SYNCHRONIZED AND GROUND-STATE SOLUTIONS TO A COUPLED SCHröDINGER SYSTEM

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Abstract. In this paper, we study the following coupled nonlinear Schrödinger system of the form
\[
\begin{cases}
-\Delta u_i - \kappa_i u_i = g_i(u_i) + \lambda \partial_i F(\vec{u}), \\
\vec{u} = (u_1, u_2, \cdots, u_m), u_i \in D_0^{1,2}(\Omega),
\end{cases}
\]
for \( m = 2, 3 \), where \( \Omega \subset \mathbb{R}^N \) is a bounded domain or \( \mathbb{R}^N \), \( N \geq 3 \), \( F(t_1, t_2, \cdots, t_m) \in C^1(\mathbb{R}^m, \mathbb{R}) \), \( \kappa_i \in \mathbb{R} \), \( g_i \in C(\mathbb{R}) \) (\( i = 1, 2, \cdots, m \)) and \( \lambda > 0 \) is large enough. In this work we mainly focus on the existence of fully nontrivial ground-state solutions and synchronized ground-state solutions under certain conditions.

1. Introduction. This paper is concerned with the existence of fully nontrivial ground-state solutions and synchronized ground-state solutions to the subcritical and critical cases of the time-independent Schrödinger equations
\[
\begin{cases}
-\Delta u_i - \kappa_i u_i = g_i(u_i) + \lambda \partial_i F(\vec{u}), \\
\vec{u} = (u_1, u_2, \cdots, u_m), u_i \in D_0^{1,2}(\Omega),
\end{cases}
\]
for \( m = 2, 3 \), where \( \Omega \subset \mathbb{R}^N \) is a bounded domain or \( \mathbb{R}^N \), \( N \geq 3 \), \( F(t_1, t_2, \cdots, t_m) \in C^1(\mathbb{R}^m, \mathbb{R}) \), \( \lambda > 0 \) and \( \kappa_i \in \mathbb{R} \), \( g_i \in C(\mathbb{R}) \) (\( i = 1, 2, \cdots, m \)). This type of system appears in mathematical models for various physics problems, especially in nonlinear optics and Bose-Einstein condensation (see for instance [1, 13, 14, 16]). Due to its importance, it has drawn much attention from many mathematicians in recent years.

A particular case of (1.1) is the following subcritical system
\[
\begin{cases}
-\Delta u_1 - \kappa_1 u_1 = g_1(u_1) + \lambda \partial_1 F(u_1, u_2), \\
-\Delta u_2 - \kappa_2 u_2 = g_2(u_2) + \lambda \partial_2 F(u_1, u_2), \\
u_1, u_2 \in H_0^1(\Omega),
\end{cases}
\]

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where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $N \geq 3$, $F(t, s) \in C^1(\mathbb{R}^2, \mathbb{R})$, $\kappa_1, \kappa_2 \in \mathbb{R}$, $\lambda > 0$ and $g_1, g_2 \in C(\mathbb{R})$. This system has an extensive literature, specifically on the cubic case of homogeneous nonlinearities ($g_i(u_i) = \mu_i|u_i|^2u_i$ ($i = 1, 2$), $F(u) = |u|^2|u|^2$) in dimension $N \leq 3$. For example for $N = 2, 3$, the existence and the qualitative description of least energy solutions have raised considerable attention in the last decades, starting from the seminal paper [17] by T.-C. Lin and J. Wei. By now, a good description of positive solutions and of least energy solutions is available.

One of the interesting features of the system (1.2) is that it admits solutions with trivial components. Thereby, this system is sometimes called weakly coupled. For a deeper discussion of introduction we refer the reader to [10].

For the critical case of (1.1), we consider the systems

$$
\begin{aligned}
-\Delta u_1 &= \mu_1 |u_1|^{2^*-2}u_1 + \lambda p_{12}|u_1|^{p_{12}-2}|u_2|^{p_{21}}u_1 + \lambda p_{13}|u_1|^{p_{13}-2}|u_3|^{p_{31}}u_1, \\
-\Delta u_2 &= \mu_2 |u_2|^{2^*-2}u_2 + \lambda p_{23}|u_2|^{p_{23}-2}|u_3|^{p_{32}}u_2, \\
-\Delta u_3 &= \mu_3 |u_3|^{2^*-2}u_3 + \lambda p_{31}|u_3|^{p_{31}-2}|u_1|^{p_{13}}u_3 + \lambda p_{32}|u_3|^{p_{32}-2}|u_2|^{p_{23}}u_3,
\end{aligned}
$$

for $1, 2, 3 \in D^{1,2}(\mathbb{R}^N)$, and

$$
\begin{aligned}
-\Delta u_1 - \kappa_1 u_1 &= \mu_1 |u_1|^{2^*-2}u_1 + \lambda p_{12}|u_1|^{p_{12}-2}|u_2|^{p_{21}}u_1 + \lambda p_{13}|u_1|^{p_{13}-2}|u_3|^{p_{31}}u_1, \\
-\Delta u_2 - \kappa_2 u_2 &= \mu_2 |u_2|^{2^*-2}u_2 + \lambda p_{23}|u_2|^{p_{23}-2}|u_3|^{p_{32}}u_2, \\
-\Delta u_3 - \kappa_3 u_3 &= \mu_3 |u_3|^{2^*-2}u_3 + \lambda p_{31}|u_3|^{p_{31}-2}|u_1|^{p_{13}}u_3 + \lambda p_{32}|u_3|^{p_{32}-2}|u_2|^{p_{23}}u_3,
\end{aligned}
$$

for $1, 2, 3 \in H_0^1(\Omega)$, (1.3)

where $\kappa_i \in \mathbb{R}$, $\mu_i > 0$, $p_{ij} > 1$, $p_{ij} + p_{ji} = 2^*$, $i \neq j$ ($i, j = 1, 2, 3$) and $\Omega$ is a bounded domain. Physically, for the cubic case ($N = 4$, $p_{ij} = p_{ji} = 2$) of the above systems, the solution $u_i$ denotes the $i^{th}$ component of the beam in Kerr-Like photorefractive media (see [1]). The positive constant $\mu_i$ is for self-focusing in the $i^{th}$ component of the beam and the coupling constant $\lambda p_{ij}$ is the interaction between the $i^{th}$ and $j^{th}$ components of the beam. As $\lambda p_{ij} > 0$, the interaction is attractive, but the interaction is repulsive if $\lambda p_{ij} < 0$. In the attractive case the components of a vector solution tend to go along each other, leading to synchronization and in the repulsive case, the components tend to segregate from each other, leading to phase separations [18].

If $\kappa_i = 0$ with $m = 2$, we have the weakly coupled critical system

$$
\begin{aligned}
-\Delta u &= \mu_1 |u|^{2^*-2}u + \lambda \alpha |u|^{\alpha-2}|u|^\beta u, \\
-\Delta v &= \mu_2 |v|^{2^*-2}v + \lambda \beta |v|^{\beta-2}|u|^\alpha v,
\end{aligned}
$$

for $u, v \in D_0^{1,2}(\Omega)$, (1.5)

where $\Omega$ is either a bounded smooth domain in $\mathbb{R}^N$ or $\Omega = \mathbb{R}^N$, $\alpha, \beta > 1$ and $\alpha + \beta = 2^*$. Recently, a prescribed number of fully nontrivial solutions for bounded domains and existence of infinitely many fully nontrivial solutions for $\mathbb{R}^N$ to (1.5) are found by Clapp and Faya [9]. For $\lambda = 0$, (1.3) reduces to the problem

$$
-\Delta w = \mu |w|^{2^*-2}w, \quad w \in D^{1,2}(\mathbb{R}^N).
$$

In [12] W.Y. Ding showed the equation (1.6) has infinitely many solutions. The existence and multiplicity of positive solutions to the corresponding subcritical case in dimension $N \leq 3$ for the system (1.4) with two components and $\kappa_i > 0$ have been extensively investigated in [2, 4, 5, 6, 8, 11]. For $\lambda = 0$, the system (1.4) can
There exist constants $c_j$ with $c_j < \kappa < c_{j+1}$ or $N \geq 5$ and $\kappa \in \mathbb{R}$ where $\kappa_j$ is the sequence of eigenvalues of the operator $-\Delta$ in $H^1_0(\Omega)$, the existence of ground-state solutions to the equation (1.7) has been established in reference [19].

We point out that our results are an improvement and generalization of [10] for the subcritical and critical cases. In this work, the functions $g_i$ and $F(\nu)$ are general functions for subcritical case. For critical case we consider three components. It is noted that it is difficult to verify Palais-Smale condition for energy functional of (1.2) when $\kappa_i$ is a positive number. In this paper we establish the existence of fully nontrivial ground-state solutions to (1.2) for the subcritical case by variational method. In particular, the Palais-Smale condition and Nehari manifold are utilized. Additionally by comparing the energy level we find a ground-state solution to (1.3) which is fully nontrivial and synchronized under some assumptions. Moreover, the system (1.4) admits a nontrivial ground-state solution by variational method. When $\kappa_1 = \kappa_2 = \kappa_3 = \kappa$, by establishing a relation between two minimization problems, we prove that all ground-state solutions to (1.4) are synchronized. It turns out that all components of ground-state solutions of (1.4) are the product of a known function and some constants which are critical points of an auxiliary function.

In order to find a solution for subcritical case (1.2), we make the following assumptions, which are needed in the proof of Theorem 1.1.

(C1) $g_i \in C(\mathbb{R}), i = 1, 2$ is odd and $F(t, s) \in C^1(\mathbb{R}^2, \mathbb{R})$ is even. Its partial derivatives $f_1(t, s) := \partial_t F(t, s), f_2(t, s) := \partial_s F(t, s)$ satisfy that $f_1(0, s) = 0, f_2(t, 0) = 0$ for any $t, s \in \mathbb{R}$.

(C2) There exist constants $\alpha, \beta > 1, c_1, c_2 > 0$ such that $p = \alpha + \beta < 2^* = 2N/(N - 2)$ and

$$c_1|t|^{\alpha}|s|^{\beta} \leq F(t, s) \leq c_2|t|^{\alpha}|s|^{\beta}, \quad |f_1(t, s)t| + |f_2(t, s)s| \leq c_2|t|^{\alpha}|s|^{\beta},$$

for any $(t, s) \in \mathbb{R}^2$.

(C3) There exist constants $\ell_i, \mu_i > 0, i = 1, 2$ and $2 < r \leq p = \alpha + \beta$ such that

$$\lim_{t \to 0} \frac{g_i(t)}{|t|^{p-2}t} = \ell_i, \quad \lim_{t \to \infty} \frac{g_i(t)}{|t|^{p-1}} = \mu_i.$$ 

(C4) $g_i(t)/|t|$ is strictly increasing on $(-\infty, 0)$ and on $(0, \infty)$.

(C5) Define $G_i(t) := \int_0^t g_i(s)ds$, $t \in \mathbb{R}, i = 1, 2$. There exists a constant $\theta > 2$ such that

$$\theta G_i(t) \leq g_i(t)t, \quad t \in \mathbb{R}, i = 1, 2,$$

and

$$\theta F(t, s) \leq f_1(t, s)t + f_2(t, s)s, \quad \forall t, s \in \mathbb{R}.$$ 

It is easy to see that the functions $g_i(t) = \ell_i|t|^{r-2}t + \mu_i|t|^{p-2}t, i = 1, 2, F(t, s) = |t|^{\alpha}|s|^{\beta}$, with $\alpha, \beta > 1, \alpha + \beta < 2^*$ satisfy the assumptions (C1) - (C5).

Throughout this paper we use the following notations. $H^1_0(\Omega)$ denotes the usual Sobolev space with the norm

$$\|u\| := (\int_\Omega |\nabla u|^2)^{1/2}, \quad \forall u \in H^1_0(\Omega).$$
Denote the norm of \( L^p(\Omega) \) by
\[
|u|_p := \left( \int_{\Omega} |u|^p \right)^{\frac{1}{p}}
\]
and \( X = H^1_0(\Omega) \times H^1_0(\Omega) \) with the norm
\[
\|\tilde{u}\| := \left( \|u\|^2 + \|v\|^2 \right)^{\frac{1}{2}}, \quad \forall \tilde{u} = (u, v) \in X.
\]
For \( \ell = 1, 2 \) define
\[
B_\ell(u, v) := \int_{\Omega} [\nabla u(x) \nabla v(x) - \kappa_\ell uv], \quad \forall (u, v) \in X
\]
and
\[
B(\tilde{u}, \tilde{v}) := B_1(u_1, v_1) + B_2(u_2, v_2)
\]
for \( \tilde{u} = (u_1, u_2), \tilde{v} = (v_1, v_2) \in X \). The energy functional associated with (1.2) is given by
\[
J_\lambda(\tilde{u}) := \frac{1}{2} B(\tilde{u}, \tilde{u}) - \int_{\Omega} \left[ G_1(u_1) + G_2(u_2) + \lambda F(\tilde{u}) \right]
\]
for \( \tilde{u} = (u_1, u_2) \in X \). Thus, the solutions of (1.2) are the critical points of \( J_\lambda \) and its partial derivatives are
\[
\langle \partial_u J_\lambda(u, v), \phi \rangle = B_1(u, \phi) - \int_{\Omega} [g_1(u) \phi(x) + \lambda f_1(u, v) \phi(x)],
\]
\[
\langle \partial_v J_\lambda(u, v), \phi \rangle = B_2(v, \phi) - \int_{\Omega} [g_2(v) \phi(x) + \lambda f_2(u, v) \phi(x)]
\]
for any \( \phi \in C_0^\infty(\Omega) \).

It follows from [20] that for \( p < 2^* \) there is a minimizer \( \omega_i \) for the energy functional
\[
I_i(u) = \frac{1}{2} B_i(u, u) - \int_{\Omega} G_i(u), \quad i = 1, 2
\]
on the associated generalized Nehari manifold. The same is true for \( p = 2^* \) if \( \kappa_i > 0 \) and either \( N \geq 5 \), or \( N = 4 \) and \( \kappa_i \) is not an eigenvalue of \(-\Delta \) in \( H^1_0(\Omega) \) (see [19] or [15]). Hence, \( \omega_i \) is a least energy nontrivial solution to the following equation
\[
-\Delta u - \kappa_i u = g_i(u), \quad u \in H^1_0(\Omega).
\]
Let
\[
c_0 := \min\{I_1(\omega_1), I_2(\omega_2)\}.
\]
Our main results are stated as follows:

**Theorem 1.1.** Suppose that \( \kappa_1, \kappa_2 \in \mathbb{R} \) and the assumptions (C1) – (C5) hold.

1. There exists a constant \( \Lambda_1 > 0 \) such that for each \( \lambda > \Lambda_1 \) the system (1.2) has a fully nontrivial ground-state solution \((u, v) \in X\).

2. For each positive integer \( k \) there is \( \Lambda_k > 0 \) such that, if \( \lambda > \Lambda_k \), then the system (1.2) has at least \( k \) fully nontrivial solutions.

