GELFAND-SHILOV SMOOTHING EFFECT FOR THE SPATIALLY INHOMOGENEOUS BOLTZMANN EQUATIONS WITHOUT CUT-OFF

WEI-XI LI
School of Mathematics and Statistics, Wuhan University & Hubei Key Laboratory of Computational Science, Wuhan University, Wuhan 430072, China

LVQIAO LIU∗
School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China

(Communicated by Changjiang Zhu)

Abstract. In this work we consider the Cauchy problem for the spatially inhomogeneous non-cutoff Boltzmann equation. For any given solution belonging to weighted Sobolev space, we will show it enjoys at positive time the Gelfand-Shilov smoothing effect for the velocity variable and Gevrey regularizing properties for the spatial variable. This improves the result of Lerner-Morimoto-Pravda-Starov-Xu [J. Funct. Anal. 269 (2015) 459-535] on one-dimensional Boltzmann equation to the physical three-dimensional case. Our proof relies on the elementary $L^2$ weighted estimate.

1. Introduction and main result. The spatially inhomogeneous Boltzmann equation is written as

$$\partial_t F + v \cdot \partial_x F = Q(F, F),$$

where $F(t, x, v)$ is a unknown probability density with $x = (x_1, x_2, \cdots, x_n)$ and $v = (v_1, v_2, \cdots, v_n)$ stand respectively for the spatial and velocity variables. Here, $Q$ is the collision operator,

$$Q(G, F)(t, x, v) = \int_{\mathbb{R}^n} \int_{S^{n-1}} B(v-v_*, \sigma)(F' G'_* - F G_*) dv_* d\sigma,$$

where we have used the conventional abbreviation $F' = F(t, x, v'), F = F(t, x, v)$, $G'_* = G(t, x, v'_*)$ and $G_* = G(t, x, v_*)$ and the pre-collisional velocities $(v, v_*)$ and the post-collisional ones $(v', v'_*)$ satisfy

$$\begin{cases}
v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \\
v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma.
\end{cases}$$

2020 Mathematics Subject Classification. Primary: 35B65; 35Q20; 35H10.

Key words and phrases. Boltzmann equation, Gelfand-Shilov regularity, subelliptic estimate, non cut-off, weighted estimate.

The first author is supported by NSF grant Nos. 11871054, 11771342; Fok Ying Tung Education Foundation (151001) and the Natural Science Foundation of Hubei Province (2019CFA007).

* Corresponding author: Lvhqiao Liu.
for $\sigma \in \mathbb{S}^2$, according to the conservation laws of momentum and energy

$$v + v_* = v' + v'_*, \quad |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2,$$

The cross-section $B(v - v_*, \sigma)$ in (2) depends on the relative velocity $|v - v_*|$ and the deviation angle $\theta$ with

$$\cos \theta = \langle (v - v_*)/|v - v_*|, \sigma \rangle.$$

Here we denote by $\langle \cdot, \cdot \rangle$ the scalar product in $\mathbb{R}^3$. Without loss of generality we may assume that $B(v - v_*, \sigma)$ is supported on the set $\theta \in [0, \pi/2]$ where $\langle v - v_*, \sigma \rangle \geq 0$, since as usual $B$ can be replaced by its symmetrized version, and furthermore we may suppose it takes the following form:

$$B(v - v_*, \sigma) = |v - v_*|^\gamma b(\cos \theta),$$

(3)

where $\gamma \in ]-3, 1[$. Recall $\gamma = 0$ is the Maxwellian molecules case and meanwhile the cases of $-3 < \gamma < 0$ and $0 < \gamma < 1$ are called respectively soft potential and hard potential. In this paper we will restrict our attention to the cases of Maxwellian molecules and hard potential, i.e., $0 \leq \gamma < 1$. Furthermore we are concerned about singular cross-sections, also called non cut-off sections, that is, the angular part $b(\cos \theta)$ has singularity near $0$ so that

$$\int_0^{\pi/2} \sin \theta b(\cos \theta) \, d\theta = +\infty.$$

Precisely, we suppose $b$ has the following expression near $\theta = 0$:

$$0 \leq \sin \theta b(\cos \theta) \approx \theta^{-1-2s},$$

(4)

where and throughout the paper $p \approx q$ means $C^{-1}q \leq p \leq Cq$ for some constant $C \geq 1$. Note that the cross-sections of type (3) include the potential of inverse power law as a typical physical model.

In this paper we will restrict our attention to the fluctuation around the Maxwellian distribution. Let

$$\mu(v) = (2\pi)^{-3/2}e^{-|v|^2/2}$$

be the normalized Maxwellian distribution. Write solution $F$ of (1) as $F = \mu + \sqrt{\mu}f$ and accordingly $F_0 = \mu + \sqrt{\mu}f_0$ for the initial datum. Then the fluctuation $f$ satisfies the Cauchy problem

$$\begin{cases}
\partial_t f + v \cdot \partial_x f - \mu^{-\frac{1}{2}}Q(\mu, \sqrt{\mu}f) - \mu^{-\frac{1}{2}}Q(\sqrt{\mu}f, \mu) = \mu^{-\frac{1}{2}}Q(\sqrt{\mu}f, \sqrt{\mu}f), \\
f|_{t=0} = f_0.
\end{cases}$$

(5)

We will use throughout the paper the notations as follows.

Define by $\mathcal{L}$ the linearized collision operator, that is

$$\mathcal{L} f = \mu^{-1/2}Q(\mu, \sqrt{\mu}f) + \mu^{-1/2}Q(\sqrt{\mu}f, \mu),$$

and denote

$$\Gamma(g, h) = \mu^{-1/2}Q(\sqrt{\mu}g, \sqrt{\mu}h).$$

Furthermore denote by $P$ the linearized Boltzmann operator:

$$P = \partial_t + v \cdot \partial_x - \mathcal{L}.$$

So the Cauchy problem (5) for the perturbation $f$ can be rewritten as

$$P f = \Gamma(f, f), \quad f|_{t=0} = f_0.$$

(6)
For the homogeneous case, the regularity theory is well developed. In this case the loss of derivatives doesn’t occur so that it behaves essentially as a fractional heat equation. Strongly related to this regularization effect is another well known Landau equation, taking into account all grazing collisions. The mathematical treatment of the regularization properties goes back to Desvillettes [14, 15] for a one-dimensional model of the Boltzmann equation. Later on, Alexandre-Desvillettes-Villani-Wennberg [1] establish the optimal regularity estimate in $v$ for the collision operator after the earlier work of Lions [25], and since then there have been extensive works on the regularity, in a wide variety of different settings of solutions to the homogeneous Boltzmann equation without angular cut-off; see for instance [1, 6, 7, 13, 12, 14, 15, 16, 17, 24, 20, 18, 29, 28, 26, 27, 30] and references therein. We refer to the very recent work of Barbaroux-Hundertmark-Ried-Vugalter [9], where they prove any weak solution of the fully non-linear homogeneous Boltzmann equation for Maxwellian molecules belongs to the Gevrey class at positive time, and the Gevrey index therein is optimal.

Compared with the homogeneous case, much less is known for the Gevrey regularization properties for spatially inhomogeneous Boltzmann equation. The main difficulty lies in the degeneracy in spatially variable since diffusion only occurs in the velocity. Note the smoothing effect in the $C^\infty$ setting is obtained by the series of works [2, 3, 4]. The analysis of the Gevrey regularizing properties of spatially inhomogeneous kinetic equations with respect to both position and velocity variables is more complicated. Up to now, there are few results expect for a very simplified model of the linearized inhomogeneous non-cutoff Boltzmann equation, and the Gevrey smoothing effect has been proven recently by [8]. In the setting of Gelfand-Shilov that is finer than Gevrey class, there are only few works on the Kac equation that is a one dimensional Boltzmann model equation, where the proof relies on the explicit computation of the collision operator in terms of the Hermite basis. So the argument therein only works for one-dimensional case and can’t apply to the three-dimensional Boltzmann equation. In this work we will investigate the Gelfand-Shilov regularizing property for three-dimensional Boltzmann equation.

**Definition 1.1.** Let $\mu + \nu \geq 1$, $\mu, \nu > 0$ and we denote Gelfand-Shilov space by $G^\mu_\nu$ the space of all the $C^\infty$ functions $u(x,v)$ satisfying that a constant $C$ exists such that

$$\forall \ |\alpha| + |\beta| \geq 0, \ |v^\beta \partial_{x}^\alpha u| \leq C^{|\alpha|+|\beta|+1}(|\alpha|!|\beta|!)^{\mu}.$$  

The Gevrey class $G^\mu(\mathbb{R}^n)$ is the set of smooth function fulfilling

$$\forall \ |\alpha| \geq 0, \ |\partial_{x}^\alpha u| \leq C^{\alpha+1}(\alpha!)^\mu.$$ 

Before stating our main Theorem we provide a presentation of the triple norm $\| \cdot \|$ defined by

$$\| u \|^2 \sim \| (a^{1/2})^w u \|_{L^2(\mathbb{R}^n)}^2$$

where $(a^{1/2})^w$ is the weyl quantization with symbol $a^{1/2}$. The definition of $a^{1/2}$ as well as some basic facts on the symbolic calculus will find in [11].

