Metastability of the Ising model on random regular graphs at zero temperature

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Abstract

We study the metastability of the ferromagnetic Ising model on a random $r$-regular graph in the zero temperature limit. We prove that in the presence of a small positive external field the time that it takes to go from the all minus state to the all plus state behaves like $\exp(\beta(r/2 + O(\sqrt{r}))n)$ when the inverse temperature $\beta \to \infty$ and the number of vertices $n$ is large enough. The proof is based on the so-called pathwise approach and bounds on the isoperimetric number of random regular graphs.

Keywords Metastability, Ising model, random graphs, pathwise approach

Mathematics Subject Classification 60K35, 82C20

1 Introduction

One of the most important models in the study of phase transitions is the Ising model. In this model, to every vertex a spin is assigned that can have value $+1$ or $-1$. These spins interact with each other and have the tendency to align: the spins of vertices that are connected by an edge tend to get the same value. This model was originally suggested by Lenz to his student Ising to study ferromagnetism [18], but later also became a model to study cooperative behavior. For a historic account of this model see [26, 27, 28].

In this paper, we focus on the dynamics of this model, more precisely on metastability. When there is a positive external field present, the stable state at very low temperature is the one where all vertices have spin $+1$. If, however, we start with a system where all spins are equal to $-1$ and let the system evolve, then it will take a very long time before the system reaches this stable state due to the strong interaction between the spins. Hence, on a short time scale the state where all spins are equal to $-1$ seems stable. This is what we call a metastable state and the time it takes for the system to reach the stable state is called the metastable time.

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Metastability in this setting where the temperature is very low has been studied in two dimensions in [24] and for three dimensions in [5]. Sharper results for both of these models were obtained in [9]. The Ising model on the two dimensional lattice where the external field is very small and the temperature is below the critical temperature, but not necessarily close to zero, is studied in [29]. These papers describe the size and shape of the critical droplet that has to be formed before the system quickly relaxes to the stable state.

When interpreting the Ising model as a model for cooperative behavior, studying it in a setting where the graph is a lattice makes less sense. Hence, in recent years there has been a large interest in studying the Ising model, and other models, on random graphs, which are itself models for complex networks, see for example [25] for an overview. We focus on one of the simplest random graphs, called the regular random graph, where all vertices have the same degree and the graph is chosen uniformly at random among all graphs with this property.

The Ising model on a random graph was first studied rigorously in [14], where the high temperature and zero temperature solution at equilibrium where obtained for the Erdős-Rényi random graph using interpolation techniques. Later, in [15] the equilibrium solution at all temperatures was obtained for random graphs that locally behave like a branching process. Such graphs include the Erdős-Rényi random graph and random regular graphs, but also other random graphs with an uncorrelated degree sequence with finite variance degrees. The latter condition was relaxed in [16] to random graphs where the degrees have strongly finite mean. The weak limit of the Ising measure in zero field is studied in more detail in [21] and [2]. In [17], the critical exponents for this model have been computed.

About the dynamics of the Ising model on random graphs, however, not much is known. In [22, 23] the mixing time for the the Erdős-Rényi random graph and random regular graphs was computed in the high temperature regime. In [19], a cut-off phenomenon was proved for high enough temperature. To the best of our knowledge, on the low temperature behavior only some simulation results have appeared in the literature [11, 12, 30].

In this paper, we present results on the dynamics of the Ising model on random regular graphs in the the zero temperature limit. We give bounds on the metastable time and show that it is exponential in the inverse temperature, the number of vertices and the degrees of the vertices.

Whereas the results on the equilibrium solution mentioned above are mainly based on the locally tree-like structure of the random graphs, our results on the metastable time are based on the structure of subsets consisting of a positive fraction of the vertices. We use results by Bollobás [6] and Alon [1] concerning the isoperimetric number of random regular graphs, which is a measure of how connected to the rest of the graph a subset consisting of half of the vertices has to be.

We combine these results with the pathwise approach to metastability. This approach was introduced in [10] and first used for the Ising model on the two dimensional lattice in [24]. We use in particular the results in [20], where the control of the exact trajectory is decoupled from the control of the metastable time. We combine this with the approach used in [13], which simplifies the necessary calculations.
2 Model definitions and preliminary results

In this section we give the definitions of random regular graphs and of the Ising model, its dynamics and metastability. We also present several preliminary results on these topics that are used in the paper.

2.1 Random regular graphs

We denote a graph \( G_n = (V_n = [n], E_n) \) where \([n] = \{1, \ldots, n\}\) by \( G_n \). For given \( r \in \mathbb{N} \) and \( n > r \) with \( nr \) even, we say that \( G_n \) is a random \( r \)-regular graph if we select \( G_n \) uniformly at random from the set of all simple \( r \)-regular graphs with \( n \) vertices.

An event \( A_n \) is said to hold with high probability (whp), if

\[
\lim_{n \to \infty} \mathbb{P}[A_n] = 1,
\]

(2.1)

where \( \mathbb{P} \) denotes the measure of selecting a random \( r \)-regular graph.

For a graph \( G_n \), the (edge) boundary of a set \( A \subseteq [n] \) equals

\[
\partial_e A = \{(i, j) \in E_n \mid i \in A, j \notin A\}.
\]

(2.2)

An important quantity of the graphs of interest is the so-called isoperimetric number, which is defined as follows:

**Definition 2.1 (Isoperimetric number).** For a graph \( G_n \), the (edge) isoperimetric number of \( G_n \) equals

\[
i_e(G_n) = \min_{A \subseteq [n]} \frac{\partial_e A}{|A|},
\]

(2.3)

where \( |A| \) denotes the cardinality of the set \( A \).