In order to state the next theorem we need some notations. Let
\[
S := \inf \left\{ \frac{\|\nabla u\|^2}{|u|^2} : u \in D^{1,2}(\mathbb{R}^N), u \neq 0 \right\},
\]
\[
S_{\infty, \lambda} := \inf_{\tilde{a} \in D^{1,2}(\mathbb{R}^N), \tilde{a} \neq 0} \frac{\|u_1\|^2 + \|u_2\|^2 + \|u_3\|^2}{\int_{\mathbb{R}^N} \left( \sum_{i=1}^3 \mu_i |u_i|^{2^*} + 2^* \lambda \sum_{i < j} |u_i|^{2^*} |u_j|^{2^*} \right)^{2/2^*}}
\]
and \( \overline{m} = \max \{ s_1^{p_{11}} s_2^{p_{21}} + s_1^{p_{12}} s_3^{p_{31}} + s_2^{p_{22}} s_3^{p_{32}} \mid s_1^2 + s_2^2 + s_3^2 = 1, s_i \geq 0, 1 \leq i \leq 3 \} \). For any \( \epsilon > 0 \), define
\[
U_\epsilon(x) := [N(N-2)]^{\frac{N-2}{4}} \left( \frac{\epsilon}{r^2 + |x|^2} \right)^{\frac{N-2}{2}}, \quad x \in \mathbb{R}^N.
\]

The energy functional corresponding to (1.3) is defined by
\[
I_{\infty,\lambda}(\bar{u}) := \frac{1}{2} (\|u_1\|^2 + \|u_2\|^2 + \|u_3\|^2) - \frac{1}{2} (\mu_1|u_1|^{2^*}_2 + \mu_2|u_2|^{2^*}_2 + \mu_3|u_3|^{2^*}_2) - \lambda \int_{\mathbb{R}^N} (|u_1|^{p_{11}}|u_2|^{p_{21}} + |u_1|^{p_{12}}|u_3|^{p_{31}} + |u_2|^{p_{22}}|u_3|^{p_{32}}),
\]
where \( \| \cdot \| \) and \( | \cdot |_{2^*} \) are the usual norms in \( D^{1,2}(\mathbb{R}^N) \) and \( L^{2^*}(\mathbb{R}^N) \) respectively. Then
\[
\langle I_{\infty,\lambda}(\bar{u}), \bar{u} \rangle = \sum_{i=1}^{3} \|u_i\|^2 - 2 \lambda \sum_{i<j} \int_{\mathbb{R}^N} |u_i|^{p_{ij}}|u_j|^{p_{ji}}.
\]

The Nehari manifold associated with (1.3) is given by
\[
M_{\infty,\lambda} := \{ \bar{u} \in [D^{1,2}(\mathbb{R}^N)]^3 : \bar{u} \neq (0, 0, 0), \langle I'_{\infty,\lambda}(\bar{u}), \bar{u} \rangle = 0 \}.
\]

**Theorem 1.2.**

1. For any \( \lambda > 0 \), then
\[
0 < S_{\infty,\lambda} := \inf_{s_i > 0, i=1,2,3} \frac{S \sum_{i=1}^{3} s_i^2}{\sum_{i=1}^{3} \mu_i s_i^{2^*_2} + 2 \lambda \sum_{i<j} s_i^{p_{ij}} s_j^{p_{ji}} |2^*/2^*_2|}.
\]

2. Let \( |p_{12} - 2^*_2| \geq \max \{|p_{13} - 2^*_2|, |p_{23} - 2^*_2|\} \) and suppose that \( \overline{m} > \left( \frac{p_{12}^2 p_{22}^2}{(2^*_2)^2} \right)^{\frac{1}{2}} \).

Then there exists \( \Lambda_1 > 0 \) such that for every \( \epsilon > 0 \) and any \( \lambda > \Lambda_1 \) (1.3) admits a solution \((s_1 U_\epsilon, s_2 U_\epsilon, s_3 U_\epsilon)\) with
\[
I_{\infty,\lambda}(s_1 U_\epsilon, s_2 U_\epsilon, s_3 U_\epsilon) = \inf_{\bar{u} \in M_{\infty,\lambda}} I_{\infty,\lambda}(\bar{u}) = \frac{1}{N} S^{N/2}_{\infty,\lambda},
\]
where \( s_1, s_2, s_3 > 0 \) are constants depending on \( \lambda \).

The energy functional associated with (1.4) is defined by
\[
J_\lambda(\bar{u}) := \frac{1}{2} \|\bar{u}\|^2 - \frac{3}{2} \sum_{i=1}^{3} \kappa_i |u_i|^{2^*_2} - \lambda \sum_{i<j} \int_{\Omega} |u_i|^{p_{ij}}|u_j|^{p_{ji}}.
\]

for \( \bar{u} = (u_1, u_2, u_3) \in [H^1_0(\Omega)]^3 \).

**Theorem 1.3.** Let \( p_{ij} + p_{ji} = 2^* \) and suppose that \( \kappa_1, \kappa_2, \kappa_3 > 0 \). If \( N \geq 5 \), then there exists \( \Lambda_1 > 0 \) such that for each \( \lambda > \Lambda_1 \) the system (1.4) has a ground-state solution \( \bar{u} = (u_1, u_2, u_3) \). The same conclusion remains valid if \( N = 4 \) and \( \kappa_1, \kappa_2, \kappa_3 \) are not eigenvalues of \( -\Delta \) in \( H^1_0(\Omega) \). Moreover,
\[
0 < J_\lambda(\bar{u}) < \min \{ B_i(\omega_i, \omega_i) \mid i = 1, 2, 3 \},
\]
where \( B_i \) is defined the same as in (1.8) and \( \omega_i \) is a ground-state solution to the equation
\[
-\Delta v - \kappa_i v = \mu_i |v|^{2^*_2 - 2} v, \quad v \in H^1_0(\Omega).
\]

**Theorem 1.4.** Let \( \kappa_1 = \kappa_2 = \kappa_3 = \kappa \in \mathbb{R} \) in (1.4) and \( 0 < \kappa_1 < \kappa_2 < \cdots < \kappa_\ell < \cdots \) be the eigenvalues of \( -\Delta \) in \( H^1_0(\Omega) \).
(1) Suppose that \( \bar{\kappa}_\ell < \kappa < \bar{\kappa}_{\ell+1} \) for some \( \ell \) and \( N = 4 \) or \( \kappa \in \mathbb{R} \) and \( N \geq 5 \), then the system (1.4) has a synchronized solution.

(2) If \( 0 < \kappa < \bar{\kappa}_1 \) and \( N \geq 4 \), then the set \( G \) which consists of all ground-state solutions for the system (1.4) can be formulated by

\[
G = \{(\pm r_1 \omega, \pm r_2 \omega, \pm r_3 \omega)|\vec{r} \in A, H(\vec{r}) = \Theta, \omega \in G_0\},
\]

where the set \( A \) is defined as

\[
A := \{\vec{r} \in [0, +\infty)^3 \setminus \{(0)\}|t_i^2 = \mu_i t_i^2 + \sum_{j \neq i}^3 \lambda_p t_i^{p_i-j} t_j^{p_j}, 1 \leq i \leq 3\},
\]

the function \( H(\vec{r}) \) is given by

\[
H(\vec{r}) = \sum_{i=1}^3 \mu_i t_i^{2*} + \sum_{1 \leq i < j \leq 3} \lambda_p t_i^{p_i-j} t_j^{p_j}, \quad \vec{r} \in A,
\]

the constant \( \Theta \) is defined by

\[
\Theta = \min\{H(\vec{r})|\vec{r} \in A\}
\]

and the set \( G_0 \) consists of all ground-state solutions of

\[
-\Delta \varphi - \kappa \varphi = |\varphi|^{2*-2} \varphi, \quad \varphi \in H^1_0(\Omega).
\]

This paper is organized as follows. In Section 2 we introduce some preliminaries and variational setting. Section 3 is dedicated to the subcritical case and Section 4 to the critical case. Finally, Section 5 is devoted to the synchronized solutions.

2. Preliminaries.

Definition 2.1. A solution \( \vec{u} = (u_1, u_2, \ldots, u_m) \) is called fully nontrivial if \( u_i \neq 0, i = 1, 2, 3, \ldots, m \), \( \vec{u} \) is called semitrivial solution if \( \vec{u} \) is not zero and \( u_i = 0 \) for some \( i \). A solution \( \vec{u} \) is called positive if \( u_i > 0 \) in \( \Omega \) for every \( i \).

Definition 2.2. A nonzero solution is called a ground-state solution if it has the least energy among all nonzero solutions.

Proposition 1. If the functional \( J_\lambda \) defined in (1.9) has a critical point \( \vec{u} \) such that \( 0 < J_\lambda(\vec{u}) < c_0 \), then \( \vec{u} \) is a fully nontrivial solution of (1.2).

Proof. If \( \vec{u} = (u_1, u_2) \) is a semitrivial solution of (1.2), then either \( u_1 = 0 \) and \( u_2 \neq 0 \) or \( u_2 = 0 \) is a solution of (1.13) for \( i = 2 \) or \( \vec{u} \neq 0 \) and \( u_2 = 0 \) is a solution of (1.13) for \( i = 1 \). For the first case, we have \( J_\lambda(\vec{u}) = I_2(u_2) \geq c_0 \). It contradicts to the assumption. Similarly, we can deal with the second case. \( \square \)

The following proposition is exactly Proposition 2.2 in [10].

Proposition 2. Let \( X \) be a Hilbert space and suppose \( J \in C^1(X, \mathbb{R}) \) is even and \( J(0) = 0 \). Suppose also there exist closed subspaces \( Y, Z \) of \( X \) and constants \( b, c_0, \rho \) such that \( Y \) has finite codimension in \( X \), \( Z \) has finite dimension, \( 0 < b < c_0 \), \( \rho > 0 \) and

\[
\inf\{J(\vec{u}) : \vec{u} \in Y, \|\vec{u}\| = \rho\} > b, \quad \sup\{J(\vec{u}) : \vec{u} \in Z\} < c_0.
\]

If \( J \) satisfies the Palais-Smale condition at all levels \( c \in (b, c_0) \) and if \( k = \dim Z - \text{codim} Y > 0 \), then \( J \) has at least \( k \) critical values in \( (b, c_0) \) or it has infinitely many critical points in \( J^{-1}(b, c_0) \).
Let \( 0 < \tilde{\kappa}_1 < \tilde{\kappa}_2 < \cdots \) be the eigenvalues of \(-\Delta\) in \(H^1_0(\Omega)\) counted with their multiplicity and let \(e_1, e_2, \cdots\) be the corresponding orthonormal eigenfunctions in \(L^2(\Omega)\). These are also the eigenfunctions of the operator \(-\Delta - \kappa_i\) in \(H^1_0(\Omega)\) but the eigenvalues are shifted by \(-\kappa_i\).

For \(i = 1, 2\), we set \(X_i := H^1_0(\Omega)\) and we write \(X_i = X_i^- \oplus X_i^0 \oplus X_i^+\) for the orthogonal decomposition corresponding to the negative, zero and positive part of the spectrum of \(-\Delta - \kappa_i\) in \(H^1_0(\Omega)\). Namely

\[
\begin{align*}
X_i^- &= \text{span}\{e_\ell : \tilde{\kappa}_\ell < \kappa_i\}, \quad (2.1) \\
X_i^0 &= \text{span}\{e_\ell : \tilde{\kappa}_\ell = \kappa_i\}, \quad (2.2) \\
X_i^+ &= \text{span}\{e_\ell : \tilde{\kappa}_\ell > \kappa_i\}. \quad (2.3)
\end{align*}
\]

The spaces \(\bar{X}_i := X_i^- \oplus X_i^0\) are finite-dimensional. Note that \(X = X^+ \oplus \bar{X}\), with

\[
X^+ := X^+_1 \times X^+_2, \quad \bar{X} := \bar{X}_1 \times \bar{X}_2.
\]

Indeed, let \(u = u^+ + u^0 \in X^+_1 \oplus \bar{X}_1\), \(v = v^+ + v^0 \in X^+_2 \oplus \bar{X}_2\), then we have

\[
(u, v) = (u^+, v^+) + (u^0, v^0) \in (X^+_1 \times X^+_2) \oplus (\bar{X}_1 \times \bar{X}_2) = X^+ \oplus \bar{X}.
\]

3. The subcritical case. Throughout this section we suppose that \(2 < p < 2^*\). Fix a positive integer \(s\) and let \(W_s = \text{span}\{e_1, \cdots, e_s\} \subset H^1_0(\Omega)\). Set

\[
Z_s := \{(w, w) : w \in W_s\} \subset X,
\]

then \(\dim Z_s = s\). In fact, let \(w = \sum_{i=1}^s c_i e_i \in W_s\), then

\[
(w, w) = \sum_{i=1}^s c_i (e_i, e_i) \in Z_s.
\]

Thus,

\[
Z_s := \text{span}\{(w, w) : w \in W_s\} \subset X.
\]

**Lemma 3.1.**

1. For any \(\epsilon > 0\) there exists a constant \(C_\epsilon > 0\) such that

\[
|g_i(t)| \leq \epsilon|t| + C_\epsilon|t|^{p-1}, \quad \forall t \in \mathbb{R} \quad (3.1)
\]

and

\[
|G_i(t)| \leq \epsilon|t|^2 + C_\epsilon|t|^p, \quad \forall t \in \mathbb{R}. \quad (3.2)
\]

2. \(g_i(t) > 0\) if \(t > 0\), \(g_i(t) < 0\) if \(t < 0\), \(i = 1, 2\).

3. \(G_i(t) > 0\), if \(t \neq 0\), \(i = 1, 2\).

4. There exists a constant \(C_0 > 0\) such that

\[
C_0(|t|^\theta - 1) \leq G_i(t), \quad i = 1, 2, \quad t \in \mathbb{R}, \quad (3.3)
\]

where \(\theta > 2\) is from the assumption (C5).

**Proof.**

1. It follows from (C3) that for any \(\epsilon > 0\) there exists a constant \(C_\epsilon > 0\) such that (3.1) holds and then by integrating (3.1) we obtain (3.2).

