Forward and backward difference equations in digital signal processing

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Abstract. The relationship between difference equations with constant coefficients, obtained by establishing forward and backward differences in the filter design concepts of digital signal processing is explored in this paper. Moreover, the correlation is established between the input-output difference equations along with definite time invariant linear system and the state space difference equations related to filter design and this leads to identify the efficient adaptive filter design and validate its function. Finally, the state space illustration of a system is provided, that is acceptable to construct and validate a new developed system between the difference equation and the digital signal processor.

1. Introduction

The difference equations representing numerous algorithms in the diverse filed digital signal processing particularly in designing of digital filters, calculating SNR ratio, noise reduction, authentication issues and so on. The exploitation and illustration of difference equations with forward and backward differences equations along with spatial state space input-output difference equations are replacing various new techniques in the communication filed through new methodologies and constructive programming algorithms [2, 5, 9, 11, 13, 15]. But the problem in the input output difference equations is, the incorrectly chosen (like trial and error) values will typically result in huge errors once the rule is enforced [16].

Aberkane et al [1], discussed about the stability of discrete time varying linear stochastic systems. They arrived systems stability even in robustness period delay. Agarwal et al [3], Wang et al [14] and Long [6] were discourse the concepts of higher of difference equations like fourth order rational and functional difference equation with boundary values. The role of difference equations in digital signal processing and filter design was discussed by Smith et al [12] and A. V. Oppenheim et al [7]. Stochastic difference equations in filter design concepts are built by Budhiraja et al [4] and Sebastian Schreiber [10]. The inverse examination of the transfer function is applied within the Z transform and the filters like Adaptive filter, and the gain rule (Kalman gain) might even be accustomed to calculating the transfer function [17].

This hypothesis structures of various systems are expanded and distributed extensively and perpetually. This strategy is applied into the load balancing in circuits and systems, and is of getting elementary importance within the analysis of the properties of various systems. Kloeden et al [8] declared various hypothesis on discrete dynamical systems.
This paper is structured as follows. Section II focused the concepts of difference equations with forward and backward differences and implementation of Z transform of these equations. In section III, state space difference equations (forward and backward) and input-output difference equations are processed. The matrix and vector form of these equations are transformed and arrived optimal result of definite time invariant linear system. Finally section IV concludes the paper.

2. Approximations of forward and backward difference equations

In arithmetic, the term difference equations neighborhood is associated within the equations between addition required performance of input and actual output, expected output and the differences of actual and expected outputs. The difference equation might even be understood as results of the function:

\[ y[n + 1] = f(y[n]) \]  

Function differences were initially represented as approximations of the performance, and subsequently represent like:

\[ f^{(n)} = \lim_{\Delta x \to 0} \frac{\Delta^nf(x)}{\Delta^nx^n} \]  

Consider the following difference within the form:

\[ \Delta f[n] = f[n + 1] - f[n] \]  

This equation defines the questionable difference within the future. Throughout this method, the equation might even be outlined in backward differences of the primary order:

\[ \Delta_s f[n] = f[n] - f[n - 1] \]  

The properties of these systems measures for the foremost represented pattern of difference equations. Likewise, the properties of the various systems might even be created by treating the difference equations in an appropriate applicable form. Having an initial confidence during a stationary linear system of nth order whose properties might even be drawn from a linear difference like Sth order equation with constants:

\[ B_s \Delta^s y[n] + B_{s-1} \Delta^{s-1} y[n] + B_{s-2} \Delta^{s-2} y[n] + B_{s-3} \Delta^{s-3} y[n] + \cdots + B_1 \Delta^1 y[n] + B_0 \Delta^0 y[n] \]

\[ = A_s \Delta^s x[n] + A_{s-1} \Delta^{s-1} x[n] + A_{s-2} \Delta^{s-2} x[n] + \cdots + A_1 \Delta^1 x[n] + A_0 \Delta^0 x[n] \]

The constants \( B_0, B_1, \ldots, B_s \) and \( A_0, A_1, \ldots, A_s \) are the coefficients of the difference equation (5), and the signal \( x[n] \) represents input value of the signal and \( y[n] \) is the corresponding signal output. If the difference system has multiple inputs and outputs, then it is important to seek out an appropriate difference equations for every individual input and every output. Therefore it is suitable to perceive the initial conditions

\[ \Delta^{s-1} y[0], \Delta^{s-2} y[0], \Delta^{s-3} y[0], \cdots, \Delta y[0], y[0] \]

