Galaxy number counts at second order in perturbation theory: a leading-order term comparison

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Abstract
The Galaxy number density is a key quantity to compare theoretical predictions to the observational data from current and future large scale structure surveys. The precision demanded by these stage IV surveys requires the use of second order cosmological perturbation theory. Based on the independent calculation published previously, we present the result of the comparison with the results of three other groups at leading order. Overall we find that the differences between the different approaches lie mostly on the definition of certain quantities, where the ambiguity of signs results in the addition of extra terms at second order in perturbation theory.

Keywords: galaxy, counts, perturbation, theories, leading

1. Introduction
Within the next couple of years next generation large scale structure (LSS) surveys such as Euclid [1], MeerKAT [2], SKA [3], the VRO/LSST [4], DESI [5], J-PAS [6] and WFIRST [7] will begin to take data of unprecedented quality and quantity. It is therefore high time for theoretical cosmologists to match the precision of their predictions to that of the forthcoming stage IV experiments. One of the key quantities in this context is the Galaxy number density,

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which allows us to compare theoretical predictions to the observational data provided by the experiments. However, the calculation of this quantity is rather involved, in particular if we take the calculation beyond first order in cosmological perturbation theory. Going beyond linear order is necessary to capture all the subtle effects affecting the number density and to achieve the precision required by the latest observational data.

Recently three different derivations of the second order number density were published [8–11]. Unfortunately it quite difficult to compare the results of the three derivations for the Galaxy number density. First of all, the final expressions for the Galaxy number density occupy several pages. Then, the notation and the break-up into separate terms is very different throughout all the derivations. An additional difficulty is that terms can be converted into each other by integration by parts, in a non-trivial way, making the comparison of even partial results tricky at best.

Because of these difficulties, there is a general question of whether or not all these derivations coincide. A first comparison of the different derivations has been done in reference [12], where the authors concentrate on the terms which dominate on sub-horizon scales, considering only the terms of the order of \( (k/H)^4 \Psi^2 \) and neglect smaller contributions to the second order number count. Here, \( k \) is the comoving wave number, \( H \) the conformal Hubble parameter and \( \Psi \) a scalar metric perturbation. In reference [12] the authors use a straight-forward derivation to reproduce the result of reference [8], finding disagreements concerning lensing terms and a double counting of volume distortion effects in references [9, 11], respectively.

Given the importance of the number density for current and future LSS surveys, we decided to have an independent calculation of this quantity up to second order in the perturbations and published the results in reference [13]. In this paper, we use the derivation of the second order number count given in reference [13] and follow a similar approach as the one made in reference [12] comparing leading terms in the aforementioned literature, however not constraining our comparison to sub-horizon scales.

**Notation.** Greek indices \((\mu, \nu, \ldots)\) denote space-time components, while Latin indices \((i, j, \ldots)\) stand for spatial components in a perturbed Friedmann–Lemaître–Robertson–Walker (FLRW) spacetime. A dash represents derivative with respect to the conformal time \( X' \equiv \frac{dX}{d\eta} \), while the derivative with respect to the affine parameter is

\[
\frac{dX}{d\lambda} = X' + n^i X_i,
\]

where \( n^i \) is the direction of observation (line of sight). For a scalar function \( X \), this implies,

\[
n^i n^j X_{ij} = n^i n^j \partial_i \partial_j X = \frac{d^2X}{d\lambda^2} - 2 \frac{dX'}{d\lambda} + X'' ,
\]

where \( \nabla_i X \) is the spatial part of the covariant derivative, and for the scalar \( \nabla_i X = \partial_i X \equiv X_j \).

