Number and amplitude of limit cycles emerging from topologically equivalent perturbed centers

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Abstract

We consider three examples of weakly perturbed centers which do not have geometrical equivalence: a linear center, a degenerate center and a non-hamiltonian center. In each case the number and amplitude of the limit cycles emerging from the period annulus are calculated following the same strategy: we reduce all of them to locally equivalent perturbed integrable systems of the form: \(dH(x, y) + \epsilon(f(x, y)dy - g(x, y)dx) = 0\), with \(H(x, y) = \frac{1}{2}(x^2 + y^2)\). This reduction allows us to find the Melnikov function, \(M(h) = \int_{H=h} fdy - gdx\), associated to each particular problem. We obtain the information on the bifurcation curves of the limit cycles by solving explicitly the equation \(M(h) = 0\) in each case.

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1 Introduction

Dissipation is present in almost every physical phenomenon. The only possibility to create self-sustained oscillations in a system is by re-injecting the energy dissipated into it. For instance, the pumping in a switched on laser must provide the necessary energy to compensate the lost energy owing to the interaction between the single frequency light beam and its surrounding (cavity, mirrors, etc.). In other context, the solar nuclear bomb must radiate to the space enough energy to maintain all the living cycles on the earth. In this case, many different frequencies are associated to the thousands of different species of animals and plants [1]. Thus, every oscillation in nature is generated by a balance between input and output energies, in definitive between amplification and dissipation. This dynamical state (characterized by a preferred period, wave form and amplitude) is represented in phase space by a isolated closed curve called limit cycle [2]. This isolated periodic motion can emerge from the perturbation of a conservative system where a continuum of periodic solutions exists. Only those closed orbits where the energetic balance after perturbation vanishes remain as periodic motions. Take, for instance, the harmonic oscillator $\ddot{x} + x = 0$. A pattern of circles filling the whole plane $(x, \dot{x})$ are all the solutions of this system. A weak nonlinear perturbation proportional to the velocity, such as that in the Liénard equation $\ddot{x} + \epsilon f(x)\dot{x} + x = 0$, destroys the geometrical aspect of the phase space. The system can balance the pumping and damping caused by the nonlinear term, $\epsilon f(x)\dot{x}$, only on some special periodic orbits. In the van der Pol case, where $f(x) = (1 - x^2)$, it has been proved the existence, uniqueness and non-algebraicity of a limit cycle for every value of the control parameter $\epsilon$ [3]. Depending on the conditions imposed over the strength of the nonlinearity $\epsilon$ and over the properties of $f(x)$: degree, parity, etc., many results on the number and amplitude of the limit cycles for general Liénard systems are scattered in the literature (see, for instance, Ref. [4, 5, 6, 7, 8, 9, 10] and references therein). This behavior is the mathematical evidence of how a small perturbation in a physical system can destroy the coexistence of infinitely many periodic motions and leaves only a finite
number of them verifying strict energetic balance conditions.

Following this line of reasoning, the aim of this work is to exploit the topological similarities of different systems presenting an infinite number of periodic oscillations which give rise to the emergence of isolated closed orbits under small perturbations. This is an alternative perspective of the problem of identifying the limit cycles growing from centers respect to other methods existing in the literature [11, 12, 13, 14]. In Section 2 we sketch the theory behind this type of systems and we develop the method to find those periodic solutions. In section 3 we apply this method to three different problems and calculate the number and amplitude of the limit cycles emerging from their perturbations. The comparison with previous investigations on these systems is also performed. Finally, we present our conclusions.

2 Geometrical and Topological Equivalence

The plane is the natural space for the representation of the integral curves of a second order one-dimensional or first order two-dimensional dynamical system. A periodic motion of this system draws a closed orbit on the plane. If the relation $H(x, y) = h$, with $h$ a fixed real number, is verified by every point $(x, y)$ on this type of orbit, a continuum set of periodic orbits is obtained when $h$ runs on an interval of real values, $h \in [h_1, h_2]$. This orbit structure receives the name of period annulus. Suppose, without lose of generality, that the boundary orbit given by $H = h_1$ is a degenerate orbit, say the origin $(0,0)$. This equilibrium point surrounded in its immediate neighborhood by closed paths is called a center [15]. Obviously, every orbit of the period annulus verifies the differential relation

$$dH = 0.$$  \hfill (1)

Depending on the time parameterization of the curves $H(x, y) = h$, different dynamical systems having the same pattern of integral paths are obtained. At a first sight, the difficulty to find the integrating factor in each case prevent us from
establishing the phase plane geometrical equivalence between them: we say that two systems are \textit{phase plane geometrically equivalent} when their integral curves on the plane are the same. Think, for instance, on the set of circles \( H_c(x, y) = \frac{1}{2}(x^2 + y^2) = h \). They obey the differential relation \( dH_c = 2xdx + 2ydy = 0 \). Different time parameterizations of this expression produce systems with different time behaviors (e.g.: (1) \( \ddot{x} + x = 0 \); (2) \( \dot{x} = ky, \dot{y} = -kx, k = cte., \) and (3) \( \dot{x} = y^2, \dot{y} = -xy \)), although with the circles as their integral curves. If the interest does not reside on the time evolution of the system, the last concept of geometrical equivalence will take importance in the study of the system.

