Conductance Fluctuations of Disordered Mesoscopic Devices

with Many Weakly-Coupled Probes

Martin R. Zirnbauer *

Wissenschaftskolleg zu Berlin, Germany

ABSTRACT

The conductance coefficients of disordered mesoscopic devices with \( n \) probes are investigated within the noninteracting electron approximation at zero temperature. The probes are eliminated from the theoretical description at the expense of introducing non-local boundary conditions at the position of the contacts. Conductors with a large number of weak contacts are analyzed in detail. The ensemble-averaged conductance coefficients are in this case given by the so-called Hauser-Feshbach formula \( \langle g_{ab} \rangle \sim p_a p_b \) where \( p_a \) is the probability for emission of an electron into lead \( a \). The conductance energy autocorrelation function is shown to deviate significantly from the conventional Lorentzian form and to have a negative tail.

published in: Nucl. Phys. A560 (1993) 95-116

* On leave from: Institut für Theoretische Physik, Universität zu Köln, Germany
1. Introduction

At temperatures in the sub-Kelvin range, disordered metallic devices with a linear size \( L \) of about 1\( \mu \)m or less display random but reproducible conductance fluctuations as a function of an externally applied magnetic field \( B \) [1]. It is now well understood [2] that these fluctuations, whose amplitude is universally of the order of \( e^2/h \), originate from the phase-coherent and diffusive quantum motion of electrons through the interior of the device, the “mesoscopic conductor”. The sample-specific fluctuations with varying magnetic field are related, by an ergodicity argument [3], to statistical fluctuations across an ensemble of conductors characterized by the diffusion constant \( D \). To estimate the correlation field \( B_c \), one uses a semiclassical path-sum argument and equates \( L^2/D \) with the time it takes in order for a typical diffusive electron path to acquire from the magnetic field \( B_c \) an extra quantum mechanical phase equal to \( \pi \).

The authors of Ref. [4] first pointed out that the value of \( B_c \) is sensitive to the contact apertures joining the mesoscopic conductor to the current-voltage probes. They noticed that the mean time an electron dwells inside the conductor before escaping into the leads, is enhanced over \( \tau_{\text{diff}} = L^2/D \) in the case of small apertures. Such an increase in dwelling time amounts to an increase in the area enclosed by a typical Feynman path, thereby leading to a reduction of \( B_c \).

Ref. [4], while arriving at conclusions that are quite sound within the specific model considered, fell short of realizing the need for explicit introduction of a separate and independent time scale in addition to the diffusion time \( \tau_{\text{diff}} \). This step was made in Ref. [5], where the importance of the decay time \( \tau_{\text{decay}} \) for the mesoscopic conductor problem was recognized. A lucid discussion of the physical meaning of \( \tau_{\text{decay}} \) is given in Ref. [6]. The new time scale is set by the decay width \( \Gamma = \hbar/\tau_{\text{decay}} \) for electron emission into one of the leads. Ref. [6] quotes the formula

\[
\Gamma = \frac{\Delta}{2\pi} \alpha N, \tag{1}
\]

where \( \Delta \) is the level spacing of the mesoscopic conductor, \( N \) is the number of open scattering channels at the Fermi energy \( E_F \) in the leads, and \( \alpha \) is a dimensionless parameter measuring the quality of the contacts. (\( \alpha \simeq 1 \) for a good contact, and \( \alpha \ll 1 \) for a weak contact.) When \( \Gamma \gg E_c = hD/L^2 \) or, equivalently, \( \tau_{\text{decay}} \ll \tau_{\text{diff}} \), it is the Thouless energy \( E_c \) that defines the correlation energy of the problem and \( \Gamma \) becomes an irrelevant parameter. On the other hand, in the opposite limit \( \Gamma \ll E_c \), roles are interchanged and it is \( \Gamma \), not \( E_c \), that sets the relevant energy scale. Mesoscopic conductors satisfying the inequality \( \Gamma \ll E_c \) form the subject of the second half of this paper and will be called “decay-width dominated” for short.

Formula (1) shows that there exist three independent ways of reaching the decay-width dominated regime: (i) reduce \( N \) by making the leads thinner; (ii) reduce \( \alpha \) by degrading
the quality of the contacts; and (iii) if the conductor is three-dimensional (resp. quasi-one-dimensional), reduce $\Delta \sim L^{-d}$ relative to $E_c \sim L^{-2}$ by increasing (resp. decreasing) $L$.

The second option is pushed to the extreme by placing tunnelling barriers at the contacts, that is to say, potential barriers which electrons must tunnel through in order to reach the conductor from one of the leads or vice versa.

The purpose of the present paper is twofold. First, we wish to formulate the mesoscopic conductor problem in a way which, like the model of Ref. [5], takes proper account of the existence of the decay width $\Gamma$, but differs from Ref. [5] in several respects: it does not exploit the connection with scattering theory but utilizes a more elementary expression for the conductance coefficients in terms of Green’s functions; it uses the standard continuum Gaussian white noise model rather than Wegner’s $n$-orbital model; and it eliminates the leads by a procedure known as R-matrix theory [7-10]. This procedure is particularly well suited for conductors with tunnelling barriers, where a treatment in terms of the scattering matrix seems awkward and may even be unfeasible.

The second purpose is to communicate some analytical results for the conductance fluctuations of devices with tunnelling barriers and, more generally, of decay-width dominated conductors in the “locally-weak absorption limit”. By this term we mean the limit in which the electron emission rate per unit area of a contact becomes small and the total area of all contacts grows large, while their product is kept fixed. A special case of this limit are devices with a large number of weak contacts. To our knowledge, the limit of locally weak absorption has not been considered before in mesoscopic conductor physics (although its nuclear physics analog has received some attention [11]), perhaps because its experimental realizability is not clear. Nevertheless, we find it instructive to consider this limit since it (i) adds to our theoretical understanding of the phenomenon of universal conductance fluctuations and (ii) stands out by some distinctive features concerning correlations. In fact, we will show that the functional form of the conductance energy autocorrelation function deviates markedly from the conventional Lorentzian one in this case.

An outline of the contents of this paper is as follows. Sect. 2 defines the microscopic model and quotes the basic expression for the conductance coefficients $g_{ab} (a, b = 1, ..., n)$ in terms of advanced and retarded Green’s functions at the Fermi energy. In Sect. 3, the problem of calculating $g_{ab}$ is reformulated. The probes are eliminated and replaced by nonlocal boundary conditions on the surfaces separating the mesoscopic conductor from the leads. In Sect. 4, it is argued that for conductors with tunnelling barriers a local approximation for the integral kernel defining the boundary conditions may be used. This leads to the formulation of a phenomenological model, whose ensemble-averaged conductance coefficients $\langle g_{ab} \rangle$ are calculated in Sect. 5 and shown to be given by the so-called Hauser-Feshbach formula known from the statistical theory of nuclear reactions. Sect. 6 is concerned with the conductance covariance function $\langle \delta g_{ab}(E_1)\delta g_{cd}(E_2) \rangle$. This function
turns negative as $|E_1 - E_2|$ increases beyond $\Gamma$ and satisfies the sum rule of having vanishing integral over $E_2$ (or $E_1$). Our results are summarized in Sect. 7.

I owe my understanding of many of the physical concepts and mathematical tools underlying this article to Hans A. Weidenmüller. It is therefore appropriate that the article be dedicated to him on the occasion of his 60$^{th}$ birthday and published in the present volume.

2. The model and its conductance coefficients

In this paper, three-dimensional disordered metallic devices of the generic type will be considered. Such devices consist of a mesoscopic conductor, roughly of linear size $L$, and of $n$ leads joined to the conductor by the contacts. Both the conductor and its leads may have arbitrary geometrical shapes.

