COLLECTIVE BEHAVIORS OF THE LOHE HERMITIAN SPHERE MODEL WITH INERTIA

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Dedicated to the celebration of the 80th birthday of Professor Shuxing Chen

Abstract. We present a second-order extension of the first-order Lohe Hermitian sphere (LHS) model and study its emergent asymptotic dynamics. Our proposed model incorporates an inertial effect as a second-order extension. The inertia term can generate an oscillatory behavior of particle trajectory in a small time interval (initial layer) which causes a technical difficulty for the application of monotonicity-based arguments. For emergent estimates, we employ two-point correlation function which is defined as an inner product between positions of particles. For a homogeneous ensemble with the same frequency matrix, we provide two sufficient frameworks in terms of system parameters and initial data to show that two-point correlation functions tend to the unity which is exactly the same as the complete aggregation. In contrast, for a heterogeneous ensemble with distinct frequency matrices, we provide a sufficient framework in terms of system parameters and initial data, which makes two-point correlation functions be close to unity by increasing the principal coupling strength.

1. Introduction. Collective behaviors of many-body systems are often observed in biological complex networks, to name a few, flocking of birds, swarming of fish, herding of sheep, synchronous firing of fireflies, neurons and pacemaker cells [1,3,6,12,13,19,20,22,24] etc. Well known models for those phenomena are the Kuramoto and Lohe matrix models. These models are extensively studied in [21,25,26]. In this paper, we are interested in the aggregation phenomena of particles on a Hermitian sphere. To motivate our discussion, we begin with the first-order LHS model. Let

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where $z_j = z_j(t)$ be the position of the $j$-th particle on the Hermitian unit sphere at time $t$. In order to fix the idea, we begin with the first-order Lohe Hamiltonian sphere model \cite{9–11} on a Hermitian sphere $\mathbb{HS}^d := \{ z \in \mathbb{C}^{d+1} : \|z\| = r \}$:

$$
\dot{z}_j = \Omega_j z_j + \kappa_0(\langle z_j, z_j \rangle z_j - \langle z_c, z_j \rangle z_j) + \kappa_1(\langle z_j, z_c \rangle - \langle z_j, z_j \rangle)z_j, \quad j = 1, \cdots, N, \quad (1.1)
$$

where $z_c := \frac{1}{N} \sum_{j=1}^{N} z_j$ and $(\langle z_1, z_2 \rangle := \sum_{\alpha=1}^{d+1} \overline{z}_{\alpha}^{\dagger} z_{\alpha})$ which is conjugate linear in the first argument and linear in the second argument. In addition, $\Omega_j$ is the skew-symmetric $(d + 1) \times (d + 1)$ matrix such that $\Omega_j^\dagger = -\Omega_j$ and $\kappa_0, \kappa_1$ are non-negative coupling strengths. The emergent dynamics of the LHS model (1.1) has been extensively studied in \cite{9–11} (see Section 2.1).

In this paper, we are interested in the large-time dynamics of the Cauchy problem to the second-order extension of (1.2) incorporating inertial effect to (1.1):

$$
\begin{align*}
&m \left( \dot{v}_j - \frac{\Omega_j}{\gamma} v_j \right) + \gamma v_j = \kappa_0(\langle z_j, z_j \rangle z_c - \langle z_c, z_j \rangle z_j) + \kappa_1(\langle z_j, z_c \rangle - \langle z_j, z_j \rangle)z_j \\
&\quad - \frac{m\|v_j\|^2}{\|z_j\|^2} z_j, \quad z_j \in \mathbb{HS}^d, \quad t > 0, \\
&v_j = \dot{z}_j - \frac{\Omega_j}{\gamma} z_j, \quad z_j(0) = z_j^{in}, \quad v_j(0) = v_j^{in}, \quad \langle z_j^{in}, v_j^{in} \rangle + \langle v_j^{in}, z_j^{in} \rangle = 0,
\end{align*}
$$

(1.2)

where $m$ is the strength of inertia which is nonnegative.

It is easy to see that for zero inertia and unit friction constant, system (1.2) reduces to the LHS model (1.1). The basic conservation law and solution splitting property will be given in Lemma 2.1 and Lemma 2.2, respectively. Before we present our main results, we first recall the concepts of complete aggregation and practical aggregation as follows.

**Definition 1.1.** Let $Z := \{z_j\}$ be a solution to (1.2).

1. The solution $Z$ exhibits (asymptotic) complete aggregation if the following estimate holds.

$$
\lim_{t \to \infty} \max_{i,j} \|z_i(t) - z_j(t)\| = 0.
$$

2. The solution $Z$ exhibits (asymptotic) practical aggregation if the following estimate holds.

$$
\lim_{\kappa_0 \to \infty} \limsup_{0 \leq t < \infty} \max_{i,j} \|z_i(t) - z_j(t)\| = 0.
$$

Next, we briefly present our two main results on the large-time emergent dynamics of (1.2).

First, we present a sufficient framework for the complete aggregation for the homogeneous ensemble with $\Omega_j = \Omega$. In this case, we may assume that $\Omega = 0$ and $z_j$ satisfies

$$
m \ddot{z}_j = -\gamma \dot{z}_j + \kappa_0(\langle z_c, z_j \rangle - \langle z_j, z_j \rangle)z_j + \kappa_1(\langle z_j, z_c \rangle - \langle z_j, z_j \rangle)z_j - m\|z_j\|^2 z_j.
$$

Under the following conditions on system parameters and initial data:

$$
\gamma \gg m, \quad G(0) \ll 1, \quad |G(0)| + G(0) \ll 1.
$$

For the detailed conditions, we refer to frameworks $(\mathcal{F}_A1)$–$(\mathcal{F}_A2)$ and $(\mathcal{F}_B1)$–$(\mathcal{F}_B2)$ in Section 4.1. Our first main result is concerned with the complete aggregation (see Theorem 4.1):

$$
\lim_{t \to \infty} \max_{i,j} |z_i(t) - z_j(t)| = 0.
$$
For this, we introduce two-point correlation functions:

\[ h_{ij} = \langle z_i, z_j \rangle, \quad g_{ij} := 1 - h_{ij}, \quad 1 \leq i, j \leq N, \quad G := \frac{1}{N^2} \sum_{i,j=1}^{N} |g_{ij}|^2, \]

and then, we also derive a differential inequality for \( G \):

\[
m\ddot{G} + \gamma \dot{G} + 4\kappa_0 \delta G \leq f(t), \quad f(t) \to 0 \text{ as } t \to \infty.\]

Then, via Gronwall’s differential inequality, we can derive the zero convergence of \( G \):

\[
\lim_{t \to \infty} G(t) = 0, \quad \text{i.e.,} \quad \lim_{t \to \infty} \langle z_i(t), z_j(t) \rangle = 1, \quad \forall i, j = 1, \ldots, N.
\]

This clearly implies the complete aggregation in the sense of Definition 1.1.

Second, we deal with a heterogeneous ensemble with distinct natural frequency matrices \( \Omega_i \). In this situation, we derive a rather weak aggregation, namely practical aggregation. For this, we propose a framework on the system parameters and initial data:

\[
\gamma \gg m, \quad G(0) \ll 1, \quad |\dot{G}(0)| + G(0) \ll 1.
\]

For the detailed conditions, we refer to frameworks \((FC1)-(FC2)\) in Section 4.2. As in the aggregation estimate to the homogeneous ensemble, we derive a second-order Gronwall’s inequality:

\[
m\ddot{G} + \gamma \dot{G} + 4\kappa_0 \delta G \leq 4\Omega^\infty + 8\kappa_1 + \frac{16m}{\gamma^2} \left[ \Omega^\infty + 2(\kappa_0 + \kappa_1) \right]^2, \quad t > 0.
\]

Then, via the second-order Gronwall lemma (Lemma 5.4) and a suitable ansatz for \( m = \frac{m_0}{\kappa_0^2} \), one can show

\[
G(t) \lesssim \max \left\{ \frac{1}{\kappa_0}, \frac{1}{\kappa_1} \right\}, \quad \text{for } t \gg 1.
\]

This clearly implies the practical aggregation in Definition 1.1. We refer to Theorem 4.2 for a detailed discussion.

The rest of this paper is organized as follows. In Section 2, we briefly introduce the second-order LHS model and basic properties of the proposed model. Moreover, we discuss it with our previous first-order LHS model and review the previous results on the first-order LHS model. In Section 3, we study the characterization and instability of some distinguished states. In Section 4, we summarize our main results on the emergent dynamics of the second-order LHS model. In Section 5 and Section 6, we provide proofs of Theorem 4.1 and Theorem 4.2. Finally, Section 7 is devoted to a brief summary of our main results and discussion on some remaining problems for a future work.

Notation: For a vector \( z = (z_1, \ldots, z_{d+1}) \in \mathbb{C}^{d+1} \) and \( w = (w_1, \ldots, w_{d+1}) \in \mathbb{C}^{d+1} \), we set the inner product \( \langle \cdot, \cdot \rangle \) and its corresponding \( \ell^2 \)-norm:

\[
\langle z, w \rangle := \sum_{i=1}^{d+1} z_i \overline{w_i}, \quad \|z\| := \sqrt{\langle z, z \rangle}, \quad \text{HS} = \text{HS}^d.
\]

For a given configuration \( C := \{(z_j, w_j := \dot{z}_j)\} \), we set state and velocity diameters as follows.

\[
\mathcal{D}(Z) := \max_{i,j} |z_i - z_j|, \quad \mathcal{D}(W) := \max_{i,j} |w_i - w_j|.
\]
2. Preliminaries. In this section, we briefly introduce the second-order LHS model (1.2) and its basic properties, and discuss its relations with other aggregation models such as the first-order LHS model and the Kuramoto model.

2.1. The second-order LHS model. In this subsection, we study basic properties of the second-order LHS model. To factor out the rotational motion, we introduce an auxiliary variable \( u_j \) on \( \mathbb{H}_N^2 \):

\[
\begin{align*}
    z_j := e^{\frac{t}{N} \Omega_j} u_j, \quad j = 1, \cdots, N.
\end{align*}
\]

By direct calculations, one has

\[
\begin{align*}
    u_j = e^{-\frac{t}{N} \Omega_j} z_j, \quad \dot{u}_j = e^{-\frac{t}{N} \Omega_j} v_j, \quad \ddot{u}_j = e^{-\frac{t}{N} \Omega_j} \left( \dot{u}_j - \frac{1}{\gamma} \Omega_j v_j \right).
\end{align*}
\]

We substitute (2.2) into (1.2) and use the fact that \( \Omega_j \) is skew-Hermitian to derive the equations for \( u_j \):

\[
\begin{align*}
    m \ddot{u}_j + \gamma \dot{u}_j &= \frac{K_0}{N} \sum_{k=1}^{N} \left( \langle u_j, e^{\frac{t}{N} \Omega_j^*} u_k \rangle \dot{u}_j - \langle e^{\frac{t}{N} \Omega_j^*} u_k, u_j \rangle u_k \right) \\
    &+ \frac{K_1}{N} \sum_{k=1}^{N} \left( \langle u_j, e^{\frac{t}{N} \Omega_j^*} u_k \rangle - \langle e^{\frac{t}{N} \Omega_j^*} u_k, u_j \rangle \right) u_j - \frac{m}{\|u_j\|^2} \|\dot{u}_j\|^2 u_j,
    \\
    u_j(0) &= u_j^0 = z_j^0, \quad \dot{u}_j(0) = \dot{u}_j^0 = v_j^0, \quad \langle u_j^0, \dot{u}_j^0 \rangle + \langle \dot{u}_j^0, u_j^0 \rangle = 0.
\end{align*}
\]

In the following lemma, we study the conservation of \( \|z_j\| \) and \( \|u_j\| \).

**Lemma 2.1.** (Conservation laws) Let \( \{z_j\} \) and \( \{u_j\} \) be global solutions of (1.2) and (2.3), respectively. Then, \( \ell^2 \)-norms \( \|z_j\| \) and \( \|u_j\| \) are conserved quantities:

\[
\frac{d}{dt} \|z_j(t)\| = 0, \quad \frac{d}{dt} \|u_j(t)\| = 0, \quad \forall \ t \geq 0, \quad j = 1, \cdots, N.
\]

**Proof.** Since \( e^{-\frac{t}{N} \Omega_j} \) is unitary, one can see

\[
\|u_j\|^2 = \langle u_j, u_j \rangle = \langle e^{-\frac{t}{N} \Omega_j} z_j, e^{-\frac{t}{N} \Omega_j} z_j \rangle = \langle e^{-\frac{t}{N} \Omega_j} \left( e^{-\frac{t}{N} \Omega_j} \right)^* z_j, z_j \rangle = \langle z_j, z_j \rangle = \|z_j\|^2.
\]

Hence, we only verify the conservation of the norm \( \|u_j\| \). Now we claim:

\[
\frac{d}{dt} \|u_j\|^2 = \langle u_j, \dot{u}_j \rangle + \langle \dot{u}_j, u_j \rangle = 0.
\]

Simple calculation yields

\[
\begin{align*}
    m \frac{d}{dt} \left( \langle u_j, \dot{u}_j \rangle + \langle \dot{u}_j, u_j \rangle \right) = 2m \|\dot{u}_j\|^2 + \langle u_j, m \ddot{u}_j \rangle + \langle m \ddot{u}_j, u_j \rangle.
\end{align*}
\]

Here, we use (2.3) and (2.4) to obtain

\[
\begin{align*}
    \langle u_j, m \ddot{u}_j \rangle &= -\gamma \langle u_j, \dot{u}_j \rangle + \frac{K_0}{N} \sum_{k=1}^{N} \left( \langle u_j, e^{\frac{t}{N} \Omega_j^*} u_k \rangle - \langle e^{\frac{t}{N} \Omega_j^*} u_k, u_j \rangle \right) \|u_j\|^2 \\
    &+ \frac{K_1}{N} \sum_{k=1}^{N} \left( \langle u_j, e^{\frac{t}{N} \Omega_j^*} u_k \rangle - \langle e^{\frac{t}{N} \Omega_j^*} u_k, u_j \rangle \right) \|u_j\|^2 - m \|\dot{u}_j\|^2.
\end{align*}
\]

