BOWMAN-BRADLEY TYPE THEOREM FOR FINITE MULTIPLE ZETA VALUES IN $\mathcal{A}_2$

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Abstract. Bowman and Bradley obtained a remarkable formula among multiple zeta values. The formula states that the sum of multiple zeta values for indices which consist of the shuffle of two kinds of the strings \{1,3,\ldots,1,3\} and \{2,\ldots,2\} is a rational multiple of a power of $\pi^2$. Recently, Saito and Wakabayashi proved that analogous but more general sums of finite multiple zeta values in an adelic ring $\mathcal{A}_1$ vanish. In this paper, we partially lift Saito-Wakabayashi’s theorem from $\mathcal{A}_1$ to $\mathcal{A}_2$. Our result states that a Bowman-Bradley type sum of finite multiple zeta values in $\mathcal{A}_2$ is a rational multiple of a special element and this is closer to the original Bowman-Bradley theorem.

1. Introduction

For positive integers $k_1,\ldots,k_r$ with $k_r \geq 2$, the multiple zeta values (MZVs) and the multiple zeta-star values (MZSVs) are defined by

$$\zeta(k_1,\ldots,k_r) := \sum_{1 \leq n_1 < \cdots < n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}},$$

$$\zeta^*(k_1,\ldots,k_r) := \sum_{1 \leq n_1 \leq \cdots \leq n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}.$$ 

By convention, we set $\zeta(\emptyset) = \zeta^*(\emptyset) = 1$ for the empty index. Let $\{a_1,\ldots,a_l\}^m$ denote the $m$-times repetition of $a_1,\ldots,a_l$, e.g. $\{2\}^2 = 2,2$ and $\{1,3\}^2 = 1,3,1,3$. For MZVs, Bowman and Bradley [1] established the following result:

**Theorem 1.1** (Bowman-Bradley [1, Corollary 5.1]). For non-negative integers $l$ and $m$, we have

$$\sum_{m_0 + \cdots + m_{2l} = m} \zeta(\{2\}^{m_0},1,\{2\}^{m_1},3,\{2\}^{m_2},\ldots,\{2\}^{m_{2l-2}},1,\{2\}^{m_{2l-1}},3,\{2\}^{m_{2l}}) = \left(\frac{2l + m}{2l}\right) \pi^{4l + 2m} \cdot (4l + 2m + 1)^m.$$ 

A similar result for MZSVs is known by Kondo-Saito-Tanaka [4] and Yamamoto [10], i.e. the similar sum for MZSVs is also a rational multiple of $\pi^{4l + 2m}$.

Let us consider counterparts of these results for finite multiple zeta values. For a positive integer $n$, we define the $\mathbb{Q}$-algebra $\mathcal{A}_n$ by

$$\mathcal{A}_n := \left(\prod_p \mathbb{Z}/p^n\mathbb{Z}\right) / \left(\bigoplus_p \mathbb{Z}/p^n\mathbb{Z}\right).$$

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where \( p \) runs over prime numbers. For positive integers \( k_1, \ldots, k_r \) and \( n \), the finite multiple zeta values (FMZVs) and the finite multiple zeta-star values (FMZSVs) in \( \mathcal{A}_n \) are defined by

\[
\zeta_{\mathcal{A}_n}(k_1, \ldots, k_r) := \left( \sum_{1 \leq n_1 < \cdots < n_r \leq p-1} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}} \mod p^n \right) \in \mathcal{A}_n,
\]

\[
\zeta_{\mathcal{A}_n}^*(k_1, \ldots, k_r) := \left( \sum_{1 \leq n_1 < \cdots < n_r \leq p-1} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}} \mod p^n \right) \in \mathcal{A}_n.
\]

