THE NON-ARCHIMEDEAN THEORY
OF DISCRETE SYSTEMS

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Abstract. In the paper, we study behaviour of discrete dynamical systems (automata) w.r.t. transitivity; that is, speaking loosely, we consider how diverse may be behaviour of the system w.r.t. variety of word transformations performed by the system: We call a system completely transitive if, given arbitrary pair $a, b$ of finite words that have equal lengths, the system $A$, while evolution during (discrete) time, at a certain moment transforms $a$ into $b$. To every system $A$, we put into a correspondence a family $F_A$ of continuous mappings of a suitable non-Archimedean metric space and show that the system is completely transitive if and only if the family $F_A$ is ergodic w.r.t. the Haar measure; then we find easy-to-verify conditions the system must satisfy to be completely transitive. The theory can be applied to analyse behaviour of straight-line computer programs (in particular, pseudo-random number generators that are used in cryptography and simulations) since basic CPU instructions (both numerical and logical) can be considered as continuous mappings of a (non-Archimedean) metric space $Z_2$ of 2-adic integers.

1. Introduction

According to the most general definition of a system (see e.g. [13]), by a discrete system (further — a system) we understand a stationary dynamical system with a discrete time $\mathbb{N}_0 = \{0, 1, 2, \ldots \}$; that is, a 5-tuple $A = \langle I, S, O, S, O \rangle$ where $I$ is a non-empty finite set, the input alphabet; $O$ is a non-empty finite set, the output alphabet; $S$ is a non-empty (possibly, infinite) set of states; $S: I \times S \to S$ is a state transition function; $O: I \times S \to O$ is an output function. Note that in literature systems are also called (synchronous) automata; however, in order to avoid misunderstanding, in the paper only initial automata are so referred. Remind that the initial automaton $A(s_0) = \langle \mathcal{I}, S, O, S, O, s_0 \rangle$ is a discrete system $A$ where one state $s_0 \in S$ is fixed; $s_0$ is called the initial state. At the moment $n = 0$ the system $A(s_0)$ is at the state $s_0$; once fed by the input symbol $\chi_0 \in I$, the system outputs the symbol $\xi_0 = O(\chi_0, s_0) \in O$ and reaches the state $s_1 = S(\chi_0, s_0) \in S$; then the system is fed by the next input symbol $\chi_1 \in I$ and repeats the routine. We stress that the definition of the automaton $A(s_0)$ is nearly the same as the one of Mealy automaton (see e.g. [9, 13]) (or of a ‘letter’ transducer, see e.g. [2, 11]), with the only important difference: the automata $A(s_0)$ we consider in the paper are not

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necessarily finite; that is, the set of states $S$ of $\mathfrak{A}(s_0)$ may be infinite. Furthermore, throughout the paper we assume that there exists a state $s_0 \in S$ such that all the states of the system $\mathfrak{A}$ are reachable from $s_0$; that is, given $s \in S$, there exists input word $w$ over alphabet $I$ such that after the word $w$ has been fed to the automaton $\mathfrak{A}(s_0)$, the automaton reaches the state $s$. To the system $\mathfrak{A}$ we put into a correspondence the family $\mathcal{F}(\mathfrak{A})$ of all automata $\mathfrak{A}(s) = (I, S, O, S, O, s)$, $s \in S$. For better exposition, throughout the paper we assume that both alphabets $I$ and $O$ are $p$-element alphabets with $p$ prime: $I = O = \{0, 1, \ldots, p - 1\} = \mathbb{F}_p$; so further the word ‘automaton’ stands for initial automaton with input/output alphabets $\mathbb{F}_p$. A typical example of an automaton of that sort is the 2-adic adding machine $\mathcal{O}(1) = (\mathbb{F}_2, \mathbb{F}_2, \mathbb{F}_2, S, O, 1)$, where $S(\chi, s) \equiv \chi s \pmod{2}$, $O(\chi, s) \equiv \chi + s \pmod{2}$ for $s \in S = \mathbb{F}_2$, $\chi \in I = \mathbb{F}_2$.

Automata often are represented by Moore diagrams. Moore diagram of the automaton $\mathfrak{A}(s_0) = (I, S, O, S, O, s_0)$ is a directed labeled graph whose vertices are identified with the states of the automaton and for every $s \in S$ and $r \in I$ the diagram has an arrow from $s$ to $S(r, s)$ labeled by the pair $(r, O(r, s))$. Figure 1 is an example of Moore diagram.

![Figure 1. Moore diagram of the 2-adic adding machine.](image)

Given an automaton $\mathfrak{A}(s) = (I, S, O, S, O, s) \in \mathcal{F}(\mathfrak{A})$, the automaton transforms input words of length $n$ into output words of length $n$; that is, $\mathfrak{A}(s)$ maps the set $W_n$ of all words of length $n$ into $W_n$; we denote corresponding mapping via $f_n.\mathfrak{A}(s)$. It is clear now that behaviour of the system $\mathfrak{A}$ can be described in terms of the mappings $f_n.\mathfrak{A}(s)$ for all $s \in S$ and all $n \in \mathbb{N}$ = \{1, 2, 3, \ldots\}. As all states of the system $\mathfrak{A}$ are reachable from the state $s_0$, it suffices to study only the mappings $f_n.\mathfrak{A}(s_0)$ for all $n \in \mathbb{N}$. Now we remind the notion of transitivity:

**Definition 1.1** (Transitivity of a family of mappings). A family $\mathcal{F}$ of mappings of a finite non-empty set $M$ into $M$ is called transitive whenever given a pair $(a, b) \in M \times M$, there exists $f \in \mathcal{F}$ such that $f(a) = b$.

It is clear that once $M$ contains more than one element, a family that consists of a single mapping $f: M \to M$ cannot be transitive in the meaning of Definition 1.1; that is why the transitivity of a single mapping is defined as follows:

**Definition 1.2** (Transitivity of a single mapping). A mapping $f: M \to M$, where $M$ is a finite non-empty set, is called transitive if it $f$ cyclically permutes elements of $M$.

In other words, a single mapping $f: M \to M$ is transitive if and only if the family $\{e, f, f^2 = f \circ f, f^3 = f \circ f \circ f, \ldots\}$ is transitive in the meaning of the Definition
Definition 1.3 (Automata transitivity). The automaton \( \mathcal{A}(s_0) \) (equivalently, the system \( \mathcal{A} \)) is said to be:

- **n-word transitive**, if the mapping \( f_n, \mathcal{A}(s_0) \) is transitive on the set \( W_n \) of all words of length \( n \);
- **word transitive**, if \( \mathcal{A}(s_0) \) is \( n \)-word transitive for all \( n \in \mathbb{N} = \{1, 2, 3, \ldots\} \);
- **completely transitive**, if for every \( n \in \mathbb{N} \), the family \( f_n, \mathcal{A}(s) \), \( s \in \mathcal{S} \), is transitive on \( W_n \);
- **absolutely transitive**, if for every \( s \in \mathcal{S} \) the automaton \( \mathcal{A}(s) \) is completely transitive; that is, if for every \( n \in \mathbb{N} \) the family \( f_n, \mathcal{A}(t) \), \( t \in \mathcal{S}_{\mathcal{A}(s)} \), is transitive on \( W_n \), where \( \mathcal{S}_{\mathcal{A}(s)} \) is the set of all reachable states of the automaton \( \mathcal{A}(s) \).

The transitivity properties may be defined in equivalent way, in terms of words; this way is more common in automata theory. We remand some notions related to words beforehand.

Given a non-empty alphabet \( \mathcal{A} \), its elements are called **symbols**, or **letters**. By the definition, a **word of length \( n \) over alphabet \( \mathcal{A} \)** is a finite string (stretching from right to left) \( \alpha_{n-1} \cdots \alpha_1 \alpha_0 \), where \( \alpha_{n-1}, \ldots, \alpha_1, \alpha_0 \in \mathcal{A} \). The empty word is a sequence of length 0, that is, the one that contains no symbols. Hereinafter the length of the word \( w \) is denoted via \( \Lambda(w) \). Given a word \( w = \alpha_{n-1} \cdots \alpha_1 \alpha_0 \), any word \( v = \alpha_{k-1} \cdots \alpha_1 \alpha_0 \), \( n \geq k \geq 1 \), is called a prefix (or, an **initial subword**) of the word \( w \), any word \( u = \alpha_{n-1} \cdots \alpha_{i+1} \alpha_i \), \( 0 \leq i \leq n - 1 \) is called a suffix of the word \( w \), and any word \( \alpha_k \cdots \alpha_{i+1} \alpha_i \), \( n - 1 \geq k \geq i \geq 0 \), is called a subword of the word \( w \). Given words \( a = \alpha_{n-1} \cdots \alpha_1 \alpha_0 \) and \( b = \beta_{k-1} \cdots \beta_1 \beta_0 \), the concatenation \( a \circ b \) is the following word (of length \( n + k \)):

\[
a \circ b = \alpha_{n-1} \cdots \alpha_1 \alpha_0 \beta_{k-1} \cdots \beta_1 \beta_0.
\]

Definition 1.4 (Automata transitivity, equivalent).

