SPACE-TIME GEOMETRY OF THREE-DIMENSIONAL YANG-MILLS THEORY

V. Radovanović
Faculty of Physics, University of Belgrade, P.O. Box 550, Belgrade, Yugoslavia

and

Dj. Šijački
Institute of Physics, P.O. Box 57, Belgrade, Yugoslavia

Abstract:
It is shown that the $SU(2)$ Yang-Mills theory in 3-dimensional Riemann-Cartan space-time can be completely reformulated as a gravity-like theory in terms of gauge invariant variables. The resulting Yang-Mills induced equations are found, and it is demonstrated that they can be derived from a torsion-square type of action.

E-mail: esijacki@ubbg.etf.bg.ac.yu  radovanovic@castor.phy.bg.ac.yu
1. Introduction Over the past thirty years quite a number of attempts to connect certain strong interactions phenomena to intricate space-time variables and/or objects have been suggested.

(a) Already in 1965, in an attempt to describe the hadron spectroscopy in terms of non-compact groups, i.e. algebras, Dothan, Gell-Mann and Ne’eman [1] conjectured that the $\Delta J = 2$ Regge trajectory rule can be accomplished by making use of the $SL(3, R)$ spectrum-generating algebra (SGA) of operators given as the time-derivatives of the gravitational quadrupoles. The $SL(3, R)$ SGA was later combined with the Lorentz algebra yielding the $SL(4, R)$ SGA [2], thus describing in one shot both the single Regge trajectory recurrences and the corresponding ‘daughter’ states.

(b) The Veneziano dual amplitudes of the late sixties were subsequently rederived in terms of the Nambu-Gotto string. The string itself, or as an approximation for the gluon-field flux-tubes, opened a new era of space-time extended objects in Physics [3]. Moreover, the string indicates that the existence of a gravity-like component within the QCD gauge should be no surprise – the truncated massless sector of the open string reduces to a $J = 1$ Yang-Mills field theory, while the same truncation for the closed string reduces to a $J = 2$ gravitational field theory (since the closed string is the contraction of two open strings).

(c) The Bag Model came as another kind of extended object – it was proposed as a candidate for the dynamical approximation of confined color [4]. It is interesting to note that $SL(3, R)$ is the invariance group of a volume, thus the states described by the corresponding excitation represent the pulsations and deformational vibrations of a fixed volume ‘bag’.

(d) Salam and others [5-7] have used a gravitational framework to describe hadrons, assuming it to be genuinely extraneous to QCD – perhaps a true short range component of gravity. The ‘strong gravity’ interactions are mediated through an exchange of the additional ‘strong metric’, that was originally recognized as the $f_2$ meson (with $J^P = 2^+$). It has been shown recently that a strong gravity type of theory – Chromogravity – can be constructed, in the IR approximation, starting from QCD itself [8-9]. In this case one finds
a $p^{-4}$ propagator for the chromometric (two-gluon) field, indicating confinement, as well as a Regge-like $m^2 \sim J$ spectrum.

(e) In the course of expressing the QCD theory entirely in terms of field strengths, Halpern [10] introduced gauge invariant field variables of a curved space-time. Recently, Lunev [11] considered to some extent the problem of expressing the $SU(2)$ Yang-Mills theory in three dimensions in terms of the gauge-invariant variables arriving at a Riemannian type of geometry. Freedman et al. [12], addressed the question of the Gauss law constraint in the Hamiltonian form of the $SU(2)$ gauge theory and found a 3-dimensional spatial geometry with $GL(3, R)$ structure that underlines the gauge-invariant configuration space of the theory.

The aim of this paper is to investigate systematically the possibility of expressing the $SU(2)$ Yang-Mills theory in 3-dimensions solely in terms of $SU(2)$ gauge-invariant space-time field variables, i.e. to view it as a kind of Riemann-Cartan space-time geometry. In order to find out the precise role and meaning of the $SU(2)$ induced gauge-invariant space-time quantities, we find it crucial to embed the Yang-Mills theory into an extrinsic curved space-time. An efficient way to do this is to study a gauge theory based on the $[T_3 \otimes SO(1,2)] \otimes SU(2)$ group, i.e. to gauge simultaneously both the 3-dimensional Poincaré symmetry and the $SU(2)$ symmetry.