One step further. It is also possible that two different planar systems (having each one a period annulus, given by \( H(x, y) = h \) and \( \bar{H}(x, y) = \bar{h} \), with \( h \in [h_1, h_2] \) and \( \bar{h} \in [\bar{h}_1, \bar{h}_2] \), respectively) which are not phase plane geometrical equivalent, are yet \textit{topologically equivalent}. By this we mean that there exists a bijective and continuous transformation of coordinates (homomorphism), \( \Gamma : \mathbb{R}^2 \to \mathbb{R}^2 \), transforming one period annulus into the other one. That is, if we represent the integral curves of the first system as \( C_h = \{(x, y) | H(x, y) = h\} \) and of the second one as \( \bar{C}_\bar{h} = \{(x, y) | \bar{H}(x, y) = \bar{h}\} \) then \( \Gamma(\bar{C}_\bar{h}) = C_h \) in a continuous and monotone way when \( h \) and \( \bar{h} \) run in their intervals of existence. In particular, the boundary conditions over \( \Gamma \) are: \( \Gamma(\bar{C}_{\bar{h}_1}) = C_{h_1} \) and \( \Gamma(\bar{C}_{\bar{h}_2}) = C_{h_2} \). Take for example the system \( \dot{x} = ay, \dot{y} = -bx \) with \( a, b > 0 \). It presents a period annulus formed by ellipses, \( H(x, y) = \frac{x^2}{a} + \frac{y^2}{b} = h \). Clearly, this system is topologically equivalent to another one having a circular period annulus: \( \bar{H}_c(x, y) = \frac{1}{2}(x^2 + y^2) = \bar{h} \). It is straightforward to verify that the function \( \Gamma(x, y) = (\frac{x}{\sqrt{a}}, \frac{y}{\sqrt{b}}) \) establishes the homomorphism between both systems. Obviously, if two systems are geometrical equivalent they are topologically equivalent because in this case \( H = \bar{H} \) and then we can choose \( \Gamma \) as the identity mapping.

The relevance of this property is the following. Imagine that we are able to find the limit cycles that emerge from the perturbation of the first system,

\[ dH + \text{perturbation} = 0. \]
Then, the application of $\Gamma$ to the second perturbed system, $d\bar{H} + \text{perturbation}' = 0$, bring this last expression to the form $[2]$, even if we do not know $\bar{H}$. Now the limit cycles of the second system can be found by solving equation $[2]$. Undoing the change of variables with $\Gamma^{-1}$, we will obtain the periodic motions of the perturbed system in the original coordinates.

3 Limit Cycles emerging from some Centers

3.1 Perturbation of a circular period annulus

As an application of the method above explained, we study here the limit cycles emerging from the perturbation of what we could consider a paradigmatic case in the center’s typology: a continuum set of circles, $H_c(x, y) = \frac{1}{2}(x^2 + y^2) = h$. The solution of many different dynamical systems is this kind of period annulus. As a simple representation of them we take the linear center:

$$\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -x,
\end{align*}$$

(3)

whose integral curves clearly verify $dH_c = x\dot{x} + y\dot{y} = 0$. This path diagram is destroyed when we slightly perturb this conservative system with two general nonlinear terms, $f(x, y)$ and $g(x, y)$, controlled by the parameter $|\epsilon| \ll 1$,

$$\begin{align*}
\dot{x} &= y + \epsilon f(x, y), \\
\dot{y} &= -x + \epsilon g(x, y).
\end{align*}$$

(4)

It is a time parameterized version of the differential relation: $dH_c + \epsilon (f\, dy - g\, dx) = 0$, in which the time variable has been removed. Following the procedure proposed in the previous section, we need to study this differential relation in order to find the limit cycles of system $[4]$. If its integral curves are represented by $y(x)$ and we define $y'(x) = dy/dx$, we obtain:

$$(yy' + x) + \epsilon[f(x, y)y' - g(x, y)] = 0.$$  

(5)

If we suppose the origin $(0, 0)$ is the only fixed point of equation $[4]$, a limit cycle $C_l \equiv (x, y_{\pm}(x))$ of it, with a positive branch $y_{+}(x) > 0$ and a negative branch
\( y_-(x) < 0 \), cut the \( x \)-axis in two points \((-a_-, 0)\) and \((a_+, 0)\) with \(a_-, a_+ > 0\). Every limit cycle \( C_l \) solution of equation (4) encloses the origin and the oscillation \( x \) runs in the interval \(-a_- < x < a_+\).

