Regularized Renormalization Group Reduction of Symplectic Maps

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Abstract

By means of the perturbative renormalization group method, we study a long-time behaviour of some symplectic discrete maps near elliptic and hyperbolic fixed points. It is shown that a naive renormalization group (RG) map breaks the symplectic symmetry and fails to describe a long-time behaviour. In order to preserve the symplectic symmetry, we present a regularization procedure, which gives a regularized symplectic RG map describing an approximate long-time behaviour successfully.

1 Introduction

There has been a long history to study an asymptotic solution of Hamiltonian flows by means of singular perturbation methods such as the averaging method and the method of multiple time-scales. A Hamiltonian flow can be reduced to a symplectic discrete map called the Poincare map, which has the lower dimension than the original flow and is, therefore, extensively studied. However, neither the averaging method nor the method of multiple time-scales may be immediately applicable to symplectic maps. The perturbative renormalization group (RG) method developed recently may be a useful tool to tackle asymptotic behaviours of discrete maps as well as flows. The original RG method is an asymptotic singular perturbation technique developed for differential equations [1]. Secular or divergent terms of perturbation solutions

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of differential equations are removed by renormalizing integral constants of the lowest order solution. The RG method is reformulated on the basis of a naive renormalization transformation and the Lie group [2]. This reformulated RG method based on the Lie group is easy to apply to discrete systems, by which asymptotic expansions of unstable manifolds of some chaotic discrete systems are obtained [3]. The extension of the RG method to discrete symplectic systems is not trivial because the symplectic structure is easily broken in naive renormalization group equations (maps) as shown in this paper, while the application of the RG method to Hamiltonian flows does not cause such a problem as the broken symplectic symmetry except a special case [4]. The application of the RG method to some non-symplectic discrete systems was tried in the framework of the envelope method [5]. However, the method, if applied to a symplectic map, would give only a naive RG map which breaks the symplectic symmetry.

The main purpose of the present paper is to present an RG procedure to preserve the symplectic structure in RG maps and to obtain symplectic RG maps. In this paper, this procedure is called a regularized RG procedure, which consists of the following two steps. First, using the reformulated RG method [2], we get a naive RG map near elliptic and hyperbolic fixed points of some symplectic discrete systems. The naive map preserves the symplectic symmetry only approximately and fails to describe a long-time behavior of the original maps. Second, in order to recover the symplectic symmetry, we introduce a process of "exponentiation" which yields a symplectic RG map. This process and a symplectic RG map thus obtained are called, respectively, regularization of an RG map and a regularized RG map.

In section 2, a long-time behavior of a simple linear map is analyzed in order to elucidate the broken symplectic symmetry in a naive RG map and our regularization process. In section 3, regularized (symplectic) RG maps are obtained near elliptic and hyperbolic fixed points of a two-dimensional nonlinear symplectic map. In section 4, the regularized RG procedure is applied to a four-dimensional symplectic map, of which elliptic-elliptic fixed point has an irrational frequency ratio.
2 Linear symplectic map

It may be instructive to analyze a linear symplectic map, which is exactly solvable, \( R^2 \ni (x_n, y_n) \mapsto (x_{n+1}, y_{n+1}) : \)

\[
\begin{align*}
x_{n+1} &= x_n + y_{n+1}, \\
y_{n+1} &= y_n - Jx_n + 2\epsilon Jx_n,
\end{align*}
\]

(1)

where \( n \in \mathbb{Z} \) and \( \epsilon \) is a small parameter. The map (1) has an elliptic fixed point at the origin \((0,0)\) for \( 0 < J < 2 \). Eliminating \( y \) from (1), we obtain a second order difference equation:

\[
Lx_n \equiv x_{n+1} - 2\cos(\theta)x_n + x_{n-1} = \epsilon 2Jx_n,
\]

(2)

where \( \cos \theta = 1 - J/2 \) and \( 0 < J < 2 \) is assumed. The linear map (2) has the following exact solution \( x_n^E \):

\[
x_n^E = A \exp \left( i \arccos(\cos \theta + \epsilon J) n \right) + c.c.,
\]

(3)

where \( A(\in \mathbb{C}) \) is a complex “integral” constant and c.c. stands for the complex conjugate to the preceding terms.