Following common practice in mesoscopic physics, we assume that the statistical features of the conductance coefficients of such a device, at zero temperature, can be modelled by an ensemble of single-electron (or mean-field) Hamiltonians of the general form

$$H = \frac{1}{2m}(p - eA)^2 + U + U_m \cdot \sigma + (p - eA) \cdot (U_{SO} \times \sigma).$$

The functions $U$, $U_m$ and $U_{SO}$ are taken to be Gaussian white-noise potentials, with their strengths determined by the mean-free times for potential scattering, $\tau$, magnetic spin-flip scattering, $\tau_m$, and spin-orbit scattering, $\tau_{SO}$, respectively [12]. The times $\tau_m$, $\tau_{SO}$ and the inverse cyclotron frequency $(e/m \times \text{rot} A)^{-1}$ are supposed to be long compared to $\tau$. To avoid complications that would otherwise appear in Sects. 5 and 6, we assume the contacts to be separated from each other by a distance of at least a few times the elastic mean-free path $\ell = v_F\tau$.

The objects of the present investigation are the conductance coefficients $G_{ab} = e^2/h \times g_{ab}$ ($a, b = 1, \ldots, n$), which determine the current response of the $n$-probe conductor to the applied electrostatic potentials. Our starting point is a formula for $g_{ab}$ which originates from linear response theory and can be found in Ref. [13]; see also Ref. [14]. To write it down, the following definitions are needed. We take $G^+$ (resp. $G^-$) to be the retarded (resp. advanced) one-particle Green’s function of $H$ at the Fermi energy $E_F$; formally:

$$G^{\pm}(x, y) = \lim_{\varepsilon \to 0^+} \left( E_F \pm i\varepsilon - H \right)^{-1}(x, y).$$

The limit $\varepsilon \to 0^+$ exists because the “openness” of the system makes the spectrum of $H$ absolutely continuous. To keep the notation simple, we adopt the convention that the symbols $x$, $y$ etc. comprise both position coordinates and spin projection. We introduce the conductivity tensor

$$\sigma_{\alpha\beta}(x, y) = (v_\alpha G^+)(x, y) (v_\beta G^-)(y, x) + (G^+ v_\beta)(x, y) (G^- v_\alpha)(y, x)$$

$$+ (v_\alpha G^+ v_\beta)(x, y) G^-(y, x) + G^+(x, y) (v_\beta G^- v_\alpha)(y, x).$$

$$\text{(3a)}$$
where \( v_\alpha \) is the \( \alpha \)-component (\( \alpha = 1, 2, 3 \)) of the velocity operator

\[
\mathbf{v} = \frac{\partial H}{\partial \mathbf{p}} = \frac{1}{m}(\mathbf{p} - e\mathbf{A}) + \mathbf{U}_{SO} \times \sigma.
\]

For \( a \in \{1, ..., n\} \), let \( c_a \) be any cross section of lead \( a \). To account for spin, we introduce two identical copies of \( c_a \), one for each spin projection: \( c_a(\uparrow) \) and \( c_a(\downarrow) \), and we set \( C_a = c_a(\uparrow) \cup c_a(\downarrow) \). Eq. (75) of Ref. [13] can then be cast in the form \( (a \neq b) \)

\[
g_{ab} = -\frac{\hbar^2}{4} \int_{C_a} d^2x \int_{C_b} d^2y \sigma_{ab}(x, y).
\]

Here \( \int_{C_a} d^2x \) stands for integration over \( c_a \) and summation over spin, and, with \( n_a \) the vector normal to \( c_a \),

\[
\sigma_{ab}(x, y) = \sum_{\alpha, \beta} n_\alpha^a(x)n_\beta^b(y)\sigma_{\alpha\beta}(x, y).
\]

(Note that Ref. [13] ignores the spin degrees of freedom but the extension is straightforward.)

3. Elimination of the leads

According to a standard assumption made in mesoscopic physics, the properties of the conductance coefficients \( g_{ab} \) are determined primarily by the phase-coherent diffusive motion of electrons inside the conductor and to a lesser extent by the details of what happens in the leads. In fact, one often takes the leads to be “ideal” or “clean” (although in reality the contacts and the leads are usually made from the very same dirty metal as is the mesoscopic conductor), i.e. one approximates the motion in the leads by free motion. Such an approximation is well justified if there is a clear geometrical division between the conductor and the leads, as effected for example by small contact apertures. In a situation where details of the motion in the leads do not matter, it makes sense to try and eliminate the leads from the theoretical formulation altogether. This can be done by a procedure known as R-matrix theory [9] in nuclear physics. (Note, however, that the roles played by the interior and the exterior of configuration space will be interchanged relative to the nuclear case.) It is applied in the present context as follows.

Let \( a \) and \( b \) two fixed elements of the set \( \{1, ..., n\} \). Evaluation of the conductance coefficient \( g_{ab} \) from Eq. (3) requires as input the Green’s functions \( G^+(x, y) \) and \( G^-(y, x) \), and their derivatives, for \( x \in C_a \) and \( y \in C_b \). We will show how to construct \( G^+ \); \( G^- \) can then be obtained from the relation \( G^-(y, x) = \overline{G^+(x, y)} \).

We observe that the total configuration space, \( V \), is partitioned by the surfaces \( C_c \) into the conductor space, \( V_0 \), and the lead spaces, \( V_c \) (\( c = 1, ..., n \)). Suppose now that we are to calculate \( G^+(x, y) \) for \( x \in C_a \) and \( y \in V_0 \). (We take \( y \) to \( C_b \) in the end.)
begin, recall that \( G^+(\cdot, y) \equiv \psi(\cdot) \) is a solution of \((E_F - H)\psi = 0\) on the region \( V - \{y\}\) with outgoing-wave boundary conditions. As a first step, we take \( G_a^-(z, x) \) to be the advanced Green’s function of \( H \) (again evaluated at \( E_F \)) with support contained in \( V_a \) and subject to the boundary condition \( G_a^-(z, x) = 0 \) for all \( z \in C_a \) and \( x \in V_a \). We write \( G_a^- (\cdot, x) \equiv G_+^- (\cdot) \) for short. If \( \mathcal{D} \) denotes the “covariant derivative”

\[
\mathcal{D} = \nabla - \frac{ie}{\hbar} \mathbf{A} + \frac{im}{\hbar} \mathbf{U}_{SO} \times \sigma,
\]

current conservation implies the validity of the relation

\[
\text{div} \left( G_+^- \cdot \mathcal{D} \psi - \overline{\mathcal{D} G_+^-} \cdot \psi \right) = 0 \tag{5}
\]
on all of \( V_a \) with the exception of the point \( x \in V_a \). The dots in Eq. (5) indicate summation over spin. Next, we integrate Eq. (5) over the region \( V_a - \{x\} \) and use Green’s theorem to obtain

\[
\psi(x) = -\frac{\hbar^2}{2m} \int_{C_a} (\mathcal{D} G_+^-)(z)\psi(z) d^2z, \tag{6}
\]
where \( \hbar \mathcal{D}_a / i = mv_a \) with \( v_a = \sum_\alpha n_a^\alpha v_\alpha \) and \( n_a \) is the vector normal to \( c_a \) as before. Now we apply \( v_a \) to both sides of (6) and take the point \( x \) to \( C_a \). Using the relation

\[
G_+^- (z) = G_a^+(x, z)
\]
we then get

\[
(v_a \psi)(x) = \int_{C_a} B_a(x, z)\psi(z) d^2z \quad (x \in C_a) \tag{7}
\]
where

\[
B_a(x, z) = \frac{i\hbar}{2} \left( v_a G_a^+ v_a \right)(x, z) \quad (x, z \in C_a). \tag{8}
\]
Eq. (7) relates the normal component of the covariant derivative of \( \psi \) on \( C_a \) to the values and the derivatives of \( \psi \) on \( C_a \). This relation can be regarded as a boundary condition satisfied by \( \psi \). We have thus arrived at an exact reformulation of the original problem: instead of solving the Schrödinger equation for \( \psi \) on the total space \( V - \{y\} \) (with outgoing-wave boundary conditions), we may solve \((E_F - H)\psi = 0\) on the restricted space \( V_0 - \{y\} \) supplemented with the boundary condition (7). The latter procedure produces all the information needed, viz. the values of \( \psi \equiv G^+(\cdot, y) \) on \( C_a \).

