NON-HAMILTONIAN ACTIONS WITH ISOLATED FIXED POINTS

SUSAN TOLMAN

Abstract. We construct a non-Hamiltonian symplectic circle action on a closed, connected, six-dimensional symplectic manifold with exactly 32 fixed points.

1. Introduction

Let the circle $S^1 \simeq \mathbb{R}/\mathbb{Z}$ act on a (non-empty) closed, connected symplectic manifold $(M, \omega)$, and let $\xi_M$ be the associated vector field on $M$. The action is symplectic if it preserves the symplectic form, equivalently, if $\iota_{\xi_M} \omega$ is closed. The action is Hamiltonian if there exists a moment map $\Psi : M \rightarrow \mathbb{R}$ satisfying
$$d\Psi = -\iota_{\xi_M} \omega,$$
equivalently, if $\iota_{\xi_M} \omega$ is exact. In this case, we can reduce the number of degrees of freedom by passing to the reduced space $M/\!/S^1 := \Psi^{-1}(t)/S^1$, which is a symplectic orbifold for all regular $t \in \mathbb{R}$. Moreover, the moment map is a perfect Morse-Bott function whose critical set is the fixed set $M^{S^1}$. Therefore, the (equivariant) cohomology and (equivariant) Chern classes of $M$ are largely determined by the fixed set; for example,
$$\sum_i \dim H^i(M; \mathbb{R}) = \sum_i \dim H^i(M^{S^1}; \mathbb{R}).$$

(1.1)

This leads to the following important question: What conditions force a symplectic action to be Hamiltonian? By the discussion above, if $H^1(M; \mathbb{R}) = 0$ then every symplectic action is Hamiltonian. In contrast, equation (1.1) implies that symplectic circle actions with no fixed points, such as the diagonal circle action on the torus $S^1 \times S^1$, are never Hamiltonian. Frankel made the first significant progress towards answering the question by proving that a Kähler circle action on a closed, connected Kähler manifold is Hamiltonian exactly if it has fixed points [Fr]. In contrast, McDuff constructed a non-Hamiltonian symplectic circle action with fixed tori on a closed, connected six dimensional symplectic manifold [Mc]. Since then, no new examples with fixed points have been constructed, but there has been a great deal of work proving that symplectic actions with fixed points must be Hamiltonian when various additional criteria are satisfied [On, Mc, Gin, LO, TWe, Gin, Go05, Go06, Fe, Ki06, Ro, PT, Li, Ja, MPR]. Nevertheless, the following question, asked by McDuff and Salomon in [MS], and often called the “McDuff conjecture”, is open: Does there exist a non-Hamiltonian...
symplectic circle action with isolated fixed points on a closed, connected symplectic manifold? The main goal of this paper is to answer that question in the affirmative. More precisely, we prove the following theorem:

**Theorem 1.** There exists a non-Hamiltonian symplectic circle action with exactly 32 fixed points on a closed, connected, six-dimensional symplectic manifold.

Now let $(M, \omega)$ fulfill the conclusions of Theorem 1. Given $n \geq 3$ and a Hamiltonian circle action on $\mathbb{CP}^{n-3}$, the diagonal action on the product $\mathbb{CP}^{n-3} \times M$ is symplectic but not Hamiltonian. Hence, Theorem 1 has the following corollary.

**Corollary 1.2.** Given $n \geq 3$, there exists a non-Hamiltonian symplectic circle action with exactly $32(n - 2)$ fixed points on a closed, connected, $2n$-dimensional symplectic manifold.

**Remark 1.3.** McDuff proved that a symplectic circle action on a closed, connected, $2n$-dimensional symplectic manifold with $n \leq 2$ is Hamiltonian exactly if it has fixed points. Thus, the example described in Theorem 1 has the lowest possible dimension.

In contrast, there probably exist examples with fewer fixed points. A non-Hamiltonian symplectic circle action on a closed symplectic $2n$-dimensional manifold cannot have exactly one fixed point. However, we can’t rule out the possibility of such an action with two fixed points unless $n \neq 3$. On the other hand, Jang ruled out the possibility of such an action with three fixed points. Moreover, if $n$ is odd the number of fixed points is even.

To elaborate on the example in Theorem 1, we need some terminology. Let a circle act symplectically on a $2n$-dimensional symplectic manifold $(M, \omega)$. Given $p \in M^{S^1}$ there is a unique multiset of integers $\{w_1, \ldots, w_n\}$, called (isotropy) weights, so that the induced symplectic representation of $S^1$ on $T_p M$ is isomorphic to the representation on $(\mathbb{C}^n, \sqrt{-1}/2 \sum_z dz_i / d\bar{z}_i)$ given by $\lambda z = (\lambda^{w_1} z_1, \ldots, \lambda^{w_n} z_n)$. A map $\Psi: M \to S^1$ is a generalized moment map exactly if $\Psi^* (dt) = -\iota_{\xi_t} \omega$. As in the Hamiltonian case, the reduced space $M/\iota S^1 := \Psi^{-1}(t)/S^1$ is a $2n - 2$ dimensional symplectic orbifold for all regular $t \in \mathbb{R}$. The Duistermaat-Heckman function of $M$ is the unique continuous function $\varphi: \Psi(M) \to \mathbb{R}$ satisfying

$$
\varphi(t) = \int_{M/\iota S^1} \omega^{n-1}_t
$$

for all regular $t \in \Psi(M)$, where $\omega_t \in \Omega^2(M/\iota S^1)$ is the reduced symplectic form. A K3 surface is a closed, connected complex surface $(X, I)$ with $H^1(X; \mathbb{R}) = \{0\}$ and trivial canonical bundle. A symplectic form $\sigma \in \Omega^2(X)$ is tamed if $\sigma(v, I(v)) > 0$ for all nonzero tangent vectors $v$; it is Kähler if, additionally, $\sigma(I(v), I(w)) = \sigma(v, w)$ for all tangent vectors $v$ and $w$. In this case, we say that the triple $(X, I, \sigma)$ is a tame (respectively, Kähler) K3 surface. Finally, $\mathbb{Z}_2$ acts holomorphically on the torus $T = \mathbb{C}^2/(\mathbb{Z}^2 + \sqrt{1} \mathbb{Z}^2)$ by the involution $[z] \mapsto [-z]$. The quotient $T/\mathbb{Z}_2$ is a Kummer surface; it is a complex orbifold with exactly 16 isolated singular points with isosity $\mathbb{Z}/(2)$.

**Theorem 1 redux.** There exists a non-Hamiltonian symplectic circle action on a closed, connected, six-dimensional symplectic manifold $(M, \omega)$ with a generalized moment map $\Psi: M \to \mathbb{R}/(4\mathbb{Z}) \simeq S^1$. The level sets $\Psi^{-1}(\pm 1)$ each contain 16
Finally, the reduced space \( M/\!/S^1 \) is symplectomorphic to a tame K3 surface for all \( t \in (1, 3) \) and diffeomorphic to the Kummer surface \( T/\mathbb{Z}_2 \) for all \( t \in (-1, 1) \).

Our proof of Theorem 1 is adapted from [Ko], where Kotschick answered another question of McDuff and Salomon [MS] by constructing a free (and therefore non-Hamiltonian) symplectic circle action on a six dimensional closed symplectic manifold with contractible orbits; see [3] and [Ko]. In particular, many of the ideas in [3] and [Ko] are taken directly from Kotschick’s paper, although we handle the technical details differently. (For example, he considers Kähler K3 surfaces, but we need to allow tame K3 surfaces because the example we construct in [3] is tame but not Kähler.)

**Proof of Theorem 4.** Clearly, \(-4 + 16t - 4t^2 > 0\) for all \( t \in [1, 3] \). Therefore, by Proposition 3.2 there exists a free circle action on a symplectic manifold \( \Psi' : M' \to (1, 3) \) so that, for all \( t \in (1, 3) \), the reduced space \( M'/\!/S^1 \) is symplectomorphic to a tame K3 surface \( (X', I', \sigma_1') \); moreover,

- \((\sigma_1', \sigma_1') = -4 + 16t - 4t^2\); and
- \([\sigma_1'] = k' - \eta'\), where \( k', \eta' \) induce a primitive embedding \( \mathbb{Z}^2 \hookrightarrow H^2(X'; \mathbb{Z}) \).

Let \((M_+, \omega_+, \Psi_+)\) and \((M_-, \omega_-, \Psi_-)\) be the symplectic manifolds with locally free circle actions and proper moment maps described in Proposition 4.1 (For now, ignore the complex structure.) The reduced space \( M_{\pm}/\!/S^1 \) is diffeomorphic to the Kummer surface \( T/\mathbb{Z}_2 \) for all \( t \in \mathbb{R} \). Moreover, the Duistermaat-Heckman function of \( M_{\pm} \) is \( 4 + 4t^2 \); see Remark 4.3. Finally, \( \Psi_{\pm}^{-1}(-1, 1) \subset M_+ \) and \( \Psi_{-}^{-1}(-1, 1) \subset M_- \) are equivariantly symplectomorphic.

Let \((\tilde{M}_+, \tilde{\omega}_+, \tilde{\Psi}_+)\) and \((\tilde{M}_-, \tilde{\omega}_-, \tilde{\Psi}_-)\) be the symplectic manifolds with circle actions and proper moment maps described in Proposition 4.1. Fix \( \epsilon > 0 \) sufficiently small. The preimages \( \tilde{\Psi}_{\pm}(\pm(-\infty, 1 + \epsilon)) \) each contain exactly 16 fixed points; each lies in \( \tilde{\Psi}_{\pm}(\pm1) \) and has weights \( \pm\{2, 1, 1\} \). Additionally, there exist \( a_{\pm} < b_{\pm} \) in \( \pm(0, 1) \) so that \( \Psi_{\pm}^{-1}(a_{\pm}, b_{\pm}) \subset M_{\pm} \) and \( \tilde{\Psi}_{\pm}^{-1}(a_{\pm}, b_{\pm}) \subset \tilde{M}_{\pm} \) are equivariantly symplectomorphic. Finally, for all \( t \in \pm(1, 1 + \epsilon) \), the reduced space \( \tilde{M}_{\pm}/\!/S^1 \) is symplectomorphic to a tame K3 surface \( (\tilde{X}, \tilde{I}, (\tilde{\sigma}_{\pm})_t) \); moreover,

- \( (\tilde{\sigma}_{\pm})_t, \tilde{\sigma}_{\pm} \) \( = -4 \pm 16t - 4t^2 \); and
- \([\tilde{\sigma}_{\pm}]_t = \tilde{k} - t\tilde{\eta}_{\pm}, \) where \( \tilde{k}, \tilde{\eta}_{\pm} \) induce a primitive embedding \( \mathbb{Z}^2 \hookrightarrow H^2(X'; \mathbb{Z}) \).

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1. An embedding \( (\ell_1, \ell_2) \to t_1k' + \ell_2\eta' \) is **primitive** if every lattice element that is a real linear combination of \( k' \) and \( \eta' \) is also an integral linear combination.
2. A Lie group \( G \) acts **locally freely** on \( M \) if the set of \( g \in G \) with non-empty fixed set \( \{x \in M \mid g \cdot x = x\} \) is discrete.
Thus, by Proposition 3.1 they imply that there exist $a'_1 < b'_1$ in $\pm(1, 1 + \epsilon)$ so that
$\Psi_t^{-1}(a'_1, b'_1) \subset M_+$ and $(\Psi')^{-1}(a'_1, b'_1) \subset M'$ are equivariantly symplectomorphic, and also $\Psi_{-1}^{-1}(a'_+, b'_+) \subset \tilde{M}_-$ and $(\Psi')^{-1}(a'_+, 4, b'_+, 4) \subset M'$ are equivariantly symplectomorphic.

Therefore, we can glue together $(\Psi')^{-1}(a'_+, b'_+ + 4) \subset M'$, $\Psi_{-1}^{-1}(a'_-, b'_-) \subset M_-$, $\tilde{\Psi}_{+}^{-1}(a'_+, b'_+) \subset \tilde{M}_+$, and $\tilde{\Psi}_{-}^{-1}(a'_-, b'_-) \subset \tilde{M}_-$ to construct a symplectic circle action on a closed, connected six-dimensional symplectic manifold $M$ with a generalized moment map $\Psi: M \to \mathbb{R}/(4\mathbb{Z})$ satisfying all requirements. In particular, the action is not Hamiltonian because there is no fixed point with all positive (or negative) weights.

