A Monopole Index for $\mathcal{N} = 4$ Chern-Simons Theories

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Abstract

We compute a certain index for an $\mathcal{N} = 4$ Chern-Simons theory with gauge group $U(N)^r$ in the large $N$ limit with taking account of monopole contribution, and compare it to the corresponding multi-particle index for M-theory in the dual geometry $\text{AdS}_4 \times X_7$. The internal space $X_7$ has non-trivial two-cycles, and M2-branes wrapped on them contribute to the multi-particle index. We establish one-to-one map between $r - 1$ independent magnetic charges on the gauge theory side and the same number of charges on the gravity side: the M-momentum and $r - 2 (= b_2(X_7))$ wrapping numbers. With a certain assumption for the wrapped M2-brane contribution, we confirm the agreement of the indices for many sectors specified by the $r - 1$ charges by using analytic and numerical methods.

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1 Introduction

Monopole operators play important roles in gauge theories for M2-branes. Taking account of them is essential to obtain correct background geometries of M2-branes as moduli spaces of the gauge theories. Let us consider the $\mathcal{N} = 8$ supersymmetric Yang-Mills theory realized on a stack of $N$ D2-branes as a simple example. The diagonal $U(1)$ subgroup of the $U(N)$ gauge group does not couple to any fields, and we can define the dual photon field $a$ by

$$da = \frac{1}{g_3^2} * \text{tr}F,$$

where $g_3$ is the gauge coupling. This scalar field describes the collective motions of the M2-branes, and is identified with “the eleventh direction” or “the M-direction.” By definition, $a$ is the canonical conjugate of $\text{tr}F$, and the operator $e^{ima}$ changes the flux $(2\pi)^{-1} \int \text{tr}F$ by $m$. Namely, $e^{ima}$ is a monopole operator carrying magnetic charge $m$. In this sense, the M-direction emerges by taking account of monopole operators.

The ABJM model\cite{1}, $U(N) \times U(N)$ Chern-Simons matter system with $\mathcal{N} = 6$ supersymmetry, was proposed as a theory describing M2-branes in $\mathbb{R}^8/\mathbb{Z}_k$. Monopole operators in ABJM model\cite{2,3,4,5,6,7,8} also play a similar role. The action of ABJM model includes Chern-Simons terms

$$S_{CS} = \frac{k}{4\pi} \int \text{tr} \left[ A_1 dA_1 + \frac{2}{3} A_1^3 - A_2 dA_2 - \frac{2}{3} A_2^3 \right]. \quad (2)$$

The component of gauge fields corresponding to the diagonal $U(1)$ subgroup of $U(N) \times U(N)$ appears in the action only through these Chern-Simons terms, and its equation of motion gives the Gauss-law constraint

$$\text{tr}F_1 - \text{tr}F_2 = 0. \quad (3)$$

Therefore, we can define one gauge invariant magnetic charge again by

$$m = \frac{1}{2\pi} \oint \text{tr}F_1 = \frac{1}{2\pi} \oint \text{tr}F_2. \quad (4)$$

Just as in the case of the D2-brane theory, this magnetic charge is identified with the Kaluza-Klein momentum along the M-direction (M-momentum). The dual photon field $a$ can be defined by solving (3) as

$$da = \text{tr}A_1 - \text{tr}A_2, \quad (5)$$

and monopole operators in the form $e^{ima}$ correspond to Kaluza-Klein modes with the M-momentum proportional to $m$. (More precisely, the operator $e^{ima}$ is not gauge invariant in the ABJM model. We can construct gauge invariant operators by combining $e^{ima}$ and matter fields, and such gauge invariant operators correspond to Kaluza-Klein modes.) See also \cite{9,10} for similar analysis for BLG model\cite{11,12,13,14,15}.

When $k = 1$ or 2, the supersymmetry of the ABJM model is expected to be enhanced from $\mathcal{N} = 6$ to $\mathcal{N} = 8$. Equivalently, the R-symmetry is enhanced from $SO(6)$ to $SO(8)$. Because the $SO(8)$ mixes the M-direction with other directions, the analysis of monopole operators is indispensable to show the existence of the $\mathcal{N} = 8$ supersymmetry\cite{16,17}.

Even in more general quiver Chern-Simons theories, the emergence of the M-direction is explained in the same way\cite{18,19,20,21}. In quiver-type theories, we always have $U(1)$ diagonal gauge field which couples no matter fields in the theory.
and its dual scalar field plays the role of the coordinate of the M-direction. In
general, however, we can define more than one magnetic charges, and only one of
them accounts for the emergence of the M-momentum. For more concrete analysis,
let us consider a $U(N)^r$ quiver Chern-Simons theory with the Chern-Simons terms

$$S_{CS} = \sum_{a=1}^{r} \frac{k_a}{4\pi} \int \left( A_a dA_a + \frac{2}{3} A_a^3 \right). \tag{6}$$

We assume that the levels $k_a$ satisfy

$$\sum_{a=1}^{r} k_a = 0. \tag{7}$$

In this theory, we can define $r$ gauge invariant magnetic charges by

$$m_a = \frac{1}{2\pi} \oint tr F_a, \quad a = 1, \ldots, r. \tag{8}$$

But there is one constraint imposed on these magnetic charges. Due to the as-
sumption (7), the diagonal $U(1)$ gauge field becomes a Lagrange multiplier, and its
equation of motion gives the constraint

$$\sum_{a=1}^{r} k_a tr F_a = 0. \tag{9}$$

By integrating this over a 2-cycle, we obtain

$$\sum_{a=1}^{r} k_a m_a = 0. \tag{10}$$

This relation decreases the number of independent magnetic charges by one. Hence
we have $r-1$ independent magnetic charges. One of them should be identified with
the M-momentum, but we still have extra $r-2$ charges. What do these charges
represent in the dual geometry $AdS_4 \times X_7$?

In [22][23], it is proposed that monopole operators carrying these extra magnetic
charges correspond to M2-branes wrapped on non-trivial two-cycles in the internal
space $X_7$. In [22], it is pointed out that for $\mathcal{N} = 4$ Chern-Simons theories[24][25][26][27], the two-cycle Betti number $b_2(X_7)$ and $r-2$, the number of independent
magnetic charges subtracted by one, agree. In [23], for Abelian $\mathcal{N} = 4$ theories, the
agreement of the spectrum of monopole operators and that of wrapped M2-branes
are partially confirmed.

The purpose of this paper is to investigate the correspondence between wrapped
M2-branes and monopole operators in more detail. The analysis in [23] is carried out
with the gauge group $U(1)^r$, and by this reason, the perfect agreement of spectrum
cannot be expected. In this paper, we consider an $\mathcal{N} = 4$ Chern-Simons theory
with gauge group $U(N)^r$, and confirm a certain index for the gauge theory in the
large $N$ limit including monopole contribution agrees with the corresponding multi-
particle index on the gravity side with taking account of contribution of wrapped
M2-branes.

Analysis of indices has been done for ABJM model in [28]. The ABJM model
has $SU(4)$ R-symmetry and $U(1)$ flavor symmetry. Let $(h_1, h_2, h_3)$ be the $SU(4)_R$
weight vector and $h_4$ be the $U(1)$ charge. $h_4$ is identified with the M-momentum
on the gravity side. See Table 2 for concrete definition of the charges $h_m$. The
superconformal indices investigated in [28] are defined by

$$I(x,y_1,y_2) = tr \left[ (-1)^F e^{-\beta' (Q,S)} x^{2(\Delta+j_3)} y_1^{h_1} y_2^{h_2} \right] \tag{11}$$

We replace the variable $x$ commonly used in the literature by $x^2$ to avoid fractional power.
We also use different numbering for the chemical potentials $y_i$ from references for convenience.
where $Q$ is the component of supercharge with R-charge $(h_1, h_2, h_3) = (0, 0, 1)$, and $S$ is its Hermitian conjugate. On the gauge theory side, the trace in (11) is regarded as the summation over gauge invariant operators, and we denote the index by $I^\text{gauge}$. On the gravity side, we can define two indices by (11). The single-particle index $I$ is defined by taking the trace over all single-particle states, while the multi-particle states including single- and no-particle states. In general, a single particle index and the corresponding multi-particle index are related by

$$I^\text{mp}(\cdot) = \exp \sum_{n=1}^\infty \frac{1}{n} I^\text{mp}(\cdot^n),$$

where “$(\cdot)$” represents the sequence of the arguments of the index, and “$(\cdot^n)$” on the right hand side is the sequence with every argument replaced by its $n$-th power.

The indices defined by (11) are independent of $\beta'$, and only operators (states) saturating the BPS bound

$$\{Q, S\} = \Delta - j_3 - h_3 \geq 0$$

contribute to the indices. Because the computation of gauge theory index in [28] is performed in the large $k$-limit, all the monopole contribution decouples. On the gravity side, this corresponds to the decoupling of Kaluza-Klein modes with non-vanishing M-momentum $h_4$. The gauge theory index obtained in [28] is

$$I^\text{gauge}(x, y_1, y_2) = \prod_{n=1}^\infty \frac{(1 - x^{4n})^2}{(1 - \frac{x^n}{y_1})(1 - \frac{x^n}{y_2})(1 - x^{2n}y_1^2)(1 - x^{2n}y_2^2)}. \quad (14)$$

The corresponding graviton index is obtained by a projection of the graviton index for $\text{AdS}_4 \times \mathbb{S}^7$. The graviton index for $\text{AdS}_4 \times \mathbb{S}^7$ [29] is

$$I^\text{grav}(x, y_1, y_2, y_3) = \tr \left[ (-1)^F e^{-\beta' (Q, S) x^{2(\Delta + j_3)}} y_1^{h_1} y_2^{h_2} y_3^{h_3} \right] \frac{\text{(numerator)}}{\text{(denominator)}}, \quad (15)$$

where the numerator and the denominator are given by

$$\text{(numerator)} = \sqrt{y_1 y_2 y_3} (1 + y_1 y_2 + y_2 y_3 + y_3 y_1)x$$
$$- \sqrt{y_1 y_2 y_3} (y_1 + y_2 + y_3 + y_1 y_2 y_3)x^7 + (y_1 y_2 + y_2 y_3 + y_3 y_1 + y_1 y_2 y_3(y_1 + y_2 + y_3))(x^6 - x^2). \quad (16)$$

$$\text{(denominator)} = (1 - x^4)(\sqrt{y_3} - x\sqrt{y_1 y_2})(\sqrt{y_1} - x\sqrt{y_2 y_3})$$
$$\times (\sqrt{y_2} - x\sqrt{y_3 y_1})(\sqrt{y_1 y_2 y_3} - x). \quad (17)$$

The single-particle index for the orbifold $\mathbb{S}^7/\mathbb{Z}_k$ in the large $k$ limit is obtained from (14) by picking up $y_3$ independent terms as

$$I^\text{mp}(x, y_1, y_2) = \frac{x^2}{y_1 - x^2} + \frac{1}{1 - x^2 y_1} + \frac{x^2}{y_2 - x^2} + \frac{1}{1 - x^2 y_2} - \frac{2}{1 - x^4}. \quad (18)$$

It is easy to see the perfect agreement between (14) and the multi-particle index $I^\text{mp}(x, y_1, y_2)$ obtained from (15) by the relation (12).

A gauge theory index for ABJM model including the monopole contribution is computed in [3]. The index is defined by

$$I^\text{gauge}(x, y_1, y_2, y_3) = \tr \left[ (-1)^F e^{-\beta' (Q, S) x^{2(\Delta + j_3)}} y_1^{h_1} y_2^{h_2} y_3^{h_3} \right], \quad (19)$$

on the gravity side, we can define two indices by (11). The single-particle index $I^\text{mp}$ is defined by taking the trace over all single-particle states, while the multi-particle states including single- and no-particle states. In general, a single particle index and the corresponding multi-particle index are related by

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where the numerator and the denominator are given by

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$$\text{(denominator)} = (1 - x^4)(\sqrt{y_3} - x\sqrt{y_1 y_2})(\sqrt{y_1} - x\sqrt{y_2 y_3})$$
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where \( y_3 \) is introduced as the chemical potential for the charge \( h_4 \), which is related to the monopole charge \( m \) by \( h_4 = km \). In [5] it is confirmed that this agrees with the multi-particle index \([12]\) with \( I^{mp} \) being replaced by \( Z_k \) projection of \([15]\).

For \( \mathcal{N} = 4 \) theories, a superconformal index without monopole contribution is computed in [30]. They consider an \( \mathcal{N} = 4 \) theory obtained as \( \mathbb{Z}_M \) orbifold of ABJM model, which includes \( M \) untwisted and \( M \) twisted hypermultiplets, and compute the index

\[
I^{\text{gauge}}(x, y_1) = \text{tr} \left[ (-1)^F e^{-\beta' (Q,S) x^2} y_1^{\beta_1} \right] \tag{20}
\]
on the gauge theory side. The result suggests that the corresponding single-particle index should be

\[
I^{mp}(x, y_1) = \frac{1}{1 - x^2 y_1} + \frac{1}{1 - x^2 / y_1} - \frac{2}{1 - x^4} + \frac{2x^2 M}{1 - x^2 M} + (M - 1) \left( \frac{1}{1 - x^2 y_1} + \frac{1}{1 - x^2 / y_1} - \frac{2}{1 - x^4} \right). \tag{21}
\]

An interesting feature of this result is that this index consists of two parts of different origins. The first line in \([21]\) is obtained from \([18]\) by the projection which leaves only terms invariant under the \( \mathbb{Z}_M \) rotation \( y_2 \rightarrow e^{2\pi i/M} y_2 \). Thus, the first line is regarded as the bulk contribution. On the other hand, the second line is interpreted as the contribution of twisted sectors. Indeed, there are two \( A_{M-1} \)-type singular loci in the internal space \( X_7 \), and we expect the existence of an \( SU(M) \) vector multiplet on each of these singular loci. Let \( G_S = SU(M) \times SU(M) \) be the gauge group realized on the loci, and \( H_S = U(1)^{2(M-1)} \) its Cartan subgroup. In \([30]\) the twisted sectors are identified with the contribution of the \( H_S \) vector multiplets, which arise as supergravity modes localized at the singular loci.

