Estimates for solutions of systems of linear equations with circulant matrices

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Abstract. In the present paper we consider the problem to estimate a solution of the system of equations with a circulant matrix in uniform norm. We give the estimate for circulant matrices with diagonal dominance. The estimate is sharp. Based on this result and an idea of decomposition of the matrix into a product of matrices associated with factorization of the characteristic polynomial, we propose an estimate for any circulant matrix.

1. Introduction

We begin with considering the following difference equation

\[ y_i = \sum_{j \in \mathbb{Z}} a_{j-i} x_j, \quad i \in \mathbb{Z}. \tag{1} \]

Equations of this type appear in the cardinal interpolation problem to reconstruct a function \( f(x), x \in \mathbb{R} \), from its values \( \{f(ih) : i \in \mathbb{Z}\} \) at points of a uniform grid with the step size \( h \). For instance, if \( f(x) \) is some linear combination of translates of another function \( \phi(x) \), i.e.,

\[ \sum_{i \in \mathbb{Z}} x_i \phi(x - ih), \]

and coefficients \( \{x_i\} \) satisfy (1) with \( y_i = f(ih) \) and \( a_i = \phi(ih), i \in \mathbb{Z} \). Another example, if the interpolation problem is replaced by the problem of finding a solution to a linear differential equation by the collocation method.

Equation (1) can be rewritten in the form of a bi-infinite system of linear algebraic equations

\[ Ax = y. \]

The coefficient matrix \( A = (a_{j-i})_{i,j=\pm\infty} \) is a bi-infinite matrix. Such matrices are called Laurent matrices (or bi-infinite Toeplitz matrices). The entries \( a_{j-i} \) of the matrix \( A \) are the coefficients in (1). Below we will also refer to (1) as a difference equation, as well a bi-infinite system of linear equations. The function

\[ p(z) = \sum_{j \in \mathbb{Z}} a_j z^j \tag{2} \]
is called the symbol of the matrix $A$.

In 1907, Toeplitz [1] found spectra of Laurent’s matrices, and he also proved that if $p(z)$ has a root on the unit circle $|z| = 1$ of the complex plane, then $A$ is not invertible. Krein [2] established that if $p(z)$ has no roots on $|z| = 1$, then there exists a unique solution of (1). Thus, the existence of the inverse matrix $A^{-1} = (\alpha_{j-i})_{i,j=\infty}^{+\infty}$ is equivalent to non-vanishing of the symbol $p(z)$ on the unit circle of the complex plane.

The explicit form of the bounded solution of difference equation (1) is given as

$$x_i = \sum_{j \in \mathbb{Z}} \alpha_{j-i} y_j, \quad i \in \mathbb{Z},$$

where $\{\alpha_j\}_{j \in \mathbb{Z}}$ are the coefficients in the Laurent expansion of the meromorphic function $1/p(z)$ on an open annulus containing the unit circle, free of zeros of $p(z)$, i.e.,

$$q(z) = \frac{1}{p(z)} = \sum_{j \in \mathbb{Z}} \alpha_j z^j.$$

This result was obtained by Krein [2], see also Subbotin [3].

When the number of nonzero diagonals of the matrix $A$ is finite, the symbol $p(z)$ turns into the characteristic polynomial of difference equation (1).

Note that the coefficients $\{\alpha_j\}_{j \in \mathbb{Z}}$ are real; this fact follows from the well-known integral representations for the coefficients of the Laurent series.

When $\phi(x)$ is a compactly supported function, for example B-spline, that is, $\phi(x) = 0$, if $x \not\in [\beta_0, \beta_1]$ where $[\beta_0, \beta_1]$ is the support of the function $\phi(x)$, the Laurent matrix $A$ is banded. In this case, we will assume that $a_i = 0$, $i < 0$, and $i > n$, for some $n \in \mathbb{N}$. Therefore, the matrix $A$ has a finite number $n + 1$ of nonzero diagonals that consist of the numbers $a_0, a_1, \ldots, a_n$. Some spline reconstruction methods of this type are given in [4, 5].

We are interesting in obtaining effective estimates for $l_\infty$-norm of a bounded solution of (1) for a bounded sequence $y$.

Although the classical Laurent theorem and other known methods allow writing explicit formulas for the Laurent coefficients, these formulas are often inconvenient for application. Note that the methods of iterative construction of solutions for infinite systems of linear equations can be found in [6].

