Monopoles in the Abelian Projection of Gluodynamics *)

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We discuss some properties of the abelian monopoles in compact $U(1)$ gauge theory and in the $SU(2)$ gluodynamics both on the lattice and in the continuum.

§1. Introduction

Abelian monopoles play a key role in the dual superconductor mechanism of confinement \(^1\) in non-abelian gauge theories. Abelian monopoles appear after the so called abelian projection \(^2\). According to the dual superconductor mechanism a condensation of abelian monopoles should give rise to the formation of an electric flux tube between the test quark and antiquark. Due to a non-zero string tension the quark and the antiquark are confined by a linear potential. This mechanism has been confirmed by many numerical simulations of the lattice gluodynamics \(^3\), \(^4\) which show that the abelian monopoles in the Maximal Abelian projection are responsible for the confinement. The $SU(2)$ string tension is well described by the contribution of the abelian monopole currents \(^5\); these currents satisfy the London equation for a superconductor \(^6\), \(^7\). In Fig. 1, taken from Ref. \(^7\), the abelian monopole currents near the center of the flux tube formed by the quark–anti-quark pair are shown. It is seen that the monopoles wind around the center of the flux tube just as the Cooper pairs wind around the center of the Abrikosov string. In Fig. 2 taken from Ref. \(^8\) we show the dependence of the value of the monopole condensate $\Phi_{inf}^c$ on $\beta$ is shown. It is clearly seen that $\Phi_{inf}^c$ vanishes at the phase transition and it plays the role of the order parameters \(^5\), \(^8\). In Ref. \(^10\) the effective lagrangian for monopoles was reconstructed from numerical data for monopole currents for $SU(2)$ gluodynamics in the Maximal Abelian gauge. It occurs that this lagrangian corresponds to the Abelian Higgs model, the monopole are condensed in the classical string tension of the Abrikosov-Nielsen-Olesen string describes well the quantum string tension of the $SU(2)$ gluodynamics. It means that the description of the gluodynamics at large distances in terms of the monopole variables can be very useful.

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Below we give a short review of the recently obtained results; for the elementary introduction to the subject see Ref. 4).

Since the abelian monopoles play a dominant role in the confinement phenomena it is important to understand how do the abelian monopoles arise in the non-abelian gauge theories. This question is well understood in the lattice gauge theory while it deserves some additional study in the continuum theory. In Section 2 we discuss the origin of the abelian monopoles in the compact electrodynamics.

In Sections 3 and 4 we discuss the procedure of abelian projection in the contin-
uum. As an example we consider the Polyakov Abelian gauge (Section 3) and the Maximal Abelian gauge (Section 4).

In Section 5 we show that the abelian monopoles carry also an abelian electric charge, this fact means that the abelian monopoles are dyons. The dyonic nature of the abelian monopoles is a consequence of the complicated topological structure of non-abelian vacuum which contains self-dual non-abelian configurations. Due to the presence of such configurations the abelian monopoles become abelian dyons.

It is possible that the abelian monopoles correspond to some non-abelian objects which populate the vacuum of gluodynamics. If there exists such correspondence one should observe a non-zero excess of energy density near the monopole world trajectories. Indeed, this excess had been found in numerical simulations of the lattice $SU(2)$ gluodynamics. In Section 6 we discuss this question in details.

In Section 7 we discuss a new three-dimensional Maximal Abelian projection which is better than the usual Maximal Abelian projection from the physical point of view.

§2. Singluar Gauge Transformations and Monopoles

At first we discuss the question how the abelian monopoles appear in the $U(1)$ gauge theory. We use the lattice regularization, the action is a periodic function of the field strength tensor:

$$S_{\text{lattice}} = \sum_{\mathbf{x}, \mu > \nu} f(Re[U_{\mathbf{x}, \mu \nu}]) = \sum_{\mathbf{x}, \mu > \nu} S_{\text{plaq}}(F_{\mathbf{x}, \mu \nu}); \quad U_{\mathbf{x}, \mu \nu} = e^{i F_{\mathbf{x}, \mu \nu}},$$

(2.1)

where $U_{\mathbf{x}, \mu \nu} = U_{\mathbf{x}, \mu}U_{\mathbf{x}+\mu, \nu}U_{\mathbf{x}+\nu, \mu}^+U_{\mathbf{x}, \nu}^+$ and sum is over all lattice plaquettes.

