Abstract

How far apart are two neural networks? This is a foundational question in their theory. We derive a simple and tractable bound that relates distance in function space to distance in parameter space for a broad class of nonlinear compositional functions. The bound distills a clear dependence on depth of the composition. The theory is of practical relevance since it establishes a trust region for first-order optimisation. In turn, this suggests an optimiser that we call Frobenius matched gradient descent—or Fromage. Fromage involves a principled form of gradient rescaling and enjoys guarantees on stability of both the spectra and Frobenius norms of the weights. We find that the new algorithm increases the depth at which a multilayer perceptron may be trained as compared to Adam and SGD and is competitive with Adam for training generative adversarial networks. We further verify that Fromage scales up to a language transformer with over $10^8$ parameters. Please find code & reproducibility instructions at:

https://github.com/jxbz/fromage.

1. Introduction

Suppose that a teacher wishes to assess a student’s learning. Traditionally, they will assign the student homework and track their progress. What if, instead, they could peer inside the student’s head and observe change directly in the synapses—would that not be better for everyone?

Neural networks are usually trained by (stochastic) gradient descent. The basic premise is that gradient descent solves:

$$
\min_{\Delta W} \left[ \mathcal{L}(W) + \nabla \mathcal{L}(W)^T \Delta W + \frac{1}{2\eta} \cdot D_E(W, W + \Delta W) \right].
$$

That is, gradient descent chooses the parameter perturbation $\Delta W$ to minimise a local linear approximation to the objective function $\mathcal{L}$, where we add the penalty $D_E$ to prevent $\Delta W$ from straying beyond the region where the gradient $\nabla \mathcal{L}(W)$ is trusted (Nocedal & Wright, 2006). For gradient descent, the penalty takes the form:

**Definition 1** (Euclidean trust).

$$
D_E(W, W + \Delta W) := \|\Delta W\|_F^2.
$$

We refer to this model as *Euclidean trust* since a quadratic penalty is akin to assuming a Euclidean structure on the parameter space. We perform a theoretical analysis and experimental study to test this model and find evidence that for multilayer perceptrons, trust is lost not quadratically but rather quasi-exponentially in the perturbation size. Figure 1 illustrates the difference.

Our analysis exposes the following mathematical structure for the trust region of a broad family of deep neural networks with layers indexed $l = 1, \ldots, L$:

**Definition 2** (Deep relative trust).

$$
D(W, W + \Delta W) := \prod_{l=1}^{L} \left( 1 + \frac{\|\Delta W_l\|_F^2}{\|W_l\|_F^2} \right) - 1.
$$

Deep relative trust has two essential features: the first is a dependence on the relative magnitude of perturbations; the
second is a product over the network’s \( L \) layers, reflecting the product structure of the network itself. These features are both absent from Euclidean trust. In our model, relative perturbations across layers compound.

The main contributions of this paper are:

1. proposing that deep relative trust is an appropriate notion of distance between neural networks based on both theoretical analysis and experimental evidence.
2. developing an optimisation theory based on deep relative trust, and using the tools of matrix perturbation theory to study the stability of learning.
3. deriving a neural network optimiser called Fromage (Algorithm 1) that exploits the new theory. The algorithm has one hyperparameter with a clear meaning.
4. benchmarking Fromage on popular machine learning problems such as image classification, generative adversarial networks and natural language transformers, revealing often favourable performance compared to standard optimisers such as Adam and SGD.

2. Entendámonos...

...so we understand each other.

The goal of this section is to review a few basics of deep learning, including heuristics commonly used in algorithm design and areas where current optimisation theory falls short. We shall also review generative adversarial learning.

We shall see that, whilst it is central to both optimisation and generative adversarial learning, finding an appropriate notion of functional distance for deep networks is not a solved problem.

Deep learning basics

Deep learning seeks to fit a neural network function \( f(W; x) \) with parameters \( W \) to a dataset of \( N \) input-output pairs \( \{x_i, y_i\}_{i=1}^N \). If we let \( L_i := L(f_i, y_i) \) measure the discrepancy between prediction \( f_i := f(W; x_i) \) and target \( y_i \), then learning proceeds by gradient descent on the loss: \( \sum_{i=1}^N L_i \).

Though various neural network architectures exist, we shall focus our theoretical effort on the multilayer perceptron, which already contains the most striking features of general neural networks: matrices, nonlinearities, and layers.

**Definition 3** (Multilayer perceptron). A multilayer perceptron is a function \( f : \mathbb{R}^{n_0} \rightarrow \mathbb{R}^{n_L} \) composed of \( L \) layers.

\[
  f(x) = \varphi \circ W_N \circ \varphi \circ W_{L-1} \circ \ldots \circ \varphi \circ W_1(x).
\]

The \( l \)th layer is a linear map \( W_l : \mathbb{R}^{n_{l-1}} \rightarrow \mathbb{R}^{n_l} \) followed by a nonlinearity \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \) that is applied elementwise.

**Algorithm 1** Fromage \( \bullet \) (a good default for \( \eta \) is 0.01).

**Input:** learning rate \( \eta \) and weight matrices \( \{W_l\}_{l=1}^L \)

**repeat**

collect gradients \( \{g_l\}_{l=1}^L \) via backpropagation

**for** layer \( l = 1 \) to \( L \) **do**

\[
  W_l \leftarrow \frac{1}{\sqrt{1+\eta^2}} \left[ W_l - \eta \cdot \frac{\|W_l\|F}{\|g_l\|F} \cdot g_l \right]
\]

**end for**

until converged

The multilayer perceptron may be described recursively in terms of the \( l \)th hidden layer \( h_l(x) \in \mathbb{R}^{n_l} \) as:

\[
  h_l(x) := \varphi(W_l h_{l-1}(x)); \quad h_0(x) := x.
\]

Since we wish to fit the network via gradient descent, we shall be interested in the gradient of the loss with respect to the \( l \)th parameter matrix. Schematically, via the chain rule:

\[
  \nabla_{W_l} \mathcal{L} = \frac{\partial \mathcal{L}}{\partial f} \cdot \frac{\partial f}{\partial h_l} \cdot \frac{\partial h_l}{\partial W_l}.
\]

Let us zoom in on the second term on the righthand side, following the treatment of Pennington et al. (2017).

