Average Angular Velocity

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Abstract

This paper addresses the problem of the separation of rotational and internal motion. It introduces the concept of average angular velocity as the moment of inertia weighted average of particle angular velocities. It extends and elucidates the concept of Jellinek and Li (1989) of separation of the energy of overall rotation in an arbitrary (non-linear) \( N \)-particle system. It generalizes the so called Koenig’s theorem on the two parts of the kinetic energy (center of mass plus internal) to three parts: center of mass, rotational, plus the remaining internal energy relative to an optimally translating and rotating frame.

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1 Introduction

The motion of a rigid body is completely characterized by its (center of mass) translational velocity and its angular velocity which describes the rotational motion. Rotational motion as a phenomenon is, however, not restricted to rigid bodies and it is then a kinematic problem to define exactly what the rotational motion of the system is. This paper introduces the new concept of ‘average angular velocity’ as the solution this problem and discusses some applications briefly.

The average angular velocity concept is closely analogous to the concept of center of mass velocity. For a system of particles the center of mass velocity is simply the mass weighted average of the particle velocities. In a similar way the average angular velocity is the moment of inertia weighted average of the angular velocities of the particle position vectors.

The separation of rotational motion from the internal degrees of freedom of a system is of interest in a wide variety of applications. Among the more obvious are vibration-rotation coupling in polyatomic molecules and the understanding of biomechanical phenomena involving falling cats, figure skaters, springboard divers etc. The basic theoretical work on the subject is Carl Eckart’s (1935) whose work has been extended and elucidated by many authors, for example Sayvetz (1939) and Essén (1979). Biomechanical aspects have been discussed by Frohlich (1979) from a multibody dynamic point of view, and by Essén (1981) from the Eckart point of view.

Considering the maturity of the subject one might think that all basic theoretical results are quite old, but a letter on the subject was published as late as 1989 by Jellinek and Li (1989). They showed that one can define an angular velocity vector and separate out a rotational part of the energy, for an arbitrary (nonlinear) system of particles, without the use of the rigid reference configuration that is needed in the Eckart construction.

In this paper I present some new developments based on ideas related to those of Jellinek and Li. In particular I introduce the concept of *average angular velocity* as the weighted average of particle angular velocities with moments of inertia as weights. Especially for the elementary, but pedagogically important, case of fixed axis rotation this leads to simple and conceptually powerful results. The relevance of these results to spinning athletes and turning cats are briefly indicated.

Later sections of the paper treat the three dimensional case. It is shown that the separation of a rotational part of the kinetic energy can be done in
a way closely analogous to the well known split of the kinetic energy to an overall center of mass translational part plus the kinetic energy of internal motion relative to the center of mass system. A second split is thus done, now to an overall rotational energy plus a remaining part corresponding to motions in a rotating center of mass system, the rotation of which is given by the average angular velocity.

Throughout this paper I consider a system of \( N \) particles with masses \( m_k \) and position vectors \( \mathbf{r}_k \) \((k = 1, \ldots, N)\). A distance from the \( Z \)-axis is denoted by \( \rho \), and a distance from the origin by \( r \), the azimuthal angle is \( \phi \) and the angle to the positive \( Z \)-axis is \( \theta \).

## 2 Average Angular Velocity Around a Fixed Axis

The \( z \)-component of the angular momentum, \( L_z \), is by definition

\[
L_z = \sum_{k=1}^{N} m_k (\mathbf{r}_k \times \mathbf{v}_k) \cdot \mathbf{e}_z = \sum_{k=1}^{N} m_k (x_k \dot{y}_k - y_k \dot{x}_k). \tag{1}
\]

Here the \( Z \)-axis has a fixed direction and is either fixed in an inertial system or fixed in the center of mass of the system. We now introduce cylindrical (polar) coordinates \( \rho, \phi, \) and \( z \). In terms of these we have for the position vectors \( \mathbf{r}_k = \rho_k \mathbf{e}_{\rho_k} + z_k \mathbf{e}_z \) and for the velocities \( \dot{\mathbf{r}}_k = \dot{\rho}_k \mathbf{e}_{\rho_k} + \rho_k \dot{\phi}_k \mathbf{e}_{\phi_k} + \dot{z}_k \mathbf{e}_z \). This gives us