We set \( \zeta_{\mathcal{A}_n}(\emptyset) = \zeta_{\mathcal{A}_n}^*(\emptyset) = 1 \). For details, see Rosen \[6\] and Seki \[9\]. Recently, Saito and Wakabayashi \[7\] obtained Bowman-Bradley type results in a strong sense for finite multiple zeta values in \( \mathcal{A}_1 \). The following is a part of their results:

**Theorem 1.2** (Saito-Wakabayashi \[7\] Theorem 1.4). Let \( a \) and \( b \) be odd positive integers and \( c \) an even positive integer. For non-negative integers \( l \) and \( m \) with \((l, m) \neq (0, 0)\), we have

\[
\sum_{m_0 + \cdots + m_{2l} = m \atop m_i \geq 0 \atop 0 \leq i \leq 2l} \zeta_{\mathcal{A}_1}\{\{c\}^{m_0}, a, \{c\}^{m_1}, b, \{c\}^{m_2}, \ldots, \{c\}^{m_{2l-2}}, a, \{c\}^{m_{2l-1}}, b, \{c\}^{m_{2l}}\} = 0.
\]

In this paper, we partially lift Saito-Wakabayashi’s result from \( \mathcal{A}_1 \) to \( \mathcal{A}_2 \). In fact, we show that the Bowman-Bradley type sum of FMZ(S)Vs in \( \mathcal{A}_2 \) for the shuffle of \( \{1, 3\}^l \) and \( \{2\}^m \) is a rational multiple of the special element \( \beta_{4l+2m+1}p \). Here, \( p \) and \( \beta_k \) are defined to be \((p \mod p^n)^{m} \) and \((B_{p-k} / k \mod p^n)^{m} \) as elements of \( \mathcal{A}_2 \), respectively, where \( B_n \) is the \( n \)th Seki-Bernoulli number and \( k \) is an integer greater than 1. Then, our main theorem is the following:

**Theorem 1.3** (Main theorem). For non-negative integers \( l \) and \( m \) with \((l, m) \neq (0, 0)\), we have

\[
\sum_{m_0 + \cdots + m_{2l} = m \atop m_i \geq 0 \atop 0 \leq i \leq 2l} \zeta_{\mathcal{A}_2}\{\{2\}^{m_0}, 1, \{2\}^{m_1}, 3, \{2\}^{m_2}, \ldots, \{2\}^{m_{2l-2}}, 1, \{2\}^{m_{2l-1}}, 3, \{2\}^{m_{2l}}\}
\]

\[
= (-1)^m \left(-1\right)^{l+2l-2l} \left(l + m \atop l\right) - 4 \left(2l + m \atop 2l\right) \beta_{4l+2m+1}p.
\]

\[
\sum_{m_0 + \cdots + m_{2l} = m \atop m_i \geq 0 \atop 0 \leq i \leq 2l} \zeta_{\mathcal{A}_2}^*\{\{2\}^{m_0}, 1, \{2\}^{m_1}, 3, \{2\}^{m_2}, \ldots, \{2\}^{m_{2l-2}}, 1, \{2\}^{m_{2l-1}}, 3, \{2\}^{m_{2l}}\}
\]

\[
= (-1)^{l+2l-2l} \left(l + m \atop l\right) \beta_{4l+2m+1}p.
\]

Saito-Wakabayashi’s theorem (Theorem 1.2) says that the sum of FMZ(S)Vs in \( \mathcal{A}_1 \) for the shuffle of \( \{a, b\}^l \) and \( \{c\}^m \) is zero for any odd positive integers \( a, b \) and any even positive integer \( c \). On the other hand, by our computer calculations, it seems that the similar sum of FMZ(S)Vs in \( \mathcal{A}_2 \) is not a rational multiple of \( \beta_{(a+b)(l+cm+1)}p \), generally. For example, it is probable that \( \zeta_{\mathcal{A}_2}(1, 5, 1, 5) \) is not a rational multiple of \( \beta_{13}p \).
Zhao conjectures that the dimension of the \( \mathbb{Q} \)-vector space spanned by MZVs of weight \( k \) coincides with the dimension of the \( \mathbb{Q} \)-vector space spanned by FMZVs in \( A_2 \) of weight \( k \) ([11] Conjecture 9.6). It seems likely that the algebraic structures of the algebra of MZVs and the algebra of FMZVs in \( A_2 \) are different, but it is worth emphasizing that there exists a similarity between Bowman-Bradley type theorems for MZ(S)Vs and FMZ(S)Vs in \( A_2 \), i.e. the sum of MZ(S)Vs for the shuffle of \( \{1,3\}^1 \) and \( \{2\}^m \) is a rational multiple of \( \pi^{4l+2m} \) and the similar sum of FMZ(S)Vs in \( A_2 \) is a rational multiple of \( \beta_{4l+2m+1}p \).

