Ageing and dynamical scaling in the critical Ising spin glass

MALTE HENKEL\(^1\) AND MICHEL PLEIMLING\(^2\)

\(^1\) Laboratoire de Physique des Matériaux (CNRS UMR 7556), Université Henri Poincaré Nancy I, B.P. 239, F – 54506 Vandœuvre lès Nancy Cedex, France

\(^2\) Institut für Theoretische Physik I, Universität Erlangen-Nürnberg, D – 91058 Erlangen, Germany

PACS. 05.70.Ln – Nonequilibrium and irreversible thermodynamics.
PACS. 75.50.Lk – Spin glasses and other random magnets.
PACS. 64.60.Ht – Dynamic critical phenomena.

Abstract. – The non-equilibrium ageing behaviour of the three-dimensional and four-dimensional critical Ising spin glass is studied for both binary and gaussian disorder. The same phenomenology of the time-dependent scaling as in non-disordered magnets is found but the non-equilibrium exponents and the universal limit fluctuation-dissipation ratio depend on the distribution of the coupling constants.

Understanding the kinetics of spin systems or glasses after a rapid quench from an initial disordered state continues to pose challenging problems. A key insight has been the observation that many of the apparently erratic and history-dependent properties of such systems can be organized in terms of a simple scaling picture [1]. The simplest way to interpret this is by assuming the existence of a single time-dependent length-scale \(L(t)\) in the problem.

For simple magnets without disorder, one typically finds a power-law scaling \(L(t) \sim t^{1/z}\), where \(z\) is the dynamical exponent. It has been realized that the ageing behaviour is more fully revealed in observables such as the two-time autocorrelation function \(C(t, s) := \langle \phi(t)\phi(s) \rangle\) or the two-time linear autoresponse function \(R(t, s) := \delta\langle \phi(t)\rangle/\delta h(s)\bigg|_{h=0}\), where \(\phi(t)\) denotes the time-dependent order-parameter, \(h(s)\) is the time-dependent conjugate magnetic field, \(t\) is referred to as observation time and \(s\) as waiting time. One says that the system undergoes ageing if \(C\) or \(R\) depend on both \(t\) and \(s\) and not merely on the difference \(\tau = t - s\). For simple magnets, these two-time functions are expected to show dynamical scaling in the ageing regime \(t, s \gg t_{\text{micro}}\) and \(t - s \gg t_{\text{micro}}\), where \(t_{\text{micro}}\) is some microscopic time scale. Then

\[
C(t, s) = s^{-b}f_C(t/s) \quad R(t, s) = s^{-1-a}f_R(t/s)
\]

where \(a, b\) are non-equilibrium exponents. The scaling functions \(f_{C,R}(y)\) should satisfy the following asymptotic behaviour

\[
f_C(y) \sim y^{-\lambda_C/z} \quad f_R(y) \sim y^{-\lambda_R/z}
\]
as \( y \to \infty \) and where \( \lambda_C \) and \( \lambda_R \), respectively, are known as the autocorrelation \([2,3]\) and autoresponse exponents \([4]\). In what follows, we shall concentrate on quenches onto the critical point. Then for simple ferromagnets \( a = b = 2\beta/\nu z = (d-2+\eta)/\nu z \) where \( \beta, \eta, \nu \) are standard equilibrium critical exponents. Furthermore, for spatially short-ranged initial correlations, one has \( \lambda_C = \lambda_R \). These expectations on critical ageing have been confirmed in a large variety of models through simulations, exact solution or field-theory calculations, see \([5–8]\) for recent reviews. Critical ageing \([1]\) also occurs in systems without detailed balance such as the contact process, but now with the exponent relations \( 1+a = b = 2\beta/\nu z \) while the autocorrelation and autoresponse exponents still coincide \([9,10]\).

