A multi-state description of roughness effects in turbulent pipe flow

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\textbf{Abstract}. Despite persistent efforts over the last century, a theory for predicting the effects of surface roughness on the mean flow is still missing. Here, we extend a recently proposed multi-state theory to incorporate roughness effects. A formula for the roughness function is constructed by applying the Lie-group invariance theory, giving excellent agreement with the fully collapsed data of both the Nikuradze sand-coated and Princeton honed pipes. A major advantage of the theory is its ability to successfully describe the non-universality—either inflectional or monotonic variations—in a transitionally rough regime with a single sharpness parameter. This model then yields an analytic prediction for the friction factor and mean velocity profiles in rough pipes, agreeing with the empirical data. Consistent with classical understanding (e.g. Townsend’s similarity hypothesis), our results confirm the multi-layer theory of wall turbulence, regardless of smooth or rough surfaces.
1. Introduction

The roughness effect on wall turbulence is of growing significance as all practical surfaces appear rough with an increasing Reynolds number \(Re\) and the viscous scale shrinks.

Experimental research on rough walls dates from the 1850s [1]. The most complete measurements so far were carried out by Nikuradze in 1933 [2], in pipes roughened with sand-grains of well-defined sizes, measuring the friction factor \(f\) over a wide range of \(Re\), sand-grain heights \(h\) and pipe radii \(\delta\). It is recognized that the flow depends on the roughness only through \(h/\delta\), denoted as \(k_s\), and when \(k_s \ll 1\), the flow changes with \(Re\). At low \(Re\), the flow is hydraulically smooth (there is no obvious effect of the roughness). As \(Re\) increases, the flow becomes transitionally rough (\(f\) rises above the smooth value), and eventually becomes fully rough (where \(f\) is a function of only \(k_s\), but independent of \(Re\)). Nikuradze’s experiments have remained the benchmark.

Nikuradze [2] further suggested that the mean velocity distribution over a rough surface can still be expressed in a log-law as [1]

\[
U^+(y; k_s) = \frac{1}{\kappa} \ln y^+ + 5.1 + \frac{\Pi}{\kappa} W(y),
\]

where \(U\) is the mean velocity at a distance \(y\) from the wall, \(\Pi W(y)/\kappa\) is the ‘wake’ component, representing the effect of the outer-layer dynamics with the wake parameter \(\Pi\) and wake function \(W(y)\), and the superscript + denotes the normalization using the friction velocity \(U_f\) and viscous length \(v/U_f\). \(\Psi(k_s)\) is assumed to be dependent only on an equivalent sand-grain roughness, \(k_s\), and \(Re\) [1], and is a normalization factor of the roughness effect. An alternative way to express equation (1) is

\[
U^{+(\text{rough})}(y; k_s) = \frac{1}{\kappa} \ln y^+ + 5.1 + \frac{\Pi}{\kappa} W(y) - \Delta U^+,
\]

where \(\Delta U^+ \equiv \kappa^{-1} \ln \Psi(k_s)\) is also called Hama’s roughness function [3]. Integrating equation (2) over the entire flow domain yields a relation for the friction coefficient between a smooth and rough pipe at the same \(Re\):

\[
\sqrt{\frac{8}{f_{\text{rough}}}} = \sqrt{\frac{8}{f_{\text{smooth}}}} - \Delta U^+,
\]
which in fact implies a way to collapse all of the roughness data of $k$ and $Re$. Here, we show that this is indeed the case, if the formula for a smooth pipe is known. The new collapse is notably superior to recent proposals by Goldenfeld [4] and Tao [5].

Furthermore, we offer an explicit expression of the roughness function $\Delta U^+$ by introducing a three-state description for roughness induced turbulence, which is accomplished with a new order function involving an effective radius for the rough pipe. The Lie-group invariance argument yields a formula with all of the parameters determined except for the transition sharpness. The resulting roughness function, as well as the friction factor, are in very good agreement with the data of both the Nikuradze sand-coated and Princeton honed pipes. A renormalized expression for the mean velocity profile (MVP) yields a better description than equation (3) and agrees well with the empirical data. The errors of the predicted MVP are uniformly bounded within 1%—typical of experimental measurements.

