Taming Non-stationary Bandits: A Bayesian Approach

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Abstract

We consider the multi armed bandit problem in non-stationary environments. Based on the Bayesian method, we propose a variant of Thompson Sampling which can be used in both rested and restless bandit scenarios. Applying discounting to the parameters of prior distribution, we describe a way to systematically reduce the effect of past observations. Further, we derive the exact expression for the probability of picking sub-optimal arms. By increasing the exploitative value of Bayes’ samples, we also provide an optimistic version of the algorithm. Extensive empirical analysis is conducted under various scenarios to validate the utility of proposed algorithms. A comparison study with various state-of-the-art algorithms is also included.

1 Introduction

Background and Motivation. Multi armed bandit (MAB) is a well known paradigm for sequential decision making under uncertainty and partial feedback. In such scenarios, there exists a tension between exploring different options at the cost of losing the optimal action and exploiting the information already acquired at the cost of ignoring the uncertainty. To minimize the loss that can possibly be incurred by choosing suboptimal options, the decision maker has to find a balance between the exploration and exploitation phases. This problem of optimally allocating information acquisition efforts to exploration and exploitation phases is originally proposed in [1] in the context of clinical trials. The formalization of the problem is done in [2], in which each action is viewed as an arm indexed by $i = 1, \ldots, K$, with an unknown probability distribution $\nu_i$, and the pay-off from arm $i$ at each instant $t$, $X_{i,t}$, as an independent draw from $\nu_i$. Depending on the assumed nature of reward structure, MAB problems can be divided into stationary bandits and non-stationary bandits.

Stationary bandit formulation assumes the underlying pay-off probability distribution of each option to be stationary. Hence, statistical properties such as mean, variance etc., remain constant for the entire period of interest. Most widely used metric to compare the desirability of an option over another is its mean pay-off. Because of the stationary behavior of pay-off, we can hope to converge to the best option, at least asymptotically. Seminal work in [3] introduced the technique of upper confidence bounds for the asymptotic analysis of loss incurred while playing a suboptimal option. Notable works in this area include Gittins indices [4] [5][6], upper confidence bound based policies [7][8], probability matching techniques [9][10]. Recent additions to this family are POKER [11], KL-UCB algorithm [12], Bayes-UCB [13] etc., which provide tight performance bounds.

There exists a different kind of bandit formulation where the reward generating process is no longer assumed to be stationary. This makes the problem harder than stationary bandits as there is no single optimal option to which an algorithm can converge. Adversarial bandit formulation is one of the strongest generalizations of this case, where the reward generating process is controlled by an adversary for worst-case per play [14]. Although it may seem hopeless to play such a game, randomizing the process of decision making is proposed as a method for minimizing regret [15][16].
Efforts have also gone into unifying these two separate worlds of stochastic and adversarial bandits [17,18]. Another formulation, commonly referred to as Non-Stationary bandits, does not assume a stationary reward generating process. Dealing with reward processes that can evolve over time, this line of work is of great importance in modelling real world processes and is currently a very active research area [19,20,21,22,23]. For more details about multi armed bandits, interested readers are referred to [24].

Thompson Sampling is one of the oldest algorithms proposed for the trade-off between exploration and exploitation [1]. It works by selecting an arm to pull according to its probability of being the optimal. Recent studies showing strong empirical results [9,25] followed by solid theoretical guarantees [10,13,26] have rekindled the interest in Thompson Sampling method. Another line of work has come up with information theoretic analysis of Thompson Sampling under stationary environments [27,28]. In [29], authors have proved that a variant of Thompson Sampling is asymptotically optimal in non-parametric reinforcement learning under countable classes of general stochastic environments. However, there exists very few results for the analysis of Thompson Sampling in non-stationary cases.

Related Works. One of the earliest works in dynamic bandits with abrupt changes in the reward generation process is the algorithm Adapt-EvE proposed in [19]. It uses a change point detection technique to detect any abrupt change in the environment and utilizes a meta bandit formulation for exploration-exploitation dilemma once change is detected. Authors of [20,30] considered a dynamic bandit setting where the reward evolves as Brownian motion and provided results of regret linear in time horizon $T$. In an effort to combine both stochastic and adversarial regimes, authors of [13] proposed EXP3++ algorithm which achieves almost optimal performance in both cases. A popular belief was that in non-stationary bandits, for high-confidence performance guarantees, the player has to sample all the arms uniformly at least $\Omega(\sqrt{T})$ times. An undesirable effect of this is the growth of regret at $O(\sqrt{T})$. But, [23] showed theoretically that this need not be the case and strong guarantees can be derived with high probability without this requirement. Based on this observation, authors proposed a variant of EXP3 algorithm with Implicit Exploration, called EXP3-IX. Interestingly, however, empirical studies showed that EXP3-IX also sampled arms roughly $\sqrt{T}$ number of times.

