Arrays of waveguide-coupled optical cavities that interact strongly with atoms

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Abstract. We describe a realistic scheme for coupling atoms or other quantum emitters with an array of coupled optical cavities. We consider open Fabry–Perot microcavities coupled to the emitters. Our central innovation is to connect the microcavities to waveguide resonators, which are in turn evanescently coupled to each other on a photonic chip to form a coupled cavity chain. In this paper, we describe the components, their technical limitations and the factors that need to be determined experimentally. This provides the basis for a detailed theoretical analysis of two possible experiments to realize quantum squeezing and controlled quantum dynamics. We close with an outline of more advanced applications.

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1. Introduction

The interaction of light and matter is of central importance to research in quantum optics [1]. The strength of this interaction depends on the intensity of the light at the location of the atom. To enhance the interaction, photons can be trapped in a high-finesse cavity of small volume, leading to high-intensity fields inside the cavity even with small photon numbers. This arrangement forms the central paradigm of cavity quantum electrodynamics (CQED) [2–4]. Early experiments in CQED used atoms that were dropped through the cavity and hence passed briefly through the region of strong interaction [5]. With the advent of laser cooling and trapping techniques, it became possible to position atoms or ions at a desired location within the cavity and to achieve ‘strong’ atom–photon coupling, i.e. a coupling rate higher than both the atomic decay rate and the cavity decay rate. Phenomena such as vacuum Rabi splitting [6], photon blockade [7, 8], and single-photon sources [9] have now been observed.

So far, experiments in CQED have almost exclusively been carried out with single cavities. In recent years, however, it has become possible to fabricate microcavities in regular arrays in various settings [10–17]. Due to their small size, these cavities all have small mode volume and correspondingly a strong interaction between light and matter. Indeed the ‘strong coupling’ regime has already been demonstrated for many of these devices [13, 18, 19].

An important next step is to find a way of coupling these cavities so that they form an array in which photons can tunnel from one cavity to another. Recently, various approaches were put forward for using such arrays of coupled cavities as a platform for quantum emulators [20–24].
Figure 1. Outline of the waveguide chip. W and C refer to the waveguide cavities and the microcavities, respectively. Circles indicate the positions of active heating elements. These tune the resonant frequency of each waveguide, and adjust the coupling between adjacent waveguides. Dotted arrows indicate the regions where waveguides are evanescently coupled to each other. The cavity array is pumped by a laser field $E_{in}$, which can be applied to any waveguide input port. Furthermore, each atom can be driven individually at Rabi frequency $\Omega$ using laser beams transverse to the array, as indicated on the left. Table 1 provides a description of the symbols used.

These include a scheme for emulating the Bose–Hubbard Hamiltonian [20, 25–27], models of interacting Jaynes–Cummings Hamiltonians [22, 23] and an effective spin Hamiltonian [28]. The phase diagrams of these models have been studied [29–32] and the existence of a glassy phase has been predicted [29].

In this paper, we describe the design for a practical device in which atoms interact strongly with high-finesse microcavities and, at the same time, photons can tunnel with low loss from one cavity to the next through an interconnecting waveguide chip. The main components of the device are shown schematically in figure 1.

This paper is organized as follows. In section 2, we describe the design of the device in detail and specify some realistic parameters. In section 3, we calculate the spectrum of one composite cavity. We then show (section 4) that the dynamics of the device is well approximated by a Jaynes–Cummings Hamiltonian. We describe experiments on a small scale to demonstrate two types of quantum emulator using coupled cavity arrays. As a first example (section 5) we show that a lattice of interacting Jaynes–Cummings Hamiltonians could be implemented in our device and analyse its steady states in a driven dissipative regime for a two-cavity setup [33–36]. In a second example (section 6), we show that our device is also suitable for implementing effective spin Hamiltonians and we study the dynamics in a two-cavity setup. Finally, in section 7, we outline future applications that can be based on these capabilities but go beyond the two-cavity setting.
Table 1. Definitions of relevant symbols. All frequencies are angular: e.g. $2\pi \times 1$ GHz means $6.28 \times 10^9$ s$^{-1}$.

| Symbol   | Definition                                      | Typical values              |
|----------|-------------------------------------------------|-----------------------------|
| $L_W, L_W$ | Length of the cavities                          | $L_C = (10–100)$ $\mu$m    |
|          |                                                 | $L_W > (10–20)$ $\text{mm}$ |
| $r_i$    | Amplitude reflection coefficients of the mirrors| $99.95\% > R_C = 99.9\% > 99.0\%$ |
|          |                                                 | $99.95\% > R_W = 99.0\%$   |
|          |                                                 | $R_{CW} \approx 98\%$      |
| $FSR_i = \pi c / L_i$ | Free spectral range | $FSR_C \approx 2\pi \times (0.2–2)$ THz |
|          |                                                 | $FSR_W \approx 2\pi \times (1–2)$ GHz |
| $F_C = \frac{\pi}{1 - \sqrt{R_C R_{CW}}}$ | Cavity finesse |                                                        |
| $F_W = \frac{\pi}{1 - \sqrt{R_W R_{CW}}}$ | Waveguide finesse |                                                        |
| $\kappa_i = FSR_i / 2F_i$ | Resonance linewidth |                                                        |
| $w_C$ | Microcavity mode waist | $w_C = 3–5$ $\mu$m |
| $w_W$ | Waveguide mode size | Ideally matched to $w_C$ |
| $g_{AC} = \sqrt{\frac{3\gamma\pi^5}{\pi^5 w_C^2 L_C}}$ | Atom–cavity coupling | $2\pi \times (0.1–1)$ GHz |
| $\gamma$ | Atom amplitude decay rate | Rb: $\gamma = 2\pi \times 3$ MHz |
| $\lambda$ | Wavelength | Rb: $\lambda = 780$ nm |
| $J_{WW}$ | Tunnelling rate between adjacent waveguide resonators | $0 < J_{WW} < 2\pi \times 2$ GHz |

2. Device

We envisage a device with quantum emitters in several separate microcavities, coupled by waveguides on a coupling chip as illustrated in figure 1. The microcavities are based on the design described in [19]. Hemispherical micro-mirrors, wet-etched into a silicon chip [15, 19] and reflection-coated, form one side of a set of Fabry–Perot microcavities. These cavities are closed by the plane reflection-coated ends of waveguides integrated on a photonic chip. The other end of each waveguide is also reflection-coated to form a second set of optical resonators. Each waveguide/microcavity pair, coupled together through the shared mirror, forms a composite cavity. Henceforth, we will sometimes simply call this a cavity, and we will speak of a microcavity or a waveguide cavity if we wish to distinguish the component parts. The microcavities are open in the transverse direction, giving access to lasers that trap and manipulate atoms at the position of maximum interaction with cavity mode. This design benefits from the intrinsic scalability of microfabrication to achieve controlled nearest-neighbour coupling of many optical cavities.

