ON FRAMINGS OF KNOTS IN 3-MANIFOLDS

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ABSTRACT. We show that the only way of changing the framing of a knot or a link by ambient isotopy in an oriented 3-manifold is when the manifold has a properly embedded non-separating $S^2$. This change of framing is given by the Dirac trick, also known as the light bulb trick. The main tool we use is based on McCullough’s work on the mapping class groups of 3-manifolds. We also relate our results to the theory of skein modules.

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1. Introduction

We show that the only way to change the framing of a knot in an oriented 3-manifold by ambient isotopy is when the manifold has a properly embedded non-separating $S^2$. More precisely the only change of framing is by the light bulb trick as illustrated in the Figure 1. Here the change of framing is very local (takes part in $S^2 \times [0, 1]$ embedded in the manifold) and is related to the fact that the fundamental group of $SO(3)$ is $\mathbb{Z}_2$. Furthermore, we use the fact that 3-manifolds possess spin structures given by the parallelization of their tangent bundles. We give a short outline of the history of the problem in Subsection 1.1.

In Section 2 we introduce the following preliminary material from 3-dimensional topology: incompressible surfaces, mapping class groups of 3-manifolds, and Dehn homeomorphisms. In Section 3 we prove our main results (see Theorems 3.1 and 3.5). Furthermore, we extend our main results to non-compact 3-manifolds. Our main tools are various results of D. McCullough about
mapping class groups of 3-manifolds. The use of the Papakyriakopoulos’ Loop theorem [Pap] and its generalization to the annulus, proved by Waldhausen in [Wal], is in the background of the proof as it was used by McCullough to prove Theorem 2.6. Additionally, in Subsection 3.2 we use spin structures on 3-manifolds to show that the framing of a knot or a link cannot be changed by an odd number of full twists. For an introduction to spin structures from the point of view used in this paper, we refer the reader to [Bar] and [Tur].

In Section 4 we describe the ramifications of our results in the context of skein modules, for example, the framing skein module, $q$-homology skein module, and the Kauffman bracket skein module.

In Section 5 we discuss future directions of research, in particular, questions about incompressible surfaces and torsion in various skein modules. Futhermore, we conjecture that an oriented atoroidal 3-manifold cannot have torsion in its Kauffman bracket skein module. We also relate our results to Witten’s conjecture on the finiteness of Kauffman bracket skein modules.

1.1. History of the problem. When the fourth author was visiting Michigan State University in East Lansing in the Fall of 1989 and was working on the Kauffman bracket skein module of lens spaces, he thought about the possibility of changing framings of knots in 3-manifolds by ambient isotopy. He found an argument that such changes of framing are impossible for irreducible 3-manifolds. Y. Rong, then a postdoc at MSU, pointed out (see [HP]) that from McCullough’s work it follows that if $M$ is a compact 3-manifold then the framing of a knot in $M$ cannot be changed by ambient isotopy if and only if $M$ has no non-separating $S^2$. The fourth author was then referred to [McC1] and [MM] by D. McCullough from which it follows that the only possibility of changing the framing of a knot in $M$ is by the Dirac trick. The fourth author stated this result several
times (for example, see [Prz2]) but without giving a detailed proof. Thus, when V. Chernov became interested in the problem he gave his own proof in [Che]¹. When the authors of this paper decided to give details of the old proof they noted that from the new paper [HM] the proof could be deduced in a relatively simple way (but still using the generalized Loop theorem for the annulus). Soon after, the authors found another paper by D. McCullough [McC2] from which the main result follows more directly and this is the proof presented in this paper.

2. Preliminaries

The following definitions and theorems in low-dimensional topology are useful in the proofs of our results.

**Definition 2.1.** Let $F$ be a properly embedded surface in a 3-manifold $M$ or be a part of the boundary of $M$. A disk $D^2$ in $M$ is called a compressing disk of $F$ if $D^2 \cap F = \partial D^2$ and $\partial D^2$ is not contractible in $F$.