Returning finally to complete notation and the specific task posed by Eqs. (3a-c), we observe that the Green’s functions \( G^\pm (p, q) \) for \( p, q \in V_0 \) are computed by solving \((E_F - H)G^\pm)(p, q) = \delta(p - q)\) together with the boundary conditions

\[
(v_c G^+)(u, q) = \int_{C_c} B_c(u, z)G^+(z, q) d^2z = (G^- v_c)(q, u) \tag{9a}
\]
\[
(v_c G^-)(u, q) = -\int_{C_c} B_c(z, u)G^-(z, q) d^2z = (G^+ v_c)(q, u) \tag{9b}
\]
for all \( u \in C_c (c = 1, ..., n) \). At the end of the computation we take \( p \to x \in C_a \) and \( q \to y \in C_b \). Let us now decompose the operators \( B_c \) as \( B_c = \text{Re} B_c + i \text{Im} B_c \) where both \( (\text{Re} B_c)(x, y) \) and \( (\text{Im} B_c)(x, y) \) satisfy the hermiticity relation \( O(x, y) = \overline{O(y, x)} \). Eqs. (9) then permit us to recast (3b) in the form

\[
g_{ab} = \hbar^2 \int_{C_a} d^2 x \int_{C_b} d^2 y \left( (\text{Re} B_a) G^+(x, y) \right) \left( (\text{Re} B_b) G^-(y, x) \right)
\]

(10a)

where \( (c = a \text{ or } c = b) \)

\[
(\text{Re} B_c) G^\pm(x, y) = \int_{C_c} (\text{Re} B_c)(x, z) G^\pm(z, y) d^2 z.
\]

(10b)

Of course, the usefulness of the above reformulation depends critically upon the possibility of finding adequate and manageable approximations to the nonlocal integral kernels \( B_a(x, y) \) \( (a = 1, ..., n) \). This possibility in turn is contingent upon our ability to control the Green’s functions \( G^+_a \), see Eq. (8).

In devising approximations to \( B_a \), one should note that \( \text{Im} B_a \) corresponds, in a rough manner of speaking, to reflection at \( C_a \). It is of minor importance and can often be neglected. On the other hand, \( \text{Re} B_a \) causes “absorption” (taking the point of view of restricted configuration space \( V_0 \)), that is to say, the loss of probability due to emission of electrons into lead \( a \). This latter effect is crucial for determining the relevant time scales of the mesoscopic conductor, and it must therefore be modelled properly.

By its definition in terms of \( G^+_a \) through Eq. (8), \( B_a \) depends on the location of the surface \( C_a \), and the difficulty of devising a good approximation varies with location. It is therefore very fortunate that current conservation leaves complete freedom in choosing the surfaces \( C_a \), thereby permitting us to optimize the choice. How to make an optimal choice is rather obvious if there exists a clear geometrical division between the conductor and the leads, and if the leads can be taken to be clean: one will then place \( C_a \) right at the contact of lead \( a \). In this case, an approximation which captures the essential features of \( B_a \) is

\[
B_a(x, y) = \pi \sum_c W^c_a(x) \overline{W^c_a(y)},
\]

(11)

where the sum runs over all open scattering channels at \( E_F \) in lead \( a \), and the amplitudes \( W^c_a(x) \) are chosen phenomenologically to fit the average rate of emission into the channels. This corresponds to the approximation used in Ref. [5] at the level of the S-matrix. Another case where \( B_a \) can be controlled with ease is analyzed in the sequel.
4. Reduced model for devices with tunnelling barriers

From now on we will restrict ourselves to the case of \( n \)-probe conductors with *weak* contacts. In other words, emission into the leads will be supposed to be inhibited by obstacles of some kind. For definiteness, we imagine these obstacles in the form of tunnelling barriers and add a barrier potential \( U_B \) to the Hamiltonian \( H \) of Eq. (2). Let us denote by \( C'_a \) (resp. \( C''_a \)) the surface separating the classically forbidden region under the barrier at contact \( a \) from the allowed region inside the conductor (resp. in lead \( a \)). An optimal choice (in the sense of the previous section) is then to put \( C_a = C'_a \) (\( a = 1, ..., n \)). We adopt this choice.

The simplification that occurs for devices of such kind is that one may use a *local* approximation to the kernels \( B_a \). This is intuitively clear and can be quantified by the following semiclassical argument.

Let \( a \in \{1, ..., n\} \). To construct \( B_a \), we must first calculate \( G^+_a(x, y) \) for two points \( x \) and \( y \) under the barrier and close to \( C_a \), then apply the derivatives \( v_a \) (see Eq. (8)) and finally take \( x \) and \( y \) to \( C_a \). For simplicity, we will assume that the motion under the barrier is determined mostly by the interplay of \( U_B \) with the kinetic energy \( p^2/2m \) and neglect the influence of all other terms in \( H \).

By applying the stationary-phase approximation [15] to Feynman’s path integral for the propagator [16], we write \( G^+_a(x, y) \) as a sum over classical paths with energy \( E_F \) connecting \( y \) with \( x \). The contribution largest in magnitude comes from the path of shortest length, which, for \( U_B \) constant in the barrier region, would be a straight-line trajectory. This path, while generating the correct Green’s function singularity as the distance \( |x - y| \) goes to zero, is uninteresting here since it is an imaginary-time path without conjugate points and its contribution to \( G^+_a \) (resp. \( B_a \)) is purely real (resp. purely imaginary).

An imaginary contribution to \( G^+_a \) arises from the path that takes off from \( y \) in the direction of lead \( a \), *bounces* off the surface \( C'_a \) and then heads straight for \( x \). If \( l_B \) denotes the thickness of the tunnelling barrier and \( \kappa_F \) the average value of \( \sqrt{2m(U_B - E_F)}/\hbar \) along this path, the path’s action is roughly given by

\[
S \sim 2i\hbar\kappa_F \sqrt{l_B^2 + |x - y|^2}/4.
\]

There exist three more classical paths of a similar kind. These are the paths that are reflected at the surface \( C_a \) before heading for \( C''_a \) and/or before arriving at \( x \). They carry opposite sign factors, and their combined contribution to the semiclassical approximation for \( G^+_a(x, y) \) vanishes as \( x \) and/or \( y \) approaches \( C_a \), as it should.

In summary, by the above argument

\[
\text{Im}G^+_a(x, y) \sim -c_0d_a(x)d_a(y)e^{-2\kappa_F \sqrt{l_B^2 + |x - y|^2}/4}
\]

where \( d_a(\cdot) \) is distance from \( C_a \), and \( c_0 \) is a positive constant. Taking derivatives and sending \( x \) and \( y \) to \( C_a \), we obtain \( \text{Re}B_a(x, x) \sim \exp(-2\kappa_F l_B) \), and we arrive at the
important conclusion that the range of $\text{Re}B_a(x, y)$ is of the order of $\sqrt{l_B/\kappa F}$. This range is to be compared to the elastic mean-free length, $\ell$, which is the smallest length scale of relevance for the diffusive motion inside the conductor. Inserting reasonable numbers ($\kappa^{-1} F > l_B \sim 1\text{nm}$ and $\ell > 10\text{nm}$), we see that $\ell \gg \sqrt{l_B/\kappa F}$. It is therefore justified, for the purpose of calculating the conductance coefficients, to suppress the length scale $\sqrt{l_B/\kappa F}$ and use a local approximation for $\text{Re}B_a(x, y)$.

