S-packing colorings of distance graphs $G(\mathbb{Z}, \{2, t\})$

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May 22, 2020

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Abstract

Given a graph $G$ and a non-decreasing sequence $S = (a_1, a_2, \ldots)$ of positive integers, the mapping $f : V(G) \to \{1, 2, \ldots, k\}$ is an $S$-packing $k$-coloring of $G$ if for any distinct vertices $u, v \in V(G)$ with $f(u) = f(v) = i$ the distance between $u$ and $v$ in $G$ is greater than $a_i$. The smallest $k$ such that $G$ has an $S$-packing $k$-coloring is the $S$-packing chromatic number, $\chi_S(G)$, of $G$. In this paper, we consider the distance graphs $G(\mathbb{Z}, \{2, t\})$, where $t > 1$ is an odd integer, which has $\mathbb{Z}$ as its vertex set, and $i, j \in \mathbb{Z}$ are adjacent if $|i - j| \in \{2, t\}$. We determine the $S$-packing chromatic numbers of the graphs $G(\mathbb{Z}, \{2, t\})$, where $S$ is any sequence with $a_i \in \{1, 2\}$ for all $i$. In addition, we give lower and upper bounds for the $d$-distance chromatic numbers of the distance graphs $G(\mathbb{Z}, \{2, t\})$, which in the cases $d \geq t - 3$ give the exact values. Implications for the corresponding $S$-packing chromatic numbers of the circulant graphs are also discussed.

Key words: $S$-packing coloring, $S$-packing chromatic number, distance graph, distance coloring.

AMS Subj. Class: 05C15, 05C12

1 Introduction

Let $G$ be a graph, and let $S = (a_1, a_2, \ldots)$ be a non-decreasing sequence of positive integers. A mapping $f : V(G) \to \{1, 2, \ldots\}$ such that for every two vertices $u, v$ with $f(u) = f(v) = i$ the distance $d_G(u, v)$ is bigger than $a_i$, is an $S$-packing coloring of a graph $G$. (The distance $d_G(u, v)$ is the length of a shortest path between $u$ and $v$ in $G$.) If such a mapping $f$ exists, then $G$ is $S$-packing colorable. If there exists an integer $k$ such that the range $R_f$ of $f$ is $\{1, 2, \ldots, k\}$, then $f$ is an $S$-packing $k$-coloring. The $S$-packing chromatic number of $G$, denoted by $\chi_S(G)$, is the smallest integer $k$ such that $G$ admits an $S$-packing $k$-coloring. For the sequence $S = (i)_{i \in \mathbb{N}}$ (of natural numbers in
the standard order) the above concepts are known under the names packing coloring, packing k-coloring and the packing chromatic number, $\chi_p(G)$, of $G$. These concepts were introduced by Goddard, S.M. Hedetniemi, S.T. Hedetniemi, Harris, and Rall [13] with a motivation coming from distributing broadcast frequencies to radio stations. The current names were given in the second paper on the topic [4], and a number of authors considered these colorings later on; see a recent survey of Brešar, Ferme, Klavžar and Rall [3]. In addition, for an integer $d$ the sequence $S = (d,d,\ldots)$ yields the $d$-distance coloring of a graph, which has also been extensively studied; see a survey [14].

A lot of attention was given to $S$-packing colorings, where $S$ contains only integers 1 and 2, since they lie between the classical colorings (where $S = (1,1,\ldots)$) and 2-distance colorings (where $S = (2,2,\ldots)$). In addition, an intriguing conjecture on the packing chromatic numbers of subdivisions of cubic graphs [2,5,12] initiated several related studies. Gastineau and Togni proved that every subcubic graph is $(1,2,2,2)$-packing colorable [12]. Brešar, Klavžar, Rall, and Wash [5] investigated which subcubic graphs are $(1,1,2,2)$-packing colorable, and this was continued by Liu, Liu, Rolek, and Yu [17]. Goddard and Xu in [14] studied $S$-packing colorings in the two-way infinite path for different sequences $S$, which was initiated already in the seminal paper [13]. For instance, they proved that $\chi_S(P_\infty) = 3$ if and only if $S$ starts with either $(1,2,3)$, $(1,3,3)$, or $(2,2,2)$; see [10] for some additional results. Goddard and Xu [15] considered $S$-packing colorings of the square lattice and some other related infinite graphs. Concerning the complexity issues, they proved that determining whether a graph is $(1,1,k)$-packing colorable for any $k \geq 1$, or $(1,2,2)$-packing colorable, are NP-hard problems [14]. These being very difficult problems in general, some authors were searching for classes of graphs and sequences $S$ in which the computational complexity of determining the $S$-packing chromatic number is polynomial; see such an investigation of Gastineau [9].

For a finite set of positive integers $D = \{d_1,\ldots,d_k\}$, the integer distance graph $G(Z,D)$ with respect to $D$ is the infinite graph with $Z$ as the vertex set and two distinct vertices $i,j \in Z$ are adjacent if and only if $|i-j| \in D$. The study of packing colorings of distance graphs was initiated by Togni [20]. He focused mainly on packing colorings of the distance graph $G(Z,D)$, where $1 \in D$. The study of the packing chromatic number of these graphs was continued by Ekstein, Holub, and Lidický [7], Ekstein, Holub, and Togni [8], and Shao and Vesel [19]. In this paper, the $S$-packing chromatic numbers of the distance graph $G(Z,D)$, where $D = \{2,t\}$ are considered. Note that if $t$ is even, then $G(Z,D)$ consists of two components, which are both isomorphic to $G(Z,\{1,t/2\})$. We restrict our attention only on the case $G(Z,\{2,t\})$, where $t > 2$ is an odd integer.

To simplify the notation in this paper, we let $G(Z,\{2,t\})$ be denoted by $G_t$, where $t \geq 3$ is an odd integer. Hence, $V(G_t) = Z$ and $i,j \in E(G_t)$ if and only if $|i-j| = 2$ or $|i-j| = t$. From results in [8,20], one can derive that $\chi_p(G_3) = 13$, and from [19,20] one can infer $14 \leq \chi_p(G_5) \leq 15$. In addition, Ekstein et al. [8] proved that for $k$ even and $t \geq 923$ and $t$ relatively prime to $k$, $\chi_p(G(Z,\{k,t\})) \leq 56$; in particular, we have $\chi_p(G_t) \leq 56$ for any odd integer $t \geq 923$. In addition, $\chi_p(G_t) \geq 12$ for $t \geq 9$. On the other hand, it appears that $S$-packing colorings of distance graphs, where $S \neq (1,2,3,\ldots)$, have not yet been considered, and we want to make the first step in this direction.
In this paper, the main focus is on \((a_1, a_2, \ldots)\)-packing colorings of graphs \(G_t\) with \(a_i \in \{1, 2\}\) for all \(i\), which is done in Section 2. We start the section by determining that the chromatic number of \(G_t\) equals 3, and then deal with the sequences \(S\) that start with either two 1s or one 1 and all other integers in the sequence are 2s. In the former case (i.e., \(S = (1, 1, 2, 2, \ldots)\)), we have \(\chi_S(G_t) = 4\), while in the latter case (i.e., \(S = (1, 2, 2, \ldots)\)), \(\chi_S(G_t) = 5\) if \(t \geq 5\) and \(\chi_S(G_3) = 6\). In Section 3 we deal with distance colorings of distance graphs. We give some lower and upper bounds in the general case of \(d\)-distance colorings for arbitrary \(d\), and in some cases when \(d \geq t - 3\), we determine the exact values of the \(d\)-distance chromatic numbers. We also prove the exact values of the 2-distance chromatic numbers of graphs \(G_t\). In the final section we give some remarks about the \(S\)-packing colorings of the circulant graphs, which are related to the constructions (patterns) presented in this paper.

2 \(S\)-packing colorings of graphs \(G_t\)

In this section, we determine the \(S\)-packing chromatic numbers of graphs \(G_t\) for all sequences \(S\) that start with 1. As it turns out, there are only a few cases that need to be considered for the sequence \(S = (a_1, a_2, \ldots)\) where \(a_1 = 1\). First, if \(a_i = 1\) for all \(i\), that is, the standard chromatic number, is dealt with in Subsection 2.1. We prove that \(\chi(G_t) = 3\), which means that only the cases when there are two 1s in \(S\) or there is just one 1 in \(S\) are left to be considered. The case when \(a_1 = 1 = a_2\) and \(a_i = 2\) for \(i \geq 3\) is considered in Subsection 2.2 while the case when \(a_1 = 1\) and \(a_i = 2\) for all \(i\) is considered in Subsection 2.3.

