Free Vibration Analysis of Rectangular FGM Plates with a Cutout

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Abstract. Rayleigh – Ritz method was introduced to analyze the free vibration characteristics of functionally graded material (FGM) rectangular plate with a cutout in complex boundary conditions. The virtual spring model was adopted to simulate the complex boundary conditions. The Improved Fourier series was chosen as the admissible functions for its great property. Combining numerical method, the strain energy and kinetic energy of the plate without the cutout and of the cutout were obtained. By doing subtraction, the energy of the plate with a cutout can be obtained. And the elastic potential energy of the border was given. Finally, comparing the results with the references, the accuracy and convergence of this method can be proved, providing a reference for the practical engineering problem.

Keywords: FGM plate; Rayleigh-Ritz method; Energy method; Complex boundary conditions; Cutouts

1. Introduction
Plate and shell structures are widely used in various engineering fields, such as aerospace, ship and ocean engineering, architecture, civil engineering, etc. nowadays. Due to various special uses, cutouts need to be made in plate and shell structures. There must be resonance problems in engineering structures. Therefore, it is very meaningful to analyze the vibration of plate and shell structures with typical cutouts.

Many scholars have studied the problems of plates and shells with different cutouts. Dae Seung Cho (2013) [1] used assumed mode method to analyze the free vibration of rectangular plates with cutouts, the comparison shows that the accuracy of that method is good. At present, the research findings of analysis of plates and shells with cutouts are of a great number [2,3].

With the development of science and technology, composite materials are used in engineering field more and more. There are scholars who have studied the vibration characteristic of functionally graded plates and shells with cutouts. K. Sivakumar (1999) [4] adopted the Ritz finite element model using a nine-noded C\textsuperscript{0} continuity, isoparametric quadrilateral element along with a higher order displacement theory which accounts for parabolic variation of transverse shear stresses to predict the dynamic behavior of composite plates in the presence of cutouts undergoing large amplitude oscillations. P. A. A. Laura (1998) [5] studied the determination of the lower natural frequencies of transverse vibration of simply supported orthotropic plates with circular openings combining the Rayleigh-Ritz method and
the finite element method using a standard code. Achchhe Lal (2012) [6] used the second order statistics of post buckling to analyze the functionally graded materials (FGMs) plates subjected to mechanical and thermal loadings without and with square and circular holes at center having random material properties. Avadesh K Sharma (2016) [7] used finite element method to investigate free vibration of functionally graded plates with circular cutouts and gave regular laws on the effects of volume fraction index, thickness ratio and different boundary conditions on the natural frequencies of plates.

In this paper, Rayleigh Ritz method was researched for solving the vibration problems of rectangular FGM plate with different cutouts. Combining the first order shear deformation theory, the accuracy can be ensured. Numerical examples were given to prove the feasibility for solving the problem. Providing a reliable method for actual engineering problems.

2. Material Properties

The material model in this paper is a mixture of two different materials with different properties. The material properties changed along thickness direction. The elastic material properties vary through the plate thickness according to the volume fractions of the constituents, which can be expressed as:

\[ X(z) = X_m V_m(z) + X_c V_c(z) \]  

(1)

Where \( X \) denotes the effective material property such as density, young's modulus etc. \( X_m \) and \( X_c \) are the properties of the two different materials respectively. \( V_m \) and \( V_c \) are the volume fraction of the material which meet the conditions of:

\[ V_m(z) + V_c(z) = 1 \]  

(2)

\[ V_m = \left(\frac{2z + h}{2h}\right)^k \quad (k \geq 0) \]  

(3)

In which \( k \) is the volume fraction exponent. Taken Eqs. (2) and Eqs. (3) into the Eqs. (1), the following equation can be obtained:

\[ X(z) = (X_m - X_c)\left(\frac{2z + h}{2h}\right)^k + X_c \]  

(4)

The relation between volume fraction and exponent is shown in Figure 1[8].

