Shear and bulk viscosities for pure glue matter

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Shear \( \eta \) and bulk \( \zeta \) viscosities are calculated in a quasiparticle model within a relaxation-time approximation for pure glueon matter. Below \( T_c \), the confined sector is described within a quasiparticle glueball model. The constructed equation of state reproduces the first-order phase transition for the glue matter. It is shown that with this equation of state, it is possible to describe the temperature dependence of the shear viscosity to entropy ratio \( \eta/s \) and the bulk viscosity to entropy ratio \( \zeta/s \) in reasonable agreement with available lattice data, but absolute values of the \( \zeta/s \) ratio underestimate the upper limits of this ratio in the lattice measurements typically by an order of magnitude.

I. INTRODUCTION

The high-energy heavy-ion collisions at Super Proton Synchrotron (SPS) and Relativistic Heavy-Ion Collider (RHIC) energies have shown evidence of a new state of matter characterized by very low shear viscosity to entropy density ratio \( \eta/s \) similar to a nearly ideal fluid [1–3]. Lattice calculations indicate that the crossover region between hadron and quark-gluon matter has been reached in these experiments. On the other hand, lattice calculations performed in gluodynamics (GD) clearly demonstrate the occurrence of there occurs the first-order phase transition.

The shear \( \eta \) and bulk \( \zeta \) viscosities are parameters that quantify dissipative processes in the hydrodynamic evolution of a fluid. It is known that the behavior of transport coefficients is sensitive to the presence of phase transitions in a medium (see [3–11] and references therein). Values of the bulk and shear viscosities near the phase-transition critical temperature \( T_c \) affect the hydrodynamic evolution of the medium and may influence observables.

Lattice quantum chromodynamics (QCD) is the most powerful technique to extract nonperturbative information on an equation of state (EoS). When experimental data are lacking, lattice data are often used to fit model parameters. For pure gluon SU(3) theory, the EoS was computed on the lattice more than a decade ago [12, 13]. Recently, much more accurate data have been obtained [13].

Among various existing phenomenological approaches, quasiparticle (QP) models are used to reproduce results obtained in the lattice QCD. In the case of GD, the QP models rely on the assumption that for the temperature \( T \) above the critical one, \( T > T_c \), the system consists of a gas of massive deconfined gluons. In the confined phase, at \( T < T_c \), the glue matter is considered as a gas of massive glueballs.

In this paper we aim to investigate the behavior of viscosity coefficients for a gluon system that exhibits a deconfinement phase transition. The phenomenological QP model is applied to describe available lattice data on the EoS. Shear and bulk viscosities are calculated within a relaxation-time approximation.

II. EQUATION OF STATE OF GLUE MATTER

In the QP approach the system of interacting gluons is treated as a gas of noninteracting quasiparticles with an effective mass \( m_g(T) \), which depends on \( T \) as [3]

\[
m_g^2(T) = \frac{N_c}{6} \frac{g^2(T) T^2}{} \tag{1}
\]

with the temperature-dependent strong interaction constant

\[
g^2(T) = \frac{48\pi^2}{11N_c \ln \frac{\lambda(T-T_\eta)/T_c}{\lambda}} \tag{2}
\]

where parameters \( T_\eta/T_c = 0.5853, \lambda = 3.3 \) are taken to fit the new lattice data, see below, and a number of colors \( N_c = 3 \). The energy density and the pressure acquire then the following forms:

\[
\varepsilon_g(T) = \frac{d_s}{2\pi^2} \int_0^\infty p^2 dp E \exp\left(\frac{E}{T}\right) - T \int_\varepsilon_\eta m_g(T) + B(T) \tag{3}
\]

\[
P_g(T) = \frac{d_s}{2\pi^2} \int_0^\infty p^2 dp \frac{p^2}{E \exp\left(\frac{E}{T}\right)} - \frac{d_s}{2\pi^2} \int_\varepsilon_\eta m_g(T) - B(T) \tag{4}
\]

where the degeneracy factor \( d_s = 2(N_c^2 - 1) = 16 \) for the SU(3) gluodynamics, \( \varepsilon_\eta \) and \( P_g^{\text{id}} \) are the energy density and the pressure of the ideal gas of massive gluons. The temperature-dependent function \( B(T) \) in Eq. (3) results from the thermodynamical identity, see Ref. [14],

\[
T \frac{dP}{dT} - P(T) = \varepsilon(T) \tag{5}
\]
which leads to the equation for $B(T)$:

$$\frac{dB(T)}{dT} = -\frac{\varepsilon_g^{id} - 3P_g^{id}}{m_g} \frac{d\rho_g}{dT}. \quad (6)$$

