Hitting Time Distribution for Skip-Free Markov Chains: A Simple Proof

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Abstract

A well-known theorem for an irreducible skip-free chain with absorbing state \( d \), under some conditions, is that the hitting (absorbing) time of state \( d \) starting from state 0 is distributed as the sum of \( d \) independent geometric (or exponential) random variables. The purpose of this paper is to present a direct and simple proof of the theorem in the cases of both discrete and continuous time skip-free Markov chains. Our proof is to calculate directly the generation functions (or Laplace transforms) of hitting times in terms of the iteration method.

Keywords: skip-free, random walk, birth and death chain, absorbing time, hitting time, eigenvalues, recurrence equation.

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1 Introduction

The skip-free Markov chain on \( \mathbb{Z}^+ \) is a process for which upward jumps may be only of unit size, and there is no restriction on downward jumps. If a chain start at 0, and we suppose \( d \) is an absorbing state. An interesting property for the chain is that the hitting time of state \( d \) is distributed as a sum of \( d \) independent geometric (or exponential) random variables.

There are many authors give out different proofs to the results. For the birth and death chain, the well-known results can be traced back to Karlin and McGregor \cite{7}, Keilson \cite{8}, \cite{9}, Kent and Longford \cite{10} proved the result for the discrete time version (nearest random walk) although they have not specified the result as usual form (section 2, \cite{10}). Fill \cite{4} gave the first stochastic proof to both nearest random walk and birth and death chain cases via duality which was established in \cite{2}. Diaconis and Miclo \cite{3} presented another probabilistic proof for birth and death chain. Very recently, Gong, Mao and Zhang \cite{6} gave a similar result in the case that the state space is \( \mathbb{Z}^+ \), they use the well established result to determine all the eigenvalues or the spectrum of the generator.

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For the skip-free chain, Brown and Shao [1] first proved the result in continuous time situation. By using the duality, Fill [5] gave a stochastic proof to both discrete and continuous time cases. The purpose of this paper is to present a direct and simple proof of the theorem in the cases of both discrete and continuous time skip-free Markov chains. Our proof is to calculate directly the generation functions (or Laplace transforms) of hitting times in terms of the iteration method.

**Theorem 1.1.** For the discrete-time skip-free random walk:
Consider an irreducible skip-free random walk with transition probability $P$ on $\{0, 1, \cdots, d\}$ started at 0, suppose $d$ is an absorbing state. Then the hitting time of state $d$ has the generation function
\[
\varphi_d(s) = \prod_{i=0}^{d-1} \left[ \frac{(1 - \lambda_i)s}{1 - \lambda_i} \right],
\]
where $\lambda_0, \cdots, \lambda_{d-1}$ are the $d$ non-unit eigenvalues of $P$.

In particular, if all of the eigenvalues are real and nonnegative, then the hitting time is distributed as the sum of $d$ independent geometric random variables with parameters $1 - \lambda_i$.

**Theorem 1.2.** For the skip-free birth and death chain:
Consider an irreducible skip-free birth and death chain with generator $Q$ on $\{0, 1, \cdots, d\}$ started at 0, suppose $d$ is an absorbing state. Then the hitting time of state $d$ has the Laplace transform
\[
\varphi_d(s) = \prod_{i=0}^{d-1} \frac{\lambda_i}{1 - \lambda_i s},
\]
where $\lambda_i$ are the $d$ non-zero eigenvalues of $-Q$.

In particular, if all of the eigenvalues are real and nonnegative, then the hitting time is distributed as the sum of $d$ independent exponential random variables with parameters $\lambda_i$.

## 2 Proof of Theorem 1.1

Define the transition probability matrix $P$ as
\[
P = \begin{pmatrix}
  r_0 & p_0 & p_1 & \cdots & \cdots & \cdots & \cdots & p_{d-1} & p_d-1 \\
  q_{1,0} & r_1 & p_1 & \cdots & \cdots & \cdots & \cdots & p_{d-1} & 1 \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  q_{d-1,0} & q_{d-1,1} & q_{d-1,2} & \cdots & r_{d-1} & p_{d-1} & 1 \\
\end{pmatrix}_{(d+1) \times (d+1)},
\]
and for $0 \leq n \leq d - 1$, $P_n$ denote the first $n + 1$ rows and first $n + 1$ lines of $P$.