![Figure 1](image)

**Figure 1.** Variation of the volume fraction through the thickness.

Two common material properties covered in this paper are given as follows: Al, Young's modulus \( E=70\text{GPa} \), density \( \rho=2720\ \text{kg/m}^3 \). \( \text{Al}_2\text{O}_3 \), Young's modulus \( E=380\text{GPa} \), density \( \rho=3800\ \text{kg/m}^3 \).
3. Computational Model
The diagram of the model is shown as Figure 2. Here we consider a rectangular FGM plate with a central circular cutout as the calculation model. To deal with the complex boundary conditions, the virtual spring model is used. Introducing displacement restrained springs and corner restrained springs with adjustable stiffness coefficient along the edges. Supposing the displacement restrained springs constants and corner restrained springs constants are $k_{ij}$ (N/m²), $K_{xij}$ (N/rad) and $K_{yij}$ (N/rad) respectively. The constant of the springs can equal to any values according to different conditions. The simulation of classical boundary conditions is described below. The specific constants are shown in Table 1.

![Figure 2. Calculation model](image)

| Clamped | Simply supported | Free |
|---------|------------------|------|
| $k_{ij}$ (N/m²) | $\infty$ | $\infty$ | 0 |
| $K_{xij}$ (N/rad) | $\infty$ | 0 | 0 |
| $K_{yij}$ (N/rad) | $\infty$ | 0 | 0 |

4. Theoretical Formulations
In the present study, the Rayleigh Ritz method was used to solve the problems. Before derived the characteristic equation, a suitable admissible functions shall be given through which the specific expression of strain energy, kinetic energy and elastic potential energy can be obtained. The first order shear theory was introduced for the sake of accuracy. For the calculation of the energy of the plate with cutout, the energy of the rectangular plate and the energy of the cutout were calculated respectively, and then using the energy of the rectangular plate to minus the energy of the cutout, then the energy of the plate with cutout can be obtain.

Supposing the admissible functions of the rectangular FGM plate with a cutout can be expressed as:

$$w(x, y) = \sum_{m=1}^{M} \sum_{n=1}^{N} A_{mn} \varphi_{mn}(x) \psi_{mn}(y) e^{i\omega t} \quad (5a)$$

$$\beta_{x}(x, y) = \sum_{m=1}^{M} \sum_{n=1}^{N} B_{mn} \varphi_{mn}(x) \psi_{mn}(y) e^{i\omega t} \quad (5b)$$

$$\beta_{y}(x, y) = \sum_{m=1}^{M} \sum_{n=1}^{N} C_{mn} \varphi_{mn}(x) \psi_{mn}(y) e^{i\omega t} \quad (5c)$$

In which $w(x, y)$ is the flexural displacement, $\beta_{x}(x, y)$ is the rotation angle of the normal line along the $xz$ plane, and $\beta_{y}(x, y)$ is the rotation angle of the normal line along the $yz$ plane, $A_{mn}$, $B_{mn}$, $C_{mn}$ are unknown coefficients, $\varphi_{mn}(x)$, $\psi_{mn}(y)$ are functions related to $x$ and $y$ respectively, $e^{i\omega t}$ is harmonic time factor. The improved Fourier series is selected as the admissible function so that when using the
spring model, arbitrary boundary conditions can be satisfied, which can be expressed as [9]:

\[ \varphi_m(x) = \sin(m\pi x) \quad 0 < m < 5 \]
\[ \varphi_m(x) = \cos([m-5]\pi x) \quad m \geq 5 \]

(6a)

\[ \psi_n(y) = \sin(n\pi x) \quad 0 < n < 5 \]
\[ \psi_n(y) = \cos([n-5]\pi x) \quad n \geq 5 \]

(6b)

Where \( m = 1, 2, 3, \ldots M; n = 1, 2, 3, \ldots N. \)

