Dispersion and damping of two-dimensional dust acoustic waves: theory and simulation

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Abstract. A two-dimensional generalized hydrodynamics (GH) model is developed to study the full spectrum of both longitudinal and transverse dust acoustic waves (DAW) in strongly coupled complex (dusty) plasmas, with memory-function-formalism being implemented to enforce high-frequency sum rules. Results are compared with earlier theories (such as quasi-localized charge approximation and its extended version) and with a self-consistent Brownian dynamics simulation. It is found that the GH approach provides a good account, not only of dispersion relations, but also of damping rates of the DAW modes in a wide range of coupling strengths, an issue hitherto not fully addressed for dusty plasmas.

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1. Introduction

Dusty, or complex plasmas have grown into a mature research field with a surprisingly broad range of interdisciplinary facets [1]–[3]. There has been significant interest in the study of dynamics of collective excitations of dusty plasmas, both in experiments [4]–[14] and through theory/numerical simulation of strongly coupled Yukawa systems [15]–[27]. Of particular interest in these studies is theoretical modeling of longitudinal and transverse dust acoustic wave (DAW) modes [15], and consequently several comparisons with experiments were made, including the ordinary hydrodynamics (OH) model [4, 11, 15], the dust-lattice (DL) model [5]–[7], [22, 23], the so-called quasi-localized charge approximation (QLCA) [10, 19, 24] and the generalized hydrodynamics (GH) model [12, 17, 18]. In addition, several assessments of how these analytical theories compare with the results from computer experiments have appeared recently [21]–[24], [26]–[28]. In particular, we have compared the Brownian dynamics (BD) simulation results for the DAW spectra in two-dimensional (2D) dusty plasmas with various theoretical models, including the OH, QLCA and DL models, as well as several ad hoc upgrades thereof [26]. It was found that an extended version of the QLCA [19, 24] provides an overall good account of the DAW dispersion relations, i.e. the peak locations in the simulation spectra for dusty plasma states characterized by the coupling strength $\Gamma$ in the range $10 \lesssim \Gamma \lesssim 1000$ and for wavelengths in the range going below the average inter-particle distance [26]. However, since the QLCA provides no information about the profile of such spectra and, consequently, gives no account of damping processes for the collective modes [19], it is desirable to explore alternative theoretical approaches to strongly coupled systems. In that context, the GH model offers the possibility to describe the full wave spectra by introducing viscoelastic effects in the dynamics of dusty plasmas in a phenomenological manner, as was demonstrated by studying DAWs in 3D dusty plasmas [16]–[18].

In contrast to the OH model, which is valid only for weakly coupled fluids with $\Gamma \ll 1$ in the long wavelength limit, or the DL model, which describes solid-like states with $\Gamma \gg 1$ and includes short wavelengths, the GH model covers an intermediate range of coupling strengths...
between those two models. A particularly suitable and physically intuitive framework for implementing the GH model is provided by descriptions of the Lennard-Jones fluids, pioneered by Ailawadi et al \[29\], Boon and Yip \[30\] and Hansen and McDonald \[31\]. This framework may be applied to a dusty plasma by asserting that dust particles form a quasi-neutral fluid with the inter-particle interactions described via a Debye–Hückel or Yukawa potential, which arises due to the screening of the charge accumulated on dust particles by the background electron and ion fluids. Successful implementations of the GH model to dusty plasma were conducted within the memory function formalism by Kaw and Sen \[17\] and Murillo \[18\] to study the wave dispersion and the shear wave cutoff in 3D dust liquids, respectively. However, comparisons of the GH model with both the MD simulation \[18, 21\] and the experimental data \[12\] for DAWs in 3D dusty plasmas were conducted only for long wavelengths. It will be shown here that, by imposing high-frequency sum rules to the GH model within the memory function formalism \[30, 31\], it is possible to extend its range of applicability to as short wavelengths as those that are accessible in the QLCA and the DL models.

Moreover, there is still great demand for a GH model for 2D strongly coupled dusty plasmas (SCDPs), especially because such systems have become particularly favored in recent laboratory experiments \[3, 10\]. Therefore, our principal motivation in this work is to develop and implement a GH model for studying the collective dynamics in 2D SCDPs. In addition, it is necessary to generalize the GH model to include the effects of collisions of dust particles with the neutral gas in the background plasma in a manner similar to that used for colloidal suspensions in describing the collisions of macro-ions with the solvent molecules \[32\]. We note that, while such collisions give rise to the classical Brownian motion of macroparticles in both these systems, it is the friction force on dust particles due to the neutrals that provides a damping mechanism for DAWs in dusty plasmas which scales with the pressure of the neutral gas. For example, it was shown experimentally that the dispersion relations and damping rates of DAWs in both 3D \[4, 12\] and 2D dusty plasmas \[7\] may depend on the background gas pressure in a significant way. While it is generally difficult to develop a consistent GH model that combines viscoelastic effects and the neutral-gas drag on an equal footing within the memory function formalism, as in the case of the colloidal suspensions \[32\], it will be shown here that progress can be made for dusty plasmas with sufficiently small neutral drag, which limits our model to the regime of low neutral gas pressure. On the other hand, keeping small but finite neutral drag in the BD simulation enables us to treat the dust particles as Brownian particles, which provides a natural and convenient way to eliminate the need for thermostatation that arises in the MD simulation of dusty plasmas \[26, 34\].

In this paper, we perform a BD simulation of strongly coupled 2D Yukawa liquids and use the results to evaluate the equilibrium radial distribution function (RDF) and the static structure factor, as well as the (power) spectral densities for both the longitudinal and transverse current densities in such systems. The former two quantities enable us to use the GH approach within the memory function formalism to evaluate both the dispersion relation and the damping rate of the DAW modes in 2D dusty plasmas by enforcing the low-order, high-frequency sum rules upon the theoretical spectral densities. As a consequence, it is found that the GH approach could provide dispersion relations that compare well with those resulting from the BD simulation spectra over broader ranges of wavelengths and coupling strengths than those employed in the previous comparisons with simulations and experiments \[12, 18, 21\]. Additional comparison with the dispersion relations from the QLCA model shows that the GH approach performs as well as the QLCA model at higher coupling strengths, and that it provides a better account of
the direct thermal effect, which is seen in the simulation data at lower coupling strengths and shorter wavelengths, but is absent in the standard QLCA model. On the other hand, a simple extension of the QLCA dispersion relation, which was recently proposed in [26], is found to be in much better agreement with both the GH results and simulation data. Most importantly, the GH results are shown to yield a good fit to the spectral density profiles from our BD simulations, thus providing semi-analytical modeling of the wavenumber-dependent damping rates of the DAW modes in SCDPs in a broad range of coupling strengths, an issue that has not been fully addressed so far [26]. Finally, we also find that the effect of collisions with the neutrals on the DAW damping rates is well accounted for, at least in the low neutral pressure regime, by introducing a local in time friction force into the GH equations.