(i) The **word transitivity** means that given two finite words \( w, w' \) whose lengths are equal one to another, \( \Lambda(w) = \Lambda(w') = n \), the word \( w \) can be transformed into \( w' \) by a sequential composition of a sufficient number of copies of \( \mathcal{A}(s_0) \):

(ii) The **complete transitivity** means that given finite words \( w, w' \) such that \( \Lambda(w) = \Lambda(w') \), there exists a finite word \( y \) (may be of length other than that of \( w \) and \( w' \)) such that the automaton \( \mathcal{A}(s_0) \) transforms the input word \( w \circ y \) (with the prefix \( y \)) to the output word \( w' \circ y' \) that has a suffix \( w' \):

\[
\text{\includegraphics[width=\textwidth]{transitivity-diagram.png}}
\]
(iii) The absolute transitivity means that given finite words \( x, w, w' \) such that \( \Lambda(w) = \Lambda(w') \) (may be \( \Lambda(x) \neq \Lambda(w) \)), there exists a finite word \( y \) such that the automaton \( \mathfrak{A}(s_0) \) transforms the input word \( w \circ y \circ x \) to the output word \( w' \circ y' \circ x' \):

![Diagram of \( \mathfrak{A} \)]

**Example 1.5** (Word transitive automaton). The 2-adic adding machine \( \mathcal{O}(1) \), which was introduced above, is word transitive: It is clear that if one treats an \( n \)-bit word as a base-2 expansion of a non-negative integer \( w \) then \( f_{n,\mathcal{O}(1)}(w) \equiv w + 1 \) (mod \( 2^n \)), \( n = 1, 2, 3, \ldots \); therefore \( f_{n,\mathcal{O}(1)}(w) \equiv w + i \) (mod \( 2^n \)) for all \( i \in \mathbb{N}_0 = \{0, 1, 2, \ldots \} \) which means that \( f \) is transitive on the set \( W_n \) of all \( n \)-bit words, cf. Definition 1.3, Definition 1.2 and Definition 1.4(i).

Note that the 2-adic adding machine \( \mathcal{O}(1) \) is not completely transitive as given \( n \in \mathbb{N} = \{1, 2, 3, \ldots \} \), the corresponding family consists of the following two mappings: \( f_{n,\mathcal{O}(1)}(w) \equiv w + 1 \) (mod \( 2^n \)) and \( f_{n,\mathcal{O}(0)}(w) \equiv w \) (mod \( 2^n \)); so none of the mappings maps the two-bit word 00 (which is a base-2 expansion of 0) to the two-bit word 10 (which is a base-2 expansion of 2).

**Example 1.6** (Absolutely transitive automaton). Let \( (\alpha_i)_{i=0}^\infty = \alpha_0, \alpha_1, \ldots \) be an infinite binary sequence such that every binary pattern \( \beta_1 \cdots \beta_n \) occurs in the sequence \( (\alpha_i)_{i=0}^\infty \) (whence, occurs infinitely many times); that is, given \( n \in \mathbb{N} = \{1, 2, 3, \ldots \} \) and \( \beta_1, \ldots, \beta_n \in \mathbb{F}_2 \), the following equalities \( \alpha_i = \beta_1, \alpha_{i+1} = \beta_2, \ldots, \alpha_{i+n-1} = \beta_n \) hold simultaneously for some (equivalently, for infinitely many) \( i \in \mathbb{N}_0 = \{0, 1, 2, \ldots \} \). Then the following automaton \( \mathcal{E}(0) \) is absolutely transitive: \( \mathcal{E}(0) = \langle \mathbb{F}_2, \mathbb{N}_0, \mathbb{F}_2, S, O, 0 \rangle \), where \( S(\chi, s) = s + 1, O(\chi, s) = \alpha_s \) for \( s \in S = \mathbb{N}_0, \chi \in J = \mathbb{F}_2 \).

Indeed, given an \( n \)-bit word \( w \), we see that \( f_{n,\mathcal{E}(s)}(w) = \alpha_{s+n-1} \cdots \alpha_s \) for every \( s \in \mathbb{N}_0 \) which by Definition 1.3(iii) (or, equivalently, by Definition 1.3) implies absolute transitivity of the automaton \( \mathcal{E}(0) \) due to the choice of the sequence \( (\alpha_i)_{i=0}^\infty \).

Note also that the automaton \( \mathcal{E}(0) \) is \( n \)-word transitive for no \( n \in \mathbb{N} = \{1, 2, 3, \ldots \} \) as \( f_{n,\mathcal{E}(0)} \) is not bijective on \( W_n \), cf. Definitions 1.2 and 1.3.

The goal of the paper is to present techniques to determine whether a system \( \mathfrak{A} \) is word transitive, or completely transitive, or absolutely transitive. For this purpose, we study how the automaton \( \mathfrak{A}(s_0) \) acts on infinite words over alphabet \( \mathbb{F}_p \). The latter words are considered as \( p \)-adic integers, and the corresponding transformation turns out to be a continuous transformation on the space of \( p \)-adic integers \( \mathbb{Z}_p \). We remind main notions of \( p \)-adic analysis in the next section where we describe our approach, first formally and then less formally.

We note that the \( p \)-adic approach (and wider the non-Archimedean one) has already been successfully applied to automata theory. Seemingly the paper [17] is the first one where the \( p \)-adic techniques is applied to study automata functions; the paper deals with linearity conditions of automata mappings. For application of the non-Archimedean methods to automata and formal languages see expository paper [18] and references therein; for applications to automata and group theory
see [12] [11]. In [21] [23] [22] the 2-adic methods are used to study binary automata, in particular, to obtain the finiteness criterion for these automata. In monograph [5] the $p$-adic ergodic theory is studied (see numerous references therein) aiming at applications to computer science and cryptography (in particular, to automata theory, to pseudorandom number generation and to stream cipher design) as well as to applications in other areas like quantum theory, cognitive sciences and genetics.

As for mathematical techniques used in the paper, these are somewhat complex: to study ergodic properties of families of automata functions related to a given discrete system, we combine $p$-adic methods, methods of real analysis and methods from automata theory.

The paper is organized as follows:

- In Section 2 we remind basic notions of $p$-adic analysis and show that automata functions (the transformations of infinite words performed by automata) are continuous (actually, 1-Lipschitz) functions w.r.t. the $p$-adic metric. In particular, we mention that basic computer instructions, both arithmetic (like addition, subtraction and multiplication of integers) and bitwise logical (like bitwise conjunction, disjunction, negation and exclusive ‘or’), as well as some other (like shifts towards higher order bits and masking) are continuous w.r.t. 2-adic metric.
- In Section 3 we remind basics of the $p$-adic ergodic theory in connection to automata functions.
- Section 4 contains main results of the paper: By plotting an automaton function into real unit square we establish the automata 0-1 law and find sufficient conditions for a system to be completely transitive or absolutely transitive.
- We conclude in Section 5.

2. The $p$-adic representation of automata functions

Every (left-)infinite word $\ldots \chi_2 \chi_1 \chi_0$ over the alphabet $\mathbb{F}_p$ can be associated to the $p$-adic integer $\chi_0 + \chi_1 p + \chi_2 p^2 + \cdots$ which is an element of the ring $\mathbb{Z}_p$ of $p$-adic integers; the ring $\mathbb{Z}_p$ is a complete compact metric space w.r.t. $p$-adic metric (we remind the notion below). The automaton $\mathcal{A}(s_0)$ maps infinite words to infinite words. Denote the corresponding mapping via $f = f_{\mathcal{A}(s_0)}$; then $f$ is a function defined on $\mathbb{Z}_p$ and valuated in $\mathbb{Z}_p$. The function $f = f_{\mathcal{A}(s_0)} : \mathbb{Z}_p \to \mathbb{Z}_p$ is called the automaton function of the automaton $\mathcal{A}(s_0)$. For instance, the automaton function $f_{\mathcal{O}(1)}$ of the 2-adic adding machine $\mathcal{O}(1)$ is the 2-adic odometer, the transformation $f_{\mathcal{O}(1)}(x) = x + 1$ of the ring $\mathbb{Z}_2$ of 2-adic integers; whereas the automaton function $f_{\mathcal{E}(0)}$ of the automaton $\mathcal{E}(0)$ from Example 1.6 is a constant function on $\mathbb{Z}_2$: $f_{\mathcal{E}(0)}(x) = \sum_{i=0}^{\infty} \alpha_i 2^i \in \mathbb{Z}_2$.

Due to the fact that at every moment $n = 0, 1, 2, \ldots$, the $n$-th output symbol may depend only on the input symbols $\chi_0, \chi_1, \ldots, \chi_n$ that have been feeded to the automaton at the moments $0, 1, \ldots, n$ respectively, the automaton function is a $p$-adic 1-Lipschitz function; that is, $f$ satisfies the $p$-adic Lipschitz condition with the constant 1 w.r.t. $p$-adic metric and thus $f$ is a $p$-adic continuous function. Vice versa, given a 1-Lipschitz function $f : \mathbb{Z}_p \to \mathbb{Z}_p$, there exists an automaton $\mathcal{A}(s_0)$ such that $f = f_{\mathcal{A}(s_0)}$, see further Theorem 2.1. Therefore to study the behavior of the system $\mathcal{A}$ we may (and will) study corresponding automata functions rather than automata themselves; and to study the behaviour of the latter functions we
may apply techniques from $p$-adic analysis and $p$-adic dynamics, see [5]. This is the key point of our approach.

