On the Suboptimality of Thompson Sampling in High Dimensions

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October 22, 2021

Abstract

In this paper we consider Thompson Sampling (TS) for combinatorial semi-bandits. We demonstrate that, perhaps surprisingly, TS is sub-optimal for this problem in the sense that its regret scales exponentially in the ambient dimension, and its minimax regret scales almost linearly. This phenomenon occurs under a wide variety of assumptions including both non-linear and linear reward functions, with Bernoulli distributed rewards and uniform priors. We also show that including a fixed amount of forced exploration to TS does not alleviate the problem. We complement our theoretical results with numerical results and show that in practice TS indeed can perform very poorly in some high dimensional situations.

1 Introduction

We consider the problem of combinatorial bandits with semi-bandit feedback. At time \( t = 1, \ldots, T \) a learner selects a decision \( x(t) \in \mathcal{X} \) where \( \mathcal{X} \subset \{0, 1\}^d \) is the set of available decisions. The environment then draws a random vector \( Z(t) \in \mathbb{R}^d \). The learner then observes \( Y(t) = x(t) \odot Z(t) \), where \( \odot \) denotes the Hadamard (elementwise) product. This setting is called semi bandit feedback. We assume that \( (Z(t))_{t \geq 1} \) are i.i.d., and that \( Z_1(t), \ldots, Z_d(t) \) are independent and distributed as \( Z_i(t) \sim \text{Bernoulli}(\theta_i) \) for all \( t, i \). Then the learner receives a reward \( f(x(t), Z(t)) \) where \( f \) is a known function.

The goal is to minimize the regret:

\[
R(T, \theta) = T \max_{x \in \mathcal{X}} \left\{ \mathbb{E}f(x, Z(t)) \right\} - \sum_{t=1}^{T} \mathbb{E}f(x(t), Z(t)).
\]

Initially \( \theta \) is unknown to the learner and minimizing regret involves exploring suboptimal decisions just enough in order to identify the optimal decision. For any decision \( x \in \mathcal{X} \), define the reward gap

\[
\Delta_x = \max_{x \in \mathcal{X}} \{\mathbb{E}f(x, Z(t))\} - \mathbb{E}f(x, Z(t)),
\]

which is the amount of regret incurred by choosing \( x \) instead of an optimal decision

\[
x^* \in \arg \max_{x \in \mathcal{X}} \left\{ \mathbb{E}f(x, Z(t)) \right\},
\]
\[ \Delta_{\text{min}} = \min_{x \in \mathcal{X} : \Delta_x > 0} \Delta_x \] the minimal gap. We define \[ m \triangleq \max_{x \in \mathcal{X}} \sum_{i=1}^{d} |x_i| \] the size of the maximal decision.

For this problem, an algorithm which has attracted a lot of interest is Thompson Sampling (TS), which at time \( t \) selects the decision maximizing \( x \mapsto f(x, V(t)) \) where \( V(t) \) is a random variable distributed as the posterior distribution of \( \theta \) knowing the information available at time \( t \), which is \( Y(1), \ldots, Y(t-1) \). The prior distribution of \( \theta \) can be chosen in various ways, the most natural being a non-informative distribution such as the uniform distribution.

TS is usually computationally simple to implement, for instance when \( f \) is linear, since it involves maximizing \( f \) over \( \mathcal{X} \). Also, for some problem instances it tends to perform well numerically. A particular case of interest is linear combinatorial semi-bandits where \( f(x, \theta) = \theta^\top x \) so that the reward is a linear function of the decision.

**Our contribution.** We show that the regret of TS in general does not scale polynomially in the ambient dimension \( d \).

(i) We provide several examples, both for linear and non-linear combinatorial bandits, where the regret of TS does not scale polynomially in the dimension \( d \) (in fact in some cases it may scale even faster than exponentially in the dimension). In some cases, we show that one must wait for an amount of time greater than \( \Omega(d^d) \) for TS to perform at least as well as random choice where one simply chooses \( x(t) \) uniformly distributed in \( \mathcal{X} \) at every round. Therefore, in high dimensions, in some instances, TS in general can perform strictly worse than random choice for all practically relevant time horizons.

(ii) We show that the minimax regret of TS scales at least as \( \Omega(T^{1-\frac{d}{d}}) \) so that it is not minimax optimal, as there exists algorithms such as CUCB and ESCB with minimax regret \( O(\text{poly}(d) \sqrt{T(\ln T)}) \). In fact, in high dimensions, the minimax regret of TS is almost linear.

(iii) We further show that adding forced exploration as an initialization step to TS does not correct the minimax problem, so that this is not an artifact due to initialization.

(iv) Using numerical experiments, we show that indeed, for reasonable time horizons, TS performs very poorly in high dimensions in some instances.

We believe that our results highlight two general characteristics of TS. First, TS tends to be much more greedy than optimistic algorithms such as ESCB and CUCB. This greedy behavior explains why the regret of TS is, in some instances, much smaller than that of optimistic algorithms. In fact it is sometimes so greedy that it misses the optimal decision. Second, TS tends to be by nature a ”risky” algorithm so that its regret exhibits very large fluctuations across runs. In some cases it finds the optimal arm very quickly and with little to no regret, while in other cases it simply misses the optimal decision and performs worse than random choice.

**Related work.** Combinatorial bandits are a generalization of classical bandits studied in [15]. Several asymptotically optimal algorithms are known for classical bandits, including the algorithm of [16], KL-UCB [4], DMED [12] and TS [20, 13]. Other algorithms include the celebrated UCB1 [2]. A large number of algorithms for combinatorial semi-bandits have been proposed, many of which naturally extend algorithms for classical bandits to the combinatorial setting. CUCB [5] [14] is a natural extension of UCB1 to the combinatorial setting. ESCB [7, 9] is an improvement of CUCB which leverages the independence of rewards between items. AESCB [8] is an approximate version of ESCB with roughly the same performance guarantees.
Table 1: Algorithms and best known regret bounds.

| Algo. | Regret |
|-------|--------|
|       | ((i) problem dependent and (ii) minimax) |
| CUCB  | $O(d \ln T / \Delta_{\min})$ [14][Theorem 4]  |
|       | $O(\sqrt{dm \ln T + dm})$ [14][Theorem 6] |
| ESCB  | $O(d \ln m^2 \ln T / \Delta_{\min} + dm^3 / \Delta_{\min}^2 + m((m^2 + 1)/\Delta_{\min})^{2+4m})$ [18][Theorem 1]  |
|       | $O(\sqrt{d \ln m^2 T / \ln T + dm})$ [2][Corollary 1] |
| TS    | (i) $O(d \ln m^2 \ln(|X|)/\Delta_{\min} + dm^3 / \Delta_{\min}^2 + m((m^2 + 1)/\Delta_{\min})^{2+4m})$ [18][Theorem 1]  |
|       | (ii) not available |

and reduced computational complexity. TS for combinatorial bandits was considered in [10, 21, 18]. Also, combinatorial semi bandits are a particular case of structured bandits, for which there exists asymptotically optimal algorithms such as OSSB [6]. Table 1 presents the best known regret upper bounds for CUCB, ESCB and TS. For completeness, we also recall the complete regret upper bound for ESCB as Theorem 10 (Appendix A).

We provide two types of bounds for TS: problem dependent bounds (sometimes called gap-dependent bounds) and minimax bounds (or gap-free bounds). The former involves $T, d, m$ and $\Delta_{\min}$, while the latter hold for any value of $\Delta_{\min}$.

An important observation is that all of the known regret upper bounds for TS [10, 21, 18] feature at least one term that does not scale polynomially with the dimension. In particular, the paper [18] shows that there exists a universal constant $C \geq 0$ such that the regret of TS is upper bounded by

$$R(T) \leq C \left[ d \ln m^2 \ln(|X|)/\Delta_{\min} + dm^3 / \Delta_{\min}^2 + m((m^2 + 1)/\Delta_{\min})^{2+4m} \right].$$

This is a general bound for all combinatorial sets of interest. The bound has a super exponential term (that does not depend on $T$) : $m((m^2 + 1)/\Delta_{\min})^{2+4m}$. Regret bounds for CUCB and ESCB do not feature this exponential dependency in the dimension. However, since TS tends to perform very well in all of the numerical experiments presented in the literature, a natural intuition would be that all known upper bounds are simply not sharp, and that the true regret of TS does not really grow exponentially with the dimension. We show in this work that this intuition is incorrect, and that the regret of TS really does scale (at least) exponentially in the dimension i.e. it suffers from the "curse of dimensionality". This directly implies that, in high dimensions, and for some combinatorial sets, CUCB and ESCB perform much better than TS.

In order to alleviate this problem, it would be natural to attempt to modify the prior distribution used by TS, since it is known to have a strong influence on its performance [5][17]. Similarly, in a related problem, [1] suggests to use correlated Thompson samples, and [11] studies the influence of the prior on numerical performance. We believe that this is an interesting open problem.
Prior knowledge to the learner: combinatorial set $\mathcal{X} \subset \{0, 1\}^d$, function $f : \mathcal{X} \times \{0, 1\}^d \to \mathbb{R}$

For $t = 1, ..., T$:

1. The learner computes statistics $A(t) = \sum_{s=1}^{t-1} Z(s) \odot x(s)$ and $B(t) = \sum_{s=1}^{t-1} x(s) - Z(s) \odot x(s)$
2. The learner draws $V(t)$ with $V_1(t), ..., V_d(t)$ independent and $V_i(t) \sim \text{Beta}(A_i(t) + 1, B_i(t) + 1)$
3. The learner chooses decision $x(t) \in \arg \max_{x \in \mathcal{X}} \{f(x, V(t))\}$
4. The environment draws $Z(t)$ with $Z_1(t), ..., Z_d(t)$ independent and $Z_i(t) \sim \text{Ber}(\theta_i)$
5. The learner observes $Z(t) \odot x(t)$ and receives the reward $f(x(t), Z(t))$

Performance metric: expected regret $R(T, \theta) = T(\max_{x \in \mathcal{X}} \{\mathbb{E} f(x, Z(t))\}) - \sum_{t=1}^{T} f(x(t), Z(t))$

Figure 1: TS for combinatorial semi-bandits with Bernoulli rewards and uniform prior.

2 Model

2.1 Problem Dependent Regret and Minimax Regret

In order to evaluate the performance of an algorithm over the set of instances $\theta \in [0, 1]^d$, there are two main figures of merit that we study in this paper. The first is the problem-dependent regret which is $R(T, \theta)$ when $\theta \in [0, 1]^d$ is fixed. The second is the minimax regret which is the worse case over $\theta$ for $T$ fixed: $\max_{\theta \in [0, 1]^d} R(T, \theta)$.

