The deformations of antibracket with even and odd deformation parameters, defined on the space $DE_1$

S.E.Konstein* and I.V.Tyutin†‡
I.E.Tamm Department of Theoretical Physics,
P. N. Lebedev Physical Institute,
119991, Leninsky Prospect 53, Moscow, Russia.

Abstract

We consider antiPoisson superalgebra realized on the smooth Grassmann-valued functions of the form $\xi f_0(x) + f_1(x)$, where $f_0$ has compact support on $\mathbb{R}^n$, and with the parity opposite to that of the Grassmann superalgebra realized on these functions. The deformations with even and odd deformation parameters of this superalgebra are found.

1 Introduction

Let $\mathbb{G}^n$ be a Grassmann algebra on $n$ indeterminates. In [1] we described the deformations of antiPoisson superalgebra realized on the space $D_n$ of smooth $\mathbb{G}^n$-valued functions with compact support on $\mathbb{R}^n$ and show that there exists either one deformation with one even deformation parameter, or one deformation with one odd parameter. During the proof of this statement, the second cohomology space $H^2(D_n, E_n)$ was calculated, where $E_n$ is the space of smooth $\mathbb{G}^n$-valued functions on $\mathbb{R}^n$, and it was shown that $\dim H^2(D_n, E_n) = 1|1$ if $n \geq 2$ and $\dim H^2(D_1, E_1) = 3|3$.

Let $DE$ be the space of $\mathbb{G}^1$-valued functions of the form $f(x, \xi) = \xi f_0(x) + f_1(x)$, where $f_0$ and $f_1$ are smooth functions on $\mathbb{R}^1$, such that $f_0$ has compact support, and $\xi$ is the only generating element of the Grassmann algebra $\mathbb{G}^1$.

Here we explore the observation, that $H^2(D_1, E_1) = H^2(D_1, DE) \subset H^2(DE, DE)$, and that $DE$ may have a deformation with several even and several odd deformation parameters.

In the present work, we found the deformations of Poisson antibracket realized on $DE$.

Particularly, we found all the deformations with one even and at most three odd deformation parameters, and all the deformation of some particular form with arbitrary number of odd parameters.

The text is organized as follows. For background, see Section 2. Section 3 contains calculation of $H^2(DE, DE)$. This space is much wider than $H^2(D_1, E_1)$ and the additional cocycles are parameterized by the elements of the quotient space $D'/(C^\infty)'$.

*E-mail: konstein@lpi.ru
†E-mail: tyutin@lpi.ru
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Theorems 6.1 and 6.2 describing the deformations are formulated in Section 6. Main Theorems (5.1 and 6.2) are proved in Appendices 2-6.

2 Preliminary and notation

Let \( K \) be either \( \mathbb{R} \) or \( \mathbb{C} \). We denote by \( D(\mathbb{R}^n) \) the space of smooth \( K \)-valued functions with compact supports on \( \mathbb{R}^n \). This space is endowed with its standard topology. We set

\[
D_n = D(\mathbb{R}^n) \otimes \mathbb{G}^n, \quad E_n = C^\infty(\mathbb{R}^n) \otimes \mathbb{G}^n,
\]

The generators of the Grassmann algebra (resp., the coordinates of the space \( \mathbb{R}^n \)) are denoted by \( \xi^\alpha, \alpha = 1, \ldots, n \) (resp., \( x^i, i = 1, \ldots, n \)). We also use collective variables \( z^A \) which is opposite to the \( \varepsilon \)-parity: \( \varepsilon = \varepsilon + 1 \).

We set \( \varepsilon_A = 0, \varepsilon_A = 1 \) for \( A = 1, \ldots, n \) and \( \varepsilon_A = 1, \varepsilon_A = 0 \) for \( A = n+1, \ldots, 2n \).

It is well known, that the bracket

\[
[f, g](z) = \sum_{i=1}^{n} \left( f(z) \frac{\overline{\partial}}{\overline{\partial} x^i} \frac{\partial}{\partial \xi^i} g(z) - f(z) \frac{\overline{\partial}}{\overline{\partial} \xi^i} \frac{\partial}{\partial x^i} g(z) \right), \quad (2.1)
\]

which we call ”antibracket”, defines the structure of Lie superalgebra on the superspaces \( D_n \) and \( E_n \) with the \( \epsilon \)-parity.

We call these Lie superalgebras antiPoisson superalgebras.1

The integral on \( D_n \) is defined by the relation \( \int dz f(z) = \int_{\mathbb{R}^n} dx \int d\xi f(z) \), where the integral on the Grassmann algebra is normed by the condition \( \int d\xi \xi^1 \ldots \xi^n = 1 \). We identify \( \mathbb{G}^n \) with its dual space \( \mathbb{G}^n \) setting \( f(g) = \int d\xi f(\xi) g(\xi), f, g \in \mathbb{G}^n \). Correspondingly, the space \( D'_n \) of continuous linear functionals on \( D_n \) is identified with the space \( D'(\mathbb{R}^n) \otimes \mathbb{G}^n \) and \( E'_n \) is identified with the space \( (C^\infty(\mathbb{R}^n))' \otimes \mathbb{G}^n \).

The value \( m(f) \) of a functional \( m \in D'_n \) on a function \( f \in D_n \) will be often written in the integral form: \( m(f) = \int dz m(z)f(z) \).

Introduce the superalgebra \( DE_n, D_n \subset DE_n \subset E_n \) by the relation

\[
DE_n = \{ f \in E_n : f - \int d\xi \xi^1 \ldots \xi^n f \in D_n \}. \quad (2.2)
\]

Clearly, if \( f \in DE_n \) and \( g \in DE_n \), then \( [f, g] \in D_n \).

Below we consider the case \( n = 1 \) only. For simplicity we will denote \( D(\mathbb{R}) \) as \( D \), \( C^\infty(\mathbb{R}) \) as \( E \), \( DE_1 \) as \( DE \). Besides we will denote \( D_1 \) as \( D \), \( E_1 \) as \( E \).

It follows from (2.2) that \( DE \) consists of the functions of the form \( f = \xi f_0(x) + f_1(x) \) where \( f_0 \in D \) and \( f_1 \in E \).

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1 We will also consider associative multiplication of the elements of considered antiPoisson superalgebras with commutation relations \( fg = (-1)^{\varepsilon(f)\varepsilon(g)} gf \).
3 Cohomology of antibrackets

Let \( U \) be either \( D \) or \( DE \). Let \( U \) acts in a \( \mathbb{Z}_2 \)-graded space \( V \) (the action of \( f \in U \) on \( v \in V \) will be denoted by \( f \cdot v \)). The space \( C_p(U, V) \) of \( p \)-cochains consists of all multilinear superantisymmetric mappings from \( U^p \) to \( V \). Superantisymmetry means, as usual, that

\[
M_p(\ldots, f_i, f_{i+1}, \ldots) = -(-1)^{\epsilon(f_i)\epsilon(f_{i+1})} M_p(\ldots, f_{i+1}, f_i, \ldots)
\]

The space \( C_p(U, V) \) possesses a natural \( \mathbb{Z}_2 \)-parity: by definition, \( M_p \in C_p(U, V) \) has the definite parity \( \epsilon_{M_p} \) if

\[
\epsilon(M_p(f_1, \ldots, f_p)) = \epsilon(f_1) + \ldots + \epsilon(f_p)
\]

for any \( f_j \in U \) with \( \epsilon \)-parities \( \epsilon(f_j) \). Sometimes we will use the Grassmann \( \epsilon \)-parity\(^2\) of cochains: \( \epsilon_{M_p} = \epsilon_{M_p} + p + 1 \). The differential \( d_p^V \) is defined to be the linear operator from \( C_p(U, V) \) to \( C_{p+1}(U, V) \) such that

\[
d_p^V M_p(f_1, \ldots, f_{p+1}) = - \sum_{j=1}^{p+1} (-1)^{j+\epsilon(f_j)\epsilon(f)|i,j-1+\epsilon(f_j)\epsilon_{M_p}} f_j \cdot M_p(f_1, \ldots, \hat{f}_j, \ldots, f_{p+1}) - \\
- \sum_{i<j} (-1)^{j+\epsilon(f_j)\epsilon(f)|i,j-1} M_p(f_1, \ldots, \hat{f}_i, \hat{f}_j, \ldots, \hat{f}_j, \ldots, f_{p+1}), \quad (3.1)
\]

for any \( M_p \in C_p(U, V) \) and \( f_1, \ldots, f_{p+1} \in U \) having definite \( \epsilon \)-parities. Here the sign \( ^\wedge \) means that the argument is omitted and the notation

\[
|\epsilon(f)|_{i,j} = \sum_{l=i}^{j} \epsilon(f_l)
\]

has been used. The differential \( d_p^V \) is nilpotent (see [2]), i.e., \( d_{p+1}^V d_p^V = 0 \) for any \( p = 0, 1, \ldots \). The \( p \)-th cohomology space of the differential \( d_p^V \) will be denoted by \( H^p \). The second cohomology space \( H_2^{\text{ad}} \) in the adjoint representation is closely related to the problem of finding formal deformations of the Lie bracket \([\cdot, \cdot]\) of the form \([f, g] = [f, g] + \hbar [f, g] + \ldots\) up to similarity transformations \([f, g]_T = T^{-1} [T f, T g] \) where continuous linear operator \( T \) from \( V[[\hbar]] \) to \( V[[\hbar]] \) has the form \( T = \text{id} + \hbar T_1 \).

The condition that \([\cdot, \cdot]_1 \) is a 2-cocycle is equivalent to the Jacobi identity for \([\cdot, \cdot]_1 \) modulo the \( h \)-order terms.

In the present paper, similarly to [3], we suppose that cochains are separately continuous multilinear mappings.

We need the cohomologies of the antiPoisson algebra \( DE \) in the adjoint representation: \( V = DE \) and \( f \cdot g = [f, g] \) for any \( f, g \in DE \). The space \( C_p(DE, DE) \) consists of separately continuous superantisymmetric multilinear mappings from \((DE)^p \) to \( DE \). In fact, we need here the case \( p = 2 \) only.

We call \( p \)-cocycles \( M_1, \ldots, M_k \) independent cohomologies if they give rise to linearly independent elements in \( H^0 \). For a multilinear form \( M_p \) taking values in \( D, E, \) or \( DE \), we write \( M_p(\ldots, f_i, \ldots, f_p) \) instead of more cumbersome \( M_p(f_1, \ldots, f_p)(z) \).

\(^2\)If \( V \) is the space of Grassmann-valued functions on \( \mathbb{R}^n \) then \( \epsilon \) defined in such a way coincides with usual Grassmann parity.
The following theorem proved in [3] and [1] describes $p = 2$ cohomology of antibracket mapping $m_2|_1, m_2|_2, m_2|_3, m_2|_4, m_2|_5,$ and $m_2|_6$ from $(D)^2$ to $E$ be defined by the relations

$$m_2|_1(z, f, g) = \int du \partial_u g(u) \partial_u^z f(u), \quad \epsilon_{m_2|_1} = 1,$$

$$m_2|_2(z, f, g) = \int du \theta(x - y)[\partial_u g(u) \partial_u^z f(u) - \partial_u f(u) \partial_u^z g(u)] + x[f] \partial_u g(z) - \{\partial_u f(z)\} \partial_u^z g(z), \quad \epsilon_{m_2|_2} = 1,$$

$$m_2|_3(z, f, g) = (-1)^{\epsilon(f)} \{1 - N_\xi \} f(z) \{1 - N_\xi \} g(z), \quad \epsilon_{m_2|_3} = 1,$$

$$m_2|_4(z, f, g) = (-1)^{\epsilon(f)} \Delta f(z) \Delta g(z), \quad \epsilon_{m_2|_4} = 0,$$

$$m_2|_5(z, f, g) = \int du (-1)^{\epsilon(f)} \partial_u f(u) \partial_u g(u), \quad \epsilon_{m_2|_5} = 0,$$

$$m_2|_6(z, f, g) = \int du \theta(x - y)(-1)^{\epsilon(f)} \partial_u f(u) \partial_u g(u), \quad \epsilon_{m_2|_6} = 0,$$

where $z = (x, \xi), u = (y, \eta), N_\xi = \xi \partial_\xi,$ and

$$\Delta = \partial_x \partial_\xi, \quad \mathcal{E}_z = 1 - \frac{1}{2}(x \partial_x + \xi \partial_\xi)$$

Then $\dim H^2(D, E) = 3|3$ and the cochains $m_2|_1(z, f, g), m_2|_2(z, f, g), m_2|_3(z, f, g), m_2|_4(z, f, g), m_2|_5(z, f, g),$ and $m_2|_6(z, f, g)$ are independent nontrivial cocycles.

Observe that in fact all the forms $m_2|_i$ ($i = 1, ..., 6$) take the value in $DE$ and can be extended from $D^2$ to $DE^2$ taking the value in $DE$. So, $H^2(D, E) = H^2(D, DE) \subset H^2(DE, DE)$.

To find $H^2(DE, DE)$ we have to determine whether $DE$ has 2-cocycles different from $m_2|_i$ ($i = 1, ..., 6$).

The following theorem answers this question.

**Theorem 3.2.**

1. Let $M \in D'$, $M \notin E'$ and let $M(f) = \int dx \mu(x)f(x)$ with some distribution $\mu$. Then bilinear form $m_2|_7(M)$ defined as

$$m_2|_7(M, f, g) = \int dz \xi \mu(x)[f, g]$$

or, equivalently,

$$m_2|_7(M, f, g) = M \left( \int d\xi \xi[f, g] \right)$$

is representative of nontrivial element of $H^2(DE, DE)$.

2. Up to exact form, each element in $H^2(DE, DE)$ has the form

$$m_2 = \sum_{i=1}^6 c_i m_2|_i + m_2|_7(M),$$

where $m_2|_i$ are listed in (3.2)-(3.7), and $M \in D'/E'$.
Note, that if $M \in E'$, then $m_{2|7}(M) = d_1 M$.

Proof.

Let $m_2$ be nontrivial cocycle in $C^2(\D, \D)$, i.e. $d_2 m_2 = 0$ and there does not exist such $m_1 \in C^\infty(\D, \D)$ that $m_2 = d_1 m_1$.

Consider the restriction $m_2|D^2$ of $m_2$ to $D^2$. As this restriction is a cocycle in $C^2(D, \D)$, this restriction has the form

$$m_2|D^2 = \sum_1^6 c_i m_{2|i} + d_1 m_1$$

with some $m_1 \in C^1(D, \D)$. Denoting $(m_2 - \sum_1^6 c_i m_{2|i})$ as $n_2$ we obtain

$$n_2|D^2 = d_1 m_1.$$ 

Let us look for such $m_1 \in C^1(D, \D)$ that $d_1 m_1$ can be extended to $C^2(\D, \D)$ and $m_1$ can not be extended to $C^1(\D, \D)$.

Straightforward calculation gives that $m_1(f) = M(\int d\xi \xi f)$, where $M \in D'$, $M \notin E'$.

Indeed, $d_1 m_1(f, g) = M(\int d\xi \xi [f, g]) \in C^2(\D, \D)$ because $[f, g] \in D$ if $f, g \in \D$.

In Appendices 2 - 6 we use the notation $\omega$ both for the linear functional $M \in D'/E'$ defining cocycle $m_{2|7}$ in Theorem 3.2 and for the kernel of this functional.

4 Deformation with even and odd parameters. Preliminary.

It occurred, that there exist odd second cohomologies with coefficients in adjoint representation. It is natural to look for the deformations associated with these odd cohomologies and having the odd deformation parameter [3, 6].

Below we consider the case, where the functions and multilinear forms may depend on outer odd parameters $\theta_i$, where $\theta$-s belong to some supercommutative associative superalgebra $\mathcal{A}$. Thus we consider a colored algebras $\D \otimes \mathcal{A}$, $C^p(\D, \D) \otimes \mathcal{A}$,... with $(\mathbb{Z}_2)^2$ grading, namely the grading of element $\theta \otimes f$ is $(\epsilon_1(\theta), \epsilon_2(f))$.

We preserve the notation $\D$ for $\D \otimes \mathcal{A}$.

We can consider $\D$ as a Lie superalgebra with the parity $\epsilon = \epsilon_1 + \epsilon_2$. One can easily check that such consideration is selfconsistent (see also [7] and discussion on Necludova and Sheunert theorems in [8].

As the second cohomology space with coefficients in adjoint representation has the dimension $3|3$, it is natural to suppose that $\mathcal{A} = \mathbb{G}^3$. For this case all the deformations are found (see Theorem 6.1 and Theorem 6.2) and listed in Subsection 6.1.

