A note on seminormality of cut polytopes

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Abstract. We prove that seminormality of cut polytopes is equivalent to normality. This settles two conjectures regarding seminormality of cut polytopes.

A cut $A|B$ in a graph $G(V,E)$ is an unordered partition $(A|B = B|A)$ of its vertices, i.e. $A \cup B = V$. A cut $A|B$ determines a point in $\mathbb{R}^{|E|}$, denoted by $\delta_{A|B}$, with value 1 on edges $e$ separated by the cut (i.e. $|e \cap A| = 1$ and $|e \cap B| = 1$) and value 0 on edges $e$ within cut parts (i.e. $e \subset A$ or $e \subset B$). The cut polytope $\text{Cut}^{□}(G)$ corresponding to a graph $G$ is the convex hull of points $\delta_{A|B}$ over all cuts $A|B$ in $G$, see e.g. [10].

A polytope $P$ is normal if for any $k \in \mathbb{N}$ every lattice point (a point that belongs to the lattice spanned by the lattice points of $P$) in $kP$ is a sum of $k$ lattice points from $P$. A slightly weaker property of a polytope is ‘very ampleness’, see e.g. [6]. A polytope $P$ is very ample if it has only finitely many gaps – lattice points in $kP$ (for some $k$) which are not a sum of $k$ lattice points from $P$. This is equivalent to the fact that for every vertex $v \in P$ the monoid of lattice points in the real cone generated by $P - v$ is generated by lattice points of $P - v$ [1 Def. 2.2.7], [7] Ex. 4.9. A polytope $P$ is seminormal [5] if for every lattice point $x$, if $2x$ and $3x$ are not gaps, then nor is $x$.

The most well-known conjecture in this area is the following.

Conjecture 1 ([10]). The cut polytope $\text{Cut}^{□}(G)$ is normal if and only if the graph $G$ has no $K_5$ minor.

The implication from the left to the right is known, as $\text{Cut}^{□}(K_5)$ is not normal [10] and normality of cut polytopes is a minor closed property [8]. The difficulty of the conjecture lies in proving that for graphs with no $K_5$ minor the cut polytope is normal. We start with a proof of a weaker property – very ampleness, for which we did not find an explicit reference and which is a corollary of [4 Corollary 1.3].

Theorem 2. Suppose a graph $G$ has no $K_5$ minor. Then the cut polytope $\text{Cut}^{□}(G)$ is very ample.
Proof. First, we show that the cut polytope is transitive. That is, for any two vertices $v_1, v_2$ of $\text{Cut}^{\Box}(G)$ there exists an affine isomorphism $\varphi$ of $\mathbb{R}^{|E|}$ such that $\varphi(\text{Cut}^{\Box}(G)) = \text{Cut}^{\Box}(G)$ and $\varphi(v_1) = v_2$. It is enough to show that when $v_1 = \delta_{|E|}$ and $v_2 = \delta_{|A|B}$ is arbitrary. Map $\varphi_{|A|B}$ is defined by: $x_e \rightarrow x_e$ when $e$ is contained in $A$ or $B$, and $x_e \rightarrow 1 - x_e$ when $e$ is separated by the cut $A|B$. Observe that

$$\varphi_{|A|B}(\delta_{C|D}) = \delta_{(|A\cap D)|\cup(B\cap C)|(A\cap C)|\cup(B\cap D)}.$$ 

In particular, $\varphi_{|A|B}(\delta_{|E|}) = \delta_{|A|B}$.

Next, we note that for every vertex $\delta_{|A|B} \in \text{Cut}^{\Box}(G)$ the monoid of lattice points in the real cone generated by $\text{Cut}^{\Box}(G) - \delta_{|A|B}$ is isomorphic, via $(\varphi_{|A|B} - \delta_{|A|B})^{-1}$, to the monoid of lattice points in the real cone generated by $\text{Cut}^{\Box}(G) - \delta_{|E|} = \text{Cut}^{\Box}(G)$. Thus, in order to show that $\text{Cut}^{\Box}(G)$ is very ample it is enough to check the second characterization of very ample polytopes for a single vertex $\delta_{|E|}$. Therefore, very ample property of the cut polytope coincides with the class $\mathcal{H}$ in [3] of graphs whose set of cuts is a Hilbert basis in $\mathbb{R}^{|E|}$.

The statement that remains is proved in [4, Corollary 1.3] and for planar graphs already in [9]. Since both rely on the four color theorem, the theorem also does. \qed

Remark that the cut polytope of $K_5$ is very ample [2]. Moreover, very ampleness of cut polytopes is a minor closed property [3]. In particular, Theorem 2 does not give a characterization of graphs with very ample cut polytopes.

Using Theorem 2 we settle Conjectures 1.2 and 4.5 from [5].

Theorem 3. The cut polytope $\text{Cut}^{\Box}(G)$ of a graph $G$ is seminormal if and only if it is normal. In particular, the class of graphs $G$ for which $\text{Cut}^{\Box}(G)$ is seminormal is minor closed.

Proof. If $\text{Cut}^{\Box}(G)$ is normal, then clearly it is seminormal.

Let $G$ be a graph such that $\text{Cut}^{\Box}(G)$ is seminormal. Then by [5, Corollary 4.4] graph $G$ has no $K_5$ minor. By Theorem 2 the cut polytope $\text{Cut}^{\Box}(G)$ is very ample. Suppose contrary, that $\text{Cut}^{\Box}(G)$ is not normal – it has gaps. Since $\text{Cut}^{\Box}(G)$ is very ample, it has only finitely many gaps. Let $x$ be a largest gap, i.e. a gap that belongs to the largest dilation $k$. Then $2x$ and $3x$ belong to larger dilations, so they are not gaps. Since $\text{Cut}^{\Box}(G)$ is seminormal, $x$ is also not a gap. A contradiction. \qed

We show how a part of Conjecture 1 is equivalent to the four color theorem.

Theorem 4. The fact that every lattice point in 3 $\text{Cut}^{\Box}(G)$ is a sum of 3 lattice points from $\text{Cut}^{\Box}(G)$ for a planar graph $G$ is equivalent to the four color theorem.

Proof. One implication, proving the four color theorem, is presented in [7, Proposition 9.4], but originally the idea is due to David Speyer. We note that this implication only uses a decomposition of one specific point in 3 $\text{Cut}^{\Box}(G)$.

For the other implication we extend the assertion to loopless multigraphs and proceed by induction on the number of edges. Let $p \in 3 \text{Cut}^{\Box}(G)$ be a lattice point. Let $E_0$ be the set of edges $e \in E(G)$ such that $p(e) = 0$. Consider the contraction $G' := G/E_0$. Notice that $G'$ may have multiple edges, but it is loopless. Indeed, if $e \in E(G)$ became a loop in $G'$, then there was a path between endpoints $x, y$ of $e$ consisting of edges from $E_0$. This is impossible, as since $p(e) > 0$ and $p$ is a convex combination of cuts, points $x, y$ were separated by some cut. Now, we may identify
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