Weak and Strong Type Weighted Estimates for Multilinear Calderón-Zygmund Operators*

Kangwei Li and Wenchang Sun†
School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, China
Email: likangwei9@mail.nankai.edu.cn, sunwch@nankai.edu.cn

Abstract

In this paper, we study the weighted estimates for multilinear Calderón-Zygmund operators from $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ to $L^p(v \vec{w})$, where $1 < p, p_1, \ldots, p_m < \infty$ with $1/p_1 + \cdots + 1/p_m = 1/p$ and $\vec{w} = (w_1, \ldots, w_m)$ is a multiple $A_\vec{P}$ weight. We give weak and strong type weighted estimates of mixed $A_p-A_\infty$ type. Moreover, the strong type weighted estimate is sharp whenever $\max_i p_i \leq p'/((mp - 1)$.

Keywords. multilinear Calderón-Zygmund operators; multiple $A_\vec{P}$ weights; weighted inequalities.

1 Introduction and Main Results

The weighted estimate for operators is an interesting topic in harmonic analysis. And it has attracted many authors in this area [4, 9, 18, 20, 24, 25]. In this paper, we study the weighted estimates for multilinear Calderón-Zygmund operators with multiple $A_\vec{P}$ weights.

Recall that $T$ is called a multilinear Calderón-Zygmund operator if $T$ is initially defined on the $m$-fold product of Schwartz spaces and taking values into the space of tempered distributions,

$$T : \mathcal{S}(\mathbb{R}^n) \times \cdots \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n),$$

and for some $1 \leq q_i < \infty$, it extends to a bounded multilinear operator from $L^{q_1} \times \cdots \times L^{q_m}$ to $L^q$, where $1/q_1 + \cdots + 1/q_m = 1/q$, and if there exists a function $K$, defined off the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, satisfying

$$T(f_1, \ldots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m$$

for all $x \notin \bigcap_{j=1}^m \text{supp } f_i$;

$$|K(y_0, y_1, \ldots, y_m)| \leq \frac{A}{\left(\sum_{k,l=0}^m |y_k - y_l|\right)^m}$$

∗This work was supported partially by the National Natural Science Foundation of China(10990012) and the Research Fund for the Doctoral Program of Higher Education.

†Corresponding author.
and
\[ |K(y_0, \cdots, y_i, \cdots, y_m) - K(y_0, \cdots, y_i', \cdots, y_m)| \leq \frac{A|y_i - y_i'|^\epsilon}{(\sum_{k,l=0}^m |y_k - y_l|)^{m+\epsilon}} \]
for some $A, \epsilon > 0$ and all $0 \leq i \leq m$, whenever $|y_i - y_i'| \leq \frac{1}{2} \max_{0 \leq k \leq m} |y_i - y_k|$.

For the theory of multilinear Calderón-Zygmund operators, we refer the readers to [4 5 6 7 8] for an overview.

The multiple $A_p$ weights introduced by Lerner, Ombrosi, Pérez, Torres and Trujillo-González [15] are defined as follows. Let $\vec{P} = (p_1, \cdots, p_m)$ with $1 \leq p_1, \cdots, p_m < \infty$ and $1/p_1 + \cdots + 1/p_m = 1/p$. Given $\vec{w} = (w_1, \cdots, w_m)$, set
\[ v_{\vec{w}} = \prod_{i=1}^m w_i^{p_i/p_i}. \]

We say that $\vec{w}$ satisfies the multilinear $A_{\vec{P}}$ condition if
\[ [\vec{w}]_{A_{\vec{P}}} := \sup_Q \left( \frac{1}{|Q|} \int_Q v_{\vec{w}} \right) \prod_{i=1}^m \left( \frac{1}{|Q|} \int_Q w_i^{1-p_i'} \right)^{p_i/p_i'} < \infty, \]
where $[\vec{w}]_{A_{\vec{P}}}$ is called the $A_{\vec{P}}$ constant of $\vec{w}$. When $p_i = 1$, $\left( \frac{1}{|Q|} \int_Q w_i^{1-p_i'} \right)^{1/p_i'}$ is understood as $(\inf_Q w_i)^{-1}$. It is easy to see that in the linear case (that is, $m = 1$) $[\vec{w}]_{A_{\vec{P}}} = [w]_{A_p}$ is the usual $A_p$ constant. Recall that $A_\infty = \bigcup_{1 \leq p < \infty} A_p$ and the $A_\infty$ constant $[w]_{A_\infty}$ is defined by
\[ [w]_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q). \]

In [18], it was shown that for $1 < p_1, \cdots, p_m < \infty$, $\vec{w} \in A_{\vec{P}}$ if and only if $w_i^{1-p_i'} \in A_{mp_i'}$ and $v_{\vec{w}} \in A_{mp_p}$.

For the linear case, i.e. $m = 1$, the $A_p$-$A_\infty$ type estimates for Calderón-Zygmund operators were investigated in [11]. Notice that the main technique in [11] is an appropriate characterization which simplifies the estimate of the weighted bounds to calculate a test condition [12, 15]. The advantage of their technique is that it does not rely upon the extrapolation. In this paper, roughly speaking, we follow the idea used in [12]. But we do not use the method such as the linearization used in that paper. Instead, we use the idea of Damián, Lerner and Pérez [3] and reduce the problem to consider the following type of operators,
\[ A_{\vec{P}, S}(\vec{f}) = \sum_{j,k} \left( \prod_{i=1}^m \frac{1}{|Q_{j,k}|} \int_{Q_{j,k}} f_i(y_i) dy_i \right) \chi_{Q_{j,k}}, \]
where $\vec{f} := (f_1, \cdots, f_m)$, $\mathcal{D}$ is a dyadic grid and $S := \{Q_{j,k}\}$ is a sparse family in $\mathcal{D}$ (see Section 2 for definitions of these notations).

