The higher order rogue wave solutions of the Gerdjikov–Ivanov equation

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Abstract

We construct higher order rogue wave solutions for the Gerdjikov–Ivanov equation explicitly in terms of a determinant expression. The dynamics of both soliton and non-soliton solutions is discussed. A family of solutions with distinct structures, which are new to the Gerdjikov–Ivanov equation, are presented.

Keywords: Darboux transformation, higher order rogue wave, Gerdjikov–Ivanov equation

(Some figures may appear in color only in the online journal)

1. Introduction

Rogue waves are a type of natural first discovered in deep ocean waves. Later, similar phenomena were observed in other physical breaches, such as optical physics, plasmas, capillary waves, etc [1–5]. There is a consensus that these rogue waves are the result of modulation instability waves. Breather solutions usually arise from the instability of small amplitude perturbations that may grow in size to disastrous proportions [6–9]. Therefore, in a mathematical understanding, rogue waves can be treated as a limit case of a Ma soliton when the space period tends to infinity or of the Akhmediev breather as the time period approaches infinity [10]. Due to both the theoretical framework and applications to reality, it is imperative to study rogue waves further, especially higher order rogue waves of different models.

The Peregrine soliton, which is located in the time–space plane, is one of the formal ways in which to explain this surprise phenomenon mathematically [11]. The solution appears from a non-zero constant and disappears into the constant background as time approaches infinity, but it develops a localized hump with peak amplitude three times that of average waves in the intermediate time. The subject of higher order rogue waves is now under intense scrutiny using different approaches such as iterated Darboux transformation [12, 13], the algebra-geometric method [14] and generalized Darboux transformation [15]. Recently, the expression of the first order rogue wave solution and the figure of the second order rogue wave for the Gerdjikov–Ivanov (GI) equation were provided by the third and fifth authors of this paper [16]. However, the higher order rogue wave solutions for this equation have not been studied. The main aim of this paper is to discuss higher order rogue wave solutions and describe their different structures.

The nonlinear Schrödinger equation is one of the most important equations in physics and it can be derived from the Ablowitz–Kaup–Newell–Segur system [17, 18]. Considering higher order nonlinear effects, the derivative nonlinear Schrödinger (DNLS) equations with a polynomial spectral problem of arbitrary order [19] are regarded as models in a wide variety of fields such as weakly nonlinear dispersive water waves [20], nonlinear optics fibers [21–23], quantum field theory [24] and plasmas [25]. The DNLS equations have three generic deformations, the DNLSI equation [26]

\[ iq_t - q_{xx} + i(q^2q^*)_x = 0, \]  

the DNLSII equation [27]

\[ iq_t + q_{xx} + iqq^*q_x = 0 \]  

and the DNLSIII equation or the GI equation [28]

\[ iq_t + q_{xx} - iq^2q^*_x + \frac{1}{2}q^3q^{*2} = 0. \]

Darboux transformation has been proved in many circumstances to be a powerful method for obtaining soliton solutions [29, 30], breather solutions and rational solutions.
The Darboux transformation and its determinant expression for the GI equations are provided in [16, 31]. However, there is not a straightforward extension to construct higher order rogue wave solutions at the same eigenvalue. In this paper, we use a limit technique with respect to degenerate eigenvalues and Taylor expansion in Darboux transformation [15, 32, 33]. Based on this explicit method, we are further able to discuss the structure of solutions for both solitons and rogue waves.

This paper is organized as follows: in section 2, we review the general n-fold Darboux transformation for the GI equation. In section 3, explicit solutions, such as soliton, breather, position solutions and higher order rogue waves with two parameters $D_1$ and $D_2$, are constructed. By choosing different values of $D_1$ and $D_2$, we show four basic patterns, fundamental structure, triangular structure, modified triangular structure and ring structure, and display their dynamical evolution in section 4. The discussion and conclusion are located in the final section.

2. Darboux transformation for the GI equation

In this section, we start with the following Lax pair to construct the Darboux transformation. Considering the spectral problem

$$\begin{align*}
\partial_x \psi &= (J \lambda^2 + Q_1(\lambda + Q_0)) \psi = U \psi, \\
\partial_t \psi &= (2J \lambda^4 + V_3 \lambda^3 + V_2 \lambda^2 + V_1 \lambda + V_0) \psi = V \psi,
\end{align*}$$

(4)

where

$$\psi = \begin{pmatrix} \phi \\ \varphi \end{pmatrix}, \quad \begin{pmatrix} \phi(x, t, \lambda) \\ \varphi(x, t, \lambda) \end{pmatrix}, \quad J = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \quad Q_0 = \begin{pmatrix} -\frac{1}{2}iqr & 0 \\ 0 & \frac{1}{2}iqr \end{pmatrix},$$

$$V_1 = 2Q_1, \quad V_2 = J qr, \quad V_1 = \begin{pmatrix} 0 & i q_t \\ -i r_t & 0 \end{pmatrix},$$

$$V_0 = \begin{pmatrix} \frac{1}{2}(r q_x - q r_x) + \frac{i}{2} q_x^2 r^2 + \frac{i}{2} q_y^2 r^2 & 0 \\ 0 & -\frac{1}{2}(r q_x - q r_x) - \frac{i}{2} q_y^2 r^2 \end{pmatrix}.\]$$

Here $\lambda \in \mathbb{C}$, $\psi$ is the eigenfunction of (4) corresponding to the eigenvalue $\lambda$. By the condition $U_t - V_x + [U, V] = 0$, we get

$$\frac{i q_t + q_x + i q_x^2 r_x + \frac{1}{2} q_y^2 r_x^2}{i r_t - r_x + i r_x^2 q_t + \frac{1}{2} q_y^2 r_x^2} = 0,$$

(5)

This system admits the reduction $r = -q^*$ and (5) becomes the GI equation (3).

