Asynchronous Distributed Optimization over Lossy Peer-to-Peer Networks via ADMM: Stability and Exponential Convergence

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Abstract

In this work we focus on the problem of minimizing the sum of convex cost functions in a distributed fashion over a peer-to-peer network. In particular, we are interested in the case in which communications between nodes are lossy and the agents are not synchronized among themselves. We address this problem by proposing a modified version of the relaxed ADMM (R-ADMM), which corresponds to the generalized Douglas-Rachford operator applied to the dual of our problem. By exploiting results from operator theory we are then able to prove the almost sure convergence of the proposed algorithm under i.i.d. random packet losses and asynchronous operation of the agents. By further assuming the cost functions to be strongly convex, we are able to prove that the algorithm converges exponentially fast in mean square in a neighborhood of the optimal solution. Moreover, we provide an upper bound to the convergence rate. Finally, we present numerical simulations of the proposed algorithm over random geometric graphs in the aforementioned lossy and asynchronous scenario.

Index Terms
distributed optimization, ADMM, asynchronous update, lossy communications, operator theory, Douglas-Rachford operator

I. INTRODUCTION

From classical control theory to more recent machine learning applications, many problems can be cast as optimization problems [1] and, in particular, as large-scale optimization problems, given the advent of Internet-of-Things and the ever-increasing growth of large-scale cyber-physical systems. Hence, stemming from classical optimization theory, in order to break down the computational complexity, parallel and distributed optimization methods have been the focus of a wide branch of research [2]. Within this vast topic, typical applications foresee computing nodes to cooperate in order to achieve a desired common goal such as

$$\min_x \sum_{i=1}^{N} f_i(x)$$

where, usually, each $f_i$ is owned by one node only.

While parallel optimization methods usually rely on a shared memory architecture to implement the communication among agents, in distributed systems a message passing architecture is employed, in which agents can exchange transmissions with a (subset) of the other agents. The message passing (or peer-to-peer) architecture however introduces some issues due to the implementation of the transmission protocols, which may suffer from faulty and time-varying communications, on top of the possible asynchronism of the agents’ operations.

A popular class of algorithms that have been proposed to solve (1) is that of distributed subgradient methods. These classes can handle non-differentiable convex cost functions, and require only the computation of local (sub-)gradients. However, they exhibit sub-linear converge rates even if the cost functions are smooth [3], [4]. The recent works [5], [6], [7], [8] derived convergence guarantees in the presence of time-varying communications. However, the need for a diminishing stepsize implicitly requires a synchronization mechanism between agents.

Another popular class of distributed optimization algorithms is the one of gradient consensus methods, which consists of strategies that exploit first and second-order derivatives for computing descent directions. For example in [9], [10] the authors apply quasi-Newton distributed descent schemes to general time-varying directed graphs. A different approach, based on computing Newton-Raphson directions through average consensus algorithms, has been proposed in [11]. Even if initially proposed for synchronous implementations, this scheme has been later extended to cope with asynchronous and faulty communication schemes [12]. However, in [12], the convergence is proved only locally and no characterization of the convergence rate is provided.

Finally, dual decomposition schemes have been widely employed to solve distributed problems, and we refer to [13] for a comprehensive tutorial. Among these algorithms, the Alternating Direction Multiplier Method (ADMM) has attracted the attention of the scientific community for its simple distributed implementation and good convergence speed. This algorithm was originally proposed in mid ’70s as a general convex optimization strategy, then exploited in the context of networked optimization [14], and recently popularized by the survey [15]. Substantial research has been dedicated in optimizing the free parameters of ADMM in order to obtain faster convergence rates, but these are mainly restricted to synchronous implementations.

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over undirected communication graphs [16], [17], [18], [19]. Some recent exceptions extend dual decomposition and ADMM to asynchronous scenarios, see [20], [21], [22], [23]. Other works have addressed the problem of random delays in the communication/update rounds in ADMM schemes [24], [25], [23]. However the strategies [24], [23] are restricted to networks to asynchronous scenarios, see [20], [21], [22], [23]. Other works have addressed the problem of random delays in the over undirected communication graphs [16], [17], [18], [19]. Some recent exceptions extend dual decomposition and ADMM nonexpansive operators, that is operators with unitary Lipschitz constant [28]. Given an operator networks, was given in [26], [27], which are preliminary versions of this paper. In the present work we extend them to allow asynchronous operations of the agents, and further study the convergence rate of the proposed algorithm.

To the best of our knowledge, the first explicit focus on the robustness of the ADMM to packet losses, in general peer-to-peer networks, was given in [26], [27], which are preliminary versions of this paper. In the present work we extend them to allow packet losses.

A. Notation and useful definitions

Consider the scalar function \( f : \mathbb{R}^n \to \mathbb{R} \cup \{ +\infty \} \). We say that \( f \) is closed if \( \forall a \in \mathbb{R} \) the set \( \{ x \in \text{dom}(f) \mid f(x) \leq a \} \) is closed. We say that \( f \) is proper if it does not attain \(-\infty\).

A function is said to be convex if \( \forall x, y \in \mathbb{R}^n \) and \( \forall \lambda \in [0, 1] \) it holds \( f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \). Moreover, we say that it is strongly convex if it is twice differentiable, \( f \in \mathcal{C}^2 \), and its Hessian is bounded from below, that is \( \nabla^2 f(x) \succ cI \) for all \( x \) with some positive scalar.

We define the convex conjugate of \( f \) as \( f^*(y) = \sup_{x \in \mathbb{R}^n} \{ \langle y, x \rangle - f(x) \} \).

By operator on \( \mathbb{R}^n \) we mean a mapping \( T : \mathbb{R}^n \to \mathbb{R}^n \) that assigns to each point \( x \in \mathbb{R}^n \) the corresponding point \( Tx \in \mathbb{R}^n \). An operator \( T \) is nonexpansive if for any two \( x, y \in \mathbb{R}^n \) it holds \( \| Tx - Ty \| \leq \| x - y \| \), and averaged if there exist \( \alpha \in \mathbb{R}_+ \) and \( R \) nonexpansive such that \( T = (1-\alpha)I + \alpha R \).

In the following we will use the class of proximal operators, defined for a closed, proper and convex function \( f \) and penalty parameter \( \rho > 0 \) as

\[
\text{prox}_{\rho f}(x) = \arg \min_y \left\{ f(y) + \frac{1}{2\rho} \| y - x \|^2 \right\}.
\]

Notice that proximal operators are nonexpansive.
B. The Relaxed-ADMM algorithm

Consider the following optimization problem, which we assume has a finite optimal solution:

\[
\min_{x \in \mathbb{R}^n, y \in \mathbb{R}^m} \{ f(x) + g(y) \}
\]

\[
s.t. \ Ax + By = c
\]

(2)

with \( f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\} \) and \( g : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+\infty\} \) closed, proper and convex functions.

Let us define the augmented Lagrangian for problem (2) as

\[
\mathcal{L}_\rho(x, y; w) = f(x) + g(y) - w^\top (Ax + By - c) + \frac{\rho}{2} \|Ax + By - c\|^2
\]

(3)

where \( \rho > 0 \) is the penalty parameter and \( w \) is the vector of Lagrange multipliers.

The Relaxed ADMM (R-ADMM) \(^{33}\) consists in alternating the following three steps

\[
y(k+1) = \arg\min_y \{ \mathcal{L}_\rho(x(k), y; w(k)) + \rho(2\alpha - 1) \langle By, (Ax(k) + By(k) - c) \rangle \}
\]

(4)

\[
w(k+1) = w(k) - \rho(Ax(k) + By(k+1) - c) - \rho(2\alpha - 1)(Ax(k) + By(k) - c)
\]

(5)

\[
x(k+1) = \arg\min_x \mathcal{L}_\rho(x, y(k+1); w(k+1)).
\]

(6)

A well known result is that the R-ADMM algorithm can be alternatively derived applying the relaxed Douglas-Rachford splitting (DRS) operator to the dual of problem (2), see \([15]\), \([33]\). The following convergence result applies.

**Proposition 1:** Assume that the functions \( f \) and \( g \) are closed, convex and proper. If \( 0 < \alpha < 1 \) and \( \rho > 0 \), then the iterates of the R-ADMM algorithm converge to the optimal solution of (2). \( \square \)

We conclude this section by observing that setting \( \alpha = 1/2 \) returns the classical ADMM, thoroughly analyzed in \([15]\).

III. DISTRIBUTED CONSENSUS PROBLEM FORMULATION

In this section we introduce the distributed optimization problem of interest.

Consider the undirected, connected graph \( G = (V, E) \) with \( N \) nodes. We are interested in solving the following optimization problem

\[
\min_{x} \sum_{i=1}^{N} f_i(x)
\]

(7)

where \( f_i : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\} \) are closed, proper and convex functions and each \( f_i \) is known only to node \( i \). In the following we denote by \( x^* \) a solution of (7) and assume that such a solution exists and is finite.

