Quantum Dissipation and Decay in Medium

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ABSTRACT

Quantum dissipation in thermal environment is investigated, using the path integral approach. The reduced density matrix of the harmonic oscillator system coupled to thermal bath of oscillators is derived for arbitrary spectrum of bath oscillators. Time evolution and the end point of two-body decay of unstable particles is then elucidated: After early transient times unstable particles undergo the exponential decay, followed by the power law decay and finally ending in a mixed state of residual particles containing contributions from both on and off the mass shell, whose abundance does not suffer from the Boltzmann suppression.
It is a long standing problem in physics to resolve how a quantum system evolves with time under influence of an environment described by some mixed state. The effect of the environment on the system dynamics is governed by the fluctuation (or the noise) and the dissipation, related by the fluctuation-dissipation theorem. The simplest, yet the most fundamental model of quantum dissipation is harmonic oscillator coupled to infinitely many oscillators that make up a bath in a mixed state. Two powerful methods to analyze this problem are the quantum Langevin equation \[1\], and the path integral approach \[2\], \[3\], \[4\].

The important and the difficult part of analysis of this problem is how to treat the non-local part of the correlation of the system variable obtained after integrating out environment variables. The non-locality is required from quantum mechanical principles. Many past studies have relied on the local form of the dissipation kernel in the path integral approach. Some exceptions are consideration in limited applications such as \[5\], \[6\]. It has been pointed out that this often-used approximation of the local friction (in the path integral sense) has a fundamental flaw; violation of the positivity of the density matrix \[7\]. This has made it difficult to extract in a reliable way a fully quantum mechanical time evolution of the small system immersed in thermal environment at early times, especially when the temperature is low. Needless to say, the approximation of the local friction is excellent in the most important phase of time evolution, thus is useful in many practical applications. But as we shall see, there are physical effects that can be understood only with the full non-locality incorporated.

In the present work we extend previous works on the non-local dissipation \[4\] and give a quantum mechanical formulation of the system evolution in thermal environment. The present model itself is not new, and indeed, is a standard model of system-environment interaction discussed repeatedly in the quantum dissipation problem. A new method is developed by fully exploiting analyticity familiar in scattering theory, to treat the general class of environment models characterized by the spectral weight consistent with general principles, but otherwise taken arbitrary. We then apply this formalism to the two-body decay of unstable particle in field theory, by identifying as the environment oscillator a composite operator of two decaying particles. We may thus discuss details of the decay law and what is left behind after decay. It is the extended analyticity of a kernel (denoted later by \(F(z)\)) that allows us to treat the whole time region of the decay process in a unified way.
Another difference of our study from most of the past ones is our assumption of the presence of a gap in the spectrum of environment oscillators. Although this appears a minor technical point, it actually leads to differences at low temperatures and also for behaviors of physical quantities at very late times. The gapless case often appears in condensed matter physics, for instance, phonons in medium. But there are cases in which one cannot neglect the gap such as in the unstable particle decay if the mass of the daughter particle is finite.

Let the system variable in question be denoted by $q$ and the environment variable by $Q_a$. The Lagrangian of our problem consists of three parts:

$$L = L_q[q] + L_{\text{int}}[q,Q] + L_Q[Q].$$

We take for the system-environment interaction the bilinear term with infinitely many harmonic oscillators for the environment:

$$L_{\text{int}} = -q \sum_a c_a Q_a,$$

$$L_q = \frac{1}{2} (\dot{q}^2 - \omega_0^2 q^2),$$

$$L_Q = \frac{1}{2} \sum_a (\dot{Q}_a^2 - \omega_a^2 Q_a^2).$$

The renormalization of the frequency will be discussed later in appropriate places. Introduction of a counter term in relation to the renormalization as often discussed in the literature is actually the problem of how to relate the parameters of the theory such as $\omega_0$ to observable quantities. We shall discuss the relevant observable parameter $\bar{\omega}$ later.

The response of the environment to the system is characterized by what we call the response weight, denoted by $r(\omega)$: For the dynamical system of harmonic oscillator of frequency $\omega_0$ coupled to environment oscillators of discrete frequency $\omega_a$, it is

$$r(\omega) = \sum_a \frac{c_a^2}{2\omega_a} \delta(\omega - \omega_a) = \int_{\omega_c}^{\infty} d\omega' \frac{c^2(\omega')}{2\omega'} D(\omega'),$$

where in the continuum limit $D(\omega)$ is the density of states per unit frequency. $\omega_c$ is the threshold for $r(\omega) \neq 0$. Although we often write expressions in the discrete distribution of frequencies, we have to consider the continuum limit. It is often useful to extend $r(\omega)$ to the region $\omega < 0$ by $r(-\omega) = -r(\omega).$
It is very useful to introduce the influence functional $\mathcal{F}[q(\tau), q'(\tau)]$, following, but slightly modifying, the original definition of Feynman and Vernon [2]. The basic idea is that one is interested in the behavior of the $q-$system alone and traces out the environment variable altogether. This way one can compute the reduced density matrix for the system that contains all informations without knowing too much details of the environment. We define the influence functional by convoluting with the initial state of the environment unlike the original one. Once the influence functional is known, one may compute the transition probability and any physical quantities of the $q-$system by convoluting dynamics of the system under study. For instance, the transition probability is given, with introduction of the density matrix $\rho^{(R)}$, by

