MUSING ON KUNEN’S COMPACT $L$-SPACE

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ABSTRACT. We present a connected version of the compact $L$-space constructed by Kenneth Kunen under CH. We show that this provides a Corson compact space $K$ such that the Banach space $C(K)$ is isomorphic to no space of continuous function on a zero-dimensional compactum.

1. INTRODUCTION

Assuming the continuum hypothesis, Kunen [18] presented a construction of a compact $L$-space — a nonseparable space which is hereditarily Lindelöf. A compactified Suslin line is also an $L$-space but its existence requires axioms stronger than CH. Besides, Kunen’s compactum has very interesting measure-theoretic properties which are not possible on linearly ordered spaces.

Saying that $\mu$ is a measure on a compact space $K$ we always mean that $\mu$ is a finite Borel measure on $K$ that is inner-regular. The basic idea that was behind Kunen’s construction was that a compact space $K$ is an $L$-space whenever it carries a probability measure $\mu$ such that

M.1 $\mu$ vanishes on singletons;
M.2 $\mu$ is strictly positive;
M.3 $\mu(B) = 0$ for every Borel set $B$ having empty interior;
M.4 every Borel set of measure zero set is metrizable.

Note that M.3 implies that $\mu(B) = \mu(B)$ for every Borel set $B$, and we can check the required properties of $K$ as follows.

The space $K$ is not separable: For every countable $A \subseteq K$ we have $\mu(A) = 0$ by M.1; hence, $\mu(\overline{A}) = 0$ so $\overline{A} \neq K$.

To see that $K$ is hereditarily Lindelöf consider any family $\mathcal{U}$ of open subsets of $K$. Writing $U = \bigcup \mathcal{U}$, we first take, using inner-regularity of $\mu$, a countable subfamily $\mathcal{V} \subseteq \mathcal{U}$ so that its union $V = \bigcup \mathcal{V}$ satisfies $\mu(V) = \mu(U)$. Hence $\mu(U \setminus V) = 0$ and $\mu(U \setminus V) = 0$.

Consequently, by M.4, $U \setminus V$ is a subset of a compact metrizable space $\overline{U \setminus V}$ so $U \setminus V$ is covered by a countable subfamily $\mathcal{V}_1 \subseteq \mathcal{U}$. This means that $\mathcal{V} \cup \mathcal{V}_1$ forms a countable subcover of $U$.

Around the same time, Haydon [14] and Talagrand [28] presented their constructions of compact spaces with measures carried out for different purposes. Those constructions,
however, shared some features of the one from \cite{18}. All the three examples were later amalgamated by Negrepontis to the Kunen-Haydon-Talagrand example solving a number of problems from topology and functional analysis, see \cite{22}, section 5.

**Definition 1.1.** A (regular Borel) measure \( \mu \) on a compact space \( K \) is normal if \( \mu(B) = 0 \) for every Borel set \( B \) with empty interior.

With such a definition we can briefly say that Kunen constructed a dense-in-itself compact space supporting a normal measure such that all the measure zero sets are metrizable. Definition 1.1 follows the tradition in functional analysis originated in Dixmier’s paper published in 1951. Recall that a normal measure on a compact space \( K \) defines a so-called normal functional on the Banach lattice \( C(K) \), one which is order continuous, see \cite{7}, section 4.7 for details. Normal measures are sometimes called hyperdiffuse, see e.g. the survey paper by Flachsmeyer and Lotz \cite{12}. We should warn the reader that Dixmier’s notion of a normal measure has little to do with the same term used widely in set theory, in the context of real-valued measurable cardinals.

Kunen’s compactum mentioned above is defined as an inverse system of Cantor sets so it is zero-dimensional. Recall that a typical example of a normal measure is the natural measure defined on the Stone space of the measure algebra \( \mathfrak{A} \) of the Lebesgue measure \( \lambda \) on \([0, 1] \). Since the algebra \( \mathfrak{A} \) is complete, its Stone space is extremally disconnected.

A couple of years ago, Garth Dales brought our attention to an old question, if a compact connected space can carry a normal probability measure, see \cite{11} and \cite{12}. The problem was partially motivated by a result of Fishel and Papert \cite{11} who proved that a compact space carries no nonzero normal probability measure whenever \( K \) is locally connected. Answering the question, we proved that there is a compact connected space of weight \( c \) admitting a normal probability measure, using some Kunen-like inductive construction of length \( c \) carried out in ZFC. The preprint \cite{25} remained unpublished but the result was included in the monograph \cite{7} (as Theorem 4.7.24).

