NORMAL CLOSURE AND INJECTIVE NORMALIZER OF A GROUP HOMOMORPHISM

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Abstract. Let \( \varphi : \Gamma \to G \) be a homomorphism of groups. We consider factorizations \( \Gamma \xrightarrow{f} M \xrightarrow{g} G \) of \( \varphi \) having certain universal properties. First we continue the investigation (see [BHS]) of the case where \( g \) is a universal normal map (our term for a crossed module). Then we introduce and investigate a seemingly new dual case, where \( f \) is a universal normal map. These two factorizations are natural generalizations of the usual normal closure and normalizer of a subgroup.

Iterating these universal factorizations yield certain towers associated to the map \( \varphi \); we prove stability results for these towers. In one of the cases we get a generalization of the stability of the automorphisms tower of a centerless group. The case where \( g \) is a universal normal map is closely related to hypercentral group extensions, Bousfield’s localizations, and the relative Schur multiplier \( H_2(G, \Gamma) = H_2(BG \cup_{B\varphi} \text{Cone}(B\Gamma)) \).

Although our constructions here have strong ties to topological constructions we take here a group theoretical point of view.

1. Introduction and main results

Starting with two standard constructions in group theory, namely the normal closure and the normalizer of a subgroup, we consider similar constructions for a general group homomorphism \( \varphi : \Gamma \to G \). We start with the free normal closure of \( \varphi \) (see below for its precise relation to earlier works), and continue with the seemingly new dual notion of injective normalizer of \( \varphi \).

To settle the terminology, we recall the notion of a crossed module, which in this paper we call a normal map, since we are trying to understand basic results about normal subgroups in the framework of general group maps. Further motivation for the latter terminology was given in [FS1] and comes from topology: These maps have a well-defined topological (or simplicial) group structure as homotopy cokernels or quotients \( G//M \).

Definition 1.1. A normal map consists of a group homomorphism

\[ n : M \to G, \]

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together with an action of \( G \) on \( M \):

\[
\ell: G \to \text{Aut}(M),
\]

which we call here a normal structure on \( n \), such that when denoting by \( a^g \) the image of \( a \in M \) under \( \ell(g) \) for \( g \in G \) (this notation will prevail throughout this paper), the following two requirement are satisfied.

\begin{align*}
\text{(NM1)} \quad (a^g)n &= (an)^g, \text{ for all } g \in G \text{ and } a \in M. \\
\text{(NM2)} \quad a^{bh} &= a^b, \text{ for all } a, b \in M.
\end{align*}

Note that \( a^g = a\ell(g) \), while \( h^g = g^{-1}hg \) and \( a^b = b^{-1}ab \), for all \( a, b \in M \) and \( h, g \in G \). Note also that here we apply maps on the right.

Thus (see Lemma \[2.1\]) \( M \) is a central extension of the normal subgroup \( n(M) \trianglelefteq G \), coupled with a group action of \( G \) (on \( M \)) satisfying (NM1) and (NM2).

The notion of a crossed module was introduced by J. H. C. Whitehead \([W1, W2, W3]\). He was motivated by attempts to capture the homotopy groups of certain quotient spaces associated to a group homomorphism. This notion is useful in many situations and has been widely looked into, see, e.g., the book \([BHS]\).

1.2. The free normal closure and the injective normalizer. Let now \( \varphi: \Gamma \to G \) be a group homomorphism. The free normal closure of \( \varphi \) denoted here by \( \Gamma^\varphi \) is a factorization

\begin{align*}
\text{(FNC)} \quad \Gamma &\xrightarrow{\varphi} \Gamma^\varphi \xrightarrow{\overline{\varphi}} G \\
\text{(IN)} \quad \Gamma &\xrightarrow{\varphi} N(\varphi) \xrightarrow{p_\varphi} G
\end{align*}

of \( \varphi \), with \( \overline{\varphi} \) a normal map, having certain universal properties (see subsection \[1.3\] below).

The injective normalizer of \( \varphi \) is a factorization

\begin{align*}
\text{(FNC)} \quad \Gamma &\xrightarrow{\varphi} \Gamma^\varphi \xrightarrow{\overline{\varphi}} G \\
\text{(IN)} \quad \Gamma &\xrightarrow{\varphi} N(\varphi) \xrightarrow{p_\varphi} G
\end{align*}

with \( \overline{\varphi} \) a normal map, having certain universal properties (see subsection \[1.10\] below).

As mentioned above the “free normal closure” was introduced and considered in a more general setup: that of induced crossed module as in \([BH]\). In fact if one takes \( M = P \) in \([BHS]\) Definition 5.2.1, p. 109], then \( f_pM \) is the present “free normal closure” for the map \( f \). Basic properties of the free normal closure were derived in \([BH]\) Proposition 9 and 10] as well as in \([BW1]\) Theorem 2.1], in chapter 5 of \([BHS]\) and in other papers. We give the definition and the construction of the free normal closure, but most of the details are deferred to Appendix \[A\]. We need the basics of the construction as we apply those in subsequent results, and to be self contained.

The notion of the injective normalizer is a dual notion. It too has strong topological background and analogues related to principal fibrations, to be considered elsewhere.

We now briefly define the notions of the free normal closure and of the injective normalizer.
1.3. The free normal closure of a group homomorphism. Throughout this subsection let 
\[ \varphi : \Gamma \to G, \]
be a group homomorphism. We associate to \( \varphi \) a factorization as in equation \( \text{FNC} \). Furthermore \( \overline{\varphi}(\Gamma^\varphi) = \langle \varphi(\Gamma)^G \rangle \), is the usual normal closure of \( \varphi(\Gamma) \) in \( G \). Thus \( \Gamma^\varphi \) is a central extension of \( \langle \varphi(\Gamma)^G \rangle \), coupled with a group action of \( G \) (on \( \Gamma^\varphi \)) satisfying (NM1) and (NM2) with respect to the map \( n = \overline{\varphi} \).

Moreover, the factorization \( \Gamma \xrightarrow{c_\varphi} \Gamma^\varphi \xrightarrow{\overline{\varphi}} G \) is universal in the sense that any factorization \( \Gamma \xrightarrow{\psi} M \xrightarrow{n} G \) of \( \varphi \), with \( n \) a normal map, defines uniquely a normal morphism \( \Gamma^\varphi \xrightarrow{\overline{\psi}} M \) of normal maps over \( G \) (see Definition 2.3) rendering the diagram

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{\psi} & M \\
\downarrow{c_\varphi} & \equiv & \downarrow{\overline{\varphi}} \\
\Gamma^\varphi & \xrightarrow{\overline{\psi}} & G \\
\end{array}
\]

Commutative. In particular the free normal closure is unique. The construction of \( \Gamma^\varphi \) is functorial for the category of maps. As an example we mention that if \( \langle \varphi(\Gamma)^G \rangle = G \), then \( \Gamma^\varphi \) is just a central extension of \( G \) together with a factorization as in equation \( \text{FNC} \). In particular we prove (see Theorem 5.4)

**Theorem 1.4.** Suppose \( \varphi : \Gamma \to G \) is a group homomorphism such the normal closure \( \langle \varphi(\Gamma)^G \rangle = G \). Then the kernel of \( \overline{\varphi} \) is the relative homology group \( H_2(G, \Gamma) \) with respect to the map \( \varphi \).

In the case where \( G \neq \langle \varphi(\Gamma)^G \rangle \), we ask

**Question 1.5.** Let \( \varphi : \Gamma \to G \) be a group homomorphism. What can be said about the structure of \( \Gamma^\varphi \)? What is the kernel of \( \overline{\varphi} : \Gamma^\varphi \to G \)?

Recall from [CDFS] the notion of A-cellularity, for an arbitrary group \( A \). In Proposition 3.10 we prove:

**Proposition 1.6.** The group \( \Gamma^\varphi \) is \( \Gamma \)-cellular.

The free normal closures tower.

Notice that the process of taking the free normal closure can be iterated; this yields the (free) normal closures tower: Let \( \varphi_1 := \varphi, \Gamma_1 = G \) and define inductively \( \varphi_{i+1} = c_{\varphi_i}, \) and \( \Gamma_{i+1} = \Gamma^{\varphi_i}, i \geq 1: \)
Notice that diagram (1.2) is commutative, the maps $\varphi_i$ are normal maps and that $\Gamma_{i+1}$ is a central extension of the normal closure of $\varphi_i(\Gamma)$ in $\Gamma_i$ for all $i \geq 1$.

Few points to note are:

(a) One can readily check (see Corollary 3.9(1)) that if $\varphi$ is surjective then $\Gamma^\varphi = \Gamma/[\Gamma, \ker \varphi]$ and $c_\varphi : \Gamma \to \Gamma/[\Gamma, \ker \varphi]$ is the canonical homomorphism. Thus if $G = 1$, then if we consider the normal closures tower we get that $\Gamma_i = \Gamma/\gamma_i(\Gamma)$, where $\Gamma = \gamma_1(\Gamma) \geq \gamma_2(\Gamma) \geq \ldots$ is the descending central series of $\Gamma$.

Thus the more challenging cases are when $\varphi$ is not surjective.

(b) In the case where $\langle \varphi(\Gamma)^G \rangle = G$, all the maps $\varphi_i$ are surjective, for integers $i \geq 1$ (see Lemma 4.2(2)), so we get a series of central extensions making diagram (1.2) commutative. We prove (see Theorem 4.1):

**Theorem 1.7.** Suppose that $\Gamma$ and $G$ are finite and that $G = \langle \varphi(\Gamma)^G \rangle$, then the normal closures tower (1.2) terminates after a finite number of steps.

Note now that Example 6.2 shows that if $\Gamma$ and $G$ are non-trivial finite abelian groups and $\varphi$ is not surjective, then the size of the (finite abelian) groups $\Gamma_i$ of diagram (1.2) grows to infinity. However we ask:

**Question 1.8.** Suppose that $\Gamma$ and $G$ are finite. Is it true that the inverse limit $\Gamma_\infty := \lim \leftarrow \Gamma_i$, where $\Gamma_i$ are as in diagram (1.2), is finite?

It is interesting to note the behaviour of the normal closures tower on abelianizations. In Proposition 3.11 we prove:

**Proposition 1.9.** Let $\Gamma_\infty := \lim \leftarrow \Gamma_i$, and let $\varphi_\infty : \Gamma \to \Gamma_\infty$ be the map obtained by the universal property of $\Gamma_\infty$. Then,

1. the map $(c_\varphi)_\text{ab} : \Gamma_{\text{ab}} \to \Gamma^\varphi_{\text{ab}}$ induced by $c_\varphi$ is injective;
2. the map $(\varphi_\infty)_\text{ab} : \Gamma_{\text{ab}} \to (\Gamma_\infty)_{\text{ab}}$ induced by $\varphi_\infty$ is injective.

Going back to question 1.8 it is reasonable to expect that the normal closures tower is pro-equivalent to a fixed finite group that gives the universal subnormal factorization $\Gamma \xrightarrow{\varphi_\infty} \Gamma_\infty \to G$, of the original map. For a general group map (not necessarily of finite groups), we should get a relative version of the nilpotent and the Bousfield completion of a group $\Gamma$ that is closely related to the tower of fundamental groups of a topological (“relative nilpotent”) completion tower.
The existence and uniqueness of the free normal closure are recalled in §3 and Appendix A. Some of its properties are given in §3.7. We note already at this early stage that if $\Gamma \triangleleft G$ and $\phi$ is inclusion, we do not always get that $\Gamma^\phi = \Gamma$ (see Example 6.2).

1.10. The injective normalizer of a group homomorphism. Here we add a construction which, in some sense, is “dual” to the construction of the free normal closure. Namely with every group map $\phi: \Gamma \to G$ we associate a factorization as in equation (IN). Further, this factorization is injective in the sense that any factorization $\Gamma \to H \to G$ of $\phi$ with $\Gamma \to H$ a normal map defines uniquely a normal morphism $H \to N(\phi)$. In particular the injective normalizer is unique. In this case the construction is functorial in the variable $G$, assuming a fixed group $\Gamma$.

The image $p_\phi(N(\phi))$ is always a subgroup of the normalizer $N_G(\phi(\Gamma))$, but is not always equal to it (see Lemma 8.2(1) and Remark 7.7(3)). As opposed to the free normal closure, $N(\phi)$ does agree with the usual normalizer $N_G(\phi(\Gamma))$ if $\phi$ is injective.