2. By the assumptions (C3) and (C4) we deduce that \(g_i(t) > 0\) if \(t > 0\) and also \(g_i(t) < 0\) if \(t < 0\).

3. By the definition of the functions \(G_i\) and part (2), it is easy to see that \(G_i(t) > 0, t \neq 0, i = 1, 2\).
(4) From the assumption (C5) and the third conclusion of this lemma we obtain
\[ \frac{\theta}{t} \leq \frac{g_i(t)}{G_i(t)} \]
for any \( t > 1 \). Integrating this inequality from 1 to \( t \) gives
\[ \ln t^\theta \leq \ln G_i(t) - \ln G_i(1). \]
It implies that
\[ |t|^{\theta} - 1 \leq t^\theta \leq \frac{G_i(t)}{G_i(1)}. \]
We set \( C_0 = G_i(1) \), then \( C_0 \) satisfies (3.3) for \( t > 1 \). The case \( t < -1 \) can be dealt with similarly by considering the inequality
\[ t \geq \frac{g_i(t)}{G_i(t)}; \forall t < -1: \]

**Lemma 3.2.** Let \( c_0 \) be as in (1.14).

(i) For each fixed \( \lambda > 0 \) there exist \( \theta > 0, 0 < b < c_0 \) such that
\[ \inf \{ J_{\lambda}(\bar{u}) : \bar{u} \in X^+, \| \bar{u} \| = \rho \} > b. \]

(ii) For any positive integer \( s \) there is \( \tilde{\Lambda}_s \) such that, if \( \lambda > \tilde{\Lambda}_s \), then
\[ \max \{ J_{\lambda}(\bar{u}) | \bar{u} \in Z_s \} < c_0. \]

**Proof.** (i) For any \( \lambda > 0 \) it follows from Lemma 3.1 that \( \int_{\Omega} G_i(u) = o(\|u\|^2), i = 1, 2 \) as \( \|u\| \to 0 \). By the assumption (C2) and Hölder’s inequality we deduce that
\[ \int_{\Omega} F(\bar{u}) = o(\|\bar{u}\|^2) \]
as \( \|\bar{u}\| \to 0 \). Therefore
\[ J_{\lambda}(\bar{u}) = \frac{1}{2} B(\bar{u}, \bar{u}) + o(\|\bar{u}\|^2) \] (3.4)
for \( \bar{u} \in X \) with \( \|\bar{u}\| \to 0 \). We observe that there exists a positive constant \( c > 0 \) depending on \( \kappa_1 \) and \( \kappa_2 \) such that
\[ \frac{1}{c} \|\bar{u}\|^2 \leq B(\bar{u}, \bar{u}) \leq c \|\bar{u}\|^2, \forall \bar{u} \in X^+. \] (3.5)
Inserting (3.5) into (3.4) gives
\[ J_{\lambda}(\bar{u}) \geq \frac{1}{2c} \|\bar{u}\|^2 + o(\|\bar{u}\|^2). \] (3.6)
Let \( \rho \in (0, \sqrt{c_0}) \) be a fixed constant satisfying
\[ |o(\|\bar{u}\|^2)| \leq \frac{1}{4c} \|\bar{u}\|^2 \] (3.7)
for any \( \|\bar{u}\| \leq \rho \). From (3.6) and (3.7) we conclude that
\[ \inf \{ J_{\lambda}(\bar{u}) : \bar{u} \in X^+, \|\bar{u}\| = \rho \} \geq \frac{\rho^2}{4c}. \] (3.8)
Set \( b = \rho^2/8c \), then \( b \) and \( \rho \) satisfy all requirements in part (i).

(ii) Suppose the assertion of part (ii) is false, then there exist \( \lambda_n \to \infty \) such that
\[ \max \{ J_{\lambda_n}(\bar{u}) | \bar{u} \in Z_s \} \geq c_0. \] (3.9)
For \( \tilde{u} = (w, w) \in Z_s \), from (1.9) we have
\[
J_{\lambda}(\tilde{u}) = \frac{1}{2} B(\tilde{u}, \tilde{u}) - \int_{\Omega} [G_1(w) + G_2(w) + \lambda F(\tilde{u})].
\] (3.10)

Note that there exists a constant \( M_1 > 0 \) such that
\[
|B(\tilde{u}, \tilde{u})| = \left[ \int_{\Omega} \left| \nabla w \right|^2 - \frac{\kappa_1 + \kappa_2}{2} w^2 \right| \leq M_1 \| w \|^2
\] (3.11)
due to the Poincaré inequality. Using Lemma 3.1 we deduce that there is a constant \( M_2 > 0 \) such that
\[
|\int_{\Omega} [G_1(w) + G_2(w)]| \leq M_2 \| w \|^2 + M_2 \| w \|^p,
\] (3.12)
thanks to the Sobolev imbedding theorem \( H^1_0(\Omega) \subset L^q(\Omega), 2 \leq q \leq 2^* \). By the assumption (C2) and the space \( W_s \) being finite-dimension, there is a constant \( M_3 > 0 \) satisfying
\[
\int_{\Omega} F(\tilde{u}) \geq c_1 \int_{\Omega} |w|^p \geq M_3 \| w \|^p,
\] (3.13)
where the constant \( c_1 \) is from the assumption (C2). Combining (3.11)-(3.13) we obtain
\[
J_{\lambda}(\tilde{u}) \leq M_1 \| w \|^2 + M_2 \| w \|^2 + M_2 \| w \|^p - \lambda M_3 \| w \|^p.
\] (3.14)
Thus, (3.14) implies that for each \( \lambda > M_2/M_3 \), there exist some \( \tilde{u}_\lambda \in Z_s \) such that
\[
J_{\lambda}(\tilde{u}_\lambda) = \max \{ J_{\lambda}(\tilde{u}) | \tilde{u} \in Z_s \}.
\]
Note that \( \lambda_n \to \infty \) as \( n \to \infty \), then there exists a \( n_0 \) such that there is a sequence of functions \( \tilde{u}_n = (w_n, w_n) \in Z_s \) satisfying
\[
J_{\lambda_n}(\tilde{u}_n) = \max \{ J_{\lambda_n}(\tilde{u}) | \tilde{u} \in Z_s \}
\] (3.15)
for any \( n \geq n_0 \). From (3.9),(3.14) and (3.15) we infer that
\[
c_0 + \lambda_n M_3 \| w_n \|^p \leq (M_1 + M_2) \| w_n \|^2 + M_2 \| w_n \|^p
\] (3.16)
for all \( n \geq n_0 \). Let \( n_1 > n_0 \) be an integer such that
\[
M_2 \leq \frac{\lambda_n M_3}{2}
\] (3.17)
for every \( n \geq n_1 \) because of \( \lim_{n \to \infty} \lambda_n = \infty \). Inserting (3.17) into (3.16) leads to
\[
c_0 + \frac{\lambda_n M_3}{2} \| w_n \|^p \leq (M_1 + M_2) \| w_n \|^2.
\] (3.18)
It reveals that
\[
\sqrt{\frac{c_0}{M_1 + M_2}} \leq \| w_n \| \leq \left( \frac{2(M_1 + M_2)}{\lambda_n M_3} \right)^{\frac{1}{p-2}}, \quad \forall n \geq n_1
\]
which is impossible as \( p > 2 \) and \( \lim_{n \to \infty} \lambda_n = \infty \). Therefore it finishes the proof of part (ii).

**Lemma 3.3.** \( J_{\lambda} \) satisfies the Palais-Smale condition in \( X \).

**Proof.** Let \( \{ \tilde{u}_n = (u_n, v_n) \} \) be a Palais-Smale sequence for \( J_{\lambda} \), then
\[
J_{\lambda}(\tilde{u}_n) = \frac{1}{2} B(\tilde{u}_n, \tilde{u}_n) - \int_{\Omega} [G_1(u_n) + G_2(v_n) + \lambda F(\tilde{u}_n)] = d + o(1)
\] (3.19)
and
\[ \langle J'_\lambda(\bar{u}_n), \bar{u}_n \rangle = B(\bar{u}_n, \bar{u}_n) - \int_\Omega [g_1(u_n)u_n + g_2(v_n)v_n + \lambda f_1(\bar{u}_n)u_n + \lambda f_2(\bar{u}_n)v_n] \]
\[ = o(1)\|\bar{u}_n\|. \quad (3.20) \]
Let \( 2 < \theta' < \theta \), then
\[ \theta' J'_\lambda(\bar{u}_n) - \langle J'_\lambda(\bar{u}_n), \bar{u}_n \rangle \]
\[ = (\frac{\theta'}{2} - 1)B(\bar{u}_n, \bar{u}_n) + \int_\Omega [g_1(u_n)u_n - \theta' G_1(u_n) + g_2(v_n)v_n - \theta' G_2(v_n)] \]
\[ + \lambda \int_\Omega [f_1(\bar{u}_n)u_n - \theta' F(u_n)] \]
\[ \geq (\frac{\theta'}{2} - 1)B(\bar{u}_n, \bar{u}_n) + (\theta' - \theta') \int_\Omega [G_1(u_n) + G_2(v_n)]. \quad (3.21) \]

It follows from part (4) of Lemma 3.1 that
\[ \theta' J'_\lambda(\bar{u}_n) - \langle J'_\lambda(\bar{u}_n), \bar{u}_n \rangle \geq (\frac{\theta'}{2} - 1)B(\bar{u}_n, \bar{u}_n) + C_0(\theta - \theta') \int_\Omega [\|u_n\|^\theta + \|v_n\|^\theta - 2] \quad (3.22) \]
for some positive constants \( C_0 \). Inserting (3.19) and (3.20) into (3.22) leads to
\[ \theta'(d + o(1)) - o(1)\|\bar{u}_n\| \geq (\frac{\theta'}{2} - 1)B(\bar{u}_n, \bar{u}_n) + C_0(\theta - \theta') \int_\Omega [\|u_n\|^\theta + \|v_n\|^\theta - 2]. \quad (3.23) \]

Let \( u_n = u_n^+ + u_n^{\leq 0}, v_n = v_n^+ + v_n^{\leq 0} \in X^+ + \bar{X} \), then \( \|u_n\|^\theta = \|u_n^+\|^\theta + \|v_n^{\leq 0}\|^\theta \) and
\[ \|v_n\|^\theta = \|v_n^+\|^\theta + \|v_n^{\leq 0}\|^\theta. \]

Note that \( L^\theta(\Omega) \subseteq L^2(\Omega) \), then
\[ |u_n^+ + v_n^{\leq 0}|^\theta \geq C|u_n^+|^\theta + |u_n^{\leq 0}|^\theta = C|u_n^+|^\theta + |u_n^{\leq 0}|^\theta \]
\[ \geq C|u_n\|^\theta. \]

Similarly, we have
\[ |v_n^+ + v_n^{\leq 0}|^\theta \geq C|v_n\|^\theta. \]

Because \( B(\cdot, \cdot)^\frac{1}{\theta} \) and \( \| \cdot \| \) are equivalent norms in \( X^+, \bar{X} \) is finite-dimension, we deduce that
\[ C_1(1 + \|u_n^{\leq 0}\|^\theta + \|v_n^{\leq 0}\|^\theta + |\bar{u}_n^n|^\theta) \geq \|\bar{u}_n^n\|^\theta + \|\bar{u}_n^{\leq 0}\|^\theta \]
due to (3.23). It implies that \( \bar{u}_n \) is bounded in \( X \) because of \( \theta > 2 \). So, passing to a subsequence,
\[ \begin{align*}
\bar{u}_n & \rightharpoonup \bar{u} \text{ weakly in } X, \\
\bar{u}_n & \rightarrow \bar{u} \text{ strongly in } L^2(\Omega) \text{ and in } L^p(\Omega), \\
\bar{u}_n & \rightarrow \bar{u} \text{ strongly in } \bar{X},
\end{align*} \quad (3.24) \]

where \( \bar{u}_n = \bar{u}_n^+ + \bar{u}_n \) and \( \bar{u} = \bar{u}^+ + \bar{u} \) with \( \bar{u}_n^+, \bar{u}^+ \in X^+ \) and \( \bar{u}_n, \bar{u} \in \bar{X} \). Also, it is easy to see that \( \bar{u} \) is a critical point of \( J_\lambda \). By the H"{o}lder inequality and the boundedness of \( \{\bar{u}_n\} \) we have
\[ \int_\Omega [g_1(u_n) - g_1(u)](u_n - u) = o(1), \]
\[ \int_\Omega [g_2(v_n) - g_2(v)](v_n - v) = o(1), \]
\[ \int_\Omega [f_1(\bar{u}_n) - f_1(\bar{u})](u_n - u) = o(1), \]
\[ \int_\Omega [f_2(\bar{u}_n) - f_2(\bar{u})](v_n - v) = o(1). \]
Hence,

\[ o(1) = \langle J'_\lambda(\bar{u}_n) - J'_\lambda(\bar{u}), \bar{u}_n - \bar{u} \rangle = B(\bar{u}_n - \bar{u}, \bar{u}_n - \bar{u}) + o(1) = B(\bar{u}_n^+ - \bar{u}^+, \bar{u}_n^+ - \bar{u}^+) + o(1). \]

It follows that \( \bar{u}_n^+ \to \bar{u}^+ \) in \( X \). Thus, \( \bar{u}_n \to \bar{u} \) in \( X \) as claimed. \( \square \)

**Lemma 3.4.** Let

\[ N_\lambda := \{ \bar{u} \in X \setminus \bar{X} : \langle J'_\lambda(\bar{u}), \bar{u} \rangle = \langle J'_\lambda(\bar{v}), \bar{v} \rangle = 0, \ \forall \bar{v} \in \bar{X} \} \]

be the generalized Nehari manifold. The set \( N_\lambda \) is closed in \( X \) and bounded away from \( \bar{X} \).

**Proof.** Arguing by contradiction, assume there exists a sequence \( \{\bar{u}_n\} \) in \( N_\lambda \) such that \( \bar{u}_n^+ \to 0 \). Write \( \bar{u}_n = \bar{u}_n^+ + \bar{u}_n^0 + \bar{u}_n^- \) with \( \bar{u}_n^+ \in X_1^+ \times X_2^+ =: X^+ \) and \( \bar{u}_n^0 \in X_1^0 \times X_2^0 =: X^0 \). Notice that \( \langle J'_\lambda(\bar{u}_n), \bar{u}_n \rangle = 0 \) can be rewritten as follows

\[ B(\bar{u}_n^+, \bar{u}_n^+) = -B(\bar{u}_n^-, \bar{u}_n^-) + \int_\Omega [g_1(u_n)u_n + g_2(v_n)v_n + \lambda f_1(\bar{u}_n)u_n + \lambda f_2(\bar{u}_n)v_n] \tag{3.26} \]

Since \( B(\bar{u}_n^+, \bar{u}_n^+) \to 0 \), by the assumptions \((C4),(C5)\) and \( B(\bar{u}_n^-, \bar{u}_n^-) \leq 0 \), (3.26) implies that \( \bar{u}_n^- \to 0 \) and

\[ \lim_{n \to \infty} \int_\Omega [g_1(u_n)u_n + g_2(v_n)v_n] = 0. \tag{3.27} \]

It follows from part (4) of Lemma 3.1 and \((C5)\) that

\[ \theta C_0 \int_\Omega (|u_n|^\theta + |v_n|^\theta - 2) \leq \int_\Omega [g_1(u_n)u_n + g_2(v_n)v_n]. \tag{3.28} \]

Therefore the sequences \( \{u_n\}, \{v_n\} \) are bounded in \( L^\theta(\Omega) \). It implies immediately that the sequences \( \{u_n\}, \{v_n\} \) are bounded in \( L^2(\Omega) \). Note that

\[ |u_n^0|^2 \leq |u_n|^2 + |u_n^+|^2 + |u_n^-|^2 \leq |u_n|^2 + C\|u_n^+\| + C\|u_n^-\|. \]

Then it reveals that the sequence \( \{u_n^0\} \) is bounded in \( L^2(\Omega) \). Similarly, the sequence \( \{v_n^0\} \) is also bounded in \( L^2(\Omega) \). Since \( \bar{X} \) is finite-dimensional, up to a subsequence there is \( \bar{u}^0 = (u^0, v^0) \in \bar{X} \) such that

\[ \lim_{n \to \infty} \bar{u}_n^0 = \bar{u}^0. \]

