INFINITE PERIODIC DISCRETE MINIMAL SURFACES
WITHOUT SELF-INTERSECTIONS

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1. INTRODUCTION

The goal of this article is to show existence of examples of discrete triply-periodic minimal surfaces that are modelled on smooth triply-periodic minimal surfaces. For each smooth minimal surface model we consider, we show how one can find a variety of corresponding discrete minimal surfaces. We restrict ourselves to discrete surfaces with a high degree of symmetry with respect to their density of vertices, and thus they have a highly discretized appearance. The advantage of such a restriction is two-fold: (1) we can give explicit mathematical proofs of minimality without relying on numerics, and (2) we can make changes in the symmetries that would not be allowed in the smooth case. In general one can also consider discrete minimal surfaces with finer triangulations, but considering only highly discretized examples is still sufficient to show existence of many discrete triply-periodic minimal surfaces corresponding to a single smooth one. For motivational purposes, we first briefly introduce smooth minimal surfaces.

1.1. Smooth minimal surfaces. Soap films that do not contain bounded pockets of air are surfaces that minimize area with respect to their boundaries. Smooth compact minimal surfaces are mathematical models for soap films, because they are (by definition) surfaces that are critical for area with respect to all smooth variations that fix their boundaries. By computing the first derivative of area for a smooth variation of a general surface, one finds that minimal surfaces are also those whose two principal curvatures at each point are equal and opposite. Since the mean curvature at a point on a surface is the average of the principal curvatures, a minimal surface is then one for which the mean curvature is zero at every point.
The simplest example of a minimal surface is the flat plane. Another example is
the catenoid, which is a surface of revolution that can be parametrized by
\[ \{(\cosh x \cos y, \cosh x \sin y, x) \in \mathbb{R}^3 \mid x \in \mathbb{R}, y \in (0, 2\pi) \subset \mathbb{R} \} , \]
where \( \mathbb{R} \) denotes the real numbers and \( \mathbb{R}^3 \) denotes Euclidean 3-space. (Note that
by restricting to soap films not containing bounded pockets of air, we have ruled
out surfaces like the sphere, which is certainly a soap film but also has nonzero
mean curvature.)

As our goal is to study discrete minimal surfaces, with smooth minimal surfaces
playing only a suggestive role, we will not go into more detail here about basic
properties of the smooth case. (Some fine general introductions to smooth minimal
surfaces are [5], [6], [7], [12], [14], [18] and [20].) We will simply go directly to this
definition:

**Definition 1.1.** A smooth minimal surface in \( \mathbb{R}^3 \) is a \( C^\infty \) immersion \( f : \mathcal{M} \to \mathbb{R}^3 \) of a 2-dimensional manifold \( \mathcal{M} \) whose mean curvature is identically zero; or
equivalently, the map \( f \) is critical for area with respect to all smooth variations
compactly-supported in the interior of \( \mathcal{M} \).

A triply-periodic smooth surface is one that is periodic in three independent
directions of \( \mathbb{R}^3 \), i.e. \( f \) and \( f + \vec{v}_j \) have equal images for three independent constant
vectors \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \) in \( \mathbb{R}^3 \). There are a wide variety of smooth triply-periodic minimal
surfaces, as can be seen by looking at papers of H. Karcher, K. Polthier, A. Schoen,
W. Fischer and E. Koch [4], [10], [12], [13], [28], for example. We show a few
examples here: the central surface in Figure 1 was named the superman surface by
W. Meeks [14] (one special case of surfaces of this type is the Schwarz D surface);
the second surface in the second row of Figure 2 is a generalized Schwarz P surface
(one special case of surfaces of this type is the original Schwarz P surface); the
smooth Schwarz CLP surface is shown in the lower-right of Figure 7 and the
smooth triply-periodic Fischer-Koch surface is shown in the lower row of Figure 11.

1.2. **Defining discrete minimal surfaces.** Recently, finding discrete analogs of
smooth objects has become an important theme in mathematics, appearing in a
variety of places in analysis and geometry. So, minimal surface theory being a
subject in geometry that relies heavily on analysis, it is natural to ask how to
define a discrete analog of smooth minimal surfaces. But there is no single definitive
answer; the definition one chooses would depend on which particular properties of
smooth minimal surfaces one would wish to emulate in the discrete case.

For example, a definition by A. Bobenko and U. Pinkall [2] uses discrete in-
tegrable systems, in analogy to smooth integrable systems properties of smooth
minimal surfaces (or, more accurately, of smooth surfaces with possibly-nonzero
constant mean curvature). Their definition is good from the viewpoint of inte-
grable systems, but does not yield discrete surfaces that are critical for area with
respect to variations of their vertices.

Here we will take a variational point of view, so we wish to consider area-
critical discrete surfaces. We choose this definition: A discrete minimal surface
in \( \mathbb{R}^3 \) is a piecewise linear triangulated surface that is critical for area with respect
to any compactly-supported boundary-fixing continuous piecewise-linear variation
that preserves its simplicial structure, as defined in [21] and [25] and Section 2 here.
Although we will define discrete minimal surfaces as just above, we should remark that, even from within the variational point of view, this is not the unique choice of a definition. For example, a broader definition by K. Polthier [24] uses "non-conforming triangulations", unlike the triangulations here which will all be conforming. Non-conforming discrete surfaces are those for which adjacent triangles are required only to intersect at midpoints of their boundary edges, not along entire edges. Then discrete minimality can again be defined variationally. This broader approach is useful for finding pairs of noncongruent isometric discrete minimal surfaces. Such pairs are called conjugate minimal surfaces in the case of smooth minimal surfaces.

1.3. Constructing discrete minimal surfaces. Just as for smooth surfaces, a triply-periodic discrete surface is one that is periodic in three independent directions of \( \mathbb{R}^3 \). As noted at the beginning of this article, we will show that one can generally construct a variety of discrete examples modelled on a single smooth example. (Hence, in this sense, there exist an even wider variety of discrete triply-periodic minimal surfaces than there are smooth ones.) There are at least two possible ways to do this:

- **Method 1** vary the choice of simplicial structure of some compact portion of the surface, and
- **Method 2** vary the choice of rigid motions of \( \mathbb{R}^3 \) that create the complete surface from some compact portion.

To explain these two methods in more detail, imagine a compact portion \( M \) of a smooth triply-periodic minimal surface with piece-wise smooth boundary \( \partial M \) consisting of smooth curves \( \gamma_1, \ldots, \gamma_n \). Suppose that each \( \gamma_j \) is either a straight line segment or a curve in a principal curvature direction of \( M \) that also lies in a plane of \( \mathbb{R}^3 \). (In the latter case we call \( \gamma_j \) a planar geodesic, since it is necessarily a geodesic of \( M \).) A larger minimal surface \( M' \) is constructed from \( M \) by including images of \( M \) under \( 180^\circ \) rotations about the lines containing linear \( \gamma_j \) and under reflections through the planes containing planar geodesic \( \gamma_j \). (The fact that the larger portion \( M' \) is still a smooth minimal surface can be shown using complex analysis, see [10], [12], [18], [20], for example.) The larger portion \( M' \) again has a piece-wise smooth boundary \( \partial M' \) consisting of line segments and planar geodesics, so this procedure
can be repeated again on $M'$. Repeating this procedure on ever-bigger pieces of the surface a countably infinite number of times, one builds the entire complete surface. $M$ is often called a fundamental domain of the complete surface.

