Entanglement constant for conformal families

Paweł Caputa$^a$ and Alvaro Veliz-Osorio$^b$

$^a$Nordita, KTH Royal Institute of Technology and Stockholm University, Roslagstullsbacken 23, SE-106 91 Stockholm, Sweden

$^b$Mandelstam Institute for Theoretical Physics, School of Physics
University of the Witwatersrand, Johannesburg, WITS 2050, South Africa

Abstract

We show that in 1+1 dimensional conformal field theories, exciting a state with a local operator increases the Rényi entanglement entropies by a constant which is the same for every member of the conformal family. Hence, it is an intrinsic parameter that characterises local operators from the perspective of quantum entanglement. In rational conformal field theories this constant corresponds to the logarithm of the quantum dimension of the primary operator. We provide several detailed examples for the second Rényi entropies and a general derivation.
Contents

1 Introduction

2 Replica trick for locally excited states
   2.1 Example: EPR-primary

3 Descendants examples
   3.1 Energy-momentum tensor
   3.2 Derivative of a primary
   3.3 Both derivatives
   3.4 Descendants at level 2

4 General derivation

5 Conclusions and discussion

A Correlators of the EPR operators

B Transformation of descendants

C Correlators at level 1

D Correlators at level 2

E Comments on large c and holography

1 Introduction

Conformal field theories (CFT) in two dimensions play a very important role in the understanding of quantum entanglement (see [1] for review). For example, in the vacuum state, one can compute entanglement measures like Rényi entropies analytically for an arbitrary interval. They are universal and determined by the central charge $c$ and in particular the von-Neumann entropy has the famous $\frac{c}{3} \log L$ scaling with the size of the interval $L$. This relation has been tested quite extensively and serves as an efficient way to numerically determine the central charge for the CFT that governs particular critical points.

The next natural step is to explore the entanglement in excited states. A particularly useful protocol to study that, is provided by global or local quenches [2] where one can follow the time evolution of entanglement measures in a state that differs from an eigenstate globally or locally respectively. Under such evolution, entanglement grows in the system and it is hard to study numerically, nevertheless, the power of (boundary) CFT allows to extract the universal features like the speed of the growth of entanglement.
Recently, inspired by the quench setup, another class of excited states by local operators was proposed in \cite{3}. They can be thought of as a milder version of a local quench and the growth of entanglement is less drastic. In fact, at late times, the Rényi entanglement entropies saturate to constants that can be used to characterize local operators by the way they change quantum entanglement in a given state. Detailed reviews can be found in \cite{6,7} and further results and applications to finite temperature and holography in \cite{8,9,10,11,12,13}.

For primary operators in 1+1 dimensional rational CFTs, the growth of the entanglement of an interval can be computed analytically and it was proved to be the logarithm of the quantum dimension of the local operator \cite{9}. Since in this setup we can uniquely decompose an operator into left and right movers (chiral, anti-chiral parts) this constant is equivalent the left-right entanglement in this excited state. Intuitively, this is explained by the quasiparticle picture for the propagation of entanglement where the excitation can be thought of as insertion of the EPR like pair. Quasiparticles propagate in opposite directions and once one of them is in the entangling region it increases the entanglement with the rest by the entanglement of the pair. This picture is supported by the one point functions of the energy in locally excited state \cite{14}.

In this work we also focus on states excited by local operators in 1+1 d CFTs with Virasoro symmetry, and in the search for universal features of entanglement we ask how the above picture changes for descendant operators. As we will show, one-point functions of the energy behave differently in the states excited by descendants. Moreover, their conformal transformations, as well as their correlators, are much more complicated than for primaries and the appearance of the quantum dimension is not obvious. Nevertheless, we will show for descendants up to level 2 and provide a general argument that the constant contribution to the Rényi entanglement entropies is the same for the entire conformal family.

This paper is organized as follows. In section 2 we briefly review the replica method with local operators in 2d CFT \cite{9} and provide an example that illustrates the CFT technology. In section 3 we compute one-point functions of the energy density and the increase in the second Renyi entropy for descendant operators up to level 2. In section 4 we outline a general argument for why the contribution to the Rényi entanglement is a characteristic of the entire conformal family. In section 5 conclude and discuss the Schmidt decomposition and possible connection to topological entanglement entropy. Several details of the analysis are moved to the appendices.

## 2 Replica trick for locally excited states

The entanglement entropy of a subsystem quantifies the amount of information that we would forfeit if we were to lose access to the rest of the system. Imagine that the system of interest is in a pure state $|\Psi\rangle$ and suppose that we wish to quantify the entanglement entropy.

\footnote{see also \cite{4,5} for a different setup with local operators}
between a subsystem $A$ and its complement $\bar{A}$. As a first step we find the reduced density matrix obtained by tracing out the degrees of freedom in $\bar{A}$

$$\rho_A = \text{Tr}_{\bar{A}} |\Psi\rangle \langle \Psi| .$$

Once we have constructed this matrix, we notice that if there is any entanglement between the degrees of freedom in $A$ and those in $\bar{A}$ the system appears, to an observer having access only to $A$, to be in a mixed state. If that is the case, then the Von Neumann entropy of $\rho_A$

$$S_A = -\text{Tr}(\rho_A \log \rho_A) ,$$

is non-vanishing. We refer to this quantity as the entanglement entropy of $A$. In practice, we use the so-called the replica trick\cite{1} and compute instead the Rényi entropies

$$S_A^{(n)} = \frac{1}{1-n} \log \text{Tr} [\rho_A^n] .$$

The entanglement entropy \cite{2,2} can be extracted from the above expression by taking the $n \to 1$ limit. These Rényi entropies provide interesting measures of entanglement on their own right e.g. the min entropy $S_A^{(\infty)}$ and the purity $S_A^{(2)}$\cite{15}.

It is natural to wonder what would be the effect on the entanglement between different parts of a system if we were to perturb it in some way. In this work we focus on the case where these perturbation is due to the insertion of a local operator. Recently, the replica method to compute the Rényi entropies has been generalized to deal with these scenarios. Below we summarize the relevant formulas and refer the reader for more details to \cite{3,6,9}.

Consider the ground state of a 1+1 dimensional CFT on a line. We split the space into a finite interval $A = [l_1, l_2]$ of length $L \equiv l_2 - l_1$ and its complement. Then at $t = 0$ we insert a local operator $O$ at $x = -l$ and let the system evolve. Without losing generality, we can shift the interval to $A = [0, L]$. The resulting density matrix is given by

$$\rho(t, l, L, \epsilon) = \mathcal{N} \cdot e^{-itH} e^{-\epsilon H} O(0, -l) |0\rangle \langle 0| O^\dagger(0, -l) e^{-\epsilon H} e^{itH}$$

$$\equiv \mathcal{N} \cdot O(w_2, \bar{w}_2) |0\rangle \langle 0| O^\dagger(w_1, \bar{w}_1) .$$

(2.4)

where $\epsilon \ll 1$ plays the role of an UV regulator\footnote{Analogously to the local quench setup, we assume that all length and time scales are much larger than $\epsilon$.} and $\mathcal{N}$ is a normalization that ensures that $\text{Tr} \rho = 1$. In the second line the \textit{insertion points} are defined as

$$w_1 = i(\epsilon - it) - l, \quad w_2 = -i(\epsilon + it) - l,$$

$$\bar{w}_1 = -i(\epsilon - it) - l, \quad \bar{w}_2 = i(\epsilon + it) - l .$$

(2.5)

Hereafter, we omit the dependence on $l, L, \epsilon$ and write simply $\rho(t)$.
The increase in the $n$-th Rényi entanglement entropy of the interval $A$ wrt to the ground state due to the local operator is given by

$$\Delta S_A^{(n)} \equiv \frac{1}{1 - n} \log \left( \frac{\langle \mathcal{O}(w_1, \bar{w}_1)\mathcal{O}(w_2, \bar{w}_2) \cdots \mathcal{O}(w_{2n-1}, \bar{w}_{2n-1})\mathcal{O}(w_{2n}, \bar{w}_{2n}) \rangle_{\Sigma_n}}{\langle \mathcal{O}(w_1, \bar{w}_1)\mathcal{O}(w_2, \bar{w}_2) \rangle_{\Sigma_1}^n} \right), \quad (2.6)$$

where $\Sigma_n$ is the $n$-sheeted surface with cuts on each copy corresponding to $A$, and $\Sigma_1$ is a single copy with an interval cut $A$. Thus, we are faced with the task of evaluating a $2n$-point function on $\Sigma_n$. As shown in [9], this can be computed by using the uniformization map

$$z^n = \frac{w}{w - L}, \quad (2.7)$$

that takes $\Sigma_n$ to the complex plane. Notice that the transformation properties of $\mathcal{O}$ under the above map are expected to play an important role.

