One-Way Functions in Worst-Case Cryptography: Algebraic and Security Properties*

Alina Beygelzimer, Lane A. Hemaspaandra, Christopher M. Homan
Department of Computer Science
University of Rochester
Rochester, NY 14627

Jörg Rothe
Institut für Informatik
Friedrich-Schiller-Universität Jena
07740 Jena, Germany

Abstract

We survey recent developments in the study of (worst-case) one-way functions having strong algebraic and security properties. According to [RS93], this line of research was initiated in 1984 by Rivest and Sherman who designed two-party secret-key agreement protocols that use strongly noninvertible, total, associative one-way functions as their key building blocks. If commutativity is added as an ingredient, these protocols can be used by more than two parties, as noted by Rabi and Sherman [RS93] who also developed digital signature protocols that are based on such enhanced one-way functions.

Until recently, it was an open question whether one-way functions having the algebraic and security properties that these protocols require could be created from any given one-way function. Recently, Hemaspaandra and Rothe [HR99] resolved this open issue in the affirmative, by showing that one-way functions exist if and only if strong, total, commutative, associative one-way functions exist.

We discuss this result, and the work of Rabi, Rivest, and Sherman, and recent work of Homan [Hom99] that makes progress on related issues.

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1 Motivation

Professor One: Hello, Professor Way! How’s life?

Professor Way: Very exciting indeed. I’ve developed some very exciting worst-case cryptographic protocols. If you read these papers and manuscripts of mine, you’ll see how intuitively attractive, interesting, and exciting my protocols are.

Professor One (spends 10 minutes skimming the papers as Professor Way waits patiently): Wow... I am attracted, interested, and excited by those protocols. But wait. Is there some catch?

Professor Way: Well, I do assume that we have, to use in the protocols, (worst-case) one-way functions that have various additional algebraic and security properties such as associativity, commutativity, and “strong” noninvertibility.

Professor One: You’re assuming WHAT!!?? Whether vanilla one-way functions exist is a major open research issue, and you’re throwing in all sorts of wild extra requirements on one-way functions? Though like many people I believe that vanilla one-way functions exist, I have no similar intuition as to whether one-way functions exist with the many extra properties you are assuming. And so, I must view your protocols as less attractive than protocols built on the assumption that vanilla one-way functions exist.

(Until recently, Professor Way would not have had any good reply at this point. However, due to the work this article is about, Professor Way does have a slam-dunk reply.)

Professor Way: Your worries are completely natural, but nonetheless unfounded. The reason is that one can now prove that all those “wild” extra properties come for free. That is, it remains an open issue whether vanilla one-way functions exist. And it also remains an open issue whether spiffy (say, strongly noninvertible, total, commutative, associative) one-way functions exist. However, they are the same open issue: Spiffy one-way functions exist if and only if vanilla one-way functions exist.
2 Organization and Definitions

Section 1 provided an example of why it may be useful to understand the interactions between one-way-ness and other properties. The present section gives the basic formal definitions. Section 3 summarizes the main results of the papers we survey. Section 4 sketches proofs of restricted cases of some of the results discussed.

We now define the concepts important to this survey.

Throughout this paper, we mainly deal with 2-ary functions, in particular functions mapping from $\Sigma^* \times \Sigma^*$ to $\Sigma^*$, where $\Sigma = \{0,1\}$ is our fixed alphabet. We use both prefix and infix notation for 2-ary functions $\sigma$, i.e., $\sigma(x,y) = x\sigma y$. Unless explicitly stated as being total or one-to-one, the functions we consider are partial and potentially many-to-one. We assume that we have a pairing function $\langle \cdot, \cdot \rangle$ mapping $\Sigma^* \times \Sigma^*$ onto $\Sigma^*$ with the standard nice properties.

Worst-case one-way functions have been studied by many researchers, see, e.g., the papers [GS88, Ko85, Sel92, RS97, HR99, Hom99]. Definition 2.1 presents the case of 2-ary one-way functions.

**Definition 2.1** (see, e.g., [RS97]) For any 2-ary function $\sigma : \Sigma^* \times \Sigma^* \rightarrow \Sigma^*$, we say:

- $\sigma$ is **honest** if $\sigma$ does not shrink its inputs more than by a polynomial amount, i.e., there is a polynomial $p$ such that for every image element $c$ of $\sigma$, there is a domain element $(a, b)$ of $\sigma$ satisfying $a\sigma b = c$ and $|a| + |b| \leq p(|c|)$;

- $\sigma$ is **(polynomial-time) invertible** if there exists a total, polynomial-time computable function $g : \Sigma^* \rightarrow \Sigma^* \times \Sigma^*$ such that for every $c$ in the image of $\sigma$, $\sigma(g(c)) = c$;

- $\sigma$ is a **one-way function** if $\sigma$ is honest, polynomial-time computable, and noninvertible.

As we will see in Section 3, if one-way functions possess certain algebraic properties such as associativity and commutativity, they may be useful as building blocks of some clever cryptographic protocols designed by Rivest, Rabi, and Sherman. The following definition is due to Hemaspaandra and Rothe [HR99].

