Real Separated Algebraic Curves, Quadrature Domains, Ahlfors Type Functions and Operator Theory

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Abstract

The aim of this paper is to inter-relate several algebraic and analytic objects, such as real-type algebraic curves, quadrature domains, functions on them and rational matrix functions with special properties, and some objects from Operator Theory, such as vector Toeplitz operators and subnormal operators. Our tools come from operator theory, but some of our results have purely algebraic formulation. We make use of Xia’s theory of subnormal operators and of the previous results by the author in this direction. We also correct (in Section 5) some inaccuracies in the works of [59], [60] by the author.

Keywords: Klein surface, quadrature domain, subnormal operator, analytic vector Toeplitz operator, Schottky double

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0 Introduction

In this paper, we discuss the inter-relations between the following objects:

1. Separated real algebraic curves in $\mathbb{C}^2$;
2. Algebraic curves in $\mathbb{C}^3$ of special type, which we will call Ahlfors type curves;
3. Quadrature domains;
4. Rational matrix functions of a certain class;
5. The corresponding (analytic) Toeplitz operators on vector Hardy spaces $H^2_m$, $m \in \mathbb{N}$;
6. Subnormal operators with finite rank self-commutators and isometries that commute with them.

Each of the objects we discuss depends on a finite number of real parameters, and some of the connections we speak about are given by explicit formulas (see, for instance, §7).

Here, in the Introduction, we will define these objects and formulate some of our main results. More background and more explanations will be given in next sections.

A polynomial $Q(z, w)$ will be called of real type if it has a form

$$Q(z, w) = \sum_{j=0}^{n} \sum_{k=0}^{n} a_{jk} z^j w^k,$$

where $n \geq 1$ is an integer, $a_{jk} \in \mathbb{C}$, and

$$a_{kj} = \bar{a}_{jk}, \quad 0 \leq j, k \leq n.$$  \hspace{1cm} (2)

The linear invertible substitution $z = x + iy$, $w = x - iy$, $x, y \in \mathbb{C}$ transforms each real-type polynomial into a real polynomial in variables $x$, $y$ (the converse also is true).

Consider the algebraic curve

$$\Delta = \{(z, w) \in \mathbb{C}^2 : Q(z, w) = 0\}$$

that corresponds to $Q$. Since $Q(\bar{w}, \bar{z}) = \overline{Q(z, w)}$, the anti-analytic involution

$$\delta = (z, w) \mapsto \delta^* \overset{\text{def}}{=} (\bar{w}, \bar{z})$$  \hspace{1cm} (3)

maps $\Delta$ onto itself. We will call $\Delta$ a real-type algebraic curve if it is given by an equation in the form (1), subject to (2). Equivalently, an algebraic curve in $\mathbb{C}^2$ is of real type if its equation is invariant under the involution (3).
The polynomial $Q$ admits a unique decomposition $Q = \prod_{j=1}^{T} Q_j^{k_j}$ into a product of irreducible polynomials [21]; we will call algebraic curves $\Delta_j = \{Q_j = 0\}$ in $\mathbb{C}^2$ the irreducible pieces of the curve $Q$ and write symbolically $\Delta = \Delta_1^{k_1} \cup \cdots \cup \Delta_T^{k_T}$. Each of $\Delta_j$ essentially is a compact (unbordered) Riemann surface of finite genus. That is, there is only a finite number of singular points on $\Delta_j$, where $\frac{\partial Q_j}{\partial z} = \frac{\partial Q_j}{\partial w} = 0$, and by picking out these points from $\Delta_j$ and adding a finite number of new (“ideal”) points one gets a compact (unbordered) abstract Riemann surface $\hat{\Delta}_j$. This procedure is unique, and this surface is called a desingularization of $\Delta_j$ (see [21]). Coordinate functions $z, w$ are globally meromorphic functions on $\hat{\Delta}_j$.

We assume the desingularizations $\hat{\Delta}_1 \ldots \hat{\Delta}_k$ to be disjoint. The formula $\delta \mapsto (z(\delta), w(\delta))$ defines a continuous “projection” of $\hat{\Delta}$ onto $\Delta$. The involution (3) is also defined on $\hat{\Delta}$.

We put $\hat{\Delta}_\mathbb{R} = \{\delta \in \hat{\Delta} : \delta = \delta^*\}$ to be the real part of $\hat{\Delta}$. (In general, a point of a compact Riemann surface with anti-analytic involution is called real if it is invariant under the involution.)

**Definitions.** 1) A component $\Delta_j$ of $\Delta$ is called degenerate if it has either the form $(z - a)^{k_j} = 0$ or $(w - \bar{a})^{k_j} = 0$, and non-degenerate in all other cases.

We put $\hat{\Delta}_{\text{deg}}$ to be the union of all degenerate components of $\hat{\Delta}$, and $\hat{\Delta}_{\text{ndeg}}$ to be the union of all its non-degenerate components (with their multiplicities).

2) An irreducible real-type algebraic curve $\hat{\Delta}$ will be called separated if the real part $\hat{\Delta}_\mathbb{R}$ separates $\hat{\Delta}$ in the topological case. We will say that a real-type algebraic curve is separated if all its non-degenerate pieces are separated.

3) Let $\hat{\Delta}$ be a separated real-type algebraic curve. We call $\hat{\Delta}$ pole definite if no pole of $z(\cdot)$ lies on the real part of $\hat{\Delta}$ and for each non-degenerate irreducible piece $\hat{\Delta}_j$ of $\hat{\Delta}$, all poles of $z(\cdot)$ belong to the same connected component of $\hat{\Delta}_j \setminus \hat{\Delta}_\mathbb{R}$.

If $\hat{\Delta}$ is separated, then the involution (3) maps each of its non-degenerate irreducible pieces onto itself. The general theory [32] of Klein surfaces (Riemann surfaces with anti-analytic involution) implies that in this case, for any piece $\hat{\Delta}_j$ of $\hat{\Delta}_{\text{ndeg}}$, the complement $\hat{\Delta}_j \setminus \hat{\Delta}_\mathbb{R}$ has exactly two connected components. In the case when $\hat{\Delta}$ is pole definite, we will call these connected components $\hat{\Delta}_j^+$ and $\hat{\Delta}_j^-$, assuming that $z(\hat{\Delta}_j^+)$ is bounded and $z(\hat{\Delta}_j^-)$ is not. The involution interchanges $\hat{\Delta}_j^+$ with $\hat{\Delta}_j^-$, so that they can be called “halves” of the piece $\hat{\Delta}_j$.

The coordinate function $\delta \mapsto z(\delta)$ is bounded and analytic on $\hat{\Delta}_j^+$. 

We put
\[ \hat{\Delta}_\pm = \bigcup \hat{\Delta}_j \]
so that we have a disjoint union
\[ \hat{\Delta}_{\text{ndeg}} = \hat{\Delta}_+ \cup \hat{\Delta}_- \cup \hat{\Delta}_\mathbb{R}. \]

**Quadrature domains.** Suppose \( \Omega \) is a bounded domain in \( \mathbb{C} \) and there are points \( z_k \in \Omega \) and complex constants \( c_{jk} \) such that an identity
\[
\int \int_{\Omega} f \, dx \, dy = \sum_{k=1}^{s} \sum_{j=0}^{r_k-1} c_{jk} f^{(j)}(z_k)
\]
holds for all analytic functions \( f \) in \( L^1(\Omega) \). Then \( \Omega \) is called a quadrature domain. We will call points \( z_k \) the nodes and the coefficients \( c_{jk} \) the weights of our domain \( \Omega \).

Quadrature domains possess many interesting and intriguing properties. After the pioneering work [3] by Aharonov and Shapiro, in the last 20-30 years, quadrature domains have been related with such diverse fields as algebraic geometry [3], [23]–[25], potential theory and different problems in fluid dynamics [39], [10]–[12], [38], moment problems [36], [37], [22], extremal problems for univalent functions (studied by Aharonov, Shapiro and Solynin, see [4] and references therein). They also have close relations with subnormal and hyponormal operators “of finite type” [50]–[57], [25], [36], [59]–[61]. We refer to the recent book [17] for more information.

A function \( w(z) \) on clos \( \Omega \) is called a Schwartz function of \( \Omega \) if \( w \) is holomorphic on \( \Omega \), except for finitely many poles, continuous on the boundary of \( \Omega \) and satisfies \( w(z) = \bar{z}, z \in \partial \Omega \). It has been known since the work [3] that \( \Omega \) is a quadrature domain if and only if it possesses a a Schwartz function. Moreover, in this case the nodes of \( \Omega \) coincide with the poles of the function \( w \). If the poles are simple, only the weights \( c_{0k} \) are present in (4), and they are proportional to the residues of \( w(z) \) at points \( z = z_k \).

The following result relates algebraic curves of the above form with quadrature domains.

**Theorem A** (Aharonov and Shapiro [3], Gustafsson [23]). If \( \Delta \) is an irreducible separated real-type algebraic curve and the coordinate function \( z \) is injective on \( \hat{\Delta}_+ \), then the image \( z(\hat{\Delta}_+) \) is a quadrature domain. Each quadrature domain is formed in this way.

In the situation of this theorem, the Schwartz function \( w(z) \) on \( \Omega \) coincides with the coordinate \( w \) on our curve. More precisely, the Schwartz function is
$w((z|\tilde{\Delta}_{+})^{-1})$. The Riemann surface $\tilde{\Delta}$ can be constructed from the domain $\Omega$ as follows. Take one more copy $\tilde{\Omega}$ of $\Omega$ and endow it with the conformal structure, provided by the function $\tilde{z}$. Then $\tilde{\Delta}$ is isomorphic as a compact Riemann surface to the so-called Schottky double of $\Omega$, which is obtained by welding $\Omega$ and $\tilde{\Omega}$ together along $\partial\Omega$. We refer to [23] for more details. The irreducibility of the curve $\tilde{\Delta}$ follows from the fact that the Schottky double is always a connected topological space.

**Rational matrix functions.** Suppose $F$ is a continuous $m \times m$ matrix function on the unit circle $\mathbb{T}$. Then eigenvalues of $F(t)$ depend continuously on the point $t \in \mathbb{T}$, that is, there are continuous functions $\zeta_1(\theta), \ldots, \zeta_m(\theta)$, $\theta \in [0,2\pi]$ such that for each $\theta$, $F(e^{i\theta})$ has eigenvalues $\zeta_1(\theta), \ldots, \zeta_m(\theta)$, counted with their multiplicities. In general, $\zeta_j(0) \neq \zeta_j(2\pi)$. For a point $z_0 \in \mathbb{C}$ such that $\det(F(t) - z_0I)$ does not vanish for $t \in \mathbb{T}$, we define the winding number of the matrix function $F$ around $z_0$ as the sum of the increments of the argument of $\zeta_j(\cdot) - z_0$:

$$\text{wind}_F(z_0) = \frac{1}{2\pi} \sum_{j=1}^{m} \Delta_{[0,2\pi]} \arg(\zeta_j(\cdot) - z_0). \quad (5)$$

The number $\text{wind}_F(z_0)$ equals to the winding number of the scalar function $\theta \mapsto \det(F(e^{i\theta}) - z_0I)$ around the origin. Hence it does not depend on the choice of continuous branches of eigenvalues $\zeta_1, \ldots, \zeta_m$.

We need some special classes of symbols.

**Definitions.** Let $F$ be a square matrix function on the (open) unit disc $\mathbb{D}$. Function $F$ will be called

- analytic, if $F$ is bounded and analytic on $\mathbb{D}$;
- normal, if the matrix $F(t)$ is defined and is normal for almost every $t \in \mathbb{T}$;
- non-degenerate, if for any constant $c$ in $\mathbb{C}$, the determinant

$$\det(F(t) - cI)$$

is not identically zero on $\mathbb{D}$.

We denote by $\mathcal{NDRN}_m$ the class of all non-degenerate rational normal matrix functions of size $m \times m$, and by $\mathcal{NDARN}_m$ the class of all non-degenerate analytic rational normal matrix functions of size $m \times m$ (so that $\mathcal{NDARN}_m$ is a subclass of $\mathcal{NDRN}_m$).

If $F$ is rational, then it is non-degenerate iff for any $c$, the above determinant is not identically zero on $\mathbb{C}$. A scalar rational function on $\mathbb{D}$ without poles on $\text{clos} \mathbb{D}$ is non-degenerate iff it is not constant.
We say that a domain \( \Omega \) in \( \mathbb{C} \) is \( p \)-connected (or has connectivity \( p \)) if the homology group \( H^1(\Omega, \mathbb{Z}) \) is isomorphic to \( \mathbb{Z}^p \). A simply connected domain has connectivity 0 and a domain with one hole has connectivity 1. Sometimes the use of this term is different.

Aharonov and Shapiro also proved in [3] the following result.

**Theorem B (Aharonov and Shapiro).** A simply connected domain \( \Omega \) is a quadrature domain if and only if there is a (scalar) rational function \( g \), which is analytic on the closed unit disc, univalent in the open unit disc \( D \) and satisfies \( g(D) = \Omega \).

In the situation of this theorem, equations \( z = g(t), \ w = \overline{g(t-1)} \), \( t \in D \) define implicitly the Schwartz function on \( \Omega \). If \( \{t_k\} \) are the poles of \( g \) on the Riemann sphere, then the nodes of \( \Omega \) are exactly the points \( g(t_k-1) \). Here we denote by \( z \) both the meromorphic function \( z(\delta) \) on \( \hat{\Delta} \) and the independent variable of the \( z \)-plane; it should not confuse the reader.

Our first result extends Theorem B to the multiply connected case.

