TORSAL OR NON LOCALLY CONNECTED MINIMAL SETS FOR
R-CLOSED SURFACE HOMEOMORPHISMS

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Abstract. Let $M$ be an orientable connected closed surface and $f$ be an $R$-
closed homeomorphism on $M$ which is isotopic to identity. Then the suspension
of $f$ satisfies one of the following condition: 1) the closure of each element of it
is toral. 2) there is a minimal set which is not locally connected. Moreover, we
show that any positive iteration of an $R$-closed homeomorphism on a compact
metrizable space is $R$-closed.

1. Preliminaries

In this paper, we will show that if $f$ is a nontrivial $R$-closed homeomorphism on
$M$ which is not periodic but isotopic to identity, then either a) $f$ is “an irrational
rotation” or b) there is a minimal set which is not locally connected. Moreover if
$M$ has genus $\geq 2$, then b) holds. Taking suspensions, we show that a) implies that
each orbit closure is a torus. In addition, we show that any positive iteration of an
$R$-closed homeomorphism on a compact metrizable space is $R$-closed.

For a subset $U$ of a topological space, $U$ is locally connected if every point of $U$
admits a neighborhood basis consisting of open connected subsets. For a (binary)
relation $E$ on a set $X$ (i.e. a subset of $X \times X$), let $E(x) := \{ y \in X \mid (x,y) \in E \}$
for an element $x$ of $X$. For a subset $A$ of $X$, we say that $A$ is $E$-saturated if
$A = \cup_{x \in A} E(x)$. Also $E$ define the relation $\bar{E}$ on $X$ with $\bar{E}(x) = \overline{E(x)}$. Recall that
$E$ is pointwise almost periodic if $\bar{E}$ is an equivalence relation and $E$ is $R$-
closed if $\bar{E}$ is closed. Note that $f$ on a locally compact Hausdorff
space is pointwise almost periodic if and only if $\bar{E}_f$ is an equivalence relation (cf.
Theorem 4.10 [GH]). We call that an equivalence relation $E$ is $L$-stable if for an
element $x$ of $X$ and for any open neighborhood $U$ of $\bar{E}(x)$ contained in $U$. In [ES], they show the following: If a
continuous mapping $f$ of a topological space $X$ itself is either pointwise recurrent
or pointwise almost periodic, then so is $f^k$ for each positive integer $k$. In general
cases, see Theorem 2.24, 4.04, and 7.04 [GH]. We show the following key lemma
which is an $R$-closed version of this fact on a compact metrizable space.
Lemma 1.1. Let $f$ an homeomorphism on a compact metrizable space $X$. If $f$ is $R$-closed, then so is $f^n$ for any $n \in \mathbb{Z}_{>0}$.

Proof. Put $E := E_f$ and $E^n := E_{f^n}$. By Corollary 1.5 [?], we have that $\hat{E}$ is an equivalence relation and so $f$ is pointwise almost periodic. Since $f$ is pointwise almost periodic, by Theorem 1[ES], we have that $f^n$ is also pointwise almost periodic. Then $\hat{E^n}$ is an equivalence relation. By Corollary 3.6[?], $E$ is $L$-stable and it suffices to show that $\hat{E^n}$ is $L$-stable. Note that $\hat{E^n}(x) \subseteq E(x)$ and so $\hat{E^n}(x) \subseteq \hat{E}(x)$. For $x \in X$ with $\hat{E^n}(x) = \hat{E}(x)$ and for any open neighborhood $U$ of $\hat{E}(x) = \hat{E^n}(x)$, since $E$ is $L$-stable, there is a $E$-saturated open neighborhood $V$ of $\hat{E}(x)$ contained in $U$. Since $\hat{E^n}(x) \subseteq E(x)$, we have that $V$ is also a $E^n$-saturated open neighborhood $V$ of $\hat{E}(x)$. Fix any $x \in X$ with $\hat{E}(x) \neq \hat{E^n}(x)$.

