Spacetime Reduction of Large N Flavor Models: A Fundamental Theory of Emergent Local Geometry?

Shyamoli Chaudhuri

214 North Allegheny Street
Bellefonte, PA 16823, USA

Abstract

We introduce a novel spacetime reduction procedure for the fields of a supergravity-Yang-Mills theory in generic curved spacetime background, and with large $N$ flavor group, to linearized forms on an infinitesimal patch of local tangent space at a point in the spacetime manifold. Our new prescription for spacetime reduction preserves all of the local symmetries of the continuum field theory Lagrangian in the resulting zero-dimensional matrix Lagrangian, thereby obviating difficulties encountered in previous matrix proposals for emergent spacetime in recovering the full nonlinear symmetries of Einstein gravity. It also obviates the challenges that must be faced by any proposal for a fundamental theory, holographic or topological, where gravity emerges instead as an induced interaction. We conjecture that the zero-dimensional matrix model obtained by this prescription for spacetime reduction of the circle-compactified type I-$I'$-IIA-IIB heterotic supergravity-Yang-Mills theory with sixteen supercharges and large $N$ flavor group, and inclusive of the full spectrum of Dpbrane charges, $-2 \leq p \leq 9$, offers a potentially complete framework for nonperturbative string/M theory. We analyze the matrix Lagrangian in detail, comparing with the results of traditional planar reduction, and clarifying the emergence of the spacetime continuum in the large $N$ limit of the zero dimensional matrix model. We explain the relationship of our conjecture for a fundamental theory of emergent local spacetime geometry to recent investigations of the hidden symmetry algebra of M theory, stressing insights that are to be gained from the algebraic perspective. We conclude with a list of open questions and directions for future work.

1Current Address: 1312 Oak Drive, Blacksburg, VA 24060. E-mail: shyamolic@yahoo.com
1 Introduction

In this paper, we present a zero-dimensional matrix Lagrangian with sixteen supercharges, and an extended symmetry group, which we will argue has the potential to describe a multi-dimensional emergent local spacetime geometry in the large $N$ limit, and which might also give a unified framework for all of the weakly-coupled string and field theory limits of nonperturbative string/M theory. This new matrix Lagrangian belongs to a much more general class of matrix models conjectured to exist on the basis of algebraic symmetries alone in [1]. The matrix Lagrangian discussed in this paper is therefore distinguished by its simplicity: the $U(N)$ symmetry is assumed to be a flavor symmetry, commuting with the global symmetries that assume the role of target spacetime (supersymmetry) $\times$ (Lorentz) $\times$ (Yang-Mills) symmetry in the continuum target-space Lagrangian obtained in the large $N$ limit.

In what follows, we will explain the appearance of such novel matrix Lagrangians with extended symmetry from the perspective of supergravity hidden symmetry algebras [3, 5], on the one hand, and from the notion of a modified prescription for the planar reduction of large $N$ field theories [2] on the other. We will show that a subset of the matrix Lagrangians introduced by us in [1] can be derived by a novel spacetime reduction procedure for the spacetime fields of a higher-dimensional supergravity-Yang-Mills field theory to linearized forms on an infinitesimal patch of local tangent space at a single point in the spacetime manifold. The result of our new prescription for spacetime reduction of a continuum Lagrangian with large $N$ flavor symmetry is in each case a zero dimensional matrix model Lagrangian, where the large $N$ index originates in the auxiliary large $N$ flavor symmetry introduced in the higher dimensional supergravity-Yang-Mills theory. We should emphasize that the quantum dynamics of the higher dimensional classical field theory Lagrangian with large $N$ flavor group is of no interest to us; the introduction of the auxiliary flavor index in the continuum Lagrangian is simply a technical device that enables straightforward derivation of a zero-dimensional matrix Lagrangian manifesting certain desired extended symmetries. Hopefully, authors other than myself will succeed in simply doing away with the “crutch” of spacetime reduction, writing down the zero-dimensional matrix Lagrangian for the proposed fundamental theory of emergent spacetime geometry from first principles on the basis of algebraic symmetries alone, as explained in Footnote 2. We reiterate that it is the large $N$ quantum dynamics of this matrix Lagrangian that is of fundamental physical significance to nonperturbative String/M theory.

In the analysis of the matrix superalgebra at finite $N$ outlined in [1], the parameters for infinitesimal supersymmetry and $SL(n, \mathbb{R})$ transformations were assumed, more generally, to transform as non-singlets under the flavor $U(N)$ group. While we know of no reason to rule out such exotic extensions from the perspective of the matrix Lagrangian, such an extension is not necessary for the problem at hand. We should clarify that the matrix Lagrangian given in [1] was guessed at by intuition alone; the detailed checks of the symmetry transformations necessary to confirm the closure of such an exotic matrix algebra from first principles are quite beyond this author’s ability, although we encourage the reader to try. An alternative route that can reproduce the form of the matrix Lagrangian first given in [1] is provided in this paper, based upon the spacetime reduction of a supergravity-Yang-Mills Lagrangian with large $N$ flavor symmetry. Such a derivation obviates the necessity for a first principles check of the matrix superalgebra at finite $N$, while also illuminating the connection to continuum physics in the large $N$ limit. Notice that, for the more general class of matrix superalgebras described in [1], the large $N$ limit would have to correspond to an exotic (nonlinear) extension of the Nahm classification of spacetime linear superalgebras in $n$ target spacetime dimensions. I am grateful to Bernard de Witt for pointing out this distinction. The idea of motivating the matrix Lagrangian from a generalized spacetime reduction procedure, combining insights from both supergravity dimensional reduction [3, 5] and Eguchi-Kawai rigid Yang-Mills planar reduction [2], was inspired by Hermann Nicolai’s comments in [15].
In what follows, we will present a novel prescription for the spacetime reduction of the continuum fields of a supergravity-Yang-Mills theory in generic curved spacetime background, and with large $N$ flavor group, to linearized forms on an infinitesimal patch of local tangent space at a single point in the spacetime manifold, and which results in a new class of zero-dimensional supermatrix models first presented by us in [1]. Our novel prescription combines insights from both supergravity dimensional reduction [3, 5] as well as the well-known Eguchi-Kawai planar reduction of rigid Yang-Mills theories [2], extended to the case of rigid supersymmetric large $N$ Yang-Mills in the IKKT matrix model [12]. We should clarify that, unlike the IKKT matrix model which corresponds to a modification of the planar reduced rigidly supersymmetric large $N$ Yang-Mills theory, the class of matrix models studied in this paper, and in [1], arise from the spacetime reduction of a higher-dimensional continuum Lagrangian with large $N$ flavor symmetry. In each case, the target spacetime (supersymmetry) $\times$ (Lorentz) $\times$ (Yang-Mills gauge) symmetries commute with $U(N)$ flavor symmetry. Notice, also, that we are proposing an entirely different origin for continuum target spacetime symmetries from that conjectured by Iso and Kawai [6]. Their idea was that continuum target spacetime symmetries, such as the finite-dimensional Yang-Mills gauge symmetry of the continuum Lagrangian, should already be present in the matrix Lagrangian, embedded within the large $N$ gauge group. This proposal is difficult to realize in practice for standard choices of the large $N$ gauge group as can be seen by simple group theoretic reasoning.\(^4\)

Let us begin with a quick survey of some pertinent insights from the relevant older works. It is well-known that the toroidally-compactified eleven-dimensional supergravity, as well as the ten-dimensional type I-I', type IIA-IIB, and the heterotic $E_8 \times E_8$ and $Spin(32)/Z_2$, string supergravities, exhibit extended global symmetries as a consequence of the presence of massless scalar fields in the dimensionally-reduced supergravity Lagrangian. In 1978, Cremmer and Julia noticed that the dimensional reduction of a $(D+n)$-dimensional theory containing gravity to $D$ dimensions necessarily results in the appearance of a $SL(n, \mathbb{R})$ global symmetry, as viewed from the perspective of the $D$-dimensional spacetime [3]. This symmetry is manifest in the form of the dimensionally-reduced Lagrangian. Including an overall scaling of the volume of the compactification manifold, the global symmetry group of the Lagrangian takes the precise form $GL(n, \mathbb{R}) \sim SL(n, \mathbb{R}) \times \mathbb{R}$; the $\mathbb{R}$ factor is, therefore, a hidden symmetry of the dimensionally-reduced Lagrangian. Notice that only the volume preserving subgroup, $SL(n, \mathbb{R})$, is relevant to the reduction of a field theory to a single spacetime point. Recall that eleven-dimensional supergravity is one of the field theoretic low energy limits of M theory. In [3], Cremmer and Julia conjectured, with partial proof, that the dimensional reduction of 11d supergravity to a Lagrangian in $11-n$ dimensions would result in the appearance of the hidden symmetry group $E_n$. For $n \geq 3$, this conjecture has since been verified by direct field-theoretic duality transformations on the fields in the dimensionally-reduced classical supergravity Lagrangian [5]. Finally, note that if we continue Cremmer and Julia’s sequence of dimensional reductions of 11d supergravity to lower dimensions to its logical endpoint, namely, to zero spacetime dimensions, we have the prediction $E_{11}$ for the hidden symmetry group. Thus, it is natural to expect that

\(^3\)By linearized pforms, we mean that each spacetime field, or associated pform, is truncated at linear order in a Taylor expansion about its value at the origin of the infinitesimal patch of tangent space, coincident with a single point in the spacetime manifold. The truncation at linear order is simply a reflection of the fact that tangent space has infinitesimal extent: higher order partial derivatives exist to all orders, but the Taylor expansion is truncated because quadratic and higher powers of the infinitesimal in tangent space are dropped. For more details, see Section 3.

\(^4\)I thank B. de Witt for a discussion of this point.
the extended symmetry group of the zero-dimensional matrix model obtained from the spacetime reduction of a theory with 32 supercharges would be $E_{11}$ [30, 32]. This simple observation will be developed, and refined further, in sections 4 and 5 of this paper.

Planar reduction was first applied to the bosonic rigid large $N$ Yang-Mills theory by Eguchi and Kawai in 1980 [2]. Dimensional reduction of a rigid $U(N)$ Yang-Mills gauge theory to a single spacetime point gives what is known as a reduced unitary matrix model: naively, we set to zero all spacetime derivatives in the Yang-Mills action, retaining the $U(N)$ trace of the square of the commutator of $N \times N$ unitary matrices. The Lagrangian gives a zero-dimensional unitary matrix model with a quartic self-interaction. Reduced matrix models arise, therefore, as the result of a dramatic thinning of the infinite number of degrees of freedom of a quantum field theory upon dimensional reduction of all spacetime fields to a single spacetime point. Remarkably, planar reduced matrix models are found to share many features of exactly solvable unitary matrix models. It should be emphasized that many of the notions familiar from continuum quantum field theory, such as renormalization, universality classes, vacuum structure, and spontaneous symmetry breaking, have their counterpart in the matrix models that follow from spacetime reduction. Likewise, supermatrix models are obtained when one dimensionally reduces a rigid supersymmetric large $N$ Yang-Mills theory to a spacetime point. Such supermatrix models have been the basis of previous conjectures for nonperturbative string/M theory [11, 12].