We prove our main theorem in [2] and [3]

2. Preliminaries

We prepare some notation and lemmas in this section. Let \( \mathcal{H} \) be the Hoffman algebra \( \mathbb{Q} + \mathbb{Q}(x, y)y \). We define two kinds of shuffle products \( \mathfrak{m} \) and \( \mathfrak{m}^\prime \) on \( \mathcal{H} \) as in [5] §2. We call a tuple of positive integers an index. Let \( X \) be an index. Let \( \xi \) coincide with the dimension of the \( \mathfrak{m} \). It seems likely that the algebraic structure of the algebra of MZVs is similar to the shuffle of \( \{1,3\}^1 \) and \( \{2\}^m \), respectively. Here, we extend \( \zeta_{A_2} \) and \( \zeta_{A_2}^\prime \) to functions on \( \mathcal{H} \), linearly.

**Lemma 2.1.** For non-negative integers \( l \) and \( m \), we have

\[
4^l \{\{1,3\}^l\} \mathfrak{m} \{\{2\}^m\} = (\{2\}^{l+m}) \mathfrak{m} \{\{2\}^l\} - \sum_{k=0}^{l-1} 4^k \left( \begin{array}{c} 2l + m - 2k \\ k \\
\end{array} \right) \{\{1,3\}^k\} \mathfrak{m} \{\{2\}^{2l+m-2k}\}.
\]

**Proof.** This follows from [5] Proposition 2 (1). \( \square \)

The following lemma is the shuffle relation for FMZVs in \( A_2 \).

**Lemma 2.2.** For indices \( k \) and \( l = (l_1, \ldots, l_s) \), we have

\[
\zeta_{A_2}(k \mathfrak{m} l) = (-1)^{l_1 + \cdots + l_s} \sum_{e_1 + \cdots + e_s = 0, 1} \prod_{j=1}^{s} \left( \begin{array}{c} l_j + e_j - 1 \\ e_j \\
\end{array} \right) \zeta_{A_2}(k, l_s + e_s, \ldots, l_1 + e_1) p^{e_1 + \cdots + e_s}.
\]

**Proof.** This follows from [3] Theorem 6.4] which is also proved independently by Jarossay in [3] Lemma 4.17 by taking \( \lim_{n \to \infty} A_n \to A_2 \). \( \square \)

**Lemma 2.3.** For a positive integer \( r \), we have

\[
(3) \quad \zeta_{A_2}^\prime(\{2\}^r) = (-1)^{r-2} 2\beta_{2r+1} p,
\]

\[
(4) \quad \zeta_{A_2}(\{2\}^r) = 2\beta_{2r+1} p.
\]

**Proof.** This is a special case of the result of [12]. \( \square \)

**Lemma 2.4** (Hessami Pilehrood-Hessami Pilehrood-Tauraso [2] Theorem 4.1). For non-negative integers \( a \) and \( b \), we have

\[
\zeta_{A_1}(\{2\}^a, 3, \{2\}^b) = \left( \frac{-1}{a+1} \right)^{a+b+2} \left( \begin{array}{c} 2a + 2b + 3 \\ 2b + 2 \\
\end{array} \right) \beta_{2a+2b+3}.
\]