In the linear case, Lerner [16, 17] investigated this type of operators and gave a simple proof for the $A_2$ conjecture. For the fundamental theory of $A_p$ weights and the history of the $A_2$ conjecture, we refer the readers to [1 2 10 13 14 22 23] for an overview.

In [3], Damián, Lerner and Pérez studied the sharp weighted bound of multilinear maximal function of mixed $A_p$-$A_\infty$ type and gave a multilinear version of the $A_2$ conjecture.
In [19], the authors estimated the weighted bound of the multilinear maximal function and Calderón-Zygmund operators in terms of $\|w\|_{A_\vec{p}}$.

In this paper, we estimate the weighted bound of multilinear Calderón-Zygmund operators of mixed $A_p$-$A_\infty$ type. We give the sharp estimate for some cases. To be precise, the main result of this paper is the following.

**Theorem 1.1** Let $T$ be a multilinear Calderón-Zygmund operator, $\vec{P} = (p_1, \ldots, p_m)$ with $1/p = 1/p_1 + \cdots + 1/p_m$ and $1 < p, p_1, \ldots, p_m < \infty$. Suppose that $\vec{w} = (w_1, \ldots, w_m)$ with $\vec{w} \in A_{\vec{P}}$. Then

$$
\|T(\vec{f})\|_{L^p(v_{\vec{w}})} \leq C_{m,n,\vec{P},T}[\vec{w}]^{1/p}_{A_{\vec{P}}} \left( \prod_{i=1}^m [\sigma_i]^{1/p_i}_{A_\infty} \right) \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)},
$$

(1.1)

where $\sigma_i = w_i^{1-p'_i}$, $i = 1, \ldots, m$. The result is sharp in the sense that the exponents cannot be improved whenever $\max_i p_i \leq p'/((mp - 1)$.

For the weak type estimates, we get a similar result.

**Theorem 1.2** Let $T$ be a multilinear Calderón-Zygmund operator, $\vec{P} = (p_1, \ldots, p_m)$ with $1/p = 1/p_1 + \cdots + 1/p_m$ and $1 < p, p_1, \ldots, p_m < \infty$. Suppose that $\vec{w} := (w_1, \ldots, w_m) \in A_{\vec{P}}$. Then we have

$$
\|T(\vec{f})\|_{L^{p,\infty}(v_{\vec{w}})} \leq C_{m,n,\vec{P},T}[\vec{w}]^{1/p}_{A_{\vec{P}}} [v_{\vec{w}}]^{1/p'}_{A_{\vec{P}}} \left( \sum_{i'=1}^m \prod_{i \neq i'} [\sigma_i]^{1/p_i}_{A_\infty} \right) \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)}.
$$

In the rest of this paper, we give proofs for the main results. To avoid cumbersome notations, we only prove Theorems 1.1 and 1.2 for the case $m = 2$. And the general case can be proved similarly but with more complicated symbols.

## 2 Preliminaries

In this section, we collect some notations and preliminary results. Recall that the standard dyadic grid in $\mathbb{R}^n$ consists of the cubes

$$
[0, 2^{-k})^n + 2^{-k} j, \quad k \in \mathbb{Z}, j \in \mathbb{Z}^n.
$$

Denote the standard dyadic grid by $\mathcal{D}$.

By a general dyadic grid $\mathcal{G}$ we mean a collection of cubes with the following properties:

(i) for any $Q \in \mathcal{G}$ its sidelength $l_Q$ is of the form $2^k$, $k \in \mathbb{Z}$; (ii) $Q \cap R \in \{Q, R, \emptyset\}$ for any $Q, R \in \mathcal{G}$; (iii) the cubes of a fixed sidelength $2^k$ form a partition of $\mathbb{R}^n$.

We say that $S := \{Q_{j,k}\}$ is a sparse family of cubes if:

(i) for each fixed $k$ the cubes $Q_{j,k}$ are pairwise disjoint;

(ii) if $\Gamma_k = \bigcup_j Q_{j,k}$, then $\Gamma_{k+1} \subset \Gamma_k$;
(iii). \( |\Gamma_{k+1} \cap Q_{j,k}| \leq \frac{1}{2}|Q_{j,k}| \).

For any \( Q_{j,k} \in \mathcal{S} \), we define \( E(Q_{j,k}) = Q_{j,k} \setminus \Gamma_{k+1} \). Then the sets \( E(Q_{j,k}) \) are pairwise disjoint and \( |E(Q_{j,k})| \geq \frac{1}{2}|Q_{j,k}| \).

In \cite{3}, Damián, Lerner and Pèrez proved that for any Banach function space \( X \) over \( \mathbb{R}^n \) equipped with Lebesgue measure,

\[
\|T(\tilde{f})\|_X \leq C \sup_{\mathcal{D}, \mathcal{S}} \| A_{\mathcal{D}, \mathcal{S}} (|\tilde{f}|) \|_X, \tag{2.1}
\]

where \( |\tilde{f}| = (|f_1|, \ldots, |f_m|) \) and the supremum is taken over arbitrary dyadic grids \( \mathcal{D} \) and sparse families \( \mathcal{S} \subset \mathcal{D} \). Specially, for \( X = L^p(v_{\vec{w}}) \), \( 1 \leq p < \infty \),

\[
\|T(\tilde{f})\|_{L^p(v_{\vec{w}})} \leq C \sup_{\mathcal{D}, \mathcal{S}} \| A_{\mathcal{D}, \mathcal{S}} (|\tilde{f}|) \|_{L^p(v_{\vec{w}})}. \tag{2.2}
\]

Let \( E^\sigma_{\mathcal{Q}} f := \sigma(\mathcal{Q})^{-1} \int_{\mathcal{Q}} f \sigma \). We introduce the principal cubes \cite{12}.

