(Non) Commutative Finsler Geometry from String/M–theory

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Abstract

We synthesize and extend the previous ideas about appearance of both non-commutative and Finsler geometry in string theory with nonvanishing B–field and/or anholonomic (super) frame structures [12, 13, 18, 50]. There are investigated the limits to the Einstein gravity and string generalizations containing locally anisotropic structures modeled by moving frames. The relation of anholonomic frames and nonlinear connection geometry to M–theory and possible non-commutative versions of locally anisotropic supergravity and D–brane physics is discussed. We construct and analyze new classes of exact solutions with noncommutative local anisotropy describing anholonomically deformed black holes (black ellipsoids) in string gravity, embedded Finsler–string two dimensional structures, solitonically moving black holes in extra dimensions and wormholes with noncommutativity and anisotropy induced from string theory.

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Contents

1 Introduction

2 String Theory and Commutative Riemann–Finsler Gravity 5

2.1 Strings in general manifolds and bundles 6

2.1.1 Generalized nonlinear sigma models (some basics) 6

2.1.2 Anholonomic frame transforms of background metrics 7

2.1.3 Anholonomic background field quantisation method 11

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| Section | Title | Page |
|---------|-------|------|
| 2.2 | Low energy string anholonomic field equations | 13 |
| 2.2.1 | Low energy string anisotropic field equations and effective action | 15 |
| 2.2.2 | Affolonomic Einstein and Finsler gravity from string theory | 16 |
| 3 | Superstrings and Anisotropic Supergravity | 19 |
| 3.1 | Locally anisotropic supergravity theories | 21 |
| 3.1.1 | $N=1, n+m=11$ anisotropic supergravity | 22 |
| 3.1.2 | Type IIA anisotropic supergravity | 23 |
| 3.1.3 | Type IIB, $n+m=10, N=2$ anisotropic supergravity | 23 |
| 3.2 | Superstring effective actions and anisotropic toroidal compactifications | 24 |
| 3.3 | 4D NS–NS anholonomic field equations | 26 |
| 3.4 | Distinguishing anholonomic Riemannian–Finsler(sup) gravities | 27 |
| 4 | Noncommutative Anisotropic Field Interactions | 29 |
| 4.1 | Basic definitions and conventions | 29 |
| 4.1.1 | Matrix algebras and noncommutativity | 29 |
| 4.1.2 | Noncommutative Euclidean space $\mathbb{R}^k$ | 30 |
| 4.1.3 | The noncommutative derivative and integral | 31 |
| 4.2 | Anholonomic frames and noncommutative spacetimes | 32 |
| 4.2.1 | Noncommutative anholonomic derivatives | 32 |
| 4.2.2 | Noncommutative anholonomic torus | 34 |
| 4.3 | Anisotropic field theories and anholonomic symmetries | 34 |
| 4.3.1 | Locally anisotropic matrix scalar field theory | 35 |
| 4.3.2 | Locally anisotropic noncommutative gauge fields | 35 |
| 5 | Anholonomy and Noncommutativity: Relations to String/M–Theory | 39 |
| 5.1 | Noncommutativity and anholonomy in string theory | 39 |
| 5.2 | Noncommutative anisotropic structures in M(atrix) theory | 43 |
| 6 | Anisotropic Gravity on Noncommutative D–Branes | 44 |
| 7 | Exact Solutions: Noncommutative and/or Locally Anisotropic Structures | 47 |
| 7.1 | Black ellipsoids from string gravity | 47 |
| 7.2 | 2D Finsler structures in string theory | 50 |
| 7.3 | Moving soliton–black hole string configurations | 52 |
| 7.3.1 | 3D solitonic deformations in string gravity | 54 |
| 7.3.2 | Solitonically propagating string black hole backgrounds | 55 |
| 7.4 | Noncommutative anisotropic wormholes and strings | 58 |
| 8 | Comments and Questions | 59 |
| A | Anholonomic Frames and N–Connections | 61 |
| A.1 | The N–connection geometry | 62 |
| A.1.1 | N–connections in vector bundles and (pseudo) Riemannian spaces | 62 |
1 Introduction

The idea that string/M–theory results in a noncommutative limit of field theory and spacetime geometry is widely investigated by many authors both from mathematical and physical perspectives [57, 38, 9] (see, for instance, the reviews [13]). It is now generally accepted that noncommutative geometry and quantum groups [8, 16, 19] play a fundamental role in further developments of high energy particle physics and gravity theory.

First of all we would like to give an exposition of some basic facts about the geometry of anholonomic frames (vielbeins) and associated nonlinear connection (N–connection) structures which emphasize surprisingly some new results: We will consider N–connections in commutative geometry and we will show that locally anisotropic spacetimes (anholonomic Riemannian, Finsler like and their generalizations) can be obtained from the string/M–theory. We shall discuss the related low energy limits to Einstein and gauge gravity. Our second goal is to extend A. Connes’ differential noncommutative geometry as to include geometries with anholonomic frames and N–connections and to prove that such 'noncommutative anisotropies' also arise very naturally in the framework of strings and extra dimension gravity. We will show that the anholonomic frame method is very useful in investigating of new symmetries and nonperturbative states and for constructing new exact solutions in string gravity with anholonomic and/or noncommutative variables. We remember here that some variables are considered anholonomic (equivalently, nonholonomic) if they are subjected to some constraints (equivalently, anholonomy conditions).

Almost all of the physics paper dealing with the notion of (super) frame in string theory do not use the well developed apparatus of E. Cartan’s ‘moving frame’ method [6] which gave an unified approach to the Riemannian and Finsler geometry, to bundle spaces and spinors, to the geometric theory of systems of partial equations and to Einstein (and the so–called Einstein–Cartan–Weyl) gravity. It is considered that very ”sophisticate” geometries like the Finsler and Cartan ones, and theirs generalizations, are less related to real physical theories. In particular, the bulk of frame constructions in string and gravity theories are given by coefficients defined with respect to coordinate frames or in abstract form with respect to some general vielbein bases. It is completely disregarded the fact that via anholonomic frames on (pseudo) Riemannian manifolds and on (co) tangent and (co) vector bundles we can model different geometries and interactions with local anisotropy even in the framework of generally
accepted classical and quantum theories. For instance, there were constructed a num-
ber of exact solutions in general relativity and its lower/higher dimension extensions
with generic local anisotropy, which under certain conditions define Finsler like ge-
ometries. It was demonstrated that anholonomic geometric constructions are inevitable in the theory of anisotropic stochastic, kinetic and thermodynamic processes in curved spacetimes and proved that Finsler like (super) geometries are contained alternatively in modern string theory.

We emphasize that we have not proposed any "exotic" locally anisotropic modifi-
cations of string theory and general relativity but demonstrated that such anisotropic
structures, Finsler like or another type ones, may appear alternatively to the Riemann-
nian geometry, or even can be modeled in the framework of a such geometry, in the low
energy limit of the string theory, if we are dealing with frame (vielbein) constructions.

One of our main goals is to give an accessible exposition of some important notions
and results of N–connection geometry and to show how they can be applied to con-
crete problems in string theory, noncommutative geometry and gravity. We hope to
convince a reader–physicist, who knows that ‘the B–field’ in string theory may result
in noncommutative geometry, that the anholonomic (super) frames could define non-
linear connections and Finsler like commutative and/ or noncommutative geometries
in string theory and (super) gravity and this holds true in certain limits to general
relativity.

We address the present work to physicists who would like to learn about some
new geometrical methods and to apply them to mathematical problems arising at
the forefront of modern theoretical physics. We do not assume that such readers
have very deep knowledge in differential geometry and nonlinear connection formalism
(for convenience, we give an Appendix outlining the basic results on the geometry of
commutative spaces provided with N–connection structures but consider that they are familiar with some more geometric approaches to gravity and string theories.

Finally, we note that the first attempts to relate Riemann–Finsler spaces (and spaces
with anisotropy of another type) to noncommutative geometry and physics were made
in Refs. where some models of noncommutative gauge gravity (in the commutative
limit being equivalent to the Einstein gravity, or to different generalizations to de
Sitter, affine, or Poincare gauge gravity with, or not, nonlinear realization of the gauge
groups) were analyzed. Further developments of noncommutative geometries with
anholonomic/ anisotropic structures and their applications in modern particle physics
lead to a rigorous study of the geometry of noncommutative anholonomic frames
with associated N–connection structure (that work should be considered as the
non–string partner of the present paper).

The paper has the following structure:

In Section 2 we consider stings in general manifolds and bundles provided with
anholonomic frames and associated nonlinear connection structures and analyze the
low energy string anholonomic field equations. The conditions when anholonomic
Einstein or Finsler like gravity models can be derived from string theory are stated.
Section 3 outlines the geometry of locally anisotropic supergravity models contained in superstring theory. Superstring effective actions and anisotropic toroidal compactifications are analyzed. The corresponding anholonomic field equations with distinguishing of anholonomic Riemannian–Finsler (super) gravities are derived.

In Section 4 we formulate the theory of noncommutative anisotropic scalar and gauge fields interactions and examine their anholonomic symmetries.

In Section 5 we emphasize how noncommutative anisotropic structures are embedded in string/M–theory and discuss their connection to anholonomic geometry.

Section 6 is devoted to locally anisotropic gravity models generated on noncommutative D–branes.

In Section 7 we construct four classes of exact solutions with noncommutative and locally anisotropic structures. We analyze solutions describing locally anisotropic black holes in string theory, define a class of Finsler–string structures containing two dimensional Finsler metrics, consider moving solitonic string–black hole configurations and give an examples of anholonomic noncommutative wormhole solution induced from string theory.

Finally, in Section 8, some additional comments and questions for further developments are presented. The Appendix outlines the necessary results from the geometry of nonlinear connections and generalized Finsler–Riemannian spaces.

2 String Theory and Commutative Riemann–Finsler Gravity

The string gravitational effects are computed from corresponding low–energy effective actions and moving equations of strings in curved spacetimes (on string theory, see monographs [15]). The basic idea is to consider propagation of a string not only of a flat 26–dimensional space with Minkowski metric $\eta_{\mu\nu}$ but also its propagation in a background more general manifold with metric tensor $g_{\mu\nu}$ from where one derived string–theoretic corrections to general relativity when the vacuum Einstein equations $R_{\mu\nu} = 0$ correspond to vanishing of the one–loop beta function in corresponding sigma model. More rigorous theories were formulated by adding an antisymmetric tensor field $B_{\mu\nu}$, the dilaton field $\Phi$ and possible other background fields, by introducing supersymmetry, higher loop corrections and another generalizations. It should be noted here that propagation of (super) strings may be considered on arbitrary (super) manifolds. For instance, in Refs. [12] [13] [17], the corresponding background (super) spaces were treated as (super) bundles provided with nonlinear connection (N–connection) structure and, in result, there were constructed some types of generalized (super) Finsler corrections to the usual Einstein and to locally anisotropic (Finsler type, or theirs generalizations) gravity theories.

The aim of this section is to demonstrate that anisotropic corrections and extensions may be computed both in Einstein and string gravity [derived for string propagation in usual (pseudo) Riemannian backgrounds] if the approach is developed following a more
rigorous geometrical formalism with off–diagonal metrics and anholonomic frames. We note that (super) frames [vielbeins] were used in general form, for example, in order to introduce spinors and supersymmetry in string theory but the anholonomic transforms with mixed holonomic–anholonomic variables, resulting in diagonalization of off–diagonal (super) metrics and effective anisotropic structures, were not investigated in the previous literature on string/M–theory.

2.1 Strings in general manifolds and bundles

2.1.1 Generalized nonlinear sigma models (some basics)

The first quantized string theory was constructed in flat Minkowski spacetime of dimension $k \geq 4$. Then the analysis was extended to more general manifolds with (pseudo) Riemannian metric $g_{\mu \nu}$, antisymmetric $B_{\mu \nu}$ and dilaton field $\Phi$ and possible other background fields, including tachyonic matter associated to a field $U$ in a tachyon state. The starting point in investigating the string dynamics in the background of these fields is the generalized nonlinear sigma model action for the maps $u : \Sigma \to M$ from a two dimensional surface $\Sigma$ to a spacetime manifold $M$ of dimension $k$,

$$S = S_{g,B} + S_{\Phi} + S_{U},$$

with

$$S_{g,B}[u,g] = \frac{1}{8\pi l^2} \int_\Sigma d\mu_g \partial_A u^\mu \partial_B u^\nu \left[ g_{[2]}^{AB}(u) + \varepsilon^{AB} B_{\mu \nu}(u) \right],$$

$$S_{\Phi}[u,g] = \frac{1}{2\pi} \int_\Sigma d\mu_g R_g \Phi(u), \quad S_{U}[u,g] = \frac{1}{4\pi} \int \Sigma d\mu_g U(u),$$

where $B_{\mu \nu}$ is the pullback of a two–form $B = B_{\mu \nu} du^\mu \wedge du^\nu$ under the map $u$, written out in local coordinates $u^\mu$; $g_{[2]}^{AB}$ is the metric on the two dimensional surface $\Sigma$ (indices $A,B = 0,1$); $\varepsilon^{AB} = \varepsilon^{AB}/\sqrt{\det |g_{AB}|}, \varepsilon^{01} = -\varepsilon^{10} = 1$; the integration measure $d\mu_g$ is defined by the coefficients of the metric $g_{AB}$, $R_g$ is the Gauss curvature of $\Sigma$. The constants in the action are related as

$$k = \frac{1}{4\pi \alpha'} = \frac{1}{8\pi l^2}, \quad \alpha' = 2\ell^2$$

where $\alpha'$ is the Regge slope parameter $\alpha'$ and $\ell \sim 10^{-33} cm$ is the Planck length scale. The metric coefficients $g_{\mu \nu}(u)$ are defined by the quadratic metric element given with respect to the coordinate co–basis $d^\mu = du^\mu$ (being dual to the local coordinate basis $\partial_\mu = \partial/\partial_\mu$),

$$ds^2 = g_{\mu \nu}(u) du^\mu du^\nu.$$

The parameter $\ell$ is a very small length–scale, compared to experimental scales $L_{\exp} \sim 10^{-17}$ accessible at present. This defines the so–called low energy, or $\alpha'$–expansion. A perturbation theory may be carried out as usual by letting $u = u_0 + \ell u_1$
for some reference configuration $u_0$ and considering expansions of the fields $g, B$ and $\Phi$, for instance,

$$g_{\mu\nu}(u) = g_{\mu\nu}(u_0) + \ell \partial_\alpha g_{\mu\nu}(u_0) u_\alpha^0 + \frac{1}{2} \ell^2 \partial_\alpha \partial_\beta g_{\mu\nu}(u_0) u_\alpha^0 u_\beta^0 + \ldots$$  \hspace{1cm} (3)

This reveals that the quantum field theory defined by the action (1) is with an infinite number of couplings; the independent couplings of this theory correspond to the successive derivatives of the fields $g, B$ and $\Phi$ at the expansion point $u_0$. Following an analysis of the general structure of the Weyl dependence of Green functions in the quantum field theory, standard regularizations schemes (see, for instance, Refs. [15]) and conditions of vanishing of Weyl anomalies, computing the $\beta$–functions, one derive the low energy string effective actions and field equations.

2.1.2 Anholonomic frame transforms of background metrics

Extending the general relativity principle to the string theory, we should consider that the string dynamics in the background of fields $g, B$ and $\Phi$ and possible another ones, defined in the low energy limit by certain effective actions and moving equations, does not depend on changing of systems of coordinates, $u^{\alpha'} \to u^{\alpha'}(u^\alpha)$, for a fixed local basis (equivalently, system, frame, or vielbein) of reference, $e_\alpha(u)$, on spacetime $M$ (for which, locally, $u = u^{\alpha} e_\alpha = u^{\alpha'} e_{\alpha'}$, $e_{\alpha'} = \partial u^{\alpha}/\partial u^{\alpha'} e_\alpha$, usually one considers local coordinate bases when $e_\alpha = \partial/\partial u^{\alpha}$) as well the string dynamics should not depend on changings of frames like $e_\alpha \to e^\alpha_{\alpha'}(u) e_\alpha$, parametrized by non–degenerated matrices $e^\alpha_{\alpha'}(u)$.

Let us remember some details connected with the geometry of moving frames in (pseudo) Riemannian spaces [6] and discuss its applications in string theory, where the orthonormal frames were introduced with the aim to eliminate non–trivial dependencies on the metric $g_{\mu\nu}$ and on the background field $u_0^\mu$ which appears in elaboration of the covariant background expansion method for the nonlinear sigma models [15, 21]. Such orthonormal frames, in the framework of a $SO(1, k - 1)$ like gauge theory are stated by the conditions

$$g_{\mu\nu}(u) = e^\mu_{\mu'}(u) e^\nu_{\nu'}(u) \eta^{\mu\nu},$$  
$$e^\mu_{\mu'} e^\nu_{\nu'} = \delta^\nu_{\mu'}, \quad e^\mu_{\mu'} e^\mu_{\nu} = \delta^\nu_{\mu},$$  \hspace{1cm} (4)

where $\eta^{\mu\nu} = diag(-1, +1, \ldots, +1)$ is the flat Minkowski metric and $\delta^\nu_{\mu}, \delta^\mu_{\nu}$ are Kronecker’s delta symbols. One considers the covariant derivative $D_\mu$ with respect to an affine connection $\Gamma$ and a corresponding spin connection $\omega^\alpha_{\mu \beta}$ for which the frame $e^\mu_{\alpha}$ is covariantly constant,

$$D_\mu e^\alpha_{\nu} \equiv \partial_\mu e^\alpha_{\nu} - \Gamma^\alpha_{\mu \nu} e^\alpha_{\alpha} + \omega^\alpha_{\mu \beta} e^\beta_{\nu} = 0.$$  

One also uses the covariant derivative

$$D_\mu e^\alpha_{\nu} = D_\mu e^\alpha_{\nu} + \frac{1}{2} H^\rho_{\mu \nu} e^\alpha_{\rho},$$  \hspace{1cm} (5)
including the torsion tensor $H_{\mu\nu\rho}$ which is the field strength of the field $B_{\nu\rho}$, given by $H = dB$, or, in component notation,

$$H_{\mu\nu\rho} \equiv \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}. \quad (6)$$

All tensors may be written with respect to an orthonormal frame basis, for instance,

$$H_{\mu\nu\rho} = e^\mu_{\mu'} e^\nu_{\nu'} e^\rho_{\rho'} H_{\mu'\nu'\rho'}$$

and

$$\mathcal{R}_{\mu\nu\rho\sigma} = e^\mu_{\mu'} e^\nu_{\nu'} e^\rho_{\rho'} e^\sigma_{\sigma'} \mathcal{R}_{\mu'\nu'\rho'\sigma'},$$

where the curvature $\mathcal{R}_{\mu\nu\rho\sigma}$ of the connection $\mathcal{D}_\mu$, defined as

$$(\mathcal{D}_\mu \mathcal{D}_\nu - \mathcal{D}_\nu \mathcal{D}_\mu) \xi^\rho \equiv [\mathcal{D}_\mu \mathcal{D}_\nu] \xi^\rho = H^\sigma_{\mu\nu} \mathcal{D}_\sigma \xi^\rho + \mathcal{R}^\sigma_{\sigma\mu\nu} \xi^\sigma,$$

can be expressed in terms of the Riemannian tensor $R_{\mu\nu\rho\sigma}$ and the torsion tensor $H^\sigma_{\mu\nu}$,

$$\mathcal{R}_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{1}{2} D_\rho H_{\sigma\mu\nu} - \frac{1}{2} D_\sigma H_{\rho\mu\nu} + \frac{1}{4} H_{\sigma\mu\alpha} H_{\sigma\nu} \alpha - \frac{1}{4} H_{\sigma\mu\alpha} H_{\rho\nu} \alpha.$$

Let us consider a generic off–diagonal metric, a non-degenerated matrix of dimension $k \times k$ with the coefficients $g_{\mu\nu}(u)$ defined with respect to a local coordinate frame like in [2]. This metric can transformed into a block $(n \times n) \oplus (m \times m)$ form, for $k = n + m$,

$$g_{\mu\nu}(u) \rightarrow \{g_{ij}(u), h_{ab}(u)\}$$

if we perform a frame map with the vielbeins

$$e^\mu_{\mu'}(u) = \begin{pmatrix} e^i_1(x^j, y^a) & N^a_i(x^j, y^a) e^\mu_a(x^j, y^a) \\ 0 & e^\mu_a(x^j, y^a) \end{pmatrix} \quad (7)$$

$$e^\nu_{\nu'}(u) = \begin{pmatrix} e^i_1(x^j, y^a) & -N^a_i(x^j, y^a) e^\nu_a(x^j, y^a) \\ 0 & e^\nu_a(x^j, y^a) \end{pmatrix}$$

which conventionally splits the spacetime into two subspaces: the first subspace is parametrized by coordinates $x^i$ provided with indices of type $i, j, k, ...$ running values from 1 to $n$ and the second subspace is parametrized by coordinates $y^a$ provided with indices of type $a, b, c, ...$ running values from 1 to $m$. This splitting is induced by the coefficients $N^a_i(x^j, y^a)$. For simplicity, we shall write the local coordinates as $u^a = (x^i, y^a)$, or $u = (x, y)$.

The coordinate bases $\partial_\alpha = (\partial_i, \partial_a)$ and theirs duals $d^\alpha = du^\alpha = (d^i = dx^i, d^a = dy^a)$ are transformed under maps [7] as

$$\partial_\alpha \rightarrow e^\alpha_\mu = e^\alpha_a(u) \partial_a, d^\alpha \rightarrow e^\alpha_a(u) d^a,$$

or, in 'N–distinguished' form,

$$e^1 = e^i_1 \partial_i - N^a_i e^k_a \partial_a, e^a_2 = e^a_2 \partial_a, \quad (8)$$

$$e^{-1} = e_i^i d^i, e^a = N^a_i e^a_2 d^i + e^a_2 d^a. \quad (9)$$
The quadratic line element (2) may be written equivalently in the form

\[ ds^2 = g_{ij}(x,y)e^i e_j + h_{ab}(x,y)e^a e^b \]  

(10)

with the metric \( g_{\mu\nu}(u) \) parametrized in the form

\[ g_{\alpha\beta} = \begin{bmatrix} g_{ij} + N_i^a N_j^b h_{ab} & h_{ab} N_i^a \\ h_{ab} N_j^b & h_{ab} \end{bmatrix} \]  

(11)

If we choose \( e_i(x_j, y^a) = \delta_{ij} \) and \( e^a(x_j, y^a) = \delta^a_a \), we may not distinguish the 'underlined' and 'non–underlined' indices. The operators (8) and (9) transform respectively into the operators of 'N–elongated' partial derivatives and differentials

\[ e_i = \delta_i = \partial_i - N_i^a \partial_a, e^a = \partial_a, \]

\[ e^i = d^i, e^a = \delta^a = d^a + N_i^a d^i \]

(12)

(which means that the anholonomic frames (8) and (9) generated by vielbein transforms (7) are, in general, anholonomic; see the respective formulas (105), (106) and (107) in the Appendix) and the quadratic line element (10) transforms in a d–metric element (see (111) in the Appendix).

The physical treatment of the vielbein transforms (7) and associated N–coefficients depends on the types of constraints (equivalently, anholonomies) we impose on the string dynamics and/or on the considered curved background. There were considered different possibilities:

- Ansatz of type (11) were used in Kaluza–Klein gravity [35], as well in order to describe toroidal Kaluza–Klein reductions in string theory (see, for instance, [25]). The coefficients \( N_i^a \), usually written as \( A_i^a \), are considered as the potentials of some, in general, non–Abelian gauge fields, which in such theories are generated by a corresponding compactification. In this case, the coordinates \( x^i \) can be used for the four dimensional spacetime and the coordinates \( y^a \) are for extra dimensions.

- Parametrizations of type (11) were considered in order to elaborate an unified approach on vector/tangent bundles to Finsler geometry and its generalizations [29, 28, 41, 40, 47, 41, 42, 43]. The coefficients \( N_i^a \) were supposed to define a non-linear connection (N–connection) structure in corresponding (super) bundles and the metric coefficients \( g_{ij}(u) \) and \( g_{ab}(u) \) were taken for a corresponding Finsler metric, or its generalizations (see formulas (127), (128), (129), (130) and related discussions in Appendix). The coordinates \( x^i \) were defined on base manifolds and the coordinates \( y^a \) were used for fibers of bundles.

- In a series of papers [45, 44, 49, 40, 53, 54, 52] the concept of N–connection was introduced for (pseudo) Riemannian spaces provided with off–diagonal metrics and/or anholonomic frames. In a such approach the coefficients \( N_i^a \) are associated
to an anholonomic frame structure describing a gravitational and matter fields dynamics with mixed holonomic and anholonomic variables. The coordinates $x^i$ are defined with respect to the subset of holonomic frame vectors, but $y^a$ are given with respect to the subset of anholonomic, $N$–ellongated, frame vectors. It was proven that by using vielbein transforms of type (7) the off–diagonal metrics could be diagonalized and, for a very large class of ansatz of type (11), with the coefficients depending on 2,3 or 4 coordinate variables, it was shown that the corresponding vacuum and non–vacuum Einstein equations may be integrated in general form. This allowed an explicit construction of new classes of exact solutions parametrized by off–diagonal metrics with some anholonomically deformed symmetries. Two new and very surprising conclusions were those that the Finsler like (and another type) anisotropies may be modeled even in the framework of the general relativity theory and its higher/lower dimension modifications, as some exact solutions of the Einstein equations, and that the anholonomic frame method is very efficient for constructing such solutions.

There is an important property of the off–diagonal metrics $g_{\mu\nu}$ (11) which does not depend on the type of space (a pseudo–Riemannian manifold, or a vector/tangent bundle) this metric is given. With respect to the coordinate frames it is defined a unique torsionless and metric compatible linear connection derived as the usual Christoffel symbols (or the Levi Civita connection). If anholonomic frames are introduced into consideration, we can define an infinite number of metric connections constructed from the coefficients of off–diagonal metrics and induced by the anholonomy coefficients (see formulas (123) and (124) and the related discussion from Appendix); this property is also mentioned in the monograph [30] (pages 216, 223, 261) for anholonomic frames but without any particularities related to associated N–connection structures. In this case there is an infinite number of metric compatible linear connections, constructed from metric and vielbein coefficients, all of them having non–trivial torsions and transforming into the usual Christoffel symbols for $N_i^a \to 0$ and $m \to 0$. For off–diagonal metrics considered, and even diagonalized, with respect to anholonomic frames and associated N–connections, we can not select a linear connection being both torsionless and metric. The problem of establishing of a physical linear connection structure constructed from metric/frame coefficients is to be solved together with that of fixing of a system of reference on a curved spacetime which is not a pure dynamical task but depends on the type of prescribed constraints, symmetries and boundary conditions are imposed on interacting fields and/or string dynamics.

In our further consideration we shall suppose that both a metric $g_{\mu\nu}$ (equivalently, a set $\{g_{ij}, g_{ab}, N_i^a\}$) and metric linear connection $\Gamma^{\alpha}_{\beta\gamma}$, i.e. satisfying the conditions $D_\alpha g_{\beta\gamma} = 0$, exist in the background spacetime. Such spaces will be called locally anisotropic (equivalently, anolonomic) because the anholonomic frames structure imposes locally a kind of anisotropy with respective constraints on string and effective string dynamics. For such configurations the torsion, induced as an anholonomic frame effect, vanishes only with respect coordinate frames. Here we note that in the string
theory there are also another type of torsion contributions to linear connections like $H^\sigma_{\mu\nu}$, see formula (3).

2.1.3 Anholonomic background field quantisation method

We revise the perturbation theory around general field configurations for background spaces provided with anholonomic frame structures (8) and (9), \( \delta_{\alpha} = (\delta_i = \partial_i - N^a_i \partial_a) \) and \( \delta^\alpha = (d^i, \delta^a = d^a + N^a_i d^i) \), with associated N–connections, \( N^a_i \), and \{\( g_{ij}, h_{ab} \)\} adapted to such structures (distinguished metrics, or d–metrics, see formula (11)). The linear connection in such locally anisotropic backgrounds is considered to be compatible both to the metric and N–connection structure (for simplicity, being a d–connection or an anholonomic variant of Levi Civita connection, both with nonvanishing torsion, see formulas (113), (123), (124), and (116), and related discussions in the Appendix). The general rule for the tensorial calculus on a space provided with N–connection structure is to split indices \( \alpha, \beta, ... \) into "horizontal", \( i, j, ... \), and "vertical", \( a, b, ... \), subsets and to apply for every type of indices the corresponding operators of N–adapted partial and covariant derivations.

The anisotropic sigma model is to be formulated by anholonomic transforms of the metric, \( g_{\mu\nu} \rightarrow \{g_{ij}, h_{ab}\} \), partial derivatives and differentials, \( \partial_\alpha \rightarrow \delta_\alpha \) and \( d_\alpha \rightarrow \delta_a \), volume elements, \( d\mu_g \rightarrow \delta\mu_g \) in the action (1)

\[
S = S_{gN,B} + S_\Phi + S_U,
\]

with

\[
S_{gN,B}[u, g] = \frac{1}{8\pi l^2} \int_\Sigma \delta\mu_g \left[ g^{AB} \left( \partial_A x^i \partial_B x^j g_{ij}(x,y) + \partial_A x^a \partial_B x^b h_{ab}(x,y) \right) \right] \\
+ \varepsilon^{AB} \partial_A u^\mu \partial_B u^\nu B_{\mu\nu}(u) \\
S_\Phi[u, g] = \frac{1}{2\pi} \int_\Sigma \delta\mu_g R_g \Phi(u), \\
S_U[u, g] = \frac{1}{4\pi} \int_\Sigma \delta\mu_g U(u),
\]

where the coefficients \( B_{\mu\nu} \) are computed for a two–form \( B = B_{\mu\nu} \delta u^\mu \wedge \delta u^\nu \).

