Analysis of wavepacket tunneling with the method of Laplace transformation

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Abstract

We use the method of Laplace transformation to determine the dynamics of a wave packet that passes a barrier by tunneling. We investigate the transmitted wave packet and find that it can be resolved into a sequence of subsequent wave packages. This result sheds new light on the Hartmann effect for the tunneling time.

1 Introduction

There are several definitions of tunneling times and the discussion about their meaning is still ongoing ([1], [2], [3], [4]). Recently, experiments with atoms stimulated by ultrashort, infrared laser pulses, the so-called attoclock experiments, reinforced the interest in the prediction of tunneling times ([4]).

In this article we investigate the dynamics of a wavepacket that tunnels through a barrier. The tunneling time we determine is the so-called group delay or phase time (see for instance [1]). It can be characterized as the time interval between the moment the peak of the incoming wave packet reaches the barrier and the arrival of the peak of the transmitted wave packet at the end of the barrier. One of the oldest results for this tunneling time was obtained by Mac Coll ([5]) in 1932 by the application of the stationary phase method. He concluded that there was "no appreciable delay in the transmission of the packet through the barrier". These calculations were later refined by Hartmann ([6], [11]), who found a finite delay time. For thicker barriers this delay time becomes independent of the thickness of the barrier and tends to a fixed value which is known as the Hartmann effect.

In these approaches the phase time is obtained by representing the solution of the time-dependent Schrödinger as the integral over stationary solutions for different energies. Under the assumption that the energy distribution of the initial wave packet is sufficiently peaked the application of the stationary phase
method yields a time for the arrival of the peak of the wavepacket at the end of the barrier.

In this article we use instead the method of Laplace transformation [7] to determine the wavepacket dynamics. We then simplify this result assuming that the initial wavepacket is sufficiently peaked in the momentum representation.

We find that what comes out of the barrier is not a single wave packet, but infinitely many, one after the other. The time each of these wavepackets needs for the tunneling is completely independent of the thickness of the barrier. Instead for thicker barriers, the later wavepackets are much more attenuated and so there remains a single transit time that equals the result Hartmann obtained as the upper limit for the tunneling time.

Bernardini (8) investigated the transmission of wavepackets with energies above the height of the barrier. He also finds that there is not only one reflected and one transmitted wave packet, but many of them that leave the barrier. This result was obtained by the decomposition of the transmission and reflection coefficients of the stationary solution in infinite series. Our result is the counterpart in the tunneling regime.

2 The solution of the time-dependent Schrödinger equation

A finite barrier is described by the potential

\[ V(x) = 0 \quad \text{for} \quad x \leq 0 \]
\[ V(x) = V \quad \text{for} \quad 0 < x < d \]
\[ V(x) = 0 \quad \text{for} \quad x \geq d. \]

The dynamics of a wave packet \( \psi(x,0) \) is determined by the Schrödinger equation

\[ - \frac{\hbar^2}{2m} \frac{\partial^2 \psi(x,t)}{\partial x^2} = i\hbar \frac{\partial \psi(x,t)}{\partial t} \quad \text{for} \quad x < 0, \quad (1) \]
\[ - \frac{\hbar^2}{2m} \frac{\partial^2 \psi(x,t)}{\partial x^2} + V\psi(x,t) = i\hbar \frac{\partial \psi(x,t)}{\partial t} \quad \text{for} \quad 0 < x < d, \quad (2) \]
\[ - \frac{\hbar^2}{2m} \frac{\partial^2 \psi(x,t)}{\partial x^2} = i\hbar \frac{\partial \psi(x,t)}{\partial t} \quad \text{for} \quad x > d, \quad (3) \]

where \( \psi(x,t) \) is supposed to be continuously differentiable everywhere and square integrable.

We will further assume that the initial wave packets are located at the left side of the barrier

\[ \psi(x,0) = 0 \quad \text{for} \quad x \geq 0. \quad (4) \]