This yields

\[
\begin{align*}
    \langle u_j, m \ddot{u}_j \rangle + \langle m \ddot{u}_j, u_j \rangle = \langle u_j, m \ddot{u}_j \rangle + \langle m \ddot{u}_j, u_j \rangle = -\gamma \left( \langle u_j, \dot{u}_j \rangle + \langle \dot{u}_j, u_j \rangle \right) - 2m \|\dot{u}_j\|^2.
\end{align*}
\]
Now, we derive Gronwall’s inequality for \( \langle u_j, \dot{u}_j \rangle + \langle \dot{u}_j, u_j \rangle \):

\[ m \frac{d}{dt} \langle u_j, u_j \rangle + \langle \dot{u}_j, u_j \rangle = 2m\|\ddot{u}_j\|^2 + \langle u_j, m\dot{u}_j \rangle + \langle m\dot{u}_j, u_j \rangle = -\gamma (\langle u_j, \dot{u}_j \rangle + \langle \dot{u}_j, u_j \rangle). \]

Gronwall’s lemma and initial conditions imply

\[ \frac{d}{dt} \|u_j\|^2 = \langle u_j(t), \dot{u}_j(t) \rangle + \langle \dot{u}_j(t), u_j(t) \rangle = e^{-\frac{\gamma}{m}t} (\langle u_j^{in}, \dot{u}_j^{in} \rangle + \langle \dot{u}_j^{in}, u_j^{in} \rangle) = 0, \quad \forall \ t > 0. \]

**Lemma 2.2.** Suppose \( \Omega_j \) satisfies

\[ \Omega_j^\dagger = -\Omega_j, \quad \Omega_j \equiv \Omega \quad \text{for all} \ j = 1, \ldots, N, \]  

where \( \dagger \) denotes the Hermitian conjugate, and let \( \{z_j\} \) be a solution to (1.2). Then, \( u_j \) defined in (2.1) satisfies

\[
\begin{align*}
\begin{cases}
\dot{u}_j + \gamma u_j = \kappa_0 (\|u_j\|^2 u_c - \langle u_c, u_j \rangle u_j) + \kappa_1 (\langle u_j, u_c \rangle - \langle u_c, u_j \rangle) u_j - m\|\dot{u}_j\|^2 u_j, \\
(u_j(0), \dot{u}_j(0)) = (u_j^{in}, \dot{u}_j^{in}), \quad (\dot{u}_j^{in}, u_j^{in}) + (\dot{u}_j^{in}, u_j^{in}) = 0,
\end{cases}
\end{align*}
\]

where \( u_c := \frac{1}{N} \sum_{k=1}^{N} u_k \).

**Proof.** We substitute the relation (2.5) into (2.3) to get a desired estimate. \( \Box \)

**Remark 2.1.** For a homogeneous ensemble with the common natural frequency \( \Omega \), one can also see that \( z_j \) satisfies

\[
\begin{align*}
\begin{cases}
m\dot{z}_j + \gamma z_j = \kappa_0 (\langle z_j, z_j \rangle z_c - \langle z_c, z_j \rangle z_j) + \kappa_1 (\langle z_j, z_c \rangle - \langle z_c, z_j \rangle) z_j - \frac{m}{\|z_j\|^2} z_j, \\
(\dot{z}_j(0), z_j(0)) = (z_j^{in}, \dot{z}_j^{in}), \quad (\dot{z}_j^{in}, z_j^{in}) + (\dot{z}_j^{in}, z_j^{in}) = 0.
\end{cases}
\end{align*}
\]

2.2. **Relation with other aggregation models.** In this subsection, we briefly discuss relations with other aggregation models with (1.2). For zero inertia and unit friction constant case:

\[ m = 0 \quad \text{and} \quad \gamma = 1 \]

system (1.2) reduces to the first-order LHS model \([10, 11]:\]

\[ \dot{z}_j = \Omega_j z_j + \kappa_0 (\langle z_j, z_j \rangle z_c - \langle z_c, z_j \rangle z_j) + \kappa_1 (\langle z_j, z_c \rangle - \langle z_c, z_j \rangle) z_j. \]

Moreover, for the special case with \( z_j = x_j \in \mathbb{S}^d \subset \mathbb{R}^{d+1} \) and \( \kappa_1 = 0 \), system (2.6) also reduces to the Lohe sphere model:

\[ \dot{x}_j = \Omega_j x_j + \kappa_0 (\langle x_j, x_j \rangle x_c - \langle x_c, x_j \rangle x_j), \quad \hat{\Omega}^T = -\hat{\Omega} \in \mathbb{R}^{(d+1) \times (d+1)}. \]

The emergent dynamics of (2.7) has been extensively studied in \([4, 8, 14-18, 23]\). In the sequel, we mainly discuss emergent behavior of the complex swarm sphere model (2.6). For this, we introduce new dependent variables: for state configuration \( \{z_j\} \),

\[ h_{ij} := \langle z_i, z_j \rangle, \quad R_{ij} := \text{Re}(h_{ij}), \quad I_{ij} := \text{Im}(h_{ij}) \quad \forall \ i, j = 1, 2, \ldots, N. \]

Note that

\[ h_{ij} = \langle z_i, z_j \rangle = 1 \iff R_{ij} = 1 \quad \text{and} \quad I_{ij} = 0, \quad \forall \ i, j = 1, 2, \ldots, N. \]

For a homogeneous ensemble with \( \Omega_j = \Omega \), we expect the formation of complete aggregation which means

\[ \lim_{t \to \infty} h_{ij} = 1. \]
Hence, it is natural to introduce a Lyapunov functional depending on the quantities:

\[ |1 - h_{ij}| = \sqrt{(1 - R_{ij})^2 + I_{ij}^2} \]

and we set

\[ J_{ij} := 4\sqrt{(1 - R_{ij})^2 + I_{ij}^2}, \quad \forall i, j \in \{1, 2, \cdots, N\}, \]

\[ J_M := \max_{i, j} J_{ij} \quad \text{and} \quad D(\Omega) := \max_{i, j} \|\Omega_i - \Omega_j\|_F. \]

Now, we close this section after brief summary of emergent behaviors for (2.6) without proofs.

**Theorem 2.1.** [9] The following assertions hold.

1. (A homogeneous ensemble): Suppose system parameters and initial data satisfy

\[ \kappa_0 > 2\kappa_1 \geq 0, \quad D(\Omega) = 0, \quad J_M(0) < \sqrt{1 - \frac{2\kappa_1}{\kappa_0}}, \]

and let \( \{z_j\} \) be the solution to (2.6) with the initial data \( \{z_j^0\} \). Then, there exists a positive constant \( \bar{\Lambda} \) such that

\[ J_M(t) \leq J_M(0) \exp\left(-\bar{\Lambda}t\right), \quad t > 0. \]

2. (A heterogeneous ensemble): Suppose system parameters and initial data satisfy

\[ \kappa_1 \geq 0, \quad D(\Omega) > 0, \]

and let \( \{z_j\} \) be a solution of (2.6) with the initial data \( \{z_j^0\} \). Then, one has a practical aggregation:

\[ \lim_{\kappa_0 \to \infty} \lim_{t \to \infty} J_M(t) = 0. \]

**Remark 2.2.** Major difference between the first-order LHS model (2.6) and the Lohe sphere model (2.7) lies in the following additional coupling term:

\[ \kappa_1 (\langle z_j, z_c \rangle - \langle z_c, z_j \rangle) z_j \]

Note that this term will disappear for the real valued case because the inner product is commutative, and by authors’ previous work [11], it is shown that this additional coupling term does not contribute to the aggregation behaviors of oscillators. More precisely, let \( z_j(t) \) be a solution of the following Cauchy problem:

\[
\begin{aligned}
\dot{z}_j &= \kappa_1 (\langle z_j, z_c \rangle - \langle z_c, z_j \rangle) z_j, \quad t > 0, \\
z_j(0) &= z_j^0, \quad |z_j| = 1, \quad j = 1, \cdots, N.
\end{aligned}
\]

Then, Theorem 3.2 in [11] guarantees the existence of a real time-dependent phase \( \{\theta_j\} \) satisfying

\[ z_j(t) = e^{i\theta_j(t)} z_j^0, \quad j = 1, \cdots, N, \]

and the dynamics of \( \theta_j \) is governed by the Kuramoto type model for identical oscillators. Hence, the condition \( \kappa_0 > 2\kappa_1 \) in the first assertion in Theorem 2.1 makes oscillators synchronize.
3. Characterization and instability of two distinguished states. In this section, we discuss the characterization and instability of two distinguished states for system (1.2) with zero frequency matrix and unit Hermitian sphere $\mathbb{S}^d$:

$$\Omega_j = 0, \quad \|z_j\| = 1, \quad j = 1, \cdots, N.$$ 

In this case, system (1.2) takes a much simpler form:

$$m\ddot{z}_j + \gamma \dot{z}_j + \kappa_0 (z_c - \langle z_c, z_j \rangle z_j) + \kappa_1 (\langle z_j, z_c \rangle - \langle z_c, z_j \rangle ) z_j - m\|\dot{z}_j\|^2 z_j.$$  \hspace{1cm} (3.1)

This can be rewritten as a first-order system by introducing an auxiliary variable $w_j = \dot{z}_j$:

$$\begin{cases} 
\dot{z}_j = w_j, & t > 0, \quad j = 1, \cdots, N, \\
\dot{w}_j = -\frac{\gamma}{m} w_j + \frac{\kappa_0}{m} (z_c - \langle z_c, z_j \rangle z_j) + \frac{\kappa_1}{m} (\langle z_j, z_c \rangle - \langle z_c, z_j \rangle ) z_j - m\|w_j\|^2 z_j. 
\end{cases} \hspace{1cm} (3.2)$$

3.1. Characterization of equilibria. Consider the algebraic equilibrium system associated with (3.2):

$$\begin{cases} 
\dot{w}_j = 0, \\
-\gamma w_j + \kappa_0 (z_c - \langle z_c, z_j \rangle z_j) + \kappa_1 (\langle z_j, z_c \rangle - \langle z_c, z_j \rangle ) z_j - m\|w_j\|^2 z_j = 0. 
\end{cases} \hspace{1cm} (3.3)$$

**Proposition 3.1.** Let $\{(z^e_j, w^e_j)\}$ be an equilibrium solution of (3.2) if and only if $(z^e_j, w^e_j)$ is a constant state satisfying

$$w^e_j = 0, \quad z^e_j = \langle z^e_c, z^e_j \rangle z^e_j, \quad \forall \ j = 1, \cdots, N,$$

where $z^e_c = \frac{1}{N} \sum_{j=1}^{N} z^e_j$.

**Proof.** ($\Rightarrow$ part): Suppose $\{(z^e_j, w^e_j)\}$ is an equilibrium state. Then, it satisfies

$$w^e_j = 0, \quad 0 = \kappa_0 (z^e_c - \langle z^e_c, z^e_j \rangle z^e_j) + \kappa_1 (\langle z^e_j, z^e_c \rangle - \langle z^e_c, z^e_j \rangle ) z^e_j.$$  \hspace{1cm} (3.4)

Now, we use the relation $\|z^e_j\| = 1$ to see

$$0 = \langle z^e_c, (3.4) \rangle = (\kappa_0 + \kappa_1) (\langle z^e_j, z^e_c \rangle - \langle z^e_c, z^e_j \rangle ).$$

Since $\kappa_0 > 0$ and $\kappa_1 \geq 0$, one has

$$0 = (z^e_j, z^e_c) - (z^e_c, z^e_j).$$  \hspace{1cm} (3.5)

Then, we substitute (3.5) into (3.4) and use $\kappa_0 > 0$ to get

$$z^e_c = \langle z^e_c, z^e_j \rangle z^e_j.$$  

($\Leftarrow$ part): Suppose a constant state $(z^e_j, w^e_j)$ satisfies relations:

$$z^e_c = \langle z^e_c, z^e_j \rangle z^e_j, \quad \forall \ t \geq 0 \quad \text{and} \quad w^e_j = 0, \quad j = 1, \cdots, N.$$  \hspace{1cm} (3.6)

We use the relation (3.6) and $\|z^e_j\| = 1$ to find

$$\langle z^e_j, z^e_c \rangle - \langle z^e_c, z^e_j \rangle = \langle z^e_c, z^e_j \rangle \langle z^e_j, z^e_c \rangle - \langle z^e_c, z^e_j \rangle \langle z^e_c, z^e_j \rangle = - \langle z^e_j, z^e_c \rangle \langle z^e_c, z^e_j \rangle.$$  \hspace{1cm} (3.7)

Finally, the relations (3.6) and (3.7) satisfy the equilibrium system (3.3).
Next, we introduce an order parameter \( \rho \) which measures the degree of aggregation. For a given configuration \( \{ (z_j, w_j) \} \), we set
\[
\rho := \left\| \frac{1}{N} \sum_j z_j \right\|, \quad \rho^\infty := \lim_{t \to \infty} \rho(t) \text{ if it does exist.} \tag{3.8}
\]
Then, \( \rho = 0, 1 \) denote the incoherent state and completely aggregated state, respectively.

As a corollary of Proposition 3.1, we show that equilibrium with positive \( \rho \) is either completely aggregated state or a bi-polar state.

