Qualitative reasoning in a two-layered framework

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Abstract

The reasoning with qualitative uncertainty measures involves comparative statements about events in terms of their likeliness without necessarily assigning an exact numerical value to these events.

The paper is divided into two parts. In the first part, we formalise reasoning with the qualitative counterparts of capacities, belief functions, and probabilities, within the framework of two-layered logics. Namely, we provide two-layered logics built over the classical propositional logic using a unary belief modality \( \mathcal{B} \) that connects the inner layer to the outer one where the reasoning is formalised by means of Gödel logic. We design their Hilbert-style axiomatisations and prove their completeness. In the second part, we discuss the paraconsistent generalisations of the logics for qualitative uncertainty that take into account the case of the available information being contradictory or inconclusive.

Keywords: qualitative probabilities; comparative belief; two-layered modal logics; Gödel logic; Belnap–Dunn logic; paraconsistent logics

1 Introduction

The main objective of this paper is to provide a logical framework for reasoning with uncertain information, where uncertainty in events is expressed qualitatively rather than quantitatively. This approach has been explored in the case of (qualitative counterparts of) probability, but much less so in the case of weaker notions of uncertainty, creating a gap we aim to fill with this paper. The most general kind of uncertainty measure usually considered in the literature is capacity — a function \( \mu : 2^W \to [0, 1] \) on the powerset of a set of events \( W \neq \emptyset \) which is monotone w.r.t. \( \subseteq \), non-trivial \( (\mu(W) > \mu(\emptyset)) \), and normalized: \( \mu(\emptyset) = 0, \mu(W) = 1 \) (cf. [21, 44]). We slightly generalize this notion by relaxing the last condition and will call the resulting function an uncertainty measure.

We will employ two-layered logical formalism consisting of two separate layers: the inner layer formalizes reasoning about events, while the outer layer formalizes reasoning about beliefs in the events, the uncertainty measure of choice acting as the interpretation of the belief modality

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acting on formulas describing events. We will use Gödel many-valued (fuzzy) logic to formalize the qualitative reasoning about the resulting beliefs on the outer layer. Apart from reasoning about events classically, we also wish to consider cases where the actual information about events available to an agent is faulty in the sense that it can be incomplete or even contradictory. Thus, in the apparatus of two-layered logics, we will employ not only classical logic but also fuzzy and paraconsistent logics to account for this possibility and reason with non-classical information in a non-trivial way.

**Weaker uncertainty measures** The most prominent examples of uncertainty measures are probability functions, however, they might be considered too strong to formalise people’s beliefs. Indeed, probabilities assume that the agents are able to express their certainty in a statement with some number from $[0, 1]$ and then conduct arithmetical operations with these numbers to obtain the probability assignments of complex events: add or subtract them, multiply them by some given constants, etc. This, however, is not always the case. Indeed, outside of scientific contexts (such as conducting medical tests, forecasting the weather, etc.), one can rarely give a number that corresponds to their certainty in a given statement. Furthermore, a layperson might not necessarily know how to obtain values of complex events, although they are usually able to compare their certainty in those.

In [7], this aspect was partially tackled with a logic of monotone coherent belief where not only the event description but also the reasoning was done in the language of the Belnap–Dunn logic (BD). BD, however, lacks the capacity to express statements such as ‘I think that the rain tomorrow is more likely than hailstorm’. Here the agent is a layperson and does not know the precise probabilities of any event. Neither are they likely to say something like ‘I am 73% sure that it is going to rain tomorrow’. On the other hand, one often can compare their degree of certainty in two given statements or at least be able to say something like ‘I don’t know whether rain or hailstorm is more likely tomorrow’.

For the classical case, one of such uncertainty measures is qualitative probability. One of the formal systems axiomatising qualitative probability is the modal logic proposed by Peter Gärdenfors in [19]. It uses a binary modality $\preccurlyeq$ such that $\phi \preccurlyeq \psi$ is interpreted as ‘$\phi$ is at most as likely as $\psi$’. Furthermore, Gärdenfors’ calculus allows for nesting of $\preccurlyeq$ which, in turn, leads to formulas of the form

$$(\tau \preccurlyeq \phi) \preccurlyeq (\chi \preccurlyeq \psi)$$

to be read as ‘it is at most as likely that $\tau$ is at most as likely as $\phi$ as that $\chi$ is at most as likely as $\psi$’. These statements, however, are quite cumbersome and rarely used in natural language reasoning when people are comparing their certainty in different statements.

This is why, in this paper, we will present a two-layered logic based on Gödel logic expanded with co-implication $b\Sigma$ that formalises qualitative reasoning with uncertainty measures. We will also show how to extend it with additional axioms corresponding to the qualitative counterparts of capacities, belief functions, and probabilities. For the last case, we will also show that our logic, in fact, interprets a certain fragment of the Gärdenfors’ calculus where the $\preccurlyeq$ does not nest.

**Two-layered logics for uncertainty** Two-layered logics consist of two languages: the one that describes the events such as ‘if it rains, then there should be puddles on the road’ (inner layer), and the one wherein the reasoning about beliefs in these events occurs. The uncertainty measure then acts as a non-nesting modality $M$. Thus, the formulas of the outer language are built from the atoms of the form $M\phi$ with $\phi$ being an inner-layer formula.

One of the first formalisations of a logic for reasoning about classical probability is provided in [16, 17]. There, the events are described using classical propositional logic. Then, the
reasoning happens with so-called weight formulas composed of the following constructions

\[ \sum_{i=1}^{n} a_i \cdot w(\phi_i) \geq c. \]

Here, \( a_i, c \in \mathbb{Z} \), \( \phi_i \) is a propositional formula, and \( w \) a weight modality denoting the probability of the formula.

Although, the formalisation in [17] is not a two-layered logic per se, it can be equivalently translated into one. This equivalence is demonstrated in [1]. There, the outer layer is expressed in the language of Łukasiewicz logic, as was first proposed in [25]. This removes the need of operating with the Boolean combinations of inequalities directly and allows for a finite and straightforward axiomatisation of the whole framework.

The use of many-valued logics (we refer the reader to [24] and [9, 10, 8] for basic exposition to and further details on many-valued and fuzzy logics) on the outer layer ties into the existing tradition of using these for reasoning about vagueness [39], beliefs [20, 40], and uncertainty [13]. Most of the previous work, however, employs logics that are capable of expressing a certain amount of arithmetic (usually, addition and subtraction). As this paper concentrates on the qualitative notion of uncertainty, we will be using Gödel logic and its expansions that can only compare values of formulas but not conduct arithmetic operations with them. Still, we will show that this is enough to have a logic that can formalise reasoning about probabilities qualitatively.

**Two-dimensional treatment of inconsistent information** If we recall the understanding of uncertainty, outlined in the first paragraphs of this section, it is clear that classical logic (unsurprisingly) is not well suited to reasoning with non-classical information. Indeed, an arbitrary statement follows classically from a contradiction. This, however, is counter-intuitive for there may be no connection between those. Likewise, even if one does not have any evidence at all whether \( \phi \) is true or false, \( \phi \lor \sim\phi \) is valid, whence, always true. In other words, classical logic cannot reason with non-classical, and most importantly, contradictory information non-trivially.

Thus, the goal of our project is to develop paraconsistent two-dimensional logics and apply them to the reasoning about uncertainty. Namely, we propose logics built upon or inspired by Belnap and Dunn’s ‘useful four-valued logic’ (alias, Belnap–Dunn logic, BD or first-degree entailment) [14, 4, 3].

The main idea of BD is to treat positive and negative information regarding a given statement\(^1\) \( \phi \) independently. This gives rise to four values comprising the lattice \( \mathbb{4} \) depicted on fig. 1. There, \( t \) stands for ‘the information only supports the truth of \( \phi \)’; \( f \) for ‘the information only supports the falsity of \( \phi \)’; \( b \) for ‘the information supports both the truth and falsity of \( \phi \)’ (i.e., the information is contradictory); \( n \) for ‘there is no information regarding \( \phi \)’. The upwards order \( x \preceq_{\mathbb{4}} y \) is construed as ‘\( x \) is at least as true and at most as false as \( y \)’.

Should we wish to account for the possibility of the non-classical information to be graded, the original lattice \( \mathbb{4} \) is extended to \([0, 1]\) \( \mathbb{4} \) (fig. 2). Two-layered logics axiomatising paraconsistent probabilities [26] as well as some other uncertainty measures including belief functions were presented in [7, 6]. There, the events are described by BD propositions which represents the possibility of our sources to provide non-classical information. The reasoning on the outer layer is formalised using a paraconsistent expansion of Łukasiewicz logic with the Belnap–Dunn negation \( \lnot \), and the uncertainty measure, in its turn, is a non-standard probability or belief function.

\(^1\)Originally, the values denote what a computer or a database might have been told regarding a statement.
Plan of the paper  The remainder of the paper is structured as follows. In section 2, we give all necessary preliminaries. In particular, we explain Belnap–Dunn logic BD, Gödel logic with co-implication biG and its paraconsistent expansions G2(→, ≲) and G2(→, →cope) (which we dub collectively as G2). We also provide their axiomatisations and establish the completeness of the calculi for G2.

In section 3, we show how to axiomatise reasoning with different qualitative uncertainty measures. We begin with the logic we dub QG that formalises the reasoning with arbitrary uncertainty measures using Gödel logic. We then consider axiomatic extensions of QG that formalise qualitative reasoning with capacities and belief functions. Finally, we provide QPG—the two-layered logic of qualitative probabilities. We also recall Gärdenfors’ logic of qualitative probability QP and establish a faithful embedding of ‘simple inequality formulas’ (i.e., formulas constructed as Boolean combinations of χ ≲ χ’ formulas of QP without nesting of ≲) into QPG.

In section 4, we develop two paraconsistent expansions of QG. We define semantics, construct calculi and prove their completeness. We use these logics to formalise comparisons of certainty in (potentially) contradictory or incomplete information.

Finally, in section 5, we recapitulate the results presented in the paper and discuss future work.

2 Logical preliminaries

2.1 Bi-Gödel logic

Gödel logic is an infinite-valued propositional logic, with its standard algebraic semantics being based on the full [0, 1] interval, where 1 is the designated value. The truth values are (densely) ordered, and, together with the semantics of Gödel implication, this makes Gödel logic suitable for formalizing comparisons. Gödel logic is one of the three basic t-norm based fuzzy logics, and it is also closely related to intuitionistic logic: it is the logic of linearly ordered Heyting algebras and can also be characterized as the logic of linearly ordered intuitionistic Kripke structures, and axiomatized by extending the intuitionistic logic with the axiom of prelinearity. A more detailed exposition of Gödel logics can be found e.g., in [10].

In this subsection, we formulate Gödel logic expanded with a co-implication connective ≲ and refer to it as biG (bi-Gödel logic), as it can naturally be obtained by extending the bi-intuitionistic logic with the axioms of prelinearity.
Definition 2.1. The bi-Gödel algebra $[0, 1]_G = ([0, 1], \land_G, \lor_G, \rightarrow_G, \sim_G)$ is defined as follows: for all $a, b \in [0, 1]$, the standard operations are given by $a \land_G b := \min(a, b)$, $a \lor_G b := \max(a, b)$, and $a \rightarrow_G b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{else}. \end{cases}$

Definition 2.2 (Language and semantics of biG). We set Prop to be a countable set of propositional variables and consider the following language $L_{biG}$.

$L_{biG} : \phi := p \in \text{Prop} \mid (\phi \land \phi) \mid (\phi \lor \phi) \mid (\phi \rightarrow_G \phi) \mid (\phi \sim_G \phi)$

We let $e : \text{Prop} \rightarrow [0, 1]$. Using bi-Gödel operations from definition 2.1, $e$ is extended to complex formulas in an expected manner: $e(\phi \circ_G \phi') = e(\phi) \circ_G e(\phi')$. We say that $\phi$ is valid if $e(\phi) = 1$ for any $e$. Furthermore, we define the entailment:

$\Gamma \models_{biG} \chi$ iff $\inf \{e(\phi) : \phi \in \Gamma\} \leq e(\chi)$ for any $e$

Convention 2.1 (Notational conventions). We will further use the following shorthands.

$\top_G := p \rightarrow_G p$ \quad $\bot_G := p \sim_G p$ \quad $\sim_G \phi := \phi \rightarrow_G \bot_G$ \quad $\Delta \phi := (\top_G \sim_G \phi) \rightarrow_G \bot_G$

Note that

$e(\top_G) = 1$ \quad $e(\bot_G) = 1$ \quad $e(\sim_G \phi) = \begin{cases} 1 & \text{iff } e(\phi) = 0 \\ 0 & \text{otherwise} \end{cases}$ \quad $e(\Delta \phi) = \begin{cases} 1 & \text{iff } e(\phi) = 1 \\ 0 & \text{otherwise} \end{cases}$

Finally, we write $e[\Gamma] = x$ iff $\inf \{e(\phi) : \phi \in \Gamma\} = x$.

Gödel logic can be interpreted as a logic of comparative truth, for the truth of a given formula depend only on the relative order of the values of its variables, not the values themselves. It is best seen in the semantics $\rightarrow_G$ and $\sim_G$.

Indeed, $\phi \rightarrow_G \phi'$ is true (has value 1) iff the value of $\phi$ is less or equal to that of $\phi'$. In simpler words, for an implication to be true, the value cannot decrease from the antecedent to the consequent. Likewise, for a co-implication $\phi \sim_G \phi'$ to be true, not only $\phi$ must be true but $\phi'$ should have a lesser value. In what follows, we will use $\sim_G$ and $\Delta$ to express statements of the form 'I am more certain in $\phi$ than in $\phi'$' (cf. examples 3.1 and 3.2 for more details).

Let us now present the Hilbert-style calculus for biG. Since biG is an extension of the bi-intuitionistic logic HB [22], we first begin with a Hilbert-style calculus for it.

