Fractal index, central charge and fractons

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Abstract

We introduce the notion of fractal index associated with the universal class $h$ of particles or quasiparticles, termed fractons, which obey specific fractal statistics. A connection between fractons and conformal field theory (CFT)-quasiparticles is established taking into account the central charge $c[\nu]$ and the particle-hole duality $\nu \leftrightarrow \frac{1}{\nu}$, for integer-value $\nu$ of the statistical parameter. In this way, we derive the Fermi velocity in terms of the central charge as $v \sim \frac{c[\nu]}{\nu+1}$. The Hausdorff dimension $h$ which labelled the universal classes of particles and the conformal anomaly are therefore related. Following another route, we also established a connection between Rogers dilogarithm function, Farey series of rational numbers and the Hausdorff dimension.

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We consider the conformal field theory (CFT)-quasiparticles (edge excitations) in connection with the concept of fractons introduced in [1]. These excitations have been considered at the edge of the quantum Hall systems which in the fractional regime assume the form of a chiral Luttinger liquid [2]. Beyond this, conformal field theories have been exploited in a variety of contexts, including statistical mechanics at the critical point, field theories, string theory, and in various branches of mathematics [3].

In this Letter, we suppose that the fractal statistics obeyed by fractons are shared by CFT-quasiparticles. Thus, the central charge, a model dependent constant is related to the universal class \( h \) of the fractons. We define the fractal index associated with these classes as

\[
if[h] = \frac{6}{\pi^2} \int_{0}^{1} \frac{d\xi}{\xi} \ln \{\Theta[Y(\xi)]\}
\]  

(1)

and after the change of variable \( \xi = x^{-1} \), we obtain

\[
if[h] = \frac{6}{\pi^2} \int_{0}^{1} \frac{dx}{x} \ln \{\Theta[Y(x^{-1})]\}
\]

(2)

where in the Eq.(1)

\[
\Theta[Y] = \frac{Y[\xi] - 2}{Y[\xi] - 1}
\]

(3)

is the single-particle partition function of the universal class \( h \) and \( \xi = \exp \{(\epsilon - \mu)/KT\} \), has the usual definition. The function \( Y[\xi] \) satisfies the equation

\[
\xi = \{Y[\xi] - 1\}^{h-1} \{Y[\xi] - 2\}^{2-h}
\]

(4)

We note here that the general solution of the algebraic equation derived from this last one is of the form

\[
Y_h[\xi] = f[\xi] + \tilde{h}
\]

or

\[
Y_{\tilde{h}}[\xi] = g[\xi] + h,
\]
where $\tilde{h} = 3 - h$, is a duality symmetry between the classes. The functions $f[\xi]$ and $g[\xi]$ at least for third, fourth degrees algebraic equation differ by plus and minus signs in some terms of their expressions.

The particles within each class $h$ satisfy specific fractal statistics

\[ n = \xi \frac{\partial}{\partial \xi} \ln \Theta[Y] \]
\[ = \frac{1}{Y[\xi] - h} \]  

(5)

and the fractal parameter (or Hausdorff dimension) $h$ defined in the interval $1 < h < 2$ is related to the spin-statistics relation $\nu = 2s$ through the fractal spectrum

\[ h - 1 = 1 - \nu, \quad 0 < \nu < 1; \quad h - 1 = \nu - 1, \quad 1 < \nu < 2; \]

(6)

etc.

For $h = 1$ we have fermions, with $Y[\xi] = \xi + 2$, $\Theta[1] = \frac{\xi}{\xi+1}$ and $i_f[1] = \frac{6}{\pi^2} \int_1^\infty \frac{d\xi}{\xi} \ln \left\{ \frac{\xi}{\xi+1} \right\} = \frac{1}{2}$.

For $h = 2$ we have bosons, with $Y[\xi] = \xi + 1$, $\Theta[2] = \frac{\xi-1}{\xi}$ and $i_f[2] = \frac{6}{\pi^2} \int_1^\infty \frac{d\xi}{\xi} \ln \left\{ \frac{\xi-1}{\xi} \right\} = 1$.

On the other hand, for the universal class $h = \frac{3}{2}$, we have fractons with $Y[\xi] = \frac{3}{2} + \sqrt{\frac{1}{4} + \xi^2}$, $\Theta\left[\frac{3}{2}\right] = \frac{\sqrt{1+4\xi^2-1}}{\sqrt{1+4\xi^2+1}}$, and $i_f\left[\frac{3}{2}\right] = \frac{6}{\pi^2} \int_1^\infty \frac{d\xi}{\xi} \ln \left\{ \frac{\sqrt{1+4\xi^2-1}}{\sqrt{1+4\xi^2+1}} \right\} = \frac{3}{5}$.

