Polychromatic Arm Exponents for the Critical Planar FK-Ising model

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Abstract
We derive the arm exponents of SLE_κ for κ ∈ (4, 8) and explain how to combine them with the convergence of the interface to obtain the arm exponents of critical FK-Ising model. We obtain six different patterns of boundary arm exponents and three different patterns of interior arm exponents of critical FK-Ising model.

Keywords: random-cluster model, critical FK-Ising model, Schramm Loewner Evolution, arm exponents.

1 Introduction
Fortuin and Kasteleyn introduced the random-cluster model in 1969. The random-cluster model is a probability measure on edge configurations where each edge is open or closed, and the probability of a configuration is proportional to

\[ p^{\# \text{open edges}} (1 - p)^{\# \text{closed edges}} q^{\# \text{clusters}}, \]

where \( p \in [0, 1] \) is the edge weight and \( q > 0 \) is the cluster weight. The random cluster model is related to various models: percolation, Ising model etc. and the readers could consult [DC13] for the background. When \( q \in (0, 4] \), the critical phase is believed to be conformally invariant and the interface at criticality is conjectured to converge to SLE_κ where

\[ \kappa = 4\pi / \arccos(-\sqrt{q}/2). \] (1.1)

This conjecture is only proved for \( q = 2 \) by the celebrated works of Chelkak and Smirnov [CS12, CDCH+14]. When \( q = 2 \), the critical random-cluster model is also called critical FK-Ising model. In this paper, we derive the polychromatic arm exponents of critical FK-Ising model.

In the random-cluster model, an arm is a primal-open path (type 1) or a dual-open path (type 0). We are interested in the decay of the probability that there are a certain number of arms in the semi-annulus \( A^+(n, N) \) or annulus \( A(n, N) \) connecting the inner boundary to the outer boundary. This probability should decay like a power in \( N \) as \( N \to \infty \), and the exponent in the power is called the critical arm exponents.

In the case of percolation, Kesten proved that [Kes87] the critical arm exponents are essential in the study of near-critical percolation and the so-called scaling relations would follow from the existence and the value of critical 1-arm exponent and 4-arm exponent. In [Sch00], Oded Schramm introduced Schramm Loewner Evolution as the candidate of the scaling limits of critical lattice models. In [Smi01], Smirnov proved the convergence of the interface in critical percolation to SLE_6, hence made it possible to calculate

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the value of the arm exponents of critical percolation through SLE6, and the value of the polychromatic arm exponents are precisely derived in [LSW01a, LSW01b, LSW02b, LSW02a, SW01].

In [SW01], the authors explained that, in order to derive the arm exponents of critical percolation, one needs three inputs: (1) the convergence of the interface to SLE6; (2) the arm exponents of SLE6; and (3) the quasi-multiplicativity (see more detail in Section 3). This strategy also works for random-cluster model. In this paper, we derive the arm exponents of FK-Ising model following this strategy. The convergence of the interface to SLE16/3 is proved in [CST12, CDCH14] and the quasi-multiplicativity is obtained in [CDCH16]. We derive the arm exponents of SLE in this paper and explain how to combine these three inputs to get the arm exponent for FK-Ising model.

We derive the arm exponents of SLEκ with κ ∈ (4, 8) in Section 2. In Section 3, we explain how to combine the convergence of the interface, the quasi-multiplicativity and the arm exponents of SLE to derive the arm exponents of FK-Ising model. In Section 3, we carefully point out the general results of the random-cluster model and the particular results of the FK-Ising model. The same proof in Section 3 could also serve as the derivation of the arm exponents for the random-cluster model with q ∈ [1, 4]. Note that q and κ are related through (1.1) and q ∈ [1, 4] corresponds to κ ∈ (4, 6], and thus the arm exponents in Theorems 1.1 and 1.3 would give the arm exponents for these q provided that the convergence of the interface and the quasi-multiplicativity are at hand.

We state the conclusion for SLE using the language from random-cluster model which is not defined for SLE by now, we will explain the precise definition for SLE in Section 2 and they will become clear then. The readers could consult Figures 1.1 and 1.2 for the idea.

**Theorem 1.1.** Fix κ ∈ (4, 8). Set α0+ = β0+ = 0 and γ0+ = γ1+ = 0. We have the following six different patterns of boundary arm exponents of SLEκ. Fix j ≥ 1.

- **Consider the wired boundary condition (11),** let σ = (010· · ·10) with length 2j − 1 (σ starts with 0 and it is followed by (j − 1) pairs of 10). The boundary arm exponents for this pattern is given by

\[ \alpha_{2j-1}^+ = j(4j + 4 - \kappa)/\kappa. \] (1.2)

- **Consider the wired boundary condition (11),** let σ = (010· · ·1) with length 2j (σ contains j pairs of 01). The boundary arm exponents for this pattern is given by

\[ \beta_{2j}^+ = j(4j + \kappa - 4)/\kappa. \] (1.3)

- **Consider the wired boundary condition (11),** let σ = (101· · ·01) with length 2j + 1 (σ starts with 1 and it is followed by j pairs of 01). The boundary arm exponents for this pattern is given by

\[ \gamma_{2j+1}^+ = (j + 1)(4j + 3\kappa - 16)/\kappa + (\kappa - 6)(\kappa - 8)/(2\kappa). \] (1.4)

- **Consider the free/wired boundary condition (01),** let σ = (10· · ·10) with length 2j (σ contains j pairs of 10). The boundary arm exponents for this pattern is given by

\[ \alpha_{2j}^+ = j(4j + 8 - \kappa)/\kappa. \] (1.5)

- **Consider the free/wired boundary condition (01),** let σ = (10· · ·101) with length 2j − 1 (σ starts with j − 1 pairs of 10 and ends with 1). The boundary arm exponents for this pattern is given by

\[ \beta_{2j-1}^+ = j(4j + \kappa - 8)/\kappa. \] (1.6)

- **Consider the free/wired boundary condition (01),** let σ = (010· · ·01) with length 2j (σ contains j pairs of 01). The boundary arm exponents for this pattern is given by

\[ \gamma_{2j}^+ = j(4j + 3\kappa - 16)/\kappa + (\kappa - 4)(\kappa - 6)/(2\kappa). \] (1.7)
**Theorem 1.2.** For the critical planar FK-Ising model on the square lattice, we have six different patterns of boundary arm exponents with the same notations as in Theorem 1.1 taking \( \kappa = \frac{16}{3} \). We list several of them here:

\[
\alpha_1^+ = \frac{1}{2}, \quad \alpha_2^+ = \frac{5}{4}, \quad \beta_1^+ = \frac{1}{4}, \quad \beta_2^+ = 1, \quad \gamma_2^+ = \frac{2}{3}, \quad \gamma_3^+ = \frac{5}{3}.
\]

**Fig. 1.1:** The six different patterns of boundary arm exponents in Theorem 1.1

**Theorem 1.3.** Fix \( \kappa \in (4, 8) \). We have the following three different patterns of interior arm exponents of \( \text{SLE}_\kappa \). Fix \( j \geq 1 \).

- Let \( \sigma = (10 \cdots 10) \) with length \( 2j \) (\( \sigma \) contains \( j \) pairs of 10). The interior arm exponent for this pattern is given by
  \[
  \alpha_{2j} = \frac{(16j^2 - (\kappa - 4)^2)}{(8\kappa)}.
  \]

- Let \( \sigma = (10 \cdots 101) \) with length \( 2j + 1 \) (\( \sigma \) starts with \( j \) pairs of 10 and ends with 1). The interior arm exponent for this pattern is given by
  \[
  \beta_{2j+1} = \frac{j(2j + \kappa - 4)}{\kappa}.
  \]

- Let \( \sigma = (0110 \cdots 10) \) with length \( 2j + 2 \) (\( \sigma \) starts with 01 and it is followed by \( j \) pairs of 10). The interior arm exponent for this pattern is given by
  \[
  \gamma_{2j+2} = \frac{(4(2j + \kappa - 4)^2 - (\kappa - 4)^2)}{(8\kappa)}.
  \]

**Theorem 1.4.** For the critical planar FK-Ising model on the square lattice, we have three different patterns of interior arm exponents with the same notations as in Theorem 1.3 taking \( \kappa = \frac{16}{3} \). We list several of them here:

\[
\alpha_2 = \frac{1}{3}, \quad \beta_3 = \frac{5}{8}, \quad \gamma_4 = 1, \quad \alpha_4 = \frac{35}{24}, \quad \beta_5 = 2, \quad \alpha_6 = \frac{10}{3}.
\]
Fig. 1.2: The three different patterns of interior arm exponents in Theorem 1.3.

(a) $\alpha_6$: $(101010)$.  
(b) $\beta_7$: $(1010110)$.  
(c) $\gamma_8$: $(10101100)$.

Fig. 1.3: The $x$-axis corresponds to $\kappa \in (4, 8)$. The blue lines are the arm exponents $\alpha_2, \alpha_4, \alpha_6, \alpha_8$ as functions of $\kappa$. The red lines are $\beta_3, \beta_5, \beta_7$ as functions of $\kappa$. The yellow lines are $\gamma_4, \gamma_6, \gamma_8$ as functions of $\kappa$.

Remark 1.5. In Theorem 1.1, if we set $\kappa = 6$ then we find all the six formulae have the same expression:

$$\alpha_j^+ = \beta_j^+ = \gamma_j^+ = \frac{j(j + 1)}{6},$$

which is the boundary arm exponents for critical percolation, the reason is that the boundary arm exponents for percolation are independent of boundary conditions and are the same over all patterns. In Theorem 1.3, if we set $\kappa = 6$ then we find all the three formulae have the same expression

$$\alpha_{2j} = \gamma_{2j} = \frac{(2j)^2 - 1}{12}, \quad \beta_{2j+1} = \frac{(2j + 1)^2 - 1}{12},$$

which is the interior arm exponents for critical percolation, the reason is that the interior arm exponents for percolation are the same over all patterns as long as they are polychromatic, i.e. $\sigma$ is not constant. These arm exponents for percolation were derived in [LSW01a, SW01]. Note that it is proved in [BN11] that the monochromatic arm exponents (i.e. $\sigma$ is constant) for percolation are distinct from the polychromatic ones, and they are still unknown.
Remark 1.6. We point out some interesting facts with the formulae in Theorems 1.1 and 1.3:

\[ \beta_2^+ = 1, \quad \beta_3^+ = 2, \quad \beta_5 = 2, \quad \forall \kappa \in (4, 8). \]

The arm exponent \( \beta_2^+ = 1 \) is a universal arm exponent of random-cluster model for \( q \in [1, 4) \). The exponent \( \beta_5 = 2 \) is expected to be the universal arm exponent for \( q \in [1, 4) \) too, but it is only proved for \( q = 2 \) in [CDCH16].

Relation to previous works. The formulae in Theorems 1.1 and 1.3 were obtained for \( \kappa = 6 \) in [LSW01a, SW01]. The formulae (1.2, 1.3, 1.5, 1.6) were obtained in [WZ16]. In [Wu16], we prove similar results as in Theorems 1.1 and 1.3 for SLE\( _\kappa(\rho) \) where \( \kappa \in (0, 4) \), and we prove similar results as in Theorems 1.2 and 1.4 for the critical planar Ising model.

The boundary 1-arm exponent \( \alpha_1^+ = 1/2 \) for FK-Ising model was obtained in [DCHN11, Proposition 5.7]. The interior 5-arm exponent \( \beta_5 = 2 \) is one of the universal arm exponents for FK-Ising which was obtained in [CDCH16, Corollary 1.5].

The 2-arm exponents \( \alpha_2 \) is related to the Hausdorff dimension of SLE which is \( 2 - \alpha_2 \). This dimension was obtained in [Bel08]. The 3-arm exponents \( \beta_3 \) is related to the Hausdorff dimension of the frontier of SLE which is \( 2 - \beta_3 \). This dimension is the same as the dimension of SLE\( _{16/\kappa} \) by duality. The 4-arm exponent \( \alpha_4 \) is related to the Hausdorff dimension of the double points of SLE which is \( 2 - \alpha_4 \). This dimension was obtained in [MW16, Theorem 1.1]. The 4-arm exponent \( \gamma_4 \) is related to the Hausdorff dimension of the cut points of SLE which is \( 2 - \gamma_4 \). This dimension is the same as the dimension of SLE\( _{16/\kappa} \) by duality. The 4-arm exponent \( \alpha_4 \) is related to the Hausdorff dimension of the frontier of SLE which is \( 2 - \beta_4 \). This dimension was obtained in [MW16, Theorem 1.2].

Moreover, the formulae (1.2) and (1.8) were predicted by KPZ in [Dup03, Equations (11.44), (11.45)].

Outline. In Section 2, we will give preliminaries on SLE and complete the proof of Theorems 1.1 and 1.3. In Section 3, we will give preliminaries on the random-cluster models and complete the proof of Theorems 1.2 and 1.4.

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2 Schramm Loewner Evolution

2.1 Preliminaries on SLE

Notations. We denote by \( f \lesssim g \) if \( f/g \) is bounded from above by universal finite constant, by \( f \gtrsim g \) if \( f/g \) is bounded from below by universal positive constant, and by \( f \asymp g \) if \( f \lesssim g \) and \( f \gtrsim g \).

We denote by

\[
\begin{align*}
  f(\epsilon) &= g(\epsilon)^{1+o(1)} & \text{if} & \lim_{\epsilon \to 0} \frac{\log f(\epsilon)}{\log g(\epsilon)} = 1.
\end{align*}
\]

For \( z \in \mathbb{C}, r > 0 \), we denote \( B(z, r) = \{ w \in \mathbb{C} : |w - z| < r \} \).

For two subsets \( A, B \subseteq \mathbb{C} \), we denote \( \text{dist}(A, B) = \inf\{|x - y| : x \in A, y \in B\} \).

\( \mathbb{H} \)-hull and Loewner chain We call a compact subset \( K \) of \( \mathbb{H} \) an \( \mathbb{H} \)-hull if \( \mathbb{H} \setminus K \) is simply connected. Riemann’s Mapping Theorem asserts that there exists a unique conformal map \( g_K \) from \( \mathbb{H} \setminus K \) onto \( \mathbb{H} \) such that

\[
\lim_{|z| \to \infty} |g_K(z) - z| = 0.
\]

We call such \( g_K \) the conformal map from \( \mathbb{H} \setminus K \) onto \( \mathbb{H} \) normalized at \( \infty \).

Lemma 2.1. Fix \( x > 0 \) and \( \epsilon > 0 \). Let \( K \) be an \( \mathbb{H} \)-hull and let \( g_K \) be the conformal map from \( \mathbb{H} \setminus K \) onto \( \mathbb{H} \) normalized at \( \infty \). Assume that

\[
x > \max(K \cap \mathbb{R}).
\]
Denote by $\gamma$ the connected component of $\mathbb{H} \cap (\partial B(x, \epsilon) \setminus K)$ whose closure contains $x + \epsilon$. Then $g_K(\gamma)$ is contained in the ball with center $g_K(x + \epsilon)$ and radius $3(g_K(x + 3\epsilon) - g_K(x + \epsilon))$, hence it is also contained in the ball with center $g_K(x + 3\epsilon)$ and radius $8\epsilon |g'_K(x + 3\epsilon)|$.

**Proof.** [Wu16, Lemma 2.1].

**Lemma 2.2.** Fix $z \in \overline{\mathbb{H}}$ and $\epsilon > 0$. Let $K$ be an $\mathbb{H}$-hull and let $g_K$ be the conformal map from $\mathbb{H} \setminus K$ onto $\mathbb{H}$ normalized at $\infty$. Assume that

$$\text{dist}(K, z) \geq 16\epsilon.$$ 

Then $g_K(B(z, \epsilon))$ is contained in the ball with center $g_K(z)$ and radius $4\epsilon |g'_K(z)|$.

**Proof.** By Koebe 1/4 theorem, we know that

$$\text{dist}(g_K(K), g_K(z)) \geq d := 4\epsilon |g'_K(z)|.$$ 

Let $h = g_K^{-1}$ restricted to $B(g_K(z), d)$. Applying Koebe 1/4 theorem to $h$, we know that

$$\text{dist}(z, \partial h(B(g_K(z), d))) \geq d|h'(g_K(z))|/4 = \epsilon.$$ 

Therefore $h(B(g_K(z), d))$ contains the ball $B(z, \epsilon)$, and this implies that $B(g_K(z), d)$ contains the ball $g_K(B(z, \epsilon))$ as desired. \hfill \Box

Loewner chain is a collection of $\mathbb{H}$-hulls $(K_t, t \geq 0)$ associated with the family of conformal maps $(g_t, t \geq 0)$ obtained by solving the Loewner equation: for each $z \in \mathbb{H}$,

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z,$$  

where $(W_t, t \geq 0)$ is a one-dimensional continuous function which we call the driving function. Let $T_z$ be the swallowing time of $z$ defined as $\sup\{t \geq 0 : \min_{s \in [0, t]} |g_s(z) - W_s| > 0\}$. Let $K_t := \{z \in \mathbb{H} : T_z \leq t\}$. Then $g_t$ is the unique conformal map from $H_t := \mathbb{H} \setminus K_t$ onto $\mathbb{H}$ normalized at $\infty$.