Kunen’s ideas from \cite{18} were further modified and developed in a number of articles, see e.g. Džamonja and Kunen \cite{9, 10}, Kunen and van Mill \cite{20} Brandsma and van Mill \cite{4, 5}, Kunen \cite{19}, Borodulin-Nadzieja and Plebanek \cite{3}, Dales and Plebanek \cite{8}. In the present note we come back to the method of \cite{18} once again to discuss the following result (see the next section for all the unexplained terminology and notation)

**Theorem 1.2.** Assuming \( \text{cf}(\mathcal{N}) = \omega_1 \), there is a compact space \( K \) such that

(i) \( K \) is connected;
(ii) \( K \) is the support of a normal probability measure \( \mu \);
(iii) every Borel set \( B \subseteq K \) which is \( \mu \)-null is metrizable;
(iv) \( K \) is a Corson compact \( L \)-space.

Moreover, one can assure that the measure \( \mu \) in question is either of countable Maharam type or of type \( \omega_1 \).
The fact that the assumption of CH can be relaxed to the present one, on the cofinality of the null ideal, was already noted by Kunen and van Mill [20, Theorem 1.2]. It is worth recalling that hardly any feature of the space $K$ as in the theorem above is possible in the usual set theory. Indeed, under Martin’s axiom and the negation of CH

- there are no compact $L$-spaces (Juhász [15]);
- every measure on a Corson compact space has a metrizable support, (see [22] and [20]);
- no first-countable compact space admits a normal probability measure (Zindulka [29]).

The fact that, depending on the shape of successor steps of the construction, the measure $\mu$ as in 1.2 can be either of type $\omega$ or $\omega_1$ was already noted in [18]. The additional property that the resulting space is connected is the main point here. As we shall see it comes as a result of a rather straightforward modification of Kunen’s argument. However, we give below a self-contained proof of Theorem 1.2 because we later analyse further properties of the space $K$ resulting from our construction and show that $K$ gives rise to an interesting Banach space $C(K)$; see the final section.

I wish to thank H. Garth Dales for our discussion concerning the role of normal measures in Banach space theory.

2. Preliminaries

2.1. Compact spaces and measures. As we have already mentioned, by a measure $\mu$ on a compact space $K$ we always mean a finite Radon measure, that is a measure defined on the Borel $\sigma$-algebra $\text{Bor}(K)$ which satisfies, for every $B \in \text{Bor}(K)$, the regularity condition

$$\mu(B) = \sup \{ \mu(F) : \overline{F} = F \subseteq B \}.$$ 

We say that the measure $\mu$ is strictly positive on $K$ or that $K$ is the support of $\mu$ if $\mu(U) > 0$ for every nonempty open set $U \subseteq K$.

Recall that if $\mu$ is a measure on $K$ (vanishing on points) then the Maharam type of $\mu$ can be defined as the density character of the corresponding Banach space $L_1(\mu)$ or, equivalently, the density character of the underlying measure algebra with respect to the Fréchet-Nikodym metric. Thus $\mu$ is of type $\omega$ if there is a countable family $C \subseteq \text{Bor}(K)$ such that

$$\inf \{ \mu(B \triangle C) : C \in C \} = 0,$$

for every $B \in \text{Bor}(K)$.

Let $\lambda$ be the Lebesgue measure on $[0,1]$ and let $\mathcal{N}$ denote the $\sigma$-ideal of Lebesgue null sets. Then $\text{cf}(\mathcal{N})$ stands for its cofinality, so $\text{cf}(\mathcal{N}) = \omega_1$ amounts to saying that there is a family $\{ N_\xi : \xi < \omega_1 \} \subseteq \mathcal{N}$ such that for every $N \in \mathcal{N}$ there is $\xi < \omega_1$ with $N \subseteq N_\xi$. Recall that basic cardinal invariants of Radon measures are determined by their corresponding measure algebras. In particular, if $\mu$ is a measure on a metrizable
compact space $K$ and $\mu$ vanishes on singletons then the cofinality of its null ideal is equal to $\text{cf}(\mathcal{N})$, see Fremlin [13].