From gauge transformation, we can construct new solutions from initial data, i.e. if there exist some non-singular $T$, such that

$$\begin{align*}
U^{[1]} &= (T_x + T U) T^{-1}, \\
V^{[1]} &= (T_x + T V) T^{-1},
\end{align*}$$

(6)

where $U^{[1]}$ and $V^{[1]}$ have the same form as $U$ and $V$ with $q$ and $r$ replaced by certain $q^{[1]}$ and $r^{[1]}$. Therefore, it is crucial to find an algebraic formula for $T$ instead of the (6).

2.1. N-fold Darboux transformation for the GI system

The n-fold Darboux transformation for $q^{[n]}$ and $r^{[n]}$ of the GI system has been given in [16]. Since we need this result for our paper, we cite the main theorem as follows.

Theorem 1. Let $\Psi_i = (\phi_i \ 0)$ $(i = 1, 2, \ldots, n)$ be distinct solutions related to $\lambda_i$ of the spectral problem, then $q^{[n]}, r^{[n]}$ given by the following formulae are new solutions of the GI system:

$$\begin{align*}
q^{[n]} &= q + 2i \frac{\Omega_{11}}{\Omega_{12}}, \\
r^{[n]} &= r - 2i \frac{\Omega_{21}}{\Omega_{22}}.
\end{align*}$$

(7)

Here, (1) for $n = 2k$ \begin{align*}
\Omega_{11} &= \begin{pmatrix} \phi_1 & \lambda_1 \psi_1 & \ldots & \lambda_n \psi_1 & \Psi_1 \\ \phi_2 & \lambda_1 \psi_2 & \ldots & \lambda_n \psi_2 & \Psi_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi_n & \lambda_n \psi_n & \ldots & \lambda_n \psi_n & \Psi_n \end{pmatrix}, \\
\Omega_{12} &= \begin{pmatrix} \phi_1 & \lambda_1 \psi_1 & \ldots & \lambda_n \psi_1 & \Psi_1 \\ \phi_2 & \lambda_1 \psi_2 & \ldots & \lambda_n \psi_2 & \Psi_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi_n & \lambda_n \psi_n & \ldots & \lambda_n \psi_n & \Psi_n \end{pmatrix}.
\end{align*}$$

(8)

(2) for $n = 2k + 1$

\begin{align*}
\Omega_{11} &= \begin{pmatrix} \phi_1 & \lambda_1 \psi_1 & \ldots & \lambda_n \psi_1 & \Psi_1 \\ \phi_2 & \lambda_2 \psi_2 & \ldots & \lambda_n \psi_2 & \Psi_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi_n & \lambda_n \psi_n & \ldots & \lambda_n \psi_n & \Psi_n \end{pmatrix}, \\
\Omega_{12} &= \begin{pmatrix} \phi_1 & \lambda_1 \psi_1 & \ldots & \lambda_n \psi_1 & \Psi_1 \\ \phi_2 & \lambda_2 \psi_2 & \ldots & \lambda_n \psi_2 & \Psi_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi_n & \lambda_n \psi_n & \ldots & \lambda_n \psi_n & \Psi_n \end{pmatrix}.
\end{align*}$$

(9)
To solve the GI equation, we need to preserve the reduction condition \( r \). We start with the soliton solutions by (3.1). Solution with vanishing boundary condition.

Example:

\begin{itemize}
  \item \( n = 1 \), \[ \lambda_1^* = -\lambda_1, \lambda_1 = \lambda_2 \text{ and } \psi_1 = (\phi_1^* \psi_1), \] then \[ q^{[1]} = 2 \beta_1 e^{-2i\beta_1 t} - (x - 2\beta_1 t) \].

This is a plane wave with constant amplitude.

\item \( n = 2 \), \[ \begin{align*}
\lambda_2 &= \lambda_1^* = -\lambda_2, \\
\lambda_1 &= \alpha_1 + i\beta_1, \text{ and } \\
\psi_1 &= (\phi_1^* \psi_1), \psi_2 &= (-\phi_1^* \phi_2) \text{, then } \\
q^{[2]} &= \frac{4\beta_1 e^{-2i\beta_1 t} - 4\beta_1 (x - i\beta_1 t)}{1 - 4i\beta_1 x + 16i\beta_1^2 t} \end{align*} \]

\end{itemize}

By this choice, we can construct soliton and breather solutions of the GI equation. But to obtain positon and higher order rogue wave solutions, we need to modify the above Darboux transformation. At that time, the eigenfunctions and eigenvalues are no longer arbitrary and only condition (2) works. We will discuss these in detail in the next section.

