A new solution of the star–triangle relation

Andrew P Kels

Institut für Mathematik, MA 8-4, Technische Universität Berlin, Str. des 17. Juni 136, D-10623 Berlin, Germany
Department of Theoretical Physics, Research School of Physical Sciences and Engineering, Australian National University, Canberra, ACT 0200, Australia

E-mail: kels@math.tu-berlin.de

Received 4 November 2013, revised 20 December 2013
Accepted for publication 20 December 2013
Published 17 January 2014

Abstract
We obtain a new solution to the star–triangle relation for an Ising-type model with two kinds of spin variables at each lattice site, taking continuous real values and arbitrary integer values, respectively. The Boltzmann weights are manifestly real and positive. They are expressed through the Euler gamma function and depend on sums and differences of spins at the ends of an edge of the lattice.

Keywords: statistical mechanics, star–triangle relation, Yang–Baxter equation

The star–triangle relation is a distinguished form of the Yang–Baxter equation for Ising-type models on two-dimensional lattices. In these models the fluctuating variables, or ‘spins’, are assigned to lattice sites, while two spins interact only if they are connected by an edge of the lattice. Remarkably, many physically interesting models in this class can be solved exactly, for instance, the 2D Ising [1] and the chiral Potts [2, 3] models (see also [4, 5] for a review of other known cases). The star–triangle relation plays the role of the integrability condition for these models.

Recently Bazhanov and Sergeev (BS) defined a very important ‘master’ solution [6] of the star–triangle relation, which contains all previously known solutions of this relation as particular cases, and provides many new interesting examples. The above master solution is expressed in terms of the elliptic gamma function, which contains two arbitrary free parameters p and q, that play the role of elliptic nomes. The spin variables for the corresponding statistical mechanical model take continuous real values on the circle. Various interesting limiting cases arise when the parameters p and q approach some special values. These limits are rather singular and their consideration requires a detailed analysis on a case by case basis (for a comprehensive account see the forthcoming paper [5]).

Considered as a mathematical identity the BS master solution is related to the elliptic beta integral of Spiridonov [7]. The latter discovery was central to the modern development
of the elliptic hypergeometric functions [8], and its many degenerations and multivariate generalizations have been extensively studied [9, 10]. Some recent works highlight further that some of these identities are connected to the integrability of models of statistical mechanics, particularly an extension of the master solution to the multivariate case [11, 12], and a remarkable correspondence to dualities in supersymmetric gauge theories [13–15]. The purpose of the present paper is to consider one particular limit of the BS master solution, that does not appear to have been considered previously, which leads to a physical model with two types of spin variables at each lattice site. Here the spin variables at each lattice site take one continuous real value and one discrete integer value, respectively.

The new model arises as a limiting case of a particular degeneration of the BS master solution, that is related to the ‘hyperbolic beta integral’ [16]. The latter degeneration results in a statistical mechanical model that is a generalization of the Faddeev–Volkov model introduced by Bazhanov, Mangazeev, and Sergeev (BMS) [17, 18]. The main difference is that the Boltzmann weights depend on the sum of two neighbouring spins, as well as their differences. The existence of this model was also previously noted by Spiridonov as a limiting case of the elliptic beta integral [13]. In our case a special normalization of the Boltzmann weights is used when taking the limit, such that the bulk free energy of the system vanishes in the thermodynamic limit. Following this we consider the ‘strong-coupling regime’ of this model, where new integer-valued spin variables dynamically arise at each lattice site. These calculations generalize results for the BMS Faddeev–Volkov model [17].

To begin we consider a model on the square lattice of \( N \) sites and assign spin variables \( x_j, j = 1, 2, \ldots, N \), taking some continuous set of values, to all sites of the lattice. Two spins interact only if they are connected by an edge of the lattice. Let \( W_\alpha(x, y) \), \( \overline{W}_\alpha(x, y) \) denote Boltzmann weights for horizontal and vertical edges, where \( x \) and \( y \) are spins at the end of the edge, as shown in figure 1.

![Figure 1. Horizontal (left) and vertical edges, and their Boltzmann weights. Here 'rapidity lines' (dashed arrows) are shown to distinguish the two types of weights.](image)

We assume that the edge weights depend on the (additive) spectral variable \( \alpha \), and are related to each other by the crossing symmetry \( \overline{W}_\alpha(x, y) = W_{\eta - \alpha}(x, y) \), where \( \eta \) is a model dependent ‘crossing parameter’. Moreover we assume that the weights are reflection symmetric \( W_\alpha(x, y) = W_\alpha(y, x) \).