The perturbation theory has to be developed by changing the usual partial derivatives into N–elongated ones, for instance, the decomposition (3) is to be written

\[
g_{\mu\nu}(u) = g_{\mu\nu}(u_0) + \ell \delta_\alpha g_{\mu\nu}(u_0) u_\alpha^i u_\beta^j + \ell^2 \delta_\alpha \delta_\beta g_{\mu\nu}(u_0) u_\alpha^a u_\beta^b + \ell^2 \delta_\beta \delta_\alpha g_{\mu\nu}(u_0) u_\alpha^a u_\beta^b + ...,
\]

where we should take into account the fact that the operators \( \delta_\alpha \) do not commute but satisfy certain anholonomy relations (see (107) in Appendix).

The action (13) is invariant under the group of diffeomorphisms on \( \Sigma \) and \( M \) (on spacetimes provided with N–connections the diffeomorphisms may be adapted to such structures) and posses a \( U(1)_B \) gauge invariance, acting by \( B \rightarrow B + \delta\gamma \) for some \( \gamma \in \Omega^{(1)}(M) \), where \( \Omega^{(1)} \) denotes the space of 1–forms on \( M \). Wayl’s conformal transformations of \( \Sigma \) leave \( S_{gN,B} \) invariant but result in anomalies under quantization. \( S_\Phi \)
and $S_U$ fail to be conformal invariant even classically. We discuss the renormalization of quantum field theory defined by the action \([13]\) for general fields $g_{ij}, h_{ab}, N^\alpha_{\mu}, B_{\mu\nu}$ and $\Phi$. We shall not discuss in this work the effects of the tachyon field.

The string corrections to gravity (in both locally isotropic and locally anisotropic cases) may be computed following some regularization schemes preserving the classical symmetries and determining the general structure of the Weyl dependence of Green functions specified by the action \([13]\) in terms of fixed background fields $g_{ij}, h_{ab}, N^\alpha_{\mu}, B_{\mu\nu}$ and $\Phi$. One can consider unnormalized correlation functions of operators $\phi_1, \ldots, \phi_p$, instead of points $\xi_1, \ldots, \xi_p \in \Sigma \ [15]$.

By definition of the stress tensor $T_{AB}$, under conformal transforms on the two dimensional hypersurface, $g_{[2]} \rightarrow \exp[25\sigma]g_{[2]}$ with support away from $\xi_1, \ldots, \xi_p$, we have

$$\Delta_\sigma < \phi_1 \ldots \phi_p > g_{[2]} = \frac{1}{2\pi} \int_\Sigma \delta_{\mu\nu} \Delta_\sigma < T^A_A \phi_1 \ldots \phi_p > g_{[2]},$$

when assuming throughout that correlation functions are covariant under the diffeomorphisms on $\Sigma$, $\nabla^A T_{AB} = 0$. The value $T^A_A$ receives contributions from the explicit conformal non–invariance of $S_\Phi$, from conformal (Weyl) anomalies which are local functions of $u$, i.e., dependent on $u$ and on finite order derivatives on $u$, and polynomial in the derivatives of $u$. For spaces provided with $N$–connection structures we should consider $N$–elongated partial derivatives, choose a $N$–adapted linear connection structure with some coefficients $\Gamma^\alpha_{\mu\nu}$ (for instance the Levi Civita connection \([12\alpha]\), or $d$–connection \([11\beta]\)). The basic properties of $T^A_A$ are the same as for trivial values of $N^\alpha_{\mu} \ [15]$, which allows us to write directly that

$$T^A_A = g^{AB}[\partial_A x^i \partial_B x_j \beta^g_{ij}(x, y) + \partial_A x^i \partial_B y^b \beta^{g, N}_{ib}(x, y) + \partial_A y^a \partial_B x^j \beta^{g, N}_{aj}(x, y) + \partial_A y^a \partial_B y^b \beta^{g, N}_{ab}(x, y) + \epsilon^{AB} \partial_A u^a \partial_B u^b \beta^B_{\alpha\beta}(x, y) + \beta^\Phi(x, y) R_g,$$

where the functions $\beta^g_{ij} = \{\beta^g_{ij}, \beta^{g, N}_{ij}\}$, $\beta^B_{\alpha\beta}$ and $\beta^\Phi(x, y)$ are called beta functions. On general grounds, the expansions of $\beta$–functions are of type

$$\beta(x, y) = \sum_{r=0}^\infty \ell^r \beta^{[2r]}(x, y).$$

One considers expansions up to and including terms with two derivatives on the fields which consideres expansions up to order $r = 0$ of $\beta^g_{ij}$ and $\beta^B_{\alpha\beta}$ and orders $s = 0, 2$ for $\beta^\Phi$. In this approximation, after cumbersome but simple calculations (similar to those given in \([13]\), in our case on locally anisotropic backgrounds)

$$\beta^g_{ij} = a_{1[1]} R_{ij} + a_{2[1]} g_{ij} + a_{3[1]} g_{ij} R + a_{4[1]} H^N_{ipq} H^N_{jqr} + a_{5[1]} g_{ij} H^N_{ipq} H^N_{jqr} + a_{6[1]} D_i D_j \Phi + a_{7[1]} g_{ij} D^2 \Phi + a_{8[1]} g_{ij} D^2 \Phi D_\rho \Phi,$$

$$\beta^g_{ib} = a_{1[2]} R_{ib} + a_{2[2]} H^N_{ipq} H_b^N_{jqr} + a_{6[2]} D_i D_b \Phi,$$

$$\beta^g_{a_j} = a_{1[3]} R_{aj} + a_{4[3]} H^N_{aip} H^N_{jqr} + a_{6[3]} D_a D_j \Phi, \ (14)$$

$$\beta^{g, N}_{ij} = a_{1[1]} R_{ij} + a_{2[1]} g_{ij} + a_{3[1]} g_{ij} R + a_{4[1]} H^N_{ipq} H^N_{jqr} + a_{5[1]} g_{ij} H^N_{ipq} H^N_{jqr} + a_{6[1]} D_i D_j \Phi + a_{7[1]} g_{ij} D^2 \Phi + a_{8[1]} g_{ij} D^2 \Phi D_\rho \Phi,$$

$$\beta^{g, N}_{ib} = a_{1[2]} R_{ib} + a_{2[2]} H^N_{ipq} H_b^N_{jqr} + a_{6[2]} D_i D_b \Phi,$$

$$\beta^{g, N}_{a_j} = a_{1[3]} R_{aj} + a_{4[3]} H^N_{aip} H^N_{jqr} + a_{6[3]} D_a D_j \Phi, \ (14)$$

12
\[ \beta_{\alpha \beta}^{\gamma N} = a_1[4] S_{ab} + a_2[4] h_{ab} + a_3[4] h_{ab} S + a_4[4] H_{\rho \sigma}^{\gamma N} H_{ab}^{\rho \sigma N} + a_5[4] h_{ab} H_{\rho \sigma}^{\gamma N} H^{\rho \sigma N} + a_6[4] D_\alpha D_\beta \Phi + a_7[4] h_{ab} D^2 \Phi + a_8[4] h_{ab} D^\rho \Phi D_\rho \Phi, \]
\[ \beta_{\alpha \beta}^R = b_1 D^\lambda H_{\lambda \mu \nu}^{N} + b_2 (D^\lambda \Phi) H_{\lambda \mu \nu}^{N}, \]
\[ \beta^\Phi = c_0 + \ell^2 \left[ c_{1[1]} \hat{R} + c_{1[2]} S + c_2 D^2 \Phi + c_3 (D^\lambda \Phi) D_\lambda \Phi + c_4 H_{\rho \sigma}^{\gamma N} H^{\rho \sigma N} \right], \]
where \( R_{\alpha \beta} = \{ R_{ij}, R_{ab}, R_{a_j}, S_{ab} \} \) and \( \hat{R} = \{ \hat{R}, S \} \) are given respectively by the formulas (118) and (119) and the B–strength \( H_{\rho \sigma}^{\gamma N} \) is computed not by using partial derivatives, like in (6), but with N–adapted partial derivatives, \( H_{\mu \nu \rho}^{N} \equiv \delta_\mu B_{\nu \rho} + \delta_\nu B_{\rho \mu} + \delta_\rho B_{\mu \nu} \).

The formulas for \( \beta \)–functions (13) are adapted to the N–connection structure being expressed via invariant decompositions for the Ricci d–tensor and curvature scalar; every such invariant object was provided with proper constants. In order to have physical compatibility with the case \( N \rightarrow 0 \) we should take
\[ a_2[1] = a_2[2] = a_2[3] = a_2[4] = a_z, \quad z = 1, 2, \ldots, 8; \]
\[ c_{1[1]} = c_{1[2]} = c_1, \]
where \( a_z \) and \( c_1 \) are the same as in the usual string theory, computed from the 1– and 2–loop \( \ell \)–dependence of graphs \( a_2 = 0, \quad a_6 = 1, \quad a_7 = a_8 = 0 \) and \( b_2 = 1/2, \quad c_3 = 2 \) and by using the background field method (in order to define the values \( a_1, a_3, a_4, a_5, b_1 \) and \( c_1, c_2, c_4 \)).

### 2.2 Low energy string anholonomic field equations

The effective action, as the generating functional for 1–particle irreducible Feynman diagrams in terms of a functional integral, can be obtained following the background quantization method adapted, in our constructions, to sigma models on spacetimes with N–connection structure. On such spaces, we can also make use of the Riemannian coordinate expansion, but taking into account that the coordinates are defined with respect to N–adapted bases and that the covariant derivative \( D \) is of type (113), (123) or (124), i. e. is d–covariant, defined by a d–connection.

For two infinitesimally closed points \( u_0^\mu = u^\mu(\tau_0) \) and \( u^\mu(\tau) \), with \( \tau \) being a parameter on a curve connected the points, we denote \( \zeta^\alpha = du^\alpha / d\tau_0 \) and write \( u^\mu = e^\zeta u_0^\mu \). We can consider diffeomorphism invariant d–covariant expansions of d–tensors in powers of \( \ell \), for instance,
\[ \Phi(u) = \Phi(u_0) + \ell D_\alpha [\Phi(u) \zeta^\alpha]_{u_0} + \frac{\ell^2}{2} D_\alpha D_\beta [\Phi(u) \zeta^\alpha \zeta^\beta]_{u_0} + o(\ell^3), \]
\[ A_{\alpha \beta}(u) = A_{\alpha \beta}(u_0) + \ell D_\alpha [A_{\alpha \beta}(u) \zeta^\alpha]_{u_0} + \frac{\ell^2}{2} (D_\alpha D_\beta [A_{\mu \nu}(u) \zeta^\alpha \zeta^\beta]_{u_0} - \frac{1}{3} R_{\alpha \beta \rho \sigma}^{(N)} A_{\mu \rho}(u) - \frac{1}{3} R_{\alpha \beta \rho \sigma}^{(N)} A_{\mu \rho}(u))_{u_0} + o(\ell^3). \]
where the Riemannian curvature d–tensor $R^{[N]}_{\alpha\mu\beta} = \{R^i_{hjk}, R^a_{bjk}, P^i_{jka}, P^c_{bka}, S^i_{jbc}, S^a_{bcd}\}$ has the invariant components given by the formulas (117) from Appendix. Putting such expansions in the action for the nonlinear sigma model (13), we obtain the decomposition

$$S_{gN, B[u, g]} = S_{gN, B[u_0, g]} + \ell \int_\Sigma \delta \mu g \delta \beta S_{\beta[u_0, g]} + \overline{S}[u, \zeta, g],$$

where $S_{\beta}$ is given by the variation

$$S_{\beta[u_0, g]} = (\det |g|)^{-1/2} \frac{\Delta S[\epsilon^x u_0, g]}{\Delta \chi^\beta} |_{x=0}$$

and the last term $\overline{S}$ is an expansion on $\ell$,

$$\overline{S} = \overline{S}[0] + \ell \overline{S}[1] + \ell^2 \overline{S}[2] + o(\ell^3),$$

with

$$\overline{S}[0] = 1 \frac{1}{8 \pi} \int_\Sigma \delta \mu g \{g^{AB}[g_{ij}(u_0)]D_A^* \zeta^i D_B^* \zeta^j + h_{ab}(u_0)D_A^* \zeta^a D_B^* \zeta^b]$$

$$+ R^{[N]}_{\mu\nu\rho\sigma}(u_0)[g^{AB} - \epsilon^{AB}] \partial_A u_0^\mu \partial_B u_0^\nu \zeta^\sigma \zeta^\rho\} ,$$

$$\overline{S}[1] = \frac{1}{24 \pi} \int_\Sigma \delta \mu g \frac{1}{2} \left[ R^{[N]}_{\mu\nu\rho\sigma}(u_0)[g^{AB} - \epsilon^{AB}] \partial_A u_0^\mu \partial_B u_0^\nu \zeta^\sigma \zeta^\rho\right] ,$$

$$\overline{S}[2] = \frac{1}{8 \pi} \int_\Sigma \delta \mu g \{H^{[N]}_{\mu\nu\rho\sigma}(u_0)[g^{AB}] \partial_A u_0^\mu \partial_B u_0^\nu \zeta^\sigma \zeta^\rho\} ,$$

The operator $D_A^* \zeta^\nu$ from (16) is defined according the rule

$$D_A^* \zeta^\nu = D_A^* \zeta^\nu + \frac{1}{2} H^{[N]}_{\mu\nu\rho\sigma} \epsilon^{BC} \partial_C u^\mu \zeta^\sigma ,$$

with $D_A^*$ being the covariant derivative on $T^* \Sigma \otimes TM$ pulled back to $\Sigma$ by the map $u^\alpha$ and acting as

$$D_A^* \partial_B u^\nu = \nabla_A \partial_B u^\nu + \Gamma^\alpha_{\mu\nu} \partial_B u^\nu \partial_A u^\nu ,$$

with a h– and v–invariant decomposition $\Gamma^\alpha_{\beta\gamma} = \{L^i_{jk}, L^a_{bjc}, C^i_{jc}, C^a_{bc}\}$, see (113) from Appendix, and the operator $R^{[N]}_{\mu\nu\rho\sigma}$ is computed as

$$R^{[N]}_{\mu\nu\rho\sigma} = R^{[N]}_{\mu\nu\rho\sigma} + \frac{1}{2} D_\rho H^{[N]}_{\sigma\mu\nu} - \frac{1}{2} D_\sigma H^{[N]}_{\rho\mu\nu} + \frac{1}{4} H^{[N]}_{\rho\mu\nu} H^{[N]}_{\sigma\nu} - \frac{1}{4} H^{[N]}_{\rho\mu\nu} H^{[N]}_{\sigma\nu} .$$

A comparative analysis of the expansion (16) with a similar one for $N = 0$ from the usual nonlinear sigma model (see, for instance, [15]) define the ‘geometric d–covariant
we may apply the same formulas as in the usual covariant expansions but with
that difference that 1) the usual spacetime partial derivatives and differentials are
substituted by N–elongated ones; 2) the Christoffel symbols of connection are changed
into certain d–connection ones, of type [113], [123] or [124]; 3) the torsion \( H_{\sigma \mu \nu} \)
is computed via N–elongated partial derivatives as in (15) and 4) the curvature \( R_{\mu \nu}^{[N]} \)
is split into horizontal–vertical, in brief, h–v–invariant, components according the the
formulas (117). The geometric d–covariant rule allows us to transform directly the
formulas for spacetime backgrounds with metrics written with respecto coordinate
frames into the respective formulas with N–elongated terms and splitting of indices
into h– and v– subsets.

2.2.1 Low energy string anisotropic field equations and effective action

Following the geometric d–covariant rule we may apply the results of the holonomic
sigma models in order to define the coefficients \( a_1, a_3, a_4, a_5, b_1 \) and \( c_1, c_2, c_4 \) of beta
functions (14) and to obtain the following equations of (in our case, anholonomic)
string dynamics,

\[
2\beta_{g,N}^{ij} = R_{ij} - \frac{1}{4} H_{ij}^{[N]} H^{[N]}_{i\rho \sigma} + 2D_i D_j \Phi = 0,
\]

\[
2\beta_{g,N}^{ib} = R_{ib} - \frac{1}{4} H_{ib}^{[N]} H^{[N]}_{i\rho \sigma} + 2D_i D_b \Phi = 0,
\]

\[
2\beta_{g,N}^{aj} = R_{aj} - \frac{1}{4} H_{aj}^{[N]} H^{[N]}_{a\rho \sigma} + 2D_a D_j \Phi = 0,
\]

\[
2\beta_{g,N}^{ab} = S_{ab} - \frac{1}{4} H_{ab}^{[N]} H^{[N]}_{a\rho \sigma} + 2D_a D_b \Phi = 0,
\]

\[
2\beta_{B}^{\alpha \beta} = -\frac{1}{2} D^\lambda H^{[N]}_{\lambda \mu \nu} + (D^\lambda \Phi) H^{[N]}_{\lambda \mu \nu} = 0,
\]

\[
2\beta_{\Phi} = \frac{n + m - 26}{3} + \ell^2 \left[ \frac{1}{12} H_{\rho \sigma \tau}^{[N]} H^{[N]}_{\rho \sigma \tau} - \hat{R} - S - 4D^2 \Phi + 4 (D^\lambda \Phi) D_\lambda \Phi \right] = 0,
\]

where \( n + m \) denotes the total dimension of a spacetime with \( n \) holonomic and \( m \)
anholonomic variables. It should be noted that \( \beta^{g,N} = \beta^B = 0 \) imply the condition
that \( \beta^\Phi = const \), which is similar to the holonomic strings. The only way to satisfy
\( \beta^\Phi = 0 \) with integers \( n \) and \( m \) is to take \( n + m = 26 \).

The equations (17) are similar to the Einstein equations for the locally anisotropic
gravity (see (121) in Appendix) with the matter energy–momentum d–tensor defined
from the string theory. From this viewpoint the fields \( B_{\alpha \beta} \) and \( \Phi \) can be viewed as
certain matter fields and the effective field equations (17) can be derived from action

\[
S (g_{ij}, h_{ab}, N_i^a, B_{\mu \nu}, \Phi) = \frac{1}{2\kappa^2} \int \delta^{26} u \sqrt{\det g_{\alpha \beta}} e^{-2\Phi} \left[ \hat{R} + S + 4(D\Phi)^2 - \frac{1}{12} H^2 \right],
\]

(18)
where $\kappa$ is a constant and, for instance, $D\Phi = D_a\Phi$, $H^2 = H_{\mu}H^\mu$ and the critical dimension $n + m = 26$ is taken. For $N \to 0$ and $m \to 0$ the metric $g_{\alpha\beta}$ is called the string metric. We shall call $g_{\alpha\beta}$ the string d–metric for nontrivial values of $N$.

Instead of action (18), a more standard action, for arbitrary dimensions, can be obtained via a conformal transform of d–metrics of type (111),

$$g_{\alpha\beta} \to \tilde{g}_{\alpha\beta} = e^{-4\Phi/(n+m-2)}g_{\alpha\beta}.$$ 

The action in d–metric $\tilde{g}_{\alpha\beta}$ (by analogy with the locally isotropic backgrounds we call it the Einstein d–metric) is written

$$S\left(\tilde{g}_{ij}, \tilde{h}_{ab}, N^a_i, B_{\mu\nu}, \Phi\right) = \frac{1}{2\kappa^2} \int \delta^{26}u \sqrt{|\text{det} \tilde{g}_{\alpha\beta}|} \left[ \tilde{R} + \tilde{S} + \frac{4}{n + m - 2} (D\Phi)^2 - \frac{1}{12} e^{-8\Phi/(n+m-2)}H^2 \right].$$

This action, for $N \to 0$ and $m \to 0$, is known in supergravity theory as a part of Chapline–Manton action, see Ref. [15] and for the so–called locally anisotropic supergravity, [43, 47]. When we deal with superstrings, the susperstring calculations to the mentioned orders give the same results as the bosonic string except the dimension. For anholonomic backrounds we have to take into account the nontrivial contributions of $N^a_i$ and splittings into h– and v–parts.

### 2.2.2 Anholonomic Einstein and Finsler gravity from string theory

It is already known that the $B$–field can be used for generation of different types of noncommutative geometries from string theories (see original results and reviews in Refs. [57, 13, 9, 38, 50]). Under certain conditions such $B$–field configurations may result in different variants of geometries with local anistropy like anholonomic Riemannian geometry, Finsler like spaces and their generalizations. There is also an alternative possibility when locally anisotropic interactions are modeled by anholonomic frame fields with arbitrary $B$–field contributions. In this subsection, we investigate both type of anisotropic models contained certain low energy limits of string theory.

### B–fields and anholonomic Einstein–Finsler structures

The simplest way to generate an anholonomic structure in a low energy limit of string theory is to consider a background metric $g_{\mu\nu} = \begin{pmatrix} g_{ij} & 0 \\ 0 & h_{ab} \end{pmatrix}$ with symmetric Christoffel symbols $\{\beta_\gamma\}$ and such $B_{\mu\nu}$, with corresponding $H^{[N]}_{\mu\nu\rho}$ from (15), as there are the nonvanishing values $H^{[N]}_{\mu\nu\rho} = \{H_{ij}^{[N]} a, H_{ij}^{[N]} a = -H_{ij}^{[N]} a\}$. The next step is to consider a covariant operator $D_\mu = D^i_\mu + \frac{1}{2} H^{[N]}_{\mu\nu\rho}$, where $\frac{1}{2} H^{[N]}_{\mu\nu\rho}$ is identified with the torsion (114). This way the torsion $H^{[N]}_{\mu\nu\rho}$ is associated to an aholonomic frame structure with non–trivial $W^a_{ij} = \delta_i N^a_j - \delta_j N^a_i$, $W^b_{ai} = -W^b_{ai} = -\partial_a N^b_i$ (108), when $B_{\mu\nu}$ is parametrized
in the form $B_{\mu\nu} = \{B_{ij} = -B_{ji}, B_{bj} = -B_{jb}\}$ by identifying

$$g_{\mu\nu} W^\nu_{\gamma\beta} = \delta_\mu B_{\gamma\beta},$$

i.e.

$$h_{ca} W^a_{ij} = \partial_c B_{ij} \text{ and } h_{ca} W^a_{bj} = \partial_c B_{bj}. \quad (19)$$

Introducing the formulas for the anholonomy coefficients (108) into (19), we find some formulas relating partial derivatives $\partial_\alpha N^a_i$ and the coefficients $N^a_i$ with partial derivatives of $\{B_{ij}, B_{bj}\}$,

$$h_{ca} \left( \partial_i N^a_j - N^b_i \partial_b N^a_j - \partial_j N^a_i + N^b_j \partial_b N^a_i \right) = \partial_c B_{ij},$$

$$-h_{ca} \partial_b N^a_j = \partial_c B_{bj}. \quad (20)$$

So, given any data $(h_{ca}, N^a_i)$ we can define from the system of first order partial derivative equations (20) the coefficients $B_{ij}$ and $B_{bj}$, or, inversely, from the data $(h_{ca}, B_{ij}, B_{bj})$ we may construct some non-trivial values $N^a_i$. We note that the metric coefficients $g_{ij}$ and the $B$–field components $B_{ab} = -B_{ba}$ could be arbitrary ones, in the simplest approach we may put $B_{ab} = 0$.

The formulas (20) define the conditions when a $B$–field may be transformed into a $N$–connection structure, or inversely, a $N$–connection can be associated to a $B$–field for a prescribed d–metric structure $h_{ca}$, (111).

The next step is to decide what type of d–connection we consider on our background spacetime. If the values $\{g_{ij}, h_{ca}\}$ and $W^\nu_{\gamma\beta}$ (defined by $N^a_i$ as in (108), but also induced from $\{B_{ij}, B_{bj}\}$ following (20)) are introduced in formulas (123) we construct a Levi Civita d–connection $D_\mu$ with nontrivial torsion induced by anholonomic frames with associated nonlinear connection structure. This spacetime is provided with a d–metric (111), $g_{\alpha\beta} = \{g_{ij}, h_{ca}\}$, which is compatible with $D_\mu$, i.e. $D_\mu g_{\alpha\beta} = 0$. The coefficients of $D_\mu$ with respect to anholonomic frames (105) and (106), $\Gamma^\tau_{\beta\gamma}$, can be computed in explicit form by using formulas (123). It is proven in the Appendix that on spacetimes provided with anholonomic structures the Levi Civita connection is not a prioritary one being both metric and torsion vanishing. We can construct an infinite number of metric connections, for instance, the canonical d–connection with the coefficients (113), or, equivalently, following formulas (124), to substitute from the coefficients (123) the values $\frac{1}{2} g^{ik} \Omega^a_{jk} h_{ca}$, where the coefficients of N–connection curvature are defined by $N^a_i$ as in (110). In general, all such type of linear connections are with nontrivial torsion because of anholonomy coefficients.

We may generate by $B$–fields an anholonomic (pseudo) Riemannian geometry if (for given values of $g_{\alpha\beta} = \{g_{ij}, h_{ca}\}$ and $N^a_i$, satisfying the conditions (20)) the metric is considered in the form (120) with respect to coordinate frames, or, equivalently, in the form (111) with respected to N–adapted frame (106). The metric has to satisfy the gravitational field equations (121) for the Einstein gravity of arbitrary dimensions with holonomic–anholonomic variables, or the equations (17) if the gravity with anholonomic constraints is induced in a low energy string dynamics. We emphasize that the Ricci
We note that the standard definition of Finsler quadratic form $g$ to consider some embeddings of Finsler d–metrics (129) of signature $(+...+)$ similarly to (pseudo) Euclidean/Riemannian metrics, or, as a second approach, quadratic forms with non–constant signatures and to generate (pseudo) Finsler geometries to include Finsler like structures in string theories. For instance, we can consider is considered to be positively definite (see (127) in Appendix). There are different possibilities to include Finsler like structures in string theories. For instance, we can consider diagonal metrics and anholonomic frames with associated $N$–connection structures in 53, 54, 52 there were constructed and investigated a number exact solutions with off–diagonal metrics and anholonomic frames with associated $N$–connection structures in the Einstein gravity of different dimensions (see also the Section 7).

Now, we discuss the possibility to generate a Finsler geometry from string theory. We note that the standard definition of Finsler quadratic form $g_{ij}^{[F]} = (1/2)\partial^2 F/\partial y^i \partial y^j$ is considered to be positively definite (see (127) in Appendix). There are different possibilities to include Finsler like structures in string theories. For instance, we can consider quadratic forms with non–constant signatures and to generate (pseudo) Finsler geometries [similarly to (pseudo) Euclidean/Riemannian metrics], or, as a second approach, to consider some embeddings of Finsler d–metrics (129) of signature $(+...+)$ into a 26 dimensional pseudo-Riemannian anholonomic background with signature $(-++...+)$.

In the last case, a particular class of Finsler background d–metrics may be chosen in the form

$$G^{[F]} = -dx^0 \otimes dx^0 + dx^1 \otimes dx^1 + g_{ij}^{[F]}(x,y)dx^i \otimes dx^j + g_{ij}^{[F]}(x,y)dy^i \otimes dy^j$$

where $i', j', ...$ run values $1, 2, ..., n' \leq 12$ for bosonic strings. The coefficients $g_{ij}^{[F]}$ are of type (127) or may take the value $+\delta_{ij'}$ for some values of $i \neq i', j \neq j'$. We may consider some static Finsler backgrounds if $g_{ij}^{[F]}$ do not depend on coordinates $(x^0, x^1)$, but, in general, we are not imposed to restrict ourselves only to such constructions. The $N$–coefficients from $\delta y^{i'} = dy^{i'} + N^{i'}_{j'} dx^{j'}$ must be of the form (128) if we want to generate in the low energy string limit a Finsler structure with Cartan nonlinear connection (there are possible different variants of nonlinear and distinguished nonlinear connections, see details in Refs. 31, 29, 4 and Appendix).

Let us consider in details how a Finsler metric can be included in a low energy string dynamics. We take a Finsler metric $F$ which generate the metric coefficients $g_{ij}^{[F]}$ and the $N$–connection coefficients $N_{ij'}^{[F]}$, respectively, via formulas (127) and (128). The Cartan’s $N$–connection structure $N_{ij'}^{[F]}$ may be induced by a $B$–field if there are some nontrivial values, let us denote them $\{B_{ij}^{[F]} , B_{ij'}^{[F]} \}$, which satisfy the conditions (20) and the Ricci d–tensor $R_{\alpha\beta} = \{R_{ij}, R_{ia}, R_{ai}, R_{ab} \}$ (118) and the torsion $H_{\mu\nu}^{[N]} = \{H_{ij}^{[N]}, H_{ij}^{[N]} = -H_{jib}^{[N]} \}$ related with $N_{ij'}^{[F]}$ as in (19), all computed for d–metric (21) are solutions of the motion equations (17) for any value of the dilaton field $\Phi$. In the Section 7 we shall consider an explicit example of a string–Finsler metric.

