ASYMPTOTIC ANALYSIS FOR THE ELECTRIC FIELD CONCENTRATION WITH GEOMETRY OF THE CORE-SHELL STRUCTURE

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Abstract. In the perfect conductivity problem arising from composites, the electric field may become arbitrarily large as $\varepsilon$, the distance between the inclusions and the matrix boundary, tends to zero. In this paper, by making clear the singular role of the blow-up factor $Q[\varphi]$ introduced in [27] for some special boundary data of even function type with $k$-order growth, we prove the optimality of the blow-up rate in the presence of $m$-convex inclusions close to touching the matrix boundary in all dimensions. Finally, we give closer analysis in terms of the singular behavior of the concentrated field for eccentric and concentric core-shell geometries with circular and spherical boundaries from the practical application angle.

1. Introduction. There appears widely various of physical field concentration including extreme electric fields, heat fluxes and mechanical loads. Motivated by the issue of material failure initiation, in this paper we focus on the investigation of blow-up phenomena arising from high-contrast fiber-reinforced composites with the densely packed fibers. It is well known that the high concentration occurs in the narrow region between two close-to-touching inclusions and the thin gap between the inclusions and the matrix boundary. We would like to emphasize that the latter will exhibit more complex singular behavior, because the solution to the perfect conductivity problem becomes irregular near the external boundary due to the interaction from given boundary data. In the past two decades, it has been actively studied in the engineering and mathematical literature since Babuška et al’s famous work [4], where the authors numerically analyzed the damage and fracture in composite materials and observed that the size of the strain tensor remains bounded as the conductivities of inclusions are away from zero and infinity.

The conductivity problem with two inclusions can be modeled by the following scalar equation with piecewise constant coefficients
\[
\begin{align*}
\text{div}(a_k(x)\nabla u_k) &= 0, & &\text{in } D, \\
u &= \varphi, & &\text{on } \partial D,
\end{align*}
\]

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where
\[ a_k(x) = \begin{cases} 
    k \in [0, 1) \cup (1, \infty], & \text{in } D_1 \cup D_2, \\
    1, & \text{in } D \setminus D_1 \cup D_2.
\end{cases} \]

It is worthwhile to mention that equation (1.1) can also be used to describe more physical phenomenon, such as dielectrics, magnetism, thermal conduction, chemical diffusion and flow in porous media. Problem (1.1) is actually a bi-parameter problem in relation to the contrast \( k \) and the distance \( \varepsilon \) between two inclusions. When \( k \) ranges from 0 to \( \infty \), the gradient of a solution to problem (1.1) has been proved to be bounded independently of the distance \( \varepsilon \). Bonnetier and Vogelius [10] first established the \( W^{1, \infty} \) estimate of the solution to the case of two touching disks in two dimensions. Li and Vogelius [30] extended this result to general second order elliptic equations with piecewise smooth coefficients and obtained more stronger \( C^{1, \alpha} \) estimates in all dimensions. The subsequent work [29] completed by Li and Nirenberg further extended the \( C^{1, \alpha} \) estimates to general second order elliptic systems including systems of linear elasticity. Note that although the upper bounds of the gradients in [10, 29, 30] depend on the elliptic coefficients, this dependence is not explicit. Recently, Dong and Li [14] solved this problem for a class of non-homogeneous elliptic equations with discontinuous coefficients by showing the clear dependence of the elliptic coefficients. In addition, Calo, Efendiev and Galvis [11] obtained an asymptotic formula of the solution for elliptic equations with respect to the contrast \( k \) as \( k \) is sufficiently small or large.

The situation will become different with the deterioration of the elliptic coefficients. That is, when the contrast \( k \) degenerates into \( \infty \), problem (1.1) turns into the perfect conductivity problem. It has been proved that the electric field, which is the gradient of a solution to the perfect conductivity equation, blows up at the rate of \( \varepsilon^{-1/2} \) in two dimensions \([2, 3, 5, 6, 18, 32, 33]\), \( |\varepsilon \ln \varepsilon|^{-1} \) in three dimensions \([6, 7, 23, 31]\), and \( \varepsilon^{-1} \) in higher dimensions \([6]\). To have a better understanding of the concentration, it is particularly important to pursue a more precise description for the asymptotic behavior of the gradient. Kang, Lim and Yun [19] were the first to establish a complete characterization of the electric field for two adjacent disks in two dimensions, and then Ammari et al. [1] extended their results to the case of two strictly convex inclusions by using disks osculating to convex domains. The corresponding characterization in three dimensions was derived by Kang, Lim and Yun [20] for two spherical conductors with the same radii. Kang, Lee and Yun [21] studied the asymptotic behavior of the stress concentration factor and showed that its limit is a certain integral of the solution to the touching case as the distance \( \varepsilon \) between two inclusions tends to zero. Subsequently, Li, Li and Yang [25] accomplished an accurate computation of the energy to obtain an asymptotic formula for any \( 2 \)-convex inclusions in dimensions two and three. Li [28] then extended the results in [25] to the case of \( m \)-convex inclusions and captured a blow-up factor different from that in [25]. As a continuation of [25], Zhao and Hao [34] further refined the idea in [25] and captured all the blow-up factors to solve the optimality of the blow-up rate of the electric field for two adjacent \( m \)-convex inclusions in all dimensions, which is only partially answered in [28]. Bonnetier and Triki [9] investigated the spectrum of the Poincaré variational problem for two neighbouring inclusions and obtained the asymptotics of its eigenvalues as the distance between these two inclusions goes to zero. Recently, Kim and Lim [22] derived an asymptotic formula of the potential function by using the single and double layer potentials with image line charges for core-shell geometry with circular boundaries in two dimensions, see Fig. 1 below.
Additionally, there has established some qualitative characterizations of the concentrated field for the perfect conductivity problem modeled by nonlinear equation in [12, 13, 15], for which their method of barriers differs from the techniques adopted in the linear equation. For more related investigations, see [8, 16, 23, 24, 26] and the references therein.

Observe that previous investigations on the asymptotics of the field concentration mainly focused on the narrow region between two inclusions, less work is devoted to investigating the asymptotic behavior of the gradient in the presence of an inclusion with core-shell geometry which consists of a core and a surrounding shell, where the core and shell are close to touching. This paper, as a continuation of [27], improves the upper and lower bound estimate of the gradient to be an asymptotic formula. Moreover, in this paper we answer the following three remaining questions in [27]:

(a) first, in the construction of the lower bound of the gradient in Theorem 2.1 of [27], the authors only show the convergence between \( Q[\varphi] \) and its limit \( Q^*[\varphi] \) but without giving concrete convergence rate;

(b) second, in the case of \( m \)-convex inclusions in Theorem 3.10 of [27], the authors don’t reveal the singular role of the blow-up factor \( Q[\varphi] \);

(c) third, some special boundary data \( \varphi \), which makes \( Q^*[\varphi] \neq 0 \), are given in Remark 2.2, where the blow-up factor \( Q^*[\varphi] \) is defined by (2.5).

We solve questions (a) and (b) in Lemma 4.1 below. In fact, for the boundary data of even function type with \( k \)-order growth defined in condition \((S1)\) below, when the convexity index \( m \) increases to a certain extent, that is, \( m \geq n + k - 1 \), the blow-up factor \( Q[\varphi] \) can possess the blow-up rate, which means that it has no limit any more as the distance tends to zero; for the boundary data of odd function in condition \((S2)\) or the boundary data in condition \((S1)\) in the case of \( m < n + k - 1 \), we prove the boundness of the blow-up factor \( Q[\varphi] \) and meanwhile solve the explicit convergence rate between \( Q[\varphi] \) and \( Q^*[\varphi] \). We then establish an asymptotic formula of the concentrated field for inclusions of a class of \( m \)-convex shapes with core-shell geometry in all dimensions. Since circular and spherical fibers are very common in the shape design of composites, then we also present two special examples of eccentric and concentric core-shell geometries with circular and spherical boundaries from an engineering point of view.

2. Formulation of the problem and main results. To formulate our problem in a precise manner, we let \( D \subset \mathbb{R}^n \) \((n \geq 2)\) be a bounded domain with \( C^{2,\alpha} \)-boundary containing a \( C^{2,\alpha} \)-subdomain \( D_1 \) with \( 0 < \alpha < 1 \). Denote the distance \( \text{dist}(D_1, \partial D) := \varepsilon \), where \( \varepsilon > 0 \) is a sufficiently small constant. By a possible translation and rotation of the coordinates if necessary, we let \( x_n \)-axis pass through the shortest line segment between \( \partial D \) and \( \partial D_1 \), and

\[
D^*_i := D_1 + (0', -\varepsilon), \quad \partial D^*_i \cap \partial D = \{0\} \subset \mathbb{R}^n.
\]

Here and throughout the paper, we use superscript prime to denote \((n-1)\)-dimensional domains and variables.

In this paper, we consider the perfect conductivity problem with inclusions close to touching the matrix boundary as follows

\[
\begin{aligned}
\Delta u = 0, & \quad \text{in } D \setminus \overline{D}_1, \\
u = C_1, & \quad \text{in } D_1, \\
\int_{\partial D_1} \frac{\partial u}{\partial \nu} \bigg|_+ = 0, & \quad \text{in } \partial D_1, \\
u = \varphi, & \quad \text{on } \partial D, \\
\end{aligned}
\]  

(2.1)
where the free constant $C_1$ is determined later by the third line of (2.1). There has established the existence, uniqueness and regularity of weak solutions to (2.1) in [6] with a minor modification. We now give the precise definition of the eccentric core-shell geometry whose interfacial boundaries are $m$-convex. Assume that there exists a small constant $R > 0$ independent of $\varepsilon$, such that the portions of $\partial D$ and $\partial D_1$ near the origin can be written as

$$x_n = \varepsilon + h_1(x') \quad \text{and} \quad x_n = h(x'), \quad x' \in B_{2R}^2,$$

where $h_1$ and $h$ satisfy that for $m \geq 2$,

\[(H1) \quad h_1(x') - h(x') = \lambda |x'|^m + O(|x'|^{m+1}),\]

\[(H2) \quad |\nabla x_h_1(x')|, |\nabla x_h(x')| \leq \kappa_1 |x'|^{m-i}, \quad i = 1, 2,\]

\[(H3) \quad \|h_1\|_{C^{2,\alpha}(B_{2R}^2)} + \|h\|_{C^{2,\alpha}(B_{2R}^2)} \leq \kappa_2,\]

where $\lambda$ and $\kappa_j, j = 1, 2,$ are three positive constants independent of $\varepsilon$. We would like to point out that our assumption condition $(H1)$ implies that

$$|(h_1 - h)(x') - \lambda |x'|^m| \leq C|x'|^{m+\beta} \leq C|x'|^{m+1},$$

for some $\beta \geq 1, x' \in B_{2R}^2$.

Thus this assumption condition is very general and covers many regular shapes, especially disks and spheres, see Example 5.1 below.

For $z' \in B_{2R}^2, 0 < t \leq 2R,$ write

$$\Omega_t(z') := \{x \in \mathbb{R}^n \mid h(x') < x_n < \varepsilon + h_1(x'), |x' - z'| < t\}.$$ 

We will use the abbreviated notation $\Omega_t$ for the domain $\Omega_t(0')$. Denote the bottom boundary of $\Omega_R$ by $\Gamma_R = \{x \in \mathbb{R}^n \mid x_n = h(x'), |x'| < R\}$. To explicitly uncover the effect of boundary data $\varphi$ on the singularities of the field, we classify $\varphi \in C^2(\partial D)$ according to its parity as follows. Suppose that for $x \in \Gamma_R$,

\[(S1) \quad \varphi \text{ satisfies the } k \text{-order growth condition, that is,}\]

$$\varphi(x) = \eta|x|^k;$$

\[(S2) \quad \varphi \text{ is odd with respect to some } x_{i_0}, i_0 \in \{1, ..., n-1\},\]

where $\eta > 0$ and $k > 1$ is a positive integer. We additionally assume that $h_1(x') - h(x')$ is even with respect to $x_{i_0}$ in $B_{2R}^2$ under case (S2).