Let us focus on the change in the second Rényi entropy $\Delta S_A^{(2)}$. In this case we are expected to evaluate the operator’s four-point function on a two-sheeted surface. We use (2.7) with $n = 2$ to map $\Sigma_2$ to the plane, where conformal symmetry ensures that these correlator can be written in terms of the cross-ratios of the four insertion points. Notice that under the uniformization map the insertion points (2.5) become $z_4 = -z_2$ and $z_3 = -z_1$. From equations (2.5) and (2.7) and in the limit $\epsilon \rightarrow 0$ the cross ratios are given by

$$z \simeq \frac{z_{12}z_{34}}{z_{13}z_{24}} \simeq 1 - \frac{L^2\epsilon^2}{4(l-t)^2(L + l - t)^2}, \quad \bar{z} \simeq \frac{L^2\epsilon^2}{4(l + t)^2(L + l + t)^2}, \quad (2.8)$$

provided that $t \in [l, L + l]$. On the other hand, if $t \notin [l, L + l]$, then the cross-ratios read

$$z \simeq \frac{L^2\epsilon^2}{4(l-t)^2(L + l - t)^2}, \quad \bar{z} = \frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{13}\bar{z}_{24}} \simeq \frac{L^2\epsilon^2}{4(l + t)^2(L + l + t)^2}, \quad (2.9)$$

Hence, there are two possibilities, $(z, \bar{z}) \simeq (1, 0)$ for $t \in [l, L + l]$, while $(z, \bar{z}) \simeq (0, 0)$ otherwise.

The authors of [9] showed that if $\mathcal{O}$ is a primary operator, then

$$\Delta S_A^{(2)}(z, \bar{z}) = -\log \left[ |z(1 - z)|^{4h} G(z, \bar{z}) \right]. \quad (2.10)$$

where $G(z, \bar{z})$ is the canonical four point function of the operators on the plane. As discussed above, we must evaluate this result for the regimes $(z, \bar{z}) \simeq (0, 0)$ and $(z, \bar{z}) \simeq (1, 0)$. In the first regime there is no increase in entanglement

$$\Delta S_A^{(2)}(0, 0) = 0. \quad (2.11)$$

Now, since the two regimes are related to one another by the modular transformation $z \rightarrow 1 - z$, in rational CFT, it follows that

$$\Delta S_A^{(2)}(1, 0) = -\log F_{00}[\mathcal{O}], \quad (2.12)$$
where $F_{00}[\mathcal{O}]$ is the component of fusion matrix for the identity $[16]$. Moreover, $F_{00}[\mathcal{O}]$ corresponds to the inverse of the quantum dimension $d_{\mathcal{O}}$ of the operator $[17]$. Therefore, one gets

$$\Delta S_A^{(2)} = \begin{cases} 
0 & \text{if } t \not\in [l, L + l] \\
\log d_{\mathcal{O}} & \text{if } t \in [l, L + l]
\end{cases}$$

(2.13)

This behavior of the Rényi entropies has been interpreted in terms of a quasi-particle picture where the insertion of the local operator corresponds to creation of the EPR-like pair whose members propagate in the opposite directions (to the left and right). A non-trivial contribution to the Rényi entropy comes from times when one of the members is inside the interval $A$ while the other one is outside.

In this work we will test these results and quasi-particle picture further by considering excitations by descendant operators.

### 2.1 Example: EPR-primary

In this section we consider a simple illustrative example of a primary operator that is a generalization of the operators studied in [9], which will also serve to check our results for descendants. Let us start with a state locally excited by the primary operator

$$\mathcal{O}_\alpha = \sqrt{a} e^{i\sqrt{2}\alpha \phi} \pm \sqrt{1 - a} e^{-i\sqrt{2}\alpha \phi},$$

(2.14)

where $\phi$ is a free massless scalar and $a \in [0, 1]$. It is easy to verify that the conformal dimension of the operator is $h = \tilde{h} = \alpha^2$. Hereafter, we refer to the operator (2.14) as the EPR-primary.

#### Replica method

For the sake of simplicity we consider the second Rényi entropy. As discussed in the previous section, since $\mathcal{O}_\alpha$ is a primary operator the increase of $S_A^{(2)}$ is given by

$$\Delta S_A^{(2)} = -\log \left[ \frac{1}{4} |z(1 - z)|^{4h} \mathcal{G}_\alpha(z, \bar{z}) \right],$$

(2.15)

where $\mathcal{G}_\alpha(z, \bar{z})$ is defined by the four-point correlator of the EPR operators on the complex plane via

$$\langle \mathcal{O}_\alpha(z_1, \bar{z}_1) \mathcal{O}_\alpha^\dagger(z_2, \bar{z}_2) \mathcal{O}_\alpha(z_3, \bar{z}_3) \mathcal{O}_\alpha^\dagger(z_4, \bar{z}_4) \rangle = |z_{13} z_{24}|^{-4h} \mathcal{G}_\alpha(z, \bar{z}),$$

(2.16)

or more explicitly we define

$$\mathcal{G}_\alpha(z, \bar{z}) = \langle \mathcal{O}_\alpha | \mathcal{O}_\alpha(z, \bar{z}) \mathcal{O}_\alpha(1, 1) | \mathcal{O}_\alpha \rangle \equiv |z(1 - z)|^{-4h} \mathcal{F}_\alpha(z, \bar{z}).$$

(2.17)

The function $\mathcal{F}_\alpha(z, \bar{z})$, which encodes the non-singular part of $\mathcal{G}_\alpha(z, \bar{z})$ can be computed explicitly for the EPR operators (see Appendix A) and it is given by

$$\mathcal{F}_\alpha(z, \bar{z}) = a^2 + (1 - a)^2 + 2a(1 - a) \left( |z|^{8h} + |1 - z|^{8h} \right).$$

(2.18)
In terms of this function, the change in the Rényi entropy takes the compact form
\[
\exp(-\Delta S_A^{(2)}) = F(a, \bar{a}).
\] (2.19)

Finally, for times \( t \notin [l, L + l] \) we have both \((z, \bar{z}) \simeq (0, 0)\) and the change in the second Rényi entropy vanishes. On the other hand for \( t \in [l, L + l] \) the cross-ratios are approximately \((z, \bar{z}) \simeq (1, 0)\), which gives a non-trivial contribution to the second Rényi entropy
\[
\Delta S_A^{(2)} \simeq \begin{cases} 
0 & \text{if} \ t \notin [l, L + l] \\
- \log [a^2 + (1 - a)^2] & \text{if} \ t \in [l, L + l]
\end{cases}
\] (2.20)

Notice that the non-vanishing contribution to the second Rényi is maximized by an EPR primary with \( a = \frac{1}{2} \).

**Left-right entanglement**

In the following, we show how the same result can be obtained from the entanglement between left and right moving sectors in the state
\[
\mathcal{O}_a \vert 0 \rangle_L \otimes \vert 0 \rangle_R = \sqrt{a} \begin{pmatrix} e^{i \sqrt{2} \alpha \phi_L} \\ e^{i \sqrt{2} \alpha \phi_R} \end{pmatrix}_L \otimes \begin{pmatrix} e^{-i \sqrt{2} \alpha \phi_L} \\ e^{-i \sqrt{2} \alpha \phi_R} \end{pmatrix}_R,
\] (2.21)

where we used the decomposition of the scalar field \( \phi(t, x) = \phi_L(t + x) + \phi_R(t - x) \). Tracing out the right-moving sector we find that the reduced density matrix of the left-movers, which is simply
\[
\rho_L = \text{diag}\{a, 1 - a\}.
\] (2.22)

Hence, the \( n \)-th Rényi entropy is given by
\[
S_L^{(n)} = \frac{1}{1 - n} \log \text{Tr} \rho_L^n = \frac{1}{1 - n} \log [a^n + (1 - a)^n],
\] (2.23)

which for \( n = 2 \) reproduces Eq. (2.20).

Clearly the left-right decomposition of the excited state gives a quick way to find the reduced density matrix. Once we have it\(^3\) we can study the Rényi entropies for different values of \( n \) and see how they are affected by the local excitation (2.14). For instance the Hartley entropy \((n = 0)\) is given by \( S_L^{(0)} = \log(2) \) and sets an \( a \)-independent upper bound for all the Rényi entropies. It corresponds to the log of the dimension of the reduced density matrix \( \rho_L \). In turn, the entanglement entropy \((n = 1)\) equals the binary entropy
\[
S_L^{(1)} = H_2(a) = -a \log(a) - (1 - a) \log(1 - a).
\] (2.24)

Finally the min entropy \((n \to \infty)\) is given by the inverse of the logarithm of the largest eigenvalue of \( \rho_L \)
\[
S_L^{(\infty)} = - \log \max [a, 1 - a].
\] (2.25)

\(^3\)Generically the left-right decomposition of the operator is not as simple as in our example and depends on the structure of the Hilbert space of the CFT.
Figure 1: Finite value of the Rényi entropies as function of $a$ for different replica numbers $n$ in a state excited by operator (2.14).

All the Rényi entropies for $n \geq 0$ are sensitive to the special points $a = 0$ and $a = 1$ for which they vanish, reflecting the fact that for these values (2.21) becomes a product state. On the other hand, for $a = \frac{1}{2}$ they saturate the inequality set by the Hartley entropy as expected from this maximally entangled mixture (see Fig 1 for comparisons).

Let us stress that for general operators like EPR with different $a$ each Rényi entropy gives a different constant dependent on $a$ and only for $a = 1/2$ they coincide. This is the case studied in [9] and corresponds to the sigma operator in the Ising model. Hence, we expect that only for the rational CFTs all the Rényi entanglement entropies will increase by the same constant.

