Higher-order Approximations of Nonlinear Oscillator with Coordinate-dependent Mass

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Abstract. This paper analyses a nonlinear oscillator with coordinate-dependent mass based on the presented methods of multi-term harmonic balance (MHB) and iterative residue harmonic balance (IRHB). The proposed methods calculate higher-order approximations. After using the MHB, a group of complicated nonlinear algebraic equations are obtained which are cumbersome to calculate analytically. This limitation is overcome in the presented other method by using the IRHB. In the solution procedure of IRHB method, the higher-order approximations to angular frequencies and periodic responses can be determined due to linear residue equations. Results show that the presented solutions give high accuracy and better results than those obtained by other existing ones from the homotopy perturbation method and the frequency-amplitude formulation. The advantage of the IRHB method is that it balances the all residues step by step and the present second-order approximations almost coincide with the corresponding exact solutions. Thus, the presented IRHB method could be applied to other strongly nonlinear oscillator systems.

1. Introduction

There are a lot of nonlinear problems in which system parameters are not assumed as small in the field of science and engineering. Traditional perturbation methods [1] are most widely applied techniques for studying nonlinear equations. However, these traditional perturbation methods have some limitations depending on the degree of nonlinearity of the equations, especially for strongly nonlinear systems. To overcome this shortcoming, a large number of analytical techniques have been successfully developed, such as parameter expansion method [2], energy balance method [3], differential transforms method [4], variational iteration method [5], homotopy perturbation method [6-7], harmonic balance method [8], and so on [9-13].

Recently Levy, Tymchyshyn and Zagorodny developed a nonlinear oscillator with coordinate-dependent mass [14]. The model equation is simplified to

\[(1 + \alpha x^2)\ddot{x} + \alpha x\dot{x}^2 - x(1 - x^2) = 0 \quad \text{with initial conditions } x(0) = A, \quad \dot{x}(0) = 0 \]  \hspace{1cm} (1)

Eq. (1) has been widely studied for the closed phase trajectories. For instance, Wu and He [15] applied homotopy perturbation method to obtain periodic solution. Ren [16] used frequency amplitude formulation to analyze the Eq. (1). Zhang etc [17] presented the dynamic frequency method to study the approximate period of Eq. (1). However, these existing approximations are not a good match with exact solution. As a widely used technique for solving strongly nonlinear oscillators, the harmonic balance method was first used harmonic balance by Mickens in truly nonlinear oscillators [7]. When the harmonic balance technique is applied to the nonlinear systems for obtaining more accurate approximation, then a set of complicated nonlinear equations which is very difficult to analytically
solve will appear. Recently, some modifications of harmonic balance method were developed, such as rational harmonic balance method [18], Newton-harmonic balance approach [19], residue harmonic balance method [20-22], and modified harmonic balance methods [23-24]. In this paper, two kinds of analytical approximations based on the harmonic balance method have been developed to obtain the periodic solutions of Eq. (1). For solving Eq. (1), the multi-term harmonic balance and iterative residue harmonic balance are developed and applied. The obtained solutions are compared with those existing results and exact ones.

2. Solution Procedure of Multi-term Harmonic Balance
Under certain instances, Eq. (1) will appear some closed phase trajectories corresponding to periodic motion. By introducing a new time variable \( \tau = \omega t \), then Eq. (1) becomes

\[
\omega^2 (1+ \alpha x^2)x'' + \alpha \omega^2 xx'^2 - x(1-x^2) = 0 \quad \text{with initial conditions } x(0) = A, \ x'(0) = 0 \tag{2}
\]

Where the unknown angular frequency \( \omega \) would be determined in the following section, a prime denote differentiation with respect to the new time variable \( \tau \). Due to Eq. (2) is a symmetric system, then the variable \( x(\tau) \) can be expressed by a set of basic functions, that

\[
x(\tau) = \sum_{k=0}^{\infty} \{a_{2k+1} \cos((2k+1)\tau)\} \tag{3}
\]

where \( a_{2k+1}, (k = 0,1,\cdots) \) are unknown coefficients and satisfy

\[
\sum_{k=0}^{\infty} a_{2k+1} = A \tag{4}
\]

The corresponding multi-term harmonic balance equations can be obtained by substituting Eqs. (3) and (4) into Eq. (2).

