ON COMMUTATIVE AND NON-COMMUTATIVE
\(C^*\)-ALGEBRAS WITH THE APPROXIMATE \(n\)-TH ROOT
PROPERTY

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Abstract. We say that a \(C^*\)-algebra \(X\) has the approximate \(n\)-th root
property \((n \geq 2)\) if for every \(a \in X\) with \(\|a\| \leq 1\) and every \(\varepsilon > 0\) there
exists \(b \in X\) such that \(\|b\| \leq 1\) and \(\|a - b^n\| < \varepsilon\). Some properties of
commutative and non-commutative \(C^*\)-algebras having the approximate \(n\)-
th root property are investigated. In particular, it is shown that there exists
a non-commutative (resp., commutative) separable unital \(C^*\)-algebra \(X\) such
that any other (commutative) separable unital \(C^*\)-algebra is a quotient of
\(X\). Also we illustrate a commutative \(C^*\)-algebra, each element of which has
a square root such that its maximal ideal space has infinitely generated first
Čech cohomology.

1. Introduction

All topological spaces in this paper are assumed to be (at least) completely
regular. A compact Hausdorff space is called a \(compatum\) for simplicity. By \(C^*\)-
algebra and homomorphisms between \(C^*\)-algebras, we mean unital \(C^*\)-algebras
and unital \(*\)-homomorphisms. For a space \(X\) and an integer \(n \geq 2\), we consider
the following conditions (\(\| \cdot \|\) denotes the supremum norm):

\((*)_n\) For each bounded continuous function \(f : X \to \mathbb{C}\) and each \(\varepsilon > 0\), there
exists a continuous function \(g : X \to \mathbb{C}\) such that \(\|f - g^n\| < \varepsilon\).

\((**)_n\) For each bounded continuous function \(f : X \to \mathbb{C}\) and each \(\varepsilon > 0\),
there exist bounded continuous functions \(g_1, \ldots, g_n : X \to \mathbb{C}\) such that
\(f = \prod_{i=1}^{i=n} g_i\) and \(\|g_i - g_j\| < \varepsilon\) for each \(i, j\).

We say that the space \(C^*(X)\) of all bounded complex-valued functions on
\(X\) has the approximate \(n\)-th root property if \(X\) satisfies condition \((*)_n\). The
results in this paper were inspired by the following theorem established by K.
Kawamura and T. Miura [10]:

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Theorem 1.1. Let $X$ be a compactum with $\dim X \leq 1$ and $n$ a positive integer. Then the following conditions are equivalent:

1. $C(X)$ has the approximate $n$-th root property.
2. $X$ satisfies condition $(\ast)_n$.
3. the first Čech cohomology $H^1(X; \mathbb{Z})$ is $n$-divisible, that is, each element of $H^1(X; \mathbb{Z})$ is divided by $n$.

Let $A(n)$ denote the class of all completely regular spaces satisfying condition $(\ast)_n$ and $A_1(n)$ is the subclass of $A(n)$ consisting of spaces $X$ with $\dim X \leq 1$. Let also $\mathcal{H}(n)$ denote the class of all compacta $X$ with $H^1(X; \mathbb{Z})$ being $n$-divisible.

In Section 2 we investigate some properties of the classes $A(n)$, $A_1(n)$ and $\mathcal{H}(n)$. In particular, the following theorem is established:

Theorem 1.2. Let $n$ be a positive integer and let $\mathcal{K}$ denote one of the classes $A(n)$, $A_1(n)$ or $\mathcal{H}(n)$. Then, for every cardinal $\tau \geq \omega$, there exists a compactum $X_\tau \in \mathcal{K}$ of weight $\leq \tau$ and a $\mathcal{K}$-invertible map $f_\mathcal{K}: X_\tau \to \prod^\tau$.

Here, a map $h: X \to Y$ is said to be invertible for the class $\mathcal{K}$ (or simply, $\mathcal{K}$-invertible) if for every map $g: Z \to Y$ with $Z \in \mathcal{K}$ there exists a map $\overline{g}: Z \to X$ such that $g = h \circ \overline{g}$.

Theorem 1.2 implies the next corollary.

Corollary 1.3. Let $n$ be a positive integer and let $\mathcal{K}$ be one of the classes $A(n)$, $A_1(n)$ or $\mathcal{H}(n)$. Then, for every $\tau \geq \omega$, there exists a compactum $X \in \mathcal{K}$ of weight $\tau$ which contains every space from $\mathcal{K}$ of weight $\leq \tau$.

It is easily seen that the modification of condition $(\ast)_n$, obtained by requiring both $f$ and $g$ to be of norm $\leq 1$, is equivalent to $(\ast)_n$. This observation leads us to consider the following classes of general (non-commutative) $C^*$-algebras. We say that a $C^*$-algebra $X$ satisfies the approximation $n$-th root property if for every $a \in X$ with $\|a\| \leq 1$ and every $\varepsilon > 0$ there exists $b \in X$ such that $\|b\| \leq 1$ and $\|a - b^n\| < \varepsilon$. The class of all $C^*$-algebras with the approximate $n$-th root property is denoted by $\mathcal{AP}(n)$. Let $\mathcal{AP}_1(n)$ be the subclass of $\mathcal{AP}(n)$ consisting of $C^*$-algebras of bounded rank $\leq 1$ (recall that bounded rank of $C^*$-algebras is a non-commutative analogue of the covering dimension $\dim$, see [5]). We also consider the class $\mathcal{HP}(n)$ of $C^*$-algebras $X$ with the following property: for every invertible element $a \in X$ with $\|a\| \leq 1$ and every $\varepsilon > 0$ there exists $b \in X$ such that $\|b\| \leq 1$ and $\|a - b^n\| < \varepsilon$.

In the sequel, $\mathcal{AP}(n)_s$ denotes the class of all separable $C^*$-algebras from $\mathcal{AP}(n)$. The notations $\mathcal{AP}_1(n)_s$ and $\mathcal{HP}(n)_s$ have the same meaning.

Recall now the concept of $\mathcal{R}$-invertibility introduced in [4], where $\mathcal{R}$ is a given class of $C^*$-algebras. A homomorphism $p: X \to Y$ is said to be $\mathcal{R}$-invertible if,
for any homomorphism \( g: X \to Z \) with \( Z \in \mathcal{R} \), there exists a homomorphism \( \overline{g}: Y \to Z \) such that \( g = \overline{g} \circ p \). We also introduce the notion of a universal \( C^* \)-algebra for a given class \( \mathcal{R} \) as a \( C^* \)-algebra \( Y \in \mathcal{R} \) such that any other \( C^* \)-algebra from \( \mathcal{R} \) is a quotient of \( Y \).

Section 3 is devoted to the classes \( \mathcal{AP}(n) \), \( \mathcal{AP}_1(n) \) and \( \mathcal{HP}(n) \). The results of this section can be considered as non-commutative counterparts of the results from Section 2. For example, Theorem 1.4 below is a non-commutative version of Theorem 1.2.

**Theorem 1.4.** Let \( n \) be a positive integer and let \( \mathcal{K} \) be one of the classes \( \mathcal{AP}(n) \), \( \mathcal{AP}_1(n) \) and \( \mathcal{HP}(n) \). Then there exists a \( \mathcal{K} \)-invertible unital *-homomorphism \( p: C^*(\mathbb{F}_\infty) \to Z_\mathcal{K} \) of \( C^*(\mathbb{F}_\infty) \) to a separable unital \( C^* \)-algebra \( Z_\mathcal{K} \in \mathcal{K} \), where \( C^*(\mathbb{F}_\infty) \) is the group \( C^* \)-algebra of the free group on countable number of generators.

