COMMUTATIVE SIMPLICIAL BUNDLES

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Abstract. In this paper we introduce a simplicial analogue of principal bundles with commutativity structure and their classifying spaces defined for topological groups. Our construction \( \overline{W}(\tau, K) \) is a variation of the \( \overline{W} \)-construction for simplicial groups. We show that the geometric realization of our \( \overline{W}(\tau, K) \) is homotopy equivalent to the topological classifying space \( B(\tau, |K|) \) and we study what objects does the simplicial set \( \overline{W}(\tau, K) \) classify.

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1. Introduction

Let $G$ be a topological group. Principal $G$-bundles with a commutativity structure on their transition functions are introduced in [AG15]. The classifying space $B(G, G)$ for such bundles, first introduced in [ACTG12], is a variant of the ordinary classifying space $BG$, which is constructed from pairwise commuting group elements. This construction is a particular case of a class of constructions, denoted by $B(\tau, G)$, that depend on a cosimplicial group $\tau^\bullet$. We assume that there exists a surjection $\eta^\bullet: F^\bullet \rightarrow G^\bullet$ where $[n] \mapsto F^n$ is the cosimplicial group free in each degree. Then the precomposition maps $\circ \eta^n: \text{Hom}(\tau^n, G) \rightarrow \text{Hom}(F^n, G) \cong G^{\times n}$ are injective, thus we can equip the set $\text{Hom}(\tau^n, G)$ of group homomorphisms with the subspace topology, getting us the simplicial topological space $B(\tau, G)$. The space $B(\tau, G)$ is the geometric realization $|B(\tau, G)|$. The main examples are $BG$, the usual classifying space, corresponding to $F^\bullet$ and $B(Z, G)$ corresponding to its level-wise abelianization $Z^\bullet$.

We say that $G$ is $\tau$-good if the simplicial space $B(\tau, G)$ is a good simplicial space of CW-complexes. For example a Lie group is $Z$-good [AG15, Proposition 9.1]. It was previously shown that for a Lie group $G$, the space $B(Z, G)$ classifies transitionally commutative principal $G$-bundles on finite CW-complexes [AG15, Theorem 2.2]. We generalize this theorem as follows:

**Theorem 5.15.** Suppose that $G$ is $\tau$-good. Then the space $B(\tau, G)$ classifies principal $G$-bundles on CW-complexes with $\tau$-atlas.

We carry over this construction to the simplicial category. Let $K$ be a simplicial group. The classifying space of $K$ is given by the bar construction $\overline{WK}$ [May67, §21]. The simplicial set $\overline{WK}$ is isomorphic to the total simplicial set $TNK$ of the nerve of $K$. The nerve $NK$ is the bisimplicial set $(NK)_{p,q} = N(K_q)_p \cong \text{Hom}(F^n, K_q)$. To get a $\tau$-version, we modify this bisimplicial set: we let $N(\tau, K)_{p,q} \cong \text{Hom}(\tau^n, K_q)$. Then we define $\overline{W}(\tau, K) = TN(\tau, K)$.

Note that the left Kan extension of the cosimplicial group $\tau^\bullet: \Delta \rightarrow \text{Grp}$ along the natural inclusion $\Delta^\bullet: \Delta \rightarrow \text{sSet}$ that sends $[n]$ to the simplicial $n$-simplex $\Delta[n]$ is a left adjoint of the functor $\text{Hom}(\tau^\bullet, ): \text{Grp} \rightarrow \text{sSet}$. Since moreover the décalage functor $\text{Dec}: \text{sSet} \rightarrow \text{ssSet}$ is a left adjoint of the total simplicial set functor $T$, we get a pair of adjoint functors

$$G(\tau, -): \text{sSet} \dashv \text{sGrp}: \overline{W}(\tau, -).$$

We compare our construction to the topological version.

**Theorem 5.4.** There is a natural homotopy equivalence $B(\tau, [K]) \rightarrow |\overline{W}(\tau, K)|$.

Thus our construction has the correct homotopy type. We also show that

**Corollary 5.10.** For a compact Lie group $G$, there is a natural homotopy equivalence

$$B(\tau, G) \rightarrow |\overline{W}(\tau, SG)|$$

where $S$ denotes the singular functor, the right adjoint of the geometric realization.

As a consequence we can import the results regarding all the interesting examples studied previously, such as [AG15, ACGV20, AGLT17, Oka18, GH19, OW20], to the simplicial category.
Our simplicial construction has the advantage of making available methods from simplicial homotopy theory that are not available in the original topological construction. It would be interesting to understand the homotopical effect of the algebraic modification using model structures. The simplicial approach has the potential for further generalization to the category of simplicial sheaves, a direction the authors plan to pursue elsewhere.

We also study what objects does the simplicial set \( \overline{\mathcal{W}}(\tau, K) \) classify. The simplicial set \( \overline{\mathcal{W}}(\tau, K) \) is usually not fibrant, so we can’t describe the morphism set \([B, \overline{\mathcal{W}}(\tau, K)]\) in the homotopy category as the set of maps \( B \to \overline{\mathcal{W}}(\tau, K) \) modulo homotopy. On the other hand, using that geometric realization is part of a Quillen equivalence and Theorem 5.4 we get bijections

\[
[B, \overline{\mathcal{W}}(\tau, K)] \cong \left\Vert B \right\Vert, \overline{\mathcal{W}}(\tau, K)] \cong \left\Vert B \right\Vert, B(\tau, |K|)].
\]

Therefore, we can take our cue from topology. Let \( \pi : P \to X \) be a topological principal \( G \)-bundle equipped with an atlas. Then we get an atlas on the pullback \( S_{\mathcal{W}} \pi \) of \( \pi \) along the inclusion \( S_{\mathcal{W}} X \to SX \) of the simplicial subset of simplices \( \sigma : \Delta^n \to X \) which map into \( U_i \) for some \( i \in I \) (see Construction 5.16), but this atlas is in general not regular. Therefore, we need a different model than \( \overline{WK} \) for the classifying object of principal \( K \)-bundles to which we can get classifying maps induced by not necessarily regular atlases.

An atlas on a principal \( K \)-bundle \( p : E \to B \) corresponds to a choice \( \sigma(b) \in p^{-1}(b) \) for all \( n \)-simplices \( b \in B_n \) for all \( n \geq 0 \). For order-preserving maps \( \theta : [m] \to [n] \) we get transition elements \( \alpha_\theta(b, \theta) \in K_n \) such that \( \sigma(\theta^\ast b) = \alpha_\theta(b, \theta) \cdot \theta^\ast \sigma(b) \). This data can be packaged as a functor between the Grothendieck constructions

\[
\alpha_\theta : \int_{\Delta^\text{op}} B \to \int_{\Delta^\text{op}} K.
\]

We give an explicit zigzag of equivalences between \( \overline{WK} \) and \( N \int K \). Moreover, we construct a sub-simplicial set \( N(\tau, \int K) \subseteq N \int K \) and extend the zigzag as follows:

**Theorem 4.14.** There exists a zigzag of weak equivalences \( N(\tau, \int K) \leftarrow \overline{\mathcal{W}}(\tau, K) \) natural in \( \tau \) and \( K \).

As a first, direct approach we say that an atlas is a \( \tau \)-atlas if the nerve \( N\alpha_\ast \) factors through the inclusion \( N(\tau, \int K) \to N \int K \). We show that this definition is compatible with the topological one:

**Proposition 5.17.** Let \( \pi : P \to X \) be a topological principal \( G \)-bundle equipped with a \( \tau \)-atlas along an open covering \( \mathcal{U} \). Then the induced atlas on the simplicial principal \( SG \)-bundle \( S_{\mathcal{W}} \pi : S_{\mathcal{W}} P \to S_{\mathcal{W}} B \) is a \( \tau \)-atlas.

Let \( p : E \to B \) be a simplicial principal \( K \)-bundle. We say that \( p \) admits a \( \tau \)-structure if the homotopy class of a classifying map \( B \to \overline{WK} \) factors through the map induced by the inclusion \( \overline{W}(\tau, K) \to \overline{WK} \). We can show that if a simplicial principal \( K \)-bundle \( p : E \to B \) admits a \( \tau \)-atlas, then it has a \( \tau \)-structure (Lemma 4.20). But getting a converse seems to be difficult. The problem is again that the simplicial set \( N(\tau, \int K) \) is not fibrant.

Therefore, we relax the notion of \( \tau \)-atlas in two different ways: A topological \( \tau \)-atlas is the \( \tau \)-atlas gotten via Proposition 5.17 from a \( \tau \)-atlas on the induced topological principal \( |K| \)-bundle \( |p| : |E| \to |B| \). A weak \( \tau \)-atlas is a \( \tau \)-atlas after a zigzag of equivariant maps along weak equivalences of simplicial groups. We can show that a topological \( \tau \)-atlas gives a weak \( \tau \)-atlas (Lemma 4.23) and having a \( \tau \)-structure implies having a topological \( \tau \)-atlas (Lemma
4.24). To close the circle of equivalences and show that having a weak $\tau$-atlas implies having a $\tau$-structure, we need:

**Conjecture 4.10.** Let $p : E \to B$ be a principal $K$-bundle and $\alpha$ be an atlas. Then the map $N_\alpha : N \int B \to N \int K$ is a classifying map for $p : E \to B$, i.e. under the relevant zigzag of equivalences the homotopy class of $N_\alpha$ coincides with the class $[\hat{r}] \in [B, W K]$ of a classifying map coming from a pseudosection.

Using this, we get:

**Theorem 4.27.** Suppose that Conjecture 4.10 holds. Let $p : E \to B$ be a simplicial principal $K$-bundle. Then the following assertions are equivalent:

1. The principal $K$-bundle $p$ admits a $\tau$-structure.
2. The principal $K$-bundle $p$ admits a topological $\tau$-atlas.
3. The principal $K$-bundle $p$ admits a weak $\tau$-atlas.

The structure of the paper is as follows. In §2 we study the total décalage and Kan’s loop group functors. The factorization $G = \pi_1 \text{Dec}$ is proved in Proposition 2.4. In §3 we introduce $W(\tau, K)$. Under some assumptions on $\tau$ we describe its set of $n$-simplices (Proposition 3.5). We also prove that after looping once the natural map $W(\tau, K) \to W K$ splits up to homotopy (Proposition 3.7). Simplicial bundles with $\tau$-structure and different notions of $\tau$-atlas are introduced and studied in §4. Assuming Conjecture 4.10 holds, we prove the circle of equivalences between different notions of $\tau$-atlas in Theorem 4.27. We study the homotopy type of the simplicial set $N(\tau, \int K)$ in ?? We give the natural zigzag of equivalences between $N(\tau, \int K)$ and $W(\tau, K)$ in Theorem 4.14. We compare our simplicial construction to the topological version $B(\tau, G)$ in §5. The homotopy equivalence $B(\tau, |K|) \simeq |W(\tau, K)|$ is proved in Theorem 5.4 and $B(\tau, G) \simeq |W(\tau, SG)|$ is proved in Corollary 5.10. We prove the generalized classification theorem of topological principal bundles with $\tau$-structure in Theorem 5.15.

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## 2. Kan’s Loop Group and Décalage

The standard $n$-simplices $\Delta[n]$ as $n$ varies can be assembled into a cosimplicial simplicial set $\Delta^\bullet$. Applying Kan’s loop group functor $G$ level-wise gives a cosimplicial simplicial group. In this section we describe this object using the factorization $G = \pi_1 \text{Dec}$. Proposition 2.4 gives an explicit morphism $G\Delta^\bullet \to \pi_1 \text{Dec} \Delta^\bullet$ of cosimplicial simplicial groups. This description will be essential later on when we introduce variations of the $W$-construction. For the properties of the loop group functor and the décalage functor we refer to [Ste12].

For a category $\mathcal{C}$ we write $s\mathcal{C}$ for the category of simplicial objects in that category.

### 2.1. Décalage

Let $\Delta$ denote the simplex category. Let $\Delta_+$ denote the category obtained from $\Delta$ by adjoining the empty ordinal denoted by $[-1]$. The category $\Delta_+$ is a monoidal
There is an isomorphism of augmented simplicial sets

**Lemma 2.1.** There is an isomorphism of augmented simplicial sets

\[
\coprod_{0 \leq i \leq k} \Delta[l] \xrightarrow{\cong} \text{Dec}_0(\Delta[k])
\]

which is natural with respect to morphisms in \(\Delta\) and maps \(\Delta[l]\) isomorphically onto the pre-image \((d_0)^{-1}(l)\).

**Proof.** \(\text{Dec}_0(\Delta[k])\) is the coproduct of \((d_0)^{-1}(l)\) over \(l\) since the augmentation map \(d_0\) is a deformation retraction [Ste12]. An \(m\)-simplex of \((d_0)^{-1}(l)\) is a functor \(\varphi : [m+1] \to [k]\) such that \(\varphi d_0^{n+1}(0) = l\), and such functors are in one-to-one correspondence with functors \([m] \to [l]\).

\[\square\]

2.2. **Total décalage.** The décalage functor \(\text{Dec}_0\) is a comonad whose structure maps are described as follows: \(\text{Dec}_0 \to (\text{Dec}_0)^2\) is induced by \([n] + [0] + [0] \to [n] + [0]\) given by the sum of the identity map on \([n]\) and \(s_0^0 : [0] + [0] \to [0]\), and the other structure map \(\text{Dec}_0 \to \text{id}\) is induced by \((d_0^0)^* : [n] \to [n] + [0]\). The total décalage of the simplicial set \(X\) is defined to be the bisimplicial set \(\text{Dec}X\) obtained from the comonadic resolution of \(X\), which in degree \(n\), is given by the simplicial set

\(\text{Dec}_nX = (\text{Dec}_0)^{n+1}X\).

If we think of \(\text{Dec}X\) as a vertical bisimplicial set with horizontal simplicial sets then the set of \((m,n)\)-simplices is \(X_{m+n+1}\). The horizontal face and degeneracy maps are given by \(d_i^m = d_i : (\text{Dec}_nX)_m \to (\text{Dec}_nX)_{m-1}\) and \(s_i^m = s_i : (\text{Dec}_nX)_m \to (\text{Dec}_nX)_{m+1}\) where \(0 \leq i \leq m\), the vertical face and degeneracy maps are given by \(d_j^m = d_{m+1+j} : (\text{Dec}_nX)_m \to (\text{Dec}_{n-1}X)_m\) and \(s_j^m = s_{m+1+j} : (\text{Dec}_nX)_m \to (\text{Dec}_{n+1}X)_m\) where \(0 \leq j \leq n\).

We will give a description of \(\text{Dec} \Delta[k]\). For this we introduce a bisimplicial set \(D[k]\) defined by

\[D_n[k] = \coprod_{\varphi : [n] \to [k]} \Delta[\varphi(0)]\]
together with the simplicial structure

\[ d_i(\varphi, \theta) = \begin{cases} (\varphi d^0, \iota \theta) & i = 0 \\ (\varphi d^i, \theta) & 0 < i \leq n \end{cases} \]

where \( \iota : \Delta[\varphi(0)] \to \Delta[\varphi(1)] \) is induced by the inclusion \( [\varphi(0)] \subset [\varphi(1)] \) and

\[ s_j(\varphi, \theta) = (\varphi s^j, \theta) \]

for all \( 0 \leq j \leq n \).

**Proposition 2.2.** There is an isomorphism of bisimplicial sets

\[ g : D[k] \to \text{Dec} \Delta[k]. \]

**Proof.** Applying Lemma 2.1 gives an isomorphism

\[ g_n : \coprod_{\varphi : [n] \to [k]} \Delta[\varphi(0)] \to \text{Dec}_n \Delta[k] \tag{2.2.1} \]

which in degree \( m \) sends \( (\varphi, \theta) \) to the functor \( [m+n+1] \to [k] \) defined using \( \theta \) on the subset \( [m] \subset [m+n+1] \) and on the rest using \( \varphi \). More precisely it is the unique functor which fits the diagram

\[ \begin{array}{ccc} [m] & \xrightarrow{\theta} & [k] \\
\downarrow & & \downarrow \\
[m+n+1] & \xrightarrow{\varphi} & \text{Dec}_n \Delta[k] \\
(d^0)m+1 & & \alpha \end{array} \]

On the other hand the inverse to this map is given by

\[ g_n^{-1} : \text{Dec}_n \Delta[k] \to \coprod_{\varphi : [n] \to [k]} \Delta[\varphi(0)] \tag{2.2.2} \]

which in degree \( m \) sends a functor \( \alpha : [m+1+n] \to [k] \) to the pair of the functors given by \( \alpha(d^0)m+1 : [m] \to [k] \) and \( \alpha' : [m] \to [\alpha(m+1)] \) that fits into the diagram

\[ \begin{array}{ccc} [m+1+n] & \xrightarrow{\alpha} & [k] \\
\uparrow & & \uparrow \\
[m] & \xrightarrow{\alpha'} & [\alpha(m+1)] \end{array} \]

It remains to check that \( \{g_n\}_{n \geq 0} \) gives a morphism of simplicial sets. It is straightforward to check this for \( s_j \) and \( d_i \) with \( i > 0 \). When \( i = 0 \) the face map \( d_0 \) changes \( \varphi(0) \) to \( \varphi(1) \), and this is accounted for by adding the inclusion \( \iota \).

\[ \square \]

**Remark 2.3.** From the definition

\[ \text{sSet}(\Delta[m], \text{Dec}_n X) = \text{sSet}(\Delta[m+n+1], X) \]
it follows that the n-th décalage is isomorphic to the disjoint union of slice simplicial sets

\[ \text{Dec}_n X \cong \bigsqcup_{\phi : \Delta[n] \to X} X_\phi. \]

In particular, \( \Delta[k]/\phi = \Delta[\phi(0)] \) if we regard an n-simplex as an ordinal map \( \phi : [n] \to [k] \). Moreover, the following assertions hold:

1. In case \( X \) is a Kan complex, the slice simplicial sets \( X_\phi \) are contractible. Therefore, for any simplicial set \( X \), the map \( s_0 : X_n \to \sqcup_{\phi \in X_n} X_\phi \) is a deformation retract with quasi-inverse the tail map.

