RATIONALLY-EXTENDED RADIAL OSCILLATORS AND LAGUERRE EXCEPTIONAL ORTHOGONAL POLYNOMIALS IN $k$TH-ORDER SUSYQM

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Abstract

A previous study of exactly solvable rationally-extended radial oscillator potentials and corresponding Laguerre exceptional orthogonal polynomials carried out in second-order supersymmetric quantum mechanics is extended to $k$th-order one. The polynomial appearing in the potential denominator and its degree are determined. The first-order differential relations allowing one to obtain the associated exceptional orthogonal polynomials from those arising in a $(k - 1)$th-order analysis are established. Some nontrivial identities connecting products of Laguerre polynomials are derived from shape invariance.

Running head: Laguerre Exceptional Orthogonal Polynomials

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1 Introduction

Since the introduction of the first families $X_1$ of Laguerre– and Jacobi-type $n$th-degree exceptional orthogonal polynomials (EOP), with $n = 1, 2, 3, \ldots$, in the context of Sturm-Liouville theory [1, 2] and the realization of their usefulness in building new exactly solvable rational extensions of known quantum potentials [3], a lot of work has been devoted to generalizing these families and the associated exactly solvable potentials, as well as to providing several different (but equivalent) approaches to the problem.

After noting [3] that the first extended potentials enlarged the class of known translationally shape invariant potentials in supersymmetric quantum mechanics (SUSYQM) [4, 5, 6], it appeared convenient to use a SUSYQM technique to construct some additional examples of EOP and potentials [7, 8], in agreement with previous works on algebraic deformations of shape invariant potentials [9, 10].

Two distinct families of Laguerre and Jacobi $X_m$ EOP, now labelled as type I and type II, were constructed for arbitrary large $m$ [11, 12] and their properties were thoroughly studied [13, 14]. In agreement with the general definition given in Refs. 1 and 2, such $n$th-degree polynomials, with $n = m, m + 1, m + 2, \ldots$, have the remarkable property of forming orthogonal and complete sets with respect to some positive-definite measure. They include as special cases the $X_1$ EOP first considered in Refs. 1, 2, 3, and 7, as well as the $X_2$ EOP introduced in Ref. 8.

In addition, the $X_m$ EOP were shown to be obtainable through several approaches, such as the Darboux-Crum transformation [15, 16, 17], the Darboux-Bäcklund one [18], and the prepotential method [19].

Some novel procedures for generating translationally shape invariant potentials without recourse to EOP were also devised and led to examples of potentials [20, 21, 22], some of which were presented as new although they had already been built in Refs. 3 and 7.

Recently, the $X_m$ EOP (and associated potentials) were generalized to multi-indexed families $X_{m_1,m_2,\ldots,m_k}$, obtained through the use of multi-step Darboux alge-
braic transformations \cite{23}, the Crum-Adler mechanism \cite{24}, higher-order SUSYQM \cite{25} or multi-step Darboux-Bäcklund transformations \cite{26}.

In Ref. 25, in particular, the procedure for constructing rationally-extended radial oscillator potentials and associated Laguerre EOP in higher-order SUSYQM was considered with special emphasis on second order. The number of distinct potentials and EOP families corresponding to a given degree $\mu$ of the polynomial arising in the potential denominator (which is some definite function of $m_1, m_2, \ldots, m_k$) was determined for some small $\mu$ values and conjectured for higher ones.

The purpose of the present work is to extend some of the results of Ref. 25 to order $k > 2$, to establish relations between successive EOP, and to comment on shape invariance and some of its consequences.

## 2 Potentials and EOP in First-order SUSYQM

To start with, let us briefly review the construction of rationally-extended radial oscillator potentials and corresponding Laguerre $X_m$ EOP in first-order SUSYQM.

As usual, the radial oscillator potential (in units $\hbar = 2m = 1$)

$$V_l(x) = \frac{1}{4} \omega^2 x^2 + \frac{l(l + 1)}{x^2},$$

where $\omega$ and $l$ denote the oscillator frequency and the angular momentum quantum number, respectively, is defined on the half-line $0 < x < \infty$. The corresponding Schrödinger equation has an infinite number of bound-state wave functions, which, up to some normalization factor, can be written as

$$\psi^{(l)}_\nu \propto x^{l+1}e^{-\frac{1}{4} \omega x^2} L^{(l+\frac{1}{2})}_\nu \left(\frac{1}{2} \omega x^2\right) \propto \eta_l(z) L^{(\alpha)}_\nu(z), \quad \nu = 0, 1, 2, \ldots,$$

with

$$z = \frac{1}{2} \omega x^2, \quad \alpha = l + \frac{1}{2}, \quad \eta_l(z) = z^{\frac{1}{2}(2\alpha+1)}e^{-\frac{1}{2}z},$$

and $L^{(\alpha)}_\nu(z)$ some Laguerre polynomial \cite{27}. The associated bound-state energies are given by $E^{(l)}_\nu = \omega(2\nu + l + \frac{3}{2}) = \omega(2\nu + \alpha + 1)$.)