Let us concentrate first on the $SU(2)$ gauge itself. We will make use of the gauge potentials $A^a_\mu$, gauge field strength $F^{a}_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc} A^b_\mu A^c_\nu$, i.e. $F^{a\rho\sigma} = g^{a\mu} g^{\sigma\nu} F^{a}_{\mu\nu}$, and its dual $F^{*a}_\mu = \frac{1}{2} \epsilon_{\mu\rho\sigma} F^{a\rho\sigma}$, where $a = 1, 2, 3$ is the $SU(2)$ index, $\mu, \nu, \ldots = 0, 1, 2$ are the world indices, $\epsilon^{abc}$ are the $SU(2)$ structure constants while $\epsilon_{\mu\rho\sigma}$ ($\epsilon_{012} = 1$) is the Levi-Civita symbol. The inverse $\bar{F}^*$ of the dual tensor is given by $F^{*a}_\mu \bar{F}^{*a}_\nu = \delta^\nu_\mu$. We can now define the following $SU(2)$-scalar second-rank symmetric tensor:

$$G_{\mu\nu} = \eta_{ab} F^{*a}_\mu F^{*b}_\nu \det(\bar{F}^*), \quad (1.1)$$

where $\eta_{ab}$ is the $SU(2)$ Cartan metric tensor, and $\det(\bar{F}^*) = \det(\bar{F}^{*a}_\mu)$.

The starting Yang-Mills theory (expressed, say, in terms of $A^a_\mu$) has 6 physical degrees of freedom that
equals the number of $G_{\mu\nu}$ components. This fact as well as the fact that the $SU(2)$ group can be naturally mapped onto the continuous part of the automorphisms group of the 3 space-time translations are in the root of the geometric reformulation of the $SU(2)$ Yang-Mills theory in terms of gauge invariant variables. In the space-time reformulation, the $G_{\mu\nu}$ tensor plays the role of a metric tensor of the ‘Yang-Mills induced’ Riemman-Cartan-like geometry.

2. Gauging and the transformation laws. Let us consider a gauge theory based on the $[T_3 \otimes SO(1,2)] \otimes SU(2)$ group. The local (anholonomic) 3-dimensional Lorentz indices are denoted by $i, j, k, \ldots$, while the world (holonomic) indices are denoted by the Greek letters. Moreover, we denote the genuine gravity variables by small letters, whereas the Yang-Mills and/or new gravity-like Yang-Mills induced variables we denote by capital letters. The covariant derivatives generating a parallel transport of a tangent-space matter-field $\psi$ are given as follows:

\begin{align*}
SU(2) & : \quad D_i \psi = e_i^\mu (\partial_\mu + iA_\mu^a T^a) \psi \\
SO(1,2) & : \quad \tilde{D}_i \psi = e_i^\mu (\partial_\mu + i\frac{1}{2}a_{jk} S_{jk}) \psi \\
SO(1,2) \otimes SU(2) & : \quad \tilde{D}_i \psi = e_i^\mu (\partial_\mu + i\frac{1}{2}a_{jk} S_{jk} + iA_\mu^a T^a) \psi,
\end{align*}

(2.1)

where $S_{jk}$ $(i, j = 1, 2, 3)$, and $T^a$ $(a = 1, 2, 3)$ are the Lorentz, and the $SU(2)$ group generators respectively. Moreover, $e_i^\mu$ (the triads), $a_{ij}^a$ and $A_\mu^a$ are the translational, Lorentz, and $SU(2)$ gauge potentials respectively. The gauge potentials (infinitesimal) transformation law is given by

\begin{align*}
\delta e_i^\mu (x) &= \omega_i^j e_j^\mu (x) + \xi_{i\mu} e_i^\nu \\
\delta a_{ij}^a (x) &= \omega_i^k a_{kj}^a + \omega_j^k a_{ik}^a - \xi_{ij} a_{ik}^a - a_{ij}^a \\
\delta A_\mu^a (x) &= \partial_\mu \theta^a + e^{abc} \theta^b A_\mu^c - \xi_{\mu\nu} A_\nu^a,
\end{align*}

