QUANTUM INVARIANTS FOR HANDLEBODY-KNOTS

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Abstract. We construct quantum invariants for handlebody-knots in a 3-sphere $S^3$. A handlebody-knot is an embedding of a handlebody in a 3-manifold. These invariants are linear sums of Yokota’s invariants for colored spatial graphs which is defined by using Kauffman bracket. We also give a non-trivial example.

1. Introduction

A spatial graph is a embedding of a finite graph into a 3-manifold. Suzuki [5] introduced the notion of the neighborhood equivalence for spatial graphs. Two spatial graphs are neighborhood equivalent if they have the same regular neighborhoods up to isotopy of the 3-manifold. Ishii [6] reformulated this notion as a handlebody-knot which is an embedding of a handlebody into a 3-manifold. If the genus of the handlebody is 1, the handlebody-knot is regarded as an ordinary knot.

A handlebody-knot is represented by a diagram of a spatial trivalent graph whose regular neighborhood is isotopic to the handlebody-knot. Ishii [6] also showed that the two diagrams representing the same handlebody-knot are transformed to each other by a sequence of six local moves called “Reidemeister moves” for handlebody-knots. Five of them are Reidemeister moves for isotopy of spatial trivalent graphs. Therefor we can make an invariant of handlebody-knots from an invariant of trivalent graphs modifying to suffice the new Reidemeister move.

In this paper we construct quantum $U_q(sl_2)$ type invariants for handlebody-knots in a 3-sphere $S^3$ via Yokota’s invariants [10], which are quantum $U_q(sl_2)$ type invariants for spatial graphs in $S^3$. Yokota’s invariants are generalized to the Relativistic invariants [1] for spatial graphs in any 3-manifolds. The arguments in this paper holds for Relativistic invariants and our invariants are generalized for any 3-manifolds. In this paper we focus on Yokota’s invariants since it is essential.

There are various invariants for handlebody-knots. Alexsander ideals are known as invariants for neighborhood equivalent classes of spatial graphs [3] that are handlebody-knots. Invariants using Fox coloring [6] and quandle cocycle invariants [6, 7] were also...
defined. However quantum type invariants for handlebody-knots have not been defined yet.

In this paper, we first introduce a handlebody-knot and its Reidemeister moves in Section 2. We then define quantum invariants for the handlebody-knot via Yokota’s invariants in Section 3. In Section 4, some examples and properties of the invariants are shown, which tell us that our invariants are nontrivial.

Throughout this paper we work in the peacewise linear category.

2. HANDLEBODY-KNOT AND HANDLEBODY-LINK

A handlebody-knot is an embedding of a handlebody into a 3-manifold and a handlebody-link is a disjoint union of embeddings of handlebodies into 3-manifold (Figure 1). Throughout this paper, the notation of a handlebody-link include a handlebody-knot. Two handlebody-links are equal if there is an isotopy of the 3-manifold that the handlebodies are embedded such that it transforms one handlebody-link to another.

![Figure 1. An embedding of a genus 2 handlebody into a 3-manifold](image)

We have a map from trivalent graphs to handlebody-links that takes a regular neighborhood of a trivalent graph (Figure 2). Here a trivalent graph may have circle components that correspond to genus 1 components of the handlebody-link. These circles don’t have vertices nor edges. Through this map, we can treat handlebody-links as diagrams of trivalent spatial graphs. However, there are infinitely many diagrams that represent the same handlebody-link (Figure 3). In [6], Ishii shows that the diagrams of trivalent graphs have one to one correspondence to handlebody-links subject to local moves called Reidemeister moves (for handlebody-links). This means,

$$\{\text{handlebody-links}\} = \{\text{trivalent graphs}\}/\{\text{Reidemeister moves}\}. $$

The Reidemeister moves for handlebody-links are as Figure 4. The moves RI-RV are the Reidemeister moves for spatial trivalent graphs.