The structure of this paper is straightforward. In Section 2 we classify tame K3 surfaces up to symplectomorphism. In Section 3 we use this classification to analyze free Hamiltonian circle actions on symplectic manifolds with reduced spaces symplectomorphic to tame K3 surfaces. Under favorable conditions, these are classified by their Duistermaat-Heckman function. In Section 4 we construct locally free Hamiltonian circle actions on symplectic manifolds with reduced spaces diffeomorphic to the Kummer surface $T/\mathbb{Z}_2$. Finally, in Section 5 we use results from [TWa] to add fixed points to the examples constructed in [I] In that paper, which is joint with Jordan Watts, we extend some important constructions and theorems from the symplectic and Kähler categories to the tame category.

2. K3 SURFACES

In this section, we classify tame K3 surfaces up to symplectomorphism. This is a fairly straightforward consequence of classification of marked Kähler K3 surfaces, which we review, closely following [BPV].

Let $L$ be the K3 lattice, that is, the even unimodular lattice with signature $(3, 19)$; let $L_\mathbb{R} := L \otimes_{\mathbb{Z}} \mathbb{R}$. If $(X, I)$ is a K3 surface, there is an isometry from $H^2(X; \mathbb{Z})$ (with the cup product pairing) to $L$ [BPV, Proposition VIII.3.2], that is, an isomorphism of groups that preserves the symmetric bilinear forms. Additionally, any two K3 surfaces are diffeomorphic [BPV, Corollary VIII.8.6]. Given a vector space $V$ with symmetric bilinear form $(\cdot, \cdot)$ and $k \in \mathbb{Z}$, let $G^k_+ (V)$ denote the manifold of oriented $k$-planes in $V$ on which $(\cdot, \cdot)$ is positive definite. The manifold $G^k_+ (L_k)$ has two components, and so we can state our classification as follows:

**Proposition 2.1.** Let $X$ be a manifold admitting a K3 structure.

1. Given $\kappa \in H^2(X; \mathbb{R})$, there is a tame K3 structure $(I, \sigma)$ on $X$ with $[\sigma] = \kappa$ if and only if $(\kappa, \kappa) > 0$.
2. Given tame K3 structures $(I_0, \sigma_0)$ and $(I_1, \sigma_1)$ on $X$ and an isometry $\phi: H^2(X; \mathbb{Z}) \to H^2(X; \mathbb{Z})$, there is a symplectomorphism $f$ from $(X, \sigma_0)$ to $(X, \sigma_1)$ satisfying $f^* = \phi$ if and only if $\phi([\sigma_1]) = [\sigma_0]$ and $\phi$ preserves the components of $G^k_+(H^2(X; \mathbb{R}))$.

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3 However, since we classify K3 surfaces up to symplectomorphism, our definition of “Kähler K3 surface” includes the Kähler form (see [I]), while [BPV] only include its cohomology class.
To prove this, we begin with a brief review of complex surfaces. Let \((X, I)\) be a complex surface, that is, a two-dimensional complex manifold; assume that \(X\) is closed and connected. Given non-negative integers \(p\) and \(q\) with \(p + q = 2\), the Dolbeault cohomology \(H^{p,q}(X)\) is naturally isomorphic to the subspace of the de Rham cohomology \(H^2(X; \mathbb{C})\) whose elements can be represented by a \(d\)-closed form of type \((p, q)\) [BPV, Theorem IV.2.9]. Moreover, under this identification,

\[
H^2(X; \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X).
\]

A class \(d\) in the Picard lattice \(H^{1,1}(X) \cap H^2(X; \mathbb{Z})\) is effective if there exists an effective divisor \(D\) so that \(c_1(\mathcal{O}_X(D)) = d\). A class \(\kappa \in H^2(X; \mathbb{R})\) is tamed (respectively, Kähler) if it contains a tamed (respectively, Kähler) form. We will need the following fact.

**Lemma 2.2.** Let \((X, I)\) be a closed connected complex surface. Given \(\kappa \in H^2(X; \mathbb{R})\), the orthogonal projection \(\hat{\kappa}\) of \(\kappa\) onto \(H^{1,1}(X; \mathbb{R})\) is tamed exactly if \(\kappa\) is.

**Proof.** By the discussion above, there exists a \(d\)-closed form \(\alpha \in \Omega^{2,0}(X)\) so that \(\kappa = \hat{\kappa} + [\alpha + \overline{\alpha}]\). Moreover, \(\alpha(v, I(v)) = \overline{\alpha}(v, I(v)) = 0\) for all \(\alpha \in \Omega^{2,0}(X)\) and all tangent vectors \(v\).

Now assume that \(\dim H^1(X; \mathbb{R})\) is even, and define

\[
H^{1,1}(X; \mathbb{R}) := H^{1,1}(X) \cap H^2(X; \mathbb{R}).
\]

Because the signature of \(H^{1,1}(X; \mathbb{R})\) is \(1, \dim H^{1,1}(X; \mathbb{R}) - 1\) [BPV, Theorem IV.2.13], the set

\[
\{ x \in H^{1,1}(X; \mathbb{R}) \mid (x, x) > 0 \}
\]

consists of two disjoint connected cones.

Finally, assume that \((X, I)\) admits a Kähler form. Since the Kähler classes form a convex subcone of \(\mathbb{R}^{2}\), they lie in one of the two cones; we call it the positive cone and denote it by \(C_X\).

Now let \((X', I')\) be another complex surface satisfying the assumptions above. A **Hodge isometry** is an isometry \(\phi: H^2(X'; \mathbb{Z}) \to H^2(X; \mathbb{Z})\) that preserves the Hodge decomposition, that is,

\[
\phi(H^{p,q}(X')) = H^{p,q}(X) \quad \text{for all } p + q = 2.
\]

A Hodge isometry \(\phi\) is effective if it preserves the positive cones and induces a bijection between the respective sets of effective classes.

By definition, every K3 surface \((X, I)\) is a closed, connected complex surface with \(\dim H^1(X; \mathbb{R}) = 0\) even. Moreover, Siu proved that every K3 surface admits a Kähler form [BPV, Theorem VIII.14.5]. Therefore, we can restate [BPV, Theorem VIII.11.1] as follows:

**Theorem 2.4** (Torelli theorem). Let \((X, I)\) and \((X', I')\) be K3 surfaces. Given an effective Hodge isometry \(\phi: H^2(X'; \mathbb{Z}) \to H^2(X, \mathbb{Z})\), there exists a biholomorphism \(g: X \to X'\) such that \(g^* = \phi\).

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4 By a slight abuse of notation, we identify \(H^2(X; \mathbb{Z})\) with its image in \(H^2(X; \mathbb{R})\) and \(H^2(X; \mathbb{R})\) with its image in \(H^2(X; \mathbb{C})\). Since the cohomology of K3 surfaces is torsion-free, we trust that this will not cause confusion.

5 Where convenient, we identify \(\phi: H^2(X'; \mathbb{Z}) \to H^2(X; \mathbb{Z})\) with its \(\mathbb{R}\)-linear and \(\mathbb{C}\)-linear extensions.
This has the following implication for tame K3 surfaces.

**Corollary 2.5.** Let \((X, I, \sigma)\) and \((X', I', \sigma')\) be tame K3 surfaces. Given a Hodge isometry \(\phi: H^2(X'; \mathbb{Z}) \to H^2(X; \mathbb{Z})\) such that \(\phi([\sigma']) = [\sigma]\), there exists a biholomorphism \(g: X \to X'\) such that \(g^* = \phi\).

**Proof.** By Lemma 2.2, there exists tamed symplectic forms \(\bar{\sigma} \in \Omega^2(X)\) and \(\bar{\sigma}' \in \Omega^2(X')\) so that \([\bar{\sigma}]\) and \([\bar{\sigma}']\) are the orthogonal projection of \([\sigma]\) and \([\sigma']\) onto \(H^{1,1}(X; \mathbb{R})\) and \(H^{1,1}(X'; \mathbb{R})\), respectively. Since \(\phi([\sigma']) = [\sigma]\) and \(\phi\) is a Hodge isometry, \(\phi([\bar{\sigma}']) = [\bar{\sigma}]\). Following [BPV] VIII.3, we let the *Kähler cone* of \(X\) be the set of \(x\) in the positive cone \(C_X\) such that \((x, d) > 0\) for all effective classes \(d\). Since the tamed classes in \(H^{1,1}(X; \mathbb{R})\) form a convex subcone of the set \((2.3)\), the class \([\bar{\sigma}]\) must lie in \(C_X\). Moreover, since \(\bar{\sigma}\) is tamed, \((\bar{\sigma}, d) > 0\) for all effective classes \(d\). Therefore, \([\bar{\sigma}]\) lies in the Kähler cone of \(X\). By a similar argument, \([\bar{\sigma}']\) lies in the Kähler cone of \(X'\). Thus, \(\phi\) is effective by Theorem VIII.3.10 in [BPV].

Consider the complex manifold

\[
\Omega := \{(\alpha) \in \mathbb{P}(L \otimes \mathbb{C}) \mid (\alpha, \alpha) = 0 \text{ and } (\alpha, \bar{\alpha}) > 0\}.
\]

A **marked K3 surface** is a K3 surface \((X, I)\) and an isometry \(H^2(X; \mathbb{Z}) \to L\). Since the canonical bundle of \(X\) is trivial, \(H^{2,0}(X) \simeq \mathbb{C}\). Thus, there is a well-defined **period point** associated to \((X, I, \phi)\):

\[
\tau_1(X, I, \phi) = [\phi(\alpha_X)] \in \Omega,
\]

where \(\alpha_X \in H^{2,0}(X)\) is any non-zero class. Additionally, there is a **refined period point** associated to each marked tame K3 surface \((X, I, \sigma, \phi)\):

\[
\tau_2(X, I, \sigma, \phi) = (\phi([\sigma]), \tau_1(X, I, \phi)) \in L_{\mathbb{R}} \times \Omega,
\]

where \(\tau_1\) is defined as in \((2.6)\).

Marked tame K3 surfaces \((X, I, \sigma, \phi)\) and \((X', I', \sigma', \phi')\) are **isomorphic** if there exists a marked biholomorphism \(g: X \to X'\) with \(g^*([\sigma']) = [\sigma]\). Here, we say that a diffeomorphism \(g: X \to X'\) is **marked** if \(g^* = \phi^{-1} \circ \phi'\). The composition \(\phi^{-1} \circ \phi': H^2(X'; \mathbb{R}) \to H^2(X; \mathbb{R})\) is a Hodge isometry that takes \([\sigma']\) to \([\sigma]\) exactly if \(\tau_2(X, I, \sigma, \phi) = \tau_2(X', I', \sigma', \phi')\). Hence, Corollary 2.5 can be reformulated to show that isomorphism classes of marked tame K3 surfaces are classified by the image on the refined period map.

**Corollary 2.8.** Two marked tame K3 surfaces \((X, I, \sigma, \phi)\) and \((X', I', \sigma', \phi')\) are isomorphic exactly if

\[
\tau_2(X, I, \sigma, \phi) = \tau_2(X', I', \sigma', \phi').
\]

Our next goal it to describe the image or the refined period map. Define manifolds

\[
\tilde{\Omega} := \{(\kappa, [\alpha]) \in L_{\mathbb{R}} \times \Omega \mid (\kappa, \kappa)(\alpha, \bar{\alpha}) > 2|\kappa, \alpha|^2\}
\]

and

\[
\Omega := \{(\kappa, [\alpha]) \in L_{\mathbb{R}} \times \Omega \mid (\kappa, \kappa) > 0 \text{ and } (\kappa, \alpha) = 0\} \subset \tilde{\Omega}.
\]

Given \(\alpha \in L \otimes \mathbb{C}\), write \(\alpha = \Re(\alpha) + \sqrt{-1}\Im(\alpha)\), where \(\Re(\alpha), \Im(\alpha) \in L_{\mathbb{R}} := L \otimes \mathbb{R}\). Then \([\alpha] \in \Omega\) exactly if

\[
(\Re(\alpha), \Im(\alpha)) = 0 \text{ and } (\Re(\alpha), \Re(\alpha)) = (\Im(\alpha), \Im(\alpha)) = (\alpha, \bar{\alpha})/2 > 0.
\]
Hence, given $\kappa \in L_\mathbb{R}$ and $[\alpha] \in \Omega$, the orthogonal projection $\tilde{\kappa}$ of $\kappa$ onto $\alpha^\perp \cap L_\mathbb{R}$ is

\begin{equation}
\tilde{\kappa} = \kappa - \frac{(\kappa, \Re(\alpha))}{(\Re(\alpha), \Re(\alpha))} \Re(\alpha) - \frac{(\kappa, \Im(\alpha))}{(\Im(\alpha), \Im(\alpha))} \Im(\alpha),
\end{equation}

and so

\begin{equation}
(\tilde{\kappa}, \hat{\kappa}) = (\kappa, \kappa) - \frac{(\kappa, \Re(\alpha))^2}{(\Re(\alpha), \Re(\alpha))} - \frac{(\kappa, \Im(\alpha))^2}{(\Im(\alpha), \Im(\alpha))} = (\kappa, \kappa) - 2\frac{(\kappa, \alpha)^2}{(\alpha, \alpha)}.
\end{equation}

Therefore, $\kappa, \Re(\alpha)$, and $\Im(\alpha)$ are linearly independent for every $(\kappa, [\alpha]) \in \tilde{K}\Omega$.

