Single-particle quantum tunneling in ionic traps

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We describe a proposal to probe the quantum tunneling mechanism of an individual ion trapped in a double-well electromagnetic potential. The time-evolution of the probability of fluorescence measurement of the electronic ground state is employed to characterize the single-particle tunneling mechanism. The proposed scheme can be used to implement quantum information devices.

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Together with nonlocality and wave-vector collapse – the phenomena behind the recent advances in quantum communication [1] and computation [2] – the tunneling mechanism is one of the most intriguing aspects of the microscopic world of particles and their interactions, setting quantum reality apart from classical physics. While nonlocality and wave-vector collapse have been pursued as clues to understanding fundamental quantum phenomena such as the uncertainty principle and the process of quantum measurement [3], quantum tunneling is a valuable mechanism for probing the transition from quantum to classical dynamics [4].

Quantum tunneling at the macroscopic level has attracted much attention in the literature [4–6], firstly owing to its application to SQUIDs (superconducting quantum interference devices), permitted by the advances in cryogenics, and recently because of the realization of Bose-Einstein condensation in dilute atomic gases. The observation of matter-wave in-

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terference fringes has demonstrated that a Bose condensate consists of “laser-like” atoms which are spatially coherent and show long-range correlations, opening the field of coherent atomic beams and of atomic Josephson effect [7].

Probed at the single-particle level, quantum tunneling is just as provocative to our intuition as its manifestation on the macroscopic scale. The experimental techniques developed over the last decade for manipulating electronic and vibrational states of trapped ions can be employed to investigate fundamental quantum effects at a level so far accessible only as collective processes [8]. We describe in this letter a scheme for probing the tunneling mechanism of an individual ion trapped in a double-well electromagnetic potential. The time evolution of the probability of a fluorescence measurement of the electronic ground state is used to probe the single-particle tunneling mechanism.

We first assume  
(i)  
that the ion is trapped in a symmetric double-well single-particle potential \( V(x) = b(x^2 - x_0^2)^2 \). (In ionic traps, the typical oscillation frequencies in the \( y \) and \( z \) directions are very much larger than that in the \( x \) direction, providing a good approximation to a one-dimensional trap.) The minima are given by \( x_0 = \pm \sqrt{d/2b} \), so that the motion can be described as approximately harmonic, with frequency \( \omega_0 = \sqrt{4d/m} \), \( m \) being the ionic mass. The harmonic approximation can be adjusted by fixing \( d \) (and consequently \( \omega_0 \)) and conveniently choosing the parameter \( b \), which is equivalent to varying the height of the barrier separating the two wells, \( h = d^2/4b \). We also assume  
(ii)  
that the parabolic approximation to the potential around each minimum is designed to contain (at least) the two lowest states of the harmonic oscillator described by the wave functions \( \phi_i^{(n)}[x - (-1)^i x_0] \), with \( n = 1, 2 \) referring to the ground and first-excited states of the harmonic oscillator and \( i = 1, 2 \) referring to both harmonic wells centered at \( x = \mp x_0 \). Finally, we assume  
(iii)  
that the potential is such that state \( \phi_1^{(n)} \) (of harmonic trap 1) is spatially close to state \( \phi_2^{(n)} \) (of harmonic trap 2). Thus, these local modes are not orthogonal, due to the overlap \( \epsilon \) between the corresponding modes of the two wells:

\[
\int dx \left( \phi_i^{(n)}[x - (-1)^i x_0] \right)^* \phi_j^{(m)}[x - (-1)^j x_0] = [\delta_{ij} + \epsilon (1 - \delta_{ij})] \delta_{nm}.
\] (1)
When $\epsilon \ll 1$, given a first order correction, these local modes are approximately orthogonal and the eigenstates of the global double-well potential may be approximated by the symmetric and asymmetric superpositions $\phi^{(n)}_{\pm}(x) \approx \left[ \phi^{(n)}_1(x + x_0) \pm \phi^{(n)}_2(x - x_0) \right] / \sqrt{2}$, with corresponding eigenvalues $E^{(n)}_{\pm} = E^{(n)}_0 \pm \mathcal{R}^{(n)}$. The coupling constants between the local states are given by the matrix elements

