A note on $\phi$-Prüfer $v$-multiplication rings

Xiaolei Zhang$^a$

a. Department of Basic Courses, Chengdu Aeronautic Polytechnic, Chengdu 610100, China
E-mail: zxrlghj@163.com

Abstract

In this note, we show that a strongly $\phi$-ring $R$ is a $\phi$-PvMR if and only if any $\phi$-torsion free $R$-module is $\phi$-$w$-flat, if and only if any divisible module is nonnil-absolutely $w$-pure module, if and only if any $h$-divisible module is nonnil-absolutely $w$-pure module, if and only if any finitely generated nonnil ideal of $R$ is $w$-projective.

Key Words: $\phi$-PvMRs; $\phi$-$w$-flat modules; nonnil-absolutely $w$-pure modules; $w$-projective modules.

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1. Introduction

Throughout this note, $R$ denotes a commutative ring with identity and all modules are unitary. We always denote by $\text{Nil}(R)$ the nilpotent radical of $R$, $\text{Z}(R)$ the set of all zero-divisors of $R$ and $\text{T}(R)$ the total ring of fractions of $R$. An ideal $I$ of $R$ is said to be nonnil if there is a non-nilpotent element in $I$. A ring $R$ is an NP-ring if $\text{Nil}(R)$ is a prime ideal, and a ZN-ring if $\text{Z}(R) = \text{Nil}(R)$. A prime ideal $p$ is said to be divided prime if $p \not\subseteq (x)$, for every $x \in R - p$. Set $\mathcal{H} = \{R | R$ is a commutative ring and $\text{Nil}(R)$ is a divided prime ideal of $R\}$. A ring $R$ is a $\phi$-ring if $R \in \mathcal{H}$. Moreover, a ZN $\phi$-ring is said to be a strongly $\phi$-ring. For a $\phi$-ring $R$, the map $\phi : \text{T}(R) \rightarrow R_{\text{Nil}(R)}$ such that $\phi(\frac{a}{b}) = \frac{b}{a}$ is a ring homomorphism, and the image of $R$, denoted by $\phi(R)$, is a strongly $\phi$-ring. The notion of Prüfer domains is one of the most famous integral domains that attract many algebraists. In 2004, Anderson and Badawi [1] extended the notion of Prüfer domains to that of $\phi$-Prüfer rings which are $\phi$-rings $R$ satisfying that each finitely generated nonnil ideal is $\phi$-invertible. The authors in [1] characterized $\phi$-Prüfer rings from the ring-theoretic viewpoint. In 2018, Zhao [20] characterized $\phi$-Prüfer rings using the homological properties of $\phi$-flat modules. Recently, Zhang and Qi [17] gave a module-theoretic characterization of $\phi$-Prüfer rings in terms of $\phi$-flat modules and nonnil-FP-injective modules.

Recall that an integral domain $R$ is called a Prüfer $v$-multiplication domain (PvMD for short) provided that every nonzero ideal of $R$ is $w$-invertible (see [12] for example). In 2014, Wang et al. [9] showed that an integral domain $R$ is a
PvMD if and only if $R_m$ is a valuation domain for any maximal $w$-ideal $m$ of $R$. In 2015, Wang et al. [13] obtained that an integral domain $R$ is a PvMD if and only if $w$-$\text{gl.dim}(R) \leq 1$, if and only if every torsion-free $R$-module is $w$-flat. In 2018, Xing et al. [16] gave a new module-theoretic characterization of Prüfer $v$-multiplication domains, i.e., an integral domain $R$ is a Prüfer $v$-multiplication domain if and only if every divisible $R$-module is absolutely $w$-pure, if and only if every $h$-divisible $R$-module is absolutely $w$-pure. In order to extend the notion of PvMDs to that of commutative rings in $\mathcal{H}$, the author of this paper and Zhao [19] introduced the notion of $\phi$-PvMRs as the $\phi$-rings in which any finitely generated non-nil ideal is $\phi$-$w$-invertible. They also gave some ring-theoretic and homology-theoretic characterizations of $\phi$-PvMRs. In this note, we mainly study the module-theoretic characterizations of $\phi$-PvMRs which can be seen a generalization of Wang’s and Xing’s results in [13] and [16] respectively.