In the following discussion, we assume a wavelength of $\lambda = 780$ nm, which is the D2 line of rubidium-87 atoms whose amplitude decay rate is $\gamma = 2\pi \times 3$ MHz. In a microcavity of length $L_C$, the maximum coupling rate $g_{AC}$ between a two-level atom and the standing-wave field of...
one photon is
\[ g_{AC} = \sqrt{\frac{3\lambda^2\gamma}{\pi^2 w_0^2 L_C}}, \]  
(1)

where \( w_0 \) is the \( 1/e^2 \) radius of the Gaussian intensity profile at the waist and \( c \) is the speed of light. Microcavities of the type envisaged can have mode waists down to \( 2 \mu\text{m} \), whereas the length of the microcavity \( L_C \) is in the range of \( 10–100 \mu\text{m} \). Therefore, the coupling rate is of the order of \( 2\pi \times 0.1–1 \text{GHz} \), and hence \( g_{AC} \gg \gamma \). The reflectivity of each mirror is given by \( R_i = 1 - (T_i + A_i) \). Assuming that the power transmission and absorption coefficients fulfil \( T_i, A_i \ll 1 \), the cavity field amplitude decays at a rate
\[ \kappa_C = \frac{c \xi_C}{2L_C}, \quad \text{with} \quad \xi_C \approx \frac{T_C + A_C + T_{CW} + A_{CW}}{2}, \]  
(2)

The subscript C denotes the concave microcavity mirror, while the subscript CW denotes the plane coupling mirror between the cavity and the waveguide. As we intend to fabricate the concave mirror by isotropic etching of silicon, we expect the losses due to surface roughness to be of the order of \( A_C \approx 10^{-4} \) without additional polishing. The coupling mirror, on the other hand, can have losses of the order of \( A_{CW} \approx 5 \times 10^{-6} \), assuming the waveguide facet is super-polished and a dielectric mirror formed by ion-assisted deposition is used. The decay rate of the microcavity could then be made as small as \( \kappa_C \approx 2\pi \times 0.01 \text{GHz} \) for a \( 100 \mu\text{m} \) long cavity. This length will give enough space to permit external optical access to the atoms. However, in order to couple effectively to the waveguide resonator, it is desirable to increase the transmission of the coupling mirror so that \( T_{CW} \gg T_C + A_C \). Therefore in practice, \( \kappa_C \approx 2\pi \times 0.1 \text{GHz} \). This does not necessarily imply higher loss for the composite cavity since photons that go through this mirror enter the waveguide and are not necessarily lost.

Similarly, the field in the waveguide resonator decays at a rate
\[ \kappa_W = \frac{c \xi_W}{2L_W}, \quad \text{with} \quad \xi_W \approx \frac{T_W + A_W + T_{CW} + A_{CW} + K_W}{2}, \]  
(3)

where \( L_W \) is the optical length of the waveguide cavity (refractive index times physical length) and \( K_W \) accounts for the waveguide propagation loss over one round-trip. While there is little use of integrated waveguides at 780 nm, we find in the literature [37] that propagation losses can be less than \( 0.02 \text{dB cm}^{-1} \) for wavelengths around 800 nm in polymer waveguides. This platform offers all the necessary technological components for our device. We estimate that to achieve the necessary coupling lengths and separations between the waveguides without incurring additional bend losses, the waveguide resonator will have a length of \( L_W > 1 \text{cm} \). We will therefore conservatively consider \( L_W = 2 \text{cm} \), which gives a fractional round-trip loss of 1.8% and a corresponding decay rate of \( \kappa_W > 2\pi \times 10 \text{MHz} \).

The coupling from the microcavity into the waveguide resonator depends on the transmission \( T_{CW} \), but also on the spatial overlap of the waveguide and microcavity modes, which will need to be optimized experimentally. In principle, this can reach unity, and we estimate that at least 90% should be achievable in practice. The photon tunnelling rate between the microcavity and the waveguide can easily exceed the free spectral range of the waveguide cavity. In this case, it is appropriate to consider the eigenmodes of the composite cavity, rather than viewing the microcavity and waveguide cavity as individual devices. We calculate the spectrum of one such coupled cavity in section 3.

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Figure 2. Calculated reflection spectrum of two composite cavities having evanescent-wave coupling between their waveguides. A pair of thermo-optics phase shifters is adjusted to give a phase shift $\delta \phi$ between the standing waves in the two waveguides. The system is pumped on waveguide number 1. At $\delta \phi = \pi/2$, the overlap integral of the standing waves is zero, so the coupling is switched off. Consequently, there is no normal mode splitting and no power is seen in waveguide number 2. When the phase shift changes to $\delta \phi = 0$ or $\pi$, the coupling is maximized and therefore so is the mode splitting. This calculation is based on an expanded version of the transfer matrix method described in [47].

Each waveguide is coupled to its two nearest neighbours by short regions of evanescent field overlap to produce a tunnelling rate $J_{WW}$. The maximum value $J_{WW}^{(\text{max})}$, determined by the maximum field overlap and the length of the coupling region, can be as high as $10^{10}$ s$^{-1}$. This overlap can be tuned using thermal phase shifters, one near each end, to move the standing wave without changing the cavity length. In this way, $J_{WW}$ can be tuned in principle over the range 0 to $J_{WW}^{(\text{max})}$. In practice, a small amount of crosstalk will be unavoidable because of scattering, tuning noise and linewidth effects, but we estimate that $J_{WW}^{(\text{min})} < 10^{-3} \times J_{WW}^{(\text{max})}$. The maximum coupling strength must therefore be chosen judiciously to ensure access to the desired range of coupling. The thermo-optic phase shifters also allow the optical length of each waveguide cavity to be tuned at a rate of up to a few wavelengths per ms. Figure 2 illustrates how two coupled cavities exhibit a normal mode splitting and shows how the evanescent-wave coupling can be adjusted by shifting the phase of one standing wave relative to the other. In particular, the coupling is switched off for a relative phase of $\pi/2$.

3. The composite cavity spectrum

As our starting point, we consider the spectrum of a single composite cavity. Several such cavities are to be concatenated in our setup to form the coupled-cavity array.

Suppose that we couple light into one of the cavities, through the top waveguide mirror as drawn in figure 1. Without the curved microcavity mirror at the bottom, the field reflected by
shows how the reflected intensity spectrum, we see a series of dips, symmetrically disposed around \( \Delta \omega \).

Without any coupling, both constituent cavities are resonant at 

\[ \frac{r_{CW} - r_C e^{2i\phi_C}}{1 - r_{CW} r_C e^{2i\phi_C}} E_{in}, \]

(4)

where \( E_{in} \) is the incident field, \( r_W \) and \( r_{CW} \) are the amplitude reflection coefficients of the two mirrors, and \( 2\phi_W = 4\pi L_W/\lambda \) is the round-trip propagation phase. Here we have assumed lossless mirrors for simplicity. Resonance occurs when \( \phi_W = m\pi \) with \( m \) being an integer. Similarly, a travelling field \( E_{in,C} \) that is incident on the microcavity produces a reflected field in the waveguide given by

\[ \frac{E_{r,C}}{E_{in,C}} = \frac{r_{CW} - r_C e^{2i\phi_C}}{1 - r_{CW} r_C e^{2i\phi_C}} \equiv \tilde{r}_C e^{i\theta}, \]

(5)

where \( 2\phi_C = 4\pi L_C/\lambda \). This ratio defines the reflection coefficient \( \tilde{r}_C \) and phase shift \( \theta \) at the bottom end of the waveguide cavity when the microcavity mirror is in place:

\[
\tilde{r}_C^2 = \frac{r_{CW}^2 + r_C^2 - 2r_{CW} r_C \cos(2\phi_C)}{1 + r_{CW}^2 r_C^2 - 2r_{CW} r_C \cos(2\phi_C)}
\]

\[
\theta = \arctan \left( \frac{(r_{CW}^2 - 1) r_C \sin(2\phi_C)}{r_{CW}(r_C^2 + 1) - r_C (r_{CW}^2 + 1) \cos(2\phi_C)} \right).
\]

(6)

On replacing \( r_{CW} \) in equation (4) by \( \tilde{r}_C e^{i\theta} \), we obtain the reflected field from a composite cavity:

\[ E_{r,CW} = \frac{r_W - \tilde{r}_C e^{i(\theta + 2\phi_W)}}{1 - r_W \tilde{r}_C e^{i(\theta + 2\phi_W)}} E_{in}, \]

(7)

which is in resonance when

\[ 2\phi_W + \theta = 2\pi n, \]

(8)

for any integer \( n \). This set of equations describes the system fully. It cannot generally be solved analytically, but we can determine when the reflected intensity becomes zero. This is of interest because the spectrum is generally sensitive to the presence of atoms under this condition. Zero reflected field is achieved when the microcavity can be tuned to make \( r_w = \tilde{r}_C^2 \). Such a tuning is possible when the following inequality is satisfied:

\[ |E_{r,C}^{\text{min}}|^2 = \frac{(r_C - r_{CW})^2}{(1 - r_C r_{CW})^2} \leq r_w^2 \leq |E_{r,C}^{\text{max}}|^2 = \frac{r_C^2 + r_{CW}^2}{1 + (r_C r_{CW})^2}, \]

(9)

and occurs when the cavity round-trip phase satisfies

\[ 2\phi_C(\text{opt}) = \arccos \left( \frac{r_C^2 (1 + (r_C r_{CW})^2) - (r_C^2 + r_{CW}^2)}{2r_{CW} r_C (r_C^2 - 1)} \right). \]

(10)

This determines the phase shift \( \theta \), and the resonance of the whole cavity is then ensured by adjusting the length of the waveguide to satisfy equation (8).

