Fast and Simple Deterministic Algorithms for Highly-Dynamic Networks

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Abstract

This paper provides a surprisingly simple method for obtaining fast (constant amortized time) deterministic distributed algorithms for a highly-dynamic setting, in which arbitrarily many edge changes may occur in each round. Among the implications of our results are deterministic algorithms that maintain solutions to many problems, including \((\text{degree} + 1)\)-coloring, maximal matching, maximal independent set and the seemingly unrelated problem of a 2-approximation for minimum weight vertex cover (2-MWVC).

These significantly improve upon prior work in various aspects, such as having \(O(1)\) amortized round complexity, using message of logarithmic size only, handling arbitrarily many concurrent topology changes, being deterministic, having correctness guarantees for intermediate rounds, and more.

The core of our work is in defining a subclass of locally-checkable labelings which we call locally-fixable labelings (LFLs). Very roughly speaking, these are labelings that allow a node to fix its neighborhood based solely on their old labels. We present a simple algorithm for LFLs with small labels, which handles multiple edge insertions/deletions while keeping the amortized round complexity at a small constant. We then extend it, for specific tasks, to handle the insertion and deletion of nodes. Moreover, we show that the same approach can also fix labeling with large labels, given that they can be made to behave as small labels.

1 Introduction

Locally checkable labelings (LCLs) is a celebrated concept in distributed computing, first defined by Naor and Stockmeyer [NS95], for stating where nodes can efficiently detect inconsistencies in solutions for distributed tasks. Its motivation is to capture settings in which nodes can handle faults or can self-stabilize. Since this pioneering work, the complexity of solving tasks that can be described as LCLs has been extensively studied in various distributed settings.

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This paper returns to the original motivation of handling a dynamic behavior of the network, and studies what problems can be fixed fast in a highly-dynamic setting in which an unbounded number of network links may appear or be dropped adversarially in every round of computation, and despite the ability of the nodes to communicate only small messages of $O(\log n)$ bits. We provide a simple example that shows that not all LCLs can be fixed locally in this setting, even when the checking radius is $r = 1$, which motivates a different definition. To this end, we define a refined concept which we call locally fixable labelings (LFLs). We show a surprisingly simple algorithm which proves that LFL tasks with small labels allow fast fixing in this rather harsh model of computation, within a constant amortized number of rounds. Here, amortization is the ratio between rounds in which nodes communicate and the number of topology changes.

We then show that fundamental LCL tasks, such as finding a $(\text{degree} + 1)$-coloring or a maximal matching, can be described as LFLs with bounded labels, which immediately implies fast fixing even with logarithmic messages. For some tasks, fast fixing is also possible with node insertions and (abrupt) deletions. For the case of finding a maximal independent set (MIS), which can be described as an LFL but not necessarily with small labels, we provide a solution in the same spirit. These results significantly improve upon prior work within the time complexity, in the assumptions about the environment, or in the guarantees for intermediate solutions, as we elaborate in Section 1.1. Moreover, we show that the task of finding a 2-approximation for the minimum weighted vertex cover can also be described as an LFL but with large labels, and we can make it behave as an LFL with small labels, and hence our algorithm handles this task as well.

Formally, we consider networks of $n$ nodes, each marked by a unique ID in $[n]$. A labeling is a function $L : [n] \to S$, where $S$ is some set of labels. We use $\mathcal{L}$ for a set of labelings, and $\mathcal{L}[G]$ for the labelings of $G$ which are in $\mathcal{L}$. Throughout, $N^r_G(v)$ is the set of nodes at distance at most $r$ from $v$ in $G$ (we sometimes omit the superscript $G$ when it is clear from the context, or the subscript $r$ when it equals to 1), and $G[U]$ is the subgraph induced by a set of nodes $U$. We begin with recalling the definition of LCLs.

\begin{definition}[r-Locally-Checkable Labelings (LCLs)]
\[\text{For an integer } r \geq 0 \text{ (called the radius), a set of labelings } \mathcal{L} \text{ is called } r\text{-locally checkable if given a graph } G \text{ and a labeling } L \text{ for } V, \text{ the following holds: if for every node } v, \text{ the restriction of } L \text{ to } G_{r,v} = G[N^r_G(v)] \text{ is in } \mathcal{L}[G_{r,v}], \text{ then } L \in \mathcal{L}[G].\]
\end{definition}

The original definition of LCLs also addresses the possibility of having inputs for the nodes of the graph, which we will indeed require for some of our results. To simplify our presentation, we choose to handle this by incorporating inputs as part of the description of the graph $G$ itself, rather than indicating them explicitly.

Some LCLs are not easily fixable, even when the radius of the LCL is $r = 1$: consider maintaining the following simple variant of a sinkless orientation \cite{BFH+16}. Each node has a label that corresponds to an orientation of its edges, such that labels at endpoints of an edge are consistent, and such that there is no node of degree larger than 1 that is a sink, i.e., has no outgoing edge. It is easy to verify that every graph has a valid labeling\footnote{In the distributed setting, distinguishing between abrupt and graceful node deletions is paramount, as requiring only graceful deletions does not allow a node to leave the system unexpectedly.} and that this is a 1-LCL. To see that...
this 1-LCL is not easily fixable, consider a graph on \( n \) nodes that evolves dynamically, creating two paths of roughly \( n/2 \) nodes each. Each path must be oriented consistently with a single sink in one of its endpoints. Inserting an edge between the sinks of the two paths forces the orientation of all of the edges in one of the sub-paths to flip, which takes \( \Omega(n) \) rounds. Deleting this edge induces again two paths with a single sink each, and repeating the process of inserting an edge between the new sinks and deleting it causes a linear number of rounds that can be attributed to only two topology changes, which implies an amortized time of \( \Theta(n) \). This holds even if topology changes do not happen concurrently, and even if messages are allowed to be large.

Yet, the key observation of this paper is that we can pinpoint some LCLs that can be fixed fast. The following new concept is the key to our approach, and is a combinatorial definition in the spirit of LCLs, rather than an algorithmic-based definition.

**Locally-Fixable Labelings (LFLs):** A set of labelings \( \mathcal{L} \) is locally-fixable if \( \mathcal{L} \) is a 1-LCL and, given two graphs \( G \) and \( G' \) whose difference is exactly a single edge \( \{v_1, v_2\} \), the following holds:

- For the restriction to \( N_{G'}(v_1) \) of any labeling \( L \in \mathcal{L}[G] \), there is a labeling \( L_1 \) that does not differ from \( L \) for any node \( z \notin N_{G'}(v_1) \), and

- For the restriction of \( L_1 \) to \( N_{G'}(v_2) \) there is a labeling \( L_2 \in \mathcal{L}[G'] \) that does not differ from \( L_1 \) for any node \( z \notin N_{G'}(v_2) \).