Denote $\Omega := D \setminus D_1$. We introduce two scalar auxiliary functions $\bar{u} \in C^2(\mathbb{R}^n)$ and $\bar{u}_0 \in C^2(\mathbb{R}^n)$ such that $\bar{u} = 1$ on $\partial D_1, \bar{u} = 0$ on $\partial D$ and

$$\bar{u}(x) = \frac{x_n - h(x')}{\varepsilon + h_1(x') - h(x')}, \quad \text{in } \Omega_{2R}, \quad \|\bar{u}\|_{C^2(\Omega; \Omega_{n})} \leq C, \quad (2.2)$$

and $\bar{u}_0 = 0$ on $\partial D_1, \bar{u}_0 = \varphi(x)$ on $\partial D,$ and

$$\bar{u}_0 = \varphi(x', h(x'))(1 - \bar{u}), \quad \text{in } \Omega_{2R}, \quad \|\bar{u}_0\|_{C^2(\Omega; \Omega_{R})} \leq C. \quad (2.3)$$

To simplify notations used in the following, for $i = 0$, and $i = k, k$ is the order of growth defined in (S1), we denote

$$\rho_i(n, m; \varepsilon) = \begin{cases} 
\varepsilon^{\frac{n+i-1}{m}} - 1, & m > n + i - 1, \\
\ln \varepsilon, & m = n + i - 1, \\
1, & m < n + i - 1,
\end{cases}$$

and

$$\Gamma\left[\frac{n+i-1}{m}\right] = \begin{cases} 
\Gamma\left(1 - \frac{n+i-1}{m}\right) \Gamma\left(\frac{n+i-1}{m}\right), & m > n + i - 1, \\
1, & m = n + i - 1,
\end{cases}$$
where $\Gamma(s) = \int_0^{+\infty} t^{s-1} e^{-t} \, dt$, $s > 0$ is the Gamma function. Denote by $\omega_{n-1}$ the volume of $(n-1)$-dimensional unit sphere. For $(x', \lambda, \kappa_1, \kappa_2, R) \in \Omega_{2R}$, denote
\[
\delta(x') := \varepsilon + h_1(x') - h(x').
\] (2.4)

Let $\Omega^* := D \setminus \Omega_1^*$. We define a linear functional with respect to $\varphi$,
\[
Q^*[\varphi] := \int_{\partial D_1^*} \frac{\partial v_0^*}{\partial \nu},
\] (2.5)

where $v_0^*$ is a solution of the following problem:
\[
\begin{cases}
\Delta v_0^* = 0, & \text{in } \Omega^*, \\
v_0^* = 0, & \text{on } \partial D_1^* \setminus \{0\}, \\
v_0^* = \varphi(x), & \text{on } \partial D.
\end{cases}
\] (2.6)

Note that the definition of $Q^*[\varphi]$ is valid under case (S2) but only valid for $m < n + k - 1$ under case (S1). For $m < n - 1$, define
\[
a_{11}^* := \int_{\Omega^*} |\nabla v_1^*|^2,
\] (2.7)

where $v_1^*$ satisfies
\[
\begin{cases}
\Delta v_1^* = 0, & \text{in } \Omega^*, \\
v_1^* = 1, & \text{on } \partial D_1^* \setminus \{0\}, \\
v_1^* = 0, & \text{on } \partial D.
\end{cases}
\] (2.8)

Observe that $a_{11}^* \neq 0$ in the case of $m < n - 1$. In fact, we see from the Hopf Lemma that
\[
\frac{\partial v_1^*}{\partial \nu} \bigg|_{\partial D_1^* \setminus \{0\}} > 0,
\]

which, in combination with integration by parts, yields that $a_{11}^* = \int_{\partial D_1} \frac{\partial v_1^*}{\partial \nu} > 0$.

For the order of the rest term, we define
\[
r_{\varepsilon} = \begin{cases}
\frac{\varepsilon^{\frac{m}{2}}}{\varepsilon^{(\frac{m}{2})}}, & m > n + k, \\
\frac{\varepsilon^{\frac{m}{2}}}{\ln \varepsilon}, & m = n + k, \\
\frac{\varepsilon^{\frac{n-k-1}{m+1}}}{\ln \varepsilon^{-1}}, & n - 1 < m < n + k - 1, \\
\frac{\varepsilon^{\min\left(\frac{m}{2}, \frac{n-1-m}{n-m}, \frac{n-k-1}{m+1}\right)}}{\ln \varepsilon^{-1}}, & m = n - 1, \\
\frac{\varepsilon^{\min\left(\frac{m}{2}, \frac{n-1-m}{n-m}, \frac{n-k-1}{m+1}\right)}}{\ln \varepsilon^{-1}}, & m < n - 1,
\end{cases}
\] (2.9)

and
\[
\tilde{r}_{\varepsilon} = \begin{cases}
\varepsilon^{\frac{m-n-2}{n-1}}, & m > n - 1, \\
\varepsilon^{\min\left(\frac{m}{2}, \frac{n-1-m}{n-m}, \frac{m+n-2}{m+n-n-2}\right)}, & m < n - 1.
\end{cases}
\] (2.10)

For the sake of convenience, for $i = 0$ and $i = k$, $k \geq 2$ is a positive constant, we denote
\[
\mathcal{M}_i = \frac{(n-1)\omega_{n-1} \Gamma\left[\frac{n+i-1}{m}\right]}{m\lambda^{\frac{n+i-1}{m}}}. 
\] (2.11)

Unless otherwise stated, in what following $C$ represents a constant, whose values may vary from line to line, depending only on $\lambda, \kappa_1, \kappa_2, R$ and an upper bound
of the $C^{2,\alpha}$ norms of $\partial D_1$ and $\partial D$, but not on $\varepsilon$. We also call a constant having such dependence a universal constant. Without loss of generality, we set $\varphi(0) = 0$. Otherwise, we substitute $u - \varphi(0)$ for $u$ throughout this paper. For simplicity of discussions, we assume that convexity index $m \geq 2$ and growth order index $k \geq 2$ are all positive integers in the following.

**Theorem 2.1.** Assume that $D_1 \subset D \subseteq \mathbb{R}^n$ $(n \geq 2)$ are defined as above, conditions (H1)-(H3) and (S1) hold, $Q^*[\varphi] \neq 0$. Let $u \in H^1(D) \cap C^1(\bar{\Omega})$ be the solution of (2.1). Then for a sufficiently small $\varepsilon > 0$ and $x \in \Omega_R$,

(i) for $m \geq n + k - 1$,

$$\nabla u = \frac{\eta M_k}{M_0} (1 + O(r_\varepsilon)) \rho_{k,0}(n,m,\varepsilon) \nabla \bar{u} + \nabla \bar{u}_0 + O(1)\delta^{\frac{n-k}{m-k}}\|\varphi\|_{C^2(\partial D)};$$

(ii) for $n - 1 \leq m < n + k - 1$,

$$\nabla u = \frac{Q^*[\varphi]}{\rho_0(n,m,\varepsilon)} \frac{1 + O(r_\varepsilon)}{\rho_0(n,m,\varepsilon)} \nabla \bar{u} + \nabla \bar{u}_0 + O(1)\delta^{\frac{n-k}{m-k}}\|\varphi\|_{C^2(\partial D)};$$

(iii) for $m < n - 1$,

$$\nabla u = \frac{Q^*[\varphi]}{a_{11}} (1 + O(r_\varepsilon)) \nabla \bar{u} + \nabla \bar{u}_0 + O(1)\delta^{\frac{n-k}{m-k}}\|\varphi\|_{C^2(\partial D)}, \quad (2.12)$$

where $\rho_{k,0}(n,m,\varepsilon) = \rho_k(n,m,\varepsilon)/\rho_0(n,m,\varepsilon)$, $\bar{u}$ and $\bar{u}_0$ are defined by (2.2) and (2.3), respectively. $\delta$ is defined by (2.4), the blow-up factor $Q^*[\varphi]$ and the energy $a_{11}$ are defined by (2.5) and (2.7), respectively, $r_\varepsilon$ is defined by (2.9), $M_i$, $i = 0,k$ are defined in (2.11).

**Remark 2.2.** In order to prove the validity of assumption condition $Q^*[\varphi] \neq 0$ in Theorems 2.1 and 2.4, we provide some examples in terms of the boundary data $\varphi$. Pick $\varphi = \sum_{i=1}^n a_i x_i^2$ on $\partial D$ with $a_i > 0$, $i = 1,2,...,n$. In view of the definition of $Q^*[\varphi]$, it follows from integration by parts that $Q^*[\varphi] = -\int_{\partial D \setminus \{0\}} \frac{\partial^2 \varphi}{\partial x_i^2} \varphi$. By using the Hopf Lemma for $v^*$, we see that $\frac{\partial^2 \varphi}{\partial x_i^2} \varphi < 0$. Since $\varphi \neq 0$ and $\varphi \geq 0$ on $\partial D$, we then obtain that $Q^*[\varphi] \neq 0$. Moreover, this conclusion holds for more generalized boundary data $\varphi \in C^2(\partial D)$ satisfying that $\varphi(0) = 0$, $\varphi \neq 0$ and $\varphi \geq 0$ $(\text{or} \leq 0)$ on $\partial D$. We also would like to point out that these examples cannot be easily extended to the elasticity problem for lack of the Hopf Lemma in linear systems of elasticity.

**Remark 2.3.** The major singularity of $\nabla \bar{u}$ arises from $\frac{\partial x_n}{a_{11}}$ with its blow-up rate $\varepsilon^{-1}$ attaining at the $(n-1)$-dimensional sphere $\{|x'| \leq \varepsilon^{\frac{k-n}{k}}\} \cap \Omega$, while the singular point $\nabla \bar{u}_0$ is determined by $\nabla \frac{\partial x_n}{a_{11}} \bar{u} = -\varphi(x',\eta(x'))\delta^{-1}(x')$ with its blow-up rate $\varepsilon^{\frac{k-n}{k}-1}$ arriving at the cylinder surface $\{|x'| = \varepsilon^{\frac{k-n}{k}}\} \cap \Omega$ only if $m > k$. Then from decomposition (3.6) below and the asymptotic results in Theorem 2.1, we see that the first part $\frac{\partial^2 \varphi}{\partial x_i^2} \nabla v_1$ of (3.6) blows up at the rate of $\rho_{k,0}(n,m,\varepsilon)\varepsilon^{-1}$, while the second part $\nabla v_0$ possesses the singularity of order $O(\varepsilon^{\frac{k-n}{k}-1})$ in the case of $m > k$ but is bounded under $m \leq k$, which implies that the singularity of the second part is no larger than that of the first part.

**Theorem 2.4.** Assume that $D_1 \subset D \subseteq \mathbb{R}^n$ $(n \geq 2)$ are defined as above, conditions (H1)-(H3) and (S2) hold, $Q^*[\varphi] \neq 0$. Let $u \in H^1(D) \cap C^1(\bar{\Omega})$ be the solution of (2.1). Then for a sufficiently small $\varepsilon > 0$ and $x \in \Omega_R$,
(i) for $m \geq n - 1$,
\[
\nabla u = \frac{Q^*[\varphi]}{\lambda_1^m} \left( 1 + O(\tilde{r}_\varepsilon) \right) \nabla \tilde{u} + \nabla \tilde{u}_0 + O(1) \delta^{m-2} \|\varphi\|_{C^2(\partial D)};
\]
(ii) for $m < n - 1$,
\[
\nabla u = \frac{Q^*[\varphi]}{a_{11}^m} \left( 1 + O(\tilde{r}_\varepsilon) \right) \nabla \tilde{u} + \nabla \tilde{u}_0 + O(1) \delta^{m-2} \|\varphi\|_{C^2(\partial D)},
\]
where $\tilde{u}$ and $\tilde{u}_0$ are defined by (2.2) and (2.3), respectively, $\delta$ is defined by (2.4), the blow-up factor $Q^*[\varphi]$ and the energy $a_{11}$ are defined by (2.5) and (2.7), respectively, $\tilde{r}_\varepsilon$ is defined in (2.10), $M_0$ is defined in (2.11).

Remark 2.5. Compared with the interior asymptotics [28], $\nabla u_0$ there is uniformly bounded but here may possess the blow-up rate. In practice, the maximum of $|\nabla \tilde{u}_0|$ here is determined by $\partial_{x_n} u_0 = -\varphi(x', h(x'))\delta^{-1}(x')$. Then its singularity can be captured by analysing the order of Taylor expansion of $\varphi(x', h(x'))$. We give an example of $\varphi = \eta x^k$ on $\Gamma_R^*$. $\eta > 0$ and $k \geq 1$ is an odd number, then the blow-up rate of $\nabla \tilde{u}_0$ is $\varepsilon^{-k-1}$ provided that $m > k$. In this case, it follows from the results in Theorem 2.4 that if $m > n + k - 1$, then the leading singularity of $\nabla u$ is determined by the second part $\nabla u_0$ of decomposition (3.6), but not by the first part $\frac{Q^*}{a_{11}^m} \nabla v_1$. This is different from the blow-up phenomenon revealed in Theorem 2.1.

According to the detailed analysis for the maximum singularity of the gradient in Remarks 2.3 and 2.5, we further consider the more generalized m-convex inclusions satisfying condition
\[
\lambda_1 |x'|^m \leq h_1(x') - h(x') \leq \lambda_2 |x'|^m, \quad \text{in } B_{2R}',
\]  
where $\lambda_1$ and $\lambda_2$ are two positive constants independent of $\varepsilon$. By applying the proofs of Theorems 2.1 and 2.4 with a slight modification, we obtain the optimal upper and lower bounds on the gradient as follows.

Corollary 2.6. Assume that $D_1 \subset D \subseteq \mathbb{R}^n (n \geq 2)$ are defined as above, conditions (2.13) and (H2)–(H3) hold, $Q^*[\varphi] \neq 0$. Let $u \in H^3(D) \cap C^1(\overline{\Omega})$ be the solution of (2.1). Then for a sufficiently small $\varepsilon > 0$,

(i) let $\varphi = \eta x^k$ on $\Gamma_R^*$, $k \geq 1$ is an integer, then for $x \in \{x' = 0'\} \cap \Omega$,
\[
\frac{\eta \lambda_1}{\lambda_2} \frac{\rho_{k,0}(n, m; \varepsilon)}{C\varepsilon} \leq |\nabla u| \leq \frac{\eta \lambda_2}{\lambda_1} \frac{\rho_{k,0}(n, m; \varepsilon)}{C\varepsilon}, \quad m \geq n + k - 1,
\]
\[
\frac{\lambda_1}{\lambda_2} \frac{Q^*[\varphi]}{C\varepsilon \rho_{0}(n, m; \varepsilon)} \leq |\nabla u| \leq \frac{C\lambda_2 \rho_{0}(n, m; \varepsilon)}{\eta \varepsilon}, \quad n - 1 \leq m < n + k - 1,
\]
\[
\frac{Q^*[\varphi]}{a_{11}^m} \frac{1}{C\varepsilon} \leq |\nabla u| \leq \frac{Q^*[\varphi]}{a_{11}^m} \frac{C}{\varepsilon}, \quad m < n - 1;
\]

(ii) let $\varphi = \eta x^k$ on $\Gamma_R^*$, $k \geq 1$ is an odd number, then for $x \in \{x' = (\sqrt{\varepsilon}, 0, ..., 0)\} \cap \Omega$,
\[
\frac{\eta}{1 + \lambda_2} \frac{1}{C\varepsilon} \leq |\nabla u| \leq \frac{\eta C}{1 + \lambda_1}, \quad m > n + k - 1,
\]
and, for $x \in \{x' = 0\} \cap \Omega$,
\[
\frac{\lambda_1}{\lambda_2} \frac{Q^*[\varphi]}{C\varepsilon \rho_{0}(n, m; \varepsilon)} \leq |\nabla u| \leq \frac{C\lambda_2 \rho_{0}(n, m; \varepsilon)}{\eta \varepsilon}, \quad n - 1 \leq m \leq n + k - 1,
\]
where \( \rho_{k,0}(n, m; \varepsilon) = \rho_k(n, m; \varepsilon)/\rho_0(n, m; \varepsilon) \), the blow-up factor \( Q^*[\varphi] \) and the energy \( a_{11} \) are defined by (2.5) and (2.7), respectively.