Dealing only with gluon degrees of freedom, one assumes that the matter at $T < T_c$ (the "hadronic" phase) consists of glueballs. While the meson scattering amplitude is parametrically suppressed as $1/N_c$, the scattering amplitude between glueballs scales as $1/N_c^2$ \cite{13} and therefore the system can be considered as a noninteracting Bose gas of glueballs. Expected glueball masses are high, $m_{gb} \sim 1$ GeV, and thereby only the lowest-lying glueball states contribute to the EOS at the temperatures of our interest. It is difficult to single out which states of the observed hadronic spectrum are glueballs because of a lack of knowledge of decay properties and the existence of a strong mixing between glueballs and quark states \cite{10}. However, using typical constant values for the lowest-lying glueball masses within a statistical model, one fails to reproduce the strong increase of thermodynamical variables near $T_c$ \cite{17}. The $T$-behavior of masses for two lowest-lying scalar $0^{++}$ and tensor $2^{++}$ glueballs was investigated on the lattice in \cite{18}. Therefore, below we follow the SU(3) lattice GD results. It was shown that the pole mass $m_{gb}(T)$, the Breit-Wigner mass $\tilde{m}_{gb}(T)$, and the thermal width $\Gamma_{gb}$ are linked as follows:

$$m_{gb}(T) \approx \tilde{m}_{gb}(T) - 2T + \sqrt{4T^2 - \Gamma_{gb}^2(T)}. \quad (7)$$

With the help of the Ansatz $\tilde{m}_{gb}(T) = m_{gb}^0$, i.e., that the glueball Breit-Wigner masses are given by the particle data group (PDG) values,

$$\Gamma_{gb} = b_{gb}(T - T_{gb}) \Theta(T - T_{gb}) \quad \text{for} \quad T_{gb} < T < T_c \quad (8)$$

and recommended parameters $b_{gb}(0^{++}) = 4.23$ and $b_{gb}(2^{++}) = 7.152$, the relation (7) reproduces quite well the lattice results in the measured range $0.5T_c < T < T_c = 265$ MeV \cite{18}. In our consideration, we limit ourselves to the two above-mentioned species of glueballs, the only ones for which lattice data are available.

With the temperature-dependent glueball masses, a statistical treatment of glueballs needs an additional requirement of thermodynamical consistency. It has been satisfied in the same way as outlined above for gluons; see Eqs. (5) and (6) above.

To describe glue matter in the whole range of temperatures, we use the first-order phase-transition model in accordance with lattice results for GD. Thus, one should conjugate the pure gluon ($g$) and the glueball ($gb$) phases by making use of the Gibbs conditions at the transition:

$$T^* = T_{gb} = T_c, \quad P_g(T_c) = P_{gb}(T_c). \quad (9)$$

We use the value $T_c = 265$ MeV for the first-order phase transition, in agreement with the lattice SU(3) GD \cite{12, 13, 18}.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{The reduced pressure (a) and the energy density (b) of the glue matter. The solid line corresponds to taking into account two glueballs, scalar $0^{++}$ with $m_{gb}^0 = 1470$ MeV and tensor $2^{++}$ with $m_{gb}^0 = 2150$ MeV. Experimental points are the old Karsch’s (filled squares) \cite{20} and the new Panero’s (circles) \cite{13} lattice results. The region near $T_c$ is enlarged in the inset.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{The trace anomaly (a) and the reduced enthalpy or entropy (b) of the glue matter. All notations are the same as in Fig. 1.}
\end{figure}

In Fig. 1 we compare the model results for the pressure and the energy density with the lattice data. Values are normalized to those in the Stefan-Boltzmann (SB) limit. Both $\varepsilon/\varepsilon_{SB}$ and $P/P_{SB}$ increase fast and monotonically with the temperature above $T_c$, but, as we see, the Stefan-Boltzmann limit is not saturated up to $3T_c$. Two sets of lattice data are similar qualitatively, but old data \cite{20} are appreciably higher than the new ones \cite{18} at $T \gtrsim 1.5T_c$. Note that the new lattice data are extended to the region of the glueball phase, $T < T_c$. Our QP model is in a reasonable agreement with the new lattice data, except for the narrow vicinity to the left of $T_c$, where model predictions are evidently below the lattice points [see insertions in Fig. 1(a) and Fig. 1(b)]. Due to large glueball masses, this result is not changed if one adds the next two to three glueball states to our two lowest-lying glueball states, although there exist statements claiming that a good agreement with lattice data near $T_c$ can be reached only if the whole high-lying glueball spectrum...
of the Hagedorn-type [21] or glueball condensate [17] is additionally included.