In first-order SUSYQM [5], one considers a pair of SUSY partners
\[ H^{(+)} = A^\dagger A = -\frac{d^2}{dx^2} + V^{(+)}(x) - E, \quad H^{(-)} = AA^\dagger = -\frac{d^2}{dx^2} + V^{(-)}(x) - E, \]
\[ A^\dagger = -\frac{d}{dx} + W(x), \quad A = \frac{d}{dx} + W(x), \quad V^{(\pm)}(x) = W^2(x) \mp W'(x) + E, \tag{2.4} \]
which intertwine with the first-order differential operators \( A \) and \( A^\dagger \) as \( AH^{(+)} = H^{(-)}A \) and \( A^\dagger H^{(-)} = H^{(+)}A^\dagger \). Here \( W(x) \) is the superpotential, which can be expressed as \( W(x) = -(\log \phi(x))' \) in terms of a (nodeless) seed solution \( \phi(x) \) of the initial Schrödinger equation
\[ \left( -\frac{d^2}{dx^2} + V^{(+)}(x) \right) \phi(x) = E\phi(x), \tag{2.5} \]
\( E \) is the factorization energy, assumed smaller than or equal to the ground-state energy \( E^{(+)}_0 \) of \( V^{(+)} \), and a prime denotes a derivative with respect to \( x \). Except in Sec. 5, we shall only consider here the case where \( E < E^{(+)}_0 \), in which occurrence \( \phi(x) \) is nonnormalizable, and we shall assume that the same holds true for \( \phi^{-1}(x) \). Then \( H^{(+)} \) and \( H^{(-)} \) turn out to be isospectral.

For \( V_l(x) \), there are two different types of seed solutions \( \phi(x) \) with such properties, namely
\[ \phi^{\dagger}_{lm}(x) = \chi^\dagger_l(z) L_m^{(\alpha)}(-z) \propto x^{l+1} e^{\frac{1}{2} \alpha x^2} L_m^{(l+\frac{1}{2})}(-\frac{1}{2} \alpha x^2), \tag{2.6} \]
\[ \phi^\dagger_{lm}(x) = \chi^\dagger_l(z) L_m^{(-\alpha)}(z) \propto x^{-l} e^{-\frac{1}{2} \alpha x^2} L_m^{(-l-\frac{1}{2})}(-\frac{1}{2} \alpha x^2), \tag{2.7} \]
with
\[ \chi^\dagger_l(z) = z^{\frac{1}{2}(2\alpha+1)} e^{\frac{1}{2} z}, \quad \chi^\dagger_l(z) = z^{-\frac{1}{2}(2\alpha-1)} e^{-\frac{1}{2} z}, \tag{2.8} \]
and corresponding energies \( E^{\dagger}_{lm} = -\omega(\alpha + 2m + 1) \), \( E^{\dagger}_{lm} = -\omega(\alpha - 2m - 1) \), respectively. Note that for type II, \( \alpha \) must be greater than \( m \).

To obtain some rationally-extended radial oscillator potentials \( V_{l,\text{ext}}(x) \) with a given \( l \), we have to start from a conventional radial oscillator potential \( V_l(x) \) with some different \( l' \). We then get
\[ V^{(+)} = V_l', \quad V^{(-)}(x) = V_{l,\text{ext}}(x) + C = V_l(x) + V_{l,\text{rat}}(x) + C, \]
\[ V_{l,\text{rat}}(x) = -2\omega \left\{ \frac{g^{(\alpha)}_m}{g^{(\alpha)}_m} + 2z \left[ \frac{g^{(\alpha)}_m}{g^{(\alpha)}_m} - \left( \frac{g^{(\alpha)}_m}{g^{(\alpha)}_m} \right)^2 \right] \right\}, \tag{2.9} \]
where a dot denotes a derivative with respect to \( z \) and \( C \) is some constant. According to the choice made for the seed function \( \phi(x) \), we may distinguish the two cases

\[
(i) \quad l' = l - 1, \quad \phi = \phi^I_{l-1,m}, \quad g_m^{(\alpha)}(z) = L_m^{(\alpha-1)}(-z), \quad C = -\omega; \quad (2.10)
\]

\[
(ii) \quad l' = l + 1, \quad \phi = \phi^H_{l+1,m}, \quad g_m^{(\alpha)}(z) = L_m^{(-\alpha-1)}(z), \quad C = \omega. \quad (2.11)
\]