3. QUANTUM INVARIANTS FOR HANDLEBODY-LINKS

In this section we define the quantum invariants for handlebody-links in 3-sphere $S^3$.}

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1A construction of quantum invariants for handlebody-knots via Yetter-Drinfeld module was announced by [9].
3.1. **Skein Space and Jones-Wenzl idempotent.** To define invariants for handlebody-links, we introduce the skein space and its special elements called Jones-Wenzl idempotents.

**Definition 3.1.** (*Skein space*) Let $F$ be a connected and oriented 2-manifold (possibly has boundaries). A link diagram in $F$ is consisted of finitely many closed curves and arcs whose end points are at the boundaries of $F$. These curves may have finitely many transverse crossings that have the upper or lower crossing information. Two link diagrams
are regarded as the same if there is an isotopy of $F$ that change one link diagram to another fixing $\partial F$.

Let $A$ be a fixed value in $\mathbb{C} \setminus \{0\}$. Skein space $S(F)$ of $F$ is the vector space of formal linear sums, over $\mathbb{C}$, of link diagrams in $F$ subject to the next relations,

\[
\bigcup D = -(A^2 + A^{-2}) \ D,
\]

\[
A = \left\langle \begin{array}{c} + A^{-1} \\ \end{array} \right\rangle,
\]

here the left-hand side of the first relation is a disjoint union of a trivial circle and a link diagram $D$ and in the second relation the figures represent the parts of the link diagrams and the complements of them are the same diagrams.

If $F = S^2$, since $\partial F = \emptyset$, all curves in $S(F)$ are closed and become a blank diagram multiplied by a scalar value in $\mathbb{C}$ because of the relations. Thus $S(S^2)$ is identified with $\mathbb{C}$. For each link diagram in $S^2$, we represent the corresponding complex value as a diagram inside the Kauffman bracket $\langle \rangle$. Note that the value of the Kauffman bracket is an invariant of framed links (it doesn’t change under the RII and RIII moves).

We fix $A = e^{2\pi i/4r}$, $3 \leq r \in \mathbb{N}$ for later use. Let $D_n$ be a 2-disc that has $2n$ points in the boundary. We regard it as a rectangle whose left edge and right edge have just $n$ points respectively.

**Definition 3.2.** *(Jones-Wenzl idempotent)* Following [4], for $0 \leq n \leq r-1$, we define special elements $JW_n \in S(D_n^2)$ illustrated as a white box in the diagrams. $JW_n$ is defined recursively as follows,

\[
\begin{array}{c}
\begin{array}{c}
\text{n+1}
\end{array}
\end{array}
\quad = \quad
\begin{array}{c}
\begin{array}{c}
\text{1}
\end{array}
\end{array}
\quad - \quad \frac{\Delta_{n-1}}{\Delta_n}
\begin{array}{c}
\begin{array}{c}
\text{1}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{1}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{n-1}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{n}
\end{array}
\end{array}
\end{array},
\]

where $JW_0$ is a blank diagram and

\[
\Delta_n = \langle \quad \rangle = \frac{(-1)^n (A^{2n+1} - A^{-2n+1})}{A^2 - A^{-2}} = (-1)^n \frac{\sin \frac{\pi}{r}(n + 1)}{\sin \frac{\pi}{r}}.
\]

In this relation the arc attached a number means that number of parallel arcs. Note that $\Delta_{r-1} = 0$.

The elements $JW_n$ are called *Jones-Wenzl idempotent* because of the following relation,

\[
\begin{array}{c}
\begin{array}{c}
\text{n}
\end{array}
\end{array}
\quad = \quad
\begin{array}{c}
\begin{array}{c}
\text{n}
\end{array}
\end{array}.
\]
3.2. Definition of the invariants. In [10], Yokota defined invariants for colored (weighted) spatial graphs by using the Kauffman bracket. Let \( I \) be a set of colors \( \{0, 1, \ldots, r - 2\} \). A coloring of a spatial graph \( \Gamma \) is a map \( E(\Gamma) \to I \) where \( E(\Gamma) \) is a set of edges and circles of \( \Gamma \). We represent the coloring by attaching numbers to edges and circles. We define the invariants for handlebody-links via Yokota’s invariants.