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1Rabi and Sherman [RS97] use a different notion dubbed “weak associativity” in [HR99]: Any 2-ary function $\sigma$ is said to be weakly associative if the equality $a\sigma(\sigma bc) = (a\sigma b)\sigma c$ holds for all $a, b, c \in \Sigma^*$ satisfying that both $(a, b)$ and $(b, c)$ are in the domain of $\sigma$ and if $(a, b)$ and $(b, c)$ are in the domain of $\sigma$ then so are $(a, \sigma b)$ and $(a\sigma b, c)$. (Rabi and Sherman actually quantify over all $a, b, c \in \Sigma^*$ satisfying that each of $(a, b)$, $(b, c)$, $(a, b\sigma c)$, and $(a\sigma b, c)$ is in the domain of $\sigma$, a phrasing that is logically equivalent with our phrasing, but that may contain terms that are not well-defined: $(a\sigma b, c)$ is not well-defined if
Definition 2.2 For any 2-ary function \( \sigma : \Sigma^* \times \Sigma^* \to \Sigma^* \), define the set \( \Gamma = \Sigma^* \cup \{ \bot \} \) and an extension \( \hat{\sigma} : \Gamma \times \Gamma \to \Gamma \) of \( \sigma \) as follows:

\[
\hat{\sigma}(a, b) = \begin{cases} 
\sigma(a, b) & \text{if } a \neq \bot \text{ and } b \neq \bot \text{ and } (a, b) \in \text{domain}(\sigma) \\
\bot & \text{otherwise.}
\end{cases}
\]

(1)

We say \( \sigma \) is associative if \( (a \hat{\sigma} b) \hat{\sigma} c = a \hat{\sigma} (b \hat{\sigma} c) \) holds for all \( a, b, c \in \Sigma^* \). We say \( \sigma \) is commutative if \( a \hat{\sigma} b = b \hat{\sigma} a \) holds for all \( a, b \in \Sigma^* \).

Rabi and Sherman [RS97] use a notion of strong noninvertibility: A 2-ary function \( \sigma \) is strongly noninvertible if even given the output and an argument, computing the other argument is not a polynomial-time achievable task.

Let us state this formally (see [RS97,HR99]).

Definition 2.3 A 2-ary function \( \sigma : \Sigma^* \times \Sigma^* \to \Sigma^* \) is said to be strong if no polynomial-time computable function \( g : \Sigma^* \to \Sigma^* \) satisfies either of the following two conditions:

- For all \( c \) in the image of \( \sigma \) and for all \( a \in \Sigma^* \), if there is some \( b \in \Sigma^* \) with \( a \sigma b = c \), then \( \sigma(a, g(\langle a, c \rangle)) = c \).
- For all \( c \) in the image of \( \sigma \) and for all \( b \in \Sigma^* \), if there is some \( a \in \Sigma^* \) with \( a \sigma b = c \), then \( \sigma(g(\langle b, c \rangle), b) = c \).

Note that strongness implies noninvertibility.

Finally, we define bounded “many-to-one”-ness. Denote the set of nonnegative integers by \( \mathbb{N} \).

Definition 2.4 Let \( h : \mathbb{N} \to \mathbb{N} \) be any total function and let \( \sigma : \Sigma^* \times \Sigma^* \to \Sigma^* \) be any function. We say \( \sigma \) is \( h(k) \)-to-one if for every \( b \) of length \( k \) in the image of \( \sigma \), the cardinality4 of the preimage of \( b \) under \( \sigma \) is at most \( h(k) \).

\( \sigma \) is not defined at \( (a, b) \). The distinction between these two notions of associativity, in brief, can be explained via Kleene’s [Kle52, pp. 327–328] distinction between complete equality and weak equality for partial functions; see [HR99] for a discussion of some weaknesses of weak associativity.

2A change made by a journal copyeditor inserted a typo into Definition 2.3 of [HR99]. Line 27 of page 651 of [HR99] should correctly read as equation (1) given here (note the occurrence of “\( a \neq \bot \)” rather than the typo “\( a \neq 1 \)”).

3Throughout this paper, for any function \( f \) (even if \( f \) happens to be one-to-one) and for any image element \( z \) of \( f \), we mean by “the preimage of \( z \) under \( f \)” the set of all domain elements mapped to \( z \) by \( \sigma \).
3 Progress on Algebraic and Security Properties for One-Way Functions in Worst-Case Cryptography

3.1 Rabi and Sherman: Weakly Associative One-Way Functions Exist If and Only If One-Way Functions Exist

The “original” result about one-way functions is:

Theorem 3.1 (see [BDG95] and [Sel92, Proposition 1]) \( P \neq NP \) if and only if one-way functions exist.

However, writers of (even worst-case) cryptographic protocols began to desire stronger building blocks than these vanilla one-way functions—in particular, one-way functions with enhanced algebraic and security properties. In fact, according to [RS93], this idea was suggested in 1984 by Rivest and Sherman with respect to secret-key agreement.