According to the first order shear theory, the strain energy of the rectangular FGM plate can be expressed as:

\[ V_p = \frac{1}{2} D(z) \int_0^1 \int_0^1 \left( \frac{\partial \beta_x}{\partial x} + \frac{\partial \beta_y}{\partial y} \right)^2 \cdot 2(1-\mu) \frac{\partial \beta_x}{\partial x} \frac{\partial \beta_y}{\partial y} \right) \] 
\[ + \frac{1}{2} k_s G(z) h \int_0^1 \int_0^1 \left( \beta_x + \frac{\partial \omega}{\partial x} \right)^2 \] 
\[ + \frac{1}{2} k_g G(z) h \int_0^1 \int_0^1 \left( \beta_y + \frac{\partial \omega}{\partial y} \right)^2 \] 
\[ \text{dxdy} \] 

(7)

In which \( D = E(z) h^3 / 12(1-\mu^2) \) is the bending stiffness, \( E(z) \) is the young's modulus varies along the thick of the plate, \( \mu \) is the poisson's ratio, \( k_s = 6/5 \) is the shear coefficient, \( G = E(z) / 2(1+\mu) \) is the shear stiffness, \( h \) is the thickness of the plate.

The kinetic energy of plate can be expressed as:

\[ T = \frac{1}{2} \rho(z) h \omega^2 \int_0^1 \int_0^1 \left( w^2 + \frac{1}{12} h^2 \left( \beta_x^2 + \beta_y^2 \right) \right) \text{dxdy} \] 

(8)

Where \( \rho(z) \) is the mass density and \( \omega \) is the angular frequency.

The strain energy of the central circular cutout can be expressed as:

\[ V_c = \frac{1}{2} D(z) \int_s \left( \frac{\partial \beta_x}{\partial x} + \frac{\partial \beta_y}{\partial y} \right)^2 \cdot 2(1-\mu) \frac{\partial \beta_x}{\partial x} \frac{\partial \beta_y}{\partial y} \right) \] 
\[ + \frac{1}{2} k_s G(z) h \int_s \left( \beta_x + \frac{\partial \omega}{\partial x} \right)^2 \] 
\[ + \frac{1}{2} k_g G(z) h \int_s \left( \beta_y + \frac{\partial \omega}{\partial y} \right)^2 \] 
\[ \text{dxdy} \] 

(9)

In which \( s \) is the domain of the cutout. The kinetic energy of the central circular cutout can be expressed as:

\[ T_c = \frac{1}{2} \rho(z) h \omega^2 \int_s \left( w^2 + \frac{1}{12} h^2 \left( \beta_x^2 + \beta_y^2 \right) \right) \text{dxdy} \] 

(10)

To simulate the boundary conditions, the elastic potential energy of the increased spring needs to be obtained, which can be expressed as:
The energy functional of the plate with cutout can be expressed as:

\[
V = \frac{1}{2} \int \left[ k_{yy} \frac{\partial^2 w}{\partial y^2} + K_{yy} \beta_y^2 + K_{yy} \beta_y^2 \right] dy + \frac{1}{2} \int \left[ k_{xx} \frac{\partial^2 w}{\partial x^2} + K_{xx} \beta_x^2 + K_{xx} \beta_x^2 \right] dx + \frac{1}{2} \int \left[ k_{xy} \frac{\partial^2 w}{\partial x \partial y} + K_{xy} \beta_x \beta_y + K_{xy} \beta_x \beta_y \right] dx
\]

The energy functional of the plate with cutout can be expressed as:

\[
\Pi = V_p + V_s - V_c - T + T_e
\]

Substituting formula (7), (8), (9), (10), (11) into equation (12), and take the total energy for partial derivatives:

\[
\frac{\partial \Pi}{\partial A_{mn}} = 0, \quad \frac{\partial \Pi}{\partial B_{mn}} = 0, \quad \frac{\partial \Pi}{\partial C_{mn}} = 0
\]

Then the vibration problem of the structure is transformed into the problem of solving eigenvalues, which can be expressed as follows:

\[
(K - \omega^2 M)X = 0
\]

In which, \( K \) is the stiffness matrix, \( M \) is the mass matrix of the structure, and \( X \) is the unknown coefficient vector.

