Rolling balls over spheres in $\mathbb{R}^n$

Božidar Jovanović

Mathematical Institute SANU, Serbian Academy of Sciences and Arts, Kneza Mihaila 36, 11000 Belgrade, Serbia

E-mail: bozaj@mi.sanu.ac.rs

Received 27 November 2017, revised 9 May 2018
Accepted for publication 23 May 2018
Published 25 July 2018

Recommended by Professor Dmitry V Treschev

Abstract

We study the rolling of the Chaplygin ball in $\mathbb{R}^n$ over a fixed $(n - 1)$-dimensional sphere without slipping and without slipping and twisting. The problems can be naturally considered within a framework of appropriate modifications of the $L + R$ and LR systems—well known systems on Lie groups with an invariant measure. In the case of the rolling without slipping and twisting, we describe the $SO(n)$-Chaplygin reduction to $S^{n-1}$ and prove the Hamiltonization of the reduced system for a special inertia operator.

Keywords: nonholonomic systems, Chaplygin Hamiltonization, invariant measure

Mathematics Subject Classification numbers: 37J60, 37J35, 70H45

1. Introduction

Let $(Q, L, D)$ be a nonholonomic Lagrangian system, where $Q$ is a $n$-dimensional manifold, $L : TQ \to \mathbb{R}$ Lagrangian, and $D$ nonintegrable $(n - k)$-dimensional distribution of constraints. Let $q = (q_1, \ldots, q_n)$ be some local coordinates on $Q$ in which the constraints are written in the form

$$\sum_{i=1}^{n} \alpha_i^j(q) \dot{q}_i = 0, \quad j = 1, \ldots, k. \tag{1}$$

The motion of the system is described by the Lagrange–d’Alembert equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} + \sum_{j=1}^{k} \lambda_j \alpha_i^j, \quad i = 1, \ldots, n, \tag{2}$$

where the Lagrange multipliers $\lambda_j$ are chosen such that the solutions $q(t)$ satisfy constraints (1). The sum $\sum_{j=1}^{k} \lambda_j \alpha_i^j$ represents the reaction force of the constraints.
The nonholonomic systems, generically, are not Hamiltonian systems. However, many constructions from the theory of Hamiltonian systems, such as Noether’s theorem and the reduction of symmetries, apply with certain modifications (e.g. see [1, 4, 16, 17, 23, 37, 38, 40, 44]). Besides, some systems have an invariant measure, which puts them rather close to Hamiltonian systems and allow the integration using the Euler–Jacobi theorem (e.g. see [1]).

The existence of invariant measure for various nonholonomic problems is studied extensively (e.g. see [29, 30, 33, 41, 42, 50]). The LR systems introduced by Veselov and Veselova [47, 48] and L + R systems introduced by Kozlov and Fedorov [25, 26] on unimodular Lie groups are one of the basic and remarkable examples.

The closely related problem is the Hamiltonization of nonholonomic systems, in particular, after the time reparametrisation by using the Chaplygin reducing multiplier (e.g. see [2, 5, 9, 14, 17, 21, 27, 32, 43, 45]). In the case of integrability, the dynamics over regular invariant m-dimensional tori, in the original time, has the form

\[ \dot{\phi}_1 = \omega_1/\Phi(\phi_1, \ldots, \phi_m), \ldots, \dot{\phi}_m = \omega_m/\Phi(\phi_1, \ldots, \phi_m), \quad \Phi > 0. \]  

(3)

Inspired by the study of the rolling of a of a balanced, dynamically asymmetric ball without slipping (after Chaplygin [18] usually called the Chaplygin ball or the marble Chaplygin ball [21]) and without slipping and twisting (referred to as the rubber Chaplygin ball in [21]) over a fixed sphere in \( \mathbb{R}^3 \), given by Borisov, Fedorov, and Mamaev [7–10–12] and Ehlers and Koiller [22], we study the associated nonholonomic problems in \( \mathbb{R}^n \): the rolling of the Chaplygin ball in \( \mathbb{R}^n \) over a fixed \((n - 1)\)-dimensional sphere without slipping (and twisting). The problems can be naturally considered within a framework of appropriate modifications of the L + R and LR systems, recently introduced in [36].

Note that \( n \)-dimensional nonholonomic rigid body problems: the Veselova problem [27], the Suslov problem [28], the rolling of the rubber Chaplygin ball [34] and the Chaplygin ball [35] over hyperplane in \( \mathbb{R}^n \) (at the zero level set of the \( SO(n - 1) \)-momentum mapping), for certain inertia operators, are Hamiltonizable systems. Moreover, all mentioned models are integrable as well, and a motion over a generic invariant tori has the form (3). In this paper we prove that the rolling of the rubber Chaplygin ball over a sphere allows Chaplygin Hamiltonization, while, however, in general the problem is not integrable.

For a given nonintegrable distribution \( \mathcal{D} \) on a Riemannian manifold \( Q \), there is an alternative, important, variational or sub-Riemannian problem, that is already Hamiltonian. The variational problem for rolling of a \((n - 1)\)-sphere on spaces of constant curvature is studied by Jurdjevic and Zimmerman [39].

1.1. Result and outline of the paper

In section 2 we consider a motion of the Chaplygin ball of radius \( \rho \) without slipping (the velocity of the contact point equals zero) over a fixed sphere in \( \mathbb{R}^n \) of radius \( \sigma \) in three variants of the problem. The first one represents the motion of the ball over outside surface of the fixed sphere, the second one is the rolling over inside surface of the fixed sphere, and the third one is the case where Chaplygin ball represents spherical shell with fixed sphere placed in its interior. The systems are described in proposition 1. In all cases the configuration space is \( SO(n) \times S^{n-1} \) and the nonholonomic distribution is diffeomorphic to \( TSO(n) \times S^{n-1} \).

It appears that these nonholonomic problems are examples of \( \varepsilon\)-modified L + R systems (see [36]) with the parameter

\( 1 \)The Suslov problem studied in [28] is an exception. There, the invariant manifolds need not be tori.
\[ \epsilon = \frac{\sigma}{\sigma \pm \rho}, \]  
\[ (4) \]
and we directly obtain an invariant measure (see theorem 2, item (i)), which takes the simpler form for the inertia operator (theorem 2, item (ii))

\[ I(E_i \wedge E_j) = \frac{D a_i a_j}{D - a_i a_j} E_i \wedge E_j. \]
\[ (5) \]
Here \( 0 < a_i a_j < D, \ i, j = 1, \ldots, n \), \( E_1, \ldots, E_n \) is the standard base of \( \mathbb{R}^n \):

\[ E_1 = (1, 0, \ldots, 0, 0)^T, \ldots, E_n = (0, 0, \ldots, 0, 1)^T, \]
\[ (6) \]
and \( D = m \rho^2 \), where \( m \) and \( \rho \) are the mass and the radius of the rolling ball, respectively.

The operator (5) is introduced in [35] in the study of a related problem of rolling of the Chaplygin ball over a horizontal hyperplane in \( \mathbb{R}^n \). Rolling over the horizontal plane can be seen as the limit case, where \( \epsilon \) becomes 1, as the radius of the fixed sphere \( \sigma \) tends to infinity. Although we have the Hamiltonization of the system for \( \epsilon = 1 \) (at the zero level set of the \( \text{SO}(n-1) \)-momentum mapping), the Hamiltonization, and eventually integrability, for \( \epsilon \neq 1 \) is still an open problem.

In section 3, we study the rolling with additional constraints determined by the non-twist condition of the ball at the contact point (the infinitesimal rotation of the ball in the tangent plane to the contact point are forbidden), referred as the rubber Chaplygin ball problem. The equations of motion are described in proposition 3. Now, the distribution of constraints is \((n - 1)\)-dimensional and represents the connection of the principal bundle

\[
\begin{array}{c}
SO(n) \\
\longrightarrow \\
SO(n) \times S^{n-1} \\
\pi \\
\downarrow \\
S^{n-1}
\end{array}
\]
\[ (7) \]
with respect to the diagonal \( SO(n) \)-action, i.e. the system is a \( SO(n) \)-Chaplygin system.

We also consider an appropriate extended system allowing the integrals that replace the non-twist condition (the rubber Chaplygin ball problem is its subsystem, section 3.3). The obtained system is an example of \( \epsilon \)-modified LR system (see [36]), implying the form of an invariant measure described in theorems 4 and 5. In particular, for the inertia operator

\[ I(E_i \wedge E_j) = (a_i a_j - D) E_i \wedge E_j, \]
\[ (8) \]
the invariant measure, as in the case of non-rubber rolling and the operator (5), significantly simplifies (see theorem 5, item (ii)).

Further, in section 4, we derive the curvature of the nonholonomic distribution (see lemma 7), describe the \( SO(n) \)-Chaplygin reduction to \( S^{n-1} \) (theorem 8), as well as the reduced invariant measure (theorem 10). Finally, we obtain the Hamiltonization of the reduced system defined by the inertia operator (8) (theorem 12, section 5).
2. Chaplygin ball in \(\mathbb{R}^n\)

2.1. Kinematics

We consider the Chaplygin ball type problem of rolling without slipping of an \(n\)-dimensional balanced ball, the mass center \(C\) coincides with the geometrical center, of radius \(\rho\) in several nonholonomic models:

(i) rolling over outer surface of the \((n-1)\)-dimensional fixed sphere of radius \(\sigma\), figure 1(a);
(ii) rolling over inner surface of the \((n-1)\)-dimensional fixed sphere of radius \(\sigma\) \((\sigma > \rho)\), figure 1(b);
(iii) rolling over outer surface of the \((n-1)\)-dimensional fixed sphere of radius \(\sigma\), but the fixed sphere is within the rolling ball \((\sigma < \rho)\), in this case, the rolling ball is actually a spherical shell, figure 1(c).

We suppose that the origin \(O\) of \(\mathbb{R}^n\) coincides with the center of the fixed sphere. The configuration space is the direct product of Lie groups \(SO(n)\) and \(\mathbb{R}^n\), where \(g \in SO(n)\) is the rotation matrix of the sphere (mapping a frame attached to the body to the space frame) and \(\mathbf{r} = \mathbf{OC} \in \mathbb{R}^n\) is the position vector of its center \(C\) (in the space frame). The vector \(\mathbf{r}\) belongs to the \((n-1)\)-dimensional constraint hypersurface \(S\) defined by the holonomic constraint

\[
(\mathbf{r}, \mathbf{r}) = (\sigma \pm \rho)^2
\]

i.e. \(S\) is a sphere \(S^{n-1}\).

As usual, for a trajectory \((g(t), \mathbf{r}(t))\) we define angular velocities of the ball in the moving and the fixed frame, and the velocity of the center \(C\) of the ball in the fixed frame by

\[
\omega = g^{-1} \dot{g}, \quad \Omega = \dot{g} g^{-1} = \text{Ad}_g(\omega), \quad \mathbf{V} = \dot{\mathbf{r}} = \frac{d}{dt} \mathbf{OC},
\]

respectively.

---

2 It would also be interesting to study a modified problem, where we assume that the ball rolls over a rotating \(n\)-dimensional sphere (for \(n = 3\), see [8, 23]). Rolling of a \(n\)-dimensional Chaplygin ball over a rotating horizontal plane is considered in [24].

