Logarithmically-concave moment measures I

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Abstract We discuss a certain Riemannian metric, related to the toric Kähler-Einstein equation, that is associated in a linearly-invariant manner with a given log-concave measure in \( \mathbb{R}^n \). We use this metric in order to bound the second derivatives of the solution to the toric Kähler-Einstein equation, and in order to obtain spectral-gap estimates similar to those of Payne and Weinberger.

1 Introduction

In this paper we explore a certain geometric structure related to the moment measure of a convex function. This geometric structure is well-known in the community of complex geometers, see, e.g., Donaldson [13] for a discussion from the perspective of Kähler geometry.

Our motivation stems from the Kannan-Lovász-Simonovits conjecture [17, Section 5], which is concerned with the isoperimetric problem for high-dimensional convex bodies. Essentially, our idea is to replace the standard Euclidean metric by a special Riemannian metric on the given convex body \( K \). This Riemannian metric has many favorable properties, such as a Poincaré inequality with constant one, a positive Ricci tensor, the linear functions are eigenfunctions of the Laplacian, etc. Perhaps this alternative geometry does not deviate too much from the standard Euclidean geometry on \( K \), and it is conceivable that the study of this Riemannian metric will turn out to be relevant to the Kannan-Lovász-Simonovits conjecture.

Let \( \mu \) be an arbitrary Borel probability measure on \( \mathbb{R}^n \) whose barycenter is at the origin. Assume furthermore that \( \mu \) is not supported in a hyperplane. It was proven in [12] that there exists an essentially-continuous convex function \( \psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \), uniquely determined up to translations, such that \( \mu \) is the moment measure.

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of $\psi$, i.e.,

$$\int_{\mathbb{R}^n} b(y) d\mu(y) = \int_{\mathbb{R}^n} b(\nabla \psi(x)) e^{-\psi(x)} dx \quad (1)$$

for any $\mu$-integrable function $b : \mathbb{R}^n \to \mathbb{R}$. In other words, the gradient map $x \mapsto \nabla \psi(x)$ pushes the probability measure $e^{-\psi(x)} dx$ forward to $\mu$. The argument in [12] closely follows the variational approach of Berman and Berndtsson [5], which succeeded the continuity methods of Wang and Zhu [29] and Donaldson [13].

Even in the case where $\mu$ is absolutely-continuous with a $C^{\infty}$-smooth density, it is not guaranteed that $\psi$ is differentiable. From the regularity theory of the Brenier map, developed by Caffarelli [9] and Urbas [28], we learn that in order to conclude that $\psi$ is sufficiently smooth, one has to assume that the support of $\mu$ is convex.

An absolutely-continuous probability measure on $\mathbb{R}^n$ is called log-concave if it is supported on an open, convex set $K \subset \mathbb{R}^n$, and its density takes the form $\exp(-\rho)$ where the function $\rho : K \to \mathbb{R}$ is convex. An important example of a log-concave measure is the uniform probability measure on a convex body in $\mathbb{R}^n$. Here we assume that $\mu$ is log-concave and furthermore, we require that the following conditions are met:

(2) The convex set $K \subset \mathbb{R}^n$ is bounded, the function $\rho$ is $C^{\infty}$-smooth, and $\rho$ and its derivatives of all orders are bounded in $K$.

Under these regularity assumptions, we can assert that

(3) The convex function $\psi$ is finite and $C^{\infty}$-smooth in the entire $\mathbb{R}^n$.

The validity of (3) under the assumption (2) was proven by Wang and Zhu [29] and by Donaldson [13] via the continuity method. Berman and Berndtsson [5] explained how to deduce (3) from (2) by using Caffarelli’s regularity theory [9]. In fact, the argument in [5] requires only the boundness of $\rho$, and not of its derivatives, see also the Appendix in Alesker, Dar and Milman [2]. Since the function $\psi$ is smooth, it follows from (1) that the transport equation

$$e^{-\rho(\nabla \psi(x))} \det \nabla^2 \psi(x) = e^{-\psi(x)} \quad (4)$$

holds everywhere in $\mathbb{R}^n$, where $\nabla^2 \psi(x)$ is the Hessian matrix of $\psi$. In the case where $\rho \equiv \text{Const}$, equation (4) is called the toric Kähler-Einstein equation. We write $x \cdot y$ for the standard scalar product of $x, y \in \mathbb{R}^n$, and $|x| = \sqrt{x \cdot x}$.

**Theorem 1.** Let $\mu$ be a log-concave probability measure on $\mathbb{R}^n$ with barycenter at the origin that satisfies the regularity conditions (2). Then, with the above notation, for any $x \in \mathbb{R}^n$,

$$\Delta \psi(x) \leq 2R^2(K)$$

where $R(K) = \sup_{x \in K} |x|$ is the outer radius of $K$, and $\Delta \psi = \sum \frac{\partial^2 \psi}{\partial x_i^2}$ is the Laplacian of $\psi$.

Theorem 1 is proven by analyzing a certain weighted Riemannian manifold. A weighted Riemannian manifold, sometimes called a Riemannian metric-measure
space, is a triple

\[ X = (\Omega, g, \mu) \]

where \( \Omega \) is a smooth manifold (usually an open set in \( \mathbb{R}^n \)), where \( g \) is a Riemannian metric on \( \Omega \), and \( \mu \) is a measure on \( \Omega \) with a smooth density with respect to the Riemannian volume measure. In this paper we study the weighted Riemannian manifold

\[ M_\mu^* = \left( \mathbb{R}^n, \nabla^2 \psi, e^{-\psi(x)} dx \right). \tag{5} \]

That is, the measure associated with \( M_\mu^* \) has density \( e^{-\psi} \) with respect to the Lebesgue measure on \( \mathbb{R}^n \), and the Riemannian tensor on \( \mathbb{R}^n \) which is induced by the Hessian of \( \psi \) is

\[ \sum_{i,j=1}^n \psi_{ij} dx^i dx^j, \tag{6} \]

where we abbreviate \( \psi_{ij} = \frac{\partial^2 \psi}{\partial x^i \partial x^j} \). There is also a dual description of \( M_\mu^* \).

Recall that the Legendre transform of \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) is the convex function

\[ f^*(x) = \sup_{y \in \mathbb{R}^n} \left[ x \cdot y - f(y) \right] \quad (x \in \mathbb{R}^n). \]

We refer the reader to Rockafellar [26] for the basic properties of the Legendre transform. Denote \( \varphi = \psi^* \). From (4) we see that the Hessian matrix of the convex function \( \psi \) is always invertible, hence it is positive-definite. Therefore \( \varphi \) is a smooth function in \( K \) whose Hessian is always positive-definite. Consequently, the map \( \nabla \varphi : K \to \mathbb{R}^n \) is a diffeomorphism, and \( \nabla \psi \) is its inverse map. One may directly verify that the weighted Riemannian manifold \( M_\mu^* \) is canonically isomorphic to

\[ M_\mu = (K, \nabla^2 \varphi, \mu), \]

with \( x \mapsto \nabla \psi(x) \) being the isomorphism map. In differential geometry, the isomorphism between \( M_\mu \) and \( M_\mu^* \) is the passage from complex coordinates to action/angle coordinates, see, e.g., Abreu [1]. Here are some basic properties of our weighted Riemannian manifold:

(i) The space \( M_\mu \) is stochastically complete. That is, the diffusion process associated with \( M_\mu \) is well-defined, it has \( \mu \) as a stationary measure and “it never reaches the boundary of \( K \”).

(ii) The Bakry-Émery-Ricci tensor of \( M_\mu \) is positive. In fact, it is at least half of the Riemannian metric tensor.

(iii) The Laplacian associated with \( M_\mu \) has an interesting spectrum: The first non-zero eigenvalue is \(-1\), and the corresponding eigenspace contains all linear functions.

Property (ii) is a particular case of the results of Kolesnikov [23, Theorem 4.3] (the notation of Kolesnikov is related to ours via \( V = \Phi = \psi \)), and properties (i) and (iii) are discussed below.
It is important to note that the construction of $M_\mu$ does not rely on the Euclidean structure: Suppose that $V$ is a real $n$-dimensional linear space and $\mu$ is a probability measure on $V$ satisfying the assumptions of Theorem 1. Then the convex function $\psi : V^* \to \mathbb{R}$ whose moment measure is $\mu$ is well-defined up to translations, and it induces the weighted Riemannian manifolds $M_\mu$ and $M^*_\mu$ via the procedure described above. The fact that $M_\mu$ is well-defined without any reference to a Euclidean structure is in sharp contrast with the Riemannian metric-measure space $(\mathbb{R}^n, |\cdot|, \mu)$ that is frequently used for the analysis of the log-concave measure $\mu$.

In the following sections we prove the assertions made in the introduction, and as a sample of possible applications, we explain below how to recover the classical Payne-Weinberger spectral gap inequality [25], up to a constant factor:

**Corollary 1.** Let $\mu$ be a log-concave probability measure on $\mathbb{R}^n$ with barycenter at the origin that satisfies the regularity conditions (2). Then, for any $\mu$-integrable, smooth function $f : K \to \mathbb{R}$,

$$\int_K f^2 d\mu - \left( \int_K f d\mu \right)^2 \leq 2R^2(K) \int_K |\nabla f|^2 d\mu.$$  \hfill (7)

The constant $2R^2(K)$ on the right-hand side of (7) is not optimal. In the case where $\mu$ is the uniform probability measure on a convex body $K \subset \mathbb{R}^n$ with a central symmetry (i.e., $K = -K$), the best possible constant is $4R^2(K)/\pi^2$, see Payne and Weinberger [25].

Throughout this note, a convex body in $\mathbb{R}^n$ is a bounded, open, convex set. We write log for the natural logarithm. A smooth function or a smooth manifold are $C^\infty$-smooth. The unit sphere is $S^{n-1} = \{ x \in \mathbb{R}^n ; |x| = 1 \}$. The five sections below use a variety of techniques, from Itô calculus to maximum principles. We tried to make each section as independent of the others as possible.

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## 2 Continuity of the moment measure

This section is concerned with the continuity of the correspondence between convex functions and their moment measures. Our main result here is Proposition 1 below. We say that a convex function $\psi : \mathbb{R}^n \to \mathbb{R}$ is centered if

$$\int_{\mathbb{R}^n} e^{-\psi(x)} dx = 1, \quad \int_{\mathbb{R}^n} x_i e^{-\psi(x)} dx = 0, \quad i = 1, \ldots, n.$$  \hfill (8)
The role of the barycenter condition in (8) is to prevent translations of $\psi$ which result in the same moment measure. It is well-known that any convex function $\psi : \mathbb{R}^n \to \mathbb{R}$ satisfying $\int e^{-\psi} = 1$ must tend to $+\infty$ at infinity. More precisely, for any such convex function $\psi$ there exist $A, B > 0$ with

$$\psi(x) \geq A|x| - B \quad (x \in \mathbb{R}^n),$$

see, e.g., [19, Lemma 2.1]).

**Proposition 1.** Let $\Omega \subset \mathbb{R}^n$ be a compact set, and let $\psi, \psi_1, \psi_2, \ldots : \mathbb{R}^n \to \mathbb{R}$ be centered, convex functions. Denote by $\mu, \mu_1, \mu_2, \ldots$ the corresponding moment measures, which are assumed to be supported in $\Omega$. Then the following are equivalent:

(i) $\psi_\ell \to \psi$ pointwise in $\mathbb{R}^n$.