2. In case \( X \) is a Kan complex, the forgetful maps \( X_\phi \to X/d_0 \phi \) are Kan fibrations. Therefore, in case \( X \) is a Kan complex, the map \( d_0 : \text{Dec}_n X \to \text{Dec}_n X \) is a fibrant resolution of the inclusion \( X_n = s_0 \text{Dec}_n X \to \text{Dec}_n X \). That is, the Kan complex \( \text{Dec}_n X \) is a model of the \((n+1)\)-fold path space of \( X \).

All these statements can be checked easily. We have found out about them at [nLa21].

2.3. Nerve functor and its adjoint. We can take the nerve of a group to obtain a simplicial set. Let \( \text{Grp} \) denote the category of groups. This construction gives a functor \( N : \text{Grp} \to \text{sSet} \). In fact the image of the functor lives in the category \( \text{sSet}_0 \) of reduced simplicial sets i.e. those simplicial sets whose set of 0-simplices is a singleton. Let us define a functor \( \pi_1 : \text{sSet} \to \text{Grp} \) by assigning to a simplicial set \( X \) the quotient of the free group \( F(X_1) \) on the set of 1-simplices by the relations \([s_0 x] = 1\) for all \( x \in X_0 \), and \([d_2 \sigma][d_0 \sigma] = [d_1 \sigma]\) for all \( \sigma \in X_2 \). The functor \( \pi_1 \) is the left adjoint of \( N \).

Let us compute \( \pi_1 \) of a k-simplex. Note that the set of 1-simplices is given by functors \([1] \to [k] \). The relation coming from \( s_0 \) will kill those which are not injective. Each 2-simplex will introduce a relation among the edges in its boundary. As a result there is an isomorphism of groups

\[ F^k \to \pi_1 \Delta[k] \]

which sends a generator \( e_i \) of the free group \( F^k \) to the 1-simplex \([1] \to [k] \) specified by the image \( \{i-1, i\} \). We will identify these groups and given an ordinal map \( \alpha : [k] \to [l] \) we will write

\[ \alpha_* : F^k \to F^l \]

for the induced map \( \pi_1 \Delta[k] \to \pi_1 \Delta[l] \). Considering all the simplices at once defines a cosimplicial group \( F^\bullet : \Delta \to \text{Grp} \) where \( F^\bullet[k] \) is the free group \( F^k = \pi_1 \Delta[k] \). At this point we also observe that the nerve functor is represented by the cosimplicial group \( F^\bullet \) in the sense that

\[ (NG)_\bullet = \text{Grp}(F^\bullet, G). \]

This approach allows us to describe the adjunction \( \pi_1 \dashv N \) from the general theory of Kan extensions.

2.4. Loop group. Given a (reduced) simplicial set \( X \) the loop group \( G_X \) is a simplicial group which homotopically behaves as the loop space. The group of n-simplices is the free group \( F(X_{n+1} - s_0 X_n) \). The face maps are defined by

\[ d_i[x] = \begin{cases} [d_1 x][d_0 x]^{-1} & i = 0 \\ [d_{i+1} x] & 0 < i \leq n \end{cases} \]
and the degeneracy maps are defined by \( s_j[x] = [s_{j+1}x] \) for all \( 0 \leq j \leq n \). In fact this construction is related to the décalage construction.

Recall that the bisimplicial set \( \text{Dec} \Delta[k] \) has a nice description whose simplicial structure is given in Proposition 2.2. Let us consider the simplicial group \( \pi_1 \text{Dec} \Delta[k] \) obtained by applying \( \pi_1 \) to each vertical degree. From the isomorphism 2.2.1 we see that the resulting simplicial group consists of the free groups

\[
(\pi_1 \text{Dec} \Delta[k])_n \cong (\pi_1 D[k])_n = \coprod_{\varphi : [n] \to [k]} F^{\varphi(0)}
\]

(2.4.1)

whose simplicial structure maps are induced by the ones of \( D[k] \). Let us describe this structure explicitly. We write \([\varphi, e_j]\) to denote the generator of \( F^{\varphi(0)} \) corresponding to the \( \varphi \) term in the coproduct. The simplicial structure maps are given by

\[
d_i[\varphi, e_j] = \begin{cases} 
[\varphi d^0, \iota_*(e_j)] & i = 0 \\
[\varphi d^i, e_j] & 0 < i \leq n
\end{cases}
\]

and \( s_i[\varphi, e_j] = [\varphi s^i, e_j] \) for the degeneracy maps. Recall that \( \iota_* \) is the map \( \pi_1 \Delta[\varphi(0)] \to \pi_1 \Delta[\varphi(1)] \) induced by the inclusion \([\varphi(0)] \subset [\varphi(1)]\).

2.5. Relation to décalage. Let \( \Delta^* : \Delta \to \text{sSet} \) denote the functor defined by sending \([k]\) to the \( k \)-simplex \( \Delta[k] = \Delta(-,[k]) \). We can think of \( \Delta^* \) as a cosimplicial object in the category of simplicial sets. Applying the functor \( \text{Dec} \) to the cosimplicial object \( \Delta^* \) in each degree and taking \( \pi_1 \) of the resulting simplicial set gives a cosimplicial object

\[
\pi_1 \text{Dec} \Delta^* : \Delta \to \text{sGrp}.
\]

The object \([n]\) is sent to the simplicial group \( \pi_1 \text{Dec} \Delta[n] \). Another cosimplicial object can be obtained by composing with Kan’s loop group functor

\[
G \Delta^* : \Delta \to \text{sGrp}.
\]

Proposition 2.4. There is a natural isomorphism of functors

\[
e^* : G \Delta^* \to \pi_1 \text{Dec} \Delta^*.
\]

Proof. Consider the cosimplicial degree \( k \) and simplicial degree \( n \). We begin by defining a group homomorphism \( (e^k)_n : G(\Delta[k])_n \to (\pi_1 \text{Dec} \Delta[k])_n \). For this recall that \( G(\Delta[k])_n \) is a quotient of the free group on the set \( \Delta[k]_{n+1} \). By 2.4.1 \( (\pi_1 \text{Dec} \Delta[k])_n \) is the free product of \( \pi_1(\Delta[\varphi(0)]) \) over the ordinal maps \( \varphi : [n] \to [k] \), hence a free group on the disjoint union \( \coprod_{\varphi} \Delta[\varphi(0)]_1 \). We define a map between the set of generators of the free groups

\[
\Delta[k]_{n+1} \to \coprod_{\varphi : [n] \to [k]} \Delta[\varphi(0)]_1
\]

by sending \( \beta : [n+1] \to [k] \) to the pair given by \( \beta d^0 : [n] \to [k] \) and \( \beta|_{[1]} \), the restriction of \( \beta \) along the natural inclusion \([1] \subset [n+1] \). This map induces a group homomorphism

\[
(e^k)_n : G(\Delta[k])_n \longrightarrow (\pi_1 \text{Dec} \Delta[k])_n
\]

since a generator \([\alpha s^0]\) for some \( \alpha : [n+1] \to [k] \) in the image of the degeneracy map \( s_0 \) is mapped to the generator given by the pair of the map \( \alpha s^0 d^0 = \alpha \) and the map obtained by restricting \( \alpha s^0 \) to \([1]\). The latter generator corresponds to \([d^0 \alpha s^0] \) in the free group \( F^{\alpha(0)} \) and is equivalent to the identity element under the identifications in \( \pi_1(\Delta[\varphi(0)]) \). Moreover,
$(e^k)_n$ is an isomorphism since both groups are free groups and the map is a bijection between the generators. For simplicity of notation we will suppress the cosimplicial degree and write $\epsilon_n = (e^k)_n$. Next we check that the map is compatible with the simplicial structure. For $0 < i \leq n$ we have $\epsilon_n d_i(\alpha) = \alpha(d^{i+1}) = \alpha d^{i+1} d^0 = \alpha d^0 d^i$, similarly for degeneracy maps we have $\epsilon_n s_j(\alpha) = \alpha d^0 s^j$ for all $0 \leq j \leq n$. Thus for the face map we have

$$\epsilon_n (d_i[\alpha]) = [\alpha d^0 d^i, \alpha d^{i+1}|_{[1]}] = d_i[\alpha d^0, \alpha|_{[1]}] = d_i \epsilon_{n+1}[\alpha]$$

and similarly $\epsilon_n$ commutes with the degeneracy maps. Finally $d_0$ on the loop group is defined by $d_0[\alpha] = [\alpha d^1][\alpha d^0]^{-1}$. Therefore we have

$$\epsilon_n (d_0[\alpha]) = \epsilon_n([\alpha d^1])\epsilon_n([\alpha d^0])^{-1} = [\alpha(d^0)^2, \alpha d^1|_{[1]}][\alpha(d^0)^2, \alpha d^0|_{[1]}]^{-1}$$

where we used the simplicial identity $d^1 d^0 = (d^0)^2$. Under $\epsilon_n$ both $[\alpha d^1]$ and $[\alpha d^0]$ map to the same free group $F(\alpha(2))$ indexed by $\alpha(d^0)^2$. Let $\sigma$ denote the restriction of $\alpha$ to $[2]$ (note that $\alpha$ lives in degree $\geq 2$). Then using the relation in the fundamental group $\pi_1 \Delta[\alpha(2)] \cong F(\alpha(2))$ we can write

$$[\alpha d^1|_{[1]}][\alpha d^0|_{[1]}]^{-1} = [d_1 \sigma][d_0 \sigma]^{-1} = [d_2 \sigma] = [\alpha|_{[1]}]$$

which is the same, after adding the coproduct index, as $d_0$ of the generator $\epsilon_{n+1}[\alpha]$.

Thus we showed that we have an isomorphism of simplicial groups

$$e^k : G\Delta[k] \to \pi_1 \text{Dec} \Delta[k]$$

where we remembered the cosimplicial index, and suppressed the simplicial index. The resulting map is compatible with the cosimplicial structure since all the constructions involved are functorial in $\Delta[k]$. \hfill $\Box$

**Remark 2.5.** One can check that the inverse of $e^k$ is induced by the map

$$\prod_{\varphi: [n] \to [k]} \Delta[\varphi(0)] \to \Delta[k]_{n+1} \quad (2.5.1)$$

which sends $(\varphi, \gamma)$ to the functor $\alpha : [n+1] \to [k]$ defined by $\alpha(0) = \gamma(0)$ and $\alpha(i) = \varphi(i-1)$ for $0 < i \leq n + 1$. An argument similar to the proof of Proposition 2.4 can be used to see that after taking the appropriate quotients of the free groups the resulting map is an isomorphism. We will denote this map simply as the inverse

$$(e^k)^{-1} : \pi_1 \text{Dec} \Delta^* \to G\Delta^*.$$

Almost immediately we obtain the following result which is also proved in [Ste12, Proposition 16], however, without an explicit isomorphism.

**Corollary 2.6.** For any simplicial set $X$ there is a natural isomorphism of simplicial groups

$$G(X) \to \pi_1 \text{Dec} (X).$$

**Proof.** An arbitrary simplicial set can be written as a colimit of $\Delta[n]$ over the simplex category. All the functors in sight $G$, $\pi_1$, and $\text{Dec}$ (see [Ste12, §3]) are left adjoints, thus they preserve colimits. Then the result follows from Proposition 2.4. \hfill $\Box$

**Remark 2.7.** In Proposition 16 of [Ste12] Kan’s loop group $G$ is compared to $\pi_1 R \text{Dec}$ where $R$ is the left adjoint of the inclusion $\mathbf{sSet}_0 \to \mathbf{sSet}$. We avoided this functor by defining $\pi_1$ for an arbitrary simplicial set not necessarily a reduced one.
2.6. Simplicial group homomorphisms. Let $K$ be a simplicial group. The set of simplicial group homomorphisms $GΔ[k] \to K$ is well-known [GJ99]. This set is precisely the set of $k$-simplices of the $\overline{W}$-construction of $K$. For variations of this construction we need an explicit description of such simplicial group homomorphisms.

By Proposition 2.4 we can instead consider a morphism $f : \pi_1 \operatorname{Dec}Δ[k] \to K$ of simplicial groups. The degree-$n$ part $f_n$ belongs to the set

$$\operatorname{Grp}(\pi_1 \operatorname{Dec}Δ[k], K_n) = \prod_{\varphi:[n] \to [k]} \operatorname{Grp}(F^\varphi(0), K_n) = \prod_{\varphi:[n] \to [k]} K_n^{\times \varphi(0)}$$

where $K_n^{\times \varphi(0)}$ is understood to be the trivial group if $\varphi(0) = 0$. Therefore $f_n$ is determined by the tuple of elements $f_n[\varphi, e_j]$ in $K_n$ where $e_j \in F^\varphi(0)$ and $1 \leq j \leq \varphi(0)$ . The map $\varphi$ factors as follows

$$[n] \xrightarrow{\varphi} [k] \xrightarrow{(d^\varphi)^{(0)}} [k - \varphi(0)]$$

where $\varphi(0) = 0$ i.e. the canonical decomposition of $\varphi$ does not involve $d^0$. The simplicial structure in 2.4.2 implies that

$$\phi^*((d^0)^{\varphi(0)}, e_j) = [(d^0)^{\varphi(0)} \phi, e_j] = [\varphi, e_j],$$

and thus $f_n[\varphi, e_j] = \phi^* f_{k-\varphi(0)}[(d^0)^{\varphi(0)}, e_j]$. As a result the elements $f_{k-\varphi(0)}[(d^0)^l, e_j]$ where $e_j \in F^l$ for $l = 1, \cdots, k$ completely determine $f_n$. It remains to consider the effect of $d^0$. As a consequence of 2.4.2 we have

$$d^0[(d^0)^l, e_j] = [(d^0)^{l+1}, d^1 e_j].$$

Note that in this case $\iota$ is induced by the inclusion $d^{l+1} : [l] \to [l + 1]$. By abuse of notation we will identify $d^{l+1} e_j$ with the generator $e_j$ in $F^{l+1}$. After this identification we see that all the other generators are determined once we fix $f_{k-l}[(d^0)^l, e_l]$ for $1 \leq l \leq k$, where $e_l$ is the top generator of $F^l$. We package this information as a diagram

$$\begin{array}{ccccccccc}
F^1 & \xrightarrow{d^2} & F^2 & \xrightarrow{d^3} & \cdots & \xrightarrow{d^k} & F^k \\
\downarrow f_{k-1} & & \downarrow f_{k-2} & & \cdots & & \downarrow f_{0} \\
K_{k-1} & \xrightarrow{d^0} & K_{k-2} & \xrightarrow{d^0} & \cdots & \xrightarrow{d^0} & K_{0}
\end{array} \tag{2.6.1}
$$

where $d^l$ is the map $d^l : [l-1] \to [l]$.

Evaluating the homomorphisms $f_{k-l}$ at the generator $e_l$ of each free group gives a $k$-tuples $(x_{k-1}, x_{k-2}, \cdots, x_0)$ where $x_{k-l}$ belongs to $K_{k-l}$. Then the images of the generators of $F^l$ are given by

$$f_{k-l}(e_j) = (d^0)^{l-j} x_{k-j} \tag{2.6.2}$$

where $1 \leq j \leq l$.

**Proposition 2.8.** There is a bijection of sets

$$\operatorname{sGrp}(\pi_1 \operatorname{Dec}Δ[k], K) \longrightarrow K_{k-1} \times K_{k-2} \times \cdots \times K_0,$$

defined by sending

$$f \mapsto (x_{k-1}, x_{k-2}, \cdots, x_0)$$

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where \( x_{k-1} = f_{k-l}[(d^0)^i, e_i] \).

**Remark 2.9.** In particular, for \( K = G\Delta[k] \) we can find the \( k \)-tuple corresponding to the map \((e^\bullet)^{-1}\) defined in Remark 2.5. Consider the restriction \( \Delta[l]_1 \rightarrow \Delta[k]_{n+1} \) of the map in 2.5.1 to the factor \( \varphi = (d^0)^l : [k - l] \rightarrow [k] \) in the coproduct. The generator \( e_l \) in \( F^l \) is represented by the map \( \tau, K \rightarrow \pi_k \) from the \( k \)-tuple corresponding to the \((d^0)^l \)-tuple in \( [k - l + 1] \rightarrow [k] \).

Thus, under the bijection in Proposition 2.8 we have
\[
(e^\bullet)^{-1} = ([id], [(d^0)^0], \ldots),
\]

### 3. Variants of the \( \overline{\mathcal{W}} \)-construction

We introduce the simplicial set \( \overline{\mathcal{W}}(\tau, K) \) that depends on an endofunctor \( \tau \) on the category of groups. Under some niceness conditions on \( \tau \) we describe the set of \( n \)-simplices. This object comes with a map \( \overline{\mathcal{W}}(\tau, K) \rightarrow \mathcal{W}K \), which we show splits up to homotopy after looping once.

When \( \tau \) is the identity functor we recover \( \mathcal{W}K \). Other examples come mainly from the descending central series, and in particular, the abelianization functor, which gives a simplicial version of \( \mathcal{B}(\mathbb{Z}, G) \).