The obvious requirement is that for the weak fields the lattice action (2.1) is reduced to the continuum action:

$$S_{\text{plaq}}|_{F_{\mathbf{x}, \mu \nu} \to 0} \rightarrow F_{\mathbf{x}, \mu \nu}^2$$

(2.2)

Since the function $e^{ix}$ is periodic, the field-strength $F_{\mathbf{x}, \mu \nu}$ is physically equivalent to $F_{\mathbf{x}, \mu \nu} + 2\pi n_{\mathbf{x}, \mu \nu}$, where $n_{\mathbf{x}, \mu \nu}$ is an arbitrary integer-valued lattice two-form.

In the continuum limit $n_{\mathbf{x}, \mu \nu}$ is a surface $\delta$-function. Thus in the continuum theory the gauge fields which correspond to

$$F_{\mu \nu}(x) \quad \text{and to} \quad F_{\mu \nu}(x) + 2\pi \varepsilon_{\mu \nu \alpha \beta} \sum_i \Sigma_{\alpha \beta}^{(i)}(x, \tilde{x}^{(i)})$$

(2.3)

are physically equivalent. Here $\Sigma_{\mu \nu}^{(i)}(x, \tilde{x}^{(i)})$ is an arbitrary surface $\delta$–function*:

$$\Sigma_{\mu \nu}^{(i)}(x, \tilde{x}^{(i)}) = \int d^2 \sigma_{\mu \nu}^{(i)}(x - \tilde{x}^{(i)}(\sigma_{(i)})),$$

$$d^2 \sigma_{\mu \nu}^{(i)} = \varepsilon^{ab} \partial_a \tilde{x}_{\mu}^{(i)} \partial_b \tilde{x}_{\nu}^{(i)} d^2 \sigma^{(i)}$$

(2.4)

the coordinates of the surface, $\tilde{x}_{\mu}^{(i)}(\sigma_{(i)})$, are parametrized by $\sigma_{a}^{(i)}$, $a = 1, 2$.

* This surface is nothing but the Dirac sheet.
Thus defined $U(1)$ gauge theory is invariant under a large class of the singular
gauge transformations, $A_\mu \rightarrow A_\mu - i \Omega^+ \partial_\mu \Omega$, where $\Omega = e^{i\alpha}$ is a regular function (an
element of the $U(1)$ group) while the function $\alpha$ may contain discontinuities $\alpha \rightarrow \alpha + 2\pi$. Under singular gauge transformations, $F_{\mu\nu}(A) \rightarrow F_{\mu\nu}(A^\Omega) = F_{\mu\nu}(A) + [\partial_\mu, \partial_\nu]\alpha$.

It may be shown that $[\partial_\mu, \partial_\nu]\alpha$ corresponds to the surface $\delta$-function, eq.(2.4). Hence the singular gauge transformation shifts the field strength tensor as follows:

$$F_{\mu\nu}(x) \rightarrow F_{\mu\nu}(x) + 2\pi \varepsilon_{\mu\nu\alpha\beta} \sum_{i} \delta^{(i)}(x, \tilde{x}^{(i)}).$$

(2.5)

Thus $S[F_{\mu\nu}(A)] = S[F_{\mu\nu}(A^\Omega)]$, where $S[F_{\mu\nu}(A)]$ is the action of the compact $U(1)$
gauge theory. The explicit example of such singular gauge transformation is the
following: let $\alpha$ be the azimuthal angle in the polar coordinate system:

$$x_1 \pm ix_2 = r \sin \gamma e^{\pm i\alpha}, \quad x_3 = r \cos \gamma$$

(2.6)

$$F_{\mu\nu}(x)[A^\Omega] = F_{\mu\nu}(x)[A] + [\partial_\mu, \partial_\nu]\alpha =$$

$$= F_{\mu\nu}(x)[A] + 2\pi (\delta_{\mu,1}\delta_{\nu,2} - \delta_{\mu,2}\delta_{\nu,1}) \delta(x_1)\delta(x_2)$$

(2.7)

The singular part in the field-strength appears due to the topological non-triviality
of the gauge matrix, but the action which corresponds to $A_\mu$ and to $A^\Omega_\mu$ is the same.

