Propagation of shear waves in viscoelastic heterogeneous layer overlying an initially stressed half space

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Abstract. The present paper is concerned with the propagation of shear waves in an isotropic, viscoelastic, heterogeneous layer lying over a homogeneous half space under initial stress. For the layer the inhomogeneity associated to rigidity, internal friction and density is assumed to be linear function of depth. The dispersion equation of shear waves has been obtained in closed form. The dimensionless phase velocity and damping velocity have been plotted against dimensionless wave number for different values of inhomogeneity parameter and initial stress. The effects of inhomogeneity and initial stress have been shown in the dispersion curves.

1. Introduction
Many problems in seismology can be solved by representing the Earth as a layered medium [1] that is, formed by layers of certain thickness and mechanical properties. Motivation behind choosing viscoelastic media for the present study lies in the fact that the Earth is composed of silicate and iron-alloy materials. These materials respond nearly elastically under the application of small-magnitude transient forces but viscously under the application of long-duration forces. Materials like coal, tar, salt, sediments, etc., which are buried beneath the Earth surface can be modeled as viscoelastic materials. Cooper [2], Shaw and Bugl [3], Borcherdt [4], Romeo [5] and Chattopadhyay [6] have studied the propagation of SH waves in viscoelastic media with and without heterogeneity [7].
As pointed out by Bullen [8] the density inside the Earth varies at different rates with different layers within the Earth. Authors [9] have taken different forms of variation, like harmonic, linear, quadratic, etc., for simulating the variation in density and other geological parameters inside the Earth. For the present study, the heterogeneity is present in rigidity, density and internal friction.
A significant quantity of initial stress may develop in a medium due to many physical causes like slow process of creep, gravity variation, difference of temperature etc. In fact, the Earth is an initially stressed medium. Authors [10, 11, 12] have employed the theory of incremental deformation formulated by Biot [13] to study seismic waves in pre-stressed elastic solids. Keeping these things in mind, we have discussed the propagation of shear waves in an inhomogeneous, viscoelastic medium overlying a homogeneous initially stressed half space.
2. Formulation of the problem
We have considered a viscoelastic, heterogeneous layer lying over a homogeneous half space under initial stress and in welded contact with each other. The inhomogeneity has been considered in density, rigidity and internal friction for the layer $M_1$. The semi-infinite medium $M_2$ is under the constant initial stress $P$ along $x$-axis. The origin has been taken at the interface between the layer and the half space. The $x$-axis is taken along the interface and the $z$-axis vertically downwards. The layer is taken from $z = 0$ to $z = -h_1$. The semi-infinite space extends from $z = 0$ to $z = \infty$.

The inhomogeneity of the layer $M_1$ is specified by $\mu_1 = \mu_{10} (1 + \alpha z)$, $\mu'_1 = \mu'_{10} (1 + \alpha z)$, $\rho_1 = \rho_0 (1 + \alpha z)$. The elastic parameters and the initial compressive stress of the homogeneous half space $M_2$ are represented by $\mu_2$, $\rho_2$, $P$. Here $\mu_1$, $\mu_2$ are rigidities, $\rho_1$, $\rho_2$ are densities of $M_1$ and $M_2$ respectively and $\mu'_1$ is the internal friction of $M_1$. The inhomogeneity parameter $\alpha$ is real positive having dimension of inverse of length.

3. Solution of the problem
For SH-type wave motion $u = w = 0$, $v = v (x, z, t)$ and in the absence of body forces the equation of motion for the layer $M_1$ is

$$
\left( \mu_1 + \mu'_1 \frac{\partial}{\partial t} \right) \frac{\partial^2 v_1}{\partial x^2} + \frac{\partial}{\partial z} \left( \mu_1 + \mu'_1 \frac{\partial}{\partial t} \right) \frac{\partial v_1}{\partial z} = \rho_1 \frac{\partial^2 v_1}{\partial t^2}.
$$

If the plane waves propagate in the direction of increasing $x$, then we put

$$
v_j (x, z, t) = V_j (z) \ e^{i k (ct - x)}, \quad j = 1, 2.
$$

Using (2) in (1), we have

$$
\frac{d^2 V_1}{dz^2} + \frac{\alpha}{1 + \alpha z} \frac{dV_1}{dz} + k^2 \left( \frac{c^2 \rho_0}{\mu_1} - 1 \right) V_1 = 0,
$$

where $\bar{\mu}_1 = \mu_{10} + ikc\mu'_{10}$.