As our work involves the $w$-operation theory, we give a quick review as below. Let $R$ be a commutative ring and $J$ a finitely generated ideal of $R$. Then $J$ is called a GV-ideal if the natural homomorphism $R \rightarrow \text{Hom}_R(J, R)$ is an isomorphism. The set of all GV-ideals is denoted by $\text{GV}(R)$. Let $M$ be an $R$-module and define $\text{tor}_{\text{GV}}(M) := \{x \in M | Jx = 0, \text{for some } J \in \text{GV}(R)\}$. An $R$-module $M$ is said to be GV-torsion (resp., GV-torsion-free) if $\text{tor}_{\text{GV}}(M) = M$ (resp., $\text{tor}_{\text{GV}}(M) = 0$). A GV-torsion free module $M$ is said to be a $w$-module if, for any $x \in E(M)$, there is a GV-ideal $J$ such that $Jx \subseteq M$ where $E(M)$ is the injective envelope of $M$. The $w$-envelope $M_w$ of a GV-torsion free module $M$ is defined by the minimal $w$-module that contains $M$. A maximal $w$-ideal for which is maximal among the $w$-submodules of $R$ is proved to be prime (see [11, Theorem 6.2.15]). The set of all maximal $w$-ideals is denoted by $w$-$\text{Max}(R)$. Let $M$ be an $R$-module and set $L(M) = (M/\text{tor}_{\text{GV}}(M))_w$. Recall from [10] that $M$ is said to be $w$-projective if $\text{Ext}^1_R(L(M), N)$ is GV-torsion for any torsion-free $w$-module $N$.

An $R$-homomorphism $f : M \rightarrow N$ is said to be a $w$-monomorphism (resp., $w$-epimorphism, $w$-isomorphism) if for any $p \in \text{Max}(R)$, $f_p : M_p \rightarrow N_p$ is a monomorphism (resp., an epimorphism, an isomorphism). Note that $f$ is a $w$-monomorphism (resp., $w$-epimorphism) if and only if $\text{Ker}(f)$ (resp., $\text{Coker}(f)$) is GV-torsion. A sequence $A \rightarrow B \rightarrow C$ is said to be $w$-exact if, for any $p \in \text{Max}(R)$, $A_p \rightarrow B_p \rightarrow C_p$ is exact. A class $C$ of $R$-modules is said to be closed under $w$-isomorphisms provided that for any $w$-isomorphism $f : M \rightarrow N$, if one of the modules $M$ and $N$ is in $C$, so is the other. An $R$-module $M$ is said to be of finite type if there exist a finitely generated free module $F$ and a $w$-epimorphism $g : F \rightarrow M$, or equivalently, if there exists a finitely generated $R$-submodule $N$ of
M such that $N_w = M_w$. Certainly, the class of finite type modules is closed under $w$-isomorphisms.

2. NONNIL-ABSOLUTELY $w$-PURE MODULES

Recall from [15], a $w$-exact sequence of $R$-modules $0 \to N \to M \to L \to 0$ is said to be $w$-pure exact if, for any $R$-module $K$, the induced sequence $0 \to K \otimes_R N \to K \otimes_R M \to K \otimes_R L \to 0$ is $w$-exact. If $N$ is a submodule of $M$ and the exact sequence $0 \to N \to M \to M/N \to 0$ is $w$-pure exact, then $N$ is said to be a $w$-pure submodule of $M$. Recall from [16], an $R$-module $M$ is called a absolutely $w$-pure module provided that $M$ is $w$-pure in every module containing $M$ as a submodule.

Let $R$ be an NP-ring and $M$ an $R$-module. Define

$$\phi\text{-}tor(M) = \{x \in M | Ix = 0 \text{ for some nonnil ideal } I \text{ of } R\}.$$ 

An $R$-module $M$ is said to be $\phi$-torsion (resp., $\phi$-torsion free) provided that $\phi\text{-}tor(M) = M$ (resp., $\phi\text{-}tor(M) = 0$). Now we generalize the notions in [15] and [16] to NP-rings. A $w$-exact sequence $0 \to M \to N \to N/M \to 0$ of $R$-modules is said to be nonnil $w$-pure exact provided that $0 \to \text{Hom}_R(T, M) \to \text{Hom}_R(T, N) \to \text{Hom}_R(T, N/M) \to 0$ is $w$-exact for any finitely presented $\phi$-torsion module $T$. In addition, if $M$ is a submodule of $N$, then we say $M$ is nonnil $w$-pure submodule in $N$.

**Definition 2.1.** Let $R$ be an NP-ring. An $R$-module $M$ is called a non-nil-absolutely $w$-pure module provided that $M$ is nonnil $w$-pure submodule in every $R$-module containing $M$ as a submodule.

Following from Xing [16, Theorem 2.6], an $R$-module $M$ is absolutely $w$-pure if and only if $\text{Ext}_R^1(F, M)$ is GV-torsion for any finitely presented module $F$, if and only if $M$ is a $w$-pure submodule in its injective envelope. Now, we give a $\phi$-version of Xing’ result.

**Proposition 2.2.** Let $R$ be an NP-ring and $M$ an $R$-module. The following statements are equivalent:

1. $M$ is a non-nil-absolutely $w$-pure module;
2. $\text{Ext}_R^1(T, M)$ is GV-torsion for any finitely presented $\phi$-torsion module $T$;
3. $M$ is a nonnil $w$-pure submodule in every injective module containing $M$;
4. $M$ is a nonnil $w$-pure submodule in its injective envelope;
5. for any diagram

$$
\begin{array}{c}
M \\
\downarrow f \\
0 \rightarrow K \xrightarrow{i} F \\
\end{array}
$$
with $F$ finitely generated projective, $K$ finitely generated and $F/K$ \( \phi \)-torsion, there is some $J \in GV(R)$ such that any given $c \in J$, there exists $g_c : F \to M$ such that $cf = g_c i$.