It is well-known that every separable \( C^* \)-algebra is a surjective image of \( C^*(\mathbb{F}_\infty) \). Therefore, if \( \mathcal{R} \) is a class of separable \( C^* \)-algebras and \( p: C^*(\mathbb{F}_\infty) \to Y_\mathcal{R} \) is a \( \mathcal{R} \)-invertible homomorphism with \( Y_\mathcal{R} \in \mathcal{R} \), then \( Y_\mathcal{R} \) is universal for the class \( \mathcal{R} \). Hence, Theorem 1.4 implies that each of the classes \( \mathcal{AP}(n)_s \), \( \mathcal{AP}_1(n)_s \) and \( \mathcal{HP}(n)_s \) has a universal element.

Let us note that there exists a non-commutative separable \( C^* \)-algebra which belongs to any one of the classes \( \mathcal{AP}(n) \), \( \mathcal{AP}_1(n) \) and \( \mathcal{HP}(n) \). Indeed, let \( X = M(m) \) be the algebra of all \( m \times m \) complex matrixes, where \( m \geq 2 \) is a fixed integer. By [1], the bounded rank of \( X \) is 0. Moreover, using the canonical Jordan form representation, one can show that if \( A \in X \) and \( n \geq 2 \), then \( A \) can be approximated by a matrix \( B \in X \) with \( C^m = B \) for some \( C \in X \). Hence, \( X \) is a common element of \( \mathcal{AP}(n) \), \( \mathcal{AP}_1(n) \) and \( \mathcal{HP}(n) \). This implies that the universal elements of \( \mathcal{AP}(n)_s \), \( \mathcal{AP}_1(n)_s \) and \( \mathcal{HP}(n)_s \) are also non-commutative.

Section 4 deals with square root closed compacta, compacta \( X \) such that, for every \( f \in C(X) \), there is \( g \in C(X) \) with \( f = g^2 \). It is known that if \( X \) is a first-countable connected compactum, then \( X \) is square-root closed if and only if \( X \) is locally connected, dim \( X \leq 1 \) and \( \tilde{H}^1(X; \mathbb{Z}) \) is trivial, see [6], [8], [10] and [12]. A topological characterization of general square root closed compacta has not been known. Here we show that a square root closed compactum \( X \) with dim \( X \leq 2 \), constructed based on the idea of B. Cole ([13], Chap.3, section 19) and M.I. Karahanjan [9] has infinitely generated first Čech cohomology \( \tilde{H}^1(X; \mathbb{Z}) \). This space \( X \) is the limit space of an inverse system \((X_\alpha, \pi^{\beta}_{\alpha}: \alpha < \omega_1)\) starting with the unit disk in the plane and such that each map \( \pi^{\beta}_{\alpha}: X_\beta \to X_\alpha \) is invertible with respect to the class of square root closed compacta. A similar construction yields a one-dimensional such compactum. This illustrates that the topological characterization of (not necessarily first countable) square root closed compacta would be rather different than the one for first-countable compacta mentioned...
above. Also, the invertibility \(\pi_0^\beta: X_\beta \to X_\alpha\) allows us to obtain a universal element for the class of square root closed compacta with arbitrarily fixed weight.

2. Some properties of the classes \(A(n), A_1(n)\) and \(H(n)\)

**Lemma 2.1.** For a compactum \(X\), the following conditions are equivalent:

1. For any \(f: X \to S^1\) and any \(\varepsilon > 0\) there exists \(g: X \to S^1\) such that \(\|f - g^n\| < \varepsilon\).
2. \(H^1(X; \mathbb{Z})\) is \(n\)-divisible.

**Proof.** When \(\varepsilon = 0\) in (1), this equivalence was established by Kawamura-Miura in [10, Lemma 3.1]. Their arguments remain also valid in the present situation because any two sufficiently close functions from \(X\) into \(S^1\) are homotopic. \(\Box\)

**Lemma 2.2.** Let \(X\) be the limit space of an inverse system \(\{X_\alpha, p_\alpha^\beta: \alpha, \beta \in A\}\) of compacta. Then, for every \(f \in C(X)\) and every \(\varepsilon > 0\), there exists \(\alpha \in A\) and \(g \in C(X_\alpha)\) such that \(g \circ p_\alpha\) is \(\varepsilon\)-close to \(f\), where \(p_\alpha: X \to X_\alpha\) is the \(\alpha\)-th limit projection. Moreover, \(g \in C(X, S^1)\) provided \(f \in C(X, S^1)\).

**Proof.** We take a finite cover \(\omega\) of \(f(X)\) consisting of open and convex subsets of \(\mathbb{C}\) each of diameter \(< \varepsilon\). Since \(X\) is compact, we can find \(\alpha\) and an open cover \(\gamma = \{U_j : j = 1, \ldots, m\}\) of \(X_\alpha\) such that \(p_\alpha^{-1}(\gamma)\) is a star-refinement of the cover \(f^{-1}(\omega)\). Without loss of generality, we can assume that each \(U_j\) is functionally open in \(X_\alpha\), i.e., \(U_j = h_j^{-1}((0, 1])\) for some function \(h_j: X_\alpha \to [0, 1]\). For any \(j\) we fix a point \(x_j \in p_\alpha^{-1}(U_j)\) and the required function \(g: X_\alpha \to \mathbb{C}\) is defined by \(g(y) = \sum_{j=1}^{m} h_j(y)f(x_j)\). When \(f \in C(X, S^1)\) and \(\varepsilon\) is sufficiently small, \(g(X_\alpha) \subset \mathbb{C}\setminus\{0\}\) and, by considering the composition of \(g\) and the usual retraction \(r: \mathbb{C}\setminus\{0\} \to S^1\), we can assume \(g \in C(X_\alpha, S^1)\). \(\Box\)

**Corollary 2.3.** Let \(K\) be one of the classes \(A(n), A_1(n)\) and \(H(n)\). If \(X\) is the limit space of an inverse system \(\{X_\alpha, p_\alpha^\beta: \alpha, \beta \in A\}\) of compacta with each \(X_\alpha \in K\), then \(X \in K\).

**Proof.** This is a direct application of Lemma 2.2 for the class \(A(n)\). Since the limit space of any inverse system of at most one dimensional compacta is of dimension \(\leq 1\), the validity of our corollary for \(A(n)\) yields its validity for \(A_1(n)\). Finally, Lemma 2.1 and Lemma 2.2 settle the proof for the class \(H(n)\). \(\Box\)

We say that a class of spaces \(K\) is factorizable if, for every map \(f: X \to Y\) of a compactum \(X \in K\), there exists a compactum \(Z \in K\) of weight \(w(Z) \leq w(Y)\) and maps \(\pi: X \to Z\) and \(p: Z \to Y\) such that \(f = p \circ \pi\).

**Proposition 2.4.** Any one of the classes \(A(n), A_1(n)\) and \(H(n)\) is factorizable.
Proof. We consider first the class $\mathcal{H}(n)$. Fix a map $f: X \to Y$ of a compactum $X \in \mathcal{H}(n)$ and assume $w(Y) \leq \tau$. Obviously, we can assume $X$ is of weight $w(X) \geq \tau$ and $Y$ is compact. By induction, we construct sequences of compacta $X_k$, dense subsets $M_k \subset C(X_k, S^1)$ of cardinality $\leq \tau$ and maps $\pi_k: X \to X_k$, $p_k^{k+1}: X_{k+1} \to X_k$, $k \geq 0$, satisfying the following conditions:

1. $X_0 = Y$, $\pi_0 = f$,
2. $p_k^{k+1} \circ \pi_{k+1} = \pi_k$, $w(X_k) \leq \tau$ and $M_k$ separates points of $X_k$ ($k \geq 0$);
3. For every $h \in M_k$ and every $\varepsilon > 0$, there exists $g \in M_{k+1}$ such that $\|h \circ p_k^{k+1} - g^n\| < \varepsilon$ ($k \geq 0$).