For a positive integer \(k\), let \([k] = \{1, \ldots, k\}\) and let \([k]_0 = \{0, 1, \ldots, k\}\). In proofs of upper bounds, we will often say that a coloring \(f : \mathbb{Z} \to [k]\) uses a pattern \(c_1 \ldots c_\ell\),

where \(c_i \in [k]\) for all \(i \in [\ell]\). By this we mean that the subsequence of colors \(c_1 \ldots c_\ell\) is repeatedly given to consecutive vertices of \(\mathbb{Z}\); for instance and without loss of generality: \(f(i + j\ell) = c_i\) for all \(i \in [\ell]\) and all \(j \in \mathbb{Z}\).

The term packing coloring comes from the concept of packing. Given a graph \(G\) and a positive integer \(i\), a set of vertices \(S \subseteq V(G)\) is an \(i\)-packing if for every distinct vertices \(u, v \in S\), \(d_G(u, v) > i\). (The integer \(i\) in an \(i\)-packing \(S\) is sometimes called the size of the packing \(S\).) In particular, a 1-packing is an independent set. Clearly, (proper) coloring of vertices of a graph is equivalent to partitioning the vertex set to independent sets. Similarly, \(S\)-packing coloring is equivalent to partitioning the vertex set into packings of sizes that appear in \(S\).

2.1 \(S = (1, 1, \ldots)\)

In this subsection, we determine the chromatic number of \(G_t\) for all \(t \geq 3\); in terms of \(S\)-packing colorings this is the \(S\)-packing chromatic number for \(S = (1, 1, 1, \ldots)\).

Theorem 2.1 For any odd integer \(t \geq 3\), \(\chi(G_t) = 3\).
Proof. To prove the lower bound $\chi(G_t) \geq 3$ it suffices to see that every $G_t$ has an odd cycle. Indeed, the cycle $C : 0, 2, \ldots, 2t, t, 0$ has $t + 2$ vertices, which is an odd integer.

Now, let us prove the upper bound for $\chi_S(G_t) \leq 3$. We use the greedy (first-fit) algorithm, so that when we color a vertex $i$ its color depends on the colors of vertices $i - 2$ and $i - t$. In the worst case these colors are distinct, and we still have one color available to color the vertex $i$. After coloring the first $t$ vertices, we do the same (by using the decreasing order, $0, -1, -2, \ldots, -t$) for the vertices left of the starting vertex, and then continue in the same way by alternating the directions. \hspace{1cm} \Box

2.2 $S = (1, 1, 2, 2, \ldots)$

Here we prove that $G_t$ is $(1, 1, 2, 2)$-packing colorable. Moreover, the $S$-packing chromatic number of $G_t$ with $S = (1, 1, 2, 2, \ldots)$ is $4$.

Theorem 2.2 If $t \geq 3$ and $S = (1, 1, 2, 2, \ldots)$, then $\chi_S(G_t) = 4$.

Proof. First, we prove that $\chi_S(G_t) \geq 4$ for $S = (1, 1, 2, 2, \ldots)$, and the proof is by contradiction. Therefore, suppose that $G_t$ is $(1, 1, 2)$-packing colorable, and let $f : V(G) \to [3]$ be a $(1, 1, 2)$-packing coloring of $G_t$. Note that the vertices $v$ with $f(v) = 1$, respectively $f(v) = 2$, form an independent set, while the vertices $v$ with $f(v) = 3$ form a 2-packing of $G_t$. Assume without loss of generality that $f(2 + t) = 3$.

Since $P : 2, 0, t, 2t, 2t + 2$ is a path in $G_t$ all vertices of which are at distance at most 2 from $2 + t$, they are colored with a color distinct from 3. We may assume without loss of generality that $f(2) = f(t) = f(2t + 2) = 1$. The path $Q : 2, 4, \ldots, 2t$ is of even length (i.e., has an even number of edges), since $t$ is odd, and we have $f(2) = 1, f(4) = 2$. It is possible that for some $k \in \{3, \ldots, t - 1\}$, there is a vertex $x = 2k$ such that $f(2k) = 3$.

Note that all vertices in the path $Q' = 2k - 2, 2k - 2 + t, 2k + t, 2k + t + 2, 2k + 2$ are at distance at most 2 from $2k$, which implies that they are not given color 3 by $f$. We infer that $f(2k - 2) = f(2k + t) = f(2k + 2)$. That is, vertices on the path $Q$ alternate between colors 1 and 2 with possible exceptions of vertices, which are colored by 3, but in that case(s) the alternating nature of the colors on $Q$ does not change. Since $Q$ is of even length, we derive $f(2) = f(2t) = 1$. This is a contradiction, since $f(2t) = 1 = f(t)$ and vertices $t$ and $2t$ are adjacent.

Next, we prove that $\chi_S(G_t) \leq 4$ for $S = (1, 1, 2, 2, \ldots)$ by forming an $S$-packing 4-coloring $f$ of the vertices of $G_t$.

Case 1. $t = 4k - 1$

Let $f(j(4k + 1)) = 3$ for all $j \in \mathbb{Z}$, $2 \nmid j$, and let $f(j(4k + 1)) = 4$ for all $j \in \mathbb{Z}$, $2|j$.

Further, let $f(j(4k + 1) + \ell) = 1$ and $f(j(4k + 1) + m) = 2$ for any $j \in \mathbb{Z}$, $\ell \equiv 1, 2 \pmod{4}$, $1 \leq \ell < 4k + 1$ and $m \equiv 3, 4 \pmod{4}$, $3 \leq m < 4k + 1$. In this way, consecutive integers in $\mathbb{Z}$ are colored with the following pattern of colors:

$$(1122)^k 3(1122)^k 4,$$

where $(1122)^k$ means the repetition of $1122$ $k$-times. We claim that the described coloring is an $S$-packing 4-coloring, where $S = (1, 1, 2, 2, \ldots)$.
First, consider any two vertices $a, b \in V(G_t)$ such that $f(a) = f(b) = 1$. This implies that there exist $j, j' \in \mathbb{Z}$ and $\ell, \ell' \in \mathbb{Z}$ with the properties that $1 \leq \ell, \ell' < (4k + 1)$ (actually $\ell, \ell' \leq 4k - 2$), $\ell \equiv 1, 2 \pmod{4}$, $\ell' \equiv 1, 2 \pmod{4}$ such that $a = j(4k+1) + \ell$ and $b = j'(4k+1) + \ell'$. Suppose that $d_{G_t}(a, b) = 1$ and without loss of generality assume that $a > b$. Then, $a - b \in \{2, t\}$. If $j = j'$, then $a - b = \ell - \ell'$. Clearly, $\ell - \ell' \neq 2$ and $\ell - \ell' \leq 4k - 3 < t$, a contradiction to our assumption. Next, let $j \geq j' + 2$. In this case, $a - b \geq 4k + 5$. Therefore, $a - b \notin \{2, t\}$, again a contradiction to our assumption. Further, consider the case when $j = j' + 1$. Then $a - b = 4k + 1 + \ell - \ell'$, which is obviously greater than 2. Now, $a - b = t$ implies that $\ell - \ell' = 2$, but this is not possible since $\ell \equiv 1, 2 \pmod{4}$ and $\ell' \equiv 1, 2 \pmod{4}$. These findings imply that any two distinct vertices of $G_t$, both colored by 1, are at distance at least 2 in $G_t$. Analogously one can prove that the same holds for any two distinct vertices of $G_t$, both colored by 2.

