Inverse boundary value problem for the dynamical heterogeneous Maxwell’s system

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Received 12 December 2011, in final form 12 July 2012
Published 23 August 2012
Online at stacks.iop.org/IP/28/095009

Abstract
We consider the inverse problem of determining the isotropic inhomogeneous electromagnetic coefficients of the non-stationary Maxwell equations in a bounded domain of \( \mathbb{R}^3 \), from a finite number of boundary measurements. Our main result is a Hölder stability estimate for the inverse problem, where the measurements are exerted only in some boundary components. For it, we prove a global Carleman estimate for the heterogeneous Maxwell system under boundary conditions.

1. Introduction
In this paper, we discuss the uniqueness and stability in determining the isotropic electromagnetic coefficients of the dynamical Maxwell equations, by boundary measurement of their solution. More precisely, given a continuous medium with a dielectric permittivity \( \lambda^{-1} \) and a magnetic permeability \( \mu^{-1} \), occupying an open, bounded and simply connected domain \( \Omega \subset \mathbb{R}^3 \) with \( C^\infty \) boundary \( \Gamma = \partial \Omega \), and \( T > 0 \), we consider the following problem for the linear system of Maxwell’s equations:

\[
\begin{align*}
D' - \text{curl } (\mu B) &= 0 \quad &\text{in } Q := \Omega \times (-T, T), \\
B' + \text{curl } (\lambda D) &= 0 \quad &\text{in } Q, \\
\text{div } D = \text{div } B &= 0 \quad &\text{in } Q, \\
D \times \nu &= 0, \quad B \cdot \nu &= 0 \quad &\text{on } \Sigma := \Gamma \times (-T, T),
\end{align*}
\]

where the prime stands for the time derivative. Here, the electric induction \( D \) and the magnetic field \( B \) are three-dimensional vector-valued functions of the time \( t \) and the space variable \( x = (x_1, x_2, x_3) \), and \( \nu = \nu(x) \) denotes the unit outward normal vector to \( \Gamma \). Moreover, we attach the following initial condition to (1.1):

\[
B(x, 0) = B_0(x), \quad D(x, 0) = D_0(x), \quad x \in \Omega.
\]
Assume that $\mu$ and $\lambda$ are scalar functions in $C^2(\Omega)$ obeying
\begin{equation}
\mu(x) \geq \mu_0, \quad \lambda(x) \geq \lambda_0, \quad x \in \Omega,
\end{equation}
for some $\lambda_0 > 0$ and $\mu_0 > 0$. Next, in view of deriving existence and uniqueness results for (1.1), introduce the following functional space:
\[ H(\text{curl}, \Omega) := \{ u \in L^2(\Omega)^3, \text{curl} u \in L^2(\Omega)^3 \}, \]
and denote by $\gamma_1$ the unique linear continuous application from $H(\text{curl}, \Omega)$ into $H^{-1/2}(\Gamma)^3$, satisfying $\gamma_1 u = u \wedge \nu$ when $u \in C_0^\infty(\overline{\Omega})^3$ (see [38] (chapter IX A, theorem 2)). Then, putting
\[ H_0(\text{curl}, \Omega) := \{ u \in H(\text{curl}, \Omega), \gamma_1 u = 0 \}, \]
we see that the operator $iA$, where
\[ A \Phi := \begin{pmatrix} 0 & \text{curl} (\mu \cdot) \\ -\text{curl} (\lambda \cdot) & 0 \end{pmatrix}, \quad \Phi = (D, B) \in \text{Dom}(A) := H_0(\text{curl}; \Omega) \times H(\text{curl}; \Omega), \]
is self-adjoint in $\mathcal{H} := L^2(\Omega)^3 \times L^2(\Omega)^3$, endowed with the scalar product
\[ \langle \Phi, \Phi \rangle_{\mathcal{H}} := \langle \lambda D, D \rangle_{L^2(\Omega)} + \langle \mu B, B \rangle_{L^2(\Omega)}, \quad \Phi = (D, B) \in \mathcal{H}, \quad \dot{\Phi} = (\dot{D}, \dot{B}) \in \mathcal{H}. \]
Further, in light of the last line of (1.1), set
\[ H(\text{div} 0, \Omega) := \{ u \in L^2(\Omega)^3, \text{div} u = 0 \} \quad \text{and} \quad H_0(\text{div} 0, \Omega) := \{ u \in H(\text{div} 0, \Omega), \gamma_0 u = 0 \}, \]
where $\gamma_0$ is the unique linear continuous mapping from $H(\text{div}, \Omega) := \{ u \in L^2(\Omega)^3, \text{div} u \in L^2(\Omega) \}$ onto $H^{-1/2}(\Gamma)$, such that $\gamma_0 u = u \cdot \nu$ when $u \in C_0^\infty(\overline{\Omega})$ (see [16] (chapter IX A, theorem 1)). Since $\mathcal{H}_0 := H(\text{div} 0; \Omega) \times H_0(\text{div} 0; \Omega)$ is a closed subspace of $\mathcal{H}$ and that $\mathcal{H}_0 \subseteq \ker A$, the restriction
\[ A_0 \Phi = A_{\mathcal{H}_0} \Phi := A \Phi, \quad \Phi \in \text{Dom}(A_0) = \text{Dom}(A) \cap \mathcal{H}_0 := \mathcal{V}, \]
is, by Stone’s theorem [17] (chapter XVII A, section 4, theorem 3), the infinitesimal generator of a unitary group of class $C^0$ in $\mathcal{H}_0$. Thus, by rewriting (1.1) and (1.2) into the equivalent form
\[ \begin{cases} 
\dot{\Phi}' = A_0 \Phi \\
\Phi(0) = \Phi_0, 
\end{cases} \quad \text{with} \quad \Phi = (D, B)^T \quad \text{and} \quad \Phi_0 = (D_0, B_0)^T, \]
we obtain the following.

**Lemma 1.1.** Given $(D_0, B_0) \in \mathcal{V}$ there exists a unique strong solution $(D, B)$ to (1.1) starting from $(D_0, B_0)$ within the following class:
\[ (D, B) \in C^0(\mathbb{R}; \mathcal{V}) \cap C^1(\mathbb{R}; \mathcal{H}). \]

Moreover, it holds true from [16] (chapter IX A, remark 1) that
\[ \mathcal{V} = H_{r,0}(\text{curl}, \text{div} 0; \Omega) \times H_{r,0}(\text{curl}, \text{div} 0; \Omega), \]
where
\[ H_{r,0}(\text{curl}, \text{div} 0; \Omega) := \{ u \in H^1(\Omega)^3, \text{div} u = 0 \text{ and } \gamma_\ast u = 0 \}, \quad \ast = \tau, n. \]
For further reference we note from lemma 1.1 that the solution $(D, B)$ to (1.1) and (1.2) actually satisfies
\[ (D, B) \in \cap_{p=0}^m C^0([-T, T]); \quad \text{Dom}(A_0^{m-p}) \text{ provided } (D_0, B_0) \in \text{Dom}(A_0^m) \text{ and } \lambda, \mu \in C^m(\overline{\Omega}), \quad m \geq 1. \]

The main purpose of this paper is to study the inverse problem of determining the dielectric permittivity $\lambda^{-1}$ and the magnetic permeability $\mu^{-1}$ from a finite number of observations on the boundary $\Gamma$ of the solution $(B, D)$ to (1.1) which corresponds to a realistic physical approach. This is an important problem not only in electromagnetics (see [41]) but also in the identification of cracks/flaws in conductors (see [19]) or the localization of lightning discharges (see [38]). On the other hand, we obtain a reconstruction result which involves only a finite number of measurements which is not the case in most of the existing results.
1.1. Inverse problem

For suitable \( B_k, D_k, k = 1, 2 \), we aim to determine \( \lambda(x), \mu(x), x \in \Omega \), from the observation of

\[
B_k^\tau(x, t), \quad D_k^\nu(x, t), \quad (x, t) \in \Sigma, \quad k = 1, 2,
\]

where \( B^\tau = B - (B \cdot \nu)\nu \) (resp. \( D^\nu = (D \cdot \nu)\nu \)) denotes the tangential (resp. normal) component of \( B \) (resp. \( D \)).

Note that only a finite number of measurements are needed in the formulation of this inverse problem. For an overview of inverse problems for the Maxwell system, see the monograph \[39\] by Romanov and Kabanikhin. For actual examples of inverse problems for the dynamical Maxwell system involving infinitely many boundary observations (this is the case when the identification of the electromagnetic coefficients is made from the Dirichlet-to-Neumann map), we refer to Belleshev and Isakov \[3\], Caro \[12\], Caro et al \[13\], Kurylev et al \[31\], Ola et al \[35\] and Salo et al \[40\]. It turns out that a small number of uniqueness and stability results for the inverse problem of determining the electromagnetic parameters of the Maxwell system with a finite number of measurements are available, such as \[33, 34\]. In both cases, their proof is based on the methodology of \[10\] or \[23\], which is by means of a Carleman estimate.