Assigning a local copy \( x_i \) of \( x \) to each agent, (7) can be equivalently formulated as

\[
\min_{x_1,\ldots,x_N} \sum_{i=1}^{N} f_i(x_i)
\]

\[
s.t. \ x_i = x_j, \ \forall (i, j) \in E
\]

(8)

By introducing for each edge \((i, j) \in E\) the two bridge variables \( y_{ij} \) and \( y_{ji} \), the constraints in (8) can be further rewritten as

\[
x_i = y_{ij}
\]

\[
x_j = y_{ji}, \ \forall (i, j) \in E.
\]

\[
y_{ij} = y_{ji}
\]

Defining \( x = [x_1^\top, \ldots, x_N^\top]^\top, f(x) = \sum_i f_i(x_i), \) and stacking all bridge variables in \( y \in \mathbb{R}^{2|E|} \), we can reformulate the problem as \(^{4}\)

\[
\min_{x} f(x)
\]

\[
s.t. \ Ax + y = 0
\]

\[
y = Py
\]

(9)

for a suitable \( A \in \mathbb{R}^{2|E| \times nN} \) matrix, and with \( P \in \mathbb{R}^{2|E| \times 2n|E|} \) being a permutation matrix that swaps \( y_{ij} \) with \( y_{ji} \). Notice that \( P \) is symmetric.

\(^{4}\)Hereafter we will use bold letters to denote the vectors and matrices stacking local quantities.
Making use of the indicator function \( \iota_{(1-P)}(y) \) which is equal to 0 if \((I-P)y = 0\), and \(+\infty\) otherwise, we can finally rewrite problem \((8)\) as

\[
\min_{x, y} \left\{ f(x) + \iota_{(1-P)}(y) \right\} \\
\text{s.t. } Ax + y = 0.
\]  

\((10)\)

Clearly \((10)\) conforms to problem \((2)\) and therefore it can be addressed by applying the R-ADMM algorithm.

**IV. R-ADMM for Distributed Optimization**

In this section we employ \((4), (5)\) and \((6)\) to solve problem \((10)\). To do so we introduce the Lagrange multipliers \(w_{ij}\) and \(w_{ji}\) which are associated to the constraints \(x_i = y_{ij}\) and \(x_j = y_{ji}\), respectively. It is possible to show that R-ADMM applied to \((10)\) is amenable of a distributed implementation as algorithmically described in Algorithm \([1]\). More precisely, observe that node \(i\) stores in memory only the variables \(x_i, y_{ij}, w_{ij}, j \in \mathcal{N}_i\), and updates them exchanging information only with its neighbors, i.e., with nodes in \(\mathcal{N}_i\).

**Algorithm 1** Distributed R-ADMM using \((4), (5), (6)\).

**Input**: Set the termination condition \(K > 0\). For each node \(i\), initialize \(x_i(0), \{y_{ij}(0), w_{ij}(0)\}_{j \in \mathcal{N}_i}\).

1: \(k \leftarrow 0\)
2: \textbf{while} \(k < K\) each agent \(i\) \textbf{do}
3: \hspace{1em} broadcast \(x_i(k), y_{ij}(k)\) and \(w_{ij}(k)\) to all neighbors \(j \in \mathcal{N}_i\)
4: \hspace{1em} collect \(\{x_j(k), y_{ji}(k), w_{ji}(k)\}\) received from each neighbor \(j \in \mathcal{N}_i\)
5: \hspace{1em} compute in order
6: \hspace{2em} \(y_{ij}(k+1) = \frac{1}{2\rho} \left[ (w_{ij}(k) + w_{ji}(k)) + 2\alpha_0(x_i(k) + x_j(k)) - \rho(2\alpha - 1)(y_{ij}(k) + y_{ji}(k)) \right] \)
7: \hspace{2em} \(w_{ij}(k+1) = \frac{1}{2} \left[ (w_{ij}(k) - w_{ji}(k)) + 2\alpha_0(x_i(k) - x_j(k)) - \rho(2\alpha - 1)(y_{ij}(k) - y_{ji}(k)) \right] \)
8: \hspace{2em} \(x_i(k+1) = \arg \min_{x_i} \left\{ f_i(x_i) + \frac{\rho}{2} \|x_i\|^2 + \left( \sum_{j \in \mathcal{N}_i} \rho(2\alpha - 1) y_{ji}(k) - 2\alpha_0 x_j(k) - w_{ji}(k) \right)^\top x_i \right\} \)
9: \textbf{end while}

Notice that, to find the solution of \((7)\), we are only interested in the trajectories of the \(x_i\) variables in Algorithm \([1]\) and not in those of the auxiliary variables \(y_{ij}\) and \(w_{ij}\). Moreover, as we mentioned above, the R-ADMM can be derived by applying the relaxed DRS operator to the dual of problem \((10)\). In the next Proposition, based on the previous observations and leveraging the particular structure of problem \((10)\), we propose a simpler, but otherwise equivalent, implementation of the distributed R-ADMM in terms of both memory and communication requirements. This new formulation, which, beside the \(x_i\)’s variables, involves only the auxiliary variables \(z_{ij}, i \in \mathcal{V}, j \in \mathcal{N}_i\), will be instrumental for the derivation of a robust and asynchronous R-ADMM.

**Proposition 2**: The trajectories of the variables \(x_i, i \in \mathcal{V}\), generated by Algorithm \([1]\), starting from a given initial condition \(x(0), y(0), w(0)\), are identical to the trajectories generated by iterating, for \(k \geq 1\), the following two updates

\[
x_i(k) = \arg \min_{x_i} \left\{ f_i(x_i) - \left( \sum_{j \in \mathcal{N}_i} z_{ji}(k) \right) x_i + \frac{\rho}{2} \|x_i\|^2 \right\}
\]  

\((11)\)

for all \(i \in \mathcal{V}\), and

\[
\begin{align*}
z_{ij}(k+1) &= (1 - \alpha) z_{ij}(k) - \alpha z_{ji}(k) + 2\alpha_0 x_i(k) \\
z_{ji}(k+1) &= (1 - \alpha) z_{ji}(k) - \alpha z_{ij}(k) + 2\alpha_0 x_j(k)
\end{align*}
\]  

\((12)\)

for all \((i, j) \in \mathcal{E}\), where the auxiliary variables are initialized as \(z_{ij}(1) = w_{ij}(1) + \rho y_{ij}(1)\).

**Proof**: See Appendix \([A]\)

The previous proposition naturally suggests an alternative distributed implementation of the ADMM Algorithm \([1]\) in which each node \(i\) stores and updates in its local memory the variables \(x_i\) and \(z_{ij}, j \in \mathcal{N}_i\), and then exchanges them with its neighbors. An equivalent implementation, which reduces the number of transmissions required, can be obtained if the node \(i\) stores and updates the variables \(z_{ji}, j \in \mathcal{N}_i\) instead of the variables \(z_{ij}, j \in \mathcal{N}_i\). This implementation is formally described in Algorithm \([2]\).

\(^2\)With the notation of \((2)\), matrix \(B\) is the identity matrix and function \(g\) is the indicator function.
Algorithm 2 Modified distributed R-ADMM.

\textbf{Input:} Set the termination condition $K > 0$. For each node $i$, initialize $x_i(0)$ and $z_{ji}(0)$, $j \in \mathcal{N}_i$.

1: $k \leftarrow 0$
2: \textbf{while} $k < K$ each agent $i$ \textbf{do}
3: \hspace{1em} compute $x_i(k)$ according to (11)
4: \hspace{1em} compute, for $j \in \mathcal{N}_i$, the temporary variable $q_{i \rightarrow j}$ as
5: \hspace{2em} $q_{i \rightarrow j} = -z_{ji}(k) + 2\rho x_i(k)$ \hspace{1.5em} (13)
6: \hspace{1em} transmit, for $j \in \mathcal{N}_i$, $q_{i \rightarrow j}$ to node $j$
7: \hspace{1em} gather $q_{j \rightarrow i}$ from each neighbor $j$
8: \hspace{1em} update the auxiliary variables as
9: \hspace{2em} $z_{ji}(k + 1) = (1 - \alpha)z_{ji}(k) + \alpha q_{j \rightarrow i}$ \hspace{1.5em} (14)
10: \hspace{1em} $k \leftarrow k + 1$
11: \textbf{end while}

Observe that, at the beginning of each iteration node $i$ updates $x_i$ based only on local information according to (11). Then it computes the temporary variable $q_{i \rightarrow j}$ which is sent to neighbor $j$. At the same time it receives the quantity $q_{j \rightarrow i}$ from neighbor $j$ and it uses this information to update $z_{ji}$ as in (14).

