Scalar products in models with $GL(3)$ trigonometric $R$-matrix. Highest coefficient

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Abstract

We study quantum integrable models with $GL(3)$ trigonometric $R$-matrix solvable by the nested algebraic Bethe ansatz. Scalar products of Bethe vectors in such models can be expressed in terms of a bilinear combination of the highest coefficients. We show that in the models with $GL(3)$ trigonometric $R$-matrix there exist two different highest coefficients. We obtain various representations for them in terms of sums over partitions. We also prove several important properties of the highest coefficients, which are necessary for the evaluation of the scalar products.

Keywords: Nested Bethe ansatz, scalar products, highest coefficient.

1 Introduction

One of the striking facts about quantum integrable systems is the possibility to find the Hamiltonian eigenvectors. Then, the knowledge of these eigenvectors allows one to have analytical insight on the form and the behavior of correlation functions for these models. The general framework for such calculation is the Quantum Inverse Scattering Method \cite{1,4}, and the use

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of the Bethe ansatz to construct the eigenvectors of the transfer matrix, which is a generating functional of all commuting Hamiltonians. Unfortunately, if the method works well for the “simplest” cases based on $GL(2)$ or $U_q(gl_2)$ symmetries, it becomes quickly very technical for models based on algebras of higher rank, and much less is known in these later cases.

In the present paper we begin a systematic study of scalar products of the Bethe vectors in quantum integrable models with $GL(3)$ trigonometric R-matrix. The role of the scalar products is extremely important in the study of correlation functions [3, 5–7]. In particular, focusing on the class of quantum integrable models where the inverse scattering problem can be solved [8, 9], one can reduce the problem of calculation of the form factors and the correlation functions of local operators to the calculation of the scalar products of the Bethe vectors [8]. Furthermore explicit and compact formulas for the scalar products sometimes allow one to study the correlation functions even in such models, for which the solution of the inverse scattering problem is not known [3, 5–7, 10]. This approach was successfully applied for the quantum integrable models with $GL(2)$-invariant or $GL(2)$ trigonometric R-matrix [11–22]. In all these works a determinant representation for the scalar products of the Bethe vectors obtained in [23] was essentially used.

The problem of the scalar products appears to be much more sophisticated in the models based on the higher rank algebras. The first results in this field were obtained by N. Reshetikhin for the models with $GL(3)$-invariant R-matrix [24]. There, a formula for the scalar product of generic Bethe vectors and a determinant representation for the norm of the transfer matrix eigenvectors were found. In the Reshetikhin representation for the scalar product, the notion of “highest coefficient” plays the most important role. This function depends on the R-matrix of the model and appears to be a rational function of the Bethe parameters. The scalar product is a bilinear combination of these highest coefficients. The knowledge of the highest coefficient allows one, in some important particular cases, to reduce this bilinear combination to a determinant representation [25–27] analogous to the one of [23].

It was shown in [24] that the highest coefficient is equal to a partition function of the 15-vertex model with special boundary conditions. Using this fact one can obtain explicit representations for the highest coefficient in models with the $GL(3)$-invariant R-matrix [23, 28]. Unfortunately, these results can not be directly extended to the case of models with $GL(3)$ trigonometric R-matrix. The main reason is that the $GL(3)$ trigonometric R-matrix is not symmetric (see (2.2)). This leads to the fact that in these models actually there are two highest coefficients, which have essentially different explicit representations. The main purpose of this paper is to derive these explicit formulas. We also establish a number of important properties of the highest coefficients, which are necessary for the calculation of the scalar products of the Bethe vectors.

In contrast to the Reshetikhin’s approach, we do not associate the highest coefficients with some partition functions. Instead we use a more direct method for their calculation. The first tool of our approach is an explicit representation for the dual Bethe vectors [29]. It is worth mentioning that in pioneer papers on the nested Bethe ansatz [2, 30, 31] no explicit formulas for the Bethe vectors and the dual ones were given. More detailed formulas were obtained in [32] in the theory of solutions of the quantum Knizhnik–Zamolodchikov equation. There the Bethe vectors were given by certain trace over auxiliary spaces of the products of monodromy matrices and R-matrices.
Explicit expressions for the Bethe vectors in terms of the monodromy matrix elements for the models with the $GL(N)$ trigonometric R-matrix were obtained in the work [33], where the realization of Bethe vectors in terms of the current generators of the quantum affine algebra $U_q(\hat{\mathfrak{g}}_{N})$ [34] was used (see also [35]).

The second tool of our method is based on the formulas of the multiple action of the monodromy matrix entries onto the Bethe vectors [36]. Using these formulas one can calculate not only the highest coefficients, but the whole scalar product of the Bethe vectors. However, the last problem is much more technical. It requires, in particular, the knowledge of several non-obvious properties of the highest coefficients. Therefore we postpone its solution to our further publication. In the present paper we restrict ourselves with the study of the highest coefficients only.

The plan of the paper is as follows. In section 2, we present the model we work with, and introduce the notations that will be used throughout the paper. We also recall some results obtained previously and needed here. In section 3, we exhibit the main result of the paper, a sum formulas for the highest coefficients. The proof of the sum formulas is given in section 4. Section 5 gathers different properties of the highest coefficients, as well as some alternative presentations for them. Appendices collect different formulas or proofs of formulas, needed in the paper.

### 2 General background

#### 2.1 The model

We consider a quantum integrable model defined by the monodromy matrix $T(u)$ with the matrix elements $T_{ij}(u)$, $i, j = 1, 2, 3$ which satisfies the commutation relation

$$R(u, v) \cdot (T(u) \otimes 1) \cdot (1 \otimes T(v)) = (1 \otimes T(v)) \cdot (T(u) \otimes 1) \cdot R(u, v),$$

with the $GL(3)$ trigonometric quantum R-matrix

$$R(u, v) = f(u, v) \sum_{1 \leq i \leq 3} E_{ii} \otimes E_{ii} + \sum_{1 \leq i < j \leq 3} (E_{ii} \otimes E_{jj} + E_{jj} \otimes E_{ii}) + \sum_{1 \leq i < j \leq 3} (u g(u, v) E_{ij} \otimes E_{ji} + v g(u, v) E_{ji} \otimes E_{ij}).$$

Here the rational functions $f(u, v)$ and $g(u, v)$ are

$$f(u, v) = \frac{qu - q^{-1}v}{u - v}, \quad g(u, v) = \frac{q - q^{-1}}{u - v},$$

where $q$ is a complex number (a deformation parameter), and $(E_{ij})_{lk} = \delta_{il} \delta_{jk}$, $i, j, l, k = 1, 2, 3$ are $3 \times 3$ matrices with unit in the intersection of $i$th row and $j$th column and zero matrix elements elsewhere. The R-matrix (2.2) is called “trigonometric” because its classical limit gives the classical trigonometric $r$-matrix [37]. The trigonometric R-matrix (2.2) is written in multiplicative variables and depends actually on the ratio $u/v$ of these multiplicative parameters.
Due to the commutation relation (2.1) the transfer matrix \( t(w) = \text{tr} T(w) = T_{11}(w) + T_{22}(w) + T_{33}(w) \) generates a set of commuting integrals of motion and the first step of the algebraic Bethe ansatz [1, 2] is the construction of the set of eigenstates for these commuting operators in terms of the monodromy matrix entries. We assume that these matrix elements act in a quantum space \( V \) and this space possesses a vector \(|0\rangle \in V \) such that

\[
T_{ij}(u)|0\rangle = 0, \quad i > j, \quad T_{ii}(u)|0\rangle = \lambda_i(u)|0\rangle, \quad \lambda_i(u) \in \mathbb{C}[[u, u^{-1}]].
\] (2.4)

We also assume that the operators \( T_{ij}(u) \) act in a dual space \( V^* \) with a vector \(|0\rangle \in V^* \) such that

\[
\langle 0|T_{ij}(u) = 0, \quad i < j, \quad \langle 0|T_{ii}(u) = \lambda_i(u)|0|,
\] (2.5)

and \( \lambda_i \) are the same as in (2.4).

The Bethe vectors \( B^{a,b}[^{\bar{u}, \bar{v}}] \) in quantum integrable models with a \( GL(3) \) trigonometric R-matrix depend on two sets of variables

\[
\bar{u} = \{u_1, \ldots, u_a\}, \quad \bar{v} = \{v_1, \ldots, v_b\},
\] (2.6)

which are called the Bethe parameters. These vectors can be constructed in the framework of the nested Bethe ansatz method formulated in [2] and are given by certain polynomials in the monodromy matrix elements \( T_{12}(u), T_{23}(u), T_{13}(u) \) depending on the Bethe parameters and applied to the vector \(|0\rangle \). They become eigenstates of the transfer matrix \( t(w) \), if the parameters \( \bar{u} \) and \( \bar{v} \) satisfy the system of Bethe equations (2.6). Such vectors sometimes are called on-shell Bethe vectors. Otherwise, if \( \bar{u} \) and \( \bar{v} \) are generic complex numbers, we deal with generic Bethe vectors.

Similarly, dual vectors \( C^{a,b}[^{\bar{u}, \bar{v}}] \) can be constructed as polynomials in \( T_{21}(u), T_{32}(u), T_{31}(u) \) applied to the vector \(|0\rangle \). They also depend on two sets of Bethe parameters \( \bar{u} \) and \( \bar{v} \) (see (2.19), (2.20) for the explicit formulas) and become eigenstates of the transfer matrix \( t(w) \), if the sets of parameters \( \bar{u} \) and \( \bar{v} \) satisfy the system of Bethe equations.

### 2.2 Notations

Below we always denote sets of variables by bar, like in (2.6). If a set of variables is multiplied by a number \( \alpha \bar{u} \) (in particular, \( \bar{u}q^{\pm2} \)), then it means that all the elements of the set are multiplied by this number

\[
\alpha \bar{u} = \{\alpha u_1, \ldots, \alpha u_a\}, \quad \bar{v}q^{\pm2} = \{v_1q^{\pm2}, \ldots, v_bq^{\pm2}\}.
\] (2.7)

To save space and simplify the presentation, we use the following convention for the products of the commuting entries of the monodromy matrix \( T_{ij}(w) \), the vacuum eigenvalues \( \lambda_i(w) \) and their ratios \( r_k(w) = \lambda_k(w)/\lambda_2(w) \), \( k = 1, 3 \). Namely, whenever such an operator or a scalar function depends on a set of variables (for instance, \( T_{ij}(\bar{u}) \), \( \lambda_i(\bar{u}) \), \( r_k(\bar{v}) \)), this means that we deal with the product of the operators or the scalar functions with respect to the corresponding set:

\[
T_{ij}(\bar{w}) = \prod_{w_k \in \bar{w}} T_{ij}(w_k); \quad \lambda_2(\bar{u}) = \prod_{u_j \in \bar{u}} \lambda_2(u_j); \quad r_k(\bar{v}_\ell) = \prod_{v_j \in \bar{v}, v_j \neq v_\ell} r_k(v_j).
\] (2.8)
Here and below the notation \( \bar{v} \) for an arbitrary set \( v \) means the set \( v \setminus v_k \). A similar convention will be used for the products of functions \( f(u, v) \)

\[
f(w_i, \bar{w}_i) = \prod_{w_j \in \bar{w}} f(w_i, w_j); \quad f(\bar{u}, \bar{v}) = \prod_{u_j \in \bar{u}} \prod_{v_k \in \bar{v}} f(u_j, v_k). \tag{2.9}
\]

Partitions of sets into two or more subsets will be noted as \( \bar{u} \Rightarrow \{\bar{u}_1, \bar{u}_\eta\} \). Here the roman numbers are used for the numeration of subsets \( \bar{u}_i \) and \( \bar{u}_\eta \). Union of sets is denoted by braces, for example, \( \{\bar{w}, \bar{u}\} = \bar{\eta} \).

In various formulas the Izergin determinant \( K_k(\bar{x}|\bar{y}) \) appears \[38\]. It is defined for two sets \( x \) and \( y \) of the same cardinality \( \#x = \#y = k \):

\[
K_k(\bar{x}|\bar{y}) = \frac{\prod_{1 \leq i,j \leq k}(qx_i - q^{-1}y_j)}{\prod_{1 \leq i < j \leq k}(x_i - x_j)(y_j - y_i)} \cdot \det \left[ \frac{q - q^{-1}}{(x_i - y_j)(qx_i - q^{-1}y_j)} \right]. \tag{2.10}
\]

Below we also use two modifications of the Izergin determinant

\[
K_k(\bar{x}|\bar{y}) = \prod_{i=1}^{k} x_i \cdot K_k(\bar{x}|\bar{y}), \quad K_k(\bar{y}|\bar{x}) = \prod_{i=1}^{k} y_i \cdot K_k(\bar{x}|\bar{y}), \tag{2.11}
\]

which we call left and right Izergin determinants respectively. Some properties of the Izergin determinant and its modifications are gathered in Appendix A.

The left and the right Izergin determinants play the role of the highest coefficients of scalar products in the models based on the \( U_q(\hat{gl}_2) \) algebra. Certainly the difference between them is very small, in particular,

\[
\prod_{i=1}^{k} x_i^{-1} \cdot K_k(\bar{x}|\bar{y}) - \prod_{i=1}^{k} y_i^{-1} \cdot K_k(\bar{y}|\bar{x}) = 0. \tag{2.12}
\]

Moreover in the \( U_q(\hat{gl}_2) \) algebra there exists a transformation that makes the R-matrix symmetric. This map, being applied to the scalar products, makes the two highest coefficients equal to each other (in this case they both are given as the original Izergin determinant \( K_k \) multiplied by the product of \( \sqrt{y_1 y_\eta} \)). However in the models based on the higher rank algebras the mentioned transformation of the R-matrix no longer exists. This leads to the fact that the analogs of the right and left highest coefficients in these models have essentially different representations.