For example, the minimal surface on the left-hand side of Figure 1 is a fundamental domain $M$ of a complete triply-periodic smooth minimal surface. The boundary $\partial M$ contains the eight vertices $p_1 = (0, 0, 0), \quad p_2 = (x, 0, 0), \quad p_3 = (x, 0, z), \quad p_4 = (x, y, z),$

$p_5 = (x, y, 0), \quad p_6 = (0, y, 0), \quad p_7 = (0, y, z), \quad p_8 = (0, 0, z),$

for some given positive reals $x, y, z > 0$. Then $\partial M$ is a polygonal loop consisting of eight line segments, from $p_j$ to $p_{j+1}$ for $j = 1, 2, ..., 7$, and finally from $p_8$ to $p_1$. One can construct the entire complete surface, as described above, using only 180° rotations about boundary line segments. A larger piece of this complete surface can be seen in the center of Figure 1. For general values of $x, y, z$, the resulting complete triply-periodic surface is a superman surface. When $x = y$ this surface represents Schwarz’ solution of Gergonne’s problem (see [10] and [13] for more on this), and when $x = y = \sqrt{2} \cdot z$ this surface is the Schwarz D surface.

To construct triply-periodic discrete minimal surfaces modelled on smooth superman surfaces, we first find a discrete version of the smooth fundamental domain $M$. One such example, found numerically using JavaView software [22], is on the right-hand side of Figure 1. However, there are many ways the simplicial structure of the discrete version can be chosen, and a number of them are shown in Figure 4. (The examples in Figure 4 all have coarser simplicial structures, and hence showing their minimality is manageable by direct computation without using a computer, as we will see in Section 4.) The corresponding complete triply-periodic discrete minimal surfaces are then constructed in the same way as in the smooth case, by 180° rotations about boundary line segments. This variety of simplicial structures all based on one smooth superman surface is an example of using Method 1.

To demonstrate Method 2, consider the left-hand minimal surface $M$ in the second row of Figure 2. Its boundary $\partial M$ is two squares in parallel planes, where
one square projects to the other by projection orthogonal to the planes. As in
the previous example, 180° rotations about boundary line segments produces a
complete triply-periodic smooth minimal surface, and a larger piece of this surface
is shown just to the right of $M$ in Figure 2. This example is a Schwarz P surface.
The original Schwarz P surface is the special case that the distance between the
two parallel planes containing $\partial M$ is $\sqrt{2}$ times shorter than the length of each edge
in $\partial M$, and is conjugate to the Schwarz D surface. (For later use, we mention that
the lower-right surface $\hat{M}$ in Figure 2 is also a portion of a Schwarz P surface, now
bounded by six planar geodesics, each lying within one face of a rectangularoid with
square base. The entire complete surface can be built from $\hat{M}$ solely by applying a
set of reflections in equally-spaced planes parallel to the faces of the rectangularoid.
The top one-fourth of $\hat{M}$ equals the bottom half of $M$. So one could choose either
$M$ or $\hat{M}$ as the fundamental piece for constructing a complete Schwarz P surface.)

As with the superman surface, to create discrete analogs of the Schwarz P surface,
one can apply Method 1 and choose amongst many different simplicial structures
for this surface. Two such possibilities are the first two discrete minimal surfaces
in the first row of Figure 2. The first one has 4 squares in parallel planes amongst
its edge set, and the second one has 5 squares in parallel planes amongst its edge
set. In fact, one can make examples with any number $\geq 3$ of such squares in its
dge set, as we will see in Section 5. In the third figure in the first row of Figure 2
a larger portion of a resulting complete discrete triply-periodic minimal surface is
shown.

But let us return to a demonstration of Method 2 on the Schwarz P surface.
Let $P$ be a plane perpendicular to the planes containing $\partial M$ so that $P$ also contains
two disjoint boundary edges in $\partial M$. Consider the surface we would have if we first
included a reflected image of $M$ across $P$, and then created a complete surface by
180° rotations about all resulting boundary line segments. A part of this surface
is shown in the third figure of the second row of Figure 2 (this part is bounded by
three boundary components, one square and two rectangles). In this part in Figure
2 there are three parallel dotted lines, along which the surface is not even a $C^1$
immersion, hence the mean curvature is not defined there, and we cannot call this a
minimal surface. So this construction is forbidden in the smooth case. However, for
the discrete analogs shown in Figure 2 such a construction actually does produce
a discrete complete triply-periodic minimal surface. This construction works in the
discrete case because the notion of ”minimality” is defined only at the vertices of
the discrete surface, as we will see in Section 2. Choosing the simplicial structure
so that there are no vertices in the interiors of the edges comprising $\partial M$ is what
makes this construction possible.

Thus we have varied the rigid motions of $\mathbb{R}^3$ that were used to create the complete
smooth surface from $M$, in a way that cannot be allowed for smooth surfaces, to
make a different kind of discrete minimal surface modelled on $M$. Part of this
surface is shown in the upper-right of Figure 2 (this part also has three boundary
components, one square and two rectangles). One can easily imagine including more
reflections (not just across a single plane $P$), to make infinitely many different kinds
of discrete minimal surfaces modelled on $M$. This is Method 2.

In Section 5 we state our main result. In Sections 4 and 5 as applications of
Method 1, we will see a variety of different simplicial structures making discrete
analogs of the smooth superman and Schwarz P surfaces. Section 6 contains examples modelled on other smooth minimal surfaces. In the last example of Section 5 and the first example of Section 6 there are further applications of Method 2.

2. Discrete Minimal Surfaces

In the smooth case, a compact minimal surface is area-critical for any variation that fixes the boundary. We wish to define discrete minimal surfaces so that they have the same variational property for the same types of variations. We begin by defining discrete surfaces and their variations, and we first give an informal definition:

**Definition 2.1.** (Informal Definition) A discrete surface in $\mathbb{R}^3$ is a $C^0$ mapping $f : \mathcal{M} \rightarrow \mathbb{R}^3$ of a 2-dimensional manifold $\mathcal{M}$ so that each face of some triangulation of $\mathcal{M}$ is mapped to a triangle in $\mathbb{R}^3$. The surface $f(\mathcal{M})$ is embedded if $f$ is an injection.