**Proposition 1** (Jacobian of the multilayer perceptron). Consider a multilayer perceptron with \( L \) layers. For \( l = 1, 2, \ldots, L \), the layer-1-to-output Jacobian is given by:

\[
  \frac{\partial f(x)}{\partial h_l} = \frac{\partial f}{\partial h_{L-1}} \cdot \frac{\partial h_{L-1}}{\partial h_{L-2}} \cdot \ldots \cdot \frac{\partial h_{l+1}}{\partial h_l} = \Phi'_L W_L \cdot \Phi'_{L-1} W_{L-1} \cdot \ldots \cdot \Phi'_{l+1} W_{l+1},
\]

where \( \Phi'_l := \text{diag} \left[ \varphi'(W_l h_{l-1}(x)) \right] \).

A key observation is that the network function \( f \) and Jacobian \( \frac{\partial f}{\partial h_0} \) share a common mathematical structure—a deep, layered composition. We shall exploit this in our theory.

**Empirical deep learning**

For the \( l \)th layer, gradient descent prescribes the update:

\[
  W_l \leftarrow W_l - \eta \cdot \nabla_{W_l} \mathcal{L},
\]

where \( \eta > 0 \) is a small perturbation parameter or learning rate chosen independent of layer.

Practitioners quickly run into a problem with this formulation known as the vanishing and exploding gradient problem, where the scale of updates becomes miscalibrated with the scale of parameters in different layers of the network. Common tricks to ameliorate the problem include careful choice of weight initialisation (Glorot & Bengio, 2010), dividing out the gradient scale (Kingma & Ba, 2015) and gradient clipping (Pascanu et al., 2013). Each of the techniques has been adopted in numerous deep learning applications.
Still, there is a cost to using heuristic techniques. For instance, techniques that rely on careful initialisation may break down by the end of training, leading to instabilities that are difficult to trace. Gradient clipping involves introducing and tuning a new parameter: the clipping threshold.

**Related work in deep learning optimisation theory**

Euclidean trust, as set up in the introduction, is commonly justified by assuming that the loss function has Lipschitz continuous gradients, meaning that:

$$\|\nabla L(W + \Delta W) - \nabla L(W)\|_F \leq \frac{2}{\eta} \|\Delta W\|_F.$$

By a standard argument (Bottou et al., 2016), this implies a quadratic or Euclidean upper bound on the loss function:

$$L(W + \Delta W) \leq L(W) + \nabla L(W)^T \Delta W + \frac{1}{2\eta} \|\Delta W\|^2_F.$$  

Gradient descent as in (2) iteratively minimises this bound.

The gradient-Lipschitz assumption is ubiquitous to the point that it is often just referred to as *smoothness* (Hardt et al., 2016). The assumption is a natural starting point for theory and it is used by: Hardt et al. (2016), Lee et al. (2016), Du et al. (2017) and Allen-Zhu (2018) in the context of deep learning optimisation; Bernstein et al. (2018) in the context of distributed training; Schaefer & Anandkumar (2019) in the context of generative adversarial networks.

The Lipschitz assumption played a central role in classical optimisation (Nesterov, 2014, Chapter 1). However, it is unclear how applicable the assumption is to deep learning—in a comprehensive review on deep learning optimisation, Sun (2019) writes that “neural network optimization problems do not have a global gradient Lipschitz constant” and that “the lack of global Lipschitz constants is a general challenge for non-linear optimization”.

The surest way to see that neural networks are not gradient-Lipschitz for all practical purposes is to measure the gradient empirically. We do this for a 16 layer multilayer perceptron, and find that the gradient grows roughly exponentially in the size of a perturbation (Figure 2). For more work of that ilk, Benjamin et al. (2019) empirically test the use of Euclidean distance as a proxy for functional distance, and find the relationship non-trivial and difficult to interpret.

Several classical optimisation frameworks study non-Euclidean models of functional distance. For example, mirror descent (Nemirovsky & Yudin, 1983) replaces $\|\Delta W\|^2_F$ by a Bregman divergence appropriate to the geometry of the problem. This framework was studied in relation to deep learning (Azizan & Hassibi, 2019; Azizan et al., 2019), but the design of good divergence measures remains an area of active research.

Another classical technique is natural gradient descent (Amari, 2016), which replaces $\|\Delta W\|^2_F$ by $\Delta W^T F \Delta W$. The Riemannian metric $F \in \mathbb{R}^{d \times d}$ should capture the geometry of the $d$-dimensional function class. Unfortunately, this technique is computationally heavy since just writing down the metric takes $O(d^2)$ space, and for neural networks $d \gg 1$. Whilst Martens & Grosse (2015) explore more efficient surrogates, natural gradient descent is fundamentally a quadratic model of trust. Our results suggest that trust is lost far more catastrophically in deep networks (Figure 1).

A final line of related work studies the effect of architectural decisions on signal propagation through the network (Saxe et al., 2014; Pennington et al., 2017; Yang & Schoenholz, 2017; Xiao et al., 2018; Anil et al., 2019), which inspired aspects of our work. Though these works neglect theoretical study of functional distance and curvature of the loss surface, they do carry out direct analyses of the deep neural network structure. Pennington & Bahri (2017), on the other hand, do study curvature of the loss surface, though they rely on random matrix models to make progress.

**Generative adversarial networks**

Neural networks can learn to generate samples from complex distributions. Generative adversarial learning (Goodfellow et al., 2014) trains a discriminator network $D$ to classify data as real or fake, and a generator network $G$ is trained to fool $D$. Competition drives learning in both networks. Letting $V$ denote the success rate of the discriminator, the learning process is described as:

$$\min_G \max_D V(G, D).$$

Defining the optimal discriminator for a given generator as $D^*(G) := \max_D V(G, D)$, then generative adversarial learning reduces to a straightforward minimisation over the parameters of the generator:

$$\min_G \max_D V(G, D) = \min_G V(G, D^*(G)).$$

In practice this is solved as an inner-loop, outer-loop optimisation procedure where $k$ steps of gradient descent are performed on the discriminator, followed by 1 step on the generator. For example, Miyato et al. (2018) take $k = 5$ and Brock et al. (2019) take $k = 2$.

For small $k$, this procedure is only well founded if the perturbation $\Delta G$ to the generator is small so as to induce a small perturbation in the optimal discriminator. In symbols, we hope that

$$\|\Delta G\| \ll 1 \implies \|D^*(G + \Delta G) - D^*(G)\| \ll 1.$$  

But what does $\|\Delta G\|$ mean? In what sense should it be small? Again, we realise that we are lacking an appropriate notion of functional distance for neural networks.
3. The distance between neural networks

We would like to establish a meaningful notion of functional distance for neural networks. The main pitfall of the Euclidean distance on parameters is that it does not reflect the product structure of the network.

