S-duality in Hyperkähler Hodge Theory

Tamás Hausel

To Nigel Hitchin for his 60th birthday.

Abstract

Here we survey questions and results on the Hodge theory of hyperkähler quotients, motivated by certain S-duality considerations in string theory. The problems include $L^2$ harmonic forms, Betti numbers and mixed Hodge structures on the moduli spaces of Yang-Mills instantons on ALE gravitational instantons, magnetic monopoles on $\mathbb{R}^3$ and Higgs bundles on a Riemann surface. Several of these spaces and their hyperkähler metrics were constructed by Nigel Hitchin and his collaborators.

1 Introduction

In this paper we survey the motivations, related results and progress made towards the following problem, raised by Hitchin in 1995:

Problem 1.1. What is the space of $L^2$ harmonic forms on the moduli space of Higgs bundles on a Riemann surface?

The moduli space $\mathcal{M}_{\text{Dol}}^d(\text{SL}_n)$ of stable rank $n$ Higgs bundles with fixed determinant of degree $d$ on a Riemann surface was introduced and studied in [32], [57] and [49]. The Betti numbers of this space for $n = 2$ were determined in [32] while for $n = 3$ in [17]. The above problem raised two new directions to study. First is the Riemannian geometry of $\mathcal{M}_{\text{Dol}}^d(\text{SL}_n)$, or more precisely the asymptotics of the natural hyperkähler metric, and its connection with Hodge theory. The second one, which can be considered the topological side of Problem 1.1, is to determine the intersection form on the middle dimensional compactly supported cohomology of $\mathcal{M}_{\text{Dol}}^d(\text{SL}_n)$. While the first question seems still out of reach, although we will report on some modest progress below, the second is more approachable and we offer a conjecture at the end of this survey.

Problem 1.1 was motivated by S-duality conjectures emerging from the string theory literature about Hodge theory on certain hyperkähler moduli spaces, which are close relatives of $\mathcal{M}_{\text{Dol}}^d(\text{SL}_n)$.

In the physics literature S-duality stands for a strong-weak duality between two quantum field theories. The interest from the physics point of view is that it gives a tool to study physical theories with a large coupling constant via a conjectured equivalence with a theory with a small coupling constant where perturbative methods give a good understanding. The $S$-duality conjecture relevant for us is based on the Montonen-Olive electro-magnetic duality proposal from 1977 in four dimensional Yang-Mills theory [46]. It was noted in [60] that this duality proposal is more likely to hold in a supersymmetric version of the theory, and in [50] it was argued that $N = 4$ supersymmetry is a good candidate. Hyperkähler Hodge theory is relevant in $N = 4$ supersymmetry as the space of differential forms on a hyperkähler manifold.
possesses an action of the $N = 4$ supersymmetry algebra via the various operators in hyperkähler Hodge theory.

In this paper our interest lies in the mathematical predictions of such $S$-duality conjectures in physics. In 1994 Sen [54], using $S$-duality arguments in $N = 4$ supersymmetric Yang-Mills theory, predicted the dimension of the spaces $H^d(\tilde{M}_k^0)$ of $L^2$ harmonic $d$-forms on the universal cover $\tilde{M}_k^0$ of the hyperkähler moduli space $M_k^0$ of certain SU(2) magnetic monopoles on $\mathbb{R}^3$. In the interpretation of [54] the $L^2$ harmonic forms on $\tilde{M}_k^0$ can be thought of as bound states of the theory, and the conjectured $S$-duality implies an action of $\text{SL}(2, \mathbb{Z})$ on $\bigoplus_k H^*(\tilde{M}_k^0)$. By further physical arguments Sen managed to predict this representation of $\text{SL}(2, \mathbb{Z})$ completely, implying the following

Conjecture 1.2. The dimension of the space of $L^2$ harmonic forms on $\tilde{M}_k^0$ is

$$\dim \left(H^d(\tilde{M}_k^0)\right) = \begin{cases} 0 & d \neq \text{mid} \\ \phi(k) & d = \text{mid} \end{cases},$$

where $\phi(k) = \sum_{i=1}^k \delta_{1,(i,k)}$ is the Euler $\phi$ function, and $\text{mid} = 2k - 2$ is half of the dimension of $\tilde{M}_k^0$.

Similar $S$-duality arguments led Vafa and Witten [59] to get a conjecture on the space of $L^2$ harmonic forms on a certain smooth completion $M_{\phi}^{k,c_1}$, constructed in [45, 48], of the moduli space of U(n) Yang-Mills instantons of first Chern class $c_1$, energy $k$ and framing $\phi$ on one of Kronheimer’s ALE spaces, which are 4-dimensional complete hyperkähler manifolds, with an asymptotically locally Euclidean metric.

Conjecture 1.3. The dimension of the space of $L^2$ harmonic forms on $M_{\phi}^{k,c_1}$ is

$$\dim \left(H^d(M_{\phi}^{k,c_1})\right) = \begin{cases} 0 & d \neq \text{mid} \\ \dim \left(\text{im}(H^{\text{mid}}(M_{\phi}^{k,c_1}) \rightarrow H^{\text{mid}}(M_{\phi}^{k,c_1}))\right) & d = \text{mid} \end{cases},$$

where $\text{mid}$ now denotes half of the dimension of $M_{\phi}^{k,c_1}$.

The paper [59] further argues that Conjecture 1.3 implies, via the work of Nakajima [48] and Kac [41], that

$$Z_{\phi}(q) = \sum_{c_1,k} q^{k-c_1/24} \dim \left(H^{\text{mid}}(M_{\phi}^{k,c_1})\right)$$

is a modular form, which, as was speculated in [59], might be a consequence of $S$-duality.

This paper will introduce the reader to various mathematical aspects of these three problems and offer mathematical techniques and results relating to them.

Acknowledgment. This paper is a write-up of the author’s talk at the Geometry Conference in Honour of Nigel Hitchin in Madrid in September 2006. Problem 1.1 was raised by Nigel Hitchin in 1995, then the author’s PhD supervisor, as a project for the author’s PhD thesis. This survey paper would like to show the impact of this modestly looking question on the author’s subsequent research. The author’s research has been supported by a Royal Society University Research Fellowship, NSF grant DMS-0604775 and an Alfred Sloan Fellowship. The visit to Madrid was supported by a Royal Society International Joint Project between the UK and Spain.
2 Hyperkähler quotients

A Riemannian manifold \((M,g)\) is hyperkähler if it is Kähler with respect to three integrable complex structures \(I,J,K \in \Gamma(\text{End}(TM))\), which satisfy \(I^2 = J^2 = K^2 = IJK = -1\), with Kähler forms \(\omega_I, \omega_J\) and \(\omega_K\). Known compact examples are scarce, see e.g. [39, §7]. Non-compact complete examples however are much more abundant. This is mostly because there is a widely applicable\(^1\) hyperkähler quotient construction, due to Hitchin-Karlhede-Lindström-Roček [35]. The construction itself is an elegant quaternionization of the Marsden-Weinstein symplectic (or more precisely Kähler) quotient construction (see [37, Chapter 8] for an introduction for the latter).

Let \(\mathbb{M}\) be a hyperkähler manifold, \(\mathcal{G}\) a Lie group, with Lie algebra \(\mathfrak{g}\), and assume \(\mathcal{G}\) acts on \(\mathbb{M}\) preserving the hyperkähler structure (i.e. it acts by triholomorphic isometries). Let us further assume that we have moment maps \(\mu_I : \mathbb{M} \to \mathfrak{g}^*\), \(\mu_J : \mathbb{M} \to \mathfrak{g}^*\) and \(\mu_K : \mathbb{M} \to \mathfrak{g}^*\) with respect to the symplectic forms \(\omega_I, \omega_J\) and \(\omega_K\) respectively. We combine them into a single hyperkähler moment map:

\[
\mu_\Xi = (\mu_I, \mu_J, \mu_K) : \mathbb{M} \to \mathbb{R}^3 \otimes \mathfrak{g}^*.
\]

One takes \(\xi \in \mathbb{R}^3 \otimes (\mathfrak{g}^*)^\mathcal{G}\) and constructs the hyperkähler quotient at level \(\xi\) by:

\[
\mathbb{M} \sslash \sslash_\xi \mathcal{G} := \mu_\Xi^{-1}(\xi)/\mathcal{G}.
\]

The main result of [35] is that the natural Riemannian metric on the smooth points of this quotient is hyperkähler.

Now we list three important examples of this construction, where the original hyperkähler manifold \(\mathbb{M}\) and Lie group \(\mathcal{G}\) are both infinite dimensional.