Furthermore, define

\[
i'_e(G_n) = \min_{|A| = \lfloor n/2 \rfloor} \frac{\partial_e A}{|A|}.
\]

(2.4)

Note that it always holds that \( i_e(G_n) \leq i'_e(G_n) \). The following lower bound on the isoperimetric number was proved in [6] by Bollobás:

**Proposition 2.2 (Lower bound on isoperimetric number).** Let \( G_n \) be a random \( r \)-regular graph with \( r \geq 3 \) and let \( \zeta \in (0, 1) \) be such that

\[
2^{4/r} < (1 - \zeta)^{1-\zeta}(1 + \zeta)^{1+\zeta}.
\]

(2.5)

Then, whp,

\[
i_e(G_n) \geq (1 - \zeta)r/2.
\]

(2.6)

In particular, whp,

\[
i_e(G_n) \geq \frac{r}{2} - \sqrt{\log 2r}.
\]

(2.7)
For small $r$ better bounds on $i_e(G_n)$ then (2.7) can be obtained as observed in [6]. For $r$ tending to infinity $r/2 - \Theta(\sqrt{r})$ turns out to be the correct scaling of the isoperimetric number, since in [1] Alon derived the following upper bound:

**Proposition 2.3** (Upper bound on isoperimetric number). There exists an absolute constant $C > 0$ such that for any $r$-regular graph $G_n$ with $n \geq 40r^9$ there exists a set $A \subset [n]$ with $|A| = \lfloor n/2 \rfloor$, such that

$$\frac{|\partial_e A|}{|A|} \leq \frac{r}{2} - C\sqrt{r}. \quad (2.8)$$

Hence,

$$i_e(G_n) \leq i'_e(G_n) \leq \frac{r}{2} - C\sqrt{r}. \quad (2.9)$$

### 2.2 Ising model, dynamics and metastability

The Ising model on a graph $G_n$ is defined as follows. To each vertex $i \in [n]$ we assign a spin $\sigma_i \in \{-1, +1\}$ and we denote $\sigma = (\sigma_i)_{i \in [n]}$. The Hamiltonian $H(\sigma)$ is then given by

$$H(\sigma) = -J \sum_{(i,j) \in E_n} \sigma_i \sigma_j - h \sum_{i \in [n]} \sigma_i, \quad (2.10)$$

where $J > 0$ is the interaction constant and $h \in \mathbb{R}$ is the external magnetic field. The Boltzmann-Gibbs measure for the Ising model on $G_n$ is then defined as

$$\mu_n(\sigma) = \frac{1}{Z_n} e^{-\beta H(\sigma)}, \quad (2.11)$$

where $\beta \geq 0$ is the inverse temperature and $Z_n$ is the normalization factor, called the partition function, i.e.,

$$Z_n = \sum_{\sigma \in \{-1, +1\}^n} e^{-\beta H(\sigma)}. \quad (2.12)$$

Without loss of generality, we assume that $J = 1$, since this is just a rescaling of $\beta$ and $h$.

For a set $A \subseteq [n]$, denote by $\sigma^A$ the configuration where

$$\sigma^A_i = \begin{cases} +1, & \text{if } i \in A, \\ -1, & \text{if } i \notin A. \end{cases} \quad (2.13)$$

We also denote $\square = \sigma^{\emptyset}$ and $\boxdot = \sigma^{[n]}$, the all minus and all plus configurations, respectively. We often identify the vertex and its spin, e.g., we say that vertex $i$ has a $+$ neighbor if there is a vertex $j$ such that $(i, j) \in E_n$ with $\sigma_j = +1$.

We let the system evolve according to *Glauber dynamics* with Metropolis rates. That is, we consider a discrete time Markov chain where the transition probability $c(\sigma^A, \sigma^B)$ from configuration $\sigma^A$ to $\sigma^B$ equals

$$c(\sigma^A, \sigma^B) = \begin{cases} \frac{1}{n} e^{-\beta[H(\sigma^B) - H(\sigma^A)]^+}, & \text{if } |A \triangle B| = 1; \\ 1 - \sum_{B:|A \triangle B| = 1} \frac{1}{n} e^{-\beta[H(\sigma^B) - H(\sigma^A)]^+}, & \text{if } A = B, \\ 0, & \text{otherwise}. \end{cases} \quad (2.14)$$
where \( A \triangle B \) is the symmetric difference between sets \( A \) and \( B \), and \([a]^+ = \max\{a, 0\}\). We denote by \( \mathbb{P}_\eta \) the law of the process starting from configuration \( \eta \).

The time at which the process visits the configuration \( \sigma \) for the first time if the process starts from \( \eta \) is called the hitting time of \( \sigma \) and is denoted by \( \tau_\sigma \). When studying metastability, the problem is to find the hitting time of the stable configuration if the system starts in a metastable configuration. We now define what it means for a configuration to be (meta)stable.

The stable state is the state for which the Hamiltonian is minimal. Throughout the rest of the paper we assume that \( h > 0 \), so that it is obvious from (2.10) that the stable state is \( \sqcup \).

