ORTHOLIC TETRAHEDRA WITH INTERSECTING EDGES

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ABSTRACT. Two tetrahedra are called orthologic if the lines through vertices of one and perpendicular to corresponding faces of the other are intersecting. This is equivalent to the orthogonality of non-corresponding edges. We prove that the additional assumption of intersecting non-corresponding edges (“orthosecting tetrahedra”) implies that the six intersection points lie on a sphere. To a given tetrahedron there exists generally a one-parametric family of orthosecting tetrahedra. The orthographic projection of the locus of one vertex onto the corresponding face plane of the given tetrahedron is a curve which remains fixed under isogonal conjugation. This allows the construction of pairs of conjugate orthosecting tetrahedra to a given tetrahedron.

1. INTRODUCTION

Ever since the introduction of orthologic triangles and tetrahedra by J. Steiner in 1827 [10] these curious pairs have attracted researchers in elementary geometry. The characterizing property of orthologic tetrahedra is concurrency of the straight lines through vertices of one tetrahedron and perpendicular to corresponding faces of the second. Alternatively, one can say that non-corresponding edges are orthogonal. Proofs of fundamental properties can be found in [7] and [8]. Quite a few results are known on orthologic tetrahedra. See for example [5, 6, 9, 11] for more information.

In this article we are concerned with orthosecting tetrahedra — orthologic tetrahedra such that non-corresponding edges intersect orthogonally. The concept as well as a few basic results will be introduced in Section 2. In Section 3 we show that the six intersection points of non-corresponding edges necessarily lie on a sphere (or a plane). While the computation of orthosecting pairs requires, in general, the solution of a system of algebraic equations, conjugate orthosecting tetrahedra can be constructed from a given orthosecting pair. This is the topic of Section 4. Our treatment of the subject is of elementary nature. The main ingredients in the proofs come from descriptive geometry and triangle geometry.

A few words on notation: By $A_1A_2A_3$ we denote the triangle with vertices $A_1$, $A_2$, and $A_3$, by $A_1A_2A_3A_4$ the tetrahedron with vertices $A_1$, $A_2$, $A_3$, and $A_4$. The line spanned by two points $A_1$ and $A_2$ is $A_1 \lor A_2$, the plane spanned by three points $A_1$, $A_2$, and $A_3$ is $A_1 \lor A_2 \lor A_3$. Furthermore, $I_n$ denotes the set of all n-tuples with pairwise different entries taken from the set $\{1, \ldots, n\}$.

2. PRELIMINARIES

Two triangles $A_1A_2A_3$ and $B_1B_2B_3$ are called orthologic, if the three lines
\begin{align*}
(1) & \quad a_i : A_i \in a_i, \quad a_i \perp B_j \lor B_k; \quad (i, j, k) \in I_3 \\
(2) & \quad b_i : B_i \in b_i, \quad b_i \perp A_j \lor A_k; \quad (i, j, k) \in I_3
\end{align*}
intersect in a point $O_A$, the orthology center of $A_1A_2A_3$ with respect to $B_1B_2B_3$. In this case, also the lines
\begin{align*}
(3) & \quad a_i : A_i \in a_i, \quad a_i \perp B_j \lor B_k \lor B_l; \quad (i, j, k, l) \in I_4 \\
(4) & \quad b_i : B_i \in b_i, \quad b_i \perp A_j \lor A_k \lor A_l; \quad (i, j, k, l) \in I_4
\end{align*}
intersect in a point $O_B$, the orthology center of $B_1B_2B_3$ with respect to $A_1A_2A_3$. The concept of orthologic tetrahedra is similar. Two tetrahedra $A = A_1A_2A_3A_4$ and $B = B_1B_2B_3B_4$ are called orthologic, if the four lines
\begin{align*}
(5) & \quad A_1 \lor A_2 \lor B_k \lor B_l; \quad (i, j, k, l) \in I_4
\end{align*}
intersect in a point $O_A$, the orthology center of $A$ with respect to $B$. In this case, also the lines $A_1 \lor A_2 \lor B_k \lor B_l; \quad (i, j, k, l) \in I_4$ intersect in a point $O_B$, the orthology center of $B$ with respect to $A$.