Nevertheless, it is possible to consider other superalgebras $\mathcal{A}$. In what follows, we consider the deformation of the form

$$C = \sum_{i=0}^{\infty} h^i C_i(\theta) \tag{4.1}$$

where $\sum_{i=0}^{\infty} h^i C_i(\theta) \in C^2(\D, \D) \otimes \mathbb{G}^{s_k}$ and $s_0 \leq s_1 \leq s_2 ...$.

During the proofs, we assume that all the $\theta$-s has the same order that $h$, and sometimes we stress this by writing $C(h, h\theta_i)$. 


5 Algebra of Jacobiators

Before formulate (and prove) the main theorem of present work, which describes the deformations of antibracket on $\text{DE}$, we have to calculate all the Jacobiators $J_{i,j} = J(m_{2|i}, m_{2|j})$ for $i, j = 1, ..., 7$.

As it was mentioned above, we define Jacobiator of bilinear forms by nonlinear way. Let $m$ and $n$ be bilinear forms. Then

$$J(m, n)(f, g, h) = (-1)^{\epsilon(f)\epsilon(h)} (m(n(f, g), h) + (-1)^{\epsilon(m)\epsilon(n)} n(m(f, g), h)) +$$

$$+ \text{cycle}(f, g, h) \text{ if } m \neq n$$

$$J(m)(f, g, h) = \begin{cases} \frac{1}{2} J(m, m)(f, g, h) & \text{if } \epsilon(m) = 0, \\ 0 & \text{if } \epsilon(m) = 1. \end{cases}$$

Jacobiator is connected with the differential by the relation

$$d_2m(f, g, h) = -(-1)^{\epsilon(f)\epsilon(h)}J(m_{2|0}, m)(f, g, h),$$

where $m_{2|0}(f, g) = [f, g]$.

The Jacobiators of each pair of cocycles $J_{i,j} = J(m_{2|i}, m_{2|j})$ are described here for $i, j = 1, ..., 7$.

In the next table stars denote all the pairs $i, j$ such that $J_{i,j} \neq 0$ and has not the form $J(m_{2|0}, m_{2|j})$ with some 2-form $n$. The sign "X" denote all the pairs $i, j$ such that we don’t need $J_{i,j}$ for deriving the deformations of $\text{DE}$:

| $i \setminus j$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----------------|---|---|---|---|---|---|---|
| 1               | 0 | 0 | * | * | 0 | 0 | 0 |
| 2               | 0 | 0 | * | * | 0 | 0 | J($m_{2|0}, m_{2|9}$) J($m_{2|0}, m_{2|10}(M)$) |
| 3               | * | * | 0 | * | * | * | * |
| 4               | * | * | * | X | * | * | * |
| 5               | 0 | 0 | * | * | 0 | 0 | 0 |
| 6               | 0 | J($m_{2|0}, m_{2|9}$) * | * | 0 | 0 | J($m_{2|0}, m_{2|8}(M)$) |
| 7               | 0 | J($m_{2|0}, m_{2|10}(M)$) * | * | 0 | J($m_{2|0}, m_{2|8}(M)$) | 0 |

Expressions for needed nonzero $J_{i,j}$ and for the forms $m_{2|8}$, $m_{2|9}$, $m_{2|10}$ are listed below.

We use in what follows the notation $\tilde{\theta}$ for smooth function such that $\tilde{\theta}(x) - \theta(x)$ has compact support.

5.0.1 $J_{4,4}$

We have no needs to calculate $J_{4,4}$ because we know from [5] and [4] that $m_{2|4}$ generates the deformations

$$[f, g]_s = [f, g] + (-1)^{\epsilon(f)} \left\{ \frac{c_4}{1 + c_4 N_z/2} \Delta f \right\} \mathcal{E}_z \ g + \left\{ \mathcal{E}_z \ f \right\} \frac{c_4}{1 + c_4 N_z/2} \Delta g$$

where $c_4$ is a series in $\hbar$ and $\hbar^2 \theta_i \theta_j$ such that $c_4|_{\hbar=0} = 0$. 

\[ J_{4,5} = \partial_x g_0(x) \int dy \partial_y h_0 \partial_y f_1 - \partial_x h_0(x) \int dy \partial_y g_0 \partial_y f_1 + \int dy f_1( \partial_y g_0 \partial_y^2 h_0 - \partial_y h_0(1 - y \partial_y)g_0) + \frac{1}{2} \int dy f_1( \partial_y^3 g_0(1 - y \partial_y)h_0 - \partial_y^2 h_0(1 - y \partial_y)g_0) + \text{cycles} \]

\[ J_{4,6} = -\frac{1}{2} \int du \theta(x-y)\eta y(-\partial_y^2 \Delta g(u) \Delta f(u) + \partial_y^2 \Delta f(u) \Delta g(u)) + \frac{1}{2} x(-\partial_x \Delta g(z) \Delta f(z) + \partial_x \Delta f(z) \Delta g(z))(1 - \xi \partial_\xi h(z) - \Delta f(z) \int du \theta(x-y)\eta \partial_\eta \partial_y^2 g(u)h(u) + \Delta g(z) \int du \theta(x-y)\eta h(u)\partial_\eta \partial_y^2 f(u) + \text{cycles} \]

\[ J_{4,7} = \int d\eta \mu(y)[(\Delta f \partial_\eta g - \partial_\eta f \Delta g)(-\partial_y + \frac{1}{2} y^2 \partial_y^2)h + (\partial_\eta f \partial_y \Delta g - \partial_y \Delta f \partial_\eta g)(1 - \frac{1}{2} y \partial_y)h] + \Delta f(z) \int d\eta \mu(y)(\partial_\eta g \partial_\eta h - \partial_\eta g \partial_\eta h) + \text{cycles} \]

\[ J_{6,7} = J(m_0, m_8), \text{ where} \]

\[ m_8 = \int du \tilde{\mu}(y)(-1)^{\epsilon(f)} \partial_y f(u) \partial_y g(u) \quad (5.4) \]

and

\[ \tilde{\mu}(x) = \int dy \mu(y) \left( \theta(y-x) - \bar{\theta}(y) \right) \quad (5.5) \]

\[ J_{1,3} = -(1 - N_\xi) h(z) \int du \partial_\eta f(u) \partial_\eta^3 g(u) + \text{cycle} \]
5.0.7 \( J_{2,3} \)

\[
J_{2,3} = (1 - N_\xi)h(z) \int du \theta(x - y)[\partial_\eta g(u)\partial_y^3 f(u) - \partial_\eta f(u)\partial_y^3 g(u)] + \\
+ (1 - N_\xi)h(z) [x\{\partial_x^2 f(z)\}\partial_x g(z) - x\{\partial_x f(z)\}\partial_x^2 g(z)] + \text{cycle}
\]

5.0.8 \( J_{1,4} \)

\[
J_{14} = \Delta h(z) \int du \eta \partial_\eta g(u)\partial_y^3 \partial_\eta f(u) + \text{cycle}
\]

5.0.9 \( J_{2,4} \)

\[
J_{24} = 2\Delta h(z) \int du \theta(x - y)\partial_\eta g(u)\partial_y^3 \partial_\eta f(u) + \text{loc} + \text{cycle}
\]

Here we denote as loc some local 2-form. We don’t need its specific form for deriving the deformation.

5.0.10 \( J_{2,6} \)

\( J_{26} = J(m_0, m_9) \) where \( m_9(f, g) = \partial_x \partial_\xi [f, g] \) \hspace{1cm} (5.6)

5.0.11 \( J_{2,7} \)

\( J_{27} = J(m_0, m_{10}) \) where

\[
m_{10} = \int dz \xi \mu(x) \left( m_{2|2} (z|f, g) - \bar{\theta}(x) \int du [\partial_\eta g(u)\partial_y^3 f(u) - \partial_\eta f(u)\partial_y^3 g(u)] \right) \hspace{1cm} (5.7)
\]

5.0.12 \( J_{3,4} \)

\[
J_{34} = (1 - \frac{1}{2} x\partial_x)[(1 - N_\xi)f(z)(1 - N_\xi)g(z)]\Delta h(z) + \\
(1 - N_\xi)h\{\Delta f(z)(1 - N_\xi)\mathcal{E}_z g - \Delta g(z)(1 - N_\xi)\mathcal{E}_z f(z)\} + \text{cycle}
\]

5.0.13 \( J_{3,5} \)

\[
J_{35} = (1 - N_\xi)h(z) \int du \partial_\eta f(u)\partial_y g(u) + \\
\int du \eta\{(1 - N_\xi)f\}(1 - N_\xi)g)\partial_\eta \partial_y^2 h + \text{cycle}
\]
5.0.14 $J_{3,6}$

\[
J_{36} = \{(1 - N_x)h\} \int du \theta(x - y) \partial_y f(u) \partial_y g(u) - \\
- \int du \theta(x - y) \eta \partial_y ((1 - N_x)f)(1 - N_x)g) \partial_y \partial_\eta h + \text{cycle}
\]

5.0.15 $J_{3,7}$

\[
J_{37} = - \int du \mu(y) \partial_y (f \cdot g) \partial_\eta h + \\
(1 - N_x)h(z) \int du \mu(y)(\partial_\eta f \partial_y g + \partial_\eta g \partial_y f) + \text{cycles}
\]

5.1 Relations between constants

From these values of Jacobiators the next theorem follows

Theorem 5.1. Let $V_2 = m_{2|0} + \sum_{i=1}^{6} c_i m_{2|i} + m_{2|7}(M)$. Then the equation for 2-form $X_2$

\[
J(m_{2|0}, X_2) = J(V_2, V_2)
\]

has the solution if and only if

\[
c_4c_i = 0, \quad i \neq 4, \\
c_4M = 0, \\
c_3c_i = 0, \\
c_3M = 0.
\]

In addition, as $c_1$, $c_2$, $c_3$ are odd Grassmannian elements, we have $c_ic_i = 0$ for $i = 1, 2, 3$.

It follows from this theorem that we have no needs to calculate the Jacobiators that have the products $c_i c_i$ ($i \neq 4$) or $c_3 c_i$ as a coefficients. Namely we have not to calculate $J_{4,8}$, $J_{4,9}$, $J_{4,10}$, $J_{3,8}$, $J_{3,9}$ and $J_{3,10}$. As well, because $c_2 c_2 = 0$ we have not to calculate $J_{2,9}$, $J_{2,10}$, $J_{9,9}$, $J_{9,10}$, $J_{10,10}$.

5.2 Jacobiators $J_{8,i}$, $J_{9,i}$, $J_{10,i}$.

The Jacobiators of each pair of 2-forms $J_{i,j} = J(m_{2|i}, m_{2|j})$ ($i = 8, 9, 10$, $j = 1, ..., 10$) are described in the following table.

| $i \backslash j$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----------------|---|---|---|---|---|---|---|---|---|---|
| 8               | 0 | a | X | X | 0 | 0 | 0 | 0 | J$(m_{2|0}, m_{2|11}(M))$ | 0 |
| 9               | 0 | X | X | X | 0 | 0 | -a| J$(m_{2|0}, m_{2|11}(M))$ | X | X |
| 10              | 0 | X | X | X | 0 | 0 | 0 | 0 | X | X |
The expression for $a$ can be written, but we don’t need it here. An important relation can be easily verified

$$J_{8,2} + J_{9,7} + J_{10,6} = 0$$  \hspace{1em} (5.9)

The 2-form $m_{2|11}$ has the form

$$m_{2|11}(z|f, g) = -\int du\eta\tilde{\mu}(y)[\partial_{\eta}f(u)\partial_{\eta}g''(u) - \partial_{\eta}f'''(u)\partial_{\eta}g(u)].$$  \hspace{1em} (5.10)

5.3 $J_{i,11}$

At last, $J_{i,11}$ are presented in the following table

| $i \setminus j$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|-----------------|---|---|---|---|---|---|---|---|---|----|----|
| 11              | 0 | X | X | X | 0 | 0 | 0 | 0 | X | X | X |

6 Deformations of antibrackets (Results)

Theorem 6.1.

Let

$$\mathcal{M}_{c_4}(f, g) = (-1)^{c(f)} \left\{ \frac{c_4}{1 + c_4 N_z/2} \Delta f \right\} \mathcal{E}_z g + \left\{ \mathcal{E}_z f \right\} \frac{c_4}{1 + c_4 N_z/2} \Delta g$$  \hspace{1em} (6.1)

where $c_4$ is a series in $\hbar$ and $\hbar^2\theta_i\theta_j$ such that $c_4|_{\hbar=0} = 0$.

Let $\tilde{\theta}$ be smooth function such that $\tilde{\theta}(x) - \theta(x)$ has compact support.

Let $M \in D/\mathcal{E}'[[h, h\theta_k]]$ be even in $\theta$s and $M|_{\hbar=0} = 0$ and let $\mu$ be the kernel of some representative of $M$. 
Let

\[ m_{2|0}(z|f,g) = [f,g], \] (6.2)

\[ m_{2|1}(z|f,g) = \int du \partial_\eta g(u) \partial_y^3 f(u), \quad \epsilon_{m_{2|1}} = 1, \] (6.3)

\[ m_{2|2}(z|f,g) = \int du \theta(x-y)[\partial_\eta g(u) \partial_y^3 f(u) - \partial_\eta f(u) \partial_y^3 g(u)] + 
+ x[\{\partial_\xi \partial_x f(z)\} \partial_\xi \partial_x g(z) - \{\partial_\xi \partial_x f(z)\} \partial_\xi \partial_x^2 g(z)], \quad \epsilon_{m_{2|2}} = 1, \] (6.4)

\[ m_{2|3}(z|f,g) = (-1)^\epsilon(f) \{(1 - N_\xi)f(z)\}(1 - N_\xi)g(z), \quad \epsilon_{m_{2|3}} = 1, \] (6.5)

\[ m_{2|5}(z|f,g) = \int du (-1)^\epsilon(f) \partial_y f(u) \partial_y g(u), \quad \epsilon_{m_{2|5}} = 0, \] (6.6)

\[ m_{2|6}(z|f,g) = \int du \theta(x-y)(-1)^\epsilon(f) \partial_y f(u) \partial_y g(u), \quad \epsilon_{m_{2|6}} = 0, \] (6.7)

\[ m_{2|7}(M|f,g) = \int du \eta \mu(y) [f(u), g(u)], \quad \epsilon_{m_{2|7}} = 0, \] (6.8)

\[ m_{2|8}(M|f,g) = \int du \bar{\mu}(y)(-1)^\epsilon(f) \partial_y f(u) \partial_y g(u) \] (6.9)

\[ m_{2|9}(z|f,g) = \partial_x \partial_\xi [f,g], \] (6.10)

\[ m_{2|10}(M|f,g) = \int dz \xi \mu(x) \left( m_{2|2}(z|f,g) - 
- \bar{\theta}(x) \int du [\partial_\eta g(u) \partial_y^3 f(u) - \partial_\eta f(u) \partial_y^3 g(u)] \right), \] (6.11)

\[ m_{2|11}(M|f,g) = \int du \bar{\mu}(y) \left( \partial_\eta g(u) \partial_y^3 f(u) - \partial_\eta f(u) \partial_y^3 g(u) \right). \] (6.12)

Let \( c_i \in \mathbb{K}[[\hbar, \hbar \theta_\xi]] \) \((i = 1, \ldots, 6)\) be formal series on even parameter \( \hbar \) and Grassmannian parameters \( \theta_\xi \); \( c_1, c_2, c_3 \) are odd and other \( c_i \) are even; \( c_i|_{\hbar=0} = 0 \). Let \( c_i \) satisfy the relations

\[ c_4 c_i = 0, \quad i \neq 4, \] (6.13)

\[ c_4 M = 0, \] (6.14)

\[ c_3 c_i = 0, \] (6.15)

\[ c_3 M = 0. \] (6.16)

Then the bilinear form

\[ C = m_{2|0} + M_{c_4} + \sum_{i=1,2,3,5,6} c_i m_{2|i} + c_6 m_{2|8}(M) - c_6 m_{2|9} - c_2 m_{2|10}(M) - c_2 c_6^2 m_{2|11}(M) \] (6.17)

is a deformation of antibracket \( m_{2|0} \) on DE.

To prove this theorem it is sufficient to check Jacobi identity for \( C \) using previous results for Jacobiators.

The solution (6.17) is complete in the cases listed in the following theorem.