We remind that the space $\mathbb{Z}_p$ is the completion of the ring $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots \}$ of (rational) integers w.r.t. the $p$-adic metric $d_p$ which is defined as follows: given $a, b \in \mathbb{Z}$, if $a \neq b$ then denote $p^{\text{ord}_p(a-b)}$ the largest power of $p$ that divides $a - b$ and put $d_p(a, b) = \|a - b\|_p = p^{-\text{ord}_p(a-b)}$, put $\|a - b\|_p = 0$ if $a = b$. The $p$-adic metric violates the Archimedean Axiom and thus is called a non-Archimedean metric (or, an ultrametric). Now we describe our approach less formally.

Multiplication and addition of infinite words over alphabet $\mathbb{F}_p$ can be defined via school-textbook-like algorithms for multiplication/addition of integers represented by base-$p$ expansions. For instance, in case of 2-adic integers (i.e., when $p = 2$) the following example shows that $-1 = \ldots 11111$ in $\mathbb{Z}_2$ (as $\ldots 0001 = 1$):

\[
\begin{array}{cccccc}
\ldots & 1 & 1 & 1 & 1 \\
+ & \ldots & 0 & 0 & 0 & 1 \\
\hline \\
\ldots & 0 & 0 & 0 & 0 & 0
\end{array}
\]

The next example shows that $\ldots 1010101 = -\frac{1}{3}$ in $\mathbb{Z}_2$ (as $\ldots 00011 = 3$):

\[
\begin{array}{cccccc}
\ldots & 0 & 1 & 0 & 1 & 0 & 1 \\
\times & \ldots & 0 & 0 & 0 & 0 & 1 & 1 \\
\hline \\
\ldots & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
+ & \ldots & 1 & 0 & 1 & 0 & 1 & 1 \\
\hline \\
\ldots & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}
\]

The set of all infinite words over the alphabet $\mathbb{F}_p$ with so defined operations (and distance) constitutes the ring (and the metric space) $\mathbb{Z}_p$. Note that $\mathbb{Z}_p$ contains the ring of all (rational) integers $\mathbb{Z}$ as well as some other elements from the field $\mathbb{Q}$ of rational numbers; so $\mathbb{Z}_p \cap \mathbb{Q} \supsetneq \mathbb{Z}$. For instance, in $\mathbb{Z}_2$ the sequences that contain only finite number of 1-s correspond to non-negative rational integers represented by their base-2 expansions (e.g., $\ldots 00011 = 3$); the sequences that contain only finite number of 0-s correspond to negative rational integers (e.g., $\ldots 111100 = -4$); the sequences that are (eventually) periodic correspond to rational numbers that can be represented by irreducible fractions with odd denominators (e.g., $\ldots 1010101 = -\frac{1}{3}$); and non-periodic sequences correspond to no rational numbers. It is also worth noticing that when $p = 2$, the 2-adic integers representing negative rational integers may be regarded as 2’s complements of the latter (cf. e.g. [19, 15]). In computer science, 2-adic representations of rational integers are also known as Hensel codes, cf. [15], after the name of German mathematician Kurt Hensel who discovered $p$-adic numbers more than a century ago.
By the definition, given two infinite words \( \cdots \chi_2 \chi_1 \chi_0 \) and \( \cdots \xi_2 \xi_1 \xi_0 \) over the alphabet \( \mathbb{F}_p \), the distance \( d_p \) between these words is \( p^{-n} \), where \( n = \min \{i = 0, 1, 2, \ldots : \chi_i \neq \xi_i \} \), and the distance is 0 if no such \( n \) exists. For instance, in the case \( p = 2 \) we have that

\[
\begin{align*}
\ldots 101010101 &= -\frac{1}{3} \\
\ldots 0000010101 &= 5
\end{align*}
\]

\[d_2 \left( -\frac{1}{3}, 5 \right) = \left\| \left( -\frac{1}{3} \right) - 5 \right\|_2 = \frac{1}{2^2} = \frac{1}{16}.
\]

In other words, \(-\frac{1}{3} \equiv 5 \mod 16\); \(-\frac{1}{3} \not\equiv 5 \mod 32\). Note that actually \( \operatorname{mod} p^k \), the reduction modulo \( p^k \), is an epimorphism of \( \mathbb{Z}_p \) onto the residue ring \( \mathbb{Z}/p^k\mathbb{Z} \) modulo \( p^k \) (we associate elements of the latter ring to \( 0 \ldots 9 \))

\[
\begin{equation}
\operatorname{mod} p^k : \mathbb{Z}_p \to \mathbb{Z}/p^k\mathbb{Z}, \quad \left( \sum_{i=0}^{\infty} \alpha_i p^i \right) \mod p^k = \sum_{i=0}^{k-1} \alpha_i p^i,
\end{equation}
\]

where \( \alpha_i \in \mathbb{F}_p \). Thus, for \( a, b \in \mathbb{Z}_p \), the following equivalences hold:

\[
\|a-b\|_p \leq p^{-k} \quad \text{if and only if} \quad a \equiv b \mod p^k.
\]

Due to equivalence (2.2), one may use congruences between \( p \)-adic numbers rather than inequalities for \( p \)-adic absolute values which is sometimes more convenient during proofs. The advantage of using congruences rather than inequalities in \( p \)-adic analysis over \( \mathbb{Z}_p \) is that one may work with congruences by applying standard number-theoretic techniques; e.g., add or multiply congruences sidewise, etc. More about this in [5].

Metrics on Cartesian powers \( \mathbb{Z}_p^n \) can be defined in a manner similar to that of the case \( n = 1 \):

\[
\| (a_1, \ldots, a_n) - (b_1, \ldots, b_n) \|_p = \max \{ \| a_j - b_j \|_p : j = 1, 2, \ldots, n \}
\]

for every \( (a_1, \ldots, a_n), (b_1, \ldots, b_n) \in \mathbb{Z}_p^n \); so \( p \)-adic continuous multi-variate functions defined on and valued at \( \mathbb{Z}_p \) can be considered as well.

Once the metric is defined, one can speak of limits, of continuous functions, of derivatives, of convergent series, etc.; that is, of \( p \)-adic Calculus. We refer to the numerous books on \( p \)-adic analysis (e.g., [10] [14] [10] [20] ) for further details.

An important example of continuous 2-adic functions are basic computer instructions, both arithmetic (addition, multiplication, subtraction) and bitwise logical (AND, the bitwise conjunction; OR, the bitwise disjunction; XOR, the bitwise exclusive ‘or’; NOT, the bitwise negation) and some others (shifts towards higher order bits, masking). All these instructions can be regarded as (univariate or two-variate) 1-Lipschitz functions defined on and valued in the space of 2-adic integers \( \mathbb{Z}_2 \), [5]. That is why the theory we develop finds numerous applications in computer science and cryptology: the straight-line programs (and more complicated ones) combined from the mentioned instructions can also be regarded as continuous 2-adic mappings; so behaviour of these programs can be analysed by techniques of the non-Archimedean dynamics, see e.g. [3] [4] [5] [6] [7] [8]. It is worth noticing here that similar approaches work effectively also in genetics, cognitive sciences, image processing, quantum theory, etc., see comprehensive monograph [5] and references therein.

Concluding the section, we now give a formal proof that the class of all automata functions \( f_{\mathfrak{A}(s_0)} \) of automata of the form \( \mathfrak{A}(s_0) = \langle \mathbb{F}_p, S, \mathcal{F}_p, S, O, s_0 \rangle \) coincides with
where

Given a \( \omega \)-function \( \text{Automata functions are 1-Lipschitz functions and vice versa} \)

Theorem 2.1

Conversely, for every 1-Lipschitz function \( f : Z_p \to Z_p \) there exists an automaton \( \mathfrak{A}(s_0) = (F_p, S, F_p, S, O, s_0) \) such that \( f = f_{\mathfrak{A}(s_0)} \).

Proof. Given a \( p \)-adic integer \( z \in Z_p \), denote via \( \delta_i(z) \in F_p \), the \( i \)-th ‘\( p \)-adic digit’ of \( z \); that is, the \( i \)-th term coefficient in the \( p \)-adic representation of \( z = \sum_{i=0}^{\infty} \delta_i(z)p^i \).

As \( s_i = S(\delta_i(z), s_{i-1}) \) for every \( i = 1, 2, \ldots \), the \( i \)-th output symbol \( \xi_i = \delta_i(f_{\mathfrak{A}}(z)) \)
depends only on input symbols \( \chi_0, \chi_1, \ldots, \chi_i \); that is

\[
\delta_i(f_{\mathfrak{A}}(z)) = \psi_i(\delta_0(z), \delta_1(z), \ldots, \delta_i(z))
\]

for all \( i = 0, 1, 2, \ldots \) and for suitable mappings \( \psi_i : F_{p+1} \to F_p \). That is, \( f = f_{\mathfrak{A}(s_0)} : Z_p \to Z_p \) is of the form

\[
f : x = \sum_{i=0}^{\infty} \chi_ip^i \mapsto f(x) = \sum_{i=0}^{\infty} \psi_i(\chi_0, \ldots, \chi_i)p^i.
\]

This means that the function \( f_{\mathfrak{A}(s_0)} \) is 1-Lipschitz by [3, Proposition 3.35] as the mentioned proposition in application to the mappings we consider here can be restated as follows: A mapping \( f : Z_p \to Z_p \) is 1-Lipschitz if and only if \( f \) can be represented in the form \( (2.3) \) for suitable mappings \( \psi_i : F_{p+1} \to F_p, i = 0, 1, 2, \ldots \).