For the formulation with a finite number of observations, Bukhgeim and Klibanov \[10\] proposed a remarkable method based on a Carleman estimate and established the uniqueness for similar inverse problems for scalar partial differential equations. See also Bellassoued \[1, 2\], Bellassoued and Yamamoto \[4, 5\], Benabdallah et al \[7\], Bukhgeim \[8\], Bukhgeim et al \[9\], Cristofol and Roques \[14\], Cristofol and Soccorsi \[15\], Imanuvilov and Yamamoto \[22, 23\], Isakov \[24\], Kha˘ıdarov \[26\], Klibanov \[27, 28\], Klibanov and Timonov \[29\], Klibanov and Yamamoto \[30\], Li and Yamamoto \[33, 34\] and Yamamoto \[43\].

A Carleman estimate is an inequality for a solution to a partial differential equation with weighted \( L^2 \)-norm and is a strong tool also for proving the uniqueness in the Cauchy problem or the unique continuation for a partial differential equation with non-analytic coefficients. Moreover, Carleman estimates have been applied essentially for estimating the energy (e.g., Kazemi and Klibanov \[25\]).

As a pioneering work concerning a Carleman estimate, we refer to Carleman’s paper \[11\] which proved what is later called a Carleman estimate and applied it for proving the uniqueness in the Cauchy problem for a two-dimensional elliptic equation. Since \[11\], the theory of Carleman estimates has been studied extensively. We refer to a general theory by Hörmander \[20\] in the case where the symbol of a partial differential equation is isotropic and functions under consideration have compact supports (that is, they and their derivatives of suitable orders vanish on the boundary of a domain). Later Carleman estimates for functions with compact supports have been obtained for partial differential operators with anisotropic symbols by Isakov \[24\]. For Carleman estimates for functions without compact supports, see Imanuvilov \[21\], Tataru \[42\]. For a direct derivation of pointwise Carleman estimates for hyperbolic equations which are applicable to functions without compact supports, see Klibanov and Timonov \[29\] and Lavrent’ev et al \[32\].

The Carleman estimate for the non-stationary Maxwell system was obtained for functions with compact supports by Eller et al \[18\]. Li and Yamamoto \[33, 34\] prove a Carleman estimate for the two-dimensional Maxwell equations in isomagnetic anisotropic media for functions with compact supports. Lemmas 2.1 and 2.2 are our Carleman estimate for the Maxwell system whose solutions have not necessarily compact supports.

By the methodology of \[10\] or \[23\] with such Carleman estimates, several uniqueness and stability results are available for the inverse problem for the Maxwell system (1.1). That
is, in [33, 34] Li and Yamamoto established the uniqueness in determining three coefficients, using a finite number of measurements.

Li and Yamamoto [34] consider nonstationary Maxwell equations in an anisotropic medium in the $(x_1, x_2, x_3)$-space, where equations of the divergences of electric and magnetic flux densities are also unknown. They address the inverse problem of determining the three independent components of the electric current density from observations on the plane $x_3 = 0$ over a time interval and prove conditional stability in the inverse problem provided the permittivity and the permeability are independent of $x_3$.

In [37], Nicaise and Pignotti consider the heterogeneous Maxwell system defined in an open bounded domain. Under checkable conditions on the coefficients of the principal part, they proved a Carleman-type estimate where some weighted $H^1$-norm of solution is dominated by the $L^2$ norm of the boundary traces $\partial_\nu U$ and $\partial_\nu U_t$, modulo an interior lower-order term. Once homogeneous boundary conditions are imposed the lower-order term can be absorbed by the standard unique continuation theorem. Unfortunately, to our knowledge, these results may not be applied directly to the linearized inverse problem associated with the original problem.

Our argument is based on a new Carleman estimate. In comparison with [33] and [37], our Carleman estimate is advantageous in the following two points.

- We show a Carleman estimate which holds over the whole domain $Q$. We need not assume that the functions under consideration have compact supports and so ours is different from the Carleman estimates presented in [33], and we can establish a Hölder estimate.
- We do not need a priori any unique continuation property and compactness/uniqueness argument to absorb the lower-order interior term. In our approach, we establish a Carleman estimate for $H^1$-solutions of the hyperbolic equation with variable coefficients. This is essential to the proof of our main result, because here our problem is involved with a source term and we cannot use the standard compactness/uniqueness argument as in [37].

1.2. Notations and statement of the main result

In this subsection, we introduce some notation used throughout this text and state the main result of this paper. Pick $x_0 \in \mathbb{R}^3 \setminus \Omega$, set $c(x) = \mu(x)\lambda(x)$ for $x \in \overline{\Omega}$, $c_0 = \mu_0\lambda_0$, where $\mu_0$ and $\lambda_0$ are the same as in (1.3), and assume that the condition

$$\frac{3}{2} |\nabla \log c(x)| |x - x_0| \leq 1 - \frac{\rho}{c_0}, \quad x \in \overline{\Omega},$$

(1.6)

holds true for some $\rho \in (0, c_0)$. This purely technical condition was imposed by the method we use to solve the inverse problem under study, which is by means of the Carleman estimate stated in lemma 2.2 for any weight function $\psi_0$ satisfying the two assumptions (A1) and (A2). More precisely, in the particular case where

$$\psi_0(x) := |x - x_0|^2, \quad x \in \Omega,$$

(1.7)

then (1.6) arises from the classical pseudo-convexity condition expressed by (2.5). The somehow non-natural condition (1.6) is thus closely related to the peculiar expression (1.7) in the sense that another choice of $\psi_0$ fulfilling (A1) and (A2) may eventually lead to a completely different condition on $c(x)$.

Next, for $M_0 > 0$ and two given functions $\mu^2, \lambda^2 \in C^2(\omega)$, where $\omega = \Omega \cap O$ for some neighborhood $O$ of $\Gamma$ in $\mathbb{R}^3$, we define the admissible set of unknown coefficients $\mu$ and $\lambda$ as

$$\Lambda_{\omega}(M_0) = \{ (\mu, \lambda) : \mu^2, \lambda^2 \in C^2(\omega), \mu, \lambda \text{ obeying } (1.3) \text{ and } (1.6); \| (\mu, \lambda) \|_{C^2(\overline{\Omega})} \leq M_0 \text{ and } (\mu, \lambda) = (\mu^2, \lambda^2) \text{ in } \omega \}.$$

(1.8)
Further, the identification of \((\lambda, \mu)\) imposing, as will appear in the sequel, that \((B, D)\) be observed twice, we consider two sets of initial data \((D_0^k, B_0^k), k = 1, 2,\)

\[
D_0^k(x) = (d_1^k(x), d_2^k(x), d_3^k(x))^\top, \quad B_0^k(x) = (b_1^k(x), b_2^k(x), b_3^k(x))^\top, \quad (1.9)
\]

and define the 12 \times 6 matrix:

\[
\mathcal{K}(x) = \begin{pmatrix}
    e_1 \times B_0^1 & e_2 \times B_0^1 & e_3 \times B_0^1 & 0 & 0 & 0 \\
    0 & 0 & 0 & e_1 \times D_0^1 & e_2 \times D_0^1 & e_3 \times D_0^1 \\
    e_1 \times B_0^2 & e_2 \times B_0^2 & e_3 \times B_0^2 & 0 & 0 & 0 \\
    0 & 0 & 0 & e_1 \times D_0^2 & e_2 \times D_0^2 & e_3 \times D_0^2 
\end{pmatrix}, \quad x \in \Omega. \quad (1.10)
\]

We then write \((B_i^k(x, t), D_i^k(x, t))\) the solution to (1.1) with initial data \((B_0^k, D_0^k), k = 1, 2,\) where \((\mu_i, \lambda_i), i = 1, 2,\) is substituted for \((\mu, \lambda)\).

Finally, noting \(\mathcal{H}(\Sigma) = H^1(\Omega) \cap H^2(\Omega)\) the Hilbert space equipped with the norm

\[
\|u\|_{\mathcal{H}(\Sigma)}^2 = \|u\|_{H^1(\Omega)}^2 + \|u\|_{H^2(\Omega)}^2, \quad u \in \mathcal{H}(\Sigma),
\]

we may now state the main result of this paper as follows.

**Theorem 1.** Let \(T > c_0^{1/2} \max_{x \in \partial \Omega} |x - x_0|\) and pick \((B_0^k, D_0^k) \in (H^2(\Omega)^3 \times \mathcal{H}(\Omega))^3 \cap \mathcal{V}\), \(k = 1, 2,\) in such a way that there exists a 6 \times 6 minor \(m(x)\) of the matrix \(\mathcal{K}(x)\) defined in (1.10), obeying

\[
m(x) \neq 0, \quad x \in \Omega \setminus \omega. \quad (1.11)
\]

Further, choose \((\mu_i, \lambda_i) \in \Lambda_{w_i}(M_0), i = 1, 2,\) so that

\[
\| (B_i^k, D_i^k) \|_{C_{(-T,T;W^2,\infty(\Omega))}} \leq M, \quad k = 1, 2, \quad (1.12)
\]

for some \(M > 0\). Then, there are two constants \(C > 0\) and \(\kappa \in (0, 1)\), depending on \(\Omega, \omega, T, M, M_0\), such that we have

\[
\|\mu_1 - \mu_2\|_{H^2(\Omega)} + \|\lambda_1 - \lambda_2\|_{H^2(\Omega)} \leq C \left( \sum_{k=1}^2 \left( \| (B_i^k, D_i^k) \|_{\mathcal{H}(\Sigma)} + \| (D_i^k, D_i^k) \|_{\mathcal{H}(\Sigma)} \right) \right)^\kappa.
\]

Note that condition (1.11), which is independent of the choice of the unknown coefficients \(\mu\) and \(\lambda\), actually relates to the initial functions in (1.2). Moreover, this condition is stable with respect to perturbations in \(C^2\). Namely, if \((B_0^k, D_0^k)\) obeys (1.11), then this is the case for \((B_i^k, D_i^k)\) as well, provided \(\max_{k=1,2} \| (B_0^k, D_0^k) - (B_i^k, D_i^k) \|_{C(\Omega)}\) is sufficiently small. Furthermore, there are actual choices of \((B_0^k, D_0^k), k = 1, 2,\) satisfying (1.11). This can be seen by taking

\[
B_0^1(x) = e_1, \quad D_0^1(x) = e_3, \quad B_0^2(x) = e_2, \quad D_0^2(x) = e_2, \quad x \in \Omega \setminus \omega
\]

and selecting the 6 \times 6 minor formed by rows 2, 3, 4, 9, 10 and 12.

