Abstract  A combinatorial problem concerning the maximum size of the (Hamming) weight set of an \([n,k]_q\) linear code was recently introduced. Codes attaining the established upper bound are the Maximum Weight Spectrum (MWS) codes. Those \([n,k]_q\) codes with the same weight set as \(F^n_q\) are called Full Weight Spectrum (FWS) codes. FWS codes are necessarily “short”, whereas MWS codes are necessarily “long”. For fixed \(k,q\) the values of \(n\) for which an \([n,k]_q\)-FWS code exists are completely determined, but the determination of the minimum length \(M(H,k,q)\) of an \([n,k]_q\)-MWS code remains an open problem. The current work broadens discussion first to general coordinate-wise weight functions, and then specifically to the Lee weight and a Manhattan like weight. In the general case we provide bounds on \(n\) for which an FWS code exists, and bounds on \(n\) for which an MWS code exists. When specializing to the Lee or to the Manhattan setting we are able to completely determine the parameters of FWS codes. As with the Hamming case, we are able to provide an upper bound on \(M(L,k,q)\) (the minimum length of Lee MWS codes), and pose the determination of \(M(L,k,q)\) as an open problem. On the other hand, with respect to the Manhattan weight we completely determine the parameters of MWS codes.

Keywords  linear codes \cdot Lee weight \cdot Manhattan weight \cdot maximum weight spectrum \cdot full weight spectrum

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1 Introduction

In 1973 Delsarte studied the number of distinct distances held by a code $C$, which in the case of linear codes equates to studying the number of distinct weights $|C|$. Discussions on the set of distinct weights of a code can be traced in \cite{25}, where the author tackled the following question. Given a set of positive integers, $S$, is it possible to construct a code whose set of non-zero weights is $S$? Partial solutions were presented, and necessary conditions were established.

In \cite{37} Shi et al. consider a combinatorial problem concerning the maximum number $L(k,q)$ of distinct non-zero (Hamming) weights a linear code over $\mathbb{F}_q$ with dimension $k$ may possess. Specifically they propose

$$L(k,q) \leq \frac{q^k - 1}{q - 1}. \quad (1)$$

In \cite{2}, linear codes meeting this bound were shown to exist for all $k$ and $q$, and such codes were named maximum weight spectrum (MWS) codes.

A further refinement was also investigated in \cite{37} by the introduction of the function $L(n,k,q)$, denoting the maximum number of non-zero (Hamming) weights a linear $[n,k]_q$ code may have. Since an arbitrary vector of length $n$ has Hamming weight at most $n$, an immediate upper bound is $L(n,k,q) \leq n$. In \cite{1}, the parameters of linear codes meeting this bound were completely determined, and were named full weight spectrum (FWS) codes.

The bound (1) was shown to be sharp for binary codes by Haily and Harzalla \cite{19}. In \cite{37}, the binary case is proved independently from \cite{19} and the bound is further shown to be sharp for all $q$-ary linear codes of dimension $k = 2$. The authors went on to conjecture that the bound is sharp for all $q$ and $k$. This conjecture was proved correct in \cite{2}, where it is shown that linear MWS codes exist for all $q$ and $k$. For fixed $k$ and $q$, only sufficient conditions on $n$ are known for which an MWS code exists \cite{2,12,26}. The questions relating to maximal weight spectra are explored in \cite{35} as they relate to cyclic codes, and in \cite{36} as they relate to quasi-cyclic codes. In both cases existence results are established. The problem of determining the minimum value of $n$ for which an $[n,k]_q$ MWS code exists has attracted some attention \cite{2,12,26} but remains open.

In this work we expand discussions to linear codes equipped with general coordinate-wise weight functions, with a view towards Lee weight and a Manhattan type weight. With linear codes, all scalar multiples of any codeword also belongs to the code. Non-zero scalar multiples of a codeword have the same Hamming weight, but in general have differing Lee weights and differing Manhattan weights. Hence, under these new weight functions we might expect bounds regarding respective weight spectra to differ from those established under the Hamming metric.

In the sequel we establish necessary and sufficient conditions on $n$ under which FWS codes with respect to these weight functions exist. We are also able to establish the existence of MWS codes for each $k$ and $q$, and for the Manhattan weight we are able to determine the minimal length $n$ for which these MWS codes exist. On the other hand, determining minimal length of an MWS code under the Lee metric remains as elusive as it is in the case of the Hamming metric.
The research on codes in the Manhattan metric is not extensive. This metric is popular in psychology and related areas but not well known in communications where such channels appear when a signal limiter is used at the receiver [17]. The literature on codes in the Lee metric is comparatively richer, though not entirely extensive, e.g., [4, 10, 16, 18, 32, 33]. Lee-codes were first introduced by C. Y. Lee in [23], and were found to be advantageous for certain types of data transmission. For example, the input alphabet may correspond to phases of a sinusoidal signal of fixed amplitude and frequency, to which white Gaussian noise is added. Such a channel has an inherent ordering of the letters of the input alphabet, and transitions between adjacent letters are much more likely than transitions between distant input letters. Corrupted digits of phase-modulated signals are more likely to have only a slightly different phase in comparison to the original signal. For such channels, the Lee metric is a much more useful notion than the Hamming metric [7, 8].

The properties of Lee-codes and, in particular, the question of existence (or nonexistence) of perfect codes in the Lee metric has been studied by numerous authors, for example, [3, 5, 14, 15, 18, 20, 29]. More recent interest in Lee codes has been spurred on due to further development of applications for these codes. Some examples include constrained and partial-response channels [32], interleaving schemes [9], orthogonal frequency-division multiplexing [34], multidimensional burst-error-correction [38], error-correction in the rank modulation scheme for flash memories [22], and VLSI decoders and fault-tolerant logic [6, 32]. There have also been more recent attempts to settle the existence question of perfect codes in the Lee metric [20, 21, 30].