The paper is organized as follows. Details of the BD simulation are given in section 2. Theoretical development of GH is discussed in section 3. Results based on two simple models for the memory function are presented in section 4, and the conclusion is given in section 5.

2. Simulation

We perform BD simulation of a 2D system consisting of \( N = 4000 \) dust particles, each carrying a constant charge \( q_d \), which are initially placed at random positions within a square simulation cell in the \( r = (x, y) \) plane with periodic boundary conditions. Evolution of the system is governed by equations that may be regarded as a stochastic generalization of the usual MD method, where the effect of collisions with neutral particles in the background plasma is modeled by a Langevin generalization of the usual Newton’s equations. Therefore, equations for the velocity and the position vectors of the \( i \)th dust particle are given by

\[
\frac{\text{d}v_i(t)}{\text{d}r} = -\gamma_a v_i(t) + \frac{1}{m_d} \sum_{j \neq i} F(r_i(t) - r_j(t)) + A_i(t),
\]

where \( m_d \) is the mass of each dust particle and \( F(r) = -(r/r)U(r)/\text{dr} \) is the pairwise force between two dust particles a distance \( r = \sqrt{x^2 + y^2} \) apart, interacting via the Yukawa potential, \( U(r) = (q_d^2/r) \exp(-r/\lambda_D) \), with \( \lambda_D \) being the Debye screening length of electrons and ions in the background plasma. Collisions of the \( i \)th dust particle with the neutrals in the plasma give rise to a systematic drag force, which we model by the Epstein drag coefficient \( \gamma_n \), and to a cumulative random force, which we model by a delta-correlated Gaussian white noise. When the system is in thermal equilibrium, the friction coefficient \( \gamma_n \) and the stochastic acceleration of the \( i \)th particle \( A_i(t) \) are related to the ambient temperature \( T_d \) via the fluctuation–dissipation theorem. Equations (1) are solved for our system by the Gear-like predictor–corrector algorithm for BD simulation [34], which was previously used in modeling both the equilibrium structure [35] and collective modes in 2D dusty plasmas [26].

Equations (1) can be expressed in terms of just three parameters [26]: the coupling strength \( \Gamma = q_d^2/(a k_B T_d) \), the reduced neutral drag coefficient \( \gamma = \gamma_n/\omega_{pd} \) and the screening parameter \( \kappa = a/\lambda_D \), where \( a \equiv 1/\sqrt{\pi \rho} \) is the average inter-particle separation with \( \rho \) being the average surface density of dust particles, and \( \omega_{pd} = \sqrt{2q_d^2/(m_d a^3)} \) is the characteristic dust–plasma frequency. In this work, we perform BD simulations using \( \gamma = 0.06 \) (which is a value chosen as a matter of convenience only) and \( \kappa = 1 \) as standard values, in a range of coupling strengths 20 \( \leq \Gamma \leq 1000 \) covering liquid-to-crystalline states of dusty plasma [26].

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Further, the current density of dust particles is defined as

\[ j(r, t) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} v_i(t) \delta(r - r_i(t)), \]

with the Fourier transform of its Cartesian component \( \alpha \) given by

\[ j_\alpha(k, t) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} v_{i\alpha}(t) e^{ik \cdot r_i(t)}. \]

Assuming that the dust collective modes propagate along the \( x \)-direction, i.e. \( k = (k, 0) \), we let \( \alpha = x \) or \( y \), allowing us to define the corresponding longitudinal or transverse current density auto-correlation functions as

\[ C_{l,t}(k, t) = \langle j_{l,t}^*(k, 0) j_{l,t}(k, t) \rangle, \]

where the angular brackets denote ensemble averaging over the initial time. The longitudinal/transverse spectral densities are then obtained as the Fourier transform of the respective current density auto-correlation functions as

\[ P_{l,t}(k, \omega) = \int_{-\infty}^{\infty} dt \ e^{i\omega t} C_{l,t}(k, t). \]

The Fourier transforms defined in equation (5) are evaluated using fast Fourier transform from simulation data. An equivalent and useful definition of the spectral densities, to be used later, is given as the real part of the Laplace transform \([31]\) of the current density auto-correlation functions,

\[ P_{l,t}(k, \omega) = 2\Re\{\mathcal{L}[C_{l,t}(k, t)]\} |_{s=i\omega}. \]

### 3. Generalized hydrodynamics (GH)

The basic approach we will take in the development of a GH model for a 2D dusty plasma is to treat the dust system as a compressible, viscous neutral fluid of particles interacting via the Yukawa potential, analogous to the way molecules in a Lennard-Jones fluid interact \([31, 33]\). In that context, we note that the OH approach provides a satisfactory account of the long-time (or low-frequency) response of a fluid in the weak coupling regime by means of the familiar Navier–Stokes equation, in which viscous effects are treated as an instantaneous (or local in time) internal friction force, related to fast thermalization of dust particles due to their mutual collisions. However, as the coupling strength increases, the effect of ‘caging’ sets in, so that individual dust particles are temporarily trapped in potential wells that migrate through the system on a slow time scale, similar to the picture invoked in the QLCA model. Therefore, the short-time (or high-frequency) response of the system is dominated by elastic effects due to the restoring forces on dust particles in the itinerant potential wells. In the GH approach to the Navier–Stokes equation, a transition from the purely viscous, fluid-like behavior to the elastic response of a solid-like system is described by postulating a non-local (in time) friction with a memory function characterized by relaxation time \( \tau \), such that the limit of a viscous fluid is recovered at times \( t \gg \tau \), whereas the solid-like elastic effects are dominant at times \( t \ll \tau \).