With the discovery of Dirichlet-pbranes by Dai, Leigh and Polchinski [8], and with their crucial role as solitonic carriers of dual electric-magnetic charge in the type I and type II string supergravities clarified by Polchinski in [9], the dimensional reduction of rigid Yang-Mills theories has found an alternative, and rather interesting, new interpretation. Recall that in open and closed string theories, $n$ successive T-duality transformations on $n$ spacetime coordinates parallel to the worldvolume of $N$ coincident D9branes in the type IB string theory carrying 10d nonabelian Yang-Mills gauge fields: $R_n \to \alpha'/R_n$, where $n \leq 10$, converts $n$ components of the worldvolume gauge bosons to the $n$ components of a scalar field in the $n$-dimensional spatial bulk orthogonal to the D(9-n)brane [8]. The vacuum expectation values of the $n$ components of the scalar field can be interpreted as the coordinate locations of the D(9-n)brane soliton in an $n$-dimensional space. In open string theory, this scalar excitation has as vertex operator $(\partial_z X^i_\mu(z))^2$, where $\mu=1, \cdots, n$, and $i=1, \cdots, N$. As first noted by Witten, this implied the tantalizing fact that the “coordinates” of space orthogonal to the $N$ D(9-n)branes arise as the $N$ eigenvalues of $n$ noncommuting, $N \times N$, unitary matrices. For example, with 9 spatial dualizations, we have 9 collective coordinates for the $N$ coincident D0brane solitons: $A^i_\mu(x^0) \leftrightarrow X^i_\mu(x^0)$, $i = 1, \cdots, N$, and $\mu = 1, \cdots, 9$, where $x^0$ is time. Here, $i$ is the Chan-Paton index, and the gauge group realized on $N$ coincident D0branes is the nonabelian group $U(N)$, of rank $N$. In the unoriented type I string theory we obtain, instead, the orthogonal group $SO(2N)$ as worldvolume gauge group [42].

More generally, the $X^\mu$ coordinate location of the $i$th D0brane is the $i$th eigenvalue of the $U(N)$
matrix $X^i_{\mu}, i=1, \cdots, N$, described above. Restricting to the Yang-Mills field theory on the one-dimensional worldvolume of the D0branes, we have a worldvolume Lagrangian that agrees precisely with the dimensional reduction of the 10d nonabelian Yang-Mills Lagrangian. This gives the familiar quartic interaction for one-dimensional $N \times N$ matrices [8, 10]. Such matrix Hamiltonians describe the quantum mechanics of, time-dependent, large $N$ unitary matrices, as in the Banks-Fischler-Shenker-Susskind proposal for M(atrix) Theory [11]. Planar reduced matrix models, akin to the Ishibashi-Kawai-Kitazawa-Tsuchiya IIB Matrix Model [12], follow as the result of taking this logic one step further: we must T-dualize all ten directions of spacetime. The coordinates $X^i_{\mu}$, with $i=1, \cdots, N$, and $\mu = 0, \cdots, 9$, can now be interpreted as the locations of $N$ Dinstanton events in a bulk ten-dimensional spacetime. Recall that the tension of a Dinstanton has mass dimension zero. Thus, such a matrix Lagrangian has no dimensionful couplings and is reminiscent of a topological theory.

It should be emphasized that the perspective we have just described views the D(9-n)branes as fundamental degrees of freedom in M theory, but also as semi-classical solitons that exist in an $n$-dimensional classical flat spacetime geometry: an underlying spacetime continuum has been assumed. Thus, the M(atrix) theory conjecture of [11] proposes an especially simple Hamiltonian for an especially simple sector of M theory: the remnant fundamental degrees of freedom that survive decoupling in the infinite momentum frame. It should be noted that both M(atrix) Theory [11], and the IIB Matrix Model [12], are therefore conjectured theories of emergent, or induced, linearized gravity: Newtonian gravity appears as an effective long-distance interaction of fundamental, pointlike, degrees of freedom, respectively, D0branes or Dinstantons, in an embedding flat spacetime background. Reconstructing the full nonlinear structure of the Einstein gravitational interaction from this simplified starting point has proven prohibitively difficult [13], as has the problem of extending the matrix model formalism to backgrounds with curved spacetime geometries [14]. Although theories for an emergent gravitational interaction, they are not fundamental theories of emergent spacetime, and do not therefore capture the full spirit of Einstein gravity. We have required more from our zero dimensional matrix Lagrangian [1]: we want both the emergence of a continuum spacetime manifold, as well as the emergence of local spacetime geometry. We wish to find this in a matrix Lagrangian that can credibly encapsulate what we know about the Duality Web of M theory, inclusive of all of its weakly coupled field and string theory limits. Most importantly, the emergence of the spacetime continuum is tied to the large N limit. This spacetime geometry could generically receive quantum corrections but, unlike the alternative proposals for quantum gravity we comment upon below, quantum dynamics is not an essential aspect of our proposal for an emergent continuum spacetime geometry. The key point is that for renormalizable and anomaly-free superstring theories with 16 supercharges, the flat spacetime background is exact, without quantum back reaction.

A brief clarification about alternative attempts, old and new, to formulate quantum theories of gravity is in order. Note that we are putting aside here the issue of whether such formulations also extend to all of physics, namely, to the Standard Model with its Yang-Mills gauge fields and chiral matter multiplets, a unification which is one of perturbative superstring theory’s undeniable successes. Loop quantum gravity is perhaps the best known attempt at writing a background independent theory of quantum gravity in terms of alternate, loop variables: note that a continuum spacetime manifold is assumed, but there is no spacetime metric [16]. Thus, this is not a theory of emergent continuum spacetime, but neither does it contain a satisfactory explanation for why one should trust a quantum field theoretic formulation of 4d quantum general relativity given the
well-known non-renormalizability of perturbative gravity. The general philosophy of identifying background independent variables for a fundamental theory of emergent spacetime geometry is nevertheless appealing, and the matrix formulation described in this paper is a beautiful illustration of this phenomenon.

Topological string theories, and topological M theory, are theories of pform fields where a topological gravity sector is conjectured to emerge from the dynamics of p-form fields [18, 17]. Notice that the loop representation of 2+1 dimensional quantum gravity can incorporate both the background spacetime manifold as well as its topological degrees of freedom [16]. But this property is special to 2+1 dimensions. Spin network, and spin foam, models have also been extensively explored as alternative formulations of quantum general relativity [19]. A recent twist on these ideas are the melting crystal configurations underlying the proposed notion of quantum foam [17], which also represents a latticization of space. This is a rough analogue of the discretized eigenvalue coordinates of the zero dimensional matrix model. But the connections of the proposals in [18, 17] to genuine M theory, or even to genuine Einstein gravity, are at best tenuous, and it is not clear to us how the observations in [17, 18] might extend to the full theory. They remain an interesting exploration of certain special spacetime backgrounds that can enter the quantum gravity path integral, representing wild fluctuations in topology, and geometry, rather than smooth spacetime metric geometries. Finally, neither can the AdS/CFT conjecture [20] be considered the basis of a true theory of emergent spacetime, except in the limited, holographic, sense that the dynamics of a (D-1)-dimensional continuum quantum field theory is being conjectured to describe all of physics in a related D-dimensional continuum spacetime.

Let us close with an outline of the paper. In section 2, we apply the simple procedure of planar reduction to a sample supergravity Lagrangian with an auxiliary large $N$ flavor group, and both with, and without, a Yang-Mills gauge sector. We present our analysis using as prototype the manifestly supersymmetric 10d Lagrangian density obtained in the low energy limit of the heterotic string theory, computed up to quartic order in the $\alpha'$ expansion in [24, 25], and inclusive of gauge-coupling dependent corrections required by closure of the supersymmetry algebra. We compare the resulting planar reduced matrix models with previously studied matrix models. Our discussion includes an especially elegant and simple result for the planar reduction of the 11d supergravity Lagrangian augmented with the large $N$ flavor symmetry. We explain why simple planar reduction a la Eguchi-Kawai [2], or the supersymmetric extension in the IKKT matrix model [12], always results in the absence of any remnant of the spectrum of supergravity pform potentials in the reduced matrix model, despite our introduction of a large $N$ flavor symmetry in order to obtain a nontrivial large N matrix model. We need a new idea.

The introduction of a large $N$ flavor index does not, in itself, suffice as a device to obtain a matrix model Lagrangian with remnant knowledge of the local spacetime symmetries of the higher dimensional field theory. In section 3, we explain the notion of reduction of spacetime fields to linearized forms on an infinitesimal patch of local tangent space at a single point in the spacetime manifold. The key new insight is to recognize that the Lagrangian density in quantum field theory satisfies spacetime locality. As a consequence, the reduction of all spacetime fields to linearized forms on the infinitesimal patch of local tangent space at a single point in the spacetime manifold, suffices to preserve all of the local symmetries of a covariant field theory Lagrangian in the corresponding reduced matrix model. We also explain in section 3 the emergence of the coordinates of
a $D$-dimensional noncompact spacetime continuum in the large $N$ limit: a large $N$ ground state of the matrix Lagrangian which is characterized by $D$, simultaneously diagonalizable, $N \times N$ matrices belonging to the zehn(elf)bein array, $E_\mu^a d\xi^a$, and where $\xi^a$ parameterizes the infinitesimal patch of local tangent space, will correspond to a background of M theory with $D$ noncompact spacetime dimensions. Such correspondences can be extended to generic curved spacetime metrics. We also check in Section 3 the self-consistency of our matrix Lagrangian with the basic relations of Riemannian geometry in continuum spacetimes, with the form of infinitesimal Lorentz, supersymmetry, and Yang-Mills, transformations, and with standard spacetime field redefinitions, properties which are required to emerge in the large $N$ limit of the matrix model. The key aspect of our prescription for spacetime reduction that enables such self-consistency is the fact that the large $N$ flavor symmetry group has been chosen to commute with the local symmetries of the original field theory Lagrangian.

In section 4, we describe our proposal for a fundamental theory of emergent spacetime that also encapsulates the insights of the string/M theory duality web, especially with regard to theories with sixteen supercharges [34, 35, 1]. Of special interest to us is the principle of generalized electric-magnetic duality in the Dirichlet pbrane spectrum, examined by us in the accompanying paper [72]. We review the intriguing fact that the full spectrum of supergravity pform potentials, $-1 \leq p \leq 10$, is represented in the nine-dimensional supergravity Lagrangian of [39], up to field redefinitions and dualities. Thus, the circle-compactified type I-1′-heterotic $E_8 \times E_8$ and Spin(32)/$Z_2$-IIA-IIB string theories appear in a single spacetime Lagrangian, and on equal footing. An explicit analysis of all of the two- and four-fermion terms in the nonmaximal 10D $N=1$ supergravity-Yang-Mills Lagrangian required by closure of the supersymmetry algebra, and up to quartic order in $\alpha'$ corrections, was performed by Bergshoeff and de Roo in [24, 25], and is the case described in Section 3. The analysis of the $D=9$ massive type IIA-IIB supergravity Lagrangian described in [39] is far less comprehensive, restricted to discussion of the bosonic sector alone, and with an absence of any discussion of the $\alpha'$ corrections from string theory. In nine dimensions, there are no Poincare self-dual forms, and it would be interesting to understand if a manifestly Poincare dual form of the Lagrangian exists, exhibiting both electric and magnetic dual p-form potentials [38]. These issues are left to future work. Instead, in sections 4 we address the complex nature of the vacuum landscape of theories with sixteen supercharges: beyond the standard component of the Moduli Space with rank 16 Yang-Mills fields explored in the familiar “Star” diagram, there are disconnected components first discovered in the CHL analysis [34, 35]. How are these theories accounted for in the matrix model framework?