In this paper we consider an \( \mathcal{N} = 4 \) Chern-Simons theory with general numbers of untwisted and twisted hypermultiplets. Let \( p \) and \( q \) be the numbers of two kinds of hypermultiplets. In this case, as we will explain later, the internal space \( X_7 \) includes an \( A_{p-1} \)-type singular locus and an \( A_{q-1} \)-type singular locus. On these loci gauge group \( G_S = SU(p) \times SU(q) \) is realized. We compute an index \( I^{\text{gauge}} \) similar to \([20]\) with taking account of the monopole contribution, and compare it to the corresponding multi-particle index \( I^{mp} \) on the gravity side. We find that they agree if we take account of the full \( G_S \) vector multiplets, which include not only the Cartan part but also \( H_S \)-charged particles. The \( H_S \) charges are represented as a vector in the \( G_S \) root lattice, and we can regard them as wrapping numbers of M2-branes on the vanishing two-cycles at the singularities. By comparing the indices, we establish the relation between \( r - 1 \) independent magnetic charges and the same number of charges on the gravity side: the \( M \)-momentum and \( r - 2 \) wrapping numbers. The \( H_S \) charges couple to the Wilson lines associated with the non-trivial fundamental groups of the singular loci, and this coupling shifts the Kaluza-Klein momenta along the cycles. Such a shift of momenta is correctly reproduced on the gauge theory side as a selection rule for charges associated with the global symmetries \( U(1) \times U(1)' \), which are defined in \([2]\) For \( U(1)' \) case, it was confirmed in \([22]\) by investigating charges associated with \( U(1) \times U(1)' \) of gauge-invariant chiral monopole operators. We show that it is also the case for \( U(N)' \) from the analysis of a monopole index in \([4]\).

This paper is organized as follows. In \([3]\) we briefly review the field contents, global symmetries, and the moduli space of the \( \mathcal{N} = 4 \) Chern-Simons theory. We also comment on the relation between the Wilson lines in the singular loci and the fivebrane linking numbers. In \([4]\) we derive a general expression for the multi-particle index from the known result of the graviton index for \( \text{AdS}_4 \times S^7 \) and a certain assumption for the index for a vector multiplet localized on \( \text{AdS}_4 \times S^3 \). In \([4]\) we investigate the gauge theory index including the monopole contribution. We
derive the selection rules for the $U(1) \times U(1)'$ charges and confirm that they agree with what is expected from the relation between the Wilson lines and the fivebrane linking numbers. We compare the gauge theory index and multi-particle index in \ref{6} for many sectors specified by the $r - 1$ charges, and confirm that they agree with each other by using analytic and numerical methods. We summarize the results in \ref{6}

\section{$\mathcal{N} = 4$ Chern-Simons theories}

\subsection{Action and symmetries}

We consider an $\mathcal{N} = 4$ Chern-Simons theory with unitary-type gauge group $U(N)'$. Such a Chern-Simons theory is described by a circular quiver diagram, in which vertices and edges represent vector multiplets $V_a$ and hypermultiplets $H_I$, respectively. We use index $a$ for vertices and $I$ for edges. In the terminology of $\mathcal{N} = 2$ superspace formalism, $\mathcal{N} = 4$ supermultiplets are represented as pairs of superfields as $V_a = (v_a, \Phi_a)$ and $H_I = (Q_I, \tilde{Q}_I)$. We define component fields for these superfields as

$$Q_I = (q_I, \psi_I), \quad \tilde{Q}_I = (\tilde{q}_I, \tilde{\psi}_I), \quad v_a = (A^a_{\mu}, \lambda_a), \quad \Phi_a = (\phi_a, \chi_a).$$

The action is

$$S = \sum_{I=1}^{r} \int d^3 x d^3 \theta \text{tr}(Q_I^\dagger e^{2v_L(I)}Q_I e^{-2v_R(I)} + \tilde{Q}_I e^{2v_L(I)}\tilde{Q}_I^\dagger e^{-2v_R(I)})$$

$$- \sum_{I=1}^{r} \left( \int d^3 x d^3 \theta \sqrt{2}\text{tr}(Q_I \Phi_{L(I)} Q_I - \tilde{Q}_I Q_I \Phi_{R(I)}) + \text{c.c.} \right)$$

$$+ \sum_{a=1}^{k_a} \frac{k_a}{2} \left[ \frac{1}{2\pi} \int d^3 x d^3 \theta \int_0^1 dt (v_a \overline{D}(e^{-2tv_a}De^{2tv_a})) - \left( \int d^3 x d^3 \theta \Phi_a^2 + \text{c.c.} \right) \right]$$

where $L(I)$ and $R(I)$ represent the vertices at the left and the right ends of an edge $I$, respectively. Similarly, we define $L(a)$ and $R(a)$ for the edges on the left and the right side of a vertex $a$. The first line in \ref{23} includes the kinetic terms of the hypermultiplets, and the second line is the standard superpotential coupling between vector and hypermultiplets. The third line is the supersymmetric completion of the Chern-Simons terms \ref{6}.

$k_a$ in \ref{23} are the Chern-Simons couplings. The gauge invariance of the action requires them to be integers. Furthermore, the existence of $\mathcal{N} = 4$ supersymmetry requires them to be given by

$$k_a = k(s_{L(a)} - s_{R(a)}), \quad k \in \mathbb{Z}, \quad s_I, 0, 1,$$

where $s_I$ are integers assigned to edges in the quiver diagram, and they take only two values 0 and 1. Corresponding to these two values, the hypermultiplets fall into two groups, untwisted and twisted hypermultiplets. If $s_I = 0$ ($s_I = 1$) the hypermultiplet is called untwisted (twisted) hypermultiplet. When we want to distinguish these two kinds of hypermultiplets, we use index $i$ for untwisted hypermultiplets, and $i'$ for twisted ones. Let $p$ and $q$ be the numbers of untwisted, and twisted hypermultiplets, respectively. Because the quiver diagram is circular, the total number of hypermultiplets and the number of vector multiplets are the same:

$$r = p + q.$$

The R-symmetry of this $\mathcal{N} = 4$ theory is

$$\text{Spin}(4)_R = SU(2) \times SU(2)'.$$
There also exist flavor symmetries
\[ U(1) \times U(1)'. \] (27)

The component fields in untwisted hypermultiplets \( H_i \) and those of twisted hypermultiplets \( H_i' \) are transformed in different ways under the global symmetries. See Table 1.

Table 1: The global symmetries

|          | \( SU(2) \) | \( SU(2)' \) | \( U(1) \) | \( U(1)' \) |
|----------|-------------|-------------|----------------|----------------|
| \((q_i, \bar{q}_i')\) | 2           | 1           | +1              | 0              |
| \((\psi_i, \psi_i')\)  | 1           | 2           | +1              | 0              |
| \((q_i', \bar{q}_i')\) | 1           | 2           | 0               | +1             |
| \((\psi_i', \psi_i')\) | 2           | 1           | 0               | +1             |

The M2-brane background corresponding to this theory is obtained as the Higgs branch of moduli spaces\(^\text{31}\). (See also \textsuperscript{32, 33}.) To obtain the background for a single M2-brane, let us consider the Abelian case with the gauge group \( U(1) \). With the terminology of \( N = 2 \) supersymmetry, the moduli space is obtained by dividing the solution of F-term conditions by the complexified gauge symmetry.

From the superpotential terms in the action \( \text{(23)} \), the F-term conditions of \( Q_I \) and \( \bar{Q}_{I'} \) give
\[ \phi_L(I) = \phi_R(I). \] (28)

(We assume that \( q_I, \bar{q}_{I'} \neq 0 \) for the Higgs branch.) This means that all \( \phi_a \) take the same value. We denote it by \( \phi \). The F-term condition for \( \Phi_a \) is
\[ q_{L(a)}\bar{q}_{L(a)} - ks_{L(a)}\phi = q_{R(a)}\bar{q}_{R(a)} - ks_{R(a)}\phi. \] (29)

This means that \( q_I\bar{q}_{I'} - ks_I\phi \) is a constant independent of the index \( I \). In other words, the product \( q_I\bar{q}_{I'} \) takes two values according to \( s_I \). We can define “meson operators” \( M \) and \( M' \) by
\[ M = q_i\bar{q}_i, \quad M' = q'_i\bar{q}'_{i}'. \] (30)

Now, we have \( 2r \) complex variables \( q_I \) and \( \bar{q}_{I'} \) constrained by \( \text{(30)} \). \( \phi_a \) are dependent fields. The number of independent complex variables is \( r + 2 \). In addition to these, we need to take account of the dual photon field \( a \). It is defined by solving the Gauss law constraint \( \text{(9)} \) as
\[ da = \sum_{a=1}^{r} k_a A_a. \] (31)

The dual photon field is combined with the scalar field \( \sigma \) in the diagonal \( U(1) \) vector multiplet to a complex scalar field belonging to a chiral multiplet. It is convenient to define \( e^{ia+\sigma} \).

Now we have \( r + 3 \) independent complex variables. We have to divide this space by complexified gauge symmetry \( U(1)_{C}^{-1} \) to obtain a complex 4-dimensional moduli space. Let us consider a gauge symmetry with parameter \( \lambda_a \), which transform the gauge fields by
\[ \delta A_a = d\lambda_a. \] (32)

This transform the complex scalar fields as
\[ q_I \rightarrow e^{i\beta_I} q_I, \quad e^{ia+\sigma} \rightarrow e^{-ik} \sum_i s_i \beta_i e^{ia+\sigma}, \] (33)
where we defined
\[ \beta_I = \lambda_{L(I)} - \lambda_{R(I)}. \]  
(34)

By definition, parameters \( \beta_I \) are constrained by
\[ \sum_{I=1}^r \beta_I = 0. \]  
(35)

Let us rewrite the parameters \( \beta_I \) by \( \varphi, \theta_i, \theta_i' \) as
\[ \beta_i = \frac{\varphi}{p} + \theta_i, \quad \beta_i' = -\frac{\varphi}{q} + \theta_i', \]  
(36)

where \( \theta_i \) and \( \theta_i' \) satisfy
\[ \sum_{i=1}^p \theta_i = \sum_{i'=1}^q \theta_i' = 0. \]  
(37)

Then, the gauge transformation becomes
\[ q_i \to e^{i\varphi/p} e^{i\theta_i} q_i, \quad q_i' \to e^{-i\varphi/q} e^{i\theta_i'} q_i', \quad e^{i\alpha + \sigma} \to e^{-ik\varphi} e^{i\alpha + \sigma} \]  
(38)

We can fix the continuous part of this gauge symmetry by
\[ e^{i\alpha + \sigma} = 1, \]  
(39)
and
\[ q_{i=1} = \cdots = q_{i=p}, \quad q_{i'=1} = \cdots = q_{i'=q}. \]  
(40)

(39) fixes \( \varphi \) transformation and two equations in (40) fix the \( \theta_i \) and \( \theta_i' \) transformations. If (40) hold, the relations in (39) guarantee
\[ q_{i=1} = \cdots = \tilde{q}_{i=p}, \quad q_{i'=1} = \cdots = \tilde{q}_{i'=q}. \]  
(41)

After the gauge fixing, we have four independent complex variables. We introduce the coordinates \( z_m \) \( (m = 1, 2, 3, 4) \) in the Higgs branch moduli space by
\[ z_1 = q_i, \quad z_2 = \tilde{q}_i, \quad z_3 = q_{i'}, \quad z_4 = \tilde{q}_{i'}. \]  
(42)

Even after the gauge fixing above, we still have residual gauge symmetry with the parameters
\[ \beta_i = \frac{2\pi N}{kp} + \frac{2\pi m}{p}, \quad \beta_i' = -\frac{2\pi N}{kq} + \frac{2\pi n}{q}, \]  
(43)
where \( N, m, n \) are arbitrary integers. Due to this residual gauge symmetry the global rotations
\[ \exp \left( \frac{2\pi i}{p} P \right), \quad \exp \left( \frac{2\pi i}{q} P' \right), \quad \exp (2\pi i P_M), \]  
(44)
are gauge equivalent to 1, \( P \) and \( P' \) are the generators of \( U(1) \) and \( U(1)' \), respectively, and their action on the coordinates are shown in Table 2. \( P_M \) is the linear combination of \( P \) and \( P' \);
\[ P_M = \frac{1}{kq} P' - \frac{1}{kp} p. \]  
(45)

The shift generated by \( P_M \) is gauge equivalent to the shift of dual photon field up to the gauge symmetry associated with the parameter \( \varphi \), and we regard \( P_M \) as the M-momentum.

By taking account of the discrete residual gauge symmetry (44), we obtain the moduli space
\[ M_{p,q,k} = \left( \mathbb{C}^2/\mathbb{Z}_p \right) \times \left( \mathbb{C}^2/\mathbb{Z}_q \right)/\mathbb{Z}_k. \]  
(46)
Table 2: Actions of generators of global symmetries on the coordinates $z_1$, $z_2$, $z_3$, and $z_4$ are shown.

|     | $T_3$ | $T'_3$ | $P$ | $P'$ | $h_1$ | $h_2$ | $h_3$ | $h_4$ |
|-----|-------|--------|-----|------|-------|-------|-------|-------|
| $z_1$ | +1/2  | 0      | +1  | 0    | +1/2  | +1/2  | −1/2  | +1/2  |
| $z_2$ | −1/2  | 0      | +1  | 0    | −1/2  | +1/2  | +1/2  | +1/2  |
| $z_3$ | 0     | +1/2   | 0   | +1   | −1/2  | +1/2  | −1/2  | −1/2  |
| $z_4$ | 0     | −1/2   | 0   | +1   | +1/2  | +1/2  | +1/2  | −1/2  |

We summarize the action of global symmetries to the coordinates $z_m$ in Table 2. $T_3$ and $T'_3$ are the Cartan generators of $SU(2)$ and $SU(2)'$, respectively. For convenience, we also include the weights $h_m$ ($m = 1, \ldots, 4$) used in the table. The charges $h_m$ are related to $P$, $P'$, $T_3$, and $T'_3$ by

$$h_1 = T_3 - T'_3, \quad h_2 = \frac{1}{2}(P + P'), \quad h_3 = -(T_3 + T'_3), \quad h_4 = \frac{1}{2}(P - P').$$  \hspace{1cm} (47)

2.2 Wilson lines and fivebrane linking numbers

By restricting the orbifold $M_{p,q,k}$ on the sphere of the unit radius, we obtain the internal space $X_7$ of the dual geometry

$$X_7 = (S^7/(\mathbb{Z}_p \times \mathbb{Z}_q)) / \mathbb{Z}_k.$$  \hspace{1cm} (48)

$\mathbb{Z}_k$ freely acts on the sphere and does not generate fixed points, while $\mathbb{Z}_p$ and $\mathbb{Z}_q$ generate three-dimensional loci of $A$-type singularities. We denote the singular loci associated with the $\mathbb{Z}_p$ and $\mathbb{Z}_q$ orbifodings by $\mathcal{S}_U$ and $\mathcal{S}_T$, respectively. $\mathcal{S}_U$ is $A_{p-1}$ singularity and an $SU(p)$ vector multiplet lives on it. Similarly, on the other locus $\mathcal{S}_T$, $SU(q)$ vector multiplet lives. We define the gauge groups

$$G_S = G_{\mathcal{S}_U} \times G_{\mathcal{S}_T} = SU(p) \times SU(q)$$  \hspace{1cm} (49)

and their Cartan parts

$$H_S = H_{\mathcal{S}_U} \times H_{\mathcal{S}_T} = U(1)^{p-1} \times U(1)^{q-1}$$  \hspace{1cm} (50)

for later convenience.

