Topological pressure for sub-additive potentials of amenable group actions

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Abstract

We introduce the topological pressure for any sub-additive potentials of a countable discrete amenable group action and any given open cover, and establish a local variational principle for it.

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1. Introduction

Entropies are fundamental to our current understanding of dynamical systems. The classical measure-theoretic entropy for an invariant measure and the topological entropy were introduced in [21] and [1] respectively, and the classical variational principle was completed in [11,12]. Since then a subject involving definition of new measure-theoretic and topological notions of entropy and study the relationship between them has gained a lot of attention in the study of dynamical systems.

Topological pressure is a generalization of topological entropy for a dynamical system. The notion was first introduced by Ruelle [27] in 1973 for expansive dynamical system and later by Walters [30] for the general case. The variational principle formulated by Walters can be stated...
precisely as follows: Let \((X, T)\) be a topological dynamical system, where \(X\) is a compact metric space and \(T : X \to X\) is a continuous map, and \(f : X \to \mathbb{R}\) is a continuous function. Let \(P(T, f)\) denote the topological pressure of \(f\) (see [31]). Then

\[
P(T, f) = \sup \left\{ h_\mu(T) + \int f \, d\mu : \mu \in \mathcal{M}(X, T) \right\},
\]

(1.1)

where \(\mathcal{M}(X, T)\) denotes the space of all \(T\)-invariant Borel probability measures on \(X\) and \(h_\mu(T)\) denotes the measure-theoretic entropy of \(\mu\).

The theory related to the topological pressure, variational principle and equilibrium states plays a fundamental role in statistical mechanics, ergodic theory and dynamical systems (see, e.g., the books [5,18,28,31]). Since the works of Bowen [6] and Ruelle [29], the topological pressure turned into a basic tool in the dimension theory related to dynamical systems. In 1984, Pesin and Pitskel [26] defined the topological pressure of additive potentials for non-compact subsets of compact metric spaces and proved the variational principle under some supplementary conditions. In 1988, Falconer [8] considered the thermodynamic formalism for sub-additive potentials for mixing repellers. He proved the variational principle for the topological pressure under some Lipschitz conditions and bounded distortion assumptions on the sub-additive potentials. In 1996, Barreira [2] extended the work of Pesin and Pitskel. He defined the topological pressure for an arbitrary sequence of continuous functions on an arbitrary subset of compact metric spaces, and proved the variational principle under a strong convergence assumption on the potentials. In 2008, Y. Cao, D. Feng and W. Huang [7] generalized Ruelle and Walters’s results to sub-additive potentials in general compact dynamical systems. Recently, G. Zhang [33] introduced the notion of measure-theoretic pressure for sub-additive potentials, and studied the relationship between topological pressure and measure-theoretic pressure.

Since notions of entropy pairs were introduced in both topological [3] and measure-theoretic system [4], much attention has been paid to the study of the local variational principle of entropy. Recently, Kerr and Li introduced various notions of independence and gave a uniform treatment of entropy pairs and sequence entropy pairs [19,20]. For an overview of local entropy theory, see the survey paper [10]. In 2007, to study the local variational principle of topological pressure, W. Huang and Y. Yi [16] introduced a new definition of topological pressure for open covers. They proved a local variational principle for topological pressure for any given open cover.

In this paper, we generalize Huang–Yi’s results to dynamical systems acting by a countable discrete amenable group. Let \((X, G)\) be an amenable group action dynamical system. We define the local topological pressure for sub-additive potentials \(\mathcal{F} = \{f_E\}_{E \in \mathcal{F}(G)}\) and establish a local variational principle between the topological pressure and measure-theoretic entropy.

Throughout the paper, let \((X, G)\) be a \(G\)-system, where \(G\) is a countable discrete amenable group and \(X\) is a compact metric space. A sub-additive potential on \((X, G)\) is a collection \(\mathcal{F} = \{f_E\}_{E \in \mathcal{F}(G)}\) of real valued continuous functions on \(X\) satisfying the following conditions:

\begin{align*}
&\text{(C1)} \quad f_{E \cup F}(x) \leq f_E(x) + f_F(x) \text{ for all } x \in X \text{ and all disjoint } E, F \in \mathcal{F}(G); \\
&\text{(C2)} \quad f_E(gx) = f_E(x) \text{ for all } x \in X, g \in G \text{ and } E \in \mathcal{F}(G); \\
&\text{(C3)} \quad C = \sup_{E \in \mathcal{F}(G)} \sup_{x \in X, g \in G} (f_E(x) - f_{E \cup \{g\}}(x)) < \infty. 
\end{align*}

Our main result is the following local variational principle.
Theorem 1.1 (Local variational principle). Let $(X, G)$ be a $G$-system, $U$ an open cover of $X$ and $\mathcal{F} = \{f_E\}_{E \in \mathcal{F}(G)}$ a sub-additive potential on $(X, G)$. Then the topological pressure $P(G, \mathcal{F}; U)$ relative to $U$ satisfies

$$P(G, \mathcal{F}; U) = \max \left\{ h_\mu(G, U) + F_*(\mu) : \mu \text{ is a } G\text{-invariant measure} \right\} \tag{1.2}$$

and the maximum can be attained by a $G$-invariant ergodic measure, if one of the following conditions holds:

1. $G$ is an Abelian group;
2. $\mathcal{F}$ is strongly sub-additive, i.e.
   $f_{E \cup F} + f_{E \cap F} \leq f_E + f_F$ for all $E, F \in \mathcal{F}(G)$, where $h_\mu(G, U)$ is the metric entropy of $\mu$ relative to $U$ and $F_*(\mu)$ is the Lyapunov exponent of $\mathcal{F}$ with respect to $\mu$.

In particular, if $f \in C(X)$ (here $C(X)$ denotes the Banach space of all real valued continuous function on $X$ endowed with the supremum norm $\| \cdot \|$), then

$$\mathcal{F}_f := \left\{ f_E = \sum_{g \in E} f \circ g : E \in \mathcal{F}(G) \right\}$$

satisfies the condition (2) in Theorem 1.1. In this case, write $P(G, f; U) = P(G, \mathcal{F}_f; U)$, then we get

Corollary 1.2. Let $(X, G)$ be a $G$-system, $U$ an open cover of $X$ and $f \in C(X)$, then

$$P(G, f; U) = \max \left\{ h_\mu(G, U) + \int_X f \, d\mu : \mu \text{ is a } G\text{-invariant measure} \right\}$$

and the maximum can be attained by a $G$-invariant ergodic measure.

The paper is organized as follows: in Section 2, we recall some notations about amenable group, and local measure-theoretic entropy for amenable group action. Moreover, we introduce the local pressure for a sub-additive potential. In Section 3, we provide some lemmas and prove Theorem 1.1. In Section 4, we give a nontrivial example of sub-additive potential.