The semi-logarithmic graph on the top left of figure 3 shows how the reflected intensity varies with a change \( \Delta \omega \) in the angular frequency of the laser for a cavity with \( L_w = 15.6 \text{ mm} \) and \( L_C = 156 \mu \text{ m} \) and with mirror reflectivities of \( R_C = 0.999 \), \( R_{CW} = 0.98 \) and \( R_W = 0.99 \). Without any coupling, both constituent cavities are resonant at \( \Delta \omega = 0 \). In the reflection spectrum, we see a series of dips, symmetrically disposed around \( \Delta \omega = 0 \). The two central dips represent normal modes of the composite cavity. These would be degenerate in the absence
Figure 3. Behaviour of composite cavity with cavity lengths $L_C = 156 \, \mu m$, $L_W = 15.6 \, cm$ and power reflectivities $R_C = 0.999$, $R_{CW} = 0.98$ and $R_W = 0.99$. Top left: calculated reflection spectrum of the composite cavity versus detuning of the laser. In the absence of coupling, each separate cavity has a resonance at $\Delta \omega = 0$. Here they are coupled to produce normal modes split by $2.5 \times 10^{10} \, s^{-1}$. Dashed curve: $\tilde{r}_C^2$. Top right: reflected intensity near one of the central resonances. The laser frequency is fixed at $\Delta \omega = 0$, while the (uncoupled) resonance frequencies of the two constituent cavities are varied. The green circle indicates the point of full contrast where the reflected intensity is zero, as calculated from equation (10). Bottom: circulating field strength inside the waveguide resonator (left) and the microcavity (right), as a function of their detuning from the laser. Green circles are as above.

of coupling but are split apart here. Further away from this doublet, the resonances are not so strongly coupled to the microcavity and their spacing approaches the free spectral range of the bare waveguide cavity. The horizontal line at 0.99 shows the reflection coefficient $r_W^2$ of the input mirror, while the dashed line indicates the value of $\tilde{r}_C^2$. These are most nearly equal at the second reflection dip on each side, making those dips the strongest ones. The width of each resonance can be understood in a simple way because the waveguide resonator is very much longer than the microcavity. Consequently, $\tilde{r}_C$ and $\theta$ are essentially constant over any given resonance line.
of the coupled system. By analogy with equation (2) this gives the cavity damping rate as
\[ \tilde{\kappa} \approx \frac{c}{2L_W} \xi, \quad \text{with} \quad \xi \approx 1 - r_W r_C. \]  
(11)

The top right graph in figure 3 explores how the reflected intensity near one of the central resonances depends on the detuning of the two constituent cavities. We see that the reflection goes through a minimum whose position, indicated by the small circle, is as expected from equation (10). The two lower graphs show how the circulating power inside the waveguide resonator (left) and the microcavity (right) varies with the same detuning of the constituent cavities. In particular, we see that the condition of minimum reflected power (again indicated by a circle) corresponds closely to having maximum power inside the microcavity, as required for good atom–cavity coupling. At this point, the energy density in the microcavity is ten times that in the waveguide (although the total energy is ten times lower because of the 100-fold disparity in lengths).

4. Approximation by a Jaynes–Cummings lattice Hamiltonian

In this section, we describe a series of approximations that allow us to obtain a description of our system in terms of atom–photon interactions of Jaynes–Cummings form in each cavity together with the tunnelling of photons between adjacent cavities.

4.1. Composite cavity Hamiltonian

The Hamiltonian for the field in one composite cavity is
\[ H_{cav} = \sum_\alpha \omega_\alpha a_{j,\alpha}^\dagger a_{j,\alpha}, \]  
(12)

where the index \( j \) labels the particular cavity and index \( \alpha \) labels its eigenmodes. The frequencies \( \omega_\alpha \) are those of the eigenmodes discussed in section 3.

4.2. Photon tunnelling between adjacent waveguides

The rate \( J_{WW} \) for photons to tunnel between two adjacent waveguides can be tuned over a wide range. Let us choose to make it small compared with the waveguide free spectral range, which also ensures that \( |J_{WW}| \ll |\omega_\alpha - \omega_\beta| \). In this regime, we can write the coupling as a tunnelling term between resonant modes of the two composite cavities,
\[ H_{WW} = \sum_{\alpha,\beta} J_{\alpha,\beta}(a_{j,\alpha}^\dagger a_{j+1,\beta} + \text{H.c.}) \approx - \sum_{\alpha} J_{\alpha,\alpha}(a_{j,\alpha}^\dagger a_{j+1,\alpha} + \text{H.c.}). \]  
(13)

When the cavity network only contains photons that are near-resonant with the normal modes \( \alpha_0 \) and when the Rabi frequencies of atom–photon coupling are also \( \ll |\omega_\alpha - \omega_\alpha'| \), only these modes are populated and \( H_{WW} \) can be simplified further to read
\[ H_{WW} \approx - J_{\alpha_0,\alpha_0}(a_{j,\alpha_0}^\dagger a_{j+1,\alpha_0} + \text{H.c.}). \]  
(13)

The tunnelling rate \( J_{\alpha_0,\alpha_0} \) is related to \( J_{WW} \). Yet, since \( J_{WW} \) is harder to determine experimentally than \( J_{\alpha_0,\alpha_0} \), we do not specify the conversion here.
4.3. Atom–photon coupling

The circulating fields in the microcavity and waveguide, $E_C$ and $E_W$, respectively, are given by

$$E_W = E_{in} \frac{t_W}{1 - r_W r_C e^{i(2\phi_W+\theta)}},$$  \hspace{1cm} (14)$$

$$E_C = E_{in} \frac{t_{CW} e^{i\phi_W}}{1 - r_C r_W e^{2i\phi_C} + r_C r_W e^{2i(\phi_C+\phi_W)} - r_{CW} r_W e^{2i\phi_W}},$$  \hspace{1cm} (15)$$

where $E_{in}$ is the input field from the top end of the waveguide; see figure 1. Since one photon in the composite cavity has a total energy content of $\bar{\hbar}\omega$, the absolute values of the circulating field amplitudes per photon in the microcavity and waveguide are

$$|E_C| = \sqrt{\frac{\hbar\omega}{2\varepsilon_0\pi w_0^2} \frac{|E_C|^2}{L_C |E_C|^2 + L_W |E_W|^2}},$$  \hspace{1cm} (16)$$

$$|E_W| = \sqrt{\frac{\hbar\omega}{2\varepsilon_0\pi w_0^2} \frac{|E_W|^2}{L_C |E_C|^2 + L_W |E_W|^2}}.$$  \hspace{1cm} (17)$$

Hence the atom–photon coupling at an antinode of the microcavity is

$$g = g_{\text{AC}} \sqrt{\frac{|E_C|^2}{|E_C|^2 + \frac{L_W}{L_C} |E_W|^2}},$$  \hspace{1cm} (18)$$

where $g_{\text{AC}}$ is given in equation (1) for a two-level atom. This formula for a two-level atom applies because we are considering the closed cycling transition $5^2S_{1/2}(F = 2, m_F = \pm 2) \leftrightarrow 5^2P_{3/2}(F' = 3, m'_{F} = \pm 3)$ on the $D_2$ line of $^{87}\text{Rb}$.