The amplitudes of oscillation \( a_-, a_+ \) identify the limit cycle. The result for \( \epsilon \neq 0 \) is a nested set of closed curves that defines the qualitative distribution of the integral curves in the plane \((x, y)\). The stability of the limit cycles is alternated. For a given stable limit cycle, the two neighboring limit cycles, the closest one in its interior and the closest one in its exterior, are unstable, and conversely (see Figure 1). If we determine the stability of the origin, then the stability of every limit cycle remains fixed by the alternation property. The stability of the origin is determined by the sign of the real part of the eigenvalues of the Jacobian at this point. Hence, its calculation shows that if \( \epsilon (f_x(0, 0) + g_y(0, 0)) < 0 \) then the origin is stable and if \( \epsilon (f_x(0, 0) + g_y(0, 0)) > 0 \) then it is unstable.

Another property of a limit cycle can be derived from the fact that the mechanical energy \( E = H_c = \frac{1}{2} (x^2 + y^2)/2 \) is conserved in a whole oscillation:

\[
\int_{C_l} dH_c = \int_{C_l} \frac{dE}{dx} dx = 0.
\]

Thus, if equation (5) is integrated along a limit cycle, between the amplitudes of oscillation, we obtain:

\[
\begin{align*}
\int_{a_-}^{a_+} [g(x, y_+(x)) - f(x, y_+(x))y_+(x)] dx + \\
\int_{a_-}^{a_+} [g(x, y_-(x)) - f(x, y_-(x))y_-(x)] dx = 0.
\end{align*}
\]  

(6)

Every couple of solutions \( \{y_+(x), y_-(x)\} \) of equation (5), vanishing in the extremes, \(y_+(a_-) = y_-(a_-) = y_+(a_+) = y_-(a_+) = 0\), and verifying equation (5), constitute the finite set of limit cycles of equation (4). As a consequence of the continuity in the parameter \( \epsilon \), these conditions are also valid for \( \epsilon = 0 \). For this parameter value, the limit cycles are circles: \( y_+(x) = \sqrt{a^2 - x^2} \) and \( y_-(x) = -\sqrt{a^2 - x^2} \). Here \( a_+ = a_- = a > 0 \) represents the amplitudes of the limit cycles. At this order of approximation, the condition (4) is verified, and condition (3) reads:

\[
\begin{align*}
\int_{a_-}^{a_+} [g(x, \sqrt{a^2 - x^2}) + \frac{zf(x, \sqrt{a^2 - x^2})}{\sqrt{a^2 - x^2}}] dx + \\
\int_{a_-}^{a_+} [g(x, -\sqrt{a^2 - x^2}) - \frac{zf(x, -\sqrt{a^2 - x^2})}{\sqrt{a^2 - x^2}}] dx = 0.
\end{align*}
\]  

(7)
that is,

\[
\beta(a) \equiv \int_{-a}^{a} \left[ \tilde{g}(x, \sqrt{a^2 - x^2}) + x \frac{\tilde{f}(x, \sqrt{a^2 - x^2})}{\sqrt{a^2 - x^2}} \right] \, dx = 0, \tag{8}
\]

where

\[
\tilde{f}(x, y) \equiv \frac{1}{2} [f(x, y) + f(x, -y) - f(-x, y) - f(-x, -y)],
\]

\[
\tilde{g}(x, y) \equiv \frac{1}{2} [g(x, y) - g(x, -y) + g(-x, y) - g(-x, -y)]. \tag{9}
\]

If \( \beta(a) \) does not vanish identically, each solution \( a > 0 \) of the equation \( \beta(a) = 0 \) is the amplitude of a limit cycle of the system (4) in the weak nonlinear regime. And conversely, at order zero in \( \epsilon \), the amplitudes of all the limit cycles emerging from the period annulus are solutions of equation (8). These results are exact for \( \epsilon = 0 \). Therefore, equations (7) or (8) determine the amplitudes of the period motions surviving to a slight perturbation of a period annulus formed by circles. A typical portrait of limit cycles in this regime is given in Figure 1.

We remark that \( \beta(a) \) is the first Melnikov function of the system (4). Here it has been obtained by following a different approach to the usual one (based on the calculation of the displacement function of the first return mapping on the period annulus). Observe also that \( \tilde{f}(x, y) \) is an odd function of \( x \) and an even function of \( y \), whereas \( \tilde{g}(x, y) \) is an even function of \( x \) and an odd function of \( y \). Therefore, if \( f(x, y) \) and \( g(x, y) \) are polynomials in \( x \) and \( y \), only the odd terms in \( x \) and even in \( y \) of \( f(x, y) \) and the even terms in \( x \) and odd in \( y \) of \( g(x, y) \) survive in (9) and contribute to \( \beta(a) \) in (8).

