ON MAXIMAL-ACCELERATION , STRINGS AND THE GROUP OF MINIMAL PLANCK-AREA RELATIVITY THEORY

Carlos Castro
Center for Theoretical Studies of Physical Systems,
Clark Atlanta University, Atlanta, GA. 30314
November , 2002

Abstract

Recently we have presented a new physical model that links the maximum speed of light with the minimal Planck scale into a maximal-acceleration Relativity principle in phase spaces. The maximal proper-acceleration bound is \( a = c^2/\Lambda \) where \( \Lambda \) is the Planck scale. The group transformation laws of this Maximal-acceleration Relativity theory under velocity and acceleration boosts are analyzed in full detail. For pure acceleration boosts it is shown that the minimal Planck-areas (maximal tension) are universal invariant quantities in any frame of reference. The implications of this minimal Planck-area (maximal tension) principle in future developments of string theory, \( W \)-geometry and Quantum Gravity are briefly outlined.

I . Introduction

In recent years there has been growing evidence that the Relativity principle should be extended to include all dimensions and signatures on the same footing. Relativity in C-spaces (Clifford manifolds) [1] is a very natural extension of Einstein’s relativity and Nottale’s scale relativity [2] where the impassible speed of light and the minimum Planck scale are the two universal invariants. An event in C-space is represented by a polyvector, or Clifford-aggregate of lines, areas, volumes, ...... which bear a one-to-one correspondence to the holographic shadows/projections (onto the embedding spacetime coordinate planes) of a nested family of \( p \)-loops (closed \( p \)-branes of spherical topology) of various dimensionalities: \( p = 0 \) represents a point; \( p = 1 \) a closed string, \( p = 2 \) a closed membrane, etc.... where \( p = 0, 1, 2, ... , D - 1 \).

The invariant “line” element associated with a polyparticle is:

\[
d\Sigma^2 = dX.dX = (d\Omega)^2 + \Lambda^{2D-2}(dx_\mu dx^\mu) + \Lambda^{2D-4}(dx_\mu dx^\nu) + ... \quad (1.1)
\]
the Planck scale appears as a natural quantity in order to match units and combine p-branes (p-loops) of different dimensions. The polyparticle lives in a target background of \(D + 1 = p + 2\) dimensions due to the fact that C-space has two times, the coordinate time \(x_0 = t\) and the \(\Omega\) temporal variable representing the proper \(p + 1\)-volume. The fact that the Planck scale is a minimum was based on the real-valued interval \(dX\) when \(dX.dX > 0\). The analog of photons in C-space are tensionless branes: \(dX.dX = 0\). Scales smaller than \(\Lambda\) yield “tachyonic” intervals \(dX.dX < 0\) [1]. Due to the matrix representation of the gamma matrices and the cyclic trace property, it can be easily seen why the line element is invariant under the C-space Lorentz group transformations:

\[
\text{Trace } X'^2 = \text{Trace } [RX^2R^{-1}] = \text{Trace } [RR^{-1}X^2] = \text{Trace } X^2, \quad (1.2)
\]

where a finite polydimensional rotation that reshuffles dimensions is characterized by the C-space “rotation” matrix:

\[
R = \exp[i(\theta I + \theta^\mu \gamma_\mu + \theta^{\mu\nu} \gamma_{\mu\nu} + ...)]. \quad (1.3)
\]

The parameters \(\theta, \theta^\mu, \theta^{\mu\nu}, ...\) are the C-space extension of the Lorentz boost parameters and for this reason the naive Lorentz transformations of spacetime are modified to be:

\[
x'^\mu = L_\nu^\mu [\theta, \theta^\mu, \theta^{\mu\nu}, ...]x^\nu + L_{\nu\rho}^\mu [\theta, \theta^\mu, \theta^{\mu\nu}, ...]x^{\nu\rho} + ..... \quad (1.4)
\]

It was argued in [1] that the extended Relativity principle in C-space may contain the clues to unravel the physical foundations of string and M-theory since the dynamics in C-spaces encompass in one stroke the dynamics of all p-branes of various dimensionalities. In particular, how to formulate a master action that encodes the collective dynamics of all extended objects.

For further details about these issues we refer to [1] and all the references therein. Like the derivation of the minimal length/time string/brane uncertainty relations; the logarithmic corrections to the black-hole area-entropy relation; the existence of a maximal Planck temperature; the origins of a higher derivative gravity with torsion; why quantum-spacetime may be truly infinite dimensional whose average dimension today is close to \(4 + \phi^3 = 4.236\) where \(\phi = 0.618\) is the Golden Mean; the construction of the p-brane propagator; the role of supersymmetry; the emergence of two
times; the reason behind a running value for \( \hbar \); the way to correctly pose the cosmological constant problem as well as other results.

In [1] we discussed another physical model that links the maximum speed of light, and the minimal Planck scale, into a maximal-acceleration principle in the spacetime tangent bundle, and consequently, in the phase space (cotangent bundle). Crucial in order to establish this link was the use of Clifford algebras in phase spaces. The maximal proper acceleration bound is \( a = c^2/\Lambda \) in full agreement with [4] and the Finslerian geometry point of view in [6]. A series of reasons why C-space Relativity is more physically appealing than all the others proposals based on kappa-deformed Poincare algebras [10] and other quantum algebras was presented. Maximal-acceleration effects within the context of kappa-deformed Poincare have been discussed in [11].

On the other hand, we argued why the truly bicovariant quantum algebras based on inhomogeneous quantum groups developed by Castellani [15] had a very interesting feature related to the T-duality in string theory; the deformation \( q \) parameter could be written as: \( q = \exp[\Lambda/L] \) and, consequently, the classical limit \( q = 1 \) is attained when the Planck scale \( \Lambda \) is set to zero, but also when the upper impassible scale \( L \) goes to infinity! This entails that there could be two dual quantum gravitational theories with the same classical limit! Nottale has also postulated that if there is a minimum Planck scale, by duality, there should be another upper impassible upper scale \( L \) in Nature [2]. For a recent discussion on maximal-acceleration and kappa-deformed Poincare algebras see [10]. It was also argued in [1] why the theories based on kappa-deformed Poincare algebras may in fact be related to a Moyal star-product deformation of a classical Lorentz algebra whose deformation parameter is precisely the Planck scale \( \Lambda = 1/\kappa \).