4.4. Jaynes–Cummings lattice

With the above approximations, the dynamics of atoms coupled to the composite cavity normal modes $a_j, a_{\alpha_0}$ can be described by a Jaynes–Cummings lattice model. Dropping the index $\alpha_0$ on the field operators, the Hamiltonian for $N$ composite cavities reads

$$H_{\text{JC array}} = \omega_{\text{C}} \sum_{j=1}^{N} \sigma_j^+ \sigma_j^- + \omega_{\text{C}} \sum_{j=1}^{N} a_j^+ a_j$$

$$+ g \sum_{j=1}^{N} (\sigma_j^+ a_j + \sigma_j^- a_j^+)$$

$$- J \sum_{j=1}^{N-1} (a_j^+ a_{j+1} + a_j a_{j+1}^+).$$  \hspace{1cm} (19)$$

Here, $\omega_{\text{C}} = \omega_{\alpha_0}$, $\sigma_j^- = |g_j\rangle\langle e_j|$ is the transition operator between the excited state $|e_j\rangle$ and the ground state $|g_j\rangle$ of the atom in cavity $j$ and $a_j$ is the annihilation operator for photons in mode $\alpha_0$ of that cavity. The losses for the system described by this Hamiltonian arise through spontaneous emission from the excited states of the atoms at rate $\gamma$ and photon loss from the normal modes $a_j$ at rate $\kappa$ given by equation (11).
5. Spectroscopy for a driven Jaynes–Cummings array

We now consider an array of composite cavities connected to each other by nearest-neighbour coupling and excited at the end of one waveguide by a laser tuned to the normal mode $a_1$. The resonant pumping is described by an additional term $H_D = \frac{\eta}{2} a_1^\dagger + \frac{\eta'}{2} a_1$ in the Hamiltonian, whereas photon leakage and spontaneous emission losses are taken into account by Markovian damping terms. The dynamics of the resulting driven dissipative system is then described by the master equation,

$$\frac{d\rho}{dt} = -i [H, \rho] + \gamma \sum_j (2\sigma_j^- \rho \sigma_j^+ - \sigma_j^+ \sigma_j^- \rho - \rho \sigma_j^+ \sigma_j^-) + \tilde{\kappa} \sum_j (2a_j \rho a_j^\dagger - a_j^\dagger a_j \rho - \rho a_j^\dagger a_j),$$

(20)

where $H = H_D + H_{\text{array}}$. For one cavity, a driving amplitude $\eta$ gives a steady-state energy in the cavity of $\hbar \omega \langle a_1 a_1^\dagger \rangle = \hbar \omega |\eta|^2/(2\tilde{\kappa}^2)$. Also, the ratio of energy in the cavity to input power $P$ is $2(|E_C|^2L_C + |E_W|^2L_W)/(|E_{\text{in}}|^2c)$, where $E_C$ and $E_W$ are the circulating fields in the cavity. Hence, the power $P$ of the driving laser is related to the driving amplitude $\eta$ by

$$P = \hbar \omega_L \frac{|\eta|^2}{\tilde{\kappa}^2} \frac{|E_{\text{in}}|^2}{2(|E_C|^2L_C + |E_W|^2L_W)}.$$

(21)

Below, we calculate the spectrum and photon statistics that can be observed at the output ports of the coupled waveguides [33–36]. These properties correspond to the most straightforward experiments that might be conducted using such a cavity array. As we shall see, they nonetheless reveal interesting physics including significantly entangled photon output states.

5.1. Composite cavity modes with strong coupling and high single-atom cooperativity

Since we are interested in having large atom–photon coupling and small photon losses, both from the cavities and from spontaneous emission, we wish to work with a mode that has high single-atom cooperativity, $C_i = g^2/(2\tilde{\kappa}\gamma)$, and is resonant with the atomic transition. The required combinations of microcavity and waveguide lengths are found by numerical optimization. In figure 4, the cavity resonances are indicated by vertical lines. The resonance near $\delta \omega = 0$, with the highest cooperativity (solid line) of $C_1 = 17.3$, coincides with the atomic transition frequency. The cavity parameters ($L_C = 156.05$ $\mu$m, $L_W = 20.000$ mm, $R_C = 99.9\%$, $R_{\text{CW}} = 98.0\%$ and $R_W = 99.8\%$) are very close to the parameters used in figure 3. This resonance corresponds to the strong reflection dip in figure 3 on the high-frequency side of the central doublet. The atom–photon coupling $g$ and photon loss rate $\tilde{\kappa}$ are $g = 0.3304 \times g_{\text{AC}} = 2\pi \times 33.04$ MHz and $\tilde{\kappa} = 2\pi \times 10.6$ MHz.

5.2. Steady state of a two-site Jaynes–Cummings array

Figure 5 shows the steady-state behaviour when two of these cavities are coupled together at a rate $J = 0.2 \times g_{\text{AC}} = 2\pi \times 20$ MHz. Each contains a rubidium atom and cavity number 1 is driven by a laser with $\eta = 0.1 \times g_{\text{AC}} = 2\pi \times 10$ MHz. The atoms are resonant with the normal modes $a_j$. Figure 5(a) shows the number of photons in each cavity: $n_1 = \langle a_1^\dagger a_1 \rangle$ and $n_2 = \langle a_2^\dagger a_2 \rangle$. The four resonance lines in the spectrum correspond to the four singly excited eigenstates of the
Figure 4. Composite cavity properties versus laser detuning. The atomic transition frequency is at $\delta \omega = 0$. The cavity parameters ($L_C = 156.05 \mu m$, $L_W = 20.000 \text{mm}$, $R_C = 99.9\%$, $R_{CW} = 98.0\%$ and $R_W = 99.8\%$) are close to those of figure 3. Vertical lines: frequencies of reflection minima due to composite cavity resonances. Solid line: single-atom cooperativity $C_1 = \frac{g^2}{(2\tilde{\kappa}\gamma)}$. At its peak value $C_1 = 17.3$. Dashed line: coupling relative to atomic decay rate, $g/\gamma$. Dotted line: coupling relative to cavity decay rate, $g/\tilde{\kappa}$. These quantities vary because the laser detuning determines how the amplitude of the electric field is distributed within the waveguide and the microcavity.

The two-site Jaynes–Cummings Hamiltonian, i.e. equation (19) with $N = 2$. Here, the resonances at $\delta \omega \approx \pm 2\pi \times 44 \text{MHz}$ correspond to states where the excitation is more likely to be found in one of the cavity modes, whereas for the resonances at $\delta \omega \approx \pm 2\pi \times 24 \text{MHz}$ the excitation is more likely to be in the atom than in the cavity field.

In figure 5(b), we study the photon density correlations $g^{(2)}(1, 1) = \langle a_1^\dagger a_1^\dagger a_1 a_1 \rangle / n_1^2$, $g^{(2)}(1, 2) = \langle a_1^\dagger a_2^\dagger a_2 a_1 \rangle / n_1 n_2$ and $g^{(2)}(2, 2) = \langle a_2^\dagger a_2^\dagger a_2 a_2 \rangle / n_2^2$. On the two outer resonances ($\delta \omega \approx \pm 2\pi \times 44 \text{MHz}$) we see small dips below unity, indicating photon anti-bunching in both cavities, $g^{(2)}(1, 1) < 1$ and $g^{(2)}(2, 2) < 1$, as well as anti-correlations between photons in distinct cavities, $|g^{(2)}(1, 2)| < 1$. This indicates that the coupling in both cavities is strong enough to generate an optical nonlinearity that converts coherent classical input light into a manifestly non-classical state of the cavity photons. For the resonances at $\delta \omega \approx \pm 2\pi \times 44 \text{MHz}$ the state of the two cavity modes is approximately a superposition of one photon in cavity 1 with cavity 2 empty and one photon in cavity 2 with cavity 1 empty, which gives rise to the anti-correlations shown by $g^{(2)}(1, 1)$, $g^{(2)}(2, 2)$ and $g^{(2)}(1, 2)$. Since cavity 2 is not directly driven, it experiences a lower intensity of incoming photons than cavity 1 and hence exhibits slightly stronger anti-bunching for the same nonlinearity. For this state of the cavity modes, the photons in the two cavities are expected to become entangled. This is confirmed by figure 5(c), which shows the entanglement between modes $a_1$ and $a_2$ as quantified by the logarithmic negativity $E_N$ [39],

