Deformed Heisenberg Algebra with a minimal length: Application to some molecular potentials

Djamil Bouaziz

Abstract. We review the essentials of the formalism of quantum mechanics based on a deformed Heisenbeg algebra, leading to the existence of a minimal length scale. We compute in this context, the energy spectra of the pseudoharmonic oscillator and Kratzer potentials by using a perturbative approach. We derive the molecular constants, which characterize the vibration-rotation energy levels of diatomic molecules, and investigate the effect of the minimal length on each of these parameters for both potentials. We confront our result to experimental data for the hydrogen molecule to estimate an order of magnitude of this fundamental scale in molecular physics.

Département de Physique, Université de Jijel, BP 98, Ouled Aissa, 18000 Jijel, Algeria
E-mail: djamilbouaziz@univ-jijel.dz

1. Introduction
This work, is a continuation of the recent studies [1, 2], where the effects of the minimal length on the vibration-rotation of diatomic molecules have been investigated through the pseudoharmonic oscillator (PHO) [1] and the Kratzer [2] interactions.

As it is well known, the minimal length is a prediction of quantum gravity [3, 4, 5] and string theory [6, 7, 8], where the Planck length is supposed to be a lower bound to all physical length scales [9, 10]. This fundamental scale is implemented by the modification of the Heisenberg uncertainty relation to the form: $(\Delta X)(\Delta P) \geq \hbar (1 + \beta (\Delta P)^2 + ...)$, where $\beta$ is a small positive parameter [4, 7, 11]. The minimal uncertainty in position (minimal length) implied by this generalized uncertainty principle (GUP) is then given by $\langle \Delta X \rangle_{\text{min}} = \hbar \sqrt{\beta}$ [12].

The study of theoretical and physical implications of the GUP is still a hot topic. Many problems with great physical interest have been studied in connection with the GUP, see, for instance, Refs. [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31].

It has been in particular concluded that the presence of the minimal length in the formalism provides a natural ultraviolet regularization in quantum mechanics [26] and in quantum field theory [15]. Furthermore, upper bounds for this elementary length have been estimated; the values differ from one application to another and, mostly, belong to the range $10^{-6} - 10^6 \text{ fm}$ [27, 28]. The minimal length seems to depend on the energy scale of the problem and might therefore characterize the size of the system under study [13, 26]. The latter finding was behind the motivation of our recent investigations [1, 2] on the GUP effects in diatomic molecules, because the spatial extension of these systems is relatively large, and the effect of the minimal length may clearly manifest.

In Ref. [1], we studied the vibration-rotation energy levels of diatomic molecules in the presence of a minimal length by addressing the Schrödinger equation with the PHO potential.
A more detailed investigation was carried out in Ref. [2], by taking the Kratzer interaction. It has been explicitly shown that the minimal length would have some physical importance in studying the spectra of diatomic molecules. Furthermore, an upper bound for the minimal length has been estimated of about 0.01 Å by confronting the correction of the GUP to an experimental result of the hydrogen molecule [2].

Here, we review the main results obtained in [1, 2]: We give the expression of the vibration-rotation energy spectrum of diatomic molecules in the presence of a minimal length by studying the deformed Schrödinger equation with the two potentials. In both cases, we apply the obtained formulas to compute the spectroscopic constants of diatomic molecules and investigate the effect of the GUP on these constants. We show how the importance that the deformation parameter can play in fitting the experimental results. Moreover, the expressions of the molecular constants derived from the energy spectrum of the PHO show the importance of the generalization of these studies to the case of a two-parameter deformed Heisenberg algebra, which is presented in Sec. II.

The rest of this paper is organized as follows. In Sec. II, we review the essentials of the formalism of quantum mechanics with a GUP. Sec. III is devoted to investigate the KP in this formalism: we derive the molecular constants, and investigate the effect of the minimal length on vibration-rotation energy levels of diatomic molecules. In Sec. VI, we perform the same study with the PHO interaction. The last section summarizes our main results and conclusions.

