Functional Equation and Its Modular Stability With and Without $\Delta_p$–Condition

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Abstract. Mixed type is a further step of development in functional equations. In this paper, the authors made an attempt to introduce such equation of the following form with its general solution

$$h(py + z) + h(py - z) + h(y + pz) + h(y - pz) = (p + p^2)[h(y + z) + h(y - z)] + 2h(py) - 2(p^2 + p - 1)h(y)$$

for all $y, z \in \mathbb{R}, p \neq 0, \pm 1$. Also, without Fatou property authors investigate its various stabilities related to Ulam problem in modular space by considering with and without $\Delta_p$–condition.

1. Introduction

For the detailed study on Ulam problem and its recent developments called generalized Hyers-Ulam-Rassias stability, one can refer [1, 8, 11]. In 1950, Nakano [7] established the modular linear spaces and further developed by many authors, one can refer [5, 6, 9]. The definitions related to our main theorem related to modular space can be referred in [3, 4].

In 2015, Abasalt Bodaghi et al.[1] investigated the stabilities of following mixed type equation

$$h(3y + z) - 5h(2y + z) + h(2y - z) + 10h(y + z) - 5h(y - z) = 10h(z) + 4h(2y) - 8h(y)$$

for all $y, z \in \mathbb{R}$.

In 2016, Pasupathi Narasimman et al.[8] introduced the equations quintic and sextic, respectively of the form

$$p[h(py - z) + h(py + z)] + h(y - pz) + h(y + pz)$$

$$= (p^4 + p^2)[h(y - z) + h(y + z)] + 2(p^6 - p^4 - p^2 + 1)h(y),$$

$$h(py - z) + h(py + z) + h(y - pz) + h(y + pz)$$

$$= (p^4 + p^2)[h(y - z) + h(y + z)] + 2(p^6 - p^4 - p^2 + 1)[h(y) + h(z)]$$

with $p \in \mathbb{R} - \{0, \pm 1\}$ also discussed their various stabilities related to Ulam problem.
In 2017, authors Hark-Mahn Kim and Young Soon Hong [2] investigated the alternative stability theorem in a modular space using Δ_3-condition of a modified quadratic equation.

In 2019, authors John Michael Rassias, Hemen Dutta and Narasimman Pasupathi [10] investigated Ulam stability problem in non-Archimedean intuitionistic fuzzy normed spaces of the generalized quartic equation

\[ h(py-z) + h(py+z) + h(y-pz) + h(y+pz) = 2p^2[h(y-z) + h(y+z)] + 2(p^2 - 1)^2[h(y) + h(z)] \]

where \( p \neq 0, \pm 1 \). Motivation from the above literature, the authors made an attempt to introduce a new mixed type equation satisfied by \( h(x) = x + x^3 \) of the form

\[ h(py-z) + h(py+z) + h(y-pz) + h(y+pz) = (p + p^3)[h(y-z) + h(y+z)] - 2(p^2 - 1)h(y) \]

for all \( y, z \in \mathbb{R} \), \( p \neq 0, \pm 1 \). Mainly, authors investigate various stabilities concerning Ulam problem in modular spaces and its general solution.

In Section-2 and Section-3, authors obtain the solution of (1) in additive case and cubic case, respectively. Authors provide the various stabilities of equation (1) in modular space in Sections-4 for additive case and in Section-5 for cubic case, and we given the conclusion in Section-6.

2. General Solution of (1): Additive Case

**Lemma 2.1.** Let \( X \) and \( Y \) are linear spaces, a mapping \( h : X \rightarrow Y \) is additive and odd if \( h \) satisfies

\[ h(py-z) + h(py+z) + h(y-pz) + h(y+pz) = (p + p^3)[h(y-z) + h(y+z)] - 2(p^2 - 1)h(y) \]

for all \( y, z \in X \).

**Proof.** Consider \( h \) satisfies (2). Replacing \((y, z)\) by \((0, 0)\) and \((y, 0)\) in (2), we get \( h(0) = 0 \) and

\[ h(y) = ph(y) \]

respectively, for all \( y \in X \). Therefore, \( h \) is an additive function. Let \((y, z) = (0, y)\) in (2) and by (3), we reached

\[ h(-y) = -h(y); \quad y \in X. \]

Thus \( h \) is an odd function. \( \square \)

**Theorem 2.2.** A function \( h : X \rightarrow Y \) is a solution of (2) iff \( A(y) \) is the diagonal of the additive symmetric map \( A : X \times X \rightarrow Y \) such that \( h \) is of the form \( h(y) = A(y) \) for all \( y \in X \).