2.1 Moduli of Yang-Mills instantons on \(\mathbb{R}^4\)

Here we follow [31] I Example 3.6, compare also with [1].

Let \(G\) be a compact connected Lie group, which will be \(U(n)\) or \(SU(n)\) in this paper. Let \(P \to \mathbb{R}^4\) be a \(G\)-principal bundle over \(\mathbb{R}^4\). Let \(\mathbb{M}\) be the space of \(G\)-connections \(A\) on \(P\) of class \(C^\infty\), such that the energy

\[
\left| \int_{\mathbb{R}^4} \text{tr}(F_A \wedge *F_A) \right| < \infty
\]

is finite. Write

\[
A = A_1 dx_1 + A_2 dx_2 + A_3 dx_3 + A_4 dx_4
\]

in a fixed gauge, where \(A_i \in \Omega^0(\mathbb{R}^4, \text{ad}(P))\). Let \(\mathcal{G} = \Omega(\mathbb{R}^4, \text{Ad}(P))\) be the gauge group of \(P\). An element \(g \in \mathcal{G}\) acts on \(A \in \mathbb{M}\) by the formula \(g(A) = g^{-1}Ag + g^{-1}dg\), preserving the hyperkähler structure. One finds that the hyperkähler moment map equation

\[
\mu_\Xi(A) = 0 \iff F_A = *F_A
\]

is just the self-dual Yang-Mills equation. Define the hyperkähler quotient \(M(\mathbb{R}^4, P) = \mu_\Xi^{-1}(0)/\mathcal{G}\), the moduli space of finite energy self-dual Yang-Mills instantons on \(P\). By its construction it has a natural hyperkähler metric.

Similar construction [45] for \(G = U(n)\) yields a hyperkähler metric on moduli spaces of \(U(n)\) Yang-Mills instantons on certain four dimensional complete hyperkähler manifolds, the ALE spaces of Kronheimer [44]. These moduli spaces will have natural completions and various components of them will be the spaces \(M^k_{\mathfrak{g}_1}\) which were mentioned in the introduction. They will resurface later as examples for Nakajima quiver varieties.

\(^1\)Some colleagues even suggest, due to the success of this construction, that HyperKähLeR is in fact just a pronouncable version of the acronym HKLR.
2.2 Moduli space of magnetic monopoles on $\mathbb{R}^3$

The following construction can be considered as a dimensional reduction of the previous example. Here we follow [31, I Example 3.5] and [3].

Assume that $G = SU(2)$ and the matrices $A_i$ are independent of $x_4$. Then we have

$$A = A_1 dx_1 + A_2 dx_2 + A_3 dx_3$$

a connection on $\mathbb{R}^3$ and $A_4 = \phi \in \Omega^0(\mathbb{R}^3, \text{ad} P)$ becomes the Higgs field. The gauge group now will be $\mathcal{G} = \Omega(\mathbb{R}^3, \text{Ad} P)$ and $\mathcal{M} = \{(A, \phi) + \text{certain boundary condition}\}$. (The boundary condition is chosen to ensure finite energy.) The gauge group $\mathcal{G}$ acts on $\mathcal{M}$ by gauge transformations, preserving the natural hyperkähler metric on $\mathcal{M}$. The corresponding hyperkähler moment map equation

$$\mu_{\mathbb{H}}(A, \phi) = 0 \iff F_A = *d_A \phi$$

is equivalent with the Bogomolny equation.

Now by construction $M = \mu_{\mathbb{H}}^{-1}(0)/\mathcal{G}$, the moduli space of magnetic monopoles on $\mathbb{R}^3$, has a natural hyperkähler metric. It has infinitely many components $M = \bigcup_{k=1}^{\infty} M_k$ labeled by the magnetic charge $k$ of the monopole.

$M_k$ is acted upon by $\mathbb{R}^3$ by translations and by $U(1)$ by rotating the phase of the monopole. The quotient $M_k^0$ is still a smooth complete hyperkähler manifold of dimension $4k - 4$, with fundamental group $\mathbb{Z}_k$. We will denote by $\widetilde{M}_k^0$ its universal cover. In [2] Atiyah and Hitchin find the hyperkähler metric explicitly on the 4-manifold $M_2^0$ and subsequently describe the scattering of two monopoles.

2.3 Hitchin moduli space

This example can be considered as a two-dimensional reduction of §2.1. We follow [32, Section 1] and [31, I Example 3.3].

Now we assume that $G = U(n)$ and the matrices $A_i$ in §2.1 are independent of $x_3, x_4$. We have now the connection $A = A_1 dx_1 + A_2 dx_2$ on the $U(n)$ principal bundle $P$ on $\mathbb{R}^2$. We introduce $\Phi = (A_3 - A_4) dz \in \Omega^{1,0} (\mathbb{R}^2, \text{ad} P \otimes \mathbb{C})$ the complex Higgs field. The gauge group now is $\mathcal{G} = \Omega(\mathbb{R}^2, \text{Ad} P)$, which acts by gauge transformations on the space $\mathcal{M} = \{(A, \Phi)\}$ preserving the natural hyperkähler metric on $\mathcal{M}$. The moment map equations

$$\mu_{\mathbb{H}}(A, \Phi) = 0 \iff F(A) = -[\Phi, \Phi^*], \quad d_A^* \Phi = 0,$$

are then equivalent with Hitchin’s self-duality equations. There are no solutions of finite energy on $\mathbb{R}^2$, but as the equations are conformally invariant, we can replace $\mathbb{R}^2$ with a genus $g$ compact Riemann surface $C$ in the above definitions, and define $\mathcal{M}(C, \mathcal{P}) = \mu_{\mathbb{H}}^{-1}(0)/\mathcal{G}$, the Hitchin moduli space, which has a natural hyperkähler metric by construction. There are different ways to think about this space with the different complex structures, which will be explained in §5.2.

3 Hodge theory

3.1 $L^2$ harmonic forms on complete manifolds

Let $M$ be a complete Riemannian manifold of dimension $n$. We say that a smooth differential $k$-form $\alpha \in \Omega^k(M)$ is harmonic if and only if $d\alpha = d^* \alpha = 0$, where $*: \Omega^k(M) \to \Omega^{n-k}(M)$ is the Hodge star...
operator. It is $L^2$ if and only if
\[
\int_M \alpha \wedge *\alpha < \infty.
\]
We denote by $\mathcal{H}^*(M)$ the space of $L^2$ harmonic forms.

A fundamental theorem of Hodge theory is the Hodge (orthogonal) decomposition theorem [12, §32 Theorem 24, §35 Theorem 26]:
\[
\Omega^*_L = d(\Omega^*_c) \oplus \mathcal{H}^* \oplus \overline{d(\Omega^*_c)},
\]
where $\delta$ is the adjoint of $d$. When $M$ is compact this implies the celebrated Hodge theorem, which says that $\mathcal{H}^*(M) \cong H^*(M)$ i.e. that there is a unique harmonic representative in every de Rham cohomology class. When $M$ is non-compact we only have a topological lower bound. Namely, the Hodge decomposition theorem implies that the composite map
\[
\mathcal{H}^*_{cpt}(M) \to \mathcal{H}^*(M) \to H^*(M)
\]
is just the forgetful map. (In the compact case these maps are isomorphisms, which gives the Hodge theorem mentioned above.) Thus
\[
\text{im}(\mathcal{H}^*_{cpt}(M) \to H^*(M))
\]
is a ”topological lower bound” for $\mathcal{H}^*(M)$. By Poincaré duality the map $\mathcal{H}^*_{cpt}(M) \to H^*(M)$ is equivalent with the intersection pairing on $\mathcal{H}^*_{cpt}(M)$.

In the cases most relevant for us $M$ will be a hyperkähler manifold (sometimes orbifold) so $\dim(M) = 4k$ and we will additionally have $\dim(H^i(M) = 0$ for $i > 2k$. Therefore the possible non-trivial image in $\text{im}(\mathcal{H}^*_{cpt}(M) \to H^*(M))$ will be concentrated in the middle $2k$ dimension. (We will use the notation $\text{mid} = \dim(M)/2$ for the middle dimension of a manifold.) For such a hyperkähler manifold we denote
\[
\chi_{L^2}(M) = \dim(\text{im}(\mathcal{H}^*_{cpt}(M) \to H^\text{mid}(M))) = \dim(\text{im}(\mathcal{H}^*_{cpt}(M) \to H^*(M)))
\]
the dimension of this image. $\chi_{L^2}(M)$ can be thought of either as a ”topological lower bound” for $\dim(\mathcal{H}^*(M))$ or the Euler characteristic of topological $L^2$ cohomology.