In section 2.1, we review once again the work in [1] and show how to derive the Nesterenko action [5] associated with a sub-maximally accelerated particle in spacetime directly from phase-space Clifford algebras and present a full-fledged C-phase-space generalization of the Nesterenko action. It was this principle of Maximal-Acceleration Phase space Relativity that allowed us to derive the exact integral equation that governs the Renormalization-Group-like scaling dependence of the fractional change of the fine structure constant as a function of the cosmological redshift factor and a cutoff scale \( L_c \) where the maximal acceleration relativistic effects are dominant. For a review of the variation of the fundamental constants see Uzan [13].
Maximal-acceleration corrections to the Lamb shifts of one-electron atoms were performed by Lambiase, Papini and Scarpetta [13].

If the cutoff scale $L_c$ was set equal to the minimal Planck scale we have shown [1] why one obtains a Cosmological model dominated entirely by the cosmological constant, with $\Omega_\Lambda = 1$. We argued why in this extreme case scenario all the matter in the Universe may have been created out of the vacuum (vacuum fluctuations) as a result of the acceleration effects, analogous to the Hawking-Unruh effect of particle production due to the accelerated motion (noninertial) with respect to a Minkowski vacuum.

In section 2.2 we review the main features of Born's Dual Relativity and the work of Low [14] in the construction of $U(1,3)$ group transformations which leave the intervals in classical Phase spaces invariant along with the construction of the unitary irreducible representations which describe the particle spectrum of the theory. In section 3 we present the explicit transformations rules of the Phase space coordinates under velocity and acceleration boosts and show explicitly why they have the required group structure to qualify for a genuine Phase Space Relativity Theory.

In section 4 we prove why pure acceleration-boost transformations leave invariant the minimal Planck-Areas, fact which can be reinterpreted as the existence of an invariant universal maximum string tension and Planck temperature in Nature.

We finalize by making some comments concerning the Conformal Group, $W$ gravity, higher conformal spin theories on $AdS$ spaces, $W_\infty$ strings.....and advocate the importance to build an Extended Relativity Theory in C-Phase-Spaces that will encompass the physics of all p-branes into one single footing by implementing the Relativity principle of minimal and invariant Planck areas, Planck-volumes, Planck-hypervolumes ...in all frames of references in Phase spaces.

2. Maximal-Acceleration Phase Space Relativity

2.1. Maximal-Acceleration from Clifford algebras

Readers familiar with the previous work may omit this subsection. We will follow closely the procedure described in the book [3] to construct the phase space Clifford algebra. For simplicity we shall begin with a two-dimensional phase space, with one coordinate and one momentum variable and afterwards we will generalize the construction to higher dimensions.
Let \( e_p e_q \) be the Clifford basis elements in a two-dimensional phase space obeying the following relations:

\[
e_p e_q \equiv \frac{1}{2} (e_q e_p + e_p e_q) = 0, \quad e_p e_p = e_q e_q = 1.
\]  

(2.1)

The Clifford product of \( e_p, e_q \) is by definition the sum of the scalar product and wedge product furnishing the unit bivector:

\[
e_p e_q \equiv e_p e_q + e_p \wedge e_q = e_p \wedge e_q = j. \quad j^2 = e_p e_q e_p e_q = -1.
\]  

(2.2)

due to the fact that \( e_p, e_q \) anticommute, eq. (2.1).

In this fashion, using Clifford algebras one can justify the origins of complex numbers without introducing them ad-hoc. The imaginary unit \( j \) is \( e_p e_q \). For example, a Clifford vector in phase space can be expanded, setting aside for the moment the issue of units, as:

\[
Q = q e_q + p e_p. \quad Q e_q = q + p e_p e_q = q + j p = z. \quad e_q Q = q + p e_q e_p = q - j p = z^*,
\]  

(2.3)

which reminds us of the creation/annihilation operators used in the harmonic oscillator case and in coherent states.

The analog of the action for a massive particle is obtained by taking the scalar product:

\[
dQ dQ = (dq)^2 + (dp)^2 \Rightarrow S = m \int \sqrt{dQ dQ} = m \int \sqrt{(dq)^2 + (dp)^2}.
\]  

(2.4)

One may insert now the appropriate length and mass parameters in order to have consistent units:

\[
S = m \int \sqrt{(dq)^2 + \left(\frac{\Lambda}{m}\right)^2 (dp)^2}.
\]  

(2.5)

where we have introduced the Planck scale \( \Lambda \) and the mass \( m \) of the particle to have consistent units, \( \hbar = c = 1 \). The reason will become clear below.

Extending this two-dimensional action to a higher \( 2n \)-dimensional phase space requires to have \( e_{p_\mu}, e_{q_\mu} \) for the Clifford basis where \( \mu = 1, 2, 3...n \). The
action in this $2n$-dimensional phase space is:

$$
S = m \int \sqrt{(dq^\mu dq_\mu) + \left(\frac{\Lambda}{m}\right)^2 dp^\mu dp_\mu} = m \int d\tau \sqrt{1 + \left(\frac{\Lambda}{m}\right)^2 \left(\frac{dp^\mu}{d\tau}\right)\left(\frac{dp_\mu}{d\tau}\right)}
$$

(2.6)

in units of $c = 1$, one has the usual infinitesimal proper time displacement $d\tau^2 = dq^\mu dq_\mu$.