$$
\mathcal{R}^{(n)} = \int dx \left[ \phi^{(n)}_1(x + x_0) \right]^* \left[ V(x) - \tilde{V}_1 [x + x_0] \right] \phi^{(n)}_2(x - x_0),
$$

where $\tilde{V}_1 [x - (-1)^l x_0]$ indicate the parabolic potential approximations around $x = (-1)^l x_0$. The tunneling frequencies $\Omega^{(n)}$ between the corresponding eigenstates in $\tilde{V}_1$ and $\tilde{V}_2$ are given by $\Omega^{(n)} = 2 |\mathcal{R}^{(n)}| / \hbar$. In order to make the first order correction ($\epsilon^1$) valid, ensuring that $\mathcal{R}^{(n)}/E_0^{(n)} \ll 1$, we further impose the condition $x_0^2/\Delta_x^2 \gg 1$. Scaling the length in units of the position uncertainty in a harmonic oscillator ground state $\Delta_x^2 = \hbar/(2m\omega_0)$, we obtain from Eq. (2) $\Omega^{(1)} = (3\omega_0x_0^2/8\Delta_x^2) \exp(-x_0^2/2\Delta_x^2)$ and $\Omega^{(2)} = (x_0^2/\Delta_x^2)^2 \Omega^{(1)}$.

The Hamiltonian describing the motional degrees of freedom of the ion trapped in the above-described double-well potential is

$$
\hat{H}_{\text{motional}} = \int dx \hat{\Psi}^\dagger \hat{H} \hat{\Psi}, \quad H = -\frac{\hbar^2}{2m} \nabla^2 + V(x),
$$

where $\hat{\Psi}(x) = \sum_{n,i} \phi^{(n)}_i(x) \hat{c}^{(n)}_i$ is the field operator which annihilates the ion in the state $\phi^{(n)}_i$, i.e., in energy level $E^{(n)}_0$ of the harmonic well $\tilde{V}_1$. From the eigenvalue equation $H \phi^{(n)}_{\pm} = E^{(n)}_{\pm} \phi^{(n)}_{\pm}$ and the assumption that $\epsilon \ll 1$, the Hamiltonian in Eq. (3) becomes

$$
\hat{H}_{\text{motional}} = \sum_n \left[ E^{(n)}_0 \left( \hat{c}^{(n)}_1 \hat{c}^{(n)}_1 + \hat{c}^{(n)}_2 \hat{c}^{(n)}_2 \right) + \mathcal{R}^{(n)} \left( \hat{c}^{(n)}_1 \hat{c}^{(n)}_2 + \hat{c}^{(n)}_2 \hat{c}^{(n)}_1 \right) \right].
$$

Next, we assume that the trapped ion has two effective electronic states, excited $|\uparrow\rangle$ and ground $|\downarrow\rangle$, separated by frequency $\omega$ and coupled by the interaction with an effective laser plane wave propagating in the $x$ direction, with wave vector $k_L = \omega_L/c$. In this configuration, only the ionic motion along the $x$ axis will be modified. The effective pumping laser beam, is detuned by $\delta \equiv \omega - \omega_L$ from the $|\uparrow\rangle \leftrightarrow |\downarrow\rangle$ transition. In NIST experiments with $^{9}\text{Be}^+$ the laser beams (with different frequencies) composing the effective pumping laser, are...
detuned from a third more excited level which, in the stimulated Raman-type configuration, is adiabatically eliminated [8]. In the Innsbruck experiments with \(^{40}\text{Ca}^+\), a coherently driving direct transition between the electronic ground and excited states is employed [9]. A probe laser, strongly coupled to the transition between the electronic ground state and a third more excited level \(|s\rangle\), is also employed in order to measure the ionic vibrational state by collecting the resonance fluorescence signal, which is the probability of the ion being found in the internal state \(|\downarrow\rangle\).