2. Deformed Heisenberg algebra with a minimal length: A review

As mentioned in Sec. I, to include a minimal length scale, the uncertainty relation can be modified to the form \[ (\Delta x)(\Delta p) \geq \frac{\hbar}{2}(1 + \beta (\Delta p)^2 + ...), \] where \( \beta \) is a small positive parameter related to the minimal length by \( (\Delta x)_{\text{min}} = \hbar \sqrt{\beta} \). This generalized uncertainty principle (GUP) implies the modification of the commutation relations between the position and the momentum operators as [12]

\[
[\hat{X}, \hat{P}] = i\hbar(1 + \beta \hat{P}^2). \quad (1)
\]

The operators \( \hat{X} \) and \( \hat{P} \) can be represented in both coordinate and momentum spaces as follows:

In momentum space, the simplest realization is [12]

\[
\hat{X} = (1 + \beta \hat{p}^2)\hat{x}, \quad \hat{P} = \hat{p}, \quad (2)
\]

where \( \hat{p} = p \) and \( \hat{x} = i\hbar \frac{\partial}{\partial \hat{p}} \) represent the position and momentum operators of ordinary quantum mechanics.

In coordinate space one has [29]

\[
\hat{X} = \hat{x}, \quad \hat{P} = (1 + \frac{1}{3} \beta \hat{p}^2)\hat{p}, \quad (3)
\]

where \( \hat{x} = x, \hat{p} = -i\hbar \frac{\partial}{\partial x} \).

The formalism of quantum mechanics with a minimal length has been extended to arbitrary dimensions \( (D) \) [12, 13, 14, 15]. The modified Heisenberg algebra reads

\[
[\hat{X}_i, \hat{P}_j] = i\hbar[(1 + \beta \hat{P}^2)\delta_{ij} + \beta' \hat{P}_i \hat{P}_j], \quad (\beta, \beta') > 0,
\]

\[
[\hat{P}_i, \hat{P}_j] = 0,
\]

\[
[\hat{X}_i, \hat{X}_j] = i\hbar \frac{2\beta - \beta' + \beta(2\beta + \beta')\hat{P}^2}{1 + \beta \hat{P}^2}(\hat{P}_j \hat{X}_i - \hat{P}_i \hat{X}_j). \quad (4)
\]
The corresponding GUP is
\[
(\Delta X_i) (\Delta P_i) \geq \frac{\hbar}{2} (1 + \beta \sum_{j=1}^{N} [(\Delta P_j)^2 + \langle \hat{P}_j \rangle^2] + \beta' [(\Delta P_i)^2 + \langle \hat{P}_i \rangle^2]),
\]
(5)
in which the minimal uncertainty in position (minimal length) is [13]
\[
(\Delta X_i)_{\min} = \sqrt{\hbar D \beta + \beta'}, \quad \forall i.
\]
(6)

The position and momentum operators are represented in momentum space as [13, 17]
\[
\hat{X}_i = i\hbar \left[ (1 + \beta p^2) \frac{\partial}{\partial p_i} + \beta' p_i p_j \frac{\partial}{\partial p_j} + \gamma p_i \right], \quad \hat{P}_i = p_i,
\]
(7)
where \( \gamma \) is a small positive parameter related to \( \beta \) and \( \beta' \).

In coordinate space, the simplest representation of the operators \( \hat{X}_i \) and \( \hat{P}_i \) is [16]
\[
\hat{X}_i = \hat{x}_i, \quad \hat{P}_i = \hat{p}_i (1 + \beta \hat{p}^2),
\]
(8)
where \( \hat{x}_i \) and \( \hat{p}_i \) satisfy the standard commutation relations of ordinary quantum mechanics.