Essentially, the definition says that \( v_1 \) and \( v_2 \) can correctly relabel themselves and their neighbors after an edge insertion or deletion based only on the neighborhood labels that they can see after the change, in any order among the two nodes. Notice two subtleties: first, since edges are not directed, the definition must hold both orderings \( v_1, v_2 \) of the endpoints of the edge. Second, the fixing is a two-step procedure: from \( L \) to \( L_1 \) and then from \( L_1 \) to \( L_2 \). This is done in order to capture cases in which the neighborhoods of the two nodes intersect and thus cannot be fixed concurrently.

The definition of LFLs is natural, in the sense that one can be easily convinced that if only a single topology change may occur in the network at a given time period, and if sizes of messages are not limited, then a single round is required for each of the two neighborhoods for fixing the labeling to be valid for the graph after the change.

Still, to fix arbitrarily many concurrent changes of the topology in a constant amortized time with small messages, some additional machinery is needed. To this end, our method is to assign a timestamp to each change in the graph and fix a node that suffered from the change only if its timestamp is a local minimum, thus avoiding conflicting concurrent fixes. To further cope with the restriction on the size of messages, we deterministically hash the timestamps into a small bounded domain so that the nodes can afford sending a hashed timestamp in a single small message, and we do so in a way that preserves the total order over timestamps. This idea is taken from literature on shared memory algorithms, and appears, e.g., in [ADS89].

With some care, this gives the required fixing for LFLs with logarithmic labels, which we call **bounded LFLs**. Our algorithm also features strong guarantees on the labels that are present during unstable rounds. Roughly, the labels are always correct for a subgraph that includes the nodes that are not neighbors of nodes for which topology changes have occurred and have not yet been fixed, and possibly some of the other edges that touch unfixd nodes, in the sense that fixing always requires changing the labels of only the immediate neighborhood. Thus, our main
algorithmic contribution lies in the proof for the following (Section 2), which holds in a model with an unbounded number of topology changes that may occur concurrently, and when only a logarithmic number of bits may be sent in a message.

**Theorem 1.** For every bounded LFL \( \mathcal{L} \), there is a deterministic dynamic distributed algorithm which fixes edge insertions/deletions in \( O(1) \) amortized rounds.

We then get that the above holds for maintaining a \( (\text{degree} + 1) \)-coloring, and maintaining a maximal matching, which directly implies also a 2-approximation for both the minimum cardinality vertex cover and the maximum cardinality matching. Theorem 1 handles changes in edges, and we provide an example for a bounded LFL in which handling node changes seems to be inherently costly. However, for some of the above tasks we prove that the same approach can also handle node insertions and deletions within the same complexity (see Section 2.1).

Curiously, the naïve labeling for maintaining an MIS is an LFL but not a bounded one. To see this, suppose an edge is inserted between two nodes in the MIS. Then, the label of one of them, say \( v \), has to change, but this requires relabeling all nodes in its neighborhood that were previously dominated only by \( v \). Since their induced subgraph may be arbitrary but is unknown to \( v \) itself without including this information in the labels (but then they are large), fixing these labels cannot be done within few rounds. The crux here is blaming previous topology changes for such a situation – for every node in the neighborhood of \( v \) which is only dominated by \( v \) there is a previous topology change which took fewer rounds to fix. This allows us to amortize the round complexity all the way down to \( O(1) \), and this also handles node insertions and deletions (see Section 2.2).

We then turn our attention to the task of maintaining a 2-approximation for the minimum weighted vertex cover task (2-MWVC). First, we prove that while this is an approximation task and not a symmetry-breaking task we can still show a simple labeling for it, which is an LFL. Our labeling is not the naïve one, which only indicates which nodes are in the cover, but rather contains information about how a solution was obtained, when one considers it being the result of a local-ratio weight reduction process. The caveat here is that the labels are then way too large. We observe, however, that for fixing the labels after a topology change, we do not need to communicate entire labels, and only parts of them suffice.

To capture this in our framework, we extend the definition of an LFL in a natural manner to handle large labels that are integrated from several parts, one for each node, for which relabeling only depends on some small excerpts of the labels in the neighborhood. Our fixing algorithm directly applies to such integrated LFLs, and allows us to obtain our fast fixing result also for 2-MWVC (see Section 3).

Before diving into formal proofs, we being with a thorough comparison with related work.

### 1.1 Related work

The end results of our work provide fast fixing for fundamental graph problems, whose static algorithmic complexity has been extensively studied in distributed settings. A full overview of the known results merits an entire survey paper on its own (see, e.g., \[Suo13, BE13\]), but we provide at the end of this section below a (very) partial picture in order to give a flavor of known complexities in the static setting.

There is a rich history of research on the important paradigm of self-stabilization (see, e.g., the book by Dolev [Dol00]) and in particular on symmetry breaking (see, e.g., the survey by Guellati and Kheddouci [GK10]). Our model greatly differs from the above. There are many additional
models of dynamic distributed computation, which are very different from the one we consider in this paper, and in what follows we focus on the more relevant related studies.

Some of the oldest works in similar models to ours are by Italiano [Ita91] and Elkin [Elk07], who provide algorithms for distance-related tasks such as shortest paths and spanners.

Extremely fast algorithms were given by Königs and Wattenhofer [KW13], who pinpointed some problems that can be fixed in a single round. Theirs is the first result that addresses symmetry breaking in this particular dynamic setting, yet their model assumes large messages and a single topology change at a time. Their work was followed by that of Censor-Hillel, Haramaty, and Karnin [CHK16], which provides a randomized MIS algorithm with small messages, still assuming a single change at a time. The latter left as an open question the complexity of maintaining an MIS in the sequential dynamic setting. This was picked up by Assadi, Onak, Schieber, and Solomon [AOSS18b, AOSS18a] and by Du and Zhang [DZ18], who provide beautiful algorithms that significantly improve upon the naive solution. They show fast amortized time complexity for maintaining an MIS, and the former two works have implications for the distributed setting. However, they handle only a single change at a time, and need to know the number of edges $m$, which is global knowledge that our work avoids assuming. In fact, if one is happy with restricting the algorithm to work only in a model with a single topology change at a time, then sending timestamps is not required, so $O(1)$-bit messages suffice in our algorithm for MIS, which resembles the properties of [AOSS18b] for that model.

Parter, Peleg, and Solomon [PPS16] provide a neat log-starization technique, which translates logarithmic static distributed algorithms into a dynamic setting such that their amortized time complexity becomes $O(\log^* n)$. They also assume that topology changes are sufficiently spaced in time in order to allow recovery before the next change, and their work allows large messages. Solomon [Sol16] shows that maximal matching (and hence 2-MVC and 2-MCM) have $O(1)$ amortized complexity, and this holds even when counting messages. However, this again assumes a single change at a time.