More interestingly, the only arbitrary parameter in our model, namely the transition sharpness, describes the non-universality in a transitionally rough regime well. While surfaces such as the mesh roughness investigated by Perry and Abell [6], the tightly packed spheres by Ligrani and Moffat [7] and the honed pipe by Allen et al [8] and Shockling et al [9], reproduce the Nikuradze-like transition, other surface finishes and roughness geometries could produce different transitional behaviours.

For example, Colebrook and White [10] collected results for several industrial pipes and found more gradual transitions of friction factors with $Re$, distinct from those inflectional behaviours in Nikuradze’s data—$f$ falls below the fully rough value before rising to the very high-$Re$ values. Colebrook [11] hence proposed a monotonic transition function, which then served as a basis for the widely-used Moody chart [12]. More recently, the Princeton team reported mean flow measurements in a honed pipe [9] and a commercial steel pipe [13], with small roughness ($k_s \approx 1/8000$) at relatively high $Re$ (up to $21 \times 10^6$). While the friction factor in the former follows the inflectional transition, it is monotonic in the latter, albeit more abrupt than the Colebrook function indicates. Our transition sharpness yields a one-parameter family of roughness functions covering all of the above mentioned cases, hence it is the first systematic theory quantifying different roughness influences, as Marusic et al [14] have hoped for.

The paper is organized as follows. In section 2, a new data collapse is achieved for a Nikuradze sand-coated pipe and Princeton honed pipe using equation (3) and our theoretical smooth pipe results. Then, a multi-state functional form for the collapsed curves is developed (section 3.1), and is shown to characterize different behaviour in a transitionally rough regime (section 3.2), and to quantify the friction diagram and mean velocity profiles (section 3.3), in good agreement with experimental data. The success of a unified parametrization of roughness effects supports a complex-system view of turbulence, proposed in a general framework of the structural ensemble dynamics (SED) theory [15]. A summary is presented in section 4.

2. Data collapse for Nikuradze rough pipes and Princeton honed pipe

Goldenfeld’s data collapse [4] is obtained by the intriguing analogy between turbulence in a rough pipe and the critical phenomena in a ferromagnet, in which the friction factor $f$, Reynolds number $Re$ and roughness $k_s$ are analogous to the magnetization $M$, temperature $T$ and external magnetic field $H$, respectively. The basis of this analogy is two scaling laws in power form, i.e. the Blasius law ($f \sim Re^{-1/4}$ asymptotically as $k_s \to 0$) and Strickler’s law ($f \sim k_s^{1/3}$ as
Then an analogue of Widom scaling implies

\[ f \, Re^{1/4} = \mathcal{G} \left( k_s, Re^{3/4} \right), \tag{4} \]

leading to the collapse of Nikuradze’s data over a certain range. Tao [5] generates another collapse, with less scatter, by postulating a composite scaling variable,

\[ f \, Re = \mathcal{G} \left( Re^{3/4} + C_s \, Re^2 \, k_s^{2/3} \right), \tag{5} \]

which takes into account the transition from small to large roughness states, retaining the asymptotic power laws.

However, recent measurements, as in the Princeton Superpipe [16], have proved that the friction factor indeed follows a logarithmic law, while the power scaling invoked above is in fact an approximation, valid only for a limited range of \( Re \) and \( k_s \). The seemingly good collapse reveals its deficiency when more data, e.g. recent Princeton honed pipe data of small \( k_s \) and large \( Re \), are included, see figure 1. While one might attribute this deficiency to unknown properties of roughness surfaces in Princeton experiments, we will show that it is indeed due to an \( Re \)-effect, satisfactorily understood in a multi-layer logarithmic theory.

In fact, the first trial to collapse Nikuradze’s data was proposed by Prandtl in 1933 [17], as

\[ \frac{1}{\sqrt{f}} - 2 \log(2/k_s) = \mathcal{G}(k_s^+), \tag{6} \]

based on the assumption that the difference of mean momentum flux due to roughness, compared with the fully rough value, is a function of the roughness Reynolds number \( k_s^+ \). Note that the scaling law in a fully rough regime is written in logarithmic form as \( f \sim (2 \log(2/k_s) + A)^{-2} \), rather than Strickler’s law in power form. With the correct asymptotic laws taken into account, Prandtl’s scaling function is therefore able to collapse both Nikuradze’s and Princeton’s data in hydraulically smooth and fully rough regimes; however, it fails in a transitionally rough regime, as shown in figure 2.