(ii) $\mu_\ell \to \mu$ weakly (i.e., $\int bd\mu_\ell \to \int bd\mu$ for any continuous function $b : \Omega \to \mathbb{R}$).

Several lemmas are required for the proof of Proposition 1. For a centered, convex function $\psi : \mathbb{R}^n \to \mathbb{R}$ we define

$$K(\psi) = \left\{ x \in \mathbb{R}^n ; \psi(x) \leq 2n + \inf_{y \in \mathbb{R}^n} \psi(y) \right\},$$

a convex set in $\mathbb{R}^n$. Since the barycenter of $e^{-\psi(x)}dx$ lies at the origin, then $\psi(0) \leq n + \inf_{x \in \mathbb{R}^n} \psi(x)$, according to Fradelizi [14]. Hence the origin is necessarily in the interior of $K(\psi)$. For $x \in \mathbb{R}^n$ consider the Minkowski functional

$$||x||_\psi = \inf \{ \lambda > 0 \, ; \, x/\lambda \in K(\psi) \}.$$

Since a convex function is continuous, then $\psi(x/||x||_\psi) = 2n + \inf \psi$ for $0 \neq x \in \mathbb{R}^n$.

**Lemma 1.** Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a centered, convex function. Then,

$$\psi(x) \geq n||x||_\psi + \psi(0) - 2n \quad (x \in \mathbb{R}^n).$$

**Proof.** Since the barycenter of $e^{-\psi(x)}dx$ lies at the origin, from Fradelizi [14],

$$\psi(0) \leq n + \inf_{x \in \mathbb{R}^n} \psi(x).$$

Whenever $x \in K(\psi)$ we have $||x||_\psi \leq 1$. Therefore (10) follows from (11) for $x \in K(\psi)$. In order to prove (10) for $x \notin K(\psi)$, we observe that for such $x$ we have $||x||_\psi \geq 1$ and hence

$$\psi(0) + n \leq \inf_{y \in \mathbb{R}^n} \psi(y) + 2n = \psi\left(\frac{x}{||x||_\psi}\right) \leq \left(1 - \frac{1}{||x||_\psi}\right) \cdot \psi(0) + \frac{1}{||x||_\psi} \cdot \psi(x),$$

see, e.g., [19, Lemma 2.1]).
due to the convexity of $\psi$. We conclude that $\psi(x) \geq \psi(0) + n\|x\|_\psi$ for any $x \not\in K(\psi)$, and (10) is proven in all cases.

Proof of the direction (i) $\Rightarrow$ (ii) in Proposition 1. Denote

$$K = \{x \in \mathbb{R}^n; \psi(x) \leq 2n + 1 + \psi(0)\},$$

a convex set containing a neighborhood of the origin. Since $e^{-\psi}$ is integrable, then $K$ must be of finite volume, hence bounded. According to Rockafellar [26, Theorem 10.8], the convergence of $\psi_\ell$ to $\psi$ is locally uniform in $\mathbb{R}^n$. In particular, the convergence is uniform on $K$, and there exists $\ell_0 \geq 1$ such that $\psi_\ell(x) > 2n + \psi(0)$ for any $x \in \partial K$ and $\ell \geq \ell_0$. Setting $M = \psi(0) - 1$ we conclude that

$$K(\psi_\ell) \subseteq K, \quad \psi_\ell(0) \geq M \quad \text{for all } \ell \geq \ell_0. \quad (12)$$

Denote $R = \sup_{x \in K} |x|$. From (12) and Lemma 1, for any $\ell \geq \ell_0$,

$$\psi_\ell(x) \geq n\|x\|_\psi + \psi(0) - 2n \geq \frac{n}{R}|x| + (M - 2n) \quad (x \in \mathbb{R}^n). \quad (13)$$

According to our assumption (i) and [26, Theorem 24.5] we have that

$$\nabla \psi_\ell(x) \xrightarrow{\ell \to \infty} \nabla \psi(x)$$

for any $x \in \mathbb{R}^n$ in which $\psi, \psi_1, \psi_2, \ldots$ are differentiable. Let $b : \Omega \to \mathbb{R}$ be a continuous function. Since a convex function is differentiable almost everywhere, we conclude that

$$b(\nabla \psi_\ell(x))e^{-\psi_\ell(x)} \xrightarrow{\ell \to \infty} b(\nabla \psi(x))e^{-\psi(x)} \quad \text{for almost any } x \in \mathbb{R}^n.$$

The function $b$ is bounded because $\Omega$ is compact. We may use the dominated convergence theorem, thanks to (13), and conclude that

$$\int_{\Omega} b d\mu_\ell = \int_{\mathbb{R}^n} b(\nabla \psi_\ell(x))e^{-\psi_\ell(x)}dx \xrightarrow{\ell \to \infty} \int_{\mathbb{R}^n} b(\nabla \psi(x))e^{-\psi(x)}dx = \int_{\Omega} bd\mu.$$

Thus (ii) is proven.

It still remains to prove the direction (ii) $\Rightarrow$ (i) in Proposition 1. A function $f : \mathbb{R}^n \to \mathbb{R}$ is $L$-Lipschitz if $|f(x) - f(y)| \leq L|x - y|$ for any $x, y \in \mathbb{R}^n$.

Lemma 2. Let $L, \varepsilon > 0$. Suppose that $\psi : \mathbb{R}^n \to \mathbb{R}$ is a centered, $L$-Lipschitz, convex function, such that

$$\int_{\mathbb{R}^n} |\nabla \psi(x) \cdot \theta|e^{-\psi(x)}dx \geq \varepsilon \quad \text{for all } \theta \in S^{n-1}. \quad (14)$$

Then,

$$\alpha|x| - \beta \leq \psi(x) \leq L|x| + \gamma \quad (x \in \mathbb{R}^n), \quad (15)$$
where \( \alpha, \beta, \gamma > 0 \) are constants depending only on \( L, \varepsilon \) and \( n \).

**Proof.** Fix \( \theta \in S^{n-1} \) and set \( H = \theta^\perp \), the hyperplane orthogonal to \( \theta \). The function

\[
m_\theta(y) = \inf_{t \in \mathbb{R}} \psi(y + t\theta) \quad (y \in H)
\]

is convex. Furthermore, for any fixed \( y \in H \), the function \( t \mapsto \psi(y + t\theta) \) is convex, \( L \)-Lipschitz and tends to \( +\infty \) as \( t \to \pm \infty \). Hence the one-dimensional convex function \( t \mapsto \psi(y + t\theta) \) attains its minimum at a certain point \( t_0 \in \mathbb{R} \), is non-decreasing on \([t_0, +\infty)\) and non-increasing on \((-\infty, t_0)\). Therefore, for any \( y \in H \),

\[
\int_{-\infty}^{\infty} \left| \frac{\partial \psi(y + t\theta)}{\partial t} \right| e^{-\psi(y + t\theta)} \, dt = \int_{-\infty}^{\infty} \left| \frac{\partial}{\partial t} e^{-\psi(y + t\theta)} \right| \, dt = 2e^{-m_\theta(y)}.
\]

We now integrate over \( y \in H \) and use Fubini’s theorem to conclude that

\[
\int_{\mathbb{R}^n} \left| \nabla \psi(x) \cdot \theta \right| e^{-\psi(x)} \, dx = 2 \int_{H} e^{-m_\theta(y)} \, dy.
\]  

Consider the interval

\[
I_\theta = \{ t \in \mathbb{R} : t\theta \in K(\psi) \}.
\]

Then,

\[
\int_{-\infty}^{\infty} e^{-\psi(t\theta)/2} \, dt \geq \int_{I_\theta} e^{-\psi(t\theta)/2} \, dt \geq e^{-n - \frac{m_\theta(0)}{2}} |I_\theta|, \quad (18)
\]

where \(|I_\theta|\) is the length of the interval \( I_\theta \). Fix a point \( y \in H \). Then there exists \( t_0 \in \mathbb{R} \) for which \( m_\theta(y) = \psi(y + t_0\theta) \). From (18) and from the convexity of \( \psi \),

\[
\int_{-\infty}^{\infty} e^{-\psi(t\theta)/2} \, dt \geq \frac{1}{2} \int_{-\infty}^{\infty} e^{-\psi(y + t_0\theta + t\theta)/2} \, dt \geq \frac{1}{2} e^{-\frac{m_\theta(y)}{2}} \int_{-\infty}^{\infty} e^{-\psi(t\theta)/2} \, dt \geq \frac{1}{2} e^{-\frac{m_\theta(y)}{2}} e^{-n |I_\theta|} \geq \frac{1}{2} e^{-m_\theta(y)} e^{-2n |I_\theta|}, \quad (19)
\]

where in the last passage we used that \( m_\theta(0) \leq \psi(0) \leq n + \inf \psi \leq n + m_\theta(y) \), because the barycenter of \( e^{-\psi(x)} \, dx \) lies at the origin. Integrating (19) over \( y \in H \),

we see that

\[
\int_{H} e^{-m_\theta(y)} \, dy \leq \frac{2e^{2n}}{|I_\theta|} \int_{H} \int_{-\infty}^{\infty} e^{-\psi(t\theta)/2} \, dt \, dy = \frac{2e^{2n}}{|I_\theta|} \int_{\mathbb{R}^n} e^{-\psi} = \frac{2e^{2n}}{|I_\theta|}.
\]

Combine the last inequality with (14) and (16). This leads to the bound

\[
|I_\theta| \leq C_n \left( \int_{\mathbb{R}^n} \left| \nabla \psi(x) \cdot \theta \right| e^{-\psi(x)} \, dx \right)^{-1} \leq \frac{C_n}{\varepsilon}, \quad (20)
\]

for some constant \( C_n \) depending only on \( n \). Recall that the origin belongs to \( K(\psi) \) and hence \( 0 \in I_\theta \). By letting \( \theta \) range over all of \( S^{n-1} \) and glancing at (17) and (20), we see that
\[
K(\psi) \subseteq B(0, C_n/\varepsilon)
\]
where \( B(x, r) = \{ y \in \mathbb{R}^n ; |y - x| \leq r \} \). From (21) and from Lemma 1,
\[
\psi(x) \geq \psi(0) - 2n + n\|x\|_\psi \geq \psi(0) - 2n + \frac{\varepsilon}{C_n}|x| \quad (x \in \mathbb{R}^n),
\]
for \( \hat{C}_n = C_n/n \). By integrating (22) we obtain
\[
1 = \int_{\mathbb{R}^n} e^{-\psi} \leq e^{-(\psi(0)-2n)} \int_{\mathbb{R}^n} e^{-\varepsilon|x|/\hat{C}_n} dx.
\]
Therefore, \( \psi(0) \leq \gamma \) for \( \gamma = 2n + \log(\int_{\mathbb{R}^n} e^{-\varepsilon|x|/\hat{C}_n} dx) \). Since \( \psi \) is \( L \)-Lipschitz, then the right-hand side inequality of (15) follows. Next, observe that
\[
1 = \int_{\mathbb{R}^n} e^{-\psi(x)} dx \geq \int_{\mathbb{R}^n} e^{-(\psi(0)-L|x|} dx = e^{-\psi(0)} \int_{\mathbb{R}^n} e^{-L|x|} dx.
\]
Hence \( \psi(0) \geq \log(\int_{\mathbb{R}^n} e^{-L|x|} dx) \), and the left-hand side inequality of (15) follows from (22).