In addition to the collisions among dust particles, it is necessary to include in the GH approach also the effect of their repeated collisions with the neutral molecules in the background
plasma. These collisions may be treated as a Gaussian white noise, giving rise to a picture of Brownian motion, where each dust particle is subject to a local (in time) friction force with the neutral drag coefficient $\gamma_n$. Progress can be made by first noting that, at times $t \ll \gamma_n^{-1}$, the Brownian motion of dust particles due to the neutrals may be considered as ballistic. Next, we limit our considerations to dusty plasmas with low enough pressure of the neutral gas, such that $\gamma_n \tau \ll 1$. Then, it is justified to ignore the effect of the neutral drag at the time scale $t \sim \tau \ll \gamma_n^{-1}$ where viscoelastic effects dominate due to interactions among the dust particles [32]. On the other hand, at long times, such that $t \sim \gamma_n^{-1} \gg \tau$, both the viscous drag and the neutral drag on dust particles may be treated by means of two additive, local in time friction forces in a generalized Navier–Stokes equation because the underlying physical mechanisms of energy dissipation are statistically independent.

Therefore, we first ignore the neutral drag and introduce a memory function formalism, which incorporates the effects of viscous relaxation into the short-time dynamics of the dust collective modes by enforcing high-frequency sum rules upon the power spectral densities of such modes [30]. The even-order sum rules are defined in terms of the corresponding frequency moments as

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \omega^{2n} P_{l,n}(k, \omega) = \langle \omega^{2n}_l(k) \rangle,$$

(7a)

whereas all odd moments of frequency vanish since $P_{l,1}(k, \omega)$ are even functions of $\omega$. For example, the zeroth-order sum rule is given by the equation [30]

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega P_{l,0}(k, \omega) = v_{th}^2,$$

(7b)

where $v_{th} = \sqrt{k_B T_d m_d}$ is the dust thermal speed. A connection with microscopic properties of the system is accomplished by expressing e.g. the second moments of frequency for the longitudinal and transverse collective modes in equation (7a) with $n = 1$ in terms of the RDF of the dust layer, $g(r)$, respectively as [30]

$$\langle \omega_l^2(k) \rangle = 3k^2 v_{th}^4 + \frac{\rho}{m_d} v_{th}^4 \int d^2 r g(r) [1 - \cos(kx)] \frac{\partial^2 U(r)}{\partial x^2},$$

(8)

$$\langle \omega_t^2(k) \rangle = k^2 v_{th}^4 + \frac{\rho}{m_d} v_{th}^4 \int d^2 r g(r) [1 - \cos(kx)] \frac{\partial^2 U(r)}{\partial y^2}.$$  

(9)

Once the initial-value properties of the relevant memory functions are fixed by enforcing the sum rules in equations (7b) and (7a), one may incorporate the effect of the neutral drag by simply adding a local dissipative force into the fluid equations of motion [17, 33], as explained in section 3.2.

### 3.1. Collective modes

The starting point for the extension of OH to GH is given by the linearized continuity equation and the Navier–Stokes equation, which may be written for the longitudinal and transverse dust current densities as [30]

$$\frac{\partial}{\partial t} \delta \rho(k, t) - i k j_l(k, t) = 0,$$

(10a)
\[ \frac{\partial}{\partial t} j_l(k, t) - \frac{ik\delta \rho(k, t)}{m_d \rho \chi_T} = -v_l k^2 j_l(k, t), \quad (10b) \]
\[ \frac{\partial}{\partial t} j_t(k, t) = -v_t k^2 j_t(k, t), \quad (10c) \]

where \( \chi_T \) is the isothermal compressibility, \( v_l = 2v_1 + v_2 \) is defined as the longitudinal viscosity, with \( v_1 \) being related to the shear viscosity and \( v_2 \) to the bulk viscosity, \( \rho \) is the average surface density, \( \delta \rho \) is a small perturbation in density and \( j_l, t(k, t) \) is a small perturbation to the longitudinal/transverse current density, assumed to be zero on average.

Inserting the integral of equation (10a) into (10b) and (10c), one can obtain integro-differential equations for the current density auto-correlation functions \( C_{l,t}(k, t) \) defined in equation (4), which must be subject to the initial condition \( C_{l,t}(k, 0) = \frac{v_{th}^2}{v_l} \). In order to model the short-time correlation effects, the resulting equations can be modified to satisfy the frequency sum rules in the following way [29, 30]. In the first step, longitudinal spectral density is forced to satisfy the zeroth frequency sum rule, equation (7b), by replacing the isothermal compressibility with the static structure factor, \( S(k) \)

\[ \chi_T \mapsto \frac{S(k)}{\rho k_B T_d}, \quad (11) \]

which generalizes a standard statistical–mechanical relation in the limit \( k \to 0 \). In the next step, generalizations of the longitudinal and shear viscosities are introduced via the as yet undefined memory functions, \( \phi_l(k, t) \) and \( K_t(k, t) \), respectively, giving integro-differential equations of the form

\[ \frac{\partial}{\partial t} C_{l,t}(k, t) = -\int_0^t dt' K_{l,t}(k, t - t') C_{l,t}(k, t'), \quad (12) \]

where

\[ K_{l,t}(k, t) = \frac{(k v_{th})^2}{S(k)} + k^2 \phi_l(k, t). \quad (13) \]

It can be shown that the second frequency sum rule, equation (7a) with \( n = 1 \), can be satisfied if the initial values of the longitudinal and transverse viscosity memory functions are expressed in terms of the corresponding second frequency moments, equations (8) and (9), respectively, as [30]

\[ \phi_l(k, 0) = \frac{\langle \omega_l^2(k) \rangle}{v_{th}^2 k^2} - \frac{v_{th}^2}{S(k)}, \quad (14a) \]
\[ K_l(k, 0) = \frac{\langle \omega_l^2(k) \rangle}{v_{th}^2}. \quad (14b) \]

3.2. Effects of neutral drag

For dusty plasmas with low pressure of the neutral gas, the effect of neutral drag on the dispersion relations of DAW modes is expected to show up only at very low frequencies, \( \omega < \gamma_n \), and we have indeed found in our BD simulations that variation in the actual value of the drag coefficient \( \gamma_n \) has very little effect on the dispersion relations in comparison to the viscoelastic
effects, as long as $\gamma_n$ is kept sufficiently small. However, the spectral widths in equations (5) or (6) are found to be affected, even by small values of the neutral drag, as discussed in section 4.

