UNIFORM RECTIFIABILITY AND $\varepsilon$-APPROXIMABILITY OF HARMONIC FUNCTIONS IN $L^p$

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Abstract. Suppose that $E \subset \mathbb{R}^{n+1}$ is a uniformly rectifiable set of codimension 1. We show that every harmonic function is $\varepsilon$-approximable in $L^p(\Omega)$ for every $p \in (1, \infty)$, where $\Omega := \mathbb{R}^{n+1} \setminus E$. For bounded $n$-ADR sets this property gives a new characterization of uniform rectifiability. Our results and techniques are generalizations of recent works of T. Hytönen and A. Rosén and the first author, J. M. Martell and S. Mayboroda.

1. Introduction

Starting from the work of N. Th. Varopoulos [Var78] and J. Garnett [Gar81], $\varepsilon$-approximability has had an important role in the development of the theory of elliptic partial differential equations. It has been used to explore the absolute continuity properties of elliptic measures [HMM16] and, very recently, give a new characterization of uniform rectifiability [HMM16, GMT16].

In this article, we extend the recent results of the first author, J. M. Martell and S. Mayboroda [HMM16] and show that if $\Omega \subset \mathbb{R}^{n+1}$ is an open set with a uniformly rectifiable boundary of codimension 1, then every harmonic function is $\varepsilon$-approximable in $L^p(\Omega)$ for every $\varepsilon \in (0, 1)$ and every $p \in (0, 1)$. The $L^p$ version of $\varepsilon$-approximability was recently introduced by T. Hytönen and A. Rosén [HR16] who showed that any weak solution to certain elliptic partial differential equations in $\mathbb{R}^{n+1}$ is $\varepsilon$-approximable in $L^p$ for every $\varepsilon \in (0, 1)$ and every $p \in (1, \infty)$.

Let us be more precise and recall the definition of $\varepsilon$-approximability. The basic idea is that a function $u$ is $\varepsilon$-approximable if it can be approximated well in $L^\infty$ sense by a function $\varphi^\varepsilon$ such that $|\nabla \varphi^\varepsilon|$ is a Carleson measure. Formally put:

Definition 1.1. Suppose that $E \subset \mathbb{R}^{n+1}$ is an $n$-dimensional ADR set (see Definition 1.3) and let $\Omega := \mathbb{R}^{n+1} \setminus E$ and $\varepsilon \in (0, 1)$. We say that a function $u$ is $\varepsilon$-approximable if there exists a constant $C_\varepsilon$ and a function $\varphi = \varphi^\varepsilon \in BV_{\text{loc}}(\Omega)$ satisfying

$$\|u - \varphi\|_{L^\infty(\Omega)} < \varepsilon \quad \text{and} \quad \sup_{x \in E, r > 0} \frac{1}{m(B(x, r) \cap \Omega)} \int_{B(x, r) \cap \Omega} |\nabla \varphi(Y)| dY \leq C_\varepsilon.$$

Sometimes $W^{1,1}_{\text{loc}}$ [HKMP15] or $C^\infty$ [Gar81] [KKPT00] is used in the definition instead of $BV_{\text{loc}}$. The first results about $\varepsilon$-approximability showed that every bounded harmonic function enjoys this approximation property for every $\varepsilon \in (0, 1)$ in the upper half-space $\mathbb{R}^{n+1}_+$ [Var78, Gar81] and in Lipschitz domains [Dah80]. This is a highly non-trivial property since there exist bounded harmonic functions $u$ such that $|\nabla u|$ is not a Carleson measure [Gar81].

If we move from $\mathbb{R}^{n+1}_+$ to the UR context (see Definition 1.4) with no assumptions on connectivity, things will not only get more complicated but we also lose many powerful tools. For example, constructing objects like Whitney regions and Carleson boxes becomes considerably more difficult and the harmonic measure no longer necessarily belongs to the class $A_\infty$ with respect to the surface measure [BJ90]. Despite these difficulties, there exists a rich theory of harmonic analysis and many results on elliptic partial differential equations on sets with UR boundaries. Uniform
rectifiability can be characterized in numerous different ways and many of these characterizations are valid in all codimensions (see the seminal work of G. David and S. Semmes [DS91, DS93]). For example, UR sets are precisely those ADR sets for which certain types of singular integral operators are bounded from $L^2$ to $L^2$.

Our main result is the following generalization of the Hytönen-Rosén approximation theorem [HR16, Theorem 1.3]:

**Theorem 1.2.** Let $E \subset \mathbb{R}^{n+1}$ be a UR set of co-dimension 1. Suppose that $u$ is a harmonic function in $\Omega := \mathbb{R}^{n+1} \setminus E$ such that $N_* u \in L^p(\Omega)$. Then for every $p \in (1, \infty)$ and $\varepsilon \in (0, 1)$ there exists a function $\varphi = \varphi^\varepsilon \in BV_{loc}(\Omega)$ such that

$$
\begin{align*}
\|N_* (u - \varphi)\|_{L^p(\Omega)} &\lesssim \varepsilon C_p \|N_* u\|_{L^p(\Omega)} \\
\|C(\nabla \varphi)\|_{L^p(\Omega)} &\lesssim \frac{\varepsilon}{p} \|N_* u\|_{L^p(\Omega)}
\end{align*}
$$

where $N_*$ is the non-tangential maximal operator, $C_p := \|M_\mathbb{Z}\|_{L^p \to L^p}$, $D_p := \|M\|_{L^p \to L^p}$ and

$$
C(\nabla \varphi)(x) := \sup_{r > 0} \frac{1}{r^n} \int_{B(x,r) \cap \Omega} |\nabla \varphi| \, dY
$$

If the set $E$ is bounded, then the usual $\varepsilon$-approximability in $L^\infty$ can be seen as a limiting case of this $L^p$ version of $\varepsilon$-approximability. Since the usual $\varepsilon$-approximability of bounded harmonic functions is enough to imply uniform rectifiability [GMT16], we also get a converse result in bounded sets (see Theorem [5.3]). It is an open problem to determine whether $\varepsilon$-approximability in $L^p$ for a fixed $p$ is enough to imply uniform rectifiability in the bounded or unbounded case.

To prove this theorem, we combine the techniques of the proof of the Hytönen-Rosén theorem with the tools and techniques from [HMM16]. Some of the techniques can be used in a straightforward way but with the rest of them we have take care of many technicalities and be careful with the details.

We start by recalling the basic definitions and some results needed in our statements and proofs. For the most part, our notation and terminology agrees with [HMM16].

### 1.1. Notation

We use the following notation.

- The set $E \subset \mathbb{R}^{n+1}$ will always be a closed set of Hausdorff dimension $n$. We denote $\Omega := \mathbb{R}^{n+1} \setminus E$.
- The letters $c$ and $C$ denote constants that depend only on the dimension, the ADR constant (see Definition 1.3), the UR constants (see Definition 1.4) and other similar parameters. We call them *structural constants*. The values of $c$ and $C$ may change from one occurrence to another. We do not track how our bounds depend on these constants and usually just write $M \lesssim N$ if $M \leq cN$ for a structural constant $c$ and $M \approx N$ if $M \lesssim N \lesssim M$.
- We use capital letters $X, Y, Z$, and so on to denote points in $\Omega$ and lowercase letters $x, y, z$, and so on to denote points in $E$.
- The $(n+1)$-dimensional Euclidean open ball of radius $r$ will be denoted $B(x, r)$ or $B(X, r)$ depending on whether the center point lies in $\Omega$ or $E$. We denote the surface ball of radius $r$ centered at $x$ by $\Delta(x, r) := B(x, r) \cap E$.
- Given a Euclidean ball $B := B(x, r)$ or a surface ball $\Delta := \Delta(x, r)$ and constant $\kappa > 0$, we denote $\kappa B := B(x, \kappa r)$ and $\kappa \Delta := \Delta(x, \kappa r)$.
- For every $X \in \Omega$ we set $\delta(X) := \text{dist}(X, E)$.
- We let $H^n$ be the $n$-dimensional Hausdorff measure and denote $\sigma := H^n|_E$. The $(n+1)$-dimensional Lebesgue measure of a measurable set $A \subset \Omega$ will be denoted by $|A|$.
- For a set $A \subset \mathbb{R}^{n+1}$, we let $1_A$ be the indicator function of $A$: $1_A(x) = 0$ if $x \notin A$ and $1_A(x) = 1$ if $x \in A$.
- The interior of a set $A$ will be denoted $\text{int}(A)$.
- For $\mu$-measurable sets $A$ with positive and finite measure we set $\int_A f \, d\mu := \frac{1}{|A|} \int f \, d\mu$.  


The Hardy-Littlewood maximal operator and its dyadic version (see Section 1.3) in $E$ will be denoted $M$ and $M_D$, respectively:

$$Mf(x) := \sup_{\Delta(y,r) \ni x} \int_{B(y,r)} |f(z)| \, d\sigma(z), \quad M_Df(x) := \sup_{Q \in \mathcal{D}, Q \ni x} \int_Q |f(z)| \, d\sigma(z).$$

1.2. ADR, UR and NTA sets.

**Definition 1.3.** We say that a closed set $E \subset \mathbb{R}^{n+1}$ is an $n$-ADR (Ahlfors-David regular) set if there exists a uniform constant $C$ such that

$$\frac{1}{C} r^n \leq \sigma(\Delta(x,r)) \leq C r^n$$

for every $x \in E$ and every $r \in (0, \text{diam}(E))$, where $\text{diam}(E)$ may be infinite.

**Definition 1.4.** Following [DS91, DS93], we say that an $n$-ADR set $E \subset \mathbb{R}^{n+1}$ is UR (uniformly rectifiable) if it contains “big pieces of Lipschitz images” (BPLI) of $\mathbb{R}^n$: there exist constants $\theta, M > 0$ such that for every $x \in E$ and $r \in (0, \text{diam}(E))$ there is a Lipschitz mapping $\rho = \rho_{x,r}: \mathbb{R}^n \to \mathbb{R}^{n+1}$, with Lipschitz norm no larger that $M$, such that

$$H^n(E \cap B(x,r) \cap \rho (\{y \in \mathbb{R}^n : |y| < r\})) \geq \theta r^n.$$

**Definition 1.5.** Following [HK82], we say that a domain $\Omega \subset \mathbb{R}^{n+1}$ is NTA (nontangentially accessible) if

- $\Omega$ satisfies the Harnack chain condition: there exists a uniform constant $C$ such that for every $\rho > 0$, $\Lambda \geq 1$ and $X, X' \in \Omega$ with $\delta(X), \delta(X') \geq \rho$ and $|X - X'| < \Lambda \rho$ there exists a chain of open balls $B_1, \ldots, B_N \subset \Omega$, $N \leq C(\Lambda)$, with $X \in B_1, X' \in B_N, B_k \cap B_{k+1} = \emptyset$ and $C^{-1} \text{diam}(B_k) \leq \text{dist}(B_k, \partial \Omega) \leq C \text{diam}(B_k)$,
- $\Omega$ satisfies the corkscrew condition: there exists a uniform constant $c$ such that for every surface ball $\Delta := \Delta(x,r)$ with $x \in \partial \Omega$ and $0 < r < \text{diam}(\partial \Omega)$ there exists a point $X_\Delta \in \Omega$ such that $B(X_\Delta, cr) \subset B(x,r) \cap \Omega$,
- $\mathbb{R}^{n+1} \setminus \overline{\Omega}$ satisfies the corkscrew condition.

1.3. Dyadic cubes.

**Theorem 1.6** (E.g. [Chr90, SW92, HK12]). Suppose that $E$ is an ADR set. Then there exists a countable collection $\mathcal{D}$,

$$\mathcal{D} := \bigcup_{k \in \mathbb{Z}} \mathcal{D}_k, \quad \mathcal{D}_k := \{Q^k_\alpha : \alpha \in A_k\}$$

of Borel sets (that we call dyadic cubes) such that

- the collection $\mathcal{D}$ is nested: if $Q, P \in \mathcal{D}$, then $Q \cap P \in \{\emptyset, Q, P\}$,
- $E = \bigcup_{Q \in \mathcal{D}} Q$ for every $k \in \mathbb{Z}$ and the union is disjoint,
- there exist constants $c_1 > 0$ and $C_1 \geq 1$ such that

$$\Delta(z^k_\alpha, c_1 2^{-k}) \subseteq Q^k_\alpha \subseteq \Delta(z^k_\alpha, C_1 2^{-k}) =: \Delta Q^k_\alpha \tag{1.7}$$

- if $Q, P \in \mathcal{D}$ and $Q \subseteq P$, then

$$\Delta Q \subseteq \Delta P, \tag{1.8}$$

- for every set $Q^k_\alpha$ there exists at most $N$ sets $Q^{k+1}_{\beta_1}$ (called the children of $Q^k_\alpha$) such that $Q^k_\alpha = \bigcup \mathcal{Q}^{k+1}_{\beta_1}$, where the constant $N$ depends only on the ADR constant of $E$.
- the cubes form a connected tree under inclusion: if $Q, P \in \mathcal{D}$, then there exists a cube $R \in \mathcal{D}$ such that $Q \cup P \subseteq R$.

**Remark 1.9.** The last property in the previous theorem does not appear in the constructions in [Chr90, SW92, HK12], but it is easy to modify the construction to get this property: we simply choose the center points $z^k_\alpha$ in such a way that $\{z^k_\alpha\}_\alpha \subseteq \{z^{k+1}_{\beta_1}\}_{\beta}$ for every $k \in \mathbb{Z}$. 