### 2.3 Multiple action of the operators \( T_{ij} \) on Bethe vectors

For the derivation of explicit representations of the highest coefficients we need to know the actions of products \( T_{ij}(\bar{w}) \) onto the Bethe vectors. They have been computed in \[36\]. We recall here the ones that we use in this paper. Below everywhere \( \{\bar{v}, \bar{w}\} = \xi, \{\bar{u}, \bar{w}\} = \bar{\eta} \) and \( \#\bar{w} = n \).

The multiple action of \( T_{21} \) is given by

\[
T_{21}(\bar{w}) \Xi^{a,b}(\bar{u}; \bar{v}) = (-q)^{n} \lambda_2(\bar{w}) \sum r_1(\bar{\eta}_1) f(\bar{\eta}_1, \bar{\eta}_\eta)f(\bar{\eta}_\eta, \bar{\eta}_\eta)f(\bar{\eta}_\eta, \bar{\eta}_1)f(\bar{\eta}_1, \bar{\eta}_\eta) \frac{f(\bar{\xi}_a, \bar{\xi}_i)}{f(\bar{\xi}_a, \bar{\eta}_1)} \\
\times K^{(r)}_n(q^{-2} \bar{w}|\bar{\eta}_\eta)K^{(l)}_n(\bar{\eta}_1|q^2 \bar{\xi}_i)K^{(l)}_n(\bar{\xi}_1|q^2 \bar{w}) \Xi^{a-n,b}(\bar{\eta}_\eta; \bar{\xi}_a). \tag{2.13}
\]
The sum is taken over partitions of: \( \tilde{\eta} \Rightarrow \{ \tilde{\eta}_1, \tilde{\eta}_n, \tilde{\eta}_m \} \) with \( \# \tilde{\eta}_1 = \# \tilde{\eta}_n = n \); and \( \tilde{\xi} \Rightarrow \{ \tilde{\xi}_1, \tilde{\xi}_a \} \) with \( \# \tilde{\xi}_1 = n \).

The multiple action of \( T_{32} \) is given by

\[
T_{32}(\tilde{w})B^{a,b}(\tilde{u}; \tilde{v}) = (-q)^{-n} \lambda_2(\tilde{w}) \sum \mathfrak{r}_3(\tilde{\xi}_1) \frac{f(\tilde{\eta}_n, \tilde{\eta}_m) f(\tilde{\xi}_a, \tilde{\xi}_m)}{f(\tilde{\xi}_1, \tilde{\eta}_n)} \times K_n^{(r)}(q^{-2} \tilde{w}| \tilde{\eta}_1) K_n^{(l)}(q^{-2} \tilde{\eta}_1| q^2 \tilde{w}) B^{a,b}(\tilde{\eta}_1; \tilde{\xi}_1). \tag{2.14}
\]

The sum is taken over partitions of: \( \tilde{\xi} \Rightarrow \{ \tilde{\xi}_1, \tilde{\xi}_a, \tilde{\xi}_m \} \) with \( \# \tilde{\xi}_1 = \# \tilde{\xi}_a = n \); and \( \tilde{\eta} \Rightarrow \{ \tilde{\eta}_1, \tilde{\eta}_n \} \) with \( \# \tilde{\eta}_1 = n \).

**Remark.** Note that the restrictions on the cardinalities of subsets in the formulas (2.13) and (2.14) are shown explicitly by the subscripts of the Izergin determinants and the superscripts of the Bethe vectors. However, for convenience we will describe such the restrictions in special comments after formulas.

If we set \( n = a \) in (2.13), then \( \tilde{\eta}_m = \emptyset \), and we obtain

\[
T_{21}(\tilde{w})B^{a,b}(\tilde{u}; \tilde{v}) = (-q)^a \lambda_2(\tilde{w}) \sum \mathfrak{r}_1(\tilde{\eta}_1) \frac{f(\tilde{\eta}_n, \tilde{\eta}_1) f(\tilde{\xi}_a, \tilde{\xi}_1)}{f(\tilde{\xi}_1, \tilde{\eta}_n)} \times K_n^{(r)}(q^{-2} \tilde{w}| \tilde{\eta}_1) K_n^{(l)}(\tilde{\eta}_1| q^2 \tilde{w}) B^{a,b}(0; \tilde{\xi}_1). \tag{2.15}
\]

If in addition \( \tilde{v} = \emptyset \) and we want to find a coefficient of \( r_1(\tilde{w}) \), then \( \tilde{\xi}_1 = \tilde{w}, \tilde{\xi}_a = \emptyset \), and we should set \( \tilde{\eta}_1 = \tilde{w}, \tilde{\eta}_n = \tilde{u} \). Using (A.3) and (A.4) we obtain

\[
T_{21}(\tilde{w})B^{a,0}(\tilde{u}; 0) = \lambda_2(\tilde{w}) r_1(\tilde{w}) K_n^{(l)}(\tilde{w}| \tilde{v})|0\rangle + \text{IT}, \tag{2.16}
\]

where IT stands for irrelevant terms, i.e. terms that do not contribute to the coefficient we consider.

Similarly, if we set \( n = b \) in (2.14), then \( \tilde{\xi}_m = \emptyset \), and we obtain

\[
T_{32}(\tilde{w})B^{a,b}(\tilde{u}; \tilde{v}) = (-q)^{-b} \lambda_2(\tilde{w}) \sum \mathfrak{r}_3(\tilde{\xi}_1) \frac{f(\tilde{\eta}_n, \tilde{\eta}_m) f(\tilde{\xi}_1, \tilde{\eta}_n)}{f(\tilde{\xi}_1, \tilde{\eta}_n)} \times K_n^{(r)}(q^{-2} \tilde{w}| \tilde{\eta}_1) K_n^{(l)}(q^{-2} \tilde{\eta}_1| q^2 \tilde{w}) B^{a,b}(0; \tilde{\xi}_1). \tag{2.17}
\]

If in addition \( \tilde{u} = \emptyset \) and we want to find a coefficient of \( r_3(\tilde{w}) \), then \( \tilde{\eta}_1 = \tilde{w}, \tilde{\eta}_n = \emptyset \), and we should set \( \tilde{\xi}_1 = \tilde{w}, \tilde{\xi}_a = \tilde{v} \). Using (A.3) and (A.4) we obtain

\[
T_{32}(\tilde{w})B^{0,b}(\emptyset; \tilde{v}) = \lambda_2(\tilde{w}) r_3(\tilde{w}) K_n^{(r)}(\tilde{w}| \tilde{v})|0\rangle + \text{IT}. \tag{2.18}
\]

Observe that the actions (2.16), (2.18) reproduce the known results for the models with \( GL(2) \) trigonometric R-matrix.

### 2.4 Dual Bethe vectors

We have mentioned already that the Bethe vectors are given by certain polynomials in the monodromy matrix elements \( T_{12}(u), T_{23}(u), T_{13}(u) \) applied to the vector \( |0\rangle \). The explicit
form of these polynomials is not essential in the formulas for the multiple action \([2.13], [2.14]\). However we need explicit representations for the dual Bethe vectors in terms of the monodromy matrix elements in order to obtain formulas for the highest coefficients. Such representations were obtained in our work \([29]\). We give two of them:

\[
\mathbb{C}^{a,b}(\bar{u}; \bar{v}) = \sum \frac{K_h^3(\bar{u}_1 | \bar{u}_3)}{\lambda_2(\bar{u}_3) \lambda_2(\bar{u})} \frac{f(\bar{v}_3, \bar{v}) f(\bar{u}_1, \bar{u}_3)}{f(\bar{v}, \bar{u})} \langle 0 | T_{32}(\bar{v}_3)T_{21}(\bar{u}_3)T_{31}(\bar{u}_1), \tag{2.19} \]

and

\[
\mathbb{C}^{a,b}(\bar{u}; \bar{v}) = \sum \frac{K_h^3(\bar{v}_1 | \bar{v}_3)}{\lambda_2(\bar{v}_3) \lambda_2(\bar{v})} \frac{f(\bar{v}_3, \bar{v}) f(\bar{u}_1, \bar{u}_3)}{f(\bar{v}, \bar{u})} \langle 0 | T_{21}(\bar{u}_3)T_{32}(\bar{v}_3)T_{31}(\bar{v}_1). \tag{2.20} \]

Here the sum goes over all partitions of the sets \(\bar{u} \Rightarrow \{\bar{u}_i, \bar{u}_3\}\) and \(\bar{v} \Rightarrow \{\bar{v}_i, \bar{v}_3\}\) such that \(#\bar{u}_i = #\bar{v}_i = k, k = 0, \ldots, \min(a, b)\).

Both of these representations are needed for our purpose. They correspond to two different embeddings of \(U_q(\hat{gl}_3)\) into \(U_q(\hat{gl}_4)\) algebra. \(1\) is also easy to check that \((2.19)\) and \((2.20)\) are related by the isomorphism \(\varphi\) described in \([29]\). This isomorphism maps the original algebra \(U_q(\hat{gl}_3)\) to the algebra \(U_{q^{-1}}(\hat{gl}_4)\)

\[
\varphi(T_{i,j}(u)) = \hat{T}_{i-j,4-i}(u), \tag{2.21} \]

where \(T(u) \in U_q(\hat{gl}_3)\) and \(\hat{T}(u) \in U_{q^{-1}}(\hat{gl}_4)\) respectively. The map \((2.21)\) is a very powerful tool for the study of the scalar products. In particular, many properties of the scalar products can be established via the mapping between \(U_q(\hat{gl}_3)\) and \(U_{q^{-1}}(\hat{gl}_4)\).

## 3 Sum formulas for the highest coefficients

The scalar products are defined as

\[
S_{a,b}(\bar{u}^C; \bar{v}^C | \bar{u}^B; \bar{v}^B) = \mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C) \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B), \tag{3.1} \]

where all the Bethe parameters are generic complex numbers. We have added the superscripts \(C\) and \(B\) to the sets \(\bar{u}, \bar{v}\) in order to stress that the vectors \(\mathbb{C}, \mathbb{B}^{a,b}\) may depend on different sets of parameters.

Knowing the explicit form of the dual Bethe vectors \((2.19), (2.20)\) and the multiple action of the operators \(T_{ij} (2.13), (2.14)\) one can formally calculate the scalar product \(S_{a,b}(\bar{u}^C; \bar{v}^C | \bar{u}^B; \bar{v}^B)\). It is clear that the result is given as a sum with respect to partitions of the sets \(\bar{u}^C, \bar{u}^B, \bar{v}^C, \) and \(\bar{v}^B\). The terms of this sum depend on the products of the vacuum eigenvalues \(T_{ij}\) as well as on the functions entering the R-matrix. In complete analogy with the case of \(GL(3)\)-invariant R-matrix (see e.g. \([21]\)) one can derive the following representation:

\[
S_{a,b}(\bar{u}^C; \bar{v}^C | \bar{u}^B; \bar{v}^B) = \sum \frac{r_1(\bar{u}_3) r_3(\bar{v}_3) r_3(\bar{v}_3) r_3(\bar{v}_3)}{f(\bar{v}^C, \bar{u}^C) f(\bar{v}^B, \bar{u}^B)} W_{\text{part}} \left( \frac{\bar{u}^C}{\bar{v}^C}, \frac{\bar{u}^B}{\bar{v}^B}, \frac{\bar{u}^C}{\bar{v}^C}, \frac{\bar{u}^B}{\bar{v}^B} \right), \tag{3.2} \]

\(^1\)For the complete calculation of the scalar product one should also know the multiple action of the operator \(T_{31}(u)\). This action can be found in \([30]\). However for the calculation of the highest coefficients this action is not needed.
Here the sum runs over all the partitions $\bar{u}^C \Rightarrow \{\bar{u}^C_1, \bar{u}^C_2\}$, $\bar{u}^B \Rightarrow \{\bar{u}^B_1, \bar{u}^B_2\}$, $\bar{v}^C \Rightarrow \{\bar{v}^C_1, \bar{v}^C_2\}$ and $\bar{v}^B \Rightarrow \{\bar{v}^B_1, \bar{v}^B_2\}$ with $\#\bar{u}^C = \#\bar{u}^B$ and $\#\bar{v}^C = \#\bar{v}^B$. The form of the functions $W_{\text{part}}$ depends on the partitions, what is shown by the subscript ‘part’. They also depend on the R-matrix entries, but not on the functions $r_1$ and $r_3$. In other words, they depend on the algebra, not on the representations one chooses.

The highest coefficients $Z_{a,b}^{(l,r)}$ are defined as particular cases of the functions $W_{\text{part}}$, corresponding to special choices of partitions:

$$Z_{a,b}^{(l)}(\bar{u}^C; \bar{u}^B|\bar{v}^C; \bar{v}^B) = W_{\text{part}} \left( \bar{u}^C, \bar{u}^B, \emptyset, \emptyset \right); \quad Z_{a,b}^{(r)}(\bar{u}^B; \bar{u}^C|\bar{v}^B; \bar{v}^C) = W_{\text{part}} \left( \emptyset, \emptyset, \bar{u}^C, \bar{u}^B \right).$$

(3.3)

Just as in the case of the Izergin determinant, we call these coefficients left and right. The subscripts of the highest coefficients shows that $\#\bar{u}^B = a$ and $\#\bar{v}^B = b$.

Similarly to the $GL(3)$-invariant case, all other coefficients $W_{\text{part}}$ in (3.2) can be expressed in terms of products of the left and the right highest coefficients. Thus, the scalar product is a bilinear combination of the highest coefficients, and that is why the role of $Z_{a,b}^{(l,r)}$ is so important.

In the case of the $GL(3)$-invariant R-matrix, the two highest coefficients coincide and are equal to a partition function of the 15-vertex model with special boundary conditions. However, as we have already mentioned, in the models with $GL(3)$ trigonometric R-matrix the left and right highest coefficients differ from each other. The main result of the paper is an explicit expression for them.