### 3. Solutions of the GI equation

#### 3.1. Solution with vanishing boundary condition

We start with the soliton solutions by (7). Let \( q = 0 \), solutions of the spectral problem (4) with eigenvalues \( \lambda_k \) are solved as

\[
\begin{bmatrix}
\phi_1 \\
\phi_2 \\
\vdots \\
\phi_n
\end{bmatrix} = \begin{bmatrix}
\lambda_1 \phi_1 \\
\lambda_2 \phi_2 \\
\vdots \\
\lambda_n \phi_n
\end{bmatrix} = (\phi_1^* \lambda_1^* \phi_2^* \ldots \lambda_n^* \phi_n).
\]

To solve the GI equation, we need to preserve the reduction condition \( r = -q^* \), i.e. under \( n \) steps of Darboux transformation, \( \psi^{[n]} = -q^{[n]} \). Therefore, we can choose eigenfunctions \( \psi_k = (\phi_k^* \psi_k) \) as follows [16]:

\begin{enumerate}
  \item \( \lambda_k = -\lambda_k^* \) and \( \psi_k = (\phi_k^* \psi_k) = (\phi_k^* \phi_k) \),
  \item \( \lambda_{2k} = \lambda_{2k-1}^* \) and \( \psi_{2k} = (\phi_{2k}^* \psi_{2k}) = (\phi_{2k}^* \phi_{2k}) \).
\end{enumerate}

By this choice, we can construct soliton and breather solutions of the GI equation. But to obtain positon and higher order rogue wave solutions, we need to modify the above Darboux transformation. At that time, the eigenfunctions and eigenvalues are no longer arbitrary and only condition (2) works. We will discuss these in detail in the next section.

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By this choice, we can construct soliton and breather solutions of the GI equation. But to obtain positon and higher order rogue wave solutions, we need to modify the above Darboux transformation. At that time, the eigenfunctions and eigenvalues are no longer arbitrary and only condition (2) works. We will discuss these in detail in the next section.

#### Example:

- \( n = 1 \), \[ \lambda_1^* = -\lambda_1, \lambda_1 = \beta_1 \text{ and } \psi_1 = (\phi_1^* \psi_1), \] then \[ q^{[1]} = 2 \beta_1 e^{-2i\beta_1 t} - (x - 2\beta_1 t) \].

This is a plane wave with constant amplitude.

- \( n = 2 \), \[ \begin{align*}
\lambda_2 &= \lambda_1^* = -\lambda_2, \\
\lambda_1 &= \alpha_1 + i\beta_1, \text{ and } \\
\psi_1 &= (\phi_1^* \psi_1), \psi_2 &= (-\phi_1^* \phi_2) \text{, then } \\
q^{[2]} &= \frac{4\beta_1 e^{-2i\beta_1 t} - 4\beta_1 (x - i\beta_1 t)}{1 - 4i\beta_1 x + 16i\beta_1^2 t} \end{align*} \]

with

\[ F_1 = -2i (\alpha_1^2 x + 2\alpha_1^4 t - 12\alpha_1^2 \beta_1 t) - \beta_1 (x - 2t \beta_1^2 + x) \] (12)

It is a line soliton, and its trajectory is

\[ x = -4t \alpha_1 - \beta_1 \] (11)

on the \((x, t)\) plane. Let \( \alpha_1 \to 0 \) in (11), it leads to a rational traveling soliton solution

\[ q^{[2]} = \frac{4\beta_1 e^{-2i\beta_1 t} - 4\beta_1 (x - 2\beta_1 t)}{1 - 4i\beta_1 x + 16i\beta_1^2 t} \] (13)

with an arbitrary real constant \( \beta_1 \). Its trajectory is defined explicitly by

\[ x = 4\beta_1^2 t \] (14)

on the \((x, t)\) plane. The above two solutions are plotted in figures 1(a) and (b).

(2) Let \( \lambda_1 = -\lambda_1^* = \beta_1 \text{ and } \lambda_2 = -\lambda_2^* = \beta_2 \text{, then } \psi_1 = (\phi_1^* \psi_1) \text{ and } \psi_2 = (\phi_2^* \phi_2) \). We obtain a soliton solution

\[ q^{[2]} = \frac{\exp(-4\beta_1 e^{-2i\beta_1 t} - 4\beta_1 (x - 2\beta_1 t))}{1 - 4i\beta_1 x + 16i\beta_1^2 t} \] (15)
Its trajectory is defined explicitly as

\[ x = 2t (\beta_1^2 + \beta_2^2) \]

on the \((x, t)\) plane. It is plotted in figure 1(c).

- **n = 3.** Let \( \lambda_i = -\lambda_i^* \), \( i = 1, 2, 3 \), we obtain a one-soliton solution. While for \( \lambda_1 = -\lambda_1^* \), \( \lambda_3 = \lambda_3^* \), we obtain a quasi-periodic solution. Since the analytic expressions are clumsy and tedious, the profile of this quasi-periodic solution is shown in figure 2(a).

- **n = 4.** Under the different reduction conditions for eigenvalues, we obtain three kinds of solution. One of them is a soliton solution under the quasi-periodic background, and the others are second soliton solutions. A soliton solution under a quasi-periodic background is plotted in figure 2(b).