This excellent, insightful idea of Rivest and Sherman led to the important 1993 paper of Rabi and Sherman ([RS93], see also the journal version [RS97]), which proposes explicit protocols that exploit such algebraic and security properties as strong noninvertibility, totality, commutativity, and weak associativity. This of course raised the issue of whether one-way functions with these properties were likely to exist. Rabi and Sherman prove the following result.

Theorem 3.3 ([RS93, RS97]) Weakly associative, commutative one-way functions exist if and only if one-way functions exist.

Interestingly, their proof technique is quite different from the techniques used to study one-to-one one-way functions.

This result is widely known and cited, but the authors have yet to find an attribution as to who first discovered it.

For the special case of one-to-one one-way functions (see the excellent survey by Selman [Sel92]), the history is much clearer. The analogous theorem for those is the following.

Theorem 3.2 ([GS88, Ko85, Ber77]) One-to-one one-way functions exist if and only if \( P \neq UP \), where \( UP \) is Valiant’s unambiguous polynomial time [Val77].

This theorem was found independently by Grollmann and Selman [GS88] and Ko [Ko85], and Berman’s thesis [Ber77] independently obtained essentially the same result (see [Sel92]).

To avoid possible confusion, we mention that though our Definition 2.1 (and this entire article) does not require one-way functions to be one-to-one, some authors do mean “one-to-one one-way function” when they write “one-way function.”
Note, however, that the proof of Theorem 3.3 does not achieve totality, associativity (as per Definition 2.2), or strongness. Another result due to Rabi and Sherman is the following.

**Theorem 3.4** [RS93,RS97] No total, one-to-one, weakly associative one-way functions exist.

### 3.2 Hemaspaandra and Rothe: Strong, Total, Commutative, Associative One-Way Functions Exist If and Only If One-Way Functions Exist

One key worry with the protocols discussed by Rabi and Sherman is that their key characterization result, Theorem 3.3, is not strong enough to ensure that (with at least the same certainty as that with which vanilla one-way functions exist) there exist one-way functions having the properties the protocols of Rabi, Rivest, and Sherman require. For example, strong noninvertibility is important for the protocols, and a lack of totality would severely decrease their applicability.

Hemaspaandra and Rothe remove this worry by proving that spiffy one-way functions are just as likely to exist as vanilla one-way functions. In particular, they prove the following result.

**Theorem 3.5** [HR99] Strong, total, commutative, associative one-way functions exist if and only if one-way functions exist.

**Professor One:** Gotcha! Theorem 3.5 is about associative one-way functions as in Definition 2.2, yet the protocols of Rivest et al. require weakly associative one-way functions. And in one of your overlong footnotes you claim that weak associativity is different than associativity by which, I suppose, you mean provably different.

**Professor Way:** That’s right. But, firstly, every associative function outright is weakly associative, so Theorem 3.3 does provide the type of one-way function needed for the protocols. Secondly, for total 2-ary functions such as those of Theorem 3.5, the two notions of associativity coincide anyway; look at [HR99, Proposition 2.4] if you don’t see why these claims hold. Thirdly, note that most results of [HR99] and of [RS97] are shown, in [HR99], to hold both for associative and weakly associative one-way functions. And finally: Motivation time is over, we are in the middle of a technical section, so the two of us shouldn’t distract the reader from reading the results and proof sketches.
The proof of Theorem 3.5, which will be partially discussed in Section 4, has two parts. One part shows how to establish strongness, associativity, and commutativity. The second part shows how the very special strong, associative, and commutative one-way function created from any given one-way function in the first part of the proof can be extended to achieve totality without destroying any of the other properties.

Note that this extension is a very specific “conversion to totality.” Another result of [HR99] addresses the issue of broader “conversions to totality.” In particular, [RS97] gives a construction, call it $C$, that it asserts lifts any nontotal, weakly associative one-way function whose domain is in $P$ to a total, weakly associative one-way function. Though it remains possible that this construction in fact always works, under a plausible complexity-theoretic hypothesis Hemaspaandra and Rothe [HR99] show that there will be cases on which it fails.

**Theorem 3.6 [HR99]** If $\text{UP} \neq \text{NP}$ then there exists a weakly associative one-way function $\tau$ such that

(a) the domain of $\tau$ is in $P$,

(b) there exists some $x \in \Sigma^*$ such that $(x, x)$ is not in the domain of $\tau$, and

(c) construction $C$ fails on $\tau$, that is, the total extension of $\tau$ yielded by $C$ is not weakly associative.

Note that, for construction $C$ to work, both condition (a) and condition (b) are required. While in [RS97], without proof, condition (b) is simply assumed to be true for every nontotal, weakly associative one-way function, there may well be counterexamples to this claim. However, for the particular function $\tau$ constructed in the proof of Theorem 3.4, condition (b) is explicitly shown to hold. Thus, construction $C$ does not fail on $\tau$ (see condition (c)) because it cannot be applied to $\tau$, but rather because $C$ does not preserve weak associativity. In contrast, $C$ does preserve associativity as defined in Definition 2.2 and so is useful in achieving the “conversion to totality” in the second part of the proof of Theorem 3.5.