### 3.2 Results on $L^2$ harmonic forms

There were few general theorems on describing $\mathcal{H}^*(M)$ for a non-compact complete manifold $M$, see however [25, Introduction] for an overview. It was thus a surprising development when Sen [54], using arguments from $S$-duality, managed to predict the dimension of $L^2$ harmonic forms on $\tilde{M}_0^k$ as was explained in Conjecture 1.2 in the Introduction. In particular, according to Sen’s Conjecture 1.2 the space $\mathcal{H}^2(\tilde{M}_0^k)$ should be one dimensional. Using the explicit description [2] of the metric on $M_0^k$ Sen in [54] was able to find an explicit $L^2$ harmonic 2-form, called the Sen 2-form, on $\tilde{M}_0^k$. This was perhaps the strongest mathematical support exhibited for Conjecture 1.2 in [54].

More general mathematical support for Conjecture 1.2 came in 1996. Segal and Selby in [53] showed that the intersection form on $\mathcal{H}^\text{mid}_{cpt}(\tilde{M}_0^k)$ is definite. Moreover they obtained for the topological lower bound (3.2) for $\mathcal{H}^\text{mid}_{cpt}(\tilde{M}_0^k)$
\[
\chi_{L^2}(\tilde{M}_0^k) = \dim(\mathcal{H}^\text{mid}_{cpt}(\tilde{M}_0^k)) = \phi(k).
\]
This agrees with the predicted dimension of $\mathcal{H}^\text{mid}_{cpt}(\tilde{M}_0^k)$ in Sen’s Conjecture 1.2.
Motivated by Problem 1.1 and Segal-Selby’s topological lower bound for Conjecture 1.2 calculated in 1998 that the intersection pairing on the g dimensional space $H^m_{cpt}(\mathcal{M}_0^{1}\text{Dol}(SL_2))$ is trivial, in other words

$$\chi_L: \left(\mathcal{M}_0^{1}\text{Dol}(SL_2)\right) = 0$$

for $g > 1$. This thus gave the surprising result that there are no $L^2$ harmonic forms on $\mathcal{M}_0^{1}\text{Dol}(SL_2)$ plainly by topological reasons. The technique used in the proof of (3.4) was imitating Kirwan’s proof of Mumford’s conjecture on the cohomology ring of the moduli space of stable rank 2 bundles of degree 1 on the Riemann surface $C$. Therefore the extension of (3.4) to higher rank Higgs bundle moduli spaces $\mathcal{M}_0^{k}\text{Dol}(SL_n)$ was not straightforward.

Next advance towards Sen’s Conjecture 1.2 came in 2000. Hitchin in [34] showed that $\mathcal{H}^d(M) = 0$ unless $d = \dim(M)/2$ for a complete hyperkähler manifold $M$ of linear growth. Examples include all our hyperkähler quotients discussed in this paper. The proofs in [34] use techniques inspired by Jost an Zuo’s extension [38] of ideas of Gromov [19]. It is interesting to note that some of the proofs in [34] also exploit the operators in hyperkähler Hodge theory, which are relevant in $N = 4$ supersymmetry. Using the symmetries of the Atiyah-Hitchin metric [34] proves Sen’s conjecture for $k = 2$, i.e. that up to a scalar the only $L^2$ harmonic form on $\bar{M}_0^0$ is Sen’s 2-form.

A more topological approach was introduced in [25] in 2004. [25] proves for fibered boundary manifolds $M$

$$\mathcal{H}^{mid}(M) \cong \text{im}(\mathcal{H}^{mid}_{\bar{M}^{\bar{m}}}(\bar{M}) \rightarrow \mathcal{H}^{mid}_{\bar{M}^{\bar{m}}}(\bar{M})),$$

where $\bar{M}$ is a certain compactification of $M$, dictated by the asymptotics of the fibered boundary metric on $M$. Moreover $\mathcal{H}^{mid}_{\bar{M}^{\bar{m}}}(\bar{M}))$ denotes the intersection cohomology in dimension $\text{mid} = \dim(M)/2$ with lower middle perversity $\bar{m}$ and $\mathcal{H}^{mid}_{\bar{M}^{\bar{m}}}(\bar{M}))$ denotes the intersection cohomology in the middle dimension with upper middle perversity $\bar{m}$ of the possibly badly singular (i.e. not necessarily a Witt space) compactification $\bar{M}$. To illustrate (3.5) we take the compactification of $\bar{M}_0$, which happens to be the smooth space $\mathbb{C}P^2$ (with the non-standard orientation), where the above cohomologies in (3.5) all coincide, giving $\mathcal{H}^2(\bar{M}_0^0) \cong H^2(\mathbb{C}P^2)$. This provides a topological explanation for the existence and uniqueness of the Sen 2-form.

The assumption that the metric is fibered boundary in [25] is fairly restrictive. Among hyperkähler quotients only a few examples satisfy this property (see the discussion in [25] §7)). Examples include all ALE gravitational instantons of [44] and all known ALF (see [10]) and some ALG gravitational instantons (see [11]). In general our hyperkähler quotients have some kind of stratified asymptotic behaviour at infinity. For example the metric on $\bar{M}_k^0$ is fibered boundary only when $k = 2$, for higher $k$ it is known to behave differently at different regions of infinity. The first result, which could handle Hodge theory on Riemannian manifolds with such a stratified behaviour at infinity appeared recently in a work [9] by Carron. It proves for a QALE space $M$ that:

$$\mathcal{H}^{mid}(M) \cong \text{im}(\mathcal{H}^{mid}_{\bar{M}^{\bar{m}}}(\bar{M}) \rightarrow \mathcal{H}^{mid}(M)).$$

A QALE space [39] §9) by definition is a certain Calabi-Yau metric on a crepant resolution of $\mathbb{C}^k/\Gamma$, where $\Gamma \subset \text{SU}(k)$ is a finite subgroup. The asymptotics of the metric on such a QALE space is reminiscent to the asymptotics of the natural hyperkähler metric on $M_\phi^{k,\epsilon}$ appearing in the Vafa-Witten Conjecture [1.3]. It is thus reasonable to hope that the Vafa-Witten Conjecture [1.3] will be decided soon.

As there have been extensive studies starting with [16] and more recently [6] on the asymptotics of the Riemannian metric on $M_\phi^0$, it is conceivable that we will have a precise understanding of the asymptotic
behaviour of this metric, and in turn the Hodge theory of $L^2$ harmonic forms on $\overline{M}_k^0$, perhaps extending techniques from [9]. Thus one may be optimistic that Sen’s Conjecture 1.2 will be decided in the foreseeable future.

Finally, one must admit that the description of the asymptotics of the metric at infinity on $M_{Dol}^d(SL_n)$ is still lacking, thus calculation of $\mathcal{H}^*(M_{Dol}^d(SL_n))$ is presently hopeless. The topological side of Problem [11] that is to determine $\chi_{L^2}(M_{Dol}^d(SL_n))$, when $(d, n) = 1$, is more reasonable. After introducing a new arithmetic technique to study Hodge structures on the cohomology of our hyperkähler manifolds, we will be able to offer a general conjecture on the intersection form on Higgs moduli spaces, in particular that (3.4) holds for any $n$.

4 Mixed Hodge theory

As explained above there have been some limited successes of calculating $\mathcal{H}^*(M)$ for a hyperkähler quotient and understanding its relation to the cohomology $H^*(M)$ or more generally the cohomology of an appropriate compactification $H^*(\overline{M})$. Another extension of Hodge theory yields some different and in some ways more detailed insight into the cohomology of our hyperkähler quotients. This technique is Deligne’s mixed Hodge structure on the cohomology of any complex algebraic variety. Instead of the global analysis on the Riemannian geometry of the complex algebraic variety it will relate to the arithmetic of the variety over finite fields.

4.1 Mixed Hodge structure of Deligne

Motivated by the (then still unproven) Weil Conjectures and Grothendieck’s ”yoga of weights”, which drew cohomological conclusions about complex varieties from the truth of those conjectures, Deligne in [13, 14] proved the existence of mixed Hodge structures on the cohomology $H^*(M, \mathbb{Q})$ of a complex algebraic variety $M$. Here we give a quick introduction, for more details see [27, §2.2] and the references therein. Deligne’s mixed Hodge structure entails two filtrations on the rational cohomology of $M$.