2. Pressure of an amenable group action

2.1. Backgrounds of a countable discrete amenable group

Let $G$ be countable discrete infinite groups and $\mathcal{F}(G)$ the set of all finite non-empty subsets of $G$. A tile $T \subseteq G$ is a finite subset that has a collection of right translates that partitions $G$, i.e., there is a set $C \subseteq G$ of tiling centers such that $\{Tc : c \in C\}$ form a disjoint family whose union $TC$ is all of $G$. Note that $T \in \mathcal{F}(G)$ is a tile of $G$ if and only if any $A \in \mathcal{F}(G)$ can be covered by disjoint right translates of $T$.

A group $G$ is said to be amenable if for each $\epsilon > 0$ and $K \in \mathcal{F}(G)$, there exists $F \in \mathcal{F}(G)$ such that

$$\frac{|F \triangle K F|}{|F|} < \epsilon,$$
where $|F|$ is the cardinality of the set $F$, $KF = \{kf: k \in K, f \in F\}$ and $F \triangle KF = (F \setminus KF) \cup (KF \setminus F)$. Let $K \in \mathcal{F}(G)$ and $\delta > 0$. Set $K^{-1} = \{k^{-1}: k \in K\}$. We say that $A \in \mathcal{F}(G)$ is $(K, \delta)$-invariant if

$$\frac{|B(A, K)|}{|A|} < \delta,$$

where $B(A, K) := \{g \in G: Kg \cap A \neq \emptyset$ and $Kg \cap (G \setminus A) \neq \emptyset\} = K^{-1}A \cap K^{-1}(G \setminus A)$. A sequence $\{F_n\}_{n \in \mathbb{N}}$ in $\mathcal{F}(G)$ is called a Følner sequence for $G$ (see [9]), if for each $K \in \mathcal{F}(G)$ and $\delta > 0$, $F_n$ is $(K, \delta)$-invariant whenever $n \in \mathbb{N}$ is sufficiently large, i.e.,

$$\lim_{n \to \infty} \frac{|K F_n \triangle F_n|}{|F_n|} = 0$$

for each $K \in \mathcal{F}(G)$. It is not hard to see that a countable group is amenable if and only if $G$ has a Følner sequence $\{F_n\}_{n \in \mathbb{N}}$. For more description of this class of groups see [13] or [25].

It is well known that the class of amenable groups contains all finite groups and Abelian groups, and it is closed under taking subgroups, quotients, extensions and inductive limits. All finitely generated groups of subexponential growth are amenable. A basic example of non-amenable groups is the free group of rank 2.

Cyclic groups have Følner sequences of tiling sets, and one can build up from them to show that all solvable groups, finite extensions thereof, increasing unions, etc., in brief the so-called class of elementary amenable groups, all have Følner sequences of tiling sets. In particular, all Abelian groups have tiling Følner sequences. It is an open problem whether all countable discrete amenable groups have Følner sequences of tiling sets [24].

Let $f: \mathcal{F}(G) \to \mathbb{R}$ be a function. We say that $f$ is

1. **monotone**, if $f(E) \leq f(F)$ for any $E, F \in \mathcal{F}(G)$ with $E \subseteq F$;
2. **non-negative**, if $f(F) \geq 0$ for any $F \in \mathcal{F}(G)$;
3. **$G$-invariant**, if $f(Fg) = f(F)$ for any $F \in \mathcal{F}(G)$ and $g \in G$;
4. **sub-additive**, if $f(E \cup F) \leq f(E) + f(F)$ for any disjoint $E, F \in \mathcal{F}(G)$;
5. **strongly sub-additive**, if $f(E \cup F) + f(E \cap F) \leq f(E) + f(F)$ for any $E, F \in \mathcal{F}(G)$, where we set $f(\emptyset) = 0$ by convention.

The following limit theorem for invariant sub-additive functions on finite subsets of amenable groups is due to Ornstein and Weiss (see [14,22,24]). It plays a central role in the definition of some dynamical invariants such as topological entropy and measure-theoretic entropy.

**Lemma 2.1 (Ornstein–Weiss).** Let $G$ be a countable amenable group. Let $f: \mathcal{F}(G) \to \mathbb{R}$ be a monotone $G$-invariant sub-additive function. Then there exists $\lambda = \lambda(G, f) \in [-\infty, +\infty)$ depending only on $G$ and $f$ such that

$$\lim_{n \to \infty} \frac{f(F_n)}{|F_n|} = \lambda$$

for all Følner sequences $\{F_n\}_{n \in \mathbb{N}}$ of $G$. 
Remark 2.2. (1) If $f$ is also strongly sub-additive, then
\[
\lim_{n \to \infty} \frac{f(F_n)}{|F_n|} = \inf_{F \in \mathcal{F}(G)} \frac{f(F)}{|F|},
\]
(see Definitions 2.2.10 and 3.1.5, Remark 3.1.7 and Proposition 3.1.9 of [23]).

(2) If $G$ admits a Følner sequence $\{F_n\}_{n \in \mathbb{N}}$ of tiling sets, then
\[
\lim_{n \to \infty} \frac{f(F_n)}{|F_n|} = \inf_{n \in \mathbb{N}} \frac{f(F_n)}{|F_n|},
\]
and the value of the limit is independent of the choice of such a Følner sequence. For details see [32].

2.2. Topological pressure for sub-additive potentials

Let $(X, G)$ be a $G$-system. Denote by $\mathcal{B}_X$ the collection of all Borel subsets of $X$. Recall that a cover of $X$ is a family of Borel subsets of $X$ whose union is $X$. An open cover is one that consists of open sets. A partition of $X$ is a cover of $X$ consisting of pairwise disjoint sets. We denote the set of finite covers, finite open covers and finite partition of $X$ by $\mathcal{C}_X$, $\mathcal{C}^0_X$ and $\mathcal{P}_X$, respectively. Given two covers $U, V \in \mathcal{C}_X$, $U$ is said to be finer than $V$ (denoted by $U \succ V$) if each element of $U$ is contained in some element of $V$. Let $U \lor V = \{U \cap V : U \in U, V \in V\}$. Given $F \in \mathcal{F}(G)$ and $U \in \mathcal{C}_X$, set $U_F = \bigvee_{g \in F} g^{-1}U$ (letting $U_\emptyset = \{X\}$).

