Temporal evolution of drift-type modes in plasma flows with strong time-dependent velocity shear

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Abstract

The linear and renormalized nonlinear analysis of the temporal evolution of drift-type modes in plasma flows with strong time-varying velocity shear is developed. Analysis is performed in the time domain without spectral decomposition in time and admits time variation of the flow velocity with time scales which may be comparable with turbulent time scales.

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I. INTRODUCTION.

A great variety of experimental results are now available[1], which confirm a suppression of turbulence by $E \times B$ velocity shear as a key feature of the regimes of improved confinement[2] of plasma and formation of transport barriers in tokamaks. The first remarkable success in theoretical investigation of this fascinating and rich topic of fusion research was achieved in Refs.[3, 4], where nonlinear enhanced decorrelation of fluctuations by shear flow had been proposed as a universal and robust mechanism for turbulence quenching and transport reduction. It was compared quite favorably with experimental data for numerous tokamaks and was confirmed by gyrofluid codes[5, 6, 7]. This theory was developed with the help of a simple ”modal” approach, under an implicit assumption that perturbations of fields, densities, currents etc. are ”normal eigenmodes” with certain frequencies and wave numbers. However shear flow itself affects the structure of the waves and their temporal evolution. The shear flow distorts wave pattern, leads to its stretching and to temporal changing with time the wave number of any separate wave[8, 9]. Under condition of strong flow shear, which is observed in edge layers of tokamaks, where velocity shearing rate $dv_0/dr$ approaches or even exceeds the drift wave frequencies, this theory has to be strongly modified.

The experimentally relevant strong shear regime was studied analytically in numerous papers applying the gyrokinetic approach (see, for example,Refs.[10], [11]; we mention also Ref.[12] as an example of a recent work). In our papers (see Refs.[13]-[17]) the investigations of linear evolution of drift waves in shear flows applied Kelvin’s method of shearing modes or a so-called non-modal approach. This treatment is performed in terms of a physical time variable without any prior spectral decomposition in time. Kelvin’s method appears to be very effective for the analytical investigations of the temporal evolution of numerous low frequency modes in plasma shear flows[13]-[17].

Using one of the simplest models available describing three-dimensional plasma turbulence at the edge of a magnetic confinement device, the Hasegawa–Wakatani equations[18], we find that under condition of strong flow shear non-modal evolution of the drift–type perturbations supersedes the modal development of linear instabilities. Drift modes are suppressed before they can grow, preventing the development of nonlinear processes and imposing its own physical character on the dynamics. The suppression of the drift resistive instability in the case of sufficiently strong flow shear is a non-modal process, during which the initial amplitude of the separate spatial Fourier mode of the perturbed electrostatic potential decreases with time as $(v_0' t)^{-2}$.

In this paper we extend the non-modal treatment of the drift modes stability onto the plasma flows with a spatially homogeneous time-dependent velocity shear. In Section II we consider linear and weak-nonlinear solutions of the nonlinear Hasegawa-Mima equation for plasma flow with strong time-dependent velocity shear. In Section III we discuss the correlation properties of drift waves in plasma shear flows and their influence on the saturation of the resistive drift instability. In that section we pay the attention to flows with a ”modest” velocity shear (of the order of or greater than the instability growth rate, but less than the drift waves frequency). In this case the stage at
which nonlinear processes determine the saturation of the drift resistive instability occurs before the development of the linear non-modal suppression of drift instability. Conclusions are given in Section IV.

II. Non-modal drift waves in plasma flow with strong time-dependent velocity shear

We investigate the temporal evolution of drift modes in time-dependent shear flow using the Hasegawa–Wakatani equations[18]. We use Cartesian coordinates aligned so that $z$ is in the direction of the mean magnetic field, $y$ is in the direction of the mean flow, and $x$ is in the direction of the inhomogeneity of the mean flow and plasma density, assuming that these coordinates correspond to toroidal ($\varphi$), poloidal ($\theta$), and the radial ($r$) coordinates, respectively, of the toroidal coordinate system. In this geometry the Hasegawa–Wakatani system of equations for the dimensionless density $n = \tilde{n}/n_e$ and potential $\phi = e\varphi/T_e$ perturbations ($n_e$ is the electron background density, $T_e$ is the electron temperature) is [18]

$$
\rho_s^2 \left( \frac{\partial}{\partial t} + V_0(x,t) \frac{\partial}{\partial y} - \frac{c}{B} \left( \frac{\partial \phi}{\partial y} \frac{\partial}{\partial x} - \frac{\partial \phi}{\partial x} \frac{\partial}{\partial y} \right) \right) \nabla^2 \phi = a \frac{\partial^2}{\partial z^2} (n - \phi),
$$

$$
\left( \frac{\partial}{\partial t} + V_0(x,t) \frac{\partial}{\partial y} - \frac{c}{B} \left( \frac{\partial \phi}{\partial y} \frac{\partial}{\partial x} - \frac{\partial \phi}{\partial x} \frac{\partial}{\partial y} \right) \right) n + v_{de} \frac{\partial \phi}{\partial y} = a \frac{\partial^2}{\partial z^2} (n - \phi),
$$

