Abstract. This is a paper about triangle cubics and conics in classical geometry with elements of projective geometry. In recent years, N.J. Wildberger has actively dealt with this topic using an algebraic perspective. Triangle conics were also studied in detail by H.M. Cundy and C.F. Parry recently. The main task of the article was to develop an algorithm for creating curves, which pass through triangle centers. During the research, it was noticed that some different triangle centers in distinct triangles coincide. The simplest example: an incenter in a base triangle is an orthocenter in an excentral triangle. This was the key for creating an algorithm. Indeed, we can match points belonging to one curve (base curve) with other points of another triangle. Therefore, we get a new interesting geometrical object. During the research were derived number of new triangle conics and cubics, were considered their properties in Euclidian space. In addition, was discussed corollaries of the obtained theorems in projective geometry, what proves that all of the discovered results could be transferred to the projective plane.

Key–Words: triangle cubics, conics, curves, projective geometry, Euclidian space.

MSC: 51A05, 51A20,14H50, 14H52

1. Introduction

Centers of triangle and central triangles were studied by Clark Kimberling [1]. We cosider curves which pase throw incenter in a base triangle is an orthocenter in an excentral triangle and other triangle curves. Also we studied some properties of Jerabek hyperbola for the mid-arc triangle.

2. Main result

Firstly, we will consider excentral triangle. Correspondence between points of the excentral and base triangles will give us significant results in developing new triangle curves. Below you may observe correspondence table between points of the base and excentral triangles.

All of the above facts could be easily proved by basic principles of classical geometry [1]. Hence, we may apply derived results for creating new triangle cubics and conics. Firstly Jerabek hyperbola was considered.

Definition 1. Jerabek hyperbola is a curve which passes through vertices of triangle, circumcenter, orthocenter, Lemoine point, isogonal conjugate of the de Longchamps point [5].

We may observe that Jerabek hyperbola for the excentral tringle has number of points which corresponde to other ones in the base triangle. The study of such matches gave us significant results [2]

Therefore, we got new triangle hyperbola [1] which passes through centers of the excircles, Bevan point, incenter, mittenpunkt and de Longhamps point. It is still rectangular as Jerabek hyperbola is. Known fact about Jerabek hypperboal is that its center is center of the Euler circle. However, Euler circle of the excentral triangle is circumscribed circle for the base triangle. In addition, Jerabek Hyperbola is isogonal conjugate to the Euler line. Meanwhile, Euler line for the excentracl triangle is line (I (incenter), O (circumcenter), Be (Bevan point), Mi’ (isogonal conjugate for the mittenpunkt with respect to the extriangle),Mi’’(isogonal conjugate for the mittenpunkt with respect to the base circle).
Table 1. Correspondence table between points of the base and excentral triangles

| Base triangle       | Excentral triangle          |
|---------------------|-----------------------------|
| $I$ (incenter)      | $H$ (orthocenter)           |
| $O$ (circumcenter)  | $E$ (nine-point center)     |
| $Be$ (Bevan point)  | $O$ (circumcenter)          |
| $Mi$ (mittenpunkt)  | $Sy$ (Lemoine point)        |
| $Mi'$ (isogonal conjugate of the mittenpunkt with respect to the base triangle) | $M$ (centroid) |
| $Sp$ (Speaker point)| $Ta$ (Taylor point)         |
| $Sy$ (Lemoine point)| $Sy(H_1H_2H_3)$ (Lemoine point of the orthic triangle) |
| $M_i''$ (isogonal conjugate point of the mittenpunkt with respect to the excentral triangle) | $GOT$ (homothetic center of the orthic and tangent triangles) |

Table 2. Matching points for Jerabek hyperbola

| Jerabek hyperbola for excentral triangle | New hyperbola for the base triangle |
|-----------------------------------------|-----------------------------------|
| $A, B, C$ (vertices of the base triangle) | $I_1, I_2, I_3$ (excenters) |
| $O$ (circumcenter)                      | $Be$ (Bevan point)               |
| $H$ (orthocenter)                      | $I$ (incenter)                   |
| $Sy$ (Lemoine point)                   | $Mi$ (mittenpunkt)               |
| $L'$ (isogonal conjugate of the de Longchamps point) | $L$ (de Longchamps point) |

Therefore, we can conclude, that our new hyperbola is isogonal conjugate to the line $(I, O, Be, Mi', Mi'')$ and its center is the circumcenter.