Finally, when evaluating quantities at the source (s) and observer (o) points, we use the notation

\[
(X^s_i = X^o_i = X_s - X_o = X(\lambda_s) - X(\lambda_o)).
\]

### 2. Basic definitions for the Galaxy number density

To exercise our notation and for convenience of use we define in this section the basic quantities required to calculate the Galaxy number density up to second order in cosmological perturbation theory. We follow very closely our previous paper, reference [13]. However, we do not
reproduce the rather lengthy calculations presented there, and focus here on the results. For a more pedagogic treatment of the Galaxy number density calculation see reference [14].

2.1. Metric perturbations

The perturbed FLRW spacetime is described in the longitudinal gauge by [15]

\[ ds^2 = a^2 \left[ - (1 + 2\Phi_1 + \Phi_2) d\eta^2 + (1 - 2\Psi_1 - \Psi_2) \delta_{ij} dx^i dx^j \right], \] (2.1)

where \( a = a(\eta) \) is the scale factor, and \( \eta \) is the conformal time; \( \delta_{ij} \) is the flat spatial metric, and we have neglected the vector and tensor modes, we also allow for first and second order anisotropic stresses. From now on we consider perturbations around a FLRW metric up to second-order.

2.2. Matter velocity field and peculiar velocities

The components of the four-velocity \( u^\mu = dx^\mu/d\eta \), up to second order, for the perturbed metric are

\[ u_0 = -a \left[ 1 + \Phi_1 + \frac{1}{2} \Phi_2 - \frac{1}{2} \Psi_1^2 + \frac{1}{2} v_1 v_1^k v_k \right], \] (2.2)

\[ u_i = a \left[ v_{1i} + \frac{1}{2} v_2 - 2 \Psi_1 v_{1k} \right], \] (2.3)

\[ u^0 = a^{-1} \left[ 1 - \Phi_1 - \frac{1}{2} \Phi_2 + \frac{3}{2} \Phi_1^2 + \frac{1}{2} v_1 v_1^k v_k \right], \] (2.4)

\[ u^i = a^{-1} \left[ v_i^1 + \frac{1}{2} v_i^2 \right], \] (2.5)

where \( v_i = \partial_i v \), and with \( v \) the velocity potential.

2.3. Photon wavevector

In a redshift survey Galaxy positions are identified through the detection of photons emitted by a source, denoted by \( s \), and detected by the observer, labelled \( o \). In a general spacetime, we consider a lightray with tangent vector \( \tilde{k}^\mu \) and affine parameter \( \lambda \), that parametrises the trajectory followed by the lightray. The source and the observer are represented by specific values of the affine parameter, \( \lambda_s \) and \( \lambda_o \), respectively. The components of the photon wavevector are

\[ \tilde{k}^\mu = \frac{dx^\mu}{d\lambda} = a^{-1} \left[ 1, n^\mu \right], \] (2.6)

where the overbar denotes background quantities. The direction of observation is given by the vector \( n^i \) which points from the observer to the source\(^3\), and obeys the normalisation condition: \( n^i n_i = 1 \).

The tangent vector is null

\[ k_\mu k^\mu = 0, \] (2.7)

\(^3\) Some authors define \( n^i \) with the opposite sign. See, for example, references [9, 10, 16].
and geodesic
\[ k^\nu \nabla_\nu k^\mu = 0, \]  
(2.8)

where \( \nabla_\nu \) is the covariant derivative defined by the metric given in equation (2.1). To make a distinction, we denote the perturbed wavevector as
\[ \delta^{(n)} k^\mu = a^{-1} \left[ \delta^{(n)} \nu, \delta^{(n)} n^\nu \right], \]  
(2.9)

where \( \delta^{(n)} \) gives the \( n \)th order perturbation, and where the usual notation is followed for the temporal component, that is \( k^0 \equiv \nu \) (see e.g. reference [17]).