We shall use the following consequence of a result due to Cichoń, Kamburelis and Pawlikowski [6] (see also [13, Proposition 6.9]).

**Theorem 2.1.** If $\text{cf}(\mathcal{N}) = \omega_1$ then for every measure $\mu$ on a compact metrizable space $K$ there is a family $\{F_\xi : \xi < \omega_1\}$ of closed nonnegligible sets such that every $B \in \text{Bor}(K)$ with $\mu(B) > 0$ contains $F_\xi$ for some $\xi < \omega$.

A zero set in a topological space $X$ is one of the form $f^{-1}[\{0\}]$ where the function $f : X \to \mathbb{R}$ is continuous. It is easy to check that a subset of a compact space is a zero set if and only if it is closed and $G_\delta$. We write $\mathcal{Z}(X)$ for the family of all closed $G_\delta$ subsets of $X$. Note that if a compact space $K$ is an $L$-space then it is perfectly normal, i.e. every closed subset $F$ of it is $G_\delta$ so $F \in \mathcal{Z}(K)$.

**Lemma 2.2.** Let $K$ be a compact space, and suppose that $\mu$ is a strictly positive measure on $K$ such that $\mu(Z) = 0$ for every $Z \in \mathcal{Z}(K)$ with empty interior. Then $\mu$ is a normal measure.

**Proof.** The assertion follows from the following observation.

**Claim.** Every closed set $F \subseteq K$ with empty interior is contained in some $Z \in \mathcal{Z}(K)$ with empty interior.

To verify the claim, consider a maximal family $\mathcal{F}$ of continuous functions $K \to [0, 1]$ such that $f|F = 0$ for $f \in \mathcal{F}$ and $f \cdot g = 0$ whenever $f, g \in \mathcal{F}$, $f \neq g$. Then $\mathcal{F}$ is necessarily countable because $K$, being the support of a measure, satisfies the countable chain condition. Write $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ and let $f = \sum_n 2^{-n}f_n$ and $Z = f^{-1}[0]$. Then the function $f$ is continuous so $Z \in \mathcal{Z}(K)$. We have $Z \supseteq F$ and the interior of $Z$ must be empty by the maximality of $\mathcal{F}$. □

If $f : K \to L$ is a continuous map and $\mu$ is a measure on $K$ then the image measure $f[\mu]$ on $L$ is defined by

$$f[\mu](B) = \mu(f^{-1}[B])$$

for every Borel set $B \subseteq L$.

### 2.2. Inverse systems.

We shall consider inverse systems of compact (metrizable) spaces with probability measures of the form

$$\langle K_\alpha, \mu_\alpha, \pi_\beta^\alpha : \beta < \alpha < \omega_1 \rangle,$$

where for all $\gamma < \beta < \alpha < \omega_1$

- **2.2(a)** $K_\alpha$ is a compact space and $\mu_\alpha$ is a probability measure on $K_\alpha$;
- **2.2(b)** $\pi_\beta^\alpha : K_\alpha \to K_\beta$ is a continuous surjection;
- **2.2(c)** $\pi_\beta^\beta \circ \pi_\beta^\alpha = \pi_\gamma^\alpha$;
- **2.2(d)** $\pi_\beta^\alpha[\mu_\alpha] = \mu_\beta$. 

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The following summarises basic facts on such inverse systems, see [7, Proposition 4.1.15] or [22, 0.33].

**Theorem 2.3.** Let $K$ be the limit of the system satisfying (2.2(a)–(d)) with uniquely defined continuous surjections $\pi_\alpha : K \to K_\alpha$ for $\alpha < \omega_1$.

(a) $K$ is a compact space and $K$ is connected whenever all the space $K_\alpha$ are connected.

(b) There is the unique probability measure $\mu$ on $K$ such that $\pi_\alpha[\mu] = \mu_\alpha$ for $\alpha < \omega_1$.

(c) If every $\mu_\alpha$ is strictly positive then $\mu$ is strictly positive.

We also need the following standard fact.

**Lemma 2.4.** Let $K$ be the limit of the system satisfying (2.2(a)–(c)). Then every continuous function $f : K \to \mathbb{R}$ can be written as $f = g \circ \pi_\alpha$ for some $\alpha < \omega_1$ and a continuous function $g : K_\alpha \to \mathbb{R}$.

