The role of beam waist in Laguerre-Gauss expansion of vortex beams

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Laguerre-Gauss (LG) modes represent an orthonormal basis set of solutions of the paraxial wave equation. LG are characterized by two integer parameters \( n \) and \( \ell \) that are related to the radial and azimuthal profile of the beam. The physical dimension of the mode is instead determined by the beam waist parameter \( w_0 \); only LG modes with the same \( w_0 \) satisfy the orthogonality relation.

Here, we derive the scalar product between two LG modes with different beam waists and show how this result can be exploited to derive different expansions of a generic beam in terms of LG modes. In particular, we apply our results to the recently introduced Circular Beams, by deriving a previously unknown expansion. We finally show how the waist parameter must be chosen in order to optimize such expansion.

I. INTRODUCTION

Laguerre-Gaussian (LG) beams are exact solutions of the free-space paraxial wave equation in circular cylindrical coordinates. They form a complete base of orthogonal modes under which any paraxial optical field can be expanded. Moreover, LG modes carry Orbital Angular Momentum (OAM) and their importance has raised together with the recent developments [1] in applications of OAM for communication [2–5], imaging [6–8] and fundamental physics [9–11].

LG beams are defined in term of two integer numbers \( n, \ell \in \mathbb{Z} \) with \( n \geq 0 \): the value of \( n \) determines the radial profile of the LG mode, while \( \ell \) is related to the OAM content of the beam [12]. LG modes are also characterized by two dimensional quantities, the beam waist \( w_0 \) and its location \( q_0 \) along the propagation axis: these two quantities can be encoded in a single complex parameter \( q_0 \equiv -d_0 + iz_0 \), where \( z_0 = \kappa w_0^2/2 > 0 \) is the so called Rayleigh range and \( \kappa \) is the wavenumber [13]. While \( d_0 \) can be arbitrarily changed by a translation on the propagation axis, the beam waist parameter \( w_0 \) determines the physical scale of the LG modes. It is worth noticing that only LG with the same \( q_0 \) form a complete basis of orthogonal modes.

In the present work we will evaluate the overlap integral of two LG modes with different complex parameters \( q_0 \) and \( q'_0 \). We will show that such overlap, besides the radial and angular parameters, depends only on a single adimensional complex variable given by a combination of \( q_0 \) and \( q'_0 \). We will show how such result can be exploited in finding new expansions of generic beams in terms of LG modes and in particular we will apply our method to the recently introduced Circular Beams [14–15].

II. SCALAR PRODUCT OF LG MODES

In this section we calculate the scalar product (defined by an overlap integral) between two LG modes with dif-

\[
\langle \text{LG}_{n,\ell}(q_0) | \text{LG}_{n',\ell'}(q'_0) \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi^*(r,\phi, z) \Phi(r,\phi, z + d) \sqrt{2} e^{i\kappa z_0} \Psi(\sqrt{2}(\kappa z_0), \phi, z) \Phi(\sqrt{2}(\kappa z_0), \phi, z + d)
\]

with \( L_n^\ell(t) \) the generalized Laguerre polynomial.

We now derive the scalar product \( \langle \text{LG}^{(q_0)}_{n,\ell} | \text{LG}^{(q'_0)}_{n',\ell'} \rangle \) between two LG modes characterized by different beam parameters \( q_0 \) and \( q'_0 \). The scalar product between two generic beams \( \Psi(x) \) and \( \Phi(x) \) is defined by the overlap integral \( \langle \Psi | \Phi \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi^*(x) \Phi(x) \). If \( \Psi(x) \) and \( \Phi(x) \) satisfy the paraxial wave equation, \( \langle \Psi | \Phi \rangle \) is invariant under the following transformations applied to both beams: the translation along the propagation axis \( T : \Psi(r, \phi, z) \rightarrow \Psi(r, \phi, z + d) \) and the rescaling of the axis \( S : \Psi(r, \phi, z) \rightarrow \sqrt{s} \Psi(\sqrt{s} r, \phi, s z) \). For the LG modes, the two above transformations can expressed
in term of the complex parameter \( q_0 \) as \( T : \text{LG}^{(q_0)}_{n,\ell} \rightarrow \text{LG}^{(q_0+\delta)}_{n,\ell} \) and \( S : \text{LG}^{(q_0)}_{n,\ell} \rightarrow \sqrt{s} \text{LG}^{(q_0/s)}_{n,\ell} \). Then, the scalar product between two LG modes must depend only on a combination of \( d_0, d_0', z_0 \) and \( z_0' \) that is invariant under the two above transformations applied to \( q_0 \) and \( q_0' \). A possible invariant combination is the following (adimensional) complex variable:

\[
\gamma = z_0' / z_0 + i (d_0' - d_0) / z_0 .
\]  