How does our proposal relate to conjectures for the hidden symmetry algebra of string/M theory? We address this question in Section 5, emphasizing how the algebraic viewpoints can lend significant insight into some key aspects of our proposal. In an accompanying paper [32] we have given a pedagogical review, presented from our own perspective, on recent work on the hidden symmetry algebra of the ten and eleven dimensional supergravities with 32 supercharges. Concrete evidence for generalized electric-magnetic duality in the worldsheet formalism of perturbative string theory has been given by us in [48, 45, 72], works done in partial collaboration with Chen and Novak, and building on the earlier results in [47]. The worldsheet computation of the tension of a D(-2)brane coupling to a (-1)form supergravity potential, the magnetic dual of the nine-form potential of massive IIA supergravity, is described in the accompanying paper [72], along with a summary of the corroborating evidence for generalized electric-magnetic duality presented by Schnakenburg and West in their analysis of the global symmetry algebra of the massive IIA supergravity [28], and
by the early brane spectrum analyses by Lavrinenko, Lu, Pope, and Stelle [44]. West’s arguments in favor of the very-extended Lorentzian Kac-Moody algebra $E_{11}$ as the hidden symmetry algebra of the ten and eleven dimensional supergravities with 32 supercharges are summarized in [30]; a review by us can be found in [32]. The relationship to our proposal for a fundamental theory of emergent local spacetime geometry [1], and one that is also rooted in the principle of generalized electric-magnetic duality as presented in [72], is briefly described in sections 4 and 5. In particular, we conjecture a plausible extension of West’s $E_{11}$ framework which can incorporate theories with sixteen supercharges, and a nontrivial Yang-Mills gauge sector, and which is also compatible with the matrix model framework proposed by us in [1]. We conclude in Section 6 with a list of open questions and some key directions for future work.

2 Planar Reduction of Theories with Flavor $U(N)$

As explained above, there is no sign in either of the previous matrix proposals for M theory [11, 12] of the global symmetries of the Einstein supergravity theories. It is natural to suspect that the dimensional reductions of locally supersymmetric Yang Mills theories would be a more relevant direction to explore in this context and a concrete suggestion to this effect was made by Nicolai in 1997-98 [15]. Notice, however, that it is not immediately obvious why performing the planar reduction of a locally supersymmetric large $N$ Yang-Mills theory is an interesting exercise: the standard prescription for planar reduction [2] requires that we drop all space and time derivatives, leaving a potential that is quartic in the spin connection. But there are only a finite number of degrees of freedom in the gravitational sector due to the absence of large $N$ symmetry: there is no room for interesting quantum dynamics in the matrix model. Thus, it becomes necessary to modify the Eguchi-Kawai prescription for planar reduction of rigid Yang-Mills theories [2] when we study the zero-dimensional reduction of a gravity theory.

Let us begin with the ten-dimensional spacetime Lagrangian of the anomaly-free heterotic supergravity coupled to Yang-Mills fields [21, 22, 23, 25]. In the supergravity literature, this is often referred to as the nonmaximal ten-dimensional supergravity, in order to distinguish it from the $N=2$ supergravities with 32 supercharges. The relevant Lagrangian was constructed in several steps, starting with the four-dimensional, $N=4$ supergravity multiplet covariantly coupled to an abelian (Maxwell) multiplet [21]. This Lagrangian can be straightforwardly interpreted as the dimensional reduction of a ten-dimensional $N=1$ supergravity theory coupled to an abelian gauge field [22]. Extension to a sector with nonabelian Yang-Mills fields requires inclusion of a crucial Kalb-Ramond term in the Lagrangian, necessitated by the mixed gauge-gravity anomaly cancellation conditions [23]. Another feature of interest in this Lagrangian is the possibility of a field-dualization: the Lagrangian can be expressed interchangeably in terms of either a two-form, or a six-form, supergravity potential [23, 25]. The former choice is natural from the perspective of the string supergravities: the low-energy spacetime effective action of the heterotic string theory has a Neveu-Schwarz sector two-form potential, $B_{ab}$, that can couple to the fundamental closed string. In the low-energy spacetime effective action of the type IIB string theory, on the other hand, the relevant two-form potential, $C_{ab}$, appears in the Ramond sector, representing the possibility of coupling to a D1brane [42].

We have taken the liberty of simplifying the notation and conventions given in the original
references [21, 22, 23, 25]. Our expressions correspond to the last of these references.\(^6\) Expressed in terms of the Einstein frame metric, and to lowest order in the inverse string tension, the \(O(\alpha'^0)\) Lagrangian takes the form [25, 1]:

\[
\mathcal{L} = \frac{1}{\kappa^2} \left( \bar{\psi}_a \Gamma^{abc} D_b(\omega) \psi_c - 4 \bar{\Lambda} \Gamma^{ab} D_a(\omega) \psi_b - 4 \bar{\Lambda} \Gamma^a D_a(\omega) \lambda \right) + \frac{1}{g^2} \bar{\chi} \Gamma^a D_a(\omega, A) \chi^i + \frac{1}{g^2} \left( \mathcal{R}(\omega, E) - \partial^a \Phi \partial_a \Phi + 3 H^{abc} H_{abc} \right) + \frac{1}{2} g^2 F^{ab}(A) F_{ab}(A) + \mathcal{L}_{2\text{-fermi}} + \mathcal{L}_{4\text{-fermi}} .
\]

(1)

Here, \(\psi_a\), \(\lambda\), and \(\chi\) denote, respectively, the gravitino, dilatino, and gaugino, of the \(N=1, D=10\), supergravity-Yang-Mills theory. The covariant derivative, \(D_a\), is covariantized with respect to both Lorentz and Yang-Mills gauge transformations. \(\mathcal{R}\) is the ten-dimensional Einstein curvature scalar, and \(\Phi\) is the dilaton. We work in Palatini’s first-order formulation for Einstein gravity with a zehnbein, \(E^a\), and spin-connection, \(\omega^{ab}_{\mu}\). \(F_{ab}\) and \(H_{abc}\) are, respectively, the two-form Yang-Mills and three-form supergravity field strength. The \(\{\Gamma^a\}\) are the rank sixteen Dirac matrices of the ten-dimensional Clifford algebra; we have suppressed all spinor indices. Finally, \(g\) and \(\kappa\) denote, respectively, the dimensionless part of the physical, ten-dimensional Yang-Mills and gravitational couplings. In addition, we must include the crucial two-fermion and four-fermion terms required by supersymmetry [22, 25]. The two-fermion terms take the form:

\[
\mathcal{L}_{2\text{-fermi}} = \bar{\psi}_a \Gamma^{a} \psi_b \partial^\Phi \Phi - 2 \bar{\psi}_a \Gamma^{a} \lambda \partial_b \Phi - \frac{1}{4} H^{def} \left[ \bar{\psi}_a \Gamma^{a} \Gamma_{def} \psi_b + 4 \bar{\psi}_a \Gamma^{a} \lambda - 4 \bar{\Lambda} \Gamma_{def} \lambda + \frac{1}{g^2} \bar{\chi} \Gamma_{def} \chi \right] + \frac{1}{4} \frac{1}{\kappa^2} \bar{\chi} \Gamma^d \Gamma^{ab} (\psi_d - \frac{3}{2} \Gamma^d \chi)(F_{ab} + \hat{F}_{ab}) .
\]

(2)

Likewise, the 4-fermion terms in the ten-dimensional covariant supergravity-Yang-Mills Lagrangian take the form:

\[
\mathcal{L}_{4\text{-fermi}} = \frac{1}{3 \kappa^2} \bar{\psi}_a \Gamma^{abc} \psi_f \left( \frac{1}{2} \bar{\psi}_a \Gamma^{a} \Gamma_{abc} \Gamma^{e} \psi_e + \bar{\psi}_d \Gamma^{a} \Gamma_{abc} \Gamma^{d} \psi_d - 2 \bar{\Lambda} \Gamma^{a} \lambda - 2 \bar{\Lambda} \Gamma^{a} \Gamma_{abc} \Gamma^{d} \psi_d \right) - \frac{1}{6 \kappa^2} \frac{1}{g^4} \bar{\chi} \Gamma^{a} \chi \left( \bar{\psi}_d (4 \Gamma^{a} \Gamma_{abc} \Gamma^{d} \lambda + 2 \bar{\Lambda} \Gamma_{abc} \lambda + 3 \cdot 2^3 H_{abc} + \bar{\chi} \Gamma_{abc} \chi) \right) .
\]

(3)

The Einstein curvature, Yang-Mills, and three-field strengths take the familiar form:

\[
\mathcal{R} = \partial_a \omega^{ab} - \partial_b \omega^{ab} + \omega^{ac} \omega_{ac} - \omega^{ac} \omega_{bc} = \partial_a A^i_b - \partial_b A^i_a + f^{ijk} A^j_a A^k_b \\
H_{abc} = \partial_{[a} B_{bc]} - \frac{1}{g^2} (A^i_{[a} \partial_b A^i_{c]} - \frac{2}{3} f^{ijk} A^i_{[a} A^j_b A^k_c) .
\]

(4)

Here, \([\tau^i, \tau^j] = i f^{ijk} \tau^k\), defines the structure constants, \(f^{ijk}\), of the Yang-Mills gauge group, \(G\), with \(i, j, k = 1, \cdots, \text{dim } G\). Notice the Chern-Simons contribution to the three-field strength. Hats on the curvatures denote the supercovariant derivatives [22, 25]. Explicitly, we have [24, 25]:

\[
\hat{D}_a \Phi = \partial_a \Phi + \sqrt{2} \bar{\psi}_a \lambda \\
\hat{D}_a(\omega) \lambda = D_a(\omega) \lambda + \frac{1}{8} \sqrt{2} \bar{\psi}_a \Gamma^{abc} D_b(\omega) \Phi - \frac{1}{8} \Gamma^{abc} \psi_a \left( \hat{H}_{abc} \right) - \frac{1}{3 \cdot 2^2 \kappa^2} \sqrt{2} \Gamma^{bcd} \bar{\psi}_a \chi^i \Gamma_{bcd} \chi^i \\
\hat{D}_a(\omega, A) \chi^i = D_a(\omega, A) \chi^i + \frac{1}{8} \Gamma^{abc} \psi_a \hat{F}_{bc} - \frac{1}{2} \left( \bar{\psi}_a \chi^i \lambda - \chi^i \bar{\psi}_a \lambda + \Gamma^b \lambda \chi^i \Gamma_0 \psi_a \right) .
\]

(5)

\(^6\)Comparing with [24, 25], note the following convenient replacements: \(\phi \rightarrow e^{\Phi/3}\), \(\sqrt{2} \lambda \rightarrow \lambda\), \(B_{ab} \rightarrow \sqrt{2} B_{ab}\), \(H_{abc} \rightarrow \sqrt{2} H_{abc}\), \(\beta \rightarrow g^2 \kappa^2\). Finally, an overall factor of \(-\frac{1}{2} \kappa^2 E e^{-\Phi}\) has been suppressed in writing Eq. (1).
and the supercovariantized field strengths take the form:

\[ F_{ab}^i = \tilde{F}_{ab}^i - \psi_{[a} \Gamma_{b]} \chi^i \]

\[ \hat{H}_{abc} = \partial_{[a} B_{bc]} - \frac{1}{4} \psi_{[a} \Gamma_{b]c} - \frac{1}{g^2} \sqrt{2} \left\{ A^i_{[a} \partial_{b} A^i_{c]} - \frac{1}{3} f_{ijk} A^i_{[a} A^j_{b} A^k_{c]} \right\} . \]  

(6)

We should reiterate that the Lagrangian described above is, strictly speaking, that of the heterotic 10d supergravity-Yang-Mills string supergravity [25]. The two-form potential \( B_{ab} \) couples to fundamental heterotic closed strings. Now consider introducing a flavor quantum number in the Einstein-Yang-Mills Lagrangian given in Eq. (1), replacing the gravitational spin-connection and Yang-Mills vector potential with \( N \times N \) arrays as follows:

\[ \mathcal{R} \rightarrow \partial_a (\omega^a_{\mu b})_{AB} - \partial_b (\omega^a_{\mu a})_{AB} + (\omega^a_{\mu c})_{AC}(\omega^b_{\nu c})_{CB} - (\omega^a_{\mu c})_{AC}(\omega^b_{\nu c})_{CB} \]

\[ F^{ab} \rightarrow \partial_a (A^b_{\mu a})_{AB} - \partial_b (A^b_{\mu b})_{AB} + f^{ijk} (A^i_{\mu a})_{AC}(A^j_{\mu b})_{CB} \]  

(7)

where the indices run from \( i=1, \ldots, \dim G \), and \( A, B=1, \ldots, N \). We will need to include a trace over the \( U(N) \) flavor group in order that the new Lagrangian density transform as a \( U(N) \) singlet. This Lagrangian will have the symmetry group \( U(N) \times G \), except that the \( U(N) \) is not a gauge symmetry. Rather, it plays the role of a flavor group. How does one give meaning to the planar reduction of a locally supersymmetric Lagrangian with a huge flavor symmetry group to a single spacetime point? And why have we introduced a large \( N \) flavor symmetry, as opposed to the usual large \( N \) gauge symmetries invoked in [2, 11, 12]?

