Trigonometric approximation of the Max-Cut polytope is star-like

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Abstract

The Max-Cut polytope appears in the formulation of many difficult combinatorial optimization problems. These problems can also be formulated as optimization problems over the so-called trigonometric approximation which possesses an algorithmically accessible description but is not convex. Hirschfeld conjectured that this trigonometric approximation is star-like. In this article, we provide a proof of this conjecture.

Keywords: Max-Cut polytope, Trigonometric Approximation

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1 Introduction

A common problem in combinatorial optimization is the maximization of a quadratic form over \{-1,1\}^n

\[
\max_{x \in \{-1,1\}^n} x^T Ax = \max_{x \in \{-1,1\}^n} \langle A, X \rangle
\]

where \(\langle ., . \rangle\) denotes the usual scalar product on real symmetric matrices of size \(n\).

The decision problem associated to this optimization problem is NP-complete. Indeed the Max-Cut problem, one of Karp’s 21 NP-complete problems, can be reduced in polynomial time to the maximization of a quadratic form over \{-1,1\}^n [2]. The reformulation in the form of (1) of several common hard combinatorial optimization problems such as vertex cover, knapsack, traveling salesman, etc, can be found in [3].

Consider the set

\[ SR = \{ X \succeq 0 \mid \text{diag} X = 1 \} \]

in the space of real symmetric \(n \times n\) matrices, where \(X \succeq 0\) means that \(X\) is a positive semidefinite matrix. It serves as a simple and convex outer approximation of the Max-Cut polytope

\[ MC = \text{conv}\{ X \in SR \mid \text{rk} X = 1 \}, \]

where conv denotes the convex envelope and \(\text{rk} X\) denotes the rank of \(X\).

Note that \(\{ X \in SR \mid \text{rk} X = 1 \} = \{ X \mid \exists x \in \{-1,1\}^n, X = xx^T \}\). Indeed a positive semidefinite matrix \(X\) has rank 1 if and only if there exists a nonzero vector \(x\) such that \(X = xx^T\). Then the condition \(\text{diag} X = 1\) implies that \(x_i^2 = 1\) for every \(i \in \{1, \ldots, n\}\), i.e., \(x_i = \pm 1\), and conversely.

The maximal value of a linear functional \(\langle A, . \rangle\) over a set \(E\) does not change if the set \(E\) is replaced by its convex envelope conv \(E\). Therefore

\[
\max_{X = xx^T} \langle A, X \rangle = \max_{X \in MC} \langle A, X \rangle.
\]

However, the Max-Cut polytope is a difficult polytope. Indeed, it has an exponential number of vertices and is defined by even more linear constraints. A good review of results on the Max-Cut polytope can be found in [4].

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Maximizing $\langle A, X \rangle$ over $\mathcal{SR}$ instead of $\mathcal{MC}$ for $A \succeq 0$ approximates the exact solution of the problem with relative accuracy $\mu = \frac{\pi}{2} - 1$ [4]:

$$\frac{2}{\pi} \max_{X \in \mathcal{SR}} \langle A, X \rangle \leq \max_{X \in \mathcal{MC}} \langle A, X \rangle \leq \max_{X \in \mathcal{SR}} \langle A, X \rangle.$$

Define a function $f : [-1, 1] \to [-1, 1]$ by $f(x) = \frac{2}{\pi} \arcsin x$. Let $\mathcal{A}$ be the operator which applies $f$ element-wise to a matrix. A non-convex inner approximation of $\mathcal{MC}$ is given by the trigonometric approximation

$$\mathcal{A} = \{ f(X) \mid X \in \mathcal{SR} \}.$$

Nesterov proved in [4, Theorem 2.5] that

$$\max_{X \in \mathcal{A}} \langle A, X \rangle = \max_{X \in \mathcal{MC}} \langle A, X \rangle.$$

Although not convex, $\mathcal{A}$ is simpler than $\mathcal{MC}$ in the sense that checking whether a matrix $X$ is in $\mathcal{A}$ can be done in polynomial time by computing $f^{-1}(X)$ and checking whether $f^{-1}(X)$ is in $\mathcal{SR}$. This allows to reformulate the initial difficult problem [1] as an optimization problem over the algorithmically accessible set $\mathcal{A}$. The complexity of the problem in this form arises solely from the non-convexity of this set.