The variable \( \gamma \) can be equivalently defined by the implicit relation \( q_0' = -d_0 + i \gamma z_0 \). A single constraint on \( \gamma \), namely \( \Re(\gamma) > 0 \), is required due to the conditions \( z_0', z_0 > 0 \).

It is easy to show that the parameter \( \gamma \) defined by Eq. (2) is invariant under the two transformations \( T \) and \( S \). Moreover, \( \gamma \) has a simple physical interpretation: its real part is the square of the beam waist ratio and it is thus related to the physical sizes of the two LG beams. Its imaginary part is the distance between the locations of the two beam waists measured in units of \( z_0 \).

The LG modes defined in Eq. (1) and with the same \( q_0 \) are orthonormal with respect to the above defined scalar product, namely \( \langle \text{LG}^{(q_0)}_{n',\ell'} | \text{LG}^{(q_0)}_{n,\ell} \rangle = \delta_{n,n'} \delta_{\ell,\ell'} \). On the other hand, as demonstrated in appendix A, the scalar product between two LG modes with different beam parameters \( q_0 \) and \( q_0' \) is given by

\[
\langle \text{LG}^{(q_0')}_{n',\ell'} | \text{LG}^{(q_0)}_{n,\ell} \rangle = \delta_{\ell,\ell'} G^{(\gamma)}_{n',n,\ell} ,
\]  

where

\[
G^{(\gamma)}_{n',n,\ell} = \sqrt{\binom{|\ell| + n}{n} \binom{|\ell| + n'}{n'}} \tau_1^{1+|\ell|} \tau_2^{n-n'} \times 2 F_1 \left(-n', -|\ell|, 1+1; \frac{\tau_1^2}{\tau_2 \tau_3} \right) ,
\]  

and the parameters \( \tau_j \) depend on the complex parameter \( \gamma \) defined in Eq. (2) as follow:

\[
\tau_1 = \frac{2 \sqrt{\Re(\gamma)}}{1 + \gamma^*} , \quad \tau_2 = \frac{1 - \gamma^*}{1 + \gamma^*} , \quad \tau_3 = \frac{1 - \gamma}{1 + \gamma} .
\]  

We recall that the Hypergeometric polynomial \( 2 F_1 \) in Eq. (4) is defined by using the Pochhammer symbol \( (a)_k \equiv \Gamma(a + k) / \Gamma(a) \) as

\[
2 F_1 \left(-n', -|\ell|, 1+1; \frac{\tau_1^2}{\tau_2 \tau_3} \right) = \sum_{k=0}^{n'} \binom{n'}{k} (-1)^k \frac{(n_0 k)_k}{k!} .
\]

If \( q_0' \neq q_0 \), LG modes with different radial numbers \( n \) and \( n' \) may become not-orthogonal (see Fig. 1), while modes with different OAM remains orthogonal due to the factor \( \delta_{\ell,\ell'} \) in Eq. (3). When \( q_0' = q_0 \), Eq. (3) reduces to the standard orthogonality relation of the LG modes, as expected. Indeed, for \( q_0' \rightarrow q_0 \), we obtain \( \gamma \rightarrow 1 \) and \( G^{(\gamma)}_{n',n,\ell} \rightarrow \delta_{n,n'} \).