We now define the topological pressure of a sub-additive potential $\mathcal{F} = \{f_E\}_{E \in \mathcal{F}(G)}$ relative to an open cover. For $E \in \mathcal{F}(G)$ and $U \in \mathcal{C}_X$, we define
\[
P_E(G, \mathcal{F}; U) := \inf \left\{ \sum_{V \in \mathcal{V}} \sup_{x \in V} e^{f_E(x)} : \mathcal{V} \in \mathcal{C}_X \text{ and } \mathcal{V} \succ U_E \right\}.
\]
(2.1)

Remark 2.3. It is easy to deduce that in the definition of (2.1) we can restrict $\mathcal{V}$ to partitions, i.e.,
\[
P_E(G, \mathcal{F}; U) = \inf \left\{ \sum_{B \in \beta} \sup_{x \in B} e^{f_E(x)} : \beta \in \mathcal{P}_X \text{ and } \beta \succ U_E \right\}.
\]
(2.2)

In fact, let $\mathcal{V} = \{V_1, \ldots, V_k\} \in \mathcal{C}_X$ with $\mathcal{V} \succ U_E$. Denote $B_i = V_i \setminus \bigcup_{j=1}^{i-1} V_j$, $1 \leq i \leq k$. Then $\beta = \{B_i : B_i \neq \emptyset, 1 \leq i \leq k\} \in \mathcal{P}_X$ with $\beta \succ \mathcal{V} \succ U_E$, and
\[
\sum_{V \in \mathcal{V}} \sup_{x \in V} e^{f_E(x)} \geq \sum_{B \in \beta} \sup_{x \in B} e^{f_E(x)}.
\]
Since $\mathcal{V}$ is arbitrary, (2.2) holds.

Lemma 2.4. The following hold:

(1) $K = \sup \{ \frac{|f_E(x)|}{|E|} : x \in X, E \in \mathcal{F}(G) \} < \infty$;
(2) Set \( \mathcal{G} = \{ f_E(x) + C \cdot |E| : E \in \mathcal{F}(G), \ X \ni x \mapsto f_E(x) \in \mathcal{F} \} \) (where \( C \) is the constant in condition (C3)), then \( \mathcal{G} \) is a monotone sub-additive potential. If \( \mathcal{F} \) is strongly sub-additive, then \( \mathcal{G} \) is also strongly sub-additive.

**Proof.** It easily follows from conditions (C1), (C2) and (C3).

It is not hard to see that \( E \in \mathcal{F}(G) \mapsto \log P_E(G, \mathcal{F}; U) \) is a monotone \( G \)-invariant sub-additive function. By Lemma 2.1,

\[
\lim_{n \to \infty} \frac{1}{|F_n|} \log P_{F_n}(G, \mathcal{F}; U) = \lim_{n \to \infty} \frac{1}{|F_n|} \log P_{F_n}(G, \mathcal{F}; U) - C
\]

is independent of the choice of the Følner sequence \( \{F_n\}_{n \in \mathbb{N}} \). Define the **topological pressure of** \( \mathcal{F} \) **relative to** \( U \) as

\[
P(G, \mathcal{F}; U) := \lim_{n \to \infty} \frac{1}{|F_n|} \log P_{F_n}(G, \mathcal{F}; U),
\]

(2.3)

where \( \{F_n\}_{n \in \mathbb{N}} \) is a Følner sequence of \( G \). The **topological pressure of** \( \mathcal{F} \) **is defined by**

\[
P(G, \mathcal{F}) := \sup_{U \in \mathcal{C}_X^U} P(G, \mathcal{F}; U).
\]

(2.4)

For a \( G \)-invariant Borel probability measure \( \mu \), denote

\[
\mathcal{F}_n(\mu) := \lim_{n \to \infty} \frac{1}{|F_n|} \int f_{F_n} \, d\mu,
\]

where \( \{F_n\}_{n \in \mathbb{N}} \) is a Følner sequence. The existence of the above limit follows from conditions (C1), (C2) and (C3). We call \( \mathcal{F}_n(\mu) \) the **Lyapunov exponent of** \( \mathcal{F} \) **with respect to** \( \mu \).

### 2.3. Measure-theoretic entropy

We recall the basic definitions about measure-theoretic entropy (see [17] for details). Let \( \mathcal{M}(X), \mathcal{M}(X, G) \) and \( \mathcal{M}_e(X, G) \) be the sets of all Borel probability measures, \( G \)-invariant Borel probability measures and ergodic \( G \)-invariant Borel probability measures, on \( X \), respectively. Note that amenability of \( G \) ensures that \( \mathcal{M}(X, G) \neq \emptyset \) and both \( \mathcal{M}(X) \) and \( \mathcal{M}(X, G) \) are convex compact metric spaces when endowed with the weak*-topology; \( \mathcal{M}_e(X, G) \) is a \( G_\beta \)-subset of \( \mathcal{M}(X, G) \).

Given \( \alpha, \beta \in \mathcal{P}_X \) and \( \mu \in \mathcal{M}(X) \), define

\[
H_\mu(\alpha) = \sum_{A \in \alpha} -\mu(A) \log \mu(A) \quad \text{and} \quad H_\mu(\alpha | \beta) = H_\mu(\alpha \vee \beta) - H_\mu(\beta).
\]

One standard fact is that \( H_\mu(\alpha | \beta) \) increases with respect to \( \alpha \) and decreases with respect to \( \beta \). When \( \mu \in \mathcal{M}(X, G) \), it is not hard to see that \( F \in \mathcal{F}(G) \mapsto H_\mu(\alpha_F) \) is a monotone non-negative
$G$-invariant sub-additive function for a given $\alpha \in \mathcal{P}_X$. The **measure-theoretic entropy of $\mu$ relative to $\alpha$** is defined by

$$h_{\mu}(G, \alpha) = \lim_{n \to \infty} \frac{1}{|F_n|} H_{\mu}(\alpha F_n) = \inf_{F \in \mathcal{F}(G)} \frac{1}{|F|} H_{\mu}(\alpha F),$$

where $\{F_n\}_{n \in \mathbb{N}}$ is a Følner sequence of $G$. The last identity follows from the fact that $F \in \mathcal{F}(G) \mapsto H_{\mu}(\alpha F)$ is strongly sub-additive (see, e.g., [17, Lemma 3.1] or [23, Proposition 4.2.9]).

The **measure-theoretic entropy of $\mu$** is defined by

$$h_{\mu}(G, X) = \sup_{\alpha \in \mathcal{P}_X} h_{\mu}(G, \alpha).$$

(2.5)

For a given $\mathcal{U} \in \mathcal{C}_X$, W. Huang, X. Ye and G. Zhang (see [17]) introduced the following two types of **measure-theoretic entropy relative to $\mathcal{U}$** as

$$h_{\mu}^-(G, \mathcal{U}) := \lim_{n \to \infty} \frac{1}{|F_n|} H_{\mu}(\mathcal{U} F_n) \quad \text{and} \quad h_{\mu}^+(G, \mathcal{U}) := \inf_{\alpha \geq \mathcal{U}, \alpha \in \mathcal{P}_X} h_{\mu}(G, \alpha),$$

where

$$H_{\mu}(\mathcal{U}) := \inf_{\alpha \geq \mathcal{U}, \alpha \in \mathcal{P}_X} H_{\mu}(\alpha).$$