**Theorem 1.** A bounded domain \( \Omega \) in \( \mathbb{C} \) is a quadrature domain if and only if there is a natural number \( m \) and a function \( F \in \text{NDARN}_m \) with continuous branches \( \zeta_1(\theta), \ldots, \zeta_m(\theta) \) of eigenvalues of \( F(e^{i\theta}) \), \( \theta \in [0, 2\pi] \) such that

1. \( \partial \Omega = \bigcup_j \zeta_j([0, 2\pi]) \);
2. \( \text{wind}_F(z) = 1 \) for \( z \in \Omega \);
3. \( \text{wind}_F(z) = 0 \) for \( z \in \mathbb{C} \setminus \text{clos} \Omega \).

If \( \Omega \) has connectivity \( p \), then one can find a function \( F \) with these properties in \( \text{NDARN}_{p+1} \).

If \( \Omega \) and \( F \) are related as in this theorem, we will say that the matrix function \( F \) generates the domain \( \Omega \).

In fact, in §1 we will associate with any function \( F \in \text{NDARN}_m \) an algebraic curve \( \Delta_{(2)}(F) \) in \( \mathbb{C}^2 \) and an algebraic curve \( \Delta_{(3)}(F) \) in \( \mathbb{C}^3 \). Theorem 3 and its Corollary assert that an algebraic curve \( \Delta \) in \( \mathbb{C}^2 \) is admissible, pole definite and separated iff \( \Delta = \Delta_{(2)}(F) \) for some \( F \in \text{NDARN}_m \) (for a certain \( m \)). Theorem 1 can be considered as a particular case of this result. For the reader’s convenience, we will prove Theorem 1 independently of Theorem 3.

Now let us pass to the operator theory objects we will need.

**Analytic vector Toeplitz operators.** Let \( m \geq 1 \) be an integer. The *vector Hardy space* is

\[
H^2_m \overset{\text{def}}{=} \left\{ f(t) = \sum_{n \geq 0} a_n t^n : a_n \in \mathbb{C}^m, \|f\|^2 \overset{\text{def}}{=} \sum_{n \geq 0} |a_n|^2 < \infty \right\}.
\]
it is a Hilbert space of $\mathbb{C}^m$-valued analytic functions in the unit disc $\mathbb{D}$. A function $f(t)$ in $H^2_m$ can also be considered as a $\mathbb{C}^m$-valued $L^2$ function on the unit circle; its values on the circle are radial limits of its values on $\mathbb{D}$ a.e.

In this interpretation, the space $H^2_m$ becomes a closed subspace of $L^2(\mathbb{C}^m)$, and in the above formula $a_n$ are the Fourier coefficients of $f$.

For $m = 1$, $H^2_1$ is a classical scalar-valued Hardy space $H^2$; in general, $H^2_m = \bigoplus_{i=1}^{m} H^2_i$. We refer to [16] for basic properties of Hardy spaces $H^p$.

The class $H^\infty$ consists of all bounded analytic functions in $\mathbb{D}$; it is equipped with the supremum norm. We denote by $H^\infty_{m \times k}$ the class of $m \times k$ matrix functions on the unit disc, whose entries are in $H^\infty$. These matrix functions, certainly, also have boundary limit values a.e. on $\mathbb{T}$.

Let $F \in H^\infty_{m \times m}$. The analytic vector Toeplitz operator on the vector Hardy space $H^2_m$ with the symbol $F$ is in fact a multiplication operator, which acts by the formula

$$T_F g(t) = F(t)g(t), \quad g \in H^2_m$$

(the general definition of a vector Toeplitz operator will be given in §3).

**Subnormal operators.** Let $H$ be a Hilbert space. Throughout the article, we will deal only with separable complex Hilbert spaces and bounded linear operators. We denote by $\mathcal{L}(H_1, H_2)$ the set of linear operators acting from $H_1$ to $H_2$ and write $\mathcal{L}(H)$ instead of $\mathcal{L}(H, H)$.

**Definitions.** A linear operator $S$ acting on a Hilbert space $H$ is called **subnormal** if there exist a larger Hilbert space $K$, $K \supset H$ and a normal operator $N$ in $\mathcal{L}(K)$ such that $NH \subset H$ and $S = N|H$. In this case, we call $N$ a **normal extension** of $S$. We will say that $S$ **has no point masses** if it has a normal extension $N$ that has no non-zero eigenvectors. We call $S$ **pure** if it has no nonzero invariant subspace, on which it is normal.

We will say that a subnormal operator $S$ is **of finite type** if it is pure and its self-commutator $[S^*, S] \overset{\text{def}}{=} S^*S - SS^*$ has finite rank.

Subnormal operators have been much investigated; we refer to the book [13] for a background.

In a general setting, a kind of the spectral theory of subnormal operators was developed by Xia in [50]–[53]. In author’s previous work [59], [60], an alternative exposition of Xia’s theory for subnormal operators of finite type was given. A strong two-sided relationship between subnormal operators of finite type and pole definite real-type algebraic curves was revealed. A discriminant curve of a subnormal operator of finite type was defined there; it is an algebraic curve of real type such that all its non-degenerate irreducible
pieces are real-type pole definite (see formulas (16), (17) below). It was shown in [52], [60] that a subnormal operator can be modeled as a multiplication operator by the coordinate $z$ on a direct sum of certain (vector) Hardy classes over the $+$ parts of the irreducible pieces of the curve. The converse statement also is true (see [60] and Theorem E in §3 below). In §§6, 7 of the present work, we will make use of these results. All necessary definitions will be repeated. We also make some small corrections to the formulations in [59], [60].

**Theorem 2.** An operator $S$ is a subnormal operator of finite type without point masses if and only if it is unitarily equivalent to a vector Toeplitz operator $T_F$ for some symbol $F$ of class NDARN$_m$ for some $m$.

It will follow from the proof that the function $F$ is not determined uniquely.

Any scalar rational nonconstant function $F$ on $\mathbb{D}$ without poles in $\text{clos} \mathbb{D}$ belongs to NDARN$_1$. Since the analytic Toeplitz operator $T_F$ on $H^2$ has a normal extension, which is the operator of multiplication by $F$ on $L^2(\mathbb{T})$, $T_F$ is subnormal. Since this normal extension has no non-zero eigenvectors, $T_F$ has no point masses. As it follows from formula (11), for any such $F$, $T_F$ is of finite type. This illustrates one of implications in Theorem 2 for this simple case.

The logic of our exposition is as follows. In §1, we discuss a special class of algebraic curves in $\mathbb{C}^3$, which we call Ahlfors type curves. We formulate Theorem 3, which gives a relationship between rational matrix functions in NDARN$_m$, pole definite curves in $\mathbb{C}^2$ and Ahlfors type curves in $\mathbb{C}^3$. In §2, among other things, we define vector Hardy spaces $H^2$ of a bordered Riemann surface. In §3, we formulate Theorem 4, which characterizes commuting pairs of operators $(S, V)$ such that $S$ is a subnormal of finite type without point masses and $V$ is a pure isometry. Theorems 1 and 4 are proven in §4, Theorem 2 in §5 and Theorem 3 in §6. The proof of Theorem 3 is based on Theorem 4 and the existence of a pure isometry $V$, which commutes with a given subnormal $S$ of finite type. This is the assertion of Lemma 6, which plays a crucial role. This lemma is derived from the structure result for subnormal operators of finite type from [52], [60].

Note that Theorem 3 has purely algebraic formulation, but our method of proving it relies on Operator Theory.

A subnormal operator of finite type is determined uniquely by two matrices. In §7, we calculate these matrix parameters of a subnormal operator $S$ without point masses in terms of a function $F$ in NDARN$_m$ such that $S$ is unitarily equivalent to $T_F$. In §8, we describe a method for constructing matrices of classes NDRN$_m$, NDARN$_m$. Some references to related fields are given in the final §9.
Some of our arguments resemble the constructions by S. Fedorov and B. Pavlov [18], [19], [34] and by Abrahamse and Bastian [1].

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1 Ahlfors type functions, Ahlfors type curves, and rational matrix functions of classes NDRN_m, NDARN_m.

Definition. Suppose that $\Delta$ is a real-type algebraic curve in $\mathbb{C}^2$, all whose irreducible pieces are nondegenerate (that is, $\hat{\Delta} = \hat{\Delta}_{\text{ndeg}}$). We call a function $\xi$ on $\hat{\Delta}$ an Ahlfors type function if $\xi$ is globally meromorphic on each irreducible piece of $\Delta$ and for $\delta \in \hat{\Delta}$, $|\xi(\delta)| = 1$ if and only if $\delta$ belongs to the real part $\hat{\Delta}_R$ of $\hat{\Delta}$.

Let $\hat{\Delta}_j$ be an irreducible piece of $\hat{\Delta}$. It follows that $\hat{\Delta}_R$ divides $\hat{\Delta}_j$ into a union of two disjoint open sets, namely, $\{|\xi(\delta)| < 1\}$ and $\{|\xi(\delta)| > 1\}$. Since $\hat{\Delta}_j \setminus \hat{\Delta}_R$ has at most two connected components, these components are exactly these two subsets. We conclude that if $\hat{\Delta}$ has an Ahlfors type function, then $\hat{\Delta}$ is separated.

By the Schwartz reflection principle, every Ahlfors type function satisfies $\xi(\delta^*) = \overline{\xi(\delta)}^{-1}$, $\delta \in \hat{\Delta}$.

For any irreducible piece $\hat{\Delta}_j$ of $\hat{\Delta}$, the function $\xi$, restricted to the component $\{\delta \in \hat{\Delta}_j : |\xi(\delta)| < 1\}$ is a branched covering of the unit disc.

As proved Ahlfors in 1950, for any compact bordered Riemann surface $\Omega$ there exists a branched covering $\xi : \Omega \to \mathbb{D}$, where $\mathbb{D}$ is the unit disc, see [5], Theorem 10. The proof made use of a certain extremal problem. If $\Omega$ has $p$ handles and $q$ boundary contours, then [5] the degree $N$ of the extremal Ahlfors function satisfies $q \leq N \leq 2p + q$.

Let $\hat{\Delta}$ be a separated real type algebraic curve without degenerate pieces, $\hat{\Delta}_j$ one of its irreducible pieces, and $\hat{\Delta}_j^+$ (any) of the connected components of the complement $\hat{\Delta}_j \setminus \hat{\Delta}_R$. Let $\xi : \hat{\Delta}_j^+ \to \mathbb{D}$ be a branched covering. Then, by the Schwartz reflection, $\xi$ continues to an Ahlfors type function on $\hat{\Delta}$. We come to the following

Proposition 1. A real type algebraic curve has an Ahlfors type function if and only if it is separated.

According to our definition, a general Ahlfors type function needs not to be related to an extremal problem, so that its degree needs not to satisfy the
above bounds. We refer to [20], Chapter 5, and to [62] and references therein for more information.

**Definition.** Let \( \Delta_3 \) be an algebraic curve in \( \mathbb{C}^3 \) and \( \hat{\Delta}_3 \) its desingularization. We call it an Ahlfors type curve if

a) The coordinate functions (denoted by \( z, w, t \) in the sequel) are non-constant on each irreducible piece of \( \Delta_3 \);

b) \( \Delta_3 \) is invariant under the anti-analytic involution \( (z, w, t) \mapsto (\bar{w}, \bar{z}, \bar{t}^{-1}) \);

c) Every point in \( \hat{\Delta}_3 \) with \( |t| = 1 \) is a fixed point of this involution.

Let \( \Delta_3 \) be an Ahlfors type curve. Then the complement of the set \( |t| = 1 \) in any irreducible piece of \( \Delta_3 \) is not connected. By the general theory of Klein surfaces, each of irreducible pieces of \( \hat{\Delta}_3 \) is separated, and the set \( |t| = 1 \) divides it into exactly two components, defined, respectively, by the inequalities \( |t| < 1 \) and \( |t| > 1 \). The following properties are straightforward.

**Proposition 2.** 1) If \( \Delta_3 \) is an Ahlfors type curve in \( \mathbb{C}^3 \), then its projection onto the plane \( zw \) is a real-type separated algebraic curve in \( \mathbb{C}^2 \). All its irreducible pieces are non-degenerate.

2) Conversely, let \( \Delta \) be a real-type non-degenerate algebraic curve in \( \mathbb{C}^2 \) without degenerate components, and let \( \xi \) be an Ahlfors type function on it. Then the graph curve

\[
\{(z, w, \xi((z, w))) \in \mathbb{C}^3 : (z, w) \in \Delta\}
\]

is an Ahlfors type curve.

If the projection of \( \Delta_3 \) onto the \( zw \) plane is pole definite, then we call \( \Delta_3 \) a pole definite Ahlfors type curve. We remark that in general, the degrees of this projection on irreducible pieces of \( \Delta_3 \) can be greater than one. If \( \Delta_3 \) is pole definite Ahlfors type curve and \( \Delta_{3,j} \) is one of its pieces, then we define its “halves” \( \Delta_{3,j}^{\pm} \) as in the Introduction, by requiring that \( z \) has no poles on the + part. Then \( |t| < 1 \) on one of the halves and \( |t| > 1 \) on the other. We put \( (\Delta_3)_{\pm} = \cup_j \Delta_{3,j}^{\pm} \).

Let \( F \) be a matrix function in \( \text{NDRN}_m \) (see the Introduction). Let \( P \) be the set of poles of \( F \), and put

\[
\Delta^0(3)(F) = \{(z, w, t) \in \mathbb{C}^3 : t \neq 0, t, \bar{t}^{-1} \notin P \text{ and } \exists \varphi \in \mathbb{C}^m, \varphi \neq 0 : (F(t) - zI)\varphi = (F^*(\bar{t}^{-1}) - wI)\varphi = 0\}.
\]

We put \( \Delta(3)(F) \) to be the closure of \( \Delta^0(3)(F) \) in \( \mathbb{C}^3 \) and

\[
\Delta(2)(F) = \{(z, w) \in \mathbb{C}^2 : \exists t \in \mathbb{C} : (z, w) \in \Delta(3)(F)\}.
\]
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For a fixed \( t \), there are finitely many pairs \((z, w)\) such that \((z, w, t) \in \Delta_{(3)}(F)\). Hence \( \Delta_{(2)}(F) \) and \( \Delta_{(3)}(F) \) always have complex dimension one.