Now we discuss how the abelian monopoles appear in the path integral
formalism. In the continuum limit the compact electrodynamics is described by the
following partition function:

$$Z = \int \mathcal{D}A \int \mathcal{D}\tilde{x} \exp \left\{-\frac{1}{4\epsilon^2} \int d^4x \left( F_{\mu\nu}(A) + 2\pi \varepsilon_{\mu\nu\alpha\beta} \Sigma_{\alpha\beta}(x, \tilde{x}) \right)^2 \right\},$$

(2.8)

this theory is manifestly invariant under the singular gauge transformations which
shift the field strength tensor $F_{\mu\nu}$ according to eq.(2.5). The integration in eq.(2.8)
is over collection of all open and closed surfaces $\Sigma$. The simplest (but not unique)
measure of the integration over the surfaces $\Sigma$ is the string integration measure
described in Ref.12.

It is simple to rewrite the partition function (2.8) in the monopole representation.
First we introduce the additional antisymmetric tensor field $G_{\mu\nu}$ and represent the
partition function as follows:

$$Z = \int \mathcal{D}A \int \mathcal{D}G \int \mathcal{D}\tilde{x} \exp \left\{-\int d^4x \left( \frac{\epsilon^2}{4} G_{\mu\nu}^2 + iG_{\mu\nu} \left( F_{\mu\nu}(A) + 2\pi \varepsilon_{\mu\nu\alpha\beta} \Sigma_{\alpha\beta}(x, \tilde{x}) \right) \right) \right\}.$$  

(2.9)

Integrating over the field $A_\mu$ we get a constraint: $\partial_\mu G_{\mu\nu} = 0$ which can be resolved by
the use of the regular field $B_\mu$: $G_{\mu\nu} = (2\pi)^{-1} \varepsilon_{\mu\nu\alpha\beta} \partial_\alpha B_\beta$. Changing the integration
in eq.(2.9) from $G$ to $B$ we get:

$$Z = \int \mathcal{D}B \int \mathcal{D}j \exp \left\{-\int d^4x \left( \frac{1}{4\epsilon^2} F_{\mu\nu}(B) + i j_\mu(x, \tilde{x}) B_\mu \right) \right\}.$$  

(2.10)
where $g = 4\pi/e$ is the charge of the monopole, and the vector $\hat{x}_{\mu}$ parametrizes the boundaries $j_{\mu}$ of the Dirac sheets $\Sigma^{(i)}_{\mu}(x, \tilde{x})$:

$$j_{\mu} = \partial_{\nu} \Sigma^{(i)}_{\mu}(x, \tilde{x}). \quad (2.11)$$

The boundary of the Dirac sheet is the world trajectory of the monopole. The field $B_{\mu}$ in eq. (2.10) plays the role of the dual gauge field.

Applying the Bardakci-Samuel formula to eq. (2.10) we get:

$$Z = \int DB \int D\Phi \exp \left\{ -\int d^4x \left( \frac{1}{4g^2} F^2_{\mu\nu}(B) + \frac{1}{2} |(\partial_{\mu} + iB_{\mu})\Phi|^2 \right) \right\}, \quad (2.12)$$

where $\Phi$ is the (complex) monopole field which carries the magnetic charge 1.

The situation with the $SU(2) (SU(N))$ lattice gauge theory is similar. There are (singular in the continuum) gauge transformations which do not change the lattice action but create a string-like singularity in the continuum limit. We discuss the physical consequences of this fact in the separate publication, below we give an explicit example of such gauge transformation.

Consider

$$\hat{\Omega}(x) = \left[ \begin{array}{cc} \cos \frac{\gamma}{2} & -\sin \frac{\gamma}{2} e^{-i\alpha} \\ \sin \frac{\gamma}{2} e^{i\alpha} & \cos \frac{\gamma}{2} \end{array} \right] \quad (2.13)$$

where $\alpha$ and $\gamma$ are respectively azimuthal and polar angles in the polar coordinate system (2.6). The action of the lattice $SU(2)$ gauge model is invariant under this gauge transformation. But in the continuum,

$$\hat{F}_{\mu\nu}(\hat{A}) = \Omega^+ \hat{F}_{\mu\nu}(\hat{A}) \Omega - i\Omega^+ [\partial_{\mu}, \partial_{\nu}] \Omega = \Omega^+ \hat{F}_{\mu\nu}(\hat{A}) \Omega - 2\pi (\delta_{\mu,1} \delta_{\nu,2} - \delta_{\mu,2} \delta_{\nu,1}) \sigma^3 \delta(x_1) \delta(x_2) \Theta(-x_3) \quad (2.14)$$