The weight of the function space $C(Y, S^1)$ is $\leq \tau$, so $C(Y, S^1)$ contains a dense subset $M_0$ of cardinality $\leq \tau$, separating points of $Y$. Suppose the spaces $X_i$, the sets $M_i$ and the maps $\pi_i$, $p_{i-1}^i$, $i \leq k$, have been constructed for some $k$. Since $X \in \mathcal{H}(n)$, for each $h \in M_k$ and each positive rational number $r \in Q^+$, there exists $g(h, r) \in C(X, S^1)$ with $\|h \circ \pi_k - g(h, r)^n\| < r$. Let $\pi_{k+1}: X \to X_k \times (S^1)^M \times Q^+ \times (S^1)^M$ be the diagonal product of $\pi_k$ and all maps $g(h, r)$ and $h \circ \pi_k$, where $h \in M_k$, $r \in Q^+$. Let $X_{k+1} = \pi_{k+1}(X)$ and $p_k^{k+1}: X_{k+1} \to X_k$ be the natural projection onto $X_k$. Since $M_k$ separates points of $X_k$ (condition (1)), $\pi_{k+1}$ is an embedding and hence every $g(h, r)$ can be represented as $g_{k+1}(h, r) \circ \pi_{k+1}$ with $g_{k+1}(h, r) \in C(X_{k+1}, S^1)$. Because $w(X_{k+1}) \leq \tau$, $C(X_{k+1}, S^1)$ contains a dense subset $M_{k+1}$ of cardinality $\leq \tau$ containing all $g_{k+1}(h, r)$, $h \in M_k$, $r \in Q^+$ and also separating points of $X_{k+1}$. Obviously, $X_{k+1}$, $M_{k+1}$ and $\pi_{k+1}$ satisfy conditions (1) and (2). Let $Z$ be the limit of the inverse sequence $\{X_k, p_k^{k+1}: k = 1, 2, \ldots, p: Z \to Y$ the first limit projection and $\pi: X \to Z$ the limit of the maps $\pi_k$. Also let $p_k: Z \to X_k$ be the $k$-th limit projection. By Lemma 2.2, for every $h \in C(Z, S^1)$ and every $\varepsilon > 0$, there exists $m$ and $g_m: X_m \to S^1$ such that $\|h - g_m \circ p_m\| < \varepsilon/3$. Now, take $h_m \in M_m$ with $\|g_m - h_m\| < \varepsilon/3$. According to our construction, $\|h_m \circ p_m^{m+1} - g^n\| < \varepsilon/3$ for some $g \in M_{m+1}$. Hence, $\|h - (g \circ p_{m+1})^n\| < \varepsilon$. Finally, by Lemma 2.1, we see $Z \in \mathcal{H}(n)$.

The same arguments remain valid when the class $\mathcal{H}(n)$ is replaced by $\mathcal{A}(n)$. The only difference is that we have to consider the function spaces $C(X_k)$ instead of $C(X_k, S^1)$. For the class $\mathcal{A}_1(n)$ we need the following modifications: all $M_k$, $k \geq 0$, are dense subsets of $C(X_k)$ of cardinality $|M_k| \leq \tau$ satisfying conditions (1) and (2), where the compactum $X_k$ is of dimension $\leq 1$ for each $k \geq 1$. It suffices to demonstrate the construction of $X_1$ and $M_1$. Using the above notations, take the diagonal product $q_1: X \to Y \times C^0 \times Q^+ \times C^0$ of $\pi_0 = f$ and all maps $g(h, r)$ and $h \circ \pi_0$, where $h \in M_0$ and $r \in Q^+$. Let also $Z_1 = q_1(X)$ and $q_0: Z_1 \to Y$ be the natural projection. Then, $w(Z_1) \leq \tau$ and, by the Mardešić factorization theorem [11], there exists a compactum $X_1$ of weight $\leq \tau$ and $\dim X_1 \leq 1$, and maps $\pi_1: X \to X_1$ and $q_2: X_1 \to Z_1$ with $q_1 = q_2 \circ \pi_1$. 
Obviously, every \( g(h, r) \) can be represented as \( g_1(h, r) \circ \pi_1 \) with \( g_1(h, r) \in C(X_1) \). We denote \( p_1^1 = q_0 \circ q_2 \) and choose a dense subset \( M_1 \subset C(X_1) \) such that \( |M_1| \leq \tau \) and \( M_1 \) contains every \( g_1(h, r) \) with \( h \in M_0 \) and \( r \in Q^+ \), and separates points of \( X_1 \). In this way we obtain the spaces \( X_k \) with \( \dim X_k \leq 1 \). The last inequalities imply that the limit space \( Z \) is also of dimension \( \leq 1 \). Moreover, by Lemma 2.2, \( Z \) satisfies \((*)_n\), so \( Z \in A_1(n) \).

**Corollary 2.5.** Let \( K \) be one of the classes \( A(n) \) and \( A_1(n) \). Then every space \( X \in K \) has a compactification \( Z \in K \) with \( w(Z) = w(X) \).

**Proof.** Obviously, \( X \in K \) implies \( \beta X \in K \). Let \( Y \) be an arbitrary compactification of \( X \) with \( w(Y) = w(X) \) and let \( f : \beta X \to Y \) the extension of the identity on \( X \). Then, by Proposition 2.4, there exists a compactum \( Z \in K \) and maps \( g : \beta X \to Z \) and \( h : Z \to Y \) with \( h \circ g = f \) and \( w(Z) = w(X) \). It remains only to observe that \( Z \) is a compactification of \( X \). \( \square \)

**Proposition 2.6.** Let \( K \) be one of the classes \( A(n), A_1(n) \) and \( H(n) \). Then every compactum \( X \in K \) can be represented as the limit space of an \( \omega \)-spectrum \( \{X_\alpha, p_\alpha^\beta : \alpha, \beta \in A \} \) of metrizable compacta with each \( X_\alpha \in K \).

**Proof.** Because of similarity of the arguments, we consider only the class \( A(n) \). First, represent \( X \) as the limit space of an \( \omega \)-spectrum \( \{X_\alpha, p_\alpha^\beta : \alpha, \beta \in \Lambda \} \) and introduce the relation \( L \) on \( \Lambda^2 \) consisting of all \( (\alpha, \beta) \in \Lambda^2 \) such that \( \alpha \leq \beta \) and for each \( f \in C(X_\alpha) \) and \( \varepsilon > 0 \) there is \( g \in C(X_\beta) \) with \( \|f \circ p_\alpha^\beta - g^n\| < \varepsilon \). The relation \( L \) has the following properties:

(i) for every \( \alpha \in \Lambda \) there exists \( \beta \in \Lambda \) with \( (\alpha, \beta) \in L \):

(ii) if \( (\alpha, \beta) \in L \) and \( \beta \leq \gamma \), then \( (\alpha, \gamma) \in L \);

(iii) if \( \{\alpha_k\} \) is a chain in \( \Lambda \) with each \( (\alpha_k, \beta) \in L \), then \( (\alpha, \beta) \in L \), where \( \alpha = \sup\{\alpha_k\} \).