The scalar product between LG modes with different beam parameters can be used to optimize the expansion of a generic beam. Indeed, if a given expansion in term of LG modes is known, e.g. \( \Psi(x) = \sum_{n,\ell} c_{n,\ell} \text{LG}^{(q_0)}_{n,\ell} (x) \), by exploiting Eq. (3) the same field can be also expressed as

\[
\Psi(x) = \sum_{n'} c_{n',\ell} \text{LG}^{(q_0')}_{n',\ell} (x) \]  

where

\[
\psi_{n',\ell} = \sum_n c_n \text{LG}^{(q_0)}_{n,\ell} .
\]

While the expansion of \( \Psi(x) \) is unique at a given \( q_0 \) (indeed the LG modes represent a complete basis), by changing the beam parameter to \( q_0' \), a different expansion is found. It is worth noticing that, if the expansion is truncated such as \( \sum_{n \leq N} \sum_{|\ell| \leq L} c_{n,\ell} \text{LG}^{(q_0)}_{n,\ell} (x) \), the resulting beam may differently approximate the original beam depending on the choice of the parameter \( q_0 \). Then, the choice of the correct value of the beam waist size \( w_0 \) and its location \( d_0 \) for the LG modes is essential for obtaining a faithful approximation in a truncated expansion. We now apply the above considerations to the Circular Beams.

### III. APPLICATION TO CIBS

Circular Beams (CiBs) represent a very general solution of the paraxial wave equation in cylindrical coordinate \( \mathbb{R}^2 \). They generalize many well known beam carrying OAM, such as the elegant LG modes \( \text{LG} \), the Hypergeometric-Gaussian (HyGG) beams \( \text{HyGG} \) or the optical vortex beam \( \text{OV} \). The CiBs are determined by three complex parameters \( \xi, q_0 \) and \( p \) and one integer parameter \( \ell \in \mathbb{Z} \) related to the OAM content \( \ell \). The parameter \( \xi \) is related to the beam shape and specific values identify some well known beams; \( q_0 \) is related to the physical size (similarly to the \( q_0 \) parameter of the LG modes) while \( p \) defines the radial index.

As demonstrated in \( \text{CiB}^{(q_0,\xi)}_{p,\ell} (x) = \sum_{n=0}^{+\infty} C_n \text{LG}^{(q_0)}_{n,\ell} (x) \),

\[
C_n = \frac{\xi^n (-p/2)_n}{\sqrt{n!(|\ell| + 1)n}}
\]

with

\[
C_n = (2 F_1 \left[-q_0^2, -\xi^2, 1 + |\ell|, |\xi|^2 \right])^{-1/2}
\]  

and \( C_0 = (2 F_1 \left[-q_0^2, -\xi^2, 1 + |\ell|, |\xi|^2 \right])^{-1/2} \) is determined by normalization and we used again the Pochhammer symbol. In the notation of \( C_n \), for simplicity we dropped the dependence on \( \xi, p \) and \( \ell \). The above expansion is correct only for some subsets of the beam parameters that correspond to a square integrable CiB \( \text{CiB}^{(q_0,\xi)}_{p,\ell} (x) \).

We note that \( (n)_k \equiv \frac{(n+k-1)!}{(n-k)!} \) and \( (-n)_k \equiv (-1)^k \frac{n!}{(n-k)!} \) for \( n, k \in \mathbb{N} \).
be explicitly evaluated as $C_n$. As an example, we will look for the value of $|\langle C_0|)$ that maximizes the probability $P_0$.

It is worth noticing that in Eq. (7), the CiB and the LG modes are defined with the same complex parameter $q_0$ (and the same $\ell$). By using the result of Eq. (8), we may now expand a CiB in terms of LG modes with a different beam parameter $q_0'$. As explained in the previous section, the new expansion can be written as

$$\text{CiB}_{p, \ell}^{(q_0, \xi)}(x) = \sum_{m=0}^{+\infty} C_m^{(q_0, \xi)} L_{m, \ell}(x),$$

where $C_m^{(q_0, \xi)} = \sum_n C_n^{(m, \ell)}$, according to Eq. (6). As demonstrated in appendix [3] the new coefficients $C_n^{(q_0, \xi)}$ can be explicitly evaluated as

$$C_n^{(q_0, \xi)} = \sum_{m=0}^{+\infty} C_m^{(m, \ell)}(x),$$

and

$$C_n^{(q_0, \xi)} = C_0 \gamma_1^{1+|\ell|}(1-\xi \tau_2)^{p/2}.$$  

with the $\tau_1$ parameters defined in Eq. (5). Eq. (9) and Eq. (10) represent, to our knowledge, a previously unknown expansion of the Circular Beam in terms of LG modes.