Consequently, for every zero set $Z \subseteq K$ there are $\alpha < \omega_1$ and $Z' \in \mathcal{Z}(K_\alpha)$ such that $Z = \pi_\alpha^{-1}[Z'].$

**Proof.** The first statement is an immediate consequence of the Stone-Weierstrass theorem; the second statement clearly follows.

2.3. **Corson compact spaces.** A compact space $K$ is Corson compact if, for some $\kappa$, it can be embedded into the space $\Sigma(\mathbb{R}^\kappa)$ of those $x \in \mathbb{R}^\kappa$ which have only countably many nonzero coordinates. The class of Corson compacta is tightly connected to several aspects of Banach space theory, see [22].

3. **A connected $L$-space**

The following lemma describes the essential part of the inductive construction leading to (1.2).

**Lemma 3.1.** Let $K$ be a metrizable compact connected space, and let $\mu$ be a strictly positive measure on $K$. If $F \subseteq K$ is a closed set with $\mu(F) > 0$, then there are a compact connected space $\hat{K} \subseteq K \times [0,1]$ and two strictly positive measures $\nu_1, \nu_2$ on $\hat{K}$ such that, writing $\pi : \hat{K} \to K$ for the projection, the following are satisfied

(i) $\text{int}((\pi^{-1}[F])) \neq \emptyset$;

(ii) $\pi[\nu_1] = \pi[\nu_2] = \mu$;

(iii) there is $B \in \text{Bor}(\hat{K})$ such that $\inf\{\nu_1(B \triangle \pi^{-1}[A]) : A \in \text{Bor}(K)\} > 0$;

(iv) $\inf\{\nu_2(B \triangle \pi^{-1}[A]) : A \in \text{Bor}(K)\} = 0$ for every $B \in \text{Bor}(\hat{K})$.

**Proof.** Let $F_0$ be the support of $\mu$ restricted to $F$, that is

$F_0 = F \setminus \bigcup \{U : U \text{ open and } \mu(F \cap U) = 0\}$.

Let $\hat{K} = \{(x,t) \in K \times [0,1] : x \in F_0 \text{ or } t = 0\}$. Then $\hat{K}$ is clearly a compact connected subspace of $K \times [0,1]$. Moreover, the set $\pi^{-1}[F]$ contains $F_0 \times [0,1]$, a set with non-empty interior so (i) is granted.
We define the required measure $\nu_1$ on $\widehat{K}$ by setting

$$\nu_1(B) = \mu \otimes \lambda(B),$$

for Borel sets $B \subseteq F_0 \times [0, 1]$, where $\lambda$ is the Lebesgue measure on $[0, 1]$. On the remaining part $(K \setminus F_0) \times \{0\}$ we just copy the measure $\mu$ from $K \setminus F_0$. It is easy to verify $\pi[\nu_1] = \mu$ and that $(iii)$ holds for $B = F_0 \times [1/2, 1]$.

To define $\nu_2$ we choose a countable base $\mathcal{U}$ of the space $F_0$ and fix an enumeration $\{(U_n, q_n) : n \in \omega\}$ of all pairs $(U, q)$ where $\emptyset \neq U \in \mathcal{U}$ and $q \in (0, 1) \cap \mathbb{Q}$. Then we inductively construct a pairwise disjoint sequence of closed sets $L_n \subseteq F_0$ such that $\mu(L_n \cap U_n) > 0$ for every $n$. Note that it is easy to carry out such a construction under the inductive assumption that ever $L_n$ has empty interior in the space $F_0$.

Now we take a function $h : F_0 \to \widehat{K}$ defined by $h(x) = (x, q_n)$ for $x \in L_n$ (and $h(x) = (x, 0)$ otherwise). Set $\nu_2$ to be the obvious copy of $\mu$ outside $F_0 \times [0, 1]$ and put $\nu_2(B) = \mu(h^{-1}[B])$ for $B \subseteq F_0 \times [0, 1]$. Then $\nu_2$ is strictly positive (by the way $h$ is defined) and $(iv)$ holds since we can take $A = h^{-1}[B]$ there. □

**Remark 3.2.** We shall later call on the following additional property of $\pi : \widehat{K} \to K$ defined as in the proof above. Namely, $\pi$ is monotone, that is $\pi^{-1}[F]$ is connected for every connected $F \subseteq K$.