We now review some related works on the Gelfand-Shilov regularizing effect. The first work goes back to Lerner-Morimoto-Starov-Xu [22] where they considered the radially symmetric solutions to the spatially homogeneous non-cutoff Boltzmann equation and proved that these solution enjoy the same Gelfand-Shilov regularizing effect as the Cauchy problem defined by the evolution equation associated to
a fractional harmonic oscillator. Later on Lerner-Morimoto-Starov-Xu [23] considered the spatially inhomogeneous non cutoff Kac equation and showed that the Cauchy problem for the fluctuation around the Maxwellian distribution admits $G^{(1+2s)/2s}$ Gelfand-Shilov regularity properties with respect to the velocity variable and $G^{(1+2s)/2s}$ Gevrey regularizing properties with respect to the spatial variable; see [10] for the improvement to the critical Besov space.

Denote by $H^k(T_x^3 \times R^3)$ the classical Sobolev space. For any $\ell \in R$ define

$$H^\ell(T_x^3 \times R^3) = \left\{ \langle \nu \rangle^\ell u \in H^k(T_x^3 \times R^3) \right\},$$

where and throughout the paper we use the notation

$$\langle \cdot \rangle = \left(1 + |\cdot|^2\right)^{1/2}.$$

We will state the main Theorem as follows:

**Theorem 1.2.** Assume that the cross-section satisfies (3) and (4) with $0 < s < \frac{1}{2}$ and $\gamma \geq 0$. Let $f \in L^\infty((0, +\infty]; H^2(T_x^3 \times R_x^3))$ be any solution to (6) such that

$$\sup_{t \geq 0} \|f(t)\|_{H^2(T_x^3 \times R_x^3)} + \sum_{|\alpha| + |\beta| \leq 2} \left( \int_0^{+\infty} \| (a^{1/2})^\nu \langle v \rangle^2 \partial_x^\alpha \partial_t^\beta f(t) \|_{L^2(T_x^3 \times R_x^3)}^2 dt \right)^{1/2} \leq \varepsilon_0$$

(7)

for some constant $\varepsilon_0 > 0$. Suppose $\varepsilon_0$ is small enough. Then there exists a constant $C_0 \geq 1$ depending only on $s$, $\gamma$ and $\varepsilon_0$ above such that for any multi-indices $\alpha$ and $\beta$ with $|\alpha| + |\beta| + |\xi| \geq 0$,

$$\sup_{t > 0} \phi(t)^{\frac{1+2s}{2s}} |\alpha| + |\beta| + |\xi| \| v^\alpha \partial_x^\beta \partial_t^\gamma f(t) \|_{L^2(T_x^3 \times R_x^3)}$$

$$\leq C_0 |\alpha| + |\beta| + |\xi| + 1 \left( (|\alpha| + |\beta| + |\gamma|)! \right)^{\frac{1+2s}{2s}}$$

with $\phi(t) \overset{def}{=} \min \{t, 1\}$.

We refer to the works [5, 19] for the global existence in the weighted Sobolev space $H^\ell$.

**Notations.** If no confusion occurs we will use $L^2$ to stand for the function space $L^2(T_x^3 \times R_x^3)$, and use $\| \cdot \|_{L^2}$ and $(\cdot, \cdot)_{L^2}$ to denote the norm and inner product of $L^2 = L^2(T_x^3 \times R_x^3)$. We will also use the notations $\| \cdot \|_{L^2(R_x^3)}$ and $(\cdot, \cdot)_{L^2(R_x^3)}$ when the variables are specified. Similarly for $H^k$ and $H^\ell$.

Given two operators $T_1$ and $T_2$ we denote by $[T_1, T_2]$ the commutator between $T_1$ and $T_2$, that is,

$$[T_1, T_2] = T_1 T_2 - T_2 T_1.$$

Throughout this paper, we use $C$ to denote a general positive constant which may different from line to line. And we sometimes write $A \lesssim B$ instead of $A \leq CB$. Likewise, $A \sim B$ means that $C_1 B \leq A \leq C_2 B$ with absolute constants $C_1$, $C_2$.

2. **Estimate on commutators.** In this section we will treat the commutators between the weight function and the collision operators.
Proposition 1. Assume that 0 < s < 1 and 0 ≤ γ ≤ 1. Let \( \Lambda_\sigma = \left( 1 + \sigma |v|^2 \right)^{1/2} \) with 0 < \( \sigma \leq 1 \) and any \( \ell \geq 2 \), we have for some suitable functions \( f, g, h \)

\[
\left| \left( \Lambda^{-1}_\sigma \langle v \rangle^\ell \Gamma(f, g) - \Gamma(\Lambda^{-1}_\sigma \langle v \rangle^\ell f, g) \right) \right|_{L^2(\mathbb{R}^3_0)} \lesssim \ell \| \langle v \rangle^{s+\gamma/2} f \|_{L^2(\mathbb{R}^3_0)} \| \Lambda^{-1}_\sigma \langle v \rangle^{\ell + \gamma/2} g \|_{L^2(\mathbb{R}^3_0)} \| h \|.
\]

A direct result from the above Lemma gives

\[
\left| \left( \Lambda^{-1}_\sigma \langle v \rangle^\ell \Gamma(f, f) - \Gamma(\Lambda^{-1}_\sigma \langle v \rangle^\ell f, \Lambda^{-1}_\sigma \langle v \rangle^\ell f) \right) \right|_{L^2(\mathbb{R}^3_0 \times \mathbb{T}^3_0)} \lesssim \ell \| \langle v \rangle^{s+\gamma/2} f \|_{H^1(\mathbb{R}^3_0 \times \mathbb{T}^3_0)} \| (\Lambda^{-1}_\sigma \langle v \rangle^\ell f) \|_{L^2(\mathbb{R}^3_0 \times \mathbb{T}^3_0)} \| \Lambda^{-1}_\sigma \langle v \rangle^{\ell + \gamma/2} f \|_{L^2(\mathbb{R}^3_0 \times \mathbb{T}^3_0)}.
\]

Before the proof of Proposition 1, we give a decomposition to \( \mu_* \). Which can be find the proof in [5].

Lemma 2.1 (see [5]). For any integer \( k \geq 2 \), we have

\[
\mu_*^{1/2} = \mu(\mu, \mu_*) + \sum_{i=1}^{k} \alpha_i^* \mu_* \mu^{b_i^*},
\]

where \( \mu(\mu, \mu_*) = \left( \mu^{a^*} - \mu^{a^2} \right)^k \sum_{i=1}^{k+2} \alpha_i^* \mu_* \mu^{b_i^*} \) and \( \alpha_i^* \) are real numbers, the other exponents are strictly positive, at the exception of \( b_1^* = 0 \), and with \( b_i^* > a_i^* \) otherwise.

Proof of Proposition 1. The idea of proof is similar to Proposition 3.13 in [5]. We here just give a brief proof, for more details, one can refer [5].

The operator \( \Gamma \) can be written as:

\[
(\Gamma(f, g), h) = (\Gamma_\mu(f, g), h) + (\Gamma_{\text{rest}}(f, g), h),
\]

where

\[
(\Gamma_\mu(f, g), h) = \iiint b \left( \frac{v - v_*}{|v - v_*|} \cdot \sigma \right) \Phi_\gamma(v - v_*) \mu(v, v_*) (f_* g' - f_* g) h \, dv \, dv_* \, d\sigma,
\]

and \( (\Gamma_{\text{rest}}(f, g), h) \) is a finite combination of terms:

\[
(\Gamma_{\text{mod}, i}(f, g), h) = \iiint b \left( \frac{v - v_*}{|v - v_*|} \cdot \sigma \right) \Phi_\gamma(v - v_*) (f_* g' - f_* g) \mu_*^{c_i} \mu^{d_i} h \, dv \, dv_* \, d\sigma
\]

\[
= (Q(\mu^{c_i} f, \mu^{c_i} g), \mu^{d_i} h),
\]

with \( d_i, c_i > 0 \) for \( i = 1, 2, 3 \).

