LORENTZIAN CR STRUCTURES

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ABSTRACT

The mathematics of a 4-dimensional renormalizable generally covariant
lagrangian model (with first order derivatives) is reviewed. The lorentzian CR
manifolds are totally real submanifolds of 4(complex)-dimensional complex
manifolds determined by four special conditions. The defining tetrad permits
the definition of a class of lorentzian metrics which admit two geodetic and
shear free congruences. These metrics permit the classification of the
structures using the Weyl tensor and the Flaherty pseudo-complex structure.

The Cartan procedure permits the definition of three relative invariants.
Viewed as a pair of two hypersurface-type 3-dimensional CR structures, the
lorentzian CR structures may be osculated on the basis of SU(1,2) group. An
osculation on the basis of SU(2,2) group reveals the Poincaré group which may
be identified with the observed group in nature. Examples of static axially
symmetric lorentzian CR structures are computed. For every lorentzian CR
manifold, a class of Kaehler metrics of the ambient complex manifold is found,
which induce the class of compatible lorentzian metrics on the submanifold.
Then the lorentzian CR manifold becomes a lagrangian submanifold in the
 corresponing ambient (Kaehler) symplectic manifold. The lorentzian CR
manifolds may be considered as dynamical processes in the context of the
Einstein-Infeld-Hofman derivation of the equations of motion.
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1 INTRODUCTION

Einstein’s theory of relativity is based on the riemannian geometry, assuming the metric tensor $g_{\mu\nu}$ as its fundamental dynamical variable. But it is actually well known that the conventional quantization of the Einstein-Hamilton action does no imply a self-consistent quantum field theory of gravity. Many years ago, trying to transfer in four dimensions the metric independence property of the linearized string action

$$I_S = \frac{1}{2} \int d^2 \xi \sqrt{-\gamma} \gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

(1.1)

I found the following four dimensional Yang-Mills action\[7\]

$$I_G = \frac{1}{2} \int d^4 z \det(g_{\alpha\tilde{\alpha}}) g_{\alpha\tilde{\beta}} g_{\gamma\tilde{\delta}} F_{j\alpha\gamma} F_{j\tilde{\beta}\tilde{\delta}} + c. \text{ conj.} = \int d^4 z F_{j01} F_{j\tilde{0}\tilde{1}} + c. c.$$ (1.2)

which does not depend on the metric. I call this lagrangian model quantum cosmodynamic. But there is an essential difference between the two and four dimensional cases. In two dimensions any orientable surface admits a metric which takes the form $ds^2 = 2dzd\tilde{z}$. In four dimensions a metric takes the convenient form $ds^2 = 2g_{\alpha\tilde{\beta}} dz^\alpha d\tilde{z}^\beta$, if it admits two geodetic and shear free null congruences. This turns out to restrict the spacetimes to a special kind of totally real submanifolds\[1\] of a complex manifold, which I will call lorentzian CR manifolds. This metric independence makes the quantum cosmodynamic model formally renormalizable. I think that it is the unique known generally covariant renormalizable model with first order derivatives in four dimensions. In this paper I will review the mathematical properties of these lorentzian CR manifolds in order to make the problems of the quantum cosmodynamic model\[7\] more comprehensive to mathematicians.

In the context of the Newman-Penrose (NP) formalism in General relativity we usually work with the tangent space basis (tetrad) $(\ell^\mu \partial_\mu, n^\mu \partial_\mu, m^\mu \partial_\mu, \overline{m}^\mu \partial_\mu)$, where the first two are real, the third one is complex and the forth one is the complex conjugate of the third vector, as the notation indicates. The following commutation relations define the NP coefficients

$$[\ell^\mu \partial_\mu, n^\nu \partial_\nu] = - (\gamma + \overline{\gamma}) \ell^\rho \partial_\rho - (\varepsilon + \overline{\varepsilon}) m^\rho \partial_\rho + (\overline{\gamma} + \gamma) m^\rho \partial_\rho + (\tau + \overline{\tau}) m^\rho \partial_\rho$$

$$[\ell^\mu \partial_\mu, m^\nu \partial_\nu] = (\overline{\pi} - \pi - \beta) \ell^\rho \partial_\rho - \kappa n^\rho \partial_\rho + (\overline{\varepsilon} + \varepsilon) m^\rho \partial_\rho + \sigma \overline{m}^\rho \partial_\rho$$

$$[n^\mu \partial_\mu, m^\nu \partial_\nu] = \overline{\pi} \ell^\rho \partial_\rho + (\pi + \beta - \tau) n^\rho \partial_\rho + (\gamma - \overline{\gamma} - \mu) m^\rho \partial_\rho - \overline{m}^\rho \partial_\rho$$

$$[m^\mu \partial_\mu, \overline{m}^\nu \partial_\nu] = (\mu - \overline{\mu}) \ell^\rho \partial_\rho + (\rho - \overline{\rho}) m^\rho \partial_\rho + (\overline{\beta} - \alpha) m^\rho \partial_\rho + (\overline{\pi} - \pi) m^\rho \partial_\rho$$

(1.3)

The corresponding basis of the cotangent space $(\ell, n, m, \overline{m})$ is defined via the relations
\ell^\mu n_\mu = 1, \ m^\nu \overline{m}_\nu = -1, \ all \ other \ contractions \ vanish \quad (1.4)

I point out that I have not yet introduced any metric. The CR structure does not need the notion of the metric. The conditions (1.4) are implied by properly inverting the $4 \times 4$ matrix $(\ell_\mu, n_\mu, m_\mu, \overline{m}_\mu)$. Then one can find the following differential forms

\[d\ell = -(\varepsilon + \overline{\varepsilon})\ell \wedge n + (\alpha + \beta - \tau)\ell \wedge m + (\overline{\alpha} + \beta - \tau)\ell \wedge \overline{m} - \\
-\kappa n \wedge m - \kappa n \wedge \overline{m} + (\rho - \overline{\rho})m \wedge \overline{m} \]

\[dn = -(\gamma + \overline{\gamma})\ell \wedge n + \nu \ell \wedge m + \overline{\nu} \ell \wedge \overline{m} + (\pi - \alpha - \overline{\beta})n \wedge m + \\
+ (\overline{\pi} - \overline{\alpha} - \beta)n \wedge \overline{m} + (\mu - \overline{\mu})m \wedge \overline{m} \]

\[dm = -(\tau + \overline{\tau})\ell \wedge n + (\gamma - \overline{\gamma} + \overline{\pi})\ell \wedge m + \overline{\lambda} \ell \wedge \overline{m} + \\
+ (\varepsilon - \overline{\varepsilon} - \rho)n \wedge m - \sigma n \wedge \overline{m} + (\beta - \overline{\beta})m \wedge \overline{m} \quad (1.5)\]

**DEFINITION:** A regular tetrad $(\det(e^a_\mu) \neq 0, \infty)$ satisfying the relations $\kappa = \sigma = \lambda = \nu = 0$ defines a lorentzian CR structure. Then we will say that the corresponding manifold admits a lorentzian CR structure and it will be called lorentzian CR manifold.

Using the tetrad, the lorentzian CR structure conditions take the form

\[(\ell^\mu m^\nu - \ell^\nu m^\mu)(\partial_\mu \ell_\nu) = 0, \quad (\ell^\mu m^\nu - \ell^\nu m^\mu)(\partial_\mu m_\nu) = 0 \quad (1.6)\]

Then Frobenius theorem states that there are four independent complex functions $(z^\alpha, \overline{z}^{\tilde{\alpha}})$, $\alpha = 0, 1$, such that

\[dz^\alpha = f_\alpha \ell_\mu dx^\mu + h_\alpha m_\mu dx^\mu, \quad dz^{\tilde{\alpha}} = f^{\tilde{\alpha}} n_\mu dx^\mu + h^{\tilde{\alpha}} \overline{m}_\mu dx^\mu \]

\[\ell = \ell_\alpha dz^\alpha, \quad m = m_\alpha dz^\alpha \]

\[n = n^{\tilde{\alpha}} dz^{\tilde{\alpha}}, \quad \overline{m} = m^{\tilde{\alpha}} dz^{\tilde{\alpha}} \quad (1.7)\]

The reality conditions of the tetrad imply that the structure coordinates $z^b = (z^\alpha, \overline{z}^{\tilde{\alpha}})$, $\alpha = 0, 1$ satisfy the relations

\[dz^0 \wedge dz^1 \wedge dz^0 \wedge dz^{\overline{1}} = 0 \]

\[dz^\delta \wedge dz^0 \wedge dz^{\overline{0}} \wedge dz^{\overline{\delta}} = 0 \]

\[dz^{\overline{\delta}} \wedge dz^{\overline{0}} \wedge dz^{\overline{0}} \wedge dz^{\overline{\delta}} = 0 \quad (1.8)\]

that is, there are two real functions $\rho_{11}, \rho_{22}$ and a complex one $\rho_{12}$, such that
\[
\rho_{11}(\overline{z}^\alpha, z^\alpha) = 0 , \quad \rho_{12}(\overline{z}^\alpha, z^{\tilde{\alpha}}) = 0 , \quad \rho_{22}(z^\alpha, z^{\tilde{\alpha}}) = 0 \tag{1.9}
\]

According to the conventional terminology, the manifold is locally a totally real submanifold of \( C^4 \). These relations and the complex structure of the ambient manifold determine a lorentzian CR structure on the 4-dimensional manifold. Notice that the defining functions do not depend on all the structure coordinates. The precise dependence of the defining functions on the structure coordinates characterizes the lorentzian CR structure from the general definition of a totally real submanifold of \( C^4 \).