Now we will assume that the bi-infinite sequence $y$ is periodic, i.e., $y_i = y_{i+m}$, $i \in \mathbb{Z}$, for some integer $m > n$, $m \in \mathbb{N}$. Then the sequence $x$, a solution of (1), is also periodic $x_i = x_{i+m}$, $i \in \mathbb{Z}$. So the problem (1) with nonzero coefficients $a_0, \ldots, a_n$ in periodic case is equivalent to the system of equations

$$C \tilde{x} = \tilde{y},$$

where $\tilde{x} = (x_0, \ldots, x_{m-1})^T$, $\tilde{y} = (y_0, \ldots, y_{m-1})^T$,

$$C = \begin{bmatrix}
a_0 & a_1 & \cdots & a_{m-1} \\
a_{m-1} & a_0 & \cdots & a_{m-2} \\
\vdots & \vdots & \ddots & \vdots \\
a_1 & a_2 & \cdots & a_0
\end{bmatrix}$$

with nonzero coefficients $a_0, \ldots, a_n$ (more precisely, only $a_0$ and $a_n$ should be nonzero) and $a_{n+1} = \cdots = a_{m-1} = 0$. If $m = n + 1$, then the matrix is free of zeros. Here $C = C(a_0, \ldots, a_{m-1})$ is a classical circulant matrix (or circulant) of order $m \times m$ [7]. The polynomial (2) also we will call the characteristic polynomial. In this case

$$p(z) = a_0 + a_1z + \ldots + a_n z^n.$$
In the present paper we consider the problem of finding estimates for solutions of the system of equations (4) with circulant matrices. In section 2 we give the estimate for circulant matrices with diagonal dominance. The estimate is sharp. In section 3 we propose an estimate for any circulant matrix based on this result and an idea of decomposition of the matrix into a product of matrices associated with factorization of the characteristic polynomial.

2. Diagonally dominant matrices

A sufficiently simple method for estimation of the infinity-norm of the inverse is known for the matrices with diagonal dominance. This method was proposed in [8] in connection with the necessity to derive some error bounds of interpolation by cubic splines. Many papers have appeared, since then the estimate from [8] was improved and refined in various particular cases.

We will say that the circulant matrix $C$ is diagonally dominant if for some $k \in \{0, 1, \ldots, n\}$, the inequality

$$|a_k| - \sum_{j=0, j\neq k}^{n} |a_j| = r > 0$$

(5)

holds.

**Theorem 1.** If the circulant matrix $C$ has a dominant diagonal, then the system of linear equations (4) has a unique solution $\tilde{x}$ for every $\tilde{y}$ and the following inequality holds

$$\|\tilde{x}\|_\infty \leq \frac{\|\tilde{y}\|_\infty}{r}.$$  

(6)

**Proof.** Suppose that the matrix $C$ has a dominant diagonal, i.e., the inequality (5) holds for some $k \in \{0, 1, \ldots, n\}$. Let $d$ be an arbitrary vector $d = (d_0, \ldots, d_{m-1})^T$ and $d_{i+m} = d_{i-m} = d_i$, $i = 0, \ldots, m - 1$. We have

$$\|Cd\|_\infty = \max_{0 \leq i \leq m-1} |a_0d_i + \ldots + a_n d_{i+n}| \geq |a_0d_i + \ldots + a_n d_{i+n}|$$

$$\geq |a_k|d_{i+k} - \|d\|_\infty (|a_0| + \ldots + |a_{k-1}| + |a_{k+1}| + \cdots + |a_n|),$$

i.e., the inequality

$$|a_k|d_{i+k} \leq \|Cd\|_\infty + \|d\|_\infty (|a_0| + \ldots + |a_{k-1}| + |a_{k+1}| + \cdots + |a_n|)$$

is true for every $i = 0, \ldots, m - 1$.

In this inequality, we pass to the supremum over all $i = 0, \ldots, m - 1$ and obtain

$$\|Cd\|_\infty \geq \left(|a_k| - \sum_{j=0, j\neq k}^{n} |a_j|\right)\|d\|_\infty = r\|d\|_\infty.$$ 

Consequently, there exists the inverse matrix $C^{-1}$ and

$$\|C^{-1}\|_\infty \leq \frac{1}{r}.$$ 

From this, for the solution $\tilde{x}$ of our system (4), we get

$$\|\tilde{x}\|_\infty \leq \|C^{-1}\tilde{y}\|_\infty \leq \|C^{-1}\|_\infty \|\tilde{y}\|_\infty \leq \frac{1}{r} \|\tilde{y}\|_\infty.$$ 


Theorem 1 is proved.