As in the case of the free normal closure we can iterate the process of taking the injective normalizer and we obtain the (injective) normalizers tower

\[
\begin{array}{cccccccc}
\Gamma^0 & \xrightarrow{\varphi_0} & \Gamma^1 & \xrightarrow{\varphi_1} & \Gamma^2 & \xrightarrow{\varphi_2} & \cdots & \xrightarrow{\varphi_\alpha} & \Gamma^\alpha \\
& G & \downarrow & & & & & \Gamma & \xrightarrow{\varphi_\alpha} & \Gamma^\alpha \\
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Remark 1.12. The map \( \varphi: \Gamma \to G \) is a normal map iff \( \varphi \) is a retract of \( \tilde{\varphi} \), i.e. there exists a section \( s: G \to N(\varphi) \) such that the following diagram

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{=} & \Gamma \\
\varphi \downarrow & & \varphi \downarrow \\
G & \xrightarrow{s} & N(\varphi) & \xrightarrow{p_{\varphi}} & G
\end{array}
\]

is commutative (so \( s \circ p_{\varphi} = \text{id}_G \)). See Lemma 7.11.

We conclude the introduction with a remark putting our work in a more general framework.

Remark 1.13. We note that one can view the two constructions in this paper as functors adjoint to the corresponding forgetful functors. However, we give and use here explicit constructions of these two adjoint functors. These constructions are the main tools used to demonstrate some of the properties of these adjoint functors.

Let \( \mathcal{NM} \) (resp. \( \mathcal{NM}^{\Gamma} \)) be the category of normal maps (resp. normal maps from a fixed group \( \Gamma \)) of groups, and \( \mathcal{G}^2 \) (resp. \( \mathcal{G}^\Gamma \)) be the category of maps of groups (resp. maps from a fixed group \( \Gamma \)).

Consider the the forgetful ("Underlying") functor to the category of group maps:

\[ U: \mathcal{NM} \to \mathcal{G}^2. \]

It is not hard to see that it commutes with inverse limits, but does not commute in general with direct limits. However the restriction \( U^\Gamma \) of \( U \) to \( \mathcal{NM}^{\Gamma} \) does commute with direct limits. Thus one expects that \( U \) has a left adjoint and that \( U^\Gamma \) has a right adjoint. The left adjoint of \( U \) namely \( \text{cl}: \mathcal{G}^2 \to \mathcal{NM} \) is called here the free normal closure and is denoted by \( (\varphi: \Gamma \to G) \mapsto (\Gamma \varphi \to G) \).

The right adjoint of \( U^\Gamma \) namely \( \text{nor}^\Gamma: \mathcal{G}^\Gamma \to \mathcal{NM}^{\Gamma} \) is call here the injective normalizer and is denoted by \( (\varphi: \Gamma \to G) \mapsto (\Gamma \to N(\varphi)) \).

The factorizations of \( \Gamma \to G \) by the normalizer and the normal closure arise as the natural augmentation of these functors.

Now for these two functors one can consider as usual “algebras” namely objects which are retract of composition \( U \circ F \) where \( F \) is one of the above adjoint functors to \( U \). It turns out (see Remark 1.12 above), that a retract of the \( \text{nor}^\Gamma \) is exactly a normal map namely such a retraction exactly equips a map with a normal structure. It is not clear what “algebra” is given by a retract of \( U \circ \text{cl} \).

2. Preliminaries: Normal maps (Crossed modules)

Recall from Definition the notion of a normal map.

Lemma 2.1. Let \( n: M \to G \) be a normal map. Then

1. \( \ker(n) \leq Z(M) \), and \( \ker(n) \) is a \( G \)-invariant subgroup of \( M \);
2. \( n(M) \leq G \) and the map \( n: M \to n(M) \) is a normal map;
(3) If $N \leq M$ is a $G$-invariant subgroup of $M$, then the restriction $n: N \to G$ is a normal map with the same normal structure, restricted to $N$;

(4) If $V$ is an abelian group, then the map (also denoted $n$) $n: M \times V \to G$, defined by $(a, v)n = an$, is a normal map with the normal structure $(a, v)^g = (a^g, v)$, for all $a \in M, v \in V$ and $g \in G$.

Proof. Part (1) is well known: if $b \in \ker n$, then, by (NM2), $a^b = a^h = a$, for all $a \in M$, so $b \in Z(M)$. Also, if $a \in \ker(n)$, then by (NM1), $(a^g)n = (an)^g = 1$, so $a^g \in \ker(n)$. Part (2) is also well known: By (NM1) we have $(n(a))^g = n(a^g)$, for all $a \in M$ and $g \in G$. It is also clear that the second part of (2) holds. Part (3) is obvious, simply observe that (NM1) and (NM2) hold with $M$ replaced by $N$.

For part (4) we check that

\[(n(a)^g)n = (a^g, v)n = (a^g)n = (an)^g = ((a, v)n)^g,\]

so (NM1) holds. Also,

\[(a, v)((b, w)n) = (a, v)^{bn} = (a^b, v) = (a, v)^{(b, w)},\]

so (NM2) holds as well. \qed

Remark 2.2. Let $n: M \to G$ be a surjective map such that $\ker(n) \leq Z(M)$. Then there is a natural action of $G$ on $M$, where $a^g = a^b$, with $a \in M, g \in G$ and $b \in M$ is an element such that $n(b) = g$. It is easy to check that this definition is independent of the choice of $b$, and that $n$ becomes a normal map over $G$. In fact the above is the unique normal structure on $n$.

We require the following well established notions of morphism between normal maps.

Definition 2.3. Let $n_i: M_i \to G_i, i = 1, 2$, be two normal maps. A normal morphism from $n_1$ to $n_2$ is a pair of maps $(\mu, \eta)$ such that the diagram

\[
\begin{array}{ccc}
M_1 & \xrightarrow{\mu} & M_2 \\
\downarrow{n_1} & & \downarrow{n_2} \\
G_1 & \xrightarrow{\eta} & G_2
\end{array}
\]

commutes, and such that $\mu(m_1^{g_1}) = (\mu(m_1))^{\eta(g_1)}$, for all $m_1 \in M_1$ and $g_1 \in G_1$. If $M_1 = M_2$, we always assume that $\mu$ is the identity map, and if $G_1 = G_2$, we always assume that $\eta$ is the identity map.
Lemma 2.4. Suppose we are given a commutative diagram

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{\varphi} & G \\
\downarrow{\mu} & & \downarrow{\eta} \\
\Gamma' & \xrightarrow{\varphi'} & G'
\end{array}
\]

with \(n'\) a normal map and let

\[
\begin{array}{ccc}
M & \xrightarrow{n} & G \\
\downarrow{\pi_2} & & \downarrow{\eta} \\
M' & \xrightarrow{n'} & G'
\end{array}
\]

be a pull back diagram.

Then \(n\) is a normal map having the normal structure \((m', h)^g = ((m')^{g_{\eta}}, h^g), \) for all \((m', h) \in M\) and all \(g \in G\). Further \(\pi_2\) is a normal morphism and there is a map \(\psi: \Gamma \to M\) such that the diagram

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{\mu} & M \\
\downarrow{\varphi} & & \downarrow{\eta} \\
\Gamma' & \xrightarrow{\varphi'} & G'
\end{array}
\]

(2.2)

commutes.

Proof. We have

\[(m', h)^g n = ((m')^{g_{\eta}}, h^g)n = h^g = ((m', h)n)^g,\]

for all \((m', h) \in M\) and \(g \in G\), so (NM1) holds for \(n\). Also

\[(m', h)^{(a', g)n} = (m', h)^g = ((m')^{g_{\eta}}, h^g) = ((m')^{(a')n'}, h^g) = ((m')^{a', h^g}) = (m', h)^{(a', g)},\]

for all \((m', h), (a', g) \in M\), so (NM2) holds for \(n\) as well, and \(n\) is a normal map with the given normal structure. It is easy to check that \(\pi_2\) is a normal morphism.

Let \(\psi: \Gamma \to M\) be defined by

\[\gamma \psi = ((\gamma)\mu\psi', (\gamma)\varphi).\]

Then, by definition \(\gamma \psi n = \gamma \varphi\), for all \(\gamma \in \Gamma\), so \(\psi \circ n = \varphi\). Also, by definition, \(\psi \circ \pi_2 = \mu \circ \psi'\). □
3. THE FREE NORMAL CLOSURE OF A MAP

In this section \( \varphi: \Gamma \to G \) is a fixed map. We recall that the free normal closure of \( \varphi \) is a factorization \( \Gamma \xrightarrow{c_\varphi} \Gamma^g \xrightarrow{\overline{\varphi}} G \) of \( \varphi \), with \( \overline{\varphi} \) a normal map, as defined below.

**Definition 3.1.** Let \( \varphi: \Gamma \to G \) be a map. A free normal closure of \( \varphi \) is a factorization of the latter via

\[
\Gamma \xrightarrow{c_\varphi} \Gamma^g \xrightarrow{\overline{\varphi}} G
\]

such that \( \varphi: \Gamma^g \to G \) is a normal map, and such that for any other factorization \( \varphi = \psi \circ n \) via a normal map \( n: M \to G \) with \( \psi: \Gamma \to M \), there exists a unique normal morphism \( \psi: \Gamma^g \to M \) rendering the diagram below commutative.

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{\varphi} & G \\
\downarrow{c_\varphi} & & \downarrow{\overline{\varphi}} \\
\Gamma^g & \xrightarrow{\psi} & M \\
\end{array}
\]

In §2 below, and Appendix A we will show that the free normal closure exists. As for uniqueness we have:

**Lemma 3.2.** The free normal closure of \( \varphi: \Gamma \to G \) is unique up to an isomorphism of normal maps over \( G \).

**Proof.** Straightforward from the universal properties.

**Lemma 3.3.** Let \( \Gamma \xrightarrow{c_\varphi} \Gamma^g \xrightarrow{\overline{\varphi}} G \), be the free normal closure of \( \varphi \). Then

1. the group \( \Gamma^g \) is generated by \( \{(c_\varphi(\Gamma))^g \mid g \in G\} \);
2. \( \overline{\varphi}(\Gamma^g) \) is the normal closure of the subgroup \( \varphi(\Gamma) \) in \( G \).

**Proof.** Let \( M \leq \Gamma^g \) be the subgroup generated by \( \{(c_\varphi(\Gamma))^g \mid g \in G\} \), and consider diagram (3.1), where \( n \) is the map \( \overline{\varphi} \) restricted to \( M \), and where \( \psi \) is the map \( c_\varphi \) (with range \( M \) in place of \( \Gamma^g \)). By Lemma 2.1(3), \( m \) is a normal morphism. Hence there exists a (unique) normal morphism

\[
(3.2) \quad \overline{\psi}: \Gamma^g \to M,
\]

rendering the diagram commutative.

Now consider again diagram (3.1), with \( \Gamma^g, \overline{\varphi}, c_\varphi \) in place of \( M, m, \psi \) respectively. Of course \( \overline{c_\varphi} \) in this case is the identity map. However the map \( \overline{\psi} \) of equation (3.2) considered as a map from \( \Gamma^g \) to \( \Gamma^g \) also renders diagram (3.1) commutative in this case. By uniqueness, \( \overline{\psi} \) is the identity map, so part (1) holds.
Next, by (1), \( \varphi(\Gamma^g) \) is generated by \( \{ \varphi(\gamma)^g | \gamma \in G \} \), since \( \varphi \) is a group homomorphism. But since \( \varphi \) is a normal map, \( \varphi(\gamma g^g) = (\varphi(\gamma))^g = \varphi(\Gamma)^g \), for all \( g \in G \). Hence (2) holds. 

\[ \square \]

3.4. A construction of the free normal closure.

The purpose of this subsection is to recall the construction of the free normal closure of a map \( \varphi: \Gamma \rightarrow G \). The detailed proofs are given in Appendix \( \Delta \).

**Theorem 3.5.** Let \( \varphi: \Gamma \rightarrow G \) be a map of groups. Then the free normal closure of \( \varphi \) exists.

