Dynamics of Schrödinger cat states

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Abstract. A review of the definition of Crystallized Schrödinger cat states and their statistical properties is presented. The associated Wigner and Husimi quasi-probability distribution functions are also calculated together with the second moment of the Husimi function. The Hamiltonian formulation of the Nöther’s theorem is used to get the linear time dependent invariants of until general quadratic Hamiltonians in the quadratures of the electromagnetic field. By means of these invariants, the evolution of Crystallized Schrödinger cats is studied, thus we have explicit time dependent expressions for the means and dispersions of the quadratures of the electromagnetic field together with the corresponding Wigner and Husimi functions. By means of these results, we establish some conditions on the parameters of the Crystallized cat states to have the description of non-classical light.

1. Introduction

The discovery of the laser in the early sixties gave birth to the research areas of quantum optics and quantum electronics. The last one implies investigations of device oriented problems, and very soon there was an explosion of works that made laser a common word for the people. The development of quantum optics was slower, and it was until the middle of the seventies and the beginning of eighties that the quantum manifestations of the radiation field become amenable to experimental verification [1, 2]. The coherent states of light introduced by Glauber are usually called classical states of light, they minimize the Heisenberg uncertainties relations of the quadratures of the electromagnetic field and have a Poisson distribution function of the number of photons [3, 4, 5].

Non-classical states of light such as Schrödinger cat states, even and odd coherent states, squeezed states, or correlated state have different statistical properties than the mentioned coherent states [6, 7]. There are several proposals to generate and detect non-classical states of light. For example, one can mention the evolution of a coherent state through an amplitude-dispersive medium [8], others use the evolution in a nonlinear birefringent optical model [9, 10], and dinamically through the evolution of an initial state formed by a two-level state direct product a standard coherent state in a optical cavity [11, 12, 13].

The motivation of this work is to study macroscopic superpositions of coherent states that carry an irreducible representation (irrep) of a finite group. These linear combinations of coherent states are written in terms of the simple characters of the elements of a given irreducible representation, and one of the purposes is to establish the conditions that they must satisfy to
have non-classical light, that is for example, when the superposition of states have a subpoissonian distribution function of the number of photons or they present the squeezing phenomena [14].

In general, if we have a group with \( n \) elements \( G : \{ g_1, \ldots, g_n \} \), and denote by \( \{ D(g_1), \ldots, D(g_n) \} \) a representation of the group. Then the trace of the representation \( \chi(g_s) = \sum_{\mu=1}^{l} D_{\mu \mu}(g_s) \) is called the characteristic of \( g_s \) in the representation \( D \), where \( l \) indicates the dimension of the representation [15]. Given two elements \( (g_i, g_j) \) of the group \( G \), they are conjugate one to another if there is an element \( g_k \) of the group such that \( g_k g_j g_k^{-1} = g_j \), then it is immediate that two elements that belong to the same class have the same characteristic. Therefore, to the set of numbers \( \{ \chi_1, \chi_2, \ldots, \chi_k \} \) with the label \( k \) denoting the number of classes of the group we are going to call it the character of the representation \( \{ D \} \).

It is well known that a finite group of \( n \) elements can be divided into classes and that the number of classes is equal to the number of irreducible representations, which are denoted by \( k \). The character associated to an irreducible representation \( \lambda \) is called simple, and thus it can also be denoted by \( \{ \chi_\rho \} \) with \( \rho = 1, 2, \ldots, k \). The simple characters satisfy the orthogonality relation

\[
\frac{1}{n} \sum_{g_s} \chi^{(\lambda)}(g_s) \chi^{*(\lambda')}(g_s) = \delta_{\lambda \lambda'},
\]

where the sum is over the elements of the group \( G \). The sum is equivalent to the sum over the classes, then if we denote by \( C_\rho \) the class and assume it has \( n_\rho \) elements then the orthogonality relation can be rewritten in the form

\[
\frac{1}{n} \sum_{\rho=1}^{k} n_\rho \chi^{(\lambda)}_\rho \chi^{*(\lambda')}_\rho = \delta_{\lambda \lambda'} .
\]

The crystallized states are defined using the simple characters of the groups [14], i.e.,

\[
\left| \psi^{(\lambda)} \right> = \sum_{s=1}^{n} \chi^{(\lambda)}(g_s) \left| \alpha_s \right>, \quad \left| \psi^{(\lambda')} \right> = \sum_{s=1}^{n} \chi^{(\lambda')} (g_s) \left| \alpha_s \right> ,
\]

where in the first row \( \chi^{(\lambda)}(g_s) \) represents the simple character for the irreducible representation \( \lambda \) corresponding to the \( g_s \) element of the group, \( \alpha_s = g_s \alpha \) defines the action of the element \( g_s \) over the complex parameter \( \alpha \) of a coherent state, and \( N_\lambda \) is the normalization constant. In the second row, we replace the sum of the elements by the sum of the classes of the group and the sum of elements in each class, and modify properly the notation of the simple characters of the group.

Notice that states that belong to different irreducible representations are, in general, not orthogonal due to the non-orthogonality property of the coherent states. As it can be seen immediately by considering the scalar product

\[
\left< \psi^{(\lambda')} | \psi^{(\lambda)} \right> = N_\lambda N_{\lambda'} \sum_{s,s'} \chi^{(\lambda)} (g_s) \chi^{*(\lambda')} (g_{s'}) E_{s's} ,
\]

where the quantity \( E_{s's} \equiv \exp \left( -\frac{1}{2} |\alpha_s|^2 - \frac{1}{2} |\alpha_{s'}|^2 + \overline{\alpha_{s'}} \alpha_s \right) \).

If we replace \( E_{s's} = \delta_{s,s'} \) in the expression (2), one can use the orthogonality properties for the simple characters and conclude that states \( \left| \psi^{(\lambda)} \right> \) belonging to different irreducible
Figure 1. The real (continuous lines) and imaginary (dashed lines) parts of the overlap $E_{sr} = \langle \alpha_s | \alpha_r \rangle$ between two coherent states are shown as a function of their phase difference $\Delta_{rs} = \varphi_r - \varphi_s$ of the complex parameters $\alpha_r$ and $\alpha_s$. At the left, we take $|\alpha_r| = |\alpha_s| = 2$ and to the right, $|\alpha_r| = |\alpha_s| = 4$.

representations are orthogonal. Then it is important to establish the conditions in the complex parameters of the coherent states to have an orthogonal basis. For the Cyclic and Dihedral groups which are the only ones considered in this work, the parameters of the coherent states $\alpha_s = \rho \exp(i \phi_s)$ have the same magnitude. Thus one has

$$E'_{s,s} = e^{-\rho^2 (1 - \cos \Delta_{s,s'} - i \sin \Delta_{s,s'})},$$

(3)

where we define the phase difference $\Delta_{s,s'} = \phi_s - \phi_{s'}$. Then it is straightforward that the scalar product is zero when $\Delta_{s,s'} \approx \pi$ and $\rho > 1$, this behaviour is shown in Fig. (1) for the parameters $|\alpha_{s'}| = |\alpha_s| = 2$ and 4. It can be seen that in a vicinity of $\Delta_{s,s'} = \pi$, the real and the imaginary parts of the overlap are very close to zero.

