ON UNIMODULAR MULTILINEAR FORMS WITH SMALL NORMS ON SEQUENCE SPACES

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Abstract. The Kahane–Salem–Zygmund inequality is a probabilistic result that guarantees the existence of special matrices with entries 1 and −1 generating unimodular m-linear forms $A_{m,n} : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{R}$ (or $\mathbb{C}$) with relatively small norms. The optimal asymptotic estimates for the smallest possible norms of $A_{m,n}$ when $\{p_1, \ldots, p_m\} \subset [2, \infty]$ and when $\{p_1, \ldots, p_m\} \subset [1, 2)$ are well-known and in this paper we obtain the optimal asymptotic estimates for the remaining case: $\{p_1, \ldots, p_m\}$ intercepts both $[2, \infty]$ and $[1, 2)$. In particular we prove that a conjecture posed by Albuquerque and Rezende is false and, using a special type of matrices that dates back to the works of Toeplitz, we also answer a problem posed by the same authors.

1. Introduction

Let $\mathbb{K}$ be the real or complex scalar field. The Kahane–Salem–Zygmund inequality (see [3, 4]) asserts that for positive integers $m, n$ and $p_1, \ldots, p_m \in [2, \infty]$, there exist a universal constant $C$ (depending only on $m$), a choice of signs 1 and $-1$, and an $m$-linear form $A_{m,n} : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$ of the type

$$A_{m,n}(z^{(1)}, \ldots, z^{(m)}) = \sum_{j_1, \ldots, j_m=1}^n \pm z_{j_1}^{(1)} \cdots z_{j_m}^{(m)},$$

such that

$$\|A_{m,n}\| \leq C n^{\frac{m+1}{2} - \frac{1}{p_1} - \cdots - \frac{1}{p_m}}.$$

An interpolation argument shows that if $p_1, \ldots, p_m \in [1, 2]$, there is a universal constant $C$ (depending only on $m$), and an $m$-linear form as above such that

$$\|A_{m,n}\| \leq C n^{1 - \frac{1}{\max\{p_1, \ldots, p_m\}}}.$$

The above estimate appears is essence in Bayart’s paper [2]. Both the multilinear and polynomial versions of the Kahane–Salem–Zygmund inequalities play a fundamental role in modern Analysis (see, for instance, [3, 5, 9] and the references therein). However, to the best of the authors’ knowledge, despite the existence of more involved abstract generalizations of the Kahane–Salem–Zygmund inequality (see [8]), the best estimate (i.e., the smallest possible exponent for $n$) for the general case ($p_1, \ldots, p_m \in [1, \infty]$) of sequence spaces is still unknown. Recently, Albuquerque and Rezende (11) have proved that, for $p_1, \ldots, p_m \in [1, \infty]$, there is a universal constant $C$ (depending only in $m$) and an $m$-linear form as above satisfying

$$\|A_{m,n}\| \leq C n^{1 - \frac{1}{\max\{p_1, \ldots, p_m\}} - \frac{1}{\gamma}},$$

with

$$\gamma := \min\{2, \max\{p_k : p_k \leq 2\}\}.$$
Note that this last estimate encompasses the previous ones. In this note we obtain the optimal solution to the general case:

**Theorem 1.1.** Let $m, n$ be positive integers and $p_1, \ldots, p_m \in [1, \infty]$. Then there exist a universal constant $C$ (depending only on $m$), a choice of signs $1$ and $-1$ and an $m$-linear form $A_{m,n} : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to K$ of the type

$$A_{m,n}(z^{(1)}, \ldots, z^{(m)}) = \sum_{j_1, \ldots, j_m = 1}^n \pm z_{j_1}^{(1)} \cdots z_{j_m}^{(m)},$$

such that

$$\|A_{m,n}\| \leq C n^{\min\left\{ \max\{2, p_k^*\} \right\}^{1} + \sum_{k=1}^m \max\left\{ \frac{1}{2} - \frac{1}{p_k}, 0 \right\}},$$

where $p_k^*$ is the conjugate of $p_k$. Moreover, the exponent $\min\left\{ \max\{2, p_k^*\} \right\}^{1} + \sum_{k=1}^m \max\left\{ \frac{1}{2} - \frac{1}{p_k}, 0 \right\}$ is optimal.