Let us derive an asymptotic solution of the map (1) for small \( \epsilon \) by means of the RG method. Substituting the expansion:

\[
x_n = x_n^{(0)} + \epsilon x_n^{(1)} + \epsilon^2 x_n^{(2)} + \cdots
\]

(4)

into Eq.(2), we have

\[
Lx_n^{(0)} = 0, \quad Lx_n^{(1)} = 2Jx_n^{(0)}, \quad Lx_n^{(2)} = 2Jx_n^{(1)} \cdots
\]

and

\[
\begin{align*}
x_n^{(0)} &= A \exp(i\theta n) + c.c., \\
x_n^{(1)} &= \frac{JA}{i \sin \theta} n \exp(i\theta n) + c.c., \\
x_n^{(2)} &= -\frac{J^2 A}{2 \sin^2 \theta} \left( n^2 + i \frac{\cos \theta}{\sin \theta} n \right) \exp(i\theta n) + c.c.,
\end{align*}
\]

3
where $A(\in \mathbb{C})$ is a complex constant. To remove secular terms ($\propto n, n^2$), we introduce a renormalization transformation $A \mapsto \tilde{A}(n)$ [2]:

$$
\tilde{A}(n) \equiv A + \epsilon \frac{J A}{\sin \theta} n + \epsilon^2 \frac{J^2 A}{2 \sin^2 \theta} \left( n^2 + i \frac{\cos \theta}{\sin \theta} n \right) + \mathcal{O}(\epsilon^3),
$$

(5)

A discrete version of the RG equation is just the first order difference equation of $\tilde{A}(n)$, whose local solution is given by Eq. (5). From Eq. (5), we have

$$
\tilde{A}(n+1) - \tilde{A}(n) = \left(-i \frac{J}{\sin \theta} - \epsilon^2 \frac{J^2}{2 \sin^2 \theta} \left( 2n + 1 + i \frac{\cos \theta}{\sin \theta} \right) \right) A + \mathcal{O}(\epsilon^3),
$$

(6)

where $A$ should be expressed in terms of $\tilde{A}(n)$. This is done by taking the inversion of the renormalization transformation (5) iteratively.

$$
A = \left( 1 + i \frac{J n}{\sin \theta} + \mathcal{O}(\epsilon^2) \right) \tilde{A}(n).
$$

(7)

Substituting (7) into (6), we obtain the following RG equation (RG map) up to $\mathcal{O}(\epsilon^2)$.

$$
\tilde{A}(n+1) = \left(1 + \frac{-i \epsilon J}{\sin \theta} + \frac{1}{2!} \left( \frac{-i \epsilon J}{\sin \theta} \right)^2 - i \epsilon^2 \frac{J^2 \cos \theta}{2 \sin^2 \theta} \right) \tilde{A}(n) + \mathcal{O}(\epsilon^3),
$$

(8)

of which solution is

$$
\tilde{A}(n) = \left(1 + \frac{-i \epsilon J}{\sin \theta} + \frac{1}{2!} \left( \frac{-i \epsilon J}{\sin \theta} \right)^2 - i \epsilon^2 \frac{J^2 \cos \theta}{2 \sin^2 \theta} + \mathcal{O}(\epsilon^3) \right)^n \tilde{A}(0).
$$

(9)

On the other hand, from Eq. (3), we have $\tilde{A}(n)$ exactly as

$$
\tilde{A}(n) = \tilde{A}(0) \exp \left[ i \left( \epsilon \frac{-J}{\sin \theta} - \epsilon^2 \frac{\cos \theta}{2 \sin \theta} \left( \frac{J}{\sin \theta} \right)^2 + \cdots \right) n \right],
$$