Thus the continuum limit of the lattice action is different from the naive continuum limit, $\lim_{a \to 0} S_{\text{lattice}} \neq \int d^4x \frac{1}{2} Tr[\hat{F}^2_{\mu\nu}]$, due to the presence of the singular fields.

§3. Abelian Projection

For the sake of simplicity we explain the Abelian projection for $SU(2)$ gauge group, the generalization to $SU(N)$ gauge group is straightforward.

The Abelian Projection suggested by ’t Hooft is a partial gauge fixing defined by the conditions of diagonalization of some functional $X[A]$ with respect to gauge transformations $\Omega$, the functional $X[A]$ belongs to the adjoint representation of the $SU(2)$ gauge group:

$$X[A] \rightarrow X[A^{(\Omega)}] = \Omega^+ X[A] \Omega. \quad (3.1)$$

In the abelian projection the matrix $X[A]$ is diagonal and the theory possesses the $U(1)$ gauge symmetry. The generator of this symmetry is the Cartan group generator $\sigma^3/2$, where $\sigma^3$ is the Pauli matrix. The diagonal nonabelian gauge field $A_{\mu}^3$ behaves as an abelian gauge field with respect to the residual $U(1)$ gauge transformations ($A_{\mu} \rightarrow A_{\mu} + i\Omega_{U(1)}^+ \partial_{\mu} \Omega_{U(1)}$, $\Omega_{U(1)} = \exp\{i\sigma^3 $$\alpha\}$).
The abelian monopoles appear as additional abelian degrees of freedom which are associated with the singularities in the gauge fixing conditions. These singularities appear at the points of the space where eigenvalues of the matrix $X[A]$ coincide. The proof that these singularities are monopoles is given in Ref. 2). Two eigenvalues of the matrix $X[A]$ coincide if three independent equations are satisfied 2) and in the four-dimensional space these singularities form closed loops §).

Consider as an example the Polyakov Abelian projection in $SU(2)$ gluodynamics. This abelian projection is defined for a finite temperature gauge theory or for the gauge theory in a finite box. The Polyakov Abelian projection corresponds to the diagonalization of the functional $P_x$ related to the Polyakov loop, $P = \frac{1}{2} \text{Tr} P_x$:

$$P_x = \mathcal{T} \exp \left\{ i \oint_{C_x} \text{d}x_0 A_0(x) \right\},$$

where the integration is over the closed path $C_x$ which starts and ends in the same point $x$ and is parallel to the "time" direction; the symbol $\mathcal{T}$ means the path ordering. The path is closed due to the periodic boundary conditions.

Now we show that in the continuum all the abelian monopoles are static in the Polyakov abelian gauge **). Consider a monopole trajectory which passes through some point $x$, thus the eigenvalues of the Polyakov loop coincide with each other in the point $x$. This means that the matrix $L_x$ belongs to the center of $SU(2)$ group, $\mathbb{Z}_2$: $\text{Tr} L_x = \pm 2$. Consider another point $y$ which lies on the same Polyakov loop (this means that $y_i = x_i$, $i = 1, 2, 3$). $\text{Tr} L_y = \text{Tr} L_x = \pm 2$ and the eigenvalues of the matrix $L_y$ coincide since this matrix belongs to the center of $SU(2)$ group. Thus, if the abelian monopole passes through the point $x = (t_0, \vec{x})$ it also passes through all points $y$ with the same spatial coordinates: $y = (t, \vec{x})$ for all $t$. Thus in the Polyakov Abelian projection all abelian monopole trajectories are static.