**Theorem 1.** New hyperbola passes though excenters, Bevan point, incenter, mittenpunkt, de Longchamps point. It is isogonal conjugate to the line $(I, O, Be, Mi', Mi'')$, and its center is circumcircle.

Similarly we studied Thomson cubic for the base triangle and matched its points with ones in the excetral triangle.
**Definition 2.** Thomson cubic is a curve that passes through vertices of the triangle, middles of triangle sides, centers of the excircles, incenter, centroid, circumcenter, Lemoine point, mittenpunkt, isogonal conjugate of the mittenpunkt \[1\].

By applying correspondence table between points of the base and excentral triangles \[2\] to the points of Thomson cubic we obtain a new triangle cubic \[2\].

| Thomson cubic for the base triangle | New cubic for the excentral triangle |
|-----------------------------------|-----------------------------------|
| \(A, B, C\) (vertices of the base triangle) | \(H_1, H_2, H_3\) (bases of the altitudes) |
| \(M_a, M_b, M_c\) (middles of the base triangle sides) | \(M_{ha}, M_{hb}, M_{hc}\) (middles of orthic triangle's sides) |
| \(I_1, I_2, I_3\) (excenters) | \(A, B, C\) (vertices) |
| \(I\) (incenter) | \(H\) (orthocenter) |
| \(M\) (centroid) | \(M(H_1H_2H_3)\) (centroid in the orthic triangle) |
| \(O\) (circumcenter) | \(E\) (nine-point center) |
| \(Sy\) (Lemoine point) | \(Sy(H_1H_2H_3)\) (Lemoine point of the orthic triangle) |
| \(Mi\) (mittenpunkt) | \(Sy\) (Lemoine point) |
| \(Mi''\) (isogonal conjugate of the mittenpunkt with respect to the excental triangle) | \(GOT\) (gomotetic center of the orthic and tangent triangles) |

**Table 3.** Matching points for Thomson cubic

According to the table, we got new cubic.

**Theorem 2.** New triangle cubic \[2\] passes through vertices of the triangle, bases of the altitudes, middles of the triangle sides in the orthic triangle, orthocenter, Euler point, centroid in orthic triangle, Lemoine point, and gomotetic center of the orthic and tangent triangles.

![Figure 2. New triangle conic based on Thomson hyperbola for the excentral triangle](image-url)

Analogically was derived new triangle cubic based on the Darboux cubic and correspondence of its points with triangle centers in the excentral triangle.
**Definition 3.** Darboux cubic is a curve that passes through vertices of the triangle, centers of the excircles, incenter, circumcenter, Bevan point \([3]\).

Triangle centers of the Darboux cubic in the base triangle were matched with points in the excentral triangle.

| Darboux cubic for the base triangle | New cubic for the excentral triangle |
|------------------------------------|-------------------------------------|
| \(A, B, C\) (vertices of the base triangle) | \(H_1, H_2, H_3\) (bases of the altitudes) |
| \(I_1, I_2, I_3\) (excenters) | \(A, B, C\) (vertices) |
| \(I\) (incenter) | \(H\) (orthocenter) |
| \(O\) (circumcenter) | \(E\) (nine-point center) |
| \(Be\) (Bevan point) | \(O\) (circumcenter) |

**Table 4.** Matching points for Darboux cubic

**Theorem 3.** Therefore, we got new cubic \([3]\), which passes through vertices of the triangle, bases of the altitudes, orthocenter, Euler center, and circumcenter.