**Energy density**

The result (2.20) has been interpreted as the effect on entanglement due to an EPR pair whose members are receding from each other at the speed of light. To test this intuition, one could compute the time evolution for the expectation value of the energy density

$$T_t(x, \bar{x}) = -(T(x) + \hat{T}(\bar{x})), \quad (2.26)$$

in the excited state corresponding to $O_\alpha$ [6]. The three-point function of the primary operators with the stress tensor is universal and the final answer is given by

$$\langle T_t(x, x) \rangle_{O_\alpha} = \frac{\langle O_\alpha^d(w_2, \bar{w}_2)T_t(x, x)O_\alpha(w_1, \bar{w}_1) \rangle}{\langle O_\alpha(w_1, \bar{w}_1)\bar{O}_\alpha^d(w_2, \bar{w}_2) \rangle} = \frac{4h \epsilon^2}{((x + l - t)^2 + \epsilon^2)^2} + \frac{4\hbar \epsilon^2}{((x + l + t)^2 + \epsilon^2)^2}. \quad (2.27)$$

Indeed, the above expression describes two wave packets of width $\epsilon$ propagating in the opposite directions from the insertion point. The total energy injected to the system with the operator is equal to $E = \frac{2\pi(h+\hbar)}{\epsilon}$.  

7
The evolution of the energy density profile strongly supports the EPR interpretation of the operator excitation since the two wave packets resemble the pair (see Fig. 2). Notice that the above computation is very robust and insensitive to the details of the primary operator, for example, the value of $\alpha$ in (2.14).

3 Descendants examples

In this section we compute the increase in the second Rényi entropy for few simple descendants. As we shall see, although some modifications on the energy density arise, the behavior of the Rényi entropy remains the same.

3.1 Energy-momentum tensor

Arguably, the most important non-primary operator in a CFT is the stress-energy tensor $T(z)$. Recall that $T(z)$ is a descendant of the identity operator with weight $h = 2$. It must be clear that the insertion of the identity operator into the system doesn’t cause any modification in the Rényi entropy. It is with the stress-energy tensor that we start our study of the effect of the insertion of descendants on entanglement. In the following, we compute the change of the second Rényi entropy $\Delta S^{(2)}$ due to the insertion of the chiral part of the stress-energy tensor $T(z)$. For this case, equation (2.6) reads

$$\exp\left(-\Delta S^{(2)}_A(z)\right) = \frac{\langle T(w_1)T(w_2)T(w_3)T(w_4)\rangle_{\Sigma_2}}{(\langle T(w_1)T(w_2)\rangle_{\Sigma_1})^2}. \quad (3.1)$$

In order to compute the numerator of the above equation, we map the two-sheeted surface to the plane using the map (2.7), under which $T(z)$ transforms as

$$T(w_i) = \frac{(1 - z_i^2)^2}{4 z_i^4 L^2} \left[T(z_i) + \frac{c}{8z_i^2}\right] \equiv \alpha_i [T(z_i) + \beta_i]. \quad (3.2)$$
On the plane, the four-point function of the energy-momentum tensor is given by

\[
\langle T(z_1)T(z_2)T(z_3)T(z_4) \rangle = (z_{13}z_{24})^{-4} G_T(z) \equiv (z_{13}z_{24}z(1-z))^{-4} \mathcal{F}_T(z),
\]  
(3.3)

where

\[
G_T(z) = \frac{c^2}{4} \left( 1 + \frac{1}{z^4} + \frac{1}{(1-z)^4} \right) + 2c \left( \frac{1-z(1-z)}{z^2(1-z)^2} \right),
\]  
(3.4)

and

\[
\mathcal{F}_T(z) = \frac{c^2}{4} \left[ z^4(1-z)^4 + (1-z)^4 + z^4 \right] + 2c z^2(1-z)^2 [1-z(1-z)] ,
\]  
(3.5)

where the latter is constructed from \( G_T(z) \) by extracting the leading singular behavior.

The contributions to (3.1) coming from the Jacobian and the denominator can be written in terms of the cross ratios and read

\[
\prod_i \alpha_i \frac{z_1^4 z_2^4}{z_{13}^4 z_{24}^4 (T(w_1)T(w_2))^2} = \frac{z^4(1-z)^4}{c^2/4},
\]  
(3.6)

Hence, we find that

\[
\exp \left( -\Delta S_A^{(2)}(z) \right) = \frac{z^4(1-z)^4}{c^2/4} \left[ G_T(z) + \frac{c^4}{16} + \frac{c^3}{4} \frac{1-z(1-z)^2}{z^2(1-z)^2} - \frac{2c^2}{z(1-z)} \right],
\]  
(3.7)

where the terms in the bracket originate from the non-vanishing correlators of products of \( T(z_i) + \beta_i \).

Once again, as we take \( \epsilon \to 0 \), we either have \( z \simeq 0 \) for \( t \notin [l,L+l] \) or \( z \simeq 1 \) for \( t \in [l,L+l] \). Notice, that the pre-factor of (3.7) cancels the divergence of \( G_T(z) \) at those points. This is, moreover, the highest degree divergence inside the brackets. To make this fact manifest, we rewrite (3.7) as

\[
\exp \left( -\Delta S_A^{(2)}(z) \right) = \frac{4}{c^2} \left[ \mathcal{F}_T(z) + z^4(1-z)^4 \left( \frac{c^4}{16} + \frac{c^3}{4} \frac{1-z(1-z)^2}{z^2(1-z)^2} - \frac{2c^2}{z(1-z)} \right) \right].
\]  
(3.8)

Therefore, we find

\[
\exp \left( -\Delta S_A^{(2)}(z) \right) \simeq \left\{ \begin{array}{ll}
\frac{4}{c^2} \mathcal{F}_T(0) & t \notin [l,l+L] \\
\frac{4}{c^2} \mathcal{F}_T(1) & t \in [l,l+L]
\end{array} \right.
\]

From equation (3.5) we see that

\[
\mathcal{F}(z \to 0) = \mathcal{F}(z \to 1) = \frac{c^2}{4},
\]  
(3.9)

thus

\[
\Delta S_A^{(2)} = 0
\]  
(3.10)
for all times. This way we confirm that both, primary $I$ and descendant $T(z)$ lead to the same behavior, $\Delta S_A^{(2)} = 0$, for any 1+1d CFT (not necessarily rational). As we will show, this phenomenon is not a coincidence and we will provide further evidence for it in the remainder of this work.

A clarification at this point is in order. From the above formulas we can notice an order of limits conflict if we consider CFTs with large central charge. Namely in order to extract the interesting constant (in this case equal to zero) we first have to take $\epsilon \to 0$ and then $c \to \infty$ (for the opposite interesting order see App E).

## 3.2 Derivative of a primary

In this section we consider a state locally excited by the first descendant of a primary operator $O$. We denote the descendant operator by

$$
\partial O(z, \bar{z}) = L^{-1} O(z, \bar{z}) = \partial z O(z, \bar{z}).
$$

Clearly, the conformal dimensions of $\partial O$ are $(h+1, \bar{h})$, where $(h, \bar{h})$ are those corresponding to the primary. Let us start by computing the evolution of the energy density in states locally excited by (3.11). As mentioned above, the expectation value of the energy density is universal. As a matter of fact, we will only need the OPE between the stress tensor and the derivative of a primary field which reads $[16]$

$$
T(x) \partial O(z_i, \bar{z}_i) \sim \frac{2h O(z_i, \bar{z}_i)}{(x - z_i)^3} + \frac{(h + 1) \partial z O(z_i, \bar{z}_i)}{(x - z_i)^2} + \frac{\partial^2 z O(z_i, \bar{z}_i)}{x - z_i},
$$

whereas the OPE with $\bar{T}(\bar{x})$ is the same as for the primary.

The correlators with the stress tensor and descendants can be effectively written in terms a differential operator acting on the correlation functions of the primary operators (see also section 3.4). More precisely, we can write the three-point correlator as a sum over the residues from the OPEs of $T$ and $\bar{T}$ with $\partial_i O_i$, $i = 1, 2$, taken inside the correlator and then pull the derivatives in front, such that the operator acts on the two-point function of the primaries $O$. Applying the differential operator and inserting the points (2.5), the expectation value of the energy density can be written as

$$
\langle T_{tt}(x, x) \partial O \rangle = \langle T_{tt}(x, x) \rangle_O + \frac{4e^2 [(1 - 4h)(x + l - t)^2 + (1 + 4h)e^2]}{(2h + 1) [(x + l - t)^2 + e^2]^3},
$$

and the total energy injected into the system equals

$$
E_{\partial O} = \int dx \langle T_{tt}(x, x) \partial O \rangle = \frac{2\pi (h + 1 + \bar{h})}{\epsilon},
$$

as expected. Once again, the evolution of the energy density corresponds to that of two wave packets propagating in opposite directions from the insertion point. However, in the present
case there is an asymmetry in the amount of energy propagating on each side. Figure 3 clearly displays the asymmetry, where the right moving wave packet is seen to carry more energy. Naively one could expect that this different behavior of the energy density will imply the different constant increase in the Rényi entanglement entropies. We will show below that this turns out not to be correct.

Figure 3: Evolution of the energy density for three different times in the state excited by the descendant (3.11). For comparison, the dashed lines show the evolution in the state excited by a primary. Plot for $l = 0$ and $\epsilon = 1$.