Definition 2.3 (HB). The calculus has the following axiom schemas and rules (for any $\phi, \chi, \psi$):

1. $(\phi \rightarrow_G \chi) \rightarrow_G ((\chi \rightarrow_G \psi) \rightarrow_G (\phi \rightarrow_G \psi))$
2. $\phi \rightarrow_G (\phi \lor \chi) : \chi \rightarrow_G (\phi \lor \chi)$
3. $(\phi \rightarrow_G \chi) \rightarrow_G ((\chi \rightarrow_G \psi) \rightarrow_G ((\phi \lor \chi) \rightarrow_G \psi))$
4. $(\phi \land \chi) \rightarrow_G \phi : (\phi \land \chi) \rightarrow_G \chi$
5. $(\phi \rightarrow_G \chi) \rightarrow_G ((\phi \rightarrow_G \psi) \rightarrow_G (\phi \rightarrow_G (\chi \land \psi)))$
6. $(\phi \rightarrow_G (\chi \rightarrow_G \psi)) \rightarrow_G ((\phi \land \chi) \rightarrow_G \psi) : ((\phi \land \chi) \rightarrow_G \psi) \rightarrow_G ((\phi \rightarrow_G (\chi \rightarrow_G \psi))$
7. $(\phi \rightarrow_G \chi) \rightarrow_G (\sim_G \chi \rightarrow_G \sim_G \phi)$
2.2 Belnap–Dunn logic

Belnap–Dunn logic (BD) or first-degree entailment was introduced in [4] as a four-valued paraconsistent logic over the following language, interpreted on the lattice $\mathcal{4}$ (fig. 1).

$\mathcal{L}_{BD} : \phi := \phi \in \text{Prop} \mid \neg \phi \mid (\phi \land \phi) \mid (\phi \lor \phi)$

Recall that the values designate four kinds of information a computer (or a database) could have regarding a given statement $\phi$, or what a computer might be told regarding it:

- told **only** that $\phi$ is **true** — $t$;
- told **both** that $\phi$ is **true and** that it is **false** — $b$;
- told **neither** that $\phi$ is true nor that it is false — $n$;
- told **only** that $\phi$ is **false** — $f$.

In this paper, we will present BD using an alternative semantics of BD using Kripke-style models.

**Definition 2.4** (Belnap–Dunn models). A **Belnap–Dunn model** is a tuple $\mathfrak{M} = \langle W, v^+, v^- \rangle$ with $W \neq \emptyset$ and $v^+, v^- : \text{Prop} \to \mathcal{P}(W)$.

Here, each state of the model can be thought of as a source of information. It is clear that regarding each statement, a source can either say that it is true, false, say nothing at all, or provide a contradictory account. This is formalised in the following definition.
Definition 2.5 (Frame semantics for BD). Let $\phi, \phi' \in \mathcal{L}_{\text{BD}}$. For a model $\mathcal{M} = (W, v^+, v^-)$, we define notions of $w \vDash^+ \phi$ and $w \vDash^- \phi$ (positive and negative supports, respectively) for $w \in W$ as follows:

\[
\begin{align*}
  w \vDash^+ p & \text{ if } w \in v^+(p) & w \vDash^- p & \text{ if } w \in v^-(p) \\
  w \vDash^+ \neg \phi & \text{ if } w \vDash^- \phi & w \vDash^- \neg \phi & \text{ if } w \vDash^+ \phi \\
  w \vDash^+ \phi \land \phi' & \text{ if } w \vDash^+ \phi \text{ and } w \vDash^+ \phi' & w \vDash^- \phi \land \phi' & \text{ if } w \vDash^- \phi \text{ or } w \vDash^- \phi' \\
  w \vDash^+ \phi \lor \phi' & \text{ if } w \vDash^+ \phi \text{ or } w \vDash^+ \phi' & w \vDash^- \phi \lor \phi' & \text{ if } w \vDash^- \phi \text{ and } w \vDash^- \phi'
\end{align*}
\]

We denote the positive and negative interpretations of a formula as follows:

\[
|\phi|^+ := \{ w \in W \mid w \vDash^+ \phi \} \quad |\phi|^− := \{ w \in W \mid w \vDash^- \phi \}.
\]

$|\phi|^+$ (resp., $|\phi|^-$) can be interpreted as the set of the sources saying that the statement $\phi$ is true (resp., false).

We say that a sequent $\phi \vdash \chi$ is valid on $\mathcal{M} = (W, v^+, v^-)$ (denoted, $\mathcal{M} \models [\phi \vdash \chi]$) iff for any $w \in W$, it holds that:

- if $w \vDash^+ \phi$, then $w \vDash^+ \chi$ as well;
- if $w \vDash^- \chi$, then $w \vDash^- \phi$ as well.

A sequent $\phi \vdash \chi$ is universally valid iff it is valid on every model. In this case, we will say that $\phi$ entails $\chi$.

Convention 2.2. To facilitate the presentation, we define $\text{Lit} = \text{Prop} \cup \{-p : p \in \text{Prop}\}$ and denote

\[
\text{Var}(\phi) = \{ p \in \text{Prop} : p \text{ occurs in } \phi \}, \quad \text{Lit}(\phi) = \{ l \in \text{Lit} : l \text{ occurs in } \phi \}.
\]

We also denote $\text{St}(\phi)$ the set of all subformulas of $\phi$.

One can see that BD inherits truth and falsity conditions from the classical logic but treats them independently. In particular, $\mathcal{W}$ claims that $\phi$ is false ($w \vDash^- \phi$) whenever it claims that its negation is true ($w \vDash^+ \neg \phi$); likewise, a conjunction is false when at least one of its members is false, etc. This is why there are no universally true, nor universally false formulas. Indeed, one can check that any $\phi(p_1, \ldots, p_n)$ is neither true nor false at $w$ if $w \vDash^+ p_i$ and $w \vDash^- p_i$ for all $p_i$’s.

Furthermore, $\neg \phi \lor \chi$ cannot be treated as a shorthand for implication. First, it lacks modus ponens: if both $\phi$ and $\neg \phi \lor \chi$ are true, it does not entail that $\chi$ is true as well (for if $\phi$ is both true and false but $\chi$ is not true, then $\neg \phi \lor \chi$ is still true). Second, the deduction theorem fails: even if $\phi$ entails $\chi$, the sequent $\neg \phi \vdash \chi$ is still not valid. In fact, even if $\phi \land \phi'$ entails $\chi$, then it is not always the case that $\phi$ entails $\neg \phi' \lor \chi$, for instance, $p \land q \vdash q$ is valid while $p \vdash q \lor \neg q$ is not. The converse fails as well: $p \vdash p \lor q$ is valid but $p \land \neg p \vdash q$ is not.

Example 2.1. Let us clarify which information corresponds to which truth value using the following example. Assume that we read an announcement about a dog being lost by its owner.

“A female golden retriever was lost on the 5th of October. The last time I saw her on the 7th of October, she had a wide blue made of leather. Any finder is kindly requested to call +33625153633.”

It is clear that it is true only that the dog lost was a golden retriever, and false only that it was male. However, it is both true and false that the owner lost her on the 5th of October: the
announcement contradicts itself saying that the owner saw the dog for the last time two days after the loss. Furthermore, since there is no information regarding the location where the dog was lost (or seen last), a statement such as ‘the dog was lost near the city theatre’ would be neither true nor false.

BD can be completely axiomatised\(^2\) using the following calculus \(R_{fde} [15]\).

\[
\begin{align*}
\phi \land \chi & \vdash \phi & \phi \vdash \phi \lor \chi \\
\phi \land \chi & \vdash \chi & \chi \vdash \phi \lor \chi \\
\neg(\phi \land \chi) & \models \neg \phi \lor \neg \chi & \neg(\phi \lor \chi) & \models \neg \phi \land \neg \chi \\
\neg \phi & \models \phi & \phi \land (\chi \lor \psi) & \vdash (\phi \land \chi) \lor (\phi \land \psi)
\end{align*}
\]

\[
\begin{align*}
\lor_{\chi} & \phi \vdash \psi \quad \chi \vdash \psi \\
\land_{\chi} & \phi \vdash \chi \quad \phi \vdash \chi \land \psi
\end{align*}
\]

Theorem 2.6. \(\phi \vdash \chi\) is provable in \(R_{fde}\) iff it is valid.

Proof. From [15], we know that \(R_{fde}\) is complete w.r.t. the four-valued semantics. We show that \(\phi \vdash \chi\) is valid on \(4\) iff it is valid w.r.t. the frame semantics. Indeed, assume, \(v : \text{Prop} \rightarrow 4\) invalidates \(\phi \vdash \chi\), i.e., \(v(\phi) \not\leq_{4} v(\chi)\). We let \(W = \{w\}\) and construct \(v^+, v^- : \text{Prop} \rightarrow \{\emptyset, \{w\}\}\) as follows.

\[
\begin{align*}
w \in v^+(p) & \text{ iff } v(p) \in \{t, b\} \\
w \in v^-(p) & \text{ iff } v(p) \in \{f, b\}
\end{align*}
\]

Now, it is easy to show by induction on \(\psi\), that

\[
\begin{align*}
w & \in v^+(\psi) \text{ iff } v(\psi) \in \{t, b\} & \text{ (for any } \psi) \\
w & \in v^-(\psi) \text{ iff } v(\psi) \in \{f, b\} & \text{ (for any } \psi)
\end{align*}
\]

But then it is clear that if \(v(\phi) \not\leq_{4} v(\chi)\), then \(\mathfrak{M}_{4} \not\models [\phi \vdash \chi]\).

For the converse, we have two options, either (i) there is \(w \in \mathfrak{M}\) s.t. \(w \vdash^+ \phi\) and \(w \nvdash^+ \chi\), or (ii) there is \(w' \in \mathfrak{M}\) s.t. \(w' \vdash^- \chi\) and \(w' \nvdash^- \phi\). Again, we use (1) to construct \(v\) on \(4\) but now, using \(v^+\) and \(v^-\). The result now follows. \(\square\)

2.3 Paraconsistent expansions of bi\(G\)

To treat support of truth and support of falsity independently, we expand the language of bi\(G\) with De Morgan negation \(\neg\). This makes the interpretation of formulas two-dimensional: the first co-ordinate stands for the support of truth and the second one for the support of falsity. In a sense, \(G^2\) is a Gödel logic interpreted on the twist-product algebra \([0, 1]^\infty\) (recall fig. 2). While De Morgan negations of \(\land\) and \(\lor\) are unambiguous, there are several options for treating the De Morgan negation of \(\rightarrow\) \([41]\).

We choose two of them: (1) an intuitive option, ‘implication is false when its antecedent is true and consequent is false’ (i.e., the way we disprove classical implication), this is the way it is done in Nelson’s logic \([30]\) (we denote this implication \(\rightarrow\)); and (2) via co-implication as it is done by Moisil\(^3\) in \([29]\) (this implication will be denoted \(\rightarrow\)). The falsity conditions of co-implications (\(\prec\) and \(\rightarrow\)) are dual to those of implications (\(\rightarrow\) and \(\rightarrow\)).

The second versions of implication and co-implication produce a completely self-dual logic.

\(^2\)Alternative axiomatisations can be found in \([32, 33, 38]\).

\(^3\)This logic was introduced several times: by Wansing \([41]\) as \(I_4C_4\) and then by Leitgeb \([28]\) as HYPE. Cf. \([31]\) for a recent and more detailed discussion.
2.3 Paraconsistent expansions of $\mathbf{biG}$

**Definition 2.7** (Language and semantics of $G^2$). We fix a countable set $\text{Prop}$ of propositional letters and consider the following language:

$$\phi := p \in \text{Prop} \mid \neg \phi \mid (\phi \land \phi) \mid (\phi \lor \phi) \mid (\phi \rightarrow \phi) \mid (\phi \leftarrow \phi) \mid (\phi \rightarrow \neg \phi)$$

Let $e_1, e_2 : \text{Prop} \rightarrow [0, 1]$. We extend them as follows.

$$
\begin{align*}
e_{1}(\neg \phi) & = e_{2}(\phi) \\
e_{1}(\phi \land \phi') & = e_{1}(\phi) \land_{G} e_{1}(\phi') \\
e_{1}(\phi \lor \phi') & = e_{1}(\phi) \lor_{G} e_{1}(\phi') \\
e_{1}(\phi \rightarrow \phi') & = e_{1}(\phi) \rightarrow_{G} e_{1}(\phi') \\
e_{1}(\phi \leftarrow \phi') & = e_{1}(\phi') \leftarrow_{G} e_{1}(\phi') \\
e_{1}(\phi \rightarrow \phi') & = e_{1}(\phi) \rightarrow_{G} e_{1}(\phi') \\
e_{1}(\phi \leftarrow \phi') & = e_{1}(\phi') \leftarrow_{G} e_{1}(\phi')
\end{align*}
$$

We will consider two separate logics: $G^{2}(\rightarrow, \leftarrow)$ and $G^{2}(\rightarrow, \rightarrow)$ and their respective languages $\mathcal{L}_{G^{2}(\rightarrow, \leftarrow)}$ and $\mathcal{L}_{G^{2}(\rightarrow, \rightarrow)}$ which have only one set of (co-)implications indicated in the brackets. We define their entailments as follows.

**Definition 2.8** ($G^2$ entailments). Let $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_{G^{2}(\rightarrow, \leftarrow)}$ and $\Delta \cup \{\chi\} \subseteq \mathcal{L}_{G^{2}(\rightarrow, \rightarrow)}$. We define:

$$
\begin{align*}
\Gamma \models_{G^{2}(\rightarrow, \leftarrow)} \phi & \text{ iff } \forall e_{1}, e_{2} : \inf\{e_{1}(\psi) : \psi \in \Gamma\} \leq e_{1}(\psi) \text{ and } \sup\{e_{2}(\psi) : \psi \in \Gamma\} \geq e_{2}(\phi) \\
\Delta \models_{G^{2}(\rightarrow, \rightarrow)} \chi & \text{ iff } \forall e_{1} : \inf\{e_{1}(\psi) : \psi \in \Delta\} \leq e_{1}(\chi)
\end{align*}
$$

As we see, $G^{2}(\rightarrow, \rightarrow)$ entailment considers only the first coordinate on $[0, 1]^{x}$ of the valuation while $G^{2}(\rightarrow, \leftarrow)$ takes into consideration both of them$^{4}$.

**Convention 2.3**. In what follows, we will write $e$ as a shorthand of $(e_{1}, e_{2})$ and $e(\phi)$ instead of $(e_{1}(\phi), e_{2}(\phi))$ when there is no risk of confusion.