**Proposition 3.1.** The left and right highest coefficients have the following representations:

$$Z_{a,b}^{(l)}(\bar{t}; \bar{s}; \bar{y}) = (-q)^{-b} \sum K_b^{(r)}(\bar{s}|\bar{w}_1 q^2) K_b^{(l)}(\bar{w}_1|\bar{t}) K_b^{(l)}(\bar{y}|\bar{w}_1) f(\bar{w}_1, \bar{w}_2),$$

(3.4)

$$Z_{a,b}^{(r)}(\bar{t}; \bar{s}; \bar{y}) = (-q)^{-b} \sum K_b^{(l)}(\bar{s}|\bar{w}_1 q^2) K_a^{(r)}(\bar{w}_1|\bar{t}) K_b^{(r)}(\bar{y}|\bar{w}_1) f(\bar{w}_1, \bar{w}_2).$$

(3.5)

Here $\bar{w} = \{\bar{s}, \bar{t}\}$. The sum is taken with respect to partitions of the set $\bar{w} \Rightarrow \{\bar{w}_1, \bar{w}_2\}$ with $\#\bar{w}_1 = b$ and $\#\bar{w}_2 = a$.

We would like to draw the attention of the reader that the difference between $Z_{a,b}^{(l)}$ and $Z_{a,b}^{(r)}$ is much more essential than the one between $K_b^{(l)}$ and $K_a^{(r)}$. To see this one can consider the explicit expressions for the highest coefficients in the simplest nontrivial case $a = b = 1$:

$$Z_{1,1}^{(l)}(t; x|s; y) = xy g(x,t)g(y,s)f(s,x) + xys g(x,s)g(s,t)g(y,x),$$

$$Z_{1,1}^{(r)}(t; x|s; y) = ts g(x,t)g(y,s)f(s,x) + tsx g(x,s)g(s,t)g(y,x).$$

(3.6)

From this we find, for example,

$$(ts)^{-1}Z_{1,1}^{(r)}(t; x|s; y) - (xy)^{-1}Z_{1,1}^{(l)}(t; x|s; y) = (q - q^{-1}) g(s,t)g(y,x).$$

(3.7)

See [28] for the explicit formula. The detailed proof will be given in a forthcoming paper.
Thus, in contrast to the case of the left and the right Izergin determinants, the difference between $Z_{a,b}^{(l)}$ and $Z_{a,b}^{(r)}$ can not be removed via simple multiplication of them by certain sets of variables, like in (2.12).

Below, to save space, we will combine the formulas for $Z^{(l)}$ and $Z^{(r)}$ into one. For instance, the equations (3.4) and (3.5) can be written as follows:

$$ Z^{(l,r)}_{a,b}(\tilde{t}; \tilde{x}; \tilde{s}; \tilde{y}) = (-q)^{\pm b} \sum_{K_b^{(l,r)}} (\tilde{s}|\tilde{w}_1 q^2) (\tilde{w}_n | t) K_a^{(l,r)} (\tilde{y}|\tilde{w}_1) f(\tilde{w}_1, \tilde{w}_n). \quad (3.8) $$

The superscript $(l, r)$ on $Z_{a,b}$ means that the equation (3.8) is valid for $Z_{a,b}^{(l)}$ and for $Z_{a,b}^{(r)}$ separately. Choosing the first or the second component of $(l, r)$ and the corresponding (up or down resp.) exponent of $(-q)^{\pm b}$ in this equation, we obtain either (3.4) or (3.5).

Similarly to the case of the $GL(3)$-invariant R-matrix there exist slightly different representations, so-called twin formula for the highest coefficients:

$$ Z^{(l,r)}_{a,b}(\tilde{t}; \tilde{x}; \tilde{s}; \tilde{y}) = (-q)^{\pm a} \sum_{K_a^{(l,r)}} (\tilde{w}_1 | x q^2) (\tilde{w}_n | t) K_a^{(l,r)} (\tilde{y}|\tilde{w}_1) f(\tilde{w}_1, \tilde{w}_n). \quad (3.9) $$

All the notations are the same as in (3.8). This formula follows from the reduction properties of the Izergin determinants (3.3). Indeed, we have

$$ (-q)^{\pm b} K_b^{(l,r)}(\tilde{s}|\tilde{w}_1 q^2) = (-q)^{\pm (a+b)} K_{a+b}^{(l,r)}(\tilde{s}, \tilde{x}) (\tilde{w}_1 q^2, \tilde{x} q^2) $$

$$ = (-q)^{\pm (a+b)} K_{a+b}^{(l,r)}(\tilde{w}_1, \tilde{w}_n) (\tilde{w}_1 q^2, \tilde{x} q^2) = (-q)^{\pm a} K_a^{(l,r)}(\tilde{w}_n | x q^2), \quad (3.10) $$

where the superscript $(l, r)$ has the same meaning as in (3.8). Due to (3.10), the equivalence of (3.8) and (3.9) becomes evident. Other representations for the highest coefficients in terms of sums over partitions are given in section 6.2.

4 Derivation of sum formulas for the highest coefficients

In order to derive (3.4), (3.5) we should calculate the scalar product $S_{a,b}(\tilde{u}^C; \tilde{v}^C | \tilde{u}^B; \tilde{v}^B)$ and find rational coefficients of the products $r_1(\tilde{u}^B)r_3(\tilde{v}^C)$ and $r_1(\tilde{u}^C)r_3(\tilde{v}^B)$.

4.1 The coefficient of $r_1(\tilde{u}^B)r_3(\tilde{v}^C)$

Here we calculate the coefficient of $r_1(\tilde{u}^B)r_3(\tilde{v}^C)$. We start with the dual Bethe vector in the form (2.19). For our goal it is enough to take only one term from the sum over partitions corresponding to $k = 0$:

$$ C_{a,b}^{(a,b)}(\tilde{u}^C; \tilde{v}^C) = \frac{(0|T_{32}(\tilde{v}^C)T_{21}(\tilde{u}^C))}{\lambda_2(\tilde{v}^C)\lambda_2(\tilde{u}^C)} f(\tilde{v}^C, \tilde{u}^C) + IT, \quad (4.1) $$

and we recall that IT means the terms that do not contribute to the coefficient we consider. Indeed, acting with $C_{a,b}^{(a,b)}(\tilde{u}^C; \tilde{v}^C)$ on the Bethe vector we want to obtain the product $r_3(\tilde{v}^C)$. This is possible if and only if the product $T_{32}(\tilde{v}^C)$ in (2.19) depends on the complete set $\tilde{v}^C$, that is $\tilde{v}^C = \tilde{v}^C$. Hence, $\tilde{u}_1^C = \tilde{v}^C = \emptyset$, and all other terms in (2.19) are not essential.
It remains to act successively with \( T_{21}(\vec{u}^C) \) and \( T_{32}(\vec{v}^C) \) on the Bethe vector. Using (2.15) we obtain
\[
C^{a,b}(\vec{u}^C; \vec{v}^C)B^{a,b}(\vec{u}^a; \vec{v}^a) = \frac{(-q)^a}{\lambda_2(\vec{v}^C)f(\vec{v}^C, \vec{u}^C)} \sum r_1(\vec{h}) \frac{f(\vec{h}, \vec{h})f(\vec{f}, \vec{f})}{f(\vec{f}, \vec{h})} \times K^{(r)}_a(q^{-2}\vec{u}^C|\vec{h}_a)K^{(l)}_a(\vec{h}_a)q^2\vec{f}_aK^{(l)}_a(\vec{f}_a)q^2\vec{u}^C(0|T_{32}(\vec{v}^C)B^{0,b}(0; \vec{f}_a) + IT.
\]

Here \( \vec{h} = \{\vec{u}^C, \vec{v}^a\} \) and \( \vec{f} = \{\vec{u}^C, \vec{v}^a\} \). The sum is taken over partitions \( \vec{h} = \{\vec{h}, \vec{h}_a\} \) and \( \vec{f} = \{\vec{f}, \vec{f}_a\} \) with \#\(\vec{h}_a = \#\vec{f}_a = a\).

The remaining action of \( T_{32}(\vec{v}^C) \) on \( B^{0,b}(0; \vec{f}_a) \) should be calculated via (2.18). This gives us
\[
C^{a,b}(\vec{u}^C; \vec{v}^C)B^{a,b}(\vec{u}^a; \vec{v}^a) = \frac{(-q)^a r_3(\vec{v}^C)}{f(\vec{v}^C, \vec{u}^C)} \sum r_1(\vec{h}) \frac{f(\vec{h}, \vec{h})f(\vec{f}, \vec{f})}{f(\vec{f}, \vec{h})} \times K^{(r)}_a(q^{-2}\vec{u}^C|\vec{h}_a)K^{(l)}_a(\vec{h}_a)q^2\vec{f}_aK^{(l)}_a(\vec{f}_a)q^2\vec{u}^C(0|T_{32}(\vec{v}^C)B^{0,b}(0; \vec{f}_a) + IT.
\]

Now we should extract from the sum (4.3) the terms proportional to \( r_1(\vec{u}^a) \). For this we should simply set \( \vec{h} = \vec{u}^a \) and \( \vec{h}_a = \vec{u}^C \). After elementary algebra based on the use of (A.3) and (A.4) we arrive at
\[
C^{a,b}(\vec{u}^C; \vec{v}^C)B^{a,b}(\vec{u}^a; \vec{v}^a) = \frac{r_1(\vec{u}^a)r_3(\vec{v}^C)}{f(\vec{v}^C, \vec{u}^C)} Z^{(a)}_{a,b}(\vec{u}^a; \vec{u}^C|\vec{v}^a; \vec{v}^C) + IT,
\]
where \( Z^{(a)}_{a,b} \) has the following form:
\[
Z^{(a)}_{a,b}(\vec{u}^a; \vec{u}^C|\vec{v}^a; \vec{v}^C) = (-q)^a \sum K^{(l)}_a(\vec{f}_a)q^2\vec{f}_aK^{(r)}_a(\vec{h}_a)q^2\vec{u}_aK^{(r)}_a(\vec{v}^C|\vec{f}_a)f(\vec{h}_a, \vec{f}_a),
\]
where \( \vec{f} = \{\vec{u}^C, \vec{v}^a\} \), and the sum is taken over partitions \( \vec{h} = \{\vec{h}, \vec{h}_a\} \) with \#\(\vec{h}_a = a\). This coincides with (3.9) up to notations.

### 4.2 The coefficient of \( r_1(\vec{u}^C)r_3(\vec{v}^a) \)

The calculation is very similar to the previous one. This time we start with the dual Bethe vector in the form (2.21). Now we want to obtain the coefficient of \( r_1(\vec{u}^C)r_3(\vec{v}^a) \), therefore we have
\[
C^{a,b}(\vec{u}^C; \vec{v}^C) = \frac{(0|T_{21}(\vec{u}^C)T_{32}(\vec{v}^C)}{\lambda_2(\vec{v}^C)f(\vec{v}^C, \vec{u}^C)} + IT.
\]

We should act successively with \( T_{32}(\vec{v}^C) \) and \( T_{21}(\vec{u}^C) \) on the Bethe vector. Using (2.17) we obtain
\[
C^{a,b}(\vec{u}^C; \vec{v}^C)B^{a,b}(\vec{u}^a; \vec{v}^a) = \frac{(-q)^{-b}}{\lambda_2(\vec{u}^C)f(\vec{v}^C, \vec{u}^C)} \sum r_3(\vec{h}) \frac{f(\vec{h}, \vec{h})f(\vec{f}, \vec{f})}{f(\vec{f}, \vec{h})} \times K^{(r)}_b(q^{-2}\vec{v}^C|\vec{h}_b)K^{(r)}_b(q^{-2}\vec{h}_b|\vec{f}_b)K^{(l)}_b(\vec{f}_b)q^2\vec{u}^C(0|T_{21}(\vec{u}^C)B^{0,b}(\vec{h}_b; \vec{0}) + IT.
\]
Here $\bar{\eta} = \{\bar{v}^C, \bar{u}^B\}$ and $\bar{\xi} = \{\bar{v}^C, \bar{u}^B\}$. The sum is taken over partitions $\bar{\eta} \Rightarrow \{\bar{\eta}_h, \bar{\eta}_n\}$ and $\bar{\xi} \Rightarrow \{\bar{\xi}_t, \bar{\xi}_n\}$ with $\#\bar{\eta}_h = \#\bar{\xi}_t = b$.

Now we act with $T_{21}(\bar{u}^C)$ on $\mathbb{B}^{a,b}(\bar{\eta}_n; \emptyset)$ via (2.16)

$$C^{a,b}(\bar{u}^C; \bar{v}^C)\mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B) = \frac{(-q)^{-b}r_1(\bar{u}^C)}{f(\bar{v}^C, \bar{v}^C)} \sum r_3(\bar{\xi}_t) \frac{f(\bar{\xi}_t, \bar{\xi}_n)}{f(\bar{\xi}_t, \bar{\eta}_n)} \times K_b^{(r)}(q^{-2}\bar{v}^C|\bar{\eta}_h)K_b^{(r)}(q^{-2}\bar{\eta}_h|\bar{\xi}_t)K_b^{(l)}(\bar{\xi}_t|q^2\bar{v}^C)K_a^{(l)}(\bar{\eta}_n|\bar{u}^C) + \text{IT}.$$  (4.8)

Setting here $\bar{\xi}_t = \bar{v}^B$ and $\bar{\xi}_n = \bar{v}^C$ we obtain after trivial algebra (and the use of (A.3), (A.4))

$$C^{a,b}(\bar{u}^C; \bar{v}^C)\mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B) = \frac{r_1(\bar{u}^C)r_3(\bar{v}^B)}{f(\bar{v}^C, \bar{v}^C)} Z^{(l)}_{a,b}(\bar{u}^C; \bar{u}^B|\bar{v}^C; \bar{v}^B) + \text{IT},$$  (4.9)

where $Z^{(l)}_{a,b}$ has the following form:

$$Z^{(l)}_{a,b}(\bar{u}^C; \bar{u}^B|\bar{v}^C; \bar{v}^B) = (-q)^{-b} \sum K_b^{(r)}(q^{-2}\bar{v}^C|\bar{\eta}_h)K_b^{(l)}(\bar{\eta}_h|\bar{u}^C)K_a^{(l)}(\bar{\eta}_n|\bar{u}^C)f(\bar{\eta}_n, \bar{\eta}_u),$$  (4.10)

and $\bar{\eta} = \{\bar{u}^B, \bar{v}^C\}$. This coincides with (3.4) up to notations.

