Network Growth via Preferential Attachment based on Prisoner’s Dilemma Game

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Abstract

In this article we discuss network growth based on Prisoner’s Dilemma Game (PDG) where players on nodes in a network play with its linked players. The players estimate total profits according to the payoff matrix of the PDG. When a new node is attached to the network, the node makes links to nodes in the network with the probabilities in proportion to the profits made by the game. Iterating this process, a network grows. We investigate properties of this type of growing networks, especially the degree distribution and time-dependent strategy distribution by running computer simulation. We also find a sort of phase transition in the strategy distributions. For these phenomena given by computer simulation, theoretical studies are also carried out.

keywords: Network Growth, Game Theory, Prisoner’s Dilemma, Degree Distribution, Phase Transition

1 Introduction

At the close of the 20th century, many empirical networks in real world turn out to be scale free or small world networks [1, 2, 3, 4, 5], rather than random networks [6]. At the same time, it becomes clear that in what way should we construct these networks. In those ways networks are mainly constructed by some outside algorithms, only depending of the topology of the networks [7]. In the growing way such as preferential attachment [1, 2], preferential nodes will forever preserve the position as prefereller. In real systems, however, such situations are not universal. Depending on a dynamics of a system, preferential nodes may change moment by moment. In this article we consider the inner dynamics on a network to influence which are preferential nodes. Thus we explore the interactions between an inner dynamics on a network and network growing. Such network growing seems to be more universal than models without some inner dynamics. Though many flexible model have been proposed, in which fitness [8], aging [9, 10, 11, 12], hierarchy [13] and so on have been considered, there are little dynamical models accompanied with interactions among nodes in a network.

As considering an inner dynamics on a networks, we need to introduce some inner degree of freedom to realize interactions. In this article we adopt Prisoner’s Dilemma game (PD) [14, 15, 16], which is used in various fields, economics, sociology, biology and so on, as a dynamics on networks. In this case, the types of strategies, cooperation (C) and defection (D), which correspond to the inner degree of freedom are two. Iteration of games played by players arranged in two dimensional space has been first considered on regular lattice by Nowark and May [17, 18]. After that many researchers have studied the subject [19]. Researchs in game theory related to complex networks, which was anticipated by Ref. [20], are summarized in Ref. [21] well. The relations between a network topology and a game dynamics on the network have been also investigated by some athors [22, 23, 24, 25, 26, 27, 28, 29, 30, 31].

In this article we use game theory for growing networks [32]. We first simulate network growing based on PDG dynamics by changing a parameter that represents payoff of PDG, and analyze the results. We first focus the time series of the population of two strategies. A phase transition like phenomenon appears when changing the parameter where all D world turns into all C world all at once. Then we investigate the degree distributions of resultant networks. They are mainly classified into two types of forms, depending on the parameter. The turning point is the critical point in the D-C transition. We also discuss these phenomena from theoretical point of view and some are quantitatively derived. A comment is also made on a relation between a shape of probability function and degree distribution function.
In the section two we formulate the algorithm for network growing studied in this article. We give simulation results in the section three and theoretical analyses are given in the next section. The last section, five, is devoted to summary.

2 Model for Network Growing

We describe the model for network growing investigated in this article.

1. Start a complete graph with \( k + 1 \) nodes on which either strategy C or D are assigned at random.

2. The nodes play PDG with their \((k + 1)\) linked nodes including themselves on the network.

3. Each node \( i \) evaluates the total payoff \( R_i \) acquired from her/his neighborhood according to the payoff matrix given by Table 1.

4. Each node \( i \) mimics the strategy of the node with the highest payoff among linked nodes.

5. Add a new node with a strategy chosen randomly.

6. Then new links from the new node to \( k \) ones among already existing nodes are connected with the probability \( P_i \) depending on the total payoffs of the existing nodes;

\[
P_i = \frac{1}{1 + \exp(-R_i/A)}
\]

where \( A \) is some positive constant.