Proof of the direction (ii) \( \Rightarrow \) (i) in Proposition 1.

Step 1. We claim that
\[
\liminf_{\ell \to \infty} \left( \inf_{\theta \in S^{n-1}} \int_{\Omega} |x \cdot \theta| d\mu_\ell(x) \right) > 0.
\]
Assume that (23) fails. Then there exist sequences \( \ell_j \in \mathbb{N} \) and \( \theta_j \in S^{n-1} \) such that
\[
\lim_{j \to \infty} \int_{\Omega} |x \cdot \theta_j| d\mu_{\ell_j}(x) = 0.
\]
Passing to a subsequence, if necessary, we may assume that \( \theta_j \to \theta_0 \in S^{n-1} \). The sequence of functions \( |x \cdot \theta_j| \) tends to \( |x \cdot \theta_0| \) uniformly in \( x \in \Omega \). Hence, from (ii) and (24),
\[
\int_{\Omega} |x \cdot \theta_0| d\mu(x) = \lim_{j \to \infty} \int_{\Omega} |x \cdot \theta_0| d\mu_{\ell_j}(x) = \lim_{j \to \infty} \int_{\Omega} |x \cdot \theta_j| d\mu_{\ell_j}(x) = 0.
\]
Therefore \( \mu \) is supported in the hyperplane \( \theta_0^\perp \). However, \( \mu \) is the moment measure of the convex function \( \psi : \mathbb{R}^n \to \mathbb{R} \), and according to [12, Proposition 1], it cannot be supported in a hyperplane. We have thus arrived at a contradiction, and (23) is proven.

Step 2. We will prove that there exist \( \alpha, \beta, \gamma > 0 \) and \( \ell_0 \geq 1 \) such that
\[
\alpha|x| - \beta \leq \psi_\ell(x) \leq L|x| + \gamma \quad (\ell \geq \ell_0, x \in \mathbb{R}^n).
\]
Indeed, according to Step 1, there exists \( \ell_0 \geq 1 \) and \( \varepsilon_0 > 0 \) such that
\[ \int_{\mathbb{R}^n} |\nabla \psi_\ell(x) \cdot \theta| e^{-\psi_\ell(x)} \, dx = \int_{\Omega} |x \cdot \theta| \, d\mu_\ell(x) > \varepsilon_0 \quad (\ell \geq \ell_0, \theta \in S^{n-1}). \]  

(26)

Denote \( L = \sup_{x \in \Omega} |x| \). The function \( \psi_\ell \) is centered and convex. Furthermore, for almost any \( x \in \mathbb{R}^n \) we know that \( \nabla \psi_\ell(x) \in \Omega \), because the moment measure of \( \psi_\ell \) is supported in \( \Omega \). Hence, for \( \ell \geq 1 \),

\[ |\nabla \psi_\ell(x)| \leq L \quad \text{for almost any } x \in \mathbb{R}^n. \]  

(27)

Since a convex function is always locally-Lipschitz, then (27) implies that \( \psi_\ell \) is \( L \)-Lipschitz, for any \( \ell \). We may now apply Lemma 2, thanks to (26), and conclude (25).

**Step 3.** Assume by contradiction that there exists \( x_0 \in \mathbb{R}^n \) for which \( \psi_\ell(x_0) \) does not converge to \( \psi(x_0) \). Then there exist \( \varepsilon > 0 \) and a subsequence \( \ell_j \) such that

\[ |\psi_{\ell_j}(x_0) - \psi(x_0)| \geq \varepsilon \quad (j = 1, 2, \ldots). \]  

(28)

From (25) we know that the sequence of functions \( \{\psi_{\ell_j}\}_{j=1,2,\ldots} \) is uniformly bounded on any compact subset of \( \mathbb{R}^n \). Furthermore, \( \psi_{\ell_j} \) is \( L \)-Lipschitz for any \( j \). According to the Arzelà-Ascoli theorem, we may pass to a subsequence and assume that \( \psi_{\ell_j} \) converges locally uniformly in \( \mathbb{R}^n \), to a certain function \( F \). The function \( F \) is convex and \( L \)-Lipschitz, as it is the limit of convex and \( L \)-Lipschitz functions. Furthermore, thanks to (25) we may apply the dominated convergence theorem and conclude that \( F \) is centered.

To summarize, the functions \( F, \psi_{\ell_1}, \psi_{\ell_2}, \ldots \) are \( L \)-Lipschitz, centered and convex. We know that \( \psi_{\ell_j} \to F \) locally uniformly in \( \mathbb{R}^n \). According to the implication (i) \( \Rightarrow \) (ii) proven above, the sequence of measure \( \{\mu_{\ell_j}\}_{j=1,2,\ldots} \) converges weakly to the moment measure of \( F \). But we assumed that \( \mu_{\ell_j} \) converges weakly to \( \mu \), and hence \( \mu \) is the moment measure of \( F \). Thus \( \psi, F : \mathbb{R}^n \to \mathbb{R} \) are two centered, convex functions with the same moment measure \( \mu \). This means that \( \psi \equiv F \), according to the uniqueness part in [12]. Therefore \( \psi_{\ell_j} \to \psi \) pointwise in \( \mathbb{R}^n \), in contradiction to (28), and the proof is complete.

\[ \square \]

3 A preliminary weak bound using the maximum principle

In this section we prove a rather weak form of Theorem 1, which will be needed for the proof of the theorem later on in Section 5. Throughout this section, \( \mu \) is a log-concave probability measure on \( \mathbb{R}^n \) with barycenter at the origin, supported on a convex body \( K \subset \mathbb{R}^n \), with density \( e^{-\rho} \) satisfying the regularity conditions (2). Also, \( \psi : \mathbb{R}^n \to \mathbb{R} \) is the smooth, convex function whose moment measure is \( \mu \), which is uniquely defined up to translation, and \( \varphi = \psi^* \) is its Legendre transform.

In this section we make the following strict-convexity assumptions:
The convex body $K$ has a smooth boundary and its Gauss curvature is positive everywhere. Additionally, there exists $\varepsilon_0 > 0$ with
\[
\nabla^2 \rho(x) \geq \varepsilon_0 \cdot \text{Id} \quad (x \in K),
\]
in the sense of symmetric matrices.

Denote by $\|A\|$ the operator norm of the matrix $A$. Our goal in this section is to prove the following:

**Proposition 2.** Under the above assumptions,
\[
\sup_{x \in \mathbb{R}^n} \|\nabla^2 \psi(x)\| < +\infty.
\]

The argument we present for the demonstration of Proposition 2 closely follows the proof of Caffarelli’s contraction theorem [10, Theorem 11]. An alternative approach to Proposition 2 is outlined in Kolesnikov [22, Section 6]. We begin the proof of Proposition 2 with the following lemma, which is due to Berman and Berndtsson [5]. Their proof is reproduced here for completeness.

**Lemma 3.** $\sup_{x \in K} \phi(x) < +\infty$.

**Proof.** Since $K$ is bounded, it suffices to show that $\phi$ is $\alpha$-Hölder for some $\alpha > 0$. According to the Sobolev inequality in the convex domain $K \subset \mathbb{R}^n$ (see, e.g., [27, Chapter 1]), it is sufficient to prove that
\[
\int_K |\nabla \phi(x)|^p dx < +\infty,
\]
for some $p > n$. Fix $p > n$. The map $x \mapsto \nabla \phi(x)$ pushes the measure $\mu$ forward to $\exp(-\psi(x)) dx$. Hence,
\[
\int_K |\nabla \phi|^p d\mu = \int_{\mathbb{R}^n} |x|^p e^{-\psi(x)} dx < +\infty,
\]
where we used the fact that $e^{-\psi}$ decays exponentially at infinity (see, e.g., (9) above or [19, Lemma 2.1]). Since $\rho$ is a bounded function on $K$ and $e^{-\rho}$ is the density of $\mu$, then (30) follows from (31). \qed

For $x \in \mathbb{R}^n$ denote $h_K(x) = \sup_{y \in K} x \cdot y$, the supporting functional of $K$. The following lemma is analogous to [10, Lemma 4].

**Lemma 4.** $\lim_{R \to \infty} \sup_{|x| \geq R} \|\nabla \psi(x) - \nabla h_K(x)\| = 0$.

**Proof.** The function $\phi : K \to \mathbb{R}$ is convex, hence bounded from below by some affine function, which in turn is greater than some constant on the bounded
set $K$. According to Lemma 3, the function $\varphi$ is also bounded from above. Set $M = \sup_{x \in K} |\varphi(x)|$. By elementary properties of the Legendre transform, for any $x \in \mathbb{R}^n$, 
\[
\psi(x) = x \cdot \nabla \psi(x) - \varphi(\nabla \psi(x)) \leq x \cdot \nabla \psi(x) + M. \tag{32}
\]
Recall that $x/|x|$ is the outer unit normal to $K$ at the boundary point $\nabla h_K(x)$ whenever $0 \neq x \in \mathbb{R}^n$, and that $\sup_{y \in K} x \cdot y = x \cdot \nabla h_K(x)$. Therefore, for any $x \in \mathbb{R}^n$, 
\[
\psi(x) = \sup_{y \in K} [x \cdot y - \varphi(y)] \geq -M + \sup_{y \in K} x \cdot y = -M + x \cdot \nabla h_K(x). \tag{33}
\]
Using (32) and (33), 
\[
(\nabla h_K(x) - \nabla \psi(x)) \cdot \frac{x}{|x|} \leq \frac{2M}{|x|} \quad (0 \neq x \in \mathbb{R}^n). \tag{34}
\]
Recall that $\nabla \psi(x) \in K$ for any $x \in \mathbb{R}^n$. Since $\partial K$ is smooth with positive Gauss curvature, inequality (34) implies that there exist $R_K, \alpha_K > 0$, depending only on $K$, with 
\[
|\nabla h_K(x) - \nabla \psi(x)| \leq \alpha_K \sqrt{\frac{2M}{|x|}} \quad \text{for } |x| \geq R_K. \tag{35}
\]
The lemma follows from (35). \hfill \Box

For $\varepsilon > 0$, $\theta \in \mathbb{R}^n$ and a function $f : \mathbb{R}^n \to \mathbb{R}$ denote 
\[
\delta^{(\varepsilon)}_{\theta \theta} f(x) = f(x + \varepsilon \theta) + f(x - \varepsilon \theta) - 2f(x) \quad (x \in \mathbb{R}^n).
\]
For a smooth $f$ and a small $\varepsilon$, the quantity $\delta^{(\varepsilon)}_{\theta \theta} f(x)/\varepsilon^2$ approximates the pure second derivative $f_{\theta \theta}(x)$. We would like to use the maximum principle for the function $\psi_{\theta \theta}(x)$, but we do not know whether or not it attains its supremum. This is the reason for using the approximate second derivative $\delta^{(\varepsilon)}_{\theta \theta} \psi(x)$ as a substitute.

**Corollary 2.** Fix $0 < \varepsilon < 1$. Then the supremum of $\delta^{(\varepsilon)}_{\theta \theta} \psi(x)$ over all $x \in \mathbb{R}^n$ and $\theta \in S^{n-1}$ is attained.