We define *embedded* in the discrete case without any conditions about nondegeneracy of $f$ (nondegeneracy is meaningless here, as $f$ is only $C^0$). However, we still use this word, to maintain the analogy to embeddedness of smooth surfaces.

We stated the above definition for the intuition it provides, but we will require a more involved definition, which we now give:

**Definition 2.2.** (Formal Definition) A discrete surface in $\mathbb{R}^3$ is a triangular mesh which has the topology of an abstract 2-dimensional locally-finite simplicial surface $K$ combined with a geometric $C^0$ realization $\mathcal{T}$ in $\mathbb{R}^3$ that is piecewise-linear on each simplex. (Because $K$ is a simplicial "surface", each 1-dimensional simplex in $K$ lies in the boundary of exactly one or two 2-dimensional simplices of $K$.) The geometric realization $\mathcal{T}$ is determined by a set of vertices $\mathcal{V} = \{p_1, p_2, \ldots\} \subset \mathbb{R}^3$ corresponding to the 0-dimensional simplices of $K$, and $\mathcal{V}$ could be either finite or countably infinite. The simplicial surface $K$ represents the connectivity of $\mathcal{T}$. The 0, 1, and 2 dimensional simplices of $K$ represent the vertices, edges, and triangles of $\mathcal{T}$.

Let $\mathcal{T} = (p, q, r)$ denote an oriented triangle of $\mathcal{T}$ with vertices $p, q, r \in \mathcal{V}$. Let $pq$ denote an edge of $\mathcal{T}$ with endpoints $p, q \in \mathcal{V}$.

For $p \in \mathcal{V}$, let $\text{star}(p)$ denote the triangles of $\mathcal{T}$ that contain $p$ as a vertex.

The boundary $\partial \mathcal{T}$ of $\mathcal{T}$ is the union of those edges bounding only a single triangle of $\mathcal{T}$. The interior vertices (respectively, boundary vertices) of $\mathcal{T}$ are those that are not contained (respectively, are contained) in $\partial \mathcal{T}$.
We say that $\mathcal{T}$ is complete if $\partial \mathcal{T}$ is empty and if $\mathcal{T}$ is complete with respect to the distance function induced by its realization in $\mathbb{R}^3$.

**Definition 2.3.** Let $\mathcal{V} = \{p_1, p_2, \ldots\}$ be the set of vertices of a discrete surface $\mathcal{T}$. A variation $\mathcal{T}(t)$ of $\mathcal{T}$ is defined as a $C^\infty$ variation of the vertices $p_i$

$p_i(t) : [0, \epsilon) \to \mathbb{R}^3$ so that $p_i(0) = p_i \forall i = 1, \ldots, m$.

The straightness of the edges and the flatness of the triangles are preserved as the vertices $p_i(t)$ move with respect to $t$.

When $\mathcal{T}$ is compact, we say that $\mathcal{T}(t)$ fixes the boundary $\partial \mathcal{T}$ if $p_i(t)$ is constant in $t$ for all $p_i \in \partial \mathcal{T}$. When $\mathcal{T}$ is complete, we say that $\mathcal{T}(t)$ is compactly supported if $p_i(t)$ is constant in $t$ for all but a finite number of vertices $p_i$.

The area of a discrete surface is

$$\text{area} \, \mathcal{T} := \sum_{T \in \mathcal{T}} \text{area} \, T,$$

where $\text{area} \, T$ denotes the Euclidean area of the triangle $T$ as a subset of $\mathbb{R}^3$.

**Lemma 2.1.** Let $\mathcal{T}(t)$ be a variation of a discrete surface $\mathcal{T}$. At each vertex $p$ of $\mathcal{T}$, the gradient of area is

$$\nabla_p \text{area} \, \mathcal{T} = \frac{1}{2} \sum_{T = (p, q, r) \in \text{star}(p)} J(r - q),$$

where $J$ is $90^\circ$ rotation in the plane of each oriented triangle $T$. The first derivative of the area is then given by the chain rule

$$\frac{d}{dt} \text{area} \, \mathcal{T}(t) \bigg|_{t=0} = \sum_{p \in \mathcal{V}} \left\langle \frac{d(p(t))}{dt} \bigg|_{t=0}, \nabla_p \text{area} \, \mathcal{T} \right\rangle.$$

**Proof.** Let $p_i(t)$ be the corresponding variation of each vertex in the vertex set $\mathcal{V}(t)$ of the variation $\mathcal{T}(t)$. Then

$$\text{area} \, \mathcal{T}(t) = \frac{1}{6} \sum_{p(t) \in \mathcal{V}(t)} \left( \sum_{(p(t), q(t), r(t)) \in \text{star}(p(t))} ||(r(t) - p(t)) \times (q(t) - p(t))|| \right),$$

and a computation implies

$$\frac{d}{dt} \text{area} \, \mathcal{T}(t) = \frac{1}{2} \sum_{p(t) \in \mathcal{V}(t)} \left( \sum_{(p(t), q(t), r(t)) \in \text{star}(p(t))} ||r(t) - q(t)|| |\eta(t)|| \right),$$

where $\eta(t)$ is the unit conormal in the plane of the triangle $(p(t), q(t), r(t))$ along the edge $r(t) - q(t)$, oriented in the same direction as $J(r(t) - q(t))$. Restricting to $t = 0$ proves the lemma.

As defined in Section 11, a smooth immersion $f : \mathcal{M} \to \mathbb{R}^3$ of a 2-dimensional complete manifold $\mathcal{M}$ without boundary is minimal if $f$ is area-critical for all compactly-supported smooth variations. In the case that $\mathcal{M}$ is compact with boundary, then $f$ is minimal if it is area-critical for all smooth variations preserving $f(\partial \mathcal{M})$.

We wish to define discrete minimal surfaces $\mathcal{T}$ so that they have the analogous properties, for variations as in Definition 2.3. So when $\mathcal{T}$ is compact, we consider variations $\mathcal{T}(t)$ of $\mathcal{T}$ that fix $\partial \mathcal{T}$; and when $\mathcal{T}$ is complete, we consider variations...
\( \mathcal{T}(t) \) of \( \mathcal{T} \) that are compactly supported. By Lemma 2.1, the condition that makes \( \mathcal{T} \) area-critical for any variation of these types is expressed in the following definition.

**Definition 2.4.** A discrete surface is minimal if

\[
\nabla_p \text{area} \mathcal{T} = 0
\]

for all interior vertices \( p \).

**Remark 2.1.** If \( \mathcal{T} \) is a discrete minimal surface that contains a discrete subsurface \( \mathcal{T}' \) lying in a plane \( \mathcal{P} \), it follows from Equations 2.1 and 4.1 that the discrete minimality of \( \mathcal{T} \) is independent of the choice of triangulation of the trace of \( \mathcal{T}' \) within \( \mathcal{P} \). Thus whenever such a planar part \( \mathcal{T}' \) occurs in the following examples, we will be free to triangulate \( \mathcal{T}' \) any way we please, within its trace in \( \mathcal{P} \).

