Crossed squares and 2-crossed modules

A. Mutlu and T. Porter

Abstract

A. M. S. Classification: 18G30, 18G55.
Key words and phrases: Simplicial Group, Crossed \(n\)-cubes, Crossed Squares, 2-Crossed Modules.

1 Introduction

Simplicial groups were first studied by D. M. Kan in the 1950s [24]. Early work by Kan himself, Moore, Milnor, and Dold showed that
(a) these objects have a well structured homotopy theory,
(b) they modelled all homotopy types of connected spaces,
(c) abelian simplicial groups were an equivalent tool to that of chain complexes and could therefore be applied within homological algebra, and
(d) in low dimensions, calculations were possible, provided, for instance, the simplicial group was free with chosen ‘\(CW\)-basis’.

Simplicially enriched groupoids have a more recent birth, but have the same sort of attributes plus being able to model non-connected homotopy types. The shortened form of their name ‘simplicial groupoid’ used by Dwyer and Kan [14] is more usually used for these objects although not strictly correct as simplicial objects in the category of groupoids form a much larger setting than do simplicially enriched groupoids. None the less we will often use that shortened form here.

Crossed modules model homotopy 2-types. Crossed squares model 3-types. Crossed \(n\)-cubes model \((n + 1)\)-types, cf. [35] and the references therein. Conduché [10] has an alternative model for 3-types namely 2-crossed modules and Baez [3] uses a variant of this, the quadratic module, in some of his work. Another model for 3-types was introduced by Brown and Gilbert, [5], (involving a particular subdivided triangular diagram in its derivation). This model, braided crossed modules and their lax counterpart, the braided categorical groups, have been further studied by members of the Granada Algebra group (cf. Carrasco and Cegarra, [9], and Garzon and Miranda, [20], for example). Similar object, Gray groupoids, have been studied by Joyal and Tierney (unpublished) and have recently started to receive some attention in the TQFT literature. The proof of the equivalence between Gray groupoids (that is, 2-groupoid enriched groupoids) and Conduché’s models is discussed in [23] and more detailed references to other work by LeRoy, Berger, Marty, and, of course, the original proof from the 1980s by Joyal and Tierney, can be found there as well as in the bibliography of this paper.
In a letter to Brown and Loday, dated in the mid 1980s, Conduché pointed out that given a crossed square,

\[
\mathcal{M} = \begin{pmatrix}
    L & M \\
    N & P
\end{pmatrix},
\]

the mapping cone complex of \(\mathcal{M}\),

\[L \to M \rtimes N \to P\]

constructed by Loday in [27], has a 2-crossed module structure. He ended his letter by pointing out that this seemed to give a canonical and direct way to link the categories of crossed squares and 2-crossed modules, and his results suggested that it should yield some sort of equivalence between the two models.

Although much more is known on this area than when Conduché’s letter was written, the background underlying structure still seems obscure. In this paper we aim to shed some light on the 2-crossed module structure given by Conduché and also on the subdivided triangular diagram of Brown and Gilbert. This, in fact, provides the key and suggests ways of generalising Conduché’s construction to higher \(n\)-types.

2 Preliminaries

2.1 Simplicial Groups and Groupoids

We assume that the reader is conversant with the basic theory of simplicial sets and simplicial groups. The following merely sets up notation and some conventions.

Let \(\mathbf{Grp}\) be the category of groups. A simplicial group \(G\) consists of a family of groups \(\{G_n\}\) together with face and degeneracy maps \(d_i = d^n_i : G_n \to G_{n-1}\), \(0 \leq i < n\) \((n \neq 0)\) and \(s_i = s^n_i : G_n \to G_{n+1}\), \(0 \leq i \leq n\), satisfying the usual simplicial identities given in [12], [24] and [25]. It can be completely described as a functor \(G : \Delta^{op} \to \mathbf{Grp}\) where \(\Delta\) is the category of finite ordinals, \([n] = \{0 < 1 < 2 < \cdots < n\}\), and increasing maps. We will denote the category of simplicial group by \(\mathbf{SimpGrps}\). We have for each \(k \geq 0\), a subcategory \(\Delta_{\leq k}\) determined by objects \([j]\) of \(\Delta\) with \(j \leq k\). A \(k\)-truncated simplicial group is a functor from \((\Delta^{op}_{\leq k})\) to \(\mathbf{Grp}\), where \((\Delta^{op}_{\leq k})\) has the obvious meaning.

Remark:

We will restrict detailed attention in the main to simplicial groups and hence to connected homotopy types. This is traditional but a bit unnatural as all the results and definitions extend with little or no trouble to simplicial groupoids in the sense of Dwyer and Kan [14] and hence to non-connected homotopy types. It should be noted that such simplicial groupoids have a fixed and constant simplicial set of objects and so are not merely simplicial objects in the category of groupoids. In this context if \(G\) is a simplicial groupoid with set of objects \(O\), the natural form of the Moore complex \(NG\) (see below) is given by the same formula as in the reduced case, interpreting Ker\(d^n_i\) as being the subgroupoid of elements in \(G_n\) whose \(i\)th face is an identity of \(G_{n-1}\). Of course if \(n \geq 1\), the resulting \(NG_n\) is a disjoint union of groups, so \(NG\) is a disjoint union of the Moore complexes of the vertex simplicial groups of
together with the groupoid $G_0$ providing elements that allow conjugation between (some of) these vertex complexes (cf. Ehlers and Porter [15]).

Consider the product $\Delta \times \Delta$ whose objects are pairs $([p], [q])$ and whose maps are pairs of weakly increasing maps. A functor $\mathcal{G} : (\Delta \times \Delta)^{\text{op}} \to \text{Grp}$ is called a bisimplicial group. To give $\mathcal{G}$ is equivalent to giving for each $(p, q)$ a group $G_{p, q}$ and homomorphisms

\[
\begin{align*}
d^h_i & : G_{p, q} \to G_{p-1, q} \\
s^h_i & : G_{p, q} \to G_{p+1, q} & i : 0, 1, \ldots, p \\
d^v_j & : G_{p, q} \to G_{p, q-1} \\
s^v_j & : G_{p, q} \to G_{p, q+1} & j : 0, 1, \ldots, p
\end{align*}
\]

such that the maps $d^h_i$, $s^h_i$ commute with $d^v_j$, $s^v_j$ and $d^h_i$, $s^h_i$ (resp. $d^v_j$, $s^v_j$) and satisfy the usual simplicial identities. Here $d^h_i$, $s^h_i$ denote the horizontal operators and $d^v_j$, $s^v_j$ denote the vertical operators. A bisimplicial group can also be thought of as a simplicial object in the category of simplicial groups.

Multisimplicial groups are similarly defined. Explicitly if $n$ is a positive integer, form the $n$-fold product $\Delta \times_n \Delta$ of $\Delta$ with itself, then an $n$-simplicial group is merely a functor $G : (\Delta \times_n \Delta)^{\text{op}} \to \text{Grp}$. Of course an $n$-simplicial group leads to an $n$-indexed family of groups $G_{p_1, \ldots, p_n}$ with face and degeneracy operators in the obvious way, generalising the case $n = 2$ discussed above.

All of this extends painlessly to simplicial groupoids and with obvious definitions of morphisms of $n$-simplicial groups and groupoids, we thus get a whole host of categories $n\text{-SimpGrps}$ and $n\text{-SimpGrds}$.

Recall from [35] that a normal chain complex of groups, $(X, d)$ means one in which each $\text{Im} d_{i+1}$ is a normal subgroup of the corresponding $X_i$. Given any normal chain complex $(X, d)$ of groups and an integer $n$, the truncation, $t_n|X$ of $X$ at level $n$ is defined by

\[
(t_n|X)_i = \begin{cases} 
X_i & \text{if } i < n \\
X_i/\text{Im}d_{n+1} & \text{if } i = n \\
0 & \text{if } i > n.
\end{cases}
\]

The differential $d$ of $t_n|X$ is that of $X$ for $i < n$, whilst $d_n$ is induced by the $n^{\text{th}}$ differential of $X$ and all other differentials are zero.

There are multicomplex and groupoid versions of this. The only points worth noting here are that any non trivial normal subgroupoid is a kernel and hence must be a disjoint union of its vertex groups and that in a normal $n$-multicomplex of groups, $(X, d_1, \ldots, d_n)$, each image in each direction is assumed normal in each dimension.

### 2.2 The Moore complex of simplicial group

The archetypal normal $n$-complex of group(oid)s is the Moore complex of an $n$-simplicial group(oid). In particular given a simplicial group $G$, the Moore complex $(NG, \partial)$ of $G$ is the normal chain complex defined by

\[
\text{NG}_n = \bigcap_{i=0}^{n-1} \text{Ker}d_i
\]
and with differential $\partial_n : NG_n \to NG_{n-1}$ induced from $d_n$ by restriction.

To form the Moore $n$-complex of an $n$-simplicial group(oid) one just applies the functor $N$ repeatedly in each direction in turn.

There is an extensive theory of the Moore complex and its links with homotopy theory. We mention that $\pi_n(G)$ can be calculated by calculating the $n^{th}$ homology group of $(NG, \partial)$. Explicitly the $n^{th}$ homotopy group $\pi_n(G)$ of $G$ is defined to be the $n^{th}$ homology of the Moore complex of $G$, i.e.,

$$\pi_n(G) \cong H_n(NG, \partial) = \bigcap_{i=0}^{n} \text{Ker} d_i^n / d_{n+1}^{n+1}(\bigcap_{i=0}^{n} \text{Ker} d_{n+1}^i).$$

We say that the Moore complex $NG$ of a simplicial group is of length $k$ if $NG_n = 1$ for all $n \geq k + 1$ so that a Moore complex of length $k$ also has length $l$ for $l \geq k$.

A simplicial map $f : G \to G'$ is called an $n$-equivalence if it induces isomorphisms $\pi_k(G) \cong \pi_k(G')$ for $k \leq n$.

