GIANT DESCENDANT TREES AND MATCHING SETS IN THE PREFERENTIAL ATTACHMENT GRAPH.

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Abstract. We study the \( \delta \)-version of the preferential attachment graph with \( m \) attachments for each of incoming vertices. We show that almost surely the scaled size of a breadth-first (descendant) tree rooted at a fixed vertex converges, for \( m = 1 \), to a limit whose distribution is a mixture of two beta distributions, and that for \( m > 1 \) the limit is 1. We also analyze the likely performance of a greedy matching algorithm for all \( m \geq 1 \) and establish an almost sure lower bound for the size of the matching set.

1. Introduction

It is widely accepted that graphs/networks are an inherent feature of life today. The classical models \( G_{n,m} \) and \( G_{n,p} \) of Erdős and Rényi [17] and Gilbert [22], respectively, lacked some salient features of observed networks. In particular, they failed to have a degree distribution that decays polynomially. Barabási and Albert [3] suggested the Preferential Attachment Model (PAM) as a more realistic model of a “real world” network. There was a certain lack of rigour in [3] and later Bollobás, Riordan, Spencer and Tusnády [6] gave a rigorous definition.

Many properties of this model have been studied. Bollobás and Riordan [7] studied the diameter and proved that with high probability (whp) PAM with \( n \) vertices and \( m > 1 \) attachments for every incoming vertex has diameter \( \approx \log n / \log \log n \). Earlier result by Pittel [30] implied that for \( m = 1 \) whp the diameter of PAM is of exact order \( \log n \). Bollobás and Riordan [9, 10] studied the effect on component size from deleting random edges from PAM and showed that it is quite robust whp. The degree distribution was studied in Mori [27, 28], Flaxman, Frieze and Fenner [18], Berger, Borgs, Chayes and Saberi [4]. Peköz, Röllin and Ross [29] established convergence, with rate, of the joint distribution of the degrees of finitely many vertices. Acan and Hitczenko [2] found an alternative proof, without rate, via a memory game. Pittel [32] used the Bollobás-Riordan pairing model to

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approximate, with explicit error estimate, the degree sequence of the first $n^{m/(m+2)}$ vertices, $m \geq 1$, and proved that, for $m > 1$, PAM is connected with probability $\approx 1 - O((\log n)^{-(m-1)/3})$. Random walks on PAM have been considered in the work of Cooper and Frieze [14, 15]. In the first paper there are results on the proportion of vertices seen by a random walk on an evolving PAM and the second paper determines the asymptotic cover time of a fully evolved PAM. Frieze and Pegden [21] used random walk in a “local algorithm” to find vertex 1, improving results of Borgs, Braunbar, Chayes, Khanna and Lucier [12]. The mixing time of such a walk was analyzed in Mihail, Papadimitriou and Saberi [23] who showed rapid mixing. Interpolating between Erdős-Rényi and preferential attachment, Pittel [31] considered birth of a giant component in a graph process $G_M$ on a fixed vertex set, when $G_{M+1}$ is obtained by inserting a new edge between vertices $i$ and $j$ with probability proportional to $[\text{deg}(i) + \delta] \cdot [\text{deg}(j) + \delta]$, with $\delta > 0$ being fixed.

The previous paragraph gives a small sample of results on PAM that can be related to its role as a model of a real world network. It is safe to say that PAM has now been accepted into the pantheon of random graph models that can be studied purely from a combinatorial aspect. For example, Cooper, Klasing and Zito [16] studied the size of the smallest dominating set and Frieze, Pérez-Giménez, Prałat and Reiniger [19] studied the existence of perfect matchings and Hamilton cycles.

2. Our Results

We study the number of descendants of a given vertex and also analyse the performance of an on-line greedy algorithm for finding a large matching. We carry out this analysis in the context of a generalization of the model from [6]. The precise model is taken from Hofstad [26, Ch. 8], and is described next.

**Preferential Attachment, $\delta$-extension:**

Vertex 1 has $m$ loops, so its degree is $2m$ initially. Recursively, vertex $t + 1$ has $m$ edges, and it uses them one at a time either to connect to a vertex $x \in [t]$ or to loop back on itself.

More precisely, at step $i \in [m]$: Step $t + 1$:

(a) vertex $t + 1$ attaches itself to $x \in [t]$, thus increasing degrees of both $x$ and $t + 1$ by 1, with probability $C \cdot (d_{t,i-1}(x) + \delta)$, where $d_{t,i-1}(x)$ is the degree of vertex $x$ just before the arrival of vertex $t + 1$ plus the number of times vertex $t + 1$ connected to vertex $x$ in the preceding $i - 1$ steps;

(b) vertex $t + 1$ loops back on itself with probability $C \cdot (d_{t,i-1}(t + 1) + 1 + \delta)/m$, thus increasing the degree of $t + 1$ by 2, where $d_{t,i-1}(t + 1)$ is the
degree of $t + 1$ after $i - 1$ attachments. (The summand 1 in the formula for the probability is the contribution to the degree of $t + 1$ coming from the $i$-th edge whose other endpoint has not been chosen yet.)

(c) $C$ is determined from the condition “the $t + 1$ probabilities add up to 1”.

In other words, denoting by $w$ the random receiving end of the $i$-th edge of $t + 1$, we have

$$\mathbb{P}(w = x) = \begin{cases} C \times (d_{t,i-1}(x) + \delta) & \text{if } x \in [t] \\ C \times (d_{t,i-1}(t + 1) + 1 + i\delta/m) & \text{if } x = t + 1, \end{cases}$$

where

$$C = \left( (t + i/m)\delta + 1 + \sum_{x=1}^{t+1} d_{t,i-1}(x) \right)^{-1} = \frac{1}{(t + i/m)\delta + 2(mt + i) - 1}.$$

**Remark 2.1.** Note that the process is well defined for $\delta \geq -m$ since for such $\delta$, all the probabilities defined in (2.1) are nonnegative and add up to 1. However, we will see that $G_{m,-m}(t)$ is the star centered at vertex 1, and the key problems we want to solve have trivial solutions in that extreme case.

We will use the notation $\{G_{m,\delta}(t)\}$ for the resulting graph process. In particular, for $m = 1$ we have: using “$|\circ|$” to indicate conditioning on pre-history,

$$\mathbb{P}(t + 1 \text{ selects } x|\circ) = \begin{cases} \frac{1 + \delta}{(2 + \delta)t + (1 + \delta)}, & x = t + 1, \\ \frac{d_t(x) + \delta}{(2 + \delta)t + (1 + \delta)}, & x \in [t]. \end{cases}$$

The total degree of $G_{m,\delta}(t)$ is $2mt$.

