Abstract. We derive sharper probabilistic concentration bounds for the Monte Carlo Empirical Rademacher Averages (MCERA), which are proved through recent results on the concentration of self-bounding functions. Our novel bounds allow obtaining sharper bounds to (Local) Rademacher Averages. We also derive novel variance-aware bounds for the special case where only one vector of Rademacher random variables is used to compute the MCERA. Then, we leverage the framework of self-bounding functions to derive novel probabilistic bounds to the supremum deviations, that may be of independent interest.

Keywords: rademacher complexity, statistical learning theory, self-bounding functions, concentration inequalities.

1 Introduction

Uniform convergence is a central problem in statistical learning theory (Vapnik, 1998). Obtaining tight and uniformly valid probabilistic bounds on the accuracy of empirical averages of sets of functions is a fundamental problem, with widespread and impactful applications in Machine Learning and Data Science (Mitzenmacher and Upfal, 2017; Shalev-Shwartz and Ben-David, 2014). Probabilistic bounds on the largest error of the empirical averages are typically obtained by adding to the empirically estimated error a term that depends on the complexity of the functions. Both distribution-free concepts of complexity, such as the VC dimension (Vapnik and Chervonenkis, 1971), and distribution and data-dependent complexities have been proposed as breakthroughs with great success for this problem. One of the most interesting notions of data-dependent measure of complexity of sets of functions is the Rademacher Complexity, extensively studied in the context of uniform convergence by Koltchinskii and Panchenko (2000), Bartlett et al. (2002), and others.

A drawback of Rademacher Averages is that the “global” error that can be obtained is characterised by the so called “slow” convergence rate of $O(m^{-1/2})$, where $m$ is the number of analysed samples; while such rate is essentially the best possible when some elements of a set of function $F$ achieve maximum variance (Boucheron et al., 2005), it may be substantially improved for the other functions, that are often more interesting to the analysis. Therefore, a rich collection of contributions (Bartlett et al., 2005; Bousquet et al., 2002; Koltchinskii, 2006; Koltchinskii and Panchenko, 2000; Massart, 2000; Mendelson, 2002) have then focused on providing local estimates of the complexity, restricting the estimation to a proper subset of $F$ that contains only functions with lower variance. This operation is also known as “peeling”, and it is often combined with scaling the elements of $F$ with weights that are functions of their variance. In such settings, one would hope to achieve sharper error bounds, with rates between $O(m^{-1/2})$ and $O(m^{-1})$.

The slow convergence rate is due to both the “global” nature of Rademacher Averages and from the application of probabilistic concentration inequalities based on the method of bounded differences, that is essentially tight only when there are elements of the set of functions under consideration that achieve maximum variance (Boucheron et al., 2013). Therefore, the study of novel concentration inequalities for the supremum of empirical processes that take advantage of smaller bounds to the variance has been a central focus of research, such as the fundamental contributions of Talagrand (1994) and many others (Boucheron et al., 2005, 2013; Bousquet, 2002).
Recent methods relying on Rademacher Averages compute *deterministic* upper bounds to it, often though results such as Massart’s Lemma (Massart, 2000). An alternative, often much sharper, approach is to **directly estimate** the Rademacher Averages with the **n-Monte Carlo Empirical Rademacher Average** (n-MCERA) (defined formally in the next Section); this quantity is computed by sampling a matrix of \( n \times m \) Rademacher random variables and then computing the corresponding \( n \) supremums over the elements of \( F \) on the \( m \) samples (Bartlett and Mendelson, 2002), and then obtaining a probabilistic upper bound to the Rademacher Complexity with concentration of measure inequalities.

Recently, [De Stefani and Upfal, 2019] used this idea to obtain error bounds in an adaptive setting: in their scenario, batch of functions are considered at successive steps, while allowing the choice of the functions to process at every iteration to be based on past information. In their case, the issue of efficient computation of the supremums is alleviated by the fact that only few functions are considered at every new iteration of the analysis, resulting in only a linear computational overhead. In other situations, in particular when the size of \( F \) is large, it is generally more expensive to compute such supremums, and it is probably the reason why this approach has not gained widespread practical consideration.

In the context of Data Mining and approximate pattern mining, [Pellegrina et al., 2020] address this challenge deriving a general and practical scheme to compute such supremums that exploits the combinatorial structure of \( F \) in a branch-and-bound strategy. In all these applications, it is critical to apply sharp concentration results to have tight error rates.

These works achieve error bounds that relate the n-MCERA to its expectation using concentration inequalities based on the bounded difference property (or, equivalently, assuming maximum variance); for this reason, such error bounds are characterised by the same slow rate of \( \mathcal{O}(nm^{-1/2}) \). While, in theory, one could use an arbitrary large number \( n \) of vectors of Rademacher random variables, and in particular \( n = m \) to achieve \( \mathcal{O}(m^{-1}) \) error rates, this would imply the computation of a large number of supremums over \( F \), something impractical in almost all situations.

The question of whether the n-MCERA can be tightly estimated without using an impractically large number of Monte Carlo trials is an unexplored question. In fact, it is not clear whether the n-MCERA enjoys the same sharp variance-aware concentration properties that have been recently leveraged to estimate other quantities involves in localised notion of complexities.

**Our contributions.** The main goal of this work is to provide a positive answer to this question: in Section 2 we derive novel concentration bounds for the n-MCERA whose convergence rates depend on characteristic quantities computable from the data, such as the empirical wimpy-variance of the set of functions, resulting in a significantly improved trade-off between the accuracy of the estimate and the number \( n \) of required vectors of Rademacher random variables. Our proofs rely on the concepts of self-bounding functions; the novel concentration inequalities we prove follow from the sharp exponential concentration inequalities that self-bounding functions have been shown to satisfy (Boucheron et al., 2009). The new bounds we derive in this work are relevant to all methods based on n-MCERA we described before and, given their generality, possibly others. In particular, we believe it would be interesting to fit our results in the framework of Localised Rademacher Averages, and that there are interesting new algorithmic applications of the n-MCERA that may benefit from our results, in particular in problems already tackled with methods based on Rademacher Averages.

Another interesting question we explore is whether the maximum difference between empirical averages and their expectation, quantities often denoted by Supremum Deviations (SDs), satisfy some form of self-bounding properties. Indeed, in Section 8 we show that the SDs are also self-bounding, for appropriate constants that depend on the maximum and minimum expected values of the functions in \( F \); consequently, we derive novel concentration inequalities for the SDs, that may be of independent interest.

Lastly, in Section 9 we study the interesting case of \( n = 1 \), and show that it is possible to directly and tightly bound the Rademacher Complexity from its estimate 1-MCERA with a single application of a novel variance-aware concentration inequality.

## 2 Preliminaries

Let \( F \) be a class of real valued functions from a domain \( \mathcal{X} \) to the bounded interval \( [a, b] \subset \mathbb{R} \), and let \( z = \max\{|a|, |b|\} \) and \( c = |b - a| \), with \( b > 0 > a \), and \( c, z > 0 \). To simply address non-negativity issues, we assume w.l.o.g. that \( F \) contains a constant function \( f_0 \) such that \( f_0(x) = 0 \), for all \( x \in \mathcal{X} \).
Let a sample $S$ be a bag $\{s_1, \ldots, s_m\}$ of size $m$, such that $s \in \mathcal{X}, \forall s \in S$. We assume that each element of $S$ is drawn i.i.d. from $\mathcal{X}$ according to an unknown probability distribution $\mu$. Our goal is to derive tight bounds on the difference between the average value of $f$, computed on the sample $S$, and its expectation $\mathbb{E}[f]$, taken w.r.t. $\mu$, that are valid for all functions $f \in \mathcal{F}$. More formally, we define the positive Supremum Deviation (SD) $D^+(\mathcal{F}, S)$ and the negative supremum deviation $D^-(\mathcal{F}, S)$ as

$$D^+(\mathcal{F}, S) = \sup_{f \in \mathcal{F}} \left\{ \sum_{i=1}^{m} f(s_i) - \mathbb{E}[f] \right\},$$

$$D^-(\mathcal{F}, S) = \sup_{f \in \mathcal{F}} \left\{ \mathbb{E}[f] - \sum_{i=1}^{m} f(s_i) \right\}.$$  

As $\mu$ is unknown, it is not possible to directly compute such supremum deviations. However, fundamental results from statistical learning theory allow to obtain probabilistic upper bounds to them only using information obtainable from $S$. We now introduce the notion of Rademacher Averages, that will be instrumental to achieve this goal.