Thus, we have proved that the coefficient of the product $r_1(\bar{u}^B)r_3(\bar{v}^C)$ is proportional to the function $Z^{(r)}_{a,b}(\bar{u}^B; \bar{v}^C|\bar{u}^B; \bar{v}^C)$, while the coefficient of the product $r_1(\bar{u}^C)r_3(\bar{v}^B)$ is proportional to the function $Z^{(l)}_{a,b}(\bar{u}^C; \bar{u}^B|\bar{v}^C; \bar{v}^B)$.

5 Properties and alternative expressions of $Z^{(l)}$ and $Z^{(r)}$

In order to obtain a complete formula for the scalar product one should know some properties of the highest coefficients. In particular, different representations for $Z^{(l,r)}_{a,b}$ are of great importance.

The description of the residues of $Z^{(l,r)}_{a,b}$ in their poles, as well as some reduction properties, also are useful. In this section we give a list of properties of the highest coefficients.

5.1 Simple properties of the highest coefficients

It is easy to see that both highest coefficients are symmetric with respect to all the permutations of variables in any of the four sets: $\bar{t}, \bar{s}, \bar{z}$, and $\bar{y}$. It also follows immediately from the definitions (3.8), (3.9) that

$$Z^{(l,r)}_{a,0}(\bar{t}; \bar{x}|\emptyset; \emptyset) = K^{(l+r)}_a(\bar{x}|\bar{t}), \quad Z^{(l,r)}_{0,b}(\emptyset; \emptyset|\bar{s}; \bar{y}) = K^{(l+r)}_b(\bar{y}|\bar{s}).$$  (5.1)

Using (A.2) one can easily see that the highest coefficients are invariant under the rescaling of all arguments

$$Z^{(l,r)}_{a,b}(\alpha \bar{t}; \alpha \bar{x}|\alpha \bar{s}; \alpha \bar{y}) = Z^{(l,r)}_{a,b}(\bar{t}; \bar{x}|\bar{s}; \bar{y}).$$  (5.2)

In order to describe more sophisticated properties one should use different representations for the highest coefficients.
5.2 Different representations of $Z^{(t)}$ and $Z^{(r)}$

Just like in the case of the $GL(3)$-invariant R-matrix there exist several representations for the highest coefficients in terms of sums over partitions. The original formula (3.3) is given in terms of the sums over partitions of the union of the sets $\{\tilde{t}, \tilde{s}\} = \tilde{w}$. There are also representations in terms of the sums over partitions of the unions of the sets $\{\tilde{t}q^{-2}, \tilde{y}\}$, $\{\tilde{t}, \tilde{x}\}$, and $\{\tilde{s}, \tilde{y}\}$. We give the complete list of these representations below.

- **Representations in terms of the partitions of $\{\tilde{t}q^{-2}, \tilde{y}\}$**.

$$Z_{a,b}^{(l,r)}(\tilde{t}, \tilde{r}; \tilde{s}, \tilde{y}) = (q)^{\pm a} f(\tilde{y}, \tilde{x}) f(\tilde{s}, \tilde{t}) \sum K_a^{(l,r)}(\tilde{t} q^{-2} | \tilde{y} q^2) K_a^{(l,r)}(\tilde{x} q^{-2} | \tilde{y} q^4) K_b^{(l,r)}(\tilde{y} | \tilde{y} q^2) f(\tilde{y}, \tilde{y}).$$

Here $\tilde{y} = \{\tilde{y}, \tilde{t} q^{-2}\}$. The sum is taken with respect to the partitions $\tilde{t} \Rightarrow \{\tilde{n}, \tilde{n}\}$ with $\# \tilde{n} = a$ and $\# \tilde{n} = b$.

These representations also have a twin formula:

$$Z_{a,b}^{(l,r)}(\tilde{t}, \tilde{r}; \tilde{s}, \tilde{y}) = (q)^{\pm b} f(\tilde{y}, \tilde{x}) f(\tilde{s}, \tilde{t}) \sum K_b^{(l,r)}(\tilde{n} q^2 | \tilde{y} q^2) K_a^{(l,r)}(\tilde{x} q^{-2} | \tilde{y} q^2) K_b^{(l,r)}(\tilde{y} | \tilde{y} q^2) f(\tilde{y}, \tilde{y}).$$

All the notations are the same as in (5.3). The twin-formula follows from (5.3) due to the identity $(q)^{\pm a} K_a^{(l,r)}(\tilde{t} q^{-2} | \tilde{y} q^2) = (q)^{\pm b} K_b^{(l,r)}(\tilde{n} q^2 | \tilde{y} q^2)$.

- **Representations in terms of the partitions of $\{\tilde{t}, \tilde{x}\}$**.

$$Z_{a,b}^{(l,r)}(\tilde{t}, \tilde{r}; \tilde{s}, \tilde{y}) = \sum (q)^{\pm n} f(\tilde{s}, \tilde{t}) f(\tilde{y}, \tilde{x}) f(\tilde{t}, \tilde{t}) f(\tilde{x}, \tilde{x}) \times K_n^{(l,r)}(\tilde{x} | \tilde{t}) K_a^{(l,r)}(\tilde{x} q^{-2} | \tilde{x} q^{2n}) K_b^{(l,r)}(\tilde{y} | \tilde{y} q^2) f(\tilde{y}, \tilde{y}).$$

The sum is taken with respect to all partitions $\tilde{t} \Rightarrow \{\tilde{t}, \tilde{t}\}$ and $\tilde{x} \Rightarrow \{\tilde{x}, \tilde{x}\}$ with $\# \tilde{t} = \# \tilde{x} = n, n = 0, 1, \ldots, a$.

- **Representations in terms of the partitions of $\{\tilde{s}, \tilde{y}\}$**.

$$Z_{a,b}^{(l,r)}(\tilde{t}, \tilde{r}; \tilde{s}, \tilde{y}) = \sum (q)^{\pm n} f(\tilde{s}, \tilde{t}) f(\tilde{y}, \tilde{t}) f(\tilde{s}, \tilde{s}) f(\tilde{y}, \tilde{y}) \times K_n^{(l,r)}(\tilde{y} | \tilde{s}) K_b^{(l,r)}(\tilde{y} q^{-2} | \tilde{s} q^{2n}) K_a^{(l,r)}(\tilde{y} | \tilde{y} q^2) f(\tilde{y}, \tilde{y}).$$

The sum is taken with respect to all partitions $\tilde{s} \Rightarrow \{\tilde{s}, \tilde{s}\}$ and $\tilde{y} \Rightarrow \{\tilde{y}, \tilde{y}\}$ with $\# \tilde{s} = \# \tilde{y} = n, n = 0, 1, \ldots, b$.

All the representations above follow from the original ones. In complete analogy with the case of the $GL(3)$-invariant R-matrix the sums over partitions in (3.3) can be presented as multiple contour integrals, where the integration contours surround the points $\tilde{w} = \{\tilde{s}, \tilde{x}\}$. Then moving these contours to the points $\{\tilde{t}, \tilde{x}\}$ or $\{\tilde{s}, \tilde{y}\}$ (depending on the specific representation) one obtains the equations (5.5) or (5.6). We refer the reader to the work [28] for the details of this derivation.
Representations (5.5) and (5.6) allow us to prove a very important property of $Z^{(l,r)}$:

$$Z^{(l,r)}_{b,a} (\bar{s}; y | \bar{t}q^{-2}; \bar{x}q^{-2}) = f^{-1}(y, \bar{x}) f^{-1}(\bar{s}, \bar{t}) Z^{(l,r)}_{a,b} (\bar{t}; \bar{x} | \bar{s}; \bar{y}).$$  \hspace{1cm} (5.7)

This formula can be obtained by substitution of the $Z^{(l,r)}_{b,a} (\bar{s}; y | \bar{t}q^{-2}; \bar{x}q^{-2})$ into (5.5). This will give us (5.5) for the $Z^{(l,r)}_{a,b} (\bar{t}; \bar{x} | \bar{s}; \bar{y})$.

The property (5.7) immediately implies the representations (5.3), (5.4). Indeed, one can easily check that using (3.8) for $\text{l.h.s.}$ the function $Z^{(l,r)}_{a,b}$ give us (5.6) for the $Z^{(l,r)}_{b,a}$.

One more property of the highest coefficients with respect to re-ordering of their arguments has the following form:

$$Z^{(l,r)}_{a,b,q-1} (\bar{t}; \bar{x} | \bar{s}; \bar{y}) = Z^{(r,l)}_{b,a,q} (\bar{y}; \bar{s} | \bar{x}; \bar{t}).$$ \hspace{1cm} (5.8)

Here we have added to the highest coefficients the subscripts $q$ and $q^{-1}$, in order to stress that in the l.h.s. the function $Z^{(l,r)}_{a,b}$ is evaluated with replacement $q$ by $q^{-1}$. On the contrary, in the r.h.s. of (5.8) the highest coefficient is evaluated at the same $q$, but with replacement left by right, $a$ by $b$, and re-ordering of the arguments. Usually we omit the additional subscript $q$ in the formulas. The proof of this property is given in appendix B.1. Using (5.8) and (A.5) one can easily check that the representation (5.6) follows from (5.5) after the replacement $q$ by $q^{-1}$.

We conclude this section by establishing the behavior of the highest coefficients as one of their arguments goes to infinity. For this, it is convenient to use the representations (5.5) and (5.6). Due to (A.7) and (A.8) one can easily convince himself that

$$Z^{(l)}_{a,b} (\bar{t}; \bar{x} | \bar{s}; \bar{y}), \hspace{1cm} Z^{(r)}_{a,b} (\bar{t}; \bar{x} | \bar{s}; \bar{y}) \rightarrow 0,$$

$$t_i \rightarrow \infty \text{ or } s_j \rightarrow \infty, \hspace{1cm} i = 1, \ldots, a,$$

$$Z^{(l)}_{a,b} (\bar{t}; \bar{x} | \bar{s}; \bar{y}) \text{ is bounded, } x_i \rightarrow \infty \text{ or } y_j \rightarrow \infty, \hspace{1cm} j = 1, \ldots, b.$$ \hspace{1cm} (5.9)

These equations together with the property (5.8) yield

$$Z^{(r)}_{a,b} (\bar{t}; \bar{x} | \bar{s}; \bar{y}), \hspace{1cm} Z^{(r)}_{a,b} (\bar{t}; \bar{x} | \bar{s}; \bar{y}) \rightarrow 0,$$

$$x_i \rightarrow \infty \text{ or } y_j \rightarrow \infty, \hspace{1cm} i = 1, \ldots, a,$$

$$Z^{(r)}_{a,b} (\bar{t}; \bar{x} | \bar{s}; \bar{y}) \text{ is bounded, } t_i \rightarrow \infty \text{ or } s_j \rightarrow \infty, \hspace{1cm} j = 1, \ldots, b.$$ \hspace{1cm} (5.10)

### 5.3 Special sums over partitions reducible to $Z^{(l,r)}_{a,b}$

All the sum formulas for $Z^{(l,r)}_{a,b}$ involve the products of three Izergin determinants. There exits more general formulas with three Izergin determinants, which are also reducible to the highest coefficients. Such formulas are necessary for the derivation of sum representations for the scalar product of the Bethe vectors. Below we give the list of these formulas.

Let $a \geq b$. Then

$$\sum K^{(r,l)}_b (\bar{t}_1 | yq^2) K^{(l,r)}_b (\bar{t}_1 | sq^2) K^{(l,r)}_{a-b} (\bar{\xi} | \bar{t}q^{-2} f(\bar{t}_1, \bar{t}_1) = (-q)^{\frac{a+b}{2}} \frac{Z^{(l,r)}_{a,b} (\bar{t}; \bar{\xi}; \bar{s}; \bar{y}q^{-2})}{f(y, \bar{t}) f(\bar{s}, \bar{t})}, \hspace{1cm} (5.11)$$

$$\sum K^{(l,r)}_b (\bar{t}_1 | yq^2) K^{(r,l)}_b (\bar{t}_1 | sq^2) K^{(l,r)}_{a-b} (\bar{\xi} | \bar{t}q^{-2} f(\bar{t}_1, \bar{t}_1) = (-q)^{\frac{a+b}{2}} \frac{Z^{(l,r)}_{a,b} (\bar{t}; \bar{\xi}; \bar{s}; \bar{y}q^{-2})}{f(y, \bar{t}) f(\bar{s}, \bar{t})}. \hspace{1cm} (5.12)$$
Here the sum is taken over partitions $\ell \Rightarrow \{\ell_1, \ell_2\}$ with $\#\ell_1 = b$ and $\#\ell_2 = a - b$.

Let now $a \leq b$. Then

\[
\sum K^{(r, l)}_a(q^{-2\ell t}|y_i)K^{(l, r)}_a(xq^{-2}|y_i)K^{(l, r)}_{b-a}(y_{a,i}|\xi) f(y_i, y_a) = (-q)^{\#a} \frac{Z^{(l, r)}_a(\tilde{q}\xi^2; \tilde{x}q; \tilde{\xi}; \tilde{y})}{f(\tilde{y}, \tilde{t})f(\tilde{y}, \tilde{x})},
\]

(5.13)

\[
\sum K^{(r, l)}_a(q^{-2\ell t}|y_i)K^{(l, r)}_a(xq^{-2}|y_i)K^{(l, l)}_{b-a}(y_{a,i}|\xi) f(y_i, y_a) = (-q)^{\#a} \frac{Z^{(r, l)}_a(\tilde{q}\xi^2; \tilde{x}q; \tilde{\xi}; \tilde{y})}{f(\tilde{y}, \tilde{t})f(\tilde{y}, \tilde{x})}.
\]

(5.14)

Here the sum is taken over partitions $\bar{y} \Rightarrow \{\bar{y}_1, \bar{y}_2\}$ with $\#\bar{y}_1 = a$ and $\#\bar{y}_2 = b - a$.

