On the relation between dynamic regret and closed-loop stability*

Marko Nonhoffa,*, Matthias A. Müllera

*aLeibniz University Hannover, Institute of Automatic Control, 30167 Hannover, Germany

Abstract

In this work, we study the relations between bounded dynamic regret and the classical notion of asymptotic stability for the case of a priori unknown and time-varying cost functions. In particular, we show that bounded dynamic regret implies asymptotic stability of the optimal steady state for a constant cost function. For the case of an asymptotically stable closed loop, we first derive a necessary condition for achieving bounded dynamic regret. Then, given some additional assumptions on the system and the cost functions, we also provide a sufficient condition ensuring bounded dynamic regret. Our results are illustrated by examples.

Keywords: Dynamic regret, asymptotic stability, time-varying optimal control, online convex optimization

1. INTRODUCTION

In recent years, there has been considerable research interest in applications of machine learning and online optimization techniques to (optimal) control problems. Along with methods and algorithms, analysis techniques that originated in the field of online learning and optimization have been applied in order to study the behavior of the closed loop consisting of a learning algorithm and a controlled system. In particular, dynamic regret

\[ \mathcal{R} := \sum_{t=0}^{T} L_t(u_t, x_t) - L_t(\hat{u}_t, \hat{x}_t) \]

characterizes the accumulated performance gap between the closed loop \((u_t, x_t)\) and some benchmark \((\hat{u}_t, \hat{x}_t)\) with respect to a (possibly time-varying) cost function \(L_t\) to be minimized over a horizon \(T\). Typically, either some part of the system’s environment, most commonly either the cost functions \(L_t\) or process noise affecting the system, or the system dynamics itself are assumed to be time-varying and/or a priori unknown and the benchmark \((\hat{u}_t, \hat{x}_t)\) is defined in hindsight, i.e., with full knowledge of the system dynamics and the time-varying environment. Thus, the dynamic regret \(\mathcal{R}\) measures the performance lost due to not knowing the environment in which the system operates a priori. It is desirable to derive an upper bound on the dynamic regret that is sublinear in \(T\), because such a bound implies that the closed loop achieves asymptotically on average the performance of the benchmark, i.e.,

\[ \lim_{T \to \infty} \mathcal{R}/T = 0. \]

In contrast to classical Lyapunov stability, which only characterizes the asymptotic behavior of the system, dynamic regret takes the closed loop’s transient performance into account. Recently, regret analysis has been performed for various control algorithms, e.g., controllers minimizing dynamic regret \([1, 2, 3, 4, 5]\), control techniques based on online convex optimization (OCO) \([6, 7, 8, 9, 10]\), and model predictive control (MPC) \([11, 12, 13, 14, 15]\) as well as moving horizon estimation (MHE) \([16]\). However, stability results only exist for some of the above mentioned algorithms. In another closely related line of research, feedback optimization as an emerging control paradigm considers a similar setting but typically only asymptotic stability is shown instead of bounds on the dynamic regret of the closed loop \([17, 18, 19, 20, 21]\).

In the works mentioned above, either time-varying cost functions (due to, e.g., time-varying and a priori unknown energy prices \([8, 11]\)), process noise (e.g., renewable energy and a priori unknown consumption in power networks \([19]\)) or unknown system dynamics are considered. In this work, we focus on the first case, i.e., time-varying and a priori unknown cost functions, and leave process noise and learning unknown system dynamics as interesting directions for future research. As discussed above, in this setting, both dynamic regret and stability are frequently applied to study the closed loop behavior of the proposed algorithms. Therefore, it is interesting to study connections between dynamic regret and the more classical notion of stability in control theory to improve comparability and enable transferring results from different approaches that analyze one of these properties. To the authors’ best knowledge, this is the first paper studying the relation between dynamic regret and stability. We show that bounded dynamic regret for time-varying cost functions implies asymptotic stability of the optimal steady state for a constant cost function under mild assumptions and that the converse implica-

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*Corresponding author
Email addresses: nonhoff@irt.uni-hannover.de (Marko Nonhoff), mueller@irt.uni-hannover.de (Matthias A. Müller)

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time-invariant cost functions

Theorem 1

Proposition 1

time-varying cost functions

Theorem 2

summable $KL$ bound

bounded dynamic regret

Figure 1: Overview of our results. Summable $KL$ functions are used to derive a necessary (Proposition 1) and a sufficient (Theorem 2) condition for achieving bounded dynamic regret.

We begin by proving that bounded dynamic regret implies asymptotic stability and proceed to show that the converse implication does not hold. More specifically, we provide a necessary condition for asymptotic stability implying bounded dynamic regret. Finally, we show that a similar condition is sufficient for ensuring bounded dynamic regret given some additional assumptions. Section 4 concludes the paper.

Notation: The set of integer numbers and real numbers are denoted by $\mathbb{Z}$ and $\mathbb{R}$, respectively. The set of integer numbers greater than or equal to $s \in \mathbb{R}$ is $\mathbb{Z}_{\geq s}$. For a vector $x \in \mathbb{R}^n$, $\|x\|_S$ denotes the Euclidean norm and for a matrix $A \in \mathbb{R}^{n \times m}$, $\|A\|$ is the corresponding induced matrix norm. For a closed set $S \subset \mathbb{R}^n$, $\|x\|_S := \inf_{y \in S} \|x - y\|$. A function $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ ($\alpha \in \mathcal{K}$) if it is continuous, strictly increasing, and $\alpha(0) = 0$. If additionally $\lim_{s \rightarrow \infty} \alpha(s) = \infty$, then $\alpha \in \mathcal{K}_\infty$. A function $\sigma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{L}$ if it is nonincreasing, and $\lim_{s \rightarrow \infty} \sigma(s) = 0$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$ for any fixed $t \in \mathbb{R}_{\geq 0}$ and $\beta(s, \cdot) \in \mathcal{L}$ for any fixed $s \in \mathbb{R}_{\geq 0}$. A function $\beta \in \mathcal{KL}$ is called a summable $\mathcal{KL}$-function ($\beta \in \mathcal{KL}^+$) if for any $s \in \mathbb{R}_{\geq 0}$ there exists $B(s) < \infty$ such that $\sum_{t=0}^{\infty} \beta(s, t) \leq B(s)$.

2. SETTING

In this work, we consider general discrete-time nonlinear systems of the form

$$x_{t+1} = f(x_t, u_t),$$

where $x_t \in \mathbb{R}^n$ and $u_t \in \mathbb{R}^m$ are the system states and inputs at time $t \in \mathbb{Z}_{\geq 0}$, respectively, and $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ are the system dynamics. The goal is to design an algorithm $A$ that computes control inputs $u_t$ online and achieves satisfactory performance with respect to the optimal control problem

$$\min_u \sum_{t=0}^{T} L_t(u_t, x_t)$$

s.t. (1),

for arbitrary $T \in \mathbb{Z}_{\geq 0}$, where the cost functions $L_t(u, x) : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$ are time-varying and a priori unknown, i.e., at each time $t$, the algorithm only has access to the previous cost functions $L_0, \ldots, L_{t-1}$. Due to the a priori unknown nature of the cost functions $L_t$, we assume that the cost functions belong to a certain suitably restricted class $\mathcal{V}$, i.e., $L_t \in \mathcal{V}$ for all $t \in \mathbb{Z}_{\geq 0}$, in order to avoid the cumulative cost $\sum_{t=0}^{T} L_t(u_t, x_t)$ becoming arbitrarily large. We denote the solution to (1), i.e., the optimal input sequence in hindsight, by $u^*(x_0) = (u^*_t(x_0))^T_{t=0}$ and the corresponding state trajectory by $x^*(x_0) = (x^*_t(x_0))^T_{t=0}$. The algorithm $A$ is given by the general mapping

$$u_t = A(I_t),$$

where $I_t = \{x_0, \ldots, x_t, u_0, \ldots, u_{t-1}, L_0, \ldots, L_{t-1}, i_0\}$ describes all the available information at time $t$ and $i_0$ includes prior information, e.g., the initialization of the algorithm. In the following, we omit the dependence on $I_t$ when it is clear from context. We make the following assumptions on the system dynamics and cost functions.