To define metastable states, we need to define the communication height between two configurations \( \sigma \) and \( \sigma' \) which is given by
\[
\Phi(\sigma, \sigma') = \min_{\omega \text{ path from } \sigma \text{ to } \sigma'} \max_{\sigma'' \in \omega} H(\sigma'') - H(\sigma),
\]
where \( \omega \) is a sequence of configurations \( \omega = (\sigma = \sigma^0, \sigma^1, \ldots, \sigma^\ell = \sigma') \) for some \( \ell \geq 1 \) and \( |A_k \triangle A_{k+1}| = 1 \) for all \( 0 \leq k < \ell \). We then define the stability level of a configuration \( \sigma \) as
\[
V_\sigma = \min_{\sigma': H(\sigma') < H(\sigma)} \Phi(\sigma, \sigma') - H(\sigma).
\]

Note that \( V_{\sqcup} = \infty \) since there are no configurations with smaller energy. The maximal stability level is defined as
\[
\Gamma = \max_{\sigma \neq \sqcup} V_\sigma,
\]
and the metastable states are those configurations \( \eta \) such that \( V_\eta = \Gamma \).

In [13], Cirillo and Nardi give an easy characterization of the maximal stability level \( \Gamma \) by looking at the communication height between the metastable and stable state, and upper bounds on the stability levels of all other states. This gives an easier way to compute \( \Gamma \), since it is no longer necessary to compute the stability level of all states exactly. We use a similar approach to give bounds on \( \Gamma \) by verifying the following conditions for some \( \Gamma_\ell \leq \Gamma_u \):

**Condition (1)** \( \Phi(\sqcup, \sqcup) - H(\sqcup) \geq \Gamma_\ell \);

**Condition (2)** \( \Phi(\sqcup, \sqcup) - H(\sqcup) \leq \Gamma_u \);

**Condition (3a)** for all \( \sigma \notin \{\sqcup, \sqcup\} \) it holds that \( V_\sigma \leq \Gamma_u \);

**Condition (3b)** for all \( \sigma \notin \{\sqcup, \sqcup\} \) it holds that \( V_\sigma < \Gamma_\ell \).

In [13], the same conditions are used with \( \Gamma_\ell = \Gamma_u \). One can prove in a similar way as in [13] that it follows from Conditions (1), (2) and (3a) that \( \Gamma_\ell \leq \Gamma \leq \Gamma_u \). Condition (3b) allows us to prove that \( \sqcup \) is the unique metastable state. By combining this with results from [20], we get the following bound on the metastable time:

**Proposition 2.4 (Metastable time).** If Conditions (1), (2) and (3a) hold then, for all \( \varepsilon > 0 \),
\[
\lim_{\beta \to \infty} \mathbb{P}_{\sqcup}[e^{-\beta(\Gamma_\ell - \varepsilon)} < \tau_{\sqcup} < e^{-\beta(\Gamma_u + \varepsilon)}] = 1.
\]

If \( \eta \) is a metastable state, then the same holds with \( \sqcup \) replaced by \( \eta \). If also Condition (3b) holds, then \( \sqcup \) is the unique metastable state.

The proof of this proposition using the methodology of [13] and the results in [20] can be found in Appendix A.
3 Main result and discussion

We can now present our main result, which gives bounds on the metastable time:

**Theorem 3.1 (Metastable time for random \( r \)-regular graphs).** Let \( G_n \) be a random \( r \)-regular graph with \( r \geq 3 \) and suppose that \( 0 < h < C_0 \sqrt{r} \) for some uniform constant \( C_0 > 0 \) small enough and \( n \) is large enough. Then, there exist uniform constants \( 0 < C_1 < \sqrt{3}/2 \) and \( C_2 < \infty \) so that, whp,

\[
\lim_{\beta \to \infty} \mathbb{P} \left[ e^{\beta(r/2 - C_1 \sqrt{r})n} < \tau_{\square} < e^{\beta(r/2 + C_2 \sqrt{r})n} \right] = 1. \tag{3.1}
\]

If \( \eta \) is a metastable state, then the same holds with \( \square \) replaced by \( \eta \). If \( r \geq 21 \), then \( \square \) is the unique metastable state.

The condition that \( C_1 < \sqrt{3}/2 \) ensures that \( r/2 - C_1 \sqrt{r} > 0 \) for all \( r \geq 3 \). The proof of this theorem can be found in the next section, where we verify Conditions (1), (2), (3a) and, for \( r \geq 21 \), (3b) with

\[
\Gamma_{\ell} = (r/2 - C_1 \sqrt{r})n \quad \text{and} \quad \Gamma_u = (r/2 + C_2 \sqrt{r})n. \tag{3.2}
\]

Unfortunately, we were not able to prove that \( \square \) is the unique metastable state for all \( r \geq 3 \), although we expect this to be the case. We show that \( V_{\sigma} \leq C_3 \sqrt{r} n \) for \( \sigma \notin \{\square, \eta\} \) and need to show that this is less then \( (r/2 - C_1 \sqrt{r})n \), but for small values of \( r \) our bounds on the constants are not sharp enough and hence, we can only prove this for \( r \) large enough.

The condition that \( h < C \sqrt{r} \) is only necessary to prove that the metastable time behaves like \( \exp(\beta(r/2 + O(\sqrt{r}))n) \) for \( r \to \infty \). For \( h < i_\epsilon(G_n) \), the same proof strategy can be used, but this comes at the expense that the bounds on the metastable time become less tight.