To guide intuition, consider a simple network that multiplies its input $x \in \mathbb{R}$ by two scalars $a, b \in \mathbb{R}$. That is $f(x) = a \cdot b \cdot x$. Also consider perturbed function $\tilde{f}(x) = \tilde{a} \cdot \tilde{b} \cdot x$ where $\tilde{a} := a + \Delta a$ and $\tilde{b} := b + \Delta b$. By expanding the square and bounding the cross-terms with Young’s inequality, we find that the relative difference obeys:

$$
\left( \frac{\tilde{f}(x) - f(x)}{f(x)} \right)^2 \leq 3 \left( 1 + \frac{\Delta a^2}{a^2} \right) \left( 1 + \frac{\Delta b^2}{b^2} \right) - 1.
$$

We flesh out this important derivation in the appendix. The following theorem, also proved in the appendix, generalises this argument to the deep, nonlinear case.

**Theorem 1** (Relative functional difference). Let $f$ be a multilayer perceptron with nonlinearity $\varphi$ and $L$ weight matrices $\{W_l\}_{l=1}^L$. Likewise consider perturbed network $\tilde{f}$ with weight matrices $\{\tilde{W}_l\}_{l=1}^L$. For convenience, we define perturbation matrices $\Delta W_l := \tilde{W}_l - W_l$.

Let the dimension of the $l$th hidden layer be $n_l$, meaning that $h_l(x) \in \mathbb{R}^{n_l}$. We define the maximum width $n^* := \max_l n_l$.

Suppose that the following conditions hold:

1. **Fixed point.** The nonlinearity satisfies $\varphi(0) = 0$.
2. **Transmission.** There exist $\alpha, \beta \geq 0$ such that $\forall x, y$:
   $$
   \alpha \cdot \|x - y\|_2^2 \leq \|\varphi(x) - \varphi(y)\|_2^2 \leq \beta \cdot \|x - y\|_2^2.
   $$
3. **Conditioning.** Each of the unperturbed weight matrices $\{W_l\}_{l=1}^N$ has condition number bounded by $\kappa$.

For all non-zero inputs $x \in \mathbb{R}^{n_0}$ we have:

$$
\left( \frac{\tilde{f}(x) - f(x)}{\|f(x)\|_2} \right)^2 \leq C_0 \left( \prod_{l=1}^L \left( 1 + \frac{\|\Delta W_l\|_F^2}{\|W_l\|_F^2} \right) - 1 \right),
$$

where we have defined $C_0 := \left( \frac{2\alpha \kappa^2}{\alpha^2} \right)^L$.

In words, Theorem 1 says that the change of a multilayer perceptron in function space is controlled by deep relative trust (Definition 2). As deep relative trust goes to zero, the relative change in function space goes to zero too.

Bounding the relative change in function $f$ in terms of the relative change in parameters $\Delta W$ is reminiscent of a concept from numerical analysis known as the relative condition number. The relative condition number of a numerical technique measures the sensitivity of the technique to input perturbations. This suggests that we may think of Theorem 1 as defining the relative condition number of a neural network with respect to parameter perturbations.

We must discuss the plausibility of the assumptions. The first two conditions are on the nonlinearity and are both satisfied by the “leaky relu” function, where for $0 \leq a \leq 1$:

$$
\varphi(x) = \begin{cases} 
  x & \text{if } x \geq 0; \\
  ax & \text{if } x < 0.
\end{cases}
$$

Setting $a = 0$ yields the “relu” function, which only satisfies the second condition with $\alpha = 0$ for which the bound diverges. We may suspect that for inputs that occur in practice, the second assumption may hold for relu with an $\alpha > 0$. We leave detailed investigation for future work.

As for the third condition, in general $\kappa$ may be infinite—rendering the bound vacuous. However, we know by smoothed analysis of the condition number (Sankar et al., 2006; Bürgisser & Cucker, 2010) that $\kappa$ is finite with probability 1 for an iid Gaussian initialisation, and continues to be so throughout training provided a small amount of iid Gaussian noise is added to the updates.

4. Breakdown of a local linear approximation

In the last section we studied the relative functional difference between two neural networks and found that it depends on deep relative trust. Here we will focus on the relative difference in gradient, so that we may establish a trust region for optimisation. We shall see that the relative functional difference and relative gradient difference are connected.

We are interested in the relative change in the gradient expression (1). Tackling the product of the three terms on the right-hand side directly is challenging, not least because the loss function $L(f, y)$ is unknown and arbitrary. As a result, we will tackle each term individually.

We will argue that both the first term $\frac{\partial L}{\partial \varphi}$ and the third term $\frac{\partial \varphi}{\partial f}$ depend on the output of a hidden layer, and since a hidden layer is itself the last layer of a sub-network, these terms are connected to deep relative trust via Theorem 1.

To realise this argument, observe that the first term depends on the network output $f$. For example, for the squared error loss we have $L(f, y) = \frac{1}{2} \|f - y\|_2^2$ and $\frac{\partial L}{\partial \varphi} = f - y$. Similarly, the third term depends on the output of layer $h_{l-1}$. To see this, note that $h_1 = \varphi(W_1 h_{l-1})$ and therefore schematically we have that $\frac{\partial h_1}{\partial W_1} = \varphi'(W_1 h_{l-1}) h_{l-1}$.

The final term to tackle is the middle term in (1): $\frac{\partial f}{\partial W_l}$. This is the layer-$l$-to-output Jacobian. As detailed in Proposition 1, it is a product of matrices. We proffer the following theorem to bound its relative change:
On the distance between two neural networks

Figure 2. Using Fromage, we train a 2-layer (left) and 16-layer (right) perceptron to classify the MNIST dataset. With the network frozen at ten different training checkpoints, we first compute the gradient of the $l$th layer $g_l$ using the full data batch. We then record the loss and full batch gradient $g_l$ after perturbing all weight matrices $W_l$ at various perturbation strengths $\eta$. We plot the relative change in gradient of the input layer $\|g_l - g_l\|_F/\|g_l\|_F$ and also the classification loss along these parameter slices. Note that these plots are on a log scale. We find that the loss and relative change in gradient grow quasi-exponentially when the perceptron is deep, suggesting that Euclidean trust is violated. As such, these results seem more consistent with deep relative trust.