We are thus led to the following reduced model. We take reduced position space, $V_0$, to consist of the interior of the device only. $\partial V_0$ denotes the total boundary of $V_0$, and $C \equiv \cup_{a=1}^n C_a$ (resp. $\partial V_0 - C$) its conducting (resp. insulating) part. We introduce a phenomenological function $\beta : \partial V_0 \to \mathbb{R}$ characterizing the penetrability of the barrier; $\beta(x) \neq 0$ for $x \in C$, and $\beta(x) = 0$ for $x \in \partial V_0 - C$. $\beta$ is assumed to be spin-independent (neglecting Zeeman splitting in the presence of a magnetic field, and magnetic spin-flip and spin-orbit scattering under the barrier). We impose the boundary condition *

$$ (v_n \psi)(x) = \beta(x) \psi(x) \quad (\text{all } x \in \partial V_0). \quad (12) $$

Here $v_n$ is the component of $\mathbf{v}$, Eq. (4), normal (in the spatial sense) to $\partial V_0$. We take the Hamiltonian $H$ to be a linear operator of the general form of Eq. (2), restricted to act on functions $\psi$ supported on $V_0$ and subject to the boundary condition (12). We define $G^+(x, y)$ to be the kernel of the operator $(E_F - H)^{-1}$ with the boundary condition (12), and we set $G^-(x, y) = G^+(y, x)$. The conductance coefficients are then calculated from

$$ g_{ab} = \hbar^2 \int_{C_a} d^2 x \int_{C_b} d^2 y \beta(x) G^+(x, y) \beta(y) G^-(y, x). \quad (13) $$

This completes the definition of the reduced model.

Of course, the Hamiltonian $H$, which was a hermitean operator in the original full space (including the leads, and with the usual square-integrability condition), becomes non-hermitean upon introduction of the boundary conditions (12), unless $\beta$ vanishes identically. This non-hermitecity is an inevitable and, in fact, essential feature of any reduced description of the present type, and is particularly evident from the following alternative formulation of the reduced model.

Let $\delta_C$ be Dirac’s $\delta$-distribution with uniform support on $C$, i.e.

$$ \int_{V_0} \delta_C f d^3 x = \int_C f d^2 x, $$

* Although Dirichlet’s boundary condition might seem more realistic for the insulating part of the surface, we feel that the internal consistency of the reduced model is enhanced by using boundary conditions of the form (12) everywhere. In any case, in the diffusive regime we intend to study, the conductance coefficients and their statistical properties are affected by the conductor-insulator boundary conditions only in a minor way.
and introduce an effective Hamiltonian

\[ H_{\text{eff}} = H - \frac{i\hbar}{2} \beta \delta_C. \quad (14) \]

Then \( G^+ \), as defined above, is also the Green’s function of the linear operator \( H_{\text{eff}} \) acting on functions \( \psi \) that satisfy the generalized Neumann boundary conditions

\[ v_n \psi \big|_{\partial V_0} = 0. \quad (15) \]

To verify this statement, integrate \((E - H_{\text{eff}})\psi = 0\) along an infinitesimal piece of curve intersecting \( C \) at a right angle, and use the boundary condition (15). From the continuity of \( \psi \) across \( C \), the boundary condition (12) is then recovered. The extra term \(-i\hbar \beta \delta_C/2\) in \( H_{\text{eff}} \) has an interpretation as an “absorptive contact potential”, giving rise to the loss of probability which is caused by the escape of flux through \( C \). Note that the escape rate at \( x \in C \), per unit area and spin projection, is given by \( \beta(x)|\psi(x)|^2 \) for a state \( \psi \) with amplitude \( \psi(x) \).

5. Average conductance coefficients of decay-width dominated devices in the locally-weak absorption limit

Given an ensemble of one-particle Hamiltonians of the form specified in Sect. 2, we will now calculate for the reduced model of Sect. 4 the ensemble average \( \langle g_{ab} \rangle (a \neq b) \) for decay-width dominated conductors in the limit of locally weak absorption.

Recall first the definition of decay-width dominated conductors by the inequality \( \tau_{\text{diff}} \ll \tau_{\text{decay}} \) where \( \tau_{\text{diff}} = L^2/D \) and, with \( \text{vol} = \int_{V_0} d^3x \),

\[ \tau_{\text{decay}}^{-1} = \text{vol}^{-1} \int_C \beta d^2x. \]

Recall also the definition of the locally-weak absorption limit by \( \beta \to 0 \) and \( \int_C d^2x \to \infty \) with \( \int_C \beta d^2x \) kept fixed. A more quantitative condition for this limit to be attained is

\[ \beta(x)\ell^2 \ll \int_C \beta d^2x \quad \text{for all} \quad x \in C \quad (16) \]

where \( \ell = v_F\tau \) is the elastic mean-free path for potential scattering. The inequality (16) is motivated by the observation that, since \( \ell \) is the lower cutoff length for diffusive motion, \( \int_C d^2x \) should be compared to \( \ell^2 \).

To compute \( \langle g_{ab} \rangle \) from Eq. (13), we require the ensemble average

\[ P(x, y) = \langle |G^+(x, y)|^2 \rangle = \langle G^+(x, y)G^-(y, x) \rangle \]

for \( x \in C_a \) and \( y \in C_b \). We will now calculate this quantity in the prescribed limit. To begin, let \( x \) and \( y \) be any two points in \( V_0 \). The inequality \( \tau_{\text{diff}} \ll \tau_{\text{decay}} \) implies that an
electron moving diffusively inside the mesoscopic conductor, traverses the distance $L$ many times before being absorbed at one of the contacts (that is to say, before being emitted into one of the leads). The density $|G^+(x, y)|^2$ of a stationary state sustained by a source at $y$ therefore becomes independent of $x$ upon ensemble averaging, with two exceptions. The first occurs whenever $x$ is within a distance of $\ell$ or less from one of the contacts, where wave functions, and consequently $P(x, y)$ as well, are depressed by absorption in general. However, the condition (16) for absorption to be locally weak precisely means that this depression can be neglected under the present circumstances. The second exception occurs when $x$ lies within a distance of $\ell$ or less from the source point $y$. In this case, the value of $P(x, y)$ is modified by coherent backscattering, as is well known from weak localization physics [17]. Summarizing this paragraph, we write

$$P(x, y) = P_d + P_c(x, y)$$  \hspace{1cm} (17)

where $P_d$ is the constant part of $P$ and $P_c$ is the short-ranged backscattering correction. The sign of $P_c(x, x)$ depends on the relative strengths of potential scattering, magnetic spin-flip scattering, spin-orbit scattering and the magnetic field [12], but its magnitude is always comparable to $P_d$. If the arguments presented in support of Eq. (17) seem too heuristic, a technical derivation is given in Appendix A.