2 Preliminaries

Let $\mathcal{A}$ be an alphabet of size $q$ which without loss of generality is the set $\{0, 1, \ldots, q-1\}$. An $(n, M)_q$-code, $C$ is a subset of $\mathcal{A}^n$ with $|C| = M$. The elements of $\mathcal{A}^n$ are called words or vectors, while elements of $C$ are called codewords. In the case that $\mathcal{A} = \mathbb{F}_q$ (the finite field with $q$ elements) and $C$ is a subspace of dimension $k$, $C$ is a linear code, denoted an $[n, k]_q$-code. We only consider nondegenerate linear codes, which is to say linear codes having no coordinate which is identically zero. The Hamming distance, $d_H(a, b)$, between two words is the number of coordinates in which they differ. A generator matrix $G$ for an $[n, k]_q$-code $C$ is a $k \times n$ matrix over $\mathbb{F}_q$ whose row vectors generate $C$. The support of a word $c$ is the set $\text{supp}(c) = \{i \mid c_i \neq 0\}$.

Throughout we shall adopt standard notation regarding vector spaces. In particular if $S$ is a set of vectors then $\langle S \rangle$ shall denote the span of $S$, by $e_i$ we shall denote the standard basis vector with $i$‘th entry 1 and all other entries 0, and given $n$ $k$-vectors $\lambda_1, \lambda_2, \ldots, \lambda_n$ we shall denote by $G = [\lambda_1^{\alpha_1} | \lambda_2^{\alpha_2} | \cdots | \lambda_n^{\alpha_n}]$ the $k \times (\alpha_1 + \alpha_2 + \cdots + \alpha_n)$ matrix obtained by repeating the $i$‘th column $\alpha_i$ times, $1 \leq i \leq n$.

2.1 Component-wise Metrics and Weight Functions, MWS and FWS Codes

For our purposes a weight function on $\mathbb{F}_q^n$ is a mapping $w_\mu : \mathbb{F}_q^n \rightarrow \mathbb{R}^+$ with $w_\mu(x) = 0$ if and only if $x = 0$; $w_\mu(u)$ is referred to as the $\mu$-weight of $u$. A weight function $w_\mu$ on
$\mathbb{F}_q$ may be extended in a natural way (and with a slight abuse of notation) to a weight function on $\mathbb{F}_q^n$ for each $n \geq 1$ by taking $w_\mu(u) = w_\mu(u_1) + w_\mu(u_2) + \cdots + w_\mu(u_n)$.

In such a case we say that $w_\mu$ is a component-wise weight function. For any subset $\mathcal{C} \subseteq \mathbb{F}_q^n$ the $\mu$-weight set of $\mathcal{C}$ is defined as

$$w_\mu(\mathcal{C}) = \{ w_\mu(c) \mid c \in \mathcal{C} \setminus \{0\} \}.$$ 

For $u \in \mathbb{F}_q^n$, the Hamming weight of $u$ is given by $w_H(u) = d_H(u, 0)$, the (Hamming) distance between $u$ and 0. Likewise, given any metric on $\mathbb{F}_q^n$ one has, mutatis mutandis an associated weight function.

If $d_\mu : \mathbb{F}_q^n \times \mathbb{F}_q^n \to \mathbb{Z}^+$ is a metric then we denote by $d_\mu(u, v)$ the $\mu$-distance between $u$ and $v$, and similarly $w_\mu(u)$ denotes the $\mu$-weight (distance from 0) of $u$. By $S_\mu(u, t)$ we denote the $\mu$-sphere of radius $t$ around $u$, and by $\mathcal{S}_\mu(u, t)$ its boundary, that is

$$S_\mu(u, t) = \{ x \mid d_\mu(u, x) \leq t \}, \quad \text{and} \quad \mathcal{S}_\mu(u, t) = \{ x \mid d_\mu(u, x) = t \}.$$

We say $d_\mu$ is a component-wise metric if $d_\mu(u, v) = \sum_{i=1}^n d_\mu(u_i, v_i)$, and note that $d_\mu$ then corresponds to a component-wise weight function $w_\mu$. The Hamming metric is an example of component-wise metric, as are both the Lee and Manhattan metrics.

Analogous to the Hamming case, for fixed $k, q$ we denote by $L_\mu(n, k, q)$ the maximum cardinality of $w_\mu(\mathcal{C})$ as $\mathcal{C}$ ranges over all $q$-ary linear codes of dimension $k$. If $n$ is also fixed then the quantity is denoted by $L_\mu(n, k, q)$. Since an $[n, k, q]$ code $\mathcal{C}$ is a subset of $\mathbb{F}_q^n$ it follows that $|w_\mu(\mathcal{C})| \leq |w_\mu([n, k, q])|$ motivating the following definition.

**Definition 1** If $\mathcal{C}$ is a linear $[n, k, q]$-code then $\mathcal{C}$ is a $\mu$-Full Weight Spectrum ($\mu$-FWS) code if $w_\mu(\mathcal{C}) = w_\mu([n, k, q])$.

Note that $|w_H([n, k, q])| = n$, so the definition above subsumes that of FWS codes with respect to Hamming weights. Clearly, if $n = k$ then $\mathcal{C}$ is necessarily FWS, so the task at hand is that of determining the maximum length $n$ of a $k$-dimensional $q$-ary $\mu$-FWS code. The following inequality also follows from the respective definitions:

$$L_\mu(n, k, q) \leq L_\mu(k, q). \quad (2)$$

An open problem posed in [37] is that of determining the sharpness of (2) for the Hamming weight; to determine parameters for which the inequality is strict. This is also of interest when widening discussions to other types of weight functions.

For a component-wise weight function on $\mathbb{F}_q^n$, a further parameter shall prove useful.

**Definition 2** If $w_\mu$ is a component-wise weight function on $\mathbb{F}_q^n$ then we define the parameter $\Delta_\mu$ by

$$\Delta_\mu = \max_{n \geq k} \{ |w_\mu([u])| \mid u \in \mathbb{F}_q^n \}. \quad (3)$$

For notational convenience, if $w_\mu$ is clear from the context then we write $\Delta_\mu = \Delta$. 

The $\mu$-component-wise weight function $w_\mu$ is necessarily FWS, so the task at hand is that of determining the $\mu$-sphere of radius 2 around $u$, and by $\mathcal{S}_\mu(u, t)$ it's boundary, that is

$$S_\mu(u, t) = \{ x \mid d_\mu(u, x) \leq t \}, \quad \text{and} \quad \mathcal{S}_\mu(u, t) = \{ x \mid d_\mu(u, x) = t \}.$$
The following is easily established.