The affine parameter of the geodesic equation is also related to the comoving distance \( (\chi) \) by
\[ \chi = \lambda_0 - \lambda_s, \]  
(2.10)

which in terms of the redshift is
\[ \chi(z) = \int_0^z \frac{dz}{(1 + \tilde{z})H(\tilde{z})}. \]  
(2.11)

2.4. The observed redshift

The photon energy measured by an observer with four-velocity \( u^\mu \) is
\[ E = -g_{\mu \nu} u^\mu k^\nu. \]  
(2.12)

This implies that the observed redshift of a source (e.g. a Galaxy) can be defined as
\[ 1 + z = \frac{E_s}{E_0}. \]  
(2.13)

This definition explicitly shows the nature of the Doppler effect on the observed redshift, a function of the velocity and the wavevector, i.e. \( z = z(k^\mu, u^\mu) \).

2.5. Angular diameter distance

A given bundle of light rays leaving a source will invariantly expand and create a distance in between the light rays that conform it, this can be projected to an area in screen space, the hypersurface perpendicular to the trajectories of the photons and the four-velocity of the observer. If we denote the area of a bundle in screen space with \( A \), then the angular diameter distance \( d_\Lambda \), and the null expansion \( \theta \) are derived from this quantity as [18],
\[ \frac{1}{\sqrt{A}} \frac{d\sqrt{A}}{d\lambda} = \frac{d \ln d_\Lambda}{d\lambda} = \frac{1}{2} \theta. \]  
(2.14)

From this we can compute how the area of the bundle changes along the geodesic trajectory followed by the photons from the source up to the observer.
2.6. Physical volume

Number counts account for the number of sources detected in a bundle of rays, for a small affine parameter displacement $\lambda$ to $\lambda + d\lambda$ at an event $P$. This defines the physical distance

$$d\ell = (k' u_{\mu}) d\lambda,$$

in the rest frame of a comoving Galaxy at said point in space $P$, and is positive if $k'$ is a tangent vector to the past directed null geodesics, so that $k' u_{\mu} > 0$.

On the other hand, the cross-sectional area of the bundle is

$$dA = a^2(\lambda) d\Omega,$$

if the geodesics subtend a solid angle $d\Omega$ at the observer.

From equations (2.15) and (2.16) the corresponding volume element at a point $P$ in space is (see e.g. reference [19])

$$dV = d\ell \, dA = (k' u_{\mu}) a^2(\lambda) d\lambda \, d\Omega = -E a^2(\lambda) d\lambda \, d\Omega.$$

These covariant definitions are used in a second order expansion of the Cosmological perturbation theory in the next section.

3. Leading-order terms at second order

To determine the leading terms of the second order number counts, we first need to analyse the result given in equation (5.17) from reference [13]. From equations (5.8) and (5.9) therein, we can see that the general expression for the dominating terms is of the form

$$\Delta_{\text{Leading}} = (1 + \delta)(1 + \delta V) \simeq (1 + \delta)(1 + \text{RSD})(1 + \kappa),$$

where $\delta$ is the matter overdensity, $\delta V$ is the perturbed volume, and we further split the volume perturbation into its dominant components; that is, the lensing contribution ($\kappa$), and redshift space distortions (RSD).

Perturbing equation (3.1) up to second order in perturbation theory, we find that the nonlinear contribution to the leading terms is given by

$$\Delta_{\text{Leading}}^{(2)} = \delta^{(2)} + \delta^{(1)} \delta^{(1)} V + \delta^{(2)} V,$$

$$= \delta^{(2)} + [\text{RSD}]^{(2)} + \kappa^{(2)} + \delta^{(1)} [\text{RSD}]^{(1)} + \delta^{(1)} \kappa^{(1)} + [\text{RSD}]^{(1)} \kappa^{(1)}.$$