The symmetry of the two tetrahedra in the definition of orthologic is a consequence of the following alternative characterization of orthologic tetrahedra. It is well-known but we give a proof which introduces concepts and techniques that will frequently be employed throughout this paper.

**Proposition 1.** The two tetrahedra $A$ and $B$ are orthologic if and only if non-corresponding edges are orthogonal:
\begin{align*}
(6) & \quad A_1 \lor A_2 \parallel B_k \lor B_l; \quad (i, j, k, l) \in I_4.
\end{align*}
3. THE SIX INTERSECTION POINTS

The new results in this paper concern pairs of orthologic tetrahedra \( A = A_1A_2A_3A_4 \) and \( B = B_1B_2B_3B_4 \) such that non-corresponding edges are not only orthogonal but also intersecting. That is, in addition to \([5]\) we also require existence of the points \( V_{ij} := (A_i \vee A_j) \cap (B_k \vee B_l) \neq \emptyset \), \( (i, j, k, l) \in J_4 \).

**Definition 2.** We call two tetrahedra \( A \) and \( B \) orthosecting if their vertices can be labelled as \( A_1A_2A_3A_4 \) and \( B_1B_2B_3B_4 \), respectively, such that \([6]\) and \([7]\) hold.

**Theorem 3.** If two tetrahedra are orthosecting, the six intersection points of non-corresponding edges lie on a sphere (or a plane, if flat tetrahedra are permitted). The sphere center is the midpoint between the orthology centers.

**Proof.** Denote the two tetrahedra by \( A = A_1A_2A_3A_4 \) and \( B = B_1B_2B_3B_4 \) such that the lines \( A_i \vee A_j \) and \( B_k \vee B_l \) intersect orthogonally in \( V_{ij} \) for \( (i, j, k, l) \in J_4 \). As in the proof of Proposition [1] we consider the orthographic projection onto the plane \( A_i \vee A_j \vee A_3 \) (Figure [1]). Clearly, \( B_4' \) equals the projection \( O_B' \) of the orthology center \( O_B \) of \( B \) with respect to \( A \). If it lies on the circumcircle of \( A_1A_2A_3 \), all perpendiculars from \( B_4' \) onto the sides of \( A_1A_2A_3 \) are parallel. In this case the tetrahedron \( B_1B_2B_3B_4 \) is flat and the theorem’s statement holds. Otherwise, the points \( V_{12}, V_{13}, \) and \( V_{23} \) define a circle \( c_4 \) — the pedal circle of the point \( B_4' \) with respect to the triangle \( A_1A_2A_3 \). By the Right-Angle Theorem the projection \( O_A' \) of the orthology center \( O_A \) of \( A \) with respect to \( B \) is the orthology center of the triangle \( A_1A_2A_3 \) with respect to the triangle \( V_{23}V_{13}V_{12} \). Moreover, from elementary triangle geometry it is known that the center \( M' \) of \( c_4 \) halves the segment between \( B_4' \) and \( O_A' \) [4, pp. 54–56]. Hence all circles \( c_i \) drawn in like manner on the faces of \( A \) have axes which intersect in the midpoint \( M \) of the two orthology centers \( O_A \) and \( O_B \). Moreover, any two of these circles share one of the points \( V_{ij} \). Hence, these circles are co-spherical and the proof is finished.

The proof of Theorem [3] can also be applied to a slightly more general configuration where only five of the six edges intersect orthogonally. We formulate this statement as a corollary:

**Corollary 4.** If \( A = A_1A_2A_3A_4 \) and \( B = B_1B_2B_3B_4 \) are two orthologic tetrahedra such that five non-corresponding edges intersect, the five intersection points lie on a sphere (or a plane).
4. THE ONE-PARAMETRIC FAMILY OF SOLUTION TETRAHEDRA

So far we have dealt with properties of a pair of orthosecting tetrahedra but we have left aside questions of existence or computation. In this section $A = A_1 A_2 A_3 A_4$ is a given tetrahedron to which an orthosecting tetrahedron $B = B_1 B_2 B_3 B_4$ is sought.

4.1. Construction of orthologic tetrahedra. At first, we consider the simpler case of orthologic pairs. Clearly, translation of the face planes of $B$ will transform an orthologic tetrahedron into a like tetrahedron (unless all planes pass through a single point). Therefore, we consider tetrahedra with parallel faces as equivalent.