The interaction measure or trace anomaly \((\varepsilon - 3P)/T^4\) and the reduced enthalpy \((\varepsilon + P)/(\varepsilon_{SB} + P_{SB})\) are demonstrated in Fig. 2(a) and Fig. 2(b), respectively. The presence of a nonzero trace for the energy-momentum tensor relates to the breaking of the scale and conformal invariance. Again, a nice overall agreement is observed between the QP model and the new set of the lattice data [13], including a region near \(T_c\), for \(T > T_c\). The reduced enthalpy in Fig. 2(b) for the pure gluon system is just the reduced entropy, \(s/s_{SB}\), which is thereby also reproduced by our QP model. Thus, we see that the developed QP model successfully describes the thermodynamic properties of the glue matter.

III. CALCULATION OF VISCOSITY COEFFICIENTS

In principle, it is possible to compute the shear and bulk viscosities directly from GD at finite temperature using Kubo formulas. However, in practice, this is quite difficult because GD is generally a strongly interacting theory with an unknown mechanism of the confinement. Essential assumptions of our kinetic approach are that quasiparticles are well defined, elementary interactions are local, and the dynamics can be described in the relaxation-time approximation.

Derivation of viscosity coefficients starts with the expression for the energy-momentum tensor for quasi-free [13] boson quasiparticles of species \(a\):

\[
T_{\mu\nu}^a = \int d\Gamma \left\{ \frac{p_{\mu}^a p_{\nu}^a}{E_a} F_a \right\},
\]

(10)

where \(d_a\) is the degeneracy factor. The QP distribution function \(F_a\) fulfills the QP kinetic equation. We assume that gluon and glueball masses are given by Eqs. (11) and (17), respectively. The QP energy is determined by

\[
E_a(p) = \sqrt{p^2 + m_a^2(T, F_a)}.
\]

(11)

Below, we consider only collisional sources of the viscosity. Applying the relaxation time approximation to the relativistic QP kinetic equation, we arrive at the expression for the variation of the energy-momentum tensor [10] near the local equilibrium state:

\[
\delta T_{\mu\nu} = -\sum_a \int d\Gamma \left\{ \tau_a \frac{p_{\mu}^a p_{\nu}^a}{E_a} p_a^\alpha \partial_\alpha F_a \right\}_{\text{loc.eq.}},
\]

(12)

where \(\tau_a\) denotes the relaxation time of the given species, which generally depends on the QP momentum \(p_a\). The local equilibrium distribution function for a boson is as follows:

\[
F_a^{\text{loc.eq.}}(p_a, x_a) = \left[ e^{p_a^\mu u_\mu/T} - 1 \right]^{-1},
\]

(13)

\(u^\mu \simeq (1, \vec{u})\) for \(|\vec{u}| \ll 1\). Performing a variation in (12), we did not vary quantities that may depend on the distribution function only implicitly, such as \(E_a\), since by doing this one may arrive at the relaxation-time form of the QP kinetic equation. Besides, in the gluon-glueball model used here, only equilibrium values \(m_a(T)\) are known and we are actually not able to find \(\delta E_a/F_a\).

The shear and bulk viscosities can be expressed through the variation of the energy-momentum tensor as follows:

\[
\delta T_{ij} = -\zeta \delta_{ij} \vec{v} \cdot \vec{u} - \eta W_{ij},
\]

(14)

with \(W_{kl} = \partial_k u_l + \partial_l u_k - \frac{2}{3} \delta_{kl} \partial_i u^i\).

Here and below, Latin indices run 1, 2, 3. To find the shear viscosity, we set \(i \neq j\) in (14). To find the bulk viscosity, we substitute \(i = j\) in (14) and use that \(T_{ij}^{\text{loc.eq.}} = 3P_{\text{loc.eq.}}\).

Taking derivatives \(\partial T_{ij}^{\text{loc.eq.}}/\partial x^\mu\) in Eq. (12) and using (13) as a definition of viscosity coefficients, by straightforward calculations we find expressions (see [22, 23]) for the shear viscosity,

\[
\eta = \frac{1}{15T} \sum_a \int d\Gamma \tau_a \frac{\vec{p}_a^4}{E_a} F_a^{\text{eq}} (1 + F_a^{\text{eq}})
\]

(15)

and for the bulk viscosity \(14\),

\[
\zeta = -\frac{1}{3T} \sum_a \int d\Gamma \tau_a \frac{\vec{p}_a^2}{E_a} F_a^{\text{eq}} (1 + F_a^{\text{eq}}) Q_a,
\]

(16)

where the EOS-dependent \(Q_a\) factor is given by

\[
Q_a = \left\{ \frac{\vec{p}_a^2}{3E_a} - c_s^2 \frac{E_a - T DE_a/DT}{E_a} \right\}
\]

(17)

and \(c_s^2 = 2P/\varepsilon\) is the speed of sound squared.