$$E_N = \log_2(\text{Tr} \sqrt{\rho_{\text{in}} \rho_{\text{in}}^\dagger}),$$

\[ (22) \]
Figure 5. Steady state of a driven array of two Jaynes–Cummings cavities for $g = 2\pi \times 33.04$ MHz, $\eta = 2\pi \times 10$ MHz, $J = 2\pi \times 20$ MHz, $\bar{\kappa} = 2\pi \times 10.6$ MHz and $\gamma = 2\pi \times 3$ MHz. Panel (a) shows $n_1$ and $n_2$, panel (b) shows $g^{(2)}(1, 1)$, $g^{(2)}(2, 2)$ and $g^{(2)}(1, 2)$ and panel (c) shows the entanglement between modes $a_1$ and $a_2$ as quantified by the logarithmic negativity $E_N$. 

where $\rho_{pt}$ is the partial transpose of the reduced density matrix of the two modes $a_1$ and $a_2$. For comparison, we note that Bell states, the maximally entangled states of two qubits, have $E_N = 1$.

On the inner resonances ($\delta\omega \approx \pm 2\pi \times 24$ MHz), by contrast, only photons involving both cavities are anti-correlated, whereas photons within each cavity bunch. This indicates that photons prefer to stick together in either of the cavities and the associated state contains superpositions of two or more photons in cavity 1 with cavity 2 empty and the reverse configuration. Hence photons in the output of one cavity tend to come in bunches, whereas if one cavity emits a photon the other cavity is unlikely to emit at the same time. Consequently, we also find entanglement between the photon modes for these resonances although less than for the resonances at $\delta\omega \approx \pm 2\pi \times 44$ MHz, which feature higher photon densities.

Close to $\delta\omega = 0$, the photons in the cavities show pronounced bunching. This emerges due to the nonlinear spectrum of our device, which here causes the laser drive to be detuned with respect to single-photon transitions but resonant with multi-photon transitions. In practice
Figure 6. Lambda level structure for the $D_2$ line of $^{87}\text{Rb}$ atoms. The state $|a\rangle$ represents $5^2S_{1/2}(F = 1, m_F = 1)$, $|b\rangle$ stands for $5^2S_{1/2}(F = 2, m_F = 1)$ and $|e\rangle$ is $5^2P_{3/2}(F' = 2, m_F = 2)$. The transitions $|a\rangle \leftrightarrow |e\rangle$ and $|b\rangle \leftrightarrow |e\rangle$ are driven both by the cavity field (green arrows) with coupling strengths $g_a$ and $g_b$, and by external coherent fields (blue arrows) with Rabi frequencies $\Omega_a$ and $\Omega_b$. $\delta_a = \omega_e - \omega_C$ and $\delta_b = \omega_e - (\omega_b - \delta) - \omega_C$. State $|b\rangle$ is stable as there is no electric dipole moment for the transition $|a\rangle \leftrightarrow |b\rangle$.

The findings shown in figures 5(b) and (c) clearly demonstrate the non-classical nature of the light fields generated in our device.

6. Effective spin Hamiltonians

In this section we describe a second experiment that could be performed using such a device. We show how effective spin–spin interactions, as proposed in [28], can be implemented. Here, each cavity interacts with one rubidium atom, whose energy levels $a, b, e$ associated with the $D_2$ line form the lambda structure depicted in figure 6. These levels are coupled both by the cavity photons (with couplings $g_a$ and $g_b$) and by external laser fields (with angular frequencies $\nu_a$, $\nu_b$ and Rabi frequencies $\Omega_a$, $\Omega_b$).

Importantly, the transition $|a\rangle \leftrightarrow |b\rangle$ ($5^2S_{1/2}(F = 1, m_F = 1) \leftrightarrow 5^2S_{1/2}(F = 2, m_F = 1)$) is dipole forbidden and level $|b\rangle$ is thus metastable. Under conditions spelled out below, the dynamics can be constrained to the subspace formed by levels $|a_j\rangle$ and $|b_j\rangle$ of each atom and we can identify $|a_j\rangle$ with spin down, $|\downarrow_j\rangle \equiv |a_j\rangle$, and $|b_j\rangle$ with spin up, $|\uparrow_j\rangle \equiv |b_j\rangle$.

6.1. Outline of the approach

Following the arguments presented in section 3, we assume once again that the atom in the $j$th composite cavity only couples to one normal mode $a_j$ (again, we skip the index $\alpha_0$), for which we maximize the single-atom cooperativity as in section 5. The photons tunnel between
adjacent composite cavities at a rate $J$. The Hamiltonian for this system can thus be written as

$$H = H_A + H_C + H_{AC},$$

where

$$H_A = \sum_{j=1}^{N} \omega_e |e_j\rangle \langle e_j| + \omega_b |b_j\rangle \langle b_j|,$$

$$H_C = \omega_C \sum_{j=1}^{N} a_j^\dagger a_j - J \sum_{j=1}^{N} (a_j^\dagger a_{j+1} + a_j a_{j+1}^\dagger),$$

$$H_{AC} = \sum_{j=1}^{N} \sum_{x=a,b} \left[ \left( \frac{\Omega_k}{2} e^{-i\omega t} + g_x a_j \right) |e_j\rangle \langle x_j| + \text{H.c.} \right].$$

Here, $H_A$ describes the atoms, $H_C$ the cavity field and $H_{AC}$ the interaction between the atoms, lasers and cavity field. Also, $\omega_e$ is the $a-e$ transition frequency and $\omega_b$ the $a-b$ transition frequency.

The Hamiltonian $H_C$ can be decomposed into non-interacting collective photon modes,

$$H_C = \sum_k \omega_k a_k^\dagger a_k,$$

where $\omega_k = \omega_C - 2J \cos k$ and $a_k = \sqrt{\frac{2}{N+1}} \sum_{j=1}^{N} \sin(kj) a_j$ with $k = \frac{\pi l}{N+1}$ and $l = 1, 2, \ldots, N$. It is helpful to move to an interaction picture,

$$H(t) = e^{iH_0 t} (H - H_0)e^{-iH_0 t},$$

where $H$ is given in equation (23) and $H_0$ reads,

$$H_0 = \sum_{j=1}^{N} \left( \omega_e |e_j\rangle \langle e_j| + (\omega_b - \delta)|b_j\rangle \langle b_j| \right) + \sum_k \omega_k a_k^\dagger a_k,$$

with the value of $\delta$ to be chosen later. In this picture, the Hamiltonian (23) becomes

$$H(t) = \delta \sum_{j=1}^{N} |b_j\rangle \langle b_j| + \sum_{j=1}^{N} \sum_{x=a,b} \sum_k \left[ \left( \frac{\Omega_k}{2} e^{i\Delta x t} + g_{x,j,k} e^{i\delta_{k} t} a_k \right) |e_j\rangle \langle x_j| + \text{H.c.} \right],$$

where $\Delta_a = \omega_e - \nu_a$, $\Delta_b = \omega_e - (\omega_b - \delta) - \nu_b$, as illustrated in figure 6, $\delta_k^a = \omega_e - \omega_k$ and $\delta_k^b = \omega_e - (\omega_b - \delta) - \omega_k$. The coupling constants $g_{a,j,k}$ and $g_{b,j,k}$ are related to the couplings $g_a$, respectively, $g_b$ via $g_{x,j,k} = \sqrt{\frac{2}{N+1}} \sin(kj) g_x$ for $x = a, b$.