First, let $\sigma$ be a $n \times m$ matrix such that each component $\sigma_{i,j}$ of index $(i, j)$ is either 1 or $-1$. The n-Monte Carlo Empirical Rademacher Average (n-MCERA) $\hat{R}^n_m(\mathcal{F}, S, \sigma)$ is defined as

$$\hat{R}^n_m(\mathcal{F}, S, \sigma) = \frac{1}{n} \sum_{j=1}^{n} \sup_{f \in \mathcal{F}} \frac{1}{m} \sum_{i=1}^{m} \sigma_{j,i} f(s_i).$$

Denote the Empirical Rademacher Average (ERA) $\hat{R}(\mathcal{F}, S)$ as the expectation of the n-MCERA w.r.t. the assignments of the Rademacher random variables $\sigma$, where each $\sigma_{i,j}$ is 1 or $-1$ independently and with equal probability:

$$\hat{R}(\mathcal{F}, S) = \mathbb{E}[\hat{R}^n_m(\mathcal{F}, S, \sigma)].$$

Then, denote the Rademacher Complexity (RC) $R(\mathcal{F}, m)$ as the expectation of the ERA over $S$,

$$R(\mathcal{F}, m) = \mathbb{E}_S [\hat{R}(\mathcal{F}, S)].$$

The following fundamental result, also known as “Symmetrization lemma”, show a precise relationship between the RC and the expected supremum deviation (Mitzenmacher and Upfal, 2017; Shalev-Shwartz and Ben-David, 2014).

**Lemma 1.**

$$\mathbb{E}_S [D^+(\mathcal{F}, S)] \leq 2R(\mathcal{F}, m),$$

$$\mathbb{E}_S [D^-(\mathcal{F}, S)] \leq 2R(\mathcal{F}, m).$$

Therefore, upper bounding the RC yields upper bounds on the expected supremum deviations; consequently, one can obtain a probabilistic upper bound on the supremum deviations on the sample $S$ with the application of concentration inequalities, important tools of probability theory. Most importantly, the RC can be estimated directly on the available data using the n-MCERA. In the next Sections we succinctly introduce the most widely used concentration inequalities methods: in Section 3 we introduce the method of bounded differences; in Section 4 we present the definitions and recent results on self-bounding functions. The concept of self-bounding functions, as we will discuss later, are essential to prove our novel bounds. We remand for a more exhaustive coverage of the topic to the book of Boucheron et al. (2013).

### 3 The Method of Bounded Differences

Let $X = (X_1, \ldots, X_n)$ be a vector of variables $X_i$, each taking values in a measurable set $\mathcal{X}$ and let $g : \mathcal{X}^n \rightarrow \mathbb{R}$ be a measurable function. We now introduce the bounded difference property, that is often easy to prove in many applications.
Definition 1 (Bounded difference property). A function \( g \) has the bounded difference property if, for each \( i, 1 \leq i \leq m \), there is a nonnegative constant \( c_i \) such that:
\[
\sup_{X_1, \ldots, X_m} \left| g(X_1, \ldots, X_m) - g(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_m) \right| \leq c_i .
\]

A central result is given by the following Theorem, that shows that \( g(X) \) is well concentrated around its mean \( \mathbb{E}[g(X)] \) (taken w.r.t. \( X \)), and that the rate of convergence depends on the constants \( c_i \) of the bounded difference property.

Theorem 1 (McDiarmid (1989)). Let \( g : \mathcal{X}^m \rightarrow \mathbb{R} \) be a function with the bounded difference property with constants \( c_i \), for \( 1 \leq i \leq m \). Let \( X_1, \ldots, X_m \) be \( m \) independent random variables taking value in \( \mathcal{X}^m \), and let \( Z = g(X) \). Then it holds
\[
\Pr(Z \geq \mathbb{E}[Z] + t) \leq \exp \left( -\frac{2t^2}{\sum_{i=1}^m c_i^2} \right) .
\]
Also, it holds
\[
\Pr(Z \leq \mathbb{E}[Z] - t) \leq \exp \left( -\frac{2t^2}{\sum_{i=1}^m c_i^2} \right) .
\]

4 Self-Bounding Functions

Self-bounding functions are an important class of functions that satisfy usually easy-to-check conditions, that imply sharp concentration inequalities of their empirical estimates w.r.t. their expected values. We report their definitions and remand to Boucheron et al. (2013) a more in-depth exposition of the subject.

Let \( X = (X_1, \ldots, X_n) \) be a vector of variables \( X_i \), each taking values in a measurable set \( \mathcal{X} \) and let \( g : \mathcal{X}^n \rightarrow \mathbb{R} \) be a non-negative measurable function. Then denote \( g_i \) a function from \( \mathcal{X}^{n-1} \rightarrow \mathbb{R} \). In the following definition, we introduce \((\alpha, \beta)\)-self-bounding functions; we note that they may also be denoted by strongly \((\alpha, \beta)\)-self-bounding functions.

Definition 2 ((\(\alpha, \beta\))-self-bounding function). A function \( g \) is a \((\alpha, \beta)\)-self-bounding function if, for all \( X \in \mathcal{X}^n \),
\[
0 \leq g(X) - g_i(X^{(i)}) \leq 1 ,
\]
and
\[
\sum_{i=1}^n \left( g(X) - g_i(X^{(i)}) \right) \leq \alpha g(X) + \beta ,
\]
where \( X^{(i)} = (X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n) \in \mathcal{X}^{n-1} \) is obtained by dropping the \( i \)-th component of \( X \).

An often convenient choice of \( g_i \) to prove that \( g \) is self-bounding is
\[
g_i(X^{(i)}) = \inf_{X' \in \mathcal{X}} g \left( X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n \right) .
\]

We now introduce weakly \((\alpha, \beta)\)-self-bounding function.

Definition 3 (Weakly \((\alpha, \beta)\)-self-bounding function). A function \( g \) is weakly \((\alpha, \beta)\)-self-bounding if, for all \( X \in \mathcal{X}^n \),
\[
\sum_{i=1}^n \left( g(X) - g_i(X^{(i)}) \right)^2 \leq \alpha g(X) + \beta .
\]
Note that a \((\alpha, \beta)\)-self-bounding function is also a weakly \((\alpha, \beta)\)-self-bounding function.

The next Theorem shows that if \(g\) is self-bounding, then it enjoys sharp concentration inequalities between its expectation \(E[g(X)]\) (taken w.r.t. \(X\)) and its value \(g(X)\).

**Theorem 2 (Boucheron et al. (2009)).** Let \(X = (X_1, \ldots, X_n)\) be a vector of independent random variables, each taking values in a measurable set \(\mathcal{X}\) and let \(g : \mathcal{X}^n \to \mathbb{R}\) be a non-negative measurable function such that \(Z = g(X)\) has finite mean \(E[Z] < +\infty\). Let \(\alpha, \beta \geq 0\), and define \(\nu = (3\alpha - 1)/6\). Denote \((\nu)_+ = \max\{\nu, 0\}\) and \((\nu)_- = \max\{-\nu, 0\}\).

