On the Real-rootedness of the Descent Polynomials of $(n - 2)$-Stack Sortable Permutations

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Abstract. Bóna conjectured that the descent polynomials on $(n - 2)$-stack sortable permutations have only real zeros. Brändén proved this conjecture by establishing a more general result. In this paper, we give another proof of Brändén’s result by using the theory of $s$-Eulerian polynomials recently developed by Savage and Visontai.

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1 Introduction

Suppose that $w = w_1 \cdots w_n$ is a permutation of a set of distinct numbers and $w_i$ is the maximal number of $\{w_1, \ldots, w_n\}$. The stack sorting operation $s$ on $w$ can be recursively defined as

$$s(w) = s(w_1 \cdots w_{i-1})s(w_{i+1} \cdots w_n)w_i.$$ 

Let $S_n$ denote the set of permutations of $[n] = \{1, 2, \ldots, n\}$. We say that $\sigma \in S_n$ is $t$-stack sortable if $s^t(\sigma)$ is the identity permutation. For more information on $t$-stack sortable permutations, see Bóna [1], Knuth [7], and West [13].

For $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in S_n$, let

$$\text{Des} \, \sigma = \{i \in [n - 1] : \sigma_i > \sigma_{i+1}\}$$

denote the set of descents of $\sigma$, and let $\text{des} \, \sigma = |\text{Des} \, \sigma|$. The Eulerian polynomials $A_n(x)$ are usually defined as the descent generating function over $S_n$, namely,

$$A_n(x) = \sum_{\sigma \in S_n} x^{\text{des} \, \sigma}. \quad (1)$$

Let $W_t(n, k)$ be the number of $t$-stack sortable permutations in $S_n$ with $k$ descents, and let

$$W_{n,t}(x) = \sum_{k=0}^{n-1} W_t(n, k)x^k$$
be the descent polynomials over $t$-stack sortable permutations. Bóna [1] showed that for fixed $n$ and $t$ the descent polynomial $W_{n,t}(x)$ is symmetric and unimodal, and proposed the following conjecture.

**Conjecture 1.1** ([1]). The descent polynomial $W_{n,t}(x)$ has only real zeros for any integer $1 \leq t \leq n - 1$.

The above conjecture is true for $t = 1, 2, n - 2$, or $n - 1$, see Brändén [2] and references therein. In fact, $W_{n,1}(x)$ are the Narayana polynomials and $W_{n,n-1}(x)$ are the Eulerian polynomials, both of which are known to be real-rooted. Based on a compact and simple form of $W_2(n, k)$ due to Jacquard and Schaeffer [6],

$$W_2(n, k) = \frac{(n + k)!(2n - k - 1)!}{(k + 1)!(n - k)!(2k + 1)!(2n - 2k - 1)!}.$$  

Brändén proved the real-rootedness of $W_{n,2}(x)$ by using the tool of multiplier sequences. For $t = n - 2$, it is easy to show that

$$W_{n,n-2}(x) = A_n(x) - x A_{n-2}(x).$$

By using certain real-rootedness preserving linear operators, Brändén proved the real-rootedness of $W_{n,n-2}(x)$. Remarkably, Brändén [2] obtained the following result.

**Theorem 1.2** ([2]). For any $n \geq 3$ and $k \geq -2$, the polynomial

$$K_n(x) = A_n(x) + kx A_{n-2}(x)$$

has only real zeros.

The main objective of this paper is to give another proof of the above result by using the theory of $s$-Eulerian polynomials recently developed by Visontai and Savage [10]. The $s$-Eulerian polynomials have proven to be a powerful tool for studying the real-rootedness of Eulerian-like polynomials, see also Yang and Zhang [14]. Instead of directly proving Theorem 1.2, we shall prove a slightly general result as shown below.

**Theorem 1.3.** For any $n > 3$ and $k \geq -n$, the polynomial $A_n(x) + kx A_{n-2}(x)$ has only real zeros.

The remainder of the paper is organized as follows. In Section 2, we shall give a brief overview of the theory of $s$-Eulerian polynomials and related results. In Section 3, we shall give a proof of Theorem 1.3. In Section 4, we shall present one open problem.
2 The \( s \)-Eulerian polynomials

The aim of this section is to review some terminology and results on \( s \)-Eulerian polynomials.