The distribution function for each class $h$ above, as we can check, are given by

\[ n[1] = \frac{1}{\xi + 1}, \]  

(7)

\[ n[2] = \frac{1}{\xi - 1}, \]  

(8)

\[ n\left[\frac{3}{2}\right] = \frac{1}{\sqrt{\frac{1}{4} + \xi^2}}, \]  

(9)

1This means that fermions($h = 1$) and bosons($h = 2$) are dual objects. As a result we have a fractal supersymmetry, since for the particle with spin $s$ within the class $h$, its dual $s + \frac{1}{2}$ is within the class $\tilde{h}$.

2In fact, we have here fractal functions as discussed in [4].

3This parameter describes the properties of the path (fractal curve) of the quantum-mechanical particle.
i.e. we have the Fermi-Dirac distribution, the Bose-Einstein distribution and the fracton
distribution of the universal class \( h = \frac{3}{2} \), respectively. Thus, our formulation generalizes
\textit{in a natural way} the fermionic and bosonic distributions for particles assuming rational or
irrational values for the spin quantum number \( s \). In this way, our approach can be understood
as a \textit{quantum-geometrical} description of the statistical laws of Nature. This means that the
(Eq. 5) captures the observation about the fractal characteristic of the \textit{quantum-mechanical}
path, which reflects the Heisenberg uncertainty principle.

The fractal index as defined has a connection with the central charge or conformal
anomaly \( c[\nu] \), a dimensionless number which characterizes conformal field theories in two
dimensions. This way, we verify that the conformal anomaly is associated with universal-
ity classes, i.e. universal classes \( h \) of particles. Now, we consider the particle-hole duality
\( \nu \leftrightarrow \frac{1}{\nu} \) for integer-value \( \nu \) of the statistical parameter in connection with the universal
class \( h \). For bosons and fermions, we have

\[
\{0, 2, 4, 6, \cdots\}_{h=2}
\]

and

\[
\{1, 3, 5, 7, \cdots\}_{h=1}
\]

such that, the central charge for \( \nu \text{ even} \) is defined by

\[
c[\nu] = i_f[h, \nu] - i_f \left[ h, \frac{1}{\nu} \right]
\]

and for \( \nu \text{ odd} \) is defined by

\[
c[\nu] = 2 \times i_f[h, \nu] - i_f \left[ h, \frac{1}{\nu} \right],
\]

where \( i_f[h, \nu] \) means the fractal index of the universal class \( h \) which contains the statistical
parameter \( \nu = 2s \) or the particles with distinct spin values which obey specific fractal
statistics. We assume that the fractal index \( i_f[h, \infty] = 0 \) (the class \( h \) is undetermined) and
we obtain, for example, the results
\[ c[0] = i_f[2, 0] - i_f[h, \infty] = 1; \]
\[ c[1] = 2 \times i_f[1, 1] - i_f[1, 1] = \frac{1}{2}; \]
\[ c[2] = i_f[2, 2] - i_f \left[ \frac{3}{2}, \frac{1}{2} \right] = 1 - \frac{3}{5} = \frac{2}{5}; \]
\[ c[3] = 2 \times i_f[1, 3] - i_f \left[ \frac{5}{3}, \frac{1}{3} \right] = 1 - 0.656 = 0.344; \]

etc,

where the fractal index for \( h = \frac{5}{3} \) is obtained from

\[ i_f \left[ \frac{5}{3} \right] = \frac{6}{\pi^2} \int_1^{\infty} \frac{d\zeta}{\zeta} \times \ln \left\{ \frac{\sqrt{\frac{1}{2\pi} + \frac{\zeta^3}{2} + \frac{1}{18}\sqrt{12\zeta^3 + 81\zeta^6} + \frac{1}{9\sqrt{\frac{1}{2\pi} + \frac{\zeta^3}{2} + \frac{1}{18}\sqrt{12\zeta^3 + 81\zeta^6}}}}}{\sqrt{\frac{1}{2\pi} + \frac{\zeta^3}{2} + \frac{1}{18}\sqrt{12\zeta^3 + 81\zeta^6} + \frac{1}{9\sqrt{\frac{1}{2\pi} + \frac{\zeta^3}{2} + \frac{1}{18}\sqrt{12\zeta^3 + 81\zeta^6}}} + \frac{1}{3}} \right\} \]

\[ = 0.656 \]