Cyclic groups

These groups are uniaxial because they have only a rotation axis as element of symmetry. For this reason, the cyclic groups are abelian, all the elements have the same period given by the order of the group, and they have only one dimensional irreducible representations. A physical realization of the cyclic groups appears in the symmetry transformations, rotations, of regular polygons. If we consider the cyclic group of $n$ elements denoted by $C_n$, a representation of the elements is obtained through the rotation matrices $R_k(\theta_k)$ with $\theta_k = \frac{2\pi k}{n}$, for $k = 0$ we have the identity matrix $I$. So the $n$ elements of the cyclic group can be given in the following form $C_n : \{ I, R_1, R_2, \cdots, R_{n-1} \}$, this two dimensional representation of the cyclic groups is not irreducible, it contains two times the irreps of the cyclic groups and so each element form a class and it has $k = n$ irreducible representations. All the elements can be generated by one element of the group, and so one can select the rotation: $R_1(\frac{2\pi}{n})$.

Then, it is immediate to determine the action on the complex parameter $\alpha$ of the coherent state, that is $\alpha_s = \mu_n^s \alpha$ with $\mu_n = \exp(2\pi i/n)$ and the simple characters of the $C_n$ group is given in Appendix A. Therefore the simple characters can be given by the expression $\chi^{(\lambda)}(g_s) = \mu_n^{(\lambda-1)s}$, with $\lambda = 1, 2, \cdots, n$. So finally the states that carry the irreducible representation $\lambda$ of $C_n$ can be written in the form

$$|\psi^{(\lambda)}\rangle = N_\lambda \sum_{s=0}^{n-1} \mu_n^{(\lambda-1)s} |\mu_n^s \alpha\rangle,$$

(4)
where the normalization constant is given by
\[
N^{-2}_\lambda = \sum_{s,r=0}^{n-1} \mu_n^{(\lambda-1)(s-r)} \exp \left\{ -|\alpha|^2 \left( 1 - \mu_n^{(s-r)} \right) \right\}.
\]

In spite of the non-orthogonality of the coherent states, it can be shown directly that states defined above for different irreducible representations are orthogonal, and thus its scalar product is given by \( \langle \psi^{(\lambda')} | \psi^{(\lambda)} \rangle = \delta_{\lambda',\lambda} \).

**Dihedral groups**

The Dihedral groups are denoted by the symbol \( D_n \), they are generated by an axis of order \( n \) and an orthogonal axis \( C_2 \). The product and power of their elements give rise to \( n \) rotations corresponding to powers of \( C_n \) and \( n \) second order rotations. This can be illustrated looking for the linear symmetry transformations of a regular polygon into itself. Thus, we establish the coordinates of the \( n \) vertices of a regular polygon in the \( XY \) plane by the expressions
\[
x_k = a \cos \theta_k, \quad y_k = a \sin \theta_k,
\]
where \( \theta_k = \frac{2\pi k}{n} \) and \( k = 0, 1, 2, \ldots, n - 1 \). If symmetry transformations of regular polygons are considered, one finds: the rotations by an angle \( \theta_k \) plus rotations of the same type followed by a reflection on the \( X \) axis (also called second order rotations).

For \( n \geq 2 \) the dihedral group can be generated by the elements \( R_1 \) and \( U_0 \), which define the rotation with an angle \( \theta_1 \) and the reflection along the \( X \) axis. The \( 2n \) elements can be written in the following form: \( \{ I, R_1, R_1^2, \ldots, R_1^{n-1}, U_0, U_0 R_1, U_0 R_1^2, \ldots, U_0 R_1^{n-1} \} \), thus the notation \( D_n : \{ I, R_1, R_2, \ldots, R_{n-1}, U_0, U_1, \ldots, U_{n-1} \} \) is adopted. We can write the two dimensional irreducible representation for both the rotations and rotations-reflections in one equation
\[
\begin{pmatrix}
\cos \theta_k & \sin \theta_k \\
-\sigma \sin \theta_k & \sigma \cos \theta_k
\end{pmatrix} = \begin{cases} R_k & \text{if } \sigma = 1, \\ U_k & \text{if } \sigma = -1. \end{cases}
\]

The action on the complex parameter of the coherent state implies that \( \alpha_s = \mu_n^{s \alpha} \) with \( s = 0, 1, 2, 3, \ldots, n - 1 \), for the rotations (including the identity) while \( \alpha_{s+n} = \mu_n^{s \alpha} \) with the same values of \( s \), for the reflections.

In this contribution, we present general expressions for the statistical properties of superpositions of coherent states together with their corresponding Wigner, Husimi function and the area of phase space associated to the Husimi functions which of course can be applied to any point groups, in particular to \( D_3, D_4, D_5 \) and \( D_6 \), which describe the symmetries of the equilateral triangle, the square, the pentagon and the hexagon, respectively. For this reason we include in the Appendix A their corresponding character tables.

**2. Hamiltonian formulation for the Nöther theorem**

By means of the Nöther theorem, we are going to construct the linear time dependent invariants of at most quadratic Hamiltonian, in the quadratures of the electromagnetic field,
\[
H = \frac{a}{2} p^2 + \frac{b}{2} (q p + p q) + \frac{c}{2} q^2 + f p + g q,
\]
where \( a, b, c, f, \) and \( g \) can be time dependent functions. Examples of its applications can be found in quantum optics [16], in molecular spectroscopy [17], and to visualize states of the motion of an ion in an electromagnetic trap. In this last case, one has to consider the following choice for the parameters \( b = f = g = 0, a = 1 \) and \( c = w^2(t) \) [18]. For this system, one can also use
directly the time dependent Schrödinger equation because their solution can be written in terms of an arbitrary gaussian wavepacket written in terms of two functions than satisfy an Ermakov system \[25\].

In the hamiltonian formulation of the Nöther theorem \[20\], the variations in the position and momentum variables

\[ q \to q + \delta q, \quad p \to p + \delta p, \]

constitute a symmetry transformation, if the corresponding variation in the functional of the action, \[ A(q, p, t) = \dot{q}p - H(q, p, t), \]

i.e.,

\[ \delta A = A(q + \delta q, p + \delta p, t) - A(q, p, t) = \frac{d}{dt}\Omega, \]

can be written as a total derivative with respect to time of a function \( \Omega(q, p, t) \). Developing the last expression by making a Taylor series expansion, one gets

\[ \delta A = \dot{q}\delta p + p(\dot{\delta q}) - \frac{\partial H}{\partial q}\delta q - \frac{\partial H}{\partial p}\delta p = \frac{d}{dt}\Omega. \quad (6) \]

Substituting the time derivatives of the function \( \Omega \) and \( \delta q(q, p, t) \) into the last expression, and equating the coefficients of \( \dot{p}, \dot{q} \) and the independent terms on both sides of the expression, one gets the following set of differential equations for the function \( \Omega \)

\[
\begin{align*}
\frac{\partial \Omega}{\partial p} &= \frac{\partial \delta q}{\partial p}p, \\
\frac{\partial \Omega}{\partial q} &= \delta p + \frac{\partial \delta q}{\partial q}p, \\
\frac{\partial \Omega}{\partial t} &= -\frac{\partial H}{\partial p}\delta p - \frac{\partial H}{\partial q}\delta q + \frac{\partial \delta q}{\partial t}p.
\end{align*}
\]