The $(\Delta + 1)$-coloring algorithm of Barenboim, Elkin, and Goldenberg [BEG18] also implies fixing in a self-stabilizing manner – after the topology stops changing, only $O(\Delta + \log^* n)$ rounds are required to obtain a valid coloring. Here there needs to be a guarantee for the bound $\Delta$, beyond which the degrees do not grow.

Perhaps the most related setting is that of Bamberger, Kuhn, and Maus [BKM18], who address a very similar highly-dynamic setting as ours. They insightfully provide a wide family of tasks that includes some of the problems addressed in our paper, for which they show fast dynamic algorithms. Roughly, problems in this family can be decomposed into packing and covering problems, in the sense that a packing condition remains true when deleting edges and a covering condition remains true when inserting edges. For example, MIS is such a problem, with independence and domination being the packing and covering conditions, respectively. A crucial innovation in their algorithms is that they give guarantees also for intermediate states of the algorithm, that is, even while the system is still changing. They show that the packing property holds for the set of edges that are present throughout the last $T$ rounds, and that the covering property holds for the set of edges that are present in either of the last $T$ rounds, for $T = O(\log n)$. Moreover, their algorithms have correct solutions if a constant neighborhood of a node does not change for a logarithmic number of rounds. In comparison to their worst-case guarantee of $O(\log n)$ rounds for a correct solution, our algorithm only gives $O(n)$ rounds in the worst case. However, our amortized complexity is

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3This means that the algorithm is guaranteed to allow enough time for fixing before another change occurs.
O(1), our messages are of logarithmic size, and our algorithm is deterministic, while the above is randomized with messages that can be polylog in size. Further, for some problems, our algorithm guarantees correctness for intermediate solutions that is similar to theirs. However, we do not know whether our LFL definition captures all of the tasks that the above work does, and it may very well be that we address incomparable sets of problems.

A brief overview of some complexities of static distributed tasks: A foundational lower bound of Linial [Lin92] requires $\Omega(\log^* n)$ rounds for computing an MIS or a $(\Delta + 1)$-coloring, which for coloring is to date the best known. For MIS, 2-MVC, and additional tasks, the ingenious lower bounds of $\Omega(\log \Delta / \log \log \Delta)$, $\Omega(\sqrt{\log n / \log \log n})$ are given by Kuhn, Moscibroda and Wattenhofer [KMW16]. Upper bounds for MIS include the cornerstone $O(\log n)$ algorithms of Luby [Lub86] and Alon, Babai, and Itai [ABI86], and the more recent $O(\log \Delta) + 2^{O(\sqrt{\log \log n})}$ algorithm of Ghaffari [Gha16], as well as the classic deterministic algorithm of Panconesi and Srinivasan [PS96]. For $(\Delta + 1)$-coloring, Barenboim, Elkin, and Goldenberg [BEG18] provide a deterministic solution within $\tilde{O}(\sqrt{\Delta} + \log^* n)$ rounds, and Harris, Schneider, and Su [HSS18] provide a randomized algorithm completing within $O(\sqrt{\log \Delta} + 2^{O(\sqrt{\log \log n})})$ rounds. A 2-approximation for the minimum vertex cover of a graph can be directly obtained from a maximal matching, which can be obtained in $O(\log \Delta + \log^3 n)$ rounds by plugging the algorithm of Fischer [Fis17] into the framework of Barenboim, Elkin, Pettie, and Schneider [BEPS16]. For the weighted case, the fastest 2-approximation is an $O(\log n \log \Delta / \log^2 \Delta)$ algorithm by Ben-Basat, Even, Kawarabayashi, and Schwartzman [BEKS18], while for a $(2 + \epsilon)$-approximation only the tight $O(\log \Delta / \log \log \Delta)$ rounds are needed, as first shown by Bar-Yehuda, Censor-Hillel, and Schwartzman [BCS17].

2 An $O(1)$ amortized dynamic algorithm for bounded LFLs

We assume that the network starts as an empty graph on $n$ nodes and evolves into the graph $G_i = (V_i, E_i)$ at the beginning of round $i$. The nodes receive indications about the topology changes of which they are a part of. Our main result is that an LFL can be maintained deterministically within a constant amortized number of rounds if the label sizes are bounded by $O(\log n)$ bits. We call this a bounded LFL.

**Theorem 1.** For every bounded LFL $\mathcal{L}$, there is a deterministic dynamic distributed algorithm which fixes edge insertions/deletions in $O(1)$ amortized rounds.

**Proof.**

**The setup:** We denote $\gamma = 5$.

Let $F_i$ be a set of topology changes (edge insertions/deletions) that occur in round $i > 0$. With each change, we associate two timestamps such that a total order is induced over the timestamps as follows: for an edge $e = \{u, v\}$ in $F_i$ we associate the timestamp $ts = (i, u, v)$, with node $u$ (similarly, we associate the timestamp $(i, v, u)$ with node $v$). Since $u$ and $v$ start round $i$ with an indication of $e$ being in $F_i$, both can deduce their timestamps at the beginning of round $i$. We say that a node $v$ is the owner of the timestamps that are associated with it. In each round, a node only stores the largest timestamp that it owns.

Notice that timestamps are of unbounded size, which renders them impossible to fit in a single message. To overcome this issue we invoke a deterministic hash function $H$ over the timestamps,

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4Here $\Delta$ is the maximum node degree. We will later elaborate about $(\Delta + 1)$-coloring vs. $(\text{degree} + 1)$-coloring.
which reduces their size to $O(\log n)$ bits, while retaining the total order over timestamps. The reason we can do this is that not every two timestamp can exist in the system concurrently. To this end, we define $h(i) = i \mod 3\gamma n$ and $H(ts) = (h(i), u, v)$ for a timestamp $ts = (i, u, v)$, and we define an order $\prec_H$ over hashed timestamps as the lexicographic order of the 3-tuple, induced by the following order $\prec_h$ over values of $h$. We say that $h(i) \prec_h h(i')$ if and only if one of the following holds:

- $0 \leq h(i) < h(i') \leq 2\gamma n$, or
- $\gamma n \leq h(i) < h(i') \leq 3\gamma n$, or
- $2\gamma n \leq h(i) < 3\gamma n$ and $0 \leq h(i') < \gamma n$.

We claim that for any two timestamps $ts, ts'$ that are stored at two nodes at some given times $i, i'$, respectively, it holds that $ts < ts'$ if and only if $H(ts) \prec_H H(ts')$. The reason for this is the property that we will later prove, that says that for every two such timestamps, it holds that $i' - i \leq \gamma n$ which will imply $h(i) \prec_h h(i')$.