An embedded surface $F$ in $M$ that admits a compressing disk is said to be compressible. If the surface $F$ is not $S^2$ or $D^2$ and it contains no compressing disk then the surface is said to be incompressible. If $F = S^2$, then $F$ is incompressible if it does not bound a 3-ball in $M$.

**Definition 2.2.** Let $M$ be an orientable manifold and $\text{Homeo}(M)$ denote the space of PL orientation preserving homeomorphisms of $M$. Then the mapping class group of $M$, denoted by $\mathcal{H}(M)$, is defined as the space of all ambient isotopy classes of $\text{Homeo}(M)$.

**Definition 2.3.** Let $(F^{n-1} \times I, \partial F^{n-1} \times I) \subset (M^n, \partial M^n)$, where $F$ is a connected codimension 1 submanifold, and $(F \times I) \cap \partial M = \partial F \times I$. Let $\langle \phi_t \rangle$ be an element of $\pi_1(\text{Homeo}(F), 1_F)$, that is, for $0 \leq t \leq 1$, $\phi_t$ is a continuous family of homeomorphisms of $F$ such that $\phi_0 = \phi_1 = 1_F$. Define a Dehn homeomorphism as $h \in \mathcal{H}(M)$ by:

$$h = \begin{cases} h(x, t) = (\phi_t(x), t) & \text{if } (x, t) \in F \times I \\ h(m) = m & \text{if } m \notin F \times I \end{cases}$$

Dehn homeomorphisms are generalizations of Dehn twist homeomorphisms of a surface. When $\pi_1(\text{Homeo}(F))$ is trivial, a Dehn homeomorphism must be isotopic to the identity. Define the Dehn subgroup $\mathcal{D}(M)$ of $\mathcal{H}(M)$ to be the subgroup generated by all Dehn homeomorphisms.

**Theorem 2.4** (Finite Mapping Class Group Theorem). [HM]

Let $M$ be a closed oriented irreducible non-Haken 3-manifold, that is, $M$ is irreducible and contains no properly embedded, incompressible, two-sided surface. Then $\mathcal{H}(M)$ is finite.

This theorem is also true for all hyperbolic manifolds.

**Theorem 2.5.** [GMT]

The mapping class group of a compact hyperbolic 3-manifold is finite.

¹In this paper Chernov wrote, “This result (for compact manifolds) was first stated by Hoste and Przytycki [HP]. They referred to the work [McC1] of McCullough on mapping class groups of 3-manifolds for the idea of the proof of this fact. However to the best of our knowledge the proof was not given in the literature. The proof we provide is based on the ideas and methods different from the ones Hoste and Przytycki had in mind.”
Theorem 2.6. [McC2]

Let $M$ be a compact orientable 3-manifold which admits a homeomorphism which is Dehn twists on the boundary about the collection $C_1, \ldots, C_n$ of simple closed curves in $\partial M$. Then for each $i$, either $C_i$ bounds a disk in $M$, or for some $j \neq i$, $C_i$ and $C_j$ cobound an incompressible annulus in $M$.

Corollary 2.7. Let $M'$ be a compact oriented 3-manifold with one of the boundary components of $M'$ being a torus. We denote this torus by $\partial_0 M'$. Let $\tilde{f} : M' \rightarrow M'$ be a homeomorphism which acts nontrivially on $\partial_0 M'$ and is constant on $\partial M' \setminus \partial_0 M'$. Then $\partial_0 M'$ has a compressing disk.

Proof. $\tilde{f}|_{\partial_0 M'}$ is generated by Dehn twists on a family $C$ of $k$ ($\neq 0$) parallel nontrivial simple closed curves on the torus $\partial_0 M'$. Therefore, by Theorem 2.6 these curves bound (compressing) disks or incompressible annuli. However, Dehn twists along an annulus with both boundary components on $\partial_0 M'$ act trivially on $\partial_0 M'$. Therefore, each curve in $C$ bounds a compressing disk. □

Theorem 2.8. [McC2]

Let $M$ be a compact orientable 3-manifold which admits a homeomorphism which is Dehn twists on the boundary about the collection $C_1, \ldots, C_n$ of simple closed curves in $\partial M$. Then there is a collection of disjoint imbedded disks and annuli in $M$, each of those boundary circles is isotopic to one of the $C_i$, for which some composition of Dehn twists about these disks and annuli is isotopic to $h$ on $\partial M$.