5. Numerical Analysis

In this section, some examples on the rectangular FGM plates with different shaped cutout compared with the reference will be given to validate the good convergence and accuracy of the present method. The material is Al/Al_2O_3. The poisson’s ratio is equal to 0.3, the formula of non-dimension frequency is \( \gamma = \omega \sqrt{\rho \gamma h / D_m} \).

Example 1. Convergence analysis is on the values of truncated number \( M, N \) in the admissible function to make sure that the calculation converges to the result. Square plate with a circular cutout was chosen as an example to test the convergence of the present method. The boundary condition is free(FFFF). The parameters of the model are as follows: the length is \( a = 1 \) m, and the thickness is \( h = 0.01 \) m, the radius of the cutout is \( r = 0.1 \). Table 2 compares the frequencies calculated by using different number of terms in the series. When \( M = N = 13 \), the natural frequencies tend to be a constant, which can be considered that the method has converged.

Example 2. The model is the same as above. The boundary condition is clamped(CCCC). Natural frequencies of the rectangular FGM plates for different values of the width are presented in Table 3 when \( k = 0, 0.5, 1, 2, 10 \). By comparing the results with the references, the accuracy of the method can be proved.

Example 3. Changing the model, the rectangular FGM plate with rectangular cutout is calculated. Table 4 shows the results of the square plates with a square cutout. The length of the plate is \( a = 1 \), the length of the cutout is \( b = 0.3 \). Different boundary conditions are given.
Table 2. Natural frequencies of rectangular plate on different $M$ and $N$ (Hz) ($\gamma = \omega^2 \sqrt{\rho_i h / D_n}$, $k=1$, FFFF)

| Mode | $M=N=8$ | $M=N=9$ | $M=N=10$ | $M=N=11$ | $M=N=12$ | $M=N=13$ | $M=N=14$ |
|------|---------|---------|---------|---------|---------|---------|---------|
| 1    | 10.177  | 10.042  | 10.012  | 9.998   | 9.970   | 9.967   | 9.907   |
| 2    | 14.511  | 14.504  | 14.503  | 14.487  | 14.479  | 14.472  | 14.466  |
| 3    | 18.443  | 18.275  | 18.268  | 18.103  | 18.028  | 17.889  | 17.820  |
| 4    | 26.691  | 26.600  | 26.580  | 26.541  | 26.523  | 26.452  | 26.411  |
| 5    | 26.692  | 26.600  | 26.580  | 26.541  | 26.524  | 26.452  | 26.412  |
| 6    | 47.005  | 46.848  | 46.825  | 46.812  | 46.771  | 46.731  | 46.648  |

Table 3. Comparison of natural frequencies of rectangular FGM plate with a central circular plate ($\gamma = \omega^2 \sqrt{\rho_i h / D_n}$, CCC)