**Proof.** Let \( h \) satisfies (2) when \( h \) is additive. We can rewrite (2) as follows

\[ h(y) + \frac{1}{2(p^2 - 1)}h(py-z) + \frac{1}{2(p^2 - 1)}h(py+z) + \frac{1}{2(p^2 - 1)}h(y+pz) \]

\[ + \frac{1}{2(p^2 - 1)}h(y-pz) - \frac{p + p^3}{2(p^2 - 1)}h(y+z) - \frac{p + p^3}{2(p^2 - 1)}h(y-z) = 0 \]

for all \( y, z \in X \). Theorems 3.5 and 3.6 in [12] implies that \( h \) is of the form

\[ h(y) = A^1(y) + A^0(y) \]

for all \( y \in X \), \( A^0(y) = A^0 \) and for \( i = 1, A^i(y) \) is the diagonal of the \( i \)-additive symmetric map \( A_i : X^i \rightarrow Y \). We get \( A^0(y) = A^0 = 0 \) and \( h \) is odd, by \( h(0) = 0 \) and \( h(-y) = -h(y) \), respectively. It follows that \( h(y) = A^1(y) \).

Conversely, \( A^1(y) \) is the diagonal of the additive symmetric map \( A_1 : X^1 \rightarrow Y \) such that \( h(y) = A^1(y) \) for all \( y \in X \). From

\[ A^1(y+z) = A^1(y) + A^1(z), \quad A^1(ry) = r^1A^1(y); \quad y, z \in X, r \in Q, \]

we see that \( h \) satisfies (2) and this completes the proof of Theorem 2.2. \( \square \)
3. General Solution of (1): Cubic Case

Lemma 3.1. Let $X$ and $Y$ are linear spaces, a mapping $h : X \to Y$ is cubic and odd if $h$ satisfies

$$h(py + z) + h(py - z) + h(y + pz) + h(y - pz) = (p + p^2)[h(y + z) + h(y - z)] + 2(p^3 - p^2 - p + 1)h(y)$$

for all $y, z \in X$.

Proof. Consider $h$ satisfies (7). Replacing $(y, z)$ by $(0, 0)$ and $(y, 0)$ in (7), we get $h(0) = 0$ and

$$h(py) = p^3h(y)$$

respectively, for all $y \in X$. Therefore, $h$ is cubic function. Let $(y, z)$ by $(0, y)$ in (7) and using (8), we obtain

$$h(-y) = -h(y); \quad y \in X.$$  

Thus $h$ is an odd function. \qed

Theorem 3.2. A function $h : X \to Y$ is a solution of (7) iff $C^3(y)$ is the diagonal of the 3-additive symmetric map $C_3 : X^3 \to Y$ such that $h(y) = C_i^3(y)$ for all $y \in X$.

Proof. Let $h$ satisfies (7) when $h$ is cubic. We can rewrite (7) as follows

$$h(y) + \frac{1}{2(p^2 - 1)}h(py + z) + \frac{1}{2(p^2 - 1)}h(py - z) + \frac{1}{2(p^2 - 1)}h(y + pz)$$

$$+ \frac{1}{2(p^2 - 1)}h(y - pz) - \frac{p + p^2}{2(p^2 - 1)}h(y + z) - \frac{p + p^2}{2(p^2 - 1)}h(y - z) = 0$$

for all $y, z \in X$. Theorems 3.5 and 3.6 in [12] implies that $h$ is of the form

$$h(y) = C_i^3(y) + C^3(y) + C^3(y)$$

for all $y \in X$, where $C^0(y) = C^0$ and $i = 1, 2, 3$, $C^3(y)$ is the diagonal of the $i$-additive symmetric map $C_i : X^3 \to Y$. We get $C^0(y) = C^0 = 0$ and $h$ is odd, by $h(0) = 0$ and $h(-y) = -h(y)$, respectively. Therefore $C^3(y) = 0$. It follows that $h(y) = C^3(y) + C^3(y)$. By (8) and $C^n(rz) = r^nC^n(y)$ for all $y \in X$ and $r \in Q$, we obtain $n^1C^i(y) = n^1C^i(y)$. Hence, $C^i(x) = 0$ for all $y \in X$. Therefore $h(y) = C^3(y)$.

Conversely, $C^3(y)$ is the diagonal of the 3–additive symmetric map $C_3 : X^3 \to Y$ such that $h(y) = C^3(y)$ for all $y \in X$. From

$$C^3(y + z) = C_3^3(y) + 3C^3(y, z) + 3C^3(y, z) + C^3(z), \quad C^3(rz) = r^3C^3(y),$$

$$C^3(y, rz) = r^1C^3(y, z), \quad C^3(y, rz) = r^2C^3(y, z), \quad C^3(y, rz) = r^3C^3(y, z)$$

for all $y, z \in X, r \in Q$, we see that $h$ satisfies (7) and this completes the proof of Theorem 3.2. \qed

4. Stability of Functional Equation (1): Additive Case

Assume that the linear space $X$, $\mu$–complete convex modular space $X_\mu$ in the following theorems and corollaries. Now, we obtain the stability of (1) called generalized Hyers–Ulam–Rassias in modular spaces without $\Delta_2$–condition and the Fatou property. Here after, we use the following notation

$$D_\mu h(y, z) = h(py + z) + h(py - z) + h(y + pz) + h(y - pz) - (p + p^2)[h(y + z) + h(y - z)] + 2(p^2 - 1)h(y)$$

for all $y, z \in X$. 