Define $\Pi: \tilde{K}\Omega \to G_3^+(L_\mathbb{R})$ by letting $\Pi(\kappa, [\alpha])$ be the oriented three plane spanned by the oriented basis $\{\kappa, \Re(\alpha), \Im(\alpha)\}$. Define spaces

$$G_3^+ (L_\mathbb{R})^\circ := \{ V \in G_3^+ (L_\mathbb{R}) \mid d \notin V^\perp \text{ for any } d \in L \text{ with } (d, d) = -2 \},$$

$$\tilde{(K\Omega)^\circ} := \{ (\kappa, [\alpha]) \in \tilde{K}\Omega \mid \Pi(\kappa, [\alpha]) \in G_3^+(L_\mathbb{R})^\circ \},$$

and

$$\tilde{K}\Omega^\circ := \{ (\kappa, [\alpha]) \in K\Omega \mid \Pi(\kappa, [\alpha]) \in G_3^+(L_\mathbb{R})^\circ \} \subset \tilde{(K\Omega)^\circ}.$$

Given $(\kappa, [\alpha]) \in L_\mathbb{R} \times \Omega$, let $\tilde{\kappa}$ be the orthogonal projection of $\kappa$ onto $\alpha^\perp \cap L_\mathbb{R}$. By \eqref{eq:2.9} and \eqref{eq:2.10},

\begin{equation}
(\kappa, [\alpha]) \in \tilde{K}\Omega \Leftrightarrow (\tilde{\kappa}, [\alpha]) \in K\Omega \text{ and } (\kappa, [\alpha]) \in \tilde{K}\Omega^\circ \Leftrightarrow (\tilde{\kappa}, [\alpha]) \in (K\Omega)^\circ.
\end{equation}

Our next step is to prove isomorphism classes of marked tame K3 surfaces are classified by $(K\Omega)^\circ$. To do this, we show that the image of the refined period map is a subset of $(K\Omega)^\circ$ in Lemma \ref{lem:2.12} and that it is surjective in Corollary \ref{cor:2.14}. In fact, we prove a stronger claim; roughly, the period map is “surjective for paths”. This reflects the fact that $(K\Omega)^\circ$ does not just classify isomorphism classes of marked Kähler K3 surfaces; it is the moduli space marked Kähler K3 surfaces. (See the proof of Theorem \ref{thm:2.13}.)

\begin{lemma}
Let $(X, I, \sigma, \phi)$ be a marked tame K3 surface. Then $\tau_2(X, I, \sigma, \phi) \in (K\Omega)^\circ$; moreover, if $\sigma$ is Kähler then $\tau_2(X, I, \sigma, \phi) \in (K\Omega)^\circ$.
\end{lemma}

\begin{proof}
Note that $H^{1,1}(X; \mathbb{R}) = \alpha_X^1 \cap H^2(X; \mathbb{R})$ for any non-zero $\alpha_X \in H^{2,0}(X)$. Hence, if $\sigma$ is Kähler then $[\sigma] \in \alpha_X^1$, and so $\tau_2(X, I, \sigma, \phi) \in K\Omega$. Similarly, returning to the general case, Lemma \ref{lem:2.12} and \eqref{eq:2.11} together imply that $\tau_2(X, I, \sigma, \phi) \in \tilde{K}\Omega$. Now consider $d \in H^2(X; \mathbb{Z})$ with $(d, d) = -2$. If $d \in \alpha_X^1$, then since $d \in H^{1,1}(X) \cap H^2(X; \mathbb{Z})$ either $d$ or $-d$ is effective \cite[Proposition VIII.3.6.i]{BPV}, and so $d \notin [\sigma]^\perp$. Therefore, $\tau_2(X, I, \sigma, \phi)$ lies in $(K\Omega)^\circ$.
\end{proof}

\begin{theorem}
Given a smooth path $\gamma: [0, 1] \to (K\Omega)^\circ$, there exist marked Kähler K3 surfaces $(X, I_\ell, \sigma_\ell, \phi_\ell)$ satisfying

$$\tau_2(X, I_\ell, \sigma_\ell, \phi) = \gamma_\ell \text{ for all } t \in [0, 1];$$

moreover, $I_\ell$ depends smoothly on $t$.
\end{theorem}

\footnote{As the notation indicates, the complex structure $I_\ell$ and the Kähler form $\sigma_\ell$ depend on $t \in [0, 1]$, but the manifold $X$ and the isometry $\phi$ do not. At this point, we do not insist that $\sigma_\ell$ depends smoothly on $t$; see Lemma \ref{lem:2.14}.}
Proof. By Theorem VIII.12.1 in \[BPV\], there exists a universal marked family \(Y \overset{s} \rightarrow M_1\) of K3-surfaces. In particular, given \(x \in M_1\), the preimage \(Y_s := p^{-1}(s)\) is a marked K3 surface. The base space \(M_1\) is not Hausdorff, but otherwise is a smooth analytic space of dimension 20. The period map \(\tau_{M} : M_1 \rightarrow \Omega\) is an open submanifold. Thus, it induces a diffeomorphism from both injective \([BPV\text{, Theorem VIII.12.3}]\) and surjective \([BPV\text{, Theorem VIII.14.1}]\).

By Lemma VIII.9.3 in \[BPV\], \(M_2 := \bigcup_{s \in M_1} H^{1,1}(Y_s) \overset{s} \rightarrow M_1\) is a real analytic vector bundle. As described in \[BPV\ §VIII.12\], the set of Kähler classes \(M_2\) in \(M_2\) is an open submanifold. Thus, \(M_2\) is real analytic manifold of dimension 60, and the refined period map \(\tau_2 : M_2 \rightarrow (K\Omega)^{\circ}\) is a real analytic submersion. Since \(\tau_2\) is both injective \([BPV\text{, Theorem VIII.12.3}]\) and surjective \([BPV\text{, Theorem VIII.14.1}]\), it induces a diffeomorphism from \(M_2\) to \((K\Omega)^{\circ}\).

Given a smooth path \(\gamma : [0, 1] \rightarrow (K\Omega)^{\circ}\), let \(\gamma_t = (\kappa_t, [\alpha_t])\). Since \(\pi \circ \tau^{-1}_2 \circ \gamma\) is a smooth path in \(M_1\), there exists marked K3 surfaces \((X, I_t, \phi)\) satisfying \(\tau_1(X, I_t, \phi) = [\alpha_t] \in \Omega\) for all \(t \in [0, 1]\). Since \(\tau^{-1}_2 \circ \gamma\) is a path in \(M_2\), there exists a Kähler form \(\sigma_t \in \Omega^2(X)\) with \(\phi[\sigma_t] = \kappa_t\) for all \(t \in [0, 1]\).

This has the following implication for tame K3 surfaces.

**Corollary 2.14.** Given a smooth path \(\gamma : [0, 1] \rightarrow (K\Omega)^{\circ}\), there exist marked tame K3 surfaces \((X, I_t, \sigma_t, \phi)\) with

\[
\tau_2(X, I_t, \sigma_t, \phi) = \gamma_t \quad \text{for all} \quad t \in [0, 1];
\]

moreover, \(I_t\) depends smoothly on \(t\).

**Proof.** Let \(\tilde{\gamma}_t = (\tilde{\kappa}_t, [\alpha_t])\) for all \(t \in [0, 1]\), where \(\tilde{\kappa}_t = (\kappa_t, [\alpha_t])\) and \(\tilde{\kappa}_t\) is the projection of \(\kappa_t\) onto \(\alpha_t \cap L_R\). Then \(\tilde{\gamma}_t\) is a smooth path in \((K\Omega)^{\circ}\) by \[2.14\]. By Theorem \[2.13\] there exists marked Kähler K3 surfaces \((X, I_t, \tilde{\sigma}_t, \phi)\) satisfying \(\tau_2(X, I_t, \tilde{\sigma}_t, \phi) = (\tilde{\kappa}_t, [\alpha_t])\) for all \(t \in [0, 1]\); moreover, \(I_t\) depends smoothly on \(t\). Hence, the claim follows by Lemma \[2.2\].

We have now classified marked tame K3 surfaces up to marked biholomorphism that preserve the tame class. To classify them up to marked symplectomorphism, we need to study the topology of \((K\Omega)^{\circ}; c.f. \[BPV\text{, VIII.9}]\).

**Lemma 2.15.** Fix \(\kappa \in L_R\) with \((\kappa, \kappa) > 0\).

1. The subspace \((\tilde{K}\Omega)^{\circ} \subset \tilde{K}\Omega\) is an open submanifold with two connected components that \(\Pi\) take to different components of \(G_2^+(L_R)\).

2. The intersection \((\{\kappa\} \times \Omega) \cap (\tilde{K}\Omega)^{\circ}\) is a submanifold with two connected components that \(\Pi\) take to different components of \(G_2^+(L_R)\).

**Proof.** The map from \(\Omega\) to \(G_2^+(L_R)\) that sends \([\alpha]\) to the oriented 2-plane with oriented basis \(\{R\alpha, \Im \alpha\}\) is a diffeomorphism, and so we may identify these spaces. By Lemma \[2.11\] under this identification \(\tilde{K}\Omega\) is the set of pairs \((\kappa, W) \in L_R \times G_2^+(L_R)\) that together span a 3-plane on which \((\cdot, \cdot)\) is positive definite. Given \(V \in G_3^+(L_R)\), the preimage \(\Pi^{-1}(V) \subset \tilde{K}\Omega\) consists of \((\kappa, W) \in V \times G_2^+(V)\) with \(\kappa \notin W\) such that \(\kappa\) and \(W\) together induce the given orientation on \(V\). If we fix a non-zero \(\kappa \in V\), the set of such \(W\) is diffeomorphic to a 2-dimensional disk. Thus \(\tilde{K}\Omega\) is a fiber bundle over \(G_2^+(L_R)\) with connected fibers.

We obtain \(G_3^+(L_R)^{\circ}\) from \(G_3^+(L_R)\) by removing \(G_2^+(d^-)\) for all \(d \in L\) with \((d, d) = -2\). Given any \(d \in L\), the set \(G_2^+(d^-)\) is a closed submanifold of \(G_3^+(L_R)\) of
codimension 3. Moreover, by the proof of [BPV] Corollary VIII.9.2, the collection of $G_3^+(d^-)$ for $d \in L$ with $(d, d) = -2$ is locally finite. Therefore, $G_3^+(L_R)$ is an open submanifold with two connected components, one in each component of $G_3^+(L_R)$. The first claim follows immediately.

Now fix $\kappa \in L_R$ with $(\kappa, \kappa) > 0$. Given $W \in G_2^+(L_R)$ with $(\kappa, W) \in \widehat{K}\Omega$, it is clear that $\kappa \in \Pi(\kappa, W)$. Conversely, as we saw above, if $V \in G_3^+(L_R)$ contains $\kappa$ then

\[
\{ W \in G_2^+(L_R) \mid (\kappa, W) \in \widehat{K}\Omega \text{ and } \Pi(\kappa, W) = V \}
\]

is diffeomorphic to a 2-dimensional disk. Thus, $(\{\kappa\} \times G_2^+(L_R)) \cap \widehat{K}\Omega$ is a fiber bundle over $\{ V \in G_2^+(L) \mid \kappa \in V \}$ with contractible fibers.