$$\int \mathcal{D}q(\tau) \int \mathcal{D}q'(\tau) \int dq_i \int dq'_i \int dq_f \int dq'_f$$

$$\psi^*(q_f) \psi^*(q'_f) \mathcal{F}[q(\tau), q'(\tau)] e^{iS[q] - iS[q']} \psi(q_i) \psi(q'_f)$$

$$\equiv \int dq_f \int dq'_f \psi^*(q_f) \psi(q'_f) \rho^{(R)}(q_f, q'_f), \quad (6)$$

where $\psi$'s are wave functions of the initial and the final $q-$states, and $S[q]$ is the action of the $q-$system.

The form of the influence functional is dictated by general principles such as probability conservation and causality. Feynman and Vernon found a closed quadratic form consistent with these,

$$\mathcal{F}[q(\tau), q'(\tau)] = \exp \left[ -\int_0^\tau d\tau \int_0^\tau ds \left( \xi(\tau)\alpha_R(\tau - s)\xi(s) + i \xi(\tau)\alpha_I(\tau - s)X(s) \right) \right], \quad (7)$$

with $\xi(\tau) = q(\tau) - q'(\tau)$, \quad $X(\tau) = q(\tau) + q'(\tau)$. \quad (8)

Thus two real functions $\alpha_i(\tau)$ are all we need to characterize the system-environment interaction. These are defined here in the range of $\tau \geq 0$. The fact that $\alpha_i$ depends on the difference of time variables, $\tau - s$, is due to the assumed stationarity of the environment.

The correlation kernels appear in the influence functional as a form of the nonlocal interaction and they are the dissipation $\alpha_I$ and the noise $\alpha_R$, related to the response weight by

$$\alpha_I(\tau) = -\frac{i}{2} \int_{-\infty}^{\infty} d\omega \frac{r(\omega)}{\omega} e^{-i\omega \tau}, \quad (9)$$

$$\alpha_R(\tau) = \frac{1}{2} \int_{-\infty}^{\infty} d\omega \coth \left( \frac{\beta\omega}{2} \right) \frac{r(\omega)}{\omega} e^{-i\omega \tau}. \quad (10)$$
In the path integral approach integration over the sum variable $X(\tau)$ is trivial for the harmonic oscillator system, since both the local part and the nonlocal action above are linear in this variable:

$$\frac{1}{2} \int_0^t \left( \dot{\xi}(\tau)\dot{X}(\tau) - \omega_0^2 \xi(\tau)X(\tau) \right) - \int_0^t d\tau \int_0^\tau ds \left( \xi(\tau)\alpha_R(\tau-s)\xi(s) - i\xi(\tau)\alpha_I(\tau-s)X(s) \right).$$

Thus result of the path integration of the system variable $X(\tau)$ gives the classical integro-differential equation that $\xi(\tau)$ must obey

$$\frac{d^2 \xi}{d\tau^2} + \omega_0^2 \xi(\tau) + 2 \int_\tau^t ds \xi(s) \alpha_I(s-\tau) = 0.$$  \hspace{1cm} (11)

The end result of the $\xi$ path integral then contains an integral of the form,

$$- \int_0^t d\tau \int_0^\tau ds \xi(\tau)\alpha_R(\tau-s)\xi(s),$$  \hspace{1cm} (12)

using the classical solution $\xi(\tau)$ with specified boundary conditions, $\xi(0) = \xi_i$, $\xi(t) = \xi_f$.

In the local approximation often used the dissipation kernel of the form is taken to have the form of

$$\alpha_I(\tau) = \delta \omega^2 \delta(\tau) + \eta \delta'(\tau),$$  \hspace{1cm} (13)

with $\delta \omega^2$ the frequency shift and the $\eta$ term representing the local friction. This choice enables one to solve the $\xi$ equation (11) by elementary means. On the other hand, the noise kernel is given by the response weight of the form,

$$r(\omega) = \frac{\eta}{\pi} \omega f\left(\frac{\omega}{\Omega}\right),$$  \hspace{1cm} (14)

with $f(x)$ some cutoff function and $\Omega$ a high frequency cutoff. At high temperatures this approximation reduces to the well known classical form of the fluctuation,

$$\alpha_R(\tau) = \frac{\eta T}{\pi} \delta(\tau).$$  \hspace{1cm} (15)

