RATIONALITY OF ALGEBRAIC CYCLES OVER FUNCTION FIELD OF $\text{SL}_1(A)$-TORSORS

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Abstract. In this note we prove a result comparing rationality of algebraic cycles over the function field of a $\text{SL}_1(A)$-torsor for a central simple algebra $A$ and over the base field.

Keywords: Chow groups, central simple algebras, principal homogeneous spaces.

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1. Introduction

Let $A$ be a central simple algebra over a field $F$ and let $\text{Nrd} : A^\times \to F^\times$ be the reduced norm homomorphism. We recall that the homomorphism $F^\times \to H^1(F, \text{SL}_1(A))$, associating to $c \in F^\times$ the $\text{SL}_1(A)$-torsor $X_c$ given by the equation $\text{Nrd} = c$, is surjective (with kernel $\text{Nrd}(A^\times)$) – see [7, Proposition 2.7.3] for instance.

The main purpose of this note is to prove the following theorem dealing with rationality of algebraic cycles over function field of $\text{SL}_1(A)$-torsors.

Theorem 1.1. Let $A$ be a central simple algebra of prime degree $p$ over a field $F$ and let $X$ be a $\text{SL}_1(A)$-torsor. Then

(i) for any equidimensional $F$-variety $Y$, the change of field homomorphism

$$\text{CH}(Y) \to \text{CH}(Y_{F(X)}),$$

where CH is the integral Chow group, is surjective in codimension $< p + 1$.

(ii) it is also surjective in codimension $p+1$ for a given $Y$ provided that the variety $X_{F(\xi)}$ does not have any closed point of prime to $p$ degree for each generic point $\xi \in Y$.

The method of proof mainly relies on the following statement. This proposition is a version of the result [3, Lemma 88.5] slightly altered to fit our situation (see also the proof of [3, Proposition 2.8]).

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Proposition 1.2 (Karpenko, Merkurjev). Let $X$ be a smooth variety, and $Y$ an equidimensional variety. Given an integer $m$ such that for any nonnegative integer $i$ and any point $y \in Y$ of codimension $i$ the change of field homomorphism

$$\text{CH}^{m-i}(X) \to \text{CH}^{m-i}(X_{F(y)})$$

is surjective, the change of field homomorphism

$$\text{CH}^m(Y) \to \text{CH}^m(Y_{F(X)})$$

is also surjective.

The proof of Theorem 1.1 is given in Section 3. In Section 4, we describe how this theorem can be related to a similar result dealing with rationality of algebraic cycles over function field of projective homogeneous varieties under some groups of exceptional type.

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2. Preliminaries

2.1. Topological filtration and Chow groups. For any smooth variety $X$ over a field $F$ (in this paper, an $F$-variety is a separated scheme of finite type over $F$), one can consider the topological filtration on the Grothendieck ring $K_0(X)$, whose term of codimension $i$ is given by

$$\tau^i(X) = \langle [\mathcal{O}_Z] | Z \hookrightarrow X \text{ and codim}(Z) \geq i \rangle,$$

where $[\mathcal{O}_Z]$ is the class in $K_0(X)$ of the structure sheaf of a closed subvariety $Z$. We write $\tau^{i/i+1}(X)$ for the successive quotients. We denote by $pr^*$ the canonical surjection

$$\text{CH}^i(X) \rightarrow \tau^{i/i+1}(X),$$

where $\text{CH}$ is the integral Chow group. By the Riemann-Roch Theorem without denominators the $i$-th Chern class induces an homomorphism in the opposite way $c_i : \tau^{i/i+1}(X) \to \text{CH}^i(X)$ such that the composition $c_i \circ pr$ is the multiplication by $(-1)^{i-1}(i-1)!$.

Note that for any prime $p$, one can also consider the topological filtration $\tau_p$ on the ring $K_0(X)/pK_0(X)$ by replacing $K_0(X)$ by $K_0(X)/pK_0(X)$ in the previous definition. In particular, we get that for any $0 \leq i \leq p$, the map $pr^*_p : \text{Ch}^i(X) \rightarrow \tau^{i/i+1}_p(X)$, where Ch is the Chow group modulo $p$, is an isomorphism.