Since the signature of $\kappa^\perp$ is $(2, 19)$, the manifold $G_2^+(\kappa^\perp)$ has two components. Moreover, the natural diffeomorphism from $G_2^+(\kappa^\perp)$ to $\{ V \in G_3^+(L_R) \mid \kappa \in V \}$ takes these components to different components of $G_3^+(L_R)$. Given $d \in L$, the intersection

\[
\{ V \in G_3^+(L_R) \mid \kappa \in V \} \cap G_3^+(d^-)
\]

is empty if $(\kappa, d) \neq 0$, and is naturally diffeomorphic to $G_2^+(\kappa^\perp \cap d^-)$ if $(\kappa, d) = 0$. Therefore, it is a closed submanifold of codimension 2. Since we have already seen that the collection of $G_3^+(d^-)$ for $d \in L$ with $(d, d) = -2$ is a locally finite collection, this implies that $\{ V \in G_3^+(L_R) \mid \kappa \in V \}$ is a submanifold whose intersection with either component of $G_3^+(L_R)$ has one connected component. The second claim follows immediately.

\[\square\]

We also need a few general lemmas about tamed symplectic forms.

**Lemma 2.16.** Let $(X, I)$ and $(X', I')$ be closed complex manifolds with tamed symplectic forms $\sigma \in \Omega^2(X)$ and $\sigma' \in \Omega^2(X')$. Given a biholomorphism $g$ from $(X, I)$ to $(X', I')$ with $g^*([\sigma']) = [\sigma]$, there is a symplectomorphism $f$ from $(X, \sigma)$ to $(X', \sigma')$ with $f^* = g^*: H^*(X'; \mathbb{Z}) \to H^*(X; \mathbb{Z})$.

**Proof.** The complex structure $I$ tames $g^*(\sigma')$ and $\sigma$, and so $\sigma_t :=tg^*(\sigma')+(1-t)\sigma$ is a tamed symplectic form for all $t \in [0, 1]$. Thus, $[g^*(\sigma')] = [\sigma]$ by assumption, the claim follows by Moser’s method. \[\square\]

**Lemma 2.17.** Let $(X, I_t)$ be a closed complex manifold with tamed symplectic form $\sigma_t \in \Omega^2(X)$ for all $t \in [0, 1]$. Assume that $I_t$ depends smoothly on $t$, and that $[\sigma_t] \in H^2(X; \mathbb{R})$ is independent of $t$. Then there is a symplectomorphism from $(X, \sigma_0)$ to $(X, \sigma_1)$ that induces the identity map on $H^*(X; \mathbb{Z})$.

**Proof.** Fix $s \in [0, 1]$. Since $I_s$ tames $\sigma_s$, $I_t$ depends smoothly on $t$, and $X$ is closed, there exists an open neighborhood $V_s$ of $s \in [0, 1]$ so that $I_t$ tames $\sigma_s$ for all $t \in V_s$. Choose a partition of unity $\{ \rho_s \}$ subordinate to $\{ V_s \}$ so that $\rho_0(0) = 1$ and $\rho_1(1) = 1$. Then $\sigma'_t := \sum_s \rho_s(t)\sigma_s$ is tamed by $I_t$ (and hence is symplectic) for all $t \in [0, 1]$. Since $\sigma'_t = \sigma_t$ for $i = 0, 1$, the claim now follow by Moser’s method. \[\square\]

We are now ready to classify marked tame K3 surfaces that are taken by $\Pi \circ \tau_2$ to a given component of $G_3^+(L_R)$, up to marked symplectomorphism.

**Lemma 2.18.**
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(1) Given $\kappa \in \mathbb{L}_\mathbb{R}$ and a component of $G^+_3(L_\mathbb{R})$, there exists a marked tame K3 surface $(X, I, \sigma, \phi)$ with $\phi([\sigma]) = \kappa$ and $\Pi_2(X, I, \sigma, \phi)$ in the given component if and only if $(\kappa, \kappa) > 0$.

(2) Given marked tame K3 surfaces $(X_0, I_0, \sigma_0, \phi_0)$ and $(X_1, I_1, \sigma_1, \phi_1)$ so that $\Pi_2(X_0, I_0, \sigma_0, \phi_0)$ and $\Pi_2(X_1, I_1, \sigma_1, \phi_1)$ lie in the same component of $G^+_3(L_\mathbb{R})$, there is a marked symplectomorphism from $(X_0, \sigma_0, \phi_0)$ to $(X_1, \sigma_1, \phi_1)$ if and only if $\phi_0([\sigma_0]) = \phi_1([\sigma_1])$.

Proof. Fix $\kappa \in \mathbb{L}_\mathbb{R}$. Since the cup product pairing is positive on tamed classes, we may assume that $(\kappa, \kappa) > 0$. Hence, claim (1) follows immediately from Lemma 2.15 (2) and Corollary 2.14.

Let $(X_0, I_0, \sigma_0, \phi_0)$ and $(X_1, I_1, \sigma_1, \phi_1)$ be marked tame K3 surfaces so that $\Pi_2(X_0, I_0, \sigma_0, \phi_0)$ and $\Pi_2(X_1, I_1, \sigma_1, \phi_1)$ lie in the same component of $G^+_3(L_\mathbb{R})$; assume that $\phi_0([\sigma_0]) = \phi_1([\sigma_1])$.

By Lemmas 2.12 and 2.15 (2), there is a smooth path $\gamma: [0,1] \to (\{\phi_0([\sigma_0])\} \times \Omega) \cap (\tilde{K\Omega})^\circ$ with

$$\tau_2(X_i, I_i, \sigma_i, \phi_i) = \gamma_i \text{ for } i = 0, 1.$$

By Corollary 2.14 there exists marked tame K3 surfaces $(X', I'_t, \sigma'_t, \phi'_t)$ with

$$\tau_2(X', I'_t, \sigma'_t, \phi'_t) = \gamma_t \text{ for all } t \in [0,1];$$

moreover, $I'_t$ depends smoothly on $t$. Hence, by Lemma 2.17 implies that there is a marked symplectomorphism $f'$ from $(X', \sigma'_0, \phi'_0)$ to $(X', \sigma'_1, \phi'_1)$. By Corollary 2.8 (2.19) and (2.20) imply that there exist marked biholomorphism $g_i$ from $(X_i, I_i, \sigma_i, \phi_i)$ to $(X', I'_t, \sigma'_t, \phi'_t)$ for $i = 0, 1$. Thus, by Lemma 2.16 there is marked symplectomorphism $f_i$ from $(X_i, \sigma_i, \phi_i)$ to $(X', \sigma'_i, \phi'_i)$ for $i = 0, 1$. Then $f^{-1}_i \circ f' \circ f_0$ is a marked symplectomorphism from $(X, \sigma_0, \phi_0)$ to $(X, \sigma_1, \phi_1)$. This proves claim (2).

Technically, we could end Section 2 here and still prove Theorem 1. More precisely, we could prove Proposition 3.1 and Proposition 3.2 by using parts (2) and (1) of Lemma 2.18 respectively, instead of the corresponding parts of Proposition 2.1.

On the other hand, the theorem below – which is a slight reformulation of a theorem of Donaldson – immediately leads to a proof of Proposition 2.1. Additionally, it completes the classification of marked tame K3 surfaces started in Lemma 2.18 by showing that there are no marked symplectomorphisms between two marked tame K3 surfaces if $\Phi \circ \tau_2$ takes them to different components of $G^+_3(L_\mathbb{R})$.

**Theorem 2.21.** Let $(X_0, I_0, \sigma_0, \phi_0)$ and $(X_1, I_1, \sigma_1, \phi_1)$ be marked tame K3 surfaces. Then there is a marked diffeomorphism from $(X_0, \phi_0)$ to $(X_1, \phi_1)$ if and only if $\Pi_2(X_0, I_0, \sigma_0, \phi_0)$ and $\Pi_2(X_1, I_1, \sigma_1, \phi_1)$ lie in the same component of $G^+_3(L_\mathbb{R})$.

Proof. Assume first that $\Pi_2(X_0, I_0, \sigma_0, \phi_0)$ and $\Pi_2(X_1, I_1, \sigma_1, \phi_1)$ lie in the same component of $G^+_3(L_\mathbb{R})$. By Lemmas 2.12 and 2.15 (1), there is a smooth path $\gamma: [0,1] \to (\tilde{K\Omega})^\circ$ with

$$\tau_2(X_i, I_i, \sigma_i, \phi_i) = \gamma_i \text{ for } i = 0, 1.$$

Hence, by Corollaries 2.14 and 2.8 there exists marked tame K3 surfaces $(X', I'_t, \sigma'_t, \phi'_t)$ for all $t \in [0,1]$, and marked biholomorphisms $g_i$ from $(X_i, I_i, \sigma_i, \phi_i)$ to $(X', I'_t, \sigma'_t, \phi'_t)$ for $i = 0, 1$. Then $g^{-1}_i \circ g_0$ is a marked diffeomorphism from $(X, \phi_0)$ to $(X, \phi_1)$. 
Now assume that there is a marked diffeomorphism from \((X_0, \phi_0)\) to \((X_1, \phi_1)\). Donaldson proves that there is no diffeomorphism \(f: X_0 \to X_0\) with \(f^*: H^2(X_0; \mathbb{Z}) \to H^2(X_0; \mathbb{Z}) = 1\). Hence, there is no marked diffeomorphism from \((X_0, \phi_0)\) to \((X_1, \phi_1)\). By the previous paragraph, this implies that \(\Pi(\tau_2(X_0, I_0, \sigma_0, \phi_0))\) and \(\Pi(\tau_2(X_1, I_1, \sigma_1, \phi_1))\) lie in different components of \(G_3^+(L_R)\), i.e., \(\Pi(\tau_2(X_0, I_0, \sigma_0, \phi_0))\) and \(\Pi(\tau_2(X_1, I_1, \sigma_1, \phi_1))\) lie in the same component.

**Proof of Proposition 2.1.** Fix \(\kappa \in H^2(X; \mathbb{R})\). As in Lemma 2.18 we may assume that \((\kappa, \kappa) > 0\). By Siu’s theorem, there exists a marked tame K3 structure \((I, \sigma, \phi)\) on \(X\). By Lemma 2.18(1), there exists a marked tame K3 surface \((X', I', \sigma', \phi')\) with \(\phi'(\sigma') = \phi(\kappa)\) so that \(\Pi(\tau_2(X, I, \sigma, \phi))\) and \(\Pi(\tau_2(X, I', \sigma', \phi'))\) lie in the same component of \(G_3^+(L_R)\). Thus, Theorem 2.21 implies that there exists a marked diffeomorphism \(f\) from \((X, \phi)\) to \((X, \phi')\). The pullback \((X, f^*(I'), f^*(\sigma'))\) is a tame K3 surface with \(\phi(\lfloor f^*(\sigma') \rfloor) = \phi'(\lfloor \sigma' \rfloor) = \phi(\kappa)\). This proves claim (1).

Let \((I_0, \sigma_0)\) and \((I_1, \sigma_1)\) be tame K3 structures on \(X\), and consider an isometry \(\phi: H^2(X; \mathbb{Z}) \to H^2(X; \mathbb{Z})\). Given a marking \(\phi_0: H^2(X; \mathbb{Z}) \to L\), let \(\phi_1 = \phi_0 \circ \phi\). Then \(\Pi(\tau_2(X, I_0, \sigma_0, \phi_0))\) and \(\Pi(\tau_2(X, I_0, \sigma_0, \phi_1))\) lie in the same component of \(G_3^+(L_R)\) exactly if \(\phi\) preserves the components of \(G_3^+(L_R)\). Hence, Theorem 2.21 implies that there is no diffeomorphism \(f\) with \(f^* = \phi\) if \(\phi\) reverses the components. On the other hand, Theorem 2.21 implies that \(\Pi(\tau_2(X, I_0, \sigma_0, \phi_1))\) and \(\Pi(\tau_2(X, I_1, \sigma_1, \phi_1))\) lie in the same component of \(G_3^+(L_R)\). Hence, Claim (2) follows immediately from Lemma 2.18(2). \(\square\)

### 3. Free Hamiltonian actions

In this section, we analyze free Hamiltonian circle actions on symplectic manifolds with reduced spaces symplectomorphic to tame K3 surfaces (that also satisfy a technical condition). In this case, the Duistermaat-Heckman function is a positive polynomial of degree at most two with even coefficients. Our main result is that these polynomials classify such symplectic manifolds.