**Definition 2.1 (Principal cubes)** We form the collection \( \mathcal{G} \) of principal cubes as follows. Let \( \mathcal{G}_0 := \{ \mathcal{Q} \} \) (the maximal dyadic cube that we consider). And inductively,

\[
\mathcal{G}_k := \bigcup_{G \in \mathcal{G}_{k-1}} \{ G' \subset G : E^\sigma_{G'} |f| > 4E^\sigma_G |f|, G' is a maximal such dyadic cube \}.
\]

Let \( \mathcal{G} := \bigcup_{k=0}^\infty \mathcal{G}_k \). For any dyadic \( Q(\subset \overline{Q}) \), we let

\[
\Gamma(Q) := the\ minimal\ principal\ cube\ containing\ Q.
\]

It follows from the definition that

\[
E^\sigma_{\mathcal{Q}} |f| \leq 4E^\sigma_{\Gamma(Q)} |f|.
\]

From the idea of principal cubes, we have the following decomposition, which is similar to the ordinary corona decomposition (See \cite{14} [21]).

Let \( \mathcal{Q} \subset \mathcal{D} \) be any collection of dyadic cubes such that for any \( Q \in \mathcal{Q} \), there exists a maximal cube \( Q_{\max} \in \mathcal{Q} \) which contains \( Q \). Let \( \sigma_1 dx \) and \( \sigma_2 dx \) be two positive measures. We call \( (\mathcal{L} : \mathcal{Q}(\mathcal{L})) : \mathcal{L} \subset \mathcal{Q} \) a \( (\sigma_1, \sigma_2) \)-corona decomposition of \( \mathcal{Q} \) if these conditions hold.

(i). For each \( Q \in \mathcal{Q} \) there is a member of \( \mathcal{L} \) that contains \( Q \). Let \( \lambda(Q) \in \mathcal{L} \) denote the minimal cube which contains \( Q \). Then we have

\[
4 \frac{\sigma_1(\lambda(Q))\sigma_2(\lambda(Q))}{|\lambda(Q)|^2} \geq \frac{\sigma_1(Q)\sigma_2(Q)}{|Q|^2}.
\]

(ii). For all \( L', L \in \mathcal{L} \) with \( L' \subset L \),

\[
\frac{\sigma_1(L')\sigma_2(L')}{|L'|^2} > 4 \frac{\sigma_1(L)\sigma_2(L)}{|L|^2}.
\]
We set \( Q(L) := \{ Q \in \mathcal{Q} : \lambda(Q) = L \} \). The collection \( Q(L) \) forms a partition of \( \mathcal{Q} \).

Note that \( \{ Q \times Q : Q \in \mathcal{Q} \} \) is a collection of dyadic cubes in \( \mathbb{R}^{2n} \). Therefore, the \((\sigma_1, \sigma_2)\)-corona decomposition of \( \mathcal{Q} \) is in fact the ordinary corona decomposition of \( \{ Q \times Q : Q \in \mathcal{Q} \} \) with respect to the measure \( \sigma_1 \times \sigma_2 \).

Now we introduce some preliminary results. The following result is obvious and we omit the proof.

**Lemma 2.2** Any sub-family of a sparse family is also sparse.

Next we give a property of \( A_\infty \) weights on sparse family.

**Lemma 2.3** Let \( w \in A_\infty \) and \( \mathcal{Q} \subset \mathcal{D} \) be a sparse family. Suppose that there is some \( S \in \mathcal{D} \) such that any cube in \( \mathcal{Q} \) is contained in \( S \). Then

\[
\sum_{Q \in \mathcal{Q}} w(Q) \leq 2 \int_S M(w1_S)(x)dx \leq 2[w]_{A_\infty} w(S).
\]

**Proof.** Set

\[ E(Q) = Q \setminus \bigcup_{Q' \in \mathcal{Q}, Q' \subset Q} Q'. \]

By the sparse property, \( E(Q) \) are disjoint and \( |E(Q)| \geq \frac{1}{2}|Q| \). Then we have

\[
\sum_{Q \in \mathcal{Q}} w(Q) \leq 2 \sum_{Q \in \mathcal{Q}} \frac{w(Q)}{|Q|} |E(Q)| \leq 2 \int_S M(w1_S)(x)dx \leq 2[w]_{A_\infty} w(S).
\]

By (2.2), we have to estimate \( \|A_{\varphi, S}(|\vec{f}|)\|_{L^p(v, \vec{w})} \). First, we consider a special case.

**Lemma 2.4** Suppose that \((w_1, w_2) \in A_{\vec{P}}\), where \( \vec{P} = (p_1, p_2) \) and \( 1/p = 1/p_1 + 1/p_2 \). Let \( \mathcal{Q} \subset \mathcal{D} \) be a sparse family. Suppose that there is some \( S \in \mathcal{D} \) such that any cube in \( \mathcal{Q} \) is contained in \( S \). Set

\[
A_{\varphi, \mathcal{Q}}(\vec{f}) = \sum_{Q \in \mathcal{Q}} \left( \prod_{i=1}^2 \frac{1}{|Q|} \int_Q f_i(y_i)dy_i \right) \chi_Q.
\]

Then we have

\[
\|A_{\varphi, \mathcal{Q}}(\sigma_11_S, \sigma_21_S)\|_{L^p(v, \vec{w})}\]

\[
\lesssim [w]_{A_{\vec{P}}}^{1/p} \left( \sum_{Q \in \mathcal{Q}} \sigma_1(Q) \right)^{1/p_1} \cdot \left( \sum_{Q \in \mathcal{Q}} \sigma_2(Q) \right)^{1/p_2}.
\]

**Proof.** Without loss of generality, assume that \( p_1 \leq p_2 \). Set

\[
\mathcal{Q}_a := \left\{ Q \in \mathcal{Q} : 2^a < \left( \frac{v_{\vec{w}}(Q)}{|Q|} \right)^{1/p} \prod_{i=1}^2 \left( \frac{\sigma_i(Q)}{|Q|} \right)^{1/p'_i} \leq 2^{a+1} \right\},
\]

\[
\{ Q \in \mathcal{Q} : 2^a \leq \left( \frac{v_{\vec{w}}(Q)}{|Q|} \right)^{1/p} \prod_{i=1}^2 \left( \frac{\sigma_i(Q)}{|Q|} \right)^{1/p'_i} \}
\]

\[
= \sum_{a=0}^{\infty} \mathcal{Q}_a.
\]