The resulting extended potential is nonsingular on the half-line for any \( m = 1, 2, \ldots \) (and \( \alpha \) large enough for type II).

The bound-state wavefunctions \( \psi^{(-)}(x) \) of \( V^{(-)} \) can be obtained by acting with \( A \) on those of \( V^{(+)} \), \( \psi^{(+)}(x) \propto \eta_l(z) L_\alpha^{(\alpha')}(-z) \), and are given by

\[
\psi^{(-)}(x) \propto \eta(z) g_m^{(\alpha)}(z), \quad n = m + \nu, \quad \nu = 0, 1, 2, \ldots, \quad (2.12)
\]

where the \( n \)th-degree polynomial \( y_n^{(\alpha)}(z) \) satisfies the differential equation

\[
\left[ z \frac{d^2}{dz^2} + \left( \alpha + 1 - z - 2z \frac{g_m^{(\alpha)}}{g_m^{(\alpha)}} \right) \frac{d}{dz} + (z - \alpha) \frac{g_m^{(\alpha)}}{g_m^{(\alpha)}} + z \frac{g_m^{(\alpha)}}{g_m^{(\alpha)}} \right] y_n^{(\alpha)}(z) = (m-n)y_n^{(\alpha)}(z). \quad (2.13)
\]

The orthonormality and completeness of \( \psi^{(-)}(x), \nu = 0, 1, 2, \ldots, \) on the half-line imply that the polynomials \( y_n^{(\alpha)}(z), n = m + \nu, \nu = 0, 1, 2, \ldots, \) form an orthogonal and complete set with respect to the positive-definite measure \( z^\alpha e^{-z} (g_m^{(\alpha)})^{-2} dz \). According to the choice made for \( \phi(x) \), such polynomials belong to the \( L_1 \) or \( L_2 \) family and are denoted by \( L_{\alpha,m,n}(z) \) or \( L_{\alpha,m,n}^H(z) \). They are conventionally normalized in such a way that their highest-degree term is given by \( (-z)^n/[(n-m)!m!] \) multiplied by \( (-1)^m \) or 1, respectively [11, 12, 14].

### 3 Potentials and EOP in \( k \)th-order SUSYQM

Going from first- to \( k \)th-order SUSYQM [28, 29, 30, 31, 32, 33] is achieved by replacing the first-order differential operators \( A, A^\dagger \) by some \( k \)th-order ones \( \mathcal{A}, \mathcal{A}^\dagger \) in such a way that the SUSY partner Hamiltonians

\[
H^{(1)} = -\frac{d^2}{dx^2} + V^{(1)}(x), \quad H^{(2)} = -\frac{d^2}{dx^2} + V^{(2)}(x) \quad (3.1)
\]

\[\text{[a]It is worth noting, however, that in Ref. 15, there is an additional factor } - (\alpha + n - 2m + 1) \text{ in the definition of } L_{\alpha,m,n}^H(z).\]
intertwine with them as \( \mathcal{A}H^{(1)} = H^{(2)}\mathcal{A} \) and \( \mathcal{A}^\dagger H^{(2)} = H^{(1)}\mathcal{A}^\dagger \). Here, we restrict ourselves to the reducible case, which could be formulated in the \( k \)th-order parasupersymmetric language as well [34]. The operator \( \mathcal{A} \) can then be factorized into a product \( A^{(k)}A^{(k-1)} \ldots A^{(1)} \) of first-order differential operators \( A^{(i)} = \frac{d}{dx} + W^{(i)}(x) \), \( i = 1, 2, \ldots, k \) (and similarly for \( \mathcal{A}^\dagger \)).

