LINEAR REPRESENTATIONS OF RANDOM GROUPS

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ABSTRACT. We show that for a fixed $k \in \mathbb{N}$, Gromov random groups with any density $d > 0$ have no non-trivial degree $k$ representations over any field, a.a.s. This is especially interesting in light of the results of Agol, Ollivier and Wise that when $d < \frac{1}{6}$ such groups have a faithful linear representation over $\mathbb{Q}$, a.a.s.

1. INTRODUCTION

Gromov random groups. Let $m \geq 2$ and let $F$ be the free group on $m$ generators $x = \{x_1, \ldots, x_m\}$. For $l \in \mathbb{N}$, let $S_l$ be the sphere of radius $l$ in the Cayley graph of $F$ with respect to $X$, i.e. the set of reduced words in $x_i^{\pm 1}$ of length $l$. Fix some $d \geq 0$ and let $R$ be a random subset of $S_l$ constructed by taking $|S_l|^d = (2m(2m-1)^{(l-1)d})$ elements of $S_l$ uniformly, independently and with repetitions. The group $\Gamma = \langle x \mid R \rangle$ i.e. the group presented by the generators $x$ and the relators $R$ is called a “Gromov random group of density $d$ with $m$ generators and relators of length $l$”. For a group property $P$, we say that Gromov random groups satisfy $P$ asymptotically almost surely (a.a.s.) if the probability of $P$ goes to 1 as $l \to \infty$. In a formula,

$$\lim_{l \to \infty} \mathbb{P}(\Gamma \text{ satisfies } P) = 1.$$ 

See [4] for an invitation to the topic. The goal of this note is to prove

Theorem 1. Let $k \geq 1$, $m \geq 2$ and $d > 0$. Then Gromov random groups $\Gamma$ at density $d$ with $m$ generators satisfy a.a.s. that for any field $F$ and any $\rho : \Gamma \to \text{GL}_k(F)$, $|\rho(\Gamma)| \leq 2$.

In fact, we prove that polynomially many relators are enough for this property, see the formulation of Theorem 8 below.

When $l$ is odd, it is easy to see that $\mathbb{Z}/2\mathbb{Z}$ is a.a.s. not a quotient of $\Gamma$ hence in fact we may strengthen Theorem 1 to state that $\rho(\Gamma) = \{1\}$. Similarly, when $l$ is even $\mathbb{Z}/2\mathbb{Z}$ is (deterministically) a quotient of $\Gamma$ so the possibility of an image of size 2 cannot be removed.

We recall the well-known result of Gromov [4, §V] that for a fixed $m \geq 2$, $\Gamma$ is a.a.s. an infinite hyperbolic group for $d < \frac{1}{2}$, while $|\Gamma| \leq 2$ for $d > \frac{1}{2}$. So our theorem is of interest only for $d \leq \frac{1}{2}$. But it is especially interesting for $d < \frac{1}{5}$.

In this case Agol [1] and Ollivier and Wise [5] proved the following remarkable result:
Theorem 2. For a fixed $m \geq 2$ and $d < \frac{1}{6}$, the random group $\Gamma$ is a.a.s. linear over $\mathbb{Z}$, i.e. has a faithful representation into $\text{GL}_k(\mathbb{Z})$, for some $k \in \mathbb{N}$.

Thus the main difference between these two results is whether $k$, the degree of linearity, is allowed to depend on $l$, the length of the relators, or not. If it is allowed, we are in the case of Theorem 2 and a representation exists. If it is fixed, we are in the case of Theorem 1 and no representation exists.

A remark on the field: while Theorem 2 constructs a representation into $\mathbb{Q}$, in fact it implies arbitrarily large representations for any field. We cannot show this by taking the representation into $\text{GL}_k(\mathbb{Z})$ modulo $p$ as that might be trivial. But we can, instead, use the fact that any subgroup of $\text{GL}_k(\mathbb{Z})$ is residually finite (simply because $\cap m \ker(\text{GL}_k(\mathbb{Z}) \to \text{GL}_k(\mathbb{Z}/m\mathbb{Z})) = \{1\}$) so has arbitrarily large finite quotients. These finite quotients may be embedded into a symmetric group, hence for some $k'$ it will embed (as permutation matrices) into $\text{GL}_{k'}(F)$ for any $F$.