Theorem 1 asserts Hölder stability in determining the principal part within the class defined by (1.8), under the assumption (1.12). Note from (1.5) that such a condition is automatically fulfilled for \(\kappa, \mu \in C^1(\overline{\Omega})\) by chosing the initial data \((B_0^k, D_0^k), k = 1, 2,\) in \(\text{Dom}(A_0^k)\) (which is a dense in \(H\)).

The proof of theorem 1 is based on a Carleman estimate stated in lemma 2.2 under the conditions (1.3) and (1.6). Note that (1.6), which is essential to our argument, is much stronger than the usual uniform ellipticity condition.

The remainder of the paper is organized as follows. A Carleman estimate for the Maxwell system (1.1) is established in section 2, while section 3 contains the proof of theorem 1.
2. Carleman estimate for Maxwell’s system

As already mentioned, this section is devoted to the derivation of a global Carleman estimate for the Maxwell system (1.1).

2.1. The settings

Let us consider the following second-order hyperbolic operator:

$$Pu = \partial_t^2 u(x, t) - \text{div} (c(x) \nabla u) + \mathcal{A}_1(x, t; \partial) u, \quad x \in \Omega, \quad t \in \mathbb{R},$$

(2.1)

where $\mathcal{A}_1$ is a first-order partial operator with $L^\infty(\Omega \times \mathbb{R})$ coefficients, and $c \in C^2(\overline{\Omega})$ obeys

$$c(x) \geq c_0, \quad x \in \overline{\Omega},$$

(2.2)

for some positive constant $c_0$. Putting

$$a(x, \xi) = c(x) |\xi|^2, \quad x \in \Omega, \quad \xi \in \mathbb{R}^3,$$

(2.3)

and recalling the definition of the Poisson bracket of two given symbols $p$ and $q$,

$$\{p, q\}(x, \xi) = \frac{\partial p}{\partial \xi} \cdot \frac{\partial q}{\partial x} - \frac{\partial p}{\partial x} \cdot \frac{\partial q}{\partial \xi} = \sum_{i=1}^n \left( \frac{\partial p}{\partial \xi_i} \frac{\partial q}{\partial x_i} - \frac{\partial p}{\partial x_i} \frac{\partial q}{\partial \xi_i} \right),$$

we introduce two assumptions.

Assumption (A1). There exists $\psi_0 \in C^2(\overline{\Omega}; \mathbb{R}^*_+)$ satisfying

$$\{a, \{a, \psi_0\}\}(x, \xi) > 0, \quad x \in \overline{\Omega}, \quad \xi \in \mathbb{R}^3 \setminus \{0\},$$

(2.4)

where $a$ is given by (2.3).

Since $\overline{\Omega}$ is compact and $a(x, \xi)$ is a homogenous function with respect to $\xi$, it is clear that (2.4) yields the existence of some constant $\varrho > 0$ such that we have

$$\frac{\varrho}{4} \{a, \{a, \psi_0\}\}(x, \xi) \geq 2 \varrho c(x) |\xi|^2, \quad x \in \overline{\Omega}, \quad \xi \in \mathbb{R}^3 \setminus \{0\}.$$

(2.5)

Assumption (A2). The function $\psi_0(x)$ has no critical points on $\overline{\Omega}$:

$$\min_{x \in \overline{\Omega}} |\nabla \psi_0(x)|^2 > 0.$$

Further, with $\varrho$ being the same as in (2.5), fix $\delta > 0$ and $\beta \in (0, \varrho)$, in such a way that, upon eventually enlarging $T$, we have

$$\beta T^2 \geq \max_{x \in \overline{\Omega}} \psi_0(x) + \delta.$$

(2.6)

Hence, picking $\beta_0 > 0$ and setting

$$\psi(x, t) = \psi_0(x) - \beta t^2 + \beta_0, \quad x \in \overline{\Omega}, \quad t \in [-T, T],$$

(2.7)

so that

$$\min_{x \in \overline{\Omega}} \psi(x, 0) \geq \beta_0,$$

we check out from (2.6) that

$$\psi(x, \pm T) \leq \beta_0 - \delta, \quad x \in \overline{\Omega}.$$

(2.8)

Note from (2.7) and (2.8) that

$$\max_{x \in \overline{\Omega}} \psi(x, t) \leq \beta_0 - \frac{\delta}{2}, \quad |t| \in (T - 2\epsilon, T],$$

(2.9)

for some constant $\epsilon \in (0, T/2)$.

In view of (2.7) we may now recall the following global Carleman estimate for second-order scalar hyperbolic equations, with weight function $\varphi : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ defined as

$$\varphi(x, t) = e^{\gamma \psi(x, t)}, \quad x \in \overline{\Omega}, \quad t \in [-T, T],$$

(2.10)

for some fixed $\gamma > 0$. 
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**Theorem 2.** Assume (A1) and (A2). Then, there exist two constants $C_0 > 0$ and $s_0 > 0$ such that for each $s \geq s_0$, the following Carleman estimate:

$$C_0 \int_Q e^{2s \psi} |\nabla v|^2 + |\partial_t v|^2 + s^2 |v|^2 \, dx \, dt \leq \int_Q e^{2s \psi} |Pv(x, t)|^2 \, dx \, dt + \int_\Sigma e^{2s \psi} \left( |\nabla v|^2 + |\partial_t v|^2 + s^2 |v|^2 \right) \, d\sigma \, dt, \tag{2.11}$$

holds true whenever $v \in H^1(Q)$ verifies $\partial_j v(\pm T, \cdot) = 0$ for $j = 0, 1$, and the right-hand side of (2.11) is finite. Here, $P$ is defined by (2.1) and (2.2) and $d\sigma$ denotes the volume form of $\Gamma$.

For the proof, see Bellassoued and Yamamoto [6], where this result is obtained from a direct computation based on integration by parts.

It is worth mentioning that there are actual examples of functions $\psi_0$ fulfilling (A1) and (A2), provided the conductivity function $\psi$ defined in (2.1) verifies (1.6) for some $x_0 \in \mathbb{R}^3 \setminus \Omega$ and $c \in (0, c_0)$, where $c_0$ is the constant defined in (2.2). Indeed, by putting $\psi_0(x) = |x - x_0|^2$ and recalling (2.3), we get through an elementary computation that

$$\frac{1}{4} \{ a, \{ a, \psi_0 \} \} (x, \xi) = 2\epsilon^2 (x) \left( 1 - \frac{\nabla c \cdot (x - x_0)}{2\epsilon} \right) |\xi|^2 + 2c (\nabla c \cdot \xi) (\xi \cdot (x - x_0)), $$

so (1.6) immediately yields

$$\frac{1}{4} \{ a, \{ a, \psi_0 \} \} (x, \xi) \geq 2\epsilon c |\xi|^2.$$

This entails (A1) by (2.2). Moreover, (A2) is evidently true as well since $\nabla \psi_0(x) \neq 0$ for each $x \in \Omega$.

### 2.2. Decoupling of the system of equations

Consider now the following Maxwell system:

$$U' - \text{curl} (\mu_1 V) = f \quad \text{in } Q, \quad V' + \text{curl} (\lambda_1 U) = g \quad \text{in } Q, \quad \text{div } U = \text{div } V = 0 \quad \text{in } Q, \quad U \times v = 0, \quad V \cdot v = 0 \quad \text{on } \Sigma, \tag{2.12}$$

where the source terms $f, g \in H^1(Q; \mathbb{R}^3)$ satisfy the boundary condition:

$$f(x, t) = g(x, t) = 0, \quad (x, t) \in \omega \times (-T, T). \tag{2.13}$$

For further reference, we recall from (2.9) that

$$\max_{x \in \Omega} \psi(x, t) \leq d_0 := e^{\gamma(t - s/2)}, \quad |t| \in [T - 2\epsilon, T), \tag{2.14}$$

and then state the main result of section 2.2.