2.2 TS

The basic TS algorithm works as follows. For $i = 1, ..., d$, define $A_i(t) = \sum_{s=1}^{t-1} Z_i(s)x_i(s)$ and $B_i(t) = \sum_{s=1}^{t-1}(1 - Z_i(s))x_i(s)$ which represent the number of successes and failures observed when getting a sample to estimate $\theta_i$. We define $N_i(t) = A_i(t) + B_i(t) = \sum_{s=1}^{t-1} x_i(s)$ the number of samples available at time $t$ to estimate $\theta_i$, and

$$\hat{\theta}_i(t) = \frac{A_i(t)}{\max(N_i(t), 1)},$$

the corresponding estimate of $\theta_i$ which is simply the empirical mean.

The TS algorithm selects decision

$$x(t) \in \arg \max_{x \in \mathcal{X}} f(x, V(t)) \text{ where } V_i(t) \sim \text{Beta}(A_i(t) + 1, B_i(t) + 1),$$

and $V_1(t), ..., V_d(t)$ are independent. Vector $V(t)$ is called the Thompson sample at time $t$. TS is based on a Bayesian argument, $V(t)$ is drawn according to the posterior distribution of $\theta$ knowing the information available at time $t$, where the prior distribution for $\theta$ is uniform over $[0, 1]^d$. Choosing decision $x(t)$ as done above should ensure that
one explores just enough to find the optimal decision. In the linear case, the decision can be computed by linear maximization over $\mathcal{X}$:

$$x(t) \in \arg \max_{x \in \mathcal{X}} \{V(t)^T x\}.$$ 

This explains the practical appeal of TS, since whenever linear maximization over $\mathcal{X}$ can be implemented efficiently, the algorithm has low computational complexity.

### 2.3 TS with Forced Exploration

A natural extension of TS is to add $\ell$ forced exploration rounds, where $\ell$ is a fixed number, in order to avoid some artifacts that could possibly occur due to the prior distribution. The algorithm operates as follows. At time $1 \leq t \leq \ell$, one selects $x(t) \in \mathcal{X}$ such that $x_{i(t)}(t) = 1$ with

$$i(t) \in \min_{i=1,...,d} N_i(t).$$

Otherwise for $t \geq \ell + 1$ one selects

$$x(t) \in \arg \max_{x \in \mathcal{X}} f(x, V(t)),$$

with $V(t)$ the Thompson sample defined above. Namely, one first performs a forced exploration during $\ell$ rounds then apply TS. This guarantees that $N_i(t) \geq \lfloor \ell/d \rfloor$ samples are available to estimate $\theta_i$ for all $i = 1,...,d$, then one subsequently applies TS. We call this variant TS with $\ell$ forced exploration rounds.

### 3 Main Results

We now state our main theoretical results. All proofs are found in the appendix.

#### 3.1 Some Combinatorial Sets of Interest

We will provide several examples of combinatorial sets where the regret of TS indeed scales exponentially with the dimension, so that this phenomenon is quite general and is not an artifact that only occurs for one particular family of combinatorial structures. We define $\mathcal{X}^p$ the set of paths of the directed acyclic graph depicted in figure [2]. This set has two disjoint decisions $(1,...,1,0,...,0)$ and $(0,...,0,1,...,1)$ of equal size $m = d/2$. We define $\mathcal{X}^m$ the set of matchings of the bipartite graph depicted in figure [2]. This graph has $d$ vertices and $d$ edges.

The combinatorial sets presented are very simple. However our results can by generalised for more complex set of interest without losing the exponential nature of the regret. For example the two path environment can be generalized to $k > 2$ paths. It can also be generalized for non disjoint paths if the optimal path does not share "a lot" of edges with all the other paths. This could be the case for real life applications like shortest path routing or in medical trials where treatments cannot be associated with
each other. With those simple examples in mind many other more complex sets that exhibit exponential regret can be found. However we do not provide formal proof for those more complex examples as the simple example of paths is sufficient to prove the suboptimality of TS here.

3.2 Linear Combinatorial Bandits

We focus on linear bandits, where the expected reward function is linear i.e. \( f(x, \theta) = x^\top \theta \). In the example of [21][Theorem 3], the Thompson sample of sub-optimal decisions has no variance. One could be lead to think that the exponential dependency of the regret on the dimension could be caused by this feature, and it is hence natural to investigate the linear case, which is not only more common, but also where the Thompson sample of any decision always has a non-null variance.

In Theorem 1 we consider a linear problem over the combinatorial set \( X^p \) which is formed of two disjoint paths. We show that the regret of TS does scale exponentially in the dimension for this problem. Therefore this phenomenon is not linked to a particular, well chosen, non-linear reward function, but also occurs for the classical case of linear reward functions.

**Theorem 1.** Consider a linear combinatorial bandit problem over combinatorial set \( X^p \) and parameter \( \theta_i = 1 \) if \( 1 \leq i \leq d/2 \) and \( \theta_i = 1 - \frac{\Delta}{m} \) otherwise. Assume that \( \frac{\Delta}{m} + \frac{1}{\sqrt{m}} < \frac{1}{2} \).

Then the regret of TS is lower bounded by

\[
R(T, \theta) \geq \frac{\Delta}{4p_\Delta} (1 - (1 - p_\Delta)^{T-1}), \quad \text{with } p_\Delta = \exp \left\{ -\frac{2m}{9} \left( \frac{1}{2} - \left( \frac{\Delta}{m} + \frac{1}{\sqrt{m}} \right) \right)^2 \right\}.
\]

Theorem 1 is proven by showing that the first time that the optimal decision is selected is exponentially large in general. The central argument can be summarized as follows. Consider \( t \) such that at times \( 1, ..., t \) only the suboptimal decision has been selected. The probability of selecting the optimal decision is

\[
\mathbb{P} \left( \sum_{i=1}^{m} V_i(t) \geq \sum_{i=m+1}^{d} V_i(t) | A(t), B(t) \right)
\]
where $V_1(t),...,V_d(t)$ are independent, distributed in $[0, 1]$, and their respective expectations are

$$
\mathbb{E}(V_i(t)|A(t), B(t)) = \begin{cases} 
\frac{1}{2}, & \text{if } 1 \leq i \leq m, \\
\frac{A_i(t)+1}{t+2}, & \text{if } m+1 \leq i \leq d.
\end{cases}
$$

Furthermore, from the law of large numbers, when $t$ is large,

$$
\sum_{i=m+1}^{d} A_i(t) \approx (1 - \Delta) t
$$

since we have sampled the sub-optimal decision $t$ times. Therefore

$$
\sum_{i=1}^{m} \mathbb{E}(V_i(t)|A(t), B(t)) = \frac{m}{2}
$$

and again because $t$ is large,

$$
\sum_{i=m+1}^{d} \mathbb{E}(V_i(t)|A(t), B(t)) \approx m - \Delta.
$$

Since $V_1(t),...,V_d(t)$ are independent and distributed in $[0, 1]$, their sums must concentrate around their expectation, and from Hoeffding’s inequality:

$$
P\left(\sum_{i=1}^{m} V_i(t) \geq \sum_{i=m+1}^{d} V_i(t)|A(t), B(t)\right) \leq O\left(e^{-um(\frac{1}{2} - \Delta/m)^2} \right),
$$

where $u > 0$ is some positive exponent related to how concentrated the Thompson samples are. This implies that, for large $t$, the probability of selecting the optimal decision is exponentially small if it has never been selected previously.

Also, we see that this phenomenon of lack of exploration by TS is a typically high dimensional phenomenon. In short, the Thompson samples of decisions will tend to concentrate around their expectation, so that TS will, most of the time, act greedily and simply select the decision maximizing the empirical reward. We can also emphasize the fact that when $m$ grows, we can have an arbitrary large gap $\Delta$ and still have exponential regret. This is unexpected, since the difficulty of a bandit problem is usually a decreasing function of the gap $\Delta$. Furthermore another version of Theorem 1 which exhibits exponential behavior can be shown with parameters $\theta_i = u$ if $1 \leq i \leq d/2$ and $\theta_i = u - \Delta/m$ otherwise, under the condition $\frac{\Delta}{m} + \frac{1}{\sqrt{m}} < u - \frac{1}{2}$ where $u \in ]\frac{1}{2}, 1]$. We chose the parameters of Theorem 1 for the sake of clarity and being at the edge of the parameter space is not a necessary condition to have exponential regret.

### 3.3 Linear Combinatorial Bandits: Small Gap Regime

Theorem 2 is another regret bound for TS which is more accurate in the regime where $\Delta$ is small, it allows to deduce a lower bound for the minimax regret as well. The proof
is a more intricate version of the proof of Theorem 1 highlighted above. Corollary 3 states that the minimax regret of TS scales as $O(T)$ for $T \leq m!$, and at least as $\Omega(T^{1-\frac{1}{d}})$ for $T > m!$. Of course, in practice, when $m$ is large we have $T \leq m!$ for any reasonable time horizon, so that the regret of TS is linear in this regime. Also, as stated by Corollary 3, TS is provably not minimax optimal, as there exist algorithms with minimax regret scaling at most as $O(d^{1/4}/T^{1/2} \ln T)$.

**Theorem 2.** Consider a linear combinatorial bandit problem over combinatorial set $X^p$ and parameter $\theta_i = 1$ if $1 \leq i \leq d/2$ and $\theta_i = 1 - \frac{\Delta}{m}$ otherwise.

Then there exists universal constants $C_2, C_3$ such that for all $T \geq T_0(m) \equiv C_2 m^2 \ln m$, and $\Delta \leq 1/6$ and $m \geq 5$, the regret of TS is lower bounded by

$$R(T, \theta) \geq \frac{\Delta}{4p_\Delta} (1 - (1 - \Delta)^{T_0 - 1}) + C_3 \Delta \left( \frac{1 - g_\Delta}{g_\Delta} - (1 - g_\Delta)^{T - T_0 + 1} \right)$$

with $g_\Delta = \frac{(2\Delta)^m}{m!}$ and $p_\Delta = \exp \left\{ -\frac{2m}{9} \left[ \frac{1}{2} - \left( \frac{\Delta}{m} + \frac{1}{\sqrt{m}} \right)^2 \right] \right\}$.

**Corollary 3.** Consider a linear combinatorial bandit problem over combinatorial set $X^p$ with $m \geq 5$. If $T > 2^m m!$ then the minimax regret of TS is lower bounded by

$$\max_{\theta \in [0, 1]^d} R(T, \theta) \geq C_4 m T^{1 - \frac{1}{d}}$$

Otherwise it is lower bounded by

$$\max_{\theta \in [0, 1]^d} R(T, \theta) \geq C_4' T$$

with $C_4, C_4' > 0$ universal constants.