**Proposition 1** Let \( w_\mu \) be a component-wise weight function on \( \mathcal{A}^n \) where \( |\mathcal{A}| = q \), and let \( m = \max w_\mu(\mathcal{A}) \).

1. \( |w_\mu(\mathcal{A}^n)| \leq n \cdot m \).
2. If \( w_\mu(\mathcal{A}) = \{1, 2, \ldots, m\} \) then \( |w_\mu(\mathcal{A}^n)| = n \cdot m \).
3. If \( w_\mu(\mathcal{A}) = \{1, 2, \ldots, m\} \) then an \((n, M)_q\) code \( \mathcal{C} \) over \( \mathcal{A} \) is \( \mu \)-FWS if and only if \( |w_\mu(\mathcal{C})| = n \cdot m \).
4. If \( \mathcal{A} = \mathbb{F}_q \) then \( L_\mu(k, q) \leq \frac{q^k - 1}{q - 1} \cdot \Delta \).

We may now generalize the notion of MWS to the more general setting of component-wise weight function.

**Definition 3** Let \( w_\mu \) be a component-wise weight function on \( \mathbb{F}_q^n \). A linear \([n, k]_q\) code \( \mathcal{C} \) is a \( \mu \)-Maximum Weight Spectrum (\( \mu \)-MWS) code if

\[
|w_\mu(\mathcal{C})| = \frac{q^k - 1}{q - 1} \cdot \Delta.
\]

**Remark 1** A few observations regarding the Proposition 1 and Definition 3:

1. If \( w_\mu(\mathcal{A}) = \{1, 2, \ldots, m\} \) then we may characterize \( \mu \)-FWS codes as those having at least one codeword of each \( \mu \)-weight from 1 to \( mn \).
2. We may characterize linear \( \mu \)-MWS codes as those linear codes in which every one dimensional subspace has a \( \mu \)-weight set of size \( \Delta \), and the \( \mu \)-weight sets of one dimensional subspaces are mutually disjoint.
3. For the Hamming weight, \( m = 1 = \Delta \), and as such the terminology is compatible with that of [37].

In the sequel we will find the following to be of use.

**Lemma 1** Let \( \mathcal{C} \) be an \([n, k]_q\) code, and let \( w_\mu \) be a component-wise weight function defined on \( \mathbb{F}_q^n \) with \( w_\mu(\mathbb{F}_q) = \{1, 2, \ldots, m\} \).

1. If \( \mathcal{C} \) is \( \mu \)-FWS then \( n \cdot m \leq L_\mu(k, q) \).
2. If \( \mathcal{C} \) is both \( \mu \)-FWS and \( \mu \)-MWS, then \( n = \frac{q^k - 1}{q - 1} \cdot \frac{\Delta}{m} \).

**Proof** The first part follows from the Eq. (2) and Proposition 1. For the second part, observe that the FWS property gives \( n = \frac{|w_\mu(\mathcal{C})|}{m} \), whereas the MWS property gives \( |w_\mu(\mathcal{C})| = \frac{q^k - 1}{q - 1} \cdot \Delta \).

**Lemma 2** Let \( \mathcal{C} \) be an \((n, M)_q\) code containing the zero codeword, let \( w_\mu \) be a component-wise weight function, and let \( r = \max_\delta |\{u \in \mathcal{C} \mid w_\mu(u) = \delta\}| \). It holds that

\[
\frac{M - 1}{r} \leq |w_\mu(\mathcal{C})| \leq m \cdot n,
\]

with equality throughout if and only if \( |\{u \in \mathcal{C} \mid w_\mu(u) = \delta\}| = r \) for all \( 1 \leq \delta \leq mn \).
### Table 1: Bounds: Hamming Weight

| Hamming Weight | Reference |
|----------------|-----------|
| \( M(H, k, 2) = 2^k - 1 \) | [1, 37] |
| \( M(H, 3, 3) \leq 32 \) | [2] |
| \( \frac{q}{2} - \frac{q-1}{q-1} \leq M(H, k, q) \) | [3] |
| \( M(H, k, q) \leq 2^k - 1 \) | [3] |
| \( M(H, 2, q) = \frac{2q^k - 1}{q-1} \) | [3] |
| \( N(H, k, q) = 2^k - 1 \) | [4] |

### Table 2: Bounds: Manhattan Weight

| Manhattan Weight | Reference |
|------------------|-----------|
| \( M(M, k, q) = \frac{q^k - 1}{q-1} \) | Thm. [5] |
| \( N(M, k, q) = \frac{q^k - 1}{q-1} \) | Thm. [6] |

### Table 3: Bounds: Lee Weight

| Lee Weight | Reference |
|------------|-----------|
| \( \frac{q^k + 1}{q+1} + \left\lfloor \frac{2q-1}{q-1} \right\rfloor \leq M(L, k, q) > 2 \) | Lem. [5] |
| \( 2^k - 1 \leq M(L, k, 2) \) | Lem. [5] |
| \( M(L, k, q) \leq \frac{q^k - 1}{q+1} \left( \frac{q^k + 1}{q+1} - 1 \right) \) | Thm. [8] |
| \( M(L, 2, 7) \leq 16 \) | [27] |
| \( M(L, 2, 11) \leq 34 \) | [27] |
| \( M(L, 2, 13) \leq 46 \) | [27] |
| \( M(L, 2, 17) \leq 76 \) | [27] |
| \( M(L, 2, 19) \leq 86 \) | [27] |
| \( M(L, 2, 23) \leq 126 \) | [27] |
| \( N(L, k, q) = \frac{q^k + 1}{q+1} - 1 \) | Lem. [9] |

**Proof** The first inequality arises by counting ordered pairs \((u, w(u))\) where \(0 \neq u \in C\) in two ways to give \( M - 1 \leq |w_u(C)| \cdot r \), and the second inequality is from Proposition [1]. The equality part follows from simple counting.