Simplifying, instead of the momentum-dependent value \(\tau_a\), one may use the averaged partial relaxation time \(\bar{\tau}_a\) related to the cross section as

\[
\bar{\tau}_a^{-1}(T) = \sum_a n_a'(T) \langle v_{aa'}(v_{aa'}) \rangle,
\]

(18)

where \(n_a'\) is the particle density of \(a'\) species, \(\sigma_a'^t = \int d\cos \theta d\sigma (aa' \rightarrow aa')/d\cos \theta (1 - \cos \theta)\) is the transport cross section, in general, accounting for in-medium effects, and \(v_{aa'}\) is the relative velocity of two colliding particles \(a\) and \(a'\) in the case of binary collisions. Angular brackets denote a quantum-mechanical statistical average over an equilibrated system. However, one should bear in mind that averaged values \(\bar{\tau}_a^{-1}\) given by Eq. (18) yield only a rough estimate for the values \(\tau_a^{-1}\).
IV. RESULTS FOR VISCOSITIES

Below, the shear and bulk viscosities are calculated with the help of Eqs. (15) and (16), respectively. The only quantity that should still be specified is the relaxation time $\tilde{\tau}_a$.

Calculations of the relaxation time $\tilde{\tau}_a$ of partons already in the lowest order in the running coupling constant $g^2$ require summation of infinitely many diagrams. Resummation of the hard thermal loops results in the width $\tilde{\tau}_g^{-1}$ of partons $\sim g^2 T \ln(1/g)$. Based on this fact, the following parametrization was used for gluons $[4, 27]$

$$\tilde{\tau}_g^{-1} = N_c \frac{g^2 T}{4\pi} \ln \frac{2c}{g^2}, \quad (19)$$

with the strong interaction coupling constant $\sigma$ and a tuning parameter $c$. The relaxation time for a mixture of scalar and tensor glueballs was estimated according to $\sigma$ assuming the glueball scattering cross section $\sigma_{gb} = 30 \text{ mb}$ to be isotropic.

Comparison between the GD lattice data $[29, 30]$ and our QP results for the shear viscosity of the glue matter is presented in Fig. 3. The magnitude of the $\eta/s$ ratio in our model is defined mainly by the value of the relaxation time $[19]$. The solid line shows results of our calculation provided we use the recommended value $c = 14.4$, though our parameters of Eq. (2) are slightly different from those used in Ref. [4]. This $c$-value was tuned in Ref. [4] to the old lattice data for thermodynamic quantities $[12]$. As we see, the $\eta/s$ ratio gets discontinuity at $T = T_c$ with more

than by an order of magnitude lower value at $T \rightarrow T_c + 0$ (in gluon phase) than at $T \rightarrow T_c - 0$ (in the glueball phase). Also, the solid curve lies reasonably close to the points (filled circles, triangles, and squares) and its value in the minimum is slightly below the $\text{AdS/CFT}$ (Anti de Sitter/Conformal Field Theory) $1/4\pi$ bound $[31]$ (compare with the dotted curve). Preserving the form of the relaxation time (19), we can still increase $\eta/s$ values by tuning the parameter $c$. Taking $c = 11.44$, we achieve the limit case $\tilde{\tau}_g^{-1} \rightarrow 0$ for $T \rightarrow T_c + 0$ (full transparency). In this case (see the dashed line in Fig. 3 for $T > T_c$), we may reach a slightly better overall agreement with the lattice data $[29, 30]$, and the 1/4$\pi$ bound is achieved at the minimum. Varying the $c$-value in the interval $11.44 < c < 14.4$, one may simulate different values of the $\eta/s$ jump at $T = T_c$, but for temperatures $T \gtrsim 1.5T_c$ the $\eta/s$ ratio changes only slightly, demonstrating a slow increase with the growing temperature. Thus, bearing in mind large error bars in the lattice data, we are able to conclude that the results of the two-phase gluon-glueball model developed here are consistent with the existing lattice results $[29, 30]$. The perturbative regime (see the shaded region) is not achieved up to very high temperatures.

