WELL-POSEDNESS AND SCATTERING FOR NONLINEAR SCHRÖDINGER EQUATIONS WITH A DERIVATIVE NONLINEARITY AT THE SCALING CRITICAL REGULARITY

HIROYUKI HIRAYAMA
Graduate School of Mathematics, Nagoya University
Chikusa-ku, Nagoya, 464-8602, Japan

Abstract. In the present paper, we consider the Cauchy problem of nonlinear Schrödinger equations with a derivative nonlinearity which depends only on \( u \). The well-posedness of the equation at the scaling subcritical regularity was proved by A. Grünrock (2000). We prove the well-posedness of the equation and the scattering for the solution at the scaling critical regularity by using \( U^2 \) space and \( V^2 \) space which are applied to prove the well-posedness and the scattering for KP-II equation at the scaling critical regularity by Hadac, Herr and Koch (2009).

1. Introduction

We consider the Cauchy problem of the nonlinear Schrödinger equations:

\[
\begin{cases}
(i\partial_t + \Delta)u = \partial_k (u^m), & (t, x) \in (0, \infty) \times \mathbb{R}^d \\
u(0, x) = u_0(x), & x \in \mathbb{R}^d
\end{cases}
\]

(1.1)

where \( m \in \mathbb{N}, \ m \geq 2, \ 1 \leq k \leq d, \ \partial_k = \partial/\partial x_k \) and the unknown function \( u \) is \( \mathbb{C} \)-valued. (1.1) is invariant under the following scaling transformation:

\[ u_\lambda(t, x) = \lambda^{-1/(m-1)}u(\lambda^{-2}t, \lambda^{-1}x), \]

and the scaling critical regularity is \( s_c = d/2 - 1/(m-1) \). The aim of this paper is to prove the well-posedness and the scattering for the solution of (1.1) in the scaling critical Sobolev space.

First, we introduce some known results for related problems. The nonlinear term in (1.1) contains a derivative. A derivative loss arising from the nonlinearity makes
the problem difficult. In fact, Mizohata ([19]) proved that a necessary condition for
the $L^2$ well-posedness of the problem:
\[
\begin{align*}
  &i\partial_t u - \Delta u = b_1(x) \cdot \nabla u, \ t \in \mathbb{R}, \ x \in \mathbb{R}^d, \\
  &u(0, x) = u_0(x), \ x \in \mathbb{R}^d
\end{align*}
\]
is the uniform bound
\[
\sup_{x \in \mathbb{R}^n, \omega \in S^{n-1}, R > 0} \left| \Re \int_0^R b_1(x + r\omega) \cdot \omega dr \right| < \infty.
\]
Furthermore, Christ ([6]) proved that the flow map of the Cauchy problem:
\[
\begin{align*}
  &i\partial_t u - \partial_x^2 u = u\partial_x u, \ t \in \mathbb{R}, \ x \in \mathbb{R}, \\
  &u(0, x) = u_0(x), \ x \in \mathbb{R}
\end{align*}
\]
is not continuous on $H^s$ for any $s \in \mathbb{R}$. While, Ozawa ([20]) proved that the local
well-posedness of (1.2) in the space of all function $\phi \in H^1$ satisfying the bounded
condition
\[
\sup_{x \in \mathbb{R}} \int_{-\infty}^{x} \phi \ < \infty.
\]
Furthermore, he proved that if the initial data $\phi$ satisfies some condition, then the
local solution can be extend globally in time and the solution scatters. For the
Cauchy problem of the one dimensional derivative Schrödinger equation:
\[
\begin{align*}
  &i\partial_t u + \partial_x^2 u = i\lambda \partial_x (|u|^2 u), \ t \in \mathbb{R}, \ x \in \mathbb{R}, \\
  &u(0, x) = u_0(x), \ x \in \mathbb{R}
\end{align*}
\]
Takaoka ([22]) proved the local well-posedness in $H^s$ for $s \geq 1/2$ by using the
gauge transform. This result was extended to global well-posedness ([8], [9], [18],
[23]). While, ill-posedness of (1.3) was obtained for $s < 1/2$ ([1], [23]). Hao ([13])
considered the Cauchy problem:
\[
\begin{align*}
  &i\partial_t u - \partial_x^2 u + i\lambda |u|^k \partial_x u, \ t \in \mathbb{R}, \ x \in \mathbb{R}, \\
  &u(0, x) = u_0(x), \ x \in \mathbb{R}
\end{align*}
\]
for $k \geq 5$ and obtained local well-posedness in $H^{1/2}$. For more general problem:
\[
\begin{align*}
  &i\partial_t u - \Delta u = P(u, \nabla u, \nabla^2 u), \ t \in \mathbb{R}, \ x \in \mathbb{R}^d, \\
  &u(0, x) = u_0(x), \ x \in \mathbb{R}^d
\end{align*}
\]
$P$ is a polynomial which has no constant and linear terms,
there are many positive results for the well-posedness in the weighted Sobolev space
([2], [3], [4], [5], [16], [21]). Kenig, Ponce and Vega ([16]) also obtained that (1.5)
is locally well-posed in $H^s$ (without weight) for large enough $s$ when $P$ has no quadratic terms.

The Benjamin–Ono equation:

$$\partial_t u + H\partial^2_x u = u\partial_x u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}$$

(1.6)
is also related to the quadratic derivative nonlinear Schrödinger equation. It is known that the flow map of (1.6) is not uniformly continuous on $H^s$ for $s > 0$ ([17]). But the Benjamin–Ono equation has better structure than the equation (1.2). Actually, Tao ([24]) proved that (1.6) is globally well-posed in $H^1$ by using the gauge transform. Furthermore, Ionescu and Kenig ([15]) proved that (1.6) is globally well-posed in $H^{s_r}$ for $s \geq 0$, where $H^{s_r}$ is the Banach space of the all real valued function $f \in H^s$.

Next, we introduce some known results for (1.1). Grünrock ([10]) proved that (1.1) is locally well-posed in $L^2$ when $d = 1$, $m = 2$ and in $H^s$ for $s > s_c$ when $d \geq 1$, $m + d \geq 4$. Recently, the author ([14]) proved that (1.1) with $d \geq 2$, $m = 2$ is globally well-posed for small data in $H^{s_c}$ (also in $\dot{H}^{s_c}$) and the solution scatters.

The results are an extension of the results by Grünrock ([10]) for $d \geq 2$, $m = 2$. The main results in this paper are an extension of the results by Grünrock ([10]) for $d \geq 1$, $m \geq 3$.

Now, we give the main results in the present paper. For a Banach space $H$ and $r > 0$, we define $B_r(H) := \{f \in H \mid \|f\|_H \leq r\}$.

**Theorem 1.1.** Assume $d \geq 1$, $m \geq 3$.

(i) The equation (1.1) is globally well-posed for small data in $\dot{H}^{s_c}$. More precisely, there exists $r > 0$ such that for all initial data $u_0 \in B_r(\dot{H}^{s_c})$, there exists a solution

$$u \in \dot{Z}^{s_c}_r([0, \infty)) \subset C([0, \infty); \dot{H}^{s_c})$$

of (1.1) on $(0, \infty)$. Such solution is unique in $\dot{Z}^{s_c}_r([0, \infty))$ which is a closed subset of $\dot{Z}^{s_c}([0, \infty))$ (see Definition 1.4 and (4.3)). Moreover, the flow map

$$S_+ : B_r(\dot{H}^{s_c}) \ni u_0 \mapsto u \in \dot{Z}^{s_c}_r([0, \infty))$$

is Lipschitz continuous.