If we turned to apply the forward difference in keeping with (3) and to rewrite equation (5), then the final Sth order difference equations might be

\[ \Delta^s f[n] = \Delta^{s-1} f[n + 1] \Delta^{s-1} f[n] = \sum_{i=0}^{s} (-1)^{s-i} \frac{g^1}{i(s-i)} f[n + s - i] \]
The subsequent from of (5) using the affirmed concepts of (6) is
\[
\begin{align*}
&b_2y[n+s] + b_{s-1}y[n+s-1] + b_{s-2}y[n+s-2] + \cdots + b_1y[n+1] + b_0y[n] \\
&= a_sx[n+s] + a_{s-1}x[n+s-1] + a_{s-2}x[n+s-2] + \cdots + a_1x[n+1] + a_0x[n]
\end{align*}
\]
(7)

This difference equation is S-th order heterogeneous linear difference equations constant coefficients are
\[
a_i, b_i, i = 0, 1, \ldots, s-1, s
\]
and its results as follows:
\[
\begin{align*}
&y[n+s] = -\frac{b_{s-1}}{b_s}y[n+s-1] - \frac{b_{s-2}}{b_s}y[n+s-2] - \cdots - \frac{b_1}{b_s}y[n+1] - \frac{b_0}{b_s}y[n] + \frac{a_s}{b_s}x[n+s] + \\
&\frac{a_{s-1}}{b_s}x[n+s-1] + \frac{a_{s-2}}{b_s}x[n+s-2] + \cdots + \frac{a_1}{b_s}x[n+1] + \frac{a_0}{b_s}x[n]
\end{align*}
\]
(8)

The complete respond of the heterogeneous equation (7) consists of
\[
y[n] = y_h[n] + y_p[n]
\]
(9)

The general resolution \(y_h[n]\) of the consistent difference equation is obtained by finding equation (7) with zero mean square error within the initial conditions like
\[
y[s-1], y[s-2], \ldots, y[1], y[0]
\]
(10)

and the particular resolution of \(y_p[n]\) rely on exact exterior values of (7) and leads to the analytical type solution. The analytical solutions for numerous kinds of difference equations is shown, by many researchers \([4]\). For the intelligent activities of the digital signal methodology there are elements with some limitations and therefore the unilateral Z transform, is remodeled to employ
\[
F(z) = Z\{f[n]\} = \sum_{n=0}^{\infty} f[n]z^{-n} \iff f[n]
\]
(11)

In the resolution the difference equation (7) through the transform Z within the sort of (11), the subsequent property of the transform is utilized in the subsequent manner,
\[
f[n+s] \iff Z\{f[n+s]\} = z^s(F(z) - \sum_{i=1}^{s-1} f(i)z^{-i})
\]
(12)

Using the transformation Z between the sort of (11) and also by the property (12), we have
\[
Y(z) = H(z)X(z) + \frac{1}{b_z(z)} \sum_{i=0}^{S-1} \{F_i(z)y[i] - G_i(z)x[i]\}
\]
(13)

here \(Y(z)\) is that the unilateral Z transform of the signal \(y[n]\), and \(y[n] \iff Y(z)\), and \(X(z)\) is that the unilateral Z transform of the signal \(x[n]\) and \(x[n] \iff X(z)\). The polynomials \(F(z)\) and \(G(z)\)’s neighborhood elements produced
\[
\begin{align*}
F_0(z) &= b_2z^s + b_{s-1}z^{s-1} + b_{s-2}z^{s-2} + \cdots + b_2z^2 + b_1z, \\
F_1(z) &= b_3z^{s-1} + b_{s-1}z^{s-2} + b_{s-2}z^{s-3} + \cdots + b_3z^2 + b_2z, \\
F_2(z) &= b_3z^{s-2} + b_{s-1}z^{s-3} + b_{s-2}z^{s-4} + \cdots + b_3z^2 + b_2z, \\
\end{align*}
\]
\vdots
\[ F_{s-3}(z) = b_s z^3 + b_{s-1} z^2 + b_{s-2} z, \]
\[ F_{s-2}(z) = b_s z^2 + b_{s-1} z, \]
\[ F_{s-1}(z) = b_s z \]
\[ \vdots \]
\[ G_0(z) = a_s z^s + a_{s-1} z^{s-1} + a_{s-2} z^{s-2} + \ldots + a_2 z^2 + a_1 z, \]
\[ G_1(z) = a_s z^{s-1} + a_{s-1} z^{s-2} + a_{s-2} z^{s-3} + \ldots + a_3 z^2 + a_2 z, \]
\[ G_2(z) = a_s z^{s-2} + a_{s-1} z^{s-3} + a_{s-2} z^{s-4} + \ldots + a_4 z^2 + a_3 z, \]
\[ \vdots \]
\[ G_{s-3}(z) = a_s z^3 + a_{s-1} z^2 + a_{s-2} z, \quad G_{s-2}(z) = a_s z^2 + a_{s-1} z, \]
\[ G_{s-1}(z) = a_s z \]  \hspace{1cm} (14)