Theorem 4.1. Let a mapping \( h : X \to X \) satisfies
\[
\mu(D_A h(y, z)) \leq v(y, z) \tag{12}
\]
and a mapping \( v : X^2 \to [0, \infty) \) such that
\[
\zeta(y, z) = \sum_{j=0}^{\infty} \frac{v(p_j y, p_j z)}{p^j} < \infty, \quad y, z \in X. \tag{13}
\]
Then there exists \( A_1 : X \to X \) a unique additive mapping defined by
\[
A_1(y) = \lim_{n \to \infty} h(p_n y) \quad \forall y \in X.
\]

Proof. Substituting \( z = 0 \) in (12), we obtain
\[
\mu(h(p y) - p h(y)) \leq \frac{1}{2} v(y, 0) \tag{15}
\]
and so
\[
\mu\left( h(y) - \frac{h(p y)}{p} \right) \leq \frac{1}{2p} v(y, 0), \quad \forall y \in X. \tag{16}
\]
By induction on \( n \), we arrive
\[
\mu\left( h(y) - \frac{h(p^n y)}{p^n} \right) \leq \frac{1}{2p} \sum_{j=0}^{n-1} \frac{v(p_j y, 0)}{p^{j+1}}, \quad \forall y \in X. \tag{17}
\]
Substituting \( y \) by \( p^m y \) in (17), we obtain
\[
\mu\left( h(p^m y) - \frac{h(p^{m+n} y)}{p^{m+n}} \right) \leq \frac{1}{2p} \sum_{j=m}^{n+m-1} \frac{v(p_j y, 0)}{p^j} \tag{18}
\]
by assumption (13) it converges to zero as \( m \to \infty \). Hence, by inequality (18) the sequence \( \left\{ \frac{h(p^m y)}{p^m} \right\}, \quad \forall y \in X \) is \( \mu \)-Cauchy and hence it is convergent in \( X_\mu \) since \( X_\mu \) is \( \mu \)-complete. Thus, a mapping \( A_1 : X \to X_\mu \) is defined by
\[
A_1(y) = \mu - \lim_{n \to \infty} \left( \frac{h(p^n y)}{p^n} \right)
\]
for all \( y \in X \), which implies
\[
\lim_{n \to \infty} \mu\left( \frac{h(p^n y)}{p^n} - A_1(y) \right) = 0, \quad \forall y \in X.
\]
Next, we claim the mapping \( A_1 \) satisfies (2). Setting \( (y, z) = (p^n y, p^n z) \) in (12), and dividing the resultant by \( p^n \), we arrive
\[
\frac{\mu(D_A h(p^n y, p^n z))}{p^n} \leq \frac{v(p^n y, p^n z)}{p^n}, \quad \forall y, z \in X.
\]
Hence, by property $\mu(au) \leq a\mu(u), 0 < a \leq 1, u \in X_\mu$, we get

$$
\mu \left( \frac{1}{4p^2 + 2p + 3} DA_1(y, z) \right)
\leq \mu \left( \frac{1}{4p^2 + 2p + 3} DA_1(y, z) - \frac{Dh(p^n y, p^n z)}{(4p^2 + 2p + 3)p^n} + \frac{Dh(p^n y, p^n z)}{(4p^2 + 2p + 3)p^n} \right)
\leq \frac{1}{4p^2 + 2p + 3} \mu \left( A_1(py + z) - \frac{h(p^n(y + z))}{p^n} \right) + \frac{1}{4p^2 + 2p + 3} \mu \left( A_1(py - z) - \frac{h(p^n(y - z))}{p^n} \right)
$$

for all $y, z \in X$ and $n$ is positive integers. We obtain $\mu \left( \frac{1}{4p^2 + 2p + 3} DA_1(y, z) \right) = 0$, if $n \to \infty$. Hence $DA_1(y, z) = 0$ for all $y, z \in X$. Thus $A_1$ satisfies (2) and hence it is additive. Since $\sum_{i=0}^n \frac{1}{p^n_i} + \frac{1}{p} \leq 1$ for all $n \in \mathbb{N}$, by the convexity of modular $\mu$ and (15), we arrive