(7)

Establishing the integrability condition for \( \Omega \), that is the equality of all the cross derivatives, for example \( \frac{\partial^2 \Omega}{\partial q \partial p} = \frac{\partial^2 \Omega}{\partial p \partial q} \) and so on, which implies the existence of the generators of the symmetry transformation, the following expressions are obtained

\[
\begin{align*}
\left( \mathcal{O} + \frac{\partial}{\partial t} \right) \delta q - \frac{\partial^2 H}{\partial p^2} \delta p - \frac{\partial^2 H}{\partial p \partial q} \delta q &= 0, \\
\left( \mathcal{O} + \frac{\partial}{\partial t} \right) \delta p + \frac{\partial^2 H}{\partial q \partial p} \delta p - \frac{\partial^2 H}{\partial q^2} \delta q &= 0.
\end{align*}
\]

(8)

with the definition \( \mathcal{O} = \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p}. \)

Then the conserved quantity associated to the symmetry transformations is given by

\[ K = p\delta q - \Omega(q, p, t). \]

(9)

We use the expressions (8) and (9) to get the linear time dependent invariants of the Hamiltonian (5), by considering the time dependent variations \( \delta q = v(t) \) and \( \delta p = u(t) \). To be symmetry transformations they must satisfy the differential equations

\[
\begin{align*}
v' - au - bu &= 0, \\
u' + bu + cv &= 0.
\end{align*}
\]

(10)

These expressions remind us the Hamilton equations for the quadratic part of (5) by means of the identifications \( u \to p \) and \( v \to q \). This implies that to get the linear time dependent invariants
one needs to do variation along the classical trajectories of motion. To determine the conserved quantity, one needs to determine \( \Omega \) through the expressions (7) and the result is

\[
K = pv - qu + \int_0^t (f(\tau)u(\tau) + g(\tau)v(\tau))d\tau.
\] (11)

Since we have a second order differential equation, there are two conserved quantities, named \((Q(t), P(t))\) expressed in matrix form

\[
\begin{pmatrix}
P \\
Q
\end{pmatrix} = \begin{pmatrix}
v^{(1)} & -u^{(1)} \\
v^{(2)} & -u^{(2)}
\end{pmatrix} \begin{pmatrix}
p \\
q
\end{pmatrix} + \begin{pmatrix}
\delta^{(1)} \\
\delta^{(2)}
\end{pmatrix}
\] (12)

where \(\delta^{(i)} = \int_0^t (fu^{(i)} + gv^{(i)})d\tau\) with \(i = 1, 2\), these terms appear due to the linear parts in the Hamiltonian. By establishing the initial conditions that the constants of the motion are equal to the quadratures of the electromagnetic field \(P(0) = p\) and \(Q(0) = q\), one gets the initial conditions for the differential equations of the variations, (10), i.e.,

\[
\begin{align*}
v^{(1)}(0) &= 1, & u^{(1)}(0) &= 0, & \delta^{(1)}(0) &= 0, \\
v^{(2)}(0) &= 0, & u^{(2)}(0) &= -1, & \delta^{(2)}(0) &= 0.
\end{align*}
\] (13)

The \(P, Q\) are called linear time dependent invariants of quadratic Hamiltonians. The expressions (12) can be extended to describe multimode correlated light [7, 20]. Also they constitute a canonical transformation in a two dimensional phase space, and their extension to the multidimensional case implies to consider \(P, Q, p\) and \(q\) as \(n\) dimensional column vectors while \(v^{(1)}, v^{(2)}, u^{(1)}\) and \(u^{(2)}\) like \(n \times n\) matrices [21].

As an example, we consider the linear time dependent invariants of a free particle under the action of a time dependent force, that is \(V(q) = f(t)q\).

\[
\begin{pmatrix}
P(t) \\
Q(t)
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
-t & 1
\end{pmatrix} \begin{pmatrix}
p \\
q
\end{pmatrix} + \begin{pmatrix}
f(t) d\tau \\
-f(t) \tau d\tau
\end{pmatrix}.
\] (14)

In this contribution, we will use these time dependent constants of the motion to show the evolution of the statistical properties of crystallized Schrödinger cats. As an explicit example, we will consider the evolution of the crystallized states under the Hamiltonian of a free particle under a constant force, that is when \(f(t) = F_0\).

3. Evolution of crystallized cats

From Eq. (12), one can get the quadratures of the electromagnetic field \(p, q\) in terms of the time dependent constants of the motion \(P, Q\) or equivalently the corresponding creation and annihilation operators \(A \equiv \frac{1}{\sqrt{2}}(Q + iP), A^\dagger \equiv \frac{1}{\sqrt{2}}(Q - iP)\), this is

\[
\begin{pmatrix}
p \\
q
\end{pmatrix} = i \begin{pmatrix}
-\lambda^*_q & \lambda^*_p \\
\lambda^*_q & -\lambda^*_p
\end{pmatrix} \begin{pmatrix}
A - \delta \\
A^\dagger - \delta^*
\end{pmatrix},
\] (15)

where the parameters carry the dynamic associated to the Hamiltonian and are given in terms of the classical solutions of the Hamilton equations, i.e.,

\[
\lambda_p = \frac{1}{\sqrt{2}}(v^{(2)} + iv^{(1)}), \quad \lambda_q = \frac{1}{\sqrt{2}}(-u^{(2)} - iu^{(1)}), \quad \delta = \frac{1}{\sqrt{2}}(\delta^{(2)} + i\delta^{(1)}).
\]
Therefore to calculate the evolution of the statistical properties, it is convenient to construct the eigenstates of the annihilation operator, that is the correlated coherent states that satisfy [7]

\[ A(t)|\alpha, t\rangle = \alpha|\alpha, t\rangle. \]

Then it is straightforward to prove that the fluctuations of the quadratures of the electromagnetic field associated to the generalized correlated states of the general quadratic Hamiltonian (5) are given by [22]

\[
\begin{align*}
\sigma_p &\equiv \langle \alpha_k, t|\hat{p}^2|\alpha_k, t\rangle - \langle \alpha_k, t|\hat{p}|\alpha_k, t\rangle^2 = |\lambda_q|^2, \\
\sigma_q &\equiv \langle \alpha_k, t|\hat{q}^2|\alpha_k, t\rangle - \langle \alpha_k, t|\hat{q}|\alpha_k, t\rangle^2 = |\lambda_p|^2, \\
\sigma_{pq} &\equiv \langle \alpha_k, t|\frac{1}{2} \{\hat{p}, \hat{q}\}|\alpha_k, t\rangle - \langle \alpha_k, t|\hat{p}|\alpha_k, t\rangle \langle \alpha_k, t|\hat{q}|\alpha_k, t\rangle = -\frac{1}{2} (\lambda_p \lambda_q^* + \lambda_q \lambda_p^*). \quad (16)
\end{align*}
\]

It is important to notice that the results mentioned above are independent of the value of the complex parameter \(\alpha_k\).