Here it should be noted that instead of a Finsler structure, in a similar manner, we may select from a string locally anisotropic dynamics a Lagrange structure if the metric
coefficients $g_{ij'}$ are generated by a Lagrange function $L(x, y)$ (130). The N–connection may be an arbitrary one, or of a similar Cartan form. We omit such constructions in this paper.

**Anholonomic Einstein–Finsler structures for arbitrary B–fields**

Locally anisotropic metrics may be generated by anholonomic frames with associated N–connections which are not induced by some $B$–field configurations.

For an anholonomic (pseudo) Riemannian background we consider an ansatz of form (120) which by anholonomic transform can be written as an equivalent d–metric (111). The coefficients $N^a_i$ and $B_{\mu\nu}$ are related only via the string motion equations (17) which must be satisfied by the Ricci d–tensor (118) computed, for instance, for the canonical d–connection (113).

A Finsler like structure, not induced directly by $B$–fields, may be emphasized if the d-metric is taken in the form (21), but the values $\delta y^i = dy^i + N^i_{j'} dx^{i'}$ being elongated by some $N^i_{j'}$ are not obligatory constrained by the conditions (20). Of course, the Finsler metric $F$ and $B_{\mu\nu}$ are not completely independent; these fields must be chosen as to generate a solution of string–Finsler equations (17).

In a similar manner we can model as some alternative low energy limits of the string theory, with corresponding nonlinear sigma models, different variants of spacetime geometries with anholonomic and N–connection structures, derived on manifold or vector bundles when the metric, linear and N–connection structures are proper for a Lagrange, generalized Lagrange or anholonomic Riemannian geometry [29, 34, 4, 24, 45, 44, 46, 51, 53].

### 3 Superstrings and Anisotropic Supergravity

The bosonic string theory, from which in the low energy limits we may generate different models of anholonomic Riemannian–Finsler gravity, suffers from at least four major problems: 1) there are tachyonic states which violates the physical causality and divergence of transitions amplitudes; 2) there are not included any fermionic states transforming under a spinor representation of the spacetime Lorentz group; 3) it is not clear why Yang–Mills gauge particles arise in both type of closed and open string theories and to what type of strings should be given priority; 4) experimentally there are 4 dimensions and not 26 as in the bosonic string theory: it must be understood why the remaining dimensions are almost invisible.

The first three problems may be resolved by introducing certain additional dynamical degrees of freedom on the string worldsheet which results in fermionic string states in the physical Hibert space and modifies the critical dimension of spacetime. One tries to solve the forth problem by developing different models of compactification.

There are distinguished five, consistent, tachyon free, spacetime supersymmetric string theories in flat Minkowski spacetime (see, for instance, [15] 25 for basic results and references on types I, IIA, IIB, Heterotic $Spin(32)/Z_2$ and Heterotic $E_8 \times E_8$ string
theories). The (super) string and (super) gravity theories in general, supersymmetric backgrounds provided with N–connection structure, and corresponding anisotropic superstring perturbation theories, were investigated in Refs. [41, 43, 47]. The goal of this Section is to illustrate how anholonomic type structures arise in the low energy limits of the mentioned string theories if the backgrounds are considered with certain anholonomic frame and off–diagonal metric structures. We shall consider the conditions when generalized Finsler like geometries arise in (super) string theories.

We would like to start with the example of the two–dimensional \(N = 1\) supergravity coupled to the dimension 1 superfields, containing a bosonic coordinate \(X^\mu\) and two fermionic coordinates, one left–moving \(\psi^\mu\) and one right moving \(\bar{\psi}^\mu\) (we use the symbol \(N\) for the supersymmetric dimension which must be not confused with the symbol \(N\) for a N–connection structure). We note that the two dimensional \(N = 1\) supergravity multiplet contains the metric and a gravitino \(\chi^A\). In order to develop models in backgrounds distinguished by a N–connection structure, we have to consider splittings into h– and v–components, i. e. to write \(X^\mu = (X^i, X^a)\) and \(\psi^\mu = (\psi^i, \psi^a)\), \(\bar{\psi}^\mu = (\bar{\psi}^i, \bar{\psi}^a)\). The spinor differential geometry on anisotropic spacetimes provided with N–connections (in brief, d–spinor geometry) was developed in Refs. [40, 55]. Here we shall present only the basic formulas, emphasizing the fact that the coefficients of d–spinors have the usual spinor properties on separated h– (v-) subspaces.

The simplest distinguished superstring model can be developed from an analog of the bosonic Polyakov action,

\[
S_P = \frac{1}{4\pi\alpha'} \int_{\Sigma} \delta \mu_g \left\{ g^{AB} \left[ \partial_A X^i \partial_B X^j g_{ij} + \partial_A X^a \partial_B X^b h_{ab} \right] \right.
\]

\[
+ \frac{i}{2} \left[ \psi^k \gamma^A \partial_A \psi_k^A + \psi^a \gamma^A \partial_A \psi^a \right] + \frac{i}{2} \left( \chi^A \gamma^B \gamma^A \psi^k \right) \left( \partial_B X^k - \frac{i}{4} \chi^B \psi^k \right)
\]

\[
+ \frac{i}{2} \left( \chi^A \gamma^B \gamma^A \psi^a \right) \left( \partial_B X^a - \frac{i}{4} \chi^B \psi^a \right)
\]

being invariant under transforms (i. e. being \(N = 1\) left–moving \((1, 0)\) supersymmetric)

\[
\Delta g_{AB} = i\epsilon \left( \gamma_A \chi_B + \gamma_B \chi_A \right), \quad \Delta \chi_A = 2 \nabla_A \epsilon, \\
\Delta X^i = i\epsilon \psi^i, \quad \Delta \psi^k = \gamma^A \left( \partial_A X^k - \frac{i}{2} \chi^a \psi^a \right) \epsilon, \quad \Delta \bar{\psi}^i = 0, \\
\Delta X^a = i\epsilon \psi^a, \quad \Delta \psi^a = \gamma^A \left( \partial_A X^a - \frac{i}{2} \chi^a \psi^a \right) \epsilon, \quad \Delta \bar{\psi}^a = 0,
\]

where the gamma matrices \(\gamma_A\) and the covariant differential operator \(\nabla_A\) are defined on the two dimensional surface, \(\epsilon\) is a left–moving Majorana–Weyl spinor. There is also a similar right–moving \((0, 1)\) supersymmetry involving a right moving Majorana–Weyl spinor \(\bar{\epsilon}\) and the fermions \(\bar{\psi}^\mu\) which means that the model has a \((1, 1)\) supersymmetry.
The superconformal gauge for the action (22) is defined as
\[ g_{AB} = e^{\Phi} \delta_{AB}, \quad \chi_A = \gamma_A \zeta, \]
for a constant Majorana spinor \( \zeta \). This action has also certain matter like supercurents \( \bar{i} \psi^\mu \partial X^\mu \) and \( \bar{i} \psi^\mu \partial X^\mu \).

We remark that the so–called distinguished gamma matrices (d–matrices), \( \gamma^\alpha = (\gamma^i, \gamma^a) \) and related spinor calculus are derived from \( \gamma \)–decompositions of the h– and v– components of d–metrics \( g^{\alpha \beta} = \{ g^{ij}, h^{ab} \} \) (111)

\[ \gamma^i \gamma^j + \gamma^j \gamma^i = -2g^{ij}, \quad \gamma^a \gamma^b + \gamma^b \gamma^a = -2h^{ab}, \]
see details in Refs. [40, 55].

In the next subsections we shall distinguish more realistic superstring actions than (22) following the geometric d–covariant rule introduced in subsection 2.2, when the curved spacetime geometric objects like metrics, connections, tensors, spinors, ... as well the partial and covariant derivatives and differentials are decomposed in invariant h– and v–components, adapted to the N–connection structure. This will allow us to extend directly the results for superstring low energy isotropic actions to backgrounds with local anisotropy.

### 3.1 Locally anisotropic supergravity theories

We indicate that many papers on supergravity theories in various dimensions are reprinted in a set of two volumes [36]. The bulk of supergravity models contain locally anisotropic configurations which can be emphasized by some vielbein transforms (7) and metric anzatz (11) with associated N–connection. For corresponding parametrizations of the d–metric coefficients, \( g_{\alpha \beta}(u) = \{ g^{ij}, h^{ab} \} \), N–connection, \( N^a_i(x,y) \), and d–connection, \( \Gamma^\alpha_{\beta \gamma} = (L^i_{jk}, L^a_{bk}, C^a_{jc}, C^a_{bc}) \), with possible superspace generalizations, we can generate (pseudo) Riemannian off–diagonal metrics, Finsler or Lagrange (super) geometries. In this subsection, we analyze the anholonomic frame transforms of some supergravity actions which can be coupled to superstring theory.

We note that the field components will be organized according to multiplets of \( Spin(1,10) \). We shall use 10 dimensional spacetime indices \( \alpha, \beta \ldots = 0, 1, 2, \ldots, 9 \) or 11 dimensional ones \( \alpha, \beta \ldots = 0, 1, 2, \ldots, 9, 10 \). The coordinate \( u^{10} \) could be considered as a compactified one, or distinguished in a non–compactified manner, by the N–connection structure. There is a general argument [31] is that 11 is the largest possible dimension in which supersymmetric multiplets can exist with spin less, or equal to 2, with a single local supersymmetry. We write this as \( n + m = 11 \), which points to possible splittings of indices like \( \alpha = (i, a) \) where \( i \) and \( a \) run respectively \( n \) and \( m \) values. A consistent superstring theory holds if \( n + m = 10 \). In this case, indices are to be decomposed as \( \alpha = (i, a) \). For simplicity, we shall consider that a metric tensor in \( n + m = 11 \) dimensions decomposes as \( g_{\alpha \beta}(u^\mu, u^{10}) \rightarrow g_{\alpha \beta}(u^\mu) \) and that in low energy approximation the fields are locally anisotropically interacting and independent on \( u^{10} \).
The antisymmetric rank 3 tensor is taken to decompose as \( A_{\alpha \beta \gamma}^{(a)} (u^\mu, u^{10}) \rightarrow A_{\alpha \beta \gamma} \). A fitting with superstring theory is to be obtained if \((A_{\alpha \beta \gamma}^{(a)}, B_{\mu \nu}) \rightarrow A_{\alpha \beta \gamma}\) and consider for spinors "dilatino" fields \((\chi_\mu^\tau, \lambda_\tau) \rightarrow \chi_\mu^\tau\), see, for instance, Refs. for details on couplings of supergravity and low energy superstrings.

3.1.1 \( \mathcal{N} = 1, n + m = 11 \) anisotropic supergravity

The field content of \( \mathcal{N} = 1 \) and 11 dimensional supergravity is given by \( g_{\alpha \beta} \) (graviton), \( A_{\alpha \beta \gamma}^{(a)} \) (U(1) gauge fields) and \( \chi_\mu^\alpha \) (gravitino). The dimensional reduction is stated by \( g_{a10} = g_{10a} = A_{\alpha}^{[1]} \) and \( g_{10 10} = e^{-2\Phi} \), where the coefficients are given with respect to an \( \mathcal{N} \)-elongated basis. We suppose that an effective action

\[
S(g_{ij}, h_{ab}, N^a_i, B_{\mu \nu}, \Phi) = \frac{1}{2\kappa^2} \int \delta \mu_{[g,h]} e^{-2\Phi} \left[ -\hat{R} - S + 4(D\Phi)^2 - \frac{1}{12} H^2 \right],
\]

is to be obtained if the values \( A_{\alpha}^{[1]}, A_{\alpha \beta \gamma}^{[a]}, \chi_\mu^\tau, \lambda_\tau \) vanish. For \( \mathcal{N} \rightarrow 0, m \rightarrow 0 \) this action results from the so-called NS sector of the superstring theory, being related to the sigma model action (18). A full \( \mathcal{N} = 1 \) and 11 dimensional locally anisotropic supergravity can be constructed similarly to the locally isotropic case but considering that \( H^{[\mathcal{N}]} = \delta B \) and \( F^{[\mathcal{N}]} = \delta A \) are computed as differential forms with respect to \( \mathcal{N} \)-elongated differentials (106),

\[
S \left( g_{ij}, h_{ab}, N^a_i, A_{\alpha}, \chi \right) = -\frac{1}{2\kappa^2} \int \delta \mu_{[g,h]} \left[ \hat{R} + S - \frac{\kappa^2}{12} F^2 + \kappa^2 \chi_\mu^\alpha \Gamma_{\mu \rho \sigma} \chi_\rho^\alpha \chi_\sigma^\alpha \right] (23)
+ \frac{\sqrt{2}\kappa^3}{384} \left( \chi_\mu^\alpha \Gamma_{\mu \rho \sigma} \chi_\rho^\rho \chi_\sigma^\sigma + 12 \Gamma_{\mu \rho \sigma} \chi_\mu^\rho \chi_\rho^\sigma \right) \left( F + \hat{F} \right)_{\rho \sigma \tau} \\
- \frac{\sqrt{2}\kappa}{81 \times 56} \int A \wedge F \wedge F,
\]

where \( \Gamma_{\mu \rho \sigma} \chi_\mu^\rho \chi_\rho^\rho \chi_\sigma^\sigma \) is the standard notation for gamma matrices for 11 dimensional spacetimes, the field \( \hat{F} = F + \chi \)-terms and \( D_\sigma \) is the covariant derivative with respect to \( \frac{1}{2} (\omega + \hat{\omega}) \) where

\[
\hat{\omega}_{\mu \rho \sigma} = \omega_{\mu \rho \sigma} + \frac{1}{8} \chi_\rho^\tau \Gamma_{\mu \rho \sigma} \chi_\sigma^\tau
\]

with \( \omega_{\mu \rho \sigma} \) being the spin connection determined by its equation of motion. We put the same coefficients in the action as in the locally isotropic case as to have compatibility for such limits. Every object (tensors, connections, connections) has a \( \mathcal{N} \)-distinguished invariant character with indices split into \( h \)- and \( v \)-subsets. For simplicity we omit here further decompositions of fields with splitting of indices.
3.1.2 Type IIA anisotropic supergravity

The action for a such model can be deduced from (23) if

\[ A_{\alpha\beta\gamma} = \kappa^{1/4} A^{[3]}_{\alpha\beta\gamma} \text{ and } A_{\alpha\beta 10} = \kappa^{-1} B_{\alpha\beta} \]

with further h– and v– decompositions of indices. The bosonic part of the type IIA locally anisotropic supergravity is described by

\[
S(g_{ij}, h_{ab}, N_\alpha^a, \Phi, A^{(1)}, A^{(3)}) = -\frac{1}{2\kappa^2} \int \delta \mu_{[g,h]} \{ e^{-2\Phi} [\hat{R} + S - 4(D\Phi)^2 + \frac{1}{12} H^2] \\
+ \sqrt{\kappa} G[A] + \frac{\sqrt{\kappa}}{12} F^2 - \frac{\kappa^{-3/2}}{288} \int B \wedge F \wedge F \},
\]

with \( G[A] = \delta A^{(1)} \), \( H = \delta B \) and \( F = \delta A^{(3)} \). This action may be written directly from the locally isotropic analogous following the d–covariant geometric rule.

3.1.3 Type IIB, \( n+m=10 \), \( N = 2 \) anisotropic supergravity

In a similar manner, geometrically, for d–objects, we may compute possible anholonomic effects from an action describing a model of locally anisotropic supergravity with a super Yang–Mills action (the bosonic part)

\[
S_{IIB} = -\frac{1}{2\kappa^2} \int \delta \mu_{[g,h]} e^{-2\Phi} [\hat{R} + S + 4(D\Phi)^2 - \frac{1}{12} \tilde{H}^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}],
\]

when the super–Yang–Mills multiplet is stated by the action

\[
S_{YM} = \frac{1}{\kappa} \int \delta \mu_{[g,h]} e^{-2\Phi} [-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \bar{\psi} \Gamma^\mu D_\mu \psi_\alpha].
\]

In these actions

\[ A = A_\mu^\alpha \delta u^\mu \]

is the gauge d–field of \( E_8 \times E_8 \) or \( Spin(32)/Z_2 \) group (with generators \( t^\alpha \) labeled by the index \( \alpha \)), having the strength

\[ F = \delta A + g_F A \wedge A = \frac{1}{2} F_{\mu\nu}^\alpha t^\alpha \delta u^\mu \wedge \delta u^\nu, \]

\( g_F \) being the coupling constant, and \( \psi \) is the gaugino of \( E_8 \times E_8 \) or \( Spin(32)/Z_2 \) group (details on constructions of locally anisotropic gauge and spinor theories can be found in Refs. [51, 50, 47, 55, 48, 40]). The action with \( B \)–field strength in (25) is defined as follows

\[ \tilde{H} = \delta B - \frac{\kappa}{\sqrt{2}} \omega_{CS}(A), \]

for

\[ \omega_{CS}(A) = tr \left( A \wedge \delta A + \frac{2}{3} g_F A \wedge A \wedge A \right). \]

Such constructions conclude in a theory with \( S_{IIB} + S_{YM} + \) fermionic terms with anholonomies and \( N = 1 \) supersymmetry.
Finally, we emphasize that the actions for supersymmetric anholonomic models can considered in the framework of (super) geometric formulation of supergravities in $n + m = 10$ and 11 dimensions on superbundles provided with N–connection structure [41, 43, 47].

### 3.2 Superstring effective actions and anisotropic toroidal compactifications

The supergravity actions presented in the previous subsection can be included in different supersymmetric string theories which emphasize anisotropic effects if spacetimes provided with N–connection structure are considered. In this subsection we analyze a model with toroidal compactification when the background is locally anisotropic. In order to obtain four–dimensional (4D) theories, the simplest way is to make use of the Kaluza–Klein idea: to propose a model when some of the dimensions are curled–up into a compact manifold, the rest of dimensions living only for non–compact manifold. Our aim is to show that in result of toroidal compactifications the resulting 4D theory could be locally anisotropic.

The action (25) can be obtained also as a 10 dimensional heterotic string effective action (in the locally isotropic variant see, for instance, Ref. [25])

$$ (\alpha')^8 S_{10-n'-m'} = \int \delta^{10} u \sqrt{|g_{\alpha\beta}|} e^{-\Phi'} [\tilde{R} + S + (D\Phi')^2 - \frac{1}{12} \tilde{H}^2 - \frac{1}{4} \tilde{F}_{\mu\nu}^\alpha \tilde{F}^{\alpha\mu\nu} + o (\alpha')] , \quad (26) $$

where we redefined $2\Phi \rightarrow \Phi'$, use the string constant $\alpha'$ and consider the $(n', m')$ as the (holonomic, anholonomic) dimensions of the compactified spacetime (as a particular case we can consider $n' + m' = 4$, or $n' + m' < 10$ for any brane configurations. Let us use parametrizations of indices and of vierbeinds: Greek indices $\alpha, \beta, \ldots \mu \ldots$ run values for a 10 dimensional spacetime and split as $\alpha = (\alpha', \tilde{\alpha})$, $\beta = (\beta', \tilde{\beta})$, ... when primed indices $\alpha', \beta', \ldots \mu'$... run values for compactified spacetime and split into $h$- and $v$–components like $\alpha' = (i', a')$, $\beta' = (j', b')$, ...; the frame coefficients are split as

$$ e^\mu_{\mu'}(u) = \begin{pmatrix} e_{\alpha'}^{\alpha'}(u^{\beta'}) & A_{\tilde{\alpha}}^{\alpha'}(u^{\beta'}) e_{\tilde{\alpha}}^{\alpha'}(u^{\beta'}) \\ 0 & e_{\tilde{\alpha}}^{\alpha'}(u^{\beta'}) \end{pmatrix} $$

where $e_{\alpha'}^{\alpha'}(u^{\beta'})$, in their turn, are taken in the form [17],

$$ e_{\alpha'}^{\alpha'}(u^{\beta'}) = \begin{pmatrix} e_{\mu}^{\mu}(x^\alpha', y^\alpha') & e_{\mu}^{\alpha'}(x^\alpha', y^\alpha') \\ 0 & e_{\mu}^{\alpha'}(x^\alpha', y^\alpha') \end{pmatrix}. \quad (27) $$

For the metric we have the recurrent ansatz

$$ g_{\alpha\beta} = \begin{bmatrix} g_{\alpha'\beta'}(u^{\beta'}) + N_{\tilde{\alpha}}^{\tilde{\alpha}}(u^{\beta'}) N_{\tilde{\beta}}^{\tilde{\beta}}(u^{\beta'}) h_{\tilde{\alpha}\tilde{\beta}}(u^{\beta'}) & h_{\tilde{\alpha}\beta}(u^{\beta'}) N_{\tilde{\alpha}}^{\tilde{\alpha}}(u^{\beta'}) & h_{\tilde{\alpha}\beta}(u^{\beta'}) N_{\tilde{\alpha}}^{\tilde{\alpha}}(u^{\beta'}) & h_{\tilde{\alpha}\beta}(u^{\beta'}) N_{\tilde{\alpha}}^{\tilde{\alpha}}(u^{\beta'}) \\ h_{\tilde{\alpha}\beta}(u^{\beta'}) N_{\tilde{\alpha}}^{\tilde{\alpha}}(u^{\beta'}) & h_{\tilde{\alpha}\beta}(u^{\beta'}) N_{\tilde{\alpha}}^{\tilde{\alpha}}(u^{\beta'}) & h_{\tilde{\alpha}\beta}(u^{\beta'}) N_{\tilde{\alpha}}^{\tilde{\alpha}}(u^{\beta'}) & h_{\tilde{\alpha}\beta}(u^{\beta'}) N_{\tilde{\alpha}}^{\tilde{\alpha}}(u^{\beta'}) \end{bmatrix}. $$
where
\[ g_{\alpha'\beta'} = \begin{bmatrix} g_{\alpha'\beta'}(u^{\beta'}) + N_{\alpha'}^{\alpha}(u^{\beta'}) N_{\beta'}^{\beta}(u^{\beta'}) h_{\alpha'\beta'}(u^{\beta'}) & h_{\alpha'\beta'}(u^{\beta'}) N_{\beta'}^{\beta}(u^{\beta'}) \\ h_{\alpha'\beta'}(u^{\beta'}) N_{\beta'}^{\beta}(u^{\beta'}) & h_{\alpha'\beta'}(u^{\beta'}) \end{bmatrix} . \] (28)

The part of action (26) containing the gravity and dilaton terms becomes
\[ (\alpha')^{n'+m'} S_{n'+m'}^{\text{heterotic}} = \int \delta^{n'+m'} u^\alpha \sqrt{\det g_{\alpha\beta}} e^{-\phi} [\hat{R} + S' + (\delta_{\mu'}^\phi)(\delta_{\mu'}^\phi) - \frac{1}{4}(\delta_{\mu'} h_{\alpha\beta}) (\delta_{\mu'} h_{\alpha\beta}) - \frac{1}{4} h_{\alpha\beta} F_{\mu'\nu'} F_{\mu'\nu'}], \] (29)

where \( \phi = \Phi' - \frac{1}{2} \log(\det h_{\alpha\beta}) \) and \( F_{\mu'\nu'}^{[A]\alpha\beta} = \delta_{\mu'} A_{\nu'} - \delta_{\nu'} A_{\mu'} \) and the h- and v-components of the induced scalar curvature, respectively, \( \hat{R}' \) and \( S' \) (see formula (119) in Appendix) are primed in order to point that these values are for the lower dimensional space. The antisymmetric tensor part may be decomposed in the form
\[ -\frac{1}{12} \int \delta^{10} u^\alpha \sqrt{\det g_{\alpha\beta}} e^{-\phi'} H_{\mu'\nu'} H_{\mu'\nu'} = -\frac{1}{4} \int \delta^{n'+m'} u^\alpha \sqrt{\det g_{\alpha'\beta'}} e^{-\phi} \times \] \[ [H_{\mu'\nu'}^{\alpha\beta} H_{\mu'\nu'}^{\alpha\beta} + H_{\mu'\nu'}^{\alpha\beta} H_{\mu'\nu'}^{\alpha\beta} + \frac{1}{3} H_{\mu'\nu'}^{\alpha\beta} H_{\mu'\nu'}^{\alpha\beta}], \] (30)

where, for instance,
\[ H_{\mu'\nu'}^{\alpha\beta} = e_{\mu'}^\mu e_{\nu'}^\nu H_{\mu'\nu'}^{\alpha\beta} \]
and we have considered \( H_{\alpha'\beta'}^{\alpha'\beta'} = 0 \). In a similar manner we can decompose the action for gauge fields \( \tilde{A}_\mu^I \) with index \( I = 1, \ldots, 32, \)
\[ \int \delta^{10} u^\alpha \sqrt{\det g_{\alpha\beta}} e^{-\phi'} \sum_{I=1}^{32} \tilde{F}_{\mu'\nu'}^I \tilde{F}_{\mu'\nu'}^I = \int \delta^{n'+m'} u^\alpha \sqrt{\det g_{\alpha'\beta'}} e^{-\phi} \sum_{I=1}^{16} \tilde{F}_{\mu'\nu'}^{I,\mu'} \tilde{F}_{\mu'\nu'}^{I,\nu'} + 2 \tilde{F}_{\mu'\nu'}^{I,\mu'} \tilde{F}_{\mu'\nu'}^{I,\nu'}, \] (31)
with
\[ Y_{\alpha}^I = A_{\alpha}^I, A_{\alpha'}^I = \tilde{A}_{\alpha'}^I - Y_{\alpha}^I A_{\mu}^I, \]
\[ \tilde{F}_{\mu'\nu'}^I = F_{\mu'\nu'}^I + Y_{\alpha}^I F_{\mu'\nu'}^{[A]\alpha}, \tilde{F}_{\mu'\nu'}^I = \delta_{\mu'} Y_{\alpha}^I, \tilde{F}_{\mu'\nu'}^I = \delta_{\mu'} A_{\mu}^I - \delta_{\nu'} A_{\mu}^I, \]
where the scalars \( Y_{\alpha}^I \) coming from the ten–dimensional vectors should be associated to a normal Higgs phenomenon generating a mass matrix for the gauge fields. They are related to the fact that a non–Abelian gauge field strength contains nonlinear terms not being certain derivatives of potentials.

After a straightforward calculus of the actions’ components (29), (30) and (31) (for locally isotropic gauge theories and strings, see a similar calculus, for instance, in Refs.
subjected to certain constraints. The induced metric describes a heterotic string effective action with local anisotropies (contained in the values

\[ R \]

perturbation, the (2p + 16) \times (2p + 16) dimensional symmetric matrix \( M \) has the structure

\[ M = \begin{pmatrix} \hat{g}^{-1} & C & Y^t \\ C^t \hat{g}^{-1} & \hat{g}^{-1} C + Y^t Y & \hat{g}^{-1} Y^t + Y^t \\ Y \hat{g}^{-1} C + Y & I_{16} + Y \hat{g}^{-1} Y^t \end{pmatrix} \]

with the block sub-matrices

\[ \hat{g} = \left( \hat{g}_{\alpha\beta} \right), \quad C = \left( C_{\alpha\beta} = B_{\alpha\beta} - \frac{1}{2} Y^I_\alpha Y^I_\beta \right), \quad Y = \left( Y^I_\alpha \right) \]

for which \( I_{16} \) is the 16 dimensional unit matrix; for instance, \( Y^t \) denotes the transposition of the matrix \( Y \). The dimension \( p \) satisfies the condition \( n' + m' - p = 16 \) relevant to the heterotic string describing \( p \) left-moving bosons and \( n' + m' \) right-moving ones with \( m' \) constrained degrees of freedom. To have good modular properties \( p - n' - m' \) should be a multiple of eight. The indices \( \overline{1}, \overline{2}, \ldots (2p + 16) \). The action \( (32) \) describes a heterotic string effective action with local anisotropies (contained in the values \( \hat{R}' = \hat{R}' + S' \) and \( \delta_{\mu}' \)) induced by the fact that the dynamics of the right-moving bosons are subjected to certain constraints. The induced metric \( g_{\alpha'\beta'} \) is of type \( (11) \) given with respect to an N-elongated basis \( (10) \) (in this case, primed), \( \delta_{\mu}' = \partial_{\mu}' + N_{\mu}' \). For \( N_{\mu}' \to 0 \) and \( m' \), i.e. for a subclass of effective backgrounds with block \( n' \times n' \oplus m' \times m' \) metrics \( g_{\alpha'\beta'} \), the action \( (32) \) transforms in the well known isotropic form (see, for instance, formula (C22), from the Appendix C in Ref. \( 25 \), from which following the 'geometric d-covariant rule' we could write down directly \( (32) \); this is a more formal approach which hides the physical meaning and anholonomic character of the components \( (29), (30) \) and \( (31) \).