**2.1. Number of Descendants.** Fix a positive integer $r$ and let $X(t)$ denote the number of descendants of $r$ at time $t$. Here $r$ is a descendant of $r$ and $x$ is a descendant of $r = O(1)$ if and only if $x$ chooses to attach itself to at least one descendant of $r$ in Step $x$. In other words, if we think of the graph as a directed graph with edges oriented towards the smaller vertices, vertex $x$ is a descendant of $r$ if and only if there is a directed path from $x$ to $r$. We prove two theorems:

**Theorem 2.2.** Suppose that $m = 1$ and $\delta > -1$ and $p(t) = X(t)/t$. Then almost surely (i.e. with probability 1), $\lim p(t)$ exists, and its distribution is the mixture of two beta-distributions, with parameters $a = 1$, $b = r - \frac{1}{2 + \delta}$ and $a = \frac{1 + \delta}{2 + \delta}$, $b = r$, weighted by $\frac{1 + \delta}{(2 + \delta)t - 1}$ and $\frac{(2 + \delta)(r - 1)}{(2 + \delta)r - 1}$ respectively. Consequently a.s. $\liminf_{t \to \infty} p(t) > 0$. 
Note. (i) The proof is based on a new family of martingales $M_\ell(t) := \frac{(X(t) + \gamma)^{(t)}}{(t + \delta)^{(t)}}$, $(z)^{(t)}$ standing for the rising factorial. This family definitely resembles the martingales Mori [27, 28] used for the vertex degrees. Our proof of the martingale property is, unsurprisingly, quite different. (ii) Whp $G_{1,\delta}$ is a forest of $\Theta(\log t)$ trees rooted at vertices with loops. For the preferential attachment tree (no loops), Janson [25] recently proved that the scaled sizes of the principal subtrees, those rooted at the root’s children and ordered chronologically, converge a.s. to the $GEM$ distributed random variables. His techniques differ significantly.

When $m > 1$ we have the somewhat surprising result that, for $r = O(1)$, almost surely all but a vanishingly small fraction of vertices are descendants of $r$, (cf. [25]).

**Theorem 2.3.** Let $m > 1$ and $\delta > -m$ and let $p_X(t) = X(t)/t$, $p_Y(t) = Y(t)/(2mt)$, where $Y(t)$ is the total degree of the descendants of $r$ at time $t$. Then almost surely $\lim p_X(t) = \lim p_Y(t) = 1$.

2.2. Greedy Matching Algorithm. We analyze a greedy matching algorithm; a.s. it delivers a surprisingly large matching set even for relatively small $m$. This algorithm generates the increasing sequence $\{M(t)\}$ of partial matchings on the sets $[t]$, with $M(1) = \emptyset$. Suppose that $X(t)$ is the set of unmatched vertices in $[t]$ at time $t$. If $t + 1$ chooses a vertex $u \in X(t)$ to attach itself to then $M(t + 1) = M(t) \cup \{\{u, t + 1\}\}$, otherwise $M(t + 1) = M(t)$. (If $t + 1$ chooses multiple vertices from $X(t)$, then we pick one of those as $u$ arbitrarily.) Let

$$h(z) = h_{m,\delta}(z) := 2 \left[1 - \left(\frac{m + \delta}{2m + \delta}\right) z\right]^{m} - z - 1$$

and let $\rho = \rho_{m,\delta}$ be the unique root $\rho = \rho_{m,\delta}$ in the interval $[0, 1]$ of $h(z) = 0$: $\rho_{m,\delta} \in (0, 1)$ if $\delta > -m$. Denoting $x(t) = X(t)/t$, we have

**Theorem 2.4.** If $\delta > -m$, then, for any $\alpha < 1/3$, almost surely,

$$\lim_{t \to \infty} t^\alpha \max\{0, x(t) - \rho_{m,\delta}\} = 0.$$ 

In consequence, the Greedy Matching Algorithm a.s. finds a sequence of nested matchings $\{M(t)\}$, with $M(t)$ of size at least $(1 - o(1))(1 - \rho_{m,\delta}) t/2$.

**Remark 2.5.** Observe that $\rho_{m,-m} = 1$, which makes it plausible that the maximum matching size is miniscule compared to $t$. In fact, by Remark 2.1, $G_{m,-m}(t)$ is the star centered at vertex 1 and hence the maximum matching size is 1.
Remark 2.6. Consider the case $\delta = 0$. Let $r_m := 1 - \rho_{m,0}$; some values of $r_m$ are:

\begin{align*}
  r_1 &= 0.5000, & r_2 &= 0.6458, & r_5 &= 0.8044, \\
  r_{10} &= 0.8863, & r_{20} &= 0.9377, & r_{70} &= 0.9803.
\end{align*}

With a bit of calculus, we obtain that $r_m = 1 - 2m^{-1} \log 2 + O(m^{-2})$.

Remark 2.7. Close to the PAM is the Uniform Attachment Model: vertex $t + 1$ selects uniformly at random (repetitions allowed) $m$ vertices from the set $[t]$. (See Acan and Pittel [1] for connectivity and bootstrap percolation results.) An argument, broadly analogous to the one for Theorem 2.4, gives the following theorem.

**Theorem 2.8.** Let $r_m$ denote a unique positive root of $2(1 - z^m) - z = 0$: $r_m = 1 - m^{-1} \log 2 + O(m^{-2})$. Then, for any $\alpha < 1/3$, almost surely

$$
\lim_{t \to \infty} t^\alpha |1 - r_m - x(t)| = 0
$$

for the uniform attachment model.

[Note that $x(t)$ is the fraction of unmatched vertices.]

Some values of $r_m$ in this case are:

\begin{align*}
  r_1 &= 0.6667, & r_2 &= 0.7808, & r_5 &= 0.8891, \\
  r_{10} &= 0.9386, & r_{20} &= 0.9674, & r_{35} &= 0.9809.
\end{align*}

3. **Proof of Theorem 2.2**

For $t \geq r > 1$, let $X(t) = X_{m,\delta}(t) = X_{m,\delta}(t,r)$, $Y(t) = Y_{m,\delta}(t) = Y_{m,\delta}(t,r)$ denote the size and the total degree of the vertices in the vertex set of the sub-tree $T(t) = T_{m,\delta}(t,r)$ rooted at $r$; so $X(r) = X_{m,\delta}(r,r) = 1$ and $Y(r) = Y_{m,\delta}(r,r) \in [m, 2m]$, $m$ ($2m$ resp.) attained when $t + 1$ forms no loops (forms $m$ loops resp.) at itself. Introduce $p(t) = p_Y(t) = \frac{Y(t)}{2m}$ and $p_X(t) = \frac{X(t)}{t}$. This notation will be used in the proof of Theorem 2.3 as well, but of course $m = 1$ in the proof of Theorem 2.2.

