show them.

Observing the waves shown in above figures, we find that the central part corresponds to a rogue wave and the background is periodic to $x$. In addition, the rogue waves in Figs. 5 and 6 are periodic to $t$. Particularly, when $k = 0$ or $k = 1$, we obtain the degenerate rogue wave solutions and show them in Fig. 8.

Based on all of the above transverse plots except in Fig. 7, we find that there exists a physical phenomenon, which is called damping. Under the influence of damping, the energy and growth in the rogue waves drained with time, leaving rogue waves dissipating over time. If we take the conclusions given in Sect. 2 into consideration together, we have an interesting discovery. Specifically, when the variable coefficient of the highest-order nonlinear term is exponential function or algebraic function, the young waves are more sensitive to MI than old waves. In other words, MI in an unstable wave can be stabilized by damping. The conclusion is similar to the results given in [9, 45, 46].

### 6.3 Rogue waves on the cn-periodic background

In order to construct the rogue waves of Eq. (3) on the cn-periodic background, we apply the onefold Darboux transformation (60) to the Jacobian elliptic function cn, take the seed solution as $q = Q(x)e^{\text{ic}(t)}$, and choose the complex eigenvalue $\lambda_1 = \lambda_3 = \frac{1}{2} (k \pm i\sqrt{1 - k^2})$ in (33). Substituting $(\psi_{11}, \psi_{21})^T = (\psi_1, \psi_2)^T$ defined by (48) into (58) and using (42), (44) and (47), we obtain the rogue wave solution of Eq. (3) on the cn-periodic background as

$$q_{\text{cn-wave}} = e^{\rho(t)} \left[ \text{cn}(x;k) + \frac{k \left(1 - 2i\text{Im}\theta_{\text{cn}} - |\theta_{\text{cn}}|^2\right) \left(\text{cn}^2(x;k) + i\sqrt{1 - k^2}\text{sn}(x;k)\text{cn}(x;k)\right)}{\left(1 + |\theta_{\text{cn}}|^2\right) \text{dn}(x;k) + 2k \text{Re}\theta_{\text{cn}} \text{sn}(x;k) \text{cn}(x;k)} \right],$$

(63)

with

$$\theta_{\text{cn}} = \left(k^2 \text{cn}^2(x;k) + i k \sqrt{1 - k^2}\right)^{-1/2} \left[-2k \left(k \pm i\sqrt{1 - k^2}\right) \int_0^x \frac{k^2 \text{cn}^2(y)}{\left(k^2 \text{cn}^2(y) + i k \sqrt{1 - k^2}\right)^2} dy + \Gamma_0 t + \gamma\right].$$

(64)
Fig. 5 Rogue waves on the dn-periodic background with $\epsilon_4(t) = \sin(t), \epsilon_5(t) = \sin(t), \epsilon_6(t) = 0, \gamma = 0, k = 0.5$. a Three-dimensional plot, c transverse plot at $x = \frac{1}{2}$. Rogue waves on the dn-periodic background with $\epsilon_4(t) = \sin(t), \epsilon_5(t) = \sin(t), \epsilon_6(t) = 0, \gamma = 0, k = 0.9$. b Three-dimensional plot, d transverse plot at $x = \frac{1}{2}$.

Fig. 6 Rogue waves on the dn-periodic background with $\epsilon_4(t) = \sin(t), \epsilon_5(t) = \exp(t), \epsilon_6(t) = 0, \gamma = 0, k = 0.5$. a Three-dimensional plot, c transverse plot at $x = -\frac{1}{7}$. Rogue waves on the dn-periodic background with $\epsilon_4(t) = \sin(t), \epsilon_5(t) = \exp(t), \epsilon_6(t) = 0, \gamma = 0, k = 0.9$. b Three-dimensional plot, d transverse plot at $x = -\frac{1}{7}$. 

© Springer
Fig. 7 Rogue waves on the dn-periodic background with $\varepsilon_4(t) = \sin(t), \varepsilon_5(t) = t, \varepsilon_6(t) = 0, \gamma = 0, k = 0.5$. a Three-dimensional plot, c transverse plot at $x = \frac{t}{2}$. Rogue waves on the dn-periodic background with $\varepsilon_4(t) = \sin(t), \varepsilon_5(t) = t, \varepsilon_6(t) = 0, \gamma = 0, k = 0.9$. b Three-dimensional plot, d transverse plot at $x = \frac{t}{2}$.