**Proof.** According to Lemma 4 and the continuity and 0-homogeneity of $\nabla h_K(x)$, 
\[
\lim_{R \to \infty} \sup_{|x| \geq R} \frac{|\nabla \psi(x_1) - \nabla \psi(x_2)|}{|x_1 - x_2|} = \lim_{R \to \infty} \sup_{|x| \geq R} \frac{|\nabla h_K(x_1) - \nabla h_K(x_2)|}{|x_1 - x_2|} = \lim_{R \to \infty} \sup_{|x| = 1} \frac{|\nabla h_K(x_1) - \nabla h_K(x_2)|}{|x_1 - x_2|} = 0, \tag{36}
\]
where $B(x, r) = \{y \in \mathbb{R}^n; |x - y| < r\}$. From Lagrange’s mean value theorem,
\[
\delta^{(\varepsilon)}_{\theta \theta} \psi(x) = (\psi(x + \varepsilon \theta) - \psi(x)) - (\psi(x) - \psi(x - \varepsilon \theta)) \\
\leq \varepsilon \sup_{x_1, x_2 \in B(x, \varepsilon)} |\nabla \psi(x_1) - \nabla \psi(x_2)|. \tag{37}
\]

According to (36) and (37),
\[
\lim_{R \to \infty} \sup_{|x| \geq R \atop \theta \in S^{n-1}} \delta^{(\varepsilon)}_{\theta \theta} \psi(x) \leq \varepsilon \lim_{R \to \infty} \sup_{|x| \geq R \atop x_1, x_2 \in B(x, \varepsilon)} |\nabla \psi(x_1) - \nabla \psi(x_2)| = 0. \tag{38}
\]

Since \( \psi \) is convex and smooth, then the function \( \delta^{(\varepsilon)}_{\theta \theta} \psi \) is non-negative and continuous in \((x, \theta) \in \mathbb{R}^n \times S^{n-1}\). It thus follows from (38) that its supremum is attained.

We shall apply the well-known matrix inequality, which states that when \( A \) and \( B \) are symmetric, positive-definite \( n \times n \) matrices, then
\[
\log \det B \leq \log \det A + Tr \left[ A^{-1}(B - A) \right] = \log \det A + Tr \left[ A^{-1}B \right] - n, \tag{39}
\]
where \( Tr(A) \) stands for the trace of the matrix \( A \). Recall that the transport equation (4) is valid, hence,
\[
\log \det \nabla^2 \psi(x) = -\psi(x) + (\rho \circ \nabla \psi)(x) \quad (x \in \mathbb{R}^n). \tag{40}
\]

In particular, \( \nabla^2 \psi(x) \) is always an invertible matrix which is in fact positive-definite. We denote its inverse by \((\nabla^2 \psi(x))^{-1} = (\psi^{ij}(x))_{i,j=1,...,n}\). For a smooth function \( u : \mathbb{R}^n \to \mathbb{R} \) denote
\[
Au(x) = Tr \left[ (\nabla^2 \psi(x))^{-1} \nabla^2 u(x) \right] = \psi^{ij}(x)u_{ij}(x) \quad (x \in \mathbb{R}^n), \tag{41}
\]
where we adhere to the Einstein convention: When an index is repeated twice in an expression, once as a subscript and once as a superscript, then we sum over this index from 1 to \( n \). According to (39) for any \( \theta \in \mathbb{R}^n \),
\[
\log \det \nabla^2 \psi(x+\theta) \leq \log \det \nabla^2 \psi(x) + \psi^{ij}(x)\psi_{ij}(x+\theta) - n \quad (x \in \mathbb{R}^n), \tag{42}
\]
with an equality for \( \theta = 0 \).

**Proof of Proposition 2.** We follow Caffarelli’s argument [10, Theorem 11]. Our assumption (29) yields that the function \( \rho(x) - \varepsilon_0 |x|^2 / 2 \) is convex. Hence, for any \( x, y \) such that \( x - y, x + y, x \in K \),
\[
\rho(x + y) + \rho(x - y) - 2\rho(x) \geq \frac{\varepsilon_0}{2} (|x + y|^2 + |x - y|^2 - 2|x|^2) = \varepsilon_0 |y|^2. \tag{43}
\]
Fix \( 0 < \varepsilon < 1 \) and abbreviate \( \delta_{\theta \theta} f = \delta^{(\varepsilon)}_{\theta \theta} f \). From (40) and (42) as well as some simple algebraic manipulations, for any \( \theta \in \mathbb{R}^n \),
\[
A(\delta_{\theta \theta} \psi) \geq \delta_{\theta \theta} \left( \log \det \nabla^2 \psi \right) = -\delta_{\theta \theta} \psi + \delta_{\theta \theta}(\rho \circ \nabla \psi). \tag{44}
\]
According to Corollary 2, the maximum of \((x, \theta) \mapsto \delta_{\theta \psi}(x)\) over \(\mathbb{R}^n \times S^{n-1}\) is attained at some \((x_0, e) \in \mathbb{R}^n \times S^{n-1}\). Since \(\psi\) is smooth, then at the point \(x_0\),

\[
0 = \nabla (\delta_{ee} \psi)(x_0) = \nabla \psi(x_0 + \epsilon e) + \nabla \psi(x_0 - \epsilon e) - 2 \nabla \psi(x_0).
\]

In other words, there exists a vector \(u \in \mathbb{R}^n\) such that

\[
\nabla \psi(x_0 + \epsilon e) = \nabla \psi(x_0) + u, \quad \nabla \psi(x_0 - \epsilon e) = \nabla \psi(x_0) - u.
\]

Setting \(v = \nabla \psi(x_0)\) and using (43), we obtain

\[
\delta_{ee} \rho \circ \nabla \psi(x_0) = \rho(v + u) + \rho(v - u) - 2 \rho(v) \geq \epsilon_0 |u|^2.
\]

The smooth function \(x \mapsto \delta_{ee} \psi(x)\) reaches a maximum at \(x_0\), hence the matrix \(\nabla^2 (\delta_{ee} \psi)(x_0)\) is negative semi-definite. Since the matrix \((\nabla^2 \psi)^{-1}(x_0)\) is positive-definite, then from the definition (41),

\[
0 \geq A(\delta_{ee} \psi)(x_0).
\]

Now, (44), (45) and (46) yield

\[
\delta_{ee} \psi(x_0) \geq \delta_{ee} (\rho \circ \nabla \psi)(x_0) \geq \epsilon_0 |u|^2.
\]

By the convexity of \(\psi\),

\[
\psi(x_0 + \epsilon e) - \psi(x_0) \leq \nabla \psi(x_0 + \epsilon e) \cdot (\epsilon e) = (v + u) \cdot (\epsilon e)
\]

and

\[
\psi(x_0 - \epsilon e) - \psi(x_0) \leq \nabla \psi(x_0 - \epsilon e) \cdot (-\epsilon e) = (v - u) \cdot (-\epsilon e).
\]

Summing the last two inequalities yields

\[
\delta_{ee} \psi(x_0) \leq (v + u) \cdot (\epsilon e) + (v - u) \cdot (-\epsilon e) = 2 \epsilon (u \cdot e) \leq 2 |u| \epsilon.
\]

The inequalities (47) and (48) imply that \(|u| \leq 2 \epsilon / \epsilon_0\) and hence from (48),

\[
\delta_{ee} (\psi)(x_0) \leq 4 \epsilon^2 / \epsilon_0.
\]

Consequently, for any \(x \in \mathbb{R}^n\) and \(\theta \in S^{n-1}\) we have \(\delta^{(\epsilon)}_{\theta \psi}(x) \leq 4 \epsilon^2 / \epsilon_0\), and hence

\[
\psi_{\theta \theta}(x) = \lim_{\epsilon \to 0^+} \frac{\delta^{(\epsilon)}_{\theta \theta} \psi(x)}{\epsilon^2} \leq \frac{4}{\epsilon_0}.
\]

Therefore \(\|\nabla^2 \psi(x)\| \leq 4 / \epsilon_0\) for any \(x \in \mathbb{R}^n\), and the proof is complete.

**Remark 1.** Our proof of Proposition 2 provides the explicit bound

\[
\sup_{x \in \mathbb{R}^n} \|\nabla^2 \psi(x)\| \leq 4 / \epsilon_0.
\]
By arguing as in [11], one may improve the right-hand side of (49) to just $1/\varepsilon_0$. We omit the straightforward details.

4 Diffusion processes and stochastic completeness

In this section we consider a diffusion process associated with transportation of measure. Our point of view owes much to the article by Kolesnikov [23], and we make an effort to maintain a discussion as general as the one in Kolesnikov’s work.

Let $\mu$ be a probability measure supported on an open set $K \subseteq \mathbb{R}^n$, with density $e^{-\rho}$ where $\rho : K \to \mathbb{R}$ is a smooth function. Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a smooth, convex function with

$$\lim_{R \to \infty} \left( \inf_{|x| \geq R} \psi(x) \right) = +\infty. \quad (50)$$

Condition (50) holds automatically when $\int e^{-\psi} < \infty$, see (9) above. Rather than requiring that the transport equation (4) hold true, in this section we make the more general assumption that

$$e^{-\rho(\nabla \psi(x))} \det \nabla^2 \psi(x) = e^{-V(x)} \quad (x \in \mathbb{R}^n) \quad (51)$$

for a certain smooth function $V : \mathbb{R}^n \to \mathbb{R}$. Clearly, when $\mu$ is the moment measure of $\psi$, equation (51) holds true with $V = \psi$ and condition (50) holds as well. The transport equation (51) means that the map $x \mapsto \nabla \psi(x)$ pushes the probability measure $e^{-V(x)} \, dx$ forward to $\mu$. In this section we explain and prove the following:

**Proposition 3.** Let $K \subseteq \mathbb{R}^n$ be an open set, and let $V, \psi : \mathbb{R}^n \to \mathbb{R}$ and $\rho : K \to \mathbb{R}$ be smooth functions with $\psi$ being convex. Assume (50) and (51), and furthermore, that

$$\inf_{x \in K} \nabla \rho(x) \cdot x > -\infty. \quad (52)$$

Then the weighted Riemannian manifold $M = (\mathbb{R}^n, \nabla^2 \psi, e^{-V(x)} \, dx)$ is stochastically complete.