The above theorem shows that the metastable time grows exponentially with \( n \). This is different from the Ising model on finite dimensional lattices, where the metastable time is uniformly bounded, see, e.g., [24] for \( d = 2 \) and [5] for \( d = 3 \).

The above result holds for random \( r \)-regular graphs. It would be interesting to generalize this result to random graphs with more general degree distributions. For this, it will be necessary to get a more detailed understanding of the structure of components that consist of a positive fraction of the graph and what measures will be of interest. For example, the isoperimetric number does not always give useful information, since this is equal to zero for disconnected graphs such as the Erdős-Rényi random graph.

Our main result only holds in the zero temperature limit. It is expected, however, that the metastable time grows exponentially in \( n \) for all temperatures below the critical temperature. It would be interesting to investigate this further.

Several alternative approaches to metastability have been used in the recent literature, among which the **potential theoretic approach** developed in [7, 8] and the **martingale approach** developed in [3, 4]. It would be interesting to see if these methods also yield useful results about the metastability of the Ising model on random graphs.

4 Proofs

In the next three subsections we verify Conditions (1), (2), (3a) and, for \( r \geq 21 \), (3b) with

\[
\Gamma_{\ell} = (r/2 - C_1 \sqrt{r})n \quad \text{and} \quad \Gamma_u = (r/2 + C_2 \sqrt{r})n. \tag{4.1}
\]
Theorem 3.1 then follows from Proposition 2.4, where it should be observed that the $\varepsilon$ can be absorbed into the constants.

### 4.1 Condition (1): lower bound on communication height

We start by deriving a lower bound on the communication height between $\square$ and $\square$, verifying Condition (1):

**Proposition 4.1** (Lower bound on communication height). Let $G_n$ be a graph and suppose that $0 < h < i_e(G_n)$. Then,

$$\Phi(\square, \square) - H(\square) \geq (i_e(G_n) - h)n.$$  \hspace{1cm} (4.2)

If $G_n$ is a random $r$-regular graph with $r \geq 3$ and $0 < h < C_0\sqrt{r}$ for some uniform constant $0 < C_0 < (\sqrt{3}/2 - \sqrt{\log(2)})$, then there exists a uniform constant $0 < C_1 < \sqrt{3}/2$ so that, whp,

$$\Phi(\square, \square) - H(\square) \geq (r/2 - C_1\sqrt{r})n.$$  \hspace{1cm} (4.3)

**Proof.** For any subset $A \subseteq [n]$ with $|A| \leq n/2$, it holds that

$$H(\sigma^A) = - \sum_{(i,j) \in E_n} \sigma_i^A \sigma_j^A - h \sum_{i \in [n]} \sigma_i^A = -(|E_n| - |\partial_e A|) + |\partial_e A| - 2h|A| + h(n - |A|)$$

$$= 2|\partial_e A| - h|A| - |E_n| + hn. \hspace{1cm} (4.4)$$

Hence, by the definition of the isoperimetric number,

$$H(\sigma^A) \geq 2(i_e(G_n) - h)|A| - |E_n| + hn. \hspace{1cm} (4.5)$$

Note that every path from $\square$ to $\square$ has to go through a configuration with $|n/2|$ plus spins. Using the above, the energy of any such configuration is at least $(i_e(G) - h)n - |E_n| + hn$. The first statement of the proposition now follows by observing that

$$H(\square) = -|E_n| + hn. \hspace{1cm} (4.6)$$

To prove the second statement, observe that whp $i_e(G_n) \geq r/2 - \sqrt{\log 2}\sqrt{r}$, so that whp

$h < i_e(G_n)$ if $h < (\sqrt{3}/2 - \sqrt{\log(2)})\sqrt{r}$ for $r \geq 3$. Hence, we can use the above results which gives, whp,

$$\Phi(\square, \square) - H(\square) \geq (i_e(G_n) - h)n \geq (r/2 - C_1\sqrt{r}), \hspace{1cm} (4.7)$$

with $C_1 = \sqrt{\log 2} + C_0 < \sqrt{3}/2$. \hspace{1cm} $\square$

**Remark.** The first statement of the proposition above holds for general graphs and not only for $r$-regular graphs. For other graphs, however, it does not always give useful information. If, for example, the graph is not connected, which is, e.g., the case whp in the Erdős-Rényi random graph, then $i_e(G_n) = 0$. Also for lattices $i_e(G_n) \to 0$ for $n \to \infty$, and hence the above bound is not useful.
4.2 Condition (2): upper bound on communication height

In the following lemma we compute when the energy decreases if we flip one spin at vertex \( i \) from + to – and vice versa.

**Lemma 4.2.** Suppose that \( G_n \) is an \( r \)-regular graph. Let \( A \subseteq [n] \). Then, for all \( i \in A \), \( H(\sigma^A \setminus i) \leq H(\sigma^A) \) if and only if \( |\partial^+ i \setminus A| \geq (r + h)/2 \).

Furthermore, for all \( i \in A^c \), \( H(\sigma^{A \setminus i}) \leq H(\sigma^A) \) if and only if \( |\partial^- i \setminus A| \geq (r - h)/2 \).