![Figure 3](image)

**Figure 3.** New triangle conic based on Darboux cubic for the base triangle in correspondence with excentral triangle

The discussed above results were obtained from considering excentral triangle, its triangle centers and correspondence between points in the excentral and basic triangle. As a result, were derived three new triangle curves, which were not discovered before. However, to get wider results were applied the same idea to other triangles. Namely was consider medial triangle.

**Definition 4.** Medial triangle is a triangle with vertices in the middles of the base triangle sides.

In the same way as before, was proven the fact that some points in the medial triangle match with some points in the base triangle\([2]\). Proof of the mentioned facts relies on the patterns of the Euclidean geometry, some of the correspondence were proved before \([1]\).

**Definition 5.** Yff hyperbola is a triangle curve which passes through centroid, orthocenter, circumcenter, and Euler center.
Points in the medial triangle | Point in the base triangle
---|---
$I$ (incenter) | $Sp$ (Speaker point)
$M$ (centroid) | $M$ (centroid)
$O$ (circumcenter) | $E$ (nine-point center)
$H$ (orthocenter) | $O$ (circumcenter)
$L$ (de Longchamps point) | $H$ (orthocenter)
$Be$ (Bevan point) | $Be(M_1M_2M_3)$ (Bevan point of the medial triangle)
$Na$ (Nagel point) | $I$ (incenter)
$G$ (Gergonne point) | $Mi$ (mittenpunkt)
$S_{YA}$ (Lemoine point of the anticomplementary triangle) | $Sy$ (Lemoine point)
$B_3$ (third Brocard point) | $M_B$ (Brocard midpoint)

Table 5. Correspondence table between points of the medial and base triangles

| Yff hyperbola for the base triangle | New hyperbola for the medial triangle |
|---|---|
| $M$ (centroid) | $M$ (centroid) |
| $H$ (orthocenter) | $L$ (de Longchamps point) |
| $O$ (circumcenter) | $H$ (orthocenter) |
| $E$ (nine-point center) | $O$ (circumcenter) |

Table 6. Matching points for the Yff hyperbola and medial triangle

We have considered Yff hyperbola for the base triangle and matched its point with triangle centers of the medial triangle, applying correspondence table 2.

We got new conic, which has vertices in the centroid and de Longchamps point, focus in the orthocenter. Directrix of the Yff hyperbola is perpendicular to the Euler line and passes through center of the Euler circle. Euler line for the medial and base triangles coincide. However, center of the Euler circle of the base triangle is circumcenter for the medial. Therefore, directrix of the new hyperbola is perpendicular to the Euler line and passes through circumcenter.

**Theorem 4.** New conic is a curve that has vertices in the centroid and de Longchamps point, focus in the orthocenter. Directrix of the new hyperbola is perpendicular to the Euler line and passes through circumcenter.

Let’s go further in our research and create more triangle cubics with the help of correspondence between triangle centers in medial and base triangles.

We observed the transformation of the Darboux cubic and under the correspondence. It led us to a new cubic.

**Figure 4.** New conic based on Yff hyperbola
**Darboux cubic for the medial triangle** | **New cubic for the base triangle**
---|---
$A, B, C$ (triangle vertices) | $M_1, M_2, M_3$ (middles of the triangle sides)
$A_1, B_2, C_3$ (antipods of the triangle) | Antipods of the medial triangle
$I$ (incenter) | $Sp$ (Speaker point)
$O$ (circumcenter) | $E$ (nine-point center)
$H$ (orthocenter) | $O$ (circumcenter)
$L$ (de Longchamps point) | $H$ (orthocenter)
$L'$ (isogonal conjugate of the de Longchamps point) | $H_A$ (complementary conjugate of the orthocenter)

**Table 7.** Matching points for the Darboux cubic and medial triangle

**Theorem 5.** We got new cubic which passes through Speaker point, center of the Euler circle, circumcenter, orthocenter, complementary conjugate of the orthocenter, middles of the triangle side, and antipodes of the medial triangle.