The four functions \( z^b \equiv (z^\alpha, z^{\tilde{\alpha}}), \alpha = 0, 1 \) are the structure coordinates of the (integrable) complex structure. The holomorphic transformations which preserve the lorentzian CR structure are

\[
z'^\alpha = f^\alpha(z^\beta) , \quad z'^{\tilde{\alpha}} = f^{\tilde{\alpha}}(z^{\tilde{\beta}}) \tag{1.10}
\]

I point out that the general holomorphic transformations \( z'^b = f^b(z^c) \) do not preserve the lorentzian CR structure!

The inverse procedure to find a tetrad \((\ell, n, m, m)\) from the defining conditions (1.9) is straightforward. It is convenient to use the notation \( \partial' f = \frac{\partial f}{\partial z^\alpha} dz^\alpha \) and \( \partial'' f = \frac{\partial f}{\partial z^{\tilde{\alpha}}} dz^{\tilde{\alpha}} \). Because of \( d\rho_{ij} = 0 \) and the special dependence of each function on the structure coordinates \((z^\alpha, z^{\tilde{\alpha}})\), we find

\[
\ell = 2i\partial \rho_{11} = 2i\partial' \rho_{11} = i(\partial'' - \overline{\partial'})\rho_{11} = -2i\overline{\partial'} \rho_{11} \\
n = 2i\partial \rho_{22} = 2i\partial'' \rho_{22} = i(\partial'' - \overline{\partial'})\rho_{22} = -2i\overline{\partial'} \rho_{22} \tag{1.11}
\]

\[
m_1 = 2i\partial\frac{\overline{\rho_{12}}}{2} = i(\partial' + \overline{\partial'})\frac{\overline{\rho_{12}}}{2} \\
m_2 = 2i\partial\frac{\overline{\rho_{12}}}{2} = i(\partial' + \overline{\partial'})\frac{\overline{\rho_{12}}}{2} 
\]

where we consider all these forms restricted on the defined submanifold, therefore they are real. The relations become simpler if we use the complex form

\[
m = m_1 + im_2 = 2i\partial' \rho_{12} = -2i\overline{\partial'} \rho_{12} = i(\partial'' - \overline{\partial''})\rho_{12} \tag{1.12}
\]

In the next subsection, the typical example of the Kerr-Newman lorentzian CR structure will be presented. In section II, the first three relative invariants of the lorentzian CR structure will be defined. In the case of non-vanishing relative invariants, the lorentzian CR structure is determined by two real and one complex vector fields. It will also become clear that the static spherically symmetric spacetimes have trivial lorentzian CR structures. In sections III and IV, we present classifications of the lorentzian CR structures based on a regular compatible metric and the Flaherty pseudo-complex structure. In section V we show
that a parameter dependent antiholomorphic transformation of a 3-dimensional CR structure defines a lorentzian CR structure. In section VI an osculation of the lorentzian CR structure is written down using the homogeneous coordinates of the $G_{2,2}$ grassmannian manifold. This reveals the Poincaré group. In section VII, we use these Poincaré transformations to find static axially symmetric lorentzian CR manifolds. In section VIII I find a class of Kaehler metrics, which induce compatible metrics on the submanifold. The lorentzian CR submanifold is a lagrangian submanifold relative to the corresponding Kaehler 2-form.

1.1 The Kerr-Newman lorentzian CR structure

The spacetimes with two geodetic and shear free null congruences are typical examples of lorentzian CR manifolds. The flat spacetime tetrad

$$L^\mu \partial_\mu = \partial_t + \partial_r$$

$$N^\mu \partial_\mu = \frac{r^2+a^2 \cos^2 \theta}{2(r^2+a^2)} \left( \partial_t - \partial_r + \frac{2a}{r^2+a^2} \partial_\phi \right)$$

$$M^\mu \partial_\mu = \frac{1}{\sqrt{2(r+ia \cos \theta)}} \left( ia \sin \theta \partial_t + \partial_\theta + \frac{i}{\sin \theta} \partial_\phi \right)$$

(1.13)

and its corresponding covariant form

$$L_\mu dx^\mu = dt - dr - a \sin^2 \theta \, d\phi$$

$$N_\mu dx^\mu = \frac{r^2+a^2}{2(r^2+a^2)} [dt + \frac{r^2+2a^2 \cos^2 \theta - a^2}{r^2+a^2} dr - a \sin^2 \theta \, d\phi]$$

$$M_\mu dx^\mu = \frac{-1}{\sqrt{2(r+ia \cos \theta)}} [ -ia \sin \theta \, (dt - dr) + (r^2+a^2 \cos^2 \theta) d\theta + i \sin \theta (r^2+a^2) d\phi]$$

(1.14)

determine a lorentzian CR structure. This definition implies that the structure is singular on the cylinder $r = 0$, $\theta = \frac{\pi}{2}$. We will see below that a more appropriate definition shows that this singularity is artificial, depending on the structure coordinates of the ambient complex manifold. Using the Kerr-Schild ansatz, we can find the following curved spacetime lorentzian CR structure

$$\ell_\mu = L_\mu \quad , \quad m_\mu = M_\mu \quad , \quad n_\mu = N_\mu + \frac{h(r)}{2(r^2+a^2 \cos^2 \theta)} L_\mu$$

$$\ell^\mu = L^\mu \quad , \quad m^\mu = M^\mu \quad , \quad n^\mu = N^\mu - \frac{h(r)}{2(r^2+a^2 \cos^2 \theta)} L^\mu$$

(1.15)

where $h(r)$ is an arbitrary function, which gives a regular spacetime. I will call it Kerr-Newman lorentzian CR structure because $h(r) = -2mr + e^2$ gives the Kerr-Newman (KN) spacetime. A set of structure coordinates of the curved complex structure are

$$z^0 = t - r + ia \cos \theta - ia \quad , \quad z^1 = e^{i\varphi} \tan \frac{\theta}{2}$$

$$\tilde{z}^0 = t + r - ia \cos \theta + ia - 2f_1 \quad , \quad \tilde{z}^1 = \frac{-r+ia}{r-ia} e^{2iaf_2} e^{-i\varphi} \tan \frac{\theta}{2}$$

(1.16)
where the two new functions are
\[ f_1(r) = \int \frac{h}{r^2 + a^2 + h} \, dr \quad , \quad f_2(r) = \int \frac{h}{(r^2 + a^2 + h)(r^2 + a^2)} \, dr \quad (1.17) \]

The following relations give Newman-Penrose spin coefficients
\[
\begin{align*}
\alpha &= \ell m \partial n + (n \partial m) - (m \partial n) - 2(n \bar{m} \partial \bar{m}) \\
\beta &= \ell m \partial n + (\bar{m} \partial m) - (n \partial \bar{m}) - 2(n \bar{m} \partial \bar{m}) \\
\gamma &= \ell n \partial \ell + (\bar{m} \partial m) - (m \partial \bar{m}) + 2(\ell \partial \ell) \\
\varepsilon &= \ell n \partial \ell - (\bar{m} \partial m) - (m \partial \bar{m}) + 2(\ell \partial \ell) \\
\mu &= -\ell m \partial m + (n \partial m) + (\bar{m} \partial \bar{m}) \\
\pi &= \ell m \partial m - (n \partial m) - (\bar{m} \partial m) \\
\rho &= \ell m \partial n + (n \partial n) - (m \partial \bar{m}) \\
\tau &= \ell n \partial \ell + (\bar{m} \partial m) - (m \partial \bar{m}) \\
\kappa &= \ell m \partial m, \quad \sigma = (\ell m \partial m) \\
\nu &= -\ell m \partial m, \quad \lambda = -(n \bar{m} \partial m) \\
\end{align*}
\]

where the symbols (\ldots) are constructed according to the rule of the following example \((\ell n \partial n) = (\ell m \nu - \ell n \mu)(\partial \mu \nu)\). These symbols can be directly computed
\[
\begin{align*}
(\ell n \partial n) &= 0 \quad , \quad (\ell m \partial n) = 0 \quad , \quad (n m \partial \ell) = \frac{\sqrt{\sin \theta} \cos \theta \sin \theta}{(r + i a \cos \theta)(r - \bar{a} \cos \theta)} \\
(m n \partial m) &= \frac{2 i a \cos \theta}{(r + a \cos \theta)^2} , \quad (\ell \partial m) = -\frac{2 a^2 \sin \theta \theta - 2 h + (r + a \cos \theta) h}{2(r + a \cos \theta)(r + a \cos \theta)^2} \\
(\ell m \partial n) &= 0 \quad , \quad (n m \partial \ell) = 0 \quad , \quad (n \bar{m} \partial m) = \frac{\sqrt{\sin \theta} \cos \theta \sin \theta}{h^2} \\
(m n \partial m) &= \frac{-\sqrt{(r - i a \cos \theta)(r + a \cos \theta)} \sin \theta}{r \cos \theta + i a} , \quad (\ell \partial m) = 0 \quad , \quad (n \bar{m} \partial m) = 2(r - i a \cos \theta)(r + a \cos \theta) \\
(m n \partial m) &= -\frac{\sqrt{(r - i a \cos \theta)(r + a \cos \theta)} \sin \theta}{(r - i a \cos \theta)^2} \\
\end{align*}
\]

as the first step to the computation of the NP spin coefficients of this null tetrad in the above Lindquist coordinates
\[
\begin{align*}
\alpha &= \frac{a^2 + 1 + s - t + r - i a \cos \theta}{2} \quad , \quad \beta = \frac{-c + r - i a \cos \theta}{2} \quad , \quad \gamma = -\frac{a^2 + 1 + s - t + r - i a \cos \theta}{2} \quad , \quad \varepsilon = 0 \\
\mu &= -\frac{a^2 + 1 + s - t + r - i a \cos \theta}{2} \quad , \quad \pi = \frac{a^2 + 1 + s - t + r - i a \cos \theta}{2} \\
\rho &= -\frac{a^2 + 1 + s - t + r - i a \cos \theta}{2} \quad , \quad \tau = \frac{a^2 + 1 + s - t + r - i a \cos \theta}{2} \\
\kappa &= 0 \quad , \quad \sigma = 0 \quad , \quad \nu = 0 \quad , \quad \lambda = 0 \\
\end{align*}
\]