Composites of $n$-equivalences are, of course, also $n$-equivalences. Two simplicial groups, $G$ and $G'$, are said to have the same $n$-type if there is a zig-zag chain of $n$-equivalences linking them. A simplicial group $G$ is an $n$-type if $\pi_i(G) = 1$ for $i > n$.

The Moore complex carries a lot of fine structure and this has been studied, e.g. by Carrasco and Cegarra, [8], Wu [38], and the present authors in earlier papers in this series, [30–34]. The specific structure of the $k$-truncation of the Moore complex for $k = 1$ and 2 is now well known. For $k = 1$ this gives a crossed module, for $k = 2$, a 2-crossed module, cf. Conduché, [10]. We summarise his theory below. The hypercrossed complex structure of Carrasco and Cegarra, [8], clearly models all homotopy types and on $k$-truncation, all $(k+1)$-types, but the structure does get unwieldy for $k$ larger than about 3 or 4.

All of this theory works for both simplicial groups and simplicially enriched groupoids with virtually identical presentations. Rather than write ‘simplicial group(oid)’ all the time we have written the exposition in terms of simplicial groups but the other case works in the same way.

### 3 2-crossed modules and simplicial groups

Crossed module techniques give a very efficient way of handling information about a homotopy type. They correspond to 2-types (see [10] and [31]). As mentioned above Conduché, [10], in 1984 introduced the notion of 2-crossed module as a model for 3-types.

Throughout this paper we denote an action of $p \in P$ on $m \in M$ by $p \cdot m = ^p m$.

A crossed module is a group homomorphism $\partial : M \to P$ together with an action of $P$ on $M$ satisfying

1. $\partial(^p m) = p\partial(m)p^{-1}$
2. $\partial^nm' = mm'm^{-1}$ for all $m, m' \in M, p \in P$.

This second condition is called the Peiffer identity. We will denote such a crossed module by $(M, P, \partial)$. 

5
A morphism of crossed modules from \((M, P, \partial)\) to \((M', P', \partial')\) is a pair of group homomorphisms, \(\phi : M \rightarrow M', \psi : P \rightarrow P'\) such that \(\phi(p)m = \psi(p)\phi(m)\) and \(\partial'\phi(m) = \psi\partial(m)\). We thus get a category \(\mathbf{XMod}\) of crossed modules.

**Examples of crossed modules**

(a) Any normal subgroup \(N\) in \(P\) gives an inclusion map, \(\text{inc}: N \rightarrow P\) which is a crossed module. Conversely given any arbitrary crossed module \(\partial : M \rightarrow P\), one can easily see that the Peiffer identity implies that \(\partial M = P\) is a normal subgroup in \(P\).

(b) Given any \(P\)-module, \(L\), the trivial homomorphism \(1 : L \rightarrow P\) is a crossed \(P\)-module for the given action of \(P\) on \(L\).

The following definition of 2-crossed modules is equivalent to that given by D.Conduché, [10].

**Definition:**

A 2-crossed module consists of a complex of groups

\[
\begin{array}{ccc}
L & \partial_2 & M & \partial_1 & N \\
& f_2 & & f_1 & \gamma \\
L' & \partial'_2 & M' & \partial'_1 & N'
\end{array}
\]

\[
\begin{array}{l}
\{m, m'\} = ( \partial_1 m m' ) (m m' - 1) m^{-1}, \\
\{\partial_2(l), \partial_2(l')\} = [l', l], \\
\{mm', m''\} = \partial_2 m \{m', m''\} \{m', m''(m')^{-1}\}, \\
\{m, m''\} = \partial_1 m \{m, m''\} \{m, m''(m')^{-1}\}, \\
\{\partial_2(l), m\} = \partial_2 m \{l\}^{-1}, \\
\{m, \partial_2(l)\} = \{m, l\} \{l, \partial_2(l)\}^{-1}
\end{array}
\]

for all \(l, l' \in L\), \(m, m', m'' \in M\) and \(n \in N\).

Here we have used \(m l\) as a shorthand for \(\{\partial_2, m\}l\) in condition 2CM3(ii) where \(l\) is \(\{m, m''\}\) and \(m\) is \(mm'(m')^{-1}\). This gives a new action of \(M\) on \(L\). Using this notation, we can split 2CM4 into two pieces, the first of which is tautological:

\[
\begin{array}{l}
2CM4: (a) \{\partial_2 l, m\} = m(l) l^{-1}, \\
(b) \{m, \partial_2 l\} = (\partial_1 m l)(m l^{-1})
\end{array}
\]

The old action of \(M\) on \(L\), via \(\partial_1\) and the \(N\)–action on \(L\), is in general distinct from this second action with \(\{m, \partial_2(l)\}\) measuring the difference (by 2CM4(b)). An easy argument using 2CM2 and 2CM4(b) shows that with this action, \(m l\) of \(M\) on \(L\), \((L, M, \partial_2)\) becomes a crossed module.

We denote such a 2-crossed module by \(\{L, M, N, \partial_2, \partial_1\}\). A morphism of 2-crossed modules is given by a diagram

\[
\begin{array}{ccc}
L & \partial_2 & M & \partial_1 & N \\
& f_2 & & f_1 & \gamma \\
L' & \partial'_2 & M' & \partial'_1 & N'
\end{array}
\]

5
where \( f_0 \partial_1 = \partial'_1 f_1, \ f_1 \partial_2 = \partial'_2 f_2, \)

\[
f_1^{(n)}(m_1) = f_0^{(n)} f_1(m_1), \quad f_2^{(n)}(l) = f_0^{(n)} f_2(l),
\]

and

\[
\{,\} f_1 \times f_1 = f_2 \{,\},
\]

for all \( l \in L, \ m_1 \in M, \ n \in N. \) These compose in an obvious way.

The groupoid analogues of these definitions are left to the reader. We will concentrate on the reduced case i.e. with groups rather than groupoids.

We thus can consider the category of 2-crossed modules denoting it as \( \mathfrak{X}_2\text{Mod}. \) Conduché [10] proved that 2-crossed modules give algebraic models of connected homotopy 3-types.

**Theorem 3.1** ([10], [32]) The category, \( \mathfrak{X}_2\text{Mod} \), of 2-crossed modules is equivalent to the category \( \text{SimpGrp}_{\leq 2} \) of simplicial groups with Moore complex of length 2. \( \square \)

### 4 \text{Cat}^2\text{-groups and crossed squares}

The following definition is due to D. Guin-Walery and J.-L. Loday, see [21] and also [27].

**Definition:**

A crossed square of groups is a commutative square of groups

\[
\begin{array}{ccc}
L & \rightarrow & M \\
\lambda' \downarrow & & \mu' \\
N & \rightarrow & P \\
\lambda \downarrow & & \mu \\
\end{array}
\]

together with actions of \( P \) on \( L, M \) and \( N. \) There are thus actions of \( N \) on \( L \) and \( M \)
via \( \mu' \) and \( M \) acts on \( L \) and \( N \) via \( \mu \) and a function \( h : M \times N \rightarrow L \) such that, for all \( l \in L, \ m_1, m \in M, \ n_1, n \in N \) and \( p \in P \) the following axioms hold:

1. the homomorphisms \( \lambda, \lambda', \mu, \mu' \) and \( \kappa = \mu \lambda = \mu' \lambda' \) are crossed modules for the corresponding actions and the morphisms of maps \((\lambda) \rightarrow (\kappa); \ (\kappa) \rightarrow (\mu); \ (\lambda') \rightarrow (\kappa); \) and \((\kappa) \rightarrow (\mu')\) are morphisms of crossed modules,

2. \( \lambda h(m, n) = m \mu'(n)m, \)

3. \( \lambda' h(m, n) = \mu(m)n(n)^{-1}, \)

4. \( h(\lambda(l), n) = l^nn^{-1}, \)

5. \( h(m, \lambda'(l)) = (m)l^{-1}, \)

6. \( h(mm_1, n) = m h(m_1, n)h(m, n), \)

7. \( h(m, nn_1) = h(m, n) n h(m, n_1), \)

8. \( h(\text{p} m, \text{p} n) = \text{p} h(m, n), \)

\text{p}
The category of crossed squares will be denoted, $\mathcal{Crs}^2$.

In the simplest examples of crossed squares (see [35]), $\mu$ and $\mu'$ are normal subgroup inclusions and $L = M \cap N$, with $h$ being the conjugation map. We also note that if

$$
\begin{array}{c}
M \cap N \\
\downarrow \\
N \\
\downarrow \\
\rightarrow \\
M \\
\end{array}
$$

is a simplicial crossed square constructed from a simplicial group $G$ and two simplicial normal subgroups $M$ and $N$ then applying $\pi_0$, the square gives a crossed square and that up to isomorphism all crossed squares arise in this way, again see [35].

Although when first defined by D. Guin-Walery and J.-L. Loday [21], the notion of crossed squares was not linked to that of $\text{cat}^2$-groups, it was in this form that Loday gave their generalisation to an $n$-fold structure, $\text{cat}^n$-groups (see [27]).

Recall from [27] that a $\text{cat}^1$-group is a triple $(G, s, t)$, where $G$ is a group and $s, t$ are endomorphisms of $G$ satisfying conditions

(i) $st = t$ and $ts = s$.
(ii) $[\text{Ker} s, \text{Ker} t] = 1$.

It was shown [27] that setting $M = \text{Ker} s$, $N = \text{Im} s$ and $\partial = t|_M$, then the action of $N$ on $M$ by conjugation within $G$ makes $\partial : M \to N$ into a crossed module. Conversely if $\partial : M \to N$ is a crossed module, then setting $G = M \rtimes N$ and letting $s, t$ be defined by

$$s(m, n) = (1, n)$$

and

$$t(m, n) = (1, \partial(m)n)$$

for $m \in M$, $n \in N$, we have that $(G, s, t)$ is a $\text{cat}^1$-group.

For a $\text{cat}^2$-group, we again have a group, $G$, but this time with two independent $\text{cat}^1$-group structures on it. Explicitly:

A $\text{cat}^2$-group is a 5-tuple $(G, s_1, t_1, s_2, t_2)$, where $(G, s_i, t_i)$, $i = 1, 2$, are $\text{cat}^1$-groups and

$$s_is_j = s_js_i, \quad t_it_j = t_jt_i, \quad s_it_j = t_js_i$$

for $i, j = 1, 2$, $i \neq j$.