We can trail the tunneling phenomenon through the time-evolution of the probability of fluorescence measurement of the electronic ground state collected in two different situations: a) by individual addressing the ion in harmonic well 1 or, what seems to be more attractive experimentally, b) considering the laser beams (the pumping and the probe lasers) to reach both local potential wells simultaneously. The ion-laser interaction Hamiltonian that describes the effective interaction of the quantized motion of the ionic center-of-mass coupled to its electronic degrees of freedom is [8]

\[
\hat{H}_{\text{ion-laser}} = \hbar g \left( \hat{\sigma}_+ e^{ik_L \hat{x} - i\omega_L t + i\phi} + \hat{\sigma}_- e^{-ik_L \hat{x} + i\omega_L t - i\phi} \right),
\]

where \(\sigma_+ = |\uparrow\rangle \langle \downarrow|\) and \(\sigma_- = |\downarrow\rangle \langle \uparrow|\) are the usual Pauli pseudo-spin operators, \(g\) is the effective Rabi frequency of the transition \(|\uparrow\rangle \leftrightarrow |\downarrow\rangle\) and \(\phi\) is the phase difference between the two lasers composing the effective beam. Since, considering the situation a), the laser beams are disposed in harmonic well 1, the position operator \(\hat{x}\) is given by

\[
\hat{x} = \sum_{n,m} \int dx \phi_1^{(n)}(x) x \phi_1^{(m)}(x) \frac{c_1^{(n)*} c_1^{(m)}}{c_1^{(n)} c_1^{(m)}}
= \Delta x \sum_n \sqrt{n+1} \left( \frac{c_1^{(n)*} c_1^{(n+1)}}{c_1^{(n)} c_1^{(n+1)}} + \frac{c_1^{(n+1)*} c_1^{(n)}}{c_1^{(n+1)} c_1^{(n)}} \right).
\]

Evidently, considering the situation b) where the laser beams reach both local wells, the position operator turn to be \(\hat{x} = \Delta x \sum_{j=1,2} \sum_n \sqrt{n+1} \left( \frac{c_j^{(n)*} c_j^{(n+1)}}{c_j^{(n)} c_j^{(n+1)}} + \frac{c_j^{(n+1)*} c_j^{(n)}}{c_j^{(n+1)} c_j^{(n)}} \right).

In what follows we consider only the situation a) since the generalization for the situation b), which will be analyzed further, is straightforward. In a frame rotating at the effective laser frequency \(\omega_L\), the ion-laser Hamiltonian is given by
\[ \hat{H}_{\text{ion-laser}} = \hbar g \left[ \hat{\sigma}_+ \exp \left( i\eta \sum_n \sqrt{n+1} \left( \hat{c}_1^{(n)} \hat{c}_1^{(n+1)} + \hat{c}_1^{(n+1)} \hat{c}_1^{(n)} \right) + i\varphi_L \right) + \text{H.c.} \right], \] (7)

where we have introduced the Lamb-Dicke parameter \( \eta \equiv k_L \Delta_x \).

In what follows we rewrite the total Hamiltonian in the interaction picture (script font labels) via the unitary transformation \( \hat{U}(t) = \exp \left( -i\hat{H}_0 t \right) \), where \( \hat{H}_0 = \sum_n E_0^{(n)} \left( \hat{c}_1^{(n)\dagger} \hat{c}_1^{(n)} + \hat{c}_2^{(n)\dagger} \hat{c}_2^{(n)} \right) + \hbar \delta \hat{\sigma}_z \) indicates the free Hamiltonian composed of the internal and motional degrees of freedom of the trapped ion, \( \sigma_z \) being the Pauli operator. The resulting Hamiltonian, including the motional and electronic degrees of freedom, reads

\[ \hat{\mathcal{H}} = \sum_n \mathcal{R}^{(n)} \left( \hat{c}_1^{(n)\dagger} \hat{c}_2^{(n)} + \hat{c}_2^{(n)\dagger} \hat{c}_1^{(n)} \right) + \hbar g \exp \left( -2\pi \hbar \left( \sigma_+ \sum_n \sqrt{n+1} \left( \hat{c}_1^{(n)\dagger} \hat{c}_1^{(n+1)} e^{-i\omega_0 t} + \hat{c}_1^{(n+1)\dagger} \hat{c}_1^{(n)} e^{i\omega_0 t} \right) \right) \right), \] (8)