**Proof.** (1) $\Rightarrow$ (3) $\Rightarrow$ (4) : Trivial.

(2) $\Rightarrow$ (1) : Let $N$ be an $R$-module containing $M$, and $T$ a finitely presented $\phi$-torsion module. Then we have the following exact sequence

$$0 \to \text{Hom}_R(T, M) \to \text{Hom}_R(T, N) \to \text{Hom}_R(T, N/M) \to \text{Ext}^1_R(T, M).$$

Since $\text{Ext}^1_R(T, M)$ is GV-torsion, we have

$$0 \to \text{Hom}_R(T, M) \to \text{Hom}_R(T, N) \to \text{Hom}_R(T, N/M) \to 0$$

is $w$-exact. Hence $M$ is a nonnil $w$-pure submodule in $N$.

(4) $\Rightarrow$ (2) : Let $E$ be the injective envelope of $M$. Then for any finitely presented $\phi$-torsion module $T$, we have the following exact sequence: $0 \to \text{Hom}_R(T, M) \to \text{Hom}_R(T, E) \to \text{Hom}_R(T, E/M) \to \text{Ext}^1_R(T, M) \to 0$. Thus we have $\text{Ext}^1_R(T, M)$ is GV-torsion by (4).

(2) $\Rightarrow$ (5) : Consider the exact sequence $0 \to K \xrightarrow{i} F \xrightarrow{\pi} F/K \to 0$ with $F/K$ finitely presented $\phi$-torsion. we have the following exact sequence: $\text{Hom}_R(F, M) \xrightarrow{i^*} \text{Hom}_R(K, M) \to \text{Ext}^1_R(F/K, M) \to 0$. Since $F/K$ is finitely presented $\phi$-torsion, $\text{Ext}^1_R(F/K, M)$ is GV-torsion by (2). Thus $i^*$ is a $w$-epimorphism. Since $f \in \text{Hom}_R(K, M)$, there exists a GV-ideal $J$ of $R$ such that $Jf \in \text{Im}(i^*)$. So, for any given $c \in J$, there exists $g_c : F \to A$ such that $g_c i = cf$.

(5) $\Rightarrow$ (2) : Let $T$ be a finitely presented $\phi$-torsion module. Then exists a short sequence $0 \to K \xrightarrow{i} F \to T \to 0$ with $F$ finitely generated projective and $K$ finitely generated. Consider the exact sequence $\text{Hom}_R(F, M) \xrightarrow{i^*} \text{Hom}_R(K, M) \to \text{Ext}^1_R(T, M) \to 0$. For any $f \in \text{Hom}_R(K, M)$, there is some $J \in GV(R)$ such that any given $c \in J$, there exists $g_c : F \to M$ such that $cf = g_c i$ by (5). So $Jf \subseteq \text{Im}(i^*)$. Thus $i^*$ is a $w$-epimorphism and so $\text{Ext}^1_R(T, M)$ is GV-torsion. \( \square \)

Recall from [17, Definition 1.2] that an $R$-module $M$ is called nonnil-$FP$-injective provided that $\text{Ext}^1_R(T, M) = 0$ for any finitely presented $\phi$-torsion module $T$. Thus we have the following result by Proposition 2.2.

**Corollary 2.3.** Let $R$ be an NP-ring. Then every nonnil-$FP$-injective module is nonnil-absolutely $w$-pure.

**Lemma 2.4.** Let $T$ be a GV-torsion module. Then $T$ is a absolutely $w$-pure module.
Proof. Let $T$ be a GV-torsion module and $F$ a finitely presented $R$-module. Considering the exact sequence $0 \to K \to P \to F \to 0$ with $P$ finitely generated projective and $K$ finitely generated, we have the following exact sequence $\text{Hom}_R(K, T) \to \text{Ext}^1_R(F, T) \to 0$. Since $K$ is finitely generated and $T$ is GV-torsion, $\text{Hom}_R(K, T)$ is GV-torsion. So $\text{Ext}^1_R(F, T)$ is GV-torsion. Consequently, $T$ is an absolutely $w$-pure module. □

Obviously, we have the following result by Proposition 2.2, [16, Theorem 2.6] and Lemma 2.4.

**Corollary 2.5.** Let $R$ be an NP-ring. Then every absolutely $w$-pure module is nonnil-absolutely $w$-pure. Consequently, every GV-torsion module is a nonnil-absolutely $w$-pure module.

In order to characterize rings over which every nonnil-absolutely $w$-pure module is absolutely $w$-pure, we recall some basic facts.

**Lemma 2.6.** [19, Lemma 1.6] Let $R$ be a $\phi$-ring and $I$ a nonnil ideal of $R$. Then $\text{Nil}(R) = I\text{Nil}(R)$.

