The Book Thickness of 1-Planar Graphs is Constant

Michael A. Bekos, Till Bruckdorfer, and Michael Kaufmann

Wilhelm-Schickhard-Institut für Informatik, Universität Tübingen, Germany

Chrysanthi N. Raftopoulou

School of Applied Mathematical & Physical Sciences, NTUA, Greece.

Abstract

In a book embedding, the vertices of a graph are placed on the “spine” of a book and the edges are assigned to “pages”, so that edges on the same page do not cross. In this paper, we prove that every 1-planar graph (that is, a graph that can be drawn on the plane such that no edge is crossed more than once) admits an embedding in a book with constant number of pages. To the best of our knowledge, the best non-trivial previous upper-bound is $O(\sqrt{n})$, where $n$ is the number of vertices of the graph.

1 Introduction

A book embedding is a special type of a graph embedding, in which (i) the vertices of the graph are restricted to a line along the spine of a book, and, (ii) the edges on the pages of the book in such a way that edges residing on the same page do not cross. The minimum number of pages required to construct such an embedding is known as book thickness or page number of a graph. An obvious upper bound on the page number of an $n$-vertex graph is $\lceil n/2 \rceil$, which is tight for complete graphs [3]. Book embeddings have a long history of research dating back to early seventies [19]. Therefore, there is a rich body of literature (see, e.g., [4] and [20]).

For the class of planar graphs, a central result is due to Yannakakis [23], who in the late eighties proved that planar graphs have book thickness at most four. It remains, however, unanswered whether the known bound of four is tight. Heath [10], for example, proves that all planar 3-trees are 3-page book embeddable. For more restricted subclasses of planar graphs, Bernhart and Kainen [3] show that the graphs with book thickness one are the outerplanar graphs, while the class of two-page embeddable graphs coincides with the class of subhamiltonian graphs (recall that subhamiltonian is a graph that is a subgraph of a planar Hamiltonian graph). Testing whether a graph is subhamiltonian is NP-complete [22]. However, several graph classes are known to be subhamiltonian (and therefore two-page book embeddable), e.g., 4-connected planar graphs [18], planar graphs without separating triangles [13], Halin graphs [7], planar graphs with maximum degree 3 or 4 [11, 2].

In this paper, we go a step beyond planar graphs. In particular, we consider 1-planar graphs and prove that their book thickness is constant. Recall that a graph is 1-planar, if it admits a drawing in which each edge is crossed at most once. To the best of our knowledge, the only (non-trivial) upper bound on the book thickness of 1-planar graphs on $n$ vertices is $O(\sqrt{n})$. This is
due to two known results: First, graphs with \( m \) edges have book thickness \( O(\sqrt{m}) \) [16]. Second, 1-planar graphs with \( n \) vertices have at most \( 4n - 8 \) edges, which is a tight bound [3] [12] [21]. Minor-closed graphs (e.g., graphs of constant treewidth [8] or genus [15]) have constant book thickness [17]. Unfortunately, however, 1-planar graphs are not closed under minors [17].

In the remainder of this paper, we will assume that a simple 1-planar drawing \( \Gamma(G) \) of the input 1-planar graph \( G \) is also specified as part of the input of the problem. This is due to a result of Grigoriev and Bodlaender [9], and, independently of Kohrzik and Mohar [14], who proved that the problem of determining whether a graph is 1-planar is NP-hard (note that the problem remains NP-hard, even if the deletion of a single edge makes the graph planar [6]). In addition, we assume biconnectivity, as it is known that the page number of a graph equals the page number of its biconnected components [3].

2 Definitions and Yannakakis Algorithm

Let \( G \) be a simple topological graph, that is, undirected and drawn in the plane. We denote by \( \Gamma(G) \) the drawing of \( G \). Unless otherwise specified, we consider simple drawings, that is, no edge crosses itself, no two edges meet tangentially and no two edges cross more than once. A drawing uniquely defines the cyclic order of the edges incident to each vertex and, therefore, specifies a combinatorial embedding. A 1-planar topological graph is called planar-maximal or simply maximal, if the addition of a non-crossed edge is not possible. The following lemma, proven in many earlier papers, shows that two crossing edges induce a \( K_4 \), as the missing edges can be added without introducing new crossings; see, e.g., [1].

**Lemma 1.** In a maximal 1-planar topological graph, the endpoints of two crossing edges are pairwise adjacent.

The base of our approach is the simple version of Yannakakis algorithm, which embeds any (internally-triangulated) plane graph in a book of five pages [23]; not four. In the following, we outline this algorithm. However, we assume basic familiarity. The algorithm is based on a “peeling” into levels approach. In particular, (i) vertices on the outerface are at level zero; (ii) vertices that are on the outerface of the graph induced by deleting all vertices of levels \( \leq i - 1 \) are at level \( i \); (iii) edges between vertices of the same (different, resp.) level are called level (binding, resp.) edges; see Fig. [1]. In a high-level description, the algorithm first embeds level zero followed by level one and the binding edges between levels zero and one. The remaining graph, that is in the interior of all level-one cycles, is embedded recursively.

2.1 The two-level case

To achieve a total of five pages, first it is proven that a graph \( G = (V, E) \), consisting only of two levels, say \( L_0 \) and \( L_1 \), is three page embeddable (it is also assumed that \( L_0 \) has no chords). The vertices, say \( u_1, u_2, \ldots, u_k \), of \( L_0 \) are called outer and appear in this order along the clockwise traversal of the outerface of \( G \). The remaining vertices are called inner (and obviously belong to \( L_1 \)). The graph induced by all outer vertices is biconnected. The biconnected components (or blocks), say \( B_1, B_2, \ldots, B_m \), of the graph induced by the inner vertices form a tree (in the absence of chords in \( L_0 \)). It is assumed that the block-tree is rooted at the block, w.l.o.g. at block \( B_1 \), that contains the so-called first inner vertex, which is uniquely defined as the third vertex of the bounded face containing the outer vertices \( u_1 \) and \( u_k \). Given a block \( B_i \), an outer vertex is said to be adjacent to \( B_i \) if it is adjacent to a vertex of it; the set of outer vertices adjacent to \( B_i \) is denoted by \( N(B_i) \), \( i = 1, 2, \ldots, m \). Furthermore, a vertex \( w \) is said to see an edge \( (x, y) \), if \( w \) is adjacent to \( x \) and \( y \) and the triangle \( x, y, w \) is a face. An outer vertex sees a block if it sees an edge of it.

The leader of a block \( B_i \) is the first vertex of block \( B_i \) that is encountered in any path from the first inner vertex to block \( B_i \) and is denoted by \( \ell(B_i) \). An inner vertex that belongs to only
one block is assigned to that block. One that belongs to more than one blocks is assigned to the “highest block” in the block-tree that contains it. Given an inner vertex $v \in L_1$, we denote by $B(v)$ the block that $v$ is assigned to. The dominator of a block $B_i$ is the first outer vertex that is adjacent to a vertex of block $B_i$ and is denoted by $dom(B_i)$, $i = 1, 2, \ldots, m$.

Let $B$ be a block of level $L_1$ and assume that $v_0, v_1, \ldots, v_k$ are the vertices of $B$ as they appear in a counterclockwise traversal of the boundary of $B$ starting $v_0 = \ell(B)$. Denote by $u_f(B)$ and $u_l(B)$ the smallest- and largest-indexed vertices of level $L_0$ that see edges $(v_0, v_k)$ and $(v_0, v_1)$, respectively. Equivalently, $u_f(B)$ and $u_l(B)$ are defined as the smallest- and largest-indexed vertices of $N(B)$. Note that $u_f(B) = dom(B)$. The path on level $L_0$ from $u_f(B)$ to $u_l(B)$ in clockwise direction along $L_0$ is denoted by $P[u_f(B) \rightarrow u_l(B)]$.