We start by considering the free group \( F \) generated by the following set of distinct symbols:

\[ \Gamma_G := \{ \gamma g | \gamma \in \Gamma, g \in G \} \]

We consider the following relations on \( F \)

\[ (R_1) \quad R_1 := \{ 1 = 1_h | g, h \in G \}. \]

\[ (R_{nm1}) \quad R_{nm1} := \{ \gamma g \delta g = (\gamma \delta)_g | \gamma, \delta \in \Gamma \text{ and } g \in G \}. \]

\[ (R_{nm2}) \quad R_{nm2} := \{ \delta_h^{-1} \gamma g \delta_h = \gamma g \varphi(\delta)_h | \gamma, \delta \in \Gamma \text{ and } h, g \in G \}. \]

The relation \( R_1 \) just mean that the identity \( 1 = 1_g \), where \( g \in G \) is an arbitrary element, is the identity of \( F \), and \( F \) is the free group on the set

\[ (\Gamma \setminus \{ 1 \})_G := \{ \gamma g | \gamma \in \Gamma \setminus \{ 1 \} \text{ and } g \in G \}. \]

We let \( \Gamma^g \) be the group defined using the relations \( R_1, R_{nm1} \) and \( R_{nm2} \) above:

\[ (3.3) \quad \Gamma^g := \text{Gr}\{ \Gamma_G | R_1, R_{nm1}, R_{nm2} \} \]

**Notation 3.6.**

1. We denote

\[ \widehat{F} := \text{Gr}\{ \Gamma_G | 1_h = 1_g, \gamma g \delta_g = (\gamma \delta)_g | \gamma, \delta \in \Gamma \text{ and } g, h \in G \}. \]

and we let

\[ \widehat{\Gamma}_g \text{ be the image of the set } \{ \gamma g | \gamma \in \Gamma \} \subseteq F \text{ in } \widehat{F}. \]

Thus \( \widehat{\Gamma}_g = \{ \widehat{\gamma}_g | g \in G \} \), and \( \widehat{\Gamma}_g \cong \Gamma \), for all \( g \in G \). Further, \( \widehat{F} \) is a free product

\[ \widehat{F} = \ast_{g \in G} \widehat{\Gamma}_g. \]

2. Let \( g \in G \). We denote by \( \overline{\gamma}_g \) the image in \( \Gamma^g \) of \( \gamma_g \in F \). We let

\[ (3.4) \quad \overline{\Gamma}_g := \{ \overline{\gamma}_g | \gamma \in \Gamma \text{ and } g \in G \} \leq \Gamma^g, \]

be the subgroup of \( \Gamma^g \) consisting of the elements \( \{ \overline{\gamma}_g | \gamma \in \Gamma \text{ and } g \in G \} \).
We define

\begin{align}
(3.5) \quad \overline{\varphi} : \Gamma^\varphi \to G : \gamma \mapsto \varphi(\gamma)^g, \quad \gamma \in \Gamma, \ g \in G.
\end{align}

Lemma A.1 shows that \(\overline{\varphi}\) is a well defined map.

Next we define an action of \(\ell : G \to \text{Aut}(\Gamma^\varphi)\) by

\begin{align}
(3.6) \quad \ell(h) \in \text{Aut}(\Gamma^\varphi) : \gamma g \mapsto \gamma gh,
\end{align}

for all \(\gamma \in \Gamma\) and \(g, h \in G\).

Lemma A.2 shows that \(\ell\) defines an action of \(G\) on \(\Gamma^\varphi\) and Lemma A.3 shows that \(\overline{\varphi} : \Gamma^\varphi \to G\) is a normal map having \(\ell\) as its normal structure. Finally, in subsection A.4 we show the universality property of \(\Gamma^\varphi\).

3.7. Examples of finiteness and cellularity.

In this subsection we prove some basic properties of the normal closure and give some examples.

**Lemma 3.8.** Let \(h, g \in G\), and \(\delta \in \Gamma\), then

1. \(\Gamma^\delta_g = \Gamma_{g\varphi(\delta)^g}\);
2. \(\Gamma_g = \Gamma_{\varphi(\delta)g}\);
3. \(\ker \overline{\varphi} \leq Z(\Gamma^\varphi)\);
4. if \(G\) is finite, then \(\Gamma^\varphi = \Pi_{g \in G} \Gamma_g\).

**Proof.** Recall the notation \(\Gamma_g\) from equation (3.4). Since \(\Gamma^\varphi\) satisfies the relations \(R_{nm2}\) (see equation \((R_{nm2})\)), we see that (1) holds. Also, by the relations \(R_{nm2}\),

\[\delta_g^{-1} \gamma_g \delta_g = \gamma_{g\varphi(\delta)^g} = \gamma_{\varphi(\delta)g}.
\]

It follows that

\[\Gamma_g = \Gamma_g^\delta = \Gamma_{\varphi(\delta)g}.
\]

This shows (2), and (3) follows from Lemma 2.1(1), since \(\overline{\varphi}\) is a normal map.

For the proof of part (4) set \(G := \{g_1, \ldots, g_s\}\), where \(s = |G|\). Recall that \(\Gamma^\varphi\) is the image of the free product \(\tilde{F} = \Gamma_{g_1} * \Gamma_{g_2} * \cdots * \Gamma_{g_s}\), where \(\Gamma_{g_i} \cong \Gamma\), for all \(i\) (see Notation 3.6). Of course \(\tilde{F}\) is equipped with a natural free product word length. For \(\overline{w} \in \Gamma^\varphi\), we let \(|\overline{w}|\) be the minimal length of a word in \(\tilde{w} \in \tilde{F}\) such that \(\overline{w}\) is the image of \(\tilde{w}\). We now show by induction on \(|\overline{w}|\), that \(\overline{w} \in \Gamma_{g_1} \Gamma_{g_2} \cdots \Gamma_{g_s}\).

If \(|\overline{w}| = 1\), this is obvious. Our induction hypothesis is that if \(|\overline{w}| < r\), then \(r \leq s + 1\), and we can write \(\overline{w} \in \Gamma_{g_1} \Gamma_{g_2} \cdots \Gamma_{g_s}\) using \(|\overline{w}|\) non-identity elements.

Assume \(|\overline{w}| = r\). From all the words \(\tilde{w} \in \tilde{F}\) of length \(r\) whose image is \(\overline{w}\), choose a word

\[\tilde{w} = (\tilde{\gamma}_1)_{g_{i_1}} (\tilde{\gamma}_2)_{g_{i_2}} \cdots (\tilde{\gamma}_r)_{g_{i_r}}.
\]

so that \(i_1\) is as small as possible. By induction, we may assume that \(i_2 < i_3 < \cdots < i_r\). Now if \(i_2 = i_1\), then \(\tilde{\gamma}_{g_{i_1}} \tilde{\gamma}_{g_{i_2}} \in \Gamma_{g_{i_1}}\), so we get that \(|\overline{w}| < r\), a contradiction. If \(i_2 < i_1\), then using
the relations $R_{nm2}$ we can write $\bar{w}$ as a word of length $r$, starting with $(\tau_{i_2})_{g_{i_2}}$, contradicting the minimality of $i_1$. Hence $i_1 < i_2$, and (4) holds. \hfill \Box

Part (2) of the following Corollary should be compared with [BHS] Theorem 5.7.1, p.124.

**Corollary 3.9.** Let $\varphi : \Gamma \to G$ be a map of groups and set $K := \ker \varphi$, then

1. if $\varphi$ is surjective, the $\Gamma^\varphi = \bar{\Gamma}_1$, and $\Gamma^\varphi \cong \Gamma/[\Gamma, K]$ as crossed modules over $G$;

2. If $\Gamma$ and $G$ are finite, then $\Gamma^\varphi$ is finite.

**Proof.** (1) Since $\varphi$ is surjective $\Gamma^\varphi = \bar{\Gamma}_1$ by Lemma 3.8(2).

Since $\Gamma/[\Gamma, K]$ is a central extension of $G$, Remark 2.2 shows that the map $n : \Gamma/[\Gamma, K] \to G$ induced by $\varphi$ is a normal map. Further the canonical map $\psi : \Gamma \to \Gamma/[\Gamma, K]$ satisfies $\psi(\gamma) = 1$ iff $\gamma \in [\Gamma, K]$ and $\psi \circ n = \varphi$.

Let $\bar{\psi} : \Gamma^\varphi \to \Gamma/[\Gamma, K]$ be the unique map of diagram (3.1). The map $c_\varphi : \Gamma \to \Gamma^\varphi$ is surjective and satisfies $c_\varphi \circ \bar{\psi} = \psi$. Thus $\bar{\psi}((\bar{\gamma}_1)) = \psi(\gamma)$. It follows that if $1 = c_\varphi(\gamma) = \bar{\gamma}_1$, then $\psi(\gamma) = 1$, so $\gamma \in [\Gamma, K]$. Thus $\ker c_\varphi \leq [\Gamma, K]$. On the other hand, $c_\varphi(K) \leq \ker \varphi$, so, since $\varphi$ is a normal map, $c_\varphi(K) \leq Z(\Gamma^\varphi)$. It follows that $[\Gamma, K] \leq \ker c_\varphi$. Thus $\ker c_\varphi = [\Gamma, K]$. Since $\bar{\psi}(\bar{\gamma}_1) = \psi(\gamma)$, we see that $\bar{\psi}$ is a normal isomorphism from $\Gamma^\varphi$ to $\Gamma/[\Gamma, K]$.

(2) This follows immediately from Lemma 3.8(4). \hfill \Box

Our next proposition shows that $\Gamma^\varphi$ is $\Gamma$-cellular (see [CDFS] for the notion of cellularity).

**Proposition 3.10.** $\Gamma^\varphi$ is $\Gamma$-cellular.

**Proof.** We show that $\Gamma^\varphi$ is the coequalizer of two maps between two free products of copies of $\Gamma$:

$$
\begin{array}{ccc}
\ast & \xrightarrow{e_1} & \Gamma_i \xrightarrow{e_2} \ast, \\
\hline
i \in I
\end{array}
$$

where $\hat{\Gamma}_g \cong \Gamma$ is as in notation 3.6 for $g \in G$. Also, $I = \Gamma \times G \times G$, and $\Gamma_i \cong \Gamma$, for $i \in I$.

We now define the maps $e_1$ and $e_2$. Let $i = (\delta; g, h) \in I$. For $\gamma \in \Gamma_i$, let $e_1(\gamma) = \hat{\gamma}_{g, \varphi(\delta) h}$. This defines the homomorphism $e_1$. To define $e_2$, let $i = (\delta; g, h) \in I$ and for $\gamma \in \Gamma_i$, define $e_2(\gamma) = (\hat{\delta}_h)^{-1} \hat{\gamma}_g \hat{\delta}_h$. This defines the homomorphism $e_2$. By the construction of $\Gamma^\varphi$, and by the definition of the coequalizer, $\Gamma^\varphi$ is the coequalizer of these maps, hence $\Gamma^\varphi$ is $\Gamma$-cellular. \hfill \Box

3.11. On abelian quotients of the normal closure.

The following results shows that the normal closures tower behaves well with respect to abelianizations. We denote $H_{ab} = H/[H, H]$ the abelianization of $H$, for any group $H$.

**Proposition 3.11.** Let $\{\Gamma_i \mid i = 1, 2, 3, \ldots\}$ be the normal closures tower of $\varphi$ (see diagram (1.2)). Let $\Gamma_\infty = \varprojlim \Gamma_i$, and let $\varphi_\infty : \Gamma \to \Gamma_\infty$ be the map obtained by the universal property of $\Gamma_\infty$, then

1. the map $(c_\varphi)_{ab} : \Gamma_{ab} \to \Gamma_{\varphi_{ab}}$ induced by $c_\varphi$ is injective; hence,

2. the map $(\varphi_\infty)_{ab} : \Gamma_{ab} \to (\Gamma_\infty)_{ab}$ induced on abelianization is injective.
Proof. Consider diagram (2.1) with \( \Gamma' = \Gamma_{ab}, G' = G_{ab} \), \( \mu \) and \( \eta \) are the natural maps and \( \varphi' = \varphi_{ab} \) is the map induced by \( \varphi \). Further let \( M' = \Gamma'_{\varphi_{ab}}, \) let \( \psi' = c_{\varphi_{ab}} \) and \( n' = \varphi_{ab} \).

By Lemma 2.4, the pullback \( M \) renders diagram (2.2) commutative. Thus by the universal property of \( \Gamma_{\varphi_{ab}} \), there exists a normal morphism \( \psi: \Gamma_{\varphi_{ab}} \to M \) rendering diagram (3.1) commutative. We thus get a commutative diagram

\[
\begin{array}{ccc}
\Gamma_{\varphi_{ab}} & \xrightarrow{\rho} & G_{ab} \\
\downarrow & & \downarrow \\
\Gamma_{\varphi} & \xrightarrow{\rho} & G
\end{array}
\]

By Example 6.2, \( \Gamma'_{\varphi_{ab}} \) is abelian and \( c_{\varphi_{ab}} \) is injective.