With the short-time dynamics of both the longitudinal and transverse dust collective modes fixed by enforcing the sum rules upon the initial-value properties of the relevant memory functions, we may now follow the suggestions of [17, 33] and introduce the long-time relaxation effect due to neutral drag by simply adding a local dissipative term in equation (12), giving

$$\frac{\partial}{\partial t} C_{\nu,1}(k, t) = - \int_0^t dt' K_{\nu,1}(k, t - t') C_{\nu,1}(k, t') - \gamma_a C_{\nu,1}(k, t). \quad (15)$$

We proceed with solving equation (15) by means of Laplace transform in order to derive expressions for the spectral density by invoking the definition equation (6). For the longitudinal mode we obtain

$$P_{\nu}(k, \omega) = 2v_{th}^2 \frac{\omega^2[k^2 \phi''_1(k, \omega) + \gamma_a]}{[\omega^2 - \omega_0^2(k) + \omega k^2 \phi''_0(k, \omega)]^2 + [\omega k^2 \phi''_0(k, \omega) + \gamma_a]^2}, \quad (16)$$

where $\phi''_1(k, \omega)$ and $\phi''_0(k, \omega)$ are the real and imaginary parts, respectively, of the Laplace transformed longitudinal viscosity memory function, $\mathcal{L}[\phi_1(k, t)]_{t=\omega}$, and where we have explicitly defined

$$\omega_0(k) = \frac{k v_{th}}{\sqrt{S(k)}}. \quad (17)$$

Similarly, we obtain for the transverse mode

$$P_{\nu}(k, \omega) = 2v_{th}^2 \frac{K''_1(k, \omega) + \gamma_a}{[\omega + K''_0(k, \omega)]^2 + [K''_0(k, \omega) + \gamma_a]^2}, \quad (18)$$

with $K''_1(k, \omega)$ and $K''_0(k, \omega)$ being the real and imaginary parts, respectively, of the Laplace transformed transverse viscosity memory function, $\mathcal{L}[K_1(k, t)]_{t=\omega}$.

3.3. Model memory functions

We note that no approximations were used in the development of the GH approach so far. Specific forms of equation (16) and (18) may now be obtained by introducing simple phenomenological models for the corresponding memory functions.

3.3.1. Exponential model. Choosing an exponential longitudinal viscosity memory function of the form $\phi_1(k, t) = \phi_1(k, 0)e^{-t/\tau_1}$, we obtain [30]

$$\phi''_1(k, \omega) = \phi_1(k, 0) \frac{\tau_1}{1 + \omega^2 \tau_1^2}, \quad (19a)$$

$$\phi''_0(k, \omega) = -\phi_1(k, 0) \frac{\omega \tau_1^2}{1 + \omega^2 \tau_1^2} \quad (19b)$$

to be used in equation (16). We first discuss the limits of very short and very long viscous relaxation times $\tau_1$.

In the limit of small relaxation time, $\omega \tau_1 \ll 1$, we recover the so-called delta-function model for the memory function by letting $\phi_1(k, 0) = v_l/\tau_1$, where $v_l$ is the longitudinal viscosity. This
situation corresponds to a fluid where viscous drag is described by a local friction force, so that equation (16) is reduced to [30]

\[ P_l(k, \omega) = 2\nu_h^2 \frac{\omega^2(k^2\nu_l + \gamma_0)}{[\omega^2 - \omega_0^2(k)]^2 + \left[\omega(k^2\nu_l + \gamma_0)\right]^2}, \]  

(20)
giving the dispersion relation as \( \omega = \omega_0(k) \) with \( \omega_0(k) \) defined in equation (17) and having the full-width at half-maximum of \( \nu_l + k^2\nu_l \). This result provides a relatively simple account of the interplay between the neutral drag and viscous damping, assuming that the quasi-static longitudinal viscosity \( \nu_l \) can be defined properly [36].

On the other hand, it is important to study the opposite limit, \( \omega \tau_l \gg 1 \), for the sake of comparison with the QLCA model, which is inherently valid in this regime [38]. In particular, we find that the viscous damping vanishes in this limit and the elastic effects in the system’s response prevail, giving the dispersion relation \( \omega = \omega_\infty^l(k) \), where

\[ \omega_\infty^l(k) = \frac{\sqrt{\langle \omega_l^2(k) \rangle}}{\nu_h}, \]  

(21)

with the second frequency moment given in terms of the RDF via equation (8). We note that this situation is analogous to that in the QLCA model with two important differences: the dispersion relation \( \omega = \omega_\infty^l(k) \) is different from that found in the QLCA model, and the neutral drag still provides a mechanism for damping of the DAW modes.

For finite values of \( \tau_l \), a dispersion relation in the exponential model for the longitudinal viscosity memory function may be obtained from the peak positions in the spectral density given by equation (16) with equations (19a) and (19b). This spectral density is fully determined by the functions \( S(k) \) and \( \langle \omega_l^2(k) \rangle \) via equations (17) and (14a), respectively, and by the longitudinal viscosity relaxation time \( \tau_l \), which may be a function of \( k \) as well. We note that the former two functions can be calculated, at least in principle, from the RDF of the dust layer by using the usual definition for \( S(k) \) and equation (8) for \( \langle \omega_l^2(k) \rangle \). However, while \( g(r) \) can be obtained directly from BD simulations or may be available from first principles, there is no simple way to determine \( \tau_l \) from first principles. One possibility to proceed is to ignore its \( k \) dependence and treat \( \tau_l \) as a free parameter, possibly dependent on \( \Gamma \), that can be determined from, e.g., fitting the peak positions of equation (16) to the peak positions of the spectral densities obtained from experiments or computer simulations. It is expected that such a procedure would yield a dispersion relation lying somewhere between \( \omega = \omega_0(k) \) and \( \omega = \omega_\infty^l(k) \), defined via equations (17) and (21), respectively.

Implementation of the exponential model for the transverse viscosity memory function \( K_i(k, t) \) is a straightforward repetition of the procedure outlined above, and it produces a result for the spectral density that is entirely equivalent to that derived by Murillo for transverse modes in 3D strongly coupled Yukawa liquids [18]. In brief, one assumes \( K_i(k, t) = K_i(k, 0)e^{-t/\tau_i} \), with the initial value given in equation (14b), where the second moment of frequency for the transverse mode may be obtained also from the RDF by using equation (9). One further obtains the real and imaginary parts of the Laplace transform \( \mathcal{L}[K_i(k, t)]_{\text{real}} \) by using expressions analogous to equations (19a) and (19b), which, upon substitution into equation (18), yield a spectral density for the transverse mode in the exponential model.