(2.2)

where $\xi^\mu$, $\omega^{ij}$, and $\theta^a$ are the group parameters corresponding to the translational $T_3$, Lorentz $SO(1,2)$, and $SU(2)$ groups respectively. The translational gauge invariance requires the action integration measure $d^3x$ to be replaced by $d^3x \sqrt{g}$, where $g = det(g_{\mu\nu})$, and where $g_{\mu\nu} = \eta_{ij} e_\mu^i e_\nu^j$ ($\eta_{ij} = diag(+1, -1, -1)$) is the genuine space-time metric.
The Levi-Civita symbol $\epsilon_{\mu\nu\rho}$ is a tensor-density with respect to the general coordinate transformations $x \to x'(x)$, i.e.

$$
\epsilon'_{\mu\nu\rho} = \frac{\partial x'}{\partial x^\lambda} \frac{\partial x'}{\partial x^\eta} \frac{\partial x'}{\partial x^\kappa} \epsilon_{\lambda\eta\kappa}.
$$

Therefore, the $SU(2)$ field strength dual tensor $F_{\mu}^{*a}$ transforms as

$$
F'_{\mu}^{*a} = \frac{\partial x'}{\partial x^\mu} F_{\nu}^{*a} \frac{\partial x'}{\partial x^{\nu}} ,
$$

(2.4a)

implying

$$
det(\bar{F}^{*}) \to \left| \frac{\partial x}{\partial x'} \right|^2 det(\bar{F}^{*}),
$$

(2.4b)

and thus we can define the following world vectors

$$
E_{\mu}^{a} = F_{\mu}^{*a} \sqrt{\text{det}(\bar{F}^{*})},
$$

(2.5)

that play the role of Yang-Mills induced triads, i.e. they convert mutually the Yang-Mills and world indices. Now we can rewrite (1.1) as follows

$$
G_{\mu\nu} = \eta_{ab} E_{\mu}^{a} E_{\nu}^{b},
$$

(2.6)

The corresponding $G_{\mu\nu}$ general-coordinate transformation-rule is

$$
G'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^{\mu}} \frac{\partial x^\sigma}{\partial x^{\nu}} G_{\rho\sigma}(x),
$$

(2.7)

and thus the $G_{\mu\nu}(x)$ tensor field represent the gauge-invariant Yang-Mills induced space-time metric.

Following [11, 12], we define the gauge-invariant Yang-Mills induced space-time connection as follows,

$$
D_{\mu} E_{\nu}^{a} = \Gamma_{\mu\nu}^{\rho} E_{\rho}^{a},
$$

(2.8)

where $D_{\mu} = e_{\mu}^{i} D_{i} = \partial_{\mu} + [A_{\mu}, \ ]$ is the $SU(2)$ covariant derivative [cf (2.1)]. Rewriting (2.8) as

$$
\partial_{\mu} E_{\nu}^{a} + \epsilon^{abc} A_{\mu}^{b} E_{\nu}^{c} - \Gamma_{\mu\nu}^{\rho} E_{\rho}^{a} = 0,
$$

(2.9)
we find that $A_{\mu}^{ab} \equiv -\epsilon^{abc} A_{\mu}^{c}$ plays exactly the role that has the spin connection in a Poincaré gauge theory. Equation (2.9) relates the spin-like and affine-like Yang-Mills induced connections. To sum up, the Yang-Mills potentials $A_{\mu}^{ab}$ and the properly normalized dual field strengths $E_{\mu}^{a}$ appear geometrically in the same way as the connections and triads of the configuration space-time appear respectively. By making use of (2.8) and (2.2), we find the affine connection transformation properties with respect to the general coordinate transformations as:

$$\Gamma_{\nu\rho}' \equiv \frac{\partial x'_{\mu}}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x^\nu} \frac{\partial x^\gamma}{\partial x^\rho} \Gamma_{\beta\gamma}^{\alpha} + \frac{\partial^2 x^\lambda}{\partial x^\nu \partial x^\rho} \frac{\partial x'_{\mu}}{\partial x^\lambda}. \quad (2.10)$$

Let us restrict now to the $SU(2)$ gauge itself. From (2.2) one has $\delta F_{\mu}^{a} = \epsilon^{abc} \theta^{b} F_{\mu}^{c}$ and $D'_{\mu} E_{\nu}^{a} = D_{\mu} E_{\nu}^{a} + \epsilon^{abc} \partial_{\mu} E_{\nu}^{b} \theta^{c} + o(\theta^2)$ as well as $\delta_{YM} det(F) = 0$, and thus we find, as expected, that

$$\delta_{YM} G_{\mu\nu} = 0,$$
$$\delta_{YM} \Gamma_{\nu\rho}' = 0. \quad (2.11)$$