One can easily recognize that this action (2.6), up to a numerical factor of $m/a$, is nothing but the action for a sub-maximally accelerated particle given by Nesterenko [5]. It is sufficient to rewrite: $dp^\mu/d\tau = md^2x^\mu/d\tau^2$ to get from eq. (2.6):

$$
S = m \int d\tau \sqrt{1 + \left(\frac{1}{a}\right)^2 \left(\frac{d^2x^\mu}{d\tau^2}\right)\left(\frac{d^2x_\mu}{d\tau^2}\right)}.
$$

(2.7)

Using the postulate that the maximal-proper acceleration is given in a consistent manner with the minimal length principle (in units of $c = 1$):

$$
a = c^2/\Lambda = 1/\Lambda \Rightarrow S = m \int d\tau \sqrt{1 + \left(\frac{1}{a}\right)^2 \left(\frac{d^2x^\mu}{d\tau^2}\right)\left(\frac{d^2x_\mu}{d\tau^2}\right)}.
$$

(2.8)

which is exactly the action of [5], up to a numerical factor of $m/a$, when the metric signature is $(+,-,-,-)$.

The proper acceleration is **orthogonal** to the proper velocity as a result of differentiating the timelike proper velocity squared:

$$
V^2 = \frac{dx^\mu}{d\tau} \frac{dx_\mu}{d\tau} = 1 = V^\mu V_\mu > 0 \Rightarrow \frac{dV^\mu}{d\tau} V_\mu = \frac{d^2x^\mu}{d\tau^2} V_\mu = 0,
$$

(2.9)

which means that if the proper velocity is timelike the proper acceleration is spacelike so that:

$$
g^2(\tau) \equiv -(d^2x^\mu/d\tau^2)(d^2x_\mu/d\tau^2) > 0 \Rightarrow S = m \int d\tau \sqrt{1 - \frac{g^2}{a^2}} = m \int d\omega ,
$$

(2.10)

where we have defined:
\[ d\omega \equiv \sqrt{1 - \frac{g^2}{a^2}} d\tau. \] (2.11)

The dynamics of a submaximally accelerated particle in Minkowski spacetime can be reinterpreted as that of a particle moving in the spacetime tangent bundle background whose Finslerian-like metric is:

\[ d\omega^2 = g_{\mu\nu}(x^\mu, dx^\nu) dx^\mu dx^\nu = (d\tau)^2(1 - \frac{g^2}{a^2}). \] (2.12)

For uniformly accelerated motion, \( g(\tau) = g = \text{constant} \) the factor:

\[ \frac{1}{\sqrt{1 - \frac{g^2}{a^2}}} \] (2.13)

plays a similar role as the standard Lorentz time dilation factor in Minkowski spacetime.

The action is real valued if, and only if, \( g^2 < a^2 \) in the same way that the action in Minkowski spacetime is real valued if, and only if, \( v^2 < c^2 \). This explains why the particle dynamics has a bound on proper accelerations. Hence, for the particular case of a uniformly accelerated particle whose trajectory in Minkowski spacetime is a hyperbola, one has an Extended Relativity of uniformly accelerated observers whose proper acceleration have an upper bound given by \( c^2/\Lambda \). Rigorously speaking, the spacetime trajectory is obtained by a canonical projection of the spacetime tangent bundle onto spacetime. The invariant time, under the pseudo-complex extension of the Lorentz group [8], measured in the spacetime tangent bundle is no longer the same as \( \tau \), but instead, it is given by \( \omega(\tau) \).

This is similar to what happens in C-spaces, the truly invariant evolution parameter is not \( \tau \) nor \( \Omega \), the Stuckelberg parameter [3], but it is \( \Sigma \) which is the world interval in C-space and that has units of length\(^D\). The group of C-space Lorentz transformations preserve the norms of the Polyvectors and these have units of hypervolumes; hence C-space Lorentz transformations are volume-preserving.

Another approach to obtain the action for a sub-maximally accelerated particle was given by [8] based on a pseudo-complexification of Minkowski spacetime and the Lorentz group that describes the physics of the spacetime
tangent bundle. This picture is not very different form the Finslerian spacetime tangent bundle point of view of Brandt [6]. The invariant group is given by a pseudo-complex extension of the Lorentz group acting on the extended coordinates $X = ax^\mu + I\nu^\mu$ with $I^2 = 1$ (pseudo-imaginary unit) where both position and velocities are unified on equal footing. The invariant line interval is $a^2d\omega^2 = (dX)^2$.

A C-phase-space generalization of these actions (for sub-maximally accelerated particles, maximum tidal forces) follows very naturally by using polyvectors:

$$Y = q^\mu e_{q_\mu} + q^{\mu\nu} e_{q_\mu} \wedge e_{q_\nu} + q^{\mu\nu\rho} e_{q_\mu} \wedge e_{q_\nu} \wedge e_{q_\rho} + ....$$

$$+ p^\mu e_{p_\mu} + p^{\mu\nu} e_{p_\mu} \wedge e_{p_\nu} + ... ,$$

(2.14)

where one has to insert suitable powers of $\Lambda$ and $m$ in the expansion to match units.

The C-phase-space action reads then:

$$S \sim \int \sqrt{dY.d\overline{Y}} = \int \sqrt{dq^\mu dq_\mu + dq^{\mu\nu} dq_{\mu\nu} + ... + dp^\mu dp_\mu + dp^{\mu\nu} dp_{\mu\nu} + .....} .$$

(2.15)

This action is the C-phase-space extension of the action of Nesterenko and involves quadratic derivatives in C-spaces which from the spacetime perspective are effective higher derivatives theories [1] where it was shown why the scalar curvature in C-spaces is equivalent to a higher derivative gravity. One should expect a similar behaviour for the extrinsic curvature of a polyparticle motion in C-spaces. This would be the C-space version of the action for rigid particles [7]. Higher derivatives are the hallmark of $W$-geometry (higher conformal spins).

Born-Infeld models have been connected to maximal-acceleration [8]. Such models admits an straightforward formulation using the geometric calculus of Clifford algebras. In particular one can rewrite all the nonlinear equations of motion in precise Clifford form [9]. This lead that author to propose the nonlinear extension of Dirac’s equation for massless particles due to the fact that spinors are nothing but right/left ideals of the Clifford algebra: i.e., columns, for example, of the Maxwell-Field strength bivector $F = F_{\mu\nu} \gamma^\mu \wedge \gamma^\nu$.