We find for the Laplace transformed wave packet \( \varphi(x,s) \):
$$\varphi(x, s) = \sqrt{\frac{m}{2i\hbar}} \left\{ u_1(x, s) \int_{-d}^{x} u_2(y, s) \psi(y, 0) dy - u_2(x, s) \int_{-d}^{x} u_1(y, s) \psi(y, 0) dy \right\}$$

$$\varphi(x, s) = \alpha(s) u_1(x, s) + \beta(s) u_2(x, s) \quad \text{for} \quad x < 0 \quad (5a)$$

$$\varphi(x, s) = \gamma(s) u_3(x, s) + \delta(s) u_4(x, s) \quad \text{for} \quad 0 < x < d \quad (5b)$$

$$\varphi(x, s) = \mu(s) u_1(x, s) + \nu(s) u_2(x, s) \quad \text{for} \quad x > d \quad (5c),$$

where the functions $u_1, u_2, u_3, u_4$ are defined by

$$u_1(x, s) = e^{i\sqrt{\frac{2m}{\hbar^2}} s} \quad u_3(x, s) = e^{i\sqrt{\frac{2m}{\hbar^2} - \frac{2mV}{\hbar^2}}} \quad (6a)$$

$$u_2(x, s) = e^{-i\sqrt{\frac{2m}{\hbar^2}} s} \quad u_4(x, s) = e^{-i\sqrt{\frac{2m}{\hbar^2} - \frac{2mV}{\hbar^2}}} \quad (6b).$$

Since $\varphi(x, s)$ must vanish for $x \to \pm\infty$, we find

$$\alpha(s) = 0, \quad \nu(s) = 0.$$

If we evaluate $\varphi(x, s)$ and its first derivative at $x = 0$ and $x = d$, we obtain

$$\beta(s) = \sqrt{\frac{m}{2i\hbar s}} (I_1 - I_2) + \sqrt{\frac{m}{2i\hbar s}} \frac{2I_2}{1 + \sqrt{1 - \frac{V}{\hbar s}}} \frac{\rho(s) - e^{-2id\sqrt{\frac{2m}{\hbar^2} - \frac{2mV}{\hbar^2}}}}{\rho^2(s) - e^{-2id\sqrt{\frac{2m}{\hbar^2}}}} \quad (7)$$

$$\gamma(s) = -\sqrt{\frac{m}{2i\hbar s}} \frac{2I_2}{1 + \sqrt{1 - \frac{V}{\hbar s}}} \frac{e^{-2id\sqrt{\frac{2m}{\hbar^2} - \frac{2mV}{\hbar^2}}}}{\rho^2(s) - e^{-2id\sqrt{\frac{2m}{\hbar^2}}}} \quad (8)$$

$$\delta(s) = \sqrt{\frac{m}{2i\hbar s}} \frac{2I_2 \rho(s)}{1 + \sqrt{1 - \frac{V}{\hbar s}}} \frac{1}{\rho^2(s) - e^{-2id\sqrt{\frac{2m}{\hbar^2}}}} \quad (9)$$

$$\mu(s) = \sqrt{\frac{m}{2i\hbar s}} \frac{2I_2 e^{-id\sqrt{\frac{2m}{\hbar^2}}}}{1 + \sqrt{1 - \frac{V}{\hbar s}}} \frac{1 - \rho(s)}{\rho^2(s) - e^{-2id\sqrt{\frac{2m}{\hbar^2} - \frac{2mV}{\hbar^2}}}} \quad (10),$$

where we have introduced the abbreviations

$$I_1 = \int_{-\infty}^{0} e^{i\sqrt{\frac{2m}{\hbar^2}}} \psi(y, 0) dy, \quad I_2 = \int_{-\infty}^{0} e^{-i\sqrt{\frac{2m}{\hbar^2}}} \psi(y, 0) dy$$
and also used
\[ \rho(s) = \frac{2}{1 + \sqrt{1 - \frac{V}{V_{si}}}} - 1. \]  
(11)

Inserting this result into (5) and applying a series expansion for
\[ \frac{1}{1 - \rho^2(s)e^{2id\sqrt{\frac{\hbar}{\epsilon}}}} \]
yields
\[ \varphi(x, s) = \sqrt{\frac{m}{2\pi \hbar}} \int_{-\infty}^{0} e^{i\sqrt{\frac{2m\epsilon}{\hbar}}x-y} \psi(y, 0)dy + \int_{0}^{\infty} e^{-i\sqrt{\frac{2m\epsilon}{\hbar}}(x+y)} \psi(y, 0)dy \cdot \rho(s) \}
\]
\[ + \sqrt{\frac{m}{2\pi \hbar}} \sum_{l=0}^{\infty} \int_{-\infty}^{0} e^{2id(l+1)\sqrt{\frac{2m\epsilon}{\hbar}} - i(x+y)\sqrt{\frac{2m\epsilon}{\hbar}}} \psi(y, 0)dy \cdot (\rho^2(s) - 1)\rho^{2l+1}(s) \]
for \( x < 0 \) \( (12a) \)