(ii) The statement in (i) remains valid if we replace the space $\dot{H}^{s_c}$, $\dot{Z}^{s_c}([0, \infty))$ and $\dot{Z}^{s_c}_r([0, \infty))$ by $H^s$, $Z^s([0, \infty))$ and $Z^s_r([0, \infty))$ for $s \geq s_c$.

**Remark 1.1.** Due to the time reversibility of the system (1.1), the above theorems also hold in corresponding intervals $(-\infty, 0)$. We denote the flow map with $t \in (-\infty, 0)$ by $S_-$. 
Corollary 1.2. Assume $d \geq 1$, $m \geq 3$.

(i) Let $r > 0$ be as in Theorem 1.1. For every $u_0 \in B_r(\dot{H}^{s_c})$, there exists $u_\pm \in \dot{H}^{s_c}$ such that
\[ S_\pm(u_0) - e^{it\Delta}u_\pm \to 0 \text{ in } \dot{H}^{s_c} \text{ as } t \to \pm \infty. \]

(ii) The statement in (i) remains valid if we replace the space $\dot{H}^{s_c}$ by $H^s$ for $s \geq s_c$.

The main tools of our results are $U_p$ space and $V_p$ space which are applied to prove the well-posedness and the scattering for KP-II equation at the scaling critical regularity by Hadac, Herr and Koch ([11], [12]).

Notation. We denote the spatial Fourier transform by $\hat{\cdot}$ or $\mathcal{F}_x$, the Fourier transform in time by $\mathcal{F}_t$ and the Fourier transform in all variables by $\hat{\cdot}$ or $\mathcal{F}_{tx}$. The free evolution $e^{it\Delta}$ on $L^2$ is given as a Fourier multiplier
\[ \mathcal{F}_x[e^{it\Delta}f](\xi) = e^{-it|\xi|^2} \hat{f}(\xi). \]

We will use $A \lesssim B$ to denote an estimate of the form $A \leq CB$ for some constant $C$ and write $A \sim B$ to mean $A \lesssim B$ and $B \lesssim A$. We will use the convention that capital letters denote dyadic numbers, e.g. $N = 2^n$ for $n \in \mathbb{Z}$ and for a dyadic summation we write $\sum_N a_N := \sum_{n \in \mathbb{Z}} a_{2^n}$ and $\sum_{N \geq M} a_N := \sum_{n \in \mathbb{Z}, 2^n \geq M} a_{2^n}$ for brevity. Let $\chi \in C_0^\infty((-2, 2))$ be an even, non-negative function such that $\chi(t) = 1$ for $|t| \leq 1$. We define $\psi(t) := \chi(t) - \chi(2t)$ and $\psi_N(t) := \psi(N^{-1}t)$. Then, $\sum_N \psi_N(t) = 1$ whenever $t \neq 0$. We define frequency and modulation projections
\[ \widehat{P}_N u(\xi) := \psi_N(\xi)\hat{u}(\xi), \quad \widehat{Q}_M^{\Delta} u(\tau, \xi) := \psi_M(\tau + |\xi|^2)\hat{u}(\tau, \xi). \]
Furthermore, we define $Q_{\geq M} := \sum_{N \geq M} Q_N^\Delta$ and $Q_{< M} := I_d - Q_{\geq M}$.

The rest of this paper is planned as follows. In Section 2, we will give the definition and properties of the $U_p$ space and $V_p$ space. In Sections 3, we will give the multilinear estimates which are main estimates in this paper. In Section 4, we will give the proof of the well-posedness and the scattering (Theorems 1.1 and Corollary 1.2).

2. $U_p$, $V_p$ spaces and their properties

In this section, we define the $U_p$ space and the $V_p$ space, and introduce the properties of these spaces which are proved by Hadac, Herr and Koch ([11], [12]).

We define the set of finite partitions $\mathcal{Z}$ as
\[ \mathcal{Z} := \{ \{t_k\}_{k=0}^K | K \in \mathbb{N}, -\infty < t_0 < t_1 < \cdots < t_K \leq \infty \} \]
and if $t_K = \infty$, we put $v(t_K) := 0$ for all functions $v : \mathbb{R} \to L^2$.

**Definition 1.** Let $1 \leq p < \infty$. For $\{t_k\}_{k=0}^K \in \mathbb{Z}$ and $\{\phi_k\}_{k=0}^{K-1} \subset L^2$ with $\sum_{k=0}^{K-1} ||\phi_k||_{L^2}^p = 1$ we call the function $a : \mathbb{R} \to L^2$ given by

$$a(t) = \sum_{k=1}^{K} 1_{[t_{k-1},t_k)}(t)\phi_{k-1}$$

a “$U^p$-atom”. Furthermore, we define the atomic space

$$U^p := \left\{ u = \sum_{j=1}^{\infty} \lambda_j a_j \middle| a_j : U^p\text{-atom}, \lambda_j \in \mathbb{C} \text{ such that } \sum_{j=1}^{\infty} |\lambda_j| < \infty \right\}$$

with the norm

$$||u||_{U^p} := \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| \middle| u = \sum_{j=1}^{\infty} \lambda_j a_j, a_j : U^p\text{-atom}, \lambda_j \in \mathbb{C} \right\}.$$  

**Definition 2.** Let $1 \leq p < \infty$. We define the space of the bounded $p$-variation

$$V^p := \{ v : \mathbb{R} \to L^2 | ||v||_{V^p} < \infty \}$$

with the norm

$$||v||_{V^p} := \sup_{\{t_k\}_{k=0}^{K} \in \mathbb{Z}} \left( \sum_{k=1}^{K} ||v(t_k) - v(t_{k-1})||_{L^2}^p \right)^{1/p}.$$  

Likewise, let $V_{p, rc}^p$ denote the closed subspace of all right-continuous functions $v \in V^p$ with $\lim_{t \to -\infty} v(t) = 0$, endowed with the same norm $|| \cdot ||_{V^p}$.

**Proposition 2.1** ([11] Proposition 2.2, 2.4, Corollary 2.6). Let $1 \leq p < q < \infty$.

(i) $U^p$, $V^p$ and $V_{p, rc}^p$ are Banach spaces.

(ii) For every $v \in V^p$, $\lim_{t \to -\infty} v(t)$ and $\lim_{t \to \infty} v(t)$ exist in $L^2$.

(iii) The embeddings $U^p \hookrightarrow V_{p, rc}^p \hookrightarrow U^q \hookrightarrow L^\infty_t(\mathbb{R}; L^2_\mathbb{R}^d)$ are continuous.

**Theorem 2.2** ([11] Proposition 2.10, Remark 2.12). Let $1 < p < \infty$ and $1/p + 1/p' = 1$. If $u \in V_{p, rc}^1$ be absolutely continuous on every compact intervals, then

$$||u||_{U^p} = \sup_{v \in V^p, ||v||_{V^p} = 1} \left| \int_{-\infty}^{\infty} (u'(t), v(t))_{L^2_\mathbb{R}^d} dt \right|.$$  

**Definition 3.** Let $1 \leq p < \infty$. We define

$$U^p_\Delta := \{ u : \mathbb{R} \to L^2 | e^{-it\Delta} u \in U^p \}$$

with the norm $||u||_{U^p_\Delta} := ||e^{-it\Delta} u||_{U^p}$,

$$V^p_\Delta := \{ v : \mathbb{R} \to L^2 | e^{-it\Delta} v \in V_{p, rc}^p \}$$

with the norm $||v||_{V^p_\Delta} := ||e^{-it\Delta} v||_{V^p}$.  