3. **Results**

For the purpose of stating our main theorem, we give the following two definitions:

**Definition 3.1.** A discrete triply-periodic minimal surface \( \mathcal{T} \) has common topology and symmetry as a smooth triply-periodic minimal immersion \( f: \mathcal{M} \to \mathbb{R}^3 \) if there exists a homeomorphism

\[
\phi : f(\mathcal{M}) \to \mathcal{T}
\]

such that the following statement holds: \( R_s: \mathbb{R}^3 \to \mathbb{R}^3 \) is a rigid motion preserving \( f(\mathcal{M}) \) if and only if there exists a rigid motion \( R_d: \mathbb{R}^3 \to \mathbb{R}^3 \) preserving \( \mathcal{T} \) so that

\[
R_d \circ \phi = \phi \circ R_s|_{f(\mathcal{M})}
\]

and furthermore \( R_s \) is a reflection (resp. translation, rotation, screw motion) if and only if \( R_d \) is a reflection (resp. translation, rotation, screw motion).

**Definition 3.2.** We say that a subsurface \( \mathcal{T}' \) of a complete discrete triply-periodic minimal surface \( \mathcal{T} \) is a fundamental domain if \( \mathcal{T}' \) can be extended to all of \( \mathcal{T} \) by a discrete group of rigid motions \( \{ R_{d,\alpha} \}_{\alpha \in \Lambda} \) generated by

1. reflections across planes containing boundary edges and
2. 180° degree rotations about boundary edges

so that each \( R_{d,\alpha} \) is a symmetry of the full surface \( \mathcal{T} \).

**Remark 3.1.** In the above definition of a fundamental domain, we do not allow rigid motions that do not fix any edges of \( \mathcal{T} \) (thus any fundamental domain of the example in Subsection 4.7 must contain at least 6 triangles, even in the most symmetric case \( x = 1 \)). Also, we do not allow rigid motions that are not symmetries of the full surface \( \mathcal{T} \) (thus any fundamental domain of the example in Subsection 5.2 must contain at least 32 triangles).

We now state our results about embedded triply-periodic discrete minimal surfaces, which involve comparisons to the following smooth minimal surfaces: the superman surfaces (Figure 1), the Schwarz P surfaces (Figure 2), the Schwarz H surfaces, the Schwarz CLP surfaces (Figure 7), A. Schoen’s I-Wp and F-Rd and H-T surfaces, and the triply-periodic Fischer-Koch surfaces (Figure 11). More complete information about these smooth surfaces can be found in [4], [5], [10], [12], [13], [14], [15], [17], [18], [27] and [28].
Theorem 3.1. The following discrete embedded triply-periodic minimal surfaces exist:

1. those with common topology and symmetry as smooth superman surfaces whose fundamental domains contain 4, 5, 6 or 8 triangles;
2. those with common topology and symmetry as smooth Schwarz P surfaces whose fundamental domains contain 1, 2, 6 or 32 triangles, and also a different class of discrete surfaces with common topology and symmetry as smooth Schwarz P surfaces whose fundamental domains contain 2n triangles for any positive integer n;
3. those with common topology and symmetry as smooth Schwarz H surfaces whose fundamental domains contain 2n triangles for any positive integer n;
4. those with common topology and symmetry as smooth Schwarz CLP surfaces whose fundamental domains contain 6 triangles;
5. one with common topology and symmetry as A. Schoen’s smooth I-Wp surface whose fundamental domain contains 5 triangles;
6. one with common topology and symmetry as A. Schoen’s smooth F-Rd surface whose fundamental domain contains 3 triangles;
7. those with common topology and symmetry as A. Schoen’s smooth H-T surfaces whose fundamental domains contain 6 triangles;
8. those with common topology and symmetry as the smooth triply-periodic surfaces of Fischer-Koch whose fundamental domains contain 8 triangles.

To prove this theorem, we need only collect the examples proven to exist in the remainder of this paper, as follows:

Proof. Embedded discrete superman surfaces whose fundamental domains contain 4 (resp. 5, 6, 8) triangles are given in Subsection 4.3 (resp. 4.4, 4.2, 4.1). A second type of embedded discrete superman surfaces whose fundamental domains contain 6 triangles are given in the second to the last paragraph of Subsection 6.1 by Method 2.

Embedded discrete Schwarz P surfaces whose fundamental domains contain 1 (resp. 2, 6, 32) triangles are given in the first example of Subsection 5.1 (resp. the second example of Subsection 5.1, Subsection 5.2, the last paragraph of Subsection 6.1 by Method 2). Other classes of embedded discrete Schwarz P surfaces are given by choosing \( k = 4 \) and \( z_0 = 0 \) and \( j_0 = n \) in Subsection 5.3 for any positive integer \( n \), and here we can choose the fundamental domains to be the 2n triangles between two adjacent meridians and below the plane \( \{(x, y, 0) \mid x, y \in \mathbb{R}\} \).

Embedded discrete Schwarz H surfaces are given by choosing \( k = 3 \) and \( z_0 = 0 \) and \( j_0 = n \) in Subsection 5.3 for any positive integer \( n \), and here again we can choose the fundamental domains to be the 2n triangles between two adjacent meridians and below the plane \( \{(x, y, 0) \mid x, y \in \mathbb{R}\} \).

Embedded discrete CLP surfaces whose fundamental domains contain 6 triangles are given in Subsection 6.1.

Embedded discrete I-Wp and F-Rd surfaces whose fundamental domains contain 5 and 3 triangles, respectively, are given in Subsection 6.2.

Embedded discrete H-T surfaces whose fundamental domains contain 6 triangles are given in Subsection 6.3.

Embedded discrete triply-periodic Fischer-Koch surfaces whose fundamental domains contain 8 triangles are given in Subsection 6.4.
4. Discrete versions of the superman surface

In Sections 4, 5 and 6 we construct discrete triply-periodic minimal surfaces. All of the surfaces we construct are embedded.

To construct examples, we always start with a compact discrete fundamental piece $T$, with given simplicial structure and boundary constraints. The complete triply-periodic discrete surface is then formed by including images of $T$ under a discrete group of rigid motions of $\mathbb{R}^3$. This group of rigid motions is generated by a finite number of $180^\circ$ rotations about lines and/or reflections across planes, and for each edge $pq$ in $\partial T$ this group contains either

- the $180^\circ$ rotation about the line containing $pq$, or
- a reflection across a plane containing $pq$.

To ensure that the resulting complete discrete triply-periodic surface is minimal, Section 2 gave us the following two approaches:

1. Use symmetries of $T$ and of the resulting complete discrete surface to show that Equation (4) holds at the vertices.
2. Locate the vertices of $T$ so that $T$ is area-critical with respect to its boundary constraints.