Assumption 1. All cost functions $L_t \in \mathcal{V}$ are positive definite with respect to a (time-varying) steady-state $(\eta_t, \theta_t)$ of (1), i.e., there exists $(\eta_t, \theta_t) \in \mathbb{R}^m \times \mathbb{R}^n$ such that $\theta_t = f(\theta_t, \eta_t)$, $L_t(\eta_t, \theta_t) = 0$, and $L_t(u, x) > 0$ for all $(u, x) \neq (\eta_t, \theta_t)$. Moreover, there exists $\lambda \in \mathcal{K}$ such that $\lambda(\|u, x\| - (\eta_t, \theta_t))) \leq L_t(u, x)$ holds for all $L_t \in \mathcal{V}$ and $(u, x) \in \mathbb{R}^m \times \mathbb{R}^n$.

In the following, we drop the subscript $t$ from the cost functions $L_t$ and the corresponding optimal steady state $(\eta_t, \theta_t)$ when we consider a time-invariant cost function. Assumption 1 is less restrictive than similar assumptions in the related literature. In particular, positive definiteness with respect to an optimal steady state is either assumed [2,10] or, for cost functions that do not satisfy Assumption 1, a strategy is chosen that tracks the optimal steady states anyway [11,17,18,19,20,21]. In the latter case, the definition of dynamic regret can be modified appropriately such that Assumption 1 holds. Moreover, the class $\mathcal{L}$ lower bound in Assumption 1 is commonly replaced by similar assumptions restricting the class of considered cost functions, e.g., strong convexity [7,11,19].

Definition 1. ([22, Definition B.8]) The (closed, positively invariant) set $S$ is globally asymptotically stable for a dynamical system $x_{t+1} = f(x_t, A)$ if there exists a function $\beta(\cdot) \in \mathcal{KL}$ such that, for any initial state $x_0 \in \mathbb{R}^n$,

$$\|x_t\|_S \leq \beta(\|x_0\|_S, t)$$

holds for all $t \in \mathbb{Z}_{\geq 0}$.
In addition, dynamic regret $R$ is defined as
\[ R_T(x_0, u) = \sum_{t=0}^T L_t(u_t, x_t) - L_t(\eta_t, \theta_t), \]
where we omit the arguments of $R_T(x_0, u)$ when they are clear from context. We note that in the literature, different comparator sequences than $(\eta_t, \theta_t)$ (compare Assumption 1) are chosen regularly [6, 8]. We choose $(\eta_t, \theta_t)$ because it is the hardest benchmark, i.e., dynamic regret with respect to any other sequence is lower than or equal to the dynamic regret with respect to $(\eta_t, \theta_t)$ by Assumption 1. Furthermore, bounded dynamic regret of the Algorithm $A$ with respect to the benchmark $(\eta_t, \theta_t)$ is equivalent to bounded dynamic regret of the Algorithm $A$ with respect to any other benchmark, if this benchmark achieves bounded regret itself, compare Remark 1 below. Then, the goal is to achieve a sublinear (in $T$) upper bound of the dynamic regret, because such a bound implies that the algorithm in a closed loop with the system achieves asymptotically on average a cost that is no worse than the optimal cost. However, 7 indicates that the best achievable bound on the dynamic regret is linear in the path length, where the path length $PL$ is given by
\[ PL = \sum_{t=1}^T \| \eta_t - \eta_{t-1} \| + \| \theta_t - \theta_{t-1} \|. \]

The path length can be seen as an indicator of the variation of the cost functions $L_t$. Thus, we define bounded dynamic regret as follows.

**Definition 2.** An algorithm $A$ achieves bounded dynamic regret if, for any initial state $x_0 \in \mathbb{R}^n$, there exist constants $C_\eta, C_\theta, C \geq 0$, independent of $T$, such that
\[ R_T(x_0, u) \leq C_\eta \sum_{t=1}^T \| \eta_t - \eta_{t-1} \| + C_\theta \sum_{t=1}^T \| \theta_t - \theta_{t-1} \| + C \]
holds for any sequence of cost functions $L_0, \ldots, L_T \in \mathcal{V}$.

**Remark 1.** (Other benchmark trajectories) As discussed above, other benchmarks than the optimal steady states $(\eta_t, \theta_t)$ are commonly applied in the related literature. Let $\hat{R}_T$ be the dynamic regret with respect to the benchmark $(\hat{u}, \hat{x})$. Moreover, assume that the benchmark trajectory $(\hat{u}, \hat{x})$ achieves bounded dynamic regret according to Definition 2, i.e., there exist $C_\eta^\star, C_\theta^\star, C^\star \geq 0$ such that
\[ \sum_{t=0}^T L_t(\hat{u}_t, \hat{x}_t) - L_t(\eta_t, \theta_t) \leq C_\eta^\star \sum_{t=1}^T \| \eta_t - \eta_{t-1} \| + C_\theta^\star \sum_{t=1}^T \| \theta_t - \theta_{t-1} \| + C^\star \]
holds for all $T \in \mathbb{Z}_{\geq 1}$ and $L_0, \ldots, L_T \in \mathcal{V}$. For example, when considering the optimal trajectory in hindsight as a benchmark, i.e., $(\hat{u}, \hat{x}) = (u^*(x_0), x^*(x_0))$, similar assumptions on the optimal achievable performance are common in the related literature, compare, e.g., [6, 8]. Then, the algorithm $A$ achieves bounded dynamic regret (i.e., $R_T \leq C_\eta \sum_{t=1}^T \| \eta_t - \eta_{t-1} \| + C_\theta \sum_{t=1}^T \| \theta_t - \theta_{t-1} \| + C$) if and only if it achieves bounded regret with respect to the benchmark trajectory $(\hat{u}, \hat{x})$ (i.e., $R_T \leq C_\eta \sum_{t=1}^T \| \eta_t - \eta_{t-1} \| + C_\theta \sum_{t=1}^T \| \theta_t - \theta_{t-1} \| + C$). This claim can be proven by noting that i) $R_T \leq \hat{R}_T$ due to Assumption 1 and ii) $R_T = \hat{R}_T + \sum_{t=0}^T L_t(\hat{u}_t, \hat{x}_t) - L_t(\eta_t, \theta_t)$, which implies the result using the above assumption that the benchmark’s dynamic regret is bounded.

### 3. MAIN RESULTS

In this section, we derive conditions under which an algorithm $A$ that achieves bounded dynamic regret according to Definition 2 (for time-varying cost functions) also asymptotically stabilizes the optimal steady state $(\eta, \theta)$ (for a time-invariant cost function) and vice versa. Note that, in the case that the cost functions are constant, i.e., $L_t(u, x) = L(u, x)$ for $t \in \mathbb{Z}_{\geq 0}$, convergence to the optimal steady state, i.e., $\lim_{t \to \infty} (u_t, x_t) = (\eta, \theta)$, immediately follows from the definition of bounded regret in Definition 2 and Assumption 1.

#### 3.1. Bounded regret implies stability

First, we analyze algorithms that are already known to achieve bounded dynamic regret. In order to also establish asymptotic stability, consider the following assumptions.

**Assumption 2.** The algorithm $A$ admits a state-space representation, i.e., it can be written as
\[ x_{t+1}^A = f_{t-1}^A(x_t^A, x_1) \\
 u_t = h_{t-1}^A(x_t^A, x_1) \]
holds for some $x^A \in \mathbb{R}^{n_A}$, $f_{t-1}^A : \mathbb{R}^{n_A} \times \mathbb{R}^n \to \mathbb{R}^{n_A}$, and $h_{t-1}^A : \mathbb{R}^{n_A} \times \mathbb{R}^n \to \mathbb{R}^n$.

Assumption 2 restricts the class of considered algorithms. As standard in OCO, we assume that the algorithm only depends on the previous cost function $L_{t-1}$ because the current cost function $L_t$ is unknown [2, 11]. However, this assumption is not too restrictive and includes a wide range of nonlinear algorithms that admit a state-space realization. Moreover, we are still able to give guarantees for algorithms that do not satisfy Assumption 2 as discussed in Remark 2 below.

Furthermore, we denote by $S_L \subseteq \mathbb{R}^{n_A}$ the largest positively invariant set of states of the algorithm $A$ that produces the control input $\eta$ for $x = \theta$ and a (constant) cost function $L \in \mathcal{V}$, i.e.,
\[ S_L := \{ x^A \mid \chi_0^A = x^A, \eta = h_0^A(\chi_0^A, \theta), \chi_{t+1}^A = f_t^A(\chi_t^A, \theta) \forall t \in \mathbb{Z}_{\geq 0} \}, \]
and we define 
\[ S_L^0 := \{(x, x^4) \mid x = \theta, x^4 \in S_L\}. \]

Then, we require the constant \( C \) in Definition 2 to depend on the initial states of the closed loop as follows.