Let us first confront this with the left-right entanglement of the excited state. Formally, we can also think of the descendant operator (3.11) as acting on the left and right vacuum so that we have a new entangled state

$$\partial_z O_\alpha |0\rangle \otimes |\bar{0}\rangle = i\sqrt{2}\alpha \left[ \sqrt{a} \left| \partial \phi e^{i\sqrt{2}\alpha \phi} \right\rangle \otimes \left| e^{i\sqrt{2}\alpha \phi} \right\rangle - \sqrt{1-a} \left| \partial \phi e^{-i\sqrt{2}\alpha \phi} \right\rangle \otimes \left| e^{-i\sqrt{2}\alpha \phi} \right\rangle \right]$$

The normalized density matrix of the holomorphic or anti-holomorphic (left or right) movers for this state becomes (2.22) and therefore the corresponding Rényi entropies are again equal to (2.23). This suggests that, despite the different evolution of the energy density, the contribution to the entanglement Rényi entropy should be the same as for the primary operator.

Now, we proceed to calculate the change in the second Rényi entropy wrt to the vacuum due to the insertion of the descendant operator $\partial O$. This can be obtained by computing

$$\exp \left( -\Delta S_A^{(2)} (z, \bar{z}) \right) = \frac{(\partial O(w_1, \bar{w}_1)\partial O(w_2, \bar{w}_2)\partial O(w_3, \bar{w}_3)\partial O(w_4, \bar{w}_4))_{\Sigma_2}}{(\langle \partial O(w_1, \bar{w}_1)\partial O(w_2, \bar{w}_2)\partial O(w_3, \bar{w}_3)\partial O(w_4, \bar{w}_4) \rangle_{\Sigma_1})^2}.$$  

---

4For simplicity we suppress the $L/R$ indices that indicate left or right movers in the Minkowski signature or the functions of $z$ or $\bar{z}$ in Euclidean.
The first step is to map the two-sheeted surface $\Sigma_2$ to the plane using the map (2.7). One must proceed with caution since descendant operators transform differently from primaries under conformal transformations (see App.B). For the first descendant of a primary we have

$$
\partial \mathcal{O}(w_i, \bar{w}_i) = (w_i')^{-(h+1)}(\bar{w}_i')^{-h} \left[ \partial \mathcal{O}(z_i, \bar{z}_i) - \frac{h}{w_i'} \mathcal{O}(z_i, \bar{z}_i) \right] \\
\equiv \alpha_i \tilde{\alpha}_i \left[ \partial \mathcal{O}(z_i, \bar{z}_i) + \beta_i \mathcal{O}(z_i, \bar{z}_i) \right].
$$

(3.17)

Schematically, the correlator in the numerator of Eq. (3.16) can be written in terms of correlators on the plane as

$$
\langle \partial \mathcal{O}(w_1, \bar{w}_1) ... \partial \mathcal{O}(w_4, \bar{w}_4) \rangle_{\Sigma_2} = \prod_{i=1}^{4} \alpha_i \tilde{\alpha}_i \left( \langle \partial \mathcal{O} \partial \mathcal{O} \partial \mathcal{O} \partial \mathcal{O} \rangle + \sum_j \beta_j \langle \mathcal{O} \partial \mathcal{O} \partial \mathcal{O} \partial \mathcal{O} \rangle + \sum_{j,k} \beta_j \beta_k \langle \mathcal{O} \mathcal{O} \partial \mathcal{O} \partial \mathcal{O} \rangle + \sum_{i=1}^{4} \beta_i \langle \mathcal{O} \mathcal{O} \mathcal{O} \mathcal{O} \rangle \right).
$$

(3.18)

The four-point function of the primary operator $\mathcal{O}$ is given by Eq. (2.16), whereas that for the first descendant takes the form

$$
\langle \partial \mathcal{O}(z_1, \bar{z}_1) \partial \mathcal{O}(z_2, \bar{z}_2) \partial \mathcal{O}(z_3, \bar{z}_3) \partial \mathcal{O}(z_4, \bar{z}_4) \rangle = (4z_1z_2)^{-2(h+1)}(4\bar{z}_1\bar{z}_2)^{-2h} \mathcal{G}_{\partial \mathcal{O}}(z, \bar{z}).
$$

(3.19)

The expression for $\mathcal{G}_{\partial \mathcal{O}}$ can be found in appendix C, equations (C.3) and (C.4). On the other hand, the two point function of $\partial \mathcal{O}$ is simply

$$
\langle \partial \mathcal{O}(w_1, \bar{w}_1) \partial \mathcal{O}(w_2, \bar{w}_2) \rangle_{\Sigma_1} = -2h(1+2h)w_{12}^{2(h+1)}\bar{w}_{12}^{-2h}.
$$

(3.20)

Now, we split the RHS of equation (3.16) into two pieces

$$
\exp \left( -\Delta S^{(2)}_\chi (z, \bar{z}) \right) = N_{\partial \mathcal{O}} (\mathcal{G}_{\partial \mathcal{O}}(z, \bar{z}) + \Xi(z, \bar{z})),
$$

(3.21)

where $N_{\partial \mathcal{O}}$ is an overall pre-factor given by

$$
N_{\partial \mathcal{O}} = \frac{(4z_1z_2)^{-2(h+1)}(4\bar{z}_1\bar{z}_2)^{-2h}}{(\langle \partial \mathcal{O}(w_1, \bar{w}_1) \partial \mathcal{O}(w_2, \bar{w}_2) \rangle_{\Sigma_1})^2},
$$

(3.22)

and $\Xi$ contains all the terms of lower order in derivatives, i.e.

$$
\Xi(z, \bar{z}) = (4z_1z_2)^{2(h+1)}(4\bar{z}_1\bar{z}_2)^{2h} \left( \sum_j \beta_j \langle \mathcal{O} \partial \mathcal{O} \partial \mathcal{O} \partial \mathcal{O} \rangle + \sum_{j,k} \beta_j \beta_k \langle \mathcal{O} \mathcal{O} \partial \mathcal{O} \partial \mathcal{O} \rangle \right)
$$

$$
+ \sum_{j,k,l} \beta_j \beta_k \beta_l \langle \mathcal{O} \mathcal{O} \mathcal{O} \mathcal{O} \rangle + \sum_{i=1}^{4} \beta_i \langle \mathcal{O} \mathcal{O} \mathcal{O} \mathcal{O} \rangle.
$$

(3.23)

---

\(^{5}\)Notice that this form assumes $z_3 = -z_1$ and $z_4 = -z_2$ and for general points one cannot write the four point correlator of descendants this way.
To find the change of the second Rényi entropy, we use the Jacobian of the uniformization map (2.7) and the two point function (3.20) to show that the pre-factor can be written in terms of the cross ratios as
\[ N_{\partial \mathcal{O}} = \frac{(1 - z)z^{2(1+h)}((1 - \bar{z})\bar{z})^{2h}}{4h^2(1 + 2h)^2}. \]  
(3.24)

Therefore, we have
\[
\exp \left( -\Delta S_A^{(2)}(z, \bar{z}) \right) = \frac{(1 - z)z^{2(1+h)}((1 - \bar{z})\bar{z})^{2h}}{4h^2(1 + 2h)^2} \left[ \mathcal{G}_{\partial \mathcal{O}}(z, \bar{z}) + \Xi(z, \bar{z}) \right].
\]  
(3.25)

For primary operators we have decomposed \( \mathcal{G}_{\mathcal{O}} \) into
\[
\mathcal{G}_{\mathcal{O}}(z, \bar{z}) = (z(1 - z))^{-2h}(\bar{z}(1 - \bar{z}))^{-2h} \mathcal{F}_{\mathcal{O}}(z, \bar{z}),
\]  
(3.26)

where \( \mathcal{F}_{\mathcal{O}} \) is regular. Analogously, for the descendant we define
\[
\mathcal{G}_{\partial \mathcal{O}}(z, \bar{z}) = (z(1 - z))^{-2(h+1)}(\bar{z}(1 - \bar{z}))^{-2h} \mathcal{F}_{\partial \mathcal{O}}(z, \bar{z}),
\]  
(3.27)

where once again \( \mathcal{F}_{\partial \mathcal{O}} \) is regular and its relationship to \( \mathcal{F}_{\mathcal{O}} \) can be found in Eqs. (C.7) and (C.8). Using these expressions, (3.25) becomes
\[
\exp \left( -\Delta S_A^{(2)}(z, \bar{z}) \right) = \frac{1}{4h^2(1 + 2h)^2} \left[ \mathcal{F}_{\partial \mathcal{O}}(z, \bar{z}) + \tilde{\Xi}(z, \bar{z}) \right],
\]  
(3.28)

where we introduced
\[
\tilde{\Xi}(z, \bar{z}) = ((1 - z)z)^{2(1+h)}((1 - \bar{z})\bar{z})^{2h} \Xi(z, \bar{z}).
\]  
(3.29)