In the following examples, either approach produces the same conditions for minimality.

As noted in the introduction, we wish to show examples here for which explicit mathematical proofs of minimality are still manageable (without the aid of a computer). So we are limited to examples with a high degree of symmetry with respect to their density of vertices, and thus with a highly discretized appearance. Discrete minimal surfaces that appear more like approximations of smooth minimal surfaces usually can only be found numerically. Numerical examples, with finer simplicial structures, of discrete versions of the superman, Schwarz P, F-Rd, I-Wp and H-T surfaces are shown in [24].

4.1. First example. The fundamental piece $T$ here has eight boundary vertices

$$p_1 = (1, 0, 0), \quad p_2 = (1, 1, 0), \quad p_3 = (1, 1, x), \quad p_4 = (0, 1, x),$$
$$p_5 = (0, 1, 0), \quad p_6 = (0, 0, 0), \quad p_7 = (0, 0, x), \quad p_8 = (1, 0, x),$$

for any given fixed $x > 0$, and has one interior vertex

$$p_9 = \left(\frac{1}{2}, \frac{1}{2}, x\right).$$

There are eight triangles in $T$, which are

$$(p_j, p_{j+1}, p_9), \quad j = 1, \ldots, 7, \quad (p_8, p_1, p_9).$$

The complete triply-periodic surface is generated by including the image of $T$ under $180^\circ$ rotations about each edge of $\partial T$, and then continuing to include the images under $180^\circ$ rotations about each resulting boundary edge until the surface is complete. It is evident from the symmetries of this surface that Equation (4) holds at every vertex. The fundamental piece $T$ and a larger part of the resulting complete surface are shown on the left-hand side and center of the first row of Figure 4 for $x = 1$. The case for some given $x \in (0, 1)$ is shown on the right-hand side of the first row of Figure 4.
4.2. **Second example.** The fundamental piece $T$ here has six boundary vertices

$p_1 = (0, 0, 0), \quad p_2 = (x, 0, 0), \quad p_3 = (x, y, 0),$

$p_4 = (x, y, 1), \quad p_5 = (0, y, 1), \quad p_6 = (0, 0, 1),$

for any given fixed $x, y > 0$, and has one interior vertex

$p_7 = \left( \frac{x}{2}, \frac{y}{2}, \frac{1}{2} \right).$
There are six triangles in $T$, which are

$$(p_j, p_{j+1}, p_7), \ j = 1, \ldots, 5, \ (p_6, p_1, p_7).$$

The complete triply-periodic surface is generated by $180^\circ$ rotations about boundary edges, just as in the previous example. In this example as well, it is evident from the symmetries of this surface that Equation (4) holds at every vertex. Two fundamental pieces $T$ of different sizes and larger parts of the resulting complete surfaces are shown in the second row of Figure 4 ($x = y = 1$ in the first case, and $x < 1 < y$ in the second case).

In the case that $x = y = 1$, this fundamental piece $T$ has the same boundary as a fundamental piece of the smooth Schwarz D surface. Furthermore, for general $x$ and $y$, this surface can be viewed as a discrete analog of the superman surface as follows: Consider the eight-straight-edged polygonal curve from the point $(0, 0, 0)$ to the point $(x, -y, -\frac{1}{2})$ and then to $(x, y, \frac{1}{2})$ and then to $(2x, 0, -\frac{1}{2})$ and then to $(x, y, -\frac{1}{2})$ and then to $(x, y, \frac{1}{2})$ and then to $(0, 0, \frac{1}{2})$ and then back to $(0, 0, -\frac{1}{2})$. This polygonal curve is contained in this discrete surface (although not in its edge set) and is also the boundary of a smooth superman surface.

4.3. Third example. The fundamental piece $T$ here has four boundary vertices

$$p_1 = (0, 0, 0), \ p_2 = (1, 1, 0), \ p_3 = (1, 1, z), \ p_4 = (1, 0, z),$$

for any given fixed $z > 0$, and has one interior vertex

$$p_5 = (a, b, c).$$

There are four triangles in $T$, which are

$$(p_j, p_{j+1}, p_5), \ j = 1, \ldots, 3, \ (p_4, p_1, p_5).$$

Reflecting $T$ across the plane $\{(x_1, 0, x_3) \in \mathbb{R}^3 \mid x_1, x_3 \in \mathbb{R}\}$ and attaching its image to $T$, one has a larger discrete surface containing eight triangles and six boundary edges. One can extend this larger discrete surface to a complete triply-periodic surface by $180^\circ$ rotations about boundary edges, just as in the previous examples. In this example, Equation (4) holds at each vertex $p_1, p_2, \ldots, p_4$ in the resulting complete surface. However, getting this to hold at $p_5$ requires proper choices of $a$ and $b$.

For simplicity, we restrict to the case $z = 1$. Then, by symmetry, we may assume $b = c$. A computation shows that Equation (4) holding at $p_5$ is equivalent to

$$a^2 - 2ab + 3b^2 = (a - b) \sqrt{(1 - a)^2 + (1 - b)^2},$$

$$a^2 - 2ab + 3b^2 = (3b - a) \sqrt{(1 - a)^2 + (1 - b)^2}.$$

The solution to this is

$$b = \frac{1}{2}, \quad a = \frac{3 - \sqrt{3}}{2}.$$  

So when $z = 1$ and $b = c$ and Equation (4) holds, the area gradient is zero at each vertex $p_j$ for $j = 1, 2, \ldots, 9$ in the extended complete triply-periodic discrete surface, and then symmetries of the surface imply the entire complete surface is minimal.

Since the above minimality condition (4)-(6) is a system of two equations in two variables $a$ and $b$, we say the minimality condition here (when $z = 1$) is two-dimensional.
The fundamental piece $\mathcal{T}$ with $z = 1$ is shown on the left-hand side of the third row of Figure 4 and a larger part of the resulting complete surface is shown just to the right of this.

4.4. Fourth example. The fundamental piece $\mathcal{T}$ here has five boundary vertices

$$
p_1 = (0, 0, 0), \quad p_2 = (1, 1, 0), \quad p_3 = (1, 1, z), \quad p_4 = (0, 1, z), \quad p_5 = (0, 0, z)
$$

for any given fixed $z > 0$, and has one interior vertex

$$
p_6 = (a, 1 - a, b).
$$

There are five triangles in $\mathcal{T}$, which are

$$(p_j, p_{j+1}, p_6), \quad j = 1, \ldots, 4, \quad (p_5, p_1, p_6).$$

The complete triply-periodic surface is generated by 180° rotations about boundary edges. In this example, Equation (4) holds at each vertex $p_1, \ldots, p_5$ in the resulting complete surface, and making it hold also at $p_6$ requires proper choices of $a$ and $b$.