Remark 2.1. The embeddings $U^p_\Delta \hookrightarrow V^p_\Delta \hookrightarrow U^q_\Delta \hookrightarrow L^\infty(\mathbb{R}; L^2)$ hold for $1 \leq p < q < \infty$ by Proposition 2.1.

Proposition 2.3 ([11] Corollary 2.18). Let $1 < p < \infty$. We have
\begin{align}
\|Q_{\geq M}^\Delta u\|_{L^2_t} &\lesssim M^{-1/2}\|u\|_{V^2_2}, \\
\|Q_{< M}^\Delta u\|_{V^2_2} &\lesssim \|u\|_{V^2_2}, \quad \|Q_{\geq M}^\Delta u\|_{V^2_2} \lesssim \|u\|_{V^2_2},
\end{align}
(2.1)
(2.2)

Proposition 2.4 ([11] Proposition 2.19). Let
$$
T_0 : L^2(\mathbb{R}^d) \times \cdots \times L^2(\mathbb{R}^d) \to L^1_{loc}(\mathbb{R}^d)
$$
be a $m$-linear operator. Assume that for some $1 \leq p, q < \infty$
\begin{align}
\|T_0(e^{it\Delta}\phi_1, \cdots, e^{it\Delta}\phi_m)\|_{L^p_t(\mathbb{R}; L^q_2(\mathbb{R}^d))} &\lesssim \prod_{i=1}^m \|\phi_i\|_{L^2(\mathbb{R}^d)}.
\end{align}

Then, there exists $T : U^p_\Delta \times \cdots \times U^p_\Delta \to L^p_t(\mathbb{R}; L^q_2(\mathbb{R}^d))$ satisfying
\begin{align}
\|T(u_1, \cdots, u_m)\|_{L^p_t(\mathbb{R}; L^q_2(\mathbb{R}^d))} &\lesssim \prod_{i=1}^m \|u_i\|_{U^p_\Delta}
\end{align}
such that $T(u_1, \cdots, u_m)(t)(x) = T_0(u_1(t), \cdots, u_m(t))(x)$ a.e.

Proposition 2.5 (Strichartz estimate). Let $(p, q)$ be an admissible pair of exponents for the Schrödinger equation, i.e. $2 \leq q \leq 2d/(d-2)$ ($2 \leq q < \infty$ if $d = 2$, $2 \leq q \leq \infty$ if $d = 1$), $2/p = d(1/2 - 1/q)$. Then, we have
\begin{align}
\|e^{it\Delta}\varphi\|_{L^p_t L^q_x} &\lesssim \|\varphi\|_{L^2_x}
\end{align}
for any $\varphi \in L^2(\mathbb{R}^d)$.

Proposition 2.4 and 2.5 imply the following.

Corollary 2.6. Let $(p, q)$ be an admissible pair of exponents for the Schrödinger equation, i.e. $2 \leq q \leq 2d/(d-2)$ ($2 \leq q < \infty$ if $d = 2$, $2 \leq q \leq \infty$ if $d = 1$), $2/p = d(1/2 - 1/q)$. Then, we have
\begin{align}
\|u\|_{L^p_t L^q_x} &\lesssim \|u\|_{U^p_\Delta}, \quad u \in U^p_\Delta.
\end{align}
(2.3)

Proposition 2.7 ([11] Proposition 2.20). Let $q > 1$, $E$ be a Banach space and $T : U^p_\Delta \to E$ be a bounded, linear operator with $\|Tu\|_E \leq C_q\|u\|_{U^q_\Delta}$ for all $u \in U^q_\Delta$. In addition, assume that for some $1 \leq p < q$ there exists $C_p \in (0, C_q]$ such that the estimate $\|Tu\|_E \leq C_p\|u\|_{U^q_\Delta}$ holds true for all $u \in U^p_\Delta$. Then, $T$ satisfies the estimate
\begin{align}
\|Tu\|_E &\lesssim C_p \left(1 + \ln \frac{C_q}{C_p}\right)\|u\|_{V^p_{\sigma, re, \Delta}}, \quad u \in V^p_{\sigma, re, \Delta};
\end{align}
where implicit constant depends only on \( p \) and \( q \).

Next, we define the function spaces which will be used to construct the solution.

**Definition 4.** Let \( s, \sigma \in \mathbb{R} \).

(i) We define \( \dot{Z}^s := \{ u \in C(\mathbb{R}; \dot{H}^s(\mathbb{R}^d)) \cap U_\Delta^2 \mid \|u\|_{\dot{Z}^s} < \infty \} \) with the norm

\[
\|u\|_{\dot{Z}^s} := \left( \sum_N N^{2s} \|P_N u\|_{U_\Delta^2}^2 \right)^{1/2}.
\]

(ii) We define \( Z^s := \{ u \in C(\mathbb{R}; H^s(\mathbb{R}^d)) \cap U_\Delta^2 \mid \|u\|_{Z^s} < \infty \} \) with the norm

\[
\|u\|_{Z^s} := \|u\|_{\dot{Z}^0} + \|u\|_{\dot{Z}^s}.
\]

(iii) We define \( \dot{Y}^s := \{ u \in C(\mathbb{R}; \dot{H}^s(\mathbb{R}^d)) \cap V_{\Delta,rc}^2 \mid \|u\|_{\dot{Y}^s} < \infty \} \) with the norm

\[
\|u\|_{\dot{Y}^s} := \left( \sum_N N^{2s} \|P_N u\|_{V_{\Delta,rc}^2}^2 \right)^{1/2}.
\]

(iv) We define \( Y^s := \{ u \in C(\mathbb{R}; H^s(\mathbb{R}^d)) \cap V_{\Delta,rc}^2 \mid \|u\|_{Y^s} < \infty \} \) with the norm

\[
\|u\|_{Y^s} := \|u\|_{\dot{Y}^0} + \|u\|_{\dot{Y}^s}.
\]

**Remark 2.2 ([11] Remark 2.23).** Let \( E \) be a Banach space of continuous functions \( f : \mathbb{R} \to H \), for some Hilbert space \( H \). We also consider the corresponding restriction space to the interval \( I \subset \mathbb{R} \) by

\[
E(I) = \{ u \in C(I,H) \mid \exists v \in E \text{ s.t. } v(t) = u(t), \ t \in I \}
\]

endowed with the norm \( \|u\|_{E(I)} = \inf\{\|v\|_E \mid v(t) = u(t), \ t \in I \} \). Obviously, \( E(I) \) is also a Banach space.

### 3. Multilinear estimates

In this section, we prove multilinear estimates which will be used to prove the well-posedness.