2.2 Born’s Dual Relativity Principle
Long time ago Max Born [14] proposed the reciprocally-conjugate Relativity principle that states that physics in Phase spaces must be invariant under the reciprocity transform:

\[
\{Q, P\} \rightarrow \{P, -Q\}.
\]  

such that the rates of change of momentum (force) is bounded by a universal constant \(b\). In units of \(\hbar = c = 1\) we have that the maximal-acceleration is given in terms of the Planck scale \(A = (1/L_P)\). Hence the maximal force subjected by an elementary particle = Planck-mass x acceleration = \((1/L_P^2)\) coincides with the maximal string tension in this system of units.

The appropriate group of dynamical symmetries that incorporates elementary particle states has been studied recently by Low [14] in terms of the canonical group \(C(1, 3)\), acting in an extended noncommuting Phase space which is given by the semidirect product of \(U(1, 3) = SU(1, 3) \times U(1)\) with the Weyl-Heisenberg group \(H(1, 3)\). An essential point that distinguishes Born’s Dual Relativity from others is that here the Canonical group \(C(1, 3)\) is traded for the naive Poincare group! In ordinary Relativity, Minkowski space is represented as the coset space of the Poincare group modulo de Lorentz group. In this dual Relativity theory the physical Noncommuting Phase space is comprised of ’’ quantum oscillators ’’, instead of ’’ points ’’, and is represented by the coset space \(C(1, 3)/SU(1, 3)\) whose metric is the invariant second Casimir of the canonical group:

\[
T^2 + \frac{E^2}{c^2 b^2} - X^2 - \frac{P^2}{b^2} + \frac{2\hbar I}{bc} (\frac{Y}{bc} - 2).
\]  

(2.17)

where \(I\) is the center of the Weyl-Heisenberg algebra and \(Y\) a \(U(1)\) generator. Hence, an element of the unitary group \(U(1, 3)\) may be represented as:

\[
e^Z, \quad \text{where} \quad Z = \frac{c^i}{c} K_i + \frac{c^i}{b} N_i + \alpha^i J_i + \theta^{ij} M_{ij} + \frac{\phi}{bc} Y.
\]  

(2.19)
The Heisenberg group is represented as:

$$e^A, \text{ where } A = \sqrt{\frac{bc}{\hbar}} (tT + \frac{e}{b}cE + \frac{q^i}{c}Q_i + \frac{p^i}{b}P_i + \frac{\eta^{\hbar}}{bc} I).$$  \tag{2.20}

For more details we refer to Low [14]. One of the most important results in [14] was the construction of unitary irreducible representations (unireps) of the canonical group using Mackey’s theory of induced representations. These unireps contain representations of $U(1,3)$ which was proposed long ago by Kalman [14] as a dynamical group for hadrons. String theory originated with the study of hadrons, hence it is not surprising to have come back to the initial starting point. The unireps contain discrete series representations that can be decomposed into infinite ladders where the rungs are representations of $U(3)$.

The most salient feature of these representations that define the particle states is that an acceleration-boost will transform a single-particle state into a composite state, which in turn, can be decomposed into a sum of single particle states representing the particle interactions in the accelerated-frame of reference. This is yet another realization of the Hawking-Unruh effect of particle creation when an observer is moving in an accelerated frame of reference with respect to a Minkowski vacuum. A thermal radiation of particles is detected whose temperature, measured with respect to an asymptotic observer in the case of particle emission by Black Holes, is linearly proportional to the acceleration $T = \frac{\hbar a}{2\pi k_Bc}$. Where $k_B$ is Boltzman constant. The maximal-acceleration Phase space Relativity principle requires a maximum Planck Temperature of the order of $10^{32}$ Kelvin.

Most importantly, for the present author, is the phenomenon that a single particle state can be boosted via acceleration-boosts into a composite multiple-particle state and which is yet another manifestation of the Bootstrap principle of String theory; i.e. all particles are made of each other. In addition, the rest and null frames automatically yielded the groups $SU(3), SU(2), U(1)$ that appear in the standard theory of the strong, weak and electromagnetic interactions without the need to compactify from higher dimensions to four dimensions.

In the next section we will study the Group properties of the Maximal-acceleration Phase space relativity in the commuting Phase space case.

3. Maximal-Acceleration Relativity and the $U(1,3)$ Group transformations
The $U(1, 3) = SU(1, 3) \otimes U(1)$ Group transformations [14] can be simplified drastically when the velocity/acceleration boosts are taken to lie in the $z = X$-direction, leaving the transverse directions $x, y, p_x, p_y$ intact; i.e., the $U(1, 1) = SU(1, 1) \otimes U(1)$ subgroup transformations that leave invariant the interval in a classical (commuting) Phase space are (in units of $\hbar = c = 1$):

$$(dZ)^2 = (dT)^2 - (dX)^2 + \frac{(dE)^2 - (dP)^2}{b^2} = \text{invariant} =$$

$$(d\tau)^2[1 + \frac{(dE/d\tau)^2 - (dP/d\tau)^2}{b^2}] = (d\tau)^2[1 - \frac{m^2g^2(\tau)}{m_P^2A_{max}^2}], \quad b^2 \equiv m_P^2A_{max}^2 \quad (3.1)$$

where $m_P$ is the Planck mass $1/L_P$ so that $b = (1/L_P)^2$, where $L_P$ is the Planck scale. The quantity $g(\tau)$ is the proper four-acceleration of a particle of mass $m$ in the $x_3$-direction which we take to be $X$. We have used the results of eqs- (2-8, 2-9, 2-10) with $(d\tau)^2 = (dT)^2 - (dX)^2$. It is crucial to notice that the particle’s mass $m$ is no longer an invariant Casimir. Only ratios of masses have physical meaning in Scale Relativity [2]. The invariant interval $(dZ)^2$ in eq- (3-1) is not the same as the interval $(d\omega)^2$ of the Nesterenko action eq- (2-10) which is invariant under a pseudo-complexification of the Lorentz group [8]. Only when $m = m_P$, the intervals agree. The interval $(dZ)^2$ described by Low [14] is $U(1, 3)$-invariant for the most general transformations in the 8D phase-space.