**Lemma 2.7.** [17, Proposition 1.5] Let $R$ be a $\phi$-ring and $M$ an FP-injective $R/\text{Nil}(R)$-module. Then $M$ is nonnil-FP-injective over $R$.

Let $R\{x\}$ be the $w$-Nagata ring of $R$, that is, the localization of $R[X]$ at the multiplicative closed set $S_w = \{f \in R[x] \mid c(f) \in \text{GV}(R)\}$, where $c(f)$ is the content of $f$ (see [10]). Then $\{m(x) \mid m \in w-\text{Max}(R)\}$ is the set of all maximal ideal of $R\{x\}$ by [10, Proposition 3.3(4)]. Set

$$E' = \prod_{m \in w-\text{Max}(R)} \text{E}_R(R\{x\}/m(x))$$

where $\text{E}_R(R\{x\}/m(x))$ is the injective envelope of the $R$-module $R\{x\}/m(x)$. Since $R\{x\}/m(x)$ is a $w$-module over $R$ by [11, Theorem 6.6.19(2)], then $E'$ is an injective $w$-module over $R$. Set

$$\tilde{E} := \text{Hom}_R(R\{x\}, E').$$

Then $\tilde{E}$ is trivially an $R\{x\}$-module. Since $R\{x\}$ is a flat $R$-module, $\tilde{E}$ is an injective $w$-module by [11, Theorem 6.1.18] and [5, Theorem 3.2.9].

**Lemma 2.8.** [18, Corollary 3.11] Let $M$ be an $R$-module. The following statements are equivalent:

1. $M$ is GV-torsion;
2. $\text{Hom}_R(M, E) = 0$ for any injective $w$-module $E$;
(3) $\text{Hom}_R(M, \bar{E}) = 0$.

**Theorem 2.9.** Let $R$ be a $\phi$-ring. Then $R$ is an integral domain if and only if any nonnil-absolutely $w$-pure module is absolutely $w$-pure.

**Proof.** If $R$ is an integral domain, then any nonnil-absolutely $w$-pure module is absolutely $w$-pure obviously.

On the other hand, we have $\text{Hom}_R(R/\text{Nil}(R), \bar{E})$ is an injective $R/\text{Nil}(R)$-module by [5, Theorem 3.1.6]. Thus by Lemma 2.7 $\text{Hom}_R(R/\text{Nil}(R), \bar{E})$ is a nonnil-FP-injective $R$-module, and so is a nonnil-absolutely $w$-pure $R$-module. Thus we have $\text{Hom}_R(R/\text{Nil}(R), \bar{E})$ is an absolutely $w$-pure $R$-module by assumption. That is,

$$\text{Ext}_1^R(F, \text{Hom}_R(R/\text{Nil}(R), \bar{E})) \cong \text{Hom}_R(\text{Tor}_1^R(F, R/\text{Nil}(R)), \bar{E})$$

is a GV-torsion module for any finitely presented $R$-module $F$ as $\bar{E}$ is an injective $R$-module. Since $\bar{E}$ is a $w$-module, $\text{Hom}_R(\text{Tor}_1^R(F, R/\text{Nil}(R)), \bar{E})$ is also a $w$-module by [11, Theorem 6.1.18]. Thus we have $\text{Hom}_R(\text{Tor}_1^R(F, R/\text{Nil}(R)), \bar{E}) = 0$. Hence $\text{Tor}_1^R(F, R/\text{Nil}(R))$ is GV-torsion by Lemma 2.8. Let $s$ be a nilpotent element in $R$ and set $F = R/\langle s \rangle$. Then $\text{Tor}_1^R(F, R/\text{Nil}(R)) = \text{Tor}_1^R(R/\langle s \rangle, R/\text{Nil}(R)) \cong \langle s \rangle \cap \text{Nil}(R)/s\text{Nil}(R) = \langle s \rangle/s\text{Nil}(R)$ is GV-torsion (see [11, Exercise 3.20]). Thus there is a GV-ideal $J$ such that $sJ \subseteq s\text{Nil}(R)$. If $J$ is a GV-ideal, then it is a nonnil ideal, thus $\text{Nil}(R) = J\text{Nil}(R)$ by Lemma 2.6. So $sJ \subseteq s\text{Nil}(R) = sJ\text{Nil}(R) \subseteq sJ$. That is, $sJ = sJ\text{Nil}(R)$. Since $sJ$ is finitely generated, $sJ = 0$ by Nakayama’s lemma. Since $J \in \text{GV}(R)$, $sR \subseteq R$ is GV-torsion free, then $s = 0$. Consequently, $\text{Nil}(R) = 0$ and so $R$ is an integral domain. \hfill $\square$

**Lemma 2.10.** Let $R$ be a ring. If $R$ is a (strongly) $\phi$-ring, then $R_p$ is a (strongly) $\phi$-ring for any prime ideal $p$ of $R$.