As an example, we integrate equation (8) when \( f(x, y) \) is a polynomial of degree \( 2n_1 \) or \( 2n_1 + 1 \) in \( x \) and degree \( 2m_1 \) or \( 2m_1 + 1 \) in \( y \) and \( g(x, y) \) is a polynomial of degree \( 2n_2 \) or \( 2n_2 + 1 \) in \( x \) and degree \( 2m_2 \) or \( 2m_2 + 1 \) in \( y \). Then,

\[
\tilde{f}(x, y) = \sum_{j=0}^{n_1} \sum_{k=0}^{m_1} a_{j,k} x^{2j} y^{2k},
\]

\[
\tilde{g}(x, y) = \sum_{j=0}^{n_2} \sum_{k=0}^{m_2} b_{j,k} x^{2j} y^{2k+1}.
\]

where \( a_{j,k} \) and \( b_{j,k} \) are real coefficients. The result is

\[
\beta(a) = \frac{a^2}{2} \sum_{j=0}^{n} \sum_{k=0}^{m} \frac{\Gamma(j + 1/2)\Gamma(k + 1/2)}{(k + j + 1)!} \left[ \left( j + \frac{1}{2} \right) a_{j,k} + \left( k + \frac{1}{2} \right) b_{j,k} \right] a^{2(k+j)},
\]

where \( n \equiv \text{Max}\{n_1, n_2\} \) and \( m \equiv \text{Max}\{m_1, m_2\} \). Here, \( a_{j,k} = 0 \) for \( j > n_1 \) or \( k > m_1 \) and \( b_{j,k} = 0 \) for \( j > n_2 \) or \( k > m_2 \). The root \( a = 0 \) of \( \beta(a) \) corresponds to the fixed
point \((0,0)\) and the factor \(a^2\) can be eliminated. Thus, the possible amplitudes \(a\) are the zeros of \(\beta(a)/a^2\), a polynomial in \(a^2\) of degree \(\text{Max}\{n_1 + m_1, n_2 + m_2\}\). There are no more than \(\text{Max}\{n_1 + m_1, n_2 + m_2\}\) different solutions \(a > 0\) and therefore, the maximum number of limit cycles in this case is \(\text{Max}\{n_1 + m_1, n_2 + m_2\}\) as it was proved by Iliev [14]. This result can also be written as \(\left[\frac{\text{Max}\{\deg f, \deg g\}-1}{2}\right]\) where \([\ ]\) means the integer part and \(\deg f\) (resp. \(\deg g\)) denotes the degree of \(f(x, y)\) (resp. of \(g(x, y)\)). If we restrict ourselves to the case of Liénard equations where \(f(x, y) = 0\) and \(g(x, y)\) is linear in \(y\), we recover the known result that the number of limit cycles is less or equal than \(n_2\) in the weak nonlinear regime. The extension of this relation for the whole range of \(\epsilon\) is the still not proved Lins-Melo-Pugh conjecture on the number of limit cycles for Liénard systems [3].

3.2 Reduction of some examples to a perturbed circular period annulus

Example 1: (a particular circular period annulus)

\[
\begin{align*}
\dot{x} &= y - \epsilon(a_1 x + a_2 x^2 + a_3 x^3), \\
\dot{y} &= -x - \epsilon b_3 x^3,
\end{align*}
\] (10)

Yuquan and Zhujun [17] have investigated the number of limit cycles of a similar and more general system. The conditions of their theorems can be translated for this particular case of weak perturbations (\(|\epsilon| \ll 1\)) as following: (i) If \(\epsilon b_3 > 0\), \(\epsilon a_1 < 0\) and \(\epsilon a_3 > 0\) then system (10) has a unique stable limit cycle (Theorem 1 in Ref. [14]); (ii) If \(\epsilon b_3 > 0\), \(\epsilon a_1 > 0\), and \(\epsilon a_3 < 0\) then system (10) has a unique unstable limit cycle (Theorem 2 in Ref. [14]).