(10)

Notice that $|\tilde{A}|^2$ is an exact constant of motion while it is merely an approximate conserved quantity of the (truncated) RG map (8). The symplectic structure is also not exactly preserved in the RG map, that is, for the truncated RG map (8) up to $\mathcal{O}(\epsilon^k)$, we have

$$
d\tilde{A}(n+1) \wedge d\tilde{A}^*(n+1) - d\tilde{A}(n) \wedge d\tilde{A}^*(n) = \mathcal{O}(\epsilon^{k+1}) \neq 0,
$$
where \( k = 1, 2, \ldots \). \( \tilde{A}^* \) is complex conjugate to \( \tilde{A} \) and should also be a canonical conjugate to \( \tilde{A} \) [3]. Although this fault of the RG map vanishes in the limit of small \( \epsilon \), it is intolerable as shown in Fig. 1 where we depict a long time behaviour of the solution for small but finite \( \epsilon \). In fact, the truncated RG map becomes a system with small but finite dissipation. In order to remedy a fault like this, we take advantage of a crucial observation that the coefficient of \( \tilde{A}(n) \) in (8) can be modified as

\[
1 + \frac{-i\epsilon J}{\sin \theta} + \frac{1}{2!} \left( \frac{-i\epsilon J}{\sin \theta} \right)^2 - i\epsilon^2 \frac{J^2 \cos \theta}{2 \sin^3 \theta}.
\]

Using this modified coefficient, the symplectic symmetry in the truncated RG map is recovered as

\[
\tilde{A}(n+1) = \exp \left( \frac{-i\epsilon J}{\sin \theta} - i\epsilon^2 \frac{J^2 \cos \theta}{2 \sin^3 \theta} \right) \tilde{A}(n),
\]

which also has an exact conserved quantity \( |\tilde{A}| \). This process is nothing but “exponentiation” of the coefficient of \( \tilde{A}(n) \) and is called regularization of an RG map. It is easy to see that the solution of the regularized RG map (11) coincides with the asymptotic expansion of the exact solution (10) and describes a long time behaviour up to \( n \sim \mathcal{O}(\epsilon^{-2}) \) in the present approximation. In Fig. (1), trajectories constructed from the naive RG map (8) and the regularized RG map (11) are depicted to be compared to an “exact” trajectory of the original map (1).
Figure 1: Trajectories constructed from the naive RG map (broken line), the regularized RG map (solid line) and an “exact” trajectory of the original map (crosses) for $J = 0.2$, $\epsilon = 0.25$ and an initial condition $(x_0 = 1.0, y_0 = 0.0)$. 
3 Two-dimensional Non-linear Symplectic Map

3.1 Elliptic Fixed Point

Let us analyze a weakly non-linear symplectic map $\mathbb{R}^2 \ni (x_n, y_n) \mapsto (x_{n+1}, y_{n+1})$:

\begin{align*}
x_{n+1} &= x_n + y_{n+1},
y_{n+1} &= y_n - Jx_n + 2\epsilon Jx_n^3,
\end{align*}

or

\begin{equation}
Lx_n = \epsilon 2Jx_n^3,
\end{equation}

where $\epsilon$ is a small parameter, $L$ is defined in (2) and $0 < J < 2$ is assumed so that Eq.(13) has an elliptic fixed point at the origin $(0,0)$. Expanding $x_n$ as a power series of $\epsilon$

\begin{equation}
x_n = x_n^{(0)} + \epsilon x_n^{(1)} + \epsilon^2 x_n^{(2)} + \cdots,
\end{equation}

we have

\begin{align*}
Lx_n^{(0)} &= 0, \quad Lx_n^{(1)} = 2Jx_n^3, \quad Lx_n^{(2)} = 6Jx_n^{(0)}x_n^{(1)}, \\
&\vdots
\end{align*}