The transformations laws of the coordinates in classical phase space are [14]:

$$T' = T \cosh \xi + (\xi \nu X + \frac{\xi_aP}{b^2}) \frac{\sinh \xi}{\xi}. \quad (3.2a)$$

$$E' = E \cosh \xi + (-\xi \nu X + \xi_aP) \frac{\sinh \xi}{\xi}. \quad (3.2b)$$

$$X' = X \cosh \xi + (\xi \nu T - \frac{\xi_aE}{b^2}) \frac{\sinh \xi}{\xi}. \quad (3.2c)$$

$$P' = P \cosh \xi + (\xi_aE + \xi_aT) \frac{\sinh \xi}{\xi}. \quad (3.2d)$$
The $\xi_v$ = velocity-boost rapidity parameter and $\xi_a$ = acceleration-boost rapidity parameter of the primed-reference frame are defined physically as:

$$\tanh \xi_v = \pm \frac{v}{c}, \quad \tanh \xi_a = \pm \frac{a}{A_{\text{max}}}. \quad (3.3)$$

The effective boost parameter $\xi$ of the $U(1,1)$ subgroup transformations appearing in eqs- (3-2) is defined in units of $\hbar = c = 1$ as:

$$\xi \equiv \sqrt{\xi_v^2 + \xi_a^2}. \quad (3.4)$$

Our definition of the rapidity parameters are different than those in [14]. Straightforward algebra allows us to verify that:

- The transformations given by eqs-(3-2) leave the interval of eq-(3-1) in classical phase space invariant.
- The transformations eqs-(3-2) are fully consistent with Born’s duality symmetry principle [14] $(Q, P) \rightarrow (P, -Q)$. By inspection we can see that under Born duality the transformations in eqs- (3-2) are rotated into each other, up to numerical $b$ factors in order to match units.
- When on sets $\xi_a = 0$ in eqs- (3-2) one recovers automatically the standard Lorentz transformations for the $X, T$ and $E, P$ variables separately, leaving invariant the intervals $dT'^2 - dX'^2 = (d\tau)^2$ and $(dE'^2 - dP'^2)/b^2$ separately. Naturally their sum will also be maintained invariant.
- When one sets $\xi_v = 0$ we obtain the transformations rules of the events in Phase space, from one reference-frame into another uniformly-accelerated frame of reference, $a = \text{constant}$, whose acceleration-rapidity parameter is in this particular case:

$$\xi \equiv \frac{\xi_a}{b}. \quad \tanh \xi = \frac{a}{A_{\text{max}}} \quad (3.5)$$

The transformations for pure acceleration-boosts in one-spatial dimension are then:

$$T' = T \cosh \xi + \frac{P}{b} \sinh \xi. \quad (3.6a)$$

$$E' = E \cosh \xi - bX \sinh \xi. \quad (3.6b)$$
\[ X' = X \cosh \xi - \frac{E}{b} \sinh \xi. \quad (3.6c) \]

\[ P' = P \cosh \xi + bT \sinh \xi. \quad (3.6d) \]

It is straightforward to verify that the transformations in eqs. (3.6) leave invariant the phase space interval (3-1) but do not leave separately invariant the proper time interval \((d\tau)^2 = dT^2 - dX^2\). Only the combination:

\[ (dZ)^2 = (d\tau)^2 \left(1 - \frac{m^2 g^2}{m_p^2 A_{ \text{max}}^2}\right). \quad (3.6e) \]

is truly left invariant under pure acceleration-boosts !.

One can verify as well that the transformations (3-7) satisfy Born’s duality symmetry

\[(T, X) \rightarrow (E, P), \quad (E, P) \rightarrow (-T, -X). \quad (3.7)\]

and \(b \rightarrow \frac{1}{b}\). The latter duality transformation is nothing but a manifestation of a large/small Tension duality principle ! reminiscent of the \(T\)-duality in string theory; i.e. namely, small/large radius duality = winding modes/Kaluza-Klein modes duality in string compactifications and Ultraviolet/Infrared entanglement in Noncommutative Field Theories. Hence, Born’s duality principle in exchanging coordinates for momenta could be the underlying physical reason behind \(T\)-duality in string theory.

The Group property of eqs-(3-6) is satisfied i.e. the composition of two successive pure acceleration-boosts is another pure acceleration-boost with acceleration rapidity given by \(\xi'' = \xi + \xi'\). The addition of proper accelerations follows the usual relativistic composition rule:

\[ \tanh \xi'' = \tanh(\xi + \xi') = \frac{\tanh \xi + \tanh \xi'}{1 + \tanh \xi \tanh \xi'} \Rightarrow a'' = \frac{a + a'}{1 + \frac{a a'}{A}}. \quad (3.8) \]

in this fashion the upper limiting proper acceleration is never surpassed like it happens with the ordinary Special Relativistic addition of velocities.

The group properties of the full combination of velocity and acceleration boosts (3-2) requires much more algebra. A careful study reveals that the composition rule of two successive transformations given by eqs-(3-2
is preserved if, and only if, the following four relationships among the \( \xi; \xi'; \xi'' \), .... parameters are obeyed:

\[
cosh\xi'' = \cosh\xi\cosh\xi' + \frac{[\xi_v\xi_v' + (\xi_a\xi_a'/b^2)]}{\xi\xi'} \sinh\xi\sinh\xi'. \tag{3.9a}
\]

\[
\frac{\xi_v''}{\xi''} \sinh\xi'' = \frac{\xi_v\cosh\xi' \sinh\xi}{\xi} + \frac{\xi_v'\cosh\xi \sinh\xi'}{\xi'}. \tag{3.9b}
\]

\[
\frac{\xi_a''}{b\xi''} \sinh\xi'' = \frac{\xi_a\cosh\xi' \sinh\xi}{b\xi} + \frac{\xi_a'\cosh\xi \sinh\xi'}{b\xi'}. \tag{3.9c}
\]

\[
\xi_a'\xi_v' - \xi_a\xi_v = 0 \Rightarrow \frac{\xi_a'}{\xi_a} = \frac{\xi_v'}{\xi_v} = \lambda. \tag{3.9d}
\]