| a/b | Mode | Pre. | Ref.[10] | Pre. | Ref.[10] | Pre. | Ref.[10] | Pre. | Ref.[10] |
|-----|------|------|----------|------|----------|------|----------|------|----------|
| 0   | 1    | 36.42| 36.42    | 31.20| 30.84    | 28.12| 27.80    | 25.57| 23.44    |
| 2   | 71.19| 71.19| 61.08    | 60.31| 55.05    | 54.35| 50.05    | 49.41| 45.88    |
| 3   | 71.19| 71.19| 61.08    | 60.31| 55.05    | 54.35| 50.05    | 49.41| 45.89    |
| 4   | 105.15| 105.15| 87.57    | 89.06| 78.97    | 80.26| 71.76    | 72.96| 65.77    |
| 5   | 127.09| 127.09| 106.32   | 107.65| 95.83   | 97.01| 87.12    | 88.19| 79.88    |
| 6   | 138.12| 138.12| 116.26   | 116.99| 104.78  | 105.43| 95.26    | 95.85| 87.33    |
| 1   | 60.97 | 60.97| 52.18    | 51.63| 47.03    | 46.52| 42.75    | 42.30| 39.22    |
| 2   | 92.79 | 92.79| 78.32    | 78.58| 70.59    | 70.82| 64.18    | 64.38| 58.85    |
| 3   | 145.66| 145.66| 123.72   | 123.37| 111.51  | 111.18| 101.38   | 101.08| 92.95    |
| 4   | 149.21| 149.21| 126.12   | 126.36| 113.66  | 113.87| 103.33   | 103.52| 94.80    |
| 5   | 176.57| 176.57| 147.01   | 149.54| 132.50  | 134.76| 120.45   | 122.52| 110.43   |
| 6   | 220.60| 220.60| 187.30   | 186.87| 168.89  | 168.41| 153.53   | 153.10| 140.60   |
| 1.5 | 1    | 98.51 | 98.51    | 83.94| 83.42    | 75.65| 75.17    | 68.77| 63.13    |
| 2   | 126.64| 126.64| 106.47   | 107.25| 95.95   | 96.65| 87.24    | 87.87| 80.01    |
| 3   | 178.55| 178.55| 151.38   | 151.21| 136.43  | 136.26| 124.03   | 123.88| 113.80   |
| 4   | 249.67| 249.67| 210.30   | 211.46| 189.54  | 190.56| 172.33   | 173.25| 157.92   |
| 5   | 251.77| 251.77| 212.70   | 213.24| 191.70  | 192.17| 174.28   | 174.71| 159.80   |
| 6   | 281.55| 281.55| 234.31   | 238.45| 211.19  | 214.88| 191.99   | 195.36| 176.08   |

Table 4. Comparison of natural frequencies of square FGM plate with a square cutout ($\gamma = \omega^2 \sqrt{\rho_i h / D_n}$)

| Boundary condition | Mode | 0    | 0.5  | 1    | 2    | 5    | 10   |
|--------------------|------|------|------|------|------|------|------|
| SSSS               | 1    | 15.12| 12.80| 11.54| 10.49| 9.94 | 9.63 |
|                    | 2    | 18.87| 16.03| 14.47| 13.15| 12.40| 11.95|
|                    | 3    | 26.36| 22.32| 20.11| 18.29| 17.34| 16.79|
|                    | 1    | 13.10| 11.13| 10.02| 9.12 | 8.61 | 8.19 |
| CSFF               | 2    | 18.33| 15.54| 14.00| 12.74| 12.05| 11.56|
|                    | 3    | 36.89| 31.31| 28.20| 25.67| 24.21| 23.42|
6. Conclusion
Combining the first order shear theory and Rayleigh - Ritz method, by introducing numerical method, the free vibration characteristic of FGM rectangular plates with a cutout was analyzed. According to the numerical examples, the rate of convergence is very fast, which can prove the convergence is very good. By comparing the result of present method with the result of the reference, the error is very small, the accuracy is fairly good. Besides, virtual spring model was adopted to simulate the complex boundary conditions, so that various boundary conditions of the plate can be calculated easily. According to the calculation of the natural frequencies of rectangular FGM plate with a central circular cutout in clamped boundary conditions, when the volume fraction exponent \( k \) is changed, the natural frequencies of the structure have a great change and the frequencies change in a rule, both rise or both fall. If the natural frequencies need to be changed, changing the material’s composition can be considered of which the volume fraction exponent \( k \) can be changed.

Acknowledgment
Authors wish to express their gratitude to the National Natural Science Foundation of China (Contract No. 51579109 and 51479079) and the Chinese Fundamental Research Funds for the Central Universities (HUST: 2016YXZD010) that have supported this work.

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