2. Algebraic geometry preliminaries

The proof uses some results from algebraic geometry. We will now survey briefly the notions and results we need, assuming only that the reader is familiar with undergraduate algebra.

Let $F$ be an algebraically closed field of any characteristic, and let $n \geq 0$. A subset $W$ of the affine space $\mathbb{A}^n := F^n$ is called an (affine) variety if

$$W = \bigcap_{i=1}^k Z(p_i) \quad Z(p) = \{x \in \mathbb{A}^n : p(x) = 0\},$$

where $p_1, \ldots, p_k$ are polynomials in $n$ variables. We will use the notations $F$ (for the underlying algebraically closed field) and $Z(p)$ throughout the paper.

A variety is called irreducible if it cannot be written as a union of two proper subvarieties. Any variety can be written as a finite union of irreducible varieties. Assuming that the representation is not redundant (i.e. if $W = \bigcup X_i$ then $X_i \nsubseteq X_j$ for any $i \neq j$), it is unique. The $X_i$ of this unique representation are called the irreducible components of $W$. See [6, theorems 1.4 & 1.5]. Let us remark that in some of the literature, including [3, 6], a variety is defined to be automatically irreducible. But for us it will be convenient to define it as above.

For any affine variety one can define its dimension, denoted by dim. Heuristically it corresponds with the natural notion of dimension, but the formal definition requires some preliminaries which we prefer to skip. The reader may consult [6, chapter 1, §6]. We will need the following properties of it:

(i) $\dim(W) \in \{-1,0,1,2,\ldots\}$.
(ii) $\dim(\mathbb{A}^n) = n$.
(iii) $\dim(W) = -1$ only for $W = \emptyset$ and $\dim(W) = 0$ implies that $W$ is finite.
(iv) If $W$ is an irreducible algebraic variety with $\dim(W) = k$ and if $p$ is a polynomial not identically zero on $W$, then any irreducible component of $W \cap Z(p)$ has dimension $k - 1$.

See [6, Corollary 1.13] for this last, and most remarkable property. (Note that theorem numbering in the third edition of Shafarevich is different from those of the previous editions).
The following result will be referred to as “Bézout’s theorem” (the literature is abound with results called “Bézout’s theorem”, some of them very close in formulation to it, so we are certainly following tradition here).

**Theorem 3 (Bézout).** Let $W$ be an affine variety defined by polynomials $f_1, f_2, \ldots, f_m$ in $n$ variables i.e. $W = \cap Z(f_i) \subset \mathbb{A}^n$. Suppose $\deg f_i \leq d$ for all $i$. Then the number of irreducible components of $W$ is bounded by $d^{\min(n,m)}$.

This result is well-known, even classic. And yet we could not find a reference to it in this form. Hence we supply a proof.

**Proof.** The literature is far more complete for *projective* varieties. Hence our first step will be to define the projective space $\mathbb{P}^n$ and show how the projective Bézout theorem implies the affine one (we hope no confusion will arise from the use of $\mathbb{P}$ irreducible, and of course $\mathbb{A}$ projective variety). The dimension of a projective variety is as for

The projective space $\mathbb{P}^n$ over a field $F$ is the space $F^{n+1} \setminus \{0\}$ (we consider the coordinates $0, \ldots, n$), modulo the relation $v \sim av$ for every $v \in F^{n+1} \setminus \{0\}$ and $a \in F \setminus \{0\}$. A projective variety is the intersection of zeroes of homogeneous polynomials. Irreducible projective varieties are defined like affine ones, and the decomposition result that allows to define irreducible components is as in the affine case ([6, page 46] claims that “the proof carries over word-for-word”). We will need two maps between subvarieties of $\mathbb{A}^n$ and $\mathbb{P}^n$. The first, restriction, takes the projective variety $\cap Z(f_i), f_i$ homogeneous polynomials in $x_0, \ldots, x_n$ and maps it to the affine variety $\cap Z'(g_i)$ where $g_i(x_1, \ldots, x_n) = f_i(1, x_1, \ldots, x_n)$. The second, homogenisation, maps an affine variety $\cap Z(g_i)$ into the projective variety $\cap Z(f_i)$ where $f_i$ are produced from $g_i$ by taking every monomial $ax_1^{b_1} \cdots x_n^{b_n}$ of $g_i$ and mapping it to $ax_0^{b_0} \cdots x_n^{b_n}$ where $b_0 = \deg g_i - (b_1 + \cdots + b_n)$, and summing those to get $f_i$. We will denote “$W$ is the restriction of $V$” by $W = V \cap \mathbb{A}^n$, and “$V$ is the homogenisation of $W$” by $V = \overline{W}$. Clearly $\overline{W} \cap \mathbb{A}^n = W$ for any affine $W$.