Multiplying (2.8) with $E_{\sigma}^{a}$ and symmetrizing ($\sigma \leftrightarrow \nu$) we have

$$\nabla_{\mu} G_{\sigma\nu} \equiv \partial_{\mu} G_{\sigma\nu} - \Gamma_{\mu\nu}^{\rho} G_{\rho\sigma} - \Gamma_{\mu\sigma}^{\rho} G_{\rho\nu} = 0. \quad (2.12)$$

The meaning of this relation is the compatibility of the Yang-Mills induced connection with the corresponding metric. In other words the space-time geometry defined by $G_{\mu\nu}$ and $\Gamma_{\mu\nu}'$ is of the Riemann-Cartan type.

3. Curvature and Torsion In building up the space-time Yang-Mills induced geometrical structure we can introduce respectively the corresponding curvature and torsion tensors:

$$R_{\sigma\mu\nu}^{\rho} = \partial_{\rho} \Gamma_{\sigma\mu\nu}^{\rho} - \Gamma_{\sigma\nu}^{\rho} \Gamma_{\mu\rho}^{\sigma} + \Gamma_{\nu\rho}^{\alpha} \Gamma_{\mu\sigma}^{\rho} - \Gamma_{\nu\rho}^{\alpha} \Gamma_{\mu\sigma}^{\rho},$$

$$T_{\nu\rho}^{\mu} = \Gamma_{\nu\rho}^{\mu} - \Gamma_{\rho\nu}^{\mu} = E_{\rho}^{a}(\partial_{\nu} E_{\mu}^{a} - \partial_{\mu} E_{\nu}^{a} + A_{\nu\rho} E_{\mu}^{c} - A_{\mu\rho} E_{\nu}^{c}), \quad (3.1)$$

where $E_{a}^{\mu} = G_{a\mu} E_{\nu}^{a}$. We raise and lower indices of the Yang-Mills originated quantities by the corresponding metric $G_{\mu\nu}$. The Riemann tensor evaluation is defined by (1.1) and
and this tensor can be determined completely at the kinematical level, while the torsion tensor, as will be seen below, is related to the Yang-Mills equations of motion.

We can find \( R_{\sigma\mu}^\rho \) by evaluating \([D_\mu, D_\nu]\) in two ways. On one hand, using \( D_\mu E^a_{\nu} = -\Gamma^\nu_{\mu\rho} E^a_{\rho} \) [cf. (2.8)], we have

\[
[D_\mu, D_\nu] E^a_{\sigma\rho} = \Gamma^\rho_{\nu\sigma} D_\mu E^a_{\alpha} + \Gamma^\rho_{\mu\alpha} D_\nu E^a_{\alpha} - E^a_{\alpha} \partial_\mu \Gamma^\rho_{\nu\alpha} + E^a_{\alpha} \partial_\nu \Gamma^\rho_{\mu\alpha} \]

\[
= E^{\sigma\tau} (\partial_\mu \Gamma^\rho_{\nu\tau} - \partial_\nu \Gamma^\rho_{\mu\tau} + \Gamma^\rho_{\mu\alpha} \Gamma^\alpha_{\nu\tau} - \Gamma^\rho_{\nu\alpha} \Gamma^\alpha_{\mu\tau}) = R^\rho_{\tau\mu\nu} E^a_{\sigma\tau}, \tag{3.2}
\]

while on the other hand we find

\[
[D_\mu, D_\nu] E^a_{\sigma\rho} = D_\mu (\partial_\nu E^a_{\rho} + \epsilon^{abc} A^b_{\mu} E^c_{\rho}) - D_\nu (\partial_\mu E^a_{\rho} + \epsilon^{abc} A^b_{\mu} E^c_{\rho})
\]

\[
= -\epsilon^{abc} F^{b}_{\mu\nu} E^c_{\rho}. \tag{3.3}
\]