Indeed, to show (i), we take a countable dense subset \( M_\alpha \subset C(X_\alpha) \) and, as in Proposition 2.4, for every \( h \in M_\alpha \) and \( r \in Q^+ \) choose \( g(h, r) \in C(X) \) with \( \|h \circ p_\alpha - g(h, r)\|^n < r \). Notice that, for each \( f \in C(X) \), there is a \( \gamma \in \Lambda \) and \( \varphi \in C(X_\alpha) \) such that \( f = \varphi \circ p_\gamma \). Applying this to \( g(h, r) \), we can find \( \beta \in \Lambda \), \( \beta > \alpha \), such that for each \( (h, r) \in M_\alpha \times Q^+ \), we have \( g(h, r) = g_\beta(h, r) \circ p_\beta \), where \( g_\beta(h, r) \in C(X_\beta) \). Then \( (\alpha, \beta) \in L \). Property (ii) follows directly and (iii) follows from Lemma 2.2 and the fact that \( X_\alpha \) is the limit space of the inverse sequence generated by \( X_\alpha \), and the projections \( p_\alpha^{\alpha_{k+1}} : X_{\alpha_{k+1}} \to X_{\alpha_k}, k = 1, \ldots \), because \( \alpha \) is supremum of the chain \( \{\alpha_k\} \).

By [2, Proposition 1.1.29], the set \( A = \{\alpha \in \Lambda : (\alpha, \alpha) \in L \} \) is cofinal and \( \omega \)-closed in \( \Lambda \). Obviously, \( X_\alpha \in A(n) \) for each \( \alpha \in A \) and \( X \) is the limit of the inverse system \( \{X_\alpha, p_\alpha^\beta : \alpha, \beta \in A \} \). \( \square \)
Proof of Theorem 1.2. We consider the family of all maps \( \{ h_\alpha : Y_\alpha \to \prod^\beta \} \) such that each \( Y_\alpha \) is a closed subset of \( \prod^\beta \) with \( Y_\alpha \in K \). Let \( Y \) be the disjoint sum of all \( Y_\alpha \) and the map \( h : Y \to \prod^\beta \) coincides with \( h_\alpha \) on every \( Y_\alpha \). We extend \( h \) to a map \( \overline{h} : \beta Y \to \prod^\gamma \). Since \( \beta Y \in K \), by Proposition 2.4, there exists a compactum \( X \) of weight \( \leq \tau \) and maps \( p : \beta Y \to X \) and \( f : X \to \prod^\gamma \) such that \( X \in K \) and \( f \circ p = \overline{h} \).

Let us show that \( f \) is \( K \)-invertible. Take a space \( Z \in K \) and a map \( g : Z \to \prod^\gamma \). Considering \( \beta Z \) and the extension \( \overline{f} : \beta Z \to \prod^\gamma \) of \( g \), we can assume that \( Z \) is compact. We also can assume that the weight of \( Z \) is \( \leq \tau \) (otherwise we apply again Proposition 2.4 to find a compact space \( T \in K \) of weight \( \leq \tau \) and maps \( g_1 : Z \to T \) and \( g_2 : T \to \prod^\gamma \) with \( g_2 \circ g_1 = g \), and then consider the space \( T \) and the map \( g_2 \) instead, respectively, of \( Z \) and \( g \)). Therefore, without loss of generality, we can assume that \( Z \) is a closed subset of \( \prod^\gamma \). According to the definition of \( Y \) and the map \( h \), there is an index \( \alpha \in \Lambda \) such that \( Z = Y_\alpha \) and \( g = h_\alpha \). The restriction \( p|Z : Z \to X \) is a lifting of \( g \), i.e. \( f \circ (p|Z) = g \).

3. \( C^* \)-algebras with the approximate \( n \)-th root property

In this Section we investigate the behavior of the classes \( \mathcal{AP}(n) \), \( \mathcal{AP}_1(n) \) and \( \mathcal{HP}(n) \) with respect to direct systems and then use the result to prove the existence of universal elements in the classes \( \mathcal{AP}(n)_s \), \( \mathcal{AP}_1(n)_s \) and \( \mathcal{HP}(n)_s \).

When we refer to a unital \( C^* \)-subalgebra of a unital \( C^* \)-algebra we always assume that the inclusion is a unital \(*\)-homomorphism. The product in the category of (unital) \( C^* \)-algebras, i.e. the \( \ell^\infty \)-direct sum, is denoted by \( \prod \{ X_t : t \in T \} \). For a given set \( Y \) and a cardinal number \( \tau \), the symbol \( \exp_{\tau} Y \) denotes the partially ordered (by inclusion) set of all subsets of \( Y \) of cardinality not exceeding \( \tau \).

Recall that a direct system \( \mathcal{S} = \{ X_\alpha, i_\alpha^\beta, A \} \) of unital \( C^* \)-algebras consists of a partially ordered directed indexing set \( A \), unital \( C^* \)-algebras \( X_\alpha \), \( \alpha \in A \), and unital \(*\)-homomorphisms \( i_\alpha^\beta : X_\alpha \to X_\beta \), defined for each pair of indexes \( \alpha, \beta \in A \) with \( \alpha \leq \beta \), and satisfying the condition \( i_\alpha^\gamma = i_\alpha^\beta \circ i_\beta^\gamma \) for each triple of indexes \( \alpha, \beta, \gamma \in A \) with \( \alpha \leq \beta \leq \gamma \). The (inductive) limit of the above direct system is a unital \( C^* \)-algebra which is denoted by \( \lim_{\longrightarrow} \mathcal{S} \). For each \( \alpha \in A \) there exists a unital \(*\)-homomorphism \( i_\alpha : X_\alpha \to \lim_{\longrightarrow} \mathcal{S} \) which will be called the \( \alpha \)-th limit homomorphism of \( \mathcal{S} \).

If \( A' \) is a directed subset of the indexing set \( A \), then the subsystem \( \{ X_\alpha, i_\alpha^\beta, A' \} \) of \( \mathcal{S} \) is denoted \( \mathcal{S}|A' \).

Let \( \tau \geq \omega \) be a cardinal number. A direct system \( \mathcal{S} = \{ X_\alpha, i_\alpha^\beta, A \} \) of unital \( C^* \)-algebras \( X_\alpha \) and unital \(*\)-homomorphisms \( i_\alpha^\beta : X_\alpha \to X_\beta \) is called a direct \( C^*_\tau \)-system [3] if the following conditions are satisfied:

(a) \( A \) is a \( \tau \)-complete set, that is, for each chain \( C \) of elements of the directed set \( A \) with \( |C| \leq \tau \), there exists an element \( \sup C \) in \( A \). See [2] for details.
(b) The density \(d(X_\alpha)\) of \(X_\alpha\) is at most \(\tau\), for each \(\alpha \in A\).

(c) The \(\alpha\)-th limit homomorphism \(i_\alpha : X_\alpha \to \varinjlim S\) is an injective \(*\)-homomorphism for each \(\alpha \in A\).

(d) If \(B = \{\alpha_t : t \in T\}\) is a chain of elements of \(A\) with \(|T| \leq \tau\) and \(\alpha = \sup B\), then the limit homomorphism \(\varinjlim \{i_{\alpha_t} : t \in T\} : \varinjlim (S|B) \to X_\alpha\) is an isomorphism.

**Proposition 3.1** (Proposition 3.2, [3]). Let \(\tau\) be an infinite cardinal number. Every unital \(C^*\)-algebra \(X\) can be represented as the limit of a direct \(C^*_\tau\)-system \(S_X = \{X_\alpha, i_\alpha^\beta, A\}\) where the index set \(A = \exp_\tau Y\) for some (any) dense subset \(Y\) of \(X\) with \(|Y| = d(X)\).

**Lemma 3.2** (Lemma 3.3, [3]). If \(S_X = \{X_\alpha, i_\alpha^\beta, A\}\) is a direct \(C^*_\tau\)-system, then \(\varinjlim S_X = \bigcup \{i_\alpha(X_\alpha) : \alpha \in A\}\).