Conversely, let \( f : Z_p \to Z_p \) be a 1-Lipschitz mapping; then by [3, Proposition 3.35] \( f \) can be represented in the form \( (2.3) \) for suitable mappings \( \psi_i : F_{p+1} \to F_p, i = 0, 1, 2, \ldots \). We now construct an automaton \( \mathfrak{A}(s_0) = (F_p, S, F_p, S, O, s_0) \) such that \( f_{\mathfrak{A}(s_0)} = f \).

Let \( F_p^* \) be a set of all non-empty finite words over the alphabet \( F_p \). We consider these words as base-\( p \)-expansions of numbers from \( \mathbb{N} = \{1, 2, 3, \ldots \} \) and enumerate all these words by integers \( 1, 2, 3, \ldots \) in radix order in accordance with the natural order on \( F_p \), \( 0 < 1 < 2 < \cdots < p-1 \):

\[
0 < 1 < 2 < \cdots < p-1 < 00 < 01 < 02 < \cdots < 0(p-1) < 10 < 11 < 12 < \cdots;
\]

so that \( \nu(0) = 1, \nu(1) = 2, \nu(2) = 3, \ldots, \nu(p-1) = p, \nu(00) = p+1, \nu(01) = p+2, \ldots \).

This way we establish a one-to-one correspondence between the words \( w \in F_p^* \) and integers \( i \in \mathbb{N} : w \leftrightarrow \nu(w), \ i \leftrightarrow \omega(i) \ (\nu(w) \in \mathbb{N}, \ \omega(i) \in F_{p^*}) \). Note that \( \nu(\omega(i)) = i, \ \omega(\nu(w)) = w \) for all \( i \in \mathbb{N} \) and all non-empty words from \( w \in F_p^* \). Define \( \omega(0) \) to be the empty word.

Now put \( S = \mathbb{N}_0 = \{0, 1, 2, 3, \ldots \} \), the set of all states of the automaton \( \mathfrak{A}(s_0) \) under construction, and take the initial state \( s_0 = 0 \). The state transition function \( S \) is defined as follows:

\[
S(r, i) = \nu(r \circ \omega(i)),
\]

where \( i = 0, 1, 2, \ldots \) and \( r \in F_p \). That is, \( S(r, i) \) is the number of the word \( r \circ \omega(i) \) which is a concatenation of the word \( \omega(i) \) (the word whose number is \( i \)), the prefix, with the single-letter word \( r \), the suffix.
Now consider a one-to-one mapping \( \theta_n(\chi_{n-1} \cdots \chi_1 \chi_0) = (\chi_0, \chi_1, \ldots, \chi_{n-1}) \) from the \( n \)-letter words onto \( \mathbb{F}_p^n \) and define the output function of the automaton \( \mathfrak{A}(0) \) as follows:

\[
O(r, i) = \psi_{\Lambda(\omega(i))}(\theta_{\Lambda(\omega(i))}+1(r \circ \omega(i))),
\]

where \( i = 0, 1, 2, \ldots \) and \( r \in \mathbb{F}_p \). Remind that we denote via \( \Lambda(w) \) the length of the word \( w \).

The idea of the construction is illustrated by Figure 2 which depicts Moore diagram of the automaton \( \mathfrak{A}(0) \) for the case \( p = 2 \):

![Figure 2. Moore diagram of the automaton \( \mathfrak{A}(0) \), \( p = 2 \); so \( \omega(0) \) is the empty word, \( \omega(1) = 0 \), \( \omega(2) = 1 \), \( \omega(3) = 00 \), \( \omega(4) = 01 \), \( \omega(5) = 10 \), \( \omega(6) = 11 \ldots \)]

Now, as both \( f \) and \( f_{\mathfrak{A}(s_0)} \) are 1-Lipschitz, thus continuous with respect to the \( p \)-adic metric, and as \( \mathbb{N}_0 \) is dense in \( \mathbb{Z}_p \), to prove that \( f = f_{\mathfrak{A}(s_0)} \) is suffices to show that

\[
f_{\mathfrak{A}(s_0)}(\tilde{w}) \equiv f(\tilde{w}) \pmod{p^{\Lambda(w)}}
\]

for all finite non-empty words \( w \in \mathbb{F}_p^* \), where \( \tilde{w} \in \mathbb{N}_0 \) stands for the integer whose base-\( p \) expansion is \( w \). We prove that (2.6) holds for all \( w \in \mathbb{F}_p^* \) once \( \Lambda(w) = n > 0 \) by induction on \( n \).

If \( n = 1 \) then \( \tilde{w} \in \mathbb{F}_p \); so once \( w \) is seeded to \( \mathfrak{A} \), the automaton reaches the state \( S(w, 0) = \nu(w) \) (cf. (2.4)) and outputs \( O(w, 0) = \psi_{\Lambda(\omega(w))}(\theta_{\Lambda(\omega(w))}+1(\nu(w))) \equiv f(\tilde{w}) \pmod{p} \) (cf. (2.4)), see (2.3). Thus, (2.6) holds in this case.

Now assume that (2.6) holds for all \( w \in \mathbb{F}_p^* \) such that \( \Lambda(w) = n < k \) and prove that (2.6) holds also when \( \Lambda(w) = n = k \). Represent \( w = r \circ v \), where \( r \in \mathbb{F}_p \) and \( \Lambda(v) = n - 1 \). By the induction hypothesis, after the word \( v \) has been seeded to \( \mathfrak{A} \), the automaton reaches the state \( \nu(v) \) and outputs the word \( v_1 \) of length \( n - 1 \) such that \( \tilde{v}_1 \equiv f(\tilde{v}) \pmod{p^{n-1}} \). Next, being fed by the letter \( r \), the automaton (which is in the state \( \nu(v) \) now) outputs the letter \( O(r, \nu(v)) = \psi_{\Lambda(\omega(\nu(v)))}(\theta_{\Lambda(\omega(\nu(v)))}+1(r \circ \omega(\nu(v))) = \psi_{\Lambda(v)}(\theta_{\Lambda(v)}+1(r \circ \nu(v))) \circ v_1 \). This means that once fed by \( w \), the automaton \( \mathfrak{A}(s_0) \) outputs the word \( v_2 = (\psi_{\Lambda(v)}(\theta_{\Lambda(v)}+1(r \circ \nu(v)))) \circ v_1 \). Now note that \( \tilde{v}_2 \equiv f(\tilde{w}) \pmod{p^n} \) by (2.3). \( \square \)
Note 2.2. From the proof of Theorem 2.1 it is clear that the mapping \( f_{n,\mathfrak{A}(s_0)}: \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z} \) is just a reduction modulo \( p^n \) of the automaton function \( f_{\mathfrak{A}(s_0)}: f_{n,\mathfrak{A}(s_0)} = f_{\mathfrak{A}(s_0)} \mod p^n \) for all \( n = 1, 2, 3, \ldots \).

Note 2.3. In automata theory, word transducers (or, asynchronous automata) are also considered; the latter are automata that allow (possibly empty) words as output for each transition. Although the automata we consider are all synchronous (i.e., letter transducers rather than word transducers), it is worth mentioning here that the automaton function of a word transducer whose input/output alphabets are \( \mathbb{F}_p \) can also be considered as a continuous (however, not necessarily 1-Lipschitz any longer) mapping from \( \mathbb{Z}_p \) to \( \mathbb{Z}_p \) once the transducer is non-degenerate, see [12, Theorem 2.4].

Further in the paper, given a 1-Lipschitz function \( f: \mathbb{Z}_p \to \mathbb{Z}_p \), via \( \mathfrak{A}_f(s_0) \) we denote an automaton \( \langle \mathbb{F}_p, \mathcal{S}, \mathbb{F}_p, \mathcal{S}, O, s_0 \rangle \) whose automaton function is \( f \); that is, \( f_{\mathfrak{A}_f(s_0)} = f \). Note that given \( f \), the automaton \( \mathfrak{A}_f(s_0) \) is not unique: There are numerous automata that have the same automaton function \( f \). Nonetheless, this non-uniqueness will not cause misunderstanding since in the paper we are mostly interested with automata functions rather than with ‘internal structure’ (e.g., with state sets, state transition and output functions, etc.) of automata themselves.

3. The \( p \)-adic ergodic theory and transitivity of automata

The ring \( \mathbb{Z}_p \) can be endowed with a probability measure \( \mu_p \): Elementary \( \mu_p \)-measurable sets are balls \( B_{p^{-r}}(a) = \{ z \in \mathbb{Z}_p : z \equiv a \pmod{p^r} \} \) of radii \( p^{-r}, r = 1, 2, \ldots \), centered at \( a \in \mathbb{Z}_p \). In other words, the ball \( B_{p^{-r}}(a) \) is a set of all infinite words over alphabet \( \mathbb{F}_p = \{ 0, 1, \ldots, p-1 \} \) that have common prefix of length \( r \). We put \( \mu_p(B_{p^{-r}}(a)) = p^{-r} \) thus endowing \( \mathbb{Z}_p \) with a probability measure \( \mu_p \) (which actually is a normalized Haar measure). Note that all 1-Lipschitz mappings \( f: \mathbb{Z}_p \to \mathbb{Z}_p \) are \( \mu_p \)-measurable (i.e., \( f^{-1}(S) \) is \( \mu_p \)-measurable once \( S \subset \mathbb{Z}_p \) is \( \mu_p \)-measurable).

