Spectrum of the Laplacian on Quaternionic Kähler Manifolds

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Abstract

Let $M^{4n}$ be a complete quaternionic Kähler manifold with scalar curvature bounded below by $-16n(n+2)$. We get a sharp estimate for the first eigenvalue $\lambda_1(M)$ of the Laplacian which is $\lambda_1(M) \leq (2n+1)^2$. If the equality holds, then either $M$ has only one end, or $M$ is diffeomorphic to $\mathbb{R} \times N$ with $N$ given by a compact manifold. Moreover, if $M$ is of bounded curvature, $M$ is covered by the quaternionic hyperbolic space $\mathbb{QH}^n$ and $N$ is a compact quotient of the generalized Heisenberg group. When $\lambda_1(M) \geq \frac{8(n+2)}{3}$, we also prove that $M$ must have only one end with infinite volume.

0 Introduction

Let $M^n$ be a complete $n$-dimensional Riemannian manifold whose Ricci curvature bounded below by $-(n-1)$. It is well known from Cheng [Ch] that the first eigenvalue $\lambda_1(M)$ satisfies

$$\lambda_1(M) \leq \frac{(n-1)^2}{4}.$$

In [LW3], Li and Wang proved an analogous theorem for complete Kähler manifolds. They showed that if $M^{2n}$ is a complete Kähler manifold of complex dimension $n$ with holomorphic bisectional curvature $B_{K_M}$ bounded below by $-1$, then the first eigenvalue $\lambda_1(M)$ satisfies

$$\lambda_1(M) \leq n^2.$$

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Here BK\(_M\geq -1\) means that
\[ R_{ij\bar{j}} \geq -(1 + \delta_{ij}) \]
for any unitary frame \(e_1, \ldots, e_n\).

In this paper, we prove the corresponding Laplacian comparison theorem for a quaterionic Kähler manifold \(M^{4n}\). As an application we get the sharp estimate \(\lambda_1(M)\) for a complete quaterionic Kähler manifold \(M^{4n}\) with scalar curvature bounded below by \(-16n(n + 2)\) as
\[ \lambda_1(M) \leq (2n + 1)^2. \]

It is an interesting question to ask what one can say about those manifolds when the above inequalities are realized as equalities. In works of Li and Wang [LW1] and [LW2], the authors obtained the following theorems. The first was a generalization of the theory of Witten-Yau [WY], Cai-Galloway [CG], and Wang [W] for conformally compact manifolds. The second was to answer the aforementioned question.

**Theorem 0.1.** Let \(M^n\) be a complete Riemannian manifold of dimension \(n \geq 3\) with Ricci curvature bounded below by \(-(n - 1)\). If \(\lambda_1(M) \geq n - 2\), then either
1. \(M\) has only one infinite volume end; or
2. \(M = \mathbb{R} \times N\) with warped product metric of the form
\[ ds_M^2 = dt^2 + \cosh^2 t \, ds_N^2, \]
where \(N\) is an \((n - 1)\)-dimensional compact manifold of Ricci curvature bounded below by \(\lambda_1(M)\).

**Theorem 0.2.** Let \(M^n\) be a complete Riemannian manifold of dimension \(n \geq 2\) with Ricci curvature bounded below by \(-(n - 1)\). If \(\lambda_1(M) \geq \frac{(n-1)^2}{4}\), then either
1. \(M\) has no finite volume end; or
2. \(M = \mathbb{R} \times N\) with warped product metric of the form
\[ ds_M^2 = dt^2 + e^{2t} \, ds_N^2, \]
where \(N\) is an \((n - 1)\)-dimensional compact manifold of nonnegative Ricci curvature.
In [LW3] and [LW5], Li and Wang also consider the Kähler case. They proved the following theorems.

**Theorem 0.3.** Let \( M^n \) be a complete Kähler manifold of complex dimension \( n \geq 1 \) with Ricci curvature bounded below by

\[
\text{Ric}_M \geq -2(n+1).
\]

If \( \lambda_1(M) > \frac{n+1}{2} \), then \( M \) must have only one infinite volume end.

**Theorem 0.4.** Let \( M^n \) be a complete Kähler manifold of complex dimension \( n \geq 2 \) with holomorphic bisectional curvature bounded by

\[
\text{BK}_M \geq -1.
\]

If \( \lambda_1(M) \geq n^2 \), then either

(1) \( M \) has only one end; or

(2) \( M = \mathbb{R} \times N \) with \( N \) being a compact manifold. Moreover the metric on \( M \) is of the form

\[
ds_M^2 = dt^2 + e^{4t} \omega_2^2 + e^{2t} \sum_{i=3}^{2n} \omega_i^2,
\]

where \( \{\omega_2, \omega_3, \ldots, \omega_{2n}\} \) are orthonormal coframe of \( N \) with \( Jdt = \omega_2 \). If \( M \) has bounded curvature, then we further conclude that \( M \) is covered by \( \mathbb{C}H^n \) and \( N \) is a compact quotient of the Heisenberg group.

In [LW5], the authors pointed out that the assumption on the lower bound of \( \lambda_1(M) \) in Theorem 0.3 is sharp, since one can construct \( M \) of the form

\[
M = \Sigma \times N
\]

satisfying

\[
\text{Ric}_M \geq -2(n+1) \tag{0.1}
\]

and

\[
\lambda_1(M) = \frac{n+1}{2} \tag{0.2}
\]

with \( N \) being a compact Kähler manifold and \( \Sigma \) being a complete surface with at least two infinite volume ends. However, it is still an open question to characterize all those complete Kähler manifolds satisfying conditions (0.1) and (0.2).

In sections 4 and 5, we will prove the following quaternionic Kähler versions of the above theorems.
Theorem 0.5. Let $(M^{4n}, g)$ be a complete quaternionic Kähler manifold with scalar curvature satisfying

$$S_M \geq -16n(n + 2).$$

If $\lambda_1(M) \geq \frac{8(n + 2)}{3}$, then $M$ must have only one infinite volume end.

Theorem 0.6. Let $(M^{4n}, g)$ be a complete quaternionic Kähler manifold with scalar curvature satisfying

$$S_M \geq -16n(n + 2).$$

If $\lambda_1(M) \geq (2n + 1)^2$, then either

1. $M$ has only one end, or
2. $M$ is diffeomorphic to $\mathbb{R} \times N$ where $N$ is a compact manifold. Moreover, the metric is given by the form

$$ds^2_M = dt^2 + e^{4t} \sum_{p=2}^{4} \omega_p^2 + e^{2t} \sum_{\alpha=5}^{4n} \omega_{\alpha}^2,$$

where $\{\omega_2, \ldots, \omega_{4n}\}$ are orthonormal coframes for $N$. If $M$ is of bounded curvature then we further conclude that $M$ is covered by the quaternionic hyperbolic space $\mathbb{QH}^n$ and $N$ is a compact quotient of the generalized Heisenberg group.

Remark 0.1. It is known that a horosphere in $\mathbb{QH}^n$ is isometric to a certain generalized Heisenberg group with three-dimensional center and left-invariant Riemannian metric. Such generalized Heisenberg groups have compact quotients. For an explicit construction see for instance Example 2.6 in [G]. We don’t have an example to show that the bounded curvature condition in Theorem 0.6 is necessary. If such an example exists, its curvature should decay at exponentially in some directions.

Perhaps it is interesting to restrict our attention to the special case when $M^{4n} = \mathbb{QH}^n/\Gamma$ is given by the quotient of the quaternionic hyperbolic space $\mathbb{QH}^n$ with a discrete group of isometries $\Gamma$. In particular, it is instructional to compare with previous results by Corlette [C2] and Corlette-Iozzi [CI] where Lie group theoretic approach was used in understanding these manifolds. For example, in [CI], the authors proved a Patterson-Sullivan type formula for
\( \lambda_1(M) \) in terms of the Hausdorff dimension \( \delta(\Gamma) \) of the limit set of \( \Gamma \). More specifically, they proved that if \( \Gamma \) is geometrically finite, then for \( \delta(\Gamma) \geq 2n+1 \) one has

\[
\lambda_1(M) = \delta(\Gamma)((4n + 2) - \delta(\Gamma)).
\]

Hence in this case, the condition in Theorem 0.6 on \( \lambda_1(M) = (2n + 1)^2 \) is equivalent to the condition \( \delta(\Gamma) = 2n + 1 \).

In [C2] (Theorem 4.4), Corlette also pointed out that by a result of Kostant \( \lambda_1(M) = 0 \) or \( \lambda_1(M) \geq 8n \). On the other hand, it was also shown in [CI] that if \( \Gamma \) is geometrically finite and torsion free, then \( M = \mathbb{Q}H^n/\Gamma \) must have at most one end with infinite volume. These two statements give an interesting comparison to Theorem 0.5 stated above.

We would also like to point out to the interested readers that in [LW4] and [LW5] Li and Wang considered a more general class of manifolds satisfying a weighted Poincaré inequality. However, since quaternionic Kähler manifolds are automatically Einstein, the same type of questions are not interesting for this class of manifolds.

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1 Preliminaries on quaternionic Kähler manifolds

In this section, we will recall basic properties of quaternionic Kähler manifolds that will be needed in the sequel. These properties were proved by Berger [B] and Ishihara [I] (also see [Be]).

Let \( (M^n, g) \) be a Riemannian manifold, \( TM \) the tangent space of \( M \) and \( \nabla \) the Levi-Civita connection. The Riemannian curvature \( R: TM \otimes TM \otimes TM \to TM \) is defined by

\[
R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z
\]

If \( \{e_1, \ldots, e_n\} \) is an orthonormal basis of \( TM \), the components of curvature tensor is defined by

\[
R_{ijkl} = \langle R(e_i,e_j)e_k,e_l \rangle,
\]
the Ricci curvature is defined by
\[ \text{Ric}_M(X,Y) = \sum_{i=1}^{n} \langle R(X,e_i)e_i, Y \rangle, \]
and the scalar curvature is defined by
\[ S_M = \sum_{i,j=1}^{n} \langle R(e_i,e_j)e_j, e_i \rangle. \]

**Definition 1.1.** A quaternionic Kähler manifold \((M,g)\) is a Riemannian manifold with a rank 3 vector bundle \(V \subset \text{End}(TM)\) satisfying

(a) In any coordinate neighborhood \(U\) of \(M\), there exists a local basis \(\{I,J,K\}\) of \(V\) such that
\[
I^2 = J^2 = K^2 = -1, \\
IJ = -JI = K, \\
JK = -KJ = I, \\
KI = -IK = J,
\]
and
\[
\langle IX, IY \rangle = \langle JX, JY \rangle = \langle KX, KY \rangle = \langle X, Y \rangle
\]
for all \(X,Y \in TM\).

(b) If \(\phi \in \Gamma(V)\), then \(\nabla_X \phi \in \Gamma(V)\) for all \(X \in TM\).

**Remark 1.1.** It follows from (a) that \(\dim M = 4n\). A well known fact about 4n-dimensional Riemannian manifold is that it is quaternionic Kähler if and only if its restricted holonomy group is contained in \(Sp(n)Sp(1)\).

The 4-dimensional Riemannian manifolds with holonomy \(Sp(1)Sp(1)\) are simply the oriented Riemannian manifolds, naturally we only consider those when \(n \geq 2\).

Notice that in general \(I, J, K\) are not defined everywhere on \(M\). For example, the canonical quaternionic projective space \(QP^n\) admits no almost complex structure.

On the other hand, the vector space generated by \(I, J, K\) is well defined at each point of \(M\) and this 3-dimensional subbundle \(V\) of \(\text{End}(TM)\) is in fact “globally parallel” under the Levi-Civita connection \(\nabla\) of \(g\). A basic fact about the connection is the following lemma.
Lemma 1.1. The condition (b) is equivalent to the following condition:
\[
\nabla_X I = c(X) J - b(X) K ,
\]
\[
\nabla_X J = -c(X) I + a(X) K ,
\]
\[
\nabla_X K = b(X) I - a(X) J ,
\]
where \( a, b, c \) are local 1-forms.