The minimax regret of ESCB for this set $X^p$ is upper bounded by:

$$\max_{\theta \in [0, 1]^d} R(T, \theta) \leq C_5 d^{\frac{1}{4}} \sqrt{T \ln T}$$

with $C_5 \geq 0$ a universal constant. Therefore TS is not minimax optimal.

### 3.4 Linear Combinatorial Bandits with Forced Exploration

We finally extend our results to show that, even when forced exploration is added, TS still provably incurs a regret growing exponentially with the dimension, as stated by Theorem 4 and Theorem 5. In particular, if the number of forced exploration rounds $\ell$ satisfies

$$\ln \left( \frac{1}{1 - \frac{\Delta}{m}} \right) \leq \frac{2}{\ell + 2} - \left( \frac{\Delta}{m} + \frac{1}{\sqrt{m}} \right)^2 \left( \frac{\ell}{2} + 3 \right)$$

then Theorem 4 implies that the regret of TS still increases exponentially in $m$, in spite of the forced exploration added to the algorithm. In fact it is impossible to set $\ell$ to
prevent exponential regret from happening, unless the learner knows the value of the gap $\Delta$ in advance. Indeed, for any fixed $\ell$, the above inequality always holds providing that $\Delta$ is small enough.

**Theorem 4.** Consider a linear combinatorial bandit problem over combinatorial set $X^p$ and parameter $\theta_i = 1$ if $1 \leq i \leq d/2$ and $\theta_i = 1 - \frac{\Delta}{m}$ otherwise.

Then if $\Delta m + \frac{1}{\sqrt{m}} < \frac{1}{4} + 2$ the regret of TS with $\ell$ forced exploration rounds is lower bounded by

$$R(T, \theta) \geq (1 - \frac{\Delta}{m}) \frac{m}{4p^\ell} \frac{\Delta}{1 - (1 - p^\ell)^{T-1}}.$$  

$$p^\ell = \exp \left\{ -2m \left( \frac{1}{2} + 2 - \left( \frac{\Delta}{m} + \frac{1}{\sqrt{m}} \right)^2 / \left( \frac{\ell}{2} + 3 \right)^2 \right) \right\}.$$  

**Theorem 5.** Consider a linear combinatorial bandit problem over combinatorial set $X^p$ with $m \geq 5$ and parameter $\theta_i = 1$ if $1 \leq i \leq d/2$ and $\theta_i = 1 - \frac{\Delta}{m}$ otherwise.

Then if $T > 4^m m!$ and $T > T_0(m, l) \equiv C_0 m^2 (\ln m) \ell^{\frac{1}{4} + \frac{1}{m}}$ the minimax regret of TS with $\ell$ forced exploration rounds is lower bounded by

$$\max_{\theta \in [0,1]^d} R(T, \theta) \geq C_7 C(\ell, m) \frac{m}{\ell} T^{-1 - \frac{1}{m}}.$$  

Otherwise it is bounded by

$$\max_{\theta \in [0,1]^d} R(T, \theta) \geq C'_7 C(\ell, m) \frac{T}{\ell}.$$  

with $C_0, C_7, C'_7 > 0$ universal constants and $C$ such that $\forall \ell \in \mathbb{N}$, $\lim_{m \to \infty} C(\ell, m) = 1$

### 3.5 Non-Linear Combinatorial Bandits

We here provide a non-linear combinatorial bandits example. The example is inspired by [21]: there are two decisions, the optimal decision has an expected reward of 1 and the other one an expected reward of $1 - \Delta$. Theorem [6] shows that the regret of TS for this problem scales super-exponentially with the dimension $d$ which is an improvement over [21][Theorem 3]. By corollary, we prove that TS does not outperform random choice (i.e. a trivial algorithm which chooses one of the two decisions uniformly at random at each time) until $t \geq T_0(m)$, where $T_0(m)$ grows super-exponentially with $m$. As an illustration of how large this number might be, for $\Delta = \frac{1}{2}$, the value of $T_0(9)$ is greater than a million, and the value of $T_0(20)$ is greater than the estimated age of the universe in seconds. Therefore, in practice as well as in theory, TS does not outperform random choice in high dimensions which is perhaps even more surprising.

The proof of Theorem [6] is based on the fact that there exists a non zero probability that the optimal decision will never be selected for an exponentially large amount of time. Indeed, if the optimal decision has never been selected, it is chosen with a probability
equal to $\mathbb{P}(\prod_{i=1}^{m} U_i \geq 1 - \Delta)$ where $U_1, \ldots, U_m$ are i.i.d. uniformly distributed on $[0, 1]$, and since this probability is exponentially small in $d$, one must wait for an exponentially large time before selecting the optimal decision and the regret must scale accordingly. It is noted that this proof technique of lower bounding the expected value of the first time the optimal decision is ever selected is very powerful and will be used many times to prove our results.

**Theorem 6.** Consider a non-linear combinatorial bandit problem over combinatorial set $X = \{ \sum_{i=1}^{m} e_i, e_{m+1} \}$ where $(e_i)_{i \in [m+1]}$ is the canonical base of $\mathbb{R}^{m+1}$ with parameter $\theta = (1, \ldots, 1)$ and reward function $f(x, \theta) = \prod_{i=1}^{m} \theta_i$ if $x = \sum_{i=1}^{m} e_i$ and $f(x, \theta) = 1 - \Delta$ otherwise.

Then the regret of TS is lower bounded by

$$R(T, \theta) \geq \frac{\Delta}{p_\Delta} (1 - (1 - p_\Delta)^T) \text{ with } p_\Delta = \frac{1}{mm!} \left[ \ln \left( \frac{1}{1 - \Delta} \right) \right]^m.$$ 

**Corollary 7.** For any $T \leq T_0(m) \equiv \frac{1}{p_\Delta}$ TS performs strictly worse than random choice in the sense that

$$R(T, \theta) \geq T \Delta \left( 1 - \frac{1}{e} \right) > T \Delta \frac{1}{2}.$$ 

It is noted that Theorem 6 is a parameter-dependent lower bound, where we consider a fixed parameter $\theta$ and we let the time horizon $T$ grow. From Theorem 6, we deduce Corollary 8 which is a lower bound on the minimax regret of TS. The minimax regret of TS scales at least as $\Omega(T^{1 - \frac{d}{2}})$, so that it is almost linear in high dimensions when $d$ is large. This also proves that, as long as the dimension $d$ is strictly greater than 2, TS is not minimax optimal, since there exists algorithms such as CUCB whose minimax regret scales at most as $O(\text{poly}(d) \sqrt{T \ln T})$. This demonstrates that TS has a tendency to be too "greedy" which prevents it from exploring enough, and while this is not a problem in low dimensions, in high dimensions this matters a great deal, and causes it to perform much worse than optimistic algorithms. Corollary 8 is proven simply by letting $\Delta = T^{-\frac{d}{2}}$ in Theorem 6 and the regret upper bound for CUCB follows directly from [14].

**Corollary 8.** Consider $\mathcal{F}$ the class of $1$-Lipschitz functions.

The minimax regret of TS is lower bounded by:

$$\max_{\theta \in [0, 1]^d, f \in \mathcal{F}} R(T, \theta, f) \geq C_1 T^{1 - \frac{d}{2}},$$

with $C_1 > 0$ a universal constant, the minimax regret of CUCB is upper bounded by

$$\max_{\theta \in [0, 1]^d, f \in \mathcal{F}} R(T, \theta, f) \leq C'_1 d \sqrt{T \ln T},$$

where $C'_1$ is a universal constant. Hence TS is not minimax optimal.
3.6 Non-Linear Combinatorial Bandits with Forced Exploration

Our results above show that the regret of TS scales exponentially in the dimension since the expectation of the first time at which the optimal decision is selected can grow exponentially in the dimension. Therefore it is natural to assume that forcing some exploration initially would alleviate the problem. Theorem 9 considers the same non-linear bandit problem as that considered in Theorem 6, and shows that, while forced exploration does bring some improvement, for any fixed value of $\ell > 0$, the regret of TS with $\ell$ forced exploration rounds still scales exponentially in the dimension. The reason for this is that once again the first time at which the optimal decision is selected can be exponentially large, even with forced exploration. Upon closer inspection of Theorem 9, one can see that, in order for the regret lower bound not to scale exponentially in the dimension one would require $\frac{1}{p} \geq 1$, which in turn would require $\frac{1}{2} \ln \left( \frac{1}{1-\Delta} \right) \geq 1$. This indicates that, unless the gap $\Delta$ is known in advance (and in general $\Delta$ is of course unknown), it is not possible to select a value of $\ell$ that prevents the regret from scaling exponentially in the dimension. This suggests that some more complex modifications need to be made to TS in order to "fix" this exponential dependency on the dimension.

**Theorem 9.** Consider a non-linear combinatorial bandit problem with over combinatorial set $X = \{ \sum_{i=1}^{m} e_i, e_{m+1} \}$ with parameter $\theta = (1, ..., 1)$ and reward function $f(x, \theta) = \prod_{i=1}^{m} \theta_i$ if $x = \sum_{i=1}^{m} e_i$ and $f(x, \theta) = 1 - \Delta$ otherwise.

Then the regret of TS with $\ell$ forced exploration rounds is lower bounded by

$$R(T, \theta) \geq \frac{\Delta}{p^{\ell}_{\Delta}} \left( 1 - (1 - p^{\ell}_{\Delta})^{T} \right) \text{ with } p^{\ell}_{\Delta} = \frac{1}{mm!^{\left( 1 + \frac{\ell}{2} \right)} \ln \left( \frac{1}{1-\Delta} \right)^{m}}.$$

4 Numerical Experiments

We now illustrate the exponential regret of TS in practical settings using numerical experiments. Due to this exponential nature, some of those experiments involve a significant amount of computing time in high dimensions. Due to limited space, we solely consider the linear case, which is the most often considered in the literature. Unless specified otherwise we use 1000 independent sample paths for averaging, and 95% confidence intervals are presented on the plots.

**First selection of the optimal decision** As shown by our theoretical results, the first time that the optimal decision is selected $\tau = \min\{ t \geq 1 : x(t) = x^* \}$ can be exponentially large, and this is what causes exponential regret. On Figure 3 we present the c.d.f. (cumulative distribution function) of $\tau$ as a function of $m$ for combinatorial sets $X^p$ and $X^m$ introduced above. The parameter values are chosen as in the previous sections $\theta_i = 1$ if $1 \leq i \leq d/2$ and $\theta_i = \frac{\Delta}{m}$ otherwise. For each sample path we generate $\tau$ by simulating TS until the optimal decision is played for the first time.