**Claim.** The restriction of an irreducible projective variety is irreducible.

**Proof.** Let $V$ be the irreducible projective variety, and let $W = V \cap \mathbb{A}^n$. Assume by contradiction that $W = W_1 \cup W_2$ in a non-trivial way. We now claim that $\overline{(W_1 \cap V)} \cup (\overline{W_2} \cap V) \cup (\{x_0 = 0\} \cap V)$ is a non-trivial decomposition of $V$. Indeed, this is clearly a decomposition of $V$, and it is non-trivial because any $(x_1, \ldots, x_n) \in W_1 \setminus W_2$ would satisfy that $(1, x_1, \ldots, x_n) \in \overline{W_1} \setminus \overline{W_2}$, and similarly for $W_2 \setminus W_1$. 

With the claim, the affine Bézout theorem follows from the projective one as follows: Let $W = \cap_{i=1}^m Z(f_i)$ with $\deg f_i \leq d$. Then $\overline{W}$ has the same structure, and hence by the projective Bézout theorem its decomposition to irreducible components $\overline{W} = X_1 \cup \cdots \cup X_K$ satisfies $K \leq d^{\min(n,m)}$. By the claim, $X_i \cap \mathbb{A}^n$ are irreducible, and of course

$$W = \overline{W} \cap \mathbb{A}^n = (X_1 \cup \cdots \cup X_K) \cap \mathbb{A}^n = (X_1 \cap \mathbb{A}^n) \cup \cdots \cup (X_K \cap \mathbb{A}^n)$$

which is a decomposition of $W$ to irreducible subvarieties (it might be redundant, but that would only means the number of components of $W$ is smaller than $K$). Thus we need only show the projective Bézout theorem.

For the projective Bézout theorem we will need the concepts of the dimension and degree of a projective variety. The dimension of a projective variety is as for
an affine variety, and has the same four properties listed above \((\dim \mathbb{P}^n = n)\), with the same references in [6]. As for the degree, heuristically if \(W \subset \mathbb{P}^n\) is some irreducible variety then \(\deg W\) is the number of intersections of \(W\) with a generic linear variety of dimension \(n - \dim W\). Again, the formal definition is different and we will skip it, the reader may consult [3, page 50]. We only need the following properties to use Theorem 4 below:

(i) \(\deg(W)\) is always a positive integer, except \(\deg(\emptyset) = 0\).

(ii) \(\deg(\mathbb{P}^n) = 1\).

For both properties, see [3, Chapter 1, Proposition 7.6]. The projective Bézout theorem follows as a corollary from the following result:

**Theorem 4.** Let \(W\) be an irreducible projective variety. Let \(f\) be a homogeneous polynomial. Let \(X_1, \ldots, X_s\) be the irreducible components of \(W \cap Z(f)\). Then

\[
\sum_{j=1}^{s} \deg(X_j) \leq \deg W \cdot \deg f
\]

Where \(\deg W\) is the degree of a projective variety just mentioned, while \(\deg f\) is the usual degree of a polynomial. See [3, Theorem 7.7 and Proposition 7.6d]. The formulation in [3] has some additional quantities, intersection multiplicities, denoted by \(i(\cdot, \cdot)\) — all we need from them is that they are at least 1, which follows because they are defined as lengths of some modules ([3], top of page 53 and the definition at page 51), and the length of a module is the maximal size of a decreasing sequence of submodules. The formulation in [3] requires that \(\dim W \geq 1\) and that \(f\) is not identically zero on \(W\), but the case \(\dim W = 0\) (i.e. \(W\) is a single point) is obvious, and so is the case \(W \subset Z(f)\).