Notice that if we were to carry out the large \( N \) extension in analogy with the planar reductions of rigid Yang-Mills Lagrangians [2, 12], namely, replace the anomaly-free Yang-Mills group with the unitary large \( N \) group: \( SO(32) \rightarrow U(N) \), where \( SO(32) \subset U(32) \subset U(N) \), and where \( U(N) \) is a fully gauged symmetry, we would find nothing of interest in the gravity sector of the Lagrangian. Suppressing all spacetime derivatives in the supergravity-Yang-Mills Lagrangian as usual, we obtain the standard quartic unitary matrix potential in the Yang-Mills sector: \( [A_\mu, A_\nu]^2 \), where \( A \) is now a matrix of rank \( N \), but the Einstein sector yields an uninteresting finite rank correction to these terms. Thus, since the zehnbein and spin connection are finite-dimensional matrix arrays when reduced to a single spacetime point, dimensional reduction of the Einstein action gives a finite number of terms of the general form, \( E_{\mu a}^a \omega^a_{\mu c} \omega_{c,\lambda} E_{\nu b}^b \). It is clear that if one desires a nontrivial modification of large \( N \) dynamics of the Yang-Mills sector the gravity variables must also scale with \( N \). This implies that the role of \( U(N) \) in the continuum field theory Lagrangian must be that of a flavor symmetry group, rather than of a gauge group.\(^7\) We will find that the introduction of \( U(N) \) as a flavor symmetry in the continuum Lagrangian enables the democratic appearance of large \( N \) scaling behavior in both the gauge and gravity sectors of the matrix model obtained upon spacetime reduction to a single point.

This motivates the first innovation introduced by us in [1]: both the vector potential, zehnbein, and spin connection, were required to transform in the adjoint representation of a large \( N \) flavor group. The same requirement was made of the other bosonic fields in the Lagrangian, namely, the dilaton and two-form potential. What about the spinors in the supergravity Lagrangian? Since we have required that the large \( N \) flavor group commute with the group of supersymmetry transformations, it is important that fields which are partners under supersymmetry belong to the

\(^7\)I would like to thank Bernard de Witt for seeking this clarification.
same $U(N)$ representation. Thus, we will require that all spinor fields gravitino, dilatino, and gaugino, also transform in the adjoint representation of the large $N$ flavor group. Keeping only the terms in Eq. (1) that remain after planar reduction, gives the following supermatrix Lagrangian:\footnote{We denote the spacetime field, $f(x)$, and its planar-reduced representative which lives at the origin of spacetime, $f(0)$, by the same symbol $f$.}

\[
\mathcal{L}_{\text{planar}}^{(10d)} = \frac{1}{2g^2} \text{tr} \left( f^{ijk} f^{ilm} A^i_{\alpha} A^{ab}_{\gamma} A^j_{\beta} A^k_{\gamma} + \bar{\chi} i \Gamma^{a} A^i_{\alpha} f^{ij} \chi^j \right) \\
+ \frac{1}{\kappa^2} \text{tr} \left( E^\mu_a \left[ \omega^a_{\mu} \omega^b_{\nu} - \omega^a_{\nu} \omega^b_{\mu} \right] \tilde{E}^\nu_b - A^i_{\alpha} \tau^i \Phi A^j_{\beta} \tau^j \Phi + \bar{\chi} i \Gamma^a \omega^b_{\alpha} \Gamma^b \chi^j \right) \\
+ \frac{1}{\kappa^2} \text{tr} \left( \bar{\psi}_a \Gamma^{abc} \omega_{bde} \Gamma^{de} \psi_c - 4 \lambda \Gamma^{ab} \omega_{ade} \Gamma^{de} \psi_b - 4 \lambda \Gamma^a \omega_{ade} \Gamma^{de} \chi \right) \\
+ \frac{4}{3} \frac{1}{g^2} \text{tr} \left( f^{ilm} f^{ijk} A^i_{\alpha} A^j_{\beta} A^k_{\gamma} \right) + \text{tr} \mathcal{L}_{2-\text{fermi}} + \text{tr} \mathcal{L}_{4-\text{fermi}} .
\]  

(8)

where $i, j, k, \ldots$ are group indices for the finite-dimensional Yang-Mills gauge group, and repeated indices are to be summed. The notation “tr” denotes, instead, the trace over the large $N$ flavor group, whose indices have been suppressed. The first line of this expression is familiar: analoguos terms appear in both the Banks-Fischler-Shenker-Susskind \cite{11} and Ishibashi-Kawai-Kitazawa-Tsuchiya \cite{12} rigid matrix models. The index structure of the terms in the first line make the symmetry of the supermatrix model manifest: the model obtained by restricting to only the terms in the first line of this expression defines the simplest possible supermatrix model consistent with this symmetry group.

Thus, the distinction between flavor, and gauged, large $N$ symmetry becomes significant only when we take into account the remaining terms in the matrix Lagrangian: we find new large $N$ matrix variables originating in the supergravity sector of the continuum Lagrangian, as well as new multi-matrix interaction terms. These include a sixth-order self-interaction for the Yang-Mills potential, a term which was absent in both the BFSS and IKKT matrix models \cite{11, 12}, and which arises from the Chern-Simons contribution to the supergravity three-form field strength. It is evident that the symmetry structure of the full supermatrix Lagrangian given in Eq. (8) is much more subtle than simply $U(N) \times G$. Knowledge of the Cremmer-Julia hidden symmetries of the continuum field theory Lagrangian becomes a useful tool for its analysis.

It is illuminating to examine the form of matrix Lagrangian obtained by the planar reduction of the 11d supergravity theory. Recall the absence of Yang-Mills gauge fields, as well as the absence of a dilaton supermultiplet, in 11d supergravity. The 11d supergravity theory does, however, include a four-form field strength. The associated three-form potential couples to the supermembrane. Introduction of a large $N$ flavor quantum number in the continuum Lagrangian, followed by spacetime reduction of the field theory to a single spacetime point, gives an elegant and especially simple matrix model:

\[
\mathcal{L}_{\text{planar}}^{(11d)} = \frac{1}{\kappa} \text{tr} \left\{ \bar{\psi}_a \Gamma^{abc} \omega_{bde} \Gamma^{de} \psi_c + \omega^a_{\alpha} \omega^b_{\beta} - \omega^a_{\alpha} \omega^b_{\beta} \right\} ,
\]

(9)

where the gravitino, $\psi_a$, is a 32-component Grassmann-valued array, and $\omega_{\alpha \beta} = \omega^a_{\alpha} \omega^b_{\beta}$ is the spin-connection. Notice that the presence of a higher p-form supergravity potential in the continuum field theory Lagrangian is, unfortunately, erased from the planar reduced matrix model: there is no analogous Chern-Simons coupling to a Yang-Mills field, as was present in Eq. (8). It may well be true that
this particular supermatrix Lagrangian falls within the class of solvable zero-dimensional multi-
matrix models, enabling a detailed analysis by well-established matrix model techniques. We should
emphasize that this model is the precise, pure gravitational, super matrix model analog of the planar
reductions of rigid supersymmetric Yang-Mills theory considered in [2, 11, 12]. However, the model
appears not to capture the full content of M theory because it lacks any knowledge of the crucial
supermembrane sector of the theory.

Thus, while the planar reduction of gravity theories with large \( N \) flavor group has led to an
interesting new class of zero-dimensional matrix models, these models appear not to capture the
full content of M theory, inclusive of the crucial brane-spectrum required by duality. This brings us
to a second innovation introduced in [1]. Notice that, strictly speaking, the Lagrangian in Eq. (1)
describes a ten-dimensional supergravity theory in generic curved spacetime background. Inherent
in this expression is the notion of a local ten-dimensional flat tangent space attached to every point
in spacetime. The naive procedure of planar reduction we have borrowed from rigid super-Yang-
Mills theories [2] has ignored this aspect of the supergravity Lagrangian. We will now show that
the spacetime reduction of all spacetime fields to linearized forms defined on the infinitesimal patch
of local tangent space at a single spacetime point, suffices to ensure that all of the local symmetries
of the continuum Lagrangian are preserved in a corresponding reduced matrix model.

3 Emergence of the Spacetime Continuum

We proceed with formulating a prescription for the spacetime reduction of a \( d \)-dimensional super-
gravity theory coupled to Yang-Mills fields to a single spacetime point together with an infinitesimal
patch of a local flat tangent space. As emphasized above, our starting point is an unusual field
theory Lagrangian with a huge flavor symmetry group: all fields, bosonic and fermionic, are re-
quired, in addition, to live in the adjoint representation of the large \( N \) unitary group, \( U(N) \). We
emphasize that \( U(N) \) is a flavor symmetry; only the finite rank anomaly-free Yang-Mills group
\( G \) has been gauged. Starting with the nonmaximal \( d=10 \) supergravity theory coupled to \( O(32) \)
Yang-Mills fields, we have the corresponding \( U(N) \) invariant Lagrangian:

\[
\mathcal{L} = \frac{1}{\kappa^2} \text{tr} \left\{ \bar{\psi}_a \Gamma^{abc} D_b(\omega) \psi_c - 4\Lambda \Gamma^{ab} D_a(\omega) \psi_b - 4\Lambda \Gamma^a D_a(\omega) \lambda \right\} \\
+ \frac{1}{g^2} \text{tr} \left\{ \bar{\chi} i\Gamma^a D_a(\omega, A) \chi^i + \frac{i}{2} F^{ab}(A) F_{ab}(A) \right\} \\
+ \frac{1}{\kappa^2} \text{tr} \left\{ \mathcal{R}(\omega, E) - \partial^a \Phi \partial_a \Phi + 3 \mathcal{H}^{abc} \mathcal{H}_{abc} \right\} \\
+ \text{tr} \{ \mathcal{L}_{2-\text{fermi}} + \mathcal{L}_{4-\text{fermi}} \} .
\]

(10)

where the notation “tr” denotes taking the trace over the large \( N \) flavor group, and the two- and
four-fermi terms are as given by Eqs. (2) and (3). Notice that each term in the Lagrangian is a
flavor singlet, and the \( U(N) \) flavor group commutes with all of the spacetime symmetries of
the Lagrangian: namely, local Lorentz and local supersymmetry transformations, in addition to
Yang-Mills gauge transformations.\(^9\)

\(^9\)In our earlier papers [1], we have pointed out a more general possibility for the matrix superalgebra. Namely,
the parameters for infinitesimal supersymmetry and \( SL(n, \mathbb{R}) \) transformations could themselves be non-singlet under
the flavor \( U(N) \). While we know of no reason to rule out such an extension, it is not necessary for the problem at
The $E^\mu_a$ are the fundamental variables appearing in the matrix Lagrangian, but they are not all independent. Assuming a flat tangent space of Minkowskian signature, $\eta_{ab}$, the usual relation for the spacetime metric tensor takes the form of a $U(N)$ identity:

$$G^{\mu\nu} = \text{tr} \left( E^\mu_a E^\nu_b \right), \quad \mu, \nu, \text{ and } a, b = 0, \cdots, 9, \quad E^\mu_a = G^{\mu\nu} E^\nu_a. \quad (11)$$