**3.1. \( \overline{\mathcal{W}} \)-construction.** The loop functor \( G \) has a well-known right adjoint \( \overline{\mathcal{W}} : \mathbf{sGrp} \rightarrow \mathbf{sSet} \). For a simplicial group \( K \) the set of \( k \)-simplices of \( \overline{\mathcal{W}}K \) consists of simplicial group homomorphisms \( G\Delta[k] \rightarrow K \), and the simplicial structure is determined by the cosimplicial structure of \( G\Delta^\bullet \). More explicitly, \( \overline{\mathcal{W}}(K) \) can be identified with the product \( K_{k-1} \times K_{k-2} \times \cdots \times K_0 \). Under this identification the simplicial structure is described as follows: the face maps are given by
\[
d_i(x_k-1, \ldots, x_0) = \begin{cases} (x_k-2, x_k-3, \ldots, x_0) & i = 0 \\ (d_{i-1}x_k-1, d_i-2x_k-2, \ldots, d_0x_{k-i-1}x_k-i-2, \ldots, x_0) & 0 < i < k \\ (d_k-1x_k-1, d_k-2x_k-2, \ldots, d_1x_1) & i = k \end{cases}
\]
and the degeneracy maps are given by
\[
s_i(x_k-1, \ldots, x_0) = \begin{cases} (1, x_k-1, \ldots, x_0) & i = 0 \\ (s_{i-1}x_k-1, s_i-2x_k-2, \ldots, s_0x_{k-i-1}, x_k-i-1, x_k-i-2, \ldots, x_0) & 0 < i \leq k. \end{cases}
\]

As a consequence of the natural isomorphism \( G\Delta^\bullet \rightarrow \pi_1\mathcal{Dc}\Delta^\bullet \) proved in Proposition 2.4 the functor \( \pi_1\mathcal{Dc} \) has also a right adjoint isomorphic to the functor \( \overline{\mathcal{W}} \). The right adjoint of \( \pi_1\mathcal{Dc} \) is defined by
\[
K \mapsto \mathbf{sGrp}(\pi_1\mathcal{Dc}^\bullet, K)
\]
where the simplicial group homomorphisms are taken level-wise. Let us describe the cosimplicial structure of \( \pi_1\mathcal{Dc}^\bullet \). Given an ordinal map \( \alpha : [l] \rightarrow [k] \) the induced map on \( \mathcal{Dc}^\bullet[\kappa] \rightarrow \mathcal{Dc}^\bullet[\kappa] \) sends a pair \((\varphi, \theta)\) to composition by \( \alpha \), namely to the pair \((\alpha^*\varphi, \alpha^*\theta)\). The map between the fundamental groups is determined when \( \theta \) is a 1-simplex of \( \Delta[\varphi(0)] \), that is, we have \( \alpha^*[\varphi, e_j] = [\alpha^*\varphi, \alpha^*(e_j)] \) where \( e_j \) belongs to the free group \( F\varphi^0 \). Given this general description for arbitrary ordinal maps let us figure out the effect of the coface maps \( d^k : [k - 1] \rightarrow [k] \) first. It suffices to consider the generators \([d^0]^m, e_m]\), where
1 ≤ m ≤ k − 1, since the rest is determined by the simplicial structure. Using the cosimplicial identity \( d^i d^0 = d^0 d^0 \) for \( i > 0 \) we obtain

\[
d^i[(d^0)^m, e_m] = \begin{cases} 
[(d^0)^i, d_{m-i}, e_m] & 1 \leq m < i \leq k \\
[(d^0)^{m+1}, e_m] \quad m = i \\
[(d^0)^{m+1}, e_{m+1}] & i < m \leq k - 1
\end{cases}
\tag{3.1.1}
\]

and for the codegeneracy maps \( s^i : [k + 1] \to [k] \), using \( s^i d^0 = d^0 s^{i-1} \) for \( i > 0 \) and \( s^0 d^0 = 1 \), we have

\[
s^i[(d^0)^m, e_m] = \begin{cases} 
[(d^0)^{m}, e_{s^m}] & 1 \leq m \leq i \leq k \\
[(d^0)^{m-1}, 1] & m = i + 1 \\
[(d^0)^{m-1}, e_{m-1}] & i + 1 < m \leq k + 1.
\end{cases}
\tag{3.1.2}
\]

**Proposition 3.1.** There is a natural identification of simplicial sets

\[ \overline{W}(K) = sGrp(\pi_1 \text{Dec}\Delta^n, K). \]

**Proof.** Using the cosimplicial structure of \( \pi_1 \text{Dec}\Delta^n \) described in 3.1.1 and 3.1.2 we check that the right adjoint is equal to the \( \overline{W} \)-construction. In Proposition 2.8 we have seen that simplicial group homomorphisms \( f : \pi_1 \text{Dec}\Delta^n[k] \to K \) are in one-to-one correspondence with \( k \)-tuples \( (x_{k-1}, x_{k-2}, \ldots, x_0) \) where \( x_i \in K_i \). The correspondence is obtained by letting \( x_{k-m} \) denote the image of \( [(d^0)^m, e_m] \) under \( f \). Using the cosimplicial structure of \( \pi_1 \text{Dec}\Delta^n \) we first compute the coface maps \( d^i : [k-1] \to [k] \):

\[
f(d^i[(d^0)^m, e_m]) = \begin{cases} 
d_{i-m} x_{k-m} & 1 \leq m < i \leq k \\
d_0(x_{k-i}) x_{k-i-1} & m = i \\
x_{k-m-1} & i < m \leq k - 1
\end{cases}
\]

whereas the codegeneracy maps give

\[
f(s^i[(d^0)^m, e_m]) = \begin{cases} 
s_{i-m} x_{k-m} & 1 \leq m \leq i \leq k \\
1 & m = i + 1 \\
x_{k-m+1} & i + 1 < m \leq k - 1.
\end{cases}
\]

This shows that the simplicial structure on the \( k \)-tuples \( (x_{k-1}, x_{k-2}, \ldots, x_0) \) is exactly the one of the \( \overline{W} \)-construction. \( \square \)

3.2. **Endofunctors.** Recall that we used the cosimplicial group \( F^\bullet \), where \( F^k = \pi_1 \Delta^n[k] \), in the definition of the nerve functor, namely \( N = Grp(F^\bullet, -) \). We will introduce a variant of this construction with respect to an endofunctor \( Grp \to Grp \).

Left Kan extension [Rie14, Chapter 1] of a cosimplicial group \( \tau^\bullet : \Delta \to Grp \) along the natural inclusion \( \Delta^\bullet : \Delta \to sSet \) gives a functor

\[ \pi_1(\tau, -) : sSet \to Grp. \]

By the general theory of left Kan extensions there is a corresponding right adjoint

\[ N(\tau, -) : Grp \to sSet \]

defined by \( N(\tau, G)_n = Grp(\tau^n, G) \). Given an endofunctor \( \tau \) we consider the cosimplicial group \( \tau^\bullet \) defined by

\[ \tau^k = \tau F^k. \]

The group \( \pi_1(\tau, \Delta^n[k]) \) is naturally isomorphic to \( \tau F^k = \tau \pi_1 \Delta^n[k] \). Note that we recover the adjunction \( \pi_1 \dashv N \) when \( \tau \) is the identity functor.
Definition 3.2. We say that $\tau$ is of quotient type, if there exists a natural transformation $\eta : id \to \tau$ such that the map of groups $\eta_n : F^n \to \tau^n$ is surjective for all $n \geq 0$. In this case we also say that $\eta$ is surjective on finitely generated free groups.

Let us give a list of endofunctors that are of interest to us, see for example [Oka14].

- Descending central series endofunctor $\Gamma^q$: The descending central series of a group $H$ is defined by
  \[
  \Gamma^1(H) = H, \quad \Gamma^q(H) = [\Gamma^{q-1}(H), H].
  \]
  We will denote the $q$-th stage $H/\Gamma^qH$ of the descending central series by $\Gamma^qH$.

- $\Gamma_2$ will have a special importance. We introduce the notation $Z^\bullet = \Gamma_2F^\bullet$.

This is a cosimplicial group sending $[n]$ to the free abelian group $Z^n$ of rank $n$. Alternatively we can think of this as the functor $[n] \mapsto H_1(\Delta[n]/\Delta[n]_0, Z)$, where $\Delta[n]/\Delta[n]_0$ is the simplicial set obtained by identifying all the vertices of $\Delta[n]$ (see [OW20, §2]).

- Mod-$p$ version $\Gamma_{p,q}$: For a group $H$ mod-$p$ descending central series is defined by
  \[
  \Gamma_p^1(H) = H, \quad \Gamma^q_p(H) = [\Gamma^{q-1}_p(H), H][\Gamma^{q-1}_p(H)]^p.
  \]
  The $q$-th stage $H/\Gamma^q_pH$ is denoted by $\Gamma_{p,q}H$.

- $\Gamma_{p,2}$ is used to define
  \[
  (\mathbb{Z}/p)^\bullet = \Gamma_{p,2}F^\bullet.
  \]

- Let $\Gamma_{p^k,2}$ denote the composition of $\Gamma_2$ with the mod-$p^k$ reduction functor that sends an abelian group to the largest $p^k$-torsion quotient. We write
  \[
  (\mathbb{Z}/p^k)^\bullet = \Gamma_{p^k,2}F^\bullet.
  \]

- Let $H \mapsto H_p^\wedge$ denote the $p$-adic completion functor. Let $\hat{\Gamma}_{p,q}$ denote the functor $H \mapsto \Gamma_q(H_p^\wedge)$.

- The $p$-adic cosimplicial group is defined by
  \[
  (\mathbb{Z}_p)^\bullet = \hat{\Gamma}_{p,2}F^\bullet.
  \]

Remark 3.3. The endofunctors $\Gamma^q$ and $\Gamma_{p^k,q}$ are of quotient type. The completed version $\hat{\Gamma}_{p,q}$ fails to satisfy the surjectivity assumption.

Example 3.4. Let $X(n)$ denote the quotient of $\Delta[n]$ by the set of vertices $\Delta[n]_0$. Then one can check that $\pi_1(\mathbb{Z}, -)$ of the inclusion
\[
\vee^n X(1) \to X(n)
\]
is the abelianization map $F^n \to \mathbb{Z}^n$. Higher simplices have the effect of abelianizing the ordinary fundamental group.
3.3. Variants of the $\mathbb{W}$-construction. We can generalize the adjunction between the $\mathbb{W}$-construction and Kan’s loop group functor $G$ with respect to a given endofunctor on the category of groups.

Let $\tau^\bullet : \Delta \to \text{Grp}$ be a cosimplicial group. We define the functor

$$G(\tau, -) : s\text{Set} \to s\text{Grp} \quad X \mapsto \pi_1(\tau, \text{Dec}X)$$

which, up to natural isomorphism, is the left Kan extension of $\pi_1(\tau, \text{Dec}\Delta^\bullet)$ along the inclusion of the simplex category into the category of simplicial sets. On a $k$-simplex this functor is given by

$$G(\tau, \Delta[k])_n = \bigsqcup_{\varphi : [n] \to [k]} \tau F^\varphi(0)$$

and the simplicial structure is induced from the one of $G\Delta[k]$.

There exists a right adjoint of the functor $G(\tau, -)$ which we denote by $\mathbb{W}(\tau, -) : s\text{Grp} \to s\text{Set}$.

We restrict our attention to cosimplicial groups $\tau^\bullet$ of the form $\tau F^\bullet$ for some endofunctor $\tau : \text{Grp} \to \text{Grp}$. Observe that these constructions recover the usual adjunction $G \dashv \mathbb{W}$ when $\tau$ is the identity functor.

**Proposition 3.5.** Let $\tau : \text{Grp} \to \text{Grp}$ be an endofunctor, and $\eta : \text{id} \to \tau$ a natural transformation that is surjective on finitely generated free groups (Definition 3.2). Then the set of $k$-simplices of $\mathbb{W}(\tau, K)$ is given by tuples

$$(x_{k-1}, x_{k-2}, \ldots, x_0) \in K_{k-1} \times K_{k-2} \times \cdots \times K_0$$

that satisfy the following property: For $1 \leq l \leq k$ the homomorphism $f_{k-l} : F^l \to K_{k-l}$ defined by $f_{k-l}(e_j) = (d_0)^{l-j} x_{k-j}$ factors through $\eta_{F^l} : F^l \to \tau F^l$.

**Proof.** The set $\mathbb{W}(K)_k$, equivalently the set of simplicial group homomorphisms $\pi_1\text{Dec}\Delta[k] \to K$, is described in Proposition 2.8. Since $\eta_{F^n}$ is surjective by assumption the induced map $\eta_* : \text{Grp}(F^n, H) \to \text{Grp}(\tau F^n, H)$ is injective for any group $H$. We see that there is an inclusion of simplicial sets $\mathbb{W}(\tau, K) \subset \mathbb{W}K$ and as a consequence of 2.6.1 the elements of $\mathbb{W}(\tau, K)_k$ correspond to diagrams

$$\begin{array}{cccccc}
\tau F^1 & \xrightarrow{d_2^1} & \tau F^2 & \xrightarrow{d_2^2} & \cdots & \xrightarrow{d_2^k} & \tau F^k \\
\downarrow f_{k-1} & & \downarrow f_{k-2} & & \cdots & & \downarrow f_0 \\
K_{k-1} & \xrightarrow{d_0} & K_{k-2} & \xrightarrow{d_0} & \cdots & \xrightarrow{d_0} & K_0
\end{array}$$

$\square$

**Example 3.6.** It is instructive to look at $\mathbb{W}(\mathbb{Z}, K)$. The set of $k$-simplices consists of

$$(x_{k-1}, x_{k-2}, \ldots, x_0)$$

such that the elements $(d_0)^{l-1} x_{k-l-1}, (d_0)^{l-2} x_{k-2}, \ldots, d_0 x_{k-l+1}, x_{k-l}$ pairwise commute for all $1 \leq l \leq k$.

This easily generalizes to $\mathbb{W}(\Gamma_q, K)$. When $K$ is discrete, i.e. $K_n = G$ for some discrete group $G$ and the simplicial maps are all identity, the geometric realization of $\mathbb{W}(\Gamma_q, G)$ is precisely the space $B(q, G)$ introduced in [ACTG12].
3.4. **Descending central series filtration.** We introduce a simplicial version of the filtration introduced in [ACTG12] for the classifying space of a topological group. This filtration is obtained from the sequence of endofunctors

\[ \text{id} =: \Gamma_\infty \to \cdots \to \Gamma_q \to \Gamma_{q-1} \to \cdots \Gamma_2 \]

associated to the stages of the descending central series. For each endofunctor we have a cosimplicial group

\[ \Gamma_q^\bullet = \Gamma_q F^\bullet. \]

We write

\[ \mathcal{W}(q, -) = \mathcal{W}(\Gamma_q, -), \quad G(q, -) = G(\Gamma_q, -). \]

The resulting sequence

\[ G(X) \to \cdots \to G(q, X) \to G(q - 1, X) \to \cdots \to G(2, X) = G(\mathbb{Z}, X) \quad (3.4.1) \]

consists of surjective simplicial group homomorphisms since \( \Gamma_q H \to \Gamma_{q-1} H \) is surjective for any group \( H \).

On the other hand, we have a sequence of inclusions of simplicial sets

\[ \mathcal{W}(\mathbb{Z}, K) = \mathcal{W}(2, K) \to \cdots \to \mathcal{W}(q - 1, K) \to \mathcal{W}(q, K) \to \cdots \to \mathcal{W}(K) \quad (3.4.2) \]

that yields a filtration of the \( \mathcal{W} \)-construction.

Applying the \( \mathcal{W} \) functor to the sequence of simplicial groups in (3.4.1) gives a cofiltration of \( X \cong \mathcal{W}G(X) \). We can use (3.4.2) to further filter each term by subspaces of the form \( \mathcal{W}(q', G(q, X)) \). The construction of such filtrations is motivated by [CS16, Problem 3].

3.5. **Kan suspension.** Let \( X \) be a pointed simplicial set. The Kan suspension of \( X \) is the simplicial set \( \Sigma X \) whose set of \( n \)-simplices is given by the wedge \( X_{n-1} \vee X_{n-2} \vee \cdots \vee X_0 \) ([GJ99, page 189]). An ordinal map \( \theta : [m] \to [n] \) induces \( \theta^* : (\Sigma X)_n \to (\Sigma X)_m \) which maps a wedge summand \( X_{n-i} \) to the base point if \( \theta^{-1}(i) \) is empty, otherwise it is determined by \( \theta_i^* : X_{n-\theta(i)} \to X_{m-i} \) where \( \theta_i \) is defined by the diagram

\[
\begin{array}{ccc}
[m-i] & \xrightarrow{(\theta^*)^i} & [m] \\
\downarrow^{\theta_i} & & \downarrow^\theta \\
[n-\theta(i)] & \xrightarrow{(\theta^*)^\theta(i)} & [n]
\end{array}
\]

Let \( K \) be a simplicial group pointed by the identity element of each \( K_n \). There is a canonical map of simplicial sets

\[ \kappa : \Sigma K \to \mathcal{W}K \]

induced by the inclusion \( K_{n-1} \vee \cdots \vee K_0 \to K_{n-1} \times \cdots \times K_0 \) at the \( n \)-th level.

**Proposition 3.7.** Assume that \( \tau \) is of quotient type and satisfies the property that \( \eta_{F_1} : F_1 \to \tau F_1 \) is an isomorphism. Then the natural map

\[ G\mathcal{W}(\tau, K) \to G(\mathcal{W}K) \]

splits (naturally) up to homotopy.

This is a simplicial analogue of [ACTG12, Theorem 6.3] that applies to the topological \( B_{\text{com}} \) construction.
Proof. First observe that under the assumption on $\tau$ we have that $\eta_C : C \to \tau C$ is an isomorphism for any cyclic group $C$. Then by the description of the simplices of $W(\tau, K)$ given in Proposition 3.5 the map $\kappa$ factors through $\kappa_\tau : \Sigma K \to \overline{W}(\tau, K)$.

The splitting is given by the following diagram

$$
\begin{array}{c}
G \overline{W}(\tau, K) \xleftarrow{G(\kappa_\tau)} G(\Sigma K) \\
\downarrow \\
G(\overline{W}K) \xrightarrow{\sim} K \xrightarrow{FK}
\end{array}
$$

Let us explain the maps. The weak equivalence is the counit of the adjunction between the loop group and bar construction. In degree $n$ the simplicial group $FK$, known as Milnor’s construction [GJ99, page 285], is the free group generated on $K_n - \{\ast\}$. The map $K \to FK$ sends an $n$-simplex to the corresponding generator of the free group. The set of $n$-simplices of $G(\Sigma K)$ is given by $F(\Sigma K)_{n+1}/F(s_0(\Sigma K)_n)$ which can be identified with the $n$-simplices of $FK$ since $s_0(\Sigma K)_n$ maps onto the wedge summands of $(\Sigma K)_{n+1}$ other than $X_n$. This isomorphism is compatible with the simplicial structure. The last two maps are already described above. Starting from $K$ the composition of the five maps gives the identity. The splitting is natural with respect to $K$ since each construction is functorial in $K$. □

Corollary 3.8. Under the assumption of Proposition 3.7 the natural map $\overline{W}(\tau, K) \to \overline{WK}$ induces a split surjection on homotopy groups.

4. Simplicial bundles with $\tau$-atlases

In this section $K$ denotes a simplicial group and $\tau^\bullet$ denotes a cosimplicial group obtained from an endofunctor $\tau$, i.e. $\tau^k = \tau F^k$, which is equipped with a natural transformation $\eta : \text{id} \to \tau$ that is surjective on finitely generated free groups (Definition 3.2).