That is, $h$ must arise in the most obvious way, by composition of Dehn twists about a collection of disjoint annuli and disks with a homeomorphism that is the identity on the boundary.

Corollary 2.9. Let $M'$ be a compact oriented 3-manifold with some boundary components, say, $\partial_1(M'), \ldots, \partial_k(M')$ being tori. Let $\tilde{f} : M' \rightarrow M'$ be a homeomorphism which acts nontrivially on every $\partial_i(M')$ and which is the identity on $\partial M' \setminus \bigcup_i \partial_i(M')$. Then either $\partial_i(M')$ has a compressing disk, say $D_i^2$, or there is some $j \neq i$ such that there is an incompressible annulus $\text{Ann}_{i,j}$ with one boundary component on $\partial_i(M')$ and the other on $\partial_j(M')$. Furthermore, one can take these disks and annuli to be disjoint. Also, $\tilde{f}$ restricted to $\partial M'$ is ambient isotopic to the composition (of some powers) of the Dehn homeomorphism along $D_i^2$ and $\text{Ann}_{i,j}$.

For the remainder of the paper we will denote a framed knot (respectively link) by $K$ (respectively $L$) and the underlying unframed knot (respectively link) by $\overline{K}$ (respectively $\overline{L}$).

3. Main Results

Theorem 3.1. Let $K$ be a framed knot in a compact oriented 3-manifold $M$. The only way of changing the framing of $K$ by an ambient isotopy of $M$ is when the manifold has a properly embedded non-separating $S^2$ and the underlying knot $\overline{K}$ intersects this $S^2$ transversely exactly once. More precisely, the only change of framing is by the light bulb trick. Equivalently, all possible changes of the framing of $K$ can be realized by even powers of Dehn homeomorphisms along a non-separating $S^2$ which is cut by $K$ exactly once.

The following theorem is the key result used in proving Theorem 3.1.

Theorem 3.2. Let $K$ be a framed knot in a compact oriented 3-manifold $M$. If $f : M \rightarrow M$ is a homeomorphism which changes the framing of $K$ and $f|_{\partial M} = \text{Id}$, then there is a non-separating $S^2$ in $M$ which is intersected transversely by $K$ exactly once. Furthermore, if $\tau$ is a Dehn homeomorphism along this $S^2$, then $f$ is ambient isotopic to a function which coincides with some power of $\tau$ on the regular neighborhood of $K$. 
Proof. Let \(f\) change the framing of \(K\) by some \(j\) \((\neq 0) \in \mathbb{Z}\). We assume that \(f(K) = K^{(j)}\) and without loss of generality, we can choose \(f\) such that \(f(V_K) = V_K\) for some regular neighborhood \(V_K\) of \(K\) (see [Hud]). Let \(M' = M \setminus \text{int}V_K\) and consider the function \(\tilde{f} = f|_{M'} : M' \rightarrow M'.\) Note that \(f|_{\partial V_K} : \partial V_K \rightarrow \partial V_K\) induces a nontrivial action on \(H_1(T^2)\). More precisely, \(\mu \mapsto \mu\) and \(\lambda \mapsto \lambda + j\mu\), where \(\mu\) is the meridian of \(V_K\) and \(\lambda\) is a fixed longitude, that is, it is a curve which intersects \(\mu\) exactly once. We now use Corollary 2.7 to conclude that \(\mu\) bounds a compressing disk, say \(D^2_\mu\), in \(M'\). Let \(D^2_\mu\) denote the meridian disk in \(V_K\) bounded by \(\mu\). Now the Dehn twist along \(\partial D^2_0\) must be the same as the Dehn twist along \(\partial D^2_\mu\), otherwise \(\tilde{f}\) cannot be extended to \(f\). So \(\partial D^2_0 = \partial D^2_\mu\), and therefore, \(D^2_0 \cup \partial D^2_\mu\) is a 2-sphere, say \(S^2_\mu\). This \(S^2_\mu\) does not separate \(M\) since \(\partial V_K\) is not separated by \(S^2_\mu \cap \partial V_K = \mu\). Furthermore, by Corollary 2.9, with \(k = 1\), the function \(f\) is ambient isotopic to a function which coincides with some power of \(\tau\) (Dehn homeomorphism along \(S^2_\mu\)) on the regular neighborhood of \(K\).