In the algorithm, the nodes chop up time into epochs, each of length $\gamma$, in a non-overlapping manner. That is, epoch $j$ consists of rounds $i = \gamma j, \ldots, \gamma(j + 1) - 1$.

The algorithm: For every epoch $j \geq 0$, we consider a set $D_j \subseteq V$ of dirty nodes at the beginning of each epoch, where initially no node is dirty ($D_0 = \emptyset$). Some nodes in $D_j$ may become clean by the end of the epoch, so at the end of the epoch the set of dirty nodes is denoted by $D'_j$, and it holds that $D'_j \subseteq D_j$. At the beginning of epoch $j + 1$, all nodes that receive any indication of an edge in $F_i$ in the previous epoch are added to the set of dirty nodes, i.e., $D_{j+1} = D'_j \cup I_j$, where $I_j$ is the set of nodes that start round $i$ with any indication about $F_i$, for any $\gamma j \leq i \leq \gamma(j + 1) - 1$.

The algorithm works as follows. In epoch $j = 0$, the nodes do not send any messages, but some of them enter $I_0$ (if they receive indications of edges in $F_i$, for $0 \leq i \leq \gamma - 1$).

For epoch $j > 0$, for 2 rounds, the nodes propagate the hashed timestamps owned by dirty nodes. That is, in round $\gamma j$, each node in $D_j$ broadcasts its hashed timestamp, and in the following round all nodes broadcast the smallest hashed timestamp that they see (with respect to the order $\prec_H$). Every node $v$ in $D_j$ which does not receive a hashed timestamp that is smaller than its own becomes active and informs this to its neighbors. Note that active nodes are at least 3 hops apart.

In the following round, each neighbor $z$ of $v$ sends its label $L(z)$ to $v$. Finally, $v$ sends each neighbor its new label $L_1(z)$ (or $L_2(z)$) and assigns itself own new label $L_1(v)$ (or $L_2(v)$). Node $v$ becomes inactive, and is not included in $D'_j$, i.e., we initialize $D'_j = D_j$ at the beginning of epoch $j$, and here $D'_j \leftarrow D'_j \setminus \{v \mid v$ is active in epoch $j\}$. Note that this description does not consider any topology changes that may occur during the epoch. If an edge is inserted during epoch $j$ then the nodes do not send information over that edge in the epoch, and if an edge is deleted during epoch $j$ then any information that the above description considers as being sent over the edge after it is deleted is in fact unsent. In particular, notice that it may be that changes are made to edges touching $v$ during this process, i.e., $v \in I_j$, in which case $v$ is again dirty for the following epoch (because $v \in D_{j+1}$).

Round complexity: We now prove that the algorithm has an amortized round complexity of $O(1)$. To this end, we show that any epoch $j$ in which messages are sent can be blamed on a different
timestamp \( ts \) and that the node \( v \) that owns \( ts \) is either clean for the next epoch (\( v \notin D_{j+1} \)) or is dirty because of a (new) change that occurs in one of its adjacent edges during epoch \( j \) (\( v \in I_j \)).

First, we claim that for every two timestamps \( ts < ts' \) with rounds \( i,i' \), respectively, that are owned by nodes at a given time it holds that \( i' - i \leq \gamma n \). Assume otherwise, and consider the first time for which this condition is violated for some timestamps \( ts < ts' \). This means that the owner \( v \) of \( ts \) does not become active for more than \( n \) epochs, which means that at round \( i \) there were more than \( n \) timestamps stored in various nodes, which were not yet handled. But there are at most \( n \) nodes and each one stores at most one timestamp so the above is impossible. Since \( i' - i \leq \gamma n \), we have that \( H(ts) <_H H(ts') \), because \( h(i) <_h h(i') \), as argued earlier.

Since the hashed timestamps are totally ordered by \( <_H \), we have that in each epoch \( j \) there is at least one dirty node \( v \) that becomes active, namely the one with the minimal timestamp. The node \( v \) is not in \( D_j \), and hence either is not in \( D_{j+1} \) or is in \( I_j \), as claimed. Since every topology change results in two timestamps, we have that the number of rounds required by the algorithm is \( 2\gamma = O(1) \), amortized over all changes.

**Correctness:** For correctness, we claim that there is a labeling \( L \) that agrees with the labels \( L_j \) of nodes that are not in \( N^{G_j}(v) \) for some dirty node \( v \) at the end of epoch \( j \), such that \( L \) is in \( L \) for \( G_{\gamma j} \).

This follows since a node that becomes clean in epoch \( j \) is the owner of a timestamp \( ts \) that is minimal in its 2-neighborhood, which implies that two concurrently active nodes do not have overlapping neighborhoods. Further, if \( v \) becomes clean at epoch \( j \) and all nodes in \( N^{G_j}(v) \) are clean at the end of the epoch, then all such nodes apart from \( v \) are clean throughout the epoch.

By the definition of LFLs, and by considering \( G \) and \( G' \) in the definition as the graph before and after the change in the edge whose associated timestamp is \( ts \), this implies by induction that there is a labeling \( L \) that is identical to \( L_j \) at the end of epoch \( j \) apart from the labels of \( N^{G_j}(u) \) for dirty nodes \( u \), such that \( L \) is in \( L \) for \( G_{\gamma j} \), as claimed.

We complete the proof by recalling that an LFL is a 1-LCL, therefore, if all nodes are clean at the end of some epoch \( j \) then the above implies that the set of all labels is in \( L \) for \( G_{\gamma j} \). \( \square \)

### 2.1 Implications for known bounded LFLs

We now apply Theorem 1 to some fundamental tasks, by showing that they are bounded LFLs. Then we show that these specific examples also allow fast fixing of node insertions and deletions.

**Lemma 1.** The problems of maintaining a (degree+1)-coloring and maintaining a maximal matching are LFLs with \( O(\log n) \)-bit labels.

