Maximal classes for families of lower and upper semicontinuous functions with a closed graph

Jolanta Kosman

Received: 12 February 2016 / Accepted: 23 August 2016 / Published online: 22 September 2016
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Abstract In this paper we characterize the following maximal classes for families of lower and upper semicontinuous functions with a closed graph: the maximal additive class, the maximal multiplicative class and the maximal classes with respect to maximum and minimum.

Keywords Functions with a closed graph · Lower semicontinuous functions · Upper semicontinuous functions · Sum of functions

Mathematics Subject Classification Primary 26A15; Secondary 54C08

1 Introduction

The letters $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{N}$ denote the real line, the set of rationals and the set of positive integers, respectively. The family of all functions from a set $X$ into $Y$ is denoted by $Y^X$. For each set $A \subset X$ its characteristic function is denoted by $\chi_A$. In particular, $\chi_{\emptyset}$ stands for the zero constant function.

Let $X$ be a topological space. The symbol $X^d$ denotes the set of all accumulation points of $X$. For each set $A \subset X$ the symbols $\text{int} A$ and $\text{cl} A$ denote the interior and the closure of $A$, respectively. The spaces $\mathbb{R}$ and $X \times \mathbb{R}$ are considered with their standard topologies.

We say that a function $f : X \to \mathbb{R}$ has a closed graph, if the graph of $f$, i.e., the set $\{(x, f(x)) : x \in X\}$ is a closed subset of the product $X \times \mathbb{R}$. We say that a function $f : X \to \mathbb{R}$ is lower (upper) semicontinuous at a point $x \in X$, if for each $\varepsilon > 0$ there is an open neighborhood $U$ of $x$ such that $f(z) > f(x) - \varepsilon$ ( $f(z) < f(x) + \varepsilon$, respectively) for each $z \in U$. If $f : X \to \mathbb{R}$ is lower (upper) semicontinuous at each point $x \in X$, then we say that the function $f$ is lower (upper, respectively) semicontinuous. Let $\mathcal{C}_{\text{const}}(X)$, $\mathcal{C}(X)$,
\( \mathcal{U}(X) \), \( lsc(X) \), \( usc(X) \) denote the class of all real-valued functions on \( X \) that are constant, continuous, have a closed graph, are lower and upper semicontinuous, respectively. Obviously \( \mathcal{C}(X) \subset \mathcal{U}(X) \) (see also e.g. [5]) and \( \mathcal{C}(X) = lsc(X) \cap usc(X) \). For \( \mathcal{F}(X) \) and \( \mathcal{G}(X) \) nonempty subsets of \( \mathbb{R}^X \) the symbol \( \mathcal{F}\mathcal{G}(X) \) denotes the class \( \mathcal{F}(X) \cap \mathcal{G}(X) \). Further denote by \( \mathcal{F}^+(X) \) the family of all nonnegative functions from \( \mathcal{F}(X) \). Let \( f \in \mathbb{R}^X \). The symbol \( \mathcal{G}(f) \) denotes the graph of \( f \) and the symbols \( \mathcal{C}(f) \) and \( \mathcal{D}(f) \) denote the sets of points of continuity and discontinuity of \( f \), respectively. For each \( y \in \mathbb{R} \) let \( [f = y] = \{ x \in X : f(x) = y \} \). Similarly we define the symbols \( [f > y] \), \( [f < y] \).