\[\square\]

Remark 3.3. For oriented 3-manifolds \(M\) for which \(\mathcal{H}(M')\) is finite, Theorem 3.2 can be deduced immediately. This is because \(f|_{\partial V_K}\) has infinite order in \(\mathcal{H}(\partial V_K)\), and thus, \(\tilde{f}\) has infinite order in \(\mathcal{H}(M')\). Hence, we get a contradiction. This happens, in particular, when \(M'\) is a hyperbolic manifold or a closed oriented irreducible non-Haken 3-manifold (see Theorems 2.4 and 2.5).

3.1. Proofs of the Main Theorems.

We begin by giving a proof of Theorem 3.1.

Proof. We have shown in Theorem 3.2 that the only way to change the framing of \(K\) is by Dehn homeomorphisms along non-separating 2-spheres each of which is intersected transversely by \(K\) exactly once. Even if these 2-spheres are not necessarily ambient isotopic, the effect of rotating one about each of them has the same effect on the framing change of \(K\), that is, the framing of \(K\) changes by one. Therefore, it is enough to consider the rotation \(\tau\) about one such sphere. Now since \(\pi_1(SO(3)) = \mathbb{Z}_2\), we get that \(\tau^2\) is ambient isotopic to the identity map. Also, \(\tau^2\) along the non-separating 2-sphere realizes the light bulb trick. Therefore, unlike in Theorem 3.2, the function which changes the framing of \(K\) is now ambient isotopic to the identity map. It follows from the existence of spin structures on every oriented 3-manifold that this function can be considered to be an even power of the rotation \(\tau\) along a non-separating 2-sphere (see Subsection 3.2, [Bar], and [Tur]). This completes the proof of Theorem 3.1.

\[\square\]

Remark 3.4. (a) If \(M\) is an integral homology sphere (respectively, rational homology sphere), then every knot in \(M\) has a preferred framing (respectively, rational framing). As mentioned before, for arbitrary oriented 3-manifolds we can only define modulo 2 framing which reflects the affine space of spin structures over \(H^1(M, \mathbb{Z}_2)\) (see Subsection 3.2).

(b) If \(M\) can be embedded in a rational homology sphere, then \(K\) has a preferred framing (depending on the embedding). In particular, the framing cannot be changed by an ambient isotopy of \(M\). Notice that this does not apply if \(M\) has a properly embedded non-separating closed surface, since then it cannot be properly embedded in any rational homology sphere.

Theorem 3.5. Let \(L\) be a framed link in a compact oriented 3-manifold \(M\). The only way of changing the framing of \(L\) by ambient isotopy while preserving the components of \(L\) is when \(M\) has a properly embedded non-separating 2-sphere and either:
(i) \( L \) intersects the non-separating 2-sphere transversely exactly once, or
(ii) \( L \) intersects the non-separating 2-sphere transversely in two points, each point belonging to a different component of \( L \).

The framing is changed by a composition of even powers of the Dehn homeomorphisms along the disjoint union of \( S_j^2 \), where \( S_j^2 \) satisfy conditions (i) or (ii). We illustrate the change of framing by the light bulb trick\(^2\) (in the regular neighborhood of \( S^2 \)) in Figure 1 if there is one point of intersection and in Figure 2 if there are exactly two points of intersection of \( L \) with \( S^2 \).