Finally, what about the issue of injectivity (i.e., one-to-one-ness) for associative one-way functions? Theorem 3.4, due to Rabi and Sherman, states that no total, weakly associative function (and so, by the above comment of Professor Way, no total, associative function) is injective. However, if one does not require totality then associative, injective one-way functions are no less likely to exist than injective one-way functions, which expands Theorem 3.3.
Theorem 3.7 [HR99] One-to-one, associative one-way functions exist if and only if one-to-one one-way functions exist.

Hemaspaandra and Rothe [HR99] also establish that equivalent to the two conditions of Theorem 3.7 (and thus to the condition “P ≠ UP,” see Theorem 3.2) is the existence of strong, commutative, associative one-way functions that satisfy a certain weak notion of injectivity called “unordered injectivity.”

Definition 3.8 A 2-ary function is unordered-injective if for all \(a, b, c, d \in \Sigma^*\) with \((a, b)\) and \((c, d)\) in the domain of \(\sigma\), \(\sigma(a, b) = \sigma(c, d)\) implies \(\{a, b\} = \{c, d\}\).

They left open the issue of whether for total, associative functions—which cannot be one-to-one by Theorem 3.4—also two-to-one-ness is precluded, and what bounds on the “many-to-one”-ness of such functions (one-way or otherwise) can be shown to hold. The next section gives an answer to the first question and reports on recent progress towards resolving the general case.

3.3 Homan: Amount of “Many-to-One”-ness and its Interaction with Algebraic and Security Properties

Suppose we can encode a message using an associative one-way function, and its intended recipient can decode it. Can the space to which the encrypted message is mapped by the decoding function be feasibly searched—or is it a haystack? What if the number of potential encodings of the encoded message is so large that it cannot be determined in polynomial time which encoding was the original message? As mentioned in Footnote 4, some researchers require one-way functions to always be one-to-one. Others merely require that the ambiguity of the possible encodings be polynomially bounded, so that they can be efficiently searched. In particular, Allender and Rubinstein [ARS86,All86] introduce “poly-to-one” one-way functions and prove an analog of Theorem 3.1 for those functions (see also [RH99] for an expansion of their result), and Watanabe [Wat88], Hemaspaandra and Hemaspaandra [HH94], and others have studied variations of constant-bounded ambiguity. But how does bounded “many-to-one”-ness, or even one-to-one-ness, interact with algebraic and security properties such as associativity and strongness?

We have already seen that—whether or not one-way-ness is involved—associativity and totality preclude one-to-one-ness (Theorem 3.4). Homan [Hom99] strengthens this result.

Theorem 3.9 [Hom99] No total, associative function is constant-to-one.
Homan also proves that this bound is tight by providing the following upper bound: For each nondecreasing, unbounded function \( g \), there exists an \( O(g) \)-to-one, total, commutative, associative function.

Now, let us throw one-way-ness in and ask again: What bounds can one prove on the “many-to-one”-ness of one-way functions having the algebraic and security properties surveyed in this article?

**Theorem 3.10 [Hom99]** If \( P \neq \text{UP} \) then there exists an \( O(n) \)-to-one, strong, total, associative one-way function.

Regarding lower bounds, Homan establishes the following result.

**Theorem 3.11 [Hom99]** For every total, honest, associative function \( \sigma \) whose output length is bounded by a polynomial in the length of the input, there exists an \( m \in \mathbb{N} \) such that \( \sigma \) is not \( o(f^{-1}) \)-to-one, where \( f(x) = \lceil 2 \log x \rceil^m \lceil \log x \rceil \).

There is a rather wide gap between this lower bound and the upper bound given in Theorem 3.10 (under a plausible complexity-theoretic hypothesis). That is, there is a gap between the slowest known growth-rate of the “many-to-one”-ness of strong, total, associative one-way functions and their slowest possible growth-rate. Closing this gap is an interesting open issue. Also open is the degree of “many-to-one”-ness for commutative, strong, total, associative one-way functions.

### 4 Proof Sketches

In this section, we present proof sketches for some of the results surveyed and give the flavor of some of the different techniques used.

#### 4.1 Proof Sketches Related to Hemaspaandra and Rothe’s Work

**Proof Sketch of Theorem 3.5.** Since every spiffy one-way function is a very particular vanilla one-way function, it is enough to show how to create, given any vanilla one-way function \( v \), a one-way function that is strong, total, commutative, and associative. By Theorem 3.1, we can just as well create this function from the assumption that \( P \neq \text{NP} \). (See Grollmann and Selman [GS88] for how to convert any given one-way function into a set in \( \text{NP} \) that is not in \( P \). Although this conversion in Grollmann and Selman is done for 1-ary one-to-one one-way functions and \( P \) versus \( \text{UP} \), the analogous approach works cleanly for the case of 2-ary many-to-one one-way functions and \( P \) versus \( \text{NP} \).)
So, given $v$, let $A_v$ be the corresponding set in NP $-$ P. We will now define the little brother—call him $\sigma$—of the spiffy one-way function we are going to construct from $A_v$. Think of $\sigma$ as a piece of Swiss cheese, full of plenty of delicious, tasty, carefully made cheese, but also full of holes. That is, $\sigma$ will be a strong, commutative, associative one-way function, but it will in fact not be total. The big brother of $\sigma$, then, will be the same piece of Swiss cheese, still delicious, tasty, and carefully made, but with its holes plugged. That is, it will be the total extension of $\sigma$—carefully preserving each of $\sigma$’s algebraic and security properties—that is yielded by construction $C$ mentioned in Section 3.2. In this survey, we restrict ourselves to making just the Swiss cheese $\sigma$ with holes.