If \( \mathcal{F} \subset \mathbb{R}^X \) is a family of functions, denote by

\[
\mathcal{F} + \mathcal{F} \overset{\text{df}}{=} \{ f \in \mathbb{R}^X : f = g + h \text{ for some } g, h \in \mathcal{F} \},
\]
\[
\mathcal{M}_a(\mathcal{F}) \overset{\text{df}}{=} \{ f \in \mathbb{R}^X : (\forall g \in \mathcal{F}) \ f + g \in \mathcal{F} \},
\]
\[
\mathcal{M}_m(\mathcal{F}) \overset{\text{df}}{=} \{ f \in \mathbb{R}^X : (\forall g \in \mathcal{F}) \ f \cdot g \in \mathcal{F} \},
\]
\[
\mathcal{M}_{\max}(\mathcal{F}) \overset{\text{df}}{=} \{ f \in \mathbb{R}^X : \max(\mathcal{F}) \in \mathcal{F} \},
\]
\[
\mathcal{M}_{\min}(\mathcal{F}) \overset{\text{df}}{=} \{ f \in \mathbb{R}^X : \min(\mathcal{F}) \in \mathcal{F} \}.
\]

The above classes \( \mathcal{M}_a(\mathcal{F}) \), \( \mathcal{M}_m(\mathcal{F}) \), \( \mathcal{M}_{\max}(\mathcal{F}) \) and \( \mathcal{M}_{\min}(\mathcal{F}) \) are called the maximal additive class for \( \mathcal{F} \), the maximal multiplicative class for \( \mathcal{F} \), the maximal class with respect to maximum and minimum for \( \mathcal{F} \), respectively.

In 1987 Menkyna [7] characterized the maximal additive and multiplicative classes for the family of functions with a closed graph. He proved that \( \mathcal{M}_a(\mathcal{U}(X)) = \mathcal{C}(X) \) for a topological space \( X \) [7, Theorem 1] and \( \mathcal{M}_m(\mathcal{U}(X)) = \{ f \in \mathcal{C}(X) : [f = 0] \text{ is an open set} \} \) for a locally compact normal topological space \( X \) [7, Theorem 2]. Let \( \mathcal{Q}(X) \) denote the family of all quasi-continuous functions from a topological space \( X \) to \( \mathbb{R} \). Recall that \( f \in \mathcal{Q}(X) \) if and only if for each \( x \in X \), \( \varepsilon > 0 \) and for each neighbourhood \( U \) of \( x \) there is a nonempty open set \( V \subset U \) such that \( |f(x) - f(y)| < \varepsilon \) for each \( y \in V \). In 2008 Sieg [8] considered real functions defined on \( \mathbb{R} \) and showed that \( \mathcal{M}_a(\mathcal{Q}(\mathbb{R})) = \mathcal{C}(\mathbb{R}) \), \( \mathcal{M}_m(\mathcal{Q}(\mathbb{R})) = \{ f \in \mathcal{C}(\mathbb{R}) : f = \chi_0 \text{ or } f(x) \neq 0 \text{ for all } x \in \mathbb{R} \} \) and \( \mathcal{M}_{\max}(\mathcal{Q}(\mathbb{R})) = \mathcal{M}_{\min}(\mathcal{Q}(\mathbb{R})) = \emptyset \). In 2014 Szczuka (see [9, 10]) characterized the following maximal classes for lower and upper semicontinuous strong Świątkowski functions and lower and upper semicontinuous extra strong Świątkowski functions: the maximal additive class, the maximal multiplicative class and the maximal classes with respect to maximum. She proved, among others, that if \( \mathcal{F} \) denotes the family of lower semicontinuous strong Świątkowski real functions defined on \( \mathbb{R} \), then \( \mathcal{M}_a(\mathcal{F}) = \mathcal{C}\text{\it const} \) [9, Theorems 3.1, \( \mathcal{M}_m(\mathcal{F}) = \mathcal{C}\text{\it const}^+ \) [9, Theorem 3.2] and \( \mathcal{M}_{\max}(\mathcal{F}) = \mathcal{C}\text{\it const} \) [9, Theorem 3.3].