At low temperatures, however, the use of the cutoff function only in the noise kernel while retaining the local form of the dissipation makes validity of this approximation dubious. Our present approach does not make this local approximation, and instead uses exact solutions of the classical equation (11). This equation has been used in other approaches [5], [6], [4], too.
Two observations are crucial for our subsequent development: First, the integro-differential equation (11) is converted to the one having the Volterra type kernel with the variable change $\tau \rightarrow t - \tau$. It can then be solved by the Laplace transform, if the Laplace transform of $\alpha_I(\tau)$ denoted by $\tilde{\alpha}(p)$ has a simple form. Second, in thermal environment $\alpha(\tau) = \alpha_R(\tau) + i\alpha_I(\tau)$ is identified \[8\] to the real-time thermal Green’s function, hence its Fourier transform has the well known analyticity property. The origin of the analyticity is ultimately traced to the causality: any physical disturbance cannot propagate faster than the light.

Let us explain this in more detail. First, the Laplace transformed kernel of evolution operator is

$$\tilde{g}(p) = \frac{1}{p^2 + \omega_0^2 + 2\tilde{\alpha}(p)}, \quad (16)$$

with real $p > 0$.

$$\tilde{\alpha}(p) = \frac{1}{2} \int_{-\infty}^{\infty} d\omega \frac{r(\omega)}{ip - \omega}, \quad (17)$$

is equal to the boundary value of the analytic function $\pi \overline{G}(ip)$. This analytic function $\overline{G}(z)$ is related \[8\] to the imaginary-time thermal Green’s function \[9\] and is regular in the complex $z$ plane except on the real axis where it has the branch point singularity with the discontinuity given by $r(\omega)$. Our notational convention here is that $\omega$ and $p$ are real and the complex variable is denoted by $z$.

We next introduce the analytically extended function by

$$F(z) = \tilde{g}(-iz) = \pi \frac{1}{-z^2 + \omega_0^2 + 2\pi \overline{G}(z)}, \quad (18)$$

$$\overline{G}(z) = \int_{-\infty}^{\infty} d\omega \frac{r(\omega)}{2\pi} \frac{1}{z - \omega}. \quad (19)$$

It can be proved from the positivity of the response weight $r(\omega)$ supplemeneted by the spectrum condition later explained that $F^{-1}(z)$ has no complex zero. \[11\] Hence $F(z)$ is regular except on the real axis. Its discontinuity across the real axis

$$H(\omega) = (F(\omega + i\epsilon) - F(\omega - i\epsilon)) / (2\pi i) \quad (20)$$

is fundamental to our subsequent analysis. In terms of two real functions, $\Pi(\omega)$ and $\Gamma(\omega)$ defined by

$$2\pi \overline{G}(\omega + i\epsilon) = \Pi(\omega) - i\omega \Gamma(\omega), \quad (21)$$

$$H(\omega) = \frac{1}{\pi} \frac{\omega \Gamma(\omega)}{(\omega^2 - \omega_0^2 - \Pi(\omega))^2 + \omega^2 \Gamma^2(\omega)}. \quad (22)$$
Furthermore, \( r(\omega) = \omega \Gamma(\omega)/\pi \), and the real part \( \Pi(\omega) \) may be written by the well known dispersion integral:

\[
\Pi(\omega) = \mathcal{P} \int_{-\infty}^{\infty} d\omega' \frac{r(\omega')}{\omega - \omega'},
\]

with \( \mathcal{P} \) denoting the principal value of integration.

As in scattering theory [10], the analytic function \( F(z) \) may be extended to the second Riemann sheet through the discontinuity formula,

\[
F(\omega + i\epsilon) - F(\omega - i\epsilon) = i2\pi r(\omega) F(\omega + i\epsilon) F(\omega - i\epsilon).
\]

In the second lower sheet one finds a finite number of poles at \( z \) obeying

\[
z^2 - \omega_0^2 - 2\pi G(z) + 2\pi i r(z) = 0.
\]

Except at these poles \( F(z) \) is regular in the unphysical sheet.

Physical interpretation of these singularities is possible in terms of the spectrum of the entire harmonic system. Due to the mixing with the environment the system spectrum is modified and is given by an eigenvalue equation for \( \lambda = \omega^2 \). When the response weight \( r(\omega) \rightarrow \) a constant at \( \omega \rightarrow \infty \), as happens often and in field theory models later discussed in particular, one has to subtract a term from integrals containing the response weight. This corresponds to renormalization of the bare frequency \( \omega_0 \) with the frequency shift,

\[
\delta\omega^2 = -2 \int_{\omega_c}^{\infty} d\omega \frac{r(\omega)}{\omega}.
\]

The renormalized frequency is given by \( \omega_R^2 = \omega_0^2 + \delta\omega^2 \). The eigenvalue equation written using \( \omega_R^2 \) is then

\[
\lambda - \omega_R^2 - \lambda \int_{\omega_c}^{\infty} d\omega \frac{2r(\omega)}{\omega (\lambda - \omega^2)} = 0.
\]