Here we discuss a little about the evolution of a point $y \in \mathbb{R}$ under $g_t$. We assume $y \leq 0$. There are two possibilities: if $y$ is not swallowed by $K_t$, then we define $Y_t = g_t(y)$; if $y$ is swallowed by $K_t$, then we define $Y_t$ to be the image of the leftmost of point of $K_t \cap \mathbb{R}$ under $g_t$. Suppose that $(K_t, t \geq 0)$ is generated by a continuous path $(\eta(t), t \geq 0)$ and that the Lebesgue measure of $\eta[0, \infty] \cap \mathbb{R}$ is zero. Then the process $Y_t$ is uniquely characterized by the following equation:

$$Y_t = y + \int_0^t \frac{2ds}{Y_s - W_s}, \quad Y_t \leq W_t, \quad \forall t \geq 0.$$ 

In this paper, we may write $g_t(y)$ for the process $Y_t$.

**SLE processes** An SLE$_\kappa$ is the random Loewner chain $(K_t, t \geq 0)$ driven by $W_t = \sqrt{\kappa}B_t$ where $(B_t, t \geq 0)$ is a standard one-dimensional Brownian motion. In [RS05], the authors prove that $(K_t, t \geq 0)$ is almost surely generated by a continuous transient curve, i.e. there almost surely exists a continuous curve $\eta$ such that for each $t \geq 0$, $H_t$ is the unbounded connected component of $\mathbb{H} \setminus [0, t]$ and that $\lim_{t \to \infty} |\eta(t)| = \infty$.

We can define an SLE$_\kappa(\rho^L; \rho^R; \rho^I)$ process with three force points $(x^L; x^R; z)$ where $\rho^L, \rho^R, \rho^I \in \mathbb{R}$ and $x^L \leq 0 \leq x^R$ and $z \in \mathbb{H}$. It is the Loewner chain driven by $W_t$ which is the solution to the following systems of SDEs:

$$dW_t = \sqrt{\kappa}dB_t + \frac{\rho^L dt}{W_t - V^L_t} + \frac{\rho^R dt}{W_t - V^R_t} + \Re \frac{\rho^I dt}{W_t - V^I_t}; \quad W_0 = 0;$$

$$dV^L_t = \frac{2dt}{V^L_t - W_t}, \quad V^L_0 = x^L; \quad dV^R_t = \frac{2dt}{V^R_t - W_t}, \quad V^R_0 = x^R; \quad dV^I_t = \frac{2dt}{V^I_t - W_t}, \quad V^I_0 = z.$$
The solution exists up to the first time that $W$ hits $V_L$, $V_R$ or $V_I$. Suppose $\rho' = 0$, when $\rho^L > -2$ and $\rho^R > -2$, the solution exists for all times, and the corresponding Loewner chain is almost surely generated by a continuous transient curve ([MS16 Section 2]). When $\rho' \neq 0$ and $\rho^L > -2$, $\rho^R > -2$, the solution exists up to the first time that $z$ is swallowed, and the corresponding Loewner chain is almost surely generated by a continuous curve ([MS13 Section 2.1]). The SLE processes satisfy the Domain Markov Property: Let $\eta$ be an SLE$_\kappa(\rho^L, \rho^R; \rho')$ process with force points $(x^L, x^R; z)$. Suppose that $\tau$ is any stopping time (before $z$ is swallowed), then the image of $\eta[\tau, \infty)$ under $g_{\tau} - W_{\tau}$ has the same law as an SLE$_\kappa(\rho^L, \rho^R; \rho')$ process with force points $(V^{L\kappa}_\tau; V^{R\kappa}_\tau; V^{I\kappa}_\tau)$.

Suppose $\rho' = 0$. There are two special values of $\rho$: $\kappa/2 - 2$ and $\kappa/2 - 4$. When $\rho^R \geq \kappa/2 - 2$, then the curve never hits $[x^R, \infty)$. When $\rho^R \leq \kappa/2 - 4$, then the curve almost surely accumulates at $x^R$ at finite time. See [Dub09, Lemma 15].

From Girsanov Theorem, it follows that the law of an SLE$_\kappa(\rho^L; \rho^R; \rho')$ process can be constructed by reweighting the law of an ordinary SLE$_\kappa$.

**Lemma 2.3.** Suppose $x^L < 0 < x^R$ and $z \in \mathbb{H}$, define
\[
M_t(x^L; x^R) = g'_t(x^L)^{\rho^L/(\rho^L+4-\kappa)/(4\kappa)}(W_t - g_t(x^L))^{\rho^L/\kappa} \times (g_t(x^R) - g_t(x^L))^{\rho^R/(2\kappa)};
\]
\[
M_t(z) = |g'_t(z)|^{\rho^L/(\rho^L+8-2\kappa)/(8\kappa)} \times g_t(z)^{\rho^R/(8\kappa)}|g_t(z) - W_t|^{\rho'/\kappa}.
\]
Then $M(x^L; x^R)$ is a local martingale for SLE$_\kappa$ and the law of SLE$_\kappa$ weighted by $M(x^L; x^R)$ (up to the first time that $W$ hits one of the force points) is equal to the law of SLE$_\kappa(\rho^L; \rho^R)$ with force points $(x^L; x^R)$. Also, $M(z)$ is a local martingale for SLE$_\kappa$ and the law of SLE$_\kappa$ weighted by $M(z)$ (up to the first time that $z$ is swallowed) is equal to the law of SLE$_\kappa(\rho')$ with force point $z$.

**Proof.** [SW03 Theorem 6].

**Lemma 2.4.** Fix $\kappa \in (0, 8)$. Let $\eta$ be an SLE$_\kappa$ in $\mathbb{H}$ from $0$ to $\infty$. For $x \geq \epsilon > 0$, there is a universal constant $C > 0$ such that
\[
P[\eta \text{ hits } B(x, \epsilon)] \leq C(\epsilon/x)^{8/\kappa - 1}.
\]

**Proof.** [AK08].

**Lemma 2.5.** Suppose that $\eta$ is an SLE$_\kappa(\rho^L; \rho^R)$ process in $\mathbb{H}$ from $0$ to $\infty$ with force points located at $x^L \leq 0 \leq x^R$. Assume that $\rho^L, \rho^R > (-2) \vee (\kappa/2 - 4)$.

Fix $\delta \in (0, 1/2)$ and define the stopping time $S_1 = \inf\{t : \eta(t) \in \partial B(i, \delta)\}$. Denote by $U(\delta)$ the $\delta$-neighborhood of the segment connecting 0 to $i$. Define the stopping time $S_2 = \inf\{t : \eta(t) \notin U(\delta)\}$. Then there exists $p_0 = p_0(\delta) > 0$, which is uniform over $x^L$ and $x^R$, such that
\[
P[S_1 < S_2] \geq p_0.
\]

For $\epsilon > 0$, denote by $S_3$ the first time that $\eta$ hits $B(\epsilon, \delta \epsilon)$. Then there exists $q_0 = q_0(\delta) > 0$, which is uniform over $x^L, x^R$ and $\epsilon > 0$, such that
\[
P[S_1 < S_3] \geq q_0.
\]

**Proof.** The relation (2.2) was proved in [MW16 Lemma 2.4]. We only prove (2.3). Since $\rho^R > \kappa/2 - 4$, there is positive chance that $\eta$ never hits $B(1, \delta)$. Consider the function
\[
f(x^L; x^R) = P[\text{dist}(\eta, 1) \geq \delta] \geq |x^L|, |x^R| \to \infty,
\]
the function $f$ converges to the probability that an SLE$_\kappa$ never hits $B(1, \delta)$, which is also positive. This implies that there is $q_0 > 0$ uniform over $x^L$ and $x^R$ such that $P[\text{dist}(\eta, 1) \geq \delta] \geq q_0$. By the scaling invariance, we have $P[\text{dist}(\eta, \epsilon) \geq \delta \epsilon] \geq q_0$ for $\epsilon > 0$, which implies (2.3).
2.2 Proof of Theorem 1.1

In this section, we will first define the crossing events which correspond to the different cases in Theorem 1.1. The formulae (1.2, 1.3, 1.5, 1.6) were derived [WZ16], and we will use these results to prove the formulae (1.4, 1.7), hence completing the proof of Theorem 1.1.

Fix \( \kappa \in (4,8) \) and let \( \eta \) be an SLE\(_{\kappa} \) in \( \mathbb{H} \) from 0 to \( \infty \). We first define the crossing events \( \mathcal{H}_{2j-1}^\alpha, \mathcal{H}_{2j}^\beta \) and \( \mathcal{H}_{2j+1}^\gamma \) for \( j \geq 1 \) which correspond to Equations (1.2, 1.3, 1.4). Suppose that \( y \leq 0 < \epsilon \leq u \leq x \) and let \( T_u \) be the first time that \( \eta \) swallows \( u \) and \( T_x \) be the first time that \( \eta \) swallows the point \( x \) which is almost surely finite when \( \kappa > 4 \). Set \( T_0 = \sigma_0 = 0 \). Let \( T_1 \) be the first time that \( \eta \) hits the ball \( B(x, \epsilon) \) and let \( \sigma_1 \) be the first time after \( T_1 \) that \( \eta \) hits \( (-\infty, y) \). For \( j \geq 1 \), let \( T_j \) be the first time after \( \sigma_{j-1} \) that \( \eta \) hits the connected component of \( \partial B(x, \epsilon) \setminus \eta[0, \sigma_{j-1}] \) containing \( x + \epsilon \) and let \( \sigma_j \) be the first time after \( T_j \) that \( \eta \) hits \( (-\infty, y) \). Define

\[
\mathcal{H}_{2j-1}^\alpha(x, y) = \{ T_j < T_x \}, \quad \mathcal{H}_{2j}^\beta(x, y) = \{ \sigma_j < T_x \}, \quad \mathcal{H}_{2j+1}^\gamma(x, y, u) = \{ \sigma_1 < T_u, \sigma_j < T_x \}.
\]

In the definitions of \( \mathcal{H}_{2j-1}^\alpha, \mathcal{H}_{2j}^\beta \) and \( \mathcal{H}_{2j+1}^\gamma \), we are interested in the case when \( y = 0 \), \( x = 1 \) and \( u = 1/2 \). Imagine that \( \eta \) is the interface in FKR-Ising model and the boundary conditions is free (0) on \( \mathbb{R}_- \) and is wired (1) on \( \mathbb{R}_+ \), then the event \( \mathcal{H}_{2j-1}^\alpha(\epsilon) \) interprets that there are \( 2j - 1 \) arms going from \( B(x, \epsilon) \) to far away place of the pattern \((010 \cdots 10)\) clockwise, the event \( \mathcal{H}_{2j}^\beta(\epsilon) \) interprets that there are \( 2j \) arms going from \( B(x, \epsilon) \) to far away place of the pattern \((01 \cdots 01)\) clockwise, and the event \( \mathcal{H}_{2j+1}^\gamma(\epsilon) \) interprets that there are \( 2j + 1 \) arms going from \( B(x, \epsilon) \) to far away place of the pattern \((10 \cdots 101)\) clockwise. See Figure 1.1(a,b,c).

It is proved in [WZ16] Theorems 1.1, 1.2] that, fix some \( \delta > 0 \), for any \( y \leq 0 < \epsilon < x \) and \( j \geq 1 \), we have

\[
\mathbb{P}[\mathcal{H}_{2j-1}^\alpha(x, y)] \asymp \left( \frac{x}{x-y} \right)^{\alpha_{2j-2}} \left( \frac{\epsilon}{x} \right)^{\alpha_{2j-1}},
\]

\[
\mathbb{P}[\mathcal{H}_{2j}^\beta(x, y)] \asymp \left( \frac{x}{x-y} \right)^{\beta_{2j-1}} \left( \frac{\epsilon}{x} \right)^{\beta_{2j}},
\]

where the constants in \( \asymp \) depend only on \( \kappa \) and \( j \). In particular, for fixed \( \delta > 0 \), we have

\[
\mathbb{P}[\mathcal{H}_{2j-1}^\alpha(x, y)] \asymp e^{\alpha_{2j-1}}, \quad \mathbb{P}[\mathcal{H}_{2j}^\beta(x, y)] \asymp e^{\beta_{2j}}, \quad \text{provided} \ \delta \leq x \leq 1/\delta, -1/\delta \leq y \leq 0,
\]

where the constants in \( \asymp \) depend only on \( \kappa, j \) and \( \delta \). Fix some \( \delta > 0 \) small, define

\[
\mathcal{F} = \{ \tau_1 < T_u, \eta[0, \tau_1] \subset B(0, 1/\delta), \text{dist}(\eta[0, \tau_1], [x-\epsilon, x+3\epsilon]) \geq \delta \epsilon \}.
\]

We will prove the following estimate for \( \mathcal{H}_{2j+1}^\gamma \):

\[
\mathbb{P}[\mathcal{H}_{2j+1}^\gamma(x, y, u) \cap \mathcal{F}] \asymp \epsilon^{\gamma_{2j+1}}, \quad \text{provided} \ \delta \leq u \leq x/2 \leq 1/\delta, -1/\delta \leq y \leq 0,
\]

where the constants in \( \asymp \) depend only on \( \kappa, j \) and \( \delta \).

Next, we define the crossing events \( \mathcal{H}_{2j}^\alpha, \mathcal{H}_{2j+1}^\beta, \mathcal{H}_{2j+2}^\gamma \) for \( j \geq 0 \) which correspond to Equations (1.5, 1.6, 1.7). Fix \( \kappa \in (4,8) \) and let \( \eta \) be an SLE\(_{\kappa} \) in \( \mathbb{H} \) from 0 to \( \infty \). Suppose that \( y \leq u \leq -\epsilon \leq \epsilon \leq x \) and let \( T_u \) be the first time that \( u \) is swallowed, \( T_x \) be the first time that \( x \) is swallowed. Set \( T_0 = \sigma_0 = 0 \). Let \( \sigma_1 \) be the first time that \( \eta \) hits \( (-\infty, y) \) and \( \sigma_j \) be the first time after \( \sigma_1 \) that \( \eta \) hits the connected component of \( \partial B(x, \epsilon) \setminus \eta[0, \sigma_1] \) containing \( x + \epsilon \). For \( j \geq 1 \), let \( \sigma_j \) be the first time after \( \sigma_{j-1} \) that \( \eta \) hits \( (-\infty, y) \) and \( \tau_j \) be the first time after \( \sigma_j \) that \( \eta \) hits the connected component of \( \partial B(x, \epsilon) \setminus \eta[0, \sigma_j] \) containing \( x + \epsilon \). Define

\[
\mathcal{H}_{2j}^\alpha(x, y) = \{ \tau_j < T_x \}, \quad \mathcal{H}_{2j+1}^\beta(x, y) = \{ \sigma_{j+1} < T_x \}, \quad \mathcal{H}_{2j+2}^\gamma(x, y, u) = \{ \sigma_1 = T_u, \sigma_{j+1} < T_x \}.
\]
In the definitions of $H_{2j}^\alpha, \ H_{2j+1}^\beta, \ H_{2j+2}^\gamma$, we are interested in the case when $y = -2, u = -\epsilon, x = \epsilon$. Imagine that $\eta$ is the interface in FK-Ising model and the boundary conditions is free (0) on $\mathbb{R}_-$ and wired (1) on $\mathbb{R}_+$. then the event $H_{2j}^\alpha(\epsilon)$ interprets that there are $2j$ arms going from $B(x, 4\epsilon)$ to far away place of the pattern $(10 \cdots 10)$ clockwise, the event $H_{2j+1}^\beta(\epsilon)$ interprets that there are $2j + 1$ arms going from $B(x, 4\epsilon)$ to far away place of the pattern $(10 \cdots 10)$ clockwise, and the event $H_{2j+2}^\gamma(\epsilon)$ interprets that there are $2j + 2$ arms going from $B(x, 4\epsilon)$ to far away place of the pattern $(01 \cdots 01)$ clockwise. See Figure 1.1(d,e,f).

It is proved in [WZ16, Theorems 1.1, 1.2] that, for any $y \leq 0 < \epsilon \leq x$ and $j \geq 1$, we have

$$P[H_{2j}^\alpha(\epsilon, x, y)] \asymp \left( \frac{x}{x-y} \right)^{\alpha_{2j}^+} \left( \frac{\epsilon}{x} \right)^{\alpha_{2j-1}^+},$$

(2.7)

$$P[H_{2j+1}^\beta(\epsilon, x, y)] \asymp \left( \frac{x}{x-y} \right)^{\beta_{2j}^+} \left( \frac{\epsilon}{x} \right)^{\beta_{2j-2}^+},$$

(2.8)

where the constants in $\asymp$ depend only on $\kappa$ and $j$. In particular, fix some $\delta > 0$, we have

$$P[H_{2j}^\alpha(\epsilon, x, y)] \asymp e^{\alpha_{2j}^+}, \quad P[H_{2j+1}^\beta(\epsilon, x, y)] \asymp e^{\beta_{2j}^+}, \quad \text{provided } \delta \epsilon \leq x \leq \epsilon/\delta, -1/\delta \leq y \leq -\delta,$$

where the constants in $\asymp$ depend only on $\kappa, j$ and $\delta$. Fix some $\delta > 0$ small, define

$$\mathcal{F} = \{3\eta(S) \geq \delta, S < T_u, S < T_x\}.$$ We will prove the following estimate for $H_{2j}^\alpha$:

$$P\left[H_{2j}^\alpha(\epsilon, x, y, u) \cap \mathcal{F}\right] \asymp e^{\gamma^+_{2j}}, \quad \text{provided } -\epsilon/\delta \leq u \leq -\epsilon, \epsilon \leq x \leq \epsilon/\delta, -1/\delta \leq y \leq -2,$$

(2.9)

where the constants in $\asymp$ depend only $\kappa, j$ and $\delta$.