![Figure 5. New cubic based on Darboux cubic for the medial triangle](image)

Simirally, we take Lucas cubic for the base cubic and match it with points of the medial triangle.

**Definition 6.** Lucas cubic is such curve which passes through triangle vertices, orthocenter, Gergone point, centroid, Nagel point, Lemoine point of the anticomplementary triangle, and vertices of the anticomplementary triangle.

We make the correspondence between points of the Lucas cubic in the medial triangle with triangle centers in the base triangle. As a result we obtain the following table.

**Lucas cubic for the medial triangle** | **New cubic for the base triangle**
---|---
$Sy_A$ (Lemoine point of the anticomplementary triangle) | $Si$ (Lemoine point)
$M$ (centroid) | $M$ (centroid)
$H$ (orthocenter) | $O$ (circumcenter)
$Ge$ (Gergonne point) | $Mi$ (mittenpunkt)
$Na$ (Nagel point) | $I$ (incenter)
$L$ (de Longchamps point) | $H$ (orthocenter)

**Table 8.** Matching points for the Lucas cubic and medial triangle
**Theorem 6.** New cubic which passes through Lemoine point, centroid, circumcenter, mittenpunkt, incenter, orthocenter, triangle vertices, and middles of the triangle sides.

**Figure 6.** New cubic based on Lucas cubic for the medial triangle

Therefore, while applying correspondence method to the medial triangle we derived one new conic and two new cubics. In addition, we observed Euler and mid-arc triangles as correspondence base.

**Definition 7.** Euler triangle is triangle with vertices in the intersection points of the triangle altitudes and nine-point circle.

**Definition 8.** Mid-arc triangle is a triangle with vertices in the middles of the arcs of the circumcircle.

Let’s firstly consider correspondence of points between Euler and base triangles.

| Points in the Euler triangle | Points in the base triangle |
|-----------------------------|----------------------------|
| $I$ (incenter)              | $M_{IH}$ (midpoint of incener and orthocenter) |
| $M$ (centroid)              | $M_{MH}$ (midpoint of centroid and orthocenter) |
| $O$ (circumcenter)          | $E$ (nine-point center) |
| $H$ (orthocenter)           | $H$ (orthocenter) |
| $N$ (Nagel point)           | $F$ (Furhman point) |
| $L$ (de Longchamps point)   | $O$ (circumcenter) |

**Table 9.** Correspondence table between points in the Euler and base triangles

According to the above table we have built the correspondence between points of the Darboux cubic for the Euler triangle and triangle centers of the base triangle.

| Darboux cubic in the Euler triangle | New cubic for the base triangle |
|------------------------------------|---------------------------------|
| $A, B, C$ (vertices)               | $E_1, E_2, E_3$ (vertices of the Euler triangle) |
| $A', B', C'$ (antipods of the triangle vertices) | $M_1, M_2, M_3$ (middles of the triangle sides) |
| $I$ (incenter)                     | $M_{IH}$ (midpoint of incenter and orthocenter) |
| $O$ (circumcenter)                 | $E$ (nine-point center) |
| $H$ (orthocenter)                  | $H$ (orthocenter) |
| $L$ (de Longchamps point)          | $O$ (circumcenter) |

**Table 10.** Matching points for the Darboux cubic and Euler triangle
**Theorem 7.** New cubic \([\mathbf{4}]\) passes through circumcenter, orthocenter, Euler center, midpoint of the incenter and the orthocenter, vertices of the Euler triangle and middles of the triangle sides.