Now, we calculate \( \Delta S_A^{(2)}(z, \bar{z}) \) after removing the cut-off. As discussed in section 2, we have two possibilities, either \( t \in [l, L + l] \) or \( t \notin [l, L + l] \), for which the cross ratios become \( (z, \bar{z}) \simeq (1, 0) \) and \( (z, \bar{z}) \simeq (0, 0) \) respectively. Expanding \( \tilde{\Xi} \) about the former yields
\[
\tilde{\Xi}(z, \bar{z}) \simeq (1 - z) \left[ \frac{4h - 2}{(1 + 2h)^2} \mathcal{F}_{\mathcal{O}}(0, 0) + \left( \frac{1 - 2h}{1 + 2h} \right)^2 \partial \mathcal{F}_{\mathcal{O}}(0, 0) \right] + \bar{z} \bar{z} \partial \mathcal{F}_{\mathcal{O}}(0, 0) + \ldots.
\]  
(3.30)

while for the latter we find
\[
\tilde{\Xi}(z, \bar{z}) \simeq z \left[ \frac{4h - 2}{(1 + 2h)^2} \mathcal{F}_{\mathcal{O}}(0, 0) + \left( \frac{1 - 2h}{1 + 2h} \right)^2 \partial \mathcal{F}_{\mathcal{O}}(0, 0) \right] + z \bar{z} \partial \mathcal{F}_{\mathcal{O}}(0, 0) + \ldots.
\]  
(3.31)

Thus, both of these contributions vanish. Finally, using Eq. (C.7) we find
\[
\Delta S_A^{(2)} \simeq \begin{cases} 
- \log \mathcal{F}_{\mathcal{O}}(0, 0) & t \notin [l, L + l] \\
- \log \mathcal{F}_{\mathcal{O}}(1, 0) & t \in [l, L + l]
\end{cases}
\]  
(3.32)

which coincides with the result obtained for primary operators. The fact that we are considering a descendant operator instead of a primary has a tangible effect. Namely, the total...
energy injected into the system is divided in different proportions amongst the left and right movers. In fact we would expect the action of each \( L_{-n} (\tilde{L}_{-n}) \) to increase the height of the right (left) moving lump. Notice, however, that this unbalance doesn’t affect the jump in the second Rényi entropy of the system. It is only important that we have a pair of lumps propagating in the opposite directions and that one of the members of the pair is inside the entangling region of an arbitrary shape (in 1d either a finite or semi-infinite interval) while the other member remains outside. This strongly hints to the topological nature of this quantity (see also [25] and the discussion section).

3.3 Both derivatives

In this section, we briefly consider the change in the second Rényi entropy due to the insertion of the second order descendant

\[
\partial \partial \mathcal{O}(z, \bar{z}) = \partial \partial \mathcal{O}(z, \bar{z}) = \frac{1}{2} \frac{\partial \mathcal{O}(z, \bar{z})}{\partial z} \frac{\partial \mathcal{O}(z, \bar{z})}{\partial \bar{z}}. \tag{3.33}
\]

The energy density profile corresponding to this descendant is given by

\[
\langle T_{tt}(x, x) \rangle_{\partial \partial \mathcal{O}} = \langle T_{tt}(x, x) \rangle_{\mathcal{O}} + \frac{4 \epsilon^2 \left[ (1 - 4h)(x + l - t)^2 + (1 + 4h)\epsilon^2 \right]}{(2h + 1) \left[ (x + l - t)^2 + \epsilon^2 \right]^3} + \frac{4 \epsilon^2 \left[ (1 - 4\bar{h})(x + l + t)^2 + (1 + 4\bar{h})\epsilon^2 \right]}{(2h + 1) \left[ (x + l + t)^2 + \epsilon^2 \right]^3}, \tag{3.34}
\]

and its time evolution is depicted in Fig. 4. Moreover, the total energy injected into the system is

\[
E_{\partial \partial \mathcal{O}} = \frac{2 \pi (h + \bar{h} + 2)}{\epsilon}. \tag{3.35}
\]

Notice that the left/right symmetry of the energy carried by the wave packets is restored.

The task of calculating the \( \Delta S_A^{(2)} \) corresponding to the insertion of \( \partial \partial \mathcal{O} \) is analogous to those performed in the previous sections. Nevertheless, there are some complications that we wish to point out. First, the transformation of the operator \( \partial \partial \mathcal{O} \) under conformal maps is more complicated, it contains contributions proportional to \( \partial \partial \mathcal{O}, \partial \mathcal{O} \) and \( \mathcal{O} \). The explicit form of this transformation can be found in Eq. (B.2). Once again, we find an expression of the form

\[
\exp \left( -\Delta S_A^{(2)}(z, \bar{z}) \right) = N_{\partial \partial \mathcal{O}} \left[ G_{\partial \partial \mathcal{O}}(z, \bar{z}) + \ldots \right], \tag{3.36}
\]

where the dots contain the elements of lower order in derivatives. Furthermore, the normalization factor reads

\[
N_{\partial \partial \mathcal{O}} = \left( \frac{(1 - z)z^{2(1+h)}}{4h^2(1 + 2h)^2} \right) \left( \frac{(1 - \bar{z})\bar{z}^{2(1+\bar{h})}}{4\bar{h}^2(1 + 2\bar{h})^2} \right). \tag{3.37}
\]
Figure 4: Evolution of the energy density for three different times in the state excited by \( \bar{\partial}\partial \mathcal{O} \). The dashed lines show the evolution in the state excited by a primary. Plot for \( l = 0 \) and \( \epsilon = 1 \).

As well, the function \( G_{\bar{\partial}\partial \mathcal{O}} \) becomes more convoluted; nevertheless, it can be expressed in terms \( G_{\mathcal{O}} \), see Eq. (C.10). Despite these complications, we can show that

\[
\exp \left(-\Delta S^{(2)}_{A}(z, \bar{z})\right) = (4h(1+2h)\bar{h}(1+2\bar{h}))^{-2} \left[ F_{\bar{\partial}\partial \mathcal{O}}(z, \bar{z}) + \ldots \right],
\]

where \( F_{\bar{\partial}\partial \mathcal{O}} \) is given by (C.12). Finally, as we remove the UV cut-off we find that

\[
\Delta S^{(2)}_{A} \simeq \begin{cases} 
-\log F_{\mathcal{O}}(0, 0) & t \notin [l, L + l] \\
-\log F_{\mathcal{O}}(1, 0) & t \in [l, L + l]
\end{cases}
\]  

(3.39)

once more. This clearly shows that the propagation of the energy in the state excited by the descendants only provides a qualitative support for the quasi-particle picture.

### 3.4 Descendants at level 2

It is always possible to write the correlators of descendants in terms of the action of certain differential operators on the correlator of the primary operators (family’s parent). Whenever the correlator in question involves only (powers of) \( \hat{L}_{-1} \) and \( \hat{\bar{L}}_{-1} \) the Virasoro generators can be pulled out of the correlator as simple derivatives. However, if there is any descendant constructed with an \( \hat{L}_{-k} \) with \( k > 1 \) the situation changes. For example, if there is one such descendant in the correlator then [16]

\[
\langle \mathcal{O}(w_1, \bar{w}_1) \ldots \hat{L}_{-k} \mathcal{O}(w_i, \bar{w}_i) \ldots \mathcal{O}(w_N, \bar{w}_N) \rangle = \mathcal{L}^{(i)}_{-k} \langle \mathcal{O}(w_1, \bar{w}_1) \ldots \mathcal{O}(w_N, \bar{w}_N) \rangle,
\]

(3.40)

where

\[
\mathcal{L}^{(i)}_{-k} = \sum_{j \neq i} \left( \frac{h(k-1)}{w_{ji}^k} - \frac{\partial_j}{w_{ji}^{k-1}} \right).
\]

(3.41)
However, if the correlator contains more than one descendant the appropriate commutation
of the Virasoro algebras at different points must be taken into account. This is manifest
already for level \( k = 2 \) and we discuss it below.

In the following we compute \( \Delta S_A^{(2)} \) due to the insertion of the operator
\[
\mathcal{O}^{(-2)}(z, \bar{z}) = \hat{L}_{-2} \mathcal{O}(z, \bar{z}).
\] (3.42)

The crucial correlation function that will contain the relevant constant is the four point
function of \( \mathcal{O}^{(-2)} \), which appears in the numerator of Eq. (2.6). We proceed in a standard
way \cite{16}, first we write the correlator as
\[
\langle \hat{L}_{-2} \mathcal{O}_1 \mathcal{O}_2^{(-2)} \mathcal{O}_3^{(-2)} \mathcal{O}_4^{(-2)} \rangle = \sum_{i=2}^{4} \int_{\mathbb{C}(w_i)} \frac{dz}{2\pi i} (z_1 - z)^{-1} \mathcal{O}_1 \ldots \left( T(z) \mathcal{O}_i^{(-2)} \ldots \mathcal{O}_4^{(-2)} \right),
\] (3.43)

and then make use of the OPE
\[
T(z) \mathcal{O}^{(-2)}(w) \sim \frac{(\frac{\bar{z}}{z} + 4h) \mathcal{O}(w)}{(z - w)^4} + \frac{3 \partial \mathcal{O}(w)}{(z - w)^3} + \frac{(h + 2) \mathcal{O}^{(-2)}(w)}{(z - w)^2} + \frac{\partial \mathcal{O}^{(-2)}(w)}{z - w}. \quad (3.44)
\]

Inserting this OPE into (3.43) and calculating the residues, we can express the four point
function in terms of correlators containing one primary and three descendants. Repeating
this procedure, it is possible to reduce the number of descendants in the RHS until the full
correlator is written in terms of the four-point function of the primary operators. The full
procedure is rather cumbersome and the result is given by (D.6).