2.2 Linear order

The linear order of the vertices along the spine is computed as follows. First, the outer vertices are embedded in the order $u_1, u_2, \ldots, u_k$. For $j = 1, 2, \ldots, k$, the blocks dominated by the outer vertex $u_j$ are embedded right next to $u_j$ one after the other in the top to down order of the block-tree. The vertices that belong to block $B_i$ are ordered along the spine in the order that appear in the counterclockwise traversal of the boundary of $B_i$ starting from $\ell(B_i)$, $i = 1, 2, \ldots, m$ (which is already placed).

2.3 Edge-to-page assignment

The edges are assigned to pages as follows. All level edges of $L_0$ are assigned to the first page. Level-one edges are assigned either to the second or to the third page, based on whether they belong to a block that is in an odd or even distance from the root of the block tree, respectively. Binding edges are further classified as forward or back. A binding edge is forward if the inner vertex precedes the outer vertex; otherwise it is back (recall that a binding edge connects an outer and an inner vertex). All back edges are assigned to the first page. A forward edge incident to a block $B_i$ is assigned to the second page, if $B_i$ is on the third page; otherwise to the third page, $i = 1, 2, \ldots, m$. 

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**Figure 1.** (a) Outer (inner) vertices are colored white (gray). Level (binding) edges are solid (dashed). Blocks are highlighted in gray. The first inner vertex is $v_1$. So, the root of the block tree is $B_1$. $N(B_3) = \{u_2, u_3\}$. Vertex $v_2$ sees $(v_3, v_7)$ and so sees $B_2$. The leaders of $B_1$, $B_2$, $B_3$, $B_4$ and $B_5$ are $v_1$, $v_3$, $v_6$, $v_5$ and $v_2$, resp. The dominators of $B_1$, $B_2$, $B_3$, $B_4$ and $B_5$ are $u_1$, $u_2$, $u_3$ and $u_4$, resp.; The red edges indicate that $u_f(B_2) = u_2$ and $u_l(B_2) = u_4$. Hence, $P[u_f(B_2) \rightarrow u_l(B_2)] = u_2 \rightarrow u_3 \rightarrow u_4$. (b) Linear order and assignment of edges to pages.
2.4 The multi-level case

Possible chords in level \( L_0 \) are assigned to the first page. Note, however, that in the presence of such chords, the blocks of level \( L_1 \) form a forest in general (i.e., not a single tree). Therefore, each block-tree of the underlying forest must be embedded according to the rules described above. Graphs with more than two layers are embedded by “recycling” the remaining available pages. More precisely, consider a block \( B \) of level \( i \) and let \( B' \) be a block of level \( i \) that is in the interior of \( B \) in the peeling order. Let \( \{p_1, \ldots, p_5\} \) be a permutation of \( \{1, \ldots, 5\} \) and assume w.l.o.g. that the boundary of block \( B \) is assigned to page \( p_1 \), while the boundary of all blocks in its interior (including \( B' \)) are assigned to pages \( p_2 \) and \( p_3 \). Then, the boundary of all blocks of level \( i + 1 \) that are in the interior of \( B' \) in the peeling order will be assigned to pages \( p_4 \) and \( p_5 \). The correctness of this strategy follows from the fact that all blocks of level \( i + 1 \) that are in the interior of a certain block of level \( i \) are always between two consecutive vertices of level \( i - 1 \). This directly implies that blocks that are by at least two levels apart in the peeling order are in a sense independent, which allows pages that have already been used by some previous levels to be reused by blocks of next levels provided that they are not consecutive. In the following we present properties that we use in the remainder of the paper.

**Lemma 2** (Yannakakis [23]). Let \( G \) be a graph consisting of two levels \( L_0 \) and \( L_1 \). Let \( B \) be a block of level \( L_1 \) and let \( v_0, \ldots, v_t \) be the vertices of \( B \) in a counterclockwise order along the boundary of \( B \) starting from \( v_0 = \ell(B) \). Then:

(i) Vertices \( v_1, \ldots, v_t \) are consecutive along the spine.

(ii) \( u_f(B) \neq u_l(B) \).

(iii) If \( u_i = u_f(B) \) and \( u_j = u_l(B) \) for some \( i < j \), then vertices \( v_1, \ldots, v_t, u_i, u_{i+1}, \ldots, u_j \) appear in this order from left to right along the spine.

(iv) Let \( G[B] \) be the subgraph of \( G \) in the interior of cycle \( P[u_f(B) \to u_l(B)] \to \ell(B) \to u_f(B) \). Then, a block \( B' \in G[B] \) if and only if \( B \) is an ancestor of \( B' \), that is, \( B' \) belongs to the block-subtree rooted at \( B \).

**Lemma 3** (Yannakakis [23]). Let \( G \) be a graph consisting of two levels \( L_0 \) and \( L_1 \) and assume that \( (u_i, u_j), i < j, \) is a chord of \( L_0 \). Denote by \( H \) the subgraph of \( G \) in the interior of the cycle \( P[u_i \to u_j] \to u_i \). Then:

(i) Vertices \( u_i \) and \( u_j \) form a separation pair in \( G \).

(ii) All vertices of \( H \) lie between \( u_i \) and \( u_j \) along the spine.

(iii) If there is a vertex between \( u_i \) and \( u_j \) that does not belong to \( H \), then this vertex belongs to a block \( B \) dominated by \( u_i \). In addition, all vertices of \( H \), except for \( u_i \) are to the right of \( B \) along the spine.

3 An upper bound on the book thickness of 1-Planar Graphs

In this section, we extend the algorithm of Yannakakis [23] to 1-planar graphs based on the “peeling” approach described in Section 2. Let \( G = (V, E) \) be a 1-planar graph and \( \Gamma(G) \) be a 1-planar drawing of \( G \). Our approach, in high level description, is as follows. Initially, we consider the case where \( \Gamma(G) \) contains no crossings incident to its unbounded face. We augment \( G \) to internally maximal 1-planar. To do so, we momentarily replace each crossing in \( \Gamma(G) \) with a so-called crossing vertex. The implied planarized graph is then triangulated (only in its interior), so that no new edge is incident to a crossing vertex. The latter restriction,
however, may lead to a non-simple graph (containing multiedges), as we will see in Section 3.3.

On the other hand, the interior of all $K_{4, 5}$ implied by Lemma 1 are free of vertices and edges.

To simplify the presentation, we initially assume that the planarized graph is simple. So, $G$

is augmented to a simple internally maximal 1-planar graph with no crossings incident to its unbounded face. Following an approach slightly different from the one of Yannakakis [23], we show how one can define the levels for such a graph. We prove that if there are only two levels, then such a graph fits in 16 pages. When the number of levels is greater than two, we prove that 39 pages suffice. Finally, we show how one can cope with the cases of multiedges and crossings on the unbounded face of $\Gamma(G)$.

Assume now that $G$ is simple and internally maximal 1-planar with no crossings incident to the unbounded face of $\Gamma(G)$. Its vertices are assigned to levels as follows: (i) vertices on the outerface of $G$ are at level zero; (ii) vertices that are at distance $i$ from the level zero vertices are at level $i$. Similarly to Yannakakis naming scheme, edges that connect vertices of the same (different, resp.) level are called level (binding, resp.) edges.

Since we have assumed that $G$ is internally maximal 1-planar and that there are no crossings incident to its unbounded face, by Lemma 1 it follows that the endpoints of every crossing pair of edges are pairwise adjacent. So, if we remove one edge from each pair of crossing edges, then the result is an internally-triangulated plane graph (which we call underlying planar structure).

We apply the following simple rule. For a pair of binding crossing edges or for a pair of level crossing edges, we choose arbitrarily one to remove. However, for a pair of crossing edges consisting of a binding edge and a level edge, we always choose to remove the level edge. The main benefit of the aforementioned approach is that, the underlying planar structure allow us to define the blocks as well as the leaders and the dominators of the blocks in the exact same way as Yannakakis does. In addition, it is not difficult to observe that if all removed edges are plugged back to the graph, then a binding edge cannot cross a block, since such a crossing would involve an edge on the boundary of the block, which by definition is a level edge (and therefore not present when the blocks are computed).