The left hand side coincides with \(-F^{-1}(\sqrt{\lambda})\). The condition of stability of the entire system requires that the smallest eigenvalue \( \lambda > 0 \), giving \( \omega_R^2 > 0 \), or equivalently

\[
\omega_0^2 > 2 \int_{\omega_c}^{\infty} d\omega \frac{r(\omega)}{\omega},
\]

if the integral on the right hand side is convergent. This is the condition needed to ensure the analyticity of \( F(z) \) as described above.
We do not consider the possibility of an isolated pole of \( F(z) \) on the real axis outside the cut, which corresponds to a stable state, implying that in the problem raised here a prepared initial configuration of the system does not completely decay. We thus focus on physical systems that have a continuous spectrum alone when interaction between the small system and the large environment is considered.

The complex pole in the second Riemann sheet describes the behavior of the system oscillator interacting with environment oscillators: The real part of the pole position \( \bar{\omega}^2 \) gives a physical frequency squared including the frequency shift, and its imaginary part is related to the decay rate of the system oscillator. The pole mass \( \bar{\omega} \neq \omega_R \). Note that the pole position \( \bar{\omega} \) is, but \( \omega_R \) is not, an observable quantity.

We assume as a physical requirement that a single pole exists in the nearby second sheet, but it should be evident that an arbitrary number of poles can readily be incorporated.

Subsequent computation frequently uses the Laplace inverted evolution operator \( g(\tau) \) which can be written in terms of \( H(\omega) \):

\[
g(\tau) = 2 \int_{\omega_c}^{\infty} d\omega H(\omega) \sin(\omega \tau) .
\]  

Solution of the integro-differential equation (11) is then given by

\[
\xi(\tau) = \xi_i \frac{g(t-\tau)}{g(t)} + \xi_f \left( \dot{g}(t-\tau) - \frac{g(t-\tau)\dot{g}(t)}{g(t)} \right),
\]

with the dot denoting derivative. Both \( g(t) \) and \( \dot{g}(t) \) can be shown to satisfy the integro-differential equation of the form \( (x = g \text{ or } \dot{g}) \),

\[
\frac{d^2 x}{dt^2} + \omega^2_0 x + 2 \int_0^t d\tau \alpha_I(t-\tau) x(\tau) = 0.
\]  

The reduced density matrix of the quantum system at any time is obtained from the action written in terms of the boundary values, \( S_{cl}(\xi_f, X_f; \xi_i, X_i) \), by convoluting with the initial density matrix. This action is computed as

\[
i S_{cl} = - \frac{U}{2} \xi_f^2 - \frac{V}{2} \xi_i^2 - W \xi_i \xi_f + \frac{i}{2} X_f \dot{\xi}_f - \frac{i}{2} X_i \dot{\xi}_i,
\]

\[
U = 2 \int_0^t d\tau \int_0^\tau ds z(\tau) \alpha_R(\tau-s) z(s),
\]

\[
V = 2 \int_0^t d\tau \int_0^\tau ds y(\tau) \alpha_R(\tau-s) y(s),
\]

\[
W = \int_0^t d\tau \int_0^\tau ds \left( y(\tau)z(s) + y(s)z(\tau) \right) \alpha_R(\tau-s),
\]
\[ y(\tau) = \frac{g(t - \tau)}{g(t)}, \] (35)
\[ z(\tau) = \dot{g}(t - \tau) - g(t - \tau) \frac{\dot{g}(t)}{g(t)}, \] (36)
\[ \dot{\xi}(\tau) = -\xi_i \frac{\dot{g}(t - \tau)}{g(t)} - \xi_f \left( \frac{\ddot{g}(t - \tau) \dot{g}(t)}{g(t)} \right). \] (37)

The same effective action as ours has been derived by Grabert et al., as summarized in [4]. Our derivation here simplifies their calculation, with a new form of the reduced density matrix, Eq. (38) that follows, by fully exploiting the extended analyticity. Our method also makes possible a unified treatment of the exponential and the power law decay, as will be made clear later.

We take as the initial state a product of thermal states, a system of temperature \( T_0 = 1/\beta_0 \) and an environment of temperature \( T = 1/\beta \). We may take \( T_0 = T \) when we apply to the decay process of excited level initially in thermal equilibrium. In the limit of \( T_0 \to 0 \) it describes the ground state of the system harmonic oscillator. (We have recently computed the reduced density matrix, starting from another pure state, the first excited level of the system oscillator. As expected, the late time behavior in this case is the same as in the present case.)