and solutions of the perturbed equations to $O(\epsilon^2)$ are given by

\begin{align*}
x_n^{(0)} &= A \exp(i\theta n) + \text{c.c.}, \\
x_n^{(1)} &= \alpha_1 A^3 \exp(3i\theta n) + i\alpha_1 R |A|^2 A_n \exp(i\theta n) + \text{c.c.}, \\
x_n^{(2)} &= i\alpha_1 \alpha_1 R |A|^4 A_n \exp(i\theta n) - \frac{\alpha_1^2 R |A|^4 A}{2} \left( n^2 + i\frac{\cos \theta}{\sin \theta} n \right) \exp(i\theta n) \\
&\quad + 3i\alpha_1 \alpha_1 R |A|^2 A^3 \left( n - i\frac{\sin(3\theta)}{\cos 3\theta - \cos \theta} \right) \exp(3i\theta n) \\
&\quad + 6\alpha_1^2 |A|^2 A^3 \exp(3i\theta n) + \frac{3J\alpha_1 A^5}{\cos 5\theta - \cos \theta} \exp(5i\theta) + \text{c.c.},
\end{align*}

where

\begin{align*}
\alpha_1 &\equiv \frac{J}{\cos 3\theta - \cos \theta}, \quad \alpha_1 R \equiv -\frac{3J}{\sin \theta},
\end{align*}

and $A(\in \mathbb{C})$ is a complex integral constant. In order to remove secular terms in the coefficient of the fundamental harmonic ($\exp(i\theta n)$), we introduce a
renormalization transformation $A \mapsto \tilde{A}(n)$:

$$\tilde{A}(n) = A + \epsilon (i\alpha_1R|A|^2An)$$

$$+ \epsilon^2 \left( \frac{-\alpha_1^2R}{2} |A|^4 A \left( n^2 + i\frac{\cos \theta n}{\sin \theta} \right) + i\alpha_1 \alpha_1 R |A|^4 A \right). \quad (15)$$

Following the same procedure as that in the preceding section, we derive a naive RG map from Eq.(15):

$$\tilde{A}(n+1) - \tilde{A}(n) = \epsilon i\alpha_1 R |\tilde{A}(n)|^2 \tilde{A}(n)$$

$$+ \epsilon^2 \left( \frac{-\alpha_1^2R}{2} |\tilde{A}(n)|^4 \tilde{A}(n) \left( 1 + i\frac{\cos \theta}{\sin \theta} \right) \right)$$

$$+ i\alpha_1 \alpha_1 R |\tilde{A}(n)|^4 \tilde{A}(n) \right), \quad (16)$$

which breaks the symplectic symmetry and does not have a constant of motion. To recover the symplectic symmetry, we regularize the naive RG map (16) by “exponentiating” the coefficient of $\tilde{A}(n)$, that is,

$$\tilde{A}(n+1) = \tilde{A}(n) \exp \left( i\epsilon |\tilde{A}(n)|^2 \alpha_1 R \right)$$

$$+ i\epsilon^2 |\tilde{A}(n)|^4 \left( \frac{-\cos \theta}{2 \sin \theta} \alpha_1^2 R + \alpha_1 \alpha_1 R \right), \quad (17)$$

which of course agrees with Eq.(16) to $O(\epsilon^2)$. It is easy to see that Eq.(17) has the symplectic symmetry

$$d\tilde{A}(n+1) \wedge d\tilde{A}^*(n+1) = d\tilde{A}(n) \wedge d\tilde{A}^*(n),$$

and a constant of motion $|\tilde{A}|$. Introducing the polar expression $\tilde{A}(n) = |\tilde{A}(n)| \exp(i\tilde{\phi}(n))$, the regularized RG map is reduced to a simple phase equation given by

$$\tilde{\phi}(n+1) = \tilde{\phi}(n) + \left( \epsilon |\tilde{A}(0)|^2 \alpha_1 R \right.$$

$$+ i\epsilon^2 |\tilde{A}(0)|^4 \left( \frac{-\cos \theta}{2 \sin \theta} \alpha_1^2 R + \alpha_1 \alpha_1 R \right), \quad (18)$$