Remark 2.1. Assume that $X$ is a $\text{SL}_1(A)$-torsor and let $p$ be a prime. One has $K_0(X) = \mathbb{Z}$ by the result [13] Theorem A] of I. Panin and consequently, for $i \geq 1$, the term $\tau^i(X)$ is equal to zero. Therefore, for any $1 \leq i \leq p$, one has $\text{Ch}^i(X) = 0$. Moreover, by the result [17] Theorem 2.7] of A. Suslin, one has $\text{Ch}^i(\text{SL}_p) = 0$ for any $i \geq 1$. Hence, for $A$ of degree $p$ (then there exists a splitting field of $A$ of degree $p$), it follows by transfer argument that $p \cdot \text{Ch}^i(X) = 0$ for any $i \geq 1$. Therefore, for $X$ a $\text{SL}_1(A)$-torsor, with $A$ of prime degree $p$, one has $\text{CH}^i(X) = 0$ for any $1 \leq i \leq p$. Note that, by Proposition 1.2, this gives Theorem 1.1(i) already.
2.2. Brown-Gersten-Quillen spectral sequence. For any smooth variety $X$ and any $i \geq 1$, the epimorphism $pr^i$ coincides with the edge homomorphism of the spectral Brown-Gersten-Quillen structure $E_{2}^{i,-i}(X) \Rightarrow K_0(X)$ (see [16, §7]), that is to say

$$pr^i : CH^i(X) \simeq E_{2}^{i,-i}(X) \rightarrow \cdots \rightarrow E_{r+1}^{i,-i}(X) = \tau^{i+1}(X).$$

Assume that $X$ is a $\text{SL}_1(A)$-torsor, with $A$ of prime degree $p$. Then it follows from Remark 2.1 that $E_{r+1}^{i,-i}(X) = 0$ for $3 \leq i \leq p$. Consequently, one has $A^1(X, K_2) = \text{E}^{1,-2}(X)\text{.}$

Moreover, by the result [11] Theorem 3.4] of A. Merkurjev, for any smooth variety $X$, every prime divisor $l$ of the order of the differential $\delta_r$ ending in $E_{p+1}^{p+1,-p-1}(X)$ is such that $l - 1$ divides $r - 1$. Therefore, for any prime $p$ and $2 \leq r \leq p - 1$, the differential $\delta_r$ is of prime to $p$ order. Assume furthermore that $X$ is a $\text{SL}_1(A)$-torsor, with $A$ of prime degree $p$. Since $p \cdot \text{CH}^{p+1}(X) = 0$ (see Remark 2.1), one deduce that, for $2 \leq r \leq p - 1$, the differential $\delta_r$ is trivial. Consequently, one has $\text{CH}^{p+1}(X) = \text{E}^{p+1,-p-1}(X)\text{.}$

Therefore, for $X$ a $\text{SL}_1(A)$-torsor, with $A$ of prime degree $p$, the differential $\delta_p$ in the BGQ-structure is a homomorphism

$$\delta : A^1(X, K_2) \rightarrow \text{CH}^{p+1}(X).$$

Remark 2.2. Let $X$ be a principal homogeneous space for a semisimple group $G$. By [6] Part II, Example 4.3.3 and Corollary 5.4, one has $E_{2}^{0,-1}(X) = A^0(X, K_1) = F^\infty$ and the composition $F^\infty = K_1(F) \rightarrow K_1(X) \rightarrow A^0(X, K_1)$ of the pullback of the structural morphism with the inclusions

$$K_1^{(0/1)}(X) = E_{\infty}^{0,-1}(X) \subset \cdots \subset E_{3}^{0,-1}(X) \subset E_{2}^{0,-1}(X)$$

given by the BGQ spectral sequence, is the identity. Therefore, for any $i \geq 1$, the differential starting from $E_{i-1}^{0,-1}(X)$ is zero, i.e for any $i \geq 2$, one has

$$E_{i}^{i,-i}(X) = \tau^{i+1}(X).$$

In particular, for $X$ a $\text{SL}_1(A)$-torsor, with $A$ of prime degree $p$, one has $E_{p+1}^{p+1,-p-1}(X) = 0$, i.e the differential $\delta : A^1(X, K_2) \rightarrow \text{CH}^{p+1}(X)$ is surjective.