All these formulas follow from the representations (5.3), (5.4) (see an example of the proof in appendix [B.2]). Note also that due to (5.8) and (A.5) the equations (5.13) and (5.14) follow from respectively (5.11) and (5.12) via the replacement $q \to q^{-1}$.

### 5.4 Poles of the highest coefficients

The highest coefficients $Z^{(l, r)}_{a, b}(\ell; x|s; y)$ have simple poles at $t_i = x_j$, $t_1 = s_k$, $x_j = y_i$, and $s_k = y_i$. Similarly to the case of the $GL(3)$-invariant R-matrix the corresponding residues can be expressed in terms of $Z^{(l, l)}_{a-1, b}(\ell; x|s, y)$ or $Z^{(r, r)}_{a, b-1}(\ell; x|s; y)$. In particular,

\[
Z^{(l, r)}_{a, b}(\ell; x|s; y)\bigg|_{s_b \to y_b} = f(y_b, s_b)f(y_b, y_b)f(y_b, x_0)f(x_0, x_a)Z^{(l, r)}_{a-1, b}(\ell; x|s_b; y_b) + \text{reg},
\]

(5.15)

where reg means regular part. We remind also that $s_b = s \setminus s_b$ and $\bar{y}_b = \bar{y} \setminus y_b$.

The residue at $t_b = x_a$ has similar form

\[
Z^{(l, r)}_{a, b}(\ell; x|s; y)\bigg|_{t_a \to x_a} = f(x_a, t_a)f(x_a, t_a)f(x_a, x_a)f(s, x_a)Z^{(l, r)}_{a-1, b}(\ell; x_a|s; y) + \text{reg}.
\]

(5.16)

It is worth mentioning that equations (5.15) and (5.16) are not independent, because they are related by the transforms (5.7) and (5.8).

The formula for the residue at $s_b = t_a$ is slightly more sophisticated. Namely,

\[
Z^{(l, r)}_{a, b}(\ell; x|s; y)\bigg|_{s_b \to t_a} = f(s_b, t_a)f(s_b, s_b)f(t_a, t_a)
\]

\[
\times \sum_{p=1}^{a} K^{(l, r)}_1(x_p|s_{p})f(x_p, x_p)Z^{(l, r)}_{a-1, b}(\ell; x_p|s_{p}; y) + \text{reg}.
\]

(5.17)

Similarly the residue at $y_b = x_a$ is given by

\[
Z^{(l, r)}_{a, b}(\ell; x|s; y)\bigg|_{y_b \to x_a} = f(y_b, x_a)f(y_b, y_b)f(x_a, x_a)
\]

\[
\times \sum_{p=1}^{b} K^{(l, r)}_1(x_a|s_{p})f(s_{p}, s_{p})Z^{(l, r)}_{a-1, b}(\ell; x_a|s_{p}; y) + \text{reg}.
\]

(5.18)

The derivation of all these formulas is exactly the same as in the case of the $GL(3)$-invariant R-matrix, therefore we refer the reader to [28] for the corresponding proofs. Note that the formulas (5.17) and (5.18) are related by the properties (5.8), (A.3), and the transform $q \to q^{-1}$.
5.5 Multiple poles

The residue formulas above imply multiple residue formulas. Namely, if \( \#\bar{z} = n \), then it follows from (5.15) and (5.16) that

\[
\lim_{\bar{z}' \to \bar{z}} f^{-1}(\bar{z}', \bar{z}) Z_{a,b+n}^{(l,r)}(\bar{t}; x|\{s, \bar{z}\}; \{\bar{y}, \bar{z}'\}) = f(\bar{z}, x)f(\bar{z}, s)f(\bar{y}, \bar{z})Z_{a,b}^{(l,r)}(\bar{t}; x|\bar{s}; \bar{y}),
\]

and

\[
\lim_{\bar{z}' \to \bar{z}} f^{-1}(\bar{z}', \bar{z}) Z_{a,n+b}^{(l,r)}(\{\bar{t}, \bar{z}'\}; \{\bar{x}, \bar{z}\}|\bar{s}; \bar{y}) = f(\bar{z}, \bar{t})f(\bar{x}, \bar{z})f(\bar{s}, \bar{z})Z_{a,b}^{(l,r)}(\bar{t}; x|\bar{s}; \bar{y}).
\]

For \( \#\bar{z} = 1 \) the equations (5.19), (5.20) are direct corollaries of (5.15), (5.16) respectively. Then one can use trivial induction over \( n = \#\bar{z} \).

The residue formula (5.17) implies the following reduction:

\[
\lim_{\bar{z}' \to \bar{z}} f^{-1}(\bar{z}, \bar{z}') Z_{a,b}^{(l,r)}(\{\bar{t}, \bar{z}'\}; \bar{x}|\{s, \bar{z}\}; \bar{y}) = f(\bar{s}, \bar{z})f(\bar{z}, \bar{t}) \times \sum K_n^{(l,r)}(\bar{x}_1|\bar{z})f(\bar{x}_n, \bar{x}_1)Z_{a-n,b}^{(l,r)}(\bar{t}; x_n|\{\bar{s}, \bar{x}_1\}; \bar{y}).
\]

The sum is taken with respect to the partitions \( \bar{x} \Rightarrow \{\bar{x}_1, \bar{x}_n\} \) with \( \#\bar{x}_1 = n \).

Similarly, starting from (5.18) one can find that

\[
\lim_{\bar{z}' \to \bar{z}} f^{-1}(\bar{z}, \bar{z}') Z_{a,b}^{(l,r)}(\bar{t}; \{\bar{x}, \bar{z}'\}|\bar{s}; \{\bar{y}, \bar{z}\}) = f(\bar{y}, \bar{z})f(\bar{z}, \bar{x}) \times \sum K_n^{(l,r)}(\bar{z}|\bar{s}_1)f(\bar{s}_n, \bar{s}_1)Z_{a,n-b}^{(l,r)}(\bar{t}; \{\bar{x}, \bar{s}_1\}|\bar{s}_n; \bar{y}).
\]

Here the sum is taken with respect to the partitions \( \bar{s} \Rightarrow \{\bar{s}_1, \bar{s}_n\} \) with \( \#\bar{s}_1 = n \).

For \( \#\bar{z} = 1 \) the equations (5.21), (5.22) follow immediately from (5.17), (5.18) respectively. Then one can proceed via induction over \( n = \#\bar{z} \), using the identities (A.11), (A.12) (see details in appendix B.3).

5.6 Reductions

The reduction formulas (5.21), (5.22) can be transformed. Namely, one can apply the transform (5.7) to these equations (see appendix B.4). In this way we arrive at the following reductions

\[
Z_{a,b}^{(l,r)}(\{\bar{t}, q^2 \bar{z}\}; \bar{x}|\{\bar{s}, \bar{z}\}; \bar{y}) = \sum K_n^{(l,r)}(\bar{y}_1|\bar{z})Z_{a,b-n}^{(l,r)}(\{\bar{t}, q^2 \bar{y}_1\}; \bar{x}|\bar{s}; \bar{y})f(\bar{y}_n, \bar{y}_1)f(\bar{y}_1, \bar{x})f(\bar{y}_1, \bar{s}.
\]

The sum is taken with respect to the partitions \( \bar{y} \Rightarrow \{\bar{y}_1, \bar{y}_n\} \) with \( \#\bar{y}_1 = n \). One more reduction has the form

\[
Z_{a,b}^{(l,r)}(\bar{t}; \{\bar{x}, \bar{z}\}|\bar{s}; \{\bar{y}, \bar{z}q^{-2}\}) = \sum K_n^{(l,r)}(\bar{z}|\bar{t}_1)Z_{a-n,b}^{(l,r)}(\bar{t}_n; \bar{x}|\bar{s}; \{\bar{y}, \bar{t}_1q^{-2}\})f(\bar{t}_n, \bar{t}_1)f(\bar{x}, \bar{t}_1)f(\bar{s}, \bar{t}_1).
\]

Here the sum is taken with respect to the partitions \( \bar{t} \Rightarrow \{\bar{t}_1, \bar{t}_n\} \) with \( \#\bar{t}_1 = n \).
These formulas have special cases, when the highest coefficients degenerate into the products of the Izergin determinants. In particular, if $b \leq a$ and $n = b$, then in (5.23) $\bar{s} = \emptyset$ and $\bar{y}_n = \emptyset$. The equation (5.23) turns into

$$Z_{a,b}^{(l,r)}(\{\bar{f}, q^{2}\bar{z}\}; \bar{x}; \bar{y}) = f(\bar{y}, \bar{x})K_{a}^{(l,r)}(\bar{y}|\bar{z})K_{b}^{(l,r)}(\bar{x}|q^{2}\bar{y})).$$  

(5.25)

Similarly, if $a \leq b$ and $n = a$, then in (5.24) $\bar{x} = \emptyset$ and $\bar{t}_a = \emptyset$. The equation (5.24) turns into

$$Z_{a,b}^{(l,r)}(\bar{f}; \bar{z}; \{\bar{y}, \bar{z}q^{-2}\}) = f(\bar{s}, \bar{f})K_{a}^{(l,r)}(\bar{z}|\bar{t})K_{b}^{(l,r)}(\{\bar{y}, \bar{t}q^{-2}\}|\bar{s}).$$  

(5.26)

We draw the attention of the reader that (5.26) is the image of (5.25) under the replacement of $q$ by $q^{-1}$ and the use of eqs. (5.3) and (A.3).

### 5.7 Summation identity for the highest coefficients

The equations (5.23), (5.24) can be considered as summation identities, which allow one to express certain sums involving $K^{(l,r)}$ and $Z^{(l,r)}$ in terms of the highest coefficient $Z^{(l,r)}$. In these identities one takes a sum with respect to partitions of one set of variables. There exist more sophisticated identities of similar type, where one takes a sum with respect to partitions of two sets of variables. In this section we give one of such the identities. It plays very important role in the calculation of scalar products.

**Proposition 5.1.** Let $a, b, n, p$ be non-negative integers and $p \leq b$. Let $\bar{t}, \bar{x}, \bar{s}, \bar{y}, \bar{w}, \bar{z}$ be six sets of generic complex variables with cardinalities

$$\#\bar{t} = a, \quad \#\bar{x} = a, \quad \#\bar{z} = n, \quad \#\bar{s} = b, \quad \#\bar{y} = p, \quad \#\bar{w} = b - p.$$  

Then

$$f(\bar{\xi}, \bar{y})Z_{a,b}^{(l,r)}(\bar{f}; \bar{x}|\bar{s}; \{\bar{y}, \bar{w}\}) = \sum (-q)^{\# \bar{h}}K_{a-k}^{(r,l)}(\{\bar{s}, q^{-2}\bar{\xi}\}|\bar{y})Z_{a,b-k}^{(l,r)}(\bar{f}; \bar{x}|\bar{s}_k; \{\bar{w}, \bar{\xi}_{1}\}) \times f(\bar{s}_k, \bar{s}_n)f(\bar{\xi}_k, \bar{\xi}_1)f(\bar{y}, \bar{s}_1)f(\bar{w}, \bar{s}_1)f^{-1}(\bar{s}_1, \bar{z}).$$  

(5.27)

Here $\bar{\xi}$ is a union of two sets: $\bar{\xi} = \{\bar{x}q^{-2}, \bar{z}q^{-2}\}$. The sum is taken over partitions of the set $\bar{s} \Rightarrow \{\bar{s}_1, \bar{s}_n\}$ with $\#\bar{s}_1 = k \in [0, \ldots, p]$ and the set $\bar{\xi} \Rightarrow \{\bar{\xi}_1, \bar{\xi}_n\}$ with $\#\bar{\xi}_1 = p - k$.

The proof of this identity is given in appendix B.5.

### Conclusion

In this paper we have obtained several explicit representations for the highest coefficients $Z^{(l,r)}$ for integrable models based on $GL(3)$ trigonometric $R$-matrix and found their properties. Of course, this result is only a first step towards the calculation of the scalar products of Bethe vectors and then of the correlation functions for local operators. The calculation of the scalar products will be done in our forthcoming publication, where we are going to use the present results. Indeed, as we have explained in section 3 the scalar product can be presented as a sum
with respect to partitions of the Bethe parameters \( \mathcal{I} \). The rational coefficients \( W_{\text{part}} \) in this equation are proportional to the product of the left and the right highest coefficients. Hence the knowledge of highest coefficients is essential in the calculation of the scalar product. To stress this fact, we announce an explicit expression for \( W_{\text{part}} \), that will be proved in our forthcoming publication.

**Proposition 5.2.** The scalar product of two Bethe vectors \( \mathcal{I} \) is given by equation \( \mathcal{I} \). For a fixed partition with \( \# \bar{u}_i^C = \# \bar{u}_i^B = k \) and \( \# \tilde{v}_i^C = \# \tilde{v}_i^B = n \), (where \( k = 0, \ldots, a \) and \( n = 0, \ldots, b \)), the rational coefficient \( W_{\text{part}} \) has the form

\[
W_{\text{part}} \left( \bar{u}_i^C, \bar{u}_i^B, \bar{v}_i^C, \bar{v}_i^B \right) = f(\bar{v}_i^B) f(\bar{u}_i^B) f(\bar{v}_i^C) f(\bar{u}_i^C) f(\bar{v}_i^B) f(\bar{u}_i^B)
\]

\[
\times Z^{(l)}_{\alpha-k,n} (\bar{u}_i^C; \bar{u}_i^B) \times Z^{(r)}_{k-b-n} (\bar{v}_i^B; \bar{v}_i^C) . \tag{5.29}
\]

In the scaling limit \( u = e^{\varepsilon u}, v = e^{\varepsilon v}, q = e^{\varepsilon c/2}, \varepsilon \to 0 \), the trigonometric R-matrix goes to the \( GL(3) \)-invariant R-matrix. Then the functions \( Z^{(l)} \) and \( Z^{(r)} \) coincide, and this formula turns into the representation obtained in \( [24] \). The last one was already found to be useful for the analysis of form factors of local operators in \( GL(3) \)-invariant integrable models. We hope that the explicit representation \( \mathcal{I} \) will be also fruitful for the study of integrable models based on the \( q \)-deformed \( GL(3) \) symmetry.