Next, suppose that there exist two distinct vertices $a = j(4k+1)$ and $b = j'(4k+1)$ of $G_t$, such that $j > j'$, $f(a) = f(b) = i \in \{3, 4\}$ and $d_{G_t}(a, b) \leq 2$. Then, $a - b = (j - j')(4k+1) \in \{2, t, 4, 2 + t, 2t, t - 2\}$. If $a - b = (j - j')(4k+1) \in \{2, 4\}$, then $4k + 1 = 1$ and hence $k = 0$, a contradiction to $t$ being a positive integer. Further, let $a - b = (j - j')(4k+1) \in \{2 + t, t, t - 2\}$. Since each of the integers from $\{2 + t, t, t - 2\}$ is odd, we derive that $j - j'$ is also odd. But this is a contradiction to the fact that $f(a) = f(b)$ which implies that both $j$ and $j'$ are divisible by 2 or both are not divisible by 2 and hence $j - j'$ is even. In the case when $a - b = 2t$, we derive that $(j - j' - 2)4k = -j + j' - 2$. Since $(j - j' - 2)4k \geq 0$ and $-j + j' - 2 < 0$, we have a contradiction. Therefore, any two distinct vertices of $G_t$, both colored by $i \in \{3, 4\}$ are at distance at least 3 in $G_t$.

**Case 2.** $t = 4k + 1$

Let $f(j(4k + 3)) = 3$ for all $j \in \mathbb{Z}$, $2 \nmid j$, and $f(j(4k + 3)) = 4$ for all $j \in \mathbb{Z}$, $2|j$. Next, let $f(j(4k + 3) + \ell) = 1$ and $f(j(4k + 3) + m) = 2$ for any $j \in \mathbb{Z}$, $\ell \equiv 2, 3 \pmod{4}$, $2 \leq \ell < 4k + 3$ and $m \equiv 0, 1 \pmod{4}$, $1 \leq m < 4k + 3$. By the described coloring, the consecutive integers of $\mathbb{Z}$ are colored with the following pattern of colors:

$$(1122)^k 132(1122)^k 142,$$

where $(1122)^k$ means the repetition of 1122 $k$-times. We prove that the described coloring is an $S$-packing 4-coloring of $G_t$, where $S = (1, 1, 2, 2, \ldots)$.

First, let $a, b \in V(G_t)$ such that $f(a) = f(b) = 1$. This implies that there exist $j, j' \in \mathbb{Z}$ and $\ell, \ell'$ with the properties that $2 \leq \ell, \ell' < (4k + 3)$, $\ell \equiv 2, 3 \pmod{4}$ and $\ell' \equiv 2, 3 \pmod{4}$ such that $a = j(4k+3) + \ell$ and $b = j'(4k+3) + \ell'$. We prove that $d_{G_t}(a, b) > 1$. Suppose to the contrary that $d_{G_t}(a, b) = 1$ and without loss of generality assume that $a > b$. This implies that $a - b \in \{2, t\}$. If $j = j'$, then $a - b = \ell - \ell'$. Clearly, $\ell - \ell' \neq 2$ and $\ell - \ell' \leq 4k < t$, a contradiction to our assumption. Next, if $j \geq j' + 2$, then $a - b \geq 4k + 6$ and hence $a - b \notin \{2, t\}$, a contradiction. Finally, let $j = j' + 1$. Clearly, $a - b \neq 2$, hence $a - b = t$. This implies that $\ell' - \ell = 2$, but this is not possible since $\ell \equiv 2, 3 \pmod{4}$ and $\ell' \equiv 2, 3 \pmod{4}$. Therefore, any two distinct vertices of $G_t$, both colored by 1, are at distance at least 2 in $G_t$. Analogously one can prove that the same holds for any two distinct vertices of $G_t$, both colored by 2.
Now, suppose that there exist two distinct vertices \( a = j(4k + 3) \) and \( b = j'(4k + 3) \) of \( G_t \) such that \( j > j' \), \( f(a) = f(b) = i \in \{3, 4\} \) and \( d_{G_t}(a, b) \leq 2 \). Then, \( a - b = (j - j')(4k + 3) \in \{2, t, 4, 2 + t, 2t, t - 2\} \). If \( a - b = (j - j')(4k + 3) \in \{2, 4\} \), then \( 4k + 3 = 1 \), which is not possible since \( k \) is a positive integer. Next, let \( a - b = (j - j')(4k + 3) \in \{2 + t, t, t - 2\} \). Since each of the integers from \( 2 + t \), \( t \), \( t - 2 \) is odd, we infer that \( j - j' \) is also odd. But, due to the fact that \( f(a) = f(b) \), both \( j \) and \( j' \) are divisible by 2 or both are not divisible by 2 and hence \( j - j' \) is even, so we have a contradiction. In the case when \( a - b = 2t \), we derive that \( (j - j' - 2)4k = -3(j - j') + 2 \). Since \( (j - j' - 2)4k \geq 0 \) and \( -3(j - j') + 2 < 0 \), we have a contradiction. These findings imply that any two distinct vertices of \( G_t \), both colored by \( i \in \{3, 4\} \) are at distance at least 3. This completes the proof. \( \square \)

2.3 \( S = (1, 2, 2, \ldots) \)

In this subsection, we prove that \( G_t \) is \((1, 2, 2, 2, 2)\)-packing colorable for every odd integer \( t > 3 \), and that the \( S \)-packing chromatic number of \( G_t \) with \( S = (1, 2, 2, 2, 2, \ldots) \) is 5. On the other hand, if \( t = 3 \), then \( \chi_S(G_t) = 6 \).

**Theorem 2.3** If \( S = (1, 2, 2, 2, 2, 2, \ldots) \), then \( \chi_S(G_3) = 6 \).

**Proof.** First, prove that \( \chi_S(G_3) \geq 6 \). Suppose to the contrary that \( \chi_S(G_3) \leq 5 \) and denote by \( f \) an \( S \)-packing 5-coloring of \( G_3 \). Suppose that there exists \( i \in V(G_3) \) such that \( f(i) = 1 = f(i + 1) \). Then note that for any two integers \( j, k \) from the set \( A = \{i - 2, i - 1, i + 2, i + 3, i + 4\} \) we have \( |j - k| \in [6] \), which implies that \( d_{G_3}(j, k) \leq 2 \). Hence, since \( f \) is an \( S \)-packing coloring and \( f(j) \neq 1 \neq f(k) \), we infer \( f(j) \neq f(k) \). Thus, the five vertices in \( A \) should get pairwise distinct colors from \( \{2, 3, 4, 5\} \), which is a contradiction.

From the above paragraph we derive that no two consecutive integers can get color 1 by \( f \). Therefore, at least five of the values \( f(0), f(1), f(2), f(3), f(4), f(5), f(6) \) are different from 1, but again the integers from \( \{0, \ldots, 6\} \) are at pairwise distance at most 2 in \( G_t \), hence those integers that are not colored by 1 should get distinct colors. This is a contradiction, which yields \( \chi_S(G_3) \geq 6 \).

Next, prove that \( \chi_S(G_3) \leq 6 \). We form an \( S \)-packing 6-coloring of \( G_3 \) in such a way that we color the consecutive vertices of \( Z \) using the following pattern of colors:

\[
1123411562113451162311456.
\]

It is easy to observe that any two distinct vertices of \( G_3 \) colored by 1 are non-adjacent. Next, for any two distinct vertices \( a, b \in V(G_t) \), both colored by \( i \in \{2, 3, 4, 5, 6\} \), we have \( |a - b| \geq 7 \), which implies that they are at distance at least 3 in \( G_3 \). Hence, the described coloring is an \( S \)-packing 6-coloring of \( G_3 \), thus \( \chi_S(G_3) \leq 6 \). \( \square \)

**Theorem 2.4** For any odd integer \( t > 3 \), \( \chi_S(G_t) = 5 \) if \( S = (1, 2, 2, 2, 2, \ldots) \).
Proof. Let \( S = (1, 2, 2, 2, \ldots) \) be a sequence. The lower bound, \( \chi_S(G_t) \geq 5 \), is easy to see. Let \( f \) be an \( S \)-packing \( k \)-coloring of \( G_t \), and let \( x \in \mathbb{Z} \) have \( f(x) = 1 \). Since \( G_t \) is 4-regular, \( x \) has four neighbors in \( G_t \), which are pairwise at distance 2. Since none of them can be colored by 1, their colors must be pairwise distinct, that is, \( f(N(x)) = \{2, 3, 4, 5\} \). Thus, \( \chi_S(G_t) \geq 5 \).

Next, prove that \( \chi_S(G_t) \leq 5 \) by forming an \( S \)-packing 5-coloring of \( G_t \). We consider three cases.