A \( \mu_p \)-measurable mapping \( f: \mathbb{Z}_p \to \mathbb{Z}_p \) is called ergodic if the two following conditions hold simultaneously:

(i) \( f \) preserves the measure \( \mu_p \); i.e., \( \mu_p(f^{-1}(S)) = \mu_p(S) \) for each \( \mu_p \)-measurable subset \( S \subset \mathbb{Z}_p \), and

(ii) \( f \) has no proper invariant \( \mu_p \)-measurable subsets: \( f^{-1}(S) = S \) implies either \( \mu_p(S) = 0 \), or \( \mu_p(S) = 1 \).

A family \( \mathcal{F} = \{ f_i : i \in I \} \) of \( \mu_p \)-measurable mappings \( f_i : \mathbb{Z}_p \to \mathbb{Z}_p \) (which are not necessarily measure-preserving mappings) is called ergodic if the mappings \( f_i, i \in I \), have no common \( \mu_p \)-measurable invariant subset other than sets of measure 0 or 1; that is, if there exists a \( \mu_p \)-measurable subset \( S \subset \mathbb{Z}_p \) such that \( f_i^{-1}(S) = S \) for all \( i \in I \), then necessarily either \( \mu_p(S) = 0 \), or \( \mu_p(S) = 1 \).

Note that in the paper speaking of ergodicity of a single mapping we always mean the mapping is measure-preserving; whereas in general ergodic theory the non-measure-preserving ergodic mappings (the ones that satisfy only the second condition (ii) of the above two) are sometimes also concerned. To illustrate the notion of ergodicity we use, consider a finite set \( M \) endowed with a natural probability measure \( \mu(A) = \#A/\#M \) for all \( A \subset M \) (where \( \#A \) stands for the number of elements in \( A \)). The measure-preservation of the mapping \( f : M \to M \) is equivalent to the bijectivity of \( f \), whereas the ergodicity of \( f \) (when respective conditions
(i) and (ii) hold simultaneously) is equivalent to the transitivity of the mapping $f$ in the meaning of Definition 1.2 and the ergodicity of the family $\mathcal{F}$ of mappings $f_i: M \rightarrow M$, $i \in I$, is equivalent to the transitivity of the family $\mathcal{F}$ in the meaning of Definition 1.1.

As in the paper we deal with the only measure $\mu_p$, so further speaking of measure-preservation (as well as of measurability and of ergodicity) we omit mentioning the respective measure. From the $p$-adic ergodic theory (see [5]) the following theorem can be deduced:

**Theorem 3.1.** A system $\mathcal{A} = \langle \mathcal{F}_p, \mathcal{S}, \mathcal{E}_p, S, O \rangle$ is word transitive if and only if the automaton function $f_{\mathcal{A}(s_0)}$ on $\mathbb{Z}_p$ is ergodic. If the system $\mathcal{A}$ is completely transitive, the family $f_{\mathcal{A}(s)}$, $s \in \mathcal{S}$, of automata functions is ergodic.

Remind that under conventions from the beginning of the paper, $s_0$ is the state of the system $\mathcal{A}$ such that all other states are reachable from $s_0$.

Theorem 3.1 implies a number of various methods to determine the word transitivity of automata: For instance, a binary automaton $\mathcal{A}$ (that is an automaton with a binary input and output; i.e., with $p = 2$) whose automaton function $f_{\mathcal{A}}$ is a polynomial with integer coefficients (i.e., $f_{\mathcal{A}} = g$ where $g(x) \in \mathbb{Z}[x]$) is word transitive if and only if it is 3-word transitive; that is, the transitivity of $\mathcal{A}$ on the set $W_3$ of all binary words of length 3 is equivalent to the transitivity of $\mathcal{A}$ on the set $W_n$ of all binary words of length $n$, for all $n = 1, 2, 3, \ldots$. Moreover, a binary automaton $\mathcal{A}$ is word transitive if and only if its automaton function is of the form $f_{\mathcal{A}}(x) = 1 + x + 2(g(x+1) - g(x))$, where $g = g_{\mathcal{A}}$ is an automaton function of some binary automaton $\mathcal{G}$. For other results of this sort and for the whole $p$-adic ergodic theory see [5]. Although complete transitivity of the system $\mathcal{A} = \langle \mathcal{F}_p, \mathcal{S}, \mathcal{E}_p, S, O \rangle$ is also related to ergodicity; however, to the ergodicity of the family of automata functions $f_{\mathcal{A}(s)}$, $s \in \mathcal{S}$, cf. Definition 1.3 and Theorem 3.1 rather than to the ergodicity of a single automaton function $f_{\mathcal{A}(s_0)}$. This is why to determine complete/absolute transitivity rather than just word transitivity we need some more sophisticated techniques that are discussed further.

4. Plots of automata functions on the real plane

Remind that under conventions from the beginning of the paper, there exists a state $s_0$ of the system $\mathcal{A}$ such that all other states are reachable from $s_0$; so although further results of the paper are stated mostly for automata, they hold for systems as well.

Given an automaton $\mathcal{A}(s_0)$, consider the corresponding automaton function $f = f_{\mathcal{A}(s_0)}: \mathbb{Z}_p \rightarrow \mathbb{Z}_p$. Denote $E_k(f)$ the set of all the following points $e_k^f(x)$ of closed Euclidean unit square $\mathbb{I}^2 = [0; 1] \times [0; 1] \subset \mathbb{R}^2$:

$$e_k^f(x) = \left(\frac{x \mod p^k}{p^k}, \frac{f(x) \mod p^k}{p^k}\right),$$

where $x \in \mathbb{Z}_p$ and $\mod p^k$ is a reduction modulo $p^k$, cf. [21]. Note that $x \mod p^k$ corresponds to the prefix of length $k$ of the infinite word $x \in \mathbb{Z}_p$, i.e., to the input word of length $k$ of the automaton $\mathcal{A}(s_0)$; while $f(x) \mod p^k$ corresponds to the respective output word of length $k$. That is, given an input word $w = \chi_{k-1} \cdots \chi_1 \chi_0$ and the corresponding output word $w' = \xi_{k-1} \cdots \xi_1 \xi_0$, we consider in $\mathbb{I}^2$ the set of
all points 

\[(\chi_{k-1}p^{-1} + \cdots + \chi_1 p^{-k+1} + \chi_0 p^{-k}, \xi_{k-1} p^{-1} + \cdots + \xi_1 p^{-k+1} + \xi_0 p^{-k}),\]

for all pairs \((w, w')\) of input/output words of length \(k\).

The set \(E_k(f)\) may be considered as a sort of a plot of the automaton function \(f\) on the real plane \(\mathbb{R}^2\). The plot characterizes behaviour of the automaton; namely, it can be observed that basically the behaviour is of two types only:

(i) as \(k \to \infty\), the point set \(E_k(f)\) is getting more and more dense (cf. Fig. 3–6, \(p = 2\)), or

(ii) \(E_k(f)\) is getting less and less dense while \(k \to \infty\), cf. Fig. 7–10 (\(p = 2\)).

It is intuitively clear that, say, for pseudorandom number generation automata of type (i) are preferable\(^1\); so we need to explain/prove the phenomenon and to develop techniques in order to determine/construct automata of type (i).

### 4.1. The automata 0-1 law

Denote \(E(f)\) the closure of the set \(E(f) = \bigcup_{k=1}^{\infty} E_k(f)\) in the topology of the real plane \(\mathbb{R}^2\). As \(E\) is closed, it is measurable with respect to the Lebesgue measure on the real plane \(\mathbb{R}^2\). Let \(\alpha(f)\) be the Lebesgue measure of \(E(f)\). It is clear that \(0 \leq \alpha(f) \leq 1\); but it turns out that in fact only two extreme cases occur: \(\alpha(f) = 0\) or \(\alpha(f) = 1\). This is the first of the main results of the paper:

**Theorem 4.1** (The automata 0-1 law). For \(f\), the following alternative holds: Either \(\alpha(f) = 0\) (equivalently, \(E(f)\) is nowhere dense in \(\mathbb{R}^2\), or \(\alpha(f) = 1\) (equivalently, \(E(f) = \mathbb{R}^2\)).

We note that although Theorem 4.1 has been already announced, see [5, Proposition 11.15], actually in [5] only part of the statement is proved (the one that concerns density of \(E(f)\)) whereas the part that concerns the value of the Lebesgue measure is not. Remind that nowhere dense sets can nevertheless have positive Lebesgue measures, cf. fat Cantor sets (e.g. the Smith-Volterra-Cantor set), also known as \(\epsilon\)-Cantor sets, see e.g. [1]. Nonetheless, Theorem 4.1 is true; a complete proof follows.

**Proof of Theorem 4.1.** Let \(\alpha(f) > 0\); we are going to prove that then \(\alpha(f) = 1\) and \(E(f) = \mathbb{R}^2\).