If \(g\) is \((\alpha, \beta)\)-self-bounding, then for all \(t > 0\),
\[
\Pr(Z \geq E[Z] + t) \leq \exp\left(-\frac{t^2}{2(\alpha E[Z] + \beta + (\nu)_+)}\right).
\]

If \(g\) is weakly \((\alpha, \beta)\)-self-bounding and for all \(i \leq n\), all \(x \in \mathcal{X}\), \(g_i(X^{(i)}) \leq g(x)\), then for all \(t > 0\),
\[
\Pr(Z \geq E[Z] + t) \leq \exp\left(-\frac{t^2}{2(\alpha E[Z] + \beta + \alpha t/2)}\right).
\]

If \(g\) is weakly \((\alpha, \beta)\)-self-bounding and \(g(X) - g_i(X^{(i)}) \leq 1\) for each \(i \leq n\) and \(x \in \mathcal{X}^n\), then for \(0 < t \leq E[Z]\),
\[
\Pr(Z \leq E[Z] - t) \leq \exp\left(-\frac{t^2}{2(\alpha E[Z] + \beta + (\nu)_-)\mu}\right).
\]

Moreover, if \(g\) is weakly \((\alpha, 0)\)-self-bounding with \(0 \leq g(X) - g_i(X^{(i)}) \leq 1\) for all \(i \leq n\) and \(X \in \mathcal{X}^n\), then
\[
\Pr(Z \leq E[Z] - t) \leq \exp\left(-\frac{t^2}{2\max\{1, \alpha E[Z]\}}\right).
\]

# 5 Standard probabilistic bounds

In this section we report standard bounds to the ERA and the SDs using the bounded difference property, and a standard bound for the RC based on the self-bounding property of the ERA. Such bounds are typically combined together with a union bound.

## 5.1 Standard Probabilistic Bound to the ERA

The first standard probabilistic upper bound to the ERA from its estimate given by the \(n\)-MCERA is obtained through the application of the bounded differences method.

**Theorem 3.**
\[
\Pr\left(\hat{R}(\mathcal{F}, \mathcal{S}) \geq \hat{R}_m(\mathcal{F}, \mathcal{S}, \sigma) + \varepsilon\right) \leq \exp\left(-\frac{nm\varepsilon^2}{2\varepsilon^2}ight).
\]

**Proof.** First, it is simple to prove that \(\hat{R}_m(\mathcal{F}, \mathcal{S}, \sigma)\) has the bounded difference property with constants \(c_i = 2z(nm)^{-1}\), for all \(1 \leq i \leq nm\). Therefore, the bound follows from Theorem [1] \(\square\)

## 5.2 Standard Probabilistic Bounds to the SDs

Here we prove standard bounds on the Supremum Deviations using the bounded difference property.

**Theorem 4.** Let \(Z = \sup_{f \in \mathcal{F}} \left\{\frac{1}{m} \sum_{j=1}^{n} f(s_j) - E[f]\right\}\). Then, it holds
\[
\Pr(Z \geq E[Z] + \varepsilon) \leq \exp\left(-\frac{2m\varepsilon^2}{\varepsilon^2}\right).
\]

The same holds for \(Z = \sup_{f \in \mathcal{F}} \left\{E[f] - \frac{1}{m} \sum_{j=1}^{n} f(s_j)\right\}\).

**Proof.** It is simple to show that \(Z\) has the bounded difference property with constants \(c_i = c/m\), for all \(1 \leq i \leq m\). Thus, the bounds follows from Theorem [1] \(\square\)
\section{Standard probabilistic bounds to the RC}

A known property of the ERA is that it is a self-bounding function (see, for instance, Example 3.12 of \cite{boucheron2013concentration}). This implies concentration bounds, proved by \cite{boucheron2000concentration}, that are often sharper than the ones obtained through the bounded difference property.

**Theorem 5.** Let, for \( x \geq -1 \), \( h(x) = (1 + x) \log(1 + x) - x \). For all \( 0 < \varepsilon \leq R(F, m) \), it holds
\[
\Pr \left( R(F, m) \geq \hat{R}(F, S) + \varepsilon \right) \leq \exp \left( - \frac{mR(F, m)}{c} h \left( - \frac{\varepsilon}{R(F, m)} \right) \right) \leq \exp \left( - \frac{m^2 \varepsilon}{2cR(F, m)} \right). \tag{6}
\]

Also, with probability \( \geq 1 - \delta \), it holds
\[
R(F, m) \leq \hat{R}(F, S) + \frac{1}{2m} \left( \sqrt{c \left( 4m \hat{R}(F, S) + c \ln \frac{1}{\delta} \right)} \ln \frac{1}{\delta} + c \ln \frac{1}{\delta} \right). \tag{7}
\]

**Proof.** Equation \( \eqref{eq:6} \) follows from the self-bounding property of the ERA, and therefore follows from Theorem 2.1 of \cite{boucheron2000concentration} (see also Theorem 6.12 of \cite{boucheron2013concentration}). Equation \( \eqref{eq:7} \) is a generalisation of Theorem 3.11 of \cite{oneto2013self}, presented by \cite{pellegrina2020self}.

\section{New probabilistic bounds to the ERA}

In this Section we show that a careful application of recent results for self-bounding functions allows to prove novel bounds to the ERA from the \( n \)-MCERA, whose convergence rates depend on usually easy-to-compute functions of the elements of \( F \) on the sample \( S \). In Section \ref{subsec:6.1} we show that the \( n \)-MCERA is, in fact, \((\alpha, \beta, \beta')\)-self-bounding for appropriate values of \( \alpha, \beta, \beta' \). In Section \ref{subsec:6.2} we show that this implies novel probabilistic bounds to the ERA.

\subsection{Self-bounding properties of the \( n \)-MCERA}

Define \( \nu_F \) as
\[
\nu_F \triangleq \frac{1}{m} \sup_{f \in F} \left\{ \sum_{i=1}^{m} |f(s_i)| \right\}.
\]

**Theorem 6.** Let the \( n \times m \) matrix \( \sigma \in \{-1, 1\}^{n \times m} \), and define the function \( g(\sigma) \) as
\[
g(\sigma) \doteq \frac{n\hat{R}_m(F, S, \sigma)}{m}. \tag{8}
\]

If \( z \leq 1/2 \), then \( g(\sigma) \) is a \((1, nm\nu_F)\)-self-bounding function.

**Proof.** Denote the function \( g_{j, i}(\sigma) \), for \( j \in [1, n] \) and \( i \in [1, m] \), as
\[
g_{j, i}(\sigma) \doteq \inf_{\sigma_{j, i} \in \{-1, 1\}} \left\{ \sum_{\substack{u=1 \atop u \neq j}}^{n} \left[ \sup_{f \in F} \sum_{h=1}^{m} \sigma_{j, h} f(s_h) \right] + \sup_{f \in F} \left\{ \sum_{\substack{h=1 \atop h \neq i}}^{m} (\sigma_{j, h} f(s_h) + \sigma'_{j, i} f(s_i)) \right\} \right\}.
\]

This function correspond to \( g(\sigma) \) where the element \( \sigma_{j, i} \) of coordinates \((i, j)\) of \( \sigma \) is replaced by \( \sigma'_{j, i} \in \{-1, 1\} \); in addition, we take the infimum over \( \sigma'_{j, i} \in \{-1, 1\} \). We remark that, even if \( \sigma \) is the argument of \( g_{j, i} \), to simplify notation, \( \sigma_{j, i} \) never appears in the definition of \( g_{j, i}(\sigma) \), as required in the definition of self-bounding functions. To show that \( g(\sigma) \) is \((\alpha, \beta)\)-self-bounding, according to the definition, we have to show that, for all \( \sigma \in \{-1, 1\}^{n \times m} \), the inequalities
\[
0 \leq g(\sigma) - g_{j, i}(\sigma) \leq 1,
\]