### 3.3 4D NS–NS anholonomic field equations

As a matter of principle, compactifications of all type in (super) string theory can be performed in such ways as to include anholonomic frame effects as in the previous subsection. The simplest way to define anisotropic generalizations or such models is to apply the 'geometric d-covariant rule' when the tensors, spinors and connections are changed into their corresponding N-distinguished analogous. As an example, we write down here the anholonomic variant of the toroidally compactified (from ten to four dimensions) NS–NS action (we write in brief NS instead of Neveu–Schwarz) \( 37 \),

\[ S = \int \delta^4u \sqrt{|g_{\alpha'\beta'}|} e^{-\varphi} [\hat{R}' + S' + (\delta_{\mu}' \phi)(\delta_{\mu}' \phi) - \frac{1}{2} (\delta_{\mu}' \beta)(\delta_{\mu}' \beta) - \frac{1}{2} e^{2\varphi}(\delta_{\mu}' \sigma)(\delta_{\mu}' \sigma)], \]

26
for a d–metric parametrized as

$$\delta s^2 = -\epsilon(x^\alpha')^2 + g_{\alpha'\beta'}\delta u^{\alpha'}\delta u^{\beta'} + e^{\beta/\sqrt{3}}\delta\phi \delta u^\beta,$$

where, for instance, $u^{\alpha'} = (x^\alpha', u^\alpha)$, $\alpha' = 1, 2, 3$ and $\tilde{\alpha}, \tilde{\beta}, ... = 4, 5, ...9$ are indices of extra dimension coordinates, $\epsilon = \pm 1$ depending on signature (in usual string theory one takes $x^\alpha' = t$ and $\epsilon = -1$), the modulus field $\beta$ is normalized in such a way that it becomes minimally coupled to gravity in the Einstein d–frame, $\sigma$ is a pseudo–scalar axion d–field, related with the anti–symmetric strength,

$$H_{\alpha'\beta'\gamma'}(u^{\alpha'}) = \varepsilon_{\alpha'\beta'\gamma'}\varphi(u^{\alpha'}) D_{\tau'}\sigma(u^{\alpha'}),$$

$\varepsilon_{\alpha'\beta'\gamma'}$ being completely antisymmetric and $\varphi(u^{\alpha'}) = \Phi'(u^{\alpha'}) - \sqrt{3}\beta(u^{\alpha'})$, with $\Phi'(u^{\alpha'})$ taken as in (20).

We can derive certain locally anisotropic field equations from the action (33) by varying with respect to N–adapted frames for massless exitations of $g_{\alpha'\beta'}, B_{\alpha'\beta'}, \beta$ and $\varphi$, which are given by

$$2 \left[R_{\mu'\nu'} - \frac{1}{2} (\tilde{R}' + S') g_{\mu'\nu'} \right] = \frac{1}{2} H_{\nu'\lambda'\tau'} H_{\mu'\lambda'\tau'} - H^2 g_{\mu'\nu'} +$$

$$\left(\delta_{\mu'}^N \delta_{\nu'}^T - \frac{1}{2} g_{\mu'\nu'} g_{\lambda'\tau'} \right) D_{\lambda'\beta} D_{\tau'\beta} - g_{\mu'\nu'}(D\varphi)^2 + 2 \left( g_{\mu'\nu'} g_{\lambda'\tau'} - \delta_{\mu'}^N \delta_{\nu'}^T \right) D_{\lambda'} D_{\tau'} \varphi = 0,$$

$$D_{\mu'} \left( e^{-\varphi} H_{\mu'\lambda'} \chi \right) = 0,$$

$$D_{\mu'} \left( e^{-\varphi} D_{\mu'} \beta \right) = 0,$$

$$2D_{\mu'} D_{\mu'} \varphi = -\tilde{R}' - S' + (D\varphi)^2 + \frac{1}{2} (D\beta)^2 + \frac{1}{12} H^2 = 0,$$

where $H^2 = H_{\nu'\lambda'\tau'} H_{\mu'\lambda'\tau'}$ and, for instance, $(D\varphi)^2 = D_{\mu'} \varphi D_{\mu'} \varphi$. We may select a consistent solution of these field equations when the internal space is static with $D_{\mu'} \beta = 0$.

The equations (34) can be decomposed in invariant h– and v–components like the Einstein d–equations (121) (we omit a such trivial calculus). We recall [15] that the NS–NS sector is common to both the heterotic and type II string theories and is comprised of the dilaton, graviton and antisymmetric two–form potential. The obtained equations (34) define respective anisotropic string corrections to the anholonomic Einstein gravity.

### 3.4 Distinguishing anholonomic Riemannian–Finsler (super) gravities

There are two classes of general anisotropies contained in supergravity and superstring effective actions:

$$27$$
• Generic local anisotropies contained in the higher dimension (11, for supergravity models, or 10, for superstring models) which can be also induced in lower dimension after compactification (like it was considered for actions (23), (24), (25) and (26)).

• Local anisotropies which are in induced on the lower dimensional spacetime (for instance, actions (32) and (33) and respective field equations).

All types of general supergravity/superstring anisotropies may be in their turn to be distinguished to be of ”pure” $B$–field origin, of ”pure” anholonomic frame origin with arbitrary $B$–field, or of a mixed type when local anisotropies are both induced in a nonlinear form by both anholonomic (super) vielbeins and $B$–field (like we considered in subsection 2.2.2 for bosonic strings). In explicit form, a model of locally anisotropic superstring corrected gravity is to be constructed following the type of parametrizations we establish for the N–coefficients, d–metrics and d–connections.

For instance, if we choose the frame ansatz (27) and corresponding metric ansatz (28) with general coefficients $g_{ij}(x^j, y^c)$, $h_{ab}(x^j, y^a)$ and $N^a_i(x^j, y^a)$ satisfying the effective field equations (34) (containing also the fields $H^{\mu\nu\tau}$, $\varphi$ and $\beta$) we define an anholonomic gravity model corrected by toroidally compactified (from ten to four dimensions) NS–NS superstring model. In four and five dimensional Einstein/ Kaluza–Klein gravities, there were constructed a number of anisotropic black hole, wormhole, solitonic, spinor wave and Taub/NUT metrics [45, 49, 46, 53, 52]; in section 7 we shall consider some generalizations to string gravity.

Another possibility is to impose the condition that $g_{ij}, h_{ab}$ and $N^a_i$ are of Finsler type, $g^{[F]}_{ij} = h^{[F]}_{ab} = \partial^2 F^2 / 2 \partial y^i \partial y^j$ (127) and $N^{[F]}_j(x, y) = \partial \left[ c^{ik}(x, y) y^i y^k \right] / 4 \partial y^j$ (128), with an effective d–metric (129). If a such set of metric/N–connection coefficients can found as a solution of some string gravity equations, we may construct a lower dimensional Finsler gravity model induced from string theory (it depends of what kind of effective action, (32) or (33), we consider). Instead of a Finsler gravity we may search for a Lagrange model of string gravity if the d–metric coefficients are taken in the form (130).

We conclude this section by a remark that we may construct various type of anholonomic Riemannian and generalized Finsler/Lagrange string gravity models, with anisotropies in higher and/or lower dimensions by prescribing corresponding parametrizations for $g_{ij}, h_{ab}$ and $N^a_i$ (for ‘higher’ anisotropies) and $g^{ij}, h_{ab}$ and $N^a_i$ (for ‘lower’ anisotropies). The anholonomic structures may be of mixed type, for instance, in some dimensions being of Finsler configuration, in another ones being with anholonomic Riemannian metric, in another one of Lagrange type and different combinations and generalizations, see explicit examples in Section 7.
4 Noncommutative Anisotropic Field Interactions

We define the noncommutative field theory in a new form when spacetimes and configuration spaces are provided with some anholonomic frame and associated N–connection structures. The equations of motions are derived from functional integrals in a usual manner but considering N–elongated partial derivatives and differentials.

4.1 Basic definitions and conventions

The basic concepts on noncommutative geometry are outlined herein a somewhat pedestrian way by emphasizing anholonomic structures. More rigorous approaches on mathematical aspects of noncommutative geometry may be found in Refs. [8, 13, 16, 18], physical versions are given in Refs. [13, 57, 9, 38] (the review [50] is a synthesis of results on noncommutative geometry, N–connections and Finsler geometry, Clifford structures and anholonomic gauge gravity based on monographs [18, 47, 55, 29]).

As a fundamental ingredient we use an associative, in general, noncommutative algebra $A$ with a product of some elements $a, b \in A$ denoted $ab = a \cdot b$, or in the connotation to noncommutative spaces, written as a ”star” product $ab = a \star b$. Every element $a \in A$ corresponds to a configuration of a classical complex scalar field on a ”space” $M$, a topological manifold, which (in our approach) can be enabled with a N–connection structure. This associated noncommutative algebra generalize the algebra of complex valued functions $C(M)$ on a manifold $M$ (for different theories we may consider instead $M$ a tangent bundle $TM$, or a vector bundle $E(M)$). We consider that all functions referring to the algebra $A$, denoted as $A(M)$, arising in reasonable physical considerations are of necessary class (continuous, smooth, subjected to certain bounded conditions etc.).

4.1.1 Matrix algebras and noncommutativity

As the most elementary examples of noncommutative algebras, which are largely applied in quantum field theory and noncommutative geometry, one considers the algebra $\text{Mat}_k(\mathbb{C})$ of complex $k \times k$ matrices and the algebra $\text{Mat}_k(C(M))$ of $k \times k$ matrices whose matrix elements are elements of $C(M)$. The last algebra may be also defined as a tensor product,

$$\text{Mat}_k(C(M)) = \text{Mat}_k(\mathbb{C}) \otimes C(M).$$

The last construction is easy to be generalized for arbitrary noncommutative algebra $A$ as

$$\text{Mat}_k(A) = \text{Mat}_k(\mathbb{C}) \otimes A,$$

which is just the algebra of $k \times k$ matrices with elements in $A$. The algebra $\text{Mat}_k(A)$ admits an authomorphism group $GL(k, \mathbb{C})$ with the action defined as $a \rightarrow \zeta^{-1}a\zeta$, for $a \in A, \zeta \in GL(k, \mathbb{C})$. One considers the subgroup $U(k) \subset GL(k, \mathbb{C})$ which is preserved by hermitian conjugations, $a \rightarrow a^+$, and reality conditions, $a = a^+$. To define the hermitian conjugation, for which the hermitian matrices $a = a^+$ have real
eigenvalues, it is considered that \((a^+)^+ = a\) and \((ca)^+ = c^*a^+\), for \(c \in \mathbb{C}\) and \(c^*\) being the complex conjugated element of \(c\), i.e. it defined an antiholomorphic involution.

4.1.2 Noncommutative Euclidean space \(\mathbb{R}^k_\theta\)

Another simple example of a noncommutative space is the ‘noncommutative Euclidean space’ \(\mathbb{R}^k_\theta\) defined by all complex linear combinations of products of variables \(x = \{x^j\}\) satisfying

\[ [x^j, x^l] = x^j x^l - x^l x^j = i\theta^{jl}, \]  

where \(i\) is the complex 'imaginary' unity and \(\theta^{jl}\) are real constants treated as some noncommutative parameters or a "Poisson tensor" by analogy to the Poisson bracket in quantum mechanics where the commutator \([\ldots]\) of hermitian operators is antihermitian.

A set of partial derivatives \(\partial_j = \partial / \partial x^i\) on \(\mathbb{R}^k_\theta\) can be defined by postulating the relations

\[ \partial_j x^n = \delta_j^n, \]
\[ [\partial_j, \partial_n] = -i\Xi_{jn} \]  

where \(\Xi_{jn}\) may be zero, but in general is non–trivial if we want to incorporate some additional magnetic fields or anholonomic relations. A simplified noncommutative differential calculus can be constructed if \(\Xi_{jn} = -(\theta^{-1})_{jn}\).

The metric structure on \(\mathbb{R}^k_\theta\) is stated by a constant symmetric tensor \(\eta_{nj}\) for which \(\partial_j \eta_{nj} = 0\).

Infinitesimal translations \(x^j \rightarrow x^j + \alpha^j\) on \(\mathbb{R}^k_\theta\) are defined as actions on functions \(\varphi\) of type \(\Delta \varphi = \alpha^j \partial_j \varphi\). Because the coordinates are noncommuting there are formally defined inner derivations as

\[ \partial_j \varphi = \left[ -i \left(\theta^{-1}\right)_{jn} x^n, \varphi \right] \]  

which result in exponended global translations

\[ \varphi \left(x^j + \epsilon^j\right) = e^{-i\theta_{ij} \epsilon^j x^j} \varphi \left(x^j\right) e^{i\theta_{ij} \epsilon^j x^j}.\]

In order to understand the symmetries of the space \(\mathbb{R}^k_\theta\) it is better to write the metric and Poisson tensor in the forms

\[ ds^2 = \sum_{A=1}^{r} dz_A d\bar{z}_A + \sum_B dy_B^2, \]  
\[ = dq_A^2 + dp_A^2 + dy_B^2, \]
\[ \theta = \frac{1}{2} \sum_{A=1}^r \theta_A \partial z_A \wedge \partial \bar{z}_A, \quad \theta_A > 0, \]  

30
where $z_A = q_A + ip_A$ and $\overline{z}_A = q_A - ip_A$ are some convenient complex coordinates for which there are satisfied the commutation rules

\[
[y_A, y_B] = [y_B, q_A] = [y_B, p_A] = 0,
\]
\[
[q_A, p_B] = i\theta_A \delta_{AB}.
\]

Now, it is obvious that for fixed types of metric and Poisson structures there are two symmetry groups on $\mathbb{R}^k$, the group of rotations, denoted $O(k)$, and the group of invariance of the form $\theta$, denoted $Sp(2r)$.

### 4.1.3 The noncommutative derivative and integral

In order to elaborate noncommutative field theories in terms of an associative noncommutative algebra $\mathcal{A}$, additionally to the derivatives $\partial_j$ we need an integral $\int Tr$ which following the examples of noncommutative matrix spaces must contain also the “trace” operator. In this case we can not separate the notations of trace and integral.

It should be noted here that the role of derivative $\partial_j$ can be played by any sets of elements $d_j \in \mathcal{A}$ which some formal derivatives as $\partial_j A = [d_j, A]$, for $A \in \mathcal{A}$; derivations written in this form are called as inner derivations while those which can not written in this form are referred to as outer derivations.

The general derivation and integration operations are defined as some general dual linear operators satisfying certain formal properties: 1) the Leibnitz rule of the derivative, $\partial_j (AB) = \partial_j (A)B + A(\partial_j B)$; 2) the integral of the trace of a total derivative is zero, $\int Tr \partial_j A = 0$; 3) the integral of the trace of a commutator is zero, $\int Tr [A, B] = 0$, for any $A, B \in \mathcal{A}$. For some particular classes of functions in some noncommutative models the condition 2) and/or 3) may be violated, see details and discussion in Ref. [13].

Given a noncommutative space induced by some relations, the algebra of functions on $\mathbb{R}^k$ is deformed on $\mathbb{R}^k_\theta$ such that

\[
f(x) \star \varphi (x) = e^{\frac{i}{2} \theta^{jk} \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^k}} f(x + \xi) \varphi(x + \zeta)|_{\xi=\zeta=0}
\]
\[
= f \varphi + \frac{i}{2} \theta^{jk} \partial_j f \partial_k \varphi + o(\theta^2),
\]

which define the Moyal bracket (product), or star product ($\star$-product), of functions which is associative compatible with integration in the sense that for matrix valued functions $f$ and $\varphi$ that vanish rapidly enough at infinity we can integrate by parts in the integrals

\[
\int Tr f \star \varphi = \int Tr \varphi \star f.
\]

In a more rigorous operator form the star multiplication is defined by considering a space $M_\theta$, locally covered by coordinate carts with noncommutative coordinates, and choosing a linear map $S$ from $M_\theta$ to $\mathcal{C}(M)$, called the ”symbol” of the operator,
when $\hat{f} \rightarrow S \left[ \hat{f} \right]$. This way, the original operator multiplication is expressed in terms of the star product of symbols as

$$\hat{f} \hat{\varphi} = S^{-1} \left[ S \left[ \hat{f} \right] \ast S \left[ \hat{\varphi} \right] \right].$$

It should be noted that there could be many valid definitions of $S$, corresponding to different choices of operator ordering prescription for $S^{-1}$. One writes, for simplicity, $\int Tr f \ast \varphi = \int Tr f \varphi$ in some special cases.

### 4.2 Anholonomic frames and noncommutative spacetimes

One may consider that noncommutative relations for coordinates and partial derivatives (35) and (36) are introduced by specific form of anholonomic relations (107) for some formal anholonomic frames of type (105) and/or (106) (see Appendix) when anholonomy coefficients are complex and depend nonlinearly on frame coefficients. We shall not consider in this work the method of complex nonlinear operator anholonomic frames with associated nonlinear connection structure, containing as particular cases various type of Finsler/Cartan and Lagrange/Hamilton geometries in complexified form, which could consist in a general complex geometric formalism for noncommutative theories but we shall restrict our analysis to noncommutative spaces for which the coordinates and partial derivatives are distinguished by a $N$–connection structure into certain holonomic and anholonomic subsets which generalize the $N$–elongated commutative differential calculus (considered in the previous Sections) to a variant of both $N$– and $\theta$–deformed one.

In order to emphasize the $N$–connection structure on respective spaces we shall write $M^N_\theta$, $T M^N_\theta$, $E^M_\theta \left( M^N_\theta \right)$, $\mathcal{C} \left( M^N \right)$, $\mathcal{A}^N$ and $\mathcal{A} \left( M^N \right)$. For a space $M^N$ provided with $N$–connection structure, the matrix algebras considered in the previous subsection may be denoted $Mat_k \left( \mathcal{C} \left( M^N \right) \right)$ and $Mat_k \left( \mathcal{A}^N \right)$.

#### 4.2.1 Noncommutative anholonomic derivatives

We introduce splitting of indices, $\alpha = (i, a)$, $\beta = (j, b)$, ..., and coordinates, $u^\alpha = (x^i, y^a)$, ..., into 'horizontal' and 'vertical' components for a space $M_\theta$ (being in general a manifold, tangent/vector bundle, or their duals, or higher order models [28, 40, 43, 47, 55]. The derivatives $\partial_i$ satisfying the conditions (36) must be changed into some $N$–elongated ones if both anholonomy and noncommutative structures are introduced into consideration.

In explicit form, the anholonomic analogous of (35) is stated by a set of coordinates $u^\alpha = (x^i, y^a)$ satisfying the relations

$$[u^\alpha, u^\beta] = i \Theta^{\alpha\beta},$$

with $\Theta^{\alpha\beta} = (\Theta^{ij}, \Theta^{ab})$ parametrized as to have a noncommutative structure locally adapted to the $N$–connection, and the analogous of (36) redefined for operators (105).
\[
\delta_\alpha u^\beta = \delta_\beta^\alpha, \text{ for } \delta_\alpha = (\delta_i = \partial_i - N^a_i \partial_a, \partial_b),
\]
\[
[\delta_\alpha, \delta_\beta] = -i \Xi_{\alpha\beta}, \quad (41)
\]

where \( \Xi_{\alpha\beta} = -(\Theta^{-1})_{\alpha\beta} \) for a simplified \( N \)-elongated noncommutative differential calculus. We emphasize that if the vielbein transforms of type (7) and frames of type (8) and (9) are considered, the values \( \Theta^{\alpha\beta} \) and \( \Xi_{\alpha\beta} \) could be some complex functions depending on variables \( u^\beta \) including also the anholonomy contributions of \( N^a_i \). In particular cases, they may be constructed by some anholonomic frame transforms from some constant real tensors.

An anholonomic noncommutative Euclidean space \( \mathbb{R}^{n+m}_{N,\theta} \) is defined as a usual one of dimension \( k = n + m \) for which a \( N \)-connection structure is prescribed by coefficients \( N^a_i(x, y) \) which states an \( N \)-elongated differential calculus. The \( d \)-metric \( \eta_{\alpha\beta} = (\eta_{ij}, \eta_{ab}) \) and Poisson \( d \)-tensor \( \Theta^{\alpha\beta} = (\Theta^{ij}, \Theta^{ab}) \) are introduced via vielbein transforms (7) depending on \( N \)-coefficients of the corresponding constant values contained in (35) and (38). As a matter of principle such noncommutative spaces are already curved.

The interior derivative (37) is to be extended on \( \mathbb{R}^{n+m}_{N,\theta} \) as

\[
\delta_\alpha \varphi = \left[ -i (\Theta^{-1})_{\alpha\beta} u^\beta, \varphi \right].
\]

In a similar form, by introducing operators \( \delta_\alpha \) instead of \( \partial_\alpha \), we can generalize the Moyal product (39) for anisotropic spaces:

\[
f(x) \star \varphi(x) = e^{\frac{i \Theta^{\alpha\beta} \delta_\alpha}{\partial \xi} \delta_\beta} f(x + \xi) \varphi(x + \xi)|_{\xi = \varsigma = 0}
\]
\[
= f \varphi + \frac{i}{2} \Theta^{\alpha\beta} \delta_\alpha f \delta_\beta \varphi + o(\theta^2)
\]

For elaborating of perturbation and scattering theory, the more useful basis is the plane wave basis, which for anholonomic noncommutative Euclidean spaces, consists of eigenfunctions of the derivatives

\[
\delta_\alpha e^{ipu} = ip_\alpha e^{ipu}, \quad pu = p_\alpha u^\alpha.
\]

In this basis, the integral can be defined as

\[
N \int Tr e^{ipu} = \delta_{p,0}
\]

where the symbol \( \int Tr \) is enabled with the left upper index \( N \) in order to emphasize that integration is to be performed on a \( N \)-deformed space (we shall briefly call this as "\( N \)-integration") and the delta function may be interpreted as usually (its value at zero represents the volume of physical space, in our case, \( N \)-deformed). There is
a specific multiplication low with respect to the plane wave basis: for instance, by operator reordering,

\[ e^{ip\cdot u} \cdot e^{ip'\cdot u} = e^{-\frac{1}{2} \Theta_{\alpha\beta} p_\alpha p'_\beta} e^{i(p+p')u}, \]

when \( \Theta_{\alpha\beta} p_\alpha p'_\beta \) may be written as \( p \times p' = \Theta_{\alpha\beta} p_\alpha p'_\beta = p \otimes p' \). There is another example of multiplication, when the N–elongated partial derivative is involved,

\[ e^{ip\cdot u} \cdot f(u) \cdot e^{-ip\cdot u} = e^{-\Theta_{\alpha\beta} p_\alpha \delta_\beta} f(u) = f \left( u^\beta - \Theta_{\alpha\beta} p_\alpha \right), \]

which shows that multiplication by a plane wave in anholonomic noncommutative Euclidean space translates and N–deform a general function by \( u^\beta \rightarrow u^\beta - \Theta_{\alpha\beta} p_\alpha \). This exhibits both the nonlocality and anholonomy of the theory and preserves the principles that large momenta lead to large nonlocality which can be also locally anisotropic.

### 4.2.2 Noncommutative anholonomic torus

Let us define the concept of noncommutative anholonomic torus, \( T_{N,\theta}^{n+m} \), i.e. the algebra of functions on a noncommutative torus with some splitting of coordinates into holonomic and anholonomic ones. We note that a function \( f \) on an anholonomic torus \( T_{N}^{n+m} \) with N–decomposition is a function on \( \mathbb{R}^{n+m} \) which satisfies a periodicity condition, \( f(u^\alpha) = f(u^\alpha + 2\pi z^\alpha) \) for d–vectors \( z^\alpha \) with integer coordinates. Then the noncommutative extension is to define \( T_{N,\theta}^{n+m} \) as the algebra of all sums of products of arbitrary integer powers of the set of distinguished \( n+m \) variables \( U_\alpha = (U_i, U_j) \) satisfying

\[ U_\alpha U_\beta = e^{-i\Theta_{\alpha\beta}} U_\beta U_\alpha. \tag{42} \]

The variables \( U_\alpha \) are taken instead of \( e^{iu^\alpha} \) for plane waves and the derivation of a Weyl algebra from \( [10] \) is possible if we take

\[ [\delta_{\alpha} U_{\beta}] = i\delta_{\alpha\beta} U_{\beta}, \]

\[ N \int \text{Tr} U_{1}^{z_{1}} ... U_{n+m}^{z_{n+m}} = \delta_{\pi,0}. \]

In addition to the usual topological aspects for nontrivial values of N–connection there is much more to say in dependence of the fact what type of topology is induced by the N–connection curvature. We omit such consideration in this paper. The introduced in this subsection formulas and definitions transform into usual ones from noncommutative geometry if \( N, m \rightarrow 0 \).

### 4.3 Anisotropic field theories and anholonomic symmetries

In a formal sense, every field theory, commutative or noncommutative, can be anholonomically transformed by changing partial derivatives into N–elongated ones and redefining the integrating measure in corresponding Lagrangians. We shall apply this rule to noncommutative scalar, gauge and Dirac fields and make them to be locally anisotropic and to investigate their anholonomic symmetries.
4.3.1 Locally anisotropic matrix scalar field theory

A generic matrix locally anisotropic matrix scalar field theory with a hermitian matrix valued field $\phi(u) = \phi^+(u)$ and anholonomically N–deformed Euclidean action

$$S = N \int \delta^{n+m} u \sqrt{|g_{\alpha \beta}|} \left[ \frac{1}{2} g^{\alpha \beta} \text{Tr} \delta_{\alpha} \phi \delta_{\beta} \phi + V(\phi) \right]$$

where $V(\phi)$ is polynomial in variable $\phi$, $g_{\alpha \beta}$ is a d–metric of type $\left\langle 11 \right\rangle$ and $\delta_{\alpha}$ are N–elongated partial derivatives $\left\langle 105 \right\rangle$. It is easy to check that if we replace the matrix algebra by a general associative noncommutative algebra $\mathcal{A}$, the standard procedure of derivation of motion equations, classical symmetries from Noether’s theorem and related physical considerations go through but with N–elongated partial derivatives and N–integration: The field equations are

$$g^{\alpha \beta} \delta_{\alpha} \delta_{\beta} \phi = \frac{\partial V(\phi)}{\partial \phi}$$

and the conservation laws

$$\delta_{\alpha} J^{\alpha} = 0$$

for the current $J^{\alpha}$ is associated to a symmetry $\Delta \phi (\epsilon, \phi)$ determined by the N–adapted variational procedure, $\Delta S = N \int \text{Tr} J^{\alpha} \delta_{\alpha} \epsilon$. We emphasize that these equations are obtained according the prescription that we at the first stage perform a usual variational calculus then we change the usual derivatives and differentials into N–elongated ones. If we treat the N–connection as an object which generates and associated linear connection with corresponding curvature we have to introduce into the motion equations and conservation laws necessary d–covariant objects curvature/torsion terms.

We may define the momentum operator

$$P_{\alpha} = -i \left( \Theta^{-1} \right)_{\alpha \beta} N \int \text{Tr} u^{\beta} T^{\alpha},$$

which follows from the anholonomic transform of the restricted stress–energy tensor’ constructed from the Noether procedure with symmetries $\Delta \phi = i[\phi, \epsilon]$ resulting in

$$T^{\alpha} = ig^{\alpha \beta} [\phi, \delta_{\beta} \phi].$$

We chosen the simplest possibility to define for noncommutative scalar fields certain energy–momentum values and their anholonomic deformations. In general, in noncommutative field theory one introduced more conventional stress–energy tensors $\left\langle 11 \right\rangle$.

4.3.2 Locally anisotropic noncommutative gauge fields

Some models of locally anisotropic Yang–Mills and gauge gravity noncommutative theories are analyzed in Refs. $\left\langle 11 \right\rangle$ $\left\langle 50 \right\rangle$. Here we say only the basic facts about such theories with possible supersymmetry but not concerning points of gauge gravity.
Anholonomic Yang–Mills actions and MSYM model

A gauge field is introduced as a one form $A_\alpha$ having each component taking values in $\mathcal{A}$ and satisfying $A_\alpha = A_\alpha^+ \mathbb{1}$ and curvature (equivalently, field strength)

$$F_{\alpha\beta} = \delta_\alpha A_\beta - \delta_\beta A_\alpha + i [A_\alpha A_\beta]$$

with gauge locally anisotropic transformation laws,

$$\triangle F_{\alpha\beta} = i [F_{\alpha\beta}, \epsilon] \quad \text{for} \quad \triangle A_\alpha = \delta_\alpha \epsilon + i [A_\alpha, \epsilon]. \quad (43)$$

Now we can introduce the noncommutative locally anisotropic Yang–Mills action

$$S = \frac{-1}{4g_{YM}} N \int Tr F^2$$

which describes the $N$–anholonomic dynamics of the gauge field $A_\alpha$. Coupling to matter field can be introduced in a standard way by using $N$–elongated partial derivatives $\delta_\alpha$,

$$\nabla_\alpha \varphi = \delta_\alpha \varphi + i [A_\alpha, \varphi].$$

Here we note that by using $Mat_Z(\mathcal{A})$ we can construct both noncommutative and anisotropic analog of $U(Z)$ gauge theory, or, by introducing supervariables adapted to $N$–connections [43] and locally anisotropic spinors [40, 55], we can generate supersymmetric Yang–Mills theories. For instance, the maximally supersymmetric Yang–Mills (MSYM) Lagrangian in ten dimensions with $N = 4$ can be deduced in anisotropic form, by corresponding dimensional reductions and anholonomic constraints, as

$$S = \frac{N}{\delta^{10} u} \int Tr \left( F_{\alpha\beta}^2 + i \nabla \overrightarrow{\nabla} \chi \right)$$

where $\chi$ is a 16 component adjoint Majorana–Weyl fermion and the spinor $d$–covariant derivating operator $\overrightarrow{\nabla}$ is written by using $N$–anholonomic frames.