Hence we have a commutative diagram (note that \( \mu \) is surjective)

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{c_{\varphi}} & \Gamma_{\varphi} \\
\downarrow & & \downarrow \\
\Gamma_{ab} & \xrightarrow{(c_{\varphi})_{ab}} & \Gamma_{\varphi_{ab}}
\end{array}
\]

Since \( c_{\varphi_{ab}} \) is injective, so is \( (c_{\varphi})_{ab} \). This shows (1). Then (2) follows from (1), since by the universality property of \( \Gamma_{\infty} \) there is a map \( \varphi_{\infty}: \Gamma \to \Gamma_{\infty} \), such that \( \varphi_{\infty} \circ \psi_{2} = \varphi_{2} \), where \( \psi_{2}: \Gamma_{\infty} \to \Gamma_{2} \) is the canonical map. Hence \( (\varphi_{\infty})_{ab} \circ (\psi_{2})_{ab} = (\varphi_{2})_{ab} \). Since \( \varphi_{2} = c_{\varphi} \), and by (1), \( (c_{\varphi})_{ab} \) is injective, (2) follows.

\[
\square
\]

4. Stability of the Normal Closures Tower

The purpose of this section is to prove:

**Theorem 4.1.** Let \( \Gamma \) and \( G \) be finite groups and let \( \varphi: \Gamma \to G \) be a homomorphism. Assume that \( G = \langle \varphi(\Gamma)^{G} \rangle \). Then the normal closures tower corresponding to \( \varphi \) terminates after a finite number of steps. Furthermore, the last term of the normal closures tower has size less or equal \( |\Gamma| \cdot f(|G|) \), where \( f \) is defined in equation (4.1) below.

We first make a general observation about the normal closures tower.
Lemma 4.2. Let \( \varphi : \Gamma \to G \) be a group homomorphism. Suppose \( G = \langle \varphi(\Gamma)^G \rangle \), then

1. \( \Gamma_i = \langle \varphi_i(\Gamma)^{i} \rangle \), for all integers \( i \geq 1 \);
2. \( \Gamma_{i+1} \) is a central extension of \( \Gamma_i \), for all \( i \geq 1 \).

where \( \Gamma_i \) are the terms of the normal closures tower as in diagram (1.2).

Proof. By Lemma 3.3(2) the image of \( \overline{\varphi_i} \) in \( \Gamma_i \) is \( \langle \varphi_i(\Gamma)^{i} \rangle \). By hypothesis \( \overline{\varphi_i} \) is surjective, that is \( \Gamma_2 \) is a central extension of \( \Gamma_1 = G \). Now let \( i \geq 2 \), and suppose that \( \Gamma_i \) is a central extension of \( \Gamma_{i-1} \). We show that \( \Gamma_{i+1} \) is a central extension of \( \Gamma_i \). Indeed, by Lemma 3.3(1), \( \Gamma_i = \Gamma_i^{\varphi_i} \) is generated by \( \{ (\varphi_i(\Gamma))^g \mid g \in \Gamma_{i-1} \} \). However, since \( \Gamma_i \) is a central extension of \( \Gamma_{i-1} \), this just means that (1) holds. Thus \( \overline{\varphi_i} \) is surjective. Since \( \overline{\varphi_i} \) is a normal map, (2) holds. \( \square \)

For Theorem 4.3 below, let us recall that the upper central series of any group \( M \) is the ascending series

\[
1 = Z_0(M) \leq Z_1(M) \leq \ldots Z_\alpha(M) \leq Z_{\alpha+1}(M) \leq \ldots Z_\delta(M) = Z_\infty(M),
\]

given by \( Z_i(M) = Z(M) \) is the center of \( M \), and recursively by \( Z_{\alpha+1}(M)/Z_\alpha(M) = Z(M/Z_\alpha(M)) \) for all ordinals \( \alpha \), and \( Z_\lambda(M) = \bigcup_{\mu < \lambda} Z_\mu \) for every limit ordinal \( \lambda \). The last term \( Z_\infty(M) \) of this series is called the hypercenter of \( M \). The group \( M \) is hypercentral if \( Z_\infty(M) = M \).

Theorem 4.3 (see Theorem B, p. 2598 in [KOS]). There exists an integer valued function \( f \) such that if \( M \) is a group satisfying \( |M/Z_\infty(M)| = t < \infty \), then \( M \) contains a finite normal subgroup \( L \), with \( |L| \leq f(t) \) and such that \( M/L \) is hypercentral. Here

\[
f(t) = t^k, \quad \text{where} \quad k = \frac{1}{2} (\log_p t + 1) \quad \text{and} \quad p \text{ is the least prime divisor of } t.
\]

Lemma 4.4. Let \( N \) be a nilpotent group. Suppose \( T \leq N \) is a subgroup such that \( N = \langle T^N \rangle \).

Then \( N = T \).

Proof. The Frattini factor group \( N/\Phi(N) \) is abelian, so since \( N = \langle T^N \rangle \), we get that \( N = T\Phi(N) \), so \( N = T \). \( \square \)

Lemma 4.5. Let \( \Gamma \) and \( G \) be finite groups. Assume that \( M \) is a finite group such that

(a) \( M/Z_\infty(M) \) is isomorphic to a quotient of \( G \).

(b) There exists a homomorphism \( c: \Gamma \to M \) such that \( M = \langle c(\Gamma)^M \rangle \).

Then \( |M| \leq |\Gamma| \cdot f(|G|) \), where \( f \) is as in equation (4.1).

Proof. By Theorem 4.3 and by (a) we can find \( L \leq M \) such that \( |L| \leq f(|G|) \) and such \( M/L \) is hypercentral (note that \( f(|G/K|) \leq f(|G|) \), for any \( K \leq G \)). Of course \( M/L \) is nilpotent, as \( M/L \) is finite. Since by (b) the normal closure of the image of \( c(\Gamma) \) in \( M/L \) is \( M/L \), Lemma 4.3 implies that \( M/L \) is equal to that image. In particular \( |M/L| \leq |\Gamma| \). This proves the lemma. \( \square \)

Proof of Theorem 4.4. Let \( i \geq 1 \). We apply Lemma 4.3 with \( M = \Gamma_i \) and with \( c = \varphi_i \). By Lemma 4.2(1), hypothesis (b) of Lemma 4.3 holds, and by Lemma 4.2(2), hypothesis (a) of Lemma 4.3 holds. Hence, \( |\Gamma_i| \leq |\Gamma| \cdot f(|G|) \). This of course proves the Theorem. \( \square \)
5. \( f \)-CENTRAL EXTENSIONS AND RELATIVE SCHUR MULTIPLIER

The purpose of this section is to prove Theorem 1.4 of the introduction. Hence we assume that \( \varphi: \Gamma \to G \) satisfies \( G = \langle \varphi(\Gamma)^G \rangle \). By Lemma 3.3(2), the map \( \overline{\varphi}: \Gamma^G \to G \) is surjective, so \( \Gamma^G \) is a central extension of \( G \). Further, for any factorization \( \Gamma \xrightarrow{\psi} M \xrightarrow{n} G \) of \( \varphi \), with \( n \) a normal map, \( M \) is a central extension of \( G \). This is because by Lemma 2.1(2), \( n(M) \) is normal in \( G \), so \( n \) is surjective since \( \varphi(\Gamma) \leq n(M) \). Also by Lemma 2.1(1), \( \ker(n) \leq Z(M) \). Thus \( \Gamma^G \) is universal amongst all central extensions \( M \) of \( G \) such that \( \varphi \) factors through \( M \to G \). Indeed, for any such \( M \) there is a unique normal morphism \( \psi: \Gamma^G \to M \) rendering diagram (3.1) commutative.

To identify \( \ker \varphi \) we show that \( (\Gamma^G, \overline{\varphi}) \) is a universal \( \varphi \)-central extension in a sense to be made precise shortly. The detailed account of this identification will appear in [FS2]. Here we only give the basic definitions and the main results of [FS2]. (We note that [FS2] proves a more general result, see Proposition 5.2 and Theorem 5.3 below.)

Let us consider central extensions

\[
0 \to A \to M \to G \to 1.
\]

of \( G \). Let us also fix a map

\[
f: \Gamma \to G
\]

The following are the basic concepts used in this section.

**Definitions 5.1.**

1. An \( f \)-central extension of \( G \) is a pair \((M, \psi)\), where \( M \) is a central extension of \( G \) with kernel \( A \), together with a map \( \psi: \Gamma \to M \) that factorizes \( f \) as in diagram (5.1) below.

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{\psi} & M \\
\downarrow{\Gamma} & & \downarrow{n} \\
0 & \to & A & \to & M & \to & G & \to & 0
\end{array}
\]

2. A map between two \( f \)-central extensions \((M, \psi)\) and \((M', \psi')\) of \( G \) with kernels \( A, A' \) respectively, is a map of the underlying extensions which is the identity on \( G \), as in the commutative diagram (5.2) below. \( M \) and \( M' \) are called equivalent if \( \tau \) below is an isomorphism and \( \kappa \) below is the identity.

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{\psi} & M \\
\downarrow{\psi} & & \downarrow{n} \\
0 & \to & A & \to & M & \to & G & \to & 0
\end{array}
\]
In the following results, for an abelian group $A$, by $H_*(G,\Gamma;A)$ we mean $H_*(BG\cup_B \text{Cone}(B\Gamma);A)$, and similarly for relative cohomology (see [FS2] for an algebraic definition and more details).

**Proposition 5.2 ([FS2]).** The equivalence classes of $f$-central extension of $G$ with a given kernel $A$ has a natural abelian group structure, and are classified by the relative cohomology group $H^2(G,\Gamma;A)$, with coefficients $A$.

**Proposition 5.2** yields the following theorem:

**Theorem 5.3 ([FS2]).** Assume that the map $f_{ab}: \Gamma_{ab} \to G_{ab}$ induced on the abelianizations is surjective. Then there exists a universal $f$-central extension $(U,\eta)$ of $G$ with kernel $H_2(G,\Gamma;\mathbb{Z})$, such that for any $f$-central extension $(E,\psi)$ of $G$ there is a unique map of $f$-central extensions (as in Definition 5.1(2)) from $U$ to $M$.

As an immediate corollary to Theorem 5.3 we get

**Theorem 5.4.** Let $\varphi: \Gamma \to G$ be a group homomorphism. Assume that $G = \langle \varphi(\Gamma)^G \rangle$. Then $\Gamma^\varphi$ is the universal $\varphi$-central extension of $G$ of Theorem 5.3. In particular the kernel of $\varphi^\varphi$ is $H_2(G,\Gamma;\mathbb{Z})$.

**Proof.** The first thing to notice is that $\varphi_{ab}: \Gamma_{ab} \to G_{ab}$ is surjective since $G = \langle \varphi(\Gamma)^G \rangle$. Further, we already noted (see Lemma 3.3(2)) that if $G = \langle \varphi(\Gamma)^G \rangle$, then $\Gamma^\varphi$ is a central extension of $G$. Also, as noted in the beginning of this section, the fact that $(M,\psi)$ is a $\varphi$-central extension of $G$ is equivalent to a factorization $\Gamma \xrightarrow{\psi} M \xrightarrow{n} G$ of $\varphi$ with $n$ a normal map. Thus the universal property that defines $\Gamma^\varphi$ is precisely the universal property that defines the universal $\varphi$-central extension (of Theorem 5.3). Hence these are isomorphic, and the remaining part of the theorem follows from Theorem 5.3. □

**Remark 5.5.** The construction in [FS2] of the universal $f$-central extension $(U,\eta)$ of Theorem 5.3 extends the Schur universal central extension of a perfect group $G$ (the case $\Gamma = 1$), to any map between groups (not necessarily perfect groups) $f: \Gamma \to G$, inducing surjection on abelianizations.
6. Further Examples of the Free Normal Closure

Throughout this section \( \varphi : \Gamma \to G \) is a map of groups and \( \Gamma \xrightarrow{\varphi} \Gamma^\varphi \xrightarrow{\gamma} \varphi M \xrightarrow{n} G \) is its universal normal closure. Further, throughout this section \( \Gamma \xrightarrow{n} M \xrightarrow{\gamma} G \) is a factorization of \( \varphi \) (i.e. \( \psi \circ n = \varphi \)) with \( n \) a normal map, as in diagram (3.1), and \( \psi : \Gamma^\varphi \to M \) is the unique normal morphism as in diagram (3.1).

As we saw in Corollary 3.9 if \( \varphi \) is surjective, then there is a normal isomorphism \( \Gamma^\varphi \to \Gamma/[\Gamma, \ker \varphi] \). We also saw that if \( \Gamma \) and \( G \) are finite, then \( \Gamma^\varphi \) is finite. We consider here some more examples, some of which appear in the literature. We give details since we need those as a starting point for some of the results above. A lemma for the case where \( \varphi \) is injective and \( \varphi(\Gamma) \) is normal in \( G \) is given in Appendix B.