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**A Properties of Izergin determinants**

Most of the properties of the left and right Izergin determinants easily follow directly from their definitions \( \mathcal{II} \). We give below a list of these properties. We remind that the superscript \( (l, r) \) on \( K \) means that the equality is valid for \( K^{(l)} \) and for \( K^{(r)} \) with appropriate choice of component (first/up or second/down) throughout the equality.

**Initial condition:**

\[
K^{(l)}_1(\bar{x}|\bar{y}) = x \, g(x, y), \quad K^{(r)}_1(\bar{x}|\bar{y}) = y \, g(x, y). \tag{A.1}
\]

**Scaling:**

\[
K^{(l,r)}_{\alpha}(\alpha x|\alpha y) = K^{(l,r)}_{\alpha}(\bar{x}|\bar{y}). \tag{A.2}
\]

**Reduction:**

\[
K^{(l,r)}_{n+1}(\{\bar{x}, q^{-2}\bar{z}\}|\{\bar{y}, z\}) = K^{(l,r)}_n(\{\bar{x}, z\}|\{\bar{y}, q^2 z\}) = -q^{+1} K^{(l,r)}_{n}(\bar{x}|\bar{y}). \tag{A.3}
\]

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Inverse order of arguments:

\[ K^{(l,r)}_n(q^{-2} \bar{x}| \bar{y}) = (-q)^{\mp n} f^{-1}(\bar{y}, \bar{x})K^{(r,l)}_n(q^n \bar{y}| \bar{x}), \quad (A.4) \]
\[ K^{(l,r)}_{n,q^{-1}}(\bar{x}| \bar{y}) = K^{(r,l)}_{n,q} (q \bar{y}| \bar{x}), \quad (A.5) \]

where \( K^{(l,r)}_{n,q^{-1}} \) means \( K^{(l,r)}_n \) with \( q \) replaced by \( q^{-1} \). As for \( Z^{(l,r)}_{a,b} \) and relation \( (3.8) \), we have put an additional index \( q^{-1} \) or \( q \) in \( (A.5) \) to stress this replacement.

Residues in the poles:

\[ K^{(l,r)}_{n+1}(\{\bar{x}, z\}|\{\bar{y}, z'\})|_{z' \to z} = f(z, z')f(z, \bar{y})f(\bar{x}, z)K^{(l,r)}_n(\bar{x}| \bar{y}) + \text{reg}, \quad (A.6) \]

where reg means the regular part.

Behavior at infinity:

\[ K^{(l)}_n(\bar{x}| \bar{y}) \sim y_i^{-1}, \quad y_i \to \infty, \quad i = 1, \ldots, n, \quad (A.7) \]
\[ K^{(r)}_n(\bar{x}| \bar{y}) \sim x_i^{-1}, \quad x_i \to \infty, \quad i = 1, \ldots, n, \quad (A.8) \]

and

\[ K^{(l)}_n(\bar{x}| \bar{y}) \text{ is bounded, } x_i \to \infty, \quad i = 1, \ldots, n, \]
\[ K^{(r)}_n(\bar{x}| \bar{y}) \text{ is bounded, } y_i \to \infty, \quad i = 1, \ldots, n. \]

**Proposition A.1.** Let \( \# \bar{x} = \# \bar{y} = n \) and \( \# \bar{z} = \# \bar{z'} = m \). Then

\[ \lim_{\bar{z}' \to \bar{z}} f^{-1}(\bar{z}, \bar{z'})K^{(l,r)}_{n+m}(\{\bar{x}, \bar{z}\}|\{\bar{y}, \bar{z}'\}) = f(\bar{x}, \bar{z})f(\bar{z}, \bar{y})K^{(l,r)}_n(\bar{x}| \bar{y}). \quad (A.9) \]

**Proof.** Using \( (A.4) \) we have

\[ K^{(l,r)}_{n+m}(\{\bar{x}, \bar{z}\}|\{\bar{y}, \bar{z}'\}) = (-q)^{\mp (m+n)}K^{(r,l)}_{n+m}(\{\bar{y}, \bar{z}'\}|\{\bar{x}q^2, \bar{z}q^2\})f(\bar{x}, \bar{y})f(\bar{z}, \bar{x})f(\bar{z}, \bar{y})f(\bar{z}, \bar{z'}). \quad (A.10) \]

The limit \( \bar{z}' \to \bar{z} \) becomes trivial, and using successively \( (A.3), (A.4) \) we arrive at \( (A.9) \).

The Izergin determinants satisfy also summation identities.

**Lemma A.1.** Let \( \bar{\gamma}, \bar{\alpha} \) and \( \bar{\beta} \) be three sets of complex variables with \( \# \alpha = m_1, \# \beta = m_2, \) and \( \# \gamma = m_1 + m_2 \). Then

\[ \sum K^{(l,r)}_{m_1}(\bar{\gamma}_1|\bar{\alpha})K^{(r,l)}_{m_2}(\bar{\beta}|\bar{\gamma}_a)f(\bar{\gamma}_a, \bar{\gamma}_1) = (-q)^{\mp m_1} f(\bar{\gamma}, \bar{\alpha})K^{(r,l)}_{m_1 + m_2}(\{\bar{\alpha}q^{-2}, \bar{\beta}\}|\bar{\gamma}). \quad (A.11) \]

The sum is taken with respect to all partitions of the set \( \bar{\gamma} \Rightarrow \{\bar{\gamma}_1, \bar{\gamma}_a\} \) with \( \# \bar{\gamma}_1 = m_1 \) and \( \# \bar{\gamma}_a = m_2 \). Due to \( (A.4) \) the equation \( (A.11) \) can be also written in the form

\[ \sum K^{(l,r)}_{m_1}(\bar{\gamma}_1|\bar{\alpha})K^{(r,l)}_{m_2}(\bar{\beta}|\bar{\gamma}_a)f(\bar{\gamma}_a, \bar{\gamma}_1) = (-q)^{\pm m_2} f(\bar{\beta}, \bar{\gamma})K^{(l,r)}_{m_1 + m_2}(\bar{\gamma}|\{\bar{\alpha}, \bar{\beta}q^2\}). \quad (A.12) \]

This statement is a simple corollary of Lemma 1 of the work [26].
B Some proofs

B.1 Proof of (5.8)

We take the representation of (5.8) and replace there q by $q^{-1}$. Using (A.5) we obtain

$$Z_{a,b,q^{-1}}^{(l,r)}(\tilde{t}; \tilde{x}|\tilde{s}; \tilde{y}) = (-q)^{\pm b} \sum K_{b}^{(l,r)}(\tilde{w}_{i}q^{-2}|\tilde{s})K_{a}^{(r,l)}(\tilde{t}|\tilde{w}_{a})K_{b}^{(r,l)}(\tilde{w}_{i}|\tilde{y})f(\tilde{w}_{a}, \tilde{w}_{i}),$$

(B.1)

where $\tilde{w} = \{\tilde{x}, \tilde{s}\}$. Here we have used the evident property of the function $f(x,y)$ under the replacement $q \rightarrow q^{-1}$: $f_{q^{-1}}(x,y) = f(y,x)$. Replacing $\tilde{w}_{i} \leftrightarrow \tilde{w}_{a}$ we find that the equation (B.1) coincides with the representation (3.3) for $Z_{b,a}^{(r,l)}(\tilde{y}; \tilde{s}|\tilde{t})$.

\[
\square
\]

B.2 Proof of (5.11)

Consider the highest coefficient $Z_{a,b}^{(l,r)}(\tilde{t}; \{\tilde{z}, \tilde{y}\}|\tilde{s}; \tilde{y}q^{-2})$ for $a \geq b$. Using (5.4) we obtain

$$(q)^{\pm b}Z_{a,b}^{(l,r)}(\tilde{t}; \{\tilde{z}, \tilde{y}\}|\tilde{s}; \tilde{y}q^{-2}) = f(\tilde{y}q^{-2}, \tilde{t})f(\tilde{y}q^{-2}, \tilde{y})f(\tilde{y}, \tilde{t}) \times \sum K_{a}^{(l,r)}(\tilde{t}|\tilde{s})K_{b}^{(l,r)}(\tilde{t}q^{-2}|\tilde{y})K_{a}^{(l,r)}(\tilde{t}q^{-2}|\tilde{y})f(\tilde{y}, \tilde{t})f(\tilde{t}, \tilde{y}).$$

(B.2)

Here $\tilde{y} = \{\tilde{t}q^{-2}, \tilde{y}q^{-2}\}$, $\#\tilde{t} = a$, $\#\tilde{y} = \#\tilde{y} = b$, $\#\tilde{z} = a - b$, and $\#\tilde{s} = a$. Consider the limit $\tilde{y} \rightarrow \tilde{y}$. Then the product $f(\tilde{y}q^{-2}, \tilde{y}) = f^{-1}(\tilde{y}, \tilde{y})$ vanishes. However the Izergin determinant $K_{a}^{(l,r)}(\tilde{t}; \{\tilde{z}, \tilde{y}\}|\tilde{y}q^{-2})$ may have poles at $\tilde{y} = \tilde{y}$. Evidently, the complete compensation of the vanishing product $f^{-1}(\tilde{y}, \tilde{y})$ occurs if and only if $\tilde{y}q^{-2} \subset \tilde{y}$. Then we can set $\tilde{y} = \{\tilde{y}q^{-2}, \tilde{t}q^{-2}\}$ and $\tilde{y} = \tilde{t}q^{-2}$. Substituting this into (B.2) we obtain

$$(q)^{\pm b}Z_{a,b}^{(l,r)}(\tilde{t}; \{\tilde{z}, \tilde{y}\}|\tilde{s}; \tilde{y}q^{-2}) = \lim_{\tilde{y} \rightarrow \tilde{y}} f^{-1}(\tilde{y}, \tilde{y})f^{-1}(\tilde{z}, \tilde{y})f^{-1}(\tilde{y}, \tilde{t}) \times \sum K_{b}^{(l,r)}(\tilde{t}q^{-2}|\tilde{y})K_{a}^{(l,r)}(\tilde{t}q^{-2}|\tilde{y})K_{a}^{(l,r)}(\tilde{t}q^{-2}|\tilde{y})f(\tilde{y}, \tilde{t})f(\tilde{t}, \tilde{y})f(\tilde{t}, \tilde{y}).$$

(B.3)

where the sum is taken over partitions $\tilde{t} \Rightarrow \{\tilde{t}, \tilde{t}\}$ with $\#\tilde{t} = b$. It remains to take the limit via (A.9), and we arrive at (5.11).

\[
\square
\]

B.3 Proof of (5.21)

Let (5.21) be valid for $\#\tilde{x} = n - 1$. Consider the case $\#\tilde{x} = n$. Taking the limit successively first for $\tilde{x}'_{n} \rightarrow \tilde{z}_{n}$, and then for $\tilde{z}'_{n} \rightarrow \tilde{z}_{n}$ we obtain

$$\lim_{\tilde{z}' \rightarrow \tilde{z}} f^{-1}(\tilde{z}, \tilde{z}')Z_{a,b}^{(l,r)}(\tilde{t}; \tilde{x}', \tilde{s}|\tilde{s}, \tilde{z}'; \tilde{y}) = f(\tilde{s}, \tilde{z}_{n})f(\tilde{z}_{n}, \tilde{t}) \sum K_{a}^{(l,r)}(\tilde{x}_{i}|\tilde{z}_{n})f(\tilde{x}_{n}, \tilde{x}_{i})$$

$$\times \lim_{\tilde{z}'_{n} \rightarrow \tilde{z}_{n}} f^{-1}(\tilde{z}_{n}, \tilde{z}'_{n})Z_{a_{n+1},b}^{(l,r)}(\tilde{t}; \tilde{x}'_{n} \cup \{\tilde{s}, \tilde{z}_{n}, \tilde{x}_{1}, \tilde{z}_{n}\}; \tilde{y})$$

$$= f(\tilde{s}, \tilde{z}_{n})f(\tilde{z}_{n}, \tilde{t}) \sum Z_{a_{n+1},b}^{(l,r)}(\tilde{t}; \tilde{x}_{n} \cup \{\tilde{s}, \tilde{x}_{1}, \tilde{z}_{n}\}; \tilde{y})$$

$$\times K_{a}^{(l,r)}(\tilde{x}_{n}|\tilde{z}_{n})f(\tilde{x}_{n}, \tilde{x}_{n})K_{a}^{(l,r)}(\tilde{x}_{n}|\tilde{z}_{n})f(\tilde{x}_{n}, \tilde{x}_{n})f(\tilde{x}_{n}, \tilde{x}_{n}).$$

(B.4)
Here we first divide $\tilde{x}$ into subsets $\{\tilde{x}_1, \tilde{x}_n\}$ with $\#\tilde{x}_1 = n - 1$, and then split the subset $\tilde{x}_n$ into sub-subsets $\{\tilde{x}_1, \tilde{x}_n\}$ with $\#\tilde{x}_1 = 1$. Setting $\{\tilde{x}_1, \tilde{x}_n\} = \tilde{x}_0$ and using $K^{(l,r)}_1(\tilde{x}_1|z_n) = (-q)^{\frac{n}{2} - 1}f(\tilde{x}_1, z_n)K^{(l,r)}_1(z_nq^{-2}\tilde{x}_1)$ we find

$$\lim_{\tilde{z} \to \tilde{x}} f^{-1}(\tilde{z}, \tilde{z}')Z^{(l,r)}_{a,b}(\{\tilde{t}, \tilde{z}'\}; \tilde{x}|\{\tilde{s}, \tilde{z}\}; \tilde{y}) = f(\tilde{s}, \tilde{z})f(\tilde{z}, \tilde{t})$$

$$\times \sum Z_{a-n,b}(\tilde{t}; \tilde{x}_n|\{\tilde{s}, \tilde{x}_0\}; \tilde{y})f(\tilde{x}_n, \tilde{x}_0)f(\tilde{x}_0, z_n)$$

$$\times (-q)^{\frac{n}{2} - 1}K^{(l,r)}_n(\tilde{x}_n|\tilde{z}_n)K^{(l,r)}_1(z_nq^{-2}\tilde{x}_1) f(\tilde{x}_1, \tilde{x}_1).$$  \hspace{1cm} (B.5)