Once more, after insertion \( z_3 = -z_1 \) and \( z_4 = -z_2 \), we define
\[
\langle \mathcal{O}_1^{(-2)} \mathcal{O}_2^{(-2)} \mathcal{O}_3^{(-2)} \mathcal{O}_4^{(-2)} \rangle = (4z_1z_2)^{-2(h+1)} (4\bar{z}_1\bar{z}_2)^{-2h} \mathcal{G}_{\mathcal{O}^{(-2)}}(z, \bar{z}),
\] (3.45)

with \( \mathcal{O}_i = \mathcal{O}(z_i, \bar{z}_i) \), as well as
\[
\mathcal{G}_{\mathcal{O}^{(-2)}}(z, \bar{z}) = (z(1 - z))^{-2(h+1)} (\bar{z}(1 - \bar{z}))^{-2h} \mathcal{F}_{\mathcal{O}^{(-2)}}(z, \bar{z}).
\] (3.46)

The increase on the second Rényi entropy can be written as
\[
\exp \left( -\Delta S_A^{(2)}(z, \bar{z}) \right) = N_{\mathcal{O}^{(-2)}} \left[ \mathcal{G}_{\mathcal{O}^{(-2)}}(z, \bar{z}) + \ldots \right],
\] (3.47)

where the pre-factor reads
\[
N_{\mathcal{O}^{(-2)}} = \frac{(4z_1z_2)^{-2(h+2)} (4\bar{z}_1\bar{z}_2)^{-2h} \prod_{i=1}^{4} \alpha_i \bar{\alpha}_i}{\left( \langle \mathcal{O}^{(-2)}(w_1, \bar{w}_1) \mathcal{O}^{(-2)}(w_2, \bar{w}_2) \rangle_{\Sigma_1} \right)^2}. \quad (3.48)
\]

Using the OPE (3.44), we can show that the two-point function is given by
\[
\langle \mathcal{O}^{(-2)}(w_1, \bar{w}_1) \mathcal{O}^{(-2)}(w_2, \bar{w}_2) \rangle_{\Sigma_1} = \frac{1}{2} \left( c + 2h(9h + 22) \right) w_{12}^{2(h+2)} \bar{w}_{12}^{-2h}. \quad (3.49)
\]

\footnote{In the limit of \((z, \bar{z}) \to (1, 0)\) all the other correlators are suppressed by powers of \((1 - z)\).}
Finally, we extract $\mathcal{F}_{\mathcal{O}(-2)}$ from Eq. (D.6), and as we remove the UV cut-off $\epsilon$ find

$$\mathcal{F}_{\mathcal{O}(-2)}(z, \bar{z}) \simeq \frac{1}{4} (c + 2h(9h + 22))^2 \begin{cases} \mathcal{F}_{\mathcal{O}(0,0)} & t \notin [l, L + l] \\ \mathcal{F}_{\mathcal{O}(1,0)} & t \in [l, L + l] \end{cases}$$  \hspace{1cm} (3.50)

Therefore, we find that for the insertion of the operator $\mathcal{O}(-2)$ we have

$$\Delta S^{(2)}_A \simeq \begin{cases} -\log \mathcal{F}_{\mathcal{O}(0,0)} & t \notin [l, L + l] \\ -\log \mathcal{F}_{\mathcal{O}(1,0)} & t \in [l, L + l] \end{cases}$$  \hspace{1cm} (3.51)

as expected.

Clearly the procedure becomes more involved once the level of the descendant increases but the increase in the Renyi entropy remains fixed by the two limits of the original $\mathcal{F}_{\mathcal{O}}(z, \bar{z})$ from the correlator of the primary operators.

4 General derivation

Based on the explicit examples in the previous sections, we can clearly deduce that the relevant contribution to the Renyi entanglement entropies is the same for the whole conformal family. We show it for the second Renyi entropy but the generalization to higher $n$ should be straightforward (though very involved computationally).

The technical reason for the constant is the following\(^7\). On one hand, under conformal transformation, every descendant of level $(n, \bar{n})$ transforms as

$$\mathcal{O}^{(n,\bar{n})}(z, \bar{z}) = (f')^{h+n} (\bar{f'})^{\bar{h}+\bar{n}} \left[ \mathcal{O}^{(n,\bar{n})}(f, \bar{f}) + \ldots \right]$$  \hspace{1cm} (4.1)

where the ellipsis stand operators of the lower dimension that in the correlation functions give rise to lower order singularity. This way, once we employ the map (2.7), the Jacobian from (4.1) with the square of the two-point function on $\Sigma_1$ (2.6) combine into

$$P = N^2_2 [-4z_1z_2]^{h+n}[4\tilde{z}_1\tilde{z}_2]^{\bar{h}+\bar{n}}[z(1-z)]^{h+n}[\tilde{z}(1-\tilde{z})]^{\bar{h}+\bar{n}},$$  \hspace{1cm} (4.2)

with $N^2_2$ being the constant in the two-point function of the descendants.

When we extract the constant contribution to the entanglement Renyi entropy we take the limit of $(z, \bar{z}) \to (1, 0)$. The only terms that survive this regime must cancel the above pre-factor and are confined to the correlation function of $(n, \bar{n})$ descendants only. On the other hand, the correlators of descendants on the plane can be obtained from the correlators of the primary operators that we define as\(^8\)

$$\langle \mathcal{O}_\alpha(z_1, \tilde{z}_1) \mathcal{O}_\alpha^\dagger(z_2, \tilde{z}_2) \mathcal{O}_\alpha(z_3, \tilde{z}_3) \mathcal{O}_\alpha^\dagger(z_4, \tilde{z}_4) \rangle \equiv |z_{13}z_{24}|^{-4h}|z(1-z)|^{-4h} \mathcal{F}_\alpha(z, \tilde{z})$$  \hspace{1cm} (4.3)

\(^7\)Similar logic has been applied in [24]

\(^8\)This is a form that we assume with $F(z, \bar{z})$ being smooth at $z \to 1$ as well as $z \to 0$. This is holds for all the cases that we are aware of.
by acting with a (complicated but in principle straightforward to derive) differential operator. In the limit of \((z, \bar{z}) \to (1, 0)\) the only contribution that has enough singularity to cancel the above-mentioned pre-factor comes from the terms in which the differential operator leaves \(\mathcal{F}_\alpha(z, \bar{z})\) intact and contains precisely the square of the norm \(\mathcal{N}_{\mathcal{O}(n, \bar{n})}\) of the two-point function.\(^9\)

This way the non-trivial constant obtained from the correlators in the limit of \((z, \bar{z}) \to (1, 0)\) is the same for all the members of a given conformal family. By the same token, one can show that as \((z, \bar{z}) \to (0, 0)\) the Rényi entanglement entropy is unchanged.

5 Conclusions and discussion

In this work we have demonstrated that, in 1+1 dimensional CFTs with Virasoro symmetry, local excitations by operators in the same conformal family increase the second Rényi entanglement entropy by the same constant. For rational conformal field theories this entanglement constant is the logarithm of the quantum dimension (obtained in [9]) of the primary operator that represents the family. We have checked this on explicit examples and outlined a general derivation. A generalization to higher Rényi entropies should be also possible and we expect it to work in exactly the same way as for the second Rényi entropy.\(^10\)

Our results strongly support the existence of universal features of entanglement for locally excited states. So far only the entanglement entropy of a block in the ground state of a local hamiltonian has been used to fix the central charge of the CFT that govern the critical points. It would be interesting to perform numerical study of the evolution of the entanglement entropy in a chain (e.g. Ising chain) excited by a local operator. Similarly to [32], by tuning the parameters close to criticality\(^11\) it should be possible to verify the CFT prediction for the constant contribution to the entropy [33].

Let us also discuss two different perspectives that we hope can lead to a simpler proof of our result and shed more light on the physical meaning of the constant contribution to the Rényi entanglement entropies.

We have stressed that the constant contribution to the Rényi entropies is equivalent to the entropy between Left and Right (chiral and anti-chiral) movers in a quantum state [3]

\[ |\psi_{LR}\rangle = \mathcal{O}(z, \bar{z}) |0\rangle_L |0\rangle_R \] \(^{(5.1)}\)\n
---

\(^9\)We have verified it up to level 2 and for arbitrary powers of derivatives

\(^10\)Even though general 2n-point correlators of descendants will have a very complicated structure, the late time behaviour of the invariant cross-ratios is universal. Therefore, at least in RCFTs, one could possibly use the factorisation and the fusion transformation for any members of a conformal family to formally demonstrate our claims for general \(n\). The same should be true for \(n < 1\) and it would be interesting to check it for the logarithmic negativity (see e.g. [30, 31]).

\(^11\)Generically, local operators on the chain, at the critical point, will correspond to primaries, descendants or the linear combinations of thereof.
Formally, this means that we can write the state in the Schmidt form

$$|\psi_{LR}\rangle = \sum_i \sqrt{p_i} |i_L\rangle |i_R\rangle \quad (5.2)$$

with Schmidt bases $|i_L\rangle$ for L and $|i_R\rangle$ for R and Schmidt coefficients satisfying

$$\sum_i p_i = 1 \quad (5.3)$$

where number of non-zero $p_i$s is called the Schmidt number of state $|\psi_{LR}\rangle$ [15]. This way the constant contribution to the $n$-th Rényi entropy can be computed as

$$\Delta S^{(n)} = \frac{1}{1-n} \log \left( \sum_i p_i^n \right) \quad (5.4)$$

and comparing (5.4) and (2.12) we get for rational CFTs: $F_{00}[O] = \sum_i p_i^2$.