3 From now on, whenever we have a sign \(\pm\), we take ‘+’ for the case (i) and ‘−’ in the cases (ii) and (iii).
Let $A$ be the point of the rolling ball at the point of contact. The condition for the ball to roll without slipping leads that the velocity of the contact point is equal to zero in the fixed reference frame\(^4\):

\[
\frac{d}{dt} \vec{OA} = \frac{d}{dt} (\vec{OC} + \vec{CA}) = V - \rho \Omega \Gamma = 0 \quad \text{(the case (i))};
\]

\[
\frac{d}{dt} \vec{OA} = \frac{d}{dt} (\vec{OC} + \vec{CA}) = V + \rho \Omega \Gamma = 0 \quad \text{(the cases (ii) and (iii))},
\]

where $\Gamma \in \mathbb{R}^n$ is the unit normal to the fixed sphere at the contact point directed outward, or, equivalently, the direction of the contact point in the fixed reference frame:

\[
\Gamma = \frac{1}{|OA|} \vec{OA} = \frac{1}{\sigma \pm \rho} \vec{r}.
\]

Therefore, the nonholonomic distribution is

\[
\mathcal{D}^\pm = \{ (\omega, V, g, r) \mid V = \pm \rho \text{Ad}_g(\omega) \Gamma = \pm \frac{\rho}{\sigma \pm \rho} \text{Ad}_g(\omega) \vec{r} \}.
\]

It is clear that $\mathcal{D}^\pm$ is diffeomorphic to the product $TSO(n) \times S^{n-1}$.

### 2.2. Dynamics in the fixed frame

In what follows we identify $so(n) \cong so(n)^*$ by an invariant scalar product

\[
\langle X, Y \rangle = \frac{1}{2} \text{tr}(XY).
\]

Let $m$ be the mass of the ball and $I : so(n) \to so(n)^* \cong so(n)$ be the inertia tensor that defines a left-invariant metric on $SO(n)$. The Lagrangian of the system is then given by

\[
L(\omega, V, g, r) = \frac{1}{2} \langle \omega, \omega \rangle + \frac{1}{2} m(\dot{V}, V),
\]

where $(\cdot, \cdot)$ is the Euclidean scalar product in $\mathbb{R}^n$.

By the use of the constraints (9) we find the form of reaction forces in the right-trivialization of $SO(n)$ in which the equation (2) become

\[
\dot{M} = - (\pm \rho \Lambda \wedge \Gamma),
\]

\[
m \dot{V} = \Lambda,
\]

\[
\dot{g} = \Omega \cdot g,
\]

\[
\dot{r} = V
\]

where $M = \text{Ad}_g(\vec{\omega}) \in so(n)^* \cong so(n)$ is the ball angular momentum in the fixed frame and $\Lambda \in \mathbb{R}^n$ is the Lagrange multiplier. Differentiating the constraints (9) and using (14), we get

\[
\Lambda = \pm m \rho (\Omega \Gamma + \Omega \Gamma).
\]

Further, (10) and (9) imply that the vector $\Gamma$ in the fixed frame satisfies the equation:

\[^4\text{Through the paper, we consider vectors in } \mathbb{R}^n \text{ as columns and } \Omega \Gamma \text{ denotes the usual matrix multiplication. The Euclidean scalar product of } x, y \in \mathbb{R}^n \text{ is simply } \langle x, y \rangle = x^T y, \text{ while the wedge product is } x \wedge y = x \otimes y - y \otimes x = xy^T - yx^T.\]
\[ \dot{\Gamma} = \frac{1}{\sigma \pm \rho} V = \pm \frac{\rho}{\sigma \pm \rho} \Omega \Gamma. \]  \hspace{1cm} (18)

Finally, from (17) and (18) we get that (13) takes the form
\[ \dot{M} = -D (\dot{\Omega} \Gamma \otimes \Gamma + \Gamma \otimes \dot{\Omega}) - D (\pm \frac{\rho}{\sigma \pm \rho}) (\Omega \Gamma \otimes \Gamma - \Gamma \otimes \Gamma \Omega \Omega), \]  \hspace{1cm} (19)

where \( D = m \rho^2 \).

2.3. Dynamics in the body frame and reduction

Both the Lagrangian \( L \) and the distribution \( D^\pm \) are invariant with respect to the left \( SO(n) \)-action
\[ a \cdot (\omega, V, g, r) = (\omega, aV, ag, ar), \quad a \in SO(n). \]  \hspace{1cm} (20)

Therefore, the system can be reduced to
\[ so(n) \times S^{n-1} \cong (TSO(n) \times S^{n-1})/SO(n) \cong D^\pm / SO(n). \]

Note that the \( SO(n) \)-action defines the principal bundle (7), where the submersion \( \pi \) is given by
\[ \gamma = \pi(g, r) = \frac{1}{\sigma \pm \rho} g^{-1} r = g^{-1} \Gamma, \]  \hspace{1cm} (21)

that is, a base point of \((g, r)\) is \( \gamma = g^{-1} \Gamma \), the unit normal at the contact point to the fixed sphere (directed outward) in the frame attached to the ball.

We can use \((g, \gamma)\) instead of \((g, r)\), for coordinates of a configuration space. Then the \( SO(n) \)-action (20) takes the form:
\[ a \cdot (\omega, \dot{\gamma}, g, \gamma) = (\omega, a\dot{\gamma}, ag, \gamma). \quad a \in SO(n). \]  \hspace{1cm} (22)

From (18), we get the kinematic equation for \( \gamma \)
\[ \dot{\gamma} = \frac{d}{dt} (g^{-1}) \Gamma + g^{-1} \dot{g} = -g^{-1} \dot{g} g^{-1} \Gamma \pm \frac{\rho}{\sigma \pm \rho} g^{-1} \Omega \Gamma = -\omega \gamma \pm \frac{\rho}{\sigma \pm \rho} \omega \gamma. \]

By introducing parameter \( \epsilon \) (see (4)), we can write it as a modified Poisson equation
\[ \dot{\gamma} = -\epsilon \omega \gamma. \]  \hspace{1cm} (23)

Let
\[ k = \kappa(\omega) = \mathbb{I} \omega + D(\omega \gamma \otimes \gamma + \gamma \otimes \omega), \quad \in so(n) \cong so(n)^* \]  \hspace{1cm} (24)

be the angular momentum of the ball relative to the contact point (see [25]).

**Proposition 1.**

(i) The complete set of equations on \( T^* SO(n) \times S^{n-1} \) in variables \((k, g, \gamma)\) is given by
\[ \dot{k} = [k, \omega], \quad \dot{g} = g \cdot \omega, \quad \dot{\gamma} = -\epsilon \omega \gamma. \]  \hspace{1cm} (25)

(ii) The reduction of the left \( SO(n) \)-symmetry (22) gives a system on \( so(n)^* \times S^{n-1} \) defined by the equations
\[ \dot{k} = [k, \omega], \quad \dot{\gamma} = -\epsilon \omega \gamma. \]  \hspace{1cm} (26)
Proof. By applying the identities
\[ \dot{\omega} = \text{Ad}_{g^{-1}}(\dot{1}), \quad I \dot{\omega} - [I\omega, \omega] = \text{Ad}_{g^{-1}}(d_t(\text{Ad}_g(I\omega))) = \text{Ad}_{g^{-1}}(\dot{M}), \]
to (19), in the left trivialization of \( SO(n) \) we obtain the equation:
\[ I \dot{\omega} - [I\omega, \omega] = -D(\dot{\omega} \gamma \otimes \gamma + \gamma \otimes \gamma \dot{\omega}) - D(\pm \rho \sigma \pm \rho)(\omega \gamma \otimes \gamma - \gamma \otimes \gamma \omega) = -D(\dot{\omega} \gamma \otimes \gamma + \gamma \otimes \gamma \dot{\omega}) + D(1 - \epsilon)[\omega \gamma \otimes \gamma + \gamma \otimes \gamma \omega, \omega]. \]  
(27)

Next, from (23) we have
\[ \frac{d}{dt}(\omega \gamma \otimes \gamma + \gamma \otimes \gamma \dot{\omega}) = \dot{\omega} \gamma \otimes \gamma + \gamma \otimes \gamma \dot{\omega} - (1 - \epsilon)[\omega \gamma \otimes \gamma + \gamma \otimes \gamma \omega, \omega]. \]  
(28)

As a result, from (27) and (28) we obtain:
\[ \dot{k} = I \dot{\omega} + D(\dot{\omega} \gamma \otimes \gamma + \gamma \otimes \gamma \dot{\omega}) + D(1 - \epsilon)[\omega \gamma \otimes \gamma + \gamma \otimes \gamma \omega, \omega] = [I\omega, \omega] + D(1 - \epsilon)[\omega \gamma \otimes \gamma + \gamma \otimes \gamma \omega, \omega] = [k, \omega]. \]
□

Remark 1. If the radius \( \sigma \) of the fixed sphere (the case (i)) tends to infinity, the parameter \( \epsilon \) tends to 1, and the above equations reduce to the equations of the rolling of the Chaplygin ball over a horizontal hyperplane in \( \mathbb{R}^n \) (see [25, 31, 35]). Also, note that the rolling of a Chaplygin ball over a sphere (26) is an example of a modified L + R system on the product of \( so(n) \) and the Stiefel variety \( V_{n,r} \) for \( r = 1 \), see section 4.1 of [36].

Remark 2. Note that the mapping
\[ \xi \mapsto (\xi \Gamma) \wedge \Gamma = \xi \Gamma \otimes \Gamma + \Gamma \otimes \Gamma \xi \]
is the orthogonal projection \( \text{pr}_\gamma : so(n) \rightarrow v \) with respect to the scalar product (11), while \( \xi \mapsto (\xi \gamma) \wedge \gamma = \xi \gamma \otimes \gamma + \gamma \otimes \xi \gamma \) is the orthogonal projection \( \text{pr}_\gamma \) to \( v_\gamma \), where the subspaces \( v \) and \( v_\gamma \) are defined by
\[ v = \mathbb{R}^n \wedge \Gamma \quad \text{and} \quad v_\gamma = \text{Ad}_{g^{-1}}(v) = \mathbb{R}^n \wedge \gamma. \]
(29)

Then we have
\[ \frac{d}{dt}\text{pr}_\gamma = \epsilon[\text{pr}_\gamma, \text{ad}_\omega]. \]
(30)

where \([\cdot, \cdot]\) is the standard Lie bracket in the space of linear operators of \( so(n) \). Thus, equivalently, we can derive (28) from the identity.
Remark 3. The operator $\kappa = I + D\text{pr}_\gamma$: $so(n) \to so(n)^\ast$ can be also defined by the use of the constrained Lagrangian

$$L = L|_{V = \pm \rho \text{Ad}_g(\omega) \Gamma} = \frac{1}{2} \langle \Gamma\omega, \omega \rangle + \frac{D}{2} \langle \text{Ad}_g(\omega) \Gamma, \text{Ad}_g(\omega) \Gamma \rangle =: \frac{1}{2} \langle \kappa(\omega), \omega \rangle,$$

(31)

which represents the kinetic energy, preserved along the flow of the system.

2.4. Invariant measure

Based on general observations given for $\epsilon$-modified $L + R$ systems (see theorems 4 and 5, [36]) we have that for the rolling over a sphere, the density of an invariant measure keeps the same form as in the case of the rolling over a horizontal hyperplane (see Fedorov and Kozlov [25, 26]).