Let now \(f_1, \ldots, f_m\) be polynomials with \(\deg f_i \leq d\), and let

\[
\bigcap_{i=1}^{m} Z(f_i) = X_1 \cup \cdots \cup X_K
\]

be the decomposition of \(\cap Z(f_i)\) into irreducible components. We claim that

\[
\sum_{j=1}^{K} \deg(X_j) d^{\dim X_j} \leq d^n.
\]

We show (2) by induction on \(m\). Indeed, \(m = 0\) is obvious. Assume (2) has been proved for \(m\) and write (using the \(X_i\) of (1))

\[
\bigcap_{i=1}^{m+1} Z(f_i) = \bigcup_{j=1}^{K} X_j \cap Z(f_{m+1}).
\]

Fix \(j\) and let \(Y_{j,k}\) be the irreducible components of \(X_j \cap Z(f_{m+1})\). By Theorem 4,

\[
\sum_k \deg(Y_{j,k}) \leq d \deg(X_j).
\]

If \(X_j \not\subset Z(f_{m+1})\) then by property 4 of the dimension, \(\dim Y_{j,k} = \dim X_j - 1\) so

\[
\sum_k \deg(Y_{j,k}) d^{\dim Y_{j,k}} \leq \deg(X_j) d^{\dim X_j}.
\]
But if $X_j \subseteq Z(f_{m+1})$ then (3) holds trivially (with no need to invoke Theorem 4). So (3) holds always. We sum (3) over $j$ to get
\[ \sum_{j} \deg(Y_{jk})d^\dim Y_{jk} \leq \sum_{j} \deg(X_j)d^\dim X_j \leq d^n \]
where the second inequality is the induction assumption. Now, $Y_{jk}$ is a decomposition of $\bigcap_{i=1}^{m+1} Z(f_i)$ to irreducible components — it may be redundant, but that only reduces the sum in (2) further. Hence (2) holds for $m+1$ and the induction is complete.

Theorem 3 now follows easily. We drop the degrees (as we may, as they are always at least 1) and get
\[ \sum_{i=1}^{K} d^\dim X_j \leq d^n \]
If $m \geq n$ Theorem 3 follows immediately. If $m < n$ it follows because then each $X_j$ has dimension at least $n - m$.

The next result we need is an effective version of the nullstellensatz. Hilbert's nullstellensatz is the following: Suppose $p_i$ are polynomials in $n$ variables with $\cap Z(p_i) = \emptyset$. Then there exists polynomials $q_i$ such that $\sum p_iq_i \equiv 1$. There is also a version of the nullstellensatz when $W := \cap Z(p_i) \neq \emptyset$. It states that if $r$ is a polynomial which is zero on every point of $W$, then there exists a $v \geq 1$ and $q_i$ such that $\sum p_iq_i = r^v$. These theorems hold for any field, but we will need them only for $\mathbb{Q}$. Multiplying by the common denominator we get a result that holds in $\mathbb{Z}$, i.e. if the $p_i$ and the $r$ have integer coefficients then one may find polynomials $q_i$ with integer coefficients, and integers $v$ and $b$ such that $\sum p_iq_i = br^v$. We will need an effective version of this result but, in fact, the only quantity we need to control is $b$. Hence the effective version is as follows:

**Theorem 5.** Let $p_1, \ldots, p_t, r \in \mathbb{Z}[x_1, \ldots, x_n]$, assume $r$ vanishes on $\cap_{i=1}^{t} Z(p_i)$. Assume also that $\deg p_i \leq d \forall i \in \mathbb{N}$, $\deg r \leq d$ and all coefficients of all $p_i$ are bounded by $h$. Then there exists $q_i \in \mathbb{Z}[x_1, \ldots, x_n], i = 1, \ldots, t$ and $b, v \in \mathbb{N}$ such that
\[ \sum_{i=1}^{t} p_iq_i = br^v \]
with the bound
\[ \log b \leq C n^{2d}(d+1)^{n(n+2)}(\log h + Cn^2 \log d). \]

Here and below $C$ and $c$ will stand for absolute constants whose value might change from line to line. We will only use the following, rough bound, which holds for a fixed $n$ and $d$ sufficiently large (i.e. $d > d_0(n)$),
\[ \log b \leq (2d)^{n(n+2)+1} \log h. \] (4)

**Proof.** We will find $q_i \in \mathbb{Q}[x_1, \ldots, x_n]$ such that $\sum p_iq_i = r^v$ and then $b$ will be bounded by the lcm of the denominators of the $q_i$. By the corollary to Theorem 1 of [2], we may take $q_i \in \mathbb{Q}[x_1, \ldots, x_n]$ with
\[ \deg q_i \leq (n+1)(n+2)(d+1)^{n+2}. \] (5)

