Using Information Theory to Study the Efficiency and Capacity of Caching in the Computer Networks

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Abstract—Nowadays computer networks use different kinds of memory whose speeds and capacities vary widely. There exist methods of a so-called caching which are intended to use the different kinds of memory in such a way that the frequently used data are stored in the faster memory, whereas the infrequent ones are stored in the slower memory. We address the problems of estimating the caching efficiency and its capacity. We define the efficiency and capacity of the caching and suggest a method for their estimation based on the analysis of kinds of the accessible memory.

Keywords network management, computer networks, future Internet design.

I. INTRODUCTION

Nowadays computer networks are equipped with different kinds of memory whose capacity and speed are different and each user can use different types of memory located in different network nodes. That is why there are many special methods intended to use the different kinds of memory in such a way that the frequently used data are stored in the fast memory. As a rule, such methods are based on a so-called caching (or cache memory). There are a lot of papers devoted to methods of caching and organizing the cache memory for computers and networks, including such practically important nets as content delivery networks; see for review [1], [4], [5], [8], [9]. Note that we mainly say about computer nets, but this consideration is valid and general networks which can contain all mobile phones, satellite channels, data centers, etc.

In this paper we suggest a new approach to investigate the performance of caching algorithms. It is worth noting that close approach was used by the first author in [6] where the estimation of the computer performance was suggested. In order to describe the main idea of the suggested approach we first shortly consider a content distribution network. Suppose there is a network with several nodes which store some files (say, a set of movies $F$) and one node (or consumer) can read files from others and besides times of reading are different from different nodes. Suppose, that if a needed file $f \in F$ is stored at several nodes, it is transmitted from a node for which the time of transmission is minimal and at one moment the node $\omega$ can read a file from one node.

Now some questions seem to be natural. Suppose, this network is used and the frequency of access of files $f \in F$ obey a certain distribution $p(f)$. Either it is a good performance or it is far from the optimum? And what is the optimum? What are the maximal performance and the capacity? There are several approaches to definitions of those concepts, see [1], [4], [5], [8].

We suggest the approach which gives a possibility to define the mentioned above notations and answer those questions basing on ideas of Information Theory. In order to describe the main idea we first give some definitions. Obviously, the main goal of the network is to transmit some files (movies, songs, etc.) to their nodes (consumers). That is why we call any sequence of files $f_1f_2...f_n$, $n \geq 1$, read by a node $\omega$ as a task and denote a set of such sequences (tasks) which can be read by $\omega$ during a time $T$ as $N(T)$.

The key observation is as follows: for the large $T$, the number of tasks which can be carried out during the time $2T$ ($|N(2T)|$) is (approximately) $|N(T)|^2$, because the set of tasks (sequences of files) $N(2T)$ contains all concatenations of sequences (tasks) carried out during the first time-interval $T$ and the second one. (Here and below $|S|$ is the number of elements of a set $S$.) In other words, the number of the file sequences which can be read during a certain time $t$ grows exponentially as a function of $t$, i.e. $|N(t)| = \exp(\alpha t)$, where $\alpha$ is a certain constant. Hence, it is natural to estimate an efficiency of the considered caching by the exponent $\alpha$ and this is the approach will be developed in this paper. Note that this approach will be extended to case of estimation of the throughput for general networks and cache memory of multi-core computers.

Let us briefly consider an example which can clarify the main idea of the approach. Let there be two networks which have an identical structure, but their speeds of transmission are different in such a way that the speed of the first net is twice more than that of the second one. From the given consideration we can see that the exponent $\alpha$ of the first net is twice more than the second one. Apparently, it should be so.

This approach is close, in spirit, to methods of Information Theory, where, for example, the capacity of a channel and the performance of an information source are defined by the rate of asymptotic growth of the number of allowed sequences of basic symbols (letters), (see [2], [7]). Besides, this approach was recently applied to the definition of the efficiency and capacity of computers [6] which, in turn, found some practical applications [3].
II. THE CAPACITY

In this part we define the capacity of a network equipped by a certain method of caching and suggest a simple algorithm of calculation. Informally, the capacity equals the maximum throughput of the network and some simple examples show how the estimates of the capacity can be applied when, say, someone plans either to build or to modernize a network.

We consider a network \( \Omega \) formed by \( n \) nodes \( \omega_1, ..., \omega_n \). Suppose that the node \( \omega_i \) stores a set of files \( F_i \) and a node \( (\omega) \) can read files from some other nodes. Let \( F \) is the set of all files, i.e. \( F = \bigcup_{i=1}^{n} F_i \). If a needed file \( f \in F \) is stored at several nodes, it is transmitted from a node for which the time of transmission is minimal and at one moment the node \( \omega \) can read a file from one node.