Here

$$
Y(t) = \begin{cases} 
  2X(t), & \text{if } r \text{ looped on itself}, \\
  2X(t) - 1, & \text{if } r \text{ selected a vertex in } [r-1].
\end{cases}
$$
(In particular, \( p_X(t) = p(t) + O(t^{-1}) \). So, by (2.2),

\[
\mathbb{P}(X(t+1) = X(t) + 1 | \sigma) = \frac{Y(t) + \delta X(t)}{(2 + \delta)t + (1 + \delta)}
\]

\[
= \begin{cases} 
(2 + \delta)X(t), & \text{if } r \text{ looped on itself}, \\
(2 + \delta)t + (1 + \delta), & \text{if } r \text{ selected a vertex in } [r - 1].
\end{cases}
\]

Thus we are led to consider the process \( X(t) \) such that

\[
\mathbb{P}(X(t+1) = X(t) + 1 | \sigma) = \frac{(2 + \delta)X(t) + \gamma}{(2 + \delta)t + (1 + \delta)},
\]

\[
\mathbb{P}(X(t+1) = X(t) | \sigma) = 1 - \mathbb{P}(X(t+1) = X(t) + 1 | \sigma);
\]

\( \gamma = 0 \) if \( r \) looped on itself, \( \gamma = -1 \) if \( r \) selected a vertex in \([r - 1]\).

**Note.** Suppose \( \delta = -1 \), (see Remark 2.1). Then, by (2.2), vertex \( r \) selects a vertex in \([r - 1]\), so that \( \gamma = -1 \). Since \( X(r) = 1 \), it follows from (3.1) that \( X(t) \equiv 1 \) for \( t \geq r \). This means that \( G_{1,-1}(t) \) is the star with vertex 1 being the star’s center, cf. [26, Exercise 8.5].

**Lemma 3.1.** Let \( (z)^{\ell} = \prod_{j=0}^{\ell-1}(z + j) \), \( \beta = \frac{1+\delta}{2+\delta} \). Then, conditioned on the attachment record during the time interval \([r, t]\), i.e. starting with attachment decision by vertex \( r \), we have

\[
\mathbb{E} \left[ \left( X(t+1) + \frac{\gamma}{2 + \delta} \right)^{\ell} \right] = \frac{t + \beta + \ell}{t + \beta} \left( X(t) + \frac{\gamma}{2 + \delta} \right)^{\ell}.
\]

Consequently \( M(t) := \frac{(X(t)+\frac{\gamma}{2+\delta})^{\ell}}{(t+\beta)^{\ell}} \) is a martingale.

For \( \delta = 0 \) this claim was proved in Pittel [32].

**Proof.** First of all,

\[
\frac{(2 + \delta)X(t) + \gamma}{(2 + \delta)t + (1 + \delta)} = \frac{X(t) + \alpha}{t + \beta}, \quad \alpha = \frac{\gamma}{2 + \delta}, \quad \beta = \frac{1 + \delta}{2 + \delta}.
\]
Introduce $Z(t) = X(t) + \alpha$. By (2.2), we have: for $k \geq 1$, and $t \geq r$,

$$
E[Z^k(t+1)|\omega] = (Z(t) + 1)^k \frac{Z(t)}{t + \beta} + Z^k(t) \left(1 - \frac{Z(t)}{t + \beta}\right)
$$

$$
= \frac{Z(t)}{t + \beta} \sum_{j=0}^{k} \binom{k}{j} Z^j(t) + Z^k(t) \left(1 - \frac{Z(t)}{t + \beta}\right)
$$

$$
= Z^k(t) + \frac{Z(t)}{t + \beta} \sum_{j=0}^{k-1} \binom{k}{j} Z^j(t)
$$

$$
= Z^k(t) \frac{t + \beta + k}{t + \beta} + \frac{1}{t + \beta} \sum_{j=1}^{k-1} \binom{k}{j-1} Z^j(t)
$$

(3.2)

Next recall that

$$
z^{(\ell)} = \sum_{k=1}^{\ell} z^k s(\ell, k),
$$

(3.3)

where $s(\ell, k)$ is the signless, first-kind, Stirling number, i.e. the number of permutations of the set $[\ell]$ with $k$ cycles. In particular,

$$
\sum_{\ell \geq 1} \eta^\ell \frac{s(\ell, k)}{\ell!} = \frac{1}{k!} \log \frac{1}{1 - \eta}, \quad |\eta| < 1,
$$

(3.4)

Comtet [13, Section 5.5]. Using (3.2) and (3.3), we have

$$
E[Z^{(\ell)}(t+1)|\omega] = \sum_{k=1}^{\ell} s(\ell, k) E[Z^k(t+1)|\omega]
$$

$$
= (t + \beta)^{-1} \sum_{k=1}^{\ell} s(\ell, k) \cdot \left((t + \beta + k)Z^k(t) + \sum_{j=0}^{k-1} \binom{k}{j-1} Z^j(t)\right)
$$

$$
= (t + \beta)^{-1} \sum_{i=1}^{\ell} \sigma(\ell, i) Z^i(t),
$$

$$
\sigma(\ell, i) = \begin{cases}
(t + \beta + \ell) s(\ell, \ell), & \text{if } i = \ell, \\
(t + \beta) s(\ell, i) + \sum_{k=i}^{\ell} s(\ell, k) \binom{k}{i-1}, & \text{if } i < \ell.
\end{cases}
$$

We need to show that $\sigma(\ell, i) = (t + \beta + \ell) s(\ell, i)$ for $k < \ell$, which is equivalent to

$$
\ell s(\ell, i) = \sum_{k=i}^{\ell} s(\ell, k) \binom{k}{i-1}.
$$
To prove the latter identity, it suffices to show that, for a fixed $i$, the exponential generating functions of the two sides coincide. By (3.4),

$$
\sum_{\ell \geq 1} \frac{\eta^\ell}{\ell!} \sum_{k=i}^\infty s(\ell, k) \binom{k}{i-1} = \sum_{k \geq i} \frac{\eta^k}{k!} \prod_{\ell \geq k} s(\ell, k)
$$

$$
= \sum_{k \geq i} \binom{k}{i-1} \frac{1}{k!} \log^k \frac{1}{1-\eta} = \frac{1}{(i-1)!} \left( \log^{-1} \frac{1}{1-\eta} \right) \sum_{s \geq 1} \frac{1}{s!} \log^s \frac{1}{1-\eta}
$$

$$
= \frac{1}{(i-1)!} \left( \log^{-1} \frac{1}{1-\eta} \right) \left( \frac{1}{1-\eta} - 1 \right) = \frac{1}{(i-1)!} \left( \log^{-1} \frac{1}{1-\eta} \right) \frac{\eta}{1-\eta}.
$$

And, using (3.4) again,

$$
\sum_{\ell \geq 1} \frac{\eta^\ell}{\ell!} s(\ell, i) = \frac{\eta^i}{1-1} \frac{1}{\eta} = \frac{1}{\eta} \left( \log^{-1} \frac{1}{1-\eta} \right) \frac{\eta}{1-\eta}.
$$

□

To identify the $\lim p(t)$, recall that the classic beta probability distribution has density

$$
f(x; a, b) = \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} x^{a-1} (1-x)^{b-1}, \quad x \in (0, 1),
$$

parametrized by two parameters $a > 0, b > 0$, and moments

$$
\int_0^1 x^\ell f(x; a, b) \, dx = \prod_{j=0}^{\ell-1} \frac{a + j}{a + b + j}.
$$