**Example 6.1.** Suppose \( G = 1 \). Let \( \varphi_1 := \varphi \). Then, by Corollary 3.9(2), the free normal closure of \( \varphi \) is the factorization \( \Gamma \xrightarrow{\varphi} \Gamma/[\Gamma, \Gamma] \to 1 \). Since \( \varphi_2 := c_{\varphi_1} \), being the canonical map is a surjective map whose kernel is \( [\Gamma, \Gamma] \), Corollary 3.9(2) applies again, and shows that the free normal closure of \( \varphi_2 \) is the factorization \( \Gamma \xrightarrow{\varphi_2} \Gamma/\gamma_3(\Gamma) \xrightarrow{\gamma_3} \Gamma/\gamma_2(\Gamma) \). Proceeding in this way we see that the normal closures tower of diagram (1.2) are the quotients \( \Gamma_i = \Gamma/\gamma_i(\Gamma) \), \( i \geq 1 \), where \( \gamma_i(\Gamma) \) are the members of its descending central series \( \Gamma = \gamma_1(\Gamma) \geq \gamma_2(\Gamma) \geq \ldots \).

The following example is well known.

**Example 6.2.** Assume that \( \Gamma \) and \( G \) are abelian groups. Let \( \varphi : \Gamma \to G \) be a homomorphism. We claim that
\[
\Gamma^\varphi = \bigoplus_{x \in G/\varphi(\Gamma)} \Gamma_x \quad \text{and} \quad c_\varphi(\gamma) = \gamma_{\varphi(\gamma)}, \quad \forall \gamma \in \Gamma.
\]
where \( \Gamma_x = \{ \gamma_x \mid \gamma \in \Gamma \} \cong \Gamma \), for \( x \in G/\varphi(\Gamma) \). The action of \( G \) on \( \Gamma^\varphi \) is given by \( \gamma g = \gamma_{\varphi(g)} \), for all \( x \in G/\varphi(\Gamma) \) and \( g \in G \). The map \( \overline{\varphi} \) takes \( \gamma_{\varphi(\gamma)} \), to \( \gamma \), for all \( \gamma \in \Gamma \) and takes \( \Gamma_x \) to 0, for all \( x \neq \varphi(\Gamma) \).

To see that equation (6.1) is correct, note the relation \( \text{(R_{num2})} \), gives \( (\overline{\delta})^{-1} \overline{\gamma} \overline{\delta} = \gamma_{\varphi(\delta)} \); for all \( \overline{\delta}, \overline{\gamma} \in \overline{\Gamma} \) (taking \( h = g \), see the notation in equation (3.4)), because \( \Gamma \) is abelian. So since \( \overline{\Gamma_g} \) is abelian we see that \( \overline{\gamma_g} = \gamma_{\varphi(\delta)} \). But now using relation \( \text{(R_{num2})} \) again we see that
\[
(\overline{\delta})^{-1} \overline{\gamma} \overline{\delta} = \overline{\gamma_{\varphi(\delta)}} = \overline{\gamma_g},
\]
for all \( \overline{\delta}, \overline{\gamma} \in \overline{\Gamma_g} \), where \( g, \overline{\gamma} \in G \), again because \( G \) is abelian. Since \( \Gamma^\varphi \) is generated by \( \{ \overline{\Gamma_g} \mid g \in G \} \) we see that \( \Gamma^\varphi \) is abelian.

Next, if \( \Gamma \xrightarrow{n} M \xrightarrow{\gamma} G \) is a factorization of \( \varphi \), with \( n \) a normal map, then \( M \) is a central extension of \( \varphi(\Gamma) \), so \( M \) is nilpotent of class at most 2. Since the normal closure of \( \psi(\Gamma) \) in \( M \) is contained in \( \psi(\Gamma)Z(M) \), and since \( \psi(\Gamma) \) is abelian, we see that \( \langle \psi(\Gamma)^m \rangle \) is abelian. Further, \( \psi(\Gamma)^{\gamma g} = \psi(\Gamma)^{\gamma} \psi(g) = \psi(\Gamma)^{\gamma} \psi(g) = \psi(\Gamma)^{\gamma} \), for all \( \gamma \in \Gamma \), so we see that \( \psi(\Gamma)^h = \psi(\Gamma)^{\gamma} \), for all \( x \in G/\varphi(\Gamma) \) and all \( g, h \in x \).
Thus the map from $\overline{\psi}: \Gamma^\varphi \to M$ taking $\gamma_h$ to $\psi(\gamma)^h$, where $\gamma \in \Gamma, x \in G/\varphi(\Gamma)$ and $h \in x$ is a well defined map making diagram (3.1) commutative. It is now routine to check that $\bigoplus_{x \in G/\varphi(\Gamma)} \Gamma_x$ is the free normal closure of $\varphi$.

**Example 6.3.** There are many examples where $\varphi$ is injective, but $\Gamma^\varphi$ is *not* the normal closure of $\varphi(\Gamma)$. Here is one.

Let $G = A_5 = \text{PSL}_2(5)$, and let $\Gamma \leq G$ be a subgroup of order 3 (thus $\varphi$ is the inclusion map). Then the normal closure of $\Gamma$ in $G$ is $G$ (since $G$ is simple). Now let $M = \text{SL}_2(5)$ be the universal perfect central extension of $G$. Let $n: M \to G$ be the natural surjection, and let $\psi: \Gamma \to M$, such that $\psi(\Gamma)$ is a subgroup of order 3 in $M$ and such that $\psi \circ n = \text{inc}$ is the inclusion map $\text{inc}: \Gamma \to G$.

Consider diagram (3.1). Since the only homomorphism $A_5 \to \text{SL}_2(5)$ is the trivial map, if $\Gamma^\varphi = G$, then $\overline{\psi}$ would be the trivial map. But $c_\varphi \circ \overline{\psi} = \psi$ which means that $\psi$ is the trivial map, a contradiction.

It is interesting to note that $\Gamma^\varphi$ in this case is $\text{SL}_2(5) \times \mathbb{Z}_3$, and $c_\varphi(\Gamma)$ is not contained in $\text{SL}_2(5)$. Thus the center of $\Gamma^\varphi$ is $\mathbb{Z}_2 \times \mathbb{Z}_3 = \mathbb{Z}_6$.

**Example 6.4.** Assume that $\Gamma$ and $G$ are finite groups. Let $C$ be the normal closure of $\varphi(\Gamma)$ in $G$. Assume that $\Gamma$ and $C$ are $\pi$-groups. Recall that a $\pi$-group, for a set of primes $\pi$, is a group $H$ such that any prime $p$ dividing the order of $H$ is in $\pi$.

We claim that $\Gamma^\varphi$ is also a finite $\pi$-group, further, if $C$ is solvable (nilpotent), so is $\Gamma^\varphi$. If $C$ and $\Gamma$ are $p$-groups so is $\Gamma^\varphi$.

By Corollary 3.9(2), $\Gamma^\varphi$ is finite. By Lemma 3.8(3), $\ker \overline{\varphi} \leq Z(\Gamma^\varphi)$, so by Lemma 3.3(2), since the normal closure of $\varphi(\Gamma)$ in $G$ is a $\pi$-group, we see that $\Gamma^\varphi / \ker \overline{\varphi}$ is a $\pi$-group. Let $\pi'$ be the complement to $\pi$ in the set of all primes. If $O_{\pi'}(Z(\Gamma^\varphi)) \neq 1$, then, by the Schur-Zassenhaus Theorem ([A (18.1), p. 70]), we would get $\Gamma^\varphi = O_{\pi}(\Gamma^\varphi) \times O_{\pi'}(\Gamma^\varphi)$. But then $c_\varphi(\Gamma) \leq O_{\pi}(\Gamma^\varphi)$, since $c_\varphi(\Gamma)$ is a $\pi$-subgroup of $\Gamma^\varphi$. Hence also $\Gamma^\varphi = \langle c_\varphi(\Gamma)^g \mid g \in G \rangle \leq O_{\pi}(\Gamma^\varphi)$. But since $\Gamma^\varphi$ is generated by $\{c_\varphi(\Gamma)^g \mid g \in G \}$, we see that $O_{\pi'}(\Gamma^\varphi) = 1$.

### 7. The injective normalizer of a map

**Definition 7.1.** Let $\varphi: \Gamma \to G$ be a map of groups. The *injective normalizer* of $\varphi$ is a factorization of the latter via

$$\Gamma \xrightarrow{\overline{\varphi}} N(\varphi) \xrightarrow{\overline{\varphi}_\varphi} G$$

such that $\overline{\varphi}: \Gamma \to N(\varphi)$ is a normal map, and such that for any other factorization

$$\varphi = n \circ f$$

via a normal map $n: \Gamma \to H$ with $f: H \to G$, there exists a unique normal morphism $\overline{f}: H \to N(\varphi)$ rendering the diagram below commutative.
(7.1) \[ \bar{\phi} \]

In subsection 7.3 below we will show that the injective normalizer exists. We have

**Lemma 7.2.** The injective normalizer of \( \phi : \Gamma \to G \) is unique up to a normal isomorphism.

*Proof.* Straight forward from the universal properties. \( \square \)

### 7.3. The construction of the injective normalizer.

Throughout this section \( \phi : \Gamma \to G \) be a map of groups.

**Definition 7.4.** Let \( \tau \in \text{Aut}(\Gamma) \) and \( g \in G \). We say that \( \tau \) and \( g \) are compatible if the following diagram is commutative,

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{\phi} & G \\
\downarrow{\tau} & & \downarrow{c_g} \\
\Gamma & \xrightarrow{\phi} & G
\end{array}
\]

where \( c_g \) is conjugation by \( G \). Thus

\[ \phi(\gamma)^g = \phi(\tau(\gamma)), \quad \forall \gamma \in \Gamma. \]

**Definition 7.5.** The injective normalizer of \( \phi \) is the subgroup of \( \text{Aut}(\Gamma) \times N_G(\phi(\Gamma)) \) given by

\[ N(\phi) := \{ (\tau, g) \mid (\tau, g) \text{ is a compatible pair} \}. \]

The next lemma shows that \( N(\phi) \) is indeed a subgroup of \( \text{Aut}(\Gamma) \times N_G(\phi(\Gamma)) \).

**Lemma 7.6.**

1. \((id, 1)\) is a compatible pair;
2. if \((\tau, g)\) is a compatible pair, then \((\tau^{-1}, g^{-1})\) is a compatible pair;
3. the product of two compatible pairs is compatible;

*Proof.*

1. We take in equation (7.2) \( \tau = id \) and \( g = 1 \) and we get
   \[ \phi(\gamma)^1 = \phi(id(\gamma)), \quad \forall \gamma \in \Gamma. \]
   Hence (1) holds.
2. Since \((\tau, g)\) is compatible we have
   \[ \phi(\gamma)^g = \phi(\tau(\gamma)), \quad \forall \gamma \in \Gamma. \]
Now replace each \( \gamma \) with \( \tau^{-1}(\gamma) \) to get
\[
\varphi(\tau^{-1}(\gamma))^g = \varphi(\gamma), \quad \forall \gamma \in \Gamma,
\]
and now conjugate both sides with \( g^{-1} \) to get
\[
\varphi(\gamma)^{g^{-1}} = \varphi(\tau^{-1}(\gamma)), \quad \forall \gamma \in \Gamma,
\]
This show (2).

(3): We have
\[
\varphi(\gamma)^v = \varphi(\rho(\gamma)), \quad \forall \gamma \in \Gamma,
\]
for \((\rho, v) = (\tau, g)\) and \((\sigma, h)\). Thus
\[
(\varphi(\gamma))^gh = \varphi((\gamma)\tau^h) = \varphi((\gamma)\tau\sigma), \quad \forall \gamma \in \Gamma.
\]
Thus \((\tau\sigma, gh)\) is a compatible pair. \(\square\)

Remarks 7.7.  
(1) If \( \varphi \) is the trivial map, then any pair \((\tau, g) \in \text{Aut}(\Gamma) \times G\) is a compatible pair, so \( N(\varphi) = \text{Aut}(\Gamma) \times G \).
(2) If \( \varphi \) is inclusion, then \( N(\varphi) = \{\varphi c_g \varphi^{-1}, g \mid g \in N_G(\Gamma)\} \), where \( c_g \) is the inner automorphism of \( G \) induced by \( g \). Hence \( N(\varphi) \cong N_G(\Gamma) \).
(3) Of course \( p_\varphi(N(\varphi)) \) does not always equal to \( N_G(\varphi(\Gamma)) \). For example, let \( \Gamma \) be an extraspecial group of order \( 2^{2n+1}, n \geq 2 \) which is a central product of \( n \) dihedral groups of order 8, and let \( G \) be a semi-direct product \( E \rtimes L \), where \( E \) is an elementary abelian group of order \( 2^n \), and \( L \cong GL(n, 2) \). Note that \( \text{Aut}(\Gamma) \) is an extension of an elementary abelian group of order \( 2^n \) by the orthogonal group \( O_+(2n, 2) \). Let \( \varphi: \Gamma \to G \), be the natural map taking \( \Gamma \) onto \( E \). Then not every \( g \in L \) lifts to an automorphism of \( \Gamma \), so \( p_\varphi(N(\varphi)) \neq G = N_G(\varphi(\Gamma)) \).