Once the degree is bounded, the coefficients of the $q_i$ are given by the solution of a system of linear equations (depending on the $p_i$, on $r$ and on $v$). Let $f(d, n)$
be the dimension of the space of polynomials with \(n\) variables and degree \(\leq d\) (so \(f(d, n) \leq (d + 1)^n\)). The system might be underdetermined, we have \(tf(Q, n)\) variables and at most \(f(Q + d, n)\) equations, one for each coefficient of one monomial in the equality \(\sum p_i q_i = r\), up to the degree of the left-hand side. Let \(R\) be the rank of this system of equations, so \(R \leq f(Q + d, n)\). Pick arbitrarily \(R\) variables and \(R\) equations such that the corresponding submatrix \(M\) is invertible and solve the restricted equations. Set the rest of the variables to zero, and the remaining equations (if any) will be fulfilled automatically. It follows that some choice of the \(q_i\) can be achieved by inverting \(M\) and applying the result to the vector of the coefficients of \(r\), themselves integers. Since \(M^{-1} = M' / \det M\), where \(M'\) has integer entries, we may bound \(b \leq \det M\). But the entries of \(M\) are simply the coefficients of the \(p_i\), all of them bounded by \(h\). Applying Hadamard’s inequality (the determinant is bounded by the product of the \(l^2\) norms of the rows) gives

\[
b \leq \left( h \sqrt{R} \right)^R
\]

or

\[
\log b \leq R(\log h + \frac{1}{2} \log R) \leq f(Q + d, n)(\log h + \frac{1}{2} \log f(Q + d, n)) \leq (Q + d + 1)^n(\log h + \frac{1}{2} n \log(Q + d + 1)) \leq \left( (n + 1)(n + 2)(d + 1)^{n+2} + d + 1 \right)^n.
\]

\[
\cdot (\log h + \frac{1}{2} n \log((n + 1)(n + 2)(d + 1)^{n+2} + d + 1)) \leq C^n n^{2n}(d + 1)^{(n+2)}(\log h + Cn^2 \log d)
\]

as claimed. \(\square\)

3. Proof of the Main Result

**Lemma 6.** Let \(G\) be any \(d\)-regular connected multigraph with \(d \geq 4\) and more than 2 vertices, and let \(x\) be some vertex of \(G\). Let \(t > 1\). Then

\[
\mathbb{P}^x(N(t) = x) \leq \frac{d - 2}{d - 1}
\]

where \(N(t)\) is a nonbacktracking random walk on \(G\) at the \(t^{th}\) step and \(\mathbb{P}^x\) denotes the probability when \(N(0) = x\).

Let us define precisely what we mean by “multigraph” and “nonbacktracking random walk”. A multigraph is a graph which might contain multiple edges and self-loops. It is \(d\)-regular if every vertex has exactly \(d\) edges connected to it, with a self-loop counted as two edges. A nonbacktracking random walk is a walk that is not allowed to traverse an edge and on the next step traverse it in the opposite direction (there are no restrictions on the first step). A self-loop can be traversed in either direction, and the nonbacktracking condition is that it cannot be traversed and then traversed backwards. When the multigraph is \(d\)-regular, this process has exactly \(d - 1\) possibilities at each step (except the first one), and it chooses each with probability \(1/(d - 1)\), independently of the past.
Proof. Fix the vertex \( x \) for the rest of the proof. Every edge of our multigraph we consider as two directed edges (a self-loop too corresponds to two directed edges), and for a directed edge \( e \) we denote by \( \overline{e} \) the inverted edge. Hence, the non-backtracking condition is that the walk is not allowed to traverse \( e \) immediately after traversing \( \overline{e} \). (note that we have a multigraph, so there can be \( e \neq \overline{e} \) that both go from vertex \( x \) to vertex \( y \). Still, we may traverse \( \overline{e} \) and then \( f \), or \( \overline{f} \) and then \( e \). It is only the couples \( \overline{e}, f \) and \( \overline{f}, e \) that are prohibited. Each self-loop corresponds to two directed edges which are \( \overline{e} \) of one another). Let \( q_t(e) \) be the probability that the edge \( e \) was traversed at time \( t \) i.e. if \( e : v \rightarrow w \) (i.e., \( e \) is from \( v \) to \( w \), we will also use the notation \( e \rightarrow v \) and \( e : v \rightarrow w \) if we do not care about the other vertex) then it is the probability that \( N(t - 1) = v \) and then the process continues through \( e \) (which means, in particular, that \( N(t) = w \)). Let \( Q(t) = \max_e q_t(e) \). Then \( Q(t) \) is non-increasing because

\[
q_{t+1}(e) = \frac{1}{d-1} \sum_{f : v \leftarrow f, f \neq \overline{e}} q_t(f) \leq Q(t) \quad \forall e : v \rightarrow .
\]