By the sparse property, \( \mathcal{Q}_a \) are disjoint and

\[
|\mathcal{Q}_a| \leq 2^{a+1} \left( \frac{v_{\vec{w}}(Q)}{|Q|} \right)^{1/p} \prod_{i=1}^2 \left( \frac{\sigma_i(Q)}{|Q|} \right)^{1/p'_i}.
\]

By the definition of \( A_{\varphi, \mathcal{Q}}(\vec{f}) \), we have

\[
\|A_{\varphi, \mathcal{Q}}(\sigma_11_S, \sigma_21_S)\|_{L^p(v, \vec{w})}\]

\[
\lesssim \sum_{a=0}^{\infty} \left( \frac{v_{\vec{w}}(Q)}{|Q|} \right)^{1/p} \prod_{i=1}^2 \left( \frac{\sigma_i(Q)}{|Q|} \right)^{1/p'_i} |\mathcal{Q}_a|.
\]

By the definition of \( A_{\varphi, \mathcal{Q}}(\vec{f}) \), we have

\[
\|A_{\varphi, \mathcal{Q}}(\sigma_11_S, \sigma_21_S)\|_{L^p(v, \vec{w})}\]

\[
\lesssim \sum_{a=0}^{\infty} \left( \frac{v_{\vec{w}}(Q)}{|Q|} \right)^{1/p} \prod_{i=1}^2 \left( \frac{\sigma_i(Q)}{|Q|} \right)^{1/p'_i} |\mathcal{Q}_a|.
\]

By the sparse property, \( \mathcal{Q}_a \) are disjoint and

\[
|\mathcal{Q}_a| \leq 2^{a+1} \left( \frac{v_{\vec{w}}(Q)}{|Q|} \right)^{1/p} \prod_{i=1}^2 \left( \frac{\sigma_i(Q)}{|Q|} \right)^{1/p'_i}.
\]

By the definition of \( A_{\varphi, \mathcal{Q}}(\vec{f}) \), we have

\[
\|A_{\varphi, \mathcal{Q}}(\sigma_11_S, \sigma_21_S)\|_{L^p(v, \vec{w})}\]

\[
\lesssim \sum_{a=0}^{\infty} \left( \frac{v_{\vec{w}}(Q)}{|Q|} \right)^{1/p} \prod_{i=1}^2 \left( \frac{\sigma_i(Q)}{|Q|} \right)^{1/p'_i} |\mathcal{Q}_a|.
\]

By the sparse property, \( \mathcal{Q}_a \) are disjoint and

\[
|\mathcal{Q}_a| \leq 2^{a+1} \left( \frac{v_{\vec{w}}(Q)}{|Q|} \right)^{1/p} \prod_{i=1}^2 \left( \frac{\sigma_i(Q)}{|Q|} \right)^{1/p'_i}.
\]

By the definition of \( A_{\varphi, \mathcal{Q}}(\vec{f}) \), we have

\[
\|A_{\varphi, \mathcal{Q}}(\sigma_11_S, \sigma_21_S)\|_{L^p(v, \vec{w})}\]

\[
\lesssim \sum_{a=0}^{\infty} \left( \frac{v_{\vec{w}}(Q)}{|Q|} \right)^{1/p} \prod_{i=1}^2 \left( \frac{\sigma_i(Q)}{|Q|} \right)^{1/p'_i} |\mathcal{Q}_a|.
\]

By the sparse property, \( \mathcal{Q}_a \) are disjoint and

\[
|\mathcal{Q}_a| \leq 2^{a+1} \left( \frac{v_{\vec{w}}(Q)}{|Q|} \right)^{1/p} \prod_{i=1}^2 \left( \frac{\sigma_i(Q)}{|Q|} \right)^{1/p'_i}.
\]

By the definition of \( A_{\varphi, \mathcal{Q}}(\vec{f}) \), we have

\[
\|A_{\varphi, \mathcal{Q}}(\sigma_11_S, \sigma_21_S)\|_{L^p(v, \vec{w})}\]

\[
\lesssim \sum_{a=0}^{\infty} \left( \frac{v_{\vec{w}}(Q)}{|Q|} \right)^{1/p} \prod_{i=1}^2 \left( \frac{\sigma_i(Q)}{|Q|} \right)^{1/p'_i} |\mathcal{Q}_a|.
\]
where $-1 \leq a \leq |\log_2[\bar{w}]|^{1/p}$. Form the $(\sigma_1, \sigma_2)$-corona decomposition of $Q_a$. We get $L_a$.

Define

$$A_{\varphi, Q_a(L)}(x) = \sum_{Q \in Q_a(L)} \frac{\sigma_1(Q)\sigma_2(Q)}{|Q|^2} \chi_Q(x).$$

We conclude that there exists some $c > 0$ such that for $L \in L_a$ and $t \geq 0$, we have

$$\left| \left\{ x \in \mathbb{R}^n : A_{\varphi, Q_a(L)}(x) > 4t \frac{\sigma_1(L)\sigma_2(L)}{|L|^2} \right\} \right| \leq 2^{-t+2}|L|, \quad (2.3)$$

$$v_\varphi \left( \left\{ x \in \mathbb{R}^n : A_{\varphi, Q_a(L)}(x) > t \frac{\sigma_1(L)\sigma_2(L)}{|L|^2} \right\} \right) \leq 2^{-c} v_\varphi(L). \quad (2.4)$$

First, we prove (2.3). It is obvious that

$$\left| \left\{ x \in \mathbb{R}^n : A_{\varphi, Q_a(L)}(x) > 4 \frac{\sigma_1(L)\sigma_2(L)}{|L|^2} \right\} \right| \leq |L|.$$

Since for $Q \in Q_a(L)$,

$$\frac{\sigma_1(Q)\sigma_2(Q)}{|Q|^2} \leq 4 \frac{\sigma_1(L)\sigma_2(L)}{|L|^2},$$

by the sparse property of $Q$, we have for any integer $\tau$,

$$\left| \left\{ x \in \mathbb{R}^n : A_{\varphi, Q_a(L)}(x) > 4\tau \frac{\sigma_1(L)\sigma_2(L)}{|L|^2} \right\} \right| \leq 2^{-\tau+1}|L|.$$

This proves (2.3).

