Examples of non-trivial contact mapping classes for overtwisted contact manifolds in all dimensions

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Abstract

The aim of the article is to construct (infinitely many) examples in all dimensions of contactomorphisms of closed overtwisted contact manifolds that are smoothly isotopic but not contact-isotopic to the identity. Hence, overtwisted contact structures in each odd dimension can have a rigid behavior as far as the problem of deformations of contactomorphisms is concerned.

1 Introduction

One of the problems in the field of contact topology is to understand the topology of the space of contactomorphisms $\mathcal{D}(V, \xi)$ of a given contact manifold $(V, \xi)$ in comparison with that of the space of diffeomorphisms $\mathcal{D}(V)$ of the underlying smooth manifold $V$ or, more specifically, the problem of understanding the map $j_* : \pi_k(\mathcal{D}(V, \xi)) \to \pi_k(\mathcal{D}(V))$ induced by the natural inclusion $j : \mathcal{D}(V, \xi) \to \mathcal{D}(V)$.

If $\Xi(V)$ denotes the space of all the contact structures on $V$, in the case of closed manifolds the natural map $\mathcal{D}(V) \to \Xi(V)$ given by $\phi \mapsto \phi_* \xi$ helps to understand the properties of the $j_*$, and shows that the relation between the topology of $\mathcal{D}(V, \xi)$ and that of $\mathcal{D}(V)$ is mediated by the topology of $\Xi(V)$. Indeed, (the proof of) Gray’s theorem implies, modulo a general fibration criterion, that this map is a locally-trivial fibration with fiber $\mathcal{D}(V, \xi)$ (see [GM17] for an explanation of this result or [Mas15] for a more detailed proof). Then, the exact long sequence of homotopy groups

$$\ldots \to \pi_{k-1}(\Xi(V)) \to \pi_k(\mathcal{D}(V, \xi)) \to \pi_k(\mathcal{D}(V)) \to \pi_k(\Xi(V)) \to \ldots$$

associated to the fibration gives a relationship between the topologies of the three spaces $\mathcal{D}(V)$, $\mathcal{D}(V, \xi)$ and $\Xi(V)$.

As far as the 3-dimensional case is concerned, the availability of classification results for the isotopy classes of tight contact structures on particular 3-manifolds $V$ gives some explicit results about the lower homotopy groups in
the long exact sequence above for these specific manifolds. The reader can consult [GGP04, Bou06, DG10, GK14, GM17] for results on $\pi_1(\Xi(\mathcal{V},\xi))$. The situation in higher dimension is more complicated, due to the lack of classification results. The only results known so far are contained in [Bou06, MN16, LZ]. In the first paper, the author gives results on some homotopy groups $\pi_k(\Xi(\mathcal{V},\xi))$, for particular contact manifolds $(\mathcal{V},\xi)$, using tools from contact homology. In [MN16], the authors give examples of contact manifolds $(\mathcal{V},\xi)$ for which $\ker(\pi_0(\mathcal{D}(\mathcal{V},\xi)) \to \pi_0(\mathcal{D}(\mathcal{V})))$ is non-trivial; these examples rely on constructions in [MNW13], which we will also use in the following. The last paper, dealing with the non-compact case, contains examples of embeddings of braid groups in the contactomorphism group of contactizations of certain non-compact symplectic manifolds.

All the examples recalled so far are given on tight contact manifolds. For the 3-dimensional case, the dichotomy tight-overtwisted is well known since [Eli89] and plays an important role in the classification results on which the cited examples are based. In the higher dimensional case, a clear definition of overtwistedness is given in [BEM15], according to which the three examples above are also tight.

We remark that some of the results cited so far show that the rigidity, characterizing the tight class as far as the problem of deformations of contact structures is concerned, can be used in particular tight manifolds to obtain rigidity results for the problem of deformations of contactomorphisms.

As far as the class of overtwisted manifolds is concerned, the only result known at the moment is the classification result of the path components of the space of contactomorphisms for all overtwisted contact structures on the 3-sphere. This result, without proof until recently, is attributed to Y. Chekanov according to [EF09, Remark 4.16]. T. Vogel published a complete proof of this classification in [Vog], where it is also proven, using 3-dimensional techniques, that the space of embeddings of overtwisted disks in one of the overtwisted contact structures on $\mathbb{S}^3$ is not path-connected. This gives in particular the first known examples of contactomorphisms of overtwisted 3-manifolds that are smoothly isotopic but not contact-isotopic to the identity (we recall that, according to [Cer68], each orientation-preserving diffeomorphism of the 3-sphere is smoothly isotopic to the identity).

In this article we give other explicit examples of overtwisted $(\mathcal{V},\xi)$ such that $\ker(\pi_0(\mathcal{D}(\mathcal{V},\xi)) \to \pi_0(\mathcal{D}(\mathcal{V})))$ is non-trivial. Though, we bypass here the problem of understanding the $\pi_0$ of the space of embeddings of overtwisted disks, about which nothing is known so far in high dimensions; the advantage of our approach is then that it gives examples in all (odd) dimensions.

More precisely, we will work with the following construction. Given an integer $n \geq 1$, let $(M^{2n-1},\alpha_+\alpha_-)$ be one of the infinitely many Liouville pairs constructed in [MNW13] (and recalled in Section 2.2 below) and consider the contact structure $\xi = \ker\left(\frac{1+\cos(s)}{2}\alpha_+ + \frac{1-\cos(s)}{2}\alpha_- + \sin(s)dt\right)$ on the manifold $M \times S^1 \times S^1$, where the notation $S^1_x$ simply denotes the choice of a coordinate $x$ on the manifold $S^1$. Denote then by $\xi'$ the overtwisted contact structure obtained from $\xi$ by a half Lutz-Mori twist (see Section 2.1) along
Figure 1: Dividing set, in red, on the torus of Example 1.2.

\[ M \times \{0\} \times \{0\} \text{ and denote by } \xi_k' \text{ its pull-back via the } k\text{-fold cover } \pi_k : M \times \mathbb{T}^2 \to M \times \mathbb{T}^2, \pi_k(p, t, s) = (p, t, ks). \text{ We will then prove the following result:} \]

Theorem 1.1. If \( k \geq 2 \), the contact transformation \( f : (M \times \mathbb{T}^2, \xi_k') \to (M \times \mathbb{T}^2, \xi_k') \) defined by \( f(p, t, s) = (p, t, s + \frac{2\pi}{k}) \) is smoothly isotopic but not contact-isotopic to the identity.

Example 1.2. In the 3-dimensional case, \((M, \alpha_{\pm}) = (S^1, \pm d\theta)\). Moreover, if \( k = 2 \), the contact structure \( \xi_2' \) on \( V := M \times \mathbb{T}^2 \) is the unique contact structure which is invariant by the left-action by multiplication of \( M = S^1 \) on \( V \), invariant by the \( f(\theta, t, s) = (\theta, t, s + \pi) \) defined in the statement and such that each torus \( \{\theta_0\} \times \mathbb{T}_2(t, s) \) is convex with dividing set as in Figure 1. Our theorem then says that \( f \) is not contact-isotopic to the identity. We remark that even in this simple setting, we didn’t find trace of this result in the literature.

We point out that the results in [Vog] and Theorem 1.1 both show that overtwisted contact structures, which are flexible from the point of view of deformations of contact structures, can on the contrary behave rigidly as far as the problem of deformations of contactomorphisms is concerned. This can be interpreted as a manifestation of the fact that in contact topology the boundary between flexibility and rigidity may vary according to the problem analyzed.

Outline Section 2.1 recalls the definition of half Lutz-Mori twist. Section 2.2 reviews the explicit constructions of Liouville pairs appearing in [MNW13]. Then Section 2.3 states Proposition 2.3, which describes the effects of a half Lutz-Mori twist on Chern classes in this context.

Section 3 is then devoted to the proof of this proposition. In Section 3.1 we first use a geometric interpretation of Chern classes in terms of sections to study how certain local modifications of the vector bundle affect these classes. We then use this study in Section 3.2 to prove Proposition 2.3.

Finally, in Section 4 we prove Theorem 1.1 by contradiction. Assuming that the contactomorphism \( f \) is contact-isotopic to the identity, we construct a contactomorphism between two contact manifolds; on the other hand, Proposition 2.3 can be used to prove that these manifolds are not isomorphic.
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2 Liouville pairs and Lutz-Mori twists

2.1 The half Lutz-Mori twist

Developing some ideas introduced by A.Mori in [Mor] in the 5-dimensional case, P. Massot, K. Niederkrüger and C. Wendl introduce in [MNW13] the notion of Lutz-Mori twist along a manifold belonging to a Liouville pair as a generalization of the well known 3-dimensional Lutz twists.

In this section we give an explicit description of how to perform the half version of the Lutz-Mori twist in particular coordinates in a neighbourhood of the contact submanifold. Let’s dive in the details.