$V_0(x,t)$ is the velocity of the sheared flow, $a = T_e/n_0e^2\eta_\parallel$, $\eta_\parallel$ is the resistivity parallel to the homogeneous magnetic field $B_\parallel z$, $\rho_s$ is the ion Larmor radius at electron temperature $T_e$, $v_{de} = cT_e/eBL_n$ is the diamagnetic drift velocity, $L_n^{-1} = -d\ln n_0e(x)/dx$. We transform Eqs.(1), (2) to new spatial variables $\xi, \eta$,

$$
t = t, \quad \xi = x, \quad \eta = y - \int V_0(x,t) \, dt, \quad z = z,
$$

which are the generalizations of the convective coordinates, used previously in systems with homogeneously sheared stationary flows[13]. In these coordinates the linear convection terms in the system of equations (1), (2) are excluded, but the Laplacian operator $\Delta$ now becomes time-dependent,

$$
\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left( \frac{\partial}{\partial \xi} - \int \frac{\partial V_0(\xi,t)}{\partial \xi} dt \frac{\partial}{\partial \eta} \right) \left( \frac{\partial}{\partial \xi} - \int \frac{\partial V_0(\xi,t)}{\partial \xi} dt \frac{\partial}{\partial \eta} \right) + \frac{\partial^2}{\partial \eta^2},
$$

leaving us with an initial value problem to solve. This time dependence is responsible for the shearing of the waves pattern by the basic flow in the convective frame of reference.

The analysis of the effect of the flow shear spatial non-uniformity on the temporal evolution of the electrostatic potential of drift modes has shown[17], that for the velocity profiles without inflection point this effect is subdominant at all stages of the drift modes evolution. Therefore we consider here the case of the spatially uniform time-dependent velocity shear, for which

$$
\int \frac{\partial V_0(\xi,t)}{\partial \xi} dt = v'_0 \alpha(t),
$$
where $v'_0 = \text{const}$ is a parameter with dimension of the velocity shear, $\alpha(t)$ is a function of time. Introducing Fourier expansion with respect to $\xi$, $\eta$ and $z$-coordinates with $k_\perp$, $l$ and $k_z$ as the wave numbers conjugate to the spatial coordinates $\xi, \eta$ and $z$, respectively, the linearized system (1), (2) with omitted nonlinear convective terms may be combined into the following equation for the potential $\phi(t, k_\perp, l, k_z)$:

$$\frac{1}{C}\frac{1}{v'_0}\frac{\partial^2}{\partial t^2} (\Delta_{\perp} \phi) + \frac{\rho_s^2}{v'_0} \frac{\partial}{\partial t} (\Delta_{\perp} \phi) - \frac{1}{v'_0} \frac{\partial \phi}{\partial t} - i S \phi = 0. \quad (6)$$

In Eq.(4) $C = ak_z^2/\rho_s l^2 v'_0 = T_e k_z^2/\rho_s l^2 v'_0 n_0 e^2 \eta_\parallel \gg 1$, $S = l v_d/v'_0$. The Laplacian operator $\Delta$ is equal to

$$\Delta (x, k_\perp, l) = -(k_\perp - lv'_0 \alpha(t))^2 - l^2. \quad (7)$$

In variables $\xi$ and $\eta$ the linear solution to Eq.(6) has a form

$$\phi(\xi, \eta, t) = \int dk_\perp \int dl \phi(k_\perp, l, 0) g(k_\perp, l, t) e^{ik_\perp \xi + il \eta}, \quad (8)$$

where $\phi(k_\perp, l, 0)$ is the initial data and $g(k_\perp, l, t)$ is the linearly unstable solution. It was shown in Ref.[13] (see, also[17]) for the case of the spatially homogeneous time-independent velocity shear, that for $v'_0 \lesssim \gamma$, where $\gamma$ is growth rate of the resistive drift instability in plasma without shear flow, the solution $g(k_\perp, l, t)$ in times $t \lesssim (v'_0 \rho_s)^{-1}$ has an ordinary modal form,

$$g(k_\perp, l, t) = e^{i\omega_d t + \gamma t} \quad (9)$$

with known frequency of drift wave $\text{Re} \omega = \omega_d = S v'_0 / (1 + \rho_s^2 (k_\perp^2 + l^2))$, and growth rate of the resistive drift instability $\gamma = \text{Im} \omega = \omega_d \rho_s^2 (k_\perp^2 + l^2) / a k_z^2 (1 + \rho_s^2 (k_\perp^2 + l^2)) \simeq v'_0 S^2 / C$, where $C = C/(k_\perp^2 + l^2)$. Using solution (9), which has a form as in plasma without any shear flow, weak nonlinear theory, which is grounded on the Hasegawa-Wakatani system (1), (2) or Hasegawa-Mima equation[19] may be easily developed[20]. This theory will be free from the known problem of the solutions singularities at critical level points $x = x_0$, where phase velocity of wave becomes equal to flow velocity, i.e. $\omega - k_y v_0(x_0) = 0$. They emerge, when linear, as well as weak-nonlinear, solutions are obtained by using spectral transform in time and variables $x$, $y$ of the laboratory frame of reference. It is worth to note, that it is unreasonable from the very beginning to apply the spectral expansion in time in the case of the time dependent flow velocity.