Denote the time of reading a file \( f \) by the node \( \omega_i \) from the node \( \omega_j \) by \( \tau_{ij}(f) \) and let \( \tau_{\omega_i}(f) \) be the minimal time needed to obtain the file \( f \) by the node \( \omega_i \), i.e.

\[
\tau_{\omega_i}(f) = \min \tau_{ij}(f).
\]

For the sake of simplification of notations, it will be convenient to let \( \tau_{ij}(f) = \infty \), if \( \omega_i \) cannot obtain \( f \) from \( \omega_j \). The following Fig 1 is an example of probably the simplest network, where \( \omega_1 \) can be a server, \( \omega_2 \) can be the end user and the server cannot read files from the end user. Here, by definition, \( \tau_{12}(f) = \infty \). Let the node \( \omega_1 \) reads a sequence of files \( f' = f_1, f_2, ..., f_s \). We call any such a sequence as a task and define the execution time \( \tau_{i}(f') \) of the \( f' \) by equation

\[
\tau_{\omega_i}(f') = \sum_{k=1}^{s} \tau_{\omega_i}(f_k),
\]

i.e. it is supposed that any file \( f_k \) is transmitted at maximum speed. For a nonnegative \( T \) we define as \( \nu_{i}(T) \) the number of tasks of the node \( \omega_i \), whose execution time equals \( T \), i.e.

\[
\nu_{i}(T) = |\{ f' : \tau_{\omega_i}(f') = T \}|.
\]

We gave an informal explanation that \( \nu_{i} \) grows exponentially, that is why we define the capacity of the node \( \omega_i \) as follows:

\[
C(\omega_i) = \limsup_{T \to \infty} \frac{\log \nu_{i}(T)}{T},
\]

(bits per time unit) and let the capacity of the network be the following sum of the capacities of the nodes:

\[
C(\Omega) = \sum_{\omega \in \Omega} C(\omega).
\]

(Here and below \( \log x \equiv \log_2 x \).) Of course, both capacities depend on the stored files and time of their transmissions.

The first question is how to estimate the capacity. The simple method for calculation \( C(\omega) \) is well-known in Information Theory and was suggested by C. Shannon in 1948 [7], when he estimated the capacity of a lossless channel. Applying his method to the considered problems, we can say that the capacity of one node \( C(\omega_i) \) is equal to the logarithm of the largest real solution \( X_0 \) of the following equation:

\[
\sum_{f \in F} X^{-\tau_{\omega_i}(f)} = 1.
\]

In other words, \( C(\omega_i) = \log X_0 \). By definition, \( C(\omega_i) = 0 \), if this equation does not have solutions (it is possible if all \( \tau_{\omega_i}(f) = \infty \)). Note that the root can be calculated by a so-called bisection method which is the simplest root-finding algorithm. As we mentioned above \( C(\Omega) = \sum_{\omega \in \Omega} C(\omega) \).

Let us consider two simple examples. First we look at the Fig. 1 and suppose that the first node \( \omega_1 \) is a library which stores \( 10 \) files, whereas \( \omega_2 \) is an end user which can store 10 files, i.e. \( |F_1| = 10^7, |F_2| = 10 \). Also suppose that \( \tau_{22}(f) = 1 \) for all \( f \in F_2 \) and \( \tau_{12}(f) = 10 \), for all \( f \in F_1 \), i.e. reading of a file from “own” memory \( F_2 \) requires 1 time-unit, whereas reading from \( \omega_1 \) requires 10 time-units. Besides, the node \( \omega_1 \) cannot read files from \( \omega_2 \), hence, \( \tau_{12}(f) = \tau_{22}(f) = \infty \).

Supposing that files in \( F_1 \) and \( F_2 \) are different \( (F_1 \cap F_2 = 0) \), from the equation \( (5) \) we obtain

\[
\frac{10}{X} + \frac{10^7}{X^{10}} = 1.
\]

Calculating, we obtain \( X_0 = 10.01, C(\omega_2) = \log(10.01) = 3.324 \) bit per time-unit. Note that \( C(\Omega) = 3.324 \), too, because \( C(\omega_2) = 0 \). (Indeed, \( \tau_{21}(f) = \tau_{22}(f) = \infty \), hence, by definition, the capacity is 0.)

Fig. 2. The three-node network.