**Proof.** For \( i \in A \), let \( k^A_i = |\partial^+_i \setminus A| \), the number of – spins connected to vertex \( i \) in configuration \( \sigma^A \). Hence, \( i \) is connected to \( r - k^A_i \) + spins. Thus, the difference in energy

\[
H(\sigma^{A \setminus i}) - H(\sigma^A) = 2(r - k^A_i) - 2k^A_i + 2h = -4k^A_i + 2(r + h),
\]

since every edge connected to \( i \) between two equal spins in \( \sigma^A \) has one + and one – spin in \( \sigma^{A \setminus i} \) and vice versa. This is indeed nonpositive iff \( k^A_i \geq (r + h)/2 \).

The proof for \( i \in A^c \) is similar. \( \square \)

**Lemma 4.3.** Suppose that \( G_n \) is a connected \( r \)-regular graph and that \( 0 < h < i_c(G_n) \). Then, for every set \( A \subset [n] \) with \( 1 \leq |A| \leq n/2 \) there exists a set \( B \subset A \) with \( |B| < |A| \) such that

\[
H(\sigma^B) < H(\sigma^A),
\]

and

\[
\Phi(\sigma^A, \sigma^B) - H(\sigma^A) \leq 2(r - 2 + h)s,
\]

with \( s = \lceil \frac{r + h - 2i_c(G_n)}{r - h} |A| \rceil \).

**Proof.** Given a set \( A \subset [n] \), we construct the set \( B \) as follows. We start from configuration \( \sigma^A \) and then flip \( s \) + spins to – one spin at the time, choosing a random + with at least one – neighbor every step. As long as there are + spins left, such a + spin always exists, since the graph is connected and \( |A| \leq n/2 \). If there are no + spins left we are done, because it follows from (4.5) that \( H(\Xi) < H(\sigma^A) \). Every step the energy can go up at most \( 2(r - 1) - 2 + 2h = 2(r - 2 + h) \).

After changing these \( s \) spins from + to –, we keep changing spins from + to –, but now selecting a spin at random at every step that has at least \( (r + h)/2 \) – neighbors. We continue doing this until such spins do not exist anymore. From Lemma 4.2 it follows that the energy can not go up in any of these steps. We call the remaining configuration \( \sigma^B \).

It is obvious that \( |B| \leq |A| - s < |A| \). From the above it also follows that \( \Phi(\sigma^A, \sigma^B) - H(\sigma^A) \leq 2(r - 2 + h)s \). It thus remains to choose \( s \) big enough so that we are sure that \( H(\sigma^B) < H(\sigma^A) \).

For this, note that

\[
H(\sigma^B) = 2|\partial^ _B| - 2h|B| - |E| + hn < 2 \left( \frac{r + h}{2} - h \right) |B| - |E| + hn \leq (r - h)(|A| - s) - |E| + hn,
\]

where in the first inequality we used that every vertex in \( B \) has strictly less then \( (r + h)/2 \) – neighbors by construction.
Hence, it follows from (4.5) that we need to choose \( s \) such that
\[
(r - h)(|A| - s) \leq 2(i_e(G) - h)|A|,
\]
which is equivalent to
\[
s \geq \frac{r + h - 2i_e(G)}{r - h}|A|.
\]

**Remark.** In the above lemma we made essential use of the fact that all degrees are equal to get a bound on \( |\partial_e B| \) in terms of \( |B| \).

We can make a similar statement if already more than half the spins are +.

**Lemma 4.4.** Suppose that \( G_n \) is a connected \( r \)-regular graph and that \( h > 0 \). Then, for every set \( A \subset [n] \) with \( n/2 \leq |A| < n \) there exists a set \( B \supset A \) with \( |B| > |A| \) such that
\[
H(\sigma^B) < H(\sigma^A),
\]
and
\[
\Phi(\sigma^A, \sigma^B) - H(\sigma^A) \leq 2(r - 2 - h)s,
\]
with \( s = \left\lceil \frac{r - h - 2i_e(G_n)}{r + h} |A^c| \right\rceil \).

The proof is similar to that of Lemma 4.3. We can now verify Condition (2):

**Proposition 4.5** (Upper bound on communication height). Suppose that \( G_n \) is a connected \( r \)-regular graph, \( 0 < h < i_e(G_n) \) and \( n \) is large enough. Then,
\[
\Phi(\bigboxtimes, \bigboxtimes) - H(\bigboxtimes) \leq (i'_e(G_n) - h)n + 2(r - 2 + h) \left\lceil \frac{r + h - 2i_e(G_n) n}{r - h} \right\rceil .
\]

If \( G_n \) is a random \( r \)-regular graph and \( 0 < h < C_0 \sqrt{r} \) for \( C_0 > 0 \) small enough, then, whp,
\[
\Phi(\bigboxtimes, \bigboxtimes) - H(\bigboxtimes) \leq (r/2 + C_2 \sqrt{r})n,
\]
for some constant \( C_2 < \infty \).