The projection \((z, w, t) \mapsto t\) maps any irreducible piece of \( \Delta_{(3)}(F) \) onto the whole \( t \)-plane \( \mathbb{C} \). For all but finitely many \((z, w, t)\) in \( \Delta_{(3)}(F)\),

\[
\text{Ker} \left( F(t) - zI \right) = \text{Ker} \left( F^*(\bar{t}^{-1}) - wI \right)
\]

(because it is so for points of \( \Delta_{(3)}(F) \) with \(|t| = 1\), and the dimension of these eigenspaces is positive. This dimension defines an integer-valued multiplicity function \( \nu(\delta) \) of a point \( \delta = (z, w, t) \in \Delta_{(3)}(F) \). It follows that \( \Delta_{(3)}(F) \) is always an Ahlfors type curve.

The desingularization of \( \Delta_{(3)}(F) \) is a finite union of irreducible pieces \( \tilde{\Delta}_j \). There exist positive integers \( \alpha_j \) such that \( \nu(\delta) \equiv \alpha_j \) on \( \Delta_j \) (except for a finite number of points).

**Theorem 3.** Let \( \Delta \) be an algebraic curve in \( \mathbb{C}^3 \). Then \( \Delta \) is a pole definite Ahlfors type algebraic curve such that \(|t| < 1\) on \( \Delta_+ \) if and only if there is a non-degenerate analytic rational normal matrix function \( F \) such that \( \Delta = \Delta_{(3)}(F) \).

**Corollary.** An algebraic curve \( \Delta \) in \( \mathbb{C}^2 \) is real-type, separated and pole definite if and only if there exist \( m \geq 1 \) and a matrix function \( F \in \text{NDARN}_m \) such that \( \Delta = \Delta_{(2)}(F) \).

This follows at once from the Theorem and from Propositions 1 and 2.

If \( F \in \text{NDRN}_1 \), then \( \Delta_{(3)}(F) \) coincides with the image of the map \( t \mapsto (t, F(t), F(\bar{t}^{-1})) \), \( t \in \mathbb{C} \). If, moreover, \( F \) is analytic and univalent on \( \mathbb{D} \), then the quadrature domain \( F(\mathbb{D}) \) equals to the \( z \)-projection of \( \Delta_{(3)}(F) \).

It would be interesting to see an algebraic proof of Theorem 3 and to extend this result to functions \( F \) in \( \text{NDARN}_m \). We will give an operator theoretic proof. The “if” part of Theorem 3 is straightforward.

It is common in the algebraic geometry to consider algebraic curves in \( \mathbb{C}^\ell \) as imbedded in the projective space \( \mathbb{P}^\ell(\mathbb{C}) \), see [21]. Here we do not make an explicit use of this point of view.

## 2 Pure Isometries and Vector Hardy Spaces

We recall that an operator \( V \) on a Hilbert space \( H \) is called an isometry if \( \|Vh\| = \|h\| \) for all \( h \in H \). If \( V \) is an isometry, it does not follow that \( V^* \)
also is; if it does, then $V$ is a unitary operator. An isometry $V$ is called \textit{pure} if it does not have a non-zero invariant subspace $H_1 \subset H$ such that $V|H_1$ is unitary.

Let $m \geq 1$. An example of a unitary operator is given by the operator of the forward shift operator on the space $l^2(\mathbb{Z}, \mathbb{C}^m)$ of two-sided vector-valued sequences:

$$U\{a_n\}_{n \in \mathbb{Z}} = \{a_{n-1}\}_{n \in \mathbb{Z}}, \quad \{a_n\} \in \mathbb{C}^m.$$ The Fourier transform

$$\mathcal{F} : \{a_n\} \in l^2(\mathbb{Z}, \mathbb{C}^m) \mapsto f(t) = \sum_{n \in \mathbb{Z}} a_n t^n \in L^2(\mathbb{T}, \mathbb{C}^m) \quad (6)$$

is a unitary isomorphism of $l^2(\mathbb{Z}, \mathbb{C}^m)$ onto $L^2(\mathbb{T}, \mathbb{C}^m)$. This transform is in fact a spectral representation of $U$ in the sense that

$$\mathcal{F} U \mathcal{F}^{-1} f(t) = tf(t), \quad f \in L^2(\mathbb{T}, \mathbb{C}^m).$$

We put $\mathbb{Z}_+ = \{n \in \mathbb{Z} : n \geq 0\}, \mathbb{Z}_- = \{n \in \mathbb{Z} : n < 0\}$. It is easy to see that $l^2(\mathbb{Z}_+, \mathbb{C}^m)$ is an invariant subspace of $U$, and $V = U|l^2(\mathbb{Z}_+, \mathbb{C}^m)$ is a pure isometry. In the spectral representation of $U$, operator $V$ takes the form

$$\mathcal{F} U \mathcal{F}^{-1} f(t) = tf(t), \quad f \in H^2_m.$$ We will also need the Hardy space

$$H^2_{-m} = \mathcal{F} l^2(\mathbb{Z}_-, \mathbb{C}^m).$$

One has an orthogonal sum decomposition $L^2(\mathbb{T}, \mathbb{C}^m) = H^2_{-m} \oplus H^2_m$. Formula (6) permits one to interpret functions in $H^2_{-m}$ as boundary values of functions $f(t)$, analytic on $\hat{\mathbb{C}} \setminus \text{clos} \mathbb{D}$ with $f(\infty) = 0$ (here $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere).

**Definition.** Let $H$ be a Hilbert space and $V : H \to H$ an isometry. Let $m \in \mathbb{N}$. An operator $R : H \to H^2_m$ will be called a \textit{Kolmogorov-Wold representation} of $V$ if $R$ is a unitary operator, which transforms $V$ into the multiplication operator by the independent variable:

$$RVR^{-1} f(t) = tf(t), \quad f \in H^2_m.$$ The following statement is a particular case of the Kolmogorov - Wold lemma, see [33].

**Proposition 3 (Kolmogorov-Wold).** Let $H$ be a Hilbert space and $V : H \to H$ an isometry. Then there exists a Kolmogorov-Wold representation $R : H \to H^2_m$ of $V$ if and only if $V$ is pure and $\dim(H \ominus VH) = m$. 


**Vector Hardy Spaces.** Let $\Omega$ be a non-compact Riemann surface with piecewise smooth boundary $\partial \Omega$ (we assume that $\text{clos} \Omega = \Omega \cup \partial \Omega$ is compact and is embedded into a larger Riemann surface without boundary and that $\Omega$ equals to the interior of $\text{clos} \Omega$). Then the Dirichlet problem in $\Omega$ is uniquely solvable, that is, for any $f \in C(\partial \Omega)$ there is a unique $h \in C(\text{clos} \Omega)$, which is harmonic in $\Omega$ and satisfies $h|_{\partial \Omega} = f$. Pick a point $p_0 \in \Omega$. Since the dual space to $C(\partial \Omega)$ equals to the space of finite Borel measures on $\partial \Omega$, it follows that there is a unique measure $d\omega = d\omega_{p_0}$ such that the formula

$$h(p_0) = \int f \, d\omega$$

holds for all functions $f$ and $h$, related as above. The measure $d\omega$ is positive and is called the harmonic measure for $\partial \Omega$ at $p_0$.

Assume that $p_0$ is fixed. Let $W$ be a $k \times k$ measurable matrix valued weight on $\partial \Omega$. We say that $W$ is admissible if $W > 0$ and $W, W^{-1}$ are essentially bounded. The corresponding weighted Hardy class $H^2_k(W, \Omega)$ consists of analytic vector-valued functions $f : \Omega \to \mathbb{C}^k$ such that the function $\|f(\cdot)\|^2$ has a harmonic majorant in $\Omega$. Each such function $f$ has boundary values a.e. on $\partial \Omega$. The norm

$$\|f\|^2 \overset{\text{def}}{=} \int_{\partial \Omega} \langle Wf, f \rangle \, d\omega$$

makes $H^2_k(W, \Omega)$ a Hilbert space (see [26], [20] or [60], §9 for more details).

For any function $g \in H^\infty(\Omega)$, the operator of multiplication by $g$ on $H^2_k(W, \Omega)$ is subnormal. (In general, we denote by $M_G$ the operator of multiplication by a function $G$: $M_G f = G \cdot f$). In particular, this applies to any bounded domain $\Omega$ in $\mathbb{C}$ with piecewise smooth boundary.

It is shown in the general theory of subnormal operators [13] that every subnormal operator $S : H \to H$ has a minimal normal extension $N : K \to \mathbb{C}^k$, $K \supset H$, in the sense that $N$ has no invariant subspace $K_1$, $H \subset H_1 \subset K$ such that the restriction of $N$ to $K_1$ is normal. The minimal normal extension is unique up to the unitary equivalence.

The relationship between subnormal operators of finite type and quadrature domains is seen from the following result.

**Theorem C** (McCarthy, Yang [30]). Let $\Omega$ be a bounded finitely connected domain in $\mathbb{C}$ and $\rho$ a scalar admissible weight on $\partial \Omega$. Then the operator

$$M_z f(z) = z f(z)$$

on $H^2(\rho, \Omega)$ satisfies $\text{rank}(M_z^* M_z - M_z M_z^*) < \infty$ if and only if $\Omega$ is a quadrature domain.
It is also easy to show (see, for instance, [60], Lemma 9.2) that the above operator $M_z$ on $H^2(\rho, \Omega)$ is pure subnormal and its minimal normal extension is the operator $M_z$ on $L^2(\partial\Omega, d\omega)$.

In the next section, we will need the following notation. Let $\Omega$ be a bordered Riemann surface as above, let $\lambda \in \mathbb{C}$ and let $\tau : \Omega \rightarrow \mathbb{C}$ be a non-constant holomorphic function. We put

$$\text{ind}_\lambda(\tau, \Omega) = \#\{\delta \in \Omega : \tau(\delta) = \lambda\}, \quad (7)$$

where the solutions $\delta$ of the equation $\tau(\delta) = \lambda$ are counted with their multiplicities. If $\tau$ is continuous on $\text{clos}\Omega$ and analytic in $\Omega$, then the function $\lambda \mapsto \text{ind}_\lambda(\tau, \Omega)$ is locally constant on $\mathbb{C}\setminus \tau(\partial\Omega)$.

3 Vector Toeplitz operators and subnormal operators

Here we introduce some extra notation and formulate Theorem 4, which will be used in the proof of Theorem 3.

For a linear operator $T$ on a Banach space $H$, consider an open set

$$\rho_l(T) = \{\lambda \in \mathbb{C} : \exists \varepsilon > 0 : \|(T - \lambda I)x\| \geq \varepsilon\|x\|, \ x \in H\}.$$

For $\lambda \in \rho_l(T)$, the image $(T - \lambda I)H$ is closed in $H$. The function $\text{ind}_T : \rho_l(T) \rightarrow \mathbb{Z}_+ \cup \{\infty\}$, defined by

$$\text{ind}_T(\lambda) = \dim \left(H \ominus (T - \lambda I)H\right)$$

is locally constant on $\rho_l(T)$.

Take any matrix-valued function $F$ in $L^\infty(\mathbb{T}, \mathbb{C}^{m \times m})$. We define the (vector) Toeplitz operator $T_F$ in $\mathcal{L}(H_m^2)$ and the (vector) Hankel operator $\Gamma_F$ in $\mathcal{L}(H_m^2, H_{-m}^2)$ by

$$T_Fx = P_+(F \cdot x), \quad \Gamma_Fx = P_-(F \cdot x).$$

The function $F$ is called the symbol of these operators. If $F \in H_m^\infty$, then $T_F = M_F$.

Let $F$ be a function in $H_m^\infty$, which is continuous on the closed unit disc (that is, we assume that the entries of $F$ are in the disc algebra). Choose the curves $\zeta_j$ as in Theorem 1, and let

$$\gamma(F) = \bigcup_j \zeta_j([0, 2\pi]) = \bigcup_{s \in [0, 2\pi]} \sigma(F(e^{is})). \quad (8)$$
It is known [9] that for a function $F \in H^\infty_{m \times m}$, continuous on the closed unit disc,

$$\rho_l(T_F) = \mathbb{C} \setminus \gamma(F), \quad \text{ind}_{T_F}(\lambda) = \text{wind}_F(\lambda), \quad \lambda \in \rho_l(T_F).$$

(9)

Many much more general facts about different types of spectrum of scalar and vector Toeplitz operators are known, see, for instance, [9] and [33].

The next fact is also well-known.

**Proposition 4.** A bounded operator $T$ on $H^2_m$ commutes with the shift operator $M_t$ on $H^2_m$ if and only if $T = T_F$ for some symbol $F \in H^\infty_{m \times m}$.

It is clear that for any normal symbol $F \in H^\infty_{m \times m}$, operator $T_F$ on $H^2_m$ is subnormal (not necessarily pure), and operator $M_F$ on $L^2_m(\mathbb{T})$ is its normal extension.

The next three lemmas will be proven in the next section.

**Lemma 1.** A vector Toeplitz operator $T_F$ with symbol $F \in H^\infty_{m \times m}$ has a finite rank self-commutator iff $F$ is normal and rational.

**Lemma 2.** Let $F \in H^\infty_{m \times m}$ be a normal symbol. The operator $T_F$ is pure subnormal if and only if $F$ is non-degenerate. If this condition is fulfilled, then the operator $M_F$ on $L^2_m(\mathbb{T})$ is the minimal normal extension of $T_F$.