**Remark 2.** Note that in the most interesting case where $V = \psi$, the weighted Riemannian manifold $M$ from Proposition 3 coincides with $M_*^\mu$ as defined in (5) and (6) above. Additionally, in the case where $\mu$ is log-concave with barycenter at the origin, condition (52) does hold true: In this case, according to Fradelizi [14], we know that $\rho(0) \leq n + \inf_{x \in K} \rho(x)$. By convexity,

$$\nabla \rho(x) \cdot x \geq \rho(x) - \rho(0) \geq -n \quad (x \in K),$$

and (52) follows. Thus Proposition 3 implies the stochastic completeness of $M_*^\mu$ when $\mu$ is a log-concave probability measure with barycenter at the origin, which satisfies the regularity conditions (2).
We now turn to a detailed explanation of stochastic completeness of a weighted Riemannian manifold. See, e.g., Grigor’yan [15] for more information. The Dirichlet form associated with the weighted Riemannian manifold $M = (\Omega, g, \nu)$ is defined as

$$\Gamma(u, v) = \int_{\Omega} g(\nabla_g u, \nabla_g v) \, d\nu,$$  \hspace{1cm} (53)

where $u, v : \Omega \to \mathbb{R}$ are smooth functions for which the integral in (53) exists. Here, $\nabla_g u$ stands for the Riemannian gradient of $u$. The Laplacian associated with $M$ is the unique operator $L$, acting on smooth functions $u : \Omega \to \mathbb{R}$, for which

$$\int_{\Omega} (Lu)v \, d\nu = -\Gamma(u, v)$$  \hspace{1cm} (54)

for any compactly-supported, smooth function $v : \Omega \to \mathbb{R}$. In the case of the weighted manifold $M = (\mathbb{R}^n, \nabla^2 \psi, e^{-V(x)} \, dx)$ from Proposition 3, we may express the Dirichlet form as follows:

$$\Gamma(u, v) = \int_{\mathbb{R}^n} (\psi^{ij} u_i v_j) e^{-V}$$  \hspace{1cm} (55)

where $\nabla^2 \psi(x)^{-1} = (\psi^{ij}(x))_{i,j=1,\ldots,n}$ and $u_i = \partial u / \partial x^i$. Note that the matrix $\nabla^2 \psi(x)$ is invertible, thanks to (51). As in Section 3 above, we use the Einstein summation convention; thus in (55) we sum over $i, j$ from 1 to $n$. We will also make use of abbreviations such as $\psi_{ijk} = \partial^3 \psi / (\partial x^i \partial x^j \partial x^k)$, and also $\psi^{i\ell}_{\,jk} = \psi_{ik} \psi_{j\ell}$. Therefore, for example,

$$(\psi^{ij})_k = \frac{\partial \psi^{ij}(x)}{\partial x^k} = -\psi^{i\ell}_{\,jk} \psi^{jm} \psi_{\ell mk} = -\psi^{ij}_k.$$  \hspace{1cm} (58)

We may now express the Laplacian $L$ associated with $M = (\mathbb{R}^n, \nabla^2 \psi, e^{-V(x)} \, dx)$ by

$$Lu = \psi^{ij} u_{ij} - (\psi^{ij}_j + \psi^{ij} V_j) u_i$$  \hspace{1cm} (56)

as may be directly verified from (55) by integration by parts.

**Lemma 5.** For any smooth function $u : \mathbb{R}^n \to \mathbb{R}$,

$$Lu = \psi^{ij} u_{ij} - \sum_{j=1}^n \rho_j (\nabla \psi(x)) u_j.$$  \hspace{1cm} (57)

**Proof.** We take the logarithmic derivative of (51) and obtain that for $\ell = 1, \ldots, n$,

$$\psi^{i\ell}_{\,i}(x) = -V_\ell(x) + \sum_{i=1}^n \rho_i (\nabla \psi(x)) \psi_{i\ell}(x) \quad (x \in \mathbb{R}^n).$$  \hspace{1cm} (58)

Multiplying (58) by $\psi^{j\ell}$ and summing over $\ell$ we see that for $j = 1, \ldots, n$, 

\[ \psi^{ij}(x) = -\psi^{kl}(x)V_i(x) + \rho_j(\nabla \psi(x)) \quad (x \in \mathbb{R}^n). \] (59)

Now (57) follows from (56) and (59).

**Lemma 6.** Under the assumptions of Proposition 3, there exists \( A \geq 0 \) such that for all \( x \in \mathbb{R}^n \),

\[ (L\psi)(x) \leq A. \]

**Proof.** Set \( A = \max \{ 0, n - \inf_{y \in K} \nabla \rho(y) \cdot y \} \), which is a finite number according to our assumption (52). From Lemma 5,

\[ L\psi(x) = \psi^{ij}\psi_{ij} - \sum_{j=1}^n \rho_j(\nabla \psi(x))\psi_j(x) = n - \sum_{j=1}^n \rho_j(\nabla \psi(x))\psi_j(x). \]

It remains to prove that \( n - \sum_{j=1}^n \rho_j(\nabla \psi(x))\psi_j(x) \leq A \), or equivalently, we need to show that

\[ \nabla \rho(y) \cdot y \geq n - A \quad \text{for all } y \in K. \] (60)

However, (60) holds true in view of the definition of \( A \) above. Therefore \( L\psi \leq A \) pointwise in \( \mathbb{R}^n \).

The Laplacian \( L \) associated with a weighted Riemannian manifold \( M \) is a second-order, elliptic operator with smooth coefficients. We say that \( M \) is **stochastically complete** if the Itô diffusion process whose generator is \( L \) is well-defined at all times \( t \in [0, \infty) \). In the particular case of Proposition 3, this means the following: Let \( (B_t)_{t \geq 0} \) be the standard \( n \)-dimensional Brownian motion. The diffusion equation with generator \( L \) as in (57) is the stochastic differential equation:

\[ dY_t = \sqrt{2} \left( \nabla^2 \psi(Y_t) \right)^{-1/2} dB_t - \nabla \rho(\nabla \psi(Y_t)) dt, \] (61)

where \( (\nabla^2 \psi(x))^{-1/2} \) is the positive-definite square root of \( (\nabla^2 \psi(x))^{-1} \). For background on stochastic calculus, the reader may consult sources such as Kallenberg [16] or Øksendal [24]. The **stochastic completeness** of \( M \) is equivalent to the existence of a solution \( (Y_t)_{t \geq 0} \) to the equation (61), with an initial condition \( Y_0 = z \) for a fixed \( z \in \mathbb{R}^n \), that does not explode in finite time. Proposition 3 therefore follows from the next proposition:

**Proposition 4.** Let \( \psi, V \) and \( \rho \) be as in Proposition 3. Fix \( z \in \mathbb{R}^n \). Then there exists a unique stochastic process \( (Y_t)_{t \geq 0} \), adapted to the filtration induced by the Brownian motion, such that for all \( t \geq 0 \),

\[ Y_t = z + \int_0^t \sqrt{2} \left( \nabla^2 \psi(Y_s) \right)^{-1/2} dB_s - \int_0^t \nabla \rho(\nabla \psi(Y_s)) ds, \] (62)

and such that almost-surely, the map \( t \mapsto Y_t \ (t \geq 0) \) is continuous in \([0, +\infty)\).
Proof. Since \( \psi(x) \) tends to \(+\infty\) when \( x \to \infty \), then the convex set \( \{ \psi \leq R \} = \{ x \in \mathbb{R}^n; \psi(x) \leq R \} \) is compact for any \( R \in \mathbb{R} \). We use Theorem 21.3 in Kallenberg [16] and the remark following it. We deduce that there exists a unique continuous stochastic process \((Y_t)_{t \geq 0}\) and stopping times \( T_k = \inf\{ t \geq 0; \psi(Y_t) \geq k \}\) such that for any \( k > \psi(z) \), \( t \geq 0 \),

\[
Y_{\min\{t,T_k\}} = z + \int_0^{\min\{t,T_k\}} \sqrt{2} \left( \nabla^2 \psi(Y_t) \right)^{-1/2} dB_t - \int_0^{\min\{t,T_k\}} \nabla \rho(\nabla \psi(Y_t)) dt.
\]

(63)

Denote \( T = \sup_k T_k \). We would like to prove that \( T = +\infty \) almost-surely. According to Dynkin’s formula and Lemma 6, for any \( k > \psi(z) \) and \( t \geq 0 \),

\[
\mathbb{E} \psi(Y_{\min\{t,T_k\}}) = \psi(z) + \mathbb{E} \int_0^{\min\{t,T_k\}} (L \psi)(Y_t) dt \leq \psi(z) + 2At,
\]

where \( A \) is the parameter from Lemma 6. Set \( \alpha = -\inf_{x \in \mathbb{R}^n} \psi(x) \), a finite number in view of (50). Then \( \psi(x) + \alpha \) is non-negative. By Markov-Chebyshev’s inequality, for any \( t \geq 0 \) and \( k > \psi(z) \),

\[
P(T_k \leq t) = P(\psi(Y_{\min\{t,T_k\}}) \geq k) \leq \frac{\mathbb{E} \psi(Y_{\min\{t,T_k\}}) + \alpha}{k + \alpha} \leq \frac{2At + \psi(z) + \alpha}{k + \alpha}.
\]

Hence, for any \( t \geq 0 \),

\[
P(T \leq t) \leq \liminf_{k \to \infty} \frac{2At + \psi(z) + \alpha}{k + \alpha} = 0.
\]

Therefore \( T = +\infty \) almost surely. We may let \( k \) tend to infinity in (63) and deduce (62). The uniqueness of the continuous stochastic process \((Y_t)_{t \geq 0}\) that satisfies (62) follows from the uniqueness of the solution to (63). \( \square \)

For \( z \in \mathbb{R}^n \) write \((Y_t^{(z)})_{t \geq 0}\) for the stochastic process from Proposition 4 with \( Y_0 = z \). Denote by \( \nu \) the probability measure on \( \mathbb{R}^n \) whose density is \( e^{-V(x)} dx \).

The lemma below is certainly part of the standard theory of diffusion processes. We were not able to find a precise reference, hence we provide a proof which relies on the existence of the heat kernel.

**Lemma 7.** There exists a smooth function \( p_t(x,y) \) \( (x,y \in \mathbb{R}^n, t > 0) \) which is symmetric in \( x \) and \( y \), such that for any \( y \in \mathbb{R}^n \) and \( t > 0 \), the random vector

\[
Y_t^{(y)}
\]

has density \( x \mapsto p_t(x,y) \) with respect to \( \nu \).

**Proof.** We appeal to Theorem 7.13 and Theorem 7.20 in Grigor’yan [15], which deal with heat kernels on weighted Riemannian manifolds. According to these theorems, there exists a heat kernel, that is, a non-negative function \( p_t(x,y) \) \( (x,y \in \mathbb{R}^n, t > 0) \) which is symmetric in \( x \) and \( y \), such that for any \( y \in \mathbb{R}^n \) and \( t > 0 \), the random vector

\[
Y_t^{(y)}
\]

has density \( x \mapsto p_t(x,y) \) with respect to \( \nu \).
\(\mathbb{R}^n, t > 0\) symmetric in \(x\) and \(y\) and smooth jointly in \((t, x, y)\), that satisfies the following two properties:

(i) For any \(y \in \mathbb{R}^n\), the function \(u(t, x) = p_t(x, y)\) satisfies

\[
\frac{\partial u(t, x)}{\partial t} = L_x u(t, x) \quad (x \in \mathbb{R}^n, t > 0)
\]

where by \(L_x u(t, x)\) we mean that the operator \(L\) is acting on the \(x\)-variables.

(ii) For any smooth, compactly-supported function \(f : \mathbb{R}^n \to \mathbb{R}\) and \(x \in \mathbb{R}^n\),

\[
\int_{\mathbb{R}^n} p_t(x, y) f(y) d\nu(y) \xrightarrow{t \to 0^+} f(x),
\]

and the convergence in (64) is locally uniform in \(x \in \mathbb{R}^n\).