As is familiar from differential geometry, $G^{\mu\nu}$ is the object that raises spacetime indices, while $\eta^{ab}$ is the object that raises indices in tangent space. The spacetime metric transforms as a $U(N)$ singlet, as does $\eta^{ab}$. The usual constraint equation relating them is automatically satisfied:

$$\eta_{ab} = \text{tr} \left( E^\mu_a E^\mu_b \right) = G^{\mu\nu} \text{tr} \left( G^\nu_\lambda \eta_{ac} E^\lambda_{\nu} E^\mu_{\mu b} \right) = \delta^a_\mu \text{tr} \left( \eta_{ac} E^\lambda_{\nu} E^\mu_{\mu b} \right) = \eta_{ab}. \quad (12)$$

Consider the dimensional reduction of the large $N$ Lagrangian of Eq. (10) to a single spacetime point, call this the origin of 10d spacetime, together with an infinitesimal patch of the local flat tangent space. In other words, instead of simply setting all spacetime derivatives to zero as in the previous section, we retain the $O(\delta \xi^a)$ terms of the continuum Lagrangian, truncating at $O((\delta \xi^a)^2)$ in the Taylor expansion on tangent space, in the infinitesimal vicinity of the spacetime origin. We have parameterized the infinitesimal patch of local tangent space at the origin by the variables $\xi^a$, $a=0, \cdots, 9$. Recall the usual relation in Riemannian differential geometry linking the partial derivative operators acting in spacetime, and in the local tangent space:

$$\partial_\mu = E^\mu_a \partial_a, \quad \mu, \nu = 0, \cdots, 9, \quad a, b = 0, \cdots, 9 \quad , \quad (13)$$

Since the zehnbein is a flavor adjoint, an $N\times N$ dimensional array, whereas $\partial/\partial \xi^a$ is the ordinary partial derivative operator acting on a continuous and differentiable space with the local geometry of $R^{10}$, it follows that the partial derivative operator in spacetime, $\partial_\mu$, is also $U(N)$ valued. In particular, consistency with the obvious identity $\partial_\mu X^\mu=1$, implies that:

$$(X^\mu)_{AB} \equiv (E^\mu_a)_{AB} \delta \xi^a, \quad (\partial_\mu)_{AB} (X^\mu)_{BC} = (1)_{AC}, \quad A, B = 1, \cdots N. \quad (14)$$

In other words, the coordinate vector, $X^\mu$, is itself $U(N)$ valued! Notice that $X^\mu$ is a dependent variable in our framework: it is derived from the zehnbein, $E^\mu_a$, which is the fundamental variable appearing in the matrix Lagrangian. Unlike tangent space, which is smooth and differentiable, at least infinitesimally, spacetime contains a single element, a single spacetime “point”. All variables defined at a single point of the spacetime manifold are $N\times N$ dimensional matrices, the fundamental degrees of freedom in the matrix model Lagrangian. Further, we will require of all pforms on tangent space that they satisfy the linearized property: while the partial derivatives at the origin, $\partial_n f(0) = 0, n \geq 2$, exist up to arbitrarily order, the higher-order terms in the Taylor expansion are absent because quadratic and higher powers of the infinitesimal, $\delta \xi^a$, in tangent space has been dropped, reflecting the fact that we have a set of pforms on a base manifold of infinitesimal extent.

Why is there a need to retain an infinitesimal patch of tangent space while performing the dimensional reduction of the gravity theory to a single spacetime point? To develop some intuition hand. Notice that, for such matrix algebras, the large $N$ limit would have to correspond to an exotic (nonlinear) extension of the Nahm classification of spacetime linear superalgebras. We thank Bernard de Witt for pointing this out.
into the $U(N)$ valued relations given above, notice that no restrictions have been placed upon the eigenvalue spectrum of the various zehnbein. In principle, one can solve for the eigenvalue spectrum of each $E^\mu_\alpha$, given the equation of motion that follows from the classical matrix Lagrangian. One of the solutions to the equation of motion corresponds to choosing the 10d Minkowskian flat space time metric as classical background:

$$< G^{\mu \nu} > = < \text{tr} (E^\mu_\alpha E^\nu_\beta) > = \eta^{\mu \nu}, \ \mu, \nu, \ \text{and} \ a, b = 0, \cdots 9 \ .$$  \hspace{1cm} (15)$$

We solve for the corresponding $< E^\mu_\mu >$, expressing them in diagonal form, and ordering the eigenvalues along the diagonal to reflect a monotonic increase. It is evident that in the large $N$ limit, the eigenvalues will crowd together forming a continuum. Of course, as a consequence of the identity in Eq. (14), the coordinate matrices, $< X^\mu >$, also take diagonal form, their entries reflecting the monotonic increase along the diagonals of individual zehnbein. It is natural to interpret the ordered continuum of eigenvalues of the coordinate matrix as coordinate-locations for the continuum of spacetime points along the coordinate axis $x^\mu$ of 10d Minkowskian spacetime. Thus, we have recovered the coordinates of the spacetime continuum by taking the large $N$ limit of the matrix model!

We are now ready to carry out the spacetime reduction of the Lagrangian given in Eq. (10) in accordance with our new prescription. Note that all fields, bosonic or fermionic, transform as adjoints under the flavor $U(N)$, and every term in the Lagrangian is a $U(N)$ singlet. The Lagrangian is manifestly invariant under local supersymmetry and local Lorentz transformations, and these symmetries commute with the flavor $U(N)$. Under spacetime reduction, every field in the Lagrangian is reduced to a linearized pform on the local tangent space, reflecting the fact that tangent space is an infinitesimal manifold. Most importantly, this also has the natural consequence that the local symmetries of the continuum Lagrangian can be made manifest in the matrix Lagrangian.

The remaining independent dynamical fields in the Lagrangian reduce to linear forms on tangent space. Our notation for a generic linearized form $f(\xi)$ is as follows: $f(\xi) = f(0) + \partial_a f(0) \delta \xi^a$, where $\partial_a f(0)$ denotes, more precisely, the partial derivative of $f$ with respect to $\xi^a$, evaluated at $\xi^a = 0$. Since every field in the continuum Lagrangian is an $N \times N$ array under flavor $U(N)$, and global symmetries are preserved under spacetime reduction, $f(0)$ and $\partial_a f(0)$ are two independent unitary matrices appearing in the matrix Lagrangian. Of course, one or other matrix array will be found to drop out of any given term in the Lagrangian. Thus, the matrix Lagrangian will turn out to have exactly the same symmetry group as the original continuum field theory Lagrangian. Listing each of the independent dynamical fields appearing in Eq. (10), we have the following result upon spacetime reduction to corresponding $N \times N$ matrix arrays defined at the origin $x = 0$, which we choose coincident with the origin of tangent space, $\xi = 0$:

$$
\begin{align*}
E^\mu_\mu(x) & \rightarrow E^\mu_\mu(0) \\
A^i_\alpha(x) & \rightarrow A^i_\alpha(0) + \partial_b A^i_\beta(0) \delta \xi^b \\
\partial_\tau A^i_\alpha(x) & \rightarrow \partial_\tau (A^i_\alpha(0)) + \partial_b A^i_\beta(0) \partial_\tau (\delta \xi^b) = \partial_\tau A^i_\alpha(0) \\
D_\tau A^i_j(x) & \rightarrow \partial_\tau A^i_j(0) - \partial_\tau A^i_j(0) + f^{ijk} A^j_\zeta A^k_\eta(0) \\
R_{abc}(x) & \rightarrow \partial_\tau B_{abc}(0) - g^2 (A^i_\alpha(0)\partial_\tau A^i_\beta(0)) - \frac{2}{3} f^{ijk} A^i_\alpha(0) A^j_\beta(0) A^k_\eta(0) \\
D_\alpha \Phi(x) & \rightarrow \partial_\tau \Phi(0) - A^i_\alpha \tau^i \Phi(0) \ .
\end{align*}
$$

(16)

where the indices have range as follows: $\mu = 0, \cdots, 9, a, b, c = 0, \cdots, 9$, and $i, j, k = 1, \cdots, \text{dim } G$. We remind the reader that each of the objects on the left-hand-side of this list is also an $N \times N$
unitary matrix, the flavor indices have simply been suppressed. Suppressing the “(0)” dependence, we obtain the matrix Lagrangian:

\[
\mathcal{L}^{(\text{mat})} = \frac{1}{\kappa} \text{tr} \left\{ \bar{\psi}_a \Gamma^{ab} D_b(\omega) \psi_c - 4 \bar{\lambda} \Gamma^{ab} D_a(\omega) \psi_b - 4 \bar{\lambda} \Gamma^a D_a(\omega) \lambda \right\} \\
+ \frac{1}{\kappa} \text{tr} \left\{ \mathcal{R}(\omega, \phi) - \partial^a \Phi \partial_a \Phi + 3 H^{abc} H_{abc} \right\} \\
+ \frac{1}{g^2} \text{tr} \left\{ \bar{\chi}i \Gamma^a D_a(\omega, A) \chi^i \right\} \\
+ \mathcal{L}^{(\text{2-fermi})} + \mathcal{L}^{(\text{4-fermi})}.
\]

(17)

In other words, the matrix Lagrangian takes precisely the same form as the original continuum Lagrangian with large \(N\) flavor group, except that all spacetime fields are restricted to their value at the origin: the infinite number of degrees of freedom in the original continuum field theory have indeed been drastically thinned to those of a zero-dimensional matrix model with \(U(N)\) flavor symmetry. But, remarkably, by the introduction of linearized forms on the local flat tangent space, this matrix Lagrangian also preserves a remnant of the local symmetries of the continuum Lagrangian.

The underlying reason why there exists a matrix Lagrangian that can make manifest the local symmetries of a given continuum field theory, is the spacetime locality property of the Lagrangian density in quantum field theory.

For completeness, let us reproduce the crucial two-fermion, and four-fermion, terms required by supersymmetry [22, 25] for the matrix Lagrangian. The two-fermi terms take the form:

\[
\mathcal{L}^{(\text{mat})}_{2-\text{fermi}} = \text{tr} \left\{ \bar{\psi}_a \Gamma^a \psi_b \partial^b \Phi - 2 \bar{\psi}_a \Gamma^a \lambda \partial_b \Phi \right\} \\
- \frac{1}{4} \text{tr} \left\{ H^{ab} \left[ \bar{\psi}_a \Gamma^a \Gamma_{def} \psi_b + 4 \bar{\psi}_a \Gamma_{def} \lambda - 4 \bar{\lambda} \Gamma_{def} \lambda + \frac{1}{g^2} \bar{\chi} \Gamma_{def} \chi \right] \right\} \\
+ \frac{1}{4} g^2 \text{tr} \left\{ \bar{\chi}^i \Gamma^a D_a(\psi_d + \frac{1}{2} \Gamma_d) (F_{ab} + \hat{F}_{ab}) \right\}.
\]

(18)

Likewise, the 4-fermi terms take the form:

\[
\mathcal{L}^{(\text{mat})}_{4-\text{fermi}} = \frac{1}{32} \text{tr} \left\{ \bar{\psi}_i \Gamma^{abc} \psi_j \left[ \frac{1}{4} \bar{\psi}_i \Gamma^d \Gamma_{abc} \Gamma^e \psi_j + \frac{1}{4} \bar{\psi}_i \Gamma^d \Gamma_{abc} \psi_j - 2 \bar{\lambda} \Gamma_{abc} \lambda - 2 \bar{\lambda} \Gamma_{abc} \Gamma^d \psi_d \right] \right\} \\
- \frac{1}{32} g^2 \text{tr} \left\{ \bar{\chi}^i \Gamma^{abc} \chi^j \left( \bar{\psi}_i \left( 4 \Gamma_{abc} \Gamma^d + 3 \Gamma^d \Gamma_{abc} \right) \lambda + 2 \bar{\lambda} \Gamma_{abc} \lambda + 3 \cdot 2^3 H_{abc} + \bar{\chi} \Gamma^{abc} \chi \right) \right\}.
\]