**Remark 2.7.** The optimal gradient estimates in Corollary 2.6, in combination with the precise asymptotic formulas in Theorems 2.1 and 2.4, give a complete characterization for the singularities of the gradient and meanwhile answer the optimality of the blow-up rate of the gradient in the presence of \( m \)-convex inclusions and the boundary data with \( k \)-order growth in all dimensions. Our results improve and make complete the gradient estimates in previous work [27] by making clear the singularities of the blow-up factor \( Q^*[\varphi] \) and capturing the explicit dependence on the growth order index \( k \) and the curvature parameters \( \lambda_i, i = 1, 2 \). In addition, we would like to point out that hypothetical conditions (2.13) and (H2)–(H3) actually cover the case of strictly convex inclusions considered in [27].

The rest of this paper is organized as follows. In Section 3, we carry out a linear decomposition of the solution \( u \) to problem (2.1) as \( v_0 \) and \( v_1 \), defined by (3.2) and (3.3) below, and we prove the correspondingly main terms \( \bar{u}_0 \) and \( \bar{u} \) constructed by (2.2) and (2.3), respectively, in Lemma 3.3 and Theorem 3.1. Based on the results obtained in Section 3, we give the proofs of Theorem 2.1 and Theorem 2.4 consisting of the asymptotics of the blow-up factors \( Q[\varphi] \) and \( \Theta \), defined by (3.4) and (3.6), respectively, in Section 4. Examples 5.1 and 6.1 are, respectively, given in Sections 5 and 6. Conclusions and discussions are presented in Section 7.

3. Preliminary.

3.1. Solution split. As in [27], we decompose the solution \( u \) of (2.1) as follows

\[
 u(x) = C_1 v_1(x) + v_0(x), \quad \text{in } D \setminus \overline{D}_1, \tag{3.1}
\]

where \( v_i, i = 0, 1, \) verify

\[
 \begin{cases}
 \Delta v_0 = 0, & \text{in } \Omega, \\
 v_0 = 0, & \text{on } \partial D_1, \\
 v_0 = \varphi(x), & \text{on } \partial D,
\end{cases}
\tag{3.2}
\]

and

\[
 \begin{cases}
 \Delta v_1 = 0, & \text{in } \Omega, \\
 v_1 = 1, & \text{on } \partial D_1, \\
 v_1 = 0, & \text{on } \partial D,
\end{cases}
\tag{3.3}
\]

respectively. Similarly as (2.5) and (2.6), we define a linear functional of \( \varphi \) as follows

\[
 Q[\varphi] = \int_{\partial D_1} \frac{\partial v_0}{\partial \nu}, \tag{3.4}
\]

where \( v_0 \) is defined by (3.2). Denote

\[
 a_{11} := \int_{\Omega} |\nabla v_1|^2 dx. \tag{3.5}
\]

Then, it follows from the third line of (2.1) and decomposition (3.1) that

\[
 C_1 \int_{\partial D_1} \frac{\partial v_1}{\partial \nu} + \int_{\partial D_1} \frac{\partial v_0}{\partial \nu} = 0.
\]
Recalling the definition of $v_1$ and making use of integration by parts, we have
\[
\nabla u = \frac{Q[c]}{a_{11}} \nabla v_1 + \nabla v_0 := \frac{Q[c]}{\Theta} \rho_0^{-1}(n, m; \varepsilon) \nabla v_1 + \nabla v_0,
\]
where $\Theta = a_{11} \rho_0^{-1}(n, m; \varepsilon)$.

### 3.2. A general boundary value problem

To prove the leading terms of $\nabla v_i$, $i = 0, 1$, we first consider the following general boundary value problem:

\[
\begin{cases}
\Delta v = 0, & \text{in } \Omega, \\
v = \psi, & \text{on } \partial D_1, \\
v = 0, & \text{on } \partial D,
\end{cases}
\]

where $\psi \in C^2(\partial D_1)$ is a given scalar function. Note that if $\psi = 1$ on $\partial D_1$, then $v_1 = v$. Extend $\psi \in C^2(\partial D_1)$ to $\psi \in C^2(\overline{\Omega})$ such that $\|\psi\|_{C^2(\Omega)} \leq C\|\psi\|_{C^2(\partial D_1)}$. Construct a cutoff function $\rho \in C^2(\overline{\Omega})$ satisfying $0 \leq \rho \leq 1$, $|\nabla \rho| \leq C$ on $\overline{\Omega}$, and
\[
\rho = 1 \text{ on } \Omega_{\frac{3}{2}R}, \quad \rho = 0 \text{ on } \overline{\Omega \setminus 2R}.
\]

For $x \in \Omega$, we define
\[
\bar{v}(x) = [\rho(x)\psi(x', \varepsilon + h_1(x')) + (1 - \rho(x))\psi(x)]\bar{u}(x),
\]
where $\bar{u}$ is defined by (2.2). Specially,
\[
\bar{v}(x) = \psi(x', \varepsilon + h_1(x'))\bar{u}(x), \quad \text{in } \Omega_R.
\]

Due to (2.2), we have
\[
\|\bar{v}\|_{C^2(\Omega; \Omega_R)} \leq C\|\psi\|_{C^2(\partial D_1)}. \tag{3.9}
\]

Similarly as in [27], we can obtain an asymptotic expansion of the gradient for problem (3.3).

**Theorem 3.1.** Assume as above. Let $v \in H^1(\Omega)$ be a weak solution of (3.7). Then, for a sufficiently small $\varepsilon > 0$,
\[
|\nabla (v - \bar{v})(x)| \leq C\delta^{\frac{m-2}{m}}(|\psi(x', \varepsilon + h_1(x'))| + \delta^{\frac{1}{m}}\|\psi\|_{C^2(\partial D_1)}), \quad \text{in } \Omega_R. \tag{3.10}
\]

Consequently, (3.10), together with choosing $\psi = 1$ on $\partial D_1$, yields that
\[
\nabla v_1 = \nabla \bar{u} + O(1)\delta^{\frac{m-2}{m}}, \quad \text{in } \Omega_R, \tag{3.11}
\]

and
\[
\|\nabla v\|_{L^\infty(\Omega; \Omega R)} \leq C\|\psi\|_{C^2(\partial D_1)},
\]
where $v_1 \in H^1(\Omega)$ is a weak solution of (3.3).

**Remark 3.2.** Note that when $m > 2$, the remainder of order $O(1)$ in [27] is improved to that of order $O(\varepsilon^{1-2/m})$ for $x \in \{x' = 0\}' \cap \Omega_R$ here.

For readers’ convenience, the detailed proof of Theorem 3.1 is left in the Appendix. Similarly, by applying Theorem 3.1, we can find that the leading term of $\nabla v_0$ is $\nabla \bar{u}_0$ in the following.
Lemma 3.3. Assume as above. Let $v_0$ be the weak solution of (3.2). Then, for a sufficiently small $\varepsilon > 0$,
\[
\nabla v_0 = \nabla \bar{u}_0 + O(1)\delta^{\frac{m-2}{m}}(\varphi(x', h(x'))) + \delta^{\frac{2}{m}}\|\varphi\|_{C^2(\partial D)}, \quad \text{in } \Omega_R,
\]
and
\[
\|\nabla x' v_0\|_{L^\infty(\Omega_R)} \leq C\|\varphi\|_{C^2(\partial D)}, \quad \|\nabla v_0\|_{L^\infty(\Omega_R)} \leq C\|\varphi\|_{C^2(\partial D)},
\]
where $\bar{u}_0$ is defined by (2.3).

Proof. Lemma 4.1. Before proving Theorems 2.1 and 2.4, we first give an expansion of $Q[\varphi]$ with respect to $\varepsilon$.

Lemma 4.1. Assume as in Theorems 2.1 and 2.4. Then, for a sufficiently small $\varepsilon > 0$,

(a) if $(S1)$ holds for $m \geq n + k - 1$ in Theorem 2.1,
\[
Q[\varphi] = \eta_M k \rho_k(n, m; \varepsilon)(1 + O(\delta_{\varepsilon, k}));
\]

(b) if $(S1)$ holds for $m < n + k - 1$ in Theorem 2.1,
\[
Q[\varphi] = Q^*[\varphi] + O(1)\varepsilon^{\frac{n+k-1-m}{mn+n-2}};
\]

(c) if $(S2)$ holds in Theorem 2.4,
\[
Q[\varphi] = Q^*[\varphi] + O(1)\varepsilon^{\frac{n+k-1-m}{mn+n-2}};
\]

where $M_k$ and $\delta_{\varepsilon, k}$ are, respectively, defined by (2.11) and (4.1) in the case of $i = k$.

Proof. Step 1. Proof of (4.2). Note that the unit outward normal $\nu$ to $\partial D_1$ is given by
\[
\nu = \frac{\nabla x' h_1(x')}{\sqrt{1 + |\nabla x' h_1(x')|^2}}, \quad \text{in } \Omega_R.
\]
In light of (H2), we obtain that for $i = 1, \ldots, n - 1$,
\[
|\nu_i| \leq C|x'|^{m-1}, \quad |\nu_n| \leq 1, \quad \text{in } \Omega_R.
\]

Recalling the definition of $Q[\varphi]$ and in view of assumptions (H1) and (S1), it follows from (3.12)–(3.13) and (4.5) that
\[
Q[\varphi] = \int_{\partial D_1} \partial_{x_n} v_0 \nu_n + \int_{\partial D_1} \sum_{i=1}^{n-1} \partial_{x_i} v_0 \nu_i = \int_{|x'| < R} \frac{\eta|x'|^k}{\varepsilon + \lambda|x'|^m} + \int_{|x'| < R} \left( \frac{\eta|x'|^k}{\varepsilon + h_1 - h} - \frac{\eta|x'|^k}{\varepsilon + \lambda|x'|^m} \right) + O(1)||\varphi||_{C^2(\partial D)}
\]
\[
\begin{align*}
&= \int_{|x'|<R} \eta|x'|^k + O(1) \int_{|x'|<R} \frac{\eta|x'|^{k+1}}{\varepsilon + \lambda|x'|^{m}} + O(1) \|\varphi\|_{C^2(\partial D)} \\
&= (n-1) \omega_{n-1} \int_{0}^{R} \eta \frac{s^{n-k-2}}{\varepsilon + \lambda s^{m}} + O(1) \int_{0}^{R} \frac{\eta s^{n-k-1}}{\varepsilon + \lambda s^{m}} + O(1) \|\varphi\|_{C^2(\partial D)} \\
&= \eta M_{k} \rho_k (n, m; \varepsilon) (1 + O(\bar{r})).
\end{align*}
\]

Then (4.2) is established.

**Step 2.** Proofs of (4.3) and (4.4). In view of the definitions of \( Q[\varphi] \) and \( Q^{*}[\varphi] \), it follows from integration by parts that

\[
Q[\varphi] = - \int_{\partial D} \frac{\partial v_1}{\partial \nu} \varphi(x), \quad Q^{*}[\varphi] = - \int_{\partial D} \frac{\partial v_{1}^{*}}{\partial \nu} \varphi(x),
\]

where \( v_{1}^{*} \) and \( v_1 \) are defined by (2.8) and (3.3), respectively. Then,

\[
Q[\varphi] - Q^{*}[\varphi] = - \int_{\partial D} \frac{\partial (v_1 - v_{1}^{*})}{\partial \nu} \cdot \varphi(x).
\]

To estimate \( v_1 - v_{1}^{*} \), we first introduce a scalar auxiliary function \( \bar{u}^{*} \) satisfying \( \bar{u}^{*} = 1 \) on \( \partial D_{1}^{*} \setminus \{0\} \), \( \bar{u}^{*} = 0 \) on \( \partial D \), and

\[
\bar{u}^{*} = \frac{x_{n} - h(x')}{h_{1}(x') - h(x')}, \quad \text{in } \Omega_{2R}, \quad \|\bar{u}^{*}\|_{C^2(\Omega \cap \Omega_{H})} \leq C,
\]

where \( \Omega_{r} := \Omega \cap \{|x'| < r\} \), \( 0 < r \leq 2R \). In view of (H2), we obtain that for \( x \in \Omega_{r} \),

\[
|\nabla_{x'}(\bar{u} - \bar{u}^{*})| \leq \frac{C}{|x'|}, \quad |\partial_{x_n}(\bar{u} - \bar{u}^{*})| \leq \frac{C \varepsilon}{|x'|^{m} (\varepsilon + |x'|^{m})}.
\]

(4.6)

Applying Theorem 3.1 to (2.8), it follows that for \( x \in \Omega_{r} \),

\[
|\nabla(v_{1}^{*} - \bar{u}^{*})| \leq C|x'|^{m-2},
\]

(4.7)

and

\[
|\nabla_{x'} v_{1}^{*} \leq \frac{C}{|x'|}, \quad |\partial_{x_n} v_{1}^{*} \leq \frac{C}{|x'|^{m}}.
\]

(4.8)

For 0 < \( r \leq R \), denote

\[
C_{r} := \left\{ x \mid -\varepsilon + \min (2h - h_{1}) \leq x_{n} \leq 2\varepsilon + \max (2h_{1} - h), \quad |x'| < r \right\}.
\]

We now divide into two steps to estimate \( |Q[\varphi] - Q^{*}[\varphi]| \).