Notation 1.10. 1) Since the set $E$ may be bounded or disconnected, we may encounter a situation where $Q_\alpha^k = Q_\beta^l$ although $k \neq l$. Thus, we use the notation $D(E)$ for the collection of all relevant cubes $Q \in D$, i.e., if $Q_\alpha^k \in D(E)$, then $C_1 2^{-k} \leq \text{diam}(E)$ and the number $k$ is maximal in the sense that there does not exist a cube $Q_\alpha^k \in D$ such that $Q_\alpha^k = Q_\beta^l$ for some $l > k$. Notice that the number $k$ is bounded for each cube since the ADR condition excludes the presence of isolated points in $E$.

2) For every cube $Q_\alpha^k := Q \in D$, we denote $\ell(Q) := 2^{-k}$ and $z_Q := z_\alpha^k$. We call $\ell(Q)$ the side length of $Q$.

3) For every $Q \in D$, we denote the collection of dyadic subcubes of $Q$ by $D_Q$.

**Definition 1.11.** We say that a collection $A \subset D$ is Carleson (or that it satisfies a Carleson packing condition) if there exists a constant $C \geq 1$ such that

$$\sum_{Q \in A, Q \subset Q_0} \sigma(Q) \leq C \sigma(Q_0)$$

for every cube $Q_0 \in D$.

**Definition 1.12.** Let $A \subset D$ be any collection of dyadic cubes. We say that a cube $P \in A$ is an $A$-maximal subcube of $Q_0$ if there does not exist any cubes $P' \in A$ such that $P \subsetneq P' \subset Q_0$.

1.4. Corona decomposition, Whitney regions and Carleson boxes.

**Definition 1.13.** We say that a subcollection $S \subset D(E)$ is coherent if the following three conditions hold:

- (a) There exists a maximal element $Q(S) \in S$ such that $Q \subset S$ for every $Q \in S$.
- (b) If $Q \in S$ and $P \in D(E)$ is a cube such that $Q \subset P \subset Q(S)$, then also $P \in S$.
- (c) If $Q \in S$, then either all children of $Q$ belong to $S$ or none of them do.

If $S$ satisfies only conditions (a) and (b), then we say that $S$ is semicoherent.

In this article, we do not work directly with Definition 1.14 but use the bilateral corona decomposition instead:

**Lemma 1.14 ([HMM16, Lemma 2.2]).** Suppose that $E \subset \mathbb{R}^{n+1}$ is a uniformly rectifiable set of codimension 1. Then for any pair of positive constants $\eta \ll 1$ and $K \gg 1$ there exists a disjoint decomposition $D(E) = G \cup B$ satisfying the following properties:

1. The “good” collections $G$ is a disjoint union of coherent stopping time regimes $S$.
2. The “bad” collection $B$ and the maximal cubes $Q(S)$ satisfy a Carleson packing condition: for every $Q \in D(E)$ we have

$$\sum_{Q' \subset Q, Q' \in B} \sigma(Q') + \sum_{S : Q(S) \subset Q} \sigma(Q(S)) \leq C_{\eta,K} \sigma(Q).$$

3. For every $S$, there exists a Lipschitz graph $\Gamma_S$, with Lipschitz constant at most $\eta$, such that for every $Q \in S$ we have

$$\sup_{x \in \Delta^*_Q} \text{dist}(x, \Gamma_S) + \sup_{y \in B^*_Q \cap \Gamma_S} \text{dist}(y, E) < \eta \ell(Q),$$

where $B^*_Q := B(z_Q, K \ell(Q))$ and $\Delta^*_Q := B^*_Q \cap E$.

The proof of this decomposition is based on the use of both the unilateral corona decomposition [DS91] and the bilateral weak geometric lemma [DS93] of David and Semmes. The decomposition plays a key role in this paper.

In [HMM16, Section 3], the bilateral corona decomposition is used to construct Whitney regions $U_Q$ and Carleson boxes $T_Q$ with respect to the dyadic cubes $Q \in D(E)$ using a dyadic Whitney decomposition of $\mathbb{R}^{n+1} \setminus E$. The Whitney regions are a substitute for the dyadic Whitney tiles $Q \times (\ell(Q)/2, \ell(Q))$ and the Carleson boxes are a substitute for the dyadic boxes $Q \times (0, \ell(Q))$ in $\mathbb{R}_n^{n+1}$. We list some of their important properties in the next lemma which we use constantly without specifically referring to it each time.
Lemma 1.15. The Whitney regions $U_Q$, $Q \in \mathbb{D}(E)$, satisfy the following properties.

- The region $U_Q$ is a union of a bounded number of slightly fattened Whitney cubes $I^* := (1 + \tau)I$ such that $\ell(Q) \approx \ell(I)$ and $\text{dist}(Q,I) \approx \ell(Q)$. We denote the collection of these Whitney cubes by $W_Q$.
- The regions have a bounded overlap property, i.e. $\sum_i |U_{Q_i}| \approx |\bigcup_i U_{Q_i}|$ for cubes $Q_i$ such that $Q_i \neq Q_j$ if $i \neq j$.
- If $U_Q \cap U_P \neq \emptyset$, then $\ell(Q) \approx \ell(P)$ and $\text{dist}(Q,P) \lesssim \ell(Q)$.
- For every $Y \in U_Q$ we have $\delta(Y) \approx \ell(Q)$.
- For every $Q \in \mathbb{D}(E)$, we have $|U_Q| \approx \ell(Q)^n \approx \ell(Q) \cdot \sigma(Q)$.
- If $Q \in \mathcal{G}$, then $U_Q$ breaks into exactly two connected components $U^+_Q$ and $U^-_Q$ such that $|U^+_Q| \approx |U^-_Q|$.
- If $Q \in \mathcal{B}$, the $U_Q$ breaks into a bounded number of connected components $U_i^Q$ such that $|U^+_Q| \approx |U^-_Q|$ for all $i$ and $j$.

For every $Q \in \mathcal{G}$, the components $U^+_Q$ and $U^-_Q$ have fixed “center points” that we denote by $X^+_Q$ and $X^-_Q$, respectively. We also set $Y^+_Q := X^+_Q$, where $\tilde{Q}$ is the dyadic parent of $Q$ unless $Q = Q(S)$, in which case we set $\tilde{Q} = Q$. We use these points in the construction in Section 5.1. For any cube $Q \in \mathcal{G}$, the collection $\mathcal{W}_Q$ breaks naturally into two disjoint subcollection $\mathcal{W}_Q^+$ and $\mathcal{W}_Q^-$.

For every $Q \in \mathbb{D}(E)$, we define the Carleson box as the set

$$
T_Q := \text{int} \left( \bigcup_{Q' \in \mathbb{D}(E)} U_Q \right).
$$

1.5. $\mathcal{C}$ and $\mathcal{C}_S$. For every $k \in \mathbb{N}$, we let $E_k$ be the ordered pair $(E,k)$. In this section, we let $Q_0 = E$ be the maximal dyadic cube if $E$ is a bounded set. We define the operators $\mathcal{C}$ and $\mathcal{C}_S$ by setting

$$
\mathcal{C}(f)(x) := \sup_{r > 0} \frac{1}{r^n} \int_{B(x,r) \setminus E} |f(Y)| \, dY, \quad \mathcal{C}_S(f)(x) := \sup_{Q \in D^*, x \in Q} \frac{1}{\ell(Q)^n} \int_{T_Q} |f(Y)| \, dY,
$$

where

$$
D^* := \begin{cases} D(E), & \text{if diam}(E) = \infty \\ D(E) \cup \{E_k : k = M, M + 1, \ldots\}, & \text{if diam}(E) < \infty \end{cases}
$$

and

$$
T_{E_k} := B(z_0, 2^k \text{diam}(E)), \quad \ell(E_k) := 2^k \text{diam}(E)
$$

for some fixed point $z_0 \in E$ and a number $M$ such that $T_{Q_0} \subset T_{E_M}$. We will call also the pairs $E_k$ cubes although their actual structure is irrelevant.

Usually these functions are not pointwise equivalent but we only have the estimate $\mathcal{C}_S(f)(x) \lesssim \mathcal{C}(f)(x)$ for every $x \in E$ (this follows from the ADR property of $E$ and the fact that $T_Q \subset B(z_Q, C\ell(Q))$ for a uniform constant $C$). However, in the $L^p$ sense, these functions are always comparable. This can be seen easily from the level set comparison formula that we prove next. This comparability is convenient for us since we construct the approximating function $\varphi$ in Theorem 1.2 with the help of the dyadic Whitney regions. Thus, it is more natural for us to prove the desired $L^p$ bound for $\mathcal{C}_S(\nabla \varphi)$ instead of $\mathcal{C}(\nabla \varphi)$. We prove the comparison formula by using well-known techniques from the proof of the corresponding formula for the Hardy-Littlewood maximal function and its dyadic version [Duong, Lemma 2.12].

Lemma 1.16. There exist uniform constants $A_1$ and $A_2$ (depending on the dimension and the ADR constant) such that for every $\lambda > 0$ we have

$$
\sigma(\{x \in E : \mathcal{C}(f)(x) > A_1 \lambda\}) \leq A_2 \cdot \sigma(\{x \in E : \mathcal{C}_S(f)(x) > \lambda\}).
$$

In particular, $\|\mathcal{C}(f)\|_{L^p(E)} \leq A_1 A_2^{1/p} \|\mathcal{C}_S(f)\|_{L^p(E)}$ for every $p \in (1, \infty)$. 
The argument to show that and $y$ can be contained in any of the cubes $Q_i$. We now claim that if $A_1$ is large enough, then
\[ \{x \in E : \mathcal{C}(f)(x) > A_1 \lambda \} \subseteq \bigcup_i 2\Delta Q_i \]  
where $\Delta Q_i$ is the surface ball (1.7). Suppose that $y \notin \bigcup_i 2\Delta Q_i$ and let $r > 0$. Let us choose $k \in \mathbb{Z}$ so that $2^{k-1} \leq r < 2^k$. Now there exist at most $M$ dyadic cubes $R_1, R_2, \ldots, R_m$ such that $\ell(R_j) = 2^k$ and $R_j \cap \Delta(y, r) \neq \emptyset$ for every $j = 1, 2, \ldots, m$. We notice that none of the cubes $R_j$ can be contained in any of the cubes $Q_i$, since otherwise we would have $y \in 2\Delta R_j \subset 2\Delta Q_i$ by (1.8). Thus, we have
\[ \frac{1}{r^n} \int_{B(y, r)} |f(Y)| \, dY = \sum_{j=1}^{m} \frac{1}{r^n} \int_{B(y, r) \cap T_{R_j}} |f(Y)| \, dY \lesssim \sum_{j=1}^{m} \frac{1}{\ell(R_j)^n} \int_{T_{R_j}} |f(Y)| \, dY \lesssim \lambda \]  
and $y \notin \{x \in E : \mathcal{C}(f)(x) > A_1 \lambda \}$ for a large enough $A_1$. In particular, (1.17) holds and we have
\[ \sigma(\{x \in E : \mathcal{C}(f)(x) > A_1 \lambda \}) \leq \sum_i \sigma(2\Delta Q_i) \]  
\[ \lesssim \sum_i \sigma(Q_i) = \sigma\left(\bigcup_i Q_i\right) = \sigma(\{x \in E : \mathcal{C}_D(f)(x) > \lambda\}). \]  
The $L^p$ comparability $\mathcal{C}(f)$ and $\mathcal{C}_D(f)$ follows immediately:
\[ \|\mathcal{C}_D(f)\|_{L^p(E)}^p = p \int_0^\infty \lambda^{p-1} \sigma(\{x \in E : \mathcal{C}(f)(x) > \lambda\}) \, d\lambda \]  
\[ \leq A_2p \int_0^\infty \lambda^{p-1} \sigma(\{x \in E : A_1 \mathcal{C}_D(f)(x) > \lambda\}) \, d\lambda \]  
\[ = A_2^p A_2 \|\mathcal{C}_D(f)\|_{L^p(E)}^p. \]  

1.6. Cones, non-tangential maximal functions and square functions. We recall from [HMM16, Section 3] that the Whitney regions $U_Q$ and the fattened Whitney regions $\tilde{U}_Q$. $Q \in \mathcal{D}$, are defined using fattened Whitney boxes $I^* := (1 + \tau)I$ and $I^{**} := (1 + 2\tau)I$ respectively, where $\tau$ is a suitable positive parameter. Let us define the regions $\tilde{U}_Q$ using even fatter Whitney boxes $I^{***} := (1 + 3\tau)W$.

**Definition 1.18.** For any $x \in E$, we define the cone at $x$ by setting
\[ \Gamma(x) := \bigcup_{Q \in \mathcal{D}(E), Q \ni x} \tilde{U}_Q. \]  

For a continuous function $u$ in $\Omega$ we define the non-tangential maximal function $N_u u$, and for $u \in W^{1,2}_{\text{loc}}(\Omega)$ we define the square function $S_u$, as follows:
\[ N_u u(x) := \sup_{Y \in \Gamma(x)} |u(Y)|, \quad x \in E, \]  
\[ S_u(x) := \left( \int_{\Gamma(x)} |\nabla u(Y)|^2 dY \right)^{1/2}, \quad x \in E. \]
The Hytönen-Rosén techniques in [HR16], Section 6, rely on the use of local $S \lesssim N$ and $N \lesssim S$ estimates from [HKMP15]. Although a local $S \lesssim N$ estimate holds also in our context, a local $N \lesssim S$ estimate does not. Thus, we cannot apply the Hytönen-Rosén techniques directly but we have to combine them with the techniques created in [HM16].

In Section 5 we consider the following modified versions of $\Gamma(x)$ and $N_*u$ to bypass some technical difficulties:

**Definition 1.20.** For every $x \in E$ and $\alpha > 0$ we define the cone of $\alpha$-aperture at $x$ $\Gamma_\alpha(x)$ by setting

$$
\Gamma_\alpha(x) := \bigcup_{Q \in \mathcal{D}(E), Q \ni x} \bigcup_{P \in \mathcal{D}(E), \ell(P) = \ell(Q), \alpha \Delta_Q \cap P \neq \emptyset} \hat{U}_P.
$$

Using the cones $\Gamma_\alpha(x)$, we define the non-tangential maximal function of $\alpha$-aperture $N^\alpha_* u$ by setting $N^\alpha_* u(x) := \sup_{Y \in \Gamma_\alpha(x)} |u(Y)|$.