In order to simplify the expression (8) we (i) adjust the ion-laser detuning to the first red sideband for the ion-laser interaction \( (\delta = \omega_0) \). In addition, (ii) the assumption of the standard Lamb-Dicke limit \( (\eta \ll 1) \), where the ionic center of mass is strongly localized with respect to the laser wavelength, enables the expansion of the Hamiltonian (8) to first order correction \( (\eta^1) \). Finally, (iii) with the optical rotating wave approximation, we obtain the expression

\[ \hat{\mathcal{H}} = \sum_n \mathcal{R}^{(n)} \left( \hat{c}_1^{(n)\dagger} \hat{c}_2^{(n)} + \hat{c}_2^{(n)\dagger} \hat{c}_1^{(n)} \right) + \hbar \eta g \exp \left( -\eta^2 \sum_n \sqrt{n+1} \left( \hat{c}_1^{(n)\dagger} \hat{c}_1^{(n+1)} e^{i\varphi_L} + \hat{c}_1^{(n+1)\dagger} \hat{c}_1^{(n)} e^{-i\varphi_L} \right) \right), \] (9)

which in the absence of the tunneling mechanism leads to the coupling between the electronic and motional degrees of freedom of the trapped ion, described by the Jaynes-Cummings Hamiltonian (JCH).

At this point, we assume that the ion is initially cooled to its motional ground state in harmonic well 1, and laser-excited to the electronic state \( |\uparrow\rangle \). Thus, the JCH induces the transition \( |1,0\rangle_1 |\uparrow\rangle \leftrightarrow |0,1\rangle_1 |\downarrow\rangle \), where the ket \( |1,0\rangle_1 \) \( (|0,1\rangle_1) \) indicates the ion in the motional ground (excited) state of \( \hat{V}_1 \). Evidently, being in the motional state \( |1,0\rangle_1 \) \( (|0,1\rangle_1) \) the ion has a finite probability of tunneling to the corresponding state of \( \hat{V}_2 \): \( |1,0\rangle_2 \) \( (|0,1\rangle_2) \).
Therefore, the motional basis states of the process are restricted to the ground and first excited states of the harmonic traps, which is why we assumed from the beginning that the parabolic approximation to the potential around each minimum is designed to contain (at least) the two lowest states of the harmonic oscillator. By collecting the resonance fluorescence signal in $\tilde{V}_1$, we can probe the tunneling mechanism through the behavior of the function $P_\downarrow(t)$: the probability of fluorescence measurement of the electronic ground state.

Considering the Lamb-Dicke limit and the fact that we have only one particle in the process, the function $\exp\left(-\eta^2 \sum_n \hat{c}_1^{(n)\dagger} \hat{c}_1^{(n)}\right)$ can be fairly approximated by unity, so the Hamiltonian governing the process has the simplified form (for $n = 1, 2$)

$$\hat{H} = \mathcal{R}^{(1)} \left( \hat{c}_1^{(1)\dagger} \hat{c}_2^{(1)} + \hat{c}_1^{(1)} \hat{c}_2^{(1)\dagger} \right) + \mathcal{R}^{(2)} \left( \hat{c}_1^{(2)\dagger} \hat{c}_2^{(2)} + \hat{c}_1^{(2)} \hat{c}_2^{(2)\dagger} \right)$$

$$+ \hbar g \left( \sigma_+ \hat{c}_1^{(1)\dagger} \hat{c}_1^{(2)} e^{+i\varphi_L} - \sigma_- \hat{c}_1^{(1)} \hat{c}_1^{(2)\dagger} e^{-i\varphi_L} \right).$$

(10)

Before proceeding further, it is worth mentioning that instead of adjusting the ion-laser detuning to the first red sideband for the ion-laser interaction ($\delta = \omega_0$), we could have chosen the carrier interaction where $\delta = 0$. In this case the effective Hamiltonian following from Eq. (8) (with the above-mentioned approximations) simplifies to

$$\hat{H}_{\text{carrier}} = \mathcal{R}^{(1)} \left( \hat{c}_1^{(1)\dagger} \hat{c}_2^{(1)} + \hat{c}_1^{(1)} \hat{c}_2^{(1)\dagger} \right) + \mathcal{R}^{(2)} \left( \hat{c}_1^{(2)\dagger} \hat{c}_2^{(2)} + \hat{c}_1^{(2)} \hat{c}_2^{(2)\dagger} \right)$$

$$+ \hbar g \left( \sigma_+ e^{+i\varphi_L} + \sigma_- e^{-i\varphi_L} \right).$$

(11)

We note that the Pauli operators act only when the ion is in harmonic well 1. The dynamics governed by Hamiltonians (10) and (11) make it possible to probe the tunneling mechanism of the ion by collecting resonance fluorescence signals, as discussed above.