In terms of \( k \) seed functions \( \phi_1, \phi_2, \ldots, \phi_k \) of the starting Hamiltonian \( H^{(1)} \), the \( k \) functions \( W^{(i)}(x), i = 1, 2, \ldots, k \), can be expressed as

\[
W^{(i)}(x) = -\left( \log \phi^{(i)}(x) \right)',
\]

where

\[
\phi^{(i)}(x) = A^{(i-1)}A^{(i-2)} \ldots A^{(1)}\phi_i = \frac{W(\phi_1, \phi_2, \ldots, \phi_i)}{W(\phi_1, \phi_2, \ldots, \phi_{i-1})}
\]

and \( W(\phi_1, \phi_2, \ldots, \phi_i) \) denotes the Wronskian of \( \phi_1(x), \phi_2(x), \ldots, \phi_i(x) \). The potentials of the two partner Hamiltonians are linked by the relationship

\[
V^{(2)}(x) = V^{(1)}(x) - 2\frac{d^2}{dx^2} \log W(\phi_1, \phi_2, \ldots, \phi_k).
\]

As in Sec. 2, we start from some conventional radial oscillator potential \( V^{(1)}(x) = V_l(x) \) to get as a partner an extended potential with a given \( l \), \( V^{(2)}(x) = V_{l,\text{ext}}(x) + C = V_l(x) + V_{l,\text{rat}}(x) + C \), up to some additive constant \( C \). Since we have two types of seed functions at our disposal and their order is irrelevant as far as the final potential is concerned, there are altogether \( k + 1 \) possibilities for the set of \( k \) seed functions, which we may denote by \( \Pi^k \), where \( q \) runs over \( 0, 1, \ldots, k \). On assuming that the first \( q \) functions \( \phi_i \) are of type I, we may write

\[
\phi_i(x) = \begin{cases} 
\chi^I_{\mu}(z)L^{(\alpha')_{m_i}}(-z) & \text{if } i = 1, 2, \ldots, q, \\
\chi^I_{\mu}(z)L^{(\alpha')_{m_i}}(-z) & \text{if } i = q + 1, q + 2, \ldots, k,
\end{cases}
\]

with \( \alpha' = l' + \frac{i}{2} \), \( 0 < m_1 < m_2 < \cdots < m_q \), and \( 0 < m_{q+1} < m_{q+2} < \cdots < m_k \). For \( \alpha' \) large enough, these functions are nodeless on the half-line.

The two pure cases \( \Pi^k \) and \( \Pi^k \), corresponding to \( q = k \) and \( q = 0 \), respectively, are very easy to deal with. For the former, on assuming \( l' = l - k \) and using some simple properties of Wronskians [35], we indeed obtain

\[
W(\phi_1, \phi_2, \ldots, \phi_k) = (\omega x)^{k(k-1)/2}(\chi^I_{l-k})^k g^{(\alpha)}(z),
\]

\[
g^{(\alpha)}(z) = \tilde{W}(L^{(\alpha')_{m_1}}(-z), L^{(\alpha')_{m_2}}(-z), \ldots, L^{(\alpha')_{m_k}}(-z)).
\]
Similarly, for the latter and \( l' = l + k \), we get

\[
\mathcal{W}(\phi_1, \phi_2, \ldots, \phi_k) = (\omega x)^{k(k-1)/2} (\chi_{l+k}^\Pi)^k g_\mu^{(\alpha)}(z),
\]

\[
g_\mu^{(\alpha)}(z) = \tilde{\mathcal{W}}(L_{m_1}^{(-\alpha')}(-z), L_{m_2}^{(-\alpha')}(-z), \ldots, L_{m_k}^{(-\alpha')}(-z)).
\]

Here \( \tilde{\mathcal{W}} \) denotes Wronskians with respect to the variable \( z \), while \( \mu \) is the degree of the polynomial \( g_\mu^{(\alpha)}(z) \):

\[
g_\mu^{(\alpha)}(z) = \mathcal{C}_\mu^{(\alpha)}z^\mu + \text{lower order terms}. \tag{3.7}
\]

The values of \( \mu \) and \( \mathcal{C}_\mu^{(\alpha)} \) can be easily found by replacing all Laguerre polynomials by their highest-degree term, e.g. \( L_m^{(\alpha)}(z) \) by \((-z)^m/m!\), in the two Wronskians and evaluating the two resulting determinants. In both cases, \( \mu = \sum_{i=1}^k m_i - \frac{1}{2}k(k-1) \) and \( \mathcal{C}_\mu^{(\alpha)} = (-1)^\sigma \Delta(m_1, m_2, \ldots, m_k)/(m_1!m_2! \cdots m_k!) \), where \( \Delta(m_1, m_2, \ldots, m_k) = \prod_{i=1}^{k-1} \prod_{j=i+1}^k (m_j - m_i) \) is a Vandermonde determinant of order \( k \). In case \( \Gamma^k, \sigma = 0 \), while in case \( \Pi^k, \sigma = m_1 + m_2 + \cdots + m_k \).