**Theorem 6.2.** Let \( C \) be some deformation of antibracket \( m_{2|0} \) on DE depending on even parameter \( \hbar \) and on \( \hbar \theta_j \) with Grassmannian generating elements \( \theta_j \). So, \( C = \sum_0^\infty \hbar^p C_p(\hbar \theta_k) \)
and $C|_{\theta_k=0} = m_{2|0}$. Let $\sum_{j} h^p C_p(h \theta_k)$ depends on generating elements of $G_{n^*} (n_1 \leq n_2 \leq n_3 \leq \ldots)$. 

Then

1. If $C|_{\theta_k=0} \neq m_{2|0}$, then $C$ has the form (6.17), and $c_i$ and $M$ satisfy (6.13)-(6.16).

2. If $C|_{\theta_k=0} = m_{2|0}$ and $n_s \leq 3$ for all $s$, then $C$ has the form (6.17), and $c_i$ and $M$ satisfy (6.13)-(6.16).

**Corollary** If $n_s \leq 3$ for all $s$, then $C$ has the form (6.17), and $c_i$ and $M$ satisfy (6.13)-(6.16).

The proof of these theorems can be found in the Appendices.

### 6.1 List of possible deformations

The relations (6.13)-(6.16) can be solved under the conditions of each item of Theorem 6.2.

Item (1) of this Theorem splits onto 2 possibilities

1. $c_4|_{\theta=0} \neq 0$.

   In this case $c_1 = c_2 = c_3 = 0, c_5 = c_6 = 0, M = 0$ and
   
   $$C = m_{2|0} + M_{c_4}$$  \hspace{1cm} (6.18)

2. At least one of $c_5|_{\theta=0}, c_6|_{\theta=0}, M|_{\theta=0}$ is not zero.

   In this case $c_3 = 0, c_4 = 0$ and
   
   $$C = m_{2|0} + \sum_{i=1,2,5,6} c_im_{2|i} + m_{2|7}(M) - c_6m_{2|8}(M) - c_2c_6m_{2|9} - c_2m_{2|10}(M) - c_2c_6^2m_{2|11}(M)$$  \hspace{1cm} (6.19)

Now, let us look at the possibilities, that follows from Item (2) of Theorem 6.2.

In this case $c_4 = h^2(c_{4,1} \theta_1 \theta_3 + c_{4,2} \theta_2 \theta_1 + c_{4,3} \theta_1 \theta_2)$, where $c_{4,k} \in K[[h]]$. Clearly, it is possible to choose a new generators of $G^3$,

$$\theta'_k = \sum_{l=1,2,3} a_{k,l}(h) \theta_l$$  \hspace{1cm} (6.20)

in such a way that $c_4 = h^2 c_{4,3}(h) \theta'_1 \theta'_2$.

So, it is sufficient to consider the coefficient $c_4$ in the form $c_4 = h^2 \alpha_4 \theta_1 \theta_2$ where $d_4 \in h^2 K[[h]]$.

This form of $c_4$ is preserved under the following changes of generators of $G^3$:

$$\begin{cases} 
\theta'_k = \sum_{l=1,2} a_{k,l}(h) \theta_l & k = 1, 2 \\
\theta'_3 = \theta_3.
\end{cases}$$  \hspace{1cm} (6.21)

Up to the transformation of the form (6.20), Item (2) of Theorem 3.1 gives two possibilities:
1. \( c_i = \hbar \sum_{j=1,2} \beta^j \theta_j + \hbar \gamma_i \theta_1 \theta_2 \theta_3, \) (i = 1, 2), \( c_3 = \hbar \gamma_3 \theta_1 \theta_2 \theta_3, \)
\( c_4 = \hbar^2 \alpha_4 \theta_1 \theta_2, \) \( c_{5,6} = \hbar^2 \sum_{k,l,m=1,2,3} \alpha_k^l \epsilon_{klm} \theta_1 \theta_2 \theta_3, \) \( M = \hbar^2 \sum_{k,l,m=1,2,3} M_k \epsilon_{klm} \theta_1 \theta_2 \theta_3, \)

where \( \alpha_4, \alpha_k^l, \beta_i, \gamma_i \in \mathbb{K}[[h]], \) \( M_k \in D/\mathbb{E}[[h]] \)

In this case

\[
C = m_{2|0} + \sum_{i=1,2,3,4,5,6} c_i m_{2|i} + m_{2|7}(M) - c_2 c_6 m_{2|9} - c_2 m_{2|10}(M) \tag{6.22}
\]

2. \( c_i = \theta_1 \tilde{c}_i, \) \( M = \theta_1 \tilde{M}. \)

In this case

\[
C = m_{2|0} + \theta_1 \left( \sum_{i=1,2,3,4,5,6} \tilde{c}_i m_{2|i} + m_{2|7}(\tilde{M}) \right). \tag{6.23}
\]

**Appendix 1. Jacobiator of cocycle and coboard**

Here we formulate and prove Lemma, which helps to calculate Jacobiators \( J_{i,7} \) in Section

**Lemma A1.1.** Let \( \epsilon \) be some parity, preserving by \( m_{2|0} = [f, g], \) i.e. \( \epsilon([f, g]) = \epsilon(f) + \epsilon(g) \)

Let 2-form \( m_2(f, g) \) be cocycle, i.e.

\[
(-1)^{\epsilon(f)\epsilon(g)} m_2([f, g], h) + \text{cycle}(f, g, h) =
= -(-1)^{\epsilon(f)\epsilon(h)} [m_2(f, g), h] + \text{cycle}(f, g, h),
\]

and let \( n(f, g) = d_1 M_1(f, g), \) \( \epsilon_n = \epsilon_{M_1}. \) Then

\[
J(m_2, d_1 M_1) = J(m_{2|0}, U_2)
\]

where

\[
U_2 = (-1)^{\epsilon m_2 \epsilon M_1} M_1(m_2(f, g)) - m_2(M_1(f), g) +
+(-1)^{\epsilon(f)\epsilon(g)} m_2(M_1(g), f), \quad \epsilon_{U_2} = \epsilon_{m_2} + \epsilon_{M_1}.
\]

**Proof.** Let, for shortness, \( R = J(m_{2|0}, U_2), \) \( J = J(m_2, d_1 M_1), \) \( \sigma(f, h) = \epsilon^{-1}(f)\epsilon(h). \)

Evaluate the difference \( J - R: \)

\[
J - R = \sigma(f, h)\{m_2([M_1(f), g], h) - \sigma(g, f)m_2([M_1(g), f], h) -
-m_2(M_1([f, g]), h) + (-1)^{\epsilon m_2 \epsilon M_1} [M_1(m_2(f, g)), h] +
+(-1)^{\epsilon M_1 \epsilon[f] + \epsilon(g)} [m_2(f, g), M_1(h)] - (-1)^{\epsilon m_2 \epsilon M_1} M_1([m_2(f, g), h]) -
-(-1)^{\epsilon m_2 \epsilon M_1} [M_1(m_2(f, g)), h] + [m_2(M_1(f), g), h] -
-\sigma(g, f)[m_2(M_1(g), f), h] -
-(-1)^{\epsilon m_2 \epsilon M_1} M_1(m_2([f, g], h)) + m_2(M_1([f, g], h)) -
-\sigma(f, h)\sigma(g, h)m_2(M_1(h), [f, g])\} + \text{cycle} \implies
\]
Appendix 2. Jacoby Identity

so

\[ J - R = \sigma(f, h)\{m_2([M_1(f), g], h) - \sigma(g, f)m_2([M_1(g), f], h) + \]
\[ +(-1)^{\epsilon_1}m_1^{[0]}[m_2(f, g), M_1(h)] - (-1)^{\epsilon_2}m_1^{[1]}M_1([m_2(f, g), h], h) + \]
\[ +[m_2((M_1(f), g), h) - \sigma(g, f)[m_2((M_1(g), f), h) - \]
\[ -(-1)^{\epsilon_2}m_1^{[1]}M_1(m_2([f, g], h)) - \sigma(f, h)\sigma(g, h)m_2(M_1(h), [f, g])\} + \]
\[ +\text{cycle} \]

and

\[ J - R = \sigma(f, h)\{m_2([M_1(f), g], h) - \sigma(g, f)m_2([M_1(g), f], h) + \]
\[ +(-1)^{\epsilon_1}m_1^{[0]}[m_2(f, g), M_1(h)] + [m_2((M_1(f), g), h) - \]
\[ -\sigma(g, f)[m_2((M_1(g), f), h) - \sigma(f, h)\sigma(g, h)m_2(M_1(h), [f, g])\} + \]
\[ +\text{cycle}, \]

where we use the relation

\[ \sigma(f, h)\{m_2([f, g], h) + [m_2(f, g), h]\} + \text{cycle} = 0. \quad (A1.1) \]

After proper cycling we obtain

\[ J - R = \sigma(g, h)m_2([M_1(h), f]), g) - \sigma(f, g)\sigma(g, h)m_2([M_1(h), g], f) + \]
\[ +\sigma(f, h)(-1)^{\epsilon_1}m_1^{[0]}[m_2(f, g), M_1(h)] + \sigma(g, h)[m_2(M_1(h), f), g] - \]
\[ -\sigma(f, g)\sigma(g, h)[m_2(M_1(h), g), f] - \sigma(g, h)m_2(M_1(h), [f, g]) + \text{cycle} = \]
\[ = (-1)^{\epsilon_1}m_1^{[0]}\{\sigma(g, M_1(h))m_2([M_1(h), f], g) + \sigma(f, g)m_2([M_1(h), g], f) + \]
\[ +\sigma(f, M_1(h))[m_2(f, g), M_1(h)] + \sigma(g, M_1(h))[m_2(M_1(h), f), g] + \]
\[ +\sigma(f, g)[m_2(g, M_1(h), f] + \sigma(f, M_1(h)m_2([f, g], M_1(h))\} + \]
\[ +\text{cycle}, \]

Because \( m \) is a cocycle, the last expression is zero before cycling:

\[ (-1)^{\epsilon_1}m_1^{[0]}\{\sigma(g, M_1(h))m_2([M_1(h), f], g) + \sigma(f, g)m_2([M_1(h), g], f) + \]
\[ +\sigma(f, M_1(h))[m_2(f, g), M_1(h)] + \sigma(g, M_1(h))[m_2(M_1(h), f), g] + \]
\[ +\sigma(f, g)[m_2(g, M_1(h), f] + \sigma(f, M_1(h)m_2([f, g], M_1(h))\} = 0 \]

So, \( J - R = 0 \). ■

Appendix 2. Jacoby Identity

We consider the deformation \( C \) as a formal series on deformation parameter \( \hbar \), each term of this series depends in its turn on generators of some Grassmann algebra \( \mathbb{G}^s \).

\[ C(z|f, g) = C^{(0)}(z|f, g) + \hbar C^{(1)}(z|f, g) + \hbar^2 C^{(2)}(z|f, g) + \hbar^3 C^{(3)}(z|f, g) + ..., \]
\[ C^{(k)}(z|f, g) = \theta^k C^{(k)}(z|f, g), \]

\[ C^{(0)}(z|f, g) = \]
The notation $\theta^k$ means $\theta_1, \theta_2, \ldots, \theta_k$. The forms $\tilde{C}^{(k)}$ depend on $h$ and don’t depend on $\theta$

$$
\varepsilon_C = 1, \varepsilon_{C^{(0)}} = \varepsilon_{C^{(p)}} = 1, \varepsilon_{A^{(p)}} = 1 + p,
$$

$$
C(z|g,f) = (-1)^{\varepsilon(f)\varepsilon(g)+\varepsilon(f)+\varepsilon(g)} C(z|f,g),
$$

$$
C^{(0)}(z|g,f) = (-1)^{\varepsilon(f)\varepsilon(g)+\varepsilon(f)+\varepsilon(g)} C^{(0)}(z|f,g),
$$

$$
C^{(p)}(z|g,f) = (-1)^{\varepsilon(f)\varepsilon(g)+\varepsilon(f)+\varepsilon(g)} C^{(p)}(z|f,g),
$$

Further,

$$
C^{(k)}(z|f,g) = \sum_l h^l C^{(k)}_l(z|f,g).
$$

Jacoby identity

$$
(-1)^{\varepsilon(f)\varepsilon(g)+\varepsilon(f)+\varepsilon(g)} C(z|C(|f,g), h) + \text{cycle}(f,g,h) = 0 \quad (A2.1)
$$

leads to the equations:

$$
(-1)^{\varepsilon(f)\varepsilon(g)+\varepsilon(f)+\varepsilon(h)} C^{(0)}(z|C^{(0)}(|f,g), h) + \text{cycle}(f,g,h) = 0, \quad (A2.2)
$$

$$
(-1)^{\varepsilon(f)\varepsilon(h)+\varepsilon(f)+\varepsilon(h)} [C^{(0)}(z|C^{(1)}(|f,g), h) + C^{(1)}(z|C^{(0)}(|f,g), h)] + \text{cycle} = 0, (A2.3)
$$

and so on.

**Appendix 3. Solution of eq. (A2.2)**

In what follows we use the notation $\omega$ both for the linear functional $M \in D'/E'$ defining cocycle $mn_{2|\ell}$ in Theorem 3.2 and for the kernel of this functional.

### 3.1. First order

Represent $C^{(0)}(z|f,g)$ in the form

$$
C^{(0)}(z|f,g) = m_{2|0}(z|f,g) + hC^{(0)}_1(z|f,g) + O(h^2), \varepsilon_{C_1^{(0)}} = 1.
$$

It follows from eq. (A2.2) that

$$
J(m_{2|0}, C_1^{(0)}; z|f,g,h) = 0.
$$

This equation was solved in Section 3 and its general solution in the sector $\varepsilon_{C_1^{(0)}} = 1$ is (up to exact forms)

$$
C^{(0)}_1(z|f,g) = c_{a|1} m_{2|a}(z|f,g) + m_{2|7|1}(z|f,g), \quad a = 4, 5, 6,
$$

$$
m_{2|7|k}(z|f,g) = m_{2|7}(z|f,g)|_{\omega \rightarrow \omega_k}.
$$

The function $C^{(0)}(z|f,g)$ can be represented in the form

$$
C^{(0)}(z|f,g) = N_{a|1}(z|f,g) + hc_{b|1} m_{2|b}(z|f,g) + hm_{2|7|1}(z|f,g) +
$$

$$
+ h^2 C^{(0)}_2(z|f,g) + O(h^3), \quad b = 5, 6, \varepsilon_{C_2^{(0)}} = 1,
$$
where \[ \mathcal{N}_c = m_{2|0} + \mathcal{M}_c, \] (A3.1)

and \( \mathcal{M}_c \) is defined by (6.1).