Starting from the initial state $|1, 0\rangle_{1} |0, 0\rangle_{2} |\uparrow\rangle$ and Hamiltonian (10) we obtain from the Schrödinger evolution the evolved state

$$|\psi(t)\rangle = \left[ C_1^{(1)}(t) |1, 0\rangle_{1} |0, 0\rangle_{2} + C_2^{(1)}(t) |0, 0\rangle_{1} |1, 0\rangle_{2} \right] |\uparrow\rangle$$

$$+ \left[ C_1^{(2)}(t) |0, 1\rangle_{1} |0, 0\rangle_{2} + C_2^{(2)}(t) |0, 0\rangle_{1} |0, 1\rangle_{2} \right] |\downarrow\rangle,$$

(12)
whose coefficients satisfy the set of coupled linear equations

\[
\begin{align*}
\frac{ih}{\hbar} \frac{d}{dt} C_1^{(n)}(t) &= \mathcal{R}^{(n)} C_2^{(n)}(t) - (-1)^n i \hbar \eta g e^{-(1)^n i \varphi_L} C_1^{(m)}(t), \\
\frac{ih}{\hbar} \frac{d}{dt} C_2^{(n)}(t) &= \mathcal{R}^{(n)} C_1^{(n)}(t),
\end{align*}
\]

(13a, 13b)

with \(n, m = 1, 2\) (\(n \neq m\)) and the initial conditions \(C_1^{(1)}(0) = 1, C_2^{(1)}(0) = C_1^{(2)}(0) = C_2^{(2)}(0) = 0\). Now, we introduce a further simplification into the present scheme. The larger the ratio \(x_0^2/\Delta_x^2\) obtained by engineering the trap, the smaller becomes the ratio of the matrix elements \(|\mathcal{R}^{(1)}/\mathcal{R}^{(2)}|\). Therefore, when \(|\mathcal{R}^{(1)}/\mathcal{R}^{(2)}| \ll 1\) we can neglect the tunneling process between the ground states in the two harmonic wells. In this case, the first term in Hamiltonian (10) can be disregarded and, by adjusting the phase of the laser pulse such that \(\varphi_L = -\pi/2\), we obtain the coefficients:

\[
\begin{align*}
C_1^{(1)}(t) &= \left[ \cos(\xi w t) + \xi^2 - 1 \right] / \xi^2, \quad C_2^{(1)}(0) = 0, \\
C_1^{(2)}(t) &= i \sin(\xi w t) / \xi, \quad C_2^{(2)}(0) = \sqrt{\xi^2 - 1} [\cos(\xi w t) - 1] / \xi^2,
\end{align*}
\]

(14a, 14b)