Finally, we observe correspondence of triangle centers between mid-arc and base triangle.

| Points for the mid-arc triangle | Points for the base triangle |
|---------------------------------|-----------------------------|
| \(O\) (circumcenter)            | \(O\) (circumcenter)       |
| \(H\) (orthocenter)             | \(I\) (incenter)           |
| \(S_y\) (Lemoine center)        | \(M_{MiI}\) (midpoint of mittenpunkt and incenter) |
| \(L\) (de Longchamps point)     | \(B_e\) (Bevan point)      |
| \(K\) (Kosnita point)           | \(S\) (Schiffler point)    |

**Table 11.** Correspondence table between points in the mid-arc and base triangles

Based on the correspondence between points of the mid-arc and base triangles we discovered new cubic which is based on Jerabek hyperbola.

| Jerabek hyperbola for mid-arc triangle | New hyperbola for the base triangle |
|----------------------------------------|------------------------------------|
| \(A, B, C\) (vertices)                | \(A_1, A_2, A_3\) (middles of the arcs of the circumcircle) |
| \(O\) (circumcenter)                  | \(O\) (circumcenter)              |
| \(H\) (orthocenter)                   | \(I\) (incenter)                  |
| \(S_y\) (Lemoine point)               | \(M_{MiI}\) (midpoint of mittenpunkt and incenter) |
| \(K\) (Kosnita point)                 | \(S\) (Schiffler point)           |
| \(L'\) (isogonal conjugate of de Longchamps point) | \(B_e'\) (isogonal conjugate of the Bevan point) |

**Table 12.** Matching points for the Jerabek hyperbola and Euler triangle

**Theorem 8.** New conic \([\mathbf{8}]\) is rectangular and passes through circumcenter, incenter, midpoint of mittenpunkt and incenter, Schiffler point, and isogonal conjugate point to the Bevan point.

Therefore, during the research of the triangle curves were derived three new triangle conics and five new triangle cubics. This is a significant result and leaves room for new investigations.

Since, geometry of conic sections and other triangle curves are broadly used in the projective geometry we looked on the obtained result through the prism of the projective geometry.
According to the Pascal’s theorem if six arbitrary points are chosen on a conic and joined by line segments in any order to form a hexagon, then the three pairs of opposite sides of the hexagon meet at three points which lie on a straight line.

Let’s consider the first derived triangle curve based on Jerabek hyperbola for the excentral triangle. New hyperbola passes though excenters, Bevan point, incenter, mittenpunkt, de Longchaps point. We built a hexagon with verices in the given triangle centers and apply Pascal’s theorem.

Let $I_1, I_2$ be excenters, and $Be, Mi, L$ be Bevan point, $Mi$ mittenpunkt, de Longchamps point, respectively. We get the following results:

**Corollary 9.** Concurrent points of $I_2Be$ and $LI_1$, $BeMi$ and $I_1L$, $MiI$ and $I_2I_1$ belong to one line.

**Corollary 10.** Concurrent points of segments $I_2Mi$ and $BeI_1$, $BeL$ and $II_2$, $MiL$ and $II_2$ lie on one line.

Similarly, we have applied the same idea for the hexagon inscribed in the new hyperbola derived from the Jerabek hyperbola for the mid-arc triangle.

Let $A_2, A_3$ be middles of the arcs of the circumcircle, and $Be, I, S, O$ be Bevan point, incenter, Speaker point, circumcenter, respectively. The following facts were discovered:

**Corollary 11.** Points of intersection of lines $A_2Be$ and $SI$, $IA_3$ and $OA_2$, $BeA_3$ and $OS$ belong to one line.

**Corollary 12.** Points of intersection of line segments $BeO$ and $A_3A_2$, $BeS$ and $IA_2$, $A_3S$ and $IA_2$ belong to a straight line.