Note that (2.6) and (2.9) are weaker than (2.4, 2.5, 2.7, 2.8), but they are sufficient to derive the arm exponents for FK-Ising model. For the convenience in Section 2.3, we will also prove the following estimate for $H_{2j}^\gamma$:

$$P\left[H_{2j}^\gamma(\epsilon, x, y, u)\right] = e^{\gamma^+_{2j} + o(1)} \quad \text{provided } -1/\delta \leq y \leq -2.$$ (2.10)

In the following of this section, we will first prove (2.6) which needs (2.8) and then prove the lower bound in (2.9) which needs (2.5), and finally prove the upper bound in (2.10).

**Lemma 2.6.** Fix $\kappa > 4$, let $\eta$ be an SLE$_\kappa$ in $\mathbb{H}$ from 0 to $\infty$. For $\epsilon > 0$, let $\tau$ be the first time that $\eta$ hits $B(1, \epsilon)$ and let $T$ be the first time that $\eta$ swallows $1/2$. For $\lambda \geq 0$, define

$$u_1(\lambda) = \frac{\kappa^2 - 6\kappa + 16}{4\kappa} + \frac{\kappa - 2}{2\kappa} \sqrt{4\kappa \lambda + (\kappa/2 - 4)^2}.$$ Fix some $\delta > 0$ small, define

$$\mathcal{G} = \{\tau < T, 3\eta(\tau) \geq \delta \epsilon\}, \quad \mathcal{F} = \mathcal{G} \cap \{\eta[0, \tau) \subset B(0, 1/\delta), \text{dist}(\eta[0, \tau), [1 - \epsilon, 1 + 3\epsilon]) \geq \delta \epsilon\}.$$ Then we have

$$e^{u_1(\lambda)} \asymp E\left[ g'_\tau(1) \lambda 1_{\mathcal{F}} \right] \leq E\left[ g'_\tau(1) \lambda 1_{\mathcal{G}} \right] \asymp e^{u_1(\lambda)},$$

where the constants in $\asymp$ depend only on $\kappa$ and $\delta$. 


Proof. Set
\[ M_t = (g_t(1/2) - W_t)^{\nu/\kappa} g_t(1)^{\rho + 4 - \kappa)/(4\kappa)} g_t(1) - W_t)^{\rho/\kappa} (g_t(1) - g_t(1/2))^{\rho/2\kappa}, \]
where
\[ \nu = \kappa - 4, \quad \rho = -\kappa/2 - \sqrt{4\kappa\lambda + (\kappa/2 - 4)^2}. \]
By [SW05, Theorem 6], we know that \( M \) is a local martingale and the law of \( \eta \) weighted by \( M \) becomes the law of SLE\(_\kappa(\nu, \rho)\) with force points \((1/2, 1)\). Denote by \( O_t^R \) the rightmost point of \( g_t(\eta[0, t]) \cap \mathbb{R} \). On the event \( \{3\eta(t) \geq \delta \epsilon\} \), we know that
\[ (g_t(1/2) - W_t) \asymp (g_t(1) - W_t) \asymp (g_t(1) - g_t(1/2)) \asymp (g_t(1) - O_t^R), \]
where the constants in \( \asymp \) depend only on \( \delta \). By Koebe 1/4 theorem, we have
\[ (g_t(1) - O_t^R) \asymp g_t(1) \epsilon. \]
Therefore, by the choice of \( \nu \) and \( \rho \), we have, on the event \( \mathcal{G} \),
\[ M_t \asymp g_t'(1) \epsilon^{-u_1(\lambda)}. \]
Thus
\[ \epsilon^{u_1(\lambda)} \mathbb{P}^* [\mathcal{F}^*] \asymp \mathbb{E} \left[ g_t'(1)^{\lambda} 1_{\mathcal{F}} \right] \leq \mathbb{E} \left[ g_t'(1)^{\lambda} 1_{\mathcal{G}} \right] \asymp \epsilon^{u_1(\lambda)} \mathbb{P}^* [\mathcal{G}^*], \]
where \( \eta^* \) is an SLE\(_\kappa(\nu, \rho)\) with force points \((1/2, 1)\), \( \mathbb{P}^* \) denotes the law of \( \eta^* \) and \( \mathcal{F}^*, \mathcal{G}^* \) are defined for \( \eta^* \) accordingly. To show the conclusion, it is sufficient to show \( \mathbb{P}^* [\mathcal{F}^*] \asymp 1 \). Define \( \varphi(z) = \epsilon z/(1 - z) \). Then \( \varphi \) is the Mobius transformation of the upper half plane that sends the triple \((1/2, 1, \infty)\) to \((\epsilon, \infty, -\epsilon)\). Let \( \tilde{\eta} = \varphi(\eta^*) \), then \( \tilde{\eta} \) is an SLE\(_\kappa(\kappa - 6 - \rho; \nu)\) with force points \((-\epsilon; \epsilon)\). Let \( \tilde{S} \) be the first time that \( \tilde{\eta} \) exits the unit disc and define
\[ \tilde{\mathcal{F}} = \{\tilde{\eta}[0, \tilde{S}] \subset U(\delta)\}, \]
where \( U(\delta) \) is the \( \delta \)-neighborhood of the segment \([0, i]\). Note that \( \nu \geq \kappa/2 - 2 \) and \( \kappa - 6 - \rho \geq \kappa/2 - 2 \), by Lemma 2.5, we have
\[ \mathbb{P}^* [\mathcal{F}^*] \geq \tilde{\mathbb{P}}[\tilde{\mathcal{F}}] \geq p_0(\delta). \]
This completes the proof.

Proof of (2, 6), Upper Bound. Fix \( \kappa \in (4, 8) \) and let \( \eta \) be an SLE\(_\kappa\) in \( \mathbb{H} \) from 0 to \( \infty \). Let \( \tau \) be the first time that \( \eta \) hits \( B(1, \epsilon) \), let \( T \) be the swallowing time of \( u \). Recall that
\[ \mathcal{F} = \{\tau < T, \eta[0, \tau] \subset B(0, 1/\delta), \text{dist}(\eta[0, \tau], [1 - \epsilon, 1 + 3\epsilon]) \geq \delta \epsilon\}. \]
Given \( \eta[0, \tau] \), let \( f = g_\tau - W_\tau \). We know that the image of \( \eta[\tau, \infty) \) under \( f \), denoted by \( \tilde{\eta} \), has the law of SLE\(_\kappa\). Define \( \tilde{\mathcal{H}}_{2j-1}^\beta \) for \( \tilde{\eta} \). Given \( \eta[0, \tau] \) and on the event \( \mathcal{F} \), we have the following observations.

- Consider the image of \( \partial B(1, \epsilon) \) under \( f \). By Lemma 2.1, we know that \( f(B(1, \epsilon)) \) is contained in the ball with center \( f(1 + \epsilon) \) and radius \( 3(f(1 + 3\epsilon) - f(1 + \epsilon)) \). On the event \( \{\text{dist}(\eta[0, \tau], [1 - \epsilon, 1 + 3\epsilon]) \geq \delta \epsilon\} \), by Koebe distortion theorem [Pom92, Chapter I Theorem 1.3], we know that there exists a universal constant \( C \) depending only on \( \delta \) such that \( f(1 + \epsilon) - f(1) \leq f(1 + 3\epsilon) - f(1) \leq C f'(1) \epsilon \). This implies that, on \( \mathcal{F} \), the image \( f(B(1, \epsilon)) \) is contained in the ball with center \( f(1) \) and radius \( C f'(1) \epsilon \) for another constant \( C \) depending only on \( \delta \).

- Consider \( f(y) \). On the event \( \{\eta[0, \tau] \subset B(0, 1/\delta)\} \), we know that \( |f(y)| \) is bounded both sizes by universal constants depending only on \( \delta \).
Combining these two observations with (2.8), we have 
\[ \mathbb{P}\left[ \mathcal{H}_{2j+1}^\beta(\epsilon, x, y) \mid \eta[0, \tau], \mathcal{F} \right] \leq \mathbb{E} \left[ (g'_\tau(1)\epsilon)^{\beta_{2j-1}^+} \mathbb{1}_\mathcal{F} \right] = (g'_\tau(1)\epsilon)^{\beta_{2j-1}^+}. \]
Therefore, by Lemma 2.6, we have 
\[ \mathbb{P}\left[ \mathcal{H}_{2j+1}^\beta(\epsilon, x, y, u) \right] \leq \mathbb{E} \left[ (g'_\tau(1)\epsilon)^{\beta_{2j-1}^+} \mathbb{1}_\mathcal{F} \right] \approx \epsilon^{u_1(\beta_{2j-1}^+)+\beta_{2j-1}^+}. \]
Note that 
\[ u_1(\beta_{2j-1}^+) + \beta_{2j-1}^+ = \gamma_{2j+1}. \]
This completes the proof.

Proof of (2.6), Lower Bound. Assume the same notations as in the proof of the upper bound. Given \( \eta[0, \tau] \) and on the event \( \mathcal{F} \), we have the following observations.

- Consider the image of \( \partial B(1, \epsilon) \) under \( f \). By Koebe 1/4 theorem, we know that \( f(B(1, \epsilon)) \) contains the ball with center \( f(1) \) and radius \( f'(1)\epsilon/4 \). On the event \( \{3\eta(\tau) \geq \delta \epsilon\} \), we know that \( f(1) \approx f'(1)\epsilon \).

- Consider \( f(\gamma) \). On the event \( \{\eta[0, \tau] \subset B(0, 1/\delta)\} \), we know that \( |f(\gamma)| \) is bounded both sizes by universal constants depending only on \( \delta \).

Combining these two facts with (2.8), we have 
\[ \mathbb{P}\left[ \mathcal{H}_{2j+1}^\beta(\epsilon, x, y, u) \mid \eta[0, \tau], \mathcal{F} \right] \geq \mathbb{P}\left[ \mathcal{H}_{2j-1}^\beta(f'(1)\epsilon/4, f(1), f(\gamma)) \right] \approx (g'_\tau(1)\epsilon)^{\beta_{2j-1}^+}. \]
Therefore, by Lemma 2.6, we have 
\[ \mathbb{P}\left[ \mathcal{H}_{2j+1}^\beta(\epsilon, x, y, u) \cap \mathcal{F} \right] \geq \mathbb{E} \left[ (g'_\tau(1)\epsilon)^{\beta_{2j-1}^+} \mathbb{1}_\mathcal{F} \right] \approx \epsilon^{\gamma_{2j+1}}. \]

Lemma 2.7. Fix \( \kappa > 4 \), let \( \eta \) be an SLE\( _\kappa \) in \( \mathbb{H} \) from 0 to \( \infty \). For \( \epsilon > 0 \) small, let \( T^L \) be the first time that \( \eta \) swallows \( -\epsilon \) and let \( T^R \) be the first time that \( \eta \) swallows \( +\epsilon \). For \( C \geq 2\epsilon \), let \( \xi \) be the first time that \( \eta \) exits \( B(0, C) \). For \( \lambda \geq 0 \), define 
\[ u_2(\lambda) = \frac{(\kappa - 2)(\kappa + 2)}{4\kappa} + \frac{(\kappa - 2)}{2\kappa} \sqrt{4\kappa\lambda + (\kappa/2 - 2)^2}. \]
Fix some \( \delta > 0 \) small, define 
\[ \mathcal{F} = \{ \xi < T^L, \xi < T^R, \exists \eta(\xi) \geq \delta C \}, \quad \mathcal{G} = \mathcal{F} \cap \{\text{dist}(\eta[0, \xi], \epsilon) \geq \epsilon/4\}. \]
Then we have 
\[ C^{-A} \epsilon^{u_2(\lambda)} \leq \mathbb{E}[g'_\xi(\epsilon)^\lambda \mathbb{1}_\mathcal{G}] \leq \mathbb{E}[g'_\xi(\epsilon)^\lambda \mathbb{1}_\mathcal{F}] \leq (\delta C)^{-A} \epsilon^{u_2(\lambda)}, \]
where \( A \) is some constant depending on \( \kappa \) and \( \lambda \), the constants in \( \lesssim \) depend only on \( \kappa \), and they are uniform over \( \epsilon, C, \delta \).

Proof. Set 
\[ M_t = (W_t - g_t(-\epsilon))^{\rho_L/\kappa} g_t(\epsilon)^{\rho_R(\rho_R + 4 - \kappa)/(4\kappa)} (g_t(\epsilon) - W_t)^{\rho_R/\kappa} (g_t(\epsilon) - g_t(-\epsilon))^{\rho_L\rho_R/(2\kappa)}, \]
where 
\[ \rho_L = \kappa - 4, \quad \rho_R = \kappa/2 - 2 + \sqrt{4\kappa\lambda + (\kappa/2 - 2)^2}. \]
By Lemma 2.3, we know that $M$ is local martingale and the law of $\eta$ weighted by $M$ becomes the law of $\text{SLE}_\kappa(\rho^L; \rho^R)$ with force points $(-\epsilon; \epsilon)$. On the event $\mathcal{F}$, by [MW16, Lemma 3.4], we have that

$$\delta C \lesssim (g_\xi(\epsilon) - W_\xi), (W_\xi - g_\xi(-\epsilon)) \leq (g_\xi(\epsilon) - g_\xi(-\epsilon)) \leq 4C.$$ 

By the choice of $\rho^R$, we have $\rho^R(\rho^R + 4 - \kappa) = 4\kappa\lambda$. Thus, on the event $\mathcal{F}$, we have

$$(\delta C)^A g_\xi(\epsilon)^\lambda \lesssim M_\xi \lesssim C^A g_\xi(\epsilon)^\lambda,$$

where

$$A = \rho^L/\kappa + \rho^R/\kappa + \rho^L\rho^R/(2\kappa).$$

Therefore

$$C^{-A} M_0 \mathbb{P}^*[\mathcal{G}^*] \lesssim \mathbb{E}[g_\xi(\epsilon)^\lambda \mathbb{1}_\mathcal{G}] \leq \mathbb{E}[g_\xi(\epsilon)^\lambda \mathbb{1}_{\mathcal{F}}] \lesssim (\delta C)^{-A} M_0,$$

where $\eta^*$ is an SLE$_\kappa(\rho^L; \rho^R)$ with force points $(-\epsilon; \epsilon)$, $\mathbb{P}^*$ denotes its law and $\mathcal{G}^*$ is defined for $\eta^*$ accordingly. Note that $M_0 = e^{k_2(\lambda)}$. To show the conclusion, it is sufficient to show

$$\mathbb{P}^*[\mathcal{G}^*] \asymp 1.$$  \hfill (2.11)

Note that $\rho^L \geq \kappa/2 - 2$ and $\rho^R \geq \kappa/2 - 2$, the curve $\eta^*$ never swallows $-\epsilon$ nor $\epsilon$. Then (2.11) is guaranteed by Lemma 2.5.

**Remark 2.8.** Taking $\lambda = 0$ in Lemma 2.7, we have that (2.9) is true for $\gamma_2^+ = (\kappa - 4)/2$.

**Proof of (2.9), Lower Bound.** Fix $\kappa \in (4, 8)$ and let $\eta$ be an SLE$_\kappa$ in $\mathbb{H}$ from $0$ to $\infty$. Let $S$ be the first time that $\eta$ exits the unit disc. Fix $x = \epsilon$ and $u = -\epsilon$ and let $T^L$ be the first time that $\eta$ swallows $-\epsilon$ and $T^R$ be the first time that $\eta$ swallows $\epsilon$. Let $\sigma$ be the first time that $\eta$ hits $(-\infty, y)$. Recall that

$$\mathcal{F} = \{ 3\eta(S) \geq \delta, S < T^L, S < T^R \}.$$ 

Set $f_S = g_S - W_S$ and $f_\sigma = g_\sigma - W_\sigma$. Given $\eta[0, \sigma]$, the image of $\eta[\sigma, \infty)$ under $f_\sigma$, denoted by $\tilde{\eta}$, has the law of SLE$_\kappa$, and we define $\tilde{\mathcal{H}}_{2j-2}$ for $\hat{\eta}$. We will control the behavior of $\eta[0, S]$ and $\eta[S, \sigma]$ separately.

- Consider $\eta[0, S]$ and define $\mathcal{G} = \mathcal{F} \cap \{ \text{dist}[\eta[0, S], x] \geq \epsilon/4 \}$. Given $\eta[0, S]$ and on the event $\mathcal{G}$, consider the image of $B(x, \epsilon)$ under $f_S$. On the event $\mathcal{G}$, by Koebe 1/4 theorem, we know that $f_S(B(x, \epsilon))$ contains the ball with center $f_S(x)$ and radius $f_S'(x)\epsilon/16$. Note that on the event $\mathcal{G}$, we know that $|f_S(x)|$ is bounded both sizes by universal constants depending only $\delta$.