\textbf{Remark 1:} As we can see, both Algorithms 1 and 2 need a single round of transmissions at each time $k$. However, they differ for the number of variables that each node has to transmit and update. The comparison between the two algorithms is reported in Table I.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
& Alg. 1 & Alg. 2 \\
\hline
\text{Store and Update} & $2|\mathcal{N}_i| + 1$ & $|\mathcal{N}_i| + 1$ \\
\hline
\text{Send} & $2|\mathcal{N}_i| + 1$ & $|\mathcal{N}_i|$ \\
\hline
\text{Temporary} & $3|\mathcal{N}_i|$ & $|\mathcal{N}_i|$ \\
\hline
\end{tabular}
\caption{Comparison of R-ADMM implementations.}
\end{table}

\textbf{Remark 2:} The R-ADMM formulation presented in Algorithms 1 and 2 is edge-based in the sense that there is a local auxiliary variable associated with each neighbor of a node $i$. Therefore the memory requirement scales in the worst case as $O(N)$. On the other hand, node-based implementations of the ADMM have been proposed [35], [36], for which the memory requirement scales as $O(1)$.

We observe that the distributed R-ADMM presented in this Section can be reformulated as a node-based algorithm if the nodes store and update, alongside $x_i$, the auxiliary variables $u_i = \sum_{j \in \mathcal{N}_i} z_{ij}$ and $v_i = \sum_{j \in \mathcal{N}_i} z_{ji}$ instead of $z_{ji}$, $j \in \mathcal{N}_i$. However, this modified version is not suitable for handling packet losses and asynchronism.

\textbf{Remark 3:} The formulation of the R-ADMM presented in Algorithm 2 is derived using the same idea employed in [25] of interpreting the R-ADMM as an application of the DRS to the dual problem. Also the R-ADMM algorithm proposed in [25] to solve problem in (1) (see Section 2.6.2), involves only the use of variables $x_i$, $z_{ij}$ but the actual implementation differs from Algorithm 2. Additionally, it is worth mentioning that the authors of [25] have derived the R-ADMM within the framework of the ARock algorithm, introduced in the context of parallel computing where agents share a common memory. Interestingly, it is shown that the ARock framework successfully handles asynchronous updates and delayed information. However, due to the reliance of the convergence proof on the common memory, ARock is not suitable to deal with arbitrary packet losses, which, instead, is the focus of the second part of the present paper.

Clearly, the equivalence between Algorithm 2 and Algorithm 1 and thus with the R-ADMM reviewed in Section II-B allows us to state the following convergence result.

\textbf{Corollary 1:} Let $(\alpha, \rho)$ be such that $0 < \alpha < 1$ and $\rho > 0$. Then, for any initial conditions, the trajectories $k \rightarrow x_i(k)$, $i \in \mathcal{V}$, generated by Algorithm 2 converge to the optimal solution of (7), i.e.,
\[
\lim_{k \rightarrow \infty} x_i(k) = x^*, \quad \forall i \in \mathcal{V},
\]

for any $x_i(0)$ and $z_{ji}(0)$, $j \in \mathcal{N}_i$.

\section{Locally Exponential Convergence}

In this section we prove the \textit{locally exponential} convergence of the distributed R-ADMM illustrated in Algorithm 2 under the assumption that the local costs are strongly convex.
We first derive a formulation of the update equations (11) and (12) as affine functions perturbed by locally vanishing functions; this derivation is inspired by the approach adopted in [19].

Thanks to the strong convexity assumption, \( f_i \) is twice differentiable and hence it is possible to write the first order optimality condition for the minimization problem (11) as

\[
\nabla f_i(x_i(k)) + \left[ A^T \right]_i P z(k) + \rho |N_i| x_i(k) = 0
\]

where matrices \( A \) and \( P \) have been introduced in (9), \( [A^T]_i \) denotes the \( i \)-th row of matrix \( A^T \) and \( z \in \mathbb{R}^{2n_i|E|} \) is the vector obtained stacking together the variables \( z_{ij}, i \in V, j \in N_i \). Moreover, when \( x_i(k) \) is sufficiently close to the (unique) optimum \( x^* \) the first order Taylor expansion for the gradient of \( f_i \) can be computed as

\[

\nabla f_i(x_i(k)) = \nabla f_i(x^*) + \nabla^2 f_i(x^*)(x_i(k) - x^*) + r_i(x_i(k) - x^*)
\]

where \( \|r_i(x)\|/\|x\| \to 0 \) when \( x \to 0 \). Now, combining equations (15) and (16) we get

\[
x_i(k) = -H_i^{-1}[A^T]_i P z(k) - H_i^{-1}(r_i(x_i(k) - x^*) + b_i)
\]

where \( H_i = \nabla^2 f_i(x^*) + \rho |N_i| I \) is invertible and \( b_i = \nabla f_i(x^*) - \nabla^2 f_i(x^*)x^* \). In compact form for the entire vector \( x(k) \), we can write

\[
x(k) = -H^{-1}A^T P z(k) - H^{-1}(r(x(k) - x^*) + \mathbf{b})
\]

where \( H = \text{diag}\{H_1, \ldots, H_N\}, \mathbf{b} = [b_1^T \cdots b_N^T]^T, r(x) = [r_1(x_1)^T \cdots r_N(x_N)^T]^T, \) and \( x^* = [(x^*_1)^T \cdots (x^*_N)^T]^T \). From (12) we have

\[
z(k+1) = ((1-\alpha)I - \alpha P)z(k) - 2\alpha \rho A x(k)
\]

where \( Q = (1-\alpha)I - \alpha P + 2\alpha \rho A H^{-1} A^T P, F = 2\alpha \rho A H^{-1} \) and \( c = -2\alpha \rho A H^{-1} b \).

The following Lemma characterizes the spectral properties of the matrix \( Q \).

**Lemma 1:** Let \( 0 < \alpha < 1 \) and \( \rho > 0 \). Let \( \gamma \) be an eigenvalue of \( Q \). Then, either \( \gamma = 1 \) or \( |\gamma| < 1 \). Moreover, if \( m \) is the algebraic multiplicity of the eigenvalue 1, then the following two properties hold true:

1) the geometric multiplicity of 1 is \( m \), that is,

\[
\dim(\text{Ker}(Q - I)) = m;
\]

2) if the vectors \( v_1, \ldots, v_m \) are such that

\[
\langle v_1, \ldots, v_m \rangle = \text{Ker}(Q - I),
\]

then

\[
A^T P v_i = 0, \quad \text{for } i = 1, \ldots, m.
\]

**Proof:** See Appendix [B]

Let now \( \Gamma(Q) \) be the spectrum of the matrix \( Q \) and let

\[
\gamma_M = \max \{ |\gamma| : \gamma \in \Gamma(Q), \gamma \neq 1 \}
\]

Observe that, from Lemma 1, it follows \( \gamma_M < 1 \). We have the following result.

**Proposition 3:** Assume that the cost functions \( f_i \) are strongly convex. Then, there exists a neighborhood \( B_{x^*} \) of the optimal point \( x^* \) such that, if \( x(0) \in B_{x^*} \), then Algorithm 2 converges exponentially fast, i.e.,

\[
\|x_i(k) - x^*\| \leq C \gamma^k \|x_i(0) - x^*\|, \quad \forall i \in V,
\]

for suitable constants \( C > 0 \) and \( 0 \leq \gamma \leq \gamma_M \).

**Proof:** See Appendix [B]

**Remark 4:** Notice that the matrix \( Q \) depends both on the topology of the communication graph, via matrices \( A \) and \( P \), and on the curvature of the cost functions \( f_i \), via the matrix of Hessians \( H \). This result is different from the case of Newton algorithms whose convergence rate depends only on the structure of the graph, see [12].
VI. ASYNCHRONOUS DISTRIBUTED R-ADMM OVER LOSSY NETWORKS

The distributed algorithms illustrated in the previous section work under the standing assumption that the communication channels are reliable, that is, no packet losses occur, and that the nodes update all at the same time, i.e., they are synchronized with each other. The goal of this section is to relax these requirements and, in particular, to show how Algorithm 2 can be modified to still guarantee convergence, under probabilistic assumptions on communication failures and asynchronous updates.

A. Robust and Asynchronous R-ADMM

The following assumption will stand throughout the remainder of the paper.

Assumption 1: During any iteration of Algorithm 2 a node $i$ performs an update with probability $q$, and a packet that is sent from the node to one of its neighbors is lost with some probability $p$. In addition, an update or packet loss event is independent of any other update and loss events occurring at any $k \in \mathbb{N}$.

Consider now Algorithm 2 and notice that node $j$ updates the variable $z_{ij}$ if and only if it receives the information $q_{i \rightarrow j}$ from node $i$. For this to happen, two events need to take place: node $i$ performs an update, which happens with probability $q$; and the packet transmitted from $i$ to $j$ is not lost, which happens with probability $1-p$. To provide a formal description of this probabilistic framework, we introduce the set of random variables $\beta_{i \rightarrow j}(k)$, $k = 0, 1, 2, \ldots$, $i \in \mathcal{V}$, $j \in \mathcal{N}_i$, such that $\beta_{i \rightarrow j}(k) = 1$ if node $i$ updates at iteration $k$ and the packet $q_{i \rightarrow j}$ is received by node $j$, $\beta_{i \rightarrow j}(k) = 0$ otherwise. Under Assumption 1 by independence of the update and packet loss events it holds

$$\mathbb{P}[^{\beta_{i \rightarrow j}(k) = 1} = (1-p) =: s. $$

In this scenario, Algorithm 2 is modified as shown in Algorithm 3. We refer to the robust and asynchronous ADMM as raR-ADMM.