**Proof.** Let $R$ be a $\phi$-ring and $p$ a prime ideal of $R$. Then $R_p/\text{Nil}(R_p) \cong (R/\text{Nil}(R))_p$ which is certainly an integral domain. So $\text{Nil}(R_p)$ is a prime ideal of $R_p$. Let $\frac{r}{s} \in R_p - \text{Nil}(R_p)$ and $\frac{r}{s} \in \text{Nil}(R_p)$. Note $r \in R - \text{Nil}(R)$ and $r_1 \in \text{Nil}(R)$. Then $r_1 = rt$ for some $t \in \text{Nil}(R)$. Thus $\frac{r_1}{s_1} = \frac{rt}{s_1} = \frac{rs_1 + ts_1}{s_1} = \frac{ts_1}{s_1} \in \langle \frac{r}{s} \rangle$. So $\text{Nil}(R_p)$ is a divided prime ideal of $R_p$. Hence $R_p$ is a $\phi$-ring. Now suppose $R$ is a strongly $\phi$-ring. Let $\frac{r}{s} \in R_p - \text{Nil}(R_p)$. Then $r$ is non-nilpotent, and thus $r$ is regular. Assume $\frac{r^m}{s} = 0$ in $R_p$. Then there exists $t \in R - p$ such that $rr_1t = 0$. Thus $r_1t = 0$. Hence $r_1$ and thus $\frac{r}{s}$ is equal to 0 since $t$ is also regular. Consequently, $R_p$ is a strongly $\phi$-ring. \hfill $\square$

**Remark 2.11.** Note that the converse of Lemma 2.10 is not true in general. Indeed, let $R$ be a von Neumann regular ring which is not a field. Then $R_p$ is a field for any prime ideal $p$ of $R$. However, $R$ is not a $\phi$-ring since $\text{Nil}(R) = 0$ is not a prime ideal in this case.
Let $R$ be an NP-ring. Recall from [21] that an $R$-module $M$ is said to be $\phi$-flat if for every monomorphism $f : A \to B$ with $\text{Coker}(f)$ $\phi$-torsion, $f \otimes_R 1 : A \otimes_R M \to B \otimes_R M$ is a monomorphism; a $\phi$-ring $R$ is said to be $\phi$-von Neumann if every $R$-module is $\phi$-flat. The authors in [21, Theorem 4.1] proved that a $\phi$-ring $R$ is $\phi$-von Neumann if and only if the Krull dimension of $R$ is 0. It was also shown in [17, Theorem 1.7] that a $\phi$-ring $R$ is $\phi$-von Neumann if and only if $R/\text{Nil}(R)$ is a field, if and only if every non-nilpotent element is invertible, if and only if every $R$-module is non-nil-FP-injective. Recall from [19, Definition 1.3] that an $R$-ring is non-nil-FP-injective. Recall from [21, Theorem 4.1] that a $\phi$-ring $R$ is $\phi$-von Neumann if and only if every $R$-module is $\phi$-w-flat. Now we give a new characterization of $\phi$-von Neumann rings.

**Lemma 2.12.** Let $R$ be a $\phi$-ring. Then $R$ is a $\phi$-von Neumann regular ring if and only if $R_m$ is a $\phi$-von Neumann regular ring for any $m \in w\text{-Max}(R)$.

**Proof.** Let $R$ be a $\phi$-von Neumann regular ring and $m$ a prime ideal. Let $\zeta \in R_m$ be a non-nilpotent element in $R_m$. Then $r$ is non-nilpotent. So $r$ is invertible by [17, Theorem 1.7]. Hence $\zeta$ is also invertible in $R_m$ implying $R_m$ is a $\phi$-von Neumann regular ring by [17, Theorem 1.7] and Lemma 2.10.

Now let $r$ be non-nilpotent element in $R$. Then $\zeta$ is a non-nilpotent element in $R_m$ for any $m \in w\text{-Max}(R)$, since $R$ is a $\phi$-ring. By [17, Theorem 1.7], $\zeta$ is invertible in $R_m$. Thus $r \not\in m$ for any $m \in w\text{-Max}(R)$. So $\langle r \rangle_w = R$, and hence $r$ is invertible by [11, Exercise 6.11(2)].

**Theorem 2.13.** Let $R$ be a $\phi$-ring. Then $R$ is a $\phi$-von Neumann regular ring if and only if every $R$-module is nonnil-absolutely $w$-pure.

**Proof.** Suppose $R$ is a $\phi$-von Neumann regular ring and $M$ is an $R$-module. Then $R/\text{Nil}(R)$ is a field. By [19, Theorem 3.3] $R$ is a $\phi$-Prüfer $v$-multiplication ring and thus $R_m$ is a $\phi$-chained ring for any maximal $w$-ideal $m$ of $R$. Let $T$ be a finitely presented $\phi$-torsion module. Then $\text{Ext}^1_T(T, M)_m = \text{Ext}^1_{R_m}(T_m, M_m) = \bigoplus_{i=1}^n \text{Ext}^1_{R_m}(R_m/R_m x_i, M_m)$ for some non-nilpotent element $x_i \in R_m$ by [11, Theorem 3.9.11]. By Lemma 2.12, $R_m$ is a $\phi$-von Neumann regular ring. Thus $x_i$ is an invertible element by [17, Theorem 1.7]. So $R_m/R_m x_i = 0$ and thus $\text{Ext}^1_{R_m}(R_m/R_m x_i, M_m) = 0$. Hence $\text{Ext}^1_T(T, M)_m = 0$ for any $m \in w\text{-Max}(R)$. It follows that $\text{Ext}^1_T(T, M)$ is GV-torsion. Consequently, $M$ is nonnil-absolutely $w$-pure.