**Remark 2.5.** (1) It is not hard to see that $h_{\mu}^-(G, \mathcal{U}) \leq h_{\mu}^+(G, \mathcal{U})$. Moreover, Huang, Ye and Zhang (see [17, Theorem 4.14]) proved that these two kinds of measure-theoretic entropy are equal to each other. Thus we denote

$$h_{\mu}(G, \mathcal{U}) = h_{\mu}^\pm(G, \mathcal{U}).$$

(2) For $\mu \in \mathcal{M}(X, G)$, the following holds (see [17, Theorem 3.5]):

$$h_{\mu}(G, X) = \sup_{\mathcal{U} \in \mathcal{C}_X^0} h_{\mu}(G, \mathcal{U}).$$

**Lemma 2.6** (Ergodic decomposition of local entropy). (See [17, Lemma 3.12].) Let $\mathcal{U} \in \mathcal{C}_X^0$ and $\mu \in \mathcal{M}(X, G)$. The local entropy function $\theta \mapsto h_{\theta}(G, \mathcal{U})$ is upper semi-continuous and affine on $\mathcal{M}(X, G)$, and

$$h_{\mu}(G, \mathcal{U}) = \int_{\mathcal{M}^e(X, G)} h_{\theta}(G, \mathcal{U}) \, d\mu(\theta),$$

where $\mu = \int_{\mathcal{M}^e(X, G)} \theta \, d\mu(\theta)$ is the ergodic decomposition of $\mu$.

**3. A local variational principle of topological pressure**

In this section, we mainly prove a local variational principle of topological pressure for sub-addition potentials.
3.1. Some lemmas

Now we give some lemmas which are needed in our proof of Theorem 1.1. The first lemma is an obvious fact and we omit the proof.

**Lemma 3.1.** Let \( T \in \mathcal{F}(G) \) be a tile of \( G \) and \( \{F_n\}_{n \in \mathbb{N}} \) a Følner sequence. For each \( n \in \mathbb{N} \), let \( C_n \) be the tiling center of \( F_n \) relative to \( T \), i.e., \( F_n \subseteq \bigcup_{c \in C_n} T_c \) and \( T_c \cap F_n \neq \emptyset \) for all \( c \in C_n \), then

\[
\lim_{n \to \infty} \frac{|TC_n|}{|F_n|} = 1.
\]

**Lemma 3.2.** Let \( f : \mathcal{F}(G) \to \mathbb{R} \) be a monotone strongly sub-additive function, \( m,k \in \mathbb{N} \), \( E,F,B,E_1,\ldots,E_k \in \mathcal{F}(G) \). Then:

1. If \( 1_E(g) = \frac{1}{m} \sum_{i=1}^{k} 1_{E_i}(g) \) holds for each \( g \in G \), then \( f(E) \leq \frac{1}{m} \sum_{i=1}^{k} f(E_i) \);
2. If \( K = \sup \{ \frac{f(E)}{|E|} : E \in \mathcal{F}(G) \} < \infty \), then

\[
f(F) \leq \sum_{g \in F} \frac{1}{|B|} f(Bg) + K \cdot |F \setminus A_{F,B}|,
\]

where, \( A_{F,B} = \{ g \in G : B^{-1}g \subseteq F \} \).

**Proof.** (1) It is a special case of [23, Lemma 2.2.16], and we give a proof for completeness. Clearly, \( \bigcup_{i=1}^{k} E_i = E \). Set \( \{A_1,\ldots,A_n\} = \bigvee_{i=1}^{k} \{E_i, E \setminus E_i\} \) (neglecting all empty elements). Set \( K_0 = \emptyset, K_i = \bigcup_{j=1}^{j} A_j, i = 1,\ldots,n \). Then \( \emptyset = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = E \). Note that if for some \( i = 1,\ldots,n \) and \( j = 1,\ldots,k \) with \( E_j \cap (K_i \setminus K_{i-1}) \neq \emptyset \), then \( K_i \setminus K_{i-1} \subseteq E_j \), and so \( K_i = K_{i-1} \cup (K_i \cap E_j) \). By strongly sub-additivity of \( f \), we have \( f(K_i) + f(K_{i-1} \cap E_j) \leq f(K_{i-1}) + f(K_i \cap E_j) \), i.e.,

\[
f(K_i) - f(K_{i-1}) \leq f(K_i \cap E_j) - f(K_{i-1} \cap E_j).
\]

Now for each \( i = 1,\ldots,n \), we pick \( k_i \in K_i \setminus K_{i-1} \), then one has

\[
f(E) = \sum_{i=1}^{n} \left( \frac{1}{m} \sum_{i=1}^{k} 1_{E_i}(k_i) \right) (f(K_i) - f(K_{i-1}))
\]

\[
= \frac{1}{m} \sum_{j=1}^{k} \sum_{1 \leq i \leq n} \left( f(K_i) - f(K_{i-1}) \right)
\]

\[
\leq \frac{1}{m} \sum_{j=1}^{k} \sum_{1 \leq i \leq n} \left( f(K_i \cap E_j) - f(K_{i-1} \cap E_j) \right)
\]
\[
\leq \frac{1}{m} \sum_{j=1}^{k} \sum_{i=1}^{n} \left( f(K_i \cap E_j) - f(K_{i-1} \cap E_j) \right) = \frac{1}{m} \sum_{j=1}^{k} f(E_j).
\]

(2) Note that for each \( l \in G \), we have \( 1_{\{ h \in BF : B^{-1}h \subseteq F \}}(l) = \frac{1}{|B|} \sum_{g \in F} 1_{\{ h \in Bg : B^{-1}h \subseteq F \}}(l) \).

Using (1), we get
\[
f\left( \{ h \in BF : B^{-1}h \subseteq F \} \right) \leq \frac{1}{|B|} \sum_{g \in F} f\left( \{ h \in Bg : B^{-1}h \subseteq F \} \right) \leq \frac{1}{|B|} \sum_{g \in F} f(Bg),
\]
which implies
\[
f(F) \leq f\left( \{ h \in BF : B^{-1}h \subseteq F \} \right) + f\left( F \setminus \{ h \in BF : B^{-1}h \subseteq F \} \right)
\leq \frac{1}{|B|} \sum_{g \in F} f(Bg) + |F \setminus \{ h \in BF : B^{-1}h \subseteq F \}| \cdot K
= \frac{1}{|B|} \sum_{g \in F} f(Bg) + K \cdot |F \setminus A_{F,B}|.
\]

The proof is complete. \( \square \)

**Lemma 3.3.** Let \( (X, G) \) be a zero-dimensional \( G \)-system, \( \mu \in \mathcal{M}(X, G) \), \( \mathcal{F} \) be a sub-additive potential and \( U \in C^0_X \). Assume that for some \( K \in \mathbb{N} \), \( \{ \alpha_l \}_l \) is a sequence of finite clopen (close and open) partitions of \( X \) which are finer than \( U \). Then for each \( E \in \mathcal{F}(G) \), there is a finite subset \( B_E \) of \( X \) such that each atom of \( (\alpha_l)_E \), \( l = 1, \ldots, K \), contains at most one point of \( B_E \), and \( \sum_{x \in B_E} e^{f_E(x)} \geq \frac{P_E(G, \mathcal{F}; U)}{K} \).