Let

$$\mu(\gamma) = \sqrt{\det(\kappa)} = \sqrt{\det(I + D\text{pr}_\gamma)},$$

(32)

and let $A = \text{diag}(a_1, \ldots, a_n)$, where $a_1, \ldots, a_n$ are parameters of the inertia operator (5). Also, by $dk$ and $d\gamma$ we denote the standard volume forms on $so(n)^\ast$ and $S^n-1$, respectively, and by $\Omega$ the canonical symplectic structure on $T^*SO(n)$, $d = \dim SO(n)$.

Theorem 2.

(i) The problem of the rolling of a ball over a sphere (25) on $T^*SO(n) \times S^{n-1}$ in variables $(k, g, \gamma)$ has an invariant measure

$$\mu^{-1} \Omega^d \wedge d\gamma = 1 / \sqrt{\det(\kappa)} \Omega^d \wedge d\gamma = 1 / \sqrt{\det(I + D\text{pr}_\gamma)} \Omega^d \wedge d\gamma,$$

(33)

while the reduced flow (26) in variables $(k, \gamma)$ has an invariant measure

$$\mu^{-1} dk \wedge d\gamma = 1 / \sqrt{\det(I + D\text{pr}_\gamma)} dk \wedge d\gamma.$$  

(34)

(ii) For the inertia operator (5), the density (32) is proportional to

$$(\gamma, A^{-1}\gamma)^{1/2(n-2)}.$$

Remark 4. Since $dk = \det(\kappa) d\omega$, the invariant measure of the reduced system considered in variables $(\omega, \gamma)$ is $\mu(\gamma) d\omega \wedge d\gamma$.

2.5. Three-dimensional case

In the case $n = 3$, under the isomorphism between $\mathbb{R}^3$ and $so(3)$

$$\vec{X} = (X_1, X_2, X_3) \mapsto X = \begin{pmatrix} 0 & -X_3 & X_2 \\ X_3 & 0 & -X_1 \\ -X_2 & X_1 & 0 \end{pmatrix},$$

(35)
from (26), we obtain the classical equations of rolling without slipping of the Chaplygin ball over a sphere

$$\frac{d}{dt}\vec{k} = \vec{k} \times \vec{\omega},$$

$$\frac{d}{dt}\vec{\gamma} = \epsilon \vec{\gamma} \times \vec{\omega},$$

(36)

where $\vec{K} = \mathbb{I} + D\vec{\omega} = D(\vec{\omega}, \vec{\gamma})\vec{\gamma}$ and $\mathbb{I} = \text{diag}(I_1, I_2, I_3)$ is the inertia operator of the ball. In the space $\mathbb{R}^6(\vec{\omega}, \vec{\gamma})$ the density (32) of an invariant measure is equal to

$$\mu(\vec{\gamma}) = \sqrt{\det(\mathbb{I} + D\mathbb{E})(1 - D(\vec{\gamma}, (\mathbb{I} + D\mathbb{E})^{-1}\vec{\gamma}))},$$

(37)

the expression given by Chaplygin for $\epsilon = 1$ [18] (see remark 1), and by Yaroshchuk for $\epsilon \neq 1$ [49]. Here $\mathbb{E} = \text{diag}(1, 1, 1)$.

The system (36) always has three integrals

$$F_1 = (\vec{\gamma}, \vec{\gamma}) = 1, \quad F_2 = \frac{1}{2}(\vec{k}, \vec{\omega}), \quad F_3 = (\vec{k}, \vec{k}).$$

(38)

For $\epsilon = 1$, there is the fourth integral $F_4 = (\vec{k}, \vec{\gamma})$ and the problem is integrable by the Euler–Jacobi theorem: the phase space is almost everywhere foliated by 2D invariant tori with quasi-periodic, non-uniform motion (3) (see Chaplygin [18]). Moreover, Borisov and Mamaev proved that the system (36) is Hamiltonizable with respect to certain nonlinear Poisson bracket on $\mathbb{R}^6$ ([9], see also [14, 46]).

Remarkably, for $\epsilon = -1$ (the case (iii) with $\rho = 2\sigma$) Borisov and Fedorov (see [7]) found the integrable case with the fourth integral

$$\tilde{F}_4 = (I_2 + I_1 - I_1^2 + I_1 - I_2 + D)k_1^2\gamma_1 + (I_1 + I_1 - I_2 + I_2 + D)k_2^2\gamma_2 + (I_1 + I_1 - I_2 + I_2 + D)k_3^2\gamma_3.$$

The system is integrated on an invariant hypersurface $\tilde{F}_4 = 0$ [11]. Furthermore, its topological analysis and a representation as a sum of two conformally Hamiltonian vector fields are given in [12] and [46], respectively. We feel that it would be very interesting to have similar results in a dimension greater than three.

3. Rolling of the Chaplygin ball without slipping and twisting

3.1. Rubber rolling

3D rubber Chaplygin ball problems are introduced in [21] and [22], while the multidimensional rubber rolling over a horizontal hyperplane is considered in [34]. For a given normal vector $\gamma = g^{-1}\Gamma$, let

$$E_1, \ldots, E_{n-1}, \Gamma, \quad \text{and} \quad e_1 = g^{-1}E_1, \ldots, e_{n-1} = g^{-1}E_{n-1}, \gamma = g^{-1}\Gamma$$

be orthonormal bases of $\mathbb{R}^n$ in the fixed frame and in the body frame, respectively. Rubber Chaplygin ball is defined as a system (9) and (12) subjected to the additional constraints

$$\phi_{ij} = \langle \Omega, E_i \wedge E_j \rangle = \langle \omega, e_i \wedge e_j \rangle = 0, \quad 1 \leq i < j \leq n - 1$$

(39)

describing the no-twist condition: the angular velocity matrix $\omega$ has rank 2 and the corresponding admissible plane of rotation contains the normal vector $\gamma$ to the rolling sphere at the contact point.

Alternatively, note that
\[ E_i \wedge E_j, \quad e_i \wedge e_j = \text{Ad}_d^{-1}(E_i \wedge E_j), \quad 1 \leq i < j \leq n - 1 \]

are the orthonormal bases of \( \mathfrak{h} \) and \( \mathfrak{h}_\gamma = \text{Ad}_{d^{-1}} \mathfrak{h} \), orthogonal complements to \( \upsilon \) and \( \upsilon_\gamma \) (see (29)) with respect to the scalar product (11). Thus, the constraints (39) can be rewritten as

\[ \text{pr}_\mathfrak{h}_\gamma \Omega = 0, \quad \text{i.e.} \quad \text{pr}_\mathfrak{h}_\gamma \omega = 0 \iff \Omega \in \upsilon, \quad \text{i.e.} \quad \omega \in \upsilon_\gamma. \]  

(40)

As a result, we obtain \((n - 1)\)-dimensional constraint distribution

\[ \mathcal{F}^\pm = \{ (\omega, V, g, r) \mid V = \pm \tfrac{\rho}{\sigma \pm \rho} \text{Ad}_d(\omega) r, \ \text{pr}_\mathfrak{h}_\gamma \omega = 0 \} \subset \mathcal{D}^\pm. \]  

(41)

Let \( E \) be the identity operator on \( \mathfrak{so}(n) \). We have the relation

\[ k = \mathcal{L}_\omega + D\omega = \mathcal{L}_\omega, \quad \text{for} \quad \omega \in \upsilon_\gamma = \mathbb{R}^n \wedge \gamma, \]  

(42)

where \( k \) is given by (24) and \( I = \mathcal{I} + E \). Let \( m = \mathcal{L}_\omega \in \mathfrak{so}(n) \cong \mathfrak{so}(n)^* \) be the angular momentum with respect to the modified inertia operator \( \mathcal{I} \). After the identification \( \mathcal{D}^\pm \cong \mathcal{T SO}(n) \times S^{n-1} \), we obtain a natural phase space of the problem:

\[ \mathcal{G} = \{ (m, g, \gamma) \in T^* \mathcal{SO}(n) \times S^{n-1} \mid \text{pr}_\mathfrak{h}_\gamma I^{-1} m = \text{pr}_\mathfrak{h}_\gamma \omega = 0 \}. \]

Using proposition 1 and (42), we can write the equations of a motion in the variables \((m, g, \gamma)\)

\[ \dot{m} = [m, \omega] + \lambda_0, \quad \dot{g} = g \cdot \omega, \quad \dot{\gamma} = -\epsilon \omega \gamma. \]  

(43)

The Lagrange multiplier \( \lambda_0 \in \mathfrak{h}_\gamma \) is determined from the condition that the angular velocity \( \omega \) satisfies (40). From (30) and the identity \( \text{pr}_\mathfrak{h}_\gamma + \text{pr}_\upsilon_\gamma = E \), we have \( \frac{d}{dt} \text{pr}_\mathfrak{h}_\gamma = \epsilon [\text{pr}_\mathfrak{h}_\gamma, \text{ad}_\omega] \). Thus,

\[ 0 = \frac{d}{dt} (\text{pr}_\mathfrak{h}_\gamma \omega) = \epsilon (\text{pr}_\mathfrak{h}_\gamma \text{ad}_\omega - \text{ad}_\omega \text{pr}_\mathfrak{h}_\gamma) \omega + \text{pr}_\mathfrak{h}_\gamma \dot{\omega} = \text{pr}_\mathfrak{h}_\gamma \frac{d}{dt} ([I^{-1} [m, \omega] + I^{-1} \lambda_0]), \]

and the multiplier \( \lambda_0 \in \mathfrak{h}_\gamma \) is the solution of the equation

\[ I^{-1} ([m, \omega] + \lambda_0) - \gamma \otimes \gamma I^{-1} ([m, \omega] + \lambda_0) - I^{-1} ([m, \omega] + \lambda_0) \gamma \otimes \gamma = 0. \]  

(44)

Thus, we obtain.

**Proposition 3.** The equations of a motion of the rubber Chaplygin ball on \( \mathcal{G} \) are given by (43), where \( m = \mathcal{L}_\omega = \mathcal{I}_\omega + D\omega \), and \( \lambda_0 \in \mathfrak{h}_\gamma \) is the solution of (44). The reduction of the left \( \text{SO}(n)\)-symmetry (22) induces a system on the space \( \mathcal{G}_0 = \mathcal{G}/\text{SO}(n) = \{ (m, \gamma) \in \mathfrak{so}(n)^* \times S^{n-1} \mid \text{pr}_\mathfrak{h}_\gamma \omega = 0 \} \) given by (23) and

\[ \dot{m} = [m, \omega] + \lambda_0. \]  

(45)

The proof of the next theorem follows from considerations given in section 3.3 below.

**Theorem 4.** The problem of the rubber rolling of a ball over a sphere (43) and the reduced system (23) and (45) possess invariant measures

\[ \mu_\gamma(\gamma) \Omega^d \wedge d\gamma|_{\mathcal{G}}, \quad \mu_\gamma(\gamma) dm \wedge d\gamma|_{\mathcal{G}}. \]
respectively, where the density $\mu_\epsilon(\gamma)$ is given by
\[
\mu_\epsilon(\gamma) = (\det I^{-1}|_{B_\gamma})^{\frac{1}{2}}. 
\]

**Remark 5.** Since $dm = \det(I)d\omega = \text{const} \cdot d\omega$, contrary to remark 4, here the reduced system considered in variables $(m, \gamma)$ has the invariant measure with the same density as in the variables $(m, \gamma)$: $\mu_\epsilon d\omega \wedge d\gamma$.