### 3.2. Second order

It follows from eq. (A2.2) that

\[
J(m_{2|0}, \tilde{C}^{(0)}_2) + c_{4|1}c_{0|1}J(m_{2|4}, m_{2|6}) + c_{4|1}J_1(m_{2|4}, m_{2|7|1}) = 0, \tag{A3.2}
\]

\[
b = 5, 6, \quad \tilde{C}^{(0)}_2 = C^{(0)}_2 + c_{6|1}C_{67|1} + c_{4|1}C_{47|1},
\]

\[
C_{67|k} = C_{67|\omega \rightarrow \omega_k}, \quad C_{47|k} = C_{47|\omega \rightarrow \omega_k}.
\]

Let us remind that

\[
J(m_{2|4}, m_{2|a}; z|\hat{f}_0, \hat{g}_0, \hat{h}_0) = J(m_{2|4}, m_{2|a}; z|\hat{f}_0, \hat{g}_0, \hat{h}_1) = J(m_{2|4}, m_{2|a}; z|\hat{f}_1, \hat{g}_1, \hat{h}_1) \equiv 0, \quad a = 5, 6, 7,
\]

\[
C_{67}(z|\hat{f}_0, \hat{g}_0) = C_{67}(z|\hat{f}_1, \hat{g}_1) = C_{47|2}(\hat{f}_0, \hat{g}_0) = C_{47|2}(\hat{f}_1, \hat{g}_1) \equiv 0.
\]

Represent the form \( M(z|f), \varepsilon_M = 0 \), in the form

\[
M(z|f) = \int du[\eta M_{(0,1)}(x|y) - \xi M_{(1,0)}(x|y)]f(u),
\]

\[
M(z|\hat{f}_0) = \int dy M_{(0,1)}(x|y)f_0(y) \equiv M_{(0,1)}(x|f_0),
\]

\[
M(z|\hat{f}_1) = \xi \int dy M_{(1,0)}(x|y)f_1(y) \equiv \xi M_{(1,0)}(x|f_1).
\]

Represent the form \( C(z|f, g) = -(-1)\varepsilon(f+1)\varepsilon(g+1)C(z|g, f), \varepsilon_C = 1, \) in the form

\[
C(z|f, g) = \int dvdu[\theta C_{(1)}(x|y_u, y_v) + \eta_a C_{(2)}(x|y_u, y_v) - \eta_a C_{(2)}(x|y_v, y_u) + \xi \eta_a \eta_b C_{(3)}(x|y_u, y_v)](-1)^{(f+1)\varepsilon(f)g(v)} + C_{(3)}(x|y_u, y_v) = C_{(3)}(x|y_u, y_v),
\]

\[
C_{(1)}(x|y_u, y_v) = -C_{(1)}(x|y_u, y_v),
\]

where

\[
C_{67}(z|\hat{f}_0, \hat{g}_0) = 0 \implies C_{67}(x|f_0, g_0) = 0,
\]

\[
C_{67}(z|\hat{f}_1, \hat{g}_1) = 0 \implies C_{67}(x|f_1, g_1) = 0,
\]

\[
C_{67}(z|\hat{f}_0, \hat{g}_1) = C_{67}(x|f_0, g_1) = \int dy \hat{\omega}(y)f_0(y)g_1(y),
\]

\[
C_{67}(z|\hat{f}_1, \hat{g}_0) = -C_{67}(x|g_0, f_1).
\]
We have for $M_{d_1}(z|f,g) = d_1^{ad}M(f,g)$:

\begin{align}
M_{d_1(3)}(z|f_0,g_0) &= 0, \quad (A3.3) \\
M_{d_1(2)}(x|f_0,g_1) &= \{\partial_x M_{(0,1)}(x|f_0)\}g_1(x) + M_{(1,0)}(x|g_1)f_0'(x) - M_{(0,1)}(x|f_0'g_1), \quad (A3.4) \\
M_{d_1(1)}(x|f_1,g_1) &= \{\partial_x M_{(0,1)}(x|f_1)\}g_1(x) - \{\partial_x M_{(1,0)}(x|g_1)\}f_1(x) + M_{(1,0)}(x|g_1)f_1'(x) - \\
&- M_{(1,0)}(x|f_1')g_1'(x) + M_{(1,0)}(x|f_1'g_1' - f_1'g_1),
\end{align}

It follows from $J(m_{2|0}, \tilde{C}_2^{(0)}; z|\hat{f}_0, \hat{g}_0, \hat{h}_0) = 0$

$$C_{2(3)}^{(0)}(x|f_0,g_0)h'_1(x) + \text{cycle}(f_0,g_0,h_0) = 0,$$

This equation was solved in [4] and [1] and we have

$$C_{2(3)}^{(0)}(x|f_0,g_0) = 0.$$

After this, we have $J(m_{2|0}, \tilde{C}_2^{(0)}; z|\hat{f}_0, \hat{g}_0, \hat{h}_1) \equiv 0$.

It follows from $J(m_{2|0}, \tilde{C}_2^{(0)}; z|\hat{f}_1, \hat{g}_1, \hat{h}_1) = 0$

\begin{align}
C_{2(1)}^{(0)}(x|f_1,g_1)h'_1(x) &= \{\partial_x C_{2(1)}^{(0)}(x|f_1,g_1)\}h_1(x) + \\
&+ C_{2(1)}^{(0)}(x|f_1g'_1 - f_1g'_1, h_1) + \text{cycle}(f_1, g_1, h_1) = 0, \quad (A3.5)
\end{align}

Here we use the property $C_{67(3)} = C_{67(1)} = M_{47|2(3)} = M_{47|2(1)} = 0$, such that we have $\tilde{C}_{2(3)}^{(0)} = C_{2(3)}^{(0)}; \tilde{C}_{2(1)}^{(0)} = C_{2(1)}^{(0)}$. Eq. (A3.5) was also solved in [4] and [1] and we have

$$C_{2(1)}^{(0)}(x|f_1,g_1) = c_4|2m_{2|4(1)}(x|f_1,g_1) + M_{d_1(1)}(x|f_1,g_1),$$

where $M(z|f)$ is an one-form.

It follows from eq. (A3.2) for $f = \hat{f}_0, g = \hat{g}_1, h = \hat{h}_1$

$$\{\partial_2 \tilde{C}_{2(2)}(x|f_0,g_1)\}h_1(x) + \tilde{C}_{2(2)}(x|f_0'g_1, h_1) + \tilde{C}_{2(2)}(x|f_0,g_1h'_1) - \\
-(g_1 \leftrightarrow h_1) - C_{2(1)}(x|g_1, h_1)f_0'(x) = c_4|c_0|1J(m_{2|4}, m_{2|6}; z|\hat{f}_0, \hat{g}_1, \hat{h}_1) + \\
+c_4|J_1(m_{2|4}, m_{2|7}; z|\hat{f}_0, \hat{g}_1, \hat{h}_1), \quad b = 5, 6,
$$

or

$$\{\partial_2 D_{2(2)}(x|f_0,g_1)\}h_1(x) + D_{2(2)}(x|f_0'g_1, h_1) + D_{2(2)}(x|f_0,g_1h'_1) - \\
-(g_1 \leftrightarrow h_1) = c_4|m_{2|4(1)}(x|g_1, h_1)f_0'(x) + c_4|c_0|1J(m_{2|4}, m_{2|6}; z|\hat{f}_0, \hat{g}_1, \hat{h}_1) + \\
+c_4|J_1(m_{2|4}, m_{2|7}; z|\hat{f}_0, \hat{g}_1, \hat{h}_1),$$

where $D_{2(2)}(x|f_0,g_1) = \tilde{C}_{2(2)}(x|f_0,g_1) - M_{(1,0)}(x|g_1)f_0'(x)

Consider the domain $[x \cup \text{supp}f_0] \cap [\text{supp}g_1 \cup \text{supp}h_1] = \emptyset$. In this domain we have $J(m_{2|4}, m_{2|6}) = J_1(m_{2|4}, m_{2|7}) = 0$ and we find

$$\hat{D}_{2(2)}(x|f_0, g_1h'_1 - g'_1h_1) = 0.$$
Let \( f_0 \in D \). Then we have

\[
D_{2(2)}(x|f_0, g_1) = \sum_{q=0}^{Q} T^q_1(x|f_0) \partial^q_x g_1(x) + \sum_{q=0}^{Q} T^q_2(x|g_1 \partial^q f_0), \quad f_0 \in D.
\]

Consider the domain \( \text{supp} f_0 \cap [x \cup \text{supp} g_1 \cup \text{supp} h_1] = \emptyset \). In this domain we have \( J(m_{2|4}, m_{2|b}^{(0)}) = J_1(m_{2|4}, m_{2|7}) = 0 \) and we find

\[
\{ \partial_x D_{2(2)}(x|f_0, g_1) \} h_1(x) + D_{2(2)}(x|f_0, g_1 h_1') - (g_1 \leftrightarrow h_1) = 0.
\]

Let \( f_0 \in D \). Then we have

\[
\sum_{q=0}^{Q} \{ \partial_x \hat{T}^q_1(x|f_0) \} [\partial^q_x g_1(x) h_1(x) - g_1(x) \partial^q_x h_1(x)] +
\]

\[
+ \sum_{q=0}^{Q} \hat{T}^q_1(x|f_0) \{ \partial^{q+1}_x g_1(x) h_1(x) - g_1(x) \partial^{q+1}_x h_1(x) \} +
\]

\[
+ \sum_{q=0}^{Q} \hat{T}^q_1(x|f_0) \partial^q_x [g_1(x) h_1'(x) - g_1'(x) h_1(x)] = 0. \tag{A3.6}
\]

Let \( g_1(x) = e^{px}, \quad h_1(x) = e^{kx} \). Then we have

\[
\sum_{q=0}^{Q} \{ \partial_x (p^q - k^q) + \hat{T}^q_1(x|f_0) [(p^{q+1} - k^{q+1} - (p + k)^q(p - k))] \} = 0 \tag{A3.7}
\]

Considering in eq. \((A3.7)\) the terms of higher \((Q + 1)\) order in \( p, k \), we obtain

\[
\hat{T}^q_1 = 0, \quad T^q_1 = \text{loc}, \quad q \geq 2, \quad f_0 \in D,
\]

and eq. \((A3.6)\) reduces to the form

\[
\partial_x \hat{T}^q_1(x|f_0) = 0, \quad \partial_x T^q_1(x|f_0) = \text{loc}, \quad f_0 \in D \implies
\]

\[
D_{2(2)}(x|f_0, g_1) = T^0_1(x|f_0) g_1(x) + [t^1_{1|1}(f_0) + t^1_{1|2}(x|f_0)] g_1'(x) +
\]

\[
+ \sum_{q=0}^{Q} T^q_2(x|g_1 \partial^q f_0) + \text{loc},
\]

\[
t^1_{1|1}(f_0) = \int dy t^1_{1|1}(y)f_0(y), \quad t^1_{1|2}(x|f_0) = \int dy \theta(x - y) t^1_{1|2}(y)f_0(y), \quad f_0 \in D.
\]

Consider the domain \([x \cup \text{supp} h_1] \cap [\text{supp} f_0 \cup \text{supp} g_1] = \emptyset \). In this domain we find

\[
\{ \partial_x \hat{D}_{2(2)}(x|f_0, g_1) \} h_1(x) + \hat{D}_{2(2)}(x|f_0 g_1, h_1) =
\]

\[
= [c_4|1c_5|1 f_0 g_1' + c_4|1c_6|1 \theta f_0 g_1'] - c_4|1\omega_1(f_0'g_1)] h_1'(x).
\]
Let \( f_0 \in D \). Then we find

\[
\sum_{q=0}^{Q} \{ \partial_{z} \tilde{T}^q_2(x|g_1 \partial^q f_0) \} h_1(x) + \tilde{T}^0_1(x|f'_0 g_1) h_1(x) + [t^1_{11}(f'_0 g_1) +
\]

\[
+ t^1_{12}(x|f'_0 g_1)] h'_1(x) = [c_{41} c_{51} \overline{f_0 g'_1} + c_{41} c_{61} \theta_x f_0 g'_1] -
\]

\[
-c_{41} \omega_1(f'_0 g_1)] h'_1(x) \implies (A3.8)
\]

It follows from eq. (A3.8)

\[
\partial_{z} \tilde{T}^q_2(x|g_1) = 0, \forall q, q \neq 1,
\]

\[
\tilde{T}^0_1(x|g_1) = - \partial_{z} \tilde{T}^1_2(x|g_1) \implies T^0_1(x|g_1) = - \partial_{z} T^1_2(x|g_1) + \text{loc},
\]

\[
t^1_{11}(f'_0 g_1) + \tilde{t}^1_{12}(x|f'_0 g_1) = c_{41} c_{51} \overline{f_0 g'_1} + c_{41} c_{61} \theta_x f_0 g'_1 - c_{41} \omega_1(f'_0 g_1). \quad (A3.9)
\]

i) Using \( \text{supp } g_1 > x \), we obtain

\[
t^1_{11}(f'_0 g_1) = c_{41} c_{51} \overline{f_0 g'_1} - c_{41} \omega_1(f'_0 g_1) = 0
\]

and then

\[
\tilde{t}^1_{12}(x|f'_0 g_1) = c_{41} c_{61} \theta_x f_0 g'_1.
\]

ii) Using \( f_0(y) = y \) for \( y \in \text{supp } g_1 \), we find

\[
\tilde{t}^1_{12}(x|g_1) = 0 \implies t^1_{12}(x|g_1) = \text{loc} \implies t^1_{12}(x|g_1) = 0,
\]

\[
c_{41} c_{51} = c_{41} c_{61} = 0, \quad t^1_{11}(g_1) = - c_{41} \omega_1(g_1).
\]

Note that

\[
T^0_1(x|f_0) g_1(x) + T^1_2(x|f_1 g'_0) = - \partial_{z} T^1_2(x|f_0) g_1(x) + T^1_2(x|f_1 g'_0) +
\]

\[
+ \text{loc} = \tilde{M}_{d_1}(x|f_0, g_1) + \text{loc}, \quad \tilde{M}(x|f_0) = - \int du \eta T^1_2(x|y) f(u),
\]

\[
\tilde{M}_{d_1}(x|f_0, g_1) = \tilde{M}_{d_1}(x|f_1, g_1) = 0.
\]

Rewrite the equation for \( \tilde{C}_2^{(0)} \) just obtained

\[
J(m_{2|0}, \tilde{C}_2^{(0)}) + c_{41} J_1(m_{2|4}, m_{2|7}) = 0. \quad (A3.10)
\]

Consider eq. (A3.10) in more details. Let the functions \( f_0, g_0, h_0 \) have compact supports. Then eq. (A3.10) reduces to the form

\[
J(m_{2|0}, P_2^{(0)}) = 0, \quad P_2^{(0)} = \tilde{C}_2^{(0)} + c_{41} C_{1|47|1}, \quad C_{1|47|k} = M_{47}|_{\omega \rightarrow \omega_k},
\]

\[
C_{1|47}(z|f, g) = \omega(f_0) \Delta g(z) - \Delta f(z) \omega(g_0). \quad (A3.11)
\]

The general solution of eq. (A3.11) is \( (\varepsilon_{P_2^{(0)}} = 1) \)

\[
\tilde{C}_2^{(0)} = c_{a|2} m_{2|a} + d^\text{ad}_1 M_1 - c_{41} C_{1|47|1}, \quad a = 4, 5, 6,
\]

\[
M_1(z|f) = M_{(0,1)}(z|f_0) + \xi M_{(1,0)}(z|f_1)
\]

\[
d^\text{ad}_1 M_1(z|f, g) = [M_1(z|f), f(z)] - [M_1(z|f), g(z)] - M_1(z|[f, g]),
\]
L.h.s. of eq. \((A3.12)\) can be extended to the case \(f, g, h \in DE\). We will study the possibility of the similar extension of r.h.s. of eq. \((A3.12)\). Rewrite eq. \((A3.12)\) in the form

\[ \tilde{C}^{(0)}_2(z|f, g) - c_{a|2}m_{2|a}(z|f, g) + M_1(z|[f, g]) + c_{4|1}\omega([f, g]) = \tilde{F}^{(0)}_2(z|f, g) = [M_1(z|f), f(z)] - [M_1(z|f), g(z)] - c_{4|1}\omega_1(f_0)\Delta g(z) + (-1)^{\epsilon(f)}\Delta f(z)\omega_1(g_0)]. \tag{A3.13} \]

L.h.s. of eq. \((A3.13)\) can be as before extended to the case \(f, g, h \in DE\). Consider eq. \((A3.13)\) for the functions \(\tilde{f}_0, \tilde{g}_1\). We have

\[ \tilde{P}^{(0)}_2(z|\tilde{f}_0, \tilde{g}_1) - \tilde{M}_{(1,0)}(x|g_1)f'_0(x) = M'_{(0,1)}(x|f_0)g_1(x) - c_{4|1}\omega_1(f_0)g'_1(x). \]

Let \(g_1(x)\) is fixed and is equal to 1 in a neighborhood \(U\) of a point \(x_0\). For \(x \in U\), we have

\[ M'_{(0,1)}(x|f_0) = \tilde{P}^{(0)}_2(z|\tilde{f}_0, \tilde{g}_1) - \tilde{M}_{(1,0)}(x|g_1)f'_0(x) \]

and we can consider this formula as an extension of \(M'_{(0,1)}(x|f_0) \equiv \tilde{M}_{(0,1)}(x|f_0)\) to \(f_0 \in E\). Let now another \(g_1(x)\) is fixed and is equal to \(x\) in a neighborhood \(V\) of a point \(x_0\). For \(x \in V\), we have

\[ c_{4|1}\omega_1(f_0) = \tilde{M}_{(0,1)}(x|f_0)x + M_{(1,0)}(x|g_1)f'_0(x) - \tilde{P}^{(0)}_2(z|\tilde{f}_0, \tilde{g}_1). \]

This relation means that the form \(c_{4|1}\omega_1(f_0)\) can be extended to \(f_0 \in E\), what is possible only if

\[ c_{4|1}\omega_1(f_0) = 0. \]

Thus we obtained

\[ c_{4|1}c_{5|1} = c_{4|1}c_{6|1} = c_{4|1}\omega_1 = 0. \]

3.3. A. \(c_{4|1} \neq 0\)

First we consider the case \(c_{4|1} \neq 0\). Then \(c_{5|1} = c_{6|1} = \omega_1 = 0\).