Equating this two expressions we arrive at \( R^\rho_{\tau\mu\nu} E^a_{\sigma\tau} = -\epsilon^{abc} F^{b}_{\mu\nu} E^c_{\rho}. \) Multiplying this relation by \( E^a_{\sigma} \), we have \( R^\rho_{\sigma\mu\nu} = -\epsilon^{abc} F^{b}_{\mu\nu} E^c_{\sigma} \) and finally we find the Yang-Mills induced Riemann tensor as follows

\[
R^\rho_{\sigma\mu\nu} = G^{\rho\sigma} (g_{\mu\sigma} g_{\nu\tau} - g_{\mu\tau} g_{\nu\sigma}). \tag{3.4}
\]

The Yang-Mills induced Ricci tensor reads

\[
R_{\sigma\nu} = \delta^\mu_\rho R^\rho_{\sigma\mu\nu} = G^{\rho\sigma} (g_{\rho\sigma} g_{\nu\tau} - g_{\rho\tau} g_{\nu\sigma}), \tag{3.5}
\]

while the Yang-Mills induced scalar curvature takes the following form

\[
R = G^{\sigma\nu} R_{\sigma\nu} = -\frac{2}{G} g^{\mu\sigma} G_{\mu\nu}. \tag{3.6}
\]

The gauge algebraic structure of relevant quantities is illustrated by the following diagram:
where, $\epsilon^i_{\mu}$, $g_{\mu\nu}$, $a^i_{\mu}$, $\gamma^\rho_{\mu\nu}$, $t^\rho_{\mu}$, and $r^\rho_{\sigma\mu\nu}$ refer to the true-gravity triads, metric, spin and affine connection, torsion and curvature respectively.

4. Space-time form of the Yang-Mills field equations

The Yang-Mills action in the external Riemann-Cartan space-time has the following well known form

$$S_{YM} = -\frac{1}{4} \int dx \sqrt{g} F^a_{\mu\nu} F^a_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma}. \quad (4.1)$$

Variation of (4.1) with respect to $A^a_{\mu}$ yields the curved-space Yang-Mills field equations

$$\partial_\sigma (\sqrt{g} F^{a\sigma\rho}) + \sqrt{g} \epsilon^{abc} A_b^\sigma F^c_{\sigma\rho} = 0,$$

that can be put in the following form

$$D_\sigma F^{a\sigma\rho} + \left\{ \begin{array}{c} \sigma \\ \tau \end{array} \right\}_g F^{a\tau\rho} = 0. \quad (4.2)$$

These equations can be expressed solely in terms of the new space-time quantities. Substituting $F^{a\rho\sigma} = \epsilon^{\rho\sigma\mu} F^{*a}_\mu$ in (4.2), making use of the fact that (true gravity) Christoffel symbol is symmetric in lower indices, antisymmetrizing in $(\sigma, \mu)$, and multiplying by $\epsilon_{\rho\alpha\beta}$ we find

$$D_\sigma F^{*d}_\mu - D_\mu F^{*d}_\sigma + \left\{ \begin{array}{c} \rho \\ \sigma \end{array} \right\}_g F^{*d}_\mu - \left\{ \begin{array}{c} \rho \\ \mu \end{array} \right\}_g F^{*d}_\sigma = 0. \quad (4.3)$$

From (2.8) one has $D_\mu F^{*d}_\nu = \Gamma^\rho_{\mu\nu} F^{*a}_\rho - \frac{1}{2} F^{*a}_\rho \partial_\nu G$. Substituting this expression in (4.3), multiplying by $\bar{F}^{*d\alpha}$, and by expressing the torsion tensor in terms of the affine connections (cf. (3.1)), we arrive at

$$T^\rho_{\sigma\mu} - \frac{1}{2} \frac{g}{G} \partial_\sigma \frac{G}{g} \delta^\rho_\mu + \frac{1}{2} \frac{g}{G} \partial_\mu \frac{G}{g} \delta^\rho_\sigma = 0. \quad (4.4)$$

The $\rho, \mu$ indices contraction in (4.4) implies the following expression for the contracted torsion

$$T^\rho_\sigma = T^\rho_{\sigma\rho} = \partial_\rho \ln \frac{G}{g}. \quad (4.5)$$

Finally, from (4.4) and (4.5) we obtain

$$T^\rho_{\sigma\mu} - \frac{1}{2} \delta^\rho_\mu T_\sigma + \frac{1}{2} \delta^\rho_\sigma T_\mu = 0. \quad (4.6)$$
Thus, the Yang-Mills field equations can be expressed solely in terms of the new gauge-invariant space-time quantities as given by (4.6) and the condition (4.5). We point out that eq. (4.6) is of the kind found in a $R+T^2$ theory of gravity. Indeed, we will demonstrate below that (4.6) can be derived from an appropriate action expressed in terms of new space-time variables only. At first glance it seems (in the absence of the matter field source on the right hand side) that eq. (4.6) yields vanishing torsion. However, its vector component is given by (4.5), while the other two irreducible components vanish. Thus, the Yang-Mills induced torsion is non-vanishing.