**Proposition 3.1.** Let the circle act freely on symplectic manifolds \((M, \omega)\) and \((M', \omega')\) with proper moment maps \(\Psi: M \to (a, b)\) and \(\Psi': M' \to (a, b)\). Assume that, for all \(t \in (a, b)\), the reduced spaces \(M/\mathbb{Z} S^1\) and \(M'/\mathbb{Z} S^1\) are symplectomorphic to tame K3 surfaces \((X, I_t, \sigma_t)\) and \((X', I'_t, \sigma'_t)\); moreover,

- \((\sigma_1, \sigma_t) = (\sigma'_1, \sigma'_t)\); and
- \(|\sigma| = \kappa - t \eta\) and \(|\sigma'| = \kappa' - t \eta'\), where \(\kappa, \eta, \kappa', \eta'\) induce primitive embeddings \(\mathbb{Z} \to H^2(X; \mathbb{Z})\) and \(\mathbb{Z} \to H^2(X'; \mathbb{Z})\).

Then every \(t \in (a, b)\) has a neighborhood \(U\) so that \(\Psi^{-1}(U)\) and \((\Psi')^{-1}(U)\) are equivariantly symplectomorphic.

In the situation described in Proposition 3.1, the Duistermaat-Heckman function of \(M\) and \(M'\) at \(t \in (a, b)\) is \((\kappa, \kappa) - 2t(\kappa, \eta) + t^2(\eta, \eta)\). Since the K3 lattice is even and \(\kappa\) and \(\eta\) are integral, this is a polynomial of degree at most two with even coefficients; it is positive on \((a, b)\).

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7 Alternatively, this is the original statement of Donaldson’s theorem [Do].
8 By Ehresmann’s lemma, we can (and do) choose the symplectomorphisms \(M/\mathbb{Z} S^1 \to X\) to induce a diffeomorphism \(M/S^1 \to X \times (a, b)\). Hence, the forms \(\sigma_t\) depend smoothly on \(t\). However, the complex structures \(I_t\) may not.
Proposition 3.2. Fix a polynomial \( P \) of degree at most two with even coefficients that is positive on \([a,b] \subset \mathbb{R}\). Then there exists a free circle action on a symplectic manifold \((M, \omega)\) with proper moment map \( \Psi : M \to (a,b) \) so that, for all \( t \in (a,b) \), the reduced space \( M/\ell S^1 \) is symplectomorphic to a tame K3 surface \((X, I_1, \sigma_1)\); moreover,

- \((\sigma_t, \sigma_t) = P(t)\); and
- \( [\sigma_t] = \kappa - t\eta \), where \( \kappa, \eta \) induce a primitive embedding \( \mathbb{Z}^2 \hookrightarrow H^2(X; \mathbb{Z}) \).

To prove these propositions, we need two standard lemmas on the uniqueness and existence of free Hamiltonian circle actions; see, for example, [MS].

Lemma 3.3. Let the circle act freely on symplectic manifolds \((M, \omega)\) and \((M', \omega')\) with proper moment maps \( \Psi : M \to (a,b) \) and \( \Psi' : M' \to (a,b) \), and fix \( t \in (a,b) \). Let \( f : M/\ell S^1 \to M'/\ell S^1 \) be a symplectomorphism so that \( f^* \) takes the Euler class of the circle bundle \((\Psi')^{-1}(t) \to M'/\ell S^1 \) to the Euler class of \( \Psi^{-1}(t) \to M/\ell S^1 \). Then \( \Psi^{-1}(U) \) and \( (\Psi')^{-1}(U) \) are equivariantly symplectomorphic for some neighborhood \( U \) of \( t \).

Lemma 3.4. Let \((X, \sigma)\) be a closed symplectic manifold; fix \( \mu \in \Omega^2(X) \) with \( [\mu] \in H^2(X; \mathbb{Z}) \). There exists \( \epsilon > 0 \), and a free circle action on a symplectic manifold \((M, \omega)\) with proper moment map \( \Psi : M \to (-\epsilon, \epsilon) \) so that the reduced space \( M/\ell S^1 \) is symplectomorphic to \((X, \sigma - t\mu)\) for all \( t \in (-\epsilon, \epsilon) \).

We also need the following fact.

Lemma 3.5. Fix \( \kappa, \eta, \kappa', \) and \( \eta' \) in the K3 lattice \( L \) satisfying:

- \((\kappa - t\eta, \kappa - t\eta') = (\kappa' - t\eta', \kappa' - t\eta')\) for all \( t \in \mathbb{R} \); and
- each pair \( \kappa, \eta \) and \( \kappa', \eta' \) induces a primitive embedding \( \mathbb{Z}^2 \hookrightarrow L \).

Then there exists an isometry of \( L \) that takes \( \kappa' \) to \( \kappa \), takes \( \eta' \) to \( \eta \), and preserves (alternatively, reverses) the components of \( G_3^+ (\mathbb{R}L) \).

Proof. The K3 lattice can be written as the orthogonal direct sum

\[
L \simeq H \oplus H \oplus H \oplus -E_8 \oplus -E_8,
\]

where \( H \) is the indefinite, even, unimodular lattice of rank 2; and \( E_8 \) is the positive definite, even, unimodular lattice of rank 8. Hence, for \( i \in \{1, 2, 3\} \), there exist \( e_i \) and \( f_i \) in the \( i \)th summand above satisfying \( (e_i, e_j) = (f_i, f_j) = 0 \) and \( (e_i, f_j) = \delta_{ij} \) for all \( i, j \in \{1, 2, 3\} \). Consider

\[
\tilde{\kappa} := e_1 + \frac{1}{2}(\kappa, \kappa)f_1 \quad \text{and} \quad \tilde{\eta} := (\kappa, \eta)f_1 + e_2 + \frac{1}{2}(\eta, \eta)f_2 \in L.
\]

The isometry \( \phi : H^2(X; \mathbb{Z}) \to H^2(X; \mathbb{Z}) \) that takes \( e_3 \) to \( -e_3 \) and \( f_3 \) to \(-f_3 \) but is the identity map on the other summands fixes \( \tilde{\kappa} \) and \( \tilde{\eta} \) but exchanges the components of \( G_3^+ (\mathbb{R}L) \). By assumption, the pairs \( \kappa, \eta; \kappa', \eta' \); and \( \tilde{\kappa}, \tilde{\eta} \) each induce a primitive embedding \( \mathbb{Z}^2 \hookrightarrow L \). Moreover, their images are isomorphic lattices. Hence, [BPV] Theorem 1.2.9 implies that there exists an isometry of \( L \) that takes \( \kappa' \) to \( \tilde{\kappa} \) and \( \eta' \) to \( \tilde{\eta} \), and another isometry that takes \( \tilde{\kappa} \) to \( \kappa \) and \( \tilde{\eta} \) to \( \eta \). By either composing these two isometries or composing these two isometries with \( \phi \) inserted between them, we construct the required isometry. \( \square \)

Using the results (and notation) from the previous section, we can now specialize to that case that the reduced spaces are symplectomorphic to tame K3 surfaces.
Proof of Proposition 3.1. Fix \( t \in (a, b) \). Since all K3 surfaces are diffeomorphic, we may assume that \( X' = X \). Since \( H^2(X; \mathbb{Z}) \cong L \), Lemma 3.5 implies that there is an isometry \( \phi: H^2(X; \mathbb{Z}) \to H^2(X; \mathbb{Z}) \) so that \( \phi(k') = k \), \( \phi(q') = q \), and \( \phi \) preserves the components of \( G^+ \). By Proposition 2.1, this implies that there exists a symplectomorphism \( f \) from \((X, \sigma)\) to \((X', \sigma')\) satisfying \( f^*(q') = q \). Since \( H^2(X; \mathbb{Z}) \) is torsion-free, \( \eta \in H^2(X; \mathbb{Z}) \) is the Euler class of the circle bundle \( \Psi^{-1}(t) \to M/\mathbb{S}^1 \) and \( \eta' \in H^2(X; \mathbb{Z}) \) is the Euler class of \((\Psi')^{-1}(t) \to M'/\mathbb{S}^1 \). Hence, the claim follows immediately from Lemma 3.3.

Proof of Proposition 3.2. Write \( P(t) = 2\ell_2 t^2 + 2\ell_1 t + 2\ell_0 \), and let \( X \) be a manifold that admits a K3 structure. Since \( H^2(X; \mathbb{Z}) \) is connected, six-dimensional symplectic manifold so that each orbit is contractible. Fix \( t \in [a, b] \). Since \( P(t) > 0 \), Lemma 2.1 implies that there exists a tame K3 structure \((I, \sigma)\) on \( X \) satisfying \( |\sigma| = |\kappa - \eta| \). Pick a closed two-form \( \mu \in \Omega^2(X) \) with \(|\mu| = \eta\). By Lemma 3.4, there exists an action on a symplectic manifold \((M, \omega)\) with proper moment map \( \Psi: M \to (t - \epsilon, t + \epsilon) \) so that the reduced space \( M/\mathbb{S}^1 \) is symplectomorphic to \((X, \sigma - (s - t)\mu)\) for all \( s \in (t - \epsilon, t + \epsilon) \). By construction,

\[
[\sigma - (s - t)\mu] = \kappa - s\eta \quad \forall s \in (t - \epsilon, t + \epsilon).
\]

Finally, since tameness is an open condition, after possible shrinking \( \epsilon \), \( I \) tames \( \sigma - (s - t)\mu \) for all \( s \in (t - \epsilon, t + \epsilon) \).

Since \([a, b]\) is compact, this implies that we can cover \([a, b]\) by open sets \( V_1, \ldots, V_k \) so that each \( V_i \) is the moment image of a circle action on a symplectic manifold \((M_i, \omega_i)\) satisfying the prescribed conditions. By Proposition 3.1, after possibly shrinking the \( M_i \) and \( V_i \), we may assume that \( M_i \) is equivariantly symplectomorphic to \( M_j \) over \( V_i \cap V_j \) for all \( i \) and \( j \). Therefore, we can construct \( M \) by gluing together the \( M_i \).

3.1. Kotschick’s theorem. As we mentioned in the introduction, our proof is adapted from [Ko]. Nevertheless, we now give a simple proof of his main theorem.

Theorem 3.6 (Kotschick). There exists a free symplectic circle action on a closed connected six-dimensional symplectic manifold so that each orbit is contractible.

Proof. Fix \( \epsilon > 0 \). By Proposition 3.2, there exists a free circle action on a symplectic manifold \((M, \omega)\) with proper moment map \( \Psi: M \to (-\epsilon, 1 + \epsilon) \) so that, for all \( t \in (-\epsilon, 1 + \epsilon) \), the reduced space \( M/\mathbb{S}^1 \) is symplectomorphic to a tame K3 surface \((X, I_t, \sigma_t)\); moreover,

- \((\sigma_t, \sigma_t) = 2; \) and
- \([\sigma_t] = \kappa - t\eta \), where \( \kappa, \eta \) induce a primitive embedding \( \mathbb{Z}^2 \hookrightarrow H^2(X; \mathbb{Z}) \).