### 3.1 The Two-Level Case

In this section, we consider the (intuitively easier) case, where the given 1-planar graph $G$

consists of two levels $L_0$ and $L_1$. We also assume that there is no level edge of $L_1$ which by the combinatorial embedding is strictly in the interior of a block of $L_1$. In addition, $G$ is simple internally maximal 1-planar and has no crossings on its unbounded face. We denote by $G_P$ the underlying planar structure and proceed to obtain a 3-page book embedding of $G_P$ using Yannakakis algorithm [23] (see Section 2). We argue that we can embed the removed edges in the linear order implied by the book embedding of $G_P$ using 11 more pages.

Before we proceed with the description of our approach, we introduce an important notion useful in “eliminating” possible crossing situations. We say that two edges $e_1$ and $e_2$ of $G$ form a strong pair if (i) they are both assigned to the same page, say $P$, and, (ii) if an edge $e$, that is assigned also to page $P$, crosses $e_i$, then it also crosses $e_j$, where $i \neq j \in \{1, 2\}$. Suppose that $e_1 \notin E[G_P]$ and $e_2 \in E[G_P]$ form a strong pair of edges. If $e_3 \in E[G_P]$, then $e_3$ can cross neither $e_1$ nor $e_2$ (due to Yannakakis’ algorithm). On the other hand, if $e_3 \notin E[G_P]$ and forms a strong pair with another edge $e_4 \in E[G_P]$, then again $e_3$ can cross neither $e_1$ nor $e_2$, as otherwise $e_4$ would also be involved in a crossing with $e_1$ or $e_2$; contradicting the correctness of Yannakakis’ algorithm as $e_4 \in E[G_P]$.

In the following, we describe six types of crossings that may occur when the removed edges are plugged back to $G$; see Fig. 2. Level edges of $L_0$ that do not belong to the planar structure $G_P$ are called outer crossing chords. Such chords may be involved in crossings with (i) other chords.

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1We will shortly adjust this choice for two special cases. However, since we do not seek to enter into further details at this point, we assume for now that the choice is, in general, arbitrary.
of \( L_0 \) that belong to \( G_P \), or, (ii) binding edges (between levels \( L_0 \) and \( L_1 \)), or, (iii) degenerated blocks (so-called block-bridges) of level \( L_1 \) that are simple edges.

Level edges of \( L_1 \) that do not belong to the planar structure are called inner crossing chords or simply 2-hops.\(^2\) We claim that 2-hops do not cross with each other. For a proof by contradiction, assume that \( e = (u, v) \) and \( e' = (u', v') \) are two 2-hops that cross. Assume w.l.o.g. that \( u, u', v \) and \( v' \) appear in this order along the boundary of the block tree of the underlying planar structure \( G_P \). Since \( G \) is maximal 1-planar, by Lemma \(^1\) it follows that \( (u, v') \) belongs to the planar structure \( G_P \). On the other hand, in the presence of this edge both vertices \( u' \) and \( v \) are not anymore at the boundary of the block tree of level \( L_1 \) of \( G_P \), which is a contradiction as \( e \) and \( e' \) are both level edges of \( L_1 \). Hence, 2-hops are involved in crossings only with binding edges. Since level edges of different levels cannot cross, the only type of crossings that we have not reported so far are those between binding edges.

Recall that binding edges are of two types: forward and back. For a pair of crossing binding edges, say \( e = (u_i, v_j) \) and \( e' = (u_{i'}, v_{j'}) \), where \( u_i, u_{i'} \in L_0 \) and \( v_j, v_{j'} \in L_1 \), we mentioned that we can arbitrarily choose which one is assigned to the underlying planar structure \( G_P \). Here, we adjust this choice: Edge \( e \) is assigned to \( G_P \) if and only if vertex \( u_i \) is lower-indexed than \( u_{i'} \), that is, \( i < i' \). As a consequence, edge \( e' \) is always forward binding. Therefore, two back binding edges cannot cross.

Similarly, for a pair of crossing level edges, we mentioned that we arbitrarily choose, which one to assign to the underlying planar structure \( G_P \). We adjust this choice in the case of two crossing level edges of level \( L_0 \) as follows: If a level edge is incident to the first outer vertex \( u_1 \) of level \( L_0 \), then it is necessarily assigned to \( G_P \).

From the above, it follows that for a pair of crossing edges, say \( e \in E[G_P] \) and \( e' \notin E[G_P] \), we have the following crossing situations, each of which is separately treated in the following lemmas (except for the last one which is more demanding):

C.1: \( e' \) is an outer crossing chord and \( e \) is a chord of \( L_0 \) that belongs to \( G_P \).

C.2: \( e' \) is an outer crossing chord and \( e \) is a binding edge.

C.3: \( e' \) is an outer crossing chord and \( e \) is a block-bridge of \( L_1 \).

C.4: \( e' \) is a forward binding edge and \( e \) is a forward binding edge.

C.5: \( e' \) is a forward binding edge and \( e \) is a back binding edge.

\(^2\)The term yields from the observation that along the boundary of the block-tree a 2-hop can “bypass” only one vertex because of maximal 1-planarity.
C.6: \( e' \) is a 2-hop and \( e \) is a binding edge.

Our approach is outlined in the proof of the following theorem, which summarizes the main result of this section.

**Theorem 1.** Any simple internally maximal 1-planar graph \( G \) with 2 levels and no crossings incident to its unbounded face admits a book embedding on 16 pages.

**Proof.** The underlying planar structure can be embedded in three pages. Case C.1 requires one extra page; due to Lemma 6 (Case C.4) the crossing edges that fall into Cases C.2 and C.3 can be accommodated on the same pages used for the underlying planar structure; see Lemma 5. Case C.4 requires two extra pages due to Lemma 6. Case C.5 requires three extra pages due to Lemma 7. Finally, Case C.6 requires seven more pages due to Lemma 8. Summing up the above yields 16 pages in total.

We start by investigating the case where \( e' \) is an outer crossing chord of \( G \). By Lemma 4 any two outer crossing chords can be placed on the same page without crossing each other, while Lemma 5 describes the placement of outer crossing chords for Cases C.1, C.2 and C.3.

**Lemma 4.** Let \( u_1, \ldots, u_n \) be the vertices of level \( L_0 \) in clockwise order along its boundary. Let \( c = (u_i, u_j) \) and \( c' = (u_{i'}, u_{j'}) \) be two chords of \( L_0 \), such that \( i < i' < j < j' \). Then, exactly one of \( c \) and \( c' \) is an outer crossing chord.

**Proof.** Since \( c = (u_i, u_j) \) and \( c' = (u_{i'}, u_{j'}) \) are chords of \( L_0 \) with \( i < i' < j < j' \), \( c \) and \( c' \) cross in the 1-planar drawing \( \Gamma(G) \) of \( G \). So, one of them would belong to the underlying planar structure of \( G \) and the other one would be an outer crossing chord.

Lemma 4 implies that outer crossing chords can be placed on one page without crossing. However, as stated in the following lemma, we choose not to do so. Recall that we use three pages for \( G_P \); \( p_1 \), \( p_2 \) and \( p_3 \). Page \( p_1 \) is devoted to level edges of \( L_0 \) and back binding edges of \( G_P \). Pages \( p_2 \) and \( p_3 \) are used for level edges of \( L_1 \) and forward binding edges.

**Lemma 5 (Cases C.1 - C.3).** Let \( e = (u, v) \in E(G_P) \) and \( e' = (u', v') \notin E(G_P) \) be two edges of \( G \) that are involved in a crossing. If \( e' \) is outer crossing chord of \( L_0 \), then:

(i) If \( e \) is a chord of \( L_0 \), then \( e' \) is placed on a universal page denoted by \( u_{p_1} \) (Case C.4).