Representation (8) satisfies the deformed algebra in the case \( \beta' = 2\beta \) up to the first order of \( \beta \). It is especially more adequate in the treatment of the minimal length as a perturbation to the Schrödinger equation for a given interaction. In this special case the minimal length reads in 3-dimensions as
\[
(\Delta X_i)_{\min} = \sqrt{\hbar 5\beta}, \quad \forall i.
\]
(9)
The Schrödinger equation with the representation (8) can be written as follows:
\[
\left( \frac{\hat{P}^2}{2\mu} + V(r) + \frac{\beta}{\mu} \hat{P}^4 \right) \psi(r) = E \psi(r),
\]
(10)
where terms of order \( \beta^2 \) have been neglected.

In the following sections, we will use equation (10) to study the Kratzer and the PHO potentials. The main goal of this study is to derive the expressions of the spectroscopic constants of diatomic molecules in the presence of a minimal length by using the deformed energy spectrum of these two interactions.

3. Energy spectrum of Kratzer potential with a minimal length
We are interested to the deformed Schrödinger equation (10) with the Kratzer’s molecular potential (KP), which has the form [33]
\[
V(r) = g_1/r^2 - g_2/r,
\]
(11)
with \( g_1 = D_e r_e^2 \) and \( g_2 = 2D_e r_e \), where \( D_e \) is the dissociation energy and \( r_e \) is the equilibrium internuclear distance of a given diatomic molecule.

It has been shown in detail in Ref. [2] that the minimal length corrections \( \Delta E_{nl} \) can be analytically computed, and the energy spectrum of the KP in the presence of a minimal length is given by the following expression:
\[
E_{nl} = -\frac{e^2 D_e}{(\lambda+n)^2} + \beta \mu D_e \left( \frac{2\gamma}{\lambda+n} \right)^4 \left\{ -\frac{3}{4} + \frac{\lambda+n}{\lambda-1/2} \left( 1 + \frac{\gamma^2}{2} \left( \frac{1}{(\lambda+n)^2} - \frac{2}{\lambda(\lambda-1)} \right) \right) \right\} + \frac{\gamma^4}{4} \left( \frac{1}{(\lambda-\frac{1}{2})(\lambda-\frac{3}{2})(\lambda+n)} \right) \left( 1 + \frac{3n(2\lambda+n)}{\lambda(2\lambda+1)} \right),
\]
(12)
with the notations
\[
\gamma = \frac{\mu}{\hbar} \sqrt{2\mu D_e}, \quad \lambda = 1/2 + \sqrt{(\ell + 1/2)^2 + \gamma^2},
\]
where \(n\) and \(\ell\) are, respectively, the radial (vibrational) and orbital (rotational) quantum numbers; and \(\mu\) is the reduced mass of the molecule.

Formula (12) shows the effect of this deformed algebra on the energy levels of KP. As it has been outlined in Ref. [2], several features of diatomic molecules can be studied by using this formula. In the following, we apply Eq. (12) to investigate the effect of the minimal length on the spectroscopic constants, which characterize the rovibrational energy levels of diatomic molecules.

**Spectroscopic constants of diatomic molecules**

As we will see, the energy spectrum of the KP is similar to the well-known spectroscopic formula which defines the vibration-rotation energy spectrum of diatomic molecules [32, 33].

To this end, we observe that the dimensionless parameter \(\gamma\), in Eq. (12), is so large for most molecules (\(\gamma \gg 1\)) [33], the levels \(E_{\ell n}\) may then be expanded into powers of \(1/\gamma\) as follow:

\[
E_{\ell n} = D_e \left(-1 + 2(n + \frac{1}{2})^2 + (\ell + 1)^2 \frac{1}{\gamma^2} - 3(n + \frac{1}{2})^2 \frac{1}{\gamma^2} + 4(n + \frac{1}{2})^3 \frac{1}{\gamma^2} \right)
+ \beta \mu D_e^2 \left(6 \{ (n + \frac{1}{2})^2 + \frac{1}{4} \} \frac{1}{\gamma^3} + 2(n + \frac{1}{2}) \{ -\frac{1}{4} + 4(\ell + 1)^2 - 15(n + \frac{1}{2})^2 \} \frac{1}{\gamma^2} \right) + \ldots
\]

This formula shows the different parts of the undeformed spectrum (\(\beta = 0\), and the corrections caused by the minimal length. The effect of this GUP on each kind of the rovibrational energy was discussed in detail in Ref. [2]. Here, we will suggest some ideas on how to get information about the order of the deformation parameter \(\beta\) from formula (14), and then apply it to extract the spectroscopic constants of diatomic molecules, which is a way to give values of \(\beta\) for any molecule.