Since \( H^2(X; \mathbb{Z}) \) is torsion-free, \( \eta \) is the Euler class of the circle bundle \( \Psi^{-1}(t) \to M/\mathbb{S}^1 \) for all \( t \in (-\epsilon, 1 + \epsilon) \). Since \( \eta \) is primitive and \( X \) is simply connected, this implies that the orbit \( \mathbb{S}^1 \cdot x \) is contractible in \( \Psi^{-1}(t) \) for all \( x \in \Psi^{-1}(t) \). Finally, Proposition 3.3 implies that, after possibly shrinking \( \epsilon \), \( \Psi^{-1}(-\epsilon, \epsilon) \) and \( \Psi^{-1}(1 - \epsilon, 1 + \epsilon) \) are equivariantly symplectomorphic. We construct the proposed manifold by identifying these subspaces.
4. Locally free Hamiltonian actions

In this section, we consider locally free Hamiltonian circle actions on symplectic manifolds with reduced spaces diffeomorphic to the Kummer surface \( Z \). (Recall that \( \mathbb{Z}_2 \) acts on \( T = \mathbb{C}^2 / (\mathbb{Z}^2 + \sqrt{-1} \mathbb{Z}^2) \) by the involution \( [z] \mapsto [-z] \).) Unlike the previous section, we don’t prove general theorems but simply construct the examples we need. Moreover, we endow our examples with congenial complex structures that we use to add fixed points in the next section. More precisely, we prove the following:

**Proposition 4.1.** There exist complex manifolds \((M_+, J_+)\) and \((M_-, J_-)\) with locally free holomorphic \( \mathbb{C}^\times \) actions, \( S^1 \subset \mathbb{C}^\times \) invariant symplectic forms \( \omega_\pm \in \Omega^2(M_\pm) \), proper moment maps \( \Psi_\pm : M_\pm \to \mathbb{R} \), and \( \mathbb{C}^\times \) invariant maps \( \pi_\pm : M_\pm \to T/\mathbb{Z}_2 \) so that the following hold:

1. For all \( t \in \mathbb{R} \), the map \( \pi_\pm \) induces a symplectomorphism from \( M_\pm // t \) to \( T/\mathbb{Z}_2 \) with \( (\sigma_\pm)_t \in \Omega^2(T/\mathbb{Z}_2) \), where

\[
(\sigma_\pm)_t = dz_1 d\bar{z}_2 + dz_2 d\bar{z}_2 \pm \sqrt{-1} t dz_1 d\bar{z}_1 \pm \sqrt{-1} t dz_2 d\bar{z}_2 \tag{4.2}
\]

2. \( \pi_\pm \) induces a biholomorphism from \( M_\pm / C^\times \) to \( T/\mathbb{Z}_2 \).
3. \( \omega_\pm (\xi_\pm, J_\pm(\xi_\pm)) > 0 \), where \( \xi_\pm \in \chi(M_\pm) \) generates the \( S^1 \) action.
4. \( J_\pm \) tames \( \omega_\pm \) on the preimage \( \Psi_\pm^{-1}(\pm (0, \infty)) \).

Moreover, there is an equivariant symplectomorphism from \( M_+ \to M_- \) that intertwines the moment maps.

**Remark 4.3.** The integral of the pull-back of \((\sigma_\pm)_t \land (\sigma_\pm)_t\) over \( T \) is \( 8 + 8t^2 \). Hence, the Duistermaat-Heckman function of \( M_\pm \) is \( 4 + 4t^2 \).

**Proof.** Fix integers \( k_1 \) and \( k_2 \) with \( k_1k_2 > 0 \). Identify \( \mathbb{C}^\times \) with the quotient \( \mathbb{C} / (\sqrt{-1} \mathbb{Z}) \), and hence \( S^1 \subset \mathbb{C}^\times \) with the quotient \( \sqrt{-1} \mathbb{R} / (\sqrt{-1} \mathbb{Z}) \). Under this identification, the group \( \mathbb{Z}_4 \) acts holomorphically on \( \mathbb{C}^2 \times \mathbb{C}^\times \) by

\[
(n_1, m_1, n_2, m_2) \cdot (z_1, z_2; u) = (z_1 + n_1 + \sqrt{-1} m_1, z_2 + n_2 + \sqrt{-1} m_2; u - k_1(n_1z_1 + n_1^2/2) - k_2(n_2z_2 + n_2^2/2))
\]

for all \((n_1, m_1, n_2, m_2) \in \mathbb{Z}_4 \) and \((z_1, z_2; u) \in \mathbb{C}^2 \times \mathbb{C}^\times \).

The quotient \((\tilde{M}, \tilde{J}) = (\mathbb{C}^2 \times \mathbb{C}^\times) / \mathbb{Z}_4 \) is a complex manifold with a free holomorphic \( \mathbb{C}^\times \cong \mathbb{C} / (\sqrt{-1} \mathbb{Z}) \) action given by

\[
[w] \cdot [z_1, z_2; u] = [z_1, z_2; u + w]
\]

for all \( w \in \mathbb{C}^\times \) and \([z_1, z_2; u] \in \tilde{M} \). Let \( \Re(x + \sqrt{-1} y) := x \) for all \( x, y \in \mathbb{R} \), and define an \( S^1 \) invariant proper function \( \tilde{\Psi} : \tilde{M} \to \mathbb{R} \) by

\[
\tilde{\Psi}([z_1, z_2; u]) = \Re(u) + k_1\Re(z_1)^2/2 + k_2\Re(z_2)^2/2
\]

for all \([z_1, z_2; u] \in \tilde{M} \). Next, define a closed real \( S^1 \) invariant \((1, 1)\)-form on \( \tilde{M} \) by

\[
\tilde{\eta} := \sqrt{-1} \partial \bar{\partial} (\tilde{\Psi}^2) = 2\sqrt{-1} \partial \tilde{\Psi} \wedge \bar{\partial} \tilde{\Psi} + \sqrt{-1} \tilde{\Psi} (k_1 dz_1 d\bar{z}_1 + k_2 dz_2 d\bar{z}_2) / 2,
\]

where

\[
\partial \tilde{\Psi} = (du + k_1 \Re(z_1) dz_1 + k_2 \Re(z_2) dz_2) / 2.
\]

\footnote{By a slight abuse of notation, we describe a form on \( T/\mathbb{Z}_2 \) by giving its pull-back to \( T \).}
The form $\tilde{\eta}$ is not symplectic. However, if $k_1$ and $k_2$ are positive (respectively, negative), the complex structure $\tilde{J}$ tames the form $\tilde{\eta}$ on the preimage $\tilde{\Psi}^{-1}(0, \infty)$ (respectively, $\tilde{\Psi}^{-1}(-\infty, 0)$).

Given a non-zero integer $k_3$, define a closed real $S^1$ invariant two-form $\tilde{\omega}$ on $\tilde{M}$ by

$$\tilde{\omega} = k_3 dz_1 dz_2 / 2 + k_3 d\bar{z}_1 d\bar{z}_2 / 2 + \tilde{\eta}.$$ 

Since

$$\tilde{\omega}^3 / 6 = -\sqrt{-1} (k_3^2 + k_1 k_2 \tilde{\Psi}^2) dz_1 d\bar{z}_1 dz_2 d\bar{z}_2 du d\sigma / 8 \neq 0,$$

the form $\tilde{\omega}$ is symplectic. The action of the circle $S^1 \subset \mathbb{C}^\times$ on $\tilde{M}$ is generated by the vector field

$$\tilde{\xi} = \sqrt{-1} \left( \frac{\partial}{\partial u} - \frac{\partial}{\partial \sigma} \right),$$

and so $i_{\tilde{\xi}} \tilde{\omega} = -d\tilde{\psi}$. Therefore, $\tilde{\Psi} : \tilde{M} \to \mathbb{R}$ is a moment map for the circle action on $(\tilde{M}, \tilde{\omega})$. For any $t \in \mathbb{R}$, the map $\tilde{\pi}$ from $\tilde{M}$ to $T$ given by $[z_1, z_2; u] \mapsto [z_1, z_2]$ induces a diffeomorphism from the reduced space $\tilde{\Psi}^{-1}(t) / S^1$ to $T$. Since the form $\partial \tilde{\psi} \wedge d\tilde{\psi} = \partial \tilde{\psi} \wedge d\tilde{\psi}$ vanishes on the level set $\tilde{\Psi}^{-1}(t)$, this identifies the reduced symplectic form with $\tilde{\sigma}_t \in \Omega^2(T)$, where

$$\tilde{\sigma}_t = k_3 dz_1 dz_2 / 2 + k_3 d\bar{z}_1 d\bar{z}_2 / 2 + \sqrt{-1} k_1 t dz_1 d\bar{z}_1 / 2 + \sqrt{-1} k_2 t dz_2 d\bar{z}_2 / 2.$$ 

Additionally, the map $\tilde{\pi}$ induces a biholomorphism from $\tilde{M} / \mathbb{C}^\times$ to $T$. It is straightforward to check that $\tilde{\omega}(\tilde{\xi}, \tilde{J}(\tilde{\xi})) > 0$. Finally, since $dz_1 dz_2 \in \Omega^{2,0}(\tilde{M})$ and $d\bar{z}_1 d\bar{z}_2 \in \Omega^{0,2}(\tilde{M})$, the preceding paragraph implies that if $k_1$ and $k_2$ are positive (respectively, negative) then the complex structure $\tilde{J}$ tames the form $\tilde{\omega}$ on the preimage $\tilde{\Psi}^{-1}(0, \infty)$ (respectively, $\tilde{\Psi}^{-1}(-\infty, 0)$); cf. Lemma 2.2.

Given an integer $\ell$, the group $\mathbb{Z}_2$ acts holomorphically on $\tilde{M}$ by the involution

$$(4.4) \quad [z; u] \mapsto [-z; u + \sqrt{-1} \ell / 2].$$

In most cases, the action is not free and the quotient $\tilde{M} / \mathbb{Z}_2$ is a complex orbifold $(M_+, J_+)$. However, if $k_1$ and $k_2$ are even and $\ell$ is odd, this action is free, and so $M_+$ is a complex manifold. In any case, since the $\mathbb{C}^\times$ action on $\tilde{M}$ commutes with the involution, it induces a locally free holomorphic $\mathbb{C}^\times$ action on $M_+$. Since the involution preserves the symplectic form $\tilde{\omega}$ and the moment map $\tilde{\Psi}$, $M_+$ inherits an $S^1$ invariant symplectic form $\omega_+$ and proper moment map $\Psi_+$. Finally, since $\tilde{\pi}$ is $\mathbb{Z}_2$ equivariant, it engenders a $\mathbb{C}^\times$ invariant map $\pi_+ : M_+ \to T / \mathbb{Z}_2$ which induces a diffeomorphism from the reduced space $M_+ / S^1$ to the orbifold $T / \mathbb{Z}_2$ for all $t \in \mathbb{R}$; this identifies the reduced symplectic form on $M_+ / S^1$ with $(\sigma_+) \in \Omega^2(T / \mathbb{Z}_2)$, where

$$(4.5) \quad (\sigma_+) = k_3 dz_1 dz_2 / 2 + k_3 d\bar{z}_1 d\bar{z}_2 / 2 + \sqrt{-1} k_1 t dz_1 d\bar{z}_1 / 2 + \sqrt{-1} k_2 t dz_2 d\bar{z}_2 / 2.$$ 

Clearly, claims (2) and (3) are satisfied; if $k_1$ and $k_2$ are positive, then claim (4) is satisfied as well.

Now repeat the process with $-k_1, -k_2, k_3$, and $\ell$ to construct another complex orbifold $(M_-, J_-)$ with locally free holomorphic $\mathbb{C}^\times$ action, $S^1$ invariant symplectic form $\omega_-$, proper moment map $\Psi_- : M_- \to \mathbb{R}$, and $\mathbb{C}^\times$ invariant map $\pi_- : M_-$
Then, for all $t \in \mathbb{R}$, the map $\pi_-$ induces a symplectomorphism from $M_-/\iota S^1$ to $T/\mathbb{Z}_2$ with $(\sigma_-)_t \in \Omega^2(T/\mathbb{Z}_2)$, where

\[(4.6) \quad (\sigma_-)_t = k_1dz_1dz_2/2 + k_3dz_1\sigma_2/2 - \sqrt{-1} k_1tdz_1\sigma_1/2 - \sqrt{-1} k_2tdz_2\sigma_2/2.\]

Again, if $k_1$ and $k_3$ are positive then claims (2), (3) and (4) are satisfied.

Define an equivariant diffeomorphism $f: M_+ \to M_-$ that intertwines the moment maps by

\[f([z_1, z_2; u]) = [\sigma_1, \sigma_2; u + k_1 R(z_1)^2 + k_3 R(z_2)^2].\]

Working in coordinates, it is straightforward to check that $f$ is a symplectomorphism.

The proposition now follows if we take $k_1 = k_2 = k_3 = 2$. \hfill \Box

Remark 4.7. More generally, given integers $k_1, k_2 > 0$ and $k_3 \neq 0$, the proof above shows that Proposition 4.1 still holds if we replace “manifolds” by “orbifolds” and replace (4.2) by (4.5) and (4.6). Given an integer $\ell$, the point $[z; u] \in \hat{M}$ is fixed by the involution (4.1) exactly if

(a) $z_i = (m_i + \sqrt{-1} n_i)/2$, where $m_i, n_i \in \mathbb{Z}$ for $i = 1, 2$, and
(b) $k_1m_1n_1 + k_2m_2n_2 = \ell \pmod{2}.