(ii) If \( e \) is a binding or a block-bridge of \( L_1 \), then \( e' \) is assigned to page \( p_1 \), that is, the page used for level edges of \( L_0 \) and back binding edges of \( G_P \) (Cases C.2 and C.3).

**Proof.** (i) Since \( e \) is chord of level \( L_0 \), \( e \) is placed on page \( p_1 \) (recall that \( e \in E(G_P) \)), and \( e' \) is placed on the universal page \( u_{p_1} \). Since \( u_{p_1} \) contains only outer crossing chords of \( G \), by Lemma 2 they do not cross with each other.

(ii) If \( e \) is a binding or a block-bridge of level \( L_1 \), then \( e' \) is assigned to page \( p_1 \). Suppose that \( e' \) is in conflict with another edge, say \( e'' \), of page \( p_1 \). By Lemma 4, edge \( e'' \) is not an outer crossing chord, that is, \( e'' \) belongs to the underlying planar structure \( G_P \) of \( G \). So, \( e'' \in E(G_P) \) and it is either: (a) a level edge of level \( L_0 \), or (b) a back binding edge of \( G_P \). In the first case, the endpoints of \( e'' \) cannot be consecutive vertices of level \( L_0 \), since that would not lead to a crossing situation. Hence, \( e'' \) must be a chord of level \( L_0 \). However, if \( e' \) is involved in such a crossing, then \( e' \) is assigned to page \( u_{p_1} \); a contradiction. In the second case, \( e'' \) is back binding of \( G_P \). So, \( e'' \) is nested by a level edge of level \( L_0 \) and, therefore, if \( e' \) crosses \( e'' \), then \( e' \) must also cross this particular level edge of level \( L_0 \), which is not possible.

**Lemma 6 (Case C.4).** All forward binding edges that are involved in crossings with forward binding edges of the underlying planar structure can be assigned to 2 new pages.
Proof. To prove the lemma, we employ a simple trick. We observe that, for a pair of crossing forward binding edges, the choice of the edge that will be assigned to the underlying planar structure affects neither the decomposition into blocks nor the choice of dominators and leaders of blocks. Therefore, it does not affect the linear order of the vertices along the spine. This ensure that two new pages suffice. 

We proceed with Case C[5], where the back binding edge \( e = (u, v) \in E[G_P] \) crosses the forward binding \( e' = (u', v') \notin E[G_P] \). Let \( P \) be the block containing \( (v, v') \) and let \( v_0, v_1, \ldots, v_t \) be the vertices of \( P \) as they appear in the counterclockwise order around \( P \) starting from \( v_0 = \ell(P) \). Since \( e \) is back binding, it follows that \( u = u_f(P) \). By definition of \( u_f(P) \), \( u \) sees edges \((v_i, v_{i+1}), \ldots, (v_{t-1}, v_t), (v_t, v_0)\) of \( P \), for some \( 1 \leq i \leq t \). Hence, edges \((u, v_0), (u, v_t), \ldots, (u, v_i)\) exist and are back binding edges. This implies that either \( v = v_i \) and \( v' = v(i+1) \) or \( v = v_0 \) and \( v' = v_1 \). In the latter case and assuming that \( u' \) is to the right of \( u \) on the spine, \( P \) is a root-block. In both cases, \((u', v)\) is forward.

**Lemma 7 (Case C[5]).** Let \( e = (u, v) \) be a back binding edge and \( e' = (u', v') \) a forward binding edge of \( G \) that cross. Let also \( v_0, v_1, \ldots, v_t \) be the vertices of block \( P \) as they appear in the counterclockwise order around \( P \) starting from \( v_0 = \ell(P) \), where \( P \) is the block containing \((v, v')\). Finally, let \( i \) be the minimum s.t. vertex \( u = u_f(P) \) sees edges \((v_i, v_{i+1}), \ldots, (v_{t-1}, v_t), (v_t, v_0)\). Then we use three new pages \( p'_1, p'_2 \) and \( p'_3 \) as follows:

(i) If \( v = v_i \) and \( v' = v(i+1) \), then edge \( e' \) is placed on a new page \( p'_j \), if and only if the forward edges incident to block \( B(v') \) are assigned to page \( p_j \), \( j = 2, 3 \).

(ii) If \( v = v_0 \) and \( v' = v_1 \), then edge \( e' \) is placed on a page \( p'_1 \).

Proof. (i) We prove a stronger result. In particular, we prove that if the forward edges incident to \( B(v') \) are on \( p_j \), then \( e' \) can also be placed on \( p_j \) without crossings. Clearly, if this is true, then the lemma follows, as one can always split one page into two. We distinguish two cases based on whether \( v = v_i \) or \( v = v_t \) for \( i < t \).

First, assume that \( v = v_i \) and \( v' = v_0 \); refer to \( e = (u_2, v_0) \) and \( e' = (u_3, v_0) \) in Fig. 3a. Let \( B = B(v) \) and \( B' = B(v') \). Then, \( B = P \) and \( B' \) is the parent-block of \( B \). W.l.o.g. assume that the boundary of \( B' \) is on \( p_2 \). We claim that \( e' \) can be placed on \( p_3 \) (together with forward edges of \( B' \)). To prove it, we show that \( e' = (u', v_0) \) and \( (v_0, v_t) \) form a strong pair. First, observe that \((v_0, v_t)\) is on \( p_3 \); \( B' \) is on \( p_2 \), so \( B \) is on \( p_3 \) and \((v_0, v_t)\) is an edges of \( B \). By Lemma 2[5], vertices \( v_1, \ldots, v_t \) and \( u' \) appear in the same order from left to right along the spine, so \((v_0, v_t)\) is nested by \( e' \). So, by Lemma 3, \( e' \) and \((v_0, v_t)\) form a strong pair.

In the case where \( v = v_i \) and \( v' = v_{i+1} \) for some \( i < t \) (refer to \( e = (u_1, v_2) \) and \( e' = (u_2, v_3) \) in Fig. 3a), we have that \( B = B' = P \). Suppose w.l.o.g. that \( P \) is placed on \( p_2 \). We claim that \((v_0, v_t)\) is a forward edge of \( G_P \).
and is therefore placed on page \( p_3 \). Since vertices \( v_i \) and \( v_{i+1} \) are consecutive along the spine, edges \( e' \) and \( (u', v_i) \) form a strong pair and the lemma follows.

(ii) In this case (refer to \( e = (u_1, v_1) \) and \( e' = (u_5, v_3) \) in Fig. 3a), we have that \( P \) is a root-block and by Lemma 1 edge \((u, u')\) \( \in E[G_P]\). Let \( e_1' \) and \( e_2' \) be two edges that are assigned to the new page \( p_1' \). We claim that they do not cross. Assume that \( e_1' = (u_i, v_i) \) crosses \( G \) with \( e_i = (u_i, v_i) \) for \( i = 1, 2 \). Then, \((u_i, u_i') \in E[G_P]\) is level edge of \( L_0 \), for \( i = 1, 2 \). We assume w.l.o.g. that \( u_2 \) and \( u_2' \) are not between \( u_1 \) and \( u_1' \) along the spine (if this was not true for neither pair of vertices, then they would cross in \( G \) and by Lemma 4 one of them would not belong to \( G_P \)). By Lemma 3 edge \( e_1' \) can’t cross with \( e_2' \).