We will use the KN CR manifold as an example for a better understanding of the properties of lorentzinian CR manifolds.
2 RELATIVE INVARIANTS OF LORENTZIAN CR STRUCTURES

Applying the conditions \( \kappa = \sigma = \lambda = \nu = 0 \) to the relations (1.5) we find that the lorentzian CR structure is determined by

\[
d\ell = Z_1 \wedge \ell + (\rho - \overline{\rho}) m \wedge \overline{m}
\]
\[
dn = Z_2 \wedge n + (\mu - \overline{\mu}) m \wedge \overline{m}
\]
\[
dm = Z \wedge m - (\tau + \overline{\tau}) \ell \wedge n
\]

where

\[
Z_{1\mu} = (\theta_1 + \mu + \overline{\mu}) \ell_\mu + (\varepsilon + \overline{\varepsilon}) n_\mu - (\alpha + \overline{\beta} - \tau) m_\mu - \frac{1}{2} \left( \gamma + \overline{\gamma} \right) n_\mu \]
\[
Z_{2\mu} = - (\gamma + \overline{\gamma}) \ell_\mu + (\theta_2 - \rho - \overline{\rho}) n_\mu - (\pi - \alpha - \overline{\beta}) m_\mu - \frac{1}{2} \left( \gamma + \overline{\gamma} \right) n_\mu \]
\[
Z_\mu = (\gamma - \overline{\gamma} + \overline{\mu}) \ell_\mu + (\varepsilon - \overline{\varepsilon} - \rho) n_\mu - (\theta_3 + \pi - \overline{\beta}) m_\mu - \frac{1}{2} \left( \gamma + \overline{\gamma} \right) n_\mu
\]

with the functions \( \theta_1, \theta_2, \theta_3 \) a priori arbitrary. Because of this particular form (2.1) we can apply Frobenius theorem and define the structure coordinates in the case of real analytic functions.

The integrability conditions of the lorentzian CR structure are invariant under the transformations

\[
\ell'_\mu = \Lambda \ell_\mu , \quad \ell'^\mu = \frac{1}{\Lambda} \ell^\mu
\]
\[
n'_\mu = N n_\mu , \quad n'^\mu = \frac{1}{\Lambda} n^\mu
\]
\[
m'_\mu = M m_\mu , \quad m'^\mu = \frac{1}{\Lambda} m^\mu
\]

which we will call tetrad-Weyl transformations. Under these transformations the NP spin coefficients transform as follows

\[
\alpha' = \frac{1}{M} \alpha + \frac{M}{4MN} \frac{\Delta}{M^2} \ln \frac{M}{AN} + \frac{1}{2N} \frac{\Delta}{M^2} \ln \frac{M}{AN}
\]
\[
\beta' = \frac{1}{M} \beta + \frac{M}{4MN} \frac{\Delta}{M^2} \ln \frac{M}{AN} + \frac{1}{2N} \frac{\Delta}{M^2} \ln \frac{M}{AN}
\]
\[
\gamma' = \frac{1}{4} \gamma + \frac{M}{4MN} \frac{\Delta}{M^2} \ln \frac{M}{AN} + \frac{1}{2N} \frac{\Delta}{M^2} \ln \frac{M}{AN}
\]
\[
\varepsilon' = \frac{1}{4} \varepsilon + \frac{M}{4MN} \frac{\Delta}{M^2} \ln \frac{M}{AN} + \frac{1}{2N} \frac{\Delta}{M^2} \ln \frac{M}{AN}
\]
\[
\mu' = \frac{1}{2M} (\mu + \overline{\mu}) + \frac{M}{2N} (\mu - \overline{\mu}) + \frac{1}{2N} \frac{\Delta}{M^2} \ln (M \overline{M})
\]
\[
\rho' = \frac{1}{2M} (\rho + \overline{\rho}) + \frac{M}{2N} (\rho - \overline{\rho}) + \frac{1}{2N} \frac{\Delta}{M^2} \ln (M \overline{M})
\]
\[
\pi' = \frac{1}{2N} (\pi + \overline{\pi}) + \frac{1}{2M} (\pi - \overline{\pi}) + \frac{1}{2N} \frac{\Delta}{M^2} \ln (M \overline{M})
\]
\[
\tau' = \frac{1}{2N} (\tau + \overline{\tau}) + \frac{1}{2M} (\tau - \overline{\tau}) + \frac{1}{2N} \frac{\Delta}{M^2} \ln (M \overline{M})
\]
\[
\kappa' = \frac{1}{MN} \kappa , \quad \sigma' = \frac{1}{2M} \sigma
\]
\[
\nu' = \frac{1}{NM} \nu , \quad \lambda' = \frac{1}{2M} \lambda
\]
We see that the following relations
\[\rho' - \rho = \frac{\lambda}{M N} (\rho - \rho)\]
\[\mu' - \mu = \frac{\lambda^2}{M N} (\mu - \mu)\]
\[\tau' + \tau = \frac{\lambda^2}{M N} (\tau + \tau)\] (2.5)
establish the corresponding quantities as relative invariants of the lorentzian CR structure. That is, the lorentzian CR structures are characterized by the annihilation or not of these three quantities. A lorentzian CR structure with vanishing one of these three quantities is not equivalent with a lorentzian CR structure. That is, the lorentzian CR structures are characterized by the corresponding quantities as relative invariants of the lorentzian structure preserving transformations (2.3)

If a ≠ 0 it has non-vanishing all the three relative invariants.

These lorentzian CR structures will be called generic and the corresponding unique tetrad normalized.

The KN lorentzian CR structure has
\[\rho' - \rho = \frac{-2i a \cos \theta}{(r + ia \cos \theta)(r - ia \cos \theta)}\]
\[\mu' - \mu = \frac{i a (r^2 + a^2 + h) \cos \theta}{(r + ia \cos \theta)(r - ia \cos \theta)}\]
\[\tau' + \tau = \frac{1}{(r + ia \cos \theta)^2(r - ia \cos \theta)}\] (2.7)

Hence their differential forms
\[F_1 = dZ_1\]
\[F_2 = dZ_2\]
\[F = dZ\] (2.9)
are CR invariants. In the generic case of non vanishing relative invariants, the gauge transformations are satisfied if
\[\theta_1 = n^\mu \partial_\mu \ln \frac{\rho - \rho}{\tau + \tau}\]
\[\theta_2 = \ell^\mu \partial_\mu \ln \frac{\mu - \mu}{\tau + \tau}\]
\[\theta_3 = \bar{m}^\mu \partial_\mu \ln (\tau - \tau)\] (2.10)

In the case of the KN lorentzian CR manifold I find
\[Z_1^\mu = \frac{\sqrt{2} a^2 \sin \theta \cos \theta}{(r + ia \cos \theta)(r - ia \cos \theta)} M_\mu + \frac{\sqrt{2} a^2 \sin \theta \cos \theta}{(r + ia \cos \theta)(r - ia \cos \theta)} M_\mu\]
\[Z_2^\mu = \frac{\sqrt{2} a^2 \sin \theta \cos \theta}{(r + ia \cos \theta)(r - ia \cos \theta)} M_\mu - \frac{\sqrt{2} a^2 \sin \theta \cos \theta}{(r + ia \cos \theta)(r - ia \cos \theta)} M_\mu\]
\[Z_3^\mu = \frac{\sqrt{2} a^2 \sin \theta \cos \theta}{(r + ia \cos \theta)(r - ia \cos \theta)} M_\mu - \frac{\sqrt{2} a^2 \sin \theta \cos \theta}{(r + ia \cos \theta)(r - ia \cos \theta)} M_\mu\]

(2.11)
Notice that the static spherically symmetric spacetimes have vanishing relative invariants. Hence their lorentzian CR structure is trivial.

Concluding the present section I want to recall the structure equations of the $U(2)$ group, written in the following appropriate form

$$dl = im \wedge \overline{m}, \quad dn = -im \wedge \overline{m}, \quad dm = i(l - n) \wedge m$$

(2.12)

Notice that it is a lorentzian CR manifold with vanishing the third relative invariant $\tau + \overline{\tau} = 0$. Hence the expression (2.1) may be viewed as a Cartan $U(2)$ osculation of the lorentzian CR structure.

3 WEYL CURVATURE CLASSIFICATION OF LORENTZIAN CR STRUCTURES

We already know that the lorentzian CR structure does not uniquely determine the tetrad $(\ell, n, m, \overline{m})$. It is defined up to a tetrad-Weyl transformation (2.3). Hence the following real lorentzian metric

$$g_{\mu\nu} = \ell_\mu n_\nu + n_\mu \ell_\nu - m_\mu \overline{m}_\nu - \overline{m}_\mu m_\nu$$

(3.1)

is not uniquely determined. But the inverse problem of ”how many lorentzian CR structures are compatible with a regular lorentzian riemannian manifold?”, finds a very interesting solution.