Algorithm 3 Robust and asynchronous distributed R-ADMM.

**Input:** Set the termination condition $K > 0$. For each node $i$, initialize $x_i(0)$ and $z_{ji}(0)$, $j \in \mathcal{N}_i$.

1: $k \leftarrow 0$
2: while $k < K$ every agent $i$ do
3: \hspace{1em} if scheduled to update then
4: \hspace{2em} compute $x_i(k)$ according to (11)
5: \hspace{2em} for all $j \in \mathcal{N}_i$, compute the quantity $q_{i \rightarrow j}$ as
6: \hspace{2em} \hspace{1em} $q_{i \rightarrow j} = -z_{ji}(k) + 2\rho x_i(k)$ \hspace{1em} (20)
7: \hspace{2em} end if
8: \hspace{1em} for $j \in \mathcal{N}_i$ if $q_{j \rightarrow i}$ was received do
9: \hspace{2em} \hspace{1em} update $z_{ji}$ as
10: \hspace{2em} \hspace{2em} $z_{ji}(k+1) = (1-\alpha)z_{ji}(k) + \alpha q_{j \rightarrow i}$ \hspace{1em} (21)
11: \hspace{1em} end for
12: \hspace{1em} $k \leftarrow k + 1$
13: end while

In the modified algorithm, at the $k$-th iteration node $i$ updates $x_i$ according to Eq. (11) if it scheduled to do so. Then, for $j \in \mathcal{N}_i$, it computes $q_{i \rightarrow j}$ as in (20) and transmits it to node $j$. If node $j$ receives $q_{i \rightarrow j}$, then it updates $z_{ij}$ as $z_{ij}(k+1) = (1-\alpha)z_{ij}(k) + \alpha q_{i \rightarrow j}$, otherwise $z_{ij}$ remains unchanged, i.e., $z_{ij}(k+1) = z_{ij}(k)$. This last step can be compactly described as

$$z_{ij}(k+1) = (1-\beta_{i \rightarrow j}(k)) z_{ij}(k) + \beta_{i \rightarrow j}(k) ((1-\alpha)z_{ij}(k) + \alpha q_{i \rightarrow j}).$$ \hspace{1em} (22)

B. Almost-sure convergence

The following Proposition characterizes the convergence properties of Algorithm 3.

Proposition 4: Consider the raR-ADMM described in Algorithm 3 and suppose that Assumption 1 holds true. Let $(\alpha, \rho)$ be such that $0 < \alpha < 1$ and $\rho > 0$. Then, for any initial conditions, the trajectories $k \mapsto x_i(k)$, $i \in \mathcal{V}$, generated by Algorithm 3 converge almost surely to the optimal solution of (7), i.e.,

$$\mathbb{P} \left[ \lim_{k \to \infty} x_i(k) = x^* \right] = 1, \quad \forall i \in \mathcal{V},$$

for any $x_i(0)$ and $z_{ji}(0)$, $j \in \mathcal{N}_i$. \hspace{1em} $\square$
Notice that, while Algorithms 1 and 2 are characterized by the same convergent behavior in the synchronous and lossless scenario, we were able to prove convergence in the asynchronous and lossy case only for the latter. In particular, recall that we derived Algorithm 2 as the application of the DRS operator to the dual of problem (10). Similarly, by casting Algorithm 3 in the framework of operator theory, we are able to exploit recent results on randomized nonexpansive operators to establish the convergence of the rR-ADMM algorithm.

Remark 5: Observe that both Proposition 1, for the case of reliable communications, and Proposition 4, for the randomized updating scenario, establish convergence provided that $0 < \alpha < 1$ and $\rho > 0$. However, these conditions are only sufficient and not necessary and, in particular, the convergence might hold also for values of $\alpha \geq 1$. Indeed, in the simulation Section VII we show that, for the case of quadratic functions $f_i$, $i \in \mathcal{V}$, the region of attraction in parameter space is larger. Moreover, despite what the intuition would suggest, the larger the packet loss probability $p$ (or, the smaller the node update probability $q$), the larger the region of convergence. However, this increased region of stability is counterbalanced by a slower convergence rate of the algorithm.

Remark 6: The R-ADMM algorithm proposed in [25] allows only a single agent to update at each iteration, while the proposed Algorithm 3 is fully parallel, i.e. guarantees convergence when an arbitrary number of agents updates simultaneously.

C. Mean Square Exponential Convergence

The results stated in Proposition 4 can be refined in case the functions $f_i$ are strongly convex; indeed, under this stronger assumption, it is possible to show that the mean-square convergence is locally exponential.

To do so, let us introduce the random diagonal matrix $B(k) \in \mathbb{R}^{2n|\mathcal{E}| \times 2n|\mathcal{E}|}$ such that

$$B(k) = \text{diag}\{\beta_{i,j}(k)I_n, \ i \in \mathcal{V}, \ j \in \mathcal{N}_i\}, \quad (23)$$

and observe that, plugging the update (18) into the randomized update (22) it is possible to derive

$$z(k+1) = (I - B(k))z(k) + B(k)(Qz(k) - F r(x(k) - x^*) + c] = G(k)z(k) + B(k)[ -F r(x(k) - x^*) + c] \quad (24)$$

where $G(k) = I - B(k) + B(k)Q$.

We are now interested in evaluating the behavior of the mean square error $\mathbb{E}[\|x(k) - x^*\|^2]$ as $k \to \infty$. In particular, we show that it converges to zero exponentially fast also providing an upper bound to the rate of convergence.

We start observing that, from Lemma 3 reported in Appendix B it follows

$$\|x(k) - x^*\|^2 \leq \|A^T P (z(k) - z^*)\|^2.$$ 

Then, by iterating (24) and taking the expected value we get the following inequality

$$\mathbb{E}\left[\|x(k) - x^*\|^2\right] \leq \mathbb{E}\left[\|A^T P \mathcal{G}(k;0)(z(0) - z^*)\|^2\right] + \sum_{\ell=0}^{k-1} \mathbb{E}\left[\|A^T P \mathcal{G}(k;\ell+1)B(\ell)Fr(x(\ell) - x^*)\|^2\right] \quad (25)$$

where

$$\mathcal{G}(k;\ell) = \begin{cases} G(k-1) \cdots G(\ell) & \text{if } \ell \leq k-1, \\ I & \text{if } \ell > k-1. \end{cases}$$

The following Lemma outlines the properties of $\mathcal{G}(k;0)$ when $k \to \infty$.

Lemma 2: Let $v_1, \ldots, v_m$ be defined as in Lemma 1. Then, it holds

$$\mathcal{G}(\infty;0) = \lim_{k \to \infty} \mathcal{G}(k;0) = v_1\epsilon_1^\top + \ldots + v_m\epsilon_m^\top \quad (26)$$

where $\epsilon_1, \ldots, \epsilon_m$ are random variables taking values in $\mathbb{R}^{2n|\mathcal{E}|}$.

Proof: See Appendix C.

Consider now the first term in the right-hand side of (25), or equivalently, the term

$$(z(0) - z^*)^\top \mathbb{E}\left[\mathcal{G}^\top(k;0)PAA^T P \mathcal{G}(k;0)\right](z(0) - z^*). \quad (27)$$

Next we show that this term converges exponentially to zero as $k \to \infty$. Notice that (27) can be rewritten as

$$(z(0) - z^*)^\top \Delta(k)(z(0) - z^*),$$

where

$$\Delta(k) = \mathbb{E}\left[\mathcal{G}^\top(k;0)PAA^T P \mathcal{G}(k;0)\right].$$

Proof: See Appendix C.
if $k \geq 1$ and where $\Delta(0) = PAA^T P$. A simple recursive argument shows that

$$\Delta(k+1) = E\left[ G^T(0) \Delta(k) G(0) \right] = \mathcal{L}(\Delta(k)).$$

This shows that $\Delta(k)$ is the evolution of a linear dynamical system which can be written in the form

$$\Delta(k+1) = \mathcal{L}(\Delta(k))$$

where $\mathcal{L} : \mathbb{R}^{N \times N} \to \mathbb{R}^{N \times N}$ is given by

$$\mathcal{L}(M) = E\left[ G^T(0) MG(0) \right].$$

By iterating it is clear that $\Delta(k) = L^k(\Delta(0))$ and by using Lemma 1 combined with Lemma 2 we obtain

$$\lim_{k \to \infty} L^k(\Delta(0)) = E\left[ \sum_{i=1}^{m} \sum_{j=1}^{m} \epsilon_i v_i^T \Delta(0) v_j^T \right] = 0.$$

Now let $R$ be the convergence rate to zero of $\Delta(k)$. If we consider the reachable subspace $\mathcal{R}$ of the pair $(\mathcal{L}, P^T A A^T P)$, namely the smallest $\mathcal{L}$-invariant subspace of $\mathbb{R}^{N \times N}$ containing $P^T A A^T P$, then we have that $R = sr(\mathcal{L}|_\mathcal{R})$, where sr denotes the spectral radius. Next we characterize the eigenspace of $\mathcal{L}$ relative to the eigenvalue 1.