In the laboratory frame solution (8) becomes nonseparable in space and time and therefore quite different from the normal mode assumption. In the case of the time dependent velocity shear it has a form

$$\phi(r, t) = \int dk_\perp \int dl \phi(k_\perp, l, 0) g(k_\perp, l, t) e^{i(k_\perp - lv'_0 \alpha(t))x + il \eta}. \quad (10)$$

Only in times, for which $|v'_0 \alpha(t)| \ll 1$ (in times $t \ll |v'_0|^{-1}$ in the case of stationary velocity shear) solution (10) has a normal mode form. Because of the time dependence $k_x = k_\perp - lv'_0 \alpha(t)$ of the
wave number component along the flow shear, the modes in the laboratory frame become increasingly one-dimensional zonal-like as the perturbed $E \times B$ velocity tilts more and more closely parallel to y-axis.

The approximate weak nonlinear solution to Eq. (11) for the potential $\phi(t,k,l)$, resulting from the enhanced by shear flow time-dependent dispersion, which does not allow the weak nonlinear three waves decay processes. For strong flow shear case, when $lv_{de} \lesssim v_0$, the first term of Eq. (6) over times $t \gtrsim (lv_{de})^{-1}$ may be omitted\cite{17}. In that case Eq. (6) reduces to the Hasegawa-Mima equation\cite{19}, which with convective nonlinearity (omitted in Eq. (6)) in the case of the time-dependent velocity shear has a form

$$\frac{\partial}{\partial t} [\sigma (k_\perp, l, t) \phi (t, k_\perp, l) + ilv_{de} \phi (\tau, k_\perp, l)] = \rho_s^2 \frac{c}{B} \int dk_{\perp 1} \int dl_1 \int dk_{\perp 2} \int dl_2 \Delta (t, k_{2\perp}, l_2) \times (k_{1\perp} l_1 - k_{2\perp} l_1) \phi (t, k_{1\perp}, l_1) \phi (t, k_{2\perp}, l_2) \delta (k_{2\perp} - k_{1\perp} - k_{2\perp}) \delta (l_1 - l_2 - l_2),$$  

where $\Delta (\tau, k_{2\perp}, l_2)$ is determined by Eq. (7) and

$$\sigma (k_\perp, l, t) = 1 + \rho_s^2 \left( l^2 + (k_\perp - v_0 \alpha (t) l)^2 \right).$$

The approximate weak nonlinear solution to Eq. (11) for the potential $\phi$ is easily obtained in the form of the power series in the amplitude of the initial data $\phi (\xi, l, 0)$, $\phi (\tau, k_\perp, l) = \phi_0 (\tau, k_\perp, l) + \phi_1 (\tau, k_\perp, l) + ...$, where $\phi_0$ and $\phi_1$ are linear and square terms with respect to $\phi (\xi, l, 0)$, with

$$\phi_0 (k_\perp, l, t) = \phi (k_\perp, l, 0) g (k_\perp, l, t),$$

and

$$\phi_1 (k_\perp, l, t) = \frac{\rho_s^2 c}{B \sigma (k_\perp, l, t)} \int_{t_0}^{t} dt_1 \exp \left[ -ilv_{de} \int_{t_0}^{t_1} dt_2 \sigma^{-1} (k_\perp, l, t_2) \right] \int dk_{\perp 1} \int dl_1 \int dk_{\perp 2} \int dl_2 \times \Delta (t_1, k_{2\perp}, l_2) (k_{1\perp} l_1 - k_{2\perp} l_1) \phi (k_{1\perp}, l_1, 0) \phi (k_{2\perp}, l_2, 0) g (k_{1\perp}, l_1, t) g (k_{2\perp}, l_2, t) \times \delta (k_{\perp} - k_{1\perp} - k_{2\perp}) \delta (l_1 - l_2 - l_2),$$

where

$$g (k_\perp, l, t) = \frac{1}{\sigma (k_\perp, l, t)} \exp \left[ -ilv_{de} \int_{t_0}^{t} dt_1 \sigma^{-1} (k_\perp, l, t_1) \right].$$

It follows from Eqs. (13) and (14) that over times, for which $|\alpha (t) v_0 l| \rho_s \gtrsim 1$ the initial perturbations of the potential with given $l \rho_s$ will be suppressed before the development of the resistive drift instability, when non-modal evolution begins at times $t < \gamma^{-1}$. For the particular case of the time-independent velocity shear solution (10) was obtained and discussed in Ref.\cite{13}. For power-like time dependence of the velocity shear with $\partial V_0 / \partial \xi = v_0' \alpha_n t^n$, $\alpha (t) = \alpha_n t^{n+1}/(n+1)$ and

$$\sigma (k_\perp, l, t) = 1 + \rho_s^2 \left( l^2 + \left( k_\perp - v_0' \alpha_n t^{n+1}/(n+1) \right)^2 \right),$$
the non-modal development begins at times \( t > (v'_0 \alpha_n l \rho_s)^{-1} \), at which linear \( \phi_0 (t, k_l, l) \) and weakly nonlinear \( \phi_1 (t, k_l, l) \) parts of the potential rapidly decay with time. The fluctuation spectrum for long time will be dominated by Fourier components with sufficiently small values of wave number component \( l \) for which condition \( |\alpha(t)v'_0 l| \rho_s \gtrsim 1 \) is not met during time considered.