- Given $\eta[0, S]$ and on $\mathcal{G}$, consider $\eta[S, \sigma]$. From the above item, we know that $f_S(B(x, \epsilon))$ contains the ball with center $w := f_S(x)$ and radius $r := f_S'(x)\epsilon/16$. Define $\mathcal{E}$ to be the event that $\sigma = T^L$ and that the distance between $f_S(\eta[S, \sigma])$ and $B(w, r)$ is at least $w/4$. Clearly, the probability of $\mathcal{E}$ is bounded from below by a universal positive constant depending only $\delta$ as long as $|f_S(y)|$ is bounded from above by a constant depending only on $\delta$, $r \leq w/16$ and $w$ is bounded from below by a universal constant depending only on $\delta$. On the event $\mathcal{E}$, note that $h := f_\sigma \circ f_S^{-1}$ is the conformal map from $\mathbb{H} \setminus f_S(\eta[S, \sigma])$ onto $\mathbb{H}$, and the image of $B(w, r)$ under $h$ contains the ball with center $h(w) = f_\sigma(x)$ and radius $rh'(u)/4$. Note that on $\mathcal{E}$, $f_\sigma(x)$ is bounded both size by universal constants depending only on $\delta$; and, by Koebe 1/4 theorem, the derivative $h'(u)$ is bounded from below by $f_\sigma(x)/(4u)$ which is therefore bounded from below by universal constant depending only on $\delta$. To summarize, given $\eta[0, \sigma]$ and on the event $\mathcal{G} \cap \mathcal{E}$, we know that $f_\sigma(B(x, \epsilon))$ contains a ball with center $f_\sigma(x)$ and radius $c_\delta f_S'(x)\epsilon$ where $f_\sigma(x)$ is bounded both size by universal constants depending only on $\delta$ and $c_\delta > 0$ depends only on $\delta$. 


Combining these two facts with (2.5), we have that
\[ P \left[ \mathcal{H}_{2j}^\gamma(x, y, u) \mid \eta[0, \sigma], \mathcal{G} \cap \mathcal{E} \right] \leq P \left[ \mathcal{H}^\beta_{2j-2}(c_3 f'_S(x) \epsilon, f_\sigma(x), f_\sigma(y)) \right] \leq (g' S(x) \epsilon)^{\beta_{2j-2}}. \]

Since the probability for \( \mathcal{E} \) is bounded from below by positive constant depending only on \( \delta \), we have
\[ P \left[ \mathcal{H}_{2j}^\gamma(x, y, u) \mid \eta[0, \sigma], \mathcal{G} \right] \geq (g' S(x) \epsilon)^{\beta_{2j-2}}. \]

Therefore, by Lemma 2.7 we have
\[ P \left[ \mathcal{H}_{2j}^\gamma(x, y, u) \cap \mathcal{G} \right] \geq E \left[ (g' S(x) \epsilon)^{\beta_{2j-2}} \right] \epsilon^{\gamma_{2j}}. \]

This completes the proof.

Lemma 2.9. Fix \( \kappa > 4 \), let \( \eta \) be an SLE_\kappa in \( \mathbb{H} \) from 0 to \( \infty \). For \( y \leq -2, u \in [-\epsilon/\delta, -\epsilon], x \in [\epsilon, \epsilon/\delta], \) let \( T^L \) be the first time that \( \eta \) swallows \( u \) and let \( T^R \) be the first time that \( \eta \) swallows \( x \). For \( C \in [2\epsilon, 1] \), let \( \xi \) be the first time that \( \eta \) exits \( B(0, C) \). Fix some \( \delta > 0 \) small, define
\[ \mathcal{F} = \{ \xi < T^L, \xi < T^R, 3\eta(\xi) \geq \delta C \}. \]

Then we have, for \( j \geq 2 \),
\[ P \left[ \mathcal{H}_{2j}^\gamma(x, y, u) \cap \mathcal{F} \right] \leq C^{-A} \delta^{-B} \epsilon^{\gamma_{2j}}. \]

where \( A, B \) are some constants depending on \( \kappa \) and \( j \), the constants in \( \leq \) depend only on \( \kappa \), and they are uniform over \( \epsilon, C, \delta \). Note that this lemma gives the upper bound in (2.9).

Proof. Given \( \eta[0, \xi] \), let \( f = g_\xi - W_\xi \). We know that the image of \( \eta[\xi, \infty) \) under \( f \), denoted by \( \tilde{\eta} \), has the same law as an SLE_\kappa. Define \( \tilde{\mathcal{H}}_{2j}^\beta \) for \( \tilde{\eta} \). We have the following observations.

- Consider the image of \( \partial B(x, \epsilon) \) under \( f \). By Lemma 2.1 we know that the image of \( \partial B(x, \epsilon) \) under \( f \) is contained in the ball with center \( f(x + 3\epsilon) \) and radius \( 8\epsilon f'(x + 3\epsilon) \). On the event \( \mathcal{F} \), we know that
  \[ \delta C \lesssim f(x + 3\epsilon) \leq 2C. \]

- Consider \( f(y) \). As long as \( y \leq -2 \), we know that \( |f(y)| \) is bounded from below by universal constant.

Combining these two facts with (2.5), we have
\[ P \left[ \mathcal{H}_{2j}^\gamma(x, y, u) \mid \eta[0, \sigma], \mathcal{F} \right] \leq P \left[ \tilde{\mathcal{H}}_{2j-2}^\beta(8\epsilon f'(x + 3\epsilon), f(x + 3\epsilon), f(y)) \right] \lesssim (\delta C)^{-A} (g' S(x + 3\epsilon) \epsilon)^{\beta_{2j-2}}, \]

where \( A \) is some constant depending on \( \kappa \) and \( j \). Therefore, by Lemma 2.7 we have
\[ P \left[ \mathcal{H}_{2j}^\gamma(x, y, u) \cap \mathcal{F} \right] \lesssim (\delta C)^{-A} E \left[ (g' S(x + 3\epsilon) \epsilon)^{\beta_{2j-2}} \right] \lesssim C^{-A} \delta^{-B} \epsilon^{\gamma_{2j}}, \]

where \( A, B \) are some constants depending only on \( \kappa, j \). This completes the proof.

Lemma 2.10. Fix \( \kappa \in (0, 8) \) and let \( \eta \) be an SLE_\kappa in \( \mathbb{H} \) from 0 to \( \infty \). Fix \( n \geq 1 \) such that \( 2^{-n} \geq 2\epsilon \).

For \( 1 \leq m \leq n \), let \( \xi_m \) be the first time that \( \eta \) exits \( B(0, 2^{m-n+1}) \). Note that \( \xi_1, ..., \xi_n \) is an increasing sequence of stopping times and \( \xi_1 \) is the first time that \( \eta \) exits \( B(0, 2^{-n}) \) and \( \xi_n \) is the first time that \( \eta \) exits \( B(0, 1/2) \). For \( 1 \leq m \leq n \), define
\[ \mathcal{F}_m = \{ \exists \eta(\xi_m) \leq 2^{m-n+1} \}. \]

There exists a function \( p : (0, 1) \to [0, 1] \) with \( p(\delta) \downarrow 0 \) as \( \delta \downarrow 0 \) such that
\[ P[\cap_m^n \mathcal{F}_m] \leq p(\delta)^n. \]
Proof. For $1 \leq m \leq n$, given $\eta[0, \xi_m]$, let $f_m = g_{\xi_m} - W_{\xi_m}$. Denote $2^{m-n+1}$ by $r$. The event $\mathcal{F}_{m+2}$ is that $\eta$ exits $B(0, 4r)$ through $B(4r, 4\delta r)$ and $B(4r, 4\delta r)$. Let $\tilde{\eta}$ be the image of $\eta[\xi_m, \infty)$ under $f_m$. Then $\mathcal{F}_{m+2}$ implies that $\tilde{\eta}$ hits $f_m(B(4r, 4\delta r)) \cup f_m(B(4r, 4\delta r))$. Consider $f_m(B(4r, 4\delta r))$. By Lemma 2.2 we know that $f_m(B(4r, 4\delta r))$ is contained in the ball with center $f_m(4r)$ and radius $16\delta r f_m(4r)$. By Corollary 3.44, we have that

$$4r \leq f_m(4r) \leq 8r, \quad f_m(4r) \asymp 1.$$ 

Thus, by Lemma 2.4 we have

$$\mathbb{P}[\mathcal{F}_{m+2} | \eta[0, \xi_m]] \leq C\delta^{8/\kappa - 1}.$$ 

Iterating this relation, we have

$$\mathbb{P}[\eta^n \mathcal{F}_m] \leq \left( C\delta^{8/\kappa - 1} \right)^{n/2}.$$ 

This implies the conclusion. \hfill \Box

Proof of (2.10), Upper Bound. Assume the same notation as in Lemma 2.10. For $1 \leq m \leq n$, by Lemma 2.9 we have that

$$\mathbb{P}\left[ \mathcal{H}^n_{2j}(\epsilon, x, y, u) \cap \mathcal{F}^m \right] \lesssim 2^n A \delta^{-B} \epsilon^{2j}.$$ 

as long as $y \leq -2$, where $A, B$ are some constants depending on $\kappa, j$. Combining with Lemma 2.10 we have, for any $n$ and $\delta > 0$ small,

$$\mathbb{P}\left[ \mathcal{H}^n_{2j}(\epsilon, x, y, u) \right] \lesssim n2^n A \delta^{-B} \epsilon^{2j} + p(\delta)^n,$$

where $p(\delta) \downarrow 0$ as $\delta \downarrow 0$. This implies the conclusion. \hfill \Box

2.3 Proof of Theorem 1.3

Fix $\kappa \in (4, 8)$ and let $\eta$ be an SLE$_{\kappa}$ in $\mathbb{H}$ from $0$ to $\infty$. Fix $z \in \mathbb{H}$ with $|z| = 1$ and suppose $y \leq 0$. Let $T_z$ be the first time that $\eta$ swallows $z$. Set $\tau_0 = \sigma_0 = 0$. Let $\tau_1$ be the first time that $\eta$ hits $B(z, \epsilon)$ and let $\sigma_1$ be the first time after $\tau_1$ that $\eta$ hits $(\infty, y)$. Given $\eta[0, \sigma_1]$ and suppose $\sigma_1 < T_z$, we know that $B(z, \epsilon) \setminus \eta[0, \sigma_1]$ has one connected component that contains $z$, denoted by $C_z$. The boundary $\partial C_z$ consists of pieces of $\eta[0, \sigma_1]$ and pieces of $\partial B(z, \epsilon)$. Consider $\partial C_z \cap \partial B(z, \epsilon)$, there may be several connected components, but there is only one which can be connected to $\infty$ in $\mathbb{H} \setminus (\eta[0, \sigma_1] \cup B(z, \epsilon))$. We denote this connected component by $C^b_z$ and orient it counterclockwise and denote the endpoint of $C^b_z$ by $X^b_z$. See Figure 2.1.

Let $T_z$ be the first time after $\sigma_1$ that $\eta$ hits $C^b_z$, and let $\sigma_2$ be the first time after $T_z$ that $\eta$ hits $(-\infty, y)$. For $j \geq 2$, let $\tau_j$ be the first time after $\sigma_{j-1}$ such that $\eta$ hits the connected component of $C^b_z \setminus \eta[0, \sigma_{j-1}]$ containing $X^b_z$ and let $\sigma_j$ be the first time after $\tau_j$ that $\eta$ hits $(-\infty, y)$. Define

$$E^\alpha_{2j}(\epsilon, z, y) = \{ \tau_j < T_z \}, \quad E^{\beta}_{2j+1}(\epsilon, z, y) = \{ \sigma_j < T_z \}.$$ 

The definition of $E^\gamma$ is a little complicated. Given $\eta[0, \tau_1]$, let $f_{\tau_1} = g_{\tau_1} - W_{\tau_1}$ and set $u = -4|g_{\tau_1}'(z)|\epsilon$. Denote $f^{-1}_{\tau_1}(u)$ by $w$ and let $T_w$ be the first time that $\eta$ swallows $w$. Define

$$E^\gamma_{2j+2} = \{ \sigma_1 = T_w, \sigma_j < T_z \}.$$ 

We will estimate the probability of $E^\alpha, E^\beta$ and $E^\gamma$, but due to technical difficulty in the proof, we need an auxiliary event. Define

$$\mathcal{F} = \{ \eta[0, \tau_1] \subset B(0, R) \},$$

where $R$ is a constant depending only on $\kappa$ and $z$ which is decided in Lemma 2.11. Assume the same notations as in Theorem 1.3, we will prove, for fixed $z \in \mathbb{H}$ with $|z| = 1$, fixed $\delta > 0$ small and for $j \geq 1$,

$$\mathbb{P}\left[ E^\alpha_{2j}(\epsilon, z, y) \cap \mathcal{F} \right] = e^{\alpha_{2j} + o(1)}, \quad \text{provided } -1/\delta \leq y \leq -2R. \quad (2.12)$$
Fig. 2.1: The gray part is the connected component of $B(z, \epsilon) \setminus \eta[0, \sigma_1]$ that contains $z$, which is denoted by $C_z$. The bold part of $\partial C_z$ is $C^b_z$ and the point $X^b_z$ is indicated in the figure.

$$\mathbb{P} \left[ \mathcal{E}^z_{2j+1}(\epsilon, z, y) \cap \mathcal{F} \right] = \epsilon^{\beta_{2j+1} + o(1)}, \quad \text{provided } -1/\delta \leq y \leq -2R. \quad (2.13)$$

$$\mathbb{P} \left[ \mathcal{E}^z_{2j+2}(\epsilon, z, y) \cap \mathcal{F} \right] = \epsilon^{\gamma_{2j+2} + o(1)}, \quad \text{provided } -1/\delta \leq y \leq -2R. \quad (2.14)$$

Lemma 2.11. Fix $\kappa > 0$ and let $\eta$ be an SLE$_\kappa$ in $\mathbb{H}$ from 0 to $\infty$. Fix $z \in \mathbb{H}$ with $|z| = 1$. For $\epsilon > 0$, let $\tau$ be the first time that $\eta$ hits $B(z, \epsilon)$. Define $\Theta_t = \arg(g_t(z) - W_t)$. For $\delta \in (0, 1/16)$, $R \geq 4$, define the event

$$\mathcal{G} = \{ \tau < \infty, \Theta_\tau \in (\delta, \pi - \delta) \}, \quad \mathcal{F} = \mathcal{G} \cap \{ \eta[0, \tau] \subset B(0, R) \}.$$

For $\lambda \geq 0$, define

$$\rho = \kappa/2 - 4 - \sqrt{4\kappa \lambda + (\kappa/2 - 4)^2}, \quad v(\lambda) = \frac{1}{2} - \frac{\kappa}{16} - \frac{\lambda}{2} + \frac{1}{8} \sqrt{4\kappa \lambda + (\kappa/2 - 4)^2}.$$

There exists a constant $R$ depending only on $\kappa$ and $z$ such that the following is true:

$$e^{v(\lambda)} \lesssim \mathbb{E} \left[ |g'_\tau(z)|^{\lambda} \mathds{1}_\mathcal{F} \right] \lesssim \mathbb{E} \left[ |g'_\tau(z)|^{\lambda} \mathds{1}_\rho \right] \lesssim e^{v(\lambda) \delta - v(\lambda) - \rho^2/(2\kappa)},$$

where the constants in $\lesssim$ depend on $\kappa, z$ and are uniform over $\epsilon, \delta$.

Proof. Similar results were proved in [VL12, Section 6.3] and [MW16, Lemmas 4.1, 4.2]. In our result, we need the precise dependence on $\delta$, so we give the proof here. Set

$$M_t = |g_t(z)|^{\rho/(\rho + 8 - 2\kappa)/(8\kappa)} \Im g_t(z)^{\rho^2/(8\kappa)} |g_t(z) - W_t|^{\rho/\kappa}.$$

Then $M$ is a local martingale and the law of $\eta$ weighted by $M$ becomes the law of SLE$_\kappa(\rho)$ with force point $z$. We introduce two other quantities:

$$\Upsilon_t = \frac{\Im g_t(z)}{|g'_t(z)|}, \quad S_t = \sin \Theta_t = \frac{\Im g_t(z)}{|g_t(z) - W_t|}.$$
Then we can rewrite $M$ as follows:

$$M_t = |g_t'(z)|^\lambda \Upsilon_t^{-v(\lambda)} S_t^{v(\lambda)+\rho^2/(8\kappa)}.$$

By Koebe 1/4 theorem, we know that $\Upsilon_t \asymp \epsilon$. On the event $\mathcal{G}$, we know that $S_t \geq \delta/2$ for $\delta < 1/16$. Thus

$$\epsilon^{v(\lambda)} \mathbb{P}^*\left[\mathcal{F}^*\right] \lesssim \mathbb{E}\left[|g_t'(z)|^\lambda \mathbb{1}_{\mathcal{F}^*}\right] \lesssim \mathbb{E}\left[|g_t'(z)|^\lambda \mathbb{1}_{\mathcal{G}}\right] \lesssim \epsilon^{v(\lambda)} \delta^{-v(\lambda)-\rho^2/(8\kappa)},$$

where $\eta^*$ is an SLE$_\kappa(\rho)$ with force point $z$, $\mathbb{P}^*$ denotes its law and $\tau^*, \Theta^*, \mathcal{F}^*$ are defined accordingly. By [MW16] Equations (4.7), (4.8), we have

$$\mathbb{P}^*[\eta^*[0, \tau^*] \subset B(0, R)] \to 1, \quad \text{as } R \to \infty,$$

and

$$\mathbb{P}^*[\Theta^*_r, \tau^* \in (1/16, \pi - 1/16)] \asymp 1.$$  

Therefore, there exists a constant $R$ depending only on $\kappa$ and $z$ such that

$$\mathbb{P}^*[\mathcal{F}^*] \geq \mathbb{P}^*[\eta^*[0, \tau^*] \subset B(0, R), \Theta^*_r, \tau^* \in (1/16, \pi - 1/16)] \asymp 1.$$  

This completes the proof.  