Definition 1.2. Let \((M, g)\) be a quaternionic Kähler manifold. We can define a 4-form by

\[
\Omega = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3 ,
\]
where

\[
\omega_1 = \langle \cdot , I \cdot \rangle ,
\]
\[
\omega_2 = \langle \cdot , J \cdot \rangle ,
\]
\[
\omega_3 = \langle \cdot , K \cdot \rangle .
\]

Let \( \{e_1, Je_1, Ke_1, \ldots , e_n, Je_n, Ke_n\} \) be an orthonormal basis of \( TM \) and \( \{\theta^1, I\theta^1, J\theta^1, K\theta^1, \ldots , \theta^n, I\theta^n, J\theta^n, K\theta^n, \} \) the dual basis. It follows that

\[
\omega_1 = \sum_{i=1}^n \left( \theta^i \wedge I\theta^i + J\theta^i \wedge K\theta^i \right) ,
\]
\[
\omega_2 = \sum_{i=1}^n \left( \theta^i \wedge J\theta^i + K\theta^i \wedge I\theta^i \right) ,
\]
\[
\omega_3 = \sum_{i=1}^n \left( \theta^i \wedge K\theta^i + I\theta^i \wedge J\theta^i \right) ,
\]
and

\[
\Omega = \sum_{i,j} \left( \theta^i \wedge I\theta^i \wedge \theta^j \wedge I\theta^j + \theta^i \wedge J\theta^i \wedge \theta^j \wedge J\theta^j + \theta^i \wedge K\theta^i \wedge \theta^j \wedge K\theta^j \right) + \sum_{i,j} \left( J\theta^i \wedge K\theta^i \wedge J\theta^j \wedge K\theta^j + K\theta^i \wedge I\theta^i \wedge K\theta^j \wedge I\theta^j 
\]
\[+ I\theta^i \wedge J\theta^i \wedge I\theta^j \wedge J\theta^j \right) + 2 \sum_{i,j} \left( \theta^i \wedge I\theta^i \wedge J\theta^j \wedge K\theta^j + \theta^i \wedge J\theta^i \wedge K\theta^j \wedge I\theta^j 
\]
\[+ \theta^i \wedge K\theta^i \wedge I\theta^j \wedge J\theta^j \right) .
\]
Lemma 1.2. The condition (b) is equivalent to the following condition:

\[
\begin{align*}
\nabla_X \omega_1 &= c(X) \omega_2 - b(X) \omega_3, \\
\nabla_X \omega_2 &= -c(X) \omega_1 + a(X) \omega_3, \\
\nabla_X \omega_3 &= b(X) \omega_1 - a(X) \omega_2.
\end{align*}
\]

where \(a, b, c\) are local 1-forms.

Proof: It follows from the identities

\[
\begin{align*}
(\nabla_X \omega_1)(Y, Z) &= \langle Y, (\nabla_X I)Z \rangle, \\
(\nabla_X \omega_2)(Y, Z) &= \langle Y, (\nabla_X J)Z \rangle, \\
(\nabla_X \omega_3)(Y, Z) &= \langle Y, (\nabla_X K)Z \rangle.
\end{align*}
\]

Using this lemma, we have that

Theorem 1.1. The condition (b) is equivalent to that \(\Omega\) is parallel, that is

\[
\nabla_X \Omega = 0
\]

for any \(X \in TM\).

In the following, we shall study the curvature of quaternionic Kähler manifold. First we have the following lemma.

Lemma 1.3. If \((M^{4n}, g)\) is a quaternionic Kähler manifold, then

\[
\begin{align*}
[R(X, Y), I] &= \gamma(X, Y) J - \beta(X, Y) K, \\
[R(X, Y), J] &= -\gamma(X, Y) I + \alpha(X, Y) K, \\
[R(X, Y), K] &= \beta(X, Y) I - \alpha(X, Y) J,
\end{align*}
\]

where \(\alpha, \beta\) and \(\gamma\) are local 2-forms given by

\[
\begin{align*}
\alpha &= da + b \wedge c, \\
\beta &= db + c \wedge a, \\
\gamma &= dc + a \wedge b.
\end{align*}
\]
Corollary 1.1. If $(M^{4n}, g)$ is a quarternionic Kähler manifold, then
\[
\langle R(X,Y)Z, IZ \rangle + \langle R(X,Y)JZ, KZ \rangle = \alpha(X,Y) |Z|^2,
\]
\[
\langle R(X,Y)Z, JZ \rangle + \langle R(X,Y)KZ, IZ \rangle = \beta(X,Y) |Z|^2,
\]
\[
\langle R(X,Y)Z, KZ \rangle + \langle R(X,Y)IZ, JZ \rangle = \gamma(X,Y) |Z|^2.
\]

The following lemma is the key for quaternionic Kähler manifolds.

Lemma 1.4. If $(M^{4n}, g)$ is a quaternionic Kähler manifold and $n \geq 2$, then
\[
\alpha(X,IY) = \beta(X,JY) = \gamma(X,KY) = -\frac{1}{n+2} \operatorname{Ric}_M(X,Y).
\] (1.1)

As applications of the above lemma, one can show the following two main theorems on curvature of quaternionic Kähler manifolds.

Theorem 1.2. If $(M^{4n}, g)$ is a quaternionic Kähler manifold and $n \geq 2$, then $(M^{4n}, g)$ is Einstein, that is, there is a constant $\delta$ such that
\[
\operatorname{Ric}_M(g) = 4(n+2)\delta g.
\]

Theorem 1.3. If $(M^{4n}, g)$ is a quaternionic Kähler manifold and $n \geq 2$, then

(1) For any tangent vector $X$, the sectional curvature satisfies
\[
\langle R(X,I\gamma)I\gamma,X \rangle + \langle R(X,J\gamma)J\gamma,X \rangle
\]
\[
+ \langle R(X,K\gamma)K\gamma,X \rangle = 12\delta |X|^4.
\]

(2) For any tangent vector $Y$ satisfying
\[
\langle Y,X \rangle = \langle Y,I\gamma \rangle = \langle Y,J\gamma \rangle = \langle Y,K\gamma \rangle = 0,
\]
the sectional curvature satisfies
\[
\langle R(X,Y)Y,X \rangle + \langle R(X,\gamma)I\gamma,Y \rangle +
\]
\[
\langle R(X,J\gamma)J\gamma,X \rangle + \langle R(X,K\gamma)K\gamma,X \rangle = 4\delta |X|^2 |Y|^2,
\]
where $4(n+2)\delta$ is the Einstein constant.
Lemma 1.5. Let $\gamma : [a, b] \to M$ be a geodesic with unit speed. If $S = 16n(n + 2)\delta$, and $X_I(t), X_J(t), X_K(t)$ are parallel vector fields along $\gamma$ such that $X_I(a) = I\gamma'(a), X_J(a) = J\gamma'(a), X_K(a) = K\gamma'(a)$, then
\[
\mathcal{K}(\gamma'(t), X_I(t)) + \mathcal{K}(\gamma'(t), X_J(t)) + \mathcal{K}(\gamma'(t), X_K(t)) = 12\delta,
\]
for all $t$ and $\gamma$.

Let $Y$ be a tangent vector at $\gamma(a)$ satisfying $\langle \gamma'(a), Y \rangle = 0$, $\langle I\gamma'(a), Y \rangle = 0$, $\langle J\gamma'(a), Y \rangle = 0$, and $\langle K\gamma'(a), Y \rangle = 0$. If we denote the parallel vector fields $Y(t), Y_I(t), Y_J(t)$, and $Y_K(t)$ along $\gamma$ with initial data $Y(a) = Y, Y_I(a) = IY, Y_J(a) = JY$, and $Y_K(a) = KY$, respectively, then
\[
\mathcal{K}(\gamma'(t), Y(t)) + \mathcal{K}(\gamma'(t), Y_I(t)) + \mathcal{K}(\gamma'(t), Y_J(t)) + \mathcal{K}(\gamma'(t), Y_K(t)) = 4\delta,
\]
for all $t$ and $\gamma$.

Proof. By the discussion above, we know the 3-dimensional vector space $E(t)$ spanned by $X(t), Y(t), Z(t)$ does not depend on the choice of $I, J, K$. Hence it is parallel under the Levi-Civita connection. We consider $\langle B(\cdot, \gamma'(t))\gamma'(t), \cdot \rangle$ as a symmetric bilinear form on $E(t)$. Then $\mathcal{K}(\gamma'(t), X(t)) + \mathcal{K}(\gamma'(t), Y(t)) + \mathcal{K}(\gamma'(t), Z(t))$ is its trace on $E(t)$ which independent of the choice of orthonormal basis. By the computation above it is equal to $12\delta$. The same argument also applies to the second part of the lemma. \hfill \Box

2 Laplacian comparison theorem

For a complete Riemannian manifold $M$ and $p \in M$, let us denote the cut locus with respect to $p$ by $\text{Cut}(p)$.

Theorem 2.1. Let $(M^{4n}, g)$ be a complete quaternionic Kähler manifold with scalar curvature $S_M \geq 16n(n + 2)\delta$ and let $r(x)$ be the distance function to a fixed point $p \in M$. Then, for $x \notin \text{Cut}(p)$,
\[
\Delta r(x) \leq \begin{cases} 
6 \coth 2r(x) + 4(n - 1) \coth r(x) & \text{when } \delta = -1 \\
(4n - 3)r^{-1}(x) & \text{when } \delta = 0 \\
6 \cot 2r(x) + 4(n - 1) \cot r(x) & \text{when } \delta = 1.
\end{cases} \quad (2.1)
\]
Proof. Let $\gamma$ be the minimizing geodesic joining $p$ to $x$. At $x$, we choose $\{e_1, e_2, \cdots, e_n\}$, and two local almost complex structures $I, J$ and $K = IJ$ such that $e_1 = \nabla r$ and

$$\{e_1, Ie_1, Je_1, Ke_1, e_2, Ie_2, Je_2, Ke_2, \cdots, e_n, Ie_n, Je_n, Ke_n\}$$

is an orthonormal frame. By parallel translating along $\gamma$ we obtain an orthonormal frame with $e_1 = \nabla r$. For convenience sake, we denote this frame by $\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{4n}\}$. Since $|\nabla r|^2 = 1$ on $M \setminus \text{Cut}(p)$, by taking covariant derivative of this equation, we have

$$0 = \frac{1}{2} \sum_{i=1}^{4n} r_{ik} r_{il} + \sum_{i=1}^{4n} r_i r_{ikl}, \quad (2.2)$$

for each $k, l = 2, \cdots, 4n$. Since

$$r_{ikl} = r_{kli} + \sum_{j=1}^{n} R_{jkl} r_j,$$

with $R_{ijkl} = (R(\varepsilon_i, \varepsilon_j)\varepsilon_l, \varepsilon_k)$, and $r_1 = 1, r_j = 0, j = 2, \cdots, 4n$, we have

$$\sum_{i=1}^{4n} r_{ik} r_{il} + r_{kl1} + R_{1k1l} = 0. \quad (2.3)$$

In particular, if $k = l$, we have

$$\sum_{i=1}^{4n} r_{ik}^2 + r_{kk1} + \mathcal{K}(\varepsilon_1, \varepsilon_k) = 0, \quad (2.4)$$

where $\mathcal{K}(\varepsilon_1, \varepsilon_k) = R_{1k1k}$ is the sectional curvature of the 2-plane section spanned by $\varepsilon_1, \varepsilon_k$. Using the inequality

$$\sum_{k=2}^{4} r_{ik}^2 \geq \frac{1}{3} \left( \sum_{k=2}^{4} r_{kk} \right)^2,$$

and setting $f(t) = \sum_{k=2}^{4} r_{kk}$, (2.4) implies that

$$f'(t) + \frac{1}{3} f^2(t) + \sum_{k=2}^{4} \mathcal{K}(\varepsilon_1, \varepsilon_k) \leq 0. \quad (2.5)$$
By Lemma 1.5, we have
\[ f'(t) + \frac{1}{3} f^2(t) + 12\delta \leq 0. \tag{2.6} \]

Since a smooth Riemannian metric is locally Euclidean, then \( \lim_{t \to 0} tf(t) = 3. \) By a standard comparison argument for ordinary differential equations, we conclude that
\[ f(t) \leq \begin{cases} 
6 \cot 2t & \text{when } \delta = 1 \\
3t^{-1} & \text{when } \delta = 0 \\
6 \coth 2t & \text{when } \delta = -1. 
\end{cases} \tag{2.7} \]

Similarly, using the inequality
\[ \sum_{k=4i+1}^{4i+4} r_{ik}^2 \geq \frac{1}{4} \left( \sum_{k=4i+1}^{4i+4} r_{kk} \right)^2 \]
for \( 1 \leq i \leq n-1, \) and setting \( h_i(t) = \sum_{k=4i+1}^{4i+4} r_{kk}, \) (2.4) implies that
\[ h_i'(t) + \frac{1}{4} h_i^2(t) + \sum_{k=4i+1}^{4i+4} K(\varepsilon_1, \varepsilon_k) \leq 0. \tag{2.8} \]

Together with Lemma 1.5 asserting that
\[ \sum_{k=4i+1}^{4i+4} K(\varepsilon_1, \varepsilon_k) = 4\delta, \]
we have
\[ h_i'(t) + \frac{1}{4} h_i^2(t) + 4\delta \leq 0. \tag{2.9} \]

Hence, as before, we conclude that
\[ h_i(t) \leq \begin{cases} 
4 \cot t & \text{when } \delta = 1 \\
4t^{-1} & \text{when } \delta = 0 \\
4 \coth t & \text{when } \delta = -1. 
\end{cases} \tag{2.10} \]

The result follows from the equation \( \Delta r(x) = f(r(x)) + \sum_{i=1}^{n-1} h_i(r(x)). \) \( \blacksquare \)

**Remark 2.1.** The estimate in Theorem 2.1 is sharp since the right hand sides are exactly the Laplacian of the distance functions of quaternionic hyperbolic space \( \mathbb{QH}^n, \) quaternionic Euclidean space \( \mathbb{Q}^n \) and quaternionic projective space \( \mathbb{QP}^n \) respectively.
Remark 2.2. We actually proved the estimate for Hessian of the distance function. In particular,

$$\sum_{k=2}^{4} r_{kk} \leq \begin{cases} 6 \cot 2t & \text{when } \delta = 1 \\ 3t^{-1} & \text{when } \delta = 0 \\ 6 \coth 2t & \text{when } \delta = -1. \end{cases} \quad (2.11)$$

Also for $1 \leq i \leq n - 1$, we have

$$\sum_{k=4i+1}^{4i+4} r_{kk} \leq \begin{cases} 4 \cot 2t & \text{when } \delta = 1 \\ 4t^{-1} & \text{when } \delta = 0 \\ 4 \coth 2t & \text{when } \delta = -1. \end{cases} \quad (2.12)$$

Corollary 2.1. Let $(M^{4n}, g)$ be a complete quaternionic Kähler manifold with scalar curvature $S_M \geq -16n(n+2)$. Then for any point $x \in M$ and $r > 0$, the area $A(r)$ of the geodesic spheres centered at $x$ satisfies

$$\frac{A'(r)}{A(r)} \leq 6 \coth 2r + 4(n-1) \coth r. \quad (2.13)$$

In particular, $A(r) \leq C(\sinh 2r)^3(\sinh r)^{4(n-1)} \leq Ce^{(4n+2)r}$.