The next proposition is a non-commutative version of Corollary 2.3

**Proposition 3.3.** Let \(K\) be one of the classes \(\AP(n), \AP_1(n)\) and \(\HP(n)\). If \(X\) is the limit of a direct system \(S = \{X_\alpha, i_\alpha^\beta, A\}\) consisting of unital \(C^*\)-algebras and unital \(*\)-inclusions with \(X_\alpha \in K\) for each \(\alpha\), then \(X \in K\).

**Proof.** We consider first the case \(K = \AP(n)\). Let \(a \in X\) with \(\|a\| \leq 1\) and \(\varepsilon > 0\). Since \(\bigcup \{X_\alpha : \alpha \in A\}\) is dense in \(X\) (we identify each \(i_\alpha(X_\alpha)\) with \(X_\alpha\)), there exist \(\alpha\) and \(y \in X_\alpha\) with \(\|a - y\| < \varepsilon/4\). Then, \(\|y\| < \|a\| + \varepsilon/4 \leq 1 + \varepsilon/4\), so \(\|1 + \varepsilon/4\| < 1\). Since \(X_\alpha \in \AP(n)\), there is \(b \in X_\alpha\) with \(\|y \\| \frac{y}{1 + \varepsilon/4} - b^n \| < \varepsilon/2\) and \(\|b\| \leq 1\). Then \(\|a - b^n\| \leq \|a - \frac{y}{1 + \varepsilon/4}\| + \|\frac{y}{1 + \varepsilon/4} - b^n\| < \varepsilon\). Hence, \(X \in \AP(n)\). The above arguments work also for the class \(\HP(n)\) because of the fact that the set of invertible elements of a \(C^*\)-algebra is open. Indeed, for an invertible element \(a\) of \(X\), the above fact allows us to choose \(y\) in the above argument as an invertible element of \(X\). Consequently, \(\frac{y}{1 + \varepsilon/4}\) is invertible in \(X_\alpha\) and, since \(X_\alpha \in \HP(n)\), there is \(b \in X_\alpha\) with the required properties. Because the limit of any direct system consisting of \(C^*\)-algebras with bounded rank \(\leq 1\) has a bounded rank \(\leq 1\) [5, Proposition 4.1], the above proof remains valid for the class \(\AP_1(n)\).

As in the commutative case (see Proposition 2.6), we can establish a decomposition theorem for the classes \(\AP(n), \AP_1(n)\) and \(\HP(n)\).

**Proposition 3.4.** Let \(K\) be one of the classes \(\AP(n), \AP_1(n)\) and \(\HP(n)\). The following conditions are equivalent for any unital \(C^*\)-algebra \(X\):

(1) \(X \in K\).
(2) $X$ can be represented as the direct limit of a direct $C^*_\omega$-system $\{X_\alpha, i_\alpha^\beta, A\}$ satisfying the following properties:

(a) The indexing set $A$ is cofinal and $\omega$-closed in the $\omega$-complete set $\exp_\omega Y$ for some (any) dense subset $Y$ of $X$ such that $|Y| = d(X)$.

(b) $X_\alpha$ is a (separable) $C^*$-subalgebra of $X$ with $X_\alpha \in \mathcal{K}$, $\alpha \in A$.

**Proof.** A similar statement holds for the class of all $C^*$-algebras of bounded rank $\leq n$ (see [5, Proposition 4.2]). So, it suffices to consider the classes $\mathcal{AP}(n)$ and $\mathcal{HP}(n)$. We suppose $\mathcal{K} = \mathcal{AP}(n)$. The implication $(2) \implies (1)$ follows from Proposition 3.3.

In order to prove the implication $(1) \implies (2)$ we first consider a direct $C^*_\omega$-system $S_X = \{X_\alpha, i_\alpha^\beta, \Lambda\}$ with the properties indicated in Proposition 3.1. Each $X_\alpha$ is identified with $i_\alpha(A_\alpha)$. We next introduce the following relation $L \subseteq A^2$:

$(\alpha, \beta) \in L^2$ if and only if $\alpha \leq \beta$ and for each $x \in X_\alpha$ with $\|x\| \leq 1$ and each $\varepsilon > 0$ there exists $y \in X_\beta$ such that $\|y\| \leq 1$ and $\|x - y^n\| < \varepsilon$.

Let us show that $L$ satisfies the following conditions:

(i) for every $\alpha \in \Lambda$ there exists $\beta \in \Lambda$ with $(\alpha, \beta) \in L$;

(ii) if $(\alpha, \beta) \in L$ and $\beta \leq \gamma$, then $(\alpha, \gamma) \in L$;

(iii) if $\{\alpha_k\}$ is a chain in $\Lambda$ with each $(\alpha_k, \beta) \in L$, then $(\alpha, \beta) \in L$, where $\alpha = \sup \{\alpha_k\}$.

To verify (i), we take $\alpha \in \Lambda$ and a countable set $M \subset X_\alpha$ which is dense in the unit ball $B_\alpha = \{x \in X_\alpha : \|x\| \leq 1\}$. Since $X \in \mathcal{AP}(n)$, for each $x \in M$ and each $r \in Q^+$, we may take (and fix) $y(x, r) \in X$ with $\|x - y(x, r)^n\| < r$ and $\|y(x, r)\| \leq 1$. By Lemma 3.2, every $y(x, r)$ belongs to some $X_{\alpha(x,r)}$. Since $\Lambda$ is $\omega$-complete, according to [2, Corollary 1.1.28], there exists $\beta \in \Lambda$ such that $\beta \geq \alpha$ and $\beta \geq \alpha(x, r)$ for each $x \in M$ and $r \in Q^+$. Then, $X_\beta$ contains all $y(x, r)$ and $(\alpha, \beta) \in L$. Condition (ii) follows directly because $\beta \leq \gamma$ implies $X_\beta \subset X_\gamma$. Let us establish condition (iii). If $\alpha$ is the supremum of the countable chain $\{\alpha_k\}$, then $X_\alpha$ is the direct limit of the direct system generated by the $C^*$-subalgebras $X_{\alpha_k}$, $k = 1, 2, \ldots$, and the corresponding inclusion homomorphisms. This fact and $(\alpha_k, \beta) \in L$ for all $k$ yield $(\alpha, \beta) \in L$.

Since $L$ satisfies the conditions (i)-(iii), we can apply [2, Proposition 1.1.29] to conclude that the set $A = \{\alpha \in \Lambda : (\alpha, \alpha) \in L\}$ is cofinal and $\omega$-closed in $\Lambda$. Note that $(\alpha, \alpha) \in L$ precisely when $X_\alpha \in \mathcal{AP}(n)$. Therefore, we obtain a direct $C^*_\omega$-system $S_X' = \{X_\alpha, i_\alpha^\beta, A\}$ consisting of $C^*$-subalgebras $X_\alpha \in \mathcal{AP}(n)$ of $X$. Clearly $\lim\limits_{\longrightarrow} S_X' = X$. This completes the proof for the class $\mathcal{AP}(n)$. The case $\mathcal{K} = \mathcal{AP}(n)$ is similar. \qed