In this paper we deal with the families of lower and upper semicontinuous functions with a closed graph. We obtain the following results:

- \( \mathcal{M}_a(\mathcal{U}lsc(X)) = \mathcal{U}lsc(X) \), where \( X \) is a topological space (Theorem 2.5),
- \( \mathcal{M}_a(\mathcal{U}usc(X)) = \mathcal{U}usc(X) \), where \( X \) is a topological space (Theorem 3.3),
- \( \mathcal{M}_m(\mathcal{U}lsc(X)) = \{ f \in \mathcal{C}(X) : [f = 0] \text{ is an open set and } f(x) \geq 0 \text{ for all } x \in X \} = \mathcal{M}_m(\mathcal{U}usc(X)) \), where \( X \) is a perfectly normal topological space such that \( X \overset{d}{=} \mathbb{R} \) (Theorems 2.7, 3.4),
- \( \mathcal{M}_{\max}(\mathcal{U}lsc(X)) = \mathcal{U}sc(X) \), where \( X \) is a topological space (Theorem 2.10),
- \( \mathcal{M}_{\min}(\mathcal{U}usc(X)) = \mathcal{U}usc(X) \), where \( X \) is a topological space (Theorem 3.5),
- \( \mathcal{M}_{\min}(\mathcal{U}lsc(X)) = \mathcal{M}_{\max}(\mathcal{U}usc(X)) = \emptyset \), where \( X \) is a perfectly normal topological space such that \( X \overset{d}{=} \emptyset \) (Corollary 2.15, Theorems 3.6).
2 Lower semicontinuous functions with a closed graph

We start with a following proposition.

**Proposition 2.1** Let \( X \) be a topological space. A function \( f : X \to \mathbb{R} \) has the closed graph if and only if for each \( x \in X \) and for each \( m \in \mathbb{N} \) there is a neighbourhood \( V \) of \( x \) such that \( f(z) \in (f(x) - 1/m, f(x) + 1/m) \cup (m, \infty) \) for each \( z \in V \).

**Proof** The implication \((\Leftarrow)\) we can find in [2] (see p. 118, lines 11–14). The implication \((\Rightarrow)\) immediately follows from [6] or [1, Proposition 1]: if \( f \in \mathcal{U}(X) \), then for each \( x \in X \) and each neighborhood \( U \) of \( f(x) \) such that \( Y \setminus U \) is compact there is an neighborhood \( V \) of \( x \) such that \( f(V) \subset U \). Now, it is sufficient to take \( U = (-\infty, -m) \cup (f(x) - 1/m, f(x) + 1/m) \cup (m, \infty) \). Observe that, the equivalence of this proposition also immediately follows from [1, Proposition 2]. \( \square \)

From above and the definitions of the class \( lsc \) we obtain:

**Lemma 2.2** Let \( X \) be a topological space. A function \( f : X \to \mathbb{R} \) is lower semicontinuous function with a closed graph if and only if for each \( x \in X \) and for each \( m \in \mathbb{N} \) there is a neighbourhood \( V \) of \( x \) such that \( f(z) \in (f(x) - 1/m, f(x) + 1/m) \cup (m, \infty) \) for each \( z \in V \).

**Proof** First, assume that for each \( x \in X \) and for each \( m \in \mathbb{N} \) there is a neighbourhood \( V \) of \( x \) such that \( f(z) \in (f(x) - 1/m, f(x) + 1/m) \cup (m, \infty) \) for each \( z \in V \). Then, by Proposition 2.1, \( f \in \mathcal{U}(X) \). Now, we will show that \( f \in lsc(X) \). Let \( x \in X \) and \( \varepsilon > 0 \). We choose \( m \in \mathbb{N} \) such that \( m \geq \max \{ \frac{1}{\varepsilon}, f(x) - \varepsilon \} \). There is a neighbourhood \( V \) of \( x \) such that \( f(z) \in (f(x) - 1/m, f(x) + 1/m) \cup (m, \infty) \subset (f(x) - \varepsilon, \infty) \) for each \( z \in V \) and consequently \( f \in lsc(X) \).

Now, let \( f \in \mathcal{U}lsc(X) \). Fix \( x \in X \) and \( m \in \mathbb{N} \). Since \( f \in lsc(X) \), there is a neighbourhood \( V_1 \) of \( x \) such that \( f(z) \in (f(x) - 1/m, \infty) \) for each \( z \in V_1 \). We consider two cases.