Finally, we consider Case C.6 where \( e' = (x, y) \) is a 2-hop of level \( L_1 \) and \( e = (u, z) \) is a binding edge of \( G_P \), where \( x, y, z \in L_1 \) and \( u \in L_0 \). Let \( x, z \) and \( y \) belong to blocks \( B_x \) and \( B_y \) respectively, that are not necessarily distinct. By Lemma 1, \( x \rightarrow z \rightarrow y \) is a path in \( L_1 \). So, \( B_x \) and \( B_y \) are at distance at most two on the block-tree of \( G \). If \( x \) and \( y \) belong to the same block (that is, \( B_x = B_y \)), then \( e \) is called simple 2-hop; see Fig. 4a. Suppose w.l.o.g. that \( B_x \) precedes \( B_y \) in the pre-order traversal of the block-tree of \( G \). Then, there exist two cases depending on whether \( B_x \) is an ancestor of \( B_y \) on the block-tree. If this is not the case, then \( B_x \) and \( B_y \) have the same parent-block, say \( B_p \). In this case, \( e' \) is called bridging 2-hop; see Fig. 4b. Suppose now that \( B_x \) is an ancestor of \( B_y \). Then, the path \( x \rightarrow z \rightarrow y \) contains the leader of \( B_y \), which is either \( x \) or \( z \). By Lemma 1 \((u, x) \) (\( u, z \)) and \((u, y) \) exist in \( G \). So, \( u \) is either \( u_l(B_y) \) or \( u_f(B_y) \). In the first subcase, \( e \) is called forward 2-hop; see Fig. 4c. In the second subcase, since \( B_x \) is ancestor of \( B_y \) and the two blocks are at distance at most two, if \( B_x \) is the parent-block of \( B_y \), then \( e' \) is called backward 2-hop; see Fig. 4d. Finally, if \( B_x \) is the grand-parent-block of \( B_y \), then \( e' \) is called long 2-hop; see Fig. 4e.

**Lemma 8** (Case C.6). All crossing 2-hops can be assigned to seven pages in total.

**Proof.** In high level description, one can prove that all simple 2-hops can be embeded in any page that contains 2-hops; see Lemma 10. All bridging 2-hops can be embeded in two new pages; see Lemma 12. Forward 2-hops can be embeded in one new page; see Lemma 15. And, finally, backward and long 2-hops can be embeded in two new pages each; see Lemma 17 and 19 respectively. Summing up the above, yields a total of seven pages for all 2-hops. \( \square \)
In the following, we show how to cope with each of the aforementioned cases described above. In particular, let $e' = (x, y)$ be a 2-hop of level $L_1$ that crosses a binding edge $e = (u, z)$ of the underlying planar structure $G_P$. Then, $x, y, z \in L_1$ and $u \in L_0$. Let $x, z$ and $y$ belong to blocks $B_x, B_z$ and $B_y$, respectively. The different types of 2-hops are summarized as follows:

1. Vertices $x$ and $y$ belong to the same block: Simple 2-hop
2. Vertices $x$ and $y$ belong to different blocks, say $B_x$ and $B_y$ respectively
   2.1. Blocks $B_x$ and $B_y$ have the same parent-block $B_p$: Bridging 2-hop
   2.2. Block $B_x$ is an ancestor of $B_y$
      2.2.1. Vertex $u$ is $u_l(B_y)$: Forward 2-hop
      2.2.2. Vertex $u$ is $u_f(B_y)$
         2.2.2.1. $B_x$ is the parent-block of $B_y$: Backward 2-hop
         2.2.2.2. $B_y$ is the grand-parent-block of $B_y$: Long 2-hop

Let $B$ be a block of level $L_1$ and assume that $v_0, v_1, \ldots, v_t$ are the vertices of $B$ as they appear in a counterclockwise traversal of the boundary of $B$ starting from the leader of $B$, say $v_0$; that is $v_0 = \ell(B)$. In our proofs, we use the notion of the trail of inner vertex $v_i$, denoted by $tr(v_i)$, which is recursively defined as follows. If $v_i$ is identified by the first inner vertex, then $tr(v_i) = v_i$. Otherwise, $tr(v_i) = tr(\ell(B)) \rightarrow v_1 \rightarrow \ldots \rightarrow v_i$, $i = 1, 2, \ldots, t$. Intuitively, the trail of inner vertex $v_i$ is the path that starts from the first inner vertex and ends to $v_i$, which (i) consists exclusively of level one vertices, and, (ii) traverses each intermediate block from the root block towards $B$ always in counterclockwise direction. In Fig. 1a, the trail of vertex $v_0$ is $tr(v_0) = v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v_4 \rightarrow v_8 \rightarrow v_9$ The trail of a block is the trail of its leader. The following lemma follows from Yannakakis algorithm [23].

**Lemma 9.** Let $B$ be a block with leader $v_0$. Consider the trail $tr(B)$ of $B$, vertices $u_f(B)$ and $u_l(B)$ as in Fig. 5. Then $G$ is partitioned into regions, where the trail $tr(B)$ belongs to region $I$. Then, along the spine vertices of region $I$ are to the left of vertices of $B$, vertices of $B$ are to the left of vertices of region $II$ and vertices of region $II$ are to the left of vertices of region $III$.

**Figure 5.** Illustration of different regions $I$, $II$ and $III$ of Lemma 9; the trail of $B$ is drawn fat.

**Lemma 10.** Let edges $e$ and $e'$ of $G$ cross such that $e$ is a binding edge of $G_P$ and $e'$ a simple 2-hop of $G$. Then, if $e'$ is placed on the same page as any other 2-hop $e''$, then $e'$ and $e''$ do not cross.

**Proof.** Let $e' = (x, y)$ where $x, y \in L_1$ and $e = (u, z)$, where $u$ is a vertex of $L_0$ and $z$ a vertex of $L_1$. By definition of simple 2-hops, vertices $x$, $z$ and $y$ belong to the same block $B$. Then,
for the leader $\ell(B)$ of $B$, it holds that $\ell(B) \notin \{x, z, y\}$. By Lemma 2(iii) vertices $x$, $z$ and $y$ are consecutive along the spine and appear in this order from left to right. If $e' = (x, y)$ is placed on a page $p_h$ and crosses with another 2-hop $e''$ on $p_h$, then $e''$ has $z$ as one endpoint. Hence, $e'$ and $e''$ would cross in the 1-planar embedding of $G$; a contradiction. \hfill \square

Recall that, by definition, if $e' = (x, y)$ is a bridging 2-hop, then $B_x$ and $B_y$ are at distance two, and they have the same parent-block $B_p$ and a common leader, say $\ell$. Since $\ell$ is a cut-vertex of $L_1$, which separates blocks $B_x$ and $B_y$, any $x-y$ path on $L_1$ must go through $\ell$. Hence, for the path $x \rightarrow z \rightarrow y$, we have that $z = \ell$. Also, from the assumption that $B_x$ precedes $B_y$ along the spine and by Lemma 1, it follows that $u = u(B_x) = u_f(B_y)$.

**Lemma 11.** Let $B$ be a block of $G$ with children-blocks $B_1$, $B_2$, $\ldots$, $B_s$, where for any $i < j$ all vertices of $B_i$ appear before all vertices of $B_j$ along the spine. Then: (i) there is at most one bridging 2-hop of $G$ with one endpoint on $B_1$, (ii) there are at most two bridging 2-hops of $G$ with one endpoint on $B_i$ for $i = 2, \ldots, s - 1$ and (iii) there is at most one bridging 2-hop of $G$ with one endpoint on $B_s$.