**Proof of Theorem 1.4.** Let $\mathcal{B} = \{f_t : C^*(F_\infty) \to X_t : t \in T\}$ denote the set of all unital *-homomorphisms on $C^*(F_\infty)$ such that $X_t \in \mathcal{K}$. We claim that the product $\prod\{X_t : t \in T\}$ belongs to $\mathcal{K}$. This is obviously true if $\mathcal{K}$ is either $\mathcal{AP}(n)$ or $\mathcal{HP}(n)$. Since the bounded rank of this product is $\leq 1$ provided
each $X_t$ is of bounded rank $\leq 1$ [5, Proposition 3.16], the claim holds for the class $\mathcal{AP}_1(n)$ as well. The $\ast$-homomorphisms $f_t$, $t \in T$, define the unital $\ast$-homomorphism $f : C^* (\mathbb{F}_\infty) \to \prod \{X_t : t \in T\}$ such that $\pi_t \circ f = f_t$ for each $t \in T$, where $\pi_t : \prod \{X_t : t \in T\} \to X_t$ denotes the canonical projection $\ast$-homomorphism onto $X_t$. By Proposition 3.4, $\prod \{X_t : t \in T\}$ can be represented as the limit of the $C^*_\alpha$-system $S = \{C_\alpha, \iota_\alpha, A\}$ such that $C_\alpha$ is a separable unital $C^*$-algebra with $C_\alpha \in K$ for each $\alpha \in A$. Suppressing the injective unital $\ast$-homomorphisms $\iota_\alpha : C_\alpha \to C_\beta$, we may assume, for notational simplicity, that $C_\alpha$’s are unital $C^*$-subalgebras of $\prod \{X_t : t \in T\}$. Let $\{a_k : k \in \omega\}$ be a countable dense subset of $C^* (\mathbb{F}_\infty)$. By Lemma 3.2, for each $k \in \omega$ there exists an index $\alpha_k \in A$ such that $f(a_k) \in C_{\alpha_k}$. Since $A$ is $\omega$-complete, there exists an index $\alpha_0 \in A$ such that $\alpha_0 \geq \alpha_k$ for each $k \in \omega$. Then $f(a_k) \in C_{\alpha_k} \subseteq C_{\alpha_0}$ for each $k \in \omega$. This observation coupled with the continuity of $f$ guarantees that $f (C^* (\mathbb{F}_\infty)) = f (\mathrm{cl} \{a_k : k \in \omega\}) \subseteq \mathrm{cl} \{f (\{a_k : k \in \omega\})\} \subseteq \mathrm{cl} C_{\alpha_0} = C_{\alpha_0}$.

Let $Z_K = C_{\alpha_0}$ and define the unital $\ast$-homomorphism $p : C^* (\mathbb{F}_\infty) \to Z_K$ as $f$, regarded as a homomorphism of $C^* (\mathbb{F}_\infty)$ into $Z_K$. Note that $f = i \circ p$, where $i : Z_K = C_{\alpha_0} \hookrightarrow \prod \{X_t : t \in T\}$ stands for the inclusion.

By construction, we see $Z_K \in K$. Let us show that $p : C^* (\mathbb{F}_\infty) \to Z_K$ is $K$-invertible. For a given unital $\ast$-homomorphism $g : C^* (\mathbb{F}_\infty) \to X$, where $X$ is a separable unital $C^*$-algebra with $X \in K$, we need to establish the existence of a unital $\ast$-homomorphism $h : Z_K \to X$ such that $g = h \circ p$. Indeed, by definition of the set $B$, we conclude that $g = f_t : C^* (\mathbb{F}_\infty) \to X_t = X$ for some index $t \in T$. Observe that $g = f_t = \pi_t \circ f = \pi_t \circ i \circ p$. This allows us to define the required unital $\ast$-homomorphism $h : Z_K \to X$ as the composition $h = \pi_t \circ i$. Hence, $p$ is $K$-invertible.

4. Example

In this section, we show that a construction due to B. Cole (cf.[13, Chap.3, section 19]) and M. Karahanjan [9, Theorem 5] yields a square root closed compact $X$ such that $H^1 (X; \mathbb{Z})$ is infinitely generated. In the sequel, we shall omit the coefficient group $\mathbb{Z}$. We will need the following theorem which is a consequence of [7, Theorem 3.2].

**Theorem 4.1.** Let $f : X \to Y$ be an open surjective map between compacta. Then $f^* : H^1 (Y) \to H^1 (X)$ is a monomorphism.

Now we outline the construction due to B. Cole. This is based on the exposition in [13, Chapter 3, §19, p.194-197]. Let $X$ be a compactum and define

$$S_X = \{(x, (z_f))_{f \in C(X)} : f(x) = z_f^2 \text{ for each } f \in C(X)\} \subset X \times \mathbb{C}^{C(X)}$$
Note that $S_X$ is a closed subset of $X \times \prod \{ f(X) | f \in C(X) \}$ and hence is a compactum. Also, it is easy to see that $S_X$ is a pull-back in the following diagram:

$$
\begin{array}{c}
S_X \ar[r] & C^C(X) \ar[d]^S \\
X \ar[r]^F & C^C(X)
\end{array}
$$

where $F : X \to \mathbb{C}$ is defined by $F(x) = (f(x))_{f \in C(X)}(x \in X)$, and $S : C^C(X) \to C^C(X)$ is defined by $S((z_f)_{f \in C(X)}) = (z_f^2)_{f \in C(X)}$.

Let $\pi : S_X \to X$ be the map defined by $\pi[(x, (z_f)_{f \in C(X)})] = x$ for all $x \in X$. Then $\pi$ is an open map with zero-dimensional fibers. The critical property of $S_X$ and $\pi$ is the following:

(*) for any $f \in C(X)$ there exists $g \in C(X)$ such that $f \circ \pi = g^2$.

Indeed, define $g : S_X \to \mathbb{C}$ by $g[(x, (z_f)_{f \in C(X)})] = z_f$.

Note that (*) implies:

(**) $\pi$ is invertible with respect to the class of square root closed compacta.

Starting with a compactum $X_0$, by transfinite induction we define an inverse spectrum $\{ X_\alpha, \pi_\alpha : X_\beta \to X_\alpha : \alpha \leq \beta < \omega_1 \}$ as follows. If $\beta = \alpha + 1$ then $X_\beta = S_{X_\alpha}$ and $\pi_\alpha = \pi : X_\beta = S_{X_\alpha} \to X_\alpha$ is the map defined above. If $\beta$ is a limit ordinal, then $X_\beta = \lim(X_\alpha, \pi_\alpha : X_\alpha \to X_\alpha : \alpha \leq \gamma < \beta)$ and, for $\alpha < \beta$, let $\pi_\alpha = \lim(\pi_\alpha : X_\gamma \to X_\alpha : \gamma < \beta)$.

We let $X_\Omega = \lim X_\alpha$. The $\alpha$-th limit projection is denoted by $\pi_\alpha : X_\Omega \to X_\alpha$.

As the length of the above spectrum is $\omega_1$, the spectrum is factorizing in the sense that each $f \in C(X_\Omega)$ is represented as $f = f_\alpha \circ \pi_\alpha$ for some $\alpha < \omega_1$ and $f_\alpha \in C(X_\alpha)$, since its length is $\omega_1$. This implies that $C(X_\Omega)$ is square root closed due to the property (*).

In what follows, the unit disk in the complex plane $\{ z \in \mathbb{C} : |z| \leq 1 \}$ is denoted by $\Delta$.

**Theorem 4.2.** $C(\Delta_\Omega)$ is square-root closed, $\dim \Delta_\Omega \leq 2$, $\hat{H}^1(\Delta_\Omega)$ is infinitely generated and 2-divisible.

Notice that for each square root closed compactum $X$, $\hat{H}^1(X)$ is 2-divisible. Hence, in view of the discussion above, we need only to show that $\hat{H}_1(\Delta_\Omega)$ is infinitely generated. To show this, we need the following.