Lemma 7.8.  
Let
\[
\bar{\varphi}: \Gamma \to N(\varphi) : \gamma \mapsto (c_\gamma, \varphi(\gamma)), \quad \forall \gamma \in \Gamma.
\]
Let \( N(\varphi) \) act on \( \Gamma \) via \( \gamma^{(\tau, g)} = (\gamma)\tau \), for all \( \gamma \in \Gamma \). Let
\[
p_\varphi: N(\varphi) \to G : (\tau, g) \mapsto g,
\]
be the projection on the second coordinate. Then \( \bar{\varphi} \) is a normal map, and \( \bar{\varphi} \circ p_\varphi = \varphi \).

Proof. First note that \( \bar{\varphi} \) is a group map from \( \Gamma \) to \( N(\varphi) \). We check (NM1). For all \( \gamma \in \Gamma \) and \((\tau, g) \in N(\varphi)\), we have
\[
(\gamma^{(\tau, g)})\bar{\varphi} = (\gamma)\tau\bar{\varphi} = (c_\gamma, (\gamma)\tau\varphi).
\]
while
\[
((\gamma)\bar{\varphi})^{(\tau, g)} = (c_\gamma, (\gamma)\varphi)^{(\tau, g)} = (c_\gamma^\tau, ((\gamma)\varphi)^g) = (c_{\gamma^\tau}, (\gamma)\tau\varphi),
\]
where the last equality follows from the fact that \((\tau, g)\) is a compatible pair. This shows (NM1).

We now check (NM2).
\[
(\gamma)^{(\delta)}\bar{\varphi} = (\gamma)^{(c_\gamma, \varphi(\delta))} = (\gamma)c_\delta = \gamma^\delta, \quad \forall \gamma, \delta \in \Gamma.
\]
This shows (NM2) and shows that $\tilde{\varphi}$ is a normal map. Clearly $\tilde{\varphi} \circ p_\varphi = \varphi$. □

**Proposition 7.9.** Let $\Gamma \xrightarrow{n} H \xrightarrow{f} G$ be a decomposition of $\varphi$, with $n$ a normal map. Then there exists a unique map $\tilde{f}: H \to N(\varphi)$ rendering diagram (7.1) commutative.

**Proof.** For each $h \in H$ we let $\ell_h \in \text{Aut}(\Gamma)$ be the automorphism that comes from the normal structure of $n$. As usual, we write $\ell_h(\gamma) = \gamma h$, for all $\gamma \in \Gamma$ and $h \in H$. Let

$$\tilde{f}: H \to N(\varphi): h \mapsto (\ell_h, f(h)).$$

We first claim that $(\ell_h, f(h))$ is a compatible pair. By equation (7.2) we must check that

$$\varphi(\gamma)^{f(h)} = \varphi(\ell_h(\gamma)), \quad \forall \gamma \in \Gamma.$$

Now

$$\varphi(\gamma^h) = f(n(\gamma^h)) = f(n(\gamma))^h = \varphi(\gamma)^{f(h)} = \varphi(\gamma)^{f(h)},$$

for all $h \in H$, thus $\tilde{f}$ is indeed a map from $H$ to $N(\varphi)$. Clearly $\tilde{f}$ is a group map. Also

$$(\gamma)n \circ \tilde{f} = \tilde{f}(n(\gamma)) = (\ell_{n(\gamma)}, f(n(\gamma))) \overset{(*)}{=} (c_{\gamma}, \varphi(\gamma)) = \tilde{\varphi}(\gamma),$$

for all $\gamma \in \Gamma$, where the equality $(*)$ follows from the fact that $n$ is a normal map.

Next

$$\tilde{f} \circ p_\varphi = (\ell_h, f(h))p_\varphi = f(h).$$

Thus $\tilde{f}$ renders diagram (3.1) commutative.

Finally, we must show that

$$\gamma^h = \gamma^{\tilde{f}(h)}, \quad \text{for all } \gamma \in \Gamma \text{ and } h \in H.$$

However, by definition

$$\gamma^{\tilde{f}(h)} = \gamma^{(\ell_h, f(h))} = (\gamma)\ell_h = \gamma^h.$$

It remains to show that $\tilde{f}$ is unique. So let

$$\hat{f}: H \to N(\varphi),$$

rendering diagram (7.1) commutative. Let $h \in H$ and let $\hat{f}(h) = (\tau, g)$. Then $f(h) = p_\varphi(\hat{f}(h)) = g$, so $g = f(h)$. Also, since $\hat{f}$ is a normal morphism,

$$\gamma^h = \gamma^{\hat{f}(h)} = (\gamma)\tau,$$

so $\tau = \ell_h$, and so $\tilde{f}$ is unique and the proof of the proposition is complete. □
**Remark 7.10.** Let $\psi: G \to G'$, and consider the diagram

\[
\begin{array}{c}
\Gamma \\
\downarrow \phi \downarrow \\
G \\
\downarrow \pi \\
N(\phi) \\
\downarrow \phi \circ \psi \\
N(\phi \circ \psi)
\end{array}
\]

Thus we see that $\psi$ induces a map $\tilde{p} \circ \psi: N(\phi) \to N(\phi \circ \psi)$.

Our next lemma shows that the injective normalizer of $\phi$ precisely detects whether $\phi$ is a normal map.

**Lemma 7.11.** The map $\phi: \Gamma \to G$ is a normal map iff the map $p_\phi: N(\phi) \to G$ has a section $s: G \to N(\phi)$ such that $\phi \circ s = \tilde{\phi}$.

**Proof.** Assume first that $\phi$ is a normal map. Then $\phi$ decomposes as $\Gamma \xrightarrow{\phi} G \xrightarrow{id} G$, with $\phi$ a normal map. Thus $H, f$ of diagram (7.1) are $G, id$ respectively. The universality property of the injective normalizer implies the existence of a map $s: G \to N(\phi)$ (the map $\tilde{f}$) rendering diagram 7.1 commutative. In particular $s \circ p_\phi = id$, and $\phi \circ s = \tilde{\phi}$.

Conversely, assume there exists $s: G \to N(\phi)$ with $s \circ p_\phi = id$ and $\phi \circ s = \tilde{\phi}$. Define an action of $G$ on $\Gamma$ as follows. Let $\pi: N(\phi) \to \text{Aut}(\Gamma)$ be the projection on the first coordinate, and let $\gamma^g = \pi(s(g))(\gamma)$, for all $\gamma \in \Gamma$ and $g \in G$. We show that this is a normal structure on $\phi$.

Let $\gamma \in \Gamma$ and $g \in G$. Then $(\gamma^g)\phi = \phi(\pi(s(g))(\gamma))$. Notice now that $s(g) = (\tau, g) \in N(\phi)$, with $\tau \in \text{Aut}(\Gamma)$, since $s \circ p_\phi = id$. Thus $(\gamma^g)\phi = \phi(\pi(s(g))(\gamma)) = \phi(\tau(\gamma))$. Since $(\tau, g)$ is a compatible pair, $\phi(\tau(\gamma)) = \phi(\gamma)^g$. Thus we see that $(\gamma^g)\phi = \phi(\gamma)^g$ and (NM1) holds.

Next for $\gamma, \delta \in \Gamma$ we have

\[
(\gamma)^{\phi}(\delta) = \pi(s(\phi(\delta)))(\gamma) = \pi(\phi(\delta))(\gamma) = c_\delta(\gamma) = \gamma^\delta,
\]

and (NM2) holds. □

## 8. Stability of the Normalizers Tower

In this section $\phi: \Gamma \to G$ is a homomorphism of groups and $\Gamma \xrightarrow{\tilde{\phi}} N(\phi) \xrightarrow{p_\phi} G$ is its injective normalizer. Theorem 8.3 below generalizes the obvious observation that the repeated normalizer of any subgroup of a finite group stabilizes (the case where $\phi$ is injective), and the well known stability of the automorphism tower of a finite group with trivial center (here: the case where $G = 1$, see Remark 7.7(1)).

The following result will be required for the proof of Theorem 8.3.
Theorem 8.1. (1) (Wielandt, [R] 13.5.2) Let $H$ be a finite group and let $K$ be a sub-normal subgroup of $H$ such that $C_H(K) = 1$. Then there is an upper bound on $|H|$ depending only on $|K|$.

(2) ([R] 13.5.3) Let $\alpha$ be any ordinal, and let $K = K_0 \subseteq K_1 \subseteq \ldots \subseteq K_\alpha \subseteq K_{\alpha+1} \subseteq \ldots$ be an ascending chain of groups such that $C_{K_{\alpha+1}}(K_\alpha) = 1$, for all $\alpha$. Then $C_{K_{\alpha+1}}(K_\alpha) = 1$.

Lemma 8.2. We have

(1) $p_\varphi(N(\varphi)) \leq N_G(\varphi(\Gamma))$;
(2) if $(\tau, g) \in N(\varphi)$, then $\tau(\ker \varphi) = \ker \varphi$, and hence $\tau$ acts on $\Gamma/\ker \varphi$;
(3) $C_{N(\varphi)}(\tilde{\varphi}(\Gamma)) = \{(\tau, g) \in N(\varphi) \mid \tau \in C_{\text{Aut}(\Gamma)}(\Gamma/\ker \varphi) \text{ and } g \in G(\varphi(\Gamma))\}$;
(4) $\ker p_\varphi = \{(\tau, 1) \in N(\varphi) \mid \tau \in C_{\text{Aut}(\Gamma)}(\Gamma/\ker \varphi) \}$ and $\tilde{\varphi}(\ker \varphi) \leq \ker p_\varphi$;
(5) $\ker \tilde{\varphi} = Z(\Gamma) \cap \ker \varphi$;
(6) if $\ker \tilde{\varphi} = 1$, then $\ker \tilde{\varphi}_1 = 1$, where $\varphi_1 := p_\varphi : N(\varphi) \to G$;
(7) set $p_\varphi = \varphi_1$ and assume that $Z(\ker \varphi) = 1$, then

(a) $C_{\ker \varphi_1}(\tilde{\varphi}(\ker \varphi)) = 1$;
(b) $Z(\ker \varphi_1) = 1$.

Proof. (1): $(\tau, g) \in N(\varphi)$, iff $\varphi(\gamma)g = \varphi(\gamma)$, for all $\gamma \in \Gamma$. Hence $\varphi(\gamma)^g = \varphi(\gamma)$, for all $\gamma \in \Gamma$, so $g \in N_G(\varphi(\Gamma))$.

(2): If $\gamma \in \ker \varphi$, then $1 = \varphi(\gamma)^g = \varphi(\gamma)$, so $\tau(\gamma) \in \ker \varphi$. Since by Lemma 7.6(2), $(\tau^{-1}, g^{-1}) \in N(\varphi)$, part (2) holds.

(3): By definition, $\tilde{\varphi}(\Gamma) = \{(c_\gamma, \varphi(\gamma)) \mid \gamma \in \Gamma\}$. Thus if $(\tau, g) \in C_{N(\varphi)}(\tilde{\varphi}(\Gamma))$, then $\tau, g \in C_{\text{Aut}(\Gamma)}(\Gamma/\ker \varphi)$ and $g \in G(\varphi(\Gamma))$. It is easy to check that $C_{\text{Aut}(\Gamma)}(\Gamma/\ker \varphi) = C_{\text{Aut}(\Gamma)}(\Gamma/\ker \varphi)$.

(4): If $(\tau, 1) \in \ker p_\varphi$, then $\varphi(\gamma) = \varphi(\gamma)^1 = \varphi(\gamma)$, so we see that this is equivalent to $\tau$ centralizing $\Gamma/\ker \varphi$. The second part of (4) follows from $\tilde{\varphi} \circ p_\varphi = \varphi$.

(5): By definition, $\ker \tilde{\varphi} = \{\gamma \in \Gamma \mid (c_\gamma, \varphi(\gamma)) = (id, 1)\}$. Thus $\gamma \in Z(\Gamma) \cap \ker \varphi$.

(6): By (5), $\ker \tilde{\varphi}_1 = Z(N(\varphi)) \cap \ker(p_\varphi)$. Let $(\tau, g) \in \ker \tilde{\varphi}_1$. By (4), $g = 1$. By (3) and (4), $\tau \in C_{\text{Aut}(\Gamma)}(\Gamma/\ker \varphi) = C_{\text{Aut}(\Gamma)}(\Gamma/\ker \varphi) = C_{\text{Aut}(\Gamma)}(\Gamma) = id$, because $1 = \ker \tilde{\varphi} = Z(\Gamma) \cap \ker \varphi$.