**Proof.** By definition, if $e' = (x, y)$ is bridging 2-hop, then $B_x$ and $B_y$ have the same parent-block $B_p$ and the same leader, say $\ell$. Therefore, if $e'$ has one endpoint on one of the children-blocks $B_i$ of $B$, then its other endpoint is also on another child-block $B_j$ of $B$ ($i \neq j$). By Lemma 2 if $j > i + 1$, i.e. $B_i$ and $B_j$ are not consecutive child-blocks of $B$, then $e'$ would cross with edges $(\ell, u_1(B_i))$, $(\ell, u_2(B_{i+1}))$, $\ldots$, $(\ell, u_1(B_{j-1}))$, i.e., with at least two edges of $G$, contradicting 1-planarity of $G$. So, $j = i - 1$ or $j = i + 1$ and the lemma follows. \hfill \square

**Lemma 12.** All bridging 2-hops of $G$ can be placed on two new pages without crossings.

**Proof.** Let $e'_1 = (x_1, y_1)$ and $e'_2 = (x_2, y_2)$ be two bridging 2-hops of $G$. W.l.o.g. assume that $x_1, x_2, y_1, y_2$ appear in this order along the spine from left to right, i.e., edges $e'_1$ and $e'_2$ cannot be placed on the same page. Since $x_2$ appears after vertex $x_1$, by Lemma 9, $x_2$ is either a vertex of $B_{x_1}$ with greater index than $x_1$, or its block $B_{x_2}$ appears after block $B_{x_1}$. Similarly, since $x_2$ appears before $y_1$ on the spine, by the same lemma, $x_2$ is either a vertex of $B_{y_1}$ with smaller index than $y_1$, or its block $B_{x_2}$ appears before block $B_{y_1}$. By definition of bridging 2-hops, $B_{x_1}$ and $B_{y_1}$ are consecutive children-blocks of a block $B$ with the same leader. Hence, the only blocks that are between $B_{x_1}$ and $B_{y_1}$ on the spine are descendant-blocks of $B_{x_1}$ on the block-tree. Combining the above restrictions, we distinguish three cases for $B_{x_2}$: (c.1) $B_{x_2} = B_{x_1}$ and $x_2$ appears after $x_1$, (c.2) $B_{x_2}$ is a descendant of $B_{x_1}$, and (c.3) $B_{x_2} = B_{y_1}$ and $x_2$ appears before $y_1$.

In the first and third case, edges $e'_1$ and $e'_2$ are bridging 2-hops with one endpoint on the same block. In the second case, again by definition, $B_{y_2}$ is also a descendant-block of $B_{x_1}$ (since $B_{x_2}$ and $B_{y_2}$ have the same parent-block). Then, $B_{y_2}$ appears before $B_{y_1}$ on the ordering of blocks; a contradiction. Hence, if two bridging 2-hops $e'_1$ and $e'_2$ cannot be placed on the same page, then they have an endpoint on the same block.

Consider now an auxiliary graph where vertices are bridging 2-hops of $G$ and an edge exists if the bridging 2-hops corresponding to its endpoints cannot be placed on the same page, i.e., if the two bridging 2-hops have an endpoint on the same block in $G$. By Lemma 11 it follows that the auxiliary graph consists of disjoint paths and is therefore bipartite. We assign bridging 2-hops that correspond to vertices of the first (second, resp.) bipartition in the auxiliary graph to first (second, resp.) page of the two available ones. It is clear that bridging 2-hops on the same page do not cross, concluding the proof of the lemma. \hfill \square

Recall that, by definition, if $e' = (x, y)$ is a forward 2-hop, then $B_x$ is an ancestor of $B_y$ and vertex $u$ is $u_1(B_y)$. Also, since $x \rightarrow z \rightarrow y$ is a path of $G$ and $x$, $y$ belong to different blocks, it follows that the index of $y$ on $B_y$ is either 1 or 2.
Lemma 13. Let $v_1$ and $v_2$ be two vertices of $L_1$ such that $v_1$ appears before $v_2$ on the spine. Let $v$ be the last common vertex of trails $tr(v_1)$ and $tr(v_2)$. Then, all vertices of $tr(v_1)$ after $v$ precede vertices of $tr(v_2)$ after $v$.

Proof. Let $B_1 = B(v_1)$, $B_2 = B(v_2)$ and $B = B(v)$. Since $v$ is the last common vertex of $tr(v_1)$ and $tr(v_2)$, it follows that $B$ is the last common ancestor of $B_1$ and $B_2$ on the block-tree. Then, on the ordering of blocks, all blocks on the path defined from $B$ to $B_1$ appear before all blocks on the path from $B$ to $B_2$. Hence, $tr(v_1)$ and $tr(v_2)$ are identical up to vertex $v$ and vertices of $tr(v_1)$ after $v$ precede vertices of $tr(v_2)$ after $v$. \hfill $\square$

Lemma 14. Let $e' = (x, y)$ be a forward 2-hop. Then, $tr(y) = tr(x) \rightarrow z \rightarrow y$.

Proof. It suffices to prove that $x, z, y$ appear in this order from left to right on the spine. Since $y$ is either the first or second vertex of $B_y$, either $z = \ell(B_y)$ or $z$ is the first vertex of $B_y$ respectively. In both cases, the desired property holds. \hfill $\square$

Lemma 15. All forward 2-hops of $G$ can be placed on a new page without crossings.

Proof. Let $e'_1 = (x_1, y_1)$ and $e'_2 = (x_2, y_2)$ be two forward 2-hops of $G$. W.l.o.g. assume that $x_1, x_2, y_1, y_2$ appear in this order along the spine from left to right, i.e., edges $e'_1$ and $e'_2$ cannot be placed on the same page. Consider the trails of $y_1$ and $y_2$. By Lemma 14, $tr(y_1) = tr(x_1) \rightarrow z_1 \rightarrow y_1$ and $tr(y_2) = tr(x_2) \rightarrow z_2 \rightarrow y_2$. Let $v_1$ be the last common vertex of the trails $tr(x_1)$ and $tr(x_2)$ and $v_y$ the last common vertex of the trails $tr(y_1)$ and $tr(y_2)$. By Lemma 14, if $v_x$ is not $x_1$, then all three vertices $x_1$, $z_1$ and $y_1$ will appear before $x_2$, $z_2$ and $y_2$ (as in this case we would have $v_x = v_y$); a contradiction. Hence, $v_x = x_1$ and $v_y \in \{x_1, z_1, y_1\}$. Similarly, let $v$ be the last common vertex of the trails $tr(x_2)$ and $tr(y_1)$. By Lemma 14, if $v$ is not $x_2$, then all three vertices $x_2$, $z_1$ and $y_2$ will appear before $y_1$ (as in this case we would have $v = v_y$); contradiction. Hence $v = x_2$ and $v_y \in \{x_2, z_2, y_2\}$. From the above, it follows that $v_y \in \{x_1, z_1, y_1\}$. By the order of the vertices along the spine, it follows that $v_y \notin \{x_1, y_2\}$. If $v_y = z_1$, then $v_y = x_2$ also (otherwise, if $v_y = z_2$ vertex $x_2$ would also be on the common part of the trails, and inevitably it would be $x_2 = x_1$; contradiction). On the other hand, if $v_y = y_1$, then $v_y = z_2$ and $z_1 = x_2$ (since they are the unique vertex on the trail before $v_y$). In the first case, $e'_1$ and $e'_2$ would also cross in the 1-planar embedding of $G$. The second case contradicts Lemma 1 since edges $(z_1, u_2)$, $(z_1, x_1)$ and $(z_1, y_1)$ are all incident to $z_1$ (recall $z_1 = x_2$). \hfill $\square$

By definition, if $e' = (x, y)$ is a backward 2-hop, then vertex $u = u_f(B_y)$ and $B_x$ is the parent-block of $B_y$ and $y$ is either the last or the second-to-last vertex of $B_y$.

Lemma 16. If $e' = (x_1, y_1)$ and $e'' = (x_2, y_2)$ are two backward 2-hops, then $B(y_1) \neq B(y_2)$.

Proof. For a backward 2-hop $e' = (x, y)$ we say that the last edge of $B_y$ is covered by edge $e'$. By Lemma 1 the last edge of $B_y$ can be covered by at most one backward 2-hop. So, $B(y_1) \neq B(y_2)$ holds for $e'_1$ and $e'_2$. \hfill $\square$

Lemma 17. All backward 2-hops of $G$ can be placed on two new pages without crossings.