\section{6.1 Self-bounding properties of the \( n \)-MCERA
and
\[\sum_{j=1}^{n} \sum_{i=1}^{m} (g(\sigma) - g_{j,i}(\sigma)) \leq \alpha g(\sigma) + \beta \] (8)
all hold for some non-negative \(\alpha\) and \(\beta\). First, \(g(\sigma) \geq g_{j,i}(\sigma)\) follows from writing \(g_{j,i}(\sigma)\) as
\[g_{j,i}(\sigma) = \min \left\{ \sum_{v=1}^{n} \sup_{f \in \mathcal{F}} \sum_{h=1}^{m} \sigma_{v,h} f(s_h) + \sup_{f \in \mathcal{F}} \sum_{h=1}^{m} (\sigma_{j,h} f(s_h) - f(s_i)) \right\} \]
and from the observation that one argument of the min is equal to \(g(\sigma)\), therefore the minimum is either equal to \(g(\sigma)\) or \(< g(\sigma)\). We now prove that, if \(z \leq 1/2\), \(g(\sigma) \leq g_{j,i}(\sigma) + 1\), for all \(\sigma\) and for all \(j\) and \(i\).

\[g_{j,i}(\sigma) = \inf_{\sigma_{j,i} \in \{-1,1\}} \left\{ \sum_{v=1}^{n} \sup_{f \in \mathcal{F}} \sum_{h=1}^{m} \sigma_{v,h} f(s_h) + \sup_{f \in \mathcal{F}} \sum_{h=1}^{m} (\sigma_{j,h} f(s_h) + \sigma_{j,i} f(s_i)) \right\} \]
\[= \sum_{v=1}^{n} \sup_{f \in \mathcal{F}} \sum_{h=1}^{m} \sigma_{v,h} f(s_h) + \inf_{\sigma_{j,i} \in \{-1,1\}} \left\{ \sum_{h=1}^{m} (\sigma_{j,h} f(s_h) + \sigma_{j,i} f(s_i)) \right\} \]
\[\geq \sum_{v=1}^{n} \sup_{f \in \mathcal{F}} \sum_{h=1}^{m} \sigma_{v,h} f(s_h) + \sup_{f \in \mathcal{F}} \sum_{h=1}^{m} (\sigma_{j,h} f(s_h)) + \inf_{\sigma_{j,i} \in \{-1,1\}} \{\sigma_{j,i} f(s_i)\} \]
\[= \sum_{v=1}^{n} \sup_{f \in \mathcal{F}} \sum_{h=1}^{m} \sigma_{v,h} f(s_h) + \sup_{f \in \mathcal{F}} \sum_{h=1}^{m} (\sigma_{j,h} f(s_h)) + \inf_{\sigma_{j,i} \in \{-1,1\}} \{\sigma_{j,i} f(s_i)\} \]

Let \(f_j^*\) be one of the functions of \(\mathcal{F}\) attaining the supremum of \(\sup_{f \in \mathcal{F}} \sum_{h=1}^{m} \sigma_{j,h} f(s_h)\). Then we continue
\[g_{j,i}(\sigma) \geq \sum_{v=1}^{n} \sup_{f \in \mathcal{F}} \sum_{h=1}^{m} \sigma_{v,h} f(s_h) + \sup_{f \in \mathcal{F}} \sum_{h=1}^{m} (\sigma_{j,h} f(s_h)) + \inf_{\sigma_{j,i} \in \{-1,1\}} \{\sigma_{j,i} f(s_i)\} \]
\[\geq \sum_{v=1}^{n} \sup_{f \in \mathcal{F}} \sum_{h=1}^{m} \sigma_{v,h} f(s_h) + \sum_{h=1}^{m} (\sigma_{j,h} f_j^*(s_h)) + \inf_{\sigma_{j,i} \in \{-1,1\}} \{\sigma_{j,i} f_j^*(s_i)\} \]
\[= \sum_{v=1}^{n} \sup_{f \in \mathcal{F}} \sum_{h=1}^{m} \sigma_{v,h} f(s_h) + \sum_{h=1}^{m} (\sigma_{j,h} f_j^*(s_h)) + \sigma_{j,i} f_j^*(s_i) - \sigma_{j,i} f_j^*(s_i) + \inf_{\sigma_{j,i} \in \{-1,1\}} \{\sigma_{j,i} f_j^*(s_i)\} \]
\[= g(\sigma) - \sigma_{j,i} f_j^*(s_i) + \inf_{\sigma_{j,i} \in \{-1,1\}} \{\sigma_{j,i} f_j^*(s_i)\} \] .
(9)

We first observe that
\[\inf_{\sigma_{j,i} \in \{-1,1\}} \{\sigma_{j,i} f_j^*(s_i)\} = \begin{cases} 0, & \text{if } f_j^*(s_i) = 0, \\ -f_j^*(s_i), & \text{if } f_j^*(s_i) > 0, \\ f_j^*(s_i), & \text{if } f_j^*(s_i) < 0, \end{cases} \]
obtaining
\[
\inf_{\sigma_{j,i} \in \{-1,1\}} \{\sigma_{j,i} f_j^*(s_i)\} = -|f_j^*(s_i)|.
\]

Therefore, we continue from (9) as follows:
\[
g_{j,i}(\sigma) \geq g(\sigma) - \sigma_{j,i} f_j^*(s_i) - |f_j^*(s_i)| \geq g(\sigma) - 2z \geq g(\sigma) - 1.
\]

We now prove (5) for \(\alpha = 1\) and \(\beta = nm\nu_F\).

\[
\sum_{j=1}^n \sum_{i=1}^m (g(\sigma) - g_{j,i}(\sigma)) \\
\leq \sum_{j=1}^n \sum_{i=1}^m (g(\sigma) - g(\sigma) + \sigma_{j,i} f_j^*(s_i) + |f_j^*(s_i)|) \\
= \sum_{j=1}^n \sum_{i=1}^m (\sigma_{j,i} f_j^*(s_i) + |f_j^*(s_i)|) \\
= g(\sigma) + \sum_{j=1}^n \sum_{i=1}^m |f_j^*(s_i)| \\
\leq g(\sigma) + n \sup_{f \in F} \left\{ \sum_{i=1}^m |f(s_i)| \right\} \\
= g(\sigma) + n \nu_F,
\]

concluding the proof.

Define \(\sigma_F^2\) as
\[
\sigma_F^2 \doteq \frac{1}{m} \sup_{f \in F} \left\{ \sum_{i=1}^m f(s_i)^2 \right\}.
\]

**Theorem 7.** Let the \(n \times m\) matrix \(\sigma \in \{-1,1\}^{n \times m}\), and define the function \(g(\sigma)\) as
\[
g(\sigma) \doteq n m \tilde{R}_m^n(F, S, \sigma).
\]

Then \(g(\sigma)\) is a weakly \((2z, 2nm\sigma_F^2)\)-self-bounding function.

**Proof.** Denote \(g_{j,i}(\sigma)\) as in the proof of Theorem 6. To prove that \(g(\sigma)\) is weakly \((\alpha, \beta)\)-self-bounding, we have to prove that, for all \(\sigma\), it holds
\[
\sum_{j=1}^n \sum_{i=1}^m (g(\sigma) - g_{j,i}(\sigma))^2 \leq \alpha g(\sigma) + \beta.
\]

From the proof of Theorem 6, we have already proved that
\[
g_{j,i}(\sigma) \geq g(\sigma) - \sigma_{j,i} f_j^*(s_i) - |f_j^*(s_i)| \geq g(\sigma) - 2z.
\]
Therefore, we observe that
\[
\sum_{j=1}^{n} \sum_{i=1}^{m} (g(\sigma) - g_{j,i}(\sigma))^2 \\
\leq \sum_{j=1}^{n} \sum_{i=1}^{m} \left( \sigma_{j,i} f_j^*(s_i) + |f_j^*(s_i)| \right)^2 \\
= \sum_{j=1}^{n} \sum_{i=1}^{m} \left( f_j^*(s_i)^2 + |f_j^*(s_i)|^2 + 2\sigma_{j,i} f_j^*(s_i) |f_j^*(s_i)| \right) \\
= \sum_{j=1}^{n} \sum_{i=1}^{m} \left( 2f_j^*(s_i)^2 + 2\sigma_{j,i} f_j^*(s_i) |f_j^*(s_i)| \right) \\
\leq 2z \sum_{j=1}^{n} \sum_{i=1}^{m} \sigma_{j,i} f_j^*(s_i) + 2 \sum_{j=1}^{n} \sum_{i=1}^{m} f_j^*(s_i)^2 \\
= 2z g(\sigma) + 2 \sum_{j=1}^{n} \sum_{i=1}^{m} f_j^*(s_i)^2 \\
\leq 2z g(\sigma) + 2n \sup_{f \in \mathcal{F}} \left\{ \sum_{i=1}^{m} f(s_i)^2 \right\} \\
= 2z g(\sigma) + 2nm\sigma^2 \, ,
\]

obtaining the statement. \qed

6.2 New probabilistic bounds on the ERA

The following Theorem establish novel probabilistic upper bounds based on the self-bounding property of the \( n\)-MCERA we proved in Theorem 6.