**Assumption 3.** The Algorithm \( A \) and the set of cost functions \( V \) are such that the constant \( C \) in Definition 2 satisfies \( C = C(x_0, x_0^4) \) and for any constant cost function \( L \in V \), there exists \( \alpha \in \mathcal{K}_\infty \) such that
\[
C(x_0, x_0^4) \leq \alpha((x_0, x_0^4)\|_{S_L^0})
\]
holds for any \((x_0, x_0^4) \in \mathbb{R}^n \times \mathbb{R}^{n^4} \).

Assumption 3 requires the constant \( C \) in Definition 2 to solely depend on the initial state \( x_0 \) and the algorithm’s initialization. It does, however, not require the closed loop to stay at \((\eta, \theta)\) for \((u_0, x_0) = (\eta, \theta)\) if \(x_0^4 \notin S_L^0\). Finally, we assume a lower bound on the dynamic regret. Therein, with a slight abuse of notation, we write \( \mathcal{R}_T(x_0, x_0^4) \) for \( \mathcal{R}_T(x, u) \), because the input sequence \( u \) is uniquely defined by the initial states \((x_0, x_0^4)\) by Assumption 2.

**Assumption 4.** The algorithm \( A \) and the set of cost functions \( V \) are such that for any constant cost function \( L \in V \), there exists a function \( \alpha \in \mathcal{K}_\infty \) that satisfies
\[
\alpha((x_0, x_0^4)\|_{S_L^0}) \leq \lim_{T \to \infty} \mathcal{R}_T(x_0, x_0^4).
\]
for any \((x_0, x_0^4) \in \mathbb{R}^n \times \mathbb{R}^{n^4} \).

Since we do not pose any continuity assumptions on the algorithm dynamics \( f^A_t \) and \( h^A_t \), we require Assumption 4 in order to avoid discontinuities such that, for a constant cost function \( L \in V \), \( \lim_{t \to \infty} (u_t, x_t) = (\eta, \theta) \), but \( \lim_{t \to \infty} (x_t, x_t^4) \notin S_L^0 \). For example, Assumption 3 is satisfied on a compact set (i.e., for all \((x_0, x_0^4) \in \mathcal{X}_0^A\), where \( \mathcal{X}_0^A \subseteq \mathbb{R}^n \times \mathbb{R}^{n^4} \) is a compact set) if the algorithm dynamics \( f^A_t \) and \( h^A_t \), the system dynamics \( f \), and the cost function \( L \) are continuous. Then, using continuity of the dynamics and the cost function together with Assumption 3 one can show by applying the uniform limit theorem [23, Theorem 21.6] that \( \lim_{t \to \infty} \mathcal{R}_T(x_0, x_0^4) \) is a continuous function, which yields the desired bound by Assumption 4.

Note that Assumptions 2 and 4 are satisfied for, e.g., the algorithms proposed in [1, 2] by choosing the predicted input sequence and the estimated optimal input and steady state, respectively, as the controller states \( x^4 \). Our first main result states that any algorithm \( A \) that achieves bounded regret as specified by Definition 2 asymptotically stabilizes the closed loop.

**Theorem 1.** Suppose Assumptions 2 and 4 are satisfied. For any constant cost function \( L \in V \), the set \( S_L^0 \) is globally asymptotically stable for the extended state \((x, x^4)\) with respect to the closed loop dynamics given by (1) and (3) if the algorithm achieves bounded dynamic regret for time-varying cost functions.

**Proof.** Fix any cost function \( L \in V \) and assume that the cost function is time invariant, i.e., \( L_t(u, x) = L(u, x) \) and \((\eta_t, \theta_t) = (\eta, \theta)\) for all \( t \in \mathbb{Z}_{\geq 0} \). We begin by choosing
\[
V(x, x^4) = \lim_{T \to \infty} \mathcal{R}_T(x, x^4),
\]
as a Lyapunov function candidate. Note that the above limit exists since \((\eta, \theta)\) holds for all \( t \in \mathbb{Z}_{\geq 0} \) and the algorithm achieves bounded dynamic regret (compare Definition 2), i.e.,
\[
\lim_{T \to \infty} \mathcal{R}_T(x, x^4) \leq C.
\]
Moreover, by Assumptions 3 and 4 there exist functions \( \alpha, \beta \in \mathcal{K}_\infty \) such that
\[
\alpha((x, x^4)\|_{S_L^0}) \leq V(x, x^4) \leq \beta((x, x^4)\|_{S_L^0}).
\]
In the following, we define \( V_t = V(x_t, x_t^4) \). Then, we get
\[
V_{t+1} - V_t = \lim_{T \to \infty} \mathcal{R}_T(x_{t+1}, x_{t+1}^4) - \lim_{T \to \infty} \mathcal{R}_T(x_t, x_t^4) \\
\leq \lim_{T \to \infty} \left( \mathcal{R}_T(x_{t+1}, x_{t+1}^4) - \mathcal{R}_T(x_t, x_t^4) \right) \\
- \lim_{T \to \infty} \left( \sum_{\tau = t+1}^{T} (L(h^A_L(x_{\tau}^4, x_{\tau}), x_{\tau}) - L(\eta, \theta)) \right) \\
= \lim_{T \to \infty} \left( L(h^A_L(x_{T+t}^4, x_{T+t+1}), x_{T+t+1}) - L(h^A_L(x_{T+t}^4, x_{T+t+1}), x_{T+t+1}) \right).
\]
As discussed above, bounded dynamic regret implies convergence, i.e., \( \lim_{t \to \infty} L(h^A_L(x_{t}^4, x_t), x_t) = L(\eta, \theta) \). Thus, we get by Assumption 4
\[
V_{t+1} - V_t = -\left( L(h^A_L(x_{t}^4, x_t), x_t) - L(\eta, \theta) \right) \leq 0.
\]
Together with (3), this implies stability of the set \( S_L^0 \) by standard Lyapunov arguments. It remains to show convergence of the extended state to \( S_L^0 \). We have \( \lim_{t \to \infty} V_t = 0 \), because the dynamic regret \( V_0 = \lim_{T \to \infty} \mathcal{R}_T(x_0, x_0^4) \) is bounded by a finite constant \( C \) by Definition 2 which implies \( \lim_{t \to \infty} \|(x_t, x_t^4)\|_{S_L^0} = 0 \) by Assumption 4.

**Remark 2.** (Algorithms that admit a state-space representation) If Assumption 2 is not satisfied, it is still possible to show asymptotic stability of the states of the system: To this end, instead of Assumption 3 we assume \( C = C(x_0) \leq \tilde{\alpha}((x_0 - \theta_0)\|_{\mathcal{K}_\infty}) \) for some \( \tilde{\alpha} \in \mathcal{K}_\infty \). Note that this is a considerably stronger assumption, because Assumption 3 allows the states \( x_t \) to initially diverge from \( \theta_0 \), even if the initial state \( x_0 \) is close to \( \theta_0 \). Thus, if the algorithm does not admit a state-space representation, we have to assume that it is initialized correctly, which is a
restrictive assumption since the pair $(y_0, \theta_0)$ is a priori unknown as discussed above. Moreover, instead of Assumption 3 we require $\alpha(||x_0 - \theta_0||) \leq \lim_{T \to \infty} R_T(x_0, u)$ for some $\alpha \in \mathcal{K}_\infty$, which is satisfied if the function $\lambda$ in Assumption 3 is of class $\mathcal{K}_\infty$ (i.e., the cost function $L$ is radially unbounded) because $\lim_{T \to \infty} R_T(x_0, u) \geq \lambda(||u_0, x_0|| - (y_0, \theta_0)) \geq \lambda(||x_0 - \theta_0||)$. Then, similar arguments to those in the proof of Theorem 1 can be applied to show that the optimal steady state $\theta$ is asymptotically stable. Moreover, for constant cost functions $L(u, x)$, the input to the system $u_t$ still converges to $\eta$ as discussed above and the deviation of $u_t$ from $\eta$ is bounded, since we have $R_T(x_0, u) \leq C$ by (2) and $R_T(x_0, u)$ is a positive definite function with respect to $\|u - \eta\|$ by Assumption 1.