**Step 2.1.** Note that \( v_1 - v_{1}^{*} \) solves

\[
\begin{cases}
\Delta (v_1 - v_{1}^{*}) = 0, & \text{in } D \setminus (D_{1} \cup D_{2}^{*}), \\
v_1 - v_{1}^{*} = 1 - v_{1}^{*}, & \text{on } \partial D_{1} \setminus D_{1}^{*}, \\
v_1 - v_{1}^{*} = v_1 - 1, & \text{on } \partial D_{2} \setminus (D_{1} \cup \{0\}), \\
v_1 - v_{1}^{*} = 0, & \text{on } \partial D.
\end{cases}
\]

We first estimate \( |v_1 - v_{1}^{*}| \) on \( \partial(D_{1} \cup D_{2}^{*}) \setminus C_{\gamma} \), where \( 0 < \gamma < 1/2 \) to be determined later. In light of the definition of \( v_{1}^{*} \), we derive

\[
|\partial_{x_n} v_{1}^{*} \leq C, \quad \text{in } \Omega_{r} \setminus \Omega_{2r}.
\]

Therefore,

\[
|v_1 - v_{1}^{*} \leq C \varepsilon, \quad \text{for } x \in \partial D_{1} \setminus D_{1}^{*}.
\]

(4.9)
It follows from (3.11) that
\[ |v_1 - v_1^*| \leq C \varepsilon^{1-m\gamma}, \quad \text{on } \partial D_1 \setminus (D_1 \cup C_{\varepsilon^3}). \quad (4.10) \]
Combining Theorem 3.1 and (4.6)–(4.7), we obtain that for \( x \in \Omega_R \cap \{|x'| = \varepsilon^3\}, \)
\[ |\partial x_n(v_1 - v_1^*)| \leq |\partial x_n(v_1 - \bar{u})| + |\partial x_n(\bar{u} - \bar{u}^*)| \leq C \left( \frac{1}{\varepsilon^{2m\gamma - 1}} + \varepsilon^{(m-2)\gamma} \right), \]
which together with the standard interior and boundary estimates, leads to that, for \( x \in \Omega_R \cap \{|x'| = \varepsilon^3\}, \)
\[ |\partial x_n(v_1 - v_1^*)| \leq C \varepsilon^{1-m\gamma} + \varepsilon^{2(m-1)\gamma}. \quad (4.11) \]
Take \( \gamma = \frac{1}{m+1}. \) Then, it follows from (4.9)–(4.11) that
\[ |v_1 - v_1^*| \leq C \varepsilon^{\frac{m}{m+1}}, \quad \text{on } \partial (D \setminus (D_1 \cup D_1^* \cup C_{\varepsilon^3})). \]
Making use of the maximum principle, we obtain
\[ |v_1 - v_1^*| \leq C \varepsilon^{\frac{m}{m+1}}, \quad \text{in } D \setminus (D_1 \cup D_1^* \cup C_{\varepsilon^3}). \]
This, together with the standard interior and boundary estimates, leads to that, for any \( \varepsilon < \frac{1}{m+1}, \)
\[ \|
abla(v_1 - v_1^*)\| \leq C \varepsilon^{m\gamma - \frac{m}{m+1}}, \quad \text{on } \partial D \setminus C_{\varepsilon^3}^{\frac{1}{m+1} - \gamma}, \]
which implies that
\[ |A^{out}| = \left| \int_{\partial D \setminus C_{\varepsilon^3}^{\frac{1}{m+1} - \gamma}} \frac{\partial (v_1 - v_1^*)}{\partial \nu} \cdot \varphi(x) \right| \leq C \|
abla \varphi\|_{L^\infty(\partial D)} \varepsilon^{m\gamma - \frac{m}{m+1}}, \quad (4.12) \]
where \( \frac{1}{m+1} \varepsilon < \frac{1}{m+1} \) to be determined later.

**Step 2.2.** We further estimate
\[ A^{in} := \int_{\partial D \cap C_{\varepsilon^3}^{\frac{1}{m+1} - \gamma}} \frac{\partial (v_1 - v_1^*)}{\partial \nu} \cdot \varphi(x) \]
\[ = \int_{\partial D \cap C_{\varepsilon^3}^{\frac{1}{m+1} - \gamma}} \frac{\partial (w_1 - w_1^*)}{\partial \nu} \cdot \varphi(x) + \int_{\partial D \cap C_{\varepsilon^3}^{\frac{1}{m+1} - \gamma}} \frac{\partial (\bar{u} - \bar{u}^*)}{\partial \nu} \cdot \varphi(x) \]
\[ = A_w + A_u, \]
where \( w_1 = v_1 - \bar{u} \) and \( w_1^* = v_1^* - \bar{u}^*. \) To begin with, applying Theorem 3.1, we obtain that
\[ |A_w| \leq C \eta \int_{\partial D \cap C_{\varepsilon^3}^{\frac{1}{m+1} - \gamma}} |x'|^{m+k-2} \leq C \eta \varepsilon^{(\frac{m}{m+1} - \gamma)(m+n+k-3)}. \quad (4.13) \]
To estimate \( A_u, \) we split it into two parts as follows:
\[ A_u = \int_{\partial D \cap C_{\varepsilon^3}^{\frac{1}{m+1} - \gamma}} \sum_{i=1}^{n-1} \partial x_n(\bar{u} - \bar{u}^*) \nu_i \varphi(x) + \int_{\partial D \cap C_{\varepsilon^3}^{\frac{1}{m+1} - \gamma}} \partial x_n(\bar{u} - \bar{u}^*) \nu_n \varphi(x) \]
\[ = A^1_u + A^2_u. \]

**Case 1.** If (S1) holds for \( m < n + k - 1 \) in Theorem 2.1, owing to (4.5)–(4.6), we obtain that
\[ |A^1_u| \leq C \eta \varepsilon^{(\frac{m}{m+1} - \gamma)(n+k+m-3)}, \quad |A^2_u| \leq C \eta \varepsilon^{(\frac{m}{m+1} - \gamma)(n+k-m-1)}. \]
Then
\[ |A_u| \leq C\eta\varepsilon^{\left(\frac{1}{n-k}\right)}(n+k-m-1). \]
This, together with (4.12)–(4.13) and picking \( \tilde{\gamma} = \frac{n+k-2}{(n+k-1)(m+1)} \), yields that
\[ |Q[\varphi] - Q^*[\varphi]| \leq C(\eta + \|\varphi\|_{L^\infty(\partial D)})\varepsilon^{\left(\frac{1}{n+k-m-1}\right)}. \]

**Case 2.** If \((S2)\) holds in Theorem 2.4, based on the fact that the integrating domain is symmetric with respect to \( x_i, i = 1, \ldots, n - 1 \), we have
\[ |A_u| \leq C\eta\varepsilon^{\left(\frac{1}{n-k}\right)}(n+m-2), \quad A_u^2 = 0. \]

Hence,
\[ |A_u| \leq C\eta\varepsilon^{\left(\frac{1}{n-k}\right)}(n+m-2). \]
This, together with (4.12)–(4.13) and taking \( \tilde{\gamma} = \frac{2m+n-3}{(2m+n-2)(m+1)} \), leads to that
\[ |Q[\varphi] - Q^*[\varphi]| \leq C(\eta + \|\varphi\|_{L^\infty(\partial D)})\varepsilon^{\left(\frac{m+n-2}{m+n-2}\right)}. \]

Consequently, we complete the proofs of (4.3) and (4.4). \( \square \)

### 4.2. Expansion of \( \Theta \)

Before stating the asymptotic of \( \Theta \) with respect to \( \varepsilon \), we first introduce a notation used in the following. Denote
\[ \Theta := \sum_{i=1}^n \left| \nabla v_i \right|^2 + 2 \int_{\Omega_R^+} \nabla \tilde{u} \cdot \nabla (v_i^* - \tilde{u}^*) + \int_{\Omega_R^+} (|\nabla (v_i^* - \tilde{u}^*)|^2 + |\nabla x \cdot \tilde{u}^*|^2). \]  

**Lemma 4.2.** Assume as in Theorems 2.1 and 2.4. Then, for a sufficiently small \( \varepsilon > 0 \),

(i) for \( m \geq n - 1 \),
\[ \Theta = \mathcal{M}_0 + O(\hat{r}_{\varepsilon,0}); \]

(ii) for \( m < n - 1 \),
\[ \Theta = a_{i1}^* + O(\varepsilon^{\min\left(\frac{1}{4}, \frac{n-1-k}{6m}\right)}), \]

where \( a_{i1}^* \) is defined by (2.7), \( \mathcal{M}_0 \) and \( \hat{r}_{\varepsilon,0} \) are, respectively, defined by (2.11) and (4.1) in the case of \( i = 0 \).

**Remark 4.3.** We here would like to remark that the computation method with respect to the energy \( a_{i1} \) was first exhibited in [25].

**Proof.** In view of the definition of \( \Theta \) in (3.6), it suffices to calculate the energy \( a_{i1} \) for the purpose of expanding \( \Theta \). Fix \( \tilde{\gamma} = \frac{1}{6m} \). We first split \( a_{i1} \) into three parts as follows:
\[ a_{i1} = \int_{\Omega_R} \left| \nabla v_i \right|^2 + \int_{\Omega_R \setminus \Omega_R} \left| \nabla v_i \right|^2 + \int_{\Omega_R \setminus \Omega} \left| \nabla v_i \right|^2 =: I + II + III. \]

**Step 1.** As for I, recalling the definition of \( \tilde{u} \) and using Theorem 3.1, we obtain that
\[ I = \int_{\Omega_R} \left| \frac{\partial z_\gamma}{\partial x^*} \tilde{u} \right|^2 + \int_{\Omega_R} \left| \nabla z^* \tilde{u} \right|^2 + 2 \int_{\Omega_R} \nabla \tilde{u} \cdot \nabla (v_i - \tilde{u}) + \int_{\Omega_R} |\nabla (v_i - \tilde{u})|^2 \]
\[ = \int_{|x'| < \tilde{r} \varepsilon} \frac{dx'}{1 + h_i(x')} + O(1)\varepsilon^{\left(\frac{n+m-3}{6m}\right)}. \]

For the second term II, we further decompose it into three parts as follows:
\[ II_1 = \int_{(\Omega_R \setminus \Omega_R)} \left| \nabla v_i \right|^2, \]
Due to the fact that the thickness of \((\Omega_R \setminus \Omega_\varepsilon) \setminus (\Omega_* \setminus \Omega_{**})\) is \(\varepsilon\), it follows from (3.11) that

\[
II_1 \leq C\varepsilon \int_{\varepsilon r < |x'| < R} \frac{dx'}{|x'|^{2m}} \leq C \begin{cases} \varepsilon^{\frac{4m+n-1}{6m}}, & m > \frac{n-1}{2}, \\ \varepsilon |\ln \varepsilon|, & m = \frac{n-1}{2}, \\ \varepsilon, & m < \frac{n-1}{2}. \end{cases} \quad (4.18)
\]

By picking \(\gamma = \frac{1}{2m}\) in Step 2.1 of the proof of Lemma 4.1, it follows from (4.9)–(4.11) and the maximum principle that

\[
|v_1 - v_1^*| \leq C\varepsilon^{\frac{1}{2}}, \quad \text{in } D \setminus (D_1 \cup D_1^* \cup C_{\varepsilon^{\frac{2}{3m}}}). \quad (4.19)
\]

For \(\varepsilon^{\frac{1}{3m}} \leq |z'| \leq R\), making a change of variable

\[
\begin{align*}
&x' - z' = |z'|^m y', \\
x_n = |z'|^m y_n,
\end{align*}
\]

\(\Omega_{|z'|+|z'|^m} \setminus \Omega_{|z'|}\) and \(\Omega_{|z'|+|z'|^m} \setminus \Omega_{*|z'|}\) become two nearly unit-size squares (or cylinders) \(Q_1\) and \(Q_1^*\), respectively. Let

\[
V_1(y) = v_1(z' + |z'|^m y', |z'|^m y_n), \quad \text{in } Q_1,
\]

and

\[
V_1^*(y) = v_1^*(z' + |z'|^m y', |z'|^m y_n), \quad \text{in } Q_1^*.
\]

Due to the fact that \(0 < V_1, V_1^* < 1\), it follows from the standard elliptic estimate that

\[
|\nabla^2 V_1| \leq C, \quad \text{in } Q_1, \quad \text{and } |\nabla^2 V_1^*| \leq C, \quad \text{in } Q_1^*.
\]

By virtue of an interpolation with (4.19), we obtain

\[
|\nabla (V_1 - V_1^*)| \leq C\varepsilon^{\frac{1}{2}} (1 - \frac{1}{2}) \leq C\varepsilon^{\frac{1}{4}}.
\]