**Remark 2.1**. Note that $0$ and $1$ s.t. $e(1) = (1, 0)$ and $e(0) = (0, 1)$ are definable.

$$
\mathcal{L}_{G^{2}(\rightarrow, \rightarrow)} : \quad 0 := (p \rightarrow p) \rightarrow (p \rightarrow p) \quad \quad \mathcal{L}_{G^{2}(\rightarrow, \leftarrow)} : \quad 0 := p \leftarrow p \\
1 := 0 \rightarrow 0 \quad \quad \quad \quad \quad 1 := p \rightarrow p
$$

Note furthermore, that $\phi \leftarrow \chi$ can be defined as $\neg(\neg\chi \rightarrow \neg\phi)$.

In the remainder of this subsection, we are going to present Hilbert calculi for $G^{2}(\rightarrow, \rightarrow)$ and $G^{2}(\rightarrow, \leftarrow)$ and prove their completeness. The calculi are straightforward expansions of $\text{HbiG}$ with De Morgan axiom.

**Convention 2.4**. In what follows, we will use several shorthands.

$$
\begin{align*}
\top_{1} & := p \rightarrow p \\
\bot_{0} & := p \leftarrow p \\
\neg_{0} p & := p \rightarrow \bot_{0}
\end{align*}
$$

Note that the definitions of $1$ and $0$ in $\mathcal{L}_{G^{2}(\rightarrow, \rightarrow)}$ coincide with $\top_{1}$ and $\bot_{0}$ which is not the case for $\mathcal{L}_{G^{2}(\rightarrow, \leftarrow)}$. When there is no risk of confusion, we will drop the subscripts and write simply $\bot$ or $\top$.

---

$^{4}$There is a technical reason for such definition as well: notice that if $v(p) = (1, 1)$, then $v(p \rightarrow p) = (1, 1)$ as well. Thus, theorems of $G^{2}(\rightarrow, \rightarrow)$ are not always evaluated at $(1, 0)$. 
Definition 2.9 ($\mathcal{H}_G^2(\rightarrow, \prec)$ and $\mathcal{H}_G^2(\rightarrow, \rightarrow)$). To obtain $\mathcal{H}_G^2(\rightarrow, \prec)$, we replace $\rightarrow_G$ with $\rightarrow$ and $\prec_G$ with $\prec$ in all axioms and rules of $\mathcal{H}_{biG}$ and add the following axiom schemas and rules (below, $\phi \leftrightarrow \chi$ is a shorthand for $(\phi \rightarrow \chi) \land (\chi \rightarrow \phi)$):

- **neg.** $\neg \neg \phi \leftrightarrow \phi$
- **DeM\&.** $\neg (\phi \land \chi) \leftrightarrow (\neg \phi \lor \neg \chi)$
- **DeMV.** $\neg (\phi \lor \chi) \leftrightarrow (\neg \phi \land \neg \chi)$
- **DeM\rightarrow.** $\neg (\phi \rightarrow \chi) \leftrightarrow (\neg \chi \land \neg \phi)$
- **DeM\prec.** $\neg (\phi \prec \chi) \leftrightarrow (\neg \chi \rightarrow \neg \phi)$

For $\mathcal{H}_G^2(\rightarrow, \rightarrow)$, we replace $\rightarrow_G$ with $\rightarrow$ and $\prec_G$ with $\rightarrow$. We also add neg and De Morgan laws for $\land$ and $\lor$ (with $\leftrightarrow$ instead of $\leftrightarrow$). Finally, the De Morgan laws for (co-)implication are as follows.

- **DeM\rightarrow.** $\neg (\phi \rightarrow \chi) \leftrightarrow (\phi \land \neg \chi)$
- **DeM\prec.** $\neg (\phi \prec \chi) \leftrightarrow (\phi \lor \chi)$

Note that $\mathcal{H}_G^2(\rightarrow, \prec)$ and $\mathcal{H}_G^2(\rightarrow, \rightarrow)$ extend $I_4C_4$ and $I_1C_1$ (Nelson’s logic with co-implication) from [41] with prelinearity axioms prel.

Completeness of both calculi w.r.t. their algebraic semantics can be established in the following sense.

**Theorem 2.10.** $\mathcal{H}_G^2(\rightarrow, \prec)$ and $\mathcal{H}_G^2(\rightarrow, \rightarrow)$ are strongly complete:

$$\Gamma \vdash \mathcal{H}_G^2(\rightarrow, \prec) \phi \iff \Gamma \models \mathcal{G}_G^2 \phi \quad \Gamma \vdash \mathcal{H}_G^2(\rightarrow, \rightarrow) \phi \iff \Gamma \models \mathcal{G}_G^2 \phi$$

**Proof.** In the appendix. 

3 Qualitative reasoning about uncertainty

In many contexts, an agent may not necessarily be able to give a precise number that corresponds to their degree of certainty in a given proposition. Indeed, ‘we are 70% certain that it is going to rain tomorrow’ is relatively natural when giving a weather forecast based on some evidence as well as on previous statistical data. On the other hand, if two different people (say, Paula and Quinn) claim that a recently found dog belongs to themselves only, a statement such as ‘I am 65% confident that the dog is Paula’s’ sounds bizarre.

Nevertheless, in such contexts, there is a chance (if Paula, Quinn, or both present some relevant evidence) for a reasoning agent to compare their claims and pick a more compelling one. In other words, even though, we are not really able to measure our certainty in a given statement numerically, we can compare evidence for it with evidence for another one.

Qualitative probability was extensively studied, starting already in the work of de Finetti [18], who provided a natural axiomatisation for them. He also asked a natural question: are these conditions sufficient for the existence of a quantitative probability measure which agrees with the qualitative ordering? More formally, a measure $\mu$ on a set of events $P(W)$ agrees with a qualitative ordering $\preceq$ on the same set iff

$$\forall X, Y \subseteq W : X \preceq Y \iff \mu(X) \leq \mu(Y)$$
3.1 Qualitative uncertainty

This remained an open problem until the 1950s when Kraft, Pratt and Seidenberg in their seminal paper [27] showed that the answer is negative. They also provided sufficient conditions, which we will discuss below. The qualitative counterparts of capacities and belief functions were studied as well (see e.g. [43, 42]). We summarise the corresponding results in the following theorem.

**Theorem 3.1** (Qualitative characterisations of uncertainty measures). Let \( W \neq \emptyset \) and let further \( \preceq \) be a linear order on \( 2^W \). Consider the following conditions on \( \preceq \) for all \( X,Y,Z \in W \).

Q1  \( \emptyset \preceq X \preceq W \).

Q2  \( \emptyset \preceq X \).

Q3  If \( X \subseteq Y \), then \( X \preceq Y \).

PM If \( X \not\subseteq Y \), \( X \sim Y \), and \( Y \cap Z = \emptyset \), then \( X \cup Z \not\prec Y \cup Z \).

KPS

For any \( m \in \mathbb{N} \) and all \( X_i, Y_i \subseteq W \) (\( i \in \{0,\ldots,m\} \)), it holds that if \( X_j \preceq Y_j \) for all \( j < m \) and if any \( w \in W \) belongs to as many \( X_i \)'s as \( Y_i \)'s, then \( X_m \succeq Y_m \).

Then, it holds that

1. the counterparts of \( \preceq \) are capacities iff \( \preceq \) satisfies Q1–Q3;
2. the counterparts of \( \preceq \) are belief functions iff \( \preceq \) satisfies Q1–Q3 and PM;
3. the counterparts of \( \preceq \) are probability measures iff \( \preceq \) satisfies Q1–Q3 and KPS

Note that uncertainty measures are also counterparts of \( \preceq \) satisfying Q1–Q3 since the only difference between a capacity and an uncertainty measure is that an uncertainty measure \( \mu \) does not have to satisfy \( \mu(\emptyset) = 1 \) and \( \mu(\emptyset) = 0 \).

Moreover, since any uncertainty measure \( \mu : 2^W \to [0,1] \) gives rise to a linear order \( \preceq_{\mu} \) on \( 2^W \), the original conditions PM and KPS

can be reformulated in terms of measures. For any uncertainty measure \( \mu \), its qualitative counterpart is an ordering corresponding to a belief function iff \( \mu \text{PM} \) holds and an ordering corresponding to a probability measure iff \( \mu \text{KPS} \) holds.

\( \mu \text{PM} \): If \( \mu(X) < \mu(Y) \), \( X \not\subseteq Y \), and \( Y \cap Z = \emptyset \), then \( \mu(X \cup Z) < \mu(Y \cup Z) \).

\( \mu \text{KPS} \): For any \( m \in \mathbb{N} \) and all \( X_i, Y_i \subseteq W \) (\( i \in \{0,\ldots,m\} \)), it holds that if \( \mu(X_j) \leq \mu(Y_j) \) for all \( j < m \) and if any \( w \in W \) belongs to as many \( X_i \)'s as \( Y_i \)'s, then \( \mu(X_m) \geq \mu(Y_m) \).

In what follows, we present two-layered logics that formalise qualitative reasoning with uncertainty measures, capacities, belief functions and probabilities.

3.1 Qualitative uncertainty in a two-layered framework

In this section, we formulate two-layered modal logic \( \mathcal{QG} = \langle \text{CPL}, \{\mathfrak{B}\}, \text{biG} \rangle \). Here, we represent the events with the classical propositional logic but reason with the beliefs concerning these events using biG — Gödel logic with co-implication. This will allow us to compare likelihoods of different events. The modality connecting two layers is denoted by \( \mathfrak{B} \). The value of \( \mathfrak{B}\phi \) (with \( \phi \in \mathcal{L}_{\text{CPL}}^\circ \)) is interpreted as the degree of \( \phi \)'s likelihood. Recall, however, that in Gödel logic a formula is true (i.e., has value 1) in virtue of the relative order of the values of its variables, not their numerical values, and that this order corresponds to the Gödel implication.

\(^5\)We use a minimalistic \( \{\sim, \land\} \) language with all other connectives being introduced via definitions.
Note that the two-layered nature of $\text{QG}$ prohibits the nesting of $\mathcal{B}$ modality. In other words, statements such as ‘it is more likely that $\phi$ is more likely than $\psi$’ cannot be expressed. We will show, however, that there is a faithful translation that preserves the validity of propositional combinations of formulas $\chi \triangleleft \chi'$ (with $\chi, \chi' \in \mathcal{L}_{\text{CPL}}$).

Let us now introduce $\text{QG}$ in a formal manner.

**Definition 3.2** (Language and semantics of $\text{QG}$). We let $\phi \in \mathcal{L}_{\text{CPL}}$ and define the following grammar.

$\mathcal{L}_{\text{QG}} : \quad \alpha := \mathcal{B}\phi \mid (\alpha \land \alpha') \mid (\alpha \lor \alpha') \mid (\alpha \Rightarrow \alpha) \mid (\alpha \triangleleft \alpha')$

An uncertainty frame (or simply, a frame) is a tuple $\mathcal{F} = \langle W, \mu \rangle$. Here, $W \neq \emptyset$ and $\mu$ is an uncertainty measure on $W$.

A $\text{QG}$ model is a tuple $\mathfrak{M} = \langle W, v, \mu, e \rangle$ with $\langle W, \mu \rangle$ being a frame and $v : \text{Prop} \to 2^W$ being extended to $\mathfrak{M}, x \vDash \phi$ for $x \in W$ and $\phi \in \mathcal{L}_{\text{QG}}$ as follows.

- $\mathfrak{M}, x \vDash p$ iff $x \in v(p)$.
- $\mathfrak{M}, x \vDash \sim \phi$ iff $\mathfrak{M}, x \nvDash \phi$.
- $\mathfrak{M}, x \vDash \phi \land \phi'$ iff $\mathfrak{M}, x \vDash \phi$ and $\mathfrak{M}, x \vDash \phi'$.

$e$ is a bi-Gödel valuation (cf. definition 2.2) s.t. $e(\phi) = \mu(\parallel \phi \parallel)$ with $\parallel \phi \parallel = \{x : \mathfrak{M}, x \vDash \phi\}$.

$\text{QG}$ entailment is defined in the same manner as that of $\text{biG}$:

$\Xi \models_\text{QG} \alpha \iff \inf\{e(\gamma) : \gamma \in \Xi\} \leq e(\alpha)$ for any $e$

Finally, $\alpha \in \mathcal{L}_{\text{QG}}$ is valid on $\mathcal{F}$ ($\mathcal{F} \models \alpha$) iff $e(\alpha) = 1$ for any $e$ and $v$ defined on $\mathcal{F}$.

Let us provide an example of how to formalise statements concerning comparisons of degrees of certainty.

**Example 3.1** (Comparing certainty in $\text{QG}$). Assume that two people, Paula and Quinn, come to you and say that a recently found stray dog belongs to them (and not to the other person). Thus, we have two events: $p \land \sim q$ (the dog belongs to Paula but not to Quinn) and $\sim p \land q$ (vice versa). Now, assume further that you trust Paula more than you trust Quinn for some reason. Thus, the following statement should be true

$I am more certain that the dog belongs to Paula than to Quinn.$

We formalise it as follows

$$\Delta(B(\sim p \land q) \Rightarrow G B(p \land \sim q)) \land \sim G \Delta(B(p \land \sim q) \Rightarrow G B(\sim p \land q)) \quad (2)$$

$L$ part establishes that your certainty in $p \land \sim q$ is at least as high as your certainty in $\sim p \land q$, while $R$ part means that the converse is not the case. But note that $\Delta(\alpha \Rightarrow G \alpha') \lor \sim G \Delta(\alpha' \Rightarrow G \alpha)$ is a $\text{biG}$ valid formula, whence (2) can be simplified to

$$\sim G \Delta(B(p \land \sim q) \Rightarrow G B(\sim p \land q)) \quad (3)$$

It is a useful property of the bi-Gödel logic that $\phi(p_1, \ldots, p_n) \in \mathcal{L}_{\text{biG}}$ is valid iff it is valid for all valuations that range over $\{0, \frac{1}{n+1}, \ldots, \frac{n}{n+1}, 1\}$ [23, Theorem 35]. We can use this fact to provide a natural language interpretation of formulas avoiding reference to numerical values altogether.
3.1 Qualitative uncertainty...

Example 3.2. Recall (3). Note that there are two $\mathcal{L}_{QPG}$ atoms: $B(p \land \neg q)$ and $B(\neg p \land q)$. Thus, we need four (numerical) values corresponding to the ordered set: $\{0, 1, 2, 3\}$. We can associate them with the following subjective values: certainly false, unlikely, likely, certainly true.