From (3.27) and \((C5)\) we see that

\[ \lim_{n \to \infty} \int_\Omega [G_1(u_n) + G_2(v_n)] = 0. \tag{3.29} \]

By part (1) of Lemma 3.1 and Dominated Convergence Theorem we have

\[ \int_\Omega [G_1(u^0) + G_2(v^0)] = 0. \tag{3.30} \]

Thus, \( u^0 = v^0 = 0 \) due to part (3) of Lemma 3.1 and \( \bar{u}_n^0 \to 0 \). It then shows that \( \bar{u}_n \to 0 \). Since \( \bar{u}_n \in N_\lambda \), we obtain

\[ \langle J'_\lambda(\bar{u}_n), \bar{u}_n \rangle = \langle J'_\lambda(\bar{u}_n), \bar{u}_n^0 \rangle = \langle J'_\lambda(\bar{u}_n), \bar{u}_n^- \rangle = 0. \tag{3.31} \]

It indicates \( \langle J'_\lambda(\bar{u}_n), \bar{u}_n^+ \rangle = 0 \). By part (1) of Lemma 3.1 we obtain

\[ B_1(u_n^+, u_n^+) = \int_\Omega [g_1(u_n)u_n^+ + \lambda f_1(\bar{u}_n)u_n^+] \]
\[ \leq C \int_\Omega [\epsilon |u_n| + C_\epsilon |u_n|^{p-1} + \lambda |u_n|^{p-1} |v_n|^2] |u_n^+|. \]  

(3.32)

Using the Hölder and the Sobolev inequalities yields

\[ B_1(u_n^+, u_n^+) \leq \epsilon \|u_n\|^2 + C_\epsilon \|\tilde{u}_n\|^p. \]

Hence, \[ \|u_n^+\|^2 \leq \epsilon \|u_n\|^2 + C_\epsilon \|\tilde{u}_n\|^p, \]

\[ \|v_n^+\|^2 \leq \epsilon \|v_n\|^2 + C_\epsilon \|\tilde{u}_n\|^p, \]

\[ \|v_n^-\|^2 \leq \epsilon \|v_n\|^2 + C_\epsilon \|\tilde{u}_n\|^p. \]

Therefore,

\[ \|u_n^+\|^2 + \|v_n^+\|^2 \leq \epsilon \|u_n\|^2 + C_\epsilon \|\tilde{u}_n\|^p. \]  

(3.33)

Let \( \tilde{w}_n = (\phi_n, \psi_n) = \tilde{u}_n/\|\tilde{u}_n\| \), then (3.33) can be rewritten as

\[ \|u_n^+\|^2 + \|\tilde{w}_n^+\|^2 \leq C_\epsilon \|\tilde{w}_n\|^p. \]

(3.34)

It indicates that \( \tilde{w}_n^+ \to 0 \) and \( \tilde{w}_n^- \to 0 \). As a result, we deduce that

\[ \tilde{w}_n^0 \to \tilde{u}^0 = (\phi^0, \psi^0) \neq (0, 0) \]

(3.35)

up to a subsequence. We observe that

\[ \langle J_1(\tilde{u}_n), (\phi^0_n, 0) \rangle = 0 \]

and

\[ \langle J_1(\tilde{u}_n), (0, \psi^0_n) \rangle = 0. \]

It follows that

\[ 0 = \int_\Omega [g_1(u_n)\phi^0_n + \lambda f_1(\tilde{u}_n)\phi^0_n] \]

(3.36)

and

\[ 0 = \int_\Omega [g_2(v_n)\psi^0_n + \lambda f_2(\tilde{u}_n)\psi^0_n]. \]

(3.37)

Set \( t_n = \|\tilde{u}_n\| \) and \( u_n = t_n \phi_n, v_n = t_n \psi_n, \tilde{u}_n = t_n \tilde{w}_n \). Inserting them into (3.36) and (3.37) gives

\[ \int_\Omega [g_1(t_n \phi_n)\phi^0_n + g_2(t_n \psi_n)\psi^0_n] = -\lambda \int_\Omega [f_1(t_n \tilde{w}_n)\phi^0_n + f_2(t_n \tilde{w}_n)\psi^0_n]. \]

(3.38)

Note that

\[ \lim_{n \to \infty} \frac{g_1(t_n \phi_n(x))\phi^0_n(x)}{t_n^{p-1}} = \ell_1 |\phi^0(x)|^r, \quad \text{a.e. } x \in \Omega, \]

(3.39)

\[ \lim_{n \to \infty} \frac{g_2(t_n \psi_n(x))\psi^0_n(x)}{t_n^{p-1}} = \ell_2 |\psi^0(x)|^r, \quad \text{a.e. } x \in \Omega. \]

(3.40)

By the inequality

\[ |g_i(t)| \leq C(|t|^{r-1} + |t|^{p-1}) \]

due to the assumptions (C1) and (C3), we obtain

\[ |g_1(t_n \phi_n)\phi^0_n + g_2(t_n \psi_n)\psi^0_n| \leq C t_n^{r-1} \left( |\phi_n|^{r-1} |\phi^0_n| + |\psi_n|^{r-1} |\psi^0_n| \right) + C t_n^{p-1} \left( |\phi_n|^{p-1} |\phi^0_n| + |\psi_n|^{p-1} |\psi^0_n| \right). \]

(3.42)
Since \( \phi_n \to \phi^0 \), \( \phi^0_n \to \phi^0 \), \( \psi_n \to \psi^0 \) and \( \psi^0_n \to \psi^0 \) in \( L^p(\Omega) \), there exist a nonnegative function \( h \in L^p(\Omega) \) and a subsequence \( n_k \) of \( n \) such that
\[
|\phi_{n_k}(x)| + |\phi^0_{n_k}(x)| + |\psi_{n_k}(x)| + |\psi^0_{n_k}(x)| \leq h(x), \quad \text{a.e. in } \Omega, \tag{3.43}
\]
\[
\phi_{n_k}(x) \to \phi^0(x) \quad \text{a.e. in } \Omega,
\]
\[
\phi^0_{n_k}(x) \to \phi^0(x) \quad \text{a.e. in } \Omega,
\]
\[
\psi_{n_k}(x) \to \psi^0(x) \quad \text{a.e. in } \Omega
\]
and
\[
\psi^0_{n_k}(x) \to \psi^0(x) \quad \text{a.e. in } \Omega.
\]

For the sake of simplicity, we use the sequence \( \{n\} \) to replace the subsequence \( \{n_k\} \). Then
\[
|g_1(t_n \phi_n) \phi^0_n + g_2(t_n \psi_n) \psi^0_n| \leq C(t^*_n + t^n_0) h^p(x).
\]

By Dominated Convergence Theorem we arrive at
\[
\lim_{n \to \infty} \frac{1}{t^*_n} \int_\Omega [g_1(t_n \phi_n) \phi^0_n + g_2(t_n \psi_n) \psi^0_n] = \int_\Omega [f_1|\phi^0(x)|^\alpha + f_2|\psi^0(x)|^\alpha]. \tag{3.44}
\]

On the other hand, by the assumption (C2) we deduce that
\[
|f_1(t_n \psi_n) \phi^0_n(x) + f_2(t_n \psi_n) \psi^0_n(x)|
\leq c_2 \left( |t_n \phi_n(x)|^{\alpha-1} |t_n \psi_n(x)|^\beta |\phi^0_n(x)| + |t_n \phi_n(x)| |t_n \psi_n(x)|^{\beta-1} |\psi^0_n(x)| \right)
\leq c_2 \left( |\phi_n(x)|^{\alpha-1} |\psi_n(x)|^\beta |\phi^0_n(x)| + |\phi_n(x)| |\psi_n(x)|^{\beta-1} |\psi^0_n(x)| \right).
\]

It follows from Hölder and the Sobolev inequalities that
\[
\int_\Omega [f_1(t_n \psi_n) \phi^0_n(x) + f_2(t_n \psi_n) \psi^0_n(x)]
\leq c_2 t^{p-1}_n \int_\Omega \left( |\phi^0_n(x)|^{\alpha-1} |\psi_n(x)|^\beta |\phi^0_n(x)| + |\phi_n(x)|^\alpha |\psi_n(x)|^{\beta-1} |\psi^0_n(x)| \right)
\leq c_2 t^{p-1}_n \left( |\phi^0_n|^{\alpha-1} |\psi_n|^{\beta} |\phi^0_n| + |\phi_n|^\alpha |\psi_n|^{\beta-1} |\psi^0_n| \right)
\leq C(t^*_n)^{-1}.
\]

Here we used \( \|\phi_n\| = \|\psi_n\| = 1 \) and the boundedness of \( \|\phi^0_n\|, \|\psi^0_n\| \). Thus
\[
\lim_{n \to \infty} \int_\Omega [f_1(t_n \psi_n) \phi^0_n(x) + f_2(t_n \psi_n) \psi^0_n(x)] = 0 \tag{3.47}
\]
because of \( r < p \). Note that
\[
f_1(t_n \psi_n) \phi^0_n(x) + f_2(t_n \psi_n) \psi^0_n(x)
= f_1(t_n \psi_n) \phi_n(x) + f_2(t_n \psi_n) \psi_n(x)
+ f_1(t_n \psi_n) (\phi^0_n(x) - \phi_n(x)) + f_2(t_n \psi_n) (\psi^0_n(x) - \psi_n(x)). \tag{3.48}
\]
By assumptions (C2) and (C5) and (3.43) we deduce that
\[
\frac{f_1(t_n \psi_n) \phi_n(x) + f_2(t_n \psi_n) \psi_n(x)}{t^{-1}_n} \geq \theta c_1 |\phi_n(x)|^\alpha |\psi_n(x)|^\beta \tag{3.49}
\]
and
\[
|f_1(t_n \psi_n) (\phi^0_n(x) - \phi_n(x)) + f_2(t_n \psi_n) (\psi^0_n(x) - \psi_n(x))| \leq 4 c_2 t^{p-1}_n h^p(x)
\]
where the positive constants $c_1, c_2$ and $\theta$ come from the assumptions (C2) and (C5). Moreover
\[
\lim_{n \to \infty} \frac{1}{t_n^{p-1}} \left[ f_1(t_n \bar{w}_n(x)) (\phi_0'(x) - \phi_n(x)) + f_2(t_n \bar{w}_n(x)) (\psi_0'(x) - \psi_n(x)) \right] = 0
\]
a.e in $\Omega$. Then Dominated Convergence Theorem leads to
\[
\lim_{n \to \infty} \left[ \int \frac{f_1(t_n \bar{w}_n(x)) (\phi_0'(x) - \phi_n(x)) + f_2(t_n \bar{w}_n(x)) (\psi_0'(x) - \psi_n(x))}{t_n^{p-1}} \right] = 0. \tag{3.50}
\]
It follows from (3.38), (3.44) and (3.47)-(3.50) that
\[
\int_\Omega [\ell_1|\phi_0'(x)|^r + \ell_2|\psi_0'(x)|^r] \leq 0 \tag{3.51}
\]
for $r \leq p$. It implies that $\phi_0' = \psi_0' = 0$ for a.e. $x \in \Omega$. It contradicts to (3.35). Thus, $N_\lambda$ is bounded away from $X$. Then it reveals that $N_\lambda$ is closed. So far, the proof is finished. \hfill $\square$

**Proof of Theorem 1.1.** We first show (2), given $s := k + \text{codim} X^+$, $Y := X^+$ and $Z := Z_s$. Define $\Lambda_k := \Lambda_s$ as in Lemma 3.2 (ii). Let $\lambda > \Lambda_k$, for this $\lambda$ we choose $b, \rho > 0, b < c_0$, as in Lemma 3.2 (i). Since $J_\lambda$ satisfies the Palais-Smale condition, it follows from Proposition 2 that the system (1.2) has at least $k$ nontrivial solutions $\bar{u}_j$ such that $J_\lambda(\bar{u}_j) \in (b, c_0)$. According to Proposition 1, $\bar{u}_j$ are fully nontrivial.

To show (1), for $\lambda > \Lambda_1$ let $\{\bar{u}_n\}$ be a sequence of nontrivial solutions to (1.2) from part (2) such that $J_\lambda(\bar{u}_n) \to \inf J_\lambda(\bar{u}) : \bar{u} \neq 0$ and $J_\lambda'(\bar{u}) = 0 < c_0$. Then $\{\bar{u}_n\}$ is a Palais-Smale sequence at the energy level less than $c_0$. Since $J_\lambda$ satisfies the Palais-Smale condition, there exists a convergent subsequence $\bar{u}_n \to \bar{u} = (u, v) \in X$. According to Lemma 3.4, $\bar{u} \in N_\lambda$. On the other hand
\[
J_\lambda(\bar{u}) - \frac{1}{2} J_\lambda'(\bar{u}) \bar{u} = \frac{1}{2} \int_\Omega [g_1(u)u + g_2(v)v] - \int_\Omega [G_1(u) + G_2(v)]
+ \lambda \frac{1}{2} \int_\Omega (\mu_1 |\bar{u}|^r + \mu_2 |\bar{v}|^r) - \int_\Omega F(\bar{u})
\geq \frac{\theta}{2} - 1 \int_\Omega [G_1(u) + G_2(v) + \lambda F(\bar{u})] \geq 0. \tag{3.52}
\]
If $\int_\Omega [G_1(u) + G_2(v)] = 0$, then $u = v = 0$. Hence, $0 < J_\lambda(\bar{u}) < c_0$ and $\bar{u}$ is a fully nontrivial ground-state solution of (1.2) due to Proposition 1. \hfill $\square$