Now we discuss the Faddeev–Popov gauge fixing procedure for the 't Hooft abelian projection. The conditions of diagonalization can be explicitly written as follows:

$$C^a[A] = 0, \quad C^a[A] = \text{Tr} (X[A] \sigma^a), \quad a = 1, 2,$$

where $\sigma^a$ are the Pauli matrices. The Faddeev–Popov determinant $\Delta_{FP}[A]$ is defined by the path integral over the gauge orbits of $SU(2)$ group:

$$1 = \Delta_{FP}[A] \int \mathcal{D} \Omega \prod_{a=1,2} \delta \left( C^a[A]^{(\Omega)} \right).$$

Straightforward evaluation of this integral yields:

$$\Delta_{FP}[A] = \text{const.} \exp \left\{ 2 \int \text{d}^4 x \ln |\lambda_1[A(x)] - \lambda_2[A(x)]| \right\}$$

$$= \text{const.} \exp \left\{ 2 \int \text{d}^4 x \ln |\text{Im}\{\lambda_1[A(x)]\}| \right\},$$

§) The closeness of the monopole loops reflects the conservation of the magnetic charge. An explicit proof of this fact is given in Ref. 14).

**) The same conclusion was independently obtained by F. Lenz, private communication.
where $\lambda_a[A(x)], a = 1, 2$ are the eigenvalues of the matrix $X[A]$. The Faddeev–Popov determinant (3.5) is explicitly gauge invariant.

Substituting unity (3.4) into the path integral

$$Z = \int DA \exp \{-S[A]\}, \quad (3.6)$$

and integrating over $\Omega$ we get the partition function in the abelian gauge (3.3):

$$Z_{g.f.} = \int DA \exp \{-S[A]\} \Delta_{FP}[A] \prod_{a=1,2} \delta(C^a[A]) \quad (3.7)$$

§ 4. Maximal Abelian Projection

The most interesting results on abelian monopoles were obtained in the Maximal Abelian (MaA) gauge. This gauge is defined by the maximization of the functional

$$\max \Omega R[\hat{A}^\Omega], \quad R[\hat{A}] = -\int d^4 x \left[ (A^1_\mu)^2 + (A^2_\mu)^2 \right], \quad (4.1)$$

The condition of a local extremum of the functional $R$ is

$$(\partial_\mu \pm igA^3_\mu)A^\pm_\mu = 0. \quad (4.2)$$

This condition (as well as the functional $R[A]$) is invariant under the $U(1)$ gauge transformations, $A_\mu \rightarrow A_\mu + \partial_\mu \alpha$. The meaning of the MaA gauge is simple: by gauge transformations we make the gauge field $\hat{A}_\mu$ as diagonal as possible.

The Maximal Abelian gauge on the lattice is defined by the condition $^{14)}$

$$\max \Omega R[U^\Omega_l], \quad R[U_l] = \sum_l Tr[\sigma_3 U^+_l \sigma_3 U_l], \quad l = \{x, \mu\}. \quad (4.3)$$

This gauge condition corresponds to an abelian gauge, since $R$ is invariant under the $U(1)$ gauge transformations.

Now we discuss the Faddeev–Popov gauge fixing procedure for the MaA projection in the continuum. We define the Faddeev–Popov unity:

$$1 = \Delta_{FP}[A; \lambda] \cdot \int D\Omega \exp\{\lambda R[A^\Omega]\}, \quad \lambda \rightarrow +\infty, \quad (4.4)$$

where $\Delta_{FP}$ is the Faddeev–Popov determinant. We substitute the unity (4.4) in the partition function (3.6), shift the fields by the regular transformation $\Omega^+: A \rightarrow A^{\Omega^+}$ and use the gauge invariance of the Haar measure, the action and the Faddeev-Popov determinant are invariant under the regular gauge transformations. Thus we get the product of the volume of the gauge orbit, $\int D\Omega$, and the partition function in the fixed gauge:

$$Z_{MaA} = \int DA \exp\left\{ -\frac{1}{4} \int d^4 x F_{\mu\nu}^2[A] + \lambda R[A] \right\} \Delta_{FP}[A; \lambda]. \quad (4.5)$$
In the non–degenerate case the FP determinant can be represented in the form:

\[
\Delta_{FP}[A; \lambda] = \det \frac{\partial^2 R(A^{\Omega}(\omega))}{\partial \omega^a(x) \partial \omega^b(y)} \bigg|_{\omega=0} \exp\left\{ -\lambda R(A^{\Omega}) \right\} + \ldots, \tag{4.6}
\]

where \( \Omega^{MaA} = \Omega^{MaA}(A) \) is the regular gauge transformation which corresponds to a global maximum of the functional \( R[A^{\Omega}] \), the dots correspond to the terms which are suppressed in the limit \( \lambda \to \infty \); and

\[
M^{ab}[A] = \frac{\partial^2 R(A^{\Omega}(\omega))}{\partial \omega^a(x) \partial \omega^b(y)} \bigg|_{\omega=0}, \tag{4.7}
\]