Let $(V, \xi)$ be a contact manifold having as a codimension-2 contact submanifold $(M, \xi^+)\subset (V, \xi)$ such that $\alpha_+$ defining $\xi^+$ belongs to a Liouville pair, defined as follows:

**Definition 2.1 ([MNW13]).** Let $M^{2m+1}$ be an oriented manifold. We call Liouville pair on $M$ a couple of contact forms $(\alpha_+, \alpha_-)$ such that $\pm \alpha_+ \wedge (d\alpha_+)^m > 0$ and such that the form $e^r\alpha_+ + e^{-r}\alpha_-$ is a Liouville form on $M \times \mathbb{R}_r$, i.e. its differential is a symplectic form on $M \times \mathbb{R}$.

We point out that the existence of Liouville pairs on closed manifolds is not trivial; at the moment, the only known examples in high dimension are given by the construction in [MNW13, Section 8], which is nonetheless a source of infinitely many non-homeomorphic manifolds with Liouville pairs in each (odd) dimension. In Section 2.2 we will recall the properties of this construction which are needed in order to prove Theorem 1.1.

We now want to find particular coordinates near the submanifold $(M, \xi^+)$. For all $\varepsilon > 0$, denote $D^2_\varepsilon$ the (open) disk of radius $\varepsilon$ centered at the origin inside $\mathbb{R}^2$. Consider then a smooth map $\Psi : D^2_\varepsilon \setminus \{0\} \to S^1 \times (0, \varepsilon)$, defined by $\Psi(r, \varphi) = (\varphi, \psi(r))$, where $(r, \varphi)$ are polar coordinates on $D^2_\varepsilon$ and $\psi : (0, \varepsilon) \to (0, \varepsilon)$ is smooth, strictly increasing, equal to $r^2$ on $(0, \varepsilon)$ and equal to $r$ on $(\frac{2}{3}\varepsilon, \varepsilon)$.

Consider now the 1-form $\alpha_0 = \frac{1 + \cos(s)}{2} \alpha_+ + \frac{1 - \cos(s)}{2} \alpha_- + \sin(s) \, dt$ on $M \times S^1_\varepsilon \times (0, \varepsilon)$. The fact that $(\alpha_+, \alpha_-)$ is a Liouville pair implies that $\alpha_0$ is a contact form; see [MNW13, Proposition 9.1] for the details. If $\Psi'$ denotes the map $(\text{Id}_M, \Psi) : M \times (D^2_\varepsilon \setminus \{0\}) \to M \times S^1_\varepsilon \times (0, \varepsilon)$, then the pull-back $(\Psi')^* \alpha_0$ can be written as $\alpha_+ + r^2 d\varphi + \gamma$, with $\gamma$ smooth on $M \times (D^2_\varepsilon \setminus \{0\})$ and

$$\gamma = \frac{\cos(r^2)}{2} - \frac{1}{2} \alpha_+ + \frac{1 - \cos(r^2)}{2} \alpha_- + \frac{\sin(r^2) - r^2}{2} d\varphi \quad \text{for} \ r < \frac{\varepsilon}{3}.$$
Hence, $(\Psi')^*\alpha_0$ naturally extends to a (smooth) contact form $\alpha$ on $M \times D^2_\varepsilon$, which moreover restricts to $\alpha_+ \mid M \times \{0\} \simeq M$.

Now, each contact submanifolds of codimension 2 having topologically trivial normal bundle also has trivial conformal symplectic normal bundle. Hence, by the contact submanifold’s neighbourhood theorem [Gei08, Theorem 2.5.15], each contact manifold $(V,\xi)$ containing $(M,\xi_+)$ as a codimension 2 contact submanifold with trivial normal bundle will also contain, for $\varepsilon > 0$ small enough, the above model $(M \times D^2_\varepsilon,\xi = \ker(\alpha))$ as a (codimension 0) contact submanifold, in such a way that $(M,\xi_+)$ coincides with $(M \times \{0\},\xi \mid_{M \times \{0\}})$.

We now describe how to modify the contact structure in this particular local coordinates around $(M,\xi_+)$ in order to perform the half twist.

Consider another smooth map $\Phi : D_\varepsilon^2 \setminus \{0\} \to S^1 \times (-\pi,\varepsilon)$, defined by $\Phi(r,\varphi) = (\varphi,\phi(r))$, where $(r,\varphi)$ are again polar coordinates on $D_\varepsilon^2$ and $\phi : (0,\varepsilon) \to (-\pi,\varepsilon)$ is again smooth, strictly increasing, equal to $r$ on $(\frac{\varepsilon}{2},\varepsilon)$, but this time equal to $r^2 - \pi$ on $(0,\frac{\varepsilon}{2})$.

As before, if $\Phi'$ denotes the map $(\text{Id}_M,\Phi) : M \times (D_\varepsilon^2 \setminus \{0\}) \to M \times S^1 \times (-\pi,\varepsilon)$, then the contact form $(\Phi')^*\alpha_0$ naturally extends to a contact form $\alpha'$ on $M \times D_\varepsilon^2$, but this time at $M \times \{0\}$ we have the contact submanifold $(M,\xi_+ = \ker(\alpha_-))$.

We remark though that the contact manifolds $(M \times D_\varepsilon^2,\xi = \ker(\alpha))$ and $(M \times D_\varepsilon^2,\xi' = \ker(\alpha'))$ coincide on the subset $\{r \geq \frac{\varepsilon}{2}\}$ of $M \times D_\varepsilon^2$. If we denote by $\overline{D_\varepsilon^2}$ the closed disk of radius $\delta := \frac{1}{12}\varepsilon$ centered at the origin inside $\mathbb{R}^2$, we can thus replace $(M \times \overline{D_\varepsilon^2},\xi)$ with $(M \times \overline{D_\varepsilon^2},\xi')$ inside $(M \times D_\varepsilon^2,\xi) \subset (V,\xi)$; this gives a contact manifold $(V,\xi')$.

**Definition 2.2.** [MNW13, Remark 9.6] We say that $(V,\xi')$ is obtained from $(V,\xi)$ by a **half Lutz-Mori twist** along the contact submanifold $(M,\xi_+ = \ker(\alpha_+))$ belonging to the Liouville pair $(\alpha_+,\alpha_-)$.

We point out that performing a half Lutz-Mori twist makes the contact manifold overtwisted. Indeed, it is explained in [MNW13, Remark 9.6] that this twist always gives a PS-overtwisted manifold, which then is also overtwisted according to [CMP, Hua].

### 2.2 Construction of Liouville pairs

We outline here the construction in [MNW13, Section 8], leaving the details that are not important for our purposes.

Consider the product manifold $\mathbb{R}^m \times \mathbb{R}^{m+1}$ with the pair of contact structures $\xi_+,\xi_-$ induced by the following pair of contact forms:

$$
\alpha_{\pm} := \pm e^{t_1 + \cdots + t_m} d\theta_0 + e^{-t_1} d\theta_1 + \cdots + e^{-t_m} d\theta_m ,
$$

where we use coordinates $(t_1, \ldots, t_m)$ on $\mathbb{R}^m$ and $(\theta_0, \ldots, \theta_m)$ on $\mathbb{R}^{m+1}$. A direct computation shows that $(\alpha_+,\alpha_-)$ is a Liouville pair on $\mathbb{R}^m \times \mathbb{R}^{m+1}$.

We now remark that there are two Lie groups acting explicitly on $\mathbb{R}^m \times \mathbb{R}^{m+1}$ by strict contact transformations for both $\alpha_+$ and $\alpha_-$. Indeed, the left action of the group $\mathbb{R}^{m+1}$ on $\mathbb{R}^m \times \mathbb{R}^{m+1}$ given by the translations

$$
(\varphi_0, \ldots, \varphi_m) \cdot (t_1, \ldots, t_m, \theta_0, \ldots, \theta_m) := (t_1, \ldots, t_m, \theta_0 + \varphi_0, \ldots, \theta_m + \varphi_m)
$$

(1)
and the left action of $\mathbb{R}^n$ given by the law

$$\left(\tau_1, \ldots, \tau_m\right) \cdot \left(t_1, \ldots, t_m, \theta_0, \ldots, \theta_m\right) := \left(t_1 + \tau_1, \ldots, t_m + \tau_m, e^{-\tau_1} \tau_2 - \tau_m, \theta_0 e^{\tau_1} \theta_1, \ldots, e^{\tau_m} \theta_m\right)$$

are Lie group left-actions on $\mathbb{R}^n \times \mathbb{R}^{n+1}$ and they both preserve the contact forms $\alpha_+$ and $\alpha_-$. Moreover, these two actions allow us to produce a compact contact manifold from $\mathbb{R}^n \times \mathbb{R}^{n+1}$. Indeed, there are lattices $\Lambda, \Lambda'$ of $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$ respectively, such that the $\Lambda$-action on $\mathbb{R}^n \times \mathbb{R}^{n+1}$ induces the action of $\mathbb{R}^n$ preserves $\mathbb{R}^n \times \Lambda'$. This implies that, by first taking the quotient of $\mathbb{R}^n \times \mathbb{R}^{n+1}$ by $\Lambda'$ and then quotienting it by the (well defined by the above property) induced action of $\Lambda$, we obtain a compact manifold $M$.