**Proof.**

(degree + 1)-Coloring: We consider the set of labels \( \{1,\ldots, n\} \), and consider two graphs \( G = (V_G, E_G) \) and \( G' = (V_G', E_G') \) that differ only in an edge \( e = \{v,u\} \) as in the definition of LFLs. If \( E_G' = E_G \setminus \{e\} \), then given labeling \( L \) that is a valid coloring for \( G \) we set \( L_1(z) = L(z) \) for all nodes except \( v \), and we set \( L_1(v) \) to be the minimal among \( \{1,\ldots, degree_{G'}(v) + 1\} \) which is not \( L_1(z) \) for any neighbor \( z \in N^{G'}(v) \). Similarly we set \( L_2(u) \). It is easy to verify that this is a valid coloring in which the color of each node is at most its degree plus 1. Assume that \( E_G' = E_G \cup \{e\} \) and there is a labeling \( L \) for \( N^{G'}(v) \) that is a valid coloring for \( G \). Define a labeling \( L_1 \) in which \( L_1(v) = c \) for the minimal \( c \) such that \( L(z) \neq c \) for all \( z \in N^{G'}(v) \setminus \{v\} \), which is guaranteed to exist due to the promise of the degrees in the graph. It is straightforward to verify that already \( L_1 \)
is a valid coloring for the graph: every two nodes \( z_1, z_2 \) that are not \( v \) have different labels in \( L_1 \) because this holds for \( L \), and the label \( L_1(v) \) is chosen to be different than the labels of all of its neighbors.

**Maximal matching:** We consider the set of labels \([n] \cup \{\bot\}\), where a label in \([n]\) corresponds to a matched node and \( \bot \) indicates a non-matched node. Suppose we have a labeling \( L \) that is valid for \( G \), and let \( e = \{v, u\} \). Assume that \( E_{G'} = E_G \cup \{e\} \). Then, if \( L(v) = L(u) = \bot \), we take \( L_1(v) = u, L_2(u) = v \) and \( L_1(z) = L_2(z) = L(z) \) for all \( z \notin \{u, v\} \). Otherwise, we take \( L_1 = L_2 = L \). The labelings \( L_1 \) and \( L_2 \) define valid matchings because \( L_1(v) = u, L_2(u) = v \) only if \( L(v) = L(u) = \bot \) which means the nodes \( u, v \) were not matched in \( L \). We argue that the matchings are also maximal for their respective graphs. Indeed, let \( e' = (u', v') \) be any other edge in \( G' \) which is not in the matching (say, \( L_1(u') = \bot \) or \( L_1(v') = \bot \)). If \( e \) and \( e' \) do not share a common endpoint then as \( L \) defines a valid maximal matching for \( G \), \( e' \) has a matched endpoint. Assume now without loss of generality that \( v' = v \) and \( u' \neq u \). If \( L(v) = L(u) = \bot \), then in \( L_1 \) we have that \( v \) is already matched with \( u \). Otherwise, it means that \( v \) is already matched in \( L \) (and so in \( L_1 \)) by an edge which is not \( e, e' \). The same applies for \( L_2 \).

Assume now that \( E_{G'} = E_G \setminus \{e\} \). If \( L(v) \neq u \) then set \( L_1(z) = L_2(z) = L(z) \) for every \( z \in N_{G'}(v) \). Then \( L_1, L_2 \) correspond to maximal matchings because \( e \) was not a matched edge in \( L \) and so deleting it does not affect this condition. If \( L(v) = u \) then, if there is a neighbor \( z \in N_{G'}(v) \setminus \{v\} \) for which \( L(z) = \bot \) we set \( L_1(v) = z, L_1(z) = v \) and \( L_1(y) = L(y) \) for other nodes. If for every \( z \in N_{G'}(v) \setminus \{v\} \) it holds that \( L(z) \neq \bot \) then we set \( L_1(y) = L(y) \) for all nodes. It is easy to verify that the labels of \( L_1 \) can be similarly modified only in \( N_{G'}(u) \), such that \( L_2 \) is valid.

Since a maximal matching trivially gives a 2-approximation of a minimum cardinality vertex cover and of a maximum cardinality matching, we get the Lemma \( \| \) directly for these tasks as well.

Combining Theorem \( \| \) and Lemma \( \| \) directly gives the following.

**Corollary 1.** For maintaining a \((\text{degree} + 1)-\text{coloring}, \) maintaining a maximal matching, and maintaining a 2-approximation of a minimum cardinality vertex cover or of a maximum cardinality matching, there are deterministic dynamic distributed algorithms which handle edge insertions/deletions in \( O(1) \) amortized rounds.

Next, we consider node insertions/deletions for the above problems. Fixing these topology changes fast does not follow from the main theorem because the only straightforward thing we can say is that a node change corresponds to many edge changes, but we cannot account for this in our amortization, which without deeper analysis grows to the order of the degrees of nodes.

The following is an example of an LFL such that node insertions/deletions may require linearly many rounds to handle, even when amortized over the number of changes. Consider a labeling for graphs with labels \([1, \ldots, n, \bot] \), where odd-degree nodes, \( V_{\text{odd}} \), must maintain a \((\Delta_{\text{odd}} + 1)-\text{coloring} \) over \( G[V_{\text{odd}}] \) (\( \Delta_{\text{odd}} \) being the maximal degree in \( G[V_{\text{odd}}] \)), and even-degree nodes \( V_{\text{even}} \) must have the label \( \bot \). One can show that this labeling is also an LFL, as only the nodes \( u, v \) corresponding to the insertion or deletion of an edge \( e = \{u, v\} \) ever need to change their labels. Now, consider a graph which consists of a connected component of a linear in \( n \) number of odd-degree nodes and the remaining nodes are isolated. After constructing this graph, the labels of the odd-degree nodes induce a valid coloring with no \( \bot \) labels. Then, inserting a node \( v \) and connecting it to all nodes in the component forces their labels to change to \( \bot \), and deleting this node then forces them to fix
to a valid coloring again, which cannot be done in $O(1)$ rounds. Repeating such a node insertion-deletion pair of changes for sufficiently many times (depending on the number of changes required for constructing the component), eventually exceeds $O(1)$ when amortized over all changes. While it may be argued that there could be a different labeling that captures this task, this example serves as motivation for why the algorithm in Theorem 1 may be insufficient for node changes.

Nevertheless, we show that a variant of the algorithm in Theorem 1 with a tweak about which nodes become dirty, fixes node insertions/deletions in $O(1)$ amortized rounds in the number of topology changes. Thus we prove the following, where we relax the coloring requirement to be a $(\Delta + 1)$-coloring, though we suspect a similar result holds also for $(\text{degree} + 1)$-coloring.

**Theorem 2.** There are deterministic dynamic distributed algorithms that maintain a $(\Delta + 1)$-coloring and a maximal matching, and handle node insertions/deletions in $O(1)$ amortized rounds.

**Proof.** To deal with node insertions/deletions in a unified way we consider an even more general setting, in which any number of edge changes may occur, and any set of edge insertions (or set of edge deletions) that have a common endpoint $v$ can be charged as a single topology change, and this change is associated with $v$ in the sense that the indication that endpoints receive consists of this information. Clearly, a node insertion or deletion falls under the above description.