**Case 1.** \( t = 4k + 1 \) for \( k \in \mathbb{N} \).

In the case when \( k = 1 \), we color the consecutive vertices of \( \mathbb{Z} \) using the following pattern of colors:

\[
1221331144155.
\]

It is clear that any two distinct vertices of \( G_5 \), colored by 1, are non-adjacent. Further, let \( a, b \in V(G_5) \), where \( a > b \), be both colored by \( i \in \{2, 3, 4, 5\} \). Since the length of the above described sequence is 14, we infer that \( a \) and \( b \) are at distance at least 3. Therefore, the described coloring is an \( S \)-packing 5-coloring of \( G_t \).

Next, let \( k \geq 2 \). Note that a graph \( G_t \) can be presented using two infinite spirals and \( t \) lines; see [7, 8]. Namely, two two-way infinite spirals are drawn in parallel and then \( t \) lines are added in such a way that each of them is orthogonal to both of the spirals. We denote the lines by \( l_0, l_1, \ldots, l_{t-1} \) and the set of intersections between each line \( l_i \), \( i \in \{0, 1, \ldots, t-1\} \), and the spirals by \( L_i = \{2i + jt, \ j \in \mathbb{Z}\} \). Note that \( L_0 = \{\ldots, -2t, -t, 0, t, 2t, \ldots\} \) and \( L_{t-1} = \{\ldots, -2, t-2, 2t-2, 3t-2, 4t-2, \ldots\} \). See Figure [1].

Now, we form the coloring \( f \) of \( G_t \) as follows. We color all vertices from \( L_0 \) one after another starting with the vertex 0 using the following pattern of colors: 2345. Next, for all \( i \in \{2, 4, \ldots, t-1\} \), let \( f(2i + jt) = 1 \) for all odd integers \( j \), and for all \( i \in \{1, 3, \ldots, t-2\} \) let \( f(2i + jt) = 1 \) for all even integers \( j \). It is clear that any two distinct vertices of \( G_t \) both colored by 1 are non-adjacent.

Further, we partition the set of still uncolored vertices of \( G_t \) into 8 subsets as follows:

\[
\begin{align*}
V_1 &= \{kt + 2i; \ i \equiv 1 \pmod{4}, k \equiv 1 \pmod{4}\}; \\
V_2 &= \{kt + 2i; \ i \equiv 1 \pmod{4}, k \equiv 3 \pmod{4}\}; \\
V_3 &= \{kt + 2i; \ i \equiv 2 \pmod{4}, k \equiv 2 \pmod{4}\}; \\
V_4 &= \{kt + 2i; \ i \equiv 2 \pmod{4}, k \equiv 0 \pmod{4}\}; \\
V_5 &= \{kt + 2i; \ i \equiv 3 \pmod{4}, k \equiv 1 \pmod{4}\}; \\
V_6 &= \{kt + 2i; \ i \equiv 3 \pmod{4}, k \equiv 3 \pmod{4}\}; \\
V_7 &= \{kt + 2i; \ i \equiv 0 \pmod{4}, k \equiv 2 \pmod{4}\}; \\
V_8 &= \{kt + 2i; \ i \equiv 0 \pmod{4}, k \equiv 0 \pmod{4}\}.
\end{align*}
\]

Clearly, the sets \( V_1, V_2, \ldots, V_8 \) are disjoint and \( V_1 \cup V_2 \cup \ldots \cup V_8 \) presents the union of all uncolored vertices of \( G_t \). Now, color all vertices from \( V_1 \cup V_3 \) with color 5, all vertices from \( V_2 \cup V_4 \) with color 3, all vertices from \( V_5 \cup V_8 \) with color 2 and all vertices from \( V_4 \cup V_7 \) with color 4 (see Figure [1]).

It remains to prove that any two vertices \( a, b \in V(G_t) \), colored with the same color \( s \in \{2, 3, 4, 5\} \), are at distance at least 3. Clearly, if \( a, b \in L_i \) for some \( i \in \{0, 1, \ldots, t-1\} \), then they are at distance at least 4. Hence, suppose that \( a \in L_m \) and
Figure 1: Representation of $G_t$ with two disjoint infinite spirals and $t$ lines. Vertices of $Z$ are in small size surrounded by brackets, while colors are shown in normal size and are as in Case 1 of the proof of Theorem 2.4.
Let $b \in L_n$, where $0 \leq m < n \leq t - 1$. If $(n - m) > 2 \pmod{t}$, then $d_{G_t}(a, b) \geq 3$. If $n - m = 1 \pmod{t}$, then we have two cases: $a \in L_0, b \in L_1$ and $a \in L_0, b \in L_{t-1}$. In either case we can write $a = jm, b = \pm 2 + jn$, $j, m \in \mathbb{Z}$. Since $f(a) = f(b)$, we have $|j - m| \geq 2$ and hence $d_{G_t}(a, b) \geq 3$. Finally, suppose that $n - m = 2 \pmod{t}$. Let $a = 2m + jm, b = 2n + jn$, $j, m \in \mathbb{Z}$. Since $f(a) = f(b)$, we have that $|j - m| \geq 2$ and thus $d_{G_t}(a, b) \geq 4$. Therefore, $f$ is a $S$-packing 5-coloring of $G_t$.

**Case 2.** $t = 4k - 1$ for $k \in \mathbb{N}$, and $3 \nmid t$.

In this case, color the consecutive vertices of $G_t$ one after another using the following pattern of colors:

$$123145.$$  

Note that for any two vertices $a, b \in V(G_t)$, both colored by 1, we have $a - b = 3m, m \in \mathbb{Z}$. This implies that $a$ and $b$ are not adjacent in $G_t$. Next, if any two vertices $c, d \in V(G_t)$ are both colored by $s \in \{2, 3, 4, 5\}$, then $c - d = 6i, i \in \mathbb{Z}$. Clearly, $c - d \notin \{2, 4, t\}$, and $c - d \notin \{t + 2, t - 2\}$, since $t + 2$ and $t - 2$ are odd integers, but $c - d$ is even. Moreover, $c - d = 6i \neq 2t$, since $3 \nmid t$. Therefore, the distance between $c$ and $d$ in $G_t$ is greater than 2, which implies that the described coloring is an $S$-packing 5-coloring of $G_t$ and hence $\chi_s(G_t) \leq 5$.

**Case 3.** $t = 4k - 1$ for $k \in \mathbb{N}, k \geq 2$, and $3 | t$.

Let $k \geq 2$ be an arbitrary positive integer and define the coloring $f$ of the vertices of $G_t$ as follows. Let $f(j(4k - 3)) = 1$ and $f(j(4k - 3) + \ell) = 1$ for every $j \in \mathbb{Z}, \ell \equiv 1 \pmod{3}, 1 \leq \ell \leq 4k - 4$ (actually, $1 \leq \ell \leq 4k - 6$). Further, for every $m \in \mathbb{Z}, m \equiv 2, 3 \pmod{6}$, $1 \leq m \leq 4k - 4$ (since $4k - 4$ is even and divisible by 3, we have $2 \leq m \leq 4k - 7$), let $f(j(4k - 3) + m) = 2$ if $2 | j$, and otherwise $f(j(4k - 3) + m) = 4$. Finally, for every $p \in \mathbb{Z}, p \equiv 0, 5 \pmod{6}, 1 \leq p \leq 4k - 4$ (5 $\leq p \leq 4k - 4$), let $f(j(4k - 3) + p) = 3$ if $2 | j$, and otherwise $f(j(4k - 3) + p) = 5$. In this way, the consecutive vertices of $\mathbb{Z}$ are colored with the following pattern of colors:

$$1(221333)^v1(144155)^v,$$

where $v = \frac{2(k - 1)}{3}$.

Now, we prove that $f$ is an $S$-packing 5-coloring of $G_t$. First, consider a vertex $a \in V(G_t)$, colored by 1. If $a = j(4k - 3)$ for some $j \in \mathbb{Z}$, then its neighbors are $j(4k - 3) + 2, j(4k - 3) - 2, j(4k - 3) + t$ and $j(4k - 3) - t$. Clearly, $f(j(4k - 3) + 2) \neq 1$ and $f(j(4k - 3) - 2) \neq 1$. Using the fact that each sequence $(122133)^v$ (respectively, $(144155)^v$) contains $t - 3$ integers, we derive that $f(j(4k - 3) + t) \neq 1$ and $f(j(4k - 3) - t) \neq 1$. Next, suppose that $a = j(4k - 3) + \ell$ for some $j \in \mathbb{Z}, \ell \equiv 1 \pmod{3}, 1 \leq \ell \leq 4k - 4$. Again, it is clear that $a + 2$ and $a - 2$ do not receive color 1 by $f$. Since $a + t = (j + 1)(4k - 3) + (\ell + 2)$ and $\ell + 2 \equiv 0 \pmod{3}$, we derive that $f(a + t) \neq 1$. Analogously, $a - t = (j - 1)(4k - 3) + (\ell - 2)$ and $\ell - 2 \equiv 2 \pmod{3}$, hence $f(a - t) \neq 1$. These findings imply that any two vertices of $G_t$, both colored by 1, are at distance at least 2.