Let $\tau_{i,i+j}$ be the hitting time of state $i+j$ starting from $i$. By the Markov property, we have
\[
\tau_{i,i+j} = \tau_{i,i+1} + \tau_{i+1,i+2} + \cdots + \tau_{i+j-1,i+j}.
\]
(2.1)

If $f_{i,i+1}(s)$ is the generation function of $\tau_{i,i+1}$,
\[
f_{i,i+1}(s) = \mathbb{E}s^{\tau_{i,i+1}} \quad \text{for } 0 \leq i \leq d - 1,
\]
(2.1) says that
\[ f_{i,i+j}(s) = f_{i,i+1}(s) \cdot f_{i+1,i+2}(s) \cdots f_{i+j-1,i+j}(s), \quad \text{for } 1 \leq j \leq d - i. \]

Let
\[ g_{0,0}(s) = 1, \quad g_{i,i+j}(s) = \frac{p_0 p_{i+1} \cdots p_{i+j-1}}{f_{i,i+j}(s)} s^j, \quad \text{for } 1 \leq j \leq d - i. \]

**Lemma 2.1.** Define \( I_n \) as a \((n + 1) \times (n + 1)\) identity matrix. We have
\[ g_{0,n+1}(s) = \det(I_n - sP_n), \quad \text{for } 0 \leq n \leq d - 1. \]  

*Proof.* We will give a key recurrence to prove this lemma. By decomposing the first step, the generation function of \( \tau_{n,n+1} \) satisfy
\[ f_{n,n+1}(s) = r_n s f_{n,n+1}(s) + p_n s + q_{n,n-1} s f_{n-1,n+1}(s) + q_{n,n-2} s f_{n-2,n+1}(s) + \cdots + q_{n,0} s f_{0,n+1}(s). \]  

Recall the definition of \( g_{i,i+j}(s) \), substitute it into the formula above, we have
\[ g_{0,n+1}(s) = (1 - r_n s) g_{0,n}(s) - q_{n,n-1} s p_{n-1} g_{0,n-1}(s) - q_{n,n-2} s p_{n-2} g_{0,n-2}(s) - \cdots - q_{n,0} s^2 g_{0,0}(s). \]  

By expanding the bottom row of the matrix, we obtain
\[ \det(I_n - sP_n) = (1 - r_n s) A_{n,n} - q_{n,n-1} s A_{n,n-1} - q_{n,n-2} s A_{n,n-2} - \cdots - q_{n,0} s A_{n,0}. \]  

By some calculation, we can deduce
\[ A_{n,n} = \det(I_n - sP_{n-1}), \quad A_{n,0} = p_0 p_1 \cdots p_{n-1} s^n, \]
and for \( 1 \leq i < n \),
\[ A_{n,n-i} = p_{n-i} p_{n-i+1} \cdots p_{n-1} s^i \det(I_{n-i-1} - sP_{n-i}). \]

Now we prove the lemma by induction. At first, \( g_{0,1}(s) = \det(I_0 - sP_0) = 1 - r_0 s \). If (2.2) holds for \( n < k \), we calculate \( g_{0,k+1}(s) \). By (2.4),
\[ g_{0,k+1}(s) = (1 - r_k s) \det(I_{k-1} - sP_{k-1}) - q_{k,k-1} s^2 p_{k-1} \det(I_{k-2} - sP_{k-2}) - q_{k,k-2} s^3 p_{k-2} \det(I_{k-3} - sP_{k-3}) - \cdots - q_{k,0} s^{k+1} p_0 p_1 \cdots p_{k-1}. \]

Formula (2.5) tell us that \( g_{0,k+1}(s) = \det(I_k - sP_k) \). The proof is complete. \( \square \)

*Proof of Theorem 1.1* Denote \( \varphi_d(s) \) the generation function of \( \tau_{0,d} \), then \( \varphi_d(s) = f_{0,d}(s) \). By (2.1) and (2.2), we have
\[ \varphi_d(s) = f_{0,1}(s) f_{1,2}(s) \cdots f_{d-1,d}(s) = \frac{p_0 p_1 \cdots p_{d-1} s^d}{g_{0,d}(s)} = \frac{p_0 p_1 \cdots p_{d-1} s^d}{\det(I_{d-1} - sP_{d-1})}. \]
It is easy to prove that 1 is the unique unit eigenvalue of $P$, and for $i = 0, 2 \cdots d - 1$, $\lambda_i$ are the $d$ non-unit eigenvalues. So
\[
\det(I_d - sP_{d-1}) = (1 - \lambda_0 s)(1 - \lambda_1 s) \cdots (1 - \lambda_{d-1} s).
\]