In what follows, we shall deal with the two operator \( \Gamma_\mu, \Gamma_{\text{rest}} \). For the commutator \( \Gamma_\mu \), note that \( \mu(v, v_*) = (\mu^c - \mu_*^c)^k \mu_*^a \) for \( k \geq 4 \) and some constants \( a, c > 0 \). It is easy to see

\[
\left| \left( \left( \Lambda^{-1}_\sigma \langle v \rangle^\ell \Gamma_\mu(f, g) - \Gamma_\mu(\Lambda^{-1}_\sigma \langle v \rangle^\ell f, g) \right), h \right) \right|_{L^2}
\]

\[
= \iiint b \Phi_\gamma(v - v_*) \mu(v, v_*) f_* g' \left( \Lambda^{-1}_\sigma \langle v \rangle^\ell - \left( \Lambda^{-1}_\sigma \langle v \rangle^\ell \right)_* \right) h \, dv \, dv_* \, d\sigma.
\]
noticing that
\[
\begin{align*}
\int\int\int b\Phi_\gamma(v - v_*)\mu(v, v_*)f'_1g' \left( \Lambda_{\sigma}^{-1} \langle v \rangle^\ell - \left( \Lambda_{\sigma}^{-1} \langle v \rangle^\ell \right)' \right) & h d\nu d\sigma, \\
= \int\int\int b\Phi_\gamma(v - v_*)\mu(v, v_*)f'_1g' \left( \Lambda_{\sigma}^{-1} \langle v \rangle^\ell - \left( \Lambda_{\sigma}^{-1} \langle v \rangle^\ell \right)' \right) (h - h') d\nu d\sigma, \\
& + \int\int\int b\Phi_\gamma(v - v_*)(\mu(v', v_*) - \mu(v, v_*))f'_1g' \left( \left( \Lambda_{\sigma}^{-1} \langle v \rangle^\ell \right)' - \left( \Lambda_{\sigma}^{-1} \langle v \rangle^\ell \right)' \right) h d\nu d\sigma, \\
& + \int\int\int b\Phi_\gamma(v - v_*)\mu(v, v_*)f'_1g' \left( \left( \Lambda_{\sigma}^{-1} \langle v \rangle^\ell \right)' - \left( \Lambda_{\sigma}^{-1} \langle v \rangle^\ell \right)' \right) h d\nu d\sigma, \\
:= A + B + C,
\end{align*}
\]
then, we shall estimate the three terms respectively, for the first term A, considering that for \( v_\tau = v' + \tau(v - v') \) for \( \tau \in [0, 1] \),
\[
\frac{\langle v \rangle}{\langle v_\tau \rangle} \lesssim \langle v_\tau \rangle \lesssim \langle v \rangle^2,
\]
which implies by direct calculation,
\[
\frac{d}{dv} \left( \Lambda_{\sigma}^{-1} \langle v \rangle^\ell \right) \lesssim \ell \Lambda_{\sigma}^{-1} \langle v \rangle^{\ell - 1},
\]
following from Proposition 3.13 in [5] gives
\[
|A| \lesssim \ell \| \langle v \rangle^{s + \gamma/2} f \|_{L^2(\mathbb{R}^3)} \| \Lambda_{\sigma}^{-1} \langle v \rangle^{\ell + \gamma/2} g \|_{L^2(\mathbb{R}^3)} \| h \|.
\]
In order to estimate B, we note a similar formula as the proof in [5],
\[
(\Phi_\gamma \tilde{\Phi}_\gamma^{-1} \{ (\mu^c - \mu_*^c)^k \mu^o_* - (\mu^c - \mu_*^c)^k \mu_*^o \})^2 \lesssim \tilde{\Phi}_\gamma (\mu_*^o + \mu^o_* \mu^c) \min \{ |v - v_*|^2 \theta^2, 1 \},
\]
and the fact \( \langle v \rangle \langle v'_* \rangle^{-2} \lesssim \langle v_\tau \rangle \lesssim \langle v \rangle \langle v'_* \rangle^2 \) in view of \( \langle v_\tau - v'_* \rangle \sim \langle v - v'_* \rangle \), which implies
\[
\left| \Lambda_{\sigma}^{-1} \langle v \rangle^\ell - \left( \Lambda_{\sigma}^{-1} \langle v \rangle^\ell \right)' \right| \lesssim \ell \left| \Lambda_{\sigma}^{-1} \langle v \rangle^{\ell - 1} \min \{ \langle v_* \rangle^{2(\ell - 1) + 2\ell}, \langle v'_* \rangle^{2(\ell - 1) + 2\ell} \} \right| \min \{ |v - v_*|, \theta, \langle v \rangle \}.
\]
By means of Cauchy-Schwarz inequality, one has
\[
|B| \lesssim \ell \left( B_1^{1/2} D_1^{1/2} + B_2^{1/2} D_2^{1/2} \right),
\]
with
\[
B_1 = \left( \int\int\int b \min \{ |v - v_*|^2 \theta^2, \langle v \rangle^2 \} \Phi_\gamma \mu_*^{o/4} |f_*|^2 |\Lambda_{\sigma}^{-1} \langle v \rangle^{\ell - 1} g|^2 d\sigma d\nu d\nu \right),
\]
\[
B_2 = \left( \int\int\int b \min \{ |v - v_*|^2 \theta^2, \langle v \rangle^2 \} \Phi_\gamma \mu_*^{o/4} |f_*|^2 |\Lambda_{\sigma}^{-1} \langle v \rangle^{\ell - 1} g|^2 d\sigma d\nu d\nu \right),
\]
and
\[
D_1 = \left( \int\int\int b \min \{ |v - v_*|^2 \theta^2, 1 \} |v - v_*|^7 \mu_*^{o/4} |h|^2 d\sigma d\nu d\nu \right),
\]
\[
D_2 = \left( \int\int\int b \min \{ |v - v_*|^2 \theta^2, 1 \} |v - v'_*|^7 \mu_*^{o/4} |h|^2 d\sigma d\nu d\nu \right).
\]
It is easy to compute
\[
\int\int\int b \min \{ |v - v_*|^2 \theta^2, \langle v \rangle^2 \} d\sigma \lesssim |v - v_*|^{2s},
\]
which gives
\[ D_1 \lesssim \|h\|_{L^{2+\gamma/2}}^2, \quad D_2 \lesssim \|h\|_{L^{2+\gamma/2}}^2. \]
In view of \(|v - v_*| \sim |v - v'_*|\), then direct calculation implies
\[ B_1 \lesssim \iint \left( \int b \min\{\langle v \rangle^2, \langle v \rangle^2 \} d\sigma \right) \mu^{a/8} |f_{x*}|^2 \langle v \rangle^\gamma |\Lambda_{\sigma}^{-1} \langle v \rangle^{\ell-1} g|^2 dv_{x*} \]
\[ \lesssim \|f\|_{L^2}^2 \|\Lambda_{\sigma}^{-1} \langle v \rangle^{\ell-1} g\|_{L^{2+\gamma/2}}^2. \]
By employing a similar argument, we have
\[ B_2 \lesssim \|f\|_{L^2}^2 \|\Lambda_{\sigma}^{-1} \langle v \rangle^{\ell-1} g\|_{L^{2+\gamma/2}}^2. \]
Thus, collecting the above estimates leads to
\[ |B| \lesssim \ell \|f\|_{L^2} \|\Lambda_{\sigma}^{-1} \langle v \rangle^{\ell-1} g\|_{L^{2+\gamma/2}}^2 \|h\|_{L^{2+\gamma/2}}^2. \]
For the last term \(C\), it can be estimated with a similar process in [5], simple calculation gives
\[ |C| \lesssim \ell \|\mu^{a/4} f\|_{L^2} \|\Lambda_{\sigma}^{-1} \langle v \rangle^{\ell+\gamma/2} g\|_{L^2(\mathbb{R}^3)} \|\langle v \rangle^{\gamma/2} h\|_{L^2}. \]
Together the estimate of \(A, B\) with \(C\), we have
\[ \left| \left( \Lambda_{\sigma}^{-1} \langle v \rangle^{\ell} \Gamma_\mu(f, g) - \Gamma_\mu(\Lambda_{\sigma}^{-1} \langle v \rangle^{\ell} f, g), h \right) \right|_{L^2} \]
\[ = \ell \|\langle v \rangle^{\gamma/2} f\|_{L^2(\mathbb{R}^3)} \|\Lambda_{\sigma}^{-1} \langle v \rangle^{\ell+\gamma/2} g\|_{L^2(\mathbb{R}^3)} \|h\|_{L^2, \gamma/2}^2 \] (8)
It remains to consider the commutator for \(\Gamma_{mod}\). Noticing that
\[ \left( \Lambda_{\sigma}^{-1} \langle v \rangle^{\ell} \Gamma_{mod}(f, g) - \Gamma_{mod}(\Lambda_{\sigma}^{-1} \langle v \rangle^{\ell} f, g), h \right) \]
\[ = \left( \Lambda_{\sigma}^{-1} \langle v \rangle^{\ell} Q(F, G) - Q(F, \Lambda_{\sigma}^{-1} \langle v \rangle^{\ell} G), H \right), \]
with \(F = \mu^c f, G = \mu^c g, H = \mu^d h\) for some \(c, d > 0\).
Consider \(F, G, H\) contain Gaussians and the same type estimate for \(Q_{\varepsilon}\) follows from (2.1.15) and (2.1.18) in [2]. It is easy to see
\[ \left( \Lambda_{\sigma}^{-1} \langle v \rangle^{\ell} Q_{\varepsilon}(F, G) - Q_{\varepsilon}(F, \Lambda_{\sigma}^{-1} \langle v \rangle^{\ell} G), H \right) \]
\[ = \iint b \Phi_c(v - v_*) F_* G' \left( \Lambda_{\sigma}^{-1} \langle v \rangle^{\ell} - \left( \Lambda_{\sigma}^{-1} \langle v \rangle^{\ell} \right)^\ell \right) H dv_{x*} d\sigma \]
\[ = \iint b \Phi_c(v - v_*) F_* G \left( \left( \Lambda_{\sigma}^{-1} \langle v \rangle^{\ell} \right)' - \Lambda_{\sigma}^{-1} \langle v \rangle^{\ell} \right) H dv_{x*} d\sigma \]
\[ + \iint b \Phi_c(v - v_*) F_* G \left( \left( \Lambda_{\sigma}^{-1} \langle v \rangle^{\ell} \right)' - \Lambda_{\sigma}^{-1} \langle v \rangle^{\ell} \right) (H' - H) dv_{x*} d\sigma \]
\[ =: J_1 + J_2. \]
Since it follows from the Taylor’s expansion of second order that
\[ \left| \left( \Lambda_{\sigma}^{-1} \langle v \rangle^{\ell} \right)' - \Lambda_{\sigma}^{-1} \langle v \rangle^{\ell} - (v' - v) \cdot \nabla \left( \Lambda_{\sigma}^{-1} \langle v \rangle^{\ell} \right) \right| \lesssim \|v - v_*\|^2 \theta^2 \langle v_* \rangle^{\ell-2} \Lambda_{\sigma}^{-1} \langle v \rangle^{\ell-2}, \]
and since a part coming from the first order term vanishes by means of the spherical symmetry, then we have