We shall also refer to the following. Take a continuous function $r$ on $[0, 1]$ with $r(0) = 0 = \int_0^1 r(t) \, dt$. If we write $p$ for the projection $\widehat{K} \to [0, 1]$ onto the second coordinate then for any continuous function $f$ on $K$ we have

$$\int_{\widehat{K}} (r \circ p) \cdot (f \circ \pi) \, d\nu_1 = \int_{F_0 \times [0, 1]} (r \circ p) \cdot (f \circ \pi) \, d\mu \otimes \lambda = 0,$$

by the Fubini theorem.

**Construction 3.3.** Let $K_0 = [0, 1]$ and let $\mu_0$ be the Lebesgue measure on $[0, 1]$. Construct an inverse system of compact metrizable spaces with measures as in 2.2 with the following bookkeeping.

Given the space $K_\alpha$ with the measure $\mu_\alpha$, using $\text{cf}(\mathcal{N}) = \omega_1$ fix an enumeration $\{N_\xi^{\alpha} : \xi < \omega_1\}$ of family of $\nu_\alpha$-null subsets of $K_\alpha$ that is cofinal the corresponding null ideal. Moreover, using Theorem 2.1 fix a family $\{F_\xi^{\alpha} : \xi < \omega_1\}$ of closed subsets of $K_\alpha$ such that $\mu_\alpha(F_\xi^{\alpha}) > 0$ for every $\xi$ and every $B \in \text{Bor}(K_\alpha)$ of positive measure contains $F_\xi^{\alpha}$ for some $\xi < \omega_1$.

Fix also a bijection $\phi : \omega_1 \times \omega_1 \to \omega_1$ such that $\phi(\beta, \xi) = \alpha$ implies $\beta \leq \alpha$ — $\phi$ will tell us that at step $\alpha$ we should take care of the set $F_\xi^{\beta}$, where $\phi(\beta, \xi) = \alpha$.

There is nothing to do at limit step $\gamma$, we simply define $K_\gamma$ to be the inverse limit of the spaces defined so far. Given $K_\alpha$ and the measure $\mu_\alpha$, we construct $K_{\alpha+1}$ as follows. The set

$$N = \bigcup_{\xi, \beta < \alpha} (\pi_\beta^{\alpha})^{-1}[N_\xi^\beta],$$

\text{to be the obvious copy of } $\mu$ outside $F_0 \times [0, 1]$. Then $\nu_2$ is strictly positive (by the way $h$ is defined) and (iv) holds since we can take $A = h^{-1}[B]$ there. □

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We shall also refer to the following. Take a continuous function $r$ on $[0, 1]$ with $r(0) = 0 = \int_0^1 r(t) \, dt$. If we write $p$ for the projection $\widehat{K} \to [0, 1]$ onto the second coordinate then for any continuous function $f$ on $K$ we have

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Given the space $K_\alpha$ with the measure $\mu_\alpha$, using $\text{cf}(\mathcal{N}) = \omega_1$ fix an enumeration $\{N_\xi^{\alpha} : \xi < \omega_1\}$ of family of $\nu_\alpha$-null subsets of $K_\alpha$ that is cofinal the corresponding null ideal. Moreover, using Theorem 2.1 fix a family $\{F_\xi^{\alpha} : \xi < \omega_1\}$ of closed subsets of $K_\alpha$ such that $\mu_\alpha(F_\xi^{\alpha}) > 0$ for every $\xi$ and every $B \in \text{Bor}(K_\alpha)$ of positive measure contains $F_\xi^{\alpha}$ for some $\xi < \omega_1$.

Fix also a bijection $\phi : \omega_1 \times \omega_1 \to \omega_1$ such that $\phi(\beta, \xi) = \alpha$ implies $\beta \leq \alpha$ — $\phi$ will tell us that at step $\alpha$ we should take care of the set $F_\xi^{\beta}$, where $\phi(\beta, \xi) = \alpha$.