Finally, this manifold $M$ naturally inherits a Liouville pair, still denoted by $(\alpha_+, \alpha_-)$, from the Liouville pair on the covering $\mathbb{R}^n \times \mathbb{R}^{n+1}$, because $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$ act on $\mathbb{R}^n \times \mathbb{R}^{n+1}$ by strict contactomorphisms for both $\alpha_+$ and $\alpha_-$. We point out that this construction actually gives an infinite number of non-homeomorphic manifolds $M$, hence an infinite number of non-isomorphic Liouville pairs, in each odd dimension bigger or equal to 3.

Indeed, the existence of the lattices $\Lambda$ and $\Lambda'$ follows from number theory arguments and the manifold $M$ obtained depends on the choice of a totally real field of real numbers $k$ with finite dimension over $\mathbb{Q}$. Now, for each dimension over $\mathbb{Q}$, there are infinitely such fields $k$ and the corresponding manifolds are non-homeomorphic. See [MNW13, Lemma 8.3] for the details.

### 2.3 Topological effects of the half twists

Using the particular construction of the previous section, we obtain the following result, whose proof is postponed to Section 3.2:

**Proposition 2.3.** Let $(V^{2m+3}, \xi)$ be a contact manifold containing the $(M, \xi_+)$ of Section 2.2 as a codimension 2 contact submanifold with trivial normal bundle. Then, if we denote by $\xi'$ the contact structure on $V$ obtained by performing a half Lutz-Mori twist along the submanifold $(M, \xi_+)$ (where we consider $M$ with the orientation given by $\xi_+$), we have the following:

1. for all $i = 2, \ldots, m+1$, $c_i(\xi') - c_i(\xi) = 0$ in $H^{2i}(V; \mathbb{Z})$;
2. $c_1(\xi') - c_1(\xi) = -2 \text{PD}(j_+ [M])$ in $H^2(V; \mathbb{Z})$, where $j : M \to V$ is the inclusion, $j_* : H_{2m+1}(M; \mathbb{Z}) \to H_{2m+1}(V; \mathbb{Z})$ is the induced map and $\text{PD}(\alpha)$ denotes the Poincaré dual of the homology class $\alpha \in H_*(V; \mathbb{Z})$.

**Remark 2.4.** This result is not in contradiction with [MNW13, Theorem 9.5], where the authors prove that the contact structures before and after a full Lutz-Mori twist (as defined in [MNW13, Section 9.1]) are homotopic through almost contact structures, hence have the same Chern classes.

Indeed, the result $\xi''$ of a full Lutz-Mori twist can be interpreted as a couple of successive half twists. More precisely, we first perform a half twist along a submanifold $(M, \xi_+)$ to obtain $\xi'$; this changes the core of the tube where we perform the twist from $(M, \xi_+)$ to $(M, \xi_-)$. We then perform another half twist, this time along the new core $(M, \xi_-)$, to obtain $\xi''$. Hence, applying Proposition
2.3 twice and using the fact that $\xi^-$ induces an orientation that is opposite to that induced by $\xi^+$, we get that $c_i(\xi'') = c_i(\xi') = c_i(\xi)$ for all $i = 2, \ldots, m+1$ and that $c_1(\xi'') = c_1(\xi') - 2 \text{PD}(j_* [-M]) = c_1(\xi) - 2 \text{PD}(j_* [M]) - 2 \text{PD}(j_* [-M]) = c_1(\xi)$, as we expected from [MNW13, Theorem 9.5].

3 Chern classes and half Lutz-Mori twists

3.1 Chern classes as Poincaré duals

Chern classes are global invariants of complex vector bundles $E$ over a manifold $V$. In our setting, we then have the following problem: it’s not clear how local modifications (i.e. over an open set $U$ of $V$) of the complex vector bundle $E$ affect its Chern classes, making hard to prove results like Proposition 2.3. The solution is hence either to use a relative version of Chern classes or to shift to another point of view more local in nature; we adopt here the second strategy. More precisely, following [ACMFaA07] we describe in this section how each Chern class of $E$ can be interpreted (almost) as the Poincaré dual of the locus of points of $V$ where a “generic” set of sections of $E$ is not linearly independent.

We point out that this generalizes the classical fact that the top Chern class of $V$ is the Poincaré dual of the zero locus of a section of $E$ which is transverse to the zero section (see [BT82, Property 20.10.6 and Proposition 12.8]).

Consider a complex vector bundle $E$ of complex rank $r$ over an oriented smooth manifold $V$. Given $k$ sections $s_1, \ldots, s_k$ of $E$, take the homomorphism of vector bundles $h : V \times \mathbb{C}^k \to E$ defined by $h(p, u_1, \ldots, u_k) = \sum_{j=1}^k u_j s_j(p)$, where $V \times \mathbb{C}^k$ is the trivial complex vector bundle of rank $k$ over $V$.

If $\tau : \text{Hom}_\mathbb{C}(V \times \mathbb{C}^k, E) \to V$, is the complex vector bundle over $V$ with fiber over $p \in V$ the vector space $\text{Hom}_\mathbb{C}(\mathbb{C}^k, E_p)$ of $\mathbb{C}$-linear maps from $\mathbb{C}^k$ to $E_p$, we can reinterpret the map $h$ as a section $s_h$ of $\tau$ given by $s_h(p)(w) := h(p, w)$ for all $w \in \mathbb{C}^k$.

Take now the complex vector bundle $\pi : \text{Hom}_\mathbb{C}(V \times \mathbb{C}^k, E) \times \mathbb{C}P^{k-1} \to V \times \mathbb{C}P^{k-1}$ defined by $\pi(f, d) = (\tau(f), d)$, for every $f \in \text{Hom}_\mathbb{C}(V \times \mathbb{C}^k, E)$ and $d \in \mathbb{C}P^{k-1}$, and consider the section $\sigma_h : V \times \mathbb{C}P^{k-1} \to \text{Hom}_\mathbb{C}(V \times \mathbb{C}^k, E) \times \mathbb{C}P^{k-1}$ given by $\sigma_h := (s_h, \text{id}_{\mathbb{C}P^{k-1}})$.

If $\phi : V \times \mathbb{C}P^{k-1} \to V$ and $\hat{\phi} : \text{Hom}_\mathbb{C}(V \times \mathbb{C}^k, E) \times \mathbb{C}P^{k-1} \to \text{Hom}_\mathbb{C}(V \times \mathbb{C}^k, E)$ are the projections on the first factor, we then have the following commutative diagram:

$$
\begin{array}{ccc}
\text{Hom}_\mathbb{C}(V \times \mathbb{C}^k, E) \times \mathbb{C}P^{k-1} & \xrightarrow{\bar{\phi}} & \text{Hom}_\mathbb{C}(V \times \mathbb{C}^k, E) \\
\sigma_h \downarrow & & \downarrow \pi \\
V \times \mathbb{C}P^{k-1} & \xrightarrow{\phi} & V
\end{array}
$$

Now, in the total space $\text{Hom}_\mathbb{C}(V \times \mathbb{C}^k, E) \times \mathbb{C}P^{k-1}$ of the bundle $\pi$ we can consider the blown-up non-injectivity locus, i.e. the subset

$$
\Sigma := \left\{(f, d) \in \text{Hom}_\mathbb{C}(V \times \mathbb{C}^k, E) \times \mathbb{C}P^{k-1} \mid d \subset \ker f \right\}.
$$
The adjective blow-up comes from the fact that $\Sigma$ is a version of the non-injectivity locus

$$S := \{ f \in \text{Hom}_C (V \times \mathbb{C}^k, E) \mid \ker (f) \neq \{0\} \}$$

where we keep track of the complex lines in the kernel.

**Proposition 3.1.** [ACMFaA07, Proposition 4, Proposition 6] $\Sigma$ is a smooth oriented submanifold of $\text{Hom}_C (V \times \mathbb{C}^k, E) \times \mathbb{C}P^{k-1}$, of codimension $2r$.

As we will need it in the following, we give a sketch of proof:

**Proof (sketch).** Let $\text{pr} : \text{Hom}_C (V \times \mathbb{C}^k, E) \times \mathbb{C}P^{k-1} \to \mathbb{C}P^{k-1}$ be the projection on the second factor and $\gamma$ the tautological line bundle over $\mathbb{C}P^{k-1}$; denote then

$$\epsilon_1 := \text{pr}^* \gamma = \{(f, d, v) \in \text{Hom}_C (V \times \mathbb{C}^k, E) \times \mathbb{C}P^{k-1} \times \mathbb{C} \mid v \in d \} .$$

If $\phi : V \times \mathbb{C}P^{k-1} \to V$ is the projection on the first factor, denote also by $\epsilon_2$ the vector bundle $\pi^* \phi^* E$ over $\text{Hom}_C (V \times \mathbb{C}^k, E) \times \mathbb{C}P^{k-1}$, where the map $\pi : \text{Hom}_C (V \times \mathbb{C}^k, E) \times \mathbb{C}P^{k-1} \to V \times \mathbb{C}P^{k-1}$ is as above.