Note that the formula of \( q^{(n)} \) in (7) degenerates when the eigenvalues share the same value, we adopt the limit technique to deal with this obstacle. For convenience, we induce a definition in the following. Let \( \psi = \psi(\lambda) \), then \( \psi[i, j, k] \) is defined by

\[
\lambda^j \psi = \psi[i, j, 0] + \psi[i, j, 1] \epsilon + \psi[i, j, 2] \epsilon^2 + \cdots + \psi[i, j, k] \epsilon^k + \cdots
\]

(15)

with

\[
\psi[i, j, k] = \frac{1}{k!} \frac{\partial^k}{\partial \epsilon^k} \left( (\lambda_i + \epsilon)^j \psi(\lambda + \epsilon) \right).
\]

In other words, \( \psi[i, j, k] \) is the coefficient of \( \epsilon^k \) if we expand \((\lambda_i + \epsilon)^j \psi(\lambda_i + \epsilon) \) at \( \lambda = \lambda_i \) by Taylor expansion.

Now, consider the degenerate case of \( n = 4 \). According to the formula

\[ q^{(4)} = \frac{2\delta_{11}}{\delta_{12}} \]

(16)

with

\[
\delta_{11} = \begin{bmatrix}
\phi_1 & \lambda_1 \psi_1 & \lambda_1^2 \phi_1 & -\lambda_1^4 \phi_1 \\
\phi_2 & \lambda_2 \psi_2 & \lambda_2^2 \phi_2 & -\lambda_2^4 \phi_2 \\
\phi_3 & \lambda_3 \psi_3 & \lambda_3^2 \phi_3 & -\lambda_3^4 \phi_3 \\
\phi_4 & \lambda_4 \psi_4 & \lambda_4^2 \phi_4 & -\lambda_4^4 \phi_4
\end{bmatrix}
\]

\[
\delta_{12} = \begin{bmatrix}
\phi_1 & \lambda_1 \psi_1 & \lambda_1^2 \phi_1 & \lambda_1^3 \psi_1 \\
\phi_2 & \lambda_2 \psi_2 & \lambda_2^2 \phi_2 & \lambda_2^3 \psi_2 \\
\phi_3 & \lambda_3 \psi_3 & \lambda_3^2 \phi_3 & \lambda_3^3 \psi_3 \\
\phi_4 & \lambda_4 \psi_4 & \lambda_4^2 \phi_4 & \lambda_4^3 \psi_4
\end{bmatrix}
\]

The main process is as follows. (i) For the first (second) row, we can substitute the eigenvalue \( \lambda_1 (\lambda_2 = \lambda_1^*) \) and eigenfunction \( \psi_1 (\psi_2) \) directly. (ii) For \( \lambda_3 \rightarrow \lambda_1 \), \( \lambda_4 \rightarrow \lambda_2 \), we first expand the elements of the third (fourth) row at \( \lambda_3 = \lambda_1 + \epsilon \) (\( \lambda_4 = \lambda_2 + \epsilon \)) by Taylor expansion, then subtract the first (second) row from the third (fourth) row. (iii) Taking \( \epsilon \rightarrow 0 \), the terms with a higher order of \( \epsilon \) will vanish and the new expression of \( q^{(4)} \) is obtained in the same form as equation (16), but with the values for \( \delta_{11} \) and \( \delta_{12} \) given by

\[
\delta_{11} = \begin{bmatrix}
\phi[1, 0, 0] & \psi[1, 0, 0] & \phi[1, 2, 0] & -\phi[1, 4, 0] \\
\phi[2, 0, 0] & \psi[2, 1, 0] & \phi[2, 2, 0] & -\phi[2, 4, 0] \\
\phi[1, 0, 1] & \psi[1, 1, 1] & \phi[1, 2, 1] & -\phi[1, 4, 1] \\
\phi[2, 0, 1] & \psi[2, 1, 1] & \phi[2, 2, 1] & -\phi[2, 4, 1]
\end{bmatrix}
\]

\[
\delta_{12} = \begin{bmatrix}
\phi[1, 0, 0] & \psi[1, 0, 0] & \phi[1, 2, 0] & \psi[1, 3, 0] \\
\phi[2, 0, 0] & \psi[2, 1, 0] & \phi[2, 2, 0] & \psi[2, 3, 0] \\
\phi[1, 0, 1] & \psi[1, 1, 1] & \phi[1, 2, 1] & \psi[1, 3, 1] \\
\phi[2, 0, 1] & \psi[2, 1, 1] & \phi[2, 2, 1] & \psi[2, 3, 1]
\end{bmatrix}
\]

(17)

Substituting \( \lambda_1 = \alpha_1 + i\beta_1 \) and the eigenfunctions of equation (10) into the above formula, we obtain a positon solution

\[ q^{(4)} = \frac{G_1 + iG_2}{G_3 + iG_4}, \]