There is nothing to do at limit step $\gamma$, we simply define $K_\gamma$ to be the inverse limit of the spaces defined so far. Given $K_\alpha$ and the measure $\mu_\alpha$, we construct $K_{\alpha+1}$ as follows. The set

$$N = \bigcup_{\xi, \beta < \alpha} (\pi_\beta^{\alpha})^{-1}[N_\xi^\beta],$$

\text{to be the obvious copy of } $\mu$ outside $F_0 \times [0, 1]$. Then $\nu_2$ is strictly positive (by the way $h$ is defined) and (iv) holds since we can take $A = h^{-1}[B]$ there. □
is a countable union of \( \mu_\alpha \)-null sets so \( \mu_\alpha(N) = 0 \). For \( \beta \) and \( \xi \) such that \( \phi(\beta, \xi) = \alpha \), we consider the set

\[
H = (\pi_\beta^\alpha)^{-1}[F_\xi^\beta],
\]

which satisfies \( \mu_\alpha(H) > 0 \) so there is a closed set \( F \subseteq H \setminus N \) with \( \mu_\alpha(F) > 0 \). Using Lemma 3.1 for such \( F \subseteq K_\alpha \) we put \( K_{\alpha+1} = \hat{K}_\alpha \) and \( \mu_{\alpha+1} = \nu_1 \).

**Proof.** (of Theorem 1.2) We shall prove that the limit \( K \) of the inverse system as in 3.3 equipped with the measure \( \mu \) of type \( \omega_1 \) and both are as required. We already know that \( \mu \) is strictly positive on \( K \).

To prove that \( \mu \) is a normal measure it is sufficient, by Lemma 2.2, to check that \( \mu(Z) > 0 \) implies that \( Z \) has nonempty interior for every zero set \( Z \). But such a set \( Z \) is of the form \( Z = \pi_\beta^\alpha[Z'] \) for some \( \beta < \omega_1 \) and a closed set \( Z' \subseteq K_\beta \) (see Lemma 2.4). If \( \mu(Z) > 0 \) then \( \mu_\beta(Z') > 0 \) so there is \( \xi < \omega_1 \) such that \( F_\xi^\beta \subseteq Z' \). Then at step \( \alpha = \varphi(\beta, \xi) \) we took care to assure that the set

\[
(\pi_\beta^{\alpha+1})^{-1}[(\pi_\beta^\alpha)^{-1}F_\xi^\beta] = (\pi_\beta^{\alpha+1})^{-1}[F_\xi^\beta]
\]
gets nonempty interior. Since \( \pi_\beta = \pi_\beta^{\alpha+1} \circ \pi_\alpha+1 \) and \( Z = \pi_\beta^{-1}[Z'] \), it follows that \( Z \) has nonempty interior too.

Once we know that \( \mu \) is a normal strictly positive measure on \( K \) (clearly, \( \mu \) vanishes on singletons), we conclude that \( K \) is an \( L \)-space using the argument from the introduction. Recall that this means that every closed subset of \( K \) is a zero set.

Take a Borel set \( B \subseteq K \) such that \( \mu(B) = 0 \); then \( \mu(\overline{B}) = 0 \) (by normality of \( \mu \)) and \( Z = \overline{B} \in Z(K) \). It follows that \( Z = \pi_\beta^{-1}[Z'] \) for some \( \beta \) and \( Z' \subseteq N_\xi^\beta \) for some \( \xi < \omega \). The set \( Z' \) remains unsplit from step \( \alpha = \varphi(\beta, \xi) \) so \( \pi_\alpha \) is one-to-one on \( Z \), i.e. \( Z \) is homeomorphic to the metric compactum \( (\pi_\beta^\alpha)^{-1}[Z'] \).

To see that the space \( K \) is Corson compact note that if \( x \in K \) then \( t = \pi_0(x) \in [0, 1] \) belongs to some \( N_\xi^\beta \) and it follows that all coordinates of \( x \) above \( \varphi(0, \xi) \) are zero.

The fact that we used the measure \( \nu_1 \) from Lemma 3.1 in the construction implies that \( \mu \) is indeed of type \( \omega_1 \), as 3.1(iii) implies that no countable family of Borel subsets of \( K \) can form a \( \Delta \)-dense family in \( Bor(K) \).

To get another space \( K \) supporting a normal measure of countable type we carry out the whole of 3.3 replacing \( \nu_1 \) by \( \nu_2 \) at each successor step when referring to Lemma 3.1. Then Lemma 3.1(iv) guarantees that the resulting measure \( \mu \) has type \( \omega \) – its measure algebra is be the same as that of the Lebesgue measure.

We can now point out an interesting property of our connected version of Kunen’s \( L \)-space.