Next we prove (2.4). For integers $b \geq 0$, we define $Q_{a,b}(L)$ to be the set consisting of $Q \in Q_a(L)$ such that

$$2^{-b+1} \frac{\sigma_1(L)\sigma_2(L)}{|L|^2} < \frac{\sigma_1(Q)\sigma_2(Q)}{|Q|^2} \leq 2^{-b+2} \frac{\sigma_1(L)\sigma_2(L)}{|L|^2}.$$

Define

$$E_b(t) := \left\{ x \in \mathbb{R}^n : A_{\varphi, Q_{a,b}(L)}(x) > 4t 2^{-b} \frac{\sigma_1(L)\sigma_2(L)}{|L|^2} \right\},$$

where

$$A_{\varphi, Q_{a,b}(L)}(x) = \sum_{Q \in Q_{a,b}(L)} \frac{\sigma_1(Q)\sigma_2(Q)}{|Q|^2} \chi_Q(x).$$

Similar arguments as the above show that $|E_b(t)| \leq 2^{-t+2}|L|$. Let $K = 4 \sum_{b \geq 0} 2^{-b/2}$. We have

$$v_\varphi \left( \left\{ x \in \mathbb{R}^n : A_{\varphi, Q_a(L)}(x) > t \frac{\sigma_1(L)\sigma_2(L)}{|L|^2} \right\} \right) \leq \sum_{b \geq 0} v_\varphi \left( \left\{ x \in \mathbb{R}^n : A_{\varphi, Q_{a,b}(L)}(x) > 4t 2^{-b/2}K^{-1} \frac{\sigma_1(L)\sigma_2(L)}{|L|^2} \right\} \right)$$

$$= \sum_{b \geq 0} v_\varphi(E_b(2^{b/2}K^{-1}t)).$$
Suppose that $E_b(2^{b/2}K^{-1}t) = \bigcup_j R_j^b$, where $R_j^b$ are pairwise disjoint maximal dyadic cubes in $E_b(2^{b/2}K^{-1}t)$. Notice that $R_j^b \in \mathcal{Q}_{a,b}(L)$. We have

$$v_{\bar{w}}(E_b(2^{b/2}K^{-1}t)) = \sum_j v_{\bar{w}}(R_j^b) \leq \sum_j \left( 2^a \frac{|R_j^b|^2}{\sigma_1(R_j^b)^{1/p'} \sigma_2(R_j^b)^{1/p}} \right)^p \leq \sum_j \frac{2^a |R_j^b|^{2p/p_2}}{\sigma_1(R_j^b)^{p/p_2} \sigma_2(R_j^b)^{p/p_2}} \frac{|R_j^b|^{p/p_2}}{|R_j^b|^{1/p_2}} \leq \frac{2^{bp/p_2}}{\sigma_1(L)^{p/p_2}} \left( \sum_j \frac{|R_j^b|^{p/p_2}}{|R_j^b|} \right)^{2p/p_2} v_{\bar{w}}(L) \leq 2^{bp/p_2} (K^{-1}2^{b/2_2+2}) \frac{2^{bp/p_2}}{\sigma_1(L)^{p/p_2}} v_{\bar{w}}(L) \leq 2^{bp/p_2} (K^{-1}2^{b/2_2+2}) \frac{2^{bp/p_2}}{\sigma_1(L)^{p/p_2}} v_{\bar{w}}(L).$$

It follows that for $t \geq 1$,

$$v_{\bar{w}} \left( \left\{ x \in \mathbb{R}^n : A_{\mathcal{Q}_a}(L)(x) > t \frac{\sigma_1(L)\sigma_2(L)}{|L|^2} \right\} \right) \leq \sum_{b \geq 0} v_{\bar{w}}(E_b(2^{b/2}K^{-1}t)) \lesssim \sum_{b \geq 0} 2^{bp/p_2} (K^{-1}2^{b/2_2+2}) \frac{2^{bp/p_2}}{\sigma_1(L)^{p/p_2}} v_{\bar{w}}(L) \lesssim \sum_{b \geq 0} 2^{bp/p_2} (K^{-1}2^{b/2_2+2}) \frac{2^{bp/p_2}}{\sigma_1(L)^{p/p_2}} v_{\bar{w}}(L) \lesssim 2^{-ct} v_{\bar{w}}(L),$$

where $c = K^{-1}p/p_2$. For $0 \leq t < 1$, it is obvious that (2.4) is correct.