**Theorem 8.** Define \( \nu_{\mathcal{F}} \) as
\[
\nu_{\mathcal{F}} = \frac{1}{m} \sup_{f \in \mathcal{F}} \left\{ \sum_{i=1}^{m} |f(s_i)| \right\} 
\]

Then, for all \( 0 < \varepsilon \leq \hat{R}(\mathcal{F}, \mathcal{S}) \),
\[
\Pr \left( \hat{R}(\mathcal{F}, \mathcal{S}) \geq \hat{R}_m(\mathcal{F}, \mathcal{S}, \sigma) + \varepsilon \right) \leq \exp \left( -\frac{nm\varepsilon^2}{4z \left( \hat{R}(\mathcal{F}, \mathcal{S}) + \nu_{\mathcal{F}} \right)} \right) . \tag{11}
\]

**Proof.** Define the set of functions
\[
\mathcal{F}' \equiv \left\{ f'(x) = f(x)/(2z) : \forall x \in \mathcal{X}, f \in \mathcal{F} \right\} 
\]

composed by all functions \( f \in \mathcal{F} \) divided by \( 2z \); clearly, \( |f'(x)| \leq 1/2, \forall x \in \mathcal{X} \). We now show that \( nm\hat{R}_m(\mathcal{F}', \mathcal{S}, \sigma) \) (consequently, also \( nm\hat{R}_m(\mathcal{F}, \mathcal{S}, \sigma) \)) is a non-negative function:
\[
nm\hat{R}_m(\mathcal{F}', \mathcal{S}, \sigma) \geq \sum_{j=1}^{n} \sup_{f' \in \mathcal{F}'} \sum_{s_i \in \mathcal{S}} \sigma_{j,i} f'(s_i) \geq \sum_{j=1}^{n} \sum_{s_i \in \mathcal{S}} \sigma_{j,i} f_0(s_i) = 0 \ . \tag{12}
\]

From Theorem 6 we have that \( nm\hat{R}_m(\mathcal{F}', \mathcal{S}, \sigma) \) is a \((1, nm\nu_{\mathcal{F}'})\)-self-bounding function. This implies that it is also a weakly \((1, nm\nu_{\mathcal{F}'})\)-self-bounding function. Then, note that \( \mathbb{E}_\sigma \left[ nm\hat{R}_m(\mathcal{F}', \mathcal{S}, \sigma) \right] = nm\hat{R}(\mathcal{F}', \mathcal{S}) \).

We combine these facts with Theorem 8 obtaining, for \( g(\sigma) = nm\hat{R}_m(\mathcal{F}', \mathcal{S}, \sigma) \),
\[
\Pr \left( nm\hat{R}(\mathcal{F}', \mathcal{S}) \geq nm\hat{R}_m(\mathcal{F}', \mathcal{S}, \sigma) + t \right) \leq \exp \left( -\frac{t^2}{2 \left( nm\hat{R}(\mathcal{F}', \mathcal{S}) + nm\nu_{\mathcal{F}'} \right)} \right) .
\]
Then, for all $\alpha > 0$ we have

$$\Pr\left(\frac{nm}{2z} \hat{R}(\mathcal{F}, S) \geq \frac{nm}{2z} \hat{R}_{m}^{n}(\mathcal{F}, S, \sigma) + \frac{z t^2}{(mn \hat{R}(\mathcal{F}, S) + nm \nu_{\mathcal{F}})}\right) \leq \exp \left(-\frac{nm \alpha^2}{4(\hat{R}(\mathcal{F}, S) + \sigma_{\mathcal{F}}^2)}\right).$$

We further substitute $t$ by $nm \alpha/(2z)$, obtaining the statement.

Theorem 9. Define $\sigma_{\mathcal{F}}^2$ as

$$\sigma_{\mathcal{F}}^2 = \frac{1}{m} \sup_{f \in \mathcal{F}} \left\{ \sum_{i=1}^{m} f(s_i)^2 \right\}.$$

Then, for all $0 < \varepsilon \leq \hat{R}(\mathcal{F}, S)$,

$$\Pr\left(\hat{R}(\mathcal{F}, S) \geq \hat{R}_{m}^{n}(\mathcal{F}, S, \sigma) + \varepsilon\right) \leq \exp \left(-\frac{nm \varepsilon^2}{4(\hat{R}(\mathcal{F}, S) + \sigma_{\mathcal{F}}^2)}\right). \quad (13)$$

Proof. Let $\mathcal{F}'$ be the same set of functions defined in the proof of Theorem 8. If we denote $g(\sigma) = nm \hat{R}_{m}^{n}(\mathcal{F}', S, \sigma)$, then, from Theorem 7, $g(\sigma)$ is a weakly $(1, 2nm \sigma_{\mathcal{F}}^2)$-self-bounding function. As before,

$$\mathbb{E}_{\sigma}\left[ nm \hat{R}_{m}^{n}(\mathcal{F}', S, \sigma) \right] = nm \hat{R}(\mathcal{F}', S).$$

We apply Theorem 7 on $g(\sigma) = nm \hat{R}_{m}^{n}(\mathcal{F}', S, \sigma)$, obtaining

$$\Pr\left(\frac{nm}{2z} \hat{R}(\mathcal{F}, S) \geq \frac{nm}{2z} \hat{R}_{m}^{n}(\mathcal{F}', S, \sigma) + \frac{t^2}{2(\hat{R}(\mathcal{F}, S) + 2nm \sigma_{\mathcal{F}}^2)}\right) \leq \exp \left(-\frac{t^2}{2(\hat{R}(\mathcal{F}, S) + 2nm \sigma_{\mathcal{F}}^2)}\right).$$

We observe that $\hat{R}(\mathcal{F}', S) = \hat{R}(\mathcal{F}, S)/(2z)$, $\hat{R}_{m}^{n}(\mathcal{F}', S, \sigma) = \hat{R}_{m}^{n}(\mathcal{F}, S, \sigma)/(2z)$, and that $\sigma_{\mathcal{F}}^2 = \sigma_{\mathcal{F}}^2/(4z^2)$. This implies that

$$\Pr\left(\frac{nm}{2z} \hat{R}(\mathcal{F}, S) \geq \frac{nm}{2z} \hat{R}_{m}^{n}(\mathcal{F}, S, \sigma) + t\right) \leq \exp \left(-\frac{t^2}{2(\hat{R}(\mathcal{F}, S) + \sigma_{\mathcal{F}}^2)}\right).$$

Replacing $t$ by $nm \alpha/(2z)$ concludes the proof.

We may observe that the denominators of the exponents of (11) and (13) are not known a priori, but depend on the quantity $\hat{R}(\mathcal{F}, S)$ we actually want to bound. We remark that plugging an upper bound to $\hat{R}(\mathcal{F}, S)$ is sufficient for the validity of the results. To this aim, we may simply observe that

$$\hat{R}(\mathcal{F}, S) = \mathbb{E}_{\sigma}\left[ \hat{R}_{m}^{n}(\mathcal{F}, S, \sigma) \right] \leq \mathbb{E}_{\sigma} \left[ \nu_{\mathcal{F}} \right] = \nu_{\mathcal{F}}.$$

We now present alternative bounds that only depend on empirical quantities, following the derivations of Bartlett et al. (2003), that may be sharper than plugging the above crude upper bound.