It is natural to try and see whether these results might be extended to glassy systems. In this letter, we shall concentrate on the Ising spin glass, with a static Hamiltonian \( H = -\sum_{i,j} J_{i,j} \sigma_i \sigma_j \). Here \( \sigma_i = \pm 1 \) are the usual Ising spins and the nearest-neighbour couplings \( J_{i,j} \) are random variables. We shall consider (i) a binary distribution where \( J_{i,j} = \pm 1 \) and (ii) a Gaussian distribution of the \( J_{i,j} \) with zero mean and variance one. The dynamics of the model is given by a master equation where the rates are chosen according to heat-bath dynamics. It is now established \([11]\) that this model undergoes in \( d > 2 \) dimensions an equilibrium phase-transition between a paramagnetic and a frustrated spin-glass phase. On the other hand, when considering a quench below the spin-glass critical temperature \( T_c \), there has been considerable debate on the precise relationship between the relevant time and length scales. It has been attempted to summarize the present state of knowledge into the form \([12]\)

\[
t(L) \sim L^z \exp \left( \frac{\Delta_0}{T} \left( \frac{L}{\xi(T)} \right)^\psi \right) \tag{3}
\]

where \( \Delta_0 \) is an energy scale of order \( T_c \), \( \psi \) is a barrier exponent and \( \xi(T) \) is the equilibrium correlation length at temperature \( T \). This form has been used to fit successfully simulational data in the \( 3D \) and \( 4D \) Edwards-Anderson model \([12,13]\). Although the typical length scales are merely of the order of a few lattice sizes, see e.g. \([14]\), the relaxation times are sufficiently large for a dynamical scaling to set in. We shall return to this below. While this expression, if correct, points towards a cross-over behaviour between a simple power-law scaling and an exponential scaling as would follow from the droplet model, it also suggests that at criticality, simple power-law scaling should prevail. The determination of the exponent \( z \) in spin glasses and in real materials is a much-studied topic, see \([11]\) and references therein. It is the aim of this letter to further test this idea in the critical Ising spin-glass, paying special attention to the dynamical scaling behaviour of two-time functions in the ageing regime. A similar analysis was recently performed in a damage-spreading context \([15]\). From now on we consider the three- and the four-dimensional Ising spin glass quenched to \( T = T_c \) (the assumed values of \( T_c \) are listed in table \( I \)) from a fully disordered initial state. We shall study the scaling behaviour of the magnetic autocorrelation \( C(t,s) = N^{-1} \sum_i \langle \sigma_i(t) \sigma_i(s) \rangle \) and of its associated linear response with respect to an external magnetic field \( h \). As usual, since response functions are too noisy to be measured directly, we study instead the thermoremanent magnetization by turning on the magnetic field immediately after the quench at time \( t = 0 \) and keeping it until after the waiting time \( s \) has elapsed. The magnetization measured at a later time \( t \) is related to the linear response function by \( M(t,s) = h \int_0^s du R(t,u) \). The systems simulated contained \( 50^3 \) and \( 20^4 \) spins, respectively. Some other system sizes were also briefly considered in order to check against finite-size effects. The autocorrelation data discussed in the following have been obtained after averaging over typically a few thousand different bond distributions. For the thermoremanent magnetization we averaged over at least ten thousand samples with different realizations of the couplings.
We first study the scaling as a function of the waiting time $s$, with $t/s$ fixed. In figure 1 we show data for the 3D binary case. We observe that both correlation (figure 1a) and integrated response (figure 1b) are consistent with a power-law scaling. Similar results were obtained also in 4D and for the gaussian case. In table I our results for the exponents $a$ and $b$ are listed. Their values are close to results inside the spin-glass phase but near to $T_c$ [16].

We observe that $a = b$ within our numerical errors for a fixed choice of the distribution of the couplings. However, the results for binary and gaussian distributions are different.

In order to check that the results of table I relate to the true long-time scaling regime, we compare with the expected relation $a = b = \beta_{EA}/(\nu z) = (d - 2 + \eta_{EA})/(2z)$ where $\beta_{EA}, \eta_{EA}$ are the equilibrium critical exponents of the Edwards-Anderson order parameter which is quadratic in the magnetization. Their values were measured many times. For the 3D binary case we quote $\eta_{EA} = -0.225(25)$ and $z = 5.65(15)$ [17] or $\eta_{EA} = -0.337(15)$ [18], leading to $a = 0.068(4)$ and $0.059(3)$, respectively. For the 3D gaussian case $\eta_{EA} = -0.42(3)$ and $z = 6.45(10)$ [19, 20] leading to $a = 0.044(3)$ whereas the result $\eta_{EA} = -0.36(6)$ [21] gives $a = 0.049(5)$. In four dimensions one has for the binary case $\eta_{EA} = -0.31(1)$ and $z = 4.45(10)$ [22] leading to $a = 0.19(1)$. Finally, for the 4D gaussian case $\eta_{EA} = -0.35(5)$ [23] or $\eta_{EA} = -0.44(2)$ [19, 20] and $z = 4.9(4)$ [19, 20] which gives $a = 0.17(1)$ or $a = 0.16(1)$. These results for $a$ agree very well with the values extracted from our dynamical simulations in three dimensions and for the 4D binary case. For the 4D gaussian case our value for $a$ is substantially larger than the value obtained when combining the literature values of $\eta_{EA}$ and $z$. The origin of this discrepancy is not clear to us.