If the unit impulse \([n] = \delta[n]\), then \(H(z)\) is turnout and is termed by the system transfer function as like,
\[ H(z) = \frac{A_s(z)}{B_s(z)} = \frac{a_s z^s + a_{s-1} z^{s-1} + a_{s-2} z^{s-2} + \ldots + a_2 z^2 + a_1 z + a_0}{b_s z^s + b_{s-1} z^{s-1} + b_{s-2} z^{s-2} + \ldots + b_2 z^2 + b_1 z + b_0} \]  \hspace{1cm} (15)

In this manner the different stationary (time-invariant) linear dynamic systems symbolize the various digital filters like IIR (Infinite Impulse Response) and FIR (Finite Impulse Response) filters as:

(a) To restrict a signal into an endorsed recurrence band as in low-pass, high-pass, and band-pass channels.

(b) To deteriorate a signal into two or more sub-bands coders and frequency multiplexers.

(c) Adjust the signal frequencies range as in equalizers and audio graphic equalizers of a telephone channel.

(d) Modeling the system’s input-output relationship

(e) the human vocal and music synthesis as telecommunication networks and so on.

In this vogue, transfer functions typically influence \(H(z)\), like
\[ H(z) = \frac{A_s(z)}{B_s(z)} = \frac{c_0 z^s + c_{s-1} z^{s-1} + c_{s-2} z^{s-2} + \ldots + c_2 z^2 + c_1 z + c_0}{d_0 + d_{s-1} z^{-1} + d_{s-2} z^{-2} + \ldots + d_2 z^{-2} + d_1 z^{-1} + d_0} \]  \hspace{1cm} (16)

If we tend to ascertain equations (15) and (16), we get:
\[ c_0 = \frac{a_s}{b_s}, \quad c_1 = \frac{a_{s-1}}{b_s}, \quad c_2 = \frac{a_{s-2}}{b_s}, \quad \ldots, \quad c_{s-1} = \frac{a_1}{b_s}, \quad c_s = \frac{a_0}{b_s}, \]
\[ d_0 = \frac{b_s}{b_s}, \quad d_1 = \frac{b_{s-1}}{b_s}, \quad d_2 = \frac{b_{s-2}}{b_s}, \quad \ldots, \quad d_{s-1} = \frac{b_1}{b_s}, \quad d_s = \frac{b_0}{b_s} \]  \hspace{1cm} (17)
At the moment we tend to have an interest in knowing the pattern of the difference equations corresponding to a procedure that begins directly from the implementation of the transfer function (16) as:

\[ Y(z) = H(z)X(z) \]  

(18)

After switch over the performance of the transfer function (16) into (18), the inverse Z transform victimizing the remodeling like

\[
d_0 y[n] + d_1 y[n-1] + d_2 y[n-2] + \cdots + d_{s-1} y[n-s+2] + d_s y[n+s] = c_0 x[n] + c_1 x[n-1] + c_2 x[n-2] + \cdots + c_{s-1} x[n-s+2] + c_s x[n+s]
\]  

(19)

or once exchange the coefficients \( a_i \) and \( b_i \):

\[
b_0 y[n] + b_{s-1} y[n-1] + b_{s-2} y[n-2] + \cdots + b_2 y[n-s+2] + b_1 y[n-s+1] + b_0 y[n-s] = a_0 x[n] + a_{s-1} x[n-1] + a_{s-2} x[n-2] + \cdots + a_2 x[n-s+2] + a_1 x[n-s+1] + a_0 x[n-s]
\]  

(20)

The continual equations are obtained by rewriting (20) as,

\[
y[n] = -\frac{b_{s-1}}{b_s} y[n-1] - \frac{b_{s-2}}{b_s} y[n-s+2] - \cdots - \frac{b_1}{b_s} y[n-s+1] - \frac{b_0}{b_s} y[n-s] + \frac{a_{s-1}}{b_s} x[n] + \frac{a_{s-2}}{b_s} x[n-2] + \cdots + \frac{a_2}{b_s} x[n-s+2] + \frac{a_1}{b_s} x[n-s+1] + \frac{a_0}{b_s} x[n-s]
\]  

(21)

The initial conditions of the complete solution of the heterogeneous equation between (20) and (21) involves

\[
y[-1], y[-2], \ldots, y[-s+1], y[-s]
\]  

(22)

This leads to drive the equations in linear differences with backward differences (4), which is an appropriate of space that's usually employed in digital filter theory. In a similar manner we can do forward differences. As a remark these forward and backward input output difference equations were used to design various digital filters in the field of digital signal processing.