4. The critical case. We begin this section by studying the system (1.3). Recall that the energy functional corresponding to (1.3) was defined by
\[
I_{\infty, \lambda}(\bar{u}) := \frac{1}{2} \left( \|u\|^2 + \|v\|^2 + \|w\|^2 \right) - \frac{1}{2} \sum_{i=1}^{N} (\mu_1 |u_1|^2 + \mu_2 |v_1|^2 + \mu_3 |w_1|^2)
- \lambda \int_{\mathbb{R}^N} (|u_1|^{p_1}|u_2|^{p_2} + |u_1|^{p_3}|u_3|^{p_4} + |u_2|^{p_5}|u_3|^{p_6}),
\]
where $\|\cdot\|$ and $|\cdot|_{2^*}$ are the usual norms in $D^{1,2}(\mathbb{R}^N)$ and $L^{2^*}(\mathbb{R}^N)$ respectively. It implies
\[
\langle I_{\infty, \lambda}'(\bar{u}), \bar{v} \rangle = \sum_{i=1}^{3} \|u_i\|^2 - \sum_{i=1}^{3} \mu_i |u_i|_{2^*}^2 - 2^* \lambda \sum_{i<j} \int_{\mathbb{R}^N} |u_i|^{p_i} |u_j|^{p_j}. \]
The Nehari manifold associated with (1.3) is given by
\[ M_{\infty, \lambda} := \{ \bar{u} \in [D^{1,2}(\mathbb{R}^N)]^3 : \bar{u} \neq (0,0,0), (I_{\infty, \lambda}'(\bar{u}), \bar{u}) = 0 \}. \]
We recall that \( M_{\infty, \lambda} \) contains all nontrivial solutions of (1.3) and also the semitrivial ones. Thus,
\[ \inf_{\bar{u} \in M_{\infty, \lambda}} I_{\infty, \lambda}(\bar{u}) = \frac{1}{N} S_{\infty, \lambda}^{N/2}. \]
To estimate \( S_{\infty, \lambda} \), from above we take \( u_1 := s_1 U_\epsilon \), \( u_2 := s_2 U_\epsilon \) and \( u_3 := s_3 U_\epsilon \) with \( s_1, s_2, s_3 \geq 0 \). Recall that \( U_\epsilon(x) := [N(N-2)]^{\frac{N-2}{2}} (\frac{\epsilon}{r^2 + |x|^2})^{\frac{N-2}{2}}. \)
As \( ||U_\epsilon||^2 = S_{\frac{N}{2}} = |u_3|^2_{H^1} \) for any \( \epsilon > 0 \), we have
\[ S_{\infty, \lambda} \leq \inf_{s_i \geq 0, (s_1, s_2, s_3) 
eq (0,0,0)} \frac{\sum_{i=1}^3 s_i^2}{\sum_{i=1}^3 \mu_i s_i^2 + 2s^2 \lambda \sum_{i<j} s_i^p s_j^p} S \]
\[ \leq \frac{3S}{(\sum_{i=1}^3 \mu_i + 3 \cdot 2^s \lambda)^{2/2^s}}, \quad (4.1) \]
where \( S \) is the best constant for the embedding \( D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N) \), that is
\[ S := \inf \left\{ \frac{||\nabla u||_2^2}{||u||_2^2} : u \in D^{1,2}(\mathbb{R}^N), u \neq 0 \right\}. \]
It follows from (4.1) that
\[ \lim_{\lambda \to \infty} S_{\infty, \lambda} = 0. \quad (4.2) \]
Hence, there is \( \Lambda_0 > 0 \) such that for any \( \lambda > \Lambda_0 \),
\[ S_{\infty, \lambda} < \min \left\{ S\mu_i^{\frac{2^s}{2}}, \bar{S}\mu_i^{\frac{2^s}{2}}, \bar{S}\mu_i^{\frac{2^s}{2}} \right\}. \quad (4.3) \]

**Proof of Theorem 1.2.** For part (1) set \( \bar{\mu} := \max \{ \mu_1, \mu_2, \mu_3 \} \), then
\[ \int_{\mathbb{R}^N} \left( \sum_{i=1}^3 \mu_i |u_i|^{2^s} + 2s \lambda \sum_{i<j} |u_i|^{p_{ij}} |u_j|^{p_{ij}} \right) \]
\[ \leq \bar{\mu} \sum_{i=1}^3 |u_i|^2 \quad + 2s \lambda \sum_{i<j} \left[ \frac{p_{ij}}{2} |u_i|^2 |u_j|^2 + \frac{p_{ij}}{2^s} |u_i|^{2^s} |u_j|^{2^s} \right] \]
\[ \leq (\bar{\mu} + 2 \cdot 2^s \lambda) \sum_{i=1}^3 |u_i|^2, \]
and
\[ \left\{ \int_{\mathbb{R}^N} \left( \sum_{i=1}^3 \mu_i |u_i|^{2^s} + 2s \lambda \sum_{i<j} |u_i|^{p_{ij}} |u_j|^{p_{ij}} \right) \right\}^{\frac{2^s}{2}} \leq (\bar{\mu} + 2 \cdot 2^s \lambda) \sum_{i=1}^3 |u_i|^2, \quad (4.4) \]
thanks to the inequality
\[ (a + b + c)^\theta \leq a^\theta + b^\theta + c^\theta, \quad a, b, c \geq 0, \quad 0 < \theta < 1. \]
Hence,
\[ S_{\infty, \lambda} = \inf_{\bar{u} \in D^{1,2}(\mathbb{R}^N), \bar{u} \neq 0} \frac{||u_1||^2 + ||u_2||^2 + ||u_3||^2}{\int_{\mathbb{R}^N} \left( \sum_{i=1}^3 \mu_i |u_i|^{2^s} + 2s \lambda \sum_{i<j} |u_i|^{p_{ij}} |u_j|^{p_{ij}} \right)^{2/2^s}}. \]
By the assumption we obtain

\[ S \geq \frac{\|u\|^2}{(\bar{\mu} + 2 \times 2^* \lambda)^{\frac{2}{2^*}}}, \]  

(4.5) by the inequality

\[ \|u\|^2 \geq S\|u\|_{2^*}, \quad \forall u \in D^{1,2}(\mathbb{R}^N), \]  

which shows that \( S_{\infty, \lambda} > 0 \) for each \( \lambda > 0 \). Additionally, by (4.6) and Hölder’s inequality we obtain

\[
\frac{S \sum_{i=1}^{3} |u_i|^2}{\left[ \sum_{i=1}^{3} \mu_i |u_i|^2 + 2^* \lambda \sum_{i<j} |u_i|^{p_{ij}} |u_j|^{p_{ji}} |2^*/2^*} \right]^{2^*/2^*}, - \frac{\sum_{i=1}^{3} \mu_i |u_i|^2 + 2^* \lambda \sum_{i<j} |u_i|^{p_{ij}} |u_j|^{p_{ji}} |2^*/2^*}}{\left[ \sum_{i=1}^{3} |u_i|^2 + \|u_2\| + \|u_3\|^2 \right]^{2^*/2^*}}.
\]

(4.7)

It leads to

\[ S_{\infty, \lambda} \geq \inf_{s_i \geq 0, (s_1, s_2, s_3) \neq (0,0,0)} S \sum_{i=1}^{3} s_i^2, \]

Consequently,

\[ S_{\infty, \lambda} = \inf_{s_i \geq 0, (s_1, s_2, s_3) \neq (0,0,0)} S \sum_{i=1}^{3} s_i^2 \left[ \sum_{i=1}^{3} \mu_i s_i^{2^*} + 2^* \lambda \sum_{i<j} s_i^{p_{ij}} s_j^{p_{ji}} |2^*/2^*} \right]^{2^*/2^*}. \]

for any \( \lambda > 0 \) due to (4.1). Set

\[ f(s_1, s_2, s_3) = \frac{S \sum_{i=1}^{3} s_i^2}{\left[ \sum_{i=1}^{3} \mu_i s_i^{2^*} + 2^* \lambda \sum_{i<j} s_i^{p_{ij}} s_j^{p_{ji}} |2^*/2^*} \right]^{2^*/2^*}. \]

Then it follows from the continuity of \( f \) that (1.15) holds true.

For part (2), by a direct calculation we deduce

\[ \max\left\{ s_i^{p_{ij}} s_j^{p_{ji}} |s_i^2 + s_j^2 = 1, s_i, s_j \geq 0 \right\} = \left( \frac{p_{ij}^{p_{ij}} p_{ji}^{p_{ji}}}{(2^* s_i^2)^{2^*}} \right)^{\frac{1}{2}}, \]

Let \( m_{ij} = \left( \frac{p_{ij}}{(2^* s_i^2)^{2^*}} \right)^{\frac{1}{2}}, i < j \). It is noted that

\[ g(x) = x^x (1 - x)^{1-x}, \quad 0 < x < 1 \]

is decreasing strictly on the interval \( (0, \frac{1}{2}) \) and increasing strictly on the interval \( (\frac{1}{2}, 1) \). From \( |p_{12} - \frac{2^*}{2^*} | \geq \max\{|p_{13} - \frac{2^*}{2^*}, |p_{23} - \frac{2^*}{2^*} | \} \) we see that

\[ m_{12} \geq \max\{m_{13}, m_{23}\}. \]

We observe that

\[ \lim_{\lambda \to \infty} \lambda^{2/2^*} S_{\infty, \lambda} = \frac{S}{(2^*m)^{2/2^*}}. \]

Recall that \( \bar{m} = \max\{s_1^{p_{12}} s_2^{p_{21}} + s_1^{p_{13}} s_3^{p_{31}} + s_2^{p_{23}} s_3^{p_{32}} |s_1^2 + s_2^2 + s_3^2 = 1, s_i \geq 0, 1 \leq i \leq 3 \}. \) and

\[ \lim_{\lambda \to \infty} \lambda^{2/2^*} \inf\{f(s_1, s_2, 0) | s_1, s_2 \geq 0, (s_1, s_2) \neq (0, 0) \} = \frac{S}{(2^*m_{12})^{2/2^*}}. \]

By the assumption \( \bar{m} > m_{12} \) there exists constant \( A_1 > 0 \) large enough such that

\[ S_{\infty, \lambda} < \inf\{f(s_1, s_2, 0) | s_1, s_2 \geq 0, (s_1, s_2) \neq (0, 0) \} \]

(4.8)
for any \( \lambda > \Lambda_1 \). Therefore, there exist some constants \( s_1, s_2, s_3 > 0 \) depending on \( \lambda \) such that

\[
S_{\infty, \lambda} = f(s_1, s_2, s_3),
\]

and \((s_1 U_\varepsilon, s_2 U_\varepsilon, s_3 U_\varepsilon)\) is a solution of (1.3). It shows that

\[
I_{\infty, \lambda}(s_1 U_\varepsilon, s_2 U_\varepsilon, s_3 U_\varepsilon) = \frac{1}{N} S_{\infty, \lambda} = \inf_{\vec{u} \in M_{\infty, \lambda}} I_{\infty, \lambda}(\vec{u}).
\]

(4.9)

**Remark 1.** If \( p_{ij} = 2^*/2 \) and \( N \geq 4 \), then all assumptions of Theorem 1.2 hold true.

Now we turn our attention to the critical system (1.4). The energy functional associated with (1.4) is defined by

\[
J_\lambda(\vec{u}) := \frac{1}{2} \|\vec{u}\|^2 - \frac{1}{2} \sum_{i=1}^{3} \kappa_i |u_i|^2 - \frac{1}{2^*} \sum_{i=1}^{3} \mu_i |u_i|^{2^*} - \lambda \sum_{i<j} \int_{\Omega} |u_i|^{p_{ij}}|u_j|^{p_{ij}}
\]

and its derivative is

\[
\langle J_\lambda'(\vec{u}), \vec{v} \rangle = \sum_{i=1}^{3} (\nabla u_i, \nabla v_i) - \sum_{i=1}^{3} \kappa_i \langle u_i, v_i \rangle - \sum_{i<j} \int_{\Omega} \mu_i |u_i|^{2^*-2} u_i v_i
\]

\[
- \lambda \sum_{i<j} \int_{\Omega} [p_{ij}|u_i|^{p_{ij}-2} u_i|u_j|^{p_{ij}} v_i + p_{ji}|u_i|^{p_{ji}}|u_j|^{p_{ji}-2} u_j v_j].
\]

(4.11)

**Lemma 4.1.** \( J_\lambda \) satisfies the Palais-Smale condition below the level \( \frac{1}{N} S_{\infty, \lambda}^{N/2} \).

**Proof.** Let \( \{\vec{u}_n\} \) be a Palais-Smale sequence for \( J_\lambda \) with \( J_\lambda(\vec{u}_n) \to c < \frac{1}{N} S_{\infty, \lambda}^{N/2} \). Then \( \{\vec{u}_n\} \) is bounded in \( H^1_0(\Omega) \) by a similar analysis to the one in Lemma 3.3. Up to a subsequence we have

\[
\begin{cases}
\vec{u}_n \rightharpoonup \vec{u} \text{ weakly in } H^1_0(\Omega), \\
\vec{u}_n \rightharpoonup \vec{v} \text{ weakly in } L^{2^*}(\Omega), \\
\vec{u}_n \to \vec{u} \text{ strongly in } L^s(\Omega), \forall \ 1 \leq s < 2^*, \\
\vec{u}_n \to \vec{u} \text{ a.e. on } \Omega.
\end{cases}
\]

(4.12)

Let \( \vec{v}_n = \vec{u}_n - \vec{u} \), it suffices to prove that \( \vec{v}_n \to 0 \) in \([H^1_0(\Omega)]^3\). Note that

\[
|u_{n,i}|^{2^*-2} u_{n,i} \to |u_i|^{2^*-2} u_i, \text{ in } L^{\frac{2N}{N+2}}(\Omega)
\]

by [22]. Therefore, we have

\[
\langle J_\lambda'(\vec{u}_n), \vec{v}_n \rangle = \sum_{i=1}^{3} (\nabla u_{n,i}, \nabla v_i) - \sum_{i=1}^{3} \kappa_i \langle u_{n,i}, v_i \rangle - \sum_{i=1}^{3} \int_{\Omega} \mu_i |u_{n,i}|^{2^*-2} u_{n,i} v_i
\]

\[
- \lambda \sum_{i<j} \int_{\Omega} [p_{ij}|u_{n,i}|^{p_{ij}-2} u_{n,i}|u_{n,j}|^{p_{ij}} v_i + p_{ji}|u_{n,i}|^{p_{ji}}|u_{n,j}|^{p_{ji}-2} u_{n,j} v_j]
\]

\[
= \sum_{i=1}^{3} |u_i|^2 - \sum_{i=1}^{3} \kappa_i |u_i|^2 - \sum_{i=1}^{3} \mu_i |u_i|^{2^*} - 2^* \lambda \sum_{i<j} \int_{\Omega} |u_i|^{p_{ij}} |u_j|^{p_{ij}} + o(1).
\]

(4.13)

It implies that \( \langle J_\lambda'(\vec{u}), \vec{u}_n \rangle = 0 \). Thus by Brezis-Lieb Lemma we deduce that

\[
\langle J_\lambda'(\vec{u}_n), \vec{u}_n \rangle
\]
\[=
\sum_{i=1}^{3} \| u_{n,i} \|^2 - \sum_{i=1}^{3} \kappa_i \| u_{n,i} \|^2 - 2 \sum_{i=1}^{3} \mu_i \| u_{n,i} \|^2 - 2^* \lambda \sum_{i<j}^{} \int_\Omega \| u_{n,i} \|^2 \| u_{n,j} \|^2 + \frac{3}{2} \sum_{i=1}^{3} \mu_i \| u_{n,i} \|^2 - 2^* \lambda \sum_{i<j}^{} \int_\Omega \| u_{n,i} \|^2 \| u_{n,j} \|^2 + o_n(1) \]

It leads to
\[=
\sum_{i=1}^{3} \| u_{n,i} \|^2 - \sum_{i=1}^{3} \mu_i \| u_{n,i} \|^2 - 2^* \lambda \sum_{i<j}^{} \int_\Omega \| u_{n,i} \|^2 \| u_{n,j} \|^2 = o(1). \]