We introduce simplicial principal bundles with $\tau$-atlas. This is the simplicial analogue of a topological definition that generalizes the transitional commutative structure defined for topological bundles, see Definition 5.13. We study the relation of simplicial principal bundles with $\tau$-atlas and weak $\tau$-atlas to the objects the simplicial set $\overline{W}(\tau, K)$ classifies.

4.1. Simplicial fiber bundles. First, following [May67], we introduce simplicial fiber bundles.

Definition 4.1. Let $F$ be a nonempty simplicial set. Then a map of simplicial sets $p : E \to B$ is an $F$-fiber bundle, if for each $n$-simplex $b : \Delta[n] \to B$, there exists a pullback square of the form

$$
\begin{array}{c}
F \times \Delta[n] \xrightarrow{\beta(b)} E \\
\downarrow \pi_2 \\
\Delta[n] \xrightarrow{b} B.
\end{array}
$$

Such a map $F \times \Delta[n] \to E$ is called a local trivialization. A collection of a local trivialization $\beta(b) : F \times \Delta[n] \to E$ for each simplex $b \in B_n$ for each $n \geq 0$ is called an atlas of $p$. 

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The automorphism simplicial group \( \text{Aut} F \subseteq \text{Map}(F,F) \) has a natural right action on the mapping simplicial set \( \text{Map}(F,E) \), which is defined as follows. For \( m \)-simplices \( \alpha : F \times \Delta[m] \to F \) of \( \text{Aut} F \) and \( \beta : F \times \Delta[m] \to E \) of \( \text{Map}(F,E) \), the product \( \beta \cdot \alpha \) is defined as the composite

\[
F \times \Delta[m] \xrightarrow{(\alpha, \pi_2)} F \times \Delta[m] \xrightarrow{\beta} E.
\]

**Proposition 4.2.** Let \( p : E \to B \) be an \( F \)-fiber bundle, \( b \in B_m \) an \( m \)-simplex of \( B \), and \( \beta(b) \in \text{Map}(F,E)_m \) a local trivialization of \( p \) over \( b \). Then the set of local trivializations of \( p \) over \( b \) is a torsor over the group \( \text{Aut}(F)_m \). That is, the following assertions hold.

1. Let \( \beta'(b) \in \text{Map}(F,E)_m \) be another local trivialization of \( p \) over \( b \). Then there exists a unique \( m \)-simplex \( \alpha \in \text{Aut}(F)_m \) such that we have \( \beta'(b) = \beta(b) \cdot \alpha \).

2. Let \( \alpha \in \text{Aut}(F)_m \) be an \( m \)-simplex. Then the \( m \)-simplex \( \beta(b) \cdot \alpha \in \text{Map}(F,E)_m \) is also a local trivialization of \( p \) over \( b \).

**Proof.** By the Pasting Lemma, in the commutative diagram below, the square below \( \beta'(b) \) is Cartesian if and only if the square below \( (\alpha, \pi_2) \) is Cartesian.

![Diagram](image_url)

**Definition 4.3.** Suppose that \( p : E \to B \) is equipped with an atlas. Let \( \theta : [n] \to [m] \) be a map of ordinals. Let the map \( \theta^* \beta(b) : F \times \Delta[n] \to E \) be defined by composition as in the upper triangle of the following commutative diagram:

\[
F \times \Delta[n] \xrightarrow{\text{id} \times \Delta[\theta]} F \times \Delta[m] \xrightarrow{\beta(b)} E
\]

By Proposition 4.2 there exists a map \( \alpha_s(b, \theta) \in \text{Aut}(F)_n \) such that

\[
\theta^* \beta(b) = \beta(\theta^* b) \cdot \alpha_s(b, \theta).
\]

We call this map \( \alpha_s(b, \theta) \) a transition map. In particular, we let \( r_i(b) = \alpha_s(b, d^i) \) for \( 0 \leq i \leq n \).

We say that the atlas \( \beta \) is normalized if we have \( \alpha_s(b, s^i) = 1 \) for all codegeneracy maps \( s^i : [n+1] \to [n] \) and all \( b \in B_n \). If the atlas is normalized, then we say that it is regular if moreover we have \( r_i(b) = 1 \) for \( i > 0 \).

**Definition 4.4.** Let \( K \leq \text{Aut} F \) be a sub-simplicial group. Then a \( K \)-atlas of \( p \) is an atlas \( \{ \beta(b) : b \in B_n, n \geq 0 \} \) such that for all order-preserving maps \( \theta : [m] \to [n] \) in \( \Delta \) and all simplices \( b \in B_n \) the transition map \( \alpha_s(b, \theta) \) is in \( K_m \leq \text{Aut}(F)_m \).
Notation 4.5. Let $B$ be a simplicial set. Then we denote by $\int B = \int^{\Delta^\text{op}} B$ the Grothendieck construction of $B : \Delta^\text{op} \to \text{Set}$. That is, the category $\int B$ over $\Delta$ is defined as follows:

- The objects are the simplicial set maps $b : \Delta[n] \to B$ for $n \geq 0$.
- The morphisms are the commuting triangles
  \[ \Delta[m] \xrightarrow{\theta} \Delta[n] \xrightarrow{\theta^* b} b \]
  \[ \Downarrow b \]
  \[ \Downarrow B. \]

We will denote these morphisms by $\theta : \theta^* b \to b$.

- We have a forgetful map $U : \int B \to \Delta$.

Notation 4.6. Let $K$ be a simplicial group. Then we denote by $\int K = \int^{\Delta^\text{op}} K$ the Grothendieck construction of $\Delta^\text{op} \xrightarrow{K} \text{Grp} \hookrightarrow \text{Cat}$. That is, the category $\int K$ over $\Delta$ is defined as follows:

- The objects are the ordinals $[n]$ where $n \geq 0$.
- The morphisms are the pairs
  \[ (\theta, k) : [m] \to [n] \]
  where $\theta$ is an ordinal map and $k \in K_m$. Composition is defined by
  \[ (\theta, k)(\phi, k') = (\theta \phi, k' \varphi^*(k)). \]

- We have a forgetful map $U : \int K \to \Delta$.

Construction 4.7. Let $p : E \to B$ be an $F$-fiber bundle, $K \leq \text{Aut} F$ a sub-simplicial group, $\beta$ a $K$-atlas for $p$, and $\alpha^*$ the corresponding collection of transition maps. Then we can construct the functor

\[
\begin{array}{ccc}
\int B & \xrightarrow{\alpha^*} & \int K \\
\downarrow U & & \downarrow U \\
\Delta & & \Delta
\end{array}
\]

by sending an $n$-simplex $b \in B_n$ to the object $[n]$ and a morphism $\theta : \theta^* b \to b$ to the pair $(\theta, \alpha^*(b, \theta))$.

Remark 4.8. Suppose that $\alpha^*(b, \theta) = 1$ for all maps $\theta : [m] \to [n]$ in $\Delta$ and $b \in B_n$. Then the atlas $\beta$ defines an isomorphism $F \times B = \text{colim}_{b \in \int B} (F \times \Delta[n]) \to E$ [May67, Theorem 19.4].

Lemma 4.9. The assignment specified by $\alpha^*$ defines a functor.

Proof. Let $[k] \xrightarrow{\phi} [m] \xrightarrow{\theta} [n]$ be a diagram in $\Delta$, and $b \in B_n$. We need to show that

\[ (\theta \circ \phi, \alpha^*(\theta \circ \phi)) = (\theta, \alpha^* \theta) \circ (\phi, \alpha^* \phi) = (\theta \circ \phi, \alpha^* \phi \cdot \phi^* \alpha^* \theta). \]

By definition, the transition map $\alpha^* \theta \in \text{Aut}(F)_m$ is the unique element such that

\[ \theta^* \beta(b) = \beta(\theta^* b) \cdot \alpha^* \theta \]

in $\text{Map}(F, E)_m$. Therefore, restricting along $\phi$ we get

\[ \phi^* \theta^* \beta(b) = \phi^* \beta(\theta^* b) \cdot \phi^* \alpha^* \theta, \]

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substituting into which the definition of $\alpha_*\phi$, we get

$$\phi^*\theta^*\beta(b) = \beta(\phi^*\theta^*b) \cdot \alpha_*\phi \cdot \phi^*\alpha_*\theta,$$

which by definition shows that $\alpha_*(\theta \circ \phi) = \alpha_*\phi \cdot \phi^*\alpha_*\theta$. \qed

Let $p : E \to B$ be a $K$-fiber bundle equipped with a $K$-atlas $\beta$. We use the nerve $N\alpha_*$ to say when is $\beta$ a $\tau$-atlas on $p$ in Definition 4.11. In Lemma 4.20 we show that if $p$ admits a $\tau$-atlas, then its classifying map $\tilde{r} : B \to \mathbb{W}K$ (see Theorem 4.15) factors through the inclusion $\mathbb{W}(\tau, K) \to \mathbb{W}K$. The following Conjecture is needed to show that if $p$ admits a weak $\tau$-atlas, then the homotopy class of $\tilde{r}$ lifts to $[B, \mathbb{W}(\tau, K)]$ (see Lemma 4.26).

**Conjecture 4.10.** Let $E \to B$ be a principal $K$-bundle and $\alpha$ be an atlas. Then the map $N\alpha_* : N\int B \to N\int K$ is a classifying map for $p : E \to B$, i.e. under the relevant zigzag of equivalences (see Section 5.2) the homotopy class of $N\alpha_*$ coincides with $[\tilde{r}] \in [B, \mathbb{W}K]$.

### 4.2. $\tau$-atlasses

Recall that the endofunctor $\tau : \text{Grp} \to \text{Grp}$ is equipped with a natural map $\eta : \text{id}_{\text{Grp}} \to \tau$ such that for all $k \geq 0$ the map $\eta_{F^k} : F^k \to \tau^k$ is surjective. In particular, for a group $G$, the induced map $\text{Grp}(\tau^k, G) \to \text{Grp}(F^k, G)$ is injective. By abuse of notation we will identify $\text{Grp}(\tau^k, G)$ with its image and write $\text{Grp}(\tau^k, G) \subseteq \text{Grp}(F^k, G)$.

Let $N(\tau, \int K)$ denote the sub-simplicial set of the nerve $N\int K$ whose $k$-simplices are given by

$$[n_0] \xrightarrow{(\theta_1,x_0)} [n_1] \xrightarrow{(\theta_2,x_1)} \cdots \xrightarrow{(\theta_k,x_{k-1})} [n_k]$$

such that for $1 \leq l \leq k$ the tuple

$$(x_{k-l}, \theta^*_{k-l+1} x_{k-l+1}, \cdots, \theta^*_{k-l+1} x_{k-l}, \theta^*_{k-l+1} x_{k-l+1}, \cdots, \theta^*_{k-l+1} x_{k-l})$$

belongs to $N(\tau, K_{n_{k-l}}) = \text{Grp}(\tau^l, K_{n_{k-l}}) \subseteq \text{Grp}(F^l, K_{n_{k-l}}) \cong K^l_{n_{k-l}}$. The simplicial structure is induced from the nerve.

**Definition 4.11.** Let $p : E \to B$ be a fiber bundle. An atlas $\beta$ for $p$ is said to be a $\tau$-atlas if the nerve $N\alpha_*$ of the associated transition functor $\alpha_*$ (4.1.1) maps into $N(\tau, \int \text{Aut}(F)) \subseteq N\int \text{Aut}(F)$.

**Remark 4.12.** Let $K \leq \text{Aut}(F)$ be a sub-simplicial group and suppose that $\beta$ is a $K$-atlas. Then as $\tau$ is an endofunctor $\beta$ is a $\tau$-atlas if and only if $N\alpha_*$ factors through $N(\tau, \int K) \subseteq N\int K$.

**Example 4.13.** Let $G$ be a topological group and $\pi : P \to X$ a principal $G$-bundle. In Definition 5.13 we define what it means for a trivialization of $\pi$ along an open cover to have a $\tau$-structure. Out of this data, in Construction 5.16 we show how to construct a principal $SG$-bundle $S\gamma P \to S\gamma X$ with a $\tau$-atlas.

**Theorem 4.14.** There exists a zigzag of weak equivalences $N(\tau, \int K) \xleftarrow{\sim} \mathbb{W}(\tau, K)$ natural in $\tau$ and $K$.

**Proof.** We refer to Appendix A for the notation. More precisely we will show that there is a zigzag of weak equivalences

$$N(\tau, \int K) \cong T\Psi^*N(\tau, K) \xleftarrow{CR} d\Psi^*N(\tau, K) \xrightarrow{BK} dN(\tau, K) \xrightarrow{CR} \mathbb{W}(\tau, K)$$

natural in $\tau$. Note that both the Bousfield–Kan (Definition A.5) and Cegarra–Remedios (Definition A.8) maps are defined for arbitrary bisimplicial sets, so it is enough to check that the Tonks isomorphism (Proposition A.10) $N \int K \rightarrow T\Psi NK$ restricts to an isomorphism $N(\tau, \int K) \rightarrow T\Psi N(\tau, K)$. Let’s take an $l$-simplex $[n_0] \xrightarrow{b_{0,k_0}} \cdots \xrightarrow{b_{l-1,k_{l-1}}} [n_l]$ of $N \int K$. We need to show that this is in $N(\tau, \int K)_l$ if and only if its image is in $(T\Psi N(\tau, K))_l$. But that follows from construction: the component of the image in $(\Psi' NK)_{l,l-1}$ is the pair
\[
([n_0] \xrightarrow{b_0} \cdots \xrightarrow{b_{l-1}} [n_l], (k_i, \theta^i k_{i+1}, \ldots, \theta^i_{l-2} k_{l-1})).
\]
\[\square\]

4.3. Simplicial principal bundles. Let $K$ be a simplicial group. Then a principal $K$-bundle is a map of simplicial sets $p : E \rightarrow B$ where $K$ acts freely on $E$ from the left and $B$ is isomorphic to the quotient space. Free action can be formulated by saying that we have a pull-back diagram
\[
\begin{array}{ccc}
K \times E & \xrightarrow{m} & E \\
\downarrow{\pi_2} & \downarrow{p} & \downarrow{p} \\
E & \xrightarrow{\pi_2} & B,
\end{array}
\]
where $\pi_2$ is the projection and $m$ is the action map. For each $b \in B_n$ and $n \geq 0$ pick an element $\sigma(b) \in p^{-1}(b)$. Then the diagram
\[
\begin{array}{ccc}
K \times \Delta[n] & \xrightarrow{\beta(b)(k, \theta)} & E \\
\downarrow{\pi_2} & \downarrow{b} & \downarrow{p} \\
\Delta[n] & \xrightarrow{\beta(b)(k, \theta)} & B
\end{array}
\]
is a pullback square. This gives a $K$-atlas for $p$ as a $K$-fiber bundle and the transition functor has the form
\[
\alpha_s : \int B \rightarrow \int K.
\]
Conversely, suppose that there exists a $K$-atlas $\{\beta(b) : b \in B_n, n \geq 0\}$ for $p$ as a $K$-fiber bundle. Then we can define an action map $m : K \times E \rightarrow E$ equipping $p$ with the structure of a principal $K$-bundle as follows: for $n \geq 0$ take $(k, x) \in K_n \times E_n$. Let $b = p(x)$. Then we have $x = \beta(b)(l, \text{id}_{\Delta[n]})$ for some $l \in K_n$. Let $m(k, x) = \beta(b)(kl, \text{id}_{\Delta[n]})$. We get mutually inverse bijections
\[
\begin{cases}
K\text{-atlas} \beta \\
\text{for } p \text{ as a } K\text{-fiber bundle}
\end{cases}
\begin{cases}
\begin{aligned}
m(k,x) &= \beta(b)(kl, \text{id}_{\Delta[n]}) \\
\sigma(b) &= \beta(b)(1, \text{id}_{\Delta[n]}) \\
\beta(b)(k, \theta) &= k \cdot \theta^s \sigma(b)
\end{aligned}
\end{cases}
\begin{cases}
\begin{aligned}
\text{Principal } K\text{-bundle structures on } p \\
\text{with a choice } \sigma(b) \in p^{-1}(b) \\
\text{for every } b \in B_n \text{ for every } n \geq 0
\end{aligned}
\end{cases}
\]

The collection $\{\sigma(b) \in p^{-1}(b) : b \in B_n, n \geq 0\}$ is called a pseudo-section if for all $b \in B_n$, $n \geq 0$ we have
\[
\begin{aligned}
s_i \sigma(b) &= \sigma(s_i b) \text{ for all } 0 \leq i \leq n \\
d_i \sigma(b) &= \sigma(d_i b) \text{ for all } 0 < i < n.
\end{aligned}
\]
Note that the collection \( \sigma \) is a pseudo-section if and only if the corresponding \( K \)-atlas \( \beta \) is regular. For \( b \in B_{n+1} \) and \( n \geq 0 \) let \( r(b) \in K_n \) be the unique element such that
\[
d_0 \sigma(b) = r(b) \cdot \sigma(d_0).
\]
In terms of the transition functions we have \( r(b) = \alpha^*_s(b, d_0) \).

Let \( H^1(B, K) \) denote the isomorphism classes of principal \( K \)-bundles over \( B \) (see Remark 4.17).

**Theorem 4.15.** [May67, §21] The map sending a map \( r : B \to \overline{W}K \) to the pullback \( WB \times_{\overline{W}K} B \) of the universal principal \( K \)-bundle \( WK \to \overline{W}K \) along \( r \) induces a bijection
\[
[B, \overline{W}K] \to H^1(B, K).
\]
The preimage of a principal \( K \)-bundle \( p : E \to B \) is the homotopy class of the adjoint \( \hat{r} : B \to \overline{W}K \) of the map \( r : GB \xrightarrow{b \mapsto r(b)} K \).

### 4.4. On the classification of principal bundles with \( \tau \)-structure.

The category of simplicial sets comes with the Quillen model structure [Qui06]. As it is well-known \( \overline{W}K \) is fibrant in this model structure i.e. a Kan complex. However, \( \overline{W}(\tau, K) \) is not necessarily fibrant. This is even the case for a discrete non-abelian group if we take \( \tau^* = Z^* \). Therefore the homotopy classes of maps \([B, \overline{W}(\tau, K)]\), the set of morphisms in the homotopy category \( \text{Ho}(s\text{Set}) \), cannot be represented as simplicial homotopy classes of maps. As a fibrant replacement we can take \( S|\overline{W}(\tau, K)| \), which by Theorem 5.4 is equivalent to \( SB(\tau, |K|) \). Then \([B, \overline{W}(\tau, K)]\) is in bijective correspondence with the set of simplicial homotopy classes of maps \( B \to SB(\tau, |K|) \).