\[ \varphi(x, s) = \sqrt{\frac{m}{2\pi \hbar}} \sum_{l=0}^{\infty} \int_{-\infty}^{0} e^{i(2(l+1))\sqrt{\frac{2m\epsilon}{\hbar}} - iy\sqrt{\frac{2m\epsilon}{\hbar}}} \psi(y, 0)dy \cdot (\rho(s) + 1)\rho^{2l}(s) - \]
\[ \sqrt{\frac{m}{2\pi \hbar}} \sum_{l=0}^{\infty} \int_{-\infty}^{0} e^{i(2(l+1)-x)\sqrt{\frac{2m\epsilon}{\hbar}} - iy\sqrt{\frac{2m\epsilon}{\hbar}}} \psi(y, 0)dy \cdot (\rho(s) + 1)\rho^{2l+1}(s) \]
for \( 0 \leq x \leq d \), \( (12b) \)

\[ \varphi(x, s) = \sqrt{\frac{m}{2\pi \hbar}} \sum_{l=0}^{\infty} \int_{-\infty}^{0} e^{id(l+1)\sqrt{\frac{2m\epsilon}{\hbar}} + iy\sqrt{\frac{2m\epsilon}{\hbar}}} \psi(y, 0)dy \cdot (-\rho^2(s) + 1)\rho^{2l}(s) \]
for \( x > d \). \( (12c) \)

We introduce the shifted momentum representation
\[ f(K, p) = \frac{1}{\sqrt{2\pi \hbar}} \int_{-\infty}^{\infty} \psi(x + K, 0)e^{-\frac{ipx}{\hbar}} dx = e^{i\frac{pK}{\hbar}} f(p), \]
(13)

where
\[ f(0, p) = f(p) \]
denotes the representation of the wave function in momentum space. Applying the abbreviations
\[ a_l(t) = \mathcal{L}^{-1} \{ \rho^{2l+1}(s)(1 - \rho^2(s)) \}, \quad b_l(t) = \mathcal{L}^{-1} \{ \rho^{2l}(s)(1 + \rho(s)) \} \]
\[ c_l(t) = \mathcal{L}^{-1} \{ \rho^{2l+1}(s)(1 + \rho(s)) \}, \quad g_l(t) = \mathcal{L}^{-1} \{ \rho^{2l}(s)(1 - \rho^2(s)) \} , \]
we find proceeding as for the asymmetric square well in [7] for the inverse Laplace
transform of (12)

\[ \psi(x, t) = \frac{\kappa}{\sqrt{2\pi t i}} \int_{-\infty}^{0} e^{\frac{i(x-y)^2}{2t}} \psi(y, 0) dy + \int_{0}^{t} \frac{\kappa}{\sqrt{2\pi(t-\tau) i}} e^{\frac{i(x+y)^2}{2(t-\tau)}} \psi(y, 0) r(\tau) d\tau + \]

\[ \sum_{l=0}^{\infty} \int_{0}^{t} \int_{-\infty}^{\infty} K(2d(l+1), p, t-\tau) f(-x, p) dp \cdot a_{l}(\tau) d\tau \]

for \( x < 0 \),

\[ \psi(x, t) = \sum_{l=0}^{\infty} \int_{0}^{t} \int_{-\infty}^{\infty} K(2dl + x, p, t-\tau) f(p) dp \cdot b_{l}(\tau) d\tau + \]

\[ \sum_{l=0}^{\infty} \int_{0}^{t} \int_{-\infty}^{\infty} K(d(2l + 1) - x, p, t-\tau) f(p) dp \cdot c_{l}(\tau) d\tau \]

for \( 0 < x < d \),

\[ \psi(x, t) = \sum_{l=0}^{\infty} \int_{0}^{t} \int_{-\infty}^{\infty} K(d(2l + 1), p, t-\tau) f(x-d, p) dp \cdot g_{l}(\tau) d\tau \]

for \( x > d \),

(14a)

(14b)

(14c)

where \( K(x, p, t) \) is defined by

\[ K(x, p, t) = \frac{1}{2\sqrt{2\pi \hbar}} e^{\frac{-ix}{\hbar} - \frac{ip^2}{2m}} \cdot \]

\[ \left\{ e^{-\frac{ix}{\hbar}} \text{Erfc}\left[ -i \sqrt{\frac{2mi}{\hbar t}} \frac{x}{2} - i \sqrt{\frac{it}{2m\hbar}} q \right] + e^{\frac{ix}{\hbar}} \text{Erfc}\left[ -i \sqrt{\frac{2mi}{\hbar t}} \frac{x}{2} + i \sqrt{\frac{it}{2m\hbar}} q \right] \right\} , \]

(15)

and we used \( \kappa = \sqrt{\frac{m}{\hbar}} \) and \( q = \sqrt{p^2 - 2mV} \).