By Lemma 2.1 and the definition of $g_{0,d}(s)$,
\[
p_0 p_1 \cdots p_{d-1} = s^{-d} f_{0,d}(s) \det(I_d - sP_{d-1}) \\
= s^{-d} f_{0,d}(s)(1 - \lambda_0 s)(1 - \lambda_1 s) \cdots (1 - \lambda_{d-1} s).
\]
Let $s = 1$, because $f_{0,d}$ is a generation function, $f_{0,d}(1) = 1$. Then
\[
p_0 p_1 \cdots p_{d-1} = (1 - \lambda_0)(1 - \lambda_1) \cdots (1 - \lambda_{d-1}).
\]
As a consequence we have
\[
\varphi_d(s) = \frac{(1 - \lambda_0)(1 - \lambda_1) \cdots (1 - \lambda_{d-1})s^d}{(1 - \lambda_0 s)(1 - \lambda_1 s) \cdots (1 - \lambda_{d-1} s)} \\
= \prod_{i=0}^{d-1} \frac{(1 - \lambda_i)s}{1 - \lambda_i s}.
\]

\[\square\]

### 3 Proof of Theorem 1.2

Denote the generator $Q$ of the skip-free Markov chain as
\[
Q = \begin{pmatrix}
-\gamma_0 & \alpha_0 \\
\beta_{1,0} & -\gamma_1 & \alpha_1 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\beta_{d-1,0} & \beta_{d-1,1} & \beta_{d-1,2} & \cdots & -\gamma_{d-1} & \alpha_{d-1} \\
0 & & & & & \gamma_{d-1} \\
\end{pmatrix}_{(d+1) \times (d+1)}.
\]
and for $0 \leq n \leq d - 1$, $Q_n$ denote the sub-matrix of the first $n + 1$ rows and first $n + 1$ lines of $Q$. $\tau_{i,i+j}$ be the hitting time of state $i + j$ starting from $i$. The idea of proof is similar as Theorem 1.1. We just give a briefly description here.

It is well known that the skip-free chain on the finite state has an simple structure. The process start at $i$, it stay there with an Exponential ($\gamma_i$) time, then jumps to $i+1$ with probability $\frac{\alpha_i}{\gamma_i}$, to $i-k$ with probability $\frac{\beta_{i-k,i}}{\gamma_i}$ (1 $\leq k \leq i$). Let $\tilde{f}_{i,i+j}(s)$ be the Laplace transform of $\tau_{i,i+j}$,
\[
\tilde{f}_{i,i+j}(s) = \mathbb{E} e^{-s\tau_{i,i+j}}.
\]
Recall that if a random variable $\xi \sim \text{Exponential} (\theta)$,
\[
\mathbb{E} e^{-s\xi} = \frac{\theta}{\theta + s}.
\]
By decomposing the trajectory at the first jump,

\[
\tilde{f}_{n+1}(s) = \frac{\gamma_n}{\gamma_n + s} \alpha_n + \frac{\gamma_n}{\gamma_n + s} \beta_{n-1} \tilde{f}_{n-1}(s) + \frac{\gamma_n}{\gamma_n + s} \beta_{n-2} \tilde{f}_{n-2}(s) + \cdots + \frac{\gamma_n}{\gamma_n + s} \beta_0 \tilde{f}_0(s)
\]

\[
= \frac{\alpha_n}{\gamma_n + s} + \frac{\beta_{n-1}}{\gamma_n + s} \tilde{f}_{n-1}(s) + \frac{\beta_{n-2}}{\gamma_n + s} \tilde{f}_{n-2}(s) + \cdots + \frac{\beta_0}{\gamma_n + s} \tilde{f}_0(s).
\]

Define

\[
\tilde{g}_{0,0}(s) = 1, \quad \tilde{g}_{i,j}(s) = \frac{\alpha_i \alpha_{i+1} \cdots \alpha_{i+j-1} f_{i,j}(s)}{f_{i,j}(s)}, \quad \text{for } 1 \leq j \leq d - i.
\]

The following lemma can be proved by use the method similar as Lemma 2.1, we omit the details.

**Lemma 3.1.**

\[
\tilde{g}_{n+1}(s) = \text{det}(sI_n - Q_n), \quad \text{for } 0 \leq n \leq d - 1.
\]

Then we can calculate the Laplace transform of \(\tau_{0,d}\), recall that \(\lambda_0, \ldots, \lambda_{d-1}\) are the \(d\) non-zero eigenvalues of \(-Q\), we can see they are not equal to 0 easily. And we have

\[
\alpha_0 \alpha_1 \cdots \alpha_{d-1} = \lambda_0 \lambda_1 \cdots \lambda_{d-1}.
\]

So

\[
\varphi_d(s) = \tilde{f}_{0,1}(s) \tilde{f}_{1,2}(s) \cdots \tilde{f}_{d-1,d}(s)
\]

\[
= \frac{\alpha_0 \alpha_1 \cdots \alpha_{d-1}}{\text{det}(sI_{d-1} - Q_{d-1})} = \prod_{i=0}^{d-1} \frac{\lambda_i}{1 - \lambda_i s}
\]

complete the proof. \(\square\)

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