**Proof.** The proof follows directly from that of [16, Lemma 4.4] and is omitted. \( \square \)

Let \( (X, G) \) and \( (Y, G) \) be two \( G \)-systems. A continuous map \( \pi : X \to Y \) is called a **homomorphism** or a **factor map** from \( (X, G) \) to \( (Y, G) \) if it is onto and \( \pi \circ g = g \circ \pi \) for all \( g \in G \). We say that \( (X, G) \) is an **extension** of \( (Y, G) \), and \( (Y, G) \) is a **factor** of \( (X, G) \). If \( \pi \) is also injective then it is called an **isomorphism**.

**Lemma 3.4.** Let \( \pi : (X, G) \to (Y, G) \) be a factor map and \( \mathcal{F} = \{ f_E \}_{E \in \mathcal{F}(G)} \) a sub-additive potential on \( (Y, G) \). If \( \mu \in \mathcal{M}(X, G) \), \( \nu = \pi_* \mu \), \( \alpha \in \mathcal{P}_Y \) and \( U \in C^0_Y \), then

1. \( h_{\mu}(G, \pi^{-1}(\alpha)) = h_{\nu}(G, \alpha) \);
2. \( P(G, \mathcal{F} \circ \pi; \pi^{-1}U) = P(G, \mathcal{F}; U) \).

**Proof.** (1) It is an obvious fact.

(2) It suffices to show that for each \( E \in \mathcal{F}(G) \)
\[
P_E(G, \mathcal{F} \circ \pi; \pi^{-1}U) = P_E(G, \mathcal{F}; U).
\]

(3.1)
Let $V \in \mathcal{C}_Y$ with $V \geq U_E$ then $\pi^{-1}V \in \mathcal{C}_X$ and $\pi^{-1}V \geq (\pi^{-1}U)_E$. Hence
\[
\sum_{V \in \mathcal{Y}} \sup_{y \in V} e^{f_E(y)} = \sum_{V \in \mathcal{Y}} \sup_{Z \in \pi^{-1}V} e^{f_E \circ \pi(Z)} \geq P_E(G, \mathcal{F} \circ \pi; \pi^{-1}U).
\]
Since $\mathcal{V}$ is arbitrary, we have that $P_E(G, \mathcal{F} \circ \pi; \pi^{-1}U) \geq P_E(G, \mathcal{F} \circ \pi; \pi^{-1}U)$. The proof of the converse inequality is similar. \qed

For a fixed $U = \{U_1, \ldots, U_M\} \in \mathcal{C}_0^X$, let $U^* = \{\{A_1, \ldots, A_M\} \in \mathcal{P}_X: A_m \subseteq U_m: 1 \leq m \leq M\}$.

The following lemma will be used in the computation of $H_\mu(U)$ and $h_\mu(T, U)$ (see [15, Lemma 2] for detail).

**Lemma 3.5.** Let $H: \mathcal{P}_X \to \mathbb{R}$ be monotone in the sense that $H(\alpha) \geq H(\beta)$ whenever $\alpha \geq \beta$. Then
\[
\inf_{\alpha \in \mathcal{P}_X, \alpha \geq U} H(\alpha) = \inf_{\alpha \in U^*} H(\alpha).
\]

**Lemma 3.6.** Let $(X, G)$ be a $G$-system, where $G$ is an Abelian group. Suppose $\{v_n\}_{n=1}^\infty$ is a sequence in $\mathcal{M}(X)$ and $\{F_n\}_{n=1}^\infty$ is a tiling Følner sequence of $G$. We form the new sequence $\{\mu_n\}_{n=1}^\infty$ by $\mu_n = \frac{1}{|F_n|} \sum g \in F_n g v_n$. Assume that $\mu_{n_i}$ converges to $\mu$ in $\mathcal{M}(X)$ for some subsequence $\{n_i\}$ of natural numbers. Then $\mu \in \mathcal{M}(X, G)$, and moreover
\[
\lim_{i \to \infty} \frac{1}{|F_{n_i}|} \int f_{F_{n_i}} \, dv_{n_i} \leq \mathcal{F}_*(\mu). \tag{3.2}
\]

**Proof.** The statement $\mu \in \mathcal{M}(X, G)$ is well known. Now we show the desired inequality. Fix $k \in \mathbb{N}$. Since $F_k$ is a tile of $G$, let $C_n$ be a tiling center of $F_n$ relative to $F_k$, i.e.,
\[
\bigsqcup_{c \in C_n} F_k c \supseteq F_n \quad \text{and} \quad F_k c \cap F_n \neq \emptyset, \quad \forall c \in C_n, \ n \in \mathbb{N}. \tag{3.3}
\]
By Lemma 3.1, for each $\epsilon > 0$, when $n$ is large enough, we have
\[
|F_k C_n| \leq |(1 + \epsilon)|F_n| \quad \text{and} \quad |F_k C_n \setminus F_n| \leq \epsilon |F_n|. \tag{3.4}
\]
Without loss of generality, we can assume that $\mathcal{F}$ is monotone, then
\[
f_{F_n a}(x) \leq f_{\bigsqcup_{c \in C_n} F_k c a}(x) \leq \sum_{c \in C_n} f_{F_k c a}(x), \quad \forall a \in F_k.
\]
Set $g_{F_n}(x) = \frac{1}{|F_k|} \sum_{a \in F_k} f_{F_n a}(x)$. Since $G$ is an Abelian group, we have
\[
g_{F_n}(x) \leq \frac{1}{|F_k|} \sum_{a \in F_k, c \in C_n} f_{F_k c a}(x) = \frac{1}{|F_k|} \sum_{g \in F_k C_n} f_{F_k g}(x).
\]
Moreover, by (3.4), we get
\[
\frac{1}{|F_n|} \int_X g_{F_n}(x) \, d\nu_n(x) \leq \frac{1}{|F_n||F_k|} \int_X \sum_{g \in F_k} f_{F_k g}(x) \, d\nu_n(x)
\]
\[
= \frac{|F_k C_n|}{|F_n||F_k|} \int_X f_{F_k}(x) \, d\bar{\mu}_n(x)
\]
\[
\leq \frac{1 + \epsilon}{|F_k|} \int_X f_{F_k}(x) \, d\bar{\mu}_n(x),
\]
where \(\bar{\mu}_n = \frac{1}{|F_k C_n|} \sum_{g \in F_k C_n} g \nu_n\). To complete the lemma, it suffices to show the following two claims.