2.3. On the group $A^1(X, K_2)$. The proof in the next section will use the work of A. Merkurjev on the Rost invariant of simply connected algebraic groups (see [6] Part II). Let $X$ be a $\text{SL}_1(A)$-torsor over $F$. The group $A^1(X_{F(X)}, K_2)$ is infinite cyclic with generator $q$ and isomorphic to $A^1(\text{SL}_n, K_2)$ under restriction (where $n = \deg(A)$). Furthermore, the restriction map $r : A^1(X, K_2) \rightarrow A^1(X_{F(X)}, K_2)$ is injective with finite cokernel of same order as the element $R_{\text{SL}_1(A)}(X)$, where

$$R_{\text{SL}_1(A)} : H^1(F, \text{SL}_1(A)) \rightarrow H^3(F, Q/Z(2))$$

is the Rost invariant of $\text{SL}_1(A)$ (see [6] Theorem 9.10]). Moreover, the homomorphism $R_{\text{SL}_1(A)}$ is of order $\exp(A)$ by [6] Theorem 11.5).

If $\text{char}(F) = l$ is prime then the modulo $l$ component $H^3(F, Z/lZ(2))$ of the Galois cohomology group $H^3(F, Q/Z(2))$ is the group $H^3_l(F)$ defined by K. Kato in [10] by means of logarithmic differential forms.
3. Proof of the Result

In this section, we prove the result of this note.

**Theorem 3.1.** Let $A$ be a central simple algebra of prime degree $p$ over a field $F$ and let $X$ be a $\text{SL}_1(A)$-torsor. Then

(i) for any equidimensional $F$-variety $Y$, the change of field homomorphism

$$\text{CH}(Y) \to \text{CH}(Y_{F(X)}),$$

where $\text{CH}$ is the integral Chow group, is surjective in codimension $< p + 1$.

(ii) it is also surjective in codimension $p + 1$ for a given $Y$ provided that the variety $X_{F(\zeta)}$ does not have any closed point of prime to $p$ degree for each generic point $\zeta \in Y$.

**Proof.** We use notations and materials introduced in the previous section. One can assume that $X$ does not have any rational point over $F$ (or equivalently $X$ does not have any closed point of prime to $p$ degree, by the result [1] Theorem 3.3] of J. Black), if else there is nothing to prove. Note that in this situation, the central simple algebra $A$ is necessarily a division algebra. We recall that conclusion (i) has already been proved (see Remark 2.1). According to Proposition 1.2, it suffices to show that $\text{CH}^{p+1}(X_{F(\zeta)}) = 0$ for each generic point $\zeta \in Y$ to get conclusion (ii). Since $X_{F(\zeta)}$ does not have any closed point of prime to $p$ degree, it is enough to prove that $\text{CH}^{p+1}(X) = 0$.

Assume on the contrary that $\text{CH}^{p+1}(X) \neq 0$. Then $\delta : A^1(X, K_2) \to \text{CH}^{p+1}(X)$ is nonzero (since $\delta$ is surjective by Remark 2.2), i.e $E_{p+1}^{1,-2}(X)$ is strictly included in $E_{p}^{1,-2}(X) = A^1(X, K_2)$. We claim that this implies that, by denoting as $q_X$ the generator of $A^1(X, K_2)$, one has $r(q_X) = q$. Indeed, otherwise one has $r(q_X) = p \cdot q$ by §2.3. Consequently, by denoting as $c$ the corestriction morphism $A^1(\text{SL}_p, K_2) \to A^1(X, K_2)$, for any $i \geq 2$, one has $c(E_{i}^{1,-2}(\text{SL}_p)) = c(A^1(\text{SL}_p, K_2)) = A^1(X, K_2)$ (where the first identity is due to $\text{CH}^i(\text{SL}_p) = 0$ for any $i \geq 2$). In particular, one has $E_{p}^{1,-2}(X) = c(E_{p+1}^{1,-2}(\text{SL}_p)) \subset E_{p+1}^{1,-2}(X)$, which is a contradiction.