First, assume that \( f(x) \geq 0 \). Since \( f \in \mathcal{U}(X) \), there is a neighbourhood \( V_2 \) of \( x \) such that \( f(z) \in (f(x) - 1/m, f(x) + 1/m) \cup (m, \infty) \) for each \( z \in V_2 \) (see Proposition 2.1). Let \( V = V_1 \cap V_2 \) and let \( z \in V \). Then \( f(z) \in (f(x) - 1/m, f(x) + 1/m) \cup (m, \infty) \).

Now, assume that \( f(x) < 0 \). We choose \( k \in \mathbb{N} \) such that \( k \geq \max \{ m, -f(x) + \frac{1}{m} \} \). Since \( f \in \mathcal{U}(X) \), there is a neighbourhood \( V_2 \) of \( x \) such that \( f(z) \in (-\infty, -k) \cup (f(x) - 1/k, f(x) + 1/k) \cup (k, \infty) \) for each \( z \in V_2 \). Let \( V = V_1 \cap V_2 \) and let \( z \in V \). Since \( k \geq m \), \( -k \leq f(x) - \frac{1}{m} \) and \( \frac{1}{k} \leq \frac{1}{m} \), we have \( f(z) \in (f(x) - 1/k, f(x) + 1/k) \cup (k, \infty) \subset \left( (f(x) - 1/m, f(x) + 1/m) \cup (m, \infty) \right) \). This completes the proof. \( \square \)

The next lemma follows from Proposition 2.1 and Lemma 2.2.

**Lemma 2.3** Let \( X \) be a topological space. Then \( \mathcal{U}^+(X) \subset \mathcal{U}lsc(X) \).

Now, we will characterize the class of the sums of lower semicontinuous functions with a closed graph.

**Lemma 2.4** Let \( X \) be a topological space. Then \( \mathcal{U}lsc(X) + \mathcal{U}lsc(X) = \mathcal{U}lsc(X) \).

**Proof** Let \( f, g \in \mathcal{U}lsc(X) \). Fix \( x \in X \) and \( m \in \mathbb{N} \). Let \( k \in \mathbb{N} \) be such that \( 1/k < 1/(2m + |f(x)| + |g(x)|) \). By Lemma 2.2, there exists a neighbourhood \( V \) of \( x \) such that \( f(z) \in \ldots \)
\( (f(x) - 1/k, f(x) + 1/k) \cup (k, \infty) \) and \( g(z) \in (g(x) - 1/k, g(x) + 1/k) \cup (k, \infty) \) for each \( z \in V \). Let \( z \in V \). We consider four cases.

If \( f(z) > k \) and \( g(z) > k \), then evidently \( (f + g)(z) > m \).

If \( f(z) > k \) and \( g(z) \in (g(x) - 1/k, g(x) + 1/k) \), then

\[
(f + g)(z) > k + g(x) - 1/k > 2m + |f(x)| + |g(x)| + g(x) - 1/k > m.
\]

Similarly, \( g(z) > k \) and \( f(z) \in (f(x) - 1/k, f(x) + 1/k) \), implies \( (f + g)(z) > m \).

Now, let \( f(z) \in (f(x) - 1/k, f(x) + 1/k) \) and \( g(z) \in (g(x) - 1/k, g(x) + 1/k) \). Then, we have

\[
|g(z) - g(x)| \leq |f(x) - f(z)| + |g(z) - g(x)| < 2/k < 1/m.
\]

It follows that \( f + g \in U\text{lsc}(X) \).

\[ \square \]

**Theorem 2.5** Let \( X \) be a topological space. Then \( \mathcal{M}_a(U\text{lsc}(X)) = U\text{lsc}(X) \).

**Proof** Since \( \chi_0 \in U\text{lsc}(X) \), we conclude that \( \mathcal{M}_a(U\text{lsc}(X)) \subset U\text{lsc}(X) \). The inclusion \( U\text{lsc}(X) \subset \mathcal{M}_a(U\text{lsc}(X)) \) follows from Lemma 2.4.