Like in the previous example, we can find a pair of explicit equations, in the variables $a$ and $b$, that represent the minimality condition. These equations are similar to those of the previous example, and are slightly more complicated. One can then show the existence of $a$ and $b$ solving this minimality condition.

Two fundamental pieces $\mathcal{T}$ of different sizes ($z = 1$ in the first case, and $z < 1$ in the second case) are shown on the left and right-hand sides of the bottom row of Figure 4. A larger part of the resulting complete surface in the case $z = 1$ is shown in the bottom-middle of Figure 4.

Figure 5. Three different discrete versions of the Schwarz P surface.
5. Discrete versions of the Schwarz P surface

5.1. First two examples. Consider the vertices

\[ p_1 = (3, 0, 6), \quad p_2 = (6, 0, 3), \quad p_3 = (6, 3, 0), \]
\[ p_4 = (3, 6, 0), \quad p_5 = (0, 6, 3), \quad p_6 = (0, 3, 6), \quad p_7 = (3, 3, 3), \]

and let \( \mathcal{T}_1 \) be the planar fundamental domain with the six triangles

\[ (p_j, p_{j+1}, p_7), \quad j = 1, ..., 5, \quad (p_6, p_1, p_7). \]

Also, consider the vertices

\[ p_1 = (3, 0, 6), \quad p_2 = (4, 0, 4), \quad p_3 = (6, 0, 3), \quad p_4 = (6, 2, 2), \]
\[ p_5 = (6, 3, 0), \quad p_6 = (4, 4, 0), \quad p_7 = (3, 6, 0), \quad p_8 = (2, 6, 2), \]
\[ p_9 = (0, 6, 3), \quad p_{10} = (0, 4, 4), \quad p_{11} = (0, 3, 6), \quad p_{12} = (2, 2, 6), \quad p_{13} = (3, 3, 3), \]

and let \( \mathcal{T}_2 \) be the fundamental domain with the twelve triangles

\[ (p_j, p_{j+1}, p_{13}), \quad j = 1, ..., 11, \quad (p_{12}, p_1, p_{13}). \]

We can extend \( \mathcal{T}_j \) (for either \( j = 1, 2 \)) to a complete discrete surface by including the images of \( \mathcal{T}_j \) under the reflections across the planes \( \{(x, y, 6k) \mid x, y \in \mathbb{R}\} \), \( \{(x, 6k, z) \mid x, z \in \mathbb{R}\} \) and \( \{(6k, y, z) \mid y, z \in \mathbb{R}\} \) for all integers \( k \). Furthermore, Equation (4) holds at all vertices of the extended surface, so it is minimal. (See the first two columns of Figure 3)

The surface produced by \( \mathcal{T}_1 \) (resp. \( \mathcal{T}_2 \)) is a simpler (resp. more complicated) version of a discrete Schwarz surface. Note that they are analogous to the bottom-right picture in Figure 2 (The second example \( \mathcal{T}_2 \) was also shown in [25].)

5.2. Third example. Consider the ten vertices

\[ p_j = (a, (-1)^ja, 1), \quad p_{j+2} = (a, 1, (-1)j+1a), \quad p_{j+4} = (a, (-1)j+1a, -1), \]
\[ p_{j+6} = (1, (-1)^ja, a), \quad p_{j+8} = (1, (-1)j+1a, -a), \quad j = 1, 2, \]

and let \( \mathcal{T} \) be the discrete surface with the eight triangles

\[ (p_1, p_2, p_7), \quad (p_2, p_8, p_7), \quad (p_2, p_3, p_8), \quad (p_3, p_4, p_8), \]
\[ (p_4, p_9, p_8), \quad (p_4, p_5, p_9), \quad (p_5, p_6, p_9), \quad (p_6, p_{10}, p_9). \]

Then let \( \mathcal{T} \) be the discrete surface, with 24 vertices and 32 faces, that is made by including the four images of \( \mathcal{T} \) under the rotations about the axis \( \{(0, 0, r) \mid r \in \mathbb{R}\} \) of angles \( 0^\circ, 90^\circ, 180^\circ \) and \( 270^\circ \). This \( \mathcal{T} \) is shown in the upper-right of Figure 3

One can then generate a complete triply-periodic surface by including the images of \( \mathcal{T} \) under reflections across the planes \( \{(x, y, k) \mid x, y \in \mathbb{R}\} \), \( \{(x, k, z) \mid x, z \in \mathbb{R}\} \) and \( \{(k, y, z) \mid y, z \in \mathbb{R}\} \) for all integers \( k \). The result of applying one such reflection is shown in the lower picture of the right-most column of Figure 3

The condition for this discrete triply-periodic surface to be minimal is that

\[ a = \frac{3\sqrt{2} - \sqrt{3}}{6\sqrt{2} - \sqrt{3}}, \]

i.e. for this value of \( a \), Equation (4) holds at every vertex of the surface.
5.3. **Examples based on discrete minimal catenoids.** Here we give two closely-related types of examples based on discrete minimal catenoids. One type is a discrete analog of the Schwarz P surface. The other type is actually an analog of the smooth Schwarz H surface, not the Schwarz P surface. To construct these examples, we will use discrete analogs of the catenoid \[25\], which are described in terms of the hyperbolic cosine function, just as the smooth catenoid was in Equation (4).

The vertices of a discrete minimal catenoid lie on congruent planar polygonal meridians, and the meridians are contained in planes that meet along a single line (the axis) at equal angles. Every meridian is the image of every other meridian by some rotation about the axis. By drawing edges between corresponding vertices of adjacent meridians (i.e., so that these edges are perpendicular to the axis), we have a piecewise linear continuous surface tessellated by planar isosceles trapezoids. We can triangulate each trapezoid any way we please without affecting minimality, as noted in Remark \[26\] so we shall triangulate each trapezoid by drawing a single diagonal edge across it.

Two examples of discrete catenoids are shown in the first two pictures in the upper row of Figure 2. Both of these pictures have adjacent meridians in planes meeting at 90° angles. The first (resp. second) one has four (resp. five) vertices in each meridian. Another example is shown in the left-most picture of Figure 6, where the adjacent meridians lie in planes meeting at 120° angles, and there are four vertices in each meridian.

To explicitly describe discrete catenoids, we need only specify:

1. The axis \(\ell\): let us fix \(\ell = \{(0,0,z) | z \in \mathbb{R}\}\).
2. The angle \(\theta\) between planes of adjacent meridians: let us fix \(\theta = \frac{2\pi}{k}\) for some integer \(k \geq 3\).
3. The locations of the vertices along one meridian.