How do we make $\sigma$? First, forget about $\sigma$ being a piece of Swiss cheese. Rather, imagine $\sigma$ to be a police officer at work.

It is a busy morning at the police department. Officer $\sigma$ has many reports on her desk describing incidents $x$ that happened last night. Our set $A_v \in \text{NP} - \text{P}$ will be the set of all incidents that are crimes. (Suppose that, every night, many crimes happen and most of them are rather difficult to solve.) A report on Officer $\sigma$’s desk may contain the description of an incident $x$ with a file copy attached to it (such a report has the form $\langle x, x \rangle$). Another report may contain the description of a crime $x$ with an eye witness’s statement $w$ attached to it (such a report has the form $\langle x, w \rangle$). There are all sorts of other reports as well.

Luckily, Officer $\sigma$ can easily tell incident descriptions apart from witness statements, so she always knows whether the report at hand is of the form $\langle x, x \rangle$ or $\langle x, w \rangle$. Also, Officer $\sigma$ can easily check how reliable a witness is, since they use lie detectors at this police department to verify each witness statement taken.

Every once in a while, Officer $\sigma$ grabs two reports $a$ and $b$ (one with her left hand and one with her right hand), reads them both, and chooses one of $a$ and $b$ to pass on to her boss, Sgt. $\hat{\sigma}$, dumping the other one. Sometimes, she dumps them both. Here is how Officer $\sigma$ makes her decision on which reports to pass on and which to dump:

- Whenever report $a$ is of the form $\langle x, w_1 \rangle$ and report $b$ is of the form $\langle x, w_2 \rangle$ (that is, both describe the same incident $x$, which appears to be a crime, for there are two—possibly identical—witness statements attached to it), Officer $\sigma$ picks one of $a$ and $b$ to pass on to Sgt. $\hat{\sigma}$, dumping the other one. In particular, she always passes on the report containing the shorter (to be more precise, the lexicographically lesser) witness statement.

- Whenever one of the reports has the form $\langle x, x \rangle$ and the other one has the form $\langle x, w \rangle$ for the same crime $x$, where $w$ is a witness statement for $x$, Officer $\sigma$ passes
Whenever the reports are not of the form described in the above two cases, Officer $\sigma$ dumps them both.

Now, let us be a bit more formal. A witness for “$x \in A_v$” is any string $w \in \Sigma^*$ encoding an accepting path of $M$ on input $x$, where $M$ is a fixed NP machine accepting $A_v$. For each $x \in A_v$, define the set of witnesses for “$x \in A_v$” by

$$\text{WIT}_M(x) = \{ w \in \Sigma^* \mid w \text{ is a witness for } "x \in A_v" \}.$$ 

We may assume that, for each $x \in A_v$, any witness $w$ for “$x \in A_v$” is of length $p(|x|)$ for some strictly increasing polynomial $p$, and the length of $w$ is strictly larger than the length of $x$. This assumption is just a technical detail that enables Officer $\sigma$ to tell input strings in $A_v$ apart from their witnesses, a property that will be useful later on.

Given any two strings $a$ and $b$ in $\Sigma^*$, define $\sigma(a, b)$ as follows:

- If there is some $x \in \Sigma^*$ for which there exist witnesses $w_1, w_2 \in \text{WIT}_M(x)$ such that $a = \langle x, w_1 \rangle$ and $b = \langle x, w_2 \rangle$, then $\sigma(a, b)$ is defined to be the string $\langle x, \min(w_1, w_2) \rangle$, where $\min(w_1, w_2)$ denotes the lexicographically smaller of $w_1$ and $w_2$.

- If there is some $x \in \Sigma^*$ for which there exists some witness $w \in \text{WIT}_M(x)$ such that $a = \langle x, x \rangle$ and $b = \langle x, w \rangle$, or $a = \langle x, w \rangle$ and $b = \langle x, x \rangle$, then $\sigma(a, b)$ is defined to be the string $\langle x, x \rangle$.

- Otherwise, $\sigma(a, b)$ is undefined, that is, there is a hole in the domain of $\sigma$ at $(a, b)$.

It remains to prove that $\sigma$ has the desired properties. That $\sigma$ is honest and commutative is immediate. That $\sigma$ is polynomial-time computable can be seen as follows. By our assumption that for each $x$ in $A_v$, the length of any witness string for “$x \in A_v$” is strictly larger than the length of $x$, there is no ambiguity in deciding whether $\sigma$’s arguments, $a$ and $b$, are of the form $\langle x, x \rangle$ or $\langle x, w \rangle$, where $w$ is a potential witness for “$x \in A_v$.” Moreover, we can of course decide in polynomial time whether a potential witness $w$ for “$x \in A_v$” indeed is a witness.