**Lemma 3.** Let $F \in \text{NDARN}_m$, and define $\gamma(F)$ by (8). Let $\lambda \notin \gamma(F)$. Then

$$\text{wind}_F(\lambda) = \dim \left( H^2_m \ominus (M_F - \lambda)H^2_m \right) = \text{ind}_\lambda(z, \Delta^{(3)}_F).$$

(10)

where the last index is defined in (7).

**Lemma 4.** A vector Toeplitz operator $T_F$ with symbol $F \in H^\infty_{m \times m}$ is a subnormal operator of finite type without point masses if and only if its symbol $F$ belongs to $\text{NDARN}_m$.

**Proof.** The “only if” part follows directly from Lemmas 1 and 2. Conversely, suppose that $F \in \text{NDARN}_m$. Then, by the same lemmas, $T_F$ is pure subnormal and has finite rank self-commutator. It is clear that for any $c \in \mathbb{C}$, the set of points $t \in \mathbb{T}$ such that $\det(F(t) - cI) = 0$ has zero measure. Hence the operator $M_F$ on $L^2_m(\mathbb{T})$ has no non-zero eigenvectors. By Lemma 2, this operator is the minimal normal extension of $T_F$. Therefore $T_F$ has no point masses. \hfill \square

**Theorem 4.** Let $V : H \to H$ be a pure isometry with $\text{codim} VH = m < \infty$, and let $S : H \to H$ be a bounded operator. Let $R : H \to H^2_m$ be the
Kolmogorov–Wold representation of $V$. Then the following conditions are equivalent:

1. $S$ is a pure subnormal operator without point masses of finite type and operators $S$ and $V$ commute;
2. $R S R^{-1} = T_F$ for some $m \times m$ matrix symbol $F$ of class NDARN$_m$.

Operators $S$ and $V$ that satisfy part (1) of this theorem are a very particular example of $n$-tuples of commuting subnormal operators. We refer to works by Xia [53], [55] and others for a general study of $n$-tuples of commuting subnormal and hyponormal operators.

4 Proofs of Theorems 1 and 4 and of auxiliary lemmas

First we will prove the theorems modulo Lemmas 1–3.

Proof of Theorem 1. a) Let $\Omega$ be a quadrature domain. Let $\tau = \tau(z)$ be an Ahlfors type function in $\Omega$, and fix some admissible weight $\rho$ on $\partial \Omega$. Notice that by [5], Theorem 10, one can find $\tau$, whose degree equals to the connectivity number of $\Omega$. Consider operators $S = M_z$ and $V = M_\tau$ on $H^2(\Omega, \rho)$. Then $S$ is a pure subnormal operator of finite type (see Theorem A), $V$ is an isometry, and these operators commute. Let $m$ be the degree of $\tau$, then it is easy to see that

$$\dim \left( H^2(\Omega, \rho) \ominus \tau H^2(\Omega, \rho) \right) = m.$$ 

Let $R : H^2(\Omega, \rho) \to H^2_m$ be the Kolmogorov–Wold representation of $V$. Put $T = R S R^{-1}$. Then $T$ commutes with the shift operator $M_t$ on $H^2_m$ (we denote by $z$ the independent variable in $\Omega$ and by $t$ the independent variable in $\mathbb{D}$). Hence $T = T_F$ for some symbol $F = F(t) \in H^\infty_{m \times m}$. It follows from Theorem C and Lemma 2 that $F \in$ NDARN$_m$.

Notice that $M_z - \lambda I$ is left invertible for all $\lambda \notin \partial \Omega$, and that

$$\text{ind}_{M_z}(\lambda) = \begin{cases} 1, & \lambda \in \Omega \\ 0, & \lambda \notin \text{clos } \Omega \end{cases}.$$ 

Since $T_F$ is unitarily equivalent to the multiplication operator $M_z$ on $H^2(\Omega, \rho)$, formula (9) implies that assertions 2) and 3) of Theorem 1 hold. By (5), 1) is a consequence of 2) and 3).

b) Conversely, suppose that $F$ is in NDARN$_m$ and $\Omega$ is related with $F$ by means of conditions 1)–3). Consider the separated real-type algebraic
curve $\Delta_2 = \Delta_{(2)}(F)$ in $\mathbb{C}^2$. Then, by (10), the projection onto the $z$ plane is one-to-one on $\Delta_2$ and $\Omega = z(\hat{\Delta}_+)$. By Theorem A, $\Omega$ is a quadrature domain.

\textbf{Proof of Theorem 4.} Let us prove that (1) implies (2). Put $T = RSR^{-1}$. Since $T$ commutes with $M_t$ on $H^2_m$, it follows that $T = T_F$ for a matrix function $F = F(t) \in H^\infty_{m \times m}$. We remind that $R$ is an isometric isomorphism. Hence $T_F$ has the same operator properties as the operator $S$. By Lemma 4, $F \in \text{NDARN}_m$.

The converse implication also follows easily from Lemma 4.

\textbf{Proof of Lemma 3.} The first equality follows from (9). Let us prove the second one. Suppose first that $\lambda$ satisfies the following assumption: the eigenspace and the root space of $F(t)$ that correspond to eigenvalue $\lambda$ coincide for all $t \in \mathbb{D}$. It is clear that $\text{wind}_F(\lambda) = \dim \ker (F(t) - \lambda I)$, counted with multiplicities. The extreme right term equals to the sum

$$\sum_{t \in \mathbb{D}} \dim \ker (F(t) - \lambda I).$$

This implies (10) for points $\lambda$ with the above property.

For all but a finite number of points $t$ in the unit disc, $F(t)$ has no non-trivial Jordan blocks. Hence only a finite number of values of $\lambda$ were excluded from our consideration. Since all terms in (10) are locally constant on $\mathbb{C} \setminus \gamma(F)$, the general case follows.

\textbf{Proof of Lemma 1.} We will use the known formula

$$T_F^* T_F - T_F T_F^* = \Gamma_F^* \Gamma_F + T_{F^* F F^*}, \quad F \in H^\infty_{m \times m}. \quad (11)$$

To prove it, notice that for any $G \in L^\infty_{m \times m}(\mathbb{T})$, $T_G^* = T_G$, and

$$T_G^* G - T_G^* T_G = \Gamma_G^* \Gamma_G.$$

Then (11) is obtained by putting $G = F$ and $G = F^*$ and taking into account that $\Gamma_F = 0$.

Fix any $x = x(t)$ in $H^2_m$. It is easy to see that $t^n x$ tends weakly to zero and that $\|\Gamma_F(t^n x)\| \to 0$ as $n \to \infty$. By general properties of Toeplitz operators (see [33]) and (11), $\|F F^* F - F F^* x\| = \lim_{n \to \infty} \|T_{F^* F F^* - F F^*} t^n x\| = 0$. Therefore $F F^* F - F F^* \equiv 0$ a.e. on $\mathbb{T}$, so that $F$ is a normal symbol. Applying (11) once again, we get that $\Gamma_F$ is a finite rank operator. By Kronecker’s lemma, $F$ is rational (see [33], p. 183 for the scalar case; the vector case has the same proof).
Before proving Lemma 2, we will need one more fact.

**Lemma 5.** Let $F$, $\theta$, $G$ be $m \times m$, $m \times s$ and $s \times s$ constant matrices, respectively, where $1 \leq s \leq m$. If $F$ is normal, $\theta^*\theta = I$ and $F^*\theta = \theta G$, then $G$ is normal and $F\theta = \theta G^*$. 

**Proof.** For any polynomial $p$, $p(F^*)\theta = \theta p(G)$. Since $F$ is normal, we can find a polynomial $p$ with simple roots such that $p(F^*) = 0$. It follows that $G$ also has no non-trivial Jordan blocks.

Let $a_1, \ldots, a_s \in \mathbb{C}^s$ be a complete family of eigenvectors of $G$ and $\lambda_1, \ldots, \lambda_s$ the corresponding eigenvalues. We assume that if $\lambda_j = \lambda_k$ and $j \neq k$, then $\langle a_j, a_k \rangle = 0$.

We have

$$(F^* - \lambda_j I)\theta a_j = \theta(G - \lambda_j I)a_j = 0.$$ 

Therefore if $\lambda_j \neq \lambda_k$, then $\langle a_j, a_k \rangle = \langle \theta a_j, \theta a_k \rangle = 0$. Hence $a_1, \ldots, a_s$ form an orthogonal basis, so that $G$ is normal. From the normality of $F$ and $G$ we get

$$F\theta a_j = \bar{\lambda}_j \theta a_j = \theta G^*a_j, \quad j = 1, \ldots, s,$$

which implies that $F\theta = \theta G^*$. \hfill \Box

Recall that a function $\Phi(t) \in H_{m \times s}^\infty$ is called *inner* if its boundary values on $\mathbb{T}$ are isometries a.e. (then it follows that $s \leq m$).

**Proof of Lemma 2.** Suppose first that $F$ is degenerate, that is, there is a constant $c \in \mathbb{C}$ such that $\det(F(t) - cI) \equiv 0$ for $t \in \mathbb{D}$. Then the same holds for a.e. $t \in \mathbb{T}$. Put

$$L \overset{\text{def}}{=} \{x \in H_m^2 : Fx = cx \text{ a.e. on } \mathbb{T}\} \subseteq \{x \in H_m^2 : F^*x = \bar{c}x \text{ a.e. on } \mathbb{T}\}.$$ 

Note that $(F - cI)x \equiv 0$ a.e. on $\mathbb{T}$ iff $(F - cI)x \equiv 0$ in $\mathbb{D}$. Put $X = (F - cI)^\sim$, where $(F - cI)^\sim$ is the transpose associate matrix to $F - cI$. Then $X \in H_{m \times m}^\infty$ and $(F - cI)X \equiv 0$. If $X \not\equiv 0$, then $L \not\equiv 0$. If $X \equiv 0$, then there exists an integer $k$, $0 \leq k < m$ and a $k \times m$ submatrix $G$ of $F - cI$ such that for a vector $x$ in $H_m^2$, $(F - cI)x \equiv 0$ in $\mathbb{D}$ if and only if $Gx \equiv 0$ in $\mathbb{D}$. We take the smallest possible $k$. Let $G'$ be any matrix in $H_{(m-k)\times m}^\infty$ such that $J \not\equiv 0$, where $J \overset{\text{def}}{=} (G G')^{-1}$. Then it is easy to see that $x \overset{\text{def}}{=} J^\sim(0, \ldots, 0, 1)^T \not\equiv 0$ is in $L$.

It follows that in all cases, the subspace $L$ is non-zero, closed and is invariant both for $T_F$ and $T_F^* = P_+ M_F^*$. Hence $T_F$ is not pure.

Let us prove the converse. Suppose that $F$ is non-degenerate. In general, if $S \in \mathcal{L}(H)$ is a subnormal operator (not necessarily pure) and $N \in \mathcal{L}(K)$
its normal extension, then the maximal invariant subspace of $S$ on which $S$ is normal is given by $H_1 = \{ x \in H : N^{sk} x \in H \quad \forall k \in \mathbb{N} \}$ (see [13]). So in our case,

$$H_1 = \{ x \in H_m^2 : F^{sk} x \in H_m^2 \quad \forall k \in \mathbb{N} \}$$

and we have to prove that $H_1 = 0$. We remark that $H_1$ is a closed $M_z$-invariant subspace of $H_m^2_m$. If $H_1 \neq 0$, then by the Beurling–Lax–Halmos theorem (see [33]), there exists a natural $s, 1 \leq s \leq m$ and a matrix function $\theta \in H^\infty_{m \times s}$ such that $\theta(t)$ is an isometry for a.e. $t \in \mathbb{T}$ and $H_1 = \theta H^2_s$. For all $r \in \mathbb{C}^s$, $F^* \theta r \in H^2_s$. Therefore there exists a function $G$ in $H^\infty_{s \times s}$ such that

$$F^* \theta = \theta G \quad \text{a.e. on } \mathbb{T}. \quad (12)$$

For any complex $c$, we can replace $F, G$ by $F - cI, G - \bar{c}I$. We will assume without loss of generality that $\det G(0) = 0$.

Lemma 5 yields

$$F \theta = \theta G^* \quad \text{a.e. on } \mathbb{T}. \quad (13)$$

The matrix $\theta$ has a $s \times s$ minor whose determinant is not identically zero on $\mathbb{T}$. Therefore there exists a constant $s \times m$ matrix $\rho$ such that $\det(\rho \theta) \neq 0$. Note that $\det(\rho \theta) \in H^\infty$. By (13),

$$\rho F^n \theta = \rho \theta G^n \quad \text{a.e. on } \mathbb{T} \quad (14)$$

for all $n \in \mathbb{N}$. Put $g = \det G$, then $g \in H^\infty$. By (14),

$$\bar{g}^n \det(\rho \theta)|\mathbb{T} \in H^\infty \quad (15)$$

for all $n \in \mathbb{N}$. We obtain from the Nevanlinna factorization that $\bar{g} = \varphi^{-1} h$ on the unit circle, where $h \in H^\infty$ and $\varphi$ is inner. It follows that for every $n$, the function $h^n \det(\rho \theta)$ has an inner multiple $\varphi^n$. This implies that $\varphi$ divides $h$. Therefore $\bar{g} = h_1$ on $\mathbb{T}$ for some $h_1$ in $H^\infty$, so that $g = \text{const}$. Since $g(0) = 0$, we get that $g \equiv 0$.

Let $a(t) \in \text{Ker} G(t)$ for $t \in \mathbb{T}$, $a(t) \neq 0$, a.e. $t \in \mathbb{T}$. By (12), $F^*(\theta a) \equiv 0$ on $\mathbb{T}$, which yields $\det F(t) \equiv 0$. This contradicts to our assumption that $F$ is non-degenerate. We have proved that $H_1 = 0$, that is, that $S$ is pure.