Here, we regard \( \beta_{2a+2b+3} \) as an element of \( A_1 \) by the projection \( A_2 \to A_1 \).
Lemma 2.5. For non-negative integers \(l\) and \(m\) with \((l, m) \neq (0, 0)\), we have
\[
\zeta_A((\{2\}^{l+m}) \, \mathfrak{m} \, (\{2\}^l)) = (-1)^m 2 \left\{ 1 - 2 \left( \frac{4l + 2m}{2l} \right) \right\} \beta_{4l+2m+1}. 
\]

Proof. By Lemma 2.2, 2.3 (3), and 2.4, we have
\[
\zeta_A((\{2\}^{l+m}) \, \mathfrak{m} \, (\{2\}^l)) = \sum_{j=0}^{l-1} \zeta_A((\{2\}^{l+m-j}) \, 3, \{2\}^{l-j}) \mathfrak{p} 
= (-1)^{m-1} \left\{ 2 \beta_{4l+2m+1} \mathfrak{p} + 4 \sum_{j=0}^{l-1} \frac{m + 2j + 1}{l + m + j + 1} \left( \frac{4l + 2m + 1}{2l - 2j} \right) \beta_{4l+2m+1} \mathfrak{p} \right\}. 
\]
Since \(\frac{a-2b}{a} = (\frac{a-1}{b}) - (\frac{a-1}{b-1})\), by putting \(a = 4l + 2m + 2\) and \(b = 2j\), we have
\[
\sum_{j=0}^{l} \frac{m + 2j + 1}{l + m + j + 1} \left( \frac{4l + 2m + 1}{2l - 2j} \right) = \sum_{j=0}^{l} 2l - 2j + 1 \left( \frac{4l + 2m + 2}{2l} \right) 
= \sum_{j=0}^{l} \left\{ \left( \frac{4l + 2m + 1}{2l} \right) - \left( \frac{4l + 2m + 1}{2l - 2j} \right) \right\} = \sum_{j=0}^{2l} (-1)^j \left( \frac{4l + 2m + 1}{2l} \right). 
\]
Hence, we obtain the desired formula. \(\square\)

Lemma 2.6. For non-negative integers \(l\) and \(m\), we have
\[
\sum_{k=0}^{l} (-1)^k \left( \frac{2l + m - 2k}{l - k} \right) \left( \frac{2l + m - k}{k} \right) = 1,
\]
\[
\sum_{k=0}^{l} 4^k \left( \frac{2l + m - 2k}{l - k} \right) \left( \frac{2l + m}{2k} \right) = \left( \frac{4l + 2m}{2l} \right). 
\]

Proof. Since \(\frac{a-b}{c-b} (\frac{c}{b}) = (-1)^{a-c} (\frac{a-c-1}{b})\), by putting \(a = 2l + m - k\), \(b = k\), and \(c = l\), we have
\[
\sum_{k=0}^{l} (-1)^k \left( \frac{2l + m - 2k}{l - k} \right) \left( \frac{2l + m - k}{k} \right) 
= (-1)^{l+m} \sum_{k=0}^{l+m} \left( \frac{l}{k} \right) \left( \frac{-1}{l - m - k} \right) = (-1)^{l+m} \left( \frac{-1}{l + m} \right) = \left( \frac{l + m}{l + m} \right) = 1 
\]
by the Chu-Vandermonde identity. Next, we prove the second equality. Let \((\frac{n}{a,b,c}) := n!/(a!b!c!)\). Since
\[
(1 + Y)^{4l+2m} = (1 + 2Y + Y^2)^{2l+m} = \sum_{a+b+c=2l+m}^{a+b+c=2l+m} \left( \frac{2l + m}{a, b, c} \right) (2Y)^{b} Y^{2c} 
\]
holds, by comparing the coefficient of $Y^{2l}$, we have

$$
\binom{4l + 2m}{2l} = \sum_{j=0}^{l} \binom{2l + m}{j + m, 2l - 2j, j} 2^{2l-2j} = \sum_{k=0}^{l} 4^k \binom{2l + m - 2k}{l - k} \binom{2l + m}{2k}.
$$