Some quantiles of $\tau$ indeed seem to increase exponentially as $m$ grows and for reasonable values of $m$, $\tau$ can be very large with appreciable probability, for instance on Fig. 3 for $m = 14$, $\tau \geq 5.10^4$ with probability greater than 0.1. Clearly, on sample
paths where this happens, TS performs worse than random choice for the first $5.10^4$ time steps which is a surprisingly poor behaviour, especially on such a simple problem. This also showcases the fact that those sample paths happen relatively often. Thus the regret of TS is not only due to very rare occasions with high regret but also because of those poor behavior that can happen quite often.

To investigate the impact of the gap $\Delta$, on Figure 4a and Figure 4b, we plot the expected first time the optimal decision is selected $E(\tau)$ as a function of $m$ for various $\delta = \frac{\Delta}{m}$, for the sets of paths $\mathcal{X}^p$ and matchings $\mathcal{X}^m$. Once again we observe an exponential growth in both figures, and this growth is particularly fast for small values of $\delta$. When the gap gets small, the exponential growth of regret is exacerbated, leading to an even worse performance.

Impact of forced exploration We now investigate if forced exploration alleviates the problem in practice, and consider $\ell$ forced exploration rounds. On figures 5a and 5b we plot the expected first time the optimal decision is selected $E(\tau)$ as a function of $m$ for various $\delta = \frac{\Delta}{m}$ and $\ell$, for the sets of paths $\mathcal{X}^p$. As predicted by Theorem 4, $E(\tau)$ still seems to scale exponentially in $m$ which causes exponential regret.

Theorem 4 states that if $\ell$ is chosen such that $\frac{\Delta}{m} < \frac{2}{\ell+1}$ then regret scales exponentially. On the other hand one could think that when $\ell$ is chosen large enough to violate this condition then regret does not grow as rapidly. Figure 5b shows that, at least numerically, this does not appear to be the case, indeed, we choose $\ell = m$, and there
Figure 4: Expectation of the first time the optimal decision is played $E(\tau)$ as a function of $m$ and various values of $\Delta_m = \delta$

are values of $\delta$ such that the regret still seem to scale exponentially in $m$.

Figure 5: Expectation of the first time the optimal decision is played $E(\tau)$ as a function of $m$ and various values of $\Delta_m = \delta$ and $\ell$

Comparison with optimistic algorithms  We now compare TS with the state-of-the-art frequentist algorithms ESCB and/or CUCB. These experiments are averaged over 40 sample paths due to computational limits. On figures 6a and 6b we present the regret as a function of $m$ for the set of paths $X^p$, $\Delta_m = 0.1$, $T = 4.10^4$ and the set of matchings $X^m$, $\Delta_m = 0.05$, $T = 4.10^4$ respectively. The results show that the regret of TS is larger than that of CUCB and/or ESCB by several orders of magnitude in high dimensions, as predicted by our theoretical results. In fact the regret of TS is so overwhelmingly large that, due to the scale of the figure, it looks like the regret of ESCB and/or CUCB does not increase with the dimension (this is of course not the case). On figures 7a and 7b we perform similar experiments but with smaller gaps, $\Delta_m = \frac{1}{m}$ and the same behaviour arises.
5 Conclusion

We have shown through both theoretical analysis as well as numerical experiments that TS can perform very poorly in high dimensions, both for both linear and non-linear problems, and for various combinatorial structures such as sets of paths and matchings (one could consider more complex combinatorial set including multiple non-disjoint paths). Introducing forced exploration does not alleviate the problem either. Therefore, this is not an artifact, but rather a general problem. In essence, Thompson performs poorly because it has a tendency to play much too greedily, and in high dimensions this sometimes leads to a complete lack of exploration and missing the optimal arm. Our work points towards a new challenging open problem which is to design better TS-like algorithms for regret minimization that can deal with high-dimensional problems, while retaining the computational efficiency of TS. Two reasonable ideas to explore would be (i) carefully designing the prior distribution (ii) enforcing forced explorations at regular intervals, possibly in an adaptive manner. Also, our work concerns the Bernoulli setting but we believe that our results can be generalized to bounded distributions. It is not obvious whether or not our results still hold for Gaussian distributions, and seems like an interesting open problem.
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A  Linear Bandits: Regret Upper Bound for ESCB

We first recall a regret upper bound for ESCB found in [8], based on the more general analysis of [9].

**Theorem 10.** Consider a linear combinatorial bandit problem. Then the regret of ESCB is upper bounded by:

\[
R(T, \theta) \leq C(m) + \frac{2dm^3}{\Delta_{\min}^2} + \frac{24d(\ln T + 4m \ln \ln T)}{\Delta_{\min}} \left[ \ln m \right]^2 + \frac{24d}{1.61}.
\]

with \(C(m)\) a positive number that depends solely on \(m\).

B  Proofs

B.1 Technical Results

We state a technical result about the product of i.i.d. random variables with Beta distribution.

**Lemma 11.** Let \(V_1, ..., V_m\) i.i.d. with distribution \(V_i \sim \text{Beta}(\alpha, 1)\). Then for all \(\Delta \in [0, 1]\):

\[
P\left( \prod_{i=1}^{m} V_i \geq 1 - \Delta \right) \leq \frac{\alpha^m}{m(m!)} \left[ \ln \left( \frac{1}{1 - \Delta} \right) \right]^m.
\]

**Proof.** Taking logarithms:

\[
P\left( \prod_{i=1}^{m} V_i \geq 1 - \Delta \right) = P\left( \sum_{i=1}^{m} \ln \frac{1}{V_i} \leq \ln \left( \frac{1}{1 - \Delta} \right) \right)
\]

Now if \(V_i \sim \text{Beta}(\alpha, 1)\) then \(\ln \frac{1}{V_i} \sim \text{Exp}(\alpha)\) and since \(V_i\) are i.i.d. we have

\[
\sum_{i=1}^{m} \ln \frac{1}{V_i} \sim \text{Erlang} (m, \alpha).
\]

Therefore:

\[
P\left( \sum_{i=1}^{m} \ln \frac{1}{V_i} \leq \ln \left( \frac{1}{1 - \Delta} \right) \right) = \frac{\alpha^m}{m!} \int_{0}^{\ln \left( \frac{1}{1 - \Delta} \right)} x^{m-1} e^{-\alpha x} dx
\]

\[
\leq \frac{\alpha^m}{m!} \int_{0}^{\ln \left( \frac{1}{1 - \Delta} \right)} x^{m-1} dx
\]

\[
= \frac{\alpha^m}{(m)!m} \left[ \ln \left( \frac{1}{1 - \Delta} \right) \right]^m.
\]

which concludes the proof.
We state another technical result about the Beta distribution near 1.

**Lemma 12.** Consider \( V \sim \text{Beta}(\alpha + 1, \beta + 1) \) with \( \alpha, \beta > 0 \). Define \( T = \alpha + \beta \) and \( M = \frac{\alpha}{\alpha + \beta} \).

For all \( c \in [0, 1[ \)
\[
P(V \leq c) \leq \frac{e^{\frac{T}{2}}}{\sqrt{2\pi TM(1-M)}} \int_0^c e^{-TD(M|x)} dx,
\]
with \( D \) the Kullback-Leibler divergence between Bernoulli distributions:
\[
D(M \mid x) = M \ln \frac{M}{x} + (1-M) \ln \frac{1-M}{1-x}.
\]

We also have the simpler bound for \( c \leq M \):
\[
P(V \leq c) \leq \frac{e^{\frac{T}{2}}}{\sqrt{2\pi}} e^{-T(M-c)^2}.
\]

**Remark 1.** This proves that for any \( \nu \in (0, 1] \)
\[
P \left( V \leq M - \sqrt{\frac{1}{T} \ln \left( \frac{e^{1/12 \sqrt{T}}}{\nu \sqrt{2\pi}} \right)} \right) \leq \nu.
\]

**Remark 2.** Consider \( V_i \sim \text{Beta}(\alpha_i + 1, \beta_i + 1) \) with \( V_1, \ldots, V_m \) independent, by negation and union bound we have for all \( c \in [0, 1[ \)
\[
P \left( \sum_{i=1}^m V_i \leq \sum_{i=1}^m c_i \right) \leq \sum_{i=1}^m P(V_i \leq c_i).
\]

so that the above bound easily extends to the multidimensional case:
\[
P \left( \sum_{i=1}^m V_i \leq \sum_{i=1}^m M_i - \sqrt{\frac{m^2}{T} \ln \left( \frac{e^{1/12 \sqrt{T}}}{\nu \sqrt{2\pi}} \right)} \right) \leq \nu.
\]

**Proof.** The density of \( V \) is given by:
\[
f(x) = \frac{(\alpha + \beta)!}{\alpha!\beta!} x^\alpha (1-x)^\beta.
\]
The Stirling approximation yields for all \( n \) (see [19])
\[
\sqrt{2\pi} n^{n+1/2} \leq \sqrt{2\pi} n^{n+1/2} e^{-\frac{1}{2} \frac{1}{n}} \leq n! \leq \sqrt{2\pi} n^{n+1/2} e^{\frac{1}{2}} \leq \sqrt{2\pi} n^{n+1/2} e^{\frac{1}{2}}.
\]
Therefore:

\[
(\alpha + \beta)! \leq \frac{\sqrt{2\pi (\alpha + \beta)^{\alpha + \beta + 1/2} e^{\frac{\alpha + \beta}{2}}}}{(2\pi)(\alpha)^{\alpha + 1/2}(\beta)^{\beta + 1/2}}
\]

\[
= \frac{T^{T + 1/2} e^{\frac{\alpha + \beta}{2}}}{\sqrt{2\pi (TM)^{TM + 1/2} (T(1 - M))^{T(1 - M) + 1/2}}}
\]

\[
= \frac{e^{\frac{\alpha + \beta}{2}}}{\sqrt{2\pi TM(1 - M)} [M^M (1 - M)^{1 - M}]^T}.
\]

Furthermore:

\[
x^{\alpha}(1 - x)^{\beta} = e^{\alpha \ln(x) + \beta \ln(1 - x)}
\]

\[
= e^{T[M \ln(x) + (1 - M) \ln(1 - x)]}
\]

\[
= M^{TM} (1 - M)^{(1 - M)T} e^{-TD(M|x)}.
\]

Replacing:

\[
f(x) \leq \frac{e^{\frac{\alpha + \beta}{2}} e^{-TD(M|x)}}{\sqrt{2\pi TM(1 - M)}}.
\]

Therefore:

\[
P(V \leq c) = \int_0^c f(x) dx \leq \frac{e^{\frac{\alpha + \beta}{2}}}{\sqrt{2\pi TM(1 - M)}} \int_0^c e^{-TD(M|x)} dx.
\]

The simpler bound comes from \( TM(1 - M) = \frac{\alpha \beta}{T} \geq \frac{1}{T} \) and using Pinsker’s inequality \( D(x \mid M) \geq 2(x - M)^2 \), for \( c \leq M \)

\[
P(V \leq c) \leq \frac{e^{\frac{\alpha + \beta}{2}} \sqrt{T}}{\sqrt{2\pi}} e^{-2T(M - c)^2}.
\]

We recall a result on the Irwin-Hall distribution.