Taking into account that the integrability conditions of a lorentzian CR structure coincide with the existence of two geodetic and shear free null congruences $\ell^\mu$ and $n^\mu$, the problem turns to that of finding the geodetic and shear free null congruences of a metric $g_{\mu\nu}$. But not all the metrics admit geodetic and shear free null congruences. Hence not all the metrics are compatible with a lorentzian CR structure.

**Definition:** If a null tetrad of a metric $g_{\mu\nu}$ satisfies the lorentzian CR structure conditions $\kappa = \sigma = \lambda = \nu = 0$, we will say that the corresponding lorentzian riemannian manifold admits a lorentzian CR structure.

It is well known that the null tetrads of a metric $g_{\mu\nu}$ transform between each other with a Poincaré transformation[2]. We will always consider regular metrics up to an appropriate coordinate transformation. These metrics always admit at least one regular null tetrad up to an appropriate coordinate transformation. This tetrad does not necessarily determine geodetic and shear free congruences. But the existence of such a regular metric and a regular null tetrad implies that all the scalar quantities of the riemannian geometry are regular functions. The vector $\ell'^\mu$ of a new null tetrad is related to $(\ell, n, m, \overline{m})$, with the relation

$$\ell'^\mu = \ell^\mu + b m^\mu + \overline{b} m^\mu$$

(3.2)

where $b(x)$ is a complex function. It is geodetic and shear free if $\kappa' = \sigma' = 0$, which implies the algebraic relation[2].
\[ \Psi_0' = \Psi_0 + 4b\Psi_1 + 6b^2\Psi_2 + 4b^3\Psi_3 + b^4\Psi_4 = 0 \] (3.3)

The quantities \( \Psi_i \) are the Weyl scalars in the NP formalism determined with the regular null tetrad. Therefore they must be regular functions on the manifold. Hence we conclude that a geodetic and shear free null congruence is determined by a four degree polynomial with regular coefficients. Taking into account that a lorentzian CR structure is determined by two geodetic and shear free congruences, we conclude to the following four cases:

**Case I**: \( \Psi_1 \neq 0, \Psi_2 \neq 0, \Psi_3 \neq 0 \)

**Case II**: \( \Psi_1 \neq 0, \Psi_2 \neq 0, \Psi_3 = 0 \)

**Case III**: \( \Psi_1 \neq 0, \Psi_2 = 0, \Psi_3 = 0 \)

**Case D**: \( \Psi_1 = 0, \Psi_2 \neq 0, \Psi_3 = 0 \)

where \( \Psi_i, i = 1, 2, 3 \) are now the non-vanishing Weyl scalars relative to the tetrad which defines the lorentzian CR structure. If the lorentzian CR structure is compatible with a conformally flat metric no restriction/classification is imposed. I want to point out that "conformal flatness" is not invariant under the tetrad-Weyl transformation (2.3). Therefore this property applies to the larger set of spacetimes which become flat after a tetrad-Weyl transformation.

## 4 Flaherty’s Pseudo-Complex Structure

Any real analytic function on a totally real submanifold can be extended to a holomorphic function on the ambient complex manifold. Applying this property, the metric (3.1) takes a special form. Using the structure coordinates \( (z^\alpha, \bar{z}^\alpha) \), \( \alpha = 0, 1 \), the compatible metric \( g_{\mu\nu} \) takes the form

\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu = 2g_{\alpha\bar{\beta}}dz^\alpha d\bar{z}^\beta \] (4.1)

where \( g_{\alpha\beta} \) are holomorphic functions of the structure coordinates and the last term is taken on the lorentzian CR manifold. Its extension in the ambient complex manifold reveals the invariance of the metric under the transformation \( z'^\alpha = f^\alpha(z^\beta), \bar{z}'^{\alpha} = f^{\bar{\alpha}}(\bar{z}^{\bar{\beta}}) \).

In order to describe these covariant transformations Flaherty introduced the integrable pseudo-complex (hermitian) structure

\[ J_\mu^\nu = i(e_\mu n^\nu - n_\mu e^\nu - m_\mu \bar{m}^\nu + \bar{m}_\mu m^\nu) \]

\[ J_\mu^\nu J_\rho^\nu = -\delta_\mu^\rho, \quad J_\mu^\nu g_{\rho\sigma} = \delta_\mu^\sigma \] (4.2)
Notice that this complex structure is not a real tensor like the ordinary complex structure. In this presentation we will not use this notion and we will stick on the more conventional notion of the CR structure. I want simply to point out that if we find a way to fix the metric then the connection $\gamma^a_{bc}$ of this pseudo-complex structure is an adequate process to provide a simple classification of the lorentzian CR structures. The non vanishing components of this connection is

$$\gamma^a_{\beta\gamma} = g^{\alpha\tilde{\alpha}} \partial_\beta g_{\gamma\tilde{\alpha}} \quad , \quad \gamma^\alpha_{\beta\gamma} = g^{a\tilde{\alpha}} \partial_\beta g_{a\gamma}$$  (4.3)

The covariant derivative is defined as usual

$$D_aT^b = \partial_a T^b + \gamma^b_{ac} T^c , \quad D_aT_b = \partial_a T_b - \gamma^c_{ab} T_c$$  (4.4)

and one can easily show that this covariant derivative annihilates the metric, $D_ag_{bc} = 0$.

Flaherty has also showed that if the torsion of this connection $T^c_{ab} = \gamma^c_{ba} - \gamma^c_{ab}$ vanishes, the complex structure is kaehlerian, $d(J_{\mu\nu} \ dx^\mu \wedge dx^\nu)$, and the vectors of the null tetrad are hypersurface orthogonal. This means that the lorentzian CR structure is trivial.

In the case of generic lorentzian CR structures we may use these covariant derivatives to formulate the conventional classification scheme. We first write in structure coordinates the unique generic metric $g_{\mu\nu}$ created from the unique generic tetrad implied by fixing the relative invariants as in (2.6). We compute the Flaherty connection, the corresponding torsion and curvature. Then we create all their covariant derivatives. All the scalars one may construct with the multiplication of these covariant tensors, properly contracted using the generic metric, are invariant to the lorentzian CR structure preserving transformations. These scalars permit us to distinguish non-equivalent lorentzian CR structures.

5 A PAIR OF 3-DIMENSIONAL CR STRUCTURES

From the forms of the of the four real relations (1.9) which define the totally real submanifold of an 8 (real) dimensional complex manifold, we see that the lorentzian CR structure contains two 3-dimensional CR manifolds of the hypersurface type, which have been extensively studied[4]. We will denote $I^+$ the 3-dimensional submanifold determined by the condition $\rho_{11}(\vec{z}^\alpha, \vec{z}^\alpha) = 0$ and $I^-$ the 3-dimensional submanifold determined by $\rho_{22}(\vec{z}^{\tilde{\alpha}}, \vec{z}^{\tilde{\alpha}}) = 0$. We see that the complex tangent spaces of these two CR manifolds coincide and they are spanned by $m^\mu \partial_\mu$ and $\overline{m}^{\mu} \partial_\mu$. This coincidence relation is implied by the complex condition $\rho_{12}(\vec{z}^\alpha, \vec{z}^{\tilde{\alpha}}) = 0$.

The classification of the 3-dimensional CR manifolds of the hypersurface type have been extensively studied[4], using either the Moser or the Cartan approach. In the present section I will study the importance of these two 3-dimensional
CR manifolds in the context of the lorentzian CR structure, considering known these two fundamental classification schemes.

We start from a 3-dimensional CR structure \( \rho_{11} (z^\alpha, z^\alpha) = 0 \) and we build the family of the CR structures implied by a parameter \( w \) dependent antiholomorphic transformation \( z^\beta = f^\beta(z^\alpha; w) \). Then removing the generic parameter \( w \), we find a complex relation \( \rho_{12} (z^\alpha, \bar{z}^\beta) = 0 \) between the structure coordinates. Hence the final output is the following lorentzian CR structure

\[
z^\beta = f^\beta \left( z^\alpha; w \right)
\]

\[
\rho_{22} (z^\alpha, \bar{z}^\beta) = \rho_{22} \left( f^\beta(z^\beta), \bar{f}^\beta(\bar{z}^\alpha) \right) = \rho_{11} (z^\alpha, z^\alpha) = 0
\]

\[
\rho_{12} (z^\beta, \bar{z}^\alpha) = 0
\]

We see that this kind of lorentzian CR structures have (anti)holomorphic \( \mathcal{I}^+ \) and \( \mathcal{I}^- \) 3-dimensional submanifold.

It is well known that we can find complex coordinates \( z^0 = u + iU \) and \( \bar{z}^0 = v + iV \), such that the embedding real functions of the 3-dimensional CR submanifolds take the form

\[
i \left( z^{\beta} - z^0 \right) - 2U \left( u, z^1, z^1 \right) = 0
\]

\[
i \left( \bar{z}^{0} - \bar{z}^{0} \right) - 2V \left( v, \bar{z}^1, \bar{z}^1 \right) = 0
\]

Then we can define two characteristic coordinate systems of the embedable lorentzian CR manifold. The \( (u, v, z^1, \bar{z}^1) \) implied by the (Frobenius) submanifold \( \mathcal{I}^+ \) and \( (u, v, z^1, \bar{z}^1) \) implied by \( \mathcal{I}^- \).