The increasing weight filtration

$$0 = W_{-1} \subseteq W_0 \subseteq \cdots \subseteq W_j = H^j(X, \mathbb{Q})$$

and a decreasing Hodge filtration

$$H^j(X, \mathbb{C}) = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^m \supseteq F^{m+1} = 0.$$ 

We can define mixed Hodge numbers obtained from this two filtrations by the following formula:

$$h^{p,q,j}(X) := \dim_{\mathbb{C}} \left( Gr^{F}_{p} Gr^{W}_{p+q} H^j(X)_{\mathbb{C}} \right). \quad (4.1)$$

From these numbers we form

$$H(M; x, y, t) = \sum_{p,q,k} h^{p,q,j}(M) x^p y^q t^k,$$

the mixed Hodge polynomial. By virtue of its definition it has the property that the specialization

$$P(M; t) = H(M; 1, 1, t)$$

gives the Poincaré polynomial of $M$. When $M$ is smooth of dimension $n$ we take another specialization

$$E(M; x, y) := x^p y^q H(1/x, 1/y, -1), \quad (4.2)$$
the so-called E-polynomial of a smooth variety $M$. Deligne’s construction of mixed Hodge structure is complex geometrical: for a smooth variety $M$ it is defined by the log geometry of a compactification $\bar{M}$ with normal crossing divisors. In particular a global analytical description, like the Hodge theory of harmonic forms on a smooth complex projective manifold, of the mixed Hodge structure on a smooth variety is missing, which causes some difficulty to find the meaning of mixed Hodge numbers in physical contexts (see the remark after Conjecture 5.8).

4.2 Arithmetic and topological content of the E-polynomial

The connection of the E-polynomial to the arithmetic of the variety is provided by the following theorem of Katz [27, Appendix]. Here we give an informal version of Katz’s result for precise formulation see [27, Theorem 6.1.2.3, Theorem 2.1.8]:

**Theorem 4.1.** Let $M$ be a smooth quasi-projective variety defined over $\mathbb{Z}$ (i.e. given by equations with integer coefficients). Assume that the number of points of $M$ over a finite field $\mathbb{F}_q$, i.e.

$$E(q) := \#(M(\mathbb{F}_q))$$

is a polynomial in $q$. Then the E-polynomial can be obtained from the count polynomial as follows:

$$E(M; x, y) = E(xy).$$

This theorem is especially useful when we further have $h^{p,q,k}(M) = 0$ unless $p + q = k$. In this case we say that the mixed Hodge structure on $H^*(M)$ is pure. In this case

$$H(M; x, y, t) = (xyt^2)^n E\left(\frac{-1}{xt}, \frac{-1}{yt}\right)$$

and so the Poincaré polynomial can be recovered from the E-polynomial as follows

$$P(M; t) = H(M; 1, 1, t) = t^{2n} E\left(\frac{-1}{t}, \frac{-1}{t}\right).$$

Examples of varieties with pure MHS on their cohomology include smooth projective varieties (in this case we get the traditional Hodge structure, which is by definition pure), the moduli space of Higgs bundles $M_{\text{Dol}}$, the moduli space of flat connections $M_{\text{DR}}$ on a Riemann surface and Nakajima’s quiver varieties.

In general we can define the pure part of $H(M; x, y, t)$ as

$$PH(M; x, y) = \text{Coeff}_{T^0} \left( H(M; xT, yT, iT^{-1}) \right).$$

More generally we can define the pure part of the cohomology of $M$ as

$$PH^*(M) := W_n H^*(M) \subset H^*(M),$$

which is a subring $PH^*(M) \subset H^*(M)$ of the cohomology of $M$. For a smooth $M$, the pure part of $H^*(M)$ is always the image of the cohomology of a smooth compactification (see [13 Corollaire 3.2.17]). It is in fact this result which can be used to show that the spaces mentioned in the previous paragraph have pure mixed Hodge structure. That is one can prove that they admit a smooth compactification which surjects on cohomology. Prototypes of such compactifications were constructed in [56] for $M_{\text{DR}}$ and in [21] for $M_{\text{Dol}}$. 
5 Applications of mixed Hodge theory

Using the method sketched in the previous section the strongest results on cohomology can be achieved when the variety has a pure MHS on its cohomology, consequently the $E$-polynomial determines the mixed Hodge polynomial, and additionally it is polynomial-count so that Theorem 4.1 gives an arithmetic way to determine the $E$-polynomial. This is the case for Nakajima quiver varieties, where our method gives complete results.

5.1 Nakajima quiver varieties

Nakajima quiver varieties are constructed [48] by a finite dimensional hyperkähler quotient construction. Here we review the affine algebraic-geometric version of this construction.

Let $\Gamma$ be a quiver (oriented graph) with vertex set $I = \{1, \ldots, n\}$ and edges $E \subset I \times I$. Let $v = (v_1, \ldots, v_n), w = (w_1, \ldots, w_n) \in \mathbb{N}^I$ be two dimension vectors and $V_i$ and $W_i$ corresponding complex vector spaces, i.e. $\dim(V_i) = v_i$ and $\dim(W_i) = w_i$. We define the vector spaces $V_{v,w} = \bigoplus_{a \in E} \text{Hom}(V_{t(a)}, V_{h(a)}) \oplus \bigoplus_{i \in I} \text{Hom}(V_i, W_i)$

of framed representations of the quiver $\Gamma$, and the action $\rho : \text{GL}(v) := \prod_{i \in I} \text{GL}(V_i) \to \text{GL}(V_v),$ with derivative $\varrho : \mathfrak{gl}(v) := \prod_{i \in I} \mathfrak{gl}(V_i) \to \mathfrak{gl}(V_v).$

The complex moment map $\mu : V_v \times V_v^* \to \mathfrak{gl}_v^*$ of $\rho$ is given at $X \in \mathfrak{gl}_v$ by $\langle \mu(v, w), X \rangle = \langle \varrho(X)v, w \rangle. \quad (5.1)$

For $\xi = 1_v \in \mathfrak{gl}(v)^{\text{GL}(v)}$ we define the (always smooth) Nakajima quiver variety by $\mathcal{M}(v, w) = \mu^{-1}(\xi) // \text{GL}(v) = \text{Spec} \left( \mathbb{C}[\mu^{-1}(\xi)]^{\text{GL}(v)} \right)$
as an affine GIT quotient. Alternatively one can construct the manifold underlying $\mathcal{M}(v, w)$ as a hyperkähler quotient of $V_v \times V_v^*$ by the maximal compact subgroup $U(v) \subset \text{GL}(v)$. This shows that $\mathcal{M}(v, w)$ possesses a hyperkähler metric. The holomorphic symplectic quotient we presented above is the one where the arithmetic technique of §4 is applicable. Before we explain that, let us recall the following fundamental theorem of [48] about the cohomology of these Nakajima quiver varieties:

**Theorem 5.1.** Assume that the quiver $\Gamma$ has no edge-loops. Then there is an irreducible representation of the Kac-Moody algebra $\mathfrak{g}(\Gamma)$ of highest weight $w$ on $\bigoplus_{x} H^\text{mid}(\mathcal{M}(v, w))$. In particular the Weyl-Kac character formula gives the middle Betti numbers of Nakajima quiver varieties. Furthermore the intersection form on $H^\text{mid}_c(\mathcal{M}(v, w))$ is definite, thus $\chi_L^c(\mathcal{M}(v, w))$ equals the middle Betti number of $\mathcal{M}(v, w)$. 

9
Remark 5.2. When $\Gamma$ is an affine Dynkin diagram $\mathcal{M}(v, w)$ could be identified with one of the spaces $M^{k,c_l}$ of certain Yang-Mills instantons on a ALE space $X_T$. Kac in [41] explains that the Weyl-Kac character formula for an affine Dynkin diagram has certain modular properties. This was the line of argument in [59] that (1.1) is a modular form provided Conjecture 1.3 holds.