\[
|J_1| \lesssim \|v-v_s\|^{\gamma+1} \left( \int b\theta^2 \, d\sigma \right) \|\langle v_s \rangle^\ell F_s\| \|\Lambda^{-1}_\sigma \langle v \rangle^{\ell-2} G\| H \, d\nu d\sigma
\]

\[
\lesssim \left( \int [\langle v \rangle^{\ell-2} G] \left( \frac{\|\langle v \rangle^\ell F_s\|}{|v-v_s|^{3/2-\varepsilon}} \right)^2 \, d\nu d\sigma \right)^{1/2}
\times \left( \int [\langle v \rangle^{\ell-2} G] \left( \frac{\|\langle v \rangle^\ell F_s\|}{|v-v_s|^{2(\gamma+1)-3/2+\varepsilon}} \right)^2 \, d\nu d\sigma \right)^{1/2}
\]

\[
:= J_{1,1} J_{1,2}^{1/2}.
\]

By virtue of the Cauchy-Schwarz inequality again, we obtain

\[
|J_{1,1}| \lesssim \|\langle v \rangle^\ell F\| \|\Lambda^{-1}_\sigma \langle v \rangle^{\ell-2} G\|.
\]

Using the Hardy-Littlewood-Sobolov inequality, Sobolev embedding theorem, we get

\[
|J_{1,2}| \lesssim \|\Lambda^{-1}_\sigma \langle v \rangle^{\ell-2} G\| H^2_{L_p}
\]

\[
\lesssim \|\langle v \rangle^{\ell-2} G\| H^2_{\dot{H}^{2-1+\varepsilon/2}},
\]

with \( \frac{1}{p} = \frac{3}{2} - \frac{-2\gamma-7/2+\varepsilon}{3} (< 1) \).

Thus, we have

\[
|J_1| \lesssim \|\langle v \rangle^\ell F\| \|\Lambda^{-1}_\sigma \langle v \rangle^{\ell-2} G\| H^s.
\]

On the other hand, using Cauchy-Schwarz inequality shows that

\[
|J_2| \lesssim \left( \int \int \int b \Phi \, F_s \, G^2 \left( \langle v_s \rangle^\ell - \Lambda^{-1}_\sigma \langle v \rangle^{\ell-2} \right) \, d\nu d\sigma \right)^{1/2}
\]

\[
\times \left( \int \int \int b \Phi \, \langle v_s \rangle^\ell \, F_s (H' - H)^2 \, d\nu d\sigma \right)^{1/2}
\]

\[
:= J_{2,1} J_{2,2}^{1/2}.
\]

Simple calculation implies

\[
\left| \left( \Lambda^{-1}_\sigma \langle v \rangle^{\ell} \right)' - \Lambda^{-1}_\sigma \langle v \rangle^\ell \right| \leq |v-v_s| \theta \Lambda^{-1}_\sigma \langle v \rangle^{\ell-1} v_s^\ell,
\]

then, one has

\[
|J_{21}| \lesssim \left( \int [\langle v \rangle^{\ell-2} G] \left( \frac{\|\langle v \rangle^\ell F_s\|}{|v-v_s|^{3/2-\varepsilon}} \right) \, d\nu d\sigma \right)
\]

\[
\lesssim \|\langle v \rangle^\ell F\| \|\Lambda^{-1}_\sigma \langle v \rangle^{\ell-2} G\|.
\]

The estimates of term \( J_{22} \) follows from Lemma 2.7 in [4], that is

\[
|J_{22}| \lesssim \|\langle v \rangle^\ell F\| \|H\|_{\dot{H}^s}^2.
\]
Collecting the above estimates, we get
\[ \left| \left( A_{\sigma}^{-1} \langle v \rangle^\ell Q_v (F, G) - Q_v (F, A_{\sigma}^{-1} \langle v \rangle^\ell G), H \right) \right| \lesssim \| \langle v \rangle^\ell F \| \| A_{\sigma}^{-1} \langle v \rangle^{\ell-2} G \| H \|_{H^r} + \| A_{\sigma}^{-1} \langle v \rangle^{\ell-2} F \| \| \langle v \rangle^\ell G \| H \|_{H^r}, \]
Combining (8) and (9), the proof is thus completed.

3. **Gelfand-Shilov smoothing effect.** In this part we will show the main result on the Gelfand-Shilov smoothing effect.

**Theorem 3.1.** Under the same assumption as in Theorem 1.2, there exists a constant \( C > 0 \), which depends only on \( s, \gamma \) and the constant \( \varepsilon_0 \) in (7), such that
\[
\forall \ |\xi| \geq 0, \quad \phi (t) \| \langle v \rangle^{\ell+2(|\alpha|+|\xi|)} \| \langle v \rangle^\ell \partial_\xi^\alpha f (t) \|_{L^2} \leq C |\alpha|+|\xi|+1 (|\alpha| + |\xi|)! \| \langle v \rangle^{\ell+2(|\alpha|+|\xi|)} \| \langle v \rangle^\ell \partial_\xi^\alpha f (t) \|_{L^2},
\]
with \( \phi (t) = \min \{ t, 1 \} \).

We will use induction to prove the above theorem. Before our proof, we will recall a crucial Proposition (one can find the proof in [11]):

**Proposition 2** (Weighted estimate [11]). Assume that the cross-section satisfies (3) and (4) with \( 0 < s < 1 \) and \( \gamma \geq 0 \). Then there exists a constant \( C > 1 \) such that for any given \( r \geq 1 \) and any function \( u \) satisfying that \( Pu \in L^2 (0,1] \times T^3_x \times \mathbb{R}^3 \) and
\[
t^{-r} u(t) \in L^\infty (0,1] \times L^2 \quad \text{and} \quad t^r (a_1^{1/2} w) u(t) \in L^2 (0,1] \times T^3_x \times \mathbb{R}^3,
\]
we have, for any \( 0 < t \leq 1 \),
\[
t^2 r \| u(t) \|^2_{L^2} + \int_0^1 t^{2r} \| \langle v \rangle^{\ell+2} u(t) \|^2_{L^2} dt + \int_0^1 t^{2r} \| (a_1^{1/2} w) u(t) \|^2_{L^2} dt \leq C \int_0^1 t^{2r} \| (Pu, u) \|_{L^2} dt + C \int_0^1 t^{2r-1} \| u \|^2_{L^2} dt.
\]
And for any function \( u \) satisfying \( Pu \in L^2 (0,1] \times T^3_x \times \mathbb{R}^3 \) and that
\[
u(t) \in L^\infty (1,\infty] \times L^2 \quad \text{and} \quad (a_1^{1/2} w) u(t) \in L^2 (0,\infty] \times T^3_x \times \mathbb{R}^3,
\]
we have, for any \( t \geq 1 \),
\[
\| u(t) \|^2_{L^2} + \int_1^\infty \| \langle v \rangle^{\ell+2} u(t) \|^2_{L^2} dt + \int_1^\infty (a_1^{1/2} w) u(t) \|^2_{L^2} dt \leq \| u(1) \|^2_{L^2} + C \int_0^1 \| (Pu, u) \|_{L^2} dt + C \int_1^\infty \| u \|^2_{L^2} dt.
\]

Based on the above results, we will prove the following Proposition:

**Proposition 3.** Let \( f \in L^\infty (0,\infty] ; H^2_x \) be a solution to the Cauchy problem (6) satisfying the condition (7). Assume there exists a constant \( C_\ast > 0 \), which depends only on \( s, \gamma \) and the constant \( \varepsilon_0 \) in but independent of \( m, \) such that for any multi-index \( \zeta \) with \( 2 \leq |\zeta| \leq 3,
\[
\sup_{t > 0} \phi (t)^{\ell \langle |\zeta| - 2 \rangle} \| \langle v \rangle^{\zeta} f (t) \|_{L^2} + \left( \int_{0}^{t} \phi (t)^{\ell \langle |\zeta| - 2 \rangle} (a_1^{1/2} w) \langle v \rangle^{\zeta} f (t) \|^2_{L^2} dt \right)^{1/2} \leq C_\ast.
\]
If for any multi-index $\zeta$ with $4 \leq |\zeta| \leq m - 1$, there exists a constant $C_0 > C_*$ which depends on $s, \gamma$ such that
\[
\sup_{t > 0} \phi(t)^{\kappa(|\zeta| - 2)} \|\langle v \rangle^{\zeta} f(t)\|_{L^2} + \int_0^{+\infty} \phi(t)^{\kappa(|\zeta| - 2)} \|a^{1/2} w\langle v \rangle^{\zeta} f(t)\|_{L^2}^2 dt \leq C_0|\zeta|^{-3} \left((|\zeta| - 4)!\right)^{\frac{1+2s}{2s}}.
\]
with $\kappa = (1 + 2s)/2s$ and any integer $m \geq 5$. Then the estimate (12) still holds for any $\zeta$ with $|\zeta| = m$.