**Remark 1.22.** If the set $E$ is bounded, then the cones (1.19) and (1.21) are also bounded since we only constructed Whitney regions $U$ such that $\text{diam}(U) \lesssim \text{diam}(E)$. Thus, if $E$ is bounded, we use the cones

$$
\tilde{\Gamma}(x) := \Gamma(x) \cup B(z_0, C \cdot \text{diam}(E))^c \quad \text{and} \quad \tilde{\Gamma}_\alpha(x) := \Gamma_\alpha(x) \cup B(z_0, C_\alpha \cdot \text{diam}(E))^c
$$

for a suitable point $z_0 \in E$ and suitable constants $C$ and $C_\alpha$ instead.

The usefulness of these modified cones and non-tangential maximal functions lies in the fact that for a suitable choice of $\alpha$ the cone $\Gamma_\alpha(x)$ contains some crucial points that may not be contained in $\Gamma(x)$ and in the $L^p$ sense the function $N_*^\alpha u$ is not too much larger than $N_* u$. We prove the latter claim in the next lemma but postpone the proof of the first claim to Section 5.

**Lemma 1.23.** Suppose that $u$ is a continuous function and let $\alpha \geq 1$. Then $\|N_* u\|_{L^p(E)} \approx_\alpha \|N^\alpha_* u\|_{L^p(E)}$ for every $p \in (0, \infty)$.

**Proof.** We only prove the claim for the case $\text{diam}(E) = \infty$ as the proof for the case $\text{diam}(E) < \infty$ is almost the same.

Since the set $E$ is ADR, measures of balls with comparable radii are comparable. Using this property makes it simple and straightforward to generalize the classical proof of C. Fefferman and E. Stein [FS72] Lemma 1 from $\mathbb{R}^{n+1}_+$ to $X$ to show that $\|N_* u\|_{L^p(E)} \approx_{\alpha, \beta} \|N^\beta_* u\|_{L^p(E)}$ where

$$
N^\gamma_* u := \sup_{Y \in \Gamma^\gamma(x)} |u(Y)|, \quad \Gamma^\gamma(x) := \{Y \in \Omega: \text{dist}(x, Y) < \gamma \cdot \delta(Y)\}.
$$

By the definition of the cones $\Gamma(x)$, there exists $\gamma_0 \geq 0$ such that $\tilde{\Gamma}_{\gamma_0}(x) \subset \Gamma(x)$ for every $x \in E$. Thus, we only need to show that $\Gamma_\alpha(x) \subset \tilde{\Gamma}_{\gamma}(x)$ for some uniform $\gamma = \gamma(\alpha)$ for all $x \in E$ since this gives us the estimate (\textdegree) in the chain

$$
\|N_* u\|_{L^p(E)} \leq \|N^\alpha_* u\|_{L^p(E)} \overset{(*)}{\leq} \|N^\alpha_* u\|_{L^p(E)} \approx_{\gamma, \gamma_0} \|N^\gamma_0 u\|_{L^p(E)} \leq \|N_* u\|_{L^p(E)}.
$$

Suppose that $Q, P \in \mathcal{D}(E), x \in Q, \ell(Q) = \ell(P)$ and $\alpha \Delta_Q \cap P \neq \emptyset$. By the construction of the Whitney regions, for every $Y \in \hat{U}_P$ we have

$$
\delta(Y) \approx \ell(P) \approx \text{dist}(Y, P).
$$

On the other hand, since $\alpha \Delta_Q \cap P \neq \emptyset$ and $\ell(P) = \ell(Q)$, we know that for any $y \in P$ we have

$$
\text{dist}(x, y) \lesssim \alpha \ell(Q) = \alpha \ell(P).
$$

Let us take any $z \in P$. Now for every $Y \in \hat{U}_P$ we have

$$
\text{dist}(x, Y) \lesssim \text{dist}(x, z) + \text{dist}(z, Y) \lesssim \alpha \ell(P) + \ell(P) \lesssim \alpha \ell(P) \approx \alpha \cdot \delta(Y).
$$

In particular, there exists a uniform constant $\gamma = \gamma(\alpha)$ such that $\Gamma_\alpha(x) \subset \tilde{\Gamma}_{\gamma}(x)$. \qed
2. Principal cubes

As in [HR16], we define the numbers $M_\mathcal{D}(N_u)(Q)$ by setting

$$M_\mathcal{D}(N_u)(Q) := \sup_{Q \subseteq R \subseteq \mathcal{D}} \int_R N_u(y) \, d\sigma(y)$$

for every $Q \in \mathcal{D}(E) = \mathbb{D}$. Suppose that $\mathcal{I} \subset \mathcal{D}(E) = \mathbb{D}$ is a collection of dyadic cubes such that

$$\mathcal{I} := \left\{ Q_i : i \in \tilde{N} \right\}, \quad Q_i \subseteq Q_{i+1} \quad \forall i, \quad \bigcup_i Q_i = E,$$  \tag{2.1}

where $\tilde{N} = \{1, 2, \ldots, n_0\}$ for some $n_0 \in \mathbb{N}$ if $E$ is bounded, and $\tilde{N} = \mathbb{N}$ otherwise. By the properties of dyadic cubes, the collection $\mathcal{I}$ is Carleson. Let us construct a collection $\mathcal{P} \subset \mathcal{D}$ of "stopping cubes" using the construction described in [HR16, Section 6.1]. We set $\mathcal{P}_0 := \mathcal{I}$ and add to $\mathcal{P}_0$ all the cubes $Q' \in \mathbb{D} \setminus \mathcal{P}_0$ such that

(a) for some $Q \in \mathcal{P}_0$ we have $Q' \subseteq Q$ and

$$M_\mathcal{D}(N_u)(Q') := \sup_{Q' \subseteq R \subseteq \mathcal{D}} \int_R N_u(y) \, d\sigma(y) > 2M_\mathcal{D}(N_u)(Q),$$  \tag{2.2}

(b) $Q'$ is not contained in any such $Q'' \subseteq Q$ such that either $Q'' \in \mathcal{P}_0$ or (2.2) holds for the pair $(Q'', Q)$.

We denote by $\mathcal{P}_1$ the collection we get by adding the suitable cubes to $\mathcal{P}_0$. We then continue this process for $\mathcal{P}_1$ in place of $\mathcal{P}_0$ and so on. We set $\mathcal{P} := \bigcup_{k=0}^\infty \mathcal{P}_k$. We also set

$$\pi_\mathcal{P}Q = \text{ the smallest cube } Q_0 \in \mathcal{P} \text{ such that } Q \subseteq Q_0.$$  

Here we mean smallest with respect to the side length. Naturally, we have $\pi_\mathcal{P}Q = Q$ for every $Q \in \mathcal{P}$, and since $\mathcal{I} \subset \mathcal{P}$, for every cube $Q \in \mathcal{D}$ there exists some cube $P_Q \in \mathcal{P}$ such that $Q \subseteq P_Q$.

**Remark 2.3.** The collection $\mathcal{P}$ is an auxiliary collection that helps us to simplify the proofs of several claims. We use it in the following way. Suppose that we have a subcollection $\mathcal{W} \subset \mathcal{P}$ and we want to show that $\mathcal{W}$ satisfies a Carleson packing condition. Let $Q_0 \in \mathcal{D}$. Now for every $Q \in \mathcal{W}$ such that $Q \subseteq Q_0$, we have either $\pi_\mathcal{P}Q = \pi_\mathcal{P}Q_0$ or $\pi_\mathcal{P}Q = P = \pi_\mathcal{P}P$ for some $P \in \mathcal{P}$ such that $P \subseteq \pi_\mathcal{P}Q_0$. In particular, we have

$$\sum_{Q \in \mathcal{W}, Q \subseteq Q_0} \sigma(Q) = \sum_{Q \in \mathcal{W}, \pi_\mathcal{P}Q = \pi_\mathcal{P}Q_0} \sigma(Q) + \sum_{P \in \mathcal{P}, P \subseteq \pi_\mathcal{P}Q_0} \sum_{Q \in \mathcal{W}, \pi_\mathcal{P}Q = P} \sigma(Q) =: I_{Q_0} + \sum_{P \in \mathcal{P}, P \subseteq \pi_\mathcal{P}Q_0} I_P.$$  

By Lemma 2.4, the collection $\mathcal{P}$ satisfies a Carleson packing condition. Thus, if we can show that $I_{Q_0} \lesssim \sigma(Q_0)$ for an arbitrary cube $Q_0 \in \mathcal{P}$, we get

$$\sum_{P \in \mathcal{P}, P \subseteq \pi_\mathcal{P}Q_0} I_P \lesssim \sum_{P \in \mathcal{P}, P \subseteq \pi_\mathcal{P}Q_0} \sigma(P) \lesssim \sigma(Q_0).$$

Thus, to show that the collection $\mathcal{W}$ satisfies a Carleson packing condition, it is enough to show that $I_{Q_0} \lesssim \sigma(Q_0)$ for every cube $Q_0 \in \mathcal{D}$. The usefulness of this simplification is that if $Q \in \mathcal{D} \setminus \mathcal{P}$ and $\pi_\mathcal{P}Q = P$, then by the construction of the collection $\mathcal{P}$ we have

$$M_\mathcal{D}(N_u)(Q) \leq 2M_\mathcal{D}(N_u)(P).$$

We use this property several times in the proofs.

For any cube $Q_0 \in \mathcal{D}$, we say that $R \in \mathcal{P}$ is a $\mathcal{P}$-proper subcube of $Q_0$ if we have $M_\mathcal{D}(N_u)(R) > 2M_\mathcal{D}(N_u)(Q_0)$ and $M_\mathcal{D}(N_u)(R') \leq 2M_\mathcal{D}(N_u)(Q_0)$ for every intermediate cube $R \subseteq R' \subseteq Q_0$.

**Lemma 2.4.** For every $Q_0 \in \mathcal{D}(E)$ we have

$$\sum_{P \in \mathcal{P}, P \subseteq Q_0} \sigma(P) \lesssim \sigma(Q_0).$$
Proof. Let us start by noting that we may assume that $Q_0 \in \mathcal{P}$ since otherwise we can simply consider the $\mathcal{P}$-maximal subcubes of $Q_0$.

Suppose first that we have a collection of disjoint cubes $Q' \subset Q$ that satisfy $M_D(N_*u)(Q') > 2M_D(N_*u)(Q)$. Then, for every such cube $Q'$ we have $M_D(N_*u)(Q') > f_Q N_* u \, d\sigma$ and thus, for every point $x \in Q'$ we get

$$M_D(1_Q N_* u)(x) = \sup_{R \in \mathcal{D}, x \in R \subset Q} \int_R N_* u \, d\sigma$$

$$\geq \sup_{R \in \mathcal{D}, Q' \supseteq R \subset Q} \int_R N_* u \, d\sigma = M_D(N_*u)(Q') > 2M_D(N_*u)(Q).$$

In particular, by the $L^1 \to L^{1,\infty}$ boundedness of $M_D$ we have

$$\sum_{Q'} \sigma(Q') \leq \sigma \left( \{ x \in E : M_D(1_Q N_* u)(x) > 2M_D(N_*u)(Q) \} \right)$$

$$\leq \frac{1}{2M_D(N_*u)(Q)} \|1_Q N_* u\|_{L^1(\sigma)} = \frac{\int_Q N_* u \, d\sigma}{M(N_*u)(Q)} \frac{\sigma(Q)}{2} \leq \frac{\sigma(Q)}{2}. \quad (2.5)$$

We notice that if $R \in \mathcal{P} \setminus \mathcal{I}$, then $R$ is a $\mathcal{P}$-proper subcube of some cube $Q \in \mathcal{P}$. To be more precise, if $R \in \mathcal{P} \setminus \mathcal{I}$, then there exists a chain of cubes $R = R_1 \supseteq R_2 \supseteq \ldots \supseteq R_k$, $R_i \in \mathcal{P}$, such that for every $i = 1, 2, \ldots, k - 1$, $R_i$ is a $\mathcal{P}$-proper subcube of $R_{i+1}$ and $R_k \in \mathcal{I}$. If such a chain of length $k$ from $R$ to $Q$ exists, we denote $R \in \mathcal{P}^k$. By using the property $(2.5)$ $k$ times, we see that for each $Q \in \mathcal{P}$ we have

$$\sum_{R \in \mathcal{P}^k_Q} \sigma(R) \leq \sum_{R \in \mathcal{P}^k_Q} \sum_{S \in \mathcal{P}^k_{Q_S}, S \subset R} \sigma(S) \leq \frac{1}{2} \sum_{R \in \mathcal{P}^{k-1}_Q} \sigma(R) \leq \ldots \leq \frac{1}{2^{k-1}} \sum_{R \in \mathcal{P}^0_Q} \sigma(R) \leq \frac{\sigma(Q)}{2^k}. \quad (2.6)$$

Now it is straightforward to prove the packing condition. We have

$$\sum_{P \in \mathcal{P}, P \subseteq Q_0} \sigma(P) = \sum_{P \in \mathcal{I}, P \subseteq Q_0} \sigma(P) + \sum_{P \in \mathcal{P} \setminus \mathcal{I}, P \subseteq Q_0} \sigma(P)$$

$$\leq C_I \sigma(Q_0) + \sum_{Q \in \mathcal{I}, Q \subseteq Q_0} \sum_{P \in \mathcal{P}^k_Q} \sigma(P) \leq C_I \sigma(Q_0) + \sum_{Q \in \mathcal{I}, Q \subseteq Q_0} \frac{\sigma(Q)}{2^k}$$

$$\leq C_I \sigma(Q_0) + \sum_{Q \in \mathcal{I}, Q \subseteq Q_0} \frac{\sigma(Q)}{2^k} \leq C_I \sigma(Q_0) + C_I \sigma(Q_0)$$

which proves the claim. \hfill \square

3. “LARGE OSCILLATION” CUBES

Before constructing the approximating function, we consider two collections of cubes that will act as the basis of our construction. In this section, we show that the union of the collection of “large oscillation” cubes

$$\mathcal{R} := \left\{ Q \in \mathbb{D} : \text{osc}_u > \varepsilon M_D(N_*u)(Q) \text{ for some } i \right\},$$

and the collection of “bad” cubes from the corona decomposition satisfies a Carleson packing condition. We apply this property in the technical estimates in Section 4.