Applying lemma A.1 to the last line of (B.5) we can take the sum with respect to the partitions $\tilde{x}_0 \to \{\tilde{x}_1, \tilde{x}_1\}$, that finally gives

$$\lim_{\tilde{z}' \to \tilde{z}} f^{-1}(\tilde{z}, \tilde{z}')Z^{(l,r)}_{a,b}(\{\tilde{t}, \tilde{z}'\}; \tilde{x}|\{\tilde{s}, \tilde{z}\}; \tilde{y}) = f(\tilde{s}, \tilde{z})f(\tilde{z}, \tilde{t})$$

$$\times \sum K^{(l,r)}_n(\tilde{x}_0|\tilde{z})f(\tilde{x}_n, \tilde{x}_0)Z^{(l,r)}_{a-n,b}(\tilde{t}; \tilde{x}_n|\{\tilde{s}, \tilde{x}_0\}; \tilde{y}).$$  \hspace{1cm} (B.6)

\[\square\]

### B.4 Proof of \((5.24)\)

Let us simply write \((5.22)\) replacing $a$ by $b$ and setting: $\tilde{t} = q^2\tilde{u}, \tilde{x} = q^2\tilde{v}, \tilde{s} = \bar{\alpha},$ and $\tilde{y} = \bar{\beta}$

$$\lim_{\tilde{z}' \to \tilde{z}} f^{-1}(\tilde{z}, \tilde{z}')Z^{(l,r)}_{a,b}(q^2\tilde{u}, \{q^2\tilde{v}, \tilde{z}'\}|\bar{\alpha}; \{\bar{\alpha}, \tilde{z}\}) = f(\bar{\beta}, \tilde{z})f(\tilde{z}, q^2\tilde{v})$$

$$\times \sum K^{(l,r)}_n(\tilde{z}|\tilde{\alpha})f(\tilde{\alpha}, \tilde{\alpha}_n)Z^{(l,r)}_{b,a-n}(q^2\tilde{u}; \{q^2\tilde{v}, \tilde{\alpha}\}|\tilde{\alpha}_n; \tilde{\beta}),$$  \hspace{1cm} (B.7)

where the sum is taken over partitions $\tilde{\alpha} \Rightarrow \{\tilde{\alpha}, \tilde{\alpha}_n\}$ with $\#\tilde{\alpha} = n$. Now we can use \((5.2)\) and \((5.7)\) to transform $Z^{(l,r)}_{b,a}$ and $Z^{(l,r)}_{b,a-n}$ in \((B.7)\). We have

$$\lim_{\tilde{z}' \to \tilde{z}} f^{-1}(\tilde{z}, \tilde{z}')Z^{(l,r)}_{b,a}(q^2\tilde{u}, \{q^2\tilde{v}, \tilde{z}'\}|\bar{\alpha}; \{\bar{\beta}, \tilde{z}\}) = \frac{f(\bar{\beta}, \tilde{z})Z^{(l,r)}_{b,a}(\bar{\alpha}; \{\bar{\beta}, \tilde{z}\}|\tilde{u}; \{\tilde{v}, q^{-2}\tilde{z}\})}{f(\tilde{u}, \bar{\alpha})f(\tilde{v}, \tilde{z})f(\tilde{v}, \bar{\beta})},$$  \hspace{1cm} (B.8)

and

$$Z^{(l,r)}_{b,a-n}(q^2\tilde{u}; \{q^2\tilde{v}, \tilde{\alpha}\}|\tilde{\alpha}_n; \bar{\beta}) = \frac{f(\bar{\beta}, \tilde{\alpha}_n)Z^{(l,r)}_{b,a-n}(\bar{\alpha}_n; \bar{\beta}|\tilde{u}; \{\tilde{v}, q^{-2}\tilde{\alpha}_n\})}{f(\tilde{u}, \bar{\alpha}_n)}.$$  \hspace{1cm} (B.9)

Substituting all this into \((B.7)\) after evident cancelations we obtain

$$Z^{(l,r)}_{a,b}(\bar{\alpha}; \{\bar{\beta}, \tilde{z}\}|\tilde{u}; \{\tilde{v}, q^{-2}\tilde{z}\}) = \sum K^{(l,r)}_n(\tilde{z}|\tilde{\alpha})f(\tilde{\alpha}, \tilde{\alpha}_n)$$

$$\times f(\bar{\beta}, \tilde{\alpha}_n)f(\tilde{u}, \tilde{\alpha}_n)Z^{(l,r)}_{a-n,b}(\tilde{\alpha}_n; \bar{\beta}|\tilde{u}; \{\tilde{v}, q^{-2}\tilde{\alpha}_n\}).$$  \hspace{1cm} (B.10)

It remains to set $\tilde{\alpha} = \tilde{t}, \bar{\beta} = \tilde{x}, \tilde{u} = \tilde{s}, \tilde{v} = \tilde{y}$, and we arrive at \((5.24)\). \[\square\]
B.5 Proof of Proposition 5.1

We use induction over \( p \). Denote the l.h.s. and the r.h.s. of (5.28) by \( F^{(lr)}_{a,b,p,n}(\bar{f}; \bar{x}|\bar{s}; \bar{w}; \bar{y}|\bar{z}) \) and \( \tilde{F}^{(lr)}_{a,b,p,n}(\bar{f}; \bar{x}|\bar{s}; \bar{w}; \bar{y}|\bar{z}) \) respectively. For \( p = 0 \) the equation (5.28) is trivial. Indeed, since \( k \leq p \) we obtain that \( s_i = \xi_i = \tilde{y} = 0 \) for \( p = 0 \). Hence, the sum over partitions in the r.h.s. of (5.28) reduces to the one term, and both sides of this equation give \( Z^{(lr)}_{a,b}(\bar{f}; \bar{x}|\bar{s}; \bar{w}) \).

Now let (5.28) be valid for \( \# \tilde{y} = p - 1 \) and arbitrary \( a, b, n \):

\[
F^{(lr)}_{a,b,p-1,n}(\bar{f}; \bar{x}|\bar{s}; \bar{w}; \bar{y}|\bar{z}) = \tilde{F}^{(lr)}_{a,b,p-1,n}(\bar{f}; \bar{x}|\bar{s}; \bar{w}; \bar{y}|\bar{z}), \quad \forall a, b, n .
\] (B.11)

The general strategy of the proof is the following. We consider both sides of this equation at \( \# \tilde{y} = p \) as functions of \( y_p \), the other variables being fixed. Obviously \( F \) and \( \tilde{F} \) are rational functions of \( y_p \). We first establish that these functions have their poles in the same points and then prove that due to the induction assumption (B.11) the residues in these poles coincide. Then it means that the difference \( F - \tilde{F} \) is a polynomial in \( y_p \). Finally taking into account the behavior of this polynomial at \( y_p \to \infty \) and \( y_p = 0 \) we conclude that it is identically equal to zero.

Obviously, the function \( F^{(lr)}_{a,b,p,n} \) has poles at \( y_p = \xi_\ell, \ell = 1, \ldots, a + n \) due to the factor \( f(\xi, \tilde{y}) \).

The highest coefficient \( Z^{(lr)}_{a,b} \) has additional poles at \( y_p = s_i, i = 1, \ldots, b \). However the poles of \( Z^{(lr)}_{a,b} \) at \( y_p = x_j, j = 1, \ldots, a \) are compensated by the zeros of the prefactor:

\[
f(\xi, \tilde{y}) = f(\bar{z}q^{-2}, \tilde{y})f(\bar{x}q^{-2}, \tilde{y}) = f^{-1}(\tilde{y}, \bar{z})f^{-1}(\tilde{y}, \bar{x}) .
\] (B.12)

It is easy to see that the r.h.s. \( \tilde{F}^{(lr)}_{a,b,p,n} \) has poles in the same points. Due to the product \( f(\bar{y}, \tilde{s}_i) \) it has poles at \( y_p = s_i \). The function \( K^{(r,l)}_p(\{ \tilde{s}_i q^{-2}, \xi_\ell | \tilde{y} \}) \) has poles at \( y_p = \xi_\ell \), however the poles at \( y_p = s_i q^{-2} \) are compensated by the product \( f(\tilde{y}, \tilde{s}_i) \).

Consider the residues of \( F^{(lr)}_{a,b,p,n} \) at \( y_p = s_i \). Using the reduction property (5.15) we obtain (for shortness here and below we omit the arguments of \( F^{(lr)}_{a,b,p,n} \) and \( \tilde{F}^{(lr)}_{a,b,p,n} \) in the l.h.s. of equations):

\[
F^{(lr)}_{a,b,p,n} \bigg|_{y_p = s_i} = f(y_p, s_i)f(s_i, \tilde{s}_i)f(\tilde{y}_p, s_i)f(\bar{w}, s_i)f(s_i, \bar{x})f(\bar{x}q^{-2}, s_i) \times f(\xi, \tilde{y}_p)Z^{(lr)}_{a,b-1}(\bar{f}; \bar{x}|\bar{s}_i; \{ \tilde{y}_p, \bar{w} \}) + \text{reg} .
\] (B.13)

The terms in square brackets cancel each other, the terms in the second line give \( F^{(lr)}_{a,b-1,p-1,n} \):

\[
F^{(lr)}_{a,b,p,n} \bigg|_{y_p = s_i} = f(y_p, s_i)f(s_i, \tilde{s}_i)f(\tilde{y}_p, s_i)f(\bar{w}, s_i)f(s_i, \bar{x})f(\bar{x}q^{-2}, s_i) \times f(\xi, \tilde{y}_p)Z^{(lr)}_{a,b-1}(\bar{f}; \bar{x}|\bar{s}_i; \{ \tilde{y}_p, \bar{w} \}) + \text{reg} .
\] (B.14)

Consider now the residue of \( \tilde{F} \) at \( y_p = s_i \). The pole occurs if and only if \( s_i \in \tilde{s}_i \). Setting \( \tilde{s}_i = \{ s_i, \tilde{s}_0 \} \) and using the property (A.3) of \( K^{(r,l)}_p \) we obtain

\[
\tilde{F}^{(lr)}_{a,b,p,n} \bigg|_{y_p = s_i} = \sum (-q)^{r(k-1)}K^{(r,l)}_{p-1}(\{ \tilde{s}_0 q^{-2}, \xi_\ell | \tilde{y}_p \})Z^{(lr)}_{a,b-1}(\bar{f}; \bar{x}|\bar{s}_i; \{ \tilde{w}, \xi_\ell \})f(s_i, \tilde{s}_0)f(s_i, \tilde{s}_a) \times f(\tilde{s}_0, \tilde{s}_a)f(\xi, \tilde{s}_i) f(y_p, s_i)f(\tilde{y}_p, s_i)f(\bar{w}, s_i)f^{-1}(s_i, \bar{z}) \times f(\tilde{y}_p, \tilde{s}_0)f(\bar{w}, \tilde{s}_0)f^{-1}(\tilde{s}_0, \bar{z}) + \text{reg} .
\] (B.15)
where the sum is taking over partitions \( \bar{s}_i \Rightarrow \{ \bar{s}_0, \bar{s}_n \} \) and \( \bar{\xi} \Rightarrow \{ \bar{\xi}_1, \bar{\xi}_n \} \). The terms in square brackets can be moved out of the sum. The product \( f(s_i, \bar{s}_0)f(s_i, \bar{s}_n) \) combines into \( f(s_i, \bar{s}_i) \) and also can be moved out of the sum. We arrive at

\[
\tilde{F}_{a,b,p,n}^{(l,r)} \bigg|_{y_p \to s_i} = f(y_p, s_i) f(s_i, \tilde{s}_i) f(\tilde{y}_p, s_i) f(\tilde{w}, s_i) f^{-1}(s_i, \tilde{z}) \sum (-q)^{\pm k_0} K_{p-1}(\{ s_0q^{-2}, \bar{\xi}_i \} | \tilde{y}_p) \\
\times Z_{a,b-1-p-1,n}^{(l,r)}(\bar{\ell}; \bar{x}| \bar{s}_n; \{ \bar{w}, \bar{\xi}_1 \}) f(\tilde{s}_0, \bar{s}_n) f(\bar{\xi}_n, \bar{s}_i) f(\bar{\xi}_n, \bar{s}_i) f(\tilde{y}_p, \tilde{s}_0) f(\tilde{w}, \tilde{s}_0) f^{-1}(\tilde{s}_0, \tilde{z}) + \text{reg}, \quad (B.16)
\]

where \( k_0 = \# \bar{s}_0 = k - 1 \). Evidently, the sum over partitions in the r.h.s. of (B.16) gives \( \tilde{F}_{a,b-1,p-1,n}^{(l,r)}(\bar{\ell}; \bar{x}| \bar{s}_i; \bar{w}; \tilde{y}_p| \tilde{z}) \) and we obtain

\[
\tilde{F}_{a,b,p,n}^{(l,r)} \bigg|_{y_p \to s_i} = f(y_p, s_i) f(s_i, \tilde{s}_i) f(\tilde{y}_p, s_i) f(\tilde{w}, s_i) f^{-1}(s_i, \tilde{z}) \tilde{F}_{a,b-1,p-1,n}^{(l,r)}(\bar{\ell}; \bar{x}| \bar{s}_i; \bar{w}; \tilde{y}_p| \tilde{z}) + \text{reg}. \quad (B.17)
\]

Comparing (B.14) and (B.17) and taking into account (B.11) we conclude that the difference \( F_{a,b,p,n}^{(l,r)} - \tilde{F}_{a,b,p,n}^{(l,r)} \) is a bounded function of \( y_p \) as \( y_p \to s_i, i = 1, \ldots, b \).