2. The proof

We begin by recalling the following estimate obtained by Albuquerque and Rezende:

**Theorem 2.1.** (see [1]) Let $m, n_1, \ldots, n_m$ be positive integers and $p_1, \ldots, p_m \in [1, \infty]$. Then there exist a constant $C$ (depending only on $m$), a choice of signs $1$ and $-1$, and an $m$-linear form $A : \ell_{p_1}^{n_1} \times \cdots \times \ell_{p_m}^{n_m} \to K$ of the form

$$A(z^1, \ldots, z^m) = \sum_{j_1 = 1}^{n_1} \cdots \sum_{j_m = 1}^{n_m} \pm z_{j_1}^{j_1} \cdots z_{j_m}^{j_m},$$

such that

$$\|A\| \leq C \left( \sum_{k=1}^m n_k \right)^{1 - \frac{1}{2}} \prod_{k=1}^m n_k^{\max\left\{ \frac{1}{2} - \frac{1}{p_k}, 0 \right\}},$$

with $\gamma := \min\{2, \max\{p_k : p_k \leq 2\}\}$.

2.1. **Proof of the inequality** [1]. We shall prove [1] following the more general environment of the above result. We will show that for positive integers $m, n_1, \ldots, n_m$ and $p_1, \ldots, p_m \in [1, \infty]$, there is a universal constant (depending only on $m$), and a $m$-linear form $A : \ell_{p_1}^{n_1} \times \cdots \times \ell_{p_m}^{n_m} \to K$ of the form

$$A(z^1, \ldots, z^m) = \sum_{j_1 = 1}^{n_1} \cdots \sum_{j_m = 1}^{n_m} \pm z_{j_1}^{j_1} \cdots z_{j_m}^{j_m},$$

such that

$$\|A\| \leq C \left( \sum_{k=1}^m n_k \right)^{1 - \frac{1}{2}} \prod_{k=1}^m n_k^{\max\left\{ \frac{1}{2} - \frac{1}{p_k}, 0 \right\}},$$

with

$$\rho := \min_{k} \left\{ \max\{2, p_k^*\} \right\}.$$

If $p_k \geq 2$, for all $k = 1, \ldots, m$, our estimate coincides with the ones of Theorem 2.1. The same happens when $p_k < 2$ for all $k = 1, \ldots, m.$
Finally, let us suppose (with no loss of generality) that $1 \leq d < m$, and $p_k \geq 2$, for all $k = 1, \ldots, d$ and $p_k < 2$ for $k = d+1, \ldots, m$. Theorem 2.1 guarantees the existence of an $m$-linear form $A : \ell_{p_1}^{n_1} \times \cdots \times \ell_{p_d}^{n_d} \times \ell_2^{n_{d+1}} \times \cdots \times \ell_{p_m}^{n_m} \to \mathbb{K}$ such that

$$
\|A\|_{L\left(\ell_{p_1}^{n_1} \times \cdots \times \ell_{p_d}^{n_d} \times \ell_2^{n_{d+1}} \times \cdots \times \ell_{p_m}^{n_m}; \mathbb{K}\right)} \leq C \left(\sum_{k=1}^{m} n_k\right)^{\frac{1}{2} \prod_{k=1}^{m} n_k} \max\left\{\frac{1}{2} - \frac{1}{p_k}, 0\right\}.
$$