**Proof:** We first consider the case for $t \in [0, 1]$.

For $0 < \sigma \ll 1$, define $f_\sigma$ as:
\[
f_\sigma = \Lambda_\sigma^{-1} f, \quad \Lambda_\sigma = \left(1 + \sigma |v|^2\right)^{1/2}.
\]
Noticing from the assumption (12), one has
\[
\begin{align*}
\kappa(m-2) - 1/2 \langle v \rangle^m f_\sigma &\in L^\infty ([0, 1]; L^2), \\
\kappa(m-2) a^{1/2} w\langle v \rangle^m f_\sigma &\in L^2 ([0, 1] \times T^3 \times \mathbb{R}^3).
\end{align*}
\]
Apply Proposition 2 for $u = \langle v \rangle^m f_\sigma$, and $r = \kappa(m - 2)$, one can get for $0 < t < 1$,
\[
\int_0^1 t^{2\kappa(m-2)} \|\langle v \rangle^m f_\sigma\|_{L^2}^2 dt + \int_0^1 t^{2\kappa(m-2)} \|\langle v \rangle^{s+\frac{1}{2}}\langle v \rangle^m f_\sigma\|_{L^2}^2 dt
\]
\[
\leq C \int_0^1 t^{2\kappa(m-2)} \|\langle P \langle v \rangle^m f_\sigma, \langle v \rangle^m f_\sigma \rangle_{L^2}\| dt + Cm \int_0^1 t^{2\kappa(m-2) - 1} \|\langle v \rangle^m f_\sigma\|_{L^2}^2 dt
\]
\[
:= A_1 + A_2.
\]
Recall $f_\sigma$ is given by (13). We will first show that, for any $\varepsilon > 0$,
\[
A_1 = C \int_0^1 t^{2\kappa(m-2)} \|\langle P \langle v \rangle^m f_\sigma, \langle v \rangle^m f_\sigma \rangle_{L^2}\| dt
\]
\[
\leq (\varepsilon + \varepsilon_0 C) \int_0^1 t^{2\kappa(m-2)} \|a^{1/2} w\langle v \rangle^m f_\sigma\|_{L^2}^2 dt + C\varepsilon C_0^{2(m-4)} [(m - 4)!]^{\frac{1+2s}{2s}}.
\]
(14)

Noticing that $f$ solves the equation $Pf = \Gamma(f, f)$ and
\[
P\langle v \rangle^m f_\sigma = \Lambda_\sigma^{-1} \langle v \rangle^m Pf + \left[ P, \Lambda_\sigma^{-1} \langle v \rangle^m \right] f.
\]
Thus, one has
\[
\int_0^1 t^{2\kappa(m-2)} \|\langle P \langle v \rangle^m f_\sigma, \langle v \rangle^m f_\sigma \rangle_{L^2}\| dt
\]
\[
= \int_0^1 t^{2\kappa(m-2)} \|\left(\Lambda_\sigma^{-1} \langle v \rangle^m Pf + \left[ P, \Lambda_\sigma^{-1} \langle v \rangle^m \right] f, \langle v \rangle^m f_\sigma \rangle_{L^2}\| dt
\]
\[
:= I + II,
\]
where
\[
I = \int_0^1 t^{2\kappa(m-2)} \|\left(\Lambda_\sigma^{-1} \langle v \rangle^m Pf, \langle v \rangle^m f_\sigma \rangle_{L^2}\| dt,
\]
and
\[
II = \int_0^1 t^{2\kappa(m-2)} \|\left[ P, \Lambda_\sigma^{-1} \langle v \rangle^m \right] f, \langle v \rangle^m f_\sigma \rangle_{L^2}\| dt.
\]
\[ II = \int_0^1 t^{2\kappa(m-2)} \left| \left[ [P, \Lambda^{-1}_\sigma \langle v \rangle^m], \langle v \rangle^m f_\sigma \right]_{L^2} \right| \, dt. \]

We now estimate the two terms respectively, for the first term, direct calculation leads to
\[ I \leq \int_0^1 t^{2\kappa(m-2)} \left| (\Lambda^{-1}_\sigma \langle v \rangle^m \Gamma(f,f) - (f, \Lambda^{-1}_\sigma \langle v \rangle^m f), \langle v \rangle^m f_\sigma \right]_{L^2} \, dt \]
\[ + \int_0^1 t^{2\kappa(m-2)} \left| (\Gamma(f, \Lambda^{-1}_\sigma \langle v \rangle^m f), \langle v \rangle^m f_\sigma \right]_{L^2} \, dt \]
\[ =: I_1 + I_2, \]
by employing Proposition 1, we have
\[ I_1 \lesssim m \int_0^1 t^{2\kappa(m-2)} \| \langle v \rangle^{s+\gamma/2} f \|_{L^2} \| (a^{1/2})^w \langle v \rangle^m f_\sigma \|_{L^2} \| \langle v \rangle^{m+\gamma/2} f_\sigma \|_{L^2^2} \, dt \]
\[ \leq \varepsilon_0 \int_0^1 t^{2\kappa(m-2)} \| (a^{1/2})^w \langle v \rangle^m f_\sigma \|_{L^2^2} \, dt \]
\[ + \varepsilon_0 m^2 \int_0^1 t^{2\kappa(m-2)} \| \langle v \rangle^{m+\gamma/2} f_\sigma \|_{L^2^2} \, dt. \]

As for the last term above we use the interpolation inequality to get, for any \( \varepsilon > 0, \)
\[ m^2 \int_0^1 t^{2\kappa(m-2) - 1} \| \langle v \rangle^{m+\gamma/2} f_\sigma \|_{L^2^2} \, dt \]
\[ \leq \varepsilon \int_0^1 t^{2\kappa(m-2)} \| (a^{1/2})^w \langle v \rangle^m f_\sigma \|_{L^2^2} \, dt \]
\[ + C_{\varepsilon} m^{s+2s} \int_0^1 t^{2\kappa(m-3)} \| \langle v \rangle^{m-1+s+\gamma/2} f_\sigma \|_{L^2^2} \, dt. \] (16)

Then, we can get
\[ I_1 \leq (\varepsilon + \varepsilon_0) \int_0^1 t^{2\kappa(m-2)} \| (a^{1/2})^w \langle v \rangle^m f_\sigma \|_{L^2^2} \, dt \]
\[ + C_{\varepsilon} m^{s+2s} \int_0^1 t^{2\kappa(m-3)} \| \langle v \rangle^{m-1+s+\gamma/2} f_\sigma \|_{L^2^2} \, dt. \]

For the second term \( I_2, \) direct calculation implies
\[ I_2 \lesssim \int_0^1 t^{2\kappa(m-2)} \| f \|_{H^2} \| (a^{1/2})^w \langle v \rangle^m f_\sigma \|_{L^2^2} \, dt \]
\[ \leq \varepsilon_0 \int_0^1 t^{2\kappa(m-2)} \| (a^{1/2})^w \langle v \rangle^m f_\sigma \|_{L^2^2} \, dt. \]