After a series of straightforward Gaussian integration we find the reduced density matrix of the form,

\[ \rho^{(R)}(X_f, \xi_f) = 2\sqrt{A} \exp\left[-AX_f^2 - B\xi_f^2 + iC X_f \xi_f \right], \] (38)
\[ A = \frac{1}{8I_1}, \quad B = \frac{1}{2} \left( I_3 - \frac{I_2^2}{I_1} \right), \quad C = \frac{I_2}{2I_1}, \] (39)
\[ I_1 = \mathcal{I}[|h(\omega, t)|^2] + \frac{1}{2\omega_0} \coth\left(\frac{\beta_0\omega_0}{2}\right)(\ddot{g}^2 + \omega_0^2 g^2), \]
\[ I_2 = \Re\mathcal{I}[h(\omega, t)k^*(\omega, t)] + \frac{1}{2\omega_0} \coth\left(\frac{\beta_0\omega_0}{2}\right)\dot{\dot{g}}(\ddot{g} + \omega_0^2 g), \]
\[ I_3 = \mathcal{I}[|k(\omega, t)|^2] + \frac{1}{2\omega_0} \coth\left(\frac{\beta_0\omega_0}{2}\right)(\ddot{g}^2 + \omega_0^2 g^2). \] (40)

\( \omega_0 \) is a reference frequency taken as the initial system state, and equated here to the initial system frequency. If one so desires, either the renormalized \( \omega_R \) or the pole \( \bar{\omega} \) may be taken as another choice. But we imagine the situation a small system was added to a large environment at some time, its mutual interaction being absent prior to the initial time. In this circumstance it is appropriate to take \( \omega_0 \) as the reference frequency. As will be made clear later, dependence on the initial state dies away.
quickly as time passes. In writing the reduced density matrix, we introduced the following notations:

\[ I[f(\omega)] \equiv \int_{\omega_c}^{\infty} d\omega \ \coth(\frac{\beta \omega}{2}) r(\omega) f(\omega), \]  

(41)

and

\[ h(\omega, t) \equiv \int_0^t d\tau g(\tau)e^{-i\omega\tau}, \]  

(42)

\[ k(\omega, t) \equiv \int_0^t d\tau \dot{g}(\tau)e^{-i\omega\tau} = g(t)e^{-i\omega t} + i\omega h(\omega, t). \]  

(43)

The density matrix \( \rho^{(R)} \) from which any physical quantity can be computed at any time has explicitly been given by the discontinuity, \( H(\omega) \) or \( r(\omega) \).

This density matrix \( \rho^{(R)} \) is positive definite and behaves acceptably at early times unlike the one in the local approximation, as will be explained elsewhere \([12]\). Our interest here is limited to the late time behavior.

It is important to know the behavior of the kernel \( g(\tau) \), which is found by deforming the contour of \( \omega \) integration into the sum of the pole contribution (at \( z = z_0 \) with \( \Im z_0 < 0 \)) in the second sheet and the contribution parallel to the imaginary axis passing through \( z = \omega_c \), both in the first (I) and in the second (II) sheet \([13]\):

\[ g(\tau) = \Im \left( Ke^{-i\Re z_0 \tau} \right) e^{\Im z_0 \tau} \]
\[ + \ \Im \left[ \frac{e^{i\omega_c \tau}}{\pi} \int_0^\infty dy e^{-yt} (F_I(\omega_c + iy) - F_{II}(\omega_c + iy)) \right], \]  

(44)

with \( K^{-1} = z_0 - \pi \mathcal{G}'(z_0) + i\pi r'(z_0) \).

As seen from this formula, the pole contribution given by the first term describes the exponential decay, which usually lasts during the most dominant phase of the decay period, while the rest of contribution gives the power law decay at very late times; \( \propto t^{-\alpha - 1} \), and well-behaved early time behavior. The power \( -\alpha - 1 \) is dictated by the threshold behavior of the response weight, \( r(\omega) \propto (\omega - \omega_c^\alpha) \). \( \alpha = 1/2 \) for the S-wave two-body decay of unstable particle \([13]\). \( \alpha > -1 \) is required as a consistency of this approach; convergence of the \( \omega \) integration.

It is often claimed that the power law behavior is never observable, since by the time this term dominates the exponential decay essentially eliminates the initial population. This is perhaps so when the environment is at zero temperature. In order to check observability of the power law decay in medium, we shall estimate the
transition time $t_*$ from the exponential to the power law period at low, but finite temperatures. Let us examine a typical case by taking the form of the response weight, $r(\omega) = c (\omega - \omega_c)^{\alpha}$, with $0 < \alpha < 1$ in the range of $\omega_c < \omega < \Omega$ ($\Omega \gg \omega_c$) and with $\bar{\omega} \gg \text{Max} (\omega_c, T)$. The behavior of $g(t)$ in the power law period is

$$g(t) \approx \frac{2c}{\bar{\omega}^4} \Gamma(\alpha + 1) \frac{\cos(\omega_c t + \frac{\pi}{2} \alpha)}{t^{\alpha + 1}}.$$

By equating this to the expression for the same quantity $g(t)$ in the exponential period, one obtains

$$t_* \approx \frac{1}{\gamma} \ln \left( \frac{\bar{\omega}^3}{2c \Gamma(\alpha + 1) \gamma^{\alpha + 1}} \right),$$

with $\gamma = -\Im z_0$ the decay rate. In the weak coupling limit the factor inside the logarithm is large ($\propto c^{2\alpha - 3}$), and by the time $t_*$ the initial population has decreased like

$$e^{-2\gamma t_*} \propto c^{4\alpha + 6}.$$

At non-zero temperatures the response weight $r(\omega)$ may depend on the environment temperature, as will be made more explicit in our application to the unstable particle decay, for instance in eq. (56). Temperature dependence of the parameters in these formulas is thus needed to check observability of the power law decay at finite temperatures.