2.1. The First-order Approximation
From Eqs. (3) and (4), a reasonable initial approximation can be taken as

\[
x_1(t) = A \cos(\omega_1 t) \tag{5}
\]

When substituting Eq. (5) into Eq. (2), then the first-order MHB solution can be easily obtained by equating the coefficient of \( \cos(\tau) \)

\[
\omega_{1,MHB} = \left( \frac{3}{4} \frac{A^2 - 1}{1 + \frac{\alpha A^2}{2}} \right)^{1/2}, \ x_{1,MHB}(t) = A \cos(\omega_{1,MHB} t) \tag{6}
\]

This result is exactly same as those obtained from the references [15] and [16].

2.2. Second-order Approximation
From Eqs. (3) and (4), the second-order approximate solution is chosen as form

\[
x_2(\tau) = A[(1-u) \cos(\tau) + u \cos(3\tau)] \tag{7}
\]

When substituting Eq. (7) into Eq. (2), the following harmonic balance equations are obtained by equating the coefficients of \( \cos(\tau) \) and \( \cos(3\tau) \)

\[
1-u - \frac{3}{4} A^2 (1-2u+3u^2-2u^3) + [1-u + \frac{1}{2} \alpha A^2 (1+7u^2-8u^3)]\omega^2 = 0 \tag{8}
\]
Eliminate \( \omega^2 \) from Eqs. (8) and (9), yields
\[
\frac{11}{2} \alpha A^6 u^6 - 17 \alpha A^4 u^4 + \left( \frac{23}{2} + 27 \alpha A^2 - 5 \alpha \right) A^2 u^2 + \left( 14 \alpha - 16 - 24 \alpha A^2 \right) A u^2 + \left( -12 \alpha A^2 + \frac{41}{4} \alpha A^4 \right) - 8 + \frac{21}{2} A^2 \right) u^2 + \left( 8 - \frac{25}{4} \right) A^2 + \frac{5}{2} \alpha A^2 - \frac{3}{2} \alpha A^4 \right) u + \frac{A^2 (1 - \alpha A^2 + 2 \alpha)}{4} = 0
\] (10)

By solving Eq. (10), the unknown coefficient \( u \) can be obtained, and then the corresponding second-order MHB solutions are as bellows
\[
\omega_2 = \sqrt{\frac{3 A^2 (1 - 2u + 3u^2 - 2u^3) + 4u - 4}{2 A^2 (1 + 7u - 8u^3) + 4 - 4u}}
\] (11)

Likewise, the higher-order approximation may be calculated theoretically according to the above method, but this will become more difficult to solve.

3. Solution Procedure of Iterative Residue Harmonic Balance
To simplify the calculating process of the aforementioned method, an iterative residue harmonic balance method is presented in this section.

Firstly, an embedding parameter \( p \) with values in the interval \([0,1]\) is introduced as a hypothesis, then transform the variable \( x(t) \) to \( px(t) \) and \( \omega(t) \) to \( \omega(p) \) are done, and
\[
x(t, p) = x_0(t) + px_1(t) + p^2 x_2(t) + p^3 x_3(t) + \cdots \omega^2(p) = \omega_0 + p \omega_1 + p^2 \omega_2 + p^3 \omega_3 + \cdots
\] (12)

Secondly, substituting Eq. (12) into Eq. (2), yields
\[
(\omega_0^2 + p \omega_1 + p^2 \omega_2 + \cdots) \left( x'' + px''_1 + p^2 x''_2 + \cdots \right) + \alpha (\omega_0^2 + p \omega_1 + p^2 \omega_2 + \cdots) (x_0 + px_1 + p^2 x_2 + \cdots)^2
\] (13)