Theorem 2 (Relative Jacobian difference). Let $f$ be a multilayer perceptron with nonlinearity $\varphi$ and $L$ weight matrices $\{W_l\}_{l=1}^L$. Likewise consider perturbed network $\tilde{f}$ with weight matrices $\{\tilde{W}_l\}_{l=1}^L$. For convenience, we define perturbation matrices $\Delta W_l := \tilde{W}_l - W_l$.

Let the dimension of the $l$th hidden layer be $n_l$, so that $h_l(x) \in \mathbb{R}^{n_l}$. We define the maximum width $n^* := \max_l n_l$.

Suppose that the following conditions hold:

1. Transmission. There exist $\alpha, \beta \geq 0$ such that $\forall x, y$:
$$\alpha \cdot \|x - y\|_2^2 \leq \|\varphi(x) - \varphi(y)\|_2^2 \leq \beta \cdot \|x - y\|_2^2.$$ 
2. Conditioning. Each of the unperturbed weight matrices $\{W_l\}_{l=1}^L$ has condition number bounded by $\kappa$.

Then we have that:
$$\frac{\left\| \frac{\partial f}{\partial x_l} - \frac{\partial \tilde{f}}{\partial x_l} \right\|_F^2}{\left\| \frac{\partial f}{\partial x_l} \right\|_F^2} \leq C_1 \prod_{l=k+1}^L C_2 \left( 1 + \frac{\|\Delta W_l\|_F}{\|W_l\|_F} \right)^2 - 1,$$

where we have defined constants:
$$C_1 := \left( 2n^* \sqrt{\frac{\beta}{\alpha}} \right)^{2(L-k)}; \quad C_2 := 1 + \left( \sqrt{\frac{\beta}{\alpha}} - 1 \right)^2.$$

Notice that the assumptions are a subset of those made in Theorem 1. The proof is given in the appendix.

Let us inspect the result itself. Up to the inclusion of the constant $C_2$, we see that deep relative trust appears on the right-hand side of Theorem 2 just as it did for Theorem 1.

5. Descent under relative trust

Up until this point in the paper, we have introduced the concept of deep relative trust and shown theoretically how it connects to both the relative functional difference and relative gradient difference for a broad class of neural networks. What significance does this have for optimisation?

The most striking prediction of the theory is that for large depth $L$, a neural network diverges quasi-exponentially in the relative size of the parameter perturbation. To see this, we compare deep relative trust to the product form of exp:
$$D(W, W + \Delta W) = \prod_{l=1}^L \left( 1 + \frac{\|\Delta W_l\|_F^2}{\|W_l\|_F^2} \right) - 1;$$
$$\exp x^2 - 1 = \lim_{L \to \infty} \prod_{l=1}^L \left( 1 + \frac{x^2}{L} \right) - 1.$$

We visualise this prediction in Figure 1. We test it by comparing the loss and gradient along parameter slices for a 2-layer and 16-layer multilayer perceptron. The results are given in Figure 2 and seem to support the idea of a catastrophic breakdown in trust.

The time has come to derive algorithms. We wish to solve:
$$\min_{\Delta W} \left[ \mathcal{L}(W) + \nabla \mathcal{L}(W)^T \Delta W + \frac{1}{2\eta} D(W, W + \Delta W) \right]. \quad (3)$$

Solving (3) exactly is challenging because of the coupling across layers. Whilst one can imagine various approximation schemes such as a mean-field theory in depth, a solution via perturbation series or even a numerical solution, we prefer to keep matters simple in this work.
We compare deep relative trust and its surrogate in Figure 3. We introduce a surrogate to deep relative trust to decouple the effect of perturbations across layers for tractability.

Definition 4 (Surrogate to deep relative trust).

\[ D'(W, W + \Delta W) := \frac{1}{L} \sum_{l=1}^{L} \left[ \frac{\|\Delta W_l\|_F^2}{\|W_l\|_F^2} \right]^L. \]

To understand the use of this surrogate, observe first that it depends on the relative size of the perturbations, second it is a polynomial of the same order as deep relative trust, and third for large perturbations of constant relative size across layers, the two concepts of trust are the same. To see this, consider perturbations of relative size \( \eta \), meaning that for all layers \( \|\Delta W_l\|_F / \|W_l\|_F = \eta \). Then as \( \eta \to \infty \):

\[ \frac{D(W, W + \Delta W)}{\eta^{2L}} \to \frac{D'(W, W + \Delta W)}{\eta^{2L}} = 1. \]

We compare deep relative trust and its surrogate in Figure 3. The comparison is for a 20 layer network assuming a fixed perturbation size \( \eta \) across layers.

Then let us replace (3) by its surrogate. We define \( g_l := \nabla_{W_l} \mathcal{L}(W) \) and obtain the following optimisation problem:

\[ \min_{\{\Delta W_l\}_{l=1}^{L}} \left[ \sum_{l=1}^{L} g_l^T \Delta W_l + \frac{1}{2\eta L} \sum_{l=1}^{L} \left[ \frac{\|\Delta W_l\|_F^2}{\|W_l\|_F^2} \right]^L \right]. \]

Notice that the optimisation problem conveniently decouples over layers. For each layer \( l = 1, \ldots, L \), we have:

\[ \min_{\Delta W_l} g_l^T \Delta W_l + \frac{1}{2\eta L} \left[ \frac{\|\Delta W_l\|_F^2}{\|W_l\|_F^2} \right]^L. \]

For the \( l \)th layer, it is clear that the minimiser is of the form \( \Delta W_l = -\gamma_l g_l \) for some \( \gamma_l \geq 0 \), since the gradient \( g_l \) is the only direction in the problem, and \( +\gamma_l g_l \) would be inappropriate. We substitute in \( \Delta W_l = -\gamma_l g_l \) and minimise over \( \gamma_l \) to obtain:

\[ \Delta W_l = -\gamma_l \frac{\|W_l\|_F}{\|g_l\|_F} \cdot g_l. \]

A natural way to obtain a depth-independent algorithm is to let the depth \( L \to \infty \). We adopt the scaling \( \eta \frac{\|W_l\|_F}{\|g_l\|_F} \to \eta \) so that \( \eta \) is kept in the limit. We arrive at:

\[ \Delta W_l = -\eta \cdot \frac{\|W_l\|_F}{\|g_l\|_F} \cdot g_l. \]

Figure 3. Comparing deep relative trust (deep pink) and our surrogate (orange). The comparison is made for perturbations of fixed relative size \( \|\Delta W_l\|_F / \|W_l\|_F = \eta \) for all layers \( l = 1, \ldots, L \). The surrogate becomes increasingly accurate for large \( \eta \).