This concludes the proof. \qed

**Lemma 2.7.** For non-negative integers $l$ and $m$, we have

$$
\zeta_{A_2}^*([1, 3]^l \{2\}^m) = \sum_{2l + k + u = 2l} (-1)^{j+k} \binom{k + n}{k} \binom{u + v}{u} \zeta_{A_2}([1, 3]^l \{2\}^l) \zeta_{A_2}^{*}([2]^{k+n}) \zeta_{A_2}([2]^{u+v}),
$$

where parameters $i, j, k, n, u, v$ are non-negative integers.

**Proof.** This follows from [10, Theorem 2.1]. \qed

### 3. Proof of the Main Theorem

**Proof of Theorem 1.3.** First, we prove (1) by induction on $l$. We see that the case $l = 0$ holds by Lemma 2.3 [3]. For the general case, let $l$ be a positive integer and $m$ a non-negative integer. By Lemma 2.1 we have

$$
\zeta_{A_2}([1, 3]^l \{2\}^m) = 4^{-l} \zeta_{A_2}([2]^{l+m}) \{2\}^l - \sum_{k=0}^{l-1} 4^{k-l} \binom{2l + m - 2k}{l - k} \zeta_{A_2}([1, 3]^{k}) \{2\}^{2l+m-2k}).
$$

Hence, by Lemma 2.5 and the induction hypothesis, we have

$$
\zeta_{A_2}([1, 3]^l \{2\}^m) = (-1)^m 4^{1-2l} \left\{ 1 - 2 \left( \frac{4l + 2m}{2l} \right) \right\} \zeta_{A_2}^{*}([2]^{l+2m+1}) - \sum_{k=0}^{l-1} 4^{k-l} \binom{2l + m - 2k}{l - k} \left\{ (-1)^{k-2} \frac{2l + m - k}{2k} - 4 \frac{2l + m}{2k} \right\} \zeta_{A_2}^{*}([2]^{l+2m+1})
$$

By Lemma 2.6 we can simplify as

$$
\zeta_{A_2}([1, 3]^l \{2\}^m) = (-1)^m 4^{1-2l} \left\{ 1 - 2 \left( \frac{4l + 2m}{2l} \right) \right\} \zeta_{A_2}^{*}([2]^{l+2m+1}) - (-1)^m 4^{1-2l} \left\{ 1 - (-1)^l \left( \frac{l + m}{l} \right) \right\} \zeta_{A_2}^{*}([2]^{l+2m+1}) + (-1)^m 4^{1-2l} \left\{ \left( \frac{4l + 2m}{2l} \right) - 4 \left( \frac{2l + m}{2l} \right) \right\} \zeta_{A_2}^{*}([2]^{l+2m+1}) = (-1)^m \left\{ (-1)^l 4^{1-2l} \left( \frac{l + m}{l} \right) - 4 \left( \frac{2l + m}{2l} \right) \right\} \zeta_{A_2}^{*}([2]^{l+2m+1}).
$$
Next, we prove (2). By the equality (1) and Lemma 2.3 (4), many terms in the right-hand side of the equality in Lemma 2.7 vanish and we have
\[
\zeta^*_{A_2}((\{1,3\}^l\bar{m}\ (\{2\}^m))) = (-1)^m\zeta_{A_2}((\{1,3\}^l\bar{m}\ (\{2\}^m))) + 2\binom{2l + m}{2l}\zeta^*_{A_2}(\{2\}^{2l+m})
\]
\[
= (-1)^l2^{1-2l}\binom{l+m}{l}\beta_{4l+2m+1p}.
\]
This finishes the proof. □

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