To this end, let $G = (V, E)$ and $G' = (V, E')$ be two graphs that differ by the edges

$$E' = (E \setminus \{\{v, u_1\}, \{v, u_2\}, \ldots, \{v, u_k\}\}) \cup \{\{v, w_1\}, \{v, w_2\}, \ldots, \{v, w_\ell\}\}.$$

**$(\Delta + 1)$-coloring:** We modify the algorithm of Theorem 1 so that the nodes $\{u_1, \ldots, u_k, w_1, \ldots, w_\ell\}$ avoid marking themselves as dirty. Correctness still holds by noticing that in the proof of Lemma 1, if we only care about $(\Delta + 1)$-coloring, then for every edge change, only a single endpoint needs to actually fix neighborhood, and in fact it only needs to fix its own label and only if this was an edge insertion. The amortized round complexity also clearly remains, as in every epoch at least one dirty node either becomes clean or is assigned a new timestamp, and every topology change of a node induces a single new timestamp.

**Maximal matching:** Here too we modify the algorithm of Theorem 1 so that the nodes $\{w_1, \ldots, w_\ell\}$ avoid marking themselves as dirty. Correctness still holds because, similarly to the proof of Lemma 1 in case of multiple edge insertions the node $v$ only needs to fix the label of itself and possibly a single node that it gets matched to. For the nodes $\{u_1, \ldots, u_k\}$, only the node that is matched with $v$ (if there exists such a node) needs to become dirty. The amortized round complexity also clearly remains, as in every epoch at least one dirty node either becomes clean or is assigned a new timestamp, and every topology change of a node induces at most two new timestamps.

### 2.2 The case of maintaining an MIS

An MIS can be easily expressed as an LFL, but we do not know whether it can be expressed as a bounded LFL. The reason for this is that when an edge is inserted between two MIS nodes one of its endpoints must fix its label to become a non-MIS node, which necessitates considering the new labels for all of its neighbors for which it was the only MIS neighbor. Such a single topology change might incur a number of dirty nodes that is the degree of this endpoint, which may be linear in $n$. Yet, we make a crucial observation here: any node that becomes dirty in this manner,
can be blamed on a previous topology change in which only one node becomes dirty. This implies a potential function for the budget of dirty nodes, to which we add 2 units for every topology change, and charge either 0, 1, 2, or \( d \) units for each topology change, in a manner that preserves the potential non-negative at all times.

**Theorem 3.** For maintaining an MIS there is a deterministic dynamic distributed algorithm which handles edge/node insertions/deletions in \( O(1) \) amortized rounds.

*Proof.* We consider the set of labels \( \{M, \bar{M}\} \), where \( M \) and \( \bar{M} \) correspond to nodes that are in the independent set and nodes that are not in the set, respectively. We consider an algorithm that is identical to that of Theorem 1, with the following modifications.

1. When an edge \( e = \{v, u\} \) is deleted, if the labels of both \( u \) and \( v \) are \( \bar{M} \) then neither of them becomes dirty, and if only one of them is in \( M \) then only this node becomes dirty.

2. When an edge \( e = \{v, u\} \) is inserted, if at least one of their labels is \( \bar{M} \) then neither of them becomes dirty, and if both are in \( M \) then only the node with smaller \( ID \) becomes dirty.

3. When a node \( v \) is inserted then only \( v \) becomes dirty.

4. When a node \( v \) is deleted then a neighbor \( z \) becomes dirty only if its label is \( \bar{M} \) and it has no neighbor with a label \( M \).

When a node becomes active, it collects the labels of its neighbors, and assigns itself the label \( M \) if all the collected labels are \( \bar{M} \), and assigns itself the label \( \bar{M} \) otherwise. The crucial point here is that not all nodes that become dirty in item (4) will actually utilize their timestamp – some will drop their timestamp before competing for becoming active, and hence we will not need to account for fixing them. That is, we add the following item:

5. When an active node \( v \) changes its label to \( M \), all of its dirty neighbors become clean.

The correctness follows from the fact that this labeling is an LFL (not necessarily a bounded LFL), which is easy to verify.

For the amortized round complexity, we use a potential function argument: we prove that the cumulative number of epochs in which any node becomes active (rather than counting dirty nodes) is bounded from above by twice the number of topology changes, which prove our claim of an amortized \( O(1) \) round complexity.

To show this, consider a node \( v \) that is deleted in round \( i \), and a set \( Z = \{z_1, \ldots, z_k\} \) of its neighbors that become dirty by satisfying the condition in item (4) above, ordered by their timestamps \( (i, v, z_\ell) \) for \( 1 \leq \ell \leq k \), as induced by this topology change. For each \( 1 \leq \ell \leq k \), if node \( z_\ell \), becomes active due to this timestamp, then by item (5) it holds that since round \( i \) none of its neighbors changes its label to \( M \). Consider the last round \( i' \) before round \( i \) in which the label of \( z_\ell \) is \( M \) (\( i' \) exists since this condition occurs initially when the graph is empty). We claim that the topology change whose associated active node relabeled \( z_\ell \) to \( \bar{M} \) in round \( i' + 1 \), is either an insertions of an edge \( \{z_\ell, u\} \) that satisfies the condition of item (2) with \( ID(z_\ell) < ID(u) \), or an insertion of the node \( z_\ell \) which connects it to at least one node that is labeled \( M \). The reason for this is that these are the only topology changes which cause \( z_\ell \) to be assigned the label \( M \). The proof is complete by noticing that both of these topology changes induce only a single dirty node, and therefore we can blame \( z_\ell \) becoming active on the corresponding topology change, and
this is an injective mapping, since any other node cannot blame these changes (they are changes 
that made $z_\ell$ dirty), and for $z_\ell$ itself any future time it becomes active again due to satisfying the 
condition in item (4) above implies that it has changed its label to $\bar{M}$ again in between. 

We exemplify through our MIS algorithm the guarantees of our approach in terms of intermedi-
ate solutions. It is easy to see that at every time, there are only two possible cases where the MIS 
constraints are violated. The first is when a dirty node has a label $M$, despite having a neighbor 
with a label $M$, which only happens when an edge is inserted, so the independence condition holds 
for all edges between clean nodes. The second case is when a nodes has a label $M$ and so do all of 
its neighbors, but this only happens when one of these nodes is a dirty node due to either deleting 
a node with a label $M$, or inserting an edge between two nodes labeled $M$ after fixing one endpoint 
to become labeled $\bar{M}$. In either case, the violating node is dirty, and hence domination holds for 
all nodes that do not neighbor dirty nodes.