Claim 1. \(\lim_{n \to \infty} \frac{1}{|F_n|} \int_X |f_{F_n}(x) - g_{F_n}(x)| \, d\nu_n(x) = 0\).

Proof. Since \(\mathcal{F}\) is monotone and sub-additive, for each \(a \in F_k\),
\[
f_{F_n}(x) \leq f_{F_n a}(x) + f_{F_n \setminus F_n a}(x) \leq f_{F_n a}(x) + K \cdot |F_n \setminus F_n a|,
\]
where \(K\) is the constant in Lemma 2.4(1). By symmetry, \(|f_{F_n}(x) - f_{F_n a}(x)| \leq K \cdot |F_n \setminus F_n a|\).

Thus,
\[
|f_{F_n}(x) - g_{F_n}(x)| = \left| \frac{1}{|F_k|} \sum_{a \in F_k} f_{F_n}(x) - f_{F_n a}(x) \right| \leq \frac{K}{|F_k|} \sum_{a \in F_k} |F_n \setminus F_n a|.
\]

Therefore,
\[
\frac{1}{|F_n|} \int_X |f_{F_n}(x) - g_{F_n}(x)| \, d\nu_n \leq \frac{K}{|F_k|} \frac{\sum_{a \in F_k} |F_n \setminus F_n a|}{|F_n|} \to 0 \quad (n \to \infty).
\]
This completes the proof of Claim 1. \(\square\)

Claim 2. With the weak\(^*\)-topology, \(\bar{\mu}_n \to \mu\).

Proof. It suffices to show that for each \(f \in C(X)\),
\[
\left| \int_X f \, d\mu_n - \int_X f \, d\bar{\mu}_n \right| \to 0 \quad (n \to \infty). \tag{3.5}
\]

By Lemma 3.1, \(\lim_{n \to \infty} \frac{|F_k C_n|}{|F_n|} = 1\). So
\[
\lim_{n \to \infty} \left| \int_X f \, d\mu_n - \int_X f \, d\bar{\mu}_n \right|
\]
\[
= \lim_{n \to \infty} \left| \frac{1}{|F_n|} \sum_{g \in F_n} \int f(gx) \, d\nu_n(x) - \frac{1}{|F_k C_n|} \sum_{g \in F_k C_n} \int f(gx) \, d\nu_n(x) \right|
\]
\[
= \lim_{n \to \infty} \left| \frac{1}{|F_n|} \sum_{g \in F_n} \int f(gx) \, d\nu_n(x) - \frac{1}{|F_n|} \sum_{g \in F_k C_n} \int f(gx) \, d\nu_n(x) \right|
\]
\[
\leq \lim_{n \to \infty} \frac{|F_k C_n \setminus F_n|}{|F_n|} \cdot \|f\| = 0.
\]

This completes the proof of Claim 2. \(\square\)

The following lemma is well known (see [31, Lemma 9.9] for a proof).

**Lemma 3.7.** Let \(a_1, a_2, \ldots, a_k\) be given real numbers. If \(p_i \geq 0, i = 1, 2, \ldots, k\), and \(\sum_{i=1}^k p_i = 1\), then

\[
\sum_{i=1}^k p_i (a_i - \log p_i) \leq \log \left( \sum_{i=1}^k e^{a_i} \right),
\]

(3.6)

and the equality holds if and only if \(p_i = \frac{e^{a_i}}{\sum_{j=1}^k e^{a_j}}\) for all \(i = 1, 2, \ldots, k\).

### 3.2. Proof of Theorem 1.1

In this section we give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** We divide the proof into four small steps:

**Step 1.** \(P(G, \mathcal{F}; U) \geq h_\mu(T, U) + \mathcal{F}_*(\mu)\) for all \(\mu \in \mathcal{M}(X, G)\).

Let \(\mu \in \mathcal{M}(X, G)\) and \(\{F_n\}_{n \in \mathbb{N}}\) be a Følner sequence of \(G\). It follows from Lemma 3.7 that for every \(\beta \in \mathcal{P}_X\) satisfying \(\beta \geq U F_n\),

\[
\log \left( \sum_{B \in \beta} \sup_{x \in B} e^{f_{F_n}(x)} \right) \geq \sum_{B \in \beta} \mu(B) \left( \sup_{x \in B} f_{F_n}(x) - \log \mu(B) \right) \quad \text{by (3.6)}
\]

\[
= H_\mu(\beta) + \sum_{B \in \beta} \sup_{x \in B} f_{F_n}(x) \cdot \mu(B)
\]

\[
\geq H_\mu(U F_n) + \int \limits_X f_{F_n} \, d\mu.
\]

By (2.2), we get

\[
\log P_{F_n}(G, \mathcal{F}; U) \geq H_\mu(U F_n) + \int \limits_X f_{F_n} \, d\mu.
\]

(3.7)

The proof of Step 1 is completed by dividing (3.7) by \(|F_n|\) and then passing to the limit as \(n \to \infty\).
Step 2. If \((X, G)\) is a zero-dimensional \(G\)-system, then there exists a \(\mu \in \mathcal{M}(X, G)\) such that
\[
P(G, \mathcal{F}; U) \leq h_\mu(G, U) + \mathcal{F}_\alpha(\mu). \tag{3.8}
\]

Let \(U = \{U_1, U_2, \ldots, U_d\}\) and define
\[
U^* = \{\alpha \in \mathcal{P}_X: \alpha = \{A_1, A_2, \ldots, A_d\}, A_m \subseteq U_m, m = 1, 2, \ldots, d\}.
\]

Since \(X\) is zero-dimensional, the family of partitions in \(U^*\), which are finer than \(U\) and consist of clopen (close and open) sets, is countable. We let \(\{\alpha_l\}_{l \in \mathbb{N}}\) denote an enumeration of this family.

Let \(\{F_n\}_{n \in \mathbb{N}}\) be a Følner sequence of \(G\) with \(|F_n| \geq n\) for each \(n \in \mathbb{N}\) (such a sequence exists since \(G\) is infinite). By Lemma 3.3, for each \(n \in \mathbb{N}\), there exists a finite subset \(B_n\) of \(X\) such that
\[
\sum_{x \in B_n} e^{f_{F_n}(x)} \geq \frac{P_{F_n}(G, \mathcal{F}; U)}{n}, \tag{3.9}
\]
and each atom of \((\alpha_l)_{F_n}\) contains at most one point of \(B_n\), for each \(l = 1, \ldots, n\). Let
\[
v_n = \sum_{x \in B_n} \lambda_n(x) \delta_x \quad \text{and} \quad \mu_n = \frac{1}{|F_n|} \sum_{g \in F_n} g v_n,
\]
where \(\lambda_n(x) = \frac{e^{f_{F_n}(x)}}{\sum_{y \in B_n} e^{f_{F_n}(y)}}\) for \(x \in B_n\). Since \(\mathcal{M}(X, G)\) is compact we can choose a subsequence \(\{n_l\} \subseteq \mathbb{N}\) such that \(\mu_{n_l}\) converges to some \(\mu \in \mathcal{M}(X, G)\). It is easy to check \(\mu \in \mathcal{M}(X, G)\).