In the mixed cases \( \Pi^k \Pi^{k-q}, 0 < q < k \), let us assume \( l' = l + k - 2q \). On taking successively into account that

\[
\chi_{l'}^\Pi(z) = \chi_{l'}^1(z)z^{-\alpha'}e^{-z}, \tag{3.8}
\]

and [27]

\[
\frac{d}{dz} L_{m}^{(\alpha')}(-z) = L_{m-1}^{(\alpha'+1)}(-z), \tag{3.9}
\]

the Wronskian can be written as [26]

\[
\mathcal{W}(\phi_1, \phi_2, \ldots, \phi_k) = (\omega x)^{k(k-1)/2} (\chi_{l'}^1)^k
\times \tilde{\mathcal{W}}(L_{m_1}^{(\alpha')}(-z), \ldots, L_{m_q}^{(\alpha')}(-z), z^{-\alpha'}e^{-z}L_{m_{q+1}}^{(-\alpha')}(-z), \ldots, z^{-\alpha'}e^{-z}L_{m_k}^{(-\alpha')}(-z))
= (\omega x)^{k(k-1)/2} (\chi_{l'}^1)^k (z^{-\alpha' - k + 1}e^{-z})^{k-q} \text{det} \Gamma_\mu^{(\alpha)}, \tag{3.10}
\]

where the matrix elements of the \( k \times k \) matrix \( \Gamma_\mu^{(\alpha)} \) are defined in terms of Pochhammer’s symbol by

\[
(\Gamma_\mu^{(\alpha)})_{ij} = \begin{cases} L_{m_j-i+1}^{(\alpha'+i-1)}(-z) & \text{if } j = 1, 2, \ldots, q, \\ (m_j + 1)i-1z^{-i}L_{m_j+i-1}^{(-\alpha'-i+1)}(z) & \text{if } j = q+1, q+2, \ldots, k, \end{cases} \tag{3.11}
\]
for any $i = 1, 2, \ldots, k$.

To put $\mathcal{W}(\phi_1, \phi_2, \ldots, \phi_k)$ into a form that interpolates between those for the pure cases, given in (3.5) and (3.6), it is convenient to factorize it as

$$
\mathcal{W}(\phi_1, \phi_2, \ldots, \phi_k) = (\omega x)^{k(k-1)/2} z^{-q(k-q)} (\chi'_{\alpha})^q (\chi_{\beta})^{k-q} g_{\mu}^{(\alpha)}(z),
$$

by using Eq. (3.8) again. Determining $\mu$ and $\mathcal{C}_{\mu}^{(\alpha)}$, as defined in (3.7), is, however, not straightforward anymore, because the replacement of the Laguerre polynomials in (3.11) by their highest-degree term leads to a matrix with $k - q$ proportional columns, whose determinant vanishes.

To avoid such a drawback, it is appropriate to transform $\tilde{\Gamma}_{\mu}^{(\alpha)}$ into another matrix $\bar{\Gamma}_{\mu}^{(\alpha)}$ with equal determinant, for which such a problem does not occur. For any $j = 1, 2, \ldots, k$, the matrix elements of the latter may be defined by

$$
(\bar{\Gamma}_{\mu}^{(\alpha)})_{ij} = \begin{cases} 
(\tilde{\Gamma}_{\mu}^{(\alpha)})_{ij} & \text{if } i = 1, 2, \ldots, q + 1, \\
\sum_{r=q+1}^{i} (i-r-1) (\tilde{\Gamma}_{\mu}^{(\alpha)})_{rj} & \text{if } i = q + 2, q + 3, \ldots, k.
\end{cases}
$$

The summation on the right-hand side of this equation can be easily carried out by using definition (3.11) and the equations \[27\]

$$
z L_{m}^{(\alpha+1)}(z) + (m + 1) L_{m+1}^{(\alpha)}(z) = (m + \alpha + 1) L_{m}^{(\alpha)}(z),
$$

$$
L_{m}^{(\alpha-1)}(z) + L_{m-1}^{(\alpha)}(z) = L_{m}^{(\alpha)}(z).
$$

It can indeed be shown by induction over $i$ running over $q + 2, q + 3, \ldots, k$ that

$$
(\bar{\Gamma}_{\mu}^{(\alpha)})_{ij} = \begin{cases} 
L_{m_j-q}^{(\alpha'+i-1)}(-z) & \text{if } j = 1, 2, \ldots, q, \\
(m_j + 1) q (m_j - \alpha' - i + q + 2)_{i-1} & \text{if } j = q + 1, q + 2, \ldots, k.
\end{cases}
$$