**Lemma 2.1.** Assume (A1) and (A2) and let $h = (f, g) \in H^1(Q; \mathbb{R}^3)^2$ obey (2.13). Then, we may find two constants $C_1 > 0$ and $s_1 > 0$, for which the Carleman estimate:

$$C_1 \int_Q e^{2s \psi} |(\nabla_x W)|^2 + s^2 |W|^2 \, dx \, dt \leq \int_Q e^{2s \psi} |(\nabla_x h)|^2 + |h|^2 \, dx \, dt + s^3 e^{2s \psi} \|W\|_{H^s(Q)}^2 \, d\sigma \, dt$$

$$+ \int_\Sigma e^{2s \psi} \left( |\nabla W|^2 + |W|^2 + s^2 |W|^2 \right) d\sigma \, dt, \tag{2.15}$$

is true for any $W = (U, V)$ solution to the Maxwell system (2.12), whenever $s \geq s_1$. \hfill 7
**Proof.** The first step of the proof involves bringing (2.12) into two independent systems of decoupled equations. Namely, by differentiating the first line in (2.12) with respect to \( t \), and then substituting \( g - \text{curl} (\lambda_1 U) \) for \( V' \) in the obtained equality, we find that
\[
U'' + \text{curl} (\mu_1 \text{curl} (\lambda_1 U)) = f' + \text{curl} (\mu_1 g) \quad \text{in } Q.
\]
This entails \( U'' + \text{curl} (\mu_1 \text{curl} (\lambda_1 U)) + \text{curl} (\mu_1 \nabla \lambda_1 \times U) = f' + \text{curl} (\mu_1 g) \), and hence \( U'' + \mu_1 \lambda_1 \text{curl} (\text{curl} U) + \nabla (\mu_1 \lambda_1) \times \text{curl} U + \text{curl} (\mu_1 \nabla \lambda_1 \times U) = f' + \text{curl} (\mu_1 g) \), in \( Q \).

From this, the well-known identity \( \text{curl} \), \( \text{curl} U = -\Delta U + \nabla \text{div} U \) and the third line of (2.12) then follows that
\[
U'' - \mu_1 \lambda_1 \Delta (U) + \mathcal{R}_1 U = f' + \text{curl} (\mu_1 g) \quad \text{in } Q,
\]
where \( \mathcal{R}_1 = \mathcal{R}_1 (x, \partial) \) is some first-order operator with bounded coefficients in \( \Omega \).

Arguing in a similar way, we find that
\[
V'' - \mu_1 \lambda_1 \Delta (V) + \mathcal{R}_1 V = g' - \text{curl} (\lambda_1 f) \quad \text{in } Q,
\]
for another first-order operator \( \mathcal{R}_1 = \mathcal{R}_1 (x, \partial) \) with bounded coefficients in \( \Omega \).

Therefore, putting (2.12) and (2.16) and (2.17) together, we end up finding that any solution \( W = (U, V) \) to the Maxwell system (2.12) verifies
\[
\begin{align*}
\begin{cases}
U'' - \mu_1 \lambda_1 \Delta (U) + \mathcal{R}_1 U = G_1 & \text{in } Q, \\
U \times v = 0, \text{ curl} (\lambda_1 U) \cdot v = 0 & \text{on } \Sigma,
\end{cases}
\end{align*}
\]
and
\[
\begin{align*}
\begin{cases}
V'' - \mu_1 \lambda_1 \Delta (V) + \mathcal{R}_1 V = G_2 & \text{in } Q, \\
V \cdot v = 0, \text{ curl} (\mu_1 V) \times v = 0 & \text{on } \Sigma,
\end{cases}
\end{align*}
\]
where \( G_1 = f' + \text{curl} (\mu_1 g) \) and \( G_2 = g' - \text{curl} (\lambda_1 f) \).

Further, consider a cut-off function \( \eta \in C^\infty (\mathbb{R}; [0, 1]) \) fulfilling
\[
\eta(t) = \begin{cases} 1 & \text{if } |t| < T - 2\epsilon, \\
0 & \text{if } |t| \geq T - \epsilon,
\end{cases}
\]
where \( \epsilon \) is the same as in (2.9), and set
\[
U_z = \eta U, \quad V_z = \eta V, \quad K_1 = \eta G_1 + 2\eta' U' + \eta'' U, \quad K_2 = \eta G_2 + 2\eta' V' + \eta'' V,
\]
in such a way that we have
\[
\begin{align*}
\begin{cases}
U_z'' - \mu_1 \lambda_1 \Delta (U_z) + \mathcal{R}_1 U_z = K_1 & \text{in } Q, \\
U_z \times v = 0, \text{ curl} (\lambda_1 U_z) \cdot v = 0 & \text{on } \Sigma
\end{cases}
\end{align*}
\]
and
\[
\begin{align*}
\begin{cases}
V_z'' - \mu_1 \lambda_1 \Delta (V_z) + \mathcal{R}_1 V_z = K_2 & \text{in } Q, \\
V_z \cdot v = 0, \text{ curl} (\mu_1 V_z) \times v = 0 & \text{on } \Sigma
\end{cases}
\end{align*}
\]
directly from (2.18). Then, each of the two above systems being a principally scalar hyperbolic system, it follows from the two identities
\[
U_z (\cdot, \pm T) = U_z' (\cdot, \pm T) = 0, \quad V_z (\cdot, \pm T) = V_z' (\cdot, \pm T) = 0,
\]
and theorem 2, that \( W_z = (U_z, V_z) \) obeys the Carleman estimate
\[
C_0 \int_Q e^{2\varphi} s(|\nabla_x W_z|^2 + s^2 |W_z|^2) \, dx \, dt
\]
\[\leq \sum_{j=1,2} \int_0^T e^{2\varphi} |K_j(x, t)|^2 \, dx \, dt
\]
\[+ \int_\Sigma s e^{2\varphi} (|\nabla W_z|^2 + |W_z'|^2 + s^2 |W_z|^2) \, d\sigma \, dt,
\]
for all \( s \geq s_0 \). Here, we have used theorem 2 for the diagonal system \( U_z'' - \mu_1 \lambda_1 \Delta (U_z) \) and that we can absorb the non-decoupled first term \( \mathcal{R}_1 U_z \).
Moreover, as \( \eta' \) and \( \eta'' \) both vanish in \((-T + 2\epsilon, T - 2\epsilon)\) by (2.14), there is a constant \( C > 0 \) such that
\[
\sum_{j=1,2} \int_Q e^{2\psi} |K_j(x,t)|^2 \, dx \, dt \leq C \int_Q e^{2\psi} (|h|^2 + |\nabla h|^2) \, dx \, dt + e^{2\delta_0s} \|W\|_{\dot{H}^1(Q)}^2,
\]
according to (2.20) and (2.21) and since
\[
\int_Q e^{2\psi} s(|\nabla h|^2 + s^2|W|^2) \, dx \, dt \leq C \int_Q e^{2\psi} s(|\nabla h|^2 + s^2|W|^2) \, dx \, dt + s^2 e^{2\delta_0s} \|W\|_{\dot{H}^1(Q)}^2,
\]
we obtain the result. \( \square \)

### 2.3. Reduction of the boundary terms

The method used to derive a global Carleman estimate for the solution to (1.1) is to replace the local boundary problem (1.1) in \( \Omega \times (-T, T) \) by an equivalent one stated on the half-space \( \mathbb{R}^3_+ \times (-T, T) \). This is possible since the boundary \( \Gamma \) can be represented as the zero level set of some \( C^\infty \) function in \( \mathbb{R}^3 \). Namely, \( \Gamma \) being a \( C^\infty \) surface, there exist \( \theta \in C^\infty (\mathbb{R}^3) \) and some neighbourhood \( \mathcal{V} \) of \( \Gamma \) in \( \mathbb{R}^3 \) such that \( \Gamma = \{ x \in \mathbb{R}^3, \theta(x) = 0 \} \). We choose \( \mathcal{V} \) so small that \( \mathcal{V} \subset \mathcal{O} \), where \( \mathcal{O} \) is defined in section 1.2, write \( y = (y_1, y_2, y_3) = (y', y_3) \) the system of normal geodesic coordinates, where \( y' = (y_1, y_2) \) are orthogonal coordinates in \( \Gamma \) and \( y_3 = \theta(x) \) is the normal coordinate, and call \( x = \Phi(y) \), where \( \Phi(y) > 0 \) for all \( y \in \mathcal{V} := \Phi^{-1}(\mathcal{V}) \), the corresponding coordinates mapping. As \( \hat{\Gamma} := \Phi^{-1}(\Gamma) = \{ y \in \mathcal{V}, y_3 = 0 \} \subset \mathbb{R}^2 \), we may assume that \( \hat{\mathcal{V}} = \Phi^{-1}(\mathcal{V}) \) is a cylinder of the form \( \hat{\mathcal{V}} \times (-r, r) \) with \( r > 0 \).