![Diagram of light bulb trick]

**Figure 2.** Dirac trick for a link with two components illustrated using a light bulb

**Proof.** We follow the proof of Theorem 3.1 with slight modifications. Let the knots \( K_1, K_2, \ldots, K_k \) be the components of the framed link \( L \). Let \( f : M \rightarrow M \) be a homeomorphism, ambient isotopic to the identity with \( f|_{K_i} = Id \). Without loss of generality, we can assume that there are regular neighborhoods \( V_{K_i} \) of \( K_i \) so that \( f(V_{K_i}) = V_{K_i} \), and that \( f|_{\partial M} = Id \). Let \( M' = M \setminus \bigcup_i int(V_{K_i}) \) and

\(^2\)For a nice computer animation of the Dirac trick we refer the reader to [DFHHKPS].
\[ \tilde{f} = f|_{M'} : M' \longrightarrow M'. \] Assume that \( \tilde{f} \) acts nontrivially on all \( \partial V_{K_i} \). By Corollary 2.7, for any \( i \) there is a compressing disk \( D_i^2 \) of \( \partial V_{K_i} \), or there exists \( j \) \((i \neq j)\) and an incompressible annulus \( \text{Ann}_{i,j} \) with one boundary component on \( \partial V_{K_i} \) and the other on \( \partial V_{K_j} \). By Corollary 2.9 we can assume that these compressing disks and incompressible annuli are disjoint. The boundary of the compressing disk \( D_i^2 \) is the same as the boundary of the meridian disk \( D_{\mu_i} \) of \( V_i \) and \( D_i^2 \cup \partial D_{\mu_i} = S_{\mu_i}^2 \) is a non-separating 2-sphere intersecting \( K_i \) transversely in one point. Similarly, \( \partial \text{Ann}_{i,j} = \partial D_{\mu_i} \sqcup \partial D_{\mu_j} \) and \( D_{\mu_i} \cup \text{Ann}_{i,j} \cup D_{\mu_j} = S_{i,j}^2 \) is a non-separating 2-sphere in \( M \) cut by \( K_i \) and \( K_j \) transversely in one point each. Furthermore, we can assume that the action of \( \tilde{f} \) on \( \partial V_{K_i} \) is the same as that given by some composition of Dehn homeomorphisms along compressing disks \( D_i^2 \) and incompressible annuli \( \text{Ann}_{i,j} \). Since \( f \) is ambient isotopic to the identity, the change of framing done by \( \tilde{f} \) on \( L \) is accomplished by some composition of even powers of Dehn homeomorphisms (rotations), say \( \tau_i^2 \) and \( \tau_{i,j}^2 \) along the spheres \( S_i^2 \) and \( S_{i,j}^2 \), respectively. This completes the proof of Theorem 3.5. The fact that \( \tau_i^2 \) twists the framing of \( K_i \) twice by ambient isotopy is illustrated in Figure 1, and that \( \tau_{i,j}^2 \) simultaneously twists the framing of \( K_i \) twice and of \( K_j \) twice but in the opposite direction, is illustrated in Figure 2.

\[ \square \]

**Corollary 3.6.** Theorems 3.1 and 3.5 also hold for non-compact oriented 3-manifolds.

**Proof.** The ambient isotopy of \( M \), which changes the framing of \( L \) can be taken to have support on a finite number of 3-balls in \( M \).\(^3\) Thus, the new ambient isotopy has a compact oriented 3-manifold as a support and the result follows from Theorems 3.1 and 3.5.

\[ \square \]

### 3.2. Spin structures and framings.

Since spin structures are invariants of 3-manifolds, the framing of a knot \( K \) in \( M \) cannot be changed by one full twist using ambient isotopy. We give a short explanation of this fact below.

Let \( M \) be a 3-manifold, \( TM \) its tangent bundle and \( V : M \longrightarrow TM \) a vector field. Assume that \( V \) is nondegenerate.

**Definition 3.7.** \( M \) is parallelizable if the tangent bundle of \( M \) is trivial, that is, there are three vector fields \( V_1, V_2 \) and \( V_3 \) which form a basis at every tangent space.

**Theorem 3.8.** [Sti]

Every 3-manifold is parallelizable.

Homotopy classes of parallelizations can be identified with spin structures and spin structures form an affine space over the \( \mathbb{Z}_2 \)-linear space \( H^1(M, \mathbb{Z}_2) \). The choice of a concrete parallelization makes the affine space of spin structures the linear space \( H^1(M, \mathbb{Z}_2) \). In particular to every framed knot \( K \subset M \), which represents an element of \( H_1(M, \mathbb{Z}_2) \), we associate an element of \( \mathbb{Z}_2 \).