Fig. 8 The degenerate rogue wave on the dn-periodic background with $\varepsilon_4(t) = \sin(t), \varepsilon_5(t) = \exp(t), \varepsilon_6(t) = 0, \gamma = 0, k = 0$. a Three-dimensional plot, c transverse plot at $x = 0$. The degenerate rogue wave on the dn-periodic background with $\varepsilon_4(t) = \sin(t), \varepsilon_5(t) = \sin(t), \varepsilon_6(t) = 0, \gamma = 0, k = 1$. b Three-dimensional plot, d transverse plot at $x = 0$. 
The obtained solution (63) is presented in Figs. 9, 10, 11, and the analysis is similar to the rogue wave solution on the dn-periodic background (61), so we will not repeat the analysis here. Particularly, when $k = 1$, we also obtain the degenerate rogue wave solution and show it in Fig. 12.
7 Conclusions

In this paper, the MI of Eq. (3) is firstly analyzed to explore the existence of rogue wave solutions. Then, rogue wave solutions of the sixth-order focusing NLS equation (3) with variable coefficients on the periodic background of Jacobian elliptic functions \(dn\) and \(cn\) are constructed. Besides, we also find a connection between damping and MI. However, all the research results are still under the framework of AKNS system. In the future, we expect to apply the method in this paper to other spectral problems and expand the periodic background to other Jacobian elliptic functions. We hope our results can provide some inspiration on the research of the rogue wave phenomena in the field of nonlinear physics.

Acknowledgements This work was supported by the National Natural Science Foundation of China (Grant Nos. 11861050, 11261037), the Natural Science Foundation of Inner Mongolia Autonomous Region, China (Grant No. 2020LJH01010) and the Inner Mongolia Normal University Graduate Students’ Research and Innovation Fund (Grant No. CXJS21119).
References