Either of the two following cases is possible: 1) Some point from \(E(f)\) have an open neighbourhood (in the unit square \(\mathbb{I}^2\)) that lies completely in \(E(f)\), or, on the contrary, 2) no such point in \(E(f)\) exists (thus, \(E(f)\) is nowhere dense in \(\mathbb{I}^2\) then). We consider the two cases separately and prove that within the first one necessarily \(\alpha(f) = 1\) while the second one is impossible (that is, if \(E(f)\) is nowhere dense in \(\mathbb{I}^2\) then necessarily \(\alpha(f) = 0\)). Given \(a, b \in \mathbb{R}, a \leq b\), during the proof we denote via \((a; b)\) (respectively, via \([a; b]\)) the corresponding open interval (respectively, closed interval).

---

\(^1\)For a deeper mathematical reasoning see [5].
segment) of the real line $\mathbb{R}$; while for $c, d \in \mathbb{R}$ we denote via $(c, d)$ the corresponding point on the real plane $\mathbb{R}^2$.

**Case 1**: In this case, there exist $u, v, u', v'$, $0 \leq u < v \leq 1$, $0 \leq u' < v' \leq 1$ such that the closed square $[u; v] \times [u'; v'] \subset \Gamma^2$ lies completely in $\mathcal{E}(f)$, and every point from the open real interval $(u'; v')$ is a limit (with respect to the standard Archimedean metric in $\mathbb{R}$) of some sequence of fractions $u' < \frac{f(a_m)}{p^m} < v'$, where $u < \frac{a_m}{p^m} < v$, $m = 1, 2, \ldots$ Thus, we can take $n \in \mathbb{N}$ and $w = \omega_0 + \omega_1 \cdot p + \cdots + \omega_{n-1} \cdot p^{n-1}$, where $\omega_i \in \{0, 1, \ldots, p-1\}$, $i = 0, 1, \ldots, n-1$, so that the square

$$S = \left[ \frac{w}{p^n}; \frac{w}{p^n} + \frac{1}{p^n} \right] \times \left[ \frac{f(w) \mod p^n}{p^n}; \frac{f(w) \mod p^n}{p^n} + \frac{1}{p^n} \right]$$
lies completely in $\mathcal{E}(f)$, and every inner point $(x, y)$ of the square $S$ is a limit as $j \to \infty$ (with respect to the standard Archimedean metric in $\mathbb{R}^2$) of a sequence of inner points 

$$(r_j, t_j) = \left(\frac{z_j + p^{N_j} \cdot w}{p^{N_j+n}}, \frac{f(z_j + p^{N_j} \cdot w) \mod p^{N_j+n}}{p^{N_j+n}}\right) \in S,$$

where $N_j \in \mathbb{N}$, $z_j \in \{0, 1, \ldots, p^{N_j} - 1\}$.

Now, as $f$ is a 1-Lipschitz mapping from $\mathbb{Z}_p$ to $\mathbb{Z}_p$, for every $z \in \{0, 1, \ldots, p^N - 1\}$ we have that $f(z + p^N \cdot w) \equiv (f(z) \mod p^N) + p^N \cdot \xi_N(z) \mod p^{N+n}$ for a suitable

---

\[\text{Figure 7. } f(x) = x + x^2 \text{OR} (-131065), \ k = 16\]

\[\text{Figure 8. Same function, } k = 17\]

\[\text{Figure 9. Same function, } k = 18\]

\[\text{Figure 10. Same function, } k = 22\]
ξ_N(z) ∈ \{0, 1, \ldots, p^n - 1\}; thus,
\[
\frac{f(z + p^n \cdot w) \mod p^{N+n}}{p^{N+n}} = \frac{f(z) \mod p^N}{p^N} + \frac{ξ_N(z)}{p^n}.
\]
Hence, ξ_{N_j}(z_j) = f(w) \mod p^n for all j = 1, 2, \ldots as all (r_j, t_j) are inner points of S. Therefore, every inner point (x, y) ∈ S, which can be represented as
\[
(x, y) = \left(\frac{w}{p^n} + \frac{\chi}{p^n}, \frac{f(w) \mod p^n}{p^n} + \frac{\gamma}{p^n}\right),
\]
where χ and γ are real numbers, 0 < χ < 1, 0 < γ < 1, is a limit (as j → ∞) of the point sequence
\[
(r_j, t_j) = \left(\frac{w}{p^n} + \frac{z_j}{p^n}, \frac{1}{p^n} \cdot \frac{f(w) \mod p^n}{p^n} + \frac{f(z_j) \mod p^{N_j}}{p^{N_j}} + \frac{1}{p^n}\right) ∈ S.
\]
From here it follows that every inner point (χ, γ) ∈ \(\mathbb{I}^2\) is a limit point of the corresponding sequence of points \(\left(\frac{z_j}{p^n}, \frac{f(z_j) \mod p^{N_j}}{p^{N_j}}\right)\) as j → ∞. This means that \(E(f) = \mathbb{I}^2\) and thus \(α(f) = 1\).

**Case 2:** No point from \(E(f)\) has an open neighbourhood that lies completely in \(E(f)\); i.e., any open neighbourhood \(U\) of any point from \(E(f)\) contains points from the subset \(\mathbb{I}^2 \setminus E(f)\), which is open in \(\mathbb{I}^2\).

Hence, \(U\) contains an open subset that lies completely in \(\mathbb{I}^2 \setminus E(f)\) (we assume that \(\mathbb{I}^2 \setminus E(f) \neq \emptyset\) since otherwise \(α(f) = 1\) and there is nothing to prove). Then there exists an open square
\[
T_m(a, b) = \left(\frac{a}{p^m}; \frac{a}{p^m} + \frac{1}{p^m}\right) \times \left(\frac{b}{p^m}; \frac{b}{p^m} + \frac{1}{p^m}\right),
\]
where \(a, b ∈ \{0, 1, \ldots, p^m - 1\}\), that lies completely in \(\mathbb{I}^2 \setminus E(f)\). That is, \(T_m(a, b)\) contains no points of the form
\[
\left(\frac{x \mod p^k}{p^k}, \frac{f(x) \mod p^k}{p^k}\right),
\]
where \(x ∈ \mathbb{Z}_p\) and \(k ∈ \mathbb{N}\).

In other words this means that there exist words \(\bar{a}, \bar{b}\) of length \(m\) in the alphabet \(\mathbb{F}_p\) (which are just base-\(p\) representations of \(a\) and \(b\), respectively) such that, whenever the automaton \(\mathfrak{A} = \mathfrak{A}_f\) is fed by any input word \(\bar{w}\) with suffix \(\bar{a}\), i.e., \(w = p^{\ell+m}a + u\) where \(u ∈ \{0, 1, \ldots, p^{\ell} - 1\}\), the corresponding output word
\[
f(w) \mod p^{\ell+m} = p^{\ell+m}t + v, v ∈ \{0, 1, \ldots, p^{\ell} - 1\},
\]
never has the suffix \(\bar{b}\), i.e., \(t \neq b\) for all \(\ell ∈ \mathbb{N}_0\) and all \(u ∈ \{0, 1, \ldots, p^{\ell} - 1\}\) (\(u\) is the empty word if \(\ell = 0\)).

It is clear now that given any numbers \(a', b' ∈ \{0, 1, \ldots, p^{m'} - 1\}, m' ≥ m\), such that \(a' ≡ a (\mod p^m), b' ≡ b (\mod p^m)\), the corresponding open square \(T_{m'}(a', b')\) lies completely outside of \(E(f)\), i.e., contains no points of the form
\[
\left(\frac{x \mod p^k}{p^k}, \frac{f(x) \mod p^k}{p^k}\right),
\]
where \(x ∈ \mathbb{Z}_p\) and \(k ∈ \mathbb{N}\). Indeed, otherwise some input word \(\bar{w}'\) with the suffix \(a'\) results in the output word with the suffix \(b'\); but, this means that the corresponding initial subword (whose suffix is \(a\)) of the word \(\bar{w}'\) results in output word whose suffix is \(b\). The latter case contradicts our choice of \(a, b\).
Now take $m' = im$ for $i = 1, 2, \ldots$ and construct inductively a collection $T_1$ that consists of $(p^{2m} - 1)^{i-1}$ disjoint open squares $T_{m'}(a', b')$. The collection $T_1$ consists of the only square $T_m(a, b)$.

Given the collection $T_{i-1}$, the collection $T_i$ consists of all open squares $T_{im}(a', b')$, where $a', b' \in \{0, 1, \ldots, p^{im} - 1\}$, $a' \equiv a \pmod{p^m}$, $b' \equiv b \pmod{p^m}$, that are disjoint from all squares from the collections $T_1, \ldots T_{i-1}$.

That is, at the first step we obtain a collection $T_1$ that consists of the only $p^{-m} \times p^{-m}$ square $T_1(a, b)$; on the second step we obtain a collection $T_2$ that consists of $p^{2m} - 1$ disjoint $p^{-2m} \times p^{-2m}$-squares; on the third step we obtain a collection $T_3$ that consists of $(p^{2m} - 1)p^{2m} - (p^{2m} - 1) = (p^{2m} - 1)^2$ disjoint $p^{3m} \times p^{3m}$-squares, etc.