In most of the previously discussed algorithms, the ability to respond to the changing environment is made possible either by resetting the algorithm at suitable points or by allowing explicit exploration. An alternate way to tackle the problem of non-stationarity is to reduce the impact of past observations in the current prediction in a systematic manner. By discounting the effect of past observations suitably, the predictions from the model can be made based on more recent samples. In the context of dynamic bandits, this concept of applying an exponential filtering to past observations is suggested in [21]. Extending the idea to Bayesian methods, [31] proposed Dynamic Thompson Sampling (Dynamic TS). By assuming a Bernoulli bandit environment where the success probability evolves as a Brownian motion, authors suggest to decay the effect of past observations in the posterior distribution of the arm being updated. This is done by applying an exponential filtering to the past observations. However, by only discounting past observations of the pulled arm, this algorithm is more suitable for a rested bandit case where the underlying distribution changes only when the arm is played. But in the case of restless bandits, where the underlying distribution of all the arms changes at every time instant, Dynamic TS may perform poorly. One of the trivial cases where this can happen is when the past optimal arm remains stationary during the game, and a previously suboptimal non-stationary arm becomes optimal. Dynamic TS will find it difficult to switch to the new optimal arm, as the statistical properties of past optimal arm remains unchanged, thus missing a chance to explore any other suboptimal arm.

One of the fundamental questions that need to be answered while moving away from the well established stationary bandits is how to ascertain the performance of candidate policies. The absence of a single optimal arm for the entire game makes it difficult to set a proper benchmark for the performance of proposed solutions. In the adversarial bandit setting, the benchmark to be compared against is taken as the best arm at hindsight, which represents a static oracle. Another benchmark that can be used is a dynamic oracle who selects the optimal option at every instant. However, the use of dynamic oracle as a benchmark is less discussed in literature because of the difficulty in mathematical analysis. As mentioned in [22], static oracle can perform quite poorly when compared to dynamic oracle. However, by assuming subtle structures in the variations of reward generating process, [22] was successful in establishing bounds on minimal achievable regret against a dynamic oracle and
developed a near optimal policy, REXP3. However, in real applications, it is difficult to know about structures in the variations of environment.

**Main Contributions.** This paper proposes a Bayesian bandit algorithm for non-stationary environments. Derived from the popular Thompson Sampling (TS) algorithm, our proposed method - *Discounted Thompson Sampling (dTS)* - works by discounting the effect of past observations. Even though a similar technique is proposed in Dynamic TS [31], our method differs from it in two aspects. First, we update parameters of all posterior distributions at every timestep, while Dynamic TS updates the parameters only for the arm it played. Next, our algorithms applies the filtering at all timesteps, but DTS applies it to arm only after the condition $\alpha_k + \beta + k > C$ is met for the played arm. This exponential filtering technique increases the variance of the prior distribution maintained for all the arms, while keeping the mean almost constant unless that arm is played. By increasing the variance of all arms, we increase the probability of picking past inferior arms for exploration. However, by keeping the mean almost constant, we restrain the algorithm from picking inferior arms too often. This makes the algorithm suitable for many non-stationary cases, including the notoriously difficult restless bandit formulation. Further, inspired from the optimistic Bayesian approaches in bandit problems, we add an optimistic version of dTS, Discounted Optimistic Thompson Sampling (dOTS). Numerical verification of the performance of the proposed algorithms is conducted and a comparison with various state-of-the-art algorithms is provided for a variety of worst-case scenarios with dynamic oracle.

## 2 Problem Formulation

Let $\mathcal{K} = \{1, \ldots, K\}$ denotes the set of arms available to the decision maker. Let the horizon of the game be denoted by $T$. Hence, at every time instant $t \in \{1, \ldots, T\}$, the decision maker has to choose one arm $k \in \mathcal{K}$ to play. Let $X_{k,t}$ denote the reward obtained by pulling the $k^{th}$ arm at $t^{th}$ time instant. This instantaneous reward $X_{k,t}$ is a Bernoulli random variable with mean, $\mu_{k,t} = E[X_{k,t}]$. Best arm at any instant $t$ is the arm with highest expected reward at that instant and is denote by $\mu^*_t = \max_{k \in \mathcal{K}} \{ \mu_{k,t} \}$.