The fact that this number is the same for all the members of the family of a give descendant indicates that only the elements of the Schmidt basis $|i_L\rangle$ ($|i_R\rangle$) can change from one descendant to another however the Schmidt coefficients remain the same.

In principle one should be able to express the excited state (5.1) in the left-right orthonormal basis of descendants and then transform to the Schmidt basis using the singular value decomposition. This can be done for some excited states that have a natural left-right decomposition (see [20, 21, 22, 23]) but it would be interesting to preform it for (5.1) even in known RCFTs.

Let us also point that the logarithm of quantum dimension appears naturally in the computation of topological entanglement entropy in 2+1 dimensions. In fact it was shown in [25] that for a given region $A$ the increase of the topological entanglement entropy due to excitation $a$ is equal

$$\Delta S_{top} = \log(d_a) \quad (5.5)$$

where $d_a$ is the quantum dimension of the excitation.

Even though rational CFT are naturally linked to topological field theories in 2+1 dimensions (e.g. FQHe), the precise link to the 2d CFT technology employed above is not obvious. However, based on the connection between the topological entropy and the boundary entropy [26], or left-right entropy and TQFT in 2+1 [22] [21], it is not unlikely that the constant contribution to Rényi entanglement entropies in 1+1 d CFT is equivalent to the change in topological entanglement entropy in 2+1d.

If this happens to be the case, our results for descendants imply that the contribution from the whole topological sector (conformal family) is the same. It would be very interesting to sharpen this connection and we leave it for the future work.

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12 See also [28] for an interesting application to black holes in $AdS_3$

13 That give rise to wave functions for excited states (see e.g. [29])
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A Correlators of the EPR operators

We have used the EPR operators

\[ O_\alpha(z_1, \bar{z}_1) = \sqrt{a} e^{i \sqrt{2} \alpha \phi(z_1, \bar{z}_1)} \pm \sqrt{1-a} e^{-i \sqrt{2} \alpha \phi(z_1, \bar{z}_1)} \equiv \sqrt{a} O_\alpha^{(+)} \pm \sqrt{1-a} O_\alpha^{(-)} \]

\[ O_\alpha^\dagger(z_1, \bar{z}_1) = \sqrt{a} e^{-i \sqrt{2} \alpha \phi(z_1, \bar{z}_1)} \pm \sqrt{1-a} e^{i \sqrt{2} \alpha \phi(z_1, \bar{z}_1)} \equiv \sqrt{a} O_\alpha^{(-)} \pm \sqrt{1-a} O_\alpha^{(+)} \]

with conformal dimension \( h = \bar{h} = \alpha^2 \) and parameter \( a \in [0, 1] \).

Here we compute the four-point correlator on the complex plane

\[ G_{O_\alpha}^4 = \langle O_\alpha(z_1, \bar{z}_1) O_\alpha^\dagger(z_2, \bar{z}_2) O_\alpha(z_3, \bar{z}_3) O_\alpha^\dagger(z_4, \bar{z}_4) \rangle \]

(A.2)

Since the non zero correlators of the vertex operators are the ones that have the zero sum of the operators dimensions

\[ \langle \prod_i e^{\sqrt{2} \alpha_i \phi(z_i, \bar{z}_i)} \rangle = \prod_{i<j} |z_i - z_j|^d_{\alpha_i \alpha_j}, \quad \sum_i \alpha_i = 0 \]

(A.3)

we denote

\[ G^{(a_1, a_2, a_3, a_4)} = \langle O_1^{(a_1)} O_2^{(a_2)} O_3^{(a_3)} O_4^{(a_4)} \rangle, \quad a_i \in \{+, -\} \]

(A.4)

And the non-zero contributions to the correlator comes from

\[ G_{O_\alpha}^4 = (a^2 + (1-a)^2) G^{(+,-,+,-)} + 2a(1-a) \left( G^{(+,+,-,-)} + G^{(+,-,-,+)} \right) \]

(A.5)

where we used

\[ G^{(+,-,+,-)} = G^{(-,+,-,+)}, \quad G^{(+,+,-,-)} = G^{(-,-,+,+)}, \quad G^{(+,+,-,+)} = G^{(-,+,-,-)} \]

(A.6)

The remaining correlators are

\[ G^{(+,-,+,-)} = \frac{1}{|z_{13} z_{24}|^{|h|} |z(1-z)|^{|h|}} \]

\[ G^{(+,+,-,-)} = \frac{|z|^{|8h|}}{|z_{13} z_{24}|^{|h|} |z(1-z)|^{|h|}} \]

\[ G^{(+,-,-,+)} = \frac{|1-z|^{|8h|}}{|z_{13} z_{24}|^{|h|} |z(1-z)|^{|h|}} \]

(A.7)
The four-point correlator of such operators on the complex plane is given by
\[ G^4_{O_\alpha} = |z_{13}z_{24}|^{-4h}G(z, \bar{z}) \]  
with the canonical 4-pt function
\[ G(z, \bar{z}) \equiv \langle O_\alpha | O_\alpha(z, \bar{z})O_\alpha(1, 1) | O_\alpha \rangle \equiv |z(1-z)|^{-4h}F(z, \bar{z}) \]
where
\[ F(z, \bar{z}) = a^2 + (1-a)^2 + 2a(1-a) (|z|^{8h} + |1-z|^{8h}). \]

\[ B.1 \]

**B Transformation of descendants**

The transformation of descendant fields \( \phi^{(-k)} \) (of a primary \( \phi \) with dimension \( h \)) under finite conformal map \( f(z) \) has been worked out in [19] (see also [5]). For the first few descendants it can be obtained from
\[
\phi^{(-k)}(z) = e^{R_0 L_0} \left( 1 + R_1 L_1 + \frac{1}{2} R_1^2 L_1^2 + R_2 L_2 + \ldots \right) \phi^{(-k)}(f)
\]
where
\[
R_0 = \log f', \quad R_1 = \frac{1}{2} R_0' = \frac{f''}{2f'}, \quad R_2 = \frac{Sf}{6}
\]
where \( Sf \) is the Schwarzian derivative of \( f \). Using the Virasoro algebra and the commutation relations of \( L \)'s with the operator modes \( \phi_n \) we can derive
\[
\begin{align*}
\phi(z) &= (f')^h \phi(f) \\
\phi^{(-1)}(z) &= (f')^{h+1} \left[ \phi^{(-1)}(f) + h \frac{f''}{f'^2} \phi(f) \right] \\
\phi^{(-1,-1)}(z) &= (f')^{h+2} \left[ \phi^{(-1,-1)}(f) + (2h+1) \frac{f''}{f'^2} \phi^{(-1)}(f) + h \left( (2h+1) \frac{f''^2}{f'^4} + \frac{Sf}{f'^3} \right) \phi(f) \right] \\
\phi^{(-2)}(z) &= (f')^{h+2} \left[ \phi^{(-2)}(f) + \frac{3f''}{2f'^2} \phi^{(-1)}(f) + \left( \frac{3h f''^2}{4f'^4} + \left( 4h + \frac{c}{2} \right) \frac{Sf}{6f'^2} \right) \phi(f) \right]
\end{align*}
\]
\[ B.1 \]

The first two descendants correspond to derivatives and the transformation rule can be also reproduced from
\[
\begin{align*}
\phi^{(-1)}(z) &= \partial_z \left[ (f')^h \phi(f) \right], \quad \phi^{(-1,-1)}(z) = \partial_z^2 \left[ (f')^h \phi(f) \right]
\end{align*}
\]
It is also easy to check that for \( h = 0 \) (the identity) the transformation of \( \phi^{(-2)}(z) \) reproduces that standard transformation for \( T(z) \).
We have also used the transformations
\[
\partial \tilde{\partial} \phi(z, \bar{z}) = (f')^{h+1} (\bar{f'})^{h+1} \left[ \partial \tilde{\partial} \phi(f, \bar{f}) + h \frac{f''}{f^2} \tilde{\partial} \phi(f, \bar{f}) + \bar{h} \frac{\bar{f}''}{\bar{f}^2} \partial \phi(f, \bar{f}) + h \bar{h} \frac{f''}{f^2} \frac{\bar{f}''}{\bar{f}^2} \phi(f, \bar{f}) \right]
\]
\[
\partial \bar{\partial} \phi(f, \bar{f}) = (f')^{-(h+1)} (\bar{f'})^{-(h+1)} \left[ \partial \bar{\partial} \phi(z, \bar{z}) - h \frac{f''}{f^2} \partial \phi(z, \bar{z}) - \bar{h} \frac{\bar{f}''}{\bar{f}^2} \bar{\partial} \phi(z, \bar{z}) + h \bar{h} \frac{f''}{f^2} \frac{\bar{f}''}{\bar{f}^2} \phi(z, \bar{z}) \right]
\]

\begin{align*}
\langle O(z_1, \bar{z}_1) O(z_2, \bar{z}_2) O(z_3, \bar{z}_3) O(z_4, \bar{z}_4) \rangle &= (z_{13} \bar{z}_{24})^{-2h} (\bar{z}_{13} \bar{z}_{24})^{-2\bar{h}} G_O(z, \bar{z}), \quad (C.1) \\
\langle \partial O(z_1, \bar{z}_1) \partial O(z_2, \bar{z}_2) \partial O(z_3, \bar{z}_3) \partial O(z_4, \bar{z}_4) \rangle &= (z_{13} \bar{z}_{24})^{-2(h+1)} (\bar{z}_{13} \bar{z}_{24})^{-2\bar{h}} G_{\partial O}(z, \bar{z}), \quad (C.2)
\end{align*}

we can show that \( G_O(z, \bar{z}) \) and \( G_{\partial O}(z, \bar{z}) \) are related by
\[
G_{\partial O}(z, \bar{z}) = \sum_{l=0}^{4} g_l(z) \partial^l G_O(z, \bar{z}) \quad (C.3)
\]
where
\[
\begin{align*}
g_0(z) &= 4h^2(1 + 2h)^2, \\
g_1(z) &= 2(1 + 2h)(2z - 1), \\
g_2(z) &= 2 + 2z(4h^2 - 7 - 2h)(1 - z), \\
g_3(z) &= -4z(1 - z)(2z - 1), \\
g_4(z) &= (1 - z)^2 z^2.
\end{align*}
\]