Concretely, let \((V_1, V_2, V_3)\) be a fixed orthonormal parallelization of \( M \), that is, \((V_1, V_2, V_3)\) are orthonormal at every tangent space. Let \( K \) be a framed knot in \( M \) \((|K| \in H_1(M, \mathbb{Z}_2))\). This implies that \(|K| \in Z_1(M, \mathbb{Z}_2)\), where \( Z_1(X) = \ker(\partial_1) \). We show that the triple \((V_1, V_2, V_3)\) defines a map from \( H_1(M, \mathbb{Z}_2) \) to \( \mathbb{Z}_2 \) and this map is an element of \( H^1(M, \mathbb{Z}_2) \). To see this, consider vectors

\(^3\)It follows from Theorem 6.2 in [Hud] that if \( C \) is a compact subset of a manifold \( M \) and \( F : M \times I \longrightarrow M \) is an ambient isotopy of \( M \) then there is another ambient isotopy \( \tilde{F} : M \times I \longrightarrow M \) such that \( F_0 = \tilde{F}_0, F_1 \setminus C = \tilde{F}_1 \setminus C \) and there exists a number \( N \) such that the set \( \{x \in M \mid (\tilde{F} \setminus \{x\}) \times (k/N, (k + 1)/N) \text{ is not constant}\} \) sits in a ball embedded in \( M \).
\[ v_i \in V_i, i = 1, 2, 3 \text{ which are incidental at a point } m_a \in M \text{ and another set of orthonormal vectors } (v_T, v_{fr}, v_{y'}) \in (V_1, V_2, V_3) \text{ which are incidental at a point } m_K \in K \subset M. v_T \text{ denotes the vector which gives the direction of travel along the knot (orientation) and } v_{fr} \text{ is the vector which gives the framing of the knot. Therefore, there is an element } q \in \text{SO}(3) \text{ which maps } (v_1, v_2, v_3) \text{ to } (v_T, v_{fr}, v_{y'}). \text{ Now when we travel along the knot we obtain an element of } \pi_1(\text{SO}(3)) = \mathbb{Z}_2. \text{ Therefore, we obtain a map from } H_1(M, \mathbb{Z}_2) \to \mathbb{Z}_2. \text{ See [Bar] and [Tur] for more details.}

4. Ramifications and Connections to Skein Modules

Our main results can be formulated in the language of skein modules as follows.

**Definition 4.1.** Let \( M \) be an oriented 3-manifold and \( \mathcal{L}^{fr} \) the set of unoriented framed links in \( M \) up to ambient isotopy. Let \( \mathcal{S}^{fr} \) be the submodule of the module \( \mathbb{Z}[q^{\pm 1}]\mathcal{L}^{fr} \) generated by the framing expressions \( L^{(1)} = qL \) for any framed link \( L \) in \( \mathcal{L}^{fr} \). Here \( L^{(1)} \) denotes the link obtained from \( L \) by twisting the framing of \( L \) by a positive full twist (see Figure 3b). The **framing skein module** of \( M \) is defined as the quotient:

\[
\mathcal{S}_0(M, q) = \mathbb{Z}[q^{\pm 1}]\mathcal{L}^{fr}/\mathcal{S}^{fr}.
\]

The following theorem for skein modules is equivalent to Theorem 3.1 as long as we work with knots (see [Prz3]).

**Theorem 4.2.** For an oriented 3-manifold \( M \), \( \mathcal{S}_0(M, q) = \mathbb{Z}[q^{\pm 1}]\mathcal{L}^{f} \oplus \bigoplus_{L \in (\mathcal{L}^{fr}\setminus \mathcal{L}^{f})} \frac{\mathbb{Z}[q]}{q^2-1} \), where \( \mathcal{L}^{f} \) is composed of links which do not intersect any 2-sphere in \( M \) transversely at exactly one point.