Let \( s = (s_1, s_2, \ldots) \) be a sequence of positive integers. Following Savage and Visontai [10], we say that an \( n \)-dimensional \( s \)-inversion sequence is a sequence \( e = (e_1, \ldots, e_n) \in \mathbb{N}^n \) such that \( e_i < s_i \) for each \( 1 \leq i \leq n \). Let \( \mathcal{I}_n(s) \) denote the set of \( n \)-dimensional \( s \)-inversion sequences. For each \( e \in \mathcal{I}_n(s) \), the ascent set of \( e \) is defined as

\[
\text{Asc } e = \{ i \in [n-1] : \frac{e_i}{s_i} < \frac{e_{i+1}}{s_{i+1}} \}.
\]

Let \( \text{asc } e = |\text{Asc } e| \), and let

\[
E_n^{(s)}(x) = \sum_{e \in \mathcal{I}_n(s)} x^{\text{asc } e}.
\]

Savage and Visontai [10] called \( E_n^{(s)}(x) \) the \( s \)-Eulerian polynomial, and obtained the following theorem.

**Theorem 2.1** ([10, Theorem 1.1]). For any positive integer sequence \( s \) and any positive integer \( n \), the \( s \)-Eulerian polynomial \( E_n^{(s)}(x) \) has only real zeros.

Their proof used a refinement of \( E_n^{(s)}(x) \) as follows. Let

\[
E_{n,i}^{(s)}(x) = \sum_{e \in \mathcal{I}_n(s)} \chi(e_n = i) x^{\text{asc } e}.
\]

It is clear that

\[
E_n^{(s)}(x) = \sum_{i=0}^{s_n-1} E_{n,i}^{(s)}(x).
\]

The key point is that these polynomials satisfy certain simple recurrence relation and certain mutually interlacing property. Let us first recall the definition of mutually interlacing. Given two real-rooted polynomials \( f(z) \) and \( g(z) \) with positive leading coefficients, let \( \{r_i\} \) be the set of zeros of \( f(z) \) and \( \{s_j\} \) the set of zeros of \( g(z) \). We say that \( g(z) \) interlaces \( f(z) \), denoted \( g(z) \preceq f(z) \), if either \( \deg f(z) = \deg g(z) = n \) and

\[
s_n \leq r_n \leq s_{n-1} \leq \cdots \leq s_2 \leq r_2 \leq s_1 \leq r_1,
\]

or \( \deg f(z) = \deg g(z) + 1 = n \) and

\[
r_n \leq s_{n-1} \leq \cdots \leq s_2 \leq r_2 \leq s_1 \leq r_1.
\]

We say that a sequence of real polynomials \((f_1(x), \ldots, f_m(x))\) with positive leading coefficients is mutually interlacing if \( f_i(x) \preceq f_j(x) \) for all \( 1 \leq i < j \leq m \). Savage and Visontai obtained the following result.
Lemma 2.2 ([10, Lemma 2.1]). For \( n > 1 \) and \( 0 \leq i < s_n \), we have

\[
E_{n,i}^{(s)}(x) = \sum_{h=0}^{t_i-1} x E_{n-1,h}^{(s)}(x) + \sum_{h=t_i}^{s_{n-1}-1} E_{n-1,h}^{(s)}(x),
\]

where \( t_i = \lceil is_{n-1}/s_n \rceil \), \( E_{1,0}^{(s)}(x) = 1 \) and \( E_{1,i}^{(s)}(x) = x \) for \( 0 < i < s_1 \). Furthermore, the sequence of polynomials \( \{E_{n,i}(x)\}_{i=0}^{n-1} \) is mutually interlacing.

Define the mapping \( \phi : \mathcal{S}_n \to \mathcal{J}_{n}^{(1,2,\ldots)} \) by setting

\[
\phi(\pi) = e = (e_1, e_2, \ldots, e_n)
\]

for \( \pi = (\pi_1, \pi_2, \ldots, \pi_n) \), where

\[
e_i = |\{j : j \in [i-1] \text{ and } \pi_j > \pi_i\}|.
\]

It is easy to show that \( \phi \) is a bijection with \( \text{Des } \pi = \text{Asc } e \). Let

\[
A_{n,i}(x) = \sum_{\pi \in \mathcal{S}_n} \chi(\pi_n = n - i) x^{\text{des } \pi}.
\]

Therefore, we have

\[
E_{n}^{(1,2,\ldots)}(x) = A_n(x),
\]

\[
E_{n,i}^{(1,2,\ldots)}(x) = A_{n,i}(x).
\]

Now, for \( s = (1, 2, \ldots) \), Lemma 2.2 can be restated as follows, which is crucial for our proof of Theorem 1.3.