Next, let $b \in V(G_t)$ such that $f(b) \in \{2, 3, 4, 5\}$. Clearly, $f(b - 2) \neq f(b), f(b + 2) \neq f(b), f(b - 4) \neq f(b)$ and $f(b + 4) \neq f(b)$. Next, since each sequence $(122133)^v$
(respectively, \((144155)^v\)) contains \(t - 3\) integers, we have \(f(b \pm (t - 2)) \neq f(b)\) and \(f(b \pm t) \neq f(b)\). Further, since \(f(b) \in \{2, 3, 4, 5\}\), \(b = j(4k - 3) + x\) for some positive integers \(j, x\). Hence, \(b + 2t = (j + 2)(4k - 3) + x + 4\), which implies that \(f(b + 2t) \neq f(b)\). Analogously, \(b - 2t = (j - 2)(4k - 3) + x - 4\), \(b + (t + 2) = (j + 1)(4k - 3) + x + 4\) and \(b - (t + 2) = (j - 1)(4k - 3) + x - 4\). Thus, \(f(b - 2t) \neq f(b)\) and \(f(b \pm (2 + t)) \neq f(b)\), which implies that \(f\) is an \(S\)-packing 5-coloring of \(G_t\) and the proof is done. \(\square\)

3 Distance coloring of graphs \(G_t\)

A distance coloring relative to distance \(d\) of a graph \(G\) is a mapping \(V(G) \to \{1, 2, 3, \ldots\}\) such that any two distinct vertices \(a, b \in V(G)\) with \(f(a) = f(b)\) are at distance greater than \(d\) in \(G\). Note that for the sequence \(S = (d, d, d, \ldots)\), an \(S\)-packing coloring presents the distance coloring relative to distance \(d\), that is, the \(d\)-distance coloring.

3.1 Lower bound for distance colorings of \(G_t\)

In this section we present a lower bound for \(\chi_S(G_t)\) with \(S = (d, d, d, \ldots)\).

**Theorem 3.1** If \(t \geq 3\) is an odd integer, \(S = (d, d, d, \ldots)\), and \(d \geq \frac{t+1}{2}\) then

\[
\chi_S(G_t) \geq 1 + t \left(d - \frac{t-3}{2}\right).
\]

**Proof.** We claim that every \(1 + t \left(d - \frac{t-3}{2}\right)\) consecutive integers in \(\mathbb{Z}\) are at pairwise distance at most \(d\) in \(G_t\). It is clear that this claim implies the bound of the theorem. Without loss of generality consider the subsequence of consecutive integers starting with 0. To prove this claim it suffices to show that all integers in the set \(V = \{t \left(d - \frac{t-3}{2}\right)\} = \{1, \ldots, t(d - \frac{t-3}{2})\}\) are at distance at most \(d\) from 0 in \(G_t\). Let \(n = d - \frac{t-3}{2}\), so that we can write \(V = [nt]\).

Suppose \(y \in [nt]\) is an odd integer. Let \(i \in \mathbb{N}_0\) such that \(t(2i - 1) < y \leq t(2i + 1)\). Note that \(i \in \{0, \ldots, \frac{n}{2}\}\). If \(y < 2it + 2\), then there exists \(r \in \left[\frac{t+1}{2}\right]_0\) such that \(y = (2i - 1)t + 2r\), and

\[
P : 0, t, \ldots, (2i - 2)t, (2i - 1)t, (2i - 1)t + 2, \ldots, (2i - 1)t + 2r - 2, y
\]

is a path between 0 and \(y\) whose length is \(2i - 1 + r\). Now,

\[
2i - 1 + r \leq n - 1 + \frac{t + 1}{2} = d - \frac{t - 3}{2} - 1 + \frac{t + 1}{2} = d + 1,
\]

hence \(d_{G_t}(0, y) \leq d\) unless \(2i - 1 = n - 1\) and \(r = \frac{t+1}{2}\). However, if \(y = (n - 1)t + 2\frac{t+1}{2} = nt + 1\), then \(y\) is not in \([nt]\).

On the other hand, if \(y > 2it + 2\), then there exists \(r \in \left[\frac{t-3}{2}\right]_0\) such that \(y = (2i + 1)t - 2r\), and

\[
P : 0, t, \ldots, 2it, (2i + 1)t, (2i + 1)t - 2, \ldots, (2i + 1)t - (2r - 2), y
\]
is a path between 0 and \( y \) whose length is \( 2i + 1 + r \). Since \( y \in [nt] \), we have \( 2i + 1 \leq n \), and so
\[
2i + 1 + r \leq n + \frac{t - 3}{2} = d - \frac{t - 3}{2} + \frac{t - 3}{2} = d,
\]
hence \( d_{G_t}(0, y) \leq d \). Note that if \( y = 2i + 2 \), then \( y \) is an even integer.

If \( y \in [nt] \) is an even integer, by a similar analysis as above one can show \( d_{G_t}(0, y) \leq d \). As noted in the beginning of the proof, we derive that every \( 1 + t \left( d - \frac{t - 3}{2} \right) \) consecutive integers in \( \mathbb{Z} \) are at distance at most \( d \), which implies that they all get distinct colors. Therefore, \( \chi_S(G_t) \geq 1 + t \left( d - \frac{t - 3}{2} \right) \).

\[ \Box \]

### 3.2 Upper bound for distance colorings of \( G_t \)

In this section we present an upper bound for \( \chi_S(G_t) \) with \( S = (d, d, d, \ldots) \).

**Theorem 3.2** If \( t \geq 3 \) is an odd integer and \( S = (d, d, d, \ldots) \), then
\[
\chi_S(G_t) \leq \begin{cases} 
1 + d(d + 1) & ; \ d \leq \frac{t+1}{2} \\
\frac{t}{4}(-t^2 + 2t + 7) & ; \ d \geq \frac{t+1}{2}.
\end{cases}
\]

**Proof.** We start with \( d \leq \frac{t+1}{2} \). We use the greedy (first-fit) algorithm so that when a vertex \( i \) is colored we use the smallest possible color that was not given to already colored vertex at distance at most \( d \) from a given vertex. After coloring the first \( t \) vertices (from 0 to \( t - 1 \)), we then do the same (by using the decreasing order \( i - 1, i - 2, \ldots \)) for \( t \) vertices left of 0, and then continue in the same way by alternating the directions.

The color of a vertex, say 0, thus depends on the colors of vertices from the following sets:

\[
V_1 = \{-2, -4, -6, \ldots, -2d\},
\]
\[
V_2 = \{-t, -2t, -3t, \ldots, -td\},
\]
\[
V_3 = \{-kt \pm 2\ell | k \in [d - 1], \ell \in [d - 1] \text{ and } k + \ell \leq d\}.
\]

Clearly, all integers in \( V_1 \cup V_2 \cup V_3 \) are left of 0, and \( |V_1| = |V_2| = d \) and \( |V_3| = \sum_{k=1}^{d-1} 2(d - k) = d(d - 1) \). Hence the color of 0 depends on the colors of \( d(d + 1) \) vertices to the left of 0. In the worst case these colors are distinct, and we still have one color available to color the vertex 0. This holds for an arbitrary vertex, hence the bound \( \chi_S(G_t) \leq 1 + d(d + 1) \) follows.

Now we prove \( \chi_d(G_t) \leq \frac{t}{4}(-t^2 + 2t + 7) \) for \( d \geq \frac{t+1}{2} \). Again we use the greedy (first-fit) algorithm by starting at an arbitrary vertex (say, 0 \( \in \mathbb{Z} \)) and color the vertices in the increasing order by using the smallest possible color that was not given to already colored vertex at distance at most \( d \) from \( x \). After coloring the first \( t \) vertices, we then do the same by using the decreasing order for the vertices left of 0, and then continue in the same way by alternating the directions. It is clear that the
color of a vertex 0 may depend only on the colors of vertices in \{-1, -2, \ldots, -td\}. Not all of these vertices are at distance at most \(d\) from 0, and we next determine which are not.