### III. Renormalized hydrodynamic theory for drift modes in plasma shear flow

With variables \( \xi, \eta \), Eqs. (1) and (2) have a form

\[
\rho_s^2 \left( \frac{\partial}{\partial t} - \frac{c}{B} \left( \frac{\partial \phi}{\partial \eta} \frac{\partial}{\partial \xi} - \frac{\partial \phi}{\partial \xi} \frac{\partial}{\partial \eta} \right) \right) \nabla^2 \phi = a \frac{\partial^2}{\partial z^2} (n - \phi),
\]

\[
\left( \frac{\partial}{\partial t} - \frac{c}{B} \left( \frac{\partial \phi}{\partial \eta} \frac{\partial}{\partial \xi} - \frac{\partial \phi}{\partial \xi} \frac{\partial}{\partial \eta} \right) \right) n + v_{de} \frac{\partial \phi}{\partial y} = a \frac{\partial^2}{\partial z^2} (n - \phi).
\]

The above calculations show that with variables \( \xi \) and \( \eta \) we exclude from Eqs. (1) and (2) the spatial inhomogeneity originated from shear flow. That gives an opportunity to investigate in linear approximation the temporal evolution of the separate spatial Fourier mode of the electrostatic potential \( \phi \) with definite wave numbers \( k_l \) and \( l \). It is interesting to note that transformation (3) conserves the \( E \times B \) convective nonlinear derivative in Eqs. (1) and (2) in the form similar to one in a plasma without any flows. The flow shear contains only in time dependent Laplacian operator and become apparent in linear non-modal effect of the enhanced dispersion during times \( |\alpha(t)v'_0 l| \rho_s \gtrsim 1 \) (or, at the case of stationary velocity shear, when \( t > (v'_0 l \rho_s)^{-1} \)) after the stage of the ordinary modal evolution[13]. With new variables \( \xi_1, \eta_1 \), which are determined by the nonlinear relations

\[
\xi_1 = \xi + c \frac{B}{t} \int_{t_0}^t \frac{\partial \phi}{\partial \eta} dt_1, \quad \eta_1 = \eta - c \frac{B}{t} \int_{t_0}^t \frac{\partial \phi}{\partial \xi} dt_1,
\]

the convective nonlinearity in Eqs. (17) and (18) becomes of the higher order with respect to the potential \( \phi \),

\[
-\frac{c}{B} \left( \frac{\partial \phi}{\partial \eta} \frac{\partial}{\partial \xi} - \frac{\partial \phi}{\partial \xi} \frac{\partial}{\partial \eta} \right) =
\]

\[
-\left( \frac{c}{B} \right)^2 \left[ \left( \frac{\partial \phi (t)}{\partial \eta} \int_{t_0}^t dt_1 \frac{\partial^2 \phi (t_1)}{\partial \eta \partial \xi} - \frac{\partial \phi (t)}{\partial \xi} \int_{t_0}^t dt_1 \frac{\partial^2 \phi (t_1)}{\partial \eta^2} \right) \frac{\partial}{\partial \xi_1} + \left( \frac{\partial \phi (t)}{\partial \xi} \int_{t_0}^t dt_1 \frac{\partial^2 \phi (t_1)}{\partial \eta \partial \xi} - \frac{\partial \phi (t)}{\partial \eta} \int_{t_0}^t dt_1 \frac{\partial^2 \phi (t_1)}{\partial \xi^2} \right) \frac{\partial}{\partial \eta_1} \right].
\]
Fourier spectrum on any separate Fourier harmonic is accounted for. The functions omitting such nonlinearity, as well as small nonlinearity of the second order in the Laplacian, resulted in instability. Consider first the dispersion tensor of random displacements analysis of the correlation properties of the nonlinear solutions to Hasegawa-Wakatani system and exponentials the integrals (22) and so on. This form of solution, however, appears very useful for the analysis of the correlation properties of the nonlinear solutions to Hasegawa-Wakatani system and for the development of the approximate renormalized solutions to Hasegawa-Wakatani system, which accounted for the effect of the turbulent motions of plasma on the saturation of the drift-resistive instability. Consider first the dispersion tensor of random displacements \( \tilde{\xi} (t) \) and \( \tilde{\eta} (t) \) of the plasma resulted from its motion in such turbulence. The variances of these displacements are determined by the relations

\[
\begin{align*}
K_{\xi \xi} (t, t_0) &= K_{\xi \xi} (t) = \left\langle \left( \tilde{\xi} (t) \right)^2 \right\rangle = \frac{c^2}{B^2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \left\langle \frac{\partial \phi (t_1)}{\partial \eta} \frac{\partial \phi (t_2)}{\partial \eta} \right\rangle, \\
K_{\xi \eta} (t, t_0) &= K_{\xi \eta} (t) = \left\langle \tilde{\xi} (t) \tilde{\eta} (t) \right\rangle = -\frac{c^2}{B^2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \left\langle \frac{\partial \phi (t_1)}{\partial \eta} \frac{\partial \phi (t_2)}{\partial \xi} \right\rangle, \\
K_{\eta \eta} (t, t_0) &= K_{\eta \eta} (t) = \left\langle \left( \tilde{\eta} (t) \right)^2 \right\rangle = \frac{c^2}{B^2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \left\langle \frac{\partial \phi (t_1)}{\partial \xi} \frac{\partial \phi (t_2)}{\partial \xi} \right\rangle,
\end{align*}
\]