These results are confirmed by the method introduced in the previous section. Moreover, as it is explained in that section, the amplitudes \(a\) of the limit cycles in the weakly nonlinear regime are the nontrivial positive solutions of the equation \(\beta(a) = 0\). In this case, \(f(x, y) = -a_1 x - a_2 x^2 - a_3 x^3\) and \(g(x, y) = -b_3 x^3\). We
obtain:

\[ \beta(a) = \frac{\pi}{2} a^2 (a_1 + \frac{3}{4} a_3 a^2), \]

which shows that the system has at most one limit cycle of amplitude

\[ a = \sqrt{-\frac{4a_1}{3a_3}}. \]

Therefore, the system has not limit cycles if \( a_1 a_3 > 0 \) and just one limit cycle if \( a_1 a_3 < 0 \). This periodic motion is stable if \( \epsilon a_1 < 0 \) and unstable if \( \epsilon a_1 > 0 \). Numerical simulations are in strong agreement with this analytical result. Observe that there are certain conditions of Yuquan and Zhujun theorems that can be removed in this regime in order to have just one limit cycle.

**Example 2: (a system topologically equivalent to a circular period annulus)**

\[
\begin{align*}
\dot{x} &= -y^{2l-1} + \epsilon P(x, y), \\
\dot{y} &= x^{2k-1} + \epsilon Q(x, y),
\end{align*}
\]  

(11)

where \( k \) and \( l \) are positive integers and \( P(x, y) \) and \( Q(x, y) \) are polynomials in \( x \) and \( y \). The number of limit cycles of this system for \( |\epsilon| \ll 1 \) has been studied in Ref. [18, 19]. The constant of motion for the unperturbed problem is \( \bar{H}(x, y) = \frac{1}{2k} x^{2k} + \frac{1}{2l} y^{2l} \). When \( k, l > 1 \) it corresponds to a degenerate center topologically equivalent to the circular period annulus obtained when \( k = l = 1 \) (then \( H_c = \frac{1}{2} (x^2 + y^2) \)). If we remove the time variable, we obtain

\[
y^{2l-1} \frac{dy}{dx} + x^{2k-1} + \epsilon \left[ Q(x, y) - P(x, y) \frac{dy}{dx} \right] = 0, \]  

(12)

In order to apply the method introduced in the previous section, we perform the change of variables \( \Gamma: (x, y) \to (X, Y) \):

\[
\begin{align*}
X &= \text{sign}(x) \frac{|x|}{\sqrt{k}}, \\
Y &= \text{sign}(y) \frac{|y|}{\sqrt{l}}.
\end{align*}
\]

Then, equation (12) reads

\[
Y \frac{dY}{dX} + X + \epsilon \left[ \frac{|x|^{(1/k)-1}}{k^{1-(1/2k)}} Q(x, y) - \frac{|y|^{(1/l)-1}}{l^{1-(1/2l)}} P(x, y) \frac{dY}{dX} \right] = 0, \]  

(13)
with
\begin{align}
x &= \text{sign}(X)(\sqrt[k]{|X|})^{1/k} \\
y &= \text{sign}(Y)(\sqrt[l]{|Y|})^{1/l}.
\end{align}

Equation (13) has the form (3) with \((x, y)\) replaced by \((X, Y)\) and
\begin{align}
f(X, Y) &= -\frac{Y^{(1/l)-1}}{1 - \frac{1}{k}} P \left( \text{sign}(X)(\sqrt[k]{|X|})^{1/k}, \text{sign}(Y)(\sqrt[l]{|Y|})^{1/l} \right) \\
g(X, Y) &= -\frac{|X|^{(1/k)-1}}{1 - \frac{1}{k}} Q \left( \text{sign}(X)(\sqrt[k]{|X|})^{1/k}, \text{sign}(Y)(\sqrt[l]{|Y|})^{1/l} \right).
\end{align}

Therefore, in these coordinates, if \(A\) is the amplitude of a limit cycle \((X, Y(X))\) in the weakly nonlinear regime, it verifies the equation \(\bar{\beta}(A) = 0\), where
\[
\bar{\beta}(A) \equiv \int_{-A}^{A} \left\{ \bar{g}(X, Y_A(X)) + \frac{X}{Y_A(X)} \bar{f}(X, Y_A(X)) \right\} dX,
\]
\(\bar{f}(X, Y)\) and \(\bar{g}(X, Y)\) are defined in (4) with \(f(X, Y)\) and \(g(X, Y)\) given in (13) and \(Y_A(X) \equiv \sqrt{A^2 - X^2}\). If we undo the change of variable \((X, Y) \rightarrow (x, y)\) in the above integral and write \(a^{2k} \equiv kA^2\), then we can write \(\bar{\beta}(A) \equiv \beta(a)\) in terms of the amplitudes \(a\) of the original variables \((x, y)\):
\[
\beta(a) \equiv \int_{-a}^{a} \left\{ \bar{Q}(x, y_a(x)) - \bar{P}(x, y_a(x)) \frac{dy_a(x)}{dx} \right\} dx,
\]
where \(y_a(x) \equiv \left[ \frac{1}{k} \left( a^{2k} - x^{2k} \right) \right]^{1/(2l)}\) and \(\bar{P}(x, y)\) and \(\bar{Q}(x, y)\) are defined in (4) for \(f(x, y) = P(x, y)\) and \(g(x, y) = Q(x, y)\). Hence, the amplitudes, \(a\), of the limit cycles in the weakly nonlinear regime are the nontrivial solutions of the equation \(\beta(a) = 0\).