In non-stationary systems, the expected rewards will evolve over time. This evolution of reward probabilities can either be abrupt or can show a trend. This sequence of rewards from $k^{th}$ arm is denoted by $\mu_k = \{ \mu_{k,t} \}_{t=1}^T$. Let $\mu = \{ \mu_k \}_{k \in \mathcal{K}}$ denote the vector of sequence of rewards of all arms. Let $\mathcal{P}$ denote the family of admissible policies and let $\pi \in \mathcal{P}$ denote a candidate policy that selects an arm to pull during the game. At each instant, policy $\pi$ selects an arm $I_{\pi}^t$ based on the initial prior $U$ and the past observations $\{X_{I_{\pi}^n,t}\}_{n=1}^{t-1}$. If we denote $\pi_t$ to be the policy at instant $t$, then

$$
\pi_t = \begin{cases} 
\pi_1(U) ; & t = 1 \\
\pi_t(U, X_{I_{\pi}^1,1}, \ldots, X_{I_{\pi}^{t-1},t-1}) ; & t \geq 2
\end{cases}
$$

(1)

Thompson Sampling proceeds by maintaining a prior distribution over the success probability of each Bernoulli arm and sampling from this prior distribution for selecting the arm to play. Beta distribution has two parameters, $\alpha$ and $\beta$, which gets updated according to the rewards seen during the game. For both dTS and dOTS, we have $\alpha, \beta \in \mathbb{R}_{>0}^{[K]}$ along with the discounting factor $\gamma \in [0, 1]$ and hence $\pi_T : [0, 1] \times \mathbb{R}_{>0}^{[K]} \times \mathbb{R}_{>0}^{[K]} \times \{0, 1\}^T \rightarrow \mathcal{K}$.

As mentioned earlier, one of the fundamental questions that arises is the optimal policy against which the candidate policies can be benchmarked. We follow the notion of dynamic oracle [22] as the optimal policy for comparing the performance of the algorithms. Dynamic oracle optimizes the expected reward at each time instant over all possible actions. Regret $\mathcal{R}^\pi(T)$, is defined as the difference between the expected cumulative reward from dynamic oracle and the expected reward from the policy under test, $\pi$. Hence, regret is defined as

$$
\mathcal{R}^\pi(T) = \sum_{t=1}^{T} \mu^*_t - E_{\pi,\mu} \left[ \sum_{t=1}^{T} X_{I_{\pi}^*,t} \right]
$$

(2)
where the expectation $\mathbb{E}_{\pi, \mu}[\cdot]$ is taken over both the randomization in the policy and the randomization in the environment. To provide a normalized version, our experiments use $\frac{1}{T} \mathcal{R}^\pi(T)$ as the performance metric.

## 3 Discounted Thompson Sampling

This section introduces the Discounted Thompson Sampling (dTS) algorithm. As mentioned earlier, the key idea used in designing the algorithm is to systematically increase the variance of the prior distributions maintained for unexplored arms. Hence, the probability of picking them will get increased. Another key feature is that, while modifying the distribution to increase its variance, the mean of the distributions are kept almost constant between the plays. Based on the Bayesian approach, the algorithm will sample from this modified distribution to select the arm to play. The mean will get modified only for the arm which is played. The algorithm is listed in Algorithm 1.

### Algorithm 1 Discounted Thompson Sampling (dTS)

**Parameters**: $\gamma \in [0, 1]$, $\alpha_0, \beta_0 \in \mathbb{R}_{\geq 0}$, $K = |K| \geq 2$

**Initialization**: $S_k = 0, F_k = 0 \quad \forall k \in \{1, \ldots, K\}$

**for** $t = 1, 2, \ldots, T$ **do**

**for** $k = 1, \ldots, K$ **do**

$\theta_k(t) \sim \text{Beta}(S_k + \alpha_0, F_k + \beta_0)$

**end for**

Play arm $I^*_t := \arg \max_k \theta_k(t)$ and observe reward $r_t$.

Perform a Bernoulli trial with success probability $\tilde{r}_t$.

Update $S_{I^*_t} \leftarrow \gamma S_{I^*_t} \cdot r_t$ and $F_{I^*_t} \leftarrow \gamma F_{I^*_t} \cdot (1 - r_t)$.