The pseudoconvex 3-dimensional CR structures are usually osculated with the hyperquadric. Therefore a typical osculation of the lorentzian CR structures with \( (\rho - \mu \neq 0 \neq \mu - \bar{\mu}) \) is the following lorentzian CR structure

\[
\left( \begin{array}{cccc}
\frac{\zeta^{01}}{\bar{\xi}} & \frac{\zeta^{11}}{\bar{\xi}} & \frac{\zeta^{21}}{\bar{\xi}} \\
\zeta^{02} & \zeta^{12} & \zeta^{22} \\
0 & 0 & -i \\
i & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{array} \right)
\left( \begin{array}{cccc}
\xi^{01} & \xi^{11} & \xi^{21} \\
0 & 2 & 0 \\
i & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{array} \right) = 0
\]

\[
z^0 = \zeta^{01} \xi^{01}, \quad z^1 = \zeta^{11} \xi^{11}, \quad \bar{z}^0 = -\zeta^{02} \xi^{02}, \quad \bar{z}^1 = -\zeta^{12} \xi^{12}
\]

Hence a general lorentzian CR structure may be formally written as

\[
\left( \begin{array}{cccc}
\xi^{01} & \xi^{11} & \xi^{21} \\
0 & 2 & 0 \\
i & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{array} \right) \left( \begin{array}{cccc}
\sigma^{01} & \sigma^{02} \\
\sigma^{11} & \sigma^{12} \\
\sigma^{21} & \sigma^{22} \\
\end{array} \right)
\]

\[
= \left( \begin{array}{cc}
\sigma_{11} (\zeta^{01}, \zeta^{01}) & \sigma_{12} (\zeta^{01}, \zeta^{12}) \\
\sigma_{12} (\zeta^{02}, \zeta^{02}) & \sigma_{22} (\zeta^{02}, \zeta^{02}) \\
\end{array} \right)
\]

13
where $\sigma_{ij}$ are homogeneous functions of their corresponding arguments. This form preserves the $SU(1, 2)$ group and vice-versa the group preserves the lorentzian CR manifold. Hence the lorentzian CR manifolds appear as transitive sets of the $SU(1, 2)$ group, which we will denote as $\Sigma[SU(1, 2)]$. In a quantum field theoretic model, which is invariant under the lorentzian CR structure preserving transformations, the $\Sigma[SU(1, 2)]$ sets will appear as multiplets of the $SU(1, 2)$ group if the group is not spontaneously broken.

6 SURFACES OF THE SU(2,2) CLASSICAL DOMAIN

In section II we found that a general spacetime admits at most four geodetic and shear free null congruences. Therefore it can be compatible with a limited number of lorentzian CR structures. This limitation does not apply to spacetimes which are flat up to a tetrad-Weyl transformation. From the twistor formalism we know that the lorentzian CR structures, which are compatible with the Minkowski metric are given by the following embedding functions

$$\bar{X}^{mi}E_{mn}X^{nj} = 0$$

$$K_1(X^{m_1}) = 0 = K_2(X^{m_2})$$

where the $K_i(X^{m_i})$ are homogeneous functions, which we will call Kerr functions. Apparently, instead of considering two different Kerr functions, one may consider two different points of the same algebraic surface of $CP^3$. The $4 \times 4$ matrix $E_{mn}$ is the $SU(2, 2)$ preserving matrix. These surfaces will be called flat lorentzian CR structures. The $4 \times 2$ matrix

$$X = \begin{pmatrix} X^{01} & X^{02} \\ X^{11} & X^{12} \\ X^{21} & X^{22} \\ X^{31} & X^{32} \end{pmatrix}$$

must have rank 2. Using the chiral representation of $E_{mn}$

$$E = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

and the notation

$$X = \begin{pmatrix} \lambda \\ -ir\lambda \end{pmatrix}$$

for $\det \lambda \neq 0$, the first relation of (6.1) takes the form

$$i(r^\dagger - r) = 0$$

That is the lorentzian CR manifold is fixed without using the Kerr functions. This manifold is the characteristic (Shilov) boundary of the $SU(2, 2)$ classical...
The Kerr functions determine the lorentzian CR structures as \( CP^3 \) sections on this precise manifold. They are in fact defined by two different points of an algebraic surface of \( CP^3 \).

The following embedding functions

\[
X^{mi}E_{mn}X^{nj} = G_{ij}(X^{m1}, X^{m2})
\]

provide a formal osculation of a general lorentzian CR structure with the Shilov boundary of the \( SU(2, 2) \) classical domain.

Using the following spinorial form of the rank-2 matrix \( X^{mj} \) in its unbounded realization

\[
X^{mj} = \begin{pmatrix} \lambda^{Aj} \\ -ir_{A'B}\lambda^{Bj} \end{pmatrix}
\]

and the null tetrad

\[
L^a = \frac{1}{\sqrt{2}} \lambda^{A'B} \sigma_a^{AB} , \quad N^a = \frac{1}{\sqrt{2}} \lambda^{A'B} \sigma_a^{AB} , \quad M^a = \frac{1}{\sqrt{2}} \lambda^{A'B} \sigma_a^{AB} \quad \epsilon_{AB} \lambda^{A1} \lambda^{B2} = 1
\]

the above relations (6.6) take the form

\[
\rho_{11} = 2\sqrt{2}y^a L_a - G_{11}(Y^{m1}, Y^{n1}) = 0
\]

\[
\rho_{12} = 2\sqrt{2}y^a M_a - G_{12}(Y^{m1}, Y^{n2}) = 0
\]

\[
\rho_{22} = 2\sqrt{2}y^a N_a - G_{22}(Y^{m2}, Y^{n2}) = 0
\]

where \( y^a \) is the imaginary part of \( r^a = x^a + iy^a \) defined by the relation \( r_{A'B} = r^a\sigma_a^{AB} \) and \( \sigma_a^{AB} \) being the identity and the three Pauli matrices. The surface satisfies the relation

\[
y^a = \frac{1}{2\sqrt{2}}[G_{22}L^a + G_{11}N^a - G_{12}M^a - G_{12}M^a]
\]

which, combined with the computation of \( \lambda^{Ai} \) as functions of \( r^a \), using the Kerr conditions \( K_i(X^{mi}) \), permits us to compute \( y^a = h^a(x) \) as functions of the real part of \( r^a \). In fact the normal form[1] of any totally real submanifold of a complex manifold is \( y^a = y^a(x) \). The characteristic property of the lorentzian CR manifolds is that they can be considered as surfaces of the \( SU(2, 2) \) symmetric classical domain with \( y^a \) and \( x^a \) related to the projective coordinates of the chiral representation. Besides \( y^a(x) \) must satisfy the relation

\[
\begin{pmatrix} y^a L_a \\ y^a M_a \\ y^a N_a \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{pmatrix}
\]

with the homogeneity factors of \( G_{ij} \) properly fixed.
Notice that this surface may always be assumed to belong into the "upper half-plane" because the tetrad may be arranged such that \( y^0 > 0 \), but this surface does not generally belong into the \( SU(2, 2) \) Siegel fundamental domain, because

\[
y^a y^b \eta_{ab} = \frac{1}{4} [G_{11} G_{22} - G_{12} G_{12}] \tag{6.12}
\]

is not always positive. The regular surfaces (with an upper bound) can always be brought inside the Siegel domain (and its holomorphic bounded classical domain) with an holomorphic complex time translation.

\[
\begin{pmatrix}
  \lambda^{A_j} \\
  -i r' B^j \lambda^{B_j}
\end{pmatrix} =
\begin{pmatrix}
  I & 0 \\
  d I & I
\end{pmatrix}
\begin{pmatrix}
  \lambda^{A_j} \\
  -i r' B^j \lambda^{B_j}
\end{pmatrix} \tag{6.13}
\]

which implies the transformation \( y'^a = y^a + (d, 0, 0, 0) \). An appropriate constant \( d \) makes \( y'^0 > 0 \) and \( y'^a y'^b \eta_{ab} > 0 \). Apparently this constant \( d \) is not uniquely determined.

It is evident that my arguments for the \( SU(1, 2) \) covariant form (5.4) may be repeated here. The osculating form (6.6) preserves the \( SU(2, 2) \) group and vice-versa the group preserves the lorentzian CR structure. Hence the lorentzian CR manifolds appear as transitive sets of the \( SU(2, 2) \) group, which we will now denote as \( \Sigma[SU(2, 2)] \). In a quantum field theoretic model, which is invariant under the lorentzian CR structure preserving transformations, the \( \Sigma[SU(2, 2)] \) sets will appear as multiplets of the \( SU(2, 2) \) group, if the group is not spontaneously broken. The exact Poincaré group is expected to emerge after an appropriate spontaneous breaking of the \( SU(2, 2) \) symmetry. Therefore the sets \( \Sigma[Poincaré] \) may present special physical interest.

6.1 Asymptotically flat lorentzian CR manifolds

The osculation of the CR manifold with the Shilov boundary of the \( SU(2, 2) \) classical domain, permits us to transfer the notion of "asymptotically flat space-times at null infinities" into the terminology of lorentzian CR structures. Using the coordinates \( t, r \), such that

\[
\begin{align*}
\mp \frac{z^0}{2} &= u = t - r \\
\mp \frac{z^5}{2} &= v = t + r
\end{align*} \tag{6.14}
\]

we may consider that the gravitational content vanishes in the two limit-surfaces \( J^+ = \{ r \to \infty , \text{ with } z^a \text{ const} \} \) and \( J^- = \{ r \to \infty , \text{ with } z^5 \text{ const} \} \). It is achieved if \( G_{11} = G_{22} = 0 \). Notice that these surfaces are always outside the classical domain because (6.12) implies \( y^a y^b \eta_{ab} < 0 \). The asymptotically flat CR manifolds are invariant under the \( SU(2, 2) \) transformations, because they preserve the conditions \( G_{11} = G_{22} = 0 \).