On the other hand, let $I$ be a finitely generated nonnil ideal of $R$. Since for any $R$-module $M$, $\text{Ext}^1_R(R/I, M)$ is GV-torsion, then $R/I$ is finitely generated $w$-projective. $R_m/I_m$ is a finitely generated projective $R_m$-module for any $m \in w\text{-Max}(R)$ by [11]
Theorem 6.7.18]. Then $I_m$ is an idempotent ideal of $R_m$ by [7, Theorem 1.2.15]. By [6, Chapter I, Proposition 1.10], $I_m$ is generated by an idempotent $e_m \in R_m$. Thus $R_m$ is a $\phi$-von Neumann regular ring by [21, Theorem 4.1] and Lemma 2.10. So $R$ is $\phi$-von Neumann regular by Lemma 2.12.

3. SOME NEW CHARACTERIZATIONS OF $\phi$-PRÜFER $v$-MULTIPLICATION RINGS

Recall from [4] that a $\phi$-ring $R$ is said to be a $\phi$-chain ring ($\phi$-CR for short) if for any $a, b \in R - \text{Nil}(R)$, either $a|b$ or $b|a$ in $R$. A $\phi$-ring $R$ is said to be a $\phi$-Prüfer ring if every finitely generated nonnil ideal $I$ is $\phi$-invertible, i.e., $\phi(I)\phi(I^{-1}) = \phi(R)$ where $I^{-1} = \{x \in T(R)|Ix \subseteq R\}$. It follows from [1, Corollary 2.10] that a $\phi$-ring $R$ is $\phi$-Prüfer, if and only if $R_m$ is a $\phi$-CR for any maximal ideal $m$ of $R$, if and only if $R/\text{Nil}(R)$ is a Prüfer domain, if and only if $\phi(R)$ is a Prüfer ring.

Let $R$ be a $\phi$-ring. Recall from [8] that a nonnil ideal $J$ of $R$ is said to be a $\phi$-GV-ideal (resp., $\phi$-$w$-ideal) of $R$ if $\phi(J)$ is a GV-ideal (resp., $w$-ideal) of $\phi(R)$. An ideal $I$ of $R$ is $\phi$-$w$-invertible if $(\phi(I)\phi(I^{-1}))_W = \phi(R)$ where $W$ is the $w$-operation of $\phi(R)$. In order to extend PvMDs to $\phi$-rings, the authors in [19] gave the notion of $\phi$-Prüfer $v$-multiplication rings: a $\phi$-ring $R$ is said to be a $\phi$-Prüfer $v$-multiplication ring ($\phi$-PvMR for short) provided that any finitely generated nonnil ideal is $\phi$-$w$-invertible. They also show that a $\phi$-ring $R$ is a $\phi$-PvMR if and only if $R_m$ is a $\phi$-CR for any $m \in w\text{-Max}(R)$, if and only if $R/\text{Nil}(R)$ is a PvMD, if and only if $\phi(R)$ is a PvMR.

Recall that an $R$-module $E$ is said to be divisible if $sM = M$ for any regular element $s \in R$, and an $R$-module $M$ is said to be $h$-divisible provided that $M$ is a quotient of an injective module. Evidently, any injective module is $h$-divisible and any $h$-divisible module is divisible. The author in [17] introduced the notion of nonnil-divisible modules $E$ in which for any $m \in E$ and any non-nilpotent element $a \in R$, there exists $x \in E$ such that $ax = m$.

**Lemma 3.1.** [17, Lemma 2.2] Let $R$ be an NP-ring and $E$ an $R$-module. Consider the following statements:

(1) $E$ is nonnil-divisible;
(2) $E$ is divisible;
(3) $\text{Ext}^1_R(R/\langle a \rangle, E) = 0$ for any $a \notin \text{Nil}(R)$.

Then we have (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3). Moreover, if $R$ is a ZN-ring, all statements are equivalent.

**Lemma 3.2.** [17, Lemma 2.4] Let $R$ be an NP-ring and $E$ a nonnil-divisible $R$-module. Then $E_p$ is a nonnil-divisible $R_p$-module for any prime ideal $p$ of $R$. 