From now on, we regard the \( i \) colored edge or circle of trivalent graphs in the diagrams as \( i \) parallel arcs inserted by the idempotent \( JW_i \) and the trivalent vertices adjacent to three colored edges as Figure 5. Here \( x = (j + k - i)/2, y = (i + k - j)/2, z = (i + j - k)/2 \). If \( i, j, k \) satisfy \( i \leq j + k, j \leq i + k, k \leq i + j \) and \( i + j + k \in 2\mathbb{Z} \), then we have a corresponding diagram in the right-hand side. Otherwise we couldn’t realize the diagram. Moreover, if \( i + j + k \leq 2(r - 2) \), then the next value is non-zero.

\[
\theta(i, j, k) = \begin{vmatrix} i \\ j \\ k \end{vmatrix} = \begin{vmatrix} x \\ y \\ z \\ x \\ y \end{vmatrix} = \frac{\Delta_{x+y+z}!\Delta_{x-1}!\Delta_{y-1}!\Delta_{z-1}!}{\Delta_{y+z-1}!\Delta_{x+z-1}!\Delta_{x+y-1}!},
\]

where \( \Delta_n! = \Delta_n \Delta_{n-1} \cdots \Delta_0 \) and \( \Delta_{-1}! = 1 \). The value \( \theta(i, j, k) \) is independent of the order of triple \((i, j, k)\).

**Definition 3.3.** (admissible) A triple \((i, j, k)\) in \( I^3 \) is called admissible if \((i, j, k)\) satisfies following conditions,

\[
i \leq j + k \quad j \leq i + k \quad k \leq i + j
\]

\[
i + j + k \in 2\mathbb{Z} \quad i + j + k \leq 2(r - 2).
\]

For an admissible \((i, j, k)\), \( \theta(i, j, k) \) is non-zero.

Yokota’s invariants for colored spatial graphs in \( S^3 \) are defined as follows, first they are defined in the case of trivalent graphs and then they are generalized to arbitrary graphs.

**Definition 3.4.** (Yokota’s invariant[10]) Let \( \Gamma \) be a spatial trivalent graph embedded to \( S^3 \). We fix a coloring to \( \Gamma \) by attaching numbers \( i_1, i_2, \ldots, i_n \in I \) to the edges and the circles so that for each vertex the triple of colors of incident edges \((i_a, i_b, i_c)\) are admissible. This coloring is called a admissible coloring for \( \Gamma \). Let \( D(i_1, i_2, \ldots, i_n) \) be a diagram of \( \Gamma(i_1, i_2, \ldots, i_n) \) and \( \overline{D}(i_1, i_2, \ldots, i_n) \) be the mirror image of \( D(i_1, i_2, \ldots, i_n) \). Then the next
value is defined,
\[
\langle \Gamma(i_1, i_2, \ldots, i_n) \rangle_Y := \langle D(i_1, i_2, \ldots, i_n) \rangle \langle \overline{D}(i_1, i_2, \ldots, i_n) \rangle / \prod_{\text{triple of colors of vertices}} \theta(i_a, i_b, i_c).
\]

We call this value Yokota’s invariant of the spatial trivalent graph for the admissible coloring.

Yokota’s invariant are generalized for arbitrary graphs with next relations at vertices,

\[
(1) \quad \langle \begin{array}{c} \cdots \end{array} \rangle_Y = \sum_i \Delta_i \langle \begin{array}{c} \cdots \end{array} \rangle_Y
\]

for an \( n \)-valent vertex \((n > 3)\) where color \( i \) moves all admissible colors for the right-hand side diagram. This relation is independent of the ways extending the edge.

\[
\langle \begin{array}{c} i \end{array} \begin{array}{c} j \end{array} \rangle_Y = \delta_{ij} \frac{\Delta_i}{\Delta_i} \langle \begin{array}{c} i \end{array} \rangle_Y
\]

for a 2-valent vertex, and

\[
\langle \begin{array}{c} i \end{array} \langle \begin{array}{c} \rangle \rangle_Y = \delta_{i0} \langle \begin{array}{c} \rangle \rangle_Y
\]

for a 1-valent vertex.