and for its dual we have

\[ i_f \left[ \frac{4}{3} \right] = \frac{6}{\pi^2} \int_1^{\infty} \frac{d\zeta}{\zeta} \times \ln \left\{ \frac{\sqrt{\frac{1}{2\pi} - \frac{\zeta^3}{2} + \frac{1}{18}\sqrt{-12\zeta^3 + 81\zeta^6} + \frac{1}{9\sqrt{\frac{1}{2\pi} - \frac{\zeta^3}{2} + \frac{1}{18}\sqrt{-12\zeta^3 + 81\zeta^6}}}}}{\sqrt{\frac{1}{2\pi} - \frac{\zeta^3}{2} + \frac{1}{18}\sqrt{-12\zeta^3 + 81\zeta^6} + \frac{1}{9\sqrt{\frac{1}{2\pi} - \frac{\zeta^3}{2} + \frac{1}{18}\sqrt{-12\zeta^3 + 81\zeta^6}}} + \frac{2}{3}} \right\} \]

\[ = 0.56. \]

From the Table I we can observe the correlation between the classes \( h \) of particles and their fractal index, so our approach manifest a robust consistence in accordance with the unitary \( c[\nu] < 1 \) representations [3].
Table I

| $h$ | $i_f[h]$ | Denomination | $\nu$ | $s$ | $c[\nu] = i_f[h, \nu]$ |
|-----|----------|--------------|------|----|------------------|
| 2   | 1        | bosons      | 0    | 0  | 1                |
| ... | ...      | fractons    | ...  | ...| ...              |
| $\frac{5}{3}$ | 0.656   | fractons    | $\frac{1}{3}$ | $\frac{1}{5}$ | 0.656          |
| ... | ...      | fractons    | ...  | ...| ...              |
| $\frac{2}{3}$ | 0.6      | fractons    | $\frac{1}{2}$ | $\frac{1}{3}$ | 0.6            |
| ... | ...      | fractons    | ...  | ...| ...              |
| $\frac{4}{3}$ | 0.56     | fractons    | $\frac{2}{3}$ | $\frac{1}{3}$ | 0.56           |
| ... | ...      | fractons    | ...  | ...| ...              |
| 1   | 0.5      | fermions    | 1    | $\frac{1}{2}$ | 0.5            |

Therefore, since $h$ is defined within the interval $1 < h < 2$, the corresponding fractal index is into the interval $0.5 < i_f[h] < 1$. However, the central charge $c[\nu]$ can assumes values less than 0.5. Thus, we distinguish two concepts of central charge, one is related to the universal classes $h$ and the other is related to the particles which belong to these classes. For the statistical parameter in the interval $0 < \nu < 1$ (the first elements of each class $h$), $c[\nu] = i_f[h, \nu]$, as otherwise we obtain different values.

In another way, the central charge $c[\nu]$ can be obtained using the Rogers dilogarithm function [6], i.e.

$$c[\nu] = \frac{L[x^\nu]}{L[1]},$$

with $x^\nu = 1 - x$, $\nu = 0, 1, 2, 3, etc.$ and

$$L[x] = -\frac{1}{2} \int_0^x \left\{ \frac{\ln(1 - y)}{y} + \frac{\ln y}{1 - y} \right\} dy, \ 0 < x < 1.$$  

Thus, taking into account the Eqs. ([10][11]), we can extract the sequence of fractal indexes (Tables II and III).
### Table II

| $h$ | $\nu$ | $s$ | $i_f[h] = c[\nu]$ | $h$ | $\nu$ | $s$ | $c[\nu]$ |
|-----|------|----|----------------|-----|------|----|--------|
| 2   | 0    | 0  | 1             | 2   | 0    | 0  | 1      |
| 0.20 | 0.20 | 0.18 | 0.858   | 0.20 | 0.20 | 0.142 |         |
| 0.17 | 0.17 | 0.15 | 0.854   | 0.19 | 0.19 | 0.146 |         |
| 0.18 | 0.18 | 0.16 | 0.849   | 0.18 | 0.18 | 0.151 |         |
| 0.17 | 0.17 | 0.14 | 0.845   | 0.17 | 0.17 | 0.155 |         |
| 0.16 | 0.16 | 0.12 | 0.84    | 0.16 | 0.16 | 0.16  |         |
| 0.15 | 0.15 | 0.10 | 0.834   | 0.15 | 0.15 | 0.166 |         |
| 0.14 | 0.14 | 0.08 | 0.829   | 0.14 | 0.14 | 0.171 |         |
| 0.13 | 0.13 | 0.06 | 0.822   | 0.13 | 0.13 | 0.178 |         |
| 0.12 | 0.12 | 0.04 | 0.814   | 0.12 | 0.12 | 0.186 |         |
| 0.11 | 0.11 | 0.02 | 0.806   | 0.11 | 0.11 | 0.194 |         |