Emergence of the power law term in the quantum Brownian motion has been noted in some specific models of Ohmic type, $[15], [16], [17]$, but we find this behavior as a general property in the presence of the non-local dissipation. For instance, the power law behavior of the two point correlation function was noted in ref. $[15], [16]$, for specific models. We have instead demonstrated the power law behavior for the kernel function $g(t)$ by separating the non-pole contribution that is essential to the power law behavior. Hence we have shown this property in more generality. We would like to stress that the presence of the branch cut singularity of the function $F^{-1}(z)$ in the $z$ plane is very important to derive the power law decay. This cut is further related to the non-analytic property of the real function $r(\omega)$ at the threshold, including the special case of the gapless $\omega_c = 0$. For instance, if one takes the Ohmic form of the response weight given by pole terms alone, such as

$$r^{(4)}(\omega) = \frac{4c\Omega\gamma\omega}{(\omega^2 - \Omega^2 + \frac{\gamma^2}{4})^2 + \Omega^2\gamma^2},$$
then one does not obtain the power law decay for \( g(t) \). For the gapless case of a fractional power \( \alpha \) the formula (45) is still valid, but the kernel \( g(t) \) does not exhibit the power law behavior for an odd integer \( \alpha \), as seen from (45) with \( \omega_c = 0 \). In this case one can define a regular function \( r(\omega) \) including \( \omega = 0 \), thus the gapless case of odd integer \( \alpha \) is exceptional in the sense that only this case does not produce the power law behavior. The critical condition for the absence of the power law behavior of \( g(t) \) is then regularity of \( r(\omega) \) at the threshold.

The asymptotic late time behavior of the reduced density matrix is determined by \( h(\omega, \infty) = \int_{0}^{\infty} d\tau \, g(\tau)e^{-i\omega \tau} \), which is shown to be equal to the boundary value, \( F(\omega - i\epsilon) \), thus giving

\[
\begin{align*}
    r(\omega)|h(\omega, \infty)|^2 &= H(\omega), \\
    r(\omega)|k(\omega, \infty)|^2 &= \omega^2 H(\omega).
\end{align*}
\]

(49) \hspace{1cm} (50)

In these computations the analytic structure and only that is important. Thus, at asymptotic late times

\[
\begin{align*}
    A &\to \frac{1}{8} \left( \int_{\omega_c}^{\infty} d\omega \coth\left(\frac{\beta \omega}{2}\right) H(\omega) \right)^{-1}, \\
    B &\to \frac{1}{2} \int_{\omega_c}^{\infty} d\omega \coth\left(\frac{\beta \omega}{2}\right) \omega^2 H(\omega), \\
    C &\to 0.
\end{align*}
\]

(51) \hspace{1cm} (52) \hspace{1cm} (53)

Note that the dependence on the initial state of the system via the \( \beta_0 = 1/T_0 \) factor disappears, hence no memory effect of the initial state remains in the asymptotic final state.

In the high temperature limit

\[
\begin{align*}
    A &\sim \frac{\omega_R^2}{8T}, \quad B \sim \frac{T}{2}.
\end{align*}
\]

(54)

This result is almost equal to the pole contribution, ignoring a minor difference between \( \omega_R \) and \( \tilde{\omega} \). Contribution during the period of the power law decay is given by the continuous integral and is numerically subdominant, suppressed by the factor \( 1/T^2 \) relative to the one from the exponential period.

On the other hand, at low temperatures the contribution from the threshold region, \( \omega \approx \omega_c \), cannot be ignored, giving the dominant contribution to the power law period.
Let us apply these considerations to the decay of unstable particle; \( \varphi \rightarrow \chi + \chi \). We assume that the decay product \( \chi \) is a part of thermal components that make up the environment. The parent particle \( \varphi \) may or may not be in thermal equilibrium with the rest of medium. Since we focus on the late time behavior, the initial state dependence disappears. Unlike the unstable particle decay in vacuum some amount of parent particles are expected to be left behind, even much later than the decay lifetime, because in thermal environment even heavier parent particles can be created by energetic particles of smaller mass. If so, what is the fraction of the parent particle left behind?