Then rescaling it back to \(v_1 - v_1^*\) and in view of \(\varepsilon^{\frac{1}{3m}} \leq |z'| \leq R\), we have

\[
|\nabla (v_1 - v_1^*)(x)| \leq C\varepsilon^{\frac{1}{2}} |z'|^{-m} \leq C\varepsilon^{\frac{2}{3m}}, \quad x \in \Omega_{|z'|+|z'|^m} \setminus \Omega_{|z'|}.
\]

That is,

\[
|\nabla (v_1 - v_1^*)| \leq C\varepsilon^{\frac{2}{3m}}, \quad \text{in } D \setminus (D_1 \cup D_1^* \cup C_{\varepsilon^{\frac{2}{3m}}}). \quad (4.20)
\]

Then combining (4.8) and (4.20), we obtain

\[
|II_2| \leq C\varepsilon^{\frac{2}{3m}}. \quad (4.21)
\]

As for \(II_3\), it follows from (4.7) and (4.8) that

\[
II_3 = \int_{\Omega_{R} \setminus \Omega_{*}} |\nabla u_1^*|^2 + 2 \int_{\Omega_{R} \setminus \Omega_{*}} \nabla u_1^* \cdot \nabla (v_1^* - \bar{u}) + \int_{\Omega_{R} \setminus \Omega_{*}} |\nabla (v_1^* - \bar{u})|^2
\]

\[
= \int_{\varepsilon < |x'| < R} \frac{dx'}{h_1(x') - h(x')} + A_R - \int_{\Omega_{*} \setminus \Omega_{R}} |\nabla v_1^*|^2 + O(1)\varepsilon^{(n+m-3)\gamma},
\]
where \( A_R^* \) is defined by (4.14). This, together with (4.18) and (4.21), leads to that

\[
II = \int_{|x'|<|x'|<R} \frac{dx'}{h_1(x') - h(x')} + A_R^* - \int_{\Omega^* \Omega_R^*} |\nabla v_1|^2 + O(1)\varepsilon^{\frac{1}{2}}. \tag{4.22}
\]

For the last term \( III \), due to the fact that \( |\nabla v_1| \) is bounded in \( D_1^* \setminus (D_1 \cup \Omega_R^*) \) and \( D_1 \setminus D_1^* \) and the fact that the volume of \( D_1 \setminus (D_1 \cup \Omega_R) \) and \( D_1 \setminus D_1^* \) is of order \( O(\varepsilon) \), it follows from (4.20) that

\[
III = \int_{D_1 \setminus (D_1 \cup D_1^* \cup \Omega_R)} |\nabla v_1|^2 + O(1)\varepsilon
\]

This, together with (4.17) and (4.22), yields that

\[
a_{11} = \int_{|x'|<|x'|<R} \frac{dx'}{h_1(x') - h(x')} + \int_{|x'|<|x'|<\varepsilon} \frac{dx'}{\varepsilon + h_1(x') - h(x')} + A_R^* + O(1)\varepsilon^{\frac{1}{2}}. \tag{4.23}
\]

**Step 2.** We now calculate the major parts of the singularities of the energy as follows:

\[
\int_{|x'|<|x'|<\varepsilon} \frac{dx'}{\varepsilon + h_1(x') - h(x')},
\]

(i) For \( m \geq n - 1 \),

\[
\int_{|x'|<R} \frac{dx'}{\varepsilon + h_1 - h} + \int_{|x'|<|x'|<R} \frac{\varepsilon dx'}{(h_1 - h)(\varepsilon + h_1 - h)}
\]

\[
= \int_{|x'|<R} \frac{1}{\varepsilon + \lambda|x'|^m} + \int_{|x'|<R} \left( \frac{1}{\varepsilon + h_1 - h} - \frac{1}{\varepsilon + h_1 - h} \right) + O(1)\varepsilon^{4m+n-1}
\]

\[
= (n-1)\omega_{n-1} \int_{0}^{R} \varepsilon^{n-2} + O(1)\varepsilon^{n-2} + O(1)\varepsilon^{n-2}.
\]

(ii) For \( m < n - 1 \),

\[
\int_{|x'|<R} \frac{dx'}{h_1 - h} - \int_{|x'|<|x'|<\varepsilon} \frac{\varepsilon dx'}{(h_1 - h)(\varepsilon + h_1 - h)} = \int_{\Omega_R^*} |\partial_{x_n} \tilde{u}^*|^2 + O(1)\varepsilon^{n-1}.
\]

Therefore, it follows from **Step 1** and **Step 2** that Lemma 4.2 holds.

**Proof of Theorems 2.1 and 2.4.** Take the proof of Theorem 2.1 for example. The proof of Theorem 2.4 is the same and thus omitted. Recalling decomposition (3.6) and combining the results derived in Theorem 3.1 and Lemma 3.3, it follows that

(i) for \( m \geq n + k - 1 \), using (4.2) and (4.15), then we have

\[
\frac{Q[\varphi]\rho_k^{-1}(n; m; \varepsilon)}{\Theta} = \frac{M_k}{M_0} \frac{1}{1 - \frac{M_k}{M_0}} + \frac{Q[\varphi]\rho_k^{-1}(n; m; \varepsilon) - M_k}{\Theta} = \frac{\eta M_k}{M_0} (1 + O(\varepsilon)),
\]
which yields that
\[
\nabla u = \frac{Q[\varphi]}{M_0} \rho_0^{-1}(n, m; \varepsilon) \nabla v_1 + \nabla v_0
\]
\[
= \frac{nM_k}{M_0} (1 + O(r_\varepsilon))(\rho_{k,0}(n, m; \varepsilon)(\nabla \bar{u} + O(\delta^{\frac{m-2}{m}})))
\]
\[
+ \nabla \bar{u}_0 + O(1)\delta^{\frac{m-2}{m}}\|\varphi\|_{C^2(\partial D)}
\]
\[
= \frac{nM_k}{M_0} (1 + O(r_\varepsilon))(\rho_{k,0}(n, m; \varepsilon)\nabla \bar{u} + \nabla \bar{u}_0 + O(1)\delta^{\frac{m-2}{m}}\|\varphi\|_{C^2(\partial D)});
\]
\(\text{(ii)} \) for \( n - 1 \leq m < n - k - 1 \), utilizing (4.3) and (4.15), then we have
\[
\frac{Q[\varphi]}{M_0} \frac{1}{1 - \frac{M_k}{M_0}} + \frac{Q[\varphi] - Q^*[\varphi]}{M_0} = \frac{Q^*[\varphi]}{M_0} (1 + O(r_\varepsilon)),
\]
which implies that
\[
\nabla u = \frac{Q[\varphi]}{a_{11}} \rho_0^{-1}(n, m; \varepsilon) \nabla v_1 + \nabla v_0
\]
\[
= \frac{Q^*[\varphi]}{a_{11}} (1 + O(r_\varepsilon))(\nabla \bar{u} + O(\delta^{\frac{m-2}{m}})) + \nabla \bar{u}_0 + O(1)\delta^{\frac{m-2}{m}}\|\varphi\|_{C^2(\partial D)}
\]
\[
= \frac{Q^*[\varphi]}{a_{11}} (1 + O(r_\varepsilon))\nabla \bar{u} + \nabla \bar{u}_0 + O(1)\delta^{\frac{m-2}{m}}\|\varphi\|_{C^2(\partial D)};
\]
\(\text{(iii)} \) for \( m < n - 1 \), in view of \( a_{11} \neq 0 \), making use of (4.3) and (4.16), then we have
\[
\frac{Q[\varphi]}{a_{11}} = \frac{Q^*[\varphi]}{a_{11}} \frac{1}{1 - \frac{a_{11}}{a_{11}} - \frac{a_{11}}{a_{11}}} + \frac{Q[\varphi] - Q^*[\varphi]}{a_{11}} = \frac{Q^*[\varphi]}{a_{11}} (1 + O(r_\varepsilon)),
\]
which reads that
\[
\nabla u = \frac{Q[\varphi]}{a_{11}} \nabla v_1 + \nabla v_0
\]
\[
= \frac{Q^*[\varphi]}{a_{11}} (1 + O(r_\varepsilon))\nabla \bar{u} + \nabla \bar{u}_0 + O(1)\delta^{\frac{m-2}{m}}\|\varphi\|_{C^2(\partial D)};
\]
where \( r_\varepsilon \) is defined in (2.9) and \( \hat{r}_{\varepsilon, i}, i = 0, k \) are defined by (4.1). Consequently, we prove that Theorem 2.1 holds. □

5. The core-shell structure with eccentric circles and spheres. This section is to consider the perfect conductivity problem in the presence of eccentric core-shell geometry with circular and spherical boundaries, see Fig. 1 for eccentric circles in two dimensions. Specially, for two positive constants \( 0 < r_1 < r_2 \), independent of \( \varepsilon \), we let
\[
D_1 := B_{r_1}(0', \varepsilon + r_1), \quad D := B_{r_2}(0', r_2).
\]
By using Taylor expansion, we see
\[
h_1(x') - h(x') = \lambda_0 |x'|^2 + O(|x'|^4), \quad \text{in } \Omega_{r_0}, \quad \text{with } \lambda_0 = \frac{1}{2} \left( \frac{1}{r_1} - \frac{1}{r_2} \right),
\]
where \( \lambda_0 \) represents the mean value of curvature difference between two spheres and \( r_0 \) is a small \( \varepsilon \)-independent constant satisfying \( 0 < r_0 < r_1 \). Then, we have
Example 5.1. Assume as above, condition (S1) or (S2) holds, $Q^*[\varphi] \neq 0$. Let $u \in H^1(D) \cap C^1(\Omega)$ ($n \geq 2$) be the solution of (2.1). Then for a sufficiently small $\varepsilon > 0$ and $x \in \Omega_{r_0}$,

(i) for $n = 2$,

$$\nabla u = \frac{\sqrt{\lambda_0}Q^*[\varphi]\sqrt{\varepsilon}}{\pi + \sqrt{\lambda_0}M_2 \sqrt{\varepsilon}}(1 + O(\varepsilon^{\frac{1}{2}}))\nabla \bar{u} + \nabla \bar{u}_0 + O(1)\|\varphi\|_{C^2(\partial D)}; \quad (5.2)$$

(ii) for $n = 3$,

$$\nabla u = \frac{\lambda_0 Q^*[\varphi]}{\pi \ln \varepsilon + \lambda_0 M_3}(1 + O(\varepsilon^{\frac{1}{2}} \ln \varepsilon^{-1}))\nabla \bar{u} + \nabla \bar{u}_0 + O(1)\|\varphi\|_{C^2(\partial D)}; \quad (5.3)$$

(iii) for $n \geq 4$,

$$\nabla u = \frac{Q^*[\varphi]}{a_{11}^*}(1 + O(\varepsilon^{\frac{1}{2}}))\nabla \bar{u} + \nabla \bar{u}_0 + O(1)\|\varphi\|_{C^2(\partial D)};$$

where $\lambda_0$ is given in (5.1) and the constants $\bar{M}_n$, $n = 2, 3$ are defined by (5.14) below.

Remark 5.2. Although the asymptotic results in Theorem 2.1 and Theorem 2.4 actually cover the case of disks and spheres in Example 5.1, we can provide a more sharp characterization in terms of the singular behavior of the electric field in virtue of the explicit geometry constants $\bar{M}_n$, $n = 2, 3$ captured in (5.2)–(5.3). These two geometry constants indicate that the dependence on radius is expressed in the form of the curvature difference. This is different from the interior asymptotics in [19, 26, 34] that for two adjacent disks in two dimensions and spheres in three dimensions, the dependence on radius is expressed as the sum of curvature. Furthermore, the geometry constants are independent of the distance parameter $\varepsilon$ and the length $r_0$ of the narrow region, which is vital for numerical computations and simulations in future investigation.

Remark 5.3. In three dimensions, when the interfacial boundaries of inclusions and the matrix are strictly convex, that is, there exist two positive constants $\lambda_i$, $i = 1, 2$ independent of $\varepsilon$ such that

$$h_1(x') - h(x') = \lambda_1 x_1^2 + \lambda_2 x_2^2, \quad x' \in B_{2R}',$$
it follows from the proof of Example 5.1 with minor modification that for $x \in \Omega_R$,
\[
\nabla u = \frac{\sqrt{\lambda_1 \lambda_2} Q^*|\varphi|}{\pi |\ln \varepsilon| + G_{\lambda_1, \lambda_2}^*} (1 + O(\varepsilon^{1/2} |\ln \varepsilon|^{-1})) \nabla \bar{u} + \nabla \bar{u}_0 + O(1) \|\varphi\|_{C^2(\partial D)},
\]
where $G_{\lambda_1, \lambda_2}^* = \sqrt{\lambda_1 \lambda_2} A_R^* + 2\pi \ln R - 2 \int_0^\pi \ln(\lambda_1^{-1} \cos^2 \theta + \lambda_2^{-1} \sin^2 \theta) d\theta$ with $A_R^*$ defined by (4.14) above. It is worth emphasizing that Li, Li and Yang [25] were the first to give the precise computation of the energy in the presence of two strictly convex inclusions.

**Remark 5.4.** Note that the constants $\tilde{\mathcal{M}}_n$, $n = 2, 3$ are retained in the asymptotic expansions (5.2)–(5.3) in Example 5.1 but there is no similar constant appearing in Theorem 2.1 and Theorem 2.4. This is primarily caused by the fact that if $m > n - 1$, the difference $\int_{|x| < R} (\frac{1}{\pi k_1} - \frac{1}{\pi k_2})$ under condition (H1) is not of constant order $O(1)$ and possesses the singularity, which lets other constants such as $A_R^*$ be contained within it and not exhibited in the expansions. The difference on the calculation of the energy can be seen in Lemma 4.2 and Lemma 5.6.