**Proposition 5:** Consider the rAR-ADMM described in Algorithm 3. Suppose Assumption 1 holds true. Moreover let $\epsilon_1, \ldots, \epsilon_m$ be the random vectors such that (26) holds. Then the eigenspace of $\mathcal{L}$ relative to 1 is $m^*$ dimensional and it is generated by $E[\epsilon_i \epsilon_j^T], i, j = 1, \ldots, m$. □

**Proof:** See Appendix C.

Clearly the reachability subspace $\mathcal{R}$ will be contained in the subspace generated by the remaining eigenvectors; this implies that $R$ will be upper-bounded by the second dominant eigenvalue of $\mathcal{L}$. Precisely, let $\Gamma(\mathcal{L})$ be the spectrum of the operator $\mathcal{L}$ and let

$$\bar{\gamma}_M = \max \{ |\gamma| : \gamma \in \Gamma(\mathcal{L}), \gamma \neq 1 \}$$

then $R \leq \bar{\gamma}_M < 1$.

The following result establishes that, locally, $E[\|x(k) - x^*\|^2]$ converges exponentially fast to zero and that $R$ and, in turn, $\bar{\gamma}_M$ are suitable upper-bounds for the convergence rate.

**Proposition 6:** Assume functions $f_i$ are strongly convex. Then, there exists a neighborhood $B_{x^*}$ of the optimal point $x^*$ such that, if $x(0) \in B_{x^*}$, then Algorithm 3 converges exponentially fast – in mean-square sense – to the optimal solution, i.e.,

$$E\left[ \|x_i(k) - x^*\|^2 \right] \leq C \bar{\gamma}^k \|x_i(0) - x^*\|^2, \quad \forall i \in \mathcal{V},$$

for suitable constants $C > 0$ and $0 \leq \bar{\gamma} \leq R \leq \bar{\gamma}_M$. □

**Proof:** See Appendix C.

### D. Computing the upper-bound $\bar{\gamma}_M$

Next, we provide a numerical way to compute $\bar{\gamma}_M$. To do so, it is useful to introduce a matrix representation of the linear operator $\mathcal{L}$. In general, given a matrix $M \in \mathbb{R}^{N \times N}$, we define vect($M$) to be the $N^2$ column vector having $M_{i,j}$ in position $(i - 1)N + j$. It is known that vect($XYZ$) = $(Z^T \otimes X)$vect($Y$), where $\otimes$ is the Kronecker product of matrices. Using the properties of the Kronecker product we can derive that the linear operator $\mathcal{L}$ is described by the matrix

$$\mathbf{L} := E\left[ G^T(0) \otimes G^T(0) \right]$$

Indeed, if we vectorize the matrix $\Delta(k)$, it holds

$$\text{vect}(\Delta(k+1)) = E\left[ G^T(0) \otimes G^T(0) \right] \text{vect}(\Delta(k)) = \mathbf{L} \text{vect}(\Delta(k)).$$

Let $\Gamma(\mathbf{L})$ be the spectrum of the matrix $\mathbf{L}$, then

$$\bar{\gamma}_M = \max \{ |\gamma| : \gamma \in \Gamma(\mathbf{L}), \gamma \neq 1 \}.$$

The next Proposition gives an explicit formula for computing the matrix $\mathbf{L}$ which can be then exploited to provide an upper bound to the convergence rate by evaluating its spectrum. Before stating the Proposition, we introduce the following matrix

$$\mathbf{S} = E[\mathbf{B}(0) \otimes \mathbf{B}(0)].$$

(29)
Observe that, from the definition of the matrix $B$ given in (23), it turns out that $S$ is a diagonal matrix whose diagonal elements are of the form $E[\beta_{i\rightarrow j}(0)\beta_{l\rightarrow m}(0)]$ and, more specifically,

\[
E[\beta_{i\rightarrow j}(0)\beta_{l\rightarrow m}(0)] = \begin{cases} 
  s^2 & \text{if } i \neq l \\
  s^2/q & \text{if } i = l, j \neq m \\
  s & \text{if } i = l, j = m.
\end{cases}
\]

We have the following result.

Proposition 7: The matrix $L = E[G^T(0) \otimes G^T(0)]$ is given by

\[
L = (1 - 2s)I \otimes I + sI \otimes Q^T + S^T - (I \otimes Q^T)S^T + sQ^T \otimes I - (Q^T \otimes I)S^T + (Q^T \otimes Q^T)S^T,
\]

where $S$ is as in (29). □

Proof: See Appendix C.

Remark 7: The derivation employed in this section extends the results reported in [37] which were tailored to randomized consensus problem.

Remark 8: Note that Propositions 3, 6 and 7 hold globally in the case of quadratic cost functions. This is a consequence of the fact that the update equation (18) holds with $r(x) = 0$ for any $x$. □

VII. SIMULATIONS

In this section we discuss some numerical results obtained applying Algorithm 3 to solve distributed consensus optimization problem (7) in lossy and asynchronous scenarios. In order to simplify the numerical analysis, we restrict to the case of quadratic cost functions of the form

\[
f_i(x_i) = a_i x_i^2 + b_i x_i + c_i
\]

where, in general, the quantities $a_i, b_i, c_i \in \mathbb{R}$ are different for each node $i$. In this case the update of the primal variables becomes linear and, in particular, Eq. (11) reduces to

\[
x_i(k) = \sum_{j \in N_i} z_{ji}(k) - b_i \over 2a_i + \rho|N_i|.
\]

Moreover, we consider the case of random geometric graphs with communication radius $r = 0.1$[p.u.] in which two nodes are connected if and only if their relative distance is less that $r$. Figure 1 shows an example of such graphs.

First of all, in order to evaluate the effect of packet losses on the algorithm, Figure 2 shows the evolution of the relative error $\log \frac{\|x(k) - x^*\|}{\|x^*\|}$ computed with respect to the unique minimizer $x^*$ and averaged over 100 Monte Carlo runs, with different values of packet loss probability $p$ and for fixed values of step size $\alpha = 0.75$, penalty $\rho = 3$ and update probability $q = 0.8$. As expected, the higher the packet loss probability, the smaller the rate of convergence.

Figure 3 represents instead the stability boundaries of the algorithm as a function of the step-size $\alpha$ and penalty $\rho$, for different packet losses and with synchronous updates. In particular, each curve in Figure 3 represents the numerical boundary below which the algorithm is found to be convergent and above which it diverges.

As predicted by Proposition 4, the convergence is guaranteed for all values of the penalty $\rho$, but the algorithm is seen to converge even for values of the step size $\alpha$ larger than one. Even more interesting is the fact that the stability boundary enlarges as the packet loss probability increases, a phenomenon that, however, is compensated by a lower convergence rate, as shown by Figure 2.
The raR-ADMM has been also tested for different values of the step-size $\alpha$ and with fixed penalty $\rho = 3$, packet loss probability $p = 0.4$ and update probability $q = 0.6$, and the relative error behavior is depicted in Figure 4. Notice that for $\alpha = 1/2$, which corresponds to the case of the classical ADMM, the convergence speed is slower than that for the cases of $\alpha = 0.75$ and $\alpha = 0.8$. Therefore it is possible to exploit the relaxation of the ADMM in order to achieve faster convergence rates, which motivates the choice of using the slightly more complex raR-ADMM over the simpler classic ADMM.

Finally, in Figure 5 we compare, for different values of $\alpha$ and $\rho$, the upper-bound $\bar{\gamma}_M$ with the empirical convergence rate of the raR-ADMM algorithm obtained averaging over 100 Monte Carlo trials. In particular, the values of $\bar{\gamma}_M$ are plotted in the transparent surface on top. Notice that in particular for smaller values of the parameter $\rho$ the upper bound is not tight, suggesting that further work could be done to improve the tightness of the bound.

VIII. CONCLUSIONS AND FUTURE DIRECTIONS

In this paper we addressed distributed convex optimization problems over a peer-to-peer network with both unreliable communications and asynchronous updates of the nodes. We proposed a modified version of the relaxed ADMM that, exploiting operator theoretical results, can be shown to converge almost surely under the same assumptions as the lossless and synchronous

\[ \alpha = 0.75, \rho = 3, P[\text{update}] = 0.8 \]

Fig. 2: Evolution, in log-scale, of the relative error of Alg. 3 computed w.r.t. the unique optimal solution $x^*$ as function of different values of packet loss probability $p$, for step size $\alpha = 0.75$, penalty $\rho = 3$ and update probability $q = 0.8$. Average over 100 Monte Carlo runs for the case of cycle graphs.