Moreover, combination of two of the discovered triangle cubics gives us very interesting corollary as well. Let’s consider new cubic derived from the Darboux cubic for the excentral triangle and new cubic constructed with the base Darboux cubic with respect to the medial triangle. The first mentioned new cubic passes through bases of altitudes, vertices, orthocenter, nine-point center, circumcenter, let’s name it $P(x, y)$. The second mentioned new cubic passes through middles of the triangle sides, Speaker point, nine-point center, circumcenter, orthocenter, let’s name it $Q(x, y)$. We may notice that this two cubics pass through three common points which are nine-point center, circumcenter and orthocenter. Moreover, this three points belong to Euler line, let it has an equation $ax + by + c = 0$. Since, we have two cubics which pass through points which belong to one line, there exists such integer $t$ such that the following holds: $P(x, y) - tQ(x, y) : ax + by + c$. Therefore, Euler line is linear component of the composition of two new cubics. In addition, points of intersection of the linear component with the curve are inflection points.

**Corollary 13.** Euler line is a linear component of the composition of new triangle cubic (passes through bases of altitudes, vertices, orthocenter, nine-point center, circumcenter) and new triangle cubic.
cubic (passes through middles of the triangle sides, Speaker point, nine-point center, circumcenter, orthocenter). Moreover, orthocenter, circumcenter, and nine-point center are inflection point of the composition of these two curves.

The above corollaries prove that the discovered in the research new triangle curves could be applied in different geometric areas and studied in advanced.

Remark 14. A further continue of our research consists in the same analysis of singularities as provided by second author in [6,10] for cubic obtained by us in the presented work.

3. Conclusion

During the research were discovered three new triangle conics and five new triangle cubics, what is very significant result for the classical geometry. In addition, was shown that proceedings of the study could be applied not only in Euclidian space, but in projective as well. However, the main result of the research was developed algorithm of deriving new triangle curves. This opens an opportunity for creating more triangle curves, while applying the method for various triangles, points, and geometric constructions.

The developed idea significantly simplifies the question of creating curves passing through triangular centers. However, it opens up a number of new questions. Which interesting properties do new curves have? What is the topological nature of these transformations? Is it possible to apply a similar idea to non-Euclidean objects? Could one use the same method over an arbitrary finite field? Can this idea be further generalized?

References

[1] Clark Kimberling. Triangle centers and central triangles, 1998, Utilitas Mathematica.
[2] N.J. Wildberger. Neuberg cubics over finite fields. Algebraic Geometry and its Applications, volume 5, 488-504, 2008.
[3] H.M. Cundy and C.F. Parry. Some cubic curves associated with triangle. Journal of Geometry, 53, 41-66, 2000
[4] Robert J. Walker. Algebraic Curves, 1978, Springer-Verlag New York.
[5] Pinkernell, G. M. "Cubic Curves in the Triangle Plane." J. Geom. 55, 141-161, 1996.
[6] Drozd, Y. A., R. V. Skuratovskii. Cubic Rings and Their Ideals. Ukrainkyi Matematychnyi Zhurnal, Vol.62, no.4, Apr.2010, pp.464
[7] R. V. Skuratovskii, Aled Williams (2019) "A solution of the inverse problem to doubling of twisted Edwards curve point over finite field", Processing, transmission and security of information - 2019 vol. 2, Wydawnictwo Naukowe Akademii Techniczno-Humanistycznej w Bialsku-Bialej
[8] RV Skuratovskii, A Williams "Irreducible bases and subgroups of a wreath product in applying to diffeomorphism groups acting on the Mobius band", 2020. Rendiconti del Circolo Matematico di Palermo Series 2, 1-19.
[9] Skuratovskii R. V. Generating set of wreath product with non faithful action. // International Journal of Analysis and Applications Volume, 2020. 18, no. 1, P. 104–116.
[10] Skuratovskii R. V., Osachyuv V. Order of Edwards and Elliptic Curves Over Finite Field. WSEAS Transactions on Mathematics, Volume 19, pp. 253-264, 2020.
[11] A. V. Akopyan and A. A. Zaslavsky. Geometry of conics, volume 26 of Mathematical World. American Mathematical Society, Providence, RI, 2007.