In [24] a simple Fourier transform technique was found to enumerate the rational points of $\mathcal{M}(v, w)$ over a finite field $\mathbb{F}_q$. The corresponding count function $E(q)$ turned out to be polynomial, and as the mixed Hodge structure is pure on $H^*(\mathcal{M}(v, w))$ the technique of §4 applies in its full strength to give a formula for the Betti numbers of the varieties $\mathcal{M}(v, w)$. The result is the following formula from [24]:

**Theorem 5.3.** For any quiver $\Gamma$, the Betti numbers of the Nakajima quiver varieties are given by the following generating function, with the notation as in [24 Theorem 3]:

$$
\sum_{\lambda \in \Pi} P_\Gamma(M(v, w)) t^{-d(v, w)} T^\lambda = \frac{\sum_{\lambda \in \Pi} T^\lambda}{\sum_{\lambda \in \Pi} \prod_{i \in E(v, w)} (\langle \lambda \rangle - 2\langle \lambda \rangle \langle \lambda \rangle)}.
$$

Remark 5.4. When $\Gamma$ has no edge-loops Nakajima’s Theorem 5.1 implies that the right hand side of (5.2) is a deformation of the Weyl-Kac character formula. Simple reasoning gives the same result about the denominator of the right hand side of (5.2) and the Kac denominator. Moreover, the Kac’s denominator formula and Hua’s formula [36, Theorem 4.9] expressing the denominator of (5.2) as an infinite product implies a conjecture of Kac, cf. [36, Corollary 4.10]. Namely, if $A_F(v, q)$ denotes the number of absolutely indecomposable representations of $\Gamma$ of dimension vector $v$ over the finite field $\mathbb{F}_q$, then it turns out to be a polynomial in $q$ and Kac’s [40, Conjecture 1] says that the constant coefficient

$$
A_F(v, 0) = m_v
$$

equals with the multiplicity of the weight $v$ in the Kac-Moody algebra $\mathfrak{g}(\Gamma)$. This can be proved, as sketched above and announced in [24], to be a consequence of (5.2) and the above mentioned results of Nakajima and Hua.

Remark 5.5. When the quiver is affine ADE and the RHS becomes an infinite product (indications that this can happen is the infinite product in [24, §3] and the infinite products in the recent [51]) we could get an alternative proof of the modularity of (1.1) in the Vafa-Witten S-duality conjecture.

In the remaining part of this survey we will motivate and study another application of the technique in §4 which will be less powerful as the mixed Hodge structure will fail to be pure, but will also open new interesting directions by the study of this more complicated mixed Hodge structure.

### 5.2 Spaces diffeomorphic to the Hitchin moduli space $\mathcal{M}(C, P_{U(n)})$

Among the spaces discussed in this paper it is the Hitchin moduli space $\mathcal{M}(C, P_{U(n)})$ as defined in §2.2 which exhibits perhaps the most plentiful structures many of which are rooted in its hyperkähler quotient origin. In particular there are three distinct complex algebraic variety structures on $\mathcal{M}(C, P_{U(n)})$. These can be thought of [56] as the three types of non-Abelian (first) cohomology: Dolbeault, De Rham and Betti, of the Riemann surface $C$. The survey paper [23] gives a quick introduction to these spaces and some of the cohomological implications to be discussed below.
In this paper the ground field is always $\mathbb{C}$ unless otherwise indicated. Following [32, 56] we define a component of the twisted $\text{GL}_n = \text{GL}_n(\mathbb{C})$ Dolbeault cohomology of $C$ as

$$
\mathcal{M}_{\text{Dol}}^d(\text{GL}_n) := \{ \text{moduli space of semistable rank } n \text{ degree } d \text{ Hitchin pairs on } C \}
$$

the $\text{GL}_n$ De Rham cohomology as

$$
\mathcal{M}_{\text{DR}}^d(\text{GL}_n) := \{ \text{moduli space of flat } \text{GL}_n\text{-connections on } C \setminus \{ p \}, \text{ with holonomy } e^{2 \pi i d} \text{Id around } p \}
$$

and the $\text{GL}_n$ Betti cohomology

$$
\mathcal{M}_b^d(\text{GL}_n) := \{ A_1, B_1, \ldots, A_g, B_g \in \text{GL}_n | A_1^{-1}B_1^{-1}A_1B_1 \ldots A_g^{-1}B_g^{-1}A_gB_g = e^{2 \pi i d} \text{Id} \}/\text{GL}_n
$$

as a twisted $\text{GL}_n$ character variety of $C$.

When $d = 0$ these three varieties are diffeomorphic to the Hitchin moduli space $\mathcal{M}(C, P_{U(n)})$. However we prefer to consider the twisted versions, when $(d, n) = 1$, because then all the varieties are smooth. In this case these three varieties are all diffeomorphic to a twisted version $\mathcal{M}^d(C, P_{U(n)})$ of Hitchin moduli space and so to each other. The mixed Hodge structure is pure on $H^*(\mathcal{M}_{\text{Dol}}^d(\text{GL}_n))$ and $H^*(\mathcal{M}_{\text{DR}}^d(\text{GL}_n))$, while it is not pure on $H^*(\mathcal{M}_b^d(\text{GL}_n))$. The mixed Hodge structure are different on $H^*(\mathcal{M}_{\text{Dol}}^d(\text{GL}_n))$ and $H^*(\mathcal{M}_b^d(\text{GL}_n))$, and so the spaces $\mathcal{M}_{\text{DR}}^d(\text{GL}_n)$ and $\mathcal{M}_b^d(\text{GL}_n)$ cannot be isomorphic as complex algebraic varieties. Nevertheless as complex analytic manifolds the Riemann-Hilbert monodromy map

$$
\mathcal{M}_{\text{Dol}}^d(\text{GL}_n) \xrightarrow{\text{RH}} \mathcal{M}_{\text{DR}}^d(\text{GL}_n)
$$

sending a flat connection to its holonomy gives an isomorphism.

We will also consider the varieties $\mathcal{M}_{\text{Dol}}^d(\text{SL}_n)$, $\mathcal{M}_{\text{DR}}^d(\text{SL}_n)$ and $\mathcal{M}_b^d(\text{SL}_n)$, which can be defined by replacing $\text{GL}_n$ with $\text{SL}_n$ in the above definitions. Moreover $\mathcal{M}_{\text{Dol}}^d(\text{GL}_1)$, $\mathcal{M}_{\text{DR}}^d(\text{GL}_1)$ and $\mathcal{M}_b^d(\text{GL}_1)$ turn out to be abelian groups. Then $\mathcal{M}_{\text{Dol}}^d(\text{GL}_1)$, $\mathcal{M}_{\text{DR}}^d(\text{GL}_1)$ and $\mathcal{M}_b^d(\text{GL}_1)$, respectively, will act on $\mathcal{M}_{\text{Dol}}^d(\text{GL}_n)$, $\mathcal{M}_{\text{DR}}^d(\text{GL}_n)$ and $\mathcal{M}_b^d(\text{GL}_n)$, respectively, by an appropriate form of tensorization. Finally we denote the corresponding (affine GIT) quotients by $\mathcal{M}_{\text{Dol}}^d(\text{PGL}_n)$, $\mathcal{M}_{\text{DR}}^d(\text{PGL}_n)$ and $\mathcal{M}_b^d(\text{PGL}_n)$. In our case, when $(d, n) = 1$, they will turn out to be orbifolds. For more details on the construction of these varieties see [23].

In the next section we explain the original motivation to consider the $E$-polynomials of these three complex algebraic varieties. The motivation is mirror symmetry, and most probably the same $S$-duality we discussed in the Introduction in connection with the Hodge cohomology of the moduli spaces of Yang-Mills instantons in four dimension and magnetic monopoles in three. $S$-duality ideas relating to mirror symmetry for Hitchin spaces have appeared in the physics literature [5, 42].

### 5.3 Topological Mirror Test

For our mathematical considerations the relationship to mirror symmetry stems from the following observation of [29]. It uses the famous Hitchin map [33], which makes the moduli space of Higgs bundles $\mathcal{M}_{\text{Dol}}$ into a completely integrable Hamiltonian system, so that the generic fibers are Abelian varieties.
Theorem 5.6. In the following diagram

\[
\begin{array}{ccc}
\mathcal{M}^d_{\text{Dol}}(\text{PGL}_n) & \xrightarrow{\chi_{\text{PGL}_n}} & \mathcal{M}^d_{\text{Dol}}(\text{SL}_n) \\
\mathcal{H}_{\text{PGL}_n} & \cong & \mathcal{H}_{\text{SL}_n},
\end{array}
\]

the generic fibers of the Hitchin maps $\chi_{\text{PGL}_n}$ and $\chi_{\text{SL}_n}$ are dual Abelian varieties.

Remark 5.7. If we change complex structures and consider $\mathcal{M}^d_{\text{DR}}(\text{PGL}_n)$ and $\mathcal{M}^d_{\text{DR}}(\text{SL}_n)$, then the Hitchin map on them becomes special Lagrangian fibrations, and consequently the pair of $\mathcal{M}^d_{\text{DR}}(\text{PGL}_n)$ and $\mathcal{M}^d_{\text{DR}}(\text{SL}_n)$ satisfies the requirements of the SYZ construction [58] for a pair of mirror symmetric Calabi-Yau manifolds (see [29] and [28] for more details).