It is often convenient to represent $X_7$ as a $T^2$-fibration over a certain 5-manifold by using the global symmetry $U(1) \times U(1)'$ to define fibers as its orbits. Then the loci $\mathcal{S}_U$ and $\mathcal{S}_T$ are subsets of the base space on which one cycle of the toric fiber shrinks. (See [22] for detailed description of the structure of $X_7$.) When we blow up the singularities, $\mathcal{S}_U$ and $\mathcal{S}_T$ split into $p$ loci $\mathcal{S}_{U_i}$ and $q$ loci $\mathcal{S}_{T_i}$, respectively. We here use indices $i$ and $i'$ just as for hypermultiplets. The reason for this becomes clear shortly. Each of the loci can be regarded as a brane which supports $U(1)$ vector multiplet on its worldvolume. (Precisely speaking, these $U(1)$ are not independent because the gauge groups on the loci are not $U$ but $SU$.) Topology of the loci $\mathcal{S}_U$ and $\mathcal{S}_T$ are $S^3/\mathbb{Z}_{kq}$ and $S^3/\mathbb{Z}_{kp}$, respectively. Both the orbifold groups are generated by the third generator in $U(1)$. Associated with the fundamental groups $\pi_1(\mathcal{S}_U) = \mathbb{Z}_{kq}$ and $\pi_1(\mathcal{S}_T) = \mathbb{Z}_{kp}$, we have in general non-trivial Wilson lines

$$\text{diag}(e^{2\pi i n_1}, \ldots, e^{2\pi i n_p}) \in U(p), \quad \text{diag}(e^{2\pi i n'_1}, \ldots, e^{2\pi i n'_p}) \in U(q),$$  \hspace{1cm} (51)

where each diagonal component of these Wilson lines corresponds to each of singular loci $\mathcal{S}_{U_i}$ or $\mathcal{S}_{T_i}$. Note that these are not elements of $SU(p)$ and $SU(q)$ because we
do not impose the condition that their determinants are one. This does not cause any problem because there are no particles coupling to the $U(1)$ part. $\eta_i$ and $\eta'_i$ must be quantized by

$$\eta_i \in \frac{1}{kq} \mathbb{Z}, \quad \eta'_i \in \frac{1}{kp} \mathbb{Z}. \quad (52)$$

When we later compute the contribution of twisted sectors to a multi-particle index, we should take account of the momentum shift due to these Wilson lines.

To compare the multi-particle index with the gauge theory index, we need to relate the Wilson lines to data of the gauge theory. For this purpose, it is convenient to interpolate the M2-brane background $M_{p,q,k}$ and the Chern-Simons theory by a type IIB brane system on which the Chern-Simons theory is realized. Let us consider $N$ coincident D3-branes wrapped around $S^1$. If the size of the $S^1$ is small, the theory realized on the D3-brane worldvolume becomes effectively three-dimensional. We can realize $U(N)^r$ gauge group by introducing $r$ fivebranes intersecting with the D3-brane worldvolume at distinct points. In type IIB theory fivebranes are characterized by two charges: the NS5 charge and the D5 charge. To realize $N = 4$ Chern-Simons theory with $p$ untwisted and $q$ twisted hypermultiplets, we use $p$ NS5-branes and $q$ $(k,1)$-fivebranes, and place them around the $S^1$ according to the quiver diagram. $p + q = r$ hypermultiplets arise from massless modes of open strings stretched between two adjacent intervals on the both sides of the corresponding fivebrane. The Chern-Simons terms \((6)\) with levels \((24)\) are induced by the boundary coupling at the ends of the intervals of D3-branes\([34, 35]\).

By the T-duality transformation along $S^1$ and M-theory lift, this brane system is transformed into $N$ M2-branes in the M-theory background $M_{p,q,k}$, and its gravity dual is $\text{AdS}_4 \times X_7$. Through this duality chain, NS5-brane $i$ and $(1,k)$-fivebrane $i'$ are mapped to singular loci $S_{U_i}$ and $S_{F'_i}$, respectively. We have already assumed implicitly this correspondence between the fivebranes and singular loci when we used indices $i$ and $i'$ to label the singular loci.

In \([36]\), the relation between the set of Wilson lines and the structure of the brane system is studied in detail for the case of $k = 1$, and it is shown that the Wilson lines are determined by the fivebrane linking numbers. The relation is easily generalized for general $k$. To define the linking numbers, we first need to choose one vertex in the circular quiver diagram to cut the diagram at the vertex to make it linear. We represent the reference vertex by $a = \bullet$. For NS5-brane $i$ and $(1,k)$-fivebrane $i'$, the linking numbers are defined by

$$l_i = \delta N_i + k \sum_{i < j'} 1, \quad l'_i = \delta N'_i - k \sum_{\bullet < j < i'} 1, \quad (53)$$

where $\delta N_i = N_{L(I)} - N_{R(I)}$ represents the number of D3-branes ending on the fivebrane $I$. When $k = 1$ these reduces to those given in \([34]\). We inserted $k$ so that these linking numbers are invariant in the brane creation processes\([37]\). By definition, these numbers are integers depending on the reference point. With these linking numbers, the Wilson line parameters are given by

$$\eta_i = \frac{1}{kq} l_i, \quad \eta'_i = \frac{1}{kp} l'_i. \quad (54)$$

These relations for $k = 1$ are given in \([36]\). We generalize them by inserting factor $k^{-1}$ so that these relations are consistent with the quantization \((52)\).

In this work, we only consider the case of $\delta N_I = 0$, and the linking numbers are multiples of $k$. 
3 Gravity side

The indices we consider in this paper are defined by

\[ I(x, z, z') = \text{tr} \left[ (-)^F e^{-\beta' (Q, S)} x^{2(D + j_3)} z^P z'^{P'} \right]. \tag{55} \]

where \( Q \) is a certain component of the supercharge and \( S \) is its Hermitian conjugate. On the gauge theory side, the trace is taken over all gauge invariant operators. This index does not depend on \( \beta' \), and only operators saturating the BPS bound

\[ \{Q, S\} = \Delta - j_3 - (T_3 + T'_3) \geq 0 \tag{56} \]

contribute. We choose \( Q \) so that \( h_3 = -(T_3 + T'_3) \) is the R-charge rotating \( Q \). The global symmetries commuting with this R-charge is generated by \( P, P' \), and \( T_3 - T'_3 \). Among these three \( U(1) \) symmetries, the last one is broken when we deform the theory by adding \( Q \)-exact kinetic terms for the vector multiplets for the purpose of taking the weak coupling limit. This is the reason why we insert chemical potentials only for the charges \( P \) and \( P' \). We compare this index with the corresponding multi-particle index for M-theory in the dual geometry AdS_4 \times X_7.

On the gravity side, the single-particle index is given as the sum of two different origins. One is the contribution of bulk particles, and the other is that of the twisted sectors, which are localized at the fixed loci in the orbifold.

3.1 Bulk sector

In this subsection, we discuss the bulk sector. In general, the index for bulk particles in an orbifold \( S^7/\Gamma \) can be obtained from the index for \( S^7 \) by the projection which leaves modes invariant under the orbifold action. The single-particle index for bulk gravitons in AdS_4 \times S^7 is given in (15).

In the case of \( X_7 \) given in (48), \( \Gamma \) is generated by the three generators in (44). To obtain \( \Gamma \)-invariant part of the index, let us first rewrite the index (15) as a function of \( x, z, \) and \( z' \). Because \( zPz'P' = (zz')h_2(z/z')h_4 \), we can change the variables by substituting

\[ y_1 = 1, \quad y_2 = zz', \quad y_3 = \frac{z}{z'}, \tag{57} \]

into (15). We obtain

\[ I^\text{grav}(x, z, z') = \frac{\frac{1}{4} + z' + z + \frac{1}{4}(x - x^7) + (2 + z' + \frac{1}{2}x' + zz' + \frac{1}{2}x)(x^6 - x^2)}{(1 - x^4)(1 - xx')(1 - xz)(1 - x/z')}(1 - x/z) \tag{58} \]

We expand this index with respect to \( z \) and \( z' \) as

\[ I^\text{grav}(x, z, z') = \sum_{P, P'} I^\text{grav}_{P, P'}(x) z^P z'^{P'}. \tag{59} \]

The coefficients \( I^\text{grav}_{P, P'}(x) \) are given by

\[ I^\text{grav}_{P, P'}(x) = (1 - \delta_{P, 0} \delta_{P', 0}) x^{P|P'|} + \delta_{P, 0} x^{P|P'|+2} \frac{x^{P'|2}}{1 - x^4} + \delta_{P', 0} x^{P'|2} \frac{x^{P+2}}{1 - x^4}. \tag{60} \]

The charges \( P \) and \( P' \) in \( \Gamma \) invariant terms must satisfy

\[ \frac{1}{p} P \in \mathbb{Z}, \quad \frac{1}{q} P' \in \mathbb{Z}, \quad P_M \in \mathbb{Z}. \tag{61} \]

The general solution to these conditions is

\[ P = pa, \quad P' = q(a + kb), \quad a, b \in \mathbb{Z}. \tag{62} \]
The integer $b$ is equal to the M-momentum $P_M$. The single-particle index for the bulk gravitons in $S^7/\Gamma$ is given by

$$I^B(x, z, z') = \sum_{a, b = -\infty}^{\infty} I^{grav}_{p_a, q(a+kb)}(x) z^{p_a} z'^{q(a+kb)}. \quad \text{(63)}$$

### 3.2 Twisted sectors

As we have already mentioned, the internal space $X_7$ includes two fixed loci $S_U$ and $S_T$, and we should take account of the twisted sectors associated with these. The two sectors can be treated in parallel ways, and we first consider the contribution of the $S_U$ sector in detail. Because $S_U$ is the $A_{p-1}$ type singularity, we expect that there exists an $SU(p)$ vector multiplet localized on the locus. With the coordinates defined in (42), $S_U$ is given by $z_1 = z_2 = 0$, and is spanned by two complex coordinates $z_3$ and $z_4$ constrained by

$$|z_3|^2 + |z_4|^2 = 1. \quad \text{(64)}$$

This equation together with the identification by the $\mathbb{Z}_{kq}$ generated by the third generator in (44) defines the Lens space $S^3/\mathbb{Z}_{kq}$. Because this orbifold does not have fixed points, we can obtain the single-particle index for a vector multiplet in this manifold by the $\mathbb{Z}_{kq}$ projection from the index for the covering space $S^3$.

The component fields in a vector multiplet in $S_U$ do not carry the charge $P$. Thus, the index should be the function only of $x$ and $z'$, and is independent of $z$. We propose the single-particle index

$$I^{vec}(x, z') = \frac{x^2}{1-x^4} \left(1 + \frac{x z'}{1-x z'} + \frac{x/ z'}{1-x/ z'}\right),$$

$$= \sum_{P' = -\infty}^{\infty} I^{vec}_{P'}(x) z'^{P'}, \quad \text{(65)}$$

for a single $U(1)$ vector multiplet in $S^3$, where the coefficients in the $z'$ expansion are given by

$$I^{vec}_m(x) = \frac{x^{|m|+2}}{1-x^4}. \quad \text{(66)}$$

This single-particle index should be directly derived from the analysis of Kaluza-Klein spectrum of a vector multiplet on $S^3$. We leave such analysis for future work, and use this as a starting point of the analysis of the twisted sectors. Once we accept that the single-particle index for $S^3$ is given by (65), the index for orbifold $S^3/\mathbb{Z}_{kq}$ is obtained by the projection which leaves only $\mathbb{Z}_{kq}$ invariant modes.

When we consider the single-particle index of the covering space of the other locus $S_T$, we should replace the variable $z'$ in (65) by $z$. Namely, it is $I^{vec}(x, z)$.

The procedure of the $\mathbb{Z}_{kq}$ and $\mathbb{Z}_{kp}$ projections is similar to what we have done for the bulk sector. An important difference is that in general there exist non-trivial Wilson lines coupling to the vector multiplets in the twisted sectors. Before considering the projection for single-particle states, let us consider that for a general multi-particle state. Let $\rho_i$ and $\rho_{i'}$ be the $H_{S_U}$ and $H_{S_T}$ charges of the multi-particle state. They are the sum of charges of constituent particles in the state. Because every particle belongs to the adjoint representation of $G_S$, these charges satisfy

$$\sum_{i=1}^{p} \rho_i = \sum_{i'=1}^{q} \rho_{i'} = 0. \quad \text{(67)}$$
When we act an element of the orbifold group which rotates the cycles in $S_U$ and $S_T$ by $r$ and $s$ times, respectively, the state picks up the phase

$$2\pi i \left( r \sum_{i=1}^{p} \rho_i \eta_i + s \sum_{i'=1}^{q} \rho_{i'} \eta_{i'} \right), \quad (68)$$

and this must be canceled by the phase factor associated with the momentum. Because $(r,s) = (0,-k), (k,0),$ and $(1,1)$ for the three generators in (44), the cancellation of the phases requires

$$\exp\left( \frac{2\pi i}{p} P \right) = \exp\left( 2\pi ik \sum_{i'=1}^{q} \rho_{i'} \eta_{i'} \right), \quad (69)$$

$$\exp\left( \frac{2\pi i}{q} P' \right) = \exp\left( -2\pi ik \sum_{i=1}^{p} \rho_i \eta_i \right), \quad (70)$$

$$\exp(2\pi i P_M) = \exp\left( -2\pi i \sum_{i=1}^{p} \rho_i \eta_i - 2\pi i \sum_{i'=1}^{q} \rho_{i'} \eta_{i'} \right), \quad (71)$$

$P$ and $P'$ satisfying these conditions are given by

$$P = p \left( a + k \sum_{i'=1}^{q} \rho_{i'} \eta_{i'} \right), \quad P' = q \left( a + kb - k \sum_{i=1}^{p} \rho_i \eta_i \right), \quad a, b \in \mathbb{Z}, \quad (72)$$

and then the M-momentum $P_M$ is

$$P_M = b - \sum_{i=1}^{p} \rho_{\alpha i} \eta_i - \sum_{i'=1}^{q} \rho_{\alpha i'} \eta_{i'}, \quad b \in \mathbb{Z}. \quad (73)$$

Unlike the case of bulk sector, the M-momentum $P_M$ is not always an integer. These conditions are imposed on any multi-particle states, including single-particle states. Actually, we obtain the momenta (62) for bulk single-particle states by simply setting $\rho_i = \rho_{i'} = 0$ in (72).