We have

\[ C^{(0)}_1(z|f, g) = c_{4|1}m_{2|4}(z|f, g). \]

The function \(C^{(0)}(z|f, g)\) can be represented in the form

\[ C^{(0)}(z|f, g) = N_{c_{4|1}}(z|f, g) + h^2C^{(0)}_2(z|f, g) + O(h^3), \quad \epsilon_{c^{(0)}_2} = 1. \]

3.3.1. Second order

It follows from eq. \((A2.2)\) that

\[ J(m_{2|0}, C^{(0)}_2) = 0 \implies . \]

\[ C^{(0)}_2(z|f, g) = c_{a|2}m_{2|a}(z|f, g) + m_{2|7|2}(z|f, g), \quad a = 4, 5, 6, \]
The function $C^{(0)}(z|f,g)$ can be represented in the form

$$C^{(0)}(z|f,g) = N_{c_{4|2}}(z|f,g) + \hbar^2 c_{4|2} m_{2|a}(z|f,g) + \hbar^2 m_{2|7|2}(z|f,g) +$$

$$+ \hbar^3 C^{(0)}_3(z|f,g) + O(\hbar^4), \ a = 5,6, \ v_{c_3} = 1, \ c_{b|k} = \sum_{l=1}^{k} \hbar^{l-1} c_{b|l}, \ b = 4,5,6.$$

### 3.3.2. Third order

It follows from eq. (A2.2) that

$$J(m_{2|0}, \tilde{C}^{(0)}_2) + c_{4|1} e_{a|2} J(m_{2|4}, m_{2|a}) + c_{4|1} J_1(m_{2|4}, m_{2|7|2}) = 0, \qquad (A3.14)$$

$$a = 5,6, \ \tilde{C}^{(0)}_2 = C^{(0)}_2 + c_{4|1} C_{4|7|2}$$

It follows from eq. (A3.14), similarly to solving eq. (A3.2), that

$$c_{5|2} = c_{6|2} = \omega_2 = 0,$$

and so on.

Thus, we find: if $c_{4|1} \neq 0$, then the solution of eq. (A2.2) has the form

$$C^{(0)}(z|f,g) = N_{c_{4|2|\infty}}(z|f,g).$$

### 3.4. B. $c_{4|1} = 0$

Now we consider the case when $c_{4|1} = 0$ and at least one of $c_{5|1}, c_{6|1}, \omega_1$ does not equal to zero.

We have

$$C_1^{(0)}(z|f,g) = c_{a|1} m_{2|a}(z|f,g) + m_{2|7|1}(z|f,g), \ a = 5,6.$$

The function $C^{(0)}(z|f,g)$ can be represented in the form

$$C^{(0)} = m_{2|0} + \hbar c_{a|1} m_{2|a} + \hbar m_{2|7|1} + \hbar^2 C^{(0)}_2 + O(\hbar^3), \ a = 5,6, \ v_{c_2} = 1.$$

### 3.4.1. Second order

It follows from eq. (A2.2) that

$$J(m_{2|0}, \tilde{C}^{(0)}_2) = 0, \ \tilde{C}^{(0)}_2 = C^{(0)}_2 + c_{6|1} C_{6|7|1} \implies$$

$$C^{(0)}_2 = c_{a|2} m_{2|a} + m_{2|7|2} - c_{6|1} C_{6|7|1}, \ a = 4,5,6.$$

The function $C^{(0)}(z|f,g)$ can be represented in the form

$$C^{(0)}(z|f,g) = N_{c_{4|2}}(z|f,g) + \hbar c_{a|2} m_{2|a}(z|f,g) + \hbar m_{2|7|2}(z|f,g) - \hbar^2 c_{6|1} C_{6|7|1} +$$

$$+ \hbar^3 C^{(0)}_3(z|f,g) + O(\hbar^4), \ a = 5,6, \ v_{c_3} = 1, \ m_{2|7|k} = \sum_{l=1}^{k} \hbar^{l-1} m_{2|7|l}. $$
3.4.2. Third order

It follows from eq. \textbf{(A2.2)} (with equalities)

\[
J(m_{2|a}, m_{2|b}) = J(m_{2|5}, m_{2|7}) = J(m_{2|7}) = J(m_{2|c}, C_{67}) = 0, \\
a, b = 5, 6, c = 5, 6, 7,
\]

(A3.15)
taken into account) that

\[
J(m_{2|0}, \tilde{C}_3^{(0)}) + c_{4|2}c_{6|1}J(m_{2|4}, m_{2|b}) + c_{4|2}J_1(m_{2|4}, m_{2|7|1}) = 0, \ b = 5, 6, \quad (A3.16)
\]
\[
\tilde{C}_3^{(0)} = C_3^{(0)} + c_{6|3}C_{67|1} + c_{4|2}C_{47|1}, \ i, j = 1, 2, \ i + j = 3.
\]

It follows from eq. \textbf{(A3.16)}, similarly to solving eq. \textbf{(A3.2)}, that

\[
c_{4|2}c_{5|1} = c_{4|2}c_{6|1} = c_{4|2}\omega_1 = 0 \implies c_{4|2} = 0 \implies C_3^{(0)} = c_{a|3}m_{2|a} + m_{2|7|3} - c_{6|3}C_{67|1}, \ a = 4, 5, 6.
\]

The function \(C^{(0)}(z|f, g)\) can be represented in the form

\[
C^{(0)}(z|f, g) = N_{c_{4|3}}(z|f, g) + h\epsilon_{a|3}m_{2|a}(z|f, g) + hm_{2|7|3}(z|f, g) - \frac{\hbar^2 c_{6|2}C_{67|2}}{\hbar^4} + \hbar^4C_4^{(0)}(z|f, g) + O(h^5), \ a = 5, 6, \quad \epsilon_{C_3^{(0)}} = 1,
\]

where

\[
C_{67|k} = \sum_{l=1}^{k} h^{l-1}C_{67|l}, \quad \sum_{l=0}^{K} h^l A_l|_{k} = \sum_{l=0}^{k} h^l A_l.
\]

3.4.3. Fourth order

It follows from eq. \textbf{(A2.2)} (with equalities \textbf{(A3.15)} and)

\[
J(C_{67}) = 0
\]

(A3.17)
taken into account) that

\[
J(m_{2|0}, \tilde{C}_4^{(0)}) + c_{4|3}c_{5|1}J(m_{2|4}, m_{2|a}) + c_{4|3}J_1(m_{2|4}, m_{2|7|1}) = 0, \ a = 5, 6, \quad (A3.18)
\]
\[
\tilde{C}_4^{(0)} = C_4^{(0)} + c_{6|3}C_{67|1} + c_{4|3}C_{47|1}.
\]

It follows from eq. \textbf{(A3.18)}, similarly to solving eq. \textbf{(A3.2)}, that

\[
c_{4|3}c_{5|1} = c_{4|3}c_{6|1} = c_{4|3}\omega_1 = 0 \implies c_{4|3} = 0 \implies C_4^{(0)} = c_{a|4}m_{2|a} + m_{2|7|4} - c_{6|3}C_{67|1}, \ a = 4, 5, 6,
\]

and so on.

Thus, we find: in the case when at least one of \(c_{5|1}, c_{6|1}, \omega_1\) does not equal to zero, the solution of eq. \textbf{(A2.2)} has the form

\[
C^{(0)}(z|f, g) = m_{2|0}(z|f, g) + h\epsilon_{a|3}m_{2|a}(z|f, g) + hm_{2|7|3}(z|f, g) - \frac{\hbar^2 c_{6|2}C_{67|2}}{\hbar^4} + \hbar^4C_4^{(0)}(z|f, g).
\]
3.5. C. \( c_4|1 = c_5|1 = c_6|1 = \omega_1 = 0 \)

Representing \( C^{(0)} \) in the form

\[
C^{(0)} = m_{2\gamma_0} + \hbar^2 C_2^{(0)} + O(\hbar^3),
\]

we obtain

\[
J(C_2^{(0)}; z|f, g, \hbar) = 0.
\]

Let in an \( \hbar^{k_0} \)-order we have

\[
C^{(0)} = m_{2\gamma_0} + \hbar^{k_0} C^{(0)}_{k_0} + O(\hbar^{k_0+1}),
\]

\[
J(m_{2\gamma_0}, C^{(0)}_{k_0}) = 0 \quad \Rightarrow \quad C^{(0)}_{k_0} = c_{a|k_0} m_{2|a}(z|f, g) + m_{2|7|k_0}(z|f, g),
\]

and at least one of the quantities \( c_{4|k_0}, c_{5|k_0}, c_{6|k_0}, \omega_{k_0} \) does not equal to zero. For the following \( C^{(0)}_{k_0+i}, l = 1, \ldots, k_0 - 1 \), we obtain as well

\[
J(m_{2\gamma_0}, C^{(0)}_{k_0+i}) = 0 \quad \Rightarrow \quad C^{(0)}_{k_0+i} = c_{a|k_0+i} m_{2|a}(z|f, g) + m_{2|7|k_0+i}(z|f, g),
\]

such that we can represent \( C^{(0)} \) in the form

\[
C^{(0)} = m_{2\gamma_0} + c_{4|2k_0-1} m_{2|a}(z|f, g) + m_{2|7|2k_0-1}(z|f, g) + \hbar^{2k_0} C_2^{(0)}_{k_0} + O(\hbar^{2k_0+1}).
\]

It follows from eq. \([A2.2]\) the equation for \( C_2^{(0)}_{k_0} \),

\[
J(m_{2\gamma_0}, \tilde{C}_2^{(0)}_{k_0}) + c_{4|k_0} c_{a|k_0} J(m_{2|4}, m_{2|a}) + c_{4|k_0} J_1(m_{2|4}, m_{2|7|k_0}) = 0, \quad a = 5, 6,
\]

\[
\tilde{C}_4^{(0)} = C_4^{(0)} + c_{6|k_0} C_6^{(0)} + c_{4|k_0} C_4|k_0 \quad \Rightarrow \quad c_{4|k_0} C_5|k_0 = c_{4|k_0} c_{6|k_0} = c_{4|k_0} \omega_{k_0} = 0.
\]

Doing similarly to previous subsections, we obtain. In general case. there are three types of the solutions of eq. \([A2.2]\)

I.

\[
C^{(0)}(z|f, g) = C^{(0)I}(z|f, g) = N_{c_4[\infty]}(z|f, g).
\]

II.

\[
C^{(0)}(z|f, g) = C^{(0)II}(z|f, g) = m_{2|0}(z|f, g) + h c_{6[\infty]} m_{2|b}(z|f, g) + h m_{2|7[\infty]}(z|f, g) - h^2 c_{6[\infty]} C_6^{(0)[\infty]}, \quad b = 5, 6,
\]

where \( c_{4[\infty]}, c_{a[\infty]}, m_{2|7[\infty]}, \) and \( C_6^{(0)[\infty]} \) are formal series in \( \hbar \).

III. We single out especially the limit case

\[
C^{(0)}(z|f, g) = C^{(0)III}(z|f, g) = m_{2|0}(z|f, g).
\]

Appendix 4. Solution of eq. \([A2.3]\)
Let \( C^{(1)}(z|f,g) = C^{(1)}_1(z|f,g) + O(h), C^{(1)}_1(z|f,g) = \theta_{\alpha|1} A^{(1)}_{\alpha|1}(z|f,g), \varepsilon_{A^{(1)}_{\alpha|1}} = 0 \), where \( \theta_{\alpha|1} \) is a finite set of odd generators of the Grassmann algebra of parameters. Sometimes, it is convenient to use for \( C^{(1)}_1(z|f,g) \) a representation

\[
C^{(1)}_1(z|f,g) = \int dvdu(-1)^{\epsilon(f)}[-a^{(1)}_{(000)|1}(x|y_u,y_v) + \xi_{\eta u}a^{(1)}_{(110)|1}(x|y_u,y_v) - \\
-\xi_{\eta u}a^{(1)}_{(101)|1}(x|y_u,y_v) + \eta_u\eta_va^{(1)}_{(001)|1}(x|y_u,y_v)]f(u)g(v),
\]

\[
a^{(1)}_{(000)|1}(x|y_v,y_u) = -a^{(1)}_{(000)|1}(x|y_u,y_v), \quad a^{(1)}_{(011)|1}(x|y_v,y_u) = a^{(1)}_{(011)|1}(x|y_u,y_v),
\]

\[
C^{(1)}_1(z|\hat{f}_0, \hat{g}_0) = a^{(1)}_{(011)|1}(x|f_0, g_0), \quad C^{(1)}_1(z|\hat{f}_1, \hat{g}_1) = \xi a^{(1)}_{(110)|1}(x|f_0, g_1),
\]

\[
a^{(1)}_{(2)|1}(x|f_1, g_1) = O(\theta)
\]

### 4.1. First order

It follows from eq. (A2.3) that

\[
J(m_{2|0}, C^{(1)}_1; z|f,g,h) = 0. \tag{A4.1}
\]

The general solution of eq. (A4.1) for \( f, g \in D_1 \) was found in Section 3

\[
C^{(1)}_1(z|f,g)_D = m^{(1)}_2(z|f,g) + m^{(2)}_2(z|f,g), \quad f, g \in D_1,
\]

\[
m^{(1)}_2(z|f,g) = c_{a|1} m^{(2)}_{2|a}(z|f,g) - M^{(1)}_{1|1}(z|[f,g]), \quad a = 1, 2, 3,
\]

\[
m^{(2)}_2(z|f,g) = |M^{(1)}_{1|1}(z|f), g(z)| - (-1)^{\epsilon(f) + 1}(\epsilon(g) + 1)|M^{(1)}_{1|1}(z|g), f(z)|,
\]

\[
M^{(1)}_{1|1}(z|\hat{f}_0) = \xi_{\nu_{(01)|1}}(x|f_0), \quad M^{(1)}_{1|1}(z|\hat{f}_1) = \nu_{(00)|1}(x|f_1),
\]

\[
C^{(1)}_1(z|f,g)_D \text{ means a restriction } C^{(1)}_1(z|f,g) \text{ to the space } f, g \in D_1. \quad \text{The form } m^{(2)}_2(z|f,g) \text{ are well defined for } f, g \in ED. \quad \text{Represent } C^{(1)}_1(z|f,g) \text{ in the form}
\]

\[
C^{(1)}_1(z|f,g) = m^{(1)}_2(z|f,g) + M^{(2)}_{2|1}(z|f,g), \quad f, g \in ED,
\]

\[
M^{(2)}_{2|1}(z|f,g)_D = m^{(2)}_2(z|f,g), \quad f, g \in D_1. \tag{A4.2}
\]

Represent \( M^{(2)}_{2|1}(z|f,g) \) in the form

\[
M^{(2)}_{2|1}(z|f,g) = \int dvdu(-1)^{\epsilon(f)}[-\mu^{(2)}_{(000)|1}(x|y_u,y_v) + \xi_{\eta u}\mu^{(1)}_{(110)|1}(x|y_u,y_v) - \\
-\xi_{\eta u}\mu^{(1)}_{(101)|1}(x|y_u,y_v) + \eta_u\eta_v\mu^{(1)}_{(001)|1}(x|y_u,y_v)]f(u)g(v),
\]

\[
M^{(2)}_{2|1}(z|\hat{f}_0, \hat{g}_0) = \mu^{(2)}_{(011)|1}(x|f_0, g_0), \quad M^{(2)}_{2|1}(z|\hat{f}_1, \hat{g}_1) = \mu^{(2)}_{(000)|1}(x|f_1, g_1),
\]

\[
M^{(2)}_{2|1}(z|\hat{f}_0, \hat{g}_1) = \xi \mu^{(2)}_{(110)|1}(x|f_0, g_1), \quad M^{(2)}_{2|1}(z|\hat{f}_1, \hat{g}_0) = -\xi \mu^{(2)}_{(101)|1}(x|g_0, f_1).
\]

Consider eq. (A4.2) in terms of different components.