Collecting the above two estimates, we have
\[ I \leq (\varepsilon + 2\varepsilon_0) \int_0^1 t^{2\kappa(m-2)} \| (a^{1/2})^w \langle v \rangle^m f_\sigma \|_{L^2^2} \, dt \]
\[ + C_{\varepsilon} m^{s+2s} \int_0^1 t^{2\kappa(m-3)} \| \langle v \rangle^{m-1+s+\gamma/2} f_\sigma \|_{L^2^2} \, dt. \] (17)
Noticing the fact \( [v \cdot \partial_x, A^{-1}_\sigma \langle v \rangle^m] = 0 \), then
\[
II = \int_0^1 t^{2\kappa(m-2)} \left| \left( [P, A^{-1}_\sigma \langle v \rangle^m]f, \langle v \rangle^m f_\sigma \right)_{L^2} \right| dt
= \int_0^1 t^{2\kappa(m-2)} \left| \left( [\mathcal{L}, A^{-1}_\sigma \langle v \rangle^m]f, \langle v \rangle^m f_\sigma \right)_{L^2} \right| dt
= II_1 + II_2,
\]
with
\[
II_1 = \int_0^1 t^{2\kappa(m-2)} \left| \left( [\mathcal{L}_1, A^{-1}_\sigma \langle v \rangle^m]f, \langle v \rangle^m f_\sigma \right)_{L^2} \right| dt,
\]
\[
II_2 = \int_0^1 t^{2\kappa(m-2)} \left| \left( [\mathcal{L}_2, A^{-1}_\sigma \langle v \rangle^m]f, \langle v \rangle^m f_\sigma \right)_{L^2} \right| dt,
\]
Similar to the estimates of \( I_1 \), it is easy to get
\[
II_1 \lesssim m \int_0^1 t^{2\kappa(m-2)} \left\| (a^{1/2})^w \langle v \rangle^m f_\sigma \right\|_{L^2} \left\| \langle v \rangle^{m+\gamma/2} f_\sigma \right\|_{L^2} dt.
\]
As to the term \( II_2 \), one has
\[
II_2 \lesssim m \int_0^1 t^{2\kappa(m-2)} \left\| (a^{1/2})^w \langle v \rangle^m f_\sigma \right\|_{L^2} \left\| \langle v \rangle^{m+\gamma/2} f_\sigma \right\|_{L^2} dt.
\]
We shall deal with the three terms respectively, for the term \( I_{21} \), by means of Proposition 1, we get
\[
II_{21} \lesssim m \int_0^1 t^{2\kappa(m-2)} \left\| (a^{1/2})^w \langle v \rangle^m f_\sigma \right\|_{L^2} \left\| \langle v \rangle^{s+\gamma/2} f_\sigma \right\|_{L^2} dt
\]
\[
\lesssim \epsilon \int_0^1 t^{2\kappa(m-2)} \left\| (a^{1/2})^w \langle v \rangle^m f_\sigma \right\|_{L^2}^2 dt + C_\epsilon m^2 \int_0^1 t^{2\kappa(m-2)} \left\| \langle v \rangle^{s+\gamma/2} f_\sigma \right\|_{L^2}^2 dt,
\]
and using a similar interpolation inequality as (16) to the last term of the above inequality, we have
\[
II_{21} \leq 2\epsilon \int_0^1 t^{2\kappa(m-2)} \left\| (a^{1/2})^w \langle v \rangle^m f_\sigma \right\|_{L^2}^2 dt
+ C_\epsilon m^{\frac{1+\gamma}{2}} \int_0^1 t^{2\kappa(m-3)} \left\| \langle v \rangle^{m-1+\frac{s+\gamma/2}{1+\gamma}} f_\sigma \right\|_{L^2}^2 dt.
\]
Hence, combining (17), (18), (15) with the assumption (12) gives (14).

For the second term $I_{22}$, interpolation inequality implies that

\[
I_{22} \lesssim \int_0^1 t^{2k(m-2)} \left| \left( \Gamma(f, \Lambda_{\sigma}^{-1} (v)^m \mu^{1/2}), \langle v \rangle^m f^{\sigma} \right)_{L^2} \right| dt
\]

\[
\lesssim \int_0^1 t^{2k(m-2)} \|(a^{1/2}) w \langle v \rangle^m f^{\sigma}\|_{L^2} \|f\|_{L^2} dt
\]

\[
\lesssim \varepsilon \int_0^1 t^{2k(m-2)} \|(a^{1/2}) w \langle v \rangle^m f^{\sigma}\|_{L^2}^2 dt + C_{\varepsilon} m^2 \int_0^1 t^{2k(m-3)} \|f\|_{L^2}^2 dt,
\]

using interpolation inequality to the last term, we can get

\[
I_{22} \leq 2\varepsilon \int_0^1 t^{2k(m-2)} \|(a^{1/2}) w \langle v \rangle^m f^{\sigma}\|_{L^2}^2 dt
\]

\[
+ C_{\varepsilon} m^{1+\frac{2s}{k}} \int_0^1 t^{2k(m-3)} \|f\|_{L^2}^2 dt.
\]

For the term $I_{23}$, the estimates is similar to $I_{22}$, direct calculation implies

\[
I_{23} \lesssim \int_0^1 t^{2k(m-2)} \left| \left( \Gamma(\Lambda_{\sigma}^{-1} (v)^m f, \mu^{1/2}), \langle v \rangle^m f^{\sigma} \right)_{L^2} \right| dt
\]

\[
\lesssim \int_0^1 t^{2k(m-2)} \|(a^{1/2}) w \langle v \rangle^m f^{\sigma}\|_{L^2} \|\Lambda_{\sigma}^{-1} (v)^m f\|_{L^2} dt
\]

\[
\lesssim \varepsilon \int_0^1 t^{2k(m-2)} \|(a^{1/2}) w \langle v \rangle^m f^{\sigma}\|_{L^2}^2 dt + C_{\varepsilon} \int_0^1 t^{2k(m-2)} \|\langle v \rangle^m f^{\sigma}\|_{L^2}^2 dt,
\]

note the last term of the above inequality can be estimate as (16), then Thus, we have

\[
I_{23} \leq 2\varepsilon \int_0^1 t^{2k(m-2)} \|(a^{1/2}) w \langle v \rangle^m f^{\sigma}\|_{L^2}^2 dt
\]

\[
+ C_{\varepsilon} m^{1+\frac{2s}{k}} \int_0^1 t^{2k(m-3)} \|\langle v \rangle^m f^{\sigma}\|_{L^2}^2 dt.
\]

Collecting the above estimates, we have

\[
II = \int_0^1 t^{2k(m-2)} \left| \left( \left\langle P, \Lambda_{\sigma}^{-1} (v)^m f, \Lambda_{\sigma}^{-1} (v)^m f^{\sigma} \right\rangle L^2 \right| \right| dt
\]

\[
\leq (7\varepsilon + 2\varepsilon_0) \int_0^1 t^{2k(m-2)} \|(a^{1/2}) w \langle v \rangle^m f^{\sigma}\|_{L^2}^2 dt
\]

\[
+ C_{\varepsilon} m^{1+\frac{2s}{k}} \int_0^1 t^{2k(m-3)} \|\langle v \rangle^m f^{\sigma}\|_{L^2}^2 dt.
\]

Hence, combining (17), (18), (15) with the assumption (12) gives (14).

For the term $A_2$, by means of interpolation inequality, for $0 < s \leq \frac{1}{2}$, one has

\[
A_2 = C m \int_0^1 t^{2k(m-3)-1} \|\langle v \rangle^m f^{\sigma}\|_{L^2}^2 dt
\]

\[
\leq \varepsilon \int_0^1 t^{2k(m-3)} \|(a^{1/2}) w \langle v \rangle^m f^{\sigma}\|_{L^2}^2 dt
\]

\[
+ C_{\varepsilon} m^{1+\frac{2s}{k}} \int_0^1 t^{2k(m-4)} \|(a^{1/2}) w \langle v \rangle^m f^{\sigma}\|_{L^2}^2 dt.
\]
Together this with assumption (11) gives
\[
A_2 \leq (\varepsilon + \varepsilon_0 C) \int_0^1 t^{2\kappa(m-2)} \| (a^{1/2})^w \langle v \rangle^m f_\sigma \|^2_{L^2} dt + C_\varepsilon C_0^{2(m-4)} [(m - 4)]^{\frac{1+2\varepsilon}{s}}.
\]  
(19)
Combining (14) and (19), we get
\[
t^{2\kappa(m-2)} \| \langle v \rangle^m f_\sigma \|^2_{L^2} + \int_0^1 t^{2\kappa(m-2)} \| (a^{1/2})^w \langle v \rangle^m f_\sigma \|^2_{L^2} \leq \mathcal{M},
\]
where
\[
\mathcal{M} = (\varepsilon + \varepsilon_0 C) \int_0^1 t^{2\kappa(m-2)} \| (a^{1/2})^w \langle v \rangle^m f_\sigma \|^2_{L^2} dt + C_\varepsilon C_0^{2(m-4)} [(m - 4)]^{\frac{1+2\varepsilon}{s}}.
\]
By taking \( \varepsilon_0 \) and \( \varepsilon \) sufficiently small, we conclude for any \( 0 < t \leq 1 \),
\[
t^{2\kappa(m-2)} \| \langle v \rangle^m f_\sigma \|^2_{L^2} + \frac{1}{2} \int_0^t t^{2\kappa(m-2)} \| (a^{1/2})^w \langle v \rangle^m f_\sigma \|^2_{L^2} \leq C C_0^{2(m-4)} [(m - 4)]^{\frac{1+2\varepsilon}{s}}.
\]
Since the constants \( C \) are independent of \( \sigma \), then letting \( \sigma \to 0 \) implies that
\[
t^{\kappa(m-2)} \langle v \rangle^m f \in L^\infty ([0, 1]; L^2),
\]
and
\[
t^{\kappa(m-2)} (a^{1/2})^w \langle v \rangle^m f \in L^2 ([0, 1] \times \mathbb{T}^3 \times \mathbb{R}^3),
\]
and thus, we get
\[
\sup_{0 < t \leq 1} t^{\kappa(m-2)} \| \langle v \rangle^m f_\sigma \|^2_{L^2} + \left( \int_0^1 t^{2\kappa(m-2)} \| (a^{1/2})^w \langle v \rangle^m f_\sigma \|^2_{L^2} \right)^{1/2} \leq C C_0^{m-2} [(m - 4)]^{\frac{1+2\varepsilon}{s}}.
\]  
(20)
Hence, we have proven (12) holds true in Proposition 3 for any \( |\beta| = m \) when \( t \in [0, 1] \).
For the case of \( t > 1 \), the proof is similar as in the case of \( 0 < t \leq 1 \), and here we will just show the key points.
By applying the estimate (10) for \( u = \langle v \rangle^m f_\sigma \), and \( r = \kappa(m - 2) \), we get for any \( t > 1 \),
\[
\| \langle v \rangle^m f_\sigma \|^2_{L^2} + \int_1^{+\infty} \| (a^{1/2})^w \langle v \rangle^m f_\sigma \|^2_{L^2} dt \\
\leq \| \langle v \rangle^m f_\sigma (1) \|^2_{L^2} + C \int_1^{+\infty} | \langle P \langle v \rangle^m f_\sigma , \langle v \rangle^m f_\sigma \rangle_{L^2} | dt \\
+ Cm \int_1^{+\infty} \| \langle v \rangle^m f_\sigma \|^2_{L^2} dt.
\]  
(21)
The first term on the right-hand side is easy to estimate by using (20)
\[
\| \langle v \rangle^m f_\sigma (1) \|^2_{L^2} \leq C_{0}^{m-3} [(m - 4)]^{\frac{1+2\varepsilon}{s}}.
\]
By means of the same argument to \( 0 < t \leq 1 \), one can deal with the second terms in a similar argument, and direct calculation shows that
\[
\int_1^{+\infty} | \langle P \langle v \rangle^m f_\sigma , \langle v \rangle^m f_\sigma \rangle_{L^2} | dt \\
\leq (\varepsilon + \varepsilon_0 C) \int_0^1 t^{2\kappa(m-2)} \| (a^{1/2})^w \langle v \rangle^m f_\sigma \|^2_{L^2} dt + C_\varepsilon C_0^{2(m-4)} [(m - 4)]^{\frac{1+2\varepsilon}{s}}.
\]  
with \( \varepsilon \) sufficiently small.
To the last term in (21), by using interpolation equality again and the assumption (12), diren calculation leads to
\[
\int_1^{+\infty} \| \langle v \rangle^m f_\sigma \|^2_{L^2} dt \leq \varepsilon \int_1^{+\infty} \| (a^{1/2})^w \langle v \rangle^m f_\sigma \|^2_{L^2} dt + C \varepsilon \int_1^{+\infty} \| \langle v \rangle^{m-1} f_\sigma \|^2_{L^2} dt \\
\leq \varepsilon \int_1^{+\infty} \| (a^{1/2})^w \langle v \rangle^m f_\sigma \|^2_{L^2} dt + C_0^{2(m-4)} \| (m-4)! \|^{1+2s}.
\]
Thus, by taking \( \varepsilon_0 \) and \( \varepsilon \) sufficiently small, combining the above estimates to get, for any \( t > 1 \),
\[
\| \langle v \rangle^m f_\sigma \|_{L^2} + \left( \int_1^{+\infty} \| (a^{1/2})^w \langle v \rangle^m f_\sigma \|^2_{L^2} dt \right)^{1/2} \leq C\varepsilon_0^{m-3} \| (m-4)! \|^{1+2s}. \quad (22)
\]
Collecting (20) and (22) yield the Proposition 3. The proof is thus completed. \( \square \)