**Remark 3.** Consider \( U_1, ..., U_m \) i.i.d. uniformly distributed in \([0, 1]\). Then their sum follows the Irwin-Hall distribution and for any \( \Delta \leq 1 \) we have that:

\[
P\left( \sum_{i=1}^m U_i \geq m - \Delta \right) = P\left( \sum_{i=1}^m U_i \leq \Delta \right) = \frac{\Delta^m}{m!}.
\]

We present a technical result on the tail behaviour of the sum of beta random variables.
Lemma 13. Consider $V_1, \ldots, V_m$ independent random variables following beta laws of parameters $(\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m)$. For $\epsilon < 1$ we have that:

$$
P\left(\sum_{i=1}^{m} V_i \geq m - \epsilon\right) \leq \frac{e^{\sum_{i=1}^{m} \beta_i}}{m! \prod_{i=1}^{m} B(\alpha_i, \beta_i)},$$

where $B(\alpha, \beta) = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)}$ is the beta function.

Proof. We define $A_\epsilon \triangleq \{(u_1, \ldots, u_m) \in [0,1]^m, m - \epsilon \leq \sum_{i=1}^{m} u_i \leq m\}$. It is noted that if $(u_1, \ldots, u_m) \in A_\epsilon$ we have that $u_i \geq 1 - \epsilon$ for all $i$. We recall that the probability density of a Beta$(\alpha_i, \beta_i)$ law is $p_i(u) = u^{\alpha_i-1}(1-u)^{\beta_i-1}/B(\alpha_i, \beta_i)$.

We have

$$
P\left(\sum_{i=1}^{m} V_i \geq m - \epsilon\right) = \int_{A_\epsilon} \prod_{i=1}^{m} p_i(u_i) du_1 \ldots du_m,$$

$$
= \int_{A_\epsilon} \prod_{i=1}^{m} u_i^{\alpha_i-1}(1-u_i)^{\beta_i-1}/B(\alpha_i, \beta_i) du_1 \ldots du_m,$$

$$
\leq \int_{A_\epsilon} \prod_{i=1}^{m} e^{\beta_i-1} B(\alpha_i, \beta_i) du_1 \ldots du_m,$$

$$
= \frac{\prod_{i=1}^{m} (\beta_i-1)}{\prod_{i=1}^{m} B(\alpha_i, \beta_i)} \int_{A_\epsilon} 1 du_1 \ldots u_m.$$

But we know that the integral $\int_{A_\epsilon} 1 du_1 \ldots u_m$ corresponds to the cumulative distribution function of the sum of $m$ uniform random variables in $[0, 1]$. This is known as the Irving Hall distribution. So we have that $\int_{A_\epsilon} 1 du_1 \ldots u_m = \frac{e^m}{m!}$.

Which proves the announced result

$$
P\left(\sum_{i=1}^{m} V_i \geq m - \epsilon\right) \leq \frac{e^{\sum_{i=1}^{m} \beta_i}}{m! \prod_{i=1}^{m} B(\alpha_i, \beta_i)}.$$

\hfill \Box

Finally we make an important remark about the link between regret and the first time the optimal decision is selected.

Remark 4. Define $\tau$ the first time the optimal decision is selected. Then we have that:

$$R(T, \theta) = \mathbb{E}\left(\sum_{t=1}^{T} \Delta_x(t)\right) \geq \Delta_{\min} \mathbb{E}\left(\sum_{t=1}^{T} \mathbf{1}\{\Delta_x(t) \neq 0\}\right) \geq \Delta_{\min} \sum_{t=1}^{T} \mathbb{P}(\tau \geq t).$$
B.2 Proof of Theorem[1]

Define \( b = 1 - \frac{\Delta}{m} \). Consider \( \epsilon > 0 \) such that \( b - \epsilon \geq \frac{1}{2} \) and denote the two decisions as \( x^1 = (1, ..., 1, 0, ..., 0) \) and \( x^2 = (0, ..., 0, 1, ..., 1) \). Consider the event where the empirical mean of decision \( x^2 \) does not deviate too much from its expectation when it is selected:

\[
A = \left\{ \exists t \geq 0 : x(t) = x^2, \sum_{i=m+1}^{d} \frac{A_i(t)}{N_i(t)} \leq (b - \epsilon)m \right\}.
\]

We decompose \( A \) as \( \bigcup_{n \geq 1} A_n \) where

\[
A_n = \left\{ \exists t \geq 0 : x(t) = x^2, N_i(t) = n, i = m + 1, ..., d, \frac{1}{n} \sum_{i=m+1}^{d} A_i(t) \leq (b - \epsilon)m \right\}.
\]

Using Hoeffding’s inequality we have that:

\[
\mathbb{P}(A) \leq \sum_{n \geq 1} \mathbb{P}(A_n) \leq \sum_{n \geq 1} \exp(-2mn\epsilon^2) = \frac{\exp(-2n\epsilon^2)}{1 - \exp(-2n\epsilon^2)}.
\]

where we have used the fact that if \( N_i(t) = n \) for \( i = m + 1, ..., d \) then \( \sum_{i=m+1}^{d} A_i(t) \) is a sum of \( mn \) i.i.d. Bernoulli variables with parameter \( b \). Let us control the probability that decision \( x^1 \) is never selected between time 0 and time \( t \), which is the probability of event:

\[
B_t = \{ x(s) = x^2 : s = 1, ..., t \}.
\]

Let us assume that \( B_t \) occurs and \( A \) does not occur. Since decisions \( x^1 \) and \( x^2 \) have been selected 0 and \( t \) times respectively, the probability of selecting \( x^2 \) is lower bounded by:

\[
\mathbb{P}(B_{t+1}|B_t, \bar{A}) \geq \mathbb{P}(\sum_{i=1}^{m} V_i(t) \leq \sum_{i=m+1}^{d} V_i(t)|B_t, \bar{A}).
\]

where \( V_1(t), ..., V_d(t) \) are independent, distributed in \([0, 1]\). For \( i = 1, ..., m \), \( V_i(t) \) is uniformly distributed in \([0, 1]\) and has mean 1/2. For \( i = m + 1, ..., d \), \( V_i(t) \) has Beta\((A_i(t) + 1, t - A_i(t) + 1)\) distribution with mean \( \frac{A_i(t)+1}{t+2} \) so that expectations verify:

\[
\sum_{i=m+1}^{d} \mathbb{E}(V_i(t)|B_t, \bar{A}) - \sum_{i=1}^{m} \mathbb{E}(V_i(t)|B_t, \bar{A}) = \sum_{i=m+1}^{d} \frac{A_i(t)+1}{t+2} - \sum_{i=1}^{m} \frac{1}{2} \geq \frac{tm(b-\epsilon) + m}{t+2} - \frac{m}{2} = \frac{mt(b-\epsilon - 1/2)}{t+2} \geq \frac{m(b-\epsilon - 1/2)}{3},
\]

since \( \sum_{i=m+1}^{d} A_i(t) \geq tm(b-\epsilon) \).
Using Hoeffding’s inequality once again we have:

\[
P\left(\sum_{i=1}^{m} V_i(t) \geq \sum_{i=m+1}^{d} V_i(t) \mid B_t, \bar{A}\right) = P\left(\sum_{i=1}^{m} V_i(t) - \sum_{i=m+1}^{d} V_i(t) \geq 0 \mid B_t, \bar{A}\right) \\
\leq \exp\left\{ -2m(b - \epsilon - 1/2)^2 / 9 \right\} \\
\equiv p_{\Delta}.
\]

We have proven that for all \( t > 1 \):

\[
P(B_{t+1} \mid B_t, \bar{A}) \geq 1 - p_{\Delta},
\]

and since \( P(B_1 \mid \bar{A}) = 1/2 \):

\[
P(B_t) \geq P(B_t, \bar{A}) = P(\bar{A}) P(B_t \mid \bar{A}) \geq P(\bar{A}) P(B_t \mid \bar{A})(1 - p_{\Delta})^{t-1} = \frac{P(\bar{A})}{2}(1 - p_{\Delta})^{t-1}.
\]

Denote by \( \tau \) the first time that \( x^1 \) is selected. If \( B_t \) occurs then \( \tau \geq t \) and using Remark 4 yields the lower bound:

\[
R(T, \theta) \geq \Delta \sum_{t=1}^{T} P(\tau \geq t) \geq \frac{\Delta P(\bar{A})}{2} \sum_{t=1}^{T} (1 - p_{\Delta})^{t-1}.
\]

Setting \( \epsilon = \frac{1}{\sqrt{m}} \) we get that

\[
P(\bar{A}) \leq \frac{e^{-2}}{1 - e^{-2}} \leq \frac{1}{2},
\]

and we get the announced result:

\[
R(T, \theta) \geq \frac{\Delta}{4} \sum_{t=1}^{T} (1 - p_{\Delta})^{t}.
\]

**B.3 Proof of Theorem 2**

Consider \( m \geq 5 \). We denote by \( N^1(t) \) and \( N^2(t) \) the number of times that decisions \( x^1 \) and \( x^2 \) have been respectively selected, and it is noted that \( N_i(t) = N^1(t) \) for \( i = 1, \ldots, m \) and \( N_i(t) = N^2(t) \) for \( i = m+1, \ldots, d \). Consider the event where the empirical mean of decision \( x^2 \) deviates significantly from its expectation when it is selected:

\[
\mathcal{A} = \left\{ \exists t \geq 0 : x(t) = x^2, \frac{1}{N^2(t)} \sum_{i=m+1}^{d} A_i(t) \leq m - \Delta - \sqrt{\frac{m \ln(2N^2(t))}{N^2(t)}} \right\}.
\]