On the other hand, for each $k \notin \{1, \ldots, d\}$, by the monotonicity of the $\ell_p$ norms, the restriction of this form to $\ell_{p_1}^{n_1} \times \cdots \times \ell_{p_d}^{n_d} \times \ell_2^{d+1} \times \cdots \times \ell_{p_m}^{n_m}$ has norm

$$
\|A\|_{L\left(\ell_{p_1}^{n_1} \times \cdots \times \ell_{p_d}^{n_d} \times \ell_2^{d+1} \times \cdots \times \ell_{p_m}^{n_m}; \mathbb{K}\right)} \leq \|A\|_{L\left(\ell_{p_1}^{n_1} \times \cdots \times \ell_{p_d}^{n_d} \times \ell_2^{n_{d+1}} \times \cdots \times \ell_{p_m}^{n_m}; \mathbb{K}\right)} \leq C \left(\sum_{k=1}^{m} n_k\right)^{\frac{1}{2} \prod_{k=1}^{m} n_k} \max\left\{\frac{1}{2} - \frac{1}{p_k}, 0\right\}.
$$

Note that in this case

$$
\rho := \min_k \{\max\{2, p_k\}\} = 2.
$$

Considering $n_1 = \cdots = n_m = n$ we obtain the proof of (1).

2.2. Proof of the optimality. The optimality of the case $p_k \geq 2$ for all $k \in \{1, \ldots, m\}$ is well-known (it is a consequence of the Hardy–Littlewood inequalities) and the constant involved does not depend on $p_1, \ldots, p_m$.

More precisely, for all unimodular forms we have

$$
\|A\| \geq \frac{1}{(\sqrt{2})^{m-1} n^{\frac{1}{2} \left(\frac{1}{2} - \frac{1}{p_1}\right) + \cdots + \left(\frac{1}{2} - \frac{1}{p_m}\right)}}.
$$

It remains only to prove the optimality of the exponents in the case in which at least one of the $p_k$ is smaller than 2. We shall split the proof in three cases:

- First case: $p_k < 2$, for all $k = 1, \ldots, m$.
- Second case: $p_k \geq 2$ for only one $k \in \{1, \ldots, m\}$.
- Third case: the complement of the previous cases.

The optimality of the first case seems to be folklore, but for the sake of completeness we shall provide a proof. In the first case the exponent of $n$ is

$$
\frac{1}{\rho} = \min_k \{\max\{2, p_k\}\} = \frac{p_j - 1}{p_j},
$$

where

$$
p_j := \max_k p_k.
$$

There is no loss of generality in supposing $j = m$. In the second case (we can also suppose $k = m$), the exponent of $n$ is also $\frac{p_m}{p_m-1}$. For all $m$-linear forms $A : \ell_{p_1}^{n_1} \times \cdots \times \ell_{p_m}^{n_m} \to \mathbb{K}$, we have

$$
\sup_{j_1, \ldots, j_m-1} \left(\sum_{j_m=1}^{n} |A(e_{j_1}, \ldots, e_{j_m})|^{p_m-1}_{p_m} \right)^{\frac{p_m-1}{p_m}} \leq \|A\| \sup_{\varphi \in B_{\ell_{p_m}^{n_m}}} \left(\sum_{j_m=1}^{n} |\varphi(e_{j_m})|^{p_m-1}_{p_m} \right)^{\frac{p_m-1}{p_m}} \leq \|A\|.
$$

Thus, for all unimodular $m$-linear forms $A : \ell_{p_1}^{n_1} \times \cdots \times \ell_{p_m}^{n_m} \to \mathbb{K}$, we have

$$
\|A\| \geq n^{\frac{p_m-1}{p_m}},
$$

and this guarantees the optimality of the exponent for the first and second cases.
It remains to prove the $m$-linear case when at least two $p_i \in [2, \infty]$ and at the same time at least one $p_i \in [1, 2)$.