3.1. Proof of theorem 3.1. Proceeding in an analogous way to the proof of Proposition 3, we have

**Proposition 4.** Assume that the cross-section satisfies (3) and (4) with \( 0 < s < \frac{1}{2} \) and \( \gamma \geq 0 \). Let \( f \in L^\infty([0, +\infty]; H^2_2(\mathbb{T}^3 \times \mathbb{R}^3)) \) be a solution to (6). Then there exists a constant \( C > 0 \), depending only on \( s, \gamma \) and the constant \( \varepsilon_0 \) in (7), such that for any multi-index \( \zeta \) with \( 2 \leq |\zeta| \leq 4 \), we have
\[
\sup_{t > 0} \phi(t)^{\kappa(|\zeta|-2)} \| \langle v \rangle^{|\zeta|} f(t) \|_{L^2} + \left( \int_0^{+\infty} \phi(t)^{\kappa(|\zeta|-2)} \| (a^{1/2})^w \langle v \rangle^{|\zeta|} f(t) \|^2_{L^2} dt \right)^{1/2} \leq C^*.
\]

**Proof.** The proof is similar to Proposition 3. In fact, under the assumption (7), by using Proposition 2 and the argument is similar to the process in Proposition 3, one has
\[
\sup_{t > 0} \| \langle v \rangle^{|\zeta|} f(t) \|_{L^2} + \sum_{|\zeta|=2} \left( \int_0^{+\infty} \| (a^{1/2})^w \langle v \rangle^{|\zeta|} f(t) \|^2_{L^2} dt \right)^{1/2} \leq C \quad (23)
\]
where the constant \( C \) depends only on \( s, \gamma \) and \( \varepsilon_0 \).

In fact, for \( 0 < t \leq 1 \), as in Proposition 3, we can also obtain by using (23)
\[
\begin{aligned}
& t^{2\kappa} \| \langle v \rangle^3 f_\sigma \|^2_{L^2} + \int_0^1 t^{2\kappa} \| (a^{1/2})^w \langle v \rangle^3 f_\sigma \|^2_{L^2} dt \\
& \quad + \int_0^1 t^{2\kappa} \| (a^{1/2})^w \langle v \rangle^3 f_\sigma \|^2_{L^2} dt \\
& \leq C \int_0^1 t^{2\kappa} \left( P \langle v \rangle^3 f_\sigma, \langle v \rangle^3 f_\sigma \right)_{L^2} dt + C \int_0^1 t^{2\kappa-1} \| \langle v \rangle^3 f_\sigma \|^2_{L^2} dt \\
& \text{and along with the similar estimates in Proposition 3, one get}
\end{aligned}
\]
\[
\sup_{0 < t \leq 1} t^\kappa \| \langle v \rangle^3 f \|_{L^2} + \left( \int_0^1 t^{2\kappa} \| (a^{1/2})^w \langle v \rangle^3 f \|^2_{L^2} dt \right)^{1/2} \leq C.
\]
for some \( C > 0 \).

Then, repeating the argument for proving Proposition 3 we can verify directly that Proposition 4 holds for \(|\zeta| = 4\). We here omit the details of the proof. \( \square \)
Proposition 4 make sure that the assumption (11) in Proposition 3 is indeed true. Proposition 3 enables to use induction to obtain that for any $|\zeta| \geq 4$, we have
\[
\sup_{t>0} \phi(t)^{1+2c(|\zeta|-2)} \|v^c \partial_t^c f(t)\|_{L^2} + \left( \int_0^{+\infty} \phi(t)^{1+2c(|\zeta|-2)} \|v^c \partial_t^c f(t)\|_{L^2}^2 dt \right)^{\frac{1}{2}} \leq C_0^{c(|\zeta|-3)} (4)\|v^c \partial_t^c f(t)\|_{L^2}^{\frac{1+2c}{2c}}.
\]
As a result, for any $t > 0$ and any $|\zeta| \geq 0$ we have
\[
\sup_{t>0} \phi(t)^{1+2c(|\zeta|)} \|v^c \partial_t^c f(t)\|_{L^2} \leq C_0^{c(|\zeta|+1)} (4)\|v^c \partial_t^c f(t)\|_{L^2}^{\frac{1+2c}{2c}}.
\]
Recall $\phi(t) = \min \{ t, 1 \}$. Then for any $\alpha, \beta \in \mathbb{Z}_+^3$, direct calculation shows that
\[
\left( \phi(t)^{1+2c(|\zeta|+\beta)} \|v^c \partial_t^\alpha \partial_x^\beta f(t)\|_{L^2} \right)^2 \leq \phi(t)^{\frac{1+2c}{2c}(|\zeta|+\alpha+\beta)} \left( \|\partial_x^\alpha v^c \partial_x^\beta f(t)\|_{L^2}^2 + \|\partial_x^\alpha \partial_x^\beta f(t)\|_{L^2} \right)
\]
\[
\leq \phi(t)^{\frac{1+2c}{2c}(|\zeta|+\alpha+\beta)} \left( \frac{1}{2} \left\|v^c \partial_x^\alpha \partial_t^\beta f(t)\right\|^2_{L^2} + \|v^c \partial_t^\beta f(t)\|_{L^2} + \|v^c \partial_x^\alpha f(t)\|_{L^2} \right) \leq \frac{1}{2} \left( \phi(t)^{1+2c(|\zeta|+\alpha+\beta)} \left( \|v^c \partial_x^\alpha \partial_t^\beta f(t)\|_{L^2}^2 + \|v^c \partial_x^\alpha f(t)\|_{L^2} \right) \right)^2 + \left( C_0^{c(|\zeta|+1)\|\partial_x^\alpha f(t)\|_{L^2}} \right)^2 \left( C_0^{c(|\zeta|+1)\|\partial_t^\beta f(t)\|_{L^2}} \right)^2,
\]
which implies
\[
\phi(t)^{\frac{1+2c}{2c}(|\zeta|+\alpha+\beta)} \|v^c \partial_t^\alpha \partial_x^\beta f(t)\|_{L^2} \leq C^{c(|\zeta|+\alpha+\beta)} \left( \|v^c \partial_x^\alpha \partial_t^\beta f(t)\|_{L^2}^2 + \|v^c \partial_x^\alpha f(t)\|_{L^2} \right)
\]
Thus, we complete the proof of Theorem 3.1.