3 Tunneling of wave packets

In order to investigate the tunneling process, we consider a wave packet that is represented in momentum space by

\[ f(p) = e^{-\frac{ipx_0}{\hbar}} F(p-p_0) \]

(16a)

where \( F(p) \) fulfills

\[ \int_{-\infty}^{\infty} F^*(p) p F(p) dp = 0 , \quad \int_{-\infty}^{\infty} F^*(p) F'(p) dp = 0 . \]

(16b)

The expectation values of \( \psi(x, 0) \) are then given by

\[ \langle \hat{x} \rangle = x_0 \, , \quad \langle \hat{p} \rangle = p_0 \, . \]
We assume that the wavepacket is concentrated within a region $0 < p < p_m < \sqrt{2mV}$, so that we can use the approximation
\[ f(p) \approx 0 \quad \text{for} \quad p > p_m \quad \text{and} \quad p < 0. \tag{17} \]
Moreover it should be sufficiently peaked around the momentum expectation value to justify the approximation
\[ F(p - p_0)(p - p_0) \approx 0. \tag{18} \]
Finally the difference $\sqrt{2mV} - p_m$ should be big enough to ensure
\[ \frac{1}{\sqrt{1 - \frac{p^2}{2mV}}} = O(1) \quad \text{for} \quad 0 \leq p \leq p_m. \tag{19} \]
We start with the solution for $x > d$ \cite{12}. If we rewrite $K(x, p, t)$ for $q^2 = p^2 - 2mV < 0$ and use
\[ X_l \equiv d(2l + 1), \tag{20} \]
we find (see section 4.4 in \cite{7} for details)
\[ K(X_l, p, t) = U_0 + U_1 - U_2, \quad \text{where} \tag{20} \]
\[ U_0 = e^{-\frac{iX_l}{k}} \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{iX_l}{2m} + \frac{iX_l}{2mV}} = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{iX_l}{2m} - \frac{X_l}{\hbar} \sqrt{2mV - p^2}}, \]
\[ U_1 = \frac{\kappa}{\sqrt{2\pi t}} \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{iX}{k}} \int_0^\infty e^{\frac{i(X_l + u)^2}{2t}} e^{\frac{i\sqrt{2mV - p^2}}{2mV}} du, \tag{21} \]
\[ U_2 = \frac{\kappa}{\sqrt{2\pi t}} \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{iX_l}{k}} \int_0^\infty e^{\frac{i(X_l - u)^2}{2t}} e^{\frac{i\sqrt{2mV - p^2}}{2mV}} du. \]
Inserting
\[ \frac{\kappa}{\sqrt{2\pi t}} e^{\frac{i(X_l \pm u)^2}{2t}} = \int_{-\infty}^\infty \frac{1}{2\pi} e^{-\frac{iX_l^2}{2mV} + \frac{iX_l}{2mV}} \frac{2i\hbar}{\sqrt{2mV - p^2 + q^2}} dq, \]
we obtain
\[ U_1 - U_2 = \frac{1}{(2\pi\hbar)^{3/2}} e^{-\frac{iX_l}{k}} \int_{-\infty}^\infty e^{-\frac{iX_l^2}{2mV} + \frac{iX_l}{2mV}} \frac{2i\hbar}{2mV - p^2 + q^2} dq. \tag{22} \]
If the initial wavefunction is sufficiently peaked to assume
\[ F(p - p_0)(p - p_0) \approx 0, \tag{18} \]
the expansion of the non-oscillating part of $U_0$ around $p_0$ yields a contribution to $\psi(x, t)$ that is proportional to
\[ e^{-\frac{X_l}{\hbar} \sqrt{2mV - p^2}} \leq e^{-\frac{d}{\hbar} \sqrt{2mV - p_0^2}}. \tag{23} \]
By contrast we find for $U_1 - U_2$, that contains no oscillatory part

$$\int_{-\infty}^{\infty} (U_1 - U_2) f(x - d, p) dp \approx 0 ,$$

(24)

since

$$\int_{-\infty}^{\infty} f(p) p^n e^{i p (x - d) / \hbar} dp = -i \hbar \frac{\partial^n}{\partial x^n} \psi(x - d, 0) = 0 \quad \text{for} \quad x > d n = 0, 1, 2, N ,$$

and so the contribution to (24) will be proportional to $\Delta p^N + 1$, if $f(p)$ decays rapidly enough so that the integrals

$$\int_{-\infty}^{\infty} f(p) p^n dp$$

exist up to $n = N + 1$.