Update $S_k \leftarrow \gamma S_k$ and $F_k \leftarrow \gamma F_k; \forall k \neq I^*_t$.

**end for**

Here, $\alpha_0$ and $\beta_0$ are used as the initial values of parameters for the Beta prior distributions. Comparing to TS, dTS discounts the past value of $S_k$ and $F_k$ before updating it with the current reward. This discounting is performed even for the arms which are not played at this time instant. By setting $\alpha_0 = 1$ and $\beta_0 = 1$ along with $\gamma = 1$, we get the traditional Thompson Sampling.

Let $[\mathcal{A}]$ denote the indicator function that event $\mathcal{A}$ has occurred. From Algorithm 1, we can get the update equation for parameters of prior distributions as $S_{k,t+1} \leftarrow \gamma \cdot S_{k,t} + [I^*_t = k][r_t = 1]$ and $F_{k,t+1} \leftarrow \gamma \cdot F_{k,t} + [I^*_t = k][r_t = 0]$. By taking expectation over the randomization in the algorithm and the environment, we can write the expected values of prior parameters as

\[
\mathbb{E}S_{k,t+1} = \gamma \mathbb{E}S_{k,t} + \mu_{k,t} \mathbb{P}(I^*_t = k) \quad \text{and} \quad \mathbb{E}F_{k,t+1} = \gamma \mathbb{E}F_{k,t} + (1 - \mu_{k,t}) \mathbb{P}(I^*_t = k),
\]

where $\mathbb{P}(I^*_t = k)$ is the probability of selecting arm $k$ at instant $t$ given all the history $\{r_{I^*_t,n}\}_{n=1}^{t-1}$.

Neglecting the effect of $\alpha_0$ and $\beta_0$, for the arms which are not played at this instant, we have the posterior mean as

\[
\mu_{k,t+1} = \frac{S_{k,t+1}}{S_{k,t+1} + F_{k,t+1}} = \frac{\gamma \cdot S_{k,t}}{\gamma \cdot S_{k,t} + \gamma \cdot F_{k,t}} = \mu_{k,t}
\]

and posterior variance as

\[
\sigma^2_{k,t+1} = \frac{S_{k,t+1} \cdot F_{k,t+1}}{(S_{k,t+1} + F_{k,t+1})^2} \geq \frac{\mu_{k,t}(1 - \mu_{k,t})}{\gamma \cdot S_{k,t} + \gamma \cdot F_{k,t} + 1} = \sigma^2_{k,t}.
\]

From (4) and (5), we can see that by discounting the past values of $S_k$ and $F_k$ before updating the parameter of the posterior distribution, dTS algorithm is able to increase the variance of the prior distributions while keeping the mean almost constant for the arms which are not pulled.
4.1 Probability of picking sub-optimal arm

Consider a two armed bandit with non-stationary reward structure. Without loss of generality, let arm with index 1 be the optimal arm. Thompson sampling selects the next arm to play by drawing two independent samples from the Beta distributions maintained for each arm. Each arm will start with initial values of $\alpha_{i,0}$ and $\beta_{i,0}, i = \{1, 2\}$. Also denote the parameter values at time instant $t$ by $\alpha_{i,t}$ and $\beta_{i,t}$. We are interested in finding the probability with which Thompson sampling will pick the suboptimal arm. Let $\theta_{1,t} \sim Beta(\alpha_{1,t}, \beta_{1,t})$ be the sample for optimal arm and $\theta_{2,t} \sim Beta(\alpha_{2,t}, \beta_{2,t})$ for suboptimal arm. We are interested in finding $\mathbb{P}(\theta_{2,t} > \theta_{1,t})$.

From [33], we have

$$\mathbb{P}\left(\frac{\theta_1}{\theta_2} = \omega\right) = \frac{B(\alpha_1 + \alpha_2, \beta_2)}{B(\alpha_1, \beta_1)B(\alpha_2, \beta_2)} \omega^{\alpha_1 - 1} \text{Hypergeom}\left[^{\alpha_1 + \alpha_2 - 1}_{\alpha_1 + \alpha_2 + \beta_2}; \omega\right] ; 0 < \omega \leq 1$$

(8)

where $B(\alpha, \beta)$ is the Beta function and $\text{Hypergeom}\left[^{\alpha_1 + \alpha_2 - 1}_{\alpha_1 + \alpha_2 + \beta_2}; \omega\right]$ is Gauss hypergeometric function.