Corollary 2.2. Let $(M^{4n}, g)$ be a complete quaternionic Kähler manifold with scalar curvature $S_M \geq -16n(n+2)$. Then for any point $x \in M$ and $0 < r_1 \leq r_2$, the volume of the geodesic balls centered at $x$ satisfies

$$\frac{V_x(r_2)}{V_x(r_1)} \leq \frac{V_{QH^n}(r_2)}{V_{QH^n}(r_1)}, \quad (2.14)$$

where $V_{QH^n}(r)$ denotes the volume of the geodesic ball of radius $r$ in $Q\mathbb{H}^n$. In particular, $\lambda_1(M) \leq (2n+1)^2$.

Corollary 2.3. Let $(M^{4n}, g)$ be a complete quaternionic Kähler manifold with scalar curvature $S_M \geq 16n(n+2)$. Then it is compact, and the diameter $d(M) \leq \frac{\pi}{2}$, which is the diameter of the model space $Q\mathbb{P}^n$. Moreover, the volume of $M$ is bounded by

$$V(M) \leq V(Q\mathbb{P}^n), \quad (2.15)$$

where $V_{Q\mathbb{P}^n}$ is the volume of $Q\mathbb{P}^n$. 

13
3 Quaternionic harmonicity

In this section we will derive an over-determined system of harmonic func-
tions with finite Dirichlet integral on a manifold with a parallel form. This
result was first proved by Siu [S] for harmonic maps in his proof of the rigidity
theorem for Kähler manifolds. Corlette [C1] gave a more systematic approach
for harmonic map with finite energy from a finite-volume quaternionic hy-
perbolic space or Cayley hyperbolic plane to a manifold with nonpositive
curvature. In [L], the second author generalized Siu’s argument to harmonic
functions with finite Dirichlet integral on a Kähler manifold. We will pro-
vide an argument that generalizes Corlette’s argument to harmonic functions
with finite Dirichlet integral on a complete manifold with a parallel form. We
believe that it should be of independent interest.

**Theorem 3.1.** Let $M$ be a complete Riemannian manifold with a parallel
$p$-form $\Omega$. Assume that $f$ is a harmonic function with its Dirichlet integral
over geodesic balls centered at $o$ of radius $R$ satisfying the growth condition

$$\int_{B_o(R)} |\nabla f|^2 dv = o(R^2)$$

as $R \to \infty$, then $f$ satisfies

$$d^* (df \wedge \Omega) = 0.$$ (3.1)

Before we prove the theorem, let us first recall the following operators
and some of the basic properties. For an oriented real vector space $V$ with
an inner product, we have the Hodge star operator

$$\ast : \wedge^p V \to \wedge^{n-p} V.$$ 

For any $\theta \in \wedge^1 V$ and $v \in V$, we also have exterior multiplication and interior
product operators

$$\varepsilon(\theta) : \wedge^p V \to \wedge^{p+1} V,$$

$$\ell(v) : \wedge^p V \to \wedge^{p-1} V.$$ 

For $\theta \in \wedge^1 V$ and $v \in V$ is the dual of $\theta$ by the inner product, if $\xi \in \wedge^p V$ we
list the following identities among the operators:
1. \( \ast \ast \xi = (-1)^p(n-p)\xi \),

2. \( \ast \varepsilon(\theta)\xi = (-1)^p\ell(v) \ast \xi \),

3. \( \varepsilon(\theta) \ast \xi = (-1)^{p-1} \ast \ell(v)\xi \),

4. \( \ast \varepsilon(\theta) \ast \xi = (-1)^{(p-1)(n-p)}\ell(v)\xi \),

5. \( \ell(v)\varepsilon(\theta')\xi + \varepsilon(\theta)\ell(v')\xi = 0 \), where \( v \perp v' \),

6. \( \ell(v)\varepsilon(\theta)\xi + \varepsilon(\theta)\ell(v)\xi = \xi \).

We are now ready to prove Theorem \( \text{3.1} \).

**Proof of Theorem 3.1.** Let \( \eta : [0, +\infty) \to \mathbb{R} \) be a smooth function satisfying \( \eta'(t) \leq 0 \), and

\[
\eta(t) = \begin{cases} 
1 & \text{when } t \in [0, 1] \\
0 & \text{when } t \in [2, +\infty].
\end{cases}
\]

For \( R \geq 1 \), we define the cut-off function \( \phi_R(x) = \eta(r(x)/R) \), where \( r(x) \) is the distance function from a fixed point \( o \in M \), then there is a positive constant \( C_1 \) depending on \( \eta \) and \( C \) such that

\[ |\nabla \phi_R(x)| \leq C_1 \frac{1}{R}. \]

Since \( d^2 = 0 \), then

\[
0 = \int_M d \{ \phi_R^2 \ast (df \wedge \Omega) \wedge d \ast (df \wedge \ast \Omega) \} \\
= \int_M d(\phi_R^2) \wedge \ast (df \wedge \Omega) \wedge d \ast (df \wedge \ast \Omega) \\
+ \int_M \phi_R^2 d \ast (df \wedge \Omega) \wedge d \ast (df \wedge \ast \Omega). \quad (3.2)
\]

We claim that

\[ \ast d \ast (df \wedge \Omega) = (-1)^{n-1} d \ast (df \wedge \ast \Omega). \quad (3.3) \]

In fact, for any point \( x \in M \), we can choose an orthonormal tangent basis \( \{e_i\}_{i=1}^m \) in a neighborhood of \( x \) such that \( \nabla_{e_i} e_j(x) = 0 \). Denote by \( \{\theta^i\}_{i=1}^m \) the dual basis of \( \{e_i\}_{i=1}^m \). Then for \( \omega \in \wedge^p(T^*M) \) we have

\[ d\omega = \varepsilon(\theta)\nabla_{e_i} \omega. \]
Hence
\[
d \ast (df \wedge \ast \Omega) = d \ast \varepsilon(df) \ast \Omega
\]
\[
= (-1)^{(p-1)(m-p)} d[\ell(\nabla f)\Omega]
\]
\[
= (-1)^{(p-1)(m-p)} \sum_{i=1}^{m} \varepsilon(\theta_i) \nabla_{e_i}(\ell(\nabla f)\Omega)
\]
\[
= (-1)^{(p-1)(m-p)} \sum_{i,j=1}^{m} \varepsilon(\theta_i)(\nabla_{e_i} \nabla_{e_j} f)(\ell(e_j)\Omega)
\]
\[
= (-1)^{(p-1)(m-p)} \sum_{i,j=1}^{m} f_{ij} \varepsilon(\theta_i)(\ell(e_j)\Omega),
\]
where \( f_{ij} = \nabla_{e_i} \nabla_{e_j} f \) and the facts \( \Omega \) is parallel and \( \nabla_{e_i} e_j(x) = 0 \) have been used. On the other hand,
\[
\ast d \ast (df \wedge \Omega) = \ast d \ast \varepsilon(df)\Omega
\]
\[
= \ast \sum_{i=1}^{m} \varepsilon(\theta_i) \nabla_{e_i}(\ast (\sum_{j=1}^{m} f_{ij} \varepsilon(\theta_j) d[\ell(\nabla f)\Omega])
\]
\[
= \ast \sum_{i,j=1}^{m} f_{ij} \varepsilon(\theta_i) \ast \varepsilon(\theta_j)(\Omega)
\]
\[
= (-1)^{p(m-p-1)} \sum_{i,j=1}^{m} f_{ij} \ell(e_i) \varepsilon(\theta_j)\Omega
\]
\[
= (-1)^{p(m-p-1)} \left( \sum_{i=1}^{m} f_{ii} \ell(e_i)\varepsilon(\theta_i)\Omega + \sum_{i \neq j}^{m} f_{ij} \ell(e_i) \varepsilon(\theta_j)\Omega \right)
\]
\[
= (-1)^{p(m-p-1)} \left( \sum_{i=1}^{m} f_{ii}[\Omega - \varepsilon(\theta_i)\ell(e_i)\Omega] - \sum_{i \neq j}^{m} f_{ij} \varepsilon(\theta_j)\ell(e_i)\Omega \right)
\]
\[
= (-1)^{p(m-p-1)} \sum_{i,j=1}^{m} f_{ij} \varepsilon(\theta_i)(\ell(e_j)\Omega),
\]
where we used \( f_{ij} = f_{ji} \) and \( \sum_{i=1}^{m} f_{ii} = 0 \). So the claim is proved. By (3.2),

16
we have
\[
\int_M \phi_R^2 |d \ast (df \wedge \Omega)|^2 dv \\
= (-1)^m \int_M d(\phi_R^2) \wedge \ast (df \wedge \Omega) \wedge d \ast (df \wedge \Omega) \\
\leq 2 \left( \int_M \phi_R^2 |d \ast (df \wedge \Omega)|^2 dv \right)^{\frac{1}{2}} \left( \int_M |d\phi_R|^2 \ast (df \wedge \Omega)|^2 dv \right)^{\frac{1}{2}} \tag{3.5}
\]

On the other hand, (3.3) and the fact that \( \omega \) is bounded imply that there exists a constant \( C_2 > 0 \), such that
\[
| \ast (df \wedge \Omega) | \leq C_2 |df| \\
|d \ast (df \wedge \ast \Omega) | = |d \ast (df \wedge \Omega)|.
\]
Hence combining with (3.5) and using the definition of \( \phi_R \) we conclude that
\[
\int_{B_o(R)} |d \ast (df \wedge \Omega)|^2 dv \leq C_1 R^{-2} \int_{B_o(2R)} |df|^2 dv.
\]

The assumption on the growth of the Dirichlet integral of \( f \) implies that the right hand side tends to zero as \( R \to \infty \). Therefore \( d \ast (df \wedge \Omega) = 0 \), and the proof is complete. \( \square \)

**Lemma 3.1.** Let \( (M^{4n}, g) \) be a quarternionic Kähler manifold and \( n \geq 2 \). If \( f \) is a function on \( M \) satisfying
\[
d \ast (df \wedge \Omega) = 0 \tag{3.6}
\]
for the 4-form \( \Omega \) determined by the quarternionic Kähler structure, then \( f \) is quarternionic harmonic, namely, for any nonzero tangent vector \( X \),
\[
f_{X,X} + f_{IX,IJX} + f_{JX,IX} + f_{KX,KKX} = 0
\]
where \( f_{X,X} = \nabla df(X,X) \).

**Proof:** Let
\[
\{e_A\}_{A=1}^{4n} = \{e_1, e_2, \cdots, e_n, Ie_1, Ie_2, \cdots, Ie_n, Je_1, Je_2, \cdots, Je_n, Ke_1, Ke_2, \cdots, Ke_n\}
\]
be an orthonormal basis of $TM$ and $\{\omega_A\}$ the dual basis with $e_1 = \frac{X}{\|X\|}$. Since $\Omega$ is parallel, by (3.4) and (3.6), we have

$$0 = \sum_{A=1}^{4n} (\nabla e_A df) \wedge \ell(e_A)\Omega$$

$$= \sum_{A,B=1}^{4n} f_{e_A,e_B} \omega_B \wedge \ell(e_A)\Omega$$

where we have used the fact that $f$ is a harmonic function. Hence equation (3.6) implies

$$\sum_{A,B=1}^{4n} f_{e_A,e_B} \omega_B \wedge \ell(e_A)\Omega = 0$$

Comparing the coefficient of $\omega_i \wedge I\omega_i \wedge J\omega_i \wedge K\omega_i$ on both sides by the explicit formula for $\Omega$ given before, we obtain that

$$6 \left( f_{e_i,e_i} + f_{Ie_i,Ie_i} + f_{Je_i,Je_i} + f_{Ke_i,Ke_i} \right) = 0$$

for all $e_i$, $(1 \leq i \leq n)$. So the proof is complete. □

The following corollary is an immediate consequence of the lemma.