Consider then the vector bundle $\Pi : \text{Hom}_C (\epsilon_1, \epsilon_2) \to \text{Hom}_C (V \times \mathbb{C}^k, E) \times \mathbb{C}P^{k-1}$ and take the section $\Psi : \text{Hom}_C (V \times \mathbb{C}^k, E) \times \mathbb{C}P^{k-1} \to \text{Hom}_C (\epsilon_1, \epsilon_2)$ of $\Pi$ defined by $\Psi (f, d) = f|_d$.

It can be shown that $\Psi$ is transverse to the zero section $0_\Pi$ of $\Pi$. In particular, $\Sigma = \Psi^{-1} (0_\Pi)$ is a smooth submanifold of $\text{Hom}_C (V \times \mathbb{C}^k, E) \times \mathbb{C}P^{k-1}$.

Finally, $\Sigma$ is oriented according to Convention 3.2 below, thanks to the fact that $\text{Hom}_C (\epsilon_1, \epsilon_2)$, $\text{Hom}_C (V \times \mathbb{C}^k, E) \times \mathbb{C}P^{k-1}$ and $0_\Pi$ are naturally oriented; indeed, the first two are complex vector bundles over an oriented base and the third is a section of a vector bundle over an oriented base.  

**Convention 3.2.** Let $f : X \to Y$ be transverse to $Z \subset Y$ at $p \in X$, with $X, Y, Z$ oriented. Take a basis $(v_1, \ldots, v_l)$ of $T_p f^{-1} (Z)$, complete it to a positive basis $(v_1, \ldots, v_l, u_1, \ldots, u_n)$ of $T_p X$ and consider a positive basis $(w_1, \ldots, w_m)$ of $T_{f(p)} Z$. Then, $(v_1, \ldots, v_l)$ is positive iff $(w_1, \ldots, w_m, d_p f (u_1), \ldots, d_p f (u_n))$ is a positive basis of $T_{f(p)} Y$.

Define now the set

$$Z (h) := \sigma_h^{-1} (\Sigma) = \left\{ (p, d) \in V \times \mathbb{C}P^{k-1} \mid d \subset \ker (h_p) \right\} ,$$

where $h_p : \mathbb{C}^k \to E_p$ is the $\mathbb{C}$-linear map defined by $h_p (.) := h (p, .)$.  

8
 Proposition 3.3. [ACMFaA07, Proposition 5] For a generic choice of vector bundles map $h : V \times \mathbb{C}^k \rightarrow E$, the section $\sigma_h : V \times \mathbb{C}P^{k-1} \rightarrow \text{Hom}_\mathbb{C} (V \times \mathbb{C}^k, E) \times \mathbb{C}P^{k-1}$ is transverse to $\Sigma \subset \text{Hom}_\mathbb{C} (V \times \mathbb{C}^k, E) \times \mathbb{C}P^{k-1}$. In particular, $Z(h)$ is a closed oriented submanifold of $V \times \mathbb{C}P^{k-1}$ of codimension $2r$.

Theorem 3.4. [ACMFaA07, Theorem 11] If the section $\sigma_h : V \times \mathbb{C}P^{k-1} \rightarrow \text{Hom}_\mathbb{C} (V \times \mathbb{C}^k, E) \times \mathbb{C}P^{k-1}$ is transverse to $\Sigma \subset \text{Hom}_\mathbb{C} (V \times \mathbb{C}^k, E) \times \mathbb{C}P^{k-1}$, then the Chern class $c_{r-k+1}(E)$ is equal to the Poincaré dual of $\phi_* [Z(h)]$, where $\phi : V \times \mathbb{C}P^{k-1} \rightarrow V$ is the projection on the first factor.

Remark 3.5. The statement of the theorem shows the advantage of using the blown-up version $\Sigma$ of $S$ instead of the non-injectivity locus itself.

Indeed, while $Z(h)$ is a smooth oriented submanifold for a generic choice of $h$, hence is has a well defined fundamental class that can be pushed in $H_*(V;\mathbb{Z})$ via $\phi_*$, the set $s^{-1}_h(S)$ is only a Whitney stratified submanifold of $V$ (hence not necessarily smooth) for a generic choice of $h$, and in particular there is no natural way to associate an homology class to $s^{-1}_h(S)$.

We point out though that in the complex analytic setting it is possible to construct a cohomology class directly from $s^{-1}_h(S)$ using the theory of currents: this is done, for example, in [GH78, Section 3.3].

As it will be useful later, we remark that there is also a relative version of Proposition 3.3. Indeed, we have the following relative transversality result: if $M$ and $N$ are smooth manifolds and $f : M \rightarrow N$ is a smooth map transverse to a submanifold $Z \subset N$ on a neighbourhood of a closed subset $C \subset M$, then $f$ can be $C^\infty$-perturbed to a map $f' : M \rightarrow N$ everywhere transverse to $Z$ and such that $f'|_C = f|_C$.

This can be proven, for example, by introducing a little modification in the proof of [Lee13, Theorem 6.36], where it is shown that, for $k \in \mathbb{N}$ big enough, there is a parametric family of functions $F : M \times \mathbb{R}^k \rightarrow N$ that is everywhere transverse to $Z$ and such that $F(.,0) = f(.)$. More precisely, the $F$ appearing in the proof of [Lee13, Theorem 6.36] should be defined in our case as $F(p,s) := r (f(p) + \chi(p) \cdot e(p) \cdot s)$, where $\chi : M \rightarrow \mathbb{R}_{\geq 0}$ has support $C$; the wanted perturbation will then be $F_s := F(.,s)$ for an $s$ given by the parametric transversality theorem [Lee13, Theorem 6.35].

In our setting, using this relative transversality result in the proof of [ACMFaA07, Proposition 5] we can achieve transversality between the map $\sigma_h$ and the submanifold $\Sigma$ of $\text{Hom}_\mathbb{C} (V \times \mathbb{C}^k, E) \times \mathbb{C}P^{k-1}$ by $C^\infty$-perturbing $h$ relative to a closed subset $C \subset V \times \mathbb{C}P^{k-1}$ near which $\sigma_h$ is already transverse to $\Sigma$.

To prove Proposition 2.3, we will actually need only the following consequence of Theorem 3.4:

Proposition 3.6. Let $E, E'$ be two complex vector bundles of complex rank $r$ over the same smooth oriented manifold $V$. Suppose also that there exist two open subsets $\mathcal{O}, \mathcal{U}$ of $V$, with $\mathcal{O}$ compactly contained in $\mathcal{U}$, such that the following are satisfied:

1. There is an isomorphism of vector bundles $\psi : E|_{\mathcal{O}^c} \rightarrow E'|_{\mathcal{O}^c}$ over $\mathcal{O}^c := V \setminus \mathcal{O}$. 

9
2. The vector bundles $E|_U$ and $E'|_U$ over $U$ admit complex sub-bundles $F, L \subset E$ and $F', L' \subset E'$ such that $E|_U = F \oplus L$ and $E'|_U = F' \oplus L'$ and verifying the following conditions:

(a) $\psi \circ s_i = s'_i$ over $U \setminus O$ for all $j = 1, \ldots, r - 1$;

(b) $L, L'$ have complex rank 1, while $F, F'$ have complex rank $r - 1$ and are trivialized by two ordered sets of everywhere $\mathbb{C}$-linearly independent sections $(s_1, \ldots, s_{r-1})$ and $(s'_1, \ldots, s'_{r-1})$ of $E|_U$ and $E'|_U$;

(c) there are two additional sections $s_r, s'_r$ of $E|_U$ and $E'|_U$ respectively, with image contained in $L$ and $L'$ and such that $s_r : U \to E|_U$ intersects transversely $F$ and $s'_r : U \to E'|_U$ intersects transversely $F'$ (here $F$ and $F'$ are seen here as submanifolds of $E|_U$ and $E'|_U$);

(d) $Z := s_r^{-1}(F)$ and $Z' := (s'_r)^{-1}(F')$, which are oriented smooth manifolds of $U$ by Hypothesis 2c, are actually compactly contained in $O$.

Then, we have the following:

1. $c_k( E' ) = c_k( E )$ in $H^{2k}( V; \mathbb{Z} )$ for all $2 \leq k \leq r$;

2. $c_1( E' ) - c_1( E ) = \text{PD}( [Z'] ) - \text{PD}( [Z] )$ in $H^2( V; \mathbb{Z} )$.