A trial Yang-Mills charged (or composite, ‘color’ neutral) particle experiences two metrics, the true gravity one $g_{\mu \nu}$ and the Yang-Mills induced one $G_{\mu \nu}$, as seen from (4.2) rewritten as follows

$$\partial_\sigma F^{\mu \sigma \rho} + \left\{ \frac{\sigma}{\sigma \nu} \right\}_g F^{\mu \nu \rho} + \left\{ \frac{\rho}{\sigma \nu} \right\}_g F^{\mu \sigma \nu} + \Gamma^\mu_{\sigma \nu} F^{\nu \sigma \rho} = 0,$$

(4.7)

where $F^{\mu \sigma \rho} \equiv F^{d \sigma \rho} E^{d \mu}$. The $SU(2)$-singlet gauge and ‘lepton-like’ particles experience $g_{\mu \nu}$ only.

5. Bianchi identities. The commutator of two most general covariant derivatives $\bar{D}_i$ of (2.1) reads

$$[\bar{D}_i, \bar{D}_j] = t_i^\mu \bar{D}_\mu + r_i^{mn} S_{mn} + F_i^a T^a.$$

(5.1)

The gauge algebra closure is achieved by requiring the Jacobi identity, i.e. $[\bar{D}_i, [\bar{D}_j, \bar{D}_k]] + cycl(i, j, k) = 0$, that yields the following Bianchi identities

$$\bar{D}_\mu t^i_{\nu \rho} + cycl(\mu, \nu, \rho) = r^i_{\mu \nu \rho},$$

$$\bar{D}_\mu r^j_{\nu m} + cycl(\mu, \nu, \rho) = 0,$$

(5.2)

$$\bar{D}_\mu F^a_{\nu \rho} + cycl(\mu, \nu, \rho) = 0.$$

Let us rewrite the third set of equations of (5.2) in terms of the Yang-Mills induced variables. By making use of $D_\mu F^a_{\nu \rho} = D_\mu (\epsilon_{\nu \rho \sigma} g^{\gamma \sigma} F^{*a}_{\gamma \mu})$, the Yang-Mills Bianchi becomes

$$D_\mu F^a_{\nu \rho} + cycl(\mu, \nu, \rho) = D_\mu (g \epsilon_{\nu \rho \sigma} g^{\sigma \gamma} F^{*a}_{\gamma \mu}) + cycl(\mu, \nu, \rho).$$

(5.3)
Multiplying (3.9) by \( \epsilon^{\mu\nu\rho} \) we get \( D_\mu (g g^{\mu\nu} F^a_{\nu}) = 0 \), implying

\[
(\partial_\mu g) g^{\mu\nu} F^a_{\nu} + g (\partial_\mu g^{\mu\nu}) F^a_{\nu} + (D_\mu F^a_{\nu}) g g^{\mu\nu} = 0.
\] (5.4)

It is easy to check (due to contorsion antisymmetry) that \( \partial_\mu \ln(\sqrt{G}) = \Gamma^\rho_\mu_\rho \) and \( \partial_\mu \ln(\sqrt{g}) = \gamma^\rho_\mu_\rho \), giving

\[
\partial_\mu g^{\mu\nu} + \Gamma^\nu_\mu_\rho g^{\mu\rho} - \Gamma^\rho_\mu_\rho g^{\nu\mu} + 2 \gamma^\rho_\mu_\rho g^{\nu\mu} = 0.
\] (5.5)