If \(k = l\), then \(\beta(a)\) reads
\[
\beta(a) = a \int_{-1}^{1} \left\{ \bar{Q}(ax, a\bar{y}(x)) + \frac{x^{2k-1}\bar{y}(x)\bar{P}(ax, a\bar{y}(x))}{1 - x^{2k}} \right\} dx,
\]
where \(\bar{y}(x) \equiv \left( 1 - x^{2k} \right)^{1/(2k)}\). If \(P(x, y)\) and \(Q(x, y)\) are polynomials of degree at most \(n\), \(a^{-2}\beta(a)\) is a polynomial of degree \(\left\lfloor \frac{n-1}{2} \right\rfloor\) in \(a^2\) and we conclude that the maximum number of limit cycles in this regime is \(\left\lfloor \frac{n-1}{2} \right\rfloor\), as it has been shown by Coll et al. [19].

Take, for instance, the particular case:
\begin{align}
\dot{x} &= -y^3 + \epsilon bxy^2, \\
\dot{y} &= x^3 + \epsilon cxy^3.
\end{align}
with \( k = l = 2 \) and \( n = 5 \). The equation \( \beta(a) = 0 \) has at most one non trivial solution given by
\[
a = \frac{(\pi / 2)^{3/4}}{\Gamma(3/4)} \sqrt{-\frac{b}{c}},
\]
In this case, the system has at most one limit cycle of amplitude \( a \) emerging from the slightly perturbed period annulus. This analytical result is in agreement with the direct verification of the dynamics by integrating the equations (16).

Example 3: (a system topologically equivalent to a circular period annulus)
\[
\begin{align*}
\dot{x} &= -y + yx^2 + \epsilon F(x, y), \\
\dot{y} &= x + xy^2 + \epsilon G(x, y),
\end{align*}
\]
(17)
where \( F(x, y) = a_0 x^3 + a_1 x^2 y + a_2 xy^2 + a_3 y^3 \) and \( G(x, y) = b_0 x^3 + b_1 x^2 y + b_2 xy^2 + b_3 y^3 \).
It has been shown that this system has, at most, two limit cycles (theorem 2.6 in Ref. [20]).

The constant of motion of equation (17) is \( H(x, y) = 1 + y^2 \frac{1}{1-x^2} = h \). A period annulus topologically equivalent to \( \tilde{H}_c = \frac{1}{2} (x^2 + y^2) = \tilde{h} \) is defined in the region \(-1 < x < 1\) when \( h \) runs from 1 to \( \infty \). The origin \((0, 0)\) is found for \( h = 1 \) and the boundaries of this region are the lines \( x = \pm 1 \) which are approached when \( h \to \infty \). The interior is formed by closed curves topologically equivalent to circles. The integral paths exterior to the period annulus are open curves obtained when \( h \) increases from \( -\infty \) to zero.

Following the same steps as in the preceding example, we proceed to remove the time variable of system (17). Writing \( x' \equiv \frac{dx}{dy} \),
\[
(1 + y^2)xx' + (1 - x^2)y + \epsilon [G(x, y)x' - F(x, y)] = 0.
\]
(18)
The homomorphism with the circular period annulus is established by means of the change of variables \( \Gamma : (x, y) \to (X, Y) \):
\[
\begin{align*}
X &= \text{Sign}(x) \sqrt{-\log(1 - x^2)}, \\
Y &= \text{Sign}(y) \sqrt{\log(1 + y^2)}.
\end{align*}
\]
Then, the equation (18) reads

\[ X \frac{dX}{d\tau} + Y + \epsilon \left( |X| \frac{dX}{d\tau} - \frac{Y}{\sqrt{1-e^{-X^2}}} \right) = 0. \]

Now this equation has the form (5) with the variables \((x, y)\) replaced by the variables \((Y, X)\) and

\[
\begin{align*}
 f(Y, X) &= |X| \frac{G(Sign(X)\sqrt{1-e^{-X^2}}, Sign(Y)\sqrt{e^{Y^2}-1})}{e^{Y^2}\sqrt{1-e^{-X^2}}} , \\
 g(Y, X) &= |Y| \frac{F(Sign(X)\sqrt{1-e^{-X^2}}, Sign(Y)\sqrt{e^{Y^2}-1})}{e^{-X^2}\sqrt{e^{Y^2}-1}}.
\end{align*}
\]