**Remark 3.** (Other benchmark trajectories) As discussed in Remark 1 Theorem 1 also holds for different benchmark trajectories in the definition of dynamic regret, if the benchmark achieves bounded dynamic regret according to Definition 2. If this is not the case, Theorem 1 still holds true if, for any constant cost function $L \in \mathcal{V}$, i) there exists an optimal solution $u^*(x_0)$ to the OCP (2) for every $x_0 \in \mathbb{R}^n$ and any $T \in \mathbb{Z}_{\geq 0} \cup \mathbb{\infty}$, ii) the benchmark is chosen to be the optimal trajectory in hindsight $(u^*(x_0), x^*(x_0))$, iii) Assumptions 1 and 4 are (for the modified definition of dynamic regret), and iv) the optimal trajectory converges to the optimal steady state for every $x_0 \in \mathbb{R}^n$, i.e., $\lim_{t \to \infty} (u^*_t(x_0), x^*_t(x_0)) = (\eta, \theta)$. To show this, we define as a Lyapunov function candidate

$$V(x) = \lim_{T \to \infty} \sum_{t=0}^{T+1} L(u_t, x_t) - L(u^*_t(x_t), x^*_t(x_t)).$$

The Lyapunov function $V(x)$ is then upper (lower) bounded by $\alpha(||x||) \in \mathcal{K}_\infty$ due to Assumptions 3 and 4 respectively. Moreover, we have $V(x_{t+1}) - V(x_t) \leq 0$ because

$$\lim_{T \to \infty} \left(\sum_{t=0}^{T+1} L(u^*_t(x_t), x^*_t(x_t)) - L(u_t, x_t) - \sum_{t=0}^{T+1} L(u^*_t(x_{t+1}), x^*_t(x_{t+1}))\right) \leq 0$$

by optimality of $(u^*(x_t), x^*(x_t))$. Thus, we can conclude stability of the optimal steady state $\theta$ by standard Lyapunov arguments. Convergence follows by similar arguments as in the proof of Theorem 1.

**3.2. A necessary condition for bounded regret**

Theorem 1 and Remark 2 show that, under certain conditions, bounded dynamic regret implies asymptotic stability. However, the converse implication does in general not hold, i.e., asymptotic stability with respect to an invariant cost function does not imply bounded regret for time-varying cost functions. This is due to the fact that asymptotic stability only characterizes the asymptotic behavior of the closed loop, whereas bounded regret requires a certain rate of convergence, which is illustrated by the next example.

**Example 1.** Consider the discrete-time integrator $x_{t+1} = x_t + u_t$. We design an algorithm given by

$$u_t = -\frac{1}{\tau + 1} x_t + \frac{1}{\tau + 1} \theta_{t-1},$$

where $\theta_{-1} = x_0$ and $\tau \in \mathbb{Z}_{\geq 0}$ is set to $\tau = 0$ at time $t = 0$ and increased by one at every time step. When a changed cost function $L(x, u)$ is revealed, we use Assumption 5 to show that, if every trajectory $(\bar{x}_t)_{t=0}^\infty$ satisfies $L(x_t, u_t) = \|x_t - \theta - \eta\|$ for $t \in \mathbb{Z}_{\geq 2}$, then the closed loop is globally asymptotically stable for any time-invariant cost function $L \in \mathcal{V}$ according to Definition 2 with $\mathcal{S} = \{\theta\}$ and $\beta(s, t) = \frac{1}{t}$ if $t > 1$ and $\beta(s, t) = s$ if $t = 0$. However, by choosing the cost function $L(u, x) = \|x - \theta\| + \|u\|$ and neglecting the term $\|u\|$, we get a lower bound for the regret

$$R = \sum_{t=0}^{T} L(u_t, x_t) \geq \sum_{t=0}^{T} \|x_t - \theta\| = \|x_0 - \theta\|^2 \left(1 + \sum_{t=1}^{T} \frac{1}{t^2}\right)$$

which cannot be bounded by a constant independent of $T$ because the sum $\sum_{t=1}^{T} \frac{1}{t^2}$ diverges.

We can generalize Example 1 in order to get a necessary condition for achieving bounded regret. For this, we require the following assumption.

**Assumption 5.** The algorithm $A$ and the set of cost functions $\mathcal{V}$ are such that, for any constant cost function $L$, there exists $\mu \in \mathbb{I}_{\geq 0}$ and $\mu \geq 1$ such that, for any $s \geq 0$, there exists a closed-loop trajectory $\bar{x} = \{\bar{x}_t\}_{t=0}^\infty$ which satisfies

$$\mu \lim_{T \to \infty} \sum_{t=0}^{T} \phi(s, t) \geq \sum_{t=0}^{T} \phi(s, t),$$

where

$$\phi(s, t) = \sup_{x_0, \tau} \|x_{t+\tau} - \theta\| \quad s.t. \quad \|x_0 - \theta\| \leq s, \tau \geq t.$$
dynamics $x_{t+1} = f(x_t, A)$. Then, there exists a function $\beta \in KL^+$ such that

$$
\|x_t - \theta\| \leq \beta(\|x_0 - \theta\|, t)
$$

(11)

holds for all $t \in \mathbb{Z}_{\geq 0}$ and $x_0 \in \mathbb{R}^n$ if and only if for any $x_0 \in \mathbb{R}^n$ there exists a constant $D(\|x_0 - \theta\|)$ such that

$$
\lim_{T \to \infty} \sum_{t=0}^{T} \|x_t - \theta\| \leq D(\|x_0 - \theta\|) < \infty.
$$

The proof of Lemma 1 is given in the Appendix.

If Assumption 5 is not satisfied, Lemma 1 does in general not hold. For example, a system that satisfies $\|x_t - \theta\| = s \left(\frac{1}{1+\sigma}\right)^t$ for a constant cost function $L$, where $s = \|x_0 - \theta\|$, is asymptotically (exponentially) stable for any $x_0 \in \mathbb{R}^n$. Moreover, $\sum_{t=0}^{\infty} \|x_t - \theta\| = \sum_{t=0}^{\infty} s \left(\frac{1}{1+\sigma}\right)^t = s(1+D(s))$ holds for any $s \geq 0$, i.e., the latter condition of Lemma 1 is satisfied. However, it can be shown that there does not exist a $\beta \in KL^+$ (but only $\beta \in KL$) that satisfies (11).

Next, we state a necessary condition for an asymptotically stabilizing algorithm achieving bounded regret.

**Proposition 1**. Let Assumptions 4 and 9 hold. Assume that, for any constant cost function $L \in \mathcal{V}$, the set $S = \{\theta\}$ is asymptotically stable with respect to the closed-loop dynamics $x_{t+1} = f(x_t, A)$. If $A$ achieves bounded dynamic regret, then it holds that for every constant cost function $L \in \mathcal{V}$ that satisfies $L(u, x) \geq \bar{\mu} \|x - \theta\|$ for any $(u, x) \in \mathbb{R}^m \times \mathbb{R}^n$ and some $\bar{\mu} > 0$, there exists $\beta \in KL^+$ such that (11) is satisfied for all $t \in \mathbb{Z}_{\geq 0}$ and $x_0 \in \mathbb{R}^n$.

**Proof.** We prove this proposition by contraposition, i.e., if (11) does not hold for any $\beta \in KL^+$ and a constant cost function $L \in \mathcal{V}$ that satisfies the requirements of Proposition 1 then $A$ cannot achieve bounded regret. Fix any constant cost function $L \in \mathcal{V}$ and $\mu > 0$ such that $L(u, x) \geq \bar{\mu} \|x - \theta\|$ for all $(u, x) \in \mathbb{R}^m \times \mathbb{R}^n$, and assume that there does not exist any $\beta \in KL^+$ that satisfies (11). By Lemma 1 there exists an initial state $x_0$ such that

$$
\lim_{T \to \infty} \sum_{t=0}^{T} \|x_t - \theta\| = \infty.
$$

The dynamic regret is then lower bounded by

$$
\lim_{T \to \infty} R_T(x_0, u) \geq \lim_{T \to \infty} \sum_{t=0}^{T} \bar{\mu} \|x_t - \theta\| = \infty,
$$

i.e., it cannot be bounded independent of $T$ since the sum diverges. Therefore, the algorithm cannot achieve bounded dynamic regret.