At last, suppose that $\det(F - cI) \neq 0$ on $\mathbb{T}$ for all $c \in \mathbb{C}$, and let us check that $M_F$ is a minimal normal extension of $T_F$. We have to prove that the subspace

$$K = \text{span} \{ F^{*n} H^2_m : n \geq 0 \}$$

of $L^2_m(\mathbb{T})$ coincides with $L^2_m(\mathbb{T})$. Suppose that for some $y$ in $L^2_m(\mathbb{T})$, we have $\langle y, F^* x \rangle = 0$ for all $x$ in $H^2_m$ and all $n \geq 0$. Then $y$ is in $H^2_{m\,-}$.
The formula $\hat{x}(t) = \bar{t}x(\bar{t})$ defines a symmetry on $L^2_m(\mathbb{T})$, which maps $H^2_m$ onto $H^2_{-m}$ and $H^2_{-m}$ onto $H^2_m$. Put $\hat{F}(t) = F^*(\bar{t})$, then $\hat{F}$ is also in $H^\infty_{m \times m}$. Applying the symmetry $x \mapsto \hat{x}$, we get $\langle \hat{F}^n \hat{y}, u \rangle = 0$ for all $u = \hat{x}$ in $H^2_{m,-}$. Therefore $\hat{F}^n \hat{y} \in H^2_m$ for all $n \geq 0$. Since $\hat{F}$ has the same properties as $F$, we conclude from the above that $\hat{y} = 0$. Hence $K = L^2_{m}(\mathbb{T})$.

5 Discriminant curve of a subnormal operator. Proof of Theorem 2

Let $S$ be a subnormal operator of finite type. Put

$$M \overset{\text{def}}{=} \text{Range } S^*S - SS^*; \quad C = C(S) \overset{\text{def}}{=} S^*S - SS^*|M, \quad \Lambda = \Lambda(S) = (S^*|M)^*.$$  \hfill (16)

It is known that $C > 0$ and $S^*M \subset M$. In [50], [51], Xia discovered the role of operators $C$, $\Lambda$ in the study of the spectral structure of the operator $S$. In [50]–[56], he constructed and studied an analytic model of a subnormal operator with the help of these operators and a certain projection-valued function, analytic outside the spectrum of the minimal normal extension of $S$ (“Xia’s mosaic”). One of the consequences of Xia’s results is that the pair $(C, \Lambda)$ of operators on $M$ completely determines a pure subnormal operator $S$.

In our context of the study of subnormal operators of finite type we associate with any operators $C = C^*$ and $\Lambda$ on a finite dimensional space $M$ the discriminant surface, given by

$$\Delta = \{(z, w) \in \mathbb{C}^2 : \det(C - (w - \Lambda^*)(z - \Lambda)) = 0\}$$ \hfill (17)

It always is an algebraic curve of real type.

If in (17), $C$ and $\Lambda$ correspond to a finite type subnormal operator $S$, then we will write $\Delta = \Delta(S)$.

In [59], conditions on $\Delta$ that are necessary and sufficient for the existence of $S$ with $C = C(S), \Lambda = \Lambda(S)$ were given. The formulations in [59] contain certain inaccuracies. The corrections are as follows.

Let $\Delta$ be given by (17). Define, as in [59], a meromorphic $L(M)$-valued function $Q$ by

$$Q(\delta) \overset{\text{def}}{=} \Pi_w(C(z - \Lambda)^{-1} + \Lambda^*), \quad \delta = (z, w) \in \Delta \setminus z^{-1}(\sigma(\Lambda)),$$

where $\Pi_w(A)$ is the Riesz projection onto the root space of a matrix $A$ corresponding to the eigenvalue $w$. The values $Q(\delta), \delta \in \Delta$ are parallel projections.
in $M$. Let $\Delta_s$ be the (finite) set of singularities of $\Delta$. Then Theorem 1 in [59] has to be formulated as follows (we conserve the numeration of formulas of [59]).

**Theorem D ([59], Theorem 1).** Let $M$ be a finite-dimensional Hilbert space and $C, \Lambda$ operators on $M$ with $C > 0$. Define $\Delta, Q$ as above. Then there exists a subnormal operator $S$ satisfying $C = C(S)$ and $\Lambda = \Lambda(S)$ if and only if the following conditions hold:

i) $\Delta$ is separated and pole definite;

ii) Put

$$
\mu(z) = \sum_{w: (z, w) \in \Delta_+} Q((z, w)), \quad z \in \mathbb{C} \setminus (\sigma(\Lambda) \cup \gamma \cup z(\Delta_s)).
$$

(4.1)

Then there exists a positive $\mathcal{L}(M)$-valued measure $de(\cdot)$ such that

$$
(L - z)^{-1}(1 - \mu(z)) = \int \frac{de(u)}{u - z}, \quad z \in \mathbb{C} \setminus (\sigma(\Lambda) \cup \gamma \cup z(\Delta_s))
$$

(4.2)

and

$$
(C - (\bar{u} - \Lambda^*)(u - \Lambda))de(u) \equiv 0.
$$

(4.3)

If (i), (ii) hold, then the measure $de(\cdot)$ is connected with the operator $S$ by the formula (1.1) (from [59]), and $\mu$ is Xia’s mosaic of $S$.

In Theorem 2 in [59], one just has to replace item i’) by the following:

i’) $\Delta$ is separated and pole definite.

Proposition 1 in [59] is erroneous, namely, it may happen that $\Delta$ is separated and pole definite, but the set $\{|\frac{dw}{dx}| = 1\}$ is strictly larger than $\hat{\Delta}_R$. One has to define $\hat{\Delta}_+$ and $\hat{\Delta}_-$ only for separated pole definite curves $\Delta$ (as we do in the present paper). Then in the proof of Theorem D formula (5.10) (see [59]) can just be taken as a definition of $\hat{\Delta}_\pm$. It follows from [59], Lemma 1 that $\Delta$ is pole definite.

We remark that for a subnormal operator $S$ of finite type, $\Delta(S)$ is always separated, but can have degenerate pieces (even if $S$ has no point masses).

The paper [60] explains how to construct the operator $S$, starting from the corresponding $C$ and $\Lambda$; this paper was based on previous work by Xia. The model of the operator $S$ was formulated in [60] in terms of weighted analytic functional classes $H^2$ of the $\pm$ halves of the components $\hat{\Delta}_j$ of the curve $\Delta(S)$. Slight modifications are to be done also in this paper. Namely,
in Lemma 11.4 and Theorem 11.5 one has to replace the word “separated” by “separated pole definite”. Then Lemma 11.1 is not necessary.

Let $N$ be the minimal normal extension of $S$. As Xia proves, the spectrum of the minimal normal extension $N$ of $S$ coincides with the $z$-projection of the real part $\hat{\Delta}_R$ of the discriminant curve (17) (with a possible exception of a finite number of points). That is,

\[ \sigma(N) \cong \{ z \in \mathbb{C} : \det \left( C - (\bar{z} - \Lambda^*)(z - \Lambda) \right) = 0 \}; \] (18)

here $A \cong B$ means that sets $A$, $B$ differ in a finite number of points.

In what follows, we repeat briefly the results of [52], [60] that will be used in the sequel.

Let $\Delta = \Delta(S)$, then $\Delta$ is a separated pole definite real type algebraic curve. Let $\Delta_{\text{ndeg}} = \bigcup \Delta_j^{k_j}$ ($\Delta$ may have degenerate components). Suppose we have admissible matrix-valued weights $W_j$ on the boundaries $\partial \hat{\Delta}_j^+$. For simplicity of notation, we denote by $W$ the collection of matrix weights $(W_1, \ldots, W_k)$, and put

\[ H^2(W, \hat{\Delta}_{\text{ndeg}}^+) = \bigoplus_j H^2_{k_j}(W_j, \hat{\Delta}_j^+). \]

For any choice of a non-degenerate separated pole definite curve $\Delta_{\text{ndeg}}$ and a weight $W$, the multiplication operator

\[ (M_z f)(\delta) = z(\delta) f(\delta) \]

is pure subnormal of finite type; moreover, its discriminant surface coincides with $\bigcup \Delta_j^{k_j}$ ([60], Lemma 11.4). Subnormal operators that are unitarily equivalent to these ones were called simple in [60].

**Theorem E (see [52], [60]).** Let $S$ be a subnormal operator without point masses. and let $\bigcup \Delta_j^{k_j}$ be the non-degenerate part of its discriminant surface $\Delta(S)$. Then there are $k_j \times k_j$ admissible matrix weights $W_j$ on $\partial \hat{\Delta}_j^+$ and a subspace

\[ \hat{H}_1 \subset H^2(W, \hat{\Delta}_{\text{ndeg}}^+) \]

of finite codimension such that $\hat{H}_1$ is invariant under operator $M_z$ and $S$ is unitarily equivalent to operator $M_z$, restricted to $\hat{H}_1$.

Conversely, for any separated pole definite curve $\Delta = \prod \Delta_j^{k_j}$ without degenerate components, any $k_j \times k_j$ matrix weights $W_j$ and any subspace $\hat{H}_1$ of $H^2(W, \hat{\Delta}_{\text{ndeg}}^+)$ with the above properties, the operator $M_z$ on $\hat{H}_1$ will be subnormal of finite type, and the non-degenerate part of the discriminant surface $\Delta(M_z)$ will be exactly equal to $\bigcup \Delta_j^{k_j}$.
A subspace \( \hat{H}_1 \) of \( H^2(W, \hat{\Delta}_{\text{ndeg}}^+) \) has the above two properties iff it has a form
\[
\hat{H}_1 = \{ x \in H^2(W, \hat{\Delta}_{\text{ndeg}}^+) : \langle x, \psi^j_{\lambda_k} \rangle = 0, \ 1 \leq k \leq r, \ 0 \leq j \leq m_k \} \tag{19}
\]
where (not necessarily distinct) points \( \lambda_k, \ 1 \leq k \leq r \) belong to \( \bigcup_j z(\Delta_j^+) \) and \( \{ \psi^j_{\lambda_k} \}_{j=0}^{m_k} \) are corresponding Jordan chains of generalized eigenvectors:
\[
(M_z^* - \lambda_k) \psi^j_{\lambda_k} = 0, \ (M_z^* - \bar{\lambda}_k) \psi^j_{\lambda_k} = \psi^{j-1}_{\lambda_k}, \ j = 1, \ldots, m_k. \]
See [60], Theorem 12.3.

**Lemma 6.** For every subnormal operator \( S \) of finite type without point masses, there exists an isometry \( V \) as in Theorem 4 such that \( SV = VS \).

**Proof.** We apply Theorem E. Let \( \Delta \) be the discriminant surface of \( S \), and fix an Ahlfors type function \( \varphi \) on its non-degenerate part \( \hat{\Delta}_{\text{ndeg}}^+ \) such that \( |\varphi| < 1 \) on \( \hat{\Delta}_{\text{ndeg}}^+ \). Replacing \( S \) by a unitarily equivalent operator, we can assume that \( Sf(\delta) = z(\delta)f(\delta) \), \( f \in \hat{H}_1 \), where \( \hat{H}_1 \) is a subspace of \( H^2(W, \Delta_+) \) of finite codimension for some admissible matrix weights \( W_j \). Representation (19) implies that there exists a natural \( N \) such that
\[
\Psi^N \cdot H^2(W, \hat{\Delta}_{\text{ndeg}}^+) \subset \hat{H}_1 \subset H^2(W, \hat{\Delta}_{\text{ndeg}}^+), \tag{20}
\]
where
\[
\Psi(\delta) = \prod_k \left( z(\delta) - \lambda_k \right).
\]
Consider the finite Blaschke product \( B(\xi) = \prod_j \frac{\xi - \varphi(\delta_j)}{1 - \overline{\varphi(\delta_j)}\xi} \), where \( \{ \delta_j \} \) are all points of \( \Delta_+ \) whose \( z \)-projection coincides with one of \( \lambda_k \). Set \( \varphi_1 = B^{N_1} \circ \varphi \), where \( N_1 \) is a natural number. If \( B \) is constant, then we put \( \varphi_1 = \varphi \). Then \( \varphi_1 \) is also an Ahlfors type function on \( \hat{\Delta}_{\text{ndeg}}^+ \). If \( N_1 \) is large enough, then, moreover, (20) implies that
\[
\varphi_1 \hat{H}_1 \subset \Psi^N H^2(W, \hat{\Delta}_{\text{ndeg}}^+) \subset \hat{H}_1.
\]
In particular, the operator of multiplication by \( \varphi_1 \) acts on \( \hat{H}_1 \). Denote this operator by \( V \). It is obviously a pure isometry, and the codimension of \( V \hat{H}_1 \) in \( \hat{H}_1 \) is finite (every function in \( \hat{H}_1 \), which has zeros of sufficiently high order in zeros of \( \varphi_1 \) belongs to \( V \hat{H}_1 \)). The equality \( SV = VS \) holds, because both are multiplication operators by scalar functions. \( \square \)

**Remark.** If \( S \) is simple in the sense of [60], then one can take \( \hat{H}_1 = H^2(W, \hat{\Delta}_{\text{ndeg}}) \). It follows that in this case one can put \( \varphi_1 = \varphi \). The degree of \( \varphi_1 \) on each piece \( \Delta_j \) of \( \hat{\Delta}_{\text{ndeg}} \) does not exceed \( 2p_j + q_j \), where \( p_j \)
stands for the number of handles and \(q_j\) stands for the number of boundary contours of \(\hat{\Delta}_j\). If \(S\) is not simple, then the minimal possible degrees of \(\varphi_1\) on irreducible pieces of \(\hat{\Delta}_{\text{ndeg}}\) can be much higher.

**Proof of Theorem 2.** (1) If \(F \in \text{NDARN}_m\), then by Lemmas 1 and 2, \(T_F\) is a subnormal operator of finite type without point masses, and the same is true for any operator unitarily equivalent to \(T_F\).