**A decoupled surrogate**

We introduce a surrogate to deep relative trust to decouple the effect of perturbations across layers for tractability.

\[ D'(W, W + \Delta W) := \frac{1}{L} \sum_{l=1}^{L} \left[ \frac{\|\Delta W_l\|_F^2}{\|W_l\|_F^2} \right]^L. \]

Figure 4. Training multilayer perceptrons at depths challenging for existing optimisers. We train multilayer perceptrons of depth \( L \) on the MNIST dataset. At each depth, we plot the training accuracy after 100 epochs. For each algorithm, we plot the best performing run over 3 learning rate settings found to be appropriate for that algorithm. We also plot trend lines to help guide the eye.

We see that our theoretical arguments have recovered a special form of “gradient clipping”. You et al. (2017) proposed a similar update rule based on empirical observations. Unfortunately, there is still an issue with this update rule, in that the update tends to increase weight norms. To see this, consider an update \( \Delta W_l \) that is orthogonal to the matrix \( W_l \). Then, by (4), the norm of the updated weights is given by:

\[ \|W_l + \Delta W_l\|_F^2 = \|W_l\|_F^2 + \|\Delta W_l\|_F^2 = (1 + \eta^2)\|W_l\|_F^2. \]

This is just Pythagoras’ theorem, as visualised in the inset figure. We see that the Frobenius norm of the parameters tends to grow by a factor \( \sqrt{1 + \eta^2} \).

This \( O(\eta^2) \) effect can be serious when the model class is invariant to the parameter scale as is the case for common weight normalisation schemes (Ioffe & Szegedy, 2015; Miyato et al., 2018). Under these schemes, the loss function provides no incentive to control the parameter scale and the norm \( \|W_l\|_F \) will grow without bound.
Figure 5. Training a class-conditional generative adversarial network on the CIFAR-10 dataset (Krizhevsky, 2009). Top: we plot the norms across layers during training. Fromage stabilises the norms whereas for Adam they wander—which can be a serious issue (Brock et al., 2019, Figure 27). Bottom: we plot the mean and shade the range of the FID score (Heusel et al., 2017) during training. We attain a state of the art FID score just by switching the optimiser to Fromage.

We present the appropriately corrected version of the algorithm in Algorithm 1 on page 2. We call it Fromage, short for Frobenius matched gradient descent.

A guide to choosing hyperparameters

One of the attractive features of Algorithm 1 is that there is only one hyperparameter and its meaning is obvious. Neglecting the second order correction, we have that for every layer $l = 1, \ldots, L$, the algorithm’s update satisfies:

$$\frac{\|\Delta W_l\|_F}{\|W_l\|_F} = \eta. \quad (5)$$

In words: the algorithm induces a relative change of $\eta$ in each layer of the neural network. If we set $\eta = 0.01$, then the weight matrices are allowed to change by 1% per iteration. In practice, we find this value to be a good default.

The contrast to SGD and Adam is stark. For these algorithms, the learning rate has little intrinsic meaning, and the effective perturbation strength depends on a complicated interplay between four factors: initial weight scale, weight decay hyperparameter, weight growth during training and the user-prescribed learning rate hyperparameter.

We may say more about Fromage by appealing to Mirsky’s theorem—a basic result in matrix perturbation theory.

**Theorem 3 (Mirsky (1960)).** Let $W$ and $W + \Delta W$ be two matrices in $\mathbb{R}^{m \times n}$. Let $\{\sigma_i\}_{i=1}^\pi$ and $\{\tilde{\sigma}_i\}_{i=1}^\pi$ respectively denote their ordered singular values. Then we have that

$$\sqrt{\sum_{i=1}^\pi (\tilde{\sigma}_i - \sigma_i)^2} \leq \|\Delta W\|_F.$$

We apply this result to the $l$th network layer. Let $\tilde{\sigma}_1 \geq \tilde{\sigma}_2 \geq \ldots \geq \tilde{\sigma}_\pi$ denote the singular values of $W_l + \Delta W_l$, and $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_\pi$ denote the singular values of $W_l$. Then dividing Theorem 3 through by the root mean square singular value $\tilde{\sigma} := \sqrt{\frac{1}{\pi} \sum_{i=1}^\pi \sigma_i^2} = \frac{1}{\sqrt{\pi}} \|W_l\|_F$, we obtain:

$$\frac{1}{\pi} \sum_{i=1}^\pi \frac{(\tilde{\sigma}_i - \sigma_i)^2}{\tilde{\sigma}^2} \leq \|\Delta W_l\|_F = \eta,$$

where we have substituted in (5). In words: the learning rate controls a relative notion of spectral shift.

Spectral instabilities were found by Brock et al. (2019) in the context of large-scale generative adversarial network training with the Adam algorithm. Fromage’s natural ability to control spectral shift therefore seems desirable.
6. Empirical study

Detailed instructions to reproduce these experiments are here: https://github.com/jxbz/fromage.

To test the main prediction of our theory—that the function and gradient of a deep network break down quasi-exponentially in the size of the perturbation—we directly study the behaviour of a multilayer perceptron trained on the MNIST dataset (Lecun et al., 1998) under parameter perturbations. Perturbing along the gradient direction, we find that the change in gradient and objective function is indeed quasi-exponential for a deep network (see Figure 2).

The theory also predicts that the geometry of trust for a deep network becomes increasingly pathological as the network gets deeper, and Fromage is specifically designed to account for this. In Figure 4, we find that Adam and SGD are unable to train multilayer perceptrons over 25 layers deep whereas Fromage was able to train up to at least depth 50.

To test the predictions about the Frobenius norm stability of Fromage, we train a class-conditional generative adversarial network (Miyato et al., 2018) on the CIFAR-10 dataset (Krizhevsky, 2009). We find (Figure 5) that Fromage almost perfectly stabilises the Frobenius norms, whereas when training with Adam the norms wander significantly.

Next, we benchmark Fromage on three canonical deep learning tasks: generative adversarial image generation, image classification and natural language processing.