Examine now the event that \( e : y \rightarrow z \) was traversed in the second step. It requires that \( N(1) = y \). Assume first (call this “case I”) that each neighbour \( y \) of \( x \) is connected to \( x \) by \( \leq d - 2 \) edges (including \( x \) itself, if there are self-loops). Then \( \mathbb{P}(N(1) = y) \leq (d - 2)/d \) for every \( y \) and hence \( Q(2) \leq (d - 2)/(d(d - 1)) \).

If \( x \) has a neighbour \( y \) to which it is connected by more than \( d - 2 \) edges (“case II”), then the requirements of regularity and more than \( 2 \) vertices say that it must be connected to \( y \) by exactly \( d - 1 \) edges, and further that it has a second neighbour to which it is connected by \( 1 \) edge. This means that \( \mathbb{P}(N(2) = x) = (d - 2)/d \) and, in particular, any vertex \( z \neq x \) we have \( \mathbb{P}(N(2) = z) \leq 2/d \). Hence

\[
Q(3) \leq \frac{\max\{2, (d - 2)\}}{d(d - 1)}
\]

Since \( d \geq 4 \) we get the same bound as in case I. Hence \( Q(t) \leq (d - 2)/(d(d - 1)) \) for all \( t \geq 3 \). But this means that

\[
\mathbb{P}(N(t) = x) = \sum_{e : x \rightarrow e} q_t(e) \leq dQ(t) \leq \frac{d - 2}{d - 1}.
\]

This covers all cases of the lemma except \( t = 2 \) in case II, but we just calculated that in this case \( \mathbb{P}(N(2) = x) = (d - 2)/d \). The lemma is thus proved.

The next lemma is quite close to the formulation of Theorem 8, the only difference is that it handles only one field.

Lemma 7. Let \( F \) be an algebraically closed field, let \( k, m \in \mathbb{N} \), \( m \geq 2 \) and let \( l \) be sufficiently large (i.e. \( l > l_0(k, m) \)). Let \( R \) be given by taking \( u \geq 15m^3k^4 \log l \) random reduced words of length \( l \) in the letters \( \{x_1, \ldots, x_m, x_1^{-1}, \ldots, x_m^{-1}\} \) independently, uniformly, with repetitions. Let \( \Gamma = \langle x_1, \ldots, x_m | R \rangle \). Then

\[
\mathbb{P}(\exists \rho : \Gamma \rightarrow \text{GL}_k(F) \text{ such that } |\rho(\Gamma)| > 2 \leq \exp(-cu/mk^2)
\]

where \( \rho \) is a group homomorphism.

(log here is the natural logarithm).
Proof. We consider $\text{GL}_k(F)$ as a subvariety of $F^{2k^2}$ by considering the first set of $k^2$ variables as the entries of the matrix and the second set of $k^2$ variables as the entries of the inverse matrix, and adding polynomial equations ($k^2$ of them, all of degree 2) that ensure that indeed, the product of the two matrices is 1. Similarly we consider $\text{GL}_k(F) \times \cdots \times \text{GL}_k(F)$ ($m$ times) as a subvariety of $F^{2mk^2}$. Denote this variety by $X$. For $A \in \text{GL}_k(F) \times \cdots \times \text{GL}_k(F)$ we denote $(A, A^{-1}) := (A_1, A_1^{-1}, \ldots, A_m, A_m^{-1}) \in X$.