To proceed, recall that we assume the contacts $a$ and $b$ to be separated from each other by a distance of at least a few times $\ell$. By Eq. (13) we then need to know $P_d$ but not $P_c(x, y)$. Multiplying both sides of Eq. (17) with $\beta(x)$, integrating over $\int_C d^2x$, and dividing by $\int_C \beta d^2x$, we have

$$P_d = \int_C d^2x \beta(x)\left(P(x, y) - P_c(x, y)\right)/\int_C \beta d^2x. \hspace{1cm} (18)$$

The contribution to the right-hand side of (18) from $P_c$ is negligible under the condition (16), since

$$\int_C d^2x \beta(x)P_c(x, y) \simeq \ell^2\beta(y)P_c(y, y) \ll P_d \int_C \beta d^2x.$$

The remaining term can be calculated by using the identity

$$i\hbar \int_C d^2y \ G^+(x, y)\beta(y)G^-(y, x) = (G^- - G^+)\langle x, x \rangle,$$  \hspace{1cm} (19)

which is a consequence of the boundary condition (12) and of current conservation. Taking the ensemble average on both sides and using

$$\langle (G^- - G^+)\langle x, x \rangle \rangle = 2\pi i\nu,$$

where $\nu$ (independent of $x$) is the local density of states, we obtain

$$P_d = \frac{2\pi \nu}{\hbar} \left(\int_C \beta d^2x\right)^{-1}.$$
We insert this relation into the ensemble-averaged version of Eq. (13). Upon introduction of the quantities

\[ \Delta^{-1} = \nu \times \text{vol}, \quad \Gamma = \sum_{a=1}^{n} \Gamma_a, \quad \Gamma_a = \frac{\hbar}{\text{vol}} \int_{C_a} \beta d^2x \quad (a = 1, \ldots, n), \]

where \( \Delta^{-1} \) is the total density of states (counting spin), \( \Gamma \) the total decay width, and \( \Gamma_a \) the partial decay width for emission into lead \( a \), the expression for \( \langle g_{ab} \rangle \) takes the form

\[ \langle g_{ab} \rangle = \frac{2\pi}{\Delta} \times \frac{\Gamma_a \Gamma_b}{\Gamma}. \]

This is the analog of what is called the “Hauser-Feshbach formula” in the statistical theory of nuclear reactions [18]. Note the interpretation of the ratio \( \Gamma_a/\Gamma = \int_{C_a} \beta d^2x/\int_{C} \beta d^2x \) as the probability for emission of an electron from the mesoscopic conductor into lead \( a \). Note also that in comparison with the Thouless formula [19], the Thouless energy has been replaced by \( 2\pi \Gamma_a \Gamma_b/\Gamma \).

Let us mention in passing that the validity of the Hauser-Feshbach formula is not confined to the locally-weak absorption limit. As a matter of fact, all that is needed in order for \( \langle g_{ab} \rangle \) to have the factorized form of Eq. (20), is the long dwelling time of a decay-width dominated conductor, causing the processes of entry from lead \( a \) (or \( b \)) and emission into lead \( b \) (or \( a \)) to be uncorrelated. Thus, formula (20) remains valid for strong absorption if the partial decay widths \( \Gamma_a \) are replaced by more complicated, nonlinear expressions in \( \beta \). We will not elaborate upon this point here.

6. Conductance covariance function of decay-width dominated devices in the locally-weak absorption limit

Consider devices of the same special kind as before, we will now calculate the correlation function

\[ \langle g_{ab}(E_F + \delta E)g_{cd}(E_F) \rangle \]

for \( a \neq b \) and \( c \neq d \). This will be done for the reduced model of Sect. 4, taking \( \beta \) to be energy-independent. * Eventually, we will specialize to the three universality classes [20,17,12] that are known to exist for the mesoscopic conductor problem. Recall [21] that these are denoted by I (orthogonal class: potential scattering), IIa (unitary class: magnetic field), IIb (unitary class: magnetic spin-flip scattering) and III (symplectic class: spin-orbit scattering).

* The justification for neglecting variations of \( g_{ab} \) due to changes in the penetrability coefficient \( \beta \) - which are rather strong in a microscopic model with tunnelling barriers - is that such variations do not concern us here. What is measured in experiments with variable magnetic fields are the disorder-induced statistical variations of \( g_{ab} \), and it is therefore the latter that we wish to calculate.
With all the preparations made in Sect. 5, we can now be rather brief. Proceeding as before, we are led to consider the ensemble average
\[ \langle G^+(x_a, x_b; E_1)G^-(x_b, x_a; E_1)G^+(x_c, x_d; E_2)G^-(x_d, x_c; E_2) \rangle \]  
for \( x_a \in C_a, \ldots, x_d \in C_d \). Here it is necessary to distinguish cases. Let us first assume that the indices \( a, b, c \) and \( d \) are all mutually different and, as before, contacts are separated from each other by a distance of at least a few times the elastic mean-free path \( \ell \). Then the argument of Sect. 5 goes through without change and expression (21) can be approximated with negligible error by a constant, independent of \( x_a, \ldots, x_d \). Introducing the conductance covariance function
\[ C_{ab,cd}(E_1, E_2) = \langle g_{ab}(E_1)g_{cd}(E_2) \rangle - \langle g_{ab}(E_1) \rangle \langle g_{cd}(E_2) \rangle \]  
and using again Eq. (19), we find
\[ C_{ab,cd}(E_1, E_2) = \langle g_{ab}(E_1)g_{cd}(E_2) \rangle R(E_1, E_2), \]  
where
\[ R(E_1, E_2) = \frac{\langle \text{Im} G^+(E_1) \times \text{Im} G^+(E_2) \rangle}{\langle \text{Im} G^+(E_1) \rangle \langle \text{Im} G^+(E_2) \rangle} - 1, \]  
and \( G^+(E) = G^-(E) \) is the trace of the Green’s function of the Hamiltonian \( H \) with the boundary condition (12).

If, on the other hand, at least two indices in (21) coincide, or if contacts are separated by a distance less than \( \ell \), a complication arises. Consider for definiteness the case \( a = d \). There now arise contributions to the double integral \( \int_{C_a} d^2x_a \int_{C_a} d^2x_d \) from the region \( |x_a - x_d| < \ell \). In this region, the value of the otherwise constant expression (21) is modified by cross contractions between \( G^+(x_a, x_b; E_1) \) and \( G^-(x_d, x_c; E_2) \). In diagrammatic language, these give rise to impurity ladders of the same “cooperon” type that cause the backscattering correction discussed earlier. (For \( a = c \) diffuson ladders appear, too.) Because of the condition \( |x_a - x_d| < \ell \), the additional terms lead to contributions carrying the factor \( \ell^2 \int_{C_a} \beta^2 d^2x \), which is to be compared to the regular contribution, Eq. (22a), carrying the factor \( (\int_{C_a} \beta d^2x)^2 \). Thus the terms missing in Eq. (22a) are small if
\[ \ell^2 \int_{C_a} \beta^2 d^2x \ll \left( \int_{C_a} \beta d^2x \right)^2 \quad (a = 1, \ldots, n). \]  
Note that this condition is much more restrictive than (16). In summary, in order for Eq. (22) to be correct, we must either suppose that the indices \( a, b, c \) and \( d \) all differ from one another (and \( |x_i - x_j| > \ell \) for all \( x_i \in C_i, x_j \in C_j \) \( i \neq j \)) or else impose the stronger condition (23).
Eq. (22) reduces the problem of calculating $C_{ab,cd}(E_1, E_2)$ to a problem in “level statistics” for an open system. Note that in contrast to Ref. [19], where a somewhat similar relation appears, $C_{ab,cd}$ is here expressed directly in terms of $R(E_1, E_2)$ and not as a double integral over energy. The possibility to do so is offered by the limit under consideration.

The calculation of (22b) is still not easy in general but in the locally-weak absorption limit and for each of the universality classes, it reduces to a problem whose solution is known. For the following it will be convenient to represent $G^+(E) = G^-(E)$ alternatively by $G^+(E) = \text{tr}(E - H_{\text{eff}})^{-1}$ with $H_{\text{eff}} = H - i\hbar\beta\delta_C/2$ and the Neumann boundary condition (15).