The strongness of $\sigma$ is shown by way of contradiction. Suppose there is a polynomial-time computable function $g$ such that, for any string $c$ in the image of $\sigma$ and for any fixed first argument $a \in \Sigma^*$ for which there is some second argument $b \in \Sigma^*$ with $ab = c$, it holds that $\sigma(a, g(\langle a, c \rangle)) = c$. Using $g$, one could then decide $A_v$ in polynomial time as follows:

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\(^5\)That in part explains why so few crimes are solved in this town.
Given any input string \( x \), to decide whether or not \( x \) is in \( A_v \), compute the string \( g((x, x), (x, x)) \). Compute the projections, say \( u \) and \( w \), of our pairing function at \( g((x, x), (x, x)) \); that is, compute the unique strings \( u \) and \( w \) for which \( \langle u, w \rangle = g((x, x), (x, x)) \). Accept \( x \) if and only if \( u = x \) and \( w \in \text{WIT}_M(x) \).

This polynomial-time algorithm for \( A_v \) contradicts our assumption that \( A_v \) is not in \( P \). Hence, \( \sigma \) cannot be inverted in polynomial time even if the first argument is given. An analogous argument shows that no polynomial-time computable function can invert \( \sigma \) even if the second argument is given. Hence, \( \sigma \) is strong.

It remains to show that \( \sigma \) is associative. Let \( a, b, c \in \Sigma^* \) be any fixed arguments for \( \sigma \). Let the projections of our pairing function at \( a \), \( b \), and \( c \) be given by \( a = \langle a_1, a_2 \rangle \), \( b = \langle b_1, b_2 \rangle \), and \( c = \langle c_1, c_2 \rangle \). Let \( k \in \{0, 1, 2, 3\} \) be the number that tells you how many of \( a_2, b_2, \) and \( c_2 \) are elements of \( \text{WIT}_M(a_1) \). For example, if \( a_2 = c_2 \in \text{WIT}_M(a_1) \), but \( b_2 \not\in \text{WIT}_M(a_1) \), then \( k = 2 \).

According to Definition 2.2, we have to show that

\[
(a \hat{\sigma} b) \hat{\sigma} c = a \hat{\sigma}(b \hat{\sigma} c),
\]

where \( \hat{\sigma} \) is the extension of \( \sigma \) from that definition.

There are two cases.

**Case 1:** Suppose \( a_1 = b_1 = c_1 \) and \( \{a_2, b_2, c_2\} \subseteq \{a_1\} \cup \text{WIT}_M(a_1) \). The intuition in this case is that \( \sigma \) decreases by one the number of witnesses that may occur in its arguments in the following way.

If zero of \( \sigma \)'s arguments contain a witness for “\( a_1 \in A \),” then \( \sigma \) is undefined, so \( \hat{\sigma} \) outputs \( \bot \).

If exactly one of \( \sigma \)'s arguments contains a witness for “\( a_1 \in A \),” then \( \sigma \) — and thus \( \hat{\sigma} \) as well — has the value \( \langle a_1, a_1 \rangle \).

If both of \( \sigma \)'s arguments contain a witness for “\( a_1 \in A \),” then \( \hat{\sigma} \) outputs \( \langle a_1, w \rangle \), where \( w \in \{a_2, b_2, c_2\} \) is the lexicographically smaller of the two witnesses.

From the above we may conclude the following.

If \( k \in \{0, 1\} \) then \( (a \hat{\sigma} b) \hat{\sigma} c = \bot = a \hat{\sigma}(b \hat{\sigma} c) \).

If \( k = 2 \) then \( (a \hat{\sigma} b) \hat{\sigma} c = \langle a_1, a_1 \rangle = a \hat{\sigma}(b \hat{\sigma} c) \).

If \( k = 3 \) then \( (a \hat{\sigma} b) \hat{\sigma} c = \langle a_1, \min(a_2, b_2, c_2) \rangle = a \hat{\sigma}(b \hat{\sigma} c) \), where \( \min(a_2, b_2, c_2) \) denotes the lexicographically smallest of \( a_2, b_2, \) and \( c_2 \).
In each case, equation (2) is satisfied.

**Case 2:** Suppose case 1 does not hold. This implies that either \( a_1 \neq b_1 \) or \( a_1 \neq c_1 \) or \( b_1 \neq c_1 \), or it holds that \( a_1 = b_1 = c_1 \) and \( \{a_2, b_2, c_2\} \not\subseteq \{a_1\} \cup \text{WIT}_M(a_1) \). In either of these two subcases of case 2, one can verify that \( (a\tilde{\sigma}b)\tilde{\sigma}c = \perp = a\tilde{\sigma}(b\tilde{\sigma}c) \).

Thus, in each subcase, equation (2) is satisfied.

Hence, \( \sigma \) is associative. This completes the proof sketch.

**Proof Sketch of Theorem 3.6.** Assuming \( \text{UP} \neq \text{NP} \), we will show that the “conversion to totality” construction of Rabi and Sherman (which was called construction \( C \) in Section 3.2) does not preserve weak associativity.