**Theorem 3.4.** Let \( K \) be the compact space resulting from Construction 3.3. If a compact zero-dimensional space \( Y \) is a continuous image of a closed subspace \( Z \) of \( K \) then \( Y \) is metrizable.
Proof. Let \( g : Z \rightarrow Y \) be a continuous surjection from a closed set \( Z \subseteq K \) onto a zero-dimensional space \( Y \). Since \( K \) is perfectly normal, \( Z \) is a zero set in \( K \) so \( Z = \pi^{-1}_\alpha[Z'] \) for some \( \alpha < \omega_1 \) and a closed set \( Z' \subseteq K_\alpha \). Using Remark 3.2 we note the following.

CLAIM. The set \( \pi^{-1}_\alpha[t] \) is connected for every \( t \in Z' \).

As \( Y \) is zero-dimensional, Claim means that \( g \) is constant on every coset \( \pi^{-1}_\alpha[t] \).

Thus, we can define a mapping \( g' : Z' \rightarrow Y \) by \( g'(t) = g(x) \) where \( \pi_\alpha(x) = t \). Then \( g = g' \circ \pi_\alpha \) and this implies that \( g' \) is a continuous surjection from \( Z' \) onto \( Y \). As \( Z' \) is compact metrizable, so is \( Y \).

\( \square \)

Note that Theorem 3.4 says in particular that the space \( K \) in question contains no nonmetrizable zero-dimensional subspaces. Recall that such a feature is somewhat delicate; only quite recently Koszmider [17] constructed a ZFC example of a nonmetrizable compact space having that property. Assuming the existence of a Lusin set, Marciszewski [21] gives another example of that kind which is an Eberlein compact space.

4. On the Banach space \( C(K) \)

Given a compact space \( K \), \( C(K) \) denotes the Banach space of continuous real-valued functions on \( K \) with the usual supremum norm. Recall that if two such Banach spaces \( C(K) \) and \( C(L) \) are isometric then by the classical Banach-Stone theorem the underlying compacta \( K \) and \( L \) must be homeomorphic. The situation changes dramatically if we ask for which pairs \( K \) and \( L \) the Banach spaces \( C(K) \) and \( C(L) \) are merely isomorphic, that is, there is a linear surjection \( T : C(K) \rightarrow C(L) \) such that \( m \cdot \|g\| \leq \|Tg\| \leq M \cdot \|g\| \) for some \( m, M > 0 \) and every \( g \in C(K) \); see [26] for further information and basic references to the subject.

By the classical Miljutin theorem, \( C([0, 1]) \) and \( C(2^\omega) \) are isomorphic as Banach spaces. It was along standing problem if there is a zero-dimensional compact space \( L \) such that \( C(K) \) and \( C(L) \) are isomorphic as Banach spaces. The first counterexample was obtained by Koszmider [16] who used his involved technology of producing Banach spaces of continuous functions with ‘few operators’. Another counterexample was presented by Avilés and Koszmider [11], as a by-product of their solution to Namioka’s problem on the class of Radon-Nikodym compacta. We show below that our connected version of Kunen’s \( L \)-space provides the third example of that kind.

Theorem 4.3 given below will be derived from our main result from [25]. Recall that by a \( \pi \)-base of a topological space we mean a family \( \mathcal{V} \) of nonempty open sets having the property that every nonempty open set contains some \( V \in \mathcal{V} \).

Theorem 4.1. Let \( K \) and \( L \) be compact spaces such that \( C(K) \) is isomorphic to \( C(L) \). Then \( K \) has a \( \pi \)-base \( \mathcal{V} \) such that for every \( V \in \mathcal{V} \), \( \overline{V} \) is a continuous image of some compact subspace of \( L \).

At this point, we have to refer to 3.3 once again to check the following.
Lemma 4.2. Let $K$ be the compact space resulting from Construction 3.3. Then for every sequence of probability measures $\nu_n$ on $K$ there is a nonzero continuous function $f : K \to \mathbb{R}$ such that $\int_K f \, d\nu_n = 0$ for every $n$.

Proof. Take any nonzero function $r : [0, 1] \to [0, 1]$ such that $r(0) = 0$ and $\int_0^1 r(t) \, dt = 0$ and consider the family of functions $g_\alpha = r \circ p_\alpha \circ \pi_{\alpha+1}$ on $K$, where, for $\alpha < \omega_1$, $p_\alpha : K_{\alpha+1} \to [0, 1]$ denotes the restriction of the projection onto the second coordinate. It is enough to check that for every measure $\nu$ on $K$ we have $\int_K g_\alpha \, d\nu = 0$ for all but countably many $\alpha < \omega_1$.