The $\eta/s$ ratio for the pure gluon phase in the range of $T \sim (1 - 2)T_c$ was also evaluated in Ref. [21]. The model employs the QP Ansatz for EOS successfully tested to describe old lattice results $[28]$. In Ref. [32], viscosity is treated by means of the kinetic theory for gluon quasiparticles. It is of interest that the model, being consistent with the old (and less accurate) lattice data for viscosity $[28]$ and thermodynamics $[20]$ that overestimate pressure at $T \gtrsim 1.5T_c$ (as follows from the comparison with new data, see Fig. 1), predicts a stronger temperature dependence of $\eta/s$ at $T > T_c$ than our model, which in turn is consistent with the new lattice data $[29, 30]$. The crucial point here is that the gluon relaxation time is defined essentially differently:

$$\tilde{\tau}_{\text{BKR}}^{-1} = a_\eta/(32\pi^2 T^4 g^4 \log(a_\eta/g)) \quad (20)$$

where $a_\eta = 6.8$. Here $\tilde{\tau}_{\text{BKR}}^{-1} \sim g^4$, coming from the $a_\eta^2 \sim g^4$ dependence of the gluon-gluon transport cross section $\sigma_{\gamma\gamma}$ in the relaxation-time expression $[18]$, as was estimated in the early work of Hosoya and Kaiantie $[30]$, whereas the above used $\tilde{\tau}_g^{-1} \sim g^2$ $[27]$. The $\eta/s$ ratio obtained with the relaxation time (20) is plotted in Fig. 3 by the dot-dashed line. This result for $T \gtrsim 2T_c$ recovers that of Ref. [22], but it differs significantly from those calculated with Eq. (19). Using recent lattice results for higher $T$ $[29]$, let us try to make a choice between two parametrizations of relaxation times $[19]$ and (20). We observe that for $T \gtrsim 10T_c$, shear viscosity calculations with (20) demonstrate a noticeable growth exceeding lattice data and even a perturbative estimate $(\eta/s)_{\text{pert}} \approx 0.8 - 1.0$ for $T \simeq (2 - 20)T_c$. The perturbative result is taken from Fig. 2 of [28] ($\eta$ was calculated in $[23]$ and $s$ in $[24]$). In contrast, predictions of our QP model with relaxation time (19) are in reasonable
agreement with the lattice results and do not contradict perturbative estimates. Recently, the relaxation time $\tau_g$ was estimated in Ref. [53] according to Eq. (18) from an analysis of cross sections of the $gg \rightarrow gg$ and $gg \rightarrow ggg$ processes. It was found that $\eta/s = 0.13$ and 0.076 for values $\alpha_s = 0.3$ and 0.6, respectively [which correspond to temperatures $T/T_c = 2.6$ and 1.36 provided Eq. (2) is used]. If these points were plotted in Fig. 3 they would not do such an additional tuning restricted by the results that we have demonstrated in Fig. 3.

The measured lattice points for the ratio of the bulk viscosity to entropy density are plotted in Fig. 4 together with different model results. In a broad range of temperatures, the global behavior of lattice data can be roughly approximated as $\zeta/s = 0.02/\sqrt{T/T_c - 1}$ (see the short-dashed curve). The reduced bulk viscosity $\zeta/s$ calculated in our two-phase gluon-glueball model following Eqs. (19) and (20) is shown by the solid line for $c = 14.4$ and by the dashed line for $\bar{\tau}_g^{-1}$ vanishing at $T_c$. Values of $\zeta/s$ for both curves noticeably underestimate the corresponding values on the approximating short-dashed curve, typically by an order of magnitude. Nevertheless, the shape of the curves is similar to that given by the approximating curve. Singularity at $T \to T_c + 0$ demonstrated by the dashed line (see insertion in Fig. 4) is due to the divergence of $\bar{\tau}_a$ in this limiting case.

The bulk viscosity (19) includes a rather complicated factor $Q_a$ depending on the EOS used. Using the energy conservation for a system with temperature-independent masses of particles, one may present the result (19) as follows:

$$\zeta = \sum_a \frac{d_a}{T} \int \frac{d^3 p}{(2\pi)^3} \tau_a F_a^{eq}(1 + F_a^{eq}) \left[ \frac{p^2}{3E_a} - c^2 E_a \right]^2$$

(21)

For a single-component gas, this expression coincides exactly with the 25-years-old result of Gavin [35].

Chakraborty and Kapusta (CK) [29] presented another expression [35].

$$\zeta_{CK} = \sum_a \frac{d_a}{T} \int \frac{d^3 p}{(2\pi)^3} \tau_a F_a^{eq}(1 + F_a^{eq}) Q_a^2$$

(22)

which differs from (19) but also reduces to (21) for $m_a = const$. Note that they also disregard the QP interaction term in the energy-momentum tensor; see Eq. (99) of their work. The reasons for the differences between (19) and (22) are discussed in the Appendix. The dash-double-dotted line in Fig. 4 demonstrates the $\zeta_{CK}/s$ ratio following Eq. (22) with the relaxation time $\bar{\tau}_g = \bar{\tau}_a$ given by Eq. (19) for $\bar{\tau}_a^{-1} \to 0$ at $T \to T_c + 0$. We see that Eq. (22) yields a strong suppression of the bulk viscosity at $T \gtrsim 1.5T_c$, as compared to that given by Eq. (19) (compare the dash-double-dotted and dashed lines in Fig. 4).