Construction \( C \) works as follows. Suppose we are given any nontotal function \( \tau : \Sigma^* \times \Sigma^* \to \Sigma^* \) satisfying that (i) the domain of \( \tau \) can be decided in polynomial time, and (ii) there exists some string \( \text{trashbin} \in \Sigma^* \) such that \( (\text{trashbin}, \text{trashbin}) \) is not in the domain of \( \tau \). Construction \( C \) converts \( \tau \) into a total function \( \tilde{\tau} : \Sigma^* \times \Sigma^* \to \Sigma^* \) defined as follows:

\[
\tilde{\tau}(a, b) = \begin{cases} 
\tau(a, b) & \text{if } (a, b) \text{ is in the domain of } \tau \\
\text{trashbin} & \text{otherwise},
\end{cases}
\]

that is, \( \text{trashbin} \) is used to dump all garbage elements of \( \tau \) (i.e., elements on which \( \tau \) is not defined).

We will now define a 2-ary function \( \tau \) that resembles Officer \( \sigma \) from the proof of Theorem 3.5. However, unlike \( \sigma \), \( \tau \) will be merely weakly associative. We then show that the total extension \( \tilde{\tau} \) that is yielded by applying construction \( C \) to \( \tau \) is not weakly associative.

Pick a set \( L \) in \( \text{NP} - \text{UP} \) and a nondeterministic polynomial-time Turing machine \( M \) accepting \( L \). We assume that all technical requirements that were useful in defining \( \sigma \) also hold in this proof. In particular, for any \( x \in L \), all witnesses for “\( x \in L \)” are of length greater than the length of \( x \), and \( \text{WIT}_M(x) \) is the set of witnesses for \( x \), defined as in the proof of Theorem 3.5.

Given any two strings \( a \) and \( b \) in \( \Sigma^* \), define \( \tau(a, b) \) as follows:

- If there is some \( x \in \Sigma^* \) for which there exists some witness \( w \in \text{WIT}_M(x) \) such that \( a = \langle x, w \rangle \) and \( b = \langle x, w \rangle \), then \( \tau(a, b) \) is defined to be the string \( \langle x, w \rangle \).
- If there is some \( x \in \Sigma^* \) for which there exists some witness \( w \in \text{WIT}_M(x) \) such that \( a = \langle x, x \rangle \) and \( b = \langle x, w \rangle \), or \( a = \langle x, w \rangle \) and \( b = \langle x, x \rangle \), then \( \tau(a, b) \) is defined to be the string \( \langle x, x \rangle \).
• Otherwise, \( \tau(a, b) \) is undefined, that is, there is a hole in the domain of \( \tau \) at \((a, b)\).

Note that \( \sigma \) and \( \tau \) differ only in the first item of their definitions. It is not difficult to see that \( \tau \) is a weakly associative one-way function. So, it remains to prove that conditions (a), (b), and (c) of Theorem 3.6 are satisfied.

Condition (a): The domain of \( \tau \) can be decided in polynomial time, since witness checking can be done in deterministic polynomial time and since we can distinguish between input strings and their potential witnesses by our length requirement.

Condition (b): Since \( L \notin \text{UP} \), we have \( L \neq \Sigma^* \); so, there must be a string \( \hat{x} \) not in \( L \).

Let \( \text{trashbin} = \langle \hat{x}, 1\hat{x} \rangle \). Note that there is no string \( x \in \Sigma^* \) for which \( \text{trashbin} = \langle x, x \rangle \), and there are no strings \( x \in \Sigma^* \) and \( w \in \text{WIT}_M(x) \) for which \( \text{trashbin} = \langle x, w \rangle \) (this holds because \( \hat{x} \notin L \) and so it does not have any witnesses). By the definition of \( \tau \), it follows that \( \tau \) is not defined at \( \langle \text{trashbin}, \text{trashbin} \rangle \).

Condition (c): Since \( L \notin \text{UP} \), there exists a string \( x_0 \in L \) that has at least two distinct witnesses. Fix the two smallest witnesses, say \( w_1 \) and \( w_2 \) with \( w_1 \neq w_2 \), for \( x_0 \in L \). Let \( a = \langle x_0, w_1 \rangle \), \( b = \langle x_0, w_2 \rangle \), and \( c = \langle x_0, x_0 \rangle \) be three given arguments of \( \tilde{\tau} \). Since \( \tilde{\tau} \) is total, each of \((a, b), (b, c), (a, b\tilde{\tau}c), \text{ and } (a\tilde{\tau}b, c) \) is in the domain of \( \tilde{\tau} \). However, it holds that

\[
\tilde{\tau}(\tilde{\tau}(a, b), c) = \tilde{\tau}(\text{trashbin}, c) = \text{trashbin} \neq \langle x_0, x_0 \rangle = \tilde{\tau}(a, \langle x_0, x_0 \rangle) = \tilde{\tau}(a, \tilde{\tau}(b, c)).
\]

Hence, \( \tilde{\tau} \) is not weakly associative or associative.