So we conclude that for sufficiently peaked wave packets, $U_0$ is the only relevant contribution.

We obtain for the wavefunction for $x > d$

$$\psi(x, t) \approx \sum_{l=0}^{\infty} \int_{0}^{t} \frac{1}{\sqrt{2\pi \hbar}} e^{-\frac{x^2}{2mV} - \frac{p^2}{2m\hbar}} \int_{-\infty}^{\infty} e^{-\frac{ip^2 (t - \tau)}{2m\hbar}} e^{i p (x - d) / \hbar} f(p) dp \cdot g_l(\tau) d\tau .$$

(25)

This result contains the free time evolution of the initial wave packet that is shifted by $d$ to the right. Proceeding as for the asymmetric square well(see [7]) we can approximate the convolution integral by

$$\int_{0}^{t} \frac{1}{\sqrt{2\pi \hbar}} \int_{-\infty}^{\infty} e^{-\frac{ip^2 (t - \tau)}{2m\hbar}} e^{i p (x - d) / \hbar} f(p) dp \cdot g_l(\tau) d\tau \approx$$

(26)

$$\int_{0}^{\infty} \frac{1}{\sqrt{2\pi \hbar}} \int_{-\infty}^{\infty} e^{-\frac{ip^2 (t - \tau)}{2m\hbar}} e^{i p (x - d) / \hbar} f(p) dp \cdot g_l(\tau) d\tau \approx$$

$$\int_{-\infty}^{\infty} e^{-\frac{ip^2}{2m\hbar}} e^{i p (x - d) / \hbar} R(p)^{2l} (1 - R(p)^2) f(p) dp \approx$$

(27)

$$\approx R(p_0)^{2l} (1 - R(p_0)^2) \cdot \frac{1}{\sqrt{2\pi \hbar}} \int_{-\infty}^{\infty} e^{-\frac{ip^2}{2m\hbar}} e^{i p (x - d) / \hbar} f(p) dp ,$$

(28)

where $R(p)$ is given by

$$R(p) = -1 + 2k - 2\sqrt{k(k - 1)} \quad \text{with} \quad k = \frac{p^2}{2mV} .$$

(29)

Here the assumptions about the concentration of the wave packet [18, 19] juc-
tifies putting $R(p)$ before the integral. Evaluating the sum in (24), we obtain

$$\psi(x,t) \approx \sum_{l=0}^{\infty} \int_{0}^{t} \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{x_{l} \sqrt{2mV - p_{0}^{2}}}{\hbar}} \int_{-\infty}^{\infty} e^{-\frac{ip^{2}(x-d)}{2m\hbar}} e^{\frac{ipl(x-d)}{\hbar}} f(p)dp \cdot g_{l}(\tau) d\tau \approx$$

$$\frac{e^{-d \sqrt{2mV - p_{0}^{2}}}}{1 - e^{-2d \sqrt{2mV - p_{0}^{2}}}} R^{2}(p_{0})$$

(30)

For the neglected part of the time integral (26)

$$\Delta_{l} \equiv \int_{t}^{\infty} \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-\frac{ip^{2}(x-d)}{2m\hbar}} e^{\frac{ipl(x-d)}{\hbar}} f(p)dp \cdot g_{l}(\tau) d\tau,$$

(31)

we find the estimates (see Appendix A)

$$|\Delta_{l}| \sim O(l) \cdot O \left( \frac{2\hbar}{Vl} \right).$$

(32)

Since

$$\sum_{l=0}^{\infty} e^{-\frac{x_{l} \sqrt{2mV - p_{0}^{2}}}{\hbar}} \Delta_{l} \sim \frac{e^{d \sqrt{2mV - p_{0}^{2}}}}{-1 + e^{-2d \sqrt{2mV - p_{0}^{2}}}} \cdot O \left( \frac{2\hbar}{Vl} \right),$$

the approximation for $\psi(x,t)$ given by (30) can be used for $t \gg \frac{\hbar}{V}$. According to this result, the wave packet leaves the barrier without any time delay, it is instantaneously transmitted through the barrier attenuated by a factor of the magnitude