We deduce the above result from Theorem 3.4 using the following:

**Lemma 3.7.** Let $E$ be a complex vector bundle of complex rank $r$ over a smooth oriented manifold $V$ with empty boundary. Let $s_1, \ldots, s_{r-1}$ be $\mathbb{C}$-linearly independent sections of $E$ and denote by $F$ the vector sub-bundle of $E$ generated by them, i.e. the vector bundle with fiber $F_p = \text{Span}_\mathbb{C} ( s_1(p), \ldots, s_{r-1}(p) ) \subset E_p$ over a point $p \in V$. Let also $L$ be a complex line sub-bundle of $E$ such that $E = F \oplus L$ and assume that $s_r : V \to E$ is an additional section with image contained in $L$ and intersecting transversely $F$ (seen as a submanifold of $E$); denote by $M$ the oriented (by Convention 3.2) submanifold $s_r^{-1}(F)$. Then, if $h : V \times \mathbb{C}^r \to E$ is defined by $h(p, u) = \sum_{i=1}^r u_i s_i( p )$ and $\sigma_h$ is obtained from $h$ as above, we have that:

1. $\sigma_h : V \times \mathbb{C}^{r-1} \to \text{Homeo}_c( V \times \mathbb{C}^r, E ) \times \mathbb{C}^{r-1}$ is transverse to the blown-up non-injectivity locus $\Sigma \subset \text{Homeo}_c( V \times \mathbb{C}^r, E ) \times \mathbb{C}^{r-1}$ and, in particular, $Z(h) := \sigma_h^{-1}(\Sigma)$ is smooth and naturally oriented;

2. the projection on the first factor $\phi : V \times \mathbb{C}^r \to V$ induces an orientation preserving diffeomorphism $\phi : Z(h) \to M$.

**Proof (Proposition 3.6):** Consider another open set $V$ of $V$, compactly contained in $U$ and containing the closure of $O$.

Take the two complex vector bundle homomorphisms $h_V : V \times \mathbb{C}^r \to E|_V$ and $h'_V : V \times \mathbb{C}^r \to E'|_V$ defined by $h_V( p, u_1, \ldots, u_r ) := \sum_{i=1}^r u_i s_i( p )$ and $h'_V( p, u_1, \ldots, u_r ) := \sum_{i=1}^r u_i s'_i( p )$ for all $p \in V$, $(u_1, \ldots, u_r) \in \mathbb{C}^r$ and $i = 1, \ldots, r$. Extend then $h_V$ and $h'_V$ to two vector bundle homomorphisms $h : V \times \mathbb{C}^r \to E$ and $h' : V \times \mathbb{C}^r \to E'$ in such a way that $\psi( h(p, u) ) = h'( p, u )$ for all $p \in O^c$, $u \in \mathbb{C}^r$ and $i = 1, \ldots, r$. Such extensions exist because $\psi \circ s_i = s'_i$ on $U \setminus O$ for all $i = 1, \ldots, r$ by Hypothesis 2a.

Given an integer $j$ between 1 and $r$ included, denote respectively by $h_j$ and $h'_j$ the restrictions of $h$ and $h'$ to the sub-bundle $V \times C^j$ of $V \times \mathbb{C}^r$, where $C^j$
is the vector subspace of $C^r$ given by the points $(u_1, \ldots, u_r) \in C^r$ such that $u_{j+1} = \ldots = u_r = 0$.

Now, $\sigma_{h_j}$ and $\sigma_{h_j}$ are transverse to the blown-up non-injectivity locus $\Sigma$ near the closed set $\overline{\mathcal{O}}$, for all $j = 1, \ldots, r$ : indeed, this follows directly from Hypothesis 2b for the case $j = 1, \ldots, r - 1$ and from Hypothesis 2c and Lemma 3.7 for the case $j = r$ (remark that in Lemma 3.7 we do not make compactness assumptions, so we can choose $V$ as base manifold $V$ in the statement of the lemma). Then, using the relative version of the genericity of the transversality condition, we can perturb $h_j, h_j'$ to $g_j, g_j'$ in such a way that $g_j = h_j, g_j' = h_j'$ over $\mathcal{O}$ and that $\sigma_{g_j}$'s and $\sigma_{g_j}$'s are everywhere transverse to $\Sigma$. Moreover, because $\psi(h_j(p, \ldots)) = h_j'(p, \ldots)$ for $p \in \mathcal{O}^c$, we can also arrange that $\psi(g_j(p, \ldots)) = g_j'(p, \ldots)$ for $p \in \mathcal{O}^c$: indeed, we can use the same perturbation for $h_j$ and $h_j'$ over $\mathcal{O}^c$ because they coincide there. Lastly, if we choose the perturbation $C^0$-small, we can arrange to have the submanifolds $Z(g_j)$ and $Z(g_j')$ compactly contained in $\mathcal{O} \times \mathbb{CP}^{j-1}$.

Now, by construction of the $g_j$'s and the $g_j$'s, if we write $Z(g_j) = Z_{\Omega}(g_j) \cup Z_{\Omega'}(g_j)$ and $Z(g_j') = Z_{\Omega}(g_j') \cup Z_{\Omega'}(g_j')$, where $Z_{\Omega}(g_j), Z_{\Omega'}(g_j) \subset \mathcal{O} \times \mathbb{CP}^{j-1}$ and $Z_{\Omega'}(g_j), Z_{\Omega'}(g_j') \subset \mathcal{O}^c \times \mathbb{CP}^{j-1}$, we have that $Z_{\Omega'}(g_j) = Z_{\Omega'}(g_j')$ for all $j = 1, \ldots, r$ and $Z_{\Omega}(g_j) = Z_{\Omega}(g_j') = \emptyset$ for $j = 1, \ldots, r - 1$. Moreover, if $\mathrm{pr}_{\nu}^\ast : V \times \mathbb{CP}^{j-1} \to V$ is the projection on the first factor for all $j = 1, \ldots, r$, by Lemma 3.7 $\phi_r := \mathrm{pr}_{\nu}^\ast|_{Z(g_j)}$ and $\phi_r' := \mathrm{pr}_{\nu}^\ast|_{Z(g_j')}$ induce orientation preserving diffeomorphisms between $Z_{\Omega}(g_j)$ and $Z$ and between $Z_{\Omega'}(g_j')$ and $Z'$ respectively.

By Theorem 3.4 and the identities above, we have that for all $j = 1, \ldots, r - 1$

$$c_{r-j+1}(E) = \text{PD} \left( (\phi_j)_\ast \left[ Z(g_j) \right] \right)$$

$$= \text{PD} \left( (\phi_j)_\ast \left[ Z(g_j) \cap \mathcal{O}^c \right] \right)$$

$$= \text{PD} \left( (\phi_j')_\ast \left[ Z(g_j') \cap \mathcal{O}^c \right] \right)$$

$$= \text{PD} \left( (\phi_j)_\ast \left[ Z(g_j') \right] \right)$$

$$= c_{r-j+1}(E') ,$$

and that

$$c_1(E) = \text{PD} \left( (\phi_r)_\ast \left[ Z(g_r) \right] \right)$$

$$= \text{PD} \left( (\phi_r)_\ast \left[ Z(g_r) \cap \mathcal{O}^c \right] \right) + \text{PD} \left( (\phi_r)_\ast \left[ Z(g_r) \cap \mathcal{O} \right] \right)$$

$$= \text{PD} \left( (\phi_r)_\ast \left[ Z_{\Omega'}(g_r) \right] \right) + \text{PD} \left( [Z] \right) ,$$

$$c_1(E') = \text{PD} \left( (\phi_r')_\ast \left[ Z(g_r') \right] \right)$$

$$= \text{PD} \left( (\phi_r)_\ast \left[ Z(g_r') \cap \mathcal{O}^c \right] \right) + \text{PD} \left( (\phi_r)_\ast \left[ Z(g_r') \cap \mathcal{O} \right] \right)$$

$$= \text{PD} \left( (\phi_r)_\ast \left[ Z_{\Omega'}(g_r') \right] \right) + \text{PD} \left( [Z'] \right) ,$$

which give $c_1(E') - c_1(E) = \text{PD} \left( [Z'] \right) - \text{PD} \left( [Z] \right)$.  

\textbf{Proof (Lemma 3.7).} Because the transversality and the orientation preserving conditions are local, we can restrict our attention to an open set $\mathcal{U}$ on which there is an everywhere non-zero section $s$ of $L|_{\mathcal{U}}$. 

11
In particular, the $r$-tuple of sections $(s_1, \ldots, s_{r-1}, s)$ trivializes $E|_U$, i.e. the map $U \times \mathbb{C}^r \to E|_U$ given by $(q, w) \mapsto \sum_{i=1}^{r-1} w_i s_i(q) + w_r s(q)$ is an isomorphism of complex vector bundles.

In this local trivialization, we can rewrite $s_r$ as

$$s_r : U \to U \times \mathbb{C}^r$$

$$q \mapsto (q, 0, \ldots, 0, v(q))$$

for a certain $v : U \to \mathbb{C}$.

Also, the fact that $s_r$ is transverse to $F$ at a certain point $q \in U$ means that $v : U \to \mathbb{C}$ is transverse to $\{0\} \subset \mathbb{C}$ at $q$. Moreover, if we denote by $\nu : U \times \mathbb{C}^r \to \mathbb{C}^r$ the projection on the second factor, in the open set $U$ the submanifold $M = s_r^{-1}(F)$ (oriented according to Convention 3.2) is actually equal to the oriented manifold $(\nu \circ s_r)^{-1}(0)$; in other words, remarking that $\nu \circ s_r = (0, \ldots, 0, v)$, we have that $M \cap U = v^{-1}(0)$ as oriented manifolds.