Further, the Euclidean metric in \( \mathbb{R}^3 \) inducing the Riemannian metric with diagonal tensor \( g \),
\[
g(y) = \Phi'(y) \Phi'(y) = \text{Diag}(g_1, g_2, g_3), \quad y \in \hat{\mathcal{V}},
\]
we use the notation of [36] and note \( \left[ \frac{1}{\sqrt{g_1}}, \frac{1}{\sqrt{g_2}}, \frac{1}{\sqrt{g_3}} \right] \) the orthonormal basis associated with \( g \) to the differential basis of vector fields \( \left[ \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_3} \right] \). For any vector field \( X(y) \) expressed with respect to the Euclidian basis \( \left[ \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_3} \right] \) as \( X(y) = \sum_{i=1}^3 \alpha_i(y) \frac{\partial}{\partial y_i} \), we have an alternative representation \( \hat{X}(y) \) with respect to the new basis vectors \( \left[ \frac{\partial}{\partial \hat{y}_1}, \frac{\partial}{\partial \hat{y}_2}, \frac{\partial}{\partial \hat{y}_3} \right] \), given by
\[
\hat{X}(y) = \sum_{i=1}^3 \hat{\alpha}_i(y) \frac{\partial}{\partial \hat{y}_i}, \quad \hat{\alpha}_i(y) = \Psi \Psi(y) \alpha(y), \quad \Psi(y) = \Phi'(y) g^{-1/2}(y), \quad y \in \hat{\mathcal{V}}.
\]
(2.22)

The divergence (resp. curl) operator of any vector field \( \hat{X}(y) = \sum_{i=1}^3 \hat{\alpha}_i(y) \frac{\partial}{\partial \hat{y}_i} \) with respect to the local coordinates \( (y_1, y_2, y_3) \) is denoted by \( \text{div} \, \hat{X} \) (resp. \( \text{curl} \, \hat{X} \)), and can be brought into the form
\[
\text{div} \, \hat{X} = \frac{1}{\sqrt{\det g}} \sum_{j=1}^3 \frac{1}{\sqrt{g_j}} \frac{\partial}{\partial y_j} \left( \sqrt{\det g} \hat{\alpha}_j(y) \right) = \sum_{j=1}^3 \frac{1}{\sqrt{g_j}} \frac{\partial \hat{\alpha}_j}{\partial y_j} + A(y) \cdot \hat{X}(y),
\]
(2.23)

where \( A(y) \) is some three-dimensional vector (resp.
\[
\text{curl} \, \hat{X} = \frac{1}{2} \sum_{i,j=1}^3 \left( \frac{1}{\sqrt{g_j}} \frac{\partial \hat{\alpha}_i}{\partial y_j} - \frac{1}{\sqrt{g_i}} \frac{\partial \hat{\alpha}_j}{\partial y_i} \right) \frac{1}{\sqrt{g_i}} \frac{\partial}{\partial y_j} \times \frac{1}{\sqrt{g_j}} \frac{\partial}{\partial y_i} + M(y) \hat{X}(y),
\]
(2.24)
where $M(y)$ is some matrix function, whereas the outward normal vector field to $\hat{\Gamma}$ at $y \in \hat{\Gamma}$ is given by

$$\mathbf{n}(y) = -\frac{1}{\sqrt{g}} \frac{\partial}{\partial y_3}. \tag{2.25}$$

In light of (2.22)–(2.25), we find out by performing the change of variable $x = \Phi(y)$ in (2.12), with the space variable $x$ being restricted to be in $\Omega \cap \mathcal{V} \subset \omega \cap \mathcal{V}$, that

$$\begin{align*}
\hat{U}' &= \text{curl}_x (\hat{\mu}_1 \hat{\nu}) = 0 \quad \text{in} \quad \hat{\nu} \times (-T, T), \\
\hat{V}' &= \text{curl}_x (\hat{\nu}_1 \hat{U}) = 0 \quad \text{in} \quad \hat{\nu} \times (-T, T), \\
\text{div}_x \hat{U} &= \text{div}_x \hat{\nu} = 0 \quad \text{in} \quad \hat{\nu} \times (-T, T), \\
\hat{U} \times \hat{\nu} &= 0, \quad \hat{V} \cdot \hat{\nu} = 0 \quad \text{on} \quad \hat{\Gamma} \times (-T, T),
\end{align*} \tag{2.26}$$

where we have set

$$\hat{\mu}_1(y) = \Psi(y) \lambda_1(\Phi(y)) \Psi(y), \quad \hat{\nu}_1(y) = \Psi(y) \lambda_1(\Phi(y)) \Psi(y), \quad y \in \hat{\nu} \subset \Phi^{-1}(\omega).$$

Here, we used the identity $h(\Phi(y), t) = (f(\Phi(y), t), g(\Phi(y), t)) \equiv 0$ for $t \in (-T, T)$ and $y \in \hat{\nu}$, arising from (2.13). Further, noting $\hat{U} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)$ and $\hat{\nu} = (\hat{v}_1, \hat{v}_2, \hat{v}_3)$, the last equation in (2.26) reads $\hat{u}_1 = \hat{u}_2 = \hat{v}_3 = 0$ on $\hat{\Gamma}$, so we find that

$$\begin{align*}
\hat{U}_\Gamma &= 0, \quad \hat{\nu}_\Gamma = 0 \quad \text{on} \quad \hat{\Gamma}, \tag{2.27}
\end{align*}$$

where $\hat{U}_\Gamma$ (resp. $\hat{\nu}_\Gamma$) denotes the tangential (resp. normal) component of $\hat{U}$ (resp. $\hat{\nu}$). From this, (2.23) and the third line in (2.26) it then follows that

$$\frac{1}{\sqrt{g_1}} \frac{\partial \hat{u}_3}{\partial y_3} = -\left( \frac{1}{\sqrt{g_1}} \frac{\partial \hat{u}_1}{\partial y_1} + \frac{1}{\sqrt{g_2}} \frac{\partial \hat{u}_2}{\partial y_2} \right) - A(y) \cdot \hat{U} = -A(y) \cdot \hat{U},$$

whence

$$\nabla \hat{u}_3 = \frac{1}{\sqrt{g_1}} \frac{\partial \hat{u}_3}{\partial y_1} + \frac{1}{\sqrt{g_2}} \frac{\partial \hat{u}_3}{\partial y_2} - (A(y) \cdot \hat{U}) \hat{e}_3 \quad \text{on} \quad \hat{\Gamma}. \tag{2.28}$$

Further, as $\nabla \hat{u}_1 = \frac{1}{\sqrt{g_1}} \frac{\partial \hat{u}_1}{\partial y_1} \hat{e}_1 + \frac{1}{\sqrt{g_3}} \frac{\partial \hat{u}_2}{\partial y_3} \hat{e}_3$ and $\nabla \hat{u}_2 = \frac{1}{\sqrt{g_1}} \frac{\partial \hat{u}_2}{\partial y_1} \hat{e}_1 - (A(y) \cdot \hat{U}) \hat{e}_2$ on $\hat{\Gamma}$, according to (2.27) and (2.28) then yields

$$|\nabla \hat{U}|^2 = \sum_{j=1}^{3} |\nabla \hat{u}_j|^2 \leq C \left( \left| \frac{\partial \hat{u}_1}{\partial y_3} \right|^2 + \left| \frac{\partial \hat{u}_2}{\partial y_3} \right|^2 + \left| \frac{\partial \hat{u}_1}{\partial y_1} \right|^2 + \left| \frac{\partial \hat{u}_2}{\partial y_2} \right|^2 + |\hat{U}_\Gamma|^2 \right) \quad \text{on} \quad \hat{\Gamma}, \tag{2.29}$$

where, for the sake of notational simplicity, we shall use the generic constant $C > 0$ in the remaining of section 2.3. On the other hand, since

$$\text{curl}_x \hat{U} \times \hat{\nu} = \left( \frac{1}{\sqrt{g_1}} \frac{\partial \hat{u}_1}{\partial y_3} - \frac{1}{\sqrt{g_2}} \frac{\partial \hat{u}_2}{\partial y_1} \right) \frac{1}{\sqrt{g_1 g_3}} \frac{\partial}{\partial y_1} - \left( \frac{1}{\sqrt{g_3}} \frac{\partial \hat{u}_3}{\partial y_1} \right) \frac{1}{\sqrt{g_2 g_3}} \frac{\partial}{\partial y_2} + M(y) \hat{U}_\Gamma,$$

by (2.24) and (2.25), we have

$$\left| \frac{\partial \hat{u}_1}{\partial y_3} \right|^2 + \left| \frac{\partial \hat{u}_2}{\partial y_3} \right|^2 \leq C \left( |\text{curl}_x \hat{U}|^2 + |\hat{U}_\Gamma|^2 \right) \quad \text{on} \quad \hat{\Gamma}.$$ 

In view of (2.29), this entails

$$|\nabla \hat{U}|^2 \leq C \left( |\text{curl}_x \hat{U}|^2 + \left| \frac{\partial \hat{u}_1}{\partial y_1} \right|^2 + \left| \frac{\partial \hat{u}_2}{\partial y_2} \right|^2 + \left| \hat{U}_\Gamma \right|^2 \right) \quad \text{on} \quad \hat{\Gamma}. \tag{2.30}$$
As a consequence, we have

$$|\nabla \hat{U}|^2 \leq C (|\text{curl}_x (\hat{\lambda} \hat{U})|^2 + |\nabla T \hat{U}|^2 + |\hat{U}|^2),$$

whence

$$|\nabla \hat{U}|^2 \leq C (|\hat{V}|^2 + |\nabla T \hat{U}|^2 + |\hat{U}|^2) \text{ on } \Gamma,$$  \hspace{1cm} (2.31)

by the second line of (2.26).