We decompose \( \mathcal{A} \) as \( \bigcup_{n \geq 1} \mathcal{A}_n \), where

\[
\mathcal{A}_n = \left\{ \exists t \geq 0 : x(t) = x^2, N^2(t) = n, \frac{1}{n} \sum_{i=m+1}^{d} A_i(t) \leq m - \Delta - \sqrt{\frac{m \ln(2n)}{n}} \right\}.
\]
Using Hoeffding’s inequality we have that:

\[ P(A) \leq \sum_{n \geq 1} P(A_n) \leq \sum_{n \geq 1} \frac{1}{(2n)^2} = \frac{\pi^2}{24} \leq \frac{1}{2}, \]

where we have used the fact that if \( N^2(t) = n \) then \( \sum_{i=m+1}^{d} A_i(t) \) is a sum of \( mn \) i.i.d. Bernoulli random variables with parameter \( 1 - \frac{\Delta}{m} \). Let us control the probability that decision \( x^1 \) is never selected between time 0 and time \( t \), which is the probability of event:

\[ B_t = \{ x(s) = x^2 : s = 1, ..., t \}. \]

We have that:

\[ P(B_{t+1} | B_t, \bar{A}) \geq P\left( \sum_{i=1}^{m} V_i(t) \leq \sum_{i=m+1}^{d} V_i(t) | B_t, \bar{A} \right) \geq (1 - p_{t,1})(1 - p_{t,2}), \]

with

\[ p_{t,1} = P\left( \sum_{i=1}^{m} V_i(t) \geq m - \Delta - h(m,t) | B_t, \bar{A} \right), \]

\[ p_{t,2} = P\left( \sum_{i=m+1}^{d} V_i(t) \leq m - \Delta - h(m,t) | B_t, \bar{A} \right), \]

\[ h(m,t) = \sqrt{\frac{m \ln(2t)}{t}} + \frac{\sqrt{2}}{t} \ln \left( \frac{e^{1/12} m^{1/2}}{\sqrt{2\pi}} \right). \]

It is noted that there exists a universal constant \( C_1 > 0 \) such that

\[ h(m,t) \leq \sqrt{\frac{C_1 m^2 (\ln t \ln m)}{t}}. \]

Let us define \( T_0 = C_0 m^2 \ln m \) with \( C_0 > 0 \) a universal constant such that the five following inequalities are true:

- \( T_0 \geq m, \)
- \( h(m,t) \leq \frac{1}{3} \) for all \( t \geq T_0, \)
- \( \frac{C_1 s e^2 \ln t}{t} \leq 1 \) for all \( t \geq T_0, \)
- \( \sum_{i=T_0}^{+\infty} \left( \frac{C_1 s e^2 \ln t}{t} \right)^{\frac{2}{3}} \leq \frac{1}{3}. \)
- \( \sum_{i=T_0}^{+\infty} \frac{1}{t^2} \leq \frac{1}{2}. \)
Consider $p_{t, 2}$, and recall that for $i = 1, \ldots, d$

$$M_i(t) \triangleq \frac{A_i(t)}{A_i(t) + B_i(t)} = \frac{A_i(t)}{N_i(t)}.$$ 

is the mode of $V_i(t)$. If event $\bar{A}$ occurs then

$$\sum_{i=m+1}^{d} M_i(t) > m - \Delta - \sqrt{\frac{m \ln(2N^2(t))}{N^2(t)}}.$$ 

So using lemma[12] and remark[2] we have that:

$$p_{t, 2} \leq \frac{1}{t^2}.$$ 

Consider $p_{t, 1}$. Since $\Delta \leq \frac{1}{6}$ and $h(m, t) \leq \frac{1}{3}$ we have

$$m - \Delta - h(m, t) \geq m - \frac{1}{2} \geq m - 1.$$ 

If event $B_t$ occurs, then $A_i(t) = B_i(t) = 0$ for all $i = 1, \ldots, m$ therefore $\sum_{i=1}^{m} V_i(t)$ follows the Irwin-Hall distribution of size $m$, so from remark[3] for $t \geq T_0$ we have:

$$p_{t, 1} = \frac{1}{m!} (\Delta + h(m, t))^m \leq \frac{1}{m!} ((2\Delta)^m + (2h(m, t))^m),$$

where we used the convexity inequality $(\frac{x+y}{2})^{m} \leq \frac{x^{m}+y^{m}}{2}$. 

We have, for $T > T_0$:

$$\frac{\mathbb{P}(B_T|\bar{A})}{\mathbb{P}(B_T|\bar{A})} = \prod_{t=T_0}^{T-1} \mathbb{P}(B_{t+1}|B_t, \bar{A}) \geq \prod_{t=T_0}^{T-1} (1 - p_{t, 1})(1 - p_{t, 2}).$$

Using the union bound and the definition of $T_0$: 

$$\prod_{t=T_0}^{T-1} (1 - p_{t, 1}) \geq 1 - \sum_{t=T_0}^{T-1} p_{t, 1} \geq 1 - \sum_{t=T_0}^{T-1} \frac{1}{t^2} \geq 1 - \sum_{t=T_0}^{\infty} \frac{1}{t^2} \geq \frac{1}{2}.$$

Now:

$$1 - p_{t, 2} = 1 - \frac{(2\Delta)^m}{m!} - \frac{(2h(m, t))^m}{m!} = (1 - \frac{(2\Delta)^m}{m!}) \frac{1 - (2h(m, t))^m}{m!} \geq (1 - \frac{(2\Delta)^m}{m!})(1 - \frac{3 (2h(m, t))^m}{m!}),$$

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where we used the fact that $\Delta \leq \frac{1}{6}$ so that $\frac{(2\Delta)^m}{m!} \leq \frac{1}{3}$.

Using the union bound once more:

$$T^{-1} \prod_{t=T_0}^{t=1} (1 - p_{t,2}) \geq T^{-1} \prod_{t=T_0}^{t=1} (1 - \frac{(2\Delta)^m}{m!})(1 - \frac{3}{2} (2h(m, t))^m)
\geq (1 - \frac{(2\Delta)^m}{m!}) T^{-T_0} (1 - \frac{3}{2} \sum_{t=T_0}^{t=1} (2h(m, t))^m)
\geq (1 - \frac{(2\Delta)^m}{m!}) T^{-T_0} (1 - \frac{3}{2} \sum_{t=T_0}^{t=1} (2h(m, t))^m).$$

We turn to the last sum in the right hand side of the equation above. Since $t \geq T_0 \geq m$ we have

$$h(m, t) \leq \sqrt{C_1 m^2 (\ln t + \ln m)} \leq \sqrt{C_1 2m^2 \ln t}.$$

Using Stirling’s approximation we have $m! \geq (m/e)^m$ so that

$$\sum_{t=T_0}^{t=1} (2h(m, t))^m \leq \sum_{t=T_0}^{t=1} \left( \frac{C_1 2e^2 \ln t}{t} \right)^{\frac{m}{2}} \leq \sum_{t=T_0}^{t=1} \left( \frac{C_1 2e^2 \ln t}{t} \right)^{\frac{m}{2}} \leq \frac{1}{3},$$

where we used twice the definition of $T_0$ and $m \geq 5 \geq 3$.

Putting things together we have proven that :

$$\frac{\mathbb{P}(B_{T_0} | \bar{A})}{\mathbb{P}(B_{T_0} | \bar{A})} \geq \frac{1}{4} \left( 1 - \frac{(2\Delta)^m}{m!} \right)^{-T_0}.$$
We have proven that:

\[ \Pr(B_{T_0}|\hat{A}) \geq \frac{1}{2}(1-p_\Delta)T_0 - 1 \geq C_2. \]

which gives

\[ \Pr(B_T) \geq C_2 \left(1 - \frac{(2\Delta)^m}{m!}\right)^{T-T_0}, \]

and applying Remark \textsuperscript{4} concludes the proof.

\section*{B.4 Proof of Corollary 3}

Using the same notation as above, we recall that

\[ \Pr(B_T) \geq C_3 \left(1 - \frac{(2\Delta)^m}{m!}\right)^T. \]

If \( B_T \) occurs, decision \( x^1 \) is never played, resulting in a regret of \( \Delta T \), therefore:

\[ R(T, \theta) \geq \Delta T \Pr(B_T) \geq C_3 \Delta T \left(1 - \frac{(2\Delta)^m}{m!}\right)^T. \]

(i) If \( T \geq 3^m m! \) let us set

\[ \Delta = \frac{1}{2} \left(\frac{m!}{T}\right)^{\frac{1}{m}}, \]

so that we have \( \Delta \leq \frac{1}{6} \) and, using Stirling’s approximation \( m! \geq (m/e)^m \) we get

\[
\max_{\theta \in [0,1]^d} R(T, \theta) \geq \frac{C_3}{3} \left(\frac{m!}{T}\right)^{\frac{1}{m}} T^{1-\frac{1}{m}} \left(1 - \frac{1}{T}\right)^T \\
\geq \frac{C_3}{3} \frac{m}{e} (1-e^{-1})T^{1-\frac{1}{m}}.
\]

this yields

\[ \max_{\theta \in [0,1]^d} R(T, \theta) \geq \mathcal{O}(mT^{1-\frac{1}{m}}). \]

(ii) If \( T \leq 3^m m! \) let us set \( \Delta = 1/6 \), which yields

\[ \max_{\theta \in [0,1]^d} R(T, \theta) \geq \mathcal{O}(T). \]

and completes the proof.