Now, let’s rewrite $\sigma_h$ using the chosen trivialization of $E$ over $U$.

Firstly, the map $h$ becomes $h : U \times \mathbb{C}^r \to U \times \mathbb{C}^r$, $h(q, w) = (q, M(q) \cdot w)$, where $\cdot$ denotes the matrix product, $M : U \to \mathcal{M}_{r \times r}(\mathbb{C})$ is with values in the space $\mathcal{M}_{r \times r}(\mathbb{C})$ of square matrices $r \times r$ with complex coefficients and is defined by

$$M = \begin{pmatrix} I_{r-1} & 0 \\ 0 & v \end{pmatrix},$$

with $I_{r-1}$ the identity matrix of dimension $(r - 1) \times (r - 1)$. In other words, $s_h : U \to U \times \mathcal{M}_{r \times r}(\mathbb{C})$ is given by $s_h(q) = (q, M(q))$.

Moreover, we remark that if $(p, d) \in U \times \mathbb{C}^r$ is such that $\sigma_h(p, d) \in \Sigma$ then $d = [0 : \cdots : 0 : 1]$. We can hence further restrict to the coordinate chart $C^{r-1} = \{(z_1, \ldots, z_{r-1} : 1) \in \mathbb{C}^{r-1}\}$ containing the point $[0 : \cdots : 0 : 1]$ and consider $\sigma_h$ as a map

$$\sigma_h : U \times C^{r-1} \to U \times \mathcal{M}_{r \times r}(\mathbb{C}) \times C^{r-1}$$

$$(q, z) \mapsto (q, M(q), z)$$

where $z = (z_1, \ldots, z_{r-1}) \in \mathbb{C}^{r-1}$.

Now, in order to study the transversality of $\sigma_h$ with respect to $\Sigma$, we have to come back at the construction of $\Sigma$ as preimage of a transverse intersection. We then consider the vector bundles $\epsilon_1, \epsilon_2$ and the sections $\Psi, \theta_1$ of $\Pi$ as in the (sketch of the) proof of Proposition 3.1 and read them in the given trivialization of $E$ over $U$ and in the chart $C^{r-1} \subset \mathbb{C}^{r-1}$. We are then in the following situation:

- $\epsilon_1$, which is globally the product of $\text{Hom}_{\mathbb{C}}(V \times \mathbb{C}^r, E)$ and the tautological line bundle $\gamma$ over $\mathbb{C}^{r-1}$, becomes the trivial line vector bundle $U \times \mathcal{M}_{r \times r}(\mathbb{C}) \times \mathbb{C}^{r-1} \times \mathbb{C}$ over $U \times \mathcal{M}_{r \times r}(\mathbb{C}) \times \mathbb{C}^{r-1}$, and the projection map is just the projection on the first three factors: indeed, $\gamma$ admits over the coordinate chart $\mathbb{C}^{r-1} \subset \mathbb{C}^{r-1}$ the trivialization $\mathbb{C}^{r-1} \times \mathbb{C} \to \gamma$ given by $(z_1, \ldots, z_{r-1}, \lambda) \mapsto ([z_1 : \cdots : z_{r-1} : 1], \lambda \overline{z})$, where $\overline{z} := (z_1, \ldots, z_{r-1}, 1)$;

- $\epsilon_2$, defined globally as $\pi^* \phi^* E$, becomes the trivial vector bundle $U \times \mathcal{M}_{r \times r}(\mathbb{C}) \times \mathbb{C}^{r-1} \times \mathbb{C}^r$ over $U \times \mathcal{M}_{r \times r}(\mathbb{C}) \times \mathbb{C}^{r-1}$, again via the projection on the first 3 factors;

12
• the projection $\Pi : \text{Hom}_\Sigma (\epsilon_1, \epsilon_2) \to \text{Hom}_\Sigma (V \times \mathbb{C}^r, E) \times \mathbb{C}P^{k-1}$ becomes locally $\Pi : U \times \mathcal{M}_r (C) \times \mathbb{C}^{r-1} \times \mathcal{M}_{r,1} (C) \to U \times \mathcal{M}_r (C) \times \mathbb{C}^{r-1}$, $(q, A, z, B) \mapsto (q, A, z)$;

• the zero section $0\Pi$ of $\Pi$ is locally the image of the inclusion

$$U \times \mathcal{M}_r (C) \times \mathbb{C}^{r-1} \hookrightarrow U \times \mathcal{M}_r (C) \times \mathbb{C}^{r-1} \times \mathcal{M}_{r,1} (C)$$

$$(q, A, z) \mapsto (q, A, z, 0)$$

• the section $\Psi$ of $\Pi$ can be rewritten locally as $\Psi : U \times \mathcal{M}_r (C) \times \mathbb{C}^{r-1} \to U \times \mathcal{M}_r (C) \times \mathbb{C}^{r-1} \times \mathcal{M}_{r,1} (C)$ and is given by $\Psi (q, A, z) \equiv (q, A, z, A \cdot z)$, where again $\Sigma = (z_1, \ldots, z_{r-1}, 1) \in \mathbb{C}^r$, if $z = (z_1, \ldots, z_{r-1})$.

Then, using the expression in Equation 3 for the matrix $M(q)$, we get

$$\Psi \circ \sigma_h : U \times \mathbb{C}^{r-1} \to U \times \mathcal{M}_r (C) \times \mathbb{C}^{r-1} \times \mathcal{M}_{r,1} (C)$$

$$(q, z) \mapsto \left( q, \begin{pmatrix} I_{r-1} & 0 \\ 0 & v(q) \end{pmatrix}, z, \begin{pmatrix} z \\ v(q) \end{pmatrix} \right)$$

Now, because $\Psi$ is transverse to $0\Pi$ and $\Sigma$ coincides with the oriented preimage $\Psi^{-1} (0\Pi)$ (see the sketch of proof of Proposition 3.1), we actually have that $\sigma_h$ is transverse to $\Sigma$ at $(q, 0) \in U \times \mathbb{C}^{r-1}$ (recall that if $\sigma_h (q, z)$ is in $\Sigma$ then $z = 0$) if and only if $\Psi \circ \sigma_h$ is transverse to $0\Pi$ at $(q, 0)$. Moreover, if we have transversality at every point, $Z(h) = \sigma_h^{-1} (\Sigma)$ equals $(\Psi \circ \sigma_h)^{-1} (0\Pi)$ as oriented manifolds.

If we denote by $\mu : U \times \mathcal{M}_r (C) \times \mathbb{C}^{r-1} \times \mathcal{M}_{r,1} (C) \to \mathcal{M}_{r,1} (C)$ the projection on the last factor, we also get that $\Psi \circ \sigma_h$ is transverse to $0\Pi$ at $(q, 0) \in U \times \mathbb{C}^{r-1}$ if and only if $\mu \circ \Psi \circ \sigma_h$ is transverse to $\{0\} \subset \mathcal{M}_{r,1} (C)$ and that, in case of transversality at every point, $Z(h) \cap (U \times \mathbb{C}^{r-1}) = (\mu \circ \Psi \circ \sigma_h)^{-1} (0)$ as oriented manifolds. In other words, using the fact that

$$\mu \circ \Psi \circ \sigma_h : U \times \mathbb{C}^{r-1} \to \mathcal{M}_{r,1} (C)$$

$$(q, z) \mapsto \begin{pmatrix} z \\ v(q) \end{pmatrix}$$

we get that $\sigma_h$ is transverse to $\Sigma$ at $(q, 0) \in U \times \mathbb{C}^{r-1}$ if and only if $v : U \to \mathbb{C}$ is transverse to $\{0\} \subset \mathbb{C}$ at $q$ and that, if there is transversality everywhere, $Z(h) \cap (U \times \mathbb{C}^{r-1}) = v^{-1} (0) \times \{0\} \subset U \times \mathbb{C}^{r-1}$ as oriented manifolds.

This concludes the proof of lemma 3.7, because $v$ is transverse to $0 \in \mathbb{C}$ (as said in the beginning), hence $\sigma_h$ is transverse to $\Sigma$ over $U \times \mathbb{C}^{r-1}$, and $\phi : V \times \mathbb{C}P^{r-1} \to V$ clearly induces an orientation preserving diffeomorphism

$$\phi : Z(h) \cap (U \times \mathbb{C}^{r-1}) = v^{-1} (0) \times \{0\} \xrightarrow{\sim} M \cap U = v^{-1} (0).$$

### 3.2 Proof of proposition 2.3

Consider the manifold $M$ and the Liouville pair $(\alpha_+, \alpha_-)$ constructed in Section 2.2 and take a contact manifold $(V, \xi)$ containing $(M, \xi_+ = \ker (\alpha_+))$ as a contact submanifold of dimension 2.