**Corollary 3.1.** Let $M^{4n}$ be a complete quaternionic Kähler manifold. Assume that $f$ is a harmonic function with its Dirichlet integral satisfying the growth condition

$$\int_{B_\epsilon(R)} |\nabla f|^2 dv = o(R^2)$$

as $R \to \infty$, then $f$ must satisfy

$$d*(df \wedge \Omega) = 0,$$

(3.7)

where $\Omega$ is the parallel 4-form determined by the quaternionic Kähler structure. Moreover, $f$ is quaternionic harmonic.
4 Uniqueness of infinite volume end

Recall that for any complete manifold if $\lambda_1(M) > 0$ then $M$ must be non-parabolic. In particular, $M$ must have at least one nonparabolic ends. It was also proved in [LW1] that under the assumption that $\lambda_1(M) > 0$, an end is nonparabolic if and only if it has infinite volume.

Let us assume that $M$ has at least two nonparabolic ends, $E_1$ and $E_2$. A construction of Li-Tam [LT] asserts that one can construct a nonconstant bounded harmonic function with finite Dirichlet integral. The harmonic function $f$ can be obtained by taking a convergent subsequence of the harmonic functions $f_R$, as $R \to +\infty$, satisfying

$$\Delta f_R = 0 \quad \text{on } B(R),$$

with boundary conditions

$$f_R = 1 \quad \text{on } \partial B(R) \cap E_1$$

and

$$f_R = 0 \quad \text{on } \partial B(R) \setminus E_1.$$ 

It follows from the maximum principle that $0 \leq f_R \leq 1$, hence $0 \leq f \leq 1$. We need the following estimates from [LW1](Lemma 1.1 and 1.2 in [LW1]), and [LW3](Lemma 5.1 in [LW3]).

**Lemma 4.1.** Let $M$ be a complete Riemannian manifold with $\lambda_1(M) > 0$. Suppose $M$ has at least two nonparabolic ends and $E$ be an end of $M$. Then for the harmonic function $f$ constructed above, it must satisfy the following growth conditions:

1. There exists a constant $a$ such that $f - a \in L^2(E)$. Moreover, the function $f - a$ must satisfy the decay estimate

$$\int_{E(R+1) \setminus E(R)} (f - a)^2 \leq C \exp(-2\sqrt{\lambda_1(E)} R)$$

for some constant $C > 0$ depending on $f$, $\lambda_1(E)$ and the dimension of $M$. 

19
2. The Dirichlet integral of the function $f$ must satisfy the decay estimate
\[
\int_{E(R+1) \setminus E(R)} |\nabla f|^2 \leq C \exp(-2\sqrt{\lambda_1(E)} R),
\]
and
\[
\int_{E(R)} \exp(-2\sqrt{\lambda_1(E)} r(x)) |\nabla f|^2 \leq CR
\]
for $R$ sufficiently large.

Lemma 4.2. Let $M$ be a complete Riemannian manifold with at least two nonparabolic ends and $\lambda_1(M) > 0$. Then for the harmonic function $f$ constructed above, for any $t \in (\inf f, \sup f)$ and $(a, b) \subset (\inf f, \sup f)$,
\[
\int_{\mathcal{L}(a,b)} |\nabla f|^2 = (b - a) \int_{l(b)} |\nabla f|,
\]
where
\[
l(t) = \{ x \in M | f(x) = t \},
\]
and
\[
\mathcal{L}(a, b) = \{ x \in M | a < f(x) < b \}.
\]
Moreover,
\[
\int_{l(t)} |\nabla f| = \int_{l(b)} |\nabla f|.
\]

We are now ready to prove Theorem 0.5.

Proof of Theorem 0.5: Suppose to the contrary that there exist two ends $E_1$ and $E_2$ with infinite volume. The assumption that $\lambda_1(M) > 0$ implies that they are nonparabolic. By the construction above, there exists a harmonic function $f$ with finite energy such that
\[
\liminf_{x \to \infty, x \in E_1} f(x) = 1
\]
and
\[
\liminf_{x \to \infty, x \in E_2} f(x) = 0.
\]

The Bochner formula implies that
\[
\frac{1}{2} \Delta |\nabla f|^2 = \text{Ric}_M(\nabla f, \nabla f) + |\nabla^2 f|^2. \tag{4.1}
\]
We now choose an orthonormal basis \( \{e_A\}_{A=1}^{4n} \) satisfying

\[
\{e_1, e_2, \ldots, e_n, Ie_1, Ie_2, \ldots, Ie_n, Je_1, Je_2, \ldots, Je_n, Ke_1, Ke_2, \ldots, Ke_n\}
\]

with \( e_1 = \frac{\nabla f}{|\nabla f|} \). Corollary 3.1 implies that

\[
\sum_{i=0}^{3} f((n+1)(in+1)) = 0.
\]

Therefore, applying the arithmetic-geometric means, we have

\[
|\nabla^2 f|^2 = \sum_{A, B=1}^{4n} f_{AB}^2 \\
\geq f_{11}^2 + \sum_{i=1}^{3} f_{(i+1)(in+1)}^2 + 2 \sum_{A=2}^{4n} f_{1A}^2 \\
\geq f_{11}^2 + \frac{1}{3} \left( \sum_{i=1}^{3} f((i+1)(in+1))^2 \right) + 2 \sum_{A=2}^{4n} f_{1A}^2 \\
\geq \frac{4}{3} |\nabla|\nabla f||^2,
\]

hence combining with (4.1) we obtain

\[
\frac{1}{2} \Delta|\nabla f|^2 \geq -4(n+2)|\nabla f|^2 + \frac{4}{3} |\nabla|\nabla f||^2.
\]

If we write \( u = |\nabla f|^2 \), then

\[
\Delta u \geq -\frac{8(n+2)}{3} u.
\]

We want to prove that the above inequality is actually an equality. The argument follows from that in [LW4] after making suitable modification to fit our situation. For any compactly supported smooth function \( \phi \) on \( M \), we
have
\[
0 \leq \int_M \phi^2 u \left( \Delta u + \frac{8(n+2)}{3} u \right) \\
\leq -2 \int_M \phi u \langle \nabla u, \nabla \phi \rangle - \int_M \phi^2 |\nabla u|^2 + \lambda_1(M) \int_M (\phi u)^2 \\
\leq -2 \int_M \phi u \langle \nabla u, \nabla \phi \rangle - \int_M \phi^2 |\nabla u|^2 + \int_M |\nabla (\phi u)|^2 \\
= \int_M |\nabla \phi|^2 u^2. \quad (4.5)
\]

Let us choose \( \phi = \psi \chi \) to be the product of two compactly supported functions. For any \( \varepsilon \in (0, \frac{1}{2}) \), we define
\[
\chi(x) = \begin{cases} 
0 & \text{on } \mathcal{L}(0, \sigma \varepsilon) \cup \mathcal{L}(1 - \frac{\varepsilon}{2}, 1) \\
(\log 2)^{-1}(\log f - \log(\frac{\varepsilon}{2})) & \text{on } \mathcal{L}(\frac{\varepsilon}{2}, \varepsilon) \cap (M \setminus E_1) \\
(\log 2)^{-1}(\log(1 - f) - \log(\frac{\varepsilon}{2})) & \text{on } \mathcal{L}(1 - \varepsilon, 1 - \frac{\varepsilon}{2}) \cap E_1 \\
1 & \text{otherwise.}
\end{cases}
\]

For \( R > 1 \) we define
\[
\psi = \begin{cases} 
1 & \text{on } B(R - 1) \\
R - r & \text{on } B(R) \setminus B(R - 1) \\
0 & \text{on } M \setminus B(R).
\end{cases}
\]

Applying to the right hand side of (4.5), we obtain
\[
\int_M |\nabla \phi|^2 u^2 \leq 2 \int_M |\nabla \psi|^2 \chi^2 |\nabla f|^\frac{2}{3} + 2 \int_M |\nabla \chi|^2 \psi^2 |\nabla f|^\frac{2}{3}. \quad (4.6)
\]

Since \( \text{Ric}_M \geq -4(n+2) \), then the local estimate of Cheng-Yau [CY] (see also [LW2]) implies that there exists a constant depending on \( n \) such that
\[
|\nabla f|(x) \leq C|1 - f(x)|.
\]

On \( E_1 \), the first term of (4.6) satisfies
\[
\int_M |\nabla \psi|^2 \chi^2 |\nabla f|^\frac{2}{3} \leq \left( \int_\Omega |\nabla f|^2 \right)^\frac{2}{3} \left( \int_\Omega 1 \right)^\frac{1}{3}, \quad (4.7)
\]

22
where \( \Omega = E_1 \cap (B(R) \setminus B(R - 1)) \cap (\mathcal{L}(1 - \varepsilon, 1 - \frac{\varepsilon}{2}) \cup \mathcal{L}(\frac{\varepsilon}{2}, \varepsilon)). \) Since
\[
\int_{\Omega} 1 \leq 4 \int_{\Omega} \frac{(1 - f)^2}{\varepsilon^2} \\
\leq \frac{4}{\varepsilon^2} \int_{\Omega} (1 - f)^2 \\
\leq 4C \varepsilon^{-2} \exp(-2 \sqrt{\lambda_1} R),
\]
where in the last inequality we have used Lemma 4.1. Again by Lemma 4.1, from (4.7) we have
\[
\int_{M} |\nabla \psi|^2 |\nabla f|^\frac{4}{3} \leq C \varepsilon^{-\frac{2}{3}} \exp(-2 \sqrt{\lambda_1} R). \tag{4.8}
\]
For the second term of (4.6) we have
\[
\int_{E_1} |\nabla \chi|^2 |\nabla f|^\frac{1}{3} \\
\leq (\log 2)^{-2} \int_{\mathcal{L}(1 - \varepsilon, 1 - \frac{\varepsilon}{2}) \cap E_1 \cap B(R)} |\nabla f|^\frac{1}{3} + (1 - f)^{-2} \\
\leq C(\log 2)^{-2} \int_{\mathcal{L}(1 - \varepsilon, 1 - \frac{\varepsilon}{2}) \cap E_1 \cap B(R)} |\nabla f|^2 (1 - f)^{-\frac{2}{3}}.
\]
Using the co-area formula and Lemma 4.2 we have
\[
\int_{\mathcal{L}(1 - \varepsilon, 1 - \frac{\varepsilon}{2}) \cap E_1 \cap B(R)} |\nabla f|^2 (1 - f)^{-\frac{2}{3}} \\
\leq \int_{1 - \varepsilon}^{1 - \frac{\varepsilon}{2}} (1 - t)^{-\frac{2}{3}} \int_{l(t) \cap E_1 \cap B(R)} |\nabla f| dA dt \\
\leq C \int_{l(b)} |\nabla f| dA \int_{1 - \varepsilon}^{1 - \frac{\varepsilon}{2}} (1 - t)^{-\frac{2}{3}} dt \\
= -3C [(1 - t)^{\frac{1}{2}}]_{1 - \varepsilon}^{1 - \frac{\varepsilon}{2}} \int_{l(b)} |\nabla f| dA \\
= 3C \varepsilon^{\frac{1}{2}} \int_{l(b)} |\nabla f| dA.
\]
Combining the above inequality with (4.8) we have
\[
\int_{E_1} |\nabla \phi|^2 u^2 \leq C(\varepsilon^{\frac{2}{3}} \exp(-2 \sqrt{\lambda_1} R) + \varepsilon^\frac{1}{3}). \tag{4.9}
\]

A similar argument using \( f \) instead of \( 1 - f \) on the other end yields the estimate
\[
\int_{M \setminus E_1} |\nabla \phi|^2 u^2 \leq C(\varepsilon^2 \exp(-2\sqrt{\lambda_1} R) + \varepsilon^{\frac{4}{3}}).
\]
Letting \( R \to \infty \) and \( \varepsilon \to 0 \), we have
\[
\Delta u = -\frac{8(n+2)}{3} u \tag{4.10}
\]
with \( \lambda_1(M) = \frac{8(n+2)}{3} \), since \( f \) is nonconstant and \( u \) cannot be identically zero. Therefore all the inequalities used to prove (4.4) are equalities. Thus there exists a function \( \mu \), such that,
\[
(f_{AB}) = \begin{pmatrix}
D_1 & D_2 \\
D_2 & D_2
\end{pmatrix},
\tag{4.11}
\]
where \( D_1 \) and \( D_2 \) are \( n \times n \) matrices defined by
\[
D_1 = \begin{pmatrix}
-3\mu & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 0
\end{pmatrix}
\]
and
\[
D_2 = \begin{pmatrix}
\mu & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 0
\end{pmatrix}.
\]
Since \( f_{1\alpha} = 0 \) for \( \alpha \neq 1 \) implies that \( |\nabla f| \) is constant along the level set of \( f \). Moreover, regularity of the equation (4.10) implies that \( |\nabla f| \) can never be zero. Hence \( M \) must be diffeomorphic to \( \mathbb{R} \times N \), where \( N \) is given by the level set of \( f \). Also \( N \) must be compact since we assume that \( M \) has at least 2 ends.