Lemma 3.1. For every $Q_0 \in \mathbb{D}(E)$ we have

$$\sum_{R \in \mathcal{R}, R \subseteq Q_0} \sigma(R) \lesssim \frac{1}{\varepsilon^2} \sigma(Q_0). \quad (3.2)$$
Proof. Notice first that we may assume that \( Q_0 \in \mathcal{R} \) since otherwise we may simply consider the \( \mathcal{R} \)-maximal subcubes of \( Q_0 \).

**Part 1: Simplification.** First, by Remark 2.3 it is enough to show that
\[
\sum_{R \in \mathcal{R}, R \subset Q_0} \sigma(R) \lesssim \frac{1}{\varepsilon^2} \sigma(Q_0).
\]

Also, since the “bad” collection in the bilateral corona decomposition is Carleson, it suffices to consider the “good” cubes in \( \mathcal{R} \). Furthermore, since the Whitney regions \( U_Q \) of the “good” cubes break into two components \( U_Q^+ \) and \( U_Q^- \), it is enough to bound the sum
\[
\sum_{R \in \mathcal{R}^+, R \subset Q_0} \sigma(R) \lesssim \sigma(Q_0), \quad \text{where} \quad \mathcal{R}^+ := \left\{ Q \in \mathcal{R} \cap \mathcal{G} : \text{osc}_{U_Q^+} > \varepsilon M_{\mathcal{D}}(N, u)(Q) \right\},
\]
as the arguments for the corresponding collection \( \mathcal{R}^- \) are the same. In particular, we may assume that \( Q_0 \in \mathcal{G} \), since otherwise we may just consider the \( \mathcal{G} \)-maximal subcubes of \( Q_0 \).

Since \( Q_0 \in \mathcal{G} \), there exists a stopping time regime \( S_0 = S_0(Q_0) \) such that \( Q_0 \in S_0 \). We note that if we have \( Q \subset Q_0 \) for a cube \( Q \in \mathcal{R}^+ \), then either \( Q \in S_0 \) or, by the coherency and disjointness of the stopping time regimes, \( Q \in S \) for such a \( S \) that \( Q(S) \subset Q_0 \). Let \( \mathcal{S} = \mathcal{S}(Q_0) \) be the collection of the stopping time regimes \( S \) such that \( Q(S) \subset Q_0 \). Then we have
\[
\sum_{R \in \mathcal{R}^+, R \subset Q_0} \sigma(R) = \sum_{R \in \mathcal{R}^+ \cap S_0, R \subset Q_0} \sigma(R) + \sum_{S \in \mathcal{S}} \sum_{R \in \mathcal{R}^+ \cap S, R \subset Q_0} \sigma(R) := I_{Q_0} + II_{Q_0}.
\]

Let us show that if \( I_{Q_0} \lesssim \sigma(Q_0) \) for every \( Q_0 \in \mathcal{D} \), then \( II_{Q_0} \lesssim \sigma(Q_0) \) for every \( Q_0 \in \mathcal{D} \). Suppose that \( Q \in \mathcal{S} \). Since \( Q(S) \subset Q_0 \), we have \( \pi_{\mathcal{R}} Q = \pi_{\mathcal{R}} Q_0 \) only if \( \pi_{\mathcal{R}} Q = \pi_{\mathcal{R}} Q(S) = \pi_{\mathcal{R}} Q_0 \). Thus, it holds that
\[
II_{Q_0} = \sum_{S \in \mathcal{S}} \sum_{R \in \mathcal{R}^+ \cap S, R \subset Q_0} \sigma(R) \leq \sum_{S \in \mathcal{S}} \sum_{R \in \mathcal{R}^+ \cap S, R \subset Q_0} \sigma(R) = \sum_{S \in \mathcal{S}} I_Q(S) \lesssim \sigma(Q(S)) \lesssim \sigma(Q_0)
\]
by the Carleson packing property of the collection \( \{ Q(S) \} \). Hence, to prove (3.2), it suffices to show \( I_{Q_0} \lesssim \sigma(Q_0) \).

**Part 2: \( \delta(Y) \lesssim D_A(Y) \) in \( \hat{U}_p^+ \).** Let \( A \subset \mathcal{G} \) be a collection of cubes, set
\[
\Omega_A := \text{int} \left( \bigcup_{Q \in A} \hat{U}_Q^+ \right) = \text{int} \left( \bigcup_{Q \in A} \bigcup_{J \in W_Q^+} J^{***} \right)
\]
and fix a cube \( P \in A \) and a point \( Y \in \hat{U}_P \bigcup_{J \in W_Q^+} J^{***} \). We claim that now \( \delta(Y) \lesssim D_A(Y) := \text{dist}(Y, \partial \Omega_A) \). We notice first that although the regions \( \hat{U}_Q^+ \) may overlap, we have \( \ell(Q) \approx \ell(Q') \approx \ell(P) \) for all overlapping regions \( \hat{U}_Q^+ \) and \( \hat{U}_{Q'}^+ \) such that \( Y \in \hat{U}_Q^+ \bigcap \hat{U}_{Q'}^+ \) (see (3.2), (3.8) and related estimates in [HMM16]). Also, the fattened Whitney boxes \( J^{***} \) may overlap, but we have \( \ell(I^{***}) \approx \ell(I) \approx \ell(J) \approx \ell(J^{***}) \approx \ell(P) \) if \( Y \in I^{***} \bigcap J^{***} \). By a simple geometrical consideration we know that
\[
\text{dist}(Y, \partial I^{***}) \approx \ell(I).
\]
It now holds that $D_A(Y) = \dist(Y, \partial I^{**})$ for some $I^{**} \ni Y$ or $D_A(Y) \geq \dist(Y, \partial I^{***})$ for every such $I^{***}$. In particular, we have

$$D_A(Y) \geq \inf_{Q \in A, Y \in U_0^+} \inf_{I \in W_0^+} \dist(Y, \partial I^{**}) \approx \inf_{Q \in A, Y \in U_0^+} \inf_{I \in W_0^+} \ell(I) \approx \inf_{Q \in A, Y \in U_0^+} \ell(Q) \approx \ell(P).$$

Now we can take any $I \in W_P^+$ such that $Y \in I^{**}$ and notice that $\ell(P) \approx \ell(I) \approx \ell(I^{**}) \approx \dist(I^{**}, \partial \Omega) \approx \dist(Y, \partial \Omega)$. Hence $D_A(Y) \geq \delta(Y)$ for every $Y \in \bar{U}_P^+$.  

**Part 3: The sum $I_{Q_0}$.** To simplify the notation, let us write

$$\mathcal{R}_0^+ := \{R \in \mathcal{R}^+ \cap \mathcal{S}_0 : R \subset Q_0, \pi_P R = \pi_P Q_0\}.$$  

We consider the region $\Omega^{***}$,

$$\Omega^{***} := \text{int} \left( \bigcup_{R \in \mathcal{R}_0^+} \hat{U}_R^+ \right),$$

and set $D(Y) := \dist(Y, \partial \Omega^{***})$ for every $Y \in \Omega$. Suppose that $R \in \mathcal{R}_0^+$. By Part 2, we know that

$$\delta(Y) \lesssim D(Y) \quad \text{for every } Y \in \bar{U}_R^+. \quad (3.3)$$

We also notice that

$$\Omega^{**} = \text{int} \left( \bigcup_{R \in \mathcal{R}_0^+} \hat{U}_R^+ \right) \subset \text{int} \left( \bigcup_{R \in \mathcal{R}_0^+} \bigcup_{x \in R} \Gamma(x) \right),$$

so we have

$$\sup_{X \in \Omega^{***}} |u(X)| = \sup_{R \in \mathcal{R}_0^+} \sup_{X \in \hat{U}_R^+} |u(X)| \leq \sup_{R \in \mathcal{R}_0^+} \inf_{x \in R} N_u(x) \leq \sup_{R \in \mathcal{R}_0^+} M_D(N_u(R)) \lesssim M_D(N_u(\pi_P Q_0)). \quad (3.4)$$

In the last inequality we used the fact that since $R \in \mathcal{R}_0^+$, the condition $\begin{array}{c}(2.2) \end{array}$ does not hold for $R$ and $Q_0$.

By [HMM16 (5.8)] or [HML14 Section 4]), we have

$$\left( \frac{\osc u_{\hat{U}_n^+}}{R_n} \right)^2 \lesssim \ell(R)^{-n} \int_{\hat{U}_n^+} |\nabla u(Y)|^2 \delta(Y) \, dY, \quad (3.5)$$

for every $R \in \mathcal{R}^+$. Using (A) the definition of the numbers $M_D(N_u(Q))$, (B) the ADR property of $E$, (C) the definition of the collection $\mathcal{R}^+$ and (D) the bounded overlap of the regions $\hat{U}_R^+$ we get

$$M_D(N_u(\pi_P Q_0)^2 I_{Q_0} \leq \sum_{R \in \mathcal{R}_0^+} M_D(N_u(R)^2 \sigma(R) \quad (3.6)$$

$$\lesssim \sum_{R \in \mathcal{R}_0^+} M_D(N_u(R)^2 \ell(R)^n \quad (3.7)$$

$$\lesssim \frac{1}{\varepsilon^2} \sum_{R \in \mathcal{R}_0^+} \int_{\hat{U}_n^+} |\nabla u(Y)|^2 \delta(Y) \, dY \quad (3.8)$$

$$\lesssim \frac{1}{\varepsilon^2} \sum_{R \in \mathcal{R}_0^+} \int_{\hat{U}_n^+} |\nabla u(Y)|^2 D(Y) \, dY \quad (3.9)$$

$$\lesssim \frac{1}{\varepsilon^2} \int_{\Omega^{***}} |\nabla u(Y)|^2 D(Y) \, dY \quad (3.10)$$
Since $Q_0 \in \mathcal{R}$, we notice that the collection $\mathcal{R}^+_0$ forms a semi-coherent subregime of $\mathcal{S}_0$. Thus, by [HMM16 Lemma 3.24], the set $\Omega^{***}$ is a chord-arc domain (i.e. NTA domain with ADR boundary). Furthermore, by [AHM+14 Theorem 1.2], $\partial \Omega^{***}$ is UR. Since $\Omega^{***} \subset B(x_{Q_0}, C((Q_0)$) for a suitable structural constant $C$ (see [HMM16 (3.14)]), the ADR property of $\partial \Omega$ and [HMM16 Theorem 1.1] give us

$$\frac{1}{\varepsilon^2} \int_{\Omega^{***}} |\nabla u(Y)|^2D(Y)\,dY \lesssim \frac{1}{\varepsilon^2} ||u||_{L^\infty(\Omega^{***})} \cdot \sigma(Q_0) \lesssim \frac{1}{\varepsilon^2} M_D(N_u)(\pi_p Q_0)^2 \cdot \sigma(Q_0).$$

(3.7)

Since the numbers $M_D(N_u)(\pi_p Q_0)^2$ cancel from (3.6) and (3.7), this concludes the proof of the lemma.

Since the bad collection $\mathcal{B}$ in the bilateral corona decomposition satisfies a Carleson packing condition, we immediately get the following corollary:

**Corollary 3.8.** For every $Q_0 \in \mathcal{D}(E)$ we have

$$\sum_{R \in (\mathcal{R} \cup \mathcal{B}), R \subseteq Q_0} \sigma(R) \lesssim \frac{1}{\varepsilon^2} \sigma(Q_0).$$

(3.9)

### 4. Generation Cubes

For every stopping time regime $S$, we construct a collection of generation cubes $G(S)$ as in [HMM16 Section 5] with the modified stopping conditions

1. $Q$ is not in $S$;
2. $|u(Y^Q_2) - u(Y^Q_0)| > \varepsilon M_D(N_u)(Q)$,
3. $|u(Y^Q_2) - u(Y^Q_0)| > \varepsilon M_D(N_u)(Q)$

Recall that by the construction we have

$$S = \bigcup_{Q \in G(S)} S'(Q)$$

(4.1)

for each stopping time regime $S$, where $S'(Q)$ is a semicoherent subregime of $S$ with maximal element $Q$. We also denote the collection of all generation cubes by $G^*$:

$$G^* := \bigcup_{S} G(S).$$

Our next goal is to prove that the collection $G^*$ satisfies a Carleson packing condition:

**Lemma 4.2.** For every $Q_0 \in \mathcal{D}$ we have

$$\sum_{S \in G^* \cap \mathcal{S}_0, S \subseteq Q_0} \sigma(S) \lesssim \frac{1}{\varepsilon^2} \sigma(Q_0).$$

(4.3)

Before the proof, let us make two observations that help us to simplify the proof.

1. By arguing as in the proof of Lemma 3.1, we may assume that $Q_0 \in G^*$ and it suffices to show that

$$\sum_{S \in G^* \cap \mathcal{S}_0, S \subseteq Q_0 \pi_p S = \pi_p Q_0} \sigma(S) \lesssim \frac{1}{\varepsilon^2} \sigma(Q_0),$$

where $\mathcal{S}_0$ is the unique stopping time regime such that $Q_0 \in \mathcal{S}_0$.