Given the lower and upper bounds inherent in the above it behooves us to investigate optimality with respect to code length. To this end we define \( M(\mu, k, q) \) and \( N(\mu, k, q) \) as follows:

\[
M(\mu, k, q) = \min\{n \mid C \text{ is an } [n,k]_q \mu - \text{MWS code} \}
\]

\[
N(\mu, k, q) = \max\{n \mid C \text{ is an } [n,k]_q \mu - \text{FWS code} \}
\]

Tables 1 - 3 will perhaps place some of the results that follow in context.

In the next section we focus on determining \( N(\mu, k, q) \).

### 3 Generalized Full Weight Spectrum Codes

We can in fact determine \( N(\mu, k, q) \) when \( w_\mu(F_q) = \{1, 2, \ldots, m\} \). To do this, the following will be of use.
Lemma 3 Suppose \( w_\mu \) is a coordinate-wise weight function on \( \mathbb{F}_q^n \), where \( \mathbb{F}_q = \{0, \alpha_1, \alpha_2, \ldots, \alpha_{q-1}\} \), and \( w_\mu(\mathbb{F}_q) = \{1, 2, \ldots, m\} \).

Let \( \mathcal{C} \) be a (linear) \([n,k]_q\) code with basis \( B = \{v_1, v_2, \ldots, v_k\} \). Let \( S = \text{supp}(v_1) \), \( T = \cup_{i=2}^{k} \text{supp}(v_i) \), \( S' = S \setminus T \), with \( s = |S|, t = |T| \), and \( s' = |S'| \).

If \( C \in \mu\text{-FWS} \) then

\[
s' \leq m \cdot t + 1.
\]

**Proof** If \( s' = 0 \) then the result holds, so assume \( s' \geq 1 \). Let \( x \in \mathcal{C} \). If \( \text{supp}(x) \cap S' = \emptyset \) then \( \text{supp}(x) \subseteq T \), so \( |\text{supp}(x)| \leq t \) giving

\[
w_\mu(x) \leq m \cdot t.
\]

If \( \text{supp}(x) \cap S' \neq \emptyset \) then it follows that \( S' \subseteq \text{supp}(x) \), so \( |\text{supp}(x)| \geq s' \) giving

\[
w_\mu(x) \geq s'.
\]

Since \( n = t + s' \) (by assumption no coordinate is identically zero) we see that if \( \mathcal{C} \) is \( \mu\text{-FWS} \) then

\[
w_\mu(\mathcal{C}) = \{1, 2, \ldots, (t + s') \cdot m\},
\]

and since \( m, s' \geq 1 \) there exists a codeword \( u \) with \( w_\mu(u) = t \cdot m + 1 \). From (6) and (7) we may conclude that

\[
w_\mu(u) = t \cdot m + 1 \geq s'
\]

giving the desired inequality.

**Theorem 1** Let \( m \) be as in Lemma 3. If \( \mathcal{C} \) is a linear \([n,k]_q\) \( \mu\text{-FWS} \) code then

\[
n \leq \sum_{i=0}^{k-1} (m + 1)^i = \frac{(m + 1)^k - 1}{m}.
\]

**Proof** Let \( \mathcal{C} \) be an \([n,k]_q\) \( \mu\text{-FWS} \) code. It suffices to show that we may select a basis \( \{\lambda_1, \lambda_2, \ldots, \lambda_k\} \) of \( \mathcal{C} \) such that

\[
\left| \bigcup_{i=1}^{t} \text{supp}(\lambda_i) \right| \leq 1 + (m + 1) + \cdots + (m + 1)^{t-1}, \text{ for } 1 \leq t \leq k.
\]

(8)

To this end first note that since \( \mathcal{C} \) is \( \mu\text{-FWS} \), there is a codeword of weight 1. Assume without loss of generality that \( w_\mu(1) = 1 \), so we may take \( \lambda_1 = e_1 \), so (8) holds for \( t = 1 \). Assume \( \lambda_1, \lambda_2, \ldots, \lambda_{k-1} \) have been selected in accordance with (8), and let \( \lambda_k \) be a codeword such that \( \{\lambda_1, \lambda_2, \ldots, \lambda_k\} \) is a basis for \( \mathcal{C} \). From the Lemma 3 we have

\[
\left| \bigcup_{i=1}^{k} \text{supp}(\lambda_i) \right| \leq \left| \bigcup_{i=1}^{k-1} \text{supp}(\lambda_i) \right| + m \cdot \left| \bigcup_{i=1}^{k-1} \text{supp}(\lambda_i) \right| + 1
\]

\[
= (m + 1) \cdot \left| \bigcup_{i=1}^{k-1} \text{supp}(\lambda_i) \right| + 1
\]

\[
\leq \sum_{i=0}^{k-1} (m + 1)^i.
\]
We now show the bound in Theorem 1 is sharp.

**Theorem 2** Let \( m \) be as in Lemma 3. There exists an \([n,k]_q\) \( \mu \)-FWS code if and only if
\[
k \leq n \leq \frac{(m+1)^k - 1}{m}.
\]
In particular \( N(\mu,k,q) = \frac{(m+1)^k - 1}{m} \).

**Proof** That \( k \leq n \leq \frac{(m+1)^k - 1}{m} \) is necessary is given in Theorem 1. For sufficiency, let \( k \) be fixed and let \( G \) be the following matrix:
\[
G = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}
= \begin{bmatrix}
e_1 | (e_2)^{m+1} | (e_3)^{(m+1)^2} | \cdots | (e_k)^{(m+1)^{k-1}}
\end{bmatrix}
\]
Denote the rows of \( G \) by \( \lambda_1, \lambda_2, \ldots, \lambda_k \) and let \( C \) be the code generated by \( G \). The code \( C \) has length \( n = \frac{(m+1)^k - 1}{m} \). Moreover, for any \( u = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{F}_q^k \),
\[
w_\mu(uG) = w_\mu(\alpha_1 \lambda_1 + \cdots + \alpha_k \lambda_k) = w_\mu(\alpha_1) + w_\mu(\alpha_2)(m+1) + \cdots + w_\mu(\alpha_k)(m+1)^{k-1} = \sum_{i=1}^k w_\mu(\alpha_i)(m+1)^{i-1}.
\]
Since \( w_\mu(\mathbb{F}_q) = \{1, 2, \ldots, m\} \) we obtain \( |w_\mu(C)| = nm(= (m+1)^k - 1) \), so \( C \) is an FWS.