\[
(\mathbf{r}_k \times \mathbf{v}_k) \cdot \mathbf{e}_z = [ (\rho_k \mathbf{e}_{\rho_k} + z_k \mathbf{e}_z) \times (\dot{\rho}_k \mathbf{e}_{\rho_k} + \rho_k \dot{\phi}_k \mathbf{e}_{\phi_k} + \dot{z}_k \mathbf{e}_z) ] \cdot \mathbf{e}_z = \rho_k^2 \dot{\phi}_k, \tag{2}
\]

so that

\[
L_z(t) = \sum_{k=1}^{N} m_k \rho_k^2(t) \dot{\phi}_k(t). \tag{3}
\]

If we now define the average angular velocity of the system, around the \( Z \)-axis, by

\[
\omega_{av}(t) \equiv \langle \dot{\phi}(t) \rangle_t = \frac{\sum_{k=1}^{N} m_k \rho_k^2(t) \dot{\phi}_k(t)}{\sum_{k=1}^{N} m_k \rho_k^2(t)}, \tag{4}
\]

and the (instantaneous) moment of inertia, with respect to the \( Z \)-axis, by

\[
J_z \equiv \sum_{k=1}^{N} m_k \rho_k^2 \tag{5}
\]
we see that we get
\[ L_z = J_z \omega_{av}. \] (6)

If all particles have the same angular velocity, \( \dot{\phi} \), then, of course, \( \omega_{av} = \dot{\phi} \). This happens, in particular, if the system of particles is rigid and rotates around the \( Z \)-axis, but also more generally for any type of motion that obeys \( \phi_k = \dot{\phi} \) with arbitrary \( \rho_k \) and \( z_k \). For these cases one gets the standard result that \( L_z = J_z \dot{\phi} \).

We can now apply these results to the \( z \)-component of the angular momentum principle \( (\dot{L} = M) \) in the form
\[ \dot{L}_z = M_z. \] (7)

The general result is that
\[ \dot{J}_z \omega_{av} + J_z \dot{\omega}_{av} = M_z. \] (8)

If the angular velocity is well defined we can replace \( \omega_{av} \) by \( \dot{\phi} \) in this expression. If we furthermore assume that the body is rigid so that \( J_z = \text{constant} \), equation (8) reduces to the standard result \( J_z \dot{\phi} = M_z \).

If there is no external moment (or ‘torque’) with respect to the \( Z \)-axis, so that \( M_z = 0 \), then the \( z \)-component of the angular momentum vector will be conserved, \( L_z = \text{constant} \), and, in view of equation (6), we find that
\[ J_z(t) \omega_{av}(t) = \text{constant}. \] (9)

This then says that a large moment of inertia implies small average angular velocity and vice versa. Here it is not assumed that there is rigidity or even a definite angular velocity. It is this more general form of the standard textbook formula, \( J_z \dot{\phi} = \text{constant} \), that is actually ‘used’ by springboard divers and figure skaters.

### 3 The Cat Landing on its Feet

For some arbitrary quantity \( \gamma_k \) the averaging of equation (4) can be written
\[ \langle \gamma(t) \rangle_t \equiv \frac{\sum_{k=1}^N m_k \rho_k^2(t) \gamma_k(t)}{\sum_{k=1}^N m_k \rho_k^2(t)}. \] (10)

A question of interest is to what extent the average angular velocity can be understood as the time derivative of the ‘average angle’, \( \langle \phi \rangle_t \), of the system.
The subscript \( t \) on the averaging bracket is meant as a reminder that the weights in the averaging, \( m_k \rho_k^2(t) \), are time dependent and this means that the time derivative of an average will not be the same as the average of a time derivative.

If we take the time derivative of

\[
\varphi_{av} \equiv \langle \varphi \rangle_t
\]  

(11)
a simple calculation shows that

\[
\omega_{av} = \frac{d\varphi_{av}}{dt} + 2\langle \dot{\rho} (\varphi_{av} - \varphi) \rangle_t.
\]  

(12)

The average angular velocity is thus not simply the time derivative of the average angle. This is, of course, essential if a cat, dropped upside-down, is to be able to land on its feet. Equation (11) shows that if \( \omega_{av} = 0 \) initially, it will remain zero in the absence of external torque. The above equation reassures one that the cat, nevertheless, can change its average angle by a proper combination of angular and radial motions.