It should be reiterated that, in the case of vanishing relaxation time, \( \tau_i = 0 \), the transverse mode is purely diffusive and the resulting spectral density in equation (18) exhibits no
dispersion [30]. In the opposite limit of \( \tau_1 \rightarrow \infty \), viscous damping vanishes, while the dispersion relation, given by \( \omega = \omega_{\infty}(k) \) with

\[
\omega_{\infty}^4(k) = \frac{\langle \omega^2(k) \rangle}{v_{\text{th}}},
\]

(22)
can be shown to exhibit a quasi-acoustic behavior in the long wavelength limit. On the other hand, when using the peak positions of the spectral density in equation (18) for the exponential model with finite \( \tau_1 \), one encounters a cutoff wavenumber, \( k_c \), in the dispersion relation for the transverse mode, which can be estimated from the condition \( \langle \omega^2(k_c) \rangle = (v_{\text{th}}/\tau_1)^2 \) in the limit of vanishing neutral drag [18, 26]. However, like in the case of the longitudinal mode, since there is no simple way to determine \( \tau_1 \) from first principles, one can again treat it as a free parameter that may be determined from an appropriate fitting, e.g. by using the cutoff values \( k_c \) observed in the experiments on shear waves in 2D Yukawa liquids [9].

3.3.2. Gaussian model. Instead of using \( \tau_1 \) and \( \tau_2 \) as free parameters, one may attempt enforcing higher-order frequency sum rules to see if a refinement of the above exponential model can be achieved. However, since the exponential model for the viscosity memory functions does not support moments higher than the second order when used in equations (16) or (18), one needs a model with different time dependence. Besides having to satisfy the second-frequency sum rule via equations (14a), it is shown in the appendix that such a model must additionally satisfy both the third-order sum rule giving \( \left[ \frac{\partial^2}{\partial t^2} \phi_1(k, t) \right]_{t=0} = 0 \) and the fourth-order sum rule giving

\[
\left[ \frac{\partial^2}{\partial t^2} \phi_1(k, t) \right]_{t=0} = \frac{\langle \omega^2(k) \rangle^2 - \langle \omega^4(k) \rangle}{k^2 v_{\text{th}}^2}.
\]

(23)

We see now that a Gaussian model for the longitudinal viscosity memory function of the form \( \phi_l(k, t) = \phi_l(k, 0) \exp[-t^2/\sigma^2_l(k)] \) will satisfy all moments up to and including the fourth if the Gaussian relaxation time \( \sigma_l(k) \) is given by

\[
\sigma^2_l(k) = \frac{2k^2 v_{\text{th}}^4 \phi_l(k, 0)}{v^2_{\text{th}} \langle \omega^2_l(k) \rangle - \langle \omega^4_l(k) \rangle^2},
\]

(24)

with the initial value of the memory function, \( \phi_l(k, 0) \), given by equation (14a). Finally, by using the Laplace transform of the Gaussian longitudinal viscosity memory function with \( s = i\omega \) in equation (16), we obtain a spectral density that is fully determined by three functions: \( S(k) \), \( \langle \omega^2_l(k) \rangle \) and \( \langle \omega^4_l(k) \rangle \) via equations (17) and (14a) without the need for free parameters. Unfortunately, analytical expressions that can be used for calculation of the fourth frequency moment are rather cumbersome and require a three-particle distribution function, which is difficult to obtain from simulations [30]. Therefore, since calculation of \( \langle \omega^4_l(k) \rangle \) from first principles is impractical, we may use the Gaussian model by computing the fourth moment numerically from equation (7a) with \( n = 2 \), where \( P_l(k, \omega) \) is obtained from simulation. For consistency, it is then desirable to also compute the second moment from equation (7a) with \( n = 1 \) using the same \( P_l(k, \omega) \) based on simulation data. In this way, we may use the Gaussian model as a simulation-based, parameter-free test of the quality of the approximation achieved in using the exponential model with a suitable choice of finite relaxation time \( \tau_1 \).

We finally note that a completely analogous development of the Gaussian model can be applied to the transverse mode.
4. Results and discussion

By limiting our BD simulations to the regime of dusty plasmas with low pressure of the neutral gas, we note that testing of several small values for the reduced neutral drag coefficient \( \gamma \) gave no noticeable dependence of the resulting spectra on \( \gamma \). Moreover, all the quantities used in the GH modeling of the collective modes (e.g. the RDF and the relaxation times in the exponential model of the memory functions for the longitudinal and transverse modes) were also found to be robustly independent of the (small) values of \( \gamma \) used in simulations. Therefore, all results will be shown for a standard value of \( \gamma = 0.06 \), along with the standard screening parameter \( \kappa = 1 \).

4.1. Longitudinal wave mode

We use coupling strength with values \( \Gamma = 20, 60, 100, 200, 600 \) and 1000 to investigate the longitudinal mode in a broad range of dusty plasma conditions, going from liquid to crystalline states.

As described above, both the static structure factor \( S(k) \) and the second frequency moments can be calculated from the equilibrium RDF \( g(r) \), which is shown in figure 1 for several values of \( \Gamma \). However, there are significant difficulties related to using \( S(k) \) for modeling the dispersion relation of the longitudinal mode at long wavelengths. Namely, when \( S(k) \) is calculated from the RDF, its small \( k \) values exhibit a very strong dependence on the fluctuations appearing in the simulation due to the finite number of dust particles [37]. This is expected to give rise to rather noisy dispersion curves, especially in the situations characterized by short relaxation times when the dispersion is dominated by the frequency \( \omega_0(k) \) defined in equation (17).
Figure 2. The static structure factor $S(k)$ as a function of the reduced wavenumber $ka$ for $\kappa = 1.0$, $\gamma = 0.06$ and $\Gamma = 20, 60, 100, 200, 600$ and $1000$, corresponding to increasing height of the principal peak, respectively.

To remedy the situation to some extent, we resort to calculating $S(k)$ directly from the simulation data by using the long-time average of the Fourier transformed particle density in equilibrium, as follows:

$$S(k) = \langle \rho^*(k) \rho(k) \rangle,$$

where

$$\rho(k) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} e^{ik \cdot r_i}.$$

In figure 2, we show the thus obtained results for $S(k)$ for several $\Gamma$ values, and note that the peak structures in that figure were found to be identical to those arising in $S(k)$ computed from the corresponding RDFs in figure 1.

In figure 3, we show the simulation data for the spectral density of the longitudinal mode, along with the dispersion curves from the exponential model, represented by three solid curves corresponding to, in descending order, infinitely long, finite and vanishing relaxation times $\tau_i$. Also shown in figure 3 by a thick black solid line is the dispersion curve obtained from the Gaussian model with the second and fourth frequency moments evaluated from equation (7a) with $n = 1$ and 2, respectively, where $P_l(k, \omega)$ was taken directly from the simulation data for spectral density. While the three curves for the exponential model extend over the full range of wavenumbers shown in figure 3, the results for the Gaussian model are shown over a reduced range of wavenumbers covering the first Brillouin zone only, $ka < 6$, because computations of the frequency moments via equation (7a) were hampered by a significant noise in the simulation spectra at larger wavenumbers and lower coupling strengths.
Figure 3. Longitudinal wave dispersion curves against simulation spectra for $\kappa = 1.0$, $\gamma = 0.06$ and $\Gamma = 20, 60, 100, 200, 600$ and $1000$. The upper (dark green), middle (red) and lower (light green) solid curves represent the exponential model with infinitely long, finite and vanishing relaxation times, respectively. The thick solid (black) curve represents the Gaussian model for wavenumbers $ka < 6$.