Fix a level set \( N_0 \) of \( f \), consider \( (-\varepsilon, \varepsilon) \times N_0 \subset M \). Note that \( \{e_A\} \) is an orthonormal basis of \( TM \) such that \( e_1 \) is the normal vector to \( N_0 \) and \( \{e_{\alpha}\} \) are the tangent vectors of \( N_0 \). We shall compute the sectional curvature
\[
K(e_1, e_{\alpha}) = \langle R(e_1, e_{\alpha})e_{\alpha}, e_1 \rangle.
\]
We claim that
\[ \nabla e_1 e_1 = 0. \]

Indeed it suffices to prove all integral curves \( \eta(t) \) of the vector field \( e_1 = \frac{\nabla f}{|\nabla f|} \) emanating from \( N_0 \) are geodesics. For any point \( \eta(t_0) \), let \( \gamma \) be the geodesic realizing the distance between \( \eta(t_0) \) and \( N_0 \). Then \( \gamma \) is perpendicular to every level set \( N_t \). So \( \gamma' \) is parallel to \( e_1 \) along \( \gamma \). This implies \( \gamma \) coincides with the integral curve of \( e_1 \).

Let \( (h_{\alpha\beta}) \) with \( 2 \leq \alpha, \beta \leq 4n \) be the second fundamental form of the level set of \( f \). Then
\[
 h_{\alpha\beta} f_1 = -f_{\alpha\beta}, \tag{4.12}
\]
and
\[
 \nabla_{e_\alpha} e_1 = -\sum_{\beta=2}^{4n} h_{\alpha\beta} e_\beta.
\]

By the definition of curvature tensor, we have
\[
 \langle R(e_1, e_\alpha) e_1, e_\alpha \rangle = \langle \nabla_{e_1} \nabla_{e_\alpha} e_1 - \nabla_{e_\alpha} \nabla_{e_1} e_1 - \nabla_{[e_1, e_\alpha]} e_1, e_\alpha \rangle \\
= \langle \nabla_{e_1} \nabla_{e_\alpha} e_1, e_\alpha \rangle - \langle \nabla_{[e_1, e_\alpha]} e_1, e_\alpha \rangle \\
= \langle \nabla_{e_1} \nabla_{e_\alpha} e_1, e_\alpha \rangle - \langle \nabla_{\nabla_{e_1} e_\alpha} - \nabla_{e_\alpha} e_1 e_1, e_\alpha \rangle \\
= \langle \nabla_{e_1} \nabla_{e_\alpha} e_1, e_\alpha \rangle - \sum_{\beta=2}^{4n} \langle \nabla_{e_\alpha} e_1, e_\beta \rangle \langle \nabla_{e_\beta} e_1, e_\alpha \rangle \\
+ \sum_{\beta=2}^{4n} \langle \nabla_{e_\alpha} e_1, e_\beta \rangle \langle \nabla_{e_\beta} e_1, e_\alpha \rangle \\
= -\sum_{\beta=2}^{4n} \langle \nabla_{e_1} (h_{\alpha\beta} e_\beta), e_\alpha \rangle + \sum_{\beta=2}^{4n} h_{\alpha\beta} (\nabla_{e_1} e_\alpha, e_\beta) + \sum_{\beta=2}^{4n} h_{\alpha\beta}^2 \\
= -\sum_{\beta=2}^{4n} \langle (e_1 h_{\alpha\beta}) e_\beta, e_\alpha \rangle - \sum_{\beta=2}^{4n} h_{\alpha\beta} \langle \nabla_{e_1} e_\beta, e_\alpha \rangle \\
+ \sum_{\beta=2}^{4n} h_{\alpha\beta} \langle \nabla_{e_1} e_\alpha, e_\beta \rangle + h_{\alpha\beta}^2 \\
= -e_1 h_{\alpha\alpha} + 2 \sum_{\beta=2}^{4n} h_{\alpha\beta} (\nabla_{e_1} e_\alpha, e_\beta) + h_{\alpha\beta}^2.
\]
Therefore
\[ K(e_1, e_\alpha) = e_1 h_{\alpha\alpha} - 2 \sum_{\beta=2}^{4n} h_{\alpha\beta} \langle \nabla_{e_1} e_\alpha, e_\beta \rangle - \sum_{\beta=2}^{4n} h_{\alpha\beta}^2. \] (4.13)

Since \( h_{\alpha\beta} \) is diagonal, this implies that
\[ K(e_1, e_\alpha) = e_1 h_{\alpha\alpha} - h_{\alpha\alpha}^2. \]
Combining with (4.11) and (4.12), we conclude that
\[ K(e_1, e_2) = K(e_1, Ie_2) = K(e_1, Je_2) = K(e_1, Ke_2) = 0 \]
which implies \( M \) is Ricci flat by Theorem 1.3. This contradicts to the assumption that \( \lambda_1 > \frac{8(n+2)}{3} > 0 \). Therefore \( M \) must have only one end with infinite volume. \( \square \)

5 Maximal first eigenvalue

In this section, we will consider the case when \( \lambda_1(M) \) is of maximal value.

Proof of Theorem 0.6: According to Theorem 0.5, we know that \( M \) has exactly one nonparabolic end. Suppose that \( M \) has more than one end. Then there must exist at least an end with finite volume. We divide the rest of the proof into several parts. The first part follows exactly as that in the proof of the corresponding theorem in the Kähler case (Theorem 3.1) in [LW5]. For completeness sake, we will give a quick outline of it.

Part 1. Assume that \( E_1 \) is such an end with finite volume given by \( M \setminus B_p(1) \). Then we can choose a ray \( \eta : [0, +\infty) \) such that \( \eta(0) = p \) and \( \eta(1, +\infty) \subset E_1 \). The Busemann function corresponding to \( \gamma \) is defined by
\[ \beta(x) = \lim_{t \to +\infty} [t - d(x, \eta(t))]. \]
The Laplacian comparison theorem, Theorem 2.1, asserts that
\[ \Delta \beta \geq -2(2n + 1) \]
in the sense of distribution. We define the function \( f = \exp((2n + 1)\beta) \), and using the fact that \( |\nabla \beta| = 1 \) almost everywhere, we have
\[ \Delta f = (2n + 1) \exp((2n + 1)\beta) \Delta \beta + (2n + 1)^2 \geq -(2n + 1)^2 f. \]

Similar to the proof of above theorem, we conclude that for any compactly supported function \( \phi \),

\[
0 \leq \int_M (\Delta f + (2n + 1)^2 f) f \phi^2 
\leq \int_M f^2 |\nabla \phi|^2.
\]

By choosing the function \( \phi \) to be

\[
\phi = \begin{cases} 
1, & \text{on } B_p(R); \\
\frac{2R - r(x)}{R}, & \text{on } B_p(2R) \setminus B_p(R); \\
0, & \text{on } M \setminus B_p(2R);
\end{cases}
\]

we obtain

\[
\int_{M \cap E_1} f^2 |\nabla \phi|^2 
\leq \frac{1}{R^2} \int_{(B_p(2R) \setminus B_p(R)) \cap E_1} f^2 
\leq \frac{1}{R^2} \sum_{i=1}^{[R]} \int_{(B_p(R+i) \setminus B_p(R+i-1)) \cap E_1} f^2 
\leq \frac{C}{R^2} \sum_{i=1}^{[R]} (V_{E_1}(R+i) - V_{E_1}(R+i-1)) \exp(2(2n + 1)(R+i))
\]

where \( V_{E_1}(R+i) \) denotes the volume of the set \( E_1 \cap B_p(R+i) \). On the other hand, the volume estimate in Theorem 1.4 of [LW1] implies that

\[ V_{E_1}(\infty) - V_{E_1}(R) \leq C \exp(-2(2n + 1)R), \]

hence

\[
V_{E_1}(R+i) - V_{E_1}(R+i-1) 
= V_{E_1}(\infty) - V_{E_1}(R+i-1) - (V_{E_1}(\infty) - V_{E_1}(R+i)) 
\leq C \exp(-2(2n + 1)(R+i)).
\]
Therefore, we conclude that
\[ \int_{M \cap E_1} f^2 |\nabla \phi|^2 \leq \frac{C}{R}. \]

Let us now denote \( E_2 = M \setminus (B_p(1) \cup E_1) \) to be the other end of \( M \). When \( x \in E_2 \), following the argument in Theorem 3.1 of [LW4], we have
\[ \beta(x) \leq -d(p,x) + 2. \]

Therefore
\[
\int_{E_2} f^2 |\nabla \phi|^2 \leq \frac{1}{R^2} \int_{(B_p(2R) \setminus B_p(R)) \cap E_2} f^2 = \frac{C}{R^2} \int_{(B_p(2R) \setminus B_p(R)) \cap E_2} \exp(-2(2n+1)(r - 2)) \leq \frac{C}{R}.
\]

Letting \( R \to +\infty \), we conclude that
\[ \Delta f + (2n + 1)^2 f = 0, \tag{5.1} \]
and all inequalities used are indeed equalities and \( f \) is smooth by regularity of the equation (5.1). Moreover, \( |\nabla \beta| = 1 \), and
\[ \Delta \beta = -2(2n+1). \]

This implies that \( M \) must be diffeomorphic to \( \mathbb{R} \times N \), where \( N \) is given by the level set of \( \beta \). We choose an orthonormal basis \( \{e_i\}_{i=1}^{4n} \) as follows
\[
\{e_1, e_2, \ldots, e_n, Je_1, Je_2, \ldots, Je_n, Ke_1, Ke_2, \ldots, Ke_n\}
\]
with \( e_1 = \nabla \beta \). Applying the Bochner formula to \( \beta \), we get
\[
0 = \frac{1}{2} \Delta|\nabla \beta|^2 = \sum_{i,j=1}^{4n} \beta_{ij}^2 + \text{Ric}_M(\nabla \beta, \nabla \beta) + \sum_{i=1}^{4n} \beta_i(\Delta \beta)_i = \sum_{i,j=1}^{4n} \beta_{ij}^2 - 4(n + 2).
\]

28
By the comparison theorem, we have,

$$
\sum_{i=0}^{3} \beta_{(in+1)(in+1)} = -6.
$$

Hence

$$
(\beta_{\alpha\beta}) = \begin{pmatrix} D_1 & D_2 \\ D_2 & D_2 \end{pmatrix},
$$

where $D_1$ and $D_2$ are $n \times n$ matrices defined by

$$
D_1 = \begin{pmatrix} 0 & -1 & \ldots & -1 \\
-1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & -1 \\
-1 & \ldots & -1 & 0 \end{pmatrix},
$$

and

$$
D_2 = \begin{pmatrix} -2 & -1 & \ldots & -1 \\
-1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & -1 \\
-1 & \ldots & -1 & -2 \end{pmatrix}.
$$

**Part 2.** For a fix level set $N_0$ of $\beta$, we consider $(-\varepsilon, \varepsilon) \times N_0 \subset M$. Note that $\{e_i\}$ is an orthonormal basis of $TM$ such that $e_1$ is the normal vector to $N_0$ and $\{e_\alpha\}$, for $2 \leq \alpha \leq 4n$, are the tangent vectors of $N_0$. We shall compute the sectional curvature

$$
K(e_1, e_\alpha) = \langle R(e_1, e_\alpha)e_\alpha, e_1 \rangle.
$$

Since $\nabla_{e_\alpha}e_1 = 0$ implies that the integral curves of $e_1$ are geodesics. Let $(h_{\alpha\gamma})$ be the second fundamental form of the level set of $\nabla\beta$. Then

$$
\begin{align*}
    h_{\alpha\gamma} &= \langle \nabla_{e_\alpha}e_\gamma, e_1 \rangle \\
    &= \langle \nabla_{e_\alpha}e_\gamma, \nabla\beta \rangle \\
    &= -\beta_{\alpha\gamma}
\end{align*}
$$

and

$$
\nabla_{e_\alpha}e_1 = -\sum_{\gamma=2}^{4n} h_{\alpha\gamma} e_\gamma. \quad (5.2)
$$
By \((4.13)\) in the proof of Theorem 0.5 we have
\[
\langle R(e_1, e_\alpha) e_1, e_\alpha \rangle = -e_1 h_{\alpha\alpha} + 2 \sum_{\gamma=2}^{4n} h_{\alpha\gamma} \langle \nabla e_1 e_\gamma, e_\beta \rangle + \sum_{\gamma=2}^{4n} h_{\alpha\gamma}^2.
\]

Since \((h_{\alpha\gamma})\) are constant and diagonal, then
\[
K(e_1, e_\alpha) = -h_{\alpha\alpha}^2.
\]

In particular, we have
\[
K(e_1, e_\alpha) = \begin{cases} -4 & \text{when } \alpha = in + 1, i = 1, 2, 3 \\ -1 & \text{otherwise.} \end{cases}
\]

On the other hand, we also have
\[
K(e_{n+1}, e_{2n+1}) + K(e_{n+1}, e_{3n+1}) = -12 - K(e_1, e_{a+1}) = -8
\]
\[
K(e_{n+1}, e_{2n+1}) + K(e_{3n+1}, e_{2n+1}) = -8
\]
\[
K(e_{3n+1}, e_{2n+1}) + K(e_{n+1}, e_{3n+1}) = -8,
\]

hence
\[
K(e_{n+1}, e_{2n+1}) = K(e_{n+1}, e_{3n+1}) = K(e_{2n+1}, e_{3n+1}) = -4.
\]

Since for \(\alpha = 2, 3, \cdots, n,\)
\[
K(Ie_1, e_\alpha) = -\langle R(Ie_1, e_\alpha) Ie_1, e_\alpha \rangle
= -\langle IR(Ie_1, e_\alpha) Ie_1, Ie_\alpha \rangle
= \langle R(Ie_1, e_\alpha) e_1, Ie_\alpha \rangle
= \langle R(e_1, Ie_\alpha) Ie_1, e_\alpha \rangle
= K(e_1, Ie_\alpha)
= -1,
\]

and \(K(Je_1, e_\alpha) = K(Ke_1, e_\alpha) = -1,\) we have
\[
K(e_{in+1}, e_\alpha) = -1,
\]
for all \(i = 0, 1, 2, 3\) and \(\alpha \neq 1, n + 1, 2n + 1, 3n + 1.\)
Let $K^N(e_\alpha, e_\gamma)$ denote the sectional curvature of the level set with induced metric. By Gaussian equation,

$$K^N(e_\alpha, e_\gamma) - K(e_\alpha, e_\gamma) = h_{\alpha\alpha} h_{\gamma\gamma},$$

it is straightforward to obtain

$$K^N(e_{n+1}, e_{2n+1}) = K^N(e_{n+1}, e_{3n+1}) = K^N(e_{2n+1}, e_{3n+1}) = 0,$$

and

$$K^N(e_{in+1}, e_{\alpha}) = 1,$$  \hspace{1cm} (5.3)

for all $i = 1, 2, 3$ and $\alpha \neq 1, n+1, 2n+1, 3n+1$.