A judicious choice of rotating frame is $\delta = \omega_b - \frac{1}{2} (\nu_a - \nu_b)$, which ensures that $\delta_k^a = \Delta_k^b - \Delta_a$ for all $k$ and allows a convenient separation of fast rotating terms from near-resonant terms in what follows. In this rotating frame, a second-order adiabatic elimination of the excited levels $|e_j\rangle$ and the photons $a_j$, see appendix A and [40], yields the effective spin-$\frac{1}{2}$ Hamiltonian

$$H_{\text{spin}} = B \sum_{j=1}^{N} \sigma_j^z + \sum_{j \neq l} (J_{j,l} \sigma_j^+ \sigma_l^- + J_{j,l}^* \sigma_j^- \sigma_l^+) + K_{j,l} \sigma_j^+ \sigma_l^-),$$
where the effective transverse field $B$ reads

$$B = \frac{\delta}{2} - \frac{|\Omega_b|^2}{8\Delta_b^2} \left[ \Delta_b - \frac{|\Omega_a|^2}{2\Delta_a} - \frac{|\Omega_a|^2}{2\Delta_a - 4(\Delta_a - \Delta_b)} - \sum_k \left( \frac{|g_{b,j,k}|^2}{\delta_k^a - \Delta_a} + \frac{|g_{a,j,k}|^2}{\delta_k^a - \Delta_b} \right) \right]$$

and the coupling constants are

$$J_{j,l} = \frac{\Omega_a\omega_b}{4\Delta_a \Delta_b} \sum_k g_{b,j,k} g_{a,l,k}, \quad K_{j,l} = \frac{|\Omega_a|^2}{4\Delta_a^2} \sum_k g_{b,l,k} g_{b,j,k} + \frac{|\Omega_b|^2}{4\Delta_b^2} \sum_k g_{a,l,k} g_{a,j,k}.$$  

The Hamiltonian (30) is an appropriate description provided that $|\Delta_x - \delta_x| \ll FSR_W$ for $x, y = a, b$, $|\frac{\Omega_b}{2\Delta_b}| \ll 1$ for $x = a, b$ and $|\frac{\Omega_a g_{l,j,k}}{2\Delta_a (\Delta_x - \delta_x)}| \ll 1$ for $x, y = a, b$, see conditions (A.5). Furthermore, all the eigenmodes of the composite cavity should be sufficiently detuned from the $|e\rangle \rightarrow |a\rangle$ and $|e\rangle \rightarrow |b\rangle$ transitions to avoid Purcell-enhanced atomic relaxation.

The sums over all photon modes, $\sum_k$, depend on the number of cavities in the array. As a starting point for experiments, we focus here on the $N = 2$ case. The explicit expressions for this case are given in appendix B.

6.2. Effective spin dynamics for two coupled cavities

As an example of dynamical evolution under this effective spin Hamiltonian, we consider once again the two coupled cavities described in section 5.2, each containing an $^{87}$Rb atom. The atom in cavity number 1 is prepared in state $|b_1\rangle$, whereas the other atom is placed in state $|a_2\rangle$ to form an initial effective spin state $|\uparrow_1, \downarrow_2\rangle$. The subsequent evolution of this state under the Hamiltonian of equation (30) is illustrated by the dotted and dash-dotted lines in figure 7, which plot the probabilities of the spin-up and spin-down states in cavity 1: $P_{\uparrow,1} = \text{Tr}(|\uparrow_1\rangle\langle\uparrow_1| \tilde{\rho})$ and $P_{\downarrow,1} = \text{Tr}(|\downarrow_1\rangle\langle\downarrow_1| \tilde{\rho})$, $\tilde{\rho}$ being the state of the effective spin system. We see that the spin oscillates in this case with a period of approximately 70 $\mu$s, determined by the coupling constant $K$ in equation (30). The symmetry of the problem ensures that $P_{\uparrow,2} = P_{\downarrow,1}$ and $P_{\downarrow,2} = P_{\uparrow,1}$. The values of $g_a$ and $g_b$ are derived from equation (18) (via equations (1), (14) and (15)), together with the appropriate weightings ($1/\sqrt{2}$ and $1/\sqrt{6}$) relative to the cycling transition. The values of these and the other relevant constants are listed in the caption of figure 7.

In deriving equation (30) we adiabatically eliminated the excited state $|e\rangle$ and therefore ignored the spontaneous emission from the atoms. In reality, this emission dephases the effective spin and causes loss of probability from the three-level system $\{a, b, e\}$. In addition, equation (30) ignores the loss of photons from the cavity. If a real experiment is to emulate the dynamics of a spin chain, as given by the Hamiltonian in equation (30), these rates of dissipation must be small enough. To test whether this is the case, we have calculated the full dynamics of the relevant atomic levels, including spontaneous emission and photon leakage using the master equation given in appendix C. The solid and dashed lines in figure 7 show the probabilities for the atom in cavity 1 to be in states $a$ and $b$: $P_{a} = \text{Tr}(|a_1\rangle\langle a_1| \tilde{\rho})$ and $P_{b} = \text{Tr}(|b_1\rangle\langle b_1| \tilde{\rho})$. These should correspond to the spin-up and -down states of the equivalent spin model, but we see that they do not because the coherence is damped, leaving an incoherent mixture of states $|a\rangle$ and $|b\rangle$. 

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Figure 7. Evolution of effective spin using the two-atom, two-cavity system described in section 5.2. The cavity details are given in the caption of figure 4. Other parameters are \( J = 2\pi \times 100.0 \text{ MHz} \), \( g_a = 2\pi \times 23.36 \text{ MHz} \), \( g_b = 2\pi \times 13.49 \text{ MHz} \), \( \kappa = 2\pi \times 10.59 \text{ MHz} \), \( \Omega_a = 2\pi \times 166.67 \text{ MHz} \), \( \Omega_b = 2\pi \times 394.16 \text{ MHz} \), \( \Delta_a = 2\pi \times 5.00 \text{ 000 GHz} \), \( \Delta_b = 2\pi \times 11.825 \text{ 500 GHz} \), \( \delta_a = 2\pi \times 11.74 \text 0 \text GHz} \), \( \delta_b = 2\pi \times 4.915 \text{ 500 GHz} \) and \( \delta = 2\pi \times 10.0 \text{ 500 GHz} \). These values lead to \( B = 2\pi \times 4.05 \text{ 500 MHz} \), \( J_{j,l} = 2\pi \times 6.188 \text{ 500 kHz} \) and \( K_{j,l} = 2\pi \times 7.145 \text{ 500 kHz} \). Dotted and dash-dotted lines show the probabilities for states \(|\downarrow_1\rangle\), \(|\uparrow_1\rangle\) according to equation (30). Dashed and solid lines show probabilities for states \(|a_1\rangle\) and \(|b_1\rangle\) according to the master equation given in appendix C. The damping has a significant effect.

The situation can be rectified, as illustrated in figure 8(a). Here, the reflectivity of the microcavity mirror has been increased from 99.9 to 99.99% (together with a slight shortening of the microcavity and a minor improvement in the waveguide mirror). This improvement in the microcavity mirror allows the real system to provide a reasonable approximation to the ideal spin evolution, albeit with some residual damping. Figure 8(b) shows the excited state population \( P_{e,1} \) on site number 1, which determines the lifetime \((P_e \times 0.02 \times 2\gamma)^{-1} \simeq 260 \mu s\) for atoms to be lost from the lambda system by spontaneous decay to the state \(S(F = 2, m_F = 2)\). Also shown is the small population \( n_1 \) of cavity photons. Significant further improvement in the coherence is possible in principle if the dissipation in the waveguide cavity is also reduced. In practice, this will require an advance in integrated waveguide technology to achieve smaller propagation losses than the current state of the art.