Therefore, every point $[z; u] \in M_+$ that satisfies (a) and (b) is a singular point of the orbifold with fundamental group $\mathbb{Z}_2$; otherwise, $M_+$ is smooth. In contrast, the stabilizer group $\{\lambda \in \mathbb{C}^\times | \lambda \cdot [z; u] = [z; u]\}$ is $\mathbb{Z}_2$ if (a) holds but (b) does not; otherwise, the action of $\mathbb{C}^\times$ on $M_+$ is free.

5. Hamiltonian actions with fixed points

In this final section, we consider Hamiltonian circle actions on symplectic manifolds with fixed points. However, each regular reduced space is still either symplectomorphic to a tame K3 surface or diffeomorphic to the Kummer surface $T/\mathbb{Z}_2$. As in the previous section, we don’t prove general theorems but simply use ideas from [TWa] to construct the examples we need. More precisely, we prove the following:

Proposition 5.1. There exist symplectic manifolds $(\hat{M}_+, \omega_\pm)$ with circle actions and proper moment maps $\Psi_\pm: \hat{M}_+ \to \mathbb{R}$, and $\epsilon > 0$ so that:

1. The preimages $\Psi_\pm^{-1}(\pm(-\infty, 1 + \epsilon))$ each contain exactly 16 fixed points; each lies in $\Psi_\pm(\pm 1)$ and has weights $\pm\{-2, 1, 1\}$.

2. Let $(M_+, \omega_\pm, \Psi_\pm)$ be the symplectic manifolds with locally free circle actions and proper moment maps described in Propositions 4.1. There exist $a_\pm < b_\pm$ in $\pm(0, 1)$ so that $\Psi_\pm^{-1}(a_\pm, b_\pm)$ and $\Psi_\pm^{-1}(a_\pm, b_\pm)$ are equivariantly symplectomorphic.

3. For all $t \in \pm(1, 1 + \epsilon)$, the reduced space $\hat{M}_+/\iota S^1$ is symplectomorphic to a tame K3 surface $(\hat{X}, \hat{f}, (\sigma_\pm)_t)$; moreover,
   - $[(\overline{\sigma}_\pm)_t, (\overline{\sigma}_\pm)_t] = -4 \pm 16t - 4t^2$; and
   - $[\overline{\sigma}_\pm] = \tilde{\kappa} - t\tilde{\eta}_\pm$, where $\tilde{\kappa}, \tilde{\eta}_\pm$ induce a primitive embedding $\mathbb{Z}^2 \hookrightarrow H^2(\hat{X}; \mathbb{Z})$.

To construct these examples, we need to introduce some notation: Given a holomorphic $\mathbb{C}^\times$ action on a complex manifold $(M, J)$, let $\xi_M$ be the vector field associated to the induced $S^1 \subset C^\times$ action, and let $\Omega^2(M)^{S^1}$ denote the set of $S^1$ invariant two-forms on $M$. 
We also need the following three results from [TWa], which is joint with J. Watts. More specifically, they correspond to [TWa] Proposition 3.1, [TWa] Proposition 7.1, and [TWa] Proposition 7.8, respectively.

**Proposition 5.2.** Let \((M, J)\) be a complex manifold with a holomorphic \(\mathbb{C}^\times\)-action, a symplectic form \(\omega \in \Omega^2(M)^{S^1}\) satisfying \(\omega(\xi_M, J(\xi_M)) > 0\) on \(M \setminus M^{S^1}\), and a moment map \(\Psi: M \to R\). If \(a \in \mathbb{R}\) is a regular value of \(\Psi\) and \(U_a := \mathbb{C}^\times \cdot \Psi^{-1}(a)\), then the following hold:

- The quotient \(U_a/C^\times\) is a naturally a complex orbifold.
- There is a complex structure \(J_a\) on the reduced space \(M/\!/_a S^1\) so that the inclusion \(\Psi^{-1}(a) \hookrightarrow U_a\) induces a biholomorphism \(M/\!/_a S^1 \to U_a/C^\times\).
- If \(J\) tames \(\omega\) near \(\Psi^{-1}(a)\), then \(J_a\) tames the reduced symplectic form on \(M/\!/_a S^1\).

**Proposition 5.3.** Fix \(a \in \mathbb{R}\). Let \((M, J)\) be a complex manifold with a holomorphic \(\mathbb{C}^\times\) action, a symplectic form \(\omega \in \Omega^2(M)^{S^1}\) that is tamed near \(\Psi^{-1}(a)\) and satisfies \(\omega(\xi_M, J(\xi_M)) > 0\) on \(M \setminus M^{S^1}\), and a proper moment map \(\Psi: M \to R\). Assume that the \(S^1\) action on \(\Psi^{-1}(a)\) is free except for \(k\) orbits with stabilizer \(\mathbb{Z}_2\). Then for sufficiently small \(\epsilon > 0\) there exists a complex manifold \((\tilde{M}, \tilde{J})\) with a holomorphic \(\mathbb{C}^\times\) action, a symplectic form \(\tilde{\omega} \in \Omega^2(\tilde{M})^{S^1}\) satisfying \(\tilde{\omega}(\xi_{\tilde{M}}, \tilde{J}(\xi_{\tilde{M}})) > 0\) on \(\tilde{M} \setminus \tilde{M}^{S^1}\), and a proper moment map \(\tilde{\Psi}: \tilde{M} \to \mathbb{R}\) so that the following hold:

1. \(\tilde{\Psi}^{-1}(a - \epsilon, a] \) contains exactly \(k\) fixed points; each lies in \(\tilde{\Psi}^{-1}(a)\) and has weights \(-2, 1, \ldots, 1\).
2. There is an equivariant symplectomorphism \(\tilde{\Psi}^{-1}(\infty, a - \epsilon/2) \to \tilde{\Psi}^{-1}(\infty, a - \epsilon/2)\) that induces a biholomorphism \(\tilde{M}/\!/_t S^1 \to M/\!/_t S^1\) for all regular \(t \in (-\infty, a - \epsilon/2)\).
3. \(\tilde{\omega}\) is tamed on \(\tilde{\Psi}^{-1}(a - \epsilon, a + \epsilon)\).

**Proposition 5.4.** Let \((M, J)\) be a complex manifold with a holomorphic \(\mathbb{C}^\times\) action, a symplectic form \(\omega \in \Omega^2(M)^{S^1}\) satisfying \(\omega(\xi_M, J(\xi_M)) > 0\) on \(M \setminus M^{S^1}\), and a proper moment map \(\Psi: M \to R\). Assume that \(\dim_{\mathbb{C}} M > 1\), and that \(\Psi^{-1}(a, b) \cap M^{S^1}\) contains exactly \(k\) fixed points; each lies in \(\Psi^{-1}(0)\) and has weights \(-2, 1, \ldots, 1\). Then there exists a complex orbifold \((X, I)\), and classes \(\kappa, \eta \in H^2(X; \mathbb{R})\), such that:

1. For all \(t \in (a, 0)\), the reduced space \(M/\!/_t S^1\) is biholomorphically symplectomorphic to \((X, I, \sigma_t)\), where \(\sigma_t \in \Omega^2(X)\) satisfies \([\sigma_t] = \kappa - t\eta \in H^2(X; \mathbb{R})\).
2. For all \(t \in (0, b)\), the reduced space \(M/\!/_t S^1\) is biholomorphically symplectomorphic to \((\tilde{X}, \tilde{I}, \tilde{\sigma}_t)\), where \(\tilde{\sigma}_t \in \Omega^2(\tilde{X})\) satisfies

\[
[\tilde{\sigma}_t] = q^* \kappa - t q^* \eta - t/2 \sum_{i=1}^k \mathcal{E}_i \in H^2(\tilde{X}; \mathbb{R}).
\]

Here, \((\tilde{X}, \tilde{I})\) is the blow-up of \((X, I)\) at isolated \(\mathbb{Z}_2\) singularities \(p_1, \ldots, p_k\), the map \(q: \tilde{X} \to X\) is the blow-down, and \(\mathcal{E}_i\) is the Poincare dual of the exceptional divisor \(q^{-1}(p_i)\) for all \(i\).

**Remark 5.6.** Alternatively, by [TWa] Remark 4.4, Proposition 5.3 is true with the following modifications: First, in claim (1), the preimage is \(\tilde{\Psi}^{-1}(a, a + \epsilon)\) instead of...
\[ \tilde{\Psi}^{-1}(a-\epsilon, a), \] and the weights are \( \{2, -1, \ldots, -1\} \) instead of \( \{-2, 1, \ldots, 1\} \). Second, in claim (2), the set \( (-\infty, a-\epsilon/2) \) is replaced by \( (a+\epsilon/2, \infty) \).

Similarly, Proposition 5.3 is true with the following modifications: Assume that the weights are \( \{2, -1, \ldots, -1\} \) at each fixed points in \( \Psi^{-1}(0) \), instead of \( \{-2, 1, \ldots, 1\} \). In this case, claim (1) holds for all \( t \in (0, b) \), not all \( t \in (a, 0) \). Claim (2) holds for all \( t \in (a, 0) \), not all \( t \in (0, b) \); moreover, [5.3] is replaced by

\[ [\tilde{s}_i] = q^* \kappa - t q^* \eta + t/2 \sum_{i=1}^k \mathcal{E}_i \in H^2(\tilde{X}; \mathbb{R}). \]

Let \( \pi: T \to T/\mathbb{Z}_2 \) be the quotient map. Let \( (\tilde{X}, \tilde{I}) \) be the K3 surface formed by blowing up the quotient \( T/\mathbb{Z}_2 \) at the 16 isolated \( \mathbb{Z}_2 \) singularities \( p_1, \ldots, p_{16} \). Finally, let \( q: \tilde{X} \to T/\mathbb{Z}^2 \) be the blow-down map, and let \( \mathcal{E}_j \in H^2(\tilde{X}; \mathbb{Z}) \) be Poincare dual to the exceptional divisor \( q^{-1}(p_i) \) for all \( i \). We will use the criteria below to prove claim (3) of Proposition 5.3.

**Lemma 5.7.** Given \( y \in H^2(T/\mathbb{Z}_2; \mathbb{R}) \) such that \( \pi^*(y/2) \) is a primitive class in \( H^2(T; \mathbb{Z}) \), the pull-back \( q^*(y) \) is a primitive class in \( H^2(\tilde{X}; \mathbb{Z}) \). If additionally \( x \in H^2(\tilde{X}; \mathbb{Z}) \) satisfies \( (\mathcal{E}_i, x) = \pm 1 \) for some \( i \in \{1, \ldots, 16\} \), then \( q^*(y), x \) induce a primitive embedding \( \mathbb{Z}^2 \to H^2(\tilde{X}; \mathbb{Z}) \).

**Proof.** Let \( \tilde{T} \) be the blow-up of \( T \) at the 16 fixed points of the \( \mathbb{Z}_2 \) action, and let \( \sigma: \tilde{T} \to T \) be the blow-down map. The involution on \( T \) induces an involution on \( \tilde{T} \) that fixes the exceptional divisors. The quotient \( \tilde{T}/\mathbb{Z}_2 \) is naturally isomorphic to \( \tilde{X} \); let \( \tilde{\pi}: \tilde{T} \to \tilde{X} \) be the quotient map. Note that \( q \circ \tilde{\pi} = \pi \circ \sigma \).

Define \( \alpha: H^*(T; \mathbb{Z}) \to H^*(\tilde{T}; \mathbb{Z}) \) by \( \alpha = \tilde{\pi}_! \circ \sigma^* \), where \( \tilde{\pi}_1: H^*(\tilde{T}; \mathbb{Z}) \to H^*(\tilde{X}; \mathbb{Z}) \) is the push-forward map. By [BPV, Corollary VIII.5.6], the lattice \( \alpha(H^2(\tilde{X}; \mathbb{Z})) \) is primitive. Moreover, \( \tilde{\pi}_!(\tilde{\pi}^*(z)) = 2z \) for all \( z \in H^*(\tilde{X}; \mathbb{Z}) \) because \( \tilde{\pi} \) has degree 2. Therefore, given \( y \in H^2(T/\mathbb{Z}_2; \mathbb{R}) \) such that \( \pi^*(y/2) \) is a primitive class in \( H^2(T; \mathbb{Z}) \), its image

\[ \alpha(\pi^*(y/2)) = \tilde{\pi}_!(\sigma^*(\pi^*(y/2))) = \tilde{\pi}_!(\tilde{\pi}^*(q^*(y)))/2 = q^*(y). \]

is a primitive class in \( H^2(\tilde{X}; \mathbb{Z}) \). Since \( (\mathcal{E}_j, q^*(y)) = 0 \) for all \( j \), the second claim follows immediately. \( \square \)

We are now ready to begin our proof.