**Theorem 4.1** [27] There is an equivalence of categories between the category of $\text{cat}^2$-groups and that of crossed squares.

We include a sketch of the proof as it contains ideas that will be needed later.

**Proof:** The $\text{cat}^1$-group $(G, s_1, t_1)$ will gives us a crossed module with $M = \text{Kers}_1$, $N = \text{Im}s_1$, and $\partial = t|_M$, but as the two $\text{cat}^1$-group structures are independent, $(G, s_2, t_2)$ restricts to give $\text{cat}^1$-group structures on $M$ and $N$ and makes $\partial$ a morphism of $\text{cat}^1$-groups. We thus get a morphism of crossed modules

$$
\begin{array}{c}
\text{Kers}_1 \cap \text{Kers}_2 \\
\downarrow \\
\text{Kers}_2 \cap \text{Im}s_1 \\
\downarrow \\
\text{Im}s_1 \cap \text{Kers}_2 \\
\end{array}
$$

and

$$
\begin{array}{c}
\text{Kers}_2 \cap \text{Im}s_1 \\
\downarrow \\
\text{Im}s_1 \cap \text{Im}s_2 \\
\end{array}
$$
where each morphism is a crossed module for the natural action, i.e., conjugation in $G$. It remains to produce an $h$-map, but this is given by the commutator within $G$ since if $x \in \text{Kers}_2 \cap \text{Im}s_1$ and $y \in \text{Im}s_2 \cap \text{Kers}_1$ then $[x, y] \in \text{Kers}_1 \cap \text{Kers}_2$. It is easy to check the axioms for a crossed square.

Conversely, if

$$
\begin{array}{c}
L \rightarrow M \\
\downarrow \quad \downarrow \\
N \rightarrow P
\end{array}
$$

is a crossed square, then we can think of it as a morphism of crossed modules

$$
\begin{array}{c}
L \\
\downarrow \\
N
\end{array} \rightarrow 
\begin{array}{c}
M \\
\downarrow \\
P
\end{array}
$$

Using the equivalence between crossed modules and cat$^1$-groups this gives a morphism

$$\partial : (L \times N, s, t) \rightarrow (M \times P, s', t')$$

of cat$^1$-groups. There is an action of $(m, p) \in M \times P$ on $(l, n) \in L \times N$ given by

$$(m, p)(l, n) = m(p_l, p_n) = (\mu(m)p)h(m, p_n), p_n).$$

Using this action, we thus form the associated cat$^1$-group with 'big' group $(L \times N) \times (M \times P)$ and induced endomorphisms, $s_1, t_1, s_2, t_2$. 

It is easy to show that cat$^1$-groups are merely a reformulation of an internal groupoid in the category $\mathbf{Grps}$ of groups, whilst cat$^2$-groups correspond similarly to double groupoid objects in $\mathbf{Grps}$.

## 5 Crossed $n$-cubes and simplicial groups

In the form given above, crossed squares were difficult to generalise to higher order structures, although cat$^2$-groups could clearly and easily be generalised to cat$^n$-groups. The following generalisation is due to Ellis and Steiner [19] and includes a reformulation of crossed squares as a special case. Let $\langle n \rangle$ denote the set $\{1, ..., n\}$.

A crossed $n$-cube of group is a family $\{\mathcal{M}_A : A \subseteq \langle n \rangle\}$ of groups, together with homomorphisms $\mu_i : \mathcal{M}_A \rightarrow \mathcal{M}_{A \setminus \{i\}}$ for $i \in \langle n \rangle$ and functions

$$h : \mathcal{M}_A \times \mathcal{M}_B \rightarrow \mathcal{M}_{A \cup B}$$
for $A, B \subseteq \langle n \rangle$, such that if $a^ib$ denotes $h(a, b)b$ for $a \in M_A$ and $b \in M_B$ with $A \subseteq B$, then for $a, a' \in M_A$ and $b, b' \in M_B, c \in M_C$ and $i, j \in \langle n \rangle$, the following axioms hold:

1) $\mu_i a = a$ if $i \notin A$,
2) $\mu_i \mu_j a = \mu_{ij} \mu_i a$,
3) $\mu_i h(a, b) = h(\mu_i a, \mu_i b)$,
4) $h(a, b) = h(\mu_i a, b) = h(a, \mu_i b)$, if $i \in A \cap B$,
5) $h(a, a') = [a, a']$,
6) $h(a, b) = h(b, a)^{-1}$,
7) $h(a, b) = 1$, if $a = 1$ or $b = 1$,
8) $h(aa', b) = a^h(a', b)h(a, b)$,
9) $h(a, bb') = h(a, b)^b h(a, b')$,
10) $a^h(b, c) = h(a^h b, a^h c)$, if $A \subseteq B \cap C$,
11) $h^h(a^{-1}, b), c) e h(a, b^{-1}, c, a) = 1$.

A morphism of crossed $n$-cubes is defined in the obvious way: It is a family of group homomorphisms, for $A \subseteq \langle n \rangle$, $f_A : M_A \rightarrow M_{A'}$ commuting with the $\mu_i$’s and $h$’s. We thus obtain a category of crossed $n$-cubes denoted by $\text{Crs}^n$, cf. Ellis and Steiner [19].

We will concentrate most attention on crossed modules and crossed squares, but will recall some of the general theory.

**Examples:**

1) For $n = 1$, a crossed 1-cube is the same as a crossed module.

2) For $n = 2$, one has a crossed square:

```
   M_2  \overset{\mu_2}{\rightarrow}  M_1
  \mu_1 \downarrow \quad \quad \downarrow \mu_1
  M_2 \overset{\mu_1}{\rightarrow} M_0.
```

Each $\mu_i$ is a crossed module, as is $\mu_1 \mu_2$. The $h$-functions give actions and a function

$$h : M_1 \times M_2 \rightarrow M_2.$$ 

The maps $\mu_2$ (or $\mu_1$) also define a map of crossed modules. In fact a crossed square can be thought of as a crossed module in the category of crossed modules.

3) Let $G$ be a simplicial group. Then the following diagram, which will be denoted $M(G, 2)$,

```
   NG_2/\partial_3 NG_3  \overset{\partial_2}{\rightarrow} NG_1
  \partial_2 \downarrow \quad \quad \downarrow \mu
  NG_1 \overset{\mu'}{\rightarrow} G_1
```

is a crossed square. Here $NG_1 = \text{Ker}d^1_0$ and $\overline{NG}_1 = \text{Ker}d^1_1$.

Since $G_1$ acts on $NG_2/\partial_3 NG_3$, $\overline{NG}_1$ and $NG_1$, there are actions of $\overline{NG}_1$ on $NG_2/\partial_3 NG_3$ and $NG_1$ via $\mu'$, and $NG_1$ acts on $NG_2/\partial_3 NG_3$ and $\overline{NG}_1$ via $\mu$. Both $\mu$ and $\mu'$ are inclusions, and all actions are given by conjugation. The h-map is

$$NG_1 \times \overline{NG}_1 \rightarrow NG_2/\partial_3 NG_3$$ 

$$(x, y) \mapsto h(x, y) = [s_1 x, s_1 y s_0 y^{-1}] \partial_3 NG_3.$$
Here $x$ and $y$ are both in $NG_1$ as there exists a bijection between $NG_1$ and $\overline{NG}_1$, the element $\overline{y}$ is the image of $y$ under this.

This last example effectively presents a functor

$$\mathcal{M} : \text{SimpGrp} \longrightarrow \text{Crs}$$

(4) Let $G$ be a group with normal subgroups $N_1, \ldots, N_n$. Let

$$\mathcal{M}_A = \bigcap \{ N_i : i \in A \} \quad \text{and} \quad \mathcal{M}_\emptyset = G$$

with $A \subseteq \langle n \rangle$. For $i \in \langle n \rangle$, $\mathcal{M}_A$ is a normal subgroup of $\mathcal{M}_{A \setminus \{i\}}$. Define

$$\mu_i : \mathcal{M}_A \longrightarrow \mathcal{M}_{A \setminus \{i\}}$$

to be the inclusion. If $A, B \subseteq \langle n \rangle$, then $\mathcal{M}_{A \cup B} = \mathcal{M}_A \cap \mathcal{M}_B$, let

$$h : \mathcal{M}_A \times \mathcal{M}_B \longrightarrow \mathcal{M}_{A \cup B}$$

$$(a, b) \longmapsto \begin{bmatrix} a & b \end{bmatrix}$$

as $[\mathcal{M}_A, \mathcal{M}_B] \subseteq \mathcal{M}_A \cap \mathcal{M}_B$, where $a \in \mathcal{M}_A$, $b \in \mathcal{M}_B$. Then

$$\{ \mathcal{M}_A : A \subseteq \langle n \rangle, \mu_i, h \}$$

is a crossed $n$-cube, called the inclusion crossed $n$-cube given by the normal $n$-ad of groups $(G; N_1, \ldots, N_n)$. The following result is then fairly easily proved, see [35].

**Proposition 5.1** Let $(G; N_1, \ldots, N_n)$ be a simplicial normal $n$-ad of subgroups of groups and define for $A \subseteq \langle n \rangle$

$$\mathcal{M}_A = \pi_0\left( \bigcap_{i \in A} N_i \right)$$

with homomorphisms $\mu_i : \mathcal{M}_A \longrightarrow \mathcal{M}_{A \setminus \{i\}}$ and h-maps induced by the corresponding maps in the simplicial inclusion crossed $n$-cube, constructed by applying the previous example to each level. Then $\{ \mathcal{M}_A : A \subseteq \langle n \rangle, \mu_i, h \}$ is a crossed $n$-cube. \hfill $\square$

Up to isomorphism, all crossed $n$-cubes arise in this way. In fact any crossed $n$-cube can be realised (up to isomorphism) as a $\pi_0$ of a simplicial inclusion crossed $n$-cube coming from a simplicial normal $n$-ad of groups.