The probability of picking a sub-optimal arm can now be written as

$$\mathbb{P}(\theta_2 > \theta_1) = \mathbb{P}\left(\frac{\theta_1}{\theta_2} < 1\right) = \int_0^1 \mathbb{P}\left(\frac{\theta_1}{\theta_2} = \omega\right) d\omega$$

$$= \frac{B(\alpha_1 + \alpha_2, \beta_2)}{B(\alpha_1, \beta_1)B(\alpha_2, \beta_2)} \int_0^1 \omega^{\alpha_1 - 1} \text{Hypergeom}\left[^{\alpha_1 + \alpha_2 - 1}_{\alpha_1 + \alpha_2 + \beta_2}; \omega\right] d\omega$$

$$= \frac{B(\alpha_1 + \alpha_2, \beta_2)}{B(\alpha_1, \beta_1)B(\alpha_2, \beta_2)} \frac{1}{\alpha_1} \text{Hypergeom}\left[^{\alpha_1 + \alpha_2 - 1}_{\alpha_1 + \alpha_2 + \beta_2}; 1\right]$$

(9)
where $pF_q(\cdot)$ is the Generalized Hypergeometric function.

These expressions hold for $\beta_0 > \frac{1}{2}$ (See supporting material for the details).

5 Numerical Analysis

This section includes results of numerical evaluation of the proposed algorithm in various synthetic environments. Results also include comparison of the proposed algorithm against various state-of-the-art algorithms. All the results provided are averaged over 5000 independent runs.

5.1 Regret over time

To study how regret grows over time, we consider three different environments. These environments are similar to the environment discussed in [22]. In all the environments, we simulated a four armed Bernoulli bandit with changing success probability. The parameters for the comparison algorithms are taken to be the optimal values for each environment as discussed in their respective papers. For more information about environment and calculation of parameter values for algorithms, refer supplementary material.

**Slow Varying Environment:** In slow varying environment, we took the success probability of each arm as a sinusoidal function in time, limited between 0 and 1. To make the environment change slowly, the period of all sinusoidals are set as 1000 timesteps. To have different success probabilities at each instant, offset of sinusoidals are taken as $0, \frac{\pi}{2}, \pi$ and $\frac{3\pi}{2}$. The results are given in Figures 1a and 1b.

**Fast Varying Environment:** For fast varying environment, the period of the sinusoidal discussed above is taken as 100. The offsets are kept as same as that for slow varying environment. Results are given in Figures 1b and 1c.

**Abruptly Varying Environment:** For abrupt variations in environment, we assumed the success probability of each arm changes abruptly once in a period, going from 0 to a higher value and stays there. We assume an environment where all the 4 arms start with $p = 0$. Arm-1 changes to $p = 0.10$ at $t = 50$, arm-2 to $p = 0.37$ at $t = 100$, arm-3 to $p = 0.63$ at $t = 150$ and arm-4 to $p = 0.90$ at $t = 200$. All arms come down to $p = 0$ at $t = 250$ and the cycle starts again. Results are given in Figures 1c and 1d.

![Figure 1](image-url)

Figure 1: Performance comparison in different scenarios

From the results in Figure 1, we can observe that the algorithms proposed for non-stationary cases - Dynamic TS, REXP3, dTS and dOTS - are able to maintain an almost constant normalized regret in
various non-stationary environments. However, Thompson Sampling, primarily being an algorithm for stationary environments, experiences a growing normalized regret as expected.

From figures 1a and 1c we can see that all algorithms except REXP3 experiences a low regret in the initial phase. Then the regret for both TS and Dynamic TS grows over the time while dTS and dOTS maintains a low average regret. REXP3, being an algorithm based on the idea of randomized exploration, experiences trouble in catching up with the dynamic oracle during initial phase. By the time it gets enough samples to confidently identify the optimal arm, the optimal arm itself changes to another and hence, again faces trouble in keeping up with the dynamic oracle. However, after sufficient number of timesteps (depending on the environment), it seems to be capable of maintaining a constant average regret.

Both dTS and dOTS are able to show a clear learning experience in all the three scenarios. As the variance of all the prior distributions keep increasing whenever it is not played, dTS and dOTS do not have a trouble in keeping up with dynamic oracle. Even though Dynamic TS also uses similar discounting technique, the absence of a mechanism to systematically increase the variance of unplayed arms hurts it during the later phase of the game, where it finds itself difficult to explore the arms.