4.2 Proof Sketch Related to Homan’s Work

We present the proof of Theorem 3.9. In fact, Theorem 3.9 follows immediately from Lemma 4.1 below.

Lemma 4.1 \[\text{Hom99}\] For every \( n \in \mathbb{N} \) and for every total, associative function (one-way or otherwise) \( \sigma : \Sigma^* \times \Sigma^* \to \Sigma^* \), there exists an element \( z \in \Sigma^* \) in the image of \( \sigma \) whose preimage under \( \sigma \) is of cardinality at least \( n \).

Proof Sketch of Lemma 4.1. Let \( \sigma : \Sigma^* \times \Sigma^* \to \Sigma^* \) be any total, associative function. For each string \( w \) in the image of \( \sigma \), define two sets \( L_w \) and \( R_w \) as follows:

\[
L_w = \{ x \in \Sigma^* \mid (x \neq w) \land (\exists y \in \Sigma^*) [\sigma(x, y) = w] \};
\]

\[
R_w = \{ y \in \Sigma^* \mid (y \neq w) \land (\exists x \in \Sigma^*) [\sigma(x, y) = w] \}.
\]

To prove the lemma, we will show that for every \( n \in \mathbb{N} \), there exists a string \( z \in \Sigma^* \) in the image of \( \sigma \) for which at least one of the following two conditions is true:
(1) the set $L_z$ has cardinality at least $n$;

(2) the set $R_z$ has cardinality at least $n$.

We use induction on $n$.

For $n = 1$, pick any two distinct strings $a, b \in \Sigma^*$. Since $\sigma$ is total, $a\sigma b$ necessarily exists. Let $z = a\sigma b$. Since $a \neq b$, either $a \neq z$ or $b \neq z$ (or both), making $z$ satisfy at least one of the conditions (1) or (2).

Let $n \geq 1$, and assume that there exists a string $z \in \Sigma^*$ such that at least one of conditions (1) and (2) holds true for $n$. Assume that condition (1) holds for $n$. (If condition (2) holds for $n$, an analogous argument works.)

We show that there is a string in $\Sigma^*$ that satisfies at least one of conditions (1) and (2) for $n+1$. If the cardinality of the set $L_z$ in condition (1) is strictly greater than $n$, we are done. So, suppose condition (1) holds with equality (for $n$). Then, there exist $n$ pairs of strings $(x_1, y_1), \ldots, (x_n, y_n) \in \Sigma^* \times \Sigma^*$ each having image $z$ under $\sigma$ and so that the $x_i$ are pairwise distinct and distinct from $z$.

Choose any distinct strings $s_1, \ldots, s_{n^2+n+1} \in \Sigma^*$ not contained in $\{x_1, \ldots, x_n, z\}$. Since $\sigma$ is total, for all $i$, $1 \leq i \leq n^2+n+1$, there exists a string $u_i \in \Sigma^*$ such that

$$u_i = z\sigma s_i = (x_1\sigma y_1)\sigma s_i = \cdots = (x_n\sigma y_n)\sigma s_i.$$  

Since $\sigma$ is associative, for all $i$, $1 \leq i \leq n^2+n+1$, we also have

$$u_i = z\sigma s_i = x_1\sigma(y_1\sigma s_i) = \cdots = x_n\sigma(y_n\sigma s_i).$$

If there exists some $i$, $1 \leq i \leq n^2+n+1$, such that the corresponding string $u_i$ is not in $\{x_1, \ldots, x_n, z\}$, then $\{x_1, \ldots, x_n, z\} \subseteq L_{u_i}$. Hence, this $u_i$ satisfies condition (1) for $n+1$. (This is the only place where we make use of the assertion “$(\forall j : 1 \leq j \leq n) [x_j \neq z]$” that follows from the definition of $L_z$.)

Otherwise, for each $i$, $1 \leq i \leq n^2+n+1$, we have $u_i \in \{x_1, \ldots, x_n, z\}$. Thus, the $n^2+n+1 = (n+1)n+1$ distinct pairs $(z, s_i)$ are mapped by $\sigma$ onto the $n+1$ strings $x_1, \ldots, x_n, z$. By the pigeon-hole principle, there must exist some $\hat{z} \in \{x_1, \ldots, x_n, z\}$ whose preimage under $\sigma$ has cardinality at least $n+1$.

We claim that $\hat{z}$ satisfies condition (2) for $n+1$. Let $\hat{S}$ be the set of all $s_i$, $1 \leq i \leq n^2+n+1$, such that $\sigma(z, s_i) = \hat{z}$. The above argument shows that the cardinality of $\hat{S}$ is at least $n+1$. Since $\hat{z} \in \{x_1, \ldots, x_n, z\}$ and

$$\{s_1, \ldots, s_{n^2+n+1}\} \cap \{x_1, \ldots, x_n, z\} = \emptyset,$$

we have $\hat{z} \neq s_i$ for each $i$, $1 \leq i \leq n^2+n+1$. Thus, $\hat{S} \subseteq R_{\hat{z}}$, which makes $\hat{z}$ satisfy condition (2) for $n+1$ and completes the proof. 

\[\blacksquare\]
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