(2) Conversely, let \(S\) be a subnormal operator of finite type without point masses. By Lemma 6, there exists a pure isometry \(V\) that commutes with \(S\). Now Theorem 4 provides a desired matrix symbol \(F\) in \(\text{NDARN}_m\) (for some \(m\)) such that \(S\) and \(T_F\) are unitarily equivalent. \(\square\)

The construction of the isometry \(V\) is far from unique. Hence the symbol \(F\) in Theorem 2 is also determined in a non-unique way.

## 6 Proof of Theorem 3

Let \(P_{z,t}\) be the projection of \(\mathbb{C}^3\) onto its coordinate subspace \(zt\):

\[
P_{z,t}(z,w,t) = (z,t), (z,w,t) \in \mathbb{C}^3.
\]

**Lemma 7.** Let \(\Delta_3\) be an Ahlfors type curve in \(\mathbb{C}^3\) (not necessarily pole definite). Then there is finite subset \(\Phi\) of \(\Delta_3\) such that \(P_{z,t}\) is one-to-one on \(\Delta_3 \setminus \Phi\).

**Proof.** The image \(P_{z,t}\Delta_3\) is an algebraic curve in \(\mathbb{C}^2\). Let \(\hat{P}_{z,t}\Delta_3\) and \(\hat{\Delta}_3\) be the desingularizations of the curves \(P_{z,t}\Delta_3\), \(\Delta_3\). Then \(P_{z,t}|\Delta_3\) is a branched covering of \(\hat{P}_{z,t}\Delta_3\). The number of preimages of a point of \(\hat{\Delta}_3\) under this covering (counted with multiplicities) is constant on each irreducible piece of \(P_{z,t}\Delta_3\). Take an irreducible piece \(K\) of \(P_{z,t}\Delta_3\). It is a projection of (at least one) irreducible piece of \(\hat{\Delta}_3\). Hence the set of solutions of the equation \(|t| = 1\) on \(K\) is a finite union of closed curves, in particular, it is infinite. Any generic point \((z,t)\) of this set has only one preimage on \(\hat{\Delta}_3\), namely, \((z,\bar{z},t)\) (see the definition of an Ahlfors type curve in §1). Therefore, in general, all but finite number of points of \(K\) have only one preimage on \(\hat{\Delta}_3\). The assertion of Lemma follows. \(\square\)

It follows from this lemma that every Ahlfors type curve \(\Delta_3\) restores in a unique way from its projection onto the plane \(zt\).

**Lemma 8.** Let \(r(\cdot,\cdot)\) be a polynomial in two variables and \(F\) be a matrix function in \(\text{NDRN}_m\) for some \(m\). Then \(r(M_F, M_t) = 0\) if and only if \(r(z,t) \equiv 0\) on \(\Delta_{(3)}(F)\).
Proof. Put $\Delta_3 = \Delta_{(3)}(F)$. For $t \in \mathbb{T}$,

$$\mathbb{C}^m = \bigoplus_{(z,t) \in P_{z,t} \Delta_3} \ker (F(t) - zI).$$

(21)

It follows that the same is true for all but a finite number of points $t \in \text{clos} \mathbb{D}$. Consider the vector bundle $\mathcal{F}$ over the open subset $\{|t| < 1\}$ of $P_{z,t} \Delta_3$ with fibers

$$\mathcal{F}((z,t)) = \ker (F(t) - zI), \quad |t| < 1$$

(the dimension $k_j$ of the fiber can be different on different irreducible pieces of $P_{z,t} \Delta_3$).

For any meromorphic cross-section $\eta$ of $\mathcal{F}$, put

$$(P\eta)(t) = \sum_{z: (z,t) \in P_{z,t} \Delta_3} \eta((z,t)), \quad |t| < 1.$$ 

It follows from (21) that for any $h \in H^2_m$, there exists a unique meromorphic cross-section $h^\sharp$ of $\mathcal{F}$ such that

$$Ph^\sharp = h.$$ 

The function $h^\sharp$ can have poles in points $(z,t)$ such that $|t| < 1$ and (21) is violated in $t$; the orders of these poles are bounded by a constant that only depends on the geometry of $\mathcal{F}$. If $h \not\equiv 0$, then $h^\sharp \not\equiv 0$.

It is easy to see that

$$r(M_F, M_t)h = r(M_F, M_t)Ph^\sharp = P(r(z,t)h^\sharp), \quad h \in H^2_m.$$ 

(22)

Therefore $r(z,t)h \equiv 0$ on $\Delta_{(3)}(F)$ implies that $r(M_F, M_t) \equiv 0$. Conversely, if $r(z,t) \not\equiv 0$, then (22) implies that $r(M_F, M_t)h \not\equiv 0$ for any non-zero $h$ in $H^2_m$.

Proof of Theorem 3. (1) Let $m \geq 1$ and $F \in \text{NDARN}_m$. Then $\Delta_{(3)}(F)$ is an Ahlfors type curve. Eigenvalues of matrices $F(t), |t| \leq 1$ are uniformly bounded. Therefore $\Delta_{(3)}(F)$ is a pole definite Ahlfors type curve and $z$ is bounded on the subset $\{|t| < 1\}$ of $\Delta_{(3)}(F)$. It follows that the + part of $\Delta_{(3)}(F)$ coincides with the subset of $\Delta_{(3)}(F)$ where $|t| < 1$.

(2) Conversely, let $\Delta$ be an Ahlfors type curve in $\mathbb{C}^3$, meeting the hypotheses of the theorem. Decompose it into irreducible curves: $\Delta = \bigcup \hat{\Delta}_j$. We can define “halves” $\hat{\Delta}_j^+$ of irreducible pieces $\hat{\Delta}_j$ of $\hat{\Delta}$ so that on $\hat{\Delta}_j^+$, $z$ is bounded and $|t| < 1$. Consider the functional class

$$H^2(W, \hat{\Delta}_+^+) = \bigoplus_{j=1}^N H^2_{k_j} (W_j, \hat{\Delta}_j^+),$$
where \(k_j \times k_j\) admissible matrix weights \(W_j\) are chosen in an arbitrary way. Consider (bounded) multiplication operators \(S = M_z\) and \(V = M_t\) on \(H^2(W, \hat{\Delta}_+).\) Then \(V\) is a pure isometry and \(S\) is a pure subnormal operator and \(V S = SV.\) By Theorem 4, there exists an integer \(m \geq 1,\) a matrix function \(F \in \text{NDARN}_m\) and an isometric isomorphism \(R : H^2(W, \hat{\Delta}_+) \to H^2_m\) such that \(R S R^{-1} = M_F,\) \(R V R^{-1} = M_t.\) We are going to prove that \(\Delta_{(3)}(F) = \Delta,\) that is, that these curves have the same irreducible pieces, and that their multiplicities also coincide.

Let us use the notation of Lemmas 7–8. Take any polynomial \(r(z,t)\) in two variables. Applying Lemma 8 and the above isomorphism, we get that \(r\) vanishes on \(\Delta\) if and only if \(r(M_F, M_t) = 0\) if and only if \(r\) vanishes on \(\Delta_{(3)}(F).\) This implies that \(\Delta_{(3)}(F)\) consists of the same irreducible components as \(\Delta: \Delta_{(3)}(F) = \Delta_{(3)}^{k_j'}\) for some numbers \(k_j' \geq 1.\)

Choose a complex constant \(\alpha \in \mathbb{C}\) such that the images of the real parts of the irreducible pieces of \(\Delta\) under the function \(z + \alpha t\) all are different. It is easy to show that it is possible (for any two fixed pieces, the set of \(\alpha\)’s such that these images coincide has empty interior). We can also suppose that \(\alpha\) is chosen so that the matrix function \(F(t) + \alpha t I\) is non-degenerate; then \(F(t) + \alpha t I \in \text{NDARN}_m.\)

We apply Lemma 3 to this matrix function. Since \(R\) transforms the pair of operators \((M_z, M_t)\) on \(H^2(W, \hat{\Delta}_+)\) into the pair \((M_F, M_t)\) on \(H^2_m,\) we get

\[
\text{ind}_\lambda (z + \alpha t, \hat{\Delta}_+) = \text{codim}((M_{F(t)} + \alpha t I) H^2_m) = \text{ind}_\lambda (z + \alpha t, \hat{\Delta}_{(3)}^+(F))
\]

for all \(\lambda \notin (z + \alpha t)(\partial \hat{\Delta}_+) = (z + \alpha t)(\partial \hat{\Delta}_{(3)}^+(F)).\) By comparing the jump of these indices on \((z + \alpha t)-\) images of the components of the curve \(\partial \hat{\Delta}_{(3)}^+(F),\) we deduce that \(k_j = k_j', j = 1, \ldots, N.\) \(\square\)

**Remarks.** (1) Suppose that operators \(S\) and \(T_F\) as in Theorem 2 are unitarily equivalent. It can be proved by the same argument as above that \(\Delta_2(F) = \Delta_{\text{ndeg}}(T_F) = \Delta_{\text{ndeg}}(S),\) and multiplicities of irreducible components are equal.

(2) Recall that if \(S : H \to H\) is a subnormal operator and \(N\) its minimal normal extension, then \(S' = N'|K \ominus H\) also is subnormal; this operator is called *dual to* \(S\) [13].

Let \(S : H \to H\) be a subnormal operator of finite type without point masses and \(N : K \to K\) its minimal normal extension. Let \(T_F : H^2_m \to H^2_m\) be a Toeplitz operator unitarily equivalent to \(S\) as in Theorems 2 and 4 and \(R : H \to H^2_m\) the corresponding isometric isomorphism such that \(R S R^{-1} =\)
$T_F$. Then $R$ extends to a unitary isomorphism $\tilde{U} : K \to L^2_m(\mathbb{T})$ such that $\tilde{U} N \tilde{U}^{-1} = M_F$. One has $\tilde{U} H' = H^2_{m,-}$ and $\tilde{U} N^* = M_{\tilde{F}} \tilde{U}$. It follows that the dual operator $S'$ is unitarily equivalent to $T_{\tilde{F}}$, where $\tilde{F}(t) = F^*(\bar{t})$.

It follows from the remark after the proof of Lemma 6 and from the above proof that if $\Delta$ is a pole definite Ahlfors type curve in $\mathbb{C}^3$, then $\Delta = \Delta_3(F)$ for some $F$ in NDARN$_m$, with

$$m \leq \sum_j k_j (2p_j + q_j),$$

where $\Delta = \bigcup \Delta_j^{k_j}$ and $p_j, q_j$ denote the number of handles and of boundary contours of $\Delta_j^+$, respectively.

7 Characterization of matrix parameters

Here we will describe matrix parameters $(C, \Lambda)$ of a finite type subnormal operator $S$ without point masses (see (16)) in terms of a matrix symbol $F$ such that $T_F$ is unitarily equivalent to $S$. First let us discuss Blaschke-Potapov products.

A Blaschke factor is a scalar function of the form $b(t) = \xi \frac{t-a}{\bar{t}-\bar{a}}$, where $a \in \mathbb{D}$ and $\xi \in \mathbb{T}$ are constants.

A matrix function $B \in H^\infty_{m \times k}$ is called inner if $B(t)$ is an isometry for a.e. $t \in \mathbb{T}$ (then it follows that $k \leq m$). It is known [35] that an $m \times m$ matrix function $B$ is rational and inner iff it can be represented as

$$B(t) = v \prod_{n=1}^M \left( b_n(t) P_n + (I - P_n) \right),$$

where $v$ is an $m \times m$ unitary constant matrix, $b_n$ are Blaschke factors, and $P_n$ are orthogonal projections in $\mathbb{C}^m$. Matrix functions $B$ of this class are called finite Blaschke–Potapov products. In particular, a scalar function $B$ is rational and inner iff it is a finite Blaschke product: $B(t) = v \prod_{n=1}^M b_n(t)$, where $v$ is a complex unimodular constant.

**Definition.** Let $\alpha, h$ be rational matrix functions in $H^\infty_{m \times m}$. We call these functions right coprime if equalities $\alpha = \alpha_1 B, h = h B$, where $B$ is a finite Blaschke–Potapov product and $\alpha_1, h_1$ are in $H^\infty_{m \times m}$ imply that $B$ is a unitary constant.
Assume that \( \det \alpha \neq 0, \det h \neq 0 \) in \( \mathbb{D} \). It is easy to see that in this case \( \alpha, h \) are right coprime iff

\[
\ker \alpha(t) \cap \ker h(t) = 0, \quad t \in \mathbb{D}.
\]

**Lemma 9.** Let \( G \) be a rational \( m \times m \) matrix function such that \( \det G \neq 0 \).

1) There is an \( m \times m \) finite Blaschke–Potapov product \( \alpha \) such that \( \ker \Gamma_G = \alpha H^2_m \).

2) The above representation of \( \ker \Gamma_G \) is equivalent to a factorization

\[
G = h\alpha^{-1},
\]

where \( h, \alpha \in H^\infty_{m \times m} \) are rational, \( \alpha \) is a finite Blaschke–Potapov product and \( h, \alpha \) are right coprime.

3) The factorization \( G = h\alpha^{-1} \) of the above form is unique, up to a substitution \( h \mapsto hu, \alpha \mapsto \alpha u \), where \( u \) is a unitary constant.

We remark that the factorization \( G(t) = h(t)\alpha^{-1}(t) \) in \( \mathbb{C} \) is equivalent to \( G(t) = h(t)\alpha^*(t), t \in \mathbb{T} \).