We find that Fromage outperforms Adam for training a class-conditional generative adversarial network on the CIFAR-10 dataset. The results are given in Figure 5. Next, when training a resnet50 network to classify the Imagenet dataset (Deng et al., 2009), Fromage outperforms SGD without weight decay and matches SGD with weight decay (Figure 6), meaning that Fromage requires less tuning in this setting. Finally, when fine-tuning a transformer on SQuAD1.0 (Rajpurkar et al., 2016), Fromage marginally outperforms Adam and SGD in evaluation score (Figure 7).

7. Limitations and future work

It is common practice in deep learning to randomly subsample data to evaluate the gradient. Our theory is limited in that it neglects this stochasticity entirely. In one of our experiments (Figure 8) we witnessed an instability in Fromage at small batch size. Whilst we found that introducing a form of momentum fixed the problem, future work could investigate the theory of stochastic Fromage more thoroughly.

Our theory is also limited in that it only applies to the multilayer perceptron—the model organism of deep learning theory. Neural networks found in the wild depart from this basic structure in several key ways. Residual connections (He et al., 2016) and batch normalisation (Ioffe & Szegedy, 2015) have been found to stabilise deep network training in numerous applications. Using our tools to analyse these techniques could be a fruitful direction in which to head.

8. Conclusion

We have written down a distance on deep neural networks and studied the implications of this distance for optimisation. We are optimistic that deep relative trust may also help in studying convergence and generalisation in deep learning.

Indeed, recent work (Wilson et al., 2017; Azizan et al., 2019) has studied the relationship between the optimisation algorithm and generalisation. Since we found that Fromage tended to generalise well in our experiments, we are curious to see how it fits into this picture.
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Appendix

We begin by fleshing out the analysis of the two-layer scalar network, since this example already goes a long way to exposing the relevant mathematical structure.

Consider \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = a \cdot b \cdot x \) for \( a, b \in \mathbb{R} \). Also consider perturbed function \( \tilde{f}(x) = \tilde{a} \cdot \tilde{b} \cdot x \) where \( \tilde{a} := a + \Delta a \) and \( \tilde{b} := b + \Delta b \). The relative difference obeys:

\[
\left( \frac{\tilde{f}(x) - f(x)}{f(x)} \right)^2 = \left( \frac{\tilde{a}bx - abx}{ab} \right)^2 = \left( \frac{(a + \Delta a)(b + \Delta b) - ab}{ab} \right)^2 = \left[ \frac{\Delta a}{a} + \frac{\Delta b}{b} + \frac{\Delta a \Delta b}{ab} \right]^2.
\]

We already see the presence of strong interactions between the two layers. But let us simplify the expression by using Young’s inequality on the cross-terms. We obtain:

\[
\left( \frac{\tilde{f}(x) - f(x)}{f(x)} \right)^2 \leq \frac{\Delta a^2}{a^2} + \frac{\Delta b^2}{b^2} + \frac{\Delta a^2 \Delta b^2}{ab} + 2 \left[ \frac{1}{2} \left( \frac{\Delta a^2}{a^2} + \frac{\Delta b^2}{b^2} \right) + \frac{1}{2} \left( \frac{\Delta a^2}{a^2} + \frac{\Delta b^2}{b^2} \right) + \frac{1}{2} \left( \frac{\Delta a^2}{a^2} + \frac{\Delta b^2}{b^2} \right) \right] = 3 \left( 1 + \frac{\Delta a^2}{a^2} \right) \left( 1 + \frac{\Delta b^2}{b^2} \right) - 1.
\]

Our two main theorems generalise this argument to far more involved cases.

**Theorem 1** (Relative functional difference). Let \( f \) be a multilayer perceptron with nonlinearity \( \varphi \) and \( L \) weight matrices \( \{W_l\}_{l=1}^L \). Likewise consider perturbed network \( \tilde{f} \) with weight matrices \( \{\tilde{W}_l\}_{l=1}^L \). For convenience, we define perturbation matrices \( \Delta W_l := \tilde{W}_l - W_l \).

Let the dimension of the \( l \)th hidden layer be \( n_l \), meaning that \( h_l(x) \in \mathbb{R}^{n_l} \). We define the maximum width \( n^* := \max_l n_l \).

Suppose that the following conditions hold:

1. **Fixed point.** The nonlinearity satisfies \( \varphi(0) = 0 \).

2. **Transmission.** There exist \( \alpha, \beta \geq 0 \) such that \( \forall x, y: \alpha \cdot \| x - y \|^2 \leq \| \varphi(x) - \varphi(y) \|^2 \leq \beta \cdot \| x - y \|^2 \).

3. **Conditioning.** Each of the unperturbed weight matrices \( \{W_l\}_{l=1}^N \) has condition number bounded by \( \kappa \).

For all non-zero inputs \( x \in \mathbb{R}^{n_{\text{in}}} \) we have:

\[
\frac{\| \tilde{f}(x) - f(x) \|^2}{\| f(x) \|^2} \leq C_0 \prod_{l=1}^L \left( 1 + \frac{\| \Delta W_l \|_F^2}{\| W_l \|_F^2} \right) - 1,
\]

where we have defined \( C_0 := \left( \frac{2\beta}{\alpha} n^* \kappa^2 \right)^L \).

To aid in the proof of this result, we shall first state and prove two useful lemmas.

**Lemma 1** (Matrix-vector conditioning). Let \( M \) be a matrix in \( \mathbb{R}^{m \times n} \) with \( \pi := \min(m, n) \) singular values \( \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_\pi \). Assume that \( \pi \) has bounded condition number \( \frac{\sigma_\pi}{\sigma_1} \leq \kappa < \infty \). Then for all \( x \in \mathbb{R}^n \),

\[
\frac{1}{\pi x^2} \| M \|_F^2 \| x \|_2^2 \leq \| Mx \|_2^2 \leq \| M \|_F^2 \| x \|_2^2.
\]
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Proof. Observe that

\[ \sigma_n^2 = \min_{y \in \mathbb{R}^n} \frac{\|My\|_2^2}{\|y\|_2^2} \leq \frac{\|Mx\|_2^2}{\|x\|_2^2} \leq \max_{y \in \mathbb{R}^n} \frac{\|My\|_2^2}{\|y\|_2^2} = \sigma_n^2. \]

Since \( \|M\|_F^2 = \sum_{i=1}^n \sigma_i^4 \), we have that \( \sigma_1^2 \leq \|M\|_F^2 \) and \( \|M\|_F^2 \leq \pi \sigma_1^2 \leq \pi \kappa^2 \sigma_2^2 \), from which the result follows.