**Corollary 2.** If the circulant matrix $C$ has a dominant diagonal, then

$$
\|C^{-1}\|_\infty \leq \frac{1}{r}.
$$

(7)

**Corollary 3.** If the circulant matrix $C$ is diagonally dominant with positive dominant diagonal entries and nonpositive other entries, then

$$
\|C^{-1}\|_\infty = \frac{1}{r}.
$$

(8)

A proof can be found in [9]. The assertion of Corollary 3 remains valid for matrices of monotone type. Recall, a matrix $C$ is of monotone kind if all entries of $C^{-1}$ are nonnegative [10].

**Remark 4.** For circulant matrices, not only the central diagonal can be dominant.

**Corollary 5.** Suppose that $n = 1$, i.e., $C$ is a circulant two-diagonal matrix. Let $|a_0| \neq |a_1|$. If $a_0$ and $a_1$ have the same sign, then the value of the diagonal dominance is equal to $r = |p(-1)|$. If $a_0$ and $a_1$ have opposite signs, then $r = |p(1)|$.

3. General circulant matrices

Now we will show the way to obtain an estimate of the solution of (4) for arbitrary nonsingular circulant matrix. Our idea is based on decomposition of the matrix $C$ into a product of matrices associated with factorization of the characteristic polynomial $p(z)$ (see [11]).

Note that the set of the coefficients $a_0, a_1, \ldots, a_n$ uniquely determines both the circulant matrix $C$ and the characteristic polynomial $p(z)$. Furthermore, the set of circulant $m \times m$ matrices $C$ and the set of polynomials $p(z)$ are isomorphic with respect to multiplication.

**Lemma 6.** Suppose that the polynomial $p(z)$, corresponding to the circulant matrix $C$, is factorized into a product of polynomials $u(z)$ and $v(z)$, corresponding to some circulant matrices $U$ and $V$, and these matrices are determined by sets of real numbers $u_0, \ldots, u_{n-k}$ and $v_0, \ldots, v_k$, respectively. Then $C = UV = VU$.

**Proof.** For polynomials of order 1 and 2, Lemma 6 is obtained by direct calculations. Proof for the general case easily follows from expansions of the polynomials into product of linear and quadratic factors.

**Theorem 7.** If all roots of the characteristic polynomial $p(z)$ are negative and not equal to $-1$, then the estimate

$$
\|\tilde{x}\|_\infty \leq \left|\frac{\|\tilde{y}\|_\infty}{p(-1)}\right|
$$

holds.

**Theorem 8.** If all roots of the characteristic polynomial $p(z)$ are positive and not equal to $+1$, then the estimate

$$
\|\tilde{x}\|_\infty \leq \left|\frac{\|\tilde{y}\|_\infty}{p(1)}\right|
$$

holds.

The proofs of Theorem 7 and 8 immediately follow from Corollary 5 and Lemma 6.
**Theorem 9.** If the characteristic polynomial is \( p(z) = p_1(z)p_2(z) \), all roots of \( p_1(z) \) are positive, all roots of \( p_2(z) \) are negative and \( p(\pm 1) \neq 0 \), then we have

\[
\|\tilde{x}\|_\infty \leq \frac{\|\tilde{y}\|_\infty}{|p_1(1)p_2(-1)|}.
\]

Note that a circulant two-diagonal matrix is totally positive for even \( m \). Recall that a matrix is called *totally positive* if all of its minors are non-negative.

**Proposition 10.** If all roots of the polynomial \( p(z) \), corresponding to the matrix \( C \) of even order \( m \), are negative and not equal to \(-1\), then \( C \) or \(-C\) is totally positive and the inequality

\[
\|C^{-1}\|_\infty = \frac{1}{|p(-1)|}
\]  

is true.

**Proof.** If all the roots of the characteristic polynomial \( p(z) \) are negative, then, by Lemma 6, the matrix \( C \) (or \(-C\)) can be represented as a product of two-diagonal matrices with non-negative entries. Since two-diagonal matrices of even order are totally positive, their product is also totally positive, i.e., \( C \) (or \(-C\)) is totally positive. In accordance with [9, Corollary to Theorem 7] equality holds. The proposition is proved.