**Proof.** 1) It is a standard fact that \( \ker \Gamma_G \) is invariant under the shift operator \( x = x(t) \mapsto tx(t) \) on \( H^2_m \). By the Beurling-Lax-Halmos theorem [33], there is an integer \( k, 0 \leq k \leq m \) and a matrix inner function \( \alpha \) of size \( m \times k \) such that \( \ker \Gamma_G = \alpha H^2_k \). In our case of rational \( G \), it is easy to find a finite scalar Blaschke product \( \varphi \) such that \( \ker \Gamma_G \supset \varphi H^2_m \). It follows that \( \varphi I = \alpha \beta \), where \( \beta \) is a matrix inner function of size \( k \times m \). Therefore \( k = m \), \( \ker \Gamma_G \) has finite codimension in \( H^2_m \), and \( \alpha \) is a finite Blaschke–Potapov product.

2) Let \( \ker \Gamma_G = \alpha H^2_m \). Then \( G\alpha H^2_m \subset H^2_m \), hence \( h \equiv G\alpha \in H^\infty_{m \times m} \). If \( h \) and \( \alpha \) were not right coprime, that is, \( h = h_1 B, \alpha = \alpha_1 B \) for a nonconstant rational inner function \( B \in H^\infty_{m \times m} \), then \( G = h\alpha^{-1} = h_1\alpha_1^{-1} \) would give \( \ker \Gamma_G \subset \alpha_1 H^2_m \), a contradiction.

Conversely, the same arguments show that a right coprime factorization \( G = h\alpha^{-1} \) implies that \( \ker \Gamma_G = \alpha H^2_m \).

Statement 3) follows from the Beurling-Lax-Halmos theorem.

For any \( m \) and any matrix function \( F \) in \( \text{NDARN}_m \), \( \det F \) does not vanish identically. Note that \( F^*(t) \) coincides for \( t \in \mathbb{T} \) with a rational matrix function, namely, with the function \( \widehat{F}(t^{-1}) \), where \( \widehat{F}(t) = F^*(t) \). By applying (11) and the above Lemma, we deduce the following statement.

**Theorem 5.** Suppose \( F \) be a matrix function in \( \text{NDARN}_m \), where \( m \geq 1 \). Let

\[
F^*(t) = h(t)\alpha^{-1}(t), \quad t \in \mathbb{T}
\]
be the right coprime factorization of $F^*$ on $\mathbb{T}$, where $\alpha$ is a Blaschke–Potapov product in $\mathbb{D}$. Then the space $M$ and matrix parameters $C$ and $\Lambda$ of the subnormal operator $S = T_F$ (see (16)) can be calculated by the formulas

$$
M = H^2_m \ominus \alpha H^2_m, \quad \Lambda^* = T_{F^*} M, \quad C = \Gamma^*_{F^*} \Gamma_{F^*} M. \quad (23)
$$

In particular, a pair $(C, \Lambda)$ of operators on a finite dimensional space $M$ gives rise to a subnormal operator of finite type without point masses iff this pair is unitarily equivalent to a pair $(C, \Lambda)$, given by the above formulas. \qed

It is easy to write down an explicit orthonormal basis of the space $H^2_m \ominus \alpha H^2_m$ (see the Malmquist-Walsh lemma in [33] and also Example 1 below). This permits one to calculate matrices $C$ and $\Lambda$ explicitly.

Notice that if $F$ is a $m \times m$ rational matrix function in $H^\infty_{m \times m}$ with $\det F \neq 0$ and $F = \alpha h^*$ on the unit circle, where $\alpha, h$ are right coprime and $\alpha$ is a Blaschke–Potapov product, then the symbol $F$ is normal (that is, $F^*F = FF^*$ on the unit circle) if and only if $F = h\theta$, where $\theta$ is a $m \times m$ rational function, which is unitary on $\mathbb{T}$, but not necessarily analytic in $\mathbb{D}$.

8 A method of constructing rational matrix functions of classes NDRN$_m$ and NDARN$_m$

Let $B$ be a Blaschke–Potapov product, and let $\psi(t, \eta)$ be a scalar rational function of two variables. Put

$$
F(t) = \psi(t, B(t)), \quad (24)
$$

and suppose that $F$ is well-defined as a meromorphic function on the complex plane. Since $B$ is unitary on the unit circle, $F$ is a rational normal matrix function. For a fixed $B$ and "most" functions $\psi$, $F$ is non-degenerate, hence a function of class NDRN$_m$. If, moreover, $F$ is analytic on the closed unit disc, then $F$ belongs to NDARN$_m$. So (24) can be useful in the construction of separated real-type algebraic curves and quadrature domains.

In fact, it is more than an example: it can be proved that, basically, any function in NDRN$_m$ can be obtained in the above way. This topic will be treated in more detail elsewhere.

For functions $F$, obtained by this rule, the Ahlfors type curve $\Delta_3(F)$ is closely related to the study of the separated algebraic curve

$$
\rho(B) = \{(t, \eta) \in \mathbb{C}^2 : \det (B(t) - \eta I) = 0\} \quad (25)
$$
with the anti-analytic involution
\[ \delta = (t, \eta) \mapsto \delta^* = (\bar{t}^{-1}, \bar{\eta}^{-1}) \]  
(26)
(in general, it can be reducible). The real part \( \rho_{\mathbb{R}}(B) \) of this curve is defined by the equation \( |t| = 1 \); it has a real dimension one. The equality \( |\eta| = 1 \) also holds true on \( \rho_{\mathbb{R}}(B) \).

Define the meromorphic function \( z = \psi(t, \eta) \) on \( \rho_{\mathbb{R}}(B) \). Then the “image” \( \gamma(F) \) of the matrix function \( F(t) \) on the unit circle \( \mathbb{T} \), which was defined by (8), equals to the \( z \)-image of the real part \( \rho_{\mathbb{R}}(B) \) of the curve \( \rho(B) \). These notions will be exploited in Example 2 of the next section.

9 Concrete examples

We put \( b_\lambda(t) = \frac{t-\lambda}{1-\lambda t} \).

Example 1 (matrix parameters of a simply connected quadrature domain). Take \( a, \beta \in \mathbb{C} \) with \( |a| > 1 \) and \( \beta \neq 0 \), and put
\[ F(t) = t + \frac{\beta}{t-a}, \quad \Omega = F(\mathbb{D}). \]

Then \( F \in \text{NDARN}_1 \). Assume that \( F \) is univalent on \( \mathbb{D} \) (for a fixed \( a \), it always can be achieved by taking a small \( \beta \) ). Then \( \Omega \) is a quadrature domain by the Aharonov–Shapiro Theorem B. It corresponds to the analytic Toeplitz operator \( S = T_F \) on the scalar \( H^2 \), which, as we know, is a subnormal operator of finite type without point masses. We are going to calculate the matrix parameters \( (C, \Lambda) \) of this subnormal operator.

By Lemma 9 and Theorem 5, the space \( M = \text{Range}(S^*S - SS^*) \) can be calculated as
\[ M = \left( \text{Ker } \mu \right)^\perp. \]
The function \( \bar{F} \) coincides on \( \mathbb{T} \) with the rational function
\[ F_*(t) = F(\bar{t}^{-1}) = t^{-1} + \frac{\bar{\beta}t}{1-\bar{a}t}. \]
This function has poles 0 and \( \bar{a}^{-1} \) in \( \mathbb{D} \) of order one. It is easy to see that \( \text{Ker } \mu = \{ x \in H^2 : x(0) = x(\bar{a}^{-1}) = 0 \} \). Hence \( M = H^2 \ominus \alpha H^2 \), where \( \alpha(t) = b_0(t)b_{\bar{a}^{-1}}(t) \), and \( \dim M = 2 \).

Next, put \( M_- = \text{Range } \mu \subset H^2 \). Then \( \dim M_- = 2 \). Choose some orthonormal bases \( \{ e_1, e_2 \} \) in \( M \) and \( \{ h_1, h_2 \} \) in \( M_- \). Then \( P_-F_*e_j \in M_- \),
\( j = 1, 2 \). Since \( S^* M \subset M, P_+ F_* e_j \in M \) for \( j = 1, 2 \). Hence there are expansions

\[
F_* e_1 = r_{11} h_1 + r_{21} h_2 + \nu_{11} e_1 + \nu_{21} e_2,
\]

\[
F_* e_2 = r_{12} h_1 + r_{22} h_2 + \nu_{12} e_1 + \nu_{22} e_2.
\]

Introduce an operator \( R = \Gamma_F | M : M \to M_- \), then by Theorem 5, \( C = R^* R \). By (23), \( R \sim \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \) and \( \Lambda^* \sim \begin{pmatrix} \nu_{11} & \nu_{12} \\ \nu_{21} & \nu_{22} \end{pmatrix} \) in the bases \( \{e_1, e_2\} \) and \( \{h_1, h_2\} \). In particular, one can take \( e_1 = 1, e_2 = kt(1-a^{-1}t)^{-1}, h_1 = t^{-1}, h_2 = kt^{-1}(t-\bar{a}^{-1})^{-1} \), where \( k = \sqrt{1-|a|^{-2}} \). After calculating coefficients \( r_{js}, \nu_{js} \) (using residues), one gets

\[
\Lambda^* = \begin{pmatrix}
-\frac{\beta}{\alpha} & k - \frac{\beta}{\alpha^2 k} \\
0 & \frac{1}{\alpha} - \frac{\beta}{k^2 \alpha^3}
\end{pmatrix}, \quad R = \begin{pmatrix}
1 - \frac{\beta}{\alpha^2} & -\frac{\beta}{k^2 \alpha^3} \\
-\frac{\beta}{\alpha^2} & -\frac{\beta}{k^2 \alpha^3}
\end{pmatrix}.
\]

(27)

Eigenvalues of \( \Lambda \) coincide with the nodes of the quadrature domain \( \Omega \) (which are the points \( F(0) \) and \( F(\bar{a}^{-1}) \)). It is also known from the theorem by Helton–Howe and Carey–Pincus (see [31]) that

\[
\text{Area}(\Omega) = \pi \text{trace}(C).
\]

In [59], we called \( \Lambda \) and \( C^{1/2} \) the matrix center and the matrix radius of \( S \).

By Lemma 2, the operator of multiplication by \( F \) on \( L^2(\mathbb{T}) \) is the minimal normal extension of the analytic Toeplitz operator \( T_F \). Since \( \sigma(T_F) = F(\mathbb{T}) \), the curve \( \gamma(F) = F(\mathbb{T}) = \partial \Omega \) can be described alternatively by the equation (18).

Representations of the boundary of a quadrature domain by an equation like (18) have been also considered in the series of papers by Putinar and Gustafsson, see [36], [25] and earlier papers. See also Xia [54], [57] and others. These papers deal with hyponormal operators, instead of subnormal ones. The domain need not be simply connected. The difference with the subnormal case is that one can always find a pair \((C, \Lambda)\) that give rise to a quadrature domain so that \( \text{rank} \ C = 1 \). In our setting, \( C \) has always a full rank. The advantage of matrix parameters of a quadrature domain in the sense of Putinar and Gustafsson is that they are always determined uniquely. If a quadrature domain is not simply connected, there are many subnormal operators that correspond to it, and they give rise to different pairs \((C, \Lambda)\). The ambiguity is codified by the so-called characters [20], [19], [60].

**Example 2 (A one-connected quadrature domain).** Consider a finite Blaschke product in \( \mathbb{C}^2 \) of degree two:

\[
B(t) = (I - Q_1 + b_{\lambda}(t)Q_1)(I - Q_2 + b_{-\lambda}(t)Q_2),
\]
where $Q_1, Q_2$ are two different rank one projections in $\mathbb{C}^2$ and $\lambda$ is a fixed point in $\mathbb{D}$ with $\text{Re} \lambda \neq 0$. Assume (without loss of generality) that $Q_j = \ell_j \otimes \ell_j$, where $\ell_1 = (1, 0)$, $\ell_2 = (c, a)$, with $a > 0$, $c \geq 0$, $a^2 + c^2 = 1$. Put

$$p(t) = 1 - \bar{\lambda}^2 t^2, \quad r(t) = t^2 - \lambda^2,$$

$$q(t) = c^2(1 - \bar{\lambda}^2)t^2 + 2a^2(1 - |\lambda|^2)t + c^2(1 - \lambda^2),$$

and

$$\psi(t, \eta) = \frac{2p(t)\eta - q(t) + L(t - \gamma_1)}{2p(t)\eta - q(t) - L(t - \gamma_1)},$$

where $L \neq 0$ is a complex constant, and $\gamma_1$ is a root of the polynomial $D(t) = q^2(t) - 4p(t)r(t)$ with $|\gamma_1| < 1$. We assert that there is a continuum of parameters $a, c, \lambda, \gamma_1, L$ such that the matrix function

$$F(t) = \psi(t, B(t))$$

belongs to NDARN$_2$ and gives rise to a one-connected quadrature domain according to the rule of Theorem 1. It is so, in particular, if one chooses $\lambda = 0.8i$, $a = \frac{5}{13}$, $c = \frac{12}{13}$, $L = i$, and the corresponding root $\gamma_1 \approx 0.0729 - 0.6467i$ of $D$. Figure 1 shows the shape of the curve $\gamma(F)$ for these concrete parameters. The explicit parametrization of the two parts of this curve is $z = z_{\pm}(t), t \in \mathbb{T}$; the functions $z_{\pm}(t)$ will be defined in (33), (34).