where the effective Rabi frequency, \(w = \eta g\), is modified by the parameter \(\xi = \left\{ 1 + \left[ \mathcal{R}^{(2)}/(\hbar \eta g) \right]^2 \right\}^{1/2} \). Next we analyze the influence of parameter \(\xi\) on the time-evolution of the probability of measuring fluorescence of the electronic ground state \(P_\downarrow(t) = |C_1^{(2)}(t)|^2\) in harmonic well 1. Evidently, when \(\mathcal{R}^{(2)} = 0\), we recover the well-known dynamics of \(P_\downarrow(t)\) for JCH [8]. For the choice \(\mathcal{R}^{(2)}/(\hbar \eta g) = 1\) (\(\xi = \sqrt{2}\)) and employing the typical values in experiments with \(^{40}\text{Ca}^+ [9,10]\) \(\eta \approx 0.1\), \(g \approx 200\) kHz, and \(\omega_0 \approx 2\) MHz (giving the estimate \(\mathcal{R}^{(2)}/(\hbar \eta g) \approx 150 \times \mathcal{R}^{(2)}/E_0^{(2)}\)), we obtain \(\mathcal{R}^{(2)}/E_0^{(2)} \approx 7 \times 10^{-3}\) as required to justify the approximation that the superpositions \(\phi^\pm_\downarrow(x)\) constitute eigenstates of the global double-well potential. In fact, remembering that \(\mathcal{R}^{(2)}/(\hbar \eta g) = (3 \omega_0 x_0^4/16 \eta g \Delta_x^4) \exp(-x_0^2/2\Delta_x^2)\), we obtain from the previous parameters the values \(x_0^2/\Delta_x^2 \approx 17.3\) and \(|\mathcal{R}^{(1)}/\mathcal{R}^{(2)}| \approx 6 \times 10^{-2}\) which are in agreement with the approximations considered above. In Fig. 1 we display the behavior of function \(P_\downarrow(t)\) for \(\mathcal{R}^{(2)} = 0\) (\(\xi = 1\), corresponding to the dynamics of the JCH) and \(\mathcal{R}^{(2)}/(\hbar \eta g) = 1\). As anticipated by observing the expression for \(P_\downarrow(t)\), the increase in the tunneling rate \(\mathcal{R}^{(2)}\) leads to an increase in the effective frequency \(\xi w\) of population inversion and,
conversely, a decrease in the amplitude of the oscillations of $P_\downarrow(t)$, clearly indicating the tunneling process. In fact, as soon as the ion reaches the excited state of local well 1, the coupling $\mathcal{R}^{(2)}$ to local well 2 entangles the motional excited states of both wells, preventing the probability $P_\downarrow(t)$ (associated with the measurement of state $|0, 1\rangle_1 |\downarrow\rangle$) from reaching unity. When the curve in Fig. 1 for $\mathcal{R}^{(2)}/(\hbar \eta g) = 1$ reaches its maxima ($\omega t = n\pi \sqrt{2}/4$, $n = 1, 2, ...$), we have the entangled state $\frac{1}{2} |1, 0\rangle_1 |0, 0\rangle_2 |\uparrow\rangle + \left(\frac{i}{\sqrt{2}} |0, 1\rangle_1 |0, 0\rangle_2 - \frac{1}{2} |0, 0\rangle_1 |0, 1\rangle_2 \right) |\downarrow\rangle$. Therefore, both characteristics, the effective frequency $\xi_w$ of population inversion and the amplitude of the oscillations of $P_\downarrow(t)$, can be used to probe the single-particle tunneling mechanism.

It is worth observing that the choice of the alternative initial state $|0, 1\rangle_1 |0, 0\rangle_2 |\downarrow\rangle$ leads to the entanglement $\left(|1, 0\rangle_1 |0, 0\rangle_2 |\uparrow\rangle + |0, 0\rangle_1 |0, 1\rangle_2 |\downarrow\rangle\right)/\sqrt{2}$. In this situation, as expected for a closed system, we can re-establish a value of unity for $P_\downarrow(t)$.

Now, turning to the Hamiltonian (11) and solving the Schrödinger equation for the initial state $|0, 1\rangle_1 |0, 0\rangle_2 |\uparrow\rangle$, we obtain the result

$$P_\downarrow(t) = \frac{\left(\lambda_1 \sin (\lambda_1 gt) - \lambda_2 \sin (\lambda_2 gt)\right)^2}{(\lambda_2^2 - \lambda_1^2)^2},$$

(15)

where the parameters modifying the Rabi frequency, $g$, are $\lambda_i = \frac{1}{\sqrt{2}} \left\{1 + 2 \left[\mathcal{R}^{(2)}/(\hbar g)\right]^2 + (-1)^i \left[1 + 4 \left[\mathcal{R}^{(2)}/(\hbar g)\right]^2\right]^{1/2}\right\}^{1/2}$. Evidently, when $\mathcal{R}^{(2)} = 0$, we get the usual dynamics for the carrier pulse, with $P_\downarrow(t) = |\sin (gt)|^2$. In Fig. 2 we display the time evolution of Eq. (15), for $\mathcal{R}^{(2)} = 0$ and $\mathcal{R}^{(2)}/(\hbar g) = 1 \left(\lambda_i = \left[(3 + (-1)^i \sqrt{5})/2\right]^{1/2}\right)$. For the typical values given above, we obtain $\mathcal{R}^{(2)}/E_0^{(2)} \approx 7 \times 10^{-2}$ (a value which match $x_0^2/\Delta x^2 \approx 10.8$ and $|\mathcal{R}^{(1)}/\mathcal{R}^{(2)}| \approx 0.1$). Similarly to the behavior displayed in Fig. 1, we observe that the frequency of population inversion is higher than the Rabi frequency $g$ associated with free carrier dynamics. However, differently from the situation in Fig. 1, the probability $P_\downarrow(t)$ (displaying the characteristic beat pattern due to Eq. (15)) can still reach unity. Since in the carrier regime the electronic states do not couple to the motional states, the excited and ground electronic states are both subject to the tunneling process (differently from the JCH case), resulting in the interference pattern shown in Fig. 2.