We now have basically two approaches to deal with the parameter \(\beta\) in formula (14): The first is to consider that \(\beta\) is independent of the molecular constants \(r_e\) and \(D_e\), which have known values in molecular spectroscopy. Therefore, an upper bound for \(\beta\) can be estimated by assuming, e.g., that the minimal length corrections are included in the gap between the experimental results and those predicted by the formula (14) in the case case \(\beta = 0\). It has been argued in Ref. [2] that a better estimate may be obtained by considering the vibrational ground-state energy \(E_{00}\) of the hydrogen molecule (H\(_2\)). This led to an upper bound for the minimal length with the value

\[
(\Delta X)_{\text{min}} = \hbar \sqrt{5\beta} < 0.01 \text{ Å}.
\]

The second viewpoint consists of looking at formula (14) as an energy spectrum of a three-parameter potential, i.e., \(D_e\), \(r_e\), and \(\beta\); so, the values of \(\beta\) depend now on \(D_e\) and \(r_e\). This viewpoint allows for adjusting the three parameters of the "deformed KP" with the spectroscopic data. To this end, we can suggest to extract, from the deformed energy spectrum (14), the spectroscopic constants for diatomic molecules.

The commonly used energy equation in molecular spectroscopy is [32, 34]

\[
E_{\ell n} = Y_{00} + \omega_e \left(n + \frac{1}{2}\right) - \omega_e x_e (n + \frac{1}{2})^2 + \omega_e y_e (n + \frac{1}{2})^3
+ B_e \ell (\ell + 1) - \alpha_e \left(n + \frac{1}{2}\right) \ell (\ell + 1),
\]

where

\[
Y_{00} = \frac{D_e}{4} \frac{\omega_e x_e}{\gamma^2} + \frac{\omega_e y_e}{12B_e} \left(1 + \frac{\omega_e y_e}{12B_e}\right).
\]
The values of the spectroscopic constants \( \omega_e, \omega_e x_e, \omega_e y_e, B_e \) and \( \alpha_e \) are determined by fitting the spectroscopic data for any molecules [32].

We write now equation (14) in the form

\[
E_{n\ell} = D_e \left[ -1 + 2(n + \frac{1}{2})\frac{1}{\gamma} + (\ell + \frac{1}{2})^2 \frac{1}{\gamma^2} - 3(n + \frac{1}{2})^2 \frac{1}{\gamma} \right. \\
+ 4(n + \frac{1}{2})^3 \frac{1}{\gamma^3} - 3(n + \frac{1}{2})(\ell + \frac{1}{2})^2 \frac{1}{\gamma^2} \right. \\
+ \beta \mu D_e^2 \left[ 6 \left( (n + \frac{1}{2})^2 + \frac{1}{4} \right) \frac{1}{\gamma} - \frac{1}{2}(n + \frac{5}{2}) \right. \\
+ 2(n + \frac{1}{2}) \left[ 4(\ell + \frac{1}{2})^2 - 15(n + \frac{1}{2})^2 \right] \frac{1}{\gamma} \right. 
+ \cdots 
\]

(18)

The identification of Eqs. (18) and (16) leads to the following deformed spectroscopic constants:

\[
\begin{align*}
Y_{00} &= \frac{1}{4} \frac{D_e}{\gamma} + 3 \frac{3}{2} \beta \mu \frac{D_e^2}{\gamma^2}, \\
\omega_e x_e &= 3 \frac{D_e}{\gamma} - 6 \beta \mu \frac{D_e^2}{\gamma^2}, \\
\omega_e y_e &= \frac{4 D_e}{\gamma} - 30 \beta \mu \frac{D_e^2}{\gamma^2}, \\
B_e &= \frac{D_e}{\gamma^2}, \\
\alpha_e &= \frac{3 D_e}{\gamma} - 8 \beta \mu \frac{D_e^2}{\gamma^2},
\end{align*}
\]

(19)

We observe that the leading corrections of the minimal length are of order \( 1/\gamma^2 \) and concerns the constant of anharmonicity of vibrations \( \omega_e x_e \) and the constant \( Y_{00} \), which does not influence the line positions in a spectrum. However, the rotational constant \( B_e \) is not affected by this deformed algebra.