For $g_{\mu}^{(\alpha)}(z) = z^{-(k-q)(k-q-1)} \det \tilde{\Gamma}_{\mu}^{(\alpha)}$ with Laguerre polynomials replaced by their highest-degree term, we now directly get

$$
\mu = \sum_{i=1}^{k} m_i - \frac{1}{2} q(q - 1) - \frac{1}{2} (k - q)(k - q - 1) + q(k - q),
$$

$$
\mathcal{C}_{\mu}^{(\alpha)} = (-1)^{\sigma} \frac{\Delta(m_1, m_2, \ldots, m_q) \Delta(m_{q+1}, m_{q+2}, \ldots, m_k)}{m_1! m_2! \ldots m_k!},
$$

$$
\sigma = \sum_{i=q+1}^{k} m_i + q(k - q),
$$

8
where Eq. (3.16) agrees with a result stated without proof in Ref. 24. As it is clear that Eqs. (3.16) and (3.17), as well as the expression for the Wronskian in (3.12), include the corresponding results for pure cases (provided $\Delta(m_1, m_2, \ldots, m_q) \equiv 1$ for $q = 0$), we may treat all $q$ values simultaneously.

It is then straightforward to see from Eq. (3.13) that the two partner potentials $V^{(1)}(x)$ and $V^{(2)}(x)$ assume a form similar to that of $V^{(+)}(x)$ and $V^{(-)}(x)$ in Eq. (2.9), where we only have to substitute $\mu$ for $m$, $l + k - 2q$ for $l'$, and $(k - 2q)\omega$ for $C$. Provided $V^{(2)}(x)$ is nonsingular, its bound-state wavefunctions $\psi^{(2)}_\nu(x)$ satisfy Eqs. (2.12) and (2.13) with $\mu$ substituted for $m$. We will now denote the orthogonal and complete EOP $y_n^{(a)}(z)$, $n = \mu + \nu$, $\nu = 0, 1, 2, \ldots$, (with respect to the positive-definite measure $z^\alpha e^{-z}(g_\mu^{(\alpha)})^{-2}dz$) by $L_{\alpha,m_1,m_2,\ldots,m_k,n}^{(q,k-q)}(z)$, where $m_1, m_2, \ldots, m_q$ belong to type I and $m_{q+1}, m_{q+2}, \ldots, m_k$ to type II. We choose to normalize them so that their highest-degree term is $(-z)^n/[(n-\mu)!m_1!m_2!\ldots m_k!]$. This agrees with the choice made in Ref. 23 for $k = 2$, $q = 0$, but differs from the conventional one for $L_{\alpha,m,n}^1(z)$ (see at the end of Sec. 2).

### 4 Relations between Successive Laguerre EOP

The purpose of the present Section is to derive the first-order differential relations expressing the Laguerre EOP obtained in $k$th-order SUSYQM in terms of those arising after $k - 1$ steps with the use of the first $k - 1$ seed functions $\phi_1$, $\phi_2$, $\ldots$, $\phi_{k-1}$. Since the additional seed function $\phi_k$ is of type I in the pure case $I_k$, but of type II in all the remaining cases, we have to treat them separately.

In the former case, where we start from $V_{l-k}(x)$, we arrive after $k - 1$ steps at an extended potential $V_{l-1,ext}(x)$, which assumes a form similar to that of $V_{l,ext}(x)$ but with $g_{\mu}^{(\alpha)}(z)$ replaced by $g_{\mu}^{(\alpha-1)}(z)$, where $\mu_1 = \sum_{i=1}^{k-1} m_i - \frac{1}{2}(k - 1)(k - 2)$. From the bound-state wavefunctions $\eta_{l-1}(z)y_{n_1}^{(\alpha-1)}(z)/g_{\mu_1}^{(\alpha-1)}(z)$, $n_1 = \mu_1$, $\mu_1 + 1$, $\ldots$, of $V_{l-1,ext}(x)$, we can obtain those of $V_{l,ext}(x)$, $\eta_l(z)y_n^{(a)}(z)/g_{\mu}^{(\alpha)}(z)$, $n = \mu$, $\mu + 1$, $\ldots$, by acting with the differential operator $A^{(k)}$ defined in Sec. 3, namely