**3.3. A sufficient condition for bounded regret**

Finally, we derive a sufficient condition for an asymptotically stabilizing algorithm to achieve bounded regret. To this end, we let $N$ be the number of time steps in the time interval $[0, T]$ where the cost function $L_i$ switches (including $t = 0$) and let $t_i, i \in [0, N - 1]$, be the time instances where the cost function $L_i$ changes, i.e., $t_0 = 0$ and

$$
L_t(u, x) \neq L_{t-1}(u, x) \text{ if and only if } t = t_i, \ i \in [1, N - 1].
$$

To simplify notation, we additionally define $t_N = T$. We require that the cost functions are not allowed to change too frequently and restrict the function $\beta \in KL$ in Definition 1 to be linear in $\|x_0 - \theta\|.$

**Assumption 6.** Denote by $\mathcal{N}(\tau_1, \tau_2)$ the number of times the cost function switches in the interval $[\tau_1, \tau_2] \subseteq \mathbb{Z}_{[0, T]}$. There exist constants $N_0 \in \mathbb{Z}_{\geq 0}$ and $\varphi > 0$ such that

$$
\mathcal{N}(\tau_1, \tau_2) < N_0 + \frac{1}{\varphi}(\tau_2 - \tau_1).
$$

(12)

**Assumption 7.** The Algorithm $A$ and the set of cost functions $\mathcal{V}$ are such that there exist $\sigma \in \mathcal{L}$, $\sigma(0) = 1$, and a constant $k \geq 1$ such that the algorithm $A$ in closed loop with system $\mathcal{H}$ satisfies

$$
\|x_t - \theta\| \leq k \|x_0 - \theta\| \sigma(t)
$$

for all $t \in \mathbb{Z}_{\geq 0}$ and any constant cost function $L \in \mathcal{V}$.

Assumption 6 is commonly applied in the literature on switched systems to guarantee stability (compare, e.g., [13]). Therein, the constants $N_0$ and $\varphi$ are referred to as chatter bound and average dwell-time, respectively. In our setting, such an assumption is necessary because, if the cost function $L_i$ was allowed to change at every time step, then Assumption 7 would not be sufficient to guarantee convergence since the constant $k$ in Assumption 7 is larger than one. However, it would be reasonable to assume that the previous bound on $\|x_t - \theta_t\|$ does not increase much if $\theta_{t+1}$ is close to $\theta_t$, i.e., the new upper bound on $\|x_t - \theta_{t+1}\|$ from Assumption 7 specifying the rate of convergence during the interval $[t_i, t_{i+2} - 1]$ should depend on $\|\theta_{t+1} - \theta_t\|$. We conjecture that such an assumption could render Assumption 6 unnecessary. We leave this problem as an interesting direction for future research.

Moreover, note that we require a bound in Assumption 7 which is uniform in $L$. In order to obtain such a bound, it is necessary to restrict the class of considered cost functions. Note that this is analogous to similar assumptions that are typically taken in the literature, which also require some uniform properties for all considered cost functions, such as, e.g., (strong) convexity (compare, e.g., [6, 7, 13, 15, 16]).

Finally, assume that the initial state and the optimal steady states are contained in compact sets $X_0, \Theta \subseteq \mathbb{R}^n$.
i.e., \( x_0 \in \mathcal{X}_0 \) and \( \theta_t \in \Theta \) for all \( \mathbb{Z}_{\geq 0} \), and that \( \varphi \) in Assumption 8 is large enough. Then, there exists a compact set \( \mathcal{X} \subseteq \mathbb{R}^n \) such that \( x_t \in \mathcal{X} \) for all \( t \). Moreover, for any function \( \beta \in \mathcal{KL} \) there exist \( \kappa \in \mathcal{K}_\infty \) and \( \sigma \in \mathcal{L} \) such that \( \beta(s, t) \leq \kappa(s)\sigma(t) \) [27, Lemma 8]. Therefore, Assumption 8 is satisfied if a uniform bound (in \( L \)) can be found such that \( \kappa(s) \) is Lipschitz at the origin.

In order to make use of Assumption 8, we need to ensure that the average dwell time \( \varphi \) is sufficiently large.

**Lemma 2.** Let Assumptions 9 and 8 be satisfied. For every \( N_0 \in \mathbb{Z}_{\geq 0} \), there exists \( \varphi > 0 \), \( P > 0 \), such that for every \( \varphi \geq \varphi \) and any \( N' \in \mathbb{Z}_{\geq N_0} \), it holds that

\[
\sum_{i=0}^{N'} \prod_{j=1}^{i} k(\Delta \sigma)_j \leq P, \tag{13}
\]

where \( (\Delta \sigma)_j = \sigma(t_j - t_{j-1}) \).

The proof is given in the Appendix.

In addition to asymptotic stability, bounded dynamic regret requires bounded inputs due to their influence on the cost functions.

**Assumption 8.** The algorithm \( A \) and the set of cost functions \( \mathcal{V} \) are such that the inputs \( u_t \) satisfy

\[
\begin{align*}
\| u_t - \eta_{t-1} \| &\leq k_u \| u_{t-1} - \eta_{t-1} \|
+ k_x \| x_t - \theta_{t-1} \| + k_\sigma \| \xi_{t-1} - \xi_{t-2} \|
\end{align*} \tag{14}
\]

for all \( t \in \mathbb{Z}_{\geq 0} \), where \( \xi_t = [\theta_t^T \eta_t] \), \( \xi_0 = \xi_\infty \), \( u_0 = \eta_0 \), \( k_u \in [0, 1] \) and \( k_x, k_\sigma \geq 0 \).

In time intervals during which the algorithm receives a constant cost function \( L_t \), i.e., \( t \in [t_i + 2, t_{i+1}] \), Assumption 8 requires the inputs \( u_t \) to converge to \( \eta_t \) at a similar rate as the states \( x_t \) converge to \( \theta_t \). When the algorithm receives a changed cost function, i.e., at time instances \( t_i + 1 \), the second term on the right hand side of (14) is nonzero and allows \( u_{t_i+1} \) to increase. Note that the bound depends on both \( \eta \) and \( \theta \), because the new cost function may be such that \( \eta_t \neq \eta_{t-1} \), but \( \theta_t \neq \theta_{t-1} \). In this case, \( u_t \) has to be allowed to diverge from \( \eta_{t-1} \) in order to reach the new optimal state \( \theta_t \).

Finally, we require the cost functions to be Lipschitz continuous, which is a common assumption in the literature (compare, e.g., [8, 11]).

**Assumption 9.** All cost functions \( L_t \in \mathcal{V} \) are Lipschitz continuous, i.e., there exists \( l > 0 \) such that

\[
\| L_t(u_1, x_1) - L_t(u_2, x_2) \| \leq l \| (u_1, x_1) - (u_2, x_2) \|
\]

for all \( t \in \mathbb{Z}_{\geq 0} \) and all \( (u_1, x_1), (u_2, x_2) \in \mathbb{R}^m \times \mathbb{R}^n \).

Finally, we are ready to state a sufficient condition for asymptotic stability of the optimal steady state implying bounded regret for time-varying cost functions.

**Theorem 2.** Let Assumptions 8 and 8 hold and let \( \varphi \geq \varphi \) with \( \varphi \) from Lemma 2. The algorithm achieves bounded dynamic regret if there exists a constant \( M \) such that

\[
\lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T} \sigma(t) \leq M. \tag{15}
\]

**Proof.** We prove the claim by deriving a regret upper bound assuming that (15) holds. First, for any initial state \( x_0 \in \mathbb{R}^n \), Assumption 9 yields

\[
R_T(x_0, u) = \sum_{t=0}^{T} L_t(u_t, x_t) - L_t(\eta_t, \theta_t)
\]

\[
\leq l \sum_{t=0}^{T} \| x_t - \theta_t \| + l \sum_{t=0}^{T} \| u_t - \eta_t \|. \tag{16}
\]