Finally, a series of iterative residue equations can be obtained by equating the terms with identical powers of the embedding parameter \( p \)
\[
p^0 : \omega_0^2 x''_0 + \omega_0^2 x''_0 x''_0 + \alpha \omega_0^2 x_0 x''_0^2 - x_0 + x_0^3 = R_0
\] (14)
\[
p^1 : \omega_0^2 x''_0 + \omega_1 x''_0 + \alpha \omega_0^2 (2x_0 x_0 x''_0 + x''_0 x''_0 + \alpha \omega_0^2 (2x_0 x_0 x''_0 + x_0 x''_0^2 + x_0 x''_0 x''_0) + \alpha \omega_1 x_0 x''_0 x''_0 x''_0 - x_0 + x_0^3 = R_0
\] (15)
\[
p^2 : \omega_0^2 x''_0 + \omega_0^2 x''_0 + \omega_0^2 x''_0 + \alpha \omega_0^2 (x''_0 x''_0 + x''_0 x''_0 + 2x_0 x_0 x''_0 + 2x_0 x_0 x''_0 + x_0 x_0 x''_0 + x_0 x_0 x''_0) + \alpha \omega_1 (x''_0 x''_0 + 2x_0 x_0 x''_0 + x_0 x''_0 + 2x_0 x_0 x''_0 + x_0 x''_0 + x_0 x''_0 x_0 x''_0) + \alpha \omega_2 (x''_0 x''_0 + 2x_0 x_0 x''_0 + x_0 x''_0) + \alpha \omega_0^2 (2x_0 x_0 x''_0 x''_0 x''_0 + x_0 x''_0 x''_0 x''_0 + x_0 x''_0 x''_0 x''_0) + \alpha \omega_1 (x''_0 x''_0 + 2x_0 x_0 x''_0 + x_0 x''_0 + 2x_0 x_0 x''_0 + x_0 x''_0 + x_0 x''_0 x_0 x''_0) + \alpha \omega_2 (x''_0 x''_0 + 2x_0 x_0 x''_0 + x_0 x''_0 x_0 x''_0 + x_0 x''_0 x_0 x''_0) + \alpha \omega_0^2 (2x_0 x_0 x''_0 x''_0 x''_0 + x_0 x''_0 x''_0 x''_0 + x_0 x''_0 x''_0 x''_0) + \alpha \omega_1 (x''_0 x''_0 + 2x_0 x_0 x''_0 + x_0 x''_0 + 2x_0 x_0 x''_0 + x_0 x''_0 + x_0 x''_0 x_0 x''_0) + \alpha \omega_2 (x''_0 x''_0 + 2x_0 x_0 x''_0 + x_0 x''_0 x_0 x''_0) = R_0
\] (16)

Where \( R_i(t)(i=0,1,2,3,\cdots) \) is \( i \)-th order residual function of the iterative equations.

3.1. The First-order Approximation
Similar to multi-term harmonic balance solution procedure, substituting Eq.(5) into Eq.(14) and then equating the coefficients of the term \( \cos(\tau) \) equal to zeros, the first-order IRHB solution is obtained as
\[ \omega_{1,IRHB} = \left( \frac{3}{4} A^2 - 1 \right)^{\frac{1}{2}}, x_{1,IRHB}(t) = A \cos(\omega_{1,IRHB} t) \]  
(17)

and

\[ R_b(\tau) = \left( \frac{1}{4} A^3 - \frac{1}{2} A^2 \omega_{1,IRHB}^2 \right) \cos(3 \tau). \]  
(18)

3.2. Second-order Approximation

Based on the initial condition (2) and the first-order approximation (17), takes the form

\[ x_i(\tau) = b_{1,i}[\cos(\tau) - \cos(3\tau)] \]  
(19)

Substituting Eqs. (5) and (17-19) into Eq. (15) and setting the coefficients of the terms \( \cos(\tau) \) and \( \cos(3\tau) \) equal to zeros, yeilds

\[ \omega_i = A_i^2(-3 + 3\alpha^2 A^2 - 8\alpha^2 A^2 + 4\alpha^2 - 4\alpha), \]  
(20)

\[ \Delta = \alpha^3 A^6(6A^2 - 10) + \alpha^2 A^4(51A^2 - 76) + \alpha A^2(126A^2 - 176) + 96A^2 - 128 \]

\[ b_{1,i} = [A_i^2(\alpha^2 A^2 + 2)(\alpha A^2 - 2\alpha - 1)][\alpha^2 A^2(6A^2 - 10) + \alpha A^2(39A^2 - 56) + 48A^2 - 64]^{-1} \]  
(21)

and

\[ R_i = (-\frac{3}{4} + \frac{9}{2} \alpha \omega_{1,IRHB}^2) A_i^2 b_{1,i} \cos(5\tau) \]  
(22)