On the other hand,
\[
J_\lambda(\bar{u}_n) = \frac{1}{2} \| \bar{u}_n \|^2 - \frac{1}{2} \sum_{i=1}^{3} \kappa_i \| u_{n,i} \|^2 - \frac{3}{2} \sum_{i=1}^{3} \mu_i \| u_{n,i} \|^2 - 2^* \lambda \sum_{i<j}^{} \int_\Omega \| u_{n,i} \|^2 \| u_{n,j} \|^2 = J_\lambda(\bar{u}) + o(1).
\]

It indicates that
\[
\frac{1}{2} \sum_{i=1}^{3} \| u_{n,i} \|^2 - \frac{3}{2} \sum_{i=1}^{3} \mu_i \| u_{n,i} \|^2 - 2^* \lambda \sum_{i<j}^{} \int_\Omega \| u_{n,i} \|^2 \| u_{n,j} \|^2 = c - J_\lambda(\bar{u}) + o(1). \]

Combining (4.15) with (4.17) we obtain
\[
\sum_{i=1}^{3} \| u_{n,i} \|^2 = N(c - J_\lambda(\bar{u})) + o(1)
\]
and
\[
\sum_{i=1}^{3} \mu_i \| u_{n,i} \|^2 + 2^* \lambda \sum_{i<j}^{} \int_\Omega \| u_{n,i} \|^2 \| u_{n,j} \|^2 = N(c - J_\lambda(\bar{u})) + o(1).
\]

Hence, \( c \geq J_\lambda(\bar{u}) \). If \( c > J_\lambda(\bar{u}) \) then it follows from
\[
S_{\infty,\lambda} \left[ \sum_{i=1}^{3} \mu_i \| u_{n,i} \|^2 + 2^* \lambda \sum_{i<j}^{} \int_\Omega \| u_{n,i} \|^2 \| u_{n,j} \|^2 \right]^{\frac{1}{2}} \leq \sum_{i=1}^{3} \| u_{n,i} \|^2
\]
that
\[
S_{\infty,\lambda} \left[ N(c - J_\lambda(\bar{u})) \right]^{\frac{1}{2}} \leq N \left[ c - J_\lambda(\bar{u}) \right].
\]

By the assumption \( c < \frac{1}{2} N S_{\infty,\lambda} \) and \( c > J_\lambda(\bar{u}) \) we get
\[
Nc < N \left[ c - J_\lambda(\bar{u}) \right].
\]
Thus, \( J_\lambda(\bar{u}) < 0 \). However
\[
J_\lambda(\bar{u}) = J_\lambda(\bar{u}) - \frac{1}{2} \langle J_\lambda'(\bar{u}), \bar{u} \rangle = \frac{1}{N} \sum_{i=1}^{3} \mu_i | u_{n,i} |^2 + \frac{2^* \lambda}{N - 2} \sum_{i<j}^{} \int_\Omega | u_i |^2 | u_j |^2
\]
thanks to \( \langle J_\lambda'(\bar{u}), \bar{u} \rangle = 0 \). It is impossible. Consequently, \( c = J_\lambda(\bar{u}) \) and \( \sum_{i=1}^{3} \| u_{n,i} \|^2 \to 0 \).
We can define $\tilde{X}_3$ for $\kappa_3$ by a similar way to $\tilde{X}_1, \tilde{X}_2$. Let

$$\tilde{X} = \tilde{X}_1 \times \tilde{X}_2 \times \tilde{X}_3.$$ 

Lemma 4.2. Let $\tilde{\Omega}$ be an open nonempty subset of $\Omega$.

(1) If $(w_1, w_2, w_3) \in \tilde{X}$ and $w_i = 0$ a.e in $\tilde{\Omega}$ for some $i$, then $w_i = 0$ a.e in $\Omega$.

Consequently, $|\cdot|_{L^2(\tilde{\Omega})}$ is a norm in $\tilde{X}$, and it is equivalent to any other norm because $\dim \tilde{X} < \infty$.

(2) There exists $C > 0$ such that

$$\int_\Omega |w_i|^{p_i}|w_j|^{p_j} \geq C \|w_i\|^{p_i} \|w_j\|^{p_j} (i \neq j, i, j = 1, 2, 3), \forall (w_1, w_2, w_3) \in \tilde{X}.$$

Proof. (1) Suppose $w_i = 0$ a.e in $\tilde{\Omega}$ and $w_i \neq 0$ in $\Omega$. Note that $w_i \in \tilde{X}_i$ and $\tilde{X}_i$ is finite dimensional, then there exist some eigenfunctions $\omega_\ell$ such that $w_i = \sum_{\ell=1}^\infty \omega_\ell$ where $-\Delta \omega_\ell = \lambda_\ell \omega_\ell$ and $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. Since $w_i = 0$ in $\tilde{\Omega}$, we deduce that

$$0 = (-\Delta - \lambda_1)w_i = \sum_{\ell=2}^n (\lambda_\ell - \lambda_1)\omega_\ell$$

in $\tilde{\Omega}$. Similarly, we obtain

$$0 = \sum_{\ell=3}^n (\lambda_\ell - \lambda_1)(\lambda_\ell - \lambda_2)\omega_\ell$$

in $\tilde{\Omega}$. After finite steps, we obtain $\omega_n = 0$ on $\tilde{\Omega}$. So this is impossible.

(2) Arguing by contradiction, assume there exist $(w_{1,n}, w_{2,n}, w_{3,n}) \in \tilde{X}$ such that $\|w_{1,n}\| = \|w_{2,n}\| = \|w_{3,n}\| = 1$ and $\int_\tilde{\Omega} |w_{1,n}|^{p_i}|w_{j,n}|^{p_j} \to 0$ for some pair $i \neq j$. Since $\tilde{X}$ is finite-dimensional, passing to a subsequence we have $w_{1,n} \to w_i$ and $w_{j,n} \to w_j$. Then, $\|w_{1,n}\| = \|w_{2,n}\| = \|w_{3,n}\| = 1$ and $\int_\tilde{\Omega} |w_{1,n}|^{p_i}|w_{j,n}|^{p_j} = 0$. Thus, $w_i w_j = 0$ a.e. in $\tilde{\Omega}$. Let $\Omega_j = \{x \in \Omega | w_j(x) \neq 0\}$, then $w_i = 0$ a.e. in $\Omega_j$. Note that $w_j$ is smooth, then $\Omega_j$ is an open nonempty subset of $\Omega$. Therefore it follows from part (1) that $w_i = 0$ a.e. in $\Omega$, which contradicts $\|w_{1,n}\| = 1$. This proves the lemma.

Without loss of generality, we assume that $0 \in \Omega$. We fix a radial cut-off function $\psi \in C^\infty_0(\Omega)$ such that $\psi = 1$ for $|x| \leq \delta, \delta > 0$ sufficiently small. The following estimates are well known; see, e.g., [19, 21].

Lemma 4.3. Set $\overline{\pi}_\epsilon := \psi U_\epsilon, \epsilon > 0$. Then, as $\epsilon \to 0$,

1. $\int_\Omega |\nabla \overline{\pi}_\epsilon|^2 = \int_{\mathbb{R}^N} |\nabla U_\epsilon|^2 + O(\epsilon^{N-2})$,
2. $\int_\Omega \overline{\pi}_\epsilon^2 = \int_{\mathbb{R}^N} U_\epsilon^2 + O(\epsilon N)$,
3. $\int_\Omega \overline{\pi}_\epsilon^{2^{-1}} = O(\epsilon^{N^{-2}}), \int_\Omega \overline{\pi}_\epsilon = O(\epsilon^{N^{-2}}), \int_\Omega |\nabla \overline{\pi}_\epsilon| = O(\epsilon^{N^{-2}})$,
4. $\int_\Omega \overline{\pi}_\epsilon^{2-2} = O(\epsilon^2) \quad \text{if } N \geq 5, \int_\Omega \overline{\pi}_\epsilon^{2-2} = O(\epsilon^2 |\ln \epsilon|) \quad \text{if } N = 4$,
5. $\int_\Omega \overline{\pi}_\epsilon^2 \geq \begin{cases} \frac{d \epsilon^2}{|\ln \epsilon|} + O(\epsilon^2) & \text{if } N = 4 \\ \frac{\epsilon^2}{2} + O(\epsilon^{N-2}) & \text{if } N \geq 5 \end{cases}$

where $d$ is a positive constant that depends on $N$.

Proof of Theorem 1.3. From Theorem 1.2 there exists a constant $\Lambda_1 > 0$. Fix $\lambda > \Lambda_1$, let $u_{\epsilon,1} := s_1 \overline{\pi}_\epsilon, u_{\epsilon,2} := s_2 \overline{\pi}_\epsilon$ and $u_{\epsilon,3} := s_3 \overline{\pi}_\epsilon$ with $\overline{\pi}_\epsilon = \psi U_\epsilon$ as above and
s_i > 0, i = 1, 2, 3 as in (1.16). Set \( \bar{u}_\varepsilon = (u_{\varepsilon,1}, u_{\varepsilon,2}, u_{\varepsilon,3}) \). We apply Proposition 2 with \( Y = X^+ \) and

\[
Z := \{ t \bar{u}_\varepsilon + \bar{w} : t \in \mathbb{R}, \ \bar{w} \in \bar{X} \}.
\]

Next we show that, for \( \varepsilon \) small enough,

\[
\sup_{Z} J_{\lambda} < \frac{1}{N} S_{\infty, \lambda}^{N/2}. \tag{4.18}
\]

Since \( \kappa_1, \kappa_2, \kappa_3 > 0 \), Lemma 4.3 and Theorem 1.2 yield the existence of a constant \( C > 0 \) such that

\[
\max_{t > 0} J_{\lambda}(t \bar{u}_\varepsilon) = \frac{1}{N} \left( \frac{\sum_{i=1}^{3} s_i^2 \int_{\Omega} |\nabla u_{\varepsilon,i}|^2 \sum_{i=1}^{3} \kappa_i s_i^2 \int_{\Omega} \lambda^{2r} (\int_{\Omega} \lambda^{2r})^{2/2r}}{\sum_{i=1}^{3} \mu_i s_i^{2r} + 2^r \lambda \sum_{i<j} s_i^{p_{ij}^r} s_j^{p_{ji}^r})^{2/2r} (\int_{\Omega} \lambda^{2r})^{2/2r}} \right)^{N/2}
\leq \left\{ \begin{array}{ll}
\frac{1}{N} \left( S_{\infty, \lambda} - C \varepsilon^2 + o(\varepsilon^2) \right)^{N/2}, & N \geq 5, \\
\frac{1}{N} \left( S_{\infty, \lambda} - C \varepsilon^2 |\ln \varepsilon| + O(\varepsilon^2) \right)^2, & N = 4.
\end{array} \right.
\tag{4.19}
\]

for every \( \varepsilon > 0 \) small enough.

Let now \( t \bar{u}_\varepsilon + \bar{w} \in Z \) with \( \bar{w} = (w_1, w_2, w_3) \in \bar{X} \setminus \{(0, 0, 0)\} \) and set \( \bar{\Omega} := \Omega \setminus \supp \psi \). We may assume \( t > 0 \). The computations below become simpler if \( w_i = 0 \) for some \( i \) as the corresponding terms will vanish. Since the inequality

\[
x^r - 1 - r x + r \geq 0
\]
holds true for \( x \in [0, \infty) \) and \( r \geq 1 \), we deduce that

\[
|a + b|^r \geq a^r + r a^{r-1} b, \quad \forall b \in \mathbb{R}, \ a \geq 0, \ r \geq 1
\]
and

\[
|1 + x|^{\sigma} |1 + y|^\tau \geq 1 + \theta x + \tau y - \theta \tau |xy|, \quad \forall x, y \in \mathbb{R}, \ \theta, \tau \geq 1.
\]
Thus,

\[
|a + b|^\theta |c + d|^\sigma \geq a^\theta c^\tau + \theta b a^{\theta-1} c^\tau + \tau d a^\theta c^{\tau-1} - \theta \tau |bd| a^{\theta-1} c^{\tau-1}
\]
for any \( a, b, c, d \in \mathbb{R}, \ a, c \geq 0, \ \theta, \tau \geq 1 \). Therefore, using Lemma 4.2 we obtain

\[
B_i(tu_{\varepsilon,i} + w_1, tu_{\varepsilon,i} + w_2, tu_{\varepsilon,i} + w_3) = B_i(tu_{\varepsilon,i}, tu_{\varepsilon,i}) + 2 B_i(tu_{\varepsilon,i}, w_1) + B_i(w_1, w_1), \ i = 1, 2, 3.
\]

\[
\int_{\Omega} |tu_{\varepsilon,i} + w_i|^2\]

\[
\geq \int_{\Omega} |tu_{\varepsilon,i}|^2 + \int_{\bar{\Omega}} |w_i|^2 + \int_{\bar{\Omega}} |w_i|^{2r} + \int_{\bar{\Omega}} 2^r t^{2r-1} u_{\varepsilon,i}^{2r-1} w_i + C_1 |w_i|^2, \ i = 1, 2, 3,
\tag{4.22}
\]

\[
\sum_{i \neq j} \int_{\Omega} |tu_{\varepsilon,i} + w_i|^{p_{ij}} |tu_{\varepsilon,j} + w_j|^{p_{ji}}
\geq \sum_{i \neq j} \int_{\Omega} |tu_{\varepsilon,i} + w_i|^{p_{ij}} |tu_{\varepsilon,j} + w_j|^{p_{ji}} + \sum_{i \neq j} \int_{\Omega} |w_i|^{p_{ij}} |w_j|^{p_{ji}}
\tag{4.23}
\]

\[
\geq \sum_{i < j} \int_{\Omega} (tu_{\varepsilon,i})^{p_{ij}} (tu_{\varepsilon,i})^{p_{ji}} + t^{2r-1} \sum_{i < j} \int_{\Omega} u_{\varepsilon,i}^{p_{ij}-1} u_{\varepsilon,j}^{p_{ji}} w_i + \sum_{i < j} \int_{\Omega} u_{\varepsilon,i}^{p_{ij}} u_{\varepsilon,j}^{p_{ji}-1} w_j
\]
\[-t^2 - 2 \sum_{i<j} p_{ij} p_{ji} \int \Omega \left| u_{e,i}^{p_{ij} - 1} u_{e,j}^{p_{ij} - 1} \right| |w_i w_j| + C_2 \sum_{i<j} \|w_i\|^{p_{ij}} \|w_j\|^{p_{ij}}, \quad (i, j = 1, 2, 3).\]

Here and hereafter \(C_i\) denotes a positive constant. From inequalities (4.21), (4.22) and Lemma 4.3 we obtain

\[J_\lambda(tu_e + \bar{w}) = \frac{1}{2} \int \Omega |\nabla(tu_e + \bar{w})|^2 - \frac{1}{2} \sum_{i=1}^3 \kappa_i |tu_{e,i}|^2 - \frac{1}{2} \sum_{i=1}^3 \mu_i |tu_{e,i} + w_i|^2 \]

\[\leq C_3 \left( t^2 + t \sum_{i=1}^3 \|w_i\| + \sum_{i=1}^3 \|w_i\|^2 + \epsilon^{\frac{5}{2}} t^{2-1} \sum_{i=1}^3 \|w_i\| \right) \]

\[-C_4 \left( t^2 + \sum_{i=1}^3 \|w_i\|^2 \right). \quad (4.24)\]

So, as \(t^2 - 1 \sum_{i=1}^3 \|w_i\| \leq 3t^2 + \sum_{i=1}^3 \|w_i\|^2\), there exists \(R > 0\) such that, for every \(\epsilon\) small enough,

\[J_\lambda(tu_e + \bar{w}) \leq 0 \quad \text{if} \quad \|w_i\| \geq R \quad \text{or} \quad t \geq R. \quad (4.25)\]

From now on, we assume that \(t \leq R\) and \(\|w_i\| \leq R\). We distinguish two cases.