### Table III

| $h$ | $\nu$ | $s$ | $i_f[h] = c[\nu]$ | $h$ | $\nu$ | $s$ | $c[\nu]$ |
|-----|------|----|----------------|-----|------|----|--------|
| 0.20 | 0.20 | 0.18 | 0.797   | 0.20 | 0.20 | 0.203 |         |
| 0.17 | 0.17 | 0.16 | 0.786   | 0.19 | 0.19 | 0.214 |         |
| 0.16 | 0.16 | 0.14 | 0.774   | 0.18 | 0.18 | 0.226 |         |
| 0.15 | 0.15 | 0.12 | 0.759   | 0.17 | 0.17 | 0.241 |         |
| 0.14 | 0.14 | 0.10 | 0.742   | 0.16 | 0.16 | 0.258 |         |
| 0.13 | 0.13 | 0.08 | 0.721   | 0.15 | 0.15 | 0.279 |         |
| 0.12 | 0.12 | 0.06 | 0.693   | 0.14 | 0.14 | 0.307 |         |
| 0.11 | 0.11 | 0.04 | 0.656   | 0.13 | 0.13 | 0.344 |         |
| 0.10 | 0.10 | 0.02 | 0.6       | 0.12 | 0.12 | 0.4  |         |
| 0.09 | 0.09 | 0.0 | 0.5     | 0.11 | 0.11 | 0.5 |         |
On the one way, we can estimate the fractal index for the dual classes of $h$ with rational values, considering a fitting of the graphics $i_f[h] \times h$ and $c[\nu] \times \nu$, plus the observation that the $i_f[h, \nu]$ diminishes in the sequence

\[
i_f[h, \nu] = i_f\left[\frac{3}{2}, \frac{1}{2}\right], \quad i_f\left[\frac{4}{3}, \frac{2}{3}\right], \quad i_f\left[\frac{5}{4}, \frac{3}{4}\right], \quad i_f\left[\frac{6}{5}, \frac{4}{5}\right], \quad i_f\left[\frac{7}{6}, \frac{5}{6}\right], \quad i_f\left[\frac{8}{7}, \frac{6}{7}\right], \quad i_f\left[\frac{9}{8}, \frac{7}{8}\right], \quad i_f\left[\frac{10}{9}, \frac{8}{9}\right], \quad i_f\left[\frac{11}{10}, \frac{9}{10}\right], \quad i_f\left[\frac{12}{11}, \frac{10}{11}\right], \quad i_f\left[\frac{13}{12}, \frac{11}{12}\right], \quad i_f\left[\frac{14}{13}, \frac{12}{13}\right], \quad i_f\left[\frac{15}{14}, \frac{13}{14}\right], \quad i_f\left[\frac{16}{15}, \frac{14}{15}\right], \quad i_f\left[\frac{17}{16}, \frac{15}{16}\right], \quad i_f\left[\frac{18}{17}, \frac{16}{17}\right], \quad i_f\left[\frac{20}{19}, \frac{18}{19}\right], \quad i_f\left[\frac{21}{20}, \frac{19}{20}\right], \quad i_f[1, 1].
\]

This way, we observe that our formulation to the universal class $h$ of particles with any values of spin $s$ establishes a connection between Hausdorff dimension $h$ and the central charge $c[\nu]$, in a manner unsuspected till now. Besides this, we have obtained a connection between $h$ and the Rogers dilogarithm function, through the fractal index defined in terms of the partition function associated with the universal class $h$ of particles. Thus, considering the Eqs.(10, 11) and the Eq.(15), we have

\[
\frac{L[x]}{L[1]} = i_f[h, \nu] - i_f\left[h, \frac{1}{\nu}\right], \quad \nu = 0, 2, 4, etc. \quad (17)
\]

\[
\frac{L[x]}{L[1]} = 2 \times i_f[h, \nu] - i_f\left[h, \frac{1}{\nu}\right], \quad \nu = 1, 3, 5, etc. \quad (18)
\]

Also in [1] we have established a connection between the fractal parameter $h$ and the Farey series of rational numbers, therefore once the classes $h$ satisfy all the properties of these series we have an infinity collection of them. In this sense, we clearly establish a connection between number theory and Rogers dilogarithm function. Given that the fractal parameter is an irreducible number $h = \frac{p}{q}$, the classes satisfy the properties [4]

P1. If $h_1 = \frac{p_1}{q_1}$ and $h_2 = \frac{p_2}{q_2}$ are two consecutive fractions $\frac{p_1}{q_1} > \frac{p_2}{q_2}$, then $|p_2q_1 - q_2p_1| = 1$.