Let $E_j \subset X$ be the collection of $(A, A^{-1})$ such that the matrices $A_1, \ldots, A_m$ satisfy the first $j$ words in $R$. Since these (random) words can be thought of as (random) polynomial equations in $2mk^2$ variables as above, $E_j$ is a (random) variety in $F^{2mk^2}$. Let $(A, A^{-1})$ be a point in $E_j$ with $|\langle A \rangle| > 2$. Examine the event that $(A, A^{-1}) \in E_{j+1}$, conditioned on $E_j$. The new reduced word $\omega$ that was added to form $E_{j+1}$ is independent of the past, and hence $\omega(A)$ is distributed like a non-backtracking random walk of length $l$ on the Cayley graph generated by $A$ (if for some $i A_i = 1$ the graph will contain one corresponding self-loop on each vertex. The two directions of this self-loop will correspond to multiplying by $A_i$ and $A_i^{-1}$. This matches with the definitions we gave around Lemma 6). This Cayley graph is a $2m$-regular multigraph, and by assumption it has more than 2 vertices. Hence we may use Lemma 6 to get

$$\mathbb{P}(\langle A, A^{-1} \rangle \in E_{j+1} | \langle A, A^{-1} \rangle \in E_j) \leq \frac{2m - 2}{2m - 1}.$$ 

In other words, with probability $\geq \frac{1}{2m - 1}$, adding one relation breaks the irreducible component containing $(A, A^{-1})$ into further irreducible components, which then must have smaller dimension, by property (iv) of the dimension (see §2).

Repeating this $\lambda$ times we get that after adding $\lambda$ words we break any fixed irreducible component with probability at least $1 - \exp(-\lambda/2m)$. By Bezout’s theorem (Theorem 3), $E_j$ has no more than $l^{2mk^2}$ irreducible components (the initial polynomial equations defining $X$ have degree 2). Hence a simple union bound shows that, for $\lambda \geq 5m^2k^2 \log l$,

$$\mathbb{P}(\text{all components are broken}) \geq 1 - l^{2mk^2} \exp\left(-\frac{\lambda}{2m}\right) \geq 1 - \exp\left(-\frac{\lambda}{10m}\right).$$

Use that for $\lambda = u/(2mk^2 + 1) \geq 5m^2k^2 \log l$ words, and get that with probability at least $1 - \exp(-cu/mk^2)$ one breaks all components. Therefore the maximal degree decreases by 1. Repeating this a further $2mk^2 + 1$ times, the maximal degree of any component which contains any $(A, A^{-1})$ with $|\langle A \rangle| > 2$ is $-1$, so they are in fact empty. We get the claim of the lemma with probability $l^{2mk^2 + 1} \exp(-cu/mk^2)$ but of course the outer $2mk^2 + 1$ can be ignored (perhaps changing the constant inside the exponent).

\begin{theorem}
Let $k, l, m \in \mathbb{N}, m \geq 2$. Let $R$ be be given by taking at least $(3l)^{7m^2k^4}$ random reduced words of length $l$ in the letters $\{x_1, \ldots, x_m, x_1^{-1}, \ldots, x_m^{-1}\}$ independently, uniformly, with repetitions. Let $\Gamma = \langle x_1, \ldots, x_m | R \rangle$. Then

$$\lim_{l \to \infty} \mathbb{P}(\exists F, \rho : \Gamma \to \text{GL}_k(F) \text{ such that } |\rho(\Gamma)| > 2) = 0,$$

where $F$ runs over the all fields, and where $\rho$ is a group homomorphism.
\end{theorem}
Proof. First apply Lemma 7 with \( F = \mathbb{C} \) and with \( 15m^3k^4|\log l| \) relators. We get that with high probability, any \( \rho : \Gamma \to \text{GL}_k(\mathbb{C}) \) has \( |\rho(\Gamma)| \leq 2 \). Of course, the image of the generators \( \{x_1, \ldots, x_m\} \) is easy to characterise: we have \( \rho(x_i)^2 = 1 \forall i \) and for some \( S \subset \{1, \ldots, m\} \) we have \( \rho(x_i) = \rho(x_j) \forall i, j \in S \) and \( \rho(x_i) = 1 \forall i \not\in S \). Compactly, \( \{|\rho(x_i)| \setminus \{1\}| \leq 1 \).

Recall from the proof of Lemma 7 the variety \( X \) in \( \mathbb{C}^{2mk^2} \) and denote \( p_1, \ldots, p_{mk^2} \) the polynomials defining it; and the notation \( (A, A^{-1}) \). The conditions that matrices \( A \) satisfy a given random word is a polynomial in \( 2mk^2 \) variables. Denote the polynomial that corresponds to the \( i \)th word by \( p_{mk^2+i} \) and let \( M = mk^2 + 15m^3k^4|\log l| \). Note that each of these polynomials has integer coefficients.