The emergence of locally anisotropic spacetime

It is well known that spacetime translations may arise from a gauge group transforms in noncommutative gauge theory (see, for instance, Refs. [43]). If the same procedure is reconsidered for $N$–elongated partial derivatives and distinguished noncommutative parameters, we can write

$$\delta A_\alpha = v^\beta \delta_\beta A_\alpha$$

as a gauge transform [43] when the parameter $\epsilon$ is expressed as

$$\epsilon = v^\alpha (\Theta^{-1})_{\alpha\beta} u^\beta = v^i (\Theta^{-1})_{ijk} x^k + v^a (\Theta^{-1})_{ab} x^b,$$

which generates

$$\triangle A_\alpha = v^\beta \delta_\beta A_\alpha + v^\beta (\Theta^{-1})_{\alpha\beta}.$$
This way the spacetime anholonomy is induced by a noncommutative gauge anisotropy. For another type of functions \( \epsilon(u) \), we may generate another spacetime locally anisotropic transforms. For instance, we can generate a Poisson bracket \( \{ \varphi, \epsilon \} \) with N–elongated derivatives,

\[
\Delta \varphi = i \left[ \varphi, \epsilon \right] = \Theta^{\alpha\beta} \delta_{\alpha} \varphi \delta_{\beta} \epsilon + o \left( \delta_{\alpha}^{2} \varphi \delta_{\beta}^{2} \epsilon \right) \rightarrow \{ \varphi, \epsilon \}
\]

which proves that at leading order the locally anisotropic gauge transforms preserve the locally anisotropic noncommutative structure of parameter \( \Theta^{\alpha\beta} \).

Now, we demonstrate that the Yang–Mills action may be rewritten as a ”matrix model” action even the spacetime background is N–deformed. This is another side of unification of noncommutative spacetime and gauge field with anholonomically deformed symmetries. We can absorb a inner derivation into a vector potential by associating the covariant operator

\[
\nabla_{\alpha} \varphi \rightarrow [C_{\alpha}, \varphi]
\]

for

\[
C_{\alpha} = (-i\Theta^{-1})_{\alpha\beta} u^{\beta} + iA_{\alpha}.
\]

(44)

As in usual noncommutative gauge theory we introduce the ”covariant coordinates” but distinguished by the N–connection,

\[
Y^{\alpha} = u^{\alpha} + \Theta^{\alpha\beta} A_{\beta}(u).
\]

For invertible \( \Theta^{\alpha\beta} \), one considers another notation, \( Y^{\alpha} = i\Theta^{\alpha\beta} C_{\beta} \). Such transforms allow to express \( F_{\alpha\beta} = i [\nabla_{\alpha}, \nabla_{\beta}] \) as

\[
F_{\alpha\beta} = i [C_{\alpha}, C_{\beta}] - (\Theta^{-1})_{\alpha\beta}
\]

for which the Yang–Mills action transform into a matrix relation,

\[
S = N \text{Tr} \sum_{\alpha, \beta} \left( i [C_{\alpha}, C_{\beta}] - (\Theta^{-1})_{\alpha\beta} \right)^{2}
\]

(45)

\[
= N \text{Tr} \left\{ [i [C_{k}, C_{j}] - (\Theta^{-1})_{kj}] \left[ i [C_{k}, C_{j}] - (\Theta^{-1})_{kj} \right] \right.
\]

\[
+ [i [C_{a}, C_{b}] - (\Theta^{-1})_{ab}] \left[ i [C_{a}, C_{b}] - (\Theta^{-1})_{ab} \right] \}
\]

where we emphasize the N–distinguished components.

The noncommutative Dirac d–operator

If we consider multiplications \( a \cdot \psi \) with \( a \in A \) on a Dirac spinor \( \psi \), we can have two different physics depending on the orders of such multiplications we consider, \( a \psi \) or \( \psi a \). In order to avoid infinite spectral densities, in the locally isotropic noncommutative gauge theory, one writes the Dirac operator as

\[
\nabla_{\gamma} \psi = \gamma^{i} \left( \nabla_{x} \psi - \psi \partial_{i} \right) = 0.
\]
In the locally anisotropic case we have to introduce $N$–elongated partial derivatives,
\[
\nabla \psi = \gamma^\alpha \left( \nabla_\alpha \psi - \psi \delta_\alpha \right) \\
= \gamma^i \left( \nabla_i \psi - \psi \delta_i \right) + \gamma^a \left( \nabla_a \psi - \psi \delta_a \right) = 0
\]
and use a $d$–covariant spinor calculus [40, 55].

**The $N$–adapted stress–energy tensor**

The action (45) produces a stress–energy $d$–tensor
\[
T_{\alpha\beta} (p) = \sum_\gamma \int_0^1 ds \ N \int Tr \ e^{ispr_{Y^r}} [C_\alpha, C_\gamma] \ e^{i(1-s)p_{r}Y^r} [C_\beta, C_\gamma]
\]
as a Noether current derived by the variation $C_\alpha \rightarrow C_\alpha + a_\alpha (p) e^{ispr_{Y^r}}$. This $d$–tensor has a property of conservation,
\[
p_r \Theta^{r\lambda} T_{\lambda\beta} (p) = 0
\]
for the solutions of field equations and seem to be a more natural object in string theory, which admits an anholonomic generalizations by "distinguishing of indices”.

**The anholonomic Seiberg–Witten map**

There are two different types of gauge theories: commutative and noncommutative ones. They may be related by the so–called Seiberg–Witten map [38] which explicitly transforms a noncommutative vector potential to a conventional Yang–Mills vector potential. This map can be generalized in gauge gravity and for locally anisotropic gravity [48, 50]. Here we define the Seiberg–Witten map for locally anisotropic gauge fields with $N$–elongated partial derivatives.

The idea is that if there exists a standard, but locally anisotropic, Yang–Mills potential $A_\alpha$ with gauge transformation laws parametrized by the parameter $\epsilon$ like in [38], a noncommutative gauge potential $\hat{A}_\alpha (A_\alpha)$ with gauge transformation parameter $\hat{\epsilon} (A, \epsilon)$, when
\[
\hat{\Delta}_\epsilon \hat{A}_\alpha = \delta_\alpha \hat{\epsilon} + i \left( \hat{A}_\alpha \ast \hat{\epsilon} - \hat{\epsilon} \ast \hat{A}_\alpha \right),
\]
should satisfy the equation
\[
\hat{A} (A) + \hat{\Delta}_\epsilon \hat{A} (A) = \hat{A} (A + \Delta \epsilon A), \tag{46}
\]
where, for simplicity, the indices were omitted. This is the Seiberg–Witten equation which, in our case, contains $N$–adapted operators $\delta_\alpha$ [105] and $d$–vector gauge potentials, respectively, $\hat{A}_\alpha = \left( \hat{A}_i, \hat{A}_a \right)$ and $A_\alpha = \left( A_i, A_a \right)$. To first order in $\Theta^{\alpha\beta} = \Delta \Theta^{\alpha\beta}$,
to the equation (46) can be solved in a usual way, by related respectively the potentials and transformation parameters,

\[ \hat{A}_\alpha (A_\alpha) - A_\alpha = -\frac{1}{4} \Delta \Theta^{\beta \lambda} [A_\beta (\delta_\lambda A_\alpha + F_{\lambda \alpha}) + (\delta_\lambda A_\alpha + F_{\lambda \alpha}) A_\beta] + o(\Delta \Theta^2), \]

\[ \hat{\epsilon} (A, \epsilon) - \epsilon = \frac{1}{4} \Delta \Theta^{\beta \lambda} (\delta_\beta \epsilon A_\lambda + A_\lambda \delta_\beta \epsilon) + o(\Delta \Theta^2), \]

from which we can also find a first order relation for the field strength,

\[ \hat{F}_{\lambda \alpha} - F_{\lambda \alpha} = \frac{1}{2} \Delta \Theta^{\beta \tau} (F_{\lambda \beta} F_{\alpha \tau} + F_{\alpha \tau} F_{\lambda \beta}) - A_\beta (\nabla_\tau F_{\lambda \alpha} + \delta_\tau F_{\lambda \alpha}) - (\nabla_\tau F_{\lambda \alpha} + \delta_\tau F_{\lambda \alpha}) A_\beta + o(\Delta \Theta^2). \]

By a recurrent procedure the solution of (46) can be constructed in all orders of \( \Delta \Theta^{\alpha \beta} \) as in the locally isotropic case (see details on recent supersymmetric generalizations in Refs. [27] which can be transformed at least in a formal form into certain anisotropic analogs following the d–covariant geometric rule.

5 Anholonomy and Noncommutativity: Relations to String/M–Theory

The aim of this Section is to discuss how both noncommutative and locally anisotropic field theories arise from string theory and M–theory. The first use of noncommutative geometry in string theory was suggested by E. Witten (see Refs. [57, 38] for details and developments). Noncommutativity is natural in open string theory: interactions of open strings with two ends contains formal similarities to matrix multiplication which explicitly results in noncommutative structures. In other turn, matrix noncommutativity is contained in off–diagonal metrics and anholonomic vielbeins with associated N–connection and anholonomic relations (see [107] and related details in Appendix) which are used in order to develop locally anisotropic geometries and field theories. We emphasize that the constructed exact solutions with off–diagonal metrics in general relativity and extra dimension gravity together with the existence of a string field framework strongly suggest that noncommutative locally anisotropic structures have a deep underlying significance in such theories [45, 46, 53, 54, 48, 50, 42, 43].

5.1 Noncommutativity and anholonomy in string theory

In this subsection, we will analyze strings in curved spacetimes with constant coefficients \{g_{ij}, h_{ab}\} of d–metric (111) (the coefficients \( N_a^i (x^k, y^a) \) are not constant and the off–diagonal metric (111) has a non–trivial curvature tensor). With respect to N–adapted frames (105) and (106) the string propagation is like in constant Neveu–Schwarz constant \( B\)–field and with \( Dp\)–branes. We work under the conditions of string and brane theory which results in noncommutative geometry [57] but the background
under consideration here is an anholonomic one. The $B$–field is a like constant magnetic field which is polarized by the N–connection structure. The rank of the matrix $B_{\alpha \beta}$ is denoted $k = n + m = 11 \leq p + 1$, where $p \geq 10$ is a constant. For a target space, defined with respect to anholonomic frames, we will assume that $B_{0\beta} = 0$ with "0" the time direction (for a Euclidean signature, this condition is not necessary). We can similarly consider another dimensions than 11, or to suppose that some dimensions are compactified. We can pick some torus like coordinates, in general anholonomic, by can similarly consider another dimensions than 11, or to suppose that some dimensions are compactified. th
The equation of motion of string in anholonomic constant background defines

$$S = \frac{1}{4\pi \alpha'} \int \delta \mu_g \left( g_{\alpha \beta} \partial_A u^\alpha \partial^A u^\beta - 2\pi \alpha' i B_{\alpha \beta} \epsilon^{AB} \partial_A u^a \partial_B u^b \right)$$

$$= \frac{1}{4\pi \alpha'} \int \delta \mu_g \left( g_{\alpha \beta} \partial_A u^\alpha \partial^A u^\beta - \frac{i}{2} \int_{\partial \Sigma} \delta \mu_g B_{\alpha \beta} \ u^\alpha \partial_{\mathrm{tan}} u^\beta \right)$$

$$= \frac{1}{4\pi \alpha'} \int \delta \mu_g \left( \partial_A x^i \partial^A x^j + h_{ab} \partial_A y^a \partial_A y^b - 2\pi \alpha' i B_{ij} \epsilon^{AB} \partial_A x^i \partial_B x^j - 2\pi \alpha' i B_{ab} \epsilon^{AB} \partial_A y^a \partial_B y^b \right)$$

$$= \frac{1}{4\pi \alpha'} \int \delta \mu_g \left( \partial_A x^i \partial^A x^j + h_{ab} \partial_A y^a \partial_A y^b \right)$$

$$- \frac{i}{2} \int_{\partial \Sigma} \delta \mu_g B_{ij} \ x^i \partial_{\mathrm{tan}} x^j - \frac{i}{2} \int_{\partial \Sigma} \delta \mu_g B_{ab} \ y^a \partial_{\mathrm{tan}} y^b ,$$

where the first variant is written by using metric ansatz $g_{\alpha \beta}$ (11) but the second variant is just the term $S_{g_{\alpha \beta}}$ from action (13) with d–metric (111) and different boundary conditions and $\partial_{\mathrm{tan}}$ is the tangential derivative along the worldsheet boundary $\partial \Sigma$. We emphasize that the values $g_{ij}, h_{ab}$ and $B_{ij}, B_{ab}$, given with respect to N–adapted frames are constant, but the off–diagonal $g_{\alpha \beta}$ and $B_{\alpha \beta}$, in coordinate base, are some functions on $(x, y)$. The worldsheet $\Sigma$ is taken to be with Euclidean signature (for a Lorentzian worldsheet the complex $i$ should be omitted multiplying $B$).

The equation of motion of string in anholonomic constant background define respective anholonomic, N–adapted boundary conditions. For coordinated $\alpha$ along the $Dp$–branes they are

$$g_{\alpha \beta} \partial_{\mathrm{norm}} u^\beta + 2\pi \alpha' B_{\alpha \beta} \partial_{\mathrm{tan}} u^\beta = 0,$$

$$g_{ij} \partial_{\mathrm{norm}} x^j + h_{ab} \partial_{\mathrm{norm}} y^b + 2\pi \alpha' B_{ij} \partial_B x^j - 2\pi \alpha' i B_{ab} \partial_{\mathrm{tan}} y^b |_{\partial \Sigma} = 0,$$

where $\partial_{\mathrm{norm}}$ is a normal derivative to $\partial \Sigma$. By transforms of type $g_{\alpha \beta} = e^\alpha (u) e^\beta (u) g_{\alpha \beta}$ and $B_{\alpha \beta} = e^\alpha (u) e^\beta (u) B_{\alpha \beta}$ we can remove these boundary conditions into a holonomic off–diagonal form which is more difficult to investigate. With respect to N–adapted frames (with non–underlined indices) the analysis is very similar to the case constant.

40
values of the metric and $B$–field. For $B = 0$, the boundary conditions are Neumann ones. If $B$ has the rank $r = p$ and $B \to \infty$ (equivalently, $g_{\alpha\beta} \to 0$ along the spatial directions of the brane, the boundary conditions become of Dirichlet type). The effect of all such type conditions and their possible interpolations can be investigated as in the usual open string theory with constant $B$–field but, in this subsection, with respect to N–adapted frames.

For instance, we can suppose that $\Sigma$ is a disc, conformally and anholonomically mapped to the upper half plane with complex variables $z$ and $\bar{z}$ and $\text{Im} z \geq 0$. The propagator with such boundary conditions is the same as in [20] with coordinates redefined to anholonomic frames,

$$< x^i(z)x^j(z') > = -\alpha'[g^{ij}\log \left| \frac{z-z'}{\bar{z}-\bar{z}'} \right| + H^{ij}\log |z - \bar{z}'|^2 + \frac{1}{2\pi\alpha'}\Theta^{ij}\log \left| \frac{z-\bar{z}}{\bar{z}-z'} \right| + Q^{ij}],$$

$$< y^a(z)y^b(z') > = -\alpha'[h^{ab}\log \left| \frac{z-z'}{\bar{z}-\bar{z}'} \right| + H^{ab}\log |z - \bar{z}'|^2 + \frac{1}{2\pi\alpha'}\Theta^{ab}\log \left| \frac{z-\bar{z}}{\bar{z}-z'} \right| + Q^{ab}],$$

where the coefficients are correspondingly computed,

$$H_{ij} = g_{ij} - (2\pi\alpha')^2 \left( Bg^{-1}B \right)_{ij}, \quad H_{ab} = h_{ab} - (2\pi\alpha')^2 \left( Bh^{-1}B \right)_{ab}, \quad (49)$$

$$H^{ij} = \left( \frac{1}{g + 2\pi\alpha'B} \right)_{[\text{sym}]}^{ij} = \left( \frac{1}{g + 2\pi\alpha'B} g \frac{1}{g - 2\pi\alpha'B} \right)^{ij},$$

$$H^{ab} = \left( \frac{1}{h + 2\pi\alpha'B} \right)_{[\text{sym}]}^{ij} = \left( \frac{1}{h + 2\pi\alpha'B} h \frac{1}{h - 2\pi\alpha'B} \right)^{ij},$$

$$\Theta^{ij} = 2\pi\alpha' \left( \frac{1}{g + 2\pi\alpha'B} \right)_{[\text{antisym}]}^{ij} = -(2\pi\alpha')^2 \left( \frac{1}{g + 2\pi\alpha'B} g \frac{1}{g - 2\pi\alpha'B} \right)^{ij},$$

$$\Theta^{ab} = 2\pi\alpha' \left( \frac{1}{g + 2\pi\alpha'B} \right)_{[\text{antisym}]}^{ab} = -(2\pi\alpha')^2 \left( \frac{1}{g + 2\pi\alpha'B} g \frac{1}{g - 2\pi\alpha'B} \right)^{ab},$$

with [$\text{sym}$] and [$\text{antisym}$] prescribing, respectively, the symmetric and antisymmetric parts of a matrix and constants $Q^{ij}$ and $Q^{ab}$ (in general, depending on $B$, but not on $z$ or $z'$) do to not play an essential role which allows to set them to a convenient value. The last two terms are signed–valued (if the branch cut of the logarithm is taken in lower half plane) and the rest ones are manifestly signe–valued.
Restricting our considerations to the open string vertex operators and interactions with real \( z = \tau \) and \( z = \tau' \), evaluating at boundary points of \( \Sigma \) for a convenient value of \( D^{\alpha\beta} \), the propagator (in non–distinguished form) becomes

\[
<u^\alpha(\tau)u^\beta(\tau')> = -\alpha' H^{\alpha\beta} \log (\tau - \tau')^2 + \frac{i}{2} \Theta^{\alpha\beta} \epsilon (\tau - \tau')
\]

for \( \epsilon (\tau - \tau') \) being 1 for \( \tau > \tau' \) and -1 for \( \tau < \tau' \). The d–tensor \( H_{\alpha\beta} \) defines the effective metric seen by the open string subjected to some anholonomic constraints being constant with respect to \( N \)–adapted frames. Working as in conformal field theory, one can compute commutators of operators from the short distance behavior of operator products (by interpreting time ordering as operator ordering with time \( \tau \)) and find that the coordinate commutator

\[
[u^\alpha(\tau), u^\beta(\tau)] = i \Theta^{\alpha\beta}
\]

which is just the relation (40) for noncommutative coordinates with constant noncommutativity parameter \( \Theta^{\alpha\beta} \) distinguished by a \( N \)–connection structure.

In a similar manner we can introduce gauge fields and consider worldsheet supersymmetry together with noncommutative relations with respect to \( N \)–adapted frames. This results in locally anisotropic modifications of the results from \[57\] via anholonomic frame transforms and distinguished tensor and noncommutative calculus (we omit here the details of such calculations).

We emphasize that even the values \( H^{\alpha\beta} \) and \( \Theta^{\alpha\beta} \) \[49\] are constant with respect to \( N \)–adapted frames the anholonomic noncommutative string configurations are characterized by locally anisotropic values \( H_{\alpha\beta} \) and \( \theta^{\alpha\beta} \) which are defined with respect to coordinate frames as

\[
H_{\alpha\beta} = e_\alpha^a(u) e_\beta^b(u) H^{ab} \quad \text{and} \quad \theta^{\alpha\beta} = e_\alpha^a(u) e_\beta^b(u) \theta^{ab}
\]

with \( e_\alpha^a(u) \) \[7\] defined by \( N_i^a \) as in \[1\], i. e.

\[
\delta_i^i, \quad e_i^a = -N_i^a(u), \quad e_i^a = \delta_a^a, \quad e_i^i = 0.
\]

Now, we make use of the standard relation between world–sheet correlation function of vertex operators, the S–matrix for string scattering and effective actions which can reproduce this low energy string physics \[15\] but generalizing them for anholonomic structures. We consider that operators in the bulk of the world–sheet correspond to closed strings, while operators on the boundary correspond to open strings and thus fields which propagate on the world volume of a D–brane. The basic idea is that each local world–sheet operator \( V_s(z) \) corresponds to an interaction with a spacetime field \( \varphi_s(z) \) which results in the effective Lagrangian

\[
\int \delta^{p+1} u \sqrt{|\det g_{\alpha\beta}|} N Tr \varphi_1 \varphi_2 ... \varphi_s
\]

which is computed by integrating on \( z_s \) following the prescribed order for the correlation function

\[
\left< \int dz_1 V_1(z_1) \int dz_2 V_2(z_2) ... \int dz_s V_s(z_s) \right>
\]
on a world–sheet $\Sigma$ with disk topology, with operators $V_s$ as successive points $z_s$ on the boundary $\partial \Sigma$. The integrating measure is constructed from $N$–elongated values and coefficients of d–metric. In the leading limit of the S–matrix with vertex operators only for the massless fields we reproduce a locally anisotropic variant of the MSYM effective action which describes the physics of a D–brane with arbitrarily large but anisotropically and slowly varying field strength,

$$S_{[anh]}^{BNI} = \frac{1}{g_s l_s^4 (2\pi l_s)^p} \int \delta^{p+1} u \sqrt{| \det(g_{\alpha\beta} + 2\pi l_s^2 (B + F))|}$$ (50)

where $g_s$ is the string coupling, the constant $l_s$ is the usual one from D–brane theory and $g_{\alpha\beta}$ is the induced d–metric on the brane world–volume. The action (50) is just the Nambu–Born–Infeld (NBI) action [20] but defined for d–metrics and d–tensor fields with coefficients computed with respect to N–adapted frames.

5.2 Noncommutative anisotropic structures in M(atrix) theory

For an introduction to M–theory, we refer to [32, 15]. Throughout this subsection we consider M–theory as to be not completely defined but with a well–defined quantum gravity theory with the low energy spectrum of the 11 dimensional supergravity theory [11], containing solitonic ”branes”, the 2–brane, or supermembrane, and five–branes and that from M–theory there exists connections to the superstring theories. Our claim is that in the low energy limits the noncommutative structures are, in general, locally anisotropic.

The simplest way to derive noncommutativity from M–theory is to start with a matrix model action such in subsection 4.3.2 and by introducing operators of type $C_\alpha$ (44) and actions (45). For instance, we can consider the action for maximally supersymmetric quantum mechanics, i. e. a trivial case with $p = 0$ of MSYM, when

$$S = \int \delta t \; NTr \sum_{\alpha=1}^{9} (D_t X^\alpha)^2 - \sum_{\alpha<\beta} \left[ X^\alpha, X^\beta \right]^2 + \chi^+ (D_t + \Gamma_\alpha X^\alpha) \chi,$$ (51)

where $D_t = \delta/\delta t + iA_0$ with d–derivative (105) with varying $A_0$ which introduces constraints in physical states because of restriction of unitary symmetry. This action is written in anholonomic variables and generalizes the approach of entering the M–theory as a regularized form of the actions for the supermembranes [56]. In this interpretation the the compact eleventh dimension does not disappear and the M–theory is to be considered as to be anisotropically compactified on a light–like circle.

In order to understand how anisotropic torus compactifications may be performed (see subsection 4.2.1) we use the general theory of D–branes on quotient spaces [39]. We consider $U_\alpha = \gamma (\beta_\alpha)$ for a set of generators of $\mathbb{Z}^{n+m}$ with $\mathcal{A} = \text{Mat}_{n+m} (\mathbb{C})$ which satisfy the equations

$$U_\alpha^{-1} X^\beta U_\alpha = X^\beta + \delta^\beta_\alpha 2\pi R_\alpha$$

43
having solutions of type

\[ X_\beta = -\frac{i\delta}{\partial \sigma^\beta} + A_\beta \]

for \( A_\beta \) commuting with \( U_\alpha \) and indices distinguished by a N–connection structure as \( \alpha = (i,a), \beta = (j,b) \). For such variables the action (51) leads to a locally anisotropic MSYM on \( T^{n+m} \times \mathbb{R} \). Of course, this construction admits a natural generalization for variables \( U_\alpha \) satisfying relations (40) for noncommutative locally anisotropic tori which leads to noncommutative anholonomic gauge theories (48, 50). In original form this type of noncommutativity was introduced in M–theory (without anisotropies) in Ref. [9].

The anisotropic noncommutativity in M–theory can related to string model via nontrivial components \( C_{\alpha\beta}^- \) of a three–form potential ("\(-"\) denotes the compact light–like direction). This potential has as a background value if the M(atrix) theory is treated as M–theory on a light–like circle as in usual isotropic models. In the IIA string interpretation of \( C_{\alpha\beta}^- \) as a Neveu–Schwarz B–field which minimally coupled to the string world–sheet, we obtain the action (47) compactified on a \( \mathbb{R} \times T^{n+m} \) spacetime where torus has constant d–metric and B–field coefficients.

6 Anisotropic Gravity on Noncommutative D–Bra-

nes

We develop a model of locally anisotropic gravity on noncommutative D–branes (see Refs. [2] for a locally isotropic variant). We investigate what kind of deformations of the low energy effective action of closed strings are induced in the presence of constant background antisymmetric field (or it anholonomic transforms) and/or in the presence of generic off–diagonal metric and associated nonlinear connection terms. It should be noted that there were proposed and studied different models of noncommutative deformations of gravity [7], which were not derived from string theory but introduced "ad hoc". Anholonomic and/or gauge transforms in noncommutative gravity were considered in Refs. [50, 48]. In this Section, we illustrate how such gravity models with generic anisotropy and noncommutativity can be embedded in D–brane physics.

We can compute the tree level bosonic string scattering amplitude of two massless closed string off a noncommutative D–brane with locally anisotropic contributions by considering boundary conditions and correlators stated with respect to anholonomic frames. By using the 'geometric d–covariant rule' of changing the tensors, spinors and connections into theirs corresponding N–distinguished d–objects we derive the locally anisotropic variant of effective actions in a straightforward manner.

For instance, the action which describes this amplitude to order of the string constant \( (\alpha')^0 \) is just the so–called DBI and Einstein–Hilbert action. With respect to the Einstein N–emphasized frame the DBI action is

\[ S_{D–brane}^{[0]} = -T_p \int \delta^{\nu+1} u \ e^{-\Phi} \sqrt{|\text{det} (e^{-\gamma^\Phi} g_{\alpha\beta} + B_{\alpha\beta} + f_{\alpha\beta})|} \] (52)
where \( g_{\alpha\beta} \) is the induced metric on the D–brane, \( B_{\alpha\beta} = B_{\alpha\beta} - 2\kappa b_{\alpha\beta} \) is the pull back of the antisymmetric d–field \( B \) being constant with respect to N–adapted frames along D–brane, \( f_{\alpha\beta} \) is the gauge d–field strength and \( \gamma = -4/(n + m - 2) \) and the constant \( T_p = C(\alpha')^2/C\rho^2 \) is taken as in Ref. [2] for usual D–brane theory (this allow to obtain in a limit the Einstein–Hilber action in the bulk). There are used such parametrizations of indices:

\[
\mu', \nu', \ldots = 0, ..., 25; \mu' = (\mu, \hat{\mu}); \hat{\mu}, \hat{\nu}, \ldots = p + 1, ..., 25; \hat{\mu} = (\hat{i}, \hat{a})
\]

where \( i \) takes \( n \)–dimensional ‘horizontal’ values and \( a \) takes \( m \)–dimensional ‘vertical’ being used for a D–brane localized at \( u^{p+1}, ... u^{25} \) with the boundary conditions given with respect to a N–adapted frame,

\[
g_{\alpha\beta} (\partial - \overline{\partial}) U^\alpha + B_{\alpha\beta} (\partial + \overline{\partial}) U|_{z = \overline{z}}^\alpha = 0,
\]

which should be distinguished in h- and v–components, and the two point correlator of string anholonomic coordinates \( U^\alpha (z, \overline{z}) \) on the D–brane is

\[
< U^\alpha_{\mu'} U^\beta_{\nu'} > = -\frac{\alpha'}{2} \{ g^{\alpha'\beta'} \log [(z_{\hat{\mu}} - z_{\hat{\nu}})(\overline{z}_{\hat{\mu}} - \overline{z}_{\hat{\nu}})] \\
+ D^{\alpha'\beta'} \log (z_{\hat{\mu}} - z_{\hat{\nu}}) + D^{\beta'\alpha'} \log (\overline{z}_{\hat{\mu}} - \overline{z}_{\hat{\nu}}) \}
\]

where

\[
D^{\beta\alpha} = 2 \left( \frac{1}{\eta + B} \right)^{\alpha\beta} - \eta^{\alpha\beta}, \quad D^{\hat{\mu}\hat{\nu}} = -\delta^{\hat{\mu}\hat{\nu}}, \quad D^{\hat{\beta}^\alpha}_{\alpha\tau} D^{\nu\alpha'} = \eta^{\beta\nu'}
\]

for constant \( \eta^{\alpha'\beta'} \) given with respect to N–adapted frames.