In what follows, we will motivate this example and give more comments and details about this quadrature domain.
A direct calculation shows that the algebraic curve (25) for our choice of $B$ has the form

$$
\rho(B) : \quad p(t)\eta^2 - q(t)\eta + r(t) = 0. \quad (32)
$$

Obviously, $D(t)$ is the discriminant of this quadratic equation in $\eta$. One observes that

$$
r(\bar{t} - 1) = t - 2p(t), \quad q(\bar{t} - 1) = t - 2q(t),
$$

which implies that $D(\bar{t}) = t - 2D(t)$. Hence the roots of $D$ are symmetric with respect to the unit circle. Denote them as $\gamma_1, \gamma_2, \bar{\gamma}_1^{-1}, \bar{\gamma}_2^{-1}$. We assume that all these roots are distinct and that $\gamma_1, \gamma_2 \in \mathbb{D}$. The algebraic curve (32) is irreducible, and by taking its normalization we can regard it as a compact Riemann surface. Since the formula of the solution of (32) is

$$
\eta_\pm = q(t) \pm \frac{\sqrt{D(t)}}{2p(t)}
$$

and $p, q$ are single-valued functions, the surface $\rho(B)$ coincides with the Riemann surface of the multi-valued function

$$
t \mapsto \sqrt{D(t)} = K \cdot \sqrt{(t - \gamma_1)(t - \gamma_2)(t - \bar{\gamma}_1^{-1})(t - \bar{\gamma}_2^{-1})}.
$$

Hence $\rho(B)$ is an elliptic curve and is homeomorphic to a torus.

Inequalities $|t| < 1$ and $|t| > 1$ define the two “halves” of the curve $\rho(B)$, which we denote as $\rho_+(B)$ and $\rho_-(B)$, respectively. Then $\rho_+(B)$ is homeomorphic to the Riemann surface of the function $\sqrt{(t - \gamma_1)(t - \gamma_2)}$, defined on the disc $|t| < 1$. It is a two-sheeted branched covering of the unit disc, and thus is homeomorphic to a sphere with two holes (or to a one-connected domain).

Following B. Gustaffson [23], pages 224–225, we can search a meromorphic function $z(\cdot)$ on $\rho(B)$, which has no poles on $\text{clos } \rho_+(B)$ and is univalent on $\rho_+(B)$. The image of $\rho_+(B)$ under any such function $z$ will be a one-connected quadrature domain.

Consider the meromorphic function $z(\cdot) = \varphi(t(\cdot), \eta(\cdot))$ on $\rho(B)$. Note that

$$
z = \psi(t, \eta) = \frac{\sigma + L(t - \gamma_1)}{\sigma - L(t - \gamma_1)}, \quad (33)
$$

where

$$
\sigma \overset{\text{def}}{=} 2p(t)\eta - q(t) = \pm \sqrt{D(t)}. \quad (34)
$$

By considering the local parameter $\sigma$ on the curve $\rho_+(B)$ in a neighborhood of the branching point $t = \gamma_1$, one gets that $z(\delta)$ has no pole at this point.

There are two global continuous branches of $\sqrt{D(t)}$ on the unit circle. Define the functions $z_+(t)$, $z_-(t)$ on $\mathbb{T}$ by putting in (33) $\sigma = \pm \sqrt{D(t)}$, respectively.

The parameters $a, c, \lambda, \gamma_1, L$ lead to a quadrature domain $\Omega \overset{\text{def}}{=} z(\rho_+(B))$ if and only if the following two conditions hold. The first condition is that
the function \( z(\delta) \) should have no poles on \( \rho_+(B) \). The second one is that \( z(\cdot) \) should be univalent on \( \rho_+(B) \) (or, equivalently, that functions \( \zeta_1(\theta) = z_-(e^{i\theta}) \), \( \zeta_2(\theta) = z_+(e^{i\theta}) \) should satisfy the topological condition of Theorem 1). If these two conditions are valid, then \( \Omega \) is a quadrature domain. In this case, we can agree that when \( t \) runs over the unit circle, \( z_+(t) \) traverses the outer boundary curve of \( \Omega \) and \( z_-(t) \) traverses the inner one. One gets from (33) that \( z_-(t) \cdot z_+(t) = 1 \) for \( |t| = 1 \).

To verify the second condition for a concrete set of values of parameters, it suffices to check, for instance, that \( \arg z_+(e^{i\theta}) \) strictly increases for \( \theta \in [0, 2\pi] \). The author has checked both conditions numerically for the parameters indicated above. It follows that close values of parameters also give a quadrature domain.

In fact, the author does not know whether the first necessary condition implies the second one.

Formulas (33), (34) were found by the analogy with the inverse Zhukovsky function.

Since the meromorphic function \( z = \psi(t, \eta) \) has no poles on \( \text{clos} \rho_+(B) \), it follows that \( F \) is analytic on \( |t| < 1 \). Hence \( F \in \text{NDARN}_2 \). Notice that for \( t \in \mathbb{T} \), \( z_\pm(t) = \psi(t, \eta_\pm) \), where \( \eta_\pm \) are the two roots of the quadratic equation (32). Hence \( F(t) \) has eigenvalues \( z_+(t) \) and \( z_-(t) \) for \( |t| = 1 \). It follows that, whenever our choice of parameters produces a quadrature domain \( \Omega \), function \( F \) generates the same quadrature domain. The spectrum of \( T_F \) coincides with the closure of \( \Omega \).

**The Schwartz function and the nodes.** The defining equation

Suppose that our parameters \( \lambda, a, c, L, \gamma_1 \) are admissible, that is, they give rise to a quadrature domain \( \Omega \). Then the Schwartz function is given by

\[
w(z) = \overline{\psi(\delta(z)^*)}, \quad z \in \text{clos} \Omega
\]

where \( \delta(z) \) is the function inverse to the function \( z|\rho_+(B) \). Function \( w(z) \) has three poles, which are the nodes of the quadrature domain (the points \( z_j \) in the formula (4)). The positions of these three nodes are indicated on Fig. 1.

It is possible (in principle) to write down explicitly the polynomial defining equation of this quadrature domain. Namely, one can derive a polynomial relation \( X(z)(t) \equiv \sum_{j=1}^{K} X_j(z)t^j = 0 \) between the functions \( t, z \) on the curve \( \rho(B) \) (here \( X_j \) are polynomials of one variable). Next, one can write down a similar polynomial relation \( Y(w)(t) \equiv \sum_{j=1}^{K} Y_j(w)t^j = 0 \) between the functions \( t, w \) on \( \rho(B) \). Then one has an explicit equation

\[
\text{Res}(X_z, Y_w) = 0,
\]
which satisfy the meromorphic “coordinates” \( z, w \) on \( \rho(B) \) and which is polynomial in \( z, w \). Here \( \text{Res}(M, N) \) is the resultant of polynomials \( M(t), N(t) \), see [49]. We recall that \( \text{Res}(M, N) \) vanishes iff \( M \) and \( N \) have a common root. It is not completely clear, however, whether this equation is a minimal one (it might have extra factors of the form \( z - z_0 \) or \( w - w_0 \)).

Numerical experiments show that for different values of admissible parameters, the quadrature domain obtained has a form of three merged circular drops, and the nodes of the domain are situated approximately in the centers of these drops. This justifies the following

**Conjecture.** If the above method leads to a quadrature domain, then this domain can be obtained alternatively as a final domain at a time \( T_0 > 0 \) from a Hele–Shaw flow with three sources, situated in the three nodes of the domain (with no liquid at the starting time \( T = 0 \)).

We refer to the book [39] for a discussion of the Hele–Shaw flows and to [45] for a good elementary introduction to the subject. In [10]–[12], [38], one can find more recent results.

A general characterization of one-connected quadrature domains was given by M. Putinar in [37], Theorem 1.3. However, he did not give concrete examples.

**Example 3 (A matrix function of class NDARN\(_3\)).** One can try to use the same ideas in order to obtain more complicated examples. For any \( m \), it is easy to get many examples of functions in NDARN\(_m\). For instance, put \( Q_j = \ell_j \otimes \ell_j \), where \( \ell_1 = (\frac{12}{13}, \frac{-5}{13}, 0), \ell_2 = (0, \frac{12}{13}, \frac{-5}{13}), \ell_3 = (\frac{-5}{13}, 0, \frac{12}{13}) \). Put \( \varepsilon_1 = i \) and \( \varepsilon_2 = \exp(\frac{2}{3} \pi i) \). Let \( \lambda = \frac{1}{10} \in \mathbb{D} \). Take the matrix Blaschke product

\[
B(t) = (I - Q_1 + \varepsilon_1 b_\lambda(t)Q_1)(I - Q_2 + \varepsilon_2 b_{-\lambda}(t)Q_2)(I - Q_3 + tQ_3)
\]

and the function \( \psi(t, \eta) = t + \eta \). It is easy to see that the matrix function \( F(t) = \psi(t, B(t)) \) is in NDARN\(_3\). The corresponding curve \( \gamma(F) \) is shown on Fig. 2. It can be deduced from this picture that the real part of the algebraic curve \( \rho(B) \) has three components. One can number the three eigenvalues of \( F(e^{i\theta}) \) so that when \( \theta \) runs over \([0, 2\pi]\), each of these eigenvalues traverses its own component of the curve \( \gamma(F) \) in the positive direction. This function \( B \) can be thought of as a perturbation of the case when \( \{\ell_j\} \) are an orthonormal basis. In the latter case, \( \rho(B) \) has three irreducible pieces, and \( \gamma(F) \) consists of three concentric circles.

If we had taken \( \varepsilon_2 = \frac{-1+i}{\sqrt{2}} \), without changing other data, then \( \rho_{\mathbb{R}}(F) \) would have only two components.
In general, it is not so easy to find out the topological types of curves \( \rho(B) \). If \( \rho_+(B) \) is homeomorphic to a multiply connected domain, one could try to construct meromorphic functions \( z = \psi(t, \eta) \) on \( \rho(B) \), which give rise to quadrature domains. It is unclear by now how to do it explicitly. It would be desirable to have some general results about possible topological types of \( \rho(B) \), depending on the size and the degree of a Blaschke–Potapov product \( B \).

A general method of constructing multiply connected quadrature domains has been suggested recently by Crowdy [11], [10], by Crowdy and Marshall in [12] and by Richardson in [38]. In particular, the work by Crowdy and Marshall contains many examples of calculation of quadrature domains of different connectivity. The method by Crowdy involves Schottky–Klein prime functions, defined as an infinite product. Richardson’s method uses Poincaré series. Both methods require to solve certain systems of nonlinear equations. It would be interesting to find relations between the methods by Crowdy, Marshall and Richardson and the results of the present article.

10 Further perspectives

One can try to apply our results to several neighboring fields. There are many unanswered questions about quadrature domains, see the collection of problems in the recent book [17]. Our Theorem 1, combined probably with some new ideas, could occasion some progress. Let us indicate two concrete problems in this connection.

1) As it is proven in [23], Thm. 12, for any \( p \geq 1 \), there is a family of \( p \)-connected domains that satisfy the same quadrature identity (4) and depend on at least \( p \) real parameters. It would be interesting to find a more or less explicit parametrization of all quadrature domains of a given connectivity satisfying the same quadrature identity.

It is not known whether there is uniqueness when one considers only simply connected domains; see [40] for a partial result.

2) A point \( z_0 \) of a quadrature domain \( \Omega \) is called special if \( w(z_0) = \bar{z}_0 \), where \( w(z) \) is the Schwartz function. If \( \Omega \) is generated by a matrix function \( F \) of class \( \text{NDARN} \), then \( z_0 \) has to be an eigenvalue of \( F(t_0) \) and \( \bar{z}_0 \) an eigenvalue of \( F^*(\bar{t}_0^{-1}) \) for a point \( t_0 \) in \( \mathbb{D} \). It is interesting to estimate the number of special points of a quadrature domain; see [24] and [41] for some results in this direction. We remark that special points play an important role in the connection between subnormal and hyponormal operators attached to \( \Omega \), see [61], Thms. 1, 2 and the proof of Lemma 3.

The connection between hyponormal operators and quadrature domains...
has been exploited in works by Gustafsson, Putinar and Xia, see [37], [36], [25], [57], [61] and references therein.

Certainly, a better understanding of ways to construct multiply connected quadrature domains would be very desirable. A concrete construction of Ahlfors functions on a multiply connected domain has been given by S. Fedorov in [18].

It is also interesting to look for numerical applications of quadrature domains. They depend on a finite number of parameters and can approximate an arbitrary bounded domain, see [22] for a discussion. Our results might help in constructing conceptually simple algorithms of dealing with quadrature domain.

There are many topics that are related to the subject of this work, which we did not touch. Hyponormal Toeplitz operators with finite rank self-commutator were studied in [14], [27] and other papers; a relationship between Toeplitz operators with rational symbol and Riemann surfaces was exploited in [58]. Multiplication operators by the independent variable and by an analytic matrix function were studied in [48], [26], [2], [43], [44], [58] and others.

Vector bundles over real algebraic curves were used here very little. The topic of Section 7, in fact, is related to the so-called determinantal representations of vector bundles of real algebraic curves and characters. See [2], [46], [47] and the references therein.

Real algebraic curves and vector bundles over them appear naturally in the theory of commuting nonselfadjoint operators and discrete and continuous linear systems with multidimensional time, which is being developed by Livšic, Alpay, Ball, Vinnikov and others. We refer to [6], [7] (and the references therein) and to the book [29]. As it was shown in [6], there are advantages in defining spaces \( H^2 \) as spaces of differentials of order \( 1/2 \) instead of spaces of functions. Some algebraic and computational aspects of this theory were developed in [42].

Curves as in (25), whose anti-analytic involution has the form \((t, \mu) \mapsto (\bar{t}^{-1}, \bar{\mu}^{-1})\) appear in the theory of commuting contractions, see [8].

In a series of works, Pavlov and Fedorov developed the harmonic analysis on multiply connected domains (see [18], [19] and others). Such topics as analogues of Muckenhoupt condition and of Carleson condition, Ahlfors type functions that generate the uniform analytic algebra in the domain, coinvariant subspaces and corresponding semigroups were studied by these authors. One of the aims of this program is to develop a kind of the Lax-Phillips approach to the investigation of resonances for a selfadjoint operator with band spectrum, see [34].

Algebraic curves also appear systematically in the study of integrable
dynamical system, which is a very vast area; we only mention the review [15]. As the work [28] suggests, this topic also has strong connections with the theory of commuting nonselfadjoint linear operators.

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