Correspondingly, in Lemma 4.24 we show that a homotopy class in \([B, \overline{W}(\tau, K)]\) induces a principal \( K \)-bundle together with an equivalence with a principal \( S|K| \)-bundle that is furthermore equipped with a \( \tau \)-atlas, which structure we call a topological \( \tau \)-atlas. Conversely, we show in Lemma 4.26 that provided Conjecture 4.10 holds if a principal \( K \)-bundle \( p : E \to B \) admits a weak \( \tau \)-atlas, which is a more homotopically correct relaxation of the notion of \( \tau \)-atlas, then it is classified by a homotopy class in \([B, \overline{W}(\tau, K)]\).

The inclusion map \( \iota : \overline{W}(\tau, K) \to \overline{W}K \) induces a map
\[
\iota_* : [B, \overline{W}(\tau, K)] \to [B, \overline{W}K] \tag{4.4.1}
\]
which is, in general, neither injective nor surjective. Under the assumption of Proposition 3.8 it is surjective when \( |B| \simeq S^n \). However, it fails to be injective: Let \( G \) be an extraspecial \( p \)-group of order \( p^{2n+1} \) where \( n \geq 2 \) as in Example 5.12. Then \([S^n, \overline{W}G]\) is trivial whereas \([S^n, \overline{W}(Z, G)]\) is a free abelian group of finite rank since the universal cover of \( \overline{W}(Z, G) \) is a wedge of finitely many \( n \)-spheres.

One would like to interpret \([B, \overline{W}(\tau, K)]\) as equivalence classes of principal \( K \)-bundles which admit a \( \tau \)-atlas as in its topological analogue Theorem [AG15, Theorem 2.2] (see Theorem 5.15). One way to proceed is to define the equivalence relation using homotopy classes of maps.

**Definition 4.16.** Let \( p : E \to B \) be a principal \( K \)-bundle and \( f : B \to \overline{W}K \) denote a classifying map. A choice of a homotopy class in \( \iota_*^{-1}([f]) \), where \( \iota_* \) is as defined in (4.4.1), is called a \( \tau \)-structure for the principal bundle \( p \). Consider two principal \( K \)-bundles \( p_i : E_i \to B, i = 0, 1 \), together with \( \tau \)-structures \([f^i_\tau]\). We say \((p_0, [f^0_\tau])\) is \( \tau \)-concordant to \((p_1, [f^1_\tau])\),
written as \((p_0, [f^0_\tau]) \sim \tau (p_1, [f^1_\tau])\), if there exists a principal \(K\)-bundle \(p : E \to B \times \Delta[1]\) with a \(\tau\)-structure \([f_\tau]\) such that
\[
p|_{B \times \{i\}} = p_i, \quad [f_\tau|_{B \times \{i\}}] = [f^i_\tau].
\]

Let \(H^i_\tau(B, K)\) denote the set of \(\tau\)-concordance classes of pairs \((p : E \to B, [f_\tau])\) consisting of a principal \(K\)-bundle and a \(\tau\)-structure.

**Remark 4.17.** When \(\tau^* = F^*\) the set \(H^1_\tau(B, K)\) coincides with the set \(H^1(B, K)\) of isomorphism classes of principal \(K\)-bundles.

**Proposition 4.18.** For \(i = 0, 1\) let \(p_i : E_i \to B\) be a principal \(K\)-bundle, let \(f_i : B \to \bigwedge K\) be a classifying map for \(p_i\) and let \([f^i_\tau] \in [B, \bigwedge(\tau, K)]\) be a \(\tau\)-structure for \(p_i\). Then the following statements are equivalent:

1. We have a \(\tau\)-concordance \((p_0, [f^0_\tau]) \sim \tau (p_1, [f^1_\tau])\).
2. The principal \(K\)-bundles \(p_0\) and \(p_1\) are equivalent, and all principal \(K\)-bundles \(p : E \to B \times \Delta[1]\) such that \(p|_{B \times \{i\}} = p_i\) for \(i = 0, 1\) admit a \(\tau\)-structure \([f_\tau]\) for \(i = 0, 1\).
3. The homotopy classes of maps \([f^0_\tau], [f^1_\tau] \in [B, \bigwedge(\tau, K)]\) are equal.

**Proof.** (1) \(\Rightarrow\) (3): Let \(p : E \to B \times \Delta[1]\) be a principal \(K\)-bundle such that \(p|_{B \times \{i\}} = p_i\) for \(i = 0, 1\) and let \([f_\tau] \in [B \times \Delta[1], \bigwedge(\tau, K)]\) be a \(\tau\)-structure for \(p\) such that \([f_\tau|_{B \times \{i\}}] = [f^i_\tau]\) for \(i = 0, 1\). Let \(i : B \to B \times \Delta[1]\) denote the maps induced by the inclusions \(\{i\} \to \Delta[1]\) for \(i = 0, 1\). Then from \([i_0] = [i_1]\) we get \([f^0_\tau] = [f^1_\tau] = [f_\tau]\) = \([f^i_\tau]\) = \([f^i_\tau]\) for \(i = 0, 1\).

(3) \(\Rightarrow\) (2): Since we have \([f^0_\tau] = [f^1_\tau]\) we get \([f_0] = [f_1]\), that is \(p_0 \sim p_1\). Let \(p : E \to B \times \Delta[1]\) be a principal \(K\)-bundle such that \(p|_{B \times \{i\}} = p_i\) for \(i = 0, 1\). Then \(p\) is classified by a homotopy \(H : B \times \Delta[1] \to \bigwedge K\) such that \(H|_{B \times \{i\}} = f_i\) for \(i = 0, 1\). Let \(\pi_B : B \times \Delta[1] \to B\) denote the projection map and \(H_0 = f_0 \circ \pi_B : B \times \Delta[1] \to \bigwedge K\) denote the homotopy with constant value \(f_0\). There is a simplicial homotopy \((B \times \Delta[1]) \times \Delta[1] \to \bigwedge K\) between \(H\) and \(H_0\) obtained by first lifting the map \(B \times \Delta^2[1] \to \bigwedge K\), defined by gluing \(H\) and \(H_0\), to \(B \times \Delta[2] \to \bigwedge K\) and precomposing this map with \(\Delta[1] \times \Delta[1] \to \Delta[2]\) induced by the map of posets \([1] \times [1] \to [2]\) defined by \((0, 0) \mapsto (1, 1), (0, 1) \mapsto (1, 1), (1, 0) \mapsto 2\). Thus we have \([H] = [H_0]\). Then \([f^0_\tau] \circ [\pi_B] \in [B \times \Delta[1], \bigwedge(\tau, K)]\) is a lift to \([B \times \Delta[1], \bigwedge(\tau, K)]\) of \([H_0]\) and thus of \([H]\). That is, it is a \(\tau\)-structure for \(p\).

(2) \(\Rightarrow\) (1): Trivial.

**Corollary 4.19.** The map
\[
H^i_\tau(B, K) \xrightarrow{[p : E \to B, [f_\tau]] \mapsto [f_\tau]} [B, \bigwedge(\tau, K)]
\]
induced by the forgetful map is a bijection.

**Proof.** Proposition 4.18 shows that the map is well defined and injective. For surjectivity, given a homotopy class \([f_\tau] \in [B, \bigwedge(\tau, K)]\) one can take a representative \(f : B \to \bigwedge K\) of its image in \([B, \bigwedge K]\) and then the principal \(K\)-bundle \(p : E \to B\) classified by \(f\).

In the rest of this section we study the relationship between \(\tau\)-structures and \(\tau\)-atlases with an eye towards improving Definition 4.16.

**Lemma 4.20.** If a principal \(K\)-bundle \(p : E \to B\) admits a \(\tau\)-atlas then the classifying map \(\hat{\tau} : B \to \bigwedge K\) factors through the inclusion \(\bigwedge(\tau, K) \subset \bigwedge K\).
Proof. We will make use of the unit \( \eta \) and the counit \( \epsilon \) of the adjunction \( G \dashv \neg \). The map \( \eta_X : X \to \neg GX \) sends \( x \in X_n \) to the tuple \( \langle [x] , [d_0 x] , \ldots , [(d_0)^{n-1} x] \rangle \) and \( \epsilon_K : \neg G \neg K \to K \) sends \( [k_n , k_{n-1} , \ldots , k_0] \) to \( k_n \in K_n \) ([GJ99, Lemma 5.3]).

Assume that \( p \) has a \( \tau \)-atlas. That is, the nerve of the functor \( \alpha_s : \int B \to \int K \) factors through the simplicial set \( N(\tau, \int K) \). The classifying map \( \hat{\eta} : B \to \neg G \neg K \) in degree \( n \) sends \( b \in B_n \) to the tuple \( (\alpha_s(b, d^0), \alpha_s(d_0 b, d^0), \ldots , \alpha_s((d_0)^{n-1} b, d^0)) \) see Theorem 4.15 or the proof of [May67, Theorem 21.7]. That is, \( \hat{\eta} \) is the composite \( X \xrightarrow{\eta_X} \neg GX \xrightarrow{\eta} \neg G \neg K \). Note that this map factors through \( \neg (\tau, K) \) as a consequence of the description given in Proposition 3.5.

One would like to have a converse of Lemma 4.20 if the factorization is required to exist in the homotopy category \( \text{Ho}(s\text{Set}) \). However, we do not know if this is correct. The main issue is that the classifying space \( \neg (\tau, K) \) is not fibrant. To overcome this difficulty we relax the notion of a \( \tau \)-atlas. We give two more general definitions. The notion of a weak \( \tau \)-atlas reflects that we can take \( S[\neg (\tau, K)] \simeq \neg SB(\tau, [K]) \) as a fibrant replacement of \( \neg (\tau, K) \). In Lemma 4.23 we show that a topological \( \tau \)-atlas is a weak \( \tau \)-atlas. In Lemma 4.24, we show that if a principal bundle has a \( \tau \)-structure, then it has a topological \( \tau \)-atlas. In Lemma 4.26 we close the circle of equivalences by showing that provided Conjecture 4.10 holds, if a principal bundle has a weak \( \tau \)-atlas, then it has a \( \tau \)-structure.

**Definition 4.21.** Let \( p : E \to B \) be a principal \( K \)-bundle. Suppose that the topological principal \( |K| \)-bundle \( |p| \) admits a \( \tau \)-atlas (see Definition 5.13) along an open cover \( \mathcal{U} \) of the realization \( |B| \). Then the corresponding topological \( \tau \)-atlas on \( p \) is the diagram

\[
\begin{array}{ccc}
E & \xrightarrow{\eta_K} & S|E| \\
\downarrow{p} & & \downarrow{S|p|} \\
B & \xrightarrow{\eta_B} & S|B|
\end{array}
\]

where \( \eta : \text{id}_{s\text{Set}} \to S \circ | \) is the unit map of the Quillen equivalence \((| , S)\) together with the \( \tau \)-atlas on the principal \( S|K| \)-bundle \( S\mathcal{U}|p| \) given by Proposition 5.17.

**Definition 4.22.** Let \( p : E \to B \) be a principal \( K \)-bundle. Then a weak \( \tau \)-atlas is the collection of:

1. a zigzag of simplicial group homomorphisms between a collection \( K_0, K_1, \ldots , K_n \) of simplicial groups

\[
K = K_0 \xrightleftharpoons{\phi_0} K_1 \xleftarrow{\phi_1} K_2 \xrightleftharpoons{\phi_2} \cdots \xleftarrow{\phi_{n-1}} K_n
\]

such that for all \( 0 \leq i < n \) the induced map \( \neg (\tau, \phi_i) : \neg (\tau, K_{i-1}) \to \neg (\tau, K_i) \) is a weak equivalence,

2. a commutative diagram

\[
\begin{array}{ccccccc}
E & \xrightarrow{F_0} & E_1 & \xleftarrow{F_1} & E_2 & \xrightarrow{F_2} & \cdots & \xleftarrow{F_{n-1}} & E_n \\
\downarrow{p} & & \downarrow{p_1} & & \downarrow{p_2} & & \cdots & & \downarrow{p_n} \\
B & \xrightarrow{f_0} & B_1 & \xleftarrow{f_1} & B_2 & \xrightarrow{f_2} & \cdots & \xleftarrow{f_{n-1}} & B_n
\end{array}
\]
where
(a) for $1 \leq i \leq n$ the map $p_i$ is a principal $K_i$-bundle,
(b) the map $F_{i-1}$ is $\phi_{i-1}$-equivariant
and
(3) a $\tau$-atlas on the principal $K_n$-bundle $p_n$.

**Lemma 4.23.** Let $p : E \to B$ be a principal $K$-bundle. Then a topological $\tau$-atlas on $p$ is a weak $\tau$-atlas.

**Proof.** Consider the diagram

$$
\begin{array}{ccc}
E & \xrightarrow{\eta_E} & S[E] \\
\downarrow{p} & & \downarrow{s[\cdot]} \\
B & \xrightarrow{\eta_B} & S[B]
\end{array}
$$

given by a topological $\tau$-atlas. The left square is commutative by naturality of the unit map. The unit map $\eta_E : E \to S[E]$ is an $\eta_K$-equivariant map: Take $x \in E_n$ and $k \in K_n$. Then we have

$$
\eta_E(kx)(u) = [u, k] = [u, k] \cdot [u, x] = (\eta_K(k) \cdot \eta_E(x))(u) \text{ for } u \in \Delta^n.
$$

The right square is a pullback square by construction, and the horizontal maps are weak equivalences [DI04, Corollary 3.5].

**Lemma 4.24.** Let $p : E \to B$ be a principal $K$-bundle. If $p$ has a $\tau$-structure then $p$ admits a topological $\tau$-atlas.

**Proof.** By definition of a topological $\tau$-atlas, we need to show that the topological principal $|K|$-bundle $|p|$ admits a $\tau$-atlas. By assumption the homotopy class of the classifying map $B \xrightarrow{\tilde{\iota}} \overline{WK}$ factors through the homotopy class of the inclusion $\overline{W}(\tau, K) \to \overline{WK}$. By Theorem 5.4 (and its proof) we have a commutative diagram

$$
\begin{array}{ccc}
|\overline{W}(\tau, K)| & \xleftarrow{\simeq} & |dN(\tau, K)| \\
\downarrow & & \downarrow \\
|\overline{WK}| & \xleftarrow{\simeq} & |dNK| \\
\end{array} \xrightarrow{\simeq} \begin{array}{ccc}
B(\tau, |K|) & \xrightarrow{\simeq} & B(\tau, |K|) \\
\downarrow & & \downarrow \\
B[K] & \xrightarrow{\simeq} & B[K].
\end{array}
$$

Therefore using Corollary 5.2 we get that the homotopy class of the classifying map of the principal $|K|$-bundle $|p|$ factors through the map $B(\tau, K) \to BK$. By Theorem 5.15 this shows that $|p|$ admits a $\tau$-atlas.

**Remark 4.25.** If the unit map $\eta_K : K \to S[K]$ has a homotopy inverse $\phi$ which is also a simplicial group homomorphism then the principal $S[K]$-bundle $S_{\mathbb{H}}|p|$ can be turned into a $K$-bundle via change of structure group along $\phi$. This is the case for $K = SG$, where $G$ is a topological group, since the simplicial group homomorphism $SG : S[SG] \to SG$ is a homotopy inverse of $\eta_{SG}$. In this situation $\overline{W}(\tau, K)$ classifies principal $K$-bundles with a stronger notion of topological $\tau$-atlas: the principal $K$-bundle on $S_{\mathbb{H}}|B|$ we get has a $\tau$-atlas so we get a zigzag as in the Proof of Lemma 4.24 but with all principal bundles being $K$-bundles and both upper horizontal maps being $K$-equivariant.
**Lemma 4.26.** Suppose that $p$ admits a weak $\tau$-atlas. If Conjecture 4.10 holds then $p$ admits a $\tau$-structure.

**Proof.** Suppose that $p : E \to B$ admits a weak $\tau$-atlas as in Definition 4.22, that is there exists a zigzag of simplicial group homomorphisms

$$K = K_0 \xrightarrow{\phi_0} K_1 \xleftarrow{\phi_1} K_2 \xrightarrow{\phi_2} \cdots \xleftarrow{\phi_{n-1}} K_n$$

such that for all $0 \leq i < n$ the induced map $\overline{W}(\tau, K_{i-1}) \to \overline{W}(\tau, K_i)$ is a weak equivalence, and a commutative diagram

$$
\begin{array}{ccccccc}
E & \overset{f_0}{\sim} & E_1 & \overset{f_1}{\sim} & E_2 & \overset{f_2}{\sim} & \cdots & \overset{f_{n-1}}{\sim} & E_n \\
\downarrow \scriptstyle{p} & & \downarrow \scriptstyle{p_1} & & \downarrow \scriptstyle{p_2} & & \cdots & & \downarrow \scriptstyle{p_n} \\
B & \overset{f_0}{\sim} & B_1 & \overset{f_1}{\sim} & B_2 & \overset{f_2}{\sim} & \cdots & \overset{f_{n-1}}{\sim} & B_n
\end{array}
$$

where for $1 \leq i \leq n$ the map $p_i$ is a principal $K_i$-bundle, the map $F_{i-1}$ is $\phi_{i-1}$-equivariant and the principal $K_n$-bundle $p_n$ has a $\tau$-atlas. By definition the last condition means that the transition functor $N \int B_n \xrightarrow{N \alpha} N \int K_n$ corresponding to the atlas factors through the inclusion $N(\tau, \int K_n) \to N \int K_n$. By Conjecture 4.10 and Theorem 4.14, this implies that $p_n$ has a $\tau$-structure, that is the homotopy class of a classifying map $B_n \xrightarrow{r_n} \overline{W}K_n$ of $p_n$ factors through the homotopy class of the inclusion $\overline{W}(\tau, K_n) \to \overline{W}K_n$.