The union $T$ of all these open squares from $T_1, T_2, \ldots$ is open, whence, measurable, and the Lebesgue measure of $T$ is

$$\frac{1}{p^{2m}} + (p^{2m} - 1) \cdot \frac{1}{p^{4m}} + (p^{2m} - 1)^2 \cdot \frac{1}{p^{6m}} + \cdots = 1$$

since all these open squares are disjoint by the construction. On the other hand, by the construction $T$ contains no points of the form $(\frac{x \pmod{k}}{p^{im}}, \frac{f(x) \pmod{k}}{p^{im}})$, where $x \in \mathbb{Z}_p$ and $k \in \mathbb{N}$. Consequently, $T \cap E(f) = \emptyset$; in turn, this implies that the Lebesgue measure of $E(f)$ must be 0, i.e., that $\alpha(f) = 0$. The latter contradicts the assumption from the beginning of the proof. This proves the theorem.

4.2. Completely transitive automata. From Theorem 4.1 we immediately derive the second main result of the paper:

**Theorem 4.2** (Criterion of complete transitivity). A system $\mathfrak{A}$ is completely transitive if and only if $\alpha(f_{\mathfrak{A}(s_0)}) = 1$.

**Proof.** Follows from Theorem 4.1 cf. equivalent definition of complete transitivity in terms of words. \qed

**Note 4.3.** Nowhere in the proofs of Theorem 4.1 and of Theorem 4.2 we used that $p$ is a prime; so both theorems are true without this limitation.

A finite system (i.e., the one whose set of states is finite) can be word transitive; the odometer $x \mapsto x + 1$ on $\mathbb{Z}_2$ serves as an example. On the other hand, by [5] Theorem 11.10, given a finite system $\mathfrak{A}$, the set $E(f_{\mathfrak{A}})$ is nowhere dense; so from Theorems 4.1 and 4.2 it follows that a finite system can not be completely transitive. Thus, $\alpha(\mathfrak{A}(s_0)) = 0$ if $\mathfrak{A}(s_0)$ is a finite automaton.

To construct automata $\mathfrak{A}$ of measure 1 (i.e., such that $\alpha(\mathfrak{A}) = 1$) the following theorem (which is the third main result of the paper) may be applied:

**Theorem 4.4** (Sufficient conditions for complete transitivity). Let $f = f_{\mathfrak{A}}: \mathbb{Z}_p \rightarrow \mathbb{Z}_p$ be the automaton function of an automaton $\mathfrak{A}$, and let $f$ be differentiable everywhere in a ball $B \subset \mathbb{Z}_p$ of a non-zero radius. The function $f$ is of measure 1 whenever the following two conditions hold simultaneously:

(i) $f(B \cap N_0) \subset N_0$;

(ii) $f$ is two times differentiable at some point $v \in B \cap N_0$, and $f''(v) \neq 0$. 

Proof. We will show that for every sufficiently large \( k \) and every \( z, u \in \{0, 1, \ldots, p^k - 1\} \) there exists \( M = M(k) \) and \( a \in \{0, 1, \ldots, p^M - 1\} \) such that

\[
\frac{|a|_{p^M} - u}{p^k} < \frac{1}{p^k} \quad \text{and} \quad \left| \frac{f(a) \mod p^M}{p^M} - z \right| < \frac{1}{p^k}.
\]

This will prove the theorem as every point from the unit square \( \mathbb{I}^2 \) can be approximated by points of the form \( \left(\frac{a}{p^k}, \frac{b}{p^k}\right) \).

Briefly, our idea of the proof is as follows: As \( v \in \mathbb{N}_0 \), there exists \( k \in \mathbb{N}_0 \) such that all terms \( \nu_i \in \{0, 1, \ldots, p - 1\} \) in the \( p \)-adic expansion \( v = \sum_{i=0}^{\infty} \nu_i \cdot p^i \) are zero, for all \( i \geq k \). We then somehow tweak \( v \): Namely, we replace zeros in the \( p \)-adic expansion at positions starting with \( \ell \)-th, \( \ell > k \), by certain other letters from \( \{0, 1, \ldots, p - 1\} \) so that the tweaked \( v \), the natural number \( a = v + p^\ell t \), satisfies inequalities (4.10) for some \( M \).

As \( f \) is differentiable everywhere in \( B \), for \( x \in B \) we have that given arbitrary \( K \in \mathbb{N} \), the following congruence holds for all \( h \in \mathbb{Z}_p \) and all sufficiently large \( L \in \mathbb{N} \):

\[
f(x + p^L h) \equiv f(x) + p^L h \cdot f'(x) \pmod{p^{K+L}}.
\]

Indeed, given \( a, b \in \mathbb{Z}_p \), the condition \( \|a - b\|_p \leq p^{-d} \) is equivalent to the condition \( a \mod p^d = b \mod p^d \), where \( \mod p^d \) is a reduction modulo \( p^d \), cf. (2.2); so (4.8) is just re-statement of a condition of differentiability of a function at a point, in terms of congruences rather than in terms of inequalities for \( p \)-adic absolute values: we just write \( a \equiv b \pmod{p^d} \) instead of \( \|a - b\|_p \leq p^{-d} \).

Let \( \|f''(v)\|_p = p^{-s} \); that is, \( f''(v) = p^s \cdot \xi \), where \( s \in \mathbb{Z} \) and \( \xi \) is a unity of \( \mathbb{Z}_p \) (in other words, \( \xi \) has a multiplicative inverse in \( \mathbb{Z}_p \)). Note that \( s \) is not necessarily non-negative since \( f''(v) \) is in \( \mathbb{Q}_p \), and not necessarily in \( \mathbb{Z}_p \); nonetheless further in the proof we assume that \( k + s > 0 \) as we may take \( k \) large enough. Remind that \( \|f'(x)\|_p \leq 1 \) as \( f \) is 1-Lipschitz; so \( f'(x) \in \mathbb{Z}_p \).

Now let \( r \in \mathbb{N} \) be an arbitrary number such that \( r > s \), \( p^r > v \), and \( p^{-r} \) is less than the radius of the ball \( B \) (it is clear that there are infinitely many choices of \( r \)). Given \( r \), consider \( n \in \mathbb{N} \) such that \( n > \max\{\log_p f(v + p^k t) : t = 0, 1, 2, \ldots, p^k - 1\} \) and \( n > 2k + 2r + 2s \) (we remind that in view of condition 2 of the theorem, all \( f(v + p^t) \) are in \( \mathbb{N}_0 \) due to our choice of \( n \)). Put

\[
\tilde{u} = 1 + p^{k+r+s} u
\]

\[
\tilde{z} = f'(v) + p^{k+r+s} \zeta,
\]

where \( \tilde{z} \in \{0, 1, \ldots, p^k - 1\} \) is such that \( \frac{\tilde{z}}{p^{k+r+s}} \mod p^k = z \). In other words, we choose \( \tilde{z} \) in such a way that the number whose base-\( p \) expansion stands in positions from \( (k + r + s) \)-th to \( (2k + r + s) \)-th in the canonical \( p \)-adic expansion of \( \tilde{z} \), is equal to \( z \). Obviously, given \( f'(v) \) and \( z \), there exists a unique \( \tilde{z} \) that satisfies this condition: \( \tilde{z} \equiv z - \frac{f'(v)}{p^{k+r+s}} \pmod{p^k} \); so

\[
\tilde{z} \mod p^{2k+r+s} = (f'(v) \mod p^{k+r+s}) + p^{k+r+s} \cdot z.
\]

As \( f \) is twice times differentiable at \( v \), for every \( \zeta \in \{0, 1, \ldots, p^k - 1\} \) we conclude that

\[
f'(v + p^{r+k} \zeta) \equiv f'(v) + p^{r+k} \zeta \cdot f''(v) \pmod{p^{2k+r+s}}
\]
for all sufficiently large $r$ (formally, we just substitute $f'$ for $f$, $v$ for $x$, $\zeta$ for $h$, $k+s$ for $K$, and $r+k$ for $L$ in (4.13)). From here we deduce that as $f$ is differentiable in $B$, the following congruence holds for all sufficiently large $n$:

$$f(v + p^{r+k} \zeta + p^n \hat{u}) \equiv f(v + p^{r+k} \zeta) + p^n \hat{u} \cdot (f'(v) + p^{r+k} \zeta \cdot f''(v)) \pmod{p^{n+2k+r+s}}.$$ 

Note that the latter congruence is obtained by combination of congruence (4.13) where $K = 2k + r + s$, $x = v + p^{r+k} \zeta$, $h = \hat{u}$ and $L = n$, with congruence (4.12).

We claim that there exists $\zeta \in \{0, 1, \ldots, p^k - 1\}$ such that

$$\hat{u} \cdot (f'(v) + p^{r+k} \zeta \cdot f''(v)) \equiv \tilde{z} \pmod{p^{2k+r+s}}.$$ 

Indeed, in view of (4.3), (4.10) this congruence is equivalent to the congruence \((1 + p^{k+r+s} u) \cdot (f'(v) + p^{r+k} \zeta \cdot f''(v)) \equiv f'(v) + p^{k+r+s} \tilde{z} \pmod{p^{2k+r+s}}\), and the latter congruence is equivalent to the congruence \(f'(v) + p^{k+r+s} \tilde{z} \equiv (1 - p^{k+r+s} u) \cdot (f'(v) + p^{k+r+s} \tilde{z}) \pmod{p^{2k+r+s}}\) as \((1 + p^{k+r+s} u)^{-1} \equiv 1 - p^{k+r+s} u \pmod{p^{2k+r+s}}\). That is, congruence (4.14) is equivalent to the congruence \(p^{k+r} \zeta \cdot f''(v) \equiv p^{k+r+s} \tilde{z} - p^{k+r+s} u \cdot f'(v) \pmod{p^{2k+r+s}}\). Further, as \(f''(v) = p^k \xi\), the latter congruence is equivalent to the congruence \(\xi \equiv -u \cdot f'(v) \pmod{p^k}\). From here we find \(\xi \equiv \xi^{-1} \cdot (\tilde{z} - u \cdot f'(v)) \pmod{p^k}\), thus proving our claim (we remind that $\xi$ is a unity of $\mathbb{Z}_p$; hence, $\xi$ has a multiplicative inverse $\xi^{-1}$ modulo $p^k$).