As already anticipated, by changing $\gamma$ (or equivalently $q_0'$), a truncated expansion of the form $T_N^{(q_0, \xi)}(x) = \sum_{k=0}^{N} C_k^{(q_0, \xi)} L_{k, \ell}^{(q_0, \xi)}(x)$ can be optimized. The larger is the value of the overlap probability $P_0 = \sum_{k=0}^{N} C_k^{(q_0, \xi)}|^2 = |(T_N^{(q_0, \xi)}(x)|^2$, the better the truncated expansion approximates the original beam. As an example, we will find the value of $\gamma$ that optimize the expansion truncated to the first terms (namely for $N = 0$); in other words, we will look for the value of $\gamma$ that maximize the probability

$$P_0^{(\gamma)} = |C_0^{(q_0, \xi)}| = |C_0| (1-\xi \tau_2)^{p/2}.$$  

Maximizing $P_0^{(\gamma)}$ corresponds in finding the LG mode with lowest radial number, namely $L_{0, \ell}^{(q_0)}$, that better approximates the CiB. As shown in appendix [3] by defining the phase of $\xi$ as $2\varphi$, (i.e. $\xi = |\xi|e^{2i\varphi}$), the value of $\gamma$ that maximize $P_0^{(\gamma)}$ is given by

$$\gamma^{\text{opt}} = \frac{\alpha + \sqrt{\alpha^2 + \beta^2(1-|\xi|^2)}}{\beta(1+|\xi|)},$$

and

$$\alpha = 1 + |\ell| + \text{Re}(p), \quad \beta = 1 + |\ell| - i\text{Im}(p).$$

A remarkable property is related to the value of the second expansion coefficient. When $\gamma = \gamma^{\text{opt}}$, in the expansion of Eq. (9) the coefficient of the LG mode with radial number $n = 1$ is vanishing. Indeed, by inserting $\gamma^{\text{opt}}$ into Eq. (10) it is possible to demonstrate that $C_1^{(\gamma^{\text{opt}})} = 0$.

An important sub-case is obtained for $|\xi| = 1$, namely when the CiBs reduce to the (generalized) HyGG modes [13]. As demonstrated in [13], the HyGG modes have a very simple profile at the plane $z = d_0 + iz_0 e^{i\phi}$, given by $|\xi| = r^{p+|\ell|} |x^{2i+|\ell|} + il\phi|$ and thus can be easily generated experimentally. For such beams, the optimal $\gamma^{\text{opt}}$ reduces to a very simple expression

$$\gamma^{\text{opt}} = 1 + \frac{2p\xi}{2 + 2|\ell| + p^* - p\xi}.$$

As an example, in Fig. 2 and 3 we show the truncated approximation of a CiB $\text{CiB}_{p, \ell}^{(q_0, \xi)}$ with $\xi = 1, \ell = -p = 2$.
and \( q_0/z_0 = 0.2 + i \) for two values of \( \gamma \). In the left plot of Fig. 2 we used LG modes with the same \( q_0 \) of the CiB (corresponding to \( \gamma = 1 \)). In the right plot of Fig. 2 we used \( \gamma = 1/3 \) (i.e. \( q_0/z_0 = 0.2 + i/3 \)), representing the optimal value obtained by Eq. (15) and corresponding to LG modes with a beam waist shrank representing the optimal value obtained by Eq. (15) and Fig. 2 we used LG modes with the same \( q_0 \), than 84% of the CiB energy is contained in the first LG mode, for two values of \( \gamma \). Indeed, in this specific case, more than 84% of the CiB energy is contained in the first LG mode with optimized \( q_0 \) (LG\(_{0,2} \)), while to obtain the same energy it necessary to sum the first six LG modes with the original \( q_0 \) parameter. This improvement holds also for higher values of \( \ell \) as shown in Fig. 4.