Now we have decided the constant $R$ in Lemma 2.11 and we will fix it in the following of the paper.

Proof of (2.12), Lower Bound. Let $\eta$ be an SLE$_\kappa$ in $\mathbb{H}$ from $0$ to $\infty$. Let $\tau$ be the first time that $\eta$ hits $B(z, \epsilon)$. Denote the centered conformal map $g_t - W_t$ by $f_t$ for $t \geq 0$. Recall that

$$\mathcal{F} = \{\eta[0, \tau] \subset B(0, R)\}.$$  

Fix some $\delta > 0$ and define

$$\mathcal{G} = \mathcal{F} \cap \{\Theta_\tau \in (\delta, \pi - \delta)\}.$$  

We run $\eta$ until the time $\tau$ and on the event $\mathcal{G}$, by Koebe 1/4 theorem, we know that $f_\tau(B(z, \epsilon))$ contains the ball with center $w := f_\tau(z)$ and radius $r := \epsilon |f_\tau'(z)|/4$ and

$$\arg(w) \in (\delta, \pi - \delta), \quad r \leq \Im w \leq 16r.$$  

We wish to apply (2.7), however this ball is centered at $w = f_\tau(z)$ which does not satisfy the conditions in (2.7). We will fix this problem by running $\eta$ for a little further and argue that there is positive chance that $\eta$ does the right thing.

Let $\tilde{\eta}$ be the image of $\eta[\tau, \infty)$ under $f_\tau$. Let $\gamma$ be the broken line from $0$ to $w$ and then to $-r$ and let $A_r$ be the $r/4$-neighborhood of $\gamma$. Let $S_1$ be the first time that $\tilde{\eta}$ exits $A_r$ and let $S_2$ be the first time that $\tilde{\eta}$ hits $(-\infty, -r)$. By [MW16] Lemma 2.5, we know that $\mathbb{P}[S_2 < S_1]$ is bounded from below by positive constant depending only on $\kappa$ and $\delta$, see Figure 2.2. On the event $\{S_2 < S_1\}$, it is clear that there exist constants $x_\delta, c_\delta > 0$ depending only on $\delta$ such that $f_{S_2}(B(z, \epsilon))$ contains the ball with center $x_\delta r$ and radius $c_\delta r$. Let $\tilde{\eta}$ be the image of $\eta[S_2, \infty)$ under $f_{S_2}$ and define $\bar{\mathcal{F}}_{2j}$ for $\tilde{\eta}$. Then, by (2.7), we have

$$\mathbb{P}\left[\mathcal{E}_{2j}^\alpha(\epsilon, z, y) | \eta[0, S_2], \mathcal{G} \cap \{S_2 < S_1\}\right] \geq \mathbb{P}\left[\bar{\mathcal{F}}_{2j-2}(c_\delta r, x_\delta r, f_{S_2}(y)) \gtrsim (|g_{S_2}'(z)|\epsilon)^{\alpha^{2j-2}}\right].$$

Since $\{S_2 < S_1\}$ has positive chance, we have

$$\mathbb{P}\left[\mathcal{E}_{2j}^\alpha(\epsilon, z, y) | \eta[0, \tau], \mathcal{G}\right] \gtrsim (|g_{S_2}'(z)|\epsilon)^{\alpha^{2j-2}}.$$  

Therefore, by Lemma 2.11, we have

$$\mathbb{P}\left[\mathcal{E}_{2j}^\alpha(\epsilon, z, y) \cap \mathcal{G}\right] \gtrsim \mathbb{E}\left[\mathbb{1}_{\mathcal{G}} \left(|g_{S_2}'(z)|\epsilon\right)^{\alpha^{2j-2}}\right] \asymp \epsilon^{\alpha^{2j}},$$  

where the constants in $\gtrsim$ and $\asymp$ depend only on $\kappa, z, j$ and $\delta$. This completes the proof.  

$\square$
Fig. 2.2: By Koebe 1/4 theorem, we know that \( f_\tau(B(z, \epsilon)) \) contains the ball \( B(w, r) \) where \( w = f_\tau(z) \) and \( r = \epsilon|f'(z)|/4 \) where \( \arg(w) \in (\delta, \pi - \delta) \), and \( r \leq 3w \leq 16r \). The event \( \{ S_2 < S_1 \} \) means that the curve \( \tilde{\eta} \) hits \((-\infty, -r)\) before exiting the tube \( A_r \). This event has positive chance which is bounded from below by constant depending only on \( \kappa \) and \( \delta \). On the event \( \{ S_2 < S_1 \} \), it is clear that \( f_{S_2}(B(z, \epsilon)) \) contains a ball with center \( x_\delta r \) and radius \( c_\delta r \) where \( x_\delta, c_\delta \) are positive constants depending only on \( \delta \).

**Lemma 2.12.** Fix \( \kappa \in (4, 8) \) and let \( \eta \) be an SLE\(_\kappa\) in \( \mathbb{H} \) from 0 to \( \infty \). Fix \( z \in \mathbb{H} \) with \( |z| = 1 \) and let \( T_z \) be the first time that \( \eta \) swallows \( z \). Let \( \Theta_t = \arg(g_t(z) - W_t) \). For \( C \geq 16 \), let \( \xi \) be the first time that \( \eta \) hits \( \partial B(z, C\epsilon) \). For \( \delta \in (0, 1/16) \), define

\[
\mathcal{F} = \{ \xi < T_z, \Theta_\xi \in (\delta, \pi - \delta), \eta[0, \xi] \subset B(0, R) \}.
\]

Then we have, for \( j \geq 1 \),

\[
\mathbb{P}\left[ \mathcal{E}_{2j+2}^\alpha(\epsilon, z, y) \cap \mathcal{F} \right] \lesssim C^A \delta^{-B} e^{\alpha(2j+2)}, \quad \text{provided } y \leq -2R.
\]

where \( A, B \) are some constants depending on \( \kappa \) and \( j \), and the constant in \( \lesssim \) depends only on \( \kappa \) and \( j \), and is uniform over \( \delta, C, \epsilon \).

**Proof.** We run the curve up to time \( \xi \) and let \( f = g_\xi - W_\xi \). We know that the image of \( \eta[\xi, \infty) \) under \( f \) has the same law as SLE\(_\kappa\), we denote it by \( \tilde{\eta} \) and define \( \mathcal{H}_{2j}^\alpha \) for \( \tilde{\eta} \). We have the following observations.

- By Lemma 2.2, we know that \( f(B(z, \epsilon)) \) is contained in the ball with center \( f(z) \) and radius \( r := 4\epsilon|f'(z)| \). Applying Koebe 1/4 theorem to \( f \), we have

\[
C \epsilon|f'(z)|/4 \leq f(z) \leq 4C \epsilon|f'(z)|.
\] (2.15)

Next, we argue that \( f(B(z, \epsilon)) \) is contained in the ball with center \( |f(z)| \in \mathbb{R} \) and radius \( 8Cr/\delta \). Since \( f((z, \epsilon)) \) is contained in the ball with center \( f(z) \) and radius \( r \), it is clear that \( f(B(z, \epsilon)) \) is contained in the ball with center \( |f(z)| \) with radius \( r + 2|f(z)| \). By (2.15), we have

\[
Cr/16 \leq |f(z)| \sin \Theta_\xi \leq Cr.
\]

Since \( \Theta_\xi \in (\delta, \pi - \delta) \), we know that, for \( \delta > 0 \) small, we have \( \sin \Theta_\xi \geq \delta/2 \). Thus, \( Cr/16 \leq |f(z)| \leq 2Cr/\delta \). Therefore, \( f(B(z, \epsilon)) \) is contained in the ball with center \( |f(z)| \) with radius \( 8Cr/\delta \). In summary, we know that \( f(B(z, \epsilon)) \) is contained in the ball with center \( |f(z)| \) and radius \( 32C \epsilon|f'(z)|/\delta \) where

\[
C \epsilon|f'(z)|/4 \leq |f(z)| \leq 8C \epsilon|f'(z)|/\delta.
\]

Since \( \{ \eta[0, \xi] \subset B(0, R) \} \) and \( y \leq -2R \), it is clear that \( |f(y)| \) is bounded from below by universal constant.
Combining these two facts with (2.7), we have

\[ P\left[ E_{2j+2}^\alpha(\epsilon, z, y) \mid \eta[0, \xi], F \right] \leq P\left[ H_{2j}^\alpha(32C\epsilon|f'(z)|/\delta, \{f(z), f(y)\}) \right] \lesssim (C\epsilon|g'_\xi(z)|/\delta)^{\alpha_{2j}}, \]

where the constant in \( \lesssim \) depends only on \( \kappa \) and is independent of \( C, \epsilon, \delta \). Thus, by Lemma 2.11, we have

\[ P\left[ E_{2j+2}^\alpha(\epsilon, z, y) \cap F \right] \lesssim (C\epsilon/\delta)^{\alpha_{2j}^+} E\left[ |g'_\xi(z)|^{\alpha_{2j}^+} 1_F \right] \lesssim \delta^{-b} (C\epsilon/\delta)^{\alpha_{2j}^+} \epsilon^{v(\alpha_{2j}^+)}, \]

where \( b \) is some constant from Lemma 2.11. Note that \( \alpha_{2j+2} = v(\alpha_{2j}^+) + \alpha_{2j}^+ \).

This completes the proof. \( \square \)

From Lemma 2.12 we see that in order to show the upper bound in (2.12), it remains to argue that \( \{\Theta_\xi \in (\delta, \pi - \delta)\} \) happens with high probability. This is guaranteed by the following lemma.

**Lemma 2.13.** Fix \( \kappa \in (0, 8) \) and let \( \eta \) be an SLE\( \kappa \) in \( \mathbb{H} \) from 0 to \( \infty \). Fix \( z \in \mathbb{H} \) with \( |z| = 1 \). Let \( T_z \) be the first time that \( \eta \) swallows \( z \) and set \( \Theta_t = \arg(g_t(z) - W_t) \). Take \( n \in \mathbb{N} \) such that \( B(z, 16\epsilon 2^n) \) is contained in \( \mathbb{H} \). For \( 1 \leq m \leq n \), let \( \xi_m \) be the first time that \( \eta \) hits \( B(z, 16\epsilon 2^{n-m+1}) \). Note that \( \xi_1, \ldots, \xi_n \) is an increasing sequence of stopping times and \( \xi_1 \) is the first time that \( \eta \) hits \( B(z, 16\epsilon 2^n) \) and \( \xi_n \) is the first time that \( \eta \) hits \( B(z, 32\epsilon) \). For \( 1 \leq m \leq n \), for \( \delta > 0 \), define

\[ F_m = \{ \xi_m < T_z, \Theta_{\xi_m} \notin (\delta, \pi - \delta) \} \]

There exists a function \( p : (0, 1) \to [0, 1] \) with \( p(\delta) \downarrow 0 \) as \( \delta \downarrow 0 \) such that

\[ P\left[ \cap_{m=1}^n F_m \right] \leq p(\delta)^n. \]

**Proof.** For \( w \in \mathbb{H} \) with \( \arg(w) \notin (\delta, \pi - \delta) \), by Lemma 2.4 we know that

\[ P[\eta \text{ hits } B(w, 3w)] \leq C\delta^{\delta/\kappa - 1}, \tag{2.16} \]

where \( C \) is some universal constant.

For \( 1 \leq m \leq n \), let \( f_m = g_{\xi_m} - W_{\xi_m} \). Note that \( \xi_m \) is the first time that \( \eta \) hits \( B(z, 16\epsilon 2^{n-m+1}) \). We denote \( e2^{n-m+1} \) by \( u \). By Lemma 2.2 we know that the ball \( f_m(B(z, u)) \) is contained in the ball with center \( f_m(z) \) and radius \( 4u|f'_m(z)| \), moreover

\[ 4u|f'_m(z)| \leq 3f_m(z) \leq 64u|f'_m(z)|. \]

Therefore, by (2.16), we have

\[ P[f_{m+4} \mid \eta[0, \xi_m]] \leq C\delta^{\delta/\kappa - 1}. \]

Iterating this inequality, we have

\[ P\left[ \cap_{m=1}^n F_m \right] \leq \left( C\delta^{\delta/\kappa - 1} \right)^{n/4}, \]

where \( C \) is some universal constant. This implies the conclusion. \( \square \)

**Proof of (2.12), Upper Bound.** Assume the same notations as in Lemma 2.13. Recall that

\[ F = \{ \eta[0, \tau_1] \subset B(0, R) \}. \]

By Lemma 2.12 we have, for \( 1 \leq m \leq n \)

\[ P\left[ E_{2j+2}^\alpha(\epsilon, z, y) \cap F \right] \lesssim 2^{nA} \delta^{-B} \epsilon^{\alpha_{2j}}, \]

where \( A, B \) are some constants depending on \( \kappa \) and \( j \). Combining with Lemma 2.13, we have, for any \( n \) and \( \delta > 0 \),

\[ P\left[ E_{2j+2}^\alpha(\epsilon, z, y) \cap F \right] \lesssim n2^{nA} \delta^{-B} \epsilon^{\alpha_{2j}} + p(\delta)^n, \]

where \( p(\delta) \downarrow 0 \) as \( \delta \downarrow 0 \). This implies the conclusion. \( \square \)
Proof of (2.13). The lower bound for (2.13) can be proved in the same way as the proof of the lower bound of (2.12).

By the same proof of Lemma 2.12 where we replace (2.7) by (2.8), we could obtain
\[ \mathbb{P} \left( \mathcal{E}_{2j+1}^\beta(\epsilon, z, y) \cap \{ \xi < T_z, \Theta \xi \in (\delta, \pi - \delta), \eta[0, \xi] \subset B(0, R) \} \right) \leq C^A \delta^{-B} e^{\beta_{2j+1}}, \]
as long as \( y \leq -2R \), where \( A, B \) are some constants depending on \( \kappa, j \). Then we can repeat the same proof of the upper bound for (2.12) to obtain the upper bound for (2.13). \( \square \)

Proof of (2.14). We can repeat the same proof of the lower bound of (2.12) to give the lower bound of (2.14). We only need to take care of the point \( u := -4\epsilon g'_\beta(z) \). Given \( \eta[0, S_2] \) and on the event \( \{ S_2 < S_1 \} \), we also have that
\[ f_{S_2} \circ f^{-1}_r(u) \asymp \epsilon |g'_\beta(z)|. \]
Then we can use the same argument to get the lower bound for (2.14).

By the same proof of Lemma 2.12 where we replace (2.7) by (2.10), we could obtain
\[ \mathbb{P} \left( \mathcal{E}_{2j+2}^\gamma(\epsilon, z, y) \cap \{ \xi < T_z, \Theta \xi \in (\delta, \pi - \delta), \eta[0, \xi] \subset B(0, R) \} \right) \leq C^A \delta^{-B} e^{\beta_{2j+1} + o(1)}, \]
as long as \( y \leq -2R \), where \( A, B \) are some constants depending on \( \kappa, j \). Then we can repeat the same proof of the upper bound for (2.12) to obtain the upper bound for (2.14). \( \square \)

3 Critical Fortuin-Kasteleyn-Ising Model

3.1 Basic Properties for the Random-Cluster Model

In this section, we focus on the square lattice \( \mathbb{Z}^2 = (V(\mathbb{Z}^2), E(\mathbb{Z}^2)) \): the vertex set \( V(\mathbb{Z}^2) \) will be identified with \( \mathbb{Z}^2 \), and the edge set is composed of pairs of nearest neighbors:
\[ \mathbb{Z}^2 = \{ x = (x_1, x_2) : x_1, x_2 \in \mathbb{Z} \}, \quad E(\mathbb{Z}^2) = \{ \{ x, y \} \subset \mathbb{Z}^2 : |x_1 - y_1| + |x_2 - y_2| = 1 \}. \]

We denote by \( \Lambda_n(x) \) the box centered at \( x \):
\[ \Lambda_n(x) = x + [-n, n]^2, \quad \Lambda_n = \Lambda_n(0). \]

We will consider finite subgraphs \( G = (V(G), E(G)) \subset \mathbb{Z}^2 \). For such a graph, we denote by \( \partial G \) the inner boundary of \( G \):
\[ \partial G = \{ x \in V(G) : \exists y \notin V(G) \text{ such that } \{ x, y \} \in E(\mathbb{Z}^2) \}. \]

A configuration \( \omega = (\omega_e : e \in E(G)) \) is an element of \( \{0, 1\}^{E(G)} \). If \( \omega_e = 1 \), the edge \( e \) is said to be open, otherwise \( e \) is said to be closed. The configuration \( \omega \) can be seen as a subgraph of \( G \) with the same set of vertices \( V(G) \), and the set of edges given by open edges \( \{ e \in E(G) : \omega_e = 1 \} \).