7. The procedures from 2. to 6. are iteratively carried out \( n \) times.

As considering preferential attachments relative to payoff, it is natural that the probability \( P_i \) that are linked with a newly joining node is proportional to acquired payoffs. Now notice that payoffs can take both positive and negative values, since total payoff is zero in the payoff matrix of PDG. Then we use Eq. (1) as an extension of simple preferential attachment in the step 6. A little discussion will be given for this later on. We mainly analyze the time series of population of nodes with C or D strategy and degree distribution by computer simulations.

| Table 1. Payoff table(matrix) of PD game on which \( t > c > d > S \) is imposed. |
|---|---|---|
| strategy | C | D |
| C | \((c, c)\) | \((s, t)\) |
| D | \((t, s)\) | \((d, d)\) |

3 Results of Computer Simulations

We choose \( k = 5 \), \( A = 30 \) and \( t = -s = 5 \) as parameters for the procedure for network growing described in the previous, and \( n = 1000 \) as long as convergence is out of question. We determine \( c + d + s + t = 0 \) so that the average payoff can be 0 in the payoff matrix. So \( c = -d \) is a variable. It is studied how the features of a network system vary according to this variable \( c \).

3.1 Degree Distribution

Main results of simulations are the following.

EFor \( c < 4 \), the strategy of the network converges into all D and then degree distribution looks like linear (See Fig.1).

EFor \( c > 4.4 \), the strategy of the network converges into all C and then degree distribution looks like exponential one (See Fig.2).
According to one’s expectation, D as a dominant strategy is prevalent as $c$ takes small values. As $c$ gets bigger, all nodes comes to take a strategy C. When the system shifts from all D to all C world at a critical $c$ value, the feature of degree distribution changes as well. The abovementioned features in degree distribution is what is presumed phenomenologically. The system becomes unstable in the middle value of $c$ where stochastic behavior appears, that is, all C on one occasion, all D on another occasion. The ratio of cases where the strategy of systems becomes all C is shown in Fig.3. While this is an average over about 10 times iterations, it is clear that a sort of phase transition occurs with an order parameter $c$. D, however, is prevalent in the early of simulations in all cases. At $c > 4$, the cases where C drastically becomes prevalent with advancing time steps take palce, which is shown in the Fig 4. We will make theoretical analyses of these in the next section.

$$P_c = c + ks = c - tk. \quad (2)$$

The payoff $P_d$ of nodes with D connected to the newly joining node is given by

$$P_d = t + xd = t - xc, \quad (3)$$

where $x$ is a degree of a node with strategy D.

By demanding $P_c > P_d$, the condition that the strategy C of the newly joining node does not turn to D leads to

$$c + ks > t + xd. \quad (4)$$

Fig. 3 $c$ v.s. probability of all C.
After all we obtain a condition
\[ c > \frac{1 + k}{1 + \frac{k}{t}} \] with \( k = t = 5 \). 

(5)

Minimal values of \( x \) which satisfy the inequality (5) for various \( c \) values are given in Table 2. The situation changes at nearly the critical value of \( c \) pointed out in the previous section. Considering that \( x \) is originally meaningful only when \( x \) is integer, because \( x \) is a degree, it leads to the fact that while for \( c > 4.3 \) a newly joining C node can change D nodes with \( x = 6 \) to C nodes when the D nodes are connect with the C nodes, for \( c < 4.3 \) it can do D nodes with only \( x = 7 \). Though the boundary value of \( x \) is yet a magic number, we find that there is a gap between regions with \( c > 4.3 \) and \( c < 4.3 \) how a newly joined strategy C can easily make a strategy D change into C.

\( c > 2 \) is also enough for this C to propagate through D-dominant world. As a natural conjecture, it is considered that the magic number \( x = 6 \) has some significance and there may be a gap in strategy distribution at a point near \( c = 4.3 \). This fact may be also supported by Fig. 3. We pursue this conjecture.

| \( c \) | 2.0 | 3.0 | 3.9 | 4.0 | 4.1 | 4.2 | 4.3 | 4.4 | 4.5 | 4.6 | 4.7 | 4.8 | 4.9 |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| \( x \) | 14  | 9.0 | 6.7 | 6.5 | 6.3 | 6.1 | 6.0 | 5.8 | 5.7 | 5.5 | 5.4 | 5.3 | 5.1 |

Table 2. Minimal values of \( x \) satisfying the inequality (5).

Fig. 4 D-C transition and the ratio of degree= 5 among all nodes in \( n = 600 \) and \( c = 4.3 \).