**Proof.** Let
\[
A = \arg \min_{A \subset [n]} \frac{|\partial_e A|}{|A|},
\]
choosing an arbitrary \( A \) among the minimizers if this minimum is not unique. Then,
\[
H(A) \leq (i'_e(G_n) - h)n - |E| + hn.
\]
From Lemma 4.3 it follows that there exists a set \( B \subset A \) such that \( |B| < |A| \) with \( H(\sigma^B) < H(\sigma^A) \) and
\[
\Phi(\sigma^A, \sigma^B) \leq H(\sigma^A) + 2(r - 2 + h) \left\lceil \frac{r + h - 2i_e(G_n) n}{r - h} \right\rceil .
\]
Now, there exists a set \( B' \subset A \) such that \(|B'| < |B|\) with \( H(\sigma^{B'}) < H(\sigma^B) \) and \( \Phi(\sigma^B, \sigma^{B'}) < \Phi(\sigma^A, \sigma^B) \), where the latter follows from Lemma 4.3. We apply this recursively until we reach the set \( \square \). Hence,

\[
\Phi(\square, \sigma^A) - H(\square) = \Phi(\sigma^A, \square) - H(\square) \leq (i_e'(G_n) - h)n + 2(r - 2 + h) \left[ \frac{r + h - 2i_e(G_n)}{r - h} \frac{n}{2} \right]. \tag{4.21}
\]

Using Lemma 4.4 and similar arguments as above can show that

\[
\Phi(\sigma^A, \square) - H(\square) \leq (i_e'(G_n) - h)n + 2(r - 2 - h) \left[ \frac{r - h - 2i_e(G_n)}{r + h} \frac{n}{2} + 1 \right], \tag{4.22}
\]

where the +1 term is necessary, because we first have to flip one spin to + to start the argument with at least \( n/2 + \) spins. Note that

\[
\frac{r - h - 2i_e(G_n)}{r + h} \frac{n}{2} + 1 = \frac{r - h + 2/n(r + h) - 2i_e(G_n)}{r + h} \leq \frac{r + h - 2i_e(G_n)}{r - h} \frac{n}{2}, \tag{4.23}
\]

where the inequality holds for \( n \) large enough. The first statement of the proposition now follows by observing that

\[
\Phi(\square, \square) - H(\square) \leq \max \{ \Phi(\square, \sigma^A), \Phi(\sigma^A, \square) \} - H(\square). \tag{4.24}
\]

To obtain the second result we use Propositions 2.2 and 2.3. Let \( C > 0 \) be a constant such that (2.9) holds. Then, whp,

\[
\Phi(\square, \square) - H(\square) \leq (r/2 - C\sqrt{r} - h)n + 2(r - 2 + h) \left[ \frac{r + h - 2(r/2 - \sqrt{\log 2}\sqrt{r}) n}{r - h} \right].
\]

\[
\leq r/2n + \frac{r + h}{r - h} \left( h + 2\sqrt{\log 2}\sqrt{r} + 2/n(r - h) \right) n. \tag{4.25}
\]

If \( h \leq C_0\sqrt{r} \) for some constant \( C_0 \) small enough, then also \( h \leq r/2 \), so that \( (r + h)/(r - h) \leq 3 \). We can also choose \( n \) large enough so that \( 2/n(r - h) < h \). Hence,

\[
\Phi(\square, \square) - H(\square) \leq r/2n + 3 \left( 2C_0\sqrt{r} + 2\sqrt{\log 2}\sqrt{r} \right) n \leq (r/2 + C_2\sqrt{r})n, \tag{4.26}
\]

for some \( C_2 < \infty \). \( \square \)

### 4.3 Conditions 3(a) and 3(b): upper bounds on stability levels

It remains to verify Conditions (3a) and (3b), which we do next, again using Lemma’s 4.3 and 4.4. We start with Condition (3a):

**Proposition 4.6** (First upper bound on stability levels). Suppose that \( G_n \) is a connected \( r \)-regular graph and that \( 0 < h < i_e(G_n) \). Then, for all \( \sigma \notin \{ \square, \square \} \),

\[
V_\sigma \leq 2(r - 2 + h) \left[ \frac{r + h - 2i_e(G_n)}{r - h} \frac{n}{2} \right]. \tag{4.27}
\]

If \( G_n \) is a random \( r \)-regular graph with \( r \) big enough and \( h < C_0\sqrt{r} \) for \( C_0 > 0 \) small enough, then, whp, for all \( \sigma \notin \{ \square, \square \} \),

\[
V_\sigma \leq (r/2 + C_2\sqrt{r})n, \tag{4.28}
\]

for some constant \( C_2 < \infty \).
Proof. The first inequality follows from Lemma 4.3 if \( \sigma = \sigma^A \) with \(|A| \leq n/2\) and from Lemma 4.4 and (4.23) if \( \sigma = \sigma^A \) with \(|A| \geq n/2\).

Hence, it follows from Proposition 2.2 that, whp,
\[
V_\sigma \leq 2(r - 2 + h) \left[ \frac{r + h - 2i_e(G_n) n}{r - h} \right] \leq 2(r - 2 + h) \left[ \frac{h + 2\sqrt{\log 2} \sqrt{r} n}{r - h} \right].
\]
(4.29)

This can be bounded from above by \( C_2 \sqrt{r}n \) as in Proposition 4.5. Hence, whp,
\[
V_\sigma \leq C_2 \sqrt{r}n \leq (r/2 + C_2 \sqrt{r})n.
\]
(4.30)

Unfortunately, the bound \( V_\sigma \leq C_2 \sqrt{r}n \) is not sufficient to prove that \( V_\sigma \leq (r/2 - C_1 \sqrt{r})n \) for small values of \( r \). Hence, we need to bound the constant \( C_2 \) more precisely to verify Condition (3b). We do this in the next proposition for \( r \geq 21 \):

**Proposition 4.7** (Second upper bound on stability levels). If \( G_n \) is a random \( r \)-regular graph with \( r \geq 21 \) and \( h < C_0 \sqrt{r} \) for \( C_0 > 0 \) small enough, then, whp, for all \( \sigma \notin \{\emptyset, \sqcup\} \),
\[
V_\sigma \leq (r/2 - C_1 \sqrt{r})n.
\]
(4.31)