The scattering amplitude of two closed strings off a D–brane is computed as the integral

\[
A = g_c^2 e^{-\lambda} \int d^2z_1 d^2z_2 < V (z_1, \overline{z}_1) V (z_2, \overline{z}_2) >,
\]

for \( g_c \) being the closed string coupling constant, \( \lambda \) being the Euler number of the world sheet and the vertex operators for the massless closed strings with the momenta \( k_{\mu'} = (k_{\mu'}, k_{\alpha'}) \) and polarizations \( \epsilon_{\mu'\nu'} \) (satisfying the conditions \( \epsilon_{\mu'\nu'} k_{\mu'}^{\mu'} = \epsilon_{\mu'\nu'} k_{\mu'}^{\nu'} = 0 \) and \( k_{\mu'} k_{\nu'} = 0 \) taken as

\[
V (z_i, \overline{z}_i) = \epsilon_{\mu'\nu'} \frac{D^{\nu'}_{\alpha'}}{\partial X^{\nu'} (z_2)} \exp [ik_{\mu} X (z_2)] ; \quad \overline{\partial} X^{\alpha'} (\overline{z}_2) \exp \left[ ik_{\alpha'} D^{\beta'}_{\tau'} X^{\tau'} (\overline{z}_2) \right] ;
\]

Calculation of such calculation functions can be performed as in usual string theory with that difference that the tensors and derivatives are distinguished by N–connections.

Decomposing the metric \( g_{\alpha\beta} \) as

\[
g_{\alpha\beta} = \eta_{\alpha\beta} + 2\kappa \chi_{\alpha\beta}
\]

45
where \( \eta_{\alpha\beta} \) is constant (Minkowski metric but with respect to N–adapted frames) and \( \chi_{\alpha\beta} \) could be of (pseudo) Riemannian or Finsler like type. Action \([52]\) can be written to the first order of \( \chi \),

\[
S_{D\text{-brane}}^{[\alpha]} = -\kappa T_p c \int \delta^{p+1} u \chi_{\alpha\beta} Q^{\alpha\beta},
\]

where

\[
Q^{\alpha\beta} = \frac{1}{2} (\eta^{\alpha\beta} + D^{\alpha\beta})
\]

and \( c = \sqrt{|\det (\eta_{\alpha\beta} + B_{\alpha\beta})|} \), which exhibits a source for locally anisotropic gravity on D–brane,

\[
T^\alpha_{\chi} = -\frac{1}{2} T_p k c \left( \eta^{\alpha\beta} + D^{\beta\alpha}_{(S)} \right),
\]

for \( D^{\beta\alpha}_{(S)} \) denoting symmetrization of the matrix \( D^{\beta\alpha} \). This way we reproduce the same action as in superstring theory \([22]\) but in a manner when anholonomic effects and anisotropic scattering can be included.

Next order terms on \( \alpha' \) in the string amplitude are included by the term

\[
S_{\text{bulk}}^{[1]} = \frac{\alpha'}{8k^2} \int \delta^{26} d' e^{\gamma\Phi} \sqrt{|g_{\nu'\nu'}} \left[ R_{h'j'i'k'k'h'} + R_{i'j'i'k'k'} R^{h'i'j'k'} + P_{j'k'k'a'} P^{j'k'k'a'} + S_{j'i'i'c'} S^{j'i'i'c'} + S_{d'i'i'c'} S^{d'i'i'c'} - 4 \left( R_{i'i'c'} + R_{i'c'i'} R^{i'i'} + P_{a'i'} P^{a'i'} + R_{a'c'} R^{a'c'} \right) + (g^{i'i'} R_{i'i'} + k^{i'i'} S_{i'i'})^2 \right]
\]

where the indices are split as \( \mu' = (i', a') \) and we use respectively the d–curvatures \([117]\), Ricci d–tensors \([118]\) and d–scalars \([119]\). Splitting of ’primed’ indices reduces to splitting of D–brane values.

The DBI action on D–brane \([53]\) is defined with a gauge field strength

\[
f_{\alpha\beta} = \delta_\alpha a_\beta - \delta_\beta a_\alpha
\]

and with the induced metric

\[
g_{\alpha\beta} = \delta_\alpha x^\mu \delta_\beta x^\nu
\]

expanded around the flat space in the static gauge \( U^\mu = u^\mu \),

\[
g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa \chi_{\mu\nu} + 2\kappa \left( \chi_{\mu\nu} \delta_\mu U^\mu + \chi_{\mu\nu} \delta_\nu U^\mu \right) + \delta_\mu U^\mu \delta_\nu U^\nu + 2\kappa \chi_{\mu\nu} \delta_\mu U^\mu \delta_\nu U^\nu.
\]

In order to describe D–brane locally anisotropic processes in the first order in \( \alpha' \) we need to add a new term to the DBI as follow,

\[
S^1 = -\frac{\alpha'T_p}{2} \int \delta^{p+1} u \sqrt{|\det q_{\mu\nu}|} \{ R_{\alpha\beta\gamma\tau} q^{\alpha\tau} - (\Psi_{\alpha\gamma} \Psi_{\beta\gamma} - \Psi_{\alpha\tau} \Psi_{\beta\gamma}) \tilde{q}^{\alpha\tau} \} \tilde{q}^{\beta\gamma}
\]
where $q_{\mu \nu} = \eta_{\mu \nu} + B_{\mu \nu} + f_{\mu \nu}$ is the inverse of $q_{\mu \nu}$, $\tilde{q}_{\mu \nu} = q_{\mu \nu} + B_{\mu \nu} + f_{\mu \nu}$ is the inverse of $\tilde{q}_{\mu \nu}$, the curvature $d$–tensor $R_{\alpha \beta \gamma \tau}$ is constructed from the induced $d$–metric by using the canonical $d$–connection (see (117) and (113)) and

$$\Psi_{\alpha \beta}^\mu = \kappa \left( -\delta_{\alpha}^{\hat{\mu}} \chi_{\alpha \beta} + \delta_{\alpha} \chi_{\alpha \beta}^{\hat{\mu}} + \delta_{\beta} \chi_{\alpha}^{\hat{\mu}} \right) + \delta_{\alpha} \delta_{\beta} U_{\mu}.$$ 

The action (56) can be related to the Einstein–Hilbert action on the D–brane if the the $B$–field is turned off. To see this we consider the field $Q_{\alpha \beta} = \eta^{\alpha \beta}$ which reduces (up to some total $d$–derivatives, which by the momentum conservation relation have no effects in scattering amplitudes, and ignoring gauge fields because they do not any contraction with gravitons because of antisymmetry of $f_{\alpha \beta}$) to

$$S_{D–brane}^{[1]} = -\frac{\alpha' T_p}{2} \int \delta^{p+1} u \sqrt{|\det g_{\mu \nu}|} \left( \hat{R} + S + \Psi_{\alpha \beta}^\mu \Psi_{\mu \alpha \beta} - \Psi_{\alpha \beta}^\mu \Psi_{\mu \alpha \beta} \right)$$

where $\hat{R}$ and $S$ are computed as $d$–scalar objects (119) and by following the relation at $O(\chi^2)$,

$$\sqrt{|\det \eta_{\mu \nu}| R_{\alpha \beta \gamma \tau} \eta^{\alpha \tau} g^{\beta \gamma} = \sqrt{|\det g_{\mu \nu}| R_{\alpha \beta \gamma \tau} g^{\alpha \tau} g^{\beta \gamma} + \text{total } d\text{–derivatives}.}$$

The action (57) transforms into the Einstein–Hilbert action as it was proven for the locally isotropic D–brane theory [10] for vanishing $N$–connections and trivial vertical (anisotropic) dimensions.

In conclusion, it has been shown in this Section that the D–brane dynamics can be transformed into a locally anisotropic one, which in low energy limits contains different models of generalized Lagrange/ Finsler or anholonomic Riemannian spacetimes, by introducing corresponding anholonomic frames with associated $N$–connection structures and $d$–metric fields (like (128) and (130) and (129)).

7 Exact Solutions: Noncommutative and/ or Locally Anisotropic Structures

In the previous sections we demonstrated that locally anisotropic noncommutative geometric structures are hidden in string/ M–theory. Our aim here is to construct and analyze four classes of exact solutions in string gravity with effective metrics possessing generic off–diagonal terms which for associated anholonomic frames and $N$–connections can be extended to commutative or noncommutative string configurations.

7.1 Black ellipsoids from string gravity

A simple string gravity model with antisymmetric two form potential field $H^{\alpha \beta \gamma}$, for constant dilaton $\phi$, and static internal space, $\beta$, is to be found for the NS–NS sector
which is common to both the heterotic and type II string theories [26]. The equations (58) reduce to

\[ R_{\mu'\nu'} = \frac{1}{4} H_{\mu'\lambda'\nu'} H_{\nu'\lambda'}, \quad (58) \]

\[ D_{\mu'} H_{\nu'\lambda'} = 0, \]

for

\[ H_{\mu'\nu'\lambda'} = \delta_{\mu'} B_{\nu'\lambda'} + \delta_{\nu'} B_{\mu'\lambda'}. \]

If \( H_{\mu'\nu'\lambda'} = \sqrt{|g_{\mu'\nu'}|} \), we obtain the vacuum equations for the gravity with cosmological constant \( \lambda \),

\[ R_{\mu'\nu'} = \lambda g_{\mu'\nu'}, \quad (59) \]

for \( \lambda = 1/4 \) where \( R_{\mu'\nu'} \) is the Ricci d–tensor \([118]\), with ”primed” indices emphasizing that the geometry is induced after a topological compactification.

For an ansatz of type

\[ \delta s^2 = g_1 (dx^1)^2 + g_2 (dx^2)^2 + h_3 \left( x^{\nu'}, y^3 \right) (\delta y^3)^2 + h_4 \left( x^{\nu'}, y^3 \right) (\delta y^4)^2; \]

\[ \delta y^3 = dy^3 + w_{\nu'} \left( x^{\nu'}, y^3 \right) dx^{\nu'}, \quad \delta y^4 = dy^4 + n_{\nu'} \left( x^{\nu'}, y^3 \right) dx^{\nu'}, \quad (60) \]

for the d–metric \([111]\) the Einstein equations \((59)\) are written (see \([49, 45]\) for details on computation)

\[ R_1 = R_2 = -\frac{1}{2g_1g_2} \left[ g_1^{\bullet} g_2^{\bullet} - \frac{g_1^{\bullet} g_2^{\bullet}}{2g_1} \left( g_2^{\bullet} \right)^2 + g_1^{\bullet} \frac{g_1^{\bullet} g_2^{\bullet}}{2g_2} - \frac{(g_1^{\bullet})^2}{2g_1} \right] = \lambda, \quad (61) \]

\[ R_3 = R_4 = -\frac{\beta}{2h_3h_4} = \lambda, \quad (62) \]

\[ R_{3\nu'} = -\frac{\beta}{2h_3} - \frac{\alpha_{\nu'}}{2h_4} = 0, \quad (63) \]

\[ R_{4\nu'} = -\frac{h_4}{2h_3} \left[ n_{\nu'}^{**} + \gamma n_{\nu'}^{**} \right] = 0, \quad (64) \]

where the indices take values \( \nu', k' = 1, 2 \) and \( a', b' = 3, 4 \). The coefficients of equations \((61) - (64)\) are given by

\[ \alpha_i = \partial_i h_4^* - h_4^* \partial_i \ln \sqrt{|h_3h_4|}, \quad \beta = h_4^* - h_4^* \ln \sqrt{|h_3h_4|}, \quad \gamma = \frac{3h_4^*}{2h_4} - \frac{h_3^*}{h_3}. \quad (65) \]

The various partial derivatives are denoted as \( a^\bullet = \partial a/\partial x^1, a' = \partial a/\partial x^2, a^* = \partial a/\partial y^3 \). This system of equations \((61) - (64)\) can be solved by choosing one of the ansatz functions (e.g. \( g_1 (x^i) \) or \( g_2 (x^i) \)) and one of the ansatz functions (e.g. \( h_3 (x^i, y^3) \) or \( h_4 (x^i, y^3) \)) to take some arbitrary, but physically interesting form. Then the other ansatz functions can be analytically determined up to an integration in terms of this choice. In this way we can generate a lot of different solutions, but we impose the condition that the
initial, arbitrary choice of the ansatz functions is “physically interesting” which means that one wants to make this original choice so that the generated final solution yield a well behaved metric.

In references [46] it is proved that for

\[
\begin{align*}
    g_1 &= -1, \quad g_2 = r^2(\xi) q(\xi), \\
    h_3 &= -\eta_3(\xi, \varphi) r^2(\xi) \sin^2 \theta, \\
    h_4 &= \eta_4(\xi, \varphi) h_{4[0]}(\xi) = 1 - \frac{2\mu}{r} + \frac{\varepsilon \Phi_4(\xi, \varphi)}{2\mu^2},
\end{align*}
\]

with coordinates \( x^1 = \xi = \int dr \sqrt{1 - \frac{2m}{r} + \frac{\varepsilon}{r^2}}, \) \( x^2 = \theta, y^3 = \varphi, y^4 = t \) (the \((r, \theta, \varphi)\) being usual radial coordinates), the ansatz (66) is a vacuum solution with \( \lambda = 0 \) of the equations (59) which defines a black ellipsoid with mass \( \mu \), eccentricity \( \varepsilon \) and gravitational polarizations \( q(\xi), \eta_3(\xi, \varphi) \) and \( \Phi_4(\xi, \varphi) \). Such black holes are certain deformations of the Schwarszchild metrics to static configurations with ellipsoidal horizons which is possible if generic off–diagonal metrics and anholonomic frames are considered. In this subsection we show that the data (66) can be extended as to generate exact black ellipsoid solutions with nontrivial cosmological constant \( \lambda = 1/4 \) which can be imbedded in string theory.

At the first step, we find a class of solutions with \( g_1 = -1 \) and \( g_2 = g_2(\xi) \) solving the equation (61), which under such parametrizations transforms to

\[
\begin{align*}
    g_2^{**} = \frac{(g_2^*)^2}{2g_2} &= 2g_2\lambda.
\end{align*}
\]

With respect to the variable \( Z = (g_2)^2 \) this equation is written as

\[
Z^{**} + 2\lambda Z = 0
\]

which can be integrated in explicit form, \( Z = Z_{[0]} \sin \left( \sqrt{2\lambda} \xi + \xi_{[0]} \right) \), for some constants \( Z_{[0]} \) and \( \xi_{[0]} \) which means that

\[
\begin{align*}
    g_2 &= -Z_{[0]}^2 \sin^2 \left( \sqrt{2\lambda} \xi + \xi_{[0]} \right).
\end{align*}
\]

(67)

parametrize a class of solution of (61) for the signature \((-,-,-,+).\) For \( \lambda \to 0 \) we can approximate \( g_2 = r^2(\xi) q(\xi) = -\xi^2 \) and \( Z_{[0]}^2 = 1 \) which has compatibility with the data (66). The solution (67) with cosmological constant (of string or non–string origin) induces oscillations in the ”horozontal” part of the d–metric.

The next step is to solve the equation (62),

\[
\begin{align*}
    h_4^{**} - h_4^* \ln \sqrt{|h_3 h_4|}^* &= -2\lambda h_3 h_4.
\end{align*}
\]

For \( \lambda = 0 \) a class of solution is given by any \( \hat{h}_3 \) and \( \hat{h}_4 \) related as

\[
\hat{h}_3 = \eta_0 \left[ \left( \sqrt{\hat{h}_4} \right)^* \right]^2
\]

49
for a constant $\eta_0$ chosen to be negative in order to generate the signature $(-, -, -, +)$.

For non–trivial $\lambda$, we may search the solution as

$$h_3 = \hat{h}_3(\xi, \varphi) \quad q_3(\xi, \varphi) \quad \text{and} \quad h_4 = \hat{h}_4(\xi, \varphi),$$

which solves \((62)\) if $q_3 = 1$ for $\lambda = 0$ and

$$q_3 = \frac{1}{4\lambda} \left[ \int \frac{\hat{h}_3 \hat{h}_4}{\hat{h}_4^*} d\varphi \right]^{-1} \quad \text{for} \quad \lambda \neq 0.$$

Now it is easy to write down the solutions of equations \((63)\) (being a linear equation for $w_i'$) and \((64)\) (after two integrations of $n_i'$ on $\varphi$),

$$w_i' = -\alpha_i' / \beta,$$

were $\alpha_i'$ and $\beta$ are computed by putting \((68)\) into corresponding values from \((65)\) (we chose the initial conditions as $w_i' \to 0$ for $\varepsilon \to 0$) and

$$n_1 = \varepsilon \hat{n}_1(\xi, \varphi)$$

where

$$\hat{n}_1(\xi, \varphi) = n_{1[1]}(\xi) + n_{1[2]}(\xi) \int d\varphi \eta_3(\xi, \varphi) / \left( \sqrt{\eta_4(\xi, \varphi)} \right)^3, \eta_4^* \neq 0; \quad (70)$$

$$= n_{1[1]}(\xi) + n_{1[2]}(\xi) \int d\varphi \eta_3(\xi, \varphi), \eta_4^* = 0;$$

$$= n_{1[1]}(\xi) + n_{1[2]}(\xi) \int d\varphi / \left( \sqrt{\eta_4(\xi, \varphi)} \right)^3, \eta_4^* = 0;$$

with the functions $n_{k[1,2]}(\xi)$ to be stated by boundary conditions.

We conclude that the set of data $g_1 = -1$, with non–trivial $g_2(\xi), h_3, h_4, w_i', n_1$ stated respectively by \((67), (68), (69), (70)\) define a black ellipsoid solution with explicit dependence on cosmological constant $\lambda$, i.e. a d–metric \((59)\), which can be induced from string theory for $\lambda = 1/4$. The stability of such string static black ellipsoids can be proven exactly as it was done in Refs. \([46]\) for the vanishing cosmological constant.

### 7.2 2D Finsler structures in string theory

There are some constructions which prove that two dimensional (2D) Finsler structures can be embedded into the Einstein’s theory of gravity \([12]\). Here we analyze the conditions when such Finsler configurations can be generated from string theory. The aim is to include a 2D Finsler metric \((127)\) into a d–metric \((111)\) being an exact solution of the string corrected Einstein’s equations \((59)\).

If

$$h_{\alpha'\beta'} = \frac{1}{2} \frac{\partial^2 F^2(x', y')} {\partial y'^\alpha \partial y'^\beta}$$

$$50$$
for \( i', j', \ldots = 1, 2 \) and \( a', b', \ldots = 3, 4 \) and following the homogeneity conditions for Finsler metric, we can write

\[
F \left( x', y^3, y^4 \right) = y^3 f \left( x', s \right)
\]

for \( s = y^4/y^3 \) with \( f \left( x', s \right) = F \left( x', 1, s \right) \), that

\[
\begin{align*}
    h_{33} &= \frac{s^2}{2}(f^2)^{**} - s(f^2)^* + f^2, \\
    h_{34} &= -\frac{s^2}{2}(f^2)^{**} + \frac{1}{2}(f^2)^*, \\
    h_{44} &= \frac{1}{2}(f^2)^{**},
\end{align*}
\]

(71)

in this subsection we denote \( a^* = \partial a/\partial s \). The condition of vanishing of the off–diagonal term \( h_{34} \) gives us the trivial case, when \( f^2 \simeq s^2 \ldots s \), i. e. Riemannian 2D metrics, so we can not include some general Finsler coefficients (71) directly into a diagonal d–metric ansatz (60). There is also another problem related with the Cartan’s N–connection (128) being computed directly from the coefficients (71) generated by a function \( f^2 \): all such values substituted into the equations (62) - (64) result in systems of nonlinear equations containing the 6th and higher derivatives of \( f \) on \( s \) which is very difficult to deal with.

We can include 2D Finsler structures in the Einstein and string gravity via additional 2D anholonomic frame transforms,

\[
h_{ab} = e_a' \left( x', s \right) e_b' \left( x', s \right) h_{a'b'} \left( x', s \right)
\]

where \( h_{a'b'} \) are induced by a Finsler metric \( f^2 \) as in (74) and \( h_{ab} \) may be diagonal, \( h_{ab} = \text{diag}[h_a] \). We also should consider an embedding of the Cartan’s N–connection into a more general N–connection, \( N_y' \subset N_y' \), via transforms \( N_y' = e_i' \left( x', s \right) N_y' \), where \( e_i' \left( x', s \right) \) are some additionan frame transforms in the off–diagonal sector of the ansatz (11). Such way generated metrics,

\[
\begin{align*}
    \delta s^2 &= g_\ell (dx')^2 + e_a' h_{a'b'} (\delta y^a)^2, \\
    \delta y^a &= dy^a + e_i' N_y' dx^i
\end{align*}
\]

may be constrained by the condition to be an exact solution of the Einstein equations with (or not) certain string corrections. As a matter of principle, any string black ellipsoid configuration (of the type examined in the previous subsect ion) can be related to a 2D Finsler configuration for corresponding coefficients \( e_a' \) and \( e_i' \). An explicit form of anisotropic configuration is to be stated by corresponding boundary conditions and the type of anholonomic transforms. Finally, we note that instead of a 2D Finsler metric (147) we can use a 2D Lagrange metric (130).
7.3 Moving soliton–black hole string configurations

In this subsection, we consider that the primed coordinates are 5D ones obtained after a string compactification background for the NS–NS sector being common to both the heterotic and type II string theories. The $u^{\alpha'} = (x'^i, y'^i)$ are split into coordinates $x^i$, with indices $i', j', k' = 1, 2, 3$, and coordinates $y^i$, with indices $a', b', c', \ldots = 4, 5$. Explicitly the coordinates are of the form

$$x'^{i} = (x^1 = \chi, \quad x^2 = \phi = \ln \hat{\rho}, \quad x^3 = \theta) \quad \text{and} \quad y^{a'} = (y^4 = v, \quad y^5 = p),$$

where $\chi$ is the 5th extra–dimensional coordinate and $\hat{\rho}$ will be related with the 4D Schwarzschild coordinate. We analyze a metric interval written as

$$ds^2 = \Omega^2(x'^i, v)\hat{g}_{\alpha'\beta'}(x'^i, v)\, du^{\alpha'} du^{\beta'}, \quad (72)$$

were the coefficients $\hat{g}_{\alpha'\beta'}$ are parametrized by the ansatz

$$g_{1} = \pm 1, \quad g_{2,3} = g_{2,3}(x^2, x^3), \quad h_{4,5} = h_{4,5}(x'^i, v) = \eta_{4,5}(x^i, v) h_{4,5[i]}(x^{k'}),$$

$$w_{\nu'} = w_{\nu'}(x'^{k'}, v), \quad n_{\nu'} = n_{\nu'}(x'^{k'}, v), \quad \zeta_{\nu'} = \zeta_{\nu'}(x'^{k'}, v), \quad \Omega = \Omega(x'^i, v). \quad (74)$$

The quadratic line element (72) with metric coefficients (73) can be diagonalized by anholonomic transforms,

$$\delta s^2 = \Omega^2(x'^i, v)[g_{1}(dx^{1})^2 + g_{2}(dx^{2})^2 + g_{3}(dx^{3})^2 + h_{4}(\delta v)^2 + h_{5}(\delta p)^2], \quad (75)$$

with respect to the anholonomic co–frame $(dx'^{i'}, \hat{\delta}v, \hat{\delta}p)$, where

$$\hat{\delta}v = dv + (w_{\nu'} + \zeta_{\nu'}) dx'^i + \zeta_{\nu'} \delta p \quad \text{and} \quad \delta p = dp + n_{\nu'} dx'^i \quad (76)$$

which is dual to the frame $(\hat{\delta}_{\nu'}, \partial_{4}, \hat{\partial}_{5})$, where

$$\hat{\delta}_{\nu'} = \partial_{\nu'} - (w_{\nu'} + \zeta_{\nu'}) \partial_{4} + n_{\nu'} \partial_{5}, \quad \hat{\partial}_{5} = \partial_{5} - \zeta_{5} \partial_{4}. \quad (77)$$

The simplest way to compute the nontrivial coefficients of the Ricci tensor for the (75) is to do this with respect to anholonomic bases (70) and (77) (see details in [49, 53]), which reduces the 5D vacuum Einstein equations to the following system (in this paper
containing a non–trivial cosmological constant):

\[
\frac{1}{2} R_1^1 = R_2^2 = R_3^3 = -\frac{1}{2g_2g_3} [g_3^3 - g_2^2 g_1^2 - (g_1^3)^2 + g_2^2 - g_2^2 g_3^2 - (g_2^2)^2] = \lambda, \quad (78)
\]

\[
R_4^4 = R_5^5 = -\frac{\beta}{2 h_4 h_5} = \lambda, \quad (79)
\]

\[
R_4^i = -w_i \beta \frac{h_5}{2 h_5} - \alpha_i \frac{\gamma h_5}{2 h_5} = 0, \quad (80)
\]

\[
R_5^i = -\frac{h_5}{2 h_4} [n_i^* + \gamma n_i^*] = 0, \quad (81)
\]

with the conditions that

\[
\Omega^{n_1/q_2} = h_4 \quad (q_1 \text{ and } q_2 \text{ are integers}), \quad (82)
\]

and \( \zeta \) satisfies the equations

\[
\partial_i \Omega - (w_i + \zeta_i) \Omega^* = 0, \quad (83)
\]

The coefficients of equations (78) - (81) are given by

\[
\alpha_i = \partial_i h_5^* - h_5^* \partial_i \ln \sqrt{|h_4 h_5|}, \quad \beta = h_5^* - h_5^* \ln \sqrt{|h_4 h_5|}, \quad \gamma = \frac{3 h_5^*}{2 h_5} - \frac{h_4^*}{h_4}. \quad (84)
\]

The various partial derivatives are denoted as \( a^* = \partial a/\partial x^2, a' = \partial a/\partial x^3, a^* = \partial a/\partial v \).

The system of equations (78) - (81) and (83) can be solved by choosing one of the ansatz functions (e.g. \( h_4 (x^1, v) \) or \( h_5 (x^1, v) \)) to take some arbitrary, but physically interesting form. Then the other ansatz functions can be analytically determined up to an integration in terms of this choice. In this way one can generate many solutions, but the requirement that the initial, arbitrary choice of the ansatz functions be “physically interesting” means that one wants to make this original choice so that the final solution generated in this way yield a well behaved solution. To satisfy this requirement we start from well known solutions of Einstein’s equations and then use the above procedure to deform this solutions in a number of ways as to include it in a string theory. In the simplest case we derive 5D locally anisotropic string gravity solutions connected to the the Schwarzschild solution in isotropic spherical coordinates \[17\] given by the quadratic line interval

\[
ds^2 = \left( \frac{\hat{\rho} - 1}{\hat{\rho} + 1} \right)^2 dt^2 - \rho^2_g \left( \frac{\hat{\rho} + 1}{\hat{\rho}} \right)^4 \left( d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2 \theta d\varphi^2 \right). \quad (85)
\]

We identify the coordinate \( \hat{\rho} \) with the re–scaled isotropic radial coordinate, \( \hat{\rho} = \rho / \rho_g \), with \( \rho_g = r_g / 4; \rho \) is connected with the usual radial coordinate \( r \) by \( r = \rho (1 + r_g / 4 \rho)^2 \); \( r_g = 2 G[4] m_0 / c^2 \) is the 4D Schwarzschild radius of a point particle of mass \( m_0; G[4] = 1/M_P[4] \) is the 4D Newton constant expressed via the Planck mass \( M_P[4] \) (in general, we
may consider that $M_{P[4]}$ may be an effective 4D mass scale which arises from a more fundamental scale of the full, higher dimensional spacetime); we set $c = 1$.

The metric (85) is a vacuum static solution of 4D Einstein equations with spherical symmetry describing the gravitational field of a point particle of mass $m_0$. It has a singularity for $r = 0$ and a spherical horizon at $r = r_g$, or at $\hat{\rho} = 1$ in the re-scaled isotropic coordinates. This solution is parametrized by a diagonal metric given with respect to holonomic coordinate frames. This spherically symmetric solution can be deformed in various interesting ways using the anholonomic frames method.

Vacuum gravitational 2D solitons in 4D Einstein vacuum gravity were originally investigated by Belinski and Zakharov [5]. In Refs. [32] 3D solitonic configurations were constructed on anisotropic Taub-NUT backgrounds. Here we show that 3D solitonic/black hole configurations can be embedded into the 5D locally anisotropic string gravity.

### 7.3.1 3D solitonic deformations in string gravity

The simplest way to construct a solitonic deformation of the off-diagonal metric in equation (73) is to take one of the “polarization” factors $\eta_4, \eta_5$ from (74) or the ansatz function $n_4$ as a solitonic solution of some particular non-linear equation. The rest of the ansatz functions can then be found by carrying out the integrations of equations (78)–(83).

As an example of this procedure we suggest to take $\eta_5(r, \theta, \chi)$ as a soliton solution of the Kadomtsev–Petviashvili (KdP) equation or (2+1) sine-Gordon (SG) equation (Refs. [58] contain the original results, basic references and methods for handling such non-linear equations with solitonic solutions). In the KdP case $\eta_5(v, \theta, \chi)$ satisfies the following equation

$$\eta_5^{**} + \epsilon (\dot{\eta}_5 - 6\eta_5 \eta_5' + \eta_5'''')' = 0, \quad \epsilon = \pm 1, \quad (86)$$

while in the most general SG case $\eta_5(v, \chi)$ satisfies

$$\pm \eta_5^{**} \pm \dot{\eta}_5 = \sin(\eta_5). \quad (87)$$

For simplicity, we can also consider less general versions of the SG equation where $\eta_5$ depends on only one (e.g. $v$ and $x_1$) variable. We use the notation $\eta_5 = \eta_5^{KP}$ or $\eta_5 = \eta_5^{SG}$ ($h_5 = h_5^{KP}$ or $h_5 = h_5^{SG}$) depending on if $(\eta_5)$ $(h_5)$ satisfies equation (86), or (87) respectively.