IV. CONCLUSIONS

The scalar product between two LG modes with different beam parameters \( q_0 \) and \( q_0' \) was explicitly evaluated (see Eq. (3)) and it was used to find new expansions of generic beams in terms of LG mode. By the above results, a previously unknown expansion of the Circular Beams is obtained (see Eq. (9) and Eq. (10)). Finally, the value of the LG beam parameter \( q_0 \) that optimizes the overlap probability of the LG\(_{0,\ell} \) mode with the CiB has been found (see Eq. (12)). Our results have important applications in OAM generation, manipulation and detection, since they allow to precisely determine and optimize the expansion of a generic beam in terms of LG modes, the fundamental beam carrying OAM. The case studied in Eq. (15) is particularly relevant for experiments since CiBs with \( |\xi| = 1 \) can be easily generated experimentally [18, 21].

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Appendix A: Laguerre polynomial integral

We now demonstrate the result stated in Eq. (3). Integration over the angular variable give rise to the \( \delta_{k,k'} \) factor in Eq. (3). After changing variables in the radial integration, \( \langle \text{LG}_{n',\ell'} | \text{LG}_{n,\ell} \rangle \) is proportional to a single integral involving two Laguerre polynomials as \( f_0^{+\infty} dt' e^{-ct'} L_{n'}^{\ell'}(at')L_{n,\ell}^{\ell'}(bt) \), with \( c = \frac{\alpha + a}{2 |q_0|^2} \), \( a = \frac{\alpha + a}{2 |q_0|^2} \) and \( b = \frac{\alpha + b}{2 |q_0|^2} \). We note that \( \Re(c) = \frac{\alpha + b}{2} > 0 \). Eq. (1) is derived by evaluating the above integral as:

\[
\frac{\alpha + b}{2 |q_0|^2} \int_0^{+\infty} dt' e^{-ct'} L_{n'}^{\ell'}(at')L_{n,\ell}^{\ell'}(bt) = \left( \frac{\ell + n}{\ell} \right) \left( \frac{\ell + n'}{\ell} \right) \times (1 - \frac{a}{c})^n (1 - \frac{b}{c})^{n'} 2F_1(-n, -n', \ell + 1; \frac{ab}{(c-a)(c-b)}).
\]

(A1)

Eq. (A1), holding when \( \Re(c) > 0 \) and \( n, n' \) and \( \ell \) are non negative integers, is obtained by expressing the two Laguerre polynomials in term of their generating function as:

\[
L_n^{\ell}(x) = \frac{1}{n!} \left( \frac{d}{d\eta} \right)^n \left[ \frac{1}{(1-\eta)^{\ell+1}} \exp \left( -x \frac{1}{1-\eta} \right) \right]_{\eta=0} \]

(A2)

and then evaluating the Gaussian integral. The final result is demonstrated by recalling that \( w(\eta, x) = (1 - \eta)^{\ell-c} / (1 - \eta + x)^{\ell} \) is the generating function of the hypergeometric polynomial, namely

\[
\frac{d^n w(\eta, x)}{d\eta^n} \bigg|_{\eta=0} = \frac{(c + n - 1)!}{(c - 1)!} 2F_1(-n, b; c; x). \]

(A3)

Appendix B: New expansion of the CiB

According to Eq. (6), the coefficient of the new expansion are given by \( C_m^{(\gamma)} = \sum_n C_n C_m^{(\gamma)} \). By using the coefficient \( C_n \) defined in Eq. (7) we obtain:

\[
C_n^{(\gamma)} = C_0 \sum_n \frac{\xi^n (-p/2)^n}{n!} G_{n,\ell}^{(\gamma)} = C_0 \tau_1 \sqrt{\tau_3} \sqrt{\left( \frac{\ell + n'}{n'} \right)} \times
\sum_{k=0}^{\infty} \frac{(-n')^k}{(\ell + 1)k!} \xi^{\tau_2} \sum_{n=k}^{+\infty} \frac{(\xi\tau_2)^n}{(n-k)!} \frac{(-p/2)^n}{(2)^n}.
\]

(B1)