**Theorem 4.3.** $\hat{H}^1(S_\Delta)$ is infinitely generated.

Note that Theorem 4.2 immediately follows from Theorems 4.1 and Theorem 4.3. The proof of Theorem 4.3 is divided into two parts.
Step 1. If $\tilde{H}^1(S_\Delta)$ is finitely generated then $\tilde{H}^1(S_\Delta) = 0$.

Step 2. $\tilde{H}^1(S_\Delta) \neq 0$.

Now we shall accomplish Steps 1 and 2.

**Proposition 4.4.** Let $Y$ be a closed subspace of a compactum $X$ such that there exists a retraction $r : X \to Y$. Let also $i : Y \hookrightarrow X$ be the inclusion. Then there exist an embedding $\overline{i} : SY \hookrightarrow SX$ and a retraction $\overline{r} : SX \to SY$ such that the following diagram is commutative.

\[
\begin{array}{cccc}
SY & \xrightarrow{\overline{i}} & SX & \xrightarrow{\overline{r}} & SY \\
\pi_Y & & \pi_X & & \pi_Y \\
Y & \xrightarrow{i} & X & \xrightarrow{r} & Y
\end{array}
\]

**Proof.** Define $\overline{i}$ by

\[\overline{i}[(y, (\eta_g)_{g \in C(Y)})] = (y, (\xi_f)_{f \in C(X)})\]

where $\xi_f = \eta_f|_Y$ for all $f \in C(X)$. Define $\overline{r}$ by

\[\overline{r}[(x, (\xi_f)_{f \in C(X)})] = (r(x), (\eta_g)_{g \in C(X)})\]

where $\eta_g = \xi_{g|_Y}$ for all $g \in C(Y)$. □

Now we are ready to accomplish Step 1. Let $\Delta_m = \{z \in \mathbb{C} : |z| \leq \frac{1}{m}\} \subset \Delta$. Let $r_n : \Delta_n \to \Delta_{n+1}$ be the radial retraction and $i_n : \Delta_{n+1} \hookrightarrow \Delta_n$ be the inclusion. Consider the following sequence of commutative diagrams.

\[
\begin{array}{cccc}
S_{\Delta_1} & \xleftarrow{i_1} & S_{\Delta_2} & \xleftarrow{i_2} & \cdots & \xleftarrow{i_n} & S_{\Delta_{n+1}} & \xleftarrow{i_{n+1}} & \lim_{\leftarrow} S_{\Delta_n} \\
\pi_1 & & \pi_2 & & \cdots & & \pi_n & & \lim_{\leftarrow} \pi_n = \pi_\infty \\
\Delta_1 & \xleftarrow{i_1} & \Delta_2 & \xleftarrow{i_2} & \cdots & \xleftarrow{i_n} & \Delta_{n+1} & \xleftarrow{i_{n+1}} & \{0\}
\end{array}
\]

It follows easily form the commutativity of the diagram that $\lim_{\leftarrow} S_{\Delta_n}$ is homeomorphic to the inverse limit of the sequence

\[
\begin{array}{cccc}
\pi_1^{-1}(0) & \xleftarrow{\pi_1^{-1}(0)} & \cdots & \xleftarrow{\pi_n^{-1}(0)} & \pi_n^{-1}(0) & \xleftarrow{\pi_{n+1}^{-1}(0)} & \cdots
\end{array}
\]
Since each fiber \( \pi^{-1}_n(0) \) is 0-dimensional, we have \( \dim \lim_{\longrightarrow} S_{\Delta_n} = 0 \). This implies that \( \bar{H}^1(\lim_{\longleftarrow} S_{\Delta_n}) = \lim_{\longrightarrow} \bar{H}^1(S_{\Delta_n}) = 0 \), which is equivalent to the following observation.

**Proposition 4.5.** For each \( \alpha \in \bar{H}^1(S_{\Delta_1}) = \bar{H}^1(S_{\Delta}) \), there exists an \( n \) such that \((\tilde{\tau}_1 \circ \cdots \circ \tilde{\tau}_n)^*\)(\(\alpha\)) = 0.

Let \( A_n \) be the annulus defined by \( A_n = \{ z \in \mathbb{C} \mid \frac{1}{m+1} \leq |z| \leq \frac{1}{m} \} \), so that \( \Delta_n = \{ 0 \} \cup (\cup \{ A_j \mid j \geq n \}) \). Let \( h : \Delta = \Delta_1 \to \Delta_2 \) be the homeomorphism which maps \( A_j \) to \( A_{j+1} \) (\( j \geq 1 \)) by “radial homeomorphisms” and such that \( h(0) = 0 \). Then the following diagram is commutative

\[
\begin{array}{ccc}
\Delta_n & \xrightarrow{h} & \Delta_{n+1} \\
\downarrow{i_n} & & \downarrow{i_{n+1}} \\
\Delta_{n+1} & \xrightarrow{h} & \Delta_{n+1}
\end{array}
\]

Define \( h_n : S_{\Delta_n} \to S_{\Delta_{n+1}} \) by \( h_n([x, (u_f)_{f \in C(\Delta_n)}]) = (h(x), (v_g)_{g \in C(\Delta_{n+1})}) \), where \( v_g = u_{goh}, g \in C(\Delta_{n+1}) \). Note that \( h_n \) is a homeomorphism.

**Proposition 4.6.** The following diagram is commutative.

\[
\begin{array}{ccc}
S_{\Delta_{n+1}} & \xrightarrow{\tilde{\tau}_{n+1}} & S_{\Delta_n} \\
\downarrow{h_{n+1}} & & \downarrow{h_n} \\
S_{\Delta_{n+2}} & \xrightarrow{\tilde{\tau}_{n+1}} & S_{\Delta_{n+1}}
\end{array}
\]

**Proof.** For each \((x_{n+1}, (z_f)_{f \in C(\Delta_{n+1})}) \in S_{\Delta_{n+1}} \) we have

\[
\tilde{\tau}_n([x_{n+1}, (z_f)_{f \in C(\Delta_{n+1})}] = (x_{n+1}, (u_f)_{f \in C(\Delta_n)})
\]

where \( u_f = z_{f|\Delta_n} = z_{f|\Delta_n}, f \in C(\Delta_n) \), and

\[
h_n([x_{n+1}, (u_f)_{f \in C(\Delta_n)}]) = (h(x_{n+1}), (v_f)_{f \in C(\Delta_{n+1})})
\]

where \( v_f = u_{foh} = z_{(foh)\Delta_n} = z_{f\Delta_n(h\Delta_n)} \). On the other hand,

\[
h_{n+1}([x_{n+1}, (z_f)_{f \in C(\Delta_{n+1})}] = (h(x_{n+1}), (u_g)_{g \in C(\Delta_{n+1})})
\]

where \( u_g = z_{goh}, g \in C(\Delta_{n+2}) \), and

\[
\tilde{\tau}_{n+1}([h(x_{n+1}), (u_g)_{g \in C(\Delta_{n+1})}] = (h(x_{n+1}), (v_f)_{f \in C(\Delta_{n+1})})
\]

where \( v_f = u_{f|\Delta_{n+1}} = z_{f|\Delta_{n+1}oh} = z_{f|\Delta_{n+1}oh} \). Since \( h \circ i_n = i_{n+1} \circ h \), we conclude that the diagram is commutative. \( \square \)
The above lemma provides a commutative diagram in cohomologies:

\[
\begin{array}{ccc}
\tilde{H}^1(S_{\Delta_n+1}) & \overset{\tilde{t}_n}{\longrightarrow} & \tilde{H}^1(S_{\Delta_n}) \\
\uparrow h^*_{n+1} & & \uparrow h^*_n \\
\tilde{H}^1(S_{\Delta_{n+2}}) & \overset{\tilde{t}^*_n}{\longrightarrow} & \tilde{H}^1(S_{\Delta_{n+1}})
\end{array}
\]