Now, we can have the following assignment using example 3.1: I find it likely that the dog belongs to Paula and not Quinn (thus, $e(B(p \land \neg q))$ = ‘likely’) and unlikely that the dog belongs to Quinn, not Paula (i.e., $e(B(\neg p \land q))$ = ‘unlikely’). Hence, we conclude that I find that the dog belongs to Paula rather than to Quinn i.e., $\sim_{QG}(B(p \land q) \rightarrow_{QG} B(\neg p \land q))$, certainly true.

We are now ready to present the calculus for $QG$ and prove its completeness.

Definition 3.3 ($\mathcal{H}QG$). The calculus $\mathcal{H}QG$ has the following axioms and rules.

nontriv: $\sim_{QG}(B\phi \rightarrow_{QG} B\chi)$ for any $\phi$ and $\chi$ s.t. CPL $\vdash \phi$ and CPL $\vdash \sim \chi$.

reg: $B\phi \rightarrow_{QG} B\phi'$ with CPL $\vdash \phi \supset \phi'$; $b\text{iG}$: instantiations of $\mathcal{H}b\text{iG}$ axioms and rules with $\mathcal{L}_{QG}$ formulas.

Theorem 3.4 (Completeness of $\mathcal{H}QG$). Let $\Xi \cup \{\alpha\} \subseteq \mathcal{L}_{QG}$. Then

$$\Xi \models_{QG} \alpha \iff \exists \Xi \models_{\mathcal{H}QG} \alpha$$

Proof. As regards to the soundness part, we just need to check that the axioms are valid. It is clear that if $\phi$ is a propositional tautology, then $w \models \phi$ for any $w \in W$, and if $\chi$ is a contradictory formula, then $w \not\models \chi$ for any $w \in W$. But then, it is clear that $\mu(\|\phi\|) > \mu(\|\chi\|)$, whence $\sim_{QG}(B\phi \rightarrow_{QG} B\chi)$ is valid. Furthermore, if $\phi \supset \phi'$ is classically valid, then $v(\phi) \subseteq v(\phi')$ in any model, whence, $B\phi \rightarrow_{QG} B\phi'$ is valid as well.

For the completeness, we reason by contraposition. Assume that $\Xi \not\models_{\mathcal{H}QG} \alpha$. Then, by the completeness theorem for bi-Gödel logic, there is a valuation $e$ s.t. $e(\Xi) > e(\alpha)$. We can now extend $\Xi$ with instances of modal axioms composed from the subformulas of $\Xi$ and $\alpha$. Denote this new set $\Xi^*$. It is clear that $\Xi^* \not\models \alpha$ either and that moreover, the same valuation $e$ has it that $e(\Xi^*) > e(\alpha)$.

It remains to construct a model falsifying the entailment. The $b\text{iG}$ valuation is already given. We proceed as follows. First, we set $W = P(\text{Var}(\Xi^* \cup \{\alpha\}))$. Then, we define $w \in v(p)$ iff $p \in w$ for any $w \in W$ and extend it to $\| \cdot \|$ in a usual fashion. Finally, for any $B\phi \in Sf[\Xi^* \cup \{\alpha\}]$, we set $\mu(\phi) = e(B\phi)$. For other $X \subseteq W$, we set $\mu(X) = \sup\{\mu(\|\phi\|) : \phi \in Sf[\Psi^* \cup \{\alpha\}], \|\phi\| \subseteq X\}$. It remains to check that $\mu$ thus defined satisfies definition 3.2.

To show that $\mu$ is monotone, let $X \subseteq X'$. If there exist $\phi, \phi' \in \subseteq_{CPL}$ s.t. $\|\phi\| = X$ and $\|\phi'\| = X'$, it is clear from the construction of $W$ that $\phi \supset \phi'$ is a classical tautology, whence, $B\phi \rightarrow_{QG} B\phi'$ is an axiom, and thus $\mu(\|\phi\|) \leq \mu(\|\phi'\|)$, as required. Otherwise, recall that

$$\mu(X) = \sup\{\mu(\|\phi\|) : \phi \in Sf[\Psi^* \cup \{\alpha\}], \|\phi\| \subseteq X\}$$

$$\mu(X') = \sup\{\mu(\|\phi'\|) : \phi' \in Sf[\Psi^* \cup \{\alpha\}], \|\phi'\| \subseteq X'\}$$

whence, clearly, $\mu(X) \leq \mu(X')$, as required. Finally, since $\sim_{QG}(B \top \rightarrow_{QG} B \bot)$ is an instance of the $\top > \bot$ axiom, we have that $\mu(W) > \mu(\emptyset)$.

Note that $K$ is not valid: $B(p \supset \bot) \rightarrow_{QG} B(p \rightarrow_{QG} B \bot)$ is easy to disprove. It is, however, valid on single-point models. Moreover, $\Delta(B \bot \rightarrow_{QG} B \phi)$ which corresponds to the second condition on $\mu$ is provable in $\mathcal{H}b\text{iG}$ from reg.
3.2 Correspondence theory for weak uncertainty measures

It is clear that QG is the logic of all uncertainty frames. It is, however, quite counter-intuitive as a logic for the classical qualitative reasoning about uncertainty even if the uncertainty measure is not taken to be a probability function.

For example, from the classical point of view, we know for certain that the event ‘it rained in Paris on 28.03.2021 or it did not’ (formally, \( r \lor \sim r \)) occurred. However, \( B(r \lor \sim r) \) is not valid in QG\(^7\) since it can be that \( \mu([r \lor \sim r]) < 1 \).

Moreover, QG allows for two incompatible (or even complementary) events to have measure 1 at the same time. Moreover, it is also possible that \( \mu(Y \cup Y') > \mu(Y) \) even if \( \mu(Y') = 0 \). In other words, it is possible that the agent’s certainty in \( \phi \lor \phi' \) is strictly higher than that in \( \phi \) even if they are completely certain that \( \phi' \) is not the case.

In this section, we will show how to axiomatise these and other conditions.

Convention 3.1. We introduce the following naming conventions for several formulas.

1compl: \( \Delta B p \leftrightarrow \sim G \sim B p \)

disj+: \( \sim G B (p \land p') \land \sim G \sim G B p' \rightarrow G (\sim G \Delta (B(p \lor p') \rightarrow G B p) \land \sim G \Delta (B(p \lor p') \rightarrow G B p')) \)

disj0: \( \sim G B p \rightarrow G \Delta (B p' \leftrightarrow G B(p \lor p')) \)

cap: \( B \top \land \sim G B \bot \)

Theorem 3.5. Let \( F = \langle W, \mu \rangle \) be an uncertainty frame. Then the following equivalences hold

\[
\begin{align*}
F \models 1\text{compl} & \iff \mu(X) = 1 \leftrightarrow \mu(W \setminus X) = 0 \quad (I) \\
F \models \text{disj}+ & \iff \mu(Y \land Y') = 0 \text{ and } \mu(Y), \mu(Y') > 0 \Rightarrow \mu(Y \lor Y') > \mu(Y), \mu(Y') \quad (II) \\
F \models \text{disj}0 & \iff \mu(Y) = 0 \Rightarrow \mu(Y \lor Y') = \mu(Y') \quad (III) \\
F \models \text{cap} & \iff \mu \text{ is a capacity} \quad (IV)
\end{align*}
\]

Proof. We consider III, the other cases can be tackled in a similar manner.

Indeed, let \( \mu(Y) = 0 \) but \( \mu(Y \lor Y') \neq \mu(Y') \) for some \( Y, Y' \subseteq W \). Now let \( v(p) = Y \) and \( v(p') = Y' \). Then, it is clear that \( e(Bp) = 0 \) but \( e(Bp') \neq e(B(p \lor p')) \). Hence, \( e(\text{disj}0) \neq 1 \), as required.

For the converse, we assume that \( e(\text{disj}0) \neq 1 \). But then, \( e(\text{disj}0) = 0 \) since disj0 is composed of \( \Delta \)-formulas and \( \sim G \)-formulas. Thus \( e(\sim G Bp) = 1 \) but \( e(\Delta (B p' \rightarrow G B(p \lor p')) = 0 \) (i.e., \( \mu([p] \lor \mu([p']]) \neq \mu([p]) \lor \mu([p']]) \)). whence \( \mu([\mu([p]) \lor \mu([p']]) ] = 0 \) but \( \mu([\mu([p]) \lor \mu([p']]) ] = 0 \), as required. \( \square \)

Now, if we want to formalise qualitative counterparts of belief functions, we need to transform \( \mu PM \) into an axiom. However, there is no two-layered formula that can formalise \( X \subseteq Y \). Thus, we have to introduce a new axiom schema.

\[
\sim G \Delta (B_X \rightarrow G B \phi) \rightarrow G \sim G \Delta (B(\chi \lor \psi) \rightarrow G B(\phi \lor \psi))
\]

with CPL \( \vdash \phi \supset \chi \), CPL \( \vdash (\chi \land \psi) \), and CPL \( \not\vdash \chi \lor \phi \) \( (QBel) \)

Theorem 3.6. Let \( F = \langle W, \mu \rangle \) be an uncertainty frame. Then \( \mu \) satisfies QPM iff \( F \models QBel \).

Proof. Let \( \mu \text{PM} \) hold, and let further \( \phi, \chi, \) and \( \psi \) be as in \( (QBel) \). Thus, \( \|\phi\| \subseteq \|\chi\| \) and \( \|\chi\| \lor \|\psi\| = \emptyset \). Note also that both \( \sim G \Delta (B_X \rightarrow G B \phi) \) and \( \sim G \Delta (B(\chi \lor \psi) \rightarrow G B(\phi \lor \psi)) \) can have values only in \( \{0, 1\} \).

\( ^7\)This shows that QG can distinguish between capacities and generic uncertainty measures: \( B \top \) is valid on a frame if its uncertainty measure is a capacity.
Now, if \( e(\sim_G \Delta(B \chi \rightarrow_G B \phi)) = 1 \), then \( \mu(\|\phi\|) < \mu(\|\chi\|) \) and, in fact, \( \|\phi\| \subset \|\chi\| \). But then \( \mu(\|\phi\| \cup \|\psi\|) < \mu(\|\chi\| \cup \|\psi\|) \), whence \( \mu(\|\phi \lor \psi\|) < \mu(\|\chi \lor \psi\|) \), and thus \( e(\sim_G \Delta(B(\chi \lor \psi) \rightarrow_G B\phi(\phi \lor \psi))) = 1 \), as well.

For the converse, let \( \mu\text{PM} \) fail for \( \mathcal{F} \), and let, in particular, \( \mu(X) < \mu(Y) \), \( X \subset Y \), and \( Y \cap Z = \emptyset \), but \( \mu(X \cup Z) \geq \mu(Y \cup Z) \). We show how to falsify \( \text{QBel} \).

We let \( \|p\| = X \), \( \|p \lor q\| = Y \), and \( \|\sim q \land r\| = Z \). Now, it is easy to see that
\[
e(\sim_G \Delta(B(p \lor q) \rightarrow_G Bp) \rightarrow_G \sim_G \Delta(B((p \lor q) \lor (\sim q \land r)) \rightarrow_G B(p \lor (\sim q \land r)))) = 0
\]
as required.

### 3.3 Logics of qualitative probabilities

The main objective of this section is to provide a two-layered axiomatisation of qualitative probabilities. To do this, we need to transform \( \mu\text{KPS}_m \) into axioms. This, however, is not straightforward. This is why, we will take a detour through the classical logic of qualitative probability \( \text{QP} \) introduced in [19]. The language of \( \text{QP} \) is given by the following grammar.

\[
\mathcal{L}_{\text{QP}} : \phi := p \in \text{Prop} \mid \sim \phi \mid (\phi \land \phi) \mid (\phi \preceq \phi).
\]

The semantics uses probabilistic frames and models built upon them.

**Definition 3.7** (Frame semantics for \( \text{QP} \) [19]). A Gärdenfors probabilistic frame is a tuple \( \mathcal{F} = \langle U, \{P_x\}_{x \in U} \rangle \) with \( U \neq \emptyset \) and \( \{P_x\}_{x \in U} \) being a family of probability measures on \( 2^U \).

A Gärdenfors model is a tuple \( \mathcal{M} = \langle U, \{P_x\}_{x \in U}, v \rangle \) with \( \langle U, \{P_x\}_{x \in U} \rangle \) being a frame and \( v : \text{Prop} \rightarrow 2^U \) being a valuation that is extended to a satisfaction relation \( \|\cdot\| \) as follows:

- \( \|p\| = v(p) \);
- \( \|\sim \phi\| = U \setminus \|\phi\| \);
- \( \|\phi \land \psi\| = \|\phi\| \cap \|\psi\| \);
- \( \|\phi \preceq \psi\| = \{x \in U \mid P_x(\|\phi\|) \leq P_x(\|\psi\|)\} \).

For any model \( \mathcal{M} \), we say that \( \phi \) is true in \( \mathcal{M} \) (\( \mathcal{M} \models \phi \)) iff \( \|\phi\| = U \). Furthermore, \( \phi \) is valid in \( \mathcal{F} \) (\( \mathcal{F} \models \phi \)) iff \( \phi \) is true in every model on \( \mathcal{F} \).

Furthermore, we can define an additional notion of satisfaction in a state.

**Definition 3.8** (Pointed model semantics). Let \( \mathcal{M} \) be a Gärdenfors model and \( x \in \mathcal{M} \). We define \( \mathcal{M}, x \models \phi \) (\( \phi \) is true at \( x \)) as follows:

- \( \mathcal{M}, x \models p \iff x \in v(p) \).
- \( \mathcal{M}, x \models \sim \phi \iff \mathcal{M}, x \not\models \phi \).
- \( \mathcal{M}, x \models \phi \land \phi' \iff \mathcal{M}, x \models \phi \) and \( \mathcal{M}, x \models \phi' \).
- \( \mathcal{M}, x \models \phi \preceq \phi' \iff P_x(\|\phi\|) \leq P_x(\|\phi'\|) \).