If $\nu$ is singular with respect to $\mu$ then $\nu$ is concentrated on a metrizable closed set $Z \subseteq K$. Since $K$ is perfectly normal, $Z$ is a zero set in $K$ so for some $\beta < \omega_1$ we have $Z = \pi^{-1}_\beta[Z']$ and $Z' \subseteq N_\xi$ for some $\xi < \omega_1$. It follows that for any $\alpha \geq \varphi(\beta, \xi)$ we have $g_\alpha = 0 \nu$-almost everywhere so $\int_K g_\alpha \, d\nu = 0$.

On the other hand, if $\nu$ is a measure on $K$ that is absolutely continuous with respect to $\mu$ then $\nu(\cdot) = \int_{(\cdot)} h \, d\mu$ for some measurable $h : K \to \mathbb{R}$. Then there is a sequence of continuous functions $f_j : K \to \mathbb{R}$ converging to $h \mu$-almost everywhere. Hence, in such a case, it remains to check that $\int g_\alpha \cdot f \, d\mu = 0$ for any continuous function $f$ and all but countably many $\alpha$. This follows from the fact that we have $f = f' \circ \pi_\beta$ for some $\beta < \omega$ and then $\int g_\alpha \cdot f \, d\mu = 0$ for any $\alpha \geq \beta$ by stochastic independence of the function $f$ and $g_\alpha$, see Remark 3.2.

The general case follows, as every measure $\nu$ on $K$ can be decomposed as $\nu = \nu' + \nu''$, where $\nu'$ is singular and $\nu''$ is absolutely continuous with respect to $\mu$. □

We are now ready for the main result of this section.

Theorem 4.3. Assuming $\text{cf}(\mathcal{N}) = \omega_1$, there is a Corson compact space $K$ such that the Banach space $C(K)$ is isomorphic to no space of the form $C(L)$ with $L$ zero-dimensional.

Proof. We shall prove that the space from Theorem 1.2 which is an inverse limit of the system 3.3 supporting the measure $\mu$ of uncountable type is as required.

Suppose that $L$ is a compact totally disconnected space such that $C(L)$ is isomorphic to $C(K)$. Our space $K$ is $\text{ccc}$ (as it supports a finite measure). Rosenthal’s result from [27] states that this is reflected by the isomorphic structure of $C(K)$, that property of $K$ is equivalent to saying that every weakly compact subset of $C(K)$ is separable. It follows that $L$ must be $\text{ccc}$ too.

By Theorem 3.1 $L$ has a $\pi$-base consisting of open sets $V$ such that $\overline{V}$ is a continuous image of a closed subspace of $K$. By Theorem 3.4 such a set $\overline{V}$ is then metrizable so, in particular, $V$ is separable. It follows that $L$ is a $\text{ccc}$ space having a $\pi$-base made of open separable subspaces and this implies that $L$ is separable itself.

On the other hand, separability of $L$ is in contradiction with Lemma 1.2. Indeed, if $\{y_n : n < \omega\}$ is a dense subset of $L$ and $T : C(K) \to C(L)$ is an isomorphism then, for
every \( n \),

\[
C(K) \ni f \rightarrow Tf(y_n),
\]
is a continuous functional on \( C(K) \) which is represented by a signed measure \( \nu_n \) on \( K \). Writing every \( \nu_n \) as \( \nu_n = \nu_n^+ - \nu_n^- \) (as a difference of two nonnegative measures), we get a countable family \( \{ \nu_n^+, \nu_n^- : n < \omega \} \) of nonnegative measures separating elements of \( C(K) \).

One can check that if we let \( K \) to be connected \( L \)-space supporting a measure \( \mu \) of countable type then \( C(K)^* \) has a weak* separable dual unit ball; in particular, Lemma 4.2 does not hold. We do not know if \( C(K) \) satisfies the assertion of Theorem 4.3. Let us recall that Talagrand's construction mentioned in the introductory section gives, under CH, a tricky space \( K \) for which \( C(K)^* \) is weak* separable while the dual unit ball in \( C(K)^* \) is not. Such an example was later constructed in the usual set theory, see [2].

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