More generally, for \( k \leq n \leq \frac{(m+1)^k - 1}{m} \) consider the \( k \times n \) generator matrix \( G' \) obtained by removing rightmost columns of \( G \) while preserving rank, say
\[
G' = \begin{bmatrix}
e_1 | (e_2)^{m+1} | \cdots | (e_t)^{(m+1)^{t-1}} | (e_{t+1})^a | e_{t+2} | \cdots | e_k
\end{bmatrix},
\]
where \( 1 \leq a \leq (m+1)^t \). Let \( C' \) denote the code with generator \( G' \).

Let \( \gamma \) be an arbitrary \( \mu \)-weight with \( 1 \leq \gamma \leq mn \). If \( \gamma \leq m(k-t-1) \) then \( \gamma = w_\mu(uG') \) for any \( u = (u_1, u_2, \ldots, u_k) \) with final \( k-t-1 \) coordinates chosen such that the sum of their \( \mu \)-weights is \( \gamma \), the remaining coordinates being 0. Now consider \( \gamma > m(k-t-1) \) and let \( a = \frac{\gamma - m(k-t-1)}{\mu} \). If \( a \leq m \) then \( b = \gamma - a\mu - m(k-t-1) \leq (m+1)^t - 1 \), so we may expand \( b \) in base \( m + 1 \):
\[
b = b_1 + b_2(m+1) + b_3(m+1)^2 + \cdots + b_t(m+1)^{t-1}.
\]
Taking \( u \in \mathbb{F}_q^n \) with \( w_\mu(u_i) = b_i \), \( 1 \leq i \leq t \), \( w_\mu(u_{t+1}) = a \), and \( w_\mu(u_i) = m \) for \( i > t + 1 \) we obtain \( w_\mu(uG') = \gamma \). Likewise, if \( a > m \) we may take \( u \in \mathbb{F}_q^n \) with \( w_\mu(u_i) = b_i \), \( 1 \leq i \leq t \), and \( w_\mu(u_i) = m \) for \( i > t \) to obtain a codeword of weight \( \gamma \).
Let \( m \) be as in Proposition 1. If \( C \) is an \([n,k,q]\) code that is both \( \mu \)-MWS and \( \mu \)-FWS, then \( \Delta = m = q - 1 \), and in particular \( n = \frac{q^k - 1}{q - 1} \).

**Proof** First observe that since \( w_\mu \) is a component-wise weight function, \( w_\mu (\mathbb{F}_q) = w_\mu (\langle e_1 \rangle) \), giving \( m \leq \Delta \). Next we observe that \( n = q^{k-1} \cdot \Delta \leq (m + 1)^k - 1 \leq q^k - 1 \). (9)

The first equality follows from Lemma 1, the first inequality follows from Theorem 2, and the final inequality holds since \( m \leq q - 1 \). From (9) it follows that \( \Delta \leq m \) whence \( \Delta = m \) and the result follows.

Corollary 1 tells us that if we seek codes that are both MWS and FWS then the particular weight function must have a rather strict structure. Via a Manhattan type weight function we will exhibit infinite families of codes that are both MWS and FWS.

### 4 Codes with the Lee and Manhattan Weight Functions

In this section we specialize discussions to the Lee and Manhattan weight functions. We introduce these weight functions in the more general setting of \( \mathbb{Z}_q^n \), where \( \mathbb{Z}_q = \{0,1,\ldots,q-1\} \) denotes the ring of integer residues modulo the positive integer \( q \). Linear codes are defined over fields, so when applying these weights to linear codes it is necessarily assumed that \( q \) is prime.

Roughly following the notation and terminology as established in [31], the Lee weight of an element \( \alpha \in \mathbb{Z}_q \), denoted by \( w_\mathcal{L} (\alpha) \) or \( |\alpha| \), takes non-negative integer values and is defined by

\[
w_\mathcal{L} (\alpha) = |\alpha| = \begin{cases} \alpha & \text{if } 0 \leq \alpha \leq q/2 \\ q - \alpha & \text{otherwise.} \end{cases}
\]

We refer to the elements 1, 2, \ldots, \lfloor q/2 \rfloor as the “positive” elements of \( \mathbb{Z}_q \) and denote the set of such elements \( \mathbb{Z}_q^+ \); the remaining elements in \( \mathbb{Z}_q \setminus \{0\} \) compose the “negative” elements, denoted \( \mathbb{Z}_q^- \).

**Example 1** In \( \mathbb{Z}_7, \mathbb{Z}_7^+ = \{1,2,3\} \) and \( |2| = |5| = 2 \), whereas in \( \mathbb{Z}_8, \mathbb{Z}_8^+ = \{1,2,3,4\} \) and \( |2| = |6| = 2 \).

For a word \( c = (c_1,c_2,\ldots,c_n) \) in \( \mathbb{Z}_q^n \), we define the Lee weight of \( c \) by

\[
w_\mathcal{L} (c) = \sum_{j=1}^n |c_j|.
\]

(with the summation being integer summation). The Lee distance (metric) between two words \( x,y \in \mathbb{Z}_q^n \) is defined as \( w_\mathcal{L} (x - y) \); we denote that distance by \( d_\mathcal{L} (x,y) \). We note that for \( q = 2,3 \) the Hamming and Lee metrics coincide.
In keeping with the notation introduced in the previous section we shall adopt the following:

\[ L_l(k, q) := \max \{|w_L(C)| : C \text{ is linear, of dimension } k \text{ over } \mathbb{Z}_q \} \quad \text{(11)} \]

and

\[ L_l(n, k, q) := \max \{|w_L(C)| : C \text{ is an } [n, k]_q \text{ code over } \mathbb{Z}_q \} \quad \text{(12)} \]