A somewhat different expression for $\zeta$ was used by Bluhm, Kämpfer, and Redlich (BKR) [32]. In their model, the bag constant $B$ is a functional of the non-equilibrium distribution function. They found

$$\zeta_{BKR} = \sum_a \frac{d_a}{3T} \int \frac{d^3 p}{(2\pi)^3} \tau_a F_a^{eq}(1 + F_a^{eq}) \times Q_a \left[ m_a^2(T) - T \frac{dm_a^2(T)}{dT} \right].$$

(23)

Here it was assumed that the QP interaction contributes to the energy-momentum tensor. Thereby, compared to (19) there appeared the second term $Tdm_a^2(T)/dT$ in the square brackets of Eq. (23). For constant masses, the latter equation is also reduced to (21). Numerical calculations with (23) (see the dot-dashed line in Fig. 4) give rise to the $\zeta_{BKR}/s$ ratio, which dramatically decreases, being in large discrepancy with both the above models and the lattice data for $T > 1.5T_c$. For $T > 1.9T_c$, Eq. (23) becomes invalid, providing negative values.

A perturbative estimate [37] gives

$$\zeta/s_{pert} = 0.02 \alpha_s^2$$

(24)

for $0.06 \leq \alpha_s \leq 0.3$. Applying the $T$-dependent coupling constant (2) for $\alpha_s = g(T)/4\pi$, we get a perturbative
estimate of the bulk viscosity to entropy density ratio (plotted by the dotted line in Fig. 4). As is seen, in the region of its applicability, Eq. (24) produces only slightly larger values of $\zeta/\eta$ than those given by our QP model. The new lattice GD calculations demonstrate a significant increase of the ratio $\zeta/\eta$ at approaching the critical point ($\zeta/\eta \approx 0.5 \div 2$ at $T = 1.02 T_c$). These values are reproduced neither by our QP model using relaxation time Eq. (19) nor by the approximating short-dashed curve exploiting a simple $T$ dependence of the $\zeta/\eta$ ratio.

Comparing results presented in Figs. 3 and 4, we see that in the gluon phase in a narrow vicinity of the critical point $[(T - T_c)/T_c \lesssim 0.1]$, the ratio $\zeta/\eta \gtrsim 0.1$ reaches the value $\zeta/\eta \approx 0.3$ for $T \rightarrow T_c + 0$. The ratio sharply decreases with an increase of the temperature up to values $\zeta/\eta \sim 10^{-2} \div 10^{-3}$ for $T > 2T_c$. The smallness of this ratio controls the violation of the conformal symmetry.

V. CONCLUSIONS

A quasiparticle approach has been applied to the $SU(3)$ glue matter with temperature-dependent masses. Matching the pure gluon and glueball phase descriptions by means of the Gibbs conditions allows one to describe successfully this system in a thermodynamically consistent way both above and below the critical temperature $T_c$. For thermodynamic characteristics, the quasiparticle-model results are in good agreement with the latest lattice data.

The constructed equation of state was used to calculate the shear and bulk viscosities in the relaxation-time approximation in a wide temperature range. The magnitudes of the shear and bulk viscosities are mainly determined by the value of the relaxation time. We exploited two different values for the relaxation time that have been used in the literature; see Eqs. (19) and (20). With the relaxation time (19), the shear viscosity to entropy density ratio $\eta/\xi$ fits rather well the scant lattice data. We found that the ratio $\eta/\xi$ undergoes a discontinuity at the critical temperature $T = T_c$. At $T$ slightly above $T_c$ the ratio $\eta/\xi$ has a minimum, the value of which is close to the AdS/CFT bound $1/4\pi$. Then $\eta/\xi$ increases with the subsequent rise of the temperature. The bulk viscosity to entropy density ratio $\zeta/\xi$ also has a break at $T_c$. Then it monotonically decreases with the temperature increase. Although the calculated $\zeta/\xi$ ratio essentially underestimates the upper limits given by the corresponding lattice data, its temperature dependence is well described.

Within our model, the ratio $\zeta/\xi \approx 0.3$ at $T \rightarrow T_c + 0$ and it sharply decreases with the rising temperature up to values $\zeta/\xi \sim 10^{-2} \div 10^{-3}$ for $T > 2T_c$.