The condition of eq- (3.9d) can be recast as a global scaling of the effective boost parameters as follows:

\[
\xi' = \frac{\xi}{\xi} = \lambda \xi. \quad \lambda \equiv \frac{\xi_a'}{\xi_a} = \frac{\xi_v'}{\xi_v} = \frac{\xi'}{\xi} = \lambda. \tag{3.10}
\]

Meaning that the primed variables are all rescaled by the same factor of \( \lambda \). From (3-10) we can infer:

\[
\frac{\xi_a'}{\xi_a} = \frac{\xi_v'}{\xi_v} \Rightarrow \frac{\xi_a'\xi_a}{b^2\xi_a \xi_v} = \left(\frac{\xi_a}{b\xi_v}\right)^2. \tag{3.11a}
\]

\[
\frac{\xi_v'}{\xi_v} = \frac{\xi_v'}{\xi_v} \Rightarrow \frac{\xi_v'\xi_v}{\xi_v \xi_v} = \left(\frac{\xi_v}{\xi_v}\right)^2. \tag{3.11b}
\]

As a result of eqs- (3-11a, 3-11b), the definition of the effective boost parameter given in eq- (3-4), and after using eq- (3-9a), we get:

\[
cosh\xi'' = \cosh\xi\cosh\xi' + \frac{[\xi_v\xi_v' + (\xi_a\xi_a'/b^2)]}{\xi\xi'} \sinh\xi\sinh\xi'.
\]

\[
cosh\xi\cosh\xi' + \sinh\xi\sinh\xi' = \cosh(\xi + \xi'). \tag{3.12a}
\]

Hence, as expected, we have found that the effective boost parameters are indeed additive: 

14
\[ \xi'' = \xi + \xi'' \Rightarrow sinh(\xi'') = sinh(\xi + \xi') = sinh\xi'\cosh\xi + \cosh\xi'\sinh\xi' \quad (3.13) \]

From eqs- (3-9, 3-10, 3-11, 3-13) we can deduce:

\[
\frac{\xi_a'}{\xi} = \frac{\xi_a}{\xi} = \frac{\xi_a''}{\xi + \xi'} = \frac{\xi_a''}{\xi(1 + \lambda)} \quad (3 - 14a)
\]

\[
\frac{\xi_v'}{\xi} = \frac{\xi_v}{\xi} = \frac{\xi_v''}{\xi + \xi'} = \frac{\xi_v''}{\xi(1 + \lambda)} \quad (3.14b)
\]

Finally we arrive at the explicit expressions for \(\xi'', \xi_v'', \xi_a''\) in terms of the other parameters

\[
\xi_v'' = \xi_v + \xi_v' = (1 + \lambda)\xi_v. \quad \xi_a'' = \xi_a + \xi_a' = (1 + \lambda)\xi_a. \quad (3.15a)
\]

\[
\xi'' = \xi + \xi' = (1 + \lambda)\xi. \quad (3.15b)
\]

This is all we need to iterate again the group transformation rules to show that the composition of two successive transformations with parameters \(\xi', \xi''\), ... yields another transformation with parameters given by \(\xi'''' = \xi'' + \xi'''';'\) ....

Concluding, the *Group* law of the transformations eqs- (3-2) has been explicitly proven. Hence, we truly have a Maximal-Acceleration Phase Space Relativity theory. In the next section we shall see why this Relativity theory can be translated as a Minimal Planck-Area Relativity Theory, for *minimal* areas \(L_P^2\). And from Relativity in Phase spaces (cotangent bundles of two-dim Riemann surfaces, for example,) we may have a new understanding of what \(\mathcal{W}\) Geometry, \(W_\infty\) strings, \(\mathcal{W}\)-gravity, Higher conformal spin field theories in Anti de Sitter spaces, .... may be telling us [15, 16, 17, 18]. This is discussed next. Once again, we emphasize that it is important to notice that our definitions for the velocity/acceleration rapidity parameters in eqs- (3-4, 3-5) are very different from those used by [14].

**4. Planck-Scale Areas are Invariant under Acceleration-Boosts**

Having displayed explicitly the Group transformations rules of the coordinates in Phase space we will show why *infinite* acceleration-boosts (which is
not the same as infinite proper acceleration! ) preserve Planck-Scale Areas as a result of the fact that \( b = (1/L_P^2) \) equals the \textit{maximal invariant} Force, or Tension, if the units of \( \hbar = c = 1 \) are used.

At Planck scale \( L_P \) intervals we have by definition ( in units of \( \hbar = c = 1 \) ) :

\[
\Delta X = \Delta T = L_P. \quad \Delta E = \Delta P = \frac{1}{L_P}. \quad b \equiv \frac{1}{L_P^2} = \text{Maximal Tension}. \quad (4.1)
\]

From eqs- ( 3-6 ) we get after a direct use of eq- ( 4-1 ) , in the \textit{infinite} boost limit \( \xi \to \infty \), :

\[
\Delta T' = L_P(\cosh \xi + \sinh \xi) \to \infty. \quad (4.2a)
\]

\[
\Delta E' = \frac{1}{L_P}(\cosh \xi - \sinh \xi) \to 0. \quad (4.2b)
\]

by a simple use of L’Hopital’s rule or by noticing that both \( \cosh \xi; \sinh \xi \) functions approach infinity at the same rate.

\[
\Delta X' = L_P(\cosh \xi - \sinh \xi) \to 0. \quad (4.2c)
\]

\[
\Delta P' = \frac{1}{L_P}(\cosh \xi + \sinh \xi) \to \infty. \quad (4.2d)
\]

where the discrete displacements of two events in Phase Space are defined :

\[
\Delta X = X_2 - X_1 = L_P. \quad \Delta E = E_2 - E_1 = \frac{1}{L_P}.
\]

\[
\Delta T = T_2 - T_1 = L_P \quad \Delta P = P_2 - P_1 = \frac{1}{L_P}. \quad (4.3)
\]