Let \(V = \{-t(d-1) - 2(x+1) \mid x \in \{1, \ldots, \frac{t+1}{2} - 2\}\}\). It is easy to see that for a vertex \(v \in V\), we have \(d_G(0, v) > d\). Note that \(|V| = \frac{t+1}{2} - 2\).

Next, let \(W = \{-t(d-y)+2(y+z) \mid y \in \{0, 1, \ldots, \frac{t-3}{2} - 1\}, z \in \{1, \ldots, t-3-2y\}\}\). We claim that \(d_G(0, w) > d\) for any \(w \in W\). Consider the following two types of paths:

\[ R: 0, -t, \ldots, -t(d-y), -t(d-y) + 2, \ldots, -t(d-y) + 2y = r \]

and

\[ S = 0, -t, \ldots, -t(d-y-2), -t(d-y-2) - 2, \ldots, -t(d-y-2) - 2(y+2) = s. \]

Note that every (distinct) path from \(R \cup S\) with length at most \(d\) starting at vertex 0 does not contain a vertex \(u\) such that \(r < u < s\) except a path \(T\), where

\[ T: 0, -t, \ldots, -t(d-y-1), -t(d-y-1) \pm 2, \ldots, -t(d-y-1) \pm 2(j+1), j \leq y. \]

But then the vertex \(t \in T\), where \(r < t < s\), are odd and vertices of \(W\) are even, or vice versa (depending on \(y\)). Hence a path \(T\) does not contain any vertex of \(W\).

Now we prove that \(r < w\) and \(w < s\) for every \(w \in W\). Since \(r = -t(d-y) + 2y\) and \(w = -t(d-y) + 2y + 2z\) reduces to \(0 < 2z\), the first inequality is correct. In the proof of the second inequality, from \(w = -t(d-y) + 2y + 2z \leq -t(d-y) + 2y + 2(t-3-2y)\), we get \(w \leq (-td + ty + 2t - 2y - 4) - 2 = -td + ty + 2t - 2y - 4 = s\). Since \(r < w < s\) for all \(w \in W\), there does not exist a path of length at most \(d\) between vertices 0 and \(w\) in \(G_t\). Hence the color of 0 does not depend on the colors of vertices in \(W\). Note that \(|W| = \sum_{y=0}^{\frac{t-3}{2} - 1} 2\left(\frac{t-3}{2} - y\right) = \frac{t-3}{2} + 1\).

We infer that the color of the vertex 0 depends on the colors of at most \(dt - |V| - |W|\) vertices, and \(dt - |V| - |W| = dt - (\frac{t+1}{2} - 2) - \frac{t-3}{2} + 1 = td + \frac{1}{4}(t^2 + 2t + 3)\). In the worst case these colors are distinct, and we still have one color available to color the vertex 0, and \(td + \frac{1}{4}(t^2 + 2t + 3) + 1 = td + \frac{1}{4}(t^2 + 2t + 7)\), which equals the announced bound.

\[\square\]

### 3.3 Some exact values of the \(d\)-distance chromatic numbers of \(G_t\)

In this subsection, we prove the exact values of \(\chi_d(G_t)\) for \(d \geq t - 3\).

If \(t = 3\), then Theorems 3.1 and 3.2 yield the exact value of \(\chi_d(G_3)\).

**Corollary 3.3** If \(d \geq 2\) is an integer, then \(\chi_d(G_3) = 3d + 1\).

**Theorem 3.4** If \(t \geq 5\) is an odd integer and \(d \geq t - 3\), then

\[\chi_d(G_t) = 1 + t \left( d - \frac{t-3}{2} \right).\]
**Proof.** Let $\ell = 1 + t \left( d - \frac{t-3}{2} \right)$. From Theorem 5.1 it follows that $\chi_d(G_t) \geq \ell$, so it remains to prove that $\chi_d(G_t) \leq \ell$. Let $f$ be a coloring of the vertices of $G_t$ obtained by using the following pattern on the consecutive vertices of $Z$:

$$123\ldots(\ell - 1)\ell.$$ 

We claim that $f$ is a $d$-distance coloring for $G_t$. First, observe that $td < 2\ell$. This means that $d_{G_t}(a, b) \leq d$ implies $|b-a| < 2\ell$ for any two integers $a$ and $b$. Therefore, for any $i \in \mathbb{Z}$, we only need to check the distances between the vertices $i, i+1, \ldots, i+2\ell-1$ in $G_t$. Moreover, from the definitions of $f$ and the $d$-distance coloring, if suffices to prove that $d_{G_t}(0, \ell) \geq d + 1$. Suppose to the contrary that $d_{G_t}(0, \ell) = d' \leq d$.

First, if $d' \leq d - \frac{t-3}{2}$, then $d't < \ell$, which means that $d_{G_t}(0, \ell) > d'$, a contradiction. Hence, let $d' = d - \frac{t-3}{2} + x$, where $x \in \{1, 2, \ldots, \frac{t-3}{2} \}$. We may write, $\ell - 0 = \ell = pt \pm 2r$, where $p$ and $r$ are positive integers and $p + r = d'$. We distinguish three cases.

**Case 1.** $p \geq d' - x + 1$.

Since $p + r = d'$, we have $r \leq x - 1$. From the fact that $pt > \ell$ we derive that only the following vertices lie on a shortest $0, \ell$-path:

$$p't - 2r', p' \in [d' - x + 1], r' \in [d' - p].$$ 

Note that only a vertex $p't - 2r'$, where $p' \geq d' - x + 1$ and $r' \in [d' - p]$, is greater than $\ell$. However, since $r \leq \frac{t-3}{2} - 1$, we have $pt - 2r > \ell$ whenever $p \geq d' - x + 1$. This implies that each of the shortest $0, \ell$-paths does not contain a vertex $\ell$, a contradiction.

**Case 2.** $p = d' - x$.

In this case we derive that $pt = \ell - 1$ and hence in each of the shortest $0, \ell$-paths $P$ only the following vertices lie:

$$p't + 2r', p' \in [d' - x], r' \in [d' - p].$$ 

Note that only a vertex $p't + 2r'$, where $p' = d' - x$ and $r' > 1$ is greater than $\ell$, but such a vertex has the same parity as $pt$. Since $pt = \ell - 1$, vertex $\ell$ has distinct parity as $pt$, and so $\ell$ does not lie on $P$, a contradiction.

**Case 3.** $p \leq d' - x - 1$.

From $pt < \ell$ we infer that on each of the shortest $0, \ell$-paths $P$ only the following vertices lie:

$$p't + 2r', p' \in [d' - x - 1], r' \in [d' - p].$$ 

Note that the biggest vertex of these paths is $pt + 2r$, where $p = d' - x - 1$ and $r = d' - p$ and with some calculation we prove that $pt + 2r \leq \ell - 2$ and hence $pt + 2r < \ell$. Since $r = d' - p$ we have

$$pt + 2r = pt + 2(d' - p),$$ 

and using $p = d' - x - 1$, we get

$$pt + 2r = (d' - x - 1)t + 2d' - 2(d' - x - 1).$$
Further we use $d' = d - \frac{t-3}{2} + x$, hence we get

$$pt + 2r = \left( d - \frac{t-3}{2} \right) t - t + 2x + 2.$$  

Since $x \leq \frac{t-3}{2}$, we derive

$$pt + 2r \leq \left( d - \frac{t-3}{2} \right) t - 1,$$

and from $(d - \frac{t-3}{2}) t - 1 = \ell - 2$ we infer the desired inequality

$$pt + 2r \leq \ell - 2 < \ell,$$

implying that $P$ does not contain $\ell$, a contradiction.

Therefore, there does not exist a 0, $\ell$-path of length $d'$ for any $d' \leq d$, and so $f$ is indeed an $S$-packing $\ell$-coloring of $G_t$. The proof is complete. $\square$

### 3.4 The 2-distance chromatic number of $G_t$

In this subsection, we consider the $S$-packing coloring of $G_t$, where $S = (2, 2, 2, \ldots)$. For a set of integers $\{a_1, \ldots, a_r\}$ where $a_i < m$ for all $i$, we write $a \equiv a_1, \ldots, a_r \pmod{m}$ if $a \equiv a_i \pmod{m}$ for some $i \in [r]$.