Another interesting observation is that, performance of dOTS is better than dTS; even in a fast varying environment. This may seem counter-intuitive because by increasing the exploitative value of each arm, dOTS tends to pick arms with better empirical mean. However it appears that this approach is helping dOTS to take better decisions, even in abruptly changing environments.

An extensive study of comparison against various algorithms is provided in the supplementary material.

5.2 Increasing the number of arms

Figure 2 shows how normalized regret behaves as the number of arms in the bandit increases. Bandit environment for this experiment is similar to the one described in previous section, except for the number of arms. For sinusoidal environments, the offset of each arm is set at equal intervals between 0 and 2π. For abrupt changes, the change points are distributed equally in the period for each arm. The results are provided for a run of 5000 timesteps.

From the figure, we can observe that both dTS and dOTS are able to maintain a good margin of regret in both slow and fast varying environments. But in abruptly changing environment, both the algorithms eventually reaches the same regret as comparison algorithms, all of them growing with increase in number of arms. Again, Thompson Sampling is showing a high regret in both slow and fast varying environments. A key point to note is that the parameter values of the none of the algorithms are recalculated for the increased number of arms. Hence, this experiment indirectly shows how strongly the values of parameters depend on the number of arms.

As the number of arms increases, REXP3 shows an increased regret in slow varying environment. This could be because of the strong dependence of parameters of REXP3 on the number of arms. In abruptly varying environment, this effect is more profound and REXP3 experiences the maximum
regret, even more that that of TS. These experiments also show that the parameter value of dTS and dOTS may not be strongly dependent on the number of arms.

6 Concluding Remarks

In this paper, we proposed a Bayesian algorithm for non-stationary bandits. Based on the idea of systematically increasing the variance of prior distribution of unplayed arms and utilizing exponential filtering to forget the past observations, Discounted Thompson Sampling (dTS) and Discounted Optimistic Thompson Sampling (dOTS) algorithms are able to perform better in different worst case scenarios. We also provided the exact expression for the probability of picking a sub-optimal arm in a two armed bandit setting. To the best of our knowledge, this is the first time this probability is discussed for non-integer parameters of Thompson Sampling. By providing a general expression for the probability of a sub-optimal arm being picked, we believe that this work will help in analyzing the popular TS algorithm in a wide range of scenarios. Future work can include analyzing the performance of the proposed algorithms and bounding the regret incurred during the game.

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Taking the Gauss Hypergeometric function is defined as

\[
\begin{align*}
_2F_1\left[ a_1, a_2; z \right] = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k}{(b_1)_k} \frac{z^k}{k!}
\end{align*}
\]

where \((q)_n = \frac{\Gamma(q + n)}{\Gamma(q)}\).

Here, \(\Gamma(\cdot)\) is the Gamma function.

By substituting \(a_1 = S_{1,t} + \alpha_0 + S_{2,t} + \alpha_0\), \(a_2 = 1 - (F_{1,t} + \beta_0)\), and \(b_1 = S_{1,t} + \alpha_0 + S_{2,t} + \alpha_0 + F_{2,t} + \beta_0\) in the condition \(b_1 - a_1 - a_2 > 0\), we get

\[
\beta_0 > \frac{1 - (F_{1,t} + F_{2,t})}{2}
\]

Taking \(F_{k,t} = 0\), we need \(\beta_0 > \frac{1}{2}\). Hence, for the probability expression to hold at all times, the \(\beta\) parameter should be greater than \(\frac{1}{2}\) for Beta prior.

For \(_3F_2\left[ a_1, a_2, a_3; z \right]\), we have

\[
_3F_2\left[ a_1, a_2, a_3; z \right] = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k(a_3)_k}{(b_1)_k(b_2)_k} \frac{z^k}{k!} |z| < 1 \land |z| = 1 \land Re\left( \sum_{j=1}^{2} b_j - \sum_{j=1}^{3} a_j \right) > 0
\]
Appendix B: Comparison with state-of-the-art algorithms

This section includes the results of extensive simulation studies we did to evaluate dTS and dOTS. First, we introduce the environment used for the simulation studies. Then, we study the impact of gamma parameter in the regret of the algorithm. Later in this section, we provide results of dTS and dOTS compared with five state-of-the-art algorithms - REXP3, Dynamic TS, Discounted-UCB, Sliding Window UCB and EXP3-IX - for non-stationary bandits.