We shall proceed by induction on $m$. The case of bilinear forms is completed by the previous steps. So, let us suppose that the result is valid for $(m-1)$-linear forms and let us prove for $m$-linear forms. So, our induction hypothesis is that for all $p_i \in [1, \infty]$ and $i = 1, \ldots, m-1$ we have (for all unimodular forms $A : \ell^m_{p_1} \times \cdots \times \ell^m_{p_{m-1}} \to \mathbb{K}$)

$$
\|A\| \geq D_{m-1} n_{\min} \left\{ \max \{2, p_k^* \} \right\}^{\sum_{k=1}^{m-1} \max \left\{ \frac{1}{2} - \frac{1}{p_k}, 0 \right\}} + \sum_{k=1}^{m-1} \max \left\{ \frac{1}{2} - \frac{1}{p_k}, 0 \right\}
$$

and we want to prove that (for all unimodular forms $A : \ell^m_{p_1} \times \cdots \times \ell^m_{p_m} \to \mathbb{K}$) we have

$$
\|A\| \geq D_{m-1} n_{\min} \left\{ \max \{2, p_k^* \} \right\}^{\sum_{k=1}^{m} \max \left\{ \frac{1}{2} - \frac{1}{p_k}, 0 \right\}}.
$$

Recalling that it just remains to prove the case when at least two $p_i \in [2, \infty]$ and at the same time at least one $p_i \in [1, 2)$, we have

$$
\rho = \min_k \{ \max \{2, p_k^* \} \} = 2.
$$

So, we shall prove that for all unimodular $m$-linear forms $A : \ell^m_{p_1} \times \cdots \times \ell^m_{p_m} \to \mathbb{K}$ (at least two $p_i \in [2, \infty]$ and at the same time at least one $p_i \in [1, 2)$) we have

$$
\|A\| \geq D_m n_{\min}^{\frac{1}{2} + \sum_{k=1}^{m} \max \left\{ \frac{1}{2} - \frac{1}{p_k}, 0 \right\}}.
$$

We can suppose that $p_m \in [1, 2)$. In this case, for any unimodular $m$-linear form $A : \ell^m_{p_1} \times \cdots \times \ell^m_{p_m} \to \mathbb{K}$ we have, by the Induction Hypothesis,

$$
\|A\| \geq \sup \left\{ \left\| A \left( x^{(1)}_{j_1}, \ldots, x^{(m-1)}_{j_{m-1}}, (1, 0, \ldots, 0) \right) \right\| : \sum_{j_k=1}^{\infty} \| x^{(k)}_{j_k} \|^{p_k} \leq 1 \text{ for all } 1 \leq k \leq m-1 \right\}
$$

$$
\geq D_{m-1} n_{\min}^{\frac{1}{2} + \sum_{k=1}^{m-1} \max \left\{ \frac{1}{2} - \frac{1}{p_k}, 0 \right\}}
$$

$$
= D_{m-1} n_{\min}^{\frac{1}{2} + \sum_{k=1}^{m} \max \left\{ \frac{1}{2} - \frac{1}{p_k}, 0 \right\}}.
$$

### 3. The Conjecture of Albuquerque–Rezende is False

The following conjecture was proposed by Albuquerque and Rezende (see [1, Conjecture 3.3]):

**Conjecture 3.1.** Let $p_1, \ldots, p_m \in [1, \infty]$. There exist $B_m, C_m > 0$ (depending only on $m$) such that

$$
(4) \quad B_m \leq \inf \left( \frac{\|A\|}{\left( \sum_{k=1}^{m} \frac{1}{n_k} \right)^{\frac{1}{2}} \cdot \prod_{k=1}^{m} n_k^{\max \left\{ \frac{1}{2} - \frac{1}{p_k}, 0 \right\}}} \right) \leq C_m,
$$

with $\gamma := \min \{ 2, \max \{ p_k : p_k \leq 2 \} \}$, and the infimum is calculated over all unimodular $m$-linear forms $A : \ell^m_{p_1} \times \cdots \times \ell^m_{p_m} \to \mathbb{K}$ and the exponents involved are sharp.