A second non-equilibrium critical exponent is given by the power-law decay of the autocorrelation for long times. As shown in figure 1c the quantity $C(t,0)$ displays a power-law behaviour over more than three decades, making a very precise determination of the exponent $\lambda_C/z$ possible, see [3], but which does depend on the distribution of the $J_{ij}$. We have checked that the distribution-dependence of $\lambda_C$ does not simply result from inaccurate determinations.
Table I – Critical temperature and universal nonequilibrium quantities of the critical Ising spin glass, for both binary and gaussian distributions of the nearest-neighbour couplings. In four dimensions even our longest runs did not permit us to reliably determine $\lambda_R/z$.

| $d$ | $T_c$ | $a$   | $b$   | $\lambda_C/z$ | $\lambda_R/z$ | $X_\infty$ |
|-----|-------|-------|-------|----------------|---------------|-------------|
| 3   | 1.19  | 0.060(4) | 0.056(3) | 0.362(5) | 0.38(2) | 0.12(1) |
| 4   | 2.0   | 0.18(1) | 0.17(1) | 0.615(10) | -     | 0.19(1) |

| $d$ | $T_c$ | $a$   | $b$   | $\lambda_C/z$ | $\lambda_R/z$ | $X_\infty$ |
|-----|-------|-------|-------|----------------|---------------|-------------|
| 3   | 0.92  | 0.044(1) | 0.043(1) | 0.320(5) | 0.33(2) | 0.09(1) |
| 4   | 1.78  | 0.22(1) | 0.23(1) | 0.68(1) | -     | 0.16(1) |

of $T_c$. In 3D, using $T_c = 1.14$ [18] we find $\lambda_C/z = 0.348(5)$ for the binary case and using $T_c = 0.95$ [21], we find $\lambda_C/z = 0.335(5)$ for the gaussian case. These values still appear to be significantly different from each other.

Summarizing, we have found evidence that the dynamic universality class may depend on the choice of the distribution of the coupling constants. For the equilibrium transition, even the very existence of a phase-transition in 3D was questioned for a long time, see [11] and the question of its universality remains controversial. For example, numerical and experimental evidence against universality was presented in [19, 20] while a recent high-temperature study of the Edwards-Anderson susceptibility with four symmetric random distributions asserted the universality of the exponent $\gamma$ [25] in four to eight dimensions.

Next, we turn to the form of the scaling functions themselves. In figure 2, we show the two-time scaling of the spin-spin autocorrelator, for several waiting times $s$. It is clear that $C(t,s)$ does not merely depend on the time difference $t - s$ and therefore the system ages. We find in all cases a nice collapse of the rescaled autocorrelator $s^b C(t,s)$ compatible with a simple power-law scaling $L(t) \sim t^{1/z}$.

In a similar way, in figure 3 we show the two-time scaling of the integrated response. Again, we observe a very nice collapse of the data in terms of a simple power-law scaling. The

---

**Fig. 2** – Scaling of the autocorrelation in the critical Edwards-Anderson spin-glass with the following dimensions and couplings: (a) 3D binary, (b) 3D gaussian, (c) 4D binary and (d) 4D gaussian.
fact that both the autocorrelator and the thermoremanent magnetization can be described in terms of such a power-law scaling is evidence in favour of the time-dependent length-scale $\lambda'$. Closer inspection of figure 3d, however, reveals an unexpected subtlety. In principle, one would like to extract an exponent $\lambda'_{R}/z$ from the slopes in that figure. It turns out that the values of $\lambda'_{R}/z$ thus obtained are significantly different (3D binary: 0.45, 3D gaussian: 0.41, 4D binary: 0.72, 4D gaussian: 0.76) from the ones found before for $\lambda_{C}/z$. Indeed, if one goes to larger values of $y = t/s$ as is shown in figure 3cd, we find that our data are systematically above the asymptotic power-law $\sim (t/s)^{-\lambda'_{R}/z}$ which we obtained from smaller values of $t/s$.