For a stated solitonic form for $h_5 = h_5^{KP,SG}$, $h_4$ can be found from

$$h_4 = h_4^{KP,SG} = h_4^{[0]} \left[ \left| |h_5^{KP,SG}(x', v)| \right| \right]^{\lambda/2} \quad (88)$$

where $h_4^{[0]}$ is a constant. By direct substitution it can be shown that equation (88) solves equation (79) with $\beta$ given by (84) when $h_5^{*} \neq 0$ but $\lambda = 0$. If $h_5^{*} = 0$, then $\hat{h}_4$ is an arbitrary function $\hat{h}_4(x', v)$. In either case we will denote the ansatz function determined
in this way as \( \hat{h}_4^{KP,SG} \) although it does not necessarily share the solitonic character of \( \hat{h}_5 \). Substituting the values \( \hat{h}_4^{KP,SG} \) and \( \hat{h}_5^{KP,SG} \) into \( \gamma \) from equation (65) gives, after two \( v \) integrations of equation (64), the ansatz functions \( n_4' = n_5^{KP,SG}(v, \theta, \chi) \). Solutions with \( \lambda \neq 0 \) can be generated similarly as in (68) by redefining \( h_4 = \hat{h}_4(x', v) \) \( q_4(x', v) \) and \( h_5 = \hat{h}_5(x', v) \), which solves (79) if \( q_4 = 1 \) for \( \lambda = 0 \) and

\[
q_4 = \frac{1}{4\lambda} \left[ \int \frac{\hat{h}_5 \hat{h}_4}{\hat{h}_5^*} dv \right]^{-1} \quad \text{for} \quad \lambda \neq 0. \tag{89}
\]

Here, for simplicity, we may set \( g_2 = -1 \) but

\[
g_3 = -Z_{[0]}^2 \sin^2 \left( \sqrt{2\lambda}x^3 + \xi_{[0]} \right), \quad Z_{[0]}, \xi_{[0]} = \text{const}, \tag{90}
\]

parametrize a class of solution of (78) for the signature \((-,-,-,-,+)\) like we constructed the solution (67). In ref. [52, 53] it was shown how to generate solutions using 2D solitonic configurations for \( g_2 \) or \( g_3 \).

The main conclusion to be formulated here is that the ansatz (73), when treated with anholonomic frames, has a degree of freedom that allows one to pick one of the ansatz functions \( \eta_4, \eta_5, \) or \( n_\nu \) to satisfy some 3D solitonic equation. Then in terms of this choice all the other ansatz functions can be generated up to carrying out some explicit integrations and differentiations. In this way it is possible to build exact solutions of the 5D string gravity equations with a solitonic character.

### 7.3.2 Solitonically propagating string black hole backgrounds

The Schwarzschild solution is given in terms of the parameterization in (73) by

\[
g_1 = \pm 1, \quad g_2 = g_3 = -1, \quad h_4 = h_4^{[0]}(x'), \quad h_5 = h_5^{[0]}(x'), \quad w_\nu = 0, \quad n_\nu = 0, \quad \zeta_\nu = 0, \quad \Omega = \Omega^{[0]}(x'), \tag{91}
\]

with

\[
h_4^{[0]}(x') = \frac{b(\phi)}{a(\phi)}, \quad h_5^{[0]}(x') = -\sin^2 \theta, \quad \Omega^{[0]}(x') = a(\phi)
\]

or alternatively, for another class of solutions,

\[
h_4^{[0]}(x') = -\sin^2 \theta, \quad h_5^{[0]}(x') = \frac{b(\phi)}{a(\phi)}, \tag{92}
\]

were

\[
a(\phi) = \rho^2 g^4 \frac{e^\phi + 1}{e^{2\phi}} \quad \text{and} \quad b(\phi) = \frac{(e^\phi - 1)^2}{(e^\phi + 1)^2}, \tag{93}
\]

\]

55
A class of 5D string gravity metrics can be constructed by parametrizing

\[ ds^2 = \pm d\chi^2 - a(\phi) \left( d\lambda^2 + d\theta^2 + \sin^2 \theta d\varphi^2 \right) + b(\phi) \, dt^2 \]  

(94)

which represents a trivial embedding of the 4D Schwarzschild metric \([83]\) into the 5D spacetime. We now want to deform anisotropically the coefficients of \([94]\) in the following way

\[
\begin{align*}
h_{4[0]}(x^\nu) & \rightarrow h_4(x^\nu, v) = \eta_4 \left( x^\nu, v \right) h_{4[0]}(x^\nu), \\
h_{5[0]}(x^\nu) & \rightarrow h_5(x^\nu, v) = \eta_5 \left( x^\nu, v \right) h_{5[0]}(x^\nu), \\
\Omega^2_{4[0]}(x^\nu) & \rightarrow \Omega^2(x^\nu, v) = \Omega^2_{4[0]}(x^\nu) \Omega^2_{5[0]}(x^\nu, v).
\end{align*}
\]

The factors \(\eta_4\) and \(\Omega^2_{5[0]}\) can be interpreted as re-scaling or “renormalizing” the original ansatz functions. These gravitational “polarization” factors \(-\eta_{4,5}\) and \(\Omega^2_{5[1]}\) – generate non–trivial values for \(w^\nu(x^\nu, v), n^\nu(x^\nu, v)\) and \(\zeta^\nu(x^\nu, v)\), via the vacuum equations \([78–83]\). We shall also consider more general nonlinear polarizations which can not be expressed as \(h \sim \eta h_{4[0]}\) and show how the coefficients \(a(\phi)\) and \(b(\phi)\) of the Schwarzschild metric can be polarized by choosing the original, arbitrary ansatz function to be some 3D soliton configuration.

The horizon is defined by the vanishing of the coefficient \(b(\phi)\) from equation \([83]\). This occurs when \(e^\phi = 1\). In order to create a solitically propagating black hole we define the function \(\tau = \phi - \tau_0(\chi, v)\), and let \(\tau_0(\chi, v)\) be a soliton solution of either the 3D KdP equation \([86]\), or the SG equation \([87]\). This redefines \(b(\phi)\) as

\[ b(\phi) \rightarrow B \left( x^\nu, v \right) = \frac{e^\tau - 1}{e^\phi + 1}. \]

A class of 5D string gravity metrics can be constructed by parametrizing \(h_4 = \eta_4 \left( x^\nu, v \right) h_{4[0]}(x^\nu)\) and \(h_5 = B \left( x^\nu, v \right) / a(\phi)\), or inversely, \(h_4 = B \left( x^\nu, v \right) / a(\phi)\) and \(h_5 = \eta_5 \left( x^\nu, v \right) h_{5[0]}(x^\nu)\). The polarization \(\eta_4 \left( x^\nu, v \right)\) (or \(\eta_5 \left( x^\nu, v \right)\)) is determined from equation \([88]\) with the factor \(q_4 \[89]\) included in \(h^2\),

\[
\left| \eta_4 \left( x^\nu, v \right) h_{4[0]}(x^\nu) \right| = h^2 \left[ \frac{B \left( x^\nu, v \right)}{a(\phi)} \right]^a
\]

or

\[
\frac{B \left( x^\nu, v \right)}{a(\phi)} = \frac{h^2 h_{5[0]}(x^\nu) \left[ \left( \sqrt{\eta_5 \left( x^\nu, v \right)} \right)^a \right]^2}{h^2 h_{5[0]}(x^\nu)}.
\]

The last step in constructing of the form for these solitonically propagating black hole
solutions is to use \( h_4 \) and \( h_5 \) in equation (53) to determine \( n_{k'} \)

\[
 n_{k'} = n_{k'[1]}(x') + n_{k'[2]}(x') \left( \frac{h_4}{(\sqrt{|h_5|})^3} \right) dv, \quad h_5^* \neq 0; \tag{95}
\]

\[
 n_{k'} = n_{k'[1]}(x') + n_{k'[2]}(x') \int h_4 dv, \quad h_5^* = 0; \tag{96}
\]

\[
 = n_{k'[1]}(x') + n_{k'[2]}(x') \left( \frac{1}{(\sqrt{|h_5|})^3} \right) dv, \quad h_5^* = 0,
\]

where \( n_{k'[1,2]}(x') \) are set by boundary conditions.

The simplest version of the above class of solutions are the so-called \( t \)-solutions (depending on \( t \)-variable), defined by a pair of ansatz functions, \( [B(x',t), h_{5(0)}] \), with \( h_5^* = 0 \) and \( B(x',t) \) being a 3D solitonic configuration. Such solutions have a spherical horizon when \( h_4 = 0 \), i.e. when \( \tau = 0 \). This solution describes a propagating black hole horizon. The propagation occurs via a 3D solitonic wave form depending on the time coordinate, \( t \), and on the 5th coordinate \( \chi \). The form of the ansatz functions for this solution (both with trivial and non-trivial conformal factors) is

\[
 t\text{-solutions : } (x^1 = \chi, \quad x^2 = \phi, \quad x^3 = \theta, \quad y^4 = v = t, \quad y^5 = p = \varphi),
\]

\[
g_1 = \pm 1, \quad g_2 = -1, \quad g_3 = -Z_0^2 \sin^2 \left( \sqrt{2}x^3 + \xi^3 \right) , \quad \tau = \phi - \tau_0(\chi, t),
\]

\[
h_4 = B/a(\phi), \quad h_5 = h_{5(0)}(x') = -\sin^2 \theta, \quad \omega = \eta_5 = 1, \quad \eta_5 = 1, \quad B(x', t) = \frac{e^t - 1}{e^\varphi + 1},
\]

\[
w_{k'} = \zeta_{k'} = 0, \quad n_{k'}(x', t) = n_{k'[1]}(x') + n_{k'[2]}(x') \int B(x', t) dt, \tag{96}
\]

where \( g_4 \) is chosen to preserve the condition \( w_{k'} = \zeta_{k'} = 0 \).

As a simple example of the above solutions we take \( \tau_0 \) to satisfy the SG equation \( \partial_{x\chi} \tau_0 - \partial_t \tau_0 = \sin(\tau_0) \). This has the standard propagating kink solution

\[
 \tau_0(\chi, t) = 4 \tan^{-1} [\pm \gamma (\chi - V t)]
\]

where \( \gamma = (1 - V^2)^{-1/2} \) and \( V \) is the velocity at which the kink moves into the extra dimension \( \chi \). To obtain the simplest form of this solution we also take \( n_{k'[1]}(x') = n_{k'[2]}(x') = 0 \). This example can be easily extended to solutions with a non-trivial conformal factor \( \Omega \) that gives an exponentially suppressing factor, \( \exp[-2k|\chi|] \), see details in Ref. [53]. In this manner one has an effective 4D black hole which propagates from the 3D brane into the non-compact, but exponentially suppressed extra dimension, \( \chi \).

The solution constructed in this subsection describes propagating 4D Schwarzschild black holes in a bulk 5D spacetime obtained from string theory. The propagation arises from requiring that certain of the ansatz functions take a 3D soliton form. In the simplest version of these propagating solutions the parameters of the ansatz functions are constant, and the horizons are spherical. It can be also shown that such propagating solutions could be formed with a polarization of the parameters and/or deformation of the horizons, see the non-string case in [53].
7.4 Noncommutative anisotropic wormholes and strings

Let us construct and analyze an exact 5D solution of the string gravity which can also considered as a noncommutative structure in string theory. The d–metric ansatz is taken in the form

\[ \delta s^2 = g_1(dx^1)^2 + g_2(dx^2)^2 + g_3(dx^3)^2 + h_4(\delta y^4)^2 + h_5(\delta y^5)^2, \]
\[ \delta y^4 = dy^4 + w_{k'} (x', v) dx^{k'}, \delta y^5 = dy^5 + n_{k'} (x', v) dx^{k'}; i', k' = 1, 2, 3, \quad (97) \]

where

\[ g_1 = 1, \quad g_2 = g_2(r), \quad g_3 = -a(r), \]
\[ h_4 = \hat{h}_4(r, \theta, \varphi) h_{4[0]}(r), \quad h_5 = \hat{h}_5(r, \theta, \varphi) h_{5[0]}(r, \theta) \]

for the parametrization of coordinate of type

\[ x^1 = t, x^2 = r, x^3 = \theta, y^4 = v = \varphi, y^5 = p = \chi \]

where \( t \) is the time coordinate, \((r, \theta, \varphi)\) are spherical coordinates, \( \chi \) is the 5th coordinate; \( \varphi \) is the anholonomic coordinate; for this ansatz there is not considered the dependence of d–metric coefficients on the second anholonomic coordinate \( \chi \). The data

\[ g_1 = 1, \quad \hat{g}_2 = -1, \quad g_3 = -a(r), \]
\[ h_{4[0]}(r) = -r_0^2 e^{2\psi(r)}, \quad \eta_4 = 1/\kappa_r^2 (r, \theta, \varphi), \quad h_{5[0]} = -a(r) \sin^2 \theta, \quad \eta_5 = 1, \]
\[ w_1 = \hat{w}_1 = \omega(r), \quad w_2 = \hat{w}_2 = 0, \quad w_3 = \hat{w}_3 = n \cos \theta / \kappa_n^2 (r, \theta, \varphi), \]
\[ n_1 = \hat{n}_1 = 0, \quad n_{2,3} = \hat{n}_{2,3} = n_{2,3}[1] (r, \theta) \int \ln |\kappa_r^2 (r, \theta, \varphi)| d\varphi \]

for some constants \( r_0 \) and \( n \) and arbitrary functions \( a(r), \psi(r) \) and arbitrary vacuum gravitational polarizations \( \kappa_r (r, \theta, \varphi) \) and \( \kappa_n (r, \theta, \varphi) \) define an exact vacuum 5D solution of Kaluza–Klein gravity [54] describing a locally anisotropic wormhole with elliptic gravitational vacuum polarization of charges,

\[ \frac{q_0^2}{4a(0) \kappa_r^2} + \frac{Q_0^2}{4a(0) \kappa_n^2} = 1, \]

where \( q_0 = 2 \sqrt{a(0) \sin \alpha_0} \) and \( Q_0 = 2 \sqrt{a(0) \cos \alpha_0} \) are respectively the electric and magnetic charges and \( 2 \sqrt{a(0) \kappa_r} \) and \( 2 \sqrt{a(0) \kappa_n} \) are ellipse’s axes.

The first aim in this subsection is to prove that following the ansatz (97) we can construct locally anisotropic wormhole metrics in string gravity as solutions of the system of equations (78) - (81) with redefined coordinates as in (99). Having the vacuum data (100) we may generalize the solution for a nontrivial cosmological constant following the method presented in subsection 7.3.2 when the new solutions are represented

\[ h_4 = \tilde{h}_4 (x', v) \quad q_4 (x', v) \quad \text{and} \quad h_5 = \tilde{h}_5 (x', v), \]

\[ (101) \]

58
with \( \hat{h}_{4,5} \) taken as in (88) which solves (79) if \( q_4 = 1 \) for \( \lambda = 0 \) and

\[
q_4 = \frac{1}{4\lambda} \left[ \int \frac{\hat{h}_5 (r, \theta, \varphi) \hat{h}_4 (r, \theta, \varphi)}{\hat{h}_5^* (r, \theta, \varphi)} d\varphi \right]^{-1}
\]

for \( \lambda \neq 0 \).

This \( q_4 \) can be considered as an additional polarization to \( \eta_4 \) induced by the cosmological constant \( \lambda \). We state \( g_2 = -1 \) but

\[
g_3 = -\sin^2 \left( \sqrt{2\lambda} \theta + \xi_{[0]} \right),
\]

which give of solution of (78) with signature \((+, -, -, -)\) which is different from the solution (57). A non–trivial \( q_4 \) results in modification of coefficients (81),

\[
\alpha_{\nu'} = \hat{\alpha}_{\nu'} + \alpha_{\nu'}^{(q)}, \quad \beta = \hat{\beta} + \beta^{(q)}, \quad \gamma = \hat{\gamma} + \gamma^{(q)},
\]

\[
\hat{\alpha}_{\nu'} = \partial_i \hat{h}_5^* - \hat{h}_5^* \partial_i \ln \sqrt{|\hat{h}_4 \hat{h}_5|}, \quad \hat{\beta} = \hat{\beta}^{**} - \hat{\beta}_5^*[\ln \sqrt{|\hat{h}_4 \hat{h}_5|}]^*, \quad \hat{\gamma} = \frac{3\hat{h}_5^*}{2\hat{h}_5} - \frac{\hat{h}_4^*}{\hat{h}_4}
\]

\[
\alpha_{\nu'}^{(q)} = -\hat{h}_5^* \partial_i \ln \sqrt{|q_4|}, \quad \beta^{(q)} = -\hat{h}_5^*[\ln \sqrt{|q_4|}]^*, \quad \gamma^{(q)} = -\frac{q_4^2}{q_4},
\]

which following formulas (80) and (81) result in additional terms to the N–connection coefficients, i. e.

\[
w_{\nu'} = \hat{w}_{\nu'} + w_{\nu'}^{(q)} \quad \text{and} \quad n_{\nu'} = \hat{n}_{\nu'} + n_{\nu'}^{(q)}, \quad (102)
\]

with \( w_{\nu'}^{(q)} \) and \( n_{\nu'}^{(q)} \) computed by using respectively \( \alpha_{\nu'}^{(q)}, \beta^{(q)} \) and \( \gamma^{(q)} \).

The N–connection coefficients (102) can be transformed partially into a \( B \)–field with \( \{B_{\nu' j'}, B_{\nu' j'}^0\} \) defined by integrating the conditions (20), i. e.

\[
B_{\nu' j'} = B_{\nu' j'}^{0} \left( x^k \right) + \int h_4 \delta_{\nu'} w_{j'} d\varphi, \quad B_{\nu' j'}^0 = B_{\nu' j'}^{0} \left( x^k \right) + \int h_4 w_{j'}^* d\varphi, \quad (103)
\]

for some arbitrary functions \( B_{\nu' j'}^{0} \left( x^k \right) \) and \( B_{\nu' j'}^{0} \left( x^k \right) \). The string background corrections are presented via nontrivial \( w_{\nu'}^{(q)} \) induced by \( \lambda = 1/4 \). The formulas (103) consist the second aim of this subsection: to illustrate how a a \( B \)–field inducing noncommutativity may be related with a N–connection inducing local anisotropy. This is an explicit example of locally anisotropic noncommutative configuration contained in string theory. For the considered class of wormhole solutions the coefficients \( n_{\nu'} \) do not contribute into the noncommutative configuration, but, in general, following (129), they can be also related to noncommutativity.

8 Comments and Questions

In this paper, we have developed the method of anholonomic frames and associated nonlinear connections from a viewpoint of application in noncommutative geometry
and string theory. We note in this retrospect that several futures connecting Finsler like generalizations of gravity and gauge theories, which in the past were considered ad hoc and sophisticated, actually have a very natural physical and geometric interpretation in the noncommutative and D–brane picture in string/M–theory. Such locally anisotropic and/ or noncommutative configurations are hidden even in general relativity and its various Kaluza–Klein like and supergravity extension. To emphasize them we have to consider off–diagonal metrics which can be diagonalized in result of certain anholonomic frame transforms which induce also nonlinear connection structures in the curved spacetime, in general, with noncompactified extra dimensions.

On general grounds, it could be said the the appearance of noncommutative and Finsler like geometry when considering \( B \)-fields, off–diagonal metrics and anholonomic frames (all parametrized, in general, by noncommutative matrices) is a natural thing. Such implementations in the presence of D–branes and matrix approaches to M–theory were proven here to have explicit realisations and supported by six background constructions elaborated in this paper:

First, both the local anisotropy and noncommutativity can be derived from considering string propagation in general manifolds and bundles and in various low energy string limits. This way the anholonomic Einstein and Finsler generalized gravity models are generated from string theory.

Second, the anholonomic constructions with associated nonlinear connection geometry can be explicitly modeled on superbundles which results in superstring effective actions with anholonomic (super) field equations which can be related to various superstring and supergravity theories.

Third, noncommutative geometries and associated differential calculi can be distinguished in anholonomic geometric form which allows formulation of locally anisotropic field theories with anholonomic symmetries.

Forth, anholonomy and noncommutativity can be related to string/M–theory following consequently the matrix algebra and geometry and/or associated to nonlinear connections noncommutative covariant differential calculi.

Fifth, different models of locally anisotropic gravity with explicit limits to string and Einstein gravity can be realized on noncommutative D–branes.

Sixth, the anholonomic frame method is a very powerful one in constructing and investigating new classes of exact solutions in string and gravity theories; such solutions contain generic noncommutativity and/or local anisotropy and can be parametrized as to describe locally anisotropic black hole configurations, Finsler like structures, anisotropic solitonic and moving string black hole metrics, or noncommutative and anisotropic wormhole structures which may be derived in Einstein gravity and/or its Kaluza–Klein and (super) string generalizations.

The obtained in this paper results have a recent confirmation in Ref. \cite{33} where the spacetime noncommutativity is obtained in string theory with a constant off–diagonal metric background when an appropriate form is present and one of the spatial direction has Dirichlet boundary conditions. We note that in Refs. \cite{49,46,52,53,54} we constructed exact solutions in the Einstein and extra dimension gravity with off–diagonal
metrics which were diagonalized by anholonomic transforms to effective spacetimes with noncommutative structure induced by anholonomic frames. Those results were extended to noncommutative geometry and gauge gravity theories, in general, containing local anisotropy, in Refs. [48, 50]. The low energy string spacetime with noncommutativity constructed in subsection 7.4 of this work is parametrized by an off–diagonal metric which is a very general (non–constant) pseudo–Riemannian one defining an exact solution in string gravity.

Finally, our work raises a number of other interesting questions:

1. What kind of anholonomic quantum noncommutative structures are hidden in string theory and gravity; how such constructions are to be modeled by modern geometric methods.

2. How, in general, to relate the commutative and noncommutative gauge models of (super) gravity with local anisotropy directly to string/M–theory.

3. What kind of quantum structure is more naturally associated to string gravity and how to develop such anisotropic generalizations.

4. To formulate a nonlinear connection theory in quantum bundles and relate it to various Finsler like quantum generalizations.

5. What kind of Clifford structures are more natural for developing a unified geometric approach to anholonomic noncommutative and quantum geometry following in various perturbative limits and non–perturbative sectors of string/M–theory and when a such geometry is to be associated to D–brane configurations.

6. To construct new classes of exact solutions with generic anisotropy and noncommutativity and analyze theirs physical meaning and possible applications.

We hope to address some of these questions in future works.

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A Anholonomic Frames and N–Connections

We outline the basic definitions and formulas on anholonomic frames and associated nonlinear connection (N–connection) structures on vector bundles [29] and (pseudo) Riemannian manifolds [45-49]. The Einstein equations are written in mixed holonomic–anholonomic variables. We state the conditions when locally anisotropic structures (Finsler like and another type ones) can be modeled in general relativity and its extra dimension generalizations. This Abstract contains the necessary formulas in coordinate form taken from a geometric paper under preparation together with a co-author.
A.1 The N–connection geometry

The concept of N–connection came from Finsler geometry (as a set of coefficients it is present in the works of E. Cartan [6], then it was elaborated in a more explicit fashion by A. Kawaguchi [23]). The global definition of N–connections in commutative spaces is due to W. Barthel [3]. The geometry of N–connections was developed in details for vector, covector and higher order bundles [29, 28, 4], spinor bundles [40, 55] and superspaces and superstrings [41, 47, 42] with recent applications in modern anisotropic kinetics and thermodynamics [44] and elaboration of new methods of constructing exact off–diagonal solutions of the Einstein equations [45, 49]. The concept of N–connection can be extended in a similar manner from commutative to noncommutative spaces if a differential calculus is fixed on a noncommutative vector (or covector) bundle or another type of quantum manifolds [50].

A.1.1 N–connections in vector bundles and (pseudo) Riemannian spaces

Let us consider a vector bundle ξ = (E, μ, M) with typical fibre \( \mathbb{R}^m \) and the map

\[ \mu^T : TE \rightarrow TM \]

being the differential of the map \( \mu : E \rightarrow M \). The map \( \mu^T \) is a fibre–preserving morphism of the tangent bundle \( (TE, \tau_E, E) \) to \( E \) and of tangent bundle \( (TM, \tau, M) \) to \( M \). The kernel of the morphism \( \mu^T \) is a vector subbundle of the vector bundle \( (TE, \tau_E, E) \). This kernel is denoted \( (VE, \tau_V, E) \) and called the vertical subbundle over \( E \). By

\[ i : VE \rightarrow TE \]

it is denoted the inclusion mapping when the local coordinates of a point \( u \in E \) are written \( u^a = (x^i, y^a) \), where the values of indices are \( i,j,k,... = 1,2,...,n \) and \( a,b,c,... = 1,2,...,m \).

A vector \( X_u \in TE \), tangent in the point \( u \in E \), is locally represented

\[ \left( x, y, X, \tilde{X} \right) = \left( x^i, y^a, X^i, X^a \right), \]

where \( (X^i) \in \mathbb{R}^n \) and \( (X^a) \in \mathbb{R}^m \) are defined by the equality

\[ X_u = X^i \partial_i + X^a \partial_a \]

\([\partial_a = (\partial_i, \partial_a)\) are usual partial derivatives on respective coordinates \( x^i \) and \( y^a \)]. For instance, \( \mu^T \left( x, y, X, \tilde{X} \right) = (x, X) \) and the submanifold \( VE \) contains elements of type \( \left( x, y, 0, \tilde{X} \right) \) and the local fibers of the vertical subbundle are isomorphic to \( \mathbb{R}^m \). Having \( \mu^T (\partial_a) = 0 \), one comes out that \( \partial_a \) is a local basis of the vertical distribution \( u \rightarrow V_u E \) on \( E \), which is an integrable distribution.
A nonlinear connection (in brief, N–connection) in the vector bundle \( \xi = (E, \mu, M) \) is the splitting on the left of the exact sequence
\[
0 \to VE \to TE/VE \to 0,
\]
i. e. a morphism of vector bundles \( N : TE \to VE \) such that \( C \circ i \) is the identity on \( VE \).

The kernel of the morphism \( N \) is a vector subbundle of \( (TE, \tau_E, E) \), it is called the horizontal subbundle and denoted by \( (HE, \tau_H, E) \). Every vector bundle \( (TE, \tau_E, E) \) provided with a N–connection structure is Whitney sum of the vertical and horizontal subbundles, i. e.
\[
TE = HE \oplus VE.
\]

It is proven that for every vector bundle \( \xi = (E, \mu, M) \) over a compact manifold \( M \) there exists a nonlinear connection \([29]\).

Locally a N–connection \( N \) is parametrized by a set of coefficients \( \{ N^a_i (u^a) = N^a_i (x^j, y^b) \} \) which transform as
\[
N^a_i \frac{\partial x^{i'}}{\partial x^i} = M^a_i N^a_i - \frac{\partial M^a_i}{\partial x^i} y^a
\]
under coordinate transforms on the vector bundle \( \xi = (E, \mu, M) \),
\[
x'^i = x^{i'} (x^i) \text{ and } y'^a = M^a_i (x) y^a.
\]

The well known class of linear connections consists a particular parametization of the coefficients \( N^a_i \) when
\[
N^a_i (x^j, y^b) = \Gamma^a_i (x^j) y^b
\]
are linear on variables \( y^b \).