Parenthetically, the well-known Colebrook formula [11] writes

\[ \frac{1}{\sqrt{f}} = -2 \log \left( \frac{k_s}{7.4 + 2.51 \, Re \sqrt{f}} \right), \tag{7} \]

which is actually a simple interpolation between two asymptotic states: the celebrated Prandtl friction law for a smooth pipe, \( (1/\sqrt{f})^{(\text{smooth})} = 2 \log(Re \sqrt{f}) - 0.8 \), as \( k_s \to 0 \); and Von Karman’s asymptotic law for a fully rough pipe, \( (1/\sqrt{f})^{(\text{rough})} = 2 \log(2/k_s) + 1.74 \), as \( Re \to \infty \), both in logarithmic form. But, as with Prandtl’s scaling function, the Colebrook formula cannot characterize the transitionally rough regime, especially inflectional transitions, as in Nikuradze’s data. The issue of non-universal transitions will be discussed in section 3.2.

Considering all of the above results, we postulate that the defect, with reference to the mean momentum flux in a smooth pipe, would be the proper quantity to characterize the effect of roughness on the friction factor. This idea is consistent with the offset function \( \Delta U^+ \) in equation (2). The reason why this collapse has been overlooked is the absence of an accurate theory for a smooth pipe over a wide range of \( Re \). Here we show that, with accurate predictions of the mean momentum flux for a smooth pipe by our newly proposed multi-layer theory, a universal scaling function, in terms of the momentum flux defect, indeed collapses both the Nikuradze and Princeton honed pipe data.
Figure 1. (a) Illustration of Goldenfeld’s (equation (4)) and (b) Tao’s (equation (5)) scaling function to collapse the Nikuradze sand-grain roughened pipe and Princeton honed pipe data. The failure to collapse Princeton data is due to incorrect power-law scaling at high \( Re \).

A recent theory [18] describes turbulent pipe flow as encompassing multiple layers—namely the sublayer, buffer layer, bulk layer and central core—each of which displays distinct local power law scaling in the mixing length as follows:

\[
\ell^+_M \propto \begin{cases} 
(y^+)^{3/2}, & y^+ \ll y^+_{sub}, \\
(y^+)^2, & y^+ \ll y^+_{buf}, \\
(y^+), & y^+_{buf} \ll y^+ \ll Re_\tau, \\
r^{-1/2}, & r \ll 1,
\end{cases}
\]
An earlier trial of data collapse by Prandtl (equation (6)). With only asymptotic states taken into account, a collapse is partly achieved for both Nikuradze’s and Princeton’s data, but only in hydraulically smooth and in fully rough regimes.

where the scaling exponents 3/2 and −1/2 can be derived using boundary constraints. The linear scaling represents the celebrated logarithmic law in the overlap region; the buffer layer scaling exponent 2 is empirically determined and will be discussed elsewhere. A bulk flow of \((1−r^5)\) is also newly discovered for the pipe flow. These layers are not only physically sound, but can also be systematically defined from energy dynamics, and hence they should not be treated as artificial fitting objects. The method developed by us to determine the other parameters (such as layer thickness) is also systematic, as presented in [18].

Finally, a generalized Lie-group symmetry analysis, involving both the mixing length \(\ell_M^+\) and its spatial gradient \(d\ell_M^+/dy^+\), connects all the layers to form the entire multi-layer profile for the mean flow. A combined analytical expression for the entire profile of the mixing length is

\[
\ell_M^+ = \rho \left( \frac{y^+}{y_{sub}} \right)^{3/2} \left( 1 + \left( \frac{y^+}{y_{sub}} \right)^4 \right)^{1/8} \left( 1 + \left( \frac{y^+}{y_{buf}} \right)^4 \right)^{-1/4} \frac{1 - r^5}{5(1 - r)} Z_{core} \left( 1 + \left( \frac{r}{r_{core}} \right)^{-2} \right)^{1/4},
\]

where \(r\) is the distance to the centre of the pipe; \(Z_{core} = (1 + r_{core}^2)^{1/4} \); \(y_{sub}^+, y_{buf}^+\) and \(r_{core}\) are characteristic thicknesses of the layers; \(\rho\) is a pre-factor determining the Karman constant. The analytic formula equation (8) fully determines the mean shear profile [19]

\[
S^+ = (-1 + \sqrt{4r\ell_M^+ + 1})/(2\ell_M^+),
\]

hence the MVP, the average momentum flux per unit mass and the friction factor can be calculated as

\[
U^+ = \int S^+ \, dy^+, \\
\overline{U^+} = 2 \int U^+ r \, dr, \\
f = 8/\overline{(U^+)^2},
\]