Fig. 5 Probability function and its enlarged drawing.

First of all, Fig. 4 gives a suggestion to study the conjecture. The ratio of nodes with low degree (\( x < 6 \)) drastically decreases in the right-hand side of Fig. 4 temporarily just when all D world turns to all C world. After that the ratio increases almost smoothly with time steps. A decrease in the population of nodes with low degree induce nodes with the strategy D to the strategy C, according the inequality condition (5). The reason is that a newly joining node is apt to mainly builds linkes with nodes with highly degree due to the decrease.

We consider the condition that a newly joining node connects with nodes with high degrees on a network with a probability of not less than 50 percent. From that, we show that critical value of \( c \) in the C-D transition can be estimated. This is due to the fact that the probability that a node links with a newly joined node depends on total payoff \( R_i \), which also depends on \( c \). Let us introduce a possibility \( p \) that a node has not less than degree \( x \). From the condition that all five \( (k \)
in general) nodes linked with a newly joining node with strategy C have not less than \(x\) degrees, we obtain a constraint on \(p\):

\[
p^5 > 0.5 \quad \Rightarrow \quad p > 0.87. \tag{6}
\]

In the meanwhile, degree distribution for all D world is a linear function, which has phenomenologically a general form (the left-hand side of Fig. 1)

\[Y = \alpha R + \beta \quad \text{with} \quad \alpha = -\frac{2n}{15^2c} < < \beta = \frac{40n}{15^2}, \quad R = kc. \tag{7}\]

Nodes with not less than \(k = 9.4\) degree have a majority in all. This value is also an average one for degree of all nodes which agrees with simulation results not explicitly demonstrated in this article.

In order to specify the condition that a newly joining node with strategy C can link with nodes with high degree with a high probability, we first estimate the normalization factor of the probability in Eq. (1). The total sum of \(P_i\) over all nodes is

\[
S(d) = \sum_{i=1}^{n} \frac{\alpha R_i + \beta}{1 + e^{-R_i/A}} \sim \int_{R_{\min}}^{R_{\max}} \frac{\alpha R + \beta}{1 + e^{-R_i/A}} dR \sim \int_{6d}^{21d} \frac{\beta}{1 + e^{-R_i/A}} dR.
\]

\[
= \beta A \left( \ln[e^{6d/A} + 1] - \ln[e^{21d/A} + 1] \right). \tag{8}
\]

By using Eq. (7), the sum over nodes is changed to the sum over payoffs in the first equality. Since the sum of the above sequence can not be analytically calculated, a continuous approximation is made in Eq. (8). Furthermore using the inequality in Eq. (7), an analytic expression for \(S(d)\) is obtained in the last equality. If the partial set \(\Delta S\) of the gross area noted by \(S\) in Fig. 5 satisfies the following inequality

\[
\frac{\Delta S(d, x)}{S(d)} > 0.87, \tag{9}
\]

degree of nodes connected to a newly joining node with C would be larger than \(x\) in all likelihood. Here the explicit representation of the fraction \(\Delta S\) is given by

\[
\Delta S(d, x) = \int_{21d}^{d(1+x)} \frac{\beta}{1 + e^{-R_i/A}} = \beta A \left( \ln[e^{d(1+x)/A} + 1] - \ln[e^{21d/A} + 1] \right). \tag{10}
\]

Finally we can derive a critical value \(d\) by solving the following equation;

\[
\frac{\Delta S(d, x = k_c)}{S(d)} = 0.87, \tag{11}
\]

Though this equation has two unknown quantities \(k_c\) and \(d\), we can be solved the equation to find the value \(d\) numerically;

\[
\left\{ \begin{array}{l}
d = 0.29 \quad \text{for} \quad x = 7, \quad \text{this is nonsense,} \\
d = -4.14 \quad \text{for} \quad x = 6. \tag{12}
\end{array} \right.
\]

Note that the solution \(d = 0.29\) for degree \(x = 7\) is meaningless, but the mean solution \(d = -4.14\), which accords with Fig.3 well, at last appears for \(x = 6\), that is a just magic number.