If a N–connection structure is associated to local frame (basis, vielbein) on \( \xi \), the operators of local partial derivatives \( \partial_\alpha = (\partial_i, \partial_a) \) and differentials \( d^\alpha = du^\alpha = \left( d^i = dx^i, d^a = dy^a \right) \) should be elongated as to adapt the local basis (and dual basis) structure to the Whitney decomposition of the vector bundle into vertical and horizontal subbundles, \((104)\):
\[
\partial_\alpha = (\partial_i, \partial_a) \to \delta_\alpha = \left( \delta_i = \partial_i - N^b_i \partial_b, \partial_a \right), \quad (105)
\]
\[
d^\alpha = (d^i, d^a) \to \delta^\alpha = \left( d^i, \delta^a = d^a + N^b_i d^b \right). \quad (106)
\]

The transforms can be considered as some particular case of frame transforms of type
\[
\partial_\alpha \to \delta_\alpha = e^\beta_\alpha \partial_\beta \text{ and } d^\alpha \to \delta^\alpha = (e^{-1})^\alpha_\beta \delta^\beta,
\]
e\( e^\beta_\alpha (e^{-1})^\gamma_\beta = \delta^\gamma_\alpha \), when the vielbein coefficients \( e^\beta_\alpha \) are constructed by using the Kronecker symbols \( \delta^b_a, \delta^i_j \) and \( N^b_i \).
The bases $\delta_{\alpha}$ and $\delta^{\alpha}$ satisfy, in general, some anholonomy conditions, for instance,
\[ \delta_{\alpha}\delta_{\beta} - \delta_{\beta}\delta_{\alpha} = W_{\alpha\beta}^{\gamma}\delta_{\gamma}, \]
where $W_{\alpha\beta}^{\gamma}$ are called the anholonomy coefficients. An explicit calculus of commutators of operators shows that there are the non–trivial values:
\[ W_{ij}^{\alpha} = R_{ij}^{\alpha} = \delta_{i}N_{j}^{\alpha} - \delta_{j}N_{i}^{\alpha}, \quad W_{ai}^{b} = -W_{ia}^{b} = -\partial_{a}N_{i}^{b}. \]

Tensor fields on a vector bundle $\xi = (E, \mu, M)$ provided with $N$–connection structure $N$ (we subject such spaces with the index $N, \xi_{N}$) may be decomposed in $N$–adapted form with respect to the bases $\delta_{\alpha}$ and $\delta^{\alpha}$, and their tensor products. For instance, for a tensor of rang $(1,1)$ as $\Gamma^{\alpha\beta} = \partial_{\alpha}\Gamma^{\beta\gamma} + \Gamma^{\alpha\beta\gamma}$ the coefficients in brief, $d$–objects, like the $d$-tensor $(109)$, $d$–connection, $d$–metric:
\[ \partial_{\alpha}\Gamma^{\beta\gamma} + \Gamma^{\alpha\beta\gamma} = \text{stated by the coefficients} \]
\[ \Gamma^{\alpha\beta\gamma} = \{ N_{bi}^{a} = \partial N_{i}^{a}(x, y)/\partial y^{b} \} \text{ which defines a covariant derivative} \]
\[ D^{(N)}_{\alpha}A^{\beta} = \delta_{\alpha}A^{\beta} + \Gamma^{(N)\beta}_{\alpha\gamma}A^{\gamma}. \]

Another important characteristic of a $N$–connection is its curvature $\Omega = \{ \Omega_{ij}^{\alpha} \}$ with the coefficients
\[ \Omega_{ij}^{\alpha} = \delta_{i}N_{j}^{\alpha} - \delta_{j}N_{i}^{\alpha} = \partial_{\alpha}N_{ij} + N_{ib}^{a}N_{ja}^{b} - N_{jb}^{a}N_{ia}^{b}. \]

In general, on a vector bundle we may consider arbitrary linear connections and metric structures adapted to the $N$–connection decomposition into vertical and horizontal subbundles (one says that such objects are distinguished by the $N$–connection, in brief, $d$–objects, like the $d$-tensor $(109)$, $d$–connection, $d$–metric:

- The coefficients of linear $d$-connections $\Gamma = \{ \Gamma^{\beta}_{\alpha\gamma} = (L_{ij}^{k}, L_{bk}^{a}, C_{ij}^{c}, C_{ac}^{b}) \}$ are defined for an arbitrary covariant derivative $D$ on $\xi$ being adapted to the $N$–connection structure as $D_{\delta_{\alpha}}(\delta_{\beta}) = \Gamma^{\gamma}_{\beta\alpha}\delta_{\gamma}$ with the coefficients being invariant under horizontal and vertical decomposition
\[ D_{\delta_{\alpha}}(\delta_{\beta}) = L_{ij}^{k}\delta_{k}, \quad D_{\delta_{\alpha}}(\partial_{\alpha}) = L_{ai}^{b}\partial_{b}, \quad D_{\delta_{\alpha}}(\delta_{\beta}) = C_{ij}^{c}\delta_{k}, \quad D_{\delta_{\alpha}}(\partial_{\alpha}) = C_{ac}^{b}\partial_{b}. \]
The operator of covariant differentiation $D$ splits into the horizontal covariant derivative $D^{[h]}$, stated by the coefficients $(L_{jk}^{i}, L_{bk}^{a})$, for instance, and the operator of vertical covariant derivative $D^{[v]}$, stated by the coefficients $(C_{ij}^{c}, C_{ac}^{b})$. For instance, for $A = A^{i}\delta_{i} + A^{a}\partial_{a} = A_{i}\delta^{i} + A_{a}\delta^{a}$ one holds the $d$–covariant derivation rules,
\[ D^{[h]}_{i}A^{k} = \delta_{i}A^{k} + L_{ij}^{k}A^{j}, \quad D^{[h]}_{i}A^{b} = \delta_{i}A^{b} + L_{jc}^{b}A^{c}, \]
\[ D^{[h]}_{i}A_{k} = \delta_{i}A_{k} - L_{ik}^{j}A_{j}, \quad D^{[h]}_{i}A_{b} = \delta_{i}A_{b} - L_{ib}^{c}A_{c}, \]
\[ D^{[v]}_{a}A^{k} = \partial_{a}A^{k} + C_{aj}^{i}A^{j}, \quad D^{[v]}_{a}A^{b} = \partial_{a}A^{b} + C_{ac}^{b}A^{c}, \]
\[ D^{[v]}_{a}A_{k} = \partial_{a}A_{k} - C_{ak}^{j}A_{j}, \quad D^{[v]}_{a}A_{b} = \partial_{a}A_{b} - C_{ab}^{c}A_{c}. \]
• The d–metric structure \( G = g_{\alpha\beta} \delta^\alpha \otimes \delta^\beta \) which has the invariant decomposition as \( g_{\alpha\beta} = (g_{ij}, g_{ab}) \) following from

\[
G = g_{ij}(x,y)dx^i \otimes dx^j + g_{ab}(x,y)\delta^a \otimes \delta^b.
\]  

(111)

We may impose the condition that a d–metric \( g_{\alpha\beta} \) and a d–connection \( \Gamma^\beta_{\alpha\gamma} \) are compatible, i. e. there are satisfied the conditions

\[
D_\gamma g_{\alpha\beta} = 0.
\]  

(112)

With respect to the anholonomic frames \( (105) \) and \( (106) \), there is a linear connection, called the canonical distinguished linear connection, which is similar to the metric connection introduced by the Christoffel symbols in the case of holonomic bases, i. e. being constructed only from the metric components and satisfying the metricity conditions \( (112) \). It is parametrized by the coefficients, \( \Gamma^\alpha_{\beta\gamma} = (L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc}) \) where

\[
L^i_{jk} = \frac{1}{2} g^{in} (\delta_k g_{nj} + \delta_j g_{nk} - \delta_n g_{jk}),
\]  

(113)

\[
L^a_{bk} = \partial_b N^a_i + \frac{1}{2} h^ac (\delta_k h_{bc} - h_{dc} \partial_b N^d_k - h_{db} \partial_c N^d_k),
\]  

\[
C^i_{jc} = \frac{1}{2} g^{ik} \partial_c g_{jk}, \quad C^a_{bc} = \frac{1}{2} h^{ad} (\partial_c h_{db} + \partial_b h_{dc} - \partial_d h_{bc}).
\]  

(114)

Instead of this connection one can consider on \( \xi \) another types of linear connections which are/or not adapted to the N–connection structure (see examples in [29]).

A.1.2 D–torsions and d–curvatures:

The anholonomic coefficients \( W_{\alpha\beta}^\gamma \) and N–elongated derivatives give nontrivial coefficients for the torsion tensor, \( T(\delta_\gamma, \delta_\beta) = T_{\beta\gamma}^\alpha \delta_\alpha \), where

\[
T_{\beta\gamma}^\alpha = \Gamma_{\beta\gamma}^\alpha - \Gamma_{\alpha\beta}^\gamma + W_{\beta\gamma}^\alpha,
\]  

(115)

and for the curvature tensor, \( R(\delta_\tau, \delta_\gamma)\delta_\beta = R_{\beta\gamma}^{\alpha\tau} \delta_\alpha \), where

\[
R_{\beta\gamma}^{\alpha\tau} = \delta_\tau \Gamma_{\beta\gamma}^\alpha - \delta_\gamma \Gamma_{\alpha\beta}^\tau + \Gamma_{\beta\tau}^\gamma \Gamma_{\alpha\gamma}^\rho - \Gamma_{\alpha\rho}^\gamma \Gamma_{\beta\gamma}^\tau + \Gamma_{\beta\gamma}^{\alpha\rho} W_{\rho\gamma}^\tau.
\]  

(116)

We emphasize that the torsion tensor on (pseudo) Riemannian spacetimes is induced by anholonomic frames, whereas its components vanish with respect to holonomic frames. All tensors are distinguished (d) by the N–connection structure into irreducible (horizontal–vertical) h–v–components, and are called d–tensors. For instance, the torsion, d–tensor has the following irreducible, nonvanishing, h–v–components, \( T_{\beta\gamma}^\alpha = \{T^i_{jk}, C^i_{ja}, S^a_{bc}, T^a_{ij}, T^a_{bi}\} \), where

\[
T^i_{jk} = T^i_{jk} = L^i_{jk} - L^j_{ik}, \quad T^i_{ja} = C^i_{ja}, \quad T^i_{aj} = -C^i_{ja};
\]  

\[
T^a_{ja} = 0, \quad T^a_{bc} = S^a_{bc} = C_{bc} - C^a_{cb};
\]  

\[
T^a_{ij} = -\Omega^a_{ij}, \quad T^a_{bi} = \partial_b N^a_i - L^a_{bi}, \quad T^a_{ib} = -T^a_{bi}.
\]  

(117)
(the d–torsion is computed by substituting the h–v–components of the canonical d–
connection (113) and anholonomy coefficients (107) into the formula for the torsion
coefficients (114)).

We emphasize that with respect to anholonomic frames the torsion is not zero even
for symmetric connections with \( \Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta} \) because the anholonomy coefficients \( W^\alpha_{\beta\gamma} \)
are contained in the formulas for the torsion coefficients (114). By straightforward
computations we can prove that for nontrivial N–connection curvatures, \( \Omega_{ij}^a \neq 0 \), even
the Levi–Civita connection for the metric (111) contains nonvanishing torsion coeffi-
cients. Of course, the torsion vanishes if the Levi–Civita connection is defined as
the usual Christoffel symbols with respect to the coordinate frames, \((\partial_i, \partial_a)\) and
\((d^i, \partial^a)\); in this case the d–metric (111) is redefined into, in general, off–diagonal metric containing
products of \( N^a_i \) and \( h_{ab} \).

The curvature d–tensor has the following irreducible, non-vanishing, h–v–components
\( R^a_{\beta\gamma} = \{ R^a_{\beta jk}, R^a_{\beta jk}, P^i_{\beta jka}, P_{\beta jka}, S^i_{\beta jka}, S^a_{\beta jka} \} \), where

\[
\begin{align*}
R^a_{\beta jk} &= \delta_k L^a_{\beta j} - \delta_j L^a_{\beta k} + L^a_{\beta j} L^m_{mk} - L^m_{\beta k} L^i_{mj} - C^i_{\alpha k} \Omega^\alpha_{jk}, \\
R^a_{\beta jk} &= \delta_k L^a_{\beta j} - \delta_j L^a_{\beta k} + L^a_{\beta j} L^m_{mk} - L^m_{\beta k} L^i_{mj} - C^i_{\alpha k} \Omega^\alpha_{jk}, \\
P^i_{\beta jka} &= \partial_k C^i_{\beta ja} + C^i_{\beta kj} T^b_{\beta ka} - (\delta_k C^i_{\beta ja} + L^i_{\beta kj} C^i_{\beta la} - L^i_{\beta ak} C^i_{\beta ja}), \\
P^e_{\beta jka} &= \partial_k C^e_{\beta ja} + C^e_{\beta kj} T^b_{\beta ka} - (\delta_k C^e_{\beta ja} + L^e_{\beta kj} C^e_{\beta la} - L^e_{\beta ak} C^e_{\beta ja}), \\
S^i_{\beta jka} &= \partial_c C^i_{\beta ja} - \partial_b C^i_{\beta jca} + C^h_{\beta jbc} C^i_{\beta ca} - C^c_{\beta jca} C^h_{\beta bc}, \\
S^a_{\beta jka} &= \partial_c C^a_{\beta ja} - \partial_b C^a_{\beta jca} + C^e_{\beta jbe} C^a_{\beta ca} - C^e_{\beta jca} C^a_{\beta be}
\end{align*}
\]

(the d–curvature components are computed in a similar fashion by using the formula
for curvature coefficients (115)).

A.1.3 Einstein equations in d–variables

In this subsection we write and analyze the Einstein equations on spaces provided with
anholonomic frame structures and associated N–connections.

The Ricci tensor \( R_{\beta\gamma} = R^a_{\beta\gamma} \) has the d–components

\[
R_{ij} = R^k_{i,jk}, \quad R_{ia} = -2 P_{ia} = -P^k_{i,ka}, \quad R_{ai} = P^b_{a,ib}, \quad R_{ab} = S^c_{a,bc}.
\]

In general, since \( 1P_{ai} \neq -2 P_{aia} \), the Ricci d–tensor is non-symmetric (we emphasize that this
could be with respect to anholonomic frames of reference because the N–connection
and its curvature coefficients, \( N^a_i \) and \( \Omega_{ij}^a \), as well the anholonomy coefficients \( W^a_{\beta\gamma} \)
and d–torsions \( T^a_{\beta\gamma} \) are contained in the formulas for d–curvatures (115)). The scalar
curvature of the metric d–connection, \( \tilde{R} = g^{\beta\gamma} R_{\beta\gamma} \), is computed

\[
\tilde{R} = G^{\alpha\beta} R_{\alpha\beta} = \tilde{R} + S,
\]

where \( \tilde{R} = g^{ij} R_{ij} \) and \( S = h^{ab} S_{ab} \).
By substituting \((118)\) and \((119)\) into the Einstein equations
\[
R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \kappa \Upsilon_{\alpha\beta}, \tag{120}
\]
where \(\kappa\) and \(\Upsilon_{\alpha\beta}\) are respectively the coupling constant and the energy–momentum tensor we obtain the h-v-decomposition by N–connection of the Einstein equations
\[

R_{ij} - \frac{1}{2} \left( \hat{R} + S \right) g_{ij} = \kappa \Upsilon_{ij}, \tag{121}
\]
\[
S_{ab} - \frac{1}{2} \left( \hat{R} + S \right) h_{ab} = \kappa \Upsilon_{ab},
\]
\[
1^P_{ai} = \kappa \Upsilon_{ai}, \quad 2^P_{ia} = \kappa \Upsilon_{ia}.
\]

The definition of matter sources with respect to anholonomic frames is considered in Refs. \[40, 47, 29\].

The vacuum locally anisotropic gravitational field equations, in invariant h– v– components, are written
\[
R_{ij} = 0, S_{ab} = 0, \quad 1^P_{ai} = 0, \quad 2^P_{ia} = 0. \tag{122}
\]

We emphasize that general linear connections in vector bundles and even in the (pseudo) Riemannian spacetimes have non–trivial torsion components if off–diagonal metrics and anholonomic frames are introduced into consideration. This is a "pure" anholonomic frame effect: the torsion vanishes for the Levi–Civita connection stated with respect to a coordinate frame, but even this metric connection contains some torsion coefficients if it is defined with respect to anholonomic frames (this follows from the \(w\)–terms in \((123)\)). For the (pseudo) Riemannian spaces we conclude that the Einstein theory transforms into an effective Einstein–Cartan theory with anholonomically induced torsion if the general relativity is formulated with respect to general frame bases (both holonomic and anholonomic).

The N–connection geometry can be similarly formulated for a tangent bundle \(TM\) of a manifold \(M\) (which is used in Finsler and Lagrange geometry \[29\]), on cotangent bundle \(T^*M\) and higher order bundles (higher order Lagrange and Hamilton geometry \[28\]) as well in the geometry of locally anisotropic superspaces \[41\], superstrings \[43\], anisotropic spinor \[40\] and gauge \[51\] theories or even on (pseudo) Riemannian spaces provided with anholonomic frame structures \[55\].

### A.2 Anholonomic Frames in Commutative Gravity

We introduce the concepts of generalized Lagrange and Finsler geometry and outline the conditions when such structures can be modeled on a Riemannian space by using anholonomic frames.
Different classes of commutative anisotropic spacetimes are modeled by corresponding parametrizations of some compatible (or even non-compatible) N–connection, d–connection and d–metric structures on (pseudo) Riemannian spaces, tangent (or cotangent) bundles, vector (or covector) bundles and their higher order generalizations in their usual manifold, supersymmetric, spinor, gauge like or another type approaches (see Refs. [45, 28, 29, 4, 40, 51, 47, 55]).

A.2.1 Anholonomic structures on Riemannian spaces

We note that the N–connection structure may be defined not only in vector bundles but also on (pseudo) Riemannian spaces [45]. In this case the N–connection is an object completely defined by anholonomic frames, when the coefficients of frame transforms, $e^\beta_\alpha (u^\gamma)$, are parametrized explicitly via certain values $(N^a_i, \delta_i^j, \delta_i^a)$, where $\delta_i^j$ and $\delta_i^a$ are the Kronecker symbols. By straightforward calculations we can compute that the coefficients of the Levi–Civita metric connection

$$\Gamma^\gamma_{\alpha\beta\gamma} = g(\delta_\alpha, \nabla_\gamma \delta_\beta) = g_{\alpha\tau} \Gamma^\gamma_{\beta\gamma},$$

associated to a covariant derivative operator $\nabla$, satisfying the metricity condition $\nabla_\gamma g_{\alpha\beta} = 0$ for $g_{\alpha\beta} = (g_{ij}, h_{ab})$ and

$$\Gamma^\gamma_{\alpha\beta\gamma} = \frac{1}{2} [\delta_\beta g_{\alpha\gamma} + \delta_\gamma g_{\beta\alpha} - \delta_\alpha g_{\gamma\beta} + g_{\alpha\tau} W^\tau_{\gamma\beta} + g_{\beta\tau} W^\tau_{\alpha\gamma} - g_{\gamma\tau} W^\tau_{\beta\alpha}], \quad (123)$$

are given with respect to the anholonomic basis (106) by the coefficients

$$\Gamma^\gamma_{\beta\gamma} = \left( L^i_{jk}, L^a_{bk}, C^a_{jc} + \frac{1}{2} g^{ik} \Omega^a_{jk} h_{ca}; C^a_{bc} \right) \quad (124)$$

when $L^i_{jk}, L^a_{bk}, C^a_{jc}, C^a_{bc}$ and $\Omega^a_{jk}$ are respectively computed by the formulas (113) and (110). A specific property of off–diagonal metrics is that they can define different classes of linear connections which satisfy the metricity conditions for a given metric, or inversely, there is a certain class of metrics which satisfy the metricity conditions for a given linear connection. This result was originally obtained by A. Kawaguchi [23] (Details can be found in Ref. [29], see Theorems 5.4 and 5.5 in Chapter III, formulated for vector bundles; here we note that similar proofs hold also on manifolds enabled with anholonomic frames associated to a N–connection structure).

With respect to anholonomic frames, we can not distinguish the Levi–Civita connection as the unique one being both metric and torsionless. For instance, both linear connections (113) and (124) contain anholonomically induced torsion coefficients, are compatible with the same metric and transform into the usual Levi–Civita coefficients for vanishing N–connection and ”pure” holonomic coordinates. This means that to an off–diagonal metric in general relativity one may be associated different covariant differential calculi, all being compatible with the same metric structure (like in the
non–commutative geometry, which is not a surprising fact because the anonomic frames satisfy by definition some non–commutative relations (107). In such cases we have to select a particular type of connection following some physical or geometrical arguments, or to impose some conditions when there is a single compatible linear connection constructed only from the metric and N–coefficients. We note that if $\Omega^a_{jk} = 0$ the connections (113) and (124) coincide, i. e. $\Gamma^\alpha_{\beta\gamma} = \Gamma^\nabla^\alpha_{\beta\gamma}$.

If an anholonomic (equivalently, anisotropic) frame structure is defined on a (pseudo) Riemannian space of dimension $(n + m)$ space, the space is called to be an anholonomic (pseudo) Riemannian one (denoted as $V^{(n+m)}$). By fixing an anholonomic frame basis and co–basis with associated N–connection $N^a_i(x,y)$, respectively, as (105) and (106), one splits the local coordinates $u^\alpha = (x^i, y^a)$ into two classes: the fist class consists from $n$ holonomic coordinates, $x^i$, and the second class consists from $m$ anholonomic coordinates, $y^a$. The $d$–metric (111) on $V^{(n+m)}$,

$$G^{[R]} = g_{ij}(x,y)dx^i \otimes dx^j + h_{ab}(x,y)\delta y^a \otimes \delta y^b$$

(125)

written with respect to a usual coordinate basis $du^\alpha = (dx^i, dy^a)$,

$$ds^2 = g_{\alpha\beta}(x,y)du^\alpha du^\beta$$

is a generic off–diagonal Riemannian metric parametrized as

$$g_{\alpha\beta} = \begin{bmatrix} g_{ij} + N^a_i N^b_j g_{ab} & h_{ab} N^a_i \\ h_{ab} N^b_j & h_{ab} \end{bmatrix}.$$ (126)

Such type of metrics were largely investigated in the Kaluza–Klein gravity [35], but also in the Einstein gravity [45]. An off–diagonal metric (126) can be reduced to a block $(n \times n) \oplus (m \times m)$ form $(g_{ij}, g_{ab})$, and even effectively diagonalized in result of a superposition of anhonomic N–transforms. It can be defined as an exact solution of the Einstein equations. With respect to anholonomic frames, in general, the Levi–Civita connection obtains a torsion component (123). Every class of off–diagonal metrics can be anholonemically equivalent to another ones for which it is not possible to a select the Levi–Civita metric defied as the unique torsionless and metric compatible linear connection. The conclusion is that if anholonomic frames of reference, which authomatically induce the torsion via anholmomy coefficients, are considered on a Riemannian space we have to postulate explicitly what type of linear connection (adapted both to the anholonomic frame and metric structure) is chosen in order to construct a Riemannian geometry and corresponding physical models. For instance, we may postulate the connection (124) or the $d$–connection (113). Both these connections are metric compatible and transform into the usual Christoffel symbols if the N–connection vanishes, i. e. the local frames became holonomic. But, in general, anholonomic frames and off–diagonal Riemannian metrics are connected with anisotropic configurations which allow, in principle, to model even Finsler like structures in (pseudo) Riemannian spaces [44, 45].
A.2.2 Finsler geometry and its almost Kahlerian model

The modern approaches to Finsler geometry are outlined in Refs. [34, 29, 28, 4, 47, 55]. Here we emphasize that a Finsler metric can be defined on a tangent bundle $TM$ with local coordinates $u^a = (x^i, y^a \rightarrow y^i)$ of dimension $2n$, with a $d$–metric (111) for which the Finsler metric, i. e. the quadratic form

$$g^{[F]}_{ij} = h_{ab} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}$$

is positive definite, is defined in this way:
1) A Finsler metric on a real manifold $M$ is a function $F : TM \rightarrow \mathbb{R}$ which on $\tilde{TM} = TM\{0\}$ is of class $C^\infty$ and $F$ is only continuous on the image of the null cross–sections in the tangent bundle to $M$. 2) $F(x, \chi y) = \chi F(x, y)$ for every $\mathbb{R}^*_+$. 3) The restriction of $F$ to $\tilde{TM}$ is a positive function. 4) $\text{rank} \left[ g^{[F]}_{ij}(x,y) \right] = n$.

The Finsler metric $F(x,y)$ and the quadratic form $g^{[F]}_{ij}$ can be used to define the Christoffel symbols (not those from the usual Riemannian geometry)

$$c^j_{ik}(x,y) = \frac{1}{2} g^{ih} \left( \partial_j g_{hk}^{[F]} + \partial_k g_{jh}^{[F]} - \partial_h g_{jk}^{[F]} \right),$$

where $\partial_j = \partial/\partial x^j$, which allows us to define the Cartan nonlinear connection as

$$N^i_{[F]j}(x,y) = \frac{1}{4} \frac{\partial}{\partial \delta y^j} \left[ c^i_{jk}(x,y) \delta y^k \right]$$

where we may not distinguish the $v$- and $h$- indices taking on $TM$ the same values.

In Finsler geometry there were investigated different classes of remarkable Finsler linear connections introduced by Cartan, Berwald, Matsumoto and other ones (see details in Refs. [34, 29, 4]). Here we note that we can introduce $g^{[F]}_{ij} = g_{ab}$ and $N^i_{[F]j}(x,y)$ in (111) and construct a $d$–connection via formulas (113).

A usual Finsler space $F^n = (M, F(x,y))$ is completely defined by its fundamental tensor $g^{[F]}_{ij}(x,y)$ and Cartan nonlinear connection $N^i_{[F]j}(x,y)$ and its chosen $d$–connection structure. But the $N$–connection allows us to define an almost complex structure $I$ on $TM$ as follows

$$I (\delta_i) = -\partial/\partial y^i \text{ and } I (\partial/\partial y^i) = \delta_i$$

for which $I^2 = -1$.

The pair $(G^{[F]}, I)$ consisting from a Riemannian metric on $TM$,

$$G^{[F]} = g^{[F]}_{ij}(x,y) dx^i \otimes dx^j + g^{[F]}_{ij}(x,y) \delta y^i \otimes \delta y^j$$

and the almost complex structure $I$ defines an almost Hermitian structure on $\tilde{TM}$ associated to a 2–form

$$\theta = g^{[F]}_{ij}(x,y) \delta y^i \wedge dx^j.$$
This model of Finsler geometry is called almost Hermitian and denoted $H^{2n}$ and it is proven \[29\] that it is almost Kahlerian, i.e. the form $\theta$ is closed. The almost Kahlerian space $K^{2n} = (\tilde{T}M, G^{[F]}, I)$ is also called the almost Kahlerian model of the Finsler space $F^n$.

On Finsler (and their almost Kahlerian models) spaces one distinguishes the almost Kahler linear connection of Finsler type, $D^{[F]}$ on $\tilde{T}M$ with the property that this covariant derivation preserves by parallelism the vertical distribution and is compatible with the almost Kahler structure $(G^{[F]}, I)$, i.e.

$$D^{[F]}_X G^{[F]} = 0 \text{ and } D^{[F]}_X I = 0$$

for every $d$–vector field on $\tilde{T}M$. This $d$–connection is defined by the data

$$\Gamma = (L^i_{jk}, L^a_{bk} = 0, C^i_{ja} = 0, C^a_{bc} \to C^i_{jk})$$

with $L^i_{jk}$ and $C^i_{jk}$ computed as in the formulas \[113\] by using $g^{[F]}_{ij}$ and $N^{i}_j$ from \[128\].

We emphasize that a Finsler space $F^n$ with a $d$–metric \[129\] and Cartan’s $N$–connection structure \[128\], or the corresponding almost Hermitian (Kahler) model $H^{2n}$, can be equivalently modeled on a Riemannian space of dimension $2n$ provided with an off–diagonal Riemannian metric \[126\]. From this viewpoint a Finsler geometry is a corresponding Riemannian geometry with a respective off–diagonal metric (or, equivalently, with an anholonomic frame structure with associated $N$–connection) and a corresponding prescription for the type of linear connection chosen to be compatible with the metric and $N$–connection structures.

**A.2.3 Lagrange and generalized Lagrange geometry**

Lagrange spaces were introduced in order to geometrize the fundamental concepts in mechanics \[24\] and investigated in Refs. \[29\] (see \[40, 51, 41, 43, 47, 55\] for their spinor, gauge and supersymmetric generalizations).

A Lagrange space $L^n = (M, L(x, y))$ is defined as a pair which consists of a real, smooth $n$–dimensional manifold $M$ and regular Lagrangian $L : TM \to \mathbb{R}$. Similarly as for Finsler spaces one introduces the symmetric $d$–tensor field

$$g^{[L]}_{ij} = h_{ab} = \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j}. \quad (130)$$

So, the Lagrangian $L(x, y)$ is like the square of the fundamental Finsler metric, $F^2(x, y)$, but not subjected to any homogeneity conditions.

In the rest me can introduce similar concepts of almost Hermitian (Kahlerian) models of Lagrange spaces as for the Finsler spaces, by using the similar definitions and formulas as in the previous subsection, but changing $g^{[F]}_{ij} \to g^{[L]}_{ij}$.

R. Miron introduced the concept of generalized Lagrange space, GL–space (see details in \[29\]) and a corresponding $N$–connection geometry on $TM$ when the fundamental metric function $g_{ij} = g_{ij}(x, y)$ is a general one, not obligatory defined as a
second derivative from a Lagrangian as in (130). The corresponding almost Hermitian (Kahlerian) models of GL–spaces were investigated and applied in order to elaborate generalizations of gravity and gauge theories \cite{29,51}.

Finally, a few remarks on definition of gravity models with generic local anisotropy on anholonomic Riemannian, Finsler or (generalized) Lagrange spaces and vector bundles. So, by choosing a d-metric (111) (in particular cases (125), or (129) with $g^{[F]}_{ij}$, or $g^{[L]}_{ij}$) we may compute the coefficients of, for instance, d–connection (113), d–torsion (116) and (117) and even to write down the explicit form of Einstein equations (121) which define such geometries. For instance, in a series of works \cite{44,45,55} we found explicit solutions when Finsler like and another type anisotropic configurations are modeled in anisotropic kinetic theory and irreversible thermodynamics and even in Einstein or low/extra–dimension gravity as exact solutions of the vacuum (121) and nonvacuum (122) Einstein equations. From the viewpoint of the geometry of anholonomic frames is not much difference between the usual Riemannian geometry and its Finsler like generalizations. The explicit form and parametrizations of coefficients of metric, linear connections, torsions, curvatures and Einstein equations in all types of mentioned geometric models depends on the type of anholonomic frame relations and compatibility metric conditions between the associated N–connection structure and linear connections we fixed. Such structures can be correspondingly picked up from a noncommutative functional model, for instance, from some almost Hermitian structures over projective modules and/or generalized to some noncommutative configurations \cite{50}.

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