4 The Average Angle Concept

The concept of an ‘average angle’ requires some comment. The value of this angle will, of course, depend on the direction of the fixed reference direction (the \( X \)-axis). It will also depend on whether one thinks of \( \varphi \) as going from \(-\pi\) to \( \pi \) or if it goes from 0 to \( 2\pi \), when one assigns initial values to the \( \varphi_k \). That is, \( \varphi_{av} \) depends on whether the necessary \( 2\pi \) jump comes at the negative or at the positive \( X \)-axis, respectively. For a cylinder the initial average angle will be zero in the former case and \( \pi \) in the latter. The actual value of the average angle therefore has little physical meaning; its significance comes from the fact that it defines a reference direction in the particle system. Then, when the system has moved, it will tell how large, on average, the net turn has been.

The time dependence of the averaging naturally vanishes if the radii, \( \rho_k \), are constant. It is interesting to note that it also vanishes in the more general case when the time dependencies of the (cylindrical) radii are of the form

\[
\rho_k(t) = f(t) d_k,
\]  

(13)
where \( f(t) \) is some (positive) function of time and the \( d_k \) are constants. The average then becomes

\[
\langle \gamma(t) \rangle_t = \frac{\sum_{k=1}^{N} m_k f^2(t) d_k^2 \gamma_k(t)}{\sum_{k=1}^{N} m_k f^2(t) d_k^2} = \frac{\sum_{k=1}^{N} m_k d_k^2 \gamma_k(t)}{\sum_{k=1}^{N} m_k d_k^2} = \langle \gamma(t) \rangle.
\]  

For this case then, when the cylindrical radial motion is a common ‘scaling’, the time derivative operator commutes with the operation of taking the average. Consequently the average angular velocity will be the time derivative of the average angle and similarly for angular acceleration.

## 5 König’s Theorem

The kinetic energy of an \( N \)-particle system is given by the sum

\[
T = \frac{1}{2} \sum_{k=1}^{N} m_k \dot{r}_k \cdot \dot{r}_k.
\]  

If one introduces the center of mass position vector

\[
\mathbf{R} \equiv \frac{\sum_{k=1}^{N} m_k \mathbf{r}_k}{\sum_{k=1}^{N} m_k},
\]

and then re-writes the position vectors of the particles as

\[
\mathbf{r}_k = \mathbf{R} + \mathbf{r}'_k,
\]

the kinetic energy is seen to fall into two parts. One part corresponds to the motion of the center of mass of the system while the remaining part is due to the motion of the particles relative to the center of mass system:

\[
T = \frac{1}{2} m \dot{\mathbf{R}} \cdot \dot{\mathbf{R}} + T'.
\]

Here \( m = \sum_{k=1}^{N} m_k \) is the total mass and \( T' \), which is given by

\[
T' = \frac{1}{2} \sum_{k=1}^{N} m_k \dot{\mathbf{r}}'_k \cdot \dot{\mathbf{r}}'_k,
\]

vanishes if the particles do not move relative to a reference frame in which the center of mass is at rest. The absence of cross terms is due to the
The sums, \( \sum_{k=1}^{N} m_k \mathbf{r}_k' = \sum_{k=1}^{N} m_k \mathbf{r}_k' = \mathbf{0} \), over the center of mass system position vectors and velocities, vanish. This result is sometimes called König’s theorem (see Synge and Griffith 1970), or the ‘law of the two parts of the kinetic energy’, if it is given any name at all. It is this result that can be taken a step further according to Jellinek and Li (1989) in the sense that \( T' \) can be split into two parts, one corresponding to an overall rotation and the rest corresponding to the motion relative to the rotating system. This is shown in the next section.