**Corollary 3.1.** For \( d = 0 \), let \( \{ (z^e_j, w^e_j) \} \) be an equilibrium solution to (3.2) with \( \rho > 0 \) and \( |z^e_j| = 1 \). Then, one has
\[
\frac{z^e_j}{z^c_e} \text{ is a real number.}
\]

**Proof.** Let \( \{ (z_j^e, w_j^e) \} \) be an equilibrium state with \( \rho > 0 \) and \( |z^e_j| = 1 \). Then, by Proposition 3.1, one has
\[
\frac{z^e_j}{z^c_e} = \langle z^e_j, z^c_e \rangle z^e_j, \quad j = 1, \cdots, N. \tag{3.9}
\]
On the other hand, since \( \rho = |z^e_c| > 0 \) we write down \( z^e_c \) and \( z^e_j \) as polar forms:
\[
z^e_j = e^{i\theta_j} \quad \text{and} \quad z^e_c = \rho e^{i\phi}, \quad \rho > 0. \tag{3.10}
\]
Now, we substitute (3.10) into (3.9) to get
\[
\frac{z^e_j}{z^c_e} = \frac{1}{\rho} e^{i(\theta_j - \phi)} = \frac{1}{\rho} e^{i(2\theta_j - \phi)}.
\]
This yields
\[
e^{2i\phi} = e^{2i\theta_j}, \quad j = 1, \cdots, N.
\]
Hence, one has
\[
either \theta_j = \phi \quad or \quad \theta_j = \phi + \pi, \quad j = 1, \cdots, N.
\]
Thus,
\[
\frac{z^e_j}{z^c_e} \in \left\{ \frac{1}{\rho}, -\frac{1}{\rho} \right\}.
\]

### 3.2. Instability of two distinguished states.
In this subsection, we study linear instabilities of two distinguished state “bi-polar state and incoherence state (\( \rho = 0 \)).

\[
\begin{cases}
\dot{z}_j = w_j, \\
\dot{w}_j = -\gamma w_j + \frac{\kappa_0}{m} (z_c - \langle z_c, z_j \rangle z_j) + \frac{\kappa_1}{m} \left( \langle z_j, z_c \rangle - \langle z_c, z_j \rangle \right) z_j - \|w_j\|^2 z_j.
\end{cases} \tag{3.11}
\]

In the sequel, we consider \( z_j \) and \( w_j \) as real vectors in \( \mathbb{R}^{2d+2} \). In other words, let \( x^\alpha_j, y^\alpha_j, a^\alpha_j, b^\beta_j \in \mathbb{R} \) be given as follows:
\[
z^\alpha_j = x^\alpha_j + iy^\alpha_j, \quad w^\alpha_j = a^\alpha_j + ib^\beta_j, \quad j = 1, \cdots, N, \quad \alpha = 1, \cdots, d + 1. \tag{3.12}
\]
where \( z_j^a \) and \( w_j^a \) are \( a \)-th component of \( z_j \) and \( w_j \), respectively. We rewrite (3.11) using (3.12) as

\[
\begin{align*}
\dot{x}_j &= a_j, \quad \dot{y}_j = b_j, \\
\dot{a}_j &= -\frac{\gamma}{m} a_j + \frac{\kappa_0}{m} \left[ x_c - (x_c, x_j) x_j + (y_c, y_j) y_j \right] \\
&\quad - \frac{2\kappa_1}{m} (y_c, x_j) - (x_c, y_j) y_j - (\|a_j\|^2 + \|b_j\|^2) x_j, \\
\dot{b}_j &= -\frac{\gamma}{m} b_j + \frac{\kappa_0}{m} \left[ y_c - (x_c, x_j) y_j - (y_c, y_j) x_j \right] \\
&\quad + \frac{2\kappa_1}{m} (y_c, x_j) - (x_c, y_j) x_j - (\|a_j\|^2 + \|b_j\|^2) y_j,
\end{align*}
\]

For stability analysis, we also define

\[
\mathcal{I} := (x_1, \cdots, x_N, y_1, \cdots, y_N, a_1, \cdots, a_N, b_1, \cdots, b_N) = (c_1, \cdots, c_{4N}) \in \mathbb{R}^{4(d+1)N},
\]

and consider the following Jacobian matrix at equilibrium \( \mathcal{I}^e \):

\[
\mathcal{M} := \left. \frac{\partial \mathcal{I}}{\partial \mathcal{I}} \right|_{\mathcal{I} = \mathcal{I}^e} = (\mathcal{M}_{ij})_{1 \leq i,j \leq 4}, \quad \mathcal{M}_{ij} := \left. \frac{\partial (\hat{c}_{(i-1)N+1}, \cdots, \hat{c}_{(j-1)N+N})}{\partial (\hat{c}_{(j-1)N+1}, \cdots, \hat{c}_{(j-1)N+N})} \right|_{\mathcal{I} = \mathcal{I}^e}.
\]

By direct calculations, one has

\[
\mathcal{M}_{11} = \mathcal{M}_{12} = \mathcal{M}_{14} = \mathcal{M}_{21} = \mathcal{M}_{23} = \mathcal{M}_{34} = \mathcal{M}_{43} = O_{(d+1)N}, \\
\mathcal{M}_{13} = \mathcal{M}_{24} = I_{(d+1)N}, \quad \mathcal{M}_{33} = \mathcal{M}_{44} = -\frac{\gamma}{m} I_{(d+1)N},
\]

where we used \( w_j = 0 \) at equilibrium to calculate \( \mathcal{M}_{33} \) and \( \mathcal{M}_{44} \). Hence, \( \mathcal{M} \) has following form:

\[
\mathcal{M} := \left. \frac{\partial \mathcal{I}}{\partial \mathcal{I}} \right|_{\mathcal{I} = \mathcal{I}^e} = \begin{pmatrix} O_{2(d+1)N} & I_{2(d+1)N} \\ \mathcal{M}_s & -\frac{\gamma}{m} I_{2(d+1)N} \end{pmatrix}, \quad \mathcal{M}_s = \begin{pmatrix} \mathcal{M}_{31} & \mathcal{M}_{32} \\ \mathcal{M}_{41} & \mathcal{M}_{42} \end{pmatrix}.
\]

We use the fact that

\[
\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \det(A - BD^{-1}C) \det(D)
\]

to observe the relation between eigenvalues of \( \mathcal{M} \) and \( \mathcal{M}_s \):

\[
\det \left( \mathcal{M} - \lambda \mathcal{I}_{(d+1)N} \right) = \begin{vmatrix} -\lambda I_{2(d+1)N} & I_{2(d+1)N} \\ \mathcal{M}_s & -\left( \frac{\gamma}{m} + \lambda \right) I_{2(d+1)N} \end{vmatrix}
\]
\[
= \det \left( \lambda \left( \frac{\gamma}{m} + \lambda \right) I_{2(d+1)N} - \mathcal{M}_s \right).
\]

It follows from the above equation that if \( \lambda_0 \) is the eigenvalue of \( \mathcal{M}_s \), then \( \lambda \) satisfying

\[
\lambda_0 = \lambda \left( \frac{\gamma}{m} + \lambda \right)
\]

is also an eigenvalue of \( \mathcal{M} \).

Suppose that \( \mathcal{M}_s \) has an eigenvalue \( \lambda_p \) whose real part is positive. Then, one can see

\[
\text{Re} \lambda_p = \text{Re} \lambda \left( \frac{\gamma}{m} + \text{Re} \lambda \right) - (\text{Im} \lambda)^2 \iff \text{Re} \lambda = \frac{-\gamma \pm \sqrt{\gamma^2 + 4m(\text{Re} \lambda_p + (\text{Im} \lambda)^2)}}{2m}.
\]
This implies that $\mathcal{M}$ has an eigenvalue which has positive real part. More precisely, one has

$$\frac{-\gamma + \sqrt{\gamma^2 + 4m(\text{Re}\lambda + (\text{Im}\lambda)^2)}}{2m} > 0.$$ 

Hence, we need further estimates on $\mathcal{M}_s$. We calculate components of $\mathcal{M}_s$ one by one.

$\mathcal{M}_{31} = (A_{jk})_{j,k}$, $\mathcal{M}_{32} = (A_{jk})_{j,k}$, $\mathcal{M}_{41} = (A_{jk})_{j,k}$, $\mathcal{M}_{42} = (A_{jk})_{j,k}$, $1 \leq j, k \leq N$, 

$A_{jk}^1 := \frac{\partial \hat{a}_j}{\partial x_k} = \left( \frac{\partial \hat{a}_j^\alpha}{\partial x_k^\alpha} \right)_{\alpha, \beta}$, $A_{jk}^2 := \frac{\partial \hat{a}_j}{\partial y_k} = \left( \frac{\partial \hat{a}_j^\alpha}{\partial y_k^\alpha} \right)_{\alpha, \beta}$, 

$A_{jk}^3 := \frac{\partial \hat{b}_j}{\partial x_k} = \left( \frac{\partial \hat{b}_j^\alpha}{\partial x_k^\alpha} \right)_{\alpha, \beta}$, $A_{jk}^4 := \frac{\partial \hat{b}_j}{\partial y_k} = \left( \frac{\partial \hat{b}_j^\alpha}{\partial y_k^\alpha} \right)_{\alpha, \beta}$, $1 \leq \alpha, \beta \leq d + 1$.

More precisely, we have

$$\frac{\partial \hat{a}_j^\alpha}{\partial x_k^\beta} = \frac{k_0}{m} \frac{\partial}{\partial x_k^\beta} \left( x_j^\alpha \right) - \frac{k_1}{m} \frac{\partial}{\partial x_k^\beta} \left( \langle x_c, x_j \rangle \right) + \left( \frac{e_\beta}{N} \right) \partial x_j^\alpha \left( \langle y_c, x_j \rangle - \langle x_c, x_j \rangle \right) + \left( \frac{e_\beta}{N} \right) \partial x_j^\alpha \left( \langle y_c, x_j \rangle - \langle x_c, x_j \rangle \right).$$

Similarly, one can see

$$\frac{\partial \hat{a}_j^\alpha}{\partial y_k^\beta} = \frac{k_0}{m} \frac{\partial}{\partial y_k^\beta} \left( x_j^\alpha \right) - \frac{k_1}{m} \frac{\partial}{\partial y_k^\beta} \left( y_j^\alpha \right) - \frac{\partial}{\partial y_k^\beta} \left( \langle x_c, x_j \rangle - \langle x_c, x_j \rangle \right) + \left( \frac{e_\beta}{N} \right) \partial y_j^\alpha \left( \langle y_c, x_j \rangle - \langle x_c, x_j \rangle \right).$$
Similarly, we can observe
\[
\frac{\partial b^\alpha_j}{\partial x^\beta_k} = -\frac{\kappa_0}{m} \left( \frac{x^\beta_j}{N} + \delta_{jk} x^\alpha_c \right) y^\alpha_j - \left( \delta_{jk} y^\beta_j - \frac{y^\beta_j}{N} \right) x^\alpha_j - \left( \langle y_c, x_j \rangle - \langle x_c, y_j \rangle \right) \frac{\partial x^\alpha_j}{\partial x^\beta_k}
\]
\[
- \frac{2\kappa_1}{m} \left( \frac{y^\beta_j}{N} - \delta_{jk} y^\beta_j \right) x^\alpha_j + \left( \langle x_c, y_j \rangle - \langle y_c, x_j \rangle \right) \frac{\partial y^\alpha_j}{\partial x^\beta_k} + \left( \frac{x^\beta_j}{N} - \delta_{jk} x^\alpha_c \right) x^\alpha_j,
\]
\[
\frac{\partial y^\alpha_j}{\partial y^\beta_k} = \frac{\kappa_0}{m} \left( \frac{\delta_{\alpha\beta}}{N} \left( \frac{y^\beta_j}{N} + \delta_{jk} y^\beta_j \right) y^\alpha_j - \left( \langle x_c, x_j \rangle + \langle y_c, y_j \rangle \right) \frac{\partial y^\alpha_j}{\partial y^\beta_k} + \left( \frac{x^\beta_j}{N} - \delta_{jk} x^\alpha_c \right) x^\alpha_j \right)
\]
\[
- \frac{2\kappa_1}{m} \left( \delta_{jk} x^\alpha_c - \frac{x^\beta_j}{N} \right) x^\alpha_j.
\]
In what follows, we study the stability of two distinguished states.
- (Instability of an incoherence state): Since the trace of a matrix is equal to the sum of its eigenvalues, we observe
\[
\text{Tr}M_s = \text{Tr}M_{31} + \text{Tr}M_{42} = \sum_{j=1}^{N} \text{Tr}A_{jj}^1 + \sum_{j=1}^{N} \text{Tr}A_{jj}^4 = \sum_{j=1}^{N} \sum_{a=1}^{d+1} \frac{\partial \hat{b}^\alpha_j}{\partial \hat{y}^\alpha_j} + \sum_{j=1}^{N} \sum_{a=1}^{d+1} \frac{\partial \hat{y}^\alpha_j}{\partial \hat{y}^\alpha_j}
\]
\[
= \frac{\kappa_0}{m} \sum_{j=1}^{N} \sum_{a=1}^{d+1} \left( \frac{1}{N} - \left( \frac{x^\alpha_j}{N} + x^\alpha_c \right) \frac{x^\alpha_j}{N} + \langle x_c, x_j \rangle + \left( \frac{y^\alpha_j}{N} - y^\alpha_c \right) y^\alpha_j \right)
\]
\[
- \frac{2\kappa_1}{m} \sum_{j=1}^{N} \sum_{a=1}^{d+1} \left( \frac{y^\alpha_j}{N} - \frac{y^\alpha_j}{N} \right) y^\alpha_j
\]
\[
+ \frac{\kappa_0}{m} \sum_{j=1}^{N} \sum_{a=1}^{d+1} \left( \frac{1}{N} - \left( \frac{y^\alpha_j}{N} + y^\alpha_c \right) y^\alpha_j + \langle y_c, y_j \rangle + \left( \frac{x^\alpha_j}{N} - x^\alpha_c \right) x^\alpha_j \right)
\]
\[
- \frac{2\kappa_1}{m} \sum_{j=1}^{N} \sum_{a=1}^{d+1} \left( \frac{x^\alpha_c - x^\alpha_j}{N} \frac{x^\alpha_j}{N} \right)
\]
\[
= \frac{\kappa_0}{m} \sum_{j=1}^{N} \sum_{a=1}^{d+1} \left( \frac{1}{N} - \left( \frac{x^\alpha_j}{N} \right)^2 + \left( \frac{y^\alpha_j}{N} \right)^2 \right) + \frac{2\kappa_1}{m} \sum_{j=1}^{N} \sum_{a=1}^{d+1} \left( \frac{y^\alpha_j}{N} \right)^2
\]
\[
+ \frac{\kappa_0}{m} \sum_{j=1}^{N} \sum_{a=1}^{d+1} \left( \frac{1}{N} - \left( \frac{y^\alpha_j}{N} \right)^2 + \left( \frac{x^\alpha_j}{N} \right)^2 \right) + \frac{2\kappa_1}{m} \sum_{j=1}^{N} \sum_{a=1}^{d+1} \left( \frac{x^\alpha_j}{N} \right)^2
\]
\[
= \frac{2(d+1)\kappa_0}{m} + \frac{2\kappa_1}{m} > 0,
\]
where we used the relations:
\[
x_c = y_c = 0.
\]
Hence, $M_s$ has at least one eigenvalue whose real part is positive and so does $M$. Therefore, we can conclude that the incoherence state is unstable.
- (Instability of bi-polar state): Suppose that there exists a point $z$ and integer $n$ such that
\[
\|z\| = 1, \quad 1 \leq n < \left\lfloor \frac{N}{2} \right\rfloor, \quad z_i = -z, \quad z_j = z, \quad 1 \leq i \leq n, \quad n+1 \leq j \leq N.
\]
Without loss of generality, we can assume that \( z = z^\infty = (0, \cdots, 1) \) using the rotational symmetry of the LHS with inertia. Then, we have

\[
z_c = \left( 0, \cdots, 0, \frac{N-2n}{N} \right).
\]