7. Outlook and summary

Following an initial demonstration of quantum coherent coupling between two neighbouring cavities in such a system, a variety of more sophisticated experiments to ascertain coherent interactions across longer chains become accessible. A stringent test of the coherence properties of transport in extended chains of cavities is provided by the transport of quantum entanglement through such a chain. Most promising in this context appears to be the generation of entanglement between an atomic qubit and the photonic degree of freedom of the cavity in which the atom resides. Once created, the coherent coupling between constituents of the cavity array will lead to transport of the state of the photonic degree of freedom as in a harmonic chain [42]. Alternatively, one may use the natural dynamics of a chain of interacting quantum systems following a sudden quench to generate highly entangled states. Significant two-particle
entanglement will, for example, build up between sites whose distance is proportional to twice the propagation speed in the chain [46]. A basic demonstration of such an experiment is already possible with three sites but may be scaled to larger arrays as the strength of the entanglement decreases only slowly with distance. The presence of entanglement between distant sites may then be verified on the basis of the measurement of simple two-site correlations [41]. An alternative that avoids the verification of entanglement would exploit the quantum coherent effect of dynamical localization [43, 44] in a harmonic chain, induced by sinusoidal modulation of the cavity resonance frequencies. Depending on the frequency of this modulation the excitation transport through the chain will exhibit sharp resonances whose depth can be used to infer the level of quantum coherence in the system without the need for tomography or entanglement verification [45].

Our proposed cavity quantum emulator could also be used to explore quantum many-body physics and to exploit it for the generation of long-distance entanglement. Indeed, systems with finite correlation length, such as the one-dimensional (1D) Heisenberg and XX models, allow sizeable ground-state end-to-end entanglement, independent of the size of the system, provided that simple patterns of site-dependent couplings are selected [48, 49]. Already a chain of four cavities together with local control of interactions would suffice to demonstrate this effect.
In summary, we have presented a practical way to realize an array of coupled cavities that could be used for quantum emulation. The device consists of open Fabry–Perot microcavities, coupled together by a waveguide chip. We have demonstrated that, under suitable conditions, the dynamical evolution is well approximated by a Jaynes–Cummings lattice Hamiltonian and that it is suitable for implementing effective spin Hamiltonians. The quantum emulator discussed here provides a very promising experimental platform for achieving controlled dynamics using photons and quantum emitters. Moreover, it has the potential to go beyond simple demonstrations to achieve significant computational power based on long-range entanglement.

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Appendix A. Derivation of effective spin interactions

The Schrödinger equation containing the Hamiltonian $H$ as in equation (29) reads

$$\frac{d}{dt}|\Psi, t\rangle = -iH(t)|\Psi, t\rangle. \quad \text{(A.1)}$$

This equation can formally be integrated to yield

$$|\Psi, t + T\rangle = |\Psi, t\rangle - i \int_t^{t+T} ds H(s)|\Psi, s\rangle. \quad \text{(A.2)}$$

Iterating the right-hand side results in

$$|\Psi, t + T\rangle = |\Psi, t\rangle + \sum_{n=1}^{\infty} (-i)^n \int_t^{t+T} dt_n H(t_n) \int_{t_{n-1}}^{t_n} dt_{n-1} H(t_{n-1}) \cdots \int_t^{t_1} dt_0 H(t_0)|\Psi, t\rangle. \quad \text{(A.3)}$$

An effective Hamiltonian that accurately describes processes which happen on a time scale $T$ can now be found by performing the time integrations and identifying the dominant terms. For our derivation, we assume that at time $t$, the excited states of the atoms are not occupied and that no photons are present,

$$\langle e|\Psi, t\rangle = 0 \quad \text{and} \quad a_k|\Psi, t\rangle = 0. \quad \text{(A.4)}$$

Furthermore, we assume for the parameters in $H(t)$

$$\left|\frac{\Omega_x}{2\Delta_x}\right| \ll 1 \quad \text{for} \quad x = a, b, \quad \text{(A.5)}$$

$$\left|\frac{\Omega_y g_{y,j,k}}{2\Delta_x (\Delta_y - \delta_y^k)}\right| \ll 1 \quad \text{for} \quad x, y = a, b. \quad \text{(A.6)}$$
We keep terms up to $n = 3$ on the right-hand side of equation (A.3) and, by virtue of equation (A.5), neglect all oscillating terms to arrive at

$$
| \Psi, t+T \rangle = \left( 1 + \sum_{\mu=1}^{4} \frac{(-iH_0T)^{\mu}}{\mu!} + \sum_{\nu=1}^{2} \frac{(-iH_1T)^{\nu}}{\nu!} - iH_2T \right) | \Psi, t \rangle,
$$

(A.7)

where

$$
H_0 = \delta \sum_{j=1}^{N} | b_j \rangle \langle b_j |,
$$

$$
H_1 = -\frac{|\Omega_b|^2}{4\Delta_b} \sum_{j=1}^{N} | b_j \rangle \langle b_j | - \frac{|\Omega_a|^2}{4\Delta_a} \sum_{j=1}^{N} | a_j \rangle \langle a_j |,
$$

$$
H_2 = \sum_{j=1}^{N} \frac{|\Omega_b|^2}{2\Delta_b} \left[ \frac{|\Omega_b|^2}{2\Delta_b} + \frac{|\Omega_a|^2}{2\Delta_a} \right] + \sum_{k} \left( \frac{|g_{b,j,k}|^2}{\delta_k^a - \Delta_b} + \frac{|g_{a,j,k}|^2}{\delta_k^b - \Delta_b} \right) | b_j \rangle \langle b_j | + \frac{|\Omega_a|^2}{2\Delta_a} \sum_{j\neq l} \frac{|g_{a,j,k}|^2 g_{a,l,k}}{\delta_k^a - \Delta_a} | b_j \rangle \langle a_j | \otimes | b_l \rangle \langle a_l | + H.c.
$$

Provided we can choose the time scale $T$ such that $||H_0T|| \ll 1$, $||H_1T|| \ll 1$ and $||H_2T|| \ll 1$ but $\Delta_j T \gg 1$ and $\delta_k^x T \gg 1$ ($x = a, b$), we can write an effective Schrödinger equation,

$$
\frac{| \Psi, t+T \rangle - | \Psi, t \rangle}{T} \approx -i \left( H_0 + H_1 + H_2 \right) | \Psi, t \rangle.
$$

(A.8)

On time scales $T$, the dynamics of our system is thus accurately described by the effective Hamiltonian $\mathcal{H} = H_0 + H_1 + H_2$, which is identical to the effective spin Hamiltonian of equation (30) up to an irrelevant global constant.

**Appendix B. Explicit expressions for the spin parameters in the $N = 2$ case**

$$
B_{N=2} = \frac{\delta}{2} - \frac{|\Omega_b|^2}{8\Delta_b^2} \left[ \Delta_b - \frac{|\Omega_b|^2}{2\Delta_b} - \frac{|\Omega_a|^2}{4(\Delta_a - \Delta_b)} - \frac{|g_b|^2(\delta_b - \Delta_b)}{(\delta_b - \Delta_a)^2 - J^2} - \frac{|g_a|^2(\delta_a - \Delta_b)}{(\delta_a - \Delta_b)^2 - J^2} \right]
$$

$$
+ \frac{|\Omega_a|^2}{8\Delta_a^2} \left[ \Delta_a - \frac{|\Omega_a|^2}{2\Delta_a} - \frac{|\Omega_b|^2}{4(\Delta_b - \Delta_a)} - \frac{|g_a|^2(\delta_a - \Delta_a)}{(\delta_a - \Delta_a)^2 - J^2} - \frac{|g_b|^2(\delta_b - \Delta_a)}{(\delta_b - \Delta_a)^2 - J^2} \right].
$$

(B.1)
\[ J_{J',J}|_{N=2} = \frac{\Omega_a \Omega_b^*}{4 \Delta_a \Delta_b} \frac{g_{a}^* g_{a} J}{(\delta_b - \Delta_a)^2 - J^2}, \quad (B.2) \]
\[ K_{J',J}|_{N=2} = \frac{|\Omega_b|^2}{4 \Delta_b^2} \frac{|g_{b}|^2 J}{(\delta_a - \Delta_b)^2 - J^2} + \frac{|\Omega_a|^2}{4 \Delta_a^2} \frac{|g_{a}|^2 J}{(\delta_b - \Delta_a)^2 - J^2}. \quad (B.3) \]

Here, \( \delta_a = \omega_e - \omega_C \) and \( \delta_b = \omega_e - (\omega_b - \delta) - \omega_C \).