We can now complete the proof of Theorem 2.2. By Lemma 3.1, we have $\mathbb{E}[M(t) | X] = M(r)$, i.e.

$$
\mathbb{E} \left[ \left( \frac{X(t) + \frac{\gamma}{2+\delta}}{(t+\beta)^{\ell}} \right)^{\ell} \bigg| X \right] = \frac{\left( 1 + \frac{\gamma}{2+\delta} \right)^{\ell}}{(r+\beta)^{\ell}}.
$$

For every $\ell \geq 1$, by martingale convergence theorem, conditioned on $X$, there exists an integrably finite $M_{Y, \ell}$ such that a.s.

$$
\lim_{\ell \to \infty} \left( \frac{X(t) + \frac{\gamma}{2+\delta}}{(t+\beta)^{\ell}} \right)^{\ell} = M_{Y, \ell}, \quad \ell \geq 0,
$$

and

$$
\mathbb{E}[M_{Y, \ell}] = \frac{\left( 1 + \frac{\gamma}{2+\delta} \right)^{\ell}}{(r+\beta)^{\ell}}.
$$
So, using the notation $p_X(t) = X(t)/t$, we have: a.s.

$$\lim_{t \to \infty} (p_X(t))^\ell = \mathcal{M}_{\gamma,t} = (\mathcal{M}_{\gamma,1})^\ell, \quad (3.6)$$

and

$$\mathbb{E}[(\mathcal{M}_{\gamma,1})^\ell] = \frac{(1 + \frac{\gamma}{2 + \delta})^{(\ell)}}{(r + \beta)^{(\ell)}} = \prod_{j=0}^{\ell-1} \frac{1 + \frac{\gamma}{2 + \delta} + j}{r + \beta + j}. \quad (3.6)$$

This means that $\mathcal{M}_{\gamma,1}$ is beta-distributed with parameters $1 + \frac{\gamma}{2 + \delta}$ and $r + \beta - 1 - \frac{\gamma}{2 + \delta}$. By the definition of $\gamma$ and (2.2), we have

$$\mathbb{P}(\gamma = 0) = \frac{1 + \delta}{(2 + \delta)(r - 1) + (1 + \delta)} = \frac{1 + \delta}{2r - 1 + \delta r}. \quad (3.7)$$

We conclude that $\lim_{t \to \infty} p(t)$ has the distribution which is the mixture of the two beta distributions, with parameters $a = 1$, $b = \frac{r - 1}{2 + \delta}$, and $a = \frac{1 + \delta}{2 + \delta}$, $b = r$, weighted by $\frac{1 + \delta}{(2 + \delta)(r - 1)}$ and $\frac{(2 + \delta)(r - 1)}{(2 + \delta)(r - 1)}$ respectively.

This completes the proof of Theorem 2.2.

3.1. Proof of Theorem 2.3. We need to derive tractable formulas/bounds for the conditional distribution of $Y(t + 1) - Y(t)$. First, let us evaluate the conditional probability that selecting the second endpoints of the $m$ edges incident to vertex $t + 1$ no loops will be formed. Suppose there has been no loop in the first $i - 1$ steps, $i \in [m]$; call this event $\mathcal{E}_{i-1}$. On event $\mathcal{E}_{i-1}$, as the $i$-th edge incident to $t + 1$ is about to attach its second end to a vertex in $[t] \cup \{t + 1\}$, the total degree of all these vertices is $2mt + i - 1 - (1 \leq i \leq m)$. So, by the definition of the transition probabilities (items (a), (b), (c)) we have

$$\mathbb{P}(\mathcal{E}_i|\circ) = \frac{2mt + i - 1 + t\delta}{2mt + 2(i - 1) + t\delta + 1 + \frac{18}{m}},$$

“$\circ$” indicating conditioning on the full record of $i - 1$ preceding attachments such that the event $\mathcal{E}_{i-1}$ holds. Crucially this conditional probability depends on $i$ only. Therefore the probability of a given full loops-free record of the $m$ attachments is equal to the corresponding probability for the “no loops in $m$ attachments process”, multiplied by

$$\Pi_m(t) := \prod_{i=1}^{m} \frac{2mt + i - 1 + t\delta}{2mt + 2(i - 1) + t\delta + 1 + \frac{18}{m}} = 1 - O(t^{-1}). \quad (3.7)$$
Lemma 3.2. If no loops are allowed in the transition from \( t \) to \( t + 1 \), then

\[
\mathbb{P}(Y(t + 1) = Y(t) + m + a | \circ) = \binom{m}{a} \frac{(Y(t) + \delta X(t))(a) \cdot (2mt - Y(t) + \delta(t - X(t))(m-a)}{((2m + \delta)t)^{(m)}}, \quad (a \in [m]),
\]

\[
\mathbb{P}(Y(t + 1) = Y(t) | \circ) = \frac{(2mt - Y(t) + \delta(t - X(t))(m)}{((2m + \delta)t)^{(m)}}.
\]

Proof. Vertex \( t + 1 \) selects, in \( m \) steps, a sequence \( \{v_1, \ldots, v_m\} \) of \( m \) vertices from \( [t] \), with \( t \) choices for every selection. The total vertex degree of \( [t] \) (of \( V(T(t)) \) respectively) right before step \( i \) is \( 2mt + i - 1 \) \((Y(t) + \mu_i) \) respectively, \( \mu_i := \{|j < i : v_j \in V(H(t))|\}) \). Conditioned on this prehistory,

\[
\mathbb{P}(v_i \in V(T(t))) = \frac{Y(t) + \delta X(t) + \mu_i}{2mt + \delta t + i - 1},
\]

\[
\mathbb{P}(v_i \in [t] \setminus V(T(t))) = \frac{2mt - Y(t) + \delta(t - X(t)) + i - 1 - \mu_i}{2mt + \delta t + i - 1}.
\]

Therefore a sequence \( v = \{v_1, \ldots, v_m\} \) will be the outcome of the \( m \)-step selection with probability

\[
\mathbb{P}(v) = \left( \prod_{i \in [m]} \left( \frac{(2m + \delta)t + i - 1}{(2m + \delta)t + i - 1}\right) \right)^{-1} \times \prod_{i : v_i \in V(T(t))} (Y(t) + \delta X(t) + \mu_i) \cdot \prod_{i : v_i \in [t] \setminus V(T(t))} (2mt - Y(t) + \delta(t - X(t)) + i - 1 - \mu_i).
\]

Furthermore, for \( a \in [m] \), on the event \( \{Y(t + 1) = Y(t) + m + a\} \) for each admissible \( v \) we have

\[
\{\mu_i\} = \{0, 1, \ldots, a-1\}, \quad \{i - 1 - \mu_i\} = \{0, 1, \ldots, m - a - 1\},
\]

so that

\[
\mathbb{P}(v) = \frac{(Y(t) + \delta X(t))(a) \cdot (2mt - Y(t) + \delta(t - X(t))(m-a)}{((2m + \delta)t)^{(m)}}.
\]