Similarly, as $\text{div}_x \hat{V} = 0$ in $\hat{V}$ from the third line of (2.26), we obtain from (2.23) that

$$\frac{1}{\sqrt{\gamma_3}} \frac{\partial \tilde{v}_3}{\partial y_3} = - \left( \frac{1}{\sqrt{\gamma_1}} \frac{\partial \tilde{u}_1}{\partial y_1} + \frac{1}{\sqrt{\gamma_2}} \frac{\partial \tilde{u}_2}{\partial y_2} \right) - A(y) \cdot \hat{V} \text{ on } \hat{\Gamma}.$$

This, combined with (2.27), yields

$$|\nabla \tilde{v}_3|^2 \leq C \left( \left| \frac{\partial \tilde{u}_1}{\partial y_1} \right|^2 + \left| \frac{\partial \tilde{u}_2}{\partial y_2} \right|^2 + \left| \tilde{V}'_1 \right|^2 \right),$$

and consequently

$$|\nabla \tilde{V}|^2 = \sum_{j=1}^3 |\nabla \tilde{v}_j|^2 \leq C \left( \left| \nabla T \tilde{V}_1 \right|^2 + \left| \tilde{V}'_1 \right|^2 + \left| \frac{\partial \tilde{u}_1}{\partial y_3} \right|^2 + \left| \frac{\partial \tilde{u}_2}{\partial y_3} \right|^2 \right)^2 \text{ on } \hat{\Gamma}. \hspace{1cm} (2.32)$$

Furthermore, in light of (2.27) and (2.30) where $\tilde{V}$ (resp. $\tilde{u}_j, j = 1, 2, 3$) is substituted for $\hat{U}$ (resp. $\hat{u}_j, j = 1, 2, 3$), we see that $\left| \frac{\partial \tilde{u}_1}{\partial y_3} \right|^2 + \left| \frac{\partial \tilde{u}_2}{\partial y_3} \right|^2$ is upper bounded, up to some multiplicative constant, by $|\text{curl}_x \tilde{V}|^2 + |\tilde{V}'_1|^2$, and hence by $|\text{curl}_x (\mu_1 \tilde{V})|^2 + |\tilde{V}'_1|^2$, on $\hat{\Gamma}$. From this and the first line of (2.26), it then follows that

$$\left| \frac{\partial \tilde{u}_1}{\partial y_3} \right|^2 + \left| \frac{\partial \tilde{u}_2}{\partial y_3} \right|^2 \leq C (|\tilde{U}'|^2 + |\tilde{V}'|^2) \text{ on } \hat{\Gamma},$$

so, we end up obtaining with the aid of (2.32):

$$|\nabla \tilde{V}|^2 \leq C (|\tilde{U}'|^2 + |\nabla T \tilde{V}_1|^2 + |\tilde{V}'|^2) \text{ on } \hat{\Gamma}. \hspace{1cm} (2.33)$$

Finally, putting (2.15), (2.31) and (2.33) together, we may state the main result of section 2.3.

**Lemma 2.2.** Assume (A1) and (A2) and put $h = (f, g)$. Then, there are two constants $C_2 > 0$ and $s_2 > 0$ such that the Carleman estimate

$$C_2 s \int_Q e^{2\nu} \left( s^2 |W|^2 + |\nabla_{x,y} W|^2 \right) \, dx \, dt \leq \int_Q e^{2\nu} \left( |h|^2 + |\nabla_{x,y} h|^2 \right) \, dx \, dt + B_{x,y}(W) + s^2 e^{2\nu} \|W\|_{H^s(x)}^2,$$

where

$$B_{x,y}(W) = \int_{\Sigma} s e^{2\nu} \left( |\nabla T \tau V|^2 + |\nabla T U_v|^2 + |U_v'|^2 + |\tau V'|^2 + s^2 (|U_v|^2 + |\tau V|^2) \right) \, d\sigma \, dt, \hspace{1cm} (2.34)$$

holds true for every solution $W = (U, V)$ to (2.12), provided $s \geq s_2$.  


3. Inverse problem

This section contains the proof of theorem 1, which is divided into five steps. Firstly, the unknown parameters \( \lambda \) and \( \mu \) are brought to the source term of the linearized system associated with (1.1), governing the variation induced on the solution to (1.1) by perturbing the permittivity by \( \lambda \) and the permeability by \( \mu \). The second step follows the idea of Bukhgeim and Klibanov presented in [10], which is to differentiate the linearized system with respect to \( t \) in order to move the unknown coefficients under the initial condition. The next step is to bound the energy of this system at time \( t = 0 \) with the aid of the Carleman inequality of theorem 2. The fourth step involves relating \( \lambda \) and \( \mu \) to the above-mentioned estimate through the Carleman inequality for stationary (\( \text{div, curl} \))-systems, stated in lemma 3.1. This is rather technical and lengthy so we proceed in a succession of the two lemmas 3.3 and 3.4. The last step, detailed in section 3.3, is to derive the desired result from the estimates established in lemmas 3.3 and 3.4.

In the remainder of this text, \( x_0 \) is a fixed point in \( \mathbb{R}^3 \setminus \overline{\Omega}_0 \), and we choose as in (1.7) \( \psi_0(x) := |x - x_0|^2 \) for each \( x \in \overline{\Omega}_0 \) and \( \psi_0(x) := \varphi(x, 0) \), where \( \varphi \) denotes the function defined by (2.7) and (2.10). Moreover, for the sake of notational simplicity, we shall use the generic constant \( C > 0 \) in the various estimates of sections 3.2 and 3.3.

3.1. Linearized inverse problem

Given \( (\mu_i, \lambda_i) \in \Lambda_{\mu_0}(M_0), i = 1, 2 \), and \( (B_{0i}^k, D_{0i}^k) \in H^2(\Omega)^3 \times H^2(\Omega)^3 \), \( k = 1, 2 \), we consider the solution \( (B_{0i}^k, D_{0i}^k) \) to the system (1.1) where \( (\lambda_i, \mu_i) \) is substituted for \( (\lambda, \mu) \), with initial condition (1.2) where \( (B_0, D_0) = (B_{0i}^k, D_{0i}^k) \). Hence, putting

\[
\mu = \mu_1 - \mu_2, \quad \lambda = \lambda_1 - \lambda_2,
\]

and setting

\[
f_k = \text{curl} (\mu B_{0i}^2), \quad g_k = -\text{curl} (\lambda D_{0i}^2),
\]

we find by a straightforward computation that \( U_k = D_{1i}^k - D_{2i}^k \) and \( V_k = B_{1i}^k - B_{2i}^k \) satisfy the system:

\[
\begin{align*}
U_k' - \text{curl} (\mu_1 V_k) &= f_k & \text{in } Q, \\
V_k' + \text{curl} (\lambda_1 U_k) &= g_k & \text{in } Q, \\
\text{div } U_k &= \text{div } V_k = 0 & \text{in } Q, \\
U_k \times v &= 0, \quad V_k \cdot v &= 0 & \text{on } \Sigma,
\end{align*}
\]

with the initial data

\[
U_k(x, 0) = 0, \quad V_k(x, 0) = 0.
\]

Further, by using the notation

\[
X_{k,j}(x, t) = \partial_t^j X_k(x, t) \quad \text{for } X = U, V, f, g \quad \text{and} \quad j \in \mathbb{N}^*,
\]

it turns out by differentiating (3.2) \( j \)-times with respect to \( t \) that

\[
\begin{align*}
U_{k,j}' - \text{curl} (\mu_1 V_{k,j}) &= f_{k,j} & \text{in } Q, \\
V_{k,j}' + \text{curl} (\lambda_1 U_{k,j}) &= g_{k,j} & \text{in } Q, \\
\text{div } U_{k,j} &= \text{div } V_{k,j} = 0 & \text{in } Q, \\
U_{k,j} \times v &= 0, \quad V_{k,j} \cdot v &= 0 & \text{on } \Sigma,
\end{align*}
\]

for \( j = 1, 2, \ldots, 7 \).
and, due to (3.1) and (3.2), that $U_{k,1}$ and $V_{k,1}$ satisfy the initial condition
\[ U_{k,1}(x, 0) = \text{curl} (\mu B^1_0), \quad V_{k,1}(x, 0) = -\text{curl} (\lambda D^1_0). \] (3.6)
As will appear in sections 3.2 and 3.3, the main benefit of dealing with (3.1)–(3.6) in the analysis of the inverse problem of determining $\lambda$ and $\mu$ is the presence of these two unknown coefficients under the initial condition (3.6).

### 3.2. Preliminary estimates

Let $j$ and $k$ be in $[1, 2]$. As $W_{k,j} = (U_{k,j}, V_{k,j})$ is a solution to (3.5), we note from lemma 2.2 that
\[
C_2 s \int_Q e^{2sp} (s^2 |W_{k,j}|^2 + |\nabla_s W_{k,j}|^2) \, dx \, dt \
\leq \int_Q e^{2sp} (|h_{k,j}|^2 + |\nabla_s h_{k,j}|^2) \, dx \, dt + \mathcal{B}_{k,\nu}(W_{k,j}) \
+ s^2 e^{2sp} \|W_{k,j}\|_{H^2(Q)}^2 := \mathcal{A}_k(s),
\] (3.7)
for each $s \geq \sigma_2$, where $h_{k,j} = (h_{k,j}, g_{k,j})$ and $\mathcal{B}_{k,\nu}$ is given by (2.34).