For the crystallized cats defined in (1), it is straightforward to evaluate the expectation values of the quadratures of the electromagnetic field,

\[
\langle q \rangle_t = N_k^2 \sum_{r,s=1}^n \chi(\lambda)(g_r) \chi(\lambda)^*(g_s) q_{sr}(t) E_{sr}, \quad \langle p \rangle_t = N_k^2 \sum_{r,s=1}^n \chi(\lambda)(g_r) \chi(\lambda)^*(g_s) p_{sr}(t) E_{sr}. \quad (17)
\]

From here on, we are going to defined the matrix element of an arbitrary observable with respect to the standard coherent states associated to the complex parameters \(\alpha_s\) y \(\alpha_r\) in the form

\[
\langle \alpha_s, t|\hat{O}|\alpha_r, t\rangle \equiv O_{sr}(t) E_{sr}.
\]

Thus in the previous expression, we use that

\[
p_{sr}(t) \equiv -i \left(\lambda_q^s(\alpha_r - \delta) - \lambda_q(\alpha_r^* - \delta^*)\right), \quad q_{sr}(t) \equiv i \left(\lambda_p^s(\alpha_r - \delta) - \lambda_p(\alpha_r^* - \delta^*)\right). \quad (18)
\]

From these expressions for \(t = 0\), one gets the behavior of the expectation values for the static properties of the crystallized cats. In this case, it is immediate from expressions (13) that \(\lambda_p(0) = i/\sqrt{2}\), \(\lambda_q(0) = 1/\sqrt{2}\), and \(\delta(0) = 0\). These imply that

\[
p_{sr}(0) = -\frac{i}{\sqrt{2}} (\alpha_r - \alpha_r^*), \quad q_{sr}(0) = \frac{1}{\sqrt{2}} (\alpha_r + \alpha_r^*), \quad (19)
\]

determining in this form the static properties of the crystallized cats.

The calculations of the corresponding dispersions \((\Delta q)^2\) and \((\Delta p)^2\) of the quadratures associated to the dynamics of the crystallized cats under the Hamiltonian (5)

\[
\begin{align*}
(p^2)_{sr}(t) &= -\lambda_q^2 (\alpha_r - \delta)^2 - \lambda_p^2 (\alpha_r^* - \delta^*)^2 + \sigma_p [2 (\alpha_r^* - \delta^*) (\alpha_r - \delta) + 1], \\
(q^2)_{sr}(t) &= -\lambda_q^2 (\alpha_r - \delta)^2 - \lambda_p^2 (\alpha_r^* - \delta^*)^2 + \sigma_q [2 (\alpha_r^* - \delta^*) (\alpha_r - \delta) + 1], \quad (20)
\end{align*}
\]

with these and the expectation values given above, one easily determines in analytic form the fluctuations in the variables \(p\) and \(q\) for the crystallized cats.

Thus the squeezing coefficients for the quadratures of the electromagnetic field are determined by the ratios of the fluctuations to the uncertainties of the ground state of the electromagnetic field,

\[
S_q = \frac{(\Delta q)^2}{1/2}, \quad S_p = \frac{(\Delta p)^2}{1/2}.
\]
which let us determine if the crystallized states present the squeezing phenomena. When \( S_q \) or \( S_p \) is less than the unity there is squeezing and the crystallized states describe non-classical light.

At the left side of Fig. (2), we present for the three irreducible representations of the point group \( C_3 \) the behaviour of the squeezing parameter \( S_p \) as a function of the magnitude of the complex parameter \( |\alpha| = \rho \) associated to the coherent state. Notice that, the squeezing parameter is independent of time and of the strength of the repulsive force \( F_0 \) considered in the Hamiltonian, besides for large values of \( \rho \) the squeezing parameter is independent of the irreducible representation of the group. A similar behaviour is shown at the left hand side of Fig. (3), where the squeezing parameter \( S_p \) as a function of \( \rho \) is displayed for all the irreps of \( C_6 \).

The correlation between \( q \) and \( p \), denoted by \((\Delta pq)\), can be written in terms of the expectation values of \( p \) and \( q \) together with the time dependent anticommutator function

\[
\{ p, q \}_s(t) = \lambda_q^* \lambda_p^* (\alpha_r - \delta)^2 + \lambda_q \lambda_p (\alpha^*_s - \delta)^2 + 2 \sigma_{pq} \left[ (\alpha^*_s - \delta^*)(\alpha_r - \delta) + \frac{1}{2} \right].
\]

The correlation factor is defined by the expression

\[ K = \frac{(\Delta pq)^2}{(\Delta p)(\Delta q)}, \]

when it is different from zero one gets a correlated state with statistical dependence in the two quadratures of the electromagnetic field. Correlated states for the one mode oscillator have been introduced in [23].

For values of \(|\alpha| \geq 2\), the scalar products \( E_{sr} \approx \delta_{s,r} \), in that case, we can write \( \langle \alpha_s | \alpha_r \rangle = \delta_{s,r} \). Therefore, the normalization constant \( N_\lambda = \frac{1}{\sqrt{n}} \) and one can get expressions for all the expectation values mentioned before. In particular, for the dispersions of the quadratures of the electromagnetic field one gets

\[
(\Delta q)^2 = \frac{1}{n} \sum_{r=1}^{n} |\chi^{(\lambda)}(g_r)|^2 (q^2)_{rr}(t) - \left( \frac{1}{n} \sum_{r=1}^{n} |\chi^{(\lambda)}(g_r)|^2 q_{rr}(t) \right)^2,
\]

\[
(\Delta p)^2 = \frac{1}{n} \sum_{r=1}^{n} |\chi^{(\lambda)}(g_r)|^2 (p^2)_{rr}(t) - \left( \frac{1}{n} \sum_{r=1}^{n} |\chi^{(\lambda)}(g_r)|^2 p_{rr}(t) \right)^2,
\]

\[
(\Delta qp) = \frac{1}{n} \sum_{r=1}^{n} |\chi^{(\lambda)}(g_r)|^2 \{ p, q \}_{rr}(t) - \frac{1}{n^2} \sum_{r', r=1}^{n} |\chi^{(\lambda)}(g_r)|^2 |\chi^{(\lambda)}(g_{r'})|^2 q_{rr}(t) p_{r'r'}(t),
\]

where the expressions of \((q^2)_{rr}(t), (p^2)_{rr}(t), q_{rr}(t), p_{r'r'}(t), \) and \( \{ p, q \}_{rr}(t) \) can be evaluated by means of the expressions (18), (19), and (20). Indeed for the one dimensional representations the absolute values of the characters are equal to the unity and so the previous expressions are even simpler.