$$
\mu \left( h(y) - A_1(y) \right) = \mu \left( h(y) - \frac{h(p^n y)}{p^n} \right) + \mu \left( \frac{h(p^n y)}{p^n} - A_1(y) \right)
\leq \frac{1}{2} \sum_{i=0}^{n-1} \frac{1}{p^{i+1}} v(p^i y, 0) + \mu \left( \frac{h(p^n y)}{p^n} - A_1(y) \right)
\leq \frac{1}{2} \sum_{i=0}^{n-1} \frac{1}{p^{i+1}} v(p^i y, 0) + \frac{1}{2} \varepsilon(y, 0)
$$

for all $y \in X$. Now, to prove the uniqueness of $A_1$, we consider that there exists a additive mapping $D_1 : X \to X_\mu$ satisfying

$$
\mu \left( h(y) - D_1(y) \right) \leq \frac{1}{2} \sum_{i=0}^{n-1} \frac{1}{p^{i+1}} v(p^i y, 0), \forall y \in X.
$$

But, if $A_1(y_0) \neq D_1(y_0)$ for some $y_0 \in X$. Then there exists a constant $\varepsilon > 0$ which is positive such that $\varepsilon < \mu(A_1(y_0) - D_1(y_0))$. By (13), there is a positive integer $n_0 \in \mathbb{N}$ such that $\sum_{i=0}^\infty \frac{1}{p^n_i} v(p^i y, 0) < \frac{\varepsilon}{2}$. Since $A_1$ and $D_1$ are additive mappings, by $A_1(p^{n_0} y_0) = p^{n_0} A_1(y_0)$ and $D_1(p^{n_0} y_0) = p^{n_0} D_1(y_0)$, we arrive

$$
\varepsilon < \mu \left( A_1(y_0) - D_1(y_0) \right)
= \mu \left( \frac{A_1(p^{n_0} y_0) - h(p^{n_0} y_0)}{p^{n_0}} + \frac{h(p^{n_0} y_0) - D_1(p^{n_0} y_0)}{p^{n_0}} \right)
\leq \frac{1}{p^{n_0}} \mu \left( A_1(p^{n_0} y_0) - h(p^{n_0} y_0) \right) + \frac{1}{p^{n_0}} \mu \left( h(p^{n_0} y_0) - D_1(p^{n_0} y_0) \right)
\leq \frac{1}{p^{n_0}} \sum_{j=0}^{n_0} \frac{v(p^{j+n_0} y_0, 0)}{p^{i+1}} \leq \sum_{j=0}^{n_0} \frac{v(p^{j+n_0} y_0, 0)}{p^{i+1}} < \varepsilon,
$$

which implies a contradiction. Therefore the mapping $A_1$ is a unique additive mapping near $h$ satisfying (14) in $X_\mu$.  \[ \square \]

Letting $\nu(y, z) = \varepsilon$ and $\nu(y, z) = \varepsilon (||y||^m + ||z||^n)$ in Theorem 4.1, we obtain Hyers-Ulam and generalized Hyers-Ulam stabilities, respectively in the following corollaries.
Corollary 4.2. Let a mapping \( h : X \rightarrow X \) satisfying
\[
\mu(D_{A}h(y,z)) \leq \epsilon, \quad \forall y, z \in X
\]
for some \( \epsilon > 0 \). Then there exists \( A_1 : X \rightarrow X \), a unique additive mapping satisfies (2) and
\[
\mu(h(y) - A_1(y)) \leq \frac{\epsilon}{2(p - 1)}
\]
for all \( y \in X \) and \( p \neq 1 \).

Corollary 4.3. If \( h : X \rightarrow X \) a mapping satisfies
\[
\mu(D_{A}h(y,z)) \leq \epsilon (\|y\|^m + \|z\|^m), \quad \forall y, z \in X, \quad m < 1
\]
a real numbers \( \epsilon > 0 \), then there exists \( A_1 : X \rightarrow X \), a unique additive mapping satisfying
\[
\mu(h(y) - A_1(y)) \leq \frac{\epsilon}{2(p - m^2)} \|y\|^m, \quad \forall y \in X
\]
where \( y \neq 0 \) and \( p^m < p \).

Assuming \( \mu \) satisfies the \( \Delta_p \)-condition and if there exists \( \beta > 0 \) defined by
\[
\mu(\beta y) \leq \beta \mu(y) \quad \text{for all} \quad y \in X.
\]
Theorem 4.4. Letting \( h : X \rightarrow X \) and \( \nu : X^2 \rightarrow [0, \infty) \) be the mappings satisfies
\[
\mu(D_{A}h(y,z)) \leq \nu(y,z)
\]
and
\[
\Psi(y,z) = \sum_{j=1}^{\infty} \frac{\beta^j}{p^j} \nu\left(\frac{y}{p^j}, \frac{z}{p^j}\right) < \infty, \quad \forall y, z \in X.
\]
Then there exists \( A_2 : X \rightarrow X \) a unique additive mapping such that \( A_2(y) = \lim_{n \to \infty} p^n h\left(\frac{y}{p^n}\right) \) which satisfies (2) and
\[
\mu(h(y) - A_2(y)) \leq \frac{1}{2p} \Psi(y,0), \quad \forall y \in X.
\]