Theorem 10. With probability $\geq 1 - \delta$ it holds

$$\hat{R}(\mathcal{F}, S) \leq \inf_{\alpha \in (0, 1)} \left\{ \frac{1}{1 - \alpha} \left[ \hat{R}_{m}^{n}(\mathcal{F}, S, \sigma) + \sqrt{\frac{4z \nu_{\mathcal{F}} \ln \left(\frac{1}{\delta}\right)}{nm}} + \frac{z \ln \left(\frac{1}{\delta}\right)}{anm} \right] \right\}. \quad (14)$$

Also, with probability $\geq 1 - \delta$, it holds

$$\hat{R}(\mathcal{F}, S) \leq \inf_{\alpha \in (0, 1)} \left\{ \frac{1}{1 - \alpha} \left[ \hat{R}_{m}^{n}(\mathcal{F}, S, \sigma) + \sqrt{\frac{4\sigma_{\mathcal{F}}^2 \ln \left(\frac{1}{\delta}\right)}{nm}} + \frac{z \ln \left(\frac{1}{\delta}\right)}{anm} \right] \right\}. \quad (15)$$
**Proof.** We prove the first statement, as proving the second is analogous. We will use the following result.

**Lemma 2.** For \( u, v \geq 0 \),
\[
\sqrt{u + v} \leq \sqrt{u} + \sqrt{v} ,
\]
and for any \( \alpha > 0 \),
\[
2\sqrt{uv} \leq \alpha u + \frac{v}{\alpha} .
\]

From Theorem 8, we have that, with probability \( \geq 1 - \delta \),
\[
\hat{R}(\mathcal{F}, \mathcal{S}) \leq \hat{R}_m^n(\mathcal{F}, \sigma) + \sqrt{\frac{4z(\nu_F + \hat{R}(\mathcal{F}, \mathcal{S})) \ln \left( \frac{1}{\delta} \right)}{nm}}.
\]

Thus, we apply Lemma 2 obtaining, for all \( \alpha \in (0, 1) \),
\[
\hat{R}(\mathcal{F}, \mathcal{S}) \leq \hat{R}_m^n(\mathcal{F}, \sigma) + \sqrt{\frac{4z\nu_F \ln \left( \frac{1}{\delta} \right)}{nm}} + \sqrt{\frac{4z\hat{R}(\mathcal{F}, \mathcal{S}) \ln \left( \frac{1}{\delta} \right)}{nm}}
\]
\[
\leq \hat{R}_m^n(\mathcal{F}, \sigma) + \sqrt{\frac{4z\nu_F \ln \left( \frac{1}{\delta} \right)}{nm}} + \alpha \hat{R}(\mathcal{F}, \mathcal{S}) + \frac{z \ln \left( \frac{1}{\delta} \right)}{\alpha nm}
\]
\[
\leq \frac{1}{1 - \alpha} \left[ \hat{R}_m^n(\mathcal{F}, \mathcal{S}, \sigma) + \sqrt{\frac{4z\nu_F \ln \left( \frac{1}{\delta} \right)}{nm}} + \frac{z \ln \left( \frac{1}{\delta} \right)}{\alpha nm} \right] .
\]

We obtain the statement considering the infimum over \( \alpha \in (0, 1) \).

We conclude remarking that finding \( \alpha^* \) that minimize the bounds of Theorem 10 is straightforward.

**Lemma 3.** It holds, for all \( u, v > 0 \),
\[
\inf_{\alpha \in (0, 1)} \left\{ \frac{1}{1 - \alpha} \left( u + \frac{v}{\alpha} \right) \right\} = \frac{1}{1 - \alpha^*} \left( u + \frac{v}{\alpha^*} \right) , \text{ with } \alpha^* = \frac{\sqrt{v^2 - uw} - v}{u} .
\]

### 6.3 Discussion

By directly comparing the bounds we derived by Theorems 8 and 9 with the one given by Theorem 3, we can conclude that the former will be tighter when at least one of the following is satisfied:
\[
\hat{R}(\mathcal{F}, \mathcal{S}) + \nu_F \leq \frac{z}{2} , \quad 2z\hat{R}(\mathcal{F}, \mathcal{S}) + 2\sigma_F^2 \leq z^2 .
\]

As discussed before, since \( \hat{R}(\mathcal{F}, \mathcal{S}) \leq \nu_F \), a sufficient condition for our results to be sharper is given by
\[
\nu_F \leq \frac{z}{4} , \quad 2z\nu_F + 2\sigma_F^2 \leq z^2 .
\]

In particular, our results allow to achieve an \( \varepsilon \) such that \( \hat{R}(\mathcal{F}, \mathcal{S}) \leq \hat{R}_m^n(\mathcal{F}, \sigma) + \varepsilon \), and where \( \varepsilon \) is upper bounded by \( O \left( \sqrt{\nu_F / nm} \right) \) (or \( O \left( \sqrt{\sigma_F^2 / nm} \right) \)), instead of the standard \( O \left( \sqrt{1/nm} \right) \) bound. We also remark that, when we bound the \( \hat{R}(\mathcal{F}, \mathcal{S}) \), both \( \nu_F \) and \( \sigma_F^2 \) are deterministic quantities since the sample \( S \) is fixed; thus, the probabilistic result to apply can be selected independently from the realisation of any random variable.
7 Variance-aware probabilistic bounds to the Supremum Deviations

In this section we state a central result in statistical learning theory, due to [Bousquet (2002)], the sharpest refinement of a number of improvements of the work of [Talagrand (1994)] on bounds on the deviation of the suprema of empirical processes. This result can be applied to derive bounds on the supremum deviations that depend on the maximum variance \( \tau \equiv \sup_{f \in \mathcal{F}} \{ \text{Var}(f) \} \) of the functions of \( \mathcal{F} \). These bounds can be dramatically sharper than the ones obtainable with the bounded differences method if \( \tau \) is sufficiently smaller than its maximum possible value (equal to \( c^2/4 \) from [Popoviciu (1935)] inequality on variances).

We first report the result of Bousquet (in the version stated by Theorem A.1 of [Bartlett et al. (2005)]); bounds to the SDs follows from it.

**Theorem 11 (Theorem 2.3, [Bousquet (2002)]).** Let \( d > 0 \), \( X_i \) be independent random variables distributed according to a probability distribution \( P \), and let \( \mathcal{G} \) be a set of functions from \( X \) to \( \mathbb{R} \). Assume that all functions \( g \in \mathcal{G} \) satisfy \( \mathbb{E}[g] = 0 \) and \( \|g\|_\infty \leq d \). Let \( \sigma^2 \geq \sup_{g \in \mathcal{G}} \text{Var}(g(X_i)) \). Then, for any \( x \geq 0 \),

\[
\Pr(Z \geq \mathbb{E}[Z] + x) \leq \exp\left( -vf\left( \frac{x}{cv} \right) \right),
\]

where \( Z = \sup_{g \in \mathcal{G}} \sum_{i=1}^n g(X_i), h(x) = (1 + x) \log(1 + x) - x \) and \( v = n\sigma^2 + 2d\mathbb{E}[Z] \).

**Theorem 12.** Let \( Z \equiv \sup_{f \in \mathcal{F}} \left\{ \frac{1}{m} \sum_{j=1}^m f(s_j) - \mathbb{E}[f] \right\} \), and define \( \tau \equiv \sup_{f \in \mathcal{F}} \{ \text{Var}(f) \} \) and the function \( h(x) \equiv (1 + x) \ln(1 + x) - x \). Then, it holds

\[
\Pr(Z \geq \mathbb{E}[Z] + \varepsilon) \leq \exp\left( -m (\tau + 2c\mathbb{E}[Z]) h\left( \frac{\varepsilon}{\tau + 2c\mathbb{E}[Z]} \right) \right). \tag{17}
\]

Also, with probability at least \( 1 - \delta \), it holds

\[
Z \leq \mathbb{E}[Z] + \sqrt{\frac{2 \ln \left( \frac{1}{\delta} \right) (\tau + 2c\mathbb{E}[Z])}{m}} + \frac{c \ln \left( \frac{1}{\delta} \right)}{3m}. \tag{18}
\]

The same results are valid for \( Z \equiv \sup_{f \in \mathcal{F}} \{ \mathbb{E}[f] - \frac{1}{m} \sum_{j=1}^m f(s_j) \} \).