Note that the estimate (3) shows that the conjecture is false. In fact, for the sake of illustration, let us choose $m = 3$, $p_1 = 3/2$ and $p_2 = p_3 = 3$. By (3) there is a universal constant $C$ such that for all $n_1, n_2, n_3$ there exist a unimodular trilinear form $A : \ell^{n_1}_{p_1} \times \ell^{n_2}_{p_2} \times \ell^{n_3}_{p_3} \to \mathbb{K}$ satisfying

$$
\|A\| \leq C \left( n_1 + n_2 + n_3 \right)^{1/2} \left( n_2 \right)^{1/3} \left( n_3 \right)^{1/6}.
$$
Thus, if (1) was valid, we would have

$$0 < 2 \left( \frac{n_1 + n_2 + n_3}{n_1 + n_2 + n_3} \right)^{1/2} \frac{n_1}{n_1 + n_2 + n_3}^{1/6}$$

for all $n_1, n_2, n_3$, and this is impossible.

We end this paper by answering a problem posed in [1] for complex-valued versions of the Kahane–Salem–Zygmund inequality. More precisely, in [1, Problem 3.6] the authors ask about the constants involved in complex-valued versions of the Kahane–Salem–Zygmund inequality, i.e., when the coefficients 1 and $-1$ are replaced by complex numbers with modulo 1. We shall show that in the bilinear case the former constant can be replaced by 1.

Let $p_1, p_2 \geq 2$ and $n$ such that $n = \max\{n_1, n_2\}$. Borrowing ideas that date back to Toeplitz [10] and Littlewood [7] (see also [6, page 609]), we consider a $n \times n$ matrix $(a_{ij})$ defined by

$$a_{ij} = e^{2\pi i \frac{ij}{n}}.$$ 

Note that

$$\sum_{t=1}^{n} a_{rt} a_{st} = n\delta_{rs}.$$ 

Define $A : \ell_{p_1}^{n_1} \times \ell_{p_2}^{n_2} \to \mathbb{C}$ by

$$A(x^{(1)}, x^{(2)}) = \sum_{i_1, i_2=1}^{n} a_{i_1 i_2} x^{(1)}_{i_1} x^{(2)}_{i_2}.$$ 

Let $x^{(1)} \in B_{\ell_{p_1}^{n_1}}$ and $x^{(2)} \in B_{\ell_{p_2}^{n_2}}$, where $B_{\ell_{p_1}^{n_1}}$ and $B_{\ell_{p_2}^{n_2}}$ are the closed unit balls of $\ell_{p_1}^{n_1}$ and $\ell_{p_2}^{n_2}$, respectively. Then, completing with zeros, if necessary, consider $y^{(1)} = (x^{(1)}_1, \ldots, x^{(1)}_{n_1}, 0, \ldots, 0)$ and $y^{(1)} = (x^{(2)}_1, \ldots, x^{(2)}_{n_2}, 0, \ldots, 0)$ in $B_{\ell_{p_1}^{n_1}}$ and $B_{\ell_{p_2}^{n_2}}$. Using the Hölder inequality, we have

$$|A(x^{(1)}, x^{(2)})| \leq \sum_{i_2=1}^{n_2} \left| \sum_{i_1=1}^{n} a_{i_1 i_2} y^{(1)}_{i_1} \right| |y^{(2)}_{i_2}|$$

$$\leq \left( \sum_{i_2=1}^{n_2} |y^{(2)}_{i_2}|^2 \right)^{1/2} \left( \sum_{i_2=1}^{n_2} \left| \sum_{i_1=1}^{n} a_{i_1 i_2} y^{(1)}_{i_1} \right|^2 \right)^{1/2}$$

$$= \left( \sum_{i_2=1}^{n_2} |x^{(2)}_{i_2}|^2 \right)^{1/2} \left( \sum_{i_2=1}^{n_2} \left| \sum_{i_1=1}^{n} a_{i_1 i_2} y^{(1)}_{i_1} \right|^2 \right)^{1/2}$$