Moreover, by the assumption (1.11) we can derive
\[
\sum_{k=1}^2 |B^1_k(x)|^2 \geq c_+, \quad \quad \sum_{k=1}^2 |D^1_k(x)|^2 \geq c_+ , \quad x \in \Omega \setminus \omega
\] (3.8)
for some positive constant $c_+$. Indeed, if $B^1_k(x) \neq 0$ or $B^2_k(x) \neq 0$ for all $x \in \overline{\Omega} \setminus \omega$ then $\sum_{k=1}^2 |B^1_k(x)|^2 > 0$ in the compact set $\overline{\Omega} \setminus \omega$. Now, if there exist $x_1 \in \overline{\Omega} \setminus \omega$ such that $B^1_k(x_1) = 0$, then by (1.11) we have $B^2_k(x) \neq 0$ for any $x \in \overline{\Omega} \setminus \omega$.

Further we recall from [36] the following Carleman estimate for stationary (\text{div}, \text{curl})-systems.

**Lemma 3.1.** There exist two positive constants $s_3$ and $C_3$ depending only on $\psi_0$ and $\Omega$, such that we have
\[
C_3 s \int_\Omega e^{2sp} |u|^2 \, dx \leq \int_\Omega e^{2sp} (|\text{curl} u|^2 + |\text{div} u|^2) \, dx,
\] (3.9)
for each $s \geq s_3$ and $u \in H^1_0(\Omega)$.

Prior to the proof of lemma 3.3, we establish the following technical result, needed in the derivation of (3.10) and (3.11).

**Lemma 3.2.** There exists a constant $s_+ > 0$ depending only on $T$ such that we have
\[
\int_{Q} |z(x, 0)|^2 \, dx \leq 2 \left(s \int_{Q} |z(x, t)|^2 \, dx \, dt + s^{-1} \int_{Q} |z'(x, t)|^2 \, dx \, dt\right),
\] for all $s \geq s_+$ and $z \in H^1(-T, T; L^2(\Omega))$.

**Proof.** Let $\eta \in C^\infty([-T, T]; [0, 1])$ fulfills (2.19) for some fixed $\epsilon \in (0, T/2)$. Then, the following identity:
\[
\int_\Omega |z(x, 0)|^2 \, dx = \int_{-T}^0 \frac{d}{dt} \left( \int_\Omega \eta^2(t) |z(x, t)|^2 \, dx \right) \, dt \\
= 2\eta \left( \int_{-T}^0 \int_\Omega \eta(t) |z(x, t)|^2 \, dx \, dt \right) + 2 \int_{-T}^0 \int_\Omega \eta(t) \eta'(t) |z(x, t)|^2 \, dx \, dt,
\]
holds true for each \( z \in H^1(-T, T; L^2(\Omega)) \). Applying Young’s inequality, this entails
\[
\int_{\Omega} |z(x, t)|^2 \, dx \leq (s + 2\|\eta'\|_{\infty}) \int_{Q} |z(x, t)|^2 \, dx \, dt + s^{-1} \int_{Q} |z' (x, t)|^2 \, dx \, dt.
\]
for each \( s > 0 \), so the result follows by taking \( s_e = 2\|\eta'\|_{\infty} \).

Having said that, we are now in position to prove

**Lemma 3.3.** There exist two constants \( C_4 > 0 \) and \( s_4 > 0 \) such that the estimates
\[
s^2 \left( C_4 s \int_{\Omega} e^{2\psi_0} (|\mu|^2 + |\lambda|^2) \, dx - \int_{\Omega} e^{2\psi_0} (|\nabla \mu|^2 + |\nabla \lambda|^2) \, dx \right) \leq \sum_{k=1}^{2} \mathcal{F}_k (s),
\]
(3.10)
and
\[
C_4 s \int_{\Omega} e^{2\psi_0} (|\nabla \mu|^2 + |\nabla \lambda|^2) \, dx - \sum_{|\alpha| = 2} \int_{\Omega} e^{2\psi_0} (|\partial^\alpha \mu|^2 + |\partial^\alpha \lambda|^2) \, dx \leq \sum_{j,k=1}^{2} \mathcal{F}_{k,j} (s),
\]
(3.11)
hold true for \( k = 1, 2 \), and \( s \geq s_4 \).

**Proof.** By applying lemma 3.2 for \( z = e^{\psi_0} U_{k,1} \), we obtain
\[
C_4 s \int_{\Omega} e^{2\psi_0} |U_{k,1}(x, t)|^2 \, dx \leq s \int_{Q} e^{2\psi_0} (s^2 |U_{k,1}(x, t)|^2 + |U_{k,2}(x, t)|^2) \, dx \, dt,
\]
provided \( s \) is large enough. In light of (3.6) and (3.7), this entails
\[
C_4 s \int_{\Omega} e^{2\psi_0} |\nabla \mu|^2 \, dx \leq \mathcal{F}_k (s).
\]
(3.12)
Further, taking into account that \( \mu B_0^k \in H^1_0(\Omega)^3 \) since \( \mu \) vanishes in \( \omega \) and \( \text{div} B_0^k = 0 \), we have
\[
C_4 s \int_{\Omega} e^{2\psi_0} |\mu B_0^k|^2 \, dx - s^2 \int_{\Omega} e^{2\psi_0} |\nabla \mu|^2 \, dx \leq 2 s \int_{\Omega} e^{2\psi_0} |\nabla (\mu B_0^k)|^2 \, dx,
\]
by (3.9), whence
\[
C_4 s \int_{\Omega} e^{2\psi_0} |\mu B_0^k|^2 \, dx - s^2 \int_{\Omega} e^{2\psi_0} |\nabla \mu|^2 \, dx \leq \mathcal{F}_k (s),
\]
(3.13)
from (3.12). Similarly, by arguing as above with \( z = e^{\psi_0} \bar{V}_{k,1} \) instead of \( e^{\psi_0} U_{k,1} \), we find some constant \( C > 0 \) for which \( C_4 s \int_{\Omega} e^{2\psi_0} |\bar{V}_{k,1}|^2 \, dx - s^2 \int_{\Omega} e^{2\psi_0} |\nabla \lambda|^2 \, dx \) can be made smaller than the right-hand side of (3.13) by taking \( s \) sufficiently large. This, (3.8) and (3.13) entails (3.10).

We turn now to showing (3.11). To do that we apply lemma 3.2 with \( z = e^{\psi_0} \bar{\partial}_i U_{k,1} \), \( i = 1, 2, 3 \), obtaining
\[
C \int_{\Omega} e^{2\psi_0} |\partial_i \bar{U}_{k,1}(x, t)|^2 \, dx \leq s \int_{Q} e^{2\psi_0} (|\nabla \bar{U}_{k,1}(x, t)|^2 + s^{-2} |\nabla \bar{U}_{k,2}(x, t)|^2) \, dx \, dt,
\]
for \( s \) large enough. This yields
\[
C \int_{\Omega} e^{2\psi_0} |\partial_i \nabla (\mu B_0^k)|^2 \, dx \leq \sum_{j=1}^{2} \mathcal{F}_{k,j} (s),
\]
(3.14)
by (3.6) and (3.7). Further, bearing in mind that \( \text{div} B_0^k = 0 \) and using that \( (\partial_i \mu) B_0^k \in H^1_0(\Omega)^3 \), we obtain
\[
C s \int_{\Omega} e^{2\psi_0} |(\partial_i \mu) B_0^k|^2 \, dx - s^{-2} \int_{\Omega} e^{2\psi_0} |\partial^\alpha \mu|^2 \, dx \leq \int_{\Omega} e^{2\psi_0} |\nabla (\partial_i \mu) B_0^k|^2 \, dx,
\]
(3.15)
by (3.9). Moreover, as curl \((\partial_\mu \mu) B_0^k\) = curl \((\mu B_0^k)\) - \(\mu\) curl \((\partial_\mu B_0^k)\), and \(\mu \in H_0^1(\Omega)\), we have by applying the Poincare inequality
\[ C \int_\Omega e^{2\psi_0} |\text{curl} (\partial_\mu \mu) B_0^k|^2 \, dx \leq \int_\Omega e^{2\psi_0} |\partial_i \text{curl} (\mu B_0^k)|^2 \, dx + \int_\Omega e^{2\psi_0} |\nabla \mu|^2 \, dx, \] (3.16)
hence
\[ C \sum_{k=1}^2 \int_\Omega e^{2\psi_0} |(\partial_\mu \mu) B_0^k|^2 \, dx = \sum_{1 \leq |\alpha| \leq 2} \int_\Omega e^{2\psi_0} |\partial_\mu \mu|^2 \, dx \leq \int_\Omega e^{2\psi_0} |\partial_i \text{curl} (\mu B_0^k)|^2 \, dx, \] (3.17)
by substituting the right-hand side of (3.16) for \(\int_\Omega e^{2\psi_0} |\text{curl} (\partial_\mu \mu) B_0^k|^2 \, dx\) in (3.15). Putting (3.14) and (3.17) together, and summing up the obtained inequality over \(i = 1, 2, 3\), we find that
\[ C_3 \sum_{|\alpha|=2} \int_\Omega e^{2\psi_0} |\nabla \mu|^2 \, dx - \sum_{1 \leq |\alpha| \leq 2} \int_\Omega e^{2\psi_0} |\partial_\mu \mu|^2 \, dx \leq \sum_{j,k=1}^2 3_k \varepsilon_j(s). \] (3.18)
Here we again used (3.8). Finally, by arguing as before with \(z = e^{\psi_0} \partial_i V_{k,1}, i = 1, 2, 3\), instead of \(\partial_i U_{k,1}\), we find that (3.18) remains true with \(\mu\) replaced by \(\lambda\). This completes the proof of (3.11).