**Theorem 3.5.** \( (\cdot)_Y \) is invariant under RI - RV moves of diagrams and well-defined as an invariant of spatial graphs.

**Remark 3.6.** Originally Yokota’s invariants are defined with a general variable \( A \in \mathbb{C} \setminus \{0\} \) and a set of colors \( \mathbb{N} \cup \{0\} \). In this paper we fix \( A = e^{2\pi i/4r} \) \((3 \leq r \in \mathbb{N})\) and restrict the set of colors to \( I \). Since \( A^{-1} = \overline{A} \), \( \langle D \rangle = \langle \overline{D} \rangle \) and the definition of Yokota’s invariants is rewritten as
\[
\langle \Gamma(i_1, i_2, \ldots, i_n) \rangle_Y = |\langle D(i_1, i_2, \ldots, i_n) \rangle|^2 / \prod_{\text{triple of colors of vertices}} \theta(i_a, i_b, i_c).
\]

Now we define the invariants for handlebody-links in \( S^3 \) by taking a linear sum of Yokota’s invariants for all possible colorings with some weight.

**Definition 3.7.** (Quantum \( \mathcal{U}_q(sl_2) \) invariants for handlebody-links) Let \( J \) be a handlebody-link and \( \Gamma \) be a corresponding trivalent spatial graph. We attach colors \( i_k \) to edges and
Then we define the value \( \langle \cdot \rangle_H \) as follows,

\[
\langle J \rangle_H := \sum_{i_1, \ldots, i_n, j_1, \ldots, j_m} \Delta_{i_1} \cdots \Delta_{i_n} \langle \Gamma(i_1, \ldots, i_n, j_1, \ldots, j_m) \rangle_Y \]

\[
= \sum_{i_1, \ldots, i_n, j_1, \ldots, j_m} \Delta_{i_1} \cdots \Delta_{i_n} |\langle D(i_1, \ldots, i_n, j_1, \ldots, j_m) \rangle| \frac{1}{\prod \theta(i_a, i_b, i_c)},
\]

where \( D(i_1, \ldots, i_n, j_1, \ldots, j_m) \) is a diagram of \( \Gamma(i_1, \ldots, i_n, j_1, \ldots, j_m) \) and each \( i_k, j_l \) moves all admissible colorings for \( \Gamma \). Usually we put a diagram of the corresponding graph inside the bracket instead of the handlebody-link itself as \( \langle D \rangle_H := \langle J \rangle_H \).

We see the well-definedness of \( \langle \cdot \rangle_H \) by the next theorem.

**Theorem 3.8.** Let \( D \) be a diagram of a spatial trivalent graph that represents a handlebody-link \( J \). Then the value \( \langle J \rangle_H = \langle D \rangle_H \) doesn’t change under Reidemeister moves RI - RVI. Hence \( \langle \cdot \rangle_H \) is an invariant for handlebody-links.

**Proof.** The invariance for RI - RV is derived from the invariance of Yokota’s invariants. We show the invariance for RVI. By relation (1),

\[
\sum_i \Delta_i \langle a \hspace{0.5cm} d \hspace{0.5cm} b \hspace{0.5cm} c \rangle_Y = \langle a \hspace{0.5cm} d \hspace{0.5cm} b \rangle_Y = \sum_j \Delta_j \langle a \hspace{0.5cm} d \hspace{0.5cm} b \rangle_Y.
\]

Thus we have,

\[
\langle a \hspace{0.5cm} d \hspace{0.5cm} b \hspace{0.5cm} c \rangle_H = \sum_{a, b, c, d, i} \Delta_a \Delta_b \Delta_c \Delta_d \Delta_i \langle a \hspace{0.5cm} d \hspace{0.5cm} i \hspace{0.5cm} c \rangle_Y \]

\[
= \sum_{a, b, c, d, j} \Delta_a \Delta_b \Delta_c \Delta_d \Delta_j \langle a \hspace{0.5cm} j \hspace{0.5cm} b \hspace{0.5cm} c \rangle_Y = \langle a \hspace{0.5cm} b \rangle_H.
\]

\( \square \)

### 4. Non-triviality of the Invariants

In this section, we show calculations of the quantum invariants in some cases and that the invariants distinguish a handlebody-knot from the trivial one. We also show some properties of the invariants.