For a single-particle state in the twisted sector on the locus $S_U$, $\rho_{i'} = P = 0$. This implies that $a$ in the first equation in (72) vanishes, and the second equation gives the momentum

$$P' = kq \left( b - \sum_{i=1}^{p} \rho_{\alpha i} \eta_i \right), \quad b \in \mathbb{Z}. \quad (74)$$

Let $\{\rho_{\alpha i}\} = \vec{\rho}_\alpha$ be the charge vector for an $SU(p)$ vector multiplet living in the locus $S_U$. $\alpha = 1, \ldots, p^2 - 1$ is the adjoint index of $SU(p)$. These vectors are nothing but the weight vectors for the adjoint representation of $SU(p)$. The single particle index for vector multiplets in the locus $S_U$ is

$$I_{S_U}^{SU}(x, z'; t) = \sum_{\alpha=1}^{p^2-1} \sum_{b=-\infty}^{\infty} I_{kq(b-\vec{\rho}_\alpha \cdot \vec{\eta})}^{\text{vec}}(x) z^{kq(b-\vec{\rho}_\alpha \cdot \vec{\eta})} t^{\rho_{\alpha 1}} \cdots t^{\rho_{\alpha p}}, \quad (75)$$

where $\deg(\vec{\rho})$ is the degeneracy for the adjoint representation at $\vec{\rho}$ in the $SU(p)$ root lattice. Namely,

$$\deg(\vec{\rho}) = \begin{cases} 1 & (|\vec{\rho}|^2 = 2) \\ p-1 & (\vec{\rho} = 0) \\ 0 & (\text{others}) \end{cases} \quad (76)$$
We here introduced new chemical potentials \( \vec{\tau} = (t_1, \ldots, t_p) \) for the \( H_{SU} \) charges \( \vec{\rho} \).

The single-particle index for the \( SU(q) \) vector multiplet localized in the locus \( S_T = S^3/\mathbb{Z}_{kp} \) is obtained in the same way. Because \( \rho_i = P^i = 0 \), the projection restrict the value of the momentum \( P \) as

\[
P = kp \left( -b + \sum_{i'=1}^q \rho_{a'i'} \eta_{i'} \right), \quad b \in \mathbb{Z}.
\]

(77)

The single-particle index for the \( SU(q) \) vector multiplet in \( S_T \) is given by

\[
I^{S_T}(x, z, \vec{t}) = \sum_{a=1}^{q^2-1} \sum_{a=-\infty}^\infty I^{vec}_{KP(a+\vec{\rho}, \vec{\eta})}(x) z^{kq(p+\vec{\rho}, \eta)} I^\rho_{\vec{a}1} \ldots I^\rho_{\vec{a}q}
\]

\[
= \sum_{\vec{\rho}} \deg(\vec{\rho}) \sum_{a=-\infty}^\infty I^{vec}_{KP(a+\vec{\rho}, \vec{\eta})}(x) z^{kq(p+\vec{\rho}, \eta)} I^\rho_{\vec{a}1} \ldots I^\rho_{\vec{a}q},
\]

(78)

where we defined the degeneracy for the \( SU(q) \) adjoint representation similarly to \( \text{vec} \).

Due to the constraint (77), these indices are invariant under the overall rescaling of \( \vec{t} \) and \( \vec{\tau} \):

\[
I^{SU}(x, z', \vec{c} \vec{t}) = I^{SU}(x, z', \vec{t}), \quad I^{S_T}(x, z, \vec{c} \vec{t}) = I^{S_T}(x, z, \vec{t}).
\]

(79)

By summing up the contribution of the bulk and the twisted sectors, we obtain

\[
I^{mp}(x, z, z'; \vec{t}, \vec{t}') = I^{B}(x, z, z') + I^{SU}(x, z', \vec{t}) + I^{S_T}(x, z, \vec{t}).
\]

(80)

In the following sections, we confirm that the corresponding multi-particle index

\[
I^{mp}(x, z, z'; \vec{t}, \vec{t}') = \exp \sum_{n=1}^{\infty} \frac{1}{n} I^{mp}(x^n, z^n, z'^n; \vec{t}^n, \vec{t}'^n)
\]

(81)

is reproduced as the monopole index on the field theory side. What we will actually do in the following section is to derive the indices as functions of \( \vec{t} \) and \( \vec{\rho} \) but to compute indices for various sectors specified by charges \( (P_M, \vec{\rho}, \vec{\rho}') \) separately. The index for each sector specified by \( (P_M, \vec{\rho}, \vec{\rho}') \) is extracted from (81) by

\[
I^{mp}(x, c^{-\vec{\rho}} z, c^{\vec{\rho}'} z', \vec{t}, \vec{t}') = \sum_{(P_M, \vec{\rho}, \vec{\rho}')} I^{mp}_{(P_M, \vec{\rho}, \vec{\rho}')}(x, z, z') c^{P_M \vec{\rho} \vec{\rho}'}.
\]

(82)

To pick up the part of specific M-momentum we inserted an auxiliary variable \( c \). The summation with respect to the vectors \( \vec{\rho} \) and \( \vec{\rho}' \) are taken over the \( SU(p) \) and \( SU(q) \) root lattices. Note that \( P_M \) is not always integer. The values \( P_M \) can take are determined by the equation (73), and depend on \( \vec{\rho} \) and \( \vec{\rho}' \).

The left and right hand sides in (80) are also expanded in a similar way, and we obtain

\[
I^{mp}_{(P_M, \vec{\rho}, \vec{\rho}')} (x, z, z'; \vec{t}, \vec{t}') = \delta_{\vec{\rho}, \vec{\rho}} \delta_{\vec{\rho}', \vec{\rho}'} I^{B}_{P_M}(x, z, z')
\]

\[
+ \delta_{\vec{\rho}, \vec{\rho}'} I^{SU}_{(P_M, \vec{\rho})}(x, z') \vec{t}'
\]

\[
+ \delta_{\vec{\rho}', \vec{\rho}'} I^{S_T}_{(P_M, \vec{\rho}')}(x, z) \vec{t}'
\]

(83)

where indices on the right hand side are defined by

\[
I^{B}_{P_M}(x, z, z') = \sum_{a=-\infty}^{\infty} I^{grav}_{pa,q(a+kP_M)}(x) z^{ha} z^{\eta(a+kP_M)}.
\]

(84)

\[
I^{SU}_{(P_M, \vec{\rho})}(x, z', \vec{t}) = \deg(\vec{\rho}) I^{vec}_{kpM}(x) z^{kqP_M} \vec{t}'
\]

(85)

\[
I^{S_T}_{(P_M, \vec{\rho}')}(x, z, \vec{t}) = \deg(\vec{\rho}') I^{vec}_{-kpM}(x) z^{-kqP_M} \vec{t}'
\]

(86)
4 Gauge theory side

4.1 Gauge theory index

The gauge theory index \( I_{\text{gauge}} \) which we study is defined by

\[
I_{\text{gauge}}(x, z, z') = \text{tr} \left[ (-)^{F e^{-\beta (Q, S)}} z^{2(\Delta + j)} z' P z'^* \right].
\] (87)

This is evaluated by the radial quantization method [38, 39]. The procedure to compute this index is essentially the same as the case of the ABJM model, which is explained in [5] in detail.

By a conformal transformation, a local operator in \( \mathbb{R}^3 \) is mapped to a state in the Fock space of the conformal field theory defined in \( S^2 \times \mathbb{R} \). The trace over all operators is replaced by the path integral in the compact three-dimensional space \( S^2 \times S^1 \), where \( S^1 \) is the compactified time direction. To carry out the path integral, we need to take a weak coupling limit in such a way that it does not change the index. If the theory had continuous coupling constants, we could take such a limit by sending them to zero. In the theory we discuss, however, we do not have such continuous parameters. In the large \( N \) limit with fixed 't Hooft coupling \( \lambda = N/k \), \( \lambda \) becomes effectively continuous, and we can take the weak coupling limit \( \lambda \to 0 \). This procedure is used in [30] to compute the neutral part of the index (20). (We mean by the neutral part the contribution of operators without magnetic charges.) However, we cannot use the same procedure because the monopole contributions are suppressed in the large \( k \) limit.

In this paper, we keep the level \( k \) finite, and take a weak coupling limit by adding Q-exact terms to the action. We can realize kinetic terms of vector and hypermultiplets as Q-exact terms, and adding such terms to the action does not affect the index because only states eliminated by \( Q \) contribute to the index. We can take the weak coupling limit by sending the coefficients of the Q-exact terms to infinity. In such a limit, we can treat all fields as free fields, and the saddle point approximation gives the exact result.

Because we want to take account of monopole operators, we should consider all backgrounds with magnetic flux through the \( S^2 \). We assume that only Goddard-Nuyts-Olive (GNO) monopoles [40] contribute to the index as saddle points in the path integral. GNO monopoles are superposition of Dirac monopoles for the Cartan part of the gauge group. For every vertex \( a \) in the quiver diagram we have \( U(1) \) subgroups. We label them by color indices \( s, t, \ldots \). Let \( m_a \in \mathbb{Z} \) be the magnetic charge of the GNO monopole for the \( s \)-th \( U(1) \) subgroup of \( U(N) \). We should consider all possible charges parameterized by \( rN \) integers \( \{m_a\} \), and the total index is given as the summation over all monopole charges

\[
I_{\text{gauge}}(x, z, z'; \vec{\tau}) = \sum_{\{m_a\}} I_{\{m_a\}}(x, z, z') \prod_{a=1}^{n} \tau_a^{m_a},
\] (88)

where we introduced chemical potentials \( \tau_a \) for the magnetic charges \( m_a \) defined by

\[
m_a = \sum_{s=1}^{N} m_{as}.
\] (89)

These are the gauge invariant monopole charges introduced in [3].

In order to compare the gauge theory index [38] with the multi-particle index [31] derived on the gravity side, we need to find the relation between magnetic charges \( m_a \) and the variables \( (P_M, \vec{p}, \vec{p}') \). We discuss this relation in the next subsection. Here we focus on the way to compute the index for each sector specified by the magnetic charges.
In the weak coupling limit, we can expand fields in the theory by harmonic functions in $S^2$ with magnetic flux (monopole harmonics), and the path integral reduces to integrals for the infinite number of modes. The Gaussian integrals associated with non-zero modes can be easily performed, and we are left with the expression with integrals with respect to the holonomy around the compact time direction
\[
\text{diag}(e^{i\alpha_1}, \ldots, e^{i\alpha_N}, \ldots, e^{i\alpha_N}) \in U(N)_a
\]
in the Cartan part of the gauge group. The expression after the integration over the massive modes is
\[
I_{\{m_{as}\}}(x, z, z') = x^{2\epsilon_0(\{m_{as}\})} \left( \prod_{a=1}^r \prod_{s=1}^N \int \frac{d\alpha_{as}}{2\pi} \right) \exp \left( i \sum_{a=1}^r \sum_{s=1}^N k_a m_{as} \alpha_{as} \right)
\]
\[
\times \prod_{a=1}^r \prod_{s,t=1}^N \exp \left[ \sum_{n=1}^\infty \frac{1}{n} f^{vec}_{ast}(\{m_{as}\}; x^n, e^{in\beta_{ast}}) \right]
\]
\[
\times \prod_{l=1}^r \prod_{s,t=1}^N \exp \left[ \sum_{n=1}^\infty \frac{1}{n} f^{hyp}_{lst}(\{m_{as}\}; x^n, (z_I e^{i\beta_{lst}})^n) \right].
\]

See [5] [8] for a detailed derivation in the case of ABJM model. Generalization to $N=4$ theories is straightforward. $\epsilon_0$ is the zero point energy due to the vacuum polarization in $S^2$, and is given by
\[
\epsilon_0(\{m_{as}\}) = -\frac{1}{2} \sum_{a=1}^r \sum_{s,t=1}^N |m_{as} - m_{at}| + \frac{1}{2} \sum_{l=1}^r \sum_{s,t=1}^N |m_{L(I)s} - m_{R(I)t}|,
\]
where the first and the second terms are the contribution of vector and hyper multiplets, respectively. $f^{vec}_{ast}$ and $f^{hyp}_{lst}$ are contributions of oscillators in the vector multiplet $V_a$ and hyper multiplet $H_I$, respectively, and given by
\[
f^{vec}_{ast}(\{m_{as}\}; x, e^{i\beta_{ast}}) = - (1 - \delta_{st}) x^{2|m_{as} - m_{at}|} e^{i\beta_{ast}},
\]
\[
f^{hyp}_{lst}(\{m_{as}\}; x, z_I e^{i\beta_{lst}}) = \frac{x^{2|m_{L(I)s} - m_{R(I)t}| + 1}}{1 + x^2} \left( e^{i\beta_{lst} z_I} + \frac{1}{e^{i\beta_{lst} z_I}} \right).
\]
$z_I$ is defined by
\[
z_I = \begin{cases} 
  z & \text{for } s_I = 0 \\
  z' & \text{for } s_I = 1
\end{cases}
\]
$\beta_{ast}$ and $\beta_{lst}$ are angular variables defined by
\[
\beta_{ast} = \alpha_{as} - \alpha_{at}, \quad \beta_{lst} = \alpha_{L(I)s} - \alpha_{R(I)t}.
\]
These angular variables are holonomies for the components of $V_a$ or $H_I$ specified by the color indices $s$ and $t$.

To obtain the gauge theory index which can be compared with the graviton index, we should take the large $N$ limit. This limit is taken by adding vanishing entries to the monopole charges $\{m_{as}\}$. For each $a$, the monopole charge is described by $N$ integers $m_{as}$ ($s = 1, \ldots, N$). Let $M_a$ be the number of non-vanishing components among them. When we take the large $N$ limit, we keep $M_a$ at $O(1)$.

For this limit to be well defined, the zero-point energy should not diverge in the limit. This is indeed easily confirmed by rewriting (92) as
\[
\epsilon_0(\{m_{as}\}_s) = -\frac{1}{2} \sum_{a=1}^r \sum_{s \in M_a} \sum_{t \in M_a} |m_{as} - m_{at}| + \frac{1}{2} \sum_{a=1}^r \sum_{s \in M_{L(I)s}} \sum_{t \in M_{R(I)t}} |m_{L(I)s} - m_{R(I)t}|
\]
\[
+ \frac{1}{2} \sum_{a=1}^r (2M_a - M_{a+1} - M_{a-1}) \sum_{s \in M_a} |m_{as}|.
\]
where \( \{ m_{as} \}_s \) is the collection of non-vanishing components in \( \{ m_{as} \} \), and \( \sum_{\alpha \in \mathcal{M}_a} \) represents the summation over \( \mathcal{M}_a \) non-vanishing components in the magnetic charges. This expression is manifestly independent of \( N \), and well behaves in the large \( N \) limit.