Now let us consider the network on the Fig. 2. Suppose, that the nodes \( \omega_1 \) and \( \omega_2 \) have the same parameters as in the previous example and \( \omega_3 \) is an identical to \( \omega_2 \). Besides, let \( \omega_2 \) can read files from \( \omega_3 \) and, vise versa, \( \omega_3 \) can read files from \( \omega_2 \). Suppose that the time of both readings is two units per file, i.e. \( \tau_{23}(f) = \tau_{32}(f) = 2 \). If the set of files \( F_1 \) and \( F_2 \) are different \( (F_1 \cap F_2 = 0) \), we obtain for the nodes \( \omega_2 \) and \( \omega_3 \) the following identical equations for calculating the capacities:

\[
\frac{10}{X} + \frac{10}{X^2} + \frac{10^7}{X^{10}} = 1.
\]

Direct calculation shows that \( C(\omega_2) = C(\omega_3) = 3.449 \) and, hence, the total capacity of the network \( \Omega \) equals \( 2 \cdot 3.449 = \)
It is interesting that if the nodes $\omega_2$ and $\omega_3$ store the same set of files ($F_2 = F_3$), the capacity of each of them will be equal the capacity of $\omega_2$ from the Fig.1. In other words, the capacity depends on content of the nodes. Naturally, the speed of transmission, the memory size of different node and topology of the network effect the capacity, that is why the suggested approach can be used if one plans to build or modernize the network. Indeed, it is easy to calculate the capacity for different versions of the net in order to find the optimal.

III. THE ENTROPY EFFICIENCY

Let a node $\omega$ from the network $\Omega$ uses files from $F = \{f\}$ with a certain probability distribution. Our goal is to describe a so-called entropy efficiency which is a measure of the performance for this case. In order to model this situation we suppose that for any node $\omega$ a sequence of read files $f_1, f_2, ..., f_n$ is generated by stationary and ergodic process (note, that nowadays this model is one of the most general and popular). Again, we use ideas and concepts of Information Theory and first give some required definitions.

Let there be a stationary and ergodic process $z$ generating letters from a finite alphabet $A$ (the definition of stationary ergodic process can be found, for example, in [2]). The $n-$order Shannon entropy and the limit Shannon entropy are defined as follows:

$$h_n(z) = -\frac{1}{n+1} \sum_{u \in A^{n+1}} P_z(u) \log P_z(u),$$

$$h_\infty(z) = \lim_{n \to \infty} h_n(z) \tag{7}$$

where $n \geq 0$, $P_z(u)$ is the probability that $z_1 z_2 ... z|u| = u$ (this limit always exists, see [2], [7]). We will also consider so-called i.i.d. sources. By definition, they are independent and identically distributed random variables from some set.

Now we can define the entropy efficiency.

**Definition 1:** Let there be a network $\Omega$ and for node $\omega \in \Omega$ a sequence of read files $f_1, f_2, ..., f_n$ is generated by a stationary ergodic process $\varphi_\omega$ and let $\tau_\omega(f''')$ be the time of reading the file $f'''$. Then the entropy efficiency is defined as follows:

$$C(\omega, \varphi_\omega) = h_\infty(f') / \sum_{f \in F} P_{\varphi_\omega}(f) \tau_\omega(f),$$

where $P_{\varphi_\omega}(f''')$ is the probability that $f_1 = f'''$. The entropy efficiency of the network $\Omega$ equals

$$C(\Omega, \tilde{\varphi}) = \sum_{\omega \in \Omega} C(\omega, \varphi_\omega)$$

where $\tilde{\varphi} = (\varphi_{\omega_1}, ..., \varphi_{\omega_{|\Omega|}})$.

Informally, the Shannon entropy is a quantity of information (per letter), which can be transmitted and the denominator in (8) is the average time of reading of one file.

The entropy efficiency can be estimated basing on statistical estimations of frequencies and it gives a possibility to investigate the influence of different parameters on the entropy efficiency.

It is easy to see that the entropy efficiency (8) is maximal, if the sequence of files $f_1 f_2 ...$ is generated by an i.i.d. source with probabilities $p^*(f) = X_0^{-\tau_\omega(f)}$, where $X_0$ is the largest real solution to the equation (5). Indeed, having taken into account that $h_\infty(\omega) = h_0(\omega)$ for i.i.d. source [2] and the definition of entropy (7), the direct calculation of $c(\omega, p^*)$ in (8) shows that $c(\omega, p^*) = \log X_0$ and, hence, $c(\omega, p^*)$ equals the capacity $C(\omega)$.

It will be convenient to combine the results about the capacity and the entropy efficiency in the following statement:

**Claim 1:** Let there be a a network $\Omega$ with a set of files $F$, a node $\omega$ and let $\tau_\omega(f)$ be the time of reading of the file $f$. Then the following equalities are valid:

i) The capacity $C(\omega)$ equals $\log X_0$, where $X_0$ is the largest real solution to the equation (5).

ii) The entropy efficiency (8) is maximal if the reading files are generated by an i.i.d. source with probabilities $p^*(f) = X_0^{-\tau_\omega(f)}$, $f \in F$.

So, we can see that the node capacity is the maximal value of the entropy efficiency, and this maximum can be attained for a certain file distribution. This fact can be useful for applications. For example, if the entropy efficiency is much less than the capacity, it could mean that a different scheme of caching should be used. In other words, information about the network capacity and the entropy efficiency can be useful for network designers and manufacturers.

ACKNOWLEDGMENT

Research was supported by Russian Foundation for Basic Research (grant no. 12-07-00125).

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