8 New probabilistic bounds to the Supremum Deviations

The analysis behind Theorem 11 is, at a very high level, based on proving concentration results for sets of functions satisfying a sub-additive property, a variant of the \((1,0)\)-self-bounding property with relaxed requirements; in fact, the supremum deviation is sub-additive (see Section 6 and Lemma C.1 of [Bousquet, 2003]), but is not, in general, \((1,0)\)-self-bounding. Still, in this Section we show that the supremum deviation is \((1,\beta)\)-self-bounding, for appropriate values of \( \beta \) that depend on the maximum and minimum expectations of the elements of \( \mathcal{F} \). Consequently, we obtain novel bounds for the supremum deviation by applying concentration results for self-bounding functions, similarly to what we did for the \(n\)-MCERA.

We first prove self-bounding properties for the supremum deviations.

**Theorem 13.** Let \( \eta = \sup_{f \in \mathcal{F}} \mathbb{E}[f] - a \) and assume \( c \leq 1 \). Let \( g(S) \) be

\[
g(S) \equiv \sup_{f \in \mathcal{F}} \left\{ \sum_{j=1}^m f(s_j) - m\mathbb{E}[f] \right\}. \tag{19}
\]

Then, \( g(S) \) is a \((1, \eta)\)-self-bounding function.

**Proof.** Let \( g_i(S) \) be

\[
g_i(S) = \inf_{s_i'} \left\{ \sup_{f \in \mathcal{F}} \left\{ \sum_{j=1}^m f(s_j) + f(s_i') - m\mathbb{E}[f] \right\} \right\}. \tag{20}
\]
Notice that, as done before, $s_i$ is ignored in the definition of $g_i(\mathcal{S})$. Let $f^*$ be one of the functions in $\mathcal{F}$ that attains the supremum for $g(\mathcal{S})$. We then have

$$
G_i(\mathcal{S}) = \inf_{s_i} \left\{ \sup_{f \in \mathcal{F}} \left\{ \sum_{j=1}^{m} f(s_j) + f(s_i') - mE[f] \right\} \right\}
$$

(21)

\[ \geq \inf_{s_i} \left\{ \sum_{j=1}^{m} f^*(s_j) + f^*(s_i') - mE[f^*] \right\} \]

(22)

\[ = \sum_{j=1}^{m} f^*(s_j) - mE[f^*] + \inf_{s_i'} \{ f^*(s_i') \} \]

(23)

\[ = \sum_{j=1}^{m} f^*(s_j) - mE[f^*] + a \]

(24)

\[ = \sum_{j=1}^{m} f^*(s_j) + f^*(s_i) - f^*(s_i) - mE[f^*] + a \]

(25)

\[ = g(\mathcal{S}) - f^*(s_i) + a . \]

(26)

We then observe that $g_i(\mathcal{S}) \geq g(\mathcal{S}) - b + a = g(\mathcal{S}) - c$; assuming that $c \leq 1$, we have $g_i(\mathcal{S}) \geq g(\mathcal{S}) - 1$. We then continue with

\[
\sum_{j=1}^{m} (g(\mathcal{S}) - g_i(\mathcal{S})) \\
\leq \sum_{j=1}^{m} (f^*(s_i) - a) \\
= \sum_{j=1}^{m} f^*(s_i) - am \\
= \sum_{j=1}^{m} f^*(s_i) - mE[f^*] + mE[f^*] - am \\
= g(\mathcal{S}) + mE[f^*] - ma \\
\leq g(\mathcal{S}) + m\eta ,
\]

obtaining the statement.

\[ \square \]

**Theorem 14.** Let $\gamma = b - \inf_{f \in \mathcal{F}} E[f]$ and assume $c \leq 1$. Let $g(\mathcal{S})$ be

$$
g(\mathcal{S}) \doteq \sup_{f \in \mathcal{F}} \left\{ mE[f] - \sum_{j=1}^{m} f(s_j) \right\} .
$$

(27)

Then, $g(\mathcal{S})$ is a $(1, m\gamma)$-self-bounding function.

**Proof.** Define the set of functions $\mathcal{F}' \doteq \{ f'(x) \doteq -f(x), f \in \mathcal{F}, x \in \mathcal{X} \}$. We have that $f' \in [-b, -a]$, that $E[f'] = -E[f]$, and that $\sum_{j=1}^{m} f'(s_j) = -\sum_{j=1}^{m} f(s_j)$. Therefore,

$$
g(\mathcal{S}) = \sup_{f \in \mathcal{F}} \left\{ mE[f] - \sum_{j=1}^{m} f(s_j) \right\} = \sup_{f' \in \mathcal{F}'} \left\{ \sum_{j=1}^{m} f'(s_j) - mE[f'] \right\} .
$$

(28)
Then, we may observe that
\[
\gamma = b - \inf_{f \in F} \mathbb{E}[f] = b + \sup_{f \in F} \mathbb{E}[f'] = \sup_{f \in F} \mathbb{E}[f'] - \min_x f'(x) .
\] (29)

Thus, we apply Theorem 13 to \(g(S)\) and \(F'\) to show that it is \((1, m\gamma)\)-self bounding, obtaining the statement.

We now apply the concentration inequalities given by Theorem 2 to obtain novel bounds on the supremum deviations.

**Theorem 15.** Let \(Z\) be
\[
Z \doteq \sup_{f \in F} \left\{ \frac{1}{m} \sum_{j=1}^{m} f(s_j) - \mathbb{E}[f] \right\} ,
\]
and let \(\eta \doteq \sup_{f \in F} \mathbb{E}[f] - a\), with \(c \leq 1\). Then,
\[
\Pr(Z \geq \mathbb{E}[Z] + \varepsilon) \leq \exp\left( -\frac{m\varepsilon^2}{2(\mathbb{E}[Z] + \eta + \varepsilon/3)} \right) .
\] (30)

Consequently, with probability \(\geq 1 - \delta\),
\[
Z \leq \mathbb{E}[Z] + \sqrt{\left( \frac{\ln \left( \frac{1}{8} \right)}{3m} \right)^2 + \frac{2 \ln \left( \frac{1}{4} \right) (\mathbb{E}[Z] + \eta)}{m} + \frac{\ln \left( \frac{1}{8} \right)}{3m}} .
\]

**Proof.** We first observe that \(g(S)\) is a non-negative function, for all \(S\), since \(f_0 \in F\). Then, \(g(S) \doteq mZ\) is \((1, m\eta)\)-self-bounding from Theorem 13, therefore, we apply Theorem 2 to obtain (30). The second statement follows from imposing the r.h.s. of (30) to be \(\leq \delta\).

**Theorem 16.** Let \(Z\) be
\[
Z \doteq \sup_{f \in F} \left\{ \mathbb{E}[f] - \frac{1}{m} \sum_{j=1}^{m} f(s_j) \right\} ,
\]
and let \(\gamma \doteq b - \inf_{f \in F} \mathbb{E}[f]\), with \(c \leq 1\). Then,
\[
\Pr(Z \geq \mathbb{E}[Z] + \varepsilon) \leq \exp\left( -\frac{m\varepsilon^2}{2(\mathbb{E}[Z] + \gamma + \varepsilon/3)} \right) .
\] (31)

Consequently, with probability \(\geq 1 - \delta\),
\[
Z \leq \mathbb{E}[Z] + \sqrt{\left( \frac{\ln \left( \frac{1}{8} \right)}{3m} \right)^2 + \frac{2 \ln \left( \frac{1}{4} \right) (\mathbb{E}[Z] + \gamma)}{m} + \frac{\ln \left( \frac{1}{8} \right)}{3m}} .
\]

**Proof.** We follow analogous steps taken in the proof of Theorem 15. First, \(g(S) \doteq mZ\) is \((1, m\gamma)\)-self-bounding from Theorem 13. (31) follows from Theorem 2. The second statement is again obtained from bounding the r.h.s. of (31) below \(\delta\).