First we mention that the \( \langle \cdot \rangle_H \) don’t distinguish a handlebody-link from its mirror image. This is clear from the definition of \( \langle \cdot \rangle_H \).
4.1. Relations. For convenience, we recall some formula for diagrams including $JW_n$.

Tetrahedra’s edge

\[
\left\langle \begin{array}{cccc} n & i & j & k \\ l & m & n \end{array} \right\rangle = \left[ \begin{array}{cccc} i & j & k & l \\ m & n & i \end{array} \right] = \frac{\mathcal{F}!}{\mathcal{E}!} \sum_{c \leq z \leq C} \frac{(-1)^z[z+1]!}{\prod_{s}[z-a_s]! \prod_{t}[b_t-z]!}.
\]

where

\[
[n] = \frac{A^{2n} - A^{-2n}}{A^2 - A^{-2}} \left( = (-1)^{n-1} \Delta_{n-1} \right) \quad [n]! = [n][n-1] \cdots [1] \quad [0]! = 1
\]

\[
\mathcal{F}! = \prod_{s,t} [b_t - a_s]! \quad \mathcal{E}! = [i]![j]![k]![l]![m]![n]!
\]

\[
a_1 = \frac{1}{2}(i + j + k) \quad b_1 = \frac{1}{2}(i + j + l + m)
\]

\[
a_2 = \frac{1}{2}(i + m + n) \quad b_2 = \frac{1}{2}(i + k + l + n)
\]

\[
a_3 = \frac{1}{2}(j + l + n) \quad b_3 = \frac{1}{2}(j + k + m + n)
\]

\[
a_4 = \frac{1}{2}(k + l + m) \quad c = \max\{a_s\} \quad C = \min\{b_t\}.
\]

Local change relations

\[
\left\langle \begin{array}{ccc} n \end{array} \right\rangle = (-1)^n A^{n^2+2n} \left\langle \begin{array}{c} n \end{array} \right\rangle.
\]

\[
\left\langle \begin{array}{ccc} n \end{array} \right\rangle^i = (-1)^i A^{2(n+1)(i+1)} - A^{-2(n+1)(i+1)} \left\langle \begin{array}{c} n \end{array} \right\rangle.
\]

(2)

\[
\left\langle \begin{array}{ccc} i & j \\ k \end{array} \right\rangle = \lambda_k^{ij} \left\langle \begin{array}{ccc} i \\ j \end{array} \right\rangle,
\]

where \( \lambda_k^{ij} = (-1)^{(i+j-k)/2} A^{i+j-k+(i^2+j^2-k^2)/2} \).

(3)

\[
\left\langle \begin{array}{ccc} i \\ k & j\\ l \end{array} \right\rangle = \frac{\theta(i, k, l)}{\Delta_i} \delta_{ij} \left\langle \begin{array}{c} i \end{array} \right\rangle.
\]
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\[

g_{ijklm} = \sum_n \left\{ i \ j \ m \ k \ l \ n \right\},
\]

where \( \left\{ i \ j \ m \ k \ l \ n \right\} = \frac{\begin{bmatrix} i & j & m \\ k & l & n \end{bmatrix}}{\theta(i, l, n) \theta(j, k, n)}. \)

\[

\langle D \rangle_H = \sum_{(i,j,k)} \Delta_i \Delta_j \Delta_k \sum_{a} \Delta_k = \frac{r^2}{4 \sin^4 \frac{\pi}{r}},
\]

In the relation (4) and (5), the colors \( m, k \) move all admissible colors of the right-hand side diagrams.