In the last equality we used the definition of the Hypergeometric polynomial and we switched the sums over \( k \) and \( n \). We now exploit the following relation \( \sum_{n=-k}^{+\infty} \frac{\xi^{-n}}{(n-k)!} (-a)_n = \frac{\xi^{-k}}{(1 - \xi^{-1})^k} (\xi^{-a}-(-a)_k) \), holding when \( |\xi| < 1 \) and \( |a| > 1 \) and \( |a| > 1 \) and \( |\xi| < 1 \). When \( \xi \geq 1 \), square integrability of the CiB requires \( p = 2n \) with \( n \in \mathbb{N} \) in this case the sum over \( n \) in Eq. (B1) converges to the same value since all the terms with \( n > m \) vanish due to the factor \( (-p/2)_n = (-m)_n \). Then we obtain

\[
C_n^{(\gamma)} = C_0 \tau_1 \sqrt{\tau_3} \sqrt{\left( \frac{\ell + n'}{n'} \right)} \times
\sum_{k=0}^{\infty} \frac{(-n')^k}{(\ell + 1)k!} \frac{1}{k!} \frac{\xi^{\tau_2}}{(\tau_3\xi\tau_2 - 1)^k}.
\]

(B2)
The sum over $k$, due to the term $(-n)_{k,}$ is limited to $n$ and it converges to the hypergeometric polynomial. We have thus demonstrated Eq. (10) of the main text.

Appendix C: $\gamma$ optimization

The value of $\gamma$ that maximize $P_{0}(\gamma)$ of Eq. (11) is found by solving the equation $\gamma^{2}(\xi + 1) + 2i3m(\gamma) - 2(1 + \frac{\ell}{|\ell|})\Re(\gamma) + \xi - 1 = 0$ obtained by imposing $\frac{\partial P_{0}}{\partial \gamma} = 0$. When $\xi = |\xi|$, the above equation is solved by $\gamma_{\pm} = -\alpha\pm\sqrt{\alpha^{2} + |\beta|^{2}(1-|\ell|^{2})}/(2|\ell|)$, with $\alpha$ and $\beta$ defined in Eq. (11). When $|\xi| \leq 1$ only the $\gamma_{+}$ solution is physical since $\gamma_{-}$ does not satisfy the condition $\Re(\gamma) > 0$. For $|\xi| > 1$ the solution $\gamma_{+}$ is physical but it becomes a saddle point and not a maximum for $P_{0}$. Then, when $\xi = |\xi|$, the value of $\gamma$ that maximize $P_{0}$ is given by $\gamma_{+}$ defined above.

When $\xi \neq |\xi|$ the solution of the maximizing equation can be found by exploiting the properties of the CiBs under generic optical transformation defined by the ABCD law [22]. As detailed in [21], the beam parameters $\xi$ and $q_{0}$ are transformed as $\xi \rightarrow \xi_{1} = \frac{M_{\xi}^{-1}}{Cq_{0}+D}$ and $q_{0} \rightarrow q_{1} = \frac{Aq_{0}+B}{Cq_{0}+D}$ when the CiB passes through an optical system defined by the matrix $M = (A\ B/C\ D)$. If $\xi \neq |\xi|$ we may use the ABCD law to transform $\xi$ into $\xi_{1}$ such that $\xi_{1}$ is real and positive. If $\xi = |\xi|e^{2i\varphi}$, the required matrix $M_{\xi}$ has parameters $A = D = 1$, $B = 0$ and $C = \frac{1}{d_{0}+\cot\varphi}$ for such values we have $\xi_{1} = |\xi_{1}|$. In this system the LG expansion is optimized with $q_{1} = \Re(q_{1}) + i\gamma_{+}3m(q_{1})$. By inverting the transformation with $M_{\xi}^{-1}$, the optimal $q_{0}$ can be found, from which $\gamma_{\text{opt}} = -i\frac{q_{0} + d_{0}}{\sin\varphi} = \frac{\gamma_{+}\cos\varphi - i\sin\varphi}{\cos\varphi - \gamma_{+}\sin\varphi}$. We have thus demonstrated Eq. (12) of the main text.