Proof. The equation (15), implies that
\[
\mu\left(h(y) - ph\left(\frac{y}{p}\right)\right) \leq \frac{1}{2} \nu\left(\frac{y}{p}, 0\right), \quad y \in X.
\]
Hence, by the convexity \( \mu \), we have
\[
\mu\left(h(y) - p^2 h\left(\frac{y}{p^2}\right)\right) \leq \frac{1}{p} \mu\left(p h(y) - p^3 h\left(\frac{y}{p^3}\right)\right) \leq \frac{\beta}{2p} \nu\left(\frac{y}{p^2}, 0\right) + \frac{\beta^2}{2p^2} \nu\left(\frac{y}{p^3}, 0\right), \forall y \in X.
\]
Then by induction on \( n > 1 \), we have
\[
\mu\left(h(y) - p^n h\left(\frac{y}{p^n}\right)\right) \leq \frac{1}{2} \sum_{j=1}^{n-1} \frac{\beta^{2j-1}}{p^j} \nu\left(\frac{y}{p^j}, 0\right) + \frac{\beta^{2j}}{2p^{j+1}} \nu\left(\frac{y}{p^{j+1}}, 0\right).
\]
for all $y \in X$. Considering (25) holds true for $n$ and we deduce the following by using the convexity of $\mu$,

$$
\mu \left( h(y) - p^{n+1}h \left( \frac{y}{p^{n+1}} \right) \right)
$$

(26)

$$
= \frac{1}{p} \mu \left( ph(y) - p^2 h \left( \frac{y}{p} \right) \right) + \frac{1}{p} \mu \left( p^2 h \left( \frac{y}{p} \right) - p^{n+2}h \left( \frac{y}{p^{n+1}} \right) \right)
$$

$$
\leq \frac{\beta}{p} \mu \left( h(y) - ph \left( \frac{y}{p} \right) \right) + \frac{\beta^2}{p} \mu \left( h \left( \frac{y}{p} \right) - p^n h \left( \frac{y}{p^n} \right) \right)
$$

$$
\leq \frac{\beta}{2p} \nu \left( \frac{y}{p^{m+1}} \right) + \frac{\beta^2}{2p} \sum_{j=1}^{n-1} \frac{\beta^{2j-1}}{p^j} \nu \left( \frac{y}{p^{n+j}} \right) + \frac{\beta^2}{2p} \frac{\beta^{2(n-1)}}{p^{n-1}} \nu \left( \frac{y}{p^{n-1}} \right)
$$

$$
= \frac{1}{2} \sum_{j=1}^{n} \frac{\beta^{2j-1}}{p^j} \nu \left( \frac{y}{p^{j}} \right) + \frac{1}{2} \frac{\beta^{2n}}{p^{n+1}} \nu \left( \frac{y}{p^{n+1}} \right).
$$

The above inequality proves (25) for $n + 1$. Substituting $y$ by $\frac{y}{p^n}$ in (25), we arrive

$$
\mu \left( p^m \left( \frac{y}{p^m} \right) - p^{n+m}h \left( \frac{y}{p^{n+m}} \right) \right)
$$

$$
\leq \frac{\beta^m}{p} \mu \left( h \left( \frac{y}{p^m} \right) - p^n h \left( \frac{y}{p^n} \right) \right)
$$

$$
\leq \frac{\beta^m}{2} \sum_{j=1}^{n-1} \frac{\beta^{2j-1}}{p^j} \nu \left( \frac{y}{p^{n+j}} \right) + \frac{\beta^m}{2} \frac{\beta^{2(n-1)}}{p^{n-1}} \nu \left( \frac{y}{p^{n-1}} \right)
$$

$$
\leq \frac{p^m}{2p^m} \sum_{j=m+1}^{n+m-1} \frac{\beta^{2j-1}}{p^j} \nu \left( \frac{y}{p^{j}} \right) + \frac{p^m}{2p^m} \frac{\beta^{2(n-m-1)}}{p^{n-m-1}} \nu \left( \frac{y}{p^{n-m-1}} \right)
$$

by (22) it converges to zero as $m \to \infty$. Hence, $\left( p^m \left( \frac{y}{p^m} \right) \right)$ is $\mu$–Cauchy for all $y \in X$ and hence it is $\mu$–convergent in $X_\mu$ since $X_\mu$ is $\mu$–complete. Hence, we have

$$
A_2(y) = \mu - \lim_{n \to \infty} p^n h \left( \frac{y}{p^n} \right)
$$

(27)

for all $y \in X$, which implies

$$
\lim_{n \to \infty} \mu \left( p^n h \left( \frac{y}{p^n} \right) - A_2(y) \right) = 0, \ \forall y \in X.
$$