As the experimental values of the spectroscopic constants are available for diatomic molecules [32], it follows that the expressions (19) can be used to give values of \( \beta \) for each molecule. This investigation is under consideration in the general case of deformed Heisenberg algebra (4), with \( \beta' \neq 2 \beta \).

4. Energy spectrum of the pseudoharmonic oscillator with a minimal length

The pseudoharmonic oscillator (PHO) is also one of the molecular interactions, which is used in the study of the vibration-rotation spectra of diatomic molecules. The form of this potential is

\[
V(r) = D_e \left( \frac{r}{r_e} - \frac{r_e}{r} \right)^2,
\]

(20)

where \( D_e \) is the dissociation energy and \( r_e \) is the equilibrium internuclear distance of a given diatomic molecule.

It has been shown in Ref. [1] that the minimal length correction of this potential can also be analytically derived by following the same procedures as in Sec. III. The deformed energy spectrum of the PHO reads [1]

\[
E_{n\ell} = E_{n\ell}^0 + 4 \mu \beta \left( (E_{n\ell}^0)^2 + 4D_e E_{n\ell}^0 + 6D_e^2 - (4D_e^2 + 2D_e E_{n\ell}^0) \frac{\lambda + 2n + 1}{\gamma} \right. \\
+ \left. D_e^2 \frac{\lambda^2 + (6n + 3)\lambda + 6(n + 1)}{\gamma^2} - 2D_e \left( 2D_e + E_{n\ell}^0 \right) \frac{\lambda}{\gamma} + \frac{D_e^2 \gamma^2 \lambda + 2n + 1}{\lambda(\lambda^2 - 1)} \right),
\]

(21)

where \( E_{n\ell}^0 \) is the undeformed spectrum, given by

\[
E_{n\ell}^0 = -2D_e \left( 1 - \frac{1}{\gamma} (2n + 1 + \lambda) \right),
\]

(22)
with the notations
\[ \lambda = \sqrt{\gamma^2 + (\ell + \frac{1}{2})^2}, \quad \gamma = \sqrt{\frac{2\mu r_e^2}{\hbar^2}}, \]
where, \( n = 0, 1, 2, \ldots \), and \( \ell = 0, 1, 2, \ldots \) are, respectively, the radial (vibrational) and orbital (rotational) quantum numbers, and \( \mu \) is the reduced mass of the molecule.

The effect of the GUP on the vibration-rotation energy levels of a given diatomic molecule with the PHO interaction was qualitatively investigated in Ref. [1] by using formula (21). Here, we use this deformed spectrum to give the expressions of the molecular constants in the presence of a minimal length.

**Spectroscopic constants of diatomic molecules**

Formula (21) might be viewed as an energy spectrum of a three-parameter potential function, i.e., \( D_e, r_e, \) and \( \beta \). One can then derive the molecular constants for diatomic molecule by expanding \( E_{n\ell} \) in Eq. (21) into powers of \( 1/\gamma \) (\( \gamma \gg 1 \)); this leads to the expression

\[
E_{n\ell} = \left( \frac{1}{4} + 6\mu \beta D_e \right) \frac{D_e}{\gamma} + \frac{4D_e}{\gamma} \left( 1 + 3\beta D_e^2 \right) (n + \frac{1}{2})^2 + \frac{D_e}{\gamma^2} \ell (\ell + 1) + \frac{16\mu \beta D_e^3}{\gamma^4} (n + \frac{1}{2})^2 (\ell + 1) + \ldots , \tag{24}
\]