For $L \in \mathcal{L}_a$ and $d \in \mathbb{Z}_+$, let

$$L_{a,d} = \left\{ x \in \mathbb{R}^n : A_{\mathcal{Q}_a}(L)(x) \in (d, d+1] \frac{\sigma_1(L)\sigma_2(L)}{|L|^2} \right\}.$$

It is obvious that $L_{a,d} \subset L$ and by (2.4),

$$v_{\bar{w}}(L_{a,d}) \lesssim 2^{-cd} v_{\bar{w}}(L).$$

By the definition of $(\sigma_1, \sigma_2)$-corona decomposition, we have

$$\sum_{L \in \mathcal{L}_a} \frac{\sigma_1(L)\sigma_2(L)}{|L|^2} \chi_{L_{a,d}}(x) \propto \left( \sum_{L \in \mathcal{L}_a} \left( \frac{\sigma_1(L)\sigma_2(L)}{|L|^2} \right)^p \right)^{1/p} \chi_{L_{a,d}}(x).$$
It follows that

\[
\|A_{\varphi, Q}(\sigma_11_S, \sigma_21_S)\|_{L^p(v_\varphi)} \\
\leq \sum_{a=-1}^{[\log_2|\vec{w}|^{1/p}_{A_{\vec{P}}}] - 1} \|A_{\varphi, Q_a}(x)\|_{L^p(v_\varphi)} \\
\leq \sum_{a=-1}^{[\log_2|\vec{w}|^{1/p}_{A_{\vec{P}}}] - 1} (d + 1) \left( \sum_{L \in \mathcal{L}_a} \frac{\sigma_1(L)\sigma_2(L)}{|L|^2} \chi_{L_{a,d}}(x) \right)_{L^p(v_\varphi)} \\
\leq \sum_{a=-1}^{[\log_2|\vec{w}|^{1/p}_{A_{\vec{P}}}] - 1} \frac{d + 1}{2^a} \left( \sum_{L \in \mathcal{L}_a} \frac{\sigma_1(L)\sigma_2(L)}{|L|^2} \right)^{1/p} (\sum_{L \in \mathcal{L}_a} \sigma_2(L)^{p/p_1})^{1/p_2} \\
\leq [\vec{w}]_{A_{\vec{P}}}^{1/p} \left( \sum_{Q \in \mathcal{Q}} \sigma_1(Q) \right)^{1/p_1} \cdot \left( \sum_{Q \in \mathcal{Q}} \sigma_2(Q) \right)^{1/p_2}.
\]

This completes the proof. \(\square\)

The following result gives another special case of \(\|A_{\varphi, S}(f(x))\|_{L^p(v_\varphi)}\). Since its proof shares some common steps with the one for Theorem 1.1, we postpone the proof to Section 3.

**Lemma 2.5** Suppose that \((w_1, w_2) \in A_{\vec{P}}\) with \(\vec{P} = (p_1, p_2)\) satisfies that \(1/p = 1/p_1 + 1/p_2\) and that \(1 < p, p_1, p_2 < \infty\). Let \(S\) be a dyadic cube and supp \(f_1 \subset S\). Then

\[
\|1_S A_{\varphi, S}(f_1|\sigma_1, \sigma_21_S)\|_{L^p(v_\varphi)} \lesssim [\vec{w}]^{1/p}_{A_{\vec{P}}} [\sigma_2]^{1/p_2}_{A_{\vec{P}}} [\vec{w}]^{1/p'}_{A_{\vec{P}}} + [\sigma_1]^{1/p_1}_{A_{\vec{P}}} \|f_1\|_{L^{p_1}(\sigma_1)\sigma_2(S)}^{1/p_2}.
\]

To prove the main result, we also need the following result on multiple weights.

**Lemma 2.6** [79] **Lemma 2.2** Suppose that \(\vec{w} = (w_1, \cdots, w_m) \in A_{\vec{P}}\) and that \(1 < p, p_1, \cdots, p_m < \infty\) with \(1/p_1 + \cdots + 1/p_m = 1/p\). Then \(\vec{v} := (w_1, \cdots, v_{i-1}, v_{i-1'}^{-p'}, w_{i+1}, \cdots, w_m) \in A_{\vec{P}_{i}}\) with \(\vec{P}_{i} = (p_1, \cdots, p_{i-1}, p', p_{i+1}, \cdots, p_m)\) and

\[
[\vec{w}]_{A_{\vec{P}_{i}}} \equiv [\vec{v}]_{A_{\vec{P}}}.\]

## 3 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Without loss of generality, we assume that \(f_1, f_2 \geq 0\). Denote \(\Omega_i := \{x \in \mathbb{R}^n : A_{\varphi, S}(f_1\sigma_1, f_2\sigma_2)(x) > 2^i\}\) and let \(\mathcal{Q}_i\) denote the set of maximal dyadic cubes in \(\Omega_i\). By the structure of \(A_{\varphi, S}\), any cube in \(\mathcal{Q}_i\) must be some cube \(Q_{j,k} \subset S\).
We have
\[
\left\| A_{\mathcal{H}S}(f_1\sigma_1, f_2\sigma_2) \right\|_{L^p(v_\omega)}^p \leq 4^p \sum_{l \in \mathbb{Z}} 2^p v_\omega(\Omega_{l+1} \setminus \Omega_{l+2})
\]
\[
= 4^p \sum_{l \in \mathbb{Z}} \sum_{Q \in Q_l} 2^p v_\omega(Q \cap \Omega_{l+1} \setminus \Omega_{l+2})
\]
\[
= 4^p \sum_{l \in \mathbb{Z}} \sum_{Q \in Q_l} 2^p v_\omega(E_l(Q)),
\]
where \( E_l(Q) = Q \cap \Omega_{l+1} \setminus \Omega_{l+2} \). By the maximal property of \( Q \in Q_l \), we have
\[
\sum_{Q_{j,k} \supset Q} \prod_{i=1}^2 \frac{1}{|Q_{j,k}|} \int_{Q_{j,k}} f_i(y_i) \sigma_1 dy_i > 2^l
\]
and
\[
\sum_{Q_{j,k} \supset Q} \prod_{i=1}^2 \frac{1}{|Q_{j,k}|} \int_{Q_{j,k}} f_i(y_i) \sigma_1 dy_i \leq 2^l.
\]
Therefore, for \( x \in E_l(Q) \), we have
\[
2^{l+1} < A_{\mathcal{H}S}(f_1\sigma_1, f_2\sigma_2)(x) \leq A_{\mathcal{H}S}(f_1\sigma_1 1_Q, f_2\sigma_2 1_Q)(x) + 2^l.
\]
That is,
\[
A_{\mathcal{H}S}(f_1\sigma_1 1_Q, f_2\sigma_2 1_Q)(x) > 2^l, \quad x \in E_l(Q).
\]
Thus, for sufficiently small \( \beta > 0 \) to be determined later, we have
\[
\left\| A_{\mathcal{H}S}(f_1\sigma_1, f_2\sigma_2) \right\|_{L^p(v_\omega)}^p \leq 4^p \sum_{l \in \mathbb{Z}, Q \in Q_l} 2^p \beta v_\omega(Q) + 4^p \sum_{l \in \mathbb{Z}, Q \in Q_l} 2^p v_\omega(E_l(Q))
\]
\[
\leq \frac{4^p \beta}{1 - 2^{-p}} \left\| A_{\mathcal{H}S}(f_1\sigma_1, f_2\sigma_2) \right\|_{L^p(v_\omega)}^p + 4^p \sum_{l \in \mathbb{Z}, Q \in Q_l} v_\omega(E_l(Q))^{1-p}
\]
\[
\cdot \left( \int_{E_l(Q)} A_{\mathcal{H}S}(f_1\sigma_1 1_Q, f_2\sigma_2 1_Q)(x) v_\omega dx \right)^p.
\]
Consequently, by setting \( \beta = 4^{-p}(1 - 2^{-p})/2 \), we get
\[
\left\| A_{\mathcal{H}S}(f_1\sigma_1, f_2\sigma_2) \right\|_{L^p(v_\omega)}^p \leq \sum_{l \in \mathbb{Z}, Q \in Q_l} v_\omega(E_l(Q))^{1-p} \left( \int_{E_l(Q)} A_{\mathcal{H}S}(f_1\sigma_1 1_Q, f_2\sigma_2 1_Q)(x) v_\omega dx \right)^p
\]
\[
\leq \sum_{l \in \mathbb{Z}, Q \in Q_l} v_\omega(E_l(Q))^{1-p} \tag{3.1}
\]
\( \sum_{l \in \mathbb{Z}, Q \in Q_l} v_{\bar{w}}(E_l(Q)) > \beta v_{\bar{w}}(Q) \)