We now present the leading terms of the calculation to second order given in equation (5.17) from reference [13], which is

$$\Delta_{\text{Leading}}^{(2)} \simeq \delta^{(2)} - \frac{1}{2\mathcal{H}} \frac{d}{d\varsigma} \left( v_{1\text{r}} n' \right) + \frac{1}{2\mathcal{H}} \frac{d}{d\varsigma} \left( v_{1\text{r}} v_{1i}' \right) - \left( v_{1i} n' \right) \frac{d}{d\varsigma} \left( v_{1i} n' \right)$$

$$+ \frac{d}{d\varsigma} \int_0^{\varsigma} d\chi \left( \Phi_1' + \Psi_1' \right) \frac{\mathcal{H}'}{\mathcal{H}} \frac{d}{d\varsigma} \int_0^{\varsigma} d\chi \left( \Phi_1' + \Psi_1' \right)$$

$$+ \frac{2}{\mathcal{H}} \frac{d}{d\varsigma} \int_0^{\varsigma} d\chi \left( \Phi_1' + \Psi_1' \right) + \frac{\delta^{(1)}}{\mathcal{H}} \frac{d}{d\varsigma} \left( v_{1i} n' \right)$$

$$+ \frac{1}{\mathcal{H}} \frac{d}{d\varsigma} \frac{1}{\chi} \int_0^{\varsigma} d\chi \left[ \nabla^2 (\Phi_1 + \Phi_1) + n' n' (\Phi_1 + \Psi_1)_{ij} + \frac{2}{\chi} \frac{d\delta^{(1)} \nu}{d\varsigma} \right]$$
\[ + \frac{\delta^{(1)}_g}{\chi} \int_{0}^{\chi} d\chi \left[ \nabla^2 (\Phi_1 + \Psi_1) + n' n'(\Phi_1 + \Psi_1)_{ij} + \frac{2}{\chi} \frac{d\delta^{(1)}_g}{d\varsigma} \right] \]
\[ + \frac{2}{\chi} \int_{0}^{\chi} d\chi \left[ \nabla^2 (\Psi_2 + \Phi_2) + n' n'(\Psi_2 + \Phi_2)_{ij} + \frac{2}{\chi} \frac{d\delta^{(2)}_g}{d\varsigma} \right], \quad (3.3) \]

These terms originate from density fluctuations\(^4\),
\[ \Delta^{(2)}_{\text{leading} - \delta} \simeq \delta^{(2)}_g, \quad (3.4) \]

RSD
\[ \Delta^{(2)}_{\text{leading} - \text{RSD}} \simeq - \frac{1}{2H} \frac{d(v_{2,t}n')}{d\varsigma} + \frac{1}{2H} \frac{d(v_{1,t}v_i)}{d\varsigma} - (v_{1,m}) \frac{d(v_{1,n'})}{d\varsigma} + \left[ 1 - \frac{H''}{H^2} + \frac{2}{H} \right] \frac{d(v_{1,m})}{d\varsigma} \int_{0}^{\chi} d\chi (\Phi_1' + \Psi_1'), \quad (3.5) \]

lensing terms
\[ \Delta^{(2)}_{\text{leading} - \kappa} \simeq \frac{2}{\chi} \int_{0}^{\chi} d\chi \left[ \nabla^2 (\Psi_2 + \Phi_2) + n' n'(\Phi_2 + \Psi_2)_{ij} + \frac{2}{\chi} \frac{d\delta^{(2)}_g}{d\varsigma} \right], \quad (3.6) \]

and cross terms
\[ \Delta^{(2)}_{\text{leading} - \delta \times \text{RSD}} \simeq \frac{\delta^{(1)}_g}{H} \frac{d(v_{1,m})}{d\varsigma}, \quad (3.7) \]
\[ \Delta^{(2)}_{\text{leading} - \delta \times \kappa} \simeq \frac{\delta^{(1)}_g}{\chi} \int_{0}^{\chi} d\chi \left[ \nabla^2 (\Phi_1 + \Psi_1) + n' n'(\Phi_1 + \Psi_1)_{ij} + \frac{2}{\chi} \frac{d\delta^{(1)}_g}{d\varsigma} \right], \quad (3.8) \]
\[ \Delta^{(2)}_{\text{leading} - \text{RSD} \times \kappa} \simeq \frac{1}{H} \frac{d(v_{1,m})}{d\varsigma} \frac{1}{\chi} \int_{0}^{\chi} d\chi \left[ \nabla^2 (\Phi_1 + \Phi_1) + n' n'(\Phi_1 + \Psi_1)_{ij} + \frac{2}{\chi} \frac{d\delta^{(1)}_g}{d\varsigma} \right]. \quad (3.9) \]