Let $\hat{r} : B \to \overline{W}K$ be a classifying map for $p$ and for $1 \leq i < n$ let $\hat{r}_i : B_i \to \overline{W}K_i$ be a classifying map for $p_i$. By the weak $\tau$-atlas data we have a homotopy commutative diagram

\[
\begin{array}{ccccccc}
\xrightarrow{\hat{r}} & \xleftarrow{\widehat{W}(\tau, \phi_0)} & \xrightarrow{\hat{r}_1} & \xleftarrow{\widehat{W}(\tau, \phi_1)} & \xrightarrow{\hat{r}_2} & \cdots & \xleftarrow{\widehat{W}(\tau, \phi_{n-1})} & \xrightarrow{\hat{r}_n} \\
\overline{W}K & \xrightarrow{\phi_0} & \overline{W}K_1 & \xrightarrow{\phi_1} & \overline{W}K_2 & \xrightarrow{\phi_2} & \cdots & \xrightarrow{\phi_{n-1}} & \overline{W}K_n
\end{array}
\]

\[
\begin{array}{ccccccc}
\xrightarrow{\hat{r}} & \xleftarrow{\widehat{W}(\tau, K_1)} & \xrightarrow{\hat{r}_1} & \xleftarrow{\widehat{W}(\tau, K_2)} & \xrightarrow{\hat{r}_2} & \cdots & \xleftarrow{\widehat{W}(\tau, K_{n-1})} & \xrightarrow{\hat{r}_n} \\
\overline{W}(\tau, K) & \xrightarrow{\tau, \phi_0} & \overline{W}(\tau, K_1) & \xrightarrow{\tau, \phi_1} & \overline{W}(\tau, K_2) & \xrightarrow{\tau, \phi_2} & \cdots & \xrightarrow{\tau, \phi_{n-1}} & \overline{W}(\tau, K_n).
\end{array}
\]

This shows that the zigzag

$$B \xrightarrow{f_0} B_1 \xleftarrow{f_1} \cdots \xleftarrow{f_{n-1}} B_n \xrightarrow{r_n} \overline{W}K_n \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_1} \overline{W}K_1 \xrightarrow{\phi_0} \overline{W}K$$

classifies $p$ and its homotopy class factors through the homotopy class of the inclusion $\overline{W}(\tau, K) \to \overline{W}K$, thus giving a $\tau$-structure to $p$.

\[\square\]

In summary:

**Theorem 4.27.** Suppose that Conjecture 4.10 holds. Let $p : E \to B$ be a simplicial principal $K$-bundle. Then the following assertions are equivalent:

1. The principal $K$-bundle $p$ admits a $\tau$-structure.
2. The principal $K$-bundle $p$ admits a topological $\tau$-atlas.
3. The principal $K$-bundle $p$ admits a weak $\tau$-atlas.

In case Conjecture 4.10 holds, Definition 4.16 can be modified by replacing the notion of $\tau$-structure with the notion of weak $\tau$-atlas which brings it closer to its topological counterpart.
In this section $G$ denotes a topological group.

We introduced $\overline{W}(\tau, K)$ and claimed that this is a simplicial analogue of $B(\tau, G)$. In this section we turn this claim into a theorem. We prove two homotopy equivalences

$$B(\tau, |K|) \simeq |\overline{W}(\tau, K)|, \quad B(\tau, G) \simeq |\overline{W}(\tau, SG)|$$

where in the latter one $G$ is required to be a compact Lie group.

5.1. Classifying spaces. The nerve construction of a discrete group can be extended to the category of topological groups [Seg68]. Given a topological group $G$ the set of $n$-simplices of $NG$ is the $n$-fold direct product $G^n = G \times \cdots \times G$, hence a topological space when $G$ has a topology. The simplicial object $NG_\bullet$ is a simplicial space, and its geometric realization is denoted by

$$BG = |NG_\bullet|$$

which is called the classifying space of $G$. We can think of the $n$-fold product as the space of group homomorphisms

$$NG_n = \text{Hom}(F^n, G).$$

We use $\text{Hom}(-, -)$, rather than $\text{Grp}(-, -)$, to emphasize the topology.

Given a simplicial group $K$, the geometric realization $|K|$ is a topological space, and we can consider $B|K|$. We would like to compare this space to the geometric realization of $\overline{W}(K)$.

Proposition 5.1. There is a natural homotopy equivalence

$$B|K| \to |\overline{W}(K)|.$$

Proof. Taking the geometric realization of Proposition A.9 we obtain a homotopy equivalence $|CR| : dNK \to |\overline{W}K|$. We can realize the bisimplicial set $NK$ in different ways. All of these are homeomorphic to each other [Qui73]. There is a sequence of natural homeomorphisms

$$|dNK| \cong |[p] \mapsto |[q] \mapsto N(K_q)_p||
= |[p] \mapsto |K_p||
\cong |[p] \mapsto |K|^p|
= B|K|$$

where we also used the fact that geometric realization preserves finite products [Mil57]. Thus we obtain the desired map $B|K| \to |\overline{W}(K)|$ as the composite $|CR|Q^{-1}$ where $Q : |dNK| \to B|K|$ is the homeomorphism described above.

Corollary 5.2. Let $p : E \to B$ be a principal $K$-bundle and $\hat{r} : B \to \overline{W}K$ denote the classifying map. Then the zigzag

$$|B| \xrightarrow{|\hat{r}|} |\overline{W}K| \xleftarrow{|CR|} |dNK| \xrightarrow{Q} B|K|$$

is a classifying map for $|p| : |E| \to |B|$.

Proof. We begin by observing that

- the homeomorphism $Q : |dNK| \to B|K|$ takes $[u, (k_i)_{i=1}^n] \in |dNK|$ where $u \in \Delta^n$ and $k_i \in K_n$ for $i = 1, \ldots, n$ to $[u, ([u, k_i])_{i=1}^n] \in B|K|$ and
the weak equivalence $CR: dNK \to WK$ takes $(k_i)_{i=1}^n \in (dNK)_n$ where $k_i \in K_n$ for $i = 1, \ldots, n$ to $(d^0 k_i)_{i=1}^n \in (WK)_n$.

To show that (5.1.1) classifies the principal bundle $|p|$ we need to lift this zigzag of maps to a zigzag of homotopy pullbacks with vertical arrows principal $|K|$-bundles and top horizontal arrows equivariant maps.

Let $\Delta^{op} \tilde{K} \to \text{Cat}$ denote the simplicial groupoid where
- for $n \geq 0$ we have $\text{ob}(\tilde{K}_n) = K_n$,
- for $k, l \in K_n$ we have $\text{Hom}_{\tilde{K}_n}(k, kl) = \{l\}$,
- for a pair of maps $k \overset{l}{\to} kl \overset{l'}{\to} kll'$ in $\tilde{K}_n$ their composite is $k \overset{l''}{\to} kl'$ and
- the simplicial structure maps are induced by those on $K$.

We write the $m$-simplex $k_0 \overset{k_1}{\to} k_0 k_1 \overset{k_2}{\to} \cdots \overset{k_m}{\to} k_0 k_1 \cdots k_m$ of $N\tilde{K}_n$ as $(k_i)_{i=0}^m$. Note that taking all objects to $\ast$ gives a morphism $\tilde{K} \to K$.

We get the commutative diagram

$$
\begin{array}{ccc}
|E| & \rightarrow & |WK| \\
\downarrow |p| & & \downarrow |\hat{r}| \\
|B| & \rightarrow & |WK| \\
\end{array}
\begin{array}{ccc}
|dN\tilde{K}| & \leftarrow & |E| |K| \\
\downarrow |\hat{Q}| & & \downarrow |\hat{Q}| \\
|dNK| & \leftarrow & |E| |K| \\
\end{array}
\begin{array}{ccc}
|\hat{Q}| & \rightarrow & |E| |K| \\
\downarrow |\hat{Q}| & & \downarrow |\hat{Q}| \\
|B| & \rightarrow & |WK|
\end{array}
$$

where
- the vertical arrows are principal $|K|$-bundles,
- the leftmost square is the image by $||$ of the pullback square

$E \rightarrow WK$

$\downarrow |p|$

$B \rightarrow WK$

- the map $\tilde{CR}: dN\tilde{K} \to WK$ is the $K$-equivariant weak equivalence that takes $(k_i)_{i=0}^n \in (dN\tilde{K})_n$ to $(d^0 k_i)_{i=0}^n \in (WK)_n$,
- and the map $\hat{Q}: |dN\tilde{K}| \rightarrow |E| |K|$ is the $|K|$-equivariant homeomorphism taking $[u, (k_i)_{i=0}^n] \in |dN\tilde{K}|$ to $[u, ([u, k_i])_{i=0}^n] \in |E| |K|$.

Therefore all the squares are homotopy pullback squares with vertical arrows principal $|K|$-bundles and top horizontal arrows $|K|$-equivariant maps, so indeed the homotopy class $[Q] \circ [CR]^{-1} \circ [\hat{r}]$ is $|[B|, B|K]|$ classifies the principal $|K|$-bundle $|p|$. □

5.2. $\tau$-version. Given an endofunctor $\tau$ on the category of groups we can define

$$B(\tau, G) = [\tau] \mapsto \text{Hom}(\tau^n, G)$$

where $\tau^* = \tau F^*$ as usual. Note that this definition works for an arbitrary cosimplicial group not necessarily coming from an endofunctor. For the rest of the section we assume that $\tau$ is of quotient type. In this case the homomorphism space $\text{Hom}(\tau^n, G)$ is topologized as a
subspace of \(G^{\times n}\). The natural transformation \(\eta\) induces an inclusion \(B(\tau, G) \subset BG\). We denote by \(E(\tau, G) \to B(\tau, G)\) the pull-back of the universal bundle \(EG \to BG\).

We would like to prove an analogue of Proposition 5.1. To prepare we need a preliminary result. Observe that the set of group homomorphisms \(\text{Hom}(\tau F^n, K_m)\) can be assembled into a simplicial set by using the simplicial structure of \(K\).

**Lemma 5.3.** There is a natural homeomorphism

\[
|\text{Hom}(\tau F^n, K_\bullet)| \to \text{Hom}(\tau F^n, |K|).
\]

**Proof.** We will use some of the basic properties of the geometric realization proved in [May67, Chapter III]. A point in the geometric realization \(|X|\) of a simplicial set \(X\) is given by an equivalence class \([u_m, x_m]\) where \(x_m\) is a non-degenerate \(m\)-simplex of \(X\) and \(u_m\) is a point in the interior of \(\Delta^n\). The natural map \(\phi : |X \times X| \to |X| \times |X|\) induced by applying || to the projection maps onto each factor of \(X \times X\) is a homeomorphism. Let \(\phi^{-1} : |X| \times |X| \to |X \times X|\) denote the inverse of this map. This map can be described explicitly (as in the proof of [May67, Theorem 14.3]), but we just need to know that a point \([(u_m, x_m), (u_m, x_m')]\) in the product is sent to the point \([u_m, (x_m^{(1)}, x_m^{(2)})]\) where \(u_m, i = 1, 2\), can be obtained from \(u_m\) by applying a sequence of codegeneracy maps and \(x_m^{(i)}\), \(i = 1, 2\), are given by applying the corresponding sequence of degeneracy maps to \(x_m\). Note that the multiplication map of \(|K|\) is given by \(|K| \times |K| \xrightarrow{\phi^{-1}} |K \times K| \xrightarrow{|\cup|} |K|\). Let \(\phi_n\) denote the map \(|K^{\times n}| \to |K|^{\times n}\) obtained by applying \(\phi\) multiple times: \(|K^{\times n}| \to |K| \times |K^{\times n-1}| \to \cdots \to |K|^{\times n}\). Let us define the following maps

\[
\iota_1 : |\text{Hom}(\tau F^n, K_\bullet)| \to |K^{\times n}|
\]

defined by \(\iota_1([u_m, \tau F^n \overset{f}{\to} K_m]) = [u_m, (f(e_1), \cdots, f(e_n))]\), and

\[
\iota_2 : \text{Hom}(\tau F^n, |K|) \to |K|^{\times n}
\]

defined by \(\iota_2(\tau F^n \overset{g}{\to} |K|) = (g(e_1), \cdots, g(e_n))\). Both of these maps are embeddings. This is clear for \(\iota_1\). For \(\iota_2\) it follows from the fact that the geometric realization functor commutes with finite limits, in particular equalizers.

We will write \(R_n\) for the kernel of \(\eta_{F^n} : F^n \to \tau F^n\). Each element of \(R_n\) determines a relation \(r(e_1, \cdots, e_n)\). A group homomorphism \(F^n \to H\) factors through \(\tau F^n \to H\) if and only if \(r(f(e_1), \cdots, f(e_n)) = 1\) in \(H\) for all \(r \in R_n\). Now, consider the composite \(\phi_n\iota_1\) which maps \([u_m, \tau F^n \overset{f}{\to} K_m]\) to the tuple \([(u_m, f(e_1)), \cdots, (u_m, f(e_n))]\). This tuple regarded as a group homomorphism \(F^n \to |K|^{\times n}\) factors through \(\tau F^n\). This follows from the fact that the elements \(x_i = [u_m, f(e_i)]\) satisfy the relations \(r(x_1, \cdots, x_n) = 1\). Note that here we are using the fact that \(\phi^{-1}([u_m, k_m], [u_m, k'_m])\) is simply \([u_m, (k_m, k'_m)]\). Therefore we obtain a commutative diagram

\[
\begin{array}{ccc}
|\text{Hom}(\tau F^n, K_\bullet)| & \longrightarrow & \text{Hom}(\tau F^n, |K|) \\
\downarrow \iota_1 & & \downarrow \iota_2 \\
|K^{\times n}| & \xrightarrow{\phi_n} & |K|^{\times n}
\end{array}
\]

We will show that the top map is a homeomorphism. Injectivity of the map is clear from the diagram. To see surjectivity let \(f : F^n \to |K|\) be a homomorphism that factors through \(\tau F^n\). Each element \(f(e_i)\) is given by an equivalence class \([u_{l_i}, k_{l_i}]\) where \(k_{l_i}\) is an \(l_i\)-simplex.
of $K$ and $u_l$ is a point in the interior of $\Delta^l$. Let $[u_m, k^{(1)}_m, \ldots, k^{(n)}_m]$ denote the element which maps to $([u_1, k_{11}], \ldots, [u_n, k_{1n}])$ under the homeomorphism $\phi_n$. Here $k^{(i)}_m$ belongs to $K_m$ and $u_m$ belongs to the interior of $\Delta^m$. Then for each $r \in R_n$ the relation $r(k^{(1)}_m, \ldots, k^{(n)}_m) = 1$ holds in $K_m$ since $r(f(e_1), \ldots, f(e_n)) = 1$ in $|K|$. Sending $e_i$ to $k^{(i)}_m$ defines a homomorphism $F^n : \tau F^n \to K_m$. The element $[u_m, f']$ in the geometric realization $|\Hom(\tau F^n, K_n)|$ maps to $f$ under $\phi_n$. This proves the surjectivity. As a result the homeomorphism $\phi_n$ restricts to a bijection between the subspaces, and consequently gives the desired homeomorphism. □

**Theorem 5.4.** There is a natural homotopy equivalence

$$B(\tau, |K|) \to |\overline{W}(\tau, K)|.$$  

**Proof.** Recall that we have a weak equivalence $dY \to TY$ for any bisimplicial set $Y$. We apply this to the bisimplicial set $N(\tau, K)$, and use the identification $\overline{W}(\tau, K) \cong TN(\tau, K)$. This gives a homotopy equivalence

$$|dN(\tau, K)| \to |\overline{W}(\tau, K)|$$

after realization. It remains to identify $|dN(\tau, K)|$ with $B(\tau, |K|)$. Using Lemma 5.3 and the homeomorphism between different ways of realizing a bisimplicial set we obtain

$$|dN(\tau, K)| \cong [p] \mapsto [q] \mapsto \Hom(\tau F^n, K_q)$$

$$\cong [p] \mapsto \Hom(\tau F^n, |K|)$$

$$= B(\tau, |K|).$$

□

Combining this result with Theorem 4.14 gives the following.

**Corollary 5.5.** There is a homotopy equivalence

$$B(\tau, |K|) \simeq |N(\tau, \int K)|.$$  

5.3. **Compact Lie groups.** In Theorem 5.4 we started from a simplicial group $K$ and compared the simplicial and topological constructions. Conversely we can do such a comparison for a topological group $G$. However, for such a comparison to work we need to restrict our attention to a nice class of groups such as compact Lie groups. We introduce the class of $\tau$-good groups in Definition 5.9.

**Remark 5.6.** The main reason for this restriction is that the geometric realization functor does not respect homotopy equivalences in general. A better behaving realization functor is the *fat realization* which is obtained by forgetting the degeneracies when gluing the simplices. For a simplicial space $X$ its fat realization is denoted by $|X|$. If the simplicial space is *good*, i.e. all degeneracy maps $s_i : X_{n-1} \to X_n$ are closed cofibrations in the sense of Hurewicz, then the natural map $|X| \to |X|$ is a homotopy equivalence. Any simplicial set is a good simplicial space, or more generally the simplicial space $[n] \mapsto |S(X_n)|$ is good where $X$ is an arbitrary simplicial space.
Remark 5.7. If $G$ is a compact Lie group then $[n] \mapsto \text{Hom}(\tau^n, G)$ is good. Thus up to homotopy we can replace geometric realization by the fat realization in the construction of $B(\tau, G)$, see [OW20]. This property will be crucial in our consideration.

We need a version of Lemma 5.3 for the singular functor. Let $H$ be a discrete group and $G$ a topological group. Recall that the mapping space $\text{Map}(H, G)$ is the set of maps $H \to G$ equipped with the compact-open topology. We equip the subset $\text{Grp}(H, G) \subseteq \text{Map}(H, G)$ with the subspace topology. In the case of $H = \tau F^n$ this is consistent with the topology induced from $G^\times n$

**Lemma 5.8.** Let $H$ be a discrete group, and $G$ a topological group. Then there is a natural isomorphism of simplicial sets

$$S(\text{Hom}(H, G)) \to \text{Hom}(H, S(G)_\bullet).$$

**Proof.** It is enough to show that the pair of mutually inverse natural isomorphisms

$$f \mapsto (h \mapsto \text{ev}_h \circ f)$$

restricts to a pair of mutually inverse natural isomorphisms

$$\text{Top}(\Delta^n, \text{Map}(H, G)) \cong \text{Set}(H, \text{Top}(\Delta^n, G))$$

$$x \mapsto (h \mapsto w(h)(x)) \leftrightarrow w$$

which shows $(h \mapsto \text{ev}_h \circ f) \in \text{Grp}(H, \text{Top}(\Delta^n, G))$.