### 3.2. Three-dimensional case

For $n = 3$, under the isomorphism (35) between $\mathbb{R}^3$ and $\mathfrak{so}(3)$ and the identification of $\vec{\gamma}$ with $\vec{e}_1 \wedge \vec{e}_2$ in (39), we have
\[
G_0 = \{(\vec{m}, \vec{\gamma}) \in \mathbb{R}^3 \times S^2 \mid \phi = (\vec{\gamma}, \vec{\omega}) = 0\} 
\]
and the reduced system (45) and (23) reads
\[
\vec{m} = \vec{m} \times \vec{\omega} + \lambda \vec{\gamma}, \quad \vec{\gamma} = \epsilon \vec{\gamma} \times \vec{\omega},
\]
where
\[
\vec{m} = (I + DE)\vec{\omega} = L\vec{\omega}, \quad \lambda = -(\vec{m}, \Gamma^{-1}(\vec{m} \times \vec{\omega}))/((\vec{\gamma}, \Gamma^{-1}\vec{\gamma})).
\]

The density (46) reduces to the well known expression
\[
\mu_\epsilon(\vec{\gamma}) = (\Gamma^{-1}\vec{\gamma}, \vec{\gamma})^n
\]
(see [21] for $\epsilon = 1$ and [22] for $\epsilon \neq 1$). Apart of the integrability of the rolling over a horizontal plane ($\epsilon = 1$) [21], as in the case of non-rubber rolling, Borisov and Mamaev proved the integrability for $\epsilon = -1$ [10]. Note that for $\epsilon = 1$, the above equations coincide with the equations of nonholonomic rigid body motion studied by Veselov and Veselova [47, 48].

The problem is Hamiltonizable for all $\epsilon$ [21, 22]. On the other hand, the rubber rolling of the ball where the mass center does not coincide with the geometrical center over a horizontal plane provides an example of the system having the following interesting property (see [6, 13]). The appropriate phase space is foliated on invariant tori, such that the foliation is isomorphic to the foliation of integrable Euler case of the rigid body motion about a fixed point, but the system itself has not analytic invariant measure and is not Hamiltonizable.

### 3.3. Extended system and a dual expression for an invariant measure

Note that we can consider equations (48) and (49) on the product $\mathbb{R}^3 \times S^2$ as well. The system also has an invariant measure with density (50) and the reduced system on (47) is its subsystem ($\phi = (\vec{\omega}, \vec{\gamma})$ is the first integral). Similarly, the system (45) and (23) can be extended and the invariant measure given in theorem 4 is the restriction to $G_0$ of an invariant measure of the extended system. In order to define the extended system such that we can use the results of [36], we need to add some additional variables.

Firstly, consider the system (45) and (23) on $G_0$. We can choose vectors $\vec{e}_i(t), i = 1, \ldots, n - 1$ along a trajectory $(m(t), \gamma(t))$, such that $\vec{e}_1(t), \ldots, \vec{e}_{n-1}(t), \vec{e}_n(t) = \gamma(t)$ is an orthonormal base of $\mathbb{R}^n$ and that
\[
\dot{\vec{e}}_i = -\epsilon \omega \vec{e}_i, \quad i = 1, \ldots, n.
\]
Indeed, we can take a base $e_1(t_0), \ldots, e_n(t_0)$ at some initial time $t_0$ (it is defined modulo the orthogonal transformations of the hyperplane $\gamma(t_0)$). From the modified Poisson equation (51) it follows that the scalar products $\{e_i(t), e_j(t)\}$ are conserved.

Further, the equation (51) imply
\[
(e_i \wedge e_j) = e[e_i \wedge e_j, \omega], \quad 1 \leq i < j \leq n.
\]
(52)

We can determine the reaction force $\lambda_0$ starting from the expression
\[
\lambda_0 = \sum_{1 \leq i < j \leq n} \lambda^{ij} e_i \wedge e_j
\]
(53)

and differentiating the constraints (39) by using (45) and (52). We get the Lagrange multipliers $\lambda^{ij}$ in the form
\[
\lambda^{ij} = - \sum_{1 \leq k < l \leq n} \langle e_k \wedge e_l, I^{-1}[m, \omega]\rangle A^{ijkl},
\]
(54)

where $A^{ijkl}$ is the inverse of the matrix $A_{ijkl} = \langle e_i \wedge e_j, e_k \wedge e_l \rangle$.

The extended system on
\[
N = \{(m, e_1, \ldots, e_n) | m \in so(n)^*, e_i \in \mathbb{R}^n, \langle e_i, e_j \rangle = \delta_{ij}, 1 \leq i, j \leq n\},
\]
is defined by the equation (45) together with (51), (53) and (54), and the functions
\[
\phi_{ij} = \langle \omega, e_i \wedge e_j \rangle, \quad 1 \leq i < j \leq n - 1
\]
(55)

are its first integrals.

On the other hand, let $\mathcal{N} = so(n)^* \times \prod_{1 \leq i < j \leq n} O(e_i \wedge e_j)$, where $O(e_i \wedge e_j)$ is the adjoint orbit of $e_i \wedge e_j$ in $so(n)$. The closed system defined by (45) and (52)–(54) on $\mathcal{N}$ is an example of a $\epsilon$-modified LR system introduced in [36]. Now, the functions (55) and $\psi_{ijkl} = \langle e_i \wedge e_j, e_k \wedge e_l \rangle$, $1 \leq i < j \leq n, 1 \leq k < l \leq n$ are its first integrals. Also, the system has an invariant measure (see theorem 1, [36]):
\[
\mu_{\epsilon} \cdot dm \wedge \bigwedge_{1 \leq i < j \leq n} d(e_i \wedge e_j)|_{\mathcal{N}},
\]

where
\[
\mu_{\epsilon} = (\det A_{ijkl})^{\frac{1}{2}} \quad (1 \leq i < j \leq n - 1, 1 \leq k < l \leq n - 1).
\]
(56)

It easily follows that the extended system has an invariant measure
\[
\mu_{\epsilon} \cdot dm \wedge \bigwedge_{1 \leq i < j \leq n} d(e_i \wedge e_j)|_{\mathcal{M}}
\]

(the replacing of equations (52) by (51) do not reflect essentially on the corresponding Liouville equation). Note that the density (56) of the extended system coincides with (46) and the above statement implies invariant measures of the equations (43) and (23), (45) given in theorem 4.

Next, by introducing the momentum
\[
m = pr_{\rho}, L\omega + pr_{\rho, \omega} = \omega + \gamma \otimes \gamma (L\omega - \omega) + (L\omega - \omega) \gamma \otimes \gamma \in so(n) \cong so(n)^*,
\]
(57)

we can describe the extended system without using additional variables $e_i$, $i = 1, \ldots, n - 1$.

We have the momentum equation (see [36], i.e. [27] for $\epsilon = 1$)
\[
m = e[m, \omega] + (1 - \epsilon)pr_{\rho, \omega}[L\omega, \omega].
\]
(58)
Thus, we obtain an alternative description of the extended system on \( so(n)^* \times S^{n-1} \) given by (23) and (58). It leads to the dual expression for an invariant measure (see theorems 2 and 4, [36]).

Let \( A = \text{diag}(a_1, \ldots, a_n) \), where \( a_1, \ldots, a_n \) are parameters of the special inertia operator (8).

**Theorem 5.**

(i) The extended system (23) and (58) of the rubber rolling of a ball over a fixed sphere in variables \((\mathbf{m}, \gamma)\) has an invariant measure \( \tilde{\mu}_e \, \mathrm{dm} \wedge d\gamma \).

\[
\tilde{\mu}_e(\gamma) = (\det \Gamma_{\gamma})^{\frac{1}{n-1}}.
\]  

(59)

(ii) For \( \Gamma \) defined by (8), i.e. \( \Gamma(E_i \wedge E_j) = a_ia_jE_i \wedge E_j \), the density (59) is proportional to

\[
(\gamma, A\gamma)(\frac{1}{n-1}(a-2)).
\]

It is also clear that the momentum equation (58), together with (23), defines extended system on \( T^*SO(n) \times S^{n-1} \) with an invariant measure \( \tilde{\mu}_e \Omega^d \wedge d\gamma \).

4. Reduction of \( SO(n) \)-symmetry

4.1. Chaplygin reduction to \( TS^{n-1} \)

As we already mentioned, the problem of the rubber rolling of a ball over a fixed sphere is a \( SO(n) \)-Chaplygin system with respect to the action (20). We have the principal bundle (7) and (21), together with the principal connection

\[
T_{(\mathbf{x}, \gamma)}SO(n) \times S^{n-1} = \mathcal{F}_{(\mathbf{x}, \gamma)}^\pm \oplus \ker \, d\sigma_{(\mathbf{x}, \gamma)},
\]

\[
\ker \, d\sigma_{(\mathbf{x}, \gamma)} = so(n) \cdot (\mathbf{g}, \mathbf{r}).
\]  

(60)

The system reduces to the tangent bundle \( TS^{n-1} \cong \mathcal{F}^\pm /SO(n) \). The procedure of reduction for rubber rolling over a sphere for \( n = 3 \) is given by Ehlers and Koiller [22]. Note that in this case the system is always Hamiltonizable due to the fact that it has an invariant measure and that the reduced configuration space is 2D. We proceed with a reduction of \( n \)-dimensional variant of the problem.

Recall that the vector in \( \mathcal{F}_{(\mathbf{x}, \gamma)}^\pm \) are called horizontal, while the vectors in \( \ker \, d\sigma_{(\mathbf{x}, \gamma)} \) vertical. The horizontal lift \( \tilde{\gamma}^h \) of the base vector \( \gamma \in T_eS^{n-1} \) to the horizontal space \( \mathcal{F}^\pm \) at the point \((g, r) \in \pi^{-1}(\gamma) \) is the unique vector in \( \mathcal{F}_{(\mathbf{x}, \gamma)}^\pm \) satisfying \( d\pi(\gamma^h) = \dot{\gamma} \).

**Lemma 6.** The reduced Lagrangian on \( TS^{n-1} = \mathcal{F}^\pm /SO(n) \) reads

\[
L_{\text{red}}(\dot{\gamma}, \gamma) = \frac{1}{2\epsilon^2}(\mathbf{I}(\gamma \wedge \dot{\gamma}), \gamma \wedge \dot{\gamma}) = -\frac{1}{4\epsilon^2}
\]

\[
\text{tr}(\mathbf{I}(\gamma \wedge \dot{\gamma})\gamma \wedge \dot{\gamma}).
\]

**Proof.** The horizontal lift \( \gamma^h|_{(x, \mathbf{r})} = (\omega, \mathbf{V}) \) is given by:

\[
\omega = \frac{1}{\epsilon} \gamma \wedge \dot{\gamma} = \frac{\sigma}{\sigma} \gamma \wedge \dot{\gamma},
\]

\[
\mathbf{V} = \dot{\mathbf{r}} = (\sigma \pm \rho) \frac{d}{dt}(g^\gamma) = (\sigma \pm \rho)(\mathbf{g} \gamma + g^\gamma) = (\sigma \pm \rho)(\mathbf{g} \frac{1}{\epsilon}(\gamma \wedge \dot{\gamma})\gamma + g^\gamma)
\]

\[
= (\sigma \pm \rho)(1 - \frac{1}{\epsilon})g^\gamma = -(\sigma \pm \rho)(\pm \frac{\rho}{\sigma})g\dot{\gamma}.
\]
As a result, the reduced Lagrangian is
\[ L_{\text{red}}(\dot{\gamma}, \gamma) = L(\dot{\gamma}^h|_{(g, r)}, g, r)_{(g, r) \in \pi^{-1}(\gamma)} = \frac{1}{2\varepsilon^2} \left( \langle \gamma \wedge \dot{\gamma} \rangle, \gamma \wedge \dot{\gamma} \right) + \frac{D}{2\varepsilon^2} \langle \dot{\gamma}, \dot{\gamma} \rangle, \]
which proves the statement.