Now we consider the photon statistics of the crystallized cats, to calculate the expectation value of the photon number and its corresponding fluctuation is convenient to write the constants of the motion in terms of the creation and annihilation operators. Thus we have that

\[
\left( \begin{array}{c} a \\ a^\dagger \end{array} \right) = \left( \begin{array}{cc} \mu_1^* & -\mu_2 \\ -\mu_2^* & \mu_1 \end{array} \right) \left( \begin{array}{c} A - \delta \\ A^\dagger - \delta^* \end{array} \right),
\]

with the definitions of the parameters \( \mu_1 = \frac{1}{\sqrt{2}} (\lambda_q - i \lambda_p) \) and \( \mu_2 = \frac{1}{\sqrt{2}} (\lambda_q + i \lambda_p) \). The \( 2 \times 2 \) matrix in the right side of the expression has determinant equal to the unity. Thus the average
Figure 2. The squeezing parameter for the point group $C_3$, associated to variable $p$, is shown at the left side as a function of $q$ while the Mandel parameter is displayed at the right side. In both cases, the crystallized states are subject to a repulsive constant force $F_0 = 10$ in arbitrary units, notice that $S_p$ is independent of time and $Q_M$ is calculated for $t = 0$. The value $Q_M = 1$ is indicating a Poissonian distribution. In both cases, continuous, dashed and dash-dotted lines correspond to the irreducible representations 1, 2, and 3 of $C_3$, respectively.

Figure 3. Squeezing and Mandel parameters are shown as functions of $q$ for the cyclic group $C_6$, when the system is evolving by means of the Hamiltonian of a repulsive constant force. We use $F_0 = 10$ in arbitrary units and the Mandel parameter was evaluated at $t = 0$. Continuous, dashed and dash-dotted black lines correspond to the representations $\lambda = 1$, 2, and 3, respectively while the gray lines are associated to the representations 4, 5 and 6, respectively.

The dispersion of the photon number operator $(\Delta \hat{n})^2$ of the crystalized states is determined by the standard expression in terms of the expectation values of $\hat{n}$ and $\hat{n}^2$, thus we give the result for the second one

$$\langle \hat{n}^2 \rangle_t = N_\lambda^2 \sum_{r,s=1}^n \chi^{(\lambda)}(g_r)\chi^{(\lambda)*}(g_s) (n^2)_{sr}(t) E_{sr},$$

where we define

$$n_{sr}(t) = -\mu_1 \mu_2 (\alpha_s^* - \delta^*)^2 - \mu_1^* \mu_2^* (\alpha_r - \delta)^2 + (|\mu_1|^2 + |\mu_2|^2) (\alpha_s^* - \delta^*) (\alpha_r - \delta) + |\mu_2|^2. \quad (24)$$
where the time dependent function takes the form

\[
(n^2)_{sr}(t) = 2|\mu_1|^2|\mu_2|^2-2(|\mu_1|^2 + |\mu_2|^2) \left[\mu_1^* \mu_2^* (\alpha_r - \delta)^2 + \mu_1 \mu_2 (\alpha^*_s - \delta^*)^2\right] \\
+ (|\mu_1|^4 + |\mu_2|^4 + 6|\mu_1|^2 |\mu_2|^2) (\alpha_r - \delta) (\alpha^*_s - \delta^*) + n^2_{sr}(t).
\]

The Mandel parameter, \(Q_M = R_M - 1\), is the ratio of the dispersion to the photon number mean value. It determines the type of statistics of the considered state. If this ratio is less than the unity, the state satisfies a sub-Poissonian statistics while for larger values correspond to a super-Poisson statistics. Of course if the result is equal to the unity, the state has a Poisson statistics. The Mandel parameter can be written in the form

\[
R_M = \frac{(\Delta n)^2}{\langle n \rangle} = \frac{\sum_{r,s=1}^{n} \chi^{(\lambda)}(g_r) \chi^{(\lambda)*}(g_s)(n^2)_{sr}(t) E_{sr}}{\sum_{r,s=1}^{n} \chi^{(\lambda)}(g_r) \chi^{(\lambda)*}(g_s)n_{sr}(t) E_{sr}} - N_{\lambda}^2 \sum_{r,s=1}^{n} \chi^{(\lambda)}(g_r) \chi^{(\lambda)*}(g_s)n_{sr}(t) E_{sr}.
\]

The corresponding result for the Mandel parameter in the limit of large \(|\alpha|^2 \geq 2\), one has the result

\[
R_M = \frac{\sum_{r=1}^{n} |\chi^{(\lambda)}(g_r)|^2 (n^2)_{rr}(t)}{\sum_{r=1}^{n} |\chi^{(\lambda)}(g_r)|^2 n_{rr}(t)} - \frac{1}{n} \sum_{r=1}^{n} |\chi^{(\lambda)}(g_r)|^2 n_{rr}(t).
\]

In the right side of Fig. (2), the plot of \(Q_M\) for the irreps of \(C_3\) as a function of \(q\) are shown for \(t = 0\). One can see a sub-Poissonian behaviour for the representations \(\lambda = 2\) and \(3\) for small values of the parameter while is super-Poissonian for \(\lambda = 1\). For large values of \(q\) all the states have a Poissonian photon distribution function. A similar result is found for \(C_6\), as it can be seen in Fig. (3) for all the irreps of the group. A sub-Poissonian distribution is happening for five of them for small values of \(q\) and a super-Poissonian for \(\lambda = 1\), newly one gets that all the states have a Poissonian distribution for large values of the parameter. In Fig. (4) for the same point groups \(C_3\) and \(C_6\) the Mandel parameter is plotted as a function of \(q\) afterwards the action of a constant force \(F_0 = 10\) arbitrary units during a time equal to \(t = 0.2\) units. One can see that all the states have now a super-Poissonian photon distributin function. Thus, one has that the constant force Hamiltonian destroys the non-classical properties of the cristallized cats. Similar results are obtained when the cristallized states are under free propagation, one gets sub-Poissonian, and super-Poissonian photon number distributions for small values of \(q\) while a Poissonian distribution for large values of the parameter.
4. Quasi-Probability distribution functions

The phase space representation of a quantum system through the Wigner and Husimi distribution functions has been very useful to establish the behavior of the crystallized Schrödinger cat states. As examples, we consider to determine the regions in the phase space where the Wigner function is negative with the well known implications or to find the occupied area in phase space by the Husimi function by means of the evaluation of its second moment.

To obtain the evolution of a Wigner function under the Hamiltonian (5), one can proceed in two forms: By means of the unitary representation of the canonical transformation from the quadratures of the electromagnetic field \((q,p)\) to the linear time dependent constants of motion \((Q,P)\), it has been proved in [24] that

\[
W(q,p,t) = W(Q,P,0).
\]

An extension of the work to get the representation of canonical transformations in time-dependent quantum mechanics has been done recently [27].

The second method considers operator equations that satisfy the generalized correlated states [26].