Let $M$ be an $R$-module. Recall from [10] that $M$ is said to have $w$-rank $n$ if, for any maximal $w$-ideal $m$ of $R$, $M_m$ is a free $R_m$-module of rank $n$. Let $\tau$ denote the trace map of $M$, that is, $\tau : M \otimes_R \text{Hom}(M, R) \to R$ defined by $\tau(x \otimes f) = f(x)$ for $x \in M$ and $f \in M$. $M$ is said to be $w$-invertible, if the trace map $\tau$ is a $w$-isomorphism. It was proved in [10] Theorem 4.13 that an $R$-module $M$ is $w$-invertible if and only if $M$ is of finite type and has $w$-rank 1, if and only if $M$ is $w$-projective of finite type and has $w$-rank 1.

**Proposition 3.3.** Let $R$ be a strongly $\phi$-ring and $I$ a finitely generated nonnil ideal of $R$. If $I$ is $w$-projective, then $I$ is $\phi$-$w$-invertible.

**Proof.** Let $I$ a finitely generated nonnil ideal of the strongly $\phi$-ring $R$. Then $I$ is a regular ideal of $R$. Let $m$ be a maximal $w$-ideal of $R$. Since $I$ is $w$-projective $R$-ideal, $I_m$ is a free ideal of $R_m$ by [11] Theorem 6.7.11]. Then $I_m \cong R_m$ or $I_m = 0$. We claim that $I_m \cong R_m$. Indeed, let $r$ be a regular element in $I$. If $I_m = 0$, then there is an element $s \in R - m$ such that $rs = 0$. So $s = 0$ which is a contradiction. Hence, $I_m$ is of rank 1 for any maximal $w$-ideal $m$ of $R$. By [10] Theorem 4.13], $\phi(I) = I$ is $w$-invertible since $R$ is a strongly $\phi$-ring. Hence, $I$ is $\phi$-$w$-invertible. $\square$

**Lemma 3.4.** [17] Proposition 2.12] Let $R$ be an NP-ring, $p$ a prime ideal of $R$ and $M$ an $R$-module. Then $M$ is $\phi$-torsion over $R$ if and only if $M_p$ is $\phi$-torsion over $R_p$.

**Lemma 3.5.** Let $R$ be an NP-ring, $M$ an $R$-module. Suppose $M$ is $\phi$-torsion free over $R$, $M_m$ is $\phi$-torsion free over $R_m$ for any maximal $w$-ideal $m$ of $R$. Moreover, if $M$ is GV-torsion free, then the converse also holds.

**Proof.** Suppose $M$ is a $\phi$-torsion free $R$-module. Let $m$ be a maximal $w$-ideal of $R$ and $\frac{m}{s} \in M_m$. Suppose $I_m$ is a nonnil ideal of $R_m$ and $I_m\frac{m}{s} = 0$ in $M_m$. then there exists $t \not\in m$ such that $tIm = 0$ in $R$. Since $I$ is nonnil in $R$ by [19] Lemma 1.1], we have $It$ is also nonnil as $t$ is non-nilpotent. Since $M$ be an $\phi$-torsion free, $m$ and thus $\frac{m}{s}$ is equal to 0.

Suppose $M$ is a GV-torsion free $R$-module such that $M_m$ is $\phi$-torsion free over $R_m$ for any maximal $w$-ideal $m$ of $R$. Let $m \in M$ such that $Im = 0$ for some nonnil ideal $I$ of $R$. Then $I_m\frac{m}{s} = 0$ in $M_m$. Since $I_m$ is nonnil in $R_m$ by [19] Lemma 1.1], $\langle m \rangle_m = 0$ for any maximal $w$-ideal $m$ of $R$. Thus $\langle m \rangle$ is GV-torsion in $M$ by [11] Theorem 6.2.15]. Since $M$ is GV-torsion free by assumption, we have $m = 0$. $\square$

It is well-known that an integral domain $R$ is a PvMD if and only if every torsion-free $R$-module is $w$-flat, if and only if every $(h)$-divisible $R$-module is absolutely $w$-pure (see [13] [16]). Recently, the authors in [17] characterized $\phi$-Prüfer rings in terms of nonnil-FP-injective modules, that is, a strongly $\phi$-ring $R$ is a $\phi$-Prüfer ring
if and only if any $\phi$-torsion free $R$-module is $\phi$-flat, if and only if any $(h)$-divisible module is nonnil-FP-injective. Now, we characterize $\phi$-PvMRs in terms of $\phi$-$w$-flat modules, nonnil-absolutely $w$-pure modules and $w$-projective modules, which can be seen as a generalization of the results in [13, 16, 17].

**Theorem 3.6.** Let $R$ be a strongly $\phi$-ring. The following statements are equivalent for $R$:

1. $R$ is a $\phi$-PvMR;
2. any $\phi$-torsion free $R$-module is $\phi$-$w$-flat;
3. any nonnil ideal of $R$ is $w$-flat;
4. any ideal of $R$ is $\phi$-$w$-flat;
5. any divisible module is nonnil-absolutely $w$-pure module;
6. any $h$-divisible module is nonnil-absolutely $w$-pure module;
7. any finitely generated nonnil ideal of $R$ is $w$-projective;
8. any finite type nonnil ideal of $R$ is $w$-projective.