Now, recall the following lemma [7, Lemma 2], which will be applied in this paper.

**Proposition 2.6** Let \( X \) be a topological space and let \( f \in C(X) \). Then the function \( g : X \rightarrow \mathbb{R} \) defined by the formula

\[
  g(x) = \begin{cases} 
  \frac{1}{|f(x)|^2}, & \text{if } x \in [f \neq 0], \\
  0, & \text{if } x \in [f = 0].
  \end{cases}
\]

has the closed graph.

**Theorem 2.7** Let \( X \) be a normal topological space such that each singleton is \( G_\delta \)-set. Then

\[ \mathcal{M}_m(U\text{lsc}(X)) = \{ f \in C(X) : [f = 0] \text{ is an open set and } [f < 0]^d = \emptyset \}. \]

**Proof** We will prove this theorem in four parts. First, we will show that \( \mathcal{M}_m(U\text{lsc}(X)) \subset C(X) \). Let \( f \in \mathcal{M}_m(U\text{lsc}(X)) \). Since \( \chi_{\mathbb{R}}, -\chi_{\mathbb{R}} \in U\text{lsc}(X) \), we have \( f \in lsc(X) \) and \( -f \in lsc(X) \). Consequently \( f \in lsc(X) \cap usc(X) = C(X) \).

Now, we assume that the function \( f \in C(X) \) and the set \( [f = 0] \) is not open. We will show that \( f \notin \mathcal{M}_m(U\text{lsc}(X)) \) (The proof of this part is similar to the second part of the proof of [7, Theorem 2]). Define the function \( g : X \rightarrow \mathbb{R} \) by the formula

\[
  g(x) = \begin{cases} 
  \frac{1}{|f(x)|^2}, & \text{if } x \in [f \neq 0], \\
  0, & \text{if } x \in [f = 0].
  \end{cases}
\]

By Proposition 2.6 the function \( g \) has the closed graph. Moreover \( g \) is non-negative function and consequently, by Lemma 2.3, \( g \in U\text{lsc}(X) \). Now, we will show that \( f \cdot g \notin U\text{lsc}(X) \).

Since the set \( [f = 0] \) is not open, there is \( x_0 \in [f = 0] \) such that for each open neighbourhood \( V \) of \( x_0 \) there is \( x_V \in V \cap [f \neq 0] \). Notice that \( (f \cdot g)(x_V) \in [-1, 1] \) for each neighbourhood \( V \) of \( x_0 \) and \( (f \cdot g)(x_0) = 0 \). By Proposition 2.1, \( f \cdot g \notin U(X) \).

In the third part of the proof, suppose that \( f \in C(X) \), the set \( [f = 0] \) is open and \( [f < 0]^d \neq \emptyset \). We will prove that \( f \notin \mathcal{M}_m(U\text{lsc}(X)) \). Let \( x_0 \in [f < 0]^d \). Then there is a net \( (x_\gamma)_{\gamma \in \Gamma} \) of elements of \( X \) such that \( x_\gamma \rightarrow x_0, x_\gamma \neq x_0 \) and \( f(x_\gamma) < 0 \) for every \( \gamma \in \Gamma \).
Notice that, since \( f \in C(X) \) and the set \([f = 0]\) is open, we have \( f(x_0) < 0 \). By Urysohn Lemma there is a continuous function \( h : X \to [0, 1] \) such that \([h = 0] = \{x_0\} \).

Define the function \( g : X \to \mathbb{R} \) by the formula

\[
g(x) = \begin{cases} \frac{1}{h(x)}, & \text{if } x \neq x_0, \\ 0, & \text{if } x = x_0. \end{cases}
\]

Observe that, by Proposition 2.6 and Lemma 2.3, \( g \in Ulsc(X) \). Moreover \( f \cdot g \notin lsc(X) \), because the net \((f \cdot g)(x_0)_{y \in \Gamma}\) diverges to \(-\infty \) (recall that \( f(x_0) < 0 \)).