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respectively. This calculation yields an exact logarithmic form

\[ U^{(\text{smooth})} = \frac{1}{\kappa} \ln Re_t + B, \quad (11) \]

where the Karman constant \( \kappa = 0.45 \) and the additive constant \( B = 4.1 - 6.84\epsilon_\rho + 0.26\epsilon_c \), found from a thorough analysis of a wide range of numerical and experimental data, leading to a full determination (not free adjustment) of all parameters in equation (8) (see [18]). \( \epsilon_\rho \) and \( \epsilon_c \) are two small parameters characterizing the low \( Re \)-effect of the buffer layer and central core, respectively. A simple empirical model of linear dependence of \( \epsilon_\rho \) and \( \epsilon_c \) on \( Re_t \), for \( Re_t < 5000 \), is

\[ \epsilon_\rho = 2.5 \times 10^{-5} \times (5000 - Re_t), \]
\[ \epsilon_c = 1.5 \times 10^{-4} \times (5000 - Re_t), \]

giving a satisfactory description of the data for all \( Re \) (down to \( Re_t = 1000 \) in the current simulation data).

As stated, equation (11) is a consequence of the multi-layer structure. Now, we state a fundamental assumption in this work, namely, that these multi-layers are physically robust and ‘topologically’ stable to surface variations, such as a rough wall. Under this assumption, the functional form of equation (11) will remain valid in rough pipes, with possible modifications of the variable (\( Re_t \)) and/or parameters (\( \kappa, B \)). Vice versa, this functional form, once confirmed by a collapse of all rough pipe data, would be a reflection of the stable multi-layer structure under the action of wall roughness. The success presented below lays indeed a solid support for this assumption, and hence the general postulate of the SED theory [20].

To further specify roughness effects, we postulate that roughness influences the mean flow through a renormalization of the effective radius by \( \Psi(k^+_s) \) which yields an effective \( Re \), namely,

\[ Re^*_t \equiv \frac{u_t \delta^*}{v} = \frac{u_t \delta}{v \Psi(k^+_s)} = \frac{Re_t}{\Psi(k^+_s)}, \quad (12) \]

such that the roughness-induced mean momentum flux can be expressed as

\[ U^{(\text{rough})} = \frac{1}{\kappa} \ln (Re^*_t) + B, \quad (13) \]

leading to

\[ U^{(\text{rough})} = \frac{1}{\kappa} \ln \left( \frac{Re_t}{\Psi(k^+_s)} \right) + B = U^{(\text{smooth})} - \frac{1}{\kappa} \ln \Psi(k^+_s). \quad (14) \]

This implies that the difference

\[ \Delta U^+ \equiv U^{(\text{smooth})} - U^{(\text{rough})} = \frac{1}{\kappa} \ln \Psi(k^+_s) \quad (15) \]

is a universal function of \( k^+_s \). Combining with equation (11) yields an analytic expression for the friction factor:

\[ \sqrt{f_{\text{rough}}} = \frac{1}{\kappa} \ln Re_t + B - \frac{1}{\kappa} \ln \Psi(k^+_s), \quad (16) \]
which provides an alternative way to collapse friction factor data of various $k_s$ and $Re$. This relationship is more natural than previous attempts using the actual mean momentum flux or friction coefficient: the defect flux (equation (15)) reflects the flow’s response to an external action—the roughness—which appears to be a simple function. This argument is critical for us to develop a formalism, as described below.

Note that equation (15) is exactly the same as equation (3). However, the theory of mean velocity profiles, even for smooth pipes, has long been inaccurate and incomplete, due to uncertainty in constants ($\kappa$ and $B$), lack of an analytical expression for the wake function and so on. As a result, the roughness function has not been analysed systematically until now. With our theory of smooth pipe, we are able to perform this test; the collapse of friction factor data in both Nikuradze and Princeton honed pipes is shown in figure 3. This collapse is superior to any achieved previously (figures 1 to 3(b)), which confirms the above inferences and, in turn, verifies our multi-layer model in smooth pipes.