Lemma 2.3. For \( n \geq 2 \) and \( i = 0, 1, \ldots, n - 1 \),

\[
A_{n,i}(x) = x \sum_{j=0}^{i-1} A_{n-1,j}(x) + \sum_{j=1}^{n-2} A_{n-1,j}(x)
\]

with the initial condition \( A_{1,0}(x) = 1 \). Furthermore, the sequence of polynomials \( \{A_{n,i}(x)\}_{i=0}^{n-1} \) is mutually interlacing.

We shall also need the following result due to Haglund, Ono, and Wagner [5].

Theorem 2.4 ([5, Lemma 8]). Let \( f_1(x), \ldots, f_m(x) \) be real-rooted polynomials with non-negative coefficients, and let \( a_1, \ldots, a_m \geq 0 \) and \( b_1, \ldots, b_m \geq 0 \) be such that \( a_i b_{i+1} \geq b_i a_{i+1} \) for all \( 1 \leq i \leq m - 1 \). If the sequence \( (f_1(x), \ldots, f_m(x)) \) is mutually interlacing, then

\[
\sum_{i=1}^{m} a_i f_i(x) \preceq \sum_{i=1}^{m} b_i f_i(x), \text{ and } \sum_{i=1}^{m} b_i f_i(x) \preceq x \sum_{i=1}^{m} a_i f_i(x).
\]
It is known that interlacing of two polynomials implies the real-rootedness of their arbitrary linear combination, see Obreschkoff [8] and Dedieu [4].

**Theorem 2.5** ([2, Corollary 2.5]). Let \( f(x), g(x) \) be real polynomials. Then \( f(x) \) interlaces \( g(x) \) or \( g(x) \) interlaces \( f(x) \) if and only if the polynomial

\[
\alpha f(x) + \beta g(x)
\]

have only real zeros for any real numbers \( \alpha \) and \( \beta \) with \( \alpha^2 + \beta^2 \neq 0 \).

Therefore, we have the following result.

**Corollary 2.6.** If \( f_i(x), a_i, b_i \) are given as in Theorem 2.4, then the polynomial

\[
\sum_{i=1}^{m} (a_i x + b_i) f_i(x)
\]

has only real zeros.

### 3 Proof of Theorem 1.3

In this section we aim to give a proof of Theorem 1.3. Note that

\[
A_{n-2}(x) = A_{n-1,0}(x).
\]

Before proving Theorem 1.3, let us first express \( A_n(x) \) in terms of \( A_{n-1,i}(x) \).

**Lemma 3.1.** For any integer \( n \geq 2 \), we have

\[
A_n(x) = \sum_{j=0}^{n-2} ((n - j - 1)x + j + 1)A_{n-1,i}(x)
\]

(13)

**Proof.** By (12), we obtain that

\[
A_n(x) = \sum_{i=0}^{n-1} A_{n,i}(x)
\]

\[
= \sum_{i=0}^{n-1} \left( x \sum_{j=0}^{i-1} A_{n-1,i,j}(x) + \sum_{j=1}^{n-2} A_{n-1,i,j}(x) \right)
\]

\[
= x \sum_{i=0}^{n-1} \sum_{j=0}^{i-1} A_{n-1,i,j}(x) + \sum_{i=0}^{n-1} \sum_{j=i}^{n-2} A_{n-1,i,j}(x).
\]
Then, by interchanging the order of summation for each double summation, we get that

\[ A_n(x) = x \sum_{j=0}^{n-2} \sum_{i=j+1}^{n-1} A_{n-1,j}(x) + \sum_{j=0}^{n-2} \sum_{i=0}^{j} A_{n-1,j}(x) \]

\[ = x \sum_{j=0}^{n-2} (n - j - 1)A_{n-1,j}(x) + \sum_{j=0}^{n-2} (j + 1)A_{n-1,j}(x) \]

\[ = \sum_{j=0}^{n-2} ((n - j - 1)x + j + 1)A_{n-1,i}(x), \]

which leads to the desired equality. This completes the proof. \(\square\)

Now we proceed to give a proof of our main theorem.