Further calculation yields

\[
\frac{\partial b_j^\alpha}{\partial x_k^\alpha} = \frac{\kappa_0}{m} \left( \frac{\delta_{\alpha \beta}}{N} - \left( \frac{x_j^\beta}{N} + \delta_{jk} x_c^\beta \right) x_j^\alpha - \langle x_c, x_j \rangle \frac{\partial x_j^\beta}{\partial x_k^\beta} \right) \quad \text{and} \quad \frac{\partial \dot{b}_j^\alpha}{\partial x_k^\alpha} = 0.
\]

We observe \((n + 1)(d + 1)\)-th column of \( \mathcal{M}_s \):

\[
\mathcal{M}_s \delta_{(n+1)(d+1)} = \left( \left( A_1^1, \cdots, A_N^1 \right)_{1,d+1}, \cdots, \left( A_1^{n+1}, \cdots, A_N^{n+1} \right)_{d+1,d+1}, 0, \cdots, 0 \right) \top
\]

where \( \delta_{(n+1)}^{(d+1)} \) is a standard basis on \( \mathbb{R}^{2(d+1)N} \). Since \( n \) is smaller than \( \lfloor N/2 \rfloor \), the matrix \( \mathcal{M}_s \) has a positive eigenvalue. Therefore, we can conclude that the bipolar state is unstable.

### 4. Presentation of main results

In this section, we briefly summarize frameworks for the emergent dynamics of the second-order extension of the first-order LHS model.

#### 4.1. Complete aggregation

In this subsection, we present an emergent dynamics of the homogeneous ensemble with the same natural frequency matrix \( \Omega_j = \Omega \). For this, we set

\[
\begin{align*}
\left\{ h_{ij} := (z_i, z_j), \quad g_{ij} := 1 - h_{ij}, \quad 1 \leq i, j \leq N, \right. \\
G := \frac{1}{N^2} \sum_{i,j=1}^N |g_{ij}|^2, \right. \\
\mathcal{R}_1(\vec{Z}) := \max_j \sum_j |\dot{z}_j|^2, \quad \mathcal{R}_2(\vec{Z}) := \max_j \sum_j |z_c - z_j|^2, \\
M_1 := \max \left\{ \frac{\kappa_0 + \kappa_1}{2}, \frac{\gamma + \sqrt{\gamma^2 - 16m\kappa_0\delta}}{2m} \right\}, \quad \nu_1 = \gamma + \frac{\sqrt{\gamma^2 - 16m\kappa_0\delta}}{2m}.
\end{align*}
\]

Then, it is easy to see that

\[
|h_{ij}| \leq 1, \quad |g_{ij}| \leq 2, \quad \bar{h}_{ij} = h_{ji} \quad \text{and} \quad \bar{g}_{ij} = g_{ji}, \quad 1 \leq i, j \leq N.
\]

Now, we set up two sufficient frameworks for complete synchronization. For a fixed \( \delta \in (0, 1) \), our first framework is given as follows.

- \((\mathcal{F}_A 1)\): system parameters \( m, \gamma, \kappa_0 \) and \( \delta \) satisfy

  \[
  \gamma^2 - 16m\kappa_0\delta > 0, \quad m, \gamma, \kappa_0 > 0, \quad \kappa_1 \geq 0.
  \]

- \((\mathcal{F}_A 2)\): initial data satisfy

  \[
  G(0) < \frac{8\kappa_1 + 16mM_1^2}{4\kappa_0\delta} < \left( \frac{1-\delta}{N} \right)^2, \quad G(0) + \nu_1 G(0) < \frac{\nu_1 (8\kappa_1 + 16mM_1^2)}{4\kappa_0\delta}.
  \]

Next, we present our second framework as follows.

- \((\mathcal{F}_B 1)\): system parameters \( m, \gamma, \kappa_0 \) and \( \delta \) satisfy

  \[
  \gamma^2 - 16m\kappa_0\delta < 0, \quad m, \gamma > 0, \quad \kappa_1 \geq 0.
  \]
• (F_B2): initial data satisfy
\[ G(0) < \frac{4m}{\gamma^2}(8\kappa_1 + 16mM_1^2) < \frac{(1 - \delta)^2}{N}, \quad \dot{G}(0) + \frac{\gamma}{2m}G(0) < \frac{2}{\gamma}(8\kappa_1 + 16mM_1^2). \]

Our first main result is concerned with the complete aggregation of a homogeneous ensemble.

**Theorem 4.1.** Suppose that the sufficient frameworks (F_A1)-(F_A2) or (F_B1)-(F_B2) hold. Moreover, assume that initial data and natural frequency satisfy
\[ \|z_j^m\| = 1, \quad \Omega_j = 0, \quad j = 1, \ldots, N, \]
and let \( \{z_j\} \) be the global solution of (1.2). Then, we have
\[ \lim_{t \to \infty} G(t) = 0, \quad \text{i.e.,} \quad \lim_{t \to \infty} h_{ij}(t) = 1, \quad \forall \ 1 \leq i, j \leq N. \]

**Proof.** Although the detailed proof can be found in Section 5, we briefly sketch some ingredients in a proof for reader's convenience. Since
\[ g_{ij} + g_{ji} = 2 - \langle z_i, z_j \rangle - \langle z_j, z_i \rangle = \|z_i - z_j\|^2, \]
one has
\[ \lim_{t \to \infty} G(t) = 0 \quad \Rightarrow \quad \lim_{t \to \infty} D(Z(t)) = 0. \]
Thus, it suffices to verify
\[ \lim_{t \to \infty} G(t) = 0. \quad (4.1) \]
By straightforward calculation to be performed in next section, we can derive second-order differential inequality for \( G \) in (3.1):
\[ m\ddot{G} + \gamma\dot{G} + 4\kappa_0 G \leq 4\kappa_0\sqrt{N}\tilde{G}^2 + 2\kappa_1 R_2(Z) + 16mR_1(\dot{Z}). \]
Next, we use sufficient frameworks to show the uniform boundedness of \( G \), which yields
\[ m\ddot{G} + \gamma\dot{G} + 4\delta\kappa_0 G \leq 2\kappa_1 R_2(Z) + 16mR_1(\dot{Z}). \quad (4.2) \]
Then, we use the relations in Proposition 5.1 and (5.15):
\[ \lim_{t \to \infty} R_2(Z(t)) = 0, \quad \lim_{t \to \infty} R_1(\dot{Z}(t)) = 0, \quad (4.3) \]
and the second-order Gronwall's inequality (4.2) together with (4.3) to derive (4.1).

4.2. **Practical aggregation.** In this subsection, we first list a framework (F_C) formulated in terms of system parameters and initial data for a practical synchronization.

First, we introduce several notation:
\[ R_3(V) := \max_j \|v_j\|, \quad \Omega^\infty := \max_j \|\Omega_j\|_F, \]
\[ U(m, \Omega^\infty, \kappa_0, \kappa_1, \gamma) := 4\Omega^\infty + 8\kappa_1 + \frac{16m}{\gamma^2} \left[ \Omega^\infty + 2(\kappa_0 + \kappa_1) \right]^2. \]

Now, we set up a sufficient framework for practical aggregation. For a fixed \( \delta \in (0, 1) \), our framework is given as follows.

• (F_C1): system parameters \( m, \gamma, \kappa_0 \) and \( \delta \) satisfy
\[ \gamma^2 - 16m\kappa_0\delta > 0, \quad m, \gamma > 0, \quad \kappa_1 \geq 0. \]
(\mathcal{FC}2): initial data satisfy
\[
\begin{cases}
R_3(V^{in}) < \frac{2}{\gamma}(\kappa_0 + \kappa_1), & \mathcal{G}(0) < \frac{1}{4\kappa_0\delta} U(m, \Omega^\infty, \kappa_0, \kappa_1, \gamma) < \frac{(1 - \delta)^2}{N}, \\
\dot{\mathcal{G}}(0) + \nu_1 \mathcal{G}(0) < \frac{\nu_1}{4\kappa_0\delta} U(m, \Omega^\infty, \kappa_0, \kappa_1, \gamma).
\end{cases}
\]

Since practical aggregation is discussed with sufficiently large \(\kappa_0\), there is no statement on \(\kappa_0\) in the framework \(\mathcal{FC}\).

Under the above framework, our second result deals with the emergence of practical aggregation for a heterogeneous ensemble.

**Theorem 4.2.** Suppose that the sufficient framework (\(\mathcal{FC}1\))-(\(\mathcal{FC}2\)) holds, and let \(\{z_j\}\) be the solution of (1.2) with \(\|z_j^{in}\| = 1, j = 1, \ldots, N\). Then, we have a practical aggregation:
\[
\lim_{\kappa_0 \to \infty} \limsup_{t \to \infty} \mathcal{G}(t) = 0.
\]

**Proof.** We briefly sketch a key idea. Detailed argument can be found in Section 6. In the course of a proof, we will derive the following differential inequality:
\[
m \dddot{\mathcal{G}} + \gamma \ddot{\mathcal{G}} + 4\kappa_0 \delta \mathcal{G} \leq 4\Omega^\infty + 8\kappa_1 + \frac{16m}{\gamma^2} [\Omega^\infty + 2(\kappa_0 + \kappa_1)]^2, \quad \forall t \in (0, T^*_\star).
\]

Then, this yields
\[
\mathcal{G}(t) < \frac{\Omega^\infty + 2\kappa_1}{\kappa_0\delta} + \frac{4m}{\gamma^2\kappa_0\delta} [\Omega^\infty + 2(\kappa_0 + \kappa_1)]^2, \quad \forall t > 0.
\]

For a sufficiently large \(\kappa_0 \geq \max \{\Omega^\infty, 2\kappa_1\}\) and a suitable ansatz for \(m\):
\[
m = \frac{m_0}{\kappa_1 + \eta}, \quad \eta > 0, \quad m_0 > 0,
\]

one has
\[
\limsup_{t \to \infty} \mathcal{G}(t) < \frac{\Omega^\infty + 2\kappa_1}{\kappa_0\delta} + \frac{64}{\gamma^2\delta} \cdot \frac{m_0}{\kappa_0^\eta}.
\]

This implies the desired result. \(\square\)

5. **Emergence of complete aggregation.** In this section, we provide estimates on the complete aggregation to second-order LHS model with inertia for a homogeneous ensemble:
\[
\Omega_j = \Omega, \quad \|z_j\| = 1, \quad j = 1, \cdots, N.
\]

Furthermore, by Lemma 2.2, without loss of generality, we may assume \(\Omega = 0\). In this situation, \(z_j\) satisfies
\[
\begin{align*}
m \dddot{z}_j = -\gamma \ddot{z}_j + \kappa_0 (z_c - \langle z_c, z_j \rangle z_j) + \kappa_1 (\langle z_c, z_c \rangle - \langle z_c, z_j \rangle) z_j - m \|\dot{z}_j\|^2 z_j, \\
z_j(0) = z_j^{in}, \quad \dot{z}_j(0) = \dot{z}_j^{in}, \quad \dot{(z_j^{in}, z_j^{in}}) + \langle z_j^{in}, z_j^{in} \rangle = 0, \quad j = 1, \cdots, N.
\end{align*}
\]

Since the proof of Theorem 4.1 is very lengthy, we briefly delineate a proof strategy in several steps.

Recall that our main purpose in this section is to derive a sufficient frameworks (setting) leading to the complete aggregation:
\[
\lim_{t \to \infty} \langle z_i, z_j \rangle = 1, \quad \text{i.e.,} \quad \lim_{t \to \infty} D(Z(t)) = 0.
\]
• Step A: we introduce an energy functional $E$ and via a time-decay estimate of it, we show that
\[
\lim_{t \to \infty} \|\dot{z}_j(t)\| = 0, \quad \lim_{t \to \infty} |\langle z_c, z_j \rangle - \langle z_j, z_c \rangle| = 0, \quad j = 1, \ldots, N.
\]
See Proposition 5.1 for details.