Theorem 7.13 in Grigor’yan [15] also guarantees that \(\int p_t(x, y) d\nu(x) \leq 1\) for any \(y\). It remains to prove that the random vector \(Y_t^{(y)}\) has density \(x \mapsto p_t(x, y)\) with respect to \(\nu\). Equivalently, we need to show that for any smooth, compactly-supported function \(f : \mathbb{R}^n \to \mathbb{R}\) and \(y \in \mathbb{R}^n, t > 0\),

\[
\mathbb{E} f \left( Y_t^{(y)} \right) = \int_{\mathbb{R}^n} f(x) p_t(x, y) d\nu(x).
\]

Denote by \(v(t, y)\) \((t > 0, y \in \mathbb{R}^n)\) the right-hand side of (65), a smooth, bounded function. We also set \(v(0, y) = f(y)\) \((y \in \mathbb{R}^n)\) by continuity, according to (ii). Then the function \(v(t, y)\) is continuous and bounded in \((t, y) \in [0, +\infty) \times \mathbb{R}^n\). Since \(f\) is compactly-supported then we may safely differentiate under the integral sign with respect to \(y\) and \(t\), and obtain

\[
\frac{\partial v(t, y)}{\partial t} = \int_{\mathbb{R}^n} f(x) \frac{\partial p_t(x, y)}{\partial t} d\nu(y), \quad L_y v(t, y) = \int_{\mathbb{R}^n} f(x) (L_y p_t(x, y)) d\nu(y).
\]

From (i) we learn that

\[
\frac{\partial v(t, y)}{\partial t} = L_y v(t, y) \quad (y \in \mathbb{R}^n, t > 0).
\]

Fix \(t_0 > 0\) and \(y \in \mathbb{R}^n\). Denote \(Z_t = v \left( t_0 - t, Y_t^{(y)} \right)\) for \(0 \leq t \leq t_0\). Then \((Z_t)_{0 \leq t \leq t_0}\) is a continuous stochastic process. From Itô’s formula and (66), for \(0 \leq t \leq t_0\),

\[
Z_t = Z_0 + \int_0^t \left[ L_y v \left( t_0 - t, Y_t^{(y)} \right) - \frac{\partial v}{\partial t} (t_0 - t, Y_t^{(y)}) \right] dt = Z_0 + R_t
\]

where \((R_t)_{0 \leq t \leq t_0}\) is a local martingale with \(R_0 = 0\). Since \(v\) is bounded, then \((Z_t)_{0 \leq t \leq t_0}\) is a bounded process, and \((R_t)_{0 \leq t \leq t_0}\) is in fact a martingale. In particular \(\mathbb{E} R_{t_0} = \mathbb{E} R_0 = 0\). Thus,
\[ \mathbb{E} f(Y_{t_0}^{(y)}) = \mathbb{E} Z_{t_0} = \mathbb{E} Z_0 = v(t_0, y) = \int_{\mathbb{R}^n} f(x)p_{t_0}(x, y)d\nu(x), \]

and (65) is proven. \(\square\)

**Corollary 3.** Suppose that \(Z\) is a random vector in \(\mathbb{R}^n\), distributed according to \(\nu\), independent of the Brownian motion \((B_t)_{t \geq 0}\) used for the construction of \((Y_t^{(z)})_{t \geq 0, z \in \mathbb{R}^n}\). Then, for any \(t \geq 0\), the random vector \(Y_t^{(Z)}\) is also distributed according to \(\nu\).

**Proof.** According to Lemma 7, for any measurable set \(A \subset \mathbb{R}^n\),

\[
\mathbb{P}\left( Y_t^{(Z)} \in A \right) = \int_{\mathbb{R}^n} \mathbb{P}\left( Y_t^{(z)} \in A \right) d\nu(z) = \int_{\mathbb{R}^n} \left( \int_A p_t(z, x)d\nu(x) \right) d\nu(z) = \int_A \left( \int_{\mathbb{R}^n} p_t(x, z)d\nu(x) \right) d\nu(z) = \nu(A). \]

**Remark 3.** Our choice to use stochastic processes in this paper is just a matter of personal taste. All of the arguments here can be easily rephrased in analytic terminology. For instance, the proof of Proposition 4 relies on the fact that \(L\psi\) is bounded from above, similarly to the analytic approach in Grigor’yan [15, Section 8.4]. Another example is the use of local martingales towards the end of Lemma 7, which may be replaced by analytic arguments as in [15, Section 7.4].

### 5 Bakry-Émery technique

In this section we prove Theorem 1. While the viewpoint and ideas of Bakry and Émery [4] are certainly the main source of inspiration for our analysis, we are not sure whether the abstract framework in [3, 4] entirely encompasses the subtlety of our specific weighted Riemannian manifold. For instance, Lemma 9 below seems related to the positivity of the carré du champ \(\Gamma_2\) and to property (ii) in Section 1 above. In the case \(\varepsilon \geq 1/2\), Lemma 9 actually follows from an application of [3, Lemma 2.4] with \(f(x) = x^1\) and \(\rho = 1/2\). Yet, in general, it appears to us advantageous to proceed by analyzing our model for itself, rather than viewing it as an abstract diffusion semigroup satisfying a curvature-dimension bound.

Let \(\mu\) be a log-concave probability measure on \(\mathbb{R}^n\) satisfying the regularity assumptions (2), whose barycenter lies at the origin. Let \(\psi : \mathbb{R}^n \rightarrow \mathbb{R}\) be convex and smooth, such that the transport equation (4) holds true. In Section 4 we proved that \(M^{\psi}_\mu\) is stochastically complete. Since \(M^{\mu}_{\mu^*}\) is isomorphic to \(M_\mu\), then \(M_\mu\) is also stochastically complete.
Let us describe in greater detail the diffusion process associated with \( M_\mu = (K, \nabla^2 \varphi, \mu) \). Recall that the Legendre transform \( \varphi = \psi^* \) is smooth and convex on \( K \), and that

\[
\varphi(x) + \psi(\nabla \varphi(x)) = x \cdot \nabla \varphi(x) \quad (x \in K).
\]

We may rephrase (4) in terms of \( \varphi = \psi^* \), and using \( (\nabla^2 \varphi(x))^{-1} = \nabla^2 \psi(\nabla \varphi(x)) \), we arrive at the equation

\[
\det \nabla^2 \varphi(x) = e^{x \cdot \nabla \varphi(x) - \varphi(x) - \rho(x)} \quad (x \in K).
\]  

The Hessian matrix \( \nabla^2 \varphi \) is invertible everywhere, so we write \( (\nabla^2 \varphi(x))^{-1} = (\varphi_{ij}^k)_{i,j=1,...,n} \), and as before we use abbreviations such as \( \varphi_{i}^j \equiv \varphi_{i}^j \varphi_{km} \varphi_{im} \). In this section, for a smooth function \( u : K \to \mathbb{R} \), denote

\[
Lu(x) = \varphi_{ij}^i u_{ij} - x^i u_i \quad \text{for } x = (x^1, \ldots, x^n) \in K.
\]  

The following lemma is “dual” to Lemma 5.

**Lemma 8.** The operator \( L \) from (68) is the Laplacian associated with the weighted Riemannian manifold \( M_\mu \).

**Proof.** By taking the logarithmic derivative of (67) and arguing as in the proof of Lemma 5, we obtain that for any \( x \in K, i = 1, \ldots, n, \)

\[
\varphi_{ij}^j = x^i - \varphi_{ij}^j \rho_j.
\]  

Integrating by parts and using (69), we see that for any two smooth functions \( u, v : K \to \mathbb{R} \) with one of them compactly-supported,

\[
\int_K \varphi_{ij}^i u_{ij} v_j d\mu = -\int_K v(\varphi_{ij}^j u_{ij} - (\varphi_{ij}^j + \varphi_{ij}^j \rho_j) u_i) e^{-\rho} = -\int_K v(Lu) d\mu. \quad \square
\]

**Lemma 9.** Fix \( \varepsilon > 0 \). For \( x \in K \) set \( f(x) = \varphi_{11}^1(x) \). Then, for the function \( f^\varepsilon(x) = f(x)^\varepsilon \) we have

\[
L(f^\varepsilon) + \varepsilon f^\varepsilon \geq 0.
\]

**Proof.** For \( i, j = 1, \ldots, n, \)

\[
f_i = (\varphi_{11}^1)_i = -\varphi_{1k}^1 \varphi_{ikt}, \quad f_{ij} = -\varphi_{ij}^{11} + 2 \varphi_{j}^{1j} \varphi_{ik}^1.
\]

Therefore,

\[
Lf = \varphi_{ij}^j f_{ij} - x^i f_i = -\varphi_{ij}^{11} + 2 \varphi_{j}^{1j} \varphi_{ij}^{1i} + x^i \varphi_{ij}^{11}.
\]  

Taking the logarithm of (67) and differentiating with respect to \( x^i \) and \( x^f \), we see that
Logarithmically-concave moment measures I

\[ \varphi_{j\ell} - \varphi_{i\ell} = -\rho_{i\ell} + \varphi_{i\ell} + x^j \varphi_{i\ell} \quad (i, \ell = 1, \ldots, n). \]

Multiplying by \( \varphi_i \varphi^{1\ell} \) and summing yields

\[ \varphi_{j1} - \varphi_{k1} = -\varphi_i \varphi^{1\ell} \rho_{i\ell} + \varphi^{11} + x^j \varphi_{j1}. \]

(71)

Since \( \rho \) is convex then its Hessian matrix is non-negative definite and \( \rho_{i\ell} \varphi_i \varphi^{1\ell} \geq 0 \). From (70) and (71),

\[ Lf = \varphi_{j1} \varphi_{j1} - \varphi_{j1} + \rho_{i\ell} \varphi_i \varphi^{1\ell} \geq \varphi_{k1} \varphi_{j1} - \varphi_{k1}^1 = \varphi_{k1} \varphi_{j1} - f. \]

(72)

The chain rule of the Laplacian is \( L(\lambda(f)) = \lambda'(f)Lf + \lambda''(f)\varphi_i f_i f_j \), as may be verified directly. Using the chain rule with \( \lambda(t) = t^\varepsilon \) we see that (72) leads to

\[ L(f^\varepsilon) = \varepsilon f^\varepsilon Lf + \varepsilon(1 - \varepsilon)f^\varepsilon \varepsilon^2 \varphi_{j1} \varphi_{j1} \geq \varepsilon f^\varepsilon \varphi_{k1} \varphi_{j1} - \varepsilon f^\varepsilon + \varepsilon(1 - \varepsilon)f^\varepsilon \varepsilon^2 \varphi_{j1} \varphi_{j1}. \]

That is,

\[ L(f^\varepsilon) + \varepsilon f^\varepsilon \geq \varepsilon f^\varepsilon \left[ \varphi_{k1} \varphi_{j1} + \varepsilon \frac{\varphi_{k1} \varphi_{j1}}{\varphi_{j1}} \right] \geq \varepsilon f^\varepsilon \left[ \varphi_{k1} \varphi_{j1} - \frac{\varphi_{k1} \varphi_{j1}}{\varphi_{j1}} \right], \]