Thus, the regularized RG map (17) is exactly solvable in contrast to the naive RG map (16) which do not have a conserved quantity. It is noted that
secular coefficients of the third harmonic \((\exp(3i\theta n))\) are also summed up to give a renormalized coefficient \(\tilde{A}_3(n)\) as:

\[
\epsilon \tilde{A}_3(n) \equiv \epsilon \alpha_1 A^3 + 
\epsilon^2 \left( 3i\alpha_1 \alpha_1 R |A|^2 A^3 \left( n - \frac{i \sin 3\theta}{\cos 3\theta - \cos \theta} \right) + 6\alpha_1^2 |A|^2 A^3 \right),
\]

Substituting an iterative expression of \(A\) in terms of \(\tilde{A}(n)\) obtained from Eq.(15) into Eq.(19), we can eliminate secular terms in the same way as the case of differential equations \[2\].

As an application of the result, we obtain an approximate but analytical expression of the rotation number near the elliptic fixed point. The definition of the rotation number is

\[
\rho(x_0, y_0) \equiv \lim_{N \to \infty} \frac{1}{2\pi N} \sum_{n=0}^{N} \phi_n,
\]

where \(\phi_n\) is an angle between the vectors \((x_n, y_n)\) and \((x_{n+1}, y_{n+1})\) \[\text{[4]}\]. Neglecting effects of the higher harmonics, an approximate expression of \(\rho(x_0, y_0)\) of the present system \[\text{(12)}\] is given as following:

\[
2\pi \rho(x_0, y_0) \approx \theta + \left( \epsilon |\tilde{A}(0)|^2 \alpha_1 R + \epsilon^2 |\tilde{A}(0)|^4 \left( \frac{-\cos \theta}{2\sin \theta} \alpha_1^2 \alpha_1 R + \alpha_1 \alpha_1 R \right) \right) + \tilde{\phi}(0),
\]

where initial values \((x_0, y_0)\) are related to \(|\tilde{A}(0)|\) and \(\tilde{\phi}(0)\). In Fig.(3), the rotation number \(\rho(x_0, y_0)\) given by \[\text{(20)}\] is depicted for initial values on the half line \(0 < x_0 < 0.8, \quad y_0 = 0\) shown in Fig.(2). The result agrees well with the “exact” rotation number obtained by numerical calculations of the original map \[\text{(12)}\] until a new resonance appears near \(x_0 \simeq 0.6\).
Figure 2: A phase portrait of the map (12) for $J = 1.2$, $\epsilon = 0.25$. The solid half line represents initial phase points $0 < x_0 < 0.8$, $y_0 = 0$ of the rotation number $\rho(x_0, y_0)$.
Figure 3: The rotation number $\rho(x_0, 0)$ versus initial phase points $(x_0, 0)$ for $J = 1.2$, $\epsilon = 0.25$. The solid curve represents the rotation number given by the regularized RG map (20), while the dashed curve is obtained by numerical calculations of the original map (12).
3.2 Hyperbolic Fixed Point

In the case \( J < 0 \) in (12), the origin (0,0) is a hyperbolic fixed point of (12) and Eq.(13) becomes

\[
L x_n = \epsilon J x_n^3,
\]

where

\[
L x_n \equiv x_{n+1} - 2 \cosh(\theta) x_n + x_{n-1}, \quad \cosh \theta = 1 - J/2 \quad (J < 0),
\]

and \( K \equiv \exp(\theta) \) is one of the eigenvalues of the linearized equation at the origin, that is

\[
K = \frac{1}{2} \left( 2 - J + \sqrt{-4J + J^2} \right), \quad \text{or} \quad K = \frac{1}{2} \left( 2 - J - \sqrt{-4J + J^2} \right).
\]