The minimum Lee distance of an \((n, M)\) code \(C\) over \(\mathbb{Z}_q\) with \(M > 1\) is defined by

\[ d_L(C) = \min_{c_1, c_2 \in C, c_1 \neq c_2} d_L(c_1, c_2). \]

We note that for a linear code, the minimum distance is also the minimum weight, that is

\[ d_L(C) = \min w_L(C). \]

The Manhattan metric (or TaxiCab metric), is defined for alphabet letters taken from the integers. For two words \(x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbb{Z}^n\) the Manhattan distance between \(x\) and \(y\) is defined as

\[ d_M(x, y) = \sum_{i=1}^{n} |x_i - y_i|. \quad \text{(13)} \]

The Manhattan weight of a vector in \(\mathbb{Z}_q^n\) is its distance from zero. As we have chosen \(\mathbb{Z}_q = \{0, 1, \ldots, q - 1\}\) we may adapt the Manhattan weight to \(\mathbb{Z}_q^n\) by treating coordinate entries as integers. Doing so we define the Manhattan weight of a vector \(x \in \mathbb{Z}_q^n\) as the integer sum:

\[ w_M(x) = \sum_{i=1}^{n} x_i. \quad \text{(14)} \]

### 4.1 Lee MWS Codes

Let \(C\) be a linear \([n, k]_q\) code over \(\mathbb{Z}_q\). From the definition of Lee weight it follows that if \(x \in \mathbb{Z}_q^n\) then \(w_L(x) = w_L(-x)\) (i.e., \(|x| = |-x|\)). With reference to Definition2 we therefore have \(\Delta \leq \frac{q-1}{2}\). One may quickly verify that since \(q\) is prime, \(|w_L([e_1])| = \frac{q-1}{2}\) whence \(\Delta = \frac{q-1}{2}\) giving the following.

**Theorem 3** A linear \([n, k]_q\) code is \(L\)-MWS if and only if \(w_L(C) = \frac{q^k - 1}{2}\).

From the definition it is clear that if \(C\) is \(L\)-MWS then two codewords \(u, v\) have the same \(L\)-weight if and only if \(u = \pm v\). Hence, in Lemma2 we have \(m = \frac{q^k - 1}{2}, r = 2, \) and \(M = q^k\) giving

\[ C \text{ an } [n, k]_q \text{ } L\text{-MWS code} \implies n \geq \frac{q^k - 1}{q-1}. \quad \text{(15)} \]

We can make a slight improvement to the lower bound \((15)\).
Lemma 4 If $\mathcal{C}$ is an $[n,k]_q$ $L$-MWS code, $q > 2$, then any two non-zero codewords have at least one support position in common.

Proof If $u, v \in \mathcal{C}$ have disjoint supports then

$$w_L(u + v) = w_L(u) + w_L(v) = w_L(u - v) = w_L(-u - v).$$

Since $q > 2$ we have $1 \neq -1$, and thus by the $L$-MWS property we must have $u = 0$ or $v = 0$.

Corollary 2 If $\mathcal{C}$ is an $[n,k]_q$ $L$-MWS code, $q > 2$, and $0 \neq c \in \mathcal{C}$, then $|\text{supp}(c)| \geq k$. In particular, $d_L(\mathcal{C}) \geq k$.

Proof Let $c$ be a codeword with minimal support in $\mathcal{C}$, say $|\text{supp}(c)| = \delta$. Since $\mathcal{C}$ is a linear code, in any fixed $t \leq k$ positions there are at least $q^{k-t}$ codewords with 0 in each of these coordinate position. If $\delta < k$ then Lemma 4 gives a contradiction.

Remark 2 In the language of [11], Lemma (4) shows that in the non-binary cases, $L$-MWS codes are intersection codes. Corollary 2 thus follows from Corollary 1 of [24] with the observation that $d_L \geq d_H$.

Lemma 5 If $\mathcal{C}$ is an $[n,k]_q$ $L$-MWS code with $q > 2$ then $n \geq \frac{q^k - 1}{q - 1} + \left\lceil \frac{2(k - 1)}{q - 1} \right\rceil$. An $[n,k]_2$ $L$-MWS code satisfies $n \geq 2^k - 1$.

Proof The case $q = 2$ follows from (15). Assume $q > 2$. The number of possible distinct non-zero Lee weights of a vector $c$ of length $n$ is $\left\lceil \frac{q^k - 1}{2} \right\rceil$, moreover, if $c \in \mathcal{C}$ then we may reduce this number by $k - 1$ (Cor. 3) and the $L$-MWS property gives

$$\frac{q^k - 1}{2} - n + 1 \geq \frac{q^k - 1}{2}.$$
4.3 Existence of $\mathcal{L}^{\ell}$-MWS codes

Having seen some examples of $\mathcal{L}^{\ell}$-MWS codes for specific values of $q$ and $k$, we now provide a general construction for $\mathcal{L}^{\ell}$-MWS codes, establishing their existence for all $q, k$. We shall find the following to be of use in our construction.

**Lemma 6** Let $u_1, v_1 \in \mathbb{Z}_r$, $1 \leq i \leq k$, where $r > 2$ is an odd integer.

1. If $|u_1 - u_2| = |u_1 + u_2|$ then $u_1 = 0$ or $u_2 = 0$.
2. If $|u_1| = |v_1|$, and $|u_i + u_j| = |v_i + v_j|$ for all $i, j$ then $(u_1, u_2, \ldots, u_k) = \pm (v_1, v_2, \ldots, v_k)$.

**Proof** For the first part we have

$$|u_1 - u_2| = |u_1 + u_2| \iff u_1 - u_2 = \pm (u_1 + u_2)$$

$$\iff 2u_2 = 0 \text{ or } 2u_1 = 0$$

$$\iff u_2 = 0 \text{ or } u_1 = 0 \quad \text{(since } r \text{ is odd)}.$$

For the second part we note that the case of $k = 1$ is patently true, so let $k = 2$. If either $u_1 = 0$ or $u_2 = 0$ then the result holds, so assume otherwise. The hypothesis becomes $v_1 = \pm u_1$, $v_2 = \pm u_2$, and

$$|u_1 + u_2| = |v_1 + v_2| = |u_1 \pm u_2|,$$

so from part 1 we obtain $(u_1, u_2) = \pm (v_1, v_2)$. Induction completes the proof.