which is of the form of the spectroscopic formula (16), with the following spectroscopic constants:

\[
Y_{00} = \left( \frac{1}{4} + 6\mu \beta D_e \right) \frac{D_e}{\gamma}, \quad \omega_e = \frac{4D_e}{\gamma} \left( 1 + 3\beta D_e^2 \right),
\omega_e x_e = -\frac{24\mu \beta D_e^2}{\gamma^2}, \quad B_e = \frac{D_e}{\gamma}, \quad \alpha_e = -\frac{16\mu \beta D_e^3}{\gamma^4}. \tag{25}
\]

The value of the constant \( \omega_e x_e \) is zero in the order \( 1/\gamma^3 \).

In contrast to ordinary quantum mechanics (\( \beta = 0 \)), where the spectrum of the PHO leads to zero values of the spectroscopic constants \( \omega_e x_e \) and \( \alpha_e \), in the presence of a minimal length, these constants depend on \( \beta \) and have nonzero values. The basic empirical terms, known in the vibration-rotation energy spectrum of diatomic molecules, are now present in this deformed version of quantum mechanics. However, the sign of \( \omega_e x_e \) and \( \alpha_e \) is conflicting compared to that of the experimental values, at least for the diatomic molecules listed in Ref. [32].

We can conclude that the PHO in this one-parameter deformed algebra could not be used to fit the deformed spectroscopic constants with the empirical results. It follows that the extension of this work to the general case \( \beta' \neq 2\beta \) is interesting; it allows us not only to adjust the parameters \( D_e, r_e, \) and \( \beta \) but also to establish a constraint between the deformation parameters \( \beta \) and \( \beta' \) by appropriately choosing the sign of the molecular constants. This study is in finalization and would be published elsewhere.

### 5. Summary and conclusions

We have investigated, in quantum mechanics with a deformed Heisenberg algebra including a minimal length \( (\Delta X)_{\text{min}}^2 = \hbar \sqrt{5\beta} \), the vibration-rotation energy spectra of diatomic molecules by two molecular interactions: the Kratzer and the PHO potentials. We have discussed how an order of the deformation parameter \( \beta \) can be evaluated from the minimal length corrections and the spectroscopic data of diatomic molecules. With the Kratzer’s potential, an upper bound of the minimal length of about 0.01 Å has been obtained by comparing the theoretical and experimental values of the vibrational ground-state energy of the molecule H₂. On the other hand, by supposing that the deformation parameter is a third parameter of the potentials, we have derived the spectroscopic constants of diatomic molecules with both interactions and we have examined the effects of the minimal length on each of these constants. We showed that in the case of the PHO the extension of this study to the general case of a two-parameter deformed
Heisenberg algebra would be mandatory and interesting because it allows to give a physical constraint between \( \beta \) and \( \beta' \).

This work is supported by the Algerian Ministry of Higher Education and Scientific Research, under the CNEPRU Project No. D01720140007. I am grateful to P. Hajigeorgious for fruitful contact.