One notes in figure 3 that, for the exponential model with vanishing relaxation time (or, equivalently, the delta-function model), the resulting dispersion relation, $\omega = \omega_0(k)$ with $\omega_0(k)$ given in equation (17), displays some noise coming from the simulation data via equation (25). We found this noise to be significantly weaker than the noise arising when $S(k)$ is calculated from the RDF. On the other hand, as the longitudinal relaxation time $\tau_l$ increases in the exponential model, there is a relative increase in the contribution of the second frequency moment $\langle \omega_l^2(k) \rangle$, which, when evaluated from the RDF via equation (8), exhibits a much smoother $k$ dependence than $\omega_0(k)$ with $S(k)$ evaluated from equation (25).

As mentioned before, the dispersion relation for finite values of the relaxation time $\tau_l$ in the exponential model for the longitudinal viscosity memory function is obtained from the peak positions in the spectral density given by equation (16) with equations (19a), (10b), (17) and (14a). We have chosen the values of $\tau_l$ that provide the best fit to the peaks in the simulation spectral densities, and we have found $\tau_l$ to be a relatively weak function of $\Gamma$ that may be reasonably well approximated by

$$
\tau_l(\Gamma) = \begin{cases} 
0.769 & \text{for } \Gamma \leq 60, \\
1.000 & \text{for } 60 < \Gamma \leq 200, \\
1.333 & \text{for } 200 < \Gamma \leq 1000.
\end{cases}
$$

We emphasize that the quality of our choice of the values for $\tau_l$ in the exponential model is confirmed through the close agreement with the dispersion curves from the Gaussian model for wavenumbers in the first Brillouin zone, as displayed in figure 3.
Figure 4. Longitudinal wave dispersion curves against simulation spectra for $\kappa = 1.0$, $\gamma = 0.06$ and $\Gamma = 20$, 60, 100, 200, 600 and 1000. The upper (black) and the lower (red) dashed curve represent the results from the EQLCA and QLCA models, respectively. The upper (dark green) and the lower (light green) solid curves represent the exponential model with infinitely long and vanishing relaxation times, respectively.

It is worth mentioning that the extreme cases of the exponential model, corresponding to the limits $\tau_1 \to 0$ and $\tau_1 \to \infty$, yield two parameter-free dispersion relations for the longitudinal mode, $\omega = \omega_0(k)$ and $\omega = \omega_\infty(k)$, respectively. Remarkably, it is noted in figure 3 that these two dispersion relations provide a good account of the simulation data by straddling the thermal noise, seen in the recorded spectra at the low-to-medium $\Gamma$ values. Moreover, at the medium-to-high $\Gamma$ values, one notes that $\omega = \omega_0(k)$ closely follows the dispersion relation from the Gaussian model for the wavenumbers in the first Brillouin zone and, in particular, reproduces the near-vanishing of frequency at about $ka \approx 4$ for $\Gamma = 600$ and 1000. This feature of the delta-function model is reminiscent of the DL model [26], but is somewhat surprising, given that the GH model is stretched well into the condensed state at those two coupling strengths.

We further compare in figure 4 the dispersion relations $\omega = \omega_0(k)$ and $\omega = \omega_\infty(k)$ (shown by the upper and lower solid curves, respectively) with the results from the QLCA model [28], shown by the lower dashed curve. It is immediately obvious that the QLCA dispersion does not reproduce the direct thermal effect, which is responsible for a quasi-linear increase in the peak frequencies of the simulation spectra at higher wavenumbers and lower coupling strengths. In a previous work, we have shown that this deficiency can be easily rectified by a simple extension of the QLCA model, labeled as the EQLCA model in [26], which is shown by the upper dashed curve in figure 4. One notes surprisingly good agreement between the EQLCA dispersion relation and the curve $\omega = \omega_\infty(k)$ from the exponential model for $\tau_1 \to \infty$. While
agreement with the QLCA dispersion relation was expected at the higher $\Gamma$ values and lower $k$ values owing to the fact that the QLCA model inherently assumes $\omega \tau_l \gg 1$ [38] and the direct thermal effect is relatively weak at low $k$ and high $\Gamma$ values [26], it is remarkable how the parameter-free dispersion relation $\omega = \omega_{\infty}(k)$ provides justification for the empirically derived EQLCA model over the broad ranges of wavenumbers and coupling strengths.

We next compare the performances of the exponential model with finite relaxation time $\tau_l$ from equation (27) and the Gaussian model with the frequency moments evaluated from the simulation spectra via equation (7a) by using these two models in equation (16) to predict the frequency-dependent spectral profiles at several fixed wavenumbers. Given that the half-width at half-maximum (HWHM) of these profiles may be directly related to the wavenumber-dependent
damping rate of the longitudinal DAW mode, in this way we demonstrate the advantage of using
the memory function formalism within the GH model in tackling the difficult issue of damping
of the collective excitation modes in dusty plasmas.

Specifically, we show in figures 5–10 the frequency dependences obtained from the spectral
density in equation (16) with the exponential and the Gaussian models for four wavenumbers,
$ka = 0.61, 1.17, 2.29$ and $3.36$, and compare them with the corresponding profiles of the
simulation spectral density for $\Gamma = 20, 60, 100, 200, 600$ and $1000$, respectively. One notes
in figures 5–10 fair agreement between the simulation data and both GH models, which is
especially good for lower $\Gamma$ values where the hydrodynamic regime is expected to be more
pronounced, while for very high coupling strengths, say, $\Gamma > 200$, the agreement begins to
deteriorate. Comparison of both GH models with the simulation data may be considered fairly
reasonable, even at high coupling strengths where the hydrodynamic model has been stretched
well into the crystalline state.