The integration with respect to angular variables \( \alpha_{as} \) associated with vanishing magnetic charges \( m_{as} \) can be carried out by introducing the variables \( \lambda_{an} \) by

\[
\lambda_{an} = \frac{1}{N - M_a} \sum_{s=M_{a+1}}^{N} e^{in\alpha_{as}}, \quad n = \pm 1, \pm 2, \ldots \tag{98}
\]

The exponential factors in the second and the third lines in (91) can be rewritten as a Gaussian factor including the first and the zeroth order terms of \( \lambda_{an} \), and the matrix \( M \) is

\[
M(x, z, z') = \begin{pmatrix}
1 & \frac{x_{z_1-1}}{1+x^2} & \frac{x_{z_2-1}}{1+x^2} & \cdots & \frac{x_{z_N-1}}{1+x^2} \\
\frac{x_{z_1-1}}{1+x^2} & 1 & \frac{x_{z_2-1}}{1+x^2} & \cdots & \frac{x_{z_N-1}}{1+x^2} \\
\frac{x_{z_1-1}}{1+x^2} & \frac{x_{z_2-1}}{1+x^2} & 1 & \cdots & \frac{x_{z_N-1}}{1+x^2} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{x_{z_1-1}}{1+x^2} & \frac{x_{z_2-1}}{1+x^2} & \frac{x_{z_3-1}}{1+x^2} & \cdots & 1
\end{pmatrix} \tag{100}
\]

After the Gaussian integral with respect to \( \lambda_{an} \), we are left with the following expression including the finite number of integrals.

\[
I_{\{m_{as}\}}(x, z, z') = I^{(0)}(x, z, z') I^{(s)}_{\{m_{as}\}}(x, z, z'), \tag{101}
\]

where \( I^{(0)} \) is the determinant factor associated with the Gaussian integral of \( \lambda_{an} \):

\[
I^{(0)}(x, z, z') = \prod_{n=1}^{\infty} \frac{1}{\det M(x^n, z^n, z'^n)}, \tag{102}
\]

and \( I^{(s)}_{\{m_{as}\}} \) is given by

\[
I^{(s)}_{\{m_{as}\}}(x, z, z') = \frac{x^{2\alpha(\{m_{as}\})}}{(\text{symmetry})} \prod_{\alpha=1}^{r} \prod_{s=1}^{M_{\alpha}} \int \frac{d\alpha_{as}}{2\pi} \exp \left( i \sum_{a=1}^{r} \sum_{s=1}^{M_{\alpha}} k_{a} m_{as} \alpha_{as} \right) 
\times \prod_{a} \prod_{s, t} \left[ \exp \sum_{n=1}^{\infty} f_{\text{vec}}(\{m_{as}\}; x^n, e^{in\beta_{ast}}) \right] \times \prod_{a} \prod_{s, t} \left[ \exp \sum_{n=1}^{\infty} f_{\text{hyp}}(\{m_{as}\}; x^n, (z t e^{i\beta_{ast}})^n) \right]. \tag{103}
\]

For later convenience we divide the magnetic charges \( \{ m_{as} \}_s \) into \( \{ m_{as} \}_+ \), the collection of positive charges, and \( \{ m_{as} \}_- \), the collection of negative charges, and represent each of \( \{ m_{as} \}_+ \) and \( \{ m_{as} \}_- \) as a set of \( r \) Young diagrams. We also introduce \( M^+_a (M^-_a) \), the number of positive (negative) components in \( m_{as} \) for each \( a \). The symmetry factor in (103) is the product of the symmetry factors of the \( 2r \)
Young diagrams. The symmetry factor for a single Young diagram is defined as the product of the factorial of the number of the lines with the same length in the Young diagram. For example, the symmetry factor for \( \begin{ydiagram}{2} \end{ydiagram} \) and \( \begin{ydiagram}{3} \end{ydiagram} \) are \( 1!2! \) and \( 2!3! \), respectively.

\[ f_{\text{vec}} \] and \( f_{\text{hyp}} \) are given by

\[ f_{\text{vec}} \left( \{m_{as}\}; x, \gamma \beta_{\text{hyp}} \right) = \left( -\left( 1 - \delta_{\text{st}} \right) x^{2|m_{as}| - m_{at}|} + x^{2|m_{as}| + |m_{at}|} \right) e^{i\beta_{\text{hyp}}} \]  \hspace{1cm} (104)

\[ f_{\text{hyp}} \left( \{m_{as}\}; x, z \gamma e^{i\beta_{\text{hyp}}} \right) = \left( x^{2|m_{L(I)}| - m_{R(I)}|} - x^{2|m_{L(I)}| + |m_{R(I)}|} \right) \]

\[ \times \frac{x}{1 + x^2} \left( z \gamma e^{i\beta_{\text{hyp}}} + \frac{1}{z \gamma e^{i\beta_{\text{hyp}}}} \right) \]  \hspace{1cm} (105)

We can easily see that \( I^{(\pm)}_{\{m_{as}\}} \) is further factorized into \( I^{(+)}_{\{m_{as}\}} + I^{(-)}_{\{m_{as}\}} \) depending only on \( \{m_{as}\}_+ \) and \( \{m_{as}\}_- \) depending only on \( \{m_{as}\}_- \). To show the factorization of the zero-point energy contribution \( x^{2\epsilon_0} \), we divide the range of all the summations of color indices \( s \) and \( t \) in (107) into two parts as \( \sum_{s \in M_a} = \sum_{s \in M_a^+} + \sum_{s \in M_a^-} \). The first term in (107) is decomposed as

\[ - \sum_{a=1}^r \sum_{s \in M_a^+} \sum_{t \in M_a^+} |m_{as} - m_{at}| - \sum_{a=1}^r \sum_{s \in M_a^-} \sum_{t \in M_a^-} |m_{as} - m_{at}| \]

\[ - \sum_{a=1}^r \left( 2M_a^- \sum_{s \in M_a^-} |m_{as}| + 2M_a^+ \sum_{s \in M_a^+} |m_{as}| \right) \]  \hspace{1cm} (106)

In the first line, the contributions of \( \{m_{as}\}_+ \) and \( \{m_{as}\}_- \) decouple from each other. The two terms in the second line depend on both \( \{m_{as}\}_+ \) and \( \{m_{as}\}_- \), and for the factorization, these terms should be canceled by other terms. Actually, these terms are precisely canceled by the mixed terms arising from the \( \sum 2M_a \sum |m_{as}| \) term in the second line in (107). In this way, all the mixed terms cancel, and the zero-point energy is represented as

\[ \epsilon_0(\{m_{as}\}_+) = \epsilon_0(\{m_{as}\}_+) + \epsilon_0(\{m_{as}\}_-) \]  \hspace{1cm} (107)

The factorization of the second and the third lines in (108) is shown by using the fact that the factor in the form \( x^{2|m_{as}| - m_{at}|} - x^{2|m_{as}| + |m_{at}|} \) appearing in (104) and (105) vanish when \( m \) and \( m' \) have opposite signatures.

Now we have shown that the gauge theory index factorizes into the three parts:

\[ I_{\{m_{as}\}}(x, z, z') = I^{(0)}(x, z, z') I^{(+)}_{\{m_{as}\}}(x, z, z') I^{(-)}_{\{m_{as}\}}(x, z, z') \]  \hspace{1cm} (108)

Because the summations over \( \{m_{as}\}_+ \) and \( \{m_{as}\}_- \) are independent in the large \( N \) limit, the total index also factorizes into three parts

\[ I^{gauge}(x, z, z'; \vec{r}) = \sum_{\{m_{as}\}_+} I^{(0)}(x, z, z') I^{(+)}(x, z, z'; \vec{r}) I^{(-)}(x, z, z'; \vec{r}) \]  \hspace{1cm} (109)

where \( I^{(\pm)}(x, z, z'; \vec{r}) \) is defined by

\[ I^{(\pm)}(x, z, z'; \vec{r}) = \sum_{\{m_{as}\}_\pm} I^{(\pm)}_{\{m_{as}\}_\pm}(x, z, z') \tau_1^{m_1} \cdots \tau_r^{m_r} \]  \hspace{1cm} (110)

We also define the index for a specific gauge invariant monopole charges \( \{m_a\} \) as the sum of contributions of all the monopole backgrounds with the same \( \{m_a\} \). For example, the index for \( \{m_a\} = \{2, 2\} \) is the sum of four contributions:

\[ I_{\{2, 2\}}^{(+)} = I_{\{2, 2\}}^{(+)} + I_{\{2, 2\}}^{(+)} + I_{\{2, 2\}}^{(+)} + I_{\{2, 2\}}^{(+)} \]  \hspace{1cm} (111)

where we used the Dynkin diagrams to represent the charges \( \{m_{as}\}_+ \).
4.2 Selection rules

The integration with respect to the angular variable $\alpha_a$ leaves only terms whose $P$ and $P'$, the numbers of $z$ and $z'$ in the terms, satisfy certain selection rules, which correspond to conditions of gauge-invariance of operators. For an operator which carries at most the diagonal U(1) magnetic charge, selection rules are expected to be

$$\frac{1}{p} P \in \mathbb{Z}, \quad \frac{1}{q} P' \in \mathbb{Z}, \quad (112)$$

which means that such an operator is invariant under the residual gauge transformations $\delta$. For another operator which carries different magnetic charges, selection rules are considered to shift from (112) [23]. In this subsection, we derive selection rules and show that these selection rules precisely reproduce the spectrum (22) of the Kaluza-Klein momenta derived on the gravity side.

Let us start from (91). For every vertex ($U(N)$ gauge group) $a$, we have $N$ angular variables $\alpha_{as}$ ($s = 1, \ldots, N$). Instead of these, let us take $\alpha_{a1}$ and $\beta_{a1s}$ ($s = 2, \ldots, N$) as $N$ independent angular variables. We replace all $\alpha_{as}$ ($s \geq 2$) in (91) by $\alpha_{a1} - \beta_{a1s}$. By this replacement, the exponential factor including the levels $k_a$ becomes

$$\exp\left(i \sum_{a=1}^{r} \sum_{s=1}^{N} k_a m_a \alpha_{as}\right) = \exp\left(i \sum_{a=1}^{r} k_a m_a \alpha_{a1}\right) \times \exp\left(-i \sum_{a=1}^{r} \sum_{s=2}^{N} k_a m_a \beta_{a1s}\right).$$

The variables $\beta_{Ist}$ in $f_I^{hyp}(x, z_I e^{i\beta_{Ist}})$ become

$$\beta_{Ist} = \beta_{I11} - \beta_{L(I)1s} + \beta_{R(I)1t}. \quad (114)$$

As a result, the parameter $z_I$ is always accompanied by $e^{i\beta_{I11}}$. After integrating out $\beta_{a1s}$ ($s \geq 2$), we obtain

$$I_{\{m_a\}}(x, z, z') = \left(\prod_{a=1}^{r} \int d\alpha_{a1}\right) \exp\left(i \sum_{a=1}^{r} k_a m_a \alpha_{a1}\right) f(z_I e^{i\beta_{I11}}), \quad (115)$$

where $f(z_I e^{i\beta_{I11}})$ is a certain function of $r$ variables $z_I e^{i\beta_{I11}}$ ($I = 1, \ldots, r$). This function also depends on $x$, but we do not take care about it here.

We now have $r$ angular variables $\alpha_{a1}$ to be integrated. Instead of these, let us use $\alpha_\bullet$ and $\beta_{I11}$ as $r$ independent variables, where $a = \bullet$ is the reference vertex we used to define the fivebrane linking numbers in (42). By definition $\beta_{I11}$ satisfy

$$\sum_{I=1}^{r} \beta_{I11} = 0. \quad (116)$$

To treat all $\beta_{I11}$ as independent variables, we insert the $\delta$ function

$$\delta(\sum_{I=1}^{r} \beta_{I11}) = \sum_{d=-\infty}^{\infty} \exp(-id \sum_{I=1}^{r} \beta_{I11}) \quad (117)$$

into (115). ($\delta(\theta)$ in this equation is the $\delta$-function for an angular variable. Namely it has the periodic support $\theta = 2\pi n$.) We rewrite the exponential factor in (115) by using

$$\sum_{a=1}^{r} k_a m_a \alpha_{a1} = k \sum_{I=1}^{r} \beta_{I11} - k \alpha_\bullet \sum_{I=1}^{r} s_I \mu_I, \quad (118)$$

where we defined

$$c_I = \sum_{\bullet < J < I} (s_I - s_J) \mu_J - m_\bullet s_I, \quad (119)$$
and the relative magnetic charges

$$\mu_I = m_{L(I)} - m_{R(I)}.$$  \hfill (120)

By definition $\mu_I$ satisfy

$$\sum_{I=1}^{r} \mu_I = 0.$$  \hfill (121)

We obtain

$$I_{(m_{\alpha})}(x, z, z') = \sum_{d=-\infty}^{\infty} \int d\alpha_{\bullet 1} \left( \prod_{I=1}^{r} \int d\beta_{I11} \right) F(z_1 e^{i\beta_{I11}}) \times \exp \left( i \sum_{I=1}^{r} (kc_I - d)\beta_{I11} - k\alpha_{\bullet 1} \sum_{I=1}^{r} s_I \mu_I \right).$$  \hfill (122)

The integration of $\alpha_{\bullet 1}$ gives the constraint

$$\sum_{I=1}^{r} s_I \mu_I = 0,$$  \hfill (123)

imposed on $\mu_I$. This is equivalent to (10). It is convenient to divide the relative magnetic charges $\mu_I$ into two sets $\mu_i$ and $\mu_{i'}$ corresponding to two kinds of hypermultiplets. (121) and (123) are equivalent to the two constraints

$$\sum_{i=1}^{p} \mu_i = 0, \quad \sum_{i'=1}^{q} \mu_{i'} = 0,$$  \hfill (124)

and they form $SU(p)$ and $SU(q)$ root lattices. It is natural to relate these lattices to the gauge groups realized on the fixed loci in $X_7$, and identify $\mu_i$ and $\mu_{i'}$ with the vectors $\rho_i$ and $\rho_{i'}$ introduced in \S 3.2:

$$\mu_i = \rho_i, \quad \mu_{i'} = \rho_{i'}.$$  \hfill (125)

For this identification to be justified, $(\vec{\mu}, \vec{\mu'})$, $P$, and $P'$ should satisfy the same relation as (72). We can easily confirm this as follows. For every $I$, the $\beta_{I11}$ integration picks up terms proportional to

$$z_i^{d-kc_i}.$$  \hfill (126)

Therefore, $P$ and $P'$, the total numbers of $z$ and $z'$, are given by

$$P = \sum_{i=1}^{p} (d - kc_i) = pd + \sum_{i'=1}^{q} l_{i'} \mu_{i'},$$  \hfill (127)

$$P' = \sum_{i'=1}^{q} (d - kc_{i'}) = qd + kqm_{\bullet} - \sum_{i=1}^{p} l_i \mu_i,$$  \hfill (128)

where $l_I$ are the linking numbers defined by (53). These equations say that the charge $P$ or $P'$ of a gauge-invariant monopole operator shifts from a multiple of $p$ or $q$, respectively, corresponding to its magnetic charges, which was pointed out in [23]. On the other hand, the selection rules (127), (128) are nothing but the relations (72) on the gravity side.

The charge $P_M$ corresponding to the M-momentum is

$$P_M = m_{\bullet} - \frac{1}{kq} \sum_{i=1}^{p} l_i \mu_i - \frac{1}{kp} \sum_{i'=1}^{q} l_{i'} \mu_{i'}.$$  \hfill (129)
The right hand side of this equation is independent of $d$, and a function of the magnetic charges $m_a$. Although each of three terms in (129) separately depends on the choice of the reference point, the sum of them is independent of the choice.