In the last part suppose that \( f \in C(X) \), the set \([f = 0]\) is open, \([f < 0]\) is not \( \emptyset \) and \( g \in Ulsc(X) \). Then, by [7, Theorem 2], \((f \cdot g) \in U(X)\) (see also the third part of the proof of [7, Theorem 2]). It is enough to show that \((f \cdot g) \in lsc(X)\). Let \( x_0 \in X \). If \( f(x_0) \leq 0 \), then the function \( f \cdot g \) is continuous at \( x_0 \) and consequently \( f \cdot g \) is a lower semicontinuous at this point. Indeed, if \( f(x_0) = 0 \), then by the assumption \([f = 0] = \text{int}[f = 0] \), we have \( x_0 \in \text{int}[f = 0] \subset \text{int}[f \cdot g = 0] \) and if \( f(x_0) < 0 \), then \( x_0 \) is an isolated point of \( X \). Finally, assume that \( f(x_0) > 0 \). Since \( f \in C(X) \), there is an open neighborhood \( U \) of \( x_0 \) such that \( U \subset [f > 0] \). Since \( g \in lsc(X) \), \( f \) is continuous and positive function on \( U \), the function \( f \cdot g \) is a lower semicontinuous at \( x_0 \). The proof is complete.

It is easy to see that from above for \( X = \mathbb{R} \) we have the following corollary.

**Corollary 2.8** \( M_n(Ulsc(\mathbb{R})) = \{f \in C(\mathbb{R}) : f = \chi_\emptyset \text{ or } f(x) > 0 \text{ for all } x \in \mathbb{R}\} \).

**Lemma 2.9** Let \( X \) be a topological space and let \( f, g \in Ulsc(X) \). Then the real function \( h = \max\{f, g\} \) defined on \( X \) is a lower semicontinuous function with a closed graph.

**Proof** Let \( f, g \in Ulsc(X) \). We will use Lemma 2.2. Fix \( x \in X \) and \( m \in \mathbb{N} \). Then there exists a neighborhood \( V \) of \( x \) such that \( f(z) \in (f(x) - 1/m, f(x) + 1/m) \cup (m, \infty) \) and \( g(z) \in (g(x) - 1/m, g(x) + 1/m) \cup (m, \infty) \) for each \( z \in V \). We assume that \( f(x) \geq g(x) \) (The case \( f(x) < g(x) \) is analogous). Then \( h(x) = f(x) \) and it is easy to see that \( h(z) \in (h(x) - 1/m, h(x) + 1/m) \cup (m, \infty) \) for each \( z \in V \). So, \( h \in Ulsc(X) \).

**Theorem 2.10** Let \( X \) be a topological space. Then \( M_{\max}(Ulsc(X)) = Ulsc(X) \).

**Proof** The inclusion \( Ulsc(X) \subseteq M_{\max}(Ulsc(X)) \) follows from Lemma 2.9. So, we will only prove that \( M_{\max}(Ulsc(X)) \subseteq Ulsc(X) \). Let \( f : X \to \mathbb{R} \) be a function such that \( f \notin Ulsc(X) \). We choose \( x_0 \in X \) and \( m \in \mathbb{N} \), such that \( m \geq f(x_0) + \frac{1}{m} \) and for each open neighborhood \( V \) of \( x_0 \) there is \( x \in V \) such that \( f(x) \leq m \) and \( f(x) \notin (f(x_0) - \frac{1}{m}, f(x_0) + \frac{1}{m}) \). We will show that \( f \notin M_{\max}(Ulsc(X)) \). Let \( c = f(x_0) - \frac{1}{m} \). Define the function \( g : X \to \mathbb{R} \) by \( g \equiv c \). Clearly \( g \in Ulsc(X) \). Denote \( h = \max\{f, g\} \). We will prove that \( h \notin Ulsc(X) \). Notice that \( h(x_0) = f(x_0) \). Observe that, for each open neighborhood \( V \) of \( x_0 \) there is \( x_V \in V \) such that \( f(x_V) \in (-\infty, c] \cup [f(x_0) + \frac{1}{m}, m] \) and consequently \( h(x_V) \in [h(x_0) - \frac{1}{m}, m] \). By Proposition 2.1, \( h \notin U(X) \). This completes the proof.