• Step B: we derive a second-order differential inequality for $G$:
\[
m\ddot{G} + \gamma \dot{G} + 4\kappa_0 \frac{G}{\sqrt{N}} \frac{G^2}{2} + f(t), \quad f(t) \to 0 \text{ as } t \to \infty.
\]

• Step C: we use a second-order Gronwall’s lemma (Lemma 5.5) and the result of Step A to derive a zero convergence of $G$:
\[
\lim_{t \to \infty} G(t) = 0,
\]
which implies (5.2).

In the following two subsections, we perform the above three steps one by one.

5.1. Zero convergence of energy functional. For a solution $\{z_j\}$ to (1.2), we define an energy functional:
\[
E := \frac{1}{N} \sum_{j=1}^{N} \left( m\|\dot{z}_j\|^2 - \frac{\kappa_0 + 2\kappa_1}{2(\kappa_0 + \kappa_1)} \|\langle z_j, \dot{z}_j \rangle\|^2 + \kappa_0 \|z_c - z_j\|^2 \right)
\]
\[
= \frac{1}{N} \sum_{j=1}^{N} m \left( \|\dot{z}_j\|^2 - \frac{\kappa_0 + 2\kappa_1}{2(\kappa_0 + \kappa_1)} \|\langle z_j, \dot{z}_j \rangle\|^2 \right) + \kappa_0 \left( 1 - \|z_c\|^2 \right).
\]

In the following lemma, we can check the following two properties of $E$:
1. $E \geq 0$,
2. $E = 0 \iff \|z_c\| = 1 \text{ and } \|\dot{z}_j\| = 0, \quad j = 1, \ldots, N$.

Lemma 5.1. Suppose the coupling strengths $\kappa_0$ and $\kappa_1$ satisfy
\[
\kappa_0 > 0 \quad \text{and} \quad \kappa_1 \geq 0,
\]
and let $\{z_j\}$ be a solution to (5.1). Then the following assertions hold.
1. The energy functional $E$ is nonnegative:
\[
E(t) \geq 0, \quad \forall \ t \geq 0.
\]
2. The energy functional $E$ is zero if and only if
\[
\|z_c\| = 1 \quad \text{and} \quad \|\dot{z}_j\| = 0, \quad j = 1, \ldots, N.
\]

Proof. (i) The first assertion follows from
\[
0 < \frac{\kappa_0 + 2\kappa_1}{2(\kappa_0 + \kappa_1)} < 1, \quad \|\langle z_j, \dot{z}_j \rangle\|^2 \leq \|\dot{z}_j\|^2, \kappa_0 > 0, \quad \|z_c\| \leq 1.
\]

(ii) Note that
\[
E = 0 \iff \|\dot{z}_j\|^2 - \frac{\kappa_0 + 2\kappa_1}{2(\kappa_0 + \kappa_1)} \|\langle z_j, \dot{z}_j \rangle\|^2 = 0, \forall \ i = 1, \ldots, N, \ 1 - \|z_c\|^2 = 0
\]
\[
\iff \dot{z}_j = 0, \forall \ j = 1, \ldots, N, \text{ and } \|z_c\| = 1.
\]

Next, we study a nonincreasing property of $E$ along system (1.2).
Lemma 5.2. Let \( \{z_j\} \) be the solution of (5.1). Then, we have
\[
\frac{d\mathcal{S}}{dt} = -\frac{2\gamma}{N} \sum_{j=1}^{N} \left( |\dot{z}_j|^2 - \frac{\kappa_0 + 2\kappa_1}{2(\kappa_0 + \kappa_1)} |(z_j, \dot{z}_j)|^2 \right) \leq 0, \quad t > 0. \tag{5.4}
\]

Proof. First, we use the relation \( ||z_j||^2 = 1 \) to see
\[
(z_j, \dot{z}_j) + \langle z_j, z_j \rangle = 0. \tag{5.5}
\]
We use (5.5) to obtain
\[
m \frac{d}{dt} ||\dot{z}_j||^2 = \langle \dot{z}_j, m\dot{z}_j \rangle + \langle m\dot{z}_j, \dot{z}_j \rangle \tag{5.6}
\]
\[
= -2\gamma ||\dot{z}_j||^2 + \left[ \kappa_0 \langle \dot{z}_j, z_c \rangle \langle z_c, \dot{z}_j \rangle \right] + (c.c)
\]
\[
+ [\kappa_1 \langle \dot{z}_j, z_c \rangle \langle z_c, z_j \rangle + \langle z_c, \dot{z}_j \rangle + (c.c)] - [m||\dot{z}_j||^2(\dot{z}_j, z_j) + (c.c)]
\]
\[
= -2\gamma ||\dot{z}_j||^2 + \kappa_0 \langle \dot{z}_j, z_c \rangle \langle z_c, \dot{z}_j \rangle - \kappa_0 \langle \dot{z}_j, z_c \rangle \langle z_c, \dot{z}_j \rangle + 2\kappa_1 \langle \dot{z}_j, z_c \rangle \langle z_c, \dot{z}_j \rangle
\]
\[
= -2\gamma ||\dot{z}_j||^2 + \kappa_0 \langle \langle \dot{z}_j, z_c \rangle \langle z_c, \dot{z}_j \rangle + (\kappa_0 + 2\kappa_1) \langle \dot{z}_j, z_c \rangle - \kappa_0 \langle z_c, z_j \rangle \langle \dot{z}_j, z_c \rangle \rangle \langle \dot{z}_j, z_j \rangle,
\]
where \((c.c)\) denotes the complex conjugate of the previous term.

We take summation (5.6) over \( j \) and divide the resulting relation by \( N \) to obtain
\[
\frac{d}{dt} \left( \frac{1}{N} \sum_{j=1}^{N} m||\dot{z}_j||^2 \right)
\]
\[
= -\frac{2\gamma}{m} \left( \frac{1}{N} \sum_{j=1}^{N} m||\dot{z}_j||^2 \right) + \kappa_0 \frac{d}{dt} ||z_c||^2 + \frac{\kappa_0 + 2\kappa_1}{N} \sum_{j=1}^{N} \left( \langle z_c, z_j \rangle - \langle z_j, z_c \rangle \right) \langle \dot{z}_j, z_j \rangle,
\]
or equivalently
\[
\frac{d}{dt} \left[ \frac{1}{N} \sum_{j=1}^{N} m||\dot{z}_j||^2 + \kappa_0 \left( 1 - ||z_c||^2 \right) \right]
\]
\[
= -\frac{2\gamma}{m} \left( \frac{1}{N} \sum_{j=1}^{N} m||\dot{z}_j||^2 \right) + \frac{\kappa_0 + 2\kappa_1}{N} \sum_{j=1}^{N} \left( \langle z_c, z_j \rangle - \langle z_j, z_c \rangle \right) \langle \dot{z}_j, z_j \rangle. \tag{5.7}
\]
On the other hand, one has
\[
m \frac{d}{dt} ||(z_j, \dot{z}_j)||^2 = m \frac{d}{dt} \langle \langle \dot{z}_j, z_j \rangle \dot{z}_j, z_j \rangle
\]
\[
= m \left[ \langle ||\dot{z}_j||^2 + \langle \dot{z}_j, z_j \rangle \dot{z}_j, z_j \rangle + \langle \dot{z}_j, \dot{z}_j \rangle \langle ||\dot{z}_j||^2 + \dot{z}_j, z_j \rangle \rangle \right]. \tag{5.8}
\]
Then, we use (5.7), (5.8) and the following relation:
\[
m \langle \langle \dot{z}_j, z_j \rangle + ||\dot{z}_j||^2 \rangle = -\gamma \langle \dot{z}_j, \dot{z}_j \rangle + (\kappa_0 + \kappa_1) \langle (z_j, z_c) - \langle z_c, z_j \rangle \rangle \langle \dot{z}_j, z_j \rangle
\]
to obtain
\[
m \frac{d}{dt} ||(z_j, \dot{z}_j)||^2 = -2\gamma ||\dot{z}_j, \dot{z}_j||^2 + 2(\kappa_0 + \kappa_1) \langle (z_j, z_c) - \langle z_c, z_j \rangle \rangle \langle \dot{z}_j, \dot{z}_j \rangle,
\]
or equivalently
\[
\langle \langle z_c, z_j \rangle - \langle z_j, z_c \rangle \rangle \langle \dot{z}_j, \dot{z}_j \rangle = \frac{m}{2(\kappa_0 + \kappa_1)} \frac{d}{dt} ||(z_j, \dot{z}_j)||^2 + \frac{\gamma}{\kappa_0 + \kappa_1} \langle (z_j, \dot{z}_j)||^2. \tag{5.9}
\]
Finally, we combine (5.7) and (5.9) to get a desired result.

**Remark 5.1.** Note that the estimate (5.4) can be rewritten as
\[
\frac{dE}{dt} = -\frac{2\gamma}{m} E + \frac{2\kappa_0 \gamma}{m} (1 - \|z_c\|^2), \quad \forall \ t > 0.
\] (5.10)

As a corollary of Lemma 5.1 and Lemma 5.2, we obtain the following result.

**Corollary 5.1.** Suppose system parameters satisfy
\[ m > 0, \quad \gamma > 0, \quad \kappa_0 > 0 \quad \text{and} \quad \kappa_1 \geq 0, \]
and \( \{z_j\} \) be the solution of (5.1). Then, we have the following estimates:

(i) \( \exists E_\infty := \lim_{t \to \infty} E(t). \)

(ii) \( \max_{1 \leq j \leq N} \|\dot{z}_j\| \leq \max \left\{ \|w_1^m\|, \ldots, \|w_N^m\|, \frac{2}{\gamma} (\kappa_0 + \kappa_1) \right\} =: M_1. \)

(iii) \( \lim_{t \to \infty} \|z_c\| = 1 \implies \lim_{t \to \infty} E(t) = 0. \)

**Proof.**

(i) Since \( E(t) \geq 0, \quad \dot{E}(t) \leq 0, \quad \forall \ t > 0, \)

\( E \) converges as \( t \to \infty. \)

(ii) It follows from (5.6) that if \( \kappa_0 > 0 \) and \( \kappa_1 \geq 0, \) we have
\[
m \frac{d}{dt} \|\dot{z}_j\|^2 \leq -2\gamma \|\dot{z}_j\|^2 + 4(\kappa_0 + \kappa_1) \|\dot{z}_j\|,
\]

or equivalently,
\[
\frac{d}{dt} \|\dot{z}_j\| \leq -\frac{\gamma}{m} \|\dot{z}_j\| + \frac{2}{m} (\kappa_0 + \kappa_1).
\]

This implies
\[
\|\dot{z}_j\| \leq \left( \|w_1^m\| - \frac{2}{\gamma} (\kappa_0 + \kappa_1) \right) e^{-\frac{\gamma}{m} t} + \frac{2}{\gamma} (\kappa_0 + \kappa_1), \quad \forall \ t > 0.
\]

Hence, for all \( j, \) we have
\[
\|\dot{z}_j\| \leq \max \left\{ \|w_1^m\|, \ldots, \|w_N^m\|, \frac{2}{\gamma} (\kappa_0 + \kappa_1) \right\}.
\]

(iii) It follows from (5.10) that
\[
E(t) = E(0) e^{-\frac{2\gamma}{m} t} + \frac{2\gamma \kappa_0}{m} \int_0^t e^{-\frac{2\gamma}{m} (t-s)} (1 - \|z_c(s)\|^2) \, ds, \quad \forall \ t \geq 0. \] (5.11)

Suppose that
\[
\lim_{t \to \infty} \|z_c(t)\| = 1.
\]

Then, for any positive small \( \varepsilon, \) there exists a positive time \( T = T(\varepsilon) > 0 \) such that
\[
1 - \|z_c(t)\|^2 < \varepsilon, \quad \forall \ t > T(\varepsilon).
\]

Therefore, (5.11) becomes
\[
E(t) = E(0) e^{-\frac{2\gamma}{m} t} + \frac{2\gamma \kappa_0}{m} \int_0^{T(\varepsilon)} e^{-\frac{2\gamma}{m} (t-s)} (1 - \|z_c(s)\|^2) \, ds
\]
\[
+ \frac{2\gamma \kappa_0}{m} \int_{T(\varepsilon)}^t e^{-\frac{2\gamma}{m} (t-s)} (1 - \|z_c(s)\|^2) \, ds
\]
of (1.2) and let $\rho$.

Corollary 5.2. Under the same assumptions in Theorem 4.1, let $\rho$.

Suppose system parameters and initial data satisfy Proposition 5.1.

This implies that for $t \gg 1$,

$$E(t) \leq 2\kappa_0\varepsilon.$$

Since $\varepsilon$ is arbitrary, we have the desired zero convergence of $E$. 

\[\square\]

Remark 5.2. Uniform boundedness of $\dot{z}_j$ provides us the uniform boundedness of $\ddot{z}_j$, since

$$m\|\ddot{z}_j\| = \| - \gamma \dot{z}_j + \kappa_0 (z_c -\langle z_c, z_j \rangle z_j) + \kappa_1 (\langle z_j, z_c \rangle -\langle z_c, z_j \rangle) z_j - m\|\dot{z}_j\|^2 z_j\|
\leq \gamma\|\dot{z}_j\| + 2(\kappa_0 + \kappa_1) m\|\dot{z}_j\|^2.$$

Similarly, one can also get the uniform boundedness of $\frac{d^2 z_j}{dt^2}$.

In next proposition, we study zero convergence of $\dot{z}_j$.