This motivates the calculation of Hodge numbers of $\mathcal{M}^d_{\text{DR}}(\text{PGL}_n)$ and $\mathcal{M}^d_{\text{DR}}(\text{SL}_n)$ to see if there is any relationship between them, which one would expect in mirror symmetry. Based on calculations in the $n = 2, 3$ cases [29] proposed:

Conjecture 5.8. For all $d, e \in \mathbb{Z}$, satisfying $(d, n) = (e, n) = 1$,

\[
E^d_{\text{st}}(x, y; \mathcal{M}^d_{\text{DR}}(\text{SL}_n)) = E^d_{\text{st}}(x, y; \mathcal{M}^d_{\text{DR}}(\text{PGL}_n)),
\]

where $B^e$ and $\hat{B}^d$ are certain gerbes on the corresponding Hitchin spaces and the $E$-polynomials above are stringy $E$-polynomials for orbifolds twisted by the relevant gerbe as defined in [29].

Morally, this conjecture should be related to the S-duality considerations of [42] and in turn to the Geometric Langlands Programme of [4]. However the lack of global analytical interpretation of the mixed Hodge numbers appearing in Conjecture 5.8 prevents a straightforward physical interpretation. Nevertheless the agreement of certain Hodge numbers for Hitchin spaces for Langlands dual groups is an interesting direction from a purely mathematical point of view. In particular, if we change our focus from $\mathcal{M}_{\text{DR}}$ and $\mathcal{M}_{\text{Dol}}$ to $\mathcal{M}_B$ we will uncover some surprising connections to the representation theory of finite groups of Lie type.

5.4 Mirror symmetry for finite groups of Lie type

As $\mathcal{M}_{\text{DR}}$ and $\mathcal{M}_B$ are complex analytically identical via the Riemann–Hilbert map (5.4), the complex analytical structure of dual special Lagrangian fibrations of Theorem 5.6 are present on the pair $\mathcal{M}^d_B(\text{SL}_n)$ and $\mathcal{M}^e_B(\text{PGL}_n)$. We might as well try to think of this pair as mirror symmetric in the SYZ picture. The mixed Hodge numbers of $\mathcal{M}_B$ are however different from the mixed Hodge numbers of $\mathcal{M}_{\text{DR}}$ so the corresponding topological mirror test [23] will also be different from Conjecture 5.8:

Conjecture 5.9. For all $d, e \in \mathbb{Z}$, satisfying $(d, n) = (e, n) = 1$,

\[
E^e_{\text{st}}(x, y, \mathcal{M}^d_B(\text{SL}_n)) = E^d_{\text{st}}(x, y, \mathcal{M}^e_B(\text{PGL}_n)).
\]

For this conjecture however there is a powerful arithmetic method to calculate these $E$-polynomials. Using this technique we have already managed to check this conjecture [23] when $n$ is a prime and $n = 4$. This arithmetic method is based on the technique explained in §4 and the following character formula from [27]:

\[12\]
Theorem 5.10. Let G = SL_n or GL_n, let G(\mathbb{F}_q) be the corresponding finite group of Lie type 

\[ E(\sqrt{q}, \sqrt{q}, M_B^t(G)) = \#\{M_B^t(G(\mathbb{F}_q))\} = \sum_{\chi \in Irr(G(\mathbb{F}_q))} \chi(\xi_{n,\beta}^t), \]

where the sum is over all irreducible characters of the finite group of Lie type G(\mathbb{F}_q).

This character formula combined with Conjecture 5.9 implies certain relationships between the character tables of PGL_n(\mathbb{F}_q) and SL_n(\mathbb{F}_q). An intriguing way to formulate it is to say that certain differences between the character tables of PGL_n(\mathbb{F}_q) and its Langlands dual SL_n(\mathbb{F}_q) are governed by mirror symmetry. This kind of consideration could be interesting because the character tables of GL_n(\mathbb{F}_q) and SL_n(\mathbb{F}_q) have just recently been completed [8, 55]. It is especially enjoyable to follow the effect of the mirror symmetry proposal of Conjecture 5.9 by comparing the character tables of GL_2(\mathbb{F}_q) and SL_2(\mathbb{F}_q) first calculated a hundred years ago by Jordan [37] and Schur [52].

5.5 Conjectural answer

Finally, we can put all our observations and conjectures together to state a conjectural answer to the topological side of Problem [11].

As we already noted the mixed Hodge structure on H^*(M_B) is not pure. Therefore we are losing information by considering only \(E(M_B; x, y)\). It turns out that it is interesting to consider the full mixed Hodge polynomial \(H(M_B; x, y, t)\). When \(n = 2\) it can be calculated via the explicit description of \(H^*(M_B)\) in [30]. We get [27, Theorem 1.1.3]:

\[ H(M_B(PGL_2); x, y, t) = \frac{(q^2t^2 + 1)^{2g}}{(q^2t^2 - 1)(q^2t^2 - 1)} \cdot \frac{1}{2} \frac{q^{g-2}t^{g-4}(qt + 1)^{2g}}{(q^2 - 1)(q^2 - 1)} - \frac{1}{2} \frac{q^{g-2}t^{g-4}(qt - 1)^{2g}}{(q + 1)(qt^2 + 1)}, \]

where \(q = xy\) and the four terms correspond to the four types of irreducible characters of GL(2, \mathbb{F}_q). When \(g = 3\) this equals:

\[ t^{12}q^{12} + t^{11}q^{10} + 6t^{12}q^8 + t^{10}q^{10} + 6t^{11}q^8 + 16t^{10}q^8 + 6t^9q^8 + t^{10}q^6 + 6t^9q^6 + 26t^8q^6 \]
\[ + 16t^8q^6 + 6t^7q^6 + t^8q^4 + t^6q^8 + 6t^7q^4 + 16t^6q^4 + 6t^5q^4 + t^4q^4 + t^4q^2 + 6t^3q^2 + t^3q^2 + t^2q^2 + 1. \]

In particular we see that the pure part is \(1 + q^2t^4 + q^4t^8\). These terms correspond to the cohomology classes 1, \(\beta\) and \(\beta^2\), and the term \(q^6t^{12}\) is not present because by the Newstead relation \(\beta^8 = \beta^2 = 0\) holds [30]. In particular there is no pure part in the middle = 12 dimensional cohomology. The same argument holds for all \(g\), which shows that there is no pure part in the middle dimensional cohomology of \(M_B^t(PGL_2)\). It is however easy to see that the intersection form on middle cohomology can only be non-trivial on the pure part and so this implies [27, Corollary 5.4.1]:

Corollary 5.11. The intersection form on \(H^*_c(M_B^t(PGL_2))\) is trivial.

This gives an alternative proof of (3.4) as the equation

\[ \chi_L^2(M_B^t(SL_2)) = \chi_L^2(M_B^t(PGL_2)) \]

is easy to prove. Moreover this approach is more promising to generalize for any \(n\). We will offer a conjecture about the pure part of the cohomology of \(M_B^t(PGL_n)\) below and in turn that will yield a conjecture for the intersection form on the middle dimensional compactly supported cohomology, answering the topological side of Problem [11].
To state our conjecture in its full generality we introduce character varieties on Riemann surfaces with $k$ punctures and parabolic type $\mu = (\mu^1, \ldots, \mu^k)$ at the punctures, where $\mu^i$ is a partition of $n$. In other words we fix semisimple conjugacy classes $C_1, \ldots, C_k \subset \text{GL}_n$, which are generic and have type $\mu$ (in other words $\mu^i_j$ is the multiplicity of the $j$th eigenvalue of a matrix in $C_i$). One can prove \cite{20} that there exists generic semisimple conjugacy classes for every type $\mu = (\mu^1, \ldots, \mu^k)$. For a generic $\{C_1, \ldots, C_k\}$ of type $\mu$ we define
\[
\mathcal{M}^\mu_B := \{A_1, B_1, \ldots, A_g, B_g \in \text{GL}_n, C_1 \in C_1, \ldots, C_k \in C_k\}
\[
[A_1, B_1] \cdots [A_g, B_g]C_1 \cdots C_k = I_n]//\text{GL}_n
\]
as an affine GIT quotient by the diagonal adjoint action of $\text{GL}_n$. The generic choice of the semisimple conjugacy classes implies that $\mathcal{M}^\mu_B$ is smooth. The torus $\text{GL}_2^g$ acts on $\mathcal{M}^\mu_B$ by multiplying the matrices $A_i$ and $B_i$ by a scalar. We can define the quotient
\[
\tilde{\mathcal{M}}^\mu_B := \mathcal{M}^\mu_B//\text{GL}_2^g
\]
as the corresponding $\text{PGL}_n$ character variety. The variety $\tilde{\mathcal{M}}^\mu_B$ is an orbifold.