**Proof of Theorem 1.3.** By (13), we have

\[ K_n(x) = A_n(x) + kx A_{n-2}(x) \]

\[ = \sum_{j=0}^{n-2} ((n - j - 1)x + j + 1)A_{n-1,i}(x) + kx A_{n-1,0}(x) \]

\[ = ((n + k - 1)x + 1)A_{n-1,0}(x) + \sum_{j=1}^{n-3} ((n - j - 1)x + j + 1)A_{n-1,i}(x) \]

\[ + (x + n - 1)A_{n-1,n-2}(x). \]

In view of that \( A_{n-1,n-2}(x) = xA_{n-1,0}(x) \), we have

\[ K_n(x) = ((n + \frac{k}{2} - 1)x + 1)A_{n-1,0}(x) \]

\[ + \sum_{j=1}^{n-3} ((n - j - 1)x + j + 1)A_{n-1,i}(x) \]

\[ + (x + n + \frac{k}{2} - 1)A_{n-1,n-2}(x). \]

Now we shall use Corollary 2.6 to obtain the real-rootedness of \( K_n(x) \). To this end, let \( m = n - 1 \), \( f_i(x) = A_{n-1,i-1}(x) \) for \( 1 \leq i \leq m \) and

\[ a_i = \begin{cases} 
 n + \frac{k}{2} - 1, & i = 1, \\
 n - i, & 2 \leq i \leq m - 1, \\
 1, & i = m,
\end{cases} \]

and

\[ b_i = \begin{cases} 
 1, & i = 1, \\
 i, & 2 \leq i \leq m - 1, \\
 n + \frac{k}{2} - 1, & i = m.
\end{cases} \]
It is routine to check that $f_i(x), a_i$ and $b_i$ satisfy the conditions of Corollary 2.6. This completes the proof. 

\section{One open problem}

We have shown that for any $n > 3$ and $k \geq -n$, the polynomial $K_n(x)$ in (2) has only real zeros. Stanley [11] advised us to further study under what conditions does the polynomial $K_n(x)$ have only real zeros.

Let $T_n$ be the $n$-th tangential or “Zag” number, see [12, A000182] and let $a(n) = T_{n+1}/T_n$. Computer evidence suggests the following conjecture.

\textbf{Conjecture 4.1.} For any $n \geq 3$, the polynomial $K_n(x)$ has only real zeros for $k \leq -n(n-1)$ and $k \geq -a([n/2])$, while it is not real-rooted when $-n(n-1) < k < -a([n/2])$. Moreover, this polynomial has only negative real zeros for $k \geq -a([n/2])$.

Note that there is a useful criterion for determining whether a polynomial of degree $n$ has $n$ distinct real zeros. Suppose that

$$f(x) = \sum_{i=0}^{n} a_{n-i}x^i$$

and

$$g(x) = \sum_{i=0}^{n} b_{n-i}x^i$$

are two polynomials with $a_0 \neq 0$. For any $1 \leq k \leq n$, let

$$\Delta_{2k}(f(x), g(x)) = \det \begin{pmatrix}
    a_0 & a_1 & a_2 & \ldots & a_{2k-1} \\
    b_0 & b_1 & b_2 & \ldots & b_{2k-1} \\
    0 & a_0 & a_1 & \ldots & a_{2k-2} \\
    0 & b_0 & a_1 & \ldots & b_{2k-2} \\
    \ldots & \ldots & \ldots & \ldots & \ldots \\
    \ldots & \ldots & \ldots & \ldots & \ldots \\
    0 & 0 & 0 & \ldots & b_k 
\end{pmatrix}_{k \times k}.$$

These determinants are known as the Hurwitz determinants of $f(x)$ and $g(x)$. Hermite showed that the real-rootedness of $f(x)$ can be uniquely determined by the signs of $\Delta_{2k}(f(x), f'(x))$. The following is essentially due to Borchardt and Hermite [9, pp. 349].

\textbf{Theorem 4.2.} Suppose that $f(x)$ is a real polynomial of degree $n$ with $a_0 > 0$. Then $f(x)$ has $n$ distinct real zeros if and only if the corresponding Hurwitz determinants satisfy

$$\Delta_{2k}(f(x), f'(x)) > 0, \quad \forall 1 \leq k \leq n.$$
Using this characterization, we have verified that, for $3 \leq n \leq 18$, the polynomial $K_n(x)$ has only real zeros when $k \leq -n(n-1)$ and $k \geq -a([n/2])$.

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