**Lemma 3.1.** Let \( d \geq 1, m \geq 2, s_c = d/2 - 1/(m - 1) \) and \( b > 1/2 \). For any dyadic numbers \( N_1 \gg N_2 \geq \cdots \geq N_m \), we have

\[
\left\| \prod_{j=1}^m P_{N_j} u_j \right\|_{L_t^2 L_x^\infty} \lesssim \|P_{N_1} u_1\|_{X^{0,b}} \prod_{j=2}^m \left( \frac{N_j}{N_1} \right)^{1/2(m-1)} N_j^{s_c} \|P_{N_j} u_j\|_{X^{0,b}}, \tag{3.1}
\]

where \( \|u\|_{X^{0,b}} := \|\langle \tau + |\xi|^2 \rangle^{b/2} u\|_{L_t^2 L_x^\infty} \).
Proof. For the case \(d = 2\) and \(m = 2\), the estimate (3.1) is proved by Colliander, Delort, Kenig, and Staffilani (Lemma 1). The proof for general case as following is similar to their argument.

We put \(g_j(\tau_j, \xi_j) := \langle \tau_j + |\xi_j|^2 \rangle \rho \tau_j(\tau_j, \xi_j) (j = 1, \cdots, m)\) and \(A_N := \{\xi \in \mathbb{R}^d | N/2 \leq |\xi| \leq 2N\}\) for a dyadic number \(N\). By the Plancherel’s theorem and the duality argument, it is enough to prove the estimate

\[
I := \left| \int_{\mathbb{R}^m} \int_{\prod_{j=1}^m A_N} g_j \left( \sum_{j=1}^m \tau_j, \sum_{j=1}^m \xi_j \right) \prod_{j=1}^m \frac{g_j(\tau_j, \xi_j)}{\tau_j(\tau_j + |\xi_j|^2)^b} d\xi_j d\tau_j \right|
\]

\[
\lesssim \left( \prod_{j=2}^m \left( \frac{N_j}{N_1} \right)^{1/2(m-1)} N_j^{s_\rho} \right) \prod_{j=0}^m \|g_j\|_{L^2_{\xi}}
\]

for \(g_j \in L^2_{\xi}\), where \(\xi_* = (\xi_1, \cdots, \xi_m), \tau_* = (\tau_1, \cdots, \tau_m)\). We change the variables \(\tau_* \mapsto \theta_* = (\theta_1, \cdots, \theta_m)\) as \(\theta_j = \tau_j + |\xi_j|^2 (j = 1, \cdots, m)\) and put

\[
G_0(\theta_*, \xi_*):= g_0 \left( \sum_{j=1}^m (\theta_j - |\xi_j|^2), \sum_{j=1}^m \xi_j \right),
\]

\[
G_j(\theta_j, \xi_j):= g_j(\theta_j - |\xi_j|^2, \xi_j) (j = 1, \cdots, m).
\]

Then, we have

\[
I \leq \left( \prod_{j=1}^m \frac{1}{(\theta_j)^b} \right) \left( \int_{\prod_{j=1}^m A_N} \left| G_0(\theta_*, \xi_*) \prod_{j=1}^m G_j(\theta_j, \xi_j) \right| d\xi_* \right) d\theta_*
\]

\[
\lesssim \left( \prod_{j=1}^m \frac{1}{(\theta_j)^b} \right) \left( \int_{\prod_{j=1}^m A_N} |G_0(\theta_*, \xi_*)|^2 d\xi_* \right)^{1/2} \prod_{j=1}^m \|G_j(\theta, \cdot)\|_{L^2_{\xi}} d\theta_*
\]

by the Cauchy-Schwartz inequality. For \(1 \leq k \leq d\), we put

\[
A_{N_1}^k := \{\xi_1 = (\xi_1^{(1)}, \cdots, \xi_1^{(d)}) \in \mathbb{R}^d | N_1/2 \leq |\xi_1| \leq 2N_1, |\xi_1^{(k)}| \geq N_1/(2\sqrt{d})\}
\]

and

\[
J_k(\theta_*):= \left( \int_{A_{N_1}^k \times \prod_{j=2}^m A_N} |G_0(\theta_*, \xi_*)|^2 d\xi_* \right).
\]

We consider only the estimate for \(J_1\). The estimates for other \(J_k\) are obtained by the same way.

Assume \(d \geq 2\). By changing the variables \((\xi_1, \xi_2) = (\xi_1^{(1)}, \cdots, \xi_1^{(d)}, \xi_2^{(1)}, \cdots, \xi_2^{(d)}) \mapsto (\mu, \nu, \eta)\) as

\[
\begin{cases}
\mu = \sum_{j=1}^m (\theta_j - |\xi_j|^2) \in \mathbb{R}, \\
\nu = \sum_{j=1}^m \xi_j \in \mathbb{R}^d, \\
\eta = (\xi_2^{(2)}, \cdots, \xi_2^{(d)}) \in \mathbb{R}^{d-1},
\end{cases}
\] (3.2)
we have
\[ d\mu d\nu d\eta = 2|\xi_1^{(1)} - \xi_2^{(1)}|d\xi_1 d\xi_2 \]
and
\[ G_0(\theta_*, \xi_*) = g_0(\mu, \nu). \]
We note that \(|\xi_1^{(1)} - \xi_2^{(1)}| \sim N_1\) for any \((\xi_1, \xi_2) \in A_{N_1} \times A_{N_2}\) with \(N_1 \gg N_2\).
Furthermore, \(\xi_2 \in A_{N_2}\) implies that \(\eta \in [-2N_2, 2N_2]^{d-1}\). Therefore, we obtain
\[
J_1(\theta_*) \lesssim \int_{\Pi_{j=3}^m A_{N_j}} \left( \int_{[-2N_2, 2N_2]^{d-1}} \int_{\mathbb{R}^d} \int |g_0(\mu, \nu)|^2 \frac{1}{N_1} d\mu d\nu d\eta \right) d\xi_3 \cdots d\xi_m
\]
\[ \sim \frac{N_2^{d-1}}{N_1} \left( \prod_{j=3}^m N_j^d \right) \|g_0\|^2_{L^2_{\xi \mu \nu}} \leq \left( \prod_{j=2}^m \left( \frac{N_j}{N_1} \right)^{1/(m-1)} N_j^{d-2/(m-1)} \right) \|g_0\|^2_{L^2_{\xi \mu \nu}} \]
since \(N_2 \geq N_j\) for \(3 \leq j \leq m\). As a result, we have
\[
\mathcal{I} \lesssim \int_{\mathbb{R}^m} \left( \prod_{j=1}^m \frac{1}{(\theta_j)^b} \right) \left( \sum_{k=1}^d J_k(\theta_*) \right)^{1/2} \prod_{j=1}^m \|G_j(\theta_j, \cdot)\|_{L^2_{\xi}} d\theta_*
\]
\[ \lesssim \left( \prod_{j=2}^m \left( \frac{N_j}{N_1} \right)^{1/2(m-1)} N_j^{s_c} \right) \prod_{j=0}^m \|g_j\|_{L^2_{\xi}} \]
by the Cauchy-Schwartz inequality and changing the variables \(\theta_* \mapsto \tau_*\) as \(\theta_j = \tau_j + |\xi_j|^2\) \((j = 1, \cdots, m)\).