We get that
\[
\bigcap_{i=1}^M Z(p_i) = \{(A, A^{-1}): A_i^2 = 1 \forall i, |A \setminus \{1\}| \leq 1\}.
\]
Denote the variety on the right by \( Y \).

We now claim that \( Y \) can by written as \( \cap Z(r_j) \) for some \( r_1, \ldots, r_K \) which depend only on \( k \) and \( m \), and in particular do not depend on the field. Here is how: the condition \( A_i^2 = 1 \) corresponds to \( k^2 \) polynomials for each \( i \). The condition
\[
\bigcup_{S} \{A_i = A_j \forall i, j \in S, A_i = 1 \forall i \not\in S\},
\]
gives us at most \( (mk^2)^{2^m} \) polynomials because for every \( S \) the corresponding variety is described by at most \( mk^2 \) polynomials, but taking union requires to take every possible choice of a polynomial for each \( S \), and multiply them out. This describes our \( r_1, \ldots, r_K \) (and gives \( K \leq mk^2 + (mk^2)^{2^m} \), but we will have no use for this fact).

Now apply the effective nullstellensatz (Theorem 5) \( K \) times as follows. In all applications the polynomials \( p_i \) from the nullstellensatz are our \( p_i \), but the polynomial \( r \) we take corresponding to the \( r_j \) above. We get corresponding \( q_{i,j,v_j} \) and \( b_j \), with the \( b_j \) all satisfying some bound, which we denote by \( B \). Recall (4). The number of variables is \( 2mk^2 \) while the maximal value of the coefficients, \( h \), can be bounded roughly by \( (2mk^2)^l \). We get,
\[
B \leq \exp \left( (2l)^{4m^2k^4 + 4mk^2 + 1} \cdot l \log(2mk^2) \right) \leq \exp \left( (2l)^{7m^2k^4} \right)
\]
which holds for \( l \) sufficiently large.

Consider now a field \( F \) of characteristic larger than \( B \). Then
\[
\sum p_i q_{i,j}^v = b_j r_j^v
\]
holds also in \( F \), and because \( \text{char } F > B \geq b \) we get that \( b_j \neq 0 \) in the field and we may divide by them. This means that whenever \( p_i = 0 \) for all \( i \) so are \( r_j \) for all \( j \), but that means that any \( A \) which satisfy our first \( 15m^3k^4|\log l| \) words must also satisfy that \( A_i^2 = 1 \forall i \) and that \( |A \setminus \{1\}| \leq 1 \). So in \( \text{GL}_k(F) \) too we get \( |\langle A \rangle| \leq 2 \).

Finally, for every prime \( \tau \) smaller than \( B \) apply Lemma 7 again, but this time with the field \( F_\tau \) being the algebraic closure of \( \mathbb{Z}/\tau \mathbb{Z} \) and with \( u = \lambda mk^2(2l)^{7m^2k^4} \) for some \( \lambda \) to be fixed soon. We get that
\[
\mathbb{P}(\exists \rho : \Gamma \to \text{GL}_k(F_\tau) \text{ s.t. } |\rho(\Gamma)| > 2) \leq \exp(-cu/mk^2) = \exp(-c\lambda(2l)^{7m^2k^4}).
\]
Summing over all $\tau < B$ gives
\[
\mathbb{P}(\exists \tau < B \exists \rho : \Gamma \to \text{GL}_k(F_\tau) \text{ such that } |\rho(\Gamma)| > 2) \leq B \exp(-c\lambda(2l)^{7w^2k^4}) \leq \exp((1-c\lambda)(2l)^{7w^2k^4}).
\]
Taking $\lambda = 2/c$ we get that this probability goes to zero. Moving from $F_\tau$ to a general field of characteristic $\tau$ is done using the (usual, non-effective) nullstellensatz: find polynomials $q_{ij} \in \mathbb{Z}/\tau\mathbb{Z}[x_1, \ldots, x_{2mk^2}]$ such that $\sum p_i q_{ij} = r_j^{1/2}$ in $\mathbb{Z}/\tau\mathbb{Z}$ with the same $p_i$ and $r_1, \ldots, r_K$ as above, and note that the existence of these $q_{ij}$ ensures that in any field $F$ of characteristic $\tau$, if $A_1, \ldots, A_m$ are in $\text{GL}_k(F)$ and satisfy all words in $R$ then $|\langle A \rangle| \leq 2$, proving the theorem. □

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