Proposition 5.1. Suppose system parameters and initial data satisfy

$$m > 0, \quad \gamma > 0, \quad \kappa_0 > 0, \quad \kappa_1 \geq 0, \quad \|z_j^m\| = 1 \quad \text{for all } j = 1, \cdots, N,$$

and let $\{z_j\}$ be the solution of (5.1). Then, we have

$$\lim_{t \to \infty} \|\dot{z}_j(t)\| = 0, \quad j = 1, \cdots, N.$$

Proof. We integrate (5.4) to find

$$E(t) + \frac{2\gamma}{N} \sum_{j=1}^N \int_0^t \left( \|\dot{z}_j(s)\|^2 - \frac{\kappa_0 + 2\kappa_1}{2(\kappa_0 + \kappa_1)} \|\langle z_j(s), \dot{z}_j(s) \rangle\| \right) ds = E^{in} < \infty.$$

This yields

$$\int_0^t \left( \|\dot{z}_j(t)\|^2 - \frac{\kappa_0 + 2\kappa_1}{2(\kappa_0 + \kappa_1)} \|\langle z_j(t), \dot{z}_j(t) \rangle\|^2 \right) dt < \infty.$$

By straightforward calculation, one can show that time-derivative of the integrand is uniformly bounded because $z_j, \dot{z}_j$ and $\ddot{z}_j$ are bounded, i.e., the integrand is uniformly continuous. Hence, we can apply Barbalat’s lemma to get

$$\lim_{t \to \infty} \left( \|\dot{z}_j(t)\|^2 - \frac{\kappa_0 + 2\kappa_1}{2(\kappa_0 + \kappa_1)} \|\langle z_j(t), \dot{z}_j(t) \rangle\|^2 \right) = 0. \quad (5.12)$$

On the other hand, note that

$$0 \leq \frac{\kappa_0}{2(\kappa_0 + \kappa_1)} \|\ddot{z}_j(t)\|^2 \leq \|\ddot{z}_j(t)\|^2 - \frac{\kappa_0 + 2\kappa_1}{2(\kappa_0 + \kappa_1)} \|\langle z_j(t), \dot{z}_j(t) \rangle\|^2. \quad (5.13)$$

Finally, we combine estimates (5.12) and (5.13) to derive the desired zero convergence of $\dot{z}_j$.

\[\square\]

Corollary 5.2. Under the same assumptions in Theorem 4.1, let $\{z_j\}$ be a solution of (1.2) and let $\rho^\infty$ be an asymptotic order parameter defined by (3.8). Then, the following trichotomy holds:

$$\rho^\infty = 0, \quad \rho^\infty = 1,$$
or there exists integer $n$ such that
\[ \rho^\infty = \frac{N - 2n}{N}, \quad 1 \leq n < \left\lfloor \frac{N}{2} \right\rfloor. \]

**Proof.** For the case $\rho^\infty = 0$ or $1$, we are done. Hence, we consider only the case:
\[ \rho^\infty \in (0, 1). \]

In Remark 5.2, we noticed that $\frac{d^2z_j}{dt^2}$ is uniformly bounded. Hence, we can apply Barbalat’s Lemma [2] and Proposition 5.1 to have
\[ \lim_{t \to \infty} \ddot{z}_j(t) = 0. \]

Then, (5.1) becomes
\[ \lim_{t \to \infty} (\kappa_0(z_c - \langle z_c, z_j \rangle z_j) + \kappa_1((z_j, z_c) - \langle z_c, z_j \rangle)z_j) = 0, \quad j = 1, \ldots, N. \quad (5.14) \]

Since $z_j$ is bounded, we can take $\langle z_j, \cdot \rangle$ to get
\[ \lim_{t \to \infty} (\langle z_j, z_c \rangle - \langle z_c, z_j \rangle) = 0, \quad j = 1, \ldots, N. \quad (5.15) \]

We combine (5.14), (5.15) with the fact that $\|z_j\| = 1$ to obtain
\[ \lim_{t \to \infty} \left( z_c - \langle z_c, z_j \rangle z_j \right) = 0, \quad j = 1, \ldots, N. \]

Again, since $z_c$ is bounded, we can take $\langle z_c, \cdot \rangle$ to get
\[ \lim_{t \to \infty} (\rho^2 - \langle z_c, z_j \rangle^2) = 0, \quad j = 1, \ldots, N, \]
or equivalently,
\[ \lim_{t \to \infty} \langle z_c, z_j \rangle = \delta_j \rho^\infty, \quad \delta_j \in \{1, -1\}, \quad j = 1, \ldots, N. \quad (5.16) \]

Then, we sum (5.16) over $j$ and divide by $N$ to obtain
\[ (\rho^\infty)^2 = \lim_{t \to \infty} \|z_c\|^2 = \frac{\rho^\infty}{N} \sum_{j=1}^{N} \delta_j. \]

This implies
\[ \rho^\infty = \frac{1}{N} \sum_{j=1}^{N} \delta_j. \]

Since $\rho^\infty > 0$, there must be an integer $n$ such that
\[ n = \left| \{ j : \delta_j = -1, \quad j = 1, \ldots, N \} \right|, \quad 1 \leq n < \left\lfloor \frac{N}{2} \right\rfloor, \]

which guarantees our desired result. \qed

**Remark 5.3.** In the results of the above lemma, we call the first two cases by
1. $\rho = 0 \iff$ incoherence state,
2. $\rho = 1 \iff$ complete aggregation.

On the other hand, the remaining case is called bi-polar state, which is defined as follows:
\[ \lim_{t \to \infty} \text{dist}(S, \{z_j\}) = 0, \quad S := \{ \{p_j\} \in (\mathbb{H}^d)^N : p_j = \pm a, \quad \exists a \in \mathbb{H}^d \}. \]
5.2. **Proof of Theorem 4.1.** In this subsection, we provide a proof of our first main result by analyzing asymptotic behaviors of angle parameter $\mathcal{G}$ and diameter functional $R_i(Z)$.

First, we recall two-point correlation functions $h_{ij}$ and $g_{ij}$:

$$h_{ij} = \langle z_i, z_j \rangle, \quad g_{ij} = 1 - h_{ij}, \quad \forall i, j = 1, \ldots, N.$$  

Next, we derive an evolution equation for $|g_{ij}|^2$.

**Lemma 5.3.** Let $\{z_j\}$ be a solution of (5.1) with $\|z_j^n\| = 1$, $j = 1, \ldots, N$. Then, $|g_{ij}|^2$ satisfies

$$m \frac{d^2}{dt^2}|g_{ij}|^2 + \gamma \frac{d}{dt}|g_{ij}|^2 + 2 \left[2\kappa_0 + m(\|\dot{z}_i\|^2 + \|\dot{z}_j\|^2)\right]|g_{ij}|^2$$

$$= \frac{\kappa_0}{N} \sum_{k=1}^N (g_{ik} + g_{ki} + g_{kj} + g_{jk})|g_{ij}|^2 + \kappa_1(\langle z_i, z_j \rangle - \langle z_j, z_i \rangle)(\langle \dot{z}_i, z_j \rangle - \langle \dot{z}_j, z_i \rangle)$$

$$+ \kappa_1(\langle z_i, z_i \rangle - \langle z_j, z_j \rangle)(\langle \dot{z}_i, \dot{z}_j \rangle + \kappa_0(\langle z_i, z_j \rangle - \langle z_j, z_i \rangle)) + 2m\dot{g}_{ij}\dot{g}_{ji}$$

$$+ m(\|\ddot{z}_i\|^2 - 2\langle \dot{z}_i, \dot{z}_j \rangle + \|\dot{z}_i\|^2)\dot{g}_{ij} + m(\|\ddot{z}_j\|^2 - 2\langle \dot{z}_i, \dot{z}_j \rangle + \|\dot{z}_j\|^2)\dot{g}_{ij}. \quad (5.17)$$

**Proof.** Recall $z_j$ satisfies

$$m\ddot{z}_j = -\gamma \dot{z}_j + \kappa_0 (z_c - \langle z_c, z_j \rangle) z_j + \kappa_1(\langle z_j, z_c \rangle - \langle z_c, z_j \rangle)z_j - m\|\dot{z}_j\|^2 z_j. \quad (5.18)$$

We use $\tilde{g}_{ij} = g_{ij}$ to find

$$m \frac{d^2}{dt^2}|g_{ij}|^2 = m \frac{d^2}{dt^2}(|g_{ij}|^2) = m\ddot{g}_{ij}g_{ji} + 2m\dot{g}_{ij}\dot{g}_{ji} + m\dot{g}_{ij}\dot{g}_{ji}.$$  

On the other hand, it follows from $g_{ij} = 1 - \langle z_i, z_j \rangle$ that

$$m\ddot{g}_{ij} = -(m\ddot{z}_i, z_j) - 2m(\dot{z}_i, \dot{z}_j) - (z_i, m\ddot{z}_j). \quad (5.19)$$

Then, we use (5.18) to get

$$\langle z_i, m\ddot{z}_j \rangle = -\gamma (\dot{z}_i, z_j) + \kappa_0 (\dot{z}_i, z_c) - \kappa_0 (\langle z_c, z_j \rangle) \langle z_i, z_j \rangle$$

$$+ \kappa_1(\langle z_j, z_c \rangle - \langle z_c, z_j \rangle) \langle z_i, z_j \rangle - m\|\dot{z}_j\|^2 \langle z_i, z_j \rangle. \quad (5.20)$$

We combine (5.19) and (5.20) to obtain

$$m\ddot{g}_{ij} = \gamma((\dot{z}_i, z_j) + (\dot{z}_i, z_j)) - \kappa_0((\dot{z}_i, z_c) + (\dot{z}_i, z_j)) + \kappa_0((\dot{z}_c, z_j) + (\dot{z}_c, z_j)) \langle z_i, z_j \rangle$$

$$- \kappa_1((\dot{z}_j, z_c) - \langle z_c, z_j \rangle) \langle z_i, z_j \rangle - \kappa_1((\dot{z}_c, z_i) - \langle z_i, z_c \rangle) \langle z_i, z_j \rangle$$

$$- m\|\dot{z}_j\|^2 \langle z_i, z_j \rangle + m\|\dot{z}_i\|^2 \langle z_i, z_j \rangle - 2m(z_i, \dot{z}_j)$$

$$= -\gamma \ddot{g}_{ij} - \kappa_0((\dot{z}_i, z_c) + (\dot{z}_c, z_j))g_{ij}$$

$$- \kappa_1((\dot{z}_j, z_c) - \langle z_c, z_j \rangle) \langle z_i, z_j \rangle - \kappa_1((\dot{z}_c, z_i) - \langle z_i, z_c \rangle) \langle z_i, z_j \rangle$$

$$- m(\|\dot{z}_i\|^2 + \|\dot{z}_j\|^2)g_{ij} + m(\|\ddot{z}_j\|^2 - 2\langle \dot{z}_i, \dot{z}_j \rangle + \|\dot{z}_j\|^2).$$

Note that

$$\langle (z_i, z_c) + (\dot{z}_c, z_j) \rangle g_{ij} = \frac{1}{N} \sum_{k=1}^N (2 - g_{ik} - g_{kj})g_{ij}.$$  

This yields

$$m\ddot{g}_{ij} = -\gamma \ddot{g}_{ij} - 2\kappa_0 g_{ij} + \frac{\kappa_0}{N} \sum_{k=1}^N (g_{ik} + g_{kj})g_{ij}$$

$$- \kappa_1((\dot{z}_j, z_c) - \langle z_c, z_j \rangle) \langle z_i, z_j \rangle - \kappa_1((\dot{z}_j, z_c) - \langle z_c, z_c \rangle) \langle z_i, z_j \rangle$$

$$- \kappa_1((\dot{z}_c, z_i) - \langle z_i, z_c \rangle) \langle z_i, z_j \rangle - \kappa_1((\dot{z}_c, z_i) - \langle z_i, z_c \rangle) \langle z_i, z_j \rangle.$$
\begin{equation}
- m(\|\dot{z}_i\|^2 + \|\dot{z}_j\|^2)g_{ij} + m(\|\dot{z}_j\|^2 - 2\langle \dot{z}_i, \dot{z}_j \rangle + \|\dot{z}_i\|^2). \tag{5.21}
\end{equation}

We multiply \((5.21)\) by \(g_{ji}\) to find
\begin{align*}
& m\ddot{g}_{ij}g_{ji} + \gamma \dot{g}_{ij}g_{ji} + \left[ 2\kappa_0 + m(\|\dot{z}_i\|^2 + \|\dot{z}_j\|^2) \right] g_{ij} \\
& = \frac{\kappa_0}{N} \sum_{k=1}^{N} (g_{ik} + g_{kj})g_{ij}^2 - \kappa_1 \langle (z_j, \dot{z}_c) - \langle z_c, z_j \rangle \rangle \langle z_i, z_j \rangle g_{ji} \\
& \quad - \kappa_1 \langle (z_c, \dot{z}_i) - \langle z_i, z_c \rangle \rangle \langle z_i, z_j \rangle g_{ji} + m(\|\dot{z}_j\|^2 - 2\langle \dot{z}_i, \dot{z}_j \rangle + \|\dot{z}_i\|^2)g_{ji}. \tag{5.22}
\end{align*}

We sum up \((5.22)\) over all \(i, j\) and its complex conjugate to obtain the desired estimate.

Next, we quote some useful Lemmas on the second-order Gronwall type differential inequality from \([5,7]\) without proofs.