We now show that MWS Lee codes exist for all $q$ and $k$.

**Theorem 4** $L_{\mathcal{L}}(k, q) = \frac{q^{2k-1} - 1}{2}$, and $M(\mathcal{L}, k, q) \leq \frac{(\frac{q+1}{2})^{2k - 3k + 1} - 1}{\frac{q+1}{2} - 1}$.

**Proof** For $q = 2$ see [1]. Assume $q > 2$. Our proof is constructive. Let $\alpha = \frac{q+1}{2}$ and define the generator matrix $G_{k, q}$ by

$$
\begin{bmatrix}
  e_1 & (e_2)^{\alpha} & \cdots & (e_k)^{\alpha^{k-1}} & \left( \sum_{j=1}^{2} e_j \right)^{\alpha^1} & \left( \sum_{j=1}^{3} e_j \right)^{\alpha^{k+1}} & \cdots & \left( \sum_{j=1}^{k} e_j \right)^{\alpha^{2k-2}}
\end{bmatrix}
$$

| Bound   | Reference | Bound   | Reference |
|----------|-----------|----------|-----------|
| $L_{\mathcal{L}}(n, 1, q) = \frac{q^n - 1}{q - 1}$ (*) | See above | $L_{\mathcal{L}}(4, 2) = 2^3 - 1$ | [17] |
| $L_{\mathcal{L}}(32, 3, 3) = 13$ (*) | [22] | $L_{\mathcal{L}}(2, 3) = 6$ | [22] |
| $L_{\mathcal{L}}(16, 2, 7) = 24$ (*) | [22] | | |
| $L_{\mathcal{L}}(46, 2, 13) = 84$ (*) | [22] | $L_{\mathcal{L}}(2, 2, 5) = 4$ | [22] |
| $L_{\mathcal{L}}(86, 2, 19) = 180$ (*) | [22] | $L_{\mathcal{L}}(3, 2, 5) = 6$ | [22] |
| $L_{\mathcal{L}}(11, 2, 3) = 12$ (*) | [22] | $L_{\mathcal{L}}(4, 2, 5) = 8$ | [22] |
| $L_{\mathcal{L}}(34, 2, 11) = 60$ (*) | [22] | $L_{\mathcal{L}}(5, 2, 5) = 8$ | [22] |
| $L_{\mathcal{L}}(16, 2, 11) = 144$ (*) | [22] | $L_{\mathcal{L}}(6, 2, 5) = 9$ | [22] |
| $L_{\mathcal{L}}(126, 2, 23) = 264$ (*) | [22] | $L_{\mathcal{L}}(7, 2, 5) = 9$ | [22] |
Recall, $e_i$ denotes the unit column vector with 1 in entry $i$ and exponents correspond to repetition numbers. The length of $C$ is $n = \sum_{t=0}^{\infty} \alpha^t$ where $t = 2k - 2$. From the form of $G_{k,q}$ it follows that the Lee weight of any codeword $uG_{k,q}$ may be expressed in base $\alpha$ as
\[
w_L(uG_{k,q}) = |u_1| + |u_2|\alpha + |u_3|\alpha^3 + \cdots + |u_1 + u_2 + \cdots + u_k|\alpha^{2k-2}.
\]
If two codewords $uG_{k,q}$ and $vG_{k,q}$ have the same weight then $|u_i| = |v_i|$, for all $i$ and $|u_1 + u_2| = |v_1 + v_2|$. From the Lemma 5 we obtain $(u_1, u_2) = \pm (v_1, v_2)$. Likewise, since $|u_1 + u_2 + u_3| = |v_1 + v_2 + v_3|$ and $|u_1 + u_2| = |v_1 + v_2|$, Lemma 5 gives $(u_1, u_2, u_3) = \pm (v_1, v_2, v_3)$. Continuing inductively we obtain $C$ is $L$-MWS.

Example 2 If $k = 2$ and $q = 5$ then the generator matrix as in the proof of Theorem 4 is
\[
G_{2,5} = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}.
\]

4.4 Lee FWS Codes

Recall from the definition that for $q$ odd, an $[n,k]_q$-code is $L$-FWS if $|w_L(C)| = n \cdot \frac{q-1}{2}$. Table 4 provides some examples of FWS codes for small values of $q$ and $k$. Based simply on the properties of a component-wise metric we get an initial upper bound on the length of $L$-FWS codes. Indeed, from part I of Lemma 1 and Proposition I we obtain $n \leq \frac{q^k-1}{q-1}$. By considering Theorem I we can get a significantly better bound than this.

Lemma 7 There exists an $[n,k]_q$ $L$-FWS code if and only if
\[
n \leq k-1 \sum_{i=0}^{k-1} \left( \frac{q+1}{2} \right)^i = \left( \frac{q+1}{2} \right)^k - 1.
\]

In particular $N(L, k, q) = \left( \frac{q+1}{2} \right)^{k-1} - 1$.

Proof Follows from Theorem I by observing that for the Lee metric $m = \Delta = \lfloor (q - 1)/2 \rfloor$.

Comparing the bounds in the Lemmata 5 and 7 we obtain the following non-existence result.

Corollary 3 There exists an $[n,k]_q$ code that is both $L$-MWS and $L$-FWS if and only if $q = 2$.

Proof The Hamming and Lee metrics are the same for binary codes, so for the binary case see I. The case $q \geq 3$ follows from Lemma 2 and Corollary I.
4.5 Manhattan MWS Codes

If \( u \in \mathbb{F}_q^n \) is a nonzero vector then \( |u| = q \), so \( w_\mathcal{M}(u) \leq q - 1 \). One may show that strict inequality may occur only if \( q \) is a divisor of \( w_\mathcal{M}(u) \). As such, an immediate bound on the \( \mathcal{M} \)-weight spectrum of a linear code is

\[
L_\mathcal{M}(k, q) \leq q^k - 1.
\]

It is clear that for \( e_1 \in \mathbb{F}_q^n \), \( w_\mathcal{M}(e_1) = q - 1 \), so for \( n \geq 1 \) we have

\[
L_\mathcal{M}(n, 1, q) = L_\mathcal{M}(1, q) = q - 1.
\]

In [27] values of \( L_\mathcal{M}(n, k, q) \) were explicitly computed for \( q = 3, 5 \) and \( k = 2 \), these values appear in Table 5. Recall that for \( q = 2 \) the Manhattan, Hamming, and Lee weights coincide, so we refer to section 4.2 for the binary case.