Denote by $(V, \xi')$ the result of a half Lutz-Mori twist on $(V, \xi)$ along $(M, \xi_+)$. According to Section 2.1, we have a tubular neighbourhood $M \times D_2^2$ of $M$ in
which we can arrange to have contact forms $\alpha, \alpha'$ for $\xi$ and $\xi'$ respectively which satisfy the following: if $(r, \varphi)$ are the polar coordinates on $D^2_\varepsilon$ and $\psi, \phi$ are as in Section 2.1, then

\[
\alpha = \frac{1+\cos(\psi(r))}{2} \alpha_+ + \frac{1-\cos(\psi(r))}{2} \alpha_- + \sin(\psi(r)) \, d\varphi,
\]
\[
\alpha' = \frac{1+\cos(\phi(r))}{2} \alpha_+ + \frac{1-\cos(\phi(r))}{2} \alpha_- + \sin(\phi(r)) \, d\varphi.
\]

Now, we have explicit expressions for $\alpha_+$ and $\alpha_-$ on the cover $\mathbb{R}^m \times \mathbb{R}^{m+1}$ of $M$, i.e. $\alpha_{\pm} := \pm e^{t_1 + \ldots + t_m} d\theta_0 + e^{-t_1} d\theta_1 + \ldots + e^{-t_m} d\theta_m$. Thus, on the cover $\mathbb{R}^m \times \mathbb{R}^{m+1} \times D^2_\varepsilon$ of the tubular neighbourhood $M \times D^2_\varepsilon$ of $M$ inside $V$, we can write in a more explicit form

1. $\alpha = e^{\sum_{i=1}^m t_i} \cos(\psi(r)) \, d\theta_0 + \sum_{i=1}^m e^{-t_i} d\theta_i + \sin(\psi(r)) \, d\varphi$,
2. $\alpha' = e^{\sum_{i=1}^m t_i} \cos(\phi(r)) \, d\theta_0 + \sum_{i=1}^m e^{-t_i} d\theta_i + \sin(\phi(r)) \, d\varphi$.

Take now the following $2m$ $\mathbb{R}$-linearly independent sections of the pullback of $\xi$ and $\xi'$ to $\mathbb{R}^m \times \mathbb{R}^{m+1} \times D^2_\varepsilon$: for $i = 1, \ldots, m$,

- $s_i := \partial_{r_i}$, $r_i := e^{-\sum_{j=1}^m t_j} \cos(\psi(r)) \, \partial_{\theta_0} - e^{t_i} \partial_{\theta_i} + \sin(\psi(r)) \, \partial_{\varphi}$ for the pullback of $\xi$;
- $s'_i := \partial_{r_i}$, $r'_i := e^{-\sum_{j=1}^m t_j} \cos(\phi(r)) \, \partial_{\theta_0} - e^{t_i} \partial_{\theta_i} + \sin(\phi(r)) \, \partial_{\varphi}$ for the pullback of $\xi'$.

Let’s also consider the following sections:

- $s_{m+1} := r \partial_r$, $r_{m+1} := \cos(\psi(r)) \, \partial_{\varphi} - \sin(\psi(r)) \, e^{-\sum_{i=1}^m t_i} \partial_{\theta_0}$ for the pullback of $\xi$;
- $s'_{m+1} := r \partial_r$, $r'_{m+1} := \cos(\phi(r)) \, \partial_{\varphi} - \sin(\phi(r)) \, e^{-\sum_{i=1}^m t_i} \partial_{\theta_0}$ for the pullback of $\xi'$.

These last two couples of sections are $\mathbb{R}$-linearly independent whenever $s_{m+1}$ and $s'_{m+1}$ are non-zero.

Moreover, for $i = 1, \ldots, m + 1$, $s_i$, $r_i$, $s'_i$ and $r'_i$ are invariant under the left-action induced on $\mathbb{R}^m \times \mathbb{R}^{m+1} \times D^2_\varepsilon$ by the left-actions of the Lie groups $\mathbb{R}^m$ and $\mathbb{R}^{m+1}$ on $\mathbb{R}^m \times \mathbb{R}^{m+1}$ described in Equations 1 and 2 of Section 2.2. Hence, they induce well defined sections of $\xi$ and $\xi'$ on $M \times D^2_\varepsilon$, which we will still denote using same notations. We also point out that each section coincide with its “primed version” near $M \times D^2_\varepsilon$.

We remark now that $\text{Span}_\mathbb{R} (s_{m+1}(p), r_{m+1}(p))$ and $\text{Span}_\mathbb{R} (s'_{m+1}(p), r'_{m+1}(p))$, a priori well defined only for $p \in M \times (D^2_\varepsilon \setminus \{0\})$, actually extend smoothly also over $M \times \{0\}$.

Indeed, consider the following couples of sections of $\xi$ and $\xi'$ respectively:

- $S := \frac{1}{r} (\cos(\varphi) s_{m+1} - \sin(\varphi) r_{m+1})$, $R := \frac{1}{r} (\sin(\varphi) s_{m+1} + \cos(\varphi) r_{m+1})$;  
- $S' := \frac{1}{r} (\cos(\varphi) s'_{m+1} + \sin(\varphi) r'_{m+1})$, $R' := \frac{1}{r} (-\sin(\varphi) s'_{m+1} + \cos(\varphi) r'_{m+1})$.  

14
These sections, defined on $M \times (D^2_x \setminus \{0\})$, can be smoothly extended to sections on all $M \times D^2_x$. For example, the section $S$ can be rewritten near $r = 0$ as follows:

\[
S = \frac{1}{r}(\cos(\varphi) s_{m+1} - \sin(\varphi) r_{m+1}) = \cos \varphi \partial_x - \frac{\sin \varphi \cos r^2}{r} \partial_x + \frac{\sin \varphi \sin r^2}{r} \partial_y + \frac{1}{r^2} \left(-y \partial_x + x \partial_y\right) + y \sin \frac{r^2}{r^2} e^{-\sum_{i=1}^{m} \ell_i} \partial_{\theta_0},
\]

and each coefficient extends smoothly to all $M \times D^2_x$. Analogue computations show that also $S', R, R'$ extend smoothly to $M \times D^2_x$. We will denote these smooth extensions still by $S, R, S', R'$.

Moreover, we point out that $(s_1, r_1, \ldots, s_m, r_m, S, R)$ are everywhere $\mathbb{R}$-linear independent sections of $\xi$, which is hence trivialized by them over $M \times D^2_x$; the analogue is true for $(s'_1, r'_1, \ldots, s'_m, r'_m, S', R')$. We remark that, unlike the couples $(s_{m+1}, r_{m+1})$ and $(s'_{m+1}, r'_{m+1})$, the $(S, R)$ and $(S', R')$ do not coincide near the boundary of $M \times D^2_x$.

Computing the differentials of $\alpha$ and $\alpha'$ thanks to the above explicit expressions 1 and 2 for their pullbacks, we can see that $d\alpha(s_i, r_i) > 0$ and $d\alpha'(s'_i, r'_i) > 0$ for all $i = 1, \ldots, m$ and that $d\alpha(S, R) > 0$ and $d\alpha'(S', R') > 0$.

Then, the identities $J(s_i) := r_i$ and $J'(s'_i) := r'_i$, for all $i = 1, \ldots, m$, and the identities $J(S) := R$, $J'(S') := R'$ give two complex structures $J$ and $J'$ on $\xi$ and $\xi'$ over $M \times D^2_x$ which are tamed by $d\alpha$ and $d\alpha'$. In particular, the sections $s_1, \ldots, s_m, S$ are $\mathbb{C}$-linearly independent on $\xi$ and the sections $s'_1, \ldots, s'_m, S'$ are $\mathbb{C}$-linearly independent on $\xi'$.

We point out that $J, J'$ satisfy also the identities $J(s_{m+1}) = r_{m+1}$ and $J'(s'_{m+1}) = r'_{m+1}$. This shows in particular that $J$ and $J'$ coincide over a neighbourhood of the boundary of $M \times D^2_x$: indeed, each section coincide with its primed version near the boundary of $M \times D^2_x$ and the span of $(s_i, r_i)_{i=1}^{m+1}$ and $(s'_i, r'_{i=1}^{m+1})$ are respectively $\xi$ and $\xi'$ on $M \times \{D^2_x \setminus \{0\}\}$.

We can now extend $J$ and $J'$ to complex structures on $\xi$ and $\xi'$ over all $V$, tamed by contact forms that extend $\alpha$ and $\alpha'$, in such a way that they coincide outside $M \times D^2_x$. We denote such extensions still with $J$ and $J'$.

We now claim that we are in the hypothesis of Proposition 3.6 if we choose as open set $\mathcal{O}$ an arbitrary open set compactly contained in $\mathcal{U} := M \times D^2_x$ and containing the support of the half Lutz-Mori twist.

Indeed, if $F, F'$ are the complex span of $(s_1, \ldots, s_m), (s'_1, \ldots, s'_m)$ and $L, L'$ are the complex lines determined by $S, S'$, then Hypothesis 1, 2a and 2b are trivially satisfied because $\xi$ and $\xi'$ coincide outside $\mathcal{O}$ and because of the choice of $s_1, \ldots, s_m$ and $s'_1, \ldots, s'_m$.

Let’s show that Hypothesis 2c and 2d are also satisfied in our case.