1. Benjamin, T.B.: The disintegration of wave trains on deep water. J. Fluid Mech. 27, 417–430 (1967)
2. Turing, A.M.: The chemical basis of morphogenesis. Philos. Trans. R. Soc. Lond. B 237, 37–72 (1952)
3. Zakharov, V.E.: Collapse of Langmuir waves. Sov. Phys. JETP 35, 908–914 (1972)
4. Hasegawa, A.: Generation of a train of soliton pulses by induced modulational instability in optical fibers. Opt. Lett. 9(7), 288–290 (1984)
5. Tai, K., Hasegawa, A., Tomita, A.: Observation of modulational instability in optical fibers. Phys. Rev. Lett. 56(2), 135–138 (1986)
6. Agrawal, G.P.: Modulation instability induced by cross-phase modulation. Phys. Rev. Lett. 59(8), 880–883 (1987)
7. Akhmediev, N., Korneev, V.I.: Modulation instability and periodic solutions of the nonlinear Schrödinger equation. Theor. Math. Phys. 69(2), 189–194 (1986)
8. Islas, A., Schober, C.M.: Rogue waves, dissipation, and downshifting. Phys. D 240(12), 1041–1054 (2011)
9. Onorato, M., Proment, D.: Approximate rogue wave solutions of the forced and damped nonlinear Schrödinger equation for water waves. Phys. Lett. A 376(45), 3057–3059 (2012)
10. Schober, C.M., Strawn, M.: The effects of wind and nonlinear damping on rogue waves and permanent downshift. Phys. D 313(1), 81–98 (2015)
11. Kharif, C., Pelinovsky, D.E., Slunyaev, A.: Rogue Waves in the Ocean. Springer, Berlin (2009)
12. Solli, D.R., Ropers, C., Koonath, P., Jalali, B.: Optical rogue waves. Nature 450(7172), 1054–1057 (2007)
13. Bludov, Y.V., Konotop, V.V., Akhmediev, N.: Matter rogue waves. Phys. Rev. A 80(3), 033610 (2009)
14. Bludov, Y.V., Konotop, V.V., Akhmediev, N.: Rogue waves as spatial energy concentrators in arrays of nonlinear waveguides. Opt. Lett. 34(19), 3015–3017 (2009)
15. Akhmediev, N., Ankiewicz, A., Taki, M.: Waves that appear from nowhere and disappear without a trace. Phys. Lett. A 373, 675–678 (2009)
16. Peregrine, D.H.: Water waves, nonlinear Schrödinger equations and their solutions. J. Aust. Math. Soc. 25(1), 16–43 (1983)
17. Cao, C.W.: Nonlinearization of the Lax system for AKNS hierarchy. Sci. China Ser. A 33(5), 528–536 (1990)
18. Wen, X.Y., Meng, X.H., Xu, X.G., Wang, J.T.: N-fold Darboux transformation and explicit solutions in terms of the determinant for the three-field Blaszak–Marciniai lattice. Appl. Math. Lett. 26(11), 1076–1081 (2013)
19. Zhaqilao, Sirendaoreji: N-soliton solutions of the KdV6 and mKdV6 equations (Article). J. Math. Phys. 51(11), 073516 (2010)
20. Zhaqilao: Darboux transformation and N-soliton solutions for a more general set of coupled integrable dispersionless system. Commun. Nonlinear Sci. Numer. Simul. 16(10), 3949–3955 (2011)
21. Zhaqilao, Qiao, Z.: Darboux transformation and explicit solutions for two integrable equations. Math. Anal. Appl. 380(2), 794–806 (2011)
22. Zhao, D., Zhaqilao: On two new types of modified short pulse equation. Nonlinear Dyn. 100(1), 615–627 (2020)
23. Chen, J.B., Pelinovsky, D.E.: Rogue periodic waves of the modified KdV equation. Nonlinearity 31(5), 1955–1980 (2018)
24. Chen, J.B., Pelinovsky, D.E.: Rogue periodic waves of the focusing nonlinear Schrödinger equation. Proc. R. Soc. A 474(2210), 20170814 (2018)
25. Chen, J.B., Pelinovsky, D.E., White, R.E.: Rogue waves on the double-periodic background in the focusing nonlinear Schrödinger equation. Phys. Rev. E 100(5), 0522199 (2019)
26. Chen, J.B., Pelinovsky, D.E., White, R.E.: Periodic standing waves in the focusing nonlinear Schrödinger equation: Rogue waves and modulation instability. Physica D 405, 132378 (2020)
27. Wang, Z.J., Zhaqilao: Rogue wave solutions for the generalized fifth-order nonlinear Schrödinger equation on the periodic background. Wave Motion 108, 102839 (2021)
28. Yue, Y.F., Huang, L.L., Chen, Y.: Modulation instability, rogue waves and spectral analysis for the sixth-order nonlinear Schrödinger equation. Commun. Nonlinear Sci. Numer. Simul. 89, 105284 (2020)
29. Zhang, H.Q., Chen, F.: Rogue waves for the fourth-order nonlinear Schrödinger equation on the periodic background. Chaos 31(2), 203129 (2021)
30. Peng, W.Q., Tian, S.F., Wang, X.B., Zhang, T.T.: Characteristics of rogue waves on a periodic background for the Hirota equation. Wave Motion 93, 102454 (2020)
31. Li, R.M., Geng, X.G.: Rogue periodic waves of the sine-Gordon equation. Appl. Math. Lett. 102, 106147 (2020)
32. Zhang, H.Q., Gao, X., Pei, Z.J., Chen, F.: Rogue periodic waves in the fifth-order Ito equation. Appl. Math. Lett. 107, 106464 (2020)
33. Sun, W.R., Wang, L.: Vector rogue waves, rogue wave-to-soliton conversions and modulation instability of the higher-order matrix nonlinear Schrödinger equation. Eur. Phys. J. Plus. 133(12), 495 (2018)
34. Kedziora, D.J., Ankiewicz, A., Chowdury, A., Akhmediev, N.: Integrable equations of the infinite nonlinear Schrödinger equation hierarchy with time variable coefficients. Chaos 25, 103114 (2015)
35. Ankiewicz, A., Kedziora, D.J., Chowdury, A., Bandelow, U., Akhmediev, N.: Infinite hierarchy of nonlinear Schrödinger equations and their solutions. Phys. Rev. E 93, 012206 (2016)
36. Anastasia, D., Iain, F., Spyridoula, S.: Non-commutative NLS-type hierarchies: dressing & solutions. Nucl. Phys. B 941(4), 376–400 (2019)
37. Hirota, R.: Exact envelope-soliton solutions of a nonlinear wave equation. J. Math. Phys. 14, 805–809 (1973)
38. Lakshmanan, M., Porsezian, K., Daniel, M.: Effect of discreteness on the continuum limit of the Heisenberg spin chain. Phys. Lett. A 133, 483–488 (1988)
39. Chowdury, A., Kedziora, D.J., Ankiewicz, A., Akhmediev, N.: Soliton solutions of an integrable nonlinear Schrödinger equation with quintic terms. Phys. Rev. E 90, 032922 (2014)
40. Akhmediev, N., Ankiewicz, A.: Solitons, Nonlinear Pulses and Beams. Chapman & Hall, London (1997)
41. Ankiewicz, A., Soto-Crespo, J.M., Akhmediev, N.: Rogue waves and rational solutions of the Hirota equation. Phys. Rev. E 81, 046602 (2010)
42. Cao, C.W., Wu, Y.T., Geng, X.G.: Relation between the Kadomtsev–Petviashvili equation and the confocal involutive system. J. Math. Phys. 40(8), 3948–3970 (1999)
43. Zhou, R.G.: Nonlinearizations of spectral problems of the nonlinear Schrödinger equation and the real-valued modified Korteweg–de Vries equation. J. Math. Phys. 48(1), 013510 (2007)
44. Gu, C.H., Hu, H.S., Zhou, Z.X.: Darboux Transformation in Integrable Systems: Theory and Their Applications To Geometry. Springer, Dordrecht (2005)
45. Kharif, C., Touboul, J.: Under which conditions the Benjamin–Feir instability may spawn an extreme wave event: a fully nonlinear approach. Eur. Phys. J. Special Topics 185, 159–168 (2010)
46. Kharif, C., Kraenkel, R.A., Manna, M.A., Thomas, R.: The modulational instability in deep water under the action of wind and dissipation. J. Fluid Mech. 664, 138–149 (2010)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.