\( \Omega(\omega) = \exp\{i\omega^a T^a\} \), \( T^a = \sigma^a/2 \) are the generators of the gauge group, \( \sigma^a \) are the Pauli matrices. In the limit \( \lambda \to +\infty \) the region of the integration over the fields \( A \) reduces to region where the gauge fixing functional \( R \) is maximal, and therefore the partition function (4.5) can be rewritten as follows:

\[
Z_{MaA} = \int \mathcal{D}A \exp\{-S(A)\} \det \frac{\partial^2}{\partial \omega^a(x) \partial \omega^b(y)} \bigg|_{\omega=0} \exp\{ -\lambda R[A^{\Omega}] \}, \tag{4.8}
\]

where \( \Gamma_{FM}R[A] \) is a characteristic function of the Fundamental Modular Region\(^{15} \) for the MaA projection\(^{15} \): \( \Gamma_{FM}R[A] = 1 \) if the field \( A \) belongs to the Fundamental Modular Region (the global maximum of the functional \( R[A] \)) and \( \Gamma_{FM}R[A] = 0 \) otherwise.

Usually in the abelian projection the \( U(1) \) gauge invariant quantities (\( O \)) are considered. Below we derive the explicit expression for the \( SU(2) \) invariant quantity \( \tilde{O} \) which corresponds to \( O \). The expectation value for the quantity \( O \) in the MaA gauge (4.1) is:

\[
< O >_{MaA} = \frac{1}{Z_{MaA}} \int \mathcal{D}A \exp\{-S(A) + \lambda R[A]\} \Delta_{FP}[A; \lambda] O(A). \tag{4.9}
\]

Shifting the fields \( U \to U^\Omega \) and integrating over \( \Omega \) both in the nominator and in the denominator of expression (4.9) we get:

\[
< O >_{MaA} = < \tilde{O} >, \quad \tilde{O}(A) = \frac{\int \mathcal{D}\Omega \exp\{\lambda R[A^\Omega]\} O(A^\Omega)}{\int \mathcal{D}\Omega \exp\{\lambda R[A^\Omega]\}}, \tag{4.10}
\]

\( \tilde{O} \) is the \( SU(2) \) invariant operator. In the limit \( \lambda \to +\infty \) we can use the saddle point method to calculate \( \tilde{O} \):

\[
\tilde{O}(A) = \frac{\sum_{j=1}^{N(A)} \det \frac{\partial^2}{\partial \omega^a(x) \partial \omega^b(y)} \bigg|_{\omega=0} M[A^\Omega^{(j)}] O(A^\Omega^{(j)})}{\sum_{k=1}^{N(A)} \det \frac{\partial^2}{\partial \omega^a(x) \partial \omega^b(y)} \bigg|_{\omega=0} M[A^\Omega^{(k)}]}, \tag{4.11}
\]

where \( \Omega^{(j)} \) are the \( N \)-degenerate global maxima of the functional \( R[A^\Omega] \) with respect to the regular gauge transformations \( \Omega \): \( R[A^\Omega^{(j)}] = R[A^\Omega^{(k)}], j, k = 1, \ldots, N \). In the case of non–degenerate global maximum (\( N = 1 \)), we get \( \tilde{O}(A) = O(A^\Omega^{(1)}) \).
§5. Abelian Monopoles Carry Electric Charge