6 The Average Angular Velocity Vector

The quantities of this section may be thought of as referring to the center of mass system but we will drop the primes of the previous section. Introduce spherical coordinates \((r_k, \theta_k, \varphi_k)\) for particle \(k\) and corresponding moving basis vectors \((e_{r_k}, e_{\theta_k}, e_{\varphi_k})\). The position vector, \(\mathbf{r}_k\), of particle \(k\) is then

\[
\mathbf{r}_k = r_k e_{r_k}.
\]

and the velocity of the particle is

\[
\dot{\mathbf{r}}_k = \dot{r}_k e_{r_k} + \mathbf{\omega}_k \times \mathbf{r}_k.
\]

Here \(\mathbf{\omega}_k\) is the angular velocity of the position vector of particle \(k\). It is given by

\[
\mathbf{\omega}_k = \dot{\varphi}_k \cos \theta_k e_{r_k} - \dot{\varphi}_k \sin \theta_k e_{\theta_k} + \dot{\theta}_k e_{\varphi_k} = \dot{\varphi}_k e_z + \dot{\theta}_k e_{\varphi_k}.
\]

The kinetic energy of the \(N\)-particle system is now

\[
T' = \frac{1}{2} \sum_{k=1}^{N} m_k (\dot{r}_k e_{r_k} + \mathbf{\omega}_k \times \mathbf{r}_k) \cdot (\dot{r}_k e_{r_k} + \mathbf{\omega}_k \times \mathbf{r}_k)
\]

\[
= \frac{1}{2} \sum_{k=1}^{N} m_k \dot{r}_k^2 + \frac{1}{2} \sum_{k=1}^{N} m_k (\mathbf{\omega}_k \times \mathbf{r}_k) \cdot (\mathbf{\omega}_k \times \mathbf{r}_k)
\]

\[
= \frac{1}{2} \sum_{k=1}^{N} m_k \dot{r}_k^2 + \frac{1}{2} \sum_{k=1}^{N} \hat{J}_k \mathbf{\omega}_k \cdot \mathbf{\omega}_k
\]

where \(\hat{J}_k\) is the contribution of particle \(k\) to the (instantaneous) inertia tensor \(\hat{J}\) of the system. The matrix components of the inertia tensor \(\hat{J}_k\) in the basis
\( (\mathbf{e}_{r_k}, \mathbf{e}_{\theta_k}, \mathbf{e}_{\phi_k}) \) are given by

\[
\hat{J}_k = \begin{pmatrix}
0 & 0 & 0 \\
0 & m_k r_k^2 & 0 \\
0 & 0 & m_k r_k^2 \\
\end{pmatrix}
\]  

(26)

Using this one easily verifies that the sum of the \( k \)th terms of the sums of formula (25) gives the usual expression for the kinetic energy of particle \( k \) in spherical coordinates: \( T_k = \frac{1}{2} m_k [\dot{r}_k^2 + r_k^2 (\dot{\phi}_k^2 \sin \theta_k + \dot{\theta}_k^2)] \).

Below we will need to manipulate sums of terms like those in the above expression for the kinetic energy. One must then remember that the position dependent basis vectors \((\mathbf{e}_{r_k}, \mathbf{e}_{\theta_k}, \mathbf{e}_{\phi_k})\) are different for each particle (as indicated by the index \( k \)). In order to proceed we therefore return to a common Cartesian basis in the expression

\[
\frac{1}{2} \sum_{k=1}^{N} \hat{J}_k \mathbf{\omega}_k \cdot \mathbf{\omega}_k
\]

and the sum of these gives

\[
\hat{J} = \sum_{k=1}^{N} \hat{J}_k = \sum_{k=1}^{N} m_k \begin{pmatrix}
y_k^2 + z_k^2 & -x_k y_k & -x_k z_k \\
-x_k y_k & x_k^2 + z_k^2 & -y_k z_k \\
-z_k y_k & -y_k z_k & x_k^2 + y_k^2 \\
\end{pmatrix}
\]  

(28)

i.e. the usual (instantaneous) inertia tensor for the system of particles.

We now define the average angular velocity vector, \( \mathbf{\omega}_{av} \), by

\[
\hat{J} \mathbf{\omega}_{av} = \sum_{k=1}^{N} \hat{J}_k \mathbf{\omega}_k,
\]  

(29)

as the inertia tensor weighted average of the individual particle angular velocity vectors. Here it is necessary that \( \hat{J} \) is invertible so that one can solve for \( \mathbf{\omega}_{av} \) by multiplying to the left by \( \hat{J}^{-1} \). This means that the particles of the system may not lie on a line since then the inertia tensor is singular.