In the remainder of the paper, we will call a tuple \( (\mathcal{M}, x) \) a pointed model.
Other connectives and modalities can be introduced in an expected fashion:

\[
\phi \lor \phi' := \sim(\sim\phi \land \sim\phi') \quad \phi \land \phi' := \sim\phi \lor \phi' \quad \phi \equiv \phi' := (\phi \land \phi') \lor (\phi' \land \phi)
\]

\[
\phi \Rightarrow \phi' := (\phi \land \phi') \lor (\phi' \land \phi) \quad \top := p \lor p \quad \phi \Leftarrow \phi' := (\phi \land \phi') \lor (\phi' \land \phi)
\]

Let us now recall the axiomatisation of QP. For this, we will introduce a specific notation that will help us express the Kraft–Pratt–Seidenberg conditions in a more concise manner.

**Convention 3.2 (E-notation).** Consider \(L_{QP}\) formulas \(\phi_1, \ldots, \phi_n\) and \(\chi_1, \ldots, \chi_n\). Let further, \(\phi^\circ \in \{\phi, \sim\phi\}\) and \(\chi^\circ \in \{\chi, \sim\chi\}\). We introduce a new operator \(E\) and write

\[
\phi_1, \ldots, \phi_n E \chi_1, \ldots, \chi_n
\]

to designate that necessarily the same number of \(\phi_i^\circ\)'s as of \(\chi_j^\circ\)'s are actually of the form \(\sim\phi_i\) and \(\sim\chi_j\), respectively. For example

\[
p_1, p_2 E q_1, q_2 := ((p_1 \land p_2 \land q_1 \land q_2) \lor (\sim p_1 \land p_2 \land \sim q_1 \land q_2) \lor (p_1 \land \sim p_2 \land q_1 \land \sim q_2) \lor (\sim p_1 \land \sim p_2 \land q_1 \land \sim q_2)) \approx \top
\]

More formally, we let \(M = \{1, \ldots, m\}\), \(K, L \subseteq M\) and set

\[
\phi_1, \ldots, \phi_m E \chi_1, \ldots, \chi_m := \left(\bigvee_{i=0}^{m} \bigvee_{|K| = i} \left(\bigwedge_{k \in K} \sim \phi_k \land \bigwedge_{k' \in M \setminus K} \phi_{k'} \land \bigwedge_{i \in L} \sim \chi_i \land \bigwedge_{i' \in M \setminus L} \chi_{i'}\right)\right) \approx \top
\]

The axiomatisation of QP which we call \(\mathcal{H}_{QP}\) expands the classical propositional rules with new axioms and rules concerning \(\preceq\).

**Definition 3.9 (\(\mathcal{H}_{QP}\)).**

(\(PC\)) All propositional tautologies.

(\(A0\)) \((\phi_1 \equiv \phi_2) \approx \top\) \land ((\psi_1 \equiv \psi_2) \approx \top) \lor ((\phi_1 \preceq \psi_1) \equiv (\phi_2 \preceq \psi_2)).

(\(A1\)) \(\perp \preceq \phi\).

(\(A2\)) \((\phi \preceq \psi) \lor (\psi \preceq \phi)\).

(\(A3\)) \(\perp \land \top\).

(\(A4\)) \((\phi_1, \ldots, \phi_m E \psi_1, \ldots, \psi_m) \land \bigwedge_{i=1}^{m-1} (\phi_i \preceq \psi_i) \lor (\psi_m \preceq \phi_m)\)

The rules are modus ponens and necessitation:

\[
\text{MP} : \frac{\phi \land \chi \phi}{\chi} \quad \text{nec} : \frac{\top \phi}{\vdash \phi \approx \top}
\]

Let us consider the modal axioms. (\(A0\)) allows for substitutions of ‘believably equivalent’ formulas. Other axioms correspond to the conditions on measures we cited in the beginning of the section. In particular, (\(A1\)) corresponds to \(Q1\); (\(A2\)) is the linearity condition; (\(A3\)) corresponds to \(Q2\). Finally, the family of axioms \((A4)_m\) corresponds to the \(KPS_m\). As expected [19,
3.3 Logics of qualitative probabilities

P.179], the expansion of the classical propositional logic with these axioms is complete w.r.t. all Gärdenfors' probabilistic frames.\(^8\)

Note that \( \mathcal{L}_{QP} \) does allow for the nesting of \( \leq \) while \( \mathcal{L}_{QG} \) prohibits the nesting of \( B \). However, a specific fragment of \( \mathcal{L}_{QP} \) can be embedded into \( \mathcal{L}_{QG} \).

**Definition 3.10** (Embedding of simple inequality formulas). We define simple inequality formulas (SIF's) using the following grammar (\( \chi \) and \( \chi' \) do not contain \( \leq \)):

\[
SIF \ni \phi := \chi \leq \chi' \mid \neg \phi \mid (\phi \land \phi) \mid (\phi \lor \phi) \mid (\phi \rightarrow \phi)
\]

We define a translation \( ^\Delta \) of SIF's into \( \mathcal{L}_{QPG} \) as follows.

\[
\begin{align*}
(\chi \leq \chi')^\Delta &= \Delta(B\chi \rightarrow G B\chi') \\
(\neg \phi)^\Delta &= \neg_G \phi^\Delta \\
(\phi \land \phi')^\Delta &= \phi^\Delta \land \phi'^\Delta \\
(\phi \lor \phi')^\Delta &= \phi^\Delta \lor \phi'^\Delta
\end{align*}
\]

**Remark 3.1.** Note, however, that not all statements about comparing beliefs can be represented as SIF's and their translations into \( \mathcal{L}_{QG} \). Indeed, \( \text{disj} \) is not a translation of a SIF. In fact, \( \sim \sim \sim \sim \) \( Bp \) stipulates that the agent’s belief in \( p \) is positive. In QP, it can only be expressed as \( p > \perp \). However, there are models of QP where the certainty of \( \perp \) is positive. Thus, \( p > \perp \) is stronger than \( \sim G \sim Bp \) for the latter is compatible with \( \Delta(Bp \rightarrow B\perp) \).

In what follows, we will say that a QG model \( \mathcal{M} = \langle W, v, \mu, e \rangle \) is a QPG model if \( \mu \) satisfies \( \mu \text{KPS}_m \). In other words, in a QPG model, the order on \( B^W \) induced by \( \mu \) is a qualitative counterpart of a probability measure.

We can now establish that the translation in definition 3.10 is indeed faithful. We do this by showing how to transform a given pointed Gärdenfors model \( \langle \mathcal{M}, x \rangle \) into a QPG model that satisfies exactly the translations of SIF's that \( \langle \mathcal{M}, x \rangle \) satisfies. And conversely, how to provide a Gärdenfors model using a given QPG model preserving all satisfied SIF's.

**Definition 3.11** (G-counterparts). Let \( \mathcal{M} = \langle U, \{ P_x \}_{x \in U}, v \rangle \) be a Gärdenfors model. A G-counterpart of a pointed model \( \langle \mathcal{M}, x \rangle \) is the QPG model \( \mathcal{M}_{QPG} = \langle U, v, P_x, e \rangle \).

**Remark 3.2.** Note first of all, that a \( \Delta \)-less translation of \( \chi \leq \chi' \) as \( B\chi \rightarrow G B\chi' \) does not preserve truth. Indeed, let \( P_x(||p||) = 0.7, P_x(||q||) = 0.6, P_x(||r||) = 0.5, P_x(||s||) = 0.4 \). Then \( (p \leq q) \lor (r \leq s) \) is true at \( x \) but \( e(B\chi \rightarrow G Bq) \rightarrow G (Br \rightarrow G Bs) = 0.4 \).

**Lemma 3.12.** Let \( \langle \mathcal{M}, x \rangle \) be a pointed Gärdenfors model and \( \mathcal{M}_{QPG} \) its G-counterpart, then \( \mathcal{M}, x \models \phi \) iff \( e(\phi^\Delta) = 1 \) for any \( \phi \in SIF \).

**Proof.** First, it is clear that for any classical formula \( \chi \), it holds that \( ||\chi|| = v(\chi) \) since \( \mathcal{M} \) and \( \mathcal{M}_{G} \) have the same valuation.

We proceed by the induction on \( \phi \). First, let \( \phi := (\chi \leq \chi') \).

\[
\mathcal{M}, x \models \chi \leq \chi' \text{ iff } P_x(||\chi||) \leq P_x(||\chi'||) \\
\quad \text{iff } P_x(v(\chi)) \leq P_x(v(\chi')) \\
\quad \text{iff } e(B\chi \rightarrow G B\chi') = 1
\]

\[
\quad \text{iff } e(\Delta(B\chi \rightarrow G B\chi')) = 1
\]

\(^8\)Gärdenfors' axiomatization is based on Segerberg's approach in [37]. There are several attempts in recent literature, e.g. Delgrande and Renne [11, 12] provide finite axiomatisation, which however relies on a modal operator which is interpreted in terms of quantitative probability. Baldi and Hosni in [2] based their approach on approximations of qualitative probability due to Savage.
For the inductive step, we consider $\phi := \phi_1 \land \phi_2$ and $\phi := \lnot \phi'$.

\[
\mathcal{M}, x \vDash \phi_1 \land \phi_2 \text{ iff } \mathcal{M}, x \vDash \phi_1 \text{ and } \mathcal{M}, x \vDash \phi_2
\]

if $e(\phi_1^{\triangle})$ and $e(\phi_2^{\triangle}) = 1$ \hspace{1cm} (by IH)

if $e(\phi_1^{\triangle} \land \phi_2^{\triangle}) = 1$

\[
\mathcal{M}, x \vDash \lnot \phi' \text{ iff } \mathcal{M}, x \nvDash \phi'
\]

if $e(\phi'^{\triangle}) \neq 1$ \hspace{1cm} (by IH)

if $e(\phi'^{\triangle}) = 0$ \hspace{1cm} ($\phi'^{\triangle}$ is a Boolean combination of $\triangle$-formulas)

if $e((\lnot \phi')^{\triangle}) = 1$

\[
\mathcal{M}, x \vDash \phi \text{ iff } \mathcal{M}, x \vDash \phi
\]

Lemma 3.15. Let $\mathcal{M} = (W, v, \mu, e)$ be a QPG model. Its QPG-counterpart is a QP pointed model $(\mathcal{M}, w)$ with $w \in W$ s.t. $\mathcal{M}_G = (W, v, \pi_{\mu_x})_{x \in W}$ and $\pi_{\mu_x}$ is a probability measure s.t. $\mu(X) \leq \mu(Y)$ if $\pi_{\mu_x}(X) \leq \pi_{\mu_x}(Y)$ for all $X, Y \subseteq W$.

Proposition 3.14. For any QPG model, there exists its QP counterpart.

Proof. Note that $\mu$ conforms to Kraft–Pratt–Seidenberg conditions [27, 36]. Thus, there is a probability measure on the same set that preserves all orders from $\mu$.

Definition 3.13 (QP-counterparts). Let $\mathcal{M} = (W, v, \mu, e)$ be a QPG model. Its QP-counterpart is a QP pointed model $(\mathcal{M}, w)$ with $w \in W$ s.t. $\mathcal{M}_G = (W, v, \{\pi_{\mu_x}\}_{x \in W})$ and $\pi_{\mu_x}$ is a probability measure s.t. $\mu(X) \leq \mu(Y)$ if $\pi_{\mu_x}(X) \leq \pi_{\mu_x}(Y)$ for all $X, Y \subseteq W$.

Lemma 3.15. Let $\mathcal{M} = (W, v, \mu, e)$ be a QPG model and $(\mathcal{M}_G, w)$ its counterpart. Then, $e(\phi^{\triangle}) = 1$ iff $\mathcal{M}_G, w \vDash \phi$ for any $\phi \in \text{SIF}$.

Proof. Let $e(\phi^{\triangle}(s_1^{\triangle}, \ldots, s_n^{\triangle})) = 1$ with $s_i = \chi_i \preceq \chi_i'$ and $s_i^{\triangle} = \triangle(B_{\chi_i} \rightarrow_G B_{\chi_i'})$. Since the measure on the QP counterpart preserves all order relations from $\mathcal{M}$, it is clear that $\mathcal{M}, w \vDash s_i$ if $e(s_i) = 1$ for all $i \leq n$. But then we have that $e(\phi^{\triangle}) = 1$ iff $\mathcal{M}_G, w \vDash \phi$ since $e(s_i^{\triangle}) \in \{0, 1\}$ and Gödel connectives behave classically on values 0 and 1.

Theorem 3.16. Let $\phi \in \text{SIF}$. Then $\phi$ is QP valid iff $\phi^{\triangle}$ is QPG valid.

Proof. Immediately from lemmas 3.12 and 3.15.

Now, observe that if we instantiate $\phi_i$‘s and $\psi_i$‘s in (A4)$_m$ with propositional formulas, these formulas are going to be SIF’s. This means that to obtain the axiomatisation of the logic complete w.r.t. QPG frames, we only need to translate (A4)$_m$ into $\mathcal{L}_{\mathcal{G}}$.

Convention 3.3 (E-notation for QPG). Consider $\mathcal{L}_{\mathcal{CPL}}$ formulas $\phi_1, \ldots, \phi_n$ and $\chi_1, \ldots, \chi_n$. Let further, $\phi^\circ \in \{\phi, \lnot \phi\}$ and $\chi^\circ \in \{\chi, \lnot \chi\}$. We introduce operator $E_{\mathcal{G}}$ and write

$$\phi_1, \ldots, \phi_n E_{\mathcal{G}} \chi_1, \ldots, \chi_n$$

to designate that necessarily the same number of $\phi_i^\circ$‘s as of $\chi_i^\circ$‘s are actually of the form $\lnot \phi_i$ and $\lnot \chi_j$, respectively.

More formally, we let $M = \{1, \ldots, m\}$, $K, L \subseteq M$, and set

$$\phi_1, \ldots, \phi_m E_{\mathcal{G}} \chi_1, \ldots, \chi_m := \bigtriangleup \left( \bigvee_{i=0}^m \bigvee_{|K|=i} \left( \bigwedge_{k \in K} \lnot \phi_k \land \bigwedge_{k \in M \setminus K} \phi_k' \land \bigwedge_{l \in L} \lnot \chi_l \land \bigwedge_{l \in M \setminus L} \chi_l' \right) \right) \leftrightarrow_G B^T$$
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\( E_G \) has the same intended meaning as \( E \). Namely, that the measure of

\[
\bigvee_{i=0}^{m} \bigvee_{|K|=i} \left( \bigwedge_{k \in K} \neg \phi_k \land \bigwedge_{k' \in M \setminus K} \phi_{k'} \land \bigwedge_{l \in L} \neg \chi_l \land \bigwedge_{\nu \in M \setminus L} \chi_{\nu} \right)
\]

is equal to the measure of \( \top \).