For \(d = 1\), we obtain the same result by changing the variables \((\xi_1, \xi_2) \mapsto (\mu, \nu)\) as \(\mu = \sum_{j=1}^m (\theta_j - |\xi_j|^2)\), \(\nu = \sum_{j=1}^m \xi_j\) instead of (3.2). \(\square\)

**Corollary 3.2.** Let \(m \geq 2, m + d \geq 4\) and \(s_c = d/2 - 1/(m-1)\). For any dyadic numbers \(N_1 \gg N_2 \geq \cdots \geq N_m\) and \(0 < \delta < 1/2(m-1)\), we have
\[
\left\| \prod_{j=1}^m P_{N_j} u_j \right\|_{L^2_{tx}} \lesssim \left\| P_{N_1} u_1 \right\|_{L^2_{\lambda}} \prod_{j=2}^m \left( \frac{N_j}{N_1} \right)^{1/(m-1)} N_j^{s_c} \left\| P_{N_j} u_j \right\|_{L^2_{\lambda}}, \quad (3.3)
\]
\[
\left\| \prod_{j=1}^m P_{N_j} u_j \right\|_{L^2_{tx}} \lesssim \left\| P_{N_1} u_1 \right\|_{L^2_{\lambda}} \prod_{j=2}^m \left( \frac{N_j}{N_1} \right)^{\delta} N_j^{s_c} \left\| P_{N_j} u_j \right\|_{L^2_{\lambda}}, \quad (3.4)
\]

**Proof.** To obtain (3.3), we use the argument of the proof of Corollary 2.21 (27) in [11]. Let \(\phi_1, \cdots, \phi_m \in L^2(\mathbb{R}^d)\) and define \(\phi_{j}^\lambda(x) := \phi_j(\lambda x)\) \((j = 1, \cdots, m)\) for \(\lambda \in \mathbb{R}\). By using the rescaling \((t, x) \mapsto (\lambda^2 t, \lambda x)\), we have
\[
\left\| \prod_{j=1}^m P_{N_j} (e^{i\Delta} \phi_j) \right\|_{L^2([-T,T] \times \mathbb{R}^d)} = \lambda^{d/2+1} \left\| \prod_{j=1}^m P_{\lambda N_j} (e^{i\Delta} \phi_j^\lambda) \right\|_{L^2([-\lambda^{-2}T, \lambda^{-2}T] \times \mathbb{R}^d}).
\]
Therefore by putting $\lambda = \sqrt{T}$ and (3.1), we have
\[
\left\| \prod_{j=1}^{m} P_{N_j} \left( e^{it\Delta} \phi_j \right) \right\|_{L^2([-T,T] \times \mathbb{R}^d)} \lesssim \sqrt{T}^{md/2} \left\| P_{\sqrt{T}N_j} \phi_j^T \right\|_{L_x^2} \prod_{j=2}^{m} \left( \frac{N_j}{N_1} \right)^{1/2(m-1)} \frac{N_j^{s_c}}{N_1^{s_c}} \left\| P_{\sqrt{T}N_j} \phi_j^T \right\|_{L_x^2}^{1/2(m-1)} \frac{N_j^{s_c}}{N_1^{s_c}} \left\| P_{N_j} \phi_j \right\|_{L_t^2}.
\]

Let $T \to \infty$, then we obtain
\[
\left\| \prod_{j=1}^{m} P_{N_j} \left( e^{it\Delta} \phi_j \right) \right\|_{L^2_{t,x}} \lesssim \left\| P_{N_1} \phi_1 \right\|_{L^2_{t,x}} \left( \prod_{j=2}^{m} \left( \frac{N_j}{N_1} \right)^{1/2(m-1)} \frac{N_j^{s_c}}{N_1^{s_c}} \left\| P_{N_j} \phi_j \right\|_{L^2_{t,x}} \right)^{1/2(m-1)}
\]
and (3.3) follows from proposition 2.4.

To obtain (3.4), we first prove the $U^{2m}$ estimate. By the Cauchy-Schwartz inequality, the Sobolev embedding $W^{s_c,2md/(md-2)}(\mathbb{R}^d) \hookrightarrow L^{m(m-1)d}(\mathbb{R}^d)$ (which holds when $m \geq 2$, $m + d \geq 4$) and (2.3), we have
\[
\left\| \prod_{j=1}^{m} P_{N_j} u_j \right\|_{L^2_{t,x}} \lesssim \left\| P_{N_1} u_1 \right\|_{L^2_{t,x}} \left( \prod_{j=2}^{m} N_j^{s_c} \left\| P_{N_j} u_j \right\|_{L^2_{t,x}} \right)^{1/2(m-1)} \frac{N_j^{s_c}}{N_1^{s_c}} \left\| P_{N_j} u_j \right\|_{U^m_{t,x}}
\]
for any dyadic numbers $N_1, \ldots, N_m \in 2^\mathbb{Z}$. We use the interpolation between (3.3) and (3.5) via Proposition 2.7. Then, we get (3.4) by the same argument of the proof of Corollary 2.21 (28) in [11].

\[\Box\]

**Lemma 3.3.** We assume that $(\tau_0, \xi_0), (\tau_1, \xi_1), \ldots, (\tau_m, \xi_m) \in \mathbb{R} \times \mathbb{R}^d$ satisfy $\sum_{j=0}^{d} \tau_j = 0$ and $\sum_{j=0}^{d} \xi_j = 0$. Then, we have
\[
\max_{0 \leq j \leq m} |\tau_j + |\xi_j|^2| \geq \frac{1}{m + 1} \max_{0 \leq j \leq m} |\xi_j|^2.
\]

**Proof.** By the triangle inequality, we obtain (3.6). \[\Box\]

The following propositions will be used to prove the key estimate for the well-posedness in the next section.

**Proposition 3.4.** Let $d \geq 1$, $m \geq 3$, $s_c = d/2 - 1/(m-1)$ and $0 < T \leq \infty$. For a dyadic number $N_1 \in 2^\mathbb{Z}$, we define the set $S(N_1)$ as
\[
S(N_1) := \{(N_2, \ldots, N_m) \in (2^\mathbb{Z})^{m-1} | N_1 \gg N_2 \geq \cdots \geq N_m\}.
\]
If $N_0 \sim N_1$, then we have

$$
\left| \sum_{S(N_1)} \int_0^T \int_{\mathbb{R}^d} \left( N_0 \prod_{j=0}^m P_{N_j} u_j \right) \, dx dt \right| 
\lesssim \| P_{N_0} u_0 \|_{V^2_\Delta} \| P_{N_1} u_1 \|_{V^2_\Delta} \prod_{j=2}^m \| u_j \|_{Y^{sc}_\Delta}.
$$

(3.7)

**Proof.** We define $u_{j,N_j,T} := 1_{[0,T)} P_{N_j} u_j$ ($j = 1, \ldots, m$) and put $M := N_0^2/4(m + 1)$. We decompose $Id = Q_{\geq M} + Q_{< M}$. We divide the integrals on the left-hand side of (3.7) into $2^m + 1$ piece of the form

$$
\int_{\mathbb{R}} \int_{\mathbb{R}^d} \left( N_0 \prod_{j=0}^m Q_j^{\Delta} u_{j,N_j,T} \right) \, dx dt
$$

(3.8)

with $Q_j^{\Delta} \in \{ Q_{\geq M}, Q_{< M} \}$ ($j = 0, \ldots, m$). By the Plancherel’s theorem, we have

$$
(3.8) = c \int_{\sum_{j=0}^m \tau_j = 0} \int_{\sum_{j=0}^m \xi_j = 0} N_0 \prod_{j=0}^m \mathcal{F}[Q_j^{\Delta} u_{j,N_j,T}](\tau_j, \xi_j),
$$

where $c$ is a constant. Therefore, Lemma 3.6 implies that

$$
\int_{\mathbb{R}} \int_{\mathbb{R}^d} \left( N_0 \prod_{j=0}^m Q_{< M} u_{j,N_j,T} \right) \, dx dt = 0.
$$

So, let us now consider the case that $Q_j^{\Delta} = Q_{\geq M}$ for some $0 \leq j \leq m$.