From the value for the position uncertainty in a harmonic oscillator ground state $\Delta x^2 =$
\( h/(2m\omega_0) \) computed for \( ^{40}\text{Ca}^+ \) with motional frequency \( \omega_0 \approx 2 \text{ MHz} \), and considering the Jaynes-Cummings regime, where \( x_0^2/\Delta_x^2 \approx 17.3 \), we obtain (using the typical values \( \eta \approx 0.1 \) and \( g \approx 200 \text{ kHz} \) [9,10]) the distance between the local minima \( 2x_0 \approx 0.16 \text{ \( \mu \)m}. \) Making the same estimates for the Carrier regime, where \( x_0^2/\Delta_x^2 \approx 10.3 \), we obtain the distance between the local minima \( 2x_0 \approx 0.13 \text{ \( \mu \)m}. \) Recently, it was reported in Ref. [10] the laser addressing of individual ions in a linear ion trap with frequency \( 125 \text{ kHz} \) when the distance between the ions is about \( 19.6 \text{ \( \mu \)m}. \) The authors of Ref. [10] argue that the addressing technique permits individual addressing when the distance between the ions is only \( 7.6 \text{ \( \mu \)m} \) with small error. Therefore, with nowadays technology it is difficult to individually address each local potential well. On the other hand, when considering the situation (\( b) \) where the laser beams reach the local wells simultaneously, the Hamiltonian (10) (in Jaynes-Cummings regime and assuming \( |\mathcal{R}^{(1)}/\mathcal{R}^{(2)}| \ll 1 \) turns to be

\[
\hat{\mathcal{H}} = \mathcal{R}^{(2)} \left( \hat{c}_1^{(2)\dagger} \hat{c}_2^{(2)} + \hat{c}_1^{(2)} \hat{c}_2^{(2)\dagger} \right) + i\hbar \eta \left[ \sigma_+ \left( \hat{c}_1^{(1)\dagger} \hat{c}_1^{(2)} + \hat{c}_2^{(1)\dagger} \hat{c}_2^{(2)} \right) e^{+i\varphi_L} - \sigma_- \left( \hat{c}_1^{(1)} \hat{c}_1^{(2)\dagger} + \hat{c}_2^{(1)} \hat{c}_2^{(2)\dagger} \right) e^{-i\varphi_L} \right].
\]

Starting from the initial state \( |1,0\rangle_1 |0,0\rangle_2 |\uparrow\rangle \) and Hamiltonian (16) we obtain from the Schrödinger evolution the evolved state described by Eq. (12) with the coefficients satisfying the coupled linear equations

\[
(i\hbar) \frac{d}{dt} C_n^{(1)}(t) = i\hbar g e^{i\varphi_L} C_n^{(2)}(t),
\]

\[
(i\hbar) \frac{d}{dt} C_n^{(2)}(t) = \mathcal{R}^{(2)} C_n^{(1)}(t) - i\hbar g e^{-i\varphi_L} C_n^{(1)}(t),
\]

with \( n, m = 1, 2 \ (n \neq m) \) and the initial conditions \( C_1^{(1)}(0) = 1, C_2^{(1)}(0) = C_1^{(2)}(0) = C_2^{(2)}(0) = 0 \). Solving the system above we obtain the coefficients:

\[
C_1^{(1)}(t) = \frac{1}{(\chi^2 + 4) \sum_{j=1,2} \left[ \chi^2 - (\lambda_j)^2 + 3 \right] \cos (\lambda_j g t)},
\]

\[
C_2^{(1)}(t) = \frac{i}{\chi(\chi^2 + 4) \sum_{j=1,2} \lambda_j \left[ (\chi^2 + 2)^2 - (\chi^2 + 2)(\lambda_j)^2 \right] \sin (\lambda_j g t)},
\]
\[ C_1^{(2)}(t) = \frac{e^{-i\phi L}}{(\chi^2 + 4)} \sum_{j=1,2} \lambda_j \left[ (\lambda_j)^2 - \chi^2 - 3 \right] \sin(\lambda_j g t), \]  