(19)

It is helpful to consider the variation of the matrix Lagrangian under an infinitesimal supersymmetry transformation, parameterized by an infinitesimal spinor, \(\eta\). As explained above, we begin by setting: \(\eta = \eta(0) + \partial_a \eta(0) \delta \xi^a\), where we choose \(\eta\) itself to transform as a \(U(N)\) singlet. The supersymmetry transformations take the form [25]:

\[
\delta_{\eta} \Phi = \bar{\eta} \lambda \\
\delta_{\eta} E^a = \frac{1}{2} \bar{\eta} \Gamma^a \psi \mu \\
\delta_{\eta} A^i = \frac{1}{2} \bar{\eta} \Gamma^i \chi \\
\delta_{\eta} B_{ab} = \frac{1}{2} \bar{\eta} \Gamma_{[a} \psi_{b]} - \frac{1}{g^2} A^i_{[a} \partial_0 A^i_{b]} \\
\delta_{\eta} \psi^\mu = (\partial_\mu - \frac{1}{2} \Omega^a_{\mu \psi} \Gamma_{ab}) \eta + \frac{1}{2} (\bar{\eta} \psi_\mu - \bar{\psi}_\mu \eta) \lambda - \frac{1}{2} (\bar{\psi}_\mu \Gamma^a \eta) \Gamma_\mu \lambda + \frac{1}{g^2} (\bar{\chi} \Gamma^{abc} \chi) \Gamma_{abc} \lambda \eta \\
\delta_{\eta} \chi^i = - \frac{1}{2} (\Gamma^{ab} F_{ab}) \eta + \frac{1}{2} (\bar{\eta} \chi^i - \bar{\chi} \eta) \lambda - \frac{1}{2} (\bar{\chi} \Gamma^a \eta) \Gamma_\mu \lambda.
\]
\[
\delta_\eta \lambda = -\frac{1}{4}(\Gamma^a D_a \Phi)\eta + \left( H_{abc} - \bar{\lambda} \Gamma_{abc} \lambda + \frac{1}{g^2} \chi^i \Gamma_{abc} \chi^i \right) \Gamma^{abc} \eta ,
\]

where the combination, \( \Omega^{ab}_{\mu+} \equiv \omega^{ab}_{\mu} + \tilde{H}_{\mu ab} \), and the supercovariant field strength is defined in Eq. (6).

4 The Theory with Sixteen Supercharges

In preceding sections, we have described a detailed prescription for the spacetime reduction of a locally supersymmetric theory with large \( N \) flavor group to a single point in spacetime, such that the resulting zero-dimensional large \( N \) matrix model Lagrangian manifests all of the local symmetries of the original continuum field theory. Our prescription is a modification of Eguchi and Kawai's well-known planar reduction procedure, which takes into account the necessity for an auxiliary local tangent space in a covariant Lagrangian formulation of a gravitational theory describing spinors in a generic curved spacetime background. As our prototype example, we have analyzed the case of the heterotic 10d \( N=1 \) supergravity-Yang-Mills Lagrangian with an anomaly-free Yang-Mills gauge group, \( E_8 \times E_8 \) or \( SO(32) \), of rank 16. In part, the reason for this is that a detailed analysis of the low energy spacetime effective action, up to quartic order in the inverse string tension, \( \alpha' \), and in the inverse Yang-Mills coupling as required by closure under supersymmetry, exists for the heterotic string supergravity [25]. This comprehensive analysis is due to Bergshoeff and Roo [24, 25], building on the earlier works of [22, 23]. The resulting Lagrangian has been presented in manifestly supersymmetric form, and in terms of component fields. The equivalence of the dual two-form and six-form formulations, at least up to quartic order in \( \alpha' \), has also been established by these authors. Partial comparisons have been made, and are in agreement with, terms in the effective action inferred from direct string amplitude calculations up to one-loop order [26].

Following the c.1995 developments in string duality, we have an enhanced appreciation of the rich structure of the vacuum landscape of theories with sixteen supercharges. Toroidal compactification of the 10d heterotic string preserves all of its supersymmetries, yielding a rich class of theories with sixteen supercharges, and anomaly-free rank \( 16+n \) Yang-Mills gauge group, in \( 10-n \) spacetime dimensions [33]. The discovery of the CHL moduli spaces [34] clarified that the vacuum landscape is not simply-connected: these models are supersymmetry preserving orbifolds of the standard toroidal compactifications of the heterotic string [35]. Thus, for example, in nine spacetime dimensions the vacuum structure of the theory with sixteen supercharges is already multiply connected: in addition to the connected vacuum landscape with 17 abelian one-forms at generic points in the moduli space, we have an isolated island universe with 17-8=9 abelian one-forms at generic points.\(^\text{10}\) This theory was first identified as an asymmetric orbifold of the circle compactification of the \( E_8 \times E_8 \)

\(^{10}\)Since the orbifold twist becomes trivial in the noncompact decompactification limit where the requisite massless gauge bosons are simultaneously recovered, the CHL orbifold is not, strictly speaking, a disconnected component of the theory with 16 supercharges [35]. But we should emphasize that, at weak coupling, and in the moduli space approximation, each moduli space describes low-energy physics in a different island universe. While nonperturbative dynamics can often be invoked to infer the possibility of tunneling to a different moduli space with fewer supersymmetries [55], no such examples are known in theories with 32 or 16 supercharges. Note that the mechanism proposed for a partial breaking of supersymmetry described in [37], giving a theory with 12 supercharges, requires assumptions about the nature of the theory in the decompactification limit.
heterotic string theory by Chaudhuri and Polchinski [35]: the $\mathbb{Z}_2$ orbifold action is a supersymmetry-preserving shift in the one-dimensional momentum lattice, accompanied by the outer automorphism exchanging the two $E_8$ lattices. The gauge symmetry at generic points in the moduli space is rank 9.

We emphasize that there is no known spacetime dynamics, field-theoretic or string-theoretic, that can repair the disconnectedness of the moduli space with sixteen supercharges. Recall that there is no Higgs mechanism in theories with sixteen supercharges. Thus, while the precise enhanced gauge group can vary from point to point, the rank of the abelian subgroup is fixed for all points in a connected component of the moduli space [34]. More precisely, as is clarified by the orbifold construction [34, 35, 36], each isolated component of the moduli space is characterized by a distinct target-space duality group entering into specification of the global symmetry algebra of that island universe. An alternative viewpoint is to realize that each island universe is an example of a flux compactification [43]: one, or more, of the supergravity pform fluxes is nontrivial, an invariant on a connected component of the moduli space. The type IIA string duals of the heterotic CHL models with constant Ramond-Ramond background one-form potential constructed by Chaudhuri and Lowe [36] were the earliest known examples of compactifications of the type II string theory with nontrivial RR sector, leading to the generic class of RR/NS flux compactifications. While the notion of isolated universes can be disconcerting, raising the spectre of the anthropic principle, and banishing hopes of a unique vacuum state for String/M theory picked by dynamics alone, we have argued elsewhere that the problem could be one of misinterpretation.

Let us move on to a different aspect of the vacuum landscape of theories with sixteen supercharges, namely, the fact that the six different string theories: type I, type I’, type IIA, type IIB, heterotic $E_8 \times E_8$, and heterotic $SO(32)$, each describe a different weakly-coupled limit of the same moduli space. Consider the circle compactifications of all six string supergravities and, for convenience, let us restrict ourselves to discussion of the standard component of the moduli space characterized by a rank 16 anomaly-free Yang-Mills gauge group. As is well-known, the Lagrangian we have described above can be mapped by a strong-weak coupling duality transformation, and suitable field identifications, into that of the type IB string theory [52]. Thus, the $SO(32)$ type I string theory is the strong-coupling dual of the heterotic string theory with identical gauge group.

In nine dimensions, and below, the $SO(32)$ and $E_8 \times E_8$ heterotic string theories are related by a target space duality: $R_9 \leftrightarrow \alpha'/R_9$. What is the type I strong-coupling dual of heterotic vacua with states in the spinor representations of the orthogonal groups, as required by the appearance of exceptional Lie algebras? Fortunately, upon compactification to nine dimensions, the type I theory can acquire nonabelian gauge symmetries of nonperturbative origin. Under the $T_9$ duality, the type I string with its 32 D9branes is mapped to a type I’ vacuum with 32 D8branes: additional massless gauge bosons can arise from the zero length limit of D0-D8brane strings. Such D0-D8brane backgrounds preserve all sixteen supersymmetries. The incorporation of D0-D8backgrounds, in addition to those with only 32 D8branes, permits identification of type I-I’ strong-weak coupling duals for all of the nine-dimensional ground states of the heterotic string theories. In particular, this includes compactifications on a circle of both the $Spin(32)/\mathbb{Z}_2$ and the $E_8 \times E_8$ heterotic string theories [52, 54, 45]. the inclusion of the D0-D8brane backgrounds also enables the identification of the type I-I’ strong coupling duals for all of the heterotic CHL moduli spaces with sixteen supersymmetries [34, 35, 36]. Furthermore, since type I’ theory compactified on $S^1$ is the same thing
as M theory compactified on $S^1 \times S^1 / \mathbb{Z}_2$, these observations are consistent with the identification of M theory on $S^1 / \mathbb{Z}_2$ as the strong coupling limit of the $E_8 \times E_8$ heterotic string theory [41, 53]. We emphasize that the analysis of D0-D8 type IB backgrounds given by us in [45], elevates the equivalence of type IB and heterotic string theories to a stringy equivalence, not merely at the level of the low-energy field theory limits. The key point is that the Yang-Mills gauge sector in either case is isomorphic to the massless modes of a chiral vertex operator algebra, namely, a chiral conformal field theory.

Most importantly, it is plausible that the success in unifying the circle-compactified heterotic and type-I$'$ theories with sixteen supercharges can be extended to incorporate the type IIA and type IIB string theories with generic RR backgrounds. There exists a nine-dimensional Lagrangian formulation of the massive type IIA-IIB supergravities due to Bergshoeff, de Roo, Green, Papadopoulos, and Townsend [39] which incorporates the full spectrum of Dbrane $p$-form potentials [9], including Roman’s IIA cosmological constant [40]. By combining field-dualizations, as well as $S$ and $T$-duality transformations on the couplings, this Lagrangian can be mapped to any of six supergravity theories: the circle-compactified type I, type IIA, or heterotic string supergravities, the Scherk-Schwarz reduction of the type IIB string supergravity, or the $S^1 \times S^1 / \mathbb{Z}_2$ compactification of eleven-dimensional supergravity. This covers all six vertices of a modified star diagram linking theories with sixteen supercharges [41, 42, 1].

With our new prescription for spacetime reduction, we have shown that the local symmetries of a given field theory Lagrangian can be preserved in the reduced matrix model. Thus, there is a precise analog for each field redefinition, or dualization, of the continuum Lagrangian in the matrix model: the matrix Lagrangian is only unique up to appropriate dualizations defined on the matrix variables [1]. On the one hand, this is a beautiful illustration of the underlying unity of the different ten-dimensional string supergravities with eleven-dimensional supergravity. But it points to the importance of understanding the global symmetry algebra: the identification of a specific, hidden symmetry algebra is what gives precise meaning to one, or other, class of supergravity/M theory toroidal compactifications. This observation has been reiterated recently by West [28, 27, 30, 31]. But it is not new to string theory, nor to supergravity: target-space duality groups, and their conjectured extension to U-dualities [41], have been the bulwark of our understanding of both string and supergravity compactifications. In section 5 of this paper, we emphasize that the notion of the global symmetry algebra also provides a precise generalization incorporating all of the ground states of String/M theory. Algebraic considerations were the chief input into discovery of the disconnected “island universes” with sixteen supercharges by us in [34]. Each of the CHL moduli spaces corresponds to a supersymmetry preserving orbifold of the toroidally compactified ten-dimensional heterotic superstrings [35], characterized by a Lorentzian self-dual lattice with definite isometries.