\textbf{Lemma 5.4.} \([5]\) Let \(y = y(t)\) be a nonnegative \(C^2\)-function satisfying the following differential inequality:
\begin{equation}
a\dddot{y} + b\dddot{y} + cy + d \leq 0, \quad t > 0,
\end{equation}
where \(a, b\) and \(c\) are positive constants. Then, we have the following assertions:

1. Suppose that \(b^2 - 4ac > 0\). Then, one has
\begin{align*}
y(t) & \leq - \frac{d}{c} + \left( y(0) + \frac{d}{c} \right) e^{-\nu_1 t} \\
& \quad + \frac{a}{\sqrt{b^2 - 4ac}} \left( \dot{y}(0) + \nu_1 y(0) + \frac{2d}{b - \sqrt{b^2 - 4ac}} \right) \left( e^{-\nu_2 t} - e^{-\nu_1 t} \right)
\end{align*}
where \(\nu_1\) and \(\nu_2\) are given as follows:
\begin{align*}
\nu_1 & := \frac{b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \nu_2 := \frac{b - \sqrt{b^2 - 4ac}}{2a}.
\end{align*}

Moreover, if the following conditions hold:
\begin{align}
y(0) + \frac{d}{c} & < 0 \quad \text{and} \quad y'(0) + \nu_1 y(0) + \frac{2d}{b - \sqrt{b^2 - 4ac}} < 0, \tag{5.23}
\end{align}
then, \(y(t)\) is uniformly bounded:
\begin{equation}
y(t) < - \frac{d}{c}.
\end{equation}

2. Suppose that \(b^2 - 4ac < 0\).

Then, one has
\begin{align*}
y(t) & \leq - \frac{4ad}{b^2} + e^{-\frac{1}{2b^2} t} \left[ y(0) + \frac{4ad}{b^2} + \left( \frac{b}{2a} y(0) + \frac{2d}{b} \right) t \right].
\end{align*}

Moreover, if the following conditions hold:
\begin{align}
y(0) & < - \frac{4ad}{b^2} \quad \text{and} \quad \frac{b}{2a} y(0) + \frac{2d}{b} < 0, \tag{5.24}
\end{align}
then, \(y(t)\) is uniformly bounded:
\begin{equation}
y(t) < - \frac{4ad}{b^2}.
\end{equation}
Lemma 5.5. [7] Let \( y = y(t) \) be a nonnegative \( C^2 \)-function satisfying the second-order differential inequality:
\[
a\dot{y} + b\dot{y} + cy \leq f, \quad t > 0,
\]
where \( a, b, c \) and \( d \) are positive constants and \( f = f(t) \) is a nonnegative \( C^1 \)-function which converges to zero as \( t \to \infty \). Then, \( y \) vanishes asymptotically:
\[
\lim_{t \to \infty} y(t) = 0.
\]

Now, we are ready to provide a proof of our first main result on the complete aggregation of (5.1).

**Proof of Theorem 4.1.** Suppose that the sufficient frameworks \((\mathcal{F}_A^1)-(\mathcal{F}_A^2)\) or \((\mathcal{F}_B^1)-(\mathcal{F}_B^2)\) hold. Moreover, we assume that initial data and natural frequency satisfy
\[
\|z_j^n\| = 1, \quad \Omega_j = 0, \quad j = 1, \cdots, N.
\]
Let \( \{z_j\} \) be the global solution of (1.2). Then, we claim:
\[
\lim_{t \to \infty} G(t) = 0.
\]

It follows from (5.17) that \( G \) satisfies
\[
m\ddot{G} + \gamma \dot{G} + 4\kappa_0 G \leq \frac{2\kappa_0}{N^3} \sum_{i,j,k=1}^{N} (|g_{ik}| + |g_{jk}|)|g_{ij}|^2 + \frac{2\kappa_1}{N} \sum_{i=1}^{N} \langle z_i, z_c \rangle - \langle z_c, z_i \rangle|^2
\]
\[
+ \frac{2m}{N^2} \sum_{i,j=1}^{N} |g_{ij}|^2 + \frac{2m}{N^2} \sum_{i,j=1}^{N} (\|\dot{z}_j\|^2 + 2\langle \dot{z}_j, \dot{z}_j \rangle + \|\dot{z}_i\|^2)
\]
\[
=: I_{11} + I_{12} + I_{13} + I_{14},
\]
where each \( I_{1i} \) is defined by
\[
I_{11} := \frac{2\kappa_0}{N^3} \sum_{i,j,k=1}^{N} (|g_{ik}| + |g_{jk}|)|g_{ij}|^2, \quad I_{12} := \frac{2\kappa_1}{N} \sum_{i=1}^{N} \langle z_i, z_c \rangle - \langle z_c, z_i \rangle|^2,
\]
\[
I_{13} := \frac{2m}{N^2} \sum_{i,j=1}^{N} |g_{ij}|^2, \quad I_{14} := \frac{2m}{N^2} \sum_{i,j=1}^{N} (\|\dot{z}_j\|^2 + 2\langle \dot{z}_j, \dot{z}_j \rangle + \|\dot{z}_i\|^2).
\]

Below, we provide estimates for \( I_{11} \) one by one.

- **Case A (Estimate of \( I_{11} \)):** We use the Cauchy-Swartz inequality to obtain
\[
\frac{2\kappa_0}{N^3} \sum_{i,j,k=1}^{N} |g_{ik}| \cdot |g_{ij}|^2
\]
\[
\leq 2\kappa_0 \sqrt{N} \left( \frac{1}{N^2} \sum_{k=1}^{N} \sum_{i=1}^{N} |g_{ik}|^2 \right) \cdot \left( \frac{1}{N^2} \sum_{j=1}^{N} \sum_{i=1}^{N} |g_{ij}|^4 \right)
\]
\[
\leq 2\kappa_0 \sqrt{N} \left( \frac{1}{N^2} \sum_{k=1}^{N} \sum_{i=1}^{N} |g_{ik}|^2 \right) \cdot \left( \frac{1}{N^2} \sum_{j=1}^{N} \sum_{i=1}^{N} |g_{ij}|^2 \right)
\]
\[
\leq 2\kappa_0 \sqrt{NG^2}.
\]

This implies
\[
I_{11} \leq 4\kappa_0 \sqrt{NG^2}. \tag{5.25}
\]
• Case B (Estimate of $I_{12}$): by (5.25), we have
$$I_{12} \leq 2\kappa_1 R_2(Z).$$
(5.26)

• Case C (Estimate of $I_{13}$): we use
$$|\dot{g}_{ij}|^2 = |\dot{z}_i, z_j|^2 \leq 2(\|\dot{z}_i\|^2 + \|\dot{z}_j\|^2) \leq 4R_1(\dot{Z})$$
to find
$$I_{13} \leq 8mR_1(\dot{Z}).$$
(5.27)

• Case D (Estimate of $I_{14}$): similarly, we use
$$\|\dot{z}_j\|^2 + 2|\dot{z}_i, \dot{z}_j| \leq 2(\|\dot{z}_i\|^2 + \|\dot{z}_j\|^2) \leq 4R_1(\dot{Z}),$$
to find
$$I_{14} \leq 8mR_1(\dot{Z}).$$
(5.28)

We combine all the estimates (5.25), (5.26), (5.27), (5.28) of $I_k$’s to obtain
$$m\ddot{\mathcal{G}} + \gamma \dot{\mathcal{G}} + 4\kappa_0 \mathcal{G} \leq 4\kappa_0 \sqrt{N} \ddot{\mathcal{G}} + 2\kappa_1 R_2(Z) + 16mR_1(\dot{Z}).$$
(5.29)

Now, we derive a uniform bound for $\mathcal{G}$ using (5.29):
$$\sup_{0 \leq t < \infty} \mathcal{G}(t) < \frac{(1 - \delta)^2}{N}.$$

We define a temporal set $\mathcal{T}$ for $\delta \in (0, 1)$:
$$\mathcal{T} := \{T \in (0, \infty) : \mathcal{G}(t) < (1 - \delta)^2/N, \quad \forall t \in (0, T)\}.$$

By initial conditions, the set $\mathcal{T}$ is nonempty. Hence we can define
$$T_* := \sup \mathcal{T}.$$

Now we claim:
$$T_* = \infty.$$

Suppose not, i.e.,
$$T_* < \infty.$$

Then, we have
$$\lim_{t \to T_*^-} \mathcal{G}(t) = \frac{(1 - \delta)^2}{N}.$$  (5.30)

On the other hand, it follows from (5.29) that, for $t \in (0, T_*)$, we have
$$m\ddot{\mathcal{G}} + \gamma \dot{\mathcal{G}} + 4\kappa_0 \delta \mathcal{G} \leq 2\kappa_1 R_2(Z) + 16mR_1(\dot{Z}).$$

We use Corollary 5.1 to obtain
$$2\kappa_1 R_2(Z) + 16mR_1(\dot{Z}) \leq 8\kappa_1 + 16mM_1^2.$$

Hence, one has
$$m\ddot{\mathcal{G}} + \gamma \dot{\mathcal{G}} + 4\kappa_0 \delta \mathcal{G} \leq 8\kappa_1 + 16mM_1^2, \quad t \in (0, T_*).$$

Note that $(\mathcal{F}_A 1)$ and $(\mathcal{F}_B 1)$ are the first and second case of Lemma 5.4, respectively. Moreover, $(\mathcal{F}_A 2)$ and $(\mathcal{F}_B 2)$ satisfy the condition (5.23) and (5.24) of Lemma 5.4, respectively. So, we apply Lemma 5.4 to obtain
$$(\mathcal{F}_A) \implies \mathcal{G}(t) < \frac{8\kappa_1 + 16mM_1^2}{4\kappa_0 \delta} < \frac{(1 - \delta)^2}{N}, \quad t \in (0, T_*),$$
$$(\mathcal{F}_B) \implies \mathcal{G}(t) < \frac{4m}{\gamma^2} (8\kappa_1 + 16mM_1^2) < \frac{(1 - \delta)^2}{N}, \quad t \in (0, T_*),$$
which contradicts to (5.30). Therefore, we have $T_s = \infty$ and
\[
m\dddot{G} + \gamma \dot{G} + 4\kappa_0 \delta \dot{G} \leq 2\kappa_1 R_2(Z) + 16m R_1(\dot{Z}), \quad \forall \ t > 0.
\]
We use Theorem 4.1 and (5.15) to see
\[
\lim_{t \to \infty} \left[ 2\kappa_1 R_2(Z) + 16m R_1(\dot{Z}) \right] = 0.
\]
Then, we can apply Lemma 5.5 to conclude that complete synchronization occurs.

\[\square\]

6. Emergence of practical aggregation. In this section, we study the emergent dynamics of (1.2) with distinct set of natural frequency matrices $\Omega_j$’s. Unlike the homogeneous ensemble in previous section, we cannot expect the emergence of complete aggregation in which all the states collapse to the same state. Instead, we study a weaker concept of aggregation estimate, namely practical aggregation introduced in Definition 1.1. The key ingredient as in a homogeneous ensemble is to derive a suitable second-order differential inequality for $G$. In fact, similar to (5.3), we can derive
\[
m\dddot{G} + \gamma \dot{G} + 4\kappa_0 \delta \dot{G} \leq 4\Omega^\infty + 8\kappa_1 + \frac{16m}{\gamma^2} \left[ \Omega^\infty + 2(\kappa_0 + \kappa_1) \right]^2, \quad \forall \ t > 0.
\]
Then, via the second-order Gronwall’s lemma (Lemma 5.4) and a suitable ansatz for $m = \frac{m}{\kappa_0}$, one can show
\[
G(t) \lesssim \max \left\{ \frac{1}{\kappa_0}, \frac{1}{\kappa_0^2} \right\}, \quad \text{for } t \gg 1.
\]
This clearly implies the desired practical aggregation estimate.

6.1. Derivation of Gronwall’s inequality for $G$. Recall that the LHS model on the unit Hermitian sphere $\|z\| = 1$:
\[
\begin{aligned}
\begin{cases}
m \dddot{z}_j & = \frac{m}{\gamma} \Omega_j \dddot{z}_j + \frac{m}{\gamma} \Omega_j v_j - \gamma \dot{z}_j + \Omega_j \dot{z}_j + \kappa_0 (\dot{z}_c, z_j) \\
+ \kappa_1 ((\dot{z}_j, z_c) - (\dot{z}_c, z_j)) z_j - m\|v_j\|^2 z_j, \\
\end{cases}
\end{aligned}
\]
(6.1)
where $v_j = \dot{z}_j - \frac{1}{\gamma} \Omega_j z_j$.
Parallel to Lemma 5.3, we derive a dynamical system of $g_{ij}$ in the following lemma.