Consider now the residues of \( F_{a,b,p,n}^{(l,r)} \) at \( y_p = \xi_\ell \). We have

\[
F_{a,b,p,n}^{(l,r)} \bigg|_{y_p \to \xi_\ell} = f(\xi_\ell, y_p) f(\xi_\ell, \xi_\ell) f(\xi_\ell, \bar{y}_p) f(\bar{w}, \xi_\ell) f^{-1}(\xi_\ell, \tilde{z}) \tilde{F}_{a,b-1,p-1,n}^{(l,r)}(\bar{\ell}; \bar{x}| \bar{s}_i; \bar{w}; \bar{y}_p| \tilde{z}) + \text{reg}. \quad (B.18)
\]

Now one should distinguish between two cases: either \( \xi_\ell \in \bar{s}q^{-2} \) or \( \xi_\ell \in \bar{\xi}q^{-2} \). Let \( \xi_\ell = z_jq^{-2} \).

Then the combination in the square brackets of (B.18) is just \( F_{a,b,p-1,n-1}^{(l,r)}(\bar{\ell}; \bar{x}| \bar{s}; \{ \bar{w}, \xi_\ell \}; \bar{y}_p| \bar{z}_j) \).

Thus, we obtain

\[
F_{a,b,p,n}^{(l,r)} \bigg|_{y_p \to z_jq^{-2}} = f(\xi_\ell, y_p) f(\xi_\ell, \xi_\ell) f(\xi_\ell, \bar{y}_p) F_{a,b,p-1,n-1}^{(l,r)}(\bar{\ell}; \bar{x}| \bar{s}; \{ \bar{w}, \xi_\ell \}; \bar{y}_p| \bar{z}_j) + \text{reg}. \quad (B.19)
\]

Let now \( \xi_\ell = x_jq^{-2} \). In this case the prefactor \( f(\xi_\ell, \bar{y}_p) \) does not compensate the pole of \( Z_{a,b}^{(l,r)} \) at \( y_p = x_j \). Therefore the combination in the squared brackets in (B.18) is not directly \( F_{a,b,p-1,n-1}^{(l,r)} \).

In order to overcome this problem we use (B.21) at \( n = 1 \). We have

\[
Z_{a,b}^{(l,r)}(\bar{\ell}; \bar{x}, x_j|x_jq^{-2}) = \sum_{i=1}^{a} K_{1}^{(l,r)}(x_j|t_i) f(t_i, \xi_\ell) f(\bar{x}_j, t_i) f(\bar{s}, t_i) \\
\times Z_{a-1,b}^{(l,r)}(\bar{t}_i; \bar{x}| \bar{s}; \{ \bar{w}, t_iq^{-2}, \bar{y}_p \}). \quad (B.20)
\]

Substituting (B.20) into (B.18) we obtain

\[
F_{a,b,p,n}^{(l,r)} \bigg|_{y_p \to x_jq^{-2}} = f(\xi_\ell, y_p) f(\xi_\ell, \xi_\ell) f(\xi_\ell, \bar{y}_p) \sum_{i=1}^{a} K_{1}^{(l,r)}(x_j|t_i) f(t_i, \xi_\ell) f(\bar{x}_j, t_i) f(\bar{s}, t_i) \\
\times f(\xi_\ell, \bar{y}_p) Z_{a-1,b}^{(l,r)}(\bar{t}_i; \bar{x}| \bar{s}; \{ \bar{w}, t_iq^{-2}, \bar{y}_p \}) + \text{reg}. \quad (B.21)
\]
hence,

\[ F_{a,b,p,n}^{(l,r)} \bigg|_{y_p \rightarrow x_j, q^{-2}} = f(\xi_l, y_p)f(\xi_l, \bar{s}_i)f(\xi_l, \bar{y}_p) \sum_{i=1}^{a} K_1^{(l,r)}(x_j|t_i)f(t_i, \bar{\xi}_l)f(\bar{s}, t_i) \times F_{a-1,b,p-1,n}^{(l,r)}(\bar{t}_i; \bar{x}_j|\bar{s}; \{\bar{w}, t_i q^{-2}\}; \bar{y}_p|\bar{z}) + \text{reg}. \quad (B.22) \]

Thus, we have reduced the residues of \( F_{a,b,p,n}^{(l,r)} \) at \( y_p = \xi_l \) to the functions \( F_{a-1,b,p-1,n-1}^{(l,r)} \) or \( F_{a-1,b,p-1,n}^{(l,r)} \).

Consider now the pole of \( \tilde{F}_{a,b,p,n}^{(l,r)} \) at \( y_p = \xi_l \). It occurs if and only if \( \xi_l \in \bar{\xi}_l \). Setting \( \bar{\xi}_l = \{\xi_l, \bar{\xi}_0\} \) and using property (A.6) of \( K^{(r,l)} \) we obtain

\[ \tilde{F}_{a,b,p,n}^{(l,r)} \bigg|_{y_p \rightarrow \xi_l} = \sum (-q)^{\pm k} K_{p-1}^{(r,l)}(\{\bar{s}_l q^{-2}, \bar{\xi}_0\}|y_p)f(\xi_l, y_p)f(\xi_l, \bar{y}_p)f(\bar{s}_l, q^{-2}, \xi_l)f(\bar{\xi}_0, \xi_l)f(\bar{s}_l, \bar{s}_n) \times Z_{a,b-k}^{(l,r)}(\bar{t}; \bar{x}|\bar{s}_n; \{\bar{w}, \xi_0, \xi_l\})f(\bar{\xi}_n, \bar{\xi}_0)f(\bar{y}_p, \bar{s}_l)f(\bar{w}, \bar{s}_l)f^{-1}(\bar{s}_l, \bar{z}) + \text{reg}, \quad (B.23) \]

where the sum is taking over partitions \( \bar{s} \Rightarrow \{\bar{s}_l, \bar{s}_n\} \) and \( \xi_l \Rightarrow \{\bar{\xi}_0, \xi_l\} \). The terms \( f(\bar{s}_l q^{-2}, \xi_l) \) and \( f(\xi_l, \bar{s}_l) \) cancel each other. The terms \( f(\xi_l, y_p)f(\xi_l, \bar{y}_p) \) can be moved out of the sum. The product \( f(\bar{\xi}_0, \xi_l)f(\bar{\xi}_n, \bar{\xi}_0) \) combines into \( f(\bar{\xi}_l, \xi_l) \) and also can be moved out of the sum. We arrive at

\[ \tilde{F}_{a,b,p,n}^{(l,r)} \bigg|_{y_p \rightarrow \xi_l} = f(\xi_l, y_p)f(\xi_l, \bar{s}_l)f(\xi_l, \bar{y}_p) \sum (-q)^{\pm k} K_{p-1}^{(r,l)}(\{\bar{s}_l q^{-2}, \bar{\xi}_0\}|y_p) \times Z_{a,b-k}^{(l,r)}(\bar{t}; \bar{x}|\bar{s}_n; \{\bar{w}, \xi_0, \xi_l\})f(\bar{\xi}_n, \bar{\xi}_0)f(\bar{y}_p, \bar{s}_l)f(\bar{w}, \bar{s}_l)f^{-1}(\bar{s}_l, \bar{z}) + \text{reg}. \quad (B.24) \]

Now we set \( \xi_l = z_j q^{-2} \). Then we simply rewrite

\[ f(\bar{w}, \bar{s}_l)f^{-1}(\bar{s}_l, \bar{z}_j) \]

and we see that the sum over partitions in (B.24) gives \( \tilde{F}_{a,b,p-1,n-1}^{(l,r)}(\bar{t}; \bar{x}|\bar{s}; \{\bar{w}, \xi_l\}; \bar{y}_p|\bar{z}_j) \):

\[ \tilde{F}_{a,b,p,n}^{(l,r)} \bigg|_{y_p \rightarrow z_j q^{-2}} = f(\xi_l, y_p)f(\xi_l, \bar{s}_l)f(\xi_l, \bar{y}_p) \tilde{F}_{a,b,p-1,n-1}^{(l,r)}(\bar{t}; \bar{x}|\bar{s}; \{\bar{w}, \xi_l\}; \bar{y}_p|\bar{z}_j) + \text{reg}. \quad (B.26) \]

It remains to consider the case \( \xi_l = x_j q^{-2} \). Due to (B.24) we have

\[ Z_{a,b-k}^{(l,r)}(\bar{t}; \bar{x}_j, x_j|\bar{s}_n; \{\bar{w}, \xi_0, x_j q^{-2}\}) = \sum_{i=1}^{a} K_1^{(l,r)}(x_j|t_i)f(t_i, \bar{\xi}_l)f(\bar{x}_j, t_i)f(\bar{s}_n, t_i) \times Z_{a-1,b}^{(l,r)}(\bar{t}_i; \bar{x}_j|\bar{s}_n; \{\bar{w}, \bar{\xi}_0, t_i q^{-2}\}). \quad (B.27) \]
Substituting (B.27) into (B.24) we obtain

\[
\tilde{F}_{a,b,p,n}(\bar{\ell}; x; \bar{w}; \bar{y})|_{y_p \to x, q^{-2}} = f(\ell, y_p) f(\ell, y_p) \sum_{i=1}^{a} K_{1}^{(l,r)}(x_j t_i) f(t_i, t_i) f(x_j, t_i) \\
\times \sum (-q)^{+k} K_{p-1}^{-1}(\{\bar{s}, q^{-2}, \bar{q}\})|\bar{y} p) Z_{a-1, b-2, k}^{(l,r)}(\bar{t}_i; x_j|\bar{s}_a; \{\bar{w}, \bar{q}_{0}, t_i q^{-2}\}) \\
\times f(\bar{s}_i, \bar{s}_p) f(\bar{q}_0) f(\bar{q}_0, \bar{s}_i) f(\bar{w}, \bar{s}_i) f(\bar{s}_p, t_i) f(\bar{s}_i, \bar{z}). \tag{B.28}
\]

Now we observe that

\[
f(\bar{w}, \bar{s}_i) f(\bar{s}_i, t_i) = \left[f(\bar{w}, \bar{s}_i) f(t_i q^{-2}, \bar{s}_i)\right] f(\bar{s}_i, t_i). \tag{B.29}
\]

Substituting this into (B.28) we arrive at

\[
\tilde{F}^{(l,r)}_{a,b,p,n}(\bar{\ell}; \bar{x}; \bar{w}; \bar{y})|_{y_p \to \ell} = f(\ell, y_p) f(\ell, y_p) \sum_{i=1}^{a} K_{1}^{(l,r)}(x_j t_i) f(t_i, t_i) f(x_j, t_i) f(\bar{s}_i, t_i) \\
\times \sum (-q)^{+k} K_{p-1}^{(l,r)}(\{\bar{s}, q^{-2}, \bar{q}\})|\bar{y} p) Z_{a-1, b-2, k}^{(l,r)}(\bar{t}_i; x_j|\bar{s}_a; \{\bar{w}, \bar{q}_{0}, t_i q^{-2}\}) f(\bar{s}_i, \bar{s}_p) f(\bar{q}_0) f(\bar{q}_0, \bar{s}_i) f(\bar{w}, \bar{s}_i) f(\bar{s}_p, t_i) f(\bar{s}_i, \bar{z}). \tag{B.30}
\]

Evidently the sum over partitions in (B.30) gives \( \tilde{F}^{(l,r)}_{a-1,b,p-1,n}(\bar{t}_i; x_j|\bar{s}_a; \{\bar{w}, t_i q^{-2}\}; \bar{y} p|\bar{z}) \), therefore

\[
\tilde{F}^{(l,r)}_{a,b,p,n}(\bar{\ell}; \bar{x}; \bar{w}; \bar{y})|_{y_p \to \ell} = f(\ell, y_p) f(\ell, y_p) \sum_{i=1}^{a} K_{1}^{(l,r)}(x_j t_i) f(t_i, t_i) f(x_j, t_i) f(\bar{s}_i, t_i) f^{(l,r)}_{a-1,b,p-1,n}(\bar{t}_i; x_j|\bar{s}_a; \{\bar{w}, t_i q^{-2}\}; \bar{y} p|\bar{z}). \tag{B.31}
\]

Comparing (B.19) with (B.26) and (B.22) with (B.31), and taking into account the induction assumption (B.11) we come to conclusion that the difference \( F^{(l,r)}_{a,b,p,n} - \tilde{F}^{(l,r)}_{a,b,p,n} \) is a bounded function of \( y_p \) as \( y_p \to \ell, \ell = 1, \ldots a+n \). Thus, we have proved that the function \( F^{(l,r)}_{a,b,p,n} - \tilde{F}^{(l,r)}_{a,b,p,n} \) has no poles neither in the points \( y_p = s_i \) nor in \( y_p = \ell \). Hence, this is a polynomial in \( y_p \). Due to (A.7) and (A.10) the polynomial \( F^{(l,r)}_{a,b,p,n} - \tilde{F}^{(l,r)}_{a,b,p,n} \) decreases as \( y_p \to \infty \). Hence, \( F^{(l,r)}_{a,b,p,n} - \tilde{F}^{(l,r)}_{a,b,p,n} = 0 \). The case of the polynomial \( F^{(l)}_{a,b,p,n} - \tilde{F}^{(l)}_{a,b,p,n} \) is slightly more sophisticated. Due to (A.8) and (5.9) we conclude that it is bounded as \( y_p \to \infty \). Hence, it does not depend on \( y_p \). It follows from the representation (5.5) that the highest coefficient \( Z^{(l)}_{a,b}(\bar{f}; x|\bar{s}; \{\bar{y}, \bar{w}\}) \) is proportional to the product of all \( y_i \). Hence, the function \( F^{(l)}_{a,b,p,n} \) vanishes at \( y_p = 0 \). On the other hand in the r.h.s. of (5.28) every term of the sum over partitions contains the right Izergin determinant \( K^{(r)}_{p}([\bar{s}, q^{-2}, \bar{q}]|\bar{y}) \). The latter also is proportional to the product of all \( y_i \), and hence, \( \tilde{F}^{(l)}_{a,b,p,n} = 0 \) at \( y_p = 0 \). Thus, we conclude that \( F^{(l)}_{a,b,p,n} - \tilde{F}^{(l)}_{a,b,p,n} = 0 \). \( \square \)
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