We point out that although our QP model describes well the thermodynamical characteristics calculated on the lattice it does not guarantee that all important physics is incorporated, especially in a description of the vicinity of the critical point, where fluctuation effects are increased. Using the averaged value for the relaxation time can be considered only as a very rough approximation. The dependence of the relaxation time on the coupling constant $g$ is also not well defined. Simplifying, all existing approaches to the evaluation of the bulk viscosity use some Ansätze reductions yielding different results. We used the kinetic approach with the collisional source of the viscosity disregarding other possible sources [10]. One such source is associated with the presence of a soft mode [3] in the vicinity of the second and weak first-order phase-transition critical points. Also, since statistical error bars are very large, new more certain lattice data are required to draw a more definite conclusion on the agreement or disagreement of the calculated $\zeta/\xi$ ratio with the lattice results.

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VI. APPENDIX

Deriving kinetic coefficients, the authors of [22–25] used the relaxation-time approximation to the kinetic equation presenting the collision integral as

$$StF_a = -\delta F_a/\tau_a[F_{\text{loc.eq}}],$$

where

$$\delta F_a = F_a(E_a[F] - F_{\text{loc.eq}}[E_a[F_{\text{loc.eq}}]]),$$

see, e.g., Eqs. (38) and (40) in [22]. Here it was assumed that the collision term should be zero for the global and local equilibrium states, i.e., for $F_a = F_{\text{loc.eq}}[E_a[F_{\text{loc.eq}}]]$.

Then after setting $F_a = F_{\text{loc.eq}}[E_a[F_{\text{loc.eq}}]]$ on the left-hand side of the kinetic equation, one finds

$$\delta F_a = -\frac{\tau_a[F_{\text{loc.eq}}]}{E_a[F_{\text{loc.eq}}]} \frac{\partial F_{\text{loc.eq}}[E_a[F_{\text{loc.eq}}]]}{\partial x_a},$$

see Eq. (2.3) in [22] and Eq. (42) in [23]. We stress that all quantities on the right-hand side of this equation including the relaxation time $\tau$ are expressed in terms of the local equilibrium distribution functions.

To derive expressions for the shear and bulk viscosities [19] and [19], one presents spatial components of the variation of the energy-momentum tensor of quasiparticles [10] as

$$\delta T^{ik} = \sum_a \int d\Gamma \frac{p_i^a p_k^a}{E_a[F]} \delta F_a - \sum_a \int d\Gamma \frac{p_i^a p_k^a}{E_a[F_{\text{loc.eq}}]} \delta E_a$$

$$\rightarrow \sum_a \int d\Gamma \frac{p_i^a p_k^a}{E_a[F_{\text{loc.eq}}]} \delta F_a.$$

(28)
To avoid cumbersome expressions we omitted antiparticle terms. The reduction done in the second line in Eq. (28) is actually an Ansatz: we vary only the distribution function and do not vary quantities that depend on the distribution function implicitly (through phase-space integrals incorporating the distribution function), i.e., $\delta E_a[\delta F]$ are set zero. This reduction is in the spirit of the relaxation-time approximation to the kinetic equation, where the momentum-dependent relaxation-time parameter is replaced in actual calculations by an averaged value. Note that dropping the $\delta E$ term, we actually ignore a sub-leading term in the case of a weak coupling constant and/or for a very dilute system, see Eq. (5.22) of [10]. The distribution function $\delta F_a$ counted from the local equilibrium value enters the expression for $\delta T^{ik}$, the expression for $\delta T^{00} = 0$ [see (45) and (50) in [25]], and expressions for the viscosities. Thus one can easily compute kinetic coefficients knowing thermodynamic quantities in the local rest frame $\vec{u} = 0$.

However, we should note that in the QP Fermi liquid theory following the work of Abrikosov and Khatatnikov [41], one usually uses a different procedure to obtain transport coefficients; see [12] for details. One exploits that in the original Landau collision term, the following combination enters:

$$
\delta \left( \sum_a E_a[F] \right) \{ F_1(E_1) F_2(E_2) (1 \mp F_3(E_3))(1 \mp F_4(E_4)) - F_3(E_3) F_4(E_4) (1 \mp F_1(E_1))(1 \mp F_2(E_2)) \},
$$

where $E_a$ are functionals of the exact non-equilibrium distribution function, $a = 1, 2, 3, 4$. The term in curly brackets is zero not only for $F_a = F^{\text{loc.eq.}}_a(E_a(F^{\text{loc.eq.}}))$ but also for $F^{\text{loc.eq.}}_a(E_a[F])$. Thereby, $\delta T F^{\text{loc.eq.}}_a(E_a[F]) = 0$. Thus introducing

$$
\tilde{\delta} F_a = F_a(E_a[F]) - F^{\text{loc.eq.}}_a(E_a[F])
$$

in the relaxation-time approximation, we may rewrite the collision term as

$$
\dot{\delta} T F_a = -\tilde{\delta} F_a / \tau_a(E[F]).
$$

The quantity $\dot{\tau}[F]$ entering Eqs. (31) and (28) depends on the unknown exact non-equilibrium distribution function, since the $\delta$-function term in the collision integral and the local equilibrium distributions there continue to depend on exact energies in this approach. If we want to calculate the value of $\tau^{\text{loc.eq.}}$ entering Eqs. (15) and (10) using Eq. (31), we should still expand $E[F]$ in (31) near the known value $E^{\text{loc.eq.}}$ everywhere including the $\delta$-function term in the collision integral.