We can obtain the relation (129) in a more direct way from (91). The reason why $P_M$ is related to $m_a$ is that the flavor rotation generated by $P_M$ is gauge equivalent to the shift of the dual photon field $a$ defined by (31). The gauge invariance of an operator requires its charges associated with these two shifts to be the same.

When the gauge group is $U(1)^r$, the gauge symmetry connecting these two shifts are parameterized by $\varphi$ defined in (30). For $U(N)^r$ gauge group we can define such a parameter $\varphi$ by

$$\partial_\varphi \beta_{rst} = \frac{1}{p}, \quad \partial_\varphi \beta_{i'st} = -\frac{1}{q}, \quad \partial_\varphi \beta_{ast} = 0. \quad (130)$$

The action of $\partial_\varphi$ on the parameters $\alpha_{as}$ is

$$\partial_\varphi \alpha_{as} = \gamma_a, \quad (131)$$

where $\gamma_a$ are constants satisfying

$$\gamma_L(i) - \gamma_R(i) = \frac{1}{p}, \quad \gamma_L(i') - \gamma_R(i') = -\frac{1}{q}. \quad (132)$$

These conditions determine $\gamma_a$ up to overall shift. Integration of $\varphi$ leaves only the contribution of operators which are invariant under the $\varphi$ gauge transformation, and reproduces the relation (129) as we see below. Let us perform the integration over $\varphi$ orbit in (91). A term proportional to $z^P z^{P'}$ is accompanied by the factor $e^{-ikP_M \varphi}$. The other factor including $\varphi$ is the exponential factor including the Chern-Simons levels. It includes

$$\exp \left( i \varphi \sum_{a=1}^{r} k_a m_a \gamma_a \right). \quad (133)$$

Therefore, for the term to survive after $\varphi$ integration, the following relation must hold.

$$P_M = \frac{1}{k} \sum_{a=1}^{r} k_a m_a \gamma_a. \quad (134)$$

This is equivalent to (129). This expression is manifestly independent of the reference point. Thanks to the constraint (10), (134) is not changed by the overall shift of $\gamma_a$, and is determined unambiguously.

We can easily show $(1/k) \sum a_k \gamma_a = 1$, and $P_M$ is a weighted average of the magnetic charges.

Now we have established the relation between quantities $(P_M, \rho_l, \rho_{l'})$ defined on the gravity side and magnetic charges $m_a$ defined on the gauge theory side.

$$P_M = \frac{1}{k} \sum_{a=1}^{r} k_a \gamma_a m_a, \quad \rho_l = m_{L(I)} - m_{R(I)}. \quad (135)$$

We use this relation when we compare the indices in the following sections.

5 Comparison between graviton index and gauge theory index

In this section we confirm the complete matching of the gauge theory index and the multi-particle index. On the previous section we show that the gauge theory index
is factorized into three parts: neutral, positive, and negative parts. In the following
we first show that the multi-particle index on the gravity side is also factorized in
the same way into three parts, and then, we confirm the agreement for each factor.

We show the agreement for the neutral part analytically. Concerning the charged
class, we use computers to compute gauge theory index for many sectors with differ-
cent charges, and we show that the gauge theory index $I^{(+)\{m_a\}}$ for monopole charges
$\{m_a\}$ agrees with the multi-particle index $I^{\text{mp}(P_M, \vec{\rho}, \vec{\rho}')}_{(\{m_a\})}$ for the charges $(P_M, \vec{\rho}, \vec{\rho}')$ cor-
responding to the magnetic charges $\{m_a\}$ through (135).

\subsection{Factorization of multi-particle index}

In the previous section, we show that the gauge theory index is factorized into three
parts. For the two indices to coincides, the multi-particle index should also have
this property. Namely, $I^{\text{mp}}$ should be factorized as

$$I^{\text{mp}} = I^{\text{mp}(0)} I^{\text{mp}(+)} I^{\text{mp}(-)}.$$  \hspace{1cm} (136)

Let us first confirm this factorization.

The factorization of the multi-particle index is equivalent to the following de-
composition of the single-particle index

$$I^{\text{sp}} = I^{\text{sp}(0)} + I^{\text{sp}(+)} + I^{\text{sp}(-)}.$$  \hspace{1cm} (137)

Let us consider a single-particle state with quantum numbers $(P_M, \vec{\rho}, \vec{\rho}')$. By
the relations in (135) we can determine the corresponding magnetic charges $m_a$. The decomposability (137) claims that the magnetic charges $m_a$ determined in this
way for every single-particle state do not include positive and negative components
at the same time. This is confirmed easily as follows.

For a bulk graviton state, which has vanishing vectors $\vec{\rho} = \vec{\rho}' = 0$, all the
components of the corresponding magnetic charge are the same and are given by

$$m_1 = \cdots = m_r = P_M,$$  \hspace{1cm} (138)

and thus they never include both positive and negative charges. This is also the
case for the Cartan part of the twisted sectors.

For an $H_S$-charged particle in a twisted sector, one of $\vec{\rho}$ and $\vec{\rho}'$ is non-vanishing.
If the particle corresponds to an $SU(p)$ root vector, $\rho_i$ has two non-vanishing components, and one of them is $+1$ and the other is $-1$. In this case the second relation
in (135) means that the minimum and the maximum components of the magnetic
charges $m_a$ differ by only one. Therefore, the $r$ magnetic charges cannot include
both positive and negative charges.

We can always classify single-particle states into neutral, positive, and negative
parts according to the magnetic charges, and correspondingly, we can decompose
the single-particle index into the three parts as (137).

\subsection{Neutral part}

For the neutral part, we can analytically prove the relation $I^{(0)} = I^{\text{mp}(0)}$ as we
demonstrate below.

On the gravity side, the neutral part of the multi-particle index $I^{\text{mp}(0)}$ is given by

$$I^{\text{mp}(0)}(x, z, z') = \exp \sum_{n=1}^{\infty} \frac{1}{n} I^{\text{sp}(0,0,0)}(x^n, z^n, z'^n).$$  \hspace{1cm} (139)
where the single-particle index for \((P_M, \vec{\rho}, \vec{\rho}') = (0, \vec{0}, \vec{0})\) is (See (83).)

\[
I_{sp}(0, \vec{0}, \vec{0})(x, z, z') = \sum_{a = -\infty}^{\infty} I_{pa, qa}^{grav}(x) z^p a z'^q a + (p - 1) I_0^{vec}(x) + (q - 1) I_0^{vec}(x)
\]

\[
= \frac{x^{p+q} z^p z'^q}{1 - x^{p+q} z^p z'^q} + \frac{x^{p+q} z^p z'^q}{1 - x^{p+q} z^p z'^q} + (p + q) \frac{x^2}{1 - x^4}. \quad (140)
\]

The corresponding multi-particle index defined by (139) is

\[
I_{mp}(0) = \prod_{i=1}^{\infty} \frac{(1 + x^2)^{i(p+q)}}{(1 - (x^{p+q} z^p z'^q)^i) (1 - (x^{p+q} z^p z'^q)^i)} \quad (141)
\]

where we used Euler’s partition identity to obtain this expression.

On the gauge theory side, the corresponding index (102) is

\[
I(0)(x, z, z') = \prod_{n=1}^{\infty} \frac{1}{\det M(x^n, z^n, z'^n)} \quad (142)
\]

where \(M\) is the matrix defined in (100). We can easily compute the determinant by rewriting the matrix \(M\) as

\[
M = \frac{1}{1 + x^2} (1 - xA)(1 - xA^{-1}) \quad (143)
\]

with the matrix

\[
A(z, z') = \begin{pmatrix}
\vdots & & \\
0 & z_{I-1} & \\
& 0 & z_I \\
& & 0 & z_{I+1} \\
& & & \ddots & \\
z_r & & & & 0 \\
\end{pmatrix}. \quad (144)
\]

The determinant

\[
\frac{1}{\det M} = \frac{(1 + x^2)^{p+q}}{(1 - x^{p+q} z^p z'^q)(1 - x^{p+q} z^p z'^q)} \quad (145)
\]

does not depend on the order of the untwisted and twisted hypermultiplets in the quiver diagram. On substituting this into (142), we see that the neutral part of the gauge theory index actually coincides with the corresponding part of the graviton index;

\[
I(0)(x, z, z') = I_{mp}(0)(x, z, z'). \quad (146)
\]

This result is consistent with the result in [30]. If we set \(z = z' = 1\) and \(p = q = M\), we reproduce the index (20) with \(y_1 = 1\) substituted.

5.3 Charged part

Next, let us confirm the agreement of the charged part:

\[
I(\pm)(x, z, z') = I_{mp}(\pm)(x, z, z'). \quad (147)
\]

We can easily show the following relations between positive and the negative parts:

\[
I^{(+)}(x, z, z') = I^{(-)}(x, z^{-1}, z'^{-1}), \quad I_{mp}^{(+))(x, z, z') = I_{mp}^{(-)}(x, z^{-1}, z'^{-1}) \quad (148)
\]
Therefore, it is enough to show the relation for the positive part of the indices:

\[ I^{(+)}_{\{m_a\}^+}(x, z, z') = I^{np(+)}_{(P_M, \bar{\rho}, \bar{\rho}')} (x, z, z'), \tag{149} \]

for \( \{m_a\}^+ \) and \( (P_M, \bar{\rho}, \bar{\rho}') \) related by \( \text{(135)} \).

Unfortunately, we have not succeeded in proving \( \text{(149)} \) analytically. In the following, we consider three examples of \( \mathcal{N} = 4 \) Chern-Simons theories specified by \( \{s_I\} = \{0, 0, 1\} \) and \( \{0, 0, 1, 1\} \). (The simplest case with \( \{s_I\} = \{0, 1\} \) (ABJM model) has already been investigated in \[5\].) For each theory we compute \( I^{(+)}_{\{m_a\}} \) numerically for many sectors specified by the charges, and confirm the agreement with \( I^{np(+)}_{(P_M, \bar{\rho}, \bar{\rho}')}. \)

5.3.1 UUT theory

In this section, we consider the theory defined by

\[ \{s_I\} = \{0, 0, 1\}. \tag{150} \]

The background geometry of this theory is \((\mathbb{C}^2 / \mathbb{Z}_2 \times \mathbb{C}^2) / \mathbb{Z}_4\). The internal space \( X_7 \) includes a \( \mathbb{Z}_2 \)-fixed singular locus, and there exists one two-cycle at the locus \( S_I \). The vectors \( \bar{\rho} = \{\rho_1\} \) and \( \bar{\rho}' = \{\rho_\nu\} \) are parameterized by a single winding number \( \rho \in \mathbb{Z} \) as

\[ \bar{\rho} = \{\rho_1, \rho_2\} = \{-\rho, \rho\}, \quad \bar{\rho}' = \{\rho_3\} = \{0\}. \tag{151} \]

We introduce chemical potential \( t \) for the charge \( \rho \). This is related to the potentials \( t_I \) introduced in \[3\] by \( t = t_2 / t_1 \). By the relations in \( \text{(135)} \), the magnetic charges are determined as

\[ \{m_1, m_2, m_3\} = \{P_M, P_M + \rho, P_M\}. \tag{152} \]

The Wilson lines \( \eta_I \) vanish up to integers, and this is consistent with the fact that there is no three-cycles in the dual geometry. The quantization rules \( \text{(127)} \) and \( \text{(128)} \) for the charges \( P \) and \( P' \) are

\[ P = 2a, \quad P' = a + kP_M, \quad a, P_M \in \mathbb{Z}. \tag{153} \]

The positive part of the single-particle index is defined by \( m_a \geq 0 \) and \( \{m_1, m_2, m_3\} \neq \{0, 0, 0\} \). These conditions mean

\[ P_M \geq 0, \quad P_M + \rho \geq 0, \quad (P_M, \rho) \neq (0, 0). \tag{154} \]

For every pair of charges \( (P_M, \rho) \) satisfying \( \text{(154)} \), we would like to confirm

\[ I^{(+)}_{\{P_M, P_M + \rho, P_M\}}(x, z, z') = I^{mp(+)}_{(P_M, \rho)} (x, z, z'). \tag{155} \]

Single-particle states exist only for \( |\rho| \leq 1 \). Eq. \( \text{(83)} \) gives

\[ I^{np}_{(P_M, 0)} = \sum_{a = -\infty}^{\infty} I^{grav}_{2a + kP_M}(x) z^{2a} z^{kP_M + a} + I^{vec}_{kP_M}(x) z^{kP_M}, \tag{156} \]

\[ I^{np}_{(P_M, \pm 1)} = I^{vec}_{kP_M}(x) z^{kP_M}. \tag{157} \]

\[ ^2\text{When we describe a set of numbers } x_a \text{ assigned to vertices in the quiver diagram, we choose a reference vertex } a = \bullet, \text{ which is also used for the definition of the linking numbers, and represent } \{x_a\} \text{ as the vector } \{x_\bullet, x_{R^2(\bullet)}, \ldots, x_{L^3(\bullet)}\}, \text{ where } R^2(\bullet) \equiv R(R(\bullet)), \text{ and } L^3(\bullet) \text{ and } R^3(\bullet) \text{ are similarly defined. For a set of numbers } y_I \text{ assigned to edges, we represent them as } \{y_{R(\bullet)}, y_{R^3(\bullet)}, \ldots, y_{L(\bullet)}\}. \]
It is relatively easy to compute indices when one of two bounds in (154) is saturated. Let us first consider $P_M = 0$ case. In this case, we should confirm
\[ I_{\{0, \rho, 0\}}(x, z, z') = I^{\text{mp}(+)}_{\{0, \rho\}}(x, z, z'). \] (158)

Because the single-particle index depends on the level $k$ only through the combination $P_M k$, the multi-particle index on the right hand side in (158) is independent of $k$. We can easily see that this is also the case for the gauge theory index on the left hand side in (158) from the expression (103).