2. For every $k \geq 0$ and $S \in G(\mathcal{S}_0)$, let $G_k(S) \subset G(\mathcal{S}_0)$ be the $k$th generation $G^*$-descendants of $S$. For each such $S$ we have

$$M_D(N_u)(S)^2 \sum_{Q \in G_k(S) \pi_p Q = \pi_p Q_0} \sigma(Q) \lesssim \frac{1}{\varepsilon^2} \int_{\Omega_{S'}} |\nabla u(Y)|^2dY,$$

(4.4)

where $S' := S' \cap \mathcal{S} \cap \{Q \in \mathbb{D} : \pi_p Q = \pi_p Q_0\}$ is a semicoherent subregime of $\mathcal{S}_0$ and $\Omega_{S'}$ is the associated sawtooth region. (4.4) is a counterpart of [HMM16 Lemma
Proof of Lemma 4.2. Let us follow the arguments in the proof of [HMM16] Lemma 5.16 and write

\[
\sum_{S \in G^* \cap S_0 \cap Q_0} \sigma(S) = \sum_{k \geq 0} \sum_{S' \in G_k(Q_0)} \sum_{\pi \tau S = \pi \tau Q_0} \sigma(S) = \sigma(Q_0) + \sum_{k \geq 1} \sum_{S' \in G_{k-1}(Q_0)} \sum_{\pi \tau S' = \pi \tau Q_0} \sigma(S) =: \sigma(Q_0) + I.
\]

Using (4.4) and the definition of the sawtooth regions gives us

\[
M_2(N_\ast u(Q_0))^2 I \lesssim \frac{1}{\varepsilon^2} \sum_{k \geq 1} \sum_{S' \in G_{k-1}(Q_0)} \int_{Q_0} |\nabla u(Y)|^2 \delta(Y) dY \lesssim \frac{1}{\varepsilon^2} \sum_{k \geq 1} \sum_{S' \in G_{k-1}(Q_0)} \sum_{\pi \tau S' = \pi \tau Q_0} \int_{Q_0} |\nabla u(Y)|^2 \delta(Y) dY \tag{4.5}
\]

Let us denote \( \Omega_0 := \bigcup_{S \in G_0} U_S \) where \( G_0^* := \{ S \in D : \pi \tau S = \pi \tau Q_0 \} \cap \bigcup_{k \geq 1} \bigcup_{S' \in G_{k-1}(Q_0)} S'(S') \). By the construction, \( \bigcup_{k \geq 1} \bigcup_{S' \in G_{k-1}(Q_0)} S'(S') \) is a coherent subregime of \( S_0 \) with maximal element \( Q_0 \) and thus, \( G_0^* \) is a semicoherent subregime of \( S_0 \). In particular, the sawtooth region \( \Omega_0 \) splits into two chord-arc domains \( \Omega_0^{*\pm} \) by [HMM16] Lemma 3.24. Furthermore, by [AHM14] Theorem 1.2, both \( \partial \Omega_0^+ \) and \( \partial \Omega_0^- \) are UR. We also note that \( \Omega_0 \subset B(x_{Q_0}, C\ell(Q_0)) \) (see [HMM16] (3.14)). Thus, since the triple sum in (4.5) runs over a disjoint collection of disjoint cubes, we can use the bounded overlap of the Whitney regions, [HMM16] Theorem 1.1 and the ADR property of \( E \) to show that

\[
\frac{1}{\varepsilon^2} \sum_{k \geq 1} \sum_{S' \in G_{k-1}(Q_0)} \sum_{S \in S'(S')} \int_{Q_0} |\nabla u(Y)|^2 \delta(Y) dY \lesssim \frac{1}{\varepsilon^2} \sum_{k \geq 1} \int_{Q_0} |\nabla u(Y)|^2 \delta(Y) dY \lesssim \frac{1}{\varepsilon^2} \|u\|_{L^\infty(\Omega_0)}^2 \sigma(Q_0).
\]

Since \( \pi \tau S = \pi \tau Q_0 \) for every \( S \in G_0^* \), we have \( M_2(N_\ast u(S)) \leq 2M_2(N_\ast u(\pi \tau Q_0)) \) for every \( S \in G_0^* \) by (2.2). In particular:

\[
\|u\|_{L^\infty(\Omega_0)}^2 \leq \sup_{S \in G_0^*} \sup_{Y \in U_S} |u(Y)|^2 \leq \sup_{S \in G_0^*} \inf_{x \in S} N_\ast u(x)^2 \leq \sup_{S \in G_0^*} M_2(N_\ast u(S))^2 \leq M_2(N_\ast u(\pi \tau Q_0))^2.
\]

Since the numbers \( M_2(N_\ast u(\pi \tau Q_0))^2 \) cancel out, we have proven the Carleson packing condition of \( G^* \). \( \square \)

5. Construction of the approximating function

Before we construct the function, we prove the following technical lemma related to the modified cones \( \Gamma_\alpha(x) \) that we defined in Section 1.6. Recall that

\[
\Gamma_\alpha(x) = \bigcup_{Q \in \mathcal{D}(E)} \bigcup_{P \in \mathcal{D}(E), \ell(P) = \ell(Q), \alpha \Delta Q \cap \Gamma P \neq \emptyset} \hat{U}_P.
\]

Lemma 5.2. There exists a uniform constant \( c_0 > 0 \) such that the following holds: if \( Q \in \mathcal{D}(E) \) is any cube and \( P \in G^* \) is a generation cube such that \( \ell(Q) \leq \ell(P) \) and \( \Omega_{S'(P)} \cap T_P \neq \emptyset \), then \( X_P^+, Y_P^+ \in \Gamma_{c_0}(x) \) for every \( x \in Q \).
Proof. We start by noticing that there exists $\alpha > 0$ (depending only on the structural constants) such that
\[ \text{if } P \text{ appears in the union } (5.1), \text{ then also } \tilde{P} \text{ appears in the same union,} \]
where $\tilde{P}$ is the dyadic parent of $P$. Indeed, if we have $Q, P \in \mathbb{D}(E), x \in Q, \ell(Q) = \ell(P)$ and $\alpha \Delta Q \cap P \neq \emptyset$, then also $x \in \tilde{Q}, \ell(\tilde{Q}) = \ell(\tilde{P})$ and $\alpha \Delta \tilde{Q} \cap \tilde{P} \neq \emptyset$. The last claim follows from the fact that $\emptyset \neq \alpha \Delta Q \cap P \subset \alpha \Delta \tilde{Q} \cap \tilde{P}$.

Let us then prove the claim of the lemma by following the argument in the proof of [HMM16, Lemma 5.20]. Since $\Omega_{S'(P)} \cap T_Q \neq \emptyset$, there exist cubes $P' \in S'(P)$ and $Q' \subset Q$ such that $U_{P'} \cap U_{Q'} \neq \emptyset$. By the properties of the Whitney regions, we have $\text{dist}(Q', P') \lesssim \ell(Q') \approx \ell(P')$.

Let us consider two cases:

i) Suppose that $\ell(P') \geq \ell(Q)$. Then there exists a cube $Q''$ such that $Q \subset Q''$ and $\ell(Q'') = \ell(P')$. Since $Q' \subset Q''$, we have $\text{dist}(Q'', P') \lesssim \ell(Q') \approx \ell(P')$. Thus, for a large enough $\alpha_0$, we have $\tilde{U}_{P'} \subset \Gamma_{\alpha_0}(x)$ for every $x \in Q$ and the claim follows from (5.3).

ii) Suppose that $\ell(P') < \ell(Q)$. Then by the semicoherence of $S'(P)$, there exists a cube $P'' \in S'(P)$ such that $P' \subset P'' \subset P$ and $\ell(P'') = \ell(Q)$. Since $P' \subset P''$ and $Q' \subset Q$, we know that $\text{dist}(P'', Q) \lesssim \ell(P', Q) \approx \ell(Q)$.

Thus, for a large enough $\alpha_0$, we have $\tilde{U}_{P'} \subset \Gamma_{\alpha_0}(x)$ for every $x \in Q$. Again, the claim follows now from (5.3).

\[ \square \]

5.1. Constructing the function in $T_{Q_0}$. In this section we adopt the terminology from other papers (including [HMM16]) and say that a component $U_{Q_0}^i$ is blue if $\text{osc}_{U_{Q_0}^i} u \leq \varepsilon M_0(N, u)(Q)$ and red if $\text{osc}_{U_{Q_0}^i} u > \varepsilon M_0(N, u)(Q)$.

We recall the construction of the local functions $\varphi_0, \varphi_1$ and $\varphi$ from [HMM16, Section 5]. We start by defining an ordered family of good cubes $\{Q_k\}_{k \geq 1}$ relative to a fixed cube $Q_0 \in \mathbb{D}$. If $Q_0 \in \mathcal{G}$, then $Q_0 \in S$ for some stopping time regime $S$ and thus, $Q_0 \in S'_1$ for some subregime in (4.1). In this case, we set $Q_1 = Q(S'_1)$. If $Q_0 \notin \mathcal{G}$, then we let $Q_1$ be any good subcube of $Q_0$ such that $Q_1$ is maximal with respect to the side length; such a cube much exist since $B$ is Carleson. Since $Q_1 \in \mathcal{G}$, we have $Q_1 \in S$ for some stopping time regime $S$, and by the coherency of $S$, we have $Q_1 = Q(S'_1)$ for some subregime in (4.1). Once the cube $Q_1$ has been chosen in these two cases, we let $Q_2$ be the subcube of maximum side length in $S_k(S'_1 \cap \mathcal{G}) \setminus S'_1$ and so on. This gives us a sequence of cubes $Q_k \in \mathcal{G}$ such that $\ell(Q_1) \geq \ell(Q_2) \geq \ell(Q_3) \geq \cdots$, $Q_k = Q(S'_k)$ and $\mathcal{G} \cap \mathbb{D}(Q_0) \subset \bigcup_{k \geq 1} S'_k$. We define recursively

\[ A_1 := \Omega_{S'_1}, \quad A_k := \Omega_{S'_k} \setminus \left( \bigcup_{j=1}^{k-1} A_j \right), \quad k \geq 2. \]

and

\[ A_1^\pm := \Omega_{S'_1}^\pm, \quad A_k^\pm := \Omega_{S'_k}^\pm \setminus \left( \bigcup_{j=1}^{k-1} A_j \right), \quad k \geq 2, \]

where

\[ \Omega_{S'_k}^\pm := \text{int} \left( \bigcup_{Q \in S'_k} U_Q^\pm \right). \]

We also set

\[ \Omega_0 := \bigcup_k \Omega_{S'_k} = \bigcup_k A_k \quad \text{and} \quad \Omega_0^\pm := \bigcup_k A_k^\pm. \]

We now define $\varphi_0$ on $\Omega_0$ by setting

\[ \varphi_0 := \sum_k \left( u(Y_{Q_k}^+) 1_{A_k^+} + u(Y_{Q_k}^-) 1_{A_k^-} \right). \]
As for the rest of the subcubes of $\mathbb{D}_{Q_0}$, we let \( \{Q(k)\}_k \) be some fixed enumeration of the cubes \((\mathcal{R} \cup \mathcal{B}) \cap \mathbb{D}_{Q_0}\) and define recursively

\[
V_1 := U_{Q(1)}, \quad V_k := U_{Q(k)} \setminus \left( \bigcup_{j=1}^{k-1} V_j \right), \quad k \geq 2.
\]

Each Whitney region $U_{Q(k)}$ splits into a uniformly bounded number of connected components $U^i_{Q(k)}$. Thus, we may further split

\[
V^i_1 := U^i_{Q(1)}; \quad V^i_k := U^i_{Q(k)} \setminus \left( \bigcup_{j=1}^{k-1} V^i_j \right), \quad k \geq 2
\]

and then define

\[
\varphi_1(Y) := \begin{cases} 
  u(Y), & \text{if } U^i_{Q(k)} \text{ is red} \\
  u(X_I), & \text{if } U^i_{Q(k)} \text{ is blue} 
\end{cases}, \quad Y \in V^i_k,
\]

on each $V^i_k$, where $X_I$ is the center of a fixed Whitney cube $I \subset U^i_{Q(k)}$. We then denote $\Omega_1 := \text{int} \left( \bigcup_{Q \in (\mathcal{B} \cup \mathcal{R}) \cap \mathbb{D}_{Q_0}} U_Q \right) = \text{int} (\bigcup_k V_k)$, set the values of $\varphi_0$ and $\varphi_1$ to be 0 outside their original domains of definition and define the function $\varphi$ on the Carleson box $T_{Q_0}$ as

\[
\varphi(Y) := \begin{cases} 
  \varphi_0(Y), & Y \in T_{Q_0} \setminus \overline{\Omega_1}, \\
  \varphi_1(Y), & Y \in \Omega_1
\end{cases}.
\]

From the point of view of $\mathcal{C}_\varphi$, the values of $\varphi$ on the boundary of $\Omega_1$ are not important since the $(n+1)$-dimensional measure of $\partial \Omega_1$ is 0. Thus, we may simply set $\varphi|_{\partial \Omega_1} = u$ since this is convenient from the point of view of $N_*(u - \varphi)$.

### 5.2. Verifying the estimates on $Q_0$

Let us fix a cube $Q_0 \in \mathbb{D}(E)$. We start by verifying the following three estimates on $Q_0$.