3 An $O(1)$ amortized dynamic algorithm for integrated LFLs

Taking the endpoints of a maximal matching is of course insufficient for obtaining a 2-approxima-
tion of the minimum weight vertex cover. Instead, we want to employ the local-ratio approach, which at 
a high level says that if we repeatedly pick an edge in the graph for which both endpoints have non-
zero remaining weights and reduce the minimum of the two remaining weights from both endpoints, 
then at the end of the process the set of nodes with remaining weight zero is a 2-approximation for 
MWVC. One way to use this approach in our setting is to have a label for each node which records 
all of this information. That is, the label of a node records not only the remaining weight, but 
also all weight reductions that have been invoked for each of its incident edges. Then, if an edge 
is inserted, its endpoints invoke a weight reduction over it and update their labels, and if an edge 
is deleted, each node locally computes edge reductions with its neighbors in a consistent manner, 
and informs each one about its respective weight reduction. These fixed labels imply that this is 
an LFL, but there is a caveat in using Theorem 1: these labels are too large.

Our critical observation here is that although the labels are large, we never need a node to send 
an entire label in a message. A node $v$ that is an endpoint of a deleted edge only needs to know the 
remaining weight of each neighbor $u$, and only needs to send to $u$ its updated weight reduction 
over this edge. Node $u$ can then compute its updated label locally. To capture this, we extend 
the definition of LFLs. We begin with defining labels that can be manipulated based on partial 
information.

Integrated Labelings: A labeling $L : [n] \rightarrow S$ is an integrated labeling if for every label $L(u)$ of a node $u$ and every node $v$ there is an excerpt $\xi_v(L(u))$ of $O(\log n)$ bits, such that the set of excerpts $\{\xi_v(L(u)) \mid v \in [n]\}$ determines $L(u)$.

Now, we generalize the definition of LFLs so that only excerpts are needed in order to determine 
the new labelings.
Integrated LFLs: An LFL \( L \) is called integrated if every \( L \in \mathcal{L} \) is integrated and given two graphs \( G \) and \( G' \) whose difference is exactly the single edge \( \{v_1, v_2\} \), the following holds:

- For the excerpts \( \{\xi_{v_1}(L(z)), \xi_z(L(z)) \mid z \in N^{G'}(v_1)\} \) of the restriction to \( N^{G'}(v_1) \) of any labeling \( L \in \mathcal{L}[G] \), there is a labeling \( L_1 \) that does not differ from \( L \) for any node \( z \notin N^{G'}(v_1) \) and does not differ from \( L \) for excerpts \( \xi_w(z) \) with \( w \neq v_1 \), \( z \) for any node \( z \in N^{G'}(v_1) \), and

- For the excerpts \( \{\xi_{v_2}(L(z)), \xi_z(L(z)) \mid z \in N^{G'}(v_2)\} \) of the restriction of \( L_1 \) to \( N^{G'}(v_2) \) there is a labeling \( L_2 \in \mathcal{L}[G'] \) that does not differ from \( L_1 \) for any node \( z \notin N^{G'}(v_2) \) and does not differ from \( L \) for excerpts \( \xi_w(z) \) with \( w \neq v_2 \), \( z \) for any node \( z \in N^{G'}(v_2) \).

While the statement of Theorem 4 requires that the labels of the LFL are of logarithmic size, it is easy to see that the proof goes through also for LFLs with large labels, given that they are integrated LFLs. Whenever a node \( z \) sends its label \( L(z) \) to its active neighbor \( v \) we replace this with sending \( (\xi_v(L(z)), \xi_z(L(z))) \), and whenever an active node \( v \) sends \( L_1(z) \) (or \( L_2(z) \)) to its neighbor \( z \) we replace this with sending \( (\xi_v(L_1(z)), \xi_z(L_1(z))) \) (or the corresponding pair of excerpts for \( L_2 \)). We thus obtain the following.

**Theorem 4.** For every integrated LFL \( L \), there is a deterministic dynamic distributed algorithm which fixes edge insertions/deletions in \( O(1) \) amortized rounds.

Notice that every LFL with labels of logarithmic size is an integrated LFL by setting \( \xi_w(L(z)) \) to be \( L(z) \) for every \( v \) and \( z \), thus Theorem 4 could be obtained as a special case of Theorem 4.

### 3.1 Maintaining a 2-MWVC

We first provide the template for using the local-ratio technique for obtaining a 2-approximation for MWVC. This template does not assume any specific computation model and only describes the paradigm and correctness. It can be proven either using the primal-dual framework [BESI], or the local-ratio framework [Bar00], which are known to be equivalent [BR05]. Here we provide the template, and the proof appears in the appendix, for completeness.

We assume a given weighted graph \( G = (V, w, E) \), where \( w : V \to \mathbb{R}^+ \) is an assignment of weights to the vertices. Let \( \delta : E \to \mathbb{R}^+ \) be a function that assigns weights to edges. We say that \( \delta \) is \( G \)-valid if for every \( v \in V \), \( \sum_{e : v \in e} \delta(e) \leq w(v) \), i.e., the sum of weights of edges that touch a vertex is at most the weight of that vertex in \( G \).

Fix any \( G \)-valid function \( \delta \). Define \( \tilde{w}_\delta : V \to \mathbb{R}^+ \) by \( \tilde{w}_\delta(v) = \sum_{e : v \in e} \delta(e) \), and let \( w'_\delta : V \to \mathbb{R}^+ \) be such that \( w'_\delta(v) = w(v) - \tilde{w}_\delta(v) \). In words, \( w'_\delta(v) \) is what remains of the weight \( w(v) \) after subtracting all the weights \( \delta(e) \) that are placed over its incident edges. Since \( \delta \) is \( G \)-valid, it holds that \( w'_\delta(v) \geq 0 \) for every \( v \in V \).

Let \( S_\delta = \{v \in V \mid w'_\delta(v) \leq 0\} \) (in fact, by the above, we can take those that are equal to 0). The following theorem states that if \( S_\delta \) is a vertex cover, then it is a 2-approximation for MWVC.

**Theorem 5.** Let \( \text{OPT} \) be the minimal weight of a vertex cover of \( G \). If \( \delta \) is a \( G \)-valid function, then \( \sum_{v \in S_\delta} w(v) \leq 2\text{OPT} \). In particular, if \( S_\delta \) is a vertex cover then it is a 2-approximation for MWVC for \( G \).

Using the above we continue to proving our main result.
Lemma 2. The problem of maintaining a 2-MWVC is an integrated LFL.