Proof. We start from the following bound, which was proved in the previous proposition:
\[
V_\sigma \leq 2(r - 2 + h) \left[ \frac{h + 2\sqrt{\log 2} \sqrt{r} n}{r - h} \right] \leq \frac{r - 2 + h}{r - h} (h + 2\sqrt{\log 2} \sqrt{r} + 2/n(r - h))n.
\]
(4.32)

We want to prove that we can choose \( C_0 \) small enough and \( n \) big enough so that, for all \( r \geq 21 \),
\[
V_\sigma \leq (r/2 - C_1 \sqrt{r})n,
\]
(4.33)

where \( C_1 = \sqrt{\log 2} + C_0 \) as observed in Proposition 4.1. Hence, we need that
\[
\frac{r - 2 + h}{r - h} (h + 2\sqrt{\log 2} \sqrt{r} + 2/n(r - h)) + \sqrt{\log 2} + C_0 \leq r/2.
\]
(4.34)

We can rewrite this as
\[
3(1 - 2/r) \sqrt{\log 2} \sqrt{r} + C_3 \leq r/2,
\]
(4.35)

where \( C_3 \) can be chosen arbitrary small by choosing \( C_0 \) small enough and \( n \) big enough. It is easy to check numerically that
\[
3(1 - 2/r) \sqrt{\log 2} \sqrt{r} < r/2,
\]
(4.36)

for \( r \geq 21 \), which is hence a sufficient condition. \( \square \)
A Metastable time

In this appendix, we prove a sequence of lemma’s that together give the proof of Proposition 2.4. We prove the lower bound on the hitting time of ⊞ separately for the process starting in ⊞ and in a metastable state η, starting with ⊞:

Lemma A.1. If Condition (1) holds, then, for all ε > 0,

\[ \lim_{\beta \to \infty} \mathbb{P}_{\square} [\tau_{\square} > e^{\beta(\Gamma - \varepsilon)}] = 1. \]  

(A.1)

Proof. To prove this lemma we need to introduce some terminology from [20]. For a non-empty set of configurations \( \mathcal{A} \) denote by \( \partial^{\text{ext}} \mathcal{A} \) the external boundary of \( \mathcal{A} \), i.e.,

\[ \partial^{\text{ext}} \mathcal{A} = \{ \sigma^B : \sigma^B \neq \mathcal{A} : |A \triangle B| = 1 \text{ for some } \sigma^A \in \mathcal{A} \}. \]  

(A.2)

We define a non-trivial cycle as a connected set of configurations \( \mathcal{C} \) such that

\[ \max_{\sigma \in \mathcal{C}} H(\sigma) < \min_{\eta \in \partial^{\text{ext}} \mathcal{C}} H(\eta). \]  

(A.3)

The depth \( D(\mathcal{C}) \) of a non-trivial cycle \( \mathcal{C} \) is defined as

\[ D(\mathcal{C}) = \min_{\eta \in \partial^{\text{ext}} \mathcal{C}} H(\eta) - \min_{\sigma \in \mathcal{C}} H(\sigma). \]  

(A.4)

We consider the cycle

\[ \mathcal{C} = \{ \sigma : \Phi(\square, \sigma) - H(\square) < \Gamma \}. \]  

(A.5)

From the definition of the communication height it follows that for any \( \sigma \in \mathcal{C} \) it holds that \( H(\sigma) < \Gamma + H(\square) \) and for any configuration \( \eta \in \partial^{\text{ext}} \mathcal{C} \) it holds that \( \eta \notin \mathcal{C} \) and hence \( H(\eta) \geq \Gamma + H(\square) \). Hence, \( \mathcal{C} \) is a non-trivial cycle.

We can thus use [20, Theorem 2.17] to conclude that, for any \( \varepsilon > 0 \), \( \delta \in (0, \varepsilon) \) and \( \sigma \in \mathcal{C} \),

\[ \mathbb{P}_{\sigma} [\tau_{\partial^{\text{ext}} \mathcal{C}} > e^{\beta(D - \varepsilon)}] \geq 1 - e^{-\beta \delta}, \]  

(A.6)

where

\[ D = D(\mathcal{C}) = \min_{\eta \in \partial^{\text{ext}} \mathcal{C}} H(\eta) - \min_{\sigma \in \mathcal{C}} H(\sigma) \geq \Gamma + H(\square) - H(\square) = \Gamma. \]  

(A.7)

Clearly, \( \square \in \mathcal{C} \), but \( \square \notin \mathcal{C} \) by Condition (1). Hence if the process starts from \( \square \) then \( \tau_{\square} \geq \tau_{\partial^{\text{ext}} \mathcal{C}} \). Hence,

\[ \mathbb{P}_{\square} [\tau_{\square} > e^{\beta(\Gamma - \varepsilon)}] \geq \mathbb{P}_{\square} [\tau_{\partial^{\text{ext}} \mathcal{C}} > e^{\beta(\Gamma - \varepsilon)}] \geq \mathbb{P}_{\square} [\tau_{\partial^{\text{ext}} \mathcal{C}} > e^{\beta(D - \varepsilon)}] \geq 1 - e^{-\beta \delta}. \]  

(A.8)

The lemma follows by taking the limit \( \beta \to \infty \).