The reduced Lagrange–d’Alembert equation describing the motion of the system on a sphere \( S^{n-1} \) takes the form
\[
\left( \frac{\partial L_{\text{red}}}{\partial \dot{\gamma}} - \frac{d}{dt} \frac{\partial L_{\text{red}}}{\partial \dot{\gamma}} \right) = \langle J_{(g, r)}(\dot{\gamma}^h), K_{(g, r)}(\dot{\gamma}^h, \dot{\gamma}^h) \rangle, \quad \xi \in T_\gamma S^{n-1},
\]
where \((g, r) \in \pi^{-1}(\gamma)\), \(K(\cdot, \cdot)\) is \(\mathfrak{so}(n)\)—valued curvature of the connection, and \(J\) is the momentum mapping of \(SO(n)\)—action (20) (see [4, 40]).

It is well known that the momentum mapping
\[ J : \mathcal{T}(SO(n) \times S^{n-1}) \to \mathfrak{so}(n) \cong \mathfrak{so}(n)^* \]
of the action (20) is given by
\[ J_{(g, r)}(\omega, \mathbf{V}) = \text{Ad}_k(\mathbf{I} \omega) + m \mathbf{V} \wedge r. \]
Therefore,
\[
J_{(g, r)}(\dot{\gamma}^h) = \frac{1}{\varepsilon} \text{Ad}_k(\mathbf{I} \dot{\gamma}^h) - \frac{m}{\varepsilon}(\sigma \pm \rho)(\pm \frac{\rho}{\sigma}) \gamma \wedge r
= \text{Ad}_k\left(\frac{1}{\varepsilon} \langle \gamma \wedge \dot{\gamma} \rangle \pm m(\sigma \pm \rho)^2 \frac{\rho}{\sigma} \langle \gamma \wedge \dot{\gamma} \rangle \right)
= \frac{1}{\varepsilon} \text{Ad}_k \left( \langle \gamma \wedge \dot{\gamma} \rangle \pm D \frac{\sigma \pm \rho}{\rho} \langle \gamma \wedge \dot{\gamma} \rangle \right)
= \frac{1}{\varepsilon} \text{Ad}_k \left( \langle \gamma \wedge \dot{\gamma} \rangle + \frac{D}{1 - \varepsilon} \langle \gamma \wedge \dot{\gamma} \rangle \right).
\]

Let \(\xi_1, \xi_2 \in \mathfrak{X}_{(g, r)}^{\pm}\). By definition, the curvature \(K_{(g, r)}(\xi_1, \xi_2)\) is the element \(\eta \in \mathfrak{so}(n)\), such that \(\eta \cdot (g, r)\) is the vertical component of the commutator of vector fields \([X_2, X_1]\) at \((g, r)\), where \(X_1\) and \(X_2\) are smooth horizontal extensions of \(\xi_1\) and \(\xi_2\).

**Lemma 7.** Let \(\xi_1, \xi_2 \in T_\gamma S^{n-1}\) and \((g, r) \in \pi^{-1}(\gamma)\). Then
\[ K_{(g, r)}(\xi_1^h, \xi_2^h) = (1 - \frac{\rho^2}{\sigma^2}) \text{Ad}_k(\xi_2 \wedge \xi_1) = \frac{2\varepsilon - 1}{\varepsilon^2} \text{Ad}_k(\xi_2 \wedge \xi_1). \]
In particular, for \(\varepsilon = 1/2\), i.e. \(\rho = \sigma\), the curvature vanish and the constraints are holonomic.

**Remark 6.** Note that the factor \(1 - \frac{\rho^2}{\sigma^2}\) equals to \(1 - K_1/K_2\) where \(K_1\) and \(K_2\) are curvatures of the fixed and rolling sphere, respectively. The same factor appears in the case of rubber rolling of arbitrary two surfaces in \(\mathbb{R}^3\) (see [15]).

Since \(\langle \gamma \wedge \dot{\gamma}, \dot{\gamma} \wedge \xi \rangle = 0\), we can replace \(J\) by \(\frac{1}{\varepsilon} \text{Ad}_k(\mathbf{I} \langle \gamma \wedge \dot{\gamma} \rangle)\) at the right hand side of (61), and we get the \(J - K\) term in the form
\[ \langle J_{(g,\gamma)}(\dot{\gamma}^h), K_{(g,\gamma)}(\dot{\xi}^h, \xi^h) \rangle = \frac{2}{\epsilon^3} \left( I(\gamma \wedge \dot{\gamma}), \xi \wedge \dot{\gamma} \right) \]
\[ = -\frac{2}{\epsilon^3} \text{tr} \left( I(\gamma \wedge \dot{\gamma}) \cdot (\xi \otimes \dot{\gamma} \otimes \xi) \right) \]
\[ = \frac{2}{\epsilon^3} \left( I(\gamma \wedge \dot{\gamma}) \dot{\gamma}, \xi \right). \]

We have
\[ \frac{\partial L_{\text{red}}}{\partial \gamma} = \frac{1}{\epsilon^3} I(\gamma \wedge \dot{\gamma}) \dot{\gamma}, \quad \frac{\partial L_{\text{red}}}{\partial \dot{\gamma}} = -\frac{1}{\epsilon^3} I(\gamma \wedge \dot{\gamma}) \gamma. \]

Therefore, we obtain the following statement.

**Theorem 8.** The Lagrange–d’Alembert equation describing the motion of the reduced system are given by
\[ \left( \frac{d}{dt} I(\gamma \wedge \dot{\gamma}) + (1 - \epsilon) I(\gamma \wedge \dot{\gamma}) \dot{\gamma}, \dot{\xi} \right) = 0, \quad \xi \in T_{\gamma}S^{n-1}. \] (62)

The above reduction slightly differs from the Chaplygin SO(n − 1)—reduction of the Veselova problem studied in [27].

**Proof of lemma 7.** In the coordinates \((g, \gamma)\), the SO(n)-action takes the form (22). Let \(\eta \in so(n)\). The associated vector field on \(SO(n) \times S^{n-1}\) with respect to the action (22) is given by
\[ \eta \cdot (g, \gamma) \cong (\text{Ad}_{g^{-1}} \eta, 0) \in T_{(g,\gamma)}SO(n) \times S^{n-1}, \]
where, as above, we use the left trivialization of TSO(n). Further, the horizontal and vertical components of the vector \((\omega, \xi) \in T_{(g,\gamma)}(SO(n) \times S^{n-1})\), respectively, simply read
\[ (\omega, \xi)^H = (\frac{1}{\epsilon} \gamma \wedge \xi, \xi), \]
\[ (\omega, \xi)^V = (\omega - \frac{1}{\epsilon} \gamma \wedge \xi, 0). \]

Now, let \(\xi_1, \xi_2\) be vector fields, the extensions of \(\xi_1, \xi_2 \in T_{\gamma_0}S^{n-1}\) defined in a neighborhood \(U\) of \(\gamma_0\), and \(X_1, X_2\) their horizontal lifts to \(SO(n) \times U\):
\[ X_i(g, \gamma) = \left( \frac{1}{\epsilon} \gamma \wedge \xi_i, \xi_i \right) = Y_i + Z_i, \quad Y_i = \left( \frac{1}{\epsilon} \gamma \wedge \xi_i, 0 \right), \quad Z_i = (0, \xi_i), \quad i = 1, 2. \]

Then, by definition
\[ \langle K_{(g,\gamma)}(\xi_1^h, \xi_2^h), \eta \rangle = \langle -[X_1, X_2]^V|_{(g,\gamma_0)}, \text{Ad}_{g^{-1}} \eta \rangle, \]
i.e.
\[ K_{(g,\gamma)}(\xi_1^h, \xi_2^h) = -\text{Ad}_g [X_1, X_2]^V|_{(g,\gamma_0)}. \] (63)

We shall prove
\[ [X_1, X_2]^V = [X_1, X_2] = \left( \frac{2}{\epsilon} - \frac{1}{\epsilon^3} \right) (\xi_1 \wedge \xi_2, 0), \] (64)
which, according to (63), proves the lemma.

Without losing a generality we may suppose that $\gamma_0 = (0, 0, \ldots, 0, 1)^T$. Let $(q_1, \ldots, q_{n-1}) \in U$ be the local coordinates on the upper half-sphere $S^{n-1}_+ = \{ \gamma \in S^{n-1} \mid \gamma_n > 0 \}$ defined by

\[
\begin{align*}
\gamma_i &= q_i, & i &= 1, \ldots, n-1, \\
\gamma_n &= \sqrt{1 - q_1^2 - \cdots - q_{n-1}^2}, \\
U &= \{(q_1, \ldots, q_{n-1}) \in \mathbb{R}^{n-1} \mid q_1^2 + \cdots + q_{n-1}^2 < 1 \}.
\end{align*}
\]

The given vectors $\xi_1, \xi_2 \in T_{\gamma}S^{n-1}_+$ have the form $(\xi^1_i, \ldots, \xi^{n-1}_i, 0)^T$, $i = 1, 2$. By taking $\xi_j = \text{const}$, $\xi_i = \gamma_n \partial / \partial q_i$, $i = 1, 2$, (65) define their natural commutative extensions to $U$. Note that $\partial / \partial q_i$ corresponds to the vector field

\[
E_i = \frac{q_i}{\sqrt{1 - q_1^2 - \cdots - q_{n-1}^2}} E_n = E_i - \frac{\gamma_i}{\gamma_n} E_n
\]

in redundant variables on $S^{n-1}_+ \subset \mathbb{R}^n$, where we consider (6) as vector fields on $\mathbb{R}^n$. Whence, in redundant variables the vector fields (65) read

\[
\xi_i = (\xi^1_i, \ldots, \xi^{n-1}_i, 0)^T = (\xi^1_i, \ldots, \xi^{n-1}_i, \frac{1}{\gamma_n} (\xi^1_i \gamma_1 + \cdots + \xi^{n-1}_i \gamma_{n-1}))^T,
\]

$i = 1, 2$, implying the identities

\[
\begin{align*}
\xi_i(\gamma_j) &= \xi_j(q_i) = \xi^j_i, & j &= 1, \ldots, n-1, \\
\xi_i(\gamma_n) &= \xi_i(\sqrt{1 - q_1^2 - \cdots - q_{n-1}^2}) = -\frac{\xi^i_1 q_1 + \cdots + \xi^{n-1}_i q_{n-1}}{\sqrt{1 - q_1^2 - \cdots - q_{n-1}^2}} = \xi^n_i. (66)
\end{align*}
\]