Since the total number of admissible sequences is \( \binom{m}{m} \), we obtain the first formula in Lemma 3.2. The second formula is the case of \( \mathbb{P}(v) \) with \( a = 0 \). □

It is clear from the proof that \( \{ \mathbb{P}_m(a) \}_{0 \leq a \leq m} \),

\[
\mathbb{P}_m(a) := \binom{m}{a} \frac{(Y(t) + \delta X(t))(a) \cdot (2mt - Y(t) + \delta(t - X(t))(m-a)}{((2m + \delta)t)^{(m)}},
\]

is a probability distribution of a random variable \( D \), a “rising-factorial” counterpart of the binomial \( D = \text{Bin}(m, p = Y(t)/2mt) \). Define the falling
factorial \( (x)_\ell = x(x-1) \cdots (x-\ell+1) \). It is well known that \( \mathbb{E}[(D)\mu] = (m)\mu^\mu, \ (\mu \leq m) \). For \( D \) we have

\[
\mathbb{E}[(D)\mu] = \sum_a (a)\mu P_m(a) = \frac{(m)\mu (Y(t) + \delta X(t))^{(\mu)}}{(2m+\delta)t} \sum_{a \geq \mu} \frac{(m-\mu)}{(a-\mu)}
\]

\[
\times \frac{(Y(t) + \delta X(t) + \mu)^{(a-\mu)}((2m+\delta)t + \mu - (Y(t) + \delta X(t) + \mu))^{(m-\mu)-(a-\mu)}}{(2mt + \mu)^{(m-\mu)}}
\]

\[
= \frac{(m)\mu (Y(t) + \delta X(t))^{(\mu)}}{(2m+\delta)t}, \quad (3.8)
\]

since the sum over \( a \geq \mu \) is \( \sum_{\nu \geq 0} P_{m-\mu}(\nu) = 1 \).

From Lemma 3.2 and (3.8) we have: if no loops during the transition from \( t \) to \( t+1 \) are allowed, then

\[
\mathbb{E}[Y(t+1) - Y(t) \mid \sigma] = \sum_{a=1}^m (a + m)P_m(a)
\]

\[
= \frac{m(Y(t) + \delta X(t))}{(2m+\delta)t} + m \left( 1 - \frac{(2m+\delta)t - Y(t) - \delta X(t))^{(m)}}{(2m+\delta)t} \right), \quad (3.9)
\]

and

\[
\mathbb{E}[X(t+1) - X(t) \mid \sigma] = 1 - \frac{(2m+\delta)t - Y(t) - \delta X(t))^{(m)}}{(2m+\delta)t}, \quad (3.10)
\]

What if the ban on loops at the vertex \( t+1 \) is lifted? From the discussion right before Lemma 3.2, we see that both \( \mathbb{E}[(Y(t+1) - Y(t)) \mathbb{1}(\text{no loops}) \mid \sigma] \) and \( \mathbb{E}[(X(t+1) - X(t)) \mathbb{1}(\text{no loops}) \mid \sigma] \) are equal to the respective RHS’s in (3.9) and (3.10) times \( \Pi_m(t) = 1 - O(t^{-1}) \). Consequently, adding the terms \( O(t^{-1}) \) to the RHS of (3.9) and to the RHS of (3.10) we obtain the sharp asymptotic formulas for \( \mathbb{E}[Y(t+1) - Y(t) \mid \sigma] \) and \( \mathbb{E}[X(t+1) - X(t) \mid \sigma] \) in the case of the loops-allowed model.

Let

\[
p(t) = \frac{2m}{2m+\delta} p_Y(t) + \frac{\delta}{2m+\delta} p_X(t),
\]

where \( p_Y(t) = \frac{Y(t)}{2mt} \) and \( p_X(t) = \frac{X(t)}{t} \) as defined in the beginning of the section. Theorem 2.3 asserts

**Theorem 3.3.** Let \( m > 1 \) and \( \delta > -m \). Then almost surely \( \lim p_Y(t) = \lim p_X(t) = 1 \).

**Proof.** First of all, we note that \( mX(t) \leq Y(t) \leq 2mX(t) \). The lower bound is obvious. The upper bound follows from induction on \( t \): Suppose \( Y(t) \leq 2mX(t) \). If \( X(t+1) = X(t) \), then \( Y(t+1) = Y(t) \leq 2mX(t) = 2mX(t+1) \).
If \( X(t + 1) = X(t) + 1 \), then \( Y(t + 1) \leq Y(t) + 2m \leq 2mX(t) + 2m = 2mX(t + 1) \). Therefore, by the definition of \( p(t) \), we have

\[
\frac{p_X(t)}{2} \leq p_Y(t) \leq p_X(t) \implies \frac{m + \delta}{2m + \delta} p_X(t) \leq p(t) \leq p_X(t); \quad (3.11)
\]
in particular, \( p(t) \in [0, 1] \) since \( \delta \geq -m \). We will also need

\[
\frac{(2m + \delta)t - Y(t) - \delta X(t))^{(m)}}{(2m + \delta)t^{(m)}} = (1 - p(t))^m + O(t^{-1}).
\]

So, using (3.9), we compute

\[
\mathbb{E}[p_Y(t + 1)|\omega] = \mathbb{E}\left[\frac{Y(t + 1)}{2mt} \cdot \frac{t}{t + 1} \middle| \omega\right]
\]

\[
= \frac{t}{t + 1} (p_Y(t) + \frac{1}{2t}[1 + p(t) - (1 - p(t))^m] + O(t^{-2})) = p_Y(t) + q_Y(t),
\]

\[
q_Y(t) := \frac{1}{2(t + 1)} [1 + p(t) - 2p_Y(t) - (1 - p(t))^m] + O(t^{-2}). \quad (3.12)
\]

Likewise

\[
\mathbb{E}[p_X(t + 1)|\omega] = p_X(t) + q_X(t), \quad (3.13)
\]

\[
q_X(t) = \frac{1}{t + 1} [1 - p_X(t) - (1 - p(t))^m] + O(t^{-2}).
\]

Multiplying the equation (3.12) by \( \frac{2m}{2m + \delta} \), the equation (3.13) by \( \frac{\delta}{2m + \delta} \), and adding them, we obtain

\[
\mathbb{E}[p(t + 1)|\omega] = p(t) + q(t),
\]

\[
q(t) = \frac{m + \delta}{(2m + \delta)(t + 1)} [1 - p(t) - (1 - p(t))^m] + O(t^{-2}). \quad (3.14)
\]