The term "chiral" refers to the representation of \( E \) as the \( \gamma^0 \) Dirac matrix and should not be related (and it will not be related) to the Dirac equation. The chiral representation of \( E \) gives the unbounded realization of the \( SU(2, 2) \).
classical domain. Its bounded realization is found using the following Dirac representation of the invariant matrix

\[ E = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \]  

(6.15)

where the projective coordinates \( z \) are denoted by

\[ X^{mj} = \begin{pmatrix} \lambda \\ z\lambda \end{pmatrix} \]  

(6.16)

for \( \det \lambda \neq 0 \). The two projective coordinates are related with

\[ r = i(I + z)(I - z)^{-1} = i(I - z)^{-1}(I + z) \]

\[ z = (r - iI)(r + iI)^{-1} = (r + iI)^{-1}(r - iI) \]  

(6.17)

Notice that the "real axis" of the chiral representation does not cover all the \( U(2) \) Shilov boundary of the Dirac representation. The two boundaries \( J^\pm \) are surfaces of \( U(2) \). \( J^+ \) is \( \tau + \rho = \pi \), \( (-\pi \leq \tau - \rho \leq \pi) \), and \( J^- \) is \( \tau - \rho = \pi \), \( (-\pi \leq \tau + \rho \leq \pi) \). This fact permits us to distinguish the lorentzian CR manifolds into periodic and non-periodic. The periodic manifolds are those permitting the identification \( J^+ = J^- \) and they will be physically interpreted as excitation modes. The non-periodic manifolds are solitonic configurations according to the general rule.

The lorentzian CR structures of the Minkowski spacetime are periodic, because all the null geodesics are periodic. The topology of the compactified Minkowski spacetime turns out to be \( M^# \sim S^3 \times S^1 \).

The Kerr-Newman lorentzian CR manifold is a solitonic configuration because it is not periodic. Around \( J^+ \) the coordinates \((u, w, \theta, \varphi)\) are used, where the integrable tetrad takes the asymptotic form

\[ \ell \simeq [du - a \sin^2 \theta \, d\varphi] \]

\[ n \simeq [w^2 \, du - 2(1 + 2mw)dw - aw^2 \sin^2 \theta \, d\varphi] \]

\[ m \simeq [iaw^2 \sin \theta \, du - (1 + a^2w^2 \cos^2 \theta) \, d\theta - i \sin \theta (1 + a^2w^2) \, d\varphi] \]  

(6.18)

The physical space is for \( w > 0 \) and the integrable tetrad is regular on \( J^+ \) up to a factor, which does not affect the congruences, and it can be regularly extended to \( w < 0 \). Around \( J^- \) the coordinates \((v, w', \theta', \varphi')\) are used with

\[ dv \simeq du + 2(1 + \frac{2m}{r}) \, dr \]

\[ dw' \simeq -dw \quad , \quad d\theta' \simeq d\theta \]  

(6.19)

\[ d\varphi' \simeq d\varphi + \frac{2a}{r} (1 + 2mr) \, dr \]
and the integrable tetrad takes the form

\[
\ell \simeq \left[ w'^2 \, dv - 2(1 - 2mw') \, dw' - aw'^2 \sin^2 \theta' \, d\varphi' \right]
\]

\[
n \simeq \left[ dv - a\sin^2 \theta' \, d\varphi' \right]
\]

\[
m \simeq \left[ iaw'^2 \sin \theta \, dv - (1 + a^2w'^2 \cos^2 \theta') \, d\theta' - i\sin \theta'(1 + aw'^2) \, d\varphi' \right]
\]

(6.20)

The physical space is for \( w < 0 \) and the integrable tetrad is regular on \( \mathcal{J}^- \) up to a factor, which does not affect the congruences, and it can be regularly extended to \( w > 0 \). If the mass term vanishes the two regions \( \mathcal{J}^+ \) and \( \mathcal{J}^- \) can be identified and the \( \ell^\mu \) and \( n^\mu \) congruences are interchanged, when \( \mathcal{J}^+ \) (\( \equiv \mathcal{J}^- \)) is crossed. When \( m \neq 0 \) these two regions cannot be identified and the lorentzian CR structure cannot be extended across \( \mathcal{J}^+ \) into \( \mathcal{J}^- \), permitting the identification these two boundaries.

Notice that if the mass of a solitonic lorentzian CR manifold is defined with this asymptotic procedure and \( m > 0 \), then it can be absorbed into the dimensional coordinates and constants as follows

\[
w = \frac{\tilde{w}}{m}, \quad u = m\tilde{u}, \quad a = m\tilde{a}
\]

(6.21)

and it is factored out of the tetrad. This means that the mass parameter is a relative invariant of the CR structure. That is only counts whether it vanishes or not. Its precise value does not affect the structure.

### 6.2 The Poincaré group

The asymptotically flat lorentzian CR submanifolds of the classical domain, transformed to each other through an \( SU(2, 2) \) transformation, have the same lorentzian CR structure. Therefore we will say that they belong into a surface representation of the \( SU(2, 2) \) group, which we already denoted \( \Sigma[SU(2, 2)] \).

On the other hand, the two boundaries \( \mathcal{J}^\pm \) of the asymptotically flat lorentzian CR manifolds are 3-dimensional surfaces of the Shilov boundary, which meet at a point at spatial infinity. A general theorem, valid for all classical domains, states that the automorphic analytic transformations, which preserve a point of the characteristic boundary in the bounded realization, become linear transformations in the unbounded realization of the classical domain[7]. In the present case of the \( SU(2, 2) \) classical domain these linear transformations form the \( [\text{Poincaré}] \times [\text{dilation}] \) subgroup. Therefore a vacuum configuration with a singularity at a point in the Shilov boundary will cause spontaneously breaking of the conformal group \( SU(2, 2) \) down to its \( [\text{Poincaré}] \times [\text{dilation}] \) subgroup.

The set of the transferred into the classical domain asymptotically flat lorentzian CR submanifolds, which transform between each other through a
Poincaré transformation have the normal form

\[ y^a = k^a + h^a(x) \]

\[ k^a k^b \eta_{ab} = m^2 \]

with the "momentum" \( k^a \) characterizing each element of the set \( \Sigma[Poincaré] \). The positive constant \( m \) is fixed by the condition

\[ m = \max_{\Sigma[P]} \left( \sqrt{(h^i)^2} - h^0 \right) \]

That is the "mass" \( m \) of an asymptotically flat set \( \Sigma[Poincaré] \) is defined to be the minimal value of \( d \) of (6.13), that transfers all the elements of \( \Sigma[Poincaré] \) inside the classical domain.

The emergence of the Poincaré group is crucial for the effective description of the dynamics of static solitons in the quantum cosmodynamic model[7]. The unitary representation of the set \( \Sigma[Poincaré] \) is a free quantum field \( \Psi(x) \) which satisfies the scalar, spinorial or vector field equations depending on the spin of the soliton. The quantum representation of the Kerr-Newman type soliton (1.15) has to satisfy the following Dirac equation

\[ (\gamma^a (i \frac{\partial}{\partial x^a} - h_a) - m) \Psi(x) = 0 \]

because it is already known that its gyromagnetic ratio is fermionic. Notice that the effective field representation \( \Psi(x) \) incorporates the solitonic form factor \( h^a(x) \) through the above equation. In the vanishing form factor approximation, the Kerr-Newman satisfies the ordinary Dirac equation. Therefore the quantum cosmodynamic model may formally generate current quantum field theories as effective theories. On the other hand non-static lorentzian CR manifolds with appropriate behavior at \( t \to \pm \infty \) describe the interaction of the asymptotic static lorentzian CR manifolds. Therefore it is very interesting to find all the regular static axially symmetric lorentzian CR manifolds.

### 7 Examples of Symmetric Lorentzian CR Manifolds

In order to better understand the lorentzian CR manifolds, we have to work with special cases. The \( SU(1, 2) \) and \( SU(2, 2) \) osculation schemes provide two groups of holomorphic transformations, which preserve the lorentzian CR structure. We will work with the \( SU(2, 2) \) osculation, which contains the physically interesting Poincaré transformations, but analogous approaches can be applied to the other subgroups of \( SU(2, 2) \), or the \( SU(1, 2) \) osculation group.

In a quantum model, the lorentzian CR structure solutions \( X^{mi}(x) \) correspond to quantum states. If the \( SU(2, 2) \) group is not spontaneously broken, the transformations of \( X^{mi}(x) \) generate a group representation in the corresponding Hilbert space of the states of lorentzian CR manifolds. This representation
includes eigenstates of the three commuting generators of the \(SU(2, 2)\) group. These eigenstates remain invariant under the transformations along the corresponding generators. This implies that the eigenstates "lorentzian CR manifolds" will correspond to solutions of the symmetric conditions (6.6). Therefore we will look for asymptotically flat lorentzian CR manifolds which satisfy conditions symmetric relative to time-translation, rotation and scale transformations.