### 4.2 Degree Distribution Function

We follow the method adopted by Brabashi et al.\[2\] in order to find a degree distribution function in this model. The corresponding equation to the one used by Brabashi et. al.\[2\] for \(k_i\) in the present model is given by

\[
\frac{\partial k_i(t)}{\partial t} = \frac{m}{S(1 + e^{-R_i/A})} = \frac{m}{S(1 + e^{-rk_i/A})}, \tag{13}
\]

where \(r = c\) or \(d\). \(S\) at a time step \(t\) for both cases, all C and all D, is a linear function of \(t\) as shown in Fig.6;

\[
S \sim \gamma t. \tag{14}
\]

where \(\gamma\) is some proportionality.
Solving the differential Eq. (13) under the initial condition of $t = t_0$, $k_i = m = 5$, we obtain

$$(k_i(t) - m) + \frac{A}{r}(e^{-rm/A} - e^{-rk_i(t)/A}) = \frac{m}{\gamma} \log_e \frac{t}{t_0}. \quad (15)$$

Though we can not give any analytic expression of $k_i(t)$, we have only to find a $t_0$ derivative of $k_i(t)$. According to Barabashi et. al.[2], a distribution function can be given by

$$P(k) = -\frac{1}{\partial k_i(t)/\partial t_0}. \quad (16)$$

So we obtain the below equation:

$$P(k) \sim \frac{\gamma t}{5} \frac{1 + e^{-rk/30}}{e^{(k-5)/5}e^{(1-k)/r}(e^{-5/r} - e^{-rk/30})},$$

$$= \frac{\gamma t}{5} \frac{\coth rk/60}{e^{(k-5)/5}e^{(1-k)/r}} \quad (17)$$

$$\sim O(1) - O(k^4) \text{ for } r = d \quad (18)$$

$$\sim e^{(5-k)/5}[O(1) - O(k^4)] \text{ for } r = c \quad (19)$$

Eq. (18) in the third line shows that $P(k)$ linearly decrease with $k$ approximately in all D world and Eq. (19) roughly shows an exponential damping with respect to $k$ with some correction at in all C world. Both of them are consistent with simulation results in section2. In fact both theoretical estimation coming from Eq. (17) and simulation results are compared for an all D case and an all C case in Fig.8 and 9, respectively. The results of simulations conform with theoretical ones for both cases well.

In such a way, a critical point and behaviour of degree distributions before and after the critical point can be explained theoretically.
4.3 $A$-dependence

We chose $A = 30$ in order to analyze in a linear part of Eq.(1) in the presented model. When the value of $A$, however, changes, how results described in the previous sections differ? In this section we investigate the effects of $A$-value, especially small $A$.

At $A = 10$, the degree distribution function is such triangular in shape as Fig.9, which shows that the degree grows larger as $k$ but drops at much larger $k$. This occurs because the probability $P$ rapidly decreases at $A = 10$ as $k$ grows larger. The movement in the function $P(k)$ at $A = 10$ and $A = 30$ when $c = 4.0$ is shown in Fig.10. When nodes with small $k$ first are connected to a newly joining node with greater probability in D-dominate world. So the population of nodes with a little larger $k$ increases. A similar phenomenon occurs every time newly joining nodes are added to the network. These propagate into larger $k$ step by step. Nodes with excessively large $k$, however, are rarely connected by links to a newly joining nodes conversely, since the probability $P$ rapidly decreases as $k$ grows larger. The population of nodes with mean values of $k$ increses due to that reason, while nodes with larger $k$ are hard to increase.

These facts show that degree distribution functions largely depend on the shape of a probability function.

![Fig.9 Degree distributions for $n = 900$, $k = 5$ and $c = 4.0$ at $A = 10$.](image)

![Fig.10 $P(k)$ at $A = 10$ (the lower curve) and $A = 30$ (the upper curve) at $c = 4.0$.](image)

5 Summary

In this article we proposed a model where game dynamics are working on a network and links are attached preferentially in proportion to total payoffs received by nodes. According to the model, computer simulations are made to estimate degree distributions and analyse time series of strategies. We have found a sort of phase transition occurs with a order parameter $c$. Though we only analyze a few phenomena in this article, but they can be quantitatively explained in the theoretical point of view. The results are sensitive to the shape of a probability function that controls preferential attachment. Since there will be much interesting phenomena that should be explored in models that consider inner interactions among nodes in network growth, much wider properties of them should be investigated in details.

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