$$e^{-d \sqrt{2mV - p_{0}^{2}}}.$$

Performing the inverse Laplace transform for $x < 0$ and $0 < x < d$ (12a, 12a) and making the same approximations as in the previous case we find for the wavepacket to the left of and within the barrier:
\[ \psi(x, t) \approx \frac{\kappa}{\sqrt{2\pi it}} \int_0^0 e^{\frac{i(x-y)^2}{2t}} \psi(y, 0) dy + \frac{1}{\sqrt{2\pi \hbar}} \int_{-\infty}^{\infty} e^{-\frac{ip^2}{2m\hbar}} R(p) f(p) dp + \sum_{l=0}^{\infty} e^{\frac{-2d(l+1)+x}{\kappa} \sqrt{2mV - \frac{p^2}{\hbar}}} \int_{-\infty}^{\infty} e^{-\frac{ip^2}{2m\hbar}} R(p) 2l+1 (R(p)^2 - 1) f(p) dp \approx \]

\[ \frac{\kappa}{\sqrt{2\pi i t}} \int_0^0 e^{\frac{i(x-y)^2}{2t}} \psi(y, 0) dy + \left\{ \frac{1 + \frac{e^{-2d\sqrt{2mV - \frac{p^2}{\hbar}}}}{1 - e^{-2d\sqrt{2mV - \frac{p^2}{\hbar}}}} \cdot (R^2(p_0) - 1) \right\} R(p_0) \frac{1}{\sqrt{2\pi \hbar}} \int_{-\infty}^{\infty} e^{-\frac{ip^2}{2m\hbar}} e^{-\frac{ipx}{\hbar}} f(p) dp \]

for \( x < 0 \), \( \quad (33) \)

\[ \psi(x, t) \approx \sum_{l=0}^{\infty} e^{\frac{-2d(l+1)+x}{\kappa} \sqrt{2mV - \frac{p^2}{\hbar}}} \int_{-\infty}^{\infty} e^{-\frac{ip^2}{2m\hbar}} R(p) 2l+1 (R(p) + 1) f(p) dp \approx \]

\[ - \sum_{l=0}^{\infty} e^{\frac{-2d(l+1)+x}{\kappa} \sqrt{2mV - \frac{p^2}{\hbar}}} \int_{-\infty}^{\infty} e^{-\frac{ip^2}{2m\hbar}} R(p) 2l+1 (R(p) + 1) f(p) dp \approx \]

\[ \left\{ \frac{1}{1 - e^{-2d\sqrt{2mV - \frac{p^2}{\hbar}}}} R(p_0)^2 \right\} - \frac{e^{-2d\sqrt{2mV - \frac{p^2}{\hbar}}}}{1 - e^{-2d\sqrt{2mV - \frac{p^2}{\hbar}}}} \cdot R(p_0) \]

\[ \cdot (R(p_0) + 1)e^{-\frac{ipx}{\hbar}} \frac{1}{\sqrt{2\pi \hbar}} \int_{-\infty}^{\infty} e^{-\frac{ip^2}{2m\hbar}} f(p) dp \]

for \( 0 < x < d \). \( \quad (34) \)

The solution within the barrier differs from the solution of the potential step (see [7]) only by a time-independent factor

\[ \psi_{\text{step}}(x, t) \approx (1 + R(p_0))e^{-\frac{2mV - \frac{p^2}{\hbar}}{\kappa}} \frac{1}{\sqrt{2\pi \hbar}} \int_{-\infty}^{\infty} e^{-\frac{ip^2}{2m\hbar}} f(p) dp. \quad (35) \]

So we see that the time it needs until the barrier is (approximately) empty again, is independent of the thickness of the barrier. The solution on the left side consists of the incoming and the reflected wave packet (33). In contrast to the potential step the reflected wavepacket experiences a permanent attenuation, since a part of the wavefunction has tunneled through the barrier. After the reflection process the wavefunction consists of a reflected and a transmitted wavepacket (30) only. An explicit calculation yields
\[
\left\{ \frac{1 + e^{-2d\sqrt{2mV - p_0^2}/\hbar}}{1 - e^{-2d\sqrt{2mV - p_0^2}/\hbar}} \cdot (R^2(p_0) - 1) \right\} R(p_0)^2 + \right.
\]
\[
\left. \frac{e^{-d\sqrt{2mV - p_0^2}/\hbar}}{1 - e^{-2d\sqrt{2mV - p_0^2}/\hbar}} (1 - R^2(p_0)) \right\}^2 = 1 ,
\]

which confirms that the integral over the probability density is conserved and our approximations are consistent.

4 The tunneling time

Within our approximations we found the tunneling time to be zero, since our solution (30) indicates that the wavepacket leaves the tunnel, shifted by d to the right. Here we have assumed, that all functions of \( R(p) \) can be pulled out of the integral (27), and therefore they do not influence the dynamics of the wave packet. If we take into account the first order contributions of \( R(p) \), we find a small, but finite tunneling time. We will from now on assume that the initial wave function is an uncorrelated function of the form

\[
f(p) = G(p)e^{-ipx_0}/\hbar ,
\]

where \( G(p) \) is a real function that yields a momentum expectation value \( p_0 \). The position expectation value is then given by \( x_0 \).