**Part 3.** There is a natural map $\varphi_t$ between the level sets $N_0$ and $N_t$ given by the gradient flow of $\beta$. Since the integral curves are geodesics, $d\varphi_t(X)$ are Jacobi fields along corresponding curves. Let $(N, g_0) = N_0$ with the induced metric. We can consider $\varphi$ as a flow on $N$. We claim that

$$d\varphi_t|_{V_1} = e^{2t} \text{id}$$

and

$$d\varphi_t|_{V_2} = e^{t} \text{id},$$

where $TN = V_1 \oplus V_2$, $V_1 = \text{span}\{Ie_1, Je_1, Ke_1\}$ and $V_2 = V_1^\perp$. Indeed for any point $q \in N_0$, denote $e_1(t) = \nabla_\beta(\varphi(t))$ and $\{e_\alpha(t)\}_{\alpha=2}^{4n}$ to be the parallel transport of the orthonormal base $\{e_\alpha\}_{\alpha=2}^{4n}$ of $N_0$ at $q$ along $\varphi_t(q)$. Since both $V_1$ and $V_2$ are $\varphi$-invariant, we have, in particular,

$$\langle \nabla_{e_1(t)} e_\alpha, e_\gamma \rangle = 0,$$  \hspace{1cm} (5.4)

when $\alpha \in \{n + 1, 2n + 1, 3n + 1\}$, and $\gamma \notin \{n + 1, 2n + 1, 3n + 1\}$.
Now we can compute $R_{1\alpha\gamma}$. Then

\[
\langle R(e_1, \varepsilon_\alpha)e_1, \varepsilon_\gamma \rangle \\
= \langle \nabla_{e_1} \nabla_{\varepsilon_\alpha} e_1 - \nabla_{\varepsilon_\alpha} \nabla_{e_1} e_1 - \nabla_{[e_1, \varepsilon_\alpha]} e_1, \varepsilon_\gamma \rangle \\
= \langle \nabla_{e_1} \nabla_{\varepsilon_\alpha} e_1, \varepsilon_\alpha \rangle - \langle \nabla_{[e_1, \varepsilon_\alpha]} e_1, \varepsilon_\gamma \rangle \\
= \langle \nabla_{e_1} \nabla_{\varepsilon_\alpha} e_1, \varepsilon_\gamma \rangle - \langle \nabla_{[e_1, \varepsilon_\alpha]} - \nabla_{\varepsilon_\alpha} e_1, \varepsilon_\gamma \rangle \\
= \langle \nabla_{e_1} \nabla_{\varepsilon_\alpha} e_1, \varepsilon_\gamma \rangle - \sum_{\tau=2}^{4n} \langle \nabla_{e_1} \varepsilon_\alpha, \varepsilon_\tau \rangle \langle \nabla_{\varepsilon_\tau} e_1, \varepsilon_\gamma \rangle \\
+ \sum_{\tau=2}^{4n} \langle \nabla_{\varepsilon_\alpha} e_1, \varepsilon_\tau \rangle \langle \nabla_{\varepsilon_\tau} e_1, \varepsilon_\gamma \rangle \\
= - \sum_{\tau=2}^{4n} \langle \nabla_{e_1} (h_{\alpha\tau} \varepsilon_\tau), \varepsilon_\gamma \rangle + \sum_{\tau=2}^{4n} h_{\gamma\tau} \langle \nabla_{e_1} \varepsilon_\alpha, \varepsilon_\tau \rangle + \sum_{\tau=2}^{4n} h_{\alpha\tau} h_{\gamma\tau} \\
= - e_1 h_{\alpha\gamma} - \sum_{\tau=2}^{4n} h_{\alpha\tau} \langle \nabla_{e_1} \varepsilon_\tau, \varepsilon_\gamma \rangle \\
+ \sum_{\tau=2}^{4n} h_{\gamma\tau} \langle \nabla_{e_1} \varepsilon_\alpha, \varepsilon_\tau \rangle + \sum_{\tau=2}^{4n} h_{\alpha\tau} h_{\gamma\tau}. \quad (5.5)
\]

We see that $(h_{\alpha\gamma})$ is diagonal and

\[
h_{\alpha\alpha} = \begin{cases} 
2, & \text{when } \alpha = n + 1, 2n + 1, 3n + 1; \\
1, & \text{otherwise.}
\end{cases}
\]

Therefore, when $\alpha \neq \gamma$, \n
\[
R_{1\alpha\gamma} = -h_{\alpha\alpha} \langle \nabla_{e_1} \varepsilon_\alpha, \varepsilon_\gamma \rangle + h_{\gamma\gamma} \langle \nabla_{e_1} \varepsilon_\alpha, \varepsilon_\gamma \rangle \\
= (h_{\gamma\gamma} - h_{\alpha\alpha}) \langle \nabla_{e_1} \varepsilon_\alpha, \varepsilon_\gamma \rangle.
\]

Since $h_{\alpha\alpha} = h_{\gamma\gamma}$ when $\alpha, \gamma \in \{n + 1, 2n + 1, 3n + 1\}$ and $\alpha, \gamma \notin \{n + 1, 2n + 1, 3n + 1\}$, using (5.4), we have

\[
R_{1\alpha\gamma} = 0, \text{ for all } \alpha \neq \gamma.
\]

Define

\[
J_\alpha(t) = \begin{cases} 
e^{-2t} \varepsilon_\alpha, & \text{when } \alpha \in \{n + 1, 2n + 1, 3n + 1\}; \\
e^{-t} \varepsilon_\alpha, & \text{when } \alpha \notin \{n + 1, 2n + 1, 3n + 1\}.
\end{cases}
\]

32
Since 
\[ \nabla_{\alpha} d\varphi_t(e_\alpha)|_{t=0} = [e_1, e_\alpha] = -\nabla e_\alpha e_1, \]
then we see that \( J_\alpha \) satisfies the Jacobi equation and initial conditions \( J_\alpha(0) = e_\alpha \) and \( J'_\alpha(0) = e_\alpha = \nabla_{\alpha} d\varphi_t(e_\alpha)|_{t=0} \). By the uniqueness theorem for the Jacobi equations, we have \( d\varphi_t(e_\alpha) = J_\alpha \). The claim is proved.

**Part 4.** We have now a family of metrics on \( N \) written as
\[
ds_t^2 = e^{4t} \sum_{i=1}^{3} \omega_{i+1}^2 + e^{2t} \sum_{i=0}^{3} \sum_{\alpha=2}^{n} \omega_{i+\alpha}^2,
\]
and the metric of \( M \) can rewritten as
\[
ds^2 = dt^2 + e^{4t} \sum_{p=2}^{4} \omega_p^2 + e^{2t} \sum_{\alpha=5}^{4n} \omega_\alpha^2,
\]
where \( \{\omega_2, \omega_3, \omega_4, \ldots, \omega_{4n}\} \) is the dual coframe to \( \{e_2, e_3, e_4, \ldots, e_{4n}\} \) at \( N_0 \).
We also choose that \( I e_{4s-3} = e_{4s-2} \), \( J e_{4s-3} = e_{4s-1} \), and \( K e_{4s-3} = e_{4s} \) for \( s = 1, \ldots, n \), with \( e_1 = \frac{\partial}{\partial t} \). In particular, the second fundamental form on \( N_t \) must be a diagonal matrix when written in terms of the basis \( \{e_i\}_{i=2}^{4n} \) with eigenvalues given by
\[
(\langle \nabla e_i, e_j, e_1 \rangle) = \begin{pmatrix} 2I_3 & 0 \\ 0 & I_{4(n-1)} \end{pmatrix}, \quad (5.7)
\]
where \( I_k \) denotes the \( k \times k \) identity matrix. Also, the sectional curvatures of the sections containing \( e_1 \) are given by
\[
\mathcal{K}(e_1, e_p) = -4 \quad \text{for} \quad 2 \leq p \leq 4
\]
and
\[
\mathcal{K}(e_1, e_\alpha) = -1 \quad \text{for} \quad 5 \leq \alpha \leq 4n.
\]
The Gauss curvature equation also asserts that
\[
R_{ijkl} = \bar{R}_{ijkl} + h_{il}h_{kj} - h_{kj}h_{lj},
\]
where $\tilde{R}_{ijkl}$ is the curvature tensor on $N_t$. In particular,

$$R_{ijkl} = \begin{cases}
\tilde{R}_{ijkl} + \delta_i \delta_k \delta_j \delta_l & \text{if } 5 \leq i, j, k, l \leq 4n \\
\tilde{R}_{ijkl} + 4 \delta_i \delta_k \delta_j \delta_l & \text{if } 2 \leq i, j, k, l \leq 4 \\
\tilde{R}_{ijkl} + 2 & \text{if } 2 \leq i = l \leq 4 \text{ and } 5 \leq k = j \leq 4n \\
\tilde{R}_{ijkl} + 2 & \text{if } 2 \leq k = j \leq 4 \text{ and } 5 \leq i = l \leq 4n \\
\tilde{R}_{ijkl} - 2 & \text{if } 2 \leq j = l \leq 4 \text{ and } 5 \leq i = k \leq 4n \\
\tilde{R}_{ijkl} - 2 & \text{if } 2 \leq k = i \leq 4 \text{ and } 5 \leq j = l \leq 4n \\
\tilde{R}_{ijkl} & \text{otherwise.}
\end{cases} \tag{5.8}
$$

We will now use (5.6) to compute the curvature tensor of $M$ and hence $N_0$. Using the orthonormal coframe

$$\eta_1 = \omega_1 = dt,$$

$$\eta_p = e^{2t} \omega_p$$

$$\eta_\alpha = e^t \omega_\alpha$$

for $2 \leq p \leq 4$ and $5 \leq \alpha \leq 4n$, we obtain the first structural equations

$$d\eta_1 = 0, \tag{5.9}$$

$$d\eta_p = 2e^{2t} \omega_1 \wedge \omega_p + e^{2t} \sum_{q=2}^{4n} \omega_{pq} \wedge \omega_q + e^{2t} \sum_{\alpha=5}^{4n} \omega_{p\alpha} \wedge \omega_\alpha = -2\eta_p \wedge \eta_1 + \sum_{q=2}^{4} \omega_{pq} \wedge \eta_q + e^t \sum_{\alpha=5}^{4n} \omega_{p\alpha} \wedge \eta_\alpha, \tag{5.10}$$

and

$$d\eta_\alpha = e^t \omega_1 \wedge \omega_\alpha + e^t \sum_{p=2}^{4} \omega_{\alpha p} \wedge \omega_p + e^t \sum_{\beta=5}^{4n} \omega_{\alpha\beta} \wedge \omega_\beta = -\eta_\alpha \wedge \eta_1 + e^{-t} \sum_{p=2}^{4} \omega_{\alpha p} \wedge \eta_p + \sum_{\beta=5}^{4n} \omega_{\alpha\beta} \wedge \eta_\beta, \tag{5.11}$$
where \( \omega_{ij} \) are the connection forms of \( N_0 \). In the above and all subsequent computations, we will adopt the convention that \( 5 \leq \alpha, \beta \leq 4n, 2 \leq i, j \leq 4n, 2 \leq o, p, q, r \leq 4, 2 \leq s, t \leq n, \) and \( 1 \leq A, B \leq 4n \).