(18)

where

\[ G_1 = \alpha_1 \beta_1, \]

\[ G_2 = -16 \cos(F_2) \alpha_1^3 \left( -32 \beta_1^2 \alpha_1^4 t \sinh(F_1) + 16 \beta_1^4 \alpha_1^4 t \sinh(F_1) + 4 \beta_1^4 \alpha_1^4 t \sinh(F_1) - 48 \beta_1^4 \alpha_1^4 t \sinh(F_1) \right) - \alpha_1 \cosh(F_1) \beta_1, \]

\[ G_3 = 16 \alpha_1^4 \beta_1^4 t^2 + 64 \alpha_1^6 \beta_1^2 t^2 + 2 \alpha_1^4 \beta_1^2 \sinh(2 F_2) \]

\[ - 384 \alpha_1^4 \beta_1^4 t - 16 \alpha_1^2 \beta_1^4 t + 2 \alpha_1^2 \beta_1^4 t \]

\[ + 64 \alpha_1^2 \beta_1^4 t, \]

\[ G_4 = -256 \alpha_1^2 x \beta_1^4 t + 32 \alpha_1^2 \beta_1^4 t + \alpha_1^4 + 64 \alpha_1^4 x^2 \alpha_1^4 t \]

\[ - 256 \alpha_1^2 x \beta_1^4 t + 256 \alpha_1^2 \beta_1^4 t^2 + 3072 \beta_1^4 \alpha_1^4 t \]

\[ + 2048 \beta_1^4 \alpha_1^4 t \]

\[ + 512 \beta_1^4 \alpha_1^4 t \]

\[ + 2048 \alpha_1^4 \beta_1^4 t + \beta_1^4 + 256 \beta_1^4 \alpha_1^4 t \]

\[ + \alpha_1^4 \cosh(2 F_2) - \beta_1^4 \cosh(2 F_2), \]

\[ F_1 = 4 \alpha_1 \beta_1 \left( x + 4 t \alpha_1^2 - 4 t \beta_1^2 \right), \]

\[ F_2 = -24 \alpha_1^2 \beta_1^2 t + 4 \beta_1^4 t + 4 \alpha_1^4 t - 2 \beta_1^2 t + 2 \alpha_1^2 t. \]

Let \( \alpha_1 \rightarrow 0 \) in the above procedure, then the second order rational solution is obtained. With these parameters, the
general solution can be given as follows:

$$q^{[n]} = \frac{G_1}{G_2 + G_3} \exp(-2i\beta_1^2(-x + 2t\beta_1^2))$$

(19)

with

$$G_1 = -8\beta_1(-12i\beta_1^4 x + 768i\beta_1^6 x^3 t + 4096i\beta_1^{12} x^3 t^3 - 64i\beta_1^6 x^3 t - 48i\beta_1^2 t - 3072i\beta_1^{10} x^2 t - 3 + 2304\beta_1^6 t + 768x\beta_1^6 t^3 + 48\beta_1^6 x^3 t^2)$$

$$G_2 = 3 - 768x\beta_1^6 t + 4096\beta_1^{10} x^3 t^3 - 24576\beta_1^{12} x^2 t^2 + 65536\beta_1^{14} x t^3 + 4608\beta_1^8 t - 65536\beta_1^{16} t^4 - 256\beta_1^8 x^4 + 96\beta_1^6 x^2 t^2,$$

$$G_3 = -12288\beta_1^{10} x t^2 + 16384\beta_1^{12} x^3 - 256\beta_1^6 x^3 - 48\beta_1^2 x^2 + 3072\beta_1^4 x^2 t^2 + 576\beta_1^4 x t^4.$$ 

Equations (18) and (19) represent the interaction of solitons and rational solitons, respectively. A simplified analysis shows that they possess phase shift when $t \to \pm \infty$, which is different from general two-soliton solutions. The profile of the soliton and rational soliton are shown in figure 3. Next, we consider these works for general $n$.

**Theorem 2.** For $n = 2k$, $\psi = \psi(\alpha)$ is the eigenfunction vector of the GI system, the expression of the $n$-positon solution in terms of determinant is obtained

$$q^{[n]} = 2i\frac{\delta_{11}}{\delta_{12}}$$

where

![Figure 3. The dynamics of the two-soliton solution and two-rational solution on the $(x, t)$ plane. (a) Equation (18) with $\alpha_1 = 0.5$ and $\beta_1 = 0.5$. (b) Equation (18) with $\alpha_1 = 0$ and $\beta_1 = 0.5.$](image)

**Proof.** For the entries in the first column of $\Omega_{11}$ (8)

$$\begin{align*}
\phi_1 &= \phi[1, 0, 0], \\
\phi_2 &= \phi[2, 0, 0], \\
\phi_3 &= \phi[1, 0, 0] + \phi[1, 0, 1] \epsilon, \\
\phi_4 &= \phi[2, 0, 0] + \phi[2, 0, 1] \epsilon, \\
&\vdots \\
\phi_{n-1} &= \phi[1, 0, 0] + \phi[1, 0, 1] \epsilon + \phi[1, 0, 2] \epsilon^2 + \cdots + \phi[1, 0, k-1] \epsilon^{k-1}, \\
\phi_n &= \phi[2, 0, 0] + \phi[2, 0, 1] \epsilon + \phi[2, 0, 2] \epsilon^2 + \cdots + \phi[2, 0, k-1] \epsilon^{k-1},
\end{align*}$$

taking a similar procedure to the other entries in $\Omega_{11}$ and $\Omega_{12}$. Finally, $q^{[n]}$ is obtained through simple calculation and a certain limit. 

\[\square\]
Let \( \alpha_1 \rightarrow 0 \) in the above formula, we obtain the \( n \)-rational soliton solution.