$$A^{(k)} = \omega x \left( \frac{d}{dz} - \frac{1}{2} - \frac{2\alpha - 1}{4z} - \frac{\dot{g}_{\mu}^{(\alpha)}}{g_{\mu}^{(\alpha)}} + \frac{\dot{g}_{\mu}^{(\alpha-1)}}{g_{\mu_1}^{(\alpha-1)}} \right). \quad (4.1)$$
On using (2.3), we get the following equations

\[ O^{(\alpha)}_{\mu_1,\mu} L^{(k-1,0)}_{\alpha-1,m_1,m_2,\ldots,m_{k-1},n_1}(z) = DL^{(k,0)}_{\alpha,m_1,m_2,\ldots,m_k,n}(z), \quad (4.2) \]

\[ O^{(\alpha)}_{\mu_1,\mu} = \frac{1}{g^{(\alpha-1)}_{\mu_1}} \left[ g^{(\alpha)}_{\mu} \left( \frac{d}{dz} - 1 \right) - g^{(\alpha)}_{\mu} \right], \quad (4.3) \]

where \( n_1 = \mu_1 + \nu, \) \( n = \mu + \nu, \) and any \( \nu = 0, 1, 2, \ldots. \) The constant \( D \) appearing in (4.2) can be derived from the normalization coefficients of the two wavefunctions and SUSYQM theory. It can alternatively be obtained by comparing the highest-degree terms on both sides of (4.2). The result reads

\[ D = (-1)^{m_k-k} \prod_{i=1}^{k-1} (m_k - m_i). \quad (4.4) \]

In the latter case, where we start from \( V_{l+k-2q}(x), \) \( 0 \leq q < k, \) we arrive after \( k-1 \) steps at an extended potential \( V_{l+1,ext}(x), \) associated with a polynomial \( g^{(\alpha+1)}_{\mu_1}(z), \)

where \( \mu_1 = \sum_{i=1}^{k-1} m_i - \frac{1}{2} q(q-1) - \frac{1}{2} (k-q-1)(k-q-2) + q(k-q-1). \) As in the previous case, from its bound-state wavefunctions \( \eta_{l+1}(z)g^{(\alpha+1)}_{\mu_1}(z)/g^{(\alpha+1)}_{\mu_1}(z), \)

\( n_1 = \mu_1, \mu_1 + 1, \ldots, \) those of \( V_{l,ext}(x) \) are obtained by application of

\[ A^{(k)} = \omega x \left( \frac{d}{dz} + \frac{1}{2} + \frac{2\nu + 1}{4z} - \frac{g^{(\alpha)}_{\mu}}{g^{(\alpha)}_{\mu_1}} + \frac{g^{(\alpha+1)}_{\mu}}{g^{(\alpha+1)}_{\mu_1}} \right). \quad (4.5) \]

The resulting equation is now

\[ O^{(\alpha)}_{\mu_1,\mu} L^{(q,k-q-1)}_{\alpha+1,m_1,m_2,\ldots,m_{k-1},n_1}(z) = DL^{(q,k-q)}_{\alpha,m_1,m_2,\ldots,m_k,n}(z), \quad (4.6) \]

with

\[ O^{(\alpha)}_{\mu_1,\mu} = \frac{1}{g^{(\alpha+1)}_{\mu_1}} \left[ g^{(\alpha)}_{\mu} \left( z \frac{d}{dz} + \alpha + 1 \right) - zg^{(\alpha)}_{\mu} \right], \quad (4.7) \]

\[ D = (-1)^{k-q-1}(\alpha + \nu + k - 2q - m_k) \prod_{i=q+1}^{k-1} (m_k - m_i). \quad (4.8) \]
5 Shape invariance and some of its consequences