References

[1] D. Bouaziz and A. Boukhellout, Mod. Phys. Lett A. 29, 1450143 (2014).
[2] D. Bouaziz, Ann. Phys. 355, 269 (2015).
[3] C. A. Mead, Phys. Rev. 135, 849 (1964).
[4] L. J. Garay, Int. J. Mod. Phys. A 10, 145 (1995).
[5] C. Rovelli and L. Smolin, Nucl. Phys. B 442, 593 (1995).
[6] D. J. Gross and P. F. Mende, Nucl. Phys. B 303, 407 (1988).
[7] D. Amati, M. Ciafaloni, and G. Veneziano, Phys. Lett. B 216, 41 (1989).
[8] K. Konishi, G. Paffuti, and P. Provero, Phys. Lett. B 234, 276 (1990).
[9] T. Padmanabhan, Gen. Rel. Grav. 17, 215 (1985).
[10] M. T. Jaekel and S. Reynaud, Phys. Lett. A 185, 143 (1993).
[11] M. Maggiore, Phys. Lett. B 319, 83 (1993).
[12] A. Kempf, G. Mangano, and R. B. Mann, Phys. Rev. D 52, 1108 (1995).
[13] A. Kempf, J. Phys. A. 30, 2093 (1997).
[14] A. Kempf, J. Math. Phys. 35, 4483 (1994); A. Kempf, Phys. Rev. D 54, 5174 (1996); Erratum-ibid. D 55, 1114 (1997).
[15] A. Kempf, J. Math. Phys. 38, 1347 (1997); A. Kempf and G. Mangano, Phys. Rev. D 55, 7909 (1997).
[16] F. Brau, J. Phys. A 32, 7691 (1999).
[17] L. N. Chang, D. Minic, N. Okamura, and T. Takeuchi, Phys. Rev. D 65, 125027 (2002).
[18] S. Benczik, L. N. Chang, D. Minic, and T. Takeuchi, Phys. Rev. A 72, 012104 (2005).
[19] M. M. Stetsko and V. M. Tkachuk, Phys. Rev. A 74, 012101 (2006); M. M. Stetsko, Phys. Rev. A 74, 062105 (2006).
[20] L. Menculini, O. Panella, and P. Roy, Phys. Rev. D 87, 065017 (2013).
[21] Y. Chargui, A. Trabelsi, and L. Chetouani, Phys. Lett. A. 374, 531 (2010).
[22] C. Quesne and V. M. Tkachuk, J. Phys. A : Math. Gen. 38, 1747 (2005).
[23] U. Harbach and S. Hossenfelder, Phys. Lett. B 632, 379 (2006); O. Panella, Phys. Rev. D 76, 045012 (2007).
[24] D. Mania, and M. Maziaievili, Phys. Lett. B 705, 521 (2011).
[25] K. Nozari and S. H. Mehdipour, Chaos, Solitons & Fractals, 32, 1637 (2007); T. V. Fityo, Phys. Lett. A 372, 5872 (2008); B. Vakili and M. A Gorji, J. Stat. Mech. 10013 (2012).
[26] D. Bouaziz and M. Bawin, Phys. Rev. A. 76, 032112 (2007); D. Bouaziz and M. Bawin, Phys. Rev. A. 78, 032110 (2008).
[27] D. Bouaziz and N. Ferkous, Phys. Rev. A. 82, 022105 (2010).
[28] F. Brau and F. Buisseret, Phys. Rev. D 74, 036002 (2006).
[29] K. Nozari and P. Pedram, Europhys. Lett. 92, 50013 (2010); P. Pedram, K. Nozari, and S. H. Taheri, J. High. Energy. Phys. 1103, 093 (2011).
[30] Bibhas Ranjan Majhi, Elias C. Vagenas, Phys. Lett. B 725, 477 (2013); M. Isi, J. Mureika, P. Nicolini, JHEP. 11, 139 (2013); M. Sprenger, P. Nicolini, and M. Bleicher, Eur. J. Phys. 33, 853 (2012).
[31] S. Hossenfelder, Living Rev. Relativity. 16, 2 (2013); C. Quesne and V. M. Tkachuk, Czech. J. Phys. 56, 1209 (2006); A. F. Ali, S. Das, and E. C. Vagenas, Phys. Lett. B 678, 497 (2009); Das S, E. C. Vagenas, and A. F. Ali, Phys. Lett. B 690, 407 (2010); K. Nozari and A. Etemadi, Phys. Rev. D 85, 104029 (2012); P. Pedram, Phys. Lett. B 714, 317 (2012).
[32] Karl K. Irikura, J. Phys. Chem. Ref. Data. V. 36, N. 2, 389 (2007).
[33] S. Flügge, Practical quantum mechanics, (Springer-Verlag, 1994), pp. 178-182.
[34] H. Lelehrre-Brion, and R. W. Field, The spectra and Dynamics of Diatomic Molecules (Elsevier Academic Press, London, 2004) p. 3.