Environment

For comparing against various state of the art algorithms, the following environments are used:

1. Fast Varying Environment: To simulate a fast varying environment, we consider a bandit with four Bernoulli arms where the expected rewards vary in a sinusoidal fashion with a period of 100 time steps. Each arm has an offset which is an integer multiple of $\pi/2$ to create variation.
2. Slow Varying Environment: For slow varying environment, a bandit similar to the one mentioned above is taken, but the period of the sinusoidal wave is taken as 1000 time steps.
3. Abruptly Varying Environment: For abruptly changing bandit, we designed a four armed Bernoulli bandit with the expected reward of each arm abruptly going from 0 to some finite value at random time and it stays at new value till the end of period. The optimal arm switches every 50 time steps and this entire procedure is repeated every 250 time steps as shown in Figure 3c.

![Figure 3](image-url)

Figure 3: Different Environments used for evaluation of algorithms

All the results provided below are averaged over 1000 independent runs. Also, unless stated otherwise, all the experiments are run for a time horizon, $T = 5000$.

Effect of $\gamma$

The discounting factor introduced in the proposed algorithm for the trade-off between remembering and forgetting plays an important role in the performance of the algorithm. A high value of $\gamma$ will make the algorithm remember the past rewards for more time and a low value of $\gamma$ will make the algorithm forget past rewards faster. Hence, setting the value of $\gamma$ has an impact on the performance of algorithm. In this section, we study the effect of setting different values for $\gamma$ and try to analyse its impact.

![Figure 4](image-url)

Figure 4 shows the variation in regret of dTS for different values of $\gamma$. Figure 5 shows the same for dOTS. Both the experiments were run for a horizon length of 5000 time steps.
From both the results, we can see that the regret behaves as a smooth function of $\gamma$ in the case of dTS and dOTS for slow and fast varying environments. But in case of abruptly varying environment (in the special environment we mentioned), regret peaks at $\gamma = 0.95$, which is a surprise. With $\gamma = 1.0$, dTS (and dOTS) acts like TS (and OTS), which is an algorithm for stationary bandit case. But the results here show that, at $\gamma = 0.95$, both dTS and dOTS have difficulty in forgetting the past.

**REXP3**

REXP3 is proposed in [22] as a near optimal policy for non-stationary bandits. The algorithm has two parameters - $\gamma \in [0, 1]$, the egalitarianism factor and $\Delta_T$, the time duration for which one arm will stay as optimal arm.

For sinusoidal environments, the optimal arm switches every $T/4$ timesteps, where $T$ is the period of sinusoidal wave. Hence $\Delta_T = T/4$ is set as the resetting period of REXP3 algorithm. For setting the $\gamma$ parameter, the following formula is used:

$$\gamma = \min\left\{ 1, \sqrt{\frac{K \log K}{(e - 1)\Delta_T}} \right\}$$  \hspace{1cm} (13)

where $K$ is the number of arms. Actual values used for simulation are listed in Table 1.

**Dynamic Thompson Sampling**

Dynamic Thompson Sampling (DTS) proposed in [31] uses the idea of exponential filtering technique to adapt to the changes in the environment. One key difference of DTS with the proposed algorithm is the way the discounting is applied. DTS applies discounting only to the arm it picks to play and also after a particular threshold is crossed($C$). Dynamic Thompson Sampling takes a simple parameter, $C$, which decides when to apply discounting ($\alpha_k + \beta_k > C$) and then how much to discount past rewards (by a factor of $\frac{C}{C+1}$).

As there is no analysis of Dynamic Thompson Sampling available, we used to following heuristic argument to set the value for $C$ in the experiments. Parameter $C$ starts affecting the algorithm only after $\alpha_k + \beta_k > C$. If we know when the change points are occurring in the environment, we can set the value of $C$ to be equal to the time interval between change points. Hence for sinusoidal environments, $C$ can be set to $T/4$ where $T$ is the period of the sinusoidal. For abruptly changing environment, $C$ is taken as the minimum interval between two change points in the environment. Values used in simulation are listed in Table 2.

Results are provided in Section 5.1 of main script.

**Table 1: Comparison with REXP3: Parameters**

| Environment     | Period $T$ | REXP3 $\Delta_T$ | dTS $\gamma$ | dOTS $\gamma$ |
|-----------------|-----------|------------------|-------------|-------------|
| Fast Varying    | 100       | 25               | 0.3593      | 0.40        | 0.40        |
| Slow Varying    | 1000      | 250              | 0.1136      | 0.75        | 0.75        |
| Abruptly Varying| 250       | 25               | 0.5000      | 0.60        | 0.60        |

Dynamic Thompson Sampling
Results are provided in Section 5.1 of main script.