We are interested in the connectivity properties of the graph \( \omega \). The maximal connected components of \( \omega \) are called clusters. Two vertices \( x \) and \( y \) are connected by \( \omega \) inside \( S \subset \mathbb{Z}^2 \) if there exists a path of vertices \( \{ v_i \}_{0 \leq i \leq k} \) in \( S \) such that \( v_0 = x, v_k = y \) and \( \{ v_i, v_{i+1} \} \) is open in \( \omega \) for \( 0 \leq i < k \). We denote this event by \( \{ x \xleftarrow{S} \rightarrow y \} \). If \( S = G \), we simply drop it from the notation. For \( A, B \subset \mathbb{Z}^2 \), set \( \{ A \xleftarrow{S} \rightarrow B \} \) if there exists a vertex of \( A \) connected in \( S \) to a vertex in \( B \).

Given a finite subgraph \( G \subset \mathbb{Z}^2 \), boundary condition \( \xi \) is a partition \( P_1 \sqcup \cdots \sqcup P_k \) of \( \partial G \). Two vertices are wired in \( \xi \) if they belong to the same \( P_i \). The graph obtained from the configuration \( \omega \) by identifying the wired vertices together in \( \xi \) is denoted by \( \omega^\xi \). Boundary conditions should be understood informally as encoding how sites are connected outside of \( G \). Let \( o(\omega) \) and \( c(\omega) \) denote the number of open can dual edges of \( \omega \) and \( k(\omega^\xi) \) denote the number of maximal connected components of the graph \( \omega^\xi \).
The probability measure $\phi^\xi_{p,q,G}$ of the random cluster model model on $G$ with edge-weight $p \in [0,1]$, cluster-weight $q > 0$ and boundary condition $\xi$ is defined by

$$\phi^\xi_{p,q,G}[\omega] := \frac{p^{\omega}(1-p)^{c(\omega)}q^{k(\omega^*)}}{Z^\xi_{p,q,G}},$$

where $Z^\xi_{p,q,G}$ is the normalizing constant to make $\phi^\xi_{p,q,G}$ a probability measure. For $q = 1$, this model is simply Bernoulli bond percolation.

If all the vertices in $\partial G$ are pairwise wired (the partition is equal to $\partial G$), it is called wired boundary conditions. The random cluster model with wired boundary conditions on $G$ is denoted by $\phi^1_{p,q,G}$. If there is no wiring between vertices in $\partial G$ (the partition is composed of singletons only), it is called free boundary conditions. The random cluster model with free boundary conditions on $G$ is denoted by $\phi^0_{p,q,G}$.

For a configuration $\xi$ on $E(\mathbb{Z}^2) \setminus E(G)$, the boundary conditions induced by $\xi$ are defined by the partition $P_1 \sqcup \cdots \sqcup P_k$, where $x$ and $y$ are in the same $P_i$ if and only if there exists an open path in $\xi$ connecting $x$ and $y$. We identify the boundary condition induced by $\xi$ with the configuration itself, and denote the random cluster model with these boundary conditions by $\phi^\xi_{p,q,G}$. As a direct consequence of these definitions, we have the Domain Markov Property of the random cluster model.

**Proposition 3.1** (Domain Markov Property). Suppose that $G' \subseteq G$ are two finite subgraphs of $\mathbb{Z}^2$. Fix $p \in [0,1], q > 0$ and $\xi$ some boundary conditions on $\partial G$. Let $X$ be a random variable which is measurable with respect to edges in $E(G')$. Then we have

$$\phi^\xi_{p,q,G}(X \mid \omega_e = \psi_e, \forall e \in E(G) \setminus E(G')) = \phi^{\psi^\xi}_{p,q,G}[X], \quad \forall \psi \in \{0,1\}^{E(G) \setminus E(G')}.$$

where $\psi^\xi$ is the partition on $\partial G'$ obtained as follows: two vertices $x, y \in \partial G'$ are wired if they are connected in $\psi^\xi$. Denote the product ordering on $\{0,1\}^E$ by $\leq$. In other words, for $\omega, \omega' \in \{0,1\}^E$, we denote by $\omega \leq \omega'$ if $\omega_e \leq \omega'_e$, for all $e \in E$. An event $A$ depending on edges in $E$ is increasing if for any $\omega \in A, \omega \leq \omega'$ implies $\omega' \in A$. We have positive association when $q \geq 1$.

**Proposition 3.2** (FKG inequality). Fix $p \in [0,1], q \geq 1$ and a finite graph $G$ and some boundary conditions $\xi$. For any two increasing events $A$ and $B$, we have

$$\phi^\xi_{p,q,G}[A \cap B] \geq \phi^\xi_{p,q,G}[A] \phi^\xi_{p,q,G}[B].$$

**Proof.** [Gri06, Theorem 3.8].

As a consequence of the FKG inequality, we have the comparison principle between boundary conditions: fix $p \in [0,1], q \geq 1$ and a finite graph $G$. For any boundary conditions $\xi \leq \psi$ and any increasing event $A$, we have

$$\phi^\xi_{p,q,G}[A] \leq \phi^\psi_{p,q,G}[A]. \quad (3.1)$$

The dual square lattice $(\mathbb{Z}^2)^*$ is the dual graph of $\mathbb{Z}^2$. The vertex set is $(1/2, 1/2) + \mathbb{Z}^2$ and the edges are given by nearest neighbors. The vertices and edges of $(\mathbb{Z}^2)^*$ are called dual-vertices and dual-edges. In particular, for each edge $e$ of $\mathbb{Z}^2$, it is associated to a dual edge, denoted by $e^*$, that it crosses $e$ in the middle. For a finite subgraph $G$, we define $G^*$ to be the subgraph of $(\mathbb{Z}^2)^*$ with edge-set $E(G^*) = \{e^* : e \in E(G)\}$ and vertex set given by the end-points of these dual-edges. A configuration $\omega$ on $G$ can be uniquely associated to a dual configuration $\omega^*$ on the dual graph $G^*$ defined as follows: set $\omega^*(e^*) = 1 - \omega(e)$ for all $e \in E(G)$. A dual-edge $e^*$ is said to be dual-open if $\omega^*(e^*) = 1$, it is dual-closed otherwise. A dual-cluster is a connected component of $\omega^*$. We extend the notion of dual-open path and the connective events in the obvious way.
If \( \omega \) is distributed according to \( \phi_{p,q,G}^\xi \), then \( \omega^* \) is distributed according to \( \phi_{p^*,q^*,G^*}^\xi \) where

\[
q^* = q, \quad \frac{pp^*}{(1-p)(1-p^*)} = q,
\]
and the boundary conditions \( \xi^* \) can be deduced from \( \xi \) in a case by case manner. In particular, \( \xi = 0 \) corresponds to \( \xi^* = 1 \) and \( \xi = 1 \) corresponds to \( \xi^* = 0 \). Note that,

\[
\text{if } p = p_c(q) := \frac{\sqrt{q}}{1 + \sqrt{q}}, \quad \text{then } p^* = p.
\]

When \( p = p_c(q) \), we have the following generalized Russo-Symour-Welsh estimates. For a rectangle \( R = [a,b] \times [c,d] \subset \mathbb{Z}^2 \), let \( C_h(R) \) be the event that there exists an open path in \( R \) from \([a] \times [c,d] \) to \([b] \times [c,d] \). Such a crossing is called a horizontal crossing of \( R \). Similarly, we define \( C_v(R) \) to be the event that there exists an open path in \( R \) from \([a,b] \times [c] \) to \([a,b] \times [d] \), and such a crossing is called a vertical crossing.

**Proposition 3.3** (RSW for rectangle). Fix \( 1 \leq q < 4 \) and \( \delta > 0 \) and denote by \( R_n \) the rectangle \([0, \delta n] \times [0, n] \), there exists \( c(\delta) > 0 \) such that for any \( n \geq 1 \),

\[
\phi_{p_c(q), q, R_n}^0 (C_h(R_n)) \geq c(\delta). \tag{3.2}
\]

Fix \( 1 \leq q \leq 4 \) and \( \delta > 0, \epsilon > 0 \) and denote by \( R_n = [0, \delta n] \times [0, n], \quad R^*_n = [-\epsilon n, (\delta + \epsilon)n] \times [-\epsilon n, (1 + \epsilon)n] \).

There exists \( c(\delta, \epsilon) > 0 \) such that for any \( n \geq 1 \),

\[
\phi_{p_c(q), q, R^*_n}^0 (C_h(R_n)) \geq c(\delta, \epsilon). \tag{3.3}
\]

**Proof.** [DCST15] Theorem 3, Theorem 4, Theorem 7. \( \square \)

It is worthwhile to spend some words on the relation between (3.2) and (3.3). The estimate (3.3) holds for \( q \in [1, 4] \) and it is weaker than the estimate (3.2) which only holds for \( q \in [1, 4] \). When \( q = 4 \), the estimate (3.2) is expected to fail. In the following, when we talk about the interior quasi-multiplicativity for monochromatic arm events, the estimate (3.3) is sufficient and the conclusion will hold for \( q \in [1, 4] \); when we talk about the boundary quasi-multiplicativity for monochromatic arm events, the border case \( q = 4 \) will cause some difficulty and thus we only discuss the situation for \( q \in [1, 4] \). As a consequence of Propositions 3.1 3.2 and the estimate (3.3), we have the following mixing property at critical.

**Corollary 3.4.** Fix \( q \in [1, 4] \), there exists \( \alpha > 0 \) such that for any \( 2k \leq n \), for any event \( A \) depending only on edges in \( \Lambda_k \), and for any boundary conditions \( \xi \) and \( \psi \), we have

\[
\left| \phi_{p_c(q), q, \Lambda_n}^\xi [A] - \phi_{p_c(q), q, \Lambda_n}^\psi [A] \right| \leq \left( \frac{k}{n} \right)^\alpha \phi_{p_c(q), q, \Lambda_n}^\xi [A].
\]

In particular, this implies that for any \( 2k \leq m \leq n \), for any event \( A \) depending only on edges in \( \Lambda_k \) and any event \( B \) depending only on edges in \( \Lambda_n \setminus \Lambda_m \), and for any boundary conditions \( \xi \), we have

\[
\left| \phi_{p_c(q), q, \Lambda_n}^\xi [A \cap B] - \phi_{p_c(q), q, \Lambda_n}^\xi [A] \phi_{p_c(q), q, \Lambda_n}^\xi [B] \right| \leq \left( \frac{k}{n} \right)^\alpha \phi_{p_c(q), q, \Lambda_n}^\xi [A] \phi_{p_c(q), q, \Lambda_n}^\xi [B].
\]

**Proof.** [DCST15] Theorem 3, Theorem 4, Theorem 5. \( \square \)
3.2 Quasi-Multiplicativity

We say that a path is of type 1 if it is a primal-open path, and we say that a path is of type 0 if it is a dual-open path. Fix \( n < N \) and the annulus \( \Lambda_N \setminus \Lambda_n \), a simple path of type 0 or type 1 connecting \( \partial \Lambda_n \) to \( \partial \Lambda_N \) is called an arm. Fix an integer \( j \geq 1 \) and \( \sigma = (\sigma_1, ..., \sigma_j) \in \{0,1\}^j \). For \( n < N \), define \( \mathcal{A}_\sigma(n,N) \) to be the event that there are \( j \) disjoint arms \( (\gamma_k)_{1 \leq k \leq j} \) connecting \( \partial \Lambda_n \) to \( \partial \Lambda_N \) in the annulus \( \Lambda_N \setminus \Lambda_n \) which are of types \( (\sigma_k)_{1 \leq k \leq j} \), where we identify two sequences \( \sigma \) and \( \sigma' \) if they are the same up to cyclic permutation and the arms are indexed in clockwise order. For each \( j \geq 1 \), there exists a smallest integer \( n_0(j) \) such that, for all \( N \geq n_0(j) \), we have \( \mathcal{A}_\sigma(n_0(j),N) \neq \emptyset \).

**Proposition 3.5.** Fix a constant \( \sigma \) and fix \( q \in [1,4) \). For all \( n_0(j) \leq n_1 < n_2 < n_3 \leq m/2 \), and for all boundary conditions \( \xi \), we have
\[
\phi^\xi \rho_{p(q),q,\Lambda} [\mathcal{A}_\sigma(n_1,n_3)] \times \phi^\xi \rho_{p(q),q,\Lambda} [\mathcal{A}_\sigma(n_1,n_2)] \phi^\xi \rho_{p(q),q,\Lambda} [\mathcal{A}_\sigma(n_2,n_3)],
\]
where the constants in \( \asymp \) are uniform over \( n_1,n_2,n_3,m \) and \( \xi \).

**Proposition 3.6.** Fix a non-constant \( \sigma \) and \( q = 2 \). For all \( n_0(j) \leq n_1 < n_2 < n_3 \leq m/2 \), and for all boundary conditions \( \xi \), we have
\[
\phi^\xi \rho_{p(2),2,\Lambda} [\mathcal{A}_\sigma(n_1,n_3)] \times \phi^\xi \rho_{p(2),2,\Lambda} [\mathcal{A}_\sigma(n_1,n_2)] \phi^\xi \rho_{p(2),2,\Lambda} [\mathcal{A}_\sigma(n_2,n_3)],
\]
where the constants in \( \asymp \) are uniform over \( n_1,n_2,n_3,m \) and \( \xi \).

Propositions 3.5 and 3.6 are called the quasi–multiplicativity of the random cluster models for monochromatic arm events and polychromatic arm events respectively. They were proved in [CDCH16], and we will sketch the proof and point out the reason why we can prove the quasi–multiplicativity for monochromatic arm events for \( q \in [1,4) \) but we can only prove it for polychromatic arm events for \( q = 2 \) for the moment.

To prove Propositions 3.5 and 3.6, we need to introduce several auxiliary subevents of \( \mathcal{A}_\sigma(n,N) \). Fix \( \sigma = (\sigma_1, ..., \sigma_j) \in \{0,1\}^j \). Fix some \( \delta > 0 \) small. Suppose \( Q = [-1,1]^2 \) is the unit square. A landing sequence \( (I_k)_{1 \leq k \leq j} \) is a sequence of disjoint sub-intervals on \( \partial Q \) in clockwise order. We denote by \( z(I_k) \) the center of \( I_k \). We say \( (I_k)_{1 \leq k \leq j} \) is \( \delta \)-separated if

- the intervals are at distance at least \( 2\delta \) from each other, and they are at distance at least \( 2\delta \) from the four corners of \( \partial Q \);
- for each \( I_k \), the length of \( I_k \) is at least \( 2\delta \).

We say that two sets are \( \sigma_k \)-connected if there is a path of type \( \sigma_k \) connecting them. Fix two \( \delta \)-separated landing sequences \( (I_k)_{1 \leq k \leq j} \) and \( (I_k')_{1 \leq k \leq j} \). We say that the arms \( (\gamma_k)_{1 \leq k \leq j} \) are \( \delta \)-well-separated with landing sequence \( (I_k)_{1 \leq k \leq j} \) on \( \partial \Lambda_n \) and landing sequence \( (I_k')_{1 \leq k \leq j} \) on \( \partial \Lambda_N \) if

- for each \( k \), the arm \( \gamma_k \) connects \( nI_k \) to \( N I_k' \);
- for each \( k \), the arm \( \gamma_k \) can be \( \sigma_k \)-connected to distance \( \delta n \) of \( \partial \Lambda_n \) inside \( \Lambda_{\delta n}(z(I_k)) \);
- for each \( k \), the arm \( \gamma_k \) can be \( \sigma_k \)-connected to distance \( \delta N \) of \( \partial \Lambda_N \) inside \( \Lambda_{\delta N}(z(I_k')) \).

We denote this event by
\[
\mathcal{A}_{\sigma}^{I,I'}(n,N).
\]

We can also define \( \delta \)-well-separated only on the inner boundary \( \partial \Lambda_n \) or only on the outer boundary \( \partial \Lambda_N \) in the similar way, and denote these events by
\[
\mathcal{A}_{\sigma}^{I'}(n,N), \quad \mathcal{A}_{\sigma}^{I''}(n,N).
\]

The proof of Propositions 3.5 and 3.6 consists of the following three lemmas.
Lemma 3.7. Fix \( j \geq 1 \) and \( \delta > 0 \) and two \( \delta \)-separated landing sequences \( (I_k)_{1 \leq k \leq j} \) and \( (I'_k)_{1 \leq k \leq j} \).

1. Fix a constant \( \sigma \) and fix \( q \in [1, 4] \). For all \( n < N \leq m/2 \) such that \( \mathcal{A}^{I/}_{\sigma}(n, N) \) is not empty, and for all boundary conditions \( \xi \), we have

\[
\phi^\xi_{p_\epsilon(q), q, \Lambda_m} \left[ \mathcal{A}^{I/}_{\sigma}(n, N) \right] \asymp \phi^\xi_{p_\epsilon(q), q, \Lambda_m} \left[ \mathcal{A}_{\sigma}(n, N) \right],
\]

where the constants in \( \asymp \) depend only on \( \delta \).