The solution of (3) is obtained as

$$
V_1 = \left[ AJ_0 \left( \frac{1}{\alpha} (1 + \alpha z) k \sqrt{\frac{c^2}{\mu_1} - 1} \right) + BY_0 \left( \frac{1}{\alpha} (1 + \alpha z) k \sqrt{\frac{c^2}{\mu_1} - 1} \right) \right],
$$

Figure 1. Geometry of the problem.
Therefore, 

\[ v_1(x, z, t) = \left[ A J_0 \left( \frac{1}{\alpha} \left( 1 + \alpha z \right) k \sqrt{\frac{c^2}{\beta_1^2} - 1} \right) + B Y_0 \left( \frac{1}{\alpha} \left( 1 + \alpha z \right) \sqrt{\frac{c^2}{\beta_1^2} - 1} \right) \right] e^{ik(c t - x)}. \]  

(5)

In the half space \( M_2 \) the equation of motion [13] under initial stress \( P \) is

\[ \left( \mu - \frac{P}{2} \right) \frac{\partial^2 v_2}{\partial x^2} + \mu \frac{\partial^2 v_2}{\partial z^2} = \rho \frac{\partial^2 v_2}{\partial t^2}. \]  

(6)

The above equation can be written as

\[ (1 - \zeta) \frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial z^2} = \frac{1}{\beta^2} \frac{\partial^2 v_2}{\partial t^2}, \]  

(7)

where \( \beta^2 = \mu / \rho_2 \) and \( \zeta = P/2 \mu_2 \).

Using (2) in (7), we have

\[ d^2 V_2 / dz^2 = m^2 V_2, \]  

(8)

where \( m = k \sqrt{(1 - \zeta) - \left( \beta^2 / \beta_2^2 \right)} \).

The appropriate solution of (8) is

\[ V_2 = C e^{-m z}, \quad z \geq 0. \]  

(9)

Therefore, \( v_2(x, z, t) \) of the lower half space is

\[ v_2(x, z, t) = C e^{-m z} e^{ik(c t - x)}. \]  

(10)

4. Boundary conditions

- The upper surface \( M_1 \) is stress free, i.e., \( (\mu_1 + \mu_1' \partial / \partial t) \partial v_1 / \partial z = 0 \) at \( z = -h_1 \).
- The displacement components are continuous at the interface, i.e., \( v_1 = v_2 \) at \( z = 0 \).
- The stresses are continuous at the interface, i.e., \( (\mu_1 + \mu_1' \partial / \partial t) \partial v_1 / \partial z = \mu_2 (\partial v_2 / \partial z) \) at \( z = 0 \).

Using (5) and (10) in all the boundary conditions and eliminating \( A, B \) and \( C \), we have

\[ \mp^2 k \sqrt{\frac{c^2}{\beta_1^2} - 1} \left[ J'_0 \left( \frac{1}{\alpha} \left( 1 - \alpha h_1 \right) k \sqrt{\frac{c^2}{\beta_1^2} - 1} \right) Y_0 \left( \frac{1}{\alpha} k \sqrt{\frac{c^2}{\beta_1^2} - 1} \right) \right. \]
\[ - J_0 \left( \frac{1}{\alpha} k \sqrt{\frac{c^2}{\beta_1^2} - 1} \right) Y'_0 \left( \frac{1}{\alpha} \left( 1 - \alpha h_1 \right) k \sqrt{\frac{c^2}{\beta_1^2} - 1} \right) \]
\[ + m \mu_2 \left[ J'_0 \left( \frac{1}{\alpha} \left( 1 - \alpha h_1 \right) k \sqrt{\frac{c^2}{\beta_1^2} - 1} \right) Y_0 \left( \frac{1}{\alpha} k \sqrt{\frac{c^2}{\beta_1^2} - 1} \right) \right. \]
\[ - J_0 \left( \frac{1}{\alpha} k \sqrt{\frac{c^2}{\beta_1^2} - 1} \right) Y'_0 \left( \frac{1}{\alpha} \left( 1 - \alpha h_1 \right) k \sqrt{\frac{c^2}{\beta_1^2} - 1} \right) \right] = 0. \]  