Consider the matrix element $M_{kl}$ of $\hbar\beta\delta_C$ between two eigenstates $\psi_k$ and $\psi_l$ of $H$:

$$M_{kl} = \hbar \int_C \overline{\psi_k(x)} \beta(x) \psi_l(x) d^2 x.$$ 

The phases of $\psi_k$ and $\psi_l$ are randomized over a correlation length of the order of the elastic mean-free path $\ell$. We therefore expect

$$\langle M_{kl} \rangle = \delta_{kl} \frac{\hbar}{\text{vol}} \int_C \beta d^2 x = \delta_{kl} \Gamma$$

and, by the law of large numbers,

$$\text{var} M_{kl} \sim O \left( \int_C d^2 x / \ell^2 \right)^{-1}.$$ 

These estimates mean that $\hbar\beta\delta_C$ acts effectively as a multiple of the unit operator under the condition (16). In other words, for the purpose of calculating $R(E_1, E_2)$ we are permitted to approximate $G^+(E)$ in Eq. (22b) by $\text{tr}(E + i\Gamma/2 - H)^{-1}$. To avoid possible misconceptions, we stress that the approximation $\hbar\beta\delta_C \simeq \Gamma \times 1$ must not be overinterpreted. For the case $\Gamma \gg \Delta$, which is included in the limit where our considerations apply, the operator $\hbar\beta\delta_C$ does cause appreciable mixing between states close in energy. However, such mixing does not modify the correlation properties of the spectrum. In particular, it does not reduce the spectral rigidity. The approximation is therefore justified when used for the purpose of calculating correlation functions such as $R(E_1, E_2)$.

For the special case $\Gamma = 0$, the two-level correlation function resulting from the substitution $G^+(E) \rightarrow \text{tr}(E + i\Gamma/2 - H)^{-1}$ in Eq. (22b) has been calculated exactly for each universality class by Efetov [22] in his work on the level statistics of small metallic particles. It is straightforward to extend Efetov’s calculation to $\Gamma \neq 0$. The result for $R(E_1, E_2)$, obtained by performing this analytic continuation, is quoted in Appendix B.

Here we refrain from considering the general case but specialize to $\Gamma \gg \Delta$, the case of an “open” system, which is also accessible (without Efetov’s results) via standard diagrammatic perturbation theory. Using either the exact result of Appendix B, or more simply
the perturbative cooperator-diffuson expansion - see Eq. (33) of Ref. [19] - one obtains

\[ R_k(E_1, E_2) = -\frac{c_k\Delta^2}{4\pi^2} \Re (E_1 - E_2 + i\Gamma)^{-2}, \quad (24) \]

where the distinction between universality classes is contained in the coefficients \( c_I = 16, c_{IIa} = 8, c_{III} = 4 \) and \( c_{IIb} = 2 \). Insertion of (24) into (22a) and use of Eq. (20) gives

\[ C_{ab,cd}(E_1, E_2) = c_k p_a p_b p_c p_d \frac{1 - (E_1 - E_2)^2/\Gamma^2}{(1 + (E_1 - E_2)^2/\Gamma^2)^2} \quad (25) \]

where \( p_a = \Gamma_a/\Gamma, p_b = \Gamma_b/\Gamma \) etc.

The universal numbers \( c_k \) (\( k = I, ..., III \)) reflect the influence of symmetries on the conductance fluctuations. Breaking of time-reversal invariance (I \( \rightarrow \) IIa or III \( \rightarrow \) IIb) causes a reduction by a factor of 2, and breaking of spin-rotation invariance (I \( \rightarrow \) III or IIa \( \rightarrow \) IIb) causes a reduction by a factor of 4. These reduction factors are not new but were already observed in [19].

What is more striking about Eq. (25) is the non-Lorentzian dependence of \( C_{ab,cd} \) on \((E_1 - E_2)/\Gamma\) and the validity of the sum rule

\[ \int_{\mathbb{R}} C_{ab,cd}(E, E') dE' = 0, \quad (26) \]

which is seen most easily from Eq. (24) and Cauchy’s theorem. We emphasize that the result (26) does not hold in general but is a special feature of the locally-weak absorption limit. (It is true, however, for any value of the decay width \( \Gamma \) of decay-width dominated conductors; see Appendix B.) To understand this feature, we recall that the rigidity of the energy spectrum of an isolated system is preserved for an open (but decay-width dominated) system satisfying the condition (16): energy levels are, roughly speaking, shifted by a common amount \( E \rightarrow E - i\Gamma/2 \). The negative tail, or “correlation hole”, of \( R(E_1, E_2) \) for \(|E_1 - E_2| > \Gamma\) is a direct consequence of this rigidity.

The conductance fluctuations result from Eq. (25) by setting \( E_1 = E_2 \):

\[ \langle \delta g_{ab}\delta g_{cd} \rangle = c_k p_a p_b p_c p_d. \]

These fluctuations are universal in the sense that they are sensitive - leaving aside the dependence on symmetry - only to the geometry of the \( n \)-probe conductor as expressed by the coefficients \( p_a \) (\( a = 1, ..., n \)). However, this is not a universality in the strictest sense of the word since \( p_a = \Gamma_a/\sum_{k=1}^n \Gamma_k \) does become smaller as the total number \( n \) of contacts increases. Strict universality is recovered by dividing the contacts into two groups, say “left” (L) and “right” (R), and summing over \( a, c \in L \) (\( a \neq c \)) and \( b, d \in R \) (\( b \neq d \)).
7. Summary

This paper was in two parts. In the first part, we addressed the practical question of how to calculate the conductance coefficients $g_{ab}$ ($a, b = 1, ..., n$) from the linear-response formula (3a-c). Supposing that the statistical properties of $g_{ab}$ are determined mostly by the diffusive motion of electrons inside the mesoscopic conductor, it is natural to try and eliminate the leads from the theoretical description in favour of suitable boundary conditions imposed on the contact surfaces $C_a$ ($a = 1, ..., n$). This idea is usually implemented in a very schematic fashion. Refs. [2-4] invoke the condition $P_{d} = 0$ (resp. $\nabla_n P_{d} = 0$) on the conducting (resp. insulating) part of the surface, at the level of constructing the diffusion propagator $P_d$. Such a condition, while reasonable for thick leads and good contacts, is too crude for thin leads and/or weak contacts. (The authors of Ref. [4] are of course aware of this limitation. This is why for the thin-lead case they adopt the trick of making a somewhat arbitrary division of the leads into clean and dirty regions.)

Inspection of Eqs. (3a-c) shows that what we need to do is to calculate the covariant derivatives $(v_a G^\pm)(x, y)$ for $x \in C_a$ ($a = 1, ..., n$). In Sect. 3 we restated the problem of making this calculation on the total space (including the leads) as an exactly equivalent but reduced problem formulated on the bounded domain of the mesoscopic conductor. The reduced formulation involves a set of integral kernels $B_a$, see Eq. (8), which define boundary conditions expressing the covariant derivative of $G^\pm$ normal to $C_a$ in terms of the values of $G^\pm$ on $C_a$ ($a = 1, ..., n$). These nonlocal boundary conditions cause the reduced (or effective) Hamiltonian to be non-hermitean and energy-dependent. $B_a$ is determined exclusively by the properties of electronic motion in the leads and is therefore easy to control when the leads are clean; see Eq. (11). (We did not translate the boundary conditions (9) with $B_a$ given by (11) into a corresponding boundary condition for the diffusion propagator $P_d$. How this is done is described at length for a different microscopic model in Ref. [5]; see also Sect. 7 of Ref. [23] for some useful mathematical details.)

Another example where $B_a$ can be controlled are conductors with tunnelling barriers. The semiclassical argument given in Sect. 4 shows that a local approximation to $B_a$ may be used in this case. This argument led to the formulation of a phenomenological model, whose diffusion propagator $P_d$ satisfies

$$-\hbar D (\nabla^2 P_d)(x, y) = 2\pi \nu \delta(x - y)$$

with the boundary condition $-D \nabla_n P_d = \beta P_d$ on the surface $C$ of the conductor.