#### 4.1.1. \( f = \hat{f}_0, \ g = \hat{g}_0 \)
We have

\[ M_{21}^{(2)}(z|\hat{f}_0, \hat{g}_0)_D = \mu_{(011)}^{(2)}(x|f_0, g_0)_D = \nu_{(11)}^{(1)}(x|f_0)g_0^\prime(x)_D + f_0^\prime(x)\nu_{(11)}^{(1)}(x|g_0)_D. \]  

(A4.3)

Let \( f_0(x) = x \) for \( x \in U \) and fixed, \( U \) is some vicinity. It follows from eq. (A4.3) for \( x \in U \):

\[ \nu_{(11)}^{(1)}(x|g_0)_D = \mu_{(011)}^{(2)}(x|f_0, g_0)_D - \nu_{(11)}^{(1)}(x|f_0)g_0^\prime(x)_D. \]  

(A4.4)

Rel. (A4.4) means that the forms \( \nu_{(11)}^{(1)}(x|g_0)_D \) can be extended to the case \( g_0 \in E \) such that we obtain that

\[ M_{21}^{(2)}(z|\hat{f}_0, \hat{g}_0) = \mu_{(011)}^{(2)}(x|f_0, g_0) = \nu_{(11)}^{(1)}(x|f_0)g_0^\prime(x) + f_0^\prime(x)\nu_{(11)}^{(1)}(x|g_0). \]

\( f = \hat{f}_1, \ g = \hat{g}_1 \) We have

\[ M_{21}^{(2)}(z|\hat{f}_1, \hat{g}_1) = \mu_{(000)}^{(2)}(x|f_1, g_1) = -\nu_{(00)}^{(1)}(x|f_1)g_1(x) + \nu_{(00)}^{(1)}(x|g_1)f_1(x). \]

\( f = \hat{f}_0, \ g = \hat{g}_1 \) We have

\[ M_{21}^{(2)}(z|\hat{f}_0, \hat{g}_1) = \mu_{(110)}^{(2)}(x|f_0, g_1) = \xi[\nu_{(11)}^{(1)}(x|f_0)g_1^\prime(x) - \nu_{(11)}^{(1)}(x|f_0)g_1(x)]. \]

Thus we obtain that

\[ C_1^{(1)}(z|f, g) = c_0 m_2(z|f, g) + d_1^{as} M_{11}(z|f, g), \ f, g \in E D_1, \ a = 1, 2, 3. \]

Note that the last summand in \( C_1^{(1)} \) gives the contribution to the function \( C(z|f, g) \) that can be canceled by a similarity transformation with \( T(z|f) = f(z) - h M_{11}(z|f) + O(h^2\theta_1^2) \) which does not change the function \( C_1^{(0)}(z|f, g) \).

The function \( C_1^{(1)}(z|f, g) \) can be represented in the form

\[ C_1^{(1)}(z|f, g) = \Theta \alpha m_2(z|f, g) + h^1 C_2^{(1)}(z|f, g) + O(h^2), \ a = 1, 2, 3, \]

\[ C_2^{(1)}(z|f, g) = \theta_1 A_0^{(1)}(z|f, g), \ e_{A_0^{(1)}} = 0, \ \Theta \alpha = \theta_1 c_0 a_{\alpha 1}. \]

4.2. I. \( C_1^{(0)}(z|f, g) = C_1^{(0)I}(z|f, g) \)

Let \( c_{4|\infty} = h^{k_{9} - 1} c_{4|k_0} + O(h^{k_9}), \ c_{4|k_0} \neq 0 \) Then we have

\[ C_1^{(0)I}(z|f, g) = \Theta p|k_0 m_2(z|f, g) + h^{k_0} C_{k_0+1}^{(0)I}(z|f, g) + O(h^{k_0+1}), \]

\[ C_{k_0+1}^{(0)I} = \theta_0 A_{\alpha_0}^{(1)I}, \ \Theta p|k_0 = \sum_{k=1}^{k_0} h^{k-1} \Theta p|k, \ \Theta p|k = \theta_0 c_{\alpha_0 p|k}, \ p = 1, 2, 3. \]
4.2.1. \((k_0 + 1)\)-th order

It follows from eq. \((A2.3)\) that

\[
J(m_{2|0}, C_{k_0+1}^{(1)I}) + c_{4|k_0} \Theta_p|_1 J(m_{2|4}, m_{2|p}) = 0, \quad p = 1, 2, 3. \tag{A4.5}
\]

Let \(f = \hat{f}_1, g = \hat{g}_1, h = \hat{h}_1\).

In this case, we have \(J(m_{2|4}, m_{2|3}; z|\hat{f}_1, \hat{g}_1, \hat{h}_1) \equiv 0\). It follows from eq. \((A4.5)\)

\[
J(m_{2|0}, C_{k_0+1}^{(1)I}; z|\hat{f}_1, \hat{g}_1, \hat{h}_1) + c_{4|k_0} \Theta_q|_1 J(m_{2|4}, m_{2|a}; z|\hat{f}_1, \hat{g}_1, \hat{h}_1) = 0, \quad q = 1, 2
\]

\[
a_{(000)|k_0+1}^{(1)I}(x|f_1 g'_1 - f'_1 g_1, h_1) = [\partial_x a_{(000)|k_0+1}(x|f_1, g_1)]h_1(x) + \text{cycle}(f_1, g_1, h_1) =
\]

\[
= - c_{4|k_0} \Theta_{q|1} J(m_{2|4}, m_{2|q}; z|\hat{f}_1, \hat{g}_1, \hat{h}_1).
\]

Let

\[
[x \cup \text{supp}(h_1)] \cap [\text{supp}(f_1) \cup \text{supp}(g_1)] = x \cap \text{supp}(h_1) = \emptyset.
\]

We have

\[
\hat{a}_{(000)|k_0+1}^{(1)I}(x|f_1 g'_1 - f'_1 g_1, h_1) = 0 \implies \hat{a}_{(000)|k_0+1}^{(1)I}(x|f_1, h_1) = 0 \implies
\]

\[
a_{(000)|k_0+1}^{(1)I}(x|f_1, h_1) = a_1(x|f_1, h_1) + a_2(x|f_1, h_1),
\]

\[
a_1(x|f_1, h_1) = \sum_{q=0}^{Q} \{ \partial^q_x f_1(x)a_1^q(x|h_1) - a_1^q(x|f_1) \partial^q_x h_1(x) \},
\]

\[
a_2(x|f_1, h_1) = \sum_{l=0}^{L} a_{2l+1}^{(1)I}(x|\partial^{2l+1} f_1)h_1(x) - f_1 \partial^{2l+1} h_1(x).
\]

Let

\[
[x \cup \text{supp}(h_1)] \cap [\text{supp}(f_1) \cup \text{supp}(g_1)] = \emptyset.
\]

We have

\[
[\partial_x a_{(000)|k_0+12}(x|f_1, g_1)h_1(x) + a_{(000)|k_0+12}(x|f_1 g_1 - f'_1 g_1, h_1)] =
\]

\[
= c_{4|k_0} \left( \Theta_{1|1} \overline{f_{1}^{m} g_1} + 2 \Theta_{2|1} \overline{\theta_x f_{1}^{m} g_1} \right) h'_1(x).
\]

or

\[
\sum_{l=0}^{L} \{ \partial_x a_{2l+1}^{(1)I}(x|\partial^{2l+1} f_1)h_1(x) - f_1 \partial^{2l+1} h_1(x) \}h_1(x) + \sum_{q=0}^{Q} a_1^q(x|f_1 g'_1 - f'_1 g_1) \partial^q_x h_1(x) =
\]

\[
= c_{4|k_0} \left( \Theta_{1|1} \overline{f_{1}^{m} g_1} + 2 \Theta_{2|1} \overline{\theta_x f_{1}^{m} g_1} \right) h'_1(x) \implies
\]

\[
\hat{a}_1^1(x|f_1 g'_1 - f'_1 g_1) = c_{4|k_0} \left( \Theta_{1|1} \overline{f_{1}^{m} g_1} + 2 \Theta_{2|1} \overline{\theta_x f_{1}^{m} g_1} \right).
\]

Let \(f_1(y) = 1\) for \(y \in \text{supp} g_1\) such that we obtain that

\[
\hat{a}_1^1(x|g'_1) = 0.
\]
Let \( f_1(y) = y \) for \( y \in \text{supp} g_1 \) such that we obtain that
\[
\hat{a}_1'(x'(yg_1)' - 2g_1) = 0 \implies \hat{a}_1'(x'g_1) = 0 \implies .
\]

\[
c_{4|k_0} \left( c_{1|1} f_1^{m} g_1 + 2 c_{2|1} \theta_x f_1^{m} g_1 \right) = 0 \implies \Rightarrow \quad c_{4|k_0} \theta_{1|1} = c_{4|k_0} \theta_{2|1} = 0 \implies (A4.6)
\]

\[
\Theta_{1|1} = \Theta_{2|1} = 0
\]

Eq. (A4.3) reduces to the form
\[
J(m_{2|0}, C_{k_0+1}^{(1)}) = -c_{4|k_0} \Theta_{3|1} J(m_{2|4}, m_{2|3}) = \text{loc.} \quad \text{(A4.7)}
\]

In the case when the support of some function does not intersect with the support of another function or some neighborhood of \( x \), eq. (A4.7) reduces to
\[
J(m_{2|0}, C_{k_0+1}^{(1)}) = 0. \quad \text{(A4.8)}
\]

For the case \( f, g, h \in D \), eq. (A4.8) was solved in [4] and [11] and we have
\[
C_{k_0+1}^{(1)} = \Theta p_{|k_0+1} m_{2|p} + d_{1}^{p} M_{1|k_0+1} + C_{k_0+1}^{(1)}, \quad p = 1, 2, 3.
\]

Eq. (A4.7) reduces to
\[
J(m_{2|0}, C_{k_0+1}^{(1)}) = -c_{4|k_0} \Theta_{3|1} J(m_{2|4}, m_{2|3}).
\]

Such equation was analized in [11]. It follows from the results of [11] that
\[
c_{4|k_0} \Theta_{3|1} = 0 \implies \Theta_{3|1} = 0, \quad \text{(A4.9)}
\]

and so on. We obtain finally \( C_{(1)}^{(1)} = 0 \) (more exactly, \( C_{(1)}^{(1)} \) can be canceled by a similarity transformation).

### 4.3. II. \( C_{(0)}^{(0)}(z|f, g) = C_{(0)}^{(0)II}(z|f, g) \)

Let \( c_{0|[\infty]} = c_{0|1} + O(h), \quad b = 5, 6, \quad m_{2|7|[\infty]} = m_{2|7|1} + O(h), \) at least one of the quantities \( c_{0|1}, m_{2|7|1} \) is not equal to zero. Then we have
\[
C_{(1)}^{(1)}(z|f, g) = \Theta p_{|1} m_{2|p} (z|f, g) + h C_{2}^{(1)II}(z|f, g) + O(h^2), \quad p^i = 1, 2, 3
\]

#### 4.3.1. Second order

Using the relations
\[
J(m_{2|1}, m_{2|d}) = J(m_{2|2}, m_{2|5}) = 0, \quad d = 5, 6, 7, \quad \text{(A4.10)}
\]
\[
J(m_{2|2}, m_{2|6}) = J(m_{2|0}, C_{26}), \quad J(m_{2|2}, m_{2|7}) = J(m_{2|0}, C_{27}), \quad \text{(A4.11)}
\]
we obtain from eq. [A2.3] that
\begin{align*}
J(m_{2|0}, \tilde{C}_2^{(1)II}) + \Theta_{3|1}[c_{0|1}J(m_{2|b}, m_{2|3}) + J(m_{2|7|1}, m_{2|3})] = 0, \quad (A4.12) \\
\tilde{C}_2^{(1)II} = C_2^{(1)II} + \Theta_{2|1}[c_{0|1}C_{06} + C_{27|1}], \quad b = 5, 6, \\
\tilde{C}_2^{(1)II}(z|f, g) = \int dvdu(-1)^{\epsilon(f)}[-a_{(000)}^{(1)II}(x|y_u, y_v) + \xi \eta \tilde{a}_{(110)}^{(1)II}(x|y_u, y_v) - \\
-\xi \eta \tilde{a}_{(110)}^{(1)II}(x|y_v, y_u) + \eta \xi \tilde{a}_{(011)}^{(1)II}(x|y_u, y_v)]f(u)g(v).
\end{align*}

Let \( f = \hat{f}_0, g = \hat{g}_0, h = \hat{h}_1. \) It follows from \( (A4.12) \) that
\begin{align*}
[a_{(001)2}^{(1)II}(x|f_0, g_0)]h_1(x) + [a_{(110)2}^{(1)II}(x|f_0, h_1)g_0(x) - \\
-\tilde{a}_{(011)2}^{(1)II}(x|f_0^0h_1, g_0) + (f_0 \leftrightarrow g_0)] = \Theta_{3|1}[c_{0|1}J(m_{2|b}, m_{2|3}; z|\hat{f}_0, \hat{g}_0, \hat{h}_1) + \\
+J(m_{2|7|1}, m_{2|3}; z|\hat{f}_0, \hat{g}_0, \hat{h}_1)], \quad b = 5, 6. \quad (A4.13)
\end{align*}

Let
\[ [x \cup \text{supp}(h_1)] \cap [\text{supp}(f_0) \cup \text{supp}(g_0)] = \emptyset \]
and \( f_0, g_0 \in D. \) It follows from \( (A4.13) \) that
\begin{align*}
\tilde{a}_{(011)2}^{(1)II}(x|f_0, g_0) = 0 & \quad \implies \\
\tilde{a}_{(011)2}^{(1)II}(x|f_0, g_0) = \partial_x \sum_{q=0}^{Q} a_q(x|f_0) \partial_x^q g_0(x) + b(x|f_0)g_0(x) + \\
+ (f_0 \leftrightarrow g_0) & \quad \implies \\
\tilde{a}_{(011)2}^{(1)II}(x|f_0, g_0) = \sum_{q=0}^{Q} a_q(x|f_0) \partial_x^q g_0(x) + \\
+ \int dy \theta(x - y)b(y|f_0)g_0(y) + d(f_0, g_0) + (f_0 \leftrightarrow g_0), \quad f_0, g_0 \in D_1.
\end{align*}

Let
\[ [x \cup \text{supp}(g_0)] \cap [\text{supp}(f_0) \cup \text{supp}(h_1)] = x \cap \text{supp}(g_0) = \emptyset, \]
f_0, g_0 \in D and \( f_0(y) = y \) for \( y \in \text{supp}h_1. \) It follows from \( (A4.13) \) that
\begin{align*}
\tilde{c}_{(011)2}^{(1)II}(x|h_1, g_0) = 0 & \quad \implies \int dy \theta(x - y)b(y|h_1)g_0(y) + \hat{d}(h_1, g_0) = 0 \quad \implies \\
\hat{d}(h_1, g_0) = 0 & \quad \implies b(y|h_1) = 0.
\end{align*}

Let
\[ [x \cup \text{supp}(g_0)] \cap [\text{supp}(f_0) \cup \text{supp}(h_1)] = \emptyset \]
and \( f_0, g_0 \in D. \) It follows from eq. \( (A4.13) \) that
\begin{align*}
\tilde{a}_{(110)2}^{(1)II}(x|f_0, h_1)g_0(x) - \sum_{q=0}^{Q} \tilde{a}_q(x|f_0^0h_1) \partial_x^q g_0(x) = \\
= \Theta_{3|1}[c_{0|1}f_0^0h_1 + c_{0|1}\theta_x f_0^1 + \omega_1(f_0^0h_1)]g_0(x) \quad \implies.
\end{align*}
3.3.2 Third order

The form

\[ \dot{a}_0(x) f'_0 h_1 = \Theta_3^2 [c_{51} f'_0 h'_1 + c_{61} \theta_x f'_0 h'_1 + \omega_1 (f'_0 h_1)] \implies \]

\[ \dot{a}_0(x) f'_0 h_1 = \Theta_{3\lambda_1} [c_{51} f'_0 h'_1 + \omega_1 (f'_0 h_1)], \quad \Theta_{3\lambda_1} c_{61} \theta_x f'_0 h'_1 = 0 \implies \]

\[ \Theta_{3\lambda_1} c_{51} = \Theta_{3\lambda_1} c_{61} = 0. \]

Eq. (A4.12) takes the form

\[ J(m_{2|0}; \tilde{C}_2^{(1)II}) + \Theta_{3\lambda_1} J(m_{2|71}; m_{2|3}) = 0. \]

Let \( f_0, g_0, h_0 \in D \). The we have

\[ J(m_{2|71}; m_{2|3}; z|f, g, h) = J(m_{2|0}, C_{3|71}; z|f, g, h) \implies \]

\[ J(m_{2|0}, \tilde{C}_2^{(1)II}) + \Theta_{3\lambda_1} C_{3|71} = 0 \implies \]

\[ \tilde{C}_2^{(1)II} + \Theta_{3\lambda_1} C_{3|71} = m_{2|2} + m_{2|3}, \quad (A4.14) \]

\[ m_{2|2}(z|f, g) = \Theta_{a|2} m_{2|a}(z|f, g) - M_{1|2}(z|[f, g]), \quad a = 1, 2, 3, \]

\[ m_{2|3} = \{ M_{1|2}(z|f), g(z) \} - (1)^{(\epsilon(f)+1)(\epsilon(g)+1)} M_{1|2}(z|g), f(z) \}, \]

\[ M_{1|2}(z|\hat{f}_0) = \xi \nu_{(11)|2}(x|f_0), \quad M_{1|2}(z|\hat{f}_1) = \nu(00)|2(x|f_1). \]

Rewrite eq. (A4.14) in the form

\[ P(z|f, g) = m_{2|2}(z|f, g) - \Theta_{3\lambda_1} C_{3|71}(z|f, g), \quad f_0, g_0 \in D, \]

\[ P(z|f, g) = \tilde{C}_2^{(1)II}(z|f, g) - m_{2|2}(z|f, g), \]

the form \( P(z|f, g) \) can be extended to \( f_0, g_0 \in E \).