4. Comparison of the leading order terms

4.1. Comparison with Di Dio et al

In the following, we identify and match terms from reference [8] to equation (5.17) from reference [13]. We start by relating the notation of one to the other, the main notational differences between our work and reference [8] are:

- Work in the geodesic light-cone gauge (GLC) to obtain their solution which is then expanded in a more conventional gauge, Poisson gauge, to second order.
- Latin indices in the GLC take only the values 1, 2.
- Maintain their integrals in terms of the conformal time \( \eta \).
- Separate the final result into an isotropic and an anisotropic part.
- Projected notation is used \( v_{||} = n'v_i \).

\(^4\)For a work discussing the second-order density perturbation in the case of interacting vacuum cosmologies see reference [20].
• There is a factor of \((-1)\) in the definition of the observation vector \(n'\).

To second order, the main result is given by equation (4.41) in their paper,

\[
\Delta^{(2)} = \Sigma - \langle \Sigma \rangle, \tag{4.1}
\]

where

\[
\Sigma = \Sigma_{1S} + \Sigma_{AS}, \tag{4.2}
\]

and the isotropic, \(\Sigma_{1S}\), and anisotropic, \(\Sigma_{AS}\), parts are given in equations (4.42) and (4.43) of reference [8], respectively.

The translation of the majority of the terms is straightforward. \(r\) is the comoving distance we call \(\chi\) and \(\partial_{\eta} = d/d\eta = d/d\zeta - n'\partial_{\eta} = -d/d\chi\). Thankfully, the authors of reference [8] include a leading order second order velocity contribution in their paper, which is

\[
(2^{\text{nd}})_{\text{Di Dio}}^{\text{Leading}} \simeq \delta^{(2)}_{\rho} + \frac{1}{H} \left( \frac{d (v_{2n'})}{d\zeta} \right) - \frac{1}{2r} \int_{\eta_0}^{\eta} \psi_2 \frac{d\eta}{\eta_0 - \eta} \Delta_2 (\Psi_2 + \Phi_2)
\]

\[
+ \frac{1}{H} \left( v_{1n'} \right)^2 (v_{1n'})^2 + \left( \frac{d (v_{1n'})}{d\zeta} \right)^2 \right) - \frac{2}{H} \left( \frac{d (v_{1n'})}{d\zeta} \right) \int_{\eta_0}^{\eta} \psi_2 \frac{d\eta}{\eta_0 - \eta} \Delta_2 \psi_2
\]

\[
- \frac{1}{2} \partial_0 \left( \int_{\eta_0}^{\eta} \psi_2 \frac{d\eta}{\eta_0 - \eta} \right) \partial_0 \left( \int_{\eta_0}^{\eta} \psi_2 \frac{d\eta}{\eta_0 - \eta} \right)
\]

\[
+ \frac{4}{r} \int_{\eta_0}^{\eta} \psi_2 \left( \gamma_{00} \partial_0 \left( \int_{\eta_0}^{\eta} \psi_2 \frac{d\eta}{\eta_0 - \eta} \right) \right)
\]

\[
+ \frac{1}{H} \left( \frac{d (v_{1n'})}{d\zeta} \right) - \frac{2}{r} \int_{\eta_0}^{\eta} \psi_2 \frac{d\eta}{\eta_0 - \eta} \Delta_2 \psi_2 + \frac{1}{H} \left( v_{1n'} \right) \frac{d\delta^{(1)}}{d\zeta}
\]

\[
- 2 \partial_0 \delta^{(1)} \int_{\eta_0}^{\eta} \psi_2 \frac{d\eta}{\eta_0 - \eta} \Delta_2 \psi_2. \tag{4.3}
\]