The lower bound on the metastable time for the process starting from a metastable state is given in the next lemma:

Lemma A.2. If \( \eta \) is a metastable configuration and Condition (1) holds, then

\[ \Gamma = V_\eta \geq \Gamma_\ell, \]  

(A.9)

and for all \( \varepsilon > 0 \)

\[ \lim_{\beta \to \infty} \mathbb{P}_{\eta} [\tau_{\square} > e^{\beta(\Gamma - \varepsilon)}] = 1. \]  

(A.10)
Proof. We prove this lemma using the same reasoning as in the proof of [13, Theorem 2.4]. Suppose that Condition (1) holds, and assume that \( \Gamma < \Gamma_\ell \). Then it holds that \( V_\sigma < \Gamma_\ell \) for all configurations \( \sigma \neq \boxplus \). Hence, if we start if we start with configuration \( \sigma_0 = \boxplus \), we can find a state \( \sigma_1 \) such that \( H(\sigma_1) < H(\sigma_0) \) and \( \Phi(\sigma_0, \sigma_1) - H(\sigma_0) < \Gamma_\ell \), i.e., there exists a path \( \omega_0 \) from \( \sigma_0 \) to \( \sigma_1 \) such that

\[
\max_{\sigma' \in \omega_0} H(\sigma') < \Gamma_\ell + H(\sigma_0). \tag{A.11}
\]

As long as the new configuration with lower energy is not equal to \( \boxplus \), we can repeat this argument and find configurations \( \sigma_2, \sigma_3, \ldots, \sigma_n \) such that \( H(\sigma_0) > H(\sigma_1) > H(\sigma_2) > \ldots > H(\sigma_n) \) and \( \Phi(\sigma_i, \sigma_{i+1}) - H(\sigma_i) < \Gamma_\ell \) for all \( i = 0, 2, \ldots, n - 1 \), i.e, there exist paths \( \omega_i \) so that

\[
\max_{\sigma' \in \omega_i} H(\sigma') < \Gamma_\ell + H(\sigma_i). \tag{A.12}
\]

Since the number of configurations is finite and the energy is strictly decreasing every step, if we choose \( n \) large enough we will end in the configuration \( \sigma_n = \boxplus \). If we let \( \omega \) be the concatenation of the paths \( \omega_0, \omega_1, \ldots, \omega_n \), then \( \omega \) is a path from \( \boxplus \) to \( \boxplus \) and

\[
\max_{\sigma' \in \omega} H(\sigma') = \max_{0 \leq i < n} \max_{\sigma' \in \omega_i} H(\sigma') < \Gamma_\ell + \max_{0 \leq i < n} H(\sigma_i) = \Gamma_\ell + H(\boxplus). \tag{A.13}
\]

Hence,

\[
\Phi(\boxplus, \boxplus) - H(\boxplus) < \Gamma_\ell, \tag{A.14}
\]

which is in contradiction with Condition (1).

So, if Condition (1) holds, then \( \Gamma \geq \Gamma_\ell \). Since, \( \eta \) is assumed to be a metastable state \( V_\eta = \Gamma \geq \Gamma_\ell \). The second statement of the lemma now immediately follows from [20, Theorem 4.1].

The upper bound on the metastable time is proved in the next lemma:

**Lemma A.3** (Upper bound on the metastable time). *If Conditions (2) and (3a) hold, then, for all \( \varepsilon > 0 \),

\[
\lim_{\beta \to \infty} \sup_{\sigma} \mathbb{P}_\sigma[\tau_\boxplus > e^{\beta(\Gamma_u + \varepsilon)}] = 0. \tag{A.15}
\]

*Proof. Let

\[
K_V = \{\sigma : V_\sigma > V\}, \tag{A.16}
\]

be the so-called *metastable set at level* \( V \). If we set \( V = \Gamma_u \), then it follows from Conditions (2) and (3a) that \( V_\sigma \leq V \) for all \( \sigma \neq \boxplus \). Hence, \( K_{\Gamma_u} = \{\boxplus\} \), since \( V_\boxplus = \infty \). It thus follows from [20, Theorem 3.1] that, for all \( \varepsilon > 0 \),

\[
\lim_{\beta \to 0} \sup_{\sigma} \mathbb{P}_\sigma[\tau_\boxplus > e^{\beta(\Gamma_u + \varepsilon)}] = 0. \tag{A.17}
\]

\( \square \)

Because of the supremum, the same bound on the hitting time of \( \boxplus \) holds for the process starting from any state \( \sigma \) and in particular for the process starting in \( \boxplus \) and in any metastable state \( \eta \).

We finally characterize when \( \boxplus \) is the unique metastable state:
Lemma A.4. If Conditions (1) and (3b) hold, then $\square$ is the unique metastable state.

Proof. We want to prove that $V_{\square} \geq \Gamma_\ell$, since then

$$\square = \arg \max_{\sigma \neq \square} V_\sigma. \quad (A.18)$$

Suppose that on the contrary $V_{\square} < \Gamma_\ell$. From Condition (3b) it follows that also $V_\sigma < \Gamma_\ell$ for all $\sigma \not\in \{\square, \square\}$. Then, by the same reasoning as in Lemma A.2, we can find a path $\omega$ from $\square$ to $\square$ such that

$$\max_{\sigma' \in \omega} H(\sigma') < \Gamma_\ell + H(\square), \quad (A.19)$$

and hence that

$$\Phi(\square, \square) - H(\square) < \Gamma_\ell, \quad (A.20)$$

which is in contradiction with Condition (1). \qed

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