**Proof.** In Theorem 3.5 we described two possibilities of changing the framing of \( L \) by an ambient isotopy. We analyze both cases for the proof.

1. If \( L \) intersects a non-separating sphere \( S^2 \) transversely in exactly one point, say by component \( K_i \), then \( r_i^2 \) is ambient isotopic to the identity and twists the framing of \( K_i \) by two full twists, and thus also the framing of \( L \). Therefore, in the framing skein module \( (q^2-1)L = 0 \).

We notice that \( \mathbb{Z}[q^{\pm 1}]\mathcal{L}^{fr} \) divided by this relation exactly gives \( \mathbb{Z}[q^{\pm 1}]\mathcal{L}^{f} \oplus \bigoplus_{L \in (\mathcal{L}^{fr}\setminus \mathcal{L}^{f})} \frac{\mathbb{Z}[q]}{q^2-1} \).

2. If exactly two components \( K_i \) and \( K_j \) of the link \( L \) intersect a non-separating 2-sphere \( S^2 \) in one point each, then \( r_{i,j}^2 \) changes the framing of \( K_i \) by 2 and of \( K_j \) by \(-2\) (as illustrated in Figure 2). Thus, even though this Dehn homeomorphism changes the framing of \( L \), it is invisible in the framing skein module which does not see which component is twisted. Therefore, the twists cancels algebraically in \( \mathcal{S}_0(M, q) \).

\[ \square \]

**Corollary 4.3.** There exists an epimorphism \( \omega \) from the framing skein module to the \( \mathbb{Z}[q]/(q^2-1) \)-group ring over the first homology with \( \mathbb{Z}_2 \) coefficients, that is

\[
\omega : \mathcal{S}_0(M, q) \longrightarrow \left( \frac{\mathbb{Z}[q]}{q^2-1} \right) H_1(M, \mathbb{Z}_2),
\]

which is noncanonical and depends on the choice of spin structure (in the form of a parallelization of a tangent bundle to \( M \)). The choice of parallelization gives a map \( b : H_1(M, \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2 \). Hence, \( \omega(K) = |K| \text{ if } b(|K|) = 0 \text{ and } \omega(K) = q|K| \text{ if } b(|K|) = 1 \). Here \( K \) is a framed knot in \( M \) and \( |K| \) denotes the homology class of \( K \) in \( H_1(M, \mathbb{Z}_2) \). See Subsection 3.2.
4.1. From the Kauffman bracket skein module to spin twisted homology.

**Definition 4.4. [Prz1]**

Let $M$ be an oriented 3-manifold, $\mathcal{L}^{fr}$ the set of unoriented framed links (including the empty link $\emptyset$) in $M$ up to ambient isotopy, $R$ a commutative ring with unity, and $A$ a fixed invertible element in $R$. In addition, let $R\mathcal{L}^{fr}$ be the free $R$-module generated by $\mathcal{L}^{fr}$ and $S_{2,\infty}^{\text{sub}}$ the submodule of $R\mathcal{L}^{fr}$ generated by all (local) skein expressions of the form:

(i) $L_+ - AL_0 - A^{-1}L_\infty$, and
(ii) $L \sqcup \bigcirc + (A^2 + A^{-2})L$,

where $\bigcirc$ denotes the trivial framed knot and the skein triple $L_+, L_0$ and $L_\infty$ denote three framed links in $M$ which are identical except in a small 3-ball in $M$ where they differ as shown:

\[
\begin{array}{ccc}
\text{L}_+ & \text{L}_0 & \text{L}_\infty \\
\end{array}
\]

The *Kauffman bracket skein module* (KBSM) of $M$ is defined as the quotient:

$$S_{2,\infty}(M; R, A) = R\mathcal{L}^{fr} / S_{2,\infty}^{\text{sub}}.$$  

Notice that $L^{(1)} = -A^3L$ in $S_{2,\infty}(M; R, A)$. This relation is called the *framing relation*. For simplicity we will use the notation $S_{2,\infty}(M)$ when $R = \mathbb{Z}[A^{\pm 1}]$.