In 1993, the second author, [35], described a functor from the category of simplicial groups to that of crossed $n$-cubes of groups. We will summarise its construction.

The functor is constructed using the décalage functor studied by Duskin [13] and Illusie [22] and is a $\pi_0$-image of a functor taking values in a category of simplicial normal $(n+1)$-ads. The décalage functor will be denoted by $\text{Dec}$. Given any simplicial group $G$, $\text{Dec} G$ is the augmented simplicial group obtained from $G$ by forgetting the zeroth face and degeneracy operators at each level and then renumbering the levels (cf. Duskin [13] or Illusie [22]). (There are two main forms of the décalage functor. The alternative forgets the last face instead of the zeroth one. This second form is used in [13]. Reversing the indexed order
of the faces used gives an equivalence between the two theories and allows for the easy translation of proofs between them.) In the convention adopted in this paper, we thus have

$$\text{Dec}G_n = G_{n+1},$$

with face operators

$$d^n_{i,\text{Dec}} = d^{n+1}_{i+1}$$

and degenerate operators

$$s^n_{i,\text{Dec}} = s^{n+1}_{i+1}.$$

The remaining degeneracy $s^{n+1}_0$ of $G$ yields a contraction of $\text{Dec}^1G$ as an augmented simplicial group and

$$\text{Dec}^1G \simeq K(G_0, 0),$$

by an explicit natural homotopy equivalence (cf. Duskin [13]). The zeroth face map will be denoted $\delta_0 : \text{Dec}^1G \to G$. This is a split epimorphism and has kernel the simplicial group, $\text{Ker}d_0$ used above. We have $\pi_0(\text{Ker} \delta) \to \pi_0(\text{Dec}G)$ is a crossed module and using that

$$\pi_0(\text{Dec}G) \to \pi_0(\text{Dec}^1G)$$

for any simplicial group $H$, we get:

**Case 1:** $H = \text{Dec}G$.

Then $NH_0 = G_1$, $NH_1 = \text{Ker}(d_1^1 : G_2 \to G_1)$ so

$$\pi_0(\text{Dec}G) \cong G_1/d_1^2(\text{Ker}d_1^2),$$

which is given by

$$\pi_0(\text{Dec}^1G) = G_0,$$

since $G_1 \cong \text{Ker}d_1^1 \times s_0(G_0)$.

**Case 2:** $H = \text{Ker} \delta$. Then $NH_0 = \text{Ker}d_0^1 = NG_1$, $NH_1 = \text{Ker}d_0^2 \cap \text{Ker}d_1^2 = NG_2$ so

$$\pi_0(\text{Ker} \delta) = NG_1/\partial_2 NG_2.$$

Iterating the $\text{Dec}$ construction gives an augmented bisimplicial group

$$\left(\ldots \text{Dec}^3G \xrightarrow{\delta_0} \text{Dec}^2G \xrightarrow{\delta_1} \text{Dec}^1G\right)$$

which in expanded form is the total décalage of $G$, (see [13] or [22] for details). The maps from $\text{Dec}^1G$ to $\text{Dec}^{i-1}G$ coming from the $i$ first face maps will be labelled $\delta_0, \ldots, \delta_{i-1}$ so that $\delta_0 = d_0$, $\delta_1 = d_1$ and so on.

For a simplicial group $G$ and a given $n$, we write $\mathcal{M}(G, n)$ for the crossed $n$-cube arising as a functor

$$\mathcal{M}(-, n) : \text{SimpGrp} \to \text{Crs}^n.$$

which is given by $\pi_0(\text{Dec}G; \text{Ker} \delta_0, \ldots, \text{Ker} \delta_{n-1})$. The following data explicitly gives this crossed $n$-cube of groups, for the details see [35]:

11
Theorem 5.2 If $G$ is a simplicial group, then the crossed $n$-cube $\mathcal{M}(G, n)$ is determined by:

(i) for $A \subseteq \langle n \rangle$,

\[
\mathcal{M}(G, n)_A = \frac{\bigcap_{j \in A} \ker d_{j-1}^n}{d_{n+1}^n(\ker d_0^{n+1} \cap \{ \bigcap_{j \in A} \ker d_j^{n+1} \})};
\]

(ii) the inclusion

\[
\bigcap_{j \in A} \ker d_{j-1}^n \longrightarrow \bigcap_{j \in A \setminus \{i\}} \ker d_{j-1}^n
\]

induces the morphism

\[
\mu_i : \mathcal{M}(G, n)_A \longrightarrow \mathcal{M}(G, n)_{A \setminus \{i\}};
\]

(iii) the functions, for $A, B \subseteq \langle n \rangle$,

\[
h : \mathcal{M}(G, n)_A \times \mathcal{M}(G, n)_B \longrightarrow \mathcal{M}(G, n)_{A \cup B}
\]

are given by

\[
h(\bar{x}, \bar{y}) = [x, y],
\]

where an element of $\mathcal{M}(G, n)_A$ is denoted by $\bar{x}$ with $x \in \bigcap_{j \in A} \ker d_j^n$.

In general, we use the $(n - 1)$-skeleton of the total décalage to form an $n$-cube and thus a simplicial inclusion crossed $n$-cube of kernels. Continuing this $n$-times gives the simplicial inclusion crossed $n$-cube corresponding to the simplicial normal $(n + 1)$-ad,

\[
\mathcal{M}(G, n) = (\mathcal{D}_{\alpha} G; \ker \delta_0, \ldots, \ker \delta_{n-1}),
\]

and its associated crossed $n$-cube is

\[
\pi_0(\mathcal{M}(G, n)) = \mathcal{M}(G, n).
\]

The results of [35] now follow by direct calculation on examining the construction of $\pi_0$ as the zeroth homology of the Moore complex of each term in the inclusion crossed $n$-cube, $\mathcal{M}(G, n)$.

Expanding this out for low values of $n$ gives:

1) For $n = 0$,

\[
\mathcal{M}(G, 0) = G_0/d_1(\ker d_0),
\]

\[
\cong \pi_0(G),
\]

\[
= H_0(G).
\]

2) For $n = 1$, $\mathcal{M}(G, 1)$ is the crossed module

\[
\mu_1 : \ker d_0^1/d_2^2(NG_2) \longrightarrow G_1/d_2^2(\ker d_0^2).
\]

Since $d_2^2(NG_2) = [\ker d_1^1, \ker d_0^1]$, this gives

\[
\mu : NG_1/[\ker d_1^1, \ker d_0^1] \longrightarrow G_0.
\]

\[
\mathcal{M}(G, 1) \cong (NG_1/\partial_2 NG_2 \longrightarrow G_0).
\]
3) For $n = 2$, $\mathcal{M}(G, 2)$ is

$$
\begin{array}{c}
\text{Ker}d_0^2 \cap \text{Ker}d_1^2 / d_3^2 (\text{Ker}d_0^3 \cap \text{Ker}d_1^3 \cap \text{Ker}d_2^3) \xrightarrow{\mu_2} \text{Ker}d_0^3 / d_3^3 (\text{Ker}d_0^3 \cap \text{Ker}d_1^3) \\
\downarrow_{\mu_1} \\
\text{Ker}d_1^2 / d_3^2 (\text{Ker}d_0^3 \cap \text{Ker}d_2^3) \xrightarrow{\mu_2} G_2 / d_3^3 (\text{Ker}d_0^3).
\end{array}
$$

As shown in [35], this is isomorphic to

$$
\begin{array}{c}
NG_2 / d_3^3 (NG_3) \xrightarrow{\mu_2} \text{Ker}d_0^1 \\
\downarrow_{\mu_1} \\
\text{Ker}d_1^1 \xrightarrow{\mu_2} G_1,
\end{array}
$$

that is

$$
\mathcal{M}(G, 2) \cong \left( \begin{array}{c}
\text{Ker}d_1 (NG_3) \xrightarrow{\text{Ker}d_0 (NG_3)} \\
\text{G_1}
\end{array} \right).
$$

Here the $h$-map is

$$
h : \text{Ker}d_0^1 \times \text{Ker}d_1^1 \longrightarrow NG_2 / d_3^3 (NG_3)
$$
given by $h(x, y) = [s_1x, s_1ys_0y^{-1}] \partial_3 NG_3$. Note if we consider the above crossed square as a vertical morphism of crossed modules we can take its kernel and cokernel within the category of crossed modules. In the above, the morphisms in the top left hand corner are induced from $d_2$ so

$$
\text{Ker} \left( \mu_1 : \frac{NG_2}{\partial_3 NG_3} \longrightarrow \text{Ker}d_1 \right) = \frac{NG_2 \cap \text{Ker}d_2}{\partial_3 NG_3} \cong \pi_2(G)
$$

whilst the other map labelled $\mu_1$ is an inclusion so has trivial kernel. Hence the kernel of this morphism of crossed modules is

$$
\pi_2(G) \longrightarrow 1.
$$

The image of $\mu_2$ is normal in both the simplicial groups on the bottom line and as $\text{Ker}d_0 = NG_1$ with the corresponding $\text{Im} \mu_1$ being $d_2 NG_2$, the cokernel is $NG_1 / \partial_2 NG_2$, whilst $G_1 / \text{Ker}d_0 \cong G_0$, i.e., the cokernel of $\mu_1$ is $\mathcal{M}(G, 1)$.

In fact of course $\mu_1$ is not only a morphism of crossed modules, it is a crossed module. This means that $\pi_2(G) \longrightarrow 1$ is in some sense a $\mathcal{M}(G, 1)$-module and that $\mathcal{M}(G, 2)$ can be thought of as a crossed extension of $\mathcal{M}(G, 1)$ by $\pi_2(G)$.

6 2-crossed modules from crossed squares

D. Conduché’s unpublished work shows that there exists an equivalence (up to homotopy) between the category of crossed squares of groups and that of 2-crossed modules of groups.
Loday defined a mapping complex of crossed squares by: if

\[ \mathcal{M} = \begin{array}{ccc}
L & \xrightarrow{\lambda} & M \\
\downarrow & & \downarrow \\
N & \xrightarrow{\mu} & P
\end{array} \]

is a crossed square, then its mapping complex is

\[ L \xrightarrow{\partial_2} M \rtimes N \xrightarrow{\partial_1} P \]

where \( \partial_2 l = (\lambda l^{-1}, \lambda l) \) and \( \partial_1 (m, n) = \mu(m) \nu(n) \).