Since \( 1 - z - (1 - z)^m \geq 0 \) on \([0, 1]\), the equation (3.14) implies that \( \sum_t \mathbb{E}[|q(t)|] < \infty \). So, a.s. there exists \( Q := \lim_{t \to \infty} \sum_{1 \leq \tau \leq t} q(\tau) \), with \( \mathbb{E}[|Q|] \leq \sum_t \mathbb{E}[|q(t)|] < \infty \), i.e. a.s. \( |Q| < \infty \). Introducing \( Q(t + 1) = \sum_{\tau \leq t} q(\tau) \), we see from (3.14) that \( \{p(t + 1) - Q(t + 1)\}_{t \geq 1} \) is a martingale with \( \sup_t |p(t + 1) - Q(t + 1)| \leq 1 + \sum_{t \geq 1} |q(\tau)| \). By the martingale convergence theorem we obtain that there exists an integrable \( \lim_{t \to \infty} (p(t) - Q(t + 1)) \), implying that a.s. there exists a random \( p(\infty) = \lim_{t \to \infty} p(t) \). The (3.14) also implies that

\[
1 \geq \mathbb{E}[p(\infty)] = \frac{m + \delta}{2m + \delta} \sum_{t \geq 1} \frac{1}{t + 1} \mathbb{E}[1 - p(t) - (1 - p(t))^m] + O(1).
\]

Since \( m + \delta > 0 \) and

\[
\lim_{t \to \infty} \mathbb{E}[1 - p(t) - (1 - p(t))^m] = \mathbb{E}[1 - p(\infty) - (1 - p(\infty))^m],
\]
and the series \( \sum_{t \geq 1} t^{-1} \) diverges, we obtain that \( \mathbb{P}(p(\infty) \in \{0, 1\}) = 1 \).

Recall that \( p(t) \geq \frac{m+\delta}{2m+\delta} p_X(t) \). If we show that a.s. \( \lim \inf_{t \to \infty} p_X(t) > 0 \), it will follow that a.s. \( p(\infty) > 0 \), whence a.s. \( p(\infty) = 1 \), implying (by \( p(t) \leq p_X(t) \)) that a.s. \( p_X(\infty) \) exists, and is 1, and consequently (by the formula for \( p(t) \)) a.s. \( p_Y(\infty) \) exists, and is 1.

So let’s prove that a.s. \( \lim \inf_{t \to \infty} p_X(t) > 0 \). Recall that we did prove the latter for \( m = 1 \). To transfer this earlier result to \( m > 1 \), we need to establish some kind of monotonicity with respect to \( m \). A coupling to the rescue!

For the B-R model, with loops allowed at every vertex, the following coupling between \( G_{m,0}(t) \) and \( G_{1,0}(mt) \) was discovered by Bollobás and Riordan [7]. Start with the \( \{G_{1,0}(t)\} \) random process and let the vertices be \( v_1, v_2, \ldots \). To obtain the random graph process \( \{G_{m,0}(t)\} \) from \( \{G_{1,0}(mt)\} \),

1. collapse the first \( m \) vertices \( v_1, \ldots, v_m \) into the first vertex \( w_1 \) of \( G_{m,0}(t) \), the next \( m \) vertices \( v_{m+1}, \ldots, v_{2m} \) into the second vertex \( w_2 \) of \( G_{m,0}(t) \), and so on;

2. keep the full record of the multiple edges and loops formed by collapsing the blocks \( \{v_{(i-1)m+1}, \ldots, v_{im}\} \) for each \( i \).

Doing this collapsing indefinitely we get the jointly defined Bollobás-Riordan graph processes \( \{G_{m,0}(t)\} \) and \( \{G_{1,0}(mt)\} \). The beauty of the \( \delta \)-extended Bollobás-Riordan model is that similarly this collapsing operation applied to the process \( \{G_{1,\delta/m}(mt)\} \) delivers the process \( \{G_{m,\delta}(t)\} \), [26]. (See Appendix for the explanation.)

**Remark 3.4.** It follows from this coupling that \( G_{m,-m}(t) \) is the star centered at vertex 1. This follows from the fact that \( G_{1,-1}(mt,1) \) is a star and confirms the claim in Remark 2.1. In the coupling, each additional vertex is joined to vertex 1 by \( m \) parallel edges.

**Lemma 3.5.** For the processes \( \{G_{m,\delta}(t)\} \) and \( \{G_{1,\delta/m}(mt)\} \) coupled this way, we have \( X_{m,\delta}(t, r) \geq m^{-1} X_{1,\delta/m}(mt, mr) \).

**Proof.** Let us simply write \( G_1 \) and \( G_m \) for the two graphs \( G_{1,\delta/m}(mt) \) and \( G_{m,\delta}(t) \), respectively. Similarly, write \( T_1 \) and \( T_m \), respectively, for the descendant tree in \( G_{1,\delta/m}(mt) \) rooted at \( mr \) and the descendant tree in \( G_{m,\delta}(t) \) rooted at \( r \). If \( v_a \in T_1 \), i.e. \( v_a \) is a descendant of \( mr \), then for \( b = \lfloor a/m \rfloor \) we have \( w_b = \{v_{mb^{-1}+i}, i \in [m] \} \ni v_a \), implying that \( w_b \) is a descendant of \( r \) in \( G_m \), i.e. \( w_b \in T_m \). (The converse is generally false: if \( w_b \) is a descendant of \( r \), it does not mean that every \( v_{mb^{-1}+i}, (i \in [m]) \), is a descendant of \( mr \).) Therefore

\[
X_{m,\delta}(t, r) = |V(T_m)| \geq m^{-1} |V(T_1)| = m^{-1} X_{1,\delta/m}(mt, mr). \]

\(\square\)
Thus, to complete the proof of the theorem, i.e. for $\delta > -m$, we (a) use Theorem 2.2, to assert that for the process $\{G_{1,\delta/m}(t)\}$, a.s. $\lim p_X(t) > 0$; (b) use Lemma 3.5, to assert that a.s. $\liminf p_X(t) > 0$ for $\{G_{m,\delta}(t)\}$ as well. □

4. Proof of Theorem 2.4

Recall that the greedy algorithm generates the increasing sequence $\{M(t)\}$ of partial matchings on the sets $[t]$, with $M(1) = \emptyset$. Given $M(t)$, let

$X(t) :=$ number of unmatched vertices at time $t$,
$Y(t) :=$ total degree of unmatched vertices at time $t$,
$U(t) :=$ number of unmatched vertices selected by $t + 1$ from $[t] \setminus M(t)$,
$x(t) := X(t)/t$,
$y(t) := Y(t)/(2mt)$.

We want to prove that, for any $\delta > -m$ and $\alpha < 1/3$, almost surely,

$$
\lim_{t \to \infty} t^\alpha \max\{0, x(t) - \rho_{m,\delta}\} = 0,
$$

where $\rho_{m,\delta}$ is the unique root in $(0, 1)$ of

$$
h(z) = h_{m,\delta}(z) := 2 \left[ - \left( \frac{m + \delta}{2m + \delta} \right) z \right]^m - z - 1. \tag{4.1}
$$

(Note that, for $\delta > -m$, the function $h(z)$ is decreasing on $(0, 1)$ and $h(z) = 0$ has a unique solution in the same interval.) We will prove this first for a slightly different model that does not allow any loops other than the first vertex. In this model, vertex 1 has $m$ loops, and the $i$-th edge of vertex $t + 1$ attaches to $u \in [t]$ with probability

$$
d_{t,i-1}(u) + \delta \quad \text{where} \quad \frac{d_{t,i-1}(u) + \delta}{2mt + 2(i-1) + t\delta}.
$$

We will need the following Chernoff bound. (See e.g. [24, Theorem 2.8].)