where brackets \( \langle \ldots \rangle \) mean an average over ensemble of random values of \( \phi (k_\perp, l, 0) \). Using in Eqs.(23)-(25) solution (21) and the Corrsin’s[21] independence hypothesis,

\[
\begin{align*}
\left\langle \phi (k_{1\perp}, l_1, t_1) \phi (k_{2\perp}, l_2, t_2) e^{ik_{1\perp} \xi + il_1 \eta - ik_{2\perp} \tilde{\xi}(t_1) - il_1 \tilde{\eta}(t_1)} \right\rangle &= \left\langle \phi (k_{1\perp}, l_1, 0) \phi (k_{2\perp}, l_2, 0) \right\rangle \\
\times g (k_{1\perp}, l_1, t_1) g (k_{2\perp}, l_2, t_2) \left\langle e^{ik_{1\perp} (\tilde{\xi}(t_1) - \tilde{\xi}(t_2)) + il_1 (\tilde{\eta}(t_1) - \tilde{\eta}(t_2))} \right\rangle \delta (k_{1\perp} + k_{2\perp}) \delta (l_1 + l_2),
\end{align*}
\]
we find for the components \(K_{\xi \xi} (t)\), \(K_{\xi \eta} (t)\) and \(K_{\eta \eta} (t)\) of the dispersion tensor the expressions

\[
K_{\xi \xi} (t) = \frac{c^2}{B^2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \int dk_\perp \int dl |\phi (k_\perp, l, 0)|^2 l^2 g (k_\perp, l, t_1) g (-k_\perp, -l, t_2) \\
\times \left\langle e^{ik_\perp (\tilde{\xi}(t_1) - \tilde{\xi}(t_2)) + i\tilde{\eta}(t_1) - \tilde{\eta}(t_2)} \right\rangle,
\]

(27)

\[
K_{\xi \eta} (t) = -\frac{c^2}{B^2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \int dk_\perp \int dl |\phi (k_\perp, l, 0)|^2 k_\perp l g (k_\perp, l, t_1) g (-k_\perp, -l, t_2) \\
\times \left\langle e^{ik_\perp (\tilde{\xi}(t_1) - \tilde{\xi}(t_2)) + i\tilde{\eta}(t_1) - \tilde{\eta}(t_2)} \right\rangle,
\]

(28)

\[
K_{\eta \eta} (t) = \frac{c^2}{B^2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \int dk_\perp \int dl |\phi (k_\perp, l, 0)|^2 l^2 g (k_\perp, l, t_1) g (-k_\perp, -l, t_2) \\
\times \left\langle e^{ik_\perp (\tilde{\xi}(t_1) - \tilde{\xi}(t_2)) + i\tilde{\eta}(t_1) - \tilde{\eta}(t_2)} \right\rangle.
\]

(29)

During times \(|\alpha(t)\nu_\eta| \lesssim 1\), the solution \(g\) to the Wasegawa-Wakatani system is of a modal form (9). Assuming that the displacements \(\tilde{\xi} (t), \tilde{\eta} (t)\) obey the Gaussian statistics with mean zero, we find in this case the following relation for \(K_{\xi \xi} (t)\):

\[
K_{\xi \xi} (t) = \frac{c^2}{B^2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \int dk_\perp \int dl |\phi (k_\perp, l, 0)|^2 l^2 \exp \left(\gamma (t_1 + t_2) + i\omega_d (t_1 - t_2)\right) \\
\times \exp \left[\frac{-1}{2} k^2 K_{\xi \xi} (t_1, t_2) - k_\perp l K_{\xi \eta} (t_1, t_2) - \frac{1}{2} l^2 K_{\eta \eta} (t_1, t_2)\right].
\]

(30)

where

\[
K_{\xi \xi} (t_1, t_2) = \left\langle \left(\tilde{\xi}(t_1) - \tilde{\xi}(t_2)\right)^2 \right\rangle,
\]

\[
K_{\xi \eta} (t_1, t_2) = \left\langle \left(\tilde{\xi}(t_1) - \tilde{\xi}(t_2)\right) \left(\tilde{\eta}(t_1) - \tilde{\eta}(t_2)\right) \right\rangle,
\]

\[
K_{\eta \eta} (t_1, t_2) = \left\langle \left(\tilde{\eta}(t_1) - \tilde{\eta}(t_2)\right)^2 \right\rangle.
\]

(31)
With time variables $\tau = t_1 - t_2$ and $\hat{\tau} = (t_1 + t_2)/2$, Eq.(30) becomes

$$K_{\xi \xi}(t) = \frac{c^2}{B^2} \left[ \int_{-t}^{t} d\tau \int_{0}^{t} d\hat{\tau} \int_{0}^{t} d\hat{\tau} \int dK_{\perp} \int d\ell |\phi(k_{\perp}, l, 0)|^2 \ell^2 \exp \left( 2\gamma (k_{\perp}, l, 0) \hat{\tau} + i\omega_d (k_{\perp}, l) \tau \right) \right]$$

$$\times \exp \left[ -\frac{c^2}{2B^2} \int_{0}^{\hat{\tau}} d\hat{\tau}_{\perp} \int d\hat{\tau}_{\|} \int dK_{\perp} \int d\ell_1 |\phi(k_{\perp}, l, 0)|^2 |[K_{\perp} \times k_{\perp}]|^2 \right]$$