The interaction of the system field \( \varphi \) and the environment field \( \chi \) is assumed to be described by a relativistic field theory of Yukawa interaction Lagrangian density, \( \frac{1}{2} \mu \varphi \chi^2 \), with \( \mu \) a coupling constant of mass dimension. We presume that interaction among created \( \chi \) particles themselves is weak enough. In this case it is possible to identify as the environment variable the two-body bilinear operator of \( \chi \),

\[
\sum_a c_a Q_a = \frac{\mu}{2} \int d^3x \chi^2(x) e^{-i\vec{k} \cdot \vec{x}},
\]

with \( \vec{k} \) the momentum of \( \varphi \) particle. Due to the assumed homogeneity each \( \vec{k} \) mode can be treated independently. The continuous label \( \omega \) in the environment variable \( Q(\omega) \) is identified here to the internal configuration of two body states with a given total momentum \( \vec{k} \). This field theory model was introduced in ref [8].

The relevant response weight for \( \varphi \rightarrow \chi + \chi \) has been given in ref [8]. We shall recapitulate the main point of that calculation. The important point is that to lowest non-trivial order of the coupling \( \mu \), \( r(\omega) \) is given by the imaginary part of the one-loop diagram of the self-energy of \( \varphi \) at finite temperatures, \( \varphi \) being off the mass shell: \( \omega^2 - \vec{k}^2 \neq \varphi \) mass\(^2 \). The result for the \( \chi \) loop diagram at finite temperatures is well known [14], [8], and physically consists of two parts; the process \( \varphi \leftrightarrow \chi + \chi \) in the region of \( \omega > \sqrt{k^2 + 4m^2} \) and the other process \( \varphi + \chi \leftrightarrow \chi \) (forbidden when all particles are on the mass shell, but allowed in thermal environment) in \( 0 < \omega < k \), where \( m \) is the daughter \( \chi \) mass. The response weight \( r(\omega) \) does not vanish for \( |\omega| > \sqrt{k^2 + 4m^2} \) and \( |\omega| < |\vec{k}| \) from the kinematics of the decay and the inverse decay of particles off the mass shell, with the constraint of the momentum conservation. Thus a gap of the spectrum exists in \( k < |\omega| < \sqrt{k^2 + 4m^2} \). The finite non-vanishing mass of the daughter particle \( (m \neq 0) \) is important for the existence of the gap and for associated physical consequences that follow.
For $\omega \geq \sqrt{k^2 + 4m^2}$ the response weight is

$$
r(\omega) = \frac{\mu^2}{32\pi^2} \left( \sqrt{1 - \frac{4m^2}{\omega^2 - k^2}} + \frac{2}{k\beta} \ln \frac{1 - e^{-\beta \omega}}{1 - e^{-\beta |\omega|}} \right),
$$

(56)

with

$$
\omega_{\pm} = \frac{\omega}{2} \pm \frac{k}{2} \sqrt{1 - 4m^2/(\omega^2 - k^2)}.
$$

(57)

For $0 < \omega < k$ only the second term in the bracket of Eq.(56) contributes.

A useful, and adequate approximation we exploit for subsequent estimate is the weak coupling scheme with correct threshold and asymptotic behaviors incorporated:

$$
F(z) = \frac{1}{-z^2 + \tilde{\omega}^2 - i\pi r(z)}.
$$

(58)

In this approximation we replaced the real part $\Pi(\omega)$ by the constant pole location $\tilde{\omega}$.

A quantity of physical interest is the fraction of remaining particles given by the occupation number at asymptotic late times,

$$
n_k = \frac{1}{2} \left( \frac{p_k^2}{\omega_k} + \omega_k q_k^2 \right) \approx \frac{B_k}{\omega_k} + \frac{\omega_k}{16A_k} - \frac{1}{2},
$$

(59)

for each $\vec{k}$ mode ($\omega_k = \sqrt{\vec{k}^2 + M^2}$). The temperature dependent part of this quantity is

$$
n_k^\beta = \int_0^\infty d\omega \frac{1}{e^{\beta \omega} - 1} (\omega_k + \omega^2) H(\omega, k),
$$

(60)

with $\omega_k$ the real part of the pole position. One must sum over $\vec{k}$ to obtain the number density of remnants.

We shall limit our discussion here to the decay that occurs when the parent $\varphi$ becomes non-relativistic,

$$
\omega_k \sim M + \frac{\vec{k}^2}{2M} \gg T.
$$

(61)

This condition is relevant in interesting cosmological problems of the neutron decay at the time of nucleosynthesis and GUT $X$ boson decay at baryogenesis [18].

Computation of the temperature dependent part of the occupation number $n_k^\beta$ may proceed by deforming the contour of $\omega$ integration, in the same way as in the discussion of $g(\tau)$ above. There are then two types of contribution: One is the
pole term that gives the usual Boltzmann suppressed contribution of $e^{-\beta\omega_k}$. When mode-summed, it gives the number density,

$$
\int \frac{d^3k}{(2\pi)^3} e^{-\beta(M+k^2/2M)} = \left( \frac{MT}{2\pi} \right)^{3/2} e^{-M/T}.
$$

(62)

This is the familiar Boltzmann suppressed formula.