\section*{B.5 Proof of Theorem 4}

To simplify notation, we assume that the \( \ell \) rounds of exploration are done before the algorithm starts, so that at time \( t = 0 \) each decision has been explored \( \ell/2 \) times and the TS algorithm starts.
We consider the following event:

\[ C = \left\{ \forall i \in [d], A_i(0) = \ell \right\}. \]

We know that \( A_i(0), i = 1, ..., d \) are independent with a Binomial(\( \ell/2, \theta_i \)) distribution so that

\[ \mathbb{P}(C) = \left( 1 - \frac{\Delta}{m} \right)^{\ell m}. \]

Define \( \epsilon = \frac{1}{\sqrt{m}} \). We consider again the event where the empirical mean of decision \( x_2 \) deviates significantly from its expectation when it is selected, accounting for the rounds of forced exploration:

\[ A = \left\{ \exists t \geq 0 : x(t) = x_2, \sum_{i=m+1}^{d} A_i(t) \leq (1 - \frac{\Delta}{m} - \epsilon)(N^2(t) - \ell/2)m + \ell/2 m \right\}. \]

We decompose \( A \) as \( \bigcup_{n \geq 1} A_n \) where

\[ A_n = \left\{ \exists t \geq 0 : x(t) = x_2, N^2(t) = n + \ell/2, \sum_{i=m+1}^{d} A_i(t) \leq (1 - \frac{\Delta}{m} - \epsilon)nm + \ell/2 m \right\}. \]

Since \( \epsilon = \frac{1}{\sqrt{m}} \), using Hoeffding’s inequality we have that:

\[ \mathbb{P}(A|C) \leq \sum_{n \geq 1} \mathbb{P}(A_n|C) \leq \sum_{n \geq 1} \exp(-2mn\epsilon^2) = \frac{\exp(-2mn\epsilon^2)}{1 - \exp(-2mn\epsilon^2)} \leq \frac{1}{2}. \]

where we have used the fact that if \( N^2(t) = n + \ell/2 \), then \( \sum_{i=m+1}^{d} A_i(t) \) equals \( \frac{\ell}{2} m \) plus the sum of \( mn \) i.i.d Bernoulli random variables with parameter \( 1 - \frac{\Delta}{m} \). Let us control the probability that decision \( x_1 \) is never selected between time 0 and time \( t \), which is the probability of event:

\[ B_t = \{ x(s) = x_2 : s = 1, ..., t \}. \]

Let us assume that \( B_t \) and \( C \) occurs but \( A \) does not occur. Since decisions \( x_1 \) and \( x_2 \) have been selected \( \ell/2 \) and \( \ell/2 + t \) times respectively, the probability of selecting \( x_2 \) is lower bounded by:

\[ \mathbb{P}(B_{t+1}|B_t, A, C) \geq \mathbb{P}\left( \sum_{i=1}^{m} V_i(t) \leq \sum_{i=m+1}^{d} V_i(t)|B_t, A, C \right). \]

where \( V_1(t), ..., V_d(t) \) are independent, distributed in \([0, 1]\). For \( i = 1, ..., m \), \( V_i(t) \) follows a Beta(\( \ell/2 \), 1) law and has mean \( \frac{\ell}{2} + 1 \). For \( i = m + 1, ..., d \), \( V_i(t) \) follows a Beta(\( A_i(t) + 1, t + \ell/2 - A_i(t) + 1 \)) distribution with mean \( \frac{A_i(t)+1}{t+\ell/2} \) so that the expectations
verify:

\[
\sum_{i=m+1}^{d} E(V_i(t) | \mathcal{B}_t, \bar{A}, \bar{C}) - \sum_{i=1}^{m} E(V_i(t) | \mathcal{B}_t, \bar{A}, \bar{C}) \geq m \left( \frac{1 - \frac{\Delta}{m} - \epsilon}{t + \ell} + 1 \right) - m \frac{\ell + 1}{t + 2}
\]

\[
\geq m \left( \frac{1 - \frac{\Delta}{m} - \epsilon}{\frac{\ell}{2}} + 1 \right) - m \frac{\ell + 1}{t + 2}
\]

\[
= m \left( \frac{\frac{1}{2} + \epsilon - \frac{\Delta}{m}}{(\frac{\ell}{2}) + 3} \right)
\]

since \( \sum_{i=m+1}^{d} A_i(t) \geq tm(1 - \frac{\Delta}{m} - \epsilon) + m \ell \). Recall that \( \epsilon = \frac{1}{\sqrt{m}} \) so that \( \frac{1}{\frac{\ell}{2} + 2} - (\frac{\Delta}{m} + \epsilon) \geq 0 \). Using Hoeffding’s inequality:

\[
\mathbb{P}\left( \sum_{i=1}^{m} V_i(t) \geq \sum_{i=m+1}^{d} V_i(t) | \mathcal{B}_t, \bar{A}, \bar{C} \right) = \mathbb{P}\left( \sum_{i=1}^{m} V_i(t) - \sum_{i=m+1}^{d} V_i(t) \geq 0 \right)
\]

\[
\leq \exp \left\{ -2m \left( \frac{\frac{1}{2} + \epsilon - \frac{\Delta}{m}}{(\frac{\ell}{2}) + 3} \right)^2 \right\} = p_\Delta^\ell.
\]

We have proven that for all \( t > 1 \):

\[
\mathbb{P}(B_{t+1} | \mathcal{B}_t, \bar{A}) \geq 1 - p_\Delta^\ell.
\]

so that:

\[
\mathbb{P}(B_t) \geq \mathbb{P}(B_t, \bar{A}, \bar{C})
\]

\[
= \mathbb{P}(\bar{A}|C)\mathbb{P}(C)\mathbb{P}(B_t | \bar{A}, \bar{C})
\]

\[
\geq \mathbb{P}(\bar{A}|C)\mathbb{P}(B_t | \bar{A}, \bar{C})(1 - p_\Delta^\ell)(1 - \frac{\Delta}{m})^\ell m
\]

\[
= \frac{\mathbb{P}(\bar{A}|C)}{2}(1 - p_\Delta^\ell)^t(1 - \frac{\Delta}{m})^\ell m.
\]

Denote by \( \tau \) the first time that \( x^1 \) is selected. If \( B_t \) occurs then \( \tau \geq t \) and using Remark \( 4 \) yields the lower bound:

\[
R(T, \theta) \geq \Delta \sum_{t=1}^{T} \mathbb{P}(\tau \geq t) \geq \Delta \frac{\mathbb{P}(\bar{A}|C)}{2}(1 - \frac{\Delta}{m})^\ell m \sum_{t=1}^{T} (1 - p_\Delta^\ell)^{t-1}.
\]

From above, \( \mathbb{P}(A|C) \leq \frac{1}{4} \), and we get the announced result:

\[
R(T, \theta) \geq \frac{\Delta}{4}(1 - \frac{\Delta}{m})^\ell m \sum_{t=1}^{T} (1 - p_\Delta^\ell)^t.
\]

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B.6 Proof of Theorem 5

To simplify notation, we assume that the $\ell$ rounds of exploration are done before the algorithm starts, so that at time $t = 0$ each decision has been explored $\ell/2$ times and the TS algorithm starts.

We consider the following event:

$$C = \left\{ \forall i \in [d], A_i(0) = \frac{\ell}{2} \right\}.$$ 

We know that $A_i(0), i = 1, \ldots, d$ are independent with a Binomial$(\ell/2, \theta_i)$ distribution so that

$$P(C) = \left(1 - \frac{\Delta}{m}\right)^{\ell/2}.$$ 

We consider the event where the empirical mean of decision $x^2$ deviates significantly from its expectation when it is selected, accounting for the rounds of forced exploration:

$$A = \left\{ \exists t \geq 0 : x(t) = x^2, \sum_{i=m+1}^{d} A_i(t) \leq (m - \Delta) \left( N_2^2(t) - \frac{\ell}{2} \right) + \frac{\ell m}{2} - \sqrt{m \ln(2(N_2^2(t) - \frac{\ell}{2}))} \right\}.$$ 

We decompose $A$ as $\bigcup_{n \geq 1} A_n$ where

$$A_n = \left\{ \exists t \geq 0 : x(t) = x^2, N_2^2(t) = n + \frac{\ell}{2}, \sum_{i=m+1}^{d} A_i(t) \leq (m - \Delta)n + \frac{\ell m}{2} - \sqrt{m \ln(2n)} \right\}.$$ 

Using Hoeffding’s inequality we have that:

$$P(A|C) \leq \sum_{n \geq 1} P(A_n|C) \leq \sum_{n \geq 1} \frac{1}{(2n)^2} = \frac{\pi^2}{24} \leq \frac{1}{2},$$

where we have used the fact that if $N_2^2(t) = n + \ell/2$, then $\sum_{i=m+1}^{d} A_i(t)$ equals $\frac{\ell m}{2} + \sum_{i=m+1}^{d} V_i(t)$ plus the sum of $mn$ i.i.d Bernoulli random variables with parameter $1 - \frac{\Delta}{m}$. Let us control the probability that decision $x^1$ is never selected between time 0 and time $t$, which is the probability of event:

$$B_t = \{ x(s) = x^2 : s = 1, \ldots, t \},$$

We have that:

$$P(B_{t+1}|B_t, \bar{A}) \geq P\left( \sum_{i=1}^{m} V_i(t) \leq \sum_{i=m+1}^{d} V_i(t)|B_t, \bar{A} \right) \geq (1 - p_{t,1})(1 - p_{t,2}),$$
with
\[ p_{t,1} = \mathbb{P} \left( \sum_{i=1}^{m} V_i(t) \geq m - \Delta - h(m, \ell, t) \mid B_t, \bar{A}, C_t \right), \]
\[ p_{t,2} = \mathbb{P} \left( \sum_{i=m+1}^{d} V_i(t) \leq m - \Delta - h(m, \ell, t) \mid B_t, \bar{A}, C_t \right), \]
\[ h(m, \ell, t) = -\frac{m\ell}{2t} + \sqrt{\frac{m \ln(2t)}{t}} + \sqrt{\frac{m^2}{t + \ell} \ln \left( \frac{e^{1/12} \sqrt{t + \ell}}{2\pi} \right)}. \]

It is noted that there exists a constant \( C_1 \geq 0 \) such that
\[ h(m, \ell, t) \leq \sqrt{C_1 m^2 (\ln m + \ln(t + \ell))}. \]

Let us define \( T_0(m, \ell) = C_0 m^2 (\ln m) \ell^{\frac{1}{4}} \) with \( C_0 \) a universal constant such that the following inequalities are true:

- \( T_0 \geq \max(m, \ell, 7) \),
- \( h(m, \ell, t) \leq \frac{1}{6\ell} \), for all \( t \geq T_0 \),
- \( \sum_{i=T_0}^{+\infty} \frac{1}{i^{3/2}} \leq \frac{1}{2} \),
- \( 4 \left( \frac{e^{\frac{\sqrt{2\pi}}{2}}}{(T_0 - 1)^{\frac{1}{4}} \sqrt{\pi}} \right) \leq \frac{1}{3} \).

First consider \( p_{t,2} \), and recall that \( \forall i \in [d], \forall t, M_i(t) \triangleq \frac{A_i(t)}{A_i(t) + B_i(t)} = \frac{A_i(t)}{N_i(t)} \) is the mode of \( V_i(t) \). If event \( \bar{A} \) occurs then
\[ \sum_{i=m+1}^{d} M_i(t) > m - \Delta - \sqrt{\frac{m \ln(2N^2(t))}{N^2(t)}} + \frac{m\ell}{2N^2(t)}. \]

So using lemma 12 and remark 2 we have that: \( p_{t,2} \leq \frac{1}{12} \).