The maybe simplest construction of an equivalence class of solutions consists of the choice of the orthology center $O_A$. This immediately yields the face normals $n_i$ of $B$ as connecting vectors of $O_A$ and $A_i$. The variety of solution classes is of dimension three, one solution to every choice of $O_A$. Since five edges determine two face planes of a tetrahedron and, in case of suitable orthogonality relations, also the orthology center $O_A$, we obtain

Theorem 5. If the vertices of two tetrahedra can be labelled such that five non-corresponding pairs of edges are orthogonal then so is the sixth.

The variety of all solution classes contains a two-parametric set of trivial solutions $n_1 = n_2 = n_3 = n_4$. They correspond to orthology centers at infinity, the solution tetrahedra are flat. Note that the possibility to label the edges such that non-corresponding pairs are orthogonal is essential for the existence of non-flat solutions. If, for example, corresponding edges are required to be orthogonal only flat solutions exist.

4.2. Conjugate pairs of orthosecting tetrahedra. Establishing algebraic equations for solution tetrahedra is straightforward. Six orthogonality conditions and six intersection condition result in a system of six linear and six quadratic equations in the twelve unknown coordinates of the vertices of $B$. Because of Theorem 5 only five of the six linear orthogonality conditions are independent. Therefore, we can expect a one-dimensional variety of solution tetrahedra. This expectation is generically true, as can be confirmed by computing the dimension of the ideal spanned by the orthosecting conditions by means of a computer algebra system.

The numeric solution of the system induced by the orthosecting conditions poses no problems. We used the software Bertini\footnote{2D. J. Bates, J. D. Hauenstein, A. J. Sommese, Ch. W. Wampler: Bertini: Software for Numerical Algebraic Geometry, http://www.nd.edu/~sommese/bertini/} for that purpose. Symbolic approaches are feasible as well. One of them will be described in Subsection 4.3. It is based on a curious conjugacy which can be defined in the set of all tetrahedra that orthosect the given tetrahedron $A$.

Assume that $B = B_1 B_2 B_3 B_4$ is a solution tetrahedron and denote the orthographic projection of $B_i$ onto the face plane $A_j \setminus A_k \setminus A_l$ by $B_i^\star (i, j, k, l) \in J_4$. By the Right-Angle theorem the pedal points of all points $B_i^\star$ on the edges of $A_i A_k A_l$ are precisely the intersection points defined in (7). Three intersection points on the same face of $A$ form a pedal triangle. This observation gives rise to

Definition 6. A pedal chain on a tetrahedron is a set of four pedal triangles, each with respect to one face triangle of the tetrahedron, such that any two pedal triangles share the vertex on the common edge of their faces (Figure 2). If all vertices of pedal triangles lie on a sphere (or a plane), we speak of a spherical pedal chain.

If $A_1 A_2 A_3 A_4$ and $B_1 B_2 B_3 B_4$ are orthosecting, the proof of Theorem\footnote{3The case of collinear or coinciding points $V_i$ leads to degenerate solution tetrahedra whose faces contain one vertex of $A$.} shows that six intersection points are the vertices of a spherical pedal chain. The converse is also true:

Theorem 7. Given the vertices $V_{ij}$ of a spherical pedal chain on a tetrahedron $A = A_1 A_2 A_3 A_4$ there exists a unique orthosecting tetrahedron $B = B_1 B_2 B_3 B_4$ such that $A_i \setminus A_j \cap B_k \setminus B_l = V_{ij}$ for all $(i, j, k, l) \in J_4$.