It is convenient to introduce single-spin weights \( S(x_j) \), which are independent of the spectral variable \( \alpha \). The partition function is defined as

\[
Z = \int \cdots \int \prod_{(ij)} W_\alpha(x_i, x_j) \prod_{(kl)} \overline{W}_{\eta - \alpha}(x_k, x_l) \prod_m S(x_m) \, dx_\alpha, \tag{1}
\]
where the first product is taken over all horizontal edges \((ij)\), the second over all vertical edges \((kl)\) and the third over all internal sites of the lattice. We will implicitly assume fixed boundary conditions. Obviously the single-spin weights \(S(x_i)\) can be included into the definition of the edge weights, but they will be kept as separate weights here. The model is integrable if the Boltzmann weights satisfy the star–triangle equation of the form

\[
\int dx_0 S(x_0)W_{q-\alpha_1}(x_1, x_0)W_{q-\alpha_2}(x_2, x_0)W_{q-\alpha_3}(x_3, x_0) = R(\alpha_1, \alpha_2, \alpha_3)W_{\alpha_1}(x_3, x_1)W_{\alpha_2}(x_1, x_2)W_{\alpha_3}(x_2, x_1),
\]

where the spectral parameters \(\alpha_1, \alpha_2, \alpha_3\) satisfy the relation \(\alpha_1 + \alpha_2 + \alpha_3 = \eta\) and the factor \(R(\alpha_1, \alpha_2, \alpha_3)\) is independent of the spins \(x_1, x_2, x_3\).

To define the BS master solution, first introduce the following values of elliptic nomes \(p, q, \) and the crossing parameter \(\eta\)

\[
q = e^{i\tau}, \quad p = e^{i\sigma}, \quad |p|, |q| < 1, \quad e^{-2\eta} = pq, \quad \eta = -\frac{i\pi}{2}(\tau + \sigma).
\]

The form of the elliptic gamma function\(^1\) used in this paper is

\[
\Phi(z) = \sum_{j=0}^{\infty} \frac{1 - e^{2\pi i j/\eta}}{1 - e^{-2\pi i j/\eta}} \exp \left\{ \sum_{n \neq 0} \frac{e^{-2\pi i n \eta}}{n(p^n - p^{-n})(q^n - q^{-n})} \right\},
\]

where the exponential representation is valid only in the strip \(|\text{Im } z| < \text{Re } \eta\). The Boltzmann weights are defined by

\[
S(x) = \frac{e^{\eta/2}}{2\pi} \partial_1(2\pi p)\partial_1(2\pi q), \quad \mathcal{W}_\alpha(x, y) = \kappa(\alpha)^{-1} \frac{\Phi(x + y + i\alpha)\Phi(x - y + i\alpha)}{\Phi(x + y - i\alpha)\Phi(x - y - i\alpha)},
\]

where \(\partial_1(z|q)\) is the Jacobi theta function \([21]\). Note that the weights are periodic under a shift of the spins by \(\pi\)

\[
\mathcal{W}_\alpha(x + \pi, y) = \mathcal{W}_\alpha(x, y + \pi) = \mathcal{W}_\alpha(x, y), \quad S(x + \pi) = S(x).
\]

Correspondingly, the spin variables in (1) take their values in the interval \(0 \leq x_i < \pi\), \((i = 1, 2, \ldots, N)\). The normalization factor \(\kappa(\alpha)\) in (5) reads

\[
\log \kappa(\alpha) = \sum_{n \neq 0} \frac{e^{i\alpha n}}{n(p^n - p^{-n})(q^n - q^{-n})}. \quad \eta = \text{real}
\]

The Boltzmann weights (5) satisfy the star–triangle relation (2), where the integration is taken over the segment \(0 \leq x_0 < \pi\). The spectral parameter \(\sigma\) is taken to lie in the domain \(0 < \alpha < \eta\), where \(\eta\) is real. This is a physical regime of the model, where the Boltzmann weights (5) take real, positive values.