Let $E_{i\ell} = (E_k \wedge E_{\ell}, 0)$. Then $Y_{i\ell} = \sum_{k < \ell} y_{ij} E_{i\ell}$, where

\[
y_{ij} = \frac{1}{\epsilon} \left( (\gamma, E_k) (\xi_i, E_{\ell}) - (\gamma, E_{\ell}) (\xi_i, E_k) \right) = \frac{1}{\epsilon} \left( \gamma_k \xi_{ij} - \gamma_{ij} \xi^k \right).
\]

Next, due to the relations

\[
[E_{i\ell}, E_{k\ell}] = ([E_i \wedge E_{\ell}, E_k \wedge E_{\ell}], 0), \quad [E_{i\ell}, Z_{\ell}] = 0, \quad [Z_1, Z_2] = 0,
\]

on $SO(n) \times U$, we get:
\[
\begin{align*}
[X_1, X_2] &= \sum_{k<l,i<j} [y_k^i E_{kl}, y_j^i E_{ij}] + \sum_{i<j} [y_i^j E_{ij}, Z_2] + \sum_{k<l} [Z_1, y_k^l E_{kl}] \\
&= \left( \frac{1}{\epsilon^2} [\gamma \wedge \xi_1, \gamma \wedge \xi_2, 0] \right) + \sum_{i<j} [y_i^j E_{ij}, Z_2] - \sum_{i<j} Z_2 (y_i^j) E_{ij} \\
&\quad + \sum_{k<l} y_k^l [Z_1, E_{kl}] + \sum_{k<l} Z_1 (y_k^l) E_{kl} \\
&= - \frac{1}{\epsilon^2} ([\xi_1, \xi_2, 0] + \sum_{i<j} (\xi_1 (y_k^l) - \xi_2 (y_i^j)) E_{ij}).
\end{align*}
\]

(67)

On the other hand, from (66) we obtain
\[
\xi_1 (y_k^l) - \xi_2 (y_i^j) = \frac{1}{\epsilon} \xi_1 (\gamma_i \xi_k^l - \gamma_k \xi_i^l) - \frac{1}{\epsilon} \xi_2 (\gamma_i \xi_k^l - \gamma_k \xi_i^l)
\]
\[
= \frac{1}{\epsilon} (\xi_i \xi_k^l - \xi_k \xi_i^l) - \frac{1}{\epsilon} (\xi_i \xi_k^l - \xi_k \xi_i^l)
\]
\[
= \frac{2}{\epsilon} ((\xi_1, E_i) (\xi_2, E_j) - (\xi_1, E_j) (\xi_2, E_i)),
\]

which together with (67) implies the relation (64).

\[\Box\]

4.2. The reduced system on \( T^* S^{n-1} \)

Consider the Legendre transformation
\[
p = \frac{\partial L_{\text{red}}}{\partial \dot{\gamma}} = - \frac{1}{\epsilon^2} \mathbf{I} (\gamma \wedge \dot{\gamma}) \gamma.
\]

(68)

The point \((p, \gamma)\) belongs to the cotangent bundle of a sphere realized as a symplectic submanifold in the symplectic linear space \( (\mathbb{R}^{2n}, \mathbf{d} p_1 \wedge \mathbf{d} q_1 + \cdots + \mathbf{d} p_n \wedge \mathbf{d} q_n) \):
\[
(\gamma, \gamma) = 1, \quad (\gamma, p) = 0.
\]

(69)

Let \(\dot{\gamma} = \dot{\gamma}(p, \gamma)\) be the inverse of the Legendre transformation and
\[
\Upsilon = \Upsilon (\gamma, p) = \frac{1}{\epsilon^2} \left( \mathbf{I} (\gamma \wedge \dot{\gamma}) \right) \dot{\gamma} \big|_{\dot{\gamma}=\dot{\gamma}(p, \gamma)}.
\]

Then we can write the equation (62) in the form
\[
(- \epsilon \dot{p} + (1 - \epsilon) \Upsilon, \xi) = 0, \quad \xi \in T^* S^{n-1},
\]

which is equivalent either to
\[
\epsilon \gamma \wedge \dot{p} + (\epsilon - 1) \gamma \wedge \Upsilon = 0,
\]

(70)

or to
\[
\dot{p} = \frac{(1 - \epsilon)}{\epsilon} \Upsilon + \mu \gamma,
\]

(71)

where the multiplier \(\mu\) is determined from the equation
\[
\frac{d}{dt} (\gamma, p) = (\dot{\gamma}, p) + (\dot{p}, \gamma) = (\dot{\gamma}, p) + \frac{(1 - \epsilon)}{\epsilon} (\Upsilon, \gamma) + \mu (\gamma, \gamma) = 0.
\]
Proposition 9. The reduced flow on on the cotangent bundle $T^*S^{n-1}$ realized with constraints \((69)\) takes the following form
\[
\dot{\gamma} = X_{\gamma}(p, \gamma), \quad \dot{p} = X_p(p, \gamma),
\]
where $X_{\gamma}$ is the inverse of the Legendre transformation \((68)\) and
\[
X_p = \left(\frac{1}{\epsilon^3} \left( I(\gamma \wedge X_{\gamma}) \right) X_{\gamma} + \left( \frac{(\epsilon - 1)}{\epsilon^3} \left( (I(\gamma \wedge X_{\gamma}) \right) (X_{\gamma}, \gamma) - (X_{\gamma}, p) \right) \right)\gamma.
\]

4.3. The momentum equation and an invariant measure
Alternatively, the reduced equation \((70)\) can be derived by using the momentum equation \((58)\).
After the reduction to the sphere $S^{n-1}$, we obtain
\[
m = \frac{1}{\epsilon} \left( I(\gamma \wedge \dot{\gamma}) \right) \gamma \wedge \gamma = \epsilon \gamma \wedge p, \quad \dot{m} = \epsilon \dot{\gamma} \wedge p + \epsilon \gamma \wedge \dot{p},
\]
\[
\omega = \frac{1}{\epsilon} \gamma \wedge \dot{\gamma}, \quad [m, \omega] = [\gamma \wedge p, \gamma \wedge \dot{\gamma}] = \dot{\gamma} \wedge p,
\]
\[
[I\omega, \omega] = \frac{1}{\epsilon^2} \left( I(\gamma \wedge \dot{\gamma}) \gamma \wedge \dot{\gamma} - \gamma \wedge \dot{\gamma} I(\gamma \wedge \dot{\gamma}) \right) = \dot{\gamma} \wedge p + \gamma \wedge \Upsilon.
\]
By putting those expressions into \((58)\) we get the equation \((70)\):
\[
\epsilon \dot{\gamma} \wedge p + \epsilon \gamma \wedge \dot{p} = (2\epsilon - 1) \dot{\gamma} \wedge p + (1 - \epsilon)(\dot{\gamma} \wedge p + \gamma \wedge \Upsilon)
\]
\[
\iff \epsilon \gamma \wedge \dot{p} + (1 - \epsilon)(\gamma \wedge \Upsilon) = 0.
\]
As a bi-product, we get the following statement.

Theorem 10. The reduced equation \((72)\) has an invariant measure
\[
(det I_{|v_\gamma})^{\frac{1}{2}} w^{n-1},
\]
where $w$ is the canonical symplectic form
\[
w = dp_1 \wedge d\gamma_1 + \cdots + dp_n \wedge d\gamma_n |_{T^*S^{n-1}}.
\]

Proof. The mapping
\[
\Phi: (\gamma, p) \mapsto (\gamma, m), \quad m = \epsilon \gamma \wedge p,
\]
together with $\omega = \frac{1}{\epsilon} \gamma \wedge \dot{\gamma}$, maps the reduced system \((72)\) to the subsystem of \((23)\) and \((58)\), and the pull-back $\Phi^*(dm \wedge d\gamma)$ is the standard volume form $w^{n-1}$ on $T^*S^{n-1}$ (up to the multiplication by a constant). Now the statement follows from theorem 5, item (i). \hfill \Box

5. Hamiltonization of the reduced system

5.1. Equations for the special inertia operator
Based on the Hamiltonization and integrability of the reduced Veselova system \cite{27}, we have the Hamiltonization and integrability of the rubber rolling of a Chaplygin ball over a
horizontal hyperplane for a special inertia operator (8) (see [34]). Namely, under the time substitution \( \text{d}r = 1/\sqrt{(A\gamma, \gamma)} \text{d}t \), the reduced system becomes an integrable Hamiltonian system describing a geodesic flow on \( S^{n-1} \) of the metric

\[
ds^2 = \frac{1}{(\gamma, A\gamma)} (\langle A\text{d}\gamma, \text{d}\gamma \rangle (A\gamma, \gamma) - (A\gamma, \text{d}\gamma)^2),
\]

(74)

where \( \text{d}\gamma = (\text{d}\gamma_1, \ldots, \text{d}\gamma_n)^T \) [34].

Now we proceed with a rolling over a sphere and, as in the case of the horizontal rolling, we suppose that the inertia operator is given by (8). Then the reduced Lagrangian \( L_{\text{red}}(\dot{\gamma}, \gamma) \) and the Legendre transformation (68) take the form

\[
L_{\text{red}} = \frac{1}{2\epsilon^2} (\langle A\dot{\gamma}, \dot{\gamma} \rangle (A\gamma, \gamma) - (A\gamma, \dot{\gamma})^2).
\]

(75)

\[
p = \frac{\partial L_{\text{red}}}{\partial \dot{\gamma}} = \frac{1}{\epsilon^2} (\gamma, A\gamma) A\dot{\gamma} - \frac{1}{\epsilon^2} (\dot{\gamma}, A\gamma) A\gamma.
\]

(76)

Under conditions (69), relations (76) can be uniquely inverted to yield

\[
\dot{\gamma} = \frac{\epsilon}{(\gamma, A\gamma)} (A^{-1}p - (p, A^{-1}\gamma)\gamma)
\]

(77)

implying that the angular velocity in terms of \((p, \gamma)\) takes the form

\[
\omega(p, \gamma) = \frac{1}{\epsilon} \gamma \wedge \dot{\gamma} = \frac{\epsilon}{(\gamma, A\gamma)} \gamma \wedge A^{-1}p,
\]

and we get:

\[
\Upsilon(p, \gamma) = \frac{1}{\epsilon^2} (\gamma, A\gamma) A\dot{\gamma} = \frac{1}{\epsilon^2} ((\dot{\gamma}, A\dot{\gamma}) A\gamma - (A\gamma, \dot{\gamma}) A\dot{\gamma})
\]

\[
= \frac{\epsilon^2}{(\gamma, A\gamma)^2} (\gamma, A\gamma) A\gamma - (A\gamma, \dot{\gamma}) A\dot{\gamma}
\]

\[
= \frac{\epsilon^2}{(\gamma, A\gamma)^2} (\gamma, A\gamma) A\gamma - (A\gamma, \dot{\gamma}) A\dot{\gamma}
\]

\[
= \frac{\epsilon^2}{(\gamma, A\gamma)^2} ((A^{-1}p, p) + (A\gamma, \gamma) (p, A^{-1}\gamma)^2) A\gamma
\]

\[
+ \frac{\epsilon^2}{(\gamma, A\gamma)^2} (p, A^{-1}\gamma) (A\gamma, \gamma) (p, A^{-1}\gamma) A\gamma,
\]

that is

\[
\Upsilon(p, \gamma) = \frac{\epsilon^2}{(\gamma, A\gamma)^2} ((A^{-1}p, p) A\gamma + (p, A^{-1}\gamma) (A\gamma, \gamma) p).
\]

In particular, \((\Upsilon(p, \gamma), \gamma) = \epsilon^2 (A^{-1}p, p)/(\gamma, A\gamma)\), and the right hand side of equation (71) reads
\[ X_p(p, \gamma) = \frac{(1 - \epsilon)}{\epsilon} \Upsilon + \frac{(\epsilon - 1)}{\epsilon} (p, \gamma) \gamma - (\gamma, p) \gamma \]
\[ = \frac{(1 - \epsilon)}{\gamma, A\gamma} \left( (A^{-1} p, p) A\gamma + (p, A^{-1} \gamma) (A\gamma, \gamma) p \right) + \frac{\epsilon (\epsilon - 1)}{\gamma, A\gamma} (A^{-1} p, p) \gamma \]
\[ - \frac{\epsilon^2}{\gamma, A\gamma} ((A^{-1} p, p) - (p, A^{-1} \gamma) (\gamma, p)) \gamma. \]

Finally, we obtain the equation
\[
\dot{p} = \epsilon \left( \frac{(1 - \epsilon)}{\gamma, A\gamma} \right) \left( (A^{-1} p, p) A\gamma + (p, A^{-1} \gamma) (A\gamma, \gamma) p \right) - \frac{\epsilon}{\gamma, A\gamma} (p, A^{-1} p) \gamma.
\]

By combining theorems 5 and 10, we get.