We claim that $s_{m+1} : M \times D^2_x \to \xi$ and $s'_{m+1} : M \times D^2_x \to \xi'$ intersect transversely $F \subset \xi$ and $F' \subset \xi'$ in $M \times \{0\}$ and $-M \times \{0\}$ (i.e. $M \times \{0\}$ but with opposite orientation).

Indeed, using the complex trivialization $(s_1, \ldots, s_m, S)$ for $\xi$ on $\mathcal{U} = M \times D^2_x$, we can write $s_{m+1} : \mathcal{U} \to \xi = \mathcal{U} \times \mathbb{C}^{m+1}$ as $s_{m+1}(q) = (q, v_1(q), \ldots, v_{m+1}(q))$, with $v_i : \mathcal{U} \to \mathbb{C}$.
More precisely, recalling that $JS = R$, that $S = \frac{1}{2} (\cos (\varphi) s_{m+1} - \sin (\varphi) r_{m+1})$ and that $R = \frac{1}{2} (\sin (\varphi) s_{m+1} + \cos (\varphi) r_{m+1})$, for each $q = (m, x, y) \in \mathcal{U} = M \times D^2_\varepsilon$, with $m \in M$, we actually have that $v_i(q) = 0$ for all $i = 1, \ldots, m$ and that $v_m(q) = x + iy$, where $(x, y) \in D^2_\varepsilon$ are the Cartesian coordinates. In particular, $d_{(m,0)}v_{m+1}(\partial_x) = \partial_y$ and $d_{(m,0)}v_{m+1}(\partial_y) = \partial_x$, i.e., $d_{(m,0)}v_{m+1}|_{\{m\} \times T_0D^2_\varepsilon} : \{m\} \times T_0D^2_\varepsilon \to T_0C$ is an orientation preserving isomorphism of vector spaces. In other words, $s_{m+1} : M \times D^2_\varepsilon \to \xi$ intersects transversely $F \subset \xi$ in $M \times \{0\}$, considered as an oriented manifold.

An analogue computation with $s'_m$ shows that we can write $s'_m : \mathcal{U} \to \xi' = \mathcal{U} \times \mathbb{C}^{m+1}$ as $s'_m(q) = (q, 0, \ldots, 0, v'_m(q))$, with $v'_m(q) = x - iy$ for each $q = (m, x, y) \in \mathcal{U} = M \times D^2_\varepsilon$.

This gives in particular that $d_{(m,0)}v'_m|_{\{m\} \times T_0D^2_\varepsilon} : \{m\} \times T_0D^2_\varepsilon \to T_0\mathbb{C}$ is an orientation reversing isomorphism of vector spaces hence that $s'_{m+1} : M \times D^2_\varepsilon \to \xi'$ intersect transversely $F' \subset \xi'$ along the oriented submanifold $-M \times \{0\}$.

At this point, Proposition 2.3 follows from Proposition 3.6.

4 Proof of the main theorem

If $m$ is a positive integer, we recall that $\xi'_m$ denotes the pullback of the contact structure $\xi'$ on $M \times \mathbb{T}^2$ defined in Section 1 via $\pi_m : M \times \mathbb{T}^2 \to M \times \mathbb{T}^2$, $(p, t, s) \mapsto (p, t, ms)$.

The proof of Theorem 1.1 is then divided in two parts:

Lemma 4.1. If $f$ is contact-isotopic to the identity, there is a contactomorphism

$$\phi : (M \times S^1_t \times S^1_s, \xi'_{kn}) \cong (M \times S^1_t \times S^1_s, \xi'_{kn+1}) .$$

Lemma 4.2. $\xi'_{kn}$ and $\xi'_{kn+1}$ are not isomorphic as almost contact structures on $M \times \mathbb{T}^2$.

Then, if by contradiction $f$ is contact-isotopic to the identity, the conclusion of Lemma 4.1 is in contradiction with that of Lemma 4.2, because the contactomorphism $\phi$ gives is in particular an isomorphism of almost contact structures. This shows that $f$ can’t be contact-isotopic to the identity, as wanted.

Before proving the two lemmas, let’s introduce a notation.

Given an integer $l \geq 1$, we denote by $\xi_l$ on $M \times S^1_t \times S^1_s$ the pull-back of

$$\xi = \ker \left( \frac{1 + \cos (s)}{2} \alpha_+ + \frac{1 - \cos (s)}{2} \alpha_- + \sin (s) dt \right)$$

via the covering map $M \times S^1_t \times S^1_s \to M \times S^1_t \times S^1_s$ induced by $\nu_l : S^1_t \to S^1_t$, $s \mapsto ls$.

We point out that $\xi'_m$ can be seen also as obtained from $\xi_m$ by performing a half Lutz-Mori twist along each of the $m$ submanifolds $M \times \{0\} \times \{\frac{2\pi l}{m}\}$, where $l = 0, \ldots, m - 1$.

Proof (Lemma 4.1). In order to find the desired contactomorphism $\phi$, we use an idea already appearing in [GGP04, MP16] which consists in cutting off contact hamiltonians on a particular cover of the manifold we are working with.
The analogous procedure for the codomain obtained is contact structure on the quotient. More precisely, the quotient contact manifold of \( M \) is a trivialization as complex vector bundle given by the following sections and structure construction in Section 2.2. In particular, all the Chern classes of where we use locally on \( M \) the coordinates \((t_1, \ldots, t_m, \theta_0, \ldots, \theta_m)\) given by the construction in Section 2.2. In particular, all the Chern classes of \( \xi \) are zero.

Then each \( \xi \) is also a trivial complex vector bundle of (complex) rank \( m \) over \( M \times \mathbb{T}^2 \) and has in particular trivial Chern classes.
Now, applying Proposition 2.3 to the couples \((\xi_{kN}, \xi_{kN+1})\) (recall the notation introduced before the proof of Lemma 4.1) we conclude the following: if we denote by \(j : M \to M \times S^1 \times S^1\) the inclusion \(j(p) = (p, 0, 0)\) and by \(j_* : H_{2m+1}(M; \mathbb{Z}) \to H_{2m+1}(M \times S^1 \times S^1; \mathbb{Z})\) the induced map in homology, then \(c_1(\xi_{kN}) = -2kN \text{PD}(j_*[M])\) and \(c_1(\xi_{kN+1}) = -2(kN + 1) \text{PD}(j_*[M])\) in \(H^2(M \times S^1 \times S^1; \mathbb{Z})\).

The homological version of Kunneth’s theorem tells us that \(H_{2m+1}(M \times S^1 \times S^1; \mathbb{Z}) \cong H_{2m+1}(M; \mathbb{Z}) \otimes N,\) for a certain \(\mathbb{Z}\)-submodule \(N\) of \(H_{2m+1}(M \times S^1 \times S^1; \mathbb{Z})\). Indeed, in the Kunneth exact sequence the Tor functor gives always zero contribution in our case because the homology of \(\mathbb{T}^2\) is a \(\mathbb{Z}^n\) in each degree.

Moreover, the isomorphism \(g\) is such that \(g^{-1}[H_{2m+1}(M; \mathbb{Z})] \oplus \{0\} = j_*\), where \(j_* : H_{2m+1}(M; \mathbb{Z}) \to H_{2m+1}(M \times S^1 \times S^1; \mathbb{Z})\) is the map induced by \(j\) in homology and where we naturally identify \(H_{2m+1}(M; \mathbb{Z}) \oplus \{0\} = H_{2m+1}(M; \mathbb{Z})\).

Now, the fundamental class \([M]\) generates \(H_{2m+1}(M; \mathbb{Z})\), hence \(j_*[M]\) generates the submodule \(H_{2m+1}(M; \mathbb{Z}) \oplus \{0\}\) of \(H_{2m+1}(M; \mathbb{Z}) \oplus N \cong H_{2m+1}(M \times S^1 \times S^1; \mathbb{Z})\). In particular, \(j_*[M]\) is primitive in \(H_{2m+1}(M \times S^1 \times S^1; \mathbb{Z})\), i.e. it can’t be written as a non trivial integer multiple of another element.

Suppose now by contradiction that there is an isomorphism of almost contact structures \(\psi : (M \times \mathbb{T}^2, \xi_{kN}) \cong (M \times \mathbb{T}^2, \xi_{kN+1}).\) In particular, \(c_1(\xi_{kN}) = \psi^*c_1(\xi_{kN+1}).\) This implies \(kN \text{PD}(j_*[M]) = (kN + 1)\psi^*(\text{PD}(j_*[M])),\) i.e. \(\psi^*(\text{PD}(j_*[M])) = kN\lambda,\) where \(\lambda := \text{PD}(j_*[M]) - \psi^*(\text{PD}(j_*[M]))\).

Though, having \(N \geq 2\) (and in particular \(kN > 1\)), this contradicts (whether \(\lambda\) is zero or not) the facts that \(\text{PD}(j_*[M])\) is primitive and \(\psi^*\) is an isomorphism, hence concluding the proof by contradiction of Lemma 4.2.

\[
\square
\]

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