4.2. Some Examples. First, we calculate for genus 2 trivial handlebody-knot 0_1 by two ways; via a theta curve diagram \( D_1 \) and via a hand-cuffs diagram \( D_2 \). Secondly, we calculate a genus 2 handlebody-knot called 4_1 (Figure 6) in the table of [8].

\begin{figure}[h]
\centering
\includegraphics{figure6}
\caption{4_1 handlebody-knot}
\end{figure}

Example 4.1. (theta curve)

\[
\langle D_1 \rangle_H = \sum_{(i,j,k)} \Delta_i \Delta_j \Delta_k \sum_{a} \Delta_k = \frac{r^2}{4 \sin^4 \frac{\pi}{r}},
\]
where in the summations \( i, j, k \) move all colors such that \((i, j, k)\) are admissible, and \( a = r - 2 - |(r - 2) - (i + j)| \).

Before the calculation for the hand-cuffs diagram, we show the next property of \( \langle \cdot \rangle_H \).

**Proposition 4.2.** (reducible splitting) Let \( D \) be a diagram of a spatial trivalent graph. If \( D \) has an edge that connects two disjoint subgraph diagrams that can be separated by a \( S_1 \) curve from each other, then we have a splitting relation

\[
\langle \begin{array}{c} A \\ \end{array} \begin{array}{c} B \\ \end{array} \rangle_H = \langle \begin{array}{c} \bigcirc \\ \end{array} \bigcirc \rangle_H \langle \begin{array}{c} \bigcirc \\ \end{array} \bigcirc \rangle_H.
\]

**Proof.** The relation directly comes from the next relation of Yokota’s invariants (See [10] Proposition 4.5 and Proposition 4.8)

\[
\langle \begin{array}{c} A \\ \end{array} \begin{array}{c} j \\ i \end{array} \\ \begin{array}{c} k \\ j_1 \end{array} \\ \begin{array}{c} k_2 \\ j_2 \end{array} \rangle_Y = \delta_{ij_1} \delta_{jk_1} \sum_{i=0}^{r-2} \Delta_i \Delta_j \langle \begin{array}{c} \bigcirc \\ \end{array} \bigcirc \rangle_Y \langle \begin{array}{c} \bigcirc \\ \end{array} \bigcirc \rangle_Y.
\]

\[
\Box
\]

**Example 4.3.** (hand-cuffs)

\[
\langle D_2 \rangle_H = \langle \begin{array}{c} \bigcirc \\ \end{array} \bigcirc \rangle_H = \langle \begin{array}{c} \bigcirc \\ \end{array} \bigcirc \rangle_H = \langle \begin{array}{c} \bigcirc \\ \end{array} \bigcirc \rangle_H
\]

\[
= \sum_{i=0}^{r-2} \langle \begin{array}{c} i \\ \end{array} \bigcirc \rangle^2 \sum_{j=0}^{r-2} \langle \begin{array}{c} j \\ \end{array} \bigcirc \rangle^2 = \sum_{i=0}^{r-2} \Delta_i^2 \sum_{j=0}^{r-2} \Delta_j^2 = \frac{r^2}{4 \sin^4 \frac{\pi}{r}}.
\]

Hence we have \( \langle D_1 \rangle_H = \langle D_2 \rangle_H \).

**Example 4.4.** (41 handlebody-knot)

\[
\langle 4_1 \rangle_H = \sum_{(i,k)} \Delta_i \Delta_j \Delta_k \langle \begin{array}{c} \bigcirc \\ \end{array} \bigcirc \rangle^2 = \sum_{(i,k)} \Delta_i \Delta_j \Delta_k \theta(i, i, k) \theta(j, j, k) \langle \begin{array}{c} \bigcirc \\ \end{array} \bigcirc \rangle^2.
\]