Therefore, we have shown that under the assumption $\text{CH}^{p+1}(X) \neq 0$, the generator $q$ of $A^1(X_{F(X)}, K_2)$ is rational. Then it follows that the generator $g$ of $\text{CH}^{p+1}(X_{F(X)})$ is also rational.

However, since $A_{F(X)}$ is a still a division algebra, by [3] Theorem 7.2 and Theorem 8.2], the cycle $g^{p-1}$ in $\text{CH}_0(\text{SL}_1(A_{F(X)}))$ is nonzero and the latter group is cyclic of order $p$ generated by the class of the identity of $\text{SL}_1(A_{F(X)})$. Thus, the degree of the rational cycle $g^{p-1}$ is prime to $p$.

It follows that $X$ has a closed point of prime to $p$ degree, which is a contradiction.

The Theorem is proved. $\square$

**Remark 3.2.** The end of the above proof shows in particular that for a division algebra $A$ of prime degree $p$ over a field $F$, the kernel of the Rost invariant $R_{\text{SL}_1(A)}$ is trivial. This is already contained in the result [12] Theorem 12.2] of A. Merkurjev and A. Suslin under the assumption $\text{char}(F) \neq p$. Indeed, let $\xi \in H^1(F, \text{SL}_1(A))$ and let $X$ be the associated $\text{SL}_1(A)$-torsor. Assume that $R_{\text{SL}_1(A)}(\xi)$ is trivial. It follows then by §2.3 that the generator of $A^1(X_{F(X)}, K_2)$ is rational. As we have seen in the above proof, this implies that $X$ has a rational point over $F$, i.e the cocycle $\xi$ is trivial.
4. Exceptional projective homogeneous varieties

In this section, we describe how Theorem 1.1 implies a similar version of it for projective homogeneous varieties under a group of type $F_4$ or $E_8$. Namely, we give an alternative proof of Theorem 4.1 below. The following proof requires the characteristic of the base field to be different from $p$, with $p = 3$ when $G$ is of type $F_4$ and $p = 5$ when $G$ is of type $E_8$, although the original result [4, Theorem 1.1] is valid for arbitrary characteristic.

Let $X$ be a nonsplit $\text{SL}_1(A)$-torus over a field $F$, with $A$ a division algebra of prime degree $p$. There exists a smooth compactification $\tilde{X}$ of $X$ such that the Chow motive $\mathcal{M}(\tilde{X}, \mathbb{Z}/p\mathbb{Z})$ decomposes as a direct sum $R_p \oplus N$, where $R_p$ is the indecomposable Rost motive associated with the symbol $[A] \cup (c) \in H^3(F, \mathbb{Z}/p\mathbb{Z}(2))$, with $c \in F^x \setminus \text{Nrd}(A^x)$ giving $X$, see [9, Theorem 1.1]. Note that the projective variety $\tilde{X}$ is a norm variety of $s$.

**Theorem 4.1.** Let $G$ be a linear algebraic group of type $F_4$ or $E_8$ over a field $F$ of characteristic different from $p$, with $p = 3$ when $G$ is of type $F_4$ and $p = 5$ when $G$ is of type $E_8$, and let $X'$ be a projective homogeneous $G$-variety. For any equidimensional variety $Y$, the change of field homomorphism

$$\text{Ch}(Y) \rightarrow \text{Ch}(Y_{F(X')}),$$

where $\text{Ch}$ is the Chow group modulo $p$, is surjective in codimension $< p + 1$.

It is also surjective in codimension $p+1$ for a given $Y$ provided that $1 \notin \text{deg } \text{Ch}_0(X'_{F(\zeta)})$ for each generic point $\zeta \in Y$.