After some integrations by parts to rewrite some terms, and rewriting what the angular Laplacian is in our notation, the only difference between our leading terms and the ones from reference [8] comes from the opposite sign in the definition of the observation vector \(n'\), and a numerical factor between the four velocities, one half.

\[
\Delta^{(2)}_{\text{Leading}} - (2^{\text{nd}})_{\text{Di Dio}}^{\text{Leading}} \simeq \left[ \delta^{(2)}_{\rho} - \delta^{(2)}_{\rho} \right] - \frac{1}{H} \left( \frac{d (v_{2n'})}{d\zeta} \right) \left[ \frac{1}{2} + 1 \right],
\]

\[
\simeq - \frac{1}{H} \left( \frac{d (v_{2n'})}{d\zeta} \right) \left[ \frac{1}{2} + 1 \right], \tag{4.4}
\]

where we can see explicitly that the difference comes from the opposite sign in the definition of the observation vector \(n'\), and a \((1/2)\)-factor in the definition of our second order velocity perturbation.
4.2. Comparison with Bertacca et al

We now proceed to identify the terms of references [9, 10], the main remarks about the notation here are:

- They work with cosmic rulers, then change to Poisson gauge.
- There is a factor of (−1) in the definition of the observation vector \( n' \).
- Projected derivatives, meaning \( \partial_i = n' \partial_i \) and \( \delta_{ij} = r^{-1} \delta_{ij} \).

The leading terms in the expression for the number counts, would be,

\[
(\Delta^{(2)}_{\text{Bertacca}})^{\text{Leading}} \approx \frac{\delta^{(2)}}{\xi} - \frac{1}{H} \frac{d^2}{dc^2} (v_{1m} n') - 2 \kappa^{(2)} + 4 [\kappa^{(1)}]^2 - 4 \delta^{(1)} \kappa^{(1)}
\]

\[
\approx \frac{\delta^{(1)}}{\xi} \frac{d^2}{dc^2} (v_{1m} n') + 2 \frac{d}{dc} \left[ \frac{d^2}{dc^2} (v_{1m} n') \right] - 2 \frac{d \delta^{(1)}}{dc} \Delta \ln a^{(1)}
\]

\[
\approx 2 \left( \frac{\delta^{(1)}}{\xi} \right) \frac{d^2}{dc^2} (v_{1m} n') - 2 \kappa^{(2)}
\]

The translation of the majority of the terms is straightforward, \( a^{(1)} \) is the first order perturbation of the scale factor taken as \( 1/(1 + z) \). To leading order, we can substitute \( \Delta \ln a^{(1)} = -\partial_\xi (v_{1m} n') \).

After several integrations by parts and translations between the leading expression in our notation and the authors from reference [9], the main difference comes in the definition of the convergence, where there is a numerical factor of difference. Note that we are not taking into account the so called ‘post-Born’ contributions.

\[
(\Delta^{(2)}_{\text{Bertacca}})^{\text{Leading}} - (\Delta^{(2)}_{\text{Leading}})^{\text{Leading}} \approx \frac{\delta^{(1)}}{\xi} \int_0^{\chi_s} d\chi \left[ \nabla^2 (\Phi_1 + \Psi_1) + n' n' (\Phi_1 + \Psi_1)_{ij} + \frac{2 \delta^{(1)\nu}}{\xi} \right] - 4 \delta^{(1)} \kappa^{(1)}
\]

\[
\approx 2 \delta^{(1)} \kappa^{(1)} - 4 \delta^{(1)} \kappa^{(1)} + \kappa^{(2)} - 2 \kappa^{(2)}
\]

In the second equality we used the definition given in reference [9] (see equations (209) and (217) therein), to rewrite our notation into their convergence, and explicitly show the difference.
that arises from the convergence, where it appears to be adding twice as much to the Galaxy overdensity.