This passage from an effective exponent $\lambda'_{R}/z$ at intermediate values of the scaling variable $y$ to the truly asymptotic value $\lambda_{R}/z$ at larger values of $y$ has also been observed in the critical ageing of the ferromagnetic Ising model with Kawasaki dynamics [26, 27]. We shall return to a more quantitative discussion of $M(t,s)$ below.

In simple ferromagnets, ageing occurs out of equilibrium. The distance from the equilibrium state may be measured through the fluctuation-dissipation ratio $X(t,s) = \frac{TR(t,s)}{\partial C(t,s)/\partial s} = \hat{X}(t/s)$.

In particular, the limit fluctuation-dissipation ratio $X_{\infty} = \lim_{y \to \infty} \hat{X}(x)$ is expected to be an universal number [6]. In figure 4 we display the fluctuation-dissipation relation between the scaled autocorrelation and thermoremanent magnetization in both 3D and 4D. For small time differences $t - s$, the autocorrelation $C(t,s)$ is large and we are in a quasiequilibrium regime with a fluctuation-dissipation ratio $X(t,s) \approx 1$. On the other hand, for well-separated times $s$ and $t$, the systems moves out of equilibrium and we can read off the limit fluctuation-dissipation ratio $X_{\infty}$. We remark that since we had to use the scaled forms $s^{a}C(t,s)$ and $s^{a}M(t,s)$ rather than the unscaled observables, $X(t,s)$ cannot merely depend on the value of the autocorrelation, in distinction with what occurs in the mean-field theory of spin glasses.
Fig. 4 – Determination of the fluctuation-dissipation ratio in the critical Edwards-Anderson spin glass with binary couplings in (a) 3D and (b) 4D. The dashed lines have slope one and show the quasi-equilibrium behaviour while the slopes of the full lines give the limit fluctuation-dissipation ratio $X_\infty$. The insets show the function $X \equiv T_c M(t,s)/(hC(t,s))$ as a function of $s/t$ for both the binary and the gaussian cases. In (c) we show $X$ in 3D for the choices of the distribution and of $T_c$: binary with $T_c = 1.19$ (black) and $T_c = 1.14$ (gray), gaussian with $T_c = 0.92$ (black) and $T_c = 0.95$ (gray).

Finally, in figure [11] we display the ratio $T_c M(t,s)/(hC(t,s))$ in 3D as a function of $s/t$ and for several choices of $T_c$. For widely separated times, $s/t \rightarrow 0$ and this ratio should converge to the limit fluctuation-dissipation ratio $X_\infty$. We therefore find (see table I) that $X_\infty$ apparently depends on the distribution of the couplings as well, and independently of the assumed $T_c$.

The qualitative dynamical scaling behaviour of the magnetic two-time observables is quite analogous to the one of the conceptually simpler magnets, up to modified values of the critical nonequilibrium exponents. We now inquire into the form of the scaling function of the thermoremanent magnetization. It is known that by extending dynamical scaling eq. (11) to a local scale-invariance with infinitesimal local scale-transformations $t \rightarrow (1 + \varepsilon)t$, $r \rightarrow (1 + \varepsilon)r$ (with an infinitesimal $\varepsilon = \varepsilon(t,r)$ which depends on both time and space), the form of $f_R$ can be found from the requirement of covariance of $R(t,s)$ under these transformations, leading to [28,29] $f_R(y) \sim y^{1+a-\lambda_R/z}(y-1)^{-1-a}$. This prediction has been reproduced in many spin systems quenched to a temperature $T \leq T_c$ and whose dynamics is described by a master equation [7, 10, 28–32]. Hence, we expect $M(t,s) = M_0 s^{-a} f_M(t/s)$, where

$$f_M(y) = y^{-\lambda_R/z} I_1 (1 + a, -a + \lambda_R/z; 1 - a + \lambda_R/z; 1/y)$$