Note that using the endomorphism \( I \) and the fact that \( \nabla I = cJ - bK \), we have

\[
\omega_{ij}(X) = \langle \bar{\nabla} X e_j, e_i \rangle = \langle I \nabla X e_j, I e_i \rangle = \langle \nabla X I e_j, I e_j \rangle + c(X) \langle e_j, Ke_i \rangle + b(X) \langle e_j, Je_i \rangle
\]

for any tangent vector \( X \) to \( N_0 \), where \( \bar{\nabla} \) denotes the connection on \( N_0 \). Hence we conclude that

\[
\omega_{ij} = \omega_{I_i I_j} + c \langle e_j, Ke_i \rangle + b \langle e_j, Je_i \rangle, \tag{5.12}
\]

where \( I_i \) denotes the index corresponding to \( I e_i = e_{I_i} \). Similarly, we have

\[
\omega_{ij} = \omega_{J_i J_j} + c \langle e_j, Ke_i \rangle + a \langle e_j, I e_i \rangle,
\]

and

\[
\omega_{ij} = \omega_{K_i K_j} + b \langle e_j, Je_i \rangle + a \langle e_j, I e_i \rangle.
\]

Together with (5.7), we conclude that

\[
\omega_{2(4s-1)}(e_{4s}) = -1 = -\omega_{2(4s)}(e_{4s-1}),
\]

\[
\omega_{2(4s-3)}(e_{4s-2}) = -1 = -\omega_{2(4s-2)}(e_{4s-3}),
\]

for all \( 2 \leq s \leq n \), and

\[
\omega_{2\alpha}(e_{\beta}) = 0 \quad \text{otherwise.}
\]

Similarly,

\[
\omega_{2\alpha}(e_p) = \langle \nabla e_p e_{\alpha}, e_2 \rangle = -\langle \nabla e_p I e_{\alpha}, e_1 \rangle = 0.
\]
These identities imply that
\[
\begin{align*}
\omega_{2(4s-3)} &= -\omega_{4s-2}, \\
\omega_{2(4s-2)} &= \omega_{4s-3}, \\
\omega_{2(4s-1)} &= -\omega_{4s}, \\
\omega_{2(4s)} &= \omega_{4s-1}.
\end{align*}
\] (5.13)

A similar calculation using the endomorphisms \(J\) and \(K\) yield
\[
\begin{align*}
\omega_{3(4s-3)} &= -\omega_{4s-1}, \\
\omega_{3(4s-2)} &= \omega_{4s}, \\
\omega_{3(4s-1)} &= \omega_{4s-3}, \\
\omega_{3(4s)} &= -\omega_{4s-2},
\end{align*}
\] (5.14)

and
\[
\begin{align*}
\omega_{4(4s-3)} &= -\omega_{4s}, \\
\omega_{4(4s-2)} &= -\omega_{4s-1}, \\
\omega_{4(4s-1)} &= \omega_{4s-2}, \\
\omega_{4(4s)} &= \omega_{4s-3}.
\end{align*}
\] (5.15)

We claim that the connection forms are given by
\[
\begin{align*}
\eta_{1p} &= -\eta_{p1} \\
&= 2\eta_p \quad \text{for} \quad 2 \leq p \leq 4,
\end{align*}
\] (5.16)

\[
\begin{align*}
\eta_{1\alpha} &= -\eta_{\alpha1} \\
&= \eta_\alpha \quad \text{for} \quad 5 \leq \alpha \leq 4n,
\end{align*}
\] (5.17)

\[
\begin{align*}
\eta_{pq} &= -\eta_{qp} = \omega_{pq},
\end{align*}
\] (5.18)

\[
\begin{align*}
\eta_{p\alpha} &= -\eta_{\alpha p} = c^t \omega_{p\alpha},
\end{align*}
\] (5.19)
\[ \eta_{(4s-1)\beta} = -\eta_{\beta(4s-1)} = \begin{cases} 
\omega_{(4s-1)\beta} - (1-e^{-2t}) \eta_2 & \text{if } \beta = 4s-1 \\\n\omega_{(4s-1)\beta} + (1-e^{-2t}) \eta_3 & \text{if } \beta = 4s-2 \\\n\omega_{(4s-1)\beta} - (1-e^{-2t}) \eta_4 & \text{if } \beta = 4s-3 \\\n\omega_{(4s-1)\beta} & \text{if } \beta \neq 4s-1, 4s-2, \text{or } 4s-3, \end{cases} \tag{5.21} \]

\[ \eta_{(4s-2)\beta} = -\eta_{\beta(4s-2)} = \begin{cases} 
\omega_{(4s-2)\beta} - (1-e^{-2t}) \eta_2 & \text{if } \beta = 4s-1 \\\n\omega_{(4s-2)\beta} + (1-e^{-2t}) \eta_3 & \text{if } \beta = 4s-2 \\\n\omega_{(4s-2)\beta} - (1-e^{-2t}) \eta_4 & \text{if } \beta = 4s-3 \\\n\omega_{(4s-2)\beta} & \text{if } \beta \neq 4s-1, 4s-2, \text{or } 4s-3, \end{cases} \tag{5.22} \]

\[ \eta_{(4s-3)\beta} = -\eta_{\beta(4s-3)} = \begin{cases} 
\omega_{(4s-3)\beta} + (1-e^{-2t}) \eta_3 & \text{if } \beta = 4s \\\n\omega_{(4s-3)\beta} + (1-e^{-2t}) \eta_4 & \text{if } \beta = 4s-1 \\\n\omega_{(4s-3)\beta} + (1-e^{-2t}) \eta_2 & \text{if } \beta = 4s-2 \\\n\omega_{(4s-3)\beta} & \text{if } \beta \neq 4s, 4s-1, \text{or } 4s-2. \end{cases} \tag{5.23} \]

Indeed, if we substitute (5.16–5.23) into the first structural equations

\[ d\eta_A = \eta_{A1} \wedge \eta_1 + \sum_{q=2}^{4} \eta_{Aq} \wedge \eta_q + \sum_{\beta=5}^{4n} \eta_{A\beta} \wedge \eta_\beta \]

we obtain (5.9), (5.10), and (5.11).
To compute the curvature, we consider the second structural equations. In particular,

\[ d\eta_p - \eta_1 \wedge \eta_{qp} - \eta_{1\alpha} \wedge \eta_{\alpha p} \]
\[ = 2d\eta_p - 2\eta_q \wedge \eta_{qp} - \eta_\alpha \wedge \eta_{\alpha p} \]
\[ = -4\eta_p \wedge \eta_1 + \eta_\alpha \wedge \eta_{\alpha p} \]
\[ = -4\eta_p \wedge \eta_1 + \epsilon^r \omega_{p\alpha} \wedge \eta_{\alpha} . \]

Hence using (5.13 - 5.15), we have

\[ R_{1p1p} = -4, \]
\[ R_{12(4s-1)(4s)} = -2 = -R_{12(4s)(4s-1)} , \]
\[ R_{12(4s-3)(4s-2)} = -2 = -R_{12(4s-2)(4s-3)} , \]
\[ R_{13(4s)(4s-2)} = -2 = -R_{13(4s-2)(4s)} , \]
\[ R_{13(4s-1)(4s-3)} = 2 = -R_{13(4s-3)(4s-1)} , \]
\[ R_{14(4s)(4s-3)} = 2 = -R_{14(4s-3)(4s)} , \]
\[ R_{14(4s-1)(4s-2)} = 2 = -R_{14(4s-2)(4s-1)} , \]

and

\[ R_{1pAB} = 0, \text{ otherwise.} \]

Also,

\[ d\eta_1 - \eta_1 \wedge \eta_{q\alpha} - \eta_{1\beta} \wedge \eta_{\beta p} \]
\[ = d\eta_1 - 2\eta_q \wedge \eta_{q\alpha} - \eta_\beta \wedge \eta_{\beta p} \]
\[ = -\eta_\alpha \wedge \eta_1 + \epsilon^r \omega_{q\alpha} \wedge \eta_q . \]

hence

\[ R_{11\alpha\alpha} = -1, \]
\[ R_{11(4s)(4s-1)/2} = -1 = -R_{11(4s-1)(4s)/2}, \]
\[ R_{11(4s)(4s-2)/3} = 1 = -R_{11(4s-2)(4s)/3}, \]
\[ R_{11(4s)(4s-3)/4} = -1 = -R_{11(4s-3)(4s)/4}, \]
\[ R_{11(4s-1)(4s-2)/3} = -1 = -R_{11(4s-3)(4s-1)/3}, \]
\[ R_{11(4s-1)(4s-2)/4} = -1 = -R_{11(4s-2)(4s-1)/4}, \]
\[ R_{11(4s-2)(4s-3)/2} = -1 = -R_{11(4s-3)(4s-2)/2}, \]

38
and
\[ R_{1\alpha AB} = 0 \quad \text{otherwise.} \]

Similarly,
\[
d\eta_{pq} - \eta_{p1} \wedge \eta_{1q} - \eta_{pr} \wedge \eta_{rq} - \eta_{p\beta} \wedge \eta_{\beta q} = d\omega_{pq} + 4\eta_{p} \wedge \eta_{q} - \omega_{pr} \wedge \omega_{rq} - e^{2t} \omega_{p\beta} \wedge \omega_{\beta q} = \bar{\Omega}_{pq} + (1 - e^{2t}) \omega_{p\beta} \wedge \omega_{\beta q} + 4\eta_{p} \wedge \eta_{q},
\]

where
\[
\bar{\Omega}_{pq} = \frac{1}{2} \bar{R}_{pqij} \omega_{j} \wedge \omega_{i}
\]
is the curvature form of \( N_{0} \). In particular, this implies that
\[
R_{pqro} = \begin{cases} 
-4 + e^{-4t} \bar{R}_{pq} & \text{if } r = p \text{ and } o = q \\
4 + e^{-4t} \bar{R}_{qq} & \text{if } r = q \text{ and } o = p \\
e^{-4t} \bar{R}_{pq} & \text{otherwise,}
\end{cases}
\]

(5.24)

\[
R_{23(4s)(4s-3)} = e^{-2t} \bar{R}_{23(4s)(4s-3)} - 2(e^{-2t} - 1),
\]
\[
R_{23(4s-1)(4s-2)} = e^{-2t} \bar{R}_{23(4s-1)(4s-2)} - 2(e^{-2t} - 1),
\]
\[
R_{24(4s)(4s-2)} = e^{-2t} \bar{R}_{24(4s)(4s-2)} - 2(e^{-2t} - 1),
\]
\[
R_{24(4s-1)(4s-3)} = e^{-2t} \bar{R}_{24(4s-1)(4s-3)} + 2(e^{-2t} - 1),
\]
\[
R_{34(4s)(4s-1)} = e^{-2t} \bar{R}_{34(4s)(4s-1)} - 2(e^{-2t} - 1),
\]
\[
R_{34(4s-2)(4s-3)} = e^{-2t} \bar{R}_{34(4s-2)(4s-3)} - 2(e^{-2t} - 1),
\]

and
\[
R_{pq\alpha\beta} = e^{-2t} \bar{R}_{pq\alpha\beta}, \quad \text{otherwise.}
\]

We now continue with our curvature computation and consider
\[
d\eta_{p\alpha} - \eta_{p1} \wedge \eta_{1\alpha} - \eta_{pq} \wedge \eta_{qa} - \eta_{p\beta} \wedge \eta_{\beta\alpha} = d(e^{t} \omega_{p\alpha}) + 2\eta_{p} \wedge \eta_{\alpha} - \omega_{pq} \wedge e^{t} \omega_{qa} - e^{t} \omega_{p\beta} \wedge \eta_{\beta\alpha} = e^{t} \eta_{1} \wedge \omega_{p\alpha} + \frac{1}{2} e^{t} \bar{R}_{pqij} \omega_{j} \wedge \omega_{i} + 2\eta_{p} \wedge \eta_{\alpha} + e^{t} \omega_{p\beta} \wedge (\omega_{\beta\alpha} - \eta_{\beta\alpha}),
\]
where \( \bar{R}_{p_0ij} \) is the curvature tensor of \( N_0 \). Using \((5.13 - 5.15)\) and \((5.20 - 5.23)\), we have

\[
\frac{1}{2} R_{2(4s)AB} \eta_B \wedge \eta_A
= \eta_1 \wedge \eta_{(4s-1)} + \frac{1}{2} e^t \bar{R}_{2(4s)ij} \omega_j \wedge \omega_i - 2 \eta_{(4s)} \wedge \eta_2 + (1 - e^{-2t}) \eta_{(4s)} \wedge \eta_2
+ (1 - e^{-2t}) \eta_{(4s-3)} \wedge \eta_3 + (1 - e^{-2t}) \eta_{(4s-2)} \wedge \eta_4
\]

\[
= \eta_1 \wedge \eta_{(4s-1)} + \frac{1}{2} e^t \bar{R}_{2(4s)ij} \omega_j \wedge \omega_i - (1 + e^{-2t}) \eta_{(4s)} \wedge \eta_2
+ (1 - e^{-2t}) \eta_{(4s-3)} \wedge \eta_3 + (1 - e^{-2t}) \eta_{(4s-2)} \wedge \eta_4.
\]

\[
\frac{1}{2} R_{2(4s-1)AB} \eta_B \wedge \eta_A
= -\eta_1 \wedge \eta_{(4s)} + \frac{1}{2} e^t \bar{R}_{2(4s-1)ij} \omega_j \wedge \omega_i - (1 + e^{-2t}) \eta_{(4s-2)} \wedge \eta_2
+ (1 - e^{-2t}) \eta_{(4s-3)} \wedge \eta_3 - (1 - e^{-2t}) \eta_{(4s-3)} \wedge \eta_4.
\]

\[
\frac{1}{2} R_{2(4s-2)AB} \eta_B \wedge \eta_A
= \eta_1 \wedge \eta_{4s-3} + \frac{1}{2} e^t \bar{R}_{2(4s-2)ij} \omega_j \wedge \omega_i - (1 + e^{-2t}) \eta_{(4s-2)} \wedge \eta_2
+ (1 - e^{-2t}) \eta_{(4s-1)} \wedge \eta_3 - (1 - e^{-2t}) \eta_{(4s)} \wedge \eta_4.
\]

\[
\frac{1}{2} R_{2(4s-3)AB} \eta_B \wedge \eta_A
= -\eta_1 \wedge \eta_{4s-2} + \frac{1}{2} e^t \bar{R}_{2(4s-3)ij} \omega_j \wedge \omega_i - (1 + e^{-2t}) \eta_{(4s-3)} \wedge \eta_2
- (1 - e^{-2t}) \eta_{(4s)} \wedge \eta_3 + (1 - e^{-2t}) \eta_{(4s-1)} \wedge \eta_4.
\]