**Theorem 4.1.** If $X_1, \ldots, X_n$ are independent Bernoulli random variables, $X = \sum_{i=1}^n X_i$, and $\lambda = \mathbb{E}[X]$, then

$$
\mathbb{P}(|X - \lambda| > \varepsilon \lambda) < 2 \exp \left( -\varepsilon^2 \lambda/3 \right) \quad \forall \varepsilon \in (0, 3/2).
$$

**Proof of Theorem 2.4 for “loops only at vertex 1”**. Let $\varepsilon = \varepsilon_t := t^{-1/3} \log t$. We will show

$$
\mathbb{P}(x(t) > \rho + \varepsilon) \leq \exp \left( -\Theta \left( \log^3 t \right) \right). \tag{4.2}
$$

Once we show (4.2), the Borel-Cantelli lemma gives

$$
\mathbb{P}(x(t) - \rho > t^{-1/3} \log t \quad \text{infinitely often}) = 0,
$$
which gives what we want. Let us prove (4.2).

Since each degree is at least \( m \), we have \( Y(t) \geq mX(t) \) and hence \( y(t) \geq x(t)/2 \). Also, since

\[
X(t + 1) = \begin{cases} 
X(t) + 1 & \text{if } U(t) = 0 \\
X(t) - 1 & \text{if } U(t) > 0, 
\end{cases}
\]

we have

\[
\mathbb{E}[X(t + 1) | \circ] = X(t) + \mathbb{P}(U(t) = 0 | \circ) - \mathbb{P}(U(t) > 0 | \circ). \tag{4.3}
\]

Since \( \mathbb{P}(\text{vertex } t + 1 \text{ has some loop}) = O(t^{-1}) \), using \( Y(t) \geq mX(t) \) in the last step below, we get

\[
\mathbb{P}(U(t) = 0 | \circ) = \mathbb{P}(U(t) = 0 \text{ and vertex } t + 1 \text{ has no loop} | \circ) + O(t^{-1})
= (1 - O(t^{-1})) \frac{(2mt - Y(t) + \delta t - \delta X(t))^{(m)}}{(2mt + \delta t)^{(m)}} + O(t^{-1})
= \frac{(2mt + \delta t - Y(t) - \delta X(t))^{m}}{(2mt + \delta t)^{m}} + O(t^{-1})
= \left(1 - \frac{2m}{2m + \delta} y(t) - \frac{\delta}{2m + \delta} x(t)\right)^{m} + O(t^{-1}) \tag{4.4}
\]

Now using (4.3) and (4.4) gives

\[
\mathbb{E}[x(t + 1) | \circ] \leq x(t) + \frac{1}{t} \left[2 \left(1 - \frac{m + \delta}{2m + \delta} x(t)\right)^{m} - x(t) - 1\right] + O(t^{-2})
= x(t) + h(x(t)) + O(t^{-2}), \tag{4.5}
\]

where \( h(z) \) is as defined in (4.1).

We know that \( x(1) = 0 \). For \( T < t \), let \( \mathcal{E}_T \) be the event that \( x(t) > \rho + \varepsilon \)
and \( T \in [1, t) \) be the last time such that \( x(\tau) \leq \rho + \varepsilon/2 \), that is,

\[
x(T) \leq \rho + \varepsilon/2; \quad x(\tau) > \rho + \varepsilon/2, \quad \forall \tau \in (T, t); \quad x(t) > \rho + \varepsilon.
\]

Now

\[
X(T) + t - T \geq X(t) > t(\rho + \varepsilon) \implies Tx(T) + t - T > t(\rho + \varepsilon),
\]

\[
\implies T(\rho + \varepsilon/2) + t - T > t(\rho + \varepsilon),
\]

implying, with a bit of algebra, that

\[
t - T \geq (1 + O(\varepsilon)) \frac{t\varepsilon}{2(1 - \rho)}.
\]
We conclude that
\[
\{x(t) > \rho + \varepsilon \} \subseteq \bigcup_{T=1}^{s} \mathcal{E}_T, \quad s = s(t) := t - \left[ \frac{t\varepsilon}{3(1-\rho)} \right].
\]

Now let us fix a \(T \in [1, s]\) and bound \(\mathbb{P}(\mathcal{E}_T)\). The main idea of the proof is that, as long as \(x(\tau) > \rho\), by Equation (4.5), the process \(\{x(\tau)\}\) has a negative drift.

Let us say that we have a “failure” at step \(i+1\) when \(X(i+1) = X(i) + 1\). On the event \(\mathcal{E}_T\) we have \(X(\tau) \geq \tau(\rho + \varepsilon/2)\), \(\tau \in (T, t]\), intuitively meaning that there are many failures between steps \(T+1\) and \(t\), despite negativity of expected shift. And this should make the event \(\mathcal{E}_T\) rather unlikely. To prove it rigorously, let \(\xi_j\) denote the indicator of the event \(\{x(j-1) > \rho + \varepsilon/2 \text{ and } X(j) = X(j-1) + 1\}\). Let \(Z_T := \xi_{T+2} + \cdots + \xi_t\). On the event \(\mathcal{E}_T\), the sum \(Z_T\) counts the total number of upward unit jumps \((X(j) - X(j-1) = 1, j \in [T+2, t])\) and therefore
\[
X(T + 1) + Z_T - [(t - T) - Z_T] = X(t) \geq t(\rho + \varepsilon).
\]

Since \(X(T + 1) = X(T) + 1 \leq T(\rho + \varepsilon/2) + 1\), we see that
\[
\frac{Z_T - [(t - T) - Z_T]}{t - T} > \rho + \varepsilon, \quad Z(T) := 1 + Z_T,
\]
or equivalently,
\[
Z_T > (t - T)(1 + \rho + \varepsilon)/2.
\]

On the other hand, for \(\tau \geq T+1\), using (4.4) and conditioning on the full record (up to and including time \(\tau\)), such that \(x(\tau) > \rho + \varepsilon/2\), we have
\[
\mathbb{P}(\xi_{\tau+1} = 1|\omega) \leq \mathbb{P}(X(\tau + 1) = X(\tau) + 1|\omega) \\
\leq \left( 1 - \left( \frac{m + \delta}{2m + \delta} \right) \left( \frac{\rho + \varepsilon}{2} \right) \right)^m + O(\tau^{-1}).
\]

Hence, the sequence \(\{\xi_\tau\}\) is stochastically dominated by the sequence of independent Bernoulli random variables \(B_\tau\) with parameters \(\min(\mu + O(\tau^{-1}), 1)\), where
\[
\mu := \left( 1 - \left( \frac{m + \delta}{2m + \delta} \right) \left( \frac{\rho + \varepsilon}{2} \right) \right)^m = \left( 1 - \left( \frac{m + \delta}{2m + \delta} \right) \rho \right)^m + O(\varepsilon).
\]