$$\times \exp \left( 2\gamma (k_{\perp}, l_1) \hat{\tau}_{\perp} + i\omega_d (k_{\perp}, l_1) \tau_1 - \frac{1}{2} \left\langle \left( k_{\perp} \cdot \hat{\mathbf{r}} \left( \hat{\tau}_{\perp} + \frac{\tau_1}{2} \right) - k_{\perp} \cdot \hat{\mathbf{r}} \left( \hat{\tau}_{\perp} - \frac{\tau_1}{2} \right) \right)^2 \right\rangle \right)$$

$$\frac{1}{2} \left( k_{\perp}^2 K_{\xi \xi} (t_1, t_2) + k_{\perp} l K_{\xi \eta} (t_1, t_2) + \ell^2 K_{\eta \eta} (t_1, t_2) \right)$$

$$\approx \frac{1}{2} \left( k_{\perp}^2 K_{\xi \xi} \left( \hat{\tau} + \frac{\tau_1}{2}, \hat{\tau} - \frac{\tau_1}{2} \right) + k_{\perp} l K_{\xi \eta} \left( \hat{\tau} + \frac{\tau_1}{2}, \hat{\tau} - \frac{\tau_1}{2} \right) + \ell^2 K_{\eta \eta} \left( \hat{\tau} + \frac{\tau_1}{2}, \hat{\tau} - \frac{\tau_1}{2} \right) \right)$$

$$= C \left( k_{\perp}, l, \hat{\tau} \right) \tau$$

$$= \frac{c^2}{B^2} \int dK_{\perp} \int d\ell_1 |\phi(k_{\perp}, l_1, \hat{\tau})|^2 |[K_{\perp} \times k_{\perp}]|^2 \int d\tau_1 e^{i(\omega_d (k_{\perp}, l_1) - C(k_{\perp}, l_1, \hat{\tau})) \tau_1}$$

$$\approx \frac{c^2}{B^2} \int dK_{\perp} \int d\ell_1 |\phi(k_{\perp}, l_1, \hat{\tau})|^2 |[K_{\perp} \times k_{\perp}]|^2 \frac{C(k_{\perp}, l_1, \hat{\tau})}{\omega_d^2(k_{\perp}, l_1, \hat{\tau})},$$

(32)
where \( |\phi(k_{1\perp}, l, \hat{t})|^2 = |\phi(k_{1\perp}, l_1, 0)|^2 e^{2\gamma(k_{1\perp}, l_1)\hat{t}} \). Then \( K_{\xi \xi}(t) \) becomes equal to

\[
K_{\xi \xi}(t) = \frac{c^2}{B^2} \left[ \int_{-\frac{t}{2}}^{0} d\tau \left( \int_{0}^{t} d\tilde{t} + \int_{t}^{0} d\tilde{t} \right) \right] \int dk_{1\perp} \int d\tilde{t} |\phi(k_{1\perp}, l, 0)|^2 l^2 e^{2\gamma(k_{1\perp}, l_1)\hat{t} + i\omega_d(k_{1\perp}, l)}
\]

\[
\times \exp \left[ -\tau \left( \frac{c^2}{B^2} \int dk_{1\perp} \int d\tilde{t} |\phi(k_{1\perp}, l_1, 0)|^2 e^{2\gamma(k_{1\perp}, l_1)\hat{t}} |[k_{1\perp} \times k_{1\perp}]|^2 \frac{C(k_{1\perp}, l_1, \hat{t})}{\omega_d^2(k_{1\perp}, l_1)}\right] \approx \frac{2c^2}{B^2} \int dk_{1\perp} \int d\tilde{t}^2 \int_{0}^{t} d\tilde{t} |\phi(k_{1\perp}, l, \hat{t})|^2 |[k_{1\perp} \times k_{1\perp}]|^2 \frac{C(k_{1\perp}, l_1, \hat{t})}{\omega_d^2(k_{1\perp}, l_1)} = 2 \int_{0}^{t} d\tilde{t} C(k_{1\perp}, l, \hat{t}). \quad (36)
\]

With Eq.(36) and similar equations for \( K_{\eta \eta}(t) \) and \( K_{\eta \eta}(t) \) we obtain a more general equation,

\[
k_{1\perp}^2 K_{\xi \xi}(t) + 2k_{1\perp} l K_{\eta \eta}(t) + t^2 K_{\eta \eta}(t)
\]

\[
= \frac{c^2}{B^2} \int dk_{1\perp} \int d\tilde{t} \int_{0}^{t} d\tilde{t} |\phi(k_{1\perp}, l_1, \hat{t})|^2 |[k_{1\perp} \times k_{1\perp}]|^2 \frac{C(k_{1\perp}, l_1, \hat{t})}{\omega_d^2(k_{1\perp}, l_1)} = 2 \int_{0}^{t} d\tilde{t} C(k_{1\perp}, l, \hat{t}). \quad (37)
\]

With Eq.(37) we obtain the renormalized form of the potential (21), in which the average effect of the random convection is accounted for,

\[
\phi(\xi, \eta, t) = \int dk_{1\perp} \int d\tilde{t} \phi(k_{1\perp}, l, 0) e^{i\omega_d t + \gamma t - \int_{0}^{t} d\tilde{t} C(k_{1\perp}, l, \hat{t}) + ik_{1\perp} \xi + il\eta}. \quad (38)
\]