We began this section by pointing out that, at the current time, we do not have a comprehensive analysis of the full covariant Lagrangian— including all of the fermionic contributions necessitated by supersymmetry, for any of the low energy effective Lagrangians other than that of the heterotic string theory [24, 25]. For theories with sixteen supercharges, this is, of course, the supergravity Lagrangian of fundamental significance.
5 Global Symmetry Algebra of Type IB Supergravity

We have alluded earlier to the existence of a hidden symmetry algebra in the matrix Lagrangian that is larger than the obvious $U(N) \times G$. In part, there is an $SL(10, \mathbb{R})$ symmetry, which is the manifest remnant under spacetime reduction to a single point in spacetime of the Lorentz symmetry group of the 10d continuum field theory Lagrangian. In the Introduction, we have already explained the simple rationale for expecting the symmetry algebra of an 11d M theory with 32 supercharges to be $E_{11} = E_8^{(3)}$, the rank eleven algebra known as the very-extension of the finite dimensional Lie algebra $E_8$ [30].

The nonmaximal 10d supergravity is a theory with sixteen supercharges. It also leads to maximal supergravity-Yang-Mills theories in ten dimensions, the low energy limits of the anomaly-free heterotic and type I superstring theories [34]. In the notation of [30, 32], we have the following symmetry generators:

\[ K^a_b, R, R^{c_1\cdots c_2}, R^{c_1\cdots c_6}, R^{c_1\cdots c_8} \quad . \tag{21} \]

The heterotic supergravity theory has zero-form dilaton and NS two-form potentials, plus their ten-dimensional Hodge duals, respectively, six-form and eight-form supergravity potentials. The $K^a_b$ are the generators of $GL(10, \mathbb{R})$ in the notation of [30]. The commutator algebra of these generators was given in [31]. Not surprisingly, we will find that it agrees precisely with the algebra that can be inferred from an appropriate chirality projection on the global symmetry algebra of the type IIB supergravity. This reflects the well-known connection between these two superstring theories, following the orientation projection to the symmetric combination left-moving and right-moving modes on the worldsheet of the type IIB superstring [42].

The global symmetry algebra of the type IB supergravity can therefore be identified by performing a chiral projection on the global symmetry algebra, $G_{\text{IIB}}$, of the 10d type IIB supergravity, which was also obtained in a recent work of Schnakenburg and West [28]. As shown in [32], by setting the extra forms to zero in Eqs. (1.1-3) of [28], we find the usual $GL(10, \mathbb{R})$ algebra, extended by translations:

\[
\begin{align*}
[K^a_b, K^c_d] &= \delta^a_c K^b_d - \delta^b_d K^c_a, \quad [K^a_b, P_c] = \delta^a_c P_b, \quad [K^a_b, R^{c_1\cdots c_p}] = \delta^p_b R^{a c_1\cdots c_p} + \cdots , \tag{22}
\end{align*}
\]

plus the simplified algebra of 0, 2, 6, and 8-form generators:

\[
[R, R^{c_1\cdots c_p}] = d_p R^{c_1\cdots c_p}, \quad [R^{c_1\cdots c_p}, R^{c_1\cdots c_q}] = c_{p,q} R^{c_1\cdots c_{p+q}} . \tag{23}
\]

Comparing with the IIB result given in Eq. (1.3) of [28], the remnant non-vanishing structure constants take the simple form:

\[
d_{q+1} = -\frac{1}{4}(q - 3), \quad q = 1, 5, \quad c_{2,6} = \frac{1}{2} . \tag{24}
\]

The algebra we obtain is in precise agreement with that of the 10d $N=1$ heterotic supergravity given in Eq. (1.4) of [31]. Let us denote this algebra as $G_{\text{IIB}}$. We emphasize that, thus far, we have not included the Yang-Mills gauge sector of the nonmaximal 10d supergravity-Yang-Mills theory.

\[11\] A pedagogical review of some recent directions in the study of the symmetry algebra of theories with 32 supercharges and, in particular, of the very-extended Lie algebras, is given in [32]. The paper addresses recent results of West and collaborators in this context [30].
Let us now address the question of how the global symmetry algebra of the heterotic-type I nonmaximal supergravity relates to the symmetry algebra of M theory with 32 supercharges. West has provided mounting evidence in favor of the conjecture that the symmetry algebra of M theory is the rank eleven very-extended algebra $E_{11}$ [30]. A review, and our own assessment, of West’s arguments appears in [32]. If we compare the generators and commutation rules given above with the Chevalley basis for the algebra $E_8^{(3)}$, written in either its IIA or IIB guise as shown in [30], we find that we are missing some of the positive root generators in either formulation. We have all of the generators, $E_a = K_a^{a+1}$, $a = 1, \ldots, 9$, of $SL(10, \mathbb{R})$. In the IIA formulation, given in Eq. (4.4) of [30], we are missing the roots corresponding to the R-R one-form, and NS-NS twoform, namely, $E_{10} = R_1^{10}$, and $E_{11} = R_{10}^{010}$. In the IIB formulation, we are missing the roots labelled $E_5 = R_1^{010}$, and $E_{10} = R_2$, arising, respectively, from the NS-NS two-form potential, and R-R scalar. It is clear we cannot build a full $E_8^{(3)}$ algebra from the restricted set of generators in $G_{IB}$.

In [31], it was pointed out that a different rank eleven very-extended algebra, namely, the very-extension of the $D_8$ subalgebra of $E_8$, can be spanned by the generators of $G_{IB}$. We should note that such a construction is somewhat unmotivated from the viewpoint of any relationship to the type II theories, to eleven-dimensional supergravity, or to M theory: the authors of [31] make the choice $E_a = K_a^{a+1}$, $a = 1, \ldots, 9$, $E_{10} = R_1^{10}$, and $E_{11} = R_{5678910}$. This choice of simple roots is shown to generate the very-extended algebra $D_8^{(3)}$. Appending a one-form generator to this set converts the $D_8^{(3)}$ algebra to $B_8^{(3)}$ [31]. However, it should be noted that, since the two-form and six-form potentials are Hodge dual to each other in ten dimensions, a construction which includes both in the simple root basis is quite different in spirit from that of the $E_8^{(3)}$ algebras underlying the IIA and IIB theories [30]. On the other hand, the generators of very-extension of $D_8$ appears to encapsulate the basic physics of the supergravity sector of the renormalizable superstring theories remarkably well: gravity, the dilaton and antisymmetric two-form potential, plus their 10D electric-magnetic Poincare-Hodge duals.

One possible direction towards extending the $D_8^{(3)}$ to a full $E_8^{(3)}$ algebraic structure in supergravity theories with sixteen supercharges is to consider compactifications. Since we already have the requisite two-form potential, respectively, the R-R, or NS-NS, two-form of the type IB, or heterotic, supergravities, our goal will be to identify a one-form potential that can play the role of the positive root generator labelled $E_{10}$ in the IIA formulation of the $E_8^{(3)}$ algebra. A hint is provided by our understanding of the Duality Web linking the zero slope limits of the circle compactifications of the type I, type IIA, IIB, and heterotic string theories, along with M theory compactified on $S^1 \times S^1 / \mathbb{Z}_2$. Upon compactification on a circle, the heterotic string theories acquire an abelian one-form potential, namely, a Kaluza-Klein gauge boson. This perturbative gauge field is known to play a crucial role in six-dimensional weak-strong IIA-heterotic string-string duality: it maps under a weak-strong coupling duality to the R-R one-form potential of the IIA string theory compactified on K3.\footnote{To be more precise, compactifications of the IIA theory on K3 are described by a (19,3) cohomology lattice characterizing classical K3 surfaces. It is the quantum extension to a (20,4) quantum cohomology lattice, as a consequence of introducing a flux for the R-R one-form potential, that completes the isomorphism of the IIA theory compactified on K3 to the heterotic string compactified on $T^4$. The quantum cohomology lattice is identified with the (20,4) Lorentzian momentum lattice of the heterotic string [50, 51, 36]. The heterotic theory has 20 abelian one-forms. One of these is distinguished as the partner of the R-R one-form of the IIA theory, and this is the Kaluza-Klein gauge field we have in mind.} As the simple root generator labelled $E_{11}$ in the IIA formulation of $E_8^{(3)}$, we might choose, respectively,
the NS-NS two-form potential of the IIA string or the two-form potential of the heterotic string. Note that these are mapped to each other under string-string duality. As an aside, the reader may wonder why we had to compactify all the way to six dimensions to see these equivalences, but this methodology is in keeping with how the Cremmer-Julia hidden symmetries of the type II theories were discovered. The full global symmetries only become manifest in the dimensionally-reduced supergravity theories, but this can be taken as a hint towards discovering a higher-dimensional correspondence. In summary, identifying the Kaluza-Klein one-form, and the two-form potential, as two of the missing positive root generators in the Chevalley basis might plausibly allow one to demonstrate an $E_8^{(3)}$ global symmetry algebra in the circle compactifications of the heterotic supergravities.

It is important to notice that the supergravity structure of the CHL orbifolds is identical with that of the circle compactifications. Indeed, if we are correct in our expectation that the circle-compactified theory has an underlying $E_8^{(3)}$ global symmetry algebra, then this would also be true of the CHL orbifold. The distinction between these two theories lies only in the Yang-Mills sector: they differ in the rank of the gauge group at generic points in the moduli space, respectively, 17, and 9, and we are arguing that the additional Kaluza-Klein gauge bosons contribute to the extension of $D_8^{(3)}$ to $E_8^{(3)}$. Are the precise nonabelian enhanced gauge symmetry groups relevant to this discussion? It has become customary to think not, since it is well-known that the precise nonabelian enhancement varies from point to point in the moduli space. Conventionally, this multiplicity of enhanced symmetry points can be inferred by analysis of the perturbative T-duality groups. Now it is clear that the full T-duality group of the toroidally compactified heterotic string can never be accommodated within $E_8^{(3)}$: we emphasize that this is unlike the smaller T-duality group of the toroidally compactified type II superstrings. As shown by Lu and Pope [4], the Cremmer-Julia group always contains the T-duality group in this case. The Yang-Mills sector of the heterotic and type I string theories therefore poses a puzzle for the Cremmer-Julia conjecture: how should one incorporate the one-form generators of the Yang-Mills sector into the hidden symmetry algebra?

In summary, in light of what we have learned from the survey of the hidden symmetry algebras of nine, ten, and eleven, dimensional supergravity theories in [?], and given the striking evidence that the rank eleven very-extension of the Lie algebra $E_8$ incorporates the full spectrum of NS-NS and R-R charges, including the crucial D8brane and D(-2)brane charges, we will conjecture that the hidden symmetry algebra of supergravity theories with sixteen supercharges takes the form $G' \times G$, where $G$ is the finite-dimensional Yang-Mills gauge symmetry, and $G'$ could be as large as $E_8^{(3)}$. We will conjecture further that nonperturbative String/M theory is the realization of this algebra on unitary $N \times N$ matrices. Further elucidation of our conjecture is left to future work.

6 Conclusions and Future Directions

A proposal as radical as that described in this paper [1] has few concrete conclusions in comparison with the Pandora’s box of fascinating questions it opens up for future investigation. The significant achievements in giving a concrete proposal for an emergent spacetime geometry are already summarized in the Introduction. Let us focus in this concluding section on the most accessible of the open questions raised by our work:
It is always helpful to gain what insight may be available from detailed calculations in toy matrix models. In this context, it is noteworthy that a refreshed perspective on nonperturbative effects in string theory has emerged from recent work on noncritical type 0 string theory in a two-dimensional target space [62], and on minimal string theory, namely, the $(p,q)$ conformal minimal models coupled to Liouville gravity [63]. A new result [64], motivated in part by these developments, is the elegant numerical computation of the nonperturbative D-instanton contribution to the perturbative genus-by-genus expansion of the free energy of the zero-dimensional matrix model, the fully nonperturbative description of two-dimensional quantum gravity coupled to unitary matter with $c < 1$ [7, 58].