Lemma 2 (Relative magnitude). Under the same conditions as Theorem 1, we have that for the \( L \)th hidden layer \( h_L(x) \):

\[
\frac{\parallel \tilde{h}_L(x) \parallel_2^2}{\parallel h_L(x) \parallel_2^2} \leq \left( \frac{2\beta}{\alpha n^* \kappa^2} \right)^L \prod_{i=1}^L \left( 1 + \frac{\parallel \Delta W_i \parallel_2^2}{\parallel W_i \parallel_2^2} \right). 
\]

Proof. First observe that a trivial consequence of the first two assumptions is that \( \alpha \cdot \|x\|_2^2 \leq \|\varphi(x)\|_2^2 \leq \beta \cdot \|x\|_2^2 \) for any \( x \).

Now recall that we have defined the maximum width of the network as \( n^* := \max_i n_i \). Then we may relax Lemma 1 to:

\[
\frac{1}{n^* \kappa^2} \| W_L \|_F^2 \parallel h_{L-1}(x) \parallel_2^2 \leq \| W_L h_{L-1}(x) \|_2^2.
\]

This fact will prove its worth in the following argument:

\[
\frac{\| \tilde{h}_L(x) \|_2^2}{\| h_L(x) \|_2^2} = \frac{\| \varphi(\tilde{W}_L \tilde{h}_{L-1}(x)) \|_2^2}{\| \varphi(W_L h_{L-1}(x)) \|_2^2}
\leq \frac{\beta}{\alpha} \frac{\| \tilde{W}_L \|_F^2 \| \tilde{h}_{L-1}(x) \|_2^2}{\| h_{L-1}(x) \|_2^2}\quad \text{(assumption on } \varphi)\]

\[
\leq \frac{\beta}{\alpha} n^* \kappa^2 \frac{\| \tilde{W}_L \|_F^2 \| \tilde{h}_{L-1}(x) \|_2^2}{\| W_L \|_F^2 \| h_{L-1}(x) \|_2^2}\quad \text{(Lemma 1)}
\leq \frac{2\beta}{\alpha} n^* \kappa^2 \frac{\| W_L \|_F^2 + \| \Delta W_L \|_F^2}{\| W_L \|_F^2} \frac{\| h_{L-1}(x) \|_2^2}{\| h_{L-1}(x) \|_2^2}
\leq \frac{2\beta}{\alpha} n^* \kappa^2 \left( 1 + \frac{\| \Delta W_L \|_F^2}{\| W_L \|_F^2} \right) \| h_{L-1}(x) \|_2^2.
\]

The lemma follows from an obvious induction on depth.

With these tools in hand, let us proceed to Theorem 1.

Proof of Theorem 1. Again observe that a trivial consequence of the first two assumptions is that \( \alpha \cdot \|x\|_2^2 \leq \|\varphi(x)\|_2^2 \leq \beta \cdot \|x\|_2^2 \) for any \( x \).

To make an inductive argument, we shall assume that the result holds for a network with \( L - 1 \) layers. Extending to depth \( L \), we have:

\[
\frac{\| \tilde{f}(x) - f(x) \|_2^2}{\| f(x) \|_2^2} = \frac{\| (\varphi \circ \tilde{W}_L) \circ \tilde{h}_{L-1}(x) - (\varphi \circ W_L) \circ h_{L-1}(x) \|_2^2}{\| (\varphi \circ W_L) \circ h_{L-1}(x) \|_2^2}
\leq \frac{\beta}{\alpha} \frac{\| \tilde{W}_L \tilde{h}_{L-1}(x) - W_L h_{L-1}(x) \|_2^2}{\| W_L h_{L-1}(x) \|_2^2}
\leq \frac{\beta}{\alpha} \frac{\| \Delta W_L \tilde{h}_{L-1}(x) + W_L \tilde{h}_{L-1}(x) - h_{L-1}(x) \|_2^2}{\| W_L h_{L-1}(x) \|_2^2}
\leq \frac{2\beta}{\alpha} n^* \kappa^2 \frac{\| \Delta W_L \tilde{h}_{L-1}(x) \|_2^2 + \| W_L \tilde{h}_{L-1}(x) - h_{L-1}(x) \|_2^2}{\| W_L \tilde{h}_{L-1}(x) \|_2^2} 
\leq \frac{2\beta}{\alpha} n^* \kappa^2 \left[ \frac{\| \Delta W_L \parallel_2^2}{\| W_L \parallel_2^2} \frac{\| h_{L-1}(x) \|_2^2}{\| h_{L-1}(x) \|_2^2} + \frac{\| h_{L-1}(x) - h_{L-1}(x) \|_2^2}{\| h_{L-1}(x) \|_2^2} \right].
\]
We may bound the first term by Lemma 2, and the second term by the inductive hypothesis. Then we obtain:

\[
\|\tilde{f}(x) - f(x)\|_F^2 \\
\leq \left(2\frac{\beta}{\alpha}n^\ast\kappa^2\right)^L \left[\|\Delta W_{L-1}F\|_F^2 \prod_{i=1}^{L-1} \left(1 + \|\Delta W_i\|_F^2 \right) \right] + \left(2\frac{\beta}{\alpha}n^\ast\kappa^2\right)^L \left[\prod_{i=1}^{L-1} \left(1 + \|\Delta W_i\|_F^2 \right) - 1 \right]
\]

Let us proceed to our second main result.

**Theorem 2 (Relative Jacobian difference).** Let \( f \) be a multilayer perceptron with nonlinearity \( \varphi \) and \( L \) weight matrices \( \{W_l\}_{l=1}^L \). Likewise consider perturbed network \( \tilde{f} \) with weight matrices \( \{\tilde{W}_l\}_{l=1}^L \). For convenience, we define perturbation matrices \( \Delta W_l := \tilde{W}_l - W_l \).

Let the dimension of the \( l \)th hidden layer be \( n_l \), so that \( h_l(x) \in \mathbb{R}^{n_l} \). We define the maximum width \( n^\ast := \max_l n_l \).

Suppose that the following conditions hold:

1. Transmission. There exist \( \alpha, \beta \geq 0 \) such that \( \forall x, y:\)

\[
\alpha \cdot \|x - y\|_2^2 \leq \|\varphi(x) - \varphi(y)\|_2^2 \leq \beta \cdot \|x - y\|_2^2.
\]

2. Conditioning. Each of the unperturbed weight matrices \( \{W_l\}_{l=1}^L \) has condition number bounded by \( \kappa \).