(11)

Equation (11) is the dispersion equation of SH wave propagating in viscoelastic, inhomogeneous layer overlying a homogeneous half space under initial stress.
5. Particular case
Consider $\alpha \to 0$, $\mu' = 0$, $P = 0$. When the initial stress $P = 0$, then $\zeta = 0$. Using asymptotic analysis [14], we obtain the standard dispersion equation of Love waves as

$$\tan \left( kh_1 \sqrt{\frac{c_1^2}{\beta_{10}^2} - 1} \right) = \left[ \frac{\mu_2}{\mu_{10}} \sqrt{1 - \frac{c_1^2}{\beta_{10}^2}} \right] / \left[ \frac{\mu_{10}}{\mu_{10}^0} \sqrt{\frac{c_1^2}{\beta_{10}^2} - 1} \right],$$

(12)

where $\beta_{10}^2 = \mu_{10}/\rho_{10}$.

6. Numerical results and discussions
Equation (11) gives the dispersion equation for shear waves propagating in a viscoelastic, heterogeneous layer lying over a homogeneous half space under initial stress. In this section, the phase velocities as well as damping velocities have been plotted from (11) for different values of the inhomogeneity parameter $h_1$ and the initial stress $\zeta$. We have taken the following data from Gubbins [15] and Caloi [16]:

$$\begin{align*}
\mu_{10} &= 7.45 \times 10^4 \text{ N/m}^2, \\
\mu_{10}/\mu_{10}^0 &= 30 \text{ sec}^{-1}, \\
\rho_0 &= 3293 \text{ Kg/m}^3, \\
\mu_2 &= 7.10 \times 10^{10} \text{ N/m}^2, \\
\rho_2 &= 3321 \text{ Kg/m}^3.
\end{align*}$$

(13)

Figure 2. Dimensionless phase velocity against dimensionless wave number when $\zeta = 0.1$.

Figure 3. Dimensionless damping velocity against dimensionless wave number when $\zeta = 0.1$.

In figures 2 and 4 the dimensionless phase velocity $c_1/\beta_1$ has been plotted against the dimensionless wave number $kh_1$ whereas the damping velocity has been plotted against dimensionless wave number in figures 3 and 5.

In figures 2 and 3, we have taken different values of inhomogeneity parameter of the layer $M_1$ ($h_1 = 0.1, 1.1, 2.1$). From figure 2 it is observed that the phase velocity $c_1/\beta_1$ decreases monotonically with the increase of $kh_1$ but increases with the inhomogeneity. In the figure 3, the effect of inhomogeneity parameter on damping velocity have same pattern as in figure 2. It has been observed from the figure 3 that due to damping the range of $kh_1$ differ from without damping case (vide figure 2).

In figures 4 and 5, it has been observed that if the values of the initial stress are increased in the layer $M_2$ then the phase velocity as well as the damping velocity increases. Also it has been noticed that phase and damping velocities decrease monotonically with the increase of dimensionless wave number.
7. Conclusion

An analytical approach is used to investigate shear wave propagation in a viscoelastic, heterogeneous layer lying over a homogeneous half space under initial stress. The dispersion equation obtained here is in agreement with the classical result of Love wave propagation when the inhomogeneity of the upper layer and the initial stress on the half space are neglected. The numerical results reflect that the phase and damping velocities decrease with wave number but increase with the inhomogeneity parameter of the upper layer and the initial stress of the half space.

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Figure 4. Dimensionless phase velocity against dimensionless wave number when ah = 0.1.

Figure 5. Dimensionless damping velocity against dimensionless wave number when ah = 0.1.