By studying the Riemann-Hilbert map on the level of cohomologies we are led \cite{20} to consider the crab-shaped quiver $\Gamma$ associated to $g$ and $\mu$. Namely, we can put $g$ loops on a central vertex, and $k$ legs of length $l(\mu^i)$. We also equip $\Gamma$ with a dimension vector $v$, which has dimension $\sum_{i=1}^k \mu^i_j$ at the $i$th vertex on the $j$th leg. Consider now the number $A_\Gamma(q, v)$ of absolutely indecomposable representations of $\Gamma$ of dimension $v$ over the finite field $\mathbb{F}_q$. Kac \cite[Proposition 1.15]{40} proved that $A_\Gamma(q, v)$ is a polynomial in $q$ with integer coefficients. We have the following conjecture from \cite{20}:

**Conjecture 5.12.** The pure part of the cohomology of $\tilde{\mathcal{M}}^\mu_B$ is given by
\[
PH(\tilde{\mathcal{M}}^\mu_B, x, y) = (x y)^d_{\mu}/A_\Gamma(v, 1/(x y)),
\]
where $(\Gamma, v)$ is the star-shaped quiver and dimension vector given by the parabolic type $\mu$, and $d_{\mu}$ is the dimension of $\mathcal{M}^\mu_B$.

This conjecture gives a cohomological interpretation of $A_\Gamma(v, q)$ and in particular implies that it has non-negative coefficients confirming \cite[Conjecture 2]{40} in the case when $\Gamma$ is crab-shaped. When $\mu$ is indivisible Conjecture 5.12 can be proved to follow from the master conjecture in \cite{20}, which expresses the mixed Hodge polynomials of all the character varieties $\tilde{\mathcal{M}}^\mu_B$ as a generating function generalizing the Cauchy formula for Macdonald polynomials. It also has the following consequence on the topological $L^2$ cohomology $\chi_{L^2}(\tilde{\mathcal{M}}^\mu_B)$ of (3.3).

**Conjecture 5.13.** The topological $L^2$ cohomology of the manifold $\tilde{\mathcal{M}}^\mu_B$ is given by
\[
\chi_{L^2}(\tilde{\mathcal{M}}^\mu_B) = 0, \text{ when } g > 1
\]
\[
\chi_{L^2}(\tilde{\mathcal{M}}^\mu_B) = 1, \text{ when } g = 1
\]
\[
\chi_{L^2}(\tilde{\mathcal{M}}^\mu_B) = m_v, \text{ when } g = 0,
\]
where $m_v$ is the multiplicity of the weight $v$ in the Kac-Moody algebra $g(\Gamma)$, which are encoded by the Kac denominator formula for the star-shaped quiver $\Gamma$.

When $g > 1$ and the parabolic type is $\mu = ((n))$, i.e. we have only one puncture with central conjugacy class, then one can identify $\tilde{\mathcal{M}}^\mu_B = \mathcal{M}^\mu_B(\text{PGL}_n)$, with some $d$ such that $(d, n) = 1$. In this case \cite{5.5} says that
\[
\chi_{L^2}(\mathcal{M}^\mu_B(\text{PGL}_n)) = 0,
\]
which appeared as [27, Conjecture 4.5.1]. It follows from the mirror symmetry Conjecture [5.8] that

\[ H^{\text{mid}}_{\text{cpt}} \left( M^d_{B}(SL_n) \right) \cong H^{\text{mid}}_{\text{cpt}} \left( M^d_{B}(PGL_n) \right) \]

and then the intersection forms also agree. This and (5.7) then imply that (5.4) holds for any \( n \), i.e. that the intersection form on the compactly supported cohomology of \( M^d_{B}(SL_n) \) is trivial. This gives a conjectural answer to the topological side of Problem 1.1.

When \( g = 1 \) the conjectured (5.6) follows from Conjecture 5.12 and the observation that the coefficient of \( q \) in the \( A \)-polynomial \( A_{\Gamma}(q) \) for a \( g = 1 \) crab-shaped quiver \( \Gamma \) is always 1.

When \( g = 0 \) the varieties \( M^\mu_{B} = \bar{M}^\mu_{B} \) coincide. Conjecture 5.12 then implies that \( \chi_{L^2}(M^\mu_{B}) = A_{\Gamma}(v, 0) \).

Conjecture (5.7) is a combination of this and the equality \( A_{\Gamma}(v, 0) = m_v \), that is Kac’s [40, Conjecture 1], which, as discussed in Remark 5.4, follows from Theorem 5.3.

Finally one can define \( \bar{M}^\mu_{Dol} \) the moduli space of stable parabolic \( PGL_n \)-Higgs bundles with quasi-parabolic type \( \mu \in \mathcal{P}(n) \) and generic weights at the \( j \)th puncture on the Riemann surface [7, 15]. Then one can prove that \( \bar{M}^\mu_{B} \) is diffeomorphic to \( \bar{M}^\mu_{Dol} \). Thus Conjecture 5.13 also calculates the intersection form on the compactly supported cohomology of the moduli space \( \bar{M}^\mu_{Dol} \) of stable parabolic \( PGL_n \)-Higgs bundles of any rank.

**Example 5.14.** Consider the genus 0 Riemann surface \( \mathbb{P}^1 \) with four punctures. Consider the moduli space \( M_{toy} \) of stable rank 2 parabolic Higgs bundles on \( \mathbb{P}^1 \), with generic parabolic weights on the full parabolic flag at the punctures (see [7]). This is a complex surface and the intersection form on \( H^2_c(M_{toy}) \) was discussed in [21, Example 2 for Theorem 7.13]. \( H^2_c(M_{toy}) \) is 5 dimensional but \( \chi_{L^2}(M_{toy}) \) is only 4. (The cohomology class of the generic fiber of the Hitchin map is the one in the kernel.)

\( M_{toy} \) is diffeomorphic to the character variety \( \bar{M}^\mu_{B} \) where \( g = 0 \) and \( \mu = ((1, 1), (1, 1), (1, 1), (1, 1)) \). Thus by Conjecture 5.13 we should be able to calculate \( \chi_{L^2}(M^\mu_{B}) \) in terms of the representation theory of the corresponding quiver \( \Gamma \). The corresponding quiver \( \Gamma \) in this case will be the affine \( \tilde{D}_4 \) Dynkin diagram, with \( v = (2, 1, 1, 1, 1) \) the minimal positive imaginary root. Its multiplicity \( m_v \) in the affine Kac-Moody algebra associated to \( \Gamma \) is known to be 4. Alternatively it is known [40, Example b to Conjecture 2] that \( A_{\Gamma}(v, q) = q + 4 \), which by (5.3) gives \( m_v = 4 \). Thus indeed \( \chi_{L^2}(M^\mu_{B}) = m_v = 4 \) checking (5.7) in this case via [21, Example 2 for Theorem 7.13].

**References**

[1] M. F. Atiyah. Geometry of Yang-Mills fields. In *Mathematical problems in theoretical physics (Proc. Internat. Conf., Univ. Rome, Rome, 1977)*, volume 80 of *Lecture Notes in Phys.*, pages 216–221. Springer, Berlin, 1978.

[2] M. F. Atiyah and N. J. Hitchin. Low energy scattering of nonabelian monopoles. *Phys. Lett. A*, 107(1):21–25, 1985.

[3] M. F. Atiyah and N. J. Hitchin. *The geometry and dynamics of magnetic monopoles*. M. B. Porter Lectures. Princeton University Press, Princeton, NJ, 1988.

[4] A. Beilinson and V. Drinfeld. Quantization of Hitchin’s integrable system and Hecke eigensheaves. (ca. 1995).
[5] M. Bershadsky, A. Johansen, V. Shadov, and C. Vafa. Topological reduction of 4D SYM to 2D $\sigma$-models. *Nuclear Phys. B*, 448(1-2):166–186, 1995.

[6] R. Bielawski. Monopoles and clusters. arXiv:hep-th/0702190

[7] H. U. Boden and K. Yokogawa. Moduli spaces of parabolic Higgs bundles and parabolic $K(D)$ pairs over smooth curves. I. *Internat. J. Math.*, 7(5):573–598, 1996.

[8] C. Bonnafé. Sur les caractères des groupes réductifs finis à centre non connexe: applications aux groupes spéciaux linéaires et unitaires. *Astérisque*, (306):vi+165, 2006.