**Lemma 5.4.** Suppose that $x \in Q_0$. Then the following estimates hold:

i) $N_*(1_{T_{Q_0}}(u - \varphi))(x) \leq \varepsilon M_2(N_*(u))(x)$,

ii) for any $Q' \in \mathbb{D}_{Q_0}$ and for any $\overrightarrow{\Psi} \in C^1_0(T_{Q'})$ such that $\|\overrightarrow{\Psi}\|_{L^\infty} \leq 1$ we have

\[
\iint_{T_{Q'} \setminus \overline{\Omega_1}} \varphi_0 \text{div} \overrightarrow{\Psi} \leq \frac{1}{\varepsilon^2} \iint_{\beta \Delta_{Q'}} M_2(N_*(u)) \, ds,
\]

iii) $C_\varphi(\nabla \varphi_1)(x) \leq \frac{1}{\varepsilon} M(M_2(N_*(u)))(x),$

where $\beta > 0$ is a uniform constant and $\alpha_0 > 0$ is the constant in Lemma 5.3.

**Proof.**

i) Let us estimate the quantity $|u(Y) - \varphi(Y)|$ for different $Y \in T_{Q_0}$.

- Suppose that $Y \in V^i_k$ such that $U^i_{Q(k)}$ is a red component. Then we have $\varphi(Y) = u(Y)$ and $|u(Y) - \varphi(Y)| = 0$.

- Suppose that $Y \in V^i_k$ such that $U^i_{Q(k)}$ is a blue component. Then $\varphi(Y) = u(X_I)$ for a Whitney cube $I \subset U^i_{Q(k)}$ and $|u(Y) - \varphi(Y)| \leq \text{osc}_{U^i_{Q(k)}} u \leq \varepsilon M_2(N_*(u))(Q(k))$.

- Suppose that $Y \in T_{Q_0} \setminus \overline{\Omega_1}$. Then $Y \in A^+_k$ for some $k$ such that $Q_k \notin \mathcal{R}$. Without loss of generality, we may assume that $Y \in A^+_k$. Now $\varphi(Y) = u(Q_k)$ and, since $Q_k \notin \mathcal{R}$, we have $|u(Y) - \varphi(Y)| \leq \text{osc}_{U^+_k} u \leq \varepsilon M_2(N_*(u))(Q_k)$. 


Combining the previous estimates gives us

\[ N_\ast(T_{2\Omega_0}(u - \varphi))(x) = \sup_{Y \in \Gamma(x) \cap T_{2\Omega_0}} |u(Y) - \varphi(Y)| = \sup_{Q \in D_{2\Omega_0}} \sup_{Y \in U_Q} |u(Y) - \varphi(Y)| \]

\[ \leq \sup_{Q \in D_{2\Omega_0}} \varepsilon M_B(N_\ast u)(Q) \]

\[ \leq \varepsilon M_B(N_\ast u)(x). \]

ii) We first notice that for each \( A_k \), the set \((T_{Q'} \cap A_k) \setminus \overline{\Omega_1}\) consists of a union of boundedly overlapping sets that are “nice” enough for integration by parts. In particular, by the divergence theorem we get

\[
\int_{T_{Q'} \setminus \overline{\Omega_1}} \varphi \div \vec{\Psi} \leq \sum_k \int_{(T_{Q'} \cap A_k) \setminus \overline{\Omega_1}} \varphi \div \vec{\Psi}
\]

\[
= \sum_k \int_{(T_{Q'} \cap A_k) \setminus \overline{\Omega_1}} \text{div}(\varphi \vec{\Psi})
\]

\[
\leq \sum_k \left( \int_{\partial((T_{Q'} \cap A_k) \setminus \overline{\Omega_1})} \varphi \vec{\Psi} \cdot \vec{N} + \int_{\partial((T_{Q'} \cap A_k) \setminus \overline{\Omega_1})} \varphi \vec{\Psi} \cdot \vec{N} \right)
\]

\[
\leq \sum_k |u(Y_{Q_k}^+) \cdot H^n(T_{Q'} \cap \partial A_k^+) + \sum_k |u(Y_{Q_k}^-) \cdot H^n(T_{Q'} \cap \partial A_k^-)\]

\[
=: I^+ + I^-.
\]

We only consider the sum \( I^+ \) since the sum \( I^- \) can be handled the same way as \( I^+ \). We get

\[ H^n(T_{Q'} \cap \partial A_k^+) \leq H^n(T_{Q'} \cap \partial A_k^+) + H^n(T_{Q'} \cap A_k^+ \cap \partial \Omega_1) \]

and thus, we have

\[ I^+ \leq \sum_k |u(Y_{Q_k}^+) \cdot H^n(T_{Q'} \cap \partial A_k^+) + \sum_k |u(Y_{Q_k}^-) \cdot H^n(T_{Q'} \cap A_k^+ \cap \partial \Omega_1)\]

\[=: I_1^+ + I_2^+.
\]

Let us consider the sum \( I_1^+ \) first. We split

\[ I_1^+ = \sum_{k: Q_k \subset Q'} |u(Y_{Q_k}^+) \cdot H^n(T_{Q'} \cap \partial A_k^+) + \sum_{k: Q_k \subset Q'} |u(Y_{Q_k}^-) \cdot H^n(T_{Q'} \cap \partial A_k^-)\]

\[=: J_1^+ + J_2^+.
\]

By \[\text{[HMM16 Proposition A.2, (5.21)]}\] we know that \( \partial A_k^+ \) satisfies an upper ADR bound. Thus, since \( \partial(T_{Q'} \cap A_k^+) \subset \overline{\Omega_1} \) and \( \text{diam}(\Omega_1) \leq \ell(Q_k) \), we get

\[ J_1^+ \leq \sum_{k: Q_k \subset Q'} |u(Y_{Q_k}^+) \cdot \ell(Q_k)^n \approx \sum_{k: Q_k \subset Q'} |u(Y_{Q_k}^+) \cdot \sigma(Q_k)\]

\[\leq \sum_{k: Q_k \subset Q'} \inf_{Q_k} N_u \cdot \sigma(Q_k)\]

\[\leq \sum_{k: Q_k \subset Q'} \int_{Q_k} N_u \, d\sigma\]

\[\leq \frac{1}{\varepsilon^2} \int_{Q'} M_2(N_u) \, d\sigma\]

by Lemma 4.2 and the discrete Carleson embedding theorem (Theorem A.1). Let us then consider the sum \( J_2^+ \). By the same argument as in \[\text{[HMM16 p. 2370]}\], we know that the
number of the cubes \( Q_k \) such that \( T_{Q'} \cap \partial A_k^+ \neq \emptyset \) and \( \ell(Q_k) \geq \ell(Q') \) is uniformly bounded. Thus, by Lemma 5.2 and the fact that \( \partial A_k^+ \) satisfies an upper ADR bound (as we noted above), we get

\[
\sum_{k : Q_k \notin Q', \ell(Q') \leq \ell(Q_k)} |u(Y_{Q_k}^+)| \cdot H^n(T_{Q'} \cap \partial A_k^+) \leq \sum_{k : Q_k \notin Q', \ell(Q') \leq \ell(Q_k)} \inf_{Q'_k} N^α u \cdot H^n(T_{Q'} \cap \partial A_k^+) \\
\leq \inf_{Q'} N^α u \cdot (\text{diam}(T_{Q'}))^n \\
\approx \inf_{Q'} N^α u \cdot \sigma(Q') \\
\leq \int_{Q'} N^α u \, d\sigma.
\]

For the cubes \( Q_k \) in \( J_2^+ \) such that \( \ell(Q_k) \leq \ell(Q') \) we may use the same argument as in [HMM13, p. 237] to see that every such cube is contained in some nearby cube \( Q'' \) of \( Q' \) of the same side length as \( Q' \) with \( \text{dist}(Q', Q'') \leq \ell(Q') \). The number of such \( Q'' \) is uniformly bounded. By using the same techniques as with the sum \( J_1^+ \), we get

\[
\sum_{k : Q_k \notin Q', \ell(Q') \geq \ell(Q_k)} |u(Y_{Q_k}^+)| \cdot H^n(T_{Q'} \cap \partial A_k^+) \leq \frac{1}{\varepsilon^2} \int_{Q''} M_2(N_\ast u) \, d\sigma \leq \frac{1}{\varepsilon^2} \int_{\beta_0 \Delta_{Q'}} M_2(N_\ast u) \, d\sigma
\]

for some uniform constant \( \beta_0 \). Thus, we get

\[
J_2^+ \leq \frac{1}{\varepsilon^2} \int_{\beta_0 \Delta_{Q'}} M_2(N_\ast u) \, d\sigma.
\]

Let us then consider the sum \( I_2^+ \). We first notice that

\[
H^n(T_{Q'} \cap A_k^+ \cap \partial \Omega_1) \leq \sum_m H^n(T_{Q'} \cap A_k^+ \cap \partial V_m).
\]

Thus, we get

\[
I_2^+ \leq \sum_m \sum_k |u(Y_{Q_k}^+)| \cdot H^n(T_{Q'} \cap A_k^+ \cap \partial V_m)
\]

\[
= \sum_{k : Q_k \subset Q', m} |u(Y_{Q_k}^+)| \cdot H^n(T_{Q'} \cap A_k^+ \cap \partial V_m) + \sum_{k : Q_k \notin Q', m} |u(Y_{Q_k}^+)| \cdot H^n(T_{Q'} \cap A_k^+ \cap \partial V_m)
\]

\[
=: J_3^+ + J_4^+.
\]

Suppose that \( A_k^+ \cap \partial V_m \neq \emptyset \). Then, by the construction we have \( \ell(Q(m)) \lesssim \ell(Q_k) \) and \( \text{dist}(Q(m), Q_k) \lesssim \ell(Q_k) \). Thus, there exists a uniform constant \( \beta_1 > 0 \) such that \( Q(m) \subset \beta_1 \Delta_{Q_k} \) and the set \( \beta_1 \Delta_{Q_k} \) can be covered by a uniformly bounded number of disjoint cubes with approximately the same side length as \( Q_k \). In particular, since \( T_{Q'} \cap A_k^+ \cap \partial V_m \) satisfies an upper ADR bound for every \( m \) by the construction and
iii) Suppose that
\[ \ell_k': \sum_{m, Q(m) \in \beta_1 \Delta_{Q_k}} \ell(Q(m))^n \]
for some uniform constant
\[ \text{Finally, let us handle the sum } J_4^+ \text{. Just as above with the sum } J_2^+, \text{ for some uniform constant } \beta_3 > 0 \text{ we get} \]
\[ \sum_{k, Q_k \in Q'} \sum_{m, \ell(Q') \leq \ell(Q_k)} |u(Y_{Q_k}^+)\| H^n(T_{Q'} \cap A_k^+ \cap \partial V_m) \leq \sum_{k, Q_k \in Q'} \sum_{m, V_m \in \beta_3 \Delta_{Q'}} |u(Y_{Q_k}^+)\| \sigma(Q(m)) \]
\[ \lesssim \frac{1}{\varepsilon^2} \sum_{k, Q_k \in Q'} |u(Y_{Q_k}^+)\| \sigma(Q') \]
\[ \lesssim \frac{1}{\varepsilon^2} \sum_{k, Q_k \in Q'} \inf_{Q'} N_{\alpha}^\alpha u \cdot \sigma(Q') \]
\[ \lesssim \frac{1}{\varepsilon^2} \int_{Q'} N_{\alpha}^\alpha u \, d\sigma, \]
where we used the fact that there exists only a uniformly bounded number of cubes \( Q_k \) that satisfy the condition of the sum by [HMM16 Lemma 5.20]. By using the same argument as with the latter half of the sum \( J_2^+ \), we get the bound
\[ \sum_{k, Q_k \in Q'} \sum_{m, \ell(Q') \geq \ell(Q_k)} |u(Y_{Q_k}^+)\| H^n(T_{Q'} \cap A_k^+ \cap \partial V_m) \lesssim \frac{1}{\varepsilon^2} \int_{\beta_3 \Delta_{Q'}} M_D(N_{\alpha}^\alpha u) \, d\sigma \]
for some uniform constant \( \beta_3 > 0 \). Thus, we have
\[ J_4^+ \lesssim \frac{1}{\varepsilon^2} \int_{\beta_3 \Delta_{Q'}} M_D(N_{\alpha}^\alpha u) \, d\sigma. \]
Combining the estimates for \( J_1^+, J_2^+, J_3^+ \) and \( J_4^+ \) gives us the claim.
Let us fix an arbitrary $\tilde{\psi} \in C^1_{\text{loc}}(\mathcal{T}_{Q'})$ such that $\|\tilde{\psi}\|_{L^\infty} \leq 1$. Since the function $\varphi_1$ is supported on $\Omega_1$, we get
\[
\frac{1}{\sigma(Q')} \int_{\mathcal{T}_{Q'}} \varphi_1 \text{div} \tilde{\psi} = \frac{1}{\sigma(Q')} \sum_{i} \int_{\mathcal{T}_{Q'} \cap V_i} \varphi_1 \text{div} \tilde{\psi} \\
= \frac{1}{\sigma(Q')} \sum_{i} \int_{\mathcal{T}_{Q'} \cap V_i} \varphi_1 \text{div} \tilde{\psi} \\
= \frac{1}{\sigma(Q')} \sum_{i} \left( \int_{\mathcal{T}_{Q'} \cap V_i} \text{div}(\varphi_1 \tilde{\psi}) - \int_{\mathcal{T}_{Q'} \cap V_i} \nabla \varphi_1 \cdot \tilde{\psi} \right) \\
\leq \frac{1}{\sigma(Q')} \sum_{i} \left( \int_{\mathcal{T}_{Q'} \cap V_i} \|\text{div}(\varphi_1 \tilde{\psi})\| + \int_{\mathcal{T}_{Q'} \cap V_i} |\nabla \varphi_1| \right).
\]