Conduché showed that this mapping complex is always a 2-crossed complex, representing the same 3-type as the original crossed square. Why?

### 6.1 From crossed squares to bisimplicial groups

Recall that from a crossed module \( \mathcal{M} = (M \xrightarrow{\mu} P) \), we can build a simplicial group whose Moore complex is trivial in dimensions 2 and above. This is the nerve of the associated cat\(^1\)-group. (Because a cat\(^1\)-group is an internal groupoid in \( \text{Grps} \), we can form the nerve of its category structure internally within \( \text{Grps} \) and hence obtain a simplicial group.) We need an explicit description of this, elementwise: The simplicial group \( \text{Ner}(\mathcal{M}) \) has

\[ \text{Ner}(\mathcal{M})_0 = P \]

\[ \text{Ner}(\mathcal{M})_n = M \rtimes (\ldots (M \rtimes P) \ldots) \]

with \( n \) semidirect factors of \( M \) (see Conduché, [10], Carrasco and Cegarra, [8], or our own papers [30, 34] for more on semidirect decompositions and simplicial groups). If \( (m, p) \in \text{Ner}(\mathcal{M})_1 \), then

\[ d_0^1(m, p) = \mu(m)p \]

\[ d_1^1(m, p) = p \]

and

\[ s_0^0(p) = (1, p). \]

If \( (m_2, m_1, p) \in \text{Ner}(\mathcal{M})_2 \), then

\[ d_0^2(m_2, m_1, p) = (m_2, \mu(m_1)p) \]

\[ d_1^2(m_2, m_1, p) = (m_2m_1, p) \]

\[ d_2^2(m_2, m_1, p) = (m_1, p) \]

\[ s_0^1(m, p) = (m_1, p) \]

\[ s_1^1(m, p) = (1, m, p) \]

and the obvious pattern continues to higher dimensions.
Now going to crossed squares, let

\[
\begin{array}{ccc}
L & \overset{\lambda}{\longrightarrow} & M \\
\downarrow & & \downarrow \\
N & \overset{\mu}{\longrightarrow} & P
\end{array}
\]

be a crossed square. The construction we will need is the bisimplicial nerve of \(\mathcal{M}\) or rather of the associated cat\(^2\)-group. That will be an internal double groupoid in \(\text{Grps}\) and so when we take the nerves in the two directions, we will get a bisimplicial group. Although again this is known, we will want explicit descriptions of elements etc. for explicit calculations of actions, pairings, etc. later on, so will give quite a lot of detail, even when it is fairly simple to check.

Considering the crossed square \(\mathcal{M}\) as a morphism /crossed module from \((L \to N)\) to \((M \to P)\), we apply the above nerve construction to its domain and codomain to get a crossed module of simplicial groups. In low dimensions:

\[
\begin{array}{ccc}
L \times L \times N & \overset{(\lambda,\lambda,\nu)}{\longrightarrow} & M \times M \times P \\
\downarrow & & \downarrow \\
L \times N & \overset{(\lambda,\nu)}{\longrightarrow} & M \times P \\
\downarrow & & \downarrow \\
N & \overset{\nu}{\longrightarrow} & P
\end{array}
\]

To check, for instance, that \((\lambda,\nu)\) is a crossed module, you need to use the \(h\)-map of \(\mathcal{M}\), as follows: First the action of \((m,p)\) on \((l,n)\) is

\[(m,p)(l,n) = m(p,\nu m^{-1} n) = (\mu(m)p h(m,\nu n),\nu n).
\]

Now writing \(\partial = (\lambda,\nu),\)

\[\partial((m,p)(l,n)) = (\lambda(\mu(m)p),\lambda h(m,\nu n),\nu(n)).
\]

but \(\lambda(h(m,\nu n)) = m^{p n m^{-1}}\) and so a routine calculation shows that this expands to

\[(m,p)(\lambda l,\nu n)(p^{-1} m^{-1},p^{-1}),
\]

i.e. to \((m,p)\lambda(l,n),\) so \(\partial\) satisfies the first crossed module axiom. We leave the second crossed module axiom (Peiffer identity) to the diligent reader. It uses \(h(\lambda l,\nu n') = l^p l^{-1}.\) The description of the ‘vertical’ face and degeneracies is as before.

Next we start building the nerve in the second direction. Writing \(\mathcal{X}\) for the resulting simplicial group, we get:

\[
\begin{align*}
\mathcal{X}_{0,0} &= P \\
\mathcal{X}_{0,1} &= M \times P \\
\mathcal{X}_{0,q} &= M \times \ldots M \times P = M^{(q)} \times P
\end{align*}
\]

and in general

\[
\mathcal{X}_{p,0} = N \times \ldots N \times P = N^{(p)} \times P
\]

with \(q\)-factors of \(M\), (note the shorthand version).

Similarly
with $p$-factors of $N$. In general $\mathcal{X}_{p,q}$ can be written:

$$\mathcal{X}_{p,q} = (L^{(q)} \rtimes N)^{(p)} \rtimes (M^{(q)} \rtimes P).$$

**Remark:**
This initially looks asymmetric but in fact is not. The result of forming the horizontal nerve first then the vertical one would give an isomorphic bisimplicial group. This is easy to show categorically, but is tedious to show elementwise as it makes repeated use of the interchange / $h$-map structure within $\mathcal{M}$.

### 6.2 From bisimplicial groups to simplicial groups.

There are two useful ways of passing from bisimplicial groups to simplicial groups. One is the diagonal, the other, due to Artin and Mazur, [2], is the ‘codiagonal’ and, for us, is more useful. In the bisimplicial group $\mathcal{X}$, the corresponding Moore bicomplex has relatively few non-zero terms. In fact, $N(\mathcal{X})_{p,q}$ will be zero if $p$ or $q$ is bigger than 1. If you take the diagonal and try to apply the Moore complex functor, the task looks horrendous. Although, in fact, this Moore complex has length 2, the individual terms are quite large with a complicated expression for the differential. The diagonal gets complicated because its $p$-simplexes are the $(p,p)$-simplices of the original bisimplicial group and so seem to correspond to $2p$-dimensional data. (In fact they need a list of $(p+1)^2$-elements to describe them.)

In the Artin-Mazur construction, if $\mathcal{X}_{*,*}$ is a bisimplicial group, we first form for each $n$, a group

$$\mathcal{X}(n) = \prod_{p+q=n} \mathcal{X}_{p,q}.$$  

Within this $\mathcal{X}(n)$, we pick out a subgroup, $\nabla(\mathcal{X})_n$ as follows: Let $\underline{x} = (x_0, \ldots, x_n) \in \mathcal{X}(n)$ with $x_p \in \mathcal{X}_{p,n-p}$. Then $\underline{x} \in \nabla(\mathcal{X})_n$ if and only if for each $p = 0, \ldots, n - 1$

$$d^v_0 x_p = d^h_{p+1} x_{p+1}.$$  

This mysterious formula can be best remembered by looking at the case $n = 2$ and the diagram

```
  x_2,0
 /|
/  |
 x_0,2  x_1,1
```

where we have expanded the notation $x_p$ to $x_{p,n-p}$ to make the changes in dimension clearer (we hope). Similar subdivided simplex diagrams can be seen to give the formulae in higher dimensions, although above $n = 3$, they cannot be so simply drawn. The link between this and the ordinal subdivision of the paper, [16], by Ehlers and the second author will be explored and exploited later.
The face and degeneracy maps of $\nabla(X)$ are built up in an obvious way from this formula:

$$d_j = d^N_j : \nabla(X)_n \to \nabla(X)_{n-1},$$

for $x = (x_0, \ldots, x_n)$, with $d^0_0 x_p = d^h_{p+1} x_{p+1}$, $p = 0, \ldots, n - 1$, then for $0 < j < n$,

$$d^N_j (x) := (d^v_{j} x_0, d^v_{j-1} x_1, \ldots, d^v_{j-1} x_{j-1}, d^h_{j} x_{j+1}, \ldots, d^h_{n} x_n);$$

$$d^0_0 (x) := (d^h_{0} x_1, d^h_{0} x_2, \ldots, d^h_{0} x_n);$$

and

$$d^N_n (x) := (d^v_{n} x_0, d^v_{n-1} x_1, \ldots, d^v_{n-1} x_{n-1}).$$

whilst

$$s^N_i (x) := (s^v_i x_0, s^v_{i-1} x_1, \ldots, s^v_{i-1} x_i, s^h_i x_i, \ldots, s^h_i x_n)$$

for $0 \leq i \leq n$.

**Remark:**

Of course the formulae for $d_0$ and $d_n$ can be considered as being special cases of that for $d_j$.

Suppose now given a bisimplicial group, $X$, such that for any fixed $p$, the Moore complex of $X_{p, \ast}$ is of length at most 1 (so is a crossed module) and that for any fixed $q$, similarly $N(X_{\ast, q})$ has length at most 1. Then we claim that $N(\nabla(X))$ has length $\leq 2$.