**Proof.** Since the $F$-variety $X'$ is $A$-trivial in the sense of [8, Definition 2.3], one can assume that $G$ has no splitting field of degree coprime to $p$. Indeed, otherwise $1 \in \text{deg } \text{Ch}_0(X')$ by corestriction and this implies that $\text{Ch}(Y) \rightarrow \text{Ch}(Y_{F(X')})$ is an isomorphism in any codimension by $A$-triviality, see [8, Lemma 2.9].

Let us now write $G = G_0 \times G_0$ for a nontrivial cocycle $\xi \in H^1(F, G_0)$, with $G_0$ a split group of the same type as $G$. Then the motive $R_p(G)$ living on the Chow motive (with coefficients in $\mathbb{Z}/p\mathbb{Z}$) of $X'$ given in [13, Theorem 5.17] is the Rost motive of the symbol $R_{G_0, p}(\xi) = [A] \cup (c) \in H^3(F, \mathbb{Z}/p\mathbb{Z}(2))$, where $R_{G_0, p}$ is the the modulo $p$ component of the Rost invariant $R_{G_0}$, $A$ is a division algebra of degree $p$ and $c \in F^x \setminus \text{Nrd}(A^x)$ – see [13, §4] and [5, §14] (here the assumption char($F$) $\neq p$ is needed).

Let us denote as $X$ the nonsplit $\text{SL}_1(A)$-torus over $F$ associated with $c$ and as $\tilde{X}$ its smooth compactification. We claim that $X'$ has a closed point of prime to $p$ degree over $F(\tilde{X})$ and vice versa.

Indeed, since $\tilde{X}$ is a norm variety for $[A] \cup (c)$, the motive $R_p(G)$ decomposes as a sum of Tate motives over $F(\tilde{X})$. Therefore, the group $G_{F(\tilde{X})}$ is split by an extension of degree coprime to $p$ and it follows that $X'$ has a closed point of prime to $p$ degree over $F(\tilde{X})$ (this is more generally true for any extension $L/F$ over which $\tilde{X}$ has a closed point of prime to $p$ degree). Moreover, the motive $R_p(G)$ decomposes as a sum of Tate motives...
over $F(X')$ because $G$ is split by $F(X')$. Consequently, $\tilde{X}$ has a closed point of prime to $p$ degree over $F(X')$.

It follows then (note that $\tilde{X}$ is $A$-trivial by [3, Example 5.7]) that the right and the bottom homomorphisms in the commutative square

$$
\begin{array}{ccc}
Ch(Y) & \to & Ch(Y_{F(X)}) \\
\downarrow & & \downarrow \\
Ch(Y_{F(\tilde{X})}) & \to & Ch(Y_{F(\tilde{X} \times X')})
\end{array}
$$

are isomorphisms. Since $F(\tilde{X}) = F(X)$, Theorem 4.1 is now a direct consequence of Theorem 1.1. □

The following was pointed out to me by Philippe Gille.

**Remark 4.2.** Let $G_0$ a split group of type $E_8$ over a 5-special field $F$ (i.e $F$ has no proper extension of degree coprime to 5) of characteristic $\neq 5$. The above proof gives rise to a new argument for the triviality of the kernel of the Rost invariant modulo 5

$$H^1(F, G_0) \to H^3(F, \mathbb{Z}/5\mathbb{Z}(2)).$$

This result is originally due to Vladimir Chernousov (under the assumption $\text{char}(F) \neq 2, 3, 5$, see [2, Theorem]).

Indeed, since $F$ is 5-special, for any nontrivial cocycle $\xi \in H^1(F, G_0)$, the group $\xi G_0$ has no splitting field of degree coprime to 5. Then, as we have seen in the proof, there is a division algebra $A$ of degree 5 such that $R_{G_0,5}(\xi)$ is equal to a symbol $[A] \cup (c)$ associated with a nonsplit $\text{SL}_1(A)$-torsor $X$. The injectivity of $R_{G_0,5}$ follows now from Remark 3.2.

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