The only non-vanishing single particle index for $P_M = 0$ contributing to $I^{\text{mp}(+)}$ is
\[ I^{\text{sp}(+)}_{\{0, 1\}} = x^2 - x^4, \] (159)
and the multi-particle index with $P_M = 0$ is defined by
\[ \sum_{\rho=0}^{\infty} I^{\text{mp}(+)}_{\{0, \rho\}}(x, z, z') t^\rho = \exp \sum_{n=1}^{\infty} \frac{1}{n} I^{\text{sp}(+)}_{\{0, 1\}}(x^n, z^n, z'^n) t^n. \] (160)

By using the identity
\[ \prod_{i=0}^{\infty} \frac{1}{1 - tx^i} = \sum_{i=0}^{\infty} t^i \prod_{j=1}^{i} \frac{1}{1 - x^j}, \] (161)
we obtain
\[ I^{\text{mp}(+)}_{\{0, \rho\}} = \prod_{i=1}^{\rho} \frac{x^2}{1 - x^i}. \] (162)

Let us confirm that the gauge theory index agrees with this for small $\rho$. For $\rho = 1$, we can easily compute the corresponding gauge theory index by hand, and confirm the agreement.
\[ I_{\{0, 1, 0\}}^{(+)} = I_{\{0, 1\}}^{(+)} = \frac{x^2}{1 - x^4}. \] (163)

For $\rho = 2$, there are two contribution with different monopole backgrounds.
\[ I_{\{0, 2, 0\}}^{(+)} = I_{\{0, 2\}}^{(+)} + I_{\{0, 0\}}^{(+)}; \] (164)

It is again easy to compute these two contributions by hand. They are
\[ I_{\{0, 2\}}^{(+)} = \frac{x^4}{1 - x^8}, \quad I_{\{0, 0\}}^{(+)} = \frac{x^8}{(1 - x^4)(1 - x^8)}, \] (165)
and the summation agrees with the multi-particle index
\[ I_{\{0, 2, 0\}}^{(+)} = \frac{x^2}{1 - x^4} \frac{x^2}{1 - x^8} = I^{\text{mp}(+)}_{\{0, 2\}}. \] (166)

As the charge becomes large, the computation of the gauge theory index becomes complicated rapidly. For $\rho \geq 3$, we use computers to generate gauge theory index as series expansion with respect to the variable $x$, and check the agreement for small
\( \rho \) up to certain order of \( x \). The result is as follows.

\[
I^{(+)}_{\{0,3,0\}} = I^{(+)}_{\{\text{even}\}} + I^{(+)}_{\{\text{odd}\}} + I^{(+)}_{\{\text{other}\}}
\]
\[
= I^{\text{mp}(+)}_{(0,3)} + O(x^{101}), \quad (167)
\]
\[
I^{(+)}_{\{0,4,0\}} = I^{(+)}_{\{\text{even}\}} + I^{(+)}_{\{\text{odd}\}} + I^{(+)}_{\{\text{other}\}} + I^{(+)}_{\{\text{other}\}}
\]
\[
= I^{\text{mp}(+)}_{(0,4)} + O(x^{101}), \quad (168)
\]
\[
I^{(+)}_{\{0,5,0\}} = I^{(+)}_{\{\text{even}\}} + I^{(+)}_{\{\text{odd}\}} + I^{(+)}_{\{\text{other}\}} + I^{(+)}_{\{\text{other}\}} + I^{(+)}_{\{\text{other}\}}
\]
\[
= I^{\text{mp}(+)}_{(0,5)} + O(x^{31}). \quad (169)
\]

All these results are consistent with (162) up to the order we have computed.

Next let us consider the case with \( P_M \geq 1 \) and \( P_M + \rho = 0 \). The relation we would like to confirm is

\[
I^{(+)}_{\{P_M, 0, P_M\}} (x, z, z') = I^{\text{mp}(+)}_{\{P_M, -P_M\}} (x, z, z'). \quad (170)
\]

The single-particle index contributing to this part is

\[
I^{\text{sp}}_{(1, -1)} = \frac{x^2 (xz')^k}{1 - x^4}. \quad (171)
\]

With the help of the identity (163) we obtain

\[
I^{\text{mp}(+)}_{\{P_M, -P_M\}} = \prod_{i=1}^{P_M} x^{k + 2 z_i t_i} \frac{1}{1 - x^{4 t_i}}. \quad (172)
\]

We have confirmed the following relations up to the indicated order of \( x \) for \( k = 1, 2, 3, 4, 5 \).

\[
I^{(+)}_{\{1,0,1\}} = I^{(+)}_{\{\text{even}\}} + I^{(+)}_{\{\text{other}\}} + I^{(+)}_{\{\text{other}\}} + I^{(+)}_{\{\text{other}\}}
\]
\[
= I^{\text{mp}(+)}_{(1,-1)} + O(x^{101}), \quad (173)
\]
\[
I^{(+)}_{\{2,0,2\}} = I^{(+)}_{\{\text{even}\}} + I^{(+)}_{\{\text{other}\}} + I^{(+)}_{\{\text{other}\}} + I^{(+)}_{\{\text{other}\}}
\]
\[
= I^{\text{mp}(+)}_{(2,-2)} + O(x^{31}), \quad (174)
\]
\[
I^{(+)}_{\{3,0,3\}} = I^{(+)}_{\{\text{even}\}} + I^{(+)}_{\{\text{other}\}} + I^{(+)}_{\{\text{other}\}} + I^{(+)}_{\{\text{other}\}} + I^{(+)}_{\{\text{other}\}}
\]
\[
= I^{\text{mp}(+)}_{(3,-3)} + O(x^{31}). \quad (175)
\]

All these results are consistent with (172).

Finally, let us consider a few examples in which all magnetic charges are positive.
For $k = 1, 2, 3, 4, 5$ we have checked

\[
I^{(+)}_{\{1,1,1\}} = I_{\{\text{a.a.a}\}^{\text{mp}(+)}}^{(+)} + \mathcal{O}(x^{31}), \quad (176)
\]

\[
I^{(+)}_{\{2,1,2\}} = I_{\{\text{a.a.a.a.a}\}^{\text{mp}(+)}}^{(+)} + I^{(+)}_{\{\text{a.a.a.a.a}\}^{\text{mp}(+)}} + I^{(+)}_{\{\text{a.a.a.a.a}\}^{\text{mp}(+)}} + I^{(+)}_{\{\text{a.a.a.a.a}\}^{\text{mp}(+)}} + I^{(+)}_{\{\text{a.a.a.a.a}\}^{\text{mp}(+)}} + \mathcal{O}(x^{21}), \quad (177)
\]

\[
I^{(+)}_{\{2,2,2\}} = I_{\{\text{a.a.a.a.a.a.a}\}^{\text{mp}(+)}}^{(+)} + I^{(+)}_{\{\text{a.a.a.a.a.a.a}\}^{\text{mp}(+)}} + I^{(+)}_{\{\text{a.a.a.a.a.a.a}\}^{\text{mp}(+)}} + I^{(+)}_{\{\text{a.a.a.a.a.a.a}\}^{\text{mp}(+)}} + \mathcal{O}(x^{11}). \quad (178)
\]

where

\[
I^{\text{mp}(+)}_{\{1,0\}} = I^{\text{sp}(+)}_{\{1,0\}} + I^{\text{sp}(+)}_{\{0,1\}} I^{\text{sp}(+)}_{\{1,-1\}}, \quad (179)
\]

\[
I^{\text{mp}(+)}_{\{2,-1\}} = I^{\text{sp}(+)}_{\{2,-1\}} + I^{\text{mp}(+)}_{\{0,1\}} I^{\text{sp}(+)}_{\{1,-1\}} + I^{\text{sp}(+)}_{\{0,1\}} I^{\text{sp}(+)}_{\{1,-1\}} + I^{\text{sp}(+)}_{\{0,1\}} I^{\text{sp}(+)}_{\{1,-1\}} \left( \frac{1}{2} I^{\text{sp}(+)}_{\{1,-1\}}^2 + \frac{1}{2} I^{\text{sp}(+)}_{\{1,-1\}}^2 \right), \quad (180)
\]

\[
I^{\text{mp}(+)}_{\{2,0\}} = I^{\text{sp}(+)}_{\{2,0\}} + I^{\text{mp}(+)}_{\{2,-1\}} I^{\text{sp}(+)}_{\{0,1\}} + I^{\text{sp}(+)}_{\{0,1\}} I^{\text{sp}(+)}_{\{1,-1\}} + I^{\text{sp}(+)}_{\{0,1\}} I^{\text{sp}(+)}_{\{1,-1\}} \left( \frac{1}{2} I^{\text{sp}(+)}_{\{0,1\}}^2 + \frac{1}{2} I^{\text{sp}(+)}_{\{0,1\}}^2 \right) + I^{\text{sp}(+)}_{\{0,1\}} I^{\text{sp}(+)}_{\{1,-1\}} I^{\text{sp}(+)}_{\{0,1\}}. \quad (181)
\]

### 5.3.2 UUTT theory

Next, let us consider the cases with $p = 2$ and $q = 2$. There are two cases with $\{s_I\} = \{0, 0, 1, 1\}$ and $\{s_I\} = \{0, 1, 0, 1\}$, which we call UUTT and UTUT theories, respectively. These are simplest examples that are distinguished by the order of two kinds of hypermultiplets in the quiver diagrams.

We first consider UUTT theory with $\{s_I\} = \{0, 0, 1, 1\}$. The inking numbers are

\[
\bar{I} = \{l_1, l_2\} = \{2k, 2k\}, \quad \bar{l} = \{l_3, l_4\} = \{-2k, -2k\}, \quad (182)
\]

and the Wilson line parameters $\eta_I$ vanishes up to integers. On the gravity side, we have two $A_1$ type singular loci. We parameterize the vectors $\vec{\rho}$ and $\vec{\rho}'$ by two integers $\rho$ and $\rho'$ as

\[
\vec{\rho} = \{\rho_1, \rho_2\} = \{-\rho, \rho\}, \quad \vec{\rho}' = \{\rho_3, \rho_4\} = \{-\rho', \rho'\}, \quad (183)
\]

We introduce chemical potentials $t$ and $t'$ for the charges $\rho$ and $\rho'$, respectively. These are related to the potentials $t_I$ introduced in §3 by $t = t_2/t_1$ and $t' = t_4/t_3$. Eq. (135) gives

\[
\{m_1, m_2, m_3, m_4\} = \{P_M, P_M + \rho, P_M, P_M + \rho'\}. \quad (184)
\]

The positive part is defined by

\[
m_a \geq 0, \quad \{m_1, m_2, m_3, m_4\} \neq \{0, 0, 0, 0\}, \quad (185)
\]

and these are equivalent to

\[
P_M \geq 0, \quad P_M + \rho \geq 0, \quad P_M + \rho' \geq 0, \quad (P_M, \rho, \rho') \neq (0, 0, 0). \quad (186)
\]
We would like to show
\[
I_{(P_M,0,0,0)}^{(+)}(x, z, z') = 0,
\]
for every set of charges \((P_M, \rho, \rho')\) satisfying (186). Eq. (89) gives the single-particle index
\[
I_{(P_M,0,0)}^{sp}(x, z, z') = \sum_{a=0}^{\infty} I_{2a}(x) z^{2a} - 2z^{2a}(P_M + a),
\]
\[
I_{(P_M,-1,0)}^{sp}(x, z, z') = I_{2kP_M}(x) z^{2kP_M},
\]
\[
I_{(P_M,0,-1)}^{sp}(x, z, z') = I_{2kP_M}(x) z^{2kP_M},
\]
\[
I_{(P_M,1,0)}^{sp}(x, z, z') = I_{2kP_M}(x) z^{2kP_M},
\]
\[
I_{(P_M,0,1)}^{sp}(x, z, z') = I_{2kP_M}(x) z^{2kP_M}.
\]

When some of the inequalities in (186) are saturated, the computation of \(I_{(P_M,0,0,0)}^{sp}(x, z, z')\) is relatively easy, and we first consider such cases. The last condition in (186) means that the first three inequalities are not saturated at the same time. If \(P_M = 0\), only single-particle states saturating the same inequality can contribute to the multi-particle index. There are only two such single-particle charges, \((P_M, \rho, \rho') = (0, 1, 0)\) and \((0, 0, 1)\), and thus the multi-particle index is given by
\[
\sum_{n=1}^{\infty} I_{(P_M,0,0,0)}^{mp(+)}(x, z, z') = \prod_{i=1}^{2} \left( \frac{x^2 - x^{2i}}{1 - x^{2i}} \right).
\]

By using the identity (161), we obtain
\[
I_{(0,0,0,0)}^{mp(+)} = I_{(0,0,0,0)}^{(+)} I_{(0,0,0,0)}^{(+)}.
\]

We can easily show that for \(I_{(0,0,0,0)}^{(+)}\) the integrals in (103) are factorized into two parts, and the relation
\[
I_{(0,0,0,0)}^{(+)} = I_{(0,0,0,0)}^{(+)} I_{(0,0,0,0)}^{(+)}
\]
holds. In general, if the cyclic sequence of the magnetic charges splits into several parts by vanishing components, the integrals in (103) are factorized, and we obtain a relation like (195). Furthermore, each of two factors in (195) is the same as the index \(I_{(0,0,0,0)}^{(+)}\) for the UUT theory. By using the results in the last subsection, we can confirm \(I_{(0,0,0,0)}^{mp(+)} = I_{(0,0,0,0)}^{(+)}\).

Next, let us consider the case in which \(P_M \geq 1\) and the second or the third bounds in (186) are saturated. Namely, \(P_M + \rho = 0\) or \(P_M + \rho' = 0\). Because there is no one particle state saturating both the bounds, the multi-particle index for such charges vanishes; \(I_{(P_M,-1,0,0)}^{mp(+)} = 0\). On the gauge theory side, we can show \(I_{(P_M,0,0,0)}^{(+)} = 0\) by using the factorization \(I_{(P_M,0,0,0)}^{(+)} = I_{(P_M,0,0,0)}^{(+)} I_{(P_M,0,0,0)}^{(+)}\) and applying the selection rules to the two factors.

When only one of \(P_M + \rho = 0\) or \(P_M + \rho' = 0\) in (186) is saturated, only single-particle states with charges \((1,0,-1)\) or \((-1,0,1)\) contribute to the multi-particle
We confirm for $\rho$ theory index for small $\rho$ and $\rho'$ as follows.

\[ I_{(\rho',0,0)}^{\text{mp}(+)} = \left( \prod_{i=1}^{\rho'} \frac{x^{2(xz-1)p_{k,i}}}{1-x^{4i}} \right), \quad I_{(\rho,-\rho,0)}^{\text{mp}(+)} = \left( \prod_{i=1}^{\rho} \frac{x^{2(xz')q_{k,i}}}{1-x^{4i}} \right). \quad (196) \]

These are easily generalized to

\[ I_{(\rho+\rho',-\rho,-\rho')}^{\text{mp}(+)} = \left( \prod_{i=1}^{\rho'} \frac{x^{2(xz-1)p_{k,i}}}{1-x^{4i}} \right) \left( \prod_{i=1}^{\rho} \frac{x^{2(xz')q_{k,i}}}{1-x^{4i}} \right). \quad (197) \]

We confirm for $k = 1, \ldots, 5$ that this index is correctly reproduced as the gauge theory index for small $\rho$ and $\rho'$ as follows.

\[ I_{\{1,1,1,0\}}^{(+)} = I_{\{1,0,-1\}}^{(+)} + O(x^{101}), \quad (198) \]

\[ I_{\{2,2,0,0\}}^{(+)} = I_{\{1,0,1\}}^{(+)} + I_{\{1,2,0,0\}}^{(+)} + I_{\{1,1,1\}}^{(+)} + I_{\{1,1,2\}}^{(+)} + I_{\{1,2,1\}}^{(+)} + I_{\{2,0,2\}}^{(+)} + O(x^{21}), \quad (199) \]

\[ I_{\{2,0,2,2\}}^{(+)} = I_{\{2,0,2,0\}}^{(+)} + O(x^{21}), \quad (200) \]

\[ I_{\{2,1,2,1\}}^{(+)} = I_{\{2,1,2,0\}}^{(+)} + O(x^{21}). \quad (201) \]

Finally, we give more examples without vanishing magnetic charges.

\[ I_{\{1,1,1,1\}}^{(+)} = I_{\{1,0,0,0\}}^{(+)} + O(x^{101}), \quad (202) \]

\[ I_{\{1,2,1,1\}}^{(+)} = I_{\{1,0,1,1\}}^{(+)} + O(x^{31}), \quad (203) \]

\[ I_{\{1,1,1,2\}}^{(+)} = I_{\{1,0,0,1\}}^{(+)} + O(x^{31}), \quad (204) \]

\[ I_{\{1,2,1,2\}}^{(+)} = I_{\{1,0,1,1\}}^{(+)} + O(x^{31}), \quad (205) \]
Then the magnetic charges are given by

\[ t = t_{(1,0,0)} + t_{(1,1,0)} + t_{(1,0,1)} + t_{(1,1,1)}, \]

These are related to the potentials \( \vec{\omega} \) and \( \vec{l} \). We parameterize \( \vec{\rho} \).