Put $\{\hat{E}_1, \ldots, \hat{E}_k\} := \{\hat{E}(f^k(x)) \mid k = 0, 1, \ldots, n - 1\}$ such that $\hat{E}_1 = \hat{E^n}(x)$ and $\hat{E}_i \cap \hat{E}_j = \emptyset$ for any $i \neq j \in \{1, \ldots, k\}$. Then $\hat{E}_1$ and $\hat{E}' = \hat{E}_2 \sqcup \cdots \sqcup \hat{E}_k$. Then $\hat{E}_1$ and $\hat{E}'$ are closed and $\hat{E} = \hat{E}_1 \sqcup \cdots \sqcup \hat{E}_k = \hat{E}_1 \sqcup \hat{E}'$. For any sufficiently small $\varepsilon > 0$, let $U_{1, \varepsilon} = B_\varepsilon(\hat{E}_1)$ (resp. $U'_{\varepsilon} = B_\varepsilon(\hat{E}')$) be the open $\varepsilon$-ball of $\hat{E}_1$ (resp. $\hat{E}'$). Since $\varepsilon$ is small and $X$ is normal, we obtain $U_{1, \varepsilon} \cap U'_{\varepsilon} = \emptyset$, $U_{\varepsilon/2} \subseteq U_{1, \varepsilon}$, and $U'_{\varepsilon/2} \subseteq U'_{\varepsilon}$. Since $E$ is $L$-stable, there are neighborhoods $V_{1, \varepsilon} \subseteq U_{1, \varepsilon/2}$ (resp. $V'_{\varepsilon} \subseteq U'_{\varepsilon/2}$) of $\hat{E}_1$ (resp. $\hat{E}'$) such that $V_{1, \varepsilon} \sqcup V'_{\varepsilon}$ is an $E$-saturated neighborhood of $\hat{E}(x)$. Since $\hat{E}_1$ and $\hat{E}'$ are $f^n$-invariant and compact, there is a small $\delta > 0$ such that $f^n(V_{1, \delta}) \subseteq V_{1, \varepsilon}$ and $f^n(V'_{\delta}) \subseteq V'_{\varepsilon}$. Since $V_{1, \delta} \sqcup V'_{\delta}$ is $f^n$-invariant and $U_{1, \varepsilon} \cap U'_{\varepsilon} = \emptyset$, we obtain $V_{1, \delta} \sqcup V'_{\delta} = f^n(V_{1, \delta} \sqcup V'_{\delta}) = f^n(V_{1, \delta}) \sqcup f^n(V'_{\delta})$, $f^n(V_{1, \delta}) \cap V'_{\varepsilon} = \emptyset$, and $f^n(V'_{\delta}) \cap V_{1, \varepsilon} = \emptyset$. Hence $V_{1, \delta} = f^n(V_{1, \delta})$ and $V'_{\delta} = f^n(V'_{\delta})$. This implies that $V_{1, \delta}$ is an $E^n$-saturated neighborhood of $\hat{E}_1 = \hat{E^n}(x)$ with $V_{1, \delta} \subseteq U_{1, \varepsilon} = B_\varepsilon(\hat{E}_1) = B_\varepsilon(\hat{E^n}(x))$. □

Note this lemma is not true for compact $T_1$ spaces. (e.g. a homeomorphism $f$ on a non-Hausdorff 1-manifold $X = \{0, \ldots, 0_+\} \cup [0, 1]$ by $f(0_+) = 0_+$ and $f([0, 1]) = \text{id}$).

2. Main results

From now on, let $M$ be an orientable connected closed surface and $f$ a nontrivial $R$-closed homeomorphism on $M$ which is not periodic but isotopic to identity. We call that $f$ on $S^2$ is a topological irrational rotation if there is an irrational number $\theta_0 \in \mathbb{R} - \mathbb{Q}$ such that $f$ is topologically conjugate to a map on a unit sphere in $\mathbb{R}^3$ with the Cylindrical Polar Coordinates by $(\rho, \theta, z) \mapsto (\rho, \theta + \theta_0, z) \in \mathbb{R}_{>0} \times S^1 \times \mathbb{R}$. Also $f$ on $T^2$ is a topological irrational rotation if there is an irrational number $\theta_0 \in \mathbb{R} - \mathbb{Q}$ such that some positive iteration of $f$ is topologically conjugate to a map $S^1 \times S^1 \to S^1 \times S^1$ by $(\theta, \varphi) \mapsto (\theta + \theta_0, \varphi)$.

Lemma 2.1. If every minimal set is locally connected, then $f$ is a topological irrational rotation on $M = S^2$ or $T^2$.

Proof. By Theorem 1 and Theorem 2 [BNW], since $f$ is pointwise almost periodic, every orbit closure is a finite subset or a finite disjoint union of simple closed curves. We will show that there is a finite disjoint union of simple closed curves. Otherwise $f$ is pointwise periodic. By [M], we have $f$ is periodic, which contradicts. By Lemma 1.1 we have that $f^n$ is also $R$-closed for any $n \in \mathbb{Z}_{>0}$. Hence there is an positive integer $n$ such that $f^n$ has a simple closed curve as a minimal set. Then Theorem 2A [Y2] implies that $M$ is either $T^2$ or $S^2$. By Corollary 2.5 [Y2], if $M = S^2$, then
$f$ has a null homotopic circle and so $n = 1$. Suppose $M = \mathbb{T}^2$ (resp. $S^2$). By Theorem 2.4 \cite{Y2}, the set $\mathcal{F}_{E_f^n}$ of orbits closures consists of essential circles (resp. two singular points and other circles). Fix any $x \in M$. Then $A := \mathbb{T}^2 - \mathring{E}_{f^n}(x)$ (resp. $A := S^2 - \text{Sing}(f^n)$) is an open annulus. By Lemma 1.5 (resp. the proof of Lemma 2.1) \cite{Y2}, we have that the restriction $f^n|_A$ to the open annulus $A$ is an irrational rotation. \hfill \square

This implies our main results.

**Theorem 2.2.** Let $M$ be an orientable connected closed surface and $f$ be a nontrivial $R$-closed homeomorphism on $M$ which is not periodic but isotopic to identity. Then one of the following holds:

1) $f$ is a topological irrational rotation.
2) there is a minimal set which is not locally connected.

Moreover 2) holds when $M$ has genus $\geq 2$.

Taking a suspension, we have a following corollary.

**Corollary 2.3.** Let $M$ be an orientable connected closed surface and $f$ be an $R$-closed homeomorphism on $M$ which is isotopic to identity. Then the suspension of $f$ satisfies one of the following condition:

1) the closure of each element of it is toral.
2) there is a minimal set which is not locally connected.

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