First, we consider the case $Q_j^{\Delta} = Q_{\geq M}$. By the Cauchy-Schwartz inequality, we have

$$
\left| \sum_{S(N_1)} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left( N_0 Q_{\geq M} u_{0,N_0,T} \prod_{j=1}^m Q_j^{\Delta} u_{j,N_j,T} \right) \, dx dt \right| 
\leq \sum_{S(N_1)} N_0 \| Q_{\geq M} u_{0,N_0,T} \|_{L^2_{tx}} \left( \prod_{j=1}^m \| Q_j^{\Delta} u_{j,N_j,T} \|_{L^2_{tx}} \right).
$$

Furthermore by (2.1) and $M \sim N_0^2$, we have

$$
\| Q_{\geq M} u_{0,N_0,T} \|_{L^2_{tx}} \lesssim N_0^{-1} \| u_{0,N_0,T} \|_{V^2_\Delta}.
$$
While by (3.4), (2.2) and the Cauchy-Schwartz inequality for the dyadic sum, we have

$$\sum_{S(N_1)} \left\| \prod_{j=1}^m Q_j^{\Delta} u_{j,N_j,T} \right\|_{L^2_{1_2}} \lesssim \|u_{1,N_1,T}\|_{V^2_3} \sum_{S(N_1)} \prod_{j=2}^m \left( \sum_{N_j \leq N_1} N_j^{2s_c} \right)^{\delta} N_j^{s_c} \|u_{j,N_j,T}\|_{V^2_3}^{1/2}$$

Therefore, we obtain

$$\left| \sum_{S(N_1)} \int \int_{\mathbb{R}^d} \left( N_0 Q_j^{\Delta} u_{0,N_0,T} \prod_{j=1}^m Q_j^{\Delta} u_{j,N_j,T} \right) dx dt \right| \lesssim \|P_{N_0} u_0\|_{V^2_3} \|P_{N_1} u_1\|_{V^2_3} M_{N_k} \prod_{j=2}^m \|u_j\|_{Y^c}$$

since $\|1_{[0,T)} u\|_{V^2_3} \lesssim \|u\|_{V^2_3}$ for any $T \in (0, \infty]$. For the case $Q_{k}^{\Delta} = Q_{\geq M}^{\Delta}$ is proved in same way.

Next, we consider the case $Q_{k}^{\Delta} = Q_{\geq M}^{\Delta}$ for some $2 \leq k \leq m$. By the Hölder’s inequality, we have

$$\left| \sum_{N_k} \int \int_{\mathbb{R}^d} \left( N_0 Q_j^{\Delta} u_{k,N_k,T} \prod_{j=0}^m Q_j^{\Delta} u_{j,N_j,T} \right) dx dt \right| \lesssim N_0 \left\| Q_j^{\Delta} u_{0,N_0,T} \right\|_{L^2_{1_2} L^{2d/(d-1)}_{x \tau}} \left\| Q_j^{\Delta} u_{1,N_1,T} \right\|_{L^2_{1_2} L^{2d/(d-1)}_{x \tau}}$$

$$\times \left\| \sum_{N_k} Q_j^{\Delta} u_{k,N_k,T} \right\|_{L^2_{1_2} L^{(m-1)d}_{x \tau}} \prod_{j=2}^m \left\| \sum_{N_j} Q_j^{\Delta} u_{j,N_j,T} \right\|_{L^\infty_{x \tau} L^{(m-1)d}_{x \tau}}.$$

By (2.3), the embedding $V^2_3 \hookrightarrow U_1^2$ and (2.2), we have

$$\|Q_j^{\Delta} u_{0,N_0,T}\|_{L^2_{1_2} L^{2d/(d-1)}_{x \tau}} \|Q_j^{\Delta} u_{1,N_1,T}\|_{L^2_{1_2} L^{2d/(d-1)}_{x \tau}} \lesssim \|u_{0,N_0,T}\|_{V^2_3} \|u_{1,N_1,T}\|_{V^2_3}.$$

While by the Sobolev embedding $\dot{H}^{s_c}(\mathbb{R}^d) \hookrightarrow L^{(m-1)d}(\mathbb{R}^d)$, $L^2$ orthogonality and (2.1), we have

$$\left\| \sum_{N_k} Q_j^{\Delta} u_{k,N_k,T} \right\|_{L^2_{1_2} L^{(m-1)d}_{x \tau}} \lesssim \left( \sum_{N_k} N_k^{2s_c} \|Q_j^{\Delta} u_{k,N_k,T}\|_{L^2_{x \tau}}^2 \right)^{1/2}$$

$$\lesssim N_0^{-1} \left( \sum_{N_k} N_k^{2s_c} \|u_{k,N_k,T}\|_{V^2_3}^2 \right)^{1/2}.$$
since $M \sim N_2^2$. Furthermore by the Sobolev embedding $\dot{H}^{s_c}(\mathbb{R}^d) \hookrightarrow L^{(m-1)d}(\mathbb{R}^d)$, $L^2$ orthogonality, $V_2^2 \hookrightarrow L^\infty(\mathbb{R}; L^2)$ and (2.2), we have

$$
\left\| \sum_{N_j} Q_j^\Delta u_{j,N_j,T} \right\|_{L^\infty_t L^1_x} \lesssim \left( \sum_{N_j} N_j^{2s_c} \|Q_j^\Delta u_{j,N_j,T}\|_{L^2_x L^2_t}^2 \right)^{1/2} \lesssim \left( \sum_{N_j} N_j^{2s_c} \|u_{j,N_j,T}\|_{V_2^2}^2 \right)^{1/2}.
$$

As a result, we obtain

$$
\left| \sum_{S(N_1)} \int \int_{\mathbb{R}^d} \left( N_0 Q_{\geq M}^\Delta u_{k,N_k,T} \prod_{j=0}^{m} Q_j^\Delta u_{j,N_j,T} \right) dx dt \right| \lesssim \|P_{N_0} u_0\|_{V_2^2} \|P_{N_1} u_1\|_{V_2^2} \prod_{j=2}^{m} \|u_j\|_{\dot{Y}^{sc}}
$$
since $\|1_{[0,T]} u\|_{V_2^2} \lesssim \|u\|_{V_2^2}$ for any $T \in (0, \infty]$.

\[ \square \]

**Proposition 3.5.** Let $d \geq 1$, $m \geq 3$, $s_c = d/2 - 1/(m-1)$ and $0 < T \leq \infty$. For a dyadic number $N_2 \in 2^{Z}$, we define the set $S_s(N_2)$ as

$$
S_s(N_2) := \{(N_3, \cdots, N_m) \in (2^Z)^{m-2} | N_2 \geq N_3 \geq \cdots \geq N_m \}.
$$

If $N_0 \lesssim N_1 \sim N_2$, then we have

$$
\left| \sum_{S_s(N_2)} \int_{0}^{T} \int \prod_{j=0}^{m} P_{N_j} u_j \left( N_0 \prod_{j=0}^{m} P_{N_j} u_j \right) dx dt \right| \lesssim \frac{N_0}{N_1} \|P_{N_0} u_0\|_{V_2^2} \|P_{N_1} u_1\|_{V_2^2} N_2^{s_c} \|P_{N_2} u_2\|_{V_2^2} \prod_{j=3}^{m} \|u_j\|_{\dot{Y}^{sc}}.
$$