We wish to show that \(\mu\) satisfies (3.8). By Lemma 3.5 and the fact that
\[
h_\mu^+(G, U) = \inf_{\beta \in U^*} h_\mu(G, \beta) = \inf_{l \in \mathbb{N}} h_\mu(G, \alpha_l),
\]
it is sufficient to show that for each \(l \in \mathbb{N}\)
\[
P(G, \mathcal{F}; U) \leq h_\mu(G, \alpha_l) + \mathcal{F}_\alpha(\mu). \tag{3.10}
\]

Fix \(l \in \mathbb{N}\). For each \(n > l\), from the construction of \(B_n\) we know that each atom of \((\alpha_l)_{F_n}\) contains at most one point of \(B_n\), and
\[
\sum_{x \in B_n} -\lambda_n(x) \log \lambda_n(x) = \sum_{x \in B_n} v_n(\{x\}) \log v_n(\{x\}) = H_{v_n}((\alpha_l)_{F_n}). \tag{3.11}
\]

Moreover, it follows from (3.9), (3.11) that
\[
\log P_{F_n}(G, \mathcal{F}; U) - \log n \leq \log \left(\sum_{x \in B_n} e^{f_{F_n}(x)}\right) = \sum_{x \in B_n} \lambda_n(x)(f_{F_n}(x) - \log \lambda_n(x))
\]
\[
= H_{v_n}((\alpha_l)_{F_n}) + \sum_{x \in B_n} \lambda_n(x) f_{F_n}(x)
\]
\[
= H_{v_n}((\alpha_l)_{F_n}) + \int_X f_{F_n}(x) \, d\nu_n(x).
\]
Hence,

\[
\log P_{F_n}(G, \mathcal{F}; U) - \log n \leq H_{\nu_n}((\alpha_l)_{F_n}) + \int_{X} f_{F_n}(x) \, d\nu_n(x).
\]  

(3.12)

Without loss of generality, we can assume that \(\mathcal{F}\) is monotone and sub-additive.

**Case 1.** \(G\) is Abelian group. We can assume that \(\{F_n\}_{n \in \mathbb{N}}\) is a tiling Følner sequence. Since \(E \in \mathcal{F}(G) \mapsto H_{\nu_n}((\alpha_l)_{E})\) is a non-negative, monotone and strongly sub-additive function (see, e.g., [17, Lemma 3.1] or [23, Proposition 4.2.9]), it follows from Lemma 3.2 that for each \(B \in \mathcal{F}(G)\), one has

\[
\frac{1}{|F_n|} H_{\nu_n}((\alpha_l)_{F_n}) \leq \frac{1}{|F_n|} \sum_{g \in F_n} \frac{1}{|B|} H_{\nu_n}((\alpha_l)_{Bg}) + \frac{|F_n \setminus A_{F_n,B}|}{|F_n|} \cdot \log |\alpha_l| 
\]

\[
= \frac{1}{|B|} \frac{1}{|F_n|} \sum_{g \in F_n} H_{\nu_n}((\alpha_l)_{B}) + \frac{|F_n \setminus A_{F_n,B}|}{|F_n|} \cdot \log |\alpha_l| 
\]

\[
\leq \frac{1}{|B|} H_{\mu_n}((\alpha_l)_{B}) + \frac{|F_n \setminus A_{F_n,B}|}{|F_n|} \cdot \log |\alpha_l|,
\]  

(3.13)

where \(A_{F_n,B} = \{g \in G : B^{-1} g \subseteq F_n\}\). Set \(B_1 = B^{-1} \cup \{e_G\}\). Note that for each \(\delta > 0\), \(F_n\) is \((B_1, \delta)\)-invariant if \(n\) is large enough and

\[
F_n \setminus A_{F_n,B} = F_n \cap B(G \setminus F_n) \subseteq (B_1)^{-1} F_n \cap (B_1)^{-1} (G \setminus F_n) = B(F_n, B_1).
\]

Letting \(n \to \infty\), we get

\[
\lim_{n \to \infty} \frac{|F_n \setminus A_{F_n,B}|}{|F_n|} \leq \lim_{n \to \infty} \frac{|B(F_n, B_1)|}{|F_n|} = 0.
\]  

(3.14)

Hence, combining Lemma 3.6, (3.12), (3.13) and (3.14) we obtain

\[
P(G, \mathcal{F}; U) = \lim_{i \to \infty} \frac{\log P_{F_{n_i}}(G, \mathcal{F}; U)}{|F_{n_i}|} 
\]

\[
\leq \limsup_{i \to \infty} \left( \frac{1}{|F_{n_i}|} H_{\nu_{n_i}}((\alpha_l)_{F_{n_i}}) + \log |n_i| + \frac{1}{|F_{n_i}|} \int_{X} f_{F_{n_i}}(x) \, d\nu_{n_i}(x) \right) 
\]

\[
\leq \frac{1}{|B|} H_{\mu}((\alpha_l)_{B}) + \mathcal{F}_*(\mu).
\]

Since \(B\) was an arbitrary element of \(\mathcal{F}(G)\), (3.10) holds.

**Case 2.** \(\mathcal{F}\) is strongly sub-additive. Now \(E \in \mathcal{F}(G) \mapsto \int_{X} f_E(x) \, d\nu(x)\) is a monotone strongly sub-additive function. By Lemma 3.2, for each \(B \in \mathcal{F}(G)\), one has
\[
\frac{1}{|F_n|} \int_X f_{F_n}(x) \, d\nu_n(x) \leq \frac{1}{|F_n|} \sum_{g \in F_n} \frac{1}{|B|} \int_X f_{Bg}(x) \, d\nu_n(x) + \frac{|F_n \setminus A_{F_n,B}|}{|F_n|} \cdot K
\]
\[= \frac{1}{|B|} \int_X f_B(x) \, d\mu_n(x) + \frac{|F_n \setminus A_{F_n,B}|}{|F_n|} \cdot K,
\]
(3.15)
where \(K\) is the constant in Lemma 2.4(1). Combining (3.12), (3.13), (3.14) and (3.15), we have
\[
P(G, \mathcal{F}; U) = \lim_{i \to \infty} \frac{\log P_{F_{n_i}}(G, \mathcal{F}; U)}{|F_{n_i}|}
\]
\[\leq \limsup_{i \to \infty} \left( \frac{1}{|F_{n_i}|} H_{\nu_{n_i}}((\alpha_i)_{F_{n_i}}) + \frac{\log n_i}{|F_{n_i}|} + \frac{1}{|F_{n_i}|} \int_X f_{F_{n_i}}(x) \, d\nu_{n_i}(x) \right)
\]
\[\leq \frac{1}{|B|} H_{\mu}((\alpha_l)_B) + \frac{1}{|B|} \int_X f_B(x) \, d\mu(x).
\]
Taking \(B = F_n\) and letting \(n \to \infty\), we get (3.10).