The saturation of the instability occurs when \( \partial (\phi(\xi, \eta, t))^2 / \partial t = 0 \), i.e. when

\[
\gamma(k_{1\perp}, l) = C(k_{1\perp}, l, t)
\]

\[
= \frac{c^2}{B^2} \int dk_{1\perp} \int d\tilde{t} |\phi(k_{1\perp}, l_1, t)|^2 |[k_{1\perp} \times k_{1\perp}]|^2 \frac{C(k_{1\perp}, l_1, t)}{\omega_d^2(k_{1\perp}, l_1)}. \quad (39)
\]

From the double Eq.(39) we obtain the equation, which determines the level of the instability saturation

\[
\gamma(k_{1\perp}, l) = \frac{c^2}{B^2} \int dk_{1\perp} \int d\tilde{t} |\phi(k_{1\perp}, l_1, t)|^2 |[k_{1\perp} \times k_{1\perp}]|^2 \frac{\gamma(k_{1\perp}, l_1)}{\omega_d^2(k_{1\perp}, l_1)}. \quad (40)
\]

The sought-for value in Eq.(40) is a time \( t_{sat} \) at which the balance of the linear growth and nonlinear damping occurs for given initial disturbance \( \phi(k_{1\perp}, l_1, 0) \) and dispersion. With obtained \( t_{sat} \) the saturation level will be equal to \( |\phi(t_{sat})|^2 \approx \int dk_{1\perp} \int d\tilde{t} |\phi(k_{1\perp}, l_1, 0)|^2 e^{2\gamma(k_{1\perp}, l_1) t_{sat}} \). Also, the well known order of value estimate [20] for the potential \( \phi \) in the saturation state is obtained easily from Eq.(40),

\[
\frac{e^\phi}{T_e} \sim \frac{1}{k_{1\perp} L_n}. \quad (41)
\]

Obtained results show that the nonlinearity of the Hasegawa-Wakatani system of equations in variables \( \xi \) and \( \eta \), with which frequency and growth rate are determined without spatially inhomogeneous Doppler shift and wave number is time independent, does not display any effects of the
enhanced decorrelations provided by flow shear. In the laboratory frame of reference such spatial Fourier modes are observed as a sheared modes with time dependent component of the wave number \( k_x = k_\perp - lv'_0 \alpha(t) \) directed along the velocity shear,

\[
\phi(r, t) = \int dk_\perp \int dl \phi(k_\perp, l, 0) e^{i(k_\perp - lv'_0 \alpha(t))x + ily + iy_d + \gamma t - ik_\perp \tilde{\xi}(t_1) - i\tilde{\eta}(t_1)}.
\] (42)

To make the next analysis simpler we consider in what follows the case of the spatially homogeneous time-independent velocity shear, \( \alpha(t) = t \). The displacements \( \tilde{\xi}(t) \) and \( \tilde{\eta}(t) \) are observed in the laboratory frame as the displacements \( \tilde{x}(t) \) and \( \tilde{y}(t) \) which are equal to

\[
\tilde{x}(t) = \tilde{\xi}(t) = -\frac{c}{B} \int_{t_0}^{t} \frac{\partial \phi}{\partial y} dt_1
\]

\[
= -i \frac{c}{B} \int_{t_0}^{t} dt_1 \int dk_\perp \int dl \phi(k_\perp, l, 0) le^{i\omega_d t_1 + i(k_\perp - lv'_0 t_1)x + ily - ik_\perp \tilde{\xi}(t_1) - i\tilde{\eta}(t_1)}
\] (43)

and

\[
\tilde{y}(t) = \int_{t_0}^{t} \tilde{\eta}(t_1) dt_1 = \frac{c}{B} \int_{t_0}^{t} dt_1 \frac{\partial \phi}{\partial x} + \int_{t_0}^{t} dt_1 \frac{\partial V_0(x, t_1)}{\partial x} \tilde{x}(t_1)
\]

\[
= i \frac{c}{B} \int dk_\perp \int dl \int_{t_0}^{t} dt_1 \phi(k_\perp, l, 0) (k_\perp - lv'_0(t - t_1))
\]

\[
\times e^{i\omega_d t_1 + i(k_\perp - lv'_0 t_1)x + ily - ik_\perp \tilde{\xi}(t_1) - i\tilde{\eta}(t_1)}
\] (44)

Applying the procedure presented above to the calculations the variants of the plasma displacements \( K_{xx}(t_1, t_2) \) in variables \( x, y \) we find that \( K_{xx}(t_1, t_2) \) has the same form as \( K_{\xi\xi}(t_1, t_2) \) in Eq.(36) in which, however, \( \omega_d \) is changed by \( \omega_d - lv'_0x \), assuming that \( \omega_d - lv'_0x \sim \omega_d \), and \( C(k_\perp, l_1, t) \) is changed by \( C_1(k_\perp, l_1, t, x) \), determined by Eq.(39), in which the replacement \( \omega_d \) by \( \omega_d - lv'_0x \sim \omega_d \) has to be done. The correlation \( K_{yy}(t) \) is