As a consequence of the normalization of the Boltzmann weights (7), the spin independent factor \(R(\alpha_1, \alpha_2, \alpha_3)\) in (2) is equal to one. For the same normalization, the Boltzmann weights (5) also satisfy the following inversion relations

\[
\mathcal{W}_\alpha(x, y)\mathcal{W}_{-\alpha}(x, y) = 1, \quad \int_0^\pi dz \mathcal{S}(z)\mathcal{W}_{-\alpha}(x, z)\mathcal{W}_{\alpha+\delta}(z, y) = \frac{1}{2S(x)}(\delta(x + y) + \delta(x - y)).
\]

These relations allow one to show that in the thermodynamic limit, when the number of lattice sites goes to infinity \(N \to \infty\), the bulk free energy of the system vanishes, \([5, 6]\)

\[
\lim_{N \to \infty} N^{-1} \log Z = 0.
\]

\(^1\) This differs from a common representation of the elliptic gamma function \(\Gamma(x; p, q)\) e.g. \([19, 20]\) by a simple change of variables \(\Phi(z) = \Gamma(e^{-2\pi i z}; p^2, q^2)\).
It should be emphasized that the result (9) is purely a consequence of the special choice of normalization for the Boltzmann weights in (7). Here the boundary spins are assumed to be kept finite in the limit \( N \to \infty \).

The Boltzmann weights (5) define the BS master solution. The discrete spin, chiral Potts and Kashiwara–Miwa models, were previously shown to be special cases of the low temperature limit of this model [6]. In this limit the continuous degrees of freedom in (1), are replaced by discrete degrees of freedom, corresponding to fluctuations of the spins between the different degenerate ground states of the system. Another interesting property of the leading order low temperature expansion of (1) was also found, where one obtains the discrete Laplace system associated to the classical integrable \( Q4 \) equation [22–24]. The model thus can be considered a quantum counterpart of the corresponding \( Q4 \) system of equations [6].

Now we consider the hyperbolic limit of the BS master solution (5). Introduce the modular parameter \( b \) to take values in one of the following two regimes

\[
i (i) \ b > 0, \quad (ii) \ |b| = 1, \quad \text{Im}(b^2) > 0.
\]

(10)

The non-compact quantum dilogarithm is defined as

\[
\varphi(z) = \exp \left\{ \frac{1}{4} \int_{\mathbb{R}+i0} \frac{e^{-2by/c} \, dy}{y\sinh(yb) \sinh(y/b)} \right\}, \quad |\text{Im}(z)| < \text{Re}(\eta'),
\]

(11)

where the singularity at the origin is taken below the contour, and the crossing parameter is given by \( \eta' = (b + b^{-1})/2 \). The primed \( \eta' \) distinguishes this variable from \( \eta \) of the master solution (3). The spins, spectral variables, and elliptic nomes of the master solution, are scaled by a positive parameter \( \epsilon \)

\[
p = e^{-b\epsilon}, \quad q = e^{-b^{-1}\epsilon}, \quad x_i = x_i\epsilon, \quad \alpha_i = \alpha_i\epsilon, \quad i = 1, 2, 3,
\]

(12)

and in the limit \( \epsilon \to 0 \) of the star–triangle relation (2), with (5), one obtains the new Boltzmann weights

\[
S(x) = 2 \sinh(2\pi x_0 b) \sinh(2\pi x_0/b), \quad \mathcal{W}_\alpha(x, y) = \kappa(\alpha)^{-1} e^{\frac{4\pi i}{3} \alpha} \varphi(x + y + i\alpha) \varphi(x - y + i\alpha)
\]

(13)

where the normalization factor now reads

\[
\log \kappa(\alpha) = \frac{1}{8} \int_{\mathbb{R}+i0} \frac{e^{by/c} \, dy}{y\sinh(yb) \sinh(y/b) \cosh(2y\eta')}, \quad \text{Im}(\eta') = \frac{\pi i}{3} \eta'^2 + \frac{\pi i}{24}.
\]

(14)

These Boltzmann weights satisfy the star–triangle relation (2), where the integration is taken over the whole real line. The choice of \( b \) in (10) is a physical regime for the model, where the spins now take arbitrary values on the real line \( x_i \in \mathbb{R}, \ (i = 1, 2, \ldots, n) \), and the spectral variable is constrained to the region \( 0 < \alpha < \eta' \).