Take $w \in \text{Grp}(H, \text{Top}(\Delta^n, G))$. Then we have

$$w(h)(x) \cdot w(h')(x) = (w(h) \cdot w(h'))(x)$$

$$= w(hh')(x)$$

which shows $(x \mapsto (h \mapsto w(h)(x))) \in \text{Top}(\Delta^n, \text{Hom}(H, G))$. \qed

**Definition 5.9.** A topological group $G$ is said to be $\tau$-good if each $B(\tau, G)_n$ is a good simplicial space consisting of CW-complexes (simplicial structure maps are morphisms of CW-complexes).

**Corollary 5.10.** Let $G$ be a topological group which is $\tau$-good, and let $K$ be a simplicial group.
(1) We have a chain of homotopy equivalences

\[ |\mathcal{W}(\tau, SG)| \simeq B(\tau, |SG|) \simeq B(\tau, G) \]  

where the first one is the one in Theorem 5.4 and the second one is induced by the counit \( \epsilon_G : |SG| \to G \).

(2) The unit \( \eta_K : K \to S|K| \) induces a weak equivalence

\[ \mathcal{W}(\tau, K) \to \mathcal{W}(\tau, S|K|). \]  

Proof. It is enough to show that the map \( B(\tau, |SG|) \to B(\tau, G) \) induced by the counit \( \epsilon_G : |SG| \to G \) is a homotopy equivalence. By Remark 5.6 we can use fat realization for good simplicial spaces. Lemma 5.3, Lemma 5.8, and the weak equivalence \( \epsilon_G : |SG| \to G \) gives us the diagram

\[
\begin{array}{ccc}
|\text{SHom}(\tau F^n, G)| & \xrightarrow{\cong} & \text{Hom}(\tau F^n, G) \\
\downarrow \cong & & \uparrow \\
|\text{Hom}(\tau F^n, (SG)*)| & \xrightarrow{\cong} & \text{Hom}(\tau F^n, |SG|)
\end{array}
\]

The top map is a homotopy equivalence since \( \text{Hom}(\tau F^n, G) \) is a CW-complex. Thus we obtain a level-wise homotopy equivalence, which after realization, gives the homotopy equivalence in (5.3.1).

To prove part (2) we will show that \( |\mathcal{W}(\tau, K)| \to |\mathcal{W}(\tau, S|K|)| \) is a homotopy equivalence. By Theorem 5.4 this amounts to showing that \( B(\tau, |K|) \to B(\tau, |S|K|) \) induced by \( |\eta_K| \) is a homotopy equivalence. First observe that \( B(\tau, |K|) \) is a good simplicial space for any simplicial group \( K \): Each degeneracy map \( s_i : \text{Hom}(\tau F^n, |K|) \to \text{Hom}(\tau F^{n+1}, |K|) \) is a closed cofibration since by Lemma 5.3 \( |s_i| : |\text{Hom}(\tau F^n, K)| \to |\text{Hom}(\tau F^{n+1}, K)| \) is an inclusion of a subcomplex of a CW-complex. Therefore similar to part (1) we can argue level-wise. Now, consider the map \( \text{Hom}(\tau F^n, |K|) \to \text{Hom}(\tau F^n, |S|K|) \) induced by \( |\eta_K| \). By Lemma 5.3 and Lemma 5.8 it is a direct calculation to check that, up to homeomorphism, this map coincides with the weak equivalence

\[ |\eta_{\text{Hom}(\tau F^n, K)}| : |\text{Hom}(\tau F^n, K)| \to |S|\text{Hom}(\tau F^n, K)|. \]

This is a homotopy equivalence since the spaces involved are CW-complexes. After geometric realization we obtain the desired homotopy equivalence.

Remark 5.11. The main class of examples for part (1) is compact Lie groups. In this case \( B(\tau, G) \) is a good simplicial space (Remark 5.7) consisting of CW-complexes (see e.g. the proof of [OW20, Proposition 2.3]). In general, there is no reason to expect that a homomorphism \( G \to G' \) which is a homotopy equivalence induces a homotopy equivalence \( B(\tau, G) \to B(\tau, G') \). An important exception is the inclusion \( K \subset G \) of a maximal compact subgroup \( K \) in a complex (or real) reductive algebraic group \( G \). In this case the map \( \text{Hom}(\mathbb{Z}^n, K) \to \text{Hom}(\mathbb{Z}^n, G) \) is a homotopy equivalence [PS13]. This implies that \( B(\mathbb{Z}, K) \to B(\mathbb{Z}, G) \) is a homotopy equivalence [AG15]. However, the homotopy equivalence between the homomorphism spaces fails when \( \mathbb{Z}^n \) is replaced by an arbitrary finitely generated group, see [PS13] for an example.
Example 5.12. We list some of the known results on the homotopy type of \( B(\tau, G) \) where \( G \) is a topological group. We can use them to deduce results about \( W(\tau, K) \). We focus on the commutative case \( \tau^* = \mathbb{Z}^* \).

It is better to consider discrete and continuous cases separately. Let \( G \) be discrete for now. Then \(|W(\tau, G)| = B(\tau, G)\) by definition. Most of the known results come from the finite case.

1. Suppose that \( G \) is a finite transitively commutative (TC) group, i.e. the commutator defines a transitive relation for elements outside the center of the group. For such groups \( B(\mathbb{Z}, G) \) is homotopy equivalent to the classifying space of \( \pi_1 B(\mathbb{Z}, G) \), as shown in [ACTG12]. Thus it is an Eilenberg–Maclane space of type \( K(\pi, 1) \). For example, the symmetric group \( \Sigma_3 \) on 3 letters is a TC group.

2. Let \( G \) be an extraspecial \( p \)-group of order \( p^{2n+1} \) where \( p \) is a prime. If \( n \geq 2 \) then the universal cover of \( B(\mathbb{Z}, G) \) is a wedge of equidimensional spheres \( S^n \) [Oka18]. Thus there are non-zero higher homotopy groups. A similar phenomenon occurs for the general linear groups \( \text{GL}_n(\mathbb{F}_q) \), \( n \leq 4 \), over a field of characteristic \( p \), and the symmetric groups \( \Sigma_k \) on \( k \) letters where \( k \geq 2^4 \) [Oka15].

Next we consider Lie groups. Let \( G \) be a compact Lie group. Using Corollary 5.10 and Theorem 5.4, we can translate the results to \( W(\mathbb{Z}, SG) \).

1. The rational cohomology of \( B(\mathbb{Z}, G) \) is computed in [AG15]. These calculations apply to the classical groups \( U(n) \), \( SU(n) \) and \( Sp(n) \). Not much is known integrally. The integral cohomology rings of \( SU(2) \), \( U(2) \) and \( O(2) \) are calculated in [ACGV20]. However, a lot more is known stably. For the stable groups \( U \), \( O \), \( Sp \) the space \( B(\tau, G) \) gives rise to generalized cohomology theories [AGLT17, Vil17]. In the commutative case its homotopy type is described in [GH19].

2. At a prime \( \ell \) the homotopy type of the \( \ell \)-completion of \( B(\mathbb{Z}, G) \) in the sense of Bousfield–Kan depends only on the mod-\( \ell \) homology of \( BG \) when \( \pi_0 G \) is a finite \( \ell \)-group [OW20]. When \( \ell \) is odd this gives a mod-\( \ell \) homology isomorphism

\[
B(\mathbb{Z}, Sp(n)) \rightarrow B(\mathbb{Z}, SO(2n + 1)).
\]

5.4. **Topological bundles with \( \tau \)-structure.** We will generalize the definition of a transitionally commutative bundle introduced in [AG15]. Let \( \pi : P \rightarrow X \) be a principal \( G \)-bundle of topological spaces. Let \( \mathcal{U} = \{U_i\}_{i \in I} \) be an open cover of \( X \) that is trivializing for \( \pi \). That is, for each \( i \in I \) we have a pullback diagram of topological spaces

\[
\begin{array}{ccc}
G \times U_i & \xrightarrow{\beta_i} & P \\
\downarrow{\pi_2} & & \downarrow{\pi} \\
U_i & \xrightarrow{\iota_i} & X
\end{array}
\]

where \( \iota_i : U_i \rightarrow X \) is the inclusion map and \( \beta_i \) is \( G \)-equivariant. The collection \( \{\beta_i\}_{i \in I} \) is called an atlas.

For an \((n + 1)\)-tuple of indices \( i = (i_0, \ldots, i_n) \in I^{\times(n+1)} \) let \( U_i = U_{i_0} \cap \cdots \cap U_{i_n} \). For indices \( i, j \in I \) we get transition maps \( \rho_{ij} : U_{ij} \rightarrow G \) such that we have \( \beta_j(g, x) = \rho_{ij}(x) \beta_i(g, x) \) for \( x \in U_{ij} \) and \( g \in G \). These maps give the Čech 1-cocycle in \( H^1(\mathcal{U}, G) \) classifying \( \pi \).
Let $U = \coprod_i U_i$ and consider the map $q : U \to X$ induced by the inclusions. The Čech nerve of $q$ is defined to be the simplicial space

$$
\check{C}(\mathcal{U})_n = \check{C}(q)_n = U \times X^{(n+1)} = \coprod_{i \in I^{(n+1)}} U_i
$$

with the obvious simplicial structure maps. The (topological) transition functions $\rho_{ij} : U_{ij} \to G$ induce a map of simplicial spaces

$$
\rho : \check{C}(q)_\bullet \to BG_\bullet,
$$
sending $x \in U_{i_0} \cap \cdots \cap U_{i_n}$ to the tuple $(\rho_{i_0,i_1}(x), \ldots, \rho_{i_{n-1},i_n}(x))$ in simplicial degree $n$.

**Definition 5.13.** A trivialization along an open cover $\{U_i : i \in I\}$ of a principal $G$-bundle $\pi : P \to X$ is said to have a $\tau$-structure if the corresponding map $\rho : \check{C}(\mathcal{U})_\bullet \to BG_\bullet$ factors through the simplicial space $B(\tau,G)_\bullet$. In this case the corresponding atlas is called a $\tau$-atlas. A $\tau$-structure for $\pi$ is a choice of a lift of $[f]$, where $f : X \to BG$ is a classifying map for the principal bundle, under the map $[X, B(\tau,G)] \to [X, BG]$ induced by the inclusion $B(\tau,G) \subseteq BG$. We say that $\pi$ has a $\tau$-structure if such a lift exists.

**Remark 5.14.** Note that when $\tau^\bullet = \mathbb{Z}^\bullet$ having a $\mathbb{Z}$-structure is the same as being a transitionally commutative cover in the sense of [AG15]. A $\mathbb{Z}$-structure is the same as a transitionally commutative structure defined in [Gri17, Definition 1.1.6].

The definition of a $\tau$-structure for a principal bundle is motivated by Theorem 5.15. Originally the statement involves transitively commutative bundles. We state it more generally for principal bundles admitting a $\tau$-atlas. In the original statement [AG15, Theorem 2.2] of Theorem 5.15 $X$ is required to be a finite CW complex. However, with a little more work the techniques of the proof can be improved to remove the finiteness condition [Góm].

**Theorem 5.15.** Let $G$ be a topological group which is $\tau$-good and $X$ be a CW complex. A principal $G$-bundle $\pi : P \to X$ admits a $\tau$-atlas if and only if the classifying map $f : X \to BG$ factors through $B(\tau,G)$ in the homotopy category of topological spaces.

Proof of this result is provided in Appendix B.

**Construction 5.16.** Let $\tau : \mathbf{Grp} \to \mathbf{Grp}$ be an endofunctor of quotient type, $G$ a topological group, $\pi : P \to X$ a principal $G$-bundle, $\mathcal{U} = \{U_i : i \in I\}$ an open cover of $X$ and $\{G \times U_i \xrightarrow{\beta_i} P : i \in I\}$ a $\tau$-atlas for $\pi$. We will now construct a principal simplicial $SG$-bundle equivalent to $S\pi$ with $\tau$-structure as promised in Example 4.13.

Let $S_{\mathcal{U}} X \subseteq SX$ denote the sub-simplicial set on simplices $\sigma : \Delta^n \to X$ such that $\sigma$ maps into $U_{i(\sigma)} \subseteq X$ for some $i(\sigma) \in I$. Then the inclusion map $S_{\mathcal{U}}X \to SX$ is a weak equivalence [DI04, Corollary 3.5]. Let $S_{\mathcal{U}}P = SP \times_{SX} S_{\mathcal{U}}X$. Restricting the trivialization $\beta_i : G \times U_i \to P$ over $U_i \to X$ along the map $\text{id}_G \times \sigma : G \times \Delta^n \to G \times U_i$ where by an abuse of notation $\sigma$ is regarded as a map $\Delta^n \to U_{i(\sigma)}$, taking the map on singular complexes, and restricting that along the map $\text{id}_{SG} \times \eta_{\Delta[n]} : SG \times \Delta[n] \to SG \times S\Delta^n$ induced by the unit map $\eta$ of the adjunction $(|\cdot|, S)$, we get the trivialization $\beta(\sigma) : SG \times \Delta[n] \to S_{\mathcal{U}}P$ over $\sigma : \Delta[n] \to S_{\mathcal{U}}X$. That is, the trivialization $\beta(\sigma) : SG \times \Delta[n] \to S_{\mathcal{U}}P$ takes $(g, \phi) : \Delta[l] \to SG \times \Delta[n]$ to the composite

$$
\Delta^l \xrightarrow{(\hat{g}, \Delta^\phi)} G \times \Delta^n \xrightarrow{id_G \times \sigma} G \times U_i \xrightarrow{\beta_i} P
$$

where $\hat{g}$ is the adjoint of $g : \Delta[l] \to SG$. 

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Proposition 5.17. The atlas \( \{ \beta(\sigma) : \sigma \in (S_{xy}X)_n, n \geq 0 \} \) of the principal \( SG \)-bundle \( S_{xy}P \to S_{xy}X \) is a \( \tau \)-atlas.

Proof. We need to show that the nerve of the transition map functor \( \alpha_* : \int S_{xy}X \to \int SG \) factors through \( N(\tau, \int SG) \). Take \( \sigma_0 \in (S_{xy}X)_m \) and \( \sigma_1 \in (S_{xy}X)_n \). Let \( i = i(\sigma_0) \) and \( j = i(\sigma_1) \). Then a morphism \( \theta : \sigma_0 \to \sigma_1 \) in the Grothendieck construction \( \int S_{xy}P \) is given by an order-preserving map \( \theta : [m] \to [n] \) such that \( \sigma_0 = \sigma_1 \circ \Delta^\theta \). In particular, the simplex \( \sigma_0 \) maps into \( U_{ij} \). Let \( \rho_{ij} : U_{ij} \to G \) denote the transition map from \( \beta_i \) to \( \beta_j \) that is the map such that we have \( \beta_j(g, x) = \rho_{ij}(x) \beta_i(g, x) \) for all \( g \in G \) and \( x \in U_{ij} \). Then we have \( \alpha_*(\sigma_1, \theta) = \rho_{ij} \circ \sigma_0 \in (SG)_m \), where we abuse notation by regarding \( \sigma_0 \) as a map \( \Delta^m \to U_{ij} \).

Let \( \sigma_0 \stackrel{\theta_0}{\to} \cdots \stackrel{\theta_{k-1}}{\to} \sigma_k \) be a \( k \)-simplex in \( N(\int S_{xy}X) \). That is, for \( r = 0, \ldots, k \) we have \( \Delta^r \stackrel{\sigma_r}{\to} U_{i_r} \subseteq X \) and for \( s = 0, \ldots, k-1 \) we have \( \theta_s : [n_s] \to [n_{s+1}] \) such that \( \sigma_s = \sigma_{s+1} \circ \Delta^{\theta_s} \). Let \( x_s = \alpha_*(\sigma_{s+1}, \theta_s) \). For \( 1 \leq l \leq k \) we need to show that
\[
(x_{k-l}, \cdots, \theta^{*}_{k-l} \cdots \theta^{*}_{k-3}x_{k-2}, \theta^{*}_{k-2} \cdots \theta^{*}_{k-1}x_{k-1}) = \\
(\rho_{i_{k-l}i_{k-l+1}} \circ \sigma_{k-l}, \cdots, \rho_{i_{k-1}i_k} \circ \sigma_{k-1} \circ \Delta^{\theta_{k-2}} \cdots \circ \Delta^{\theta_{k-l}}) = \\
(\rho_{i_{k-l}i_{k-l+1}} \circ \sigma_{k-l}, \cdots, \rho_{i_{k-2}i_{k-1}} \circ \sigma_{k-1} \circ \rho_{i_{k-1}i_k} \circ \sigma_{k-1} \circ \cdots \circ \Delta^{\theta_{k-1}})
\]
is contained in \( \text{Grp}(\tau, (SG)_{n_0}) \). This follows from the fact that \( \rho \) has a \( \tau \)-structure since this implies
\[
U_{i_{k-l} \cdots i_k}) \to \text{Grp}(\tau, G) \subseteq G^{\times l}.
\]

The set \( H^1_\tau(X, G) \) also makes sense in \( \text{Top} \) and is defined using the \( \tau \)-concordance relation whose definition is completely analogous to the simplicial version given in Definition 4.16. Note that a \( \tau \)-structure for a principal \( G \)-bundle in the topological context is a choice of a lift of the classifying map under the natural map \( [X, B\tau, G] \to [X, BG] \). Theorem 5.15 says that a principal \( G \)-bundle admits a \( \tau \)-atlas if and only if the principal \( G \)-bundle has a \( \tau \)-structure, i.e. up to homotopy the classifying map factors through \( B(\tau, G) \). As in Corollary 4.19 we can identify \( H^1_\tau(X, G) \) with the set \( [X, B(\tau, G)] \).