The impact of an additional factor \( Z(p) \) on the initial wavepacket \( f(p) \) is twofold

\[
Z(p)f(p) = z(p)e^{ip\mu(p)}f(p) , \text{ where } z(p) = |Z(p)| .
\]

The probability density in momentum space is only affected by the absolute value \( z(p) \). If we restrict ourselves to first order contributions in \( p - p_0 \),

\[
z(p) \approx z(p_0) + z_1(p - p_0)
\]

we find that we can neglect the momentum shift:

\[
N^2 \equiv \int_{-\infty}^{\infty} |f(p)|^2 z(p)^2 dp \approx z(p_0)^2 \int_{-\infty}^{\infty} |f(p)|^2 dp = z(p_0)^2
\]

\[
\langle \hat{p} \rangle = N^{-2} \int_{-\infty}^{\infty} pf(p)^2 z(p)^2 dp
\]

\[
\approx p_0 + 2(z(p_0))^{-1} z_1 \int_{-\infty}^{\infty} |f(p)|^2 p(p - p_0) dp \approx p_0 .
\]

Using also a linear approximation for \( \mu'(p) \)

\[
\mu'(p) \approx \mu_1 + \mu_2(p - p_0)
\]

10
we find for the position expectation value
\[
\tilde{x}_0 = N^{-2} \int_{-\infty}^{\infty} z(p) f^*(p) e^{-i\mu(p)} \left( i\hbar \frac{\partial}{\partial p} \right) z(p) f(p) e^{i\mu(p)} dp = \quad (42)
\]
\[
N^{-2}x_0 \int_{-\infty}^{\infty} G(p)^2 z(p)^2 dp - N^{-2}h \int_{-\infty}^{\infty} G(p)^2 z(p)^2 \mu'(p) dp \quad (43)
\]
\[
\approx x_0 - \hbar \mu_1 . \quad (44)
\]
Therefore the phase of the functions
\[
R(p)^{2l}(1 - R(p)^2) \quad (45)
\]
in (44) yields a shift of the position of each particular wave packet of (30). We find
\[
\text{Arg} \left[ 1 - R(p)^2 \right] = \text{Arctan} \left[ \frac{-1 + 2l}{2\sqrt{k - k^2}} \right] \approx \text{Arctan} \left[ \frac{-1 + 2k_0}{2\sqrt{k_0 - k_0^2}} \right] + \frac{2}{\sqrt{2mV - p_0^2}} \cdot (p - p_0) \quad (46)
\]
\[
\text{Arg} [R(p)] = -\text{Arccos} \left[ 1 - 2k \right] \approx -\text{Arccos} \left[ 1 - 2k_0 \right] + \frac{2}{\sqrt{2mV - p_0^2}} \cdot (p - p_0) , \quad (47)
\]
where we have used the definitions
\[
k = \frac{p^2}{2mV} , \quad k_0 = \frac{p_0^2}{2mV} .
\]
So we see that the lth term of (30) will experience a shift of
\[
\frac{2(1 + 2l)}{\sqrt{2mV - p_0^2}} ,
\]
corresponding to a translation by
\[
\frac{2(1 + 2l)\hbar}{\sqrt{2mV - p_0^2}}
\]
to the left. So the delay time will be
\[
T_l = \frac{2(1 + 2l)m\hbar}{p_0 \sqrt{2mV - p_0^2}}
\]
instead of zero. For thick barriers, if
\[
d\frac{\sqrt{2mV - p_0^2}}{\hbar} \gg 1 , \quad (48)
\]
the first term will dominate the sum (30), and the tunneling time will be given by
\[
T_0 = \frac{2m\hbar}{\sqrt{2mV - p_0 p_0}}
\]
This is exact the time Hartmann [6] found as upper limit for the tunneling time through thick barriers. If the barrier gets thinner the other wavepackets for \( l > 0 \) will also come into play. Each of them leaves the barrier at a different time \( T_l \). But since they appear very shortly after each other they may appear as one smeared out wavepacket with a delay time bigger than \( T_0 \).

5 Discussion and conclusions

Our result for the tunneling time (48) is not an exact reproduction of Hartmann’s result ([6],[1]) which predicts an increasing tunneling time with the thickness of the barrier before saturation takes place. According to our calculations contributions to the tunneling times that are proportional to the thickness of the barrier could only come from higher order contributions in \( p - p_0 \) to (38) that might yield a correction to smaller momenta.