The time translation transformation is

\[
\delta X^{mi} = i\epsilon^0 [P_0]_{mn} X^{ni} \tag{7.1}
\]

where \(P_\mu = -\frac{1}{2} \gamma_\mu (1 + \gamma_5)\). In the chiral representation, we have

\[
\delta X^{0i} = 0 \quad , \quad \delta X^{1i} = 0 \tag{7.2}
\]

\[
\delta X^{2i} = -i\epsilon^0 X^{0i} \quad , \quad \delta X^{3i} = -i\epsilon^0 X^{1i}
\]

The rotation transformations around the z-axis is

\[
\delta X^{mi} = i\epsilon^{12} [\Sigma_{12}]_{m} X^{ni} \tag{7.3}
\]

where \(\Sigma_{\mu\nu} = \frac{1}{2} \sigma_{\mu\nu} = \frac{1}{4} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)\). Then we have

\[
\delta X^{0i} = -i\epsilon^{12} X^{0i} \quad , \quad \delta X^{1i} = i\epsilon^{12} X^{1i} \tag{7.4}
\]

\[
\delta X^{2i} = -i\epsilon^{12} X^{2i} \quad , \quad \delta X^{3i} = i\epsilon^{12} X^{3i}
\]

Taking into account that the algebraic surfaces of \(CP^3\) are locally determined by polynomials (Chow’s theorem), we will investigate quadratic polynomial surfaces, which remain invariant under the above transformations. The two solutions of the quadratic polynomial will determine the coordinates \(z^\alpha\) and \(z^\beta\) respectively. I actually found the following two invariant quadratic surfaces

\[
CASE I : \quad Z^1 Z^2 - Z^0 Z^3 + 2a Z^0 Z^1 = 0 \tag{7.5}
\]

\[
CASE II : \quad Z^0 Z^1 = 0
\]

I will first consider the surface of CASE I. In this case the convenient structure coordinates are

\[
z^0 = i \frac{X^{21}}{X^{10}} \quad , \quad z^1 = \frac{X^{11}}{X^{01}} \quad , \quad z^\bar{0} = i \frac{X^{32}}{X^{12}} \quad , \quad z^\bar{1} = -\frac{X^{02}}{X^{12}} \tag{7.6}
\]

which under time translation and z-rotation transform as follows

\[
\delta z^0 = \epsilon^0 \quad , \quad \delta z^1 = 0 \quad , \quad \delta z^\bar{0} = \epsilon^0 \quad , \quad \delta z^\bar{1} = 0 \tag{7.7}
\]

\[
\delta z^0 = 0 \quad , \quad \delta z^1 = i\epsilon^{12} z^1 \quad , \quad \delta z^\bar{0} = 0 \quad , \quad \delta z^\bar{1} = -i\epsilon^{12} z^\bar{1}
\]
I found the following invariant lorentzian CR surface

\[ z^0 = (t - r) + iU \quad , \quad z^0 = (t + r) + iV \]

\[ U = -2a \frac{z^1}{1 + z^1 \overline{z^1}} \quad , \quad V = 2a \frac{z^0}{1 + z^0 \overline{z^0}} \] \tag{7.8}

where \( f(r) \) is an arbitrary real function, and \( f(r) = 0 \) gives a flat lorentzian CR structure. These are not the most general invariant lorentzian CR manifolds. The additional condition \( U + V = 0 \) has been assumed. A simple investigation shows that this lorentzian CR manifold is (in different coordinates) the static solution (1.15) found in section II using the Kerr-Schild ansatz. One may easily compute the corresponding tetrad up to their arbitrary factors \( N_1, N_2 \) and \( N_3 \).

\[ \ell = N_1 [dt - dr - a \sin^2 \theta \, d\varphi] \]

\[ n = N_2 [dt + \left( \frac{r^2 + a^2 \cos^2 \theta}{r^2 - a^2} - 2a \sin^2 \theta \, \frac{df}{dr} \right)dr - a \sin^2 \theta \, d\varphi] \] \tag{7.9}

\[ m = N_3 [ia \sin \theta \, (dt - dr) + (r^2 + a^2 \cos^2 \theta) d\theta + i(r^2 + a^2) \sin \theta d\varphi] \]

where \( z^1 = \tan \frac{\theta}{2} e^{i\varphi} \). The corresponding projective coordinates \( r^a \) are found using the relations

\[ r_0 - r^3 = r_{00}' \quad \text{and} \quad r_0 + r^3 = r_{11}' \]

\[ (-r^1 + ir^3 = r_{01}' \quad \text{and} \quad -r^1 - ir^3 = r_{10}' \]

\[ r^0 + r^3 = r_{11}' \]

\[ \text{The dilation transformation is} \]

\[ \delta X^m = -i\varepsilon [D]_n^m X^n \] \tag{7.11}

with \( D = -\frac{i}{2} \gamma_5 \). Then we have

\[ \delta X^0 = -\frac{i}{2} X^0 \quad , \quad \delta X^1 = -\frac{i}{2} X^1 \]

\[ \delta X^2 = \frac{i}{2} X^2 \quad , \quad \delta X^3 = \frac{i}{2} X^3 \] \tag{7.12}

and subsequently

\[ \delta z^0 = \epsilon z^0 \quad , \quad \delta z^1 = 0 \]

\[ \delta z^0 = \epsilon z^0 \quad , \quad \delta z^1 = 0 \] \tag{7.13}
We find that the condition of scale invariance makes the static lorentzian CR manifold completely trivial with \( a = 0 \) and \( f(r) = \text{const} \). This indicates that the static lorentzian CR manifold cannot be a state in a Hilbert space with non-broken scale invariance.

We will now consider the CASE II (degenerate) algebraic surface \( Z^0 Z^1 = 0 \) which implies the conditions \( X^{11} = 0 = X^{02} \). In this case the convenient structure coordinates are
\[
z^0 = i \frac{X^{21}}{X^{01}} , \quad z^1 = \frac{X^{31}}{X^{01}} , \quad z^\tilde{0} = i \frac{X^{32}}{X^{12}} , \quad z^\tilde{1} = \frac{X^{22}}{X^{12}} \quad (7.14)
\]
which transform as follows
\[
\delta z^0 = \epsilon^0 , \quad \delta z^1 = 0 , \quad \delta z^\tilde{0} = \epsilon^0 , \quad \delta z^\tilde{1} = 0
\]
\[
\delta z^0 = 0 , \quad \delta z^1 = i \epsilon^{12} z^1 , \quad \delta z^\tilde{0} = 0 , \quad \delta z^\tilde{1} = - i \epsilon^{12} z^\tilde{1} \quad (7.15)
\]
under time translation and z-rotation. I found the following invariant lorentzian CR surface
\[
z^0 = u = t - r , \quad z^\tilde{0} = v = t + r \quad (7.16)
\]
\[
z^\tilde{1} = f(r) z^\tilde{1}
\]
where \( f(r) \) is an arbitrary complex function. One may easily compute the corresponding tetrad up to their arbitrary factors \( N_1, N_2 \) and \( N_3 \).
\[
\ell = N_1 du \\
n = N_2 dv \\
m = N_3 [-z^1 \frac{df}{dr} dv + 2f dz^1] \quad (7.17)
\]
Computing the differential forms of this tetrad we find that the \( \ell \) and \( n \) relative invariants vanish. Hence these lorentzian CR structures cannot be equivalent to those of the CASE I.

The dilation transformation of the new structure coordinates is
\[
\delta z^0 = \epsilon z^0 , \quad \delta z^1 = \epsilon z^1 \\
\delta z^\tilde{0} = \epsilon z^\tilde{0} , \quad \delta z^\tilde{1} = \epsilon z^\tilde{1} \quad (7.18)
\]
We find that the condition of scale invariance makes this lorentzian CR manifold completely trivial too with \( f(r) = \text{const} \). This indicates that the static lorentzian CR manifold cannot be a state in a Hilbert space with non-broken scale invariance.

We will now consider stationary configurations \( X^{mi} \) which move with the light velocity. Then \( X^{mi} \) satisfy the relations
\[
\delta X^{mi} = i \frac{\epsilon}{2} [P_0 + P_3]^m_{\, n} X^{ni} \quad (7.19)
\]
which imply
\[ \delta X^{0i} = 0 , \quad \delta X^{1i} = 0 \]
\[ \delta X^{2i} = 0 , \quad \delta X^{3i} = -i \epsilon X^{1i} \]  
(7.20)

In this case the most general quadratic polynomial, which is invariant under the above transformations is
\[ (bZ^0 + Z^2)Z^1 = 0 \]
(7.21)

The two solutions are \( X^{11} = 0 \) and \( X^{22} = -bX^{02} \). In this case we cannot use the (7.14) definitions of the structure coordinates. Instead we may use the following structure coordinates
\[ z^0 = i \frac{X^{21}}{X^{02}} , \quad z^1 = -i \frac{X^{31}}{X^{02}} , \quad \tilde{z}^0 = i \frac{X^{32}}{X^{02}} , \quad \tilde{z}^1 = \frac{X^{02}}{X^{02}} \]
(7.22)

Then they transform as follows under the above (7.20) and the z-rotation (7.24) transformations
\[ \delta z^0 = 0 , \quad \delta z^1 = 0 , \quad \delta z^\tilde{0} = \epsilon , \quad \delta z^\tilde{1} = 0 \]
\[ \delta z^0 = 0 , \quad \delta z^1 = i \epsilon z^1 , \quad \delta z^\tilde{0} = 0 , \quad \delta z^\tilde{1} = -i \epsilon z^\tilde{1} \]  
(7.23)

The asymptotic flatness conditions imply the following lorentzian CR manifold
\[ U = 0 , \quad V = b z^\tilde{1} \bar{z}^\tilde{1} \]
(7.24)

with \( b \) a real constant and \( f(u) \) an arbitrary complex function. One may easily compute the corresponding tetrad up to their arbitrary factors \( N_1, N_2 \) and \( N_3 \).
\[ \ell = N_1 du \]

\[ n = N_2 [dv + ib(f \bar{f} - f \bar{f})z^\tilde{1} \bar{z}^\tilde{1} du + ib \bar{f}(z^\tilde{1} dz^1 - z^1 d\bar{z}^\tilde{1})] \]  
(7.25)

\[ m = N_3 dz^1 \]

Computing the differential forms of this tetrad we find that the relative invariants, which correspond to \( \ell \) and \( m \) forms, vanish.