We proceed to bound the two sums in (16) separately. Since the cost function \( L_t \) is revealed to the algorithm only at time instance \( t + 1 \), i.e., with one time step delay, the algorithm stabilizes \( \theta_t \) according to Assumption 8 during the interval \( t \in [t_i + 1, t_{i+1} + 1] \). Note that \( x_{t_i+1} \) is both, the starting point of the sequence converging to \( \theta_t \) and the endpoint of the sequence converging to \( \theta_{t_i} \). Thus, we have for \( i \in [0, N - 1] \),

\[
\sum_{t=t_i+1}^{t_{i+1}} \| x_t - \theta_t \| \leq \| x_{t_i+1} - \theta_{t_i} \| + \sum_{t=t_i+1}^{t_{i+1}} \| x_t - \theta_t \| \tag{17}
\]

and, by Assumption 9

\[
\sum_{t=t_i+1}^{t_{i+1}} \| x_t - \theta_t \| \leq k \sum_{t=t_i+1}^{t_{i+1}} \sigma(t - t_i - 1) \leq k \sum_{t=t_i+1}^{t_{i+1}} \sigma(t - t_i - 1).
\]

Repeatingly applying Assumption 8 yields

\[
\sum_{t=t_i+1}^{t_{i+1}} \| x_t - \theta_t \|
\leq k \sum_{t=t_i+1}^{t_{i+1}} \| x_t - \theta_{t_i} \| + k \sum_{t=t_i+1}^{t_{i+1}} \sigma(t - t_i - 1)
\leq k \sum_{t=t_i+1}^{t_{i+1}} \| x_t - \theta_0 \| \prod_{j=1}^{i} k(\Delta \sigma)_j
\]

\[
+ k \sum_{j=1}^{i} (\| \theta_j - \theta_{t_i-1} \| \prod_{j=1}^{i-1} k(\Delta \sigma)_{j+1}).
\]

Summing over \( i \in [0, N - 1] \) yields

\[
\sum_{i=0}^{N-1} \sum_{t=t_i+1}^{t_{i+1}} \| x_t - \theta_t \| \leq k \sum_{i=0}^{N-1} \left( \prod_{j=1}^{i} k(\Delta \sigma)_j \right)
+ k \sum_{i=0}^{N-1} \sum_{j=1}^{i} (\| \theta_j - \theta_{t_i-1} \| \prod_{j=1}^{i-1} k(\Delta \sigma)_{j+1}).
\]
By using a case distinction on whether $N \geq N_0 + 1$ and defining $C_{N_0} = \sum_{i=0}^{N_0} k^i$, we get

$$\sum_{i=0}^{N-1} \sum_{t=t_i+1}^{t_{i+1}} \|x_t - \theta_t\| \leq kM \max(C_{N_0}, P) \|x_0 - \theta_0\|$$

$$+ kM \sum_{j=1}^{N-1} \left( \|\theta_{t_j} - \theta_{t_{j-1}}\| \sum_{i=0}^{N-1-j} \prod_{s=1}^{i} k(\Delta \sigma)_{s+j} \right)$$

$$\leq \hat{C} \left( \|x_0 - \theta_0\| + \sum_{i=0}^{T} \|\theta_{t_i} - \theta_{t_{i-1}}\| \right)$$

$$\leq \hat{C} \|x_0 - \theta_0\| + \hat{C} T \sum_{t=1}^{T} \|\theta_t - \theta_{t-1}\|,$$  

(18)

where $\hat{C} = kM \max(C_{N_0}, P)$. Combining the above results we get

$$\sum_{t=0}^{T} \|x_t - \theta_t\| = \|x_0 - \theta_0\| + \sum_{i=0}^{T} \|x_t - \theta_t\|$$

$$\leq \|x_0 - \theta_0\| + \sum_{i=0}^{T} \|\theta_{t_i} - \theta_{t_{i-1}}\| + \sum_{i=0}^{T} \sum_{t=t_i+1}^{t_{i+1}} \|x_t - \theta_t\|$$

$$\leq C_0 + (1 + \hat{C}) \sum_{t=1}^{T} \|\theta_t - \theta_{t-1}\|,$$  

(19)

where $C_0 = \|x_0 - \theta_0\| + \hat{C} \|x_1 - \theta_0\|$. Note that $\|x_0 - \theta_0\|$ and $\|x_1 - \theta_0\|$ are constants which only depend on the initialization $(u_0, x_0)$. It remains to bound $\sum_{t=0}^{T} \|u_t - \eta_t\|$. Assumption 8 yields

$$\sum_{t=0}^{T} \|u_t - \eta_t\| \leq \sum_{t=0}^{T} \|u_t - \eta_{t-1}\| + \sum_{t=0}^{T} \|\eta_t - \eta_{t-1}\|$$

$$\leq k_u \sum_{t=0}^{T} \|u_{t-1} - \eta_{t-1}\| + k_u \sum_{t=0}^{T} \|x_t - \theta_{t-1}\|$$

$$+ k_\zeta \sum_{t=1}^{T} \|\zeta_{t-1} - \zeta_{t-2}\| + \sum_{t=1}^{T} \|\eta_t - \eta_{t-1}\|.$$  

By Assumption 8 $u_{-1} = \eta_0$. Thus, rearranging yields

$$\sum_{t=0}^{T} \|u_t - \eta_t\| \leq \frac{k_u (2 + \hat{C}) + k_\zeta}{1 - k_u} \sum_{i=1}^{T} \|\theta_t - \theta_{t-1}\|$$

$$+ \frac{1 + k_\zeta}{1 - k_u} \sum_{t=1}^{T} \|\eta_t - \eta_{t-1}\| + \frac{k_\zeta}{1 - k_u} C_0$$

(20)

The desired regret bound then follows from inserting (19) and (20) into (16).

Theorem 2 provides a sufficient condition for achieving bounded dynamic regret (for time-varying cost functions) if the closed loop is already known to be asymptotically stable (for any constant cost function). We note that the sufficient condition in Theorem 2 and the necessary condition in Proposition 1 are similar, but not the same. More specifically, both conditions require the upper bound in Definition 1 to be summable, but Theorem 2 additionally assumes that the upper bound $\beta \in \mathcal{K}[\mathcal{L}]$ is linear in $s = \|x_0 - \theta\|$. Thus, it is possible that there exists a $\beta \in \mathcal{K}[\mathcal{L}]$ such that the optimal steady state is asymptotically stable according to Definition 1, thereby satisfying the necessary condition in Proposition 1 but every linear bound $k \|x_0 - \theta\| \sigma(t)$, even if it exists, is not summable, i.e., does not satisfy the sufficient condition of Theorem 2. Combining Proposition 1 and Theorem 2 in order to obtain a necessary and sufficient condition is an interesting direction for future research.

Remark 4. (Algorithms that admit a state-space representation) If Assumption 2 is satisfied, i.e., the Algorithm $\mathcal{A}$ admits a state-space representation, some of the requirements for Theorem 2 can be relaxed. More specifically, we assume that the extended state $(x, x^A)$ is asymptotically stable with respect to the set $\mathcal{S}_L = \{(\theta, \theta^A)\}$ according to Assumption 7, where $\theta^A$ are the unique controller states $x^A$ such that $x^A \in \mathcal{S}_L$. Thereby, we allow the system states to diverge from $\theta$ initially if the algorithm states $x^A$ are not initialized correctly (compare Remark 2). Moreover, instead of Assumption 8 we require that $h^2_\eta(x^A, x)$ in Assumption 2 is uniformly (in $L \in \mathcal{L}$) Lipschitz continuous, and that the variation of the optimal steady-state controller states $\theta^A$ is bounded by the variation of the optimal steady states and inputs, i.e., there exist constants $k^A_\theta, k^A_\eta > 0$ such that for any two cost functions $L_1, L_2 \in \mathcal{L}$, the corresponding steady states $(\eta_1, \theta_1), (\eta_2, \theta_2)$ and controller states $\theta^A_1, \theta^A_2$ satisfy $\|\theta^A_1 - \theta^A_2\| \leq k^A_\theta \|\theta_1 - \theta_2\| + k^A_\eta \|\eta_1 - \eta_2\|$. Then, we have bounded regret as in Theorem 2 by similar arguments as above. Additionally, as discussed above Lemma 2 if there exists a compact set $X \subset \mathbb{R}^n$ such that $x_t \in X$ for all $t \in \mathbb{Z}_{\geq 0}$ and if there exists $\kappa \in \mathcal{K}_\infty$, $\sigma \in \mathcal{L}$ such that $\|\eta_t, x^A_t\|_{\mathcal{S}_L} \leq \kappa(\|x_t - x^A_t\|_{\mathcal{S}_L} + \sigma(t))$ for any constant cost function $L \in \mathcal{L}$ with $\kappa(s)$ Lipschitz at the origin, then Assumption 7 is satisfied. Moreover, if the algorithm admits a state-space representation, then Lipschitz continuity at the origin of $\kappa(s)$ implies the sufficient condition in Theorem 2 i.e., (15), is satisfied as well in Proposition 2.