### 3.2. Solutions with a non-vanishing boundary condition

In this section, we consider solutions from a non-trivial seed, which will provide a higher order rogue wave. In general, we start with \( q = e^{i \phi} = \lambda_2^{\mu} = 1 \), \( a, c \in \mathbb{C} \). Then the corresponding eigenfunctions \( \psi_k \) associated with \( \lambda_k \) are

\[
\left( \begin{array}{c}
\phi_k(\lambda_k) \\
\psi_k(\lambda_k)
\end{array} \right) = \left( \begin{array}{c}
D_1 \sigma_1(\lambda_k)[1, k] + D_2 \sigma_2(\lambda_k)[1, k] \\
-D_1 \sigma_1^*(\lambda_k^*)[2, k] - D_1 \sigma_2(\lambda_k)[2, k] \\
D_1 \sigma_1(\lambda_k)[2, k] + D_2 \sigma_2(\lambda_k)[2, k] \\
+ D_2 \sigma_1^*(\lambda_k^*)[1, k] + D_1 \sigma_2^*(\lambda_k^*)[1, k]
\end{array} \right),
\]

where

\[
\sigma_1(\lambda_k) = \begin{pmatrix}
\sigma_1(\lambda_k)[1, 1] \\
\sigma_1(\lambda_k)[2, 1] \\
\sigma_1(\lambda_k)[1, 2] \\
\sigma_1(\lambda_k)[2, 2]
\end{pmatrix}, \quad \sigma_2(\lambda_k) = \begin{pmatrix}
\sigma_2(\lambda_k)[1, 1] \\
\sigma_2(\lambda_k)[1, 2] \\
\sigma_2(\lambda_k)[2, 1] \\
\sigma_2(\lambda_k)[2, 2]
\end{pmatrix},
\]

\[
c_1 = \frac{-c^4 + 2c^2a - a^2 - 4a^2c^2 - 4\lambda_k^2}{2}.
\]

In this subsection, we set \( D_1 = 1 \) and \( D_2 = 1 \). Under these circumstances, Xu and He [16] have obtained two kinds of breather solution. One is a time periodic breather and the other is a space periodic breather. In addition, the first order and the second order rogue wave solutions have also been obtained by a certain limit from the breather solution. However, the method is difficult for calculating the higher order rogue waves. Here, we take limit technique in the determinant expression of the solution for the GI equation directly, which is the same method as we used for the NLS equation [33].

**Proposition 1.** Let \( n = 2k \), \( \psi = (\psi) \) is eigenfunction vector, assuming

\[
\lambda_1 = \frac{1}{2} \sqrt{c^2 - 2a} - i \frac{1}{2} c
\]

and

\[
\lambda_2 = \frac{1}{2} \sqrt{c^2 - 2a} + i \frac{1}{2} c
\]

the formula of the \( n \)th order rogue solution is obtained

\[
q^{(n)} = q + \frac{2}{\delta_{11}} \delta_{12},
\]

where
The first order rogue wave
Higher order rogue waves.

Proof. For the entries in the first column of \(\Omega_{11}\)

\[
\begin{align*}
\phi_1 &= \phi[1, 0, 1]e, \\
\phi_2 &= \phi[2, 0, 1]e, \\
\phi_3 &= \phi[1, 0, 1]e + \phi[1, 0, 2]e^2, \\
\phi_4 &= \phi[2, 0, 1]e + \phi[2, 0, 2]e^2, \\
&\quad \vdots \\
\phi_n &= \phi[1, 0, 1]e + \phi[1, 0, 2]e^2 + \cdots + \phi[1, 0, k]e^k,
\end{align*}
\]

taking a similar procedure to the other entries in \(\Omega_{11}\) and \(\Omega_{12}\). Finally, \(q^{[n]}\) is obtained through simple calculation and a certain limit.

Next, we provide some explicit solutions as applications.

- **The first order rogue wave.** For \(n = 2\), we obtain the first order rogue wave

\[
q^{[2]} = \frac{G_1}{G_2} \exp \left( \frac{1}{2} \left( 2ax - 2ta^2 - 2tc^2a + tc^4 \right) \right)
\]

with

\[
G_1 = -8c^3a^3 + 12a^3c^4 + 8c^2a^4tx - 8c^4tax - 2c^3ax^2 + 12ac^2t - 6ac^6t^2 - 3 - 2ic^2x + 2c^4x^3 - 6ic^4t + 2c^8t^2,
\]

\[
G_2 = -8c^3a^3 + 12a^3c^4 + 8c^2a^4tx - 8c^4tax - 2c^3ax^2 + 4ic^2tx - 6ac^6t^2 + 1 - 2ic^2x + 2c^4x^3 + 2ic^4t + 2c^8t^2.
\]

By applying this method, we also obtained a similar result (second order rogue wave) in [16]. Moreover, our method is general for producing higher order rogue waves. Assuming \(x \to \infty, t \to \infty\), then \(|q^{[2]}| \to c\). The maximum amplitude of \(|q^{[2]}|\) occurs at the origin and is equal to \(3c\). A first order rogue wave with particular parameters is shown in figure 4.

- **Higher order rogue waves.** Generally, the expression for rogue waves becomes more complicated with increasing \(n\). For convenience, we let \(a = 0\) and \(c = 1\) in the following.