From (4.5) we get \( \Gamma^\rho_\sigma_\rho + \Gamma^\rho_\rho_\sigma = 2 \gamma^\rho_\sigma_\rho \), and thus we finally obtain the Yang-Mills Bianchi identities in terms of the new gauge-invariant variables in the following form

\[
\nabla_\mu g^{\mu\nu} \equiv \partial_\mu g^{\mu\nu} + \Gamma^\nu_\mu_\sigma g^{\mu\sigma} + \Gamma^\mu_\mu_\sigma g^{\nu\sigma} = 0.
\] (5.6)

This expression states that the contracted metricity of the genuine space-time metric \( g_{\mu\nu} \) with respect to the Yang-Mills induced connection \( \Gamma^\rho_\mu_\nu \) vanishes. Note that an analogous relation is required in the bimetric theory of gravity as an additional constraint relating two kinds of gravity [13]. In our case, (5.6) is a kinematical constraint.

6. The Action. The Yang-Mills field equations, rewritten in terms of new variables, as given by (4.6) mimic the form of the \( R + T^2 \) Poincaré gauge theory equations. We will show now that these equations can be obtained from the following action \( (e \) is the interaction constant)\n
\[
S = \frac{1}{e^2} \int dx \sqrt{g} (\frac{1}{4} T_{\mu\nu\rho} T^{\mu\nu\rho} + \frac{1}{2} T_{\mu\nu\rho} T^{\nu\rho\mu} + \frac{1}{2} T_\nu T^\nu)
+ \int dx \sqrt{g} \lambda^{\mu_\sigma} (E^a_\mu - \frac{1}{2} \epsilon_{\mu\nu\rho} F^{a\nu\rho} \sqrt{G})(E^a_\tau - \frac{1}{2} \epsilon_{\tau\sigma\kappa} F^{a\sigma\kappa} \sqrt{G}).
\] (6.1)

The last term in (6.1) is of no importance in derivation of the new form of the Yang-Mills equations (4.6). However this term allows us to transform (4.6) back to the original Yang-Mills form.
The variation of (6.1) with respect to $A^a_{ab}$ gives
\[
- E^a E^b \frac{\delta S}{\delta A^a_{ab}} = T_{\mu \nu} + \frac{1}{2} (G_{\mu \nu} T^\rho - G_{\mu \rho} T^\nu)
+ \lambda^{\alpha \tau} \left( E^c_\alpha - \frac{1}{2} \epsilon_{\alpha \beta \gamma} E^c_\beta E^\gamma \sqrt{G} \right) (-\frac{1}{2} \sqrt{G} \epsilon_{\tau \sigma \kappa} \frac{\delta F^{\sigma \kappa}}{\delta A^a_{ab}}) E^a E^b = 0. 
\] (6.2)

The variation of (6.1) with respect to $E^a_\alpha$ yields
\[
E^a_\lambda \frac{\delta S}{\delta E^a_\alpha} = \sqrt{g} \left( -D_\lambda (T^{\mu \epsilon \alpha} E^a_\mu) E^a_\lambda + T^{\alpha \nu \rho} T_{\lambda \nu \rho} + T_{\mu \lambda \rho} T^{\mu \alpha \rho} 
+ D_\nu (T^{\nu \sigma \rho} E^a_\sigma) E^a_\lambda - D_\rho (T^{\sigma \mu \rho} E^a_\mu) E^a_\lambda - T^{\alpha \sigma \rho} T_{\mu \lambda \rho} 
- 2T^{\mu \alpha \nu} T_{\lambda \nu \rho} - T^{\mu \nu \alpha} T_{\nu \lambda \mu} + D_\epsilon (T^\epsilon E^a_\alpha) b^a_\lambda 
- D_\epsilon (T^{\sigma \epsilon} E^a_\epsilon) E^a_\lambda - 2T^{\beta \sigma} T_{\lambda \nu} - T^{\alpha} T_{\lambda \nu} \right) 
+ \frac{\partial_\epsilon \sqrt{g}}{e^2} \left( T^{\epsilon \alpha \mu} G_{\mu \lambda} - T^{\alpha \epsilon \mu} G_{\mu \lambda} - T^{\mu \epsilon \alpha} G_{\mu \lambda} + T^{\sigma} \delta^\alpha_\lambda - T^{\alpha} \delta^\sigma_\lambda \right) 
- 2E^a_\lambda \sqrt{g} (E^b_\mu - \frac{1}{2} \epsilon_{\mu \nu \rho} F^{b \nu \rho} \sqrt{G}) \left( \delta_\epsilon^\alpha \delta^\mu_\lambda - \frac{1}{2} \sqrt{G} E^{a \sigma \kappa} \epsilon_{\tau \sigma \kappa} F^{b \sigma \kappa} \right) \lambda^{\alpha \tau} = 0
\] (6.3)