3.2. Proof of theorem 1.2. For any $\alpha, \beta, \zeta \in \mathbb{Z}_+^3$, we use Theorem 3.1 and the Theorem 3.1 in [11] as well as the fact that $(m+n)! \leq 2^{m+n} m! n!$ for any positive integers $m$ and $n$, we have
\[
\left( \phi(t)^{1+2c(|\zeta|+|\beta|)} \|v^c \partial_x^\alpha \partial_t^\beta \partial_x^\gamma f(t)\|_{L^2} \right)^2 \leq \left( \phi(t)^{1+2c(|\zeta|+|\beta|)} \left( \|v^c \partial_x^\alpha \partial_t^\beta \partial_x^\gamma f(t)\|_{L^2}^2 + \|v^c \partial_t^\beta \partial_x^\gamma f(t)\|_{L^2} \right) \right)^2 + \left( C_0^{c(|\zeta|+1)\|\partial_x^\alpha \partial_t^\beta \partial_x^\gamma f(t)\|_{L^2}} \right)^2 \left( C_0^{c(|\zeta|+1)\|\partial_t^\beta \partial_x^\gamma f(t)\|_{L^2}} \right)^2,
\]
where, we use the fact,
\[
\phi(t)^{1+2c(|\zeta|+|\beta|+|\gamma|)} \|v^c \partial_x^\alpha \partial_t^\beta \partial_x^\gamma f(t)\|_{L^2} \leq \left( \phi(t)^{1+2c(|\zeta|+|\beta|+|\gamma|)} \|v^c \partial_x^\alpha \partial_t^\beta \partial_x^\gamma f(t)\|_{L^2} \right)^2 + \left( C_0^{c(|\zeta|+1)\|\partial_x^\alpha \partial_t^\beta \partial_x^\gamma f(t)\|_{L^2}} \right)^2 \left( C_0^{c(|\zeta|+1)\|\partial_t^\beta \partial_x^\gamma f(t)\|_{L^2}} \right)^2
\]
\[
\leq C^{c(|\zeta|+2|\beta|+2|\gamma|)} \left( \|v^c \partial_x^\alpha \partial_t^\beta \partial_x^\gamma f(t)\|_{L^2}^2 + \|v^c \partial_t^\beta \partial_x^\gamma f(t)\|_{L^2} \right)
\]
\[
\leq C^{c(|\zeta|+2|\beta|+2|\gamma|)} \left( \|v^c \partial_x^\alpha \partial_t^\beta \partial_x^\gamma f(t)\|_{L^2}^2 + \|v^c \partial_t^\beta \partial_x^\gamma f(t)\|_{L^2} \right)
\]
\[
\leq C^{c(|\zeta|+2|\beta|+2|\gamma|)} \left( \|v^c \partial_x^\alpha \partial_t^\beta \partial_x^\gamma f(t)\|_{L^2}^2 + \|v^c \partial_t^\beta \partial_x^\gamma f(t)\|_{L^2} \right)
\]
\[
\leq C^{c(|\zeta|+2|\beta|+2|\gamma|+2)} \left( \|v^c \partial_x^\alpha \partial_t^\beta \partial_x^\gamma f(t)\|_{L^2}^2 + \|v^c \partial_t^\beta \partial_x^\gamma f(t)\|_{L^2} \right)
\]
Thus, we complete the proof of Theorem 1.2.

Acknowledgements. The work was supported by NSFC (Nos. 11871054, 11771342), Fok Ying Tung Education Foundation (151001) and the Natural Science Foundation of Hubei Province (2019CFA007).

REFERENCES

[1] R. Alexandre, L. Desvillettes, C. Villani and B. Wennberg, Entropy dissipation and long-range interactions, Arch. Ration. Mech. Anal., 152 (2000), 327–355.
[2] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu and T. Yang, Regularizing effect and local existence for the non-cutoff Boltzmann equation, Arch. Ration. Mech. Anal., 198 (2010), 39–123.
[3] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu and T. Yang, Uncertainty principle and kinetic equations, J. Funct. Anal., 255 (2008), 2013–2066.
[4] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu and T. Yang, The Boltzmann equation without angular cutoff in the whole space: Qualitative properties of solutions, Arch. Ration. Mech. Anal., 202 (2011), 599–661.
[5] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu and T. Yang, The Boltzmann equation without angular cutoff in the whole space: I, global existence for soft potential, J. Funct. Anal., 262 (2012), 915–1010.
[6] R. Alexandre and M. Safadi, Littlewood-Paley theory and regularity issues in Boltzmann homogeneous equations. I. Non-cutoff case and Maxwellian molecules, Math. Models Methods Appl. Sci., 15 (2005), 907–920.
[7] R. Alexandre and M. Safadi, Littlewood-Paley theory and regularity issues in Boltzmann homogeneous equations. II. Non cutoff case and non Maxwellian molecules, Discrete Contin. Dyn. Syst., 24 (2009), 1–11.
[8] R. Alexandre, F. Hérau and W.-X. Li, Global hypoelliptic and symbolic estimates for the linearized Boltzmann operator without angular cutoff, J. Math. Pures Appl. (9), 126 (2019), 1–71.
[9] J.-M. Barbaroux, D. Hundertmark, T. Ried and S. Vugalter, Gevrey smoothing for weak solutions of the fully nonlinear homogeneous Boltzmann and Kac equations without cutoff for Maxwellian molecules, Archive for Rational Mechanics and Analysis, 225 (2017), 601–661.
[10] H. Cao, H.-G. Li and J. Xu, Analytic smoothness effect of solutions for spatially homogeneous Landau equation, J. Differential Equations, 248 (2010), 77–94.
[11] L. Desvillettes, About the regularizing properties of the non-cut-off Kac equation, Comm. Math. Phys., 168 (1995), 417–440.
[12] L. Desvillettes, Regularization properties of the 2-dimensional non-radially symmetric non-cutoff spatially homogeneous Boltzmann equation for Maxwellian molecules, Transport Theory Statist. Phys., 26 (1997), 341–357.
[17] L. Desvillettes and B. Wennberg, Smoothness of the solution of the spatially homogeneous Boltzmann equation without cutoff, *Comm. Partial Differential Equations*, 29 (2004), 133–155.

[18] L. Glangetas, H.-G. Li and C.-J. Xu, Sharp regularity properties for the non-cutoff spatially homogeneous Boltzmann equation, *Kinet. Relat. Models*, 9 (2016), 299–371.

[19] P. T. Gressman and R. M. Strain, Global classical solutions of the Boltzmann equation without angular cut-off, *J. Amer. Math. Soc.*, 24 (2011), 771–847.

[20] Z. Huo, Y. Morimoto, S. Ukai and T. Yang, Regularity of solutions for spatially homogeneous Boltzmann equation without angular cutoff, *Kinet. Relat. Models*, 1 (2008), 453–489.

[21] N. Lerner, *Metrics on the Phase Space and Non-selfadjoint Pseudo-differential Operators*, Birkhäuser Verlag, Basel, 2010.

[22] N. Lerner, Y. Morimoto, K. Pravda-Starov and C.-J. Xu, Gelfand-Shilov smoothing properties of the radially symmetric spatially homogeneous Boltzmann equation without angular cutoff, *J. Differential Equations*, 256 (2014), 797–831.

[23] N. Lerner, Y. Morimoto, K. Pravda-Starov and C.-J. Xu, Gelfand-Shilov and Gevrey smoothing effect for the spatially inhomogeneous non-cutoff Kac equation, *J. Funct. Anal.*, 269 (2015), 459–535.

[24] H.-G. Li and C.-J. Xu, The Cauchy problem for the radially symmetric homogeneous Boltzmann equation with Shubin class initial datum and Gelfand-Shilov smoothing effect, *J. Differential Equations*, 263 (2017), 5120–5150.

[25] P.-L. Lions, Régularité et compacité pour des noyaux de collision de Boltzmann sans troncature angulaire, *C. R. Acad. Sci. Paris Sér. I Math.*, 326 (1998), 37–41.

[26] Y. Morimoto and S. Ukai, Gevrey smoothing effect of solutions for spatially homogeneous non-linear Boltzmann equation without angular cutoff, *J. Pseudo-Differ. Oper. Appl.*, 1 (2010), 139–159.

[27] Y. Morimoto and T. Yang, Smoothing effect of the homogeneous Boltzmann equation with measure valued initial datum, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 32 (2015), 429–442.

[28] Y. Morimoto, S. Ukai, C.-J. Xu and T. Yang, Regularity of solutions to the spatially homogeneous Boltzmann equation without angular cutoff, *Discrete Contin. Dyn. Syst.*, 24 (2009), 187–212.

[29] Y. Morimoto and C.-J. Xu, Ultra-analytic effect of Cauchy problem for a class of kinetic equations, *J. Differential Equations*, 247 (2009), 596–617.

[30] C. Villani, Regularity estimates via the entropy dissipation for the spatially homogeneous boltzmann equation without cut-off, *Rev. Mat. Iberoam.*, 15 (1999), 335–352.

Received January 2020; revised May 2020.

*E-mail address*: wei-xi.li@whu.edu.cn
*E-mail address*: lvqiaoliu@whu.edu.cn