appearing in all these sums and we omit it in the rest of this section;

(iii). by the monotone convergence theorem, we may also assume that all appearing cubes are contained in some maximal dyadic cube \( \bar{Q} \). Then we can use the technique of principal cubes.

Before further estimates, we give two lemmas. The first can be proved with similar arguments as that in [12, pp. 20-21] and we omit the details.

**Lemma 3.1** Let \( \mathcal{G} \) be the principal cubes with respect to \( f_1 \) and \( \sigma_1 \), and \( \tilde{\mathcal{G}} \) be the principal cubes with respect to \( f_2 \) and \( \sigma_2 \). Suppose that \( \Gamma(Q) \) and \( \tilde{\Gamma}(Q) \) are defined as that in Definition 2.1. Then

\[
\sum_{l \in \mathbb{Z}} \sum_{Q \in Q_l} \sum_{R \in Q_{l+2}, \tilde{R} \subset Q_{l+2}, R \cap \tilde{R} \cap Q_{l+2} \neq \emptyset} \sigma_1(R)(E^\sigma_1 f_1)^{p_1} \lesssim \|f_1\|_{L^{p_1}(\sigma_1)}^{p_1}. \tag{3.3}
\]

\[
\sum_{l \in \mathbb{Z}} \sum_{Q \in Q_l} \sum_{R \in Q_{l+2}, \tilde{R} \subset Q_{l+2}, R \cap \tilde{R} \cap Q_{l+2} \neq \emptyset} \sigma_2(\tilde{R})(E^\sigma_2 f_2)^{p_2} \lesssim \|f_2\|_{L^{p_2}(\sigma_2)}^{p_2}. \tag{3.4}
\]

And the second one can be seen from the definition of principal cubes.

10
Lemma 3.2 Let $\mathcal{G}$ be the principal cubes with respect to $f_1$ and $\sigma_1$ and $\bar{\mathcal{G}}$ be the principal cubes with respect to $f_2$ and $\sigma_2$. Then

$$\sum_{G \in \mathcal{G}} (\mathbb{W}^p_G f_1)^p_1 \sigma_1(G) \lesssim \|f_1\|_{L^p_{\sigma_1}(\sigma_1)}^p. \quad (3.5)$$

$$\sum_{G \in \bar{\mathcal{G}}} (\mathbb{W}^p_G f_2)^p_2 \sigma_2(\bar{G}) \lesssim \|f_2\|_{L^p_{\sigma_2}(\sigma_2)}^p. \quad (3.6)$$

Next we give a proof for Lemma 2.5.

Proof of Lemma 2.5. Without loss of generality, assume that $f_1 \geq 0$. Set

$$A^1_{\mathcal{G},S}(f_1\sigma_1,1,S\sigma_2) = \sum_{Q_{j,k} \ni S} \frac{\int_{Q_{j,k}} f_1(y_1)\sigma_1 dy_1\sigma_2(Q_{j,k} \cap S)}{|Q_{j,k}|^2} \chi_{Q_{j,k}}$$

and

$$A^2_{\mathcal{G},S}(f_1\sigma_1,1,S\sigma_2) = \sum_{Q_{j,k} \subset S} \frac{\int_{Q_{j,k}} f_1(y_1)\sigma_1 dy_1\sigma_2(Q_{j,k} \cap S)}{|Q_{j,k}|^2} \chi_{Q_{j,k}}.$$

It is easy to see that

$$1_S A^1_{\mathcal{G},S}(f_1\sigma_1,1,S\sigma_2) = \sum_{Q_{j,k} \ni S} \frac{\int_{Q_{j,k}} f_1(y_1)\sigma_1 dy_1\sigma_2(Q_{j,k} \cap S)}{|Q_{j,k}|^2} \chi_S$$

$$\lesssim \int_S f_1(y_1)\sigma_1 dy_1\sigma_2(S) |S|^2 \chi_S$$

$$\lesssim \|f_1\|_{L^p_{\sigma_1}(\sigma_1)} \sigma_1(S)^{1/\mu_1} \sigma_2(S) |S|^2 \chi_S.$$