**Step 3.** For \(G\)-system \((X, G)\), there exists a \(\mu \in \mathcal{M}(X, G)\) such that (3.8) holds. It is well known that there exists a factor map \(\pi : (Z, G) \to (X, G)\), where \((Z, G)\) is a zero-dimensional \(G\)-system (see the proof of Theorem 5.1 in [17], for example). Using Step 2, there is \(\nu \in \mathcal{M}(Z, G)\) such that
\[
P(G, \mathcal{F} \circ \pi; \pi^{-1} U) \leq h_{\nu}(G, \pi^{-1} U) + (\mathcal{F} \circ \pi)_*(\nu).
\]
Let \(\mu = \pi_* \nu\). By Lemma 3.4, we get
\[
h_{\mu}(G, U) + \mathcal{F}_*(\mu) = \inf_{\alpha \in \mathcal{P}_X, \alpha \gg U} (h_{\mu}(G, \alpha) + \mathcal{F}_*(\mu))
\]
\[= \inf_{\alpha \in \mathcal{P}_X, \alpha \gg U} \left( h_{\nu}(G, \pi^{-1}(\alpha) + (\mathcal{F} \circ \pi)_*(\nu) \right)
\]
\[\geq h_{\nu}(G, \pi^{-1}U) + (\mathcal{F} \circ \pi)_*(\nu).
\]
\[\geq P(G, \mathcal{F} \circ \pi; \pi^{-1} U) = P(G, \mathcal{F}; U).
\]

**Step 4.** We will show that the maximum of (1.2) can be attained in \(\mathcal{M}^e(X, G)\). Let \(\mu = \int_{\mathcal{M}^e(X, G)} \theta \, d\mu(\theta)\) be the ergodic decomposition of \(\mu\). Note that \(\theta \mapsto \mathcal{F}_*(\theta)\) is Borel measurable and \(\sup \{ |f_{F_n}(x)|/|F_n| : x \in X, \ F_n \in \mathcal{F}(G)\} < \infty\). Then, by Lebesgue’s Dominated Convergence Theorem,
\[
\int_{\mathcal{M}^e(X, G)} \mathcal{F}_*(\theta) \, d\mu(\theta) = \int_{\mathcal{M}^e(X, G)} \lim_{n \to \infty} \frac{1}{|F_n|} \int_X f_{F_n}(x) \, d\theta(x) \, d\mu(\theta)
\]
\[= \lim_{n \to \infty} \frac{1}{|F_n|} \int_X f_{F_n}(x) \, d\theta(x) \, d\mu(\theta)
\]
Combining Lemma 2.6, (3.16) and (1.2),

\[ P(G, \mathcal{F}; \mathcal{U}) = h_\mu(G, \mathcal{U}) + \mathcal{F}_*(\mu) \]

\[ = \int_{\mathcal{M}^e(X,G)} h_\theta(G, \mathcal{U}) \, dm(\theta) + \int_{\mathcal{M}^e(X,G)} \mathcal{F}_*(\theta) \, dm(\theta) \]

Hence there exists \( \theta \in \mathcal{M}^e(X,G) \) such that

\[ P(G, \mathcal{F}; \mathcal{U}) \leq h_\theta(G, \mathcal{U}) + \mathcal{F}_*(\theta), \]

which completes the proof of Theorem 1.1. \( \square \)

4. An example

In this section, we give a nontrivial example of sub-additive potentials.

**Example 4.1.** Let \((X, G)\) be a \(G\)-system and \(M : X \to \mathbb{R}^{n \times n}\) a strictly positive continuous matrix function of \(X\), i.e., \(M = (M_{i,j})_{n \times n}\), where \(M_{i,j}\) are strictly positive continuous of \(X\) for all \(i, j = 1, 2, \ldots, n\).

Now we define \(\mathcal{F} = \{ f_E \}_{E \in \mathcal{F}(G)}\) as follows: for each \(x \in X\) and \(E \in \mathcal{F}(G)\),

\[ f_E(x) := \min \log \| M(g_1 x) M(g_2 x) \cdots M(g_{|E|} x) \|, \]

where \(g_1, \ldots, g_{|E|}\) range over enumerations of elements of \(E\) and \(\| M(x) \| = \sum_{i,j=1}^{n} M_{i,j}(x)\).

We will show that \(\mathcal{F}\) satisfies conditions (C1), (C2) and (C3).

- \(f_{E \cup F} \leq f_E + f_F\) for all \(E, F \in \mathcal{F}(G)\) with \(E \cap F = \emptyset\);

- Given \(E \in \mathcal{F}(G)\) and \(g \notin E\). Then there exist an enumeration \(g_1, \ldots, g_{|E|}\) of element of \(E\) and \(0 \leq i \leq |E|\) such that

\[ f_{E \cup \{g\}}(x) = \log \| M(g_1 x) \cdots M(g_{i-1} x) M(g x) M(g_{i+1} x) \cdots M(g_{|E|} x) \|. \]

Set \(A = M(g_1 x) \cdots M(g_{i-1} x),\)

\(B = M(g x)\) and \(C = M(g_{i+1} x) \cdots M(g_{|E|} x)\). Thus,

\[ f_E(x) - f_{E \cup \{g\}}(x) \leq \log \frac{\| AC \|}{\| ABC \|}. \]

Let

\[ K_1 = \min_{1 \leq i, j \leq n} \min_{x \in X} \frac{M_{i,j}(x)}{\max_{1 \leq i, j \leq n} \max_{x \in X} M_{i,j}(x)}, \quad K_2 = \min_{1 \leq i, j \leq n} \min_{x \in X} M_{i,j}(x). \]
Then $K_1, K_2 \in (0, +\infty)$ and $M(x) - \frac{K_1}{n} EM(x)$ is a non-negative matrix, where $E = (E_{i,j})$, $E_{i,j} \equiv 1$. Hence,

$$\|ABC\| \geq \left\| A \frac{K_1}{n} EBC \right\| = \frac{K_1}{n} \|A\|\|BC\|$$

$$\geq \left( \frac{K_1}{n} \right)^2 \|A\|\|BEC\| = \left( \frac{K_1}{n} \right)^2 \|A\|\|B\|\|C\|$$

$$\geq \left( \frac{K_1}{n} \right)^2 n^2 K_2 \|A\|\|C\| = K_1^2 K_2 \|A\|\|C\|,$$

which implies that

$$\frac{\|AC\|}{\|ABC\|} \leq \frac{1}{K_1^2 K_2} < \infty.$$

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