Proof. Let $e'_1 = (x_1, y_1)$ and $e'_2 = (x_2, y_2)$ be two backward 2-hops of $G$. W.l.o.g. assume that $x_1, x_2, y_1, y_2$ appear in this order along the spine from left to right, i.e., edges $e'_1$ and $e'_2$ cannot be placed on the same page. Since $x_2$ appears after vertex $x_1$, by Lemma 8 $x_2$ is either a vertex of $B_{x_1}$ with greater index than $x_1$, or its block $B_{x_2}$ appears after block $B_{x_1}$. Similarly, since $x_2$ appears before $y_1$ on the spine, by the same lemma, $x_2$ is either a vertex of $B_{y_1}$ with smaller index than $y_1$, or its block $B_{x_2}$ appears before block $B_{y_1}$. Combining the above restrictions, we distinguish three cases for $B_{x_2}$: (c.1) $B_{x_2} = B_{x_1}$ and $x_2$ appears after $x_1$, (c.2) $B_{x_2}$ is a descendant of $B_{x_1}$, and (c.3) $B_{x_2} = B_{y_1}$ and $x_2$ appears before $y_1$.\hfill 12
In the first two cases, since \( B_{y_2} \) is child-block of \( B_{x_2} \), \( B_{y_2} \) will appear before \( B_{y_1} \) and \( y_2 \) will be to the left of \( y_1 \); a contradiction. Hence, if two backward 2-hops \( e_1' \) and \( e_2' \) cannot be placed on the same page, then \( B_{x_2} = B_{y_1} \).

We say that a backward 2-hop \( e' = (x, y) \) is charged to block \( B_x \) (i.e., to the parent-block). Then, whenever \( e_1' \) and \( e_2' \) cannot be placed on the same page, we have that they are charged to blocks of consecutive levels on the block-tree. Then, we can place backward 2-hops that are charged on blocks of odd level on one of the two new available pages and those charged on blocks of even level on the other available page. It is clear that backward 2-hops on the same page do not cross, concluding the proof of the lemma.

By definition, if \( e' = (x, y) \) is a long 2-hop, then \( u = u_f(B_y) = u_f(B_z), \) \( B_y \) is the first child-block of \( B_z \) and \( B_z \) is a child-block of \( B_x \).

**Lemma 18.** Let \( e' = (x, y) \) be a long 2-hop of \( G \). We say that \( B_z \) is the middle block of \( e' \), and \( B_y \) the ending block of \( e' \). Then, a block \( B \) can be middle (ending) block of at most one long 2-hop \( e' \). Also, \( B \) cannot be middle block of a long 2-hop \( e' \) and ending block of another long 2-hop \( e'' \) at the same time.

**Proof.** The proof is similar to the one of Lemma \[16\]. We say that the last edge of a block \( B \) is covered by a long 2-hop \( e' = (x, y) \), if \( B \) is the middle or ending block of \( e' \). By Lemma \[1] the last edge of \( B \) can be covered by at most one long 2-hop, and the lemma follows.

**Lemma 19.** All long 2-hops of \( G \) can be placed on two new pages without crossings.

**Proof.** Let \( e_1' = (x_1, y_1) \) and \( e_2' = (x_2, y_2) \) be two long 2-hops of \( G \). W.l.o.g. assume that \( x_1, x_2, y_1, y_2 \) appear in this order along the spine from left to right, i.e., edges \( e_1' \) and \( e_2' \) cannot be placed on the same page. Since \( x_2 \) appears after vertex \( x_1 \), by Lemma \[9\], \( B_{y_1} \) is the first child-block of \( B_{x_2} \), we have that \( x_2 \) is either a vertex of \( B_{x_1} \), with greater index than \( x_1 \), or its block \( B_{x_2} \) appears after block \( B_{x_1} \). Similarly, since \( x_2 \) appears before \( y_1 \), by the same lemma, we have that \( x_2 \) is either a vertex of \( B_{y_1} \) with smaller index than \( y_1 \), or its block \( B_{x_2} \) appears before block \( B_{y_1} \). Combining the above restrictions, we distinguish four cases for \( B_{x_2} \): (c.1) \( B_{x_2} = B_{x_1} \) and \( x_2 \) appears after \( x_1 \), (c.2) \( B_{x_2} \) is a descendant of \( B_{x_1} \), before \( B_{x_1} \), (c.3) \( B_{x_2} = B_{y_1} \), and (c.4) \( B_{x_2} = B_{y_1} \) and \( x_2 \) appears before \( y_1 \).

In the first two cases, since \( B_{y_2} \) is grand-child-block of \( B_{x_2} \), \( B_{y_2} \), will appear before \( B_{y_1} \) and \( y_2 \) will be to the left of \( y_1 \); a contradiction. Hence, if two long 2-hops \( e_1' \) and \( e_2' \) cannot be placed on the same page, then \( B_{x_2} = B_{y_1} \), or \( B_{x_2} = B_{y_1} \). We say that a long 2-hop \( e_1' \) is indirectly in conflict with \( e_2' \) if \( B_{x_2} = B_{y_2} \) and directly in conflict with \( e_2' \) if \( B_{x_2} = B_{x_1} \). Note that Lemma \[18\] in terms of conflicts means that a long 2-hop cannot be simultaneously indirectly and directly in conflict with other 2-hops. In the following, we use induction on the number of blocks to show that two pages are sufficient.

Start with block \( B_1 \), the root-block of the block-tree. Long 2-hop edges \( e' = (x, y) \) with \( B_x = B_1 \) are not in conflict with each other and can, therefore, be placed on the same page. Suppose that we have placed all long 2-hop edges \( e' = (x, y) \) with \( B_x \leq B_4 \) on two pages, say \( p^1 \) and \( p^2 \).

Let \( e' = (x, y) \) with \( B_x = B_{x+1} \) be a long 2-hop. We claim that it can be placed on one of the two pages without introducing any crossings. We have that \( e' \) is indirectly in conflict with a long 2-hop \( e_1' = (x_1, y_1) \), if \( B_x = B_{y_1} \) and directly in conflict with a long 2-hop \( e_2' = (x_2, y_2) \), if \( B_x = B_{x_2} \). By Lemma \[18\] \( e' \) cannot be simultaneously indirectly and directly in conflict with other long 2-hops. So, assume first that \( e' \) is only indirectly in conflict with other long 2-hops of \( G \). As already stated, if \( e' \) is indirectly in conflict with \( e_1' = (x_1, y_1) \), then \( B_x = B_{y_1} \). Then, \( B_{x_1} \) is the parent-block of \( B_x \) and \( B_{x_1} \) the grand-parent-block of \( B_x \). These blocks are uniquely defined and \( e_1' \) is the only long 2-hop that \( e' \) is indirectly in conflict with. Clearly, if \( e'_i \) is on page \( p^i \), then \( e' \) can go on the opposite page \( p^j \), where \( i \neq j \). In the case where \( e' \) is only
directly in conflict with other long 2-hops of \( G \), we have that if \( e'_2 = (x_2, y_2) \) is such an edge, then \( B_x = B_{y_2} \). By Lemma 18, \( B_x \) can be middle block of at most one long 2-hop, that is, \( e'_2 \) is the only long 2-hop that \( e' \) is directly in conflict with. So, if \( e'_2 \) is on page \( p' \), then \( e' \) can go on the opposite page \( p' \), where \( i \neq j \). By induction, it follows that all long 2-hop edges of \( G \) can be placed on two new pages. 

\[ \square \]

3.2 The Multi-Level Case

In this section, we consider the general case, according to which the given 1-planar graph \( G \) consists of more than two levels, say \( L_0, L_1, \ldots, L_\lambda; \lambda \geq 2 \). We still assume that \( \Gamma(G) \) is simple internally maximal 1-planar drawing and has no crossings on its unbounded face.

Lemma 20. Any simple internally maximal 1-planar graph \( G \) with \( \lambda \geq 2 \) levels and no crossings incident to its unbounded face admits a book embedding on 34 pages.

Proof. We first embed in 5 pages the underlying planar structure \( G_P \) of \( G \) using the algorithm of Yannakakis [23]. This implies that all vertices of a block of level \( i \), except possibly for its leader, are between two consecutive vertices of level \( i - 1, i = 1, \ldots, \lambda \). So, for outer crossing chords that are involved in crossing with level edges of \( G_P \) (Case C[1]), we need a total of 2 pages suffice, since such chords are not incident to block-leaders.