For the crystallized cats, we establish

\[
A(t)|\alpha_r,t\rangle\langle\alpha_s,t| = \alpha_r |\alpha_r,t\rangle\langle\alpha_s,t|, \quad |\alpha_r,t\rangle\langle\alpha_s,t| A^\dagger(t) = \alpha^*_s |\alpha_r,t\rangle\langle\alpha_s,t|. \quad (26)
\]

Then the constants of the motion \(A(t) = (\lambda_p \hat{p} + \lambda_q \hat{q} + \delta)\) and \(A^\dagger(t) = (\lambda_p^* \hat{p} + \lambda_q^* \hat{q} + \delta^*)\) are substituted into the operator expressions. We apply the Weyl transform to the resulting expressions, that is multiply by the bra \((q-y)\) from the left, by the ket \(|q+y\rangle\) from the right, by the exponential \(e^{2i\mu y}\) and integrate the variable \(y\). In this manner, if one defines the operator

\[
\hat{\rho}_{rs}(t) = |\alpha_r,t\rangle\langle\alpha_s,t|,
\]

the following Weyl transforms are obtained

\[
\hat{p} \hat{\rho}_{rs}(t) \rightarrow \left(p - \frac{i}{2} \frac{\partial}{\partial q}\right) W_{rs}, \quad \hat{\rho}_{rs}(t) \hat{p} \rightarrow \left(p + \frac{i}{2} \frac{\partial}{\partial q}\right) W_{rs},
\]

\[
\hat{q} \hat{\rho}_{rs}(t) \rightarrow \left(q + \frac{i}{2} \frac{\partial}{\partial p}\right) W_{rs}, \quad \hat{\rho}_{rs}(t) \hat{q} \rightarrow \left(q - \frac{i}{2} \frac{\partial}{\partial p}\right) W_{rs}. \quad (27)
\]

Using the expressions (26) and (27), the following system of differential equations for the Wigner function is obtained

\[
(\lambda_p p + \lambda_q q + \delta) W_{rs}(t) + \frac{i}{2} \left(\lambda_q \frac{\partial}{\partial p} - \lambda_p \frac{\partial}{\partial q}\right) W_{rs}(t) = \alpha_r W_{rs}(t),
\]

\[
(\lambda_p^* p + \lambda_q^* q + \delta^*) W_{rs}(t) - \frac{i}{2} \left(\lambda_q^* \frac{\partial}{\partial p} - \lambda_p^* \frac{\partial}{\partial q}\right) W_{rs}(t) = \alpha^*_s W_{rs}(t).
\]

Making the change of variables \(z = \lambda_p p + \lambda_q q + \delta\) and its complex conjugate, it is straightforward to get

\[
\left(z + \frac{1}{2} \frac{\partial}{\partial z}\right) W_{rs}(t) = \alpha_r W_{rs}(t), \quad \left(z^* + \frac{1}{2} \frac{\partial}{\partial z}\right) W_{rs}(t) = \alpha^*_s W_{rs}(t).
\]

This system of differential equations has the following solution:

\[
W_{rs}(t) = W_0(\alpha_r, \alpha_s) e^{2\alpha_r z^* + 2\alpha^*_s z - 2|z|^2},
\]
where $W_0(\alpha_r, \alpha_s)$ is an arbitrary function of the variables $\alpha_r$, $\alpha_s$, and its complex conjugate expressions. The time dependence is due to the evolution of the state in the considered Hamiltonian, whose action is reflected in the parameters $\lambda_p$, $\lambda_q$, and $\delta$.

Then quasi-probabilistic Wigner distribution function for the crystallized Schrödinger cat states can be written as follows

$$W(q, p, t) = N^2 \chi^{(\lambda)}(g_r) \chi^{(\lambda)*}(g_s) W_0(\alpha_r, \alpha_s) e^{2\alpha_r z^* + 2\alpha_s^* z^2 - 2|z|^2}. \quad (28)$$

Substituting $z$, $z^*$, and integrating the expression with respect to the phase space variables $q$ and $p$, we get

$$1 = N^2 \pi \sum_{r,s=1}^n \chi^{(\lambda)}(g_r) \chi^{(\lambda)*}(g_s) W_0(\alpha_r, \alpha_s) e^{2\alpha_r^* \alpha_r},$$

comparing the result with (2) when the irreducible representations of the point group are equal one has the result that

$$W_0(\alpha_r, \alpha_s) = \frac{1}{\pi} E_{sr} e^{-2\alpha_r^* \alpha_r}.$$

Therefore, we find that the time dependent Wigner function associated to the crystallized cats can be written in the following form

$$W(q, p, t) = N^2 \chi^{(\lambda)}(g_r) \chi^{(\lambda)*}(g_s) E_{sr} e^{\left\{-2\sigma_q \tilde{p}_{sr}(t) - 2\sigma_p \tilde{q}_{sr}(t) + 4\sigma_{pq} \tilde{p}_{sr}(t) \tilde{q}_{sr}(t)\right\}}, \quad (29)$$

where we define $\tilde{p}_{sr}(t) = p - p_{sr}(t)$ and $\tilde{q}_{sr}(t) = q - q_{sr}(t)$, the expressions for $q_{sr}(t)$ and $p_{sr}(t)$ are given in the expression (18). Thus, one can see that each term in the Wigner distribution function is changing in time according to the evolution of the state in the considered Hamiltonian, whose action is reflected in the parameters $\lambda_p$, $\lambda_q$, and $\delta$.

The expression of the Wigner distribution function can be simplified one step further by noticing that the argument in the exponential is a quadratic form and thus it can be diagonalized through a rotation in the phase space by an angle

$$\theta(t) = \frac{1}{2} \arctan \left( \frac{2 \sigma_{pq}}{\sigma_q - \sigma_p} \right) + \theta_0, \quad (30)$$

where the initial value is used to guarantee that $\theta(t = 0) = 0$. Then the Wigner function, in terms of the rotated variables, is given by

$$W(q, p, t) = N^2 \chi^{(\lambda)}(g_r) \chi^{(\lambda)*}(g_s) E_{sr} e^{\left\{-\frac{\sigma^2}{(\Delta q')^2} - \frac{\sigma^2}{(\Delta p')^2}\right\}},$$

where the fluctuations in the new variables are independent of the complex parameters $\alpha_r$ and $\alpha_s$ because they are given as follows:

$$(\Delta q')^2 = \left( \sigma_p + \sigma_q + \sqrt{(\sigma_p + \sigma_q)^2 - 1} \right)^{\frac{1}{2}},$$

$$(\Delta p')^2 = \left( \sigma_p + \sigma_q - \sqrt{(\sigma_p + \sigma_q)^2 - 1} \right)^{\frac{1}{2}}.$$
In this variables one can see that the phase space area of \((q', p')\) stays constant because the product \((\Delta q')^2 (\Delta p')^2 = 1\).

In conclusion, each term of the Wigner distribution function of a crystallized Schrödinger cat state is moving according the matrix elements of the quadratures of the electromagnetic field and rotating with an angle \(\theta(t)\). Besides in the rotating phase space the fluctuations are changing in time but the phase space area is a constant.