Consider a (anti-) self–dual configuration of the $SU(2)$ gauge field:

$$F_{\mu\nu}(A) = \pm \frac{1}{2} \epsilon_{\mu
u\alpha\beta} F_{\alpha\beta}(A) \equiv \pm \tilde{F}_{\mu\nu},$$

where $F_{\mu\nu}(A) = \partial_{[\mu} A_{\nu]} + i [A_{\mu}, A_{\nu}]$. In the MaA projection the commutator term $\text{Tr}(\sigma^3[A_{\mu}, A_{\nu}])$ of the field strength tensor $F^3_{\mu\nu}$ is suppressed, since the MaA projection is defined \(^{14}\) by the maximization condition (4.1). Therefore, in the MaA projection eq.(5.1) yields \(^{17}\):

$$f_{\mu\nu}(A) = \partial_{\nu} A_{3\mu} - \partial_{\mu} A_{3\nu} \approx \pm \tilde{f}_{\mu\nu}(A).$$

Thus, the abelian monopole currents must be accompanied by the electric currents:

$$J_{e\mu} = \partial_{\nu} f_{\mu\nu}(A) \approx \pm \partial_{\nu} \tilde{f}_{\mu\nu}(A) = \pm J_{m\mu}.$$ 

Thus, in the MaA projection the abelian monopoles are dyons for (anti) self-dual $SU(2)$ field configurations \(^{17}\). Below we show that in the real (not cooled) vacuum of lattice gluodynamics the abelian monopole currents are correlated with the electric currents \(^{18}\).

In order to study the relation of electric and magnetic currents, we have to calculate connected correlators of these currents. The simplest correlator $\langle \langle J^m_{\mu} J^c_{\mu} \rangle \rangle \equiv \langle J^m_{\mu} J^c_{\mu} \rangle - \langle J^m_{\mu} \rangle \langle J^c_{\mu} \rangle$ is zero, since $\langle J^m_{\mu} J^c_{\mu} \rangle = 0$ due to the opposite parities of the operators $J^m$ and $J^c$, and $\langle J^m_{\mu} \rangle = 0$ due to the Lorentz invariance. The simplest non–trivial (normalized) correlator is

$$\tilde{G} = \frac{1}{\rho^e \rho^m} \langle J^m_{\mu}(y) J^c_{\mu}(y) q(y) \rangle,$$

where $q(x)$ is the sign of the topological charge density at the point $x$ and $\rho_{m,e} = \sum_t |J^m_{\mu}| / (4V)$ are the densities of the magnetic and the electric charges, $V$ is the lattice volume (total number of sites).

![Fig. 3. The dependence of the correlator $\tilde{G}$ on $\beta$](image)

We perform the numerical calculation of the correlator (5.2) in the $SU(2)$ lattice gauge theory on the $8^4$ lattice with periodic boundary conditions. We use 100
statistically independent gauge field configurations for each value of $\beta$.

The dependence of the correlator $\bar{G}$ on $\beta$ is shown in Fig. 3. This correlator is positive for all values of $\beta$. Therefore, the abelian monopoles in the MaA projection carry an electric charge. According to definition (5.2), the sign of the electric charge of the monopole coincides with the product of the magnetic charge and the topological charge. Thus, in the gluodynamic vacuum the abelian monopoles become abelian dyons due to a non-trivial topological structure of the vacuum gauge fields.

§6. **Abelian Monopole Currents are Correlated with $SU(2)$ Action Density**

Abelian monopoles appear as singularities in the gauge transformations (see also Sections 2-4). On the other hand, the monopole currents reproduce the $SU(2)$ string tension (5). Thus, monopoles are likely to be related to some physical objects. A physical object is something which carries action. Below we study the local correlations of the abelian monopoles with the density of the magnetic and the electric parts of the $SU(2)$ action (the global correlation was found in Ref. 19). We show that the monopoles are physical objects but it does not mean that they have to propagate in the Minkowsky space; a chain of instantons can produce a similar effect: an enhancement of the action density along a line in Euclidean space. The simplest quantities which can show this correlation are the relative excess of the magnetic and the electric action densities $\eta^{M,E} = (S_{m,E} - S)/S$ in the region near the monopole current. Here $S$ is the expectation value of the lattice plaquette action, $S_P = \langle (1 - \frac{1}{2}Tr U_P) \rangle$. The quantities $S_{m,E}$ are, respectively, the magnetic and the electric parts of the $SU(2)$ action density, which are calculated on plaquettes closest to the monopole current.