Lemma 6.1. Let $\{z_j\}$ be a global solution of (6.1). Then, $|g_{ij}|^2$ satisfies
\[
\begin{aligned}
m \frac{d^2}{dt^2} |g_{ij}|^2 & + \gamma \frac{d}{dt} |g_{ij}|^2 + 2 \left[ 2\kappa_0 + m(\|v_i\|^2 + \|v_j\|^2) \right] |g_{ij}|^2 \\
& = - \frac{m}{\gamma} \left( (\dot{z}_j, \Omega_j v_i) + (\Omega_j v_j, z_j) + (z_j, \Omega_j \dot{z}_j) + (\Omega_j \dot{z}_j, z_j) \right) g_{ji} - \left( (\dot{z}_i, \Omega_j z_j) + (\Omega_j z_j, \dot{z}_i) \right) g_{ji} \\
& - \frac{\kappa_0}{\gamma} \left( (\dot{z}_j, \Omega_j v_i) + (\Omega_j v_j, z_j) + (z_j, \Omega_j \dot{z}_j) + (\Omega_j \dot{z}_j, z_j) \right) g_{ij} - \left( (\dot{z}_j, \Omega_j z_i) + (\Omega_j z_i, \dot{z}_j) \right) g_{ij} \\
& + \frac{\kappa_1}{N} \sum_{k=1}^{N} (g_{ik} + g_{ki} + g_{kj} + g_{jk}) |g_{ij}|^2 + \kappa_1 ((\dot{z}_c, z_j) - (z_j, \dot{z}_c)) (\dot{z}_i, z_j) - (z_i, \dot{z}_j) \\
& + \kappa_1 ((\dot{z}_i, z_j) - (z_i, \dot{z}_j)) (\dot{z}_i, z_j) - (z_i, \dot{z}_j) + 2m \dot{g}_{ij} \dot{g}_{ji} \\
& - 2m(\dot{z}_j, \dot{z}_j) g_{ji} + (\dot{z}_i, \dot{z}_i) g_{ij} + m(\|v_i\|^2 + \|v_j\|^2) (g_{ji} + g_{ij}).
\end{aligned}
\]
Proof. We use (6.1) to obtain
\[
\langle z_i, m\hat{z}_j \rangle
\]
\[
= \frac{m}{\gamma} \langle z_i, \Omega_j v_j \rangle + \frac{m}{\gamma} \langle z_i, \Omega_j \hat{z}_j \rangle - \gamma \langle z_i, \hat{z}_j \rangle + \langle z_i, \Omega_j z_j \rangle + \kappa_0 \langle z_i, z_c \rangle \langle z_i, z_j \rangle + \kappa_1 (\langle z_i, z_c \rangle - \langle z_i, z_j \rangle) \langle z_i, z_j \rangle - m \|v_j\|^2 \langle z_i, z_j \rangle.
\]
Then, we have
\[
m\hat{y}_{ij} = -\langle z_i, m\hat{z}_j \rangle - 2m\langle \hat{z}_i, \hat{z}_j \rangle - \langle m\hat{z}_i, z_j \rangle
\]
\[
= -\frac{m}{\gamma} (\langle z_i, \Omega_j v_j \rangle + \langle z_i, \Omega_j \hat{z}_j \rangle + \langle z_i, \hat{z}_j \rangle + \langle z_i, \hat{z}_j \rangle)
\]
\[
- \langle z_i, \Omega_j z_j \rangle - \langle z_i, \Omega_j \hat{z}_j \rangle - \gamma \hat{y}_{ij} - \kappa_0 \langle z_i, z_c \rangle g_{ij} - \kappa_0 \langle z_c, z_j \rangle g_{ij}
\]
\[
- \kappa_1 (\langle z_i, z_c \rangle - \langle z_i, z_j \rangle) \langle z_i, z_j \rangle - \kappa_1 (\langle z_c, z_i \rangle - \langle z_i, z_j \rangle) \langle z_i, z_j \rangle
\]
\[
- 2m \langle \hat{z}_i, \hat{z}_j \rangle + m (\|v_i\|^2 + \|v_j\|^2) - m (\|v_i\|^2 + \|v_j\|^2) g_{ij}.
\] (6.3)
We substitute
\[
\langle z_i, z_c \rangle = \frac{1}{N} \sum_{k=1}^{N} (1 - g_{ik}) = 1 - \frac{1}{N} \sum_{k=1}^{N} h_{ik},
\]
into (6.3) and multiply both sides of (6.3) by $h_{ij}$ to get
\[
m\hat{y}_{ij} g_{ij} + \gamma \hat{y}_{ij} g_{ij} + [2\kappa_0 + m (\|v_i\|^2 + \|v_j\|^2)] g_{ij}
\]
\[
= -\frac{m}{\gamma} (\langle z_i, \Omega_j v_j \rangle + \langle z_i, \Omega_j \hat{z}_j \rangle + \langle z_i, \hat{z}_j \rangle + \langle z_i, \hat{z}_j \rangle)
\]
\[
- \langle z_i, \Omega_j z_j \rangle - \langle z_i, \Omega_j \hat{z}_j \rangle - \gamma \hat{y}_{ij} - \kappa_0 \langle z_i, z_c \rangle g_{ij} - \kappa_0 \langle z_c, z_j \rangle g_{ij}
\]
\[
- \kappa_1 (\langle z_i, z_c \rangle - \langle z_i, z_j \rangle) \langle z_i, z_j \rangle g_{ij} - 2m \langle \hat{z}_i, \hat{z}_j \rangle g_{ij} + m (\|v_i\|^2 + \|v_j\|^2) g_{ij}.
\] (6.4)
We sum (6.4) and its complex conjugate to obtain the desired result. \qed

As in Corollary 5.1, we observe the uniform bound of $\|v_j\|$. Since
\[
\langle v_j, m\hat{v}_j \rangle = \frac{m}{\gamma} \langle v_j, \Omega_j v_j \rangle - \gamma \|v_j\|^2 + \kappa_0 \langle v_j, z_c \rangle - \kappa_0 \langle z_c, z_j \rangle \langle v_j, z_j \rangle
\]
\[
+ \kappa_1 (\langle z_j, z_c \rangle - \langle z_i, z_j \rangle) \langle v_j, z_j \rangle - m \|v_j\|^2 \langle v_j, z_j \rangle,
\]
we have
\[
m \frac{d}{dt} \|v_j\|^2 \leq -2\gamma \|v_j\|^2 + 4(\kappa_0 + \kappa_1) \|v_j\|^2,
\] (6.5)
where we use $\langle z_j, v_j \rangle + \langle v_j, z_j \rangle = 0$. Then, the relation (6.5) implies
\[
\|v_j(t)\| \leq \left( \|v_j^0\| - \frac{2(\kappa_0 + \kappa_1)}{\gamma} \right) e^{-\frac{\gamma}{2}t} + \frac{2}{\gamma} (\kappa_0 + \kappa_1).
\]
Hence, we can obtain the uniform bound of $\|v_j\|$
\[
\|v_j\| \leq \max \left\{ \|v_j^0\|, \cdots, \|v_N^0\|, \frac{2}{\gamma} (\kappa_0 + \kappa_1) \right\}, \quad j = 1, \cdots, N.
\] (6.6)
Also, we have the uniform bound of $\|\hat{z}_j\|$
\[
\|\hat{z}_j\| \leq \|v_j\| + \frac{\|\Omega_j\| F}{\gamma} \leq \max \left\{ \|v_1^m\|, \cdots, \|v_N^m\|, \frac{2}{\gamma} (\kappa_0 + \kappa_1) \right\} + \frac{\Omega^\infty}{\gamma}, \quad j = 1, \cdots, N,
\] (6.7)
where $\Omega^\infty := \max_j \|\Omega_j\| F$. 

6.2. **Proof of Theorem 4.2.** In this subsection, we provide a proof of our second main result on the emergence of practical aggregation. First, we begin with the derivation of a uniform bound for \(G\).

- **Step A (Derivation of uniform bound for \(G\)):** Suppose the framework \((\mathcal{FC}_1)-(\mathcal{FC}_2)\) hold, and let \(Z = (z_1, \ldots, z_N)\) be a solution of (6.1). Then, one has

\[
\sup_{0 \leq t < \infty} G(t) < \frac{(1 - \delta)^2}{N}.
\]

It follows from (6.2) that

\[
m\ddot{G} + \gamma \dot{G} + 4\kappa_0 G \leq 4m \frac{\Omega^\infty}{\gamma} \left[ R_3(V) + \sqrt{R_1(\dot{z})} \right]
+ 12m R_1(\dot{z}) + 4m R_3(V)^2 + 4\Omega^\infty + 8\kappa_1. \tag{6.8}
\]

We define a temporal set \(T\) for \(\delta \in (0,1)\):

\[
T := \{ T \in (0,\infty) : G(t) < \frac{(1 - \delta)^2}{N}, \quad \forall \ t \in (0, T) \}.
\]

By initial conditions, the set \(T\) is nonempty. Hence we can define

\[
T_* := \sup T.
\]

Now we claim:

\[
T_* = \infty.
\]

Suppose not, i.e.,

\[
T_* < \infty.
\]

Then, we have

\[
\lim_{t \to T_* -} G(t) = \frac{(1 - \delta)^2}{N}. \tag{6.9}
\]

On the other hand, it follows from (6.8) that, for \(t \in (0, T_*),\) we have

\[
m\ddot{G} + \gamma \dot{G} + 4\kappa_0 G \leq 4m \frac{\Omega^\infty}{\gamma} \left[ R_3(V) + \sqrt{R_1(\dot{z})} \right]
+ 12m R_1(\dot{z}) + 4m R_3(V)^2 + 4\Omega^\infty + 8\kappa_1. \tag{6.10}
\]

We use (6.6), (6.7) and \((\mathcal{FC}_2)_1\) to obtain

\[
\frac{4m \Omega^\infty}{\gamma} R_3(V) + 4m R_3(V)^2
\leq \frac{8m \Omega^\infty (\kappa_0 + \kappa_1)}{\gamma^2} + \frac{16m (\kappa_0 + \kappa_1)^2}{\gamma^2}
= \frac{8m}{\gamma^2} (\kappa_0 + \kappa_1) \left[ \Omega^\infty + 2(\kappa_0 + \kappa_1) \right],
\]

and

\[
\Omega^\infty \gamma \sqrt{R_1(\dot{z})} + 12m R_1(\dot{z})
\leq \frac{4m \Omega^\infty \left[ \Omega^\infty + 2(\kappa_0 + \kappa_1) \right]}{\gamma^2} + \frac{16m \left[ \Omega^\infty + 2(\kappa_0 + \kappa_1) \right]^2}{\gamma^2}
= \frac{4m}{\gamma^2} \left[ \Omega^\infty + 2(\kappa_0 + \kappa_1) \right] \left[ 4\Omega^\infty + 6(\kappa_0 + \kappa_1) \right].
\]

Hence, one has

\[
\frac{4m \Omega^\infty}{\gamma} \left[ R_3(V) + \sqrt{R_1(\dot{z})} \right] + 12m R_1(\dot{z}) + 4m R_3(V)^2 \leq \frac{16m}{\gamma^2} \left[ \Omega^\infty + 2(\kappa_0 + \kappa_1) \right]^2. \tag{6.11}
\]
Now, we combine (6.10) and (6.11) to get
\[ m \ddot{G} + \gamma \dot{G} + 4\kappa_0 \delta G \leq 4\Omega^\infty + 8\kappa_1 + \frac{16m}{\gamma^2} [\Omega^\infty + 2(\kappa_0 + \kappa_1)]^2, \quad t \in (0, T_\ast). \]
Note that \((F_{C1})\) is the first case of Lemma 5.4 and \((F_{C2})_2\) and \((F_{C2})_3\) satisfy the condition (5.23) of Lemma 5.4. So, we apply Lemma 5.4 to obtain
\[ G(t) < \frac{1}{4\kappa_0 \delta} \left( 4\Omega^\infty + 8\kappa_1 + \frac{16m}{\gamma^2} [\Omega^\infty + 2(\kappa_0 + \kappa_1)]^2 \right) < \frac{(1-\delta)^2}{N}, \quad t \in (0, T_\ast), \]
which contradicts to (6.9). Therefore, we have \(T_\ast = \infty\).
Now, we are ready to provide a proof of Theorem 4.2.
• Step B (Derivation of practical aggregation estimate): It follows from (6.12) that
\[ G(t) < \frac{\Omega^\infty + 2\kappa_1}{\kappa_0 \delta} + \frac{4m[\Omega^\infty + 2(\kappa_0 + \kappa_1)]^2}{\gamma^2 \kappa_0 \delta^2}, \quad \forall \ t > 0. \]
For the case \(\kappa_0 \geq \max \{\Omega^\infty, 2\kappa_1\}\), we have
\[ G(t) < \frac{\Omega^\infty + 2\kappa_1}{\kappa_0 \delta} + \frac{4m\kappa_0}{\gamma^2 \delta} \cdot \frac{\Omega^\infty + 2\kappa_1}{\kappa_0 \delta} \leq \frac{\Omega^\infty + 2\kappa_1}{\kappa_0 \delta} + \frac{64}{\gamma^2 \delta} \cdot \frac{m\kappa_0}{\kappa_0^2}, \quad \forall \ t > 0. \]
Hence as \(m\kappa_0 \rightarrow 0\) and \(\kappa_0 \rightarrow \infty\), one has a practical synchronization. To satisfy the constraints (6.14), we assume that there exist \(m_0 > 0\) and \(\eta > 0\) such that
\[ m = \frac{m_0}{\kappa_0^{1+\eta}}. \]
Then, it follows from (6.13) and (6.15) that, for \(\kappa_0 \geq \max \{\Omega^\infty, 2\kappa_1\}\),
\[ G(t) < \frac{\Omega^\infty + 2\kappa_1}{\kappa_0 \delta} + \frac{64m_0}{\gamma^2 \delta} \cdot \frac{m_0}{\kappa_0^2}, \quad \forall \ t > 0, \]
which implies the desired result. \(\square\)

Remark 6.1. One can question about the possibility for second inequality of \((F_{C2})_2\). We verified that it holds for a sufficiently large \(\kappa_0\). We substitute (6.15) into the second inequality of \((F_{C2})_2\) to obtain
\[ \frac{\Omega^\infty + 2\kappa_1}{\kappa_0 \delta} + \frac{4m_0[\Omega^\infty + 2(\kappa_0 + \kappa_1)]^2}{\delta \gamma^2 \kappa_0^{2+\eta}} < \frac{(1-\delta)^2}{N}. \]
One can observe that left-hand side of (6.16) converges to zeros, as \(\kappa_0\) goes to infinity.

7. Conclusion. In this work, we have studied emergent behaviors of the second-order LHS model which can be realized as a second-order extension of the first-order LHS model introduced in authors’ earlier works [9–11]. For a homogeneous ensemble with the same natural frequency matrix \(\Omega_j = \Omega\), we showed emergence of complete aggregation in the sense that all states aggregate to the same state asymptotically. For this, under a suitable set of system parameters and initial data with a finite energy, we show that the two-point correlation functions between states tend to zero asymptotically, which means the formation of complete aggregation. By linear
stability analysis, we also showed that the incoherent state and bi-polar state are linearly unstable. In contrast, for a heterogeneous ensemble, we provided a sufficient framework leading to practical aggregation which means that state diameter can be made small by increasing the principle coupling strength. Of course, there are several issues to be discussed in a future work. For example, we only considered positive all-to-all coupling strengths (attractive couplings) in this work. However, the coupling strengths can be nonpositive, i.e., repulsive or zero couplings or they can be time-dependent or state-dependent which make asymptotic dynamics more richer. We leave these interesting issues in a future work.

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