Next, we consider the outer crossing chords that are involved in crossings with binding edges or bridge-blocks of \( G \) (Cases C[2] and C[3]). Recall that such a chord \( c_{i,j} = (v_i, v_j) \) of a block \( B \) is on the same page as the leader and the non-crossing chords of \( B \). The path \( P[v_i \rightarrow v_j] \) on the boundary of \( B \) joins the endpoints of the crossing chord. Hence, if another edge of the same page crosses with \( c_{i,j} \), it must also cross with an edge of \( B \); a contradiction. Therefore, such chords do not require additional pages.

For binding edges of Case C[4], 5 pages in total suffice; one page for each page of \( G_P \). For binding edges of Case C[5] however, we need a different argument: Since a binding edge between levels \( L_i \) and \( L_{i+2} \) can not cross with a binding edge between levels \( L_{i+1} \) and \( L_{i-2} \), \( i = 2, \ldots, \lambda - 2 \), it follows that binding edges that bridge pairs of levels at distance at least 3 are independent. So, for binding edges of Case C[5] we need a total of 3 * 3 = 9 pages.

Similarly, all blocks of level \( i + 1 \) that are in the interior of a certain block of level \( i \) are always between two consecutive vertices of level \( i - 1, i = 1, 2, \ldots, \lambda - 1 \). Hence, 2-hops that are by at least two levels apart in the peeling order are independent, which implies that for 2-hops we need a total of 2 * 7 = 14 pages (Case C[6]). Summing up we need 5 + 1 + 5 + 9 + 14 = 34 pages for a multi-level simple maximal 1-planar graph \( G \).

\[ \square \]

3.3 Coping with non-maximal 1-planar graphs:

In case of a planar topological graph, one can add edges to make it maximal planar (and simultaneously preserve simplicity). Unfortunately, this is not always possible in the case of 1-planar topological graphs (without losing simplicity), as the produced graph may contain multiedges. We can assume, however, that all multi-edges are crossing-free, that is, they belong to the underlying planar structure. Indeed, if a multi-edge contains an edge that is involved in a crossing, then this particular edge can be safely removed from the graph (as it can be “replaced” by any of the corresponding crossing-free edges that were added during the triangulation at the beginning of the algorithm). In the following, we describe how to cope with a non-simple maximal 1-planar graph \( G \).

Let \((v, w)\) be a double edge of \( G \). Denote by \( G_{in}[(v, w)] \) the so-called interior subgraph of \( G \) w.r.t. \((v, w)\) bounded by the double edge \((v, w)\) in \( \Gamma(G) \). By \( G_{ext}[(v, w)] \) we denote the so-called exterior subgraph of \( G \) w.r.t. \((v, w)\), derived from \( G \) by substituting \( G_{in}[(v, w)] \) by a single edge; see Fig. 4. Clearly, \( G_{ext}[(v, w)] \) stays maximal 1-planar and simultaneously has fewer multiedges...
then it suffices to place
and (maximal 1-planar, we must also triangulate the unbounded face of the planarized graph implied
with edges of $G$ resp.). Let $v,w$ (maximal 1-planar and simultaneously has fewer multiedges than $G$.
our aim is to modify it appropriately, so as to reduce the number of its multiedges by one. To do so, we will “remove” the multiedge $(v,w)$ that defines the boundary of $G_{in}((v,w))$, so as to be able to recursively embed it (again we seek to employ Lemma 20 in the base of the recursion). Let $e_i(v)$ ($e_i(w)$, resp.) be the $i$-th edge incident to vertex $v$ ($w$, resp.) in clockwise direction that is strictly between the two edges that form the double edge $(v,w)$. We replace vertex $v$ ($w$, resp.) by a path of $d(v)$ ($d(w)$, resp.) vertices, say $v_1,v_2,...,v_{d(v)}$ ($w_1,w_2,...,w_{d(w)}$, resp.), such that vertex $v_i$ ($w_i$, resp.) is the endpoint of edge $e_i(v)$ ($e_i(w)$, resp.).
Let $G_{in}((v,w))$ be the implied graph, which can be augmented to internally maximal 1-planar and simultaneously has fewer multiedges than $G_{in}((v,w))$. Since $G_{in}((v,w))$ has no crossings incident to its unbounded face, it can be embedded recursively.

It remains to describe how to plug the embedding of $G_{in}((v,w))$ to the embedding of $G_{ext}((v,w))$. Suppose that $(v,w)$ of $G_{ext}((v,w))$ is on page $p$. Clearly, $p$ is one of the pages used to embed the planar structure of $G_{ext}((v,w))$, since $(v,w)$ is not involved in crossings in $G_{ext}((v,w))$. We assume w.l.o.g. that the boundary of $G_{in}((v,w))$ is also on page $p$. Since $(v,w)$ is already present in the embedding of $G_{ext}((v,w))$, it suffices to plug in the embedding of $G_{ext}((v,w))$ only the interior of $G_{in}((v,w))$, which is the same as the one of $G_{in}((v,w))$. Suppose w.l.o.g. that in the embedding of $G_{ext}((v,w))$ vertex $v$ appears before $w$ along the spine from left to right. Then, we place the interior subgraph of $G_{in}((v,w))$ to the right of $v$. The edges connecting the interior of $G_{in}((v,w))$ with $v$ are assigned to page $p$, while the ones connecting it to $w$ on page $p'$, which is a new page. In such a way, we need 5 more pages, one for each page of the planar structure.

Next, we prove that no crossings are introduced. By restricting the boundary of $G_{in}((v,w))$ on page $p$, all edges incident to $v$ towards $G_{in}((v,w))$ become back edges of $G_{in}((v,w))$. So, edges that join $v$ with vertices in the interior of $G_{in}((v,w))$ do not cross with other edges in the interior of $G_{in}((v,w))$. Since $G_{in}((v,w))$ is placed next to $v$, edges incident to $v$ do not cross with edges of $G_{ext}((v,w))$ on page $p$. Similarly, we can prove that edges incident to $w$ towards the interior of $G_{in}((v,w))$ do not cross with other edges in the interior of $G_{in}((v,w))$ on page $p'$. We claim that potential crossings posed by different double edges, say $(v,w)$ and $(v',w')$ with $v$ to the left of $v'$, do not occur. In the case where $v \neq v'$ such a crossing would imply that $(v,w)$ and $(v',w')$ cross; a contradiction. If on the other hand $v = v'$ (and w.l.o.g. $w$ to the left of $w'$), then it suffices to place $G_{in}((v,w))$ before $G_{in}((v',w'))$.

3.4 Coping with crossings on graph’s unbounded face:
If there exist crossings incident to the unbounded face of $G$, then, when we augment $G$ to maximal 1-planar, we must also triangulate the unbounded face of the planarized graph implied
by replacing all crossings of $G$ with crossing vertices (recall the first step of our algorithm). This procedure may lead to a maximal 1-planar graph whose unbounded face is a double edge, say $(v, w)$. In this case, $G_{\text{ext}}([v, w])$ consists of two vertices and a single edge between them; $G_{\text{in}}([v, w])$ is treated as described above. So, we are now ready to state the main result of our work.

**Theorem 2.** Any 1-planar graph admits a book embedding in a book of 39 pages.

### 4 Conclusions and Open Problems

In this paper, we proved that 1-planar graphs can be embedded in books with constant number of pages, improving the previous known bound, which was $O(\sqrt{n})$ for graphs with $n$ vertices. To keep the description simple, we decided not to “slightly” reduce the page number by more complicated arguments. So, a reasonable question is whether the number of pages can be further reduced, e.g., to less than 20 pages. This is question of importance even for optimal 1-planar graphs, i.e., graphs with $n$ vertices and exactly $4n - 8$ edges. Other classes of non-planar graphs that fit in books with constant number of pages are also of interest.

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