Here,

\[
\langle \begin{array}{c} \bigcirc \\ \end{array} \bigcirc \rangle^2 = \sum_{(i,k,l)} \Delta_i \Delta_m \frac{\lambda^k_{i}}{\lambda^l_{i}} \theta(i, k, l) \langle \begin{array}{c} \bigcirc \\ \end{array} \bigcirc \rangle^2 \langle \begin{array}{c} \bigcirc \\ \end{array} \bigcirc \rangle^2
\]

\[
= \sum_{(i,k,l)} \Delta_i \Delta_m \frac{\lambda^k_{i}}{\lambda^l_{i}} \theta(i, k, l) \langle \begin{array}{c} \bigcirc \\ \end{array} \bigcirc \rangle^2
\]
\[
\Delta_l \Delta_m \Delta_s \Delta_t \frac{\lambda_{ik}^j \lambda_{lm}^j \lambda_{ks}^j \lambda_{kt}^j}{\theta(i,k,l)\theta(j,k,m)\theta(i,i,s)\theta(k,k,s)\theta(j,j,t)\theta(k,k,t)} \times \left[ \begin{array}{ccc} i & k & l \\ k & i & s \end{array} \right] \left[ \begin{array}{ccc} j & k & m \\ j & k & j \end{array} \right] \langle i j k l \rangle
\]

Thus, we get
\[
\langle 4_1 \rangle_H = \sum_{(i,k,l),(j,k,m)} (\lambda_{ik}^j \lambda_{lm}^j \lambda_{ks}^j \lambda_{kt}^j)^{-2} \frac{\Delta_l \Delta_m \Delta_s \Delta_t}{\theta(i,k,l)\theta(j,k,m)\theta(i,i,s)\theta(k,k,s)\theta(j,j,t)\theta(k,k,t)} \times \left[ \begin{array}{ccc} i & k & l \\ k & i & s \end{array} \right] \left[ \begin{array}{ccc} j & k & m \\ k & j & k \end{array} \right] \langle i j k l \rangle^2.
\]

4.3. **Table of values of** \( \langle \cdot \rangle_H \). We did numerical calculations of the values of \( \langle \cdot \rangle_H \) in case of the trivial genus 2 handlebody-knot 0_1 and 4_1 for some \( r \). The results are in Table 1. When \( r \geq 5 \), the values of 4_1 are different from ones of 0_1. Hence \( \langle \cdot \rangle_H \) distinguishes 4_1 from the trivial handlebody-knot of genus 2 and \( \langle \cdot \rangle_H \) is not a trivial invariant.

| \( r \) | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|
| 0_1 | 4 | 16 | 52.36067977 | 144 | 345.6547999 |
| 4_1 | 4 | 16 | 84.72135954 | 216 | 499.4856577 |

**Table 1.** The values of \( \langle \cdot \rangle_H \)
REFERENCES

[1] J. W. Barrett, J. M. García-Islas and J. F. Martins Observables in the Turaev-Viro and Crane-Yetter models, J. Math. Phys. 48 093508 (2007), doi:10.1063/1.2759440.

[2] L. H. Kauffman and S. L. Lins Temperley-Lieb recoupling theory and invariants of 3-manifolds, Ann. Math. Study, Vol. 134 (1994), Princeton University Press, Princeton.

[3] S. Kinoshita On elementary ideals of polyhedra in the 3-sphere, Pacific J. Math. 42 (1972), 89-98.

[4] W. B. Lickorish An introduction to knot theory, Graduate Texts in Mathematics Vol. 175 (1997), Springer-Verlag, NY.

[5] S. Suzuki On linear graphs in 3-space, Osaka J. Math. 7 (1970), 375-396.

[6] A. Ishii Moves and invariants for knotted handlebodies, Algebr. Geom. Topol. 8 (2008), 1403-1418.

[7] A. Ishii and M. Iwakiri Quandle cocycle invariants for spatial graphs and knotted handlebodies, to appear in Canad. J. Math.

[8] A. Ishii, K. Kishimoto, H. Moriuchi and M. Suzuki A table of genus two handlebody-knots up to six crossings, to appear in J. Knot Theory Ramifications.

[9] A. Ishii and A. Masuoka A handlebody-knot and a Yetter-Drinfeld module, in workshop ‘Quantum invariants of knots and their categorifications’ at Waseda Univ. (8. 2011).

[10] Y. Yokota Topological invariants of graphs in 3-space, Topology 35 (1996), 77-87.

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