Therefore, the second-order IRHB approximation is obtained as

\[ \omega_{2,IRHB} = (\omega_{1,IRHB}^2 + \omega_{1,IRHB}^{\frac{1}{2}}), x_{2,IRHB}(t) = (A + b_{1,i})\cos(\omega_{2,IRHB} t) - b_{1,i} \cos(3 \omega_{2,IRHB} t) \]  
(23)

where \( \omega_{1,IRHB}, \omega_{1,IRHB}, \text{ and } b_{1,i} \) are given in Eqs.(17) and (20-21).

3.3. Higher-order Approximation

For further obtaining higher-order accurate approximation, according to Eq. (2) and IRHB method, we may assume

\[ x_i(\tau) = b_{1,i}[\cos(\tau) - \cos(3\tau)] + \cdots + b_{i,j}[\cos(\tau) - \cos([(2i + 1)\tau]), i = 1, 2, \ldots. \]  
(24)

From Eqs. (14-24) and use harmonic balance method, the \( k \) -order approximations to the frequency and periodic response can be successively obtained.

\[ \omega = \omega_{k,IRHB} = (\omega_{k,IRHB}^2 + \cdots + \omega_{k}^{\frac{1}{2}}) \]  
(25)

\[ x(t) = x_{k,IRHB}(t) = (A + \sum_{m=1}^{k} \sum_{j=1}^{m} b_{m,j}) \cos(\omega_{k,IRHB} t) - \sum_{m=1}^{k} b_{m,1} \cos(3 \omega_{k,IRHB} t) - \cdots - b_{k,k} \cos([(2k + 1)\omega_{k,IRHB} t]). \]  
(26)

It should be noted that the present the iterative residue Eqs. (15-16) etc are linear with respect to unknowns \( b_{1,i}, b_{2,j}, \ldots b_{j,j}, \omega_{j}, \text{ j = 1, 2, 3, \ldots}, \) which implies that the higher-order approximate solutions to any desired accuracy can be easily calculated.
4. Comparisons and Discussions

Due to the Eq. (1) is a conservative system under certain given conditions, we integrate Eq. (1) by using the initial condition (2), and then have

\[
\frac{1}{2} \left(1 + \alpha x^2\right) \left(\frac{dx}{dt}\right)^2 - \frac{1}{2} x^2 + \frac{1}{4} x^4 = \frac{1}{4} A^4 - \frac{1}{2} A^2
\]

(27)

Then,

\[
\frac{dx}{dt} = \pm \left(\frac{A^4 - x^4}{2(A^2 - x^2)}\right)^{\frac{1}{2}} \left(2(1 + \alpha x^2)\right)^{-\frac{1}{2}}
\]

(28)

Thus, the exact period \( T_e(A) \) can be calculated as

\[
T_e(A) = 4 \int_{0}^{\frac{2\pi}{\omega_e(A)}} \sqrt{\frac{2(1 + \alpha x^2)}{(A^4 - x^4) - 2(A^2 - x^2)}} \, dx, \quad \omega_e(A) = \frac{2\pi}{T_e(A)}
\]

(29)

In this section, the efficiency of solution procedures about the presented MHB and IRHB are illustrated by comparing the other existing results and exact ones. Tables 1-2 compare the present approximate frequencies corresponding to the exact frequency for cases of \( \alpha = 1 \) and \( \alpha = 100 \) respectively, the relative errors are defined as \((\omega - \omega_e)\omega_e^{-1} \times 100\% \). From these tables, the proposed solution procedures provide solutions with much higher accuracy than the existing methods from homotopy perturbation [15] and frequency amplitude formulation [16]. In most instances, the IRHB results are more accurate than the MHB ones. It is notable that the solution procedure of the MHB method is more cumbersome and laborious especially for calculating the higher-order approximate equations than the IRHB method.