We examine an element $x \in N(\nabla(X))_n$ for $n \geq 3$. Since $d_0 (x) = 1$, we have

$$x_p \in Kerd^h_0 \quad \text{for all } p > 0.$$

Now for each $0 < j < n$, $d_j (x) = 1$, so

$$d^v_j x_0 = d^v_{j-1} x_1 = \ldots d^v_{j-1} x_{j-1} = 1,$$

whilst

$$d^h_{j} x_{j+1} = \ldots = d^h_{j} x_n = 1.$$

Looking at $x_n$, $d^h_{j} x_n = 1$ for $j = 0, \ldots, n - 1$, i.e. $x_n \in N(X_{\ast, 0})_n$. If $n \geq 2$, then this group is trivial, so $x_n = 1$, but $d^h_{n} x_n = d^h_{0} x_{n-1}$, so $d^h_{0} x_{n-1} = 1$ as well. Turning to $x_{n-1} \in X_{n-1, 1}$, we have $d^h_{0} x_{n-1} = 1$. From $d^0_0$, we have $d^h_{j} x_{n-1} = 1$ for $j = 1, \ldots, n - 2$, i.e. $x_{n-1} \in N(X_{\ast, 1})_{n-1}$. As $n \geq 3$, $n - 1 \geq 2$, so $N(X_{\ast, 1})_{n-1} = 1$, i.e. $x_{n-1} = 1$ and hence $d^0_0 (x_{n-1}) = 1$. Continuing like this, we get $x_k = 1$ for all $k \geq 2$: for each index, we already know

$$d^h_{j} x_k = 1 \quad \text{for } j \leq k - 1,$$

i.e. $x_k \in N(X_{\ast, n-k})_{k} = 1$ if $k \geq 2$. This leaves us with just $x_0$ and $x_1$ to examine:

$$d^v_0 x_1 = d^h_2 x_2 = 1,$$

whilst $d^v_j x_1 = 1$ (from $d^N_{j+1} x = 1$ so again $x_1 \in N(X_{1, \ast})_{n-1}$ which is trivial as $n - 1 \geq 2$, i.e. $x_1 = 1$, so $d^h_0 x_0 = 1$ as well. Finally $d^v_j x_0 = 1$ for all $0 < j < n$ (from $d^N_0 x = 1$), so $x_0 \in N(X_{0, \ast})_n = 1$.

We have thus proved:

**Proposition 6.1** If $X_{\ast, \ast}$ is a bisimplicial group such that for any $p$, $N(X_{p, \ast})_q = 1$ for $q \geq 2$ and for any $q$, $N(X_{\ast, q})_p = 1$ for $p \geq 2$, then $N(\nabla(X))_n = 1$ for $n \geq 3$. □
We have in fact proved a bit more. If $\mathcal{X}_*$ satisfies the conditions of the proposition and $\underline{x} = (x_0, x_1, x_2) \in N(\nabla(\mathcal{X}))_2$, then $x_2 = 1$ since $x_2 \in N(\mathcal{X}_*0)_2 = 1$. Of course $N(\nabla(\mathcal{X}))_n = 1$ for $n \geq 3$ implies that $N(\nabla(\mathcal{X}))$ is a 2-crossed module. We will examine this in some detail.

**Remark:**
The above proposition suggests two questions. Firstly if we weaken the condition that each direction gives a crossed module as it Moore complex to that it gives a crossed complex, is it true that $N(\nabla(\mathcal{X}))$ is a 2-crossed complex? Also, is there a generalisation to higher order multisimplicial groups? We will later on show that the answer to the second one is positive.

For the moment we will look in more detail at the case when $\mathcal{X}$ is the binerve of a crossed square.

### 6.3 The 2-crossed module structure of $N(\nabla(\mathcal{X}(\mathcal{M})))$.

As before, we let $\mathcal{M}$ be a crossed square and will write $\mathcal{X}(\mathcal{M})$ for $\operatorname{Ner}^h(\operatorname{Ner}^s(\mathcal{M}))$, i.e. for the binerve of $\mathcal{M}$. For convenience we will often write $G$ instead of $\nabla(\mathcal{X}(\mathcal{M}))$.

We know $N(G)_n = 1$ if $n \geq 3$. It is also clear that $G_0 = P$, so $N(G)_0 = P$ as well. We thus only need to work out $N(G)_k$ for $k = 1$ and 2, together with explicit formulae for the boundary maps and the Peiffer pairing from $N(G)_1 \times N(G)_1$ to $N(G)_2$.

Suppose $x \in G_1$, then $\underline{x} = (x_0, x_1)$ with $x_0 \in \mathcal{M}_{0,1}$, $x_1 \in \mathcal{M}_{1,0}$. We have

$$\mathcal{M}_{0,1} = M \rtimes P, \quad \mathcal{M}_{1,0} = N \rtimes P,$$

so $x_0 = (m, p)$, $x_1 = (n, p')$ and, since $\underline{x} \in \nabla \mathcal{X}(\mathcal{M})_1$,

$$d^0_0(m, p) = \mu(m)p = p' = d^1_1(n, p'),$$

i.e. $p'$ is determined by $(m, p)$. The assignment

$$\underline{x} = ((m, p), (n, \mu(m)p)) \rightarrow (n, m, p)$$

is easily checked to give an isomorphism

$$G_1 \cong N \rtimes (M \rtimes P),$$

where $M$ acts on $N$ via $P$, $m^*n = \mu(m)n$. Identifying $G_1$ with $N \rtimes (M \rtimes P)$, $d_0$ and $d_1$ have the descriptions:

$$d_0(n, m, p) = \nu(n)\mu(m)p, \quad d_1(n, m, p) = p.$$

Thus $\underline{x} \in N(G_1)$ if and only if $p = \mu(m)^{-1}\nu(n)^{-1}$ and it is again easily verified that

$$NG_1 \cong M \rtimes N,$$

where the isomorphism is given by

$$(n^{-1}, m^{-1}, \mu(m)\nu(n)) \rightarrow (n, m).$$
We note that via this isomorphism, we get

\[ \partial_1 : NG_1 \to NG_0 \]

is given by

\[ \partial_1(n, m) = \mu(m)\nu(n). \]

This does look strange (inverses do not work well with conditions for homomorphisms), but does work as is easily checked.

Turning to \( NG_2 \), \( x \in \nabla X_2 \) will have the form \((x_0, x_1, x_2)\) with

\[
\begin{align*}
x_0 &= (m_2, m_1, p) \in M \times (M \times P) \\
x_1 &= (l, n, m, p') \in (L \times N) \times (M \times P) \\
x_2 &= (n_2, n_1, p'') \in N \times (N \times P)
\end{align*}
\]

The equations \( d_0^* x_0 = d_1^* x_1 \) and \( d_0^* x_1 = d_2^* x_2 \) give relations between the individual coordinates implying that \( m_2 = m, p' = \mu(m_1)p, \lambda(l)n = n_1 \) and \( \mu(m)p' = p'' \). (This is best seen on a diagram as above, but such a diagram is best left to the reader to draw!) Now suppose \( x \in N(\nabla X)_2 \), then in addition one gets \( p = 1, m_2m_1 = 1, \) etc. and it follows that all the ‘coordinates’ depend only on \( l \). In fact,

\[
x = (((\lambda(l))^{-1}, (\lambda(l), 1)), ((l, \lambda(l))^{-1}), (\lambda(l))^{-1}, \mu\lambda(l))), (1, (1, 1)))
\]

with \( d_2(x) = ((\lambda(l), 1), (\lambda(l), \mu\lambda(l))) \), which identifies to \((\lambda(l))^{-1}, \lambda(l)) \in M \rtimes N \). Thus the normal chain complex \( NG \) is isomorphic to

\[
\begin{array}{ccc}
L & \xrightarrow{\partial_2} & M \rtimes N \xrightarrow{\partial_1} P \\
& & \\
\end{array}
\]

with

\[
\begin{align*}
\partial_2(l) &= ((\lambda(l))^{-1}, \lambda(l)) \\
\partial_1(l) &= \mu(m)^{-1}\nu(n)^{-1}.
\end{align*}
\]

To complete the description, we should really specify the Peiffer pairing

\[
\{ \ , \ } : NG_1 \times NG_1 \to NG_2.
\]

We can use the formula given in our earlier paper, \([31]\), derived from work of Conduché:

\[
\{x, y\} = s_0(x)s_1(y)s_0(x)^{-1}s_1(y^{-1}x^{-1})
\]

Since \( s_0(x) = s_0(x_0, x_1) = (s_0^0 x_0, s_0^0 x_0, s_0^1 x_1) \) and \( s_1(x) = s_1(x_0, x_1) = (s_1^0 x_0, s_1^0 x_1, s_1^1 x_1) \), we can get an explicit description of \( \{x, y\} \). In fact, that description is not that useful. Its exact form is dependent on the order of multiplication used. These forms are equivalent via the relations for expanding \( h(m, m', n) \) in terms of \( h(m, n) \) and \( h(m', n) \), but at the risk of lengthening the expression. These different forms are fairly complicated and so have been omitted here, but can be retrieved from the original formula.
Remarks

(i) In Conduché’s original letter, he gives a much simpler form, namely $h(m, n!b^{-1})$. Our efforts to reduce the above to something as simple as this have so far failed! This is almost certainly due to the change in convention on the Moore complex, but we have not managed fully to understand the reason for the greater complication. Conduché’s elegant form does work as is very easily checked. What is not clear is its relationship with the ‘canonical’ form coming from the Peiffer pairings.

(ii) The above shows a subtle difficulty encountered when working with elements in models for homotopy $n$-types for $n > 1$. With (free) groups, we are used to handling normal forms of elements, but even with free crossed modules, the Peiffer identity makes calculating with representatives of elements much more difficult. This emphasises the need for a higher order version of both combinatorial and computational group theory, adapting the methods of the classical case to these higher dimensional situations. Some success has been achieved in this area by Alp and Wensley, [1], and by Ellis, [18].

6.4 Squared complexes and 2-crossed complexes

In 1993, Ellis defined the notion of a squared complex, [17]. A crossed complex combines a crossed module at its ‘base’ with a continuation by a chain complex of modules further up. They thus have good descriptive power, being able to model 2-types via their crossed module part and also a certain amount of higher homotopy information, typically thought of as generalising chains on the universal cover of a space.

Squared complexes are one of the possible notions that generalise crossed complexes to include the so-called quadratic information available in a 3-type. Other versions include double crossed complexes (cf. Tonks, [36]), 2-crossed complexes (cf. the authors, [33]) and quadratic complexes, (cf. Baues, [3]). We will need to examine 2-crossed complexes in detail later.