Hence by the $\Delta$–condition, we arrive the following by taking $n \to \infty$.

$$
\mu \left( h(y) - A_2(y) \right)
$$

$$
\leq \frac{1}{p} \mu \left( ph(y) - p^{n+1}h \left( \frac{y}{p^n} \right) \right) + \frac{1}{p} \mu \left( p^{n+1}h \left( \frac{y}{p^n} \right) - p A_2(y) \right)
$$

$$
\leq \frac{\beta}{p} \mu \left( h(y) - ph \left( \frac{y}{p} \right) \right) + \frac{\beta}{p} \mu \left( p^2 h \left( \frac{y}{p^2} \right) - A_2(y) \right)
$$

$$
\leq \frac{\beta}{2p} \sum_{j=1}^{n-1} \frac{\beta^{2j-1}}{p^j} \nu \left( \frac{y}{p^{j}} \right) + \frac{\beta}{2p} \frac{\beta^{2(n-1)}}{p^{n-1}} \nu \left( \frac{y}{p^{n-1}} \right) + \frac{\beta}{p} \mu \left( p^n h \left( \frac{y}{p^n} \right) - A_2(y) \right)
$$

$$
\leq \frac{1}{2p} \sum_{j=1}^{\infty} \frac{\beta^{2j}}{p^j} \nu \left( \frac{y}{p^j} \right) \leq \frac{1}{2p} W(y, 0).
$$
Next, we prove $A_2$ satisfies (2). Assuming $(y, z) = \left( \frac{x}{p^n}, \frac{z}{p^n} \right)$ in (21), and multiplying the resultant by $p^n$, we obtain

$$
\mu \left( p^n D_A h \left( \frac{y}{p^n}, \frac{z}{p^n} \right) \right) \leq p^n \nu \left( \frac{y}{p^n}, \frac{z}{p^n} \right) \leq p^{2n} \nu \left( \frac{y}{p^n}, \frac{z}{p^n} \right)
$$

as $n \to \infty$, which tends to zero. Hence, the property $\mu(y u) \leq \gamma \mu(u), 0 < \gamma \leq 1, u \in \mathcal{X}_\mu$ implies that

$$
\mu \left( \frac{1}{4p^2 + 2p + 3} D_A A_2(y, z) \right) \leq \mu \left( \frac{1}{4p^2 + 2p + 3} D_A A_2(y, z) - p^n D_A h \left( \frac{y}{p^n}, \frac{z}{p^n} \right) \right) \leq \mu \left( \frac{p + p^2}{4p^2 + 2p + 3} \right) A_2(y, z) - p^n h \left( \frac{y}{p^n}, \frac{z}{p^n} \right)
$$

As the limit $n \to \infty$, we obtain

$$
\mu \left( \frac{1}{4p^2 + 2p + 3} D_A A_2(y, z) \right) = 0
$$

for all $y, z \in X$. Hence, $D_A A_2(y, z) = 0$ and $A_2$ satisfies (2). Hence, it is additive. To prove the uniqueness of $A_2$, assume that $D_2 : X \to \mathcal{X}_\mu$, a additive mapping satisfies

$$
\mu(h(y) - D_2(y)) \leq \frac{1}{2p} \sum_{j=1}^{\infty} \frac{\beta^{j+1}}{p^j} \nu \left( \frac{y}{p^j}, 0 \right), \ \forall y \in X.
$$

Since $A_2$ and $D_2$ are additive mappings and $p^n A_2 \left( \frac{x}{p^n} \right) = A_2(x), p^n D_2 \left( \frac{x}{p^n} \right) = D_2(x)$ implies that

$$
\mu(D_2(y) - A_2(y))
$$

for all $y \in X$ and as $n \to \infty$ it tends to zero. Therefore, $A_2$ satisfying (23) and is a unique additive mapping. ∎
Considering \( v(y, z) = \epsilon \) and \( v(y, z) = \epsilon (\|y\|^m + \|z\|^m) \) in Theorem 4.4, we obtain the following Hyers-Ulam and Hyers-Ulam-Rassias stabilities, respectively.

**Corollary 4.5.** Let a mapping \( h : X \rightarrow X \) satisfying
\[
\mu(D_A h(y, z)) \leq \epsilon
\]
for all \( y, z \in X, \epsilon > 0 \). Hence there exists a unique additive mapping \( A_2 : X \rightarrow X \) which satisfies (2) and
\[
\mu(h(y) - A_2(y)) \leq \frac{e\beta^2}{2(p - \beta^2)}
\]
for all \( y \in X \) and for some \( \beta^2 < p \).