**Theorem 3.5** If $t > 3$ is an odd integer, then

$$\chi_2(G_t) = \begin{cases} 5 & t \equiv 1, 9 \pmod{10} \\ 6 & t \equiv 3, 5, 7 \pmod{10} \end{cases},$$

and $\chi_2(G_3) = 7$.

**Proof.** The lower bound $\chi_2(G_t) \geq 5$ is trivial. Indeed, since $G_t$ is 4-regular, there are five vertices in the closed neighborhood $N[v]$ of a vertex $v \in V(G_t)$, and they must receive pairwise distinct colors.

First, consider the case $t \equiv 1, 9 \pmod{10}$. Let $f : \mathbb{Z} \to [5]$ be a coloring produced by the following pattern of colors:

$$12345.$$

Clearly, for any two vertices $u, v \in \mathbb{Z}$ with $f(u) = f(v)$ we get $d_G(u, v) = 5k$ for some $k \in \mathbb{N}$. On the other hand, note that two distinct vertices $u, v \in \mathbb{Z}$ are at distance at most 2 in $G_t$ only if $|u - v| \equiv 1, 2, 3, 4, 7, 8, 9 \pmod{10}$, since either $t \equiv 1 \pmod{10}$ or $t \equiv 9 \pmod{10}$.

Second, consider the case $t \equiv 3, 5, 7 \pmod{10}$ and $t > 3$. For the upper bound 6, we deal with small cases separately, that is, when $t \in \{5, 7, 13\}$ take a coloring $f$ produced by the following pattern of colors:

$$123456.$$
Clearly, for any two vertices \( u, v \in \mathbb{Z} \) with \( f(u) = f(v) \) we get \( |u - v| = 6k \) for some \( k \in \mathbb{N} \). On the other hand, for \( t = 5 \), two vertices \( u, v \in \mathbb{Z} \) are at distance at most 2 in \( G_t \) only if \( |u - v| \in \{2, 3, 4, 5, 7, 10\} \); for \( t = 7 \), two vertices \( u, v \in \mathbb{Z} \) are at distance at most 2 in \( G_t \) only if \( |u - v| \in \{2, 4, 5, 7, 9, 14\} \); and, for \( t = 13 \), two vertices \( u, v \in \mathbb{Z} \) are at distance at most 2 in \( G_t \) only if \( |u - v| \in \{2, 4, 11, 13, 15, 26\} \). Since none of the integers in the distance sets is divisible by 6, the coloring \( f \) is a \((2, 2, 2, 2, 2, 2)\)-packing coloring of \( G_t \) for all \( t \in \{5, 7, 13\} \).

Now, let \( t \geq 15 \), and let \( a \) and \( \ell \) be the unique integers such that \( t + 1 = 5a + \ell \). Next, let \( k = a - \ell \). Consider the following pattern of colors given to consecutive integers:

\[(12345)^t (123456)^k \cdot \]

Note that the basic unit of the pattern \((12345)^t (123456)^k\) consist of exactly \( t + 1 \) colors, hence an integer \( x \) colored by a color \( i \in [6] \) is at \( \mathbb{Z}\)-distance \( t \) to a vertex whose color precedes or follows the place of the color of \( x \) in the basic unit of the pattern; that is, \( i - 1 \) or \( i + 1 \) (with respect to either modulo 5 or modulo 6 depending in which place of the basic unit of the pattern the color is taken). Similarly, an integer \( x \) is at \( \mathbb{Z}\)-distance \( 2t \) to a vertex whose color is two after or two before the place of the color of \( x \) in the basic unit of the pattern, and the same holds, of course, also for the distance 2. It is also easy to see that an integer \( x \) is at \( \mathbb{Z}\)-distance \( t \pm 2 \) to an integer which is either three places before, or one place before, or one place after, or three places after the place of the color of \( x \) in the basic unit of the pattern. Combining these observations with the implication

\[d_{G_t}(x, y) \leq 2 \implies d_{\mathbb{Z}}(x, y) \in \{2, t \pm 2, 2t\},\]

we infer that the color of \( x \) is different from the colors of integers at distance at most 2 from \( x \) in \( G_t \).

Now we prove the lower bound of \( \chi_2(G_t) \geq 6 \) for \( t \equiv 3, 5, 7 \pmod{10} \) and \( t > 3 \). The proof is by contradiction, thus suppose that \( G_t \) is \((2, 2, 2, 2, 2)\)-packing colorable. Let \( H_t \) be the graph obtained from \( P_2 \square P_t \) by adding the edge \((2, 1)\)(4, \( t \)), where \( V(H_t) = \{(i, j) : i \in [4], j \in [t] \} \) and edges are defined in the natural way; see Figure 3. Note that \( H_t \) is a subgraph of \( G_t \) (see Figure 2) in which \( H_t \) is embedded in \( G_t \). Hence, \( H_t \) is \((2, 2, 2, 2, 2)\)-packing colorable too. Let \( f : V(H_t) \to [5] \) be a \((2, 2, 2, 2, 2)\)-packing coloring. Without loss of generality, assume \( f(2, 2) = 2 \), \( f(1, 2) = 4 \), \( f(3, 2) = 5 \), \( f(2, 1) = 1 \) and \( f(2, 3) = 3 \), as shown in Figure 3.

We consider the following three cases for the colors of vertices \((1, 1)\) and \((3, 1)\) given by \( f \):

1. \( f(3, 1) = 4 \), \( f(1, 1) = 5 \),
2. \( f(3, 1) = 4 \), \( f(1, 1) = 3 \),
3. \( f(3, 1) = 3 \).

Note that after coloring the vertices of \( A = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2)\} \), colors of vertices in \( V(P_4 \square P_t) - (A \cup \{(4, t)\}) \) are uniquely determined, since \( f \) is assumed to be a \((2, 2, 2, 2, 2)\)-packing coloring.

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Figure 2: Subgraph $H_t$ (with thick edges) of the graph $G_t$.

Figure 3: Subgraph $H_t$ of the graph $G_t$. 
Case 1. \( f((3,1)) = 4, f((1,1)) = 5 \).
Note that the colors of vertices in \( A \) imply \( f(1,3) = 1 \). Since \( f(3,1) = 4 \), we infer that 
\((3,3)\) is at distance at most 2 in \( H_t \) from any color in [5], which is a contradiction.

Case 2. \( f((3,1)) = 4, f((1,1)) = 3 \).
As mentioned above, the colors of vertices in \( V(P_t \square P_t) - (A \cup \{(4,t)\}) \) are uniquely

determined by the colors given to vertices of \( A \) by \( f \). More precisely, \( f \) partitions

the vertices of \( V(P_t \square P_t) \) into the sets \( V_\ell \), where \( \ell \in [5] \), and \( f(i,j) = \ell \) if and only

\((i,j) \in V_\ell \), as follows:

\[ V_1 = \{(1, j \equiv 4 \mod 5), (2, j \equiv 1 \mod 5), (3, j \equiv 3 \mod 5), (4, j \equiv 0 \mod 5)\}, \]
\[ V_2 = \{(1, j \equiv 0 \mod 5), (2, j \equiv 2 \mod 5), (3, j \equiv 4 \mod 5), (4, j \equiv 1 \mod 5)\}, \]
\[ V_3 = \{(1, j \equiv 1 \mod 5), (2, j \equiv 3 \mod 5), (3, j \equiv 0 \mod 5), (4, j \equiv 2 \mod 5)\}, \]
\[ V_4 = \{(1, j \equiv 2 \mod 5), (2, j \equiv 4 \mod 5), (3, j \equiv 1 \mod 5), (4, j \equiv 3 \mod 5)\}, \]
\[ V_5 = \{(1, j \equiv 3 \mod 5), (2, j \equiv 0 \mod 5), (3, j \equiv 2 \mod 5), (4, j \equiv 4 \mod 5)\}. \]

Clearly, the sets \( V_1, \ldots, V_5 \) are pairwise disjoint and \( V_1 \cup \cdots \cup V_5 = V(H_t) \). Note that

in \( H_t \) the vertex \((4,t)\) is at distance at most 2 from all vertices in \( \{(1,1), (2,1), (2,2), (3,1)\} \),

and these vertices are colored by colors 1, 2, 3 and 4. Hence the only color possibly

available to color \((4,t)\) is 5. According to the 2-distance coloring \( f \) obtained in

Case 2, the vertex \((4,t)\) is colored by 5 if and only if \( t \equiv 4,9 \mod 10 \), which contradicts

\( t \equiv 3, 5, 7 \mod 10 \). Hence, \( H_t \) is not \((2,2,2,2,2)-\)colorable, and so \( G_t \) is not \((2,2,2,2,2)-\)colorable.