Finally, we define

\[ KPS_m: (\phi_0, \ldots, \phi_m E_G \chi_0, \ldots, \chi_m) \land \bigwedge_{i=0}^{m-1} \triangle(B\phi_i \rightarrow_G B\chi_i) \rightarrow_G \triangle(B\chi_m \rightarrow_G B\phi_m). \]

The next statement is a straightforward corollary from theorem 3.16.

**Theorem 3.17.** Let \( KPS = \{KPS_m : m \in \mathbb{N}\} \) and \( F = (W, \mu) \). Then \( F \models KPS \iff \mu \) satisfies Kraft–Pratt-Seidenberg conditions. Furthermore, \( \mathcal{HGQ} \oplus KPS \) is the logic of QPG frames.

Note that \( \mathcal{HGQ} \) has A0 as an axiom while \( \mathcal{HQP} \) lacks it. In fact, we can prove \((A0)\Delta \) (i.e., the translation of A0 for SIF's) in \( \mathcal{HQP} \) as follows.

1. \( \triangle(B(\phi_1 \equiv \phi_2) \leftrightarrow_G B\top) \) (alternatively written, \( \phi_1 E_G \phi_2 \))
2. \( \triangle(B(\psi_1 \equiv \psi_2) \leftrightarrow_G B\top) \) (alternatively written, \( \psi_1 E_G \psi_2 \))
3. \( \triangle(B\phi_1 \rightarrow_G B\phi_2) \) — from 1 via KPS
4. \( \triangle(B\phi_2 \rightarrow_G B\phi_1) \) — from 1 via KPS
5. \( \triangle(B\phi_1 \leftrightarrow_G B\phi_2) \) — from 3 and 4 in biG
6. \( \triangle(B\psi_1 \rightarrow_G B\psi_2) \) — from 2 via KPS
7. \( \triangle(B\psi_2 \rightarrow_G B\psi_1) \) — from 2 via KPS
8. \( \triangle(B\psi_1 \leftrightarrow_G B\psi_2) \) — from 6 and 7 in biG
9. \( \triangle(B\phi_1 \rightarrow_G B\psi_1) \leftrightarrow_G \triangle(B\phi_2 \rightarrow_G B\psi_2) \) — from 5 and 8 in biG

Note that we do not need to use \( \text{reg} \) in the above proof.

Furthermore, observe that \( \mathcal{HQP} \vdash \phi \preceq \chi \) if \( \text{CPL} \vdash \phi \supset \chi \). So, we expect \( \text{reg} \) to be derivable from \( KPS_m \) in \( \mathcal{HQP} \). However, an application of \( \text{nec} \) is required in the \( \mathcal{HQP} \) proof. Instead, we will use two axiom schemes \( \text{cap}' \): \( B\phi \) and \( \neg \exists B\chi \) for any \( \phi \) and \( \chi \) s.t. \( \text{CPL} \vdash \phi \) and \( \text{CPL} \vdash \neg \chi \). We denote the resulting calculus \( \mathcal{HQP}^\top \) and prove \( \text{reg} \) from \( KPS_m \) and \( \text{cap}' \) in two steps. First, we show that a qualitative counterpart of additivity holds. Then, we use additivity and \((A0)\Delta \) to prove \( \text{reg} \).

For additivity, we show that \( \triangle(B(\phi \lor \psi) \rightarrow_G B(\chi \lor \psi)) \) follows from \( \triangle(B\phi \rightarrow_G B\chi) \), \( \text{CPL} \vdash \neg(\phi \land \psi) \), and \( \text{CPL} \vdash \neg(\chi \lor \psi) \).

1. \( \phi, (\chi \lor \psi) E_G \chi, (\phi \lor \psi) \) — from CPL using \( \text{cap} \)
2. \( \triangle(B\phi \rightarrow_G B\chi) \) — assumption
3. \( (\phi, (\chi \lor \psi) E_G \chi, (\phi \lor \psi)) \land \triangle(B\phi \rightarrow_G B\chi) \rightarrow_G \triangle(B(\phi \lor \psi) \rightarrow_G B(\chi \lor \psi)) \) — an instance of \( KPS \)
4. \( \triangle(B(\phi \lor \psi) \rightarrow_G B(\chi \lor \psi)) \) — from 1–3.
Now, note that \( \text{cap'} \) derives \( B\chi \rightarrow G B\phi \) for any \( \text{CPL} \vdash \sim \chi \) since \( \sim G B\chi \) is a theorem. We show that if \( \text{CPL} \vdash \phi \supset \chi \), then \( \mathcal{HQPG} \vdash B\phi \rightarrow G B\chi \).

1. \( \Box (B \bot \rightarrow G B(\sim \phi \land \chi)) \) — from \( \text{cap} \) since \( \sim G B \bot \) is a theorem.
2. \( \Box (B(\bot \lor \phi) \rightarrow G B(\phi \lor (\sim \phi \land \chi))) \) from 1 by additivity.
3. \( (\bot \lor \phi)E_G \phi \) — by \( \text{cap} \) since \( \text{CPL} \vdash (\phi \lor \bot) \equiv \phi \)
4. \( (\phi \lor (\sim \phi \land \chi))E_G \chi \) — by \( \text{cap} \) since \( \text{CPL} \vdash (\phi \lor (\sim \phi \land \chi)) \equiv \chi \) if \( \text{CPL} \vdash \phi \supset \chi \)
5. \( \Box (B\phi \rightarrow G B\chi) \) — from 3 and 4 by \( (A0)^\Delta \)
6. \( B\phi \rightarrow G B\chi \) — from 5 in \( \text{biG} \)

4 Paraconsistent logics of monotone comparative belief

As we have discussed in the previous section, it is not necessary that an agent be able to assign a number to their certainty in a given statement. Furthermore, it is possible that the evidence regarding a given statement is contradictory or incomplete, whence if we want to compare our certainty in different statements, we need to treat evidence in favour and evidence against independently. Consider, e.g., the following situation: all sources give contradictory information regarding \( \phi \) but no information regarding \( \chi \), both our certainty that \( \phi \) is true and our certainty in its falsity should be greater than our certainty in either truth or falsity of \( \chi \).

These characteristics of evidence can be illustrated in the context of court proceedings. Indeed, the evidence in court has the features listed above: it is rare that one can reliably measure one’s certainty in any given piece thereof, instead, the court tries to establish whose claims are more compelling; the evidence presented by witnesses can be incomplete or inconsistent; in any court proceeding there are two parties, and the non-contradictory evidence can be treated as favouring one of them (or irrelevant to the process).

This is why we analyse these contexts using paraconsistent Gödel logics introduced in [5] on the outer layer (while \( \text{BD} \) is the inner-layer logic). Since Gödel logic can be seen as a logic of comparative truth, its paraconsistent expansion with a De Morgan negation \( \sim \) can be seen as a logic of comparative truth and falsity.

We will treat each atomic modal formula \( C\phi \) (read ‘the agent is certain in the truth of \( \phi \)’) as a piece of evidence. The first coordinate supports one party, and the second the other. Pieces of evidence can be combined in different fashions: we can compare our certainty therein using (co-)implication; choose the more or less certain one with \( \lor \) and \( \land \), etc. Gödelian negation represents the countering of a given statement — we show that it is absurd. Finally, \( \sim \) is the operator that swaps support of truth and support of falsity. But in the context of a court session, if a statement is used as an argument for one party, then its negation is actually an argument for the other. Thus, we posit that \( \sim C\phi \) is equivalent to \( C \sim \phi \).

Furthermore, there is a difference between criminal and civil proceedings (as well as arbitrations). Namely, during a criminal proceeding, a defendant is pronounced innocent as long as they were able to present conclusive evidence in their favour or counter the evidence of the prosecution. Furthermore, contradictions are usually interpreted in favour of the defendant. Thus, the two parties in the proceeding are not equal in this respect. On the other hand, both parties in a civil court (say, two relatives settle an inheritance dispute in court) or an arbitration present evidence in their own favour, after which the court determines whose evidence was more compelling.
This difference can be formalised if we recall two \( \mathcal{G}^2 \) logics and their entailments. \( \mathcal{G}^2(\rightarrow, \prec) \) takes into account both coordinates of a given valuation which makes it closer to the reasoning demonstrated in a civil process. On the other hand, \( \mathcal{G}^2(\rightarrow, \sim) \) takes into account only the first coordinate. Thus, we can associate each coordinate to a party of the process (for \( \mathcal{G}^2(\rightarrow, \sim) \), the first coordinate stands for the defence, and the second one for the prosecution).

Now, if \( \Gamma \cup \{ \phi \} \) is a set of statements concerning some evidence, then entailment relations can be interpreted as preservation of the degree of certainty from premises to the conclusion. We will call \( \mathcal{C} \) modality of ‘monotone comparative belief’. Here, ‘monotone’ means that \( \mathcal{C} \) conforms to the underlying \( \mathcal{BD} \) entailment in the sense that if \( \phi \vdash \chi \) is valid in \( \mathcal{BD} \), \( \mathcal{C} \phi \) implies \( \mathcal{C} \chi \) in the outer-layer logic. ‘Comparative’ relates to the fact that we use Gödel logic which can be thought of as a logic of comparative truth.

4.1 Language and semantics

In this section, we provide two logics of monotone comparative belief based on \( \mathcal{G}^2 \): \( \mathcal{MCB} = (\mathcal{BD}, \mathcal{C}, \mathcal{G}^2(\rightarrow, \sim)) \) (stands for ‘monotone comparative belief’) and the Nelson-like \( \mathcal{MCB} \) designated \( \mathcal{NMCB} = (\mathcal{BD}, \mathcal{C}, \mathcal{G}^2(\rightarrow, \sim)) \).

**Definition 4.1 (Languages of MCB and NMCB).** The languages of MCB and NMCB (\( \mathcal{L}_{\mathcal{MCB}} \) and \( \mathcal{L}_{\mathcal{NMCB}} \), respectively) are defined via the following grammars.

\[
\mathcal{L}_{\mathcal{MCB}} : \alpha ::= \mathcal{C} \phi \mid \neg \alpha \mid \alpha \circ \alpha \ (\circ \in \{\land, \lor, \rightarrow, \sim\}, \phi \in \mathcal{L}_{\mathcal{BD}})
\]

\[
\mathcal{L}_{\mathcal{NMCB}} : \alpha ::= \mathcal{C} \phi \mid \neg \alpha \mid \alpha \circ \alpha \ (\circ \in \{\land, \lor, \rightarrow, \sim\}, \phi \in \mathcal{L}_{\mathcal{BD}})
\]

**Definition 4.2 (Semantics of MCB and NMCB).** An MCB (NMCB) Kripke model is a tuple \( \mathcal{M} = (W, v^+, v^-, \pi, e_1, e_2) \) with \( (W, v^+, v^-) \) being a BD model (cf. definition 2.5), \( \pi : 2^W \rightarrow [0, 1] \) being an uncertainty measure.

Semantic conditions of atomic \( \mathcal{L}_{\mathcal{MCB}} \) and \( \mathcal{L}_{\mathcal{NMCB}} \) formulas are as follows.

\[
e_1(\mathcal{C} \phi) = \pi \left( \{w : w \models^+ \phi\} \right) = \pi(\{\phi^+\})
\]

\[
e_2(\mathcal{C} \phi) = \pi \left( \{w : w \models^- \phi\} \right) = \pi(\{\phi^-\})
\]

Values of complex formulas are computed according to definition 2.7.

For a frame \( F = (W, \pi) \) on an MCB (NMCB) model \( \mathcal{M} \), we say that \( \alpha \in \mathcal{L}_{\mathcal{MCB}} \) (\( \beta \in \mathcal{L}_{\mathcal{NMCB}} \)) is valid on \( F \) (\( F \models \alpha \) and \( F \models \beta \), respectively) iff

Finally, for \( \Psi \cup \{\alpha\} \subseteq \mathcal{L}_{\mathcal{MCB}} \) and \( \Omega \cup \{\beta\} \subseteq \mathcal{L}_{\mathcal{NMCB}} \), we define the same entailment relations as in definition 2.8.

**Convention 4.1.** In what follows, we will use \( \rightarrow \) and \( \leftrightarrow \) — congruential versions of \( \rightarrow \) and \( \leftrightarrow \) — defined as follows:

\[
\alpha \rightarrow \beta \overset{\text{def}}{=} (\alpha \rightarrow \beta) \land (\neg \beta \rightarrow \neg \alpha)
\]

\[
\alpha \leftrightarrow \beta \overset{\text{def}}{=} (\alpha \rightarrow \beta) \land (\beta \rightarrow \alpha)
\]

One can check that

\[
e_1(\alpha \rightarrow \beta) = 1 \text{ iff } e_1(\alpha) \leq e_1(\beta) \text{ and } e_2(\alpha) \geq e_2(\beta)
\]

\[
e_1(\alpha \leftrightarrow \beta) = 1 \text{ iff } e_1(\alpha) = e_1(\beta) \text{ and } e_2(\alpha) = e_2(\beta)
\]
4.2 Axiomatisation

Note that there is a very important difference between MCB and NMCB on one hand and QPG on the other hand. Namely, in QPG an agent can compare their beliefs in any two given statements. This, however, is not the case in MCB and NMCB.

To see this, we define two operators for \( \alpha \in L_{\text{MCB}} \) and \( \beta \in L_{\text{NMCB}} \).

\[
\Delta^1 \alpha := \sim_0 (\top_1 \land \alpha) \land \sim_0 \sim_0 (\top_1 \land \alpha) \quad \Delta^N \beta := \sim_N (\top_N \land \beta)
\]

One can check that

\[
e(\Delta^1 \alpha) = \begin{cases} (1, 0) & \text{iff } e(\alpha) = (1, 0) \\ (0, 1) & \text{otherwise} \end{cases} \quad e_1(\Delta^N \beta) = \begin{cases} 1 & \text{iff } e_1(\beta) = 1 \\ 0 & \text{otherwise} \end{cases}
\]

Thus, in contrast to \( \Delta(\alpha \rightarrow_G \alpha') \lor \Delta(\alpha' \rightarrow_G \alpha) \), neither \( \Delta^1(\alpha \rightarrow \alpha') \lor \Delta^1(\alpha' \rightarrow \alpha) \) nor \( \Delta_N(\alpha \rightarrow \alpha') \lor \Delta_N(\alpha' \rightarrow \alpha) \) are valid.