Using the singular functor we can define
\[
S : H^1_\tau(X, G) \to H^1_\tau(SX, SG)
\]
by sending the class of a bundle \( P \to X \) to the class of \( SP \to SX \). Conversely, for a simplicial set \( Y \) we can define
\[
R : H^1_\tau(Y, K) \to H^1_\tau(|Y|, |K|)
\]
by sending a simplicial principal bundle \( E \to Y \) to it geometric realization. Both of these maps are bijections as a consequence of Theorem 5.4 and Corollary 5.10.

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**Appendix A. Homotopy type of $N(\tau, \int K)$**

Let $X$ be a bisimplicial set. We can view it as a simplicial diagram of horizontal simplicial sets $\Delta^\op \times [n] \to sSet$. In this section, we will recall some constructions of its homotopy colimits to be used in the proof of Theorem 4.14.
Notation A.1. Let $\Delta \xrightarrow{D} \text{sSet}$ be a cosimplicial diagram of simplicial sets. Then the tensor product $D \otimes_{\Delta^{\text{op}}} X$ is the coend $\int^{\Delta} D^n \times X_{\star, n}$, that is the coequalizer of the pair of maps
\[
\prod_{\theta : [m] \to [n]} D^m \times X_{\star, n} \xrightarrow{\theta_* \times \text{id}_{X_{\star, n}}} \prod_{[n] \in \Delta} D^n \times X_{\star, n}.
\]

Proposition A.2. [GJ99, IV, Exercise 1.6] In case $D$ is the simplex functor $D^n = \Delta[n]$, the map $\prod_{[n] \in \Delta} \Delta[n] \times X_{\star, n} \to dX$ taking an $l$-simplex $(\lfloor l \rfloor \to [n], x)$ of $\Delta[n] \times X_{\star, n}$ to the $l$-simplex $\phi_n x$ of $dX$ is a coequalizer. That is, we get an isomorphism $\Delta[\cdot] \otimes_{\Delta^{\text{op}}} X \cong dX$.

Notation A.3. Let $\Psi X$ denote the bispimlicial set defined as follows:

1. The $(p, q)$-simplices are pairs of
   (a) a $q$-simplex $\sigma := [n_0] \xrightarrow{\theta_0} \cdots \xrightarrow{\theta_{q-1}} [n_q]$ of $N\Delta$ and
   (b) a $(p, n_q)$-simplex $x \in X_{p, n_q}$.

2. Let $[p'] \xrightarrow{\phi} [p]$ be a map in $\Delta$. Then the horizontal pullback of $(\sigma, x) \in (\Psi X)_{p, q}$ along $\phi$ is $((\sigma, \phi^*_n x) \in X_{p', n_q})$.

3. For $i < q$, we have $d_i^q(\sigma, x) = (d_i \sigma, x)$.

4. We have $d_q^q(\sigma, x) = (d_q \sigma, \phi_{q-1}^* x)$.

5. We have $s_i^q(\sigma, x) = (s_i \sigma, x)$.

We let $\Psi' X$ denote the transpose of $\Psi X$, that is the bispimlicial set with $(\Psi' X)_{p, q} = (\Psi X)_{q, p}$.

Proposition A.4. [Ton93, Corollary 3.2.6] In case $D$ is the fat simplex functor $D^n = N \int \Delta[n]$, the map $\prod_{[n] \in \Delta} N \int \Delta[n] \times X_{\star, n} \to d\Psi X$ sending an $l$-simplex
\[
([n_0] \xrightarrow{\theta_0} \cdots \xrightarrow{\theta_{l-1}} [n_l] \xrightarrow{\phi} [n], x)
\]
of $N \int \Delta[n] \times X_{\star, n}$ to the $l$-simplex
\[
([n_0] \xrightarrow{\theta_0} \cdots \xrightarrow{\theta_{l-1}} [n_l], \phi^*_n x)
\]
of $d\Psi X$ is a coequalizer map. That is, we get an isomorphism $(N \int \Delta[\cdot]) \otimes_{\Delta^{\text{op}}} X \cong d\Psi X$.

Definition A.5. The Bousfield–Kan map is the morphism of cosimplicial simplicial sets $N \int \Delta[\cdot] \to \Delta[\cdot]$ taking an $l$-simplex $[n_0] \xrightarrow{\theta_0} \cdots \xrightarrow{\theta_{l-1}} [n_l] \xrightarrow{\phi} [n]$ of $N \int \Delta[n]$ to the $l$-simplex $[\phi_{\theta_{l-1}, \ldots, \theta_0 n_0, \ldots, n_l}]$ of $\Delta[n]$. It induces the map $BK : d\Psi X \to dX$ taking an $l$-simplex
\[
([n_0] \xrightarrow{\theta_0} \cdots \xrightarrow{\theta_{l-1}} [n_l], x)
\]
of $d\Psi X$ to the $l$-simplex $(\theta_{l-1} \cdot \cdots \cdot \theta_0 n_0, \ldots, n_l)_* x$ of $dX$, which we will also call a Bousfield–Kan map.

Proposition A.6. [Hir03, Corollary 18.7.5] The Bousfield–Kan map $BK : d\Psi X \to dX$ is a weak equivalence.

Proposition A.7. [Ste12, pp. 777 and Lemma 5.2] The total simplicial set $TX$ has the following explicit description:

- The set of $n$-simplices is
\[
(TX)_n = \{(x_i)_{i=0}^n \in \prod_{i=0}^n X_{i, n-i} : d_i^h x_i = d_i^b x_{i+1} \text{ for all } 0 \leq i < n\}.
\]
For $0 \leq i \leq n$ the face and degeneracy maps are:

\[
\begin{align*}
  d_i(0, \ldots, x_n) &= (d_i^0 x_0, \ldots, d_i^0 x_{i-1} d_i^1 x_{i+1}, \ldots, d_i^h x_n), \\
  s_i(0, \ldots, x_n) &= (s_i^v x_0, \ldots, s_i^v x_i, s_i^h x_i, \ldots, x_i^h x_n).
\end{align*}
\]

Suppose $X = NK$ for a simplicial group $K$. Then there is an isomorphism $\overline{WK} \to TNK$ (see Proposition 3.1) defined by sending an $n$-simplex $(k_{n-1}, \ldots, k_0)$ of $\overline{WK}$ to the $n$-simplex $(x_0, \ldots, x_n)$ of $TNK$ where

\[
x_i = \begin{cases} 
  1 & i = 0, \\
  (d_i^{0-1} k_{n-1}, \ldots, d_i^{k_{n-i+1}}, k_{n-i}) & i > 0.
\end{cases}
\]

**Definition A.8.** The Cegarra–Remedios map is the map $CR : dX \to TX$ taking an $l$-simplex $x \in X_l$ of $dX$ to the $l$-simplex $((d_1^l x, (d_2^l)^{-1} d_0^l x, \ldots, (d_0^l)^l) x)$ of $TX$.

**Proposition A.9.** [CR05, Theorem 1.1] The Cegarra–Remedios map $CR : dX \to TX$ is a weak equivalence.

**Proposition A.10.** [Ton93, Theorem 3.2.12] Suppose that $X$ is the nerve $NF$ of a simplicial diagram of categories $\Delta^\text{op} \xrightarrow{F} \text{Cat}$. Then the map $N \int F \to T\Psi'NF$ taking an $l$-simplex

\[
(n_0, x_0) \xrightarrow{\theta_0 f_0} \cdots \xrightarrow{\theta_{l-1} f_{l-1}} (n_l, x_l)
\]

of $N \int F$ to the $l$-simplex

\[
([n_0] \xrightarrow{\theta_0} \cdots \xrightarrow{\theta_{l-1}} [n_l], x_i \xrightarrow{f_i} \cdots \xrightarrow{\theta_{l-1} f_{l-1}} \theta_1^l \cdots \theta_{l-1}^l x_l)_{i=0}^l
\]

of $T\Psi'X$ is an isomorphism.

**Remark A.11.** It was originally proven in the Thomason homotopy colimit theorem [Tho79, Theorem 1.2] that we have a weak equivalence $(N \int \Delta[\cdot]) \otimes_{\Delta^\text{op}} NF \simeq N \int F$.

### Appendix B. Proof of Theorem 5.15

**Construction B.1.** Let $G$ be a topological group which is $\tau$-good (Definition 5.9). Let $p : \|E(\tau, G)\| \to \|B(\tau, G)\|$ denote the pull-back of the universal bundle $\|EG\| \to \|BG\|$ along the inclusion $\|B(\tau, G)\| \subset \|BG\|$. We will construct a $\tau$-atlas for the principal $G$-bundle $p$. For this, we begin by defining an open cover for $\Delta^n$. For a subset $I = \{i_0 < i_1 < \cdots < i_q\}$ of $[n]$ let $\delta^I$ denote the composite $\delta^q \cdots \delta^{i_0}$ and $d_I = d_{i_q} \cdots d_{i_0}$. We will write $I^c = [n] - I$ when $[n]$ is clear from the context. Let $b_n$ denote the barycenter of $\Delta^n$, i.e. the point with coordinates $(\frac{1}{n+1}, \cdots, \frac{1}{n+1})$. Given $v \in \mathbb{R}^{n+1}$ and a non-empty subset $S \subset \mathbb{R}^{n+1}$ we will write $J_v(S) \subset \mathbb{R}^{n+1}$ for the subspace that is the union of the line segments from $v$ to the points in $S$. By convention we set $J_v(\emptyset) = \{v\}$. We define an open cover $\mathcal{U}^n = \{W^n_q\}_{q=0}^n$ for $\Delta^n$ as follows

- $W^0_0 = \Delta^0$,
- $W^q_n = \Delta^n - J_{b_n} (\bigcup_{j=0}^q \delta^j(\Delta^{n-1} - W_{q-1}^n))$ where $0 \leq q < n$,
- $W^n_n = \Delta^n - \bigcup_{j=0}^n \delta^j(\Delta^{n-1})$.
Each $W^{|I|^n}_q$ is a disjoint union of connected open sets $W^{|I|^n}_p$ where $I \subset [n]$ such that $|I| = q + 1$. More explicitly, $W^{|I|^n}_p$ is the connected component of $W^{|I|^n}_q$ such that $t_j > 0$ for all $j \in I^c$, that is when $I$ is a proper subset

$$W^{|I|^n}_I = \{ w \in \Delta^n \mid \min_{i \in I} d(w, \delta^i \Delta^{n-1}) > \max_{i' \in I^c} d(w, \delta^{i'} \Delta^{n-1}) \} \quad (B.0.1)$$

where $d(\cdot, \cdot)$ denotes the Euclidean distance, and $W^{|I|^n}_{[n]}$ is the interior of $\Delta^n$.

By construction we have a commutative diagram

$$\begin{array}{ccc}
\Delta^n & \xrightarrow{\delta^j} & \Delta^{n+1} \\
\uparrow & & \uparrow \\
W^{|I|^n}_I & \xrightarrow{\delta^j} & W^{|I|^n+1}_{d^j(\cdot)} \\
\end{array} \quad (B.0.2)
$$

Let $U^{|I|^n}_I$ denote the image of $W^{|I|^n}_I \times \text{Hom}(\tau F^n, G) \rightarrow |B(\tau, G)_\bullet|$ and $U^{|I|^n}_{q} \doteq$ denote the union of $U^{|I|^n}_I$ as $I$ runs over subsets of $[n]$ of size $q + 1$. We define a map $\beta^{|I|^n}_I : G \times U^{|I|^n}_I \rightarrow |E(\tau, G)_\bullet|$ by the formula

$$\beta^{|I|^n}_I(g_0, [u, (g_1, \ldots, g_n)]) = [u, (g_0 \cdot \sigma^n_I(g_1, \ldots, g_n), g_1, \ldots, g_n)].$$

Here $\sigma^n_I : G^{\times n} \rightarrow G$ is defined by $(\pi_0 d_{I^c}(1, g_1, \ldots, g_n))^{-1}$ where $\pi_0$ is projection onto the first coordinate. Note that $\sigma^n_I(g_1, \ldots, g_n) = (g_1 \cdots g_{i_0})^{-1}$ (here $g_1 \cdots g_{i_0} = 1$ if $i_0 < 1$ ). With these definitions one can verify that

$$\beta^{|I|^n+1}_{d^j(\cdot)}(g_0, [d^j u, (g_1, \ldots, g_{n+1})]) = \beta^{|I|^n}_I(g_0, [u, d_j(g_1, \ldots, g_{n+1})]). \quad (B.0.3)$$

This follows from the following observations:

- For $j > 0$ we have

$$\sigma^n_I(d_j(g_1, \ldots, g_{n+1})) = (\pi_0 d_{I^c} d_j(1, g_1, \ldots, g_{n+1}))^{-1} = \sigma^{n+1}_{d^j(\cdot)}(g_1, \ldots, g_{n+1})$$

as a consequence of the simplicial identity $d_{|n+1|-|I|} = d_{I^c}d_j$. 

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• For \( j = 0 \) we have
\[
\beta^n_0(g_0, [u, d_0(g_1, \ldots, g_{n+1})]) = [u, (g_0(g_2 \cdot \cdot \cdot g_{i_0+1})^{-1}, g_2, \ldots, g_n)]
\]
and
\[
\beta^{n+1}_0(g_0, [\delta^0u, (g_1, \ldots, g_{n+1})]) = [\delta^0u, (g_0(g_1 \cdot \cdot \cdot g_{i_0+1})^{-1}g_1^{-1}, g_1, \ldots, g_{n+1})]
\]
\[
= [u, d_0(g_0(g_2 \cdot \cdot \cdot g_{i_0+1})^{-1}g_1^{-1}, g_1, \ldots, g_{n+1})]
\]
\[
= [u, (g_0(g_2 \cdot \cdot \cdot g_{i_0+1})^{-1}, g_2, \ldots, g_n)].
\]
Let \( U_q \) denote the union \( \cup_{n=q} U^n_q \). This is an open subset of \( \|B(\tau, G)\| \). As a consequence of (B.0.3) the collection of maps \( \{\beta^n_I \mid n \geq q, I \subseteq [n]\} \) induces a \( G \)-equivariant map \( \beta_q : G \times U_q \rightarrow \|E(\tau, G)\| \) such that
\[
G \times U_q \xrightarrow{\beta_q} \|E(\tau, G)\|
\]
is a pull-back diagram. Therefore \( \{\beta_q \mid q \geq 0\} \) is an atlas for the open cover \( \mathcal{U} = \{U_q \mid q \geq 0\} \).

We will show that this atlas is a \( \tau \)-atlas. For this we need to verify that the associated map \( \rho : \check{C}(\mathcal{U}) \rightarrow BG \) factors through \( B(\tau, G) \). Let \( u = [u_n, (g_1, \ldots, g_n)] \) be a point in \( U_q \cap U_r \) where \( r \leq q \). We can assume that \( u_n \) is in the interior of \( \Delta^n \). Then there exists \( I, J \subseteq [n] \) with \( |I| = q + 1 \) and \( |J| = r + 1 \) such that \( u \in U^n_I \cap U^J_J \). This implies that \( J \subseteq I \). If \( J = [n] \) this is clear, otherwise it follows from (B.0.1). We get \( j_0 = i_t \) for some \( 0 \leq t \leq q \). Then we have
\[
\beta_I(g_0, [u, (g_1, \ldots, g_n)]) = [u, (g_0(g_1 \cdot \cdot \cdot g_{i_0+1})^{-1}, g_1, \ldots, g_n)]
\]
and
\[
\beta_J(g_0, [u, (g_1, \ldots, g_n)]) = [u, (g_0(g_1 \cdot \cdot \cdot g_{i_0+1})^{-1}, g_1, \ldots, g_n)].
\]

Therefore we get
\[
\beta_I(g_0 \cdot \rho_{IJ}([u, (g_1, \ldots, g_n)]), [u, (g_1, \ldots, g_n)]) = \beta_J(g_0, [u, (g_1, \ldots, g_n)])
\]
with
\[
\rho_{IJ}([u, (g_1, \ldots, g_n)]) = (g_{i_0+2} \cdot \cdot \cdot g_{i+t+1})^{-1}.
\]
This gives a function \( \rho_{IJ} : U^n_I \cap U^J_J \rightarrow G \). Now, consider a collection \( \{U_{q_k}\}_{k=1}^s \) and a point \( u = [u_n, (g_1, \ldots, g_n)] \) in the intersection \( \cap_{k=1}^s U_{q_k} \) with \( u_n \) in the interior of \( \Delta^n \). There exists \( I_k \subseteq [n] \) such that \( u \in \cap_{k=1}^s U^n_I_k \). This implies that after relabeling \( I_1 \subseteq \cdot \cdot \cdot \subseteq I_s \). Let \( \rho(u) : F^s \rightarrow G \) denote the homomorphism which sends \( e_k \) to \( \rho_{I_{k-1}, I_k}(u) \). Then \( \rho(u) \) factors as \( F^s \xrightarrow{\check{\rho}} F^n \xrightarrow{\eta_{F^n}} G \) as a consequence of the formula (B.0.4). Since \( (g_1, \ldots, g_n) \in B(\tau, G)_n \) we have
\[
F^s \xrightarrow{\check{\rho}} F^n \xrightarrow{\eta_{F^n}} G
\]
and thus \( (\rho_{I_{k-1}, I_k}(u))^s_{k=1} \) belongs to \( B(\tau, G)_s \). This proves that \( \{\beta_q \mid q \geq 0\} \) is a \( \tau \)-atlas.
Proof of Theorem 5.15. Assume that the classifying map factors through $\tilde{f} : X \to B(\tau, G)$. We will show that $\pi$ admits a $\tau$-atlas. Let $\mathcal{U}$ denote the open cover defined in Construction B.1. Consider the composite $\tilde{f} : X \xrightarrow{\bar{f}} B(\tau, G) \simeq \|B(\tau, G)\|$ and the open cover $\mathcal{U}'$ of $X$ consisting of $\tilde{f}^{-1}(U)$ where $U \in \mathcal{U}$. Using $\{\beta_i\}$ we can construct a $\tau$-atlas for $\mathcal{U}'$.

Conversely, assume that $\pi : P \to X$ admits a $\tau$-atlas. We can refine the cover so that whenever an intersection is non-empty it is also contractible. Then the following zigzag gives the desired factorization

$$X \xleftarrow{\cdot} \|\check{C}(\mathcal{U})\| \xrightarrow{[\rho]} \|B(\tau, G)\| \simeq B(\tau, G).$$

□

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