**Proof of Proposition 5.7.** Let \( (M_+, J_+) \) be the complex manifold with locally free holomorphic \( \mathbb{C}^\times \) action, symplectic form \( \omega_+ \in \Omega^2(M_+)^{S^1} \) satisfying \( \omega_+((J_+ \xi M_+) > 0 \), proper moment map \( \Psi_+: M_+ \to \mathbb{R} \), and \( \mathbb{C}^\times \) invariant map \( \pi_+: M_+ \to T/\mathbb{Z}_2 \) described in Proposition 4.4. For all \( t \in \mathbb{R} \), the map \( \pi_+ \) induces a symplectomorphism from the reduced space \( M_+//t S^1 \) to the Kummer surface \( T/\mathbb{Z}_2 \) with \( (\sigma_+)_t \in \Omega^2(T/\mathbb{Z}_2) \), where

\[ (\sigma_+)_t = dz_1 d\bar{z}_2 + d\sigma_1 d\sigma_2 + \sqrt{-1} t dz_1 d\bar{\sigma}_1 + \sqrt{-1} t d\bar{z}_2 d\bar{\sigma}_2. \]

In particular, the \( S^1 \) action on \( \Psi_+^{-1}(t) \) is free except for 16 orbits with stabilizer \( \mathbb{Z}_2 \). Moreover, \( \pi_+ \) induces a biholomorphism from \( M_+//\mathbb{C}^\times \) to \( T/\mathbb{Z}_2 \), and so \( U_t := \mathbb{C}^\times \cdot \Psi_+^{-1}(t) = M_+ \) for all \( t \in \mathbb{R} \). Hence, Proposition 5.2 implies that there is a natural complex structure on the reduced space \( M_+//t S^1 \), and the symplectomorphism
from $M_+//S^1$ to $T/Z_2$ described above is biholomorphism. Finally, $J_+$ tames $\omega_+$ on $\Psi_+^{-1}(0, \infty)$.

Therefore, we can apply Proposition 5.3 with $a = 1$ to construct a complex manifold $(\tilde{M}_+, \tilde{J}_+)$ with a holomorphic $\mathbb{C}^\times$ action, a symplectic form $\tilde{\omega}_+ \in \Omega^2(\tilde{M}_+)^{S^1}$ satisfying $\tilde{\omega}_+(\xi_{\tilde{M}_+}, \tilde{J}_+(\xi_{\tilde{M}_+})) > 0$ on $\tilde{M}_+ \setminus \tilde{M}_+^{S^1}$, and a proper moment map $\tilde{\Psi}_+ : \tilde{M}_+ \to \mathbb{R}$. Since we may assume that $\epsilon < 1$, claim (2) implies that their exists a non-empty open set $U_+ \subset (0, 1)$ and an equivariant symplectomorphism from $\tilde{\Psi}_+^{-1}(U_+)$ to $\Psi_+^{-1}(U_+)$ that induces a biholomorphism from $\tilde{M}_+//t S^1$ to $M_+//t S^1$ for all $t \in U_+$. In particular, the reduced space $\tilde{M}_+//t S^1$ is biholomorphically symplectomorphic to the Kummer surface $T/Z_2$ with the symplectic form $(\sigma_+) \in \Omega^2(T/Z_2)$ for all $t \in U_+$. Since $\tilde{M}_+$ has no fixed points, claims (1) and (2) together imply that, after possibly shrinking $\epsilon$, the preimage $\tilde{\Psi}_+^{-1}(-\infty, \epsilon)$ contains exactly 16 fixed points; each lies in $\tilde{\Psi}_+^{-1}(1)$ and has weights $\{-2, 1, 1\}$. Finally, by claim (3), the symplectic form $\tilde{\omega}_+$ is tamed on $\tilde{\Psi}_+^{-1}(1 - \epsilon, 1 + \epsilon)$.

Hence, by Proposition 5.4 for all $t \in (1, 1 + \epsilon)$, the reduced space $\tilde{M}_+//t S^1$ is biholomorphically symplectomorphic to the K3 surface $(\tilde{X}, \tilde{I})$ with symplectic form $(\tilde{\sigma}_+) \in \Omega^2(T/Z_2)$, where

\begin{equation}
(5.9) \quad [(\tilde{\sigma}_+)_t] = q^*([(\sigma_+)_{\epsilon}]) - (t - 1)/2 \sum_{i=1}^{16} \mathcal{E}_i.
\end{equation}

Moreover, by Proposition 5.2, the reduced complex structure $\tilde{I}$ tames $(\tilde{\sigma}_+)_t$ for all $t \in (1, 1 + \epsilon)$. Combining (5.8) and (5.9), we see that $[(\tilde{\sigma}_+)_t] = \tilde{\kappa} - t\tilde{\eta}_+$ for all $t \in (1, 1 + \epsilon)$, where

\begin{align*}
\tilde{\kappa} &= q^*([dz_1dz_2 + d\bar{z}_1d\bar{z}_2]) + 1/2 \sum_{i=1}^{16} \mathcal{E}_i, \quad \text{and} \\
\tilde{\eta}_+ &= -q^*([\sqrt{-1}dz_1d\bar{z}_1 + \sqrt{-1}dz_2d\bar{z}_2]) + 1/2 \sum_{i=1}^{16} \mathcal{E}_i.
\end{align*}

A direct calculation shows that

\[
\int_{T/Z_2} (dz_1dz_2 + d\bar{z}_1d\bar{z}_2) \wedge (dz_1dz_2 + d\bar{z}_1d\bar{z}_2) = \frac{1}{2} \int_T (dz_1dz_2 + d\bar{z}_1d\bar{z}_2) \wedge (dz_1dz_2 + d\bar{z}_1d\bar{z}_2) = 4.
\]

Therefore, the cup product of $q^*([dz_1dz_2 + d\bar{z}_1d\bar{z}_2]) \in H^2(\tilde{X})$ with itself is 4. Since $q^*([dz_1dz_2 + d\bar{z}_1d\bar{z}_2], \mathcal{E}_i) = 0$ and $\mathcal{E}_i, \mathcal{E}_j = -2\delta_{ij}$ for all $i$ and $j$, this implies that

\[
(\tilde{\kappa}, \tilde{\kappa}) = 4 + 16(-2)/2^2 = -4.
\]

By similar arguments,

\[
(\tilde{\eta}_+, \tilde{\eta}_+) = -4 \quad \text{and} \quad (\tilde{\eta}_+, \tilde{\kappa}) = -8.
\]

Thus, $(\tilde{\kappa} - \tilde{\eta}_+, \tilde{\kappa} - t\tilde{\eta}_+) = -4t^2 + 16t - 4$, as required.
Finally, since
\[ 1/2 \left[ dz_1 d\bar{z}_2 + d\bar{z}_1 dz_2 + \sqrt{-1} dz_1 d\bar{z}_1 + \sqrt{-1} d\bar{z}_2 dz_2 \right] \]
is a primitive integral in \( H^2(T; \mathbb{Z}) \), Lemma 5.7 implies that \( \hat{\kappa} - \hat{\eta}_+ \) is a primitive class in \( H^2(\hat{X}; \mathbb{Z}) \). Moreover, since \( \hat{\eta}_+ \) is Euler class of the circle bundle \( \hat{\Psi}_+^{-1}(t) \rightarrow \hat{M}_+ \setminus S^1 \simeq \hat{X} \) for all \( t \in (1,1+\epsilon) \), the class \( \hat{\eta}_+ \) lies in the integral cohomology \( H^2(\hat{X}; \mathbb{Z}) \). Hence, since the cup product \( (\hat{\eta}_+, \xi_i) = -1 \) for all \( i \), Lemma 5.7 implies that \( \hat{\kappa}, \hat{\eta}_+ \) induce a primitive embedding, as required.

The construction of \( \hat{M}_- \) is analogous, except that in this case we begin with the example \( M_- \) given in Proposition 4.1 instead of taking \( M_+ \). We then apply the variants of Proposition 5.3 and 5.4 suggested in Remark 5.6 and in the former case we take \( a = -1 \), instead of \( a = 1 \). In this case, for all \( t \in (-1-\epsilon, -1) \), the reduced space \( \hat{M}_- / t S^1 \) is biholomorphically symplectomorphic to the tame K3 surface \( (\hat{X}, \hat{I}, (\hat{\omega}_-) t) \) with \( (\hat{\omega}_-) t = \hat{\kappa} - t \hat{\eta}_- \), where
\[ \hat{\eta}_- = -q^* (\sqrt{-1} dz_1 d\bar{z}_1 + \sqrt{-1} d\bar{z}_2 dz_2) - 1/2 \sum_{i=1}^{16} \xi_i. \]

\[ \square \]

References

[BPV] W. Barth, C. Peters, and A. Van de Ven, *Compact Complex Surfaces*, Springer-Verlag, Berlin Heidelberg New York Tokyo, 1984.

[Do] S.K. Donaldson, *Polynomial invariant for smooth four-manifolds*, Topology 29 (1990), 257-315.

[DH] J.J. Duistermaat and G.J. Heckman, *On the variation in the cohomology of the symplectic form on the reduced phase space*, Invent. Math. 69 (1982), 259-268.

[Fe] K.E. Feldman, *Hirzebruch genus of a manifold supporting a Hamiltonian circle action*, Russian Math. Surveys 56 (2001), 978-979.

[Fr] T. Frankel, *Fixed points and torsion on Kähler manifolds*, Ann. of Math. 70 (1959), 1-8.

[Gia] A. Giacobbe, *Convexity of multi-valued momentum maps*, Geom. Dedicata 111 (2005), 1-22.

[Gin] V. Ginzburg, *Some remarks on symplectic actions of compact groups*, Math. Z. 210 (1992), 625-640.

[Go05] L. Godinho, *On certain symplectic circle actions*, J. Symplectic Geom. 3 (2005), 357-383.

[Go06] L. Godinho, *Semifree symplectic circle actions on 4-orbifolds*, Trans. Amer. Math. Soc. 358 (2006), 4919-4933.

[Ja] D. Jang, *Symplectic periodic flows with exactly three equilibrium points*, Ergodic Theory Dynam. Systems 34 (2014), 1930-1963.

[Ki06] M. Kyu Kim, *Frankel’s theorem the symplectic category*, Trans. Amer. Math. Soc. 358 (2006), 4367-4377.

[Ko] D. Kotschick, *Free circle actions with contractible orbits on symplectic manifolds*, Math. Z. 252 (2006), 19-25.

[Li] P. Li *The rigidity of Dolbeault-type operators and symplectic circle actions*, Proc. Amer. Math. Soc. 140 (2012) 1987-1995.

[LO] G. Lupton and J. Oprea, *Cohomologically symplectic spaces: toral actions and the Gottlieb group*, Trans. Amer. Math. Soc. 347 (1995), 261-288.

[MFR] R. Mazzeo, A. Pelayo, and T. Ratiu, *L^2-cohomology and complete Hamiltonian manifolds*, J. Geom. and Phys. 87 (2015), 305-313.

[Mc] D. McDuff, *The moment map for circle actions on symplectic manifolds*, J. Geom. Phys. 5 (1988), 49-160.

[MS] D. McDuff and D. Salamon, *Introduction to symplectic topology*, Oxford University Press, Oxford, 1998.
[On] K. Ono, Some remarks on group actions in symplectic geometry, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 35 (1988), 431-437.

[PT] A. Pelayo and S. Tolman, Fixed points of symplectic periodic flows, Ergodic Theory Dynam. Systems 31 (2011), 1237-1247.

[Ro] F. Rochon, Rigidity of Hamiltonian actions, Canad. Math. Bull. 46 (2006) 277-290.

[TWe] S. Tolman and J. Weitsman, Semifree symplectic circle action with isolated fixed points, Topology 39 (2000), 299-309.

[TWa] S. Tolman and J. Watts, Tame circle actions, preprint (2015), arXiv:1510.01721

Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801

E-mail address: stolman@math.uiuc.edu