Let \( f = f_0, g = \hat{g}_0 \) and \( y < x \) for \( y \in \text{supp} g_0 \) and \( y \in \text{supp} g_0 \). We have

\[ \Theta_{3\lambda_1} \omega_1(f_0 g_0) = P(z|\hat{f}_0, \hat{g}_0). \quad (A4.15) \]

It follows from eq. (A4.15) that the form \( \Theta_{3\lambda_1} \omega_1(f_0 g_0) \) can be extended to \( f_0, g_0 \in E \) what is possible for the case \( \Theta_{3\lambda_1} \omega_1 = 0 \) only. Thus we obtain that

\[ \Theta_{3\lambda_1} c_{51} = \Theta_{3\lambda_1} c_{61} = \Theta_{3\lambda_1} \omega_1 = 0 \implies \Theta_{3\lambda_1} = 0 \implies \]

\[ J(m_{2|0}, \tilde{C}_2^{(1)II}) = 0 \implies \]

\[ C_2^{(1)II} = \Theta_{p|2} m_{2|p} - \Theta_{2|1} c_{6|1} C_{26} + C_{27|1}, \quad p = 1, 2, 3. \]

Represent \( C^{(1)II} \) in the form

\[ C^{(1)II}(z|f, g) = \Theta_{p|2} m_{2|p}(z|f, g) - h \Theta_{2|1} c_{6|1} C_{26} + C_{27|1} + \]

\[ + h^2 C_3^{(1)II}(z|f, g) + O(h^3), \quad p = 1, 2, 3, \quad \Theta_{3\lambda_1} = 0 \]

4.3.2 Third order

It follows from eq. (A2.3) that

\[ J(m_{2|0}, \tilde{C}_3^{(1)II}) + \Theta_{3|2} c_{6|1} J(m_{2|b}, m_{2|3}) + J(m_{2|71}, m_{2|3}) = 0, \]

\[ \tilde{C}_3^{(1)II} = C_3^{(1)II} + h \Theta_{2|2} c_{6|2} C_{26} + C_{27|2} + O(h^3) \]

\[ b = 5, 6, \]
where we used relations (A4.10), (A4.11), and

\[ J(m_{2|1}, C_{67}) = J(m_{2|b}, C_{26}) = J(m_{2|b}, C_{27}) = J(m_{2|7}, C_{27}) = J(m_{2|7}) = 0, \]
\[ J(m_{2|7}, C_{26}) + J(m_{2|2}, C_{67}) = 0, \]
\[ b = 5, 6, \]

(A4.16)

and \( A(h) \) means the term of the \( h^n \)-order of the Taylor series of \( A(h) \).

Using the results of previous subsubsec., we find

\[ \Theta_{3|2} = 0, \]
\[ C_3^{(1)II} = \Theta_{a|3}m_{2|a} - \hbar^{-1} \left( \Theta_{2|[2]}[c_{6|[2]}C_{26} + C_{27|[2]}] \right)_{[1]}, \]
\[ a = 1, 2, 3. \]

Represent \( C^{(1)II} \) in the form

\[ C^{(1)II}(z|f, g) = \Theta_{p|[3]}m_{2|[p]}(z|f, g) - \hbar \left( [\Theta_{2|[2]}(c_{6|[2]}C_{26} + C_{27|[2]})] \right)_{[1]} + \]
\[ + \hbar^3 C_4^{(1)II}(z|f, g) + O(h^4), \]
\[ p = 1, 2, 3, \]
\[ \Theta_{3|[2]} = 0, \]

where \( A(h) \) means the part of the Taylor series of \( A(h) \) up to terms of \( h^n \)-order.

### 4.3.3. Fourth order

It follows from eq. (A2.3) that

\[ J(m_{2|0}, \bar{C}_4^{(1)II}) + \Theta_{3|3}[c_{6|1}J(m_{2|b}, m_{2|3}) + J(m_{2|7}, m_{2|3})] = 0, \]
\[ \bar{C}_4^{(1)II} = C_4^{(1)II} + \hbar^{-2} \left( \Theta_{2|[3]}[c_{6|[3]}C_{26} + C_{27|[3]}] \right)_{[2]} + \Theta_{2|[1]}c_{6|[1]}C_{267|[1]}, \]
\[ b = 5, 6, \]

(A4.17)

where we used relations (A4.10), (A4.11), (A4.16), and

\[ J(C_{26}, C_{67}) = J(m_{2|0}, C_{2667}), \]
\[ J(C_{27}, C_{67}) = 0. \]

(A4.18)

It follows from (A4.17) that \( C^{(1)II} \) can be represented in the form

\[ C^{(1)II}(z|f, g) = \Theta_{p|[4]}m_{2|[p]}(z|f, g) - \hbar \left( [\Theta_{2|[3]}(c_{6|[3]}C_{26} + C_{27|[3]})] \right)_{[2]} - \]
\[ - \hbar^3 \Theta_{2|[1]}c_{6|[1]}C_{267|[1]} + \hbar^4 C_5^{(1)II}(z|f, g) + O(h^5), \]
\[ p = 1, 2, 3, \]
\[ \Theta_{3|[3]} = 0, \]

where \( A(h) \) means the part of the Taylor series of \( A(h) \) up to terms of \( h^n \)-order.

### 4.3.4. Fifth order

It follows from eq. (A2.3) that

\[ J(m_{2|0}, \bar{C}_5^{(1)II}) + \Theta_{3|[4]}[c_{6|1}J(m_{2|b}, m_{2|3}) + J(m_{2|7}, m_{2|3})] = 0, \]
\[ \bar{C}_5^{(1)II} = C_5^{(1)II} + \hbar^{-3} \left( \Theta_{2|[4]}[c_{6|[4]}C_{26} + C_{27|[4]}] \right)_{[3]} + \]
\[ + \hbar^{-1} \left( \Theta_{2|[4]}c_{6|[4]}C_{267|[4]} \right)_{[1]}, \]
\[ b = 5, 6, \]

(A4.19)

where we used relations (A4.10), (A4.11), (A4.16), (A4.18) and

\[ J(m_{2|d}, C_{2667}) = J(C_{67}, C_{2667}) = 0, \]
\[ d = 5, 6, 7. \]
It follows from (A4.19) that $C^{(1)I}$ can be represented in the form

$$C^{(1)I}(z|f, g) = \Theta_{p|[5]}m_{2|[p]}(z|f, g) - \hbar \left( [\Theta_{2|[4]}(c_{6|[4]}C_{26} + C_{27|[4]})]_{[3]} - h^3 \left( \Theta_{2|[4]}c_{6|[4]}^2 C_{2667|[4]}_{[1]} \right) + h^5 C^{(1)I}(z|f, g) + O(\hbar^6), \ p = 1, 2, 3, \ \Theta_{3|[4]} = 0, \right)$$

and so on.

Finally, we obtain: general solution of eq. (A2.3) for the case $C^{(0)} = C^{(0)I}$ and at least one of the quantities $c_{6|[1]}, b = 5, 6, m_{2|[7]}$ is not equal to zero is

$$C^{(1)I}(z|f, g) = \Theta_{a|[∞]}m_{2|[a]}(z|f, g) - \hbar \Theta_{2|[∞]}(c_{6|[∞]}C_{26} + C_{27|[∞]}) - h^3 \Theta_{2|[∞]}c_{6|[∞]}^2 C_{2667|[∞]}, \ q = 1, 2.$$

(A4.20)

It can be analogously proved that general solution of eq. (A2.3) has as well form (A4.20) in the case $c_{a|[∞]} = \hbar^{k_0-1}c_{a|k_0} + O(\hbar^{k_0}), m_{2|[7]} = \hbar^{k_0-1}m_{2|[7|k_0} + O(\hbar^{k_0}), k_0 > 1,$ and at least one of the quantities $c_{a|k_0}, m_{2|[7|k_0}$ is not equal to zero.

4.4. III. $C^{(0)}(z|f, g) = C^{(0)II}(z|f, g)$

In this case, eq. (A2.3) takes the form

$$J(m_{2|0}, C^{(1)II}) = 0,$$

such that we obtain

$$C^{(1)II}(z|f, g) = \Theta_{a|[∞]}m_{2|[a]}(z|f, g), \ a = 1, 2, 3.$$

Appendix 5. Extension to exact solution

5.1. Case I

In this case, the form $C = C^{(0)}I = N_{c_{4|[∞]}}$ is an exact solution.

5.2. Case II

In this case, the form $C = C^{(0)II} + \hbar C^{(1)II}$ is an exact solution. When proving, we used
the relations

\[ J(m_{2|b}, m_{2|b'}) = J(m_{2|5}, m_{2|7}) = 0, \ J(m_{2|6}, m_{2|7}) = J(m_{2|0}, C_{67}), \]
\[ J(m_{2|b}, m_{2|1}) = J(m_{2|5}, m_{2|2}) = 0, \ J(m_{2|6}, m_{2|2}) = J(m_{2|0}, C_{26}), \]
\[ J(m_{2|b}, C_{26}) = J(m_{2|b}, C_{27}) = J(m_{2|b}, C_{267}) = 0, \]
\[ J(m_{2|7}) = J(m_{2|7}, C_{67}) = J(m_{2|7}, m_{2|1}) = 0, \ J(m_{2|7}, m_{2|2}) = J(m_{2|0}, C_{27}), \]
\[ J(m_{2|7}, C_{27}) = J(m_{2|7}, C_{267}) = 0, \ J(C_{67}) = J(C_{67}, m_{2|1}) = 0 \]
\[ J(m_{2|7}, C_{26}) + J(C_{67}, m_{2|2}) = 0, \ J(C_{67}, C_{26}) = J(m_{2|0}, C_{267}), \]
\[ J(C_{67}, C_{27}) = J(C_{67}, C_{267}) = 0, \]
\[ J(m_{2|1}, C_{26}) = J(m_{2|1}, C_{27}) = J(m_{2|1}, C_{267}) = 0. \]

5.3. Case \( \text{III} \)

We find a solution in the form

\[ C^{\text{III}} = m_{2|0} + \hbar \Theta_{p[\infty]} m_{2|p} + \hbar^2 C^{(2)\text{III}}, \ C^{(2)\text{III}} = \gamma^i C^{(2)\text{III}}_i, \ p, i = 1, 2, 3, \]
\[ \gamma^1 = \Theta_{2[\infty]} \Theta_3[\infty], \ \gamma^2 = \Theta_3[\infty] \Theta_1[\infty], \ \gamma^3 = \Theta_1[\infty] \Theta_2[\infty], \ \varepsilon C^{(2)\text{III}}_i = 1. \]

We have

\[ J(m_{2|0}, C^{(2)\text{III}}) + J(\Theta_5 m_{2|q}, \Theta_3 m_{2|3}) = 0, \ q = 1, 2, \]

or

\[ \gamma^i J(m_{2|0}, C^{(2)\text{III}}_i) + \gamma^2 J(m_{2|1}, m_{2|3}) + \gamma^1 J(m_{2|3}, m_{2|2}) = 0, \quad (A5.1) \]

where we used relation \( J(m_{2|1}, m_{2|2}) = 0. \)

Consider eq. (A5.1) for the functions \( f = \hat{f}_0, \ g = \hat{g}_1, \ h = \hat{h}_1, \)

\[ [x \cup \text{supp}(f_0)] \cap [\text{supp}(g_1) \cup \text{supp}(h_1)] = \emptyset, \]

and \( f_0(x) = 1. \) We have

\[ \gamma^i \hat{C}^{(2)\text{III}} \hat{g}_1 h_1 \eta \hat{f}_0 + \gamma^2 \hat{g}_1 h_1 \eta = 0 \quad (A5.2) \]

Let \( g(y) = 1 \) for \( y \in \text{supp}(h_1). \) It follows from eq. (A5.2) that

\[ \gamma^i \hat{C}^{(2)\text{III}} \hat{g}_1 h_1 \eta \hat{f}_0 = 0. \]

Let \( g(y) = y \) for \( y \in \text{supp}(h_1). \) It follows from eq. (A5.2) that

\[ \gamma^i \hat{C}^{(2)\text{III}} \hat{g}_1 h_1 \eta \hat{f}_0 = 0 \Rightarrow \gamma^2 \hat{g}_1 h_1 \eta = 0 \Rightarrow \gamma^1 = \gamma^2 = 0, \]

or

\[ \Theta_1[\infty] \Theta_3[\infty] = \Theta_2[\infty] \Theta_3[\infty] = 0 \quad (A5.3) \]
Eq. (A5.3) has two types of solutions.

\[ III_1: \Theta_{3[\infty]} = 0, \Theta_{1[\infty]} \text{ and } \Theta_{2[\infty]} \text{ are arbitrary.} \]

\[ III_2: \]

\[ \Theta_{3[\infty]} \neq 0, \quad \Theta_{q[\infty]} = c_q[\infty] \Theta_{3[\infty]}, \quad q = 1, 2 \]

Respectively, we have two types of exact solutions.

\[ III_1: \quad C^{III_1} = m_{2[0]} + h \Theta_{q[\infty]} m_{2[q]}, \quad q = 1, 2. \]

\[ III_2: \quad C^{III_2} = m_{2[0]} + h \Theta_{3[\infty]} (c_q[\infty] m_{2[q]} + m_{2[3]}), \quad q = 1, 2. \]

### Appendix 6. Evaluation of general solution of eq. (A2.1)

#### 6.1. Case I

In this case, we can represent the form \( C \) as \( C = C' + \hbar^2 C^{(2)} + O(\theta^3) = \mathcal{N}_{c_4[\infty]} + \hbar^2 C^{(2)} + O(\theta^3) \) where \( C^{(2)} \) has the properties \( \varepsilon_{C^{(2)}} = 1, C^{(2)} = O(\theta^2) \), and satisfies the equation

\[ J(\mathcal{N}_{c_4[\infty]}, C^{(2)}) = 0. \]

Let \( c_4[\infty] = h^{k_0-1}c_{4|k_0} + O(h^{k_0}), \quad c_4|k_0 \neq 0 \) Then, in orders \( k \leq k_0, C^{(2)}_k \) satisfy the equation

\[ J(m_{2[0]}, C^{(2)}_k) = 0 \]

and we have

\[ C^{(2)} = \Theta^{(2)}_{a|k_0} m_{2|a} + m_{2|7}^{(2)}, \quad \Theta^{(2)}_{a|k_0} = O(\theta^2), \quad a = 4, 5, 6. \]

\[ m_{2|7}^{(2)} = \sum_{k=1}^{k_0} \hbar^{k-1} m_{2|7}^{(2)}, \quad m_{2|7} = O(\theta^2) \]

Represent \( C' \) in the form

\[ C' = \mathcal{N}_{c_4[\infty]}^{(2)} + \hbar^2 \Theta^{(2)}_{b|k_0} m_{2|b} + h^2 m_{2|7}^{(2)} + \hbar^{k_0+2} C^{(2)}_{k_0+1} + O(\theta^3), \]

\[ c_4[\infty]^{(2)} = c_4[\infty] + \hbar^2 \Theta^{(2)}_{4|k_0}, \quad b = 5, 6, \quad J(\mathcal{N}_{c_4[\infty]}^{(2)}) = 0. \]

#### 6.1.1. \((k_0 + 1)\)-th order

It follows from eq. (A2.1) that

\[ J(m_{2[0]}, C^{(2)}_{k_0+1}) + c_{4|k_0} \Theta^{(2)}_{b|1} J(m_{2|4}, m_{2|b}) + c_{4|k_0} J(m_{2|4}, m_{2|7}^{(2)}) = 0, \quad b = 5, 6. \]
As we seen above, see subsec 2 of sec 3, it follows from eq. (A6.1) that
\[
\Theta_{b|1}^{(2)} = m_{2|7}^{(2)|1} = 0, \quad C_{k_0+1}^{(2)I} = \Theta_{b|k_0+1}^{(2)} m_{2|b} + m_{2|7}^{(2)|k_0+1}.
\]

Represent \( C^I \) in the form
\[
C^I = N_{c_4|\infty}^{(2)|k_0+1} + \hbar^2 \Theta_{b|k_0+1}^{(2)} m_{2|b}(z|f, g) + \hbar^2 m_{2|7}^{(2)|k_0+1} + \hbar^{k_0+3} C_{k_0+2}^{(2)I} + O(\hbar^{k_0+4}) + O(\theta^3),
\]
where
\[
c_{4|\infty}^{(2)} = c_{4|\infty} + \hbar^2 \Theta_{4|k_0+1}, \quad b = 5, 6,
\]
and so on.