Finally, in figure 11, we show a comparison between the wavenumber-dependent HWHM
values, obtained from the simulation spectral density profiles and from equation (16) where we
used both the exponential model with finite $\tau_l$ values given in equation (27) and the Gaussian model with the frequency moments evaluated from the simulation spectra via equation (7a). One notes that all three sets of data are noisy due to the noise in the simulation spectra (one recalls that, in the case of the exponential model, the noise stems from $\omega_0(k)$ with $S(k)$ evaluated from equation (25)). As expected, the HWHM values from the Gaussian model are systematically somewhat larger than those from the exponential model, but they are both seen to be in reasonably good agreement with the simulation data, even though the agreement becomes blurred by the increased noise at the highest $\Gamma$ values, say $\Gamma > 200$. One also notes in figure 11 that all damping rates roughly follow a quadratic dependence on $k$, with an increasing opening of the parabola as $\Gamma$ increases, and with the limiting value as $k \rightarrow 0$ being close to the damping rate due to the neutral drag, given by $\gamma/2 = 0.03$. For example, in the case $\Gamma = 1000$, the HWHM curve from simulation is practically constant with the value $\approx \gamma/2 = 0.03$, which may be
rationalized by the fact that viscous damping should vanish at coupling strengths characteristic of the solid state in dusty plasmas. On the other hand, the two corresponding model curves are rather noisy and/or perhaps show some structure that stems from deficiencies of the exponential and Gaussian model memory functions at such high $\Gamma$ values. Therefore, one may conclude that: (i) the choice of a model for the memory function may become a critical issue at high coupling strengths and (ii) the neutral drag may play a very important role in the regime of high pressure of the background gas, especially for long wavelengths and/or high coupling strengths.

4.2. Transverse wave mode

Transverse or shear waves are expected to exist only for higher coupling strengths $\Gamma$ [9, 18]. In the following we compare the simulation results with the GH approach for a typical case of $\Gamma = 100$.

Using equation (9) to evaluate the second frequency moment from the RDF, we have obtained a dispersion relation from the peak positions of the spectral density in equation (18) for the exponential model with $\tau_\ell = 5.0$ (note that typical values of $\tau_\ell$ for the transverse mode

---

**Figure 11.** HWHM of the spectral profile curves versus the reduced wavenumber $\kappa a$ for $\kappa = 1.0$, $\gamma = 0.06$ and $\Gamma = 20, 60, 100, 200, 600$ and $1000$, with the (blue) dots representing the simulation data, the (red) noisy solid curve representing the exponential model with finite relaxation time, and the squares representing the Gaussian model.
Figure 12. Transverse wave dispersion curves against the simulation spectra for \( \kappa = 1.0, \gamma = 0.06 \) and \( \Gamma = 100 \), with the upper solid (green) curve and the lower solid (blue) curve representing the exponential model with infinitely long and finite relaxation times, respectively.

are generally found to be larger than those for the longitudinal mode, in agreement with the remarks of Boon and Yip [30]). The results are shown in figure 12, along with the dispersion relation in the limit of infinite relaxation time, \( \tau_i \to \infty \), given by \( \omega = \omega_{\infty}^i(k) \) with equation (22), and are compared with the simulation spectrum for \( \Gamma = 100 \). Both theoretical dispersion curves show good agreement with the simulation data for \( \kappa a < 4 \) and, while the simulation spectrum is not conclusive about the cutoff wavenumber, the model with finite \( \tau_i \) clearly exhibits a cutoff at \( k_c \approx 0.5/a \) [9, 18].

Further, several profile plots of the spectral density are shown in figure 13 for \( \Gamma = 100 \), where the results from equation (18) for both the exponential model with \( \tau_i = 5.0 \) and the Gaussian model with the frequency moments evaluated from the simulation data via equation (7a) are compared with the simulation profiles for \( \kappa a = 0.61, 1.17, 2.29 \) and \( 3.36 \). One notes that the agreement of the exponential model with the simulation spectral profiles is quite satisfactory for the lower two wavenumbers, but the simulation profiles at the higher two wavenumbers are seen to be much broader than the corresponding peaks from the exponential model. On the other hand, the Gaussian model does not reproduce the dispersion of the transverse mode at all, since the corresponding profile curves peak at zero frequency for all wavenumbers, as displayed in figure 13. Nevertheless, the broadening of the simulation profiles at the two higher wavenumbers in figure 13 seems to be better echoed by the Gaussian than by the exponential spectral profile curves.
Figure 13. Transverse spectral density profile curves versus reduced frequency $\omega/\omega_{pd}$ for $\kappa = 1.0$, $\gamma = 0.06$, $\Gamma = 100$ and $ka = 0.61$, 1.17, 2.29 and 3.36. Simulation data are shown by the noisy (blue) solid curve, the exponential model with finite relaxation time is represented by the dashed (red) curve, and the Gaussian model is represented by the smooth solid (black) curve.

5. Concluding remarks

We have carried out BD simulation of a 2D layer of SCDP and extracted the equilibrium RDF, the static structure factor and the spectral densities for both the longitudinal and transverse DAW modes.

We have then shown that the memory function formalism of the theory of GH, when used in conjunction with the high-frequency sum rules, provides good semi-analytic results to model these wave modes. In particular, following the approach of Boon and Yip [30], we have developed an exponential model for the viscosity memory functions that satisfies the zeroth and the second frequency sum rules, with the relaxation time being used as a fitting parameter. With such an exponential model we have obtained good fits to the simulation data for the longitudinal dispersion curves over a wide range of coupling strengths, $20 \leq \Gamma \leq 1000$, and for wavelengths of the order of the average inter-particle separation and longer. We have also obtained reasonable match with the spectral density profiles for several values of the wavenumber $k$ in the first Brillouin zone. Next, we have extended the theory in a straightforward manner to satisfy the third and fourth frequency sum rules, providing a parameter-free Gaussian model for the viscosity memory function. The results using this model provided a successful test for our choice of the relaxation time in the exponential model in modeling both the dispersion relations and the spectral density profiles.
Moreover, we pointed out that the limits of infinitely long and infinitesimally short relaxation times in the exponential model represent two parameter-free models that are fully determined by the equilibrium RDF and the static structure factor of the system. It was then shown that these two limits provide good upper and lower bounds for the thermal noise observed in the simulation spectra for the longitudinal mode. Comparisons of the dispersion relations obtained from these two limits of the exponential model with the dispersion relations obtained from both the QLCA and the extended QLCA (EQLCA) showed that the limit of an infinitely long relaxation time reproduces remarkably well all the features of the EQLCA model, including the direct thermal effect. On the other hand, the limit of an infinitesimally short relaxation time was found to reproduce the near-vanishing of the dispersion relation, which is seen in the simulation spectra at certain short wavelengths for large coupling strengths and is often well reproduced by the DL model, but not by the QLCA model.