3. A multi-state model of roughness effect

The multi-state model postulates that roughness generated flows are characterized by a series of effective shear flows, which undergo multiple Lie-group invariant state transitions throughout the whole inflectional transition regime. Here, the scaling function $\Psi(k^+_s)$ plays a similar role as the order function, which fully characterizes the turbulent state at all $k_s$ and $Re$. Similar to the multi-layer theory for the mixing length of smooth pipe, the new multi-state model here is also quantified in terms of characteristic roughness sizes (compared to viscous scales), dilation-invariant scaling exponents and transitional sharpness values. We will show that these multi-state parameters offer an accurate description of the scaling function for inflectional transition regimes, and also describe the monotonic transition of Colebrook. Hence, the new model provides a full parametrization of all classic rough regimes.

3.1. Formulating the scaling function

The collapsed empirical scaling function $\Psi(k^+_s)$, which characterizes the roughness-induced wake flow interaction with wall-shear generated turbulent motions, demonstrates a clear multi-state picture, varying with the roughness Reynolds number $k^+_s$. Note that $k^+_s$ can also be regarded as a characteristic scale for roughness-induced wake flows. Specifically, three regimes are identified, consistent with traditional points of view:

(i) Hydraulically smooth regime ($k^+_i < k^+_1$): the roughness is buried within the viscous sublayer, thus the roughness effect is negligible with the mean flow unchanged. From figure 4, we speculate that $k^+_1 \approx 5.5$ (which is very close to the value 5 in textbooks; see [21]).

(ii) Transitionally rough regime ($k^+_1 < k^+_s < k^+_2$): as the viscous scale gets smaller, the roughness encroaches upon the buffer layer; roughness-induced wakes interfere with near-wall structures, modifying the viscous generation cycle and shaping a new scaling of the mean flow.

(iii) Fully rough regime ($k^+_i > k^+_2$): the roughness-induced motions take over, totally destroying the near wall viscous cycle; the mean flow is dominated by roughness, independent of

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Figure 3. Collapse of both Nikuradze and Princeton rough pipe data, with the flux defect as a function of roughness Reynolds number, using both (a) our smooth pipe model (equation (11)) and (b) Prandtl’s friction law. Prandtl’s result shows more scatter for large Re data in the transitionally rough regime (see insets). In addition, a multi-state formula, equation (19), is proposed to parametrize the scaling function.

From figure 4, we found $k^+ ≈ 150$, which happens to be the beginning of the quasi-balance region observed in the direct numerical simulation (DNS) data.

A theoretical model for these characteristic scales deserves a more detailed study.
Figure 4. The multi-state picture in rough pipes as unveiled by the scaling function.

As shown in figure 4, the scaling function $\Psi(k^+_i)$ reveals the following asymptotic forms in each regime:

\[
\Psi(k^+_i) \approx \begin{cases} 
(k^+_i)^0, & k^+_i < k^+_1, \\
\alpha(k^+_i)^n, & k^+_1 < k^+_i < k^+_2, \\
\beta(k^+_i)^1, & k^+_i > k^+_2, 
\end{cases}
\]  

(17)

where $n$ is the transitionally rough scaling exponent, while the (first) hydraulically smooth regime with an exponent zero and the fully rough regime with an exponent one, are well-known (also see [21]). Although we cannot fully specify $n$, it is possible to argue that $n > 1$. It is known that the second layer ($k^+_1 < k^+_i < k^+_2$) contains intense coherent vortices responsible for the self-sustaining cycle [1], while the third layer ($k^+_i > k^+_2$) is composed of more random motions. The reduction of flux is due to the interaction between the roughness-induced wakes and the fluctuation structures at $y^+ \approx k^+_i$. This interaction would be more ‘destructive’ if it involved coherent vortices instead of random motions; hence $\Delta \overline{U}^x(k^+_i) \propto \kappa^{-1} \ln k^+_i$ in the transitional rough regime would be larger than $\Delta \overline{U}^x(k^+_i) \propto \kappa^{-1} \ln k^+_i$ in the fully rough regime, i.e. $n > 1$.