(7): (a) Note that $\tilde{\varphi}(\ker \varphi) = \{(c_\gamma, 1) \mid \gamma \in \ker \varphi\} \leq \ker p_\varphi$. Let now $(\tau, 1) \in C_{\ker p_\varphi}(\tilde{\varphi}(\ker \varphi))$. By (2), $\tau(\ker \varphi) = \ker \varphi$. It follows that the restriction of $\tau$ to $\ker \varphi$ centralizes $\ker \varphi/\ker \varphi$. Since $Z(\ker \varphi) = 1$, we see that $\tau \in C_{\text{Aut}(\Gamma)}(\ker \varphi)$. Thus, by (4),

$$\tau \in C_{\text{Aut}(\Gamma)}(\Gamma/\ker \varphi) \cap C_{\text{Aut}(\Gamma)}(\ker \varphi).$$

However, $C_{\text{Aut}(\Gamma)}(\Gamma/\ker \varphi) \cap C_{\text{Aut}(\Gamma)}(\ker \varphi) \cong Z(\Gamma/\ker \varphi) \cong Z(\ker \varphi) (\text{see, e.g. [H] Satz I.4.4})$.

Since $Z(\ker \varphi) = 1$, equation (8.1) implies that $\tau = id$. This shows part (a).

(b): Let $(\tau, 1) \in Z(\ker p_\varphi)$, then, in particular, $(\tau, 1) \in C_{\ker p_\varphi}(\tilde{\varphi}(\ker \varphi))$, so by (a), $\tau = id$. □

As a corollary we get the following
**Theorem 8.3.** Assume that $Z(\ker \varphi) = 1$ and that $\Gamma$ and $G$ are finite. Then the normalizers tower (see equation 1.3) terminates after a finite number of steps.

**Proof.** (1): Consider the normalizers tower of diagram (1.3). Let $K_i = \ker \varphi_i$, for all non-negative integers $i$.

Note that by induction on $i$, by our assumption that $Z(\ker \varphi) = 1$, and by Lemma 8.2(7b), we have $Z(\ker \varphi_i) = 1$, for all non-negative integers $i$. By Lemma 8.2(5&6), $\tilde{\varphi}_i$ is injective, and by Lemma 8.2(4), $\tilde{\varphi}_i(K_i) \leq K_{i+1}$, for all non-negative integers $i$.

Thus after appropriate identifications, and using the fact that $\tilde{\varphi}_i$ is a normal map (so its image is normal in $\Gamma_{i+1}$), we have $K_i \triangleleft K_{i+1}$, for all non-negative integers $i$, and furthermore, $C_{K_{i+1}}(K_i) = 1$, by Lemma 8.2(7a).

We can now apply Theorem 8.1 as in the proof of Wielant’s Theorem that the automorphism tower of a group with trivial center terminates after finitely many steps (see, e.g., [R, 13.5.4]). Namely, by Theorem 8.1(2), $C_{K_{i+1}}(K_0) = 1$, for all positive integers $i$, and hence, by Theorem 8.1(1), there exists $s$ such that $K_s = K_{s+1} = K_{s+2} \ldots$.

However, by definition, $\varphi_i(\Gamma^i) \leq \varphi_{i+1}(\Gamma^{i+1})$, for all $i \geq 0$. Hence there exists $t$ such that $\varphi_t(\Gamma^t) = \varphi_{t+1}(\Gamma^{t+1}) = \varphi_{t+2}(\Gamma^{t+2}) \ldots$.

It follows that for $m = \max\{t, s\}$ we get that $\Gamma^m \simeq \Gamma^{m+1} \simeq \Gamma^{m+2} \ldots$. □

**Appendix A. Details of the construction of the free normal closure**

Recall the definition of the group $F$ from the beginning of §3.4. We define a map of groups

$$\varphi : F \to G : \gamma_g \mapsto \varphi(\gamma)^g, \quad \gamma \in \Gamma, \ g \in G.$$  

First we show that the map $\varphi$ respects the relations $R_1, R_{nm1}$ and $R_{nm2}$.

**Lemma A.1.** The map $\varphi$ respects the relations $R_1, R_{nm1}, R_{nm2}$.

**Proof.** For the relations $R_1$ we have

$$\varphi(1_g) = \varphi(1)^g = 1^g = 1 \in G, \quad \forall g \in G.$$  

For the relations $R_{nm1}$, let $\gamma, \delta \in \Gamma$ and $g \in G$, then we get

$$\varphi(\gamma_g \delta_g) = \varphi(\gamma_g)\varphi(\delta_g) = \varphi(\gamma)^g\varphi(\delta)^g = \varphi(\gamma\delta)^g = \varphi((\gamma\delta)_g).$$

Hence

$$\varphi(\gamma_g \delta_g) = \varphi((\gamma\delta)_g).$$
Finally for the relations $R_{nm2}$, let $\gamma, \delta \in \Gamma$ and $h, g \in G$, then
\[
\varphi(\delta^{-1} \gamma g \delta h) = \varphi(\delta^{-1} \gamma g) \varphi(\delta h) = (\varphi(\delta)^h)^{-1} \varphi(\gamma g) \varphi(\delta)^h = \varphi(\gamma g \varphi(\delta)^h).
\]
Hence
\[
\varphi(\delta^{-1} \gamma g \delta h) = \varphi(\gamma g \varphi(\delta)^h),
\]
and the lemma holds. \hfill \Box

By Lemma A.1 the map $\varphi$ induces a map, which we continue to denote by $\varphi$, as in equation (3.5).

**Lemma A.2.** The map $\ell: G \to \Gamma^\varphi$ defined in equation (3.6) is a well defined homomorphism of groups.

**Proof.** We start by defining an action of $G$ on $F$. For each $h \in G$ we define an automorphism $\ell(h) \in \text{Aut}(F)$. We denote the image of $\gamma g \in F$ under $\ell(h)$ by $\gamma h g$, and we define:
\[
\gamma h g = \gamma gh,
\]
for all $\gamma \in \Gamma$, $g \in G$.

Since $F$ is a free group on the generators $(\Gamma \setminus \{1\})G$, $\ell(h)$ determines a unique automorphism of $F$.

Next we show that the automorphism $\ell(h)$ of $F$ preserves the relations $R_{nm1}$ and $R_{nm2}$.

For the relations $R_{nm1}$ we need show that
\[
(\gamma g)^h(\delta g)^h = ((\gamma \delta) g)^h, \quad \forall \gamma, \delta \in \Gamma \text{ and } \forall g \in G,
\]
is a relation in $R_{nm1}$. Indeed
\[
(\gamma g)^h(\delta g)^h = \gamma g h \delta g h \quad \text{while} \quad ((\gamma \delta) g)^h = (\gamma \delta) g h,
\]
and
\[
\gamma g h \delta g h = (\gamma \delta) g h,
\]
is a relation in $R_{nm1}$.

For the relations $R_{nm2}$ we need show that
\[
(\delta_t^{-1} \gamma g \delta_t)^h = (\gamma g \varphi(\delta)^{t h})^h, \quad \forall \gamma, \delta \in \Gamma \text{ and } \forall h, g, t \in G,
\]
is a relation in $R_{nm2}$. Indeed
\[
(\delta_t^{-1} \gamma g \delta_t)^h = \delta_t^{-1} \gamma g h \delta_t h \quad \text{while} \quad (\gamma g \varphi(\delta)^{t h})^h = (\gamma g \varphi(\delta)^{t h}),
\]
and
\[
\delta_t^{-1} \gamma g h \delta_t h = \gamma g h \varphi(\delta)^{t h} \quad (= \gamma g \varphi(\delta)^{t h}),
\]
is a relation in $R_{nm2}$.

Since $\ell(h) \in \text{Aut}(F)$ preserves the relations $R_{nm1}$ and $R_{nm2}$, we can define the automorphism $\tilde{\ell}$ of equation (3.6) on the generators $\{\gamma g \mid \gamma \in \Gamma \text{ and } g \in G\}$ of $\Gamma^\varphi$, and extend $\tilde{\ell}$ to an automorphism of $\Gamma^\varphi$. Since $\ell(h)$ has an inverse $\ell(h^{-1})$, it is indeed an automorphism of $\Gamma^\varphi$. Further $(\gamma g)^{hh'} = \gamma g h h' = (\gamma g)^h h'$, Thus $\tilde{\ell}: G \to \text{Aut}(\Gamma^\varphi)$ is a homomorphism. \hfill \Box
Lemma A.3. The map \( \overline{\varphi} : \Gamma^\varphi \to G \) is a normal map, having the normal structure \( \overline{\ell} : G \to \text{Aut}(\Gamma^\varphi) \).

Proof. Notice that we have

\[
((\overline{\gamma}_g)^h)\overline{\varphi} = (\overline{\gamma}_gh)\overline{\varphi} = \varphi(\gamma)^h = ((\overline{\gamma}_g)\overline{\varphi})^h,
\]

for all \( \gamma \in \Gamma \) and \( g, h \in G \). That is

\[
((\overline{\gamma}_g)^h)\overline{\varphi} = ((\overline{\gamma}_g)\overline{\varphi})^h, \quad \forall g, h \in G.
\]

This shows that (NM1) in the definition of a normal map is satisfied by \( \overline{\varphi} \), on the generators of \( \Gamma \) and hence (NM1) holds for \( \overline{\varphi} \).

Next we wish to show that (NM2) in the definition of a normal map is satisfied by \( \overline{\varphi} \). It suffices to show this for the generators of \( \Gamma \), that is we need to show that

\[
(\overline{\gamma}_g(\delta)h)\overline{\varphi} = (\overline{\gamma}_g)^h, \quad \forall \gamma, \delta \in \Gamma \text{ and } \forall g, h \in G.
\]

By the definition of \( \overline{\varphi} \) equation (nm2) means

\[
(\overline{\gamma}_g)^h = (\overline{\gamma}_g)^h, \quad \forall g, h \in G.
\]

Since the last equality holds in \( \Gamma^\varphi \), the lemma holds. \( \square \)

A.4. The universality of \( \Gamma^\varphi \).

We first define

\[
c_\varphi : \Gamma \to \Gamma^\varphi : \gamma \mapsto \overline{\gamma}_1.
\]

Let now \( n : M \to G \) be a normal map such that there exists \( \psi : \Gamma \to M \) with \( \psi \circ n = \varphi \). We define a map

\[
\overline{\psi} : F \to M : \gamma_g \mapsto \psi(\gamma)^g,
\]

where recall that \( M \) is a crossed module, so there is an action of \( G \) on \( M \) and \( \psi(\gamma)^g \) is the image of \( \psi(\gamma) \) under the action of \( g \). Now

\[
(\gamma_g\delta_g)\overline{\psi} = \psi(\gamma)^g\psi(\delta)^g = \psi(\gamma\delta)^g = ((\gamma\delta)_g)\overline{\psi}.
\]

Further

\[
(\gamma_g\psi(\delta)^h)\overline{\psi} = \psi(\gamma)^{gh^{-1}\psi(\delta)^h}
\]

\[
= \psi(\gamma)^{gh^{-1}(\psi(n))h} \quad \text{(because } \varphi = \psi \circ n)\]

\[
= \psi(\gamma)^{gh^{-1}(\psi(\delta))h} \quad \text{(because } n \text{ is a normal map)}
\]

\[
= \psi(\gamma)^{\psi(\delta)^h}.
\]
while

\[
(\delta_h^{-1}\gamma_g\delta_h)\overline{\psi} = ((\delta_h)\overline{\psi})^{-1} \cdot ((\gamma_g)\overline{\psi}) \cdot (\delta_h)\overline{\psi} = (\psi(\delta)^h)^{-1} \psi(\gamma) c(\delta)^h = \psi(\gamma)(g\psi(\delta))^h.
\]

It follows that $\overline{\psi}$ induces a homomorphism which we also denote by $\overline{\psi}$

\[
\overline{\psi}: \Gamma^\varphi \to M : \gamma_g \mapsto \psi(\gamma)^g.
\]

We have

\[
(\gamma)c_\varphi \circ \overline{\psi} = (\overline{\gamma_1})\overline{\psi} = \psi(\gamma)^{1} = \psi(\gamma).
\]

Thus

\[
c_\varphi \circ \overline{\psi} = \psi.
\]

Similarly

\[
c_\varphi \circ \overline{\varphi} = \varphi.
\]

Next let

\[
\overline{\mu}: \Gamma^\varphi \to M, \text{ with } c_\varphi \circ \overline{\mu} = \psi,
\]

where $\overline{\mu}$ is a normal morphism. Then

\[
(\gamma)c_\varphi \circ \overline{\mu} = (\overline{\gamma_1})\overline{\mu} = (\gamma)\psi.
\]

Also

\[
\overline{\mu}(\overline{\gamma}_g) = \overline{\mu}(\overline{\gamma_1})^g = (\overline{\mu}(\overline{\gamma}_1))^g = \psi(\gamma)^g, \quad \forall \gamma \in \Gamma \text{ and } \forall g \in G.
\]

Since $\Gamma^\varphi$ is generated by \{\overline{\gamma}_g\}, we conclude that $\overline{\mu} = \overline{\psi}$, and $\overline{\psi}$ is unique.