**Theorem 11.** The reduced flow of the rubber Chaplygin ball rolling over a sphere with a inertia operator (8) on the cotangent bundle \( T^*S^n - 1 \) realized with constraints (69) is given by equations (77) and (78). The system has an invariant measure
\[
(A\gamma, \gamma) \frac{a^n + 2 - n}{2n} w^{a-1}.
\]

**5.2. The Chaplygin reducing multiplier**

The Hamiltonian function of the reduced system takes the form
\[
H = \frac{\epsilon^2}{2} \left( (p, A^{-1} p) \right) \frac{(\gamma, A\gamma)}{2},
\]
which is unique only on the subvariety (69).

At the points of \( T^*S^n - 1 \), the system (77) and (78) can be written in the almost Hamiltonian form
\[
\dot{x} = X_H = (X_p, X_\gamma), \quad i_{X_H} (w + \Sigma) = dH,
\]
where \( \Sigma \) is a semi-basic perturbation term, determined by the J–K term at the right hand side of (61) (e.g. see \[17, 21, 44, 45\]). The form \( w + \Sigma \) is non-degenerate, but, in general, it is not closed.

The **Chaplygin multiplier** is a nonvanishing function \( \nu \) such that \( \check{w} = \nu (w + \Sigma) \) is closed. The Hamiltonian vector field \( \check{X}_H \) of the function \( H \) on \( (T^*S^n - 1, \check{w}) \) is proportional to the original vector field:
\[
\check{X}_H = \frac{1}{\nu} X_H, \quad i_{\check{X}_H} \check{w} = dH.
\]

Thus, applying the time substitution \( d\tau = \nu dt \), the system (81) becomes the Hamiltonian system
\[
\frac{d}{d\tau} x = \check{X}_H.
\]

On the other hand, a classical way to introduce the Chaplygin reducing multiplier for our system is as follows (e.g. see [19, 27]). Consider the time substitution \( d\tau = \nu(\gamma) dt \), and denote \( \gamma' = d\gamma/d\tau = \dot{\gamma}/\nu \). Then the Lagrangian function transforms to \( L^* (\gamma', \gamma) = L_{red}(\nu \gamma', \gamma) \) and we have the new momenta \( \check{p} = \partial L^* / \partial \gamma' = \nu p \). The factor \( \nu \) is Chaplygin reducing multiplier...
if under the above time reparameterization the equations (77) and (78) become Hamiltonian in the coordinates ($\tilde{p}, \gamma$).

The existence of the Chaplygin reducing multiplier $\nu$ implies that the original system has an invariant measure $\nu^{n-2}w^{n-1}$ (e.g. see theorem 3.5, [27]). From the expression of an invariant measure (79) we get the form of a possible Chaplygin multiplier:

$$\nu(\gamma) = \text{const} \cdot (A\gamma, \gamma)^{\frac{1}{n}-1}.$$ 

Remarkably, we have.

**Theorem 12.** Under the time substitution $d\tau = \epsilon(A\gamma, \gamma)^{\frac{1}{n}-1}dt$ and an appropriate change of momenta, the reduced system (77) and (78) becomes a Hamiltonian system describing a geodesic flow on $S^{n-1}$ with the metric

$$ds_{\lambda,\epsilon}^2 = (\gamma, A\gamma)^{\frac{1}{n}-2}((A\gamma, d\gamma)(A\gamma, \gamma) - (A\gamma, \gamma)^2).$$

(82)

**Remark 7.** Note that, while reductions and invariant measures of considered nonholonomic systems are given for arbitrary inertia tensors (sections 2 and 3), the Hamiltonization is performed only for the special one (8). This assumption implies that $I = I + DE$ preserves the subset of bivectors in $so(n)$. For $n \geq 4$, it is a restrictive property, while for $n = 3$ an arbitrary inertia operator can be written in the form (8) and we reobtain the result of Ehlers and Koiller [22]. This is expected since only if the reduced configuration space is 2D, the existence of an invariant measure is equivalent to the existence of a Chaplygin multiplier (e.g. see [14]).

**Proof.** We take $\nu(\gamma) = \epsilon(A\gamma, \gamma)^{\frac{1}{n}-1}$, so the Lagrangian (75) in the new time becomes

$$L^*(\gamma', \gamma) = \frac{1}{2}(\gamma, A\gamma)^{\frac{1}{n}-2}((A\gamma', \gamma')(A\gamma, \gamma) - (A\gamma, \gamma)^2).$$

(83)

Following the method of Chaplygin reducing multiplier, we introduce the new momenta by considering the mapping

$$(p, \gamma) \mapsto (\tilde{p}, \gamma), \quad \tilde{p} = \nu p = \epsilon(A\gamma, \gamma)^{\frac{1}{n}-1}p.$$  

(84)

Under (84), the Hamiltonian (80) transforms to

$$H(\tilde{p}, \gamma) = \frac{1}{2}(\gamma, A\gamma)^{\frac{1}{n}+\frac{1}{n}}(\tilde{p}, A^{-1}\tilde{p}).$$

(85)

Now, we realize the cotangent bundle $T^*S^{n-1}$ within $\mathbb{R}^{2n}(\tilde{p}, \gamma)$:

$$\psi_1 = (\gamma, \gamma) = 1, \quad \psi_2 = (\tilde{p}, \gamma) = 0.$$  

(86)

endowed with the symplectic structure

$$\tilde{w} = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n|_{T^*S^{n-1}}.$$  

It is convenient to obtain the Hamiltonian vector field $\tilde{X}_H = (\tilde{X}_p, \tilde{X}_\gamma)$ of $H$ on $(T^*S^{n-1}, \tilde{w})$ by using the Lagrange multipliers (e.g. see [1]). Let

$$\mathcal{H} = H - \lambda \psi_1 - \mu \psi_2.$$
Then the equations of the geodesic flow of the metric $ds^2_{A,\epsilon}$ can be written as

$$\gamma' = \tilde{X}_\gamma = \frac{\partial H}{\partial \tilde{p}} = (\gamma, A\gamma)^{1/2} A^{-1} \tilde{p} - \mu \gamma,$$

$$\tilde{p}' = \tilde{X}_{\tilde{p}} = -\frac{\partial H}{\partial \gamma} = \frac{1-\epsilon}{\epsilon} (\gamma, A\gamma)^{-1/2} (\tilde{p}, A^{-1} \tilde{p}) A\gamma + 2\lambda \gamma + \mu \tilde{p},$$

where the multipliers $\lambda$ and $\mu$ are determined by taking the derivative of the constraints (86). The straightforward calculations yield

\begin{align*}
\psi'_1 &= 2(\gamma, A\gamma)^{1/2} (A^{-1} \tilde{p}, \gamma) - 2\mu(\gamma, \gamma) = 0, \\
\psi'_2 &= (\gamma, A\gamma)^{1/2} (A^{-1} \tilde{p}, \tilde{p}) - \mu(\gamma, \tilde{p}) \\
&\quad + \frac{1-\epsilon}{\epsilon} (\gamma, A\gamma)^{-1/2} (\tilde{p}, A^{-1} \tilde{p}) (\tilde{p}, \gamma) + 2\lambda(\gamma, \gamma) + \mu(\tilde{p}, \gamma) = 0,
\end{align*}

implying

$$\mu = (\gamma, A\gamma)^{1/2} (A^{-1} \tilde{p}, \gamma),$$

$$2\lambda = - (\gamma, A\gamma)^{1/2} (A^{-1} \tilde{p}, \tilde{p}) + \frac{1-\epsilon}{\epsilon} (\gamma, A\gamma)^{-1/2} (\tilde{p}, A^{-1} \tilde{p})$$

$$= - \frac{1}{\epsilon} (\gamma, A\gamma)^{1/2} (\tilde{p}, A^{-1} \tilde{p}).$$

Therefore, the Hamiltonian flow of $H$ on $(T^*\mathbb{S}^{n-1}, \tilde{\omega})$ takes the form

$$\gamma' = (\gamma, A\gamma)^{1/2} \left( A^{-1} \tilde{p} - (A^{-1} \tilde{p}, \gamma) \gamma \right),$$

$$\tilde{p}' = \frac{1-\epsilon}{\epsilon} (\gamma, A\gamma)^{-1/2} (\tilde{p}, A^{-1} \tilde{p}) A\gamma$$

$$- \frac{1}{\epsilon} (\gamma, A\gamma)^{1/2} (\tilde{p}, A^{-1} \tilde{p}) \gamma + (\gamma, A\gamma)^{1/2} (A^{-1} \tilde{p}, \gamma) \tilde{p}. \quad (88)$$

In the time $t$, after inverting the mapping (84), the equation (87) takes the form

$$\dot{\gamma} \cdot \frac{1}{\epsilon} (A\gamma, A\gamma)^{1-\frac{1}{\epsilon}} = \epsilon (A\gamma, A\gamma)^{\frac{1}{\epsilon} - 1} (A^{-1} p - (A^{-1} p, \gamma) \gamma),$$

which coincides with (77). Further, from

$$\frac{d}{dt} \tilde{p} = \frac{d}{dt} \left( \epsilon (A\gamma, A\gamma)^{-1/\epsilon} p \right) = \frac{d}{dt} \left( \epsilon (A\gamma, A\gamma)^{-1/\epsilon} p \right) \frac{1}{\epsilon} (A\gamma, A\gamma)^{1-\frac{1}{\epsilon}}$$

$$= \left( \frac{d}{dt} (A\gamma, A\gamma)^{-1/\epsilon} + \dot{p} (A\gamma, A\gamma)^{-1/\epsilon} \right) (A\gamma, A\gamma)^{1-\frac{1}{\epsilon}},$$

and

$$\frac{d}{dt} (A\gamma, A\gamma)^{-1/\epsilon} = 2\left( \frac{1}{2\epsilon} - 1 \right) (A\gamma, A\gamma)^{-2/\epsilon} (A\gamma, \dot{\gamma}) = (2\epsilon - 1) \epsilon (A\gamma, A\gamma)^{-1/\epsilon} (p, A^{-1} \gamma),$$