The dilation transformation of the new structure coordinates is
\[ \delta z^0 = \epsilon z^0 , \quad \delta z^1 = \epsilon z^1 \]
\[ \delta z^\tilde{0} = \epsilon z^\tilde{0} , \quad \delta z^\tilde{1} = 0 \]  
(7.26)

We find that the condition of scale invariance makes this lorentzian CR manifold completely trivial too (with \( b = 0 \)). This indicates that this stationary lorentzian CR manifold cannot be a state in a Hilbert space with non-broken scale invariance too.
In section III, we saw that regularity of the compatible class of metrics restricts the number of sheets up to four. Hence regular gravitational content is expected to be derived from up to four degree Kerr polynomials. Looking for static axially symmetric four degree polynomials, I found only some degenerate ones. These cases seem to exhaust the static and stationary axially symmetric lorentzian CR manifolds, which could correspond to "stable particles" at the quantum level.

Apparently the next step was to look for non-static axially symmetric lorentzian CR manifolds, which could correspond to "unstable particles". This turned out to be too complicated.

8 ON GEOMETRIC QUANTIZATION OF LORENTZIAN CR MANIFOLDS

The 4-dimensional lorentzian CR manifolds are totally real submanifolds of an 8(real)-dimensional complex manifold. In the present section I will describe a class of Kaehler metrics which is reduced to the class of lorentzian metrics in the manifold. Besides the lorentzian CR manifolds are lagrangian submanifolds to the corresponding symplectic manifold. These two properties suggest to consider the ambient complex manifold as the phase space of the lorentzian CR manifold and look for possible applications of the geometric quantization[8].

It is well known that the phase space of a free particle is the cotangent bundle of a 4-dimensional space with the symplectic form

\[ \omega = dp_a \wedge dq^a = d(p_a dq^a) \] (8.1)

Considering the vertical polarization with leaves \( q^a = \text{const} \), the geometric quantization generates a Hilbert space of square-integrable functions \( \psi(q) \), where the operators are

\[ \hat{q}^a = q^a, \quad \hat{p}_a = i\hbar \frac{\partial}{\partial q^a} \] (8.2)

The hamiltonian \( \hat{h} = \eta^{ab} p_a p_b \) describes the submanifolds which preserve the Poincaré representations with a given mass. These are the particle trajectories in the phase space. The eigenvector equation of the quantized hamiltonian \( \hat{h} \) is the Klein-Gordon equation. The corresponding quantum scalar field describes the Poincaré representation in the precise Hilbert space.

We may define the complex coordinates \( r^a = q^a + i\eta^{ab} p_b \). Then the symplectic form becomes

\[ \omega = \frac{i}{2} \eta_{ab} dr^a \wedge d\overline{r}^b = i \frac{\partial^2 S}{\partial r^a \partial \overline{r}^b} dr^a \wedge d\overline{r}^b \]

\[ S = \frac{(r^a - r^a)(r^b - r^b)}{4} \eta_{ab} \]

The phase space is a Kaehler manifold and the corresponding complex polarization is not positive definite.
In the ambient complex manifold of a general lorentzian CR structure \([1.9]\), I consider the following Kaehler metric and corresponding symplectic form

\[ ds^2 = \frac{\partial^2 S}{\partial \bar{z}^a \partial z^b} dz^a d\bar{z}^b, \quad \omega = i \frac{\partial^2 S}{\partial \bar{z}^a \partial z^b} dz^a \wedge d\bar{z}^b \]

where \(A\) and \(B\) are arbitrary non vanishing functions of the structure coordinates. Using the derivation relations \([1.11]\) of the tetrad of the lorentzian CR structure, we find that the induced metric belongs to the class of compatible metrics. Besides the symplectic form \(\omega\) vanishes on the lorentzian CR submanifold. Hence this submanifold is lagrangian relative to this class of symplectic forms. In fact I considered the precise Kaehler metric in order the lorentzian CR manifold to become lagrangian submanifold of the ambient complex manifold. My ultimate goal is to apply geometric quantization with a polarization induced by the lorentzian CR manifold. But first we must find a way to fix the Kaehler potential \(S\).

The osculation \([6.6]\) of the lorentzian CR manifolds imply that the Kerr functions provide an holomorphic expression of the CR structure coordinates \(z^a = f^a(r^b)\) relative to the projective coordinates of \(G_{2,2}\). These holomorphic transformations do not preserve the lorentzian character of the CR structure, but they reveal the Poincaré transformations and they permit us to apply the unique canonical form \(y^a = h^a(x)\) of the CR manifold. Therefore we can make the holomorphic transformation from the structure coordinates \(z^a\) to the \(G_{2,2}\) grassmannian projective coordinates \(r^a\). Taking into account that the defining conditions of the lorentzian CR manifold can always be locally written\([1]\) as \(\rho_{ij} = e^{(i)}(y^a - h^a)\) the Kaehler potential takes the form

\[ S = (y^a - h^a(x)) (y^b - h^b(x)) \gamma_{ab} \]

where \(\gamma_{ab}\) contains the arbitrary functions \(A(x, y)\) and \(B(x, y)\). Notice that in the vanishing gravity approximation \(h^a(x) \simeq 0\), the assumption \(\gamma_{ab} = \eta_{ab}\) gives the particle phase space. But in the general case the formalism becomes too complicated. The natural foliation \(x^a = \text{const}\) of the ambient complex manifold is not generally a polarization. On the other hand, the complex polarization is not generally quantizable because the scalar product

\[ (\Phi, \Psi) = \int \overline{\Phi} \Psi e^{-S} \omega^a \]

is not generally an integrable quantity \((S\) is not positive). A solution to this problem could be the possibility to holomorphically transfer the lorentzian CR manifolds inside the \(SU(2, 2)\) classical domain, which would permit us to restrict the integration to a finite region of \(\mathbb{C}^4\). But I have not yet found a way to choose a physically interesting regular Kaehler potential \(S\).
9 DISCUSSION OF PROBLEMS

After the Einstein-Infeld-Hofman derivation of the equations of motion from the Bianchi identity, many general relativists started to believe that the elementary particles are manifestations (solitons) of the gravitational field. The Wheeler geometrodynamical ideas were targeting to this goal. The computation of a fermionic gyromagnetic ratio \( g = 2 \) of the Kerr-Newman manifold pushed many researchers to identify this manifold with the electron. But they were quickly disappointed, because they could not fit current phenomenology. In the quantum cosmodynamic model the dynamical variable \( g_{\mu\nu} \) of General Relativity is replaced by the lorentzian CR structure. This solves the renormalizability problem of the Hilbert action and it implies the same geometrodynamical physical picture for the ”geometric” sector of the model. It gives straightforward solutions to many phenomenological problems of the original geometrodynamical idea, but many problems remain to be solved. Among the solved problems are:

1) The static Schwarzschild and Reissner-Nordstrom spacetimes are not observed as particles. The static lorentzian CR structures with vanishing spin \( a = 0 \) are compatible with the Minkowski metric without any gravitational content. Therefore no scalar static soliton is expected in the cosmodynamic model.

2) The asymptotically flat riemannian spacetimes belong to representations of the BMS group. The open (because of a singularity at spatial infinity) asymptotically flat lorentzian CR manifolds belong to the Poincaré group, which is a subgroup of the \( SU(2,2) \) group. A singularity at spatial infinity "opens" the manifold and breaks the \( SU(2,2) \) group down to its Poincaré subgroup.

3) The solitonic lorentzian CR manifolds are classified into 3+1(peculiar) "leptonic families" characterized by the sheets of the structure.

4) The particle-antiparticle distinction could not be understood. In the lorentzian CR structure context, the particle-antiparticle pair may be identified with the complex CR structure and its complex conjugate, because they are not equivalent.

4) The gluon field preexists in the quantum field theoretic action of the model, so there is no need to be derived from the geometry. The quadratic part of the action generates a linear static gluon potential, which implies perturbative confinement for all "colored" states. These gluonic modes excite the "leptonic" pure geometric (lorentzian CR manifolds) solitons of the model, therefore we expect a certain correspondence between the "hadronic families" and the "leptonic families". On the other hand the "particle-antiparticle imbalance" should be counted before "hadronization".

But there are many other problems to be solved or understood. Among these unsolved problems are:

1) The energy-momentum conservation problem. In any generally covariant lagrangian, the Noether theorem cannot derive conserved quantities. Instead, the well-known Dirac first-class constraints emerge for each diffeomorphic parameter. This failure does not permit us to properly define the vacuum and even to choose the appropriate metric of the static solitons.
2) The electromagnetic field. In the first geometrodynamical ideas, the Rainich conditions were presented as a way to determine a non-null electromagnetic field from the Ricci tensor of the riemannian manifold, while no effort has been made to derive the other bosonic particles of the Standard Model. If the quantum cosmodynamic model has any relation to reality, the stationary photon and the other unstable bosonic particles of the Standard Model have to emerge as periodic (modes) lorentzian CR manifolds.

3) While the emergence of the Poincaré group is very promising, the exact breaking mechanism of the $SU(2,2)$ has to be better understood in order to derive explicit numerical results.

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