Proof. We define $u_{j,N_j,T} := 1_{[0,T]} P_{N_j} u_j$ $(j = 1, \cdots, m)$ and put $M := N_1^2 / 4(m+1)$. We decompose $Id = Q_{\geq M}^\Delta + Q_{< M}^\Delta$. We divide the integrals on the left-hand side of (3.9) into $2^{m+1}$ piece of the form

$$
\int \int \left( N_0 \prod_{j=0}^{m} Q_j^\Delta u_{j,N_j,T} \right) dx dt
$$

with $Q_j^\Delta \in \{Q_{\geq M}^\Delta, Q_{< M}^\Delta \}$ $(j = 0, \cdots, m)$. By the Plancherel’s theorem, we have

$$
(3.10) = c \int_{\sum_{j=0}^{m} \tau_j = 0} \int_{\sum_{j=0}^{m} \xi_j = 0} N_0 \prod_{j=0}^{m} \mathcal{F}[Q_j^\Delta u_{j,N_j,T}](\tau_j, \xi_j),
$$
where $c$ is a constant. Therefore, Lemma 3.3 implies that
\[ \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left( N_0 \prod_{j=0}^m Q^\Delta_{M,j,N_j,T} u_j \right) dx dt = 0. \]

So, let us now consider the case that $Q^\Delta_{j} = Q^\Delta_{M,j}$ for some $0 \leq j \leq m$.

We consider only for the case $Q^\Delta_0 = Q^\Delta_{M,j}$ since the case $Q^\Delta_1 = Q^\Delta_{M,j}$ is similar argument and the cases $Q^\Delta_k = Q^\Delta_{M,j}$ ($k = 2, \ldots, m$) are similar to the argument in the proof of Proposition 3.4. By the H"older’s inequality and we have
\[ \left| \sum_{s \in (N_2)} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left( N_0 Q^\Delta_{M,j} u_0,N_0,T \prod_{j=1}^m Q^\Delta_{j} u_j,N_j,T \right) dx dt \right| \lesssim N_0 \left\| Q^\Delta_{M,j} u_0,N_0,T \right\|_{L^2_t L^s_x} \left( \prod_{j=3}^m \left\| Q^\Delta_{j} u_j,N_j,T \right\|_{L^\infty_t L^2_x} \right) \]
\[ \lesssim N_0 \left\| Q^\Delta_{M,j} u_0,N_0,T \right\|_{L^2_t L^s_x} \left\| Q^\Delta_{1} u_1,N_1,T \right\|_{L^2_t L^2_x} \left\| Q^\Delta_{2} u_2,N_2,T \right\|_{L^2_t L^2_x} \]
\[ \times \prod_{j=3}^m \left\| Q^\Delta_{j} u_j,N_j,T \right\|_{L^\infty_t L^2_x}. \]

By the Sobolev embedding $\dot{H}^{s_c}(\mathbb{R}^d) \hookrightarrow L^{(m-1)d}(\mathbb{R}^d)$ and (2.1), we have
\[ \left\| Q^\Delta_{M,j} u_0,N_0,T \right\|_{L^2_t L^s_x} \lesssim N_0 \left\| Q^\Delta_{M,j} u_0,N_0,T \right\|_{L^\infty_t L^2_x} \]
\[ \lesssim N_0^{-1} N_1^{-1} \left\| P_{0} u_0 \right\|_{V^2_\Delta} \]
since $M \sim N_1^2$ and $N_0 \lesssim N_2$. While by (2.3), the embedding $V^2_\Delta \hookrightarrow U^4_\Delta$ and (2.2), we have
\[ \left\| Q^\Delta_{1} u_1,N_1,T \right\|_{L^2_t L^2_x} \lesssim \left\| u_1,N_1,T \right\|_{V^2_\Delta} \left\| u_2,N_2,T \right\|_{V^2_\Delta}. \]

Furthermore by the Sobolev embedding $\dot{H}^{s_c}(\mathbb{R}^d) \hookrightarrow L^{(m-1)d}(\mathbb{R}^d)$, $L^2$ orthogonality, $V^2_\Delta \hookrightarrow L^\infty(\mathbb{R}; L^2)$ and (2.2), we have
\[ \left\| Q^\Delta_{j} u_j,N_j,T \right\|_{L^\infty_t L^s_x} \lesssim \left( \sum_{N_j} N_j^{2s_c} \left\| Q^\Delta_{j} u_j,N_j,T \right\|_{L^\infty_t L^2_x}^2 \right)^{1/2} \]
\[ \lesssim \left( \sum_{N_j} N_j^{2s_c} \left\| u_j,N_j,T \right\|_{V^2_\Delta}^2 \right)^{1/2}. \]

As a result, we obtain
\[ \left| \sum_{s \in (N_2)} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left( N_0 Q^\Delta_{M,j} u_0,N_0,T \prod_{j=1}^m Q^\Delta_{j} u_j,N_j,T \right) dx dt \right| \]
\[ \lesssim \frac{N_0}{N_1} \left\| P_0 N_0 u_0 \right\|_{V^2_\Delta} \left\| P_{1} u_1 \right\|_{V^2_\Delta} \left\| P_{2} u_2 \right\|_{V^2_\Delta} \prod_{j=2}^m \left\| u_j \right\|_{V^2_\Delta}. \]
since \( \|1_{[0, T]}u\|_{V^2} \lesssim \|u\|_{V^2} \) for any \( T \in (0, \infty) \).

\[ \square \]

4. Proof of the well-posedness and the scattering

In this section, we prove Theorem 1.1 and Corollary 1.2. We define the map \( \Phi_{T, \varphi} \) as

\[ \Phi_{T, \varphi}(u)(t) := e^{it\Delta} \varphi - iI_T(u, \cdots, u)(t), \]

where

\[ I_T(u_1, \cdots, u_m)(t) := \int_0^t 1_{[0, T]}(t') e^{i(t-t')\Delta} \partial_k \left( \prod_{j=1}^m u_j(t') \right) dt'. \]

To prove the well-posedness of (1.1), we prove that \( \Phi_{T, \varphi} \) is a contraction map on a closed subset of \( Z^s([0, T]) \) or \( Z^s([0, T]) \). Key estimate is the following:

**Proposition 4.1.** We assume \( d \geq 1, m \geq 3 \). Then for \( s_c = d/2 - 1/(m - 1) \) and any \( 0 < T \leq \infty \), we have

\[ \|I_T(u_1, \cdots, u_m)\|_{\tilde{Z}^s} \lesssim \prod_{j=1}^m \|u_j\|_{\dot{Y}^{s_c}}. \]  

(4.1)

**Proof.** We show the estimate

\[ \|I_T(u_1, \cdots, u_m)\|_{\tilde{Z}^s} \lesssim \sum_{k=1}^m \left( \|u_k\|_{\dot{Y}^s} \prod_{j=1}^m \|u_j\|_{\dot{Y}^{s_c}} \right) \]

(4.2)

for \( s \geq 0 \). (4.1) follows from (4.2) with \( s = s_c \). We decompose

\[ I_T(u_1, \cdots, u_m) = \sum_{N_1, \cdots, N_m} I_T(P_{N_1}u_1, \cdots P_{N_m}u_m). \]