**Proof.**

1 $\Rightarrow$ 2: Let $m$ be a maximal $w$-ideal of $R$, $M$ a $\phi$-torsion free $R$-module. By Lemma 3.5, $M_m$ is $\phi$-torsion free over $R_m$. Since $R$ is a $\phi$-PvMR, $R_m$ is a $\phi$-CR by [19, Theorem 3.3]. Then $M_m$ is $\phi$-flat by [20, Theorem 4.3], and thus $M$ is $\phi$-$w$-flat by [19, Theorem 1.4].

2 $\Rightarrow$ 4: It follows from $R$ is $\phi$-torsion free since $R$ is a strongly $\phi$-ring (see [20, Proposition 2.2]).

4 $\iff$ 3: Let $J$ be a nonnil ideal of $R$, $I$ an ideal of $R$. We have

$$\text{Tor}_1^R(R/J, I) \cong \text{Tor}_2^R(R/J, R/I) \cong \text{Tor}_1^R(R/I, J).$$

The result follows the statement that any ideal of $R$ is $\phi$-$w$-flat is equivalence to the statement that any nonnil ideal of $R$ is $w$-flat.

4 $\Rightarrow$ 1: See [19, Theorem 3.8].

1 $\Rightarrow$ 5: Let $T$ be a finitely presented $\phi$-torsion module and $m$ a maximal $w$-ideal of $R$. Then by Lemma 3.4, $T_m$ is a finitely presented $\phi$-torsion $R_m$-module. By [19, Theorem 3.3], $R_m$ is a $\phi$-chained ring. Thus, by [20, Theorem 4.1], $T_m \cong \bigoplus_{i=1}^{n} R_m/R_m x_i$ for some regular element $x_i \in R_m$ as $R_m$ is a strongly $\phi$-ring by Lemma 2.10. Let $E$ be a divisible module. Then $E_m$ is a divisible module over $R_m$ by Lemma 3.1 and Lemma 3.2. Thus $\text{Ext}_R^1(T, E)_m = \text{Ext}_{R_m}^1(T_m, E_m) = \bigoplus_{i=1}^{n} \text{Ext}_{R_m}^1(R_m/R_m x_i, E_m) = 0$ by Lemma 3.1 and [11, Theorem 3.9.11]. It follows that $\text{Ext}_R^1(T, E)$ is a GV-torsion module. Therefore, $E$ is a nonnil-absolutely $w$-pure module.
(5) ⇒ (6) and (8) ⇒ (7): Trivial.

(6) ⇒ (7): Let $N$ be an $R$-module, $I$ a finitely generated nil ideal of $R$. The short exact sequence $0 \to I \to R \to R/I \to 0$ induces a long exact sequence as follows:

$$0 = \text{Ext}^1_R(R, N) \to \text{Ext}^1_R(I, N) \to \text{Ext}^2_R(R/I, N) \to \text{Ext}^2_R(R, N) = 0.$$ 

Let $0 \to N \to E \to K \to 0$ be an exact sequence where $E$ is the injective envelope of $N$. There exists a long exact sequence as follows:

$$0 = \text{Ext}^1_R(R/I, E) \to \text{Ext}^1_R(R/I, K) \to \text{Ext}^2_R(R/I, N) \to \text{Ext}^2_R(R/I, E) = 0.$$ 

Thus $\text{Ext}^1_R(I, N) \cong \text{Ext}^2_R(R/I, N) \cong \text{Ext}^1_R(R/I, K)$ is a GV-torsion module as $K$ is nonnil-absolutely $w$-pure by (6). It follows that $I$ is a $w$-projective ideal of $R$ by [14 Corollary 2.5].

(7) ⇒ (1): It follows from Proposition 3.3.

(7) ⇒ (8): Let $I$ be a finite type nil ideal of $R$, then there is a finitely generated sub-ideal $K$ of $I$ such that $K/I$ is GV-torsion (see [11 Proposition 6.4.2(3)]). Then $I$ is $w$-isomorphic to $K$. We claim that $K$ is a nonnil ideal. Indeed, since $I$ is nil, there is a non-nilpotent element $s \in I$. Thus there is a GV-ideal $J$ of $R$ such that $Js \subseteq K$. Since $J$ is nonnil and $R$ is a $\phi$-ring, we have $K$ is a nonnil ideal of $R$. By (7), $K$ is $w$-projective. And thus $I$ is $w$-projective by [11 Proposition 6.7.8(1)]. □

Obviously, every nonnil-FP-injective module is nonnil-absolutely $w$-pure. The following example shows that the converse does not hold in general.

**Example 3.7.** Let $D$ be a PvMD but not a Prüfer domain, $K$ the quotient field of $D$. Then the idealization $R = D(+)K$ is a $\phi$-PvMR but not a $\phi$-Prüfer ring. Note that $R$ is a strongly $\phi$-ring by [2] Remark 1. Thus there is a nonnil-absolutely $w$-pure divisible module $M$ which is not nonnil-FP-injective by Theorem 3.6 and [17] Theorem 2.13.

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