Under the assumptions in Example 5.1, we know that $m = 2$ and $n, k \geq 2$. This implies
\[
\frac{n + k - 1 - m}{(n + k - 1)(m + 1)} = \frac{1}{3} - \frac{2}{3(n + k - 1)} \geq \frac{1}{9},
\]
and
\[
\frac{m + n - 2}{(m + 1)(2m + n - 2)} = \frac{1}{3} - \frac{2}{3(n + 2)} \geq \frac{1}{6}.
\]
Then recalling the asymptotic results (4.3)–(4.4) of the blow-up factor $Q[\varphi]$, we can obtain a unified expansion as follows:

**Corollary 5.5.** Assume as in Example 5.1. Then for a sufficiently small $\varepsilon > 0,$
\[
Q[\varphi] = Q^*[\varphi] + O(1)\varepsilon^{1/2}. \tag{5.4}
\]

Note that in this example the constant $\mathcal{M}_0$ defined in (2.11) becomes
\[
\mathcal{M}_0 = \pi \lambda_0^{\frac{1}{m-n}}, \quad \text{with} \quad \lambda_0 = \frac{1}{2} \left( \frac{1}{r_1} - \frac{1}{r_2} \right).
\]
With regard to the asymptotic expansion of the energy $a_{11}$, we can give a more precise calculation of the energy in the presence of spherical inclusions than that of general $m$-convex conditions (H1)–(H3).

**Lemma 5.6.** Assume as in Example 5.1. Then for a sufficiently small $\varepsilon > 0$,

(i) for $n = 2, 3$,
\[
a_{11} = \mathcal{M}_0 \rho_0(n, 2; \varepsilon) + \tilde{\mathcal{M}}_n + O(1)\varepsilon^{1/2}; \tag{5.5}
\]

(ii) for $n \geq 4$,
\[
a_{11} = a_{11}^* + O(1)\varepsilon^{1/2}, \tag{5.6}
\]
where $\lambda_0$ is defined in (5.1) and the constant $\tilde{\mathcal{M}}_n$ is defined by (5.14) below.

**Proof.** By the same argument as in (4.23), we have
\[
a_{11} = \int_{|x'| < |x'| < r_0} \frac{dx'}{h_1(x') - h(x')} + \int_{|x'| < |x'| < r_0} \frac{dx'}{\varepsilon + h_1(x') - h(x')}.
\]
\[ A_r^* := \int_{\Omega^* \setminus \Omega^*_r} |\nabla v_i^*|^2 + 2 \int_{\Omega^*_r} \nabla u^* \cdot \nabla (v_i^* - u^*) + \int_{\Omega^*_r} (|\nabla (v_i^* - u^*)|^2 + |\nabla v_i^*|^2). \]

Then in order to derive an expansion of the energy \( a_{11} \), it suffices to calculate the singular parts as follows:

\[
\int_{\varepsilon^n < |x'| < \varepsilon^n} \frac{dx'}{h_1(x') - h(x')} + \int_{|x'| < \varepsilon^n} \frac{dx'}{\varepsilon + h_1(x') - h(x')}. \]  

To begin with, it follows from (5.1) that

\[
\int_{\varepsilon^n < |x'| < \varepsilon^n} \left( \frac{1}{h_1 - h} - \frac{1}{\lambda_0|x'|^2} \right) dx' = \int_{|x'| < \varepsilon^n} O(1) dx' = C^* + O(1) \varepsilon^{n-3},
\]

where \( C^* \) depends on \( \lambda_0, n, r_0 \) but not \( \varepsilon \). Then

\[
\int_{|x'| < \varepsilon^n} \frac{dx'}{h_1 - h} = \int_{|x'| < \varepsilon^n} \frac{dx'}{\lambda_0|x'|^2} + C^* + O(1) \varepsilon^{n-1}.
\]

Analogously, we have

\[
\int_{|x'| < \varepsilon^n} \frac{dx'}{\varepsilon + h_1 - h} = \int_{|x'| < \varepsilon^n} \frac{dx'}{\varepsilon + \lambda_0|x'|^2} + O(1) \varepsilon^{n-1}.
\]

Then \( a_{11} \) becomes

\[
a_{11} = \int_{|x'| < \varepsilon^n} \frac{dx'}{\lambda_0|x'|^2} + \int_{|x'| < \varepsilon^n} \frac{dx'}{\varepsilon + \lambda_0|x'|^2} + M_{r_0}^* + O(1) \varepsilon^{n-1},
\]

where \( M_{r_0}^* = A_r^* + C^*. \) We divide into three cases to discuss as follows:

(i) if \( n = 2 \), then

\[
\begin{align*}
\pi & \int_{\varepsilon^n < |x'| < \varepsilon^n} \frac{dx'}{\lambda_0|x|^2} + \int_{|x'| < \varepsilon^n} \frac{dx'}{\varepsilon + \lambda_0|x|^2} \\
= & 2 \left( \int_{0}^{+\infty} \frac{1}{\varepsilon + \lambda_0 x^2} + \int_{\varepsilon^n}^{+\infty} \frac{1}{\lambda_0 x^2} + \int_{\varepsilon^n}^{+\infty} \frac{1}{\lambda_0 x^2} \right) \\
= & \frac{\pi}{\sqrt{\lambda_0}} \frac{1}{\sqrt{\varepsilon}} + \frac{2}{\lambda_0 r_0^2} + O(1) \varepsilon^{\frac{3}{2}};
\end{align*}
\]

(ii) if \( n = 3 \), then

\[
\begin{align*}
\pi & \int_{\varepsilon^n < |x'| < \varepsilon^n} \frac{dx'}{\lambda_0|x|^2} + \int_{|x'| < \varepsilon^n} \frac{dx'}{\varepsilon + \lambda_0|x|^2} \\
= & \int_{|x'| < r_0} \frac{dx'}{\varepsilon + \lambda_0|x|^2} + \int_{|x'| < r_0} \frac{dx'}{\lambda_0|x|^2} \frac{\varepsilon}{\varepsilon + \lambda_0 s^2} \\
= & 2\pi \int_{0}^{s} \frac{1}{\varepsilon + \lambda_0 s^2} + O(1) \varepsilon^{\frac{3}{2}} \\
= & \frac{\pi}{\lambda_0} |\ln \varepsilon| + \frac{\pi (\ln \lambda_0 + 2 \ln r_0)}{\lambda_0} + O(1) \varepsilon^{\frac{3}{2}};
\end{align*}
\]

(iii) if \( n \geq 4 \), from (5.7), we see

\[
\int_{\varepsilon^n < |x'| < \varepsilon^n} \frac{dx'}{h_1(x') - h(x')} + \int_{|x'| < \varepsilon^n} \frac{dx'}{\varepsilon + h_1(x') - h(x')}
\]
3.1 \quad 5.7
\text{Lemma 5.4} \quad 5.5 \quad 5.10
\text{Proof of Example 5.1}. \text{On one hand, for } n = 2, 3, \text{denote}
\[
\tilde{M}_n = \frac{M_n}{M_0}.
\]
In view of decomposition (3.6), it follows from (5.4) and (5.5) that
\[
\frac{Q[\varphi]}{\Theta} - \frac{Q^*[\varphi]}{M_0} = \frac{Q^*[\varphi]}{M_0} \frac{- M_n^* \rho_0^{-1}(n, 2; \varepsilon) + O(\varepsilon \delta \rho_0^{-1}(n, 2; \varepsilon))}{M_n^*} + O(\varepsilon \frac{\delta \rho_0^{-1}(n, 2; \varepsilon)}{\Theta})
= \frac{Q^*[\varphi]}{M_0} \frac{- M_n^* \rho_0^{-1}(n, 2; \varepsilon) + O(\varepsilon \delta \rho_0^{-1}(n, 2; \varepsilon))}{1 + M_n^* \rho_0^{-1}(n, 2; \varepsilon)} + O(\varepsilon \rho_0^{-1}(n, 2; \varepsilon, \varepsilon^{\frac{1}{2}})),
\]
which yields that
\[
\frac{Q[\varphi]}{\Theta} = \frac{Q^*[\varphi]}{M_0} \frac{1}{1 + M_n^* \rho_0^{-1}(n, 2; \varepsilon)} + O(\varepsilon \delta \rho_0^{-1}(n, 2; \varepsilon, \varepsilon^{\frac{1}{2}})).
\]
This, together with decomposition (3.6), \(Q^*[\varphi] \neq 0\), Theorem 3.1 and Lemma 3.3, leads to that
\[
\nabla u = \frac{Q^*[\varphi]}{M_0} \frac{1 + O(\max\{\varepsilon \frac{\delta \rho_0^{-1}(n, 2; \varepsilon, \varepsilon^{\frac{1}{2}})\})}{\rho_0(n, 2; \varepsilon) + M_n^*} (\nabla \bar{u} + O(\delta \rho_0^{-1})) + \nabla \bar{u}_0 + O(\delta \rho_0^{-1} \|\varphi\|_{C^2(\partial D)})
= \frac{Q^*[\varphi]}{M_0} \frac{1 + O(\max\{\varepsilon \frac{\delta \rho_0^{-1}(n, 2; \varepsilon, \varepsilon^{\frac{1}{2}})\})}{\rho_0(n, 2; \varepsilon) + M_n^*} \nabla \bar{u} + \nabla \bar{u}_0 + O(\delta \rho_0^{-1} \|\varphi\|_{C^2(\partial D)}).
\]
Thus, in view of (5.7) and (5.9), it follows from (5.10)–(5.12) that for \(n = 2, 3\),
\[
a_{11} = M_0 \rho_0(n, 2; \varepsilon) + \tilde{M}_n + O(1) \varepsilon^{\frac{1}{2}};
\]
for \(n \geq 4\),
\[
a_{11} = a_{11}^* + O(1) \varepsilon^{\frac{1}{2}},
\]
where
\[
\tilde{M}_n = \begin{cases} 
M_n^* - \frac{2}{\lambda_0 r_0}, & n = 2, \\
M_n^* + \frac{2}{\pi (\ln \lambda_0 + 2 \ln r_0)}, & n = 3.
\end{cases} (5.14)
\]
We claim that the constant \(\tilde{M}_n\) is independent of \(r_0\). If not, assume that there exist \(\tilde{M}_n(r_1^*)\) and \(\tilde{M}_n(r_2^*)\), \(r_i^* > 0, i = 1, 2, r_1^* \neq r_2^*\), both independent of \(\varepsilon\), such that (5.13) holds. Thus, we have
\[
\tilde{M}_n(r_1^*) - \tilde{M}_n(r_2^*) = O(1) \varepsilon^{\frac{1}{2}},
\]
which shows that \(\tilde{M}_n(r_1^*) = \tilde{M}_n(r_2^*)\). \(\square\)
That is, we complete the proofs of (5.2)–(5.3).

On the other hand, for \( n \geq 4 \), since \( Q^*[^\varphi] \neq 0 \), it follows from (5.4) and (5.6) that
\[
\frac{Q[^\varphi]}{\Theta} = \frac{Q^*[^\varphi]}{a_{11}} \frac{1}{1 - \frac{a_{11}}{a_{11}^*}} + \frac{Q[^\varphi] - Q^*[^\varphi]}{a_{11}^*} = \frac{Q^*[^\varphi]}{a_{11}^*} (1 + O(\varepsilon^{\frac{1}{n}})),
\]
which, in combination with decomposition (3.6), Theorem 3.1, Lemma 3.3, yields
\[
\nabla u = \frac{Q^*[^\varphi]}{a_{11}^*} (1 + O(\varepsilon^{\frac{1}{n}})) (\nabla \bar{u} + 0(1)) + \nabla \bar{u}_0 + O(1) \| \varphi \|_{C^2(\partial D)} + O(1) \| \varphi \|_{C^2(\partial D)}.
\]
Consequently, Example 5.1 is established.

6. The core-shell structure with concentric circles and spheres. In this section, we consider the case when \( D := B_{r+\varepsilon}(0) \) and \( D_1 := B_r(0) \) for a positive constant \( r \) independent of \( \varepsilon \), see Fig. 2 for concentric circles in dimension two. For simplicity, denote \( B_{r+\varepsilon} := B_{r+\varepsilon}(0) \) and \( B_r := B_r(0) \). In the presence of concentric circles and spheres \( B_r \) and \( B_{r+\varepsilon} \), the perfect conductivity problem (2.1) becomes
\[
\begin{align*}
\Delta u &= 0, & \text{in } B_{r+\varepsilon} \setminus \overline{B}_r, \\
u &= C_1, & \text{on } \overline{B}_r, \\
\int_{\partial B_r} \frac{\partial u}{\partial \nu} \big|_{+} &= 0, \\
u &= \varphi, & \text{on } \partial B_{r+\varepsilon}.
\end{align*}
\]

Then we state the principal result in this section as follows.