**Table 5** Optimal bounds for for the Manhattan Weight

| Bound         | \( \mathcal{M} \)-MWS | \( \mathcal{M} \)-FWS |
|---------------|------------------------|-----------------------|
| \( L_\mathcal{M}(3, 2, 3) = 6 \) | No                     | Yes                   |
| \( L_\mathcal{M}(4, 2, 3) = 8 \) | Yes                    | Yes                   |
| \( L_\mathcal{M}(3, 2, 5) = 12 \) | No                     | Yes                   |
| \( L_\mathcal{M}(4, 2, 5) = 16 \) | No                     | Yes                   |
| \( L_\mathcal{M}(5, 2, 5) = 20 \) | No                     | Yes                   |
| \( L_\mathcal{M}(6, 2, 5) = 24 \) | Yes                    | Yes                   |

Since the Manhattan weight is a coordinate-wise weight function, the maximum weight of an \( n \)-vector is \( n(q - 1) \). If \( C \) is an \([n, k]_q \) code then \( |w_\mathcal{M}(C)| \leq \max w_\mathcal{M}(C) \leq n(q - 1) \), giving the following.

**Lemma 8** If \( C \) is an \([n, k]_q \) code then \( n \geq \frac{|w_\mathcal{M}(C)|}{q-1} \). In particular, if \( C \) is an \([n, k]_q \) \( \mathcal{M} \)-MWS code then \( n \geq \frac{L_\mathcal{M}(k, q)}{q-1} \) with equality if and only if \( w_\mathcal{M}(C) = \{1, 2, \ldots, n(q - 1)\} \).

**Theorem 5** For \( k \geq 1 \),

\[
L_\mathcal{M}(k, q) = q^k - 1, \quad \text{and} \quad M(\mathcal{M}, k, q) = \frac{q^k - 1}{q - 1}.
\]

**Proof** For \( q = 2 \) see Table 5. Assume \( q > 2 \). Our proof is constructive. Define the generator matrix \( G(k, q) \) by

\[
G_{k,q} = \begin{bmatrix} e_1 | (e_2)^q | \cdots | (e_k)^{q-1} \end{bmatrix}
\]

An argument entirely similar to that in the proof of Theorem 4 shows that two code-words \( uG_{k,q} \) and \( vG_{k,q} \) have the same weight if and only if \( u = v \), whence \( C \) is \( \mathcal{M} \)-MWS. For the second part we observe that the length of the code generated by \( G \) is \( \frac{q^k - 1}{q - 1} \) and appeal to Lemma 8.
4.6 Manhattan FWS codes

It is readily verified that \( w_M(F^n_q) = n(q - 1) \), so \( \mathcal{M} \)-FWS codes are those with \( w_M(C) = n(q - 1) \). Table 5 exhibits examples of such codes for small \( k, q \). The following shows that in contrast to MWS codes, we may completely determine the optimal lengths of FWS codes.

**Theorem 6** There exists an \([n, k]_q \mathcal{M}\)-FWS code if and only if \( k \leq n \leq \frac{q^k - 1}{q - 1} \). In particular,

\[
N(\mathcal{M}, k, q) = \frac{q^k - 1}{q - 1}.
\]

**Proof** That \( n \leq \frac{q^k - 1}{q - 1} \) follows directly from part 1 of Lemma 1 and Proposition 1 by observing that \( m = q - 1 = \Delta \). The codes with generator \( G_{k,q} \) as described in the proof of Theorem 5 are \([n, k]_q\)-codes with \( w_M(C) = \frac{q^k - 1}{q - 1} = n \) and are thus \( \mathcal{M} \)-FWS. Generator matrices obtained by deleting columns of \( G_{k,q} \) in the manner described in the proof of Theorem 2 show the existence of \( \mathcal{M} \)-FWS codes for \( k \leq n \leq \frac{q^k - 1}{q - 1} \).

5 Conclusion, future work, and open problems

Here we have investigated the combinatorial problem of maximizing the number of distinct weights that a linear code may obtain. The literature relating to this problem has thus far focused on FWS and MWS codes with respect to Hamming weight. Under the Hamming metric, FWS codes are necessarily “short”, and MWS codes are necessarily “long”. The existence question and in particular the maximum length \( N(\mathcal{H}, k, q) \) of an \([n, k]_q\) FWS code has been determined. However, the determination of \( M(\mathcal{H}, k, q) \), the minimum length of an \([n, k]_q\) MWS code remains an open problem. Here, we have generalized to the setting of component-wise weight functions, with a particular focus on the Lee, and Manhattan type weights. For a certain class of component-wise weight functions (which subsumes the Hamming, Lee, and Manhattan weight functions) we determine the necessary and sufficient conditions for the existence of \( \mu \)-FWS codes (Theorem 2). We establish through construction the existence of \( \mu \)-MWS codes for all \( q, k \) (Theorem 4), and thus an upper bound on \( M(\mathcal{L}, k, q) \). Computational examples show our bound on \( M(\mathcal{L}, k, q) \) is not generally tight, thus the determination of \( M(\mathcal{L}, k, q) \) remains an open problem. On the other hand, for Manhattan weights we were able to determine both \( N(\mathcal{M}, k, q) \) and \( M(\mathcal{M}, k, q) \) and provided a construction showing that in fact \( N(\mathcal{M}, k, q) = M(\mathcal{M}, k, q) \) (Theorem 5).

The determination of \( M(H, k, q) \), \( M(\mathcal{L}, k, q) \), and more generally \( M(\mu, k, q) \) (for an arbitrary component-wise weight function \( w_\mu \) on \( \mathbb{F}_q^n \)) may be an attractive endeavour for the community.
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