We close this section by two examples illustrating the application of Theorem 2. First, we continue Example 1 and modify the proposed algorithm such that it achieves bounded regret. Thereafter, in Example 2 we design an exponentially stabilizing algorithm and derive an average dwell time for this special case that ensures bounded dynamic regret.

Example 1. (continued) Recall that we consider the discrete-time integrator $x_{t+1} = x_t + u_t$. We design an improved version of our algorithm given by

$$u_t = -\frac{2r+1}{(\tau+1)^2} x_t + \frac{2r+1}{(\tau+1)^2} \theta_{t-1},$$

where $r = \|\mathcal{P}(x, x^A)\|_{\mathcal{S}_L}$.
where $\theta_{-1} = x_0$ and $\tau$ as described above, such that, for a constant minimizer $\theta$, the closed loop achieves $\|x_t - \theta\| = \frac{1}{\tau} \|x_0 - \theta\|$ for $t \in \mathbb{Z}_{\geq 1}$ satisfying Assumption 2 with $\sigma(0) = 1, \sigma(t) = \frac{1}{\tau}$ for $t \in \mathbb{Z}_{\geq 1}$, and $k = 1$. Since $(\eta, \theta_1)$ are a steady state, we get $\eta = 0$ for all $t$, which implies

$$\|u_t - \eta_{t-1}\| = \|u_t\| = \frac{2\tau + 1}{\tau + 1} \|x_t - \theta_{t-1}\|,$$

i.e., Assumption 8 is satisfied with $k_k = k_u = 0$ and $k_s = 1$ since $\tau \geq 0$. Thus, the improved algorithm achieves bounded regret if the cost functions satisfy the remaining assumptions of Theorem 2, i.e., Assumption 2 with $\varphi \geq \varphi$ and Assumption 2. In order to ensure the former condition, we show that we can choose any $\varphi > 1$, i.e., (13) holds for any $\varphi > 1$ and $N_0 > 0$. To this end, fix $N''$ to be the smallest integer greater than or equal to $(\varphi N_0 + 1)/(\varphi - 1)$ which implies $t_{N''} > \varphi(N'' - N_0) \geq N'' + 1$, where the first inequality follows from Assumption 2. Then, we get

$$\prod_{j=1}^{N''} k(\Delta \sigma)_j \leq \frac{1}{4}$$

since $(\Delta \sigma)_j \leq \frac{1}{4}$ for at least one $j \in [1,N'')$. For any $N' \in \mathbb{Z}_{>N_0}$ we have $N' \leq rN''$ for some $r \in \mathbb{Z}_{>0}$ and

$$\sum_{i=0}^{N'} \prod_{j=1}^{i} (\Delta \sigma)_j \leq \sum_{i=0}^{rN''} \prod_{j=1}^{i} (\Delta \sigma)_j \leq N'' \sum_{i=0}^{r} \left(\frac{1}{4}\right)^i \leq \frac{4N''}{3},$$

i.e., (13) holds. Thus, if the cost functions do not change (on average) at each time step and satisfy the remaining assumptions in Theorem 3, the improved algorithm achieves bounded dynamic regret. \qed

The last example illustrates how exponential stability implies bounded regret.

**Example 2.** Consider a linear system $x_{t+1} = Ax_t + Bu_t$. We design an algorithm $A$ that, at each time instance $t$, solves an optimization problem to obtain $(\eta_{t-1}, \theta_{t-1})$ and applies

$$u_t = K(x_t - \theta_{t-1}) + \eta_{t-1},$$

where $K$ is chosen such that the closed-loop dynamics $(A + BK)$ are Schur stable. Then, for a constant cost function $L \in \mathcal{V}$, we have $\theta = A\theta + B\eta$ by Assumption 8 and

$$\|x_t - \theta\| = \|A(x_{t-1} - \theta) + B(u_{t-1} - \eta)\| \leq \|A + BK\| \|x_0 - \theta\|.$$

Since $(A + BK)$ is Schur stable, there exist constants $c \geq 1$ and $\lambda \in (0,1)$ such that

$$\|x_t - \theta\| \leq c\lambda^t \|x_0 - \theta\|,$$

i.e., the closed loop is exponentially stable with respect to $\theta$ and Assumption 8 is satisfied with $\sigma(t) = \lambda^t$ and $k = c$. Moreover, since

$$\|u_t - \eta_{t-1}\| = \|K(x_t - \theta_{t-1})\| \leq \|K\| \|x_t - \theta_{t-1}\|,$$

Assumption 8 is satisfied with $k_k = \|K\|$ and $k_s = k_u = 0$. Finally, we derive an explicit expression for $\varphi$ to ensure that (13) is satisfied. To this end, note that Lemma 2 holds for $\varphi = \frac{-\ln(k)}{\ln(A)} + \varphi_0 > 0$ and any $\varphi_0 > 0$, since we get $t_i > \varphi(i - N_0)$ by Assumption 8 and, therefore,

$$\sum_{i=0}^{N'} \prod_{j=0}^{i} k(\Delta \sigma)_j = \sum_{i=0}^{N'} k^i \lambda^i \leq \sum_{i=0}^{N'} e^{\ln(k)i} e^{-\ln(k)(i-N_0)} \lambda^{2\varphi_0(i-N_0)} \leq kN_0 \lambda^{-\varphi_0 N_0} \sum_{i=0}^{N'} \lambda^{2\varphi_0 i} \leq kN_0 \lambda^{-\varphi_0 N_0} \frac{\lambda^{-\varphi_0 N_0}}{1 - \lambda^{2\varphi_0}},$$

i.e., (13) holds. Thus, if the cost functions satisfy the remaining assumptions of Theorem 3, then the algorithm $A$ achieves bounded dynamic regret.

4. CONCLUSION

In this paper, we study the relation between dynamic regret and asymptotic stability. Loosely speaking, our results suggest that achieving bounded dynamic regret is a stronger property than asymptotic stability, because bounded dynamic regret requires a certain minimal rate of convergence, whereas asymptotic stability only characterizes the asymptotic behavior of the closed loop. In particular, we show that, under rather mild assumptions, bounded dynamic regret implies asymptotic stability (compare Theorem 1), whereas asymptotic stability does, in general, not imply bounded dynamic regret (compare Example 1 and Proposition 1). In order to derive sufficient conditions for the case of bounded regret implying asymptotic stability, however, we require additional, more restrictive assumptions (compare Theorem 3). Therefore, future work includes relaxing these additional assumptions. In particular, we conjecture that Assumption 8 could be relaxed as discussed above. Moreover, one could consider local asymptotic stability and constraints instead of the global results provided in this paper. Another interesting possibility for future work is considering other common applications of dynamic regret, e.g., with respect to a priori unknown process noise or unknown system dynamics instead of time-varying cost functions. Finally, future work includes studying the relation between asymptotic stability and dynamic regret for benchmarks that do not achieve bounded dynamic regret with respect to the optimal steady state themselves (compare Remark 1).
Appendix A. Proof of Lemma 1

If there exists \( \beta \in KL^+ \) such that (11) holds, then we have for any constant cost function \( L \in V \), initial state \( x_0 \in \mathbb{R}^n \), and \( s = \|x_0 - \theta\| \)
\[
\lim_{T \to \infty} \sum_{t=0}^{T} \|x_t - \theta\| \leq \lim_{T \to \infty} \sum_{t=0}^{T} \beta(s, t) \leq B(s) =: D(s),
\]
where the second inequality follows from \( \beta \in KL^+ \).

For the converse implication, fix any constant cost function \( L \in V \) and assume that for any initial state \( x_0 \in \mathbb{R}^n \) there exists a finite constant \( D(||x_0 - \theta||) \geq 0 \) such that
\[
\lim_{T \to \infty} \sum_{t=0}^{T} \|x_t - \theta\| \leq D(||x_0 - \theta||) < \infty. \tag{A.1}
\]
Moreover, there exists \( \tilde{\beta} \in KL \) (but not necessarily in \( KL^+ \)) such that (11) holds because the closed loop is asymptotically stable. Then, we construct \( \beta \in KL^+ \) such that (11) is satisfied.