Moreover, if we define the functions
\[
F_O(z, \bar{z}) = (z(1 - z))^{2h} (\bar{z}(1 - z))^{2\bar{h}} G_O(z, \bar{z}), \quad (C.5)
\]
and
\[
F_{\partial O}(z, \bar{z}) = (z(1 - z))^{2(h+1)} (\bar{z}(1 - z))^{2\bar{h}} G_{\partial O}(z, \bar{z}), \quad (C.6)
\]
we can show that
\[
F_{\partial O}(z, \bar{z}) = \sum_{l=0}^{4} f_l(z) \partial^l F_O(z, \bar{z}), \quad (C.7)
\]

\footnote{This form assumes \( z_3 = -z_1 \) and \( z_4 = -z_2 \)}
where

\begin{align*}
    f_0(z) &= 4h^2[(1 + (-1 + z)z)^2 + 4h^2(1 + 3(-1 + z)z)^2 \\
    &\quad - 4h(-1 + (-1 + z)z(-2 + 5(-1 + z)z)) \\
    f_1(z) &= -2(-1 + z)z(-1 + 2z)(z + 10h(-1 + z)z - z^2 \\
    &\quad - 40h^2(-1 + z)z + 16h^3(1 + 3(-1 + z)z)) \\
    f_2(z) &= 2(-1 + z)^2z^2(1 + 7(-1 + z)z + h(-6 - 34(-1 + z)z) \\
    &\quad + 4h^2(3 + 11(-1 + z)z)) \\
    f_3(z) &= -(1 + z)3z^3(-1 + 2z) \\
    f_4(z) &= -(1 + z)^4z^4.
\end{align*}

(C.8)

Note: At the points \((z, \bar{z}) \to (0, 0), (1, 0)\), the only contribution to \(\mathcal{F}_{\bar{\partial}\partial\mathcal{O}}(z, \bar{z})\) that survives comes from the \(f_0(z)\) term in (C.7), which in terms of the coefficients of (C.3) reads

\[ \mathcal{F}_{\bar{\partial}\partial\mathcal{O}}(z, \bar{z}) \to 2h(1 + 2h)[g_3(z) + 2(1 + h)((3 + 2h)\gamma_4(z) - \gamma_3(z))] \mathcal{F}_\mathcal{O}(z, \bar{z}), \]

(C.9)

where \(g_3(z) = z(1 - z)\gamma_3(z)\) and \(g_4(z) = z^2(1 - z)^2\gamma_4(z)\).

On the other hand, for the descendant \(\bar{\partial}\partial\mathcal{O}\) we have

\[ \mathcal{G}_{\bar{\partial}\partial\mathcal{O}}(z, \bar{z}) = \sum_{k,l=0}^4 g_{k,l}(z, \bar{z}) \partial^k \bar{\partial}^l \mathcal{G}_\mathcal{O}(z, \bar{z}) \]

(C.10)

where

\[ g_{k,l}(z, \bar{z}) = g_k(z)g_l(\bar{z}), \]

(C.11)

where the functions \(g_k(z)\) are defined in (C.4).

\[ \mathcal{F}_{\bar{\partial}\partial\mathcal{O}}(z, \bar{z}) = (z(1 - z))^{2(\hat{h} + 1)}(\bar{z}(1 - \bar{z}))^{2(\hat{\bar{h}} + 1)} \mathcal{G}_{\bar{\partial}\partial\mathcal{O}}(z, \bar{z}), \]

(C.12)

D Correlators at level 2

In this appendix we present some details of the computation of the correlation functions of descendents \(\mathcal{O}^{(-2)}(w, \bar{w})\). The key ingredient (for any operator) is the OPE with the energy momentum tensor that in this case reads

\[ T(z)\mathcal{O}^{(-2)}(w) \sim \frac{\left(\frac{c}{2} + 4h\right)\mathcal{O}(w)}{(z - w)^3} + \frac{3\partial\mathcal{O}(w)}{(z - w)^3} + \frac{(h + 2)\mathcal{O}^{(-2)}(w)}{(z - w)^2} + \frac{\partial\mathcal{O}^{(-2)}(w)}{z - w} \]

(D.1)

Denoting \(\mathcal{O}^{(-2)}(w_i) \equiv \mathcal{O}_i^{(-2)}\) and \(\mathcal{O}(w_i) \equiv \mathcal{O}\), we compute on the complex plane

\[ \langle \mathcal{O}_1^{(-2)}\mathcal{O}_2^{(-2)} \rangle = \frac{1}{2} \left[ c + 2h(9h + 22) \right] w_{12}^{-2(\hat{h} + 2)} \]

(D.2)
where \( w_{ij} = w_i - w_j \). The products of operators should be understood as acting from left to right on the correlators. When taking the contribution from the residues (see e.g. 6.6 in \[16\]) there are always few ways to start evaluating the correlator and we use the convention from left to right as well.

Analogously the three-point function can be written
\[
\langle O_1^{(-2)} O_2^{(-2)} O_3^{(-2)} \rangle = \left[ D_{1,2} L_{-2}^{(3)} + D_{1,3} L_{-2}^{(2)} + \mathcal{H}_{(1)}^{(1)} D_{2,3} + \mathcal{H}_{(1)}^{(2)} L_{-2}^{(3)} \right] \langle O_1 O_2 O_3 \rangle ,
\]
where
\[
D_{i,j} = \frac{\xi}{w_{ij}^3} + \frac{4h}{w_{ij}^3} + \frac{3\partial_j}{w_{ij}^3}, \quad L_{-2}^{(i)} = \sum_{j \neq i} \left( \frac{h}{w_{ji}^2} - \frac{\partial_j}{w_{ji}} \right)
\]
and
\[
\mathcal{H}_{(k)}^{(i)} = \sum_{j \neq i} \left( \frac{h_j}{w_{ij}^5} - \frac{\partial_j}{w_{ij}} \right) , \quad h_k = h, \quad h_k \neq h_m = h + 2.
\]

Finally, the four point function can be written as
\[
\langle O_1^{(-2)} O_2^{(-2)} O_3^{(-2)} O_4^{(-2)} \rangle = \left[ D_{1,2} \left( D_{3,4} + \mathcal{I}_{(1,2)}^{(3)} L_{-2}^{(4)} \right) + D_{1,3} \left( D_{2,4} + \mathcal{I}_{(1,3)}^{(2)} L_{-2}^{(4)} \right) 
+ D_{1,4} \left( D_{2,3} + \mathcal{I}_{(1,4)}^{(2)} L_{-2}^{(3)} \right) + \mathcal{H}_{(1)}^{(1)} D_{4,3} L_{-2}^{(2)} + D_{4,3} L_{-2}^{(2)} 
+ \mathcal{H}_{(1)}^{(2)} L_{-2}^{(3)} \right] \langle O_1 O_2 O_3 O_4 \rangle ,
\]
where
\[
\mathcal{I}_{(k,l)}^{(i)} = \sum_{j \neq i} \left( \frac{h_j}{w_{ij}^5} - \frac{\partial_j}{w_{ij}} \right) , \quad h_k = h_l = h \quad \text{and} \quad h_p = h + 2 \quad \text{for} \quad p \notin \{k,l\}.
\]

One can verify that setting \( h = 0 \) in the above operator reproduces the four-point function \[3.3\].

### E Comments on large c and holography

The late time value of the Rényi entanglement entropies is non-perturbative. To illustrate this fact we consider again the state excited by the descendant of identity \( T(z) \). In the computation of the late time value (semi-inifinite interval) of the second Rényi entanglement entropy we use the \[3.7\]. If we first take the limit of large central charge, the contribution to the second Rényi entropy becomes
\[
\Delta S^{(2)} \simeq 8 \log \frac{2t}{\epsilon} - 2 \log \frac{C}{2}
\]
This is the logarithmic growth with time obtained in \[6\] (the negative part comes form the normalisation) and it should be possible to perform a similar computation for the entanglement entropy and match it with gravity using \[34\] as in \[7\].

On the other hand, as we saw in the text, if we know the answer for the full correlator (all quantum gravity effects taken into account) we can first take \( \epsilon \rightarrow 0 \) and obtain the entanglement constant.

24
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