From the left-hand side of the kinetic equation, one gets

$$
\delta \tilde{F}_a = -\frac{\dot{\tau}_a[F]}{E_a[F]} \frac{\partial F^{\text{loc.eq.}}_a(E_a[F])}{\partial x_a^{\mu}}.
$$

Then one may use a simple expression for $\delta T^{ik}$,

$$
\delta T^{ik} = \sum_a \int d\tau \frac{\partial x_a^{\mu}}{E_a[F]} \delta \tilde{F}_a
$$

since now variations are always performed at fixed $E_a$. As above, the QP interaction term is omitted. Comparing the second line of Eqs. (28) and (33), we see that disregarding the implicit dependence $E[\delta F]$, Refs. [22–25] actually do not distinguish between distributions $\delta F$ and $\tilde{\delta} F$.

Then in both considered approaches, one uses the exact relation $E_a[F] = \delta T^{00}/\delta F_a$, i.e., that the variation of the energy is determined through $\delta F_a$ as

$$
\delta T^{00} = \sum_a \int d\tau E_a[F] \delta F_a = \sum_a \int d\tau E_a^{[\text{loc.eq.}]} \delta F_a + O(\delta F^2). \quad (34)
$$

Following [28] and [30], we have

$$
\delta F - \tilde{\delta} F = \frac{\partial E^{[\text{loc.eq.}]}(E[F^{\text{loc.eq.}}])}{\partial E} \delta E.
$$

Furthermore, instead of using a complicated implicit dependence $\delta E[\delta F]$ with $\delta F$ given by Eq. (27), which would be the fully correct procedure, Ref. [30] uses the Ansatz relations [see Eq. (102) of that work]

$$
\tilde{\delta} F = \exp \left\{ \frac{E^{[\text{loc.eq.}]}(E^{[\text{loc.eq.}]})}{T[F^{\text{loc.eq.}}]} \right\} E^{[\text{loc.eq.}]} \frac{\delta E}{\delta T}
$$

which assume that the distribution function in a nonequilibrium state has the form $F = e^{-E[F]/T[F]}$ in the Boltzmann limit $F \ll 1$. Thus, although Ref. [30] distinguishes between distributions $\delta F$ and $\tilde{\delta} F$, it uses very special relations [30], which might be incompatible with $\delta E(m[\delta F])$ as it follows from Eqs. (32) and (11). To find bulk viscosity, one further expresses $F_a = F^{\text{loc.eq.}}(E_a[F])(1 - A_a \partial_x u^a)$, see Ref. [30], and one observes that the shift of the solution $A(E) \rightarrow A(E) - bE$ generates new solutions of the Landau kinetic equation for arbitrary constant $b$. Then one chooses $b$ to explicitly fulfill the Landau-Lifshitz condition $\partial_x T^{\mu\nu} = 0$. Note that this modification of the solution is quite unnecessary provided one guarantees that the condition $\delta T^{00} = 0$ holds in the local rest frame. We have checked that this condition is satisfied in our QP model.

Finally, within this approach one arrives at the expression (28) for the bulk viscosity, which is explicitly positive definite, whereas positive definiteness of Eq. (10) is not seen explicitly. However, we stress once more that all quantities in (28) still depend on exact energies, while the way in which the quantities in (10) depend on unknown exact distribution functions is hidden. Thus explicit positive definiteness of expression (28) for $\zeta$ presents actually only an apparent improvement. In any case, to use Eq. (28) in practical calculations, where only equilibrium quantities are known, one should replace $E[F]$ by $E[F^{\text{loc.eq.}}]$. 


Moreover, we should stress that the values of the relaxation time in \(\text{(16)}\) and \(\text{(22)}\) are different. Since we do not perform complicated microscopic calculations of the relaxation time but only estimate its average value, we actually cannot determine whether \(\text{(16)}\) or \(\text{(22)}\) is more preferable, and we may use both of them.

Note that \(\text{(23)}\) is derived for a different model, where the QP interaction contributes to the energy-momentum tensor. Also, authors of \(\text{(22)}\) and \(\text{(23)}\) lead to different values of the bulk viscosity, as is shown in Fig. 4.

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