We may observe that the novel bounds we proved are less versatile than the result of Bousquet, as they may give faster convergence rates (w.r.t. the bounded difference method) for only one side of the deviation at a time instead of both simultaneously; this is because, for the same \(F\), \(\eta\) and \(\gamma\) cannot be both small. We note that it is not trivial to directly compare the two bounds, in particular (17) as it is implicit. However, we may observe that they are slightly sharper than Bousquet’s for some range of the values of the quantities involved in the bounds. Therefore, the combination of Theorem 12 and our new results gives opportunities to obtain sharper bounds.
We may also observe that these results depend on, respectively, the maximum or minimum expected values of elements of $F$, while Bousquet’s inequality requires an upper bound to their maximum variance; a problem in applications is how to handle the cases where these quantities are not known in advance.

One intuitive solution is to estimate such quantities from the data: if they are sufficiently smaller than the maximum, one obtains sharper error rates. Regarding the maximum variance

$$\tau = \sup_{f \in F} \text{Var}(f),$$

we point out that the bounds $\gamma$ and $\eta$ to the expectations of $f \in F$ may be sufficient to handle it; in fact, from Bhatia and Davis (2000), one has that, for all $f$,

$$\text{Var}(f) \leq (b - \mathbb{E}[f])(\mathbb{E}[f] - a).$$

Consequently, we have that

$$\tau \leq \sup \left\{ \frac{(b - x)(x - a)}{\inf_{f \in F} \mathbb{E}[f], \sup_{f \in F} \mathbb{E}[f]} \right\} \leq \gamma \eta.$$

Therefore, bounds to $\gamma$ and $\eta$ suffice for the application of our results and Bousquet’s inequality.

9 New special bounds for $n = 1$

An interesting case in applications is when only $n = 1$ vector of Rademacher random variable is used to compute the $n$-MCERA. In addition of being faster to compute than $n > 1$, Pellegrina et al. (2020) show that in this case one may obtain a sharper bound to the SDs (but also to the RC) with only one and direct application of the bounded difference method, considering pairs composed by Rademacher random variables and samples of $\mathcal{S}$ as i.i.d. random variables form the appropriate joint distribution. We now present a variant of the result they prove, that upper bounds the RC instead of the SDs, that is useful to us to be compared with the novel result we prove with Theorem 18.

Theorem 17 (Theorem 4.6, Pellegrina et al. (2020)). It holds

$$\Pr \left( \mathcal{R}(\mathcal{F}, m) \geq \hat{\mathcal{R}}_{m}^{1}(\mathcal{F}, \mathcal{S}, \sigma) + \varepsilon \right) \leq \exp \left( -\frac{m\varepsilon^{2}}{2z^{2}} \right),$$

thus, with probability $\geq 1 - \delta$, it holds

$$\mathcal{R}(\mathcal{F}, m) \leq \hat{\mathcal{R}}_{m}^{1}(\mathcal{F}, \mathcal{S}, \sigma) + z\sqrt{\frac{2\ln \left( \frac{1}{\delta} \right)}{m}}.$$

They also remark that applying the result to the range centralised set of functions

$$\mathcal{G} = \left\{ g : g(x) \equiv f(x) - \frac{c}{2}, f \in \mathcal{F}, x \in \mathcal{X} \right\}$$

is often convenient as it gives the sharpest constants in the bound.

We now derive an analogous but significantly sharper bound when the maximum expected square of the functions in $\mathcal{F}$ is small. Our proof is based on a left tail of Bousquet’s inequality.

Theorem 18. Let $\rho = \sup_{f \in \mathcal{F}} \{\mathbb{E}[f^2]\}$. Then, with probability $\geq 1 - \delta$, it holds

$$\mathcal{R}(\mathcal{F}, m) \leq \hat{\mathcal{R}}_{m}^{1}(\mathcal{F}, \mathcal{S}, \sigma) + \sqrt{\frac{2(2z\mathcal{R}(\mathcal{F}, m) + \rho)\ln \left( \frac{1}{\delta} \right)}{m}} + \frac{z\ln \left( \frac{1}{\delta} \right)}{8m}. \quad (32)$$

Proof. Define the set of functions $\mathcal{G}$ as

$$\mathcal{G} = \left\{ g : g(x, \sigma) \equiv \sigma f(x), f \in \mathcal{F}, x \in \mathcal{X}, \sigma \in \{-1, 1\} \right\},$$

where $\sigma$ is a Rademacher random variable. Therefore, we observe that

$$\mathbb{E}[g] = \mathbb{E}[f]\mathbb{E}[\sigma] = 0, \sup_{g \in \mathcal{G}} \text{Var}(g) = \rho, \|g\|_{\infty} \leq z.$$

We now need the following left tail bound of Bousquet’s inequality.
Corollary 1 (Corollary 12.2, Boucheron et al. (2013)). Consider the setup of Theorem 11. Then, for all $t \geq 0$, it holds

$$\Pr \left( Z \leq \mathbb{E}[Z] - \sqrt{2vt} - \frac{dt}{8} \right) \leq \exp(-t).$$

Thus, we apply Corollary 1 to $G$ to obtain the statement.

We note that this result can be naturally combined with Theorem 12 to obtain sharp variance-aware bounds on the SDs. Furthermore, (32) may be sharper than the combined application of (13) and (6) for $n = 1$, since one does not have to apply a union bound over two events, and since some of the constants of (32) are smaller than the ones in (13). On the other hand, one should have (or compute on the data) an upper bound to $\sup_{f \in F} \{\mathbb{E}[f^2]\}$ to apply the result, while Theorem 3 only requires to compute its empirical counterpart.

Finally, we observe that the application of Bousquet’s inequality to obtain a result analogous to Theorem 32 for directly bounding the SDs, or the RC using $n > 1$, should both be possible, provided that a careful analysis of the maximum variance of the modified set of functions (similar to the set $G$ defined in the proof of Theorem 18) is handled. De Stefani and Upfal (2019) follow this idea (in Thm. 2 and Thm. 4) to directly bound the RC or SDs with appropriate bounds on the covariances of the random variables involved in their martingales; we believe that combining those derivations with our application of Bousquet’s inequality is an interesting direction to explore.

10 Conclusions

In this work we studied the self-bounding properties of the $n$-MCERA, and show that they allow to derive novel sharper concentration bounds w.r.t. its expectation. Obtaining tight error rates on the MCERA is of central importance to obtain tight probabilistic upper bounds on the Rademacher Averages and, therefore, uniform deviation bounds to the maximum deviation between empirical means and their expectations of sets of functions.

While in this work we focused on deriving concentration results valid with high probability in finite samples, another interesting direction is to combine the self-bounding properties we proved with asymptotical concentration results, such as the Central Limit Theorem for martingales (Hall and Heyde, 2014). In fact, De Stefani and Upfal (2019) (Thm. 6) have shown how to apply this result to bound the Supremum Deviation from the $n$-MCERA; as they discuss, in many applications asymptotic bounds may be preferred as they may be sharper than their finite-sample counterparts, in particular when the size of the analysed data is sufficiently large and the convergence to the normal distribution is reasonably accurate. The self-bounding properties we proved in this work imply tighter bounds on the variance of the random processes modelled by such martingales (see Chapter 6.11 of Boucheron et al. (2013)); therefore, an interesting question is whether our results could enable a sharper application of the Central Limit Theorem for martingales.

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