The existence of the local scaling is a pre-requisite for defining dilation-group invariant states for roughness-induced turbulent motions, implying that the transition between different scaling states satisfies the generalized Lie-group invariance property that we discovered previously. Specifically, a universal ansatz, i.e. $(1 + y^p)^{n/p}$, is a good candidate for describing the transition from one scaling state to another, corresponding to a universal relation between the dilation invariants of the order function and its gradient [18]. The complete multi-state formula for the scaling function can then be written as

\[
\Psi(k^+_i; k^+_1, k^+_2, n, p) = \left(1 + \left(\frac{k^+_i}{k^+_1}\right)^p\right)^{n/p} \left(1 + \left(\frac{k^+_i}{k^+_2}\right)^p\right)^{(1-n)/p},
\]

(18)

where the transition sharpness $p$ characterizes the transitional behaviour between different states. Hence, $\alpha = (k^+_1)^{-n}$ and $\beta = (k^+_1)^{-n}(k^+_2)^{n-1}$.
Fixing the two characteristic roughness scales, \( k_1^+ = 5.5 \) and \( k_2^+ = 150 \), following previous studies [18], the only parameters left to determine are \( n \) and \( p \), which we choose by a best fit of the Nikuradze data: \( n = 1.3 \) and \( p = 4 \). Then, we have

\[
\Delta U^+ = \frac{1}{0.45} \Psi \left( k^+_s; 5.5, 150, 1.3, 4 \right),
\]

in good agreement with the empirical data, as shown in figure 3.

3.2. Parametrization of several classes of transitionally rough behaviours

A major attribute of this multi-state model is its ability to describe several classes of transitionally rough behaviours in a unified way, which has never been possible before. This is particularly significant since more recent efforts are concerned more with the different effects of the different types of surface roughness, in particular, the non-universal transition in the transitionally rough regime. In previous theories, either in Colebrook’s interpolation formula or Goldenfeld’s critical phenomena analogy, only two asymptotic states (hydraulically smooth and fully rough) are considered, which are incapable of describing the diversity in a transitionally rough regime. In contrast, our multi-state model offers a three-state description, which will be shown to be able to capture the subtle variations in the various surface conditions.

Our parametrization involves only one parameter, the transitional sharpness \( p \), which is introduced in the following natural way. Industrial roughness surfaces differ from Nikuradze’s rough pipe in that the roughness elements have a wider size distribution. Consequently, the transitional sharpness from one state to another is softened, since the transition would begin far before the characteristic roughness size reaches the critical roughness scale \( (k_1^+ \text{ or } k_2^+) \). A softer transition corresponds to a smaller \( p \). Indeed, we find that

\[
\Delta U^+ = \frac{1}{k} \ln \Psi \left( k^+_s; 5.5, 150, 1.3, p \right),
\]

(20)

describes well several industrial rough pipe scaling functions, when

\[
\begin{align*}
p^{(\text{sand-grain})} &= 4, \\
p^{(\text{honed})} &= 4, \\
p^{(\text{Princeton-commercial})} &= 3/2, \\
p^{(\text{tar-coated-iron})} &= 1, \\
p^{(\text{galvanized-iron})} &= 3/4, \\
p^{(\text{wrought-iron})} &= 3/4.
\end{align*}
\]

The agreement with empirical results is shown in figure 5.

As we can see, the value of transition sharpness \( p \) is closely associated with the distribution and/or geometry of surface roughness. Qualitatively speaking, the more irregular the surface, the smaller the transition sharpness. Take the Princeton honed and commercial pipes for example: the surface roughness of the latter exhibits more irregularity, as roughness amplitudes are obviously more disperse, even with bimodality; consequently, the latter displays...
Figure 5. An illustration of our uniform multi-layer formulation of non-universal transitional behaviours in the transitionally rough regime, where solid lines in different colors stand for different values of transition sharpness in equation (20).

A more gradual transition than the former, resulting in a smaller $p$ value. Thus, the uniform parametrization has set forth a basis for further quantitative research on the influence of the roughness of various materials, geometries and distributions. Note that non-universal transitions don’t seem to affect the critical scales $k^+$ and transitional scaling $n$, unveiling the robustness of multi-layer structures in rough wall turbulence, supporting our fundamental assumption.

The mechanism for the gradual build-up of the roughness effect over irregular surfaces, as suggested by Colebrook [11], is that they contain irregularities of widespread sizes, and that each element becomes active when it individually reaches a critical $Re$. The overall smooth evolution of the drag is the sum of these individual transitions. A mechanistic theory concerning more details of the roughness distribution function is yet to be developed; this model provides a guide for the development of such a theory.