Expanding the \( x_n \) as (14), we get the following perturbed solution

\[
x_n^{(0)} = A_+ K^n + A_- K^{-n},
\]

\[
x_n^{(1)} = 2J \left( \frac{A_+^3 K^{3n}}{D(K)} + \frac{A_-^3 K^{-3n}}{D(K)} + \frac{3 A_+^2 A_-}{K - K^{-1}} n K^n + \frac{3 A_+ A_-^2}{K - K^{-1}} n K^{-n} \right),
\]

\[
x_n^{(2)} = 12 J^2 \left( \frac{3 A_+^3 A_-^2}{2(K - K^{-1})^2} \left( n^2 - a(K)n \right) + \frac{A_+^3 A_-^2}{(K - K^{-1}) D(K)} \right) K^n
\]

\[+12 J^2 \left( \frac{3 A_-^3 A_+^2}{2(K - K^{-1})^2} \left( n^2 + a(K)n \right) + \frac{A_-^3 A_+^2}{(K - K^{-1}) D(K)} \right) K^{-n} + \text{n.r.},
\]

where

\[
D(K) \equiv K^3 - (K + K^{-1}) + K^{-3} = D(K^{-1}) \in \mathbb{R},
\]

\[
a(K) \equiv \frac{K + K^{-1}}{K - K^{-1}} = -a(K^{-1}) \in \mathbb{R}.
\]

\( A_+ \) and \( A_- \) are real “integral” constants and n.r. denotes non-resonant terms which are proportional to \( K^{3n}, K^{-3n}, K^{5n}, K^{-5n} \). Introducing a renormalization transformation \( A_+ \mapsto \tilde{A}_+(n) \) and \( A_- \mapsto \tilde{A}_-(n) \) to remove secular terms in the coefficient of \( K^n \), we have a set of naive RG equations:

\[
\tilde{A}_+(n + 1) = \tilde{A}_+(n) - 2 \epsilon J \frac{3 \tilde{A}_+^3(n) \tilde{A}_-(n)}{K - K^{-1}}
\]

\[+12 \epsilon^2 J^2 \left( \frac{3 \tilde{A}_+^3(n) \tilde{A}_-^2(n)}{2(K - K^{-1})^2} \left( 1 - a(K) \right) \right) + \text{n.r.}
\]

\[12\]
\[ \begin{align*}
\tilde{A}_+(n+1) &= \tilde{A}_-(n) - 2\epsilon J \frac{3\tilde{A}_2(n)\tilde{A}_+^2(n)}{K^{-1}-K} \\
&\quad + 12\epsilon^2 J^2 \left( \frac{3\tilde{A}_3(n)\tilde{A}_+^2(n)}{2(K^{-1}-K)^2} \left( 1 + a(K) \right) \\
&\quad + \frac{\tilde{A}_3(n)\tilde{A}_+^2(n)}{(K^{-1}-K)D(K)} \right),
\end{align*} \tag{22} \]

\[ \begin{align*}
\tilde{A}_-(n+1) &= \tilde{A}_-(n) - 2\epsilon J \frac{3\tilde{A}_2(n)\tilde{A}_+^2(n)}{K^{-1}-K} \\
&\quad + 12\epsilon^2 J^2 \left( \frac{3\tilde{A}_3(n)\tilde{A}_+^2(n)}{2(K^{-1}-K)^2} \left( 1 + a(K) \right) \\
&\quad + \frac{\tilde{A}_3(n)\tilde{A}_+^2(n)}{(K^{-1}-K)D(K)} \right).
\end{align*} \tag{23} \]

The set of Eqs. (22) and (23) do not have the symplectic symmetry but is regularized by the following exponentiation procedure.