Let us first assume that $U^i_{Q(l)}$ is a blue component. Then, by the definition of $\varphi_1$ and the divergence theorem, we have
\[
\left| \int_{\mathcal{T}_{Q'} \cap V_i} \text{div}(\varphi_1 \tilde{\psi}) \right| + \int_{\mathcal{T}_{Q'} \cap V_i} |\nabla \varphi_1| = \left| \int_{\mathcal{T}_{Q'} \cap V_i} \text{div}(\varphi_1 \tilde{\psi}) \right| \leq \int_{\mathcal{T}_{Q'} \cap \partial V_i} |u(X_{\ell(l, l)})| \\
\leq \inf_{Q(l)} \|N_u \cdot \sigma(Q(l))\| \\
\leq Q(l) \int_{Q(l)} N_u \, d\sigma.
\]

Suppose then that $U^i_{Q(l)}$ is a red component. Since $\partial V_i \subset \Gamma(y)$ for every $y \in Q(l)$, we get $\int_{\mathcal{T}_{Q'} \cap V_i} \text{div}(u \tilde{\psi}) \leq \int_{Q(l)} N_u \, d\sigma$ by the same argument as above. Also, by the definition of the function $\varphi_1$ and Caccioppoli’s inequality, we have
\[
\int_{\mathcal{T}_{Q'} \cap V_i} |\nabla \varphi_1| = \int_{V_i} |\nabla u| \lesssim \left( \int_{V_i} |\nabla u|^2 \right)^{1/2} \ell(Q(l))^{(n+1)/2} \\
\lesssim \frac{1}{\ell(Q(l))} \left( \int_{\mathcal{G}_{Q(l)}} |u|^2 \right)^{1/2} \ell(Q(l))^{(n+1)/2} \\
\lesssim \frac{1}{\ell(Q(l))} \left( \int_{\mathcal{G}_{Q(l)}} \inf_{Q(l)} (N_u)^2 \right)^{1/2} \ell(Q(l))^{(n+1)/2} \\
\lesssim \frac{1}{\ell(Q(l))} \inf_{Q(l)} (N_u) \ell(Q(l))^{n+1} \\
\approx \sigma(Q(l)) \cdot \inf_{Q(l)} (N_u) \leq \int_{Q(l)} N_u \, d\sigma.
\]

Thus, since every Whitney region $U_Q$ has only a uniformly bounded number of components $U_Q$, we get
\[
\int_{\mathcal{T}_{Q'}} |\nabla \varphi_1| \lesssim \sum_{i} \int_{Q(l)} N_u.
\]

Since $V_i$ meets $T_{Q'}$, we know that $\text{dist}(Q(l), Q') \lesssim \ell(Q')$. In particular, all the relevant cubes $Q(l)$ are contained in some nearby cubes $Q''$ such that $\ell(Q'') \approx \ell(Q')$ and $\text{dist}(Q'', Q') \lesssim \ell(Q')$. The number of such $Q''$ is uniformly bounded. Thus, by the Carleson packing condition of the cubes $\mathcal{R} \cup \mathcal{B}$ and the discrete Carleson embedding theorem
Proposition 5.6. The following (a.e) pointwise bound holds on $Q_0$:

$C_0(\nabla \varphi)(x) \lesssim \frac{1}{\varepsilon^2} M(M_0(N_0^n u))(x).$

Proof. Let $x \in Q' \in D_{Q_0}$. We get

\[
\iint_{T_{Q'}} |\nabla \varphi| = \sup_{\Psi \in C^0_0(T_{Q'})} \iint_{T_{Q'}} \varphi \div \Psi
\leq \sup_{\Psi \in C^0_0(T_{Q'})} \iint_{T_{Q'} \cap \Omega} \varphi_0 \div \Psi + \sup_{\Psi \in C^0_0(T_{Q'})} \iint_{T_{Q'} \cap \Omega} \varphi_1 \div \Psi.
\]

Let us fix an arbitrary $\tilde{\Psi} \in C^0_0(T_{Q'})$ such that $||\tilde{\Psi}||_{L^\infty} \leq 1$. For the first integral, we can simply use the part ii) of Lemma 5.4. For the second integral we get

\[
\iint_{T_{Q'} \cap \Omega} \varphi_1 \div \tilde{\Psi} = \sum_k \iint_{V_k \cap T_{Q'}} \varphi_1 \div \tilde{\Psi}
= \sum_k \left( \iint_{V_k \cap T_{Q'}} \div (\varphi_1 \tilde{\Psi}) - \iint_{V_k \cap T_{Q'}} \nabla \varphi_1 \cdot \tilde{\Psi} \right)
\leq \sum_k \iint_{V_k \cap T_{Q'}} |\nabla \varphi_1|.
\]

The second sum is just as in the proof of part iii) of Lemma 5.4 and thus, we can bound it by $\int_{Q'} M_0(N_0^n u) d\sigma$. For the first sum, we use the divergence theorem and get

\[
\sum_k \left| \iint_{V_k \cap T_{Q'}} \div (\varphi_1 \tilde{\Psi}) \right| \leq \sum_k \left| \iint_{\partial (V_k \cap T_{Q'})} \varphi_1 \tilde{\Psi} \cdot \nu \right|
\leq \sum_k \sup_{\Omega(Q(k))} |u| \cdot H^n(V_k \cap \partial T_{Q'})
\leq \sum_{k: \text{dist}(Q(k),Q') \leq \ell(Q')} \inf_{Q(k)} N_0 u \sigma(Q(k))
\leq \sum_{k: \text{dist}(Q(k),Q') \leq \ell(Q')} \int_{Q(k)} N_0 u d\sigma.
\]

By the structure of the Whitney regions, we know that there exists a uniform constant $\beta_1 > 0$ such that $Q(k) \subset \beta_1 \Delta Q'$ for every such $k$. We may cover $\beta_1 \Delta Q'$ by a uniformly bounded number of disjoint cubes $P_k$ such that $\ell(P_k) \approx \ell(Q')$. Since $Q(k) \in R \cup B$ for all $k$, by the Carleson packing property of the collection $R \cup B$ and the discrete Carleson embedding theorem (Theorem A.1), we
get

$$
\sum_{k: \text{dist}(Q(k), Q') \leq \ell(Q')} \int_{Q(k)} N_s u \, d\sigma = \sum_j \sum_{k: Q(k) \subset P_j} \int_{Q(j)} N_s u \, d\sigma \leq \frac{1}{\varepsilon^2} \sum_j \int_{P_j} M_D(N_s u) \, d\sigma \leq \frac{1}{\varepsilon^2} \int_{\beta_2 \Delta_{Q'}} M_D(N_s u) \, d\sigma
$$

for some uniform constant $\beta_2 \geq \beta_1$. Since $\sigma(Q') \approx \sigma(\beta_2 \Delta_{Q'})$, we get

$$
\frac{1}{\sigma(Q')} \frac{1}{\varepsilon^2} \int_{\beta_2 \Delta_{Q'}} M_D(N_s u) \, d\sigma \leq \frac{1}{\varepsilon^2} M(M_D(N_s u))(x),
$$

which finishes the proof. \qed

**Remark 5.7.** We notice that the previous proposition holds also in the following form: If we have cubes $Q_1, Q_2, Q_3 \in \mathbb{D}_{Q_0}$, then

$$
\left\| \nabla \varphi \right\|_\infty \leq \frac{1}{\varepsilon^2} \min \left\{ \int_{\beta_2 \Delta_{Q_1}} M_D(N_s u) \, d\sigma, \int_{\beta_2 \Delta_{Q_2}} M_D(N_s u) \, d\sigma \right\}
$$

for some uniform constant $\beta_2$. Indeed, in the previous two proofs, we needed only the upper ADR estimates for the boundaries of $A_m$ and $V_k$ and these estimates remain valid if we remove a finite number of pieces whose boundaries satisfy an upper ADR estimate. By [HMM16, Proposition A.2], $\partial T_Q$ is ADR for every $Q \in D(E)$. Also, by the structure of the regions, these modified sets are “nice” enough to justify integration by parts that we used in the proofs.

### 5.3. From local to global.

Let us now construct the global approximating function. Although our construction is a little different than the construction in [HMM16, p. 2373], the basic ideas are the same.

#### 5.3.1. $E$ is a bounded set.

Let us first assume that $\text{diam}(E) < \infty$. In this case, we have a cube $Q_0 \in \mathbb{D}(E)$ such that $E = Q_0$ and $\ell(Q_0) \approx \text{diam}(E)$. We now set

$$
\varphi(X) := \begin{cases} 
\varphi_{Q_0}(X), & \text{if } X \in T_{Q_0} \\
\omega(X), & \text{if } X \in \Omega \setminus T_{Q_0}
\end{cases}
$$

where $\varphi_{Q_0}$ is the function constructed in Section 5.1. By part i) of Lemma 5.4, we have $N_s(u - \varphi)(x) \leq \varepsilon M_D(N_s u)(x)$ on $E$. As for the $\mathcal{C}_0$ bound, we first notice that if $Q \in \mathbb{D}_{Q_0}$, then by Proposition 5.6, we have

$$
\frac{1}{\sigma(Q)} \int_{T_Q} \left| \nabla \varphi \right| \leq \frac{1}{\varepsilon^2} M(M_D(N_s u))(x)
$$

(5.8)

for every $x \in Q$. Let us now fix a cube $E_k \in \mathbb{D}^*$ (recall the definition of $\mathbb{D}^*$ in Section 1.5) and modify the argument in [HMM16, p. 2353]. We denote $R := 2^k \text{diam}(E)$ and thus have $T_{E_k} = B(z_0, R)$. By a suitable choice of parameters in the construction of the Whitney regions in [HMM16], the Carleson box $T_{Q_0}$ is so large that we may fix a ball $B(z_0, r) \subset T_{Q_0}$ such that $r \geq 2\text{diam}(E)$. Because of this, we may fix a uniform constant $\alpha_1$ such that a small enlargement of $B(z_0, R) \setminus B(z_0, r)$ is contained in $\Gamma_{\alpha_1}(x)$ (recall the definition of $\Gamma_{\alpha_1}(x)$ in Section 1.6) for every
\( x \in E \). Thus, by Hölder’s inequality and Caccioppoli’s inequality, we get
\[
\int_{T_{E_k} \setminus T_{Q_0}} |\nabla \varphi| = \int_{B(z_0, R) \setminus B(z_0, r)} |\nabla u| \leq \int_{B(z_0, R) \setminus B(z_0, r)} |\nabla u| \leq \left( \int_{B(z_0, R) \setminus B(z_0, r)} |\nabla u|^2 \right)^{1/2} R^{(n+1)/2}
\]
\[
\leq \left( \sum_{0 \leq j \leq \log_2(R/r)} \int_{2^j r \leq |z - x| < 2^{j+1} r} |\nabla u(X)|^2 \right)^{1/2} R^{(n+1)/2}
\]
\[
\lesssim \inf_{E} N_{\alpha}^a u \left( \sum_{0 \leq j \leq \log_2(R/r)} (2^j)^{n-1} \right)^{1/2} R^{(n+1)/2}
\]
\[
\lesssim \inf_{E} N_{\alpha}^a u R^{(n-1)/2} R^{(n+1)/2}
\]
\[
\leq R^n M(N_{\alpha}^a u)(x)
\]
for every \( x \in Q_0 \). In particular, we get
\[
\frac{1}{R^n} \int_{T_{Q_0}} |\nabla \varphi| + \frac{1}{R^n} \int_{T_{E_k} \setminus T_{Q_0}} |\nabla \varphi| \leq \frac{1}{\varepsilon^2} M(M_0(N_{\alpha}^a u))(x) + M(N_{\alpha}^a u)(x)
\]
\[
\leq \frac{1}{\varepsilon^2} M(M_0(N_{\alpha}^a u))(x)
\]
for every a.e. \( x \in Q_0 \) and \( \alpha := \max \{\alpha_0, \alpha_1\} \). The a.e. bound for \( C_\mathbb{D}(\nabla \varphi)(x) \) follows now from the previous calculation and by handling the jumps across the boundary of \( T_{Q_0} \) as in the proof of Proposition 5.6.

5.3.2. \( E \) is an unbounded set. Suppose then that \( \text{diam}(E) = \infty \). We fix a sequence of cubes \( Q_i \in \mathbb{D}(E), i \in \mathbb{N} \), such that \( \bigcup_i Q_i = E \) and \( Q_i \subset Q_{i+1} \) and \( \ell(Q_i) < \gamma_0 \ell(Q_{i+1}) \) for every \( i \), where we fix the value of the constant \( \gamma_0 \) later. We set
\[
W_1 := T_{Q_1}, \quad W_k := T_{Q_k} \setminus T_{Q_{k-1}}
\]
and
\[
\varphi_k := 1_{W_k} \varphi_{Q_k}, \quad \varphi := \sum_k \varphi_k.
\]
The sets \( W_k \) cover the whole space \( \Omega \) and since \( T_{Q_i} \subset T_{Q_{i+1}} \) for every \( i \), they are also pairwise disjoint. Let us consider the pointwise bound for \( N_{\alpha}(u - \varphi) \). Fix \( x \in E \) and let \( Q_m \) be the smallest of the previously chosen cubes such that \( x \in Q_m \). Now, if \( \Gamma(x) \cap T_{Q_j} = \emptyset \) for every \( j = 1, 2, \ldots, m - 1 \), then the pointwise bound follows directly from part i) of Lemma 5.4. Suppose then that there exists a point \( Y \in \Gamma(x) \cap T_{Q_i} \) for some \( j < m \). We may assume that \( Y \notin T_{Q_i} \) for all \( i < j \). By the structure of the sets, there exist now cubes \( P_1 \subset Q_m \) and \( P_2 \subset Q_j \) such that \( \ell(P_1) \approx \ell(P_2) \), \( \text{dist}(P_1, P_2) \lesssim \ell(P_1) \), \( Y \in U_{P_1} \cap U_{P_2} \) and \( \varphi(Y) = \varphi|_{U_{P_2}}(Y) \). By the considerations in the proof of part i) of Lemma 5.4 we know that \( |u(Y) - \varphi(Y)| \leq \varepsilon M_0(N_{\alpha} u)(P_2) \). By the properties of \( P_1 \) and \( P_2 \), there exists a uniform constant \( \beta_0 \) such that \( P_1 \subset \beta_0 \Delta Q \) for any \( Q \in \mathbb{D}(E) \) such that \( Q \supset P_2 \). In particular,
\[
\varepsilon M_0(N_{\alpha} u)(P_2) = \varepsilon \sup_{Q \in \mathbb{D}(E), P_2 \subset Q} \frac{1}{\lambda\Delta Q} \int_Q N_{\alpha} u \, ds \lesssim \varepsilon \sup_{Q \in \mathbb{D}(E), P_2 \subset Q} \frac{1}{\lambda\Delta Q} \int_Q N_{\alpha} u \, ds \lesssim \varepsilon M(N_{\alpha} u)(x).
\]
Thus,
\[
N_{\alpha}(u - \varphi)(x) = \sup_{Y \in \Gamma(x)} |u(Y) - \varphi(Y)| = \sup_{k \in \mathbb{N}, Y \in \Gamma(x) \cap W_k} |u(Y) - \varphi(Y)| \lesssim \varepsilon M_0(N_{\alpha} u)(x).
\]

Let us then prove the \( C_\mathbb{D} \) estimate. We fix a point \( x \in E \) and a cube \( Q \in \mathbb{D}(E) \) such that \( x \in Q \) and split the proof to three different cases. Below, \( \beta \) and \( \alpha \) are uniform constants and \( m \) is the
smallest such number that $T_Q \subset T_{Q_m}$, In cases 2) and 3), we omit the full technical details since most of them are just a repetition of the arguments in the proof of Proposition 5.6.