We showed how to get estimates of a solution of the system of equations (4) with a circulant matrix if the roots of the characteristic polynomial \( p(z) \) are arbitrary reals. But in the general case the roots can also be complex. Let the characteristic polynomial has the form

\[
p(z) = (z - z_1)(z - z_2)
\]

with complex roots, i.e., \( z_1, z_2 \in C \). Note if \( p(z) \) has no roots on \( |z| = 1 \) for the circulant matrix \( C \), then \( |z_1| = |z_2| \neq 1 \) and Corollary 5 is true with complex coefficients. Hence, for this case, the inequality

\[
\|C^{-1}\|_\infty \leq \frac{1}{(|z_1| - 1)^2}
\]

holds.

Thus, we have

**Theorem 11.** Suppose that the characteristic polynomial \( p(z) \), corresponding to the circulant matrix \( C \), has no roots on \( |z| = 1 \) and \( p(z) = a_n(z - z_1) \ldots (z - z_n) \) where the roots can be complex. Then

\[
\|C^{-1}\|_\infty \leq \frac{1}{|a_n|(|z_1| - 1) \ldots (|z_n| - 1)}.
\]

**Corollary 12.** Suppose that the characteristic polynomial \( p(z) \), corresponding to the circulant matrix \( C \), has no roots on \( |z| = 1 \) and \( p(z) = (z - z_1) \ldots (z - z_k) p_1(z) p_2(z) \), where all roots of \( p_1(z) \) are positive, and all roots of \( p_2(z) \) are negative. Then

\[
\|C^{-1}\|_\infty \leq \frac{1}{(|z_1| - 1) \ldots (|z_k| - 1)p_1(1)p_2(-1)}.
\]
4. Conclusions
In the present paper we proposed an approach that allowed to get estimates of a solution of the system of equations (4) with arbitrary nonsingular circulant matrix, if the corresponding characteristic polynomial $p(z)$ has no roots on $|z| = 1$. Our method is based on the concept of diagonal dominance and on an idea of decomposition of the matrix into a product of matrices associated with factorization of the characteristic polynomial. Corollary 3 and Proposition 10 show that our estimates are sharp. But if the characteristic polynomial $p(z)$ has complex roots, then the estimate (10) is probably not sharp.

Example 13. Let $a_0 = 4$, $a_1 = 1$, $a_2 = 1$, $n = 2$. The polynomial $p(z)$ has no real roots. Using Theorem 11, we obtain the estimate $\|\tilde{x}\|_\infty \leq \|\tilde{y}\|_\infty$. But the matrix $C$ is diagonally dominant and the diagonal corresponding $a_0$ is dominant with $r = 2$. Therefore, Theorem 1 gives a more accurate estimate $\|\tilde{x}\|_\infty = \|\tilde{y}\|_\infty/2$.

In 1973 Albasiny and Hoskins [12] have found attainable estimates for the infinity norm of the inverse of the matrix $C_{2k-1}$ arising from interpolation with an odd degree $2k - 1$ periodic polynomial spline on a uniform mesh

$$\|C_{2k-1}^{-1}\|_\infty \leq \frac{(2k)!}{2^{2k}(2^{2k} - 1)|B_{2k}|},$$

where $B_{2k}$ is the $2k$th Bernoulli number. Here $n = 2k - 2$. The inequality turns into equality for even order matrix $m$. Their methods were generalized by Hoskins and Meek [13] and applied to a larger class of circulant matrices which includes the matrices $C_{2k}$ arising from interpolation with even degree $2k$ periodic polynomial splines on a uniform mesh

$$\|C_{2k}^{-1}\|_\infty \leq \frac{(2k)!}{E_{2k}},$$

where $E_{2k}$ is the $2k$th Euler number and $n = 2k$. In [13] considered arbitrary symmetric circulant with nonnegative numbers $a_i$.

Note that in accordance with [14, 15], inequalities (11) and (12) for splines of degree $k$ can be written as a common formula through the Favard constants

$$\|C_{k}^{-1}\|_\infty \leq \left(\frac{\pi}{2}\right)^k K_k^{-1},$$

where $K_k$ is the $k$th Favard constant.

The theorems proved above can be extended to the general case of bi-infinite matrices (for solving the difference equation (3)). But now due to Subbotin [3, 16] there are estimates for the case when the characteristic polynomial $p(z)$ has only simple negative roots. Subbotin’s theorem has been repeatedly applied in approximation theory. In particular, Subbotin [3,16,17], Shevaldin [18], Novikov and Shevaldin [19,20] used it for obtaining sharp results in problems of extremal interpolation.

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