A squared complex consists of a diagram of group homomorphisms

\[
\begin{array}{c}
\cdots \rightarrow C_4 \overset{\partial_4}{\rightarrow} C_3 \overset{\partial_3}{\rightarrow} L \\
\downarrow \quad \downarrow \quad \downarrow \\
N \quad \mu \quad \overset{\lambda}{\rightarrow} \quad M \\
\downarrow \quad \downarrow \quad \downarrow \\
P \quad \overset{\mu'}{\rightarrow} \quad \overset{\lambda'}{\rightarrow} \quad \overset{\lambda}{\rightarrow} \quad \overset{\mu}{\rightarrow} \\
M \quad \overset{\mu}{\rightarrow} \quad \overset{\lambda'}{\rightarrow} \quad \overset{\lambda}{\rightarrow} \quad \overset{\mu}{\rightarrow} \\
P \\
\end{array}
\]


together with actions of $P$ on $L, N, M$ and $C_i$ for $i \geq 3$, and a function $h : M \times N \rightarrow L$. The following axioms need to be satisfied.

(i) The square \[
\begin{pmatrix}
\lambda \\
\mu
\end{pmatrix}
\]
is a crossed square;

(ii) The group $C_n$ is abelian for $n \geq 3$

(iii) The boundary homomorphisms satisfy $\partial_n \partial_{n+1} = 1$ for $n \geq 3$, and $\partial_3(C_3)$ lies in the intersection $\ker \lambda \cap \ker \lambda'$;

(iv) The action of $P$ on $C_n$ for $n \geq 3$ is such that $\mu M$ and $\mu' N$ act trivially. Thus each $C_n$
is a \( \pi_0 \)-module with \( \pi_0 = P/\mu M\mu' N \).

(v) The homomorphisms \( \partial_n \) are \( \pi_0 \)-module homomorphisms for \( n \geq 3 \).

This last condition does make sense since the axioms for crossed squares imply that \( \ker \mu' \cap \ker \mu \) is a \( \pi_0 \)-module.

A morphism of squared complexes

\[
\Phi : \left( C_\ast, \begin{pmatrix} L \rightarrow N \cr X \downarrow \downarrow M \rightarrow \mu P \end{pmatrix} \right) \rightarrow \left( C'_\ast, \begin{pmatrix} L' \rightarrow N' \cr X' \downarrow \downarrow M' \rightarrow \mu' P' \end{pmatrix} \right)
\]

consists of a morphism of crossed squares \( (\Phi_L, \Phi_N, \Phi_M, \Phi_P) \), together with a family of equivariant homomorphisms \( \Phi_n \) for \( n \geq 3 \) satisfying \( \Phi_L \partial_3 = \partial'_3 \Phi_L \) and \( \Phi_n \partial_n = \partial'_n \Phi_n \) for \( n \geq 4 \).

There is clearly a category \( \mathcal{SqComp} \) of squared complexes. This exists in both group and groupoid based versions.

A squared complex is thus a crossed square with a ‘tail’ attached. The same process can be applied to 2-crossed modules and leads to the notion of a 2-crossed complex, [33].

A 2-crossed complex of group(oid)s is a sequence of group(oid)s

\[
C : \ldots \rightarrow C_n \partial_3 C_{n-1} \rightarrow \ldots \rightarrow C_2 \partial_2 C_1 \partial_1 C_0
\]

in which

(i) \( C_n \) is abelian for \( n \geq 3 \);
(ii) \( C_0 \) acts on \( C_n \), \( n \geq 1 \), the action of \( \partial C_1 \) being trivial on \( C_n \) for \( n \geq 3 \);
(iii) each \( \partial_n \) is a \( C_0 \)-group(oid) homomorphism and \( \partial_i \partial_{i+1} = 1 \) for all \( i \geq 1 \); and
(iv) \( C_2 \partial_2 C_1 \partial_1 C_0 \) is a 2-crossed module.

The following is an easy consequence of our earlier work.

**Theorem 6.2** If

\[
\begin{array}{ccc}
\cdots & - & C_4 \partial_4 C_3 \partial_3 L \rightarrow & N \rightarrow \mu P \rightarrow & M \rightarrow \mu' \cdots \\
& N \downarrow \downarrow \downarrow \downarrow & \nu & \nu' \downarrow & \\
& C_4 \rightarrow & C_3 \rightarrow & L \rightarrow & P \rightarrow \\
& \lambda & \lambda & \lambda & \lambda & \lambda
\end{array}
\]

is a squared complex then

\[
\cdots \rightarrow C_n \rightarrow C_{n-1} \rightarrow \ldots \rightarrow C_3 \rightarrow L \rightarrow N \rightarrow M \rightarrow P
\]

is a 2-crossed complex. \( \square \)

## 7 Homotopy groups

It is folklore that any bisimplicial group \( X_{s,s} \), the diagonal, \( \text{diag}(X) \) and the codiagonal \( \nabla X \) have the same homotopy type. The original work of Loday, [27] and Conduché, [10], together
with the second author’s [35] all used the diagonal so showing that the homotopy groups of \( \text{diag}(\mathcal{M}) \) were those of \( G \) when \( \mathcal{M} = \mathfrak{M}(G, 2) \). The use of the mapping complex construction by Loday, again in [27], also gave the same homotopy groups. Here we will briefly look at the homotopy groups of \( \nabla(\mathcal{X}(\mathfrak{M}(G, 2))) \) directly.

The crossed square \( \mathfrak{M}(G, 2) \) as we recalled earlier has form

\[
\mathfrak{M}(G, 2) \cong \begin{pmatrix}
NG_2/\partial_3(NG_3) & \mu_2 \\
\mu_1 & \mu_1 \\
\text{Ker } d_1 & \mu_2 \\
\end{pmatrix} \rightarrow \text{Ker } d_0 \times \text{Ker } d_1 \rightarrow G_1.
\]

The horizontal kernel is \( \pi_2(G) \rightarrow 1 \) and the cokernel “is” \( \mathfrak{M}(G, 1) \), i.e. isomorphic to

\[
NG_1/\partial_2NG_2 \rightarrow G_0,
\]

which, in turn, has kernel \( \pi_1(G) \) and cokernel \( \pi_0G \).

The corresponding simplicial group \( \nabla(\mathcal{X}(\mathfrak{M}(G, 2))) \) has Moore complex

\[
NG_2/\partial_3(NG_3) \xrightarrow{(\mu_2^{-1}, \mu_1)} \text{Ker } d_0 \times \text{Ker } d_1 \xrightarrow{\mu_1 \mu_2} G_1.
\]

The homomorphisms \( \mu_2 \) and \( \mu_1 \) from \( NG_2/\partial_3(NG_3) \) in the crossed square are both induced from \( d_2 \), so it is immediate that \( \text{Ker}(\mu_2^{-1}, \mu_1) \) is \( \pi_2(G) \). The other two \( \mu_s \) (bottom and right of the square) are inclusions. Since \( G_0 \cong G_1/\mu_2(\text{Kerd}_1) \), we again easily check that \( \pi_0 \) of the complex, i.e. \( G_1/\text{Im}(\mu_1\mu_2) \), is \( \pi_0(G) \).

Finally

\[
(m, n) \in \text{Ker}(\mu_1\mu_2)
\]

if and only if \( \mu_1(m) = \mu_2(n)^{-1} \), but as \( \mu_1 \) and \( \mu_2 \) are inclusions, this amounts to \( \text{Ker}(\mu_1\mu_2) \) being

\[
\{(m, m^{-1}) : m \in \text{Kerd}_0 \cap \text{Kerd}_1\}.
\]

Then the image of \( (\mu_2^{-1}, \mu_1) \) identifies as

\[
\{(m, m^{-1}) : m \in \partial_2(NG_2)\}
\]

and again a routine calculation shows that

\[
\frac{\text{Ker}(\mu_1\mu_2)}{\text{Im}(\mu_2^{-1}, \mu_1)} \cong \pi_1(G),
\]

as expected.

These calculations do not involve the Peiffer lifting so are not conclusive about homotopy type, however a neat argument noted by Conduché shows that there is an (obvious) epimorphism

\[
\begin{array}{cccccc}
NG_2/\partial_3(NG_3) & \rightarrow & \text{Ker } d_0 & \times & \text{Ker } d_1 & \rightarrow & G_1 \\
& = & & & & & \\
NG_2/\partial_3(NG_3) & \rightarrow & \text{Ker } d_0 & \rightarrow & G_0
\end{array}
\]
with acyclic kernel. This still does not quite clinch the argument, since it would be better
to check that there was an acyclic fibration

\[ \nabla(\mathcal{M}(G, 2)) \to t_2G \]

where \( t_2G \) here denotes the homotopy truncation of \( G \) (essentially its 2-coskeleton). However
we have not attempted to give this here.

8 Higher dimensions

Loday’s mapping complex was defined for cat\(^n\)-groups (see [27]) and results of Bullejos,
Cegarra and Duskin, [7] suggest that a similar multiple codiagonal would give his mapping
complex. This raises the question of what would be the result on taking a crossed \( n \)-cube \( \mathcal{M} \)
and forming its \( n \)-fold nerve \( Ner^{(n)} \mathcal{M} \), which will be an \( n \)-simplicial group (just generalise the
construction of section 6.1). Again Bullejos, Cegarra and Duskin use an inductive argument
to derive a result that would suggest the multicodiodiagonal of \( Ner^{(n)} \mathcal{M} \) should have Moore
complex of length \( n \).