**Example 6.1.** For given \( \varphi \in C^2(\partial B_{r+\varepsilon}) \) and \( \varphi \neq C \) on \( \partial B_{r+\varepsilon} \), let \( u \in H^1(B_{r+\varepsilon}) \cap C^1(\overline{B}_{r+\varepsilon} \setminus B_r) \) be the solution of (6.1). Then for a sufficiently small \( \varepsilon > 0 \) and \( x \in B_{r+\varepsilon} \setminus \overline{B}_r \),
\[
\nabla u = \left( \varphi \frac{r + \varepsilon}{|x|} - \int_{\partial B_{r+\varepsilon}} \frac{\varphi}{\omega_n r^{n-1}} \frac{x}{|x|} + O(1) \varepsilon^{\frac{1-n}{n}} \| \varphi \|_{C^2(\partial B_{r+\varepsilon})} \right),
\]
where \( \omega_n \) is the volume of \( n \)-dimensional unit sphere.
Remark 6.2. From the result in Example 6.1, we see that the blow-up rate of $|\nabla u|$ is $\varepsilon^{-1}$ under the condition of $\varphi \not\equiv C$ on $\partial B_{r^\varepsilon}$. But if $\varphi \equiv C$ on $\partial B_{r^\varepsilon}$, then we decompose the solution $u$ of (6.1) as follows:

$$u = (C_1 - C)v_1 + C,$$

where $v_1$ is defined in (3.3) above. By using the third line of (6.1), we obtain that

$$(C_1 - C) \int_{\partial B_r} \frac{\partial v_1}{\partial n} = 0,$$

which together with the Hopf Lemma yields that $C = C_1$. That is, $u \equiv C$ in $B_{r^\varepsilon} \setminus B_r$ and thus there appears no blow-up.

Remark 6.3. The asymptotic expansion in Example 6.1 indicates that the electric field blows up at the rate of $\varepsilon^{-1}$ in the whole region $B_{r^\varepsilon} \setminus B_r$ in the presence of concentric circles and spheres. This is quite different from the above results in Theorems 2.1 and 2.4, Corollary 2.6 and Example 5.1, which show that the electric field concentrates only in a narrow region between the inclusion and the matrix boundary for the eccentric core-shell geometry. In addition, to the best of our knowledge, this paper is the first to consider the core-shell structure with concentric circles and spheres and to establish an asymptotic formula of the concentrated field. Although the proof of Example 6.1 is simple, the idea and result are novel.

Construct two radial auxiliary functions as follows:

$$\bar{v}_0(x) = \varphi \left( \frac{r + \varepsilon}{|x|} \right) (1 - \bar{v}_1(x)), \quad \bar{v}_1(x) = \frac{r + \varepsilon - |x|}{\varepsilon}, \quad x \in B_{r^\varepsilon} \setminus B_r. \quad (6.2)$$

We here would like to emphasize that the gradients of these two radial auxiliary functions are actually the leading terms of $\nabla \bar{v}_0$ and $\nabla \bar{v}_1$ defined by (3.2)–(3.3), respectively. This fact is critical to the establishment of the asymptotic expansion of the electric field, which will be demonstrated in the following.

Before proving Example 6.1, we first recall a classical result, which is Corollary 9.10 in [17].

Lemma 6.4. Let $\Omega$ be a domain in $\mathbb{R}^n$, $u \in W^{2,p}_0(\Omega)$, $1 < p < \infty$. Then

$$\|D^2u\|_{L^p(\Omega)} \leq C\|\Delta u\|_{L^p(\Omega)},$$

where $C = C(n,p)$.

6.1. The proof of Example 6.1. By utilizing decomposition (3.6), radial auxiliary functions $\bar{v}_0$ and $\bar{v}_1$ in (6.2), $L^p$ estimate in Lemma 6.4 and the Sobolev embedding theorem, we give the proof of Example 6.1 in the following.

The proof of Example 6.1. Step 1. Proof of

$$\nabla \bar{v}_0 = \nabla v_0 + O(1)\varepsilon^{\frac{1-n}{n}}\|\varphi\|_{C^2(\partial B_{r^\varepsilon})}, \quad \nabla \bar{v}_1 = \nabla v_1 + O(1)\varepsilon^{\frac{1-n}{n}}, \quad (6.3)$$

where the leading terms $\bar{v}_i$, $i = 0, 1$ are defined in (6.2).

For $x \in B_{r^\varepsilon} \setminus B_r$, denote $w_i := v_i - \bar{v}_i$, $i = 0, 1$. Then we have

$$\begin{cases}
\Delta w_i = -\Delta \bar{v}_i, & \text{in } B_{r^\varepsilon} \setminus B_r, \\
w_i = 0, & \text{on } \partial(B_{r^\varepsilon} \setminus B_r). 
\end{cases} \quad (6.4)$$

In view of (6.2), a direct calculation yields that for $x \in B_{r^\varepsilon} \setminus B_r$,

$$\Delta \bar{v}_1 = \frac{1-n}{\varepsilon|x|}. \quad (6.5)$$
and
\[ |\Delta \tilde{v}_0| = \left| \Delta \left[ \varphi \left( \frac{r + \varepsilon}{|x|} \right) \right] (1 - \tilde{v}_1) - 2 \nabla \left[ \varphi \left( \frac{r + \varepsilon}{|x|} \right) \right] \nabla \tilde{v}_1 - \varphi \left( \frac{r + \varepsilon}{|x|} \right) \Delta \tilde{v}_1 \right| \leq \frac{C \|\varphi\|_{C^2(\partial B_{r+\varepsilon})}}{\varepsilon^2}, \tag{6.6} \]
where we used the fact that \( \nabla [\varphi \left( \frac{r + \varepsilon}{|x|} \right)] \nabla \tilde{v}_1 = 0 \) in virtue of
\[ \partial_{x_i} \left[ \varphi \left( \frac{r + \varepsilon}{|x|} \right) \right] \partial_{x_i} \tilde{v}_1 = - \frac{(r + \varepsilon) x_i}{\varepsilon |x|^2} \partial_{x_i} \varphi \left( \frac{r + \varepsilon}{|x|} \right) + \sum_{j=1}^n \partial_j \varphi \left( \frac{r + \varepsilon}{|x|} \right) \frac{(r + \varepsilon) x_j^2 x_j}{\varepsilon |x|^4}, \quad i = 1, 2, \ldots, n. \]

In view of (6.4)–(6.6), it follows from the Sobolev embedding theorem and Lemma 6.4 that for some \( p > n \), there exists a positive constant \( C = C(n, p, r) \) such that
\[ \|\nabla w_1\|_{L^\infty(B_{r+\varepsilon} \setminus B_r)} \leq C(n, p, r) \|\nabla |w_1|\|_{L^p(B_{r+\varepsilon} \setminus B_r)} \]
\[ \leq C(n, p, r) \|\nabla \Delta \tilde{v}_1\|_{L^p(B_{r+\varepsilon} \setminus B_r)} \]
\[ \leq \frac{C(n, p, r)}{\varepsilon^{1-1/n}}. \tag{6.7} \]

By the same argument as before, we deduce
\[ \|\nabla w_0\|_{L^\infty(B_{r+\varepsilon} \setminus B_r)} \leq \frac{C(n, p, r) \|\varphi\|_{C^2(\partial B_{r+\varepsilon})}}{\varepsilon^{1-1/n}}, \quad \text{for some } p > n. \tag{6.8} \]

Then combining (6.7)–(6.8), we obtain that for \( x \in B_{r+\varepsilon} \setminus B_r \),
\[ \nabla v_1 = \Delta \tilde{v}_1 + O(\varepsilon^{1-n}) = - \frac{x}{\varepsilon |x|} + O(\varepsilon^{1-n}), \]
and
\[ \nabla v_0 = \nabla \tilde{v}_0 + O(\varepsilon^{1-n}) \|\varphi\|_{C^2(\partial B_{r+\varepsilon})} = \frac{x}{\varepsilon |x|} \varphi \left( \frac{r + \varepsilon}{|x|} \right) + O(\varepsilon^{1-n} \|\varphi\|_{C^2(\partial B_{r+\varepsilon})}). \]

That is, (6.3) holds.

**Step 2.** On one hand, recalling the definition of \( Q[\varphi] \) in (3.4), it follows from (6.3) and integration by parts that
\[ Q[\varphi] = - \int_{\partial B_{r+\varepsilon}} \frac{\partial v_1}{\partial n} \varphi = - \sum_{i=1}^n \int_{\partial B_{r+\varepsilon}} \frac{x_i \partial_{x_i} v_1}{|x|} \varphi + O(\varepsilon^{1-n}) \|\varphi\|_{L^\infty(\partial B_{r+\varepsilon})}) \]
\[ = - \sum_{i=1}^n \int_{\partial B_{r+\varepsilon}} \frac{x_i \partial_{x_i} \tilde{v}_1}{|x|} \varphi + O(\varepsilon^{1-n}) \|\varphi\|_{L^\infty(\partial B_{r+\varepsilon})}) \]
\[ = \frac{1}{\varepsilon} \sum_{i=1}^n \int_{\partial B_{r+\varepsilon}} \varphi + O(\varepsilon^{1-n}) \|\varphi\|_{L^\infty(\partial B_{r+\varepsilon})}). \tag{6.9} \]

On the other hand, using (6.3) again, we deduce
\[ a_{11} = \int_{B_{r+\varepsilon} \setminus B_r} |\nabla \tilde{v}_1|^2 + 2 \int_{B_{r+\varepsilon} \setminus B_r} \nabla (v_1 - \tilde{v}_1) \nabla \tilde{v}_1 + \int_{B_{r+\varepsilon} \setminus B_r} |\nabla (v_1 - \tilde{v}_1)|^2 \]
\[ = \frac{|B_{r+\varepsilon} \setminus B_r|}{\varepsilon^2} + O(1) |B_{r+\varepsilon} \setminus B_r| \varepsilon^{1/n-2} \]
\[ = \frac{\omega_n r_n^{n-1}}{\varepsilon} + O(1) \varepsilon^{1/n-1}. \]
This, together with (6.9), yields that
\[ Q[\varphi] \bigg|_{a_{11}} = \int_{\partial B_{r+\varepsilon}} \frac{\varphi}{\omega_n r^{n-1}} + \frac{\varepsilon Q[\varphi] - \int_{\partial B_{r+\varepsilon}} \varphi}{\varepsilon a_{11}} = \int_{\partial B_{r+\varepsilon}} \frac{\varphi}{\omega_n r^{n-1}} (1 + O(\varepsilon^{1/n})). \tag{6.10} \]

Consequently, in light of decomposition (3.6), it follows from (6.3) and (6.10) that
\[ \nabla u = \frac{Q[\varphi]}{a_{11}} \nabla v_1 + \nabla v_0 \]
\[ = \int_{\partial B_{r+\varepsilon}} \frac{\varphi}{\omega_n r^{n-1}} (1 + O(\varepsilon^{1/n})) (\nabla \bar{v}_1 + O(\varepsilon^{1/n})) \]
\[ + \nabla \bar{v}_2 + O(\varepsilon^{1/n} \| \varphi \|_{C^2(\partial B_{r+\varepsilon})}) \]
\[ = \left( \varphi \left( \frac{r + \varepsilon}{|x|} - \frac{\int_{\partial B_{r+\varepsilon}} \varphi}{\omega_n r^{n-1}} \right) \frac{x}{\varepsilon|x|} \right) + O(\varepsilon^{1/n} \| \varphi \|_{C^2(\partial B_{r+\varepsilon})}). \]

7. Discussions and conclusions. In this paper, we have analyzed the asymptotic behavior of the gradient of a solution to the perfect conductivity problem in the presence of a class of \( m \)-convex inclusions with core-shell geometry. Our asymptotic results in Theorem 2.1, Theorem 2.4 and Corollary 2.6 solve the optimality of the blow-up rate for any \( m, n, k \geq 2 \) and meanwhile make clear the dual role of the blow-up factor \( Q[\varphi] \): first, if we pick the boundary data of even function type with \( k \)-order growth introduced in condition (S1), the blow-up factor \( Q[\varphi] \) has no limit any more as the distance \( \varepsilon \) goes to zero and possesses the singularity of order \( O(\rho_k(n, m; \varepsilon)) \) in the case of \( m \geq n + k - 1 \); second, if we pick the boundary data of odd function in condition (S2) or the boundary data of condition (S1) in the case of \( m < n + k - 1 \), the blow-up factor \( Q[\varphi] \) converges to a new factor \( Q^*[\varphi] \) which consists of some certain integral of the solution to the case when the inclusion is touching the external boundary.

Since circular and spherical fibers are widely used in the practical design of composite structures due to the radial symmetry, then it is essential to pursue a sharp characterization in terms of the singularities of the concentrated field for core-shell geometry with circular and spherical boundaries. The first example is to consider eccentric circles and spheres and in Example 5.1 we capture the explicit geometry constants \( \mathcal{M}_n, n = 2, 3 \), defined in (5.14). These geometry constants show the explicit dependence on radius of disks and spheres, but not on the distance parameter \( \varepsilon \) and the length parameter \( r_0 \) of the narrow region. Further, the computation of the constants \( \mathcal{M}_n \) is actually a numerical problem and their values are definite only if we fix the radius of our domain. The blow-up factor \( Q^*[\varphi] \) is another physical quantity needed to be measured from an engineering point of view. In contrast to the geometry constants \( \mathcal{M}_n \), the numerical calculation of \( Q^*[\varphi] \) not only depend on the specific domain but also on the explicit boundary data \( \varphi \). Although we have given some examples in term of the boundary data \( \varphi \) which makes \( Q^*[\varphi] \neq 0 \) in Remark 2.2, it still be an interesting and important problem to explore more types of boundary data by utilizing numerical computation and simulation according to the needs of industrial applications.

The second example is devoted to revealing the blow-up feature of the gradient for the core-shell structure with concentric circles and spheres. As seen in Example 6.1, we find that the electric field concentrates highly in the whole matrix \( B_{r+\varepsilon} \setminus \overline{B}_r \).
This is different from the blow-up phenomenon occurring in the eccentric core-shell structure.