\[
K_{yy}(t) = \frac{c^2}{B^2} \left[ \int dt_1 \int dt_2 \phi(k_\perp, l, 0) \right] \left[ \int d\tau \int d\tilde{\tau} \left| \phi(k_\perp, l, \tilde{\tau}) \right|^2 e^{i(\omega_d(k_\perp, l) - lv'_0x)\tau - C_1|\tau|}
\]

\[
\times \left( k_\perp - lv'_0\left(t - \tilde{\tau} - \frac{\tau}{2}\right) \right) \left( k_\perp - lv'_0\left(t - \tilde{\tau} + \frac{\tau}{2}\right) \right)
\]

\[
\approx \frac{2c^2}{B^2} \int dk_\perp \int dl \int dt_1 \left| \phi(k_\perp, l, \tilde{\tau}) \right|^2 \left( k_\perp - lv'_0(t - \tilde{\tau}) \right)^2 \frac{C_1(k_\perp, l, \tilde{\tau}, x)}{(\omega_d(k_\perp, l) - lv'_0x)^2}.
\] (45)

The calculation of the integral over \( \tilde{\tau} \) needs the knowledge of the \( C_1(\tilde{\tau}) \) dependence. In the case of the stationary perturbations with \( \gamma = 0 \) the integral over \( \tilde{\tau} \) is easily calculated and we obtain for
times $t \gg \tau_{\text{correlation}} \sim (\omega_d - lv'_0 x)^{-1}$ the well known result[22]

$$K_{yy}(t) = \frac{c^2}{B^2} \text{Re} \int dk_\perp \int dl \vert \phi(k_\perp, l, 0) \vert^2 \frac{C_1(k_\perp, l, \hat{t}, x)}{(\omega_d(k_\perp, l) - lv'_0 x)^2} \times \left( \frac{2}{3} (lv'_0)^2 t^3 - 2k_\perp lv'_0 t^2 + 2k_\perp^2 t \right), \quad (46)$$

which displays the effect of the anisotropic dispersion conditioned by flow shear, observed in the laboratory frame of reference - dispersion increases much faster along flow than in the direction of the flow shear.

**VI. Conclusions**

In this paper we present the results of the analytical investigations of the linear and nonlinear evolution of drift modes in shear flows with time dependent velocity shear. The transformation (3) to convective frame of reference gives an opportunity to obtain the exact solution for time dependent flow shear. The investigations demonstrate that there are different time scales in the dynamics of drift waves in shear flows. Right with convective variables $\xi, \eta$ (3) during times, for which $|\alpha(t) v'_0| \rho_\perp \ll 1$ (or, for the stationary velocity shear, when $t \ll (l \rho_\perp v'_0)^{-1}$) the solution (9) of the modal type is obtained. That modal solution already during early times, for which $|\alpha(t) v'_0| > 1$, is observed in the laboratory frame as a sheared mode with time-dependent wave number and frequency. In this time domain in the convective variables the dispersive properties of the plasma displacements in drift turbulence are the same as in plasmas without shear flow. Using the developed two-time scale procedure of the calculation of the dispersion tensor of random displacements of the plasma we obtain the renormalized form (38) of the solution for the electrostatic potential of the unstable drift waves. This solution gives the nonlinear integral balance equation (39), which determines the level of drift turbulence in the steady state, resulted from random turbulent motion of plasma in the unstable drift turbulence. Balance equation (39) governs the process and level of the drift turbulence saturation. It does not include any effects associated with flow shear over times during which the solution (9) is valid.

The effect of the anisotropic dispersion of plasma displacements is presented by Eq.(46). It presents the variance of the plasma displacements resulted from the modes of the ordinary modal form (9), *observed in the laboratory frame*. This effect has nothing in common with effect of the "enhanced suppression of the instability by shear flow". The saturation of the instability, as it was demonstrated above, resulted from the balance (39) of the growth rate and nonlinear damping in the frame of reference, where the solution in the ordinary modal form is determined.

This result may be extended to other fluid models of plasma. The fluid equations, obtained in drift approximation, in which all nonlinearities other than $E \times B$ nonlinearity are ignored, do not include among the nonlinear mechanisms of the instability saturation the process of the enhanced
nonlinear decorrelation by velocity shear. It is important to note, that the same conclusions concerning ”suppression of the instability by the enhanced nonlinear decorrelation by velocity shear” are completely applicable to stratified shear flows of incompressible fluids, where the nonlinear convective derivative
\[
v \cdot \nabla = \frac{\partial \psi}{\partial z} \frac{\partial}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial}{\partial z}
\]  
(47)
presented in terms of the stream function \( \psi = e_y \psi \), with which \( \mathbf{v} = \nabla \times \psi \) and where basic flow along axis \( x \) with velocity gradient along axis \( z \) is assumed. Eq.(47) also conserves its form under transformation (3). The process of the instabilities suppression by the enhanced nonlinear decorrelation by velocity shear is treated appropriately with nonlinear kinetic (gyrokinetic) theory of plasma with shear flow (see, for example, nonlinear gyrokinetic turbulence simulations of \( E \times B \) shear quenching of transport in Refs.[6], [7]; we mention also Ref.[23], [24, 25] as examples of a recent works). We have obtained analytically in Ref.[25] that nonlinear effect of the suppression instability by shear flow deduced straightforwardly in Vlasov-Poisson system of equations for the magnetized plasma as the effect of the enhanced by shear flow nonlinear resonance broadening and it may be dominant in the processes of the instabilities saturation.

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