The Boltzmann weights (13) define a physical model of statistical mechanics, rather similar to the BMS Faddeev–Volkov model [18, 25]. Importantly here the normalization is chosen for the Boltzmann weights (7), such that the above limiting procedure does not affect the bulk free energy of the system, and the relation (9) continues to hold for the new model. Interestingly the result (14) is identical to that found for the BMS Faddeev–Volkov model, in that case it was obtained directly from the Boltzmann weights using the ‘inversion relation’ method for the partition function per site in the thermodynamic limit.

An interesting property of this model, is that the leading order low temperature expansion \( (\eta' \to \infty) \) of the partition function (1) corresponds to the hyperbolic circle pattern functional of Bobenko and Springborn [26]. In this limit, the spins of the model are related to the radii of the corresponding circle pattern\(^2\), spectral parameters correspond to intersection angles between

\^2\ Specifically for a spin \( x_i, \ \exp(x_i) = \tanh(r_i/2), \) where \( r_i \) is the radius of the circle centred at site \( i \) of the lattice.
circles, and the critical point of the functional corresponds to solutions of the hyperbolic circle pattern problem with fixed boundary conditions. For finite \( \eta' \) the model thus describes fluctuations of the circle pattern (fluctuations of spins) away from this classical solution. Using Baxter’s Z-invariance property of the partition function [27], one can extend this geometric correspondence to lattices that are more general than the square lattice considered here. More details of this construction can be found for the Euclidean case, arising in the same limit for the BMS Faddeev–Volkov model [18], which can be generalized for the model here (13). Another consequence of this is, that the model gives a quantum counterpart of the discrete Laplace system associated to classical Q3\(^{1/2}\) integrable equations [24].

Now consider a further specialization of the above generalized Faddeev–Volkov model (13), when the parameter \( b \to i \), or equivalently, \( \eta' \to 0 \). This is a singular limit of the model, where the Boltzmann weights \( \mathcal{W}_0(x, y) \) develop a periodic series of delta function like peaks, for integer values of the sum \( x + y \in \mathbb{Z} \) or differences \( x - y \in \mathbb{Z} \) of the arguments \( x, y \). A correct limiting procedure capturing a fine structure of these peaks requires a redefinition of the spin variables.

For integer \( n \), and real valued \( x \), the asymptotics of the non-compact quantum dilogarithm \( \varphi(z) \) and the function \( \kappa(\alpha) \) are

\[
\varphi(n + x\eta') = e^{-\frac{\eta' \pi i}{24}(4\pi \eta')^n} \frac{\Gamma\left(\frac{1-n+i\eta}{2}\right)}{\Gamma\left(\frac{1-n-i\eta}{2}\right)}, \quad \kappa(\beta\eta') = \left(8\pi \eta'\right)^{\frac{1}{2}} \frac{\Gamma\left(\frac{1+\beta}{2}\right)}{\Gamma\left(\frac{1-\beta}{2}\right)},
\]

(15)
as \( \eta' \to 0 \). Here the spins should satisfy \( x \ll \eta'^{-1} \), and the new spectral variable \( \beta \) is restricted to the range \( 0 < \beta < 1 \). Using this expansion in the Boltzmann weights (13), one obtains the following \( \eta' \to 0 \) asymptotics,

\[
\mathcal{S}(n + x\eta') \to 8\pi^2(x^2 + n^2)\eta'^2, \quad \mathcal{W}_{\beta g}(m + x\eta', n + y\eta') \to 2^{-5\beta} \frac{\Gamma\left(\frac{1+\beta}{2}\right) \Gamma\left(\frac{1-\beta-(m-n)\bar{\kappa}(x+y)}{2}\right) \Gamma\left(\frac{1-\beta-(m-n)\bar{\kappa}(x-y)}{2}\right)}{(\pi \eta')^{-2\beta} \Gamma\left(\frac{1-\beta}{2}\right) \Gamma\left(\frac{1+\beta-(m-n)\bar{\kappa}(x+y)}{2}\right) \Gamma\left(\frac{1+\beta-(m-n)\bar{\kappa}(x-y)}{2}\right)},
\]