Consequently, \(Z_T\) is stochastically dominated by \(1 + \sum_{j=T+2}^{t} B_j\), and
\[
\lambda := \sum_{j=T+2}^{t} \mathbb{E}[B_j] = \mu(t - T) + O(\log t).
\]

Note that, since
\[
\left( 1 - \left( \frac{m + \delta}{2m + \delta} \right) \rho \right)^m = \frac{\rho + 1}{2},
\]
we have
\[
\frac{\rho + 1 + \varepsilon}{2} = \frac{\rho + 1}{2} \left( 1 + \frac{\varepsilon}{\rho + 1} \right) = \left( 1 + \frac{\varepsilon}{\rho + 1} \right) \left( 1 - \left( \frac{m + \delta}{2m + \delta} \right) \rho \right)^m \geq \left( 1 + \frac{\varepsilon}{2} \right) \mu.
\]
Thus, by Theorem 4.1, we have
\[
\mathbb{P}(Z_T > (t - T)(1 + \rho + \varepsilon)/2) \leq \mathbb{P} \left( 1 + B_{T+2} + \cdots + B_t > (t - T)(\rho + \varepsilon + 1/2) \right)
\]
\[
\leq \mathbb{P} \left( 1 + B_{T+2} + \cdots + B_t > \left( 1 + \frac{\varepsilon}{2} \right) (t - T) \mu \right)
\]
\[
\leq \exp \left( -\Theta(\varepsilon^2(t - T)) \right) \leq e^{-\Theta(\log^3 t)}.
\]
Using the union bound on $T$ finishes the proof of (4.2) and the theorem. □

**Loops allowed everywhere.** The above analysis is carried over to this more complicated case via an argument similar to the one for the descendant trees in the subsections 1.1. Here is a proof sketch. First, the counterpart of (4.4) is:
\[
\mathbb{P} \{ U(t) = 0 \} \cap \{ \text{no loops at } t + 1 \} \mid \sigma
\]
\[
= \Pi_{m}(t) \prod_{j=0}^{m-1} \left( \frac{2mt - Y(t) + \delta t - \delta X(t) + j}{2mt + 2j + 1 + \delta t + (j + 1) \delta / m} \right)
\]
\[
\leq \Pi_{m}(t) \left[ \left( 1 - \frac{m + \delta}{2m + \delta} x(t) \right)^m + O(t^{-1}) \right]
\]
\[
= (1 - O(t^{-1})) \left[ \left( 1 - \frac{m + \delta}{2m + \delta} x(t) \right)^m + O(t^{-1}) \right]
\]
\[
= \left( 1 - \frac{m + \delta}{2m + \delta} x(t) \right)^m + O(t^{-1});
\]
see (3.7) for $\Pi_{m}(t)$. Therefore we obtain again the equation (4.5). The rest of the proof remains the same.

**Remark 4.2.** Let $r = r_{m,\delta} := 1 - \rho_{m,\delta}$, where $\rho_{m,\delta}$ is the unique root in $(0, 1)$ of
\[
h(z) = h_{m,\delta}(z) := 2 \left[ 1 - \left( \frac{m + \delta}{2m + \delta} \right) z \right]^m - z - 1.
\]
Then, $r$ is the unique root in $(0, 1)$ of
\[
f(z) = f_{m,\delta}(z) := 2 - z - 2 \left( \frac{m}{2m + \delta} + \frac{m + \delta}{2m + \delta} z \right)^m.
\]
Thus, by Theorem 2.4, we have
\[
\lim \inf (1 - x(t)) \geq r
\]
almost surely, where $1 - x(t)$ is the fraction of the vertices in $L(t)$. See (2.3) for various $r$ values when $\delta = 0$.

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In order to show that the coupling described in Section 3 really works, we can compute the probability that the $i$-th edge of vertex $w_{t+1}$ connects to vertex $w_x$ in the coupling and compare it with the probability in (2.1). Let us denote by $\{G'_t(t)\}$ the process obtained by collapsing the vertices of $\{G_{1,\delta/m}(mt)\}$. Note that $(mt + i)$-th edge of the $\{G_{1,\delta/m}(mt)\}$-process becomes the $i$-th edge of $w_{t+1}$ after the collapsing. Hence the $i$-th edge of vertex $w_{t+1}$ connects to $w_x$ $(x \leq t)$ if and only if the $(mt + i)$-th edge of $\{G_{1,\delta/m}(mt)\}$-process connects $v_{mt+i}$ with one of the vertices $v_{m(x-1)+1}, \ldots, v_{mx}$. Let us denote by $d_{mt+i−1}(v_y)$ the degree of $v_y$ $(y \leq mt+i)$ just before the $(mt + i)$-th edge of $\{G_{1,\delta/m}\}$-process is drawn. Also, let $D_{t,i−1}(w_x)$ denote the degree of $w_x$ at the exact same time. Hence, by definition,

$$D_{t,i−1}(w_x) = \begin{cases} \sum_{y=mx-m+1}^{mx} d_{mt+i−1}(v_y), & x \leq t \\ \sum_{y=mt+i}^{mt+i+1} d_{mt+i−1}(v_y), & x = t + 1. \end{cases}$$

By (2.2), for $x \leq t$, the probability that $v_{mt+i}$ connects to one of the vertices $v_{m(x-1)+1}, \ldots, v_{mx}$ (equivalently, the probability that the $i$-th edge of $w_{t+1}$
connects to $w_x$) is

$$\frac{\sum_{y=mx-m+1}^{mx} (d_{mt+i-1}(v_y) + \delta/m)}{(2 + \delta/m)(mt + i - 1) + 1 + \delta/m} = \frac{\delta + \sum_{y=mx-m+1}^{mx} d_{mt+i-1}(v_y)}{\delta(t + i/m) + 2mt + 2i - 1}$$

$$= \frac{\delta + D_{t,i-1}(w_x)}{\delta(t + i/m) + 2mt + 2i - 1}.$$ 

Similarly, the probability that $v_{mt+i}$ selects one of $v_{mt+1}, \ldots, v_{mt+i}$ (equivalently, the probability that the $i$-th edge of $w_{t+1}$ is a loop) is

$$\frac{1 + i\delta/m + \sum_{j=1}^{i-1} d_{mt+i-1}(v_{mt+j})}{\delta(t + i/m) + 2mt + 2i - 1} = \frac{1 + i\delta/m + D_{t,i-1}(w_{t+1})}{\delta(t + i/m) + 2mt + 2i - 1}.$$

Note that the two probabilities above are the same as those in (2.1) if we replace $D_{t,i-1}(w_x)$ with $d_{t,i-1}(x)$. Moreover, the two processes, $\{G'_{m,\delta}(t)\}$ and $\{G_{m,\delta}(t)\}$ as defined by (2.1), both start with $m$ loops on the first vertex, which implies $d_{1,0}(\cdot) = D_{1,0}(\cdot)$. This gives us that $\{G_{m,\delta}(t)\}$ and $\{G'_{m,\delta}(t)\}$ are equivalent processes, that is, at every stage, they produce the same random graph.