Finally we have
\[
C^I = N_{c_4|\infty}^{(2)|\infty} + \hbar^3 C^{(3)I} + O(\theta^4),
\]

(3). \( C^{(3)I} \) satisfies the equation
\[
J(N_{c_4|\infty}, C^{(3)I}) = 0.
\]

We have
\[
C^{(3)I} = \Theta_{p|k_0}^{(3)} m_{2|p} + \hbar^{k_0} C_{k_0+1}^{(3)I} + O(\hbar^{k_0+1}),
\]
where
\[
\Theta_{p|k}^{(3)} = \sum_{k=1}^{k_0} \hbar^{k-1} \Theta_{p|k}, \quad \Theta_{p|k}^{(3)} = O(\theta^3), \quad p = 1, 2, 3.
\]

In \((k_0 + 1)\)-th order we obtain
\[
J(m_{2|0}, C^{(3)I}_{k_0+1}) + c_{4|k_0}^{(3)I} J(m_{2|4}, m_{2|p}) = 0, \quad p = 1, 2, 3. \tag{A6.2}
\]

As we seen above, see subsubsec. 1 of subsec.2 of sec 4, it follows from eq. (A6.2) that
\[
\Theta_{p|1}^{(3)} = 0, \quad C_{k_0+1}^{(3)I} = \Theta_{p|k_0+1}^{(3)} m_{2|p}.
\]

and so on. Thus we find \( C^{(3)I} = 0 \) and
\[
C^I = N_{c_4|\infty}^{(2)|\infty} + \hbar^4 C^{(4)I} + O(\theta^5), \quad J(N_{c_4|\infty}, C^{(4)I}) = 0.
\]

Finally we obtain
\[
C^I = N_{c_4|\infty}^{(2)|\infty} + c_{4|\infty}^{(2)|\infty} = \sum_{k,l=0}^{\infty} \hbar^{k+2l-1} \theta^{2l} c_{4|k}^{(l)},
\]
\[
c_{4|k}^{(0)} = 0, \quad 0 \leq k \leq k_0 - 1, \quad c_{4|k_0}^{(0)} = c_{4|k_0} \neq 0.
\]

### 6.2. Case II
(2). In this case, we can represent the form \( C \) as \( C = C^{HI} = C(0)^{HI} + hC(1)^{HI} + h^2C(2)^{HI} + O(\theta^3) \), where \( C(0)^{HI} \) is given by eq. (A3.19), \( C(1)^{HI} \) is given by eq. (A4.20), \( J(C^{0})^{HI} + C(1)^{HI} \) = 0, \( C(2)^{HI} \) has the properties \( \varepsilon_{C(2)^{HI}} = 1 \), \( C(2)^{HI} = O(\theta^2) \), and satisfies the equation

\[
J(C(0)^{HI}, C(2)^{HI}) = 0.
\]

Introduce notation

\[
C^{0}^{HI} + hC^{1}^{HI} = \mathcal{N}^{HI}(c_{b[\infty]}, m_{2|7}[\infty], \Theta_{q[\infty]}),
\]

\[
b = 5, 6, \quad q = 1, 2.
\]

Let \( c_{b[\infty]} = h^{k_0-1}c_{b|k_0} + O(h^{k_0}), \) \( m_{2|7}[\infty] = h^{k_0-1}m_{2|7}k_0 + O(h^{k_0}), \) \( k_0 \geq 1, \) \( b = 5, 6, \) and at least one of the quantities \( c_{b|k_0}, \) \( m_{2|7}k_0 \) is not equal to zero. Then, in orders \( k \leq k_0, \) \( C_k^{(2)HI} \) satisfy the equation

\[
J(m_{2|0}, C_k^{(2)HI}) = 0
\]

and we have

\[
C_k^{(2)HI} = \Theta_{a[k]}(2)m_{2|a} + m_{2|7}[k_0] + h^{k_0}C_{k_0+1}^{(2)HI} + O(h^{k_0+1}),
\]

\[
\Theta_{a[k]}(2) = \sum_{k=1}^{k_0} h^{k-1}\Theta_{a[k]}(2)\Theta_{a[k]}(2) = O(\theta^2), \quad a = 4, 5, 6.
\]

\[
m_{2|7}[k_0] = \sum_{k=1}^{k_0} h_{k-1}m_{2|7}[k_0], \quad m_{2|7}[k_0] = O(\theta^2)
\]

Represent \( C^{HI} \) in the form

\[
C^{HI} = \mathcal{N}^{HI}(c_{b[\infty]}, m_{2|7}[\infty], \Theta_{q[\infty]} + h^2\Theta_{a[4|k]}(2)m_{2|4} + h^{k_0+2}C_{k_0+1}^{(2)HI}(z, f, g) + O(h^{k_0+3}) + O(\theta^3),
\]

\[
c_{b[\infty]} = c_{b|\infty} + h^2\Theta_{b[k_0]}(2), \quad m_{2|7}[\infty] = m_{2|7}[\infty] + h^2m_{2|7}[k_0], \quad b = 5, 6.
\]

6.2.1. \( (k_0 + 1) \)-th order

It follows from eq. (A2.1) that

\[
J(m_{2|0}, C_k^{(2)HI}) + c_{b|k_0}\Theta_{4|1}^{(2)}J(m_{2|8}, m_{2|4}) + \Theta_{4|1}^{(2)}J(m_{2|7}k_0, m_{2|4}) = 0, \quad b = 5, 6. \quad (A6.3)
\]

As we seen above, see subsec. 2 of subsec. 4 of sec. 3, it follows from eq. (A6.3) that

\[
\Theta_{4|1}^{(2)} = 0, \quad C^{(2)HI} = \Theta_{a[k_0+1]}^{(2)}m_{2|a} + m_{2|7}[k_0+1].
\]

Represent \( C^{HI} \) in the form

\[
C^{HI} = \mathcal{N}^{HI}(c_{b[\infty]}, m_{2|7}[\infty], \Theta_{q[\infty]} + h^2\Theta_{a[4|k_0+1]}(2)m_{2|4} +
\]

\[
+ h^{k_0+3}C_{k_0+2}^{(2)HI}(z, f, g) + O(h^{k_0+4}) + O(\theta^3),
\]

\[
c_{b[\infty]} = c_{b|\infty} + h^2\Theta_{b[k_0]}(2),
\]

\[
m_{2|7}[\infty] = m_{2|7}[\infty] + h^2m_{2|7}[k_0], \quad b = 5, 6, \quad \Theta_{4|1}^{(2)} = 0.
\]
and so on.

Finally we have
\[ C^{II} = \mathcal{N}^{II}(c_{b}^{[2]||[\infty]], m_{2}^{[2]||[\infty]}) + h^{3}c_{q}^{[3]||[\infty]} + O(\theta^{4}), \]

(3) \( C^{(3)II} \) satisfies the equation
\[ J(C^{(0)II}, C^{(3)II}) = 0. \]

We have
\[ C^{(3)II} = \Theta^{(3)}_{p|k_{0}}m_{2}^{[2]||[\infty]} + h^{k_{0}}C^{(3)I}_{k_{0}+1} + O(h^{k_{0}+1}), \]
\[ \Theta^{(3)}_{p|k_{0}} = \sum_{k=1}^{k_{0}} h^{k-1}\Theta_{p|k}, \quad \Theta^{(3)}_{p|k} = O(\theta^{3}), \quad p = 1, 2, 3. \]

Represent \( C^{II} \) in the form
\[ C^{II} = \mathcal{N}^{II}(c_{b}^{[2]||[\infty]], m_{2}^{[2]||[\infty]}, \Theta^{(3)}_{q|[\infty]} + \Theta^{(3)}_{q|[k_{0}+1]}, m_{2}^{[2]||[\infty]} + \]
\[ h^{k_{0}+2}C^{(3)II}_{k_{0}+1} + O(h^{k_{0}+3}) + O(\theta^{4}), \]
\[ \Theta^{(3)}_{q|[k_{0}]} = \Theta_{q|[\infty]} + \Theta_{q|[k_{0}]}, \quad q = 1, 2. \]

In \((k_{0}+1)\)-th order we obtain
\[ J(m_{2}^{[0]||[k_{0}+1]}, C^{(3)II}_{k_{0}+1}) + c_{b|k_{0}}\Theta^{(3)}_{q|[1]}, J(m_{2}^{[0]||[k_{0}+1]}, m_{2}^{[2]||[k_{0}+1]}) = 0, \quad 5, 6. \]

As we seen above, see subsubsec. 1 of subsec. 3 of sec 4, it follows from eq. (A6.4) that
\[ \Theta^{(3)}_{q|[1]} = 0, \quad C^{(3)II}_{k_{0}+1} = \Theta^{(3)}_{p|k_{0}+1}m_{2}^{[2]||[\infty]} \]
\[ C^{II} = \mathcal{N}^{II}(c_{b}^{[2]||[\infty]], m_{2}^{[2]||[\infty]}, \Theta^{(3)}_{q|[\infty]} + \Theta^{(3)}_{q|[k_{0}+1]}, m_{2}^{[2]||[\infty]} + \]
\[ h^{k_{0}+3}C^{(3)II}_{k_{0}+1} + O(h^{k_{0}+4}) + O(\theta^{4}), \quad \Theta^{(3)}_{q|[k_{0}]} = 0, \]

and so on, such that we obtain.
\[ C^{II} = \mathcal{N}^{II}(c_{b}^{[2]||[\infty]], m_{2}^{[2]||[\infty]}, \Theta^{(3)}_{q|[\infty]} + h^{3}C^{(4)II} + O(\theta^{4}), \]
\[ J(C^{(0)II}, C^{(4)II}) = 0. \]

Finally we find
\[ C^{II} = \mathcal{N}^{II}(c_{b}^{[\infty]|[\infty]}, m_{2}^{[\infty]|[\infty]}, \Theta^{[\infty]|[\infty])}, \]
\[ c_{b}^{[\infty]|[\infty]} = \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} h^{k+2l-1}\theta^{2l}c_{b|k}, \quad c_{b|k} = 0, \quad 0 \leq k \leq k_{0} - 1, \]
\[ m_{2}^{[\infty]|[\infty]} = \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} h^{k+2l-1}\theta^{2l}m_{2|7|k}, \quad m_{2|7|k} = 0, \quad 0 \leq k \leq k_{0} - 1, \]
\[ \Theta_{q|[\infty]}^{(2)} = \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} h^{k+2l-1}\theta^{2l+1}c_{q|k}, \quad q = 1, 2, \quad b = 5, 6, \]
where, at least, one of the numbers $c^0_{b|k_0}$, $m^0_{2|7|k_0}$ is not equal to zero.

### 6.3. Case III, only three Grassmann generators $\theta_p$, $p = 1, 2, 3$

In this case, we have

$$ C = C^{III} = m_{2|0} + \theta_p c_{p|p'} m_{2|p'} + \gamma_i C_i^{III(2)} + \beta C_i^{III(3)}, $$

$$ \gamma^i = \frac{1}{2} \varepsilon^{ijk} \theta_j \theta_k, \quad \beta = \theta_1 \theta_2 \theta_3, \quad p, p', i, j, k = 1, 2, 3. $$

It follows from eq. (A2.1) that

$$ J(m_{2|0}, C_i^{III(2)}) + \varepsilon^{ijk} c_{j|q} c_{k|3} J(m_{2|q}, m_{2|3}) = 0, \quad \text{(A6.5)} $$

$$ J(m_{2|0}, C_i^{III(3)}) + c_{i|p} J(m_{2|p}, C_i^{III(2)}) = 0, \quad \text{(A6.6)} $$

where we used the equalities $\theta_i \theta_j = \varepsilon^{ijk} \gamma^k$, $\theta_i \gamma^k = \delta_i^k \beta$.

It follows from eq. (A6.5) (see subsec. 3 of sec. 5)

$$ \varepsilon^{ijk} c_{j|q} c_{k|3} = 0, $$

$$ J(m_{2|0}, C_i^{III(2)}) \Rightarrow C_i^{III(2)} = d_{i|a} m_{2|a} + m_{2|7,i}, \quad a = 4, 5, 6. $$

Eq. (A6.6) reduces to the form

$$ J(m_{2|0}, \tilde{C}^{III(3)}) + c_{i|q} d_{i|4} J(m_{2|q}, m_{2|4}) + c_{i|3} J(m_{2|3}, d_{i|a} m_{2|a} + m_{2|7,i}) = 0, \quad q = 1, 2, a = 4, 5, 6, \quad \tilde{C}^{III(3)} = C^{III(3)} + c_{i|2} (d_{i|6} C_{26} + C_{27,i}). $$

Let $f = \tilde{f}_1$, $g = \tilde{g}_1$, $h = \tilde{h}_1$. In this case $J(m_{2|3}, d_{i|a} m_{2|a} + m_{2|7,i}) = 0$ and we obtain that

$$ J(m_{2|0}, \tilde{C}^{III(3)}) + c_{i|q} d_{i|4} J(m_{2|q}, m_{2|4}) = 0, $$

and we find that (see subsubsec. 1 of subsec. 2 of sec. 4, eq. (A4.6))

$$ c_{i|q} d_{i|4} = 0, $$

$$ J(m_{2|0}, \tilde{C}^{III(3)}) + c_{i|3} J(m_{2|3}, d_{i|a} m_{2|a} + m_{2|7,i}) = 0. $$

Let $f = \tilde{f}_0$, $g = \tilde{g}_0$, $h = \tilde{h}_1$ and their domains have the properties $[x \cup \supp(h_1)] \cap [\supp(f_0) \cup \supp(g_0)] = \emptyset$, or $[x \cup \supp(g_0)] \cap [\supp(f_0) \cup \supp(h_1)] = x \cap \supp(g_0) = \emptyset$, or $x \cup \supp(g_0) \cap [\supp(f_0) \cup \supp(h_1)] = \emptyset$. In this case $J(m_{2|3}, m_{2|4}) = 0$ and we obtain that

$$ J(m_{2|0}, \tilde{C}^{III(3)}) + c_{i|3} J(m_{2|3}, d_{i|b} m_{2|b} + m_{2|7,i}) = 0, \quad b = 5, 6. \quad \text{(A6.7)} $$

It follows from (A6.7) (see subsubsec. 1 of subsec. 3 of sec. 4) that

$$ c_{i|3} d_{i|b} = c_{i|3} \omega_i = 0, $$

$$ J(m_{2|0}, \tilde{C}^{III(3)}) + c_{i|3} d_{i|4} J(m_{2|3}, m_{2|4}) = 0. \quad \text{(A6.8)} $$
It follows from (A6.8) (see subsubsec. 1 of subsec. 2 of sec.4, eq. (A4.9)) that
\[ c_{i|3}d_{i|4} = 0, \]
\[ J(m_{2|0}, \tilde{C}^{III}(3)) = 0 \implies \tilde{C}^{III}(3) = e_p m_{2|p}. \]
Thus, we obtain that the general solution of the type of $C^{III}$ with only three generators $\theta_i$ has the form
\[ C^{III} = m_{2|0} + \hbar\theta_p c_{p|p'} m_{2|p'} + \hbar^2 \gamma^i (d_{i|a} m_{2|a} + m_{2|7,i}) + \]
\[ + \hbar^3 \beta [e_p m_{2|p} - c_{i|2}(d_{i|6} C_{26} + C_{27,i})], \]
\[ p, p' = 1, 2, 3, \quad q = 1, 2, \quad a = 4, 5, 6, \]
where $c_{p|p'}$, $d_{i|a}$, $e_p$, and $\omega_i$ are arbitrary series in $\hbar$ satisfying the conditions
\[ \varepsilon^{ijk} c_{j|q} c_{k|3} = c_{i|q} d_{i|4} = c_{i|3} d_{i|a} = c_{i|3} \omega_i = 0. \]

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