2. Fix a non-constant \( \sigma \) and \( q = 2 \). For all \( n < N \leq m/2 \) such that \( \mathcal{A}^{I/}_{\sigma}(n, N) \) is not empty, and for all boundary conditions \( \xi \), we have

\[
\phi^\xi_{p_\epsilon(2), 2, \Lambda_m} \left[ \mathcal{A}^{I/}_{\sigma}(n, N) \right] \asymp \phi^\xi_{p_\epsilon(2), 2, \Lambda_m} \left[ \mathcal{A}_{\sigma}(n, N) \right],
\]

where the constants in \( \asymp \) depend only on \( \delta \).

Lemma 3.8. Fix \( \sigma \) and \( q \in [1, 4] \). For all \( n < N \leq m/4 \) such that \( \mathcal{A}^{I/}_{\sigma}(n/2, N) \) is not empty, and for all boundary conditions \( \xi \), we have

\[
\phi^\xi_{p_\epsilon(q), q, \Lambda_m} \left[ \mathcal{A}^{I/}_{\sigma}(n/2, N) \right] \asymp \phi^\xi_{p_\epsilon(q), q, \Lambda_m} \left[ \mathcal{A}^{I/}_{\sigma}(n, 2N) \right] \asymp \phi^\xi_{p_\epsilon(q), q, \Lambda_m} \left[ \mathcal{A}^{I/}_{\sigma}(n, N) \right],
\]

where the constants in \( \asymp \) depend only on \( \delta \).

Lemma 3.9. Fix \( \sigma \) and \( q \in [1, 4] \). For all \( n_0(3) \leq n_1 < n_2 < n_3 \leq m/2 \), and for all boundary conditions \( \xi \), we have

\[
\phi^\xi_{p_\epsilon(q), q, \Lambda_m} \left[ \mathcal{A}^{I/}_{\sigma}(n_1, n_2) \right] \phi^\xi_{p_\epsilon(q), q, \Lambda_m} \left[ \mathcal{A}^{I/}_{\sigma}(n_2, n_3) \right] \asymp \phi^\xi_{p_\epsilon(q), q, \Lambda_m} \left[ \mathcal{A}_{\sigma}(n_1, n_3) \right],
\]

where the constant in \( \asymp \) depends only on \( \delta \).

Assuming Lemmas 3.7 to 3.9, we could complete the proof of quasi-multiplicativity.

Proof of Propositions 3.5 and 3.6. Since \( q, p_\epsilon(q), \Lambda_m \) and \( \xi \) are fixed, we eliminate them from the notations. We may assume \( n_3 \geq 2n_2 \).

\[
\phi \left[ \mathcal{A}_{\sigma}(n_1, n_3) \right] \leq \phi \left[ \mathcal{A}_{\sigma}(n_1, n_2) \cap \mathcal{A}_{\sigma}(2n_2, n_3) \right]
= \phi \left[ \mathcal{A}_{\sigma}(n_1, n_2) \vline \mathcal{A}_{\sigma}(2n_2, n_3) \right] \phi \left[ \mathcal{A}_{\sigma}(2n_2, n_3) \right] 
\times \phi \left[ \mathcal{A}_{\sigma}(n_1, n_2) \right] \phi \left[ \mathcal{A}_{\sigma}(2n_2, n_3) \right] \quad \text{(by Corollary 3.4)}
\times \phi \left[ \mathcal{A}_{\sigma}(n_1, n_2) \right] \phi \left[ \mathcal{A}^{I/}_{\sigma}(2n_2, n_3) \right] \quad \text{(by Lemma 3.7)}
\times \phi \left[ \mathcal{A}_{\sigma}(n_1, n_2) \right] \phi \left[ \mathcal{A}^{I/}_{\sigma}(n_2, n_3) \right] \quad \text{(by Lemma 3.8)}
\times \phi \left[ \mathcal{A}_{\sigma}(n_1, n_2) \right] \phi \left[ \mathcal{A}_{\sigma}(n_2, n_3) \right] ; \quad \text{(by Lemma 3.7)}
\phi \left[ \mathcal{A}_{\sigma}(n_1, n_3) \right] \geq \phi \left[ \mathcal{A}^{I/}_{\sigma}(n_1, n_2) \right] \phi \left[ \mathcal{A}^{I/}_{\sigma}(n_2, n_3) \right]
\times \phi \left[ \mathcal{A}_{\sigma}(n_1, n_2) \right] \phi \left[ \mathcal{A}_{\sigma}(n_2, n_3) \right]. \quad \text{(by Lemma 3.7)}
\]

These complete the proof.

Next, we discuss the proofs for Lemmas 3.7 to 3.9. The proofs for Lemma 3.7 Item (1) and Lemmas 3.8 and 3.9 are standard and only require the inputs from Section 3.1, see [Nol08, Section 4] and [CDCH16, Section 5]. However, the proof of Lemma 3.7 Item (2) cannot be obtained in the similar way and its proof requires a stronger version of RSW—Proposition 3.10—which is only proved for \( q = 2 \), based on
discrete complex analysis introduced in [Che16]. Given a discrete topological rectangle \((\Omega, a, b, c, d)\) (a bounded simply-connected subdomain of \(\mathbb{Z}^2\) with four marked boundary points), the four points are in counterclockwise order and \((ab)\) denotes the arc of \(\partial \Omega\) from \(a\) to \(b\). We denote by \(d_\Omega((ab), (cd))\) the \textit{discrete extremal distance} between \((ab)\) and \((cd)\) in \(\Omega\), see [Che16, Section 6]. The discrete extremal distance is uniformly comparable to and converges to its continuous counterpart—the classical extremal distance.

**Proposition 3.10.** Fix \(q = 2\). For each \(L > 0\) there exists \(c(L) > 0\) such that, for any topological rectangle \((\Omega, a, b, c, d)\) and any boundary conditions \(\xi\), the following holds:

- if \(d_\Omega((ab), (cd)) \leq L\), then
  \[
  \phi^\xi_{p_\xi,2,\Omega}[(ab) \leftrightarrow (cd)] \geq c(L);
  \]
- if \(d_\Omega((ab), (cd)) \geq 1/L\), then
  \[
  \phi^\xi_{p_\xi,2,\Omega}[(ab) \leftrightarrow (cd)] \leq 1 - c(L).
  \]

**Proof.** [CDCH16] Theorem 1.1. \(\qed\)

With Proposition 3.10, the authors proved Lemma 3.7 Item (2) in [CDCH16] Corollary 1.4 and hence completed the proof of Propositions 3.6.

Next, we state similar conclusions for the boundary arm events. We denote by \(\Lambda_n^+(x)\) the box in \(\mathbb{H}\) centered at \(x \in \mathbb{R}\):

\[
\Lambda_n^+(x) = x + [-n, n] \times [0, n], \quad \Lambda_n^+ = \Lambda^+_n(0).
\]

Fix an integer \(j \geq 1\) and \(\sigma = (\sigma_1, \ldots, \sigma_j) \in \{0, 1\}^j\). For \(n < N\), define \(\mathcal{A}_\sigma^+(n, N)\) to be the event that there are \(j\) disjoint arms \((\gamma_k)_{1 \leq k \leq j}\) connecting \(\partial \Lambda_n\) to \(\partial \Lambda_N\) in the semi-annulus \(\Lambda_N^+ \setminus \Lambda_n^+\) which are of types \((\sigma_k)_{1 \leq k \leq j}\) and the arms are indexed in clockwise order. For each \(j\), there exists a smallest integer \(n_0^+(j)\) such that \(\mathcal{A}_\sigma^+(n_0^+(j), N) \neq \emptyset\) for all \(N \geq n_0^+(j)\).

**Proposition 3.11.** Fix a constant \(\sigma\) and fix \(q \in [1, 4]\). For all \(n_0^+(j) \leq n_1 < n_2 < n_3 \leq m\), and for all boundary conditions \(\xi\), we have

\[
\phi^\xi_{p_\xi, q, \Lambda_m^+} [\mathcal{A}_\sigma^+(n_1, n_3)] \asymp \phi^\xi_{p_\xi, q, \Lambda_m^+} [\mathcal{A}_\sigma^+(n_1, n_2)] \phi^\xi_{p_\xi, q, \Lambda_m^+} [\mathcal{A}_\sigma^+(n_2, n_3)],
\]

where the constants in \(\asymp\) are uniform over \(n_1, n_2, n_3, m\) and \(\xi\).

We only state the quasi-multiplicativity of monochromatic arm events for \(q \in [1, 4]\), not including \(q = 4\). The reason is explained after Proposition 3.3.

**Proposition 3.12.** Fix a non-constant \(\sigma\) and \(q = 2\). For all \(n_0^+(j) \leq n_1 < n_2 < n_3 \leq m\), and for all boundary conditions \(\xi\), we have

\[
\phi^\xi_{p_\xi, 2, \Lambda_m^+} [\mathcal{A}_\sigma^+(n_1, n_3)] \asymp \phi^\xi_{p_\xi, 2, \Lambda_m^+} [\mathcal{A}_\sigma^+(n_1, n_2)] \phi^\xi_{p_\xi, 2, \Lambda_m^+} [\mathcal{A}_\sigma^+(n_2, n_3)],
\]

where the constants in \(\asymp\) are uniform over \(n_1, n_2, n_3, m\) and \(\xi\).

Suppose \(Q^+ = [-1, 1] \times [0, 1]\) is the semi unit square. A landing sequence \((I_k)_{1 \leq k \leq j}\) is a sequence of disjoint subintervals on \([-1] \times [0, 1] \cup [-1, 1] \times \{1\} \cup \{1\} \times [0, 1]\). We can define \(\delta\)-separated landing sequence in the similar way. Given two \(\delta\)-separated landing sequences \((I_k)_{1 \leq k \leq j}\) and \((I_k')_{1 \leq k \leq j}\), we can define \(\delta\)-well-separated arm events in the similar way and denote it by

\[
\mathcal{A}^\delta_{\sigma, 1/1'}(n, N).
\]

**Lemma 3.13.** Fix \(j \geq 1\) and \(\delta > 0\) and two \(\delta\)-separated landing sequences \((I_k)_{1 \leq k \leq j}\) and \((I_k')_{1 \leq k \leq j}\).
1. Fix a constant \( \sigma \) and fix \( q \in [1, 4) \). For all \( n < N \leq m \) such that \( A_\sigma^{+, I'/I} (n, N) \) is not empty, and for all boundary conditions \( \xi \), we have

\[
\phi_{p_c(q), \Lambda_m^+} \left[ A_\sigma^{+, I'/I} (n, N) \right] \asymp \phi_{p_c(q), \Lambda_m^+} \left[ A_\sigma^+ (n, N) \right],
\]

where the constants in \( \asymp \) depend only on \( \delta \).

2. Fix a non-constant \( \sigma \) and \( q = 2 \). For all \( n < N \leq m \) such that \( A_\sigma^{+, I'/I} (n, N) \) is not empty, and for all boundary conditions \( \xi \), we have

\[
\phi_{p_c(2), \Lambda_m^+} \left[ A_\sigma^{+, I'/I} (n, N) \right] \asymp \phi_{p_c(2), \Lambda_m^+} \left[ A_\sigma^+ (n, N) \right],
\]

where the constants in \( \asymp \) depend only on \( \delta \).

### 3.3 Proof of Theorems 1.2 and 1.4

Consider random cluster model with edge weight \( p \in [0, 1] \) and cluster weight \( q > 0 \) on the square lattice. We call critical FK-Ising model the random cluster model with

\[ q = 2, \quad p = p_c(2). \]

In this section, we will first introduce the exploration path in the random cluster model in Dobrushin domains, then state the convergence of the exploration path for \( q = 2 \) and \( p = p_c(2) \) on the square lattice and finally explain how to prove Theorems 1.2 and 1.4 by combining the convergence of the exploration path, the quasi-multiplicativity in Section 3.2, with the arm exponents of SLE.

In Section 3.1, we have introduced the square lattice \( Z^2 \) and the dual square lattice \( (Z^2)^* = (1/2, 1/2) + Z^2 \). The medial lattice \( (Z^2)^\circ \) is the graph with the centers of edges of \( Z^2 \) as vertex set, and edges connecting nearest vertices. This lattice is a rotated and rescaled version of \( Z^2 \), see Figure 3.1. The vertices and edges of \( (Z^2)^\circ \) are called medial-vertices and medial-edges. We identify the faces of \( (Z^2)^\circ \) with the vertices of \( Z^2 \) and \( (Z^2)^* \). A face of \( (Z^2)^\circ \) is said to be black if it corresponds to a vertex of \( Z^2 \) and white if it corresponds to a vertex of \( (Z^2)^* \). Dobrushin domains are discrete analogue of simply connected domains with two marked points on their boundary.

![Lattices](image)

(a) The square lattice. (b) The dual square lattice. (c) The medial lattice.

**Fig. 3.1**: The lattices.

Fix a Dobrushin domain \( (\Omega, a, b) \) and consider a configuration \( \omega \) together with its dual-configuration \( \omega^* \). The Dobrushin boundary condition is given by taking edges of \( \partial_\omega \) to be open and the dual-edges of \( \partial_{\omega^*} \) to be dual-open. Through every vertex of \( \Omega^\circ \), there passes either an open edge of \( \Omega \) or a dual open
edge of $\Omega^\circ$. Draw self-avoiding loops on $\Omega^\circ$ as follows: a loop arriving at a vertex of the medial lattice always makes a $\pm \pi/2$ turn so as not to cross the open or dual open edges through this vertex, see Figure 3.2. The loop representation contains loops together with a self-avoiding path going from $a^\circ$ to $b^\circ$. This curve is called the exploration path. See [DC13, Section 6.1] for more details.

**(a)** The configuration $\omega$ and its dual $\omega^\ast$.

**(b)** The loop representation of $\omega$.

Fig. 3.2: The loop representation of the configuration.

For $u > 0$, we consider the rescaled square lattice $u\mathbb{Z}^2$. The definitions of dual and medial Dobrushin domains extend to this context. Dobrushin domains on $u\mathbb{Z}^2$, $(u\mathbb{Z}^2)^\ast$ and $(u\mathbb{Z}^2)^\circ$ will be denoted by $(\Omega_u, a_u, b_u)$, $(\Omega_u^\ast, a_u^\ast, b_u^\ast)$, $(\Omega_u^\circ, a_u^\circ, b_u^\circ)$.

Let $(\Omega, a, b)$ be a simply connected domain with two marked points on its boundary. Consider a sequence of Dobrushin domains $(\Omega_u, a_u, b_u)$. We say that $(\Omega_u, a_u, b_u)$ converges to $(\Omega, a, b)$ in the Carathéodory sense if

$$f_u \to f$$
onumber

on any compact subset $K \subset \mathbb{H}$,

where $f_u$ (resp. $f$) is the unique conformal map from $\mathbb{H}$ to $\Omega_u$ (resp. $\Omega$) satisfying $f_u(0) = a_u, f_u(\infty) = b_u$ and $f_u'(\infty) = 1$ (resp. $f(0) = a, f(\infty) = b, f'(\infty) = 1$).

Let $X$ be the set of continuous parameterized curves and $d$ be the distance on $X$ defined for $\eta_1 : I \to \mathbb{C}$ and $\eta_2 : J \to \mathbb{C}$ by

$$d(\eta_1, \eta_2) = \min_{\varphi_1: [0,1] \to I, \varphi_2: [0,1] \to J} \sup_{t \in [0,1]} |\eta_1(\varphi_1(t)) - \eta_2(\varphi_2(t))|,$$

where the minimization is over increasing bijective functions $\varphi_1, \varphi_2$. Note that $I$ and $J$ can be equal to $\mathbb{R}_+ \cup \{\infty\}$. The topology on $(X,d)$ gives rise to a notion of weak convergence for random curves on $X$.

**Theorem 3.14.** Let $\Omega$ be a simply connected domain with two marked points $a$ and $b$ on its boundary. Let $(\Omega_u, a_u, b_u)$ be a family of Dobrushin domains converging to $(\Omega, a, b)$ in the Carathéodory sense. The exploration path of the critical FK-Ising model with Dobrushin boundary conditions in $(\Omega_u, a_u, b_u)$ converges weakly to SLE$_{16/3}$ as $u \to 0$.

**Proof.** [CDCH+14].

Now, we are ready to prove Theorems 1.2 and 1.4.

**Proof of Theorem 1.2.** We only give the proof for $\gamma_3^+$ and the other cases can be proved similarly.

Consider $\Lambda_m^\circ$ with two boundary points $a_m = (-m, 0)$ and $b_m = (m, m)$. Fix the Dobrushin boundary condition: the edges along $\partial \Lambda_m^\circ$ from $b_m$ to $a_m$ (counterclockwise) are primal-open and the dual edges
(a) \( A^{+,1/1}(n, N) \) is the well-separated arm event.  

(b) The four gray parts are \( R_1 \) to \( R_4 \) respectively.