**Discounted-UCB**

Discounted-UCB is proposed in [21] for non-stationary bandit problems. Specifically, the bandit assumed in this case has reward distributions remaining constant over an epoch and which changes at unknown time instants. The algorithm works on the principles of upper confidence bound based policies introduced in [7] and uses a discounting factor $\gamma \in [0, 1]$ to reduce the effect of past rewards on current action selection.

For simulation purposes, the discounting factor $\gamma$ is selected according to (14).

$$
\gamma = 1 - (4B)^{-1} \sqrt{\frac{\Upsilon_T}{T}} \quad (14)
$$

where $\Upsilon_T$ is the number of change points in time time horizon $T$ and $B$ is the bound on the reward. Exact values used for simulation is given in Table 3. All experiments are conducted with $\xi = 0.5$ for D-UCB. Actual values used in simulation are provided in Table 3 and results are shown in Figures 6 - 8.

| Environment       | Time Horizon | D-UCB | dTS | dOTS |
|-------------------|--------------|-------|-----|------|
| Fast Varying      | 500          | 20    | 0.9500 | 0.40  | 0.40  |
| Slow Varying      | 2500         | 10    | 0.9842 | 0.75  | 0.75  |
| Abruptly Varying  | 1000         | 20    | 0.9646 | 0.60  | 0.60  |
Sliding Window - UCB

Sliding Window UCB (SW-UCB) is also proposed in [21] for non-stationary bandit problems. SW-UCB considers only the reward obtained in a past window of time. The length of this window is denoted by $\tau$ and is calculated as

$$\tau = 2B \sqrt{\frac{T \log T}{Y_T}}. \quad (15)$$

Table 4 contains the actual values used for simulation and results are shown in Figures 9-11.

| Environment       | Time Horizon $T$ | SW-UCB $Y_T$ | dTS $\tau$ | dOTS $\gamma$ | dOTS $\gamma$ |
|-------------------|-----------------|--------------|-------------|---------------|---------------|
| Fast Varying      | 500             | 20           | 24          | 0.40          | 0.40          |
| Slow Varying      | 2500            | 10           | 89          | 0.75          | 0.75          |
| Abruptly Varying  | 1000            | 20           | 37          | 0.60          | 0.60          |
EXP3-IX

EXP3-IX is proposed in [23] for non-stochastic bandits. It showed that, explicit exploration is not necessary to achieve high probability regret bounds in non-stochastic bandits. EXP3-IX has two non negative parameters - $\gamma$, similar to the egalitarianism parameter in EXP3 and $\gamma$, which is the implicit exploration parameter. For high
probability performance bounds, the values of $\eta$ and $\gamma$ is calculated as

$$\eta = \sqrt{\frac{2 \log K}{KT}} \quad \text{and} \quad \gamma = \frac{\eta}{2}$$  \hspace{1cm} (16)$$

Table 4 contains the actual values used for simulation and results are shown in Figures 12-14.

| Environment        | Time Horizon $T$ | EXP3-IX $\eta$ | dTS $\gamma$ | dOTS $\gamma$ |
|--------------------|------------------|----------------|--------------|---------------|
| Fast Varying       | 500              | 0.0263         | 0.132        | 0.40          |
| Slow Varying       | 2500             | 0.01665        | 0.00832      | 0.75          |
| Abruptly Varying   | 1000             | 0.0263         | 0.132        | 0.60          |

Figure 12: Comparison against EXP3-IX in fast varying environment

Figure 13: Comparison against EXP3-IX in slow varying environment
Discussion

From all results shown above, we can observe that dTS and dOTS is able to perform better than the state of the art algorithms. One interesting case in these experiments is the behaviour of Discounted-UCB in abruptly varying environment (Figure 8). We can see that the regret of DUCB is almost comparable to that of dTS, still dOTS being the better. DUCB also uses similar type of discounting as used in dTS and that could be the reason for its good performance in abruptly varying environment. But in case of slow varying and fast varying environments, this effect is not observed.

Even though EXP3-IX is proposed for a non-stochastic environment, its regret is the highest among all the algorithms compared, for fast and slow varying cases. It’s worth exploring the impact of Implicit Exploration factor (γ) in this context to analyse the cause.

Figure 14: Comparison against EXP3-IX in abruptly varying environment