An important general implication of the analysis in [64] relevant for critical string/M theory, is that the matrix path integral formalism preserves the normalization of the D-instanton action: this is nonperturbative data, above and beyond the form of the action inferred from the asymptotic large order behavior of the solution to the matrix model loop equations [61].

This observation has a striking parallel in the worldsheet formalism of perturbative string theory: the Weyl-invariant path integral formalism for open and closed string amplitudes preserves the normalizations of string amplitudes [65, 66, 46], a powerful framework that goes beyond what can be extracted from the operator formalism of conformal field theory, and of light-cone gauge string theory. Thus, both because of its powerful computational elegance, demonstrated by analyses such as [64], and because it enables addressing issues in quantum gravity, and quantum cosmology, that can only be sensibly formulated in an off-shell, or semi-classical, framework eschewing the use of S-matrices, the development of mathematical techniques that make the Lagrangian matrix path integral directly amenable to analysis needs to be a central goal of future work in matrix model techniques. The strong evidence for a hidden symmetry algebra in the spacetime reduced matrix Lagrangians of relevance to M theory suggests that we need to develop a completely new strategy towards their analytic solution, and which is likely to be rooted in this algebraic structure. The necessary mathematical techniques could turn out quite different from methods that have proven effective in the toy matrix models.

Having motivated the necessity to focus attention on the hidden symmetry algebra of the theory with sixteen supercharges, let us highlight the gaps in our current understanding that remain to be filled. The identification of the global symmetry algebra of the type I-I' heterotic string theories with correct incorporation of the Yang-Mills sector needs to be completed. In particular, West has made the interesting observation that the IIA-IIB T-duality transformation simply reflects the bifurcation symmetry of the $E_8^{(3)}$ Dynkin diagram at its central node, interchanging the Dynkin diagrams of its two inequivalent $A_9$ subalgebras [30]. This argument can clearly be adapted to the T-duality symmetry relating the type I and type I' string theories. Or, to that relating the two circle-compactified heterotic string theories. The details of our broad conjectures need to be verified.

\footnote{We note in passing that it should be possible to carry out a similar analysis to [64] for the rather interesting zero-dimensional reduced matrix model describing the coupling to $c = 1$ matter. This exactly solvable matrix model is known as the Penner model [59], and the exact solution to the Lagrangian matrix path integral obtained by us using the orthogonal polynomial method appears in [60].}
It should be noted that our new perspective on the hidden symmetry algebra takes seriously the strong-weak dualities linking the heterotic, type IB, and type IIA, string theories: a theory of sixteen supercharges is self-dual, and it would be meaningless to have different hidden symmetry algebras pertaining to the different string theories. Thus, while the perturbative T-duality group of the type II theory is, in fact, incorporated in $E_{11}$, the Lorentzian extension familiar from the toroidally-compactified heterotic string cannot be contained within $E_{11}$. Our conjecture is that the hidden symmetry algebra of the type I-heterotic supergravity with sixteen supercharges, $G'$, will be unchanged for all of the CHL theories. Of course, the abelian subgroup of the nonabelian gauge symmetry characterizing generic points in a given moduli space, and hence $G$, will be different for each of the latter CHL theories. It just so happens that in nine dimensions there are no Wilson lines that permit either a breaking, or enhancement, of the $E_8$ Yang-Mills gauge symmetry.\footnote{I thank Arjan Keurentjes for requesting this clarification.}

Once the precise nature of the hidden symmetry algebra of the type I-I’-heterotic theories with sixteen supercharges has been pinned down, and which we have conjectured will take the form $G' \times G$, what new physics becomes accessible? It is a remarkable fact that there exists a \textit{unique} assignment of phases in the bosonic $E_8^{(3)}$ algebra corresponding to an eleven-dimensional theory with, respectively, Minkowskian (1,10), or Euclidean (0,11) signature \cite{68, 67}. As has been shown by Keurentjes \cite{67}, every other self-consistent choice of phases for $E_{11}$ results in a spacetime with two, or more, timelike directions. Furthermore, the Euclidean case also corresponds to a bosonic $E_{11}$ algebra with natural physical interpretation as the symmetry algebra of M theory at finite temperature. This Euclidean symmetry algebra might be of great interest in the context of M theory cosmology, as we now explain.

We will offer a suggestive interpretation for the principle subalgebra of the Euclidean signature bosonic $E_{11}$ algebra. As expected from generic considerations \cite{29}, every Lorentzian Kac-Moody algebra has a principal $SO(1,2)$ subalgebra, and it is natural to seek its physical interpretation. Based on our understanding of the String/M duality web in 11, 10, and 9, spacetime dimensions, and given the pivotal role played by the nine-form potential and its (-1)-form dual, it is natural to identify the parameters of the $SO(1,2)$ subalgebra, roughly, as follows.\footnote{We should emphasize that this is very rough intuition. Thus, we are not making any clear statement on the precise topology of the group, discrete identifications might be necessitated.} Labelling them as $R_0$, $R_{10}$, and $R_9$, respectively, suggests a natural identification with inverse temperature, $\beta$, string coupling, $g$, and cosmological constant $M$. The latter is Roman’s mass parameter, later interpreted by Polchinski as D8brane charge \cite{40, 9}. Notice that the two-parameter subspace ($\beta,g$), whose rough correspondence with the radius of coordinates ($X^0, X^{10}$) is well-known, is also the classic phase space parameterization relevant for the study of the dynamics of a finite temperature gauge theory \cite{49}. Supplementing this with the cosmological constant gives a natural three-parameter phase space relevant for discussions of M theory cosmology: the thermal dynamics of the Universe, inclusive of gravity \cite{49}. It should be emphasized that the principal $SO(1,2)$ algebra should not be confused with the corresponding subalgebra of the spacetime Lorentz algebra.
In recent work on string thermodynamics [49], we have pointed out that there is a fundamental conceptual barrier to proposals for a microcanonical description of the string ensemble: perturbative string theory is inherently a background-dependent theory. Thus, we cannot escape the “heat-bath” represented by the spacetime geometry, and additional background fields: any self-consistent discussion of string thermodynamics must therefore be relegated to the canonical ensemble. Fortunately, there is a first-principles framework for the canonical ensemble provided by the world-sheet path integral formalism, originally pointed out in [65]. On the other hand, from the perspective of quantum cosmology, the ensemble of interest is the microcanonical ensemble of the fundamental degrees of freedom: the Universe is a closed system, and there is no “heat bath” one can point to. The reduced matrix models we have described in this paper offer a self-consistent starting point in which to formulate the microcanonical ensemble of the fundamental degrees of freedom. This opens up the exciting possibility of a genuinely nonperturbative formulation for black hole thermodynamics and quantum cosmology.

• The detailed understanding of the fermionic sector of the ten and eleven dimensional supergravities with 32, and 16, supercharges in terms of the hidden symmetry algebra, $E_8^{(3)}$, or $G \in E_8^{(3)}$, respectively, needs to be completed. Preliminary work constructing the standard double-sided, spinorial representations of the SO(10,1), or SO(9,1), Lorentz algebras from the appropriate representation of the Cartan sub-algebra of the Lorentzian Kac-Moody algebra is under way [69, 71].

• As an aside, we should note that an important issue raised both by our focus on the principal three-parameter subgroup of $E_8^{(3)}$, and by its realization in a locally supersymmetric unitary matrix model, is the possibility of an undiscovered relation to the famous supermembrane theory, a conjectured theory of fundamental supermembranes [56, 57]. Does the locally supersymmetric matrix model represent a regularization of the three parameter manifold of the principal subgroup, analogous to the regularization of the worldvolume of the supermembrane provided by the rigid unitary matrix model [57]? These are open questions that might shed light on the large N continuum limit of the reduced matrix model.

• Finally, coming to the crucial open questions in the matrix model framework, there is the issue of what comes beyond leading order in large N in the matrix model: what is the significance of the off-diagonal elements of the variables in the reduced matrix model? Notice that there is an obvious extension to the notion of the double-scaling limit familiar from the $c = 1$ matrix model: namely, $\lim_{N \to \infty, g \to 0}$, with $g^\alpha N^\beta$ held fixed. The parameters $(\alpha, \beta)$ take an appropriate range of values for members in the discrete series of the gravitationally-dressed unitary conformal field theories with central charge $c \leq 1$ [7]. The generalization to large N limits with multiple-scaling was pointed out in our earlier works [1]. Since we have a full range of background fields, $(g = e^{\phi}, \bar{A}_{c_1}, \bar{C}_{c_1}, \bar{C}_{c_2}, \cdots, \bar{C}_{c_{10}})$, where $g = (M_{11} R_{10})^{3/2}$, and

---

\[ We should note here the recent paper [70] which focuses on the rank 10 hyperbolic Kac-Moody algebra $E_{10}$: $E_{10}$ has an $A_9$ subalgebra, but not the full $A_{10}$ expected in a theory that incorporates a background with 11d supergravity, and the generalized electric-magnetic duality described in [72]. Not surprisingly, the analysis of [70] cannot incorporate a space-filling D9brane since there is no corresponding rank ten generator, nor its magnetic dual, and $E_{10}$ does not therefore appear to be a viable candidate for the hidden symmetry algebra of string/M theory. I thank H. Nicolai for clarifications. \]
the single mass scale, \( M_{11} \), there are many possible inequivalent, multiple-scaling limits: a suitable combination of powers of \( N, M_{11} \), and the background fields, can be held fixed, in the limit that we take \( N \to \infty \). Here, \( M_{11} \) has been taken to be the eleven-dimensional Planck mass. The precise powers of \( M_{11} \) that enter into taking the large \( N \) limit can vary, depending on whether we wish to match to an eleven, or ten-dimensional, continuum field theory. For example, the ten-dimensional string mass scale is related as follows: 

\[
m_s = \alpha'^{-1/2} = \frac{M_{11}^{3/2} R_{10}^{1/2}}{2}.
\]

We should emphasize the fact that it was essential that the matrix Lagrangian framework allow for a wide range of inequivalent large \( N \) limits, since it would not otherwise be possible to explain the multitude of known effective dualities relating M theory ground states.

- Perhaps the most important open question is the comparison of corrections to the large \( N \) limit of the matrix model, calculated with the specific choice of scaling appropriate for matching to a particular string supergravity, with the higher order in \( \alpha' \) corrections to the string spacetime effective Lagrangian. We should remind the reader that the precise form of the low energy spacetime effective Lagrangian for string theory has not been systematically calculated beyond quartic order in the inverse string tension \([24, 25]\), and that too only in the case of the heterotic string. This is unfortunate, given that the techniques for the systematic derivation of these terms from string amplitude calculations, or based on duality symmetries of the effective action, have been known for many years. In the past, this was explained by the absence of any direct physical motivation for a comprehensive analysis. For example, it was common to focus on the particular subset of terms that had the potential to mediate some new physics beyond the standard model. But given our current understanding of String/M theory, it would now seem that there is strong motivation for a renewed effort at obtaining a comprehensive analysis of the spacetime effective Lagrangian. We emphasize that it is only at higher orders in the \( \alpha' \) expansion that we can successfully test any conjectured nonperturbative proposal for String/M theory beyond agreement with the supergravity prediction.

In summary, we believe this could be the beginning of an exciting period in the search for a more fundamental description of String/M theory that transcends its weakly-coupled perturbative limits.

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