It is easy to see that if a rationally-extended radial oscillator potential \( V_{l, \text{ext}}(x) \) of the type considered in Secs. 3 and 4 has a ground-state wavefunction equal to \( \eta_l(z)g_{\mu}^{(\alpha+1)}(z)/g_{\mu}^{(\alpha)}(z) \) up to some constant multiplicative factor, then it is (translationally) shape invariant. Such a property is based on the fact that if we consider a pair of partner potentials \( \bar{V}^{(+)}(x), \bar{V}^{(-)}(x) \) with \( \bar{V}^{(+)}(x) = V_{l, \text{ext}}(x) \) and if we take the above-mentioned wavefunction as factorization function and its eigenvalue \( E^0_l \) as factorization energy, then the corresponding superpotential \( \bar{W}(x) \) can be separated into \( \bar{W}(x) = \bar{W}_1(x) + \bar{W}_2(x) \) with

\[
\bar{W}_1(x) = \frac{1}{2} \omega x - \frac{l + 1}{x}, \quad \bar{W}_2(x) = -\omega x \left( \frac{g_{\mu}^{(\alpha+1)}}{g_{\mu}^{(\alpha+1)}} - \frac{g_{\mu}^{(\alpha)}}{g_{\mu}^{(\alpha)}} \right), \tag{5.1}
\]

and the partner potential is such that

\[
\bar{V}^{(-)}(x) - E^0_l = \left[ V_{l+1, \text{ext}}(x) - E^{(l+1)}_0 \right] + 2\omega. \tag{5.2}
\]

Conversely, if Eq. \((5.2)\) is satisfied, then the lowest-degree EOP \( g_{\mu}^{(\alpha)}(z) \) is proportional to \( g_{\mu}^{(\alpha+1)}(z) \).

Since the shape invariance of \( V_{l, \text{ext}}(x) \) has been explicitly shown in Ref. 26, in all cases \( 0 \leq q \leq k \), we may write

\[
L_{\alpha,m_1,m_2,\ldots,m_k,\mu}^{(q,k-q)}(z) = \frac{(-1)^q \sum_{k=1}^{l} \frac{1}{2} g_{\mu}^{(\alpha)}(z) - \frac{1}{2} (k-q)(k-q-1)}{\Delta(m_1,m_2,\ldots,m_q)\Delta(m_{q+1},m_{q+2},\ldots,m_k)} g_{\mu}^{(\alpha+1)}(z), \tag{5.3}
\]

where the multiplicative factor has been found by comparing the highest-degree terms on both sides of the equation.

We can now combine this result with Eqs. \((4.2), (4.3), (4.4), (4.6), (4.7), \) and \((4.8)\), where we now assume \( \nu = 0, n_1 = \mu_1, n = \mu \), to derive the two identities

\[
\left[ g_{\mu}^{(\alpha)} \left( \frac{d}{dz} - 1 \right) - \dot{g}_{\mu}^{(\alpha)} \right] g_{\mu_1}^{(\alpha)} = -g_{\mu_1}^{(\alpha-1)} g_{\mu}^{(\alpha+1)}, \tag{5.4}
\]

\[
\left[ g_{\mu}^{(\alpha)} \left( \frac{d}{dz} + \alpha + 1 \right) - z \dot{g}_{\mu}^{(\alpha)} \right] g_{\mu_1}^{(\alpha+2)} = (\alpha + k - 2q - m_k) g_{\mu_1}^{(\alpha+1)} g_{\mu}^{(\alpha+1)}, \tag{5.5}
\]

11
valid for $q = k$ and $0 \leq q < k$, respectively. These are nontrivial $(\mu_1 + \mu)$th-degree relations connecting products of Laguerre polynomials. They can be easily checked for $k = 1$ and $k = 2$. In the former case, for instance, with $g^{(a)}_{\mu_1} = 1$ and $g^{(a)}_{\mu} = g^{(a)}_m$, as given in (2.10) or (2.11), the identities reduce to $-L^{(\alpha-1)}_m(-z) - L^{(\alpha-1)}_m(-z) = -L^{(\alpha)}_m(-z)$ and $(\alpha + 1)L^{(-\alpha-1)}_m(z) - zL^{(-\alpha-1)}_m(z) = (\alpha + 1 - m)L^{(-\alpha-2)}_m(z)$ for $q = 1$ and $q = 0$, respectively.

6 Conclusion

In the present work, we have determined the isospectral rationally-extended radial oscillator potentials resulting from conventional ones in $k$th-order SUSYQM. We have found, in particular, the polynomial $g^{(a)}_{\mu}(z)$ appearing in their denominator, as well as its degree $\mu$. We have also established the first-order differential relations allowing one to obtain the corresponding Laguerre EOP from those arising in $(k - 1)$th-order SUSYQM. Finally, we have proved some nontrivial identities connecting products of Laguerre polynomials, which are direct consequences of the extended potential shape invariance.

A similar study could be carried out for Jacobi EOP and associated rationally-extended potentials. We hope to come back to this problem in a forthcoming work.

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