(16)

where \( m, n \in \mathbb{Z} \) and \( x, y \in \mathbb{R} \). Using these asymptotics one can take the \( \eta' \to 0 \) limit of the star–triangle relation (2), to obtain

\[
\sum_{n_0 \in \mathbb{Z}} \int_{-\infty}^{\infty} \mathcal{S}(x_0, n_0) \mathcal{W}_{1-\beta_1}(x_1, n_1 | x_0, n_0) \mathcal{W}_{1-\beta_2}(x_2, n_2 | x_3, n_3) \mathcal{W}_{1-\beta_3}(x_3, n_3 | x_1, n_1)
= \mathcal{W}_{\beta_1}(x_1, n_2 | x_3, n_3) \mathcal{W}_{\beta_2}(x_2, n_1 | x_3, n_3) \mathcal{W}_{\beta_3}(x_3, n_2 | x_1, n_1),
\]

(17)

where \( \beta_1 + \beta_2 + \beta_3 = 1 \), and the new Boltzmann weights are given by

\[
\mathcal{S}(x, n) = \frac{1}{2\pi} (x^2 + n^2), \quad \mathcal{W}_{\beta}(x, n | y, m) = \frac{\Gamma\left(\frac{1+\beta}{2}\right) \Gamma\left(\frac{1-\beta-(m-n)\bar{\kappa}(x+y)}{2}\right) \Gamma\left(\frac{1-\beta-(m-n)\bar{\kappa}(x-y)}{2}\right)}{\Gamma\left(\frac{1-\beta}{2}\right) \Gamma\left(\frac{1+\beta-(m-n)\bar{\kappa}(x+y)}{2}\right) \Gamma\left(\frac{1+\beta-(m-n)\bar{\kappa}(x-y)}{2}\right)},
\]

(18)

These Boltzmann weights are normalized such that they satisfy the following boundary conditions

\[
\mathcal{W}_0(x, n | y, m) = 1, \quad \mathcal{W}_{\beta}(x, n | y, m) \big|_{\beta \to 1} = \frac{1}{2\mathcal{S}(x, n)} \left( \delta_{n,m} \delta(x-y) + \delta_{n,-m} \delta(x+y) \right),\]

(19)

\[3\] Here we use the compact notation, where the \( \pm \) symbol in the argument of the gamma function indicates the product of gamma functions with both signs are taken as e.g. \( \Gamma(a \pm b) = \Gamma(a+b)\Gamma(a-b) \).
which along with the star–triangle relation (17) give the following inversion relations\(^4\)

\[
W_\beta(x, n|y, m) W_{-\beta}(x, n|y, m) = 1,
\]

\[
\sum_{n_0 \in \mathbb{Z}} \int_{-\infty}^{\infty} dx_0 S(x_0, n_0) W_{1-\beta}(x_1, n_1|n_0, n_0) W_{1+\beta}(x_0, n_0|x_2, n_2)
\]

\[
= \frac{1}{2S(x_1, n_1)} \left( \delta_{n_1,n_0} \delta(x_1 - x_2) + \delta_{n_1,-n_0} \delta(x_1 + x_2) \right).
\]

(20)

The star–triangle relation (17), is for the Ising type model given by the Boltzmann weights (18), where each site of the lattice is now assigned the following ‘dual spin’ variables \(x\)

\[
x = (x, n), \quad x \in \mathbb{R}, \quad n \in \mathbb{Z},
\]

(21)

taking some arbitrary real and integer values, respectively. Nearest neighbour interactions of the model, for two spins \(x, y\) connected by an edge as in figure 1, are described by the Boltzmann weights (18). These Boltzmann weights also satisfy the spin reflection symmetry on exchanging the two spins, \(x \leftrightarrow y\). The bulk free energy (9) of the corresponding 2D lattice model remains unaffected by the limiting procedure.

The choice of spin variables given in (21), is a manifestly physical regime of the lattice model defined by the Boltzmann weights (18). This is a particularly nice feature of this limit, since there exists some other non-physical, ‘rational’ limits of (5), or (13), that also give Boltzmann weights expressed through the Gamma function \(\Gamma(z)\) and that satisfy the star–triangle relation (2) [5]. These rational limits of (13) usually involve a non uniform shift in the spin variables, for example one spin in (2) might be shifted off the real axis, and one of the Boltzmann weights then gains a non vanishing imaginary component.