When \(n = 4\), we can obtain the second order rogue wave according to the formula (23)

\[
q^{[4]} = \frac{G_1}{G_2} \exp \left( \frac{1}{2}i\theta \right)
\]

with

\[
G_1 = 45 + 8t^6 - 24it^4x - 144it^3x^2 - 48it^2x^3 - 72itx^4 + 576it^2x^2 + 288itx^3 - 144t^3x^2 - 504t^2x^3
\]

\[
- 198t^4x - 246t^2 - 60t^4x^2 + 60t^2x^3 + 8t^6,
\]

\[
G_2 = 9 + 48t^4x^2 - 216t^2x^2 + 48t^4x^2 + 24t^6 - 24ix^5 + 24it^4x^2 + 48ix^3 + 198it - 24it^4x + 180t^4
\]

\[
+ 24t^2x^4 + 24t^2x^3 + 8t^6 + 8t^6 - 12x^4 + 66t^6
\]

\[
+ 90t^2 - 72tx + 336t^3 - 48ix^3 + 54ix + 288it^2x - 48it^2x^3.
\]

In addition, we can also obtain the \(k\)th \((k = 3, 4, 5, 6, 7)\) order rogue wave solution of the GI equation. Since their analysis expressions are too cumbersome, we omit them. Their dynamical evolutions are displayed in figure 5. From the figures, we find that these local peaks all have a high amplitude at their center and there are many small peaks located around the central peak. Through detailed analysis, we find that the amplitude of the \(k\)th order rogue wave solution of the GI equation is \((2k + 1)c\) (\(c\) is the boundary condition of the seed solution).
4. The dynamics of rogue waves

In the section above, we obtained the fundamental structure of the higher order rogue wave solution for the GI equation with $D_1 = 1$ and $D_2 = 1$ in (21). Actually, $D_1$ and $D_2$ are arbitrarily constant (or go to constant). In this section, we set $D_1$ and $D_2$ as follows:

$$
\begin{align*}
D_1 &= \exp(-ic_1(S_0 + S_1\epsilon + S_2\epsilon^2 + S_3\epsilon^3 + \cdots + S_{k-1}\epsilon^{k-1})), \\
D_2 &= \exp(ic_1(S_0 + S_1\epsilon + S_2\epsilon^2 + S_3\epsilon^3 + \cdots + S_{k-1}\epsilon^{k-1})).
\end{align*}
$$

(26)

Here $S_1, S_2, \ldots, S_{k-1} \in \mathbb{C}$. Although the terms with non-zero orders of $\epsilon$ vanish in the $\epsilon \to 0$ limit, analysis and numerics prove that their coefficients $S_i$ ($i = 1, 2, \ldots, k-1$) have a crucial effect on the dynamics of higher order rogue waves. In the following, our main task is to discuss how these parameters control these different spatial–temporal structures at the same order $k$.

4.1. Solutions with one parameter

- $S_0$: fundamental pattern. When $S_0 = 0$ ($i \neq 0$), we obtain a trivial translation. A special case for the first order rogue wave is shown in figure 6. It simply alters the location of the rogue wave. In fact, we can shift the location of rogue wave to an arbitrary position by altering the values of $S_0$. This case for the NLS equation had been given in [34]. Moreover, the $k$ order rogue waves have $k(k+1)-1$ waves. Starting from a large negative $t$, $k$ small peaks, then a row of $k-1$ larger peaks etc, the central high amplitude wave appears. This process is reversed in positive $t$. The dynamical evolution can be observed distinctly for the NLS equation in [35].

- $S_1$: the two triangular structure.

1) Triangular structure. Let all coefficients $S_i = 0$ except $S_1$. The higher order rogue wave solutions of the GI equation are split into a triangular structure. This structure of the $k$th order rogue wave contains $k(k+1)/2$ first order fundamental patterns, which make up $k$ successive arrays, and these arrays possess $k, k-1, \ldots, 1$ peaks respectively. These structures of the $k$th ($k = 2, 5, 7$) order rogue wave are shown in figure 7. From figure 7(c), we observe seven arrays, each of which has 7, 6, 5, 4, 3, 2, 1 local maxima, respectively. The orientation of these triangular structures in the $(x, t)$-plane remains the same. Our result is a general case of [36].

2) Modified triangular structure. In fact, the outer triangle is independent of the inner triangle in the above triangular structure. For instance, when $k = 5$, the outer 12 local maxima and the inner 3 humps both make up a triangle, but these two triangles are irrelevant. If we alter the appearance of (21) and mix coefficients $D_1$ and $D_2$ as follows:

$$
\begin{align*}
\phi_1(\lambda_k) &= \begin{pmatrix}
D_1 \sigma_1(\lambda_k)[1, k] + D_1 \sigma_2(\lambda_k)[1, k] \\
-2D_2 \sigma_1^*(\lambda_k)[2, k] - D_2 \sigma_2^*(\lambda_k)[2, k] \\
D_1 \sigma_1(\lambda_k)[2, k] + D_1 \sigma_2(\lambda_k)[2, k] \\
+2D_2 \sigma_1^*(\lambda_k)[1, k] + D_2 \sigma_2^*(\lambda_k)[1, k]
\end{pmatrix},
\end{align*}
$$

(27)

with

$$
\begin{align*}
D_1 &= \exp(-ic_1^2(S_0 + S_1\epsilon + S_2\epsilon^2 + S_3\epsilon^3 + \cdots + S_{k-1}\epsilon^{k-1})), \\
D_2 &= \exp(ic_1^2(S_0 + S_1\epsilon + S_2\epsilon^2 + S_3\epsilon^3 + \cdots + S_{k-1}\epsilon^{k-1})).
\end{align*}
$$

(28)

Remark. The expression of (27) is different with (21).