This last equation indicates that the difference with reference [9] lies in factors of the first and second order lensing contributions, both of which represent relativistic effects. As we shall discuss below, the lensing terms may dominate the bispectrum signal at specific configurations, where the terms in equation (4.7) must be considered carefully to avoid spurious contributions.

4.3. Comparison with Yoo and Zaldarriaga

We now proceed to identify the terms of reference [11], which has to be put together from several different contributions since the full final result is not written down in closed form. The way the results are presented is one by one, so the main result—the Galaxy overdensity—is not written explicitly; it is only shown in terms of previously found expressions, that the reader needs to find and then ‘stitch-together’ in order to obtain the full expression. The main differences between our notation and reference [11] are:

- Latin indices go from 0 to 3, greek indices go from 1 to 3 (the opposite of our notation).
- Metric perturbations are called $A_{\alpha\beta}$ and $C_{\alpha\beta}$, where $C_{\alpha\beta} = \Psi_{\delta\alpha\beta}$ in our notation, for purely scalar perturbations. The second order perturbations are defined with no factor 1/2 at second order.
- All perturbation orders are left implicit.
- Work in a different basis that depends on angles, and leave some factors of $(\sin \theta)$ between expressions, which are not present in our derivation.

From equation (3.2), we have that for reference [11] the leading terms are

$$(2) \Delta_{\text{Yoo Leading}}^{(2)} = \delta_{g}^{(2)} + \delta_{V}^{(2)} + \delta_{g}^{(1)} \delta_{V}^{(1)}. \quad (4.8)$$

The first order volume perturbation in reference [11] is given by

$$\delta_{V}^{(1)} = -2 \kappa^{(1)} + H_{c} \partial_{c} \delta_{r}^{(1)} = -2 \kappa^{(1)} + \frac{1}{H} \frac{d (v_{1}^{n})}{d \varsigma}, \quad (4.9)$$

where $\partial_{c} = H_{c}^{-1} \partial_{c}$. In reference [11] the second order perturbation of the volume is

$$\delta_{V}^{(2)} = \delta D_{\text{L}}^{(2)} + H_{c} \partial_{c} \delta_{r}^{(2)} = 2 H_{c} \kappa^{(1)} \partial \delta_{r}^{(1)} + \Delta x^{(1)b} \partial_{b} \delta_{V}^{(1)} \delta_{V}^{(1)}, \quad (4.10)$$

where $\delta D_{\text{L}}^{(2)}$ is the luminosity distance, which is related to $\delta^{(2)} d_{A}$ through the Etherington’s reciprocity theorem, which is expressed as $d_{L} = (1 + z)^{2} d_{A}$. Lastly we need to expand

$$\Delta x^{(1)b} \partial_{b} \delta_{V}^{(1)} = -2 \nabla^{a} \Phi_{1} \nabla_{a} \kappa^{(1)} + \frac{1}{H} \nabla^{a} \Phi_{1} \nabla_{a} \frac{d (v_{1}^{n})}{d \varsigma} + \frac{1}{H^{2}} \left( \frac{d (v_{1}^{n})}{d \varsigma} \right) \left( \frac{d^{2} (v_{1}^{n})}{d \varsigma^{2}} \right). \quad (4.11)$$