We can place one meridian in the plane \(\{(x,0,z) | x,z \in \mathbb{R}\}\), and locating its vertices at the following points will ensure minimality of the surface (see \[25\]):

\[
p_j = (r \cosh \left( \frac{1}{r} a (z_0 + j\delta) \right), 0, z_0 + j\delta)
\]

with \(j = j_0, j_0 + 1, ..., j_1\) for some integers \(j_0\) and \(j_1\) \((j_0 < j_1)\), and with

\[a = \frac{r}{\delta} \arccosh \left( 1 + \frac{1}{r^2} \frac{\delta^2}{1 + \cos \theta} \right),\]

where \(r > 0\) and \(\delta > 0\) and \(z_0 \in \mathbb{R}\) are constant. The edges along this meridian are \(p_j p_{j+1}\) for \(j\) between \(j_0\) and \(j_1 - 1\).

For our application, we shall restrict to either \(k = 4\), as in Figure 2, or to \(k = 3\), as in Figure 6. We shall further assume that either

- \(z_0 = 0\) and \(j_0 = -j_1 < 0\), or
- \(z_0 = \frac{\delta}{2}\) and \(j_0 = -j_1 - 1 < -1\).

Either of these conditions will produce a discrete minimal surface \(T\) whose trace has dihedral symmetry. One can then extend \(T\) by 180° rotation about boundary lines to a complete embedded discrete surface in \(\mathbb{R}^3\). To conclude minimality of this complete surface, it remains only to check that Equation (4) holds at any vertex contained in any edge about which a 180° rotation was made, and this is clear from the symmetry of the surface.
Figure 6. Discrete version of the Schwarz H surface.

The case when $k = 4$ and $z_0 = 0$ and $j_0 = -j_1 = -2$ is shown in the second picture of the first row of Figure 2, and a larger portion of the resulting complete minimal surface is shown in the picture just to the right of it. The case when $k = 3$ and $z_0 = \frac{1}{2}$ and $j_0 = -j_1 - 1 = -2$ is shown in the left-most picture of Figure 6, and a larger portion of the resulting complete minimal surface is shown in the middle of Figure 6. When $k = 4$, the analogy to the smooth Schwarz P surface is clear. When $k = 3$, one can imagine a smooth embedded minimal annulus with the same boundary as $\mathcal{T}$, and this surface is called the Schwarz H surface.

As explained in Section 1, there are infinitely many different ways (by using combinations of reflections and $180^\circ$ rotations that are not allowed in the smooth case) to extend $\mathcal{T}$ to a complete discrete minimal surface. Two such ways are shown in the upper right of Figure 2 and another two ways are shown in the center and right-hand side of Figure 6. The two examples in Figure 2 and the central one in Figure 6 can be extended to complete triply-periodic discrete minimal surfaces by $180^\circ$ rotations about boundary edges. The right-most example in Figure 6 can be extended to a complete triply-periodic discrete minimal surface by using horizontal translations perpendicular to the axis $\ell$ that generate a 2-dimensional hexagonal grid, and then by applying vertical translations parallel to $\ell$ of length $2\delta(j_1 - j_0)$. The upper-right examples in both Figures 2 and 6 are applications of Method 2.

Remark 5.1. When $k = 4$ and $j_0 = -j_1 = 1$, and $r$ and $\delta$ are chosen properly, this $\mathcal{T}$ can produce the same surface as $T_1$ produced in Subsection 5.1. The way of triangulating the planar isosceles trapezoids was different in Subsection 5.1, but by Remark 2.1 this is irrelevant to the minimality of the surfaces, and the two examples are the same in the sense that they have the same traces in $\mathbb{R}^3$.

6. Other examples

6.1. Discrete Schwarz CLP surface. The fundamental piece $\mathcal{T}$ here has six boundary vertices

\[ p_1 = (x, 0, 0), \; p_2 = (0, 0, 0), \; p_3 = (0, y, 0), \]
\[ p_4 = (0, y, 1), \; p_5 = (0, 0, 1), \; p_6 = (x, 0, 1) \]

for any given fixed $x, y > 0$, and has one interior vertex

\[ p_7 = (a, b, \frac{1}{2}) \, . \]

There are six triangles in $\mathcal{T}$, which are

\[ (p_j, p_{j+1}, p_7), \; j = 1, ..., 5, \; (p_6, p_1, p_7) \, . \]
The complete triply-periodic surface is generated by 180° rotations about boundary edges, continuing to make such rotations until the surface is complete. Every vertex in \( \partial \mathcal{T} \), and every vertex that is an image of a vertex in \( \partial \mathcal{T} \) under these rotations, satisfies Equation (4), because of the symmetry of the surface. The condition for Equation (4) to hold at the interior vertex \( p_7 \) and all images of \( p_7 \) under these rotations is that

\[
\frac{2ya}{\sqrt{a^2 + \frac{1}{4}}} + \frac{a}{\sqrt{a^2 + (y-b)^2}} + \frac{a-x}{\sqrt{b^2 + (x-a)^2}} = 0,
\]

\[
\frac{2xb}{\sqrt{b^2 + \frac{1}{4}}} + \frac{b}{\sqrt{b^2 + (x-a)^2}} + \frac{b-y}{\sqrt{a^2 + (y-b)^2}} = 0.
\]

When \( x = y = \sqrt{2} \), one explicit solution is \( a = b = \frac{\sqrt{2} - 1}{2} \). The fundamental piece \( \mathcal{T} \) and a larger part of the resulting complete surface are shown in the left-most column of Figure 7 for these values of \( x, y, a, \) and \( b \).

For general choices of \( x \) and \( y \), there is always a solution to the above system of two equations with respect to the two variables \( a \) and \( b \), thus giving \( \nabla_{pr} \text{area } \mathcal{T} = 0 \). Thus, for general \( x \) and \( y \), the minimality condition for this example is two-dimensional. Fundamental pieces \( \mathcal{T} \) for other choices of \( x \) and \( y \) are shown in the

---

**Figure 7.** Discrete and smooth Schwarz CLP surfaces.
upper-center and upper-right of Figure 7 ($x = y$ in the center and $x \neq y$ on the right).

We can also apply **Method 2** here. For example, suppose we include the reflection of $T$ across the plane $P$ containing the three points $(x,0,0)$, $(x,1,0)$, $(x,0,1)$ along with $T$ to get a discrete minimal surface $T_1$ with twelve triangles, see the left-hand side of Figure 8. (Such a reflection across $P$ would not be allowed for the smooth Schwarz CLP surface.) We can then extend $T_1$ to a complete triply-periodic discrete minimal surface by $180^\circ$ rotations about boundary edges, and this surface is yet another discrete superman surface.

For a second example of applying **Method 2**, suppose we include the reflection of $T_1$ across the plane $Q$ containing the three points $(0, y, 0)$, $(1, y, 0)$, $(0, y, 1)$ along with $T_1$ to get a discrete minimal surface $T_2$ with twenty-four triangles, see the right-hand side of Figure 8. (Such a reflection again would not be allowed in the smooth case.) We can then extend $T_2$ to a complete triply-periodic discrete minimal surface by $180^\circ$ rotations about boundary edges, and this gives yet another discrete Schwarz P surface.