In the second part of the paper, the phenomenological model of Sect. 4 was analyzed for decay-width dominated conductors in the locally-weak absorption limit. The technical simplification occurring for decay-width dominated conductors is that the spectral expansion of $(-\hbar D \nabla^2)^{-1}$ is dominated by the smallest eigenvalue, $\lambda_0$, of $-\hbar D \nabla^2$ (with the specified boundary condition) in this case. $\lambda_0$ and the next eigenvalue, $\lambda_1$, are given by

$$\lambda_0 \simeq \Gamma = \frac{\hbar}{\text{vol}} \int_C \beta d^2 x,$$

$$\lambda_1 \simeq \Gamma + \text{const} \times E_c \quad (E_c = \hbar D/L^2).$$
Note that for a three-dimensional mesoscopic conductor the limit $\lambda_0 \ll \lambda_1$ is approached by keeping the conducting part of the surface constant in size and making $L$ larger.

The locally-weak absorption limit is defined technically by the condition (16). It guarantees the correlation properties of the eigenvalues of the effective Hamiltonian $H_{\text{eff}} = H - i\hbar\beta\delta_C/2$ to be the same as those of $H - i\Gamma/2$. In other words, the eigenvalues of the Hamiltonian $H$ of the isolated system are simply shifted, roughly speaking, by a constant amount $-i\Gamma/2$.

By expressing the conductance fluctuations as a density-of-states correlation function in Eq. (22) - which is rigorously justified in the prescribed limit and for $a, b, c, d$ all mutually different - we obtained

$$\langle \delta g_{ab}(E_1)\delta g_{cd}(E_2) \rangle \sim \frac{1 - (E_1 - E_2)^2/\Gamma^2}{(1 + (E_1 - E_2)^2/\Gamma^2)^2}.$$  

The interesting feature of this expression is that it turns negative at $|E_1 - E_2| > \Gamma$. The negative tail can be understood as being a consequence of the spectral rigidity of $H$.

Finally, let us translate the correlation energy $\Gamma$ into the corresponding magnetic-field correlation scale $B_c$ measurable in experiments on a single sample. To do that, we will follow a procedure described in Ref. [4]. A difference in magnetic field $B \rightarrow B + \delta B$ couples into the equation for the diffusion propagator by the substitution $\nabla \rightarrow \nabla - ie\delta A/\hbar$ where $\delta B = \text{rot}\delta A$. We need to calculate the correction to $\lambda_0$ in lowest order of $\delta B$. Perturbation theory gives $\lambda_0 \rightarrow \lambda_0 + \text{const} \times E_c(\delta \phi/\phi_0)^2$ where $\delta \phi/\phi_0$ is the change in magnetic flux through the conductor, measured in units of the flux quantum $\phi_0 = \hbar/e$. By equating the correction to $\Gamma$, we obtain

$$B_c L^2 \sim \phi_0 \sqrt{\Gamma/E_c}.$$  

Thus, in comparison with the case of thick-lead conductors with good contacts, $B_c$ is reduced by a factor of $\sqrt{\Gamma/E_c}$.

**Acknowledgment.** This work was supported in part by the Sonderforschungsbereich 341 Köln-Aachen-Jülich. I thank M. Büttiker and H.A. Weidenmüller for useful discussions.
Appendix A

To establish Eq. (17) on a rigorous level, we find it convenient to make use of the elegant and powerful technique of Efetov [21], who has shown that $P(x, y) = \langle |G^+(x, y)|^2 \rangle$ can - in the diffusive regime - be regarded as the propagator of certain Q-matrix superfields with “free energy” functional $\mathcal{F}[Q]$. It is not difficult to adapt Efetov’s treatment to the reduced model with boundary conditions (12) or the alternative formulation in terms of the effective Hamiltonian (14). For simplicity, we assume that electrons are spinless and subject to potential scattering only. In this case, one finds that Eq. (3.53) of Ref. [21] is replaced by

$$\mathcal{F}[Q] = -\frac{\hbar \pi \nu}{8\tau} \int_{V_0} \text{str} Q^2(x) d^3 x + \frac{1}{2} \int_{V_0} \left( \text{str} \ln(E_F - \frac{p^2}{2m} + \frac{i\hbar}{2\tau} Q + \frac{i\hbar}{2} \beta \Lambda \delta_C) \right)(x, x) d^3 x,$$

and $P(x, y)$ is a sum of diffusion (d) and cooperon (c) contributions, $P = P_d + P_c$, where

$$P_d(x,y) = \int \mathcal{D}Q \; G_d^+(x,x;Q)G_d^-(y,y;Q)e^{-\mathcal{F}[Q]}, \quad (A.1a)$$

$$P_c(x,y) = \int \mathcal{D}Q \; G_c^+(x,y;Q)G_c^-(x,y;Q)e^{-\mathcal{F}[Q]}, \quad (A.1b)$$

and

$$G(x,y;Q) = \left( E_F - \frac{p^2}{2m} + \frac{i\hbar}{2\tau} Q + \frac{i\hbar}{2} \beta \Lambda \delta_C \right)^{-1}(x,y).$$

The notations used are either self-explanatory or those of Ref. [21]. In the next step, one makes the usual saddle-point approximation followed by a gradient expansion [21], valid in the metallic regime $k_F \ell \gg 1$. Retaining in $\mathcal{F}$ only terms up to linear order in $\beta$, one obtains

$$\mathcal{F}[Q] \simeq \frac{\hbar \pi \nu}{8} \left( D \int_{V_0} \text{str}(\nabla Q)^2 d^3 x + 2 \int_C \beta \text{str}\Lambda Q d^2 x \right) \quad (A.2)$$

where $Q$ satisfies the nonlinear constraint $Q^2 = 1$. Eq. (A.2) omits such terms as

$$\text{const} \times (\hbar \pi \nu)^2 \ell^2 \int_C \beta^2 \text{str}(\Lambda Q)^2 d^2 x.$$

These are negligible under the locally-weak absorption condition (16). With the same degree of accuracy, one may use the approximation $G(x,x;Q) \simeq -i\pi \nu Q(x)$, which leads to

$$P_d(x,y) = -\pi^2 \nu^2 \int \mathcal{D}Q \; Q_d^+(x)Q_d^-(y)e^{-\mathcal{F}[Q]}.$$

To analyze this expression further, one observes that the condition $\tau_{\text{diff}} \ll \tau_{\text{decay}}$ for a conductor to be decay-width dominated can be written

$$\frac{\hbar D}{L^2} \gg \frac{\hbar}{\text{vol}} \int_C \beta d^2 x.$$
A glance at Eq. (A.2) then shows that the dominant contribution to $P_d(x, y)$ comes from spatially constant $Q$-fields, spatially varying $Q$-fields being separated from the constant ones by a large gap in free energy. This proves the independence of $P_d(x, y)$ of $x$ and $y$.

The other properties used in Sect. 5 are $P_c(x, x) \sim P_d$ and the short range of $P_c(x, y)$. These follow from Eq. (A.1b) and

$$G(x, y; Q) \simeq \left(E_F - \frac{p^2}{2m} + i\hbar \frac{Q}{2\tau}\right)^{-1}(x, y) \simeq f(x - y)Q((x + y)/2)$$

where $f(0) = -i\pi\nu$, and $f$ decays over a length scale of the order of the elastic mean-free path.

Finally, we observe that the various perturbations in $H$, Eq. (2), which break time-reversal and spin-rotation symmetry, do not affect the diffusion degrees of freedom, whose free energy is always given by a functional of the form (A.2). Therefore, $P_d$ is constant in general.