4027
we get
\[
\frac{d}{dt} \tilde{p} = (2\epsilon - 1)\epsilon (A\gamma, \gamma)^{-1}(p, A^{-1}\gamma)p + \dot{p}.
\] (89)

Finally, by combining (89) with the right hand side of (88) written in variables \((p, \gamma)\),
\[
\tilde{X}_\tilde{p}(p, \gamma) = \epsilon (1 - \epsilon)(\gamma, A\gamma)^{-2}(p, A^{-1}p)A\gamma - \epsilon (A\gamma, \gamma)^{-1}(p, A^{-1}p)\gamma + \epsilon^2 (A\gamma, \gamma)^{-1}(A^{-1}p, \gamma)p,
\]
we obtain the equation (78):
\[
\dot{p} = (1 - 2\epsilon)\epsilon (A\gamma, \gamma)^{-1}(p, A^{-1}\gamma)p + \epsilon (1 - \epsilon)(\gamma, A\gamma)^{-2}(p, A^{-1}p)A\gamma - \epsilon (A\gamma, \gamma)^{-1}(p, A^{-1}p)\gamma + \epsilon^2 (A\gamma, \gamma)^{-1}(A^{-1}p, \gamma)p.
\]

We proved that the vector field defining the motion is proportional to the Hamiltonian vector field
\[
X_H = (X_p, X_\gamma) = \epsilon (A\gamma, \gamma)^{-1/2}(\tilde{X}_\tilde{p}, \tilde{X}_\gamma) = \epsilon (A\gamma, \gamma)^{-1/2-1}\tilde{X}_H,
\]
and, whence, \(\nu = \epsilon (A\gamma, \gamma)^{1/2-1}\) is the Chaplygin multiplier of the system.

**Remark 8.** For \(\epsilon = 1\), (82) becomes the metric for the horizontal rolling (74). The geodesic flow of the metric (74) is completely integrable [27]. As in the 3D case, it is possible to prove the complete integrability of the reduced systems for \(\epsilon = -1\) and arbitrary \(A\), as well as for \(\epsilon \neq -1\) with matrices \(A\) having additional symmetries. We shall consider the integrability aspects of the problem and a geometrical setting by using nonholonomic connections following [3, 20, 40] in a separate paper.

**Acknowledgments**

The author is very grateful to Yuri Fedorov, Borislav Gajić, and the referees for many valuable suggestions that helped the author to improve the exposition of the results. The research was supported by the Serbian Ministry of Science Project 174020, Geometry and Topology of Manifolds, Classical Mechanics and Integrable Dynamical Systems.

**ORCID iDs**

Božidar Jovanović @ https://orcid.org/0000-0002-3393-4323

**References**

[1] Arnold V I, Kozlov V V and Neishtadt A I 1989 *Mathematical Aspects of Classical and Celestial Mechanics* (Encyclopedia of Mathematical Sciences vol 3) (Berlin: Springer)
[2] Balseiro P and García-Naranjo L 2012 Gauge transformations, twisted Poisson brackets and hamiltonization of nonholonomic systems, *Arch. Ration. Mech. Anal.* **205** 267–310

[3] Bakša A 1975 On geometrisation of some nonholonomic systems *Mat. Vesnik* **27** 233–40 (in Serbian)

Bakša A 2017 *Theor. Appl. Mech.* **44** 133–40 (Engl. transl.)

[4] Bloch A M, Krishnaprasad P S, Marsden J E and Murray R M 1996 Nonholonomic mechanical systems with symmetry *Arch. Ration. Mech. Anal.* **136** 21–99

[5] Bolsinov A V, Borisov A V and Mamaev I S 2011 Hamiltonization of non-holonomic systems in the neighborhood of invariant manifolds *Regular Chaotic Dyn.* **16** 443–64

[6] Bolsinov A V, Borisov A V and Mamaev I S 2012 Rolling of a ball without spinning on a plane: the absence of an invariant measure in a system with a complete set of integrals *Regular Chaotic Dyn.* **17** 571–9

[7] Borisov A V and Fedorov Y N 1995 On two modified integrable problems in dynamics *Mosc. Univ. Mech. Bull.* **50** 16–8 (Russian)

[8] Borisov A V, Mamaev I S and Kilin A A 2002 The rolling motion of a ball on a surface. New integrals and hierarchy of dynamics *Regular Chaotic Dyn.* **7** 201–19

[9] Borisov A and Mamaev I 2001 Chaplygin’s ball rolling problem is Hamiltonian *Math. Notes* **70** 720–3

[10] Borisov A V and Mamaev I S 2007 Rolling of a non-homogeneous ball over a sphere without slipping and twisting *Regular Chaotic Dyn.* **12** 153–9

[11] Borisov A V, Fedorov Y N and Mamaev I S 2008 Chaplygin ball over a fixed sphere: an explicit integration *Regular Chaotic Dyn.* **13** 557–71

[12] Borisov A V and Mamaev I S 2013 Topological analysis of an integrable system related to the rolling of a ball on a sphere *Regular Chaotic Dyn.* **18** 356–71

[13] Borisov A V, Mamaev I S and Bizyaev I A 2013 The hierarchy of dynamics of a rigid body rolling without slipping and spinning on a plane and a sphere *Regular Chaotic Dyn.* **18** 277–328

[14] Borisov A V, Mamaev I S and Tsiganov A V 2014 Nonholonomic dynamics and Poisson geometry *Russ. Math. Surv.* **69** 481–538

[15] Bryant R and Hsu L 1993 Rigidity of integral curves of rank 2 distributions *Inventiones Math.* **114** 435–61

[16] Cantrijn F, de Leon M, Martín de Diego D and Marrero J C 1998 Reduction of nonholonomic mechanical systems with symmetries *Rep. Math. Phys.* **42** 25–45

[17] Cantrijn F, Cortes J, de Leon M and Martín de Diego D 2002 On the geometry of generalized Chaplygin systems *Math. Proc. Camb. Phil. Soc.* **132** 323–51

[18] Chaplygin S A 1903 On a rolling sphere on a horizontal plane *Mat. Sbornik* **24** 139–68 (Russian)

[19] Chaplygin S A 1911 On the theory of the motion of nonholonomic systems. Theorem on the reducing multiplier *Mat. Sbornik* **25** 303–14 (Russian)

[20] Dragović V and Gajić B 2003 The Wagner curvature tensor in nonholonomic mechanics *Regular Chaotic Dyn.* **8** 105–23

[21] Ehlers K, Koiller J, Montgomery R and Rios P 2005 Nonholonomic systems via moving frames: Cartan’s equivalence and Chaplygin Hamiltonization *The Breadth of Symplectic and Poisson Geometry (Progress in Mathematics vol 232)* (Boston: Birkhäuser) pp 75–120

[22] Ehlers K and Koiller J 2007 Rubber rolling over a sphere *Regular Chaotic Dyn.* **12** 127–52

[23] Fasso F and Sansonetto N 2016 Conservation of ‘moving’ energy in nonholonomic systems with affine constraints and integrability of spheres on rotating surfaces *J. Nonlinear Sci.* **26** 519–44

[24] Fasso F, García-Naranjo L C and Sansonetto N 2018 Moving energies as first integrals of nonholonomic systems with affine constraints *Nonlinearity* **31** 755–83

[25] Fedorov Yu N and Kozlov V V 1995 Various aspects of n-dimensional rigid body dynamics *Am. Math. Soc. Transl. Ser. 2* **168** 141–71

[26] Fedorov Yu 1996 Dynamical systems with an invariant measure on the Riemannian symmetric pairs (GL(N), SO(N)), *Regular Chaotic Dyn.* **1** 38–44 (Russian)

[27] Fedorov Yu N and Jovanović B 2004 Nonholonomic LR systems as generalized Chaplygin systems with an invariant measure and geodesic flows on homogeneous spaces *J. Nonlinear Sci.* **14** 341–81

[28] Fedorov Yu N and Jovanović B 2006 Quasi-chaplygin systems and nonholonomic rigid body dynamics *Lett. Math. Phys.* **76** 215–30

[29] Fedorov Yu N, García-Naranjo L C and Marrero J C 2015 Unimodularity and preservation of volumes in nonholonomic mechanics *J. Nonlinear Sci.* **25** 203–46
[30] Grabowski J 2012 Modular classes of skew symmetric relations Transformation Groups 17 989–1010
[31] Hochgerner S and García-Naranjo L 2009 G-Chaplygin systems with internal symmetries, truncation, and an (almost) symplectic view of Chaplygin’s ball J. Geom. Mech. 1 35–53
[32] Hochgerner S 2010 Chaplygin systems associated to Cartan decompositions of semi-simple Lie groups Differ. Geom. Appl. 28 436–53
[33] Jovanović B 1998 Nonholonomic geodesic ows on Lie groups and the integrable Suslov problem on SO(4) J. Phys. A: Math. Gen. 31 1415–22
[34] Jovanović B 2009 LR and L + R systems J. Phys. A: Math. Theor. 42 225202
[35] Jovanović B 2010 Hamiltonization and integrability of the Chaplygin sphere in R^n J. Nonlinear Sci. 20 569–93
[36] Jovanović B 2015 Invariant measures of modified LR and L + R systems Regular Chaotic Dyn. 20 542–52
[37] Jovanović B 2016 Noether symmetries and integrability in Hamiltonian time-dependent mechanics J. Geom. Phys. 115 255–73
[38] Jordanovic V 2008 Symmetries of line bundles and Noether theorem for time-dependent nonholonomic systems J. Geom. Mech. 10 173–87
[39] Kozlov V 1988 Invariant measures of the Euler–Poincare equations on Lie algebras Funct. Anal. Appl. 22 58–9
[40] Kupka I and Oliva W M 2001 The non-holonomic mechanics J. Differ. Equ. 169 169–89
[41] Ohsawa T, Fernandez O E, Bloch A M and Zenkov D V 2011 Nonholonomic Hamilton–Jacobi theory via Chaplygin hamiltonization J. Geom. Phys. 61 1263–91
[42] Stanchenno S 1989 Nonholonomic Chaplygin systems J. Appl. Math. Mech. 53 11–7
[43] Tatarinov Y 2003 Equations of classical mechanics in a new form Mosc. Univ. Mech. Bull. 58 13–22
[44] Tsiganov A V 2012 On the Poisson structures for the nonholonomic Chaplygin and Veselova problems Regular Chaotic Dyn. 17 439–50
[45] Veselov A P and Veselova L E 1986 Flows on Lie groups with nonholonomic constraint and integrable non-Hamiltonian systems Funct. Anal. Appl. 20 308–9
[46] Veselov A P and Veselova L E 1988 Integrable nonholonomic systems on Lie groups Math. Notes 44 810–9
[47] Yaroshchuk V A 1992 New cases of the existence of an integral invariant in a problem on the rolling of a rigid body Vestnik Moskov. Univ. Ser. I Mat. Mekh. 26–30 (Russian)
[48] Zenkov D V and Bloch A M 2003 Invariant measures of nonholonomic flows with internal degrees of freedom Nonlinearity 16 1793–807