In addition, we have also tested the memory function formalism within the theory of GH against the simulation results obtained for transverse wave modes at the coupling strength of \( \Gamma = 100 \) and found that the exponential model with relaxation time used as a fitting parameter can provide a reasonable estimate for the wave dispersion relation, including the appearance of a cutoff wavenumber. Both the peak positions and the widths in the spectral density profiles from the simulation were well reproduced by the exponential model at long wavelengths, whereas the broadening of these spectra at short wavelengths was better echoed by the Gaussian model.

The most important finding of the present analysis is that the wavenumber-dependent damping rates, which were extracted from the simulation spectra, were found to be in good overall agreement with the results from both the exponential and Gaussian models, testifying to the strengths and potentials of using the memory-function formalism within the GH theory in describing the damping of the longitudinal DAW in strongly coupled dusty plasmas. However, the agreement between the theoretical and simulation damping rates was found to deteriorate at higher coupling strengths, say \( \Gamma > 200 \), which probably limits the applicability of our theory to the range of experimental parameters characteristic of strongly coupled liquids with the \( \Gamma \) values up to approximately the freezing point.

Moreover, we have included in the GH model the effect of the neutral drag due to collisions of dust particle with the neutral molecules in the background plasma via a (local in time) friction force. This is justified when the neutral drag coefficient is small, which limits our considerations to the regime of dusty plasmas with low pressure of the neutral gas. While we did not find small neutral drag to affect our modeling of dispersion relations for both the longitudinal and transverse waves in any significant way, the overall damping rates for the longitudinal wave showed possibly important roles of the neutral drag at long wavelengths and high coupling strengths, where viscous damping is expected to be reduced. Therefore, one of the challenges that are left for future developments is to extend the present theory to the regime of larger neutral drag.

We further note that direct comparison of the present approach to the GH model with experiments is complicated by the fact that most of the experimental results for dispersion and damping of the DAWs are given in the form of frequency-dependent complex wavenumbers that are not easily extracted from our BD simulation data. This task is left for future work.

In conclusion, the memory function formalism of the GH, when used in conjunction with the high-frequency sum rules, provides a very promising route to the analytical modeling of spectral density of collective modes in strongly coupled 2D dusty plasmas and, in particular, it provides a good account of viscoelastic effects that are not well described by other analytical
methods. However, one of the weaknesses of such a theory is the need for model memory functions with empirical parameters. In this respect, we mention that Donko et al [39] have recently conducted a series of MD and BD simulations to evaluate the frequency- and wavenumber-dependent complex viscosity coefficient in 3D Yukawa liquids directly from microscopic definition of this quantity. Such \textit{ab initio} calculations could provide the fundamental basis for establishing various models of the memory function to be used in the GH theory of 2D dusty plasmas, by even including the effects of the neutral drag.

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\section*{Appendix. Frequency sum rules}

Initial-value properties of the current density auto-correlation functions are related to the even-order frequency moments via \cite{30}

\begin{equation}
\langle \omega_{kl}^{2n}(k) \rangle = (-1)^{n} \left[ \frac{\partial^{2n}}{\partial t^{2n}} C_{kl}(k, t) \right]_{t=0},
\end{equation}

whereas all odd-order moments must vanish. We extend here the development outlined by Boon and Yip \cite{30} by introducing the third and fourth frequency sum rules to the GH theory of spectral density. Since these sum rules are used to fix the short-time properties of the model memory functions, we may again initially neglect the neutral drag.

To illustrate this procedure for the longitudinal mode, we differentiate equation (12) twice to obtain the third derivative as

\begin{equation}
\frac{\partial^{3}}{\partial t^{3}} C_{l}(k, t) = -k^{2} v_{th}^{2} \frac{\partial}{\partial t} C_{l}(k, t) - k^{2} \int_{0}^{t} dt' C_{l}(k, t') \frac{\partial^{2}}{\partial t^{2}} \phi_{l}(k, t - t')
\end{equation}

\begin{equation}
- k^{2} \left[ C_{l}(k, t') \frac{\partial}{\partial t} \phi_{l}(k, t - t') \right]_{t'=t} - k^{2} \phi_{l}(k, 0) \frac{\partial}{\partial t} C_{l}(k, t),
\end{equation}

which, on evaluating at \( t = 0 \), yields

\begin{equation}
\left[ \frac{\partial^{3}}{\partial t^{3}} C_{l}(k, t) \right]_{t=0} = -k^{2} \left[ C_{l}(k, t') \frac{\partial}{\partial t} \phi_{l}(k, t - t') \right]_{t'=t=0}.
\end{equation}

Since the third moment must vanish, we have the condition

\begin{equation}
-k^{2} C_{l}(k, 0) \left[ \frac{\partial}{\partial t} \phi_{l}(k, t - t') \right]_{t'=t=0} = 0,
\end{equation}

showing that the first derivative of the longitudinal viscosity memory function must vanish at initial time. This is to be expected because a memory function should be viewed as being just another time auto-correlation function with vanishing derivatives of odd order at the initial
time [30]. Next, evaluating the fourth moment, we find
\[
\frac{\partial^4}{\partial t^4} C_1(k, t) = -\frac{k^2 v_{th}^2}{S(k)} \frac{\partial^2}{\partial t^2} C_1(k, t) - k^2 \int_0^t dt' C_1(k, t') \frac{\partial^3}{\partial t^3} \phi_1(k, t - t') - k^2 \left[ C_1(k, t') \frac{\partial^2}{\partial t^2} \phi_1(k, t - t') \right]_{t'=t} - k^2 \phi_1(k, 0) \frac{\partial^2}{\partial t^2} C_1(k, t).
\]
Again setting \( t = 0 \), we obtain
\[
\left[ \frac{\partial^4}{\partial t^4} C_1(k, t) \right]_{t=0} = -\frac{k^2 v_{th}^2}{S(k)} \left[ \frac{\partial^2}{\partial t^2} C_1(k, t) \right]_{t=0} - k^2 C_1(k, 0) \left[ \frac{\partial^2}{\partial t^2} \phi_1(k, t - t') \right]_{t'=t=0} - k^2 \phi_1(k, 0) \left[ \frac{\partial^2}{\partial t^2} C_1(k, t) \right]_{t=0}.
\]
Substituting \( \phi_1(k, 0) \) from equation (14) and invoking equation (A.1) with \( n = 1 \) and 2, we obtain equation (23) in the main text.

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