Appendix B

With $\Gamma = \hbar \int_C \beta d^2x/\text{vol}$ the decay width, let $\rho_\Gamma(E)$,

$$\rho_\Gamma(E) = -\frac{1}{\pi} \text{Im} \text{tr}(E + i\Gamma/2 - H)^{-1},$$

be the total density of states (counting spin) of the open system. We will write down exact expressions for the two-level correlation function

$$R(E_1, E_2; \Gamma) = \frac{\langle \rho_\Gamma(E_1)\rho_\Gamma(E_2) \rangle}{\langle \rho_\Gamma(E_1) \rangle \langle \rho_\Gamma(E_2) \rangle} - 1$$

for all universality classes, obtained by analytic continuation of Efetov’s results [22].

We introduce the functions $(C \to C)$

$$f(z) = -\frac{e^{i\pi z} \sin \pi z}{i \pi^2 z^2},$$  \hspace{1cm} (B.1a)

$$g(z) = \left(\frac{1}{\pi} \frac{\partial}{\partial z} \frac{e^{i\pi z}}{i\pi z}\right) \int_0^1 \frac{\sin \pi z t}{t} dt,$$  \hspace{1cm} (B.1b)

$$h(z) = \left(-\frac{1}{\pi} \frac{\partial}{\partial z} \frac{\sin \pi z}{\pi z}\right) \int_1^\infty \frac{e^{i\pi z t}}{it} dt,$$  \hspace{1cm} (B.1c)

and, with $\Delta^{-1} = \langle \rho_\Gamma(E_F) \rangle$, we set

$$F_1(z\Delta) = f(z/2) + g(z/2),$$

$$F_{1a}(z\Delta) = f(z/2), \quad F_{1b}(z\Delta) = f(z),$$

$$F_{III}(z\Delta) = f(z) + h(z).$$  \hspace{1cm} (B.2)
$R$ is then given by

$$R_k(E_1, E_2; \Gamma) = \frac{1}{2} \left( F_k(E_1 - E_2 + i\Gamma) + F_k(E_2 - E_1 + i\Gamma) \right) \quad (B.3)$$

in all cases ($k = I, IIa, IIb, III$).

Note that the function $f$, Eq. (B.1a), is holomorphic for $z \in \mathbb{C} - \{0\}$ and vanishes as $z^{-2}$ at infinity in the upper half of the complex plane. Hence, by Cauchy’s theorem,

$$\int_{R} f(x + i\gamma) dx = 0 \quad \text{for} \quad \gamma > 0.$$  

The same statements apply to the functions $g$ and $h$, Eqs. (B.2b) and (B.2c), which leads to the sum rule

$$\int_{R} R(E, E'; \Gamma) dE' = 0.$$  

The result (24) for $R(E_1, E_2; \Gamma)$ quoted in Sect. 6 follows from the asymptotic expansions

$$f(i\gamma) = \frac{1}{2\pi^2\gamma^2} \left( 1 - e^{-2\pi\gamma} \right), \quad (B.4a)$$
$$g(i\gamma) = \frac{1}{2\pi^2\gamma^2} \left( 1 + \frac{2}{\pi\gamma} + O(\gamma^{-2}) \right), \quad (B.4b)$$
$$h(i\gamma) = \frac{1}{2\pi^2\gamma^2} \left( 1 - \frac{2}{\pi\gamma} + O(\gamma^{-2}) \right), \quad (B.4c)$$

for $\gamma \gg 1$. Note that Eq. (B.4a) shows the corrections to $R_k$, Eq. (24), for $k = IIa$ and $IIb$ to be exponentially small and therefore not calculable by perturbation expansion in $(E_1 - E_2 + i\Gamma)^{-1}$. Furthermore, from the exact expressions (B.1-3) one can obtain the conductance fluctuations $\langle \delta g_{ab} \delta g_{cd} \rangle$ for small values of $\Gamma$, which are not accessible by perturbation theory either. Both $f(i\gamma)$ and $g(i\gamma)$ behave for $\gamma \to 0$ as $(\pi\gamma)^{-1}$, while $h(i\gamma) \simeq \frac{1}{3} \pi\gamma \ln \pi\gamma$. Insertion of these limiting forms into Eqs. (B.2), (B.3) and (22a) gives

$$\langle \delta g_{ab} \delta g_{cd} \rangle \simeq \text{const} \times p_a p_b p_c p_d \times \Gamma / \Delta.$$  

This shows that the conductance fluctuations go to zero for $\Gamma \to 0$, which is the limit of an isolated system. In other words, the behavior of $\langle \delta g_{ab} \delta g_{cd} \rangle$ is dominated by the decrease of $\langle g_{ab} \rangle \langle g_{cd} \rangle \sim \Gamma^2 / \Delta^2$ for $\Gamma \to 0$, overpowering the increase in relative fluctuations due to the appearance of isolated resonances. This conclusion is contrary to the one reached in Ref. [4]. We attribute the discrepancy to overextension, in Ref. [4], of perturbation theory to a regime where it does not apply.
References

[1] S. Washburn and R.A. Webb, Adv. Phys. 35 (1986) 375.
[2] P.A. Lee and A.D. Stone, Phys. Rev. Lett. 55 (1985) 1622; B.L. Al’tshuler, Pis’ma Zh. Eksp. Teor. Fiz. 41 (1985) 530 [JETP Lett. 41 (1985) 648].
[3] P.A. Lee, A.D. Stone and H. Fukuyama, Phys. Rev. B35 (1987) 1039.
[4] R.A. Serota, S. Feng, C. Kane and P.A. Lee, Phys. Rev. B36 (1987) 5031.
[5] S. Iida, H.A. Weidenmüller and J.A. Zuk, Ann. Phys. 200 (1990) 219; A. Altland, Z. Phys. B82 (1991) 105.
[6] H.A. Weidenmüller, Physica A167 (1990) 28.
[7] P.L. Kapur and R.E. Peierls, Proc. Roy. Soc. (London) A166 (1938) 277.
[8] L. Eisenbud and E.P. Wigner, Nuclear Structure, Princeton University Press, Princeton (1958).
[9] A.M. Lane and R.G. Thomas, Rev. Mod. Phys. 30 (1958) 257.
[10] C. Mahaux and H.A. Weidenmüller, Shell-Model Approach to Nuclear Reactions, North-Holland Publ. Co., Amsterdam (1969).
[11] H.L. Harney and A. Hüpper, Z. Phys. A328 (1987) 327; H.L. Harney, A. Hüpper, M. Mayer and A. Müller, Z. Phys. A335 (1990) 293; A. Müller and H.L. Harney, Z. Phys. A337 (1990) 465.
[12] P.A. Lee and T.V. Ramakrishnan, Rev. Mod. Phys. 57 (1985) 287.
[13] H. Baranger and A.D. Stone, Phys. Rev. B40 (1989) 8169.
[14] M. Janssen, Sol. State Commun. 79 (1991) 1073.
[15] M.C. Gutzwiller, Chaos in Classical and Quantum Mechanics, Springer, New York (1990).
[16] R.P. Feynman and A.R. Hibbs, Quantum Mechanics and Path Integrals, McGraw-Hill, New York (1965).
[17] G. Bergmann, Phys. Rep. 107 (1984) 1.
[18] W. Hauser and H. Feshbach, Phys. Rev. 87 (1952) 366.
[19] B.L. Al’tshuler and B.I. Shklovskii, Zh. Eksp. Teor. Fiz. 91 (1986) 220 [Sov. Phys. JETP 64 (1986) 127].
[20] S. Hikami, A.I. Larkin and Y. Nagaoka, Prog. Theor. Phys. 63 (1980) 707.
[21] K.B. Efetov, Adv. Phys. 32 (1983) 53.
[22] K.B. Efetov, Sov. Phys. JETP 56 (1982) 467.
[23] J.J.M Verbaarschot, H.A. Weidenmüller, M.R. Zirnbauer, Phys. Rep. 129 (1985) 367.