P2. If $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}$ are three consecutive fractions $\frac{p_1}{q_1} > \frac{p_2}{q_2} > \frac{p_3}{q_3}$, then $\frac{p_2}{q_2} = \frac{p_1 + p_3}{q_1 + q_3}$. 
P3. If \( \frac{p_1}{q_1} \) and \( \frac{p_2}{q_2} \) are consecutive fractions in the same sequence, then among all fractions between the two, \( \frac{p_1+p_2}{q_1+q_2} \) is the unique reduced fraction with the smallest denominator.

For example, consider the Farey series of order 6, denoted by the \( \nu \) sequence

\[
(h, \nu) = \left( \frac{11}{6}, \frac{1}{6} \right) \rightarrow \left( \frac{9}{5}, \frac{1}{5} \right) \rightarrow \left( \frac{7}{4}, \frac{1}{4} \right) \rightarrow \left( \frac{5}{3}, \frac{1}{3} \right) \rightarrow \left( \frac{8}{5}, \frac{2}{5} \right) \rightarrow \left( \frac{3}{2}, \frac{1}{2} \right) \rightarrow \left( \frac{7}{5}, \frac{2}{5} \right) \rightarrow \left( \frac{4}{3}, \frac{2}{3} \right) \rightarrow \left( \frac{5}{3}, \frac{3}{3} \right) \rightarrow \ldots.
\]

(19)

Using the fractal spectrum (Eq.6), we can obtain other sequences which satisfy the Farey properties and for the classes

\[
h = \frac{11}{6}, \frac{9}{5}, \frac{7}{4}, \frac{8}{5}, \frac{3}{2}, \frac{5}{3}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \frac{7}{6}, \ldots,
\]

and (note that these ones are dual classes, \( \tilde{h} = 3 - h \)) we can calculate the fractal index taking into account the Rogers dilogarithm function or the partition function associated with each \( h \).

Now, in [1] we also considered free fractons and an equation of state at low temperatures was obtained

\[
P = \frac{h \rho^2}{2\gamma} + \gamma (KT)^2 C_1(h),
\]

(20)

where \( \gamma = \frac{m(\nu+1)}{4\pi h^2} \) (\( h \) is the Planck constant), \( \rho \) is the particle density and

\[
C_1(h) = -\int_{1(T=0)}^{\infty(T=\infty)} \frac{d\gamma'}{(\gamma'-1)(\gamma'-2)} \ln \left\{ \frac{\gamma'-1}{\gamma'-2} \right\} = \frac{\pi^2}{6}.
\]

(21)

Thus, for the fracton systems we obtain the specific heat \( C \) as

\[
\frac{C}{L^2} = \frac{m}{4\pi h^2} K^2 (\nu + 1) \frac{\pi^2}{3}.
\]

(22)

On the other hand, the specific heat of a conformal field theory is given by [3]

\[
\frac{C}{L} = \frac{1}{2\pi h \nu} K^2 T \frac{\pi^2}{3} c[\nu].
\]

(23)

Comparing the expressions, we obtain the Fermi velocity as...
\[ v \sim \frac{c[\nu]}{\nu + 1}, \]  

so for \( \nu = 0 \), \( c[0] = 1 \), \( v \sim 1 \); \( \nu = 1 \), \( c[1] = \frac{1}{2} \), \( v \sim 0.25 \); \( \nu = 2 \), \( c[2] = \frac{2}{5} \), \( v \sim 0.133 \); \( \nu = 3 \), \( c[3] = 0.344 \), \( v \sim 0.086 \); etc. We observe that fractons are objects defined in 2+1-dimensions (see [1] for more details).

In summary, we have obtained a connection between fractons and CFT-quasiparticles. This was implemented with the notion of the fractal index associated with the universal class \( h \) of the fractons. This way, fractons and CFT-quasiparticles satisfy a specific fractal statistics. We also have obtained an expression for the Fermi velocity in terms of the conformal anomaly and the statistical parameter [8]. A connection between the Rogers dilogarithm function, Farey series of rational numbers and Hausdorff dimension \( h \) also was established. The idea of fractons as quasiparticles has been explored in the contexts of the fractional quantum Hall effect [1], Luttinger liquids [8] and high-\( T_c \) superconductivity [9]. Finally, a connection between fractal statistics and black hole entropy also was exploited in [10] and a fractal-deformed Heisenberg algebra for each class of fractons was introduced in [11].

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