**Theorem 2.11** Let \( X \) be a topological space such that \( U(X) \neq C(X) \). Then \( M_{\min}(Ulsc(X)) = \emptyset \).

**Proof** Let \( f \in \mathbb{R}^X \). We will show that there is a function \( g \in Ulsc(X) \) such that the function \( h = \min\{f, g\} \notin Ulsc(X) \).

Let \( g_1 : X \to \mathbb{R} \) be a function with a closed graph and let \( x_0 \in D(g_1) \). Put \( g_2 = |g_1| \). Then \( g_2 \in Ulsc \) and there is a net \((x_{\gamma})_{\gamma \in \Gamma}\) of elements of \( X \) which converges to the point \( x_0 \) and a net \((g_2(x_{\gamma}))_{\gamma \in \Gamma}\) diverges to \( \infty \). We consider two cases.
If \( x_0 \in C(f) \), we define the function \( g : X \to \mathbb{R} \) by \( g(x) \overset{\text{df}}{=} g_2(x) - g_2(x_0) + f(x_0) - 1 \). It is easy to see that \( g \in \text{lsc}(X) \). Let \( h = \min\{f, g\} \). Then \( h(x_0) = g(x_0) = f(x_0) - 1 \) and there is \( \gamma_0 \in \Gamma \) such that \( h(x_\gamma) = f(x_\gamma) \) for each \( \gamma > \gamma_0 \). Consequently \( (x_0, f(x_0)) \in \text{cl} G(h) \setminus G(h) \) and \( h \not\in \mathcal{U}(X) \).

Now, let \( x_0 \in D(f) \). There is \( \varepsilon > 0 \) such that for each neighbourhood \( V \) of \( x_0 \) there is \( z \in V \) such that \( f(z) \not\in (f(x_0) - \varepsilon, f(x_0) + \varepsilon) \). Define the function \( g : X \to \mathbb{R} \) by \( g(x) \overset{\text{df}}{=} f(x_0) + \varepsilon \). Let \( h = \min\{f, g\} \). Then \( h(x_0) = f(x_0) \) and for each neighbourhood \( V \) of \( x_0 \) there is \( z \in V \) such that \( h(z) \in (-\infty, h(x_0) - \varepsilon] \cup \{h(x_0) + \varepsilon\} \). By Proposition 2.2, \( h \not\in \text{lsc}(X) \) \( \square \)

It is easy to see that

\textbf{Remark 1} Let \( X \) be a topological space such that \( \mathcal{U}(X) = \mathcal{C}(X) \). Then \( \mathcal{M}_{\min}(\text{lsc}(X)) = \mathcal{C} \).

Now, we recall the definition of a \( P \)-space [4, pp. 62–63] and two propositions given by Wójtowicz and Sieg [11, Theorem 1 and Corollary 1].

\textbf{Definition 1} We say that a completely regular (Tychonoff) space \( X \) is a \( P \)-space if every \( G_\delta \)-subset (\( F_\sigma \)-subset) of \( X \) is open (closed); equivalently, every co-zero subset of \( X \) is closed.

\textbf{Proposition 2.12} Let \( X \) be a completely regular space. Then \( \mathcal{U}(X) = \mathcal{C}(X) \) if and only if \( X \) is a \( P \)-space.

\textbf{Proposition 2.13} Let \( X \) be a perfectly normal or first countable space, or a locally compact space. Then \( \mathcal{U}(X) \neq \mathcal{C}(X) \) if and only if \( X \) is non-discrete.