Let \( \phi = h^*_1 \circ i^*_1 : \tilde{H}^1(S_\Delta) \to \tilde{H}^1(S_\Delta) \). Since \( \overline{r}_1 \circ i_1 = \text{id}_{S_{\Delta}} \) we have \( \tilde{t}^*_1 \circ \overline{r}_1 = \text{id}_{\tilde{H}^1(S_\Delta)} \) and hence \( \phi \) is an epimorphism. We use diagram (†) to obtain the following diagram, in which all vertical arrows are isomorphisms.

\[
\begin{array}{ccc}
\tilde{H}^1(S_\Delta) & \overset{i^*_1}{\longrightarrow} & \tilde{H}^1(S_{\Delta_2}) \\
& \phi \downarrow & \downarrow h^*_1 \\
\tilde{H}^1(S_\Delta) & \overset{i^*_2}{\longrightarrow} & \tilde{H}^1(S_{\Delta_3}) \\
& \phi \downarrow & \downarrow h^*_2 \\
\tilde{H}^1(S_\Delta) & \overset{i^*_3}{\longrightarrow} & \tilde{H}^1(S_{\Delta_4}) \\
& \phi \downarrow & \downarrow h^*_3 \\
\vdots & & \vdots
\end{array}
\]

The above diagram together with Proposition 4.5 imply that, for each \( \alpha \in \tilde{H}^1(S_\Delta) \), there exists \( n \) such that \( \phi^n(\alpha) = 0 \). If \( \tilde{H}^1(S_\Delta) \) were finitely generated, we then would have \( \tilde{H}^1(S_\Delta) = 0 \) because of the following observation.

**Proposition 4.7.** Let \( A \) be a finitely generated Abelian group. If there exists an epimorphism \( f : A \to A \) such that for any \( a \in A \) there exists \( n \) with \( f^n(a) = 0 \), then \( A \) is trivial.

**Proof.** Note that \( f \otimes 1_\mathbb{Q} : A \otimes \mathbb{Q} \to A \otimes \mathbb{Q} \) is an epimorphism of a vector space \( A \otimes \mathbb{Q} \), which is finite-dimensional over \( \mathbb{Q} \). Hence \( f \otimes 1_\mathbb{Q} \) is an isomorphism with
the property in the hypothesis. This implies \( \text{rank} \, A = 0 \) and therefore \( A \) is a finite Abelian group. Then \( f \) is an isomorphism and therefore \( A = 0 \). □

Thus Step 1 is completed and we proceed to Step 2.

**Proposition 4.8.** For a continuous function \( f \in C(X) \), let 
\[ S_f = \{ (x, z) : f(x) = z^2 \text{ for each } x \in X \} \subset X \times \mathbb{C}. \]
Let also \( \pi_f : S_f \to X \) be the projection. Then the natural map \( p_f : S_X \to S_f, (x, (z_g)_{g \in C(X)}) \mapsto (x, z_f) \) is open. Thus we have the following diagram.

![Diagram]

**Proof.** Consider \( g_1, g_2, \ldots, g_n \in C(X) \) and open subset \( U_X \subset X, V_f, V_{g_1}, \ldots, V_{g_n} \subset \mathbb{C} \). It suffices to show that 
\[ p_f[(U_X \times V_f \times V_{g_1} \times \cdots \times V_{g_n} \times \prod_{g \neq g_1, \ldots, g_n, f} \mathbb{C}) \cap S_X] \]
is open in \( S_f \). Take a point 
\[ (x, z_f, (z_{g_i})_{i=1}^n, (z_g)_{g \neq f, g_1, \ldots, g_n}) \in U_X \times V_f \times V_{g_1} \times \cdots \times V_{g_n} \times \prod_{g \neq g_1, \ldots, g_n, f} \mathbb{C} \]
and choose \( \epsilon > 0 \) such that \( B(z_f, \epsilon) = \{ w \in \mathbb{C} : |w - z_f| < \epsilon \} \subset V_f \) and \( B(z_g, \epsilon) \subset V_{g_i} \) for all \( i = 1, 2, \ldots, n \). Let \( a = f(x), a_i = g_i(x), i = 1, 2, \ldots, n \). There exists \( \delta > 0 \) such that if \( |b - a| < \delta \) and \( |b_i - a_i| < \delta, i = 1, \ldots, n \), then the equations
\[ 
\begin{align*}
z^2 - b &= 0 \\
z_i^2 - b_i &= 0, \ i = 1, \ldots, n
\end{align*}
\]
have solutions \( z_b \) and \( z_{b_i} \) respectively such that \( |z_b - z_f| < \epsilon, |z_{b_i} - z_{g_i}| < \epsilon \). Choose a neighborhood \( N \) of \( x \) such that \( |f(y) - f(x)| < \delta \) and \( |g_i(y) - g_i(x)| < \delta \) for all \( y \in N \) and \( i = 1, \ldots, n \). We claim that 
\[ N \times B(z_f, \epsilon) \subset p_f[(U_X \times V_f \times V_{g_1} \times \cdots \times V_{g_n} \times \prod_{g \neq g_1, \ldots, g_n, f} \mathbb{C}) \cap S_X] \]
Indeed, for each pint \((y, w) \in N \times B(z_f, \epsilon) \subset N \times V_f\) we have \(|g_i(y) - g_i(x)| < \delta, i = 1, 2, \ldots, n\) by the choice of \(N\). Then we can find \(z_i \in B(z_{g_i}, \epsilon)\) such that \(z_i^2 = g_i(y)\). Now for arbitrary choice of \(z_g\), where \(g \neq f, g_1, g_2, \ldots, g_n\) with \(z_g^2 = g(x)\), we have

\[(y, w, (z_i)_{i=1}^n, (z_g)) \in U_X \times V_f \times V_{g_1} \times \cdots \times V_{g_n} \times \prod_{g \neq g_1, \ldots, g_n, f} \mathbb{C}
\]

and \(p_f[(y, w, (z_i)_{i=1}^n, (z_g))] = (y, w)\). This proves the claim and hence completes the proof of the proposition.

By Proposition 4.8 and Theorem 4.1, the statement of the Step 2 follows from the next observation.

**Proposition 4.9.** There exists a mapping \(f : \Delta \rightarrow \mathbb{C}\) such that \(\mathcal{H}^1(S_f) \neq 0\).

**Proof.** Let \(f(x, y) = (-2|x| + \sqrt{1 - y^2}, y)\) for all \((x, y) \in \Delta\). Then \(S_f\) is homeomorphic to cylinder \(S^1 \times I\). \(\Box\)

This completes the proof of Theorem 4.2.

The above construction is carried out word by word for disks of arbitrary dimensions. In particular, applying the above to the one-dimensional disk \([-1, 1]\), we have the following corollary which suggests that a topological characterization of general square root closed compacta could be rather different than the one for first-countable such compacta by [8] and [12].

**Corollary 4.10.** There exists an one-dimensional square root closed compactum \(X\) with infinitely generated first Čech cohomology.

For an infinite cardinal \(\tau \geq \omega\), we consider \((\mathbb{I})\Omega\) and the limit projection \(\pi_\Omega : (\mathbb{I})\Omega \rightarrow \mathbb{I}\). By the invertibility property (**) of \(\pi : S_X \rightarrow X\) for arbitrary compactum \(X\) and the standard spectral argument, it follows easily that \(\pi_\Omega\) is also invertible with respect to the class of square root closed compacta. Hence we have

**Proposition 4.11.** The square root closed compactum \((\mathbb{I})\Omega\) contains every square root closed compactum of weight \(\leq \tau\).

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