Fig. 3: Evolution, in log-scale, of the relative error of Alg. 3 computed w.r.t. the unique optimal solution $x^*$ as function of different values of packet loss probability $p$ for step size $\alpha = 0.75$ and penalty $\rho = 3$. Average over 100 Monte Carlo runs for the case of cycle graphs.
Fig. 4: Evolution, in log-scale, of the relative error of Alg. 3 computed w.r.t. the unique optimal solution $x^*$ as function of different values of the step-size $\alpha$, for penalty $\rho = 3$, packet loss probability $p = 0.4$ and update probability $q = 0.6$. Average over 100 Monte Carlo runs for the case of cycle graphs.

Fig. 5: Empirical and theoretical convergence rates as a function of the tunable parameters $\alpha$ and $\rho$, averaged over 100 Monte Carlo iterations, with packet loss $p = 0.4$ and update probability $q = 0.6$.

version.
Moreover, by further assuming the local costs to be strongly convex, we proved local exponential mean-square convergence of the proposed algorithm.
Finally, numerical results complement the theoretical analysis and show the performance of the algorithm.
APPENDIX A

PROOF OF PROPOSITION 2

The following derivation of the ADMM is standard, however we report it in its entirety because it is instrumental for proving the equivalence of the formulation in Algorithm 1 and Equations (11)-(12).

We start by showing how the updates (11)-(12) can be derived applying the Douglas-Rachford splitting operator to the dual of the distributed problem (10). The following derivation resembles the one proposed in [25].

The dual of (10) is defined as

$$\min_{w} \{d_f(w) + d_g(w)\}$$

(32)

with

$$d_f(w) = f^*(A^Tw), \quad \text{and} \quad d_g(w) = \iota_{(1-P)}(w).$$

Applying the relaxed Douglas-Rachford splitting to (32) consists in iterating the following equations

$$w(k) = \prox_{\rho d_y}(z(k))$$

(33)

$$\xi(k) = \prox_{\rho d_y}(2w(k) - z(k))$$

(34)

$$z(k + 1) = z(k) + 2\alpha(\xi(k) - w(k)),$$

(35)

where $\xi$ and $z$ are two auxiliary variables. In particular we can compute (33) by performing the following two updates

$$y(k) = \arg\min_y \left\{ \iota_{(1-P)}(y) - z^T(k)y + \frac{\rho}{2}\|y\|^2 \right\}$$

(36)

$$w(k) = z(k) - \rho y(k).$$

(37)

This fact can be shown by observing that

$$\prox_{\rho d_y}(z) = \arg\min_u \left\{ \iota_{(1-P)}(u) + \frac{1}{2\rho}\|u - z\|^2 \right\}$$

where the minimum on the right-hand side can be computed as

$$\min_u \left\{ \iota_{(1-P)}(u) + \frac{1}{2\rho}\|u - z\|^2 \right\} = \max_v \left\{ \min_u \left\{ u^T u + \frac{1}{2\rho}\|u - y\|^2 \right\} - \iota_{(1-P)}(v) \right\},$$

where this equality follows from standard algebraic manipulations. The solution to $\min_u \{v^T u + \frac{1}{2\rho}\|u - y\|^2\}$ is $u^* = z - \rho v$, therefore it follows that

$$\min_u \left\{ \iota_{(1-P)}(u) + \frac{1}{2\rho}\|u - z\|^2 \right\} = -\min_v \left\{ \iota_{(1-P)}(v) - z^T v + \frac{\rho}{2}\|v\|^2 \right\}$$

and by the definition of the proximal operator we can derive (36)-(37). A similar derivation can be applied to the update (34) to prove that it is equivalent to

$$x(k) = \arg\min_x \left\{ f(x) - (2w(k) - z(k))^T A x + \frac{\rho}{2}\|A x\|^2 \right\}$$

(38)

$$\xi(k) = 2w(k) - z(k) - \rho A x(k).$$

(39)

Therefore the ADMM is characterized by the equations (36)-(37), (38)-(39) and (35).

The next goal is to show that the formulation of Algorithm 1 is equivalent to the formulation we derived above. We show this fact by proving that the updates derived above are equivalent to the equations (4)-(6), which particularized for the distributed problem (10) give Algorithm 1.

By rewriting (37) as $z(k) = w(k) + \rho y(k)$ and substituting it into (39) we obtain

$$\xi(k) = w(k) - \rho(A x(k) + y(k)).$$

(40)

Rearranging (39) as $2w(k) - z(k) = \xi(k) + \rho A x(k)$ and substituting (40) into it yields $2w(k) - z(k) = w(k) - \rho y(k)$.

Plugging this result into (38) gives (6):

$$x(k) = \arg\min_x \left\{ f(x) - (w(k) - \rho y(k))^T A x + \frac{\rho}{2}\|A x\|^2 \right\}$$

$$= \arg\min_x \left\{ f(x) + \iota_{(1-P)}(y(k)) - w^T(k)(A x + y(k)) + \frac{\rho}{2}\|A x + y(k)\|^2 \right\}$$

$$= \arg\min_x \mathcal{L}(x, y(k); w(k))$$

where the second equality was derived adding the following terms independent of $x$: $\iota_{(1-P)}(y(k))$, $w(k)^T y(k)$ and $(\rho/2)\|y(k)\|^2$; and the last using the definition of augmented Lagrangian (3)
Using (41) into (36) finally we derive (4):

\[ z(k+1) = z(k) - 2\alpha\rho(Ax(k) + y(k)) \]
\[ = w(k) + \rho y(k) - 2\alpha\rho(Ax(k) + y(k)) \]
\[ = w(k) - \rho(2\alpha - 1)(Ax(k) + y(k)) - \rho A x(k) \]

(41)

where the second equality was derived using \( z(k) = w(k) + \rho y(k) \) and the third by adding and subtracting \( \rho A x(k) \).

Evaluating now (37) at time \( k + 1 \) we obtain \( w(k + 1) = z(k + 1) - y(k + 1) \), and combining it with (41) we obtain (5):

\[ w(k + 1) = w(k) - \rho(Ax(k) + y(k + 1)) - \rho(2\alpha - 1)(Ax(k) + y(k)). \]

Using (41) into (36) finally we derive (4):

\[ y(k + 1) = \arg \min_{y} \left\{ f(x(k)) + \epsilon(I - P)y - w(k)^T(Ax(k) + y) + \frac{\rho}{2} \|Ax(k) + y\|^2 + \rho(2\alpha - 1)y^T(Ax(k) + y) \right\} \]

where we added the terms – independent on \( y - f(x(k)), -w(k)^T A x(k) \) and \( \rho \|A x(k)\|^2 \).

Now that we have shown the equivalence of the two ADMM formulations (36)–(37), (38)–(39) and (35) and (41)–(6), we particularize the former to the distributed problem of interest, which yields (11)–(12); a similar derivation can be done to compute Algorithm 2.

The KKT conditions for (36) are

\[ (I - P)y = 0 \quad \text{and} \quad -z(k) + \rho y + (I - P)\nu = 0, \]

with \( \nu \) the vector of Lagrange multipliers. Rearranging the second yields \( y = (1/\rho)(z(k) - (I - P)\nu) \) and substituting it into the right-hand side of \( y = Py \) gives

\[ y = P\frac{1}{\rho}(z(k) - (I + P)\nu) = \frac{1}{\rho}(Pz(k) + (I - P)\nu) \]  

(42)

since by the properties of \( P \) it holds \( P(I - P) = -(I - P) \). Finally, summing \( y = (1/\rho)(z(k) - (I - P)\nu) \) with (42) yields

\[ y(k) = \frac{1}{2\rho}(I + P)z(k). \]  

(43)

Substituting (43) into (37) and the resulting equation for \( w(k) \) into (39) gives now

\[ w(k) = \frac{1}{2}(I - P)z(k) \]

\[ \xi(k) = -Pz(k) - \rho A x(k) \]

and thus (38) becomes

\[ x(k) = \arg \min_{x} \left\{ f(x) + (Pz(k))^T Ax + \frac{\rho}{2} \|Ax\|^2 \right\} \]

(44)

and equation (35)

\[ z(k + 1) = (1 - \alpha)z(k) - \alpha Pz(k) - 2\alpha\rho A x(k). \]

(45)

Consider the inner product \((Pz(k))^T Ax\), exploiting the particular structure of the two matrices we can compute \((Pz(k))^T Ax = -\sum_{i=1}^{N} \left( \sum_{j \in N_i} z^i_j(k) \right) x_i \). Moreover, since each primal variable \( x_i \) appears in exactly \( |N_i| \) constraints, we have that \( \|Ax\|^2 = \sum_{i=1}^{N} |N_i| \|x_i\|^2 \). Substituting these results into (44) gives (11), while using the structure of \( A \) and \( P \) in (45) yields (12). Finally, with the derivation above we have shown that the variables \( x \) in both algorithms are equivalent. Thus if they start from the same initial condition, the trajectories \( k \mapsto x(k) \) that they generate are equal; this can be ensured by requiring that (37) hold at time \( k = 1 \), that is \( z(1) = w(1) + \rho y(1) \).

\begin{center}
\textbf{APPENDIX B}
\end{center}

\textbf{PROOFS OF SECTION V}

\textbf{A. Proof Lemma 7}

Let us start by assuming that functions \( f_i \) are quadratic and, more specifically, that Eq. 16 holds true with the residual equal to 0, \( i.e., \)

\[ \nabla f_i(x) = \nabla f_i(x^*) + \nabla^2 f_i(x^*)(x_i - x^*). \]

(46)

In this case (18) simplifies to

\[ z(k + 1) = Qz(k) + c \]

(47)
We characterize now the properties of matrix $Q$, which will extend to the more general case of strongly convex functions thus proving the Lemma. By Proposition 1, we know that the sequence \( \{z(k)\}_{k \in \mathbb{N}} \) converges to a fixed point $z^∗$ for the DRS operator, which thus must satisfy $z^∗ = Qz^∗ + c$. Therefore, the eigenvalues of $Q$ must lie either in one or strictly inside the unitary circle, and all eigenvalues in one must be semisimple.