\[ \begin{align*}
\tilde{A}_+(n+1) &= \tilde{A}_+(n) \exp \left\{ \epsilon \frac{6J\tilde{A}_+(n)\tilde{A}_-(n)}{K^{-1}-K} \\
&\quad + \epsilon^2 12J^2 \left( -a(K) \frac{3\tilde{A}_2(n)\tilde{A}_-^2(n)}{2(K^{-1}-K)^2} + \frac{\tilde{A}_+(n)\tilde{A}_-(n)}{(K^{-1}-K)D(K)} \right) \right\}, \tag{24} \\
\tilde{A}_-(n+1) &= \tilde{A}_-(n) \exp \left\{ \epsilon \frac{6J\tilde{A}_-(n)\tilde{A}_+(n)}{K^{-1}-K} \\
&\quad + \epsilon^2 12J^2 \left( +a(K) \frac{3\tilde{A}_2(n)\tilde{A}_+^2(n)}{2(K^{-1}-K)^2} + \frac{\tilde{A}_-(n)\tilde{A}_+(n)}{(K^{-1}-K)D(K)} \right) \right\}. \tag{25} \\
\end{align*} \]

This regularized RG map has a constant of motion

\[ P \equiv \tilde{A}_+(n)\tilde{A}_-(n) = \tilde{A}_+(0)\tilde{A}_-(0), \]

and is also exactly solvable. In terms of \( P \), a general solution of Eqs. (24) and (25) is given as

\[ \begin{align*}
\tilde{A}_+(n) &= \tilde{A}_+(0) \exp \left( Q(P;\epsilon)n \right), \\
\tilde{A}_-(n) &= \tilde{A}_-(0) \exp \left( -Q(P;\epsilon)n \right),
\end{align*} \]

where \( Q(P;\epsilon) \) is a polynomial of \( P \):

\[ \begin{align*}
Q(P;\epsilon) &\equiv \epsilon \frac{6JP}{K^{-1}-K} \\
&\quad + \epsilon^2 12J^2 \left( -\frac{3a(K)P^2}{2(K^{-1}-K)^2} + \frac{P}{(K^{-1}-K)D(K)} \right).
\end{align*} \]
4 Four-dimensional Symplectic Map

Let us consider a coupled map of two symplectic maps (12)
\[ R^4 \ni (x, y, x', y') \mapsto (x_{n+1}, y_{n+1}, x'_{n+1}, y'_{n+1}) : \]
\[
x_{n+1} = \frac{\partial F}{\partial y_n}, \quad y_n = \frac{\partial F}{\partial x_n},
\]
\[
x'_{n+1} = \frac{\partial F}{\partial y'_{n+1}}, \quad y'_{n+1} = \frac{\partial F}{\partial x'_{n+1}};
\]
where \( F \) is a generating function,
\[
F(x, y; x', y') = x_n y_n + \frac{1}{2} (y_{n+1}^2 + y'_{n+1}^2) + \frac{1}{2} (J x_n^2 + J' x'_n^2) - 2 \epsilon \left( \frac{J x_{n+1}^4}{4} + \frac{J' x'_{n+1}^4}{4} + a x_n^2 y_{n+1}^2 \right).
\]
The origin \((0, 0, 0, 0)\) is an elliptic fixed point of (26) and (27) for \(0 < J < 2\) and \(0 < J' < 2\). Eliminating the \(y\) and \(y'\) from Eqs. (26) and (27), we obtain coupled second order difference equations:
\[
x_{n+1} - 2 \cos(\theta) x_n + x_{n-1} = 2 \epsilon (J x_n^3 + 2 a x_n x_n^2),
\]
\[
x'_{n+1} - 2 \cos(\theta') x'_n + x'_{n-1} = 2 \epsilon (J' x'_n^3 + 2 a x_n x'_n),
\]
where \(\cos \theta = 1 - J/2\), \(\cos \theta' = 1 - J'/2\). Now, we concentrate on the case that the ratio of frequencies of the leading order solution of (28) and (29), i.e. \(\theta/\theta'\) is an irrational number. Then, straightforward calculations yield perturbation solutions of (28) and (29) as
\[
x^{(0)}_n = A \exp(i \theta n) + c.c.,
\]
\[
x^{(0)}'_n = B \exp(i \theta' n) + c.c.,
\]
\[
x^{(1)}_n = \frac{-i}{\sin \theta} (3 J |A|^2 A + 4 a |B|^2 A) n \exp(i n \theta) + c.c. + n.r.,
\]
\[
x^{(1)}'_n = \frac{-i}{\sin \theta} (3 J' |B|^2 B + 4 a |A|^2 B) n \exp(i n \theta') + c.c. + n.r.,
\]
where \(A \in \mathbb{C} \) and \(B \in \mathbb{C} \) are integral constants. In order to remove secular terms in (30) and (31), we introduce a RG transformation \(A \mapsto \tilde{A}(n), B \mapsto \tilde{B}(n)\):
\[
\tilde{A}(n) \equiv A + \epsilon \frac{-i}{\sin \theta} (3 J |A|^2 + 4 a |B|^2) A n,
\]
\[
\tilde{B}(n) \equiv B + \epsilon \frac{-i}{\sin \theta} (3 J' |B|^2 + 4 a |A|^2) B n,
\]
from which a naive RG map is obtained as