From Proposition 2.12, Theorem 2.11 and Remark 1 we obtain the following Corollary.

\textbf{Corollary 2.14} Let \( X \) be a nonempty completely regular space. Then \( \mathcal{M}_{\min}(\text{lsc}(X)) = \emptyset \) if and only if \( X \) is not a \( P \)-space.

Moreover, using Proposition 2.13 and Theorem 2.11 we conclude that

\textbf{Corollary 2.15} Let \( X \) be a non-discrete perfectly normal or first countable space, or a locally compact space. Then \( \mathcal{M}_{\min}(\text{lsc}(X)) = \emptyset \).

Finally, observe that we can extend the lists (see e.g. [11, Theorem 1]) of equivalent conditions for \( X \) to be a \( P \)-space as follows:

\textbf{Corollary 2.16} Let \( X \) be a nonempty completely regular space. Then \( X \) is a \( P \)-space if and only if \( \mathcal{M}_{\min}(\text{lsc}(X)) \neq \emptyset \).

3 Upper semicontinuous functions with a closed graph

First, we recall some basic property of the functions with a closed graph [3, Proposition 2]

\textbf{Proposition 3.1} Let \( X \) be a topological space. Let \( \alpha \) be a real number. If \( f \in \mathcal{U}(X) \), then \( \alpha \cdot f \in \mathcal{U}(X) \).

From above and the definitions of the classes \( \text{lsc}(X) \) and \( \text{usc}(X) \) we obtain:
Proposition 3.2 Let $X$ be a topological space. For each function $f \in \mathbb{R}^X$ we have $f \in \mathcal{U}usc(X)$ if and only if $(-f) \in \mathcal{U}lsc(X)$.

Now, we will characterize the following maximal classes for the family of upper semicontinuous functions with a closed graph: the maximal additive class, the maximal multiplicative class and the maximal classes with respect to maximum and minimum.

Theorem 3.3 Let $X$ be a topological space. Then $\mathcal{M}_a(\mathcal{U}usc(X)) = \mathcal{U}usc(X)$.

Proof Observe that, by Proposition 3.2, $f \in \mathcal{M}_a(\mathcal{U}usc(X))$ if and only if $-f \in \mathcal{M}_a(\mathcal{U}lsc(X))$. Using Theorem 2.5 and again Proposition 3.2, we conclude that $\mathcal{M}_a(\mathcal{U}usc(X)) = \mathcal{U}usc(X)$.

The next theorem follows from Proposition 3.2.

Theorem 3.4 Let $X$ be a topological space. Then $\mathcal{M}_m(\mathcal{U}usc(X)) = \mathcal{M}_m(\mathcal{U}lsc(X))$.

Theorem 3.5 Let $X$ be a topological space. Then $\mathcal{M}_{\min}(\mathcal{U}usc(X)) = \mathcal{U}usc(X)$.

Proof Since $- \min\{f, g\} = \max\{-f, -g\}$ for each functions $f, g \in \mathbb{R}^X$, by Proposition 3.2, we conclude that $f \in \mathcal{M}_{\min}(\mathcal{U}usc(X))$ if and only if $-f \in \mathcal{M}_{\max}(\mathcal{U}lsc(X))$. Now, using Theorem 2.10 and again Proposition 3.2, we obtain that $\mathcal{M}_{\min}(\mathcal{U}usc(X)) = \mathcal{U}usc(X)$.

It is easy to see that using Theorem 2.11, Remark 1 and the equivalence $f \in \mathcal{M}_{\max}(\mathcal{U}usc(X))$ if and only if $-f \in \mathcal{M}_{\min}(\mathcal{U}lsc(X))$, we conclude that:

Theorem 3.6 Let $X$ be a topological space. Then $\mathcal{M}_{\max}(\mathcal{U}usc(X)) = \mathcal{M}_{\min}(\mathcal{U}lsc(X))$.

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