But for thicker barriers the conclusion of both calculations is that the tunneling time for sufficiently peaked wave packets is given by (48). This is as far interesting as the results were obtained by completely different methods. Moreover we ensured in our calculations that the initial wavepacket is only located at the left side of the barrier [4] which is not clearly guaranteed by Hartmann’s approach. So our result makes sure that the Hartmann time (48) is not an artifact of the stationary phase approximation or some relic of the parts of the initial wavepacket that were at the right hand side of the barrier from the beginning.

We also found out that the approximate solution within a finite barrier differs from the solution within the potential step only by a time-independent factor (34,35) which also indicates that important dynamical properties are independent of the thickness of the barrier. It would be especially interesting if this is also true for more general tunneling processes as the tunneling out of a potential well that could model radioactive decay or tunneling out of atoms as provided by the attoclock experiment [4]. Moreover an application of the method of Laplace transformation to relativistic wave equations would yield a picture of the reflection and tunneling processes in the relativistic case.

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A Estimation of the convolution integral

According to [9], the inverse Laplace transform of \( \rho(s)^l \) is given by

\[
L^{-1} (\rho(s)^l) = \frac{l}{iVt} J_l \left( \frac{Vt}{2\hbar} \right) e^{-iVt/2\hbar}.
\]  (49)
Therefore we find

\[ g_l(t) = \mathcal{L}^{-1} \left( \rho l 2l(s)(1 - \rho^2(s)) \right) = \frac{l}{i2l t} J_{2l} \left( \frac{V t}{2\hbar} \right) e^{-\frac{iV t}{2\hbar}} \frac{l}{i2(l + 1)t} J_{2(l+1)} \left( \frac{V t}{2\hbar} \right) e^{-\frac{iV t}{2\hbar}}. \]

(50)

For an integral of the form

\[ u(l,t) = \int_t^\infty e^{\frac{iV t}{2\hbar}} e^{-\frac{V^2 t}{2m\hbar}} e^{-\frac{iV l}{l\hbar}} \frac{l}{t} J_l \left( \frac{V\tau}{2\hbar} \right) d\tau, \]

(51)

we get the following estimate

\[ |u(l,t)| = \left| \int_t^\infty e^{\frac{iV t}{2\hbar}} e^{-\frac{V^2 t}{2m\hbar}} e^{-\frac{iV l}{l\hbar}} \frac{l}{t} J_l \left( \frac{V\tau}{2\hbar} \right) d\tau \right| \leq \int_t^\infty \left| \frac{l}{t} J_l (y) \right| dy \leq \left( \int_t^\infty \frac{l}{y^{1+2\epsilon}} dy \right)^\frac{1}{2} \left( \int_0^\infty \frac{(J_l(y))^2}{y^{1-2\epsilon}} dy \right)^\frac{1}{2} \leq \frac{l}{\sqrt{2\epsilon}} \left( \frac{2\epsilon}{V t} \right)^\epsilon \left( \frac{\Gamma[1 - 2\epsilon]\Gamma[l + \epsilon]}{2\Gamma[1 - \epsilon]^2\Gamma[1 + l - \epsilon]} \right)^\frac{1}{2}. \]

(52a)

\[ \left( \frac{\sqrt{\epsilon}}{2\epsilon} \right)^\epsilon \left( \frac{\Gamma[1 - 2\epsilon]\Gamma[l + \epsilon]}{2\Gamma[1 - \epsilon]^2\Gamma[1 + l - \epsilon]} \right)^\frac{1}{2} \leq \frac{l}{\sqrt{2\epsilon}} \left( \frac{2\epsilon}{V t} \right)^\epsilon \left( \frac{\Gamma[1 - 2\epsilon]}{2\Gamma[1 - \epsilon]^2} \right)^\frac{1}{2} \text{ with } 0 < \epsilon < \frac{1}{2}, \]

(52b)

where we have applied the Schwarz inequality and the integral formula (104)

\[ \int_0^\infty \frac{(J_l(y))^2}{y^{1-2\epsilon}} dy = 2\epsilon \frac{\Gamma[1 - 2\epsilon]\Gamma[l + \epsilon]}{2\Gamma[1 - \epsilon]^2\Gamma[1 + l - \epsilon]} \]

(53)

So we conclude setting \( \epsilon = 1/4 \) for \( \Delta_l \)

\[ |\Delta_l| \sim O(l) \cdot O \left( \left( \frac{2\epsilon}{V t} \right)^\frac{1}{2} \right) \]

(54)

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