This theory is the simplest example with the non-trivial Wilson lines on the singular loci. We introduce chemical potentials \( t \) and \( t' \) for the charges \( \rho \) and \( \rho' \), respectively. These are related to the potentials \( t_i \) introduced in \( 33 \) by \( t = t_3/t_1 \) and \( t' = t_4/t_2 \).

Then the magnetic charges are given by

\[ m = \{ P_M - \rho + \rho'/2, P_M + \rho - \rho'/2, P_M + \rho + \rho'/2, P_M - \rho - \rho'/2 \}, \]

where \( m_\bullet \) is the magnetic charge for the reference vertex, and is related to \( P_M \) by

\[ P_M = m_\bullet + \frac{1}{2}(\rho + \rho'). \]

The relation we would like to confirm is

\[ \tilde{I} = \{ P_M - \rho + \rho'/2, P_M + \rho - \rho'/2, P_M + \rho + \rho'/2, P_M - \rho - \rho'/2 \}(x, z, z') = I_{(P_M, \rho, \rho')}^{mp}(x, z, z'). \]

The positive part of the single particle index is defined by

\[ m_a \geq 0, \quad \{ m_1, m_2, m_3, m_4 \} \neq \{0, 0, 0, 0\}, \]

and these are equivalent to

\[ m_\bullet \geq 0, \quad m_\bullet + \rho \geq 0, \quad m_\bullet + \rho' \geq 0, \quad (m_\bullet, \rho, \rho') \neq (0, 0, 0). \]
The single particle index is given by
\[
I_{\text{sp}}(m, 0, 0)(x, z, z') = \sum_{a=-\infty}^{\infty} I_{2a, 2(km + a)}^{\text{grav}}(x)z^{2a}z'^{2(km + a)} + I_{2km}^{\text{vec}}(x)(z^{2km} + z^{-2km}), \tag{219}
\]
\[
I_{\text{sp}}(m, 1, 0)(x, z, z') = I_{k(2m - 1)}^{\text{vec}}(x)z^{k(2m - 1)}, \tag{220}
\]
\[
I_{\text{sp}}(m, 0, 1)(x, z, z') = I_{-k(2m + 1)}^{\text{vec}}(x)z^{-k(2m + 1)}, \tag{221}
\]
\[
I_{\text{sp}}(m, 0, 0)(x, z, z') = \sum_{\rho, \rho'} I_{\text{mp}}^{(+)}(\rho + \rho' 2, \rho, \rho')(x, z, z')t^\rho t'^{\rho'} = \exp \left( \sum_{n \geq 1} \frac{1}{n} \left[ I_{(1/2, 1, 0)}^{\text{sp}}(x^n, z^n, z'^n)t^n + I_{(1/2, 0, 1)}^{\text{sp}}(x^n, z^n, z'^n)t^n \right] \right). \tag{224}
\]

By using the identity (161), we obtain
\[
I_{\text{mp}}^{(+)}(z, z', \rho, \rho') = \prod_{i=1}^{\rho} x^i (x z'^i) \prod_{i'=1}^{\rho'} x^i (x z'^{-i}) \tag{225}
\]

For \( k = 1, \ldots, 5 \) and small \( \rho \) and \( \rho' \), we confirmed that this multi-particle index is...
reproduced as the gauge theory index

\[ I_{\{0,0,1\}}^{(+)} = I^{(+)}_{\{\,\alpha\}} \]
\[ = I^{\text{mp}(+)}_{(1/2,0,1)} + O(x^{101}), \]
\[ I_{\{0,0,2\}}^{(+)} = I^{(+)}_{\{\,\alpha\alpha\}} + I^{(+)}_{\{\,\alpha\bar{\alpha}\}} + I^{(+)}_{\{\,\bar{\alpha}\alpha\}} + I^{(+)}_{\{\,\bar{\alpha}\bar{\alpha}\}} \]
\[ = I^{\text{mp}(+)}_{(1,0,2)} + O(x^{31}), \quad (226) \]
\[ I_{\{0,0,3\}}^{(+)} = + I^{(+)}_{\{\,\alpha\alpha\bar{\alpha}\}} + I^{(+)}_{\{\,\alpha\bar{\alpha}\bar{\alpha}\}} + I^{(+)}_{\{\,\bar{\alpha}\alpha\bar{\alpha}\}} + I^{(+)}_{\{\,\bar{\alpha}\bar{\alpha}\bar{\alpha}\}} + I^{(+)}_{\{\,\alpha\alpha\bar{\alpha}\bar{\alpha}\}} + I^{(+)}_{\{\,\alpha\bar{\alpha}\alpha\bar{\alpha}\}} + I^{(+)}_{\{\,\bar{\alpha}\alpha\alpha\bar{\alpha}\}} + I^{(+)}_{\{\,\bar{\alpha}\bar{\alpha}\alpha\bar{\alpha}\}} \]
\[ = I^{\text{mp}(+)}_{(3/2,0,3)} + O(x^{11}), \quad (227) \]
\[ I_{\{0,1,0\}}^{(+)} = I^{(+)}_{\{\,\alpha\}} \]
\[ = I^{\text{mp}(+)}_{(1/2,1,0)} + O(x^{101}), \quad (228) \]
\[ I_{\{0,2,0\}}^{(+)} = I^{(+)}_{\{\,\alpha\alpha\}} + I^{(+)}_{\{\,\alpha\bar{\alpha}\}} + I^{(+)}_{\{\,\bar{\alpha}\alpha\}} + I^{(+)}_{\{\,\bar{\alpha}\bar{\alpha}\}} \]
\[ = I^{\text{mp}(+)}_{(1,2,0)} + O(x^{31}), \quad (229) \]
\[ I_{\{0,3,0\}}^{(+)} = I^{(+)}_{\{\,\alpha\alpha\bar{\alpha}\}} + I^{(+)}_{\{\,\alpha\bar{\alpha}\bar{\alpha}\}} + I^{(+)}_{\{\,\bar{\alpha}\alpha\bar{\alpha}\}} + I^{(+)}_{\{\,\bar{\alpha}\bar{\alpha}\bar{\alpha}\}} + I^{(+)}_{\{\,\alpha\alpha\bar{\alpha}\bar{\alpha}\}} + I^{(+)}_{\{\,\alpha\bar{\alpha}\alpha\bar{\alpha}\}} + I^{(+)}_{\{\,\bar{\alpha}\alpha\alpha\bar{\alpha}\}} + I^{(+)}_{\{\,\bar{\alpha}\bar{\alpha}\alpha\bar{\alpha}\}} \]
\[ = I^{\text{mp}(+)}_{(3/2,3,0)} + O(x^{11}), \quad (230) \]
\[ I_{\{1,0,1\}}^{(+)} = I^{(+)}_{\{\,\alpha\alpha\}} + I^{(+)}_{\{\,\alpha\bar{\alpha}\}} \]
\[ = I^{\text{mp}(+)}_{(1,1,1)} + O(x^{41}), \quad (231) \]
\[ I_{\{1,0,2\}}^{(+)} = I^{(+)}_{\{\,\alpha\alpha\alpha\alpha\}} + I^{(+)}_{\{\,\alpha\bar{\alpha}\alpha\bar{\alpha}\}} + I^{(+)}_{\{\,\bar{\alpha}\alpha\alpha\bar{\alpha}\}} + I^{(+)}_{\{\,\bar{\alpha}\bar{\alpha}\alpha\bar{\alpha}\}} \]
\[ = I^{\text{mp}(+)}_{(3/2,2,1)} + O(x^{21}), \quad (232) \]
\[ I_{\{1,0,3\}}^{(+)} = I^{(+)}_{\{\,\alpha\alpha\bar{\alpha}\bar{\alpha}\}} + I^{(+)}_{\{\,\alpha\bar{\alpha}\alpha\bar{\alpha}\}} + I^{(+)}_{\{\,\bar{\alpha}\alpha\alpha\bar{\alpha}\}} + I^{(+)}_{\{\,\bar{\alpha}\bar{\alpha}\alpha\bar{\alpha}\}} \]
\[ = I^{\text{mp}(+)}_{(3/2,1,2)} + O(x^{21}). \quad (233) \]

We also check in some sectors with magnetic charges without vanishing components the gauge theory index correctly reproduces the corresponding multi-particle index for \( k = 1, \ldots, 5 \).

\[ I_{\{1,1,1\}}^{(+)} = I^{(+)}_{\{\,\alpha\alpha\alpha\}} \]
\[ = I^{\text{mp}(+)}_{(1,1,0)} + O(x^{101}), \quad (235) \]
\[ I_{\{1,2,1\}}^{(+)} = I^{(+)}_{\{\,\alpha\alpha\alpha\alpha\}} + I^{(+)}_{\{\,\alpha\bar{\alpha}\alpha\bar{\alpha}\}} + I^{(+)}_{\{\,\bar{\alpha}\alpha\alpha\bar{\alpha}\}} + I^{(+)}_{\{\,\bar{\alpha}\bar{\alpha}\alpha\bar{\alpha}\}} \]
\[ = I^{\text{mp}(+)}_{(3/2,1,0)} + O(x^{31}), \quad (236) \]
\[ I_{\{1,1,2\}}^{(+)} = I^{(+)}_{\{\,\alpha\alpha\bar{\alpha}\bar{\alpha}\}} + I^{(+)}_{\{\,\alpha\bar{\alpha}\alpha\bar{\alpha}\}} + I^{(+)}_{\{\,\bar{\alpha}\alpha\alpha\bar{\alpha}\}} + I^{(+)}_{\{\,\bar{\alpha}\bar{\alpha}\alpha\bar{\alpha}\}} \]
\[ = I^{\text{mp}(+)}_{(3/2,0,1)} + O(x^{31}), \quad (237) \]
where

\[
I_{(1,0,0)}^{mp(+)} = I_{(1,0,0)}^{sp} + I_{(1/2,-1,0)}^{sp} I_{(1/2,1,0)}^{sp} + I_{(1/2,0,-1)}^{sp} I_{(1/2,0,1)}^{sp},
\]

\[
I_{(3/2,1,0)}^{mp(+)} = I_{(3/2,1,0)}^{sp} + I_{(1,0,0)}^{sp} I_{(1/2,1,0)}^{sp} + I_{(1/2,0,1)}^{sp} I_{(1/2,0,1)}^{sp}
+ \left( \frac{1}{2} I_{(1/2,1,0)}^{sp} \right)^2 + \frac{1}{2} I_{(1/2,0,1)}^{sp} (-2) \right) I_{(1/2,-1,0)}^{sp},
\]

\[
I_{(3/2,0,1)}^{mp(+)} = I_{(3/2,0,1)}^{sp} + I_{(1,0,0)}^{sp} I_{(1/2,0,1)}^{sp} + I_{(1/2,0,1)}^{sp} I_{(1/2,0,1)}^{sp}
+ \left( \frac{1}{2} I_{(1/2,0,1)}^{sp} \right)^2 + \frac{1}{2} I_{(1/2,0,1)}^{sp} (-2) \right) I_{(1/2,0,-1)}^{sp}.
\]

6 Conclusions

In this paper we have computed the index (87) for \( \mathcal{N} = 4 \) Chern-Simons theories with taking account of monopole contribution, and compared it with the multi-particle index for M-theory in the background \( \text{AdS}_4 \times X_7 \), where \( X_7 = (S^7/(\mathbb{Z}_p \times \mathbb{Z}_q))/\mathbb{Z}_k \).

When we calculated the gauge theory index \( I_{\text{gauge}} \), we took the large \( N \) limit with the Chern-Simons couplings \( k_a \) fixed.

On the gravity side, the internal space \( X_7 \) includes fixed loci on which \( G_S = SU(p) \times SU(q) \) vector multiplets live. We conjectured the single-particle index \( I^{\text{vec}} \) in (238) for a vector multiplet in \( \text{AdS}_4 \times S^7 \), and derived the contribution of the twisted sectors by the orbifold projection from \( I^{\text{vec}} \). We also derived the bulk sector index as the orbifold projection of the known graviton index for \( \text{AdS}_4 \times S^7 \).

We combined them to obtain the multi-particle index \( I^{mp} \).

Both the gauge theory index \( I_{\text{gauge}} \) and the graviton index \( I^{mp} \) are factorized into three parts: neutral, positive, and negative parts. We analytically proved the agreement of the neutral part of these indices. The agreement of the negative part follows from that of the positive part. To compute the positive part of the gauge theory index we used numerical methods. We considered three \( \mathcal{N} = 4 \) Chern-Simons theories with gauge group \( U(N)^3 \) or \( U(N)^4 \) as examples, and for each theory we numerically computed the gauge theory index for various sectors specified by the magnetic charges \( \{m_a\} \). The magnetic charges \( \{m_a\} \) are related to the M-momentum \( P_M \) and the \( H_S \)-charges \( \rho_I \), where \( H_S = U(1)^{1/2} \) is the Cartan subgroup of \( G_S \). We conjectured the one-to-one correspondence (135) between \( \{m_a\} \) and \( \{P_M, \rho_I\} \), and confirmed that the gauge theory index for a sector with magnetic charges \( \{m_a\} \) agrees with the multi-particle index for the sector with corresponding charges \( \{P_M, \rho_I\} \).

The \( H_S \) charges \( \rho_I \) can be regarded as the winding numbers of M2-branes on non-trivial two-cycles, and these results strongly suggest that (a part of) monopole operators correspond to wrapped M2-branes.

We also confirmed that the relation between the fivebrane linking numbers and the \( H_S \) Wilson lines on the singular loci is reproduced on the gauge theory side by analyzing the selection rules for the charges of global symmetry \( U(1) \times U(1)' \).

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