\[ \tilde{A}(n+1) = \tilde{A}(n) \left\{ 1 + \epsilon \frac{-i}{\sin \theta} (3J|\tilde{A}|^2 + 4a|\tilde{B}|^2) \right\}, \]

\[ \tilde{B}(n+1) = \tilde{B}(n) \left\{ 1 + \epsilon \frac{-i}{\sin \theta} (3J'|\tilde{B}|^2 + 4a|\tilde{A}|^2) \right\}. \]

To regularize this map, we scale the renormalized variables as

\[ \alpha(n) \equiv \frac{\tilde{A}(n)}{\sqrt{\sin \theta}}, \quad \beta(n) \equiv \frac{\tilde{B}(n)}{\sqrt{\sin \theta}}, \]

where \( \sin \theta > 0 \) and \( \sin \theta' > 0 \). Finally we “exponentiate” the resultant naive RG map as

\[ \alpha(n+1) = \alpha(n) \exp \left\{ \epsilon(-i) \left( 3J \frac{\sin \theta'}{\sin \theta} |\alpha(n)|^2 + 4a|\beta(n)|^2 \right) \right\}, \quad (32) \]

\[ \beta(n+1) = \beta(n) \exp \left\{ \epsilon(-i) \left( 3J' \frac{\sin \theta}{\sin \theta'} |\beta(n)|^2 + 4a|\alpha(n)|^2 \right) \right\}. \quad (33) \]

It is easy to see Eqs.(32) and (33) have the sympletic symmetry

\[ d\alpha(n+1) \wedge d\alpha^*(n+1) + d\beta(n+1) \wedge d\beta^*(n+1) = d\alpha(n) \wedge d\alpha^*(n) + d\beta(n) \wedge d\beta^*(n), \]

and two constants of motion \(|\alpha|^2\) and \(|\beta|^2\).

5 conclusion

We present a regularized RG procedure to preserve the symplectic structure in RG maps near elliptic and hyperbolic fixed points of some symplectic discrete systems. The regularization is accomplished by simple exponentiation of a naive RG map and gives a symplectic RG map, which successfully describes a long-time asymptotic behaviour of the original system. Without such regularization, a naive (truncated) RG map fails to describe a long-time behaviour especially near elliptic fixed points. Furthermore, all regularized RG maps obtained here have constants of motion and can be exactly solvable, while non-regularized (nonlinear) RG maps have no conserved quantities and are not solvable analytically. As an application of the regularized RG map, we give an approximate but analytical expression of the rotation number around an elliptic fixed point.
It is easy to see that the present regularization process is also applicable to
general two-dimensional symplectic maps and higher dimensional symplectic
maps, which have an elliptic fixed point with incommensurate frequencies.
However, construction of regularized RG maps of general higher dimensional
symplectic maps is still an open problem to be studied in future.

6 Acknowledgements

The authors would like to thank Prof. T. Konishi and Dr. Y. Hirata for
fruitful discussions.

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