Due to the identity :

\[
\infty \times 0 = (\cosh \xi + \sinh \xi)(\cosh \xi - \sinh \xi) = \cosh^2 \xi - \sinh^2 \xi = 1. \quad (4.4)
\]

one can see from eqs- ( 4.2 ) that the Planck-scale Areas are truly \textit{invariant} under the \textit{infinite} acceleration-boosts \( \xi = \infty \) :
\[ \Delta X^i \Delta P^i = 0 \times \infty = \Delta X \Delta P(\cosh^2 \xi - \sinh^2 \xi) = \Delta X \Delta P = \frac{L_P}{\bar{L}_P} = 1. \quad h = c = 1. \quad (4.5) \]

\[ \Delta T^i \Delta E^i = \infty \times 0 = \Delta T \Delta E(\cosh^2 \xi - \sinh^2 \xi) = \Delta T \Delta E = \frac{L_P}{L_P} = 1. \quad h = c = 1. \quad (4.6) \]

\[ \Delta X^i \Delta T^i = 0 \times \infty = \Delta X \Delta T(\cosh^2 \xi - \sinh^2 \xi) = \Delta X \Delta T = (L_P)^2. \quad (4.7) \]

\[ \Delta P^i \Delta E^i = \infty \times 0 = \Delta P \Delta E(\cosh^2 \xi - \sinh^2 \xi) = \Delta P \Delta E = \frac{1}{L_P^2}. \quad (4.8) \]

It is crucial to emphasize that the invariance property of the minimal Planck-scale Areas ( maximal Tension ) is not an exclusive property of infinite acceleration boosts \( \xi = \infty \), but, as a result of the identity \( \cosh^2 \xi - \sinh^2 \xi = 1 \), for all values of \( \xi \), the minimal Planck-scale Areas are always invariant under any acceleration-boosts transformations !. Meaning physically, in units of \( h = c = 1 \), that the Maximal Tension ( or maximal Force ) \( b = \frac{1}{L_P} \) is a true physical invariant universal quantity ! [ 15 ] . Also we notice that the Phase-space areas, or cells, in units of \( h \), are also invariant ! The pure-acceleration boosts transformations are ” symplectic “.

The infinite acceleration-boosts are closely related to the infinite redshift effects when light signals barely escape Black hole Horizons reaching an asymptotic observer with an infinite redshift factor. The important fact is that the Planck-scale Areas are truly maintained invariant under acceleration-boosts. This could reveal very important information about Black-holes Entropy and Holography. The logarithmic corrections to the Black-Hole Area-Entropy relation were obtained directly from Clifford-algebraic methods in C-spaces [ 1 ], in addition to the derivation of the maximal Planck temperature condition and the Schwarzchild radius in terms of the Thermodynamics of a gas of p-loop-oscillators quanta : area-bits, volume-bits, ... hyper-volume-bits in Planck scale units.
Concluding Remarks

To finalize we make some important remarks pertaining the Conformal group, the physics of branes, $W$ algebras, Higher conformal-spin field theories, ... within the context of Relativity of Phase spaces and the group of minimal Planck-Area Relativity.

The conformal algebra $SO(4,2)$ in four-dimensions can be extracted directly from the $D = 4$ Clifford algebra without the need to recur to $5 - \text{dim}$ hyperboloids embedded in $D = 6$ [1]. The conformal algebra in $D = 4$ is isomorphic to the isometry algebra of $AdS_5$. The conformal group $SO(4,2)$ has 15 generators like the group $SU(1,3)$ has. In fact, pure acceleration boosts play a similar role as conformal-boosts (special conformal transformations) since uniformly-accelerated trajectories in flat space are given by hyperbolas.

The conformal algebra in $D = 2$ is infinite-dimensional and further extensions of the conformal algebra in $D = 2$ exist which are given by $W$ algebras. The latter are deeply ingrained with the algebra of symplectic-diffs (area-preserving). Higher conformal spins $W_\infty, W_{1+\infty}$ - algebras (in $D = 2$) [17, 18] are associated with the area-preserving diffs of a plane and cylinder respectively.

The $SU(\infty)$ algebras are the area-preserving diffs of sphere in a suitable basis dependent limit [19]. Yang-Mills theories in $D = 4$ are conformally invariant and in [21] we have shown why $p$-brane actions can be obtained from a Moyal deformation quantization of (Generalized) $SU(N)$ Yang-Mills theories, where the auxiliary phase space variables required in the Moyal deformation procedure are later identified with the world-volume coordinates of the $p$-branes. The large $N$ limit is equivalent to the classical $\hbar = 1/N \to 0$ of the Moyal-bracket algebra.

Infinite-dimensional extensions of the finite-dim conformal algebra in $D > 2$ exist and were constructed by Vasiliev et al [15]. The construction of higher spin theories on $AdS_D$ spaces can be attained also by a Noncommutative Moyal-like star product deformation of the symplectic algebras (oscillator algebras) in $D > 2$, whose deformation parameter is the inverse of the throat size of $AdS_D$ space.

The fact that $U(1,3)$ is the symmetry group of classical phase space Relativity [14], and that the canonical group $C(1,3)$ contains the oscillator algebra in four dimensions is very appealing [14]. It is is also consistent
with the fact that the $U(2,2)$ tensor-operator algebras of higher conformal-spin field theories on $AdS_D$ spaces have been proposed by Calixto [16] as the higher-dimensional candidates of $W$-like algebras that are essential in the construction of induced conformal gravities in higher dimensions. This would be a generalization of the WZNW (Wess, Zumino, Novikov, Witten) models to higher dimensions.

The $W$-geometry of the cotangent bundle (phase space) of two-dim (complexified) Riemann surfaces [17] was shown to be directly connected to the Fedosov-deformation quantization procedure of symplectic manifolds [20], which is required when the Phase spaces are curved. A curved Phase space would be the extension of General Relativity in ordinary spacetimes. Instead of studying ordinary strings one may be forced to look deeply into $W_\infty$ strings moving in curved backgrounds, like $AdS$ spaces. Noncritical $W_\infty$ bosonic (super) strings are effective 3D theories devoid of BRST anomalies in $D = 27, 11$ dimensions, respectively [20], which coincide with the allegedly critical dimensions of the bosonic (super) membrane. Hence, noncritical $W_\infty$ strings, effective 3D theories, behave like critical membranes [20]. And moreover, they live on the (projective) 3D conformal boundary of $AdS_4$.