$$\leq \left( \sum_{i_2=1}^{n_2} |1|^{2/2} \left( \sum_{i_2=1}^{n_2} |x^{(2)}_{i_2}|^{p_2} \right)^{1/p_2} \left( \sum_{i_2=1}^{n_2} \left| \sum_{i_1=1}^{n} a_{i_1 i_2} y^{(1)}_{i_1} \right|^2 \right)^{1/2} \right)^{1/2}$$

$$\leq \frac{n_2^{1/(2-p)}}{p_2} \left( \sum_{i_2=1}^{n_2} \left| \sum_{i_1=1}^{n} a_{i_1 i_2} y^{(1)}_{i_1} \right|^2 \right)^{1/2}.$$
Since
\[
\left( \sum_{i_2=1}^{n} \sum_{i_1=1}^{n} a_{i_1i_2} y_{i_1}^{(1)} \right)^{\frac{1}{2}} = \left( \sum_{i_2=1}^{n} \sum_{i_1=1}^{n} y_{i_1}^{(1)} y_{j_1}^{(1)} a_{i_1i_2} a_{j_1j_2} \right)^{\frac{1}{2}} = \left( \sum_{i_2=1}^{n} y_{i_1}^{(1)} y_{j_1}^{(1)} \sum_{i_1=1}^{n} a_{i_1i_2} a_{j_1j_2} \right)^{\frac{1}{2}},
\]
we have
\[
\left| A \left( x^{(1)}, x^{(2)} \right) \right| \leq n_2^{\frac{1}{2}} \left( \sum_{i_1=1}^{n} y_{i_1}^{(1)} n a_{i_1i_1} \right)^{\frac{1}{2}} \leq n_1^{\frac{1}{2}} n_2^{\frac{1}{2}} \left( \sum_{i_1=1}^{n} y_{i_1}^{(1)} \right)^{\frac{1}{2}} \cdot \left( \sum_{i_1=1}^{n} x_{i_1}^{(1)} \right)^{\frac{1}{2}} \leq n_1^{\frac{1}{2}} n_2^{\frac{1}{2}} \left( \sum_{i_1=1}^{n} x_{i_1}^{(1)} \right)^{\frac{1}{2}} \leq n_1^{\frac{1}{2}} n_2^{\frac{1}{2}} \left( \sum_{i_1=1}^{n} a_{i_1i_1} \right)^{\frac{1}{2}}.
\]
Thus
\[
\| A \| \leq n_1^{\frac{1}{2}} n_2^{\frac{1}{2}} \left( \sum_{i_1=1}^{n} a_{i_1i_1} \right)^{\frac{1}{2}} \leq n_1^{\frac{1}{2}} n_2^{\frac{1}{2}} \left( \sum_{i_1=1}^{n} y_{i_1}^{(1)} n a_{i_1i_1} \right)^{\frac{1}{2}} \leq n_1^{\frac{1}{2}} n_2^{\frac{1}{2}} \left( \sum_{i_1=1}^{n} x_{i_1}^{(1)} \right)^{\frac{1}{2}} \leq n_1^{\frac{1}{2}} n_2^{\frac{1}{2}} \left( \sum_{i_1=1}^{n} x_{i_1}^{(1)} \right)^{\frac{1}{2}} \leq n_1^{\frac{1}{2}} n_2^{\frac{1}{2}} \left( \sum_{i_1=1}^{n} a_{i_1i_1} \right)^{\frac{1}{2}}.
\]
In [1] it is proved that
\[
\inf \left( \sum_{i_1=1}^{n} a_{i_1i_1} \right)^{\frac{1}{2}} \leq 8 \sqrt{2} \ln 9 \approx 16.8.
\]
For the complex case, our result shows that
\[
\inf \left( \sum_{i_1=1}^{n} a_{i_1i_1} \right)^{\frac{1}{2}} \leq \inf \left( \sum_{i_1=1}^{n} a_{i_1i_1} \right)^{\frac{1}{2}} \leq 1.
\]
The constant 1 that we have just obtained is optimal in a certain sense: if we fix, for instance, \( n_1 = 1 \), then it is simple to see that the infimum on the right-hand-side is precisely 1.

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