**Remark A.5.** Let $f: \Gamma \to \Gamma'$, and consider the diagram

Thus we see that $f$ induces a normal morphism $\overline{f \circ \varphi}: (\Gamma')^{f \circ \varphi} \to \Gamma^\varphi$. 
Appendix B. The Normal Closure of an Inclusion of a Normal Subgroup

In this appendix we prove a lemma about $\varphi^\gamma$ in the situation where $\varphi : \Gamma \to G$ is an inclusion of a normal subgroup.

Lemma B.1. Assume that $\varphi$ is injective and that $\varphi(\Gamma) \leq G$. Then

1. $c_\varphi(\Gamma)$ is a retract of $\Gamma^\varphi$, and $\Gamma^\varphi = c_\varphi(\Gamma) \times \ker \varphi$;
2. we can take $\Gamma^\varphi = \varphi(\Gamma) \times \ker \varphi$, with the action

\[(\varphi(\gamma), v)^\varphi = (\varphi(\delta), \varphi(\gamma)v), \quad \text{where } c_\varphi(\gamma)^\varphi = c_\varphi(\delta)v, \quad \delta, \gamma \in \Gamma \text{ and } v \in \ker \varphi,\]

for all $\gamma \in \Gamma$ and $v \in \ker \varphi$. We take $c_\varphi$ to be the map defined by $\gamma \mapsto (\varphi(\gamma), 0)$, and $\varphi$ the map defined by $(\varphi(\gamma), v) \mapsto \varphi(\gamma)$, for all $\gamma \in \Gamma$ and $v \in \ker \varphi$. In particular, if $\Gamma^\varphi = c_\varphi(\Gamma)$, then we can take $c_\varphi = \varphi$ and consequently $\Gamma^\varphi = \varphi(\Gamma)$ and $\varphi = \text{inc}$ is the inclusion map.

3. The following are equivalent
   (a) $\Gamma^\varphi = c_\varphi(\Gamma)$.
   (b) $c_\varphi(\Gamma)$ is $G$-invariant in $\Gamma^\varphi$.
   (c) $\psi(\Gamma)$ is $G$-invariant in $M$, for all factorization $\Gamma \xrightarrow{\psi} M \xrightarrow{m} G$ of $\varphi$, with $m$ a normal map.

4. The following are equivalent
   (a) $\Gamma^\varphi = \varphi(\Gamma)$.
   (b) For all abelian groups $V$ and any action of $G$ on $\varphi(\Gamma) \times V$ such that $(\varphi(\gamma), 0)^{\varphi(\delta)} = (\varphi(\gamma^\delta), 0)$, for all $\gamma, \delta \in \Gamma$, the subgroup $\{((\varphi(\gamma), 0) \mid \gamma \in \Gamma\}$ of $\varphi(\Gamma) \times V$ is $G$-invariant.

Proof. (1): Since by Lemma 3.3(2) $(\Gamma^\varphi)^\varphi = (\Gamma)^\varphi$, so the map $\varphi \circ \varphi^{-1} \circ c_\varphi$ is a well defined map on $\Gamma^\varphi$ and $(\varphi(c_\varphi(\gamma))) \varphi = (\varphi(c_\varphi(\gamma))) \varphi^{-1} \circ c_\varphi = (\gamma) \varphi \circ \varphi^{-1} \circ c_\varphi = (\gamma)c_\varphi$, for all $\gamma \in \Gamma$. This shows that $(\Gamma)c_\varphi$ is a retract of $\Gamma^\varphi$. Let us now compute the kernel of the retraction $\varphi \circ \varphi^{-1} \circ c_\varphi$. Since $\varphi$ (and hence $c_\varphi$) are injective, we see that this kernel is $\ker \varphi$. By Lemma 3.3(3), $\ker \varphi \leq Z(\Gamma^\varphi)$, Hence (1) holds.

(2): Let $\tilde{\Gamma} = \varphi(\Gamma) \times \ker \varphi$. Note that the action of $G$ on $\tilde{\Gamma}$ is defined by $(\varphi(\gamma), v)^\varphi = \tau((\varphi^{-1}(\varphi(\gamma), v))^\varphi)$, where $\tau : \Gamma^\varphi \to \tilde{\Gamma}$ is the isomorphism defined by $c_\varphi(\gamma)v \mapsto (\varphi(\gamma), v)$ (see part (1)). Note further that we are using Lemma 2.1(1), so that $v^\varphi \in \ker \varphi$, for all $v \in \ker \varphi$. Hence we indeed defined an action of $G$ on $\tilde{\Gamma}$.

Let $\tilde{c}_\varphi$ be the map defined by $\gamma \mapsto (\varphi(\gamma), 0)$, and $\tilde{\varphi}$ be the map defined by $(\varphi(\gamma), v) \mapsto \varphi(\gamma)$, for all $\gamma \in \Gamma$ and $v \in \ker \varphi$.

Assume that $\Gamma \xrightarrow{\psi} M \xrightarrow{\nu} G$ is a factorization of $\varphi$. Let $\tilde{\psi} : \Gamma^\varphi \to \tilde{\Gamma}$ be the unique map of diagram (3.1), and set

$$\tilde{c}_\varphi : \tilde{\Gamma} \to M : (\varphi(\gamma), v) \mapsto \tilde{\psi}(c_\varphi(\gamma)v) = \psi(\gamma)\tilde{\psi}(v).$$
Let \((\varphi(\gamma), v) \in \tilde{\Gamma}\), let \(g \in G\) and write \(c_{\varphi}(\gamma)^g = c_{\varphi}(\delta)u\), with \(u \in \ker \overline{\varphi} \). Then
\[
\overline{c}_\varphi((\varphi(\gamma), v)^g) = \overline{c}_\varphi(\varphi(\delta), uv^g) = \overline{\varphi}(c_{\varphi}(\delta)u) = \overline{\varphi}(c_{\varphi}(\gamma)^g) = (\overline{\varphi}(c_{\varphi}(\gamma))^g) = (\overline{\varphi}(\varphi(\gamma))^g).
\]
This shows that \(\overline{c}_\varphi\) is an initial normal morphism over \(G\).

Also,
\[
\overline{\varphi}((\varphi(\gamma), v)^g) = \overline{\varphi}((\varphi(\delta), uv^g) = \varphi(\delta) = (\delta)(c_{\varphi} \circ \overline{\varphi}) = \overline{\varphi}(c_{\varphi}(\delta))
\]
\[
= \overline{\varphi}(c_{\varphi}(\gamma)^gu^{-1}) = \overline{\varphi}(c_{\varphi}(\gamma)^g) = \overline{\varphi}(c_{\varphi}(\gamma))^g = \varphi(\gamma)^g
\]
\[
= (\overline{\varphi}(\varphi(\gamma))^g).
\]

So \(\overline{\varphi}\) satisfies (NM1). Further,
\[
(B.1) \quad c_{\varphi}(\gamma)^{\varphi(\delta)} = c_{\varphi}(\gamma)^{c_{\varphi}(\delta)} = c_{\varphi}(\gamma)^{c_{\varphi}(\delta)} = c_{\varphi}(\gamma^\delta), \quad \forall \gamma, \delta \in \Gamma.
\]

Notice also that for \(v \in \ker \overline{\varphi}\) and \(\delta \in \Gamma\) we have
\[
(B.2) \quad v^{\varphi(\delta)} = v^{c_{\varphi}(\delta)} = v^{c_{\varphi}(\delta)} = v,
\]
because \(v \in Z(\Gamma^\varphi)\). Hence, by the definition of \(\overline{\varphi}\) and of the action of \(G\) on \(\tilde{\Gamma}\) we get using equation \((B.1)\) and \((B.2)\),
\[
(\varphi(\gamma), v)^{\varphi(\delta), u} = (\varphi(\gamma), v)^{\varphi(\delta)} = (\varphi(\gamma), v)^{\varphi(\delta)} = (\varphi(\gamma), v)^{(\varphi(\delta), u)},
\]
and we see that \(\overline{\varphi}\) satisfies (NM2) as well. Thus \(\overline{\varphi}\) is a normal map. Next
\[
(\gamma)\overline{c}_\varphi \circ \overline{\psi} = (\varphi(\gamma), 0)\overline{\psi} = (\gamma)\varphi,
\]
and
\[
(\varphi(\gamma), v)\overline{\psi} = n = n(\psi(\gamma)\overline{\psi}(v)) = n(\psi(\gamma))n(\overline{\psi}(v)) = \varphi(\gamma)\overline{\psi}(v) = \varphi(\gamma),
\]
for all \(\gamma \in \Gamma\) and \(v \in \ker \overline{\varphi}\).

It follows that we can replace in diagram \((B.1)\) \(\Gamma^\varphi, c_{\varphi}, \overline{\varphi}\) with \(\tilde{\Gamma}, \overline{c}_\varphi, \overline{\varphi}\) respectively, and then the map \(\overline{\psi}\) render the diagram commutative. The proof of the uniqueness of \(\overline{\psi}\) is similarly, so (2) holds.

(3): (a)\(\Leftrightarrow\) (b): If \(\Gamma^\varphi = c_{\varphi}(\Gamma)\), then of course \(c_{\varphi}(\Gamma)\) is \(G\)-invariant, while if \(c_{\varphi}(\Gamma)\) is \(G\)-invariant, then by Lemma \((B.3)1\), \(\Gamma^\varphi = c_{\varphi}(\Gamma)\).

(b)\(\Leftrightarrow\) (c): Assume that \(\psi(\Gamma)\) is \(G\)-invariant for every factorization as in (c). Then, in particular, taking \(\psi = c_{\varphi}, M = \Gamma^\varphi\) and \(n = \overline{\varphi}_1\), (c) implies that \(\Gamma^\varphi = c_{\varphi}(\Gamma)\). Conversely, assume that \(\Gamma^\varphi = c_{\varphi}(\Gamma)\). Then by the commutativity of diagram \((B.1)\), and since \(\overline{\varphi}\) is a normal morphism over \(G\), we have that \((\Gamma^\varphi)\overline{\psi}\) is \(G\)-invariant in \(M\). That is \((\Gamma)c_{\varphi} \circ \overline{\psi} = (\Gamma)\psi\) is \(G\)-invariant in \(M\), and (e) holds.

(4): (a)\(\Rightarrow\) (b): Assume (a) holds. Let \(M := \varphi(\Gamma) \times V\) be as in (b). Let \(\psi: \Gamma \to M\) and \(n: M \to G\) be defined by \(\psi(\gamma) = (\varphi(\gamma), 0)\), and \(n(\varphi(\gamma), v) = \varphi(\gamma)\), for all \(\gamma \in \Gamma\) and \(v \in V\). It is easy to check that \(n\) is a normal map and clearly \(\psi \circ n = \varphi\). By the equivalence of (a) and (c) in (2), the subgroup \(\{(\varphi(\gamma), 0) | \gamma \in \Gamma\}\) must be \(G\)-invariant.
(b)⇒(a): Assume that (b) holds. Suppose $\Gamma^\varphi \neq \varphi(\Gamma)$. By (2) we may assume that $\Gamma^\varphi = \varphi(\Gamma) \times V$, with $V = \ker \varphi$, so $V$ is abelian (see Lemma 2.1(1)). Let $(\varphi(\gamma), v) \in \Gamma^\varphi$ and let $\varphi(\alpha) \in \varphi(\Gamma)$. Note that $c_{\varphi}(\gamma) c(\alpha) = c_{\varphi}(\gamma) \varphi(c_{\varphi}(\alpha)) = c_{\varphi}(\gamma^\alpha)$, because $\varphi$ is a normal map. By the definition of the action of $G$ on $\Gamma^\varphi$ we have

$$(\varphi(\gamma), 0) c(\alpha) = (\varphi(\gamma^\alpha), 0),$$

for all $\alpha, \gamma \in \Gamma$. Further, by Lemma 3.3(1), $\Gamma^\varphi$ is generated by $c_{\varphi}(\Gamma) = \{(\varphi(\gamma), 0) \mid \gamma \in \Gamma\}$. Thus $M = \Gamma^\varphi$ violates (b). □

Remark B.2. We do not know how to fully characterize those groups $\Gamma$ and $G$ that satisfy the property in 4(b) of Lemma B.1. However, if $\Gamma$ is a perfect group, then clearly this property holds.

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