**Appendix C. Master equation for the dynamics of two three-level atoms**

The three levels of \(^{87}\text{Rb} \) depicted in figure 6 are two of the \( 5s^2 S_{1/2} \) ground states, \(|a\rangle = |F = 1, m_F = 1\rangle \), \(|b\rangle = |F = 2, m_F = 1\rangle \), and the \( 5p^2 P_{3/2} \) excited state \(|e\rangle = |F' = 2, m_F' = 2\rangle \). In order to compute the dynamical evolution of this system in the presence of light (here we consider only the \( \sigma^+ \) component of the fields), one needs to include spontaneous emission on the \( \pi \) transition to the \( 5s^2 S_{1/2} \) ground state \(|x\rangle = |F = 2, m_F = 2\rangle \), which is outside the three-level system under consideration. The dynamics is thus described by the master equation

\[
\frac{d\rho}{dt} = -i[H, \rho] + \frac{\gamma}{2} \sum_j (2\sigma_j^{ae} \rho \sigma_j^{ae} - \sigma_j^{ae} \rho - \rho \sigma_j^{ae}) + \frac{\gamma}{6} \sum_j (2\sigma_j^{be} \rho \sigma_j^{be} - \sigma_j^{be} \rho - \rho \sigma_j^{be}) + \frac{\gamma}{3} \sum_j (2\sigma_j^{xe} \rho \sigma_j^{xe} - \sigma_j^{xe} \rho - \rho \sigma_j^{xe}) + \kappa \sum_j (2a_j \rho a_j^\dagger - a_j^\dagger a_j \rho - \rho a_j^\dagger a_j), \quad (C.1)
\]

where \( \sigma_j^{ab} = |a_j\rangle \langle b_j| \) and the atomic transition rates are weighted according to the squares of the respective dipole matrix elements.

**References**

[1] Walls D F and Milburn G 2008 *Quantum Optics* (Berlin: Springer)
[2] Walther H, Varcoe B T H, Englert B-H and Becker T 2006 *Rep. Prog. Phys.* **69** 1325
[3] Raimond J M, Brune M and Haroche S 2001 *Rev. Mod. Phys.* **73** 565
[4] Mabuchi H and Doherty A 2002 *Science* **298** 1372
[5] Thompson R J, Rempe G and Kimble H J 1992 *Phys. Rev. Lett.* **68** 1132
[6] Boca A, Miller R, Birnbaum K M, Boozer A D, McKeever J and Kimble H J 2004 *Phys. Rev. Lett.* **93** 233603
[7] Birnbaum K M, Boca A, Miller R, Boozer A D, Northup T E and Kimble H J 2005 *Nature* **436** 87
[8] Dayan B, Parkins A S, Aoki T, Ostby E P, Vahala K J and Kimble H J 2008 *Science* **319** 1062
[9] Hijioka M, Weber B, Specht H P, Webster S C, Kuhn A and Rempe G 2007 *Nat. Phys.* **3** 253
[10] Armani D K, Kippenberg T J, Spillane S M and Vahala K J 2003 *Ultra-high-Q toroid micro-cavity on a chip* *Nature* **421** 925
[11] Barclay P E, Srinivasan K, Painter O, Lev B and Mabuchi H 2006 *Appl. Phys. Lett.* **89** 131108
[12] Akahane Y, Asano T, Song B-S and Noda S 2003 *Nature* **425** 944
[13] Hennessy K, Badolato A, Winger M, Gerace D, Atatüre M, Gulde S, Fält S, Hu E L and Imamoğlu A 2007 *Nature* **445** 896
[14] Wallraff A, Schuster D I, Blais A, Frunzio L, Huang R-S, Majer J, Kumar S, Girvin S M and Schoelkopf R J 2004 *Nature* **431** 162
[15] Trupke M, Eriksson S, Curtis E A, Mokdadir Z, Kukharenka E and Kraft M 2005 *Appl. Phys. Lett.* **87** 211106
[16] Altug H and Vuckoic J 2004 *Appl. Phys. Lett.* **84** 161

*New Journal of Physics* **13** (2011) 113002 (http://www.njp.org/)
[17] Song B-S, Noda S, Asano T and Akahane Y 2005 Nat. Mater. 4 207
[18] Aoki T, Dayan B, Wilcut E, Bowen W P, Parkins A S, Kippenberg T J, Vahala K J and Kimble H J 2006 Nature 443 671–4
[19] Trupke M, Goldwin J, Darquié B, Dutrier G, Eriksson S, Ashmore J and Hinds E A 2007 Phys. Rev. Lett. 99 063601
[20] Hartmann M J, Brandão F G S L and Plenio M B 2006 Nat. Phys. 2 849
[21] Hartmann M J, Brandão F G S L and Plenio M B 2008 Laser Photonics Rev. 2 527
[22] Angelakis D G, Santos M F and Bose S 2007 Phys. Rev. A 76 R031805
[23] Greentree A D, Tahan C, Cole J H and Hollenberg L C L 2006 Nat. Phys. 2 856
[24] Na N, Utsunomiya S, Tian L and Yamamoto Y 2008 Phys. Rev. A 77 031803
[25] Hartmann M J and Plenio M B 2007 Phys. Rev. Lett. 99 103601
[26] Brandão F G S L, Hartmann M J and Plenio M B 2008 New J. Phys. 10 043010
[27] Hartmann M J, Brandão F G S L and Plenio M B 2008 New J. Phys. 10 033011
[28] Hartmann M J, Brandão F G S L and Plenio M B 2007 Phys. Rev. Lett. 99 160501
[29] Rossini D and Fazio R 2007 Phys. Rev. Lett. 99 186401
[30] Aichhorn M, Hohenadler M, Tahan C and Littlewood P B 2008 Phys. Rev. Lett. 100 216401
[31] Koch J and Le Hur K 2009 Phys. Rev. A 80 023811
[32] Rossini D, Fazio R and Santoro G 2008 Europhys. Lett. 83 47011
[33] Hartmann M J 2010 Phys. Rev. Lett. 104 113601
[34] Leib M and Hartmann M J 2010 New J. Phys. 12 093031
[35] Knap M, Arrigoni E, von der Linden W and Cole J H 2011 Phys. Rev. A 83 023821
[36] Ferré G, Andreani L C, Türeci H E and Gerace D 2010 Phys. Rev. A 82 013841
[37] Ma H, Jen A K-Y and Dalton L R 2002 Adv. Mater. 14 1339–65
[38] Colombe Y, Steinmetz T, Dubois G, Linke F, Hunger D and Reichel J 2007 Nature 450 272
[39] Plenio M B and Virmani S 2007 Quantum Inf. Comput. 7 1
[40] Plenio M B 2005 New J. Phys. 12 090503
Eisert J 2001 PhD Thesis University of Potsdam
[41] James D F V 2000 Fortschr. Phys. 48 823
[42] Audenaert K M R and Plenio M B 2006 New J. Phys. 8 266
[43] Plenio M B, Hartley J and Eisert J 2004 New J. Phys. 6 36
[44] Dunlap D H and Krenke V M 1986 Phys. Rev. B 34 3625
[45] Holthaus M and Hone D W 1996 Phil. Mag. B 74 105–37
[46] Vaziri A and Plenio M B 2010 New J. Phys. 12 085004
[47] Eisert J, Plenio M B, Bose S and Hartley J 2004 Phys. Rev. Lett. 93 190402
[48] Poon J, Chak P, Choi M and Yariv A 2007 J. Opt. Soc. Am. B 24 2763
[49] Giampaolo S M and Illuminati F 2009 Phys. Rev. A 80 050301
Giampaolo S M and Illuminati F 2010 New J. Phys. 12 025019