with a normalization constant $M_0$. At $T = T_c$, however, a second-order $\varepsilon$-expansion produces a deviation from that prediction [8]. As discussed previously we observe in critical spin glasses a passage from an effective exponents $\lambda_R/z$ at intermediate values of $y = t/s$ to the truly asymptotic value for larger $y$. Hence we cannot expect equation (14) to describe the scaling function $f_M(y)$ for all values of $y = t/s$. In figure 3b we compare the numerical data, in both 3D and 4D with the prediction [14] where we have inserted the values of the exponents $a$ and $\lambda_R/z$ which have been determined earlier. While we find a nice agreement of the prediction [14] of local scale-invariance with our data for $y = t/s$ not too large, for very large arguments the precise behaviour of the scaling function $f_M(y)$ cannot be fully reproduced.
In summary, we have studied the dynamical scaling behaviour of critical three- and four-dimensional Ising spin glasses in the ageing regime. We find for these disordered systems evidence of a dynamical power-law scaling analogous to non-disordered critical magnets. The measured values of several independent universal non-equilibrium critical quantities suggest that the dynamic universality class may depend on the choice of the distribution of the random coupling constants.

∗ ∗ ∗

We thank A.J. Bray, C. Godrèche, J.-M. Luck, M. Moore and F. Ricci-Tersenghi for discussions. This work was supported by the Bayerisch-Französisches Hochschulzentrum (BFHZ), by CINES Montpellier (projet pmn2095), and by NIC Jülich (Projekt Her10). MP acknowledges the support by the Deutsche Forschungsgemeinschaft through grant no. PL 323/2.

REFERENCES

[1] L.C.E. Struik, Physical ageing in amorphous polymers and other materials, Elsevier (Amsterdam 1978).
[2] D.S. Fisher and D.A. Huse, Phys. Rev. B38, 373 (1988).
[3] D.A. Huse, Phys. Rev. B40, 304 (1989).
[4] A. Picone and M. Henkel, J. Phys. A35, 5575 (2002).
[5] L.F. Cugliandolo, in Slow Relaxation and non equilibrium dynamics in condensed matter, J-L Barrat, J Dalibard, J Kurchan, M V Feigel’man eds (Springer, 2003).
[6] C. Godrèche and J.-M. Luck, J. Phys. Cond. Matt. 14, 1589 (2002).
[7] M. Henkel, Adv. Solid State Phys. 44, 389 (2004).
[8] P. Calabrese and A. Gambassi, cond-mat/0410357.
[9] J.J. Ramasco, M. Henkel, M.A. Santos and C.S. Santos, J. Phys. A37, 10497 (2004).
[10] T. Enss, M. Henkel, A. Picone and U. Schollwöck, J. Phys. A37, 10479 (2004).
[11] N. Kawashima and H. Rieger, in Frustrated magnetic systems, H. Diep (ed.) (World Scientific, 2004).
[12] J.-P. Bouchaud, V. Dupuis, J. Hammann and E. Vincent, Phys. Rev. B65, 024439 (2001).
[13] L. Berthier and J.-P. Bouchaud, Phys. Rev. B66, 054404 (2002).
[14] H. Yoshino, K. Hukusghima and H. Takayama, Phys. Rev. B66, 064431 (2002).
[15] Y. Ozeki and N. Ito, Phys. Rev. B64, 024416 (2001).
[16] J. Kisker, L. Santen, M. Schreckenberg and H. Rieger, Phys. Rev. B53, 6418 (1996).
[17] P.O. Mari and I.A. Campbell, Phys. Rev. B65, 184409 (2002).
[18] H.G. Ballesteros et al., Phys. Rev. B62, 14237 (2000).
[19] I.A. Campbell et al., J. Phys. Soc. Jpn. Suppl. A69, 186 (2000).
[20] H.G. Katzgraber and I.A. Campbell, Phys. Rev. B69, 094413 (2004).
[21] E. Marinari, G. Parisi and J. Ruiz-Lorenzo, Phys. Rev. B58, 14852 (1998).
[22] L. Bernardi and A. Aharony, Eur. Phys. J. B41, 231 (2004).
[23] C. Godrèche, F. Krzakala and F. Ricci-Tersenghi, J. Stat. Mech.: Theor. Exp. (2004) P04007.
[24] C. Sire, Phys. Rev. Lett. 93, 130602 (2004).
[25] M. Henkel, Nucl. Phys. B641, 405 (2002).
[26] M. Henkel, M. Pleimling, C. Godrèche and J.-M. Luck, Phys. Rev. Lett. 87, 265701 (2001).
[27] M. Henkel and M. Pleimling, Phys. Rev. E68, 065101(R) (2003).
[28] A. Picone and M. Henkel, Nucl. Phys. B688, 217 (2004).
[29] M. Pleimling, Phys. Rev. B70, 104401 (2004).