Although the Wigner function is very useful sometimes it is convenient to have a non-negative distribution function and for this reason the Husimi function was introduced [27]. The Husimi quasiprobability distribution can be obtained directly by means of the expression

\[
Q_H(q, p) = \frac{1}{N^2} \sum_{\alpha, \beta} \chi^{\alpha}(q) \chi^{\beta}(p) \langle \alpha | \beta \rangle,
\]

where \(\beta = \frac{1}{\sqrt{2}} (q + ip)\) and in the second row, we substitute the expression for the crystallized cat state, (1). Another form of establishing the Husimi function is smoothing out the Wigner function by averaging over a coarse graining function [28]

\[
Q_H(\beta, \beta^*) = \frac{1}{\pi} \int \int \phi_B(x, x', p) W_{\psi}(x + x', p) dx dx' dp,
\]

with

\[
\Phi(\beta, \beta^*) = \frac{1}{\sqrt{\pi}} e^{ip(x-x')} - \frac{1}{\pi^2} (\beta^2 + \beta^{*2} + 2|\beta|^2) - \frac{1}{2} (x^2 + x'^2 + \sqrt{2}(\beta x + \beta^* x')).
\]

Thus if we substitute the expression for the Wigner function of the crystallized Schrödinger cat states given in (29) and make the integrations indicated in the last expression, one gets

\[
Q_H(q, p, t) = N^2 \sum_{r, s=1}^n \chi^{r}(q) \chi^{s}(p) E_{sr} Q_{sr}(t),
\]

where we have defined

\[
Q_{sr}(t) = \frac{1}{2\pi} \sqrt{\frac{2}{\sigma_q + \sigma_p + 1}} e^{\frac{-(2\sigma_q + 1)(q - q_{sr}(t))^2 - (2\sigma_p + 1)(p - p_{sr}(t))^2 + 4 \sigma_p (q - q_{sr}(t))(p - p_{sr}(t))}{2(\sigma_q + \sigma_p + 1)}}.
\]

Finally we calculate the second moment of the Husimi distribution function, because it has been proposed as a measure of the complexity of quantum pure states and has similar properties to the Wehrl entropy [29]. Its inverse represents the effective volume in phase space occupied by the Husimi quasiprobability distribution function.

When a quantum state is expanded in a basis for example

\[
|\psi\rangle = \sum_k c_k |\phi_k\rangle,
\]

the information obtained when one measures the probability \(p_k = |c_k|^2\) is obtained through the calculation of the Shannon entropy

\[
S = -\sum_k p_k \ln p_k.
\]
In addition, the moments of the distribution $M_j = \sum_k p_k^j$ give a measure of the localization with respect to the used basis, in particular the inverse of the second moment determines the number of principal components. As the Husimi function can be considered as a probability distribution function, we are going to calculate its second moment and show the results for the inverse expression, the area, associated to the crystallized cat states. Thus, one has for the second moment of the Husimi function

$$M_H = A \int \int_{-\infty}^{\infty} Q^2 (q,p) \ dq \ dp$$

$$= N^2 \sum_r \sum_{s,s'} \chi^{(\lambda)}(g_r) \chi^{(\lambda)}(g_{s'}) \chi^{(\lambda)*}(g_s) \chi^{(\lambda)*}(g_{s'}) M_{sr}(t) E_{sr} E_{sr'}$$

with $A = 4\pi \sqrt{2}$ being a constant that we adjust to make the area of a coherent state equal to the unity. The function $M_{sr}(t)$ can be written as follows

$$M_{sr}(t) = \frac{1}{2\pi \sqrt{2(1 + \sigma p + \sigma q)}} e^{-\frac{1}{4(1+\sigma p+\sigma q)} \{(q_{sr}(t)-q_{sr'}(t))^2(1+2\sigma p)+(p_{sr}(t)-p_{sr'}(t))^2(1+2\sigma q)\}}$$

$$\times e^{-\frac{1}{4(1+\sigma p+\sigma q)} \{-4\sigma pq(q_{sr}(t)-q_{sr'}(t))(p_{sr}(t)-p_{sr'}(t))\}}.$$

In Fig. (5) we plot the area in phase space associated to crystallized states that carry the irreducible representations of the point groups $C_3$ and $C_6$, for $t = 0$. One can see for small values of $q$ that they have the same area as a coherent state, that $A = 1$ while for larger values the area increases one have a three times the area of a coherent state according to the number of components of the crystallized state for $C_3$ while one gets $A = 6$ for $C_6$. This is due to the behavior of the scalar product of two coherent states that they become orthogonal when $|\alpha| >> 1$ as it was shown in Fig. (1).

5. Summary and Conclusions

First of all, we have calculated the expectation values of the quadratures of the electromagnetic field with respect to the crystallized states of cyclic groups under the action of a constant force, and the results are

$$\langle p \rangle = -F_0 t, \quad \langle q \rangle = -F_0 \frac{t^2}{2}.$$
Figure 6. Poincare invariants are shown as functions of $\varrho$ for the point groups $C_3$ and $C_6$, when the systems are subject a repulsive constant force $F_0 = 10$ in arbitrary units. They are independent of time, and we use the same code lines than in Figs. (2) and (3).

which have the same value for all the states carrying the irreducible representation of a cyclic group. Their values correspond to the classical evolution of the quadratures. Another interesting result is that the quotient of the fluctuations

$$\frac{\Delta q}{\Delta p} = \sqrt{1 + t^2},$$

have a universal value for all the irreducible representation basis states of the cyclic groups.

Another quantity of interest, the universal Poincare-Cartan invariant which is constructed in terms of the quadrature fluctuations of the electromagnetic field, that is

$$I_{PC} = (\Delta q)^2(\Delta p)^2 - (\Delta pq)^2.$$  \hfill (36)

For correlated states that are solutions of the time dependent Schrödinger equation of the Hamiltonian (5) the calculation of this quantity can be easily done. The result is given by $I_{PC} = 1/4$ as can be proved by means of the expressions (16) and using that $\lambda_p \lambda_q^* - \lambda_p^* \lambda_q = i$. Then the correlated states minimize the Schrödinger-Robertson uncertainty relations [7]. In Fig. (6), the Poincare-Cartan invariants for $C_3$ and $C_6$ are displayed as function of $\varrho = |\alpha|$ when the corresponding cristallized states are evolving under a constant force Hamiltonian. Of course they are independent of time, always larger than $1/4$, and for large values of $\varrho$ the states of the different irreducible representations have the same bahavior.

The linear time dependent constants of the motion can be obtained by means of the Nöther’s theorem. The $(P(t), Q(t))$ constitute a set of inhomogeneous linear canonical transformations and its unitary representation can be used to find the evolution of any wave packet in phase space. For the case of a Wigner distribution function, the unitary representation has the form of a product of Dirac Delta functions. The crystallized Schrödinger cat states can present the squeezing phenomena, correlation between the quadratures, and a subpoissonian photon distribution function.

The evolution of a Wigner function under the action of a general at most quadratic Hamiltonian in the quadratures of the electromagnetic field was established. It moves in phase space according to the matrix elements of the position and momentum operators with respect to coherent states of the parameters $\alpha_r$ associated to the action of the considered point group, it is also rotating by an angle determined by the fluctuations associated to the corresponding generalized correlated states, and in the rotating frame of reference the dispersions are changing but keeping the same area in phase space. This result can be extended to the Wigner function of
the crystallized Schrödinger cat states, in this case for each term of the Wigner function one has a similar behavior by replacing the expectation values for the matrix elements, that is the functions \( q_{sr}(t) \) and so on. We have the purpose to extend this work to the multimodal case, to be able to calculate quantities of interest in quantum information theory as for example the Von Neumann entropy to measure the entanglement properties of the two modes of the electromagnetic field. calculating the entanglement of cat states, manly to see how changes under evolution of time dependent Hamiltonians.