We are now interested in proving point 2) in the Lemma. Since the solution is unique by strong convexity, we know using (17) that

\[
x^∗ = -H^{-1}A^TPz^* - H^{-1}b.
\] (48)

Moreover, the vector $z^∗ + c_i v_i$, $c_i \in \mathbb{R}$, must satisfy

\[
x^∗ = -H^{-1}A^TP(z^∗ + c_i v_i) - H^{-1}b
\] (49)

since $(z^∗ + c_i v_i) = Q(z^∗ + c_i v_i) + c$ and hence $(z^∗ + c_i v_i)$ is a fixed point. Imposing equivalence between the right-hand sides of both (48) and (49) yields

\[
c_iH^{-1}A^TPv_i = 0
\]

which, by invertibility of $H$, implies $v_i \in \ker(A^TP)$. Since this result does not depend on the particular eigenvector $v_i$ chosen, then 2) is true and the Lemma is proved.

\[\text{B. Proof Proposition 3}\]

We start by stating the following Lemma which will be instrumental for concluding the proof.

\textbf{Lemma 3:} The updates (11) can be rewritten as

\[
x_i(k) = \rho |N_i| \operatorname{prox}_{h_i} \left( \frac{1}{\rho |N_i|} [A^T]_i P z(k) \right)
\]

where $h_i(x_i) = f(x_i)/(\rho |N_i|)$ and where $[A^T]_i$ denotes the $i$-th row of the matrix $A^T$.

\textbf{Proof:} Recalling the definition of $A$ and $P$, we can see that

\[
\sum_{j \in N_i} z_{ji}(k) = [A^T]_i P z(k)
\]

where the $i$-th row of $A$ selects the auxiliary variables relative to neighbors of $i$ only. Therefore we can write

\[
x_i(k) = \arg \min_{x_i} \left\{ f_i(x_i) - \langle [A^T]_i P z(k), x_i \rangle + \frac{\rho |N_i| \|x_i\|^2}{2} \right\}
\]

\[
= \arg \min_{x_i} \left\{ f_i(x_i) + \frac{\rho |N_i|}{2} \left\| x_i - \frac{[A^T]_i P z(k)}{\rho |N_i|} \right\|^2 \right\}
\]

\[
= \rho |N_i| \operatorname{prox}_{h_i} \left( \frac{1}{\rho |N_i|} [A^T]_i P z(k) \right)
\]

where $h_i(x_i) = f_i(x_i)/(\rho |N_i|)$ and the term $\left\| [A^T]_i P z(k)/(\rho |N_i|) \right\|^2$ is independent from $x_i$ — was added to derive the second equality.

We provide now the proof of Proposition 3 which has been inspired by the one presented in [19] for a different formulation of the ADMM.

We can exploit Lemma 3 to give a formula for $Q^k$. In particular, using the spectral decomposition theorem [38] p. 603-604 we can write

\[
Q = \sum_{i=1}^m \frac{v_iw_i^T}{w_i^Tv_i} + \sum_{i=m+1}^r (\gamma_i V_i + N_i)
\]

where $w_i$ is a left eigenvector of the $i$-th eigenvalue in one, while for the remaining $m - r$ distinct eigenvalues, $V_i$ is the spectral projector and $N_i = (Q - \gamma_i I)V_i$ are nilpotent matrices. Using the properties of the $V_i$ and $N_i$ matrices, it holds that

\[
Q^k = \sum_{i=1}^m \left( \frac{v_iw_i^T}{w_i^Tv_i} \right)^k + \sum_{i=m+1}^r (\gamma_i V_i + N_i)^k
\]

and moreover

\[
(\gamma_i V_i + N_i)^k = \gamma_i^k V_i + \sum_{j=1}^{k-1} \binom{k}{j} \gamma_i^{k-j} V_i N_i^j
\]

where $k_i$ is the index of $\gamma_i$, that is the smallest integer such that $\operatorname{rk}((Q - \gamma_i I)^{k_i}) = \operatorname{rk}((Q - \gamma_i I)^{k_i+1})$. 

By Lemma 3 and the nonexpansiveness of the proximal operator it follows that
\[ \| x(k) - x^* \| \leq \| A^T P(x(k) - z^*) \|. \]
Substituting the explicit formula for \( z(k) - z^* \) and using the triangle and Cauchy-Schwarz inequalities we obtain
\[ \| x(k) - x^* \| \leq \| A^T P Q^k \| \| z(0) - z^* \| + \sum_{\ell=0}^{k-1} \| A^T P Q^{k-\ell-1} \| \| F \| \| r(x(\ell) - x^*) \|. \] (50)
Notice that \( A^T P Q^k = \sum_{i=m+1}^{k} A^T P(\gamma_i V_i + N_i) \) by property 2) in Lemma 1 and thus depends only on the eigenvalues inside the unitary circle. Therefore there exists \( C_M > 0 \) such that
\[ \| A^T P Q^k \| \leq C_M^k. \]
Since the Douglas-Rachford operator is nonexpansive, it follows that \( \| x(k) - x^* \| \) is bounded.

To formalize this, let \( \hat{u} = \{ b_i, \xi_i \} \) be the set of indices enumerating the coordinates of \( z(0) - z^* \) such that, if \( x(0) \) belongs to it then \( \| r(x(0) - x^*) \| \) is bounded. Therefore picking at each time \( k \) such that \( \| x(k) - x^* \| \leq \delta_k \) for all \( k \). Moreover
\[ e(k) = \prod_{h=1}^{k-1} (a + b_h) e(1), \quad k \geq 2, \]
from which, since \( a + b_h < 1 \) for all \( h \), we get that \( \lim_{k \to \infty} e(k) = 0 \). We conclude the proof by observing that, for any \( x \) it holds
\[ \lim_{k \to \infty} \frac{\prod_{h=1}^{k} (a + b_h)}{(a + \epsilon)^k} = 0. \]

\[ \Box \]

**APPENDIX C**

**PROOFS OF SECTION VI**

A. **Proof Proposition 4**

We prove the convergence of Algorithm 3 by resorting to the stochastic Krasnosel’ski-Mann (KM) iteration studied in [22]. In particular, first we briefly review the stochastic KM and then we show that the proposed algorithm conforms to it. Consider the classic KM iteration
\[ z(k + 1) = (1 - \alpha) z(k) + \alpha T z(k) \]
where \( T \) is an averaged operator and where at each iteration all the components of \( z \) are updated. The idea underlying the stochastic KM is to allow for only a random subset of coordinates to be updated at each iteration.

To formalize this, let \( I = \{ 1, \ldots, M \} \) be the set of indices enumerating the coordinates of \( z \), and let \( \xi \subseteq I \). We introduce the operator \( \hat{T}(\xi) : \mathcal{X} \to \mathcal{X} \) whose i-th element is given by \( \hat{T}_i(\xi) z = T_i z \) if the coordinate \( i \) is to be updated (i.e., \( i \in \xi \)), \( \hat{T}_i(\xi) z = z_i \) otherwise (i.e., \( i \notin \xi \)). Therefore picking at each time \( k \) a random subset \( \xi_k \) of \( I \) we allow the stochastic KM to update only some of the coordinates in \( z \). On a probability space \( (\Omega, \mathcal{F}, P) \), we define the random i.i.d. sequence \( \{ \xi_k \}_{k \in \mathbb{N}} \), with \( \xi_k : \Omega \to 2^\Omega \), to keep track of which coordinates are updated at each instant. The stochastic KM iteration is finally defined as
\[ z(k + 1) = (1 - \alpha) z(k) + \alpha \hat{T}(\xi_k) z(k) \] (52)
and consists of the $\alpha$-averaging of a stochastic operator.

The stochastic iteration satisfies the following convergence result, which is a particular case of \cite{ref} Theorem 3.

**Proposition 8:** Let $T$ be a nonexpansive operator with at least a fixed point, and let the step size be $\alpha \in (0, 1)$. Let $\{\xi_k\}_{k \in \mathbb{N}}$ be a random i.i.d. sequence on $2^\mathcal{I}$ such that

$$\forall i \in \mathcal{I}, \exists i \in 2^\mathcal{I} \text{ s.t. } i \in I \text{ and } \mathbb{P}[\xi_1 = I] > 0.$$ 

Then for any deterministic initial condition $z(0)$ the stochastic KM iteration \cite{ref} converges almost surely to a random variable with support in the set of fixed points of $T$.