Finally, we establish

**Lemma 3.4.** There are two constants \(C_3 > 0\) and \(s_5 > 0\) such that we have
\[ C_3 \sum_{|\alpha|=2} \int_\Omega e^{2\psi_0} |\nabla \mu|^2 \, dx - \sum_{1 \leq |\alpha| \leq 2} \int_\Omega e^{2\psi_0} |\partial_\mu \mu|^2 \, dx \leq \sum_{j,k=1}^2 3_k \varepsilon_j(s), \]
for all \(s \geq s_5\).

**Proof.** In light of the two following basic identities \(U_{k,1}(x,0) = \mu\) curl \(B_0^k + \nabla \mu \times B_0^k\) and \(V_{k,1}(x,0) = -\mu\) curl \(D_0^k - \nabla \lambda \times D_0^k\), \(k = 1, 2,\) arising from (3.6), we have
\[ K(x) \left( \frac{\nabla \mu}{\nabla \lambda} \right) = A(x) \left( \frac{\mu}{\lambda} \right) - \begin{pmatrix} U_{1,1}(x,0) \\ -V_{1,1}(x,0) \\ U_{2,1}(x,0) \\ -V_{2,1}(x,0) \end{pmatrix} \quad \text{with} \quad A(x) = \begin{pmatrix} \text{curl} B_0^1 & 0 \\ 0 & \text{curl} D_0^1 \\ 0 & \text{curl} B_0^2 \\ 0 & \text{curl} D_0^2 \end{pmatrix}, \]
hence
\[ K(x) \left( \frac{\partial_\mu \mu}{\partial_\lambda \lambda} \right) = \partial_i A \left( \frac{\mu}{\lambda} \right) + A \left( \frac{\partial_\mu \mu}{\partial_\lambda \lambda} \right) - \partial_\lambda K \left( \frac{\nabla \mu}{\nabla \lambda} \right) = \begin{pmatrix} \partial_\mu U_{1,1}(x,0) \\ -\partial_\lambda V_{1,1}(x,0) \\ \partial_\mu U_{2,1}(x,0) \\ -\partial_\lambda V_{2,1}(x,0) \end{pmatrix}, \]
for every \(i = 1, 2, 3\). From this and (1.11) it then follows that
\[ \sum_{|\alpha|=2} (|\partial_\mu \mu|^2 + |\partial_\lambda \lambda|^2) \leq C \left( \sum_{k=1}^2 (|\nabla U_{k,1}(x,0)|^2 + |\nabla V_{k,1}(x,0)|^2) + \sum_{|\alpha|=1} (|\partial_\mu \mu|^2 + |\partial_\lambda \lambda|^2) \right). \] (3.19)
Further, by multiplying (3.19) by \(e^{2\psi_0}\), integrating over \(\Omega\), and upper bounding \(\int_\Omega e^{2\psi_0} |\nabla W_{k,1}(x,0)|^2 \, dx\) with the aid of lemma 3.2, we find that
\[ C \sum_{|\alpha|=2} \int_\Omega e^{2\psi_0} (|\partial_\mu \mu|^2 + |\partial_\lambda \lambda|^2) \, dx - \int_\Omega e^{2\psi_0} (|\nabla \mu|^2 + |\nabla \lambda|^2) \, dx \]
\[ \leq s \sum_{k=1}^2 \int_Q e^{2\psi_0} |\nabla W_{k,1}(x,t)|^2 \, dx \, dt + s^{-2} \int_Q e^{2\psi_0} |\nabla W_{k,2}(x,t)|^2 \, dx \, dt. \] (3.20)
Here, we took advantage of the fact that both $\mu$ and $\lambda$ belong to $H^1_0(\Omega)$ in order to get rid of the integral $\int_\Omega e^{2s\varphi}(|\mu|^2 + |\lambda|^2) \, dx$ by applying the Poincaré inequality. Evidently the result now follows from (3.20) and lemma 2.2.

\[ \square \]

3.3. Completion of the proof of the main result

In light of (3.11) and lemma 3.4, we may find $C > 0$ such that

\[ C \sum_{|\alpha| \leq 2} \int_\Omega e^{2s\varphi}(|\partial^\alpha \mu|^2 + |\partial^\alpha \lambda|^2) \, dx \leq \sum_{j,k=1}^2 \delta_{j,k}(s), \]  

(3.21)

upon taking $s$ sufficiently large. Moreover, due to (3.1), we have

\[ \int_Q e^{2s}\varphi(|h_{jk}|^2 + |\nabla h_{jk}|^2) \, dx \, dt \leq \| (B^2_t, D^2_t) \|_{C([-T,T;W^{2,\infty}(\Omega))}^2 \left( \sum_{|\alpha| \leq 2} \int_Q e^{2s\varphi}(|\partial^\alpha \mu|^2 + |\partial^\alpha \lambda|^2) \, dx \, dt \right), \]

for each $(j, k) \in \{1, 2\}^2$, from which we obtain

\[ \sum_{|\alpha| \leq 2} \left( C \int_\Omega e^{2s\varphi}(|\partial^\alpha \mu|^2 + |\partial^\alpha \lambda|^2) \, dx - \int_Q e^{2s\varphi}(|\partial^\alpha \mu|^2 + |\partial^\alpha \lambda|^2) \, dx \, dt \right) \]

\[ \leq \sum_{j,k=1}^2 (\vartheta_{s,\varphi}(W_{jk}) + s^3 e^{2\delta s} \| W_{jk} \|^2_{H^1(t)}), \]  

(3.22)

by combining (3.7) and (3.21). Further, by recalling (2.7) and (2.10) we see for each $\alpha \in \{1, 2, 3\}^2$ with $|\alpha| = 2$ that

\[ \int_\Omega e^{2s\varphi}(|\partial^\alpha \mu|^2 + |\partial^\alpha \lambda|^2) \, dx = \int_\Omega e^{2s\varphi}(|\partial^\alpha \mu|^2 + |\partial^\alpha \lambda|^2) G_\alpha(x) \, dx, \]  

(3.23)

where

\[ G_\alpha(x) := \int_{-T}^T e^{-2s(\varphi(x) - \varphi(x,t))} \, dt \leq \int_{-T}^T e^{-2s(1 - \tau(t))} \, dt := g(s), \quad \tau(t) := e^{-\gamma \beta t^2}. \]  

(3.24)

As $\lim_{s \to +\infty} g(s) = 0$ by Lebesgue’s theorem, we thus obtain from (3.22)–(3.24) that

\[ C \sum_{|\alpha| \leq 2} \int_\Omega e^{2s\varphi}(|\partial^\alpha \mu|^2 + |\partial^\alpha \lambda|^2) \, dx \leq \sum_{j,k=1}^2 (\vartheta_{s,\varphi}(W_{jk}) + s^3 e^{2\delta s} \| W_{jk} \|^2_{H^1(t)}), \]  

(3.25)

for $s$ sufficiently large.

Furthermore, bearing in mind that $x_0 \in \mathbb{R}^3 \setminus \overline{\Omega}$, we note from (2.7) and (2.10) that

\[ \varphi_0(x) \geq \min_{x \in \overline{\Omega}} e^{r(x - x_0)^2 + \rho_0} \geq d_1 > d_0, \quad x \in \overline{\Omega}, \]  

(3.26)

where $d_0$ is defined in (2.14). From this and (3.25) it then follows that

\[ C \sum_{|\alpha| \leq 2} \int_\Omega e^{2s\varphi}(|\partial^\alpha \mu|^2 + |\partial^\alpha \lambda|^2) \, dx \leq e^{C s} \sum_{j,k=1}^2 \vartheta(W_{jk}) + s^3 e^{-2\delta s} \| W \|^2_{H^1(t)}, \]  

(3.27)

where $W$ is the same as in (1.12), and

\[ \vartheta(W) = \int_\Sigma (|\nabla_v v_1|^2 + |\nabla_v u_1|^2 + |u_1|^2 + |v_1|^2 + |u_1|^2 + |v_1|^2) \, d\sigma \, dt, \quad W = (U, V). \]  

(3.28)
In view of (3.26), (3.27) and (3.28) then yield
\[ \sum |\alpha| \leq 2 \int_{\Omega} (|\partial^\alpha \mu|^2 + |\partial^\alpha \lambda|^2) \, dx \leq C \left( \sum_{j,k=1}^{2} \mathcal{D}(W_{k,j}) \right)^\kappa, \]
for some \( \kappa \in (0, 1) \), proving theorem 1.

Acknowledgments

The authors are truly indebted to the unknown referee of this paper for suggesting several improvements. Part of this work was done while the first author was visiting the Université d’Aix-Marseille. He gratefully acknowledges the hospitality of the Université d’Aix-Marseille.

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