Let \(N \geq 5\). Since \(B_i(w_i, w_i) \leq 0\), using Lemma 4.3, the inequalities (4.21)-(4.23) and \(\max_{\tau > 0} (\tau t - \tau \gamma) \leq C_\gamma r^{\gamma \gamma} \tau^{\gamma} \) if \(\gamma > 1\),

\[\max_{0 \leq t_1, t_2 \leq R} (rt_1 t_2 - t_1^\gamma t_2^\gamma) \leq C_{R, \gamma, \gamma_1, \gamma_2} \max \{r^{\gamma_i} |i = 1, 2\} \quad \text{if} \quad \gamma_1, \gamma_2 > 1, \quad (4.27)\]

which hold true for every \(r \geq 0\) (\(C_\gamma\) and \(C_{R, \gamma, \gamma_1, \gamma_2}\) are positive constants that depend only on their subindices), we get

\[J_\lambda(tu_e + \bar{w}) \leq J_\lambda(tu_e) + O(\epsilon^{\frac{N-2}{2}}) \sum_{i=1}^3 \|w_i\| - C_5 \sum_{i=1}^3 \|w_i\|^2 \]

\[+ O(\epsilon^3) \sum_{i<j} \|w_i\| \|w_j\| - C_6 \sum_{i<j} \|w_i\|^{p_{ij}} \|w_j\|^{p_{ij}} \]

\[\leq J_\lambda(tu_e) + O(\epsilon^{\frac{5(N-2)}{2}}) + o(\epsilon^2). \quad (4.28)\]

Note that \(\frac{N(N-2)}{N+2} > 2\) for \(N \geq 5\). So (4.19) and (4.28) give

\[J_\lambda(tu_e + \bar{w}) \leq \frac{1}{N} \left( S_{\infty, \lambda} - C \epsilon^2 + o(\epsilon^2) \right)^{N/2} + o(\epsilon^2) \]

for all \(t \in [0, R]\), \(\|w_i\| \leq R\). This inequality, together with (4.25), yields (4.18) for \(\epsilon\) small enough.

Let now \(N = 4\). Since \(\kappa_1, \kappa_2, \kappa_3\) are not eigenvalues of \(-\Delta\) in \(H^1_0(\Omega)\), there exists \(C_7 > 0\) such that \(-B_i(w_i, w_i) \geq C_7 \|w_i\|^2\) for all \(w_i \in X_i, \ i = 1, 2, 3\). Thus, from Lemma 4.3 and the inequalities (4.21)-(4.23), (4.26) and (4.27), we obtain

\[J_\lambda(tu_e + \bar{w}) \leq J_\lambda(tu_e) + O(\epsilon) \sum_{i=1}^3 \|w_i\| - C_8 \sum_{i=1}^3 \|w_i\|^2 \]
\[ J_\lambda(tu_\epsilon + \bar{w}) \leq J_\lambda(tu_\epsilon) + O(\epsilon^2) + O(\epsilon^2), \]

Combining this inequality with (4.19) gives
\[ J_\lambda(tu_\epsilon + \bar{w}) \leq J_\lambda(tu_\epsilon) + O(\epsilon^2), \]

for all \( t \in (0, R], \|w_i\| \leq R \). This inequality, together with (4.25), yields (4.18) for \( \epsilon \) small enough.

Utilizing Proposition 2 and Lemma 4.1, and analysis similar to that in the proof of Theorem 1.1 we can obtain a nontrivial solution at some level \( c(\lambda) < \frac{1}{N}S_{\infty, \lambda}^{N/2} \) for \( \lambda > \Lambda_1 \) and then show that there is a ground-state solution \( \bar{u} \). Recall that \( \omega_i \) is a ground-state solution to the equation (1.17). Let
\[ \tilde{c}_0 = \min \{ \phi_i(\omega_i) | i = 1, 2, 3 \}, \] (4.30)

where
\[ \phi_i(v) = \frac{1}{2}(|\nabla v|^2_2 - \kappa_i|v|^2_2) - \frac{1}{2}\mu_i|v|^2_2^*, \quad \forall v \in H^1_0(\Omega). \]

We observe that
\[ \phi_i(\omega_i) = \frac{\mu_i}{N}|\omega_i|^2_2^*, \]

then \( \tilde{c}_0 > 0 \). By (4.2) we have
\[ \lim_{\lambda \to \infty} S_{\infty, \lambda} = 0. \] (4.31)

Then \( \frac{1}{N}S_{\infty, \lambda}^{N/2} < \tilde{c}_0 \) for enough large \( \lambda \). So
\[ J_\lambda(\bar{u}) \leq c(\lambda) < \frac{1}{N}S_{\infty, \lambda}^{N/2} < \tilde{c}_0. \]

It implies that the ground-state solution \( \bar{u} \) has one zero components at most. It is noted that for \( \bar{u} = (u_1, u_2, u_3) \), we have
\[ J_\lambda(\bar{u}) = J_\lambda(\bar{u}) - \frac{1}{2}\langle J_\lambda'(\bar{u}), \bar{u} \rangle \]
\[ = \frac{1}{N} \left[ \sum_{i=1}^{3} \mu_i|u_i|^2_2 + 2\lambda \sum_{i<j} \int_{\Omega} |u_i|^{p_{ij}}|u_j|^{p_{ji}} \right] > 0. \] (4.32)

Moreover, it is easily seen from (4.30) that
\[ J_\lambda(\bar{u}) < \tilde{c}_0 \leq \phi_i(\omega_i) = \phi_i(\omega_i) - \frac{1}{2}\langle \phi_i'(\omega_i), \omega_i \rangle \]
\[ = \frac{1}{N}(|\nabla \omega_i|^2_2 - \kappa_i|\omega_i|^2_2) = \frac{1}{N}B_i(\omega_i, \omega_i). \] (4.33)

Hence, we get
\[ 0 < J_\lambda(\bar{u}) < \frac{1}{N} \min \{ B_i(\omega_i, \omega_i) | i = 1, 2, 3 \}. \]
\[ \square \]
5. **Synchronized solutions.** If $\kappa_1 = \kappa_2 = \kappa_3 = \kappa$, we aim to find synchronized ground-state solutions to the system (1.4), that is, solutions of the form $(t_1\omega, t_2\omega, t_3\omega)$ with $t_1, t_2, t_3 \in \mathbb{R}$. It is well known that

$$- \Delta \phi - \kappa \phi = |\phi|^{2^* - 2} \phi, \quad x \in \Omega, \quad \phi \in H^1_0(\Omega)$$

has a ground-state solution $\omega \in H^1_0(\Omega)$ under the conditions $\lambda_m < \kappa < \lambda_{m+1}$ and $N \geq 4$ or any $\kappa \in \mathbb{R}$ and $N \geq 5$ (see Theorem 3.6 in [19]). Thus, $(t_1\omega, t_2\omega, t_3\omega)$ is a solution of (1.4) if and only if

$$|t_i|^2 = \mu_i |t_i|^{2^*} + \sum_{j \neq i}^3 \lambda \rho_{ij} |t_i|^{p_{ij}} |t_j|^{p_{ji}}, \quad i = 1, 2, 3.$$

(5.2)

The energy functional associated with the system (1.4) is given by

$$E(u_1, u_2, u_3) = \frac{1}{2} \sum_{i=1}^3 (|\nabla u_i|^2 - \kappa |u_i|^2) - \frac{1}{2} \sum_{i=1}^3 \mu_i |u_i|^2 - \lambda \sum_{i<j} \int_\Omega |u_i|^{p_{ij}} |u_j|^{p_{ji}}.$$  

Then the energy functional of any solution with the form $(t_1\omega, t_2\omega, t_3\omega)$ is

$$E(t_1\omega, t_2\omega, t_3\omega) = \frac{1}{N} \|H(t)B_0(\omega, \omega),$$

(5.3)

where

$$H(t) = \sum_{i=1}^3 \mu_i t_i^{2^*} + \lambda 2^* \sum_{1 \leq i < j \leq 3} t_i^{p_{ij}} t_j^{p_{ji}}, \quad t \in [0, +\infty)^3$$

(5.4)

and

$$B_0(\phi, \psi) = \int_\Omega (\nabla \phi \nabla \psi - \kappa \phi \psi), \quad \forall \phi, \psi \in H^1_0(\Omega).$$

Define

$$A := \{t \in [0, +\infty)^3 \setminus \{0\} | t_i^2 = \mu_i t_i^{2^*} + \sum_{j \neq i}^3 \lambda \rho_{ij} t_i^{p_{ij}} t_j^{p_{ji}}, \quad 1 \leq i \leq 3\}.$$  

(5.5)

Let $\mathcal{K}$ be the 3 times Cartesian of $H^1_0(\Omega)$. We consider the following minimizing problem

$$\rho = \inf \{ \frac{B(\bar{u}, \bar{u})}{J(\bar{u})^2} | \bar{u} \in \mathcal{K}, \bar{u} \neq 0 \},$$

(5.6)

where the functional $J : \mathcal{K} \to \mathbb{R}$ is defined by

$$J(\bar{u}) := \left[ \sum_{i=1}^3 \mu_i |u_i|^2 + 2^* \lambda \sum_{i<j} \int_\Omega |u_i|^{p_{ij}} |u_j|^{p_{ji}} \right]^\frac{1}{2^*}$$

and

$$B(\bar{u}, \bar{v}) = \sum_{i=1}^3 B_0(u_i, v_i).$$

Now let us introduce some sets,

$$\mathcal{N} = \{ \bar{u} \in \mathcal{K} \setminus \{0\} | B(\bar{u}, \bar{u}) = [J(\bar{u})]^2 \}$$

be the Nehari manifold of system (1.4) that contains all non zero solutions. The least energy of system (1.4) is given by

$$\mathcal{E} = \inf \{ E(\bar{u}) | \bar{u} \in \mathcal{N} \},$$
where \( \mathcal{A} \) denotes all non zero solutions to the system (1.4). Then the set of ground-state solutions is
\[
G = \{ \tilde{u} \in \mathcal{A} | E(\tilde{u}) = \mathcal{E} \}.
\]
As usual it is useful to consider the least energy on Nehari manifold, set
\[
\mathcal{E}_N = \inf \{ E(\tilde{u}) | \tilde{u} \in \mathcal{A} \}.
\]

**Lemma 5.1.** The set \( A \) is nonempty and let
\[
\Theta = \min \{ H(\tilde{t}) | \tilde{t} \in A \}
\]
and
\[
\Upsilon = \max \{ H(\tilde{t}) | \sum_{i=1}^{3} t_i^2 = 1, \tilde{t} \in [0, +\infty)^3 \}.
\]
Then
\[
\Upsilon = \Theta - 2N^{\frac{2}{3}}.
\]

**Proof.** Let \( \Upsilon = H(\bar{x}) \) for some \( \bar{x} = (x_1, x_2, x_3) \in [0, +\infty)^3 \) with \( \sum_{i=1}^{3} x_i^2 = 1 \). Then there exists a constant \( \lambda_0 \in \mathbb{R} \) such that
\[
\nabla H(\bar{x}) = \lambda_0 \bar{x}.
\]
By a straightforward calculation, one has
\[
\begin{align*}
\lambda_0 x_i^2 &= 2^* \mu_1 x_i^{2^*} + 2^* \lambda \sum_{j \neq i}^{3} p_{ij} x_j^{p_{ij}} \\
\lambda_0 x_i^2 &= 2^* \mu_2 x_i^{2^*} + 2^* \lambda \sum_{j \neq i}^{3} p_{ij} x_j^{p_{ij}} \\
\lambda_0 x_i^2 &= 2^* \mu_3 x_i^{2^*} + 2^* \lambda \sum_{j \neq i}^{3} p_{ij} x_j^{p_{ij}}
\end{align*}
\]
Summing up these equations, it gives us
\[
\lambda_0 = 2^* \sum_{i=1}^{3} \mu_i x_i^{2^*} + (2^*)^2 \lambda \sum_{1 \leq i < j \leq 3} x_i^{p_{ij}} x_j^{p_{ji}} = 2^* \Upsilon.
\]
Set
\[
\sigma = \Upsilon^{\frac{1}{2^*}}, \quad \tilde{y} = \sigma^{-1} \bar{x}.
\]
Then \( \tilde{y} \in A \). In particular, the set \( A \) is not empty. By the definition of \( \Theta \), we have
\[
\Theta \leq H(\tilde{y}) = \sigma^{-2^*} H(\bar{x}) = \sigma^{-2^*} \Upsilon.
\]
It indicates that
\[
\Theta \leq \Upsilon^{\frac{1}{2^*}} = \Upsilon^{1 - \frac{N}{2^*}}.
\]
On the other hand, we claim that
\[
A \subset \left\{ \tilde{t} \in [0, +\infty)^3 | r \leq \sum_{i=1}^{3} t_i^2 \leq R \right\}
\]
for some \( 0 < r < R < \infty \). In fact, for any \( \tilde{t} \in A \) one has
\[
C \sum_{i=1}^{3} t_i^2 \leq \sum_{i=1}^{3} t_i^2 \leq \frac{1}{C} \sum_{i=1}^{3} t_i^2
\]
for some positive constant \( C \) independent of \( \tilde{t} \) due to Young’s inequality. Moreover, we have
\[
\left( \sum_{i=1}^{3} t_i^2 \right)^{\frac{2}{3}} \leq \sum_{i=1}^{3} t_i^2 \leq 3^\frac{2}{3} \left( \sum_{i=1}^{3} t_i^2 \right)^{\frac{2}{3}}
\]
by convexity and Hölder’s inequality. It follows from (5.9) and (5.10) that
\[
C \frac{N^2}{2} \leq \sum_{i=1}^{3} t_i^2 \leq \frac{1}{C} \left( \sum_{i=1}^{3} t_i^2 \right)^{2^*}.
\]
Thus
\[
C \frac{N^2}{2} \leq \sum_{i=1}^{3} t_i^2 \leq 3C \frac{2^N}{2}.
\]
Then the claim is proved by taking \( r = C \frac{N^2}{2} \) and \( R = 3C \frac{2^N}{2} \). Thus \( A \) is compact in \( \mathbb{R}^3 \). Then by continuity we can assume that \( \Theta = H(\bar{s}) \) for some \( \bar{s} = (s_1, s_2, s_3) \in A \). Then we obtain
\[
3 \sum_{i=1}^{3} s_i^2 = 3 \sum_{i=1}^{3} \mu_i s_i^{2^*} + 2^* \lambda \sum_{1 \leq i < j \leq 3} s_i^{P_{ij}} s_j^{P_{ji}}.
\]
Thus
\[
\sum_{i=1}^{3} (s_i^2)^2 = H(\bar{s}) = \Theta.
\]
Set \( \delta = \Theta^{\frac{1}{2}} \), then
\[
\sum_{i=1}^{3} (s_i^2)^2 = 1.
\]
It implies that
\[
\Upsilon \geq H(\delta \bar{s}) = \delta^{2^*} \Theta = \Theta^{\frac{2^N}{2}}.
\]
Consequently (5.7) follows from (5.11) and (5.8). \( \square \)

**Corollary 1.** Suppose that \( \lambda_m < \kappa < \lambda_{m+1} \) and \( N \geq 4 \) or any \( \kappa \in \mathbb{R} \) and \( N \geq 5 \). Then the system (1.4) has a synchronized solution. In particular, \( \mathcal{A} \neq \emptyset \) and \( \mathcal{N} \neq \emptyset \).