By symmetry, it is enough to consider the summation for \( N_1 \geq \cdots \geq N_m \). We put

\[ S_1 := \{(N_1, \cdots, N_m) \in (2^\mathbb{Z})^m | N_1 \gg N_2 \geq \cdots \geq N_m \} \]

\[ S_2 := \{(N_1, \cdots, N_m) \in (2^\mathbb{Z})^m | N_1 \sim N_2 \geq \cdots \geq N_m \} \]

and

\[ J_k := \left\| \sum_{s_k} I_T(P_{N_1}u_1, \cdots P_{N_m}u_m) \right\|_{\tilde{Z}^s} (k = 1, 2). \]
First, we prove the estimate for $J_1$. By Theorem 2.2 and the Plancherel’s theorem, we have

$$J_1 \leq \left\{ \sum_{N_0} N_0^{2s} \left( e^{-it\Delta} P_{N_0} \sum_{S_1} I_T(P_{N_1} u_1, \cdots P_{N_m} u_m) \right) \right\}_{U^2}^{1/2} $$

$$\lesssim \left\{ \sum_{N_0} N_0^{2s} \sum_{N_1 \sim N_0} \left( \sup_{\|u_0\|_{V_0} = 1} \left| \sum_{S(N_1)} \int_0^T \int_{\mathbb{R}^d} \left( N_0 \prod_{j=0}^m P_{N_j} u_j \right) \, dx \, dt \right| \right)^2 \right\}^{1/2} .$$

Therefore by Proposition 3.3, we have

$$J_1 \lesssim \left\{ \sum_{N_0} N_0^{2s} \sum_{N_1 \sim N_0} \left( \sup_{\|u_0\|_{V_0} = 1} \|P_{N_0} u_0\|_{V_0^2} \|P_{N_1} u_1\|_{V_2} \prod_{j=2}^m \|u_j\|_{Y^s} \right)^2 \right\}^{1/2} $$

$$= \|u_1\|_{Y^s} \prod_{j=2}^m \|u_j\|_{Y^s} .$$

Next, we prove the estimate for $J_2$. By Theorem 2.2 and the Plancherel’s theorem, we have

$$J_2 \leq \sum_{N_1} \sum_{N_2 \sim N_1} \left( \sum_{N_0} N_0^{2s} \left. e^{-it\Delta} P_{N_0} \sum_{S_0(N_2)} I_T(P_{N_1} u_1, \cdots P_{N_m} u_m) \right) \right)_{U^2}^{1/2} $$

$$= \sum_{N_1} \sum_{N_2 \sim N_1} \left( \sum_{N_0 \leq N_1} N_0^{2s} \sup_{\|u_0\|_{V_0} = 1} \left| \sum_{S_0(N_2)} \int_0^T \int_{\mathbb{R}^d} \left( N_0 \prod_{j=0}^m P_{N_j} u_j \right) \, dx \, dt \right| \right)^{2, 1/2} .$$

Therefore by Proposition 3.3 and Cauchy-Schwartz inequality for dyadic sum, we have

$$J_2 \lesssim \sum_{N_1} \sum_{N_2 \sim N_1} \left( \sum_{N_0 \leq N_1} N_0^{2s} \left( \frac{N_0}{N_1} \|P_{N_1} u_1\|_{V_2} N_2^{s_c} \|P_{N_2} u_2\|_{V_2} \prod_{j=3}^m \|u_j\|_{Y^s} \right)^2 \right)^{1/2} $$

$$\lesssim \left( \sum_{N_1} N_1^{2s} \|P_{N_1} u_1\|_{V_2}^2 \right)^{1/2} \left( \sum_{N_2} N_2^{2s_c} \|P_{N_2} u_2\|_{V_2}^2 \right)^{1/2} \prod_{j=3}^m \|u_j\|_{Y^s} $$

$$= \|u_1\|_{Y^s} \prod_{j=2}^m \|u_j\|_{Y^s} .$$
The estimates (4.2) with $s = 0$ and with $s = s_c$ imply the following.

**Corollary 4.2.** We assume $d \geq 1$, $m \geq 3$. Then for $s \geq s_c \left( = d/2 - 1/(m - 1) \right)$ and any $0 < T \leq \infty$, we have

$$||I_T(u_1, \cdots, u_m)||_{Z^s} \lesssim \prod_{j=1}^{m} ||u_j||_{Y^s}. $$

**Proof of Theorem 1.1.** We prove only the homogeneous case. The inhomogeneous case is also proved by the same way. For $r > 0$, we define

$$ \dot{Z}^s_r(I) := \left\{ u \in \dot{Z}^s(I) \mid ||u||_{\dot{Z}^s(I)} \leq 2r \right\} $$

which is a closed subset of $\dot{Z}^s(I)$. Let $u_0 \in B_r(\dot{H}^{s_c})$ be given. For $u \in \dot{Z}^{s_c}_r([0, \infty))$, we have

$$||\Phi_{T, u_0}(u)||_{\dot{Z}^{s_c}([0, \infty))} \leq ||u_0||_{\dot{H}^{s_c}} + C||u||_{\dot{Z}^{s_c}([0, \infty))}^m \leq r \left( 1 + 2^m Cr^{m-1} \right) $$

and

$$||\Phi_{T, u_0}(u) - \Phi_{T, u_0}(v)||_{\dot{Z}^{s_c}([0, \infty))} \leq C(||u||_{\dot{Z}^{s_c}([0, \infty))} + ||v||_{\dot{Z}^{s_c}([0, \infty))})^m ||u - v||_{\dot{Z}^{s_c}([0, \infty))} \leq 4^{m-1} Cr^{m-1} ||u - v||_{\dot{Z}^{s_c}([0, \infty))} $$

by Proposition 4.1 and

$$||e^{it\Delta} \varphi||_{\dot{Z}^{s_c}([0, \infty))} \leq ||1_{[0, \infty)} e^{it\Delta} \varphi||_{\dot{Z}^{s_c}} \leq ||\varphi||_{\dot{H}^{s_c}}, $$

where $C$ is an implicit constant in (4.1). Therefore if we choose $r$ satisfying

$$ r < (4^{m-1} C)^{-1/(m-1)}, $$

then $\Phi_{T, u_0}$ is a contraction map on $\dot{Z}^{s_c}_r([0, \infty))$. This implies the existence of the solution of (1.1) and the uniqueness in the ball $\dot{Z}^{s_c}_r([0, \infty))$. The Lipschitz continuously of the flow map is also proved by similar argument. \(\square\)

**Proof of Corollary 1.2.** We prove only the homogeneous case. The inhomogeneous case is also proved by the same way. By Proposition 4.1, the global solution $u \in \dot{Z}^{s_c}([0, \infty))$ of (1.1) which was constructed in Theorem 1.1 satisfies

$$ N^{s_c} e^{-it\Delta} P_N I_\infty(u, \cdots, u) \in V^2 $$

for each $N \in 2 \mathbb{Z}$. This implies that

$$ u_+ := \lim_{t \to \infty} (u_0 - e^{-it\Delta} I_\infty(u, \cdots, u)(t)) $$

exists in $\dot{H}^{s_c}$ by Proposition 2.1 (4). Then we obtain

$$ u - e^{it\Delta} u_+ \to 0 $$
in $\dot{H}^{s_c}$ as $t \to \infty$.

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(H. Hirayama)

E-mail address, H. Hirayama: m08035f@math.nagoya-u.ac.jp