**Corollary 4.6.** If \( h : X \rightarrow X \) a mapping satisfies
\[
\mu(D_A h(y, z)) \leq \epsilon (\|y\|^m + \|z\|^m)
\]
for all \( y, z \in X \). Then there exists \( A_2 : X \rightarrow X \) a unique additive mapping such that
\[
\mu(h(y) - A_2(y)) \leq \frac{e\beta^2}{2(p(p^{m+1} - \beta^2))}
\]
for all \( y \in X, y \neq 0 \), for given real numbers \( \beta^2 < p^{m+1} \) and \( \epsilon > 0 \).

5. Stability of Functional Equation (1): Cubic Case

We obtain generalized Hyers-Ulam-Rassias stability of (1) in modular spaces without \( \Delta_p \)–condition and the Fatou property. Hereafter, we use the following notation
\[
D_C h(y, z) = h(py + z) + h(py - z) + h(y + pz) + h(y - pz) - (p + p^2)[h(y + z) + h(y - z)] - 2(p^3 - p^2 - p + 1)h(y)
\]
for all \( y, z \in X \).

**Theorem 5.1.** Considering \( h : X \rightarrow X \) a mapping satisfies
\[
\mu(D_C h(y, z)) \leq v(y, z)
\]
and a mapping \( v : X^2 \rightarrow [0, \infty) \) satisfies
\[
\zeta(y, z) = \sum_{j=0}^{\infty} \frac{v(p^j y, p^j z)}{p^j} < \infty, \quad \forall y, z \in X.
\]
Then there exists \( C_1 : X \rightarrow X \) a unique cubic mapping defined by \( C_1(y) = \lim_{n \rightarrow \infty} \frac{h(p^ny)}{p^n}, y \in X \) which satisfies the equation (7) and
\[
\mu(h(y) - C_1(y)) \leq \frac{1}{2p^3} \zeta(y, 0), \quad \forall y \in X
\]

**Proof.** Assuming \( y = 0 \) in (30), we obtain
\[
\mu(h(py) - p^3 h(y)) \leq \frac{1}{2} v(y, 0)
\]
and hence
\[ \mu \left( h(y) - h\left( \frac{py}{p^3} \right) \right) \leq \frac{1}{2p^3} \nu(y,0), \quad \forall y \in X. \]  
(34)

Generalizing, we arrive
\[ \mu \left( h(y) - h\left( \frac{p^m y}{p^{3m}} \right) \right) \leq \frac{1}{2} \sum_{j=0}^{n-1} \frac{\nu(p^j y,0)}{p^{3(j+1)}}, \quad \forall y \in X. \]  
(35)

Substituting \( y \) by \( p^m y \) in (35), we obtain
\[ \mu \left( h(p^m y) - h\left( \frac{p^{m+n} y}{p^{3(n+m)}} \right) \right) \leq \frac{1}{2p^3} \sum_{j=m}^{n+m-1} \frac{\nu(p^j y,0)}{p^{3j}}, \quad \forall y \in X. \]  
(36)

by the assumption (31) it converges to zero as \( m \to \infty \). Hence (36) implies that the sequence \( \{ h(p^m y) \} \) is \( \mu \)-Cauchy and therefore it is convergent in \( X_\mu \) since the \( X_\mu \) is \( \mu \)-complete. Hence we define \( C_1 : X \to X_\mu \) as
\[ C_1(y) = \mu - \lim_{n \to \infty} \left\{ \frac{h(p^m y)}{p^{3m}} \right\}, \quad \forall y \in X, \]
which implies
\[ \lim_{n \to \infty} \mu \left( \frac{h(p^m y)}{p^{3m}} - C_1(y) \right) = 0, \quad \forall y \in X. \]

Here after we complete this proof by similar way of Theorem 4.1. \( \square \)

Assuming \( \nu(y,z) = \epsilon \) and \( \nu(y,z) = \epsilon (\|y\|^m + \|z\|^m) \) in Theorem 5.1, we obtain the following stabilities called Hyers-Ulam and Hyers-Ulam-Rassias respectively.

**Corollary 5.2.** Let a mapping \( h : X \to X_\mu \) satisfying
\[ \mu(DC_h(y,z)) \leq \epsilon \]
for all \( y, z \in X \). Then there exists \( C_1 : X \to X_\mu \) a unique cubic mapping which satisfies (7) and
\[ \mu(h(y) - C_1(y)) \leq \frac{\epsilon}{2(p^3 - 1)} \]  
(37)
for all \( y \in X \), for some \( \epsilon > 0 \) and \( p^3 > 1 \).

**Corollary 5.3.** If \( h : X \to X_\mu \) a mapping satisfies
\[ \mu(DC_h(y,z)) \leq \epsilon (\|y\|^m + \|z\|^m), \quad \forall y, z \in X, \]
then there exists a unique cubic mapping \( C_1 : X \to X_\mu \) such that
\[ \mu(h(y) - C_1(y)) \leq \frac{\epsilon}{2(p^3 - p^m)} \|y\|^m \]  
(38)
for all \( y \in X, y \neq 0 \), for given real numbers \( m < 3 \) and \( \epsilon > 0 \).