![Fig. 4. The relative excess of the magnetic (triangles, from Refs. 11) and the electric (boxes) action density near the monopole current.](image-url)
In the continuum notation, the quantities $S_{m}^{M,E}$ have the following form:

$$S_{m}^{M} = \frac{1}{2} < \text{Tr}(n_{\mu}(x) \tilde{F}_{\mu\nu}(x))^{2} >, \quad S_{m}^{E} = \frac{1}{2} < \text{Tr}(n_{\mu}(x) F_{\mu\nu}(x))^{2} >, \quad (6.1)$$

$n_{\mu}(x)$ is the unit vector in the direction of the current: $n_{\mu}(x) = j_{\mu}(x)/|j_{\mu}(x)|$, if $j_{\mu}(x) \neq 0$, and $n_{\mu}(x) = 0$ if $j_{\mu}(x) = 0$. It is easy to see that for a static monopole ($j_{0} \neq 0; \ j_{i} = 0, i = 1, 2, 3$) $S_{m}^{M}$ ($S_{m}^{E}$) corresponds to the chromomagnetic action density $(B^{a}_{i})^{2}$ (chromoelectric action density $(E^{a}_{i})^{2}$) near the monopole current.

We calculate the quantities $\eta^{M}$ and $\eta^{E}$ on the lattice $24^{4}$ with periodic boundary conditions. In Fig. 4 we show the quantities $\eta^{M,E}$ vs. $\beta$. The monopole currents are calculated in the MaA projection. In Fig. 4 the statistical errors are smaller than the size of the symbols. It is clearly seen that the abelian monopoles are correlated with the magnetic and the electric parts of the $SU(2)$ action density. Note that the correlation of the monopole current with the magnetic action density is larger than the correlation of the monopole current with the electric action density.

The similar results are obtained with extended monopoles which are defined on the cubes of the size $N \times N \times N$. In Fig. 5 we show the quantities $\eta^{M}$ vs. $\beta$ for the extended monopoles of the sizes $n^3$, $n = 1, 2$ on the lattice $24^{4}$.

§7. 3D Maximal Abelian Gauge and Effective Monopole Potential

As we already discussed the largest part of the numerical calculations is performed in the MaA projection. Usually the expectation values of abelian operators (operators constructed from the abelian gauge fields, diagonal gluons) are calculated in this projection. But abelian operators correspond to nonlocal in time operators in terms of the original $SU(N)$ fields $U_{x,\mu}$ (see eq. (4.11)). This nonlocality occurs
since the gauge fixing condition (4.1) contains time like links $U_{x,4}$. For time nonlocal operators there are obvious problems with the transition from the Euclidean to Minkowsky space–time. Thus there are problems with physical interpretation of the results obtained for abelian operators in the MaA projection.

Below we present first results of calculations in the 3D MaA projection which is defined by the same maximization condition as the usual MaA projection (4.1), but the summation in $R (4.1)$ is over the space–like links\(^*\). Since the time–like links are excluded from the gauge fixing condition, the abelian operators in the 3D MaA projection correspond to local in time operators constructed from the nonabelian fields. In ref. \(^8\) the effective potential for the monopole creation operator was calculated in the MaA projection. As we discussed in the Introduction in the confinement region (below the critical temperature) this potential is of the Higgs type, above the critical temperature this potential has minimum at the zero value of the monopole field. Also this behavior of the potential is very reasonable it is important to prove the monopole condensation in the 3D MaA gauge. In Fig. 6 (a,b) we show the effective monopole potential for the confinement phase ($\beta = 1.5$, $12^3 \cdot 4$ lattice) and for the deconfinement phase ($\beta = 2.5$, $12^3 \cdot 4$ lattice). It is clearly seen that the minimum of the potential is at the nonzero value of the monopole field and is at the zero value for the deconfinement phase. Our definition of the effective potential $V(\Phi)$ is the same as in Ref. \(^8\):

$$e^{-V(\Phi)} = <\delta(\Phi - \Phi_{\text{mon}}(x))>, \quad (7.1)$$

here $\Phi_{\text{mon}}(x)$ is the monopole creation operator, defined in ref. \(^8\). We discuss the dependence of the minimum of the effective potential on the temperature for the 3D MaA projection in a separate publication.

\(^*\) This gauge was discussed by U.-J. Wiese in 1990, was recently rediscovered by D. Zwanzinger (private communication to M.I.P.), and discussed by M. Müller-Preussker at this school.
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