Then we have that:

\[
\left\| \frac{\partial \tilde{f}}{\partial \tilde{w}_l} - \frac{\partial f}{\partial w_l} \right\|_F^2 \leq C_1 \left[ \prod_{j=k+1}^L C_2 \left(1 + \|\Delta W_j\|_F^2 \right) \right],
\]

where we have defined constants:

\[
C_1 := \left(2n^\ast \sqrt{\frac{\beta}{\alpha}}\right)^{2(L-k)}; \quad C_2 := 1 + \left(\sqrt{\frac{\beta}{\alpha}} - 1\right)^2.
\]

Before we present a proof, we shall state and prove three more useful lemmas.

**Lemma 3 (Freshman’s dream).** For a collection of \( k \) matrices \( \{A_l\}_{l=1}^k \in \mathbb{R}^{m \times n} \), we have

\[
\left\| \sum_{l=1}^k A_l \right\|_F^2 \leq k \sum_{l=1}^k \|A_l\|_F^2.
\]

**Proof.** Consider \( \frac{1}{k} \sum_{l=1}^k A_l \) to be the expectation of a matrix picked uniformly at random from the collection. Apply Jensen’s inequality with convex function \( \|\cdot\|_F^2 \). □

**Lemma 4 (Matrix-matrix conditioning).** Consider matrices \( M \in \mathbb{R}^{m,n} \) and \( X \in \mathbb{R}^{n,p} \) where \( M \) has condition number \( \kappa \). Let \( \pi := \min(n, m) \). Then we have that:

\[
\frac{1}{\pi\kappa^2} \|M\|_F^2 \|X\|_F^2 \leq \|MX\|_F^2 \leq \|M\|_F^2 \|X\|_F^2.
\]

**Proof.** \( \|MX\|_F^2 = \sum_{j=1}^p \|Mx_j\|_2^2 \) where \( x_j \) are the columns of \( X \). Sum Lemma 1 over \( x_j \) to get the result. □
Lemma 5 (Relative difference of a matrix product). Consider $L$ real matrices $W_1, W_2, ..., W_L$ and perturbations $\Delta W_1, \Delta W_2, ..., \Delta W_L$. Assume that for each $l = 1, 2, ..., L$, $W_l$ has condition number bounded by $\kappa_l$.

Let $W_l, \Delta W_l \in \mathbb{R}^{n_l \times n_{l-1}}$. Then we define $n_l := \min(n_l, n_{l-1})$ to be the minimum dimension of each matrix. Then we have

$$
\frac{\left\| \prod_{l=1}^{L} (W_l + \Delta W_l) - \prod_{l=1}^{L} W_l \right\|_F^2}{\left\| \prod_{l=1}^{L} W_l \right\|_F^2} \leq 2^L \prod_{l=1}^{L} n_l \kappa_l^2 \left( \prod_{l=1}^{L} \left( 1 + \frac{\|\Delta W_l\|_F^2}{\|W_l\|_F^2} \right) - 1 \right).
$$

Proof. First consider the numerator. We may multiply out the brackets to yield the sum of $2^L - 1$ terms. The observation is that after applying Lemma 3 and recursively applying the upper bound of Lemma 4, the product has the same algebraic structure as the original numerator except with the norms pulled inside. In symbols, we have that

$$
\left\| \prod_{l=1}^{L} (W_l + \Delta W_l) - \prod_{l=1}^{L} W_l \right\|_F^2 \leq (2^L - 1) \left[ \prod_{l=1}^{L} \left( \|W_l\|_F^2 + \|\Delta W_l\|_F^2 \right) - \prod_{l=1}^{L} \|W_l\|_F^2 \right].
$$

To handle the denominator, we may apply the lower bound of Lemma 4 recursively to yield

$$
\left\| \prod_{l=1}^{L} W_l \right\|_F^2 \geq \prod_{l=1}^{L} \frac{\|W_l\|_F^2}{n_l \kappa_l^2}.
$$

Combining the bounds on the numerator and denominator and using that $2^L - 1 < 2^L$ yields the result.

Proof of Theorem 2. By Proposition 1, the layer-$l$-to-output Jacobian satisfies:

$$
\frac{\partial f(x)}{\partial h_k} = \Phi'_L W_L \cdot \Phi'_{L-1} W_{L-1} \cdot ... \cdot \Phi'_{k+1} W_{k+1}
$$

where $\Phi'_l := \text{diag}\left[ \varphi'(W_l h_{l-1}(x)) \right]$.

Denote the perturbed version by the product:

$$
\frac{\partial \tilde{f}(x)}{\partial h_k} = (\Phi'_L + \Delta \Phi'_L)(W_L + \Delta W_L) \cdot ... \cdot (\Phi'_{k+1} + \Delta \Phi'_{k+1})(W_{k+1} + \Delta W_{k+1}).
$$

Taking limits of the assumption on the nonlinearity, we realise that $\sqrt{\alpha} \leq \varphi'(x) \leq \sqrt{\beta}$. Therefore the condition number of $\Phi'_l$ is bounded by $\sqrt{\beta/\alpha}$. By assumption on the weight matrices, the condition number of $W_l$ is bounded by $\kappa$. This allows us to apply Lemma 5, which yields:

$$
\frac{\left\| \frac{\partial \tilde{f}}{\partial h_k} - \frac{\partial f}{\partial h_k} \right\|_F^2}{\left\| \frac{\partial f}{\partial h_k} \right\|_F^2} \leq \left( 2n^* \sqrt{\frac{\beta}{\alpha}} \right)^{2^L} \prod_{l=k+1}^{L} \left( 1 + \frac{\|\Delta \Phi'_l\|_F^2}{\|\Phi'_l\|_F^2} \right) \left( 1 + \frac{\|\Delta W_l\|_F^2}{\|W_l\|_F^2} \right) - 1.
$$

By assumption on the nonlinearity, we have that $\frac{\|\Delta \Phi'_l\|_F^2}{\|\Phi'_l\|_F^2} \leq \frac{(\sqrt{\beta} - \sqrt{\alpha})^2}{\sqrt{\alpha}} = \left( \sqrt{\frac{\beta}{\alpha}} - 1 \right)^2$ and substituting this in yields the result. 

\qed