Hirschfeld studied $\mathcal{A}$ in [2, Section 4]. In this work, we prove that $\mathcal{A}$ possesses an additional beneficial property. Namely, we prove the conjecture of Hirschfeld that it is starlike, i.e., for every $X \in \mathcal{A}$ and every $\lambda \in [0, 1]$, the convex combination $\lambda X + (1 - \lambda)I$ of $X$ and the central point $I$, the identity matrix, is in $\mathcal{A}$.

2 Hirschfeld’s conjecture

In this section, we describe the conjecture and related results which have been obtained by Hirschfeld in his thesis [2, Section 4.3].

In order to show that $\mathcal{A}$ is star-like, one has to prove that

$$\forall X \in \mathcal{SR}, \ f^{-1}(\lambda f X + (1 - \lambda)I) \in \mathcal{SR}.$$

Note that the operator acting on $X$ is nearly an element-wise one, defined by the function

$$f_\lambda : \ [-1, 1] \to \ [-1, 1] \ x \mapsto f^{-1}(\lambda f(x)) = \sin(\lambda \arcsin x)$$

acting on the off-diagonal elements, while the diagonal elements remain equal to 1, contrary to $f_\lambda(1) = f^{-1}(\lambda) = \sin \frac{\pi \lambda}{2}$. Thus one has to show that

$$\forall x \in \mathcal{SR}, \ f_\lambda(X) + \left( 1 - \sin \frac{\pi \lambda}{2} \right) I \succeq 0.$$

A sufficient condition is that $f_\lambda(X) \succeq 0$ for all $X \in \mathcal{SR}$, i.e., the element-wise operator $f_\lambda$ is positivity preserving. Hirschfeld conjectured that this sufficient condition is verified [2, Conjecture 4.9].

Lemma 2.1.

$$\forall X \in \mathcal{SR}, \ f_\lambda(X) \succeq 0$$

A sufficient (and necessary) condition for an operator of this type to be positivity preserving is that all of the Taylor coefficients of $f_\lambda$ are nonnegative [5].

Lemma 2.1 proves the following theorem.

Theorem 2.2. $\mathcal{A}$ is star-like.
3 Proof of the conjecture

In this section, we prove Lemma 2.1.

**Proof.** Let \( \lambda \in [0,1] \) and write \( f_\lambda \) as a power series

\[
 f_\lambda(x) = \sum_{n \in \mathbb{N}} a_n(\lambda)x^n.
\]

The first two derivatives of \( f_\lambda \) are given by

\[
 f'_\lambda(x) = \frac{\lambda}{\sqrt{1 - x^2}} \cos(\lambda \arcsin x)
\]

and

\[
 f''_\lambda(x) = \frac{x}{1 - x^2} \frac{\lambda \cos(\lambda \arcsin x)}{\sqrt{1 - x^2}} - \frac{\lambda^2}{1 - x^2} \sin(\lambda \arcsin x).
\]

Hence \( f_\lambda \) is a solution on \((-1,1)\) of the differential equation

\[
 (1 - x^2)f''_\lambda - xf'_\lambda + \lambda^2 f_\lambda = 0.
\]

Therefore, the Taylor coefficients of \( f_\lambda \) verify the recurrence relation

\[
 (n + 2)(n + 1)a_{n+2}(\lambda) - n(n - 1)a_n(\lambda) - na_n(\lambda) + \lambda^2 a_n(\lambda) = 0
\]

which can be re-expressed as

\[
 a_{n+2}(\lambda) = \frac{n^2 - \lambda^2}{(n + 2)(n + 1)} a_n(\lambda) \tag{2}
\]

with initial conditions

\[
 \begin{align*}
 a_0(\lambda) &= 0 \\
 a_1(\lambda) &= \lambda.
\end{align*}
\]

Given that \( \lambda \in [0,1] \), a trivial induction shows that

\[
 \forall n \in \mathbb{N}, \ a_n(\lambda) \geq 0.
\]

Recursion (2) also proves that the roots of the polynomials \( a_n(\lambda) \) are located at 0, \( \pm 1, \ldots, \pm n \) and are given by the polynomials \( \tilde{P}_n(\lambda) \) [2 eq. 4.23], as also conjectured by Hirschfeld.

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