Similar formulas for the curvature tensors of the form \( R_{3\alpha AB} \) and \( R_{4\alpha AB} \). Continuing with our computation of the second structural equations using
By (5.13 – 5.15), we have

\[
d d\eta_{(4s-1)(4s)} - \eta_{(4s-1)1} \land \eta_{1(4s)} - \eta_{(4s-1)q} \land \eta_{q(4s)} - \eta_{(4s-1)b} \land \eta_{b(4s)} \\
= d\omega_{(4s-1)(4s)} + 2e^{-2t} \eta_1 \land \eta_2 + (1 - e^{-2t}) d\eta_2 \\
+ \eta_{(4s-1)} \land \eta_{(4s)} - e^{2t} \omega_{(4s-1)1} \land \omega_{1(4s)} \\
- (\omega_{(4s-1)(4s-2)} - (1 - e^{-2t}) \eta_4) \land (\omega_{(4s-2)(4s)} - (1 - e^{-2t}) \eta_3) \\
- (\omega_{(4s-3)(4s-3)} - (1 - e^{-2t}) \eta_3) \land (\omega_{(4s-3)(4s)} + (1 - e^{-2t}) \eta_4) \\
= \frac{1}{2} \bar{R}_{(4s-1)(4s)} \omega_j \land \omega_i + (1 - e^{-2t}) \omega_{(4s-1)q} \land \omega_{q(4s)} + 2\eta_1 \land \eta_2 \\
+ (1 - e^{-2t}) \omega_{2q} \land \eta_q + e^t (1 - e^{-2t}) \omega_{2\beta} \land \eta_{(4s-1)} \\
+ (1 - e^{-2t}) \omega_{(4s-1)(4s-2)} \land \eta_3 + (1 - e^{-2t}) \eta_4 \land \omega_{(4s-2)(4s)} \\
+ (1 - e^{-2t}) \eta_3 \land \omega_{(4s-3)(4s)} - (1 - e^{-2t}) \omega_{(4s-1)(4s-3)} \land \eta_4 \\
+ 2(1 - e^{-2t})^2 \eta_3 \land \eta_4 \\
= \frac{1}{2} \bar{R}_{(4s-1)(4s)} \omega_j \land \omega_i + (2 - e^{-2t}) \eta_{(4s-1)} \land \eta_{(4s)} - 2(1 - e^{-2t}) \eta_{(4s-3)} \land \eta_{(4s-2)} \\
+ 2\eta_1 \land \eta_2 + (1 - e^{-2t}) \omega_{2q} \land \eta_q + 2(1 - e^{-2t}) \eta_{(4s-3)} \land \eta_{(4s-2)} \\
+ 2(1 - e^{-2t}) \eta_{(4s-1)} \land \eta_{(4s-2)} + (1 - e^{-2t}) (\omega_{(4s-1)(4s-2)} - \omega_{(4s-3)(4s)}) \land \eta_3 \\
- (1 - e^{-2t}) (\omega_{(4s-2)(4s)} + \omega_{(4s-1)(4s-3)}) \land \eta_4 + 2(1 - e^{-2t})^2 \eta_3 \land \eta_4. \\
(5.25)
\]

Note that (5.22) asserts that

\[
(1 - e^{-2t}) \omega_{2q} \land \eta_q = (1 - e^{-2t}) (-\omega_{14} \land \eta_3 + c \land \eta_4 + \omega_{13} \land \eta_4 - b \land \eta_4) \\
= (1 - e^{-2t}) (4e^{-2t} \eta_3 \land \eta_4 + c \land \eta_3 - b\eta_4),
\]

\[
(1 - e^{-2t}) (\omega_{(4s-1)(4s-2)} - \omega_{(4s-3)(4s)}) \land \eta_3 \\
= -(1 - e^{-2t}) c \land \eta_3,
\]

and

\[
-(1 - e^{-2t}) (\omega_{(4s-2)(4s)} + \omega_{(4s-1)(4s-3)}) \land \eta_4 \\
= (1 - e^{-2t}) b \land \eta_4.
\]
Hence substituting into (5.25), we obtain

\[
\frac{1}{2} R_{(4s-1)(4s)AB} \eta_B \wedge \eta_A
\]

\[
= \frac{1}{2} R_{(4s-1)(4s)ij} \omega_j \wedge \omega_i + (2 - e^{-2t}) \eta_{(4s-1)} \wedge \eta_{(4s)} - 2(1 - e^{-2t}) \eta_{(4s-3)} \wedge \eta_{(4s-2)} + 2\eta_1 \wedge \eta_2
\]

\[
+ 2(1 - e^{-2t}) \eta_{(4r-3)} \wedge \eta_{(4r-2)} + 2(1 - e^{-2t}) \eta_{(4r-1)} \wedge \eta_{(4r)} + 2(1 - e^{-4t}) \eta_3 \wedge \eta_4
\]

\[
= 2\eta_1 \wedge \eta_2 + \frac{1}{2} \tilde{R}_{(4s-1)(4s)pq} e^{-4t} \eta_q \wedge \eta_p + 2(1 - e^{-4t}) \eta_3 \wedge \eta_4 + \tilde{R}_{(4s-1)(4s)\alpha\beta} e^{-3t} \eta_\alpha \wedge \eta_\beta
\]

\[
+ \frac{1}{2} \tilde{R}_{(4s-1)(4s)\alpha\beta} e^{-2t} \eta_\alpha \wedge \eta_\beta + (2 - e^{-2t}) \eta_{(4s-1)} \wedge \eta_{(4s)} - 2(1 - e^{-2t}) \eta_{(4s-3)} \wedge \eta_{(4s-2)}
\]

\[
+ 2(1 - e^{-2t}) \eta_{(4r-3)} \wedge \eta_{(4r-2)} + 2(1 - e^{-2t}) \eta_{(4r-1)} \wedge \eta_{(4r)}.
\]

A similar computation yields the curvature tensor of the form \( R_{(4s-1)(4s-2)AB}, R_{(4s-1)(4s-3)AB}, R_{(4s-2)(4s-3)AB}, R_{(4s-2)(4s)AB}, \) and \( R_{(4s-3)(4s)AB}. \) It remains to compute

\[
\frac{1}{2} R_{(4s-3)(4r)AB} \eta_B \wedge \eta_A
\]

\[
= d\eta_{(4s-3)(4r)} - \eta_{(4s-3)1} \wedge \eta_{(4r)} - \eta_{(4s-3)q} \wedge \eta_{(4r)} - \eta_{(4s-3)1} \wedge \eta_{(4r)}
\]

\[
= d\omega_{(4s-3)(4r)} + \eta_{(4s-3)} \wedge \eta_{(4r)} - e^{2t} \omega_{(4s-3)q} \wedge \omega_{(4r)} - \eta_{(4s-3)3} \wedge \eta_{(4r)}
\]

\[
= \frac{1}{2} \tilde{R}_{(4s-3)(4r)ij} \omega_j \wedge \omega_i + (1 - e^{2t}) \omega_{(4s-3)q} \wedge \omega_{(4r)} - (1 - e^{-2t}) (\eta_4 \wedge \omega_{(4s)}(4r)
\]

\[
+ \eta_3 \wedge \omega_{(4s-1)\alpha} - (1 - e^{-2t}) (\eta_2 \wedge \omega_{(4s-2)}(4r) + \omega_{(4s-3)}(4r) \wedge \eta_2
\]

\[
- \omega_{(4s-3)(4r-2)} \wedge \eta_3 + \omega_{(4s-3)(4r-3)} \wedge \eta_4) + \eta_{(4s-3)} \wedge \eta_{(4r)}.
\]

Using (5.12 − 5.14), we can write

\[
\omega_{(4s-3)q} \wedge \omega_{(4r)} = -\eta_{(4s-2)} \wedge \eta_{(4r-1)} + \eta_{(4s-1)} \wedge \eta_{(4r-2)} - \eta_{(4s)} \wedge \eta_{(4r-3)}.
\]

Also using (5.12) asserts that

\[
\omega_{(4s-3)(4r-1)} = \omega_{(4s-2)(4r)},
\]

\[
\omega_{(4s-3)(4r-2)} = -\omega_{(4s-1)(4r)}
\]

\[
\omega_{(4s-3)(4r-3)} = \omega_{(4s)(4r)}.
\]

Hence (5.26) becomes
\[
\frac{1}{2} R_{(4s-3)(4r)AB} \eta_B \wedge \eta_A \\
= \frac{1}{2} \tilde{R}_{(4s-3)(4r)pq} e^{-4t} \eta_q \wedge \eta_p + \tilde{R}_{(4s-3)(4r)p\beta} e^{-3t} \eta_{\beta} \wedge \eta_p + \frac{1}{2} \tilde{R}_{(4s-3)(4r)\alpha\beta} e^{-2t} \eta_{\beta} \wedge \eta_{\alpha} \\
- (1 - e^{2t}) \eta_{(4s-2)} \wedge \eta_{(4r-1)} + (1 - e^{-2t}) \eta_{(4s-1)} \wedge \eta_{(4r-2)} \\
- (1 - e^{-2t}) \eta_{(4s)} \wedge \eta_{(4r-3)} + \eta_{(4s-3)} \wedge \eta_{4r}.
\]

So we have determined all curvature tensors of \(M\). Note that the quaternionic curvatures satisfy

\[K(e_1, e_2) + K(e_1, e_3) + K(e_1, e_4) = -12\]
\[K(e_2, e_1) + K(e_2, e_3) + K(e_2, e_4) = -12 + e^{-2t} (K^N(e_2, e_3) + K^N(e_2, e_4))\]
\[K(e_3, e_1) + K(e_3, e_2) + K(e_3, e_4) = -12 + e^{-2t} (K^N(e_3, e_2) + K^N(e_3, e_4))\]
\[K(e_4, e_1) + K(e_4, e_2) + K(e_4, e_3) = -12 + e^{-2t} (K(e_4, e_2) + K^N(e_4, e_3)).\]

In particular, this implies that

\[K^N(e_2, e_3) = K^N(e_2, e_4) = K^N(e_3, e_4) = 0.\]

Also, for \(2 \leq p \leq 4\), we have

\[
\sum_{i=0}^{3} K(e_1, e_{(4s-i)}) = -4
\]
\[
\sum_{i=0}^{3} K(e_p, e_{(4s-i)}) = -4 + e^{-2t} (\sum_{i=0}^{3} \tilde{K}(e_p, e_{(4s-i)}) - 4)
\]

implying

\[
\sum_{i=0}^{3} K^N(e_p, e_{(4s-i)}) = 4.
\]

We also have

\[
\sum_{i=1}^{3} K(e_{(4s)}, e_{(4s-i)}) = -12 + e^{-2t} (\sum_{i=1}^{3} K^N(e_{(4s)}, e_{(4s-i)}) + 9)
\]

implying

\[
\sum_{i=1}^{3} K^N(e_{(4s)}, e_{(4s-i)}) = -9.
\]
Lastly,
\[ \sum_{i=0}^{3} K(e_{(4s)}, e_{(4r-i)}) = -4 + e^{-2t} \sum_{i=0}^{3} K^N(e_{(4s)}, e_{(4r-i)}) \]

implying
\[ \sum_{i=0}^{3} K^N(e_{(4s)}, e_{(4r-i)}) = 0. \]

The above computation determined the whole curvature tensor for \( M \) and \( N_0 \). In particular, if \( M \) has bounded curvature, then from the formulas about the components of curvature tensors of \( M \), all curvature components are determined as those of \( \mathbb{Q}H^n \). So it must be covered by \( \mathbb{Q}H^n \). \( \square \)

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