**Proposition 4.5.** There exists an epimorphism

$$\phi : S_{2,\infty}(M) \rightarrow \left( \frac{\mathbb{Z}[A]}{A^4 + A^2 + 1} \right) H_1(M, \mathbb{Z}_2)$$

which is not canonical and depends on the choice of spin structure on $M$ (here in the form of a parallelization of a tangent bundle to $M$). The codomain of $\phi$ is called *spin twisted homology*. Compare with Corollary 4.3 and [Bar, PS].

**Proof.** Using the parallelization of the tangent bundle of $M$, we have a map

$$\bar{\phi} : \mathbb{Z}[A^{\pm 1}] \mathcal{L}^{fr} \rightarrow \left( \frac{\mathbb{Z}[A^{\pm 1}]}{A^4 + A^2 + 1} \right) H_1(M, \mathbb{Z}_2).$$

We check that $\bar{\phi}$ is zero on the Kauffman bracket skein expressions:

1. Since $\bar{\phi}(L_+) = -A^{-3}\bar{\phi}(L_0) = -A^{-3}\bar{\phi}(L_\infty)$, then $\bar{\phi}(L_+ - AL_0 - A^{-1}L_\infty) = 0$.
2. Since $L \sqcup \bigcirc = (-A^2 - A^{-2})L$ in $S_{2,\infty}(M)$, then $(A^4 + A^2 + 1)L = 0$ in $H_1(M, \mathbb{Z}_2)$.

\[\square\]

4.2. The $q$-homology skein module.

**Definition 4.6.** Let $M$ be an oriented 3-manifold, $\mathcal{L}^{fr}$ denote the set of ambient isotopy classes of oriented framed links in $M$ and let $R = \mathbb{Z}[q^{\pm 1}]$. Let us denote by $S_{2,fr}^{\text{sub}}$ the submodule of $R\mathcal{L}^{fr}$ generated by the skein expressions $L_+ - qL_0$ and $L^{(1)} - qL$, illustrated in Figure 3.

The quotient $R\mathcal{L}^{fr} / S_{2,fr}^{\text{sub}}$ is called the *$q$-homology skein module*, also known as the second skein module, and it is denoted by $S_2(M, q)$. 


**Proposition 4.7.** There exists an epimorphism

$$\psi : S_2(M, q) \longrightarrow \left( \frac{\mathbb{Z}[q]}{q^2 - 1} \right) H_1(M, \mathbb{Z}_2).$$

which is not canonical and depends on the choice of spin structure on $M$. As before, compare with Corollary 4.3.

We summarize our propositions with the following diagram:

5. **Future Directions**

As we have proven in Theorem 4.2, the presence of a non-separating $S^2$ in $M$ always yields torsion in $S_0(M, q)$. More generally, skein modules that have framing relations always have torsion which is obtained using the light bulb trick. Usually, however, skein modules have more torsion than what the framing relations give us. For example, in the $q$-homology skein module, torsion is related to the presence of closed non-separating surfaces [Prz2]. In the Kauffman bracket skein module torsion is obtained using incompressible spheres and tori (see [Kir, Oht]). We conjecture that this is the only situation in which torsion arises.
Conjecture 5.1. (1) If a compact oriented 3-manifold \( M \) is atoroidal, that is, \( M \) is irreducible such that every incompressible torus in \( M \) is parallel to the boundary of \( M \), then \( S_{2,\infty}(M) \) contains no torsion (see [Kir] and [Oht]).

(2) If a compact oriented 3-manifold \( M \) contains an incompressible 2-sphere or torus which is non-parallel to the boundary, then \( S_{2,\infty}(M) \) contains torsion (this was conjectured by the fourth author; see [Kir]).

The following statement was conjectured by Witten (a proof was announced in [GJS]).

Conjecture 5.2. The Kauffman bracket skein module for any closed oriented 3-manifold over \( \mathbb{C}(A) \), the field of rational functions in the variable \( A \), is always finite dimensional.

We can combine this conjecture with Conjecture 5.1 to obtain the following conjecture.

Conjecture 5.3. \( S_{2,\infty}(M) \) of oriented atoroidal 3-manifolds has finite rank.

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