Three inner peaks inversely form a second fundamental pattern. It is a new structure which has
Figure 8. The modified triangular structure with a higher order rogue wave located in the center. (a) An overall profile with $S_1 = 1000$. (b) A local central profile of the left panel.

Figure 9. The ring structure of higher order rogue waves. (a) The third order rogue wave with $S_3 = 50 000$. (b) The fifth order rogue wave with $S_5 = 5 000 000$. (c) The seventh order rogue wave with $S_7 = 5 \times 10^7$.

Figure 10. The ring–triangle structure of higher order rogue waves. (a) The fourth order rogue wave with $S_4 = 5 \times 10^7$ and $S_1 = 500$. (b) The fifth order rogue wave with $S_5 = 5 \times 10^9$ and $S_1 = 500$. (c) The sixth order rogue wave with $S_6 = 5 \times 10^{13}$ and $S_1 = 1000$. (d) The seventh order rogue wave with $S_6 = 1 \times 10^{11}$ and $S_1 = 150$.

Figure 11. The multi-ring model of the sixth order rogue wave with $S_5 = 8 \times 10^{9}$ and $S_3 = 80 000$, the inner peak is still a higher order wave.

never before been obtained for the NLS equation. Its dynamical evolution is shown in figure 8. For a $k$th order rogue wave, we conjecture that there are $3k - 3$ first order rogue waves located on the outer triangular shell and a $(k - 3)$th order rogue wave located in the center.

- **$S_{k-1}$: ring structure.** If $S_i = 0$ ($i \neq k - 1$), we obtain a ring structure. A high maximum peak is surrounded by some local maxima. By simple analysis, we find that the peaks located on the outer shell are all first order rogue waves, and the number is $2k - 1$. In addition, the inner peak is a higher order rogue wave (except $k = 3$), whose order is $k - 2$ (when $k = 3$, the inner peak is a first order rogue wave). For example, when $k = 7$, there are 13 first order rogue waves located on the outer shell and a fifth order rogue waves located in the center. Some of these structures are shown in figure 9.

(1) $S_{k-1}, S_1$: ring-triangle. If $S_1$ is also non-zero, the inner higher order rogue wave will become a triangular structure. For instance, when $k = 5$, the inner second order rogue wave is split into a triangle with $S_4 = 5 \times 10^9$, $S_1 = 500$. In addition, when $k = 6, 7$, similar structures are also obtained. Therefore, we can reasonably conclude that the inner higher order rogue wave is able to be split into a triangular structure when $S_{k-1}$ is big enough and $S_1 \neq 0$. We display some of these special models in figure 10. Notably, the orientation of the triangle in the seventh order ring-triangle model is different from the others.

(2) $S_{k-1}, S_{k-3}$: multi-ring. Being similar to the above case, the inner higher order rogue wave can also be split into a ring structure. Keeping $S_{k-1} \gg 0$ and setting $S_{k-3} \neq 0$ ($k > 3$), we obtain a ring structure. In this case, if the central peak is still a higher order rogue wave, we can continue splitting it into
In general, there are \( k - 1 \) free parameters for the \( k \)th order rogue wave solution, which has \( 2^k \) combinations. By allocating proper values to these parameters, we can obtain a hierarchy of higher order rogue waves of the GI equation. Note that these results of the GI equation can be thought of as natural extensions of the NLS equation [33, 36, 37], thus the GI equation may also admit ‘super-regular solitonic solutions’ derived from small perturbations at a certain moment of time for the NLS equation, which describe the nonlinear stage of the modulation instability of the condensate [9].

5. Conclusion

In this paper, we modify the generalized Darboux transformation to obtain explicit solutions for the GI equation. In the case of a vanishing boundary condition, we obtain a soliton, a rational traveling soliton, a breather type soliton and a soliton colliding with a breather type soliton. The last two kinds of solution are new for the GI equation. Moreover, we obtain the formula by applying the Taylor expansion and limit technique, when eigenvalues share the same value. Under the condition of a non-vanishing boundary condition, we provide the formula of the \( N \)-rogue wave solution for the GI equation in proposition 1. We provide the expressions of higher order rogue wave solutions as applications and discuss their structures. Furthermore, we obtain a hierarchy of solutions with different structures by adjusting the free parameters \( D_1 \) and \( D_2 \). In summary, there are four basic patterns for higher order rogue waves of the GI equation: the fundamental pattern, the triangular structure, the modified triangular structure and the ring structure. By choosing proper parameters, we can obtain solutions with a combination of the basic structures.

Our results clearly show the connection between shift parameters and structures of rogue waves due to the explicit formula. These types of solutions are new to the GI equation. Moreover, some basic models such as the fundamental pattern, ring structure and triangular structure have been found in higher rogue waves for other equations such as the NLS equation [13], but the modified triangular structure is new to the GI equation. Meanwhile, we obtain a family of solutions with a combination of structures. All of these results will help us study the universal properties for rogue waves and gain a better understanding of ‘waves that appear from nowhere and disappear without a trace’ [10].

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