[1] G. Molina-Terriza, J. P. Torres, and L. Torner, “Twisted photons,” Nat. Phys. 3, 305–310 (2007).
[2] A. Vaziri, G. Weihs, and A. Zeilinger, “Experimental Two-Photon, Three-Dimensional Entanglement for Quantum Communication,” Phys. Rev. Lett. 89, 240401 (2002).
[3] F. Tamburini, E. Mari, A. Sponselli, B. Thidé, A. Bianchini, and F. Romanato, “Encoding many channels on the same frequency through radio vorticity: first experimental test,” New Journal of Physics 14, 033001 (2012).
[4] N. Bozinovic, Y. Yue, Y. Ren, M. Tur, P. Kirstensen, H. Huang, A. E. Willner, and S. Ramachandran, “Terabit-scale orbital angular momentum mode division multiplexing in fibers,” Science (New York, N.Y.) 340, 1545 (2013).
[5] G. Vallone, V. D’Ambrosio, A. Sponselli, S. Slussarenko, L. Marrucci, F. Sciarrino, and P. Villoresi, “Free-Space Quantum Key Distribution by Rotation-Invariant Twisted Photons,” Phys. Rev. Lett. 113, 060503 (2014).
[6] S. Fürhapter, A. Jesacher, S. Bernet, and M. Ritsch-Marte, “Spiral phase contrast imaging in microscopy,” Opt. Exp. 13, 689 (2005).
[7] E. Mari, F. Tamburini, G. a. Swartzlander, A. Bianchini, C. Barbieri, F. Romanato, and B. Thidé, “Sub-Rayleigh optical vortex coronagraphy,” Opt. Exp. 20, 2445 (2012).
[8] S. W. Hell, “Noble Lecture: Nanoscopy with freely propagating light,” Reviews of Modern Physics 87, 1169–1181 (2015).
[9] F. Tamburini, B. Thidé, G. Molina-Terriza, and G. Anzolin, “Twisting of light around rotating bWe thanks Paolo Villoresi of the University of Padova, Filippo Cardano and Lorenzo Marrucci of the University Federico II of Napoli for useful suggestions and discussions. holes,” Nat. Phys. 7, 195 (2011).
[10] V. D’Ambrosio, I. Herbauts, E. Amselem, E. Nagali, M. Bourennane, F. Sciarrino, and A. Cabello, “Experimental Implementation of a Kochen-Specker Set of Quantum Tests,” Phys. Rev. X 3, 011012 (2013).
[11] V. D’Ambrosio, F. Bisesto, F. Sciarrino, J. F. Barra, G. Lima, and A. Cabello, “Device-independent certification of high-dimensional quantum systems,” Phys. Rev. Lett. 112, 140503 (2014).
[12] L. Allen, M. W. Beijersbergen, R. J. C. Spreeuw, and J. P. Woerdman, “Orbital angular momentum of light and the transformation of Laguerre-Gaussian laser modes,” Phys. Rev. A 45, 8185 (1992).
[13] A. E. Siegman, Lasers (University Science, 1986).
[14] M. A. Bandres and J. C. Gutiérrez-Vega, “Circular beams,” Optics Letters 33, 177 (2008).
[15] G. Vallone, “On the properties of circular beams: normalization, Laguerre-Gaussian expansion, and free-space divergence,” Opt. Lett. 40, 1717 (2015).
[16] G. Vallone, G. Parisi, F. Spinello, E. Mari, F. Tamburini, and P. Villoresi, “General theorem on the divergence of vortex beams,” Phys. Rev. A 94, 023802 (2016).
[17] A. Wünsche, “Generalized Gaussian beam solutions of paraxial optics and their connection to a hidden symmetry,” J. Opt. Soc. Am. A 6, 1320 (1989).
[18] E. Karimi, G. Zito, B. Piccirillo, L. Marrucci, and E. Santamato, “Hypergeometric-Gaussian modes,” Opt. Lett. 32, 3053 (2007).
[19] E. Karimi, B. Piccirillo, L. Marrucci, and E. Santamato, “Improved focusing with Hypergeometric-Gaussian type-II optical modes,” Opt. Exp. 16, 21069–21075 (2008).
[20] M. V. Berry, “Optical vortices evolving from helicoidal integer and fractional phase steps,” J. Opt. A: Pure Appl. Opt. 6, 259–268 (2004).
[21] G. Vallone, A. Sponselli, V. D’Ambrosio, L. Marrucci, F. Sciarrino, and P. Villoresi, “Birth and evolution of an optical vortex,” Optics Express 24, 16390 (2016).
[22] B. E. A. Saleh and M. C. Teich, Fundamentals of Photonics (Wiley, 1991).