8. Appendix: The proof of Theorem 3.1. In light of assumptions \((H1)\) and \((H2)\), it follows from a direct calculation that for \(i = 1, \ldots, n - 1, x \in \Omega_{2R}\),

\[
|\partial_x \bar{v}| \leq C|\psi(x', \varepsilon + h_1(x'))| \frac{1}{\sqrt{\varepsilon + |x'|^m}} + C\|\nabla \psi\|_{L^\infty(\partial D_1)}, \tag{8.1}
\]

\[
|\partial_{x_n} \bar{v}| = \frac{|\psi(x', \varepsilon + h_1(x'))|}{\delta(x')}, \quad \partial_{x_n} \bar{v} = 0, \tag{8.2}
\]

and

\[
|\Delta \bar{v}| \leq C|\psi(x', \varepsilon + h_1(x'))| \frac{1}{(\varepsilon + |x'|^m)^\frac{1}{2}} + C\|\nabla \psi\|_{L^\infty(\partial D_1)} + C\|\nabla^2 \psi\|_{L^\infty(\partial D_1)}. \tag{8.3}
\]

For simplicity, we use \(\|\psi\|_{C^2}\) to denote \(\|\psi\|_{C^2(\partial D_1)}\) in the following.

Define

\[
w := v - \bar{v}. \tag{8.4}
\]

**Step 1.** Let \(v \in H^1(\Omega)\) be a weak solution of (3.7). Then

\[
\int_\Omega |\nabla w|^2 \, dx \leq C\|\psi\|_{C^2}^2. \tag{8.5}
\]

Invoking (8.4), \(w\) satisfies

\[
\begin{cases}
\Delta w = -\Delta \bar{v}, & \text{in } \Omega, \\
w = 0, & \text{on } \partial \Omega.
\end{cases} \tag{8.6}
\]

Multiplying the equation in (8.6) by \(w\) and integrating by parts, it follows from the Poincaré inequality, Sobolev trace embedding theorem, (3.9) and (8.1)–(8.2) that

\[
\int_\Omega |\nabla w|^2 = \int_{\Omega_R} w\Delta \bar{v} + \int_{\Omega \setminus \Omega_R} w\Delta \bar{v}
\]

\[
\leq \sum_{i=1}^{n-1} \int_{\Omega_R} w\partial_{x_i} \partial_{x_i} \bar{v} + C\|\psi\|_{C^2} \int_{\Omega \setminus \Omega_R} |w|
\]

\[
\leq C\|\nabla w\|_{L^2(\Omega_R)} \|\nabla x_i \bar{v}\|_{L^2(\Omega_R)} + C\|\psi\|_{C^2} \|\nabla w\|_{L^2(\Omega \setminus \Omega_R)}
\]

\[
\leq C\|\psi\|_{C^2} \|\nabla w\|_{L^2(\Omega)}.
\]

Then (8.5) is proved.

**Step 2.** Proof of

\[
\int_{\Omega_t(z')} |\nabla w|^2 \, dx \leq C\delta^{n+2} + \delta^2 \left( |\psi(z', \varepsilon + h_1(z'))|^2 + \delta \frac{1}{\varepsilon + |x'|^m} |\psi|^2_{C^2} \right), \tag{8.7}
\]

where \(\delta\) is defined by (2.4). For \(0 < t < s < R\), let \(\eta\) be a smooth cutoff function satisfying \(\eta(x') = 1\) if \(|x' - z'| < t\), \(\eta(x') = 0\) if \(|x' - z'| > s\), \(0 \leq \eta(x') \leq 1\) if \(t \leq |x' - z'| \leq s\), and \(|\nabla \eta(x')| \leq \frac{3}{s-t}\). Then multiplying the equation in (8.6) by \(w\eta^2\) and integrating by parts, we obtain the iteration formula as follows:

\[
\int_{\Omega_t(z')} |\nabla w|^2 \, dx \leq \frac{C}{(s-t)^2} \int_{\Omega_t(z')} |w|^2 \, dx + C(s-t)^2 \int_{\Omega_t(z')} |\Delta \bar{v}|^2 \, dx.
\]

We next divide into two cases to prove (8.7).
Case 1. If \(|z'| < \varepsilon^\frac{1}{m}\), \(0 < s < \varepsilon^\frac{1}{m}\), we have \(\varepsilon \leq \delta(x') \leq C\varepsilon\) in \(\Omega_{c\varepsilon}(z')\). In light of (8.3), we derive

\[
\int_{\Omega_{c\varepsilon}(z')} |\Delta \bar{w}|^2 \leq C|\psi(z', \varepsilon + h_1(z'))|^2 \frac{s^{n-1}}{\varepsilon^{4-m}} + C s^{n-1}\varepsilon^{1-\frac{m}{2}} \|\psi\|^2_{C^2},
\]

while, due to the fact that \(w = 0\) on \(\Gamma_R := \{x \in \mathbb{R}^n | x_n = h(x'), |x'| < R\}\),

\[
\int_{\Omega_{c\varepsilon}(z')} |w|^2 \leq C\varepsilon^2 \int_{\Omega_{c\varepsilon}(z')} |\nabla w|^2.
\]

Denote

\[F(t) := \int_{\Omega_{c\varepsilon}(z')} |\nabla w|^2.\]

It follows from (8.8) and (8.9) that for \(0 < t < s < \varepsilon^\frac{1}{m}\),

\[
F(t) \leq \left(\frac{c_1\varepsilon}{s-t}\right)^2 F(s) + C(s-t)^2 s^{n-1} \left(\frac{|\psi(z', \varepsilon + h_1(z'))|^2}{\varepsilon^{4-m}} + \frac{\|\psi\|^2_{C^2}}{\varepsilon^{2-m}}\right),
\]

where \(c_1\) and \(C\) are universal constants.

Pick \(k = \left[\frac{1}{s-t}\right] + 1\) and \(t_i = \delta + 2c_1i\varepsilon\), \(i = 0, 1, 2, ..., k\). Then, (8.10), together with \(s = t_{i+1}\) and \(t = t_i\), leads to

\[
F(t_i) \leq \frac{1}{4} F(t_{i+1}) + C(i+1)^{n-1}\varepsilon^{n+2} \left[|\psi(z', \varepsilon + h_1(z'))|^2 + \varepsilon^{\frac{m}{2}} \|\psi\|^2_{C^2}\right].
\]

It follows from \(k\) iterations and (8.5) that for a sufficiently small \(\varepsilon > 0\),

\[
F(t_0) \leq C\varepsilon^{n+2} \left[|\psi(z', \varepsilon + h_1(z'))|^2 + \varepsilon^{\frac{m}{2}} \|\psi\|^2_{C^2}\right].
\]

Case 2. If \(\varepsilon^\frac{1}{m} \leq |z'| \leq R\) and \(0 < s < \frac{2|z'|}{3}\), we have \(\frac{|z'|^m}{C} \leq \delta(x') \leq C|z'|^m\) in \(\Omega_{2|z'|}(z')\). Similar to (8.8) and (8.9), we obtain

\[
\int_{\Omega_{2|z'|}(z')} |\Delta \bar{w}|^2 \leq C|\psi(z', \varepsilon + h_1(z'))|^2 \frac{s^{n-1}}{|z'|^{4-m}} + C s^{n-1}|z'|^{m-2} \|\psi\|^2_{C^2},
\]

and

\[
\int_{\Omega_{2|z'|}(z')} |w|^2 \leq C|z'|^{2m} \int_{\Omega_{2|z'|}(z')} |\nabla w|^2.
\]

Moreover, for \(0 < t < s < \frac{2|z'|}{3}\), estimate (8.10) becomes

\[
F(t) \leq \left(\frac{c_2|z'|^m}{s-t}\right)^2 F(s) + C(s-t)^2 s^{n-1} \left(\frac{|\psi(z', \varepsilon + h_1(z'))|^2}{|z'|^{4-m}} + |z'|^{m-2} \|\psi\|^2_{C^2}\right).
\]

Similarly as above, pick \(k = \left[\frac{1}{s-t}\right] + 1\), \(t_i = \delta + 2c_2i|z'|^m\), \(i = 0, 1, 2, ..., k\) and take \(s = t_{i+1}\), \(t = t_i\). Then, we obtain

\[
F(t_i) \leq \frac{1}{4} F(t_{i+1}) + C(i+1)^{n-1}|z'|^{m(n+2)-4} \left(|\psi(z', \varepsilon + h_1(z'))|^2 + |z'|^2 \|\psi\|^2_{C^2}\right).
\]

Likewise, by using \(k\) iterations, we have

\[
F(t_0) \leq C|z'|^{m(n+2)-4} \left(|\psi(z', \varepsilon + h_1(z'))|^2 + |z'|^2 \|\psi\|^2_{C^2}\right).
\]

Consequently, (8.12), together with (8.11), yields that (8.7) holds.

Step 3. Proof of

\[
|\nabla w(x)| \leq C\delta^{\frac{m-2}{m}} (|\psi(x', \varepsilon + h_1(x'))| + \delta^{\frac{m}{2}} \|\psi\|_{C^2}), \quad \text{in } \Omega_R.
\]
By utilizing the following scaling and translating of variables in $\Omega_{\delta}(z')$:
\[
\begin{align*}
x' - z' &= \delta y', \\
x_n &= \delta y_n,
\end{align*}
\]
we rescale $\Omega_{\delta}(z')$ into $Q_1$, where, for $0 < r \leq 1$,
\[
Q_r = \left\{ y \in \mathbb{R}^n \left| \frac{1}{\delta} h(\delta y' + z') < y_n < \frac{\varepsilon}{\delta} + \frac{1}{\delta} h_1(\delta y' + z'), \ |y'| < r \right. \right\}.
\]
(8.14)
The top and bottom boundaries of $Q_r$ can be written as
\[
\tilde{\Gamma}^+_r = \left\{ y \in \mathbb{R}^n \left| y_n = \hat{h}_1(y') := \frac{\varepsilon}{\delta} + \frac{1}{\delta} h_1(\delta y' + z'), \ |y'| < r \right. \right\},
\]
and
\[
\tilde{\Gamma}^-_r = \left\{ y \in \mathbb{R}^n \left| y_n = \hat{h}(y') := \frac{1}{\delta} h(\delta y' + z'), \ |y'| < r \right. \right\},
\]
respectively. Then,
\[
\hat{h}_1(0') - \hat{h}(0') = 1,
\]
and by using condition (H2), we have
\[
|\nabla^j_x \hat{h}_1(0')| \leq C|z'|^{m-j}, \ |\nabla^j_x \hat{h}_1(0')| \leq C|z'|^{m-j}, \ j = 1, 2.
\]
Since $R$ is a small positive constant, $\|\hat{h}\|_{C^{1,1}((-1,1)^{n-1})}$ and $\|\hat{h}_1\|_{C^{1,1}((-1,1)^{n-1})}$, are small and then $Q_1$ is essentially of unit size as far as applications of Sobolev embedding theorems and classical $L^p$ estimates for elliptic equations are concerned.

For $y \in Q_1$, we denote
\[
W(y', y_n) := w(\delta y' + z', \delta y_n), \quad \bar{V}(y', y_n) := \hat{v}(\delta y' + z', \delta y_n).
\]
Then $W(y)$ solves
\[
\begin{align*}
\Delta W &= -\Delta \bar{V}, \quad \text{in } Q_1, \\
W &= 0, \quad \text{on } \tilde{\Gamma}^\pm_1.
\end{align*}
\]
In light of $W = 0$ on $\tilde{\Gamma}^\pm_1$, we see from the Poincaré inequality that
\[
\|W\|_{H^1(Q_1)} \leq C\|\nabla W\|_{L^2(Q_1)}.
\]
This, together with the Sobolev embedding theorem and classical $W^{2,p}$ estimates for elliptic systems, yields that for some $p > n$,
\[
\|\nabla W\|_{L^{\infty}(Q_{1/2})} \leq C\|W\|_{W^{2,p}(Q_{1/2})} \leq C \left( \|\nabla W\|_{L^2(Q_1)} + \|\Delta \bar{V}\|_{L^\infty(Q_1)} \right).
\]
Then, rescaling back to $w$ and $\bar{v}$, we have
\[
\|\nabla w\|_{L^{\infty}(\Omega_{\delta/\delta}(z'))} \leq C \left( \delta^{1-\frac{2}{n}} \|\nabla w\|_{L^2(\Omega_{\delta}(z'))} + \delta^{2} \|\Delta \bar{v}\|_{L^{\infty}(\Omega_{\delta}(z'))} \right).
\]
In view of (8.3) and (8.7), we obtain that for $|z'| \leq R$,
\[
\delta \|\Delta \bar{v}\|_{L^{\infty}(\Omega_{\delta}(z'))} \leq C \delta^{\frac{m-2}{2}} \|\psi(z', \varepsilon + h_1(z'))\|_{C^2},
\]
and
\[
\delta^{-\frac{2}{m-2}} \|\nabla w\|_{L^2(\Omega_{\delta}(z'))} \leq C \delta^{\frac{m-2}{2}} \|\psi(z', \varepsilon + h_1(z'))\|_{C^2}.
\]
Consequently, for $h(z') < z_n < \varepsilon + h_1(z')$,
\[
|\nabla w(z', z_n)| \leq C \delta^{-\frac{2}{m-2}} \|\psi(z', \varepsilon + h_1(z'))\|_{C^2}.
\]
Estimate (3.10) is established. On the other hand, it follows from the standard interior estimates and boundary estimates for the Laplace equation that

$$\|\nabla v\|_{L^\infty(\Omega \setminus \Omega_R)} \leq C\|\psi\|_{C^2}.$$  

Thus, Theorem 3.1 is proved.

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