Consider \( p_{t,1} \). Since \( \Delta < \frac{t}{6\ell} \) and for \( t \geq T_0 \) we have \( h(m, \ell, t) \leq \frac{1}{6\ell} \), hence
\[ m - \Delta - h(m, \ell, t) \geq m - \frac{1}{3\ell}. \]

If event \( B_t \) and \( C_t \) occurs then \( A_i(t) = \frac{t}{2} \) and \( B_i(t) = 0 \) for all \( i = 1, \ldots, m \) therefore we may control the tail behaviour of \( \sum_{i=m+1}^{d} V_i(t) \) thanks to lemma 13. So we have for \( t \geq T_0 \):
\[ p_{t,1} \leq \frac{\ell m}{2^m m!} (\Delta + h(m, \ell, t))^m \leq \frac{1}{2(m!)^m} ((\ell\Delta)^m + (\ell h(m, \ell, t))^m). \]
where we used the convexity inequality \((\frac{x+y}{2})^m \leq \frac{x^m+y^m}{2}\).

We have, for \(T > T_0\):

\[
\frac{\mathbb{P}(B_T | \bar{A}, C)}{\mathbb{P}(B_{T_0} | \bar{A}, C)} = \prod_{t=T_0}^{T-1} \mathbb{P}(B_{t+1} | B_t, \bar{A}, C) \geq \prod_{t=T_0}^{T-1} (1 - p_{t,1})(1 - p_{t,2}).
\]

Using the union bound and the definition of \(T_0\):

\[
\prod_{t=T_0}^{T-1} (1 - p_{t,2}) \geq 1 - \sum_{t=T_0}^{T-1} p_{t,2} \geq 1 - \sum_{t=T_0}^{T-1} \frac{1}{T^2} \geq 1 - \sum_{t=T_0}^{+\infty} \frac{1}{T^2} \geq \frac{1}{2}.
\]

Now:

\[
1 - p_{t,1} = 1 - \frac{(\ell \Delta)^m}{m!} - \frac{(\ell h(m, \ell, t))^m}{m!} = (1 - \frac{(\ell \Delta)^m}{m!})(1 - \frac{(\ell h(m, \ell, t))^m}{m!}) \geq \frac{1}{3}.
\]

where we used the fact that \(\Delta \leq \frac{1}{6}\) so that \(\frac{(\ell \Delta)^m}{m!} \leq \frac{1}{3}\).

Using the union bound once more:

\[
\prod_{t=T_0}^{T-1} (1 - p_{t,2}) \geq \prod_{t=T_0}^{T-1} \frac{1}{3}(1 - \frac{3(\ell h(m, \ell, t))^m}{m!})
\]

\[
\geq (1 - \frac{(\ell \Delta)^m}{m!})^{T-T_0}(1 - \frac{3}{2} \sum_{t=T_0}^{T-1} \frac{(\ell h(m, \ell, t))^m}{m!})
\]

\[
\geq (1 - \frac{(\ell \Delta)^m}{m!})^{T-T_0}(1 - \frac{3}{2} \sum_{t=T_0}^{+\infty} \frac{(\ell h(m, \ell, t))^m}{m!}).
\]

We turn to the last sum in the right hand side of the equation above. It is noted that \(\log(2t) \leq \sqrt{t}\) for all \(t \geq 7\). Since \(t \geq T_0 \geq \max(m, \ell, 7)\) we have

\[
h(m, \ell, t) \leq \sqrt{\frac{C_1 m^2}{t} \ln(t+\ell) + \ln m} \leq \sqrt{\frac{C_1 2m^2 \ln(2t)}{t}} \leq m \sqrt{2C_1} t^{-\frac{1}{4}}.
\]

We now upper bound the sum as follows:

\[
\sum_{t=T_0}^{+\infty} \frac{(\ell h(m, t))^m}{m!} \leq \frac{(\ell m \sqrt{2C_1})^m}{m!} \sum_{t=T_0}^{+\infty} t^{-\frac{m}{2}} \leq \frac{(\ell e \sqrt{2C_1})^m}{m!} \sum_{t=T_0}^{+\infty} t^{-\frac{m}{2}} \leq \frac{(\ell e \sqrt{2C_1})^m}{m!} \left(\frac{T_0-1}{T_0-1} \right)^{1-\frac{m}{2}} \leq \frac{1}{3}.
\]
where we used (i) the bound above, (ii) Stirling’s approximation \(m! \geq (m/e)^m\), (iii) the following sum-integral comparison, for any \(m \geq 5\):

\[
\sum_{t=T_0}^{+\infty} t^{-\frac{m}{T}} \leq \int_{T_0}^{+\infty} t^{-\frac{m}{T}} \, dt = \frac{(T_0 - 1)^{1 - \frac{m}{T}}}{m - 1} \leq 4(T_0 - 1)^{1 - \frac{m}{T}},
\]

and (iv) the definition of \(T_0\).

Putting things together we have proven that

\[
\frac{\mathbb{P}(B_T | \tilde{A}, C)}{\mathbb{P}(B_{T_0} | \tilde{A})} \geq \frac{1}{4} \left(1 - \frac{(\ell \Delta)^m}{m!}\right)^{T - T_0}.
\]

We showed previously that

\[
\mathbb{P}(B_{T_0} | \tilde{A}) \geq \frac{1}{2} (1 - p\Delta)^{T_0 - 1} \geq C_2(\ell, m),
\]

with

\[
C_2(\ell, m) = \frac{1}{2} (1 - p\Delta)^{T_0 - 1},
\]

and it is noted that \(\lim_{m \to \infty} C_2(\ell, m) = 1\) for any fixed \(\ell \geq 0\).

Therefore

\[
\mathbb{P}(B_T | \tilde{A}, C) \geq C_2(\ell, m) \frac{1}{4} \left(1 - \frac{(\ell \Delta)^m}{m!}\right)^{T - T_0}.
\]

Since \(\mathbb{P}(C) = (1 - \Delta \frac{\ell m}{T})^{\frac{\ell m}{T}}\) we get

\[
\mathbb{P}(B_T | \tilde{A}) \geq \frac{C_2(\ell, m)}{4} \left(1 - \Delta \frac{\ell m}{T}ight)^{\frac{\ell m}{T}} \left(1 - \frac{(\ell \Delta)^m}{m!}\right)^{T - T_0},
\]

and applying Remark 4 concludes the first part of the proof.

Now choose:

\[
\Delta = \frac{1}{\ell} \min \left(\frac{m!}{T}, \frac{1}{6}\right),
\]

and lower bound the regret by

\[
\max_{\theta \in [0,1]^d} R(T, \theta) \geq \Delta T \mathbb{P}(B_T).
\]

If \(\Delta = \frac{1}{\ell} \left(\frac{m!}{T}\right)^{\frac{1}{6}}\), then replacing

\[
\max_{\theta \in [0,1]^d} R(T, \theta) \geq \Delta T \frac{C_2(\ell, m)}{4} \left(1 - \frac{\Delta}{m}\right)^{\frac{\ell m}{T}} \left(1 - \frac{(\ell \Delta)^m}{m!}\right)^{T}
= \frac{(m!)^{\frac{1}{6}} T^{1 - \frac{1}{6}}}{\ell} C_2(\ell, m) \frac{1}{4} \left(1 - \frac{\Delta}{m}\right)^{\frac{\ell m}{T}} \left(1 - \frac{1}{T}\right)^{T}.
\]

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Using the facts that (i) \((1 - \frac{1}{m})^T \geq e\), that (ii) \((m!)^{\frac{1}{m}} \geq \frac{m}{e}\), which follows from Stirling’s approximation \(m! \geq \left(\frac{m}{e}\right)^m\) and that (iii) since \(\Delta \leq \frac{1}{6\ell}\):

\[
\left(1 - \frac{\Delta}{m}\right)^\frac{e m}{\ell} \geq \left(1 - \frac{1}{6m\ell}\right)^\frac{e m}{\ell} \geq e^{-\frac{1}{12}},
\]

which yields the minimax regret bound

\[
\max_{\theta \in [0, 1]} R(T, \theta) \geq O \left( C_2(\ell, m) \frac{T}{\ell} \right).
\]

Otherwise \(\Delta = \frac{1}{6\ell}\), and we simply have

\[
\max_{\theta \in [0, 1]} R(T, \theta) \geq O \left( C_2(\ell, m) \frac{T}{\ell} \right).
\]

which completes the proof.

**B.7 Proof of Theorem 6**

At round \(t \geq 1\), if the optimal decision has never been played then \(A_i(t) = B_i(t) = 0\) for \(i = 1, ..., m\). In turn the samples \(V_i(t)\) are independent and uniformly distributed in \([0, 1]\) for \(i = 1, ..., m\).

Lemma 11 shows that at time \(t\) the optimal decision is played with probability

\[
P\left( \prod_{i=1}^{m} V_i(t) \geq 1 - \Delta \right) \leq p_{\Delta} \equiv \frac{1}{m m!} \left[ \ln \left( \frac{1}{1 - \Delta} \right) \right]^m.
\]

So the distribution of the first time the optimal decision is played

\[
\tau = \min\{t \geq 1 : x(t) = x^*\},
\]

is lower bounded by a geometric law

\[
P(\tau \geq t) \geq (1 - p_{\Delta})^{t-1}.
\]

Combining this with Remark 4 yields the announced regret bound

\[
R(T, \theta) \geq \Delta \sum_{t=1}^{T} P(\tau \geq t) \geq \sum_{t=1}^{T} (1 - p_{\Delta})^{t-1} = \frac{\Delta}{p_{\Delta}} (1 - (1 - p_{\Delta})^T).
\]

**B.8 Proof of Theorem 9**

At round \(t \geq 1\), if the optimal decision has been played \(\ell/2\) times then \(A_i(t) = B_i(t) = 1\) for \(i = 1, ..., m\). In turn the samples \(V_i(t)\) are independent with distribution \(Beta(1 + \ell/2, 1)\) for \(i = 1, ..., m\).
Lemma 11 shows that at time $t$ the optimal decision is played with probability

$$P(\prod_{i=1}^{m} V_i(t) \geq 1 - \Delta) \leq p_{\Delta}^t = \frac{1}{m^m!} \left[\left(1 + \frac{\ell}{2}\right) \ln \left(\frac{1}{1 - \Delta}\right)\right]^m.$$ 

So the distribution of the first time the optimal decision is played

$$\tau = \min\{t \geq 1 : x(t) = x^*\},$$

is lower bounded by a geometric law

$$P(\tau \geq t) \geq (1 - p_{\Delta}^t)^{t-1}.$$ 

Combining this with remark 4 yields the announced regret bound

$$R(T, \theta) \geq \Delta \sum_{t=1}^{T} P(\tau \geq t) \geq \sum_{t=1}^{T} (1 - p_{\Delta}^t)^{t-1} = \frac{\Delta}{p_{\Delta}} (1 - (1 - p_{\Delta})^T).$$