Now we put $M = n + 2k + r + s$ and $a = v + p^{r+k} \zeta + p^n \cdot (1 + p^{k+r+s} u)$; then

$$\frac{a}{p^M} = \frac{u}{p^k} + \frac{v + p^{r+k} \zeta + p^n}{p^{n+2k+r+s}},$$

so \(\left| \frac{a}{p^M} - \frac{u}{p^k} \right| < \frac{1}{p^r}\), since $v < p^r$, $\zeta < p^k$, and $n > 2r + 2s + 2k$. Also, combining (4.13), (4.10), (4.11), and (4.13), we see that

$$\frac{f(a) \mod p^M}{p^M} = \frac{z}{p^k} + \frac{f(v + p^{r+k} \zeta)}{p^n} \cdot \frac{1}{p^{2k+r+s}} + \frac{f'(v) \mod p^{k+r+s}}{p^{k+r+s}} \cdot \frac{1}{p^k},$$

since \(f(a) \mod p^M = f(v + p^{r+k} \zeta) + p^n (f'(v) \mod p^{k+r+s}) + p^{n+k+r+s} z\) (the number in the right-hand side is less than $p^M$ due to our choice of $n$). Now from (4.15) it follows that \(\left| \frac{f(a) \mod p^M}{p^M} - \frac{z}{p^k} \right| < \frac{1}{p^r}\) since \(0 \leq f(v + p^{r+k} \zeta) \leq p^n - 1\) due to our choice of $n$. \(\square\)

**Note 4.5.** We note that $\alpha(f(x)) = \alpha(-f(x)) = \alpha(f(-x))$ for every 1-Lipschitz function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ of variable $x$; so we may replace condition 1 of Theorem 4.3 by either of conditions $f(B \cap -N_0) \subset N_0$, $f(B \cap N_0) \subset -N_0$, or $f(B \cap -N_0) \subset -N_0$, where $-N_0 = \{0, 1, \ldots, -2, \ldots\}$.

Indeed, for every $c \in \mathbb{N}$ and every $n \in \mathbb{N}$ we have that \(\frac{c \mod p^n}{p^n} = p^n - \frac{c \mod p^n}{p^n} = 1 - \frac{c \mod p^n}{p^n}\). Thus, a symmetry with respect to the axis $y = \frac{1}{2}$ of the unit square $\mathbb{I}^2 \subset \mathbb{R}^2$ maps the subset

$$E(f) = \left\{ \left( \frac{x \mod p^n}{p^n}, \frac{f(x) \mod p^n}{p^n} \right) : x \in \mathbb{Z}_p, n \in \mathbb{N} \right\} \subset \mathbb{I}^2$$

onto the subset $E(-f)$ and vice versa; so $\alpha(f(x)) = \alpha(-f(x))$. A similar argument proves that $\alpha(f(x)) = \alpha(f(-x))$. 
By using Theorem [4.4] one may construct numerous automata (and systems) that are completely transitive. For instance, given \( c \in \{2, 3, 4, \ldots \} \), listed below are examples of automata functions \( f_{\mathcal{A}(s_0)} = f \) that satisfy Theorem [4.4] so the corresponding automata \( \mathcal{A}(s_0) \) are completely transitive:

- \( f(x) = cx + c^2 \) if \( c \neq 1 \), \( c \equiv 1 \pmod{p} \);
- \( f(x) = (x \ \text{AND} \ c) + ((x^2) \ \text{OR} \ c) \) if \( p = 2 \).

Note that the first of these automata is word transitive while the second one is not.

With the use of Theorem [4.4] new types of absolutely transitive automata can be constructed as well. The following corollary from Theorem [4.4] is a key tool in the construction of these:

**Corollary 4.6.** Let an automaton function \( f = f_{\mathcal{A}(s_0)} \) map \( \mathbb{N}_0 \) into \( \mathbb{N}_0 \), let \( f \) be two times differentiable on \( \mathbb{Z}_p \), and let \( f''(x) \) have no more than a finite number of zeros in \( \mathbb{N}_0 \). Then the automaton \( \mathcal{A}(s_0) \) is absolutely transitive.

**Proof.** Given a finite non-empty word \( \tilde{g} \) (say, of length \( m > 0 \)) over the alphabet \( \mathbb{F}_p \), take a finite word \( \tilde{v} \) whose prefix is \( \tilde{g} \) and such that the corresponding non-negative rational integer \( \ell \) is a non-zero of \( f'' \): \( f''(v) \neq 0 \). The word \( \tilde{v} \) that satisfies these conditions simultaneously exists as \( f'' \) has no more than a finite number of zeros in \( \mathbb{N}_0 \) (fixing arbitrary \( \tilde{g} \) means that only some less significant digits in the base-\( p \) expansion of \( v \) are fixed); so by taking \( v \) whose base-\( p \) expansion is sufficiently long (thus making \( v \) large enough), we find \( v \in \mathbb{N}_0 \) such that \( f''(v) \neq 0 \) and the \( m \)-letter prefix of the word \( \tilde{v} \) is \( \tilde{g} \).

In other words, given an arbitrary finite word \( \tilde{g} \) over the alphabet \( \mathbb{F}_p \), by properly choosing \( r \in \mathbb{N}_0 \) we find a positive rational integer \( v = g + p^mr \) (where \( g \in \{0, 1, \ldots, p^m - 1\} \), \( \tilde{g} \) is a base-\( p \) expansion of \( g \)) such that \( f''(v) \neq 0 \). This is possible due to the finiteness of a number of zeros of \( f'' \) in \( \mathbb{N}_0 \). We see that both \( f \) and the so constructed \( v \) satisfy conditions of Theorem [4.4] just assume that the ball \( B \) from the conditions of the mentioned theorem is the whole space \( \mathbb{Z}_p \).

Now note that the claim stated at the very beginning of the proof of Theorem [4.4] is just a re-statement of (ii) from Definition [4.4]. Indeed, under notation of Definition [4.4] and the one from the beginning of the proof of Theorem [4.4] the concatenation \( w \circ y \) corresponds to \( a \), \( w \) corresponds to \( u \), \( w' \) corresponds to \( z \), and \( w \) is a \( k \)-letter suffix of the output word which is a base-\( p \) expansion for \( f(a) \mod p^M \), whereas \( M \) is the length of the word \( w \circ y \). Up to these correspondences, condition [4.7] is equivalent to (ii) from Definition [4.4]. Furthermore, as the word \( \tilde{v} \) has an arbitrarily chosen prefix \( \tilde{g} \), and as the condition [4.7] holds for \( a = v + p^t \) from the proof of Theorem [4.4] (as the whole Theorem [4.4] holds for \( f \) and \( v \), (ii) from Definition [4.4] holds for input word with arbitrarily chosen prefix \( \tilde{g} \), up to all mentioned correspondences. This means that (iii) from Definition [4.4] also holds for \( x = \tilde{g} \) in the case under consideration. The latter finally proves Corollary 4.6. \( \square \)

We remark that Note [4.5] can be applied to Corollary [4.6] as well.

Note also that the only type of absolutely transitive automata \( \mathcal{A}(s_0) \) were known earlier: The ones whose automata functions are polynomials over \( \mathbb{Z} \) of degree greater than 1, see [5, Theorem 11.11]. The latter assertion follows from Corollary [4.6].

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3The one whose base \( p \)-expansion is \( \tilde{v} \); remind that according to our conventions words are read from right to left, that is the rightmost letters of \( \tilde{v} \) correspond to low order digits in the base-\( p \) expansion of \( v \).
many other types automata can be proved to be absolutely transitive as well by using the corollary. For instance, an automaton whose both input and output alphabets are $\mathbb{F}_2$ and whose automata function is $f(x) = a + bx + (x^2 \text{ OR } c)$, where $a, b, c \in \mathbb{N}_0$, is absolutely transitive: This easily follows from Corollary 4.6 as $f(\mathbb{N}_0) \subset \mathbb{N}_0$ and $f''(x) = 2$ for all $x \in \mathbb{Z}_2$.

5. Discussion

In the paper, by combining tools from $p$-adic and real analysis and automata theory we have shown that discrete systems (automata) with respect to the transitivity of their actions on finite words constitute two classes, the systems whose real plots have Lebesgue measures 1 (equivalently, the completely transitive systems; i.e. such that given two arbitrary words $w$, $w'$ of equal lengths, the system transforms $w$ into $w'$) and systems whose real plots have Lebesgue measures 0. Also we have found conditions for complete transitivity of a system; the conditions yield a method to construct numerous completely transitive automata and respective automata functions, especially the ones that are combined from standard computer instructions and thus are easily programmable. The ergodic completely transitive automata are preferable in constructions of various pseudo-random number generators aimed at cryptographic and/or simulation usage; e.g., in stream ciphers and quasi Monte-Carlo methods.

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