Proof. We assume here a graph $G = (V, w, E)$ in which the node weights are integers in the set $[W]$, where $W$ is polynomial in $n$. Each label $L(z)$ is an $n$-tuple in which each coordinate is an excerpt $\xi_v(L(z))$, such that $\xi_z(L(z))$ corresponds to the remaining weight of the node $z$, and for every $v \neq z$, $\xi_u(L(z))$ corresponds to a reduction $\delta\{v, z\}$ of weight from the edge $\{v, z\}$, if it exists. A labeling $L$ is valid if for every node $z$ it holds that

1. $0 \leq \xi_z(L(z)) \leq w(z)$,
2. for every two nodes $v, z$, $\xi_v(L(z)) = \xi_z(L(v))$, and
3. $\sum_{v \in [n]} \xi_v(L(z)) = w(z)$.

Suppose $G$ and $G'$ differ only in the edge $\{v, u\}$ and let $L$ be a labeling that is valid for $G$. Given the excerpts $\{\xi_v(L(z)), \xi_z(L(z)) | z \in N^G(v)\}$, we set the new excerpts $\{\xi_v(L_1(z)), \xi_z(L_1(z)) | z \in N^{G'}(v)\}$ as follows (the case of $L_2$ is identical).

If $\{v, u\}$ appears in $G'$, then for every $z \in N^{G'}(v)$ such that $z \neq v, u$, we set $\xi_v(L_1(z)) = \xi_v(L(z))$ and $\xi_z(L_1(z)) = \xi_z(L(z))$. Denoting $\delta_{v, u} = \min\{\xi_v(L(v)), \xi_u(L(u))\}$, we set $\xi_v(L_1(v)) = \xi_v(L(v)) - \delta_{v, u}$, and set $\xi_u(L_1(v)) = \delta_{v, u}$ (and similarly set $\xi_u(L_2(v)) = \delta_{v, u}$).

If $\{v, u\}$ appears in $G$, then denote $\delta = \xi_u(L(v))$, and set $\xi_v(L_1(v)) = 0$ and let $T(v) = \xi_v(L(v)) + \delta$. We go over $z \in N^{G'}(v)$ for every $z \neq v, u$ in an arbitrary order and do the following: let $\delta_{v, z} = \min\{T(v), \xi_z(L(z))\}$, set $\xi_v(L_1(v)) = T(v) - \delta_{v, u}$, $\xi_z(L_1(z)) = \xi_z(L(z)) - \delta_{v, z}$ and $\xi_z(L_1(v)) = \xi_v(L_1(z)) = \delta_{v, z}$, and update $T(v) = T(v) - \delta_{v, z}$ (and we do the same for $L_2$).

The above yields a 2-approximation for MWVC, where the cover at any point in time is $S = \{v \in V | \xi_v(L(v)) = 0\}$. To see that this is indeed a cover we use a simple induction over the steps of the procedure as follows. Given that $S$ is indeed a cover, for any edge $\{v', u'\}$ it cannot be the case that both $\xi_{v'}(L(v')) > 0$ and $\xi_{u'}(L(u')) > 0$. Therefore, upon an insertion of an edge $\{v, u\}$ this remains a cover for every $\{v', u'\} \neq \{v, u\}$ as the values $\xi_{v'}(L(v'))$, $\xi_{u'}(L(u'))$ do not increase in $L_1, L_2$, and for the inserted edge $\{v, u\}$ the above procedure guarantees that the new labels satisfy $\xi_v(L_1(v)) = 0$ or $\xi_u(L_2(u)) = 0$. Upon a deletion of an edge these values may increase, but as all edges are considered in the fixing process, then if the edge $\{u, v\}$ is not the one removed our process guarantees that $\xi_v(L_1(v)) = 0$ or $\xi_u(L_2(u)) = 0$. Defining $\delta\{v, u\} = \xi_v(L(u))$ (recalling that $\xi_v(L(u)) = \xi_u(L(v))$), gives that $\delta$ is a $G$-valid function, and Theorem 3 then implies the approximation guarantee.

For node insertions and deletions, here too we cannot make a general claim, as the amortized round complexity may become as large as the degrees of nodes in the graph. Yet, for the specific task of maintaining a 2-MWVC we can handle node insertions. The reason for this is that an insertion of a node $v$ requires only $v$ to become dirty, and when it becomes active it can assign new excerpts to its neighbors based on edge weight reductions that it computes given the excerpts that they send it. However, notice that when a node is deleted, all of its neighbors may have to increase their remaining weight and become dirty in order to fix the weight reductions for their incident edges when they get active. We leave the question of handling node deletions open, but the above discussion, together with Lemma 2 and Theorem 4 does imply the following.
Corollary 2. For maintaining a 2-approximation of a minimum weight vertex cover there is a deterministic dynamic distributed algorithm which handles edge insertions/deletions and node insertions in $O(1)$ amortized rounds.

4 Discussion

Many questions arise from our definition of LFL, and perhaps the most intriguing one is whether it correctly characterizes 1-LCLs that are fixable within a constant amortized number of rounds. In particular, it seems that the family of problems addressed in the work of Bamberger, Kuhn, and Maus [BKM18] is not directly comparable with ours, because our characterization aims at being non-algorithmic (but rather through the labelings only). Undoubtedly, our works share some aspects and definitely battle the same severe model.

Additional particular questions are which other tasks can be described as LFLs or as integrated LFLs, and whether there is a generic algorithm that can always address node insertions and deletions for some definition of labelings.

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A Proof of Theorem 5

Proof. For every $v \in V$ we have that $w'_\delta(v) = w(v) - \tilde{w}_\delta(v)$, which implies that $w(v) = w'_\delta(v) + \tilde{w}_\delta(v)$. For every $v \in S_\delta$ it holds that $w'_\delta(v) \leq 0$, and therefore $w(v) \leq \tilde{w}_\delta(v)$. This gives:

\[
\sum_{v \in S_\delta} w(v) \leq \sum_{v \in S_\delta} \tilde{w}_\delta(v) \\
\leq \sum_{v \in S_\delta} \sum_{e : v \in e} \delta(e) \\
\leq \sum_{v \in V} \sum_{e : v \in e} \delta(e) \\
\leq 2 \sum_{e \in E} \delta(e).
\]

It remains to prove that $\sum_{e \in E} \delta(e) \leq \text{OPT}$. To this end, let $S_{\text{OPT}}$ be a cover of minimal weight, and associate each edge $e \in E$ with its endpoint $v_e$ in $S_{\text{OPT}}$ (choose an arbitrary endpoint if both are in $S_{\text{OPT}}$). The weight $w(v)$ of each $v \in S_{\text{OPT}}$ is at least $\sum_{e : v_e = v} \delta(e)$, because it is at least $\sum_{e : v_e \in e} \delta(e)$. Hence, $\text{OPT} = \sum_{v \in S_{\text{OPT}}} w(v) \geq \sum_{v \in S_{\text{OPT}}} \sum_{e : v_e = v} \delta(e) = \sum_{e \in E} \delta(e)$, as desired. \hfill \qed