Recently, similar lattice models have been proposed where the spin variables take continuous values as in (21), as well as integer values restricted to \(n \in \mathbb{Z}_r\) [15]. The main difference is that these models contain elliptic Boltzmann weights, that satisfy the star–star relation rather than the star–triangle relation. The former relation arises there as a consequence of the Seiberg duality for 4-d \(\mathcal{N} = 1\) quiver gauge theories. In our case the existence of the star–triangle relation given in (17) implies that the underlying lattice model also satisfies a star–star relation. The form of the star–star relation obtained from the Boltzmann weights (18) is similar to that proposed for the elliptic models [15], with their variable \(r \rightarrow \infty\). It would be interesting to examine further exactly how these models are related, particularly if there is a gauge theory interpretation of our model given by (17), (18).

In this paper the star–triangle relation (17), is shown to arise in the strong-coupling limit of the generalized Faddeev–Volkov model (13), the latter model arising as a hyperbolic degeneration of the master solution (5). From either of the models (5), (13), other new solutions of the star–triangle relation might be found as different singular limits of the elliptic nome (3), and modular parameter \(b\) (10) respectively. The existence of real valued, positive Boltzmann weights needs to be checked on a case by case basis, and in general this is not easily satisfied. Also for each limiting case, the spins and spectral parameters should be suitably redefined, as was done in (12) and (16), in order to obtain convergent limits where possible.

Acknowledgments

I thank Vladimir Bazhanov for suggesting the problem and useful advice. This work was partially supported by the Australian Research Council. The author is supported by the DFG Collaborative Research Center TRR 109, ‘Discretization in Geometry and Dynamics’.

\(^4\) cf equation (8).
References

[1] Baxter R J 1982 Exactly Solved Models in Statistical Mechanics (London: Academic)
[2] Au-Yang H, McCoy B M, Perk J H H, Tang S and Yan M-L 1987 Commuting transfer matrices in the chiral Potts models: solutions of star-triangle equations for genus > 1 Phys. Lett. A 123 219–23
[3] Baxter R J, Perk J H H and Au-Yang H 1988 New solutions of the star triangle relations for the chiral Potts model Phys. Lett. A 128 138–42
[4] Baxter R J 2002 A Rapidity-Independent Parameter in the Star-Triangle Relation (Progress in Mathematical Physics vol 23) (Boston, MA: Birkhäuser)
[5] Bazhanov V V, Kels A P and Sergeev S M 2013 Quasi-classical expansion of the star-triangle relation and integrable systems on quad-graphs in preparation
[6] Bazhanov V V and Sergeev S M 2012 A master solution of the quantum Yang–Baxter equation and classical discrete integrable equations Adv. Theor. Math. Phys. 16 65–95
[7] Spiridonov V P 2001 On the elliptic beta function Usp. Mat. Nauk 56 181–2
[8] Spiridonov V P 2008 Essays on the theory of elliptic hypergeometric functions Usp. Mat. Nauk 63 3–72
[9] Rains E M 2006 Limits of elliptic hypergeometric integrals Ramanujan J. 18 257–306
[10] Bazhanov V V and Sergeev S M 2012 Elliptic gamma-function and multi-spin solutions of the Yang–Baxter equation Nucl. Phys. B 856 475–96
[11] Bazhanov V V, Kels A P and Sergeev S M 2013 Comment on star-star relations in statistical mechanics and elliptic gamma-function identities J. Phys. A: Math. Theor. 46 152001
[12] Spiridonov V P 2012 Elliptic beta integrals and solvable models of statistical mechanics Contemp. Math. 563 181–211
[13] Yamazaki M 2012 Quivers, YBE and 3-manifolds J. High Energy Phys. JHEP05(2012)
[14] Yamazaki M 2013 New integrable models from the gauge/YBE correspondence arXiv: 1307.1128 [hep-th]
[15] Stokman J V 2004 Hyperbolic beta integrals Adv. Math. 190 119–60
[16] Bobenko A I and Springborn B A 2004 Variational principles for circle patterns and Koebe’s theorem Trans. Am. Math. Soc. 356 659–89
[17] Baxter R J 1978 Solvable eight-vertex model on an arbitrary planar lattice Phil. Trans. R. Soc. Lond. A 289 315–46

J. Phys. A: Math. Theor. 47 (2014) 055203 A P Kels