3. Embedding State space difference equations along with input-output difference equations

In the heterogeneous difference equations between the explicit interval of input output difference equations say (5), (7) and (8) exclusively remains the correlation between the signal \( x[n] \) and also the signal \( y[n] \) of an individual system. The form of the input-output difference equation is acceptable for writing algorithms as a result of its trust worthysolutions to ascertain even for the sub-bands coders and frequency multiplexers. These initial conditions distributed over a period of time and so influence the optimal strategy.

In the filter design, certainly, it's not usual way to do an analysis of the input output difference equations and in most cases, the difference equations are solved within the domain of constraints. This model is then known as the state space model, since the results could arrive the values of the inner state space variables along with its moment and accuracy in design and the way of implementation by the communication engineers. This permits the mathematicians to focus on a very easy methodology to clarify the outline of the inner state space of the input output difference equations. These kinds of state variables can be declare like

\[
v_1[n] = b_s y[n] - a_s x[n]
\]
\[ v_2[n] = b_{s-1} y[n] + b_s y[n+1] - a_{s-1} x[n] - a_s x[n+1] \]
\[ v_3[n] = b_{s-2} y[n] + b_{s-1} y[n+1] + b_s y[n+2] - a_{s-2} x[n] - a_{s-1} x[n+1] - a_s x[n+2] \]
\[ \vdots \]
\[ v_s[n] = b_1 y[n] + b_2 y[n+1] + b_3 y[n+2] + \cdots + b_{s-1} y[n+s-2] + b_s y[n+s-1] - a_1 x[n] - a_2 x[n+1] - a_3 x[n+2] + \cdots - a_{s-1} x[n+s-2] - a_s x[n+s-1] \]
\[ v_{s+1}[n] = b_0 y[n] + b_1 y[n+1] + b_2 y[n+2] + \cdots + b_s y[n+s-1] + b_{s+1} y[n+s] - a_0 x[n] - a_1 x[n+1] - a_2 x[n+2] + \cdots - a_{s-1} x[n+s-1] - a_s x[n+s] \]

From (23) we calculate the output signal as
\[ y[n] = \frac{1}{b_s} v_1[n] + \frac{a_s}{b_s} x[n] \] (24)

and later the subsequent input output state space equations can be rewritten like,
\[ v_1[n+1] = v_2[n] - \frac{b_{s-1}}{b_s} v_1[n] + \left( a_{s-1} - b_{s-1} \frac{a_s}{b_s} \right) x[n], \]
\[ v_2[n+1] = v_3[n] - \frac{b_{s-2}}{b_s} v_1[n] + \left( a_{s-2} - b_{s-2} \frac{a_s}{b_s} \right) x[n], \]
\[ \vdots \]
\[ v_{s-1}[n+1] = v_s[n] - \frac{b_1}{b_s} v_1[n] + \left( a_1 - b_1 \frac{a_s}{b_s} \right) x[n], \]
\[ v_s[n+1] = -\frac{b_0}{b_s} v_1[n] + \left( a_0 - b_0 \frac{a_s}{b_s} \right) x[n], \] (25)

Extend these in terms of matrix, we get
\[ v[n+1] = A v[n] + B x[n], \] (26)

where the unit space is delineate in the vector forms follows:
\[ v[n+1] = [v_s[n+1] v_{s-1}[n+1] \ldots v_2[n+1] v_1[n+1]]^T \]
\[ \text{and } v[n] = [v_s[n] v_{s-1}[n] \ldots v_2[n] v_1[n]]^T. \]

Hence the dominated matrices A and B can be written like
Matrix notation for the state space input output difference equations have the advantage of constructing several inputs and outputs which is used to construct and develop the filter in an efficient manner. Now consider the $y[n]$ signal and drive the slope using matrix form like:

$$y[n] = C \cdot v[n] + D \cdot x[n]$$

(28)

the matrices $C$ and $D$ surfaces among the module:

$$C = [0 \ 0 \ 0 \ ... \ 0 \ 1], \quad D = \frac{a_x}{b_x}$$

These state space representation of difference equations provides the powerful representation of a definite time-invariant linear system with many totally different types of inputs and outputs. Applying the following possible initial conditions to state space vector $v[n]$,

$$v[0] = [v_5 \ 0 \ v_{s-1} \ 0 \ ... \ v_2 \ 0 \ v_1 \ 0]^T$$

To follow this dynamic modification between the various properties of the system reaches of the resultant instant of its own. The values of the signal are determined at any time throughout the retardation itself according to the state space vector $v[n]$.

4. Result and Discussion

It is ought to be noted that the input output state space difference equations among the separate systemstowards definite time invariant linear system. Forward and backward difference equations along with many variable coefficients are used to architect a digital filter and validate the functionality of the filter. The shape of the impulse responses corresponds to the general to the transfer function explores the state space input output difference equations. Heterogeneous linear difference equations with constant coefficients are considered and validated with huge inputs and outputs.

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