[9] G. Carron. Cohomologie $L^2$ des variétés QALE. arXiv:math.DG/0501290.

[10] S. A. Cherkis and A. Kapustin. Singular monopoles and gravitational instantons. *Comm. Math. Phys.*, 203(3):713–728, 1999.

[11] S. A. Cherkis and A. Kapustin. Hyper-Kähler metrics from periodic monopoles. *Phys. Rev. D (3)*, 65(8):084015, 10, 2002.

[12] G. de Rham. *Differentiable manifolds*, volume 266 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1984.

[13] P. Deligne. Théorie de Hodge. II. *Inst. Hautes Études Sci. Publ. Math.*, (40):5–57, 1971.

[14] P. Deligne. Théorie de Hodge. III. *Inst. Hautes Études Sci. Publ. Math.*, (44):5–77, 1974.

[15] O. García-Prada, P. Gothen, and V. Muñoz. Betti numbers of the moduli space of rank 3 parabolic Higgs bundles. *Memoirs of the American Mathematical Society*, 187(no. 879), 2007.

[16] G. W. Gibbons and N. S. Manton. The moduli space metric for well-separated BPS monopoles. *Phys. Lett. B*, 356(1):32–38, 1995.

[17] P. B. Gothen. The Betti numbers of the moduli space of stable rank 3 Higgs bundles on a Riemann surface. *Internat. J. Math.*, 5(6):861–875, 1994.

[18] J. A. Green. The characters of the finite general linear groups. *Trans. Amer. Math. Soc.*, 80:402–447, 1955.

[19] M. Gromov. Kähler hyperbolicity and $L^2$-Hodge theory. *J. Differential Geom.*, 33(1):263–292, 1991.

[20] T. Hausel. Arithmetic harmonic analysis, Macdonald polynomials and the topology of the Riemann-Hilbert monodromy map. (in preparation).

[21] T. Hausel. Compactification of moduli of Higgs bundles. *J. Reine Angew. Math.*, 503:169–192, 1998.

[22] T. Hausel. Vanishing of intersection numbers on the moduli space of Higgs bundles. *Adv. Theor. Math. Phys.*, 2(5):1011–1040, 1998.

[23] T. Hausel. Mirror symmetry and Langlands duality in the non-abelian Hodge theory of a curve. In *Geometric methods in algebra and number theory*, volume 235 of *Progr. Math.*., pages 193–217. Birkhäuser Boston, Boston, MA, 2005.
[24] T. Hausel. Betti numbers of holomorphic symplectic quotients via arithmetic Fourier transform. *Proc. Natl. Acad. Sci. USA*, 103(16):6120–6124, 2006.

[25] T. Hausel, E. Hunsicker, and R. Mazzeo. Hodge cohomology of gravitational instantons. *Duke Math. J.*, 122(3):485–548, 2004.

[26] T. Hausel, E. Letellier, and F. Rodriguez-Villegas. Arithmetic harmonic analysis on character and quiver varieties. (in preparation).

[27] T. Hausel and F. Rodriguez-Villegas. Mixed Hodge polynomials of character varieties. arXiv:math.AG/0612668

[28] T. Hausel and M. Thaddeus. Examples of mirror partners arising from integrable systems. *C. R. Acad. Sci. Paris Sér. I Math.*, 333(4):313–318, 2001.

[29] T. Hausel and M. Thaddeus. Mirror symmetry, Langlands duality, and the Hitchin system. *Invent. Math.*, 153(1):197–229, 2003.

[30] T. Hausel and M. Thaddeus. Relations in the cohomology ring of the moduli space of rank 2 Higgs bundles. *J. Amer. Math. Soc.*, 16(2):303–327 (electronic), 2003.

[31] N. J. Hitchin. *Monopoles, minimal surfaces and algebraic curves*, volume 105 of *Séminaire de Mathématiques Supérieures*. Presses de l’Université de Montréal, Montreal, QC, 1987.

[32] N. J. Hitchin. The self-duality equations on a Riemann surface. *Proc. London Math. Soc. (3)*, 55(1):59–126, 1987.

[33] N. J. Hitchin. Stable bundles and integrable systems. *Duke Math. J.*, 54(1):91–114, 1987.

[34] N. J. Hitchin. $L^2$-cohomology of hyperkähler quotients. *Comm. Math. Phys.*, 211(1):153–165, 2000.

[35] N. J. Hitchin, A. Karlhede, U. Lindström, and M. Roček. Hyper-Kähler metrics and supersymmetry. *Comm. Math. Phys.*, 108(4):535–589, 1987.

[36] J. Hua. Counting representations of quivers over finite fields. *J. Algebra*, 226(2):1011–1033, 2000.

[37] H. E. Jordan. Group-Characters of Various Types of Linear Groups. *Amer. J. Math.*, 29(4):387–405, 1907.

[38] J. Jost and K. Zuo. Vanishing theorems for $L^2$-cohomology on infinite coverings of compact Kähler manifolds and applications in algebraic geometry. *Comm. Anal. Geom.*, 8(1):1–30, 2000.

[39] D. D. Joyce. *Compact manifolds with special holonomy*. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2000.

[40] V. G. Kac. Root systems, representations of quivers and invariant theory. In *Invariant theory (Montecatini, 1982)*, volume 996 of *Lecture Notes in Math.*, pages 74–108. Springer, Berlin, 1983.

[41] V. G. Kac. *Infinite-dimensional Lie algebras*. Cambridge University Press, Cambridge, third edition, 1990.

[42] A. Kapustin and E. Witten. Electric-magnetic duality and the geometric Langlands program. *Commun. Number Theory Phys.* 1(1):1–236, 2007. arXiv:hep-th/0604151
[43] F. Kirwan. The cohomology rings of moduli spaces of bundles over Riemann surfaces. *J. Amer. Math. Soc.*, 5(4):853–906, 1992.

[44] P. B. Kronheimer. The construction of ALE spaces as hyper-Kähler quotients. *J. Differential Geom.*, 29(3):665–683, 1989.

[45] P. B. Kronheimer and H. Nakajima. Yang-Mills instantons on ALE gravitational instantons. *Math. Ann.*, 288(2):263–307, 1990.

[46] C. Montonen and C. D. I. Olive. Magnetic monopoles as gauge particles? *Phys. Lett. B*, B72:117–120, 1977.

[47] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2)*. Springer-Verlag, Berlin, third edition, 1994.

[48] H. Nakajima. Quiver varieties and Kac-Moody algebras. *Duke Math. J.*, 91(3):515–560, 1998.

[49] N. Nitsure. Moduli space of semistable pairs on a curve. *Proc. London Math. Soc. (3)*, 62(2):275–300, 1991.

[50] H. Osborn. Topological charges for $N=4$ supersymmetric gauge theories and monopoles of spin 1. *Phys. Lett. B83*, (321.), 1979.

[51] T. Sasaki. $O(−2)$ blow-up formula via instanton calculus on $\mathbb{P}^2/\mathbb{Z}_2$ and Weil conjecture. arXiv:hep-th/0603162.

[52] I. Schur. Untersuchungen über die darstellung der endlichen gruppen durch gebrochene lineare substitutionen. *J. Reine Angew. Math.*, 132:85, 1907.

[53] G. Segal and A. Selby. The cohomology of the space of magnetic monopoles. *Comm. Math. Phys.*, 177(3):775–787, 1996.

[54] A. Sen. Dyon-monopole bound states, self-dual harmonic forms on the multi-monopole moduli space, and $SL(2, \mathbb{Z})$ invariance in string theory. *Phys. Lett. B*, 329(2-3):217–221, 1994.

[55] T. Shoji. Lusztig’s conjecture for finite special linear groups. *Represent. Theory*, 10:164–222 (electronic), 2006.

[56] C. Simpson. The Hodge filtration on nonabelian cohomology. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 217–281. Amer. Math. Soc., Providence, RI, 1997.

[57] C. T. Simpson. Nonabelian Hodge theory. In *Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990)*, pages 747–756, Tokyo, 1991. Math. Soc. Japan.

[58] A. Strominger, S.-T. Yau, and E. Zaslow. Mirror symmetry is $T$-duality. *Nuclear Phys. B*, 479(1-2):243–259, 1996.

[59] C. Vafa and E. Witten. A strong coupling test of $S$-duality. *Nuclear Phys. B*, 431(1-2):3–77, 1994.

[60] E. Witten and D. Olive. Supersymmetry algebras that include topological charges. *Phys. Lett. 78B*, (97.), 1978.