Table 1. Comparison of the present approximate frequencies with the exact one for case \( \alpha = 1 \).

| \( A \) | \( \omega_e \) | \( \omega_{1,TBH} \), \( \omega_{1,IRHB} \) (Er\%) | \( \omega_{2,TBH} \) (Er\%) | \( \omega_{2,IRHB} \) (Er\%) |
|-------|-------------|---------------------------------|-----------------|-----------------|
| 1.5   | 0.51612     | 0.56880 (10.2069\%)             | 0.50679 (-1.8077\%) | 0.52109 (0.9630\%) |
| 1.6   | 0.61425     | 0.63522 (3.4139\%)             | 0.61146 (-0.4542\%) | 0.61392 (-0.0537\%) |
| 1.7   | 0.68689     | 0.69102 (0.6013\%)             | 0.68652 (-0.0539\%) | 0.68662 (-0.0393\%) |
| 1.8   | 0.74637     | 0.73878 (-1.0169\%)            | 0.74673 (0.0482\%)  | 0.74707 (0.0938\%)  |
| 1.9   | 0.79708     | 0.78021 (-2.1165\%)            | 0.79725 (0.0213\%)  | 0.79893 (0.2321\%)  |
| 2.0   | 0.84125     | 0.81650 (-2.9421\%)            | 0.84063 (-0.0737\%) | 0.84425 (0.3566\%)  |
Table 2. Comparison of the present approximate frequencies with the exact one for case $\alpha = 100$.

| $A$ | $\omega_{1}$ | $\omega_{1,\text{MHB}}, \omega_{1,\text{IRHB}}$ (Er%) | $\omega_{2,\text{MHB}}$ (Er%) | $\omega_{2,\text{IRHB}}$ (Er%) |
|-----|--------------|---------------------------------|-----------------|-----------------|
| 1.5 | 0.08732      | 0.07783 (-10.8681%)             | 0.08318 (-4.7412%) | 0.08628 (-1.1910%) |
| 1.6 | 0.09665      | 0.08445 (-12.6229%)            | 0.09139 (-5.4423%) | 0.09660 (-0.0517%) |
| 1.7 | 0.10317      | 0.08958 (-13.1724%)            | 0.09714 (-5.8447%) | 0.10350 (0.3199%) |
| 1.8 | 0.10814      | 0.09366 (-13.3900%)            | 0.10153 (-6.1124%) | 0.10863 (0.4531%) |
| 1.9 | 0.11210      | 0.09699 (-13.4790%)            | 0.10503 (-6.3069%) | 0.11265 (0.4906%) |
| 2.0 | 0.11534      | 0.09975 (-13.5166%)            | 0.10788 (-6.4678%) | 0.11591 (0.4942%) |

Finally, Figure 1 illustrates the comparisons of the presented periodic response with numerical solutions [25] for cases $\alpha=1, \ A=1.6$ and $\alpha=100, \ A=1.6$ respectively.

![Figure 1](image)

Figure 1. Comparison of the present approximations for $\alpha=1, \ A=1.6$ and $\alpha=100, \ A=1.6$ (given in Eq. (11) denoting by plusses, Eq. (23) denoting by stars) with numerical ones (denoting by solid lines) and the solutions from homotopy perturbation or frequency amplitude formulation solution (denoting by circles).

It is evident that the present IRHB solutions are in good agreement with the corresponding numerical solutions. The solution method is very valid for nonlinear oscillators with negative linear terms, and higher-order approximations can be easily obtained.

5. Conclusions

For determining accurate vibration frequency and response of nonlinear vibration model, many analytical technique such as the residue harmonic balance have been developed and used to study the tapered beam, rigid rod models. In this paper, two kinds of analytical techniques (MHB and IRHB) have been developed based on the method of harmonic balance to analyse and determine higher order approximations for nonlinear oscillator with coordinate-dependent mass. Compared with the existing solutions, the higher-order approximate solution procedures are given from MHB and IRHB, the obtained first-order approximate frequency is identical to the results from the existing ones, the second-order MHB and IRHB solutions are more accurate than the existing ones. Remarkably, it is evident that the IRHB method is more simple and efficient for obtaining higher-order approximations due to linear equations. Thus, the presented MHB and IRHB methods have great potential that could be applied to other nonlinear oscillatory problems.
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