Proof. If a solution tetrahedron $B$ exists at all it must be unique since its faces lie in the planes $\beta_i := V_{ij} \setminus V_{ik} \setminus V_{il}$ (i, j, k, l ∈ {1, 2, 3, 4} pairwise different).
In order to prove existence, we have to show that the lines $A_i \lor A_j$ and $\beta_i \cap \beta_j$ are, indeed, orthogonal for all pairwise different $i, j \in \{1, 2, 3, 4\}$. We denote the point from which the pedal triangle on the face $A_iA_jA_k$ originates (the "anti-pedal point") by $B_i^*$ and show orthogonality between $A_1 \lor A_2$ and $\beta_1 \cap \beta_2$ for $(i, j, k, l) \in J_4$. Relabelling according to $P_{00} := V_{13}$, $P_{01} := A_1$, $P_{02} := V_{14}$, $P_{10} := B_1^*$, $P_{11} := V_{12}$, $P_{12} := B_2^*$, $P_{20} := V_{23}$, $P_{21} := A_2$, $P_{22} := V_{24}$ (Figure 3) we obtain a net of points $P_{ij}$. In every elementary quadrilateral the angle measure at two opposite vertices equals $\pi/2$. Thus, the net is circular. Such structures are extensively studied in the context of discrete differential geometry [2]. Our case is rather special since two pairs of quadrilaterals span the same plane. This does, however, not hinder application of [2, Theorem 4.21] which states that our assumptions on the co-spherical (or co-planar) position of the points $P_{00}$, $P_{02}$, $P_{11}$, $P_{20}$, and $P_{22}$ is equivalent to the fact that the net $P_{ij}$ is a discrete isothermic net. These nets have many remarkable characterizing properties. One of them, stated in [2, Theorem 2.27], says that the planes $P_{00} \lor P_{11} \lor P_{02}, P_{10} \lor P_{11} \lor P_{12},$ and $P_{20} \lor P_{11} \lor P_{22}$ have a line in common. In our original notation this means that the line $\beta_1 \lor \beta_2$ intersects the face normal of $A_1 \lor A_3 \lor A_4$ through $B_1^*$ and the face normal of $A_1 \lor A_2 \lor A_3$ through $B_3^*$. Therefore, it is orthogonal to $A_1 \lor A_2$. \[ \square \]

As a consequence of Theorem 7 it can be shown that tetrahedra which orthosect A come in conjugate pairs: Given A and an orthosecting tetrahedron B it is possible to construct a second orthosecting tetrahedron C. The same construction with C as input yields the tetrahedron B. This conjugacy is related to the pedal chain originating from B. The key ingredient is the following result from elementary triangle geometry [4, pp. 54–56]:

**Proposition 8.** If P is a point in the plane of the triangle $A_1A_2A_3$ and $c$ its pedal circle, the reflection $Q$ of P in the center M of $c$ has the same pedal circle $c$ (Figure 4).

Suppose that A and B are orthosecting tetrahedra. The orthographic projections $B_i^*$ of the vertices of B onto corresponding face planes of A are points whose pedal triangles form a spherical pedal chain. By reflecting $B_i^*$ in the centers of the pedal circles on the faces of A we obtain points $C_i^*$ which, according to Proposition 8, give rise to a second spherical pedal chain (with the same sphere of vertices) and, by Theorem 7, can be used to construct a second orthosecting tetrahedron C (Figure 5).

The points P and Q of Proposition 8 are called isogonal conjugates with respect to the triangle $A_1A_2A_3$. The above considerations lead immediately to

**Theorem 9.** Given a tetrahedron $A = A_1A_2A_3A_4$, the orthographic projection of all vertices $B_i^*$ of orthosecting tetrahedra onto the face plane $A_1 \lor A_2 \lor A_3 \lor A_4$ is a curve which is isogonally self-conjugate with respect to the triangle $A_1A_2A_3$.

4.3. Computational issues. We continue with a few remarks on the actual computation of the isogonal self-conjugate curves of Theorem 9 with the help of a computer algebra system. Our first result concerns the construction of pedal chains.

**Theorem 10.** Consider a tetrahedron $A = A_1A_2A_3A_4$ and six points $V_{ij} \in A_i \lor A_j$, $(i, j, k, l) \in J_4$. If three of the four triangles $V_{ij}V_{ik}V_{kl}$, with $(i, j, k, l) \in J_4$, are pedal triangles with respect to the triangle $A_1A_2A_3$, then this is also true for the fourth.