1) $T_Q \subset T_{Q_m}$ such that $T_Q \cap T_{Q_k} = \emptyset$ for every $k < m$. Now we simply have

$$
\int_{T_Q} |\nabla \varphi| = \int_{T_Q} |\nabla \varphi| \lesssim \frac{1}{\varepsilon^2} \int_{\beta \Delta_Q} M_D(N_0 u) \, d\sigma
$$

by Proposition 5.6.

2) $T_Q \subset T_{Q_m}$ and $Q_k \subset Q$ for every $k < m$. Now

$$
\int_{T_Q \setminus T_{Q_{m-1}}} |\nabla \varphi| + \sum_{i=1}^{m-2} \int_{T_{Q_{m-1}} \setminus T_{Q_{m-(i+1)}}} |\nabla \varphi| + \int_{T_{Q_m} \setminus T_{Q_{1}}} |\nabla \varphi|
\lesssim \frac{1}{\varepsilon^2} \int_{\beta \Delta_Q} M_D(N_0 u) \, d\sigma + \sum_{i=1}^{k-1} \frac{1}{\varepsilon^2} \int_{\beta \Delta_{Q_i}} M_D(N_0 u) \, d\sigma
$$

by Remark 5.7. We note that the balls $\beta \Delta_{Q_i}$ form an increasing sequence with respect to inclusion. If we choose the constant $\gamma_0$ to be large enough, the balls $\beta \Delta_{Q_i}$ satisfy a Carleson packing condition. Thus, for a large enough $\gamma_0$, we get

$$
\frac{1}{\varepsilon^2} \int_{\beta \Delta_Q} M_D(N_0 u) \, d\sigma + \sum_{i=1}^{k-1} \frac{1}{\varepsilon^2} \int_{\beta \Delta_{Q_i}} M_D(N_0 u) \, d\sigma \lesssim \frac{1}{\varepsilon^2} \int_{\beta \Delta_Q} M_D(M_D(N_0 u)) \, d\sigma
$$

by the discrete Carleson embedding theorem (Theorem [A.1]).

3) $T_Q \subset T_{Q_m}$, $Q_k \subset Q$ for every $k < m$ and $T_Q \cap T_{Q_{m-1}} \neq \emptyset$. Without loss of generality, we may assume that $\ell(Q) \approx \ell(Q_{m-1})$. We get

$$
\int_{T_Q \setminus T_{Q_{m-1}}} |\nabla \varphi| + \sum_{i=1}^{m-2} \int_{(T_Q \setminus T_{Q_{m-1}}) \setminus T_{Q_{m-(i+1)}}} |\nabla \varphi| + \int_{T_Q \setminus T_{Q_{1}}} |\nabla \varphi|
\lesssim \frac{1}{\varepsilon^2} \int_{\beta \Delta_Q} M_D(N_0 u) \, d\sigma + \sum_{i=1}^{k-1} \frac{1}{\varepsilon^2} \int_{\beta \Delta_{Q_i}} M_D(N_0 u) \, d\sigma
$$

by Remark 5.7. Again, if we choose the constant $\gamma_0$ to be large enough, we get

$$
\frac{1}{\varepsilon^2} \int_{\beta \Delta_Q} M_D(N_0 u) \, d\sigma + \sum_{i=1}^{k-1} \frac{1}{\varepsilon^2} \int_{\beta \Delta_{Q_i}} M_D(N_0 u) \, d\sigma \lesssim \frac{1}{\varepsilon^2} \int_{\beta \Delta_Q} M_D(M_D(N_0 u)) \, d\sigma
$$

by the discrete Carleson embedding theorem (Theorem [A.1]).

Thus, we have

$$
\frac{1}{\sigma(Q)} \int_{T_Q} |\nabla \varphi| \lesssim \frac{1}{\varepsilon^2} \frac{1}{\sigma(Q)} \int_{\beta \Delta_Q} M_D(M_D(N_0 u)) \, d\sigma \lesssim \frac{1}{\varepsilon^2} M(D(M_D(N_0 u))(x),
$$

which proves the desired a.e. pointwise $C_D$ bound.

5.3.3. Combining the results. We can now combine all the previous results. Since our global approximating function $\varphi$ satisfies the a.e. pointwise bounds

$$
N_*(u - \varphi)(x) \leq \varepsilon M_D(N_0 u)(x) \quad \text{and} \quad C_D(\nabla \varphi)(x) \lesssim \frac{1}{\varepsilon^2} M(D(M_D(N_0 u))(x)
$$

for some uniform constant $\alpha$, by the $L^p$-boundedness of the Hardy-Littlewood maximal operator, we get

$$
\|N_*(u - \varphi)\|_{L^p(E)} \lesssim \varepsilon \|M_D(N_0 u)\|_{L^p(E)} \leq C_p \varepsilon \|N_0 u\|_{L^p(E)}
$$

and

$$
\|C(\nabla \varphi)\|_{L^p(E)} \lesssim \|C_D(\nabla \varphi)\|_{L^p(E)} \lesssim \frac{1}{\varepsilon^2} M(D(M_D(N_0 u)))(x) \lesssim \frac{C_p}{\varepsilon^2} \|N_0 u\|_{L^p(E)} \lesssim \frac{C_p}{\varepsilon^2} \|N_0 u\|_{L^p(E)}.
$$
This finishes the proof of Theorem 1.2.

6. Converse result for bounded sets

We do not know if \( \varepsilon \)-approximability in \( L^p \) is enough to characterize uniform rectifiability in general cases for a fixed \( p \) or all \( p \) nor do we know what is the optimal dependency on \( p \) in Theorem 1.2. However, if the set \( E \) is bounded and the bounds depend on \( p \) as in Theorem 1.2, then we can use simple limiting arguments to see that every bounded harmonic function is \( \varepsilon \)-approximable in \( L^{\infty} \) (see Definition 1.1). This implies uniform rectifiability by [GMT16, Theorem 1.1].

Proposition 6.1. Suppose that \( E \subset \mathbb{R}^{n+1} \) is an \( n \)-ADR set such that \( \sigma(E) < \infty \) and \( \Omega := \mathbb{R}^{n+1} \setminus E \) satisfies the corkscrew condition. Then the following two conditions are equivalent:

1) The set \( E \) is uniformly rectifiable.
2) Every harmonic function \( u \) in \( \Omega = \mathbb{R}^{n+1} \setminus E \) is \( \varepsilon \)-approximable in \( L^p(\Omega) \) in the sense of Theorem 1.2 for all \( p \in (1, \infty) \).

Proof. Suppose that 2) holds. We recall the well-known bound

\[
\|Mf\|_{L^p \to L^p} \leq p',
\]

where \( p' = \frac{p}{p-1} \). Since \( \sigma \) is a doubling measure, we may dominate the maximal function \( Mf \) by a bounded number of dyadic maximal functions (see [HK12, Proposition 7.9] or [HT14, Theorem 5.9]), i.e.

\[
Mf(x) \lesssim \sum_{\alpha=1}^{K} M_{D^{\alpha}} f(x).
\]

Thus, we also have \( \|M\|_{L^p \to L^p} \lesssim p' \). In particular, we get

\[
i) \lim_{p \to \infty} \|M\|_{L^p \to L^p} \lesssim \lim_{p \to \infty} \|M_0\|_{L^p \to L^p} \lesssim \lim_{p \to \infty} p' = 1,
\]

\[
ii) \lim_{p \to \infty} \|C(\nabla \varphi)\|_{L^p(E)} = \|C(\nabla \varphi)\|_{L^{\infty}(E)} \approx \|\sup_{x \in E, r>0} \frac{1}{r} \int_{B(x,r) \setminus E} |\nabla \varphi(Y)| \, dY,\]

\[
iii) \lim_{p \to \infty} \|N_{\ast} f\|_{L^p(E)} = \|N_{\ast} f\|_{L^{\infty}(E)} \approx \|f\|_{L^{\infty}(\Omega)}.
\]

Hence, the \( \varepsilon \)-approximability in \( L^{\infty}(\Omega) \) for bounded harmonic functions in \( \Omega \) follows. \qed

Appendix A. Discrete Carleson embedding theorem

For the convenience of the reader, we prove here the version of the Carleson embedding theorem that we used several times in the proofs.

Theorem A.1. Suppose that \( \mu \) is a Borel measure and \( \mathbb{D} \) is a dyadic system in a (quasi)metric space \( X \). Let \( f \geq 0 \) be a locally integrable function. If \( \mathcal{A} \subset \mathbb{D} \) is a collection that satisfies a Carleson packing condition with a constant \( \Lambda \geq 1 \), then

\[
\sum_{Q \in \mathbb{D}, Q \subset Q_0} \int_Q f \, d\mu \leq \Lambda \int_{Q_0} M_{\mathbb{D}} f \, d\mu
\]

for any \( Q_0 \in \mathbb{D} \).

Proof. For every \( m \in \mathbb{Z} \), we define the averaging operator \( T_m \) by setting

\[
T_m f(x) = \sum_{Q' \in \mathbb{D}, \ell(Q') = 2^{-m}} 1_{Q'}(x) \int_{Q'} f \, d\mu,
\]

and we define the measure \( \nu \) by setting

\[
d\nu(x, m) = \left( \sum_{Q' \in \mathcal{A}, \ell(Q') = 2^{-m}} 1_{Q'}(x) \right) d\mu(x).
\]
Now we have
\[
\sum_{Q \in \mathcal{D}, Q \subseteq Q_0} \int_Q f \, d\mu = \sum_{Q \in \mathcal{D}, Q \subseteq Q_0} \mu(Q) \int_Q f \, d\mu
\]
\[
= \sum_{m: 2^{-m} \leq \ell(Q_0)} \sum_{Q' \in A, \ell(Q') = 2^{-m}} \int_{Q_0} 1_{Q'} \left( \int_{Q'} f \right) \, d\mu
\]
\[
= \sum_{m: 2^{-m} \leq \ell(Q_0)} \int_{Q_0} T_m f(x) \, d\nu(x, m)
\]
\[
= \int_0^\infty \nu(E_\lambda^*) \, d\lambda,
\]
where \( E_\lambda^* := \{ (x, m) : x \in Q_0, 2^{-m} \leq \ell(Q_0), T_m f(x) > \lambda \} \). Thus, to prove the claim, we only need to show that \( \nu(E_\lambda^*) \leq \Lambda \mu(E_\lambda) \), where \( E_\lambda := \{ x \in Q_0 : \sup_m T_m f(x) > \lambda \} \).

We notice that if \( x \in E_\lambda \), then there exists a subcube \( Q' \subset Q_0 \) such that \( x \in Q' \) and \( f_{Q'} f \, d\mu > \lambda \). By the definition of \( T_m \), we also have \( y \in E_\lambda \) for every \( y \in Q' \). In particular, we have maximal disjoint subcubes \( R_j \subset Q_0 \) such that \( E_\lambda = \bigcup_j R_j \). We further observe the following two things:

- If \( x \in Q_0 \setminus \bigcup_j R_j \), then by the maximality of the cubes \( R_j \) we have \( \sup_m T_m f(x) \leq \lambda \).
- If \( x \in Q \subset Q_0 \) and \( T_m f(x) > \lambda \) for some \( m \) such that \( 2^{-m} \geq \ell(Q) \), then there exists a cube \( \tilde{Q} \supseteq Q \) such that \( f_{\tilde{Q}} f \, d\mu > \lambda \).

Based on these observations, we have
\[
E_\lambda^* \subset \bigcup_j R_j \times \{ m : 2^{-m} \leq \ell(R_j) \}.
\]

By the Carleson packing condition, we get
\[
\nu(R_j \times \{ m : 2^{-m} \leq \ell(R_j) \}) = \sum_{m: 2^{-m} \leq \ell(R_j)} \sum_{Q' \subset R_j, Q' \in A, \ell(Q') = 2^{-m}} \mu(Q') \leq \Lambda \mu(R_j)
\]
for every \( j \). In particular, since the cubes \( R_j \) are disjoint, we get
\[
\nu(E_\lambda^*) \leq \sum_j \nu(R_j \times \{ m : 2^{-m} \leq \ell(R_j) \}) \leq \sum_j \Lambda \mu(P_j) = \Lambda \mu(E_\lambda),
\]
which completes the proof. \( \square \)

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