In view of all these interesting connections related to the algebras of area-preserving diffs in Phase-Space, it is warranted to study the full C-Phase-Space Relativity Theory, and its algebra, in order to construct a unified theory of all p-branes. We believe that C-space Extended Scale Relativity Theory is the appropriate arena to study the physics of p-loops = closed p-branes, for all values of $p$ [1].

Concluding, it seems that Quantum Gravity is deeply linked to the Geometry of Noncommuting Phase spaces rather than with the naive Quantization of a spacetime, and its metric, in spacetimes of fixed dimension. The dimensions and signatures are also variables in C-space Extended Relativity Theory [1], i.e. all dimensions and signatures are treated on equal footing. Since the notion of a "point" is lost due to the minimal Planck length principle, the notion of fixed dimension and fixed Topology is also naturally lost. Roughly speaking, instead of points living a particular space of fixed dimension, we have "Dimensional" and "Topological" fluctuations/oscillations within all the p-loop oscillations of C-space.

**Acknowledgements**
We are kindly indebted to M. Bowers, H. Rosu and J. Mahecha for their assistance in preparing the manuscript and hospitality in Santa Barbara where this work was completed. We thank S. Low for sending us reference [14].

References

1 - C. Castro, “The programs of the Extended Relativity in C-spaces, towards the physical foundations of String theory”, hep-th/0205065. To appear in the proceedings of the NATO advanced workshop on the nature of time, geometry, physics and perception. Tatranska Lomnica, Slovakia, May 2002. Kluwer Academic Publishers. Noncommutative Quantum Mechanics and Geometry from the quantization of C-spaces”, hep-th/0206181. Maximal-Acceleration Phase Space Relativity from Clifford Algebras hep-th/0208138. Variable Fine structure constant from Maximal-Acceleration Phase space Relativity” hep-th/0210

C. Castro, M. Pavsic: Higher Derivative Gravity and Torsion from the Geometry of C-spaces Phys. Lett. B 539 (2002) 133. hep-th/0110079. The Clifford Algebra of spacetime and the conformal group hep-th/0203194.

2 - L. Nottale, “Fractal Spacetime and Microphysics, towards Scale Relativity”, World Scientific, Singapore, 1992; “La Relativite dans tous ses etats”, Hachette Literature Paris, 1999.

3 - M. Pavsic, “The landscape of Theoretical Physics: A global view from point particles to the brane world and beyond, in search of a unifying principle”, Kluwer Academic Publishers 119, 2001.

4 - E. Caianiello, “Is there a maximal acceleration?”, Lett. Nuovo Cimento 32 (1981) 65.

5 - V. Nesterenko, Class. Quant. Grav. 9 (1992) 1101; Phys. Lett. B 327 (1994) 50; V. Nesterenko, A. Feoli, G. Scarpetta, “Dynamics of relativistic particle”

6-H. Brandt: Contemporary Mathematics 196 (1996) 273. Chaos, Solitons and Fractals 10 (2-3) (1999).

7-M. Pavsic: Phys. Lett B 205 (1988) 231; Phys. Lett B 221 (1989) 264. H. Arodz, A. Sitarz, P. Wegrzyn: Acta Physica Polonica B 20 (1989) 921.

8- F. Schuller: “Born-Infeld Kinematics and corrections to the Thomas precession” hep-th/0207047, Annals of Phys. 299 (2002) 174.
9. A. Chernitskii: “Born-Infeld electrodynamics, Clifford numbers and spinor representations” hep-th/0009121.

10. J. Lukierski, A. Nowicki, H. Ruegg, V. Tolstoy: Phys. Lett 264 (1991) 331. J. Lukierski, H. Ruegg, W. Zakrzewski: Ann. Phys 243 (1995) 90.

G. Amelino-Camelia: Phys. Lett B 510 (2001) 255. Int. J. Mod. Phys D 11 (2002) 35, gr-qc/0012051

11. S. Rama: “Classical velocity in kappa-deformed Poincare algebra and a Maximal Acceleration” hep-th/0209129.

12. J.P. Uzan, “The fundamental constants and their variations: observational status and theoretical motivations” hep-ph/0205340.

13. G. Lambiase, G. Papini, G. Scarpetta: “Maximal Acceleration Corrections to the Lamb Shift of one Electron Atoms” hep-th/9702130.

14. M. Born: Proc. Royal Society A 165 (1938) 291; Rev. Mod. Physics 21 (1949) 463.

S. Low: Jour. Phys A Math. Gen 35 (2002) 5711.

C. Kalman: Can. Jour. Physics 51 (1973) 1573.

15. M. Vasiliev: “Higher Spin Gauge theories, Star Products and AdS spaces” hep-th/9910096.

16. M. Calixto: “Higher \( U(2, 2) \) spin fields and higher-dim \( W \)-gravities ...” hep-th/0102111.

E. Nissimov, S. Pacheva, I. Vaysburd: “\( W_\infty \) Gravity, a geometric approach” hep-th/9207048.

17. C. Hull: Phys. Letts B 269 (1991) 257.

18. P. Bouwkgnet, K. Scouetens: “\( W \) symmetry in Conformal Field Theory” Phys. Reports 223 (1993) 183-276.

E. Segin: “Aspects of \( W_\infty \) symmetry, hep-th/9112025.

19. J. Hoppe: “Quantum theory of a Relativistic Surface: MIT, Ph.D thesis 1982.

20. C. Castro, “\( W \)-geometry from Fedosov deformation quantization: J. Geometry and Physics 33 (2000) 173. J. Chaos, Solitons and Fractals 7 (1996) 711. “A New Realization of Holography” hep-th/0207231.

21. C. Castro: “Branes from Moyal deformation quantization of Generalized Yang-Mills” hep-th/9908115.

S. Ansoldi, C. Castro, E. Spallucci: Class. Quant. Grav 18 (2001) 2865.