**Proof.** Assume that the triangles $V_{12}V_{23}V_{14}, V_{23}V_{24}V_{34},$ and $V_{13}V_{24}V_{14}$ are pedal triangles of their respective face triangles. We have to show that $V_{12}V_{23}V_{13}$ is a pedal triangle of $A_1A_2A_3$. As usual, the anti-pedal points are denoted by $B_1^*, B_2^*, B_3$. Clearly, we
have $B_i^* \perp A_k \forall A_4$ for $(i, j, k, 4) \in J_4$. Denote by $B_i^*$ a point in the intersection of the three planes incident with $V_{ij}$ and perpendicular to $A_i \vee A_j$, $(i, j, k) \in J_3$. By Proposition 9 the tetrahedra $A$ and $B_i^* B_j^* B_k^* B_4^*$ are orthologic. Therefore, the face normals $n_1$ of $A_i \vee A_j \vee A_k$ through $B_i^*$ have a point $B_4$ in common ($l \neq 4, (i, j, k, l) \in J_4$). By the Right-Angle Theorem, the intersection point $B_i^*$ of the orthographic projections of $n_1, n_2$, and $n_3$ onto $A_1 \vee A_2 \vee A_3$ has $V_{12} V_{23} V_{13}$ as its pedal triangle.

In order to construct a pedal chain on a tetrahedron $A = A_1 A_2 A_3 A_4$ on can proceed as follows:

1. Prescribe an arbitrary pedal triangle, say $V_{12} V_{23} V_{13}$.
2. Choose one anti-pedal point, say $B_1^*$, on a neighbouring face. It is restricted to the perpendicular to $A_1 \vee A_2$ through $V_{12}$.
3. The remaining pedal points are determined. Theorem 10 guarantees that the final completion of $V_{34}$ is possible without contradiction.

In order to construct a spherical pedal chain, the choice of $B_1^*$ and $B_2^*$ needs to be appropriate. A simple computation shows that there exist two possible choices (in algebraic sense) for $B_1^*$ such that the points $V_{12}, V_{13}, V_{24}$, and $V_{24}$ are co-spherical (or co-planar). Demanding that the remaining vertex $V_{34}$ lies on the same sphere yields an algebraic condition on the coordinates of $B_1^*$ — the algebraic equation of the isogonally self-conjugate curve $i_4$ from Theorem 9. We are currently not able to carry out the last elimination step in full generality. Examples suggest, however, that $i_4$ is of degree nine. Once a point on $i_4$ is determined, the computation of the corresponding orthosecting tetrahedron is trivial.

5. Conclusion and future research

We introduced the concept of orthosecting tetrahedra and presented a few results related to them. In particular we characterized the six intersection points as vertices of a spherical pedal chain on either tetrahedron. This characterization allows the construction of conjugate orthosecting tetrahedra to a given tetrahedron $A$.

In general, there exists a one-parametric family of tetrahedra which orthosect $A$. The orthographic projection of their vertices on the plane of a face triangle of $A$ is an isogonally self-conjugate algebraic curve. Maybe it is worth to study other loci related to the one-parametric family of orthosecting tetrahedra. Since every sphere that carries vertices of one pedal chain also carries the vertices of a second pedal chain, the locus of their centers might have a reasonable low algebraic degree.

Moreover, other curious properties of orthosecting tetrahedra seem likely to be discovered. For example, the repeated construction of conjugate orthosecting tetrahedra yields an infinite sequence $(B_n)_{n \in \mathbb{Z}}$ of tetrahedra such that $B_{n-1}$ and $B_{n+1}$ form a conjugate orthosecting pair with respect to $B_n$ for every $n \in \mathbb{Z}$. All intersection points of non-corresponding edges lie on the same sphere and only two points serve as orthology centers for any orthosecting pair $B_n, B_{n+1}$. General properties and special cases of this sequence might be a worthy field of further study.

Finally, we would like to mention two possible extensions of this article’s topic. It seems that, with exception of Steiner’s result on orthologic triangles on the sphere, little is known on orthologic triangles and tetrahedra in non-Euclidean spaces. Moreover, one might consider a relaxed “orthology property” as suggested by the anonymous reviewer: It requires that the four lines $a_1, a_2, a_3, a_4$ defined in (9) lie in a regulus (and not necessarily in a linear pencil). This concept is only useful if the regulus position of the lines $a_i$ also implies regulus position of the lines $b_i$.
of (4). We have some numerical evidence that this is, indeed, the case.

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