Therefore, we can apply the method of the previous section to this equation. Then, the amplitude \(A\) of a limit cycle \((Y, X(Y))\) of this system in the weakly nonlinear regime, \(X(Y) \approx \sqrt{A^2 - Y^2}\), satisfies \(\bar{\beta}(A) = 0\), with

\[ \bar{\beta}(A) \equiv \int_{-A}^{A} \left\{ \frac{Y|F\left(\sqrt{1-e^{-Y^2-A^2}},\sqrt{e^{Y^2}-1}\right)}{e^{Y^2-A^2}\sqrt{e^{Y^2}-1}} + \frac{YG\left(\sqrt{1-e^{-Y^2-A^2}},\sqrt{e^{Y^2}-1}\right)}{e^{Y^2}\sqrt{1-e^{-Y^2-A^2}}} \right\} dY, \]

\(\tilde{F}(x, y) \equiv a_0 x^3 + a_2 x y^2\) and \(G(x, y) \equiv b_1 x^2 y + b_3 y^3\). If we undo the change of variable \((Y, X) \to (x, y)\) in the above integral and write \(A^2 = \log(1 + a^2)\), then we can write \(\bar{\beta}(A) \equiv \beta(a)\) in terms of the amplitudes \(a\) of the original variables \((x, y)\):

\[
\beta(a) \equiv \int_{-a}^{a} \left\{ \frac{y \sqrt{1 + a^2} G\left(\sqrt{\frac{a^2 - y^2}{1 + a^2}}, y\right)}{\sqrt{a^2 - y^2}} + (1 + a^2) \tilde{F}\left(\sqrt{\frac{a^2 - y^2}{1 + a^2}}, y\right) \right\} \frac{dy}{(1 + y^2)^2}.
\]

After straightforward computations we obtain that the real roots of \(\beta(a)\) are the real solutions of the equation \(\tilde{\beta}(a) = 0\), where

\[
\tilde{\beta}(a) \equiv [b_1 + a_2(1 + a^2)] [2 + a^2 - 2\sqrt{1 + a^2}] + b_3 [2(1 + a^2)^{3/2} - 3a^2 - 2] + a_0 [2\sqrt{1 + a^2} - 2 - a^2 + a^4].
\]

We replace the variable \(a\) in this equation by the variable \(B \equiv \sqrt{1 + a^2}\). We look for solutions \(a > 0\), that is, \(B > 1\). The two only possible solutions are

\[
B^\pm \equiv \frac{-(a_0 + b_3) \pm \sqrt{(a_0 + b_3)^2 - (a_0 + a_2)(b_1 + b_3)}}{a_0 + a_2}.
\]

Writing

\[
u \equiv \frac{a_0 + b_3}{a_0 + a_2}, \quad v \equiv \frac{b_1 + b_3}{a_0 + a_2},
\]

(20)
we have that $B^\pm = -u \pm \sqrt{u^2 - v}$. Then, the number of limit cycles in the weakly nonlinear regime is 0, 1 or 2 depending on $a_0, a_2, b_1, b_3$ (see figure 2):

(i) If $u > -1$ and $v + 2u + 1 > 0$ or $v > u^2$, neither $B^+$ neither $B^-$ are greater than 1. Therefore, the system has not limit cycles.

(ii) If $v + 2u + 1 < 0$, then $B^+ > 1, B^- < 1$. Therefore, the system has a unique limit cycle with amplitude $a = \sqrt{(B^+)^2 - 1}$.

(iii) If $v + 2u + 1 > 0$ and $v < u^2$, then $B^+, B^- > 1$. Therefore, the system has two limit cycles with amplitudes $a^+ = \sqrt{(B^+)^2 - 1}$ and $a^- = \sqrt{(B^-)^2 - 1}$.

Computer integration of equation (17) is in agreement with these analytical results. Theorem 2.6 of Ref. [20] states that the maximum number of limit cycles of this system is two. By applying the method proposed in Section 2 we obtain a more detailed information: the exact number and amplitudes of the limit cycles of this system as a function of the parameters $\{a_k, b_k | k = 1, 2, 3\}$.

4 Conclusions

Nowadays it is well known that the inclusion of a slight dissipation in a conservative system destroys almost completely the geometrical aspect of the phase space of the unperturbed motion. But the analytical methods available for the study of such perturbed systems suffer a lack of predictive power when the nonlinearities start to dominate the dynamics.

In this work, we have carried out the calculation of the precise number and amplitude of the periodic orbits surviving to a weak perturbation of different topologically equivalent period annulus. First, the equation $\beta(a) = 0$ containing that information about the limit cycles emerging from a general slight perturbation of a continuum set of circles is stated. Essentially this equation corresponds to the first Melnikov function associated to the system, although it has been obtained by
an alternative line of reasoning. Second, the dynamical equations of other topologically equivalent perturbed centers have been reduced to equivalent equations of the former case. Then, the calculation of the number and amplitude of their limit cycles has also been carried out in the same way. The numerical integration of these different systems supports the analytical calculations and confirms the theoretical predictions.