### 6.2. Discrete I-Wp and F-Rd surfaces.

The fundamental piece $T$ of this I-Wp example has six vertices

$$p_1 = (b,0,b), \quad p_2 = (b,0,0), \quad p_3 = (b,b,0),$$
$$p_4 = (1,1,a), \quad p_5 = (1,a,1), \quad p_6 = p_1p_4 \cap p_3p_5 .$$

There are five triangles in $T$, which are

$$(p_1,p_2,p_3), \quad (p_1,p_3,p_6), \quad (p_3,p_4,p_6), \quad (p_4,p_5,p_6), \quad (p_5,p_1,p_6).$$

By including the two images of $T$ under the two reflections across the planes $\{(x,x,z) \mid x,z \in \mathbb{R}\}$ and $\{(x,y,x) \mid x,y \in \mathbb{R}\}$, we have a larger discrete surface with fifteen triangles. Reflecting this larger piece across all planes of the form $\{(x,y,k) \mid x,y \in \mathbb{R}\}$, $\{(x,k,z) \mid x,z \in \mathbb{R}\}$, $\{(k,y,z) \mid y,z \in \mathbb{R}\}$ for all integers $k$, we arrive at a complete embedded triply-periodic surface in $\mathbb{R}^3$. See the right-hand side of Figure 9.

The minimality condition that Equation (4) holds at each vertex of the complete triply-periodic surface is

$$1 + a + a^2 - 3b - 2ab + 2b^2 = 0 ,$$

![Figure 8. Variants of the discrete Schwarz CLP surface.](image-url)
Thus, to make the surface minimal, we must find $a$ and $b$ satisfying Equations (8)-(9), so the minimality condition is two-dimensional. Equation (8) holds if
\[ a = \frac{1}{2} \left( 2b - 1 + \sqrt{-3 + 8b - 4b^2} \right), \]
and then Equation (9) will hold if $b$ satisfies
\[ \left( 3 - \sqrt{-3 + 8b - 4b^2} \right) (1 - b) = \sqrt{2} \sqrt{3 - 4b + 2b^2 + \sqrt{-3 + 8b - 4b^2}}. \]
One can find such a real number $b$ in a completely explicit form (although not in such simple forms like in Subsections 4.3, 5.2 and 6.1).

One can similarly find a discrete analog, shown on the left-hand side of Figure 9, of the smooth triply-periodic minimal F-Rd surface. With the simplicial structure chosen in Figure 9 one can again explicitly solve the minimality condition, in the same way as we did for the I-Wp example.

6.3. Trigonal example. The fundamental piece $T$ of this H-T example has six boundary vertices
\[
p_1 = (a \frac{\sqrt{3}a}{2}, b), \quad p_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2}, c), \quad p_3 = (2 - \frac{3s}{2}, \frac{\sqrt{3}s}{2}, 0),
\]
\[
p_4 = (2 - \frac{3s}{2}, -\frac{\sqrt{3}s}{2}, 0), \quad p_5 = (\frac{1}{2}, -\frac{\sqrt{3}}{2}, c), \quad p_6 = (a \frac{\sqrt{3}a}{2}, b).
\]
for any given fixed $b > 0$, and has one interior vertex
\[ p_7 = \frac{1}{2} (p_2 + p_5). \]
There are six triangles in $T$, which are
\[(p_j, p_{j+1}, p_7), \quad j = 1, ..., 5, \quad (p_6, p_1, p_7).\]
Including the images of $T$ under the $120^\circ$ and $240^\circ$ rotations about the axis \( \{(0, 0, z) \mid z \in \mathbb{R} \} \), and also including the images of $T$ and these two rotated copies of $T$ under reflection across the plane \( \{(x, y, 0) \mid x, y \in \mathbb{R} \} \), one has the larger piece...
shown on the left-hand side of Figure 10. This larger piece has five boundary components, each contained in a plane, and these five planes bound a trigonal prism (a prism of height $2b$ over an equilateral triangle with edge-lengths $2\sqrt{3}$). Including the images of this larger piece by reflecting across these five planes, and also by including all subsequent images of reflections across planes containing subsequent boundary components, one arrives at a triply-periodic discrete surface, which is embedded when $a, s \in (0, 1)$ and $c \in (0, b)$. A larger portion of this complete discrete surface is shown in the central figure of Figure 10.

The minimality condition involves three equations in the three variables $a, c, s$, and so is three-dimensional. We will not show the equations here, but they can be solved explicitly. For example, when $b = 1$, the following choices ensure minimality:

$$a = s = \frac{2 + \sqrt{2}}{4}, \quad c = \frac{3}{4} \quad \text{(and } b = 1).$$

In fact, these choices also ensure that all of the vertices of $T$ lie in the same plane, and hence the fundamental piece $T$ is planar and could be freely triangulated within its trace (see Remark 2.1).

Furthermore, rather than using a portion of the complete surface within a trigonal prism as a building block for the complete surface, one could have instead used a portion within a hexagonal prism as the building block. The figure on the right-hand side of Figure 10 will produce exactly the same complete surface (again by reflecting across planes containing boundary components). In the case of smooth H-T surfaces, this same duality exists between building blocks in trigonal and hexagonal prisms, as noted in [10].

6.4. Discrete Fischer-Koch example. An interesting triply-periodic smooth embedded minimal surface was found recently by W. Fischer and E. Koch [4], and is shown in the bottom row of Figure 11. Here we give a discrete minimal analog of this surface, shown in the top row of Figure 11.

The fundamental piece $T$ of this example has eight boundary vertices

$$p_1 = (0, 0, -1), \quad p_2 = (0, 0, -2), \quad p_3 = (a, 0, -2), \quad p_4 = (a, 0, 1),$$

$$p_5 = (0, 0, 1), \quad p_6 = (0, 0, 2), \quad p_7 = \left(\frac{a}{2}, \frac{\sqrt{3}a}{2}, 2\right), \quad p_8 = \left(\frac{a}{2}, \frac{\sqrt{3}a}{2}, -1\right).$$
for any given fixed \( a > 0 \), and has one interior vertex

\[ p_9 = \left( \frac{\sqrt{3}b}{2}, \frac{b}{2}, 0 \right) \]

with \( 0 < b < a \). There are eight triangles in \( T \), which are

\( (p_j, p_{j+1}, p_9), \ j = 1, \ldots, 7, \ (p_8, p_1, p_9) \).

The complete triply-periodic surface is generated by 180° rotations about boundary edges. In this example, the symmetry Equation (4) holds at each vertex \( p_1, \ldots, p_8 \) in the resulting complete triply-periodic surface. However, getting this to hold at \( p_9 \) requires the proper choice of \( b \). This minimality condition at \( p_9 \) is one-dimensional, and one can prove existence of a value \( b \in (0, a) \) solving it.

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