**Proof.** From [19] it follows that the equation (5.1) has a ground-state solution \( \omega \). Let \( \omega \) be fixed. By the analysis before Lemma 5.1, we deduce that \( (t_1 \omega, t_2 \omega, t_3 \omega) \in \mathcal{A} \) if and only if \( \bar{t} = (t_1, t_2, t_3) \in A \). Notice that the set \( A \) defined in (5.5) is nonempty. Then there exists a non zero synchronized solution to (1.4). Therefore \( \mathcal{A} \neq \emptyset \). Thus, the corollary follows from that \( \mathcal{A} \subset \mathcal{N} \). \( \square \)

**Lemma 5.2.**
\[
\mathcal{E}_N = \frac{1}{N} \rho^{\frac{2^*}{2}}.
\]

**Proof.** On one hand, let \( \bar{u} \in \mathcal{N} \), then we have
\[
\rho \leq \frac{B(\bar{u}, \bar{u})}{J(\bar{u})^{2^*}} = \left[ NE(\bar{u}) \right]^{\frac{2^*}{2}},
\]
where the identities \( B(\bar{u}, \bar{u}) = |J(\bar{u})|^{2^*} = NE(\bar{u}) \) are used for any \( \bar{u} \in \mathcal{N} \). It implies that
\[
\frac{1}{N} \rho^{\frac{2^*}{2}} \leq \mathcal{E}_N.
\]
On the other hand, let \( \{ \bar{u}_n \} \) be a minimizing sequence of \( \rho \). Then
\[
\lim_{n \to \infty} \frac{B(\bar{u}_n, \bar{u}_n)}{|J(\bar{u}_n)|^{2^*}} = \rho.
\]
Set
\[ a_n = \frac{B(\bar{u}_n, \bar{u}_n)}{|J(\bar{u}_n)|^{\frac{2}{p}}, \quad n \geq 1, \]
then we obtain \( a_n \bar{u}_n \in \mathcal{N} \). It indicates that
\[ \mathcal{E}_\mathcal{N} \leq E(a_n \bar{u}_n) = \frac{1}{N} a_n^2 B(\bar{u}_n, \bar{u}_n) = \frac{1}{N} \left( \frac{B(\bar{u}_n, \bar{u}_n)}{|J(\bar{u}_n)|^{\frac{2}{p}}} \right)^{\frac{N}{2}}. \]
Let \( n \) tend to infinity, we see that
\[ \mathcal{E}_\mathcal{N} \leq \frac{1}{N} \lim_{n \to \infty} \left( \frac{B(\bar{u}_n, \bar{u}_n)}{|J(\bar{u}_n)|^{\frac{2}{p}}} \right)^{\frac{N}{2}} = \frac{1}{N} \rho^{\frac{N}{2}}. \]
Hence, the lemma follows from (5.13) and (5.14).

In what follows we assume that \( 0 < \kappa < \bar{\kappa}_1 \). Define
\[ \bar{S} := \inf \left\{ \frac{B_0(\phi, \psi)}{\| \phi \|_{L^2}^2} : \phi \in H_0^1(\Omega), \phi \neq 0 \right\}. \]
Then it follows from [7] that each ground-state to equation (5.1) is a minimizer of \( \bar{S} \) and
\[ B_0(\omega, \omega) = \bar{S}^{\frac{N}{2}}. \]

**Proposition 3.**
\[ \rho = \bar{S} Y^{\frac{N}{2}}. \]
Moreover, each minimizer of \( \rho \) is synchronized.

**Proof.** Fix a ground-state solution \( \omega \) to (5.1), then \( B_0(\omega, \omega) > 0 \). Thus, from (5.3), Lemma 5.1 and (5.15) we obtain
\[ \mathcal{E}_\mathcal{N} \leq \frac{\bar{S}^{\frac{N}{2}}}{N} \Theta, \]
it gives
\[ \rho \leq \bar{S} Y^{\frac{N}{2}} \]
due to Lemma 5.1 and Lemma 5.2. On the other hand, for any \( \bar{u} = (u_1, u_2, u_3) \in \mathcal{N} \setminus \{ 0 \} \), set
g and \( \bar{t} = (t_1, t_2, t_3) \). By the Hölder inequality we infer that
\[ \int_{\Omega} |u_i|^{p_j} |u_j|^{p_j} \leq \bar{t}_i^{p_j} \bar{t}_j^{p_j}, \quad i, j = 1, 2, 3. \]
Hence,
\[ [J(\bar{u})]^{2^*} \leq H(\bar{t}) \leq \gamma \left( \sum_{i=1}^3 \bar{t}_i^2 \right)^{\frac{2^*}{2}}. \]
It shows that
\[ [J(\bar{u})]^{2^*} \leq \gamma \left( \sum_{i=1}^3 \bar{t}_i^2 \right)^{\frac{2^*}{2}}. \]
By the Sobolev inequality, we have
\[ \sum_{i=1}^3 \bar{t}_i^2 \leq \bar{S}^{-1} \sum_{i=1}^3 \int_{\Omega} (\nabla u_i \cdot \nabla u_i - \kappa u_i u_i) = \bar{S}^{-1} B(\bar{u}, \bar{u}). \]
Combining (5.19) and (5.20) gives
\[ |J(\tilde{u})|^2 \leq Y \tilde{S} B(\tilde{u}, \tilde{u}). \]
It is easy to see that
\[ B(\tilde{u}, \tilde{u}) \geq \tilde{S} Y^{-\frac{N}{2}} |J(\tilde{u})|^2. \]
Hence, by definition of \( \rho \) we have
\[ \rho \geq \tilde{S} Y^{-\frac{N}{2}}. \tag{5.21} \]
From (5.17) and (5.21), we get (5.16). For any fixed minimizer \( \tilde{u} = (u_1, u_2, u_3) \) of \( \rho \),
\[ B(\tilde{u}, \tilde{u}) = \rho |J(\tilde{u})|^2 = \tilde{S} Y^{-\frac{N}{2}} |J(\tilde{u})|^2. \]
Therefore, the inequalities (5.18), (5.19) and (5.20) become equalities. So, Hölder's inequality implies that there exist constants \( c_i, 1 \leq i \leq 3 \) and a nonnegative function \( \phi(x) \) such that \( |u_i(x)| = c_i \phi(x), 1 \leq i \leq 3 \). Set \( \tilde{u} = (|u_1|, |u_2|, |u_3|) \), note that
\[ B(\tilde{u}, \tilde{u}) \leq B(\tilde{u}, \tilde{u}), \text{ thus } B(\tilde{u}, \tilde{u}) = B(\tilde{u}, \tilde{u}). \]
It implies that \( \tilde{u} \) is also a minimizer of \( \rho \). By a direct computation, we deduce that
\[ \rho = \frac{B(\tilde{u}, \tilde{u})}{|J(\tilde{u})|^2} = \frac{1}{(H(\tilde{x}))^\frac{N}{2}} \frac{B_0(\phi, \phi)}{\phi^2}, \geq Y^{-\frac{N}{2}} S, \]
then \( \phi(x) \) is a minimizer for \( S \). Hence \( \phi(x) \) is positive by maximum principle. Set
\[ \alpha_i(x) = u_i(x)/\phi(x), i = 1, 2, 3. \]
Then \( |\alpha_i(x)| = c_i \). So, \( \alpha_i(x) \nabla \alpha_i(x) \) vanishes. From the fact \( |\nabla u_i|^2 = |\nabla v_i|^2 \) we obtain \( \nabla \alpha_i(x) = 0 \) a.e. in \( \Omega \). Thus, \( \alpha_i(x) \) is a constant function because \( \alpha_i(x) \in W^{1,1}_{\text{loc}}(\Omega) \). Therefore \( u_i(x) = a_i \phi(x) \) for some constants \( a_i, 1 \leq i \leq 3 \).

**Proof of Theorem 1.4.** It is noted that part (1) of Theorem 1.4 follows directly from the Corollary 1. For part (2) let
\[ \tilde{u} = (\tau_1 \omega, \tau_2 \omega, \tau_3 \omega), \]
where \( \tilde{\tau} = (\tau_1, \tau_2, \tau_3) \in A \) satisfies \( H(\tilde{\tau}) = \Theta \). It follows from Lemmas 5.1, 5.2, Proposition 3, (5.3) and (5.15) that
\[ E(\tilde{u}) = \frac{\tilde{S}^N}{N} H(\tilde{\tau}) = \frac{\tilde{S}^N}{N} \Theta = \mathcal{E}_\nu \leq \mathcal{E}. \]
Thus \( \tilde{u} \in G \).

Moreover, let \( \tilde{u} = (u_1, u_2, u_3) \) be any ground-state solution to (1.4). Then \( \tilde{u} \) is a minimizer of \( \rho \) by the proof of Lemma 5.2. Thus Proposition 3 reveals that there exist constants \( t_j \in \mathbb{R}, 1 \leq j \leq 3 \) and a function \( v \geq 0 \) such that
\[ u_j(x) = t_j v(x). \]
Thereby, we obtain that \( v\tilde{\tau} = (t_1 v, t_2 v, t_3 v) \) is also a ground-state solution of (1.4). In particular, \( v \) is a non-negative solution to
\[ -\Delta \varphi - k \varphi = \gamma |\varphi|^{2^*-2} \varphi, \quad x \in \Omega, \varphi \in H^1_0(\Omega) \tag{5.22} \]
for some positive constant \( \gamma \) (see [7]). It follows from maximum principle that \( v(x) > 0 \) for all \( x \in \Omega \) (see [3]). Let \( v = \gamma^{\frac{4-N}{2}} \xi \), then we deduce that
\[ -\Delta \varphi - k \varphi = |\varphi|^{2^*-2} \varphi. \]
Consequently,

\[ u_j(x) = t_j v(x) = t_j \gamma^{\frac{2-N}{4}} \varphi(x), \quad \forall x \in \Omega, \ j = 1, 2, 3. \]

Define \( \tau_j = |t_j| \gamma^{\frac{2-N}{4}}, j = 1, 2, 3 \), then

\[ u_j(x) = \pm \tau_j \varphi(x), \quad \forall x \in \Omega, j = 1, 2, 3. \]

Finally, by calculating the energy of \( \bar{u} \), one can deduce that \( H(\bar{\varphi}) = \Theta, \bar{\varphi} \in \cal{A} \) and \( \varphi \) is a ground-state solution of (5.1). \( \square \)

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**REFERENCES**

[1] N. Akhmediev and A. Ankiewicz, Partially coherent solitons on a finite background, *Phys. Rev. Lett.*, 82 (1999), 2661–2664.

[2] A. Ambrosetti and E. Colorado, Standing waves of some coupled nonlinear Schrödinger equations, *J. Lond. Math. Soc.*, 75 (2007), 67–82.

[3] A. Ambrosetti and A. Malchiodi, *Nonlinear Analysis and Semilinear Elliptic Problems*, Cambridge Univ. Press, 2006.

[4] T. Bartsch, N. Dancer and Z. Q. Wang, A Liouville theorem, a-priori bounds, and bifurcating branches of positive solutions for a nonlinear elliptic system, *Calc. Var. Partial Differ. Equ.*, 37 (2010), 345–361.

[5] T. Bartsch and Z. Q. Wang, Note on ground states of nonlinear Schrödinger systems, *J. Partial Differ. Equ.*, 19 (2006), 200–207.

[6] T. Bartsch, Z. Q. Wang and J. Wei, Bound states for a coupled Schrödinger system, *J. Fixed Point Theory. Appl.*, 2 (2007), 353–367.

[7] H. Brézis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Commun. Pure Appl. Math.*, 36 (1983), 437–477.

[8] Z. Chen and W. Zou, An optimal constant for the existence of least energy solutions of a coupled Schrödinger system, *Calc. Var. Partial Differ. Equ.*, 48 (2013), 695–711.

[9] M. Clapp and J. Faya, Multiple solutions to a weakly coupled purely critical elliptic system in bounded domains, *Discrete Contin. Dyn. Syst.*, 39 (2019), 3265–3289.

[10] M. Clapp and A. Szulkin, Solutions to indefinite weakly coupled cooperative elliptic systems, preprint, arXiv:2003.12343v1.

[11] N. Dancer, J. Wei and T. Weth, A priori bounds versus multiple existence of positive solutions for a nonlinear Schrödinger system, *Ann. Inst. H. Poincaré Anal. Non Linéaire.*, 27 (2010), 953–969.

[12] W. Y. Ding, On a conformally invariant elliptic equation on \( \mathbb{R}^n \), *Commun. Math. Phys.*, 107 (1986), 331–335.

[13] B. Esry, C. Greene, J. Burke and J. Bohn, Hartree-Fock theory for double condensates, *Phys. Rev. Lett.*, 78 (1997), 3594–3597.

[14] D. J. Frantzeskakis, Dark solitons in atomic Bose-Einstein condensates: from theory to experiments, *J. Phys. A*, 43 (2010), 213001.

[15] F. Gazzola and B. Ruf, Lower-order perturbations of critical growth nonlinearities in semilinear elliptic equations, *Adv. Differ. Equ.*, 2 (1997), 555–572.

[16] Yu. S. Kivshar and B. Luther-Davies, Dark optical solitons: physics and applications, *Phys. Reports*, 298 (1998), 81–107.

[17] T. C. Lin and J. Wei, Ground state of \( N \) coupled nonlinear Schrödinger equations in \( \mathbb{R}^n \), \( n \leq 3 \), *Commun. Math. Phys.*, 255 (2005), 629–653.

[18] S. Peng, Q. Wang and Z. Q. Wang, On coupled nonlinear Schrödinger systems with mixed couplings, *Trans. Amer. Math. Soc.*, 371 (2019), 7559–7583.

[19] A. Szulkin, T. Weth and M. Willem, Ground state solutions for a semilinear problem with critical exponent, *Differ. Integral Equ.*, 22 (2009), 913–926.
[20] A. Szulkin and T. Weth, Ground state solutions for some indefinite variational problems, J. Funct. Anal., 257 (2009), 3802–3822.
[21] M. Willem, Minimax Theorems, Birkhäuser Boston, Inc., Boston, MA, 1996.
[22] M. Willem, Analyse Harmonique Réelle, Hermann, Paris, 1995.

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