3.3. Quantifying friction factor diagram and mean velocity profiles

With the scaling function formulated above, the friction factor in rough pipes can be calculated as

$$
\sqrt{\frac{8}{f(\text{rough})}} = \frac{1}{0.45} \ln Re_{\tau} + 4.1 + \epsilon - \frac{1}{0.45} \ln \Psi \left(k^+; 5.5, 150, 1.3, p \right),
$$

where $p$ accounts for the influence of different surface conditions, and $\epsilon$ equals $-0.66 \left(1 - Re_{\tau}/5000\right)$ for $Re_{\tau} < 5000$ and zero otherwise [18]. The predicted drag curves are shown in figure 6; agreement with the experimental data is excellent. Since the non-universal transitional behaviours, such as the widely reported inflectional transitions, are not included in the prevalent Moody chart, our new formula is significant for engineering applications.
A significant output of this theory is its ability to predict the MVP of rough pipes. Using the effective radius, $\delta^* \equiv \delta / \Psi(k^+_s)$, we define an effective Reynolds number as

$$Re^*_\tau \equiv \frac{U_\tau \delta^*}{v} = \frac{U_\tau \delta}{v} \cdot \frac{1}{\Psi(k^+_s)} = Re_\tau \left(1 + \left(\frac{k^+_s}{5.5}\right)^p\right)^{1.3/p} \left(1 + \left(\frac{k^+_s}{150}\right)^p\right)^{-0.3/p}.$$  \hspace{1cm} (22)

Thus, the MVP in a rough pipe can be obtained from that in a smooth pipe,

$$U^{*(\text{rough})}(Re_\tau; r) = U^{*(\text{smooth})}(Re^*_\tau; r).$$  \hspace{1cm} (23)

Combining with equations (8)–(10), we fully specify all rough pipe MVPs. In figure 7, the predicted MVP are compared to the Princeton honed pipe data ($p = 4$) for a wide range of $Re$, and the agreement is excellent with errors bounded mostly with 1%. This provides extra evidence for the validity of Townsend’s similarity hypothesis [22], which has been broadly upheld by recent experiments [23, 24].

4. Conclusions

A new data collapse is achieved for both Nikuradze sand-grain roughened pipes and the Princeton honed pipe (of small relative roughness at high $Re$), verifying the accuracy of our model for smooth pipe flows. We then bring forward a multi-state formula, parallel to the multi-layer model of the mixing length for smooth pipe, for the parametrization of the scaling function, which characterizes various uniformly transitional rough behaviours (i.e. the inflectional and monotonic transitions). This model quantifies the friction factor diagram, offering a potential choice to substitute the widely-used Moody chart in engineering. In addition, the predicted mean velocity profiles for rough pipes, through mapping from the smooth results, support again Townsend’s similarity hypothesis.

Finally, we comment on several other recent approaches to modeling friction coefficients in the presence of roughness. A recent work by Yang and Joseph [25] adopts an entirely
Figure 7. Prediction of the mean velocity profiles in outer flows of rough pipe, equation (23), compared with measurement in Princeton honed pipe (a), and their relative errors (b), which are mostly bounded within 1%.

empirical fitting procedure using fractional polynomials in the log-log coordinates; good parametrization of Nikuradze’s friction factor data is obtained, individually for each roughness size, with sufficient numbers of coefficients. The obtained formula unfortunately lacks physical interpretation. However, an apparent similarity might be drawn between our approach and Yang and Joseph [25] in representing the variations through several transitions. The difference is that in our theory, the layers (states) are motivated from the energy dynamics. For example, the three states in this description are hydraulically smooth, transitional rough and fully rough, for which \( k_1^+ \approx 5.5 \) and \( k_2^+ \approx 150 \) are characteristic scales—which can be determined from the energy budget equation of smooth wall turbulence. Gioia and Chakraborty [26] and Gioia et al [27] used a momentum transfer assumption to model the Reynolds stress from an energy
spectrum. In their calculation, an idea of multi-domain is also adopted, corresponding to the spectral ranges, such as the inertial and dissipation ranges. We believe that this association is simplistic; a quantitative theory is hard to accomplish. Indeed, only qualitative features of the friction factor diagram and mean velocity profiles are presented, with no hint of the connection between the friction curves at different roughness sizes. Nevertheless, the effort to connect the mean profiles with the energy spectra is important.

So far, our work has only accomplished a phenomenological (but quantitative) description based on symmetry arguments; a detailed mechanism on the interaction of roughness and wall-layer structures, with an additional prediction of the spectral energy distribution, will lead to a viable theory towards an a priori prediction of the roughness effect on mean flow quantities.

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