\[ C_2^{(2)}(t) = \frac{-ie^{-i\phi L}}{\chi(\chi^2 + 4)} \sum_{j=1,2} \left[ 2 (\lambda_j)^2 - \chi^2 - 2 \right] \cos(\lambda_j g t), \]  

where \( \lambda_j = \left\{ 1 + \left[ \chi^2 + (-1)^j \chi \sqrt{4 + \chi^2} \right] / 2 \right\}^{1/2} \) and \( \chi = \mathcal{R}^{(2)}/\hbar \eta g \). The probability of measuring the electronic ground state \( |\downarrow\rangle \) of the ion in both local wells 1 and 2 is given by 

\[ P_{\downarrow}(t) = \left| C_1^{(2)}(t) + C_2^{(2)}(t) \right|^2. \]  

In Fig. 3 we display the behavior of function \( P_{\downarrow}(t) \) for \( \mathcal{R}^{(2)} = 0 \) (corresponding to the well know Jaynes-Cummings dynamics) and \( \mathcal{R}^{(2)}/(\hbar \eta g) = 1 \) (\( \lambda_1 = \sqrt{3 - \sqrt{5}} / 2 \) and \( \lambda_2 = \sqrt{3 + \sqrt{5}} / 2 \)). Similarly to the behavior displayed in Fig. 1, we observe that the frequency of population inversion in Fig. 3 (full line) is higher than the Rabi frequency \( g \) associated with the free JCH dynamics whereas the amplitude of the oscillations of \( P_{\downarrow}(t) \) is smaller than that for the free JCH dynamics. Both characteristics clearly indicate the tunneling process in Fig. 3, without the requirement of the individual addressing of each local potential well.

In conclusion, we have presented a scheme for probing the tunneling process of a single ion trapped in a double-well electromagnetic potential. The tunneling dynamics is characterized by the behavior of the probability of collecting an electronic ground state fluorescence signal in one of the local potential wells. Two situations were analyzed, when collecting fluorescence measurement a) by individual addressing the ion in harmonic well 1, and b) considering the laser beams to reach the local wells simultaneously. Although the situation a) is beyond nowadays technology, the single-particle quantum tunneling can clearly be probed through the situation b). Considering the possibility of individual addressing of local wells, the present proposal can be extended to the implementation of the fundamental controlled-NOT two-bit gate by adding another laser beam in harmonic well 2 and storing the quantum bits in the motional states of both local wells [11]. We also note that our proposal can be implemented in spite of the decoherence processes coming from the coupling of the motional modes with the residual background gas [12] and with classical stochastic electric field [13], in addition to the finite lifetime of the electronic levels [13,14]. From the experimental results
reported in [8,9] we observe that several Rabi oscillations are visible within the decoherence time making it possible to observe the signature of the tunneling process. However, we stress that the designing of the double-well potential would require that the trap electrodes be only few micron away from the ion; then it is likely that the heating rate in the trap becomes higher than that observe in single traps, increasing the decoherence rate [15]. Besides, the required trap structure makes difficult to obtain high trap frequencies, about MHz, and consequently, makes difficult the cooling process of the ion to its motional ground state [15]. Here we note that the cooling process could be implemented in a single trap structure which could be adiabatically modified to a double-well potential.

Although the design of such a double-well trap may turn out to be a considerable technical challenge, the fundamental principles discussed here can be implemented by engineering two-mode interactions in ion traps [16], in which the motional states of the ion in two different directions can be coupled, analogously to the dynamics for the double-well. Besides, the proposal here presented might provide a motivation for future experimental work.

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Figure Captions

Figure 1. The evolution of function $P_\downarrow(t)$, in the Jaynes-Cumming regime and individual addressing of local well 1, for the values $R^{(2)}/(\hbar\eta g) = 0$ (dotted line) and 1 (full line).

Figure 2. The evolution of function $P_\downarrow(t)$, in the carrier regime and individual addressing of local well 1, for the values $R^{(2)}/(\hbar g) = 0$ (dotted line) and 1 (full line).

Figure 3. The evolution of function $P_\downarrow(t)$, in the Jaynes-Cumming regime and the laser beams reaching both local wells simultaneously, for the values $R^{(2)}/(\hbar\eta g) = 0$ (dotted line) and 1 (full line).