Case 3. \( f((3,1)) = 3 \).
This case is very similar to the case 2, and we omit the details.

Finally, note that the pattern

\[
1234567
\]

for coloring the elements of \( \mathbb{Z} \) works in \( G_3 \). Indeed, \( d_{G_3}(x,y) \leq 2 \) if and only if \( d_{\mathbb{Z}}(x,y) \in \{1,2,3,4,5,6\} \), and by the above pattern, two integers \( x \) and \( y \) receive the

same color only if \( |x - y| \) is divisible by 7. Hence, \( \chi_2(G_3) \leq 7 \). The converse, that we

need at least 7 colors for a 2-distance coloring of \( G_3 \) uses a similar argument. Since

for two vertices \( x \) and \( y \) with the same color, \( d_{\mathbb{Z}}(x,y) \notin \{1,2,3,4,5,6\} \), we infer that

every seven consecutive integers must receive pairwise distinct colors in any 2-distance

coloring of \( G_t \), yielding \( \chi_2(G_3) \geq 7 \).

\[ \square \]

4 Concluding remarks

Some of the results of this paper can be used for determining the \( S \)-packing chromatic numbers of certain circulant graphs. Recall that a circulant graph with parameters \( n, s_1, \ldots, s_k \) is the graph \( G = C_n(s_1, \ldots, s_k) \) with \( V(G) = \{0, \ldots, n-1\} \) and vertex

\( i \in V(G) \) is adjacent to \( i + s_j \mod n \) for all \( j \in [k] \). Note that the adjacencies

are defined in the same way as in distance graphs, yet the circulant graphs are finite.

See [11] for an introductory study of circulant graphs and a recent paper [18], where the

so-called perfect colorings of circulant graphs are studied (in fact, the circulant graphs
in the latter paper are infinite and correspond to the distance graphs. It is well-known that \( C_n(s_1, \ldots, s_k) \) is connected if and only if \( \gcd(n, s_1, \ldots, s_k) = 1 \).

The patterns used in some upper bounds for the \( S \)-packing chromatic numbers of graphs \( G_t = G(\mathbb{Z}, \{2, t\}) \) in this paper can be directly applied for \( S \)-packing colorings of the circulant graphs \( C_n(2, t) \), whenever \( n \) is divisible by the length of the pattern. On the other hand, lower bounds (obtained in this paper) for the \( S \)-packing chromatic number of a given distance graph directly imply the same lower bound for the \( S \)-packing coloring of the corresponding circulant graph. We restrict our attention to circulant graphs \( C_n(2, t) \) and may assume without loss of generality that \( n \geq 2t \).

For the sequence \( S = (1, 1, 1) \), Theorem 2.1 provides the lower bound 3. However, in the proof of the theorem we did not use a pattern, but a greedy algorithm. We now present the patterns that can be used for coloring of \( C_n(2, t) \) when \( n \) is divisible by \( t + 2 \).

If \( t = 4k + 1 \) (resp. \( t = 4k - 1 \)), then we consider the following pattern of colors given to consecutive vertices \( C_n(2, t) \):

\[(1122)^k 132 \quad \text{(resp.} \quad (1122)^k 3)\]  

It is easy to see that this yields a proper coloring of \( C_n(2, t) \), when \( n \) is divisible by the length of the pattern, which is \( t + 2 \) in both cases.

**Proposition 4.1** If \( t \geq 3 \) is an odd integer and \( n = (t + 2)m \), where \( m \in \mathbb{N} - \{1\} \), then \( \chi(C_n(2, t)) = 3 \).

Let \( S = (1, 1, 2, 2) \). Theorem 2.2 and its proof providing the patterns for \( S \)-packing colorings yield the following result.

**Corollary 4.2** If \( t \geq 3 \), \( S = (1, 1, 2, 2) \) and \( n = 2(t + 2)m \), where \( m \in \mathbb{N} \), then \( \chi_S(C_n(2, t)) = 4 \).

Indeed, the patterns used when \( t = 4k + 1 \) (resp. \( t = 4k - 1 \)) were:

\[(1122)^k 132(1122)^k 142 \quad \text{(resp.} \quad (1122)^k 3(1122)^k 4)\]

the length of each of which is \( 2(t + 2) \).

In Theorem 2.3 we proved that \( \chi_S(G_3) = 6 \) for \( S = (1, 2, 2, 2, 2, 2, \ldots) \), where for the coloring we used the pattern 112341156211345116231145 6. The pattern has length 25, and we derive the following result.

**Corollary 4.3** If \( S = (1, 2, 2, 2, 2) \) and \( n = 25m \), where \( m \in \mathbb{N} \), then \( \chi_S(C_n(2, 3)) = 6 \).

When \( t > 3 \), the following result is a consequence of Theorem 2.4 and the patterns used in its proof.
Corollary 4.4 For any odd integer $t > 3$ and $S = (1, 2, 2, 2, 2)$, then $\chi_S(C_n(2, t)) = 5$ if

(i) $t = 5$, $n = 14m$, where $m \in \mathbb{N}$ or,

(ii) $t = 4k - 1, 3 \nmid t$ and $n = 6m > 2t$, where $m \in \mathbb{N}$ or,

(iii) $t = 4k - 1, 3 \nmid t$ and $n = 2(4k - 3)m$, where $m \in \mathbb{N} - \{1\}$.

The $S$-packing 5-coloring when $t = 4k + 1$, where $k > 1$, is more complex and cannot be directly applied to color $C_n(2, t)$.

Now we list some consequences of our results for distance colorings of $C_n(2, t)$. Corollary 3.3 implies the following result (the pattern is simply $12\ldots(3d + 1)$).

Corollary 4.5 If $d \geq 2$ is an integer, and $n = (3d + 1)m$, where $m \in \mathbb{N}$, then $\chi_d(C_n(2, 3)) = 3d + 1$.

For $t \geq 5$, Theorem 3.4 implies the following result.

Corollary 4.6 If $t \geq 5$ is an odd integer, $d \geq t - 3$ and $n = m(1 + t \left(d - \frac{t - 3}{2}\right))$, where $m \in \mathbb{N}$, then $\chi_d(C_n(2, t)) = 1 + t \left(d - \frac{t - 3}{2}\right)$.

For the upper bound we use the pattern for consecutive vertices of $C_n(2, t)$, which contains integers from 1 to $1 + t \left(d - \frac{t - 3}{2}\right)$ in the natural order (each integer appears once).

Combining Theorem 3.5 and its proof with Corollary 4.5 we get:

Corollary 4.7 If $t > 3$ is an odd integer, then

$$
\chi_2(C_n(2, t)) = \begin{cases} 
5 & \text{if } t \equiv 1, 9 \pmod{10} \text{ and } n = 5m \geq 4t, m \in \mathbb{N} \\
6 & \text{if } t \in \{5, 7, 13\} \text{ and } n = 6m \geq 4t, m \in \mathbb{N} \\
6 & \text{if } t \equiv 3, 5, 7 \pmod{10}, t \geq 15 \text{ and } n = sm \geq 4t, m \in \mathbb{N}, \end{cases}
$$

where $s = 5k + 6\ell, \ell < 5, t + 1 = 5a + \ell, k = a - \ell$. Furthermore, $\chi_2(C_n(2, 3)) = 7$ for $n = 7m$, where $m \in \mathbb{N}$.

We conclude the paper by proposing an investigation of $S$-packing colorings of circulant graphs. The case $C_n(2, t)$ could be a starting point, where one should resolve the missing cases, which are not covered in this section. The next natural candidates to consider are the circulant graphs $C_n(1, t)$.

Acknowledgements

We thank Přemysl Holub for initial discussions on the topic. B.B. and J.F. acknowledge the financial support of the Slovenian Research Agency (research core funding No. P1-0297 and projects J1-9109 and J1-1693). K.K. was partially supported by the project GA2009525S of the Czech Science Foundation.
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