The second one is contribution from the continuous complex path that gives the power law behavior of temperature dependence. A part of this contribution in the region $\omega > \sqrt{k^2 + 4m^2}$ is analytically calculable by using $r(\omega) = \frac{\mu^2}{16\pi} + O[m^2]$, valid for a small daughter mass $m$. It is

$$
n \approx \frac{\mu^2}{64\pi^4M^3} \int_0^\infty dk \frac{k^2}{\int_k^\infty d\omega \frac{1}{e^{\omega/T}-1}}
= \frac{1}{2880} \frac{\mu^2T^4}{M^3}.
$$

(63)

This calculation however ignores complicated logarithmic factors in $r(\omega)$.

We numerically computed all terms including the logarithmic factor in $r(\omega)$ along with $O[m^2]$ corrections. It turns out that the total contribution is ten times larger than the analytic result above; in the $m \to 0$ limit,

$$
n \approx 3.8 \times 10^{-3} \frac{\mu^2 T^4}{M^3}.
$$

(64)

The main part of this large contribution comes from $|\omega| < k$. With a dimensionless constant introduced by $\mu = gM$, this gives, relative to the photon number density,

$$
\frac{n}{T^3} \approx 4 \times 10^{-13} \left( \frac{g}{G_F m_N^2} \right)^2 \frac{T}{M}.
$$

(65)

We wrote here the numerical value with $G_F$ the weak interaction constant of mass dimensions $-2$ ( $G_F m_N^2 \approx 10^{-5}$ ), as if it were relevant to the neutron decay.

One may estimate the equal time temperature $T_{eq}$ at which the power contribution becomes equal to the Boltzmann suppressed number density, to give

$$
\frac{T_{eq}}{M} \approx \frac{1}{35}, \quad \frac{n}{T_{eq}^3} \approx 1 \times 10^{-14},
$$

(66)

taking as an example $\mu = 10^{-5} M$, the weak interaction strength. This number is in an interesting range to affect nucleosynthesis, but we should keep in mind that we did not work out the relevant three body decay, $n \to p + e + \bar{\nu}_e$. 

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The physical interpretation of the pole term is a conventional one in terms of the remnant created by the inverse decay $\chi + \chi \rightarrow \varphi$, with all relevant particles on the mass shell, hence suppressed kinematically by the Boltzmann factor $e^{-M/T}$. On the other hand, the contribution from the continuous contour can only be interpreted as remnant particles far off the mass shell that may exist in thermal equilibrium. The usual kinetic approach such as the Boltzmann-like equation is based on the rates computed from S-matrix elements on the mass shell and gives the Boltzmann suppressed abundance in equilibrium for $M \gg T$. Our fully quantum mechanical approach yields a different result.

We note that the local friction approximation is equivalent to the pole model (with identification of $\omega_0^2 + \delta \omega^2 = (\mathcal{R}z_0)^2$, $\eta = -2 \mathfrak{I}z_0$) that ignores the continuum integral around the threshold, hence the model fails to describe the off-shell remnant.

The effect of the off shell remnants seems to play important roles at least in two places: in nuclear matter and in the early universe. We shall refer to [19] for some recent attempts to estimate medium effects in nucleus.

We shall mention another application of immediate interest in cosmology; the heavy $X$ boson decay for GUT baryogenesis. It has been argued [18] that there exists a severe mass bound of order, $m_X > O[\alpha_X m_{pl}] \approx 10^{16}$ GeV, to block the inverse process of the $X$ boson decay so that generation of the baryon asymmetry proceeds with sufficient abundance of the parent $X$ particles. The usual estimate of the mass bound mentioned above however is based on the on-shell Boltzmann equation. More appropriate formula in this estimate is our remnant number density,

$$n_X \approx O[4 \times 10^{-3}] g_X^2 \frac{T^4}{m_X}.$$  \hspace{1cm} (68)

(In a more realistic estimate one should consider the $X$ boson decay into quarks and leptons. But for an order of magnitude estimate difference in statistics is not important.) With the GUT coupling of $g_X^2/4\pi = 1/40$, the equal temperature is roughly

$$T_{eq} \approx \frac{M}{10}, \quad \frac{n}{T_{eq}^3} \approx 1 \times 10^{-4}.$$  \hspace{1cm} (69)

Thus, at temperature of about a tenth of the $X$ mass the Boltzmann suppressed formula is replaced by the power formula. The kinematical condition for baryogenesis must be reconsidered in view of our off-shell formula.
In summary, we extended previous works on the theory of quantum dissipation in the linear harmonic environment, with emphasis on exact treatment using the reduced density matrix. When applied to the quantum system that decays via interaction with the environment, the power law decay is a generic feature towards the asymptotic late time limit. The remnant fraction in the asymptotic limit has contribution from particles off the mass shell that does not suffer from the Boltzmann suppression factor.

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