**Fig. 3.3:** The explanation of the proof of Theorem 1.2

along \( \partial \Lambda^+_m \) from \( a_m \) to \( b_m \) are dual-open. Since we fix \( q = 2 \), \( p = p_c(2) \), the boundary condition and \( \sigma = (101) \), we will eliminate them from the notations. We will prove that, for \( n < N \leq m/2 \),

\[
\phi_{\Lambda^+_m} [A^+(n, N)] = N^{-\gamma_3^+ + o(1)}, \quad \text{as } N \to \infty.
\]  (3.4)

Fix the landing sequence \( I = (I_1, I_2, I_3) \) where

\[
I_1 = \{-1\} \times [1/2, 3/4], \quad I_2 = [-1/2, 1/2] \times \{1\}, \quad I_3 = \{1\} \times [1/2, 3/4].
\]

Recall that \( A^{+,1/1}(n, N) \) is the 1/8-well-separated arm events with the landing sequence \( nI \) on \( \partial \Lambda^+_m \) and \( NI \) on \( \partial \Lambda^+_N \). The three arms in \( A^+(n, N) \) are denoted by \( (\gamma_1, \gamma_2, \gamma_3) \) where \( \gamma_1, \gamma_3 \) are primal-open and \( \gamma_2 \) is dual-open.

Let \( R_1 \) be the rectangle \( R_1 := [-9N/8, -N] \times [0, 3N/4] \). Define \( C_1 \) to be the event that \( \gamma_1 \) is connected to the bottom side of \( R_1 \) in \( R_1 \) by a primal-open path. Let \( R_2 \) be the rectangle \( R_2 := [-n, n] \times [n/4, 7n/8] \). Define \( C_2 \) to be the event that \( \gamma_1 \) is connected to \( \gamma_3 \) in \( R_2 \) by a primal-open path. For \( \delta \in (0, 1/8) \), let \( R_3(\delta) \) be the semi-annulus \( R_3(\delta) := [0, 4n] \times [0, n/4] \setminus [n, 3n] \times [0, \delta n] \). Define \( C_3(\delta) \) to be the event that there is a primal-open path in \( R_3(\delta) \) connecting the left bottom side to the right bottom side of \( R_3(\delta) \). Let \( R_4 \) be the rectangle \( R_4 := [-N, N] \times [N, 2N] \). Define \( C_4^* \) to be the event that \( \gamma_2 \) is connected to the top of \( R_4 \) in \( R_4 \) by a dual-open path. We need the following two estimates in the proof:

\[
\phi_{\Lambda^+_m} [A^+(n, N)] \asymp \phi_{\Lambda^+_m} [A^+(n, N)],
\]  (3.5)

where the constants in \( \asymp \) are uniform over \( n, N \) and \( m \geq 2N \); and, there exists some \( \delta > 0 \) such that

\[
\phi_{\Lambda^+_m} [A^{+,1/1}(n, N)] \asymp \phi_{\Lambda^+_m} [A^{+,1/1}(n, N) \cap C_1 \cap C_2 \cap C_3(\delta) \cap C_4^*],
\]  (3.6)

where the constants in \( \asymp \) are uniform over \( n, N \). The relation \((3.5)\) is true by Corollary 3.4. We will prove \((3.6)\) after this proof. Assume it is true, then we can complete the proof.

Let \( \mathbb{P}_N \) be the probability measure \( \phi_{\Lambda^+_m} \) where the square lattice is scaled by \( 1/N \) and let \( \mathbb{P}_\infty \) be the law of SLE_{16/3} in \([-2, 2] \times [0, 2] \) from \((-2, 0)\) to \((2, 2)\). On the event \( A^{+,1/1}(\epsilon N, N) \cap C_1 \cap C_2 \cap C_3(\delta) \cap C_4^* \), consider the exploration path \( \eta \) from \( a_{2N} \) to \( b_{2N} \). Let \( \tau \) be the first time that \( \eta \) hits \( \partial \Lambda^+_m \). The event \( C_1 \) guarantees that \( \eta[0, \tau] \) does not hit the interval \([-N, -n] \). The event \( C_4^* \) guarantees that \( \eta[0, \tau] \) is bounded away from \( b_{2N} \). The event \( C_3(\delta) \) guarantees that \( \eta[0, \tau] \) is bounded away from the interval \([n, 3n] \). See Figure 3.3.
By Theorem 3.14 for $\epsilon > 0$ small, we have
\[
\limsup_{N \to \infty} \mathbb{P}_N \left[ A_n^{+}/(\epsilon N, N) \cap C_1 \cap C_2 \cap C_3(\delta) \cap C_4^* \right] \leq \mathbb{P}_\infty \left[ \mathcal{H}_3^*(\epsilon) \cap \mathcal{F} \right] \leq \liminf_{N \to \infty} \mathbb{P}_N \left[ A_n^{+}(\epsilon N, N) \right].
\]
Here we abuse $\mathcal{H}_3^*(\epsilon)$ to indicate the event defined in Section 2.2 with appropriate $x, y, u$ and $\mathcal{F}$ is defined analogously as in (2.6). Combining with Lemma 3.13 (3.6) and (2.6), we have
\[
\liminf_{N \to \infty} \mathbb{P}_N \left[ A_n^{+}(\epsilon N, N) \right] \leq \limsup_{N \to \infty} \mathbb{P}_N \left[ A_n^{+}(\epsilon N, N) \right] \leq \epsilon^{\gamma_3^+}.
\]
By (3.5), we have
\[
\liminf_{N \to \infty} \phi_{A_m^+} \left[ A_n^{+}(\epsilon N, N) \right] \geq \limsup_{N \to \infty} \phi_{A_m^+} \left[ A_n^{+}(\epsilon N, N) \right] \geq \epsilon^{\gamma_3^+} \tag{3.7}
\]
where the constants in $\approx$ are uniform over $\epsilon$ and $m \geq 2N$.
Suppose $N = n\epsilon^{-K}$ for some integer $K$. By Proposition 3.12 for $m \geq 2N$, we have
\[
\phi_{A_m^+} \left[ A_n^{+}(n, N) \right] \leq C^K \prod_{j=1}^{K} \phi_{A_m^+} \left[ A_n^{+}(n\epsilon^{-j+1}, n\epsilon^{-j}) \right],
\]
where $C$ is some universal constant. Thus
\[
\frac{\log \phi_{A_m^+} \left[ A_n^{+}(n, N) \right]}{\log N} \leq \frac{K \log C}{\log N} + \frac{1}{\log N} \sum_{j=1}^{K} \log \phi_{A_m^+} \left[ A_n^{+}(n\epsilon^{-j+1}, n\epsilon^{-j}) \right].
\]
By (3.7), we have
\[
\limsup_{j \to \infty} \phi_{A_m^+} \left[ A_n^{+}(n\epsilon^{-j+1}, n\epsilon^{-j}) \right] \leq \epsilon^{\gamma_3^+}.
\]
Therefore,
\[
\limsup_{K \to \infty} \frac{\log \phi_{A_m^+} \left[ A_n^{+}(n, N) \right]}{\log N} \leq \frac{\tilde{C}}{\log(1/\epsilon)} - \gamma_3^+,
\]
where $\tilde{C}$ is some universal constant. Let $\epsilon \to 0$, we have
\[
\limsup_{N \to \infty} \frac{\log \phi_{A_m^+} \left[ A_n^{+}(n, N) \right]}{\log N} \leq -\gamma_3^+.
\]
We could prove the lower bound similarly:
\[
\liminf_{N \to \infty} \frac{\log \phi_{A_m^+} \left[ A_n^{+}(n, N) \right]}{\log N} \geq -\gamma_3^+.
\]
These imply (3.4) and complete the proof. \qed

Proof of (3.6). Since $\Lambda_{2N}^+$ is fixed, we eliminate it from the notations. Let $\mathcal{F}$ be information of the configuration inside $\Lambda_{2N}^\uparrow \setminus (R_1 \cup R_2 \cup R_3)$. Inside $R_1$, we start the exploration from the top of $R_1$ until we find the highest horizontal crossing of primal-open path that connects $\gamma_1$ to the left side of $R_1$. Denote this information by $\mathcal{F}_1$. Inside $\Lambda_{n/8}(-n + 5ni/8)$ (note that $-n + 5ni/8$ is the center of $nI_1$), we start the exploration from the bottom until we find the lowest horizontal crossing of primal-open path that connects $\gamma_1$ to the right side; inside $\Lambda_{n/8}(n + 5ni/8)$ (note that $n + 5ni/8$ is the center of $nI_3$), we start the exploration from the bottom until we find the lowest horizontal crossing of primal-open path that connects $\gamma_3$ to the left side. Denote this information by $\mathcal{F}_2$. Inside $\Lambda_{N/8}(Ni)$ (note that $Ni$ is the center of $NI_2$), we start the exploration from the left side of $\Lambda_{N/8}(Ni)$ until we find the leftmost vertical-crossing of
dual open path that connects \( \gamma_2 \) to the top. Denote this information by \( \mathcal{F}_4 \). Then the events \( \mathcal{A}^{+,I/I}(n,N) \) and \( \mathcal{C}_3(\delta) \) are measurable with respect to \( \mathcal{G} := \mathcal{F} \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_4 \). Thus,

\[
\begin{align*}
\phi \left[ \mathcal{A}^{+,I/I}(n,N) \cap \mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{C}_3(\delta) \cap \mathcal{C}_4^* \right] \\
= \phi \left[ 1_{\mathcal{A}^{+,I/I}(n,N) \cap \mathcal{C}_3(\delta)} \phi \left[ \mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{C}_4^* \mid \mathcal{G} \right] \right] \\
\geq \phi \left[ 1_{\mathcal{A}^{+,I/I}(n,N) \cap \mathcal{C}_3(\delta)} \phi \left[ \mathcal{C}_1 \mid \mathcal{G} \right] \phi \left[ \mathcal{C}_2 \mid \mathcal{G} \right] \phi \left[ \mathcal{C}_4^* \mid \mathcal{G} \right] \right].
\end{align*}
\]

(by Corollary 3.4)

By Propositions 3.1, 3.3 and (3.1), we have, on the event \( \mathcal{A}^{+,I/I}(n,N) \cap \mathcal{C}_3(\delta) \),

\[
\phi \left[ \mathcal{C}_1 \mid \mathcal{G} \right] \asymp 1, \quad \phi \left[ \mathcal{C}_2 \mid \mathcal{G} \right] \asymp 1, \quad \phi \left[ \mathcal{C}_4^* \mid \mathcal{G} \right] \asymp 1.
\]

Therefore, we have

\[
\phi \left[ \mathcal{A}^{+,I/I}(n,N) \cap \mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{C}_3(\delta) \cap \mathcal{C}_4^* \right] \asymp \phi \left[ \mathcal{A}^{+,I/I}(n,N) \cap \mathcal{C}_3(\delta) \right].
\]

It remains to show, there exists some \( \delta > 0 \) such that

\[
\phi \left[ \mathcal{A}^{+,I/I}(n,N) \cap \mathcal{C}_3(\delta) \right] \asymp \phi \left[ \mathcal{A}^{+,I/I}(n,N) \right]. \quad (3.8)
\]

We introduce two other events: Let \( \mathcal{B} \) be the event that \( \mathcal{A}^{+,I/I}(n,N) \) occurs and the three arms do not touch \([0,6n] \times [0,4\delta n]\). Let \( \mathcal{D}(\delta) \) be the event that there is a primal-open path inside the semi-annulus \([0,4n] \times [0,2\delta] \setminus [n,3n] \times [0,\delta] \). We have the following observations.

- By the similar proof for Lemma 3.8 we have

\[
\phi \left[ \mathcal{A}^{+,I/I}(8n,N) \right] \asymp \phi \left[ \mathcal{B} \right].
\]

- By Corollary 3.4 and Proposition 3.3 we have

\[
\phi \left[ \mathcal{B} \cap \mathcal{D}(\delta) \right] \geq c(\delta) \phi \left[ \mathcal{B} \right],
\]

where \( c(\delta) > 0 \) only depends on \( \delta \).

Combining these two facts, we have

\[
\begin{align*}
\phi \left[ \mathcal{A}^{+,I/I}(n,N) \cap \mathcal{C}_3(\delta) \right] &\geq \phi \left[ \mathcal{B} \cap \mathcal{D}(\delta) \right] \\
&\geq c(\delta) \phi \left[ \mathcal{B} \right] \\
&\asymp c(\delta) \phi \left[ \mathcal{A}^{+,I/I}(8n,N) \right].
\end{align*}
\]

(by Lemma 3.8)

This completes the proof of (3.8).

\[\square\]

**Proof of Theorem 1.4.** We only give the proof for \( \alpha_4 \) and the other cases can be proved similarly.

Consider \( \Lambda_m \) with two boundary points \( a_m = (-m,-m) \) and \( b_m = (m,m) \). Fix the Dobrushin boundary condition: the edges along \( \partial \Lambda_m \) from \( b_m \) to \( a_m \) (counterclockwise) are primal-open and the dual edges along \( \partial \Lambda_m \) from \( a_m \) to \( b_m \) are dual-open. Since we fix \( q = 2, p = p_c(2) \) and the boundary condition, and \( \sigma = (1010) \), we eliminate them from the notations. We will prove that, for \( n < N \leq m/2 \),

\[
\phi_{\Lambda_m} [\mathcal{A}(n,N)] = N^{-\alpha_4+o(1)}, \quad \text{as } N \to \infty.
\]

(3.9)
Fix the landing sequence $I = (I_1, I_2, I_3, I_4)$ where
$$I_1 = [-1/2, 1/2] \times \{-1\}, \quad I_2 = \{-1\} \times [-1/2, 1/2], \quad I_3 = [-1/2, 1/2] \times \{1\}, \quad I_4 = \{1\} \times [-1/2, 1/2].$$
Recall that $\mathcal{A}^{I/I}(n, N)$ is the 1/8-well-separated arm events with the landing sequence $nI$ on $\partial \Lambda_n$ and $NI$ on $\Lambda_n$. The four arms in $\mathcal{A}(n,N)$ are denoted by $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ where $\gamma_1$ and $\gamma_3$ are primal-open and $\gamma_2$ and $\gamma_4$ are dual-open.

Let $R_1$ be the rectangle $R_1 := [-3N/4, 3N/4] \times [-2N, -N]$. Define $C_1$ to be the event that $\gamma_1$ is connected to the bottom of $R_1$ in $R_1$ by a primal-open path. Let $R_2$ be the rectangle $R_2 := [-2N, -N] \times [-3N/4, 3N/4]$. Define $C_2$ to be the event that $\gamma_2$ is connected to the left of $R_2$ in $R_2$ by a dual-open path. Let $R_3$ be the rectangle $R_3 := [-3N/4, 3N/4] \times [N, 2N]$. Define $C_3$ to be the event that $\gamma_3$ is connected to the top of $R_3$ in $R_3$ by a primal-open path. Let $R_4$ be the rectangle $R_4 := [N, 9N/8] \times [-N/2, 2N]$. Define $C_4$ to be the event that $\gamma_4$ is connected to the top of $R_4$ in $R_4$ by a dual-open path. Let $R_5$ be the rectangle $R_5 := [-3n/4, 3n/4] \times [-n, n]$. Define $C_5$ to be the event that $\gamma_1$ is connected to $\gamma_3$ in $R_5$ by a primal-open path. By Corollary 3.4, we have
$$\phi_{\Lambda_m} [A(n,N)] \asymp \phi_{\Lambda_{2N}} [A(n,N)],$$
where the constants in $\asymp$ are uniform over $n, N$ and $m \geq 2N$. By a similar proof of 3.6, we have
$$\phi_{\Lambda_{2N}} \left[ A^{I/I}(n,N) \right] \asymp \phi_{\Lambda_{2N}} \left[ \mathcal{A}^{I/I}(n,N) \cap C_1 \cap C_2^* \cap C_3 \cap C_4^* \cap C_5 \right],$$
where the constants in $\asymp$ are uniform over $n, N$.

Let $\mathbb{P}_N$ be the probability measure $\phi_{\Lambda_{2N}}$ where the square lattice is scaled by $1/N$ and let $\mathbb{P}_\infty$ be the law of SLE$_{16/3}$ in $[-2,2] \times [-2,2]$ from $(-2,-2)$ to $(2,2)$. On the event $\mathcal{A}^{I/I}(n,N) \cap C_1 \cap C_2^* \cap C_3 \cap C_4^* \cap C_5$, consider the exploration path from $a_{2N}$ to $b_{2N}$. Let $\tau_1$ be the first time that $\eta$ hits $\partial \Lambda_n$. The event $C_1 \cap C_2^*$ guarantees that $\eta[0, \tau_1]$ is bounded away from the target $b_{2N}$. The event $C_3 \cap C_5$ guarantees that, after $\tau_1$, the path $\eta$ hits the boundary at some time $\sigma_1$. The event $C_4^*$ guarantees that, after $\sigma_1$, the path $\eta$ hits $\partial \Lambda_n$ again. See Figure 3.4. Therefore, for $\epsilon > 0$, we have
$$\limsup_{N \to \infty} \mathbb{P}_N \left[ A^{I/I}(\epsilon N, N) \cap C_1 \cap C_2^* \cap C_3 \cap C_4^* \cap C_5 \right] \leq e^{c_{\epsilon} + o(1)} \leq \liminf_{N \to \infty} \mathbb{P}_N \left[ A(\epsilon N, N) \right].$$
Then we can repeat the same proof for Theorem 1.2 to obtain (3.9).

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