(73)

where we used the fact that \( \varphi_{j1} \varphi_{j1} \geq 0 \) in the last passage (or more generally, \( \varphi_{ij} h_i h_j \geq 0 \) for any smooth function \( h \)). It remains to show that the right-hand side of (73) is non-negative. Denote \( A = (\varphi_{k,j}^{1})_{j,k=1,\ldots,n} \). The matrix \( B = (\varphi_{j,k}^{1})_{j,k=1,\ldots,n} \) is a symmetric matrix, since \( \varphi_{j,k}^{1} = \varphi^{1k} \varphi_{j}^{m \ell} \varphi_{m \ell} \). We have \( A = (\nabla^2 \varphi)B \), and hence

\[ \varphi_{k1} \varphi_{j1} = Tr(A^2) = Tr \left[ \left( \nabla^2 \varphi \right)^{1/2} B \left( \nabla^2 \varphi \right)^{1/2} \right]^2 \]

\[ = \left\| \left( \nabla^2 \varphi \right)^{1/2} B \left( \nabla^2 \varphi \right)^{1/2} \right\|_{HS}^2, \]

since the matrix \( (\nabla^2 \varphi)^{1/2} B (\nabla^2 \varphi)^{1/2} \) is symmetric, where \( \|T\|_{HS} \) stands for the Hilbert-Schmidt norm of the matrix \( T \). We will use the fact that the Hilbert-Schmidt norm is at least as large as the operator norm, that is, \( \|T\|_{HS} \geq \|Tx\|_{2}/|x|_{2} \) for any \( 0 \neq x \in \mathbb{R}^n \). Setting \( e_1 = (1, 0, \ldots, 0) \), we conclude that

\[ \varphi_{k1} \varphi_{j1} \geq \left[ \left( \nabla^2 \varphi \right)^{1/2} B \left( \nabla^2 \varphi \right)^{1/2} \right]_{ij}^2 \left[ \left( \nabla^2 \varphi \right)^{-1/2} e_1 \right]_{j}^2 \]

\[ = \varphi_{11} \varphi_{1} \varphi_{1} \frac{\varphi_{11} \varphi_{11} j}{\varphi_{11}} = \varphi_{11} \varphi_{11} j. \]

Therefore the right-hand side of (73) is non-negative, and the lemma follows. \( \Box \)
Let \((B_t)_{t \geq 0}\) be the standard \(n\)-dimensional Brownian motion. From the results of Section 4, the diffusion process whose generator is \(L\) from (68) is well-defined. That is, there exists a unique stochastic process \((X_t(z))_{t \geq 0, z \in K}\), continuous in \(t\) and adapted to the filtration induced by the Brownian motion, such that for all \(t \geq 0\),

\[
X_t(z) = z + \int_0^t \sqrt{2} \left( \nabla^2 \varphi \left( X_t(z) \right) \right)^{-1/2} dB_t - \int_0^t X_t(z) \, dt. \tag{74}
\]

Our proof of Theorem 1 relies on a few lemmas in which the main technical obstacle is to prove the integrability of certain local martingales.

**Lemma 10.** Fix \(z \in K\) and set \(X_t = X_t(z)\) \((t \geq 0)\). Then for any \(t \geq 0\),

\[
\mathbb{E}X_t = e^{-t}z, \tag{75}
\]

and for any \(\theta \in S^{n-1}\),

\[
e^{2t} \mathbb{E}(X_t \cdot \theta)^2 \geq (z \cdot \theta)^2 + 2 \int_0^t e^{2s} \mathbb{E} \left[ (\nabla^2 \varphi)^{-1}(X_s) \theta \cdot \theta \right] \, ds. \tag{76}
\]

**Proof.** From Itô's formula and (74),

\[
d(e^t X_t) = e^t dX_t + e^t X_t dt = \sqrt{2} e^t \left( \nabla^2 \varphi (X_t) \right)^{-1/2} dB_t.
\]

Therefore \((e^t X_t)_{0 \leq t \leq T}\) is a local martingale, for any fixed number \(T > 0\). However, \(e^t X_t \in e^T K\) for \(0 \leq t \leq T\), and \(K \subset \mathbb{R}^n\) is a bounded set. Therefore \((e^t X_t)_{0 \leq t \leq T}\) is a bounded process, and hence it is a martingale. We conclude that

\[
\mathbb{E}e^t X_t = \mathbb{E}e^0 X_0 = z \quad (t \geq 0),
\]

and (75) is proven. It remains to prove (76). Without loss of generality we may assume that \(\theta = e_1 = (1, 0, \ldots, 0)\). Denoting \(Y_t = X_t \cdot e_1\), we obtain from (74) that

\[
dY_t = \sqrt{2} \left( \nabla^2 \varphi (X_t) \right)^{-1/2} e_1 \cdot dB_t - Y_t dt.
\]

Set \(Z_t = e^{2t} Y_t^2 = e^{2t} (X_t \cdot e_1)^2\). According to Itô's formula,

\[
dZ_t = 2e^{2t} Y_t^2 dt + 2e^{2t} Y_t dY_t + \frac{1}{2} \cdot (2e^{2t}) \cdot 2 \varphi^{11}(X_t) dt = 2e^{2t} \varphi^{11}(X_t) dt + dM_t
\]

where \((M_t)_{t \geq 0}\) is a local martingale with \(M_0 = 0\). This implies that for any \(t \geq 0\),

\[
Z_t = (z \cdot e_1)^2 + M_t + \int_0^t \left( 2e^{2s} \varphi^{11}(X_s) \right) \, ds. \tag{77}
\]

Since \(\varphi^{11}\) is positive, then for any \(t \geq 0\),

\[
Z_t - (z \cdot e_1)^2 \geq M_t. \tag{78}
\]
The convex body $K$ is bounded, and hence $(Z_t)_{0 \leq t \leq T}$ is a bounded process for any number $T > 0$. According to (78), the local martingale $(M_t)_{0 \leq t \leq T}$ is bounded from above, and by Fatou’s lemma it is a sub-martingale. In particular $\mathbb{E}M_t \geq \mathbb{E}M_0 = 0$ for any $t$. From (77),

$$\mathbb{E}Z_t \geq (z \cdot e_1)^2 + 2\mathbb{E} \int_0^t e^{2s} \varphi^{11}(X_s) ds \quad (t \geq 0).$$

Since $\mathbb{E}Z_t < +\infty$ and $\varphi^{11}$ is positive, we may use Fubini’s theorem to conclude that for any $t \geq 0$,

$$\mathbb{E}Z_t \geq (z \cdot e_1)^2 + 2\int_0^t e^{2s} \mathbb{E}\varphi^{11}(X_s) ds. \quad \square$$

**Remark 4.** Once Theorem 1 is established, we can prove that equality holds in (76). Indeed, it follows from Theorem 1 and (77) that $(M_t)_{0 \leq t \leq T}$ is a bounded process and hence a martingale.

**Lemma 11.** Assume that the convex body $K$ has a smooth boundary and that its Gauss curvature is positive everywhere. Assume also that there exists $\varepsilon_0 > 0$ with

$$\nabla^2 \rho(x) \geq \varepsilon_0 \cdot \text{Id} \quad (x \in K) \tag{79}$$

in the sense of symmetric matrices. Fix $z \in K$ and set $X_t = X_t^{(z)}$ ($t \geq 0$). Denote $f(x) = \varphi^{11}(x)$ for $x \in K$. Then, for any $t, \varepsilon > 0$,

$$f(z) \leq e^{t} \left( \mathbb{E} f^{\varepsilon}(X_t) \right)^{1/\varepsilon}. \tag{80}$$

**Proof.** Our assumptions enable the application of Proposition 2. According to the conclusion of Proposition 2, there exists $M > 0$ such that

$$\nabla^2 \psi(y) \leq M \cdot \text{Id} \quad (y \in \mathbb{R}^n).$$

Since $(\nabla^2 \varphi)^{-1}(x) = \nabla^2 \psi(\nabla \varphi(x))$, then,

$$f(x) = \varphi^{11}(x) \leq M \quad (x \in K). \tag{81}$$

From Itô’s formula and (74),

$$e^{t \varepsilon} f^\varepsilon(X_t) = f^\varepsilon(z) + M_t + \int_0^t e^{s \varepsilon} \left[ (Lf^\varepsilon)(X_s) + \varepsilon f^\varepsilon(X_s) \right] ds, \tag{82}$$

where $M_t$ is a local martingale with $M_0 = 0$. According to (82) and Lemma 9, for any $t \geq 0$, 


\[ e^{xt} f^\varepsilon(X_t) \geq f^\varepsilon(z) + M_t. \]  

(83)

We may now use (81) and (83) in order to conclude that the local martingale \((M_t)_{0 \leq t \leq T}\) is bounded from above, for any number \(T > 0\). Hence it is a submartingale, and \(\mathbb{E}M_t \geq \mathbb{E}M_0 = 0\) for any \(t \geq 0\). Now (80) follows by taking the expectation of (83).

\[\square\]

**Remark 5.** We will only use (80) for \(\varepsilon = 1\), even though the statement for a small \(\varepsilon\) is much stronger. In the limit where \(\varepsilon\) tends to zero, it is not too difficult to prove that the right-hand side of (80) approaches \(\exp(t + \mathbb{E}\log f(X_t))\).

The covariance matrix of a square-integrable random vector \(Z = (Z_1, \ldots, Z_n) \in \mathbb{R}^n\) is defined to be

\[ Cov(Z) = (\mathbb{E}Z_i Z_j - \mathbb{E}Z_i \cdot \mathbb{E}Z_j)_{i,j=1,\ldots,n}. \]

**Corollary 4.** Assume that the convex body \(K\) has a smooth boundary and that its Gauss curvature is positive everywhere. Assume also that there exists \(\varepsilon_0 > 0\) with

\[\nabla^2 p(x) \geq \varepsilon_0 \cdot \text{Id} \quad (x \in K). \quad (84)\]

Then for any \(z \in K\) and \(t > 0\),

\[ (\nabla^2 \varphi(z))^{-1} \leq \frac{e^{2t}}{2(e^t - 1)} \cdot Cov\left(X_t^{(z)}\right) \]

in the sense of symmetric matrices.

**Proof.** Fix \(z \in K\), \(t > 0\) and \(\theta \in S^{n-1}\). We need to prove that

\[ (\nabla^2 \varphi(z))^{-1} \theta \cdot \theta \leq \frac{e^{2t}}{2(e^t - 1)} \text{Var}(X_t^{(z)} \cdot \theta). \quad (85)\]

Without loss of generality we may assume that \(\theta = e_1 = (1, 0, \ldots, 0)\). We use Lemma 10 and also Lemma 11 with \(\varepsilon = 1\), and obtain

\[ e^{2t} \mathbb{E}(X_t^{(z)} \cdot e_1)^2 \geq (z \cdot e_1)^2 + 2 \int_0^t e^{2s} \mathbb{E}\varphi^{11}(X_s^{(z)}) ds \geq (z \cdot e_1)^2 + 2\varphi^{11}(z) \int_0^t e^s ds. \]

Recall that \(\mathbb{E}X_t^{(z)} = -e^{-t}z\), according to Lemma 10. Consequently,

\[ \varphi^{11}(z) \leq \frac{e^{2t}}{2(e^t - 1)} \left( \mathbb{E}(X_t^{(z)} \cdot e_1)^2 - (e^{-t}z \cdot e_1)^2 \right) = \frac{e^{2t}}{2(e^t - 1)} \text{Var}(X_t^{(z)} \cdot e_1), \]

and (85) is proven for \(\theta = e_1\). \[\square\]

**Proof of Theorem 1.** Assume first that the convex body \(K\) has a smooth boundary, that its Gauss curvature is positive everywhere, and that there exists \(\varepsilon_0\) for which
(84) holds true. We apply Corollary 4 with \( t = \log 2 \), and conclude that for any \( z \in K \),

\[
Tr \left[ (\nabla^2 \varphi)^{-1}(z) \right] \leq 2Tr \left[ \text{Cov}(X_t^{(z)}) \right] \leq 2\mathbb{E} \left| X_t^{(z)} \right|^2 \leq 2R^2(K)
\]

as \( X_t^{(z)} \in K \) almost surely. Therefore, for any \( x \in \mathbb{R}^n \), setting \( z = \nabla \psi(x) \) we have

\[
\Delta \psi(x) = Tr \left[ (\nabla^2 \psi(x))^2 \right] = Tr \left[ (\nabla^2 \varphi)^{-1}(z) \right] \leq 2R^2(K).
\]

It still remains to eliminate the extra strict-convexity assumptions. To that end, we select a sequence of smooth convex bodies \( K_\ell \subset \mathbb{R}^n \), each with a positive Gauss curvature, that converge in the Hausdorff metric to \( K \). We then consider a sequence of log-concave probability measures \( \mu_\ell \) with barycenter at the origin that converge weakly to \( \mu \), such that \( \mu_\ell \) is supported on \( K_\ell \) and such that the smooth density of \( \mu_\ell \) satisfies (84) with, say, \( \varepsilon_0 = 1/\ell \). We also assume that \( \mu_\ell \) and \( K_\ell \) satisfy the regularity conditions (2).