And finally, variation of (6.1) with respect to the Lagrange multipliers $\lambda^{\mu \tau}$ implies
\[
\frac{\delta S}{\delta \lambda^{\mu \tau}} = \sqrt{g} (E^a_\mu - \frac{1}{2} \epsilon_{\mu \nu \rho} F^{a \nu \rho} \sqrt{G}) (E^a_\tau - \frac{1}{2} \epsilon_{\tau \sigma \kappa} F^{a \sigma \kappa} \sqrt{G}) = 0. 
\] (6.4)

Equation (6.4) can be factorized (with an appropriate orthogonal transformation) producing the following relation
\[
E^a_\mu - \frac{1}{2} \epsilon_{\mu \nu \rho} F^{a \nu \rho} \sqrt{G} = 0. 
\] (6.5)

First, we substitute (6.5) into (6.2) and (6.3) getting rid of the terms proportional to $\lambda$. Second, we substitute thus modified (6.2) into (6.3) arriving at an identity ($0 = 0$), and finally we are left with the modified (no $\lambda$ terms) equation (6.2) that is identical to (4.4).

The main results of this paper can be schematically presented as follows:
\[
-\frac{1}{4} \int dx \sqrt{g} F^2 \quad \implies \quad (4.2) \quad \iff \quad (4.6) \quad \iff \quad -\frac{1}{e^2} \int dx \sqrt{g} T^2.
\]

In other words, on one hand we start from a 3-dimensional $SU(2)$ gauge theory in an external Riemann-Cartan space-time (or a flat space-time and gauged Poincaré symmetry),
and rewrite the theory in terms of gauge invariant variables (the Yang-Mills induced metric and connection). The original Yang-Mills equations (4.2) are shown to be given in terms of the induced torsion, eq. (4.6), while the Yang-Mills Bianchi identities are given as the Poincaré metricity condition for the gauge invariant connection (5.6). On the other hand we start with a gravity-like torsion-square action (6.1), and derive eq. (4.6), that can be (making use of the Lagrange multipliers given relations) recast back into the original Yang-Mills form.
References:

1. Y. Dothan, M. Gell-Mann and Y. Ne’eman, Phys. Lett. 17 (1965) 148.
2. Dj. Šijački, Ph.D Thesis, Duke University 1974).
3. M. Jacob, *Dual Theory*, (North-Holland, 1974); J. Scherk, Rev. Mod. Phys. 47 (1975) 123; M.B. Green, J.H. Schwarz and E. Witten, *Superstring Theory*, (Cambridge U.P., 1987).
4. A. Chodos, et al. Phys. Rev. D 9 (1974) 3471.
5. C.J. Isham, A. Salam and J. Strathdee, Phys. Rev. D 8 (1973) 2600; 9 (1974) 1702; C. Sivaram and K. Sinha, Phys. Rep. 51 (1979) 111.
6. Y. Ne’eman and Dj. Šijački, Ann. Phys. (NY) 120 (1979) 292.
7. P. Caldirola, M. Pavšič and E. Recami, Nuovo Cimento B 48 (1978) 205; Phys. Lett. 66A (1978) 9.
8. Dj. Šijački, Y. Ne’eman, Phys. Lett. B 247 (1990) 571.
9. Y. Ne’eman, Dj. Šijački, Phys. Lett. B 276B (1992) 173.
10. M. B. Halpern, Phys. Rev. D, 16 (1977) 1798; 16 (1977) 3515; 19 (1979) 517.
11. F.A. Lunev, Phys. Lett. 295B (1992) 92.
12. D.Z. Freedman, P.E. Haagensen, K. Johnson and J.I. Latorre, MIT CTP-2238 preprint (August 1993); M. Bauer, D.Z. Freedman and P.E. Haagensen, CERN-TH.7238/94 preprint (May 1994).
13. N. Rosen, in *Cosmology and Gravitation*, P.G. Bergman and V.de Sabbata eds., (Plenum Press, 1980).