Intuitively, this failure of comparability is justified. First, \( \alpha \) and \( \alpha' \) can be irrelevant to one another. Indeed, we cannot always answer conclusively what we consider more likely: that it will rain tomorrow or that we will find our lost dog. Second, even if the events are related, we are not necessarily able to compare our confidence in them when the evidence is of different nature.

Recall the situation of Paula and Quinn claiming that the dog is theirs. Assume now that Paula shows a photo of her with the dog on the leash and Quinn shows the (same or at least very similar) leash. Neither piece of evidence is conclusive and without further investigation, it might not be clear whether one is stronger than the other.

Third, if the events are described classically (as done in QP, QG, and QPG), then all contradictory events have measure 0 (or the least possible positive measure). However, if an agent tries to align their beliefs with what they are told by their sources, this is not necessarily the case. Indeed, if I do not have any information at all regarding \( p \), then \( \pi([p]^+) = 0 \) and \( \pi([p]^-) = 0 \), whence \( e(\mathcal{C}(p \land \neg p)) = (0, 0) \). On the other hand, if I have somewhat reliable sources claiming that \( q \) is true and some others (less trusted ones) that it is false, then I can posit that \( \pi([q]^+) = 0.5 \) and \( \pi([q]^-) = 0.3 \), whence \( e(\mathcal{C}(q \land \neg q)) = (0.3, 0.5) \). But then my certainty in \( p \land \neg p \) is incomparable to that in \( q \land \neg q \). Finally, if I know for certain that \( r \) is true (i.e., \( \pi([r]^+) = 1 \) and \( \pi([r]^-) = 0 \)), then \( e(\mathcal{C}(r \land \neg r)) = (0, 1) \). Thus my certainty in \( r \land \neg r \) is strictly below that in both \( p \land \neg p \) and \( q \land \neg q \).

4.2 Axiomatisation

Let us now introduce Hilbert-style calculi for MCB and NMCB.

**Definition 4.3 (HMCB).** The calculus consists of the following axioms and rules (\( \phi, \chi \in L_{\text{BD}} \) and \( \alpha, \beta \in L_{\text{MCB}} \)).

\( \text{HMCB}_{\text{BD}} \)
- \( \mathcal{C}\phi \rightarrow \mathcal{C}\chi \) for any \( \phi, \chi \in L_{\text{BD}} \) s.t. \( \phi \vdash \chi \) is BD valid.

\( \text{HMCB}_{\text{MCB}} \)
- \( \mathcal{C}\neg\phi \leftrightarrow \neg\mathcal{C}\phi \).

\( \text{G}^2(\rightarrow, \land) \) all theorems and rules of \( \mathcal{H}G^2(\rightarrow, \land) \) instantiated with MCB formulas.

**Definition 4.4 (H(NMCB)).** The calculus consists of the following axioms and rules (below, \( \phi, \chi \in L_{\text{BD}} \) and \( \alpha, \beta \in L_{\text{NMCB}} \)).

\( \text{\[\text{Recall, that both in } bG \text{ and } G^2, \text{ the exact numbers assigned to our certainty in a given event are of little importance. What matters is that (in this case) I have some information that suggests that } q \text{ is true and some that it is false.}\]}

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The logics of monotone comparative belief provided in the previous subsection were, in a sense, minimal. It is thus instructive to consider their extensions with additional axioms corresponding to additional conditions imposed on \( \pi \).

First, observe that since BD lacks tautologies and universally false formulas, cap does not have any analogues in MCB, nor in NMCB. In fact, one can prove theorem 4.5 even without requiring that \( \pi(W) = 1 \) and \( \pi(\emptyset) = 0 \). This shows that the truth and falsity of MCB and NMCB formulas depend only on the order relations between different uncertainty measures of \( \pi \).

\(^{10}\)Note, however, that in this case, \( \pi \) is not going to be a measure.
different events, not on the values of these measures. However, in contrast to QP, it is not problematic: if one describes events using BD, there is no event whose uncertainty measure one could know a priori just by its description. This, again, is in line with that MCB and NMCB formalise reasoning with uncertainty when the agent tries to build their beliefs using only the information provided by their sources as we discussed in section 4.1. Furthermore, since \(\neg\) is not related to the set-theoretic complement, KPS\(_m\) axioms cannot be meaningfully translated either.

Still, we may assert that two events \(\phi\) and \(\phi'\) are incompatible if \(\pi(||\phi \land \phi'||) = 0\) and \(\pi(||\phi \land \phi'|\neg) = 1\). Indeed, this statement corresponds to \(\sim_0C(\phi \land \phi')\) having value \((1,0)\) or to the formula \(\Delta^1\sim_0C(\phi \land \phi')\). To express incompatibility in NMCB, we define the following connective:

\[
\triangleleft_\alpha := \Delta^N(1 \rightarrow \alpha)
\]

Now \(\triangleleft_\alpha\) can be used to express that the agent is completely certain in \(\phi\) as follows: \(\triangleleft_\alpha \land C(\phi \land \phi')\). It is clear that

\[
e_1(\triangleleft_\alpha) = \begin{cases} 1 & \text{iff } e(\alpha) = (1,0) \\ 0 & \text{otherwise} \end{cases}
\]

Thus, \(\triangleleft_\alpha\) corresponds to the incompatibility of \(\phi\) and \(\phi'\) in NMCB.

The next statement establishes some correspondence results for MCB and NMCB.

**Convention 4.2.** We introduce the following naming conventions.

\[
\text{disj}+: \Delta^1\sim_0C(p \land p') \land \sim_0\Delta^1\sim_0Cp \land \sim_0\Delta^1\sim_0Cp' \rightarrow (\sim_0\Delta^1(C(p \lor p') \rightarrow Cp) \land \sim_0\Delta^1(C(p \lor p') \rightarrow Cp'))
\]

\[
\text{disj}0^-: \Delta^1\sim_0Cp \rightarrow \Delta^1(Cp' \leftrightarrow C(p \lor p'))
\]

\[
\text{disj}+^N: \Delta^N\sim_0C(p \land p') \land \sim_0\Delta^N\sim_0Cp \land \sim_0\Delta^N\sim_0Cp' \rightarrow (\sim_0\Delta^N(C(p \lor p') \rightarrow Cp) \land \sim_0\Delta^N(C(p \lor p') \rightarrow Cp'))
\]

\[
\text{disj}0^N: \Delta^N\sim_0Cp \rightarrow \Delta^N(Cp' \leftrightarrow C(p \lor p'))
\]

**Theorem 4.6.** Let \(F = (\mathcal{W}, \pi)\) be a frame. Then the following statements hold.

\(F \models \text{disj}+^-\) if \(\pi(X \land X') = 0\), \(\pi(Y \lor Y') = 1\), \(\pi(X), \pi(X') \neq 0\), \(\pi(Y), \pi(Y') \neq 1\)

\[
\Rightarrow \pi(X \land X') > \pi(X), \pi(X') \lor \pi(Y \land Y') < \pi(Y), \pi(Y')
\]

(I)

\(F \models \text{disj}0^-\) if \(\pi(X) = 0\) and \(\pi(Y) = 1\) \(\Rightarrow \pi(X \land X') = \pi(X')\) and \(\pi(Y \land Y') = \pi(Y')\) (II)

\(F \models \text{disj}+^N\) if \(Y \land Y' = \emptyset\) and \(\mu(Y), \pi(Y') > 0\) \(\Rightarrow \pi(Y \lor Y') > \pi(Y), \pi(Y')\) (III)

\(F \models \text{disj}0^N\) if \(\pi(Y) = 0\) \(\Rightarrow \pi(Y \lor Y') = \pi(Y')\) (IV)

**Proof.** Analogously to theorem 3.5. □

The formulas from the previous theorem are paraconsistent analogues of those from theorem 3.5. Note, first of all, that we do not translate 1compl. It tells that if an agent is completely certain in some statement, then they should completely disbelieve its negation. In a paraconsistent setting, however, it might be the case that all sources provide contradictory information about \(p\), whence this principle is not justified.

Moreover, there is a considerable difference in the expressivity of MCB and NMCB. The former takes into account both support of truth and support of falsity, while the latter only the support of truth. It means that the properties of the uncertainty measures that can be axiomatised using MCB are considerably weaker than those axiomatised in QPG or NMCB because every outer-layer formula corresponds not to one but two subsets of the carrier.

---

\(^{11}\)One can, however express in MCB and NMCB that the agent is completely certain in a given statement.
5 Conclusion

In this paper, we considered the two-layered logic of qualitative probabilities QPG and its paraconsistent expansions MCB and NMCB that formalise reasoning with uncertain, contradictory, or inconclusive information. For these logics, we provided Hilbert-style axiomatisations and proved their completeness. We showed that the SIF fragment of Garšenfors’ logic of qualitative probabilities QP can be faithfully embedded into QPG. We also defined generalisations of both QPG and its paraconsistent expansions and proved some results regarding the correspondence between properties of uncertainty measures and two-layered modal formulas.

Still, several questions remain open. First and foremost, Wong conditions and Kraft–Pratt–Seidenberg conditions we gave in section 3 show which uncertainty measures can be extended to a belief function or a probability measure, respectively, while preserving all orders. In [26] and [6], paraconsistent counterparts of probabilities and belief functions were provided. A natural next step would be to establish their qualitative characterisations and provide a two-layered axiomatisation with $G^2$ as the outer logic.

In [7, 26], paraconsistent counterparts of belief functions and probability measures were introduced under the titles of ‘non-standard belief functions’ and ‘non-standard probabilities’. Neither MCB nor NMCB, however, have non-standard belief functions or probabilities as their belief measures. This is why it would be instructive to provide a similar characterisation of non-standard belief functions and probabilities and develop a logic of non-standard qualitative probabilities.

Second, we established a faithful translation of SIF’s into $L_{QG}$. A natural question is whether there are any other classes of $L_{QP}$ formulas that can be faithfully translated into $L_{QG}$. The most evident candidates would be formulas without nesting of $\preceq$ and formulas where each variable is in the scope of $\preceq$.

Finally, QP has finite model property and is decidable. Since proofs in QG and its expansions, as well as those in MCB and NMCB, are just derivations from assumptions in $biG$ and $G^2$, QG, MCB, and NMCB and their expansions are decidable as well. Thus, a straightforward and explicit decision procedure along with the complexity evaluation would be desirable to obtain.

References

[1] P. Baldi, P. Cintula, and C. Noguera. Classical and fuzzy two-layered modal logics for uncertainty: Translations and proof-theory. International Journal of Computational Intelligence Systems, 13(1):988–1001, 2020.

[2] P. Baldi and H. Hosni. Logical approximations of qualitative probability. In A. Cano, J. de Bock, E. Miranda, and S. Moral, editors, Proceedings of the Twelfth International Symposium on Imprecise Probability: Theories and Applications, volume 147 of Proceedings of Machine Learning Research, pages 12–21. PMLR, 06–09 Jul 2021.

[3] N.D. Belnap. A Useful Four-Valued Logic. In J. Michael Dunn and George Epstein, editors, Modern Uses of Multiple-Valued Logic, pages 5–37, Dordrecht, 1977. Springer Netherlands.

[4] N.D. Belnap. How a computer should think. In H. Omori and H. Wansing, editors, New Essays on Belnap-Dunn Logic, volume 418 of Synthese Library (Studies in Epistemology, Logic, Methodology, and Philosophy of Science). Springer, Cham, 2019.

[5] M. Bilková, S. Frittella, and D. Kožemiachenko. Constraint tableaux for two-dimensional fuzzy logics. In Automated Reasoning with Analytic Tableaux and Related Methods.
 REFERENCES

*TABLEAUX 2021*, volume 12842 of *Lecture notes in computer science*, pages 20–37. Springer International Publishing, Cham, 2021.

[6] M. Bílková, S. Frittella, D. Kozhemicheenko, O. Majer, and S. Nazari. Reasoning with belief functions over Belnap–Dunn logic. https://arxiv.org/abs/2203.01060, 2022.

[7] M. Bílková, S. Frittella, O. Majer, and S. Nazari. Belief based on inconsistent information. In *International Workshop on Dynamic Logic*, pages 68–86. Springer, 2020.

[8] P. Cintula, C.G. Fermüller, and C. Noguera. *Handbook of Mathematical Fuzzy Logic 3*. Studies in Logic 58. College Publications, 2015.

[9] P. Cintula, P. Hájek, and C. Noguera. *Handbook of Mathematical Fuzzy Logic 1*. Studies in Logic 37. College Publications, 2011.

[10] P. Cintula, P. Hájek, and C. Noguera. *Handbook of Mathematical Fuzzy Logic 2*. Studies in Logic 38. College Publications, 2011.

[11] J. Delgrande and B. Renne. The logic of qualitative probability. In *Twenty-Fourth International Joint Conference on Artificial Intelligence*, 2015.

[12] J.P. Delgrande, B. Renne, and J. Sack. The logic of qualitative probability. *Artificial Intelligence*, 275:457–486, October 2019.

[13] D. Dubois, H. Prade, and S. Schockaert. Generalized possibilistic logic: Foundations and applications to qualitative reasoning about uncertainty. *Artif. Intell.*, 252:139–174, November 2017.

[14] J.M. Dunn. Intuitive semantics for first-degree entailments and ‘coupled trees’. *Philosophical Studies*, 29(3):149–168, 1976.

[15] J.M. Dunn. Partiality and Its Dual. *Studia Logica*, 66(1):5–40, Oct 2000.

[16] R. Fagin, J. Y. Halpern, and N. Megiddo. A logic for reasoning about probabilities. *Information and Computation*, 87:78–128, 1990.

[17] R. Fagin and J.Y. Halpern. Uncertainty, belief, and probability. *Computational Intelligence*, 7(3):160–173, August 1991.