We must stress that, on the one hand, our method has given the correct maximum number of limit cycles in all the examples analyzed. On the other hand, it has allowed us to predict the exact number and amplitude of the limit cycles emerging from different topologically equivalent perturbed period annulus. It seems that in order to find the number and amplitude of limit cycles, it is not important to maintain the condition of a planar system to be a polynomial system. Perhaps it would be more interesting to try to identify the class of all its geometrical and topologically equivalent systems and to perform the calculations in the simplest system of this class.

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References

[1] A. Winfree, *The timing of the biological clocks*, Scientific American Library, W.H. Freeman Co. (1986).

[2] A.A. Andronov, A.A. Vitt and S.E. Khaikin, *Theory of Oscillators*, Dover, New York (1989).

[3] K. Odani, The limit cycle of the van der Pol equation is not algebraic, J. Differential Equations **115**, 146-152 (1995).

[4] G.S. Rychkov, The maximum number of limit cycles of the system $\dot{y} = -x$, $\dot{x} = y - \sum_{i=0}^{2} a_i x^{2i+1}$ is two, *Differential Equations* **11**, 301-302 (1975).

[5] A. Lins, W. de Melo and C.C. Pugh, On Liénard’s Equation, *Lectures Notes in Math.*, Vol. 597, p. 355 (Springer-Verlag) (1977).

[6] N.G. Lloyd, Liénard systems with several limit cycles, *Math. Proc. Camb. Phil. Soc.* **102**, 565-572 (1987).

[7] H. Giacomini and S. Neukirch, Number of limit cycles of the Liénard equation, *Phys. Rev. E* **56**, 3809-3813 (1997).

[8] J.L. López and R. López-Ruiz, The limit cycles of Liénard equations in the strongly nonlinear regime, *Chaos, Solitons & Fractals* **11**, 747-756 (2000).

[9] R. López-Ruiz and J.L. López, Bifurcation curves of limit cycles in some Liénard systems, *Int. J. of Bifurcation and Chaos* **10**, 971-980 (2000).

[10] M.C. Depassier and J. Mura, Variational approach to a class of nonlinear oscillators with several limit cycles, *Phys. Rev. E* **64**, 056217(6) (2001).

[11] V.A. Gaiko, Geometric methods of qualitative analysis and global bifurcation theory, *Nonlin. Phenom. Compl. Syst.* **5**, 1-20 (2002).

[12] C. Chicone, On bifurcation of limit cycles from centers, *Lecture Notes in Mathematics* **1455**, 20-43 (1991).
[13] T.R. Blows and L.M. Perko, Bifurcation of limit cycles from centers and separatrix cycles of planar analytic systems, *SIAM Review* 36, 341-376 (1994).

[14] H. Giacomini, J. Llibre and M. Viano, On the shape of limit cycles that bifurcate from Hamiltonian centers, *Nonlinear Analysis* 41, 523-537 (2000).

[15] J. Chavarriga and M. Sabatini, A survey of isochronous centers, *Qualitative Theory of Dynamical Systems* 1, 1-70 (1999).

[16] I.D. Iliev, The number of limit cycles due to polynomial perturbations of the harmonic oscillator, *Math. Proc. Camb. Phil. Soc.* 127, 317-322 (1999).

[17] W. Yuquan and J. Zhujun, Cubic Liénard equation with quadratic damping (I), *Acta Math. Appl. Sinica* 16, 42-52 (2000).

[18] J. Llibre and X. Zhang, The number of limit cycles of some perturbed Hamiltonian polynomial systems, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.* 8, 161-181 (2001).

[19] B. Coll, A. Gasull and R. Prohens, Bifurcation of limit cycles from two families of centers, *Preprint CRM*, no. 491 (2002).

[20] C. Li, W. Li, J. Llibre and Z. Zhang, On the limit cycles of polynomial differential systems with homogeneous nonlinearities, *Proc. Edinburgh Math. Soc.* 43, 529-543 (2000).
Figure Captions

Figure 1: A typical phase portrait of equation (5). The limit cycles of amplitudes \((-a_1, a_2), (-b_1, b_2), \ldots\), enclose the origin. Stable and unstable limit cycles alternate. The order of this alternation depends on the stability of the origin.

Figure 2: The complete bifurcation diagram of system (17). The system has no periodic solutions in the white region, one limit cycle in the grey region and two limit cycles in the black region. See example 3 for details of the calculation of \((u, v)\) from the original parameters of equation (17).