All these terms are already accounted for in $\delta D_{\text{L}}^{(2)}$ and in $\delta_{r}^{(2)}$. To avoid the cumbersome rewriting of the full term, we refrain from rewriting the terms here. For reference they are given in equations (78) and (50) in reference [11], respectively. Here, in agreement with reference [12], it can be seen that with these two terms, there is a double-counting effect, so this is the
only difference between our leading terms, if we ignore this, then we are in agreement:

\[
\begin{align*}
\Delta_{\text{Leading}}^{(2)} & - \Delta_{\text{Yoo}}^{(2)} 
\simeq -2 \nabla^a \Phi \nabla_a \kappa^{(1)}(1) + \frac{1}{H} \nabla^a \Phi \nabla_a \frac{d(v_{1i} n^i)}{d\varsigma} \\
& + \frac{1}{H^2} \left( \frac{d(v_{1i} n^i)}{d\varsigma} \right) \left( \frac{d^2(v_{1i} n^i)}{d\varsigma^2} \right).
\end{align*}
\]

(4.12)

Thus we show explicitly that, while the leading terms are in complete agreement, the above terms are accounted for twice inside the luminosity distance and the radial perturbation.

Note that, in this case, our discrepancy with the approximation of reference [11] lies in three terms, each with an independent origin and effect in observables. The first term in equation (4.12) corresponds to the product of the first order lensing convergence times the transverse displacement. The second and third terms are contributions from the first order RSD combined with the transverse displacement, and a first order Taylor expansion of the RSD itself, correspondingly.

An important observable where the second order number counts are relevant is the matter bispectrum. The contribution from the dominant terms discussed in this section to such observable has been computed numerically in [21]. There, it is shown that for triangulations computed at a single redshift (or a narrow redshift bin), the result is dominated by standard Newtonian terms, that is the second order density and RSD terms of equations (3.4), (3.5) and their cross contribution in equation (3.7). However, for triangulations with vertices in more than one redshift, and redshifts separated by \( \Delta z > 0.1 \) the Newtonian terms rapidly decay and the bispectrum is dominated by the lensing contributions, mostly those of equation (3.6) (see also [22]). In this sense the differences of equation (4.7), between our number counts estimation and those of reference [9], would be manifest in the bispectrum calculated in triangles comprising more than one redshift bin. On the other hand, the RSD contributions to the difference with reference [11] may result in spurious contributions to the bispectrum computed at a single redshift. A more detailed, numerical comparison including all contributions to the second order number counts is left for future work.

5. Discussion

In this paper we performed a comparison of the main result of reference [13], the Galaxy number count at second order in perturbation theory, with other derivations in the literature. The approaches taken by different groups lead to slight differences beyond linear order. We provide a comparison of the leading terms at second order in section 4, finding that our approach and the previous works differ by numerical factors with one group, and that another group double count some of the effects, leading to a more significant difference with our expression. Once we take these differences into account and correct for these typos, our results are in agreement with the other groups (at leading order). Although we do not provide a full comparison of the complete expression, we will return to this issue in the future. Our result is in agreement with the previous comparison made in reference [12] concentrated on the terms which are dominant in powers of \( k/H \), i.e. of the order \( (k/H)^4 \).

Since the expressions found in the literature for the Galaxy number count agree with our result at leading order—once the different notations and conventions have been taken into account and allowing for the typos discussed in section 4—all of these results can be used for future and current surveys if the leading order meets the required precision. On the other hand, beyond leading order there are differences in the Galaxy number count results, which
must be carefully considered when trying to estimate the signal associated to primordial non-Gaussianity encoded in the powerspectrum and bispectrum (see e.g. reference [23]). To what extent such differences affect the signal estimation is a task best addressed numerically given the complexity of the expressions. This lies beyond the scope of the present paper and we shall return to this point in the future.

In conclusion, our comparison shows that the differences between our approach and previous studies presented in the literature lie mostly on some of the geometrical definitions used on each calculation, particularly with the direction of observation that could affect the description of the peculiar velocities, which could bring extra (or reduced) terms in the second order expansions, and result in the addition of double terms of the velocities perturbations when contrasted against our number counts. While such differences are not relevant yet, given the current precision in Galaxy catalogues, it is important to be prepared for future surveys.

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Data availability statement

The data that support the findings of this study are openly available at the following URL/DOI: 10.1088/1361-6382/abd95c.

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