[18] B. de Finetti. La prévision: ses lois logiques, ses sources subjectives. In *Annales de l’institut Henri Poincaré*, volume 7, pages 1–68, 1937.

[19] P. Gärdenfors. Qualitative probability as an intensional logic. *Journal of Philosophical Logic*, pages 171–185, 1975.

[20] L. Godo, P. Hájek, and F. Esteva. A fuzzy modal logic for belief functions. *Fundamenta Informaticae*, 57(2–4):127–146, 2003.

[21] M. Grabisch. Belief functions on lattices. *International Journal of Intelligent Systems*, 24(1):76–95, January 2009.

[22] R. Grigolia, T. Kiseliova, and V. Odisharia. Free and projective bimodal symmetric Gödel algebras. *Studia Logica*, 104(1):115–143, 2016.

[23] R. Hähnle. Advanced many-valued logics. In D.M. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic*, volume 2nd, pages 297–395, Dordrecht, 2001. Springer Netherlands.
[24] P. Hájek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic*. Kluwer, 1998.

[25] P. Hájek, L. Godo, and F. Esteva. Fuzzy logic and probability. In Philippe Besnard and Steve Hanks, editors, *UAI '95: Proceedings of the Eleventh Annual Conference on Uncertainty in Artificial Intelligence, Montreal, Quebec, Canada, August 18-20, 1995*, pages 237–244. Morgan Kaufmann, 1995.

[26] D. Klein, O. Majer, and S. Rafiee Rad. Probabilities with gaps and gluts. *Journal of Philosophical Logic*, 50(5):1107–1141, October 2021.

[27] C.H. Kraft, J.W. Pratt, and A. Seidenberg. Intuitive probability on finite sets. *The Annals of Mathematical Statistics*, 30(2):408–419, 1959.

[28] H. Leitgeb. Hype: A system of hyperintensional logic (with an application to semantic paradoxes). *Journal of Philosophical Logic*, 48(2):305–405, 2019.

[29] G.M. Moisil. Logique modale. *Disquisitiones mathematicae et physicae*, 2:3–98, 1942.

[30] D. Nelson. Constructible falsity. *The Journal of Symbolic Logic*, 14(1):16–26, 1949.

[31] S. Odintsov and H. Wansing. Routley star and hyperintensional. *Journal of Philosophical Logic*, 50:33–56, 2021.

[32] H. Omori and H. Wansing. 40 years of FDE: An introductory overview. *Studia Logica*, 105(6):1021–1049, 2017.

[33] A. Přenosil. *Reasoning with Inconsistent Information*. PhD thesis, Charles University, Prague, 2018.

[34] C. Rauszer. Applications of kripke models to heyting-brouwer logic. *Studia Logica*, 36(1/2):61–71, 1977.

[35] C. Rauszer. *An algebraic and Kripke-style approach to a certain extension of intuitionistic logic*. dissertation, Institute of Mathematics, Polish Academy of Sciences, 1980.

[36] D. Scott. Measurement structures and linear inequalities. *Journal of mathematical psychology*, 1(2):233–247, 1964.

[37] K. Segerberg. Qualitative probability in a modal setting. In *Studies in Logic and the Foundations of Mathematics*, volume 63, pages 341–352. Elsevier, 1971.

[38] Y. Shramko. Hilbert-style axiomatization of first-degree entailment and a family of its extensions. *Annals of Pure and Applied Logic*, 172(9):103011, October 2021.

[39] N.J.J. Smith. *Vagueness and Degrees of Truth*. Oxford University Press, USA, 2008.

[40] F. Touazi, C. Cayrol, and D. Dubois. Possibilistic reasoning with partially ordered beliefs. *J. Appl. Log.*, 13(4):770–798, December 2015.

[41] H. Wansing. Constructive negation, implication, and co-implication. *Journal of Applied Non-Classical Logics*, 18(2–3):341–364, 2008.

[42] S.K.M. Wong, Y. Yao, and P. Bollmann. Characterization of comparative belief structures. *International journal of man-machine studies*, 37(1):123–133, 1992.
A Completeness Proof of $\mathcal{H} G^2(\rightarrow, \prec)$ and $\mathcal{H} G^2(\neg\rightarrow, \neg\prec)$

In this section, we prove the completeness of $\mathcal{H} G^2(\rightarrow, \prec)$ and $\mathcal{H} G^2(\neg\rightarrow, \neg\prec)$ via the equivalence between their algebraic and frame semantics.

**Definition A.1** ($G^2$-frames). A $G^2$-frame is a tuple $F = (W, \preceq)$ with $W \neq \emptyset$ and $\preceq$ being a reflexive linear order on $W$.

**Definition A.2** (Models and semantics). A model on the frame $F$ is a tuple $M = (F, v^+, v^-)$ with $v^+, v^- : \text{Prop} \to 2^W$ (positive and negative valuations) such that if $s \in v^+(p)$ and $s \preceq s'$, then $s \in v^+(p)$ (and likewise for $v^-$). Using these maps, the positive and negative support of formulas at state $s \in W$ is defined as follows.

$$
\begin{align*}
M, s \models p & \iff w \in v^+(p) \\
M, s \models \neg p & \iff w \in v^-(p) \\
M, s \models \phi \land \psi & \iff M, s \models \phi \land M, s \models \psi \\
M, s \models \phi \lor \psi & \iff M, s \models \phi \lor M, s \models \psi \\
M, s \models \phi \rightarrow \psi & \iff M, s \models \phi \rightarrow M, s \models \psi \\
M, s \models \phi_1 \rightarrow \phi_2 & \iff \forall s' \preceq s, M, s' \models \phi_1 \Rightarrow M, s' \models \phi_2
\end{align*}
$$

**Definition A.3** (Entailment). $\Gamma$ (locally) entails $\phi$ in $G^2$ — denoted $\Gamma \models \phi$ — iff for any $M$ and $s \in M$ it holds that

$$
\text{if } M, s \models \Gamma \text{ then } M, s \models \phi
$$

**Theorem A.4** ($\mathcal{H} G^2(\rightarrow, \prec)$ completeness). Let $\Gamma \cup \{ \phi \} \subseteq \mathcal{L}_{G^2(\rightarrow, \prec)}$. Then $\Gamma \models \phi$ iff there is a derivation of $\phi$ from $\Gamma$ in $\mathcal{H} G^2(\rightarrow, \prec)$ s.t. nec is applied only to $\mathcal{H} G^2(\rightarrow, \prec)$ theorems.

**Proof.** Recall first, that adding prel axioms to $\mathcal{H} \mathcal{B} \mathcal{B}$ produces a calculus for Gödel logic with co-implication which is complete w.r.t. linear frames. We use De Morgan laws to transform every formula into its negation normal form. Now let $\phi^*$ be a negation normal form of $\phi$ where each literal $\neg p$ is substituted with a fresh variable $p^*$. We know from [41, Lemma 12] that $\phi$ is falsified on a $G^2(\rightarrow, \prec)$ model iff $\phi^*$ is falsified on an $\mathcal{H} \mathcal{B}$ model over the same frame. Thus, $\mathcal{H} G^2(\rightarrow, \prec)$ is complete w.r.t. the class of linear frames.

The completeness result for $\mathcal{H} G^2(\neg\rightarrow, \neg\prec)$ can be proved in the same manner.

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[43] S.K.M. Wong, Y.Y. Yao, P. Bollmann, and H.C. Bürger. Axiomatization of qualitative belief structure. *IEEE Transactions on Systems, Man, and Cybernetics*, 21(4):726–734, 1991.

[44] C. Zhou. Belief functions on distributive lattices. *Artificial Intelligence*, 201:1–31, 2013.
Lemma A.7. We proceed by induction on construction.

Proof. For the induction steps, we consider only the case of $\phi \rightarrow \psi$. The other ones are straightforward or can be obtained in a similar manner. In the following, we let $v_0^\phi$ to stand for the counterpart of $v_j$.

$$v_1(\psi \rightarrow \psi') = 1 \text{ iff } v_1(\psi) \leq v_1(\psi')$$

if $v_0^\phi(\psi) \subseteq v_0^\phi(\psi')$

(by IH)

$$v_2(\psi \rightarrow \psi') = 1 \text{ iff } v_1(\psi) = 1 \text{ and } v_2(\psi') = 1$$

if $v_0^\phi(\psi) = v_0^\phi(\psi')$

(by IH)

$$v_1(\psi \rightarrow \psi') \leq v_j(\phi') \text{ iff } \begin{cases} v_1(\psi) \leq v_1(\psi') \text{ and } v_j(\phi') \geq v_2(0) \\ v_0^\phi(\psi) \subseteq v_0^\phi(\psi') \end{cases}$$

or

$$\begin{cases} v_1(\psi) > v_1(\psi') \text{ and } v_1(\psi') \leq v_j(\phi') \\ v_0^\phi(\psi) \supseteq v_0^\phi(\psi') \end{cases}$$

(by IH)

$$v_2(\psi \rightarrow \psi') \leq v_j(\phi') \text{ iff } \begin{cases} v_0^\phi(\psi) \subseteq v_0^\phi(\psi') \text{ and } v_0^\phi(\psi') \supseteq v_0^\phi(0) \\ v_0^\phi(\psi) \subseteq v_0^\phi(\psi') \end{cases}$$

or

$$\begin{cases} v_0^\phi(\psi) \supseteq v_0^\phi(\psi') \text{ and } v_0^\phi(\psi') \subseteq v_0^\phi(0) \\ v_0^\phi(\psi) \supseteq v_0^\phi(\psi') \end{cases}$$

(by IH)

Theorem A.5 ($HG^2(\rightarrow, \neg)$ completeness). Let $\Gamma \cup \{\phi\} \subseteq LG^2(\rightarrow, \neg)$. Then $\Gamma \models \phi$ iff there is a derivation of $\phi$ from $\Gamma$ in $HG^2(\rightarrow, \neg)$ s.t. nec is applied only to $HG^2(\rightarrow, \neg)$ theorems.

It now remains to show that every $G^2$ valuations $e_1$ and $e_2$ on $[0,1]^\infty$ can be faithfully transformed into valuations on some linear model and vice versa.

Definition A.6 (Model counterpart of a $G^2$ valuation). Let $v$ be a $G^2$ valuation on $[0,1]^\infty$. A model $M = (\mathbb{Q}, \leq, v_0^+, v_0^-)$ is a counterpart of $v$ if for its valuations $v_0^+$ and $v_0^-$ it holds that:

$$v_0^+(p) = \mathbb{Q} \text{ iff } v_1(p) = 1$$

$$v_0^-(p) = \emptyset \text{ iff } v_2(p) = 0$$

$$v_0^+(p) \subseteq v_0^+(q) \text{ iff } v_1(p) \leq v_1(q)$$

$$v_0^-(p) \subseteq v_0^-(q) \text{ iff } v_2(p) \leq v_2(q)$$

$$v_0^+(p) \subseteq v_0^+(q) \text{ iff } v_2(p) \leq v_1(q)$$

$$v_0^-(p) \subseteq v_0^-(q) \text{ iff } e_1(p) \leq e_2(q)$$

Lemma A.7. Let $\phi, \phi' \in LG^2(\rightarrow, \neg) \cup LG^2(\rightarrow, \neg) \cup \{ \top, \bot \}$. $v$ be a valuation on $[0,1]^\infty$, and $M_v$ be a counterpart of $v$. Then it holds that:

$$v_1(\phi) = 1 \text{ iff } M_v \models^+ \phi$$

$$v_2(\phi) = 1 \text{ iff } M_v \models^- \phi$$

$$v_1(\phi) \leq v_1(\phi') \text{ iff } v_0^+(\phi) \subseteq v_0^+(\phi')$$

$$v_2(\phi) \leq v_2(\phi') \text{ iff } v_0^-(\phi) \subseteq v_0^-(\phi')$$

$$v_1(\phi) \leq v_2(\phi') \text{ iff } v_0^+(\phi) \subseteq v_0^+(\phi')$$

$$v_2(\phi) \leq v_1(\phi') \text{ iff } v_0^-(\phi) \subseteq v_0^-(\phi')$$

Proof. We proceed by induction on $\phi$. Then the basis cases of variables and constants hold by construction.

For the induction steps, we consider only the case of $\phi = \psi \rightarrow \psi'$. The other ones are straightforward or can be obtained in a similar manner. In the following, we let $v_0^\phi$ to stand for the counterpart of $v_j$.
Lemma A.9. Let $\phi, \phi' \in \mathcal{L}_{G^2}^{\rightarrow, \neg}$. Then, for any $G^2$-model $\mathfrak{M} = \langle F, \vplus, \vminus \rangle$ and any valuations $v^\mathfrak{M}_1$ and $v^\mathfrak{M}_2$ induced by $\mathfrak{M}$, it holds that:

\[ \mathfrak{M} \models \vplus \phi \iff v^\mathfrak{M}_1(\phi) = 1 \]
\[ \mathfrak{M} \models \neg \phi \iff v^\mathfrak{M}_2(\phi) = 1 \]
\[ v^\mathfrak{M}_1(\phi) \subseteq v^\mathfrak{M}_1(\phi') \iff v^\mathfrak{M}_1(\phi') \subseteq v^\mathfrak{M}_1(\phi) \]
\[ v^\mathfrak{M}_2(\phi) \subseteq v^\mathfrak{M}_2(\phi') \iff v^\mathfrak{M}_2(\phi') \subseteq v^\mathfrak{M}_2(\phi) \]

Proof. Analogously to lemma A.7.

We can now finally prove theorem 2.10.

Proof of theorem 2.10. The soundness part can be easily proved by a routine check of axioms and rules. Now, assume that $\Gamma \not\models_{G^2}^{\rightarrow, \neg} \phi$. Then, by [5, Proposition 5], there is a $G^2$ valuation $\epsilon_1[\Gamma] > \epsilon_1(\phi)$. Hence, by lemma A.7, there is a model $\mathfrak{M}$ and $w \in \mathfrak{M}$ s.t. $\mathfrak{M}, w \models [\Gamma]$ but $\mathfrak{M}, w \not\models \phi$. Thus, by theorem A.4, we obtain that $\phi$ is not $HG^2(\rightarrow, \neg)$ derivable from $\Gamma$.

The case of $HG^2(\rightarrow, \neg)$ can be tackled in a similar manner.