We first note that the proof of Proposition 6.1 shows the following:

**Proposition 8.1** If \( \mathcal{X}_{*,*} \) is a bisimplicial group such that for any \( p \), \( N(\mathcal{X}_{p,*})_q = 1 \) for \( q \geq 2 \),
whilst for any \( q \), \( N(\mathcal{X}_{*,q})_p = 1 \) for \( p \geq m \), then

\[ N(\nabla \mathcal{X})_n = 1 \quad \text{for } n \geq m + 1. \]

We next need a categorical description of the multidiodiagonal:

Let \( or : \Delta \times \Delta \to \Delta \) be the ordinal sum functor, then for a bisimplicial group \( \mathcal{X} \), it is
well known that \( \nabla \mathcal{X} \) has a description as a coend

\[ (\nabla \mathcal{X})_n = \int^{[p],[q]} \Delta([n],[p or [q]]) \times \mathcal{X}_{p,q}, \]

(cf. for example, Cordier-Porter, [11]). A corresponding codiagonal for a \( m \)-fold simplicial
group \( \mathcal{X}_{*,*} \) an \( m \)-fold index, is

\[ (\nabla^{(m)} \mathcal{X})_n = \int^{\underline{p}} \Delta([n], or \underline{p}) \times \mathcal{X}_{\underline{p}}, \]

where by abuse of notation, we indicate by

\[ or : \Delta^{\times m} \to \Delta, \]

the \( m \)-fold ordinal sum, \( \underline{p} = ([p_1], \ldots, [p_m]) \) an \( m \)-fold index and have used as a shorthand

\[ or \underline{p} = [p_1] or \ldots or [p_m], \]

which causes no ambiguity as \( or \) is associative.
As \( \text{orp} = ([p_1] \text{or} \ldots \text{or} [p_{m-1}]) \text{or} [p_m] =: (\text{orp'}) \text{or} [p_m] \)

\[
(\nabla^{(m)} X)_n = \int^{p_m} \int^{[p_{m-1}]} \Delta([n], (\text{orp'}) \text{or} [p_m]) \times X_{p', p_m},
\]

but then ‘integrating’ over all \( p' \) for each \( n \) produces a new description of \( (\nabla^{(m)} X) \) as \( \nabla(\nabla^{(m-1)} X, \text{orp'}) \) i.e. an iterative description. (The proof uses the fact that any mapping from \([n]\) to \((\text{orp'}) \text{or} [p_m]\) effectively partitions \([n]\) into an initial segment mapping to \(\text{orp}'\) and a second part mapping to \([p_m]\). Fixing the latter part, we form the coend with \([n]\) replaced by the first segment of the partition.) Now assume that the \( m \)-simplicial group \( X_{\ast} \) is obtained as the \( m \)-fold nerve of a crossed \( m \)-cube (or \( \text{cat}^m \)-group), then applying the above proposition repeatedly we obtain:

**Theorem 8.2** If \( \mathcal{M} \) is a crossed \( m \)-cube with \( m \)-fold nerve, the \( m \)-simplicial group, \( X(\mathcal{M}) \), then

\[
N(\nabla X(\mathcal{M}))_n = 1 \quad \text{for} \quad n \geq m + 1
\]

There is a ‘complex’ form of this result as well. If one takes the obvious notion of \( m \)-cube complex, generalising the squared complexes considered above, then Theorem 6.2 generalises without bother to give a notion of \( m \)-crossed complex and a construction that generalises \( \nabla X \) above. The Moore complex that one gets is related to complexes considered by Duskin and Nan Tie as well as the hypercrossed complexes of [8].

The structural maps of such hypercrossed complexes and the related Peiffer pairings considered in earlier papers of the series could be given explicit algebraic formulae in terms of the \( h \)-maps in the \( m \)-cube complex, but the problem of the complexity of these formulae raises doubts as to their usefulness.

**References**

[1] M. Alp and C.D. Wensley, XMOD, share package for GAP, available from http://www.informatics.bangor.ac.uk/public/mathematics/research/preprints/preprint.html, preprint no. 97.14 or from http://www-gap.dcs.st-and.ac.uk/ gap/Info/share.html (see the list there).

[2] M. Artin and B. Mazur, On the Van Kampen theorem, Topology, 5, (1966), 179-189.

[3] H. J. Baues, Combinatorial homotopy and 4-dimensional complexes, Walter de Gruyter, (1991).

[4] C. Berger, Double loop spaces, braided monoidal categories and algebraic 3-types of spaces, preprint, Nice 1997.

[5] R. Brown and N. D. Gilbert, Algebraic models of 3-types and automorphism structures for crossed modules, Proc. London Math. Soc. (3) 59, (1989), 51-73.
[6] R. Brown and J.-L. Loday, Van Kampen Theorems for Diagram of Spaces, *Topology*, **26**, (1987), 311-335

[7] M. Bullejos, A. M. Cegarra, and J. Duskin, On cat"-groups and homotopy types, *Jour. Pure Appl. Algebra*, **86** (1993) 135-154.

[8] P. Carrasco and A. M. Cegarra, Group-theoretic Algebraic Models for Homotopy Types, *Jour. Pure Appl. Algebra*, **75**, (1991), 195-235.

[9] P. Carrasco and A. M. Cegarra, (Braided ) tensor structures on homotopy groupoids and nerves of (braided) categorical groups, *Comm. in Algebra*, **24 (3)** (1996), 3995-4058.

[10] D. Conduché, Modules Croisés Généralisés de Longueur 2, *Jour.Pure Appl.Algebra*, **34**, (1984), 155-178.

[11] J.-M. Cordier and T. Porter, Homotopy Coherent Category Theory, *Trans. Amer. Math. Soc.* 349 (1997) 1-54.

[12] E. B. Curtis, Simplicial Homotopy Theory, *Adv. in Math.*, **6**, (1971), 107-209.

[13] J. Duskin, Simplicial Methods and the Interpretation of Triple Cohomology, *Memoir A.M.S.*, Vol. 3, **163**, (1975).

[14] W. Dwyer and D. M. Kan, Homotopy Theory and Simplicial Groupoids, *Proc. Konink. Neder. Akad.* **87**, (1987), 379-389.

[15] P. J. Ehlers and T. Porter, Varieties of Simplicial Groupoids, I: Crossed Complexes. *Jour. Pure Appl. Algebra*, **120**, (1997), 221-233; plus: Correction, same journal **134**, (1999), 207-209.

[16] P. J. Ehlers and T. Porter, Ordinal Subdivision, (in preparation).

[17] G. J. Ellis, Crossed Squares and Combinatorial Homotopy, *Math. Z.*, **214**, (1993), 93-110.

[18] G. J. Ellis, Various software packages, available at [http://hamilton.nuigalway.ie/](http://hamilton.nuigalway.ie/).

[19] G. J. Ellis and R. Steiner, Higher Dimensional Crossed Modules and the Homotopy Groups of (n+1)-ads., *Jour. Pure Appl. Algebra*, **46**, (1987), 117-136.

[20] A.R. Garzon and J. G. Miranda, Homotopy theory for (braided) Cat-groups, *Cahiers de Top. et Géom. Diff. cat.*

[21] D. Guin-Waléry and J.-L. Loday, Obstructions à l’Excision en K-théorie Algébrique, *Springer Lecture Notes in Math.*, **854**, (1981), 179-216.

[22] L. Illusie, Complex Cotangent et Déformations I, II, *Springer Lecture Notes in Math.*, I, **239**, (1971), II, **283** (1972).
[23] K. H. Kamps and T. Porter, 2-groupoid enrichments in homotopy theory and algebra, em K-theory, 25, (2002), 3773 - 409.

[24] D. M. Kan, A Combinatorial Definition of Homotopy Groups, Annals of Maths., 61, (1958), 288-312.

[25] D. M. Kan, A relation between CW-complex and free c.s.s groups. Amer. Jour. of Maths.Soc., 81, (1959), 512-528.

[26] O. LeRoy, Sur la notion de 3-catégorie adaptée à l’homotopie, preprint, AGATA, Univ. Montpellier II, 1994.

[27] J.-L. Loday, Spaces having finitely many non-trivial homotopy groups, Jour. Pure Appl. Algebra, 24, (1982), 179-202.

[28] F. Marty, Approche en dimension supérieure des 3-catégories augmentées d’Olivier LeRoy, Thèse, Univ. Montpellier II, 1999.

[29] A. Mutlu, Peiffer Pairings in the Moore Complex of a Simplicial Group, Thesis, University of Wales Bangor, (1997); Bangor Preprint 97.11. Available via http://www.bangor.ac.uk/ma/research/preprints/97prep.html

[30] A. Mutlu and T. Porter, Iterated Peiffer pairings in the Moore complex of a simplicial group, Applied Categorical Structures, 9, (2001) 111-130.

[31] A. Mutlu and T. Porter, Applications of Peiffer pairings in the Moore complex of a simplicial group, Theory and Applications of Categories, 4, No. 7, (1998), 148-173, previously as Bangor Preprint 97.17, available via http://www.bangor.ac.uk/ma/research/preprints/97prep.html.

[32] A. Mutlu and T. Porter, Free crossed resolutions from simplicial resolutions with given CW-basis, Cahiers Top. Gom. Diff. catégoriques, 50 (1999) 261-283, previously as Bangor Preprint 97.18, available via http://www.bangor.ac.uk/ma/research/preprints/97prep.html.

[33] A. Mutlu and T. Porter, Freeness Conditions for 2-Crossed Modules and Complexes, Theory and Applications of Categories, 4, No.8, (1998), 174-194; previously as Bangor Preprint 97.19, available via http://www.bangor.ac.uk/ma/research/preprints/97prep.html.

[34] A. Mutlu and T. Porter, Freeness Conditions for Crossed Squares and Squared Complexes, K-Theory, 20, (2000) 345 - 368, previously as Bangor Preprint 99.01, available via http://www.bangor.ac.uk/ma/research/preprints/99prep.html.

[35] T. Porter, n-Types of simplicial groups and crossed n-cubes, Topology, 32, 5-24, (1993).

[36] A. Tonks, Theory and applications of crossed complexes: the Eilenberg-Zilber theorem and homotopy colimits, Ph.D. thesis, University of Wales (1994), available via http://www.bangor.ac.uk/ma/research/tonks/thesis.ps

26
[37] J. H. C. Whitehead, Combinatorial Homotopy I and II. Bull. Amer. Math. Soc. 55, (1949), 231-245 and 453-496.

[38] J. Wu, On combinatorial descriptions of \( \pi_*(\Sigma K(\pi, 1)) \), MSRI preprint 069, (1995).

A. Mutlu
Department of Mathematics
Faculty of Science
University of Celal Bayar
Manisa, Turkey
e-Mail: amutlu@spil.bayar.edu.tr

T. Porter
School of Informatics
University of Wales Bangor,
Gwynedd, LL57 1UT, UK.
e-Mail: t.porter@bangor.ac.uk