If we now denote by \( \mathbf{\omega}_k' \) the angular velocity vector of particle \( k \) relative to the reference system that rotates with the average angular velocity we have

\[
\mathbf{\omega}_k = \mathbf{\omega}_{av} + \mathbf{\omega}_k',
\]  

(30)
since angular velocity vectors are additive. These relative angular velocities fulfill
\[ \sum_{k=1}^{N} \tilde{J}_k \omega'_k = 0, \]  
(31)
so a calculation completely analogous to that leading to König’s theorem gives
\[ T' = \frac{1}{2} \hat{J} \omega_{av} \cdot \omega_{av} + T''. \]  
(32)

Here
\[ T'' = \frac{1}{2} \sum_{k=1}^{N} m_k \dot{r}_k^2 + \frac{1}{2} \sum_{k=1}^{N} \tilde{J}_k \omega'_k \cdot \omega'_k = \frac{1}{2} \sum_{k=1}^{N} m_k \mathbf{v}_k'' \cdot \mathbf{v}_k'' \]  
(33)
is the kinetic energy relative to a reference frame that rotates with the average angular velocity (around the fixed center of mass) and \( \mathbf{v}_k'' \) is the velocity of particle \( k \) as measured in this frame.

7 Remarks on the Conservation Laws

One notes that formula (29) is nothing but the total angular momentum, \( \mathbf{L} \), of the system (with respect to the center of mass):
\[ \hat{J} \omega_{av} = \sum_{k=1}^{N} \tilde{J}_k \omega_k = \mathbf{L}. \]  
(34)

It is thus analogous to the formula
\[ m \ddot{\mathbf{R}} = \sum_{k=1}^{N} m_k \ddot{\mathbf{r}}_k = \mathbf{p} \]  
(35)
for the total linear momentum, \( \mathbf{p} \), of the system. Using the linear and angular momenta the total kinetic energy of a system can now be written
\[ T = \frac{1}{2m} \mathbf{p} \cdot \mathbf{p} + \frac{1}{2} \hat{J}^{-1} \mathbf{L} \cdot \mathbf{L} + T''. \]  
(36)

Here \( m \) is the total mass, which is always constant, and \( \mathbf{p} \) the total linear momentum, which is constant in the absence of external force. Thus in the absence of a net external force on the system the first term in this expression for \( T \) is constant. What about the second term? In the absence of a net
external moment (of force) on the system $\mathbf{L}$ is constant, but the (inverse) inertia tensor $\hat{J}^{-1}$ is, in general, not since it depends on the ‘shape’ of the system.

In an isolated body, such as a star or planet, one can expect that internal dissipative forces, in the long run, will make the internal relative motions zero so that $T'' \rightarrow 0$. Such bodies will thus end up having only center of mass translational and average rotational energy and both will then be constant.

8 Conclusions

The fixed axis formalism at the beginning of this paper is simple and useful enough to be included in even fairly elementary texts, though I believe it is presented here for the first time in the international literature. I have introduced part of it in course notes that are used in the school of engineering physics at KTH. The point of view that the Jellinek and Li (1989) separation of a rotational part of the kinetic energy is analogous to the well known separation of center of mass translational energy, is pedagogically useful at higher levels, and the general ideas deserve to be better known. It is the opinion of the author that most of the equations and results of this paper, in fact, belong in a comprehensive advanced mechanics course.
Acknowledgments
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References

[1] Eckart C 1935 Some Studies Concerning Rotating Axes and Polyatomic Molecules Phys. Rev. 47 552-558

[2] Essén H 1979 Vibration-Rotation Coupling in Polyatomic Molecules; Additions to the Eckart Conditions Chem. Phys. 44 373-388

[3] Essén H 1981 The Cat Landing on its Feet Revisited or Angular Momentum Conservation and Torque-free Rotations of Nonrigid Mechanical Systems Am. J. Phys. 49 756-758

[4] Frohlich C 1979 Do Springboard Divers Violate Angular Momentum Conservation? Am. J. Phys. 47 583-592

[5] Jellinek J and Li D H 1989 Separation of the Energy of Overall Rotation in Any N-Body System Phys. Rev. Lett. 62 241-244

[6] Sayvetz A 1939 The Kinetic Energy of Polyatomic Molecules J. Chem. Phys. 7 383-389

[7] Synge J L and Griffith B A 1970 Principles of Mechanics (McGraw-Hill Book Co, Singapore) third edition, pp. 177-178.