Recall from Section VI that the update equation of Algorithm 3 can be compactly written as

$$z(k+1) = (I - B(k))z(k) + B(k)(1 - \alpha)z(k) + \alpha T_{DRS}(z(k))$$

where $T_{DRS}$ denotes the Douglas-Rachford operator applied to the problem in hand. Equivalently we have

$$z(k+1) = (I - \alpha)z(k) + \alpha[(I - B(k))z(k) + B(k)T_{DRS}(z(k))].$$

Therefore Proposition 8 applies with

$$\bar{T}^{(\xi_k)}z(k) = (I - B(k))z(k) + B(k)T_{DRS}(z(k)).$$

\[\square\]

**B. Proof Lemma 2**

As done in Lemma 1 suppose the functions $f_i$ are quadratic and, in turn, the update rule we consider is

$$z(k+1) = G(k)z(k) + B(k)\mathbf{c}.$$ 

Let $z^*$ be a fixed point of the synchronous iteration $z(k+1) = Qz(k) + \mathbf{c}$, that is $z^* = Qz^* + \mathbf{c}$. Observe that it holds also that

$$z^* = G(k)z^* + B(k)\mathbf{c}. \quad (53)$$

Consider now the update

$$z(k+1) - z^* = G(k)z(k) - z^* + B(k)\mathbf{c},$$

and, notice that, from (53) it follows

$$z(k) - z^* = G(k)(z(k) - z^*),$$

and, in turn,

$$z(k) - z^* = \mathbf{g}(k; 0)(z(0) - z^*).$$

Since we know that $z(k)$ converges almost surely to some fixed point $\bar{z}$, $\bar{z} = Q\bar{z}^* + \mathbf{c}$ , then we have

$$\bar{z}^* - z^* = \sum_{i=1}^{m} \alpha_i \mathbf{v}_i$$

where $\alpha_i$ depends on the particular sequence of matrices $B(0), B(1), B(2), \ldots$, and on the initial condition $z(0)$. This concludes the proof.

\[\square\]

**C. Proof Proposition 5**

Let $\bar{z}(t)$ be such that

$$\bar{z}(k+1) = G(k)\bar{z}(k).$$

Hence

$$\bar{z}(k) = \mathbf{g}(k; 0)\bar{z}(0).$$

and, in turn, from Lemma 2

$$\lim_{k \to \infty} \bar{z}(k) = \left(\sum_{h=1}^{m} \mathbf{v}_h \mathbf{e}_h^T\right)\bar{z}(0). \quad (54)$$

Now observe that

$$\bar{z}^T(0)\mathbf{L}^k(\Delta)\bar{z}(0) = \mathbb{E}\left[\bar{z}^T(k)\Delta\bar{z}(k)\right].$$
From (54) it follows that
\[
\mathbb{E} \left[ \tilde{z}^T(k) \Delta \tilde{z}(k) \right] = \tilde{z}^T(0) \mathbb{E} \left[ \left( \sum_{h=1}^{m} v_h \epsilon_h^T \right)^T \Delta \left( \sum_{h=1}^{m} v_h \epsilon_h \right) \right] \tilde{z}(0)
\]
\[
= \tilde{z}^T(0) \left[ \sum_{h=1}^{m} \sum_{\ell=1}^{m} (v_h^T \Delta v_\ell) \mathbb{E} \left[ \epsilon_h \epsilon_\ell^T \right] \right] \tilde{z}(0)
\]
Hence
\[
\lim_{k \to \infty} C^k(\Delta) = \sum_{h=1}^{m} \sum_{\ell=1}^{m} (v_h^T \Delta v_\ell) \mathbb{E} \left[ \epsilon_h \epsilon_\ell^T \right].
\]
This concludes the proof.

**D. Proof Proposition 6**

By the results reported in VI-B we know that there exists $\tilde{C}_M > 0$ such that we can bound
\[
\mathbb{E} \left[ \| A^T \mathcal{P} G(k; \ell + 1) \|^2 \right] \leq \tilde{C}_M \gamma_M^k \ell^{-1},
\]
where recall that $\gamma_M < 1$. Moreover by definition of $r(\cdot)$ in a neighborhood of the optimum for $\delta \in (0, 1)$ we have
\[
\mathbb{E} \left[ \| r(x(\ell) - x^*) \| \right]^2 \leq \delta \mathbb{E} \left[ \| x(\ell) - x^* \| \right]^2.
\]
As a consequence we can bound the mean square error as follows
\[
\mathbb{E} \left[ \| x(k) - x^* \| \right]^2 \leq \tilde{C}_M^k \gamma_M^k \gamma_M^k \sum_{\ell=0}^{k-1} \gamma_M^{k-\ell-1} \mathbb{E} \left[ \| r(x(\ell) - x^*) \| \right]^2
\]
with $\tilde{C}_M' = \tilde{C}_M \| z(0) - z^* \| ^2$ and $\tilde{C}_M = \mathbb{E} \left[ \| B(0) F \| \right] \tilde{C}_M$. Notice that $\tilde{C}_M'$ can be defined in this way since the matrices $B(k)$ have the same expected value, i.e., for all $k$ it holds $\mathbb{E} \left[ \| B(k) \| \right] = \mathbb{E} \left[ \| B(0) \| \right].$

We can now conclude the same argument used to prove Proposition 3.

**E. Proof Proposition 7**

The goal is to compute the matrix $L = \mathbb{E} \left[ G(0)^T \otimes G(0)^T \right]$ where $G(0) = I - B(0) + B(0)Q$ with $G(0) = \text{diag} \{ \beta(0) \}$ and $\beta$ the vector of all $\beta_{i \to j}$ random variables. Recall that the $\beta_{i \to j}$ random variables are all Bernoulli with probability $s = q(1 - p)$ but not all independent of each other.

For simplicity, we compute $L^T$ as follows. Omitting the time index we can write
\[
L^T = \mathbb{E} \left[ (I - B + BQ) \otimes (I - B + BQ) \right] = \mathbb{E} \left[ I \otimes I - I \otimes B + I \otimes BQ - B \otimes I + B \otimes B - B \otimes BQ + BQ \otimes I - BQ \otimes B + BQ \otimes BQ \right],
\]
and we can focus on each term separately by linearity of the expectation. The first term is clearly equal to itself, so let us analyze the following terms.

- We have $\mathbb{E} [I \otimes B] = I \otimes \mathbb{E} [B] = sI \otimes I$, and similarly $\mathbb{E} [B \otimes I] = sI \otimes I$.
- $S = \mathbb{E} [B \otimes B]$ is a block diagonal matrix with the block relative to edge $(i,j)$ equal to $\mathbb{E} [\beta_{i \to j} \text{diag} \{ \beta \}]$. The diagonal elements of each block are of the type $\mathbb{E} [\beta_{i \to j} \beta_{j \to k}]$. If $i \neq \ell$ then $\beta_{i \to j}$ and $\beta_{\ell \to k}$ are independent and $\mathbb{E} [\beta_{i \to j} \beta_{\ell \to k}] = s^2$. If $i = \ell$ and $j = k$ then $\mathbb{E} [\beta_{i \to j}^2] = s$, while if $i = \ell$ but $j \neq k$ it is $\mathbb{E} [\beta_{i \to j} \beta_{i \to k}] = s^2 / q$. This last result can be derived by considering that the product $\beta_{i \to j} \beta_{i \to k}$ is equal to one if and only if the packets $q_{i \to j}$ and $q_{i \to k}$ are both received and node $i$ performed an update; this event happens with probability $q(1 - p)^2 = s^2 / q$.

The remaining terms can be computed using the property of the following Kronecker product $(AC \otimes BD) = (A \otimes B)(C \otimes D)$ for matrices $A, B, C, D$ of suitable dimensions.

- $\mathbb{E} [I \otimes BQ] = \mathbb{E} [(I \otimes B)(I \otimes Q)] = s(I \otimes I)(I \otimes Q) = sI \otimes Q$.
- $\mathbb{E} [BQ \otimes I] = \mathbb{E} [(B \otimes I)(Q \otimes I)] = s(I \otimes I)(Q \otimes I) = sQ \otimes I$.
- $\mathbb{E} [B \otimes BQ] = \mathbb{E} [(B \otimes B)(I \otimes Q)] = \mathbb{E} [(B \otimes B)](I \otimes Q) = S(I \otimes Q)$.
- $\mathbb{E} [BQ \otimes B] = \mathbb{E} [(B \otimes B)(Q \otimes I)] = S(Q \otimes I)$.
- $\mathbb{E} [BQ \otimes BQ] = \mathbb{E} [(B \otimes B)(Q \otimes Q)] = S(Q \otimes Q)$.

Summing all terms and transposing the result we get the expression for $L$ reported in Proposition 7.