It is not very difficult to construct the \( \mu_\ell \)'s: For instance, convolve \( \mu \) with a tiny Gaussian (this preserves log-concavity), multiply the density by \( \exp(-|x|^2/\ell) \), truncate with \( K_\ell \) and translate a little so that the barycenter would lie at the origin. This way we obtain a sequence of smooth, convex functions \( \psi_\ell : \mathbb{R}^n \rightarrow \mathbb{R} \) such that \( \mu_\ell \) is the moment measure of \( \psi_\ell \). We may translate, and assume that \( \psi \) and each of the \( \psi_\ell \)'s are centered, in the terminology of Section 2. According to (86), we know that

\[
\Delta \psi_\ell(x) \leq 2R^2(K_\ell) \quad (x \in \mathbb{R}^n, \ell \geq 1).
\]

Furthermore, \( \mu_\ell \rightharpoonup \mu \) weakly, and by Proposition 1, also \( \psi_\ell \rightharpoonup \psi \) pointwise in \( \mathbb{R}^n \). Since \( \psi_\ell \) is smooth, then [26, Theorem 24.5] implies that

\[
\nabla \psi_\ell(x) \xrightarrow{\ell \rightarrow \infty} \nabla \psi(x) \quad (x \in \mathbb{R}^n).
\]

The function \( \psi \) is \( R(K_\ell) \)-Lipschitz, and \( R(K_\ell) \rightarrow R(K) \). Hence \( \sup_{x \in \mathbb{R}^n} |\nabla \psi_\ell(x)| \) is finite. By the bounded convergence theorem, for any \( x_0 \in \mathbb{R}^n \) and \( \varepsilon > 0 \),

\[
\int_{B(x_0, \varepsilon)} \Delta \psi_\ell = \int_{\partial B(x_0, \varepsilon)} \nabla \psi_\ell \cdot N \xrightarrow{\ell \rightarrow \infty} \int_{\partial B(x_0, \varepsilon)} \nabla \psi \cdot N = \int_{B(x_0, \varepsilon)} \Delta \psi, \quad (88)
\]

where \( N \) is the outer unit normal. From (87) and (88) we conclude that for any \( x_0 \in \mathbb{R}^n \) and \( \varepsilon > 0 \),

\[
\int_{B(x_0, \varepsilon)} \Delta \psi \leq Vol_n(B(x_0, \varepsilon)) \cdot \limsup_{\ell \rightarrow \infty} 2R^2(K_\ell) = 2Vol_n(B(x_0, \varepsilon))R^2(K),
\]

where \( Vol_n \) is the Lebesgue measure in \( \mathbb{R}^n \). Since \( \psi \) is smooth, then we may let \( \varepsilon \) tend to zero and conclude that \( \Delta \psi(x_0) \leq 2R^2(K) \), for any \( x_0 \in \mathbb{R}^n \). \( \Box \)
Posteriori, we may strengthen Corollary 4 and eliminate the strict-convexity assumptions. These assumptions were used only in the proof of Lemma 11, to deduce the existence of some number $M > 0$ for which $\nabla^2 \psi(x) \leq M \cdot I_d$, for all $x \in \mathbb{R}^n$. Theorem 1 provides such a number $M = 2R^2(K)$, without any strict-convexity assumptions on $\rho$ or $K$. We may therefore upgrade Corollary 4, and conclude that

**Corollary 5.** Suppose that $\mu$ is a log-concave probability measure in $\mathbb{R}^n$ with barycenter at the origin, satisfying the regularity conditions (2). Let $(X^{(z)}_t)_{t \geq 0, z \in K}$ be the stochastic process given by (74). Then this process is well-defined and bounded, and for any $z \in K$ and $t > 0$,

$$(\nabla^2 \varphi)^{-1}(z) \leq \frac{e^{2t}}{2(e^t - 1)} \cdot \text{Cov} \left(X^{(z)}_t\right)$$

in the sense of symmetric matrices.

**6 The Brascamp-Lieb inequality as a Poincaré inequality**

We retain the assumptions and notation of the previous section. That is, $\mu$ is a log-concave probability measure on $\mathbb{R}^n$, with barycenter at the origin, that satisfies the regularity assumptions (2). The measure $\mu$ is the moment-measure of the smooth and convex function $\psi : \mathbb{R}^n \to \mathbb{R}$. Equation (4) holds true, and we denote $\varphi = \psi^*$. According to the Brascamp-Lieb inequality [8], for any smooth function $u : \mathbb{R}^n \to \mathbb{R}$ such that $ue^{-\psi}$ is integrable,

$$\int_{\mathbb{R}^n} ue^{-\psi} = 0 \implies \int_{\mathbb{R}^n} u^2 e^{-\psi} \leq \int_{\mathbb{R}^n} \left[(\nabla^2 \varphi)^{-1} \nabla u \cdot \nabla u \right] e^{-\psi}. \quad (89)$$

Equality in (89) holds when $u(x) = \nabla \psi(x) \cdot \theta$ for some $\theta \in \mathbb{R}^n$. Note that (89) is precisely the Poincaré inequality with the best constant of the weighted Riemannian manifold $M_\mu^*$. By using the isomorphism between $M_\mu$ and $M_\mu^*$, we translate (89) as follows: For any smooth function $f : K \to \mathbb{R}$ which is $\mu$-integrable,

$$\text{Var}_\mu(f) \leq \int_K (\varphi ij f_i f_j) \, d\mu, \quad (90)$$

where $\text{Var}_\mu(f) = \int f^2 d\mu - (\int f d\mu)^2$. Equality in (90) holds when $f(x) = A + x \cdot \theta$, for some $\theta \in \mathbb{R}^n$ and $A \in \mathbb{R}$. This is in accordance with the fact that linear functions are eigenfunctions, i.e.,

$$Lx^i = -x^i \quad (i = 1, \ldots, n)$$

where $Lu = \varphi ij u_{ij} - x^i u_i$ is the Laplacian of the weighted Riemannian manifold $M_\mu$. In fact, (90) means that the spectrum of the (Friedrich extension of the) operator $L$ cannot intersect the interval $(-1, 0)$, and that the restriction of $-L$ to the subspace
of mean-zero functions is at least the identity operator, in the sense of symmetric operators.

Theorem 1 states that \( \Delta \psi (x) \leq 2R^2(K) \) everywhere in \( \mathbb{R}^n \). A weak conclusion is that \( \nabla^2 \psi (x) \leq 2R^2(K) \cdot \text{Id} \), or rather, that \( (\nabla^2 \varphi (x))^{-1} \leq 2R^2(K) \cdot \text{Id} \). By substituting this information into (90), we see that for any smooth function \( f \in L^1(\mu) \),

\[
\text{Var}_\mu (f) \leq 2R^2(K) \int_K |\nabla f|^2 d\mu.
\]

(91)

This completes the proof of Corollary 1. See [20, 21] for more Poincaré-type inequalities that are obtained by imposing a Riemannian structure on the convex body \( K \). The Kannan-Lovász-Simonovits conjecture speculates that \( R^2(K) \) in (91) may be replaced by a universal constant times \( \|\text{Cov}(\mu)\| \), where \( \text{Cov}(\mu) \) is the covariance matrix of the random vector that is distributed according to \( \mu \), and \( \| \cdot \| \) is the operator norm.

A potential way to make progress towards the Kannan-Lovász-Simonovits conjecture is to try to bound the matrices \( (\nabla^2 \varphi)^{-1}(x) \) \( (x \in K) \) in terms of \( \text{Cov}(\mu) \).

The following proposition provides a modest step in this direction:

**Proposition 5.** Fix \( \theta \in S^{n-1} \) and denote

\[
V = \int_{\mathbb{R}^n} (x \cdot \theta)^2 d\mu(x).
\]

Then, for any \( p \geq 1 \),

\[
\left( \int_K \left| \frac{(\nabla^2 \varphi)^{-1} \cdot \theta}{V} \right|^p d\mu \right)^{1/p} \leq 4p^2.
\]

**Proof.** Without loss of generality, assume that \( \theta = e_1 = (1, 0, \ldots, 0) \). According to Corollary 5, for any \( z \in K \) and \( t > 0 \),

\[
\varphi^{11}(z) \leq \frac{e^{2t}}{2(e^t - 1)} \text{Var} \left( X_t(z) \cdot e_1 \right) \leq \frac{e^{2t}}{2(e^t - 1)} \mathbb{E} \left( X_t(z) \cdot e_1 \right)^2.
\]

(92)

Let \( Z \) be a random vector that is distributed according to \( \mu \), independent of the Brownian motion used in the construction of the process \( (X_t(z))_{t \geq 0, z \in K} \). It follows from Corollary 3 that for any fixed \( t \geq 0 \) the random vector \( X_t(Z) \) is also distributed according to \( \mu \). By setting \( t = \log 2 \) in (92) and applying Hölder’s inequality, we see that for any \( p \geq 1 \),

\[
\mathbb{E} |\varphi^{11}(Z)|^p \leq 2^p \mathbb{E} \left| X_t(Z) \cdot e_1 \right|^{2p} = 2^p \mathbb{E} |Z \cdot e_1|^{2p}.
\]

(93)

The random vector \( Z \) has a log-concave density. According to the Berwald inequality [6, 7],
\[
\left( \mathbb{E} |Z \cdot e_1|^{2p} \right)^{1/(2p)} \leq \frac{\Gamma(2p + 1)^{1/(2p)}}{\Gamma(3)^{1/2}} \sqrt{\mathbb{E} |Z \cdot e_1|^2} \leq \frac{2p}{\sqrt{2}} \sqrt{V}.
\]

(The Berwald inequality is formulated in [6, 7] for the uniform measure on a convex body, but it is well-known that it applies for all log-concave probability measures. For instance, one may deduce the log-concave version from the convex-body version by using a marginal argument as in [18]). The proposition follows from (93) and (94).

\[\square\]

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