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Appendix A. Character tables
The character tables for the cyclic group \( C_n \), in this case the irreducible representations are one dimensional and then the characters are given by

\[
\begin{array}{cccccccc}
\chi^{(0)} & I & R_1 & R_2 & \ldots & R_{n-1} \\
& 1 & 1 & 1 & \ldots & 1 \\
\chi^{(1)} & 1 & \mu_n & \mu_n^2 & \ldots & \mu_n^{n-1} \\
\chi^{(2)} & 1 & \mu_n^2 & \mu_n^4 & \ldots & \mu_n^{2(n-1)} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\chi^{(n-1)} & 1 & \mu_n^{n-1} & \mu_n^{2(n-1)} & \ldots & \mu_n^{(n-1)(n-1)} \\
\end{array}
\]

For the dihedral group \( D_n \) we have two cases that behave different: \( n \) even and \( n \) odd. In the \( n \) even case the inversions are divided into two different classes and the rotations of \( \pi \) radians always form its own class. For the \( n \) odd case, all the inversions belong to the same class while the rotations form two or more classes. For completeness of the contribution, we give the character tables for the groups: \( D_3, D_4, D_5, \) and \( D_6 \) taken from the reference [30].

For the dihedral group \( D_3 \):

\[
\begin{array}{cccc}
I & \{ R_{2\pi/3}, R_{4\pi/3} \} & \{ U_1, U_2, U_3 \} \\
\chi^{(0)} & 1 & 1 & 1 \\
\chi^{(1)} & 1 & 1 & -1 \\
\chi^{(2)} & 2 & -1 & 0 \\
\end{array}
\]

For the dihedral group \( D_4 \):

\[
\begin{array}{cccccc}
I & \{ R_{\pi}, R_{3\pi/2} \} & R_{\pi} & \{ U_1, U_3 \} & \{ U_2, U_4 \} \\
\chi^{(0)} & 1 & 1 & 1 & 1 & 1 \\
\chi^{(1)} & 1 & 1 & -1 & -1 & -1 \\
\chi^{(2)} & 1 & -1 & 1 & 1 & -1 \\
\chi^{(3)} & 1 & -1 & 1 & -1 & 1 \\
\chi^{(4)} & 2 & 0 & -2 & 0 & 0 \\
\end{array}
\]

For the dihedral group \( D_5 \):
with $\gamma = 2 \cos \left(\frac{2\pi}{5}\right) = \left(\sqrt{5} - 1\right)/2$ the golden ratio and $\gamma_2 = 2 \cos \left(\frac{4\pi}{5}\right) = -\left(\sqrt{5} + 1\right)/2$.

Finally we consider the dihedral group $D_6$:

| $\chi^{(i)}$ | $I$ | $\left\{ R_{x \frac{2\pi}{3}}, R_{z \frac{2\pi}{3}} \right\}$ | $\left\{ R_{x \frac{2\pi}{3}}, R_{y \frac{2\pi}{3}} \right\}$ | $\left\{ R_{y \frac{2\pi}{3}}, R_{z \frac{2\pi}{3}} \right\}$ | $R_{\pi}$ | $\left\{ U_1, U_3, U_5 \right\}$ | $\left\{ U_2, U_4, U_6 \right\}$ |
|---|---|---|---|---|---|---|---|
| $\chi^{(0)}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi^{(1)}$ | 1 | 1 | 1 | 1 | -1 | -1 | -1 |
| $\chi^{(2)}$ | 2 | $\gamma$ | $\gamma_2$ | 0 | 0 | 0 | 0 |
| $\chi^{(3)}$ | 2 | $\gamma_2$ | $\gamma$ | 0 | 0 | 0 | 0 |

References

[1] Sargent M, Scully M O, and Lamb W E Jr 1975 Laser Physics (Reading: Addison-Wesley)
[2] Meystre P and Walls D F 1991 Nonclassical effects in Quantum Optics 4 (New York: American Institute of Physics)
[3] Glauber R J 1963 Phys. Rev. 130 2529
[4] Sudarshan E C G 1963 Phys. Rev. Lett. 10 227
[5] Klauder J R 1963 J. Math. Phys. 4 1055
[6] Dodonov V V, Malkin I A, and Man’ko V I 1974 Physica 72 597
[7] Dodonov V V and Man’ko V I 1989 Proceedings of the Lebedev Physical Institute 183 (Commack, NY: Nova Science)
[8] Yurke B and Stoler D 1986 Phys. Rev. Lett. 57 13
[9] Mecozzi A and Tombesi P 1987 Phys. Rev. Lett. 58 1055
[10] Tombesi P and Mecozzi A 1987 J. Opt. Soc. Am. B 4 1700
[11] Gea-Banacloche J 1990 Phys. Rev. Lett. 65 3385
[12] Buzek V and Hladky B 1993 J. Mod. Phys. 40 1309
[13] Gea-Banacloche J 1991 J. Phys. A 44 5913
[14] Castaños O, López-Peña R and Man’ko V I 1995 J. Russ. Laser Research 16 477
[15] Littlewood D E 1962 University Algebra (Reading, MA: Addison-Wesley)
[16] Delgado F, Mielnik C B and Reyes M A 1998 Phys. Lett. A 237 359
[17] Doktorov E V, Malkin I A and Man’ko V I 1977 J. Mol. Spectrosc. 64 302
[18] Schuch D and Moshinsky M 2008 SIGMA 4 054
[19] Schuch D and Moshinsky M 2008 SIGMA 4 054
[20] Castaños O and López-Peña R 1995 in: Quantum-like Models and Coherent Effects ed R Fedela and P K Shukla (Singapore: World Scientific) p 3
[21] Salamanca M and Quesne C 1972 J. Math. Phys. 12 1772.
[22] Padilla E 1996 Generalized coherent states of quantum systems (México: Bachelor thesis of the UNAM)
[23] Dodonov V V, Kurnyshév E V and Man’ko V I 1980 J. Phys. A 13 L85
[24] Schuch D and Moshinsky M 2008 SIGMA 4 054
[25] Moshinsky M and Quesne C 1972 J. Math. Phys. 12 1772.
[26] Padilla E 1996 Generalized coherent states of quantum systems (México: Bachelor thesis of the UNAM)
[27] Dodonov V V, Kurnyshév E V and Man’ko V I 1980 J. Phys. A 13 L85
[28] Schuch D and Moshinsky M 2008 SIGMA 4 054
[29] Moshinsky M and Quesne C 1972 J. Math. Phys. 12 1772.
[30] Padilla E 1996 Generalized coherent states of quantum systems (México: Bachelor thesis of the UNAM)
[31] Dodonov V V, Kurnyshév E V and Man’ko V I 1980 J. Phys. A 13 L85
[32] Schuch D and Moshinsky M 2008 SIGMA 4 054