To this end, for any \( s \geq 0 \) and \( t \in \mathbb{Z}_{\geq 0} \), consider \( \phi(s, t) \) as defined in (11). Note that \( \phi(0, t) = 0 \), \( \phi(\cdot, t) \) is nondecreasing for any fixed \( t \), and \( \phi(\cdot, t) \) is upper bounded by \( \beta(\cdot, t) \in K \) because the closed loop is asymptotically stable by the assumptions of Lemma 1. Hence, for every \( t \in \mathbb{Z}_{\geq 0} \), \( \phi(\cdot, t) \) is continuous at the origin. Therefore, for any \( \epsilon > 0 \), there exists a function \( \phi_\epsilon : \mathbb{R}_{[0,]} \times \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0} \) such that, for any fixed \( t \in \mathbb{Z}_{\geq 0} \), \( \phi_\epsilon(\cdot, t) \) is continuous, nondecreasing, \( \phi_\epsilon(0, t) = 0 \), \( \phi_\epsilon(t, \cdot) = \phi(2t, \cdot) \), and \( \phi_\epsilon(s, t) \geq \phi(s, t) \) for all \( s \in [0, \epsilon] \), and \( \phi_\epsilon(s, \cdot) \) is nonincreasing for any fixed \( s \in [0, \epsilon] \). Next, we define the function

\[
\Phi(s, t) = \begin{cases} 
\phi_\epsilon(s, t) & \text{if } s \in [0, \epsilon] \\
\phi(\epsilon^i, t) + \frac{\epsilon^i - (i-1)\epsilon}{\epsilon} \Delta \phi(\epsilon^i, t) & \text{if } s \in ((i-1)\epsilon, i\epsilon], \quad i = 2, 3, \ldots 
\end{cases}
\]

where \( \Delta \phi(\epsilon^i, t) = \phi((i+1)\epsilon, t) - \phi(\epsilon^i, t) \). The function \( \Phi \) is piece-wise linear for any fixed \( t \in \mathbb{Z}_{\geq 0} \) and \( s \geq \epsilon \) and it holds that \( \phi(s + 2\epsilon, t) \geq \Phi(s, t) \geq \phi(s, t) \) for any \( s \geq 0 \) and \( t \in \mathbb{Z}_{\geq 0} \). In the following, we will show that for any fixed \( s \geq 0 \), \( \Phi(s, \cdot) \in L \), and for any fixed \( t \in \mathbb{Z}_{\geq 0} \), \( \Phi(\cdot, t) \) is nondecreasing, continuous, and \( \Phi(0, t) = 0 \). First, fix any \( s \geq 0 \). Then, \( \Phi(s, \cdot) \) is nonincreasing because both \( \phi(\cdot, \cdot) \) and \( \phi_\epsilon(\cdot, \cdot) \) are nonincreasing by definition. Moreover, \( 0 \leq \lim_{t \to \infty} \Phi(s, t) \leq \lim_{t \to \infty} \phi(s + 2\epsilon, t) \leq \lim_{t \to \infty} \beta(s + 2\epsilon, t) = 0 \), i.e., \( \Phi(s, \cdot) \in L \). Second, \( \Phi(\cdot, t) \) is continuous by definition for any fixed \( t \in \mathbb{Z}_{\geq 0} \) and \( \Phi(0, t) = 0 \) because \( \phi_\epsilon(0, t) = 0 \). Additionally, \( \Phi(\cdot, t) \) is nondecreasing since both \( \phi_\epsilon(\cdot, t) \) and \( \phi(\cdot, \cdot) \) are nondecreasing.

Finally, choose any function \( \beta' \in KL^+ \) which satisfies, for any \( s \geq 0 \), \( 2\beta'(s, t) \geq \beta'(s, t) \) for all \( t < T \) and \( \beta'(s, T - 1) \geq \Phi(s, T) \), where \( T \) is from Assumption 5. Then, we define

\[
\beta(s, t) = \begin{cases} 
2\beta'(s, t) & \text{if } t < T \\
\beta'(s, t) + \Phi(s, t) & \text{if } t \geq T.
\end{cases}
\]

We claim that \( \beta \) is the desired function, i.e., \( \beta \in KL^+ \) and it satisfies (11). Note that \( \beta \in KL \) because \( \beta' \in KL \), \( \Phi(s, T) \leq \beta'(s, T - 1) \) for any \( s \), and by the properties of \( \Phi \) discussed above. We first show that (11) is satisfied for any initial state \( x_0 \in \mathbb{R}^n \). Let \( s = ||x_0 - \theta|| \). We have
\[
\|x_t - \theta\| \leq \beta(s, t) \leq 2\beta'(s, t) = \beta(s, t)
\]
for \( t < T \) and for \( t \geq T \) we get
\[
\|x_t - \theta\| \leq \phi(s, t) \leq \Phi(s, t) \leq \beta(s, t).
\]
Second, we have for any \( s \geq 0 \)
\[
\lim_{T \to \infty} \sum_{t=0}^{T} \beta(s, t) = C_T + \lim_{T \to \infty} \sum_{t=0}^{T} \Phi(s, t)
\]
\[
\leq C_T + \lim_{T \to \infty} \sum_{t=0}^{T} \phi(s + 2\epsilon, t),
\]
where \( C_T = \sum_{t=0}^{T-1} \beta'(s, t) + \lim_{T \to \infty} \sum_{t=0}^{T} \beta'(s, t) \) is finite because \( \beta' \in KL^+ \). Finally, by Assumption 5, there exists a closed-loop trajectory \( x = \{x_t\}_{t=0}^{\infty} \) such that we get
\[
\lim_{T \to \infty} \sum_{t=0}^{T} \beta(s, t) \leq C_T + \mu \lim_{T \to \infty} \sum_{t=0}^{T} \|x_t - \theta\|
\]
\[
\leq C_T + \mu D(||x_0 - \theta||) \leq B(s),
\]
i.e., \( \beta \in KL^+ \).

Appendix B. Proof of Lemma 2

First, we claim that for any \( N_0 \in \mathbb{Z}_{\geq 0} \) there exists \( \varphi \) and \( 0 < \delta < 1 \) such that
\[
\prod_{j=1}^{N''} \delta(k(\Delta \sigma)_j) \leq \delta \tag{B.1}
\]
for any \( N'' \in [N_0 + 1, 2N_0 + 1] \) and \( \varphi \geq \varphi \). Let
\[
\delta(N'') := \min_{\{t_j\}_{j=1}^{N''}} \prod_{j=1}^{N''} k(\Delta \sigma)_j \quad \text{s.t.} \quad \| \sigma(t) \| = \varphi \tag{B.2}
\]
be the smallest upper bound of the above product for any switching sequence \( \{t_j\}_{j=1}^{N''} \) that satisfies Assumption 6, which exists and satisfies \( \delta(N'') > 0 \) due to \( \sigma \in L \). Since \( \sigma(\cdot) \) is nonincreasing, \( \delta(N'') \) is nonincreasing with respect to \( \varphi \). Moreover, since \( \lim_{\varphi \to 0} \sigma(t) = 0 \) by Assumption 4, there exists \( \varphi' \) such that \( \delta < 1 \) holds for any \( N'' \in [N_0 + 1, 2N_0 + 1] \). The next step is to show that for any \( \sigma' \) that satisfies Assumption 6, there exists \( \varphi'' \) such that \( \delta < 1 \) holds for any \( N'' \in [N_0 + 1, 2N_0 + 1] \). Thus, \( 1 \geq (\Delta \sigma)_t > 0 \)
and \( k > 1 \) lead to

\[
\sum_{i=0}^{N'} \prod_{j=1}^{i} k(\Delta \sigma)_j = \sum_{i=0}^{N''} \prod_{j=1}^{i} k(\Delta \sigma)_j
\]

\[
= \sum_{i=0}^{N''-1} \prod_{j=1}^{i} k(\Delta \sigma)_j + \sum_{i=N''}^{N'} \prod_{j=1}^{i} k(\Delta \sigma)_j
\]

\[
\leq 2N_0 k^j + \sum_{i=0}^{r+2} (N_0 + 1)^i \delta^i
\]

\[
\leq 1 - \frac{k^{2N_0+1}}{1 - k} \quad \text{for } N_0 + 1 \leq 1 - \delta
\]

If \( k = 1 \), the first sum in (B.2) is bounded by \( 2N_0 + 1 \). \( \square \)

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