Assuming a nontrivial convex modular \( \mu \) satisfies the \( \Delta_p \)-condition if there exists \( \beta > 0 \) such that
\[ \mu(p^3 y) \leq \beta \mu(y) \]  
for all \( y \in X_\mu \), where \( \beta \geq p \) and hence \( \mu(p^3 y) \leq M \mu(y) \)
Theorem 5.4. If a mapping \( h : X \rightarrow \mathcal{X}_\mu \) satisfies
\[
\mu(D_{ch}(y, z)) \leq \nu(y, z)
\]
and \( \nu : X^2 \rightarrow [0, \infty) \) is a mapping such that
\[
\Psi(y, z) = \sum_{j=1}^{\infty} \frac{M_{2j}^p}{p^j} \nu \left( \frac{y}{p^j}, \frac{z}{p^j} \right) < \infty, \ \forall y, z \in X.
\]
Then a unique cubic mapping \( C_2 : X \rightarrow \mathcal{X}_\mu \) exists and defined by \( C_2(y) = \lim_{n \to \infty} p^n h \left( \frac{y}{p^n} \right), y \in X \), which satisfies (7) and
\[
\mu(h(y) - C_2(y)) \leq \frac{1}{2p} \Psi(y, 0), \forall y \in X.
\]
Proof. Equation (33) implies that
\[
\mu \left( h(y) - p^n h \left( \frac{y}{p^n} \right) \right) \leq \frac{1}{2p} \nu \left( \frac{y}{p^n}, 0 \right), \ \forall y \in X.
\]
Hence by the convexity \( \mu \), we arrive
\[
\begin{align*}
\mu \left( h(y) - (p^n)^2 h \left( \frac{y}{p^n} \right) \right) & \leq \frac{1}{p^n} \mu \left( p^n h(y) - (p^n)^2 h \left( \frac{y}{p^n} \right) \right) + \frac{1}{p^n} \mu \left( (p^n)^2 h \left( \frac{y}{p^n} \right) - (p^n)^3 h \left( \frac{y}{p^n} \right) \right) \\
& \leq \frac{M}{2p^3} \nu \left( \frac{y}{p^n}, 0 \right) + \frac{M^2}{2p^3} \nu \left( \frac{y}{p^n}, 0 \right), \ \forall y \in X.
\end{align*}
\]
Generalizing, we obtain
\[
\begin{align*}
\mu \left( h(y) - (p^n)^n h \left( \frac{y}{p^n} \right) \right) & \leq \frac{1}{2} \sum_{j=1}^{n-1} \frac{M_{2j-1}^p}{p^{j-1}} \nu \left( \frac{y}{p^n}, 0 \right) + \frac{1}{2} \frac{M^{2(n-1)}}{p^{2n-2}} \nu \left( \frac{y}{p^n}, 0 \right)
\end{align*}
\]
for all \( y \in X \). The rest of proof is similar to that of Theorem 4.4. \( \square \)

Assuming \( \nu(y, z) = \epsilon \) and \( \nu(y, z) = \epsilon (||y||^m + ||z||^m) \) in Theorem 5.4, we obtain the following stabilities called Hyers-Ulam and Hyers-Ulam-Rassias respectively.

Corollary 5.5. If a mapping \( h : X \rightarrow \mathcal{X}_\mu \) satisfying
\[
\mu(D_{ch}(y, z)) \leq \epsilon, \ \forall y, z \in X,
\]
then there exists \( C_2 : X \rightarrow \mathcal{X}_\mu \), a unique cubic mapping which satisfies (7) and
\[
\mu(h(y) - C_2(y)) \leq \frac{\epsilon M^2}{2p(p^3 - M^2)}, \ \forall y \in X,
\]
for some \( \epsilon > 0 \) and \( M^2 < p^3 \).

Corollary 5.6. If \( h : X \rightarrow \mathcal{X}_\mu \) a mapping satisfies
\[
\mu(D_{ch}(y, z)) \leq \epsilon (||y||^m + ||z||^m), \ \forall y, z \in X,
\]
then a unique cubic mapping \( C_2 : X \rightarrow \mathcal{X}_\mu \) exists such that
\[
\mu(h(y) - C_2(y)) \leq \frac{\epsilon M^2}{2p(p^{m+3} - M^2)} ||y||^m, \ \forall y \in X,
\]
where \( y \neq 0 \), for given real numbers \( M^2 < p^{m+3} \) and \( \epsilon > 0 \).
6. Conclusion

We introduced a generalized mixed type of additive and cubic functional equation with its general solution and various stabilities concerning Ulam problem in modular spaces by considering with and without $\Delta_p$–condition.

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