Hence

$$\|1_S A^1_{\mathcal{G},S}(f_1\sigma_1,1,S\sigma_2)\|_{L^p(v_\bar{w})} \lesssim \frac{\|f_1\|_{L^p_{\sigma_1}(\sigma_1)} \sigma_1(S)^{1/\mu_1} \sigma_2(S)}{|S|^2} v_{\bar{w}}(S)^{1/p}$$

$$\leq \|\bar{w}\|_{A_p}^{1/p} \|f_1\|_{L^p_{\sigma_1}(\sigma_1)} \sigma_2(S)^{1/p^2}.$$ 

It remains to estimate $A^2_{\mathcal{G},S}(f_1\sigma_1,1,S\sigma_2)$. Without loss of generality, assume that all cubes in $S$ are contained in $S$. By the previous arguments, we only need to estimate (3.1) in the special case $f_2 = 1_S$. We have

$$\|A_{\mathcal{G},S}(f_1\sigma_1,1,S\sigma_2)\|_{L^p(v_\bar{w})}^p$$

$$\lesssim \sum_{\{Q \subset Q_{\mathcal{G}} \cap \mathbb{R}\}} v_\bar{w}(E_1(Q))^{1-p} \left( \int_{E_1(Q)} A_{\mathcal{G},S}(f_1\sigma_11_Q,1,S\sigma_21_Q)(x)v_\bar{w}dx \right)^p$$

$$\lesssim \sum_{\{Q \subset Q_{\mathcal{G}} \cap \mathbb{R}\}} v_\bar{w}(E_1(Q))^{1-p}$$

$$\times \left( \int_{E_1(Q)} A_{\mathcal{G},S}(f_1\sigma_11_Q,1,S\sigma_21_Q)(x)v_\bar{w}dx \right)^p.$$
\[ + \sum_{i \in \mathbb{Z}, Q \in \mathcal{Q}_i, v_{\omega}(E_i(Q)) > \beta v_{\omega}(Q)} v_{\omega}(E_i(Q))^{1-p} \times \left( \int_{E_i(Q)} A_{\varphi, S}(f_1 \sigma_{1,1} Q \cap \Omega_{i+2}, 1 S \sigma_{2,1} Q)(x) v_{\omega} dx \right)^p \]

\[ := J_1 + J_2. \]

In the following, we also use the convention (ii) to omit \( v_{\omega}(E_i(Q)) > \beta v_{\omega}(Q) \).

First, we estimate \( J_1 \). Let \( S_l(Q) := \bigcup_{R \in \mathcal{Q}_{l+2}} \{ Q_{j,k} \in \mathcal{S} : R \subsetneq Q_{j,k} \subset Q \} \), if \( \mathcal{Q}_{l+2} \neq \emptyset \)

and \( S_l(Q) := \{ Q_{j,k} \in \mathcal{S} : Q_{j,k} \subset Q \} \), if \( \mathcal{Q}_{l+2} = \emptyset \).

By Lemma 2.4 \( S_l(Q) \) is sparse. For big cubes \( Q_{j,k} \supseteq Q \) and \( x \in E_i(Q) \),

\[ \sum_{Q_{j,k} \supseteq Q} \frac{\int_{Q \cap \Omega_{i+2}} f_1 \sigma_{1} dx \cdot \sigma_{2}(Q)}{|Q_{j,k}|^2} \leq \frac{\int_{Q \cap \Omega_{i+2}} f_1 \sigma_{1} dx \cdot \sigma_{2}(Q)}{|Q|^2}. \]

Hence for \( x \in E_i(Q) \), we have

\[ A_{\varphi, S}(f_1 \sigma_{1,1} Q \cap \Omega_{i+2}, \sigma_{2,1} Q)(x) \leq 2 A_{\varphi, S_l(Q)}(f_1 \sigma_{1,1} Q \cap \Omega_{i+2}, \sigma_{2,1} Q)(x), \quad (3.7) \]

where

\[ A_{\varphi, S_l(Q)}(f_1, f_2)(x) = \sum_{Q_{j,k} \in S_l(Q)} \left( \prod_{i=1}^2 \frac{1}{|Q_{j,k}|} \int_{Q_{j,k}} f_i(y_i) dy_i \right) \chi_{Q_{j,k}}(x). \]

Observe that \( S_l(Q) \) is an empty set if and only if \( Q \in \mathcal{Q}_l \cap \mathcal{Q}_{l+2} \). When the condition is satisfied, we have

\[ A_{\varphi, S}(f_1 \sigma_{1,1} Q \cap \Omega_{i+2}, \sigma_{2,1} Q)(x) = A_{\varphi, S_l(Q)}(f_1 \sigma_{1,1} Q \cap \Omega_{i+2}, \sigma_{2,1} Q)(x) = 0, \]

which means that \( (3.7) \) is correct even if \( S_l(Q) \) is empty. So \( \{ S_l(Q) : l \in 2\mathbb{Z}, Q \in \mathcal{Q}_l \} \) and \( \{ S_l(Q) : l \in 2\mathbb{Z} + 1, Q \in \mathcal{Q}_l \} \) are pairwise disjoint, respectively. Denote \( S_l(Q) = \{ Q_{l,\eta} \}_{\eta} \).

We have

\[ \int_{E_i(Q)} A_{\varphi, S}(f_1 \sigma_{1,1} Q \cap \Omega_{i+2}, 1 S \sigma_{2,1} Q)(x) v_{\omega} dx \]

\[ \lesssim \int_{E_i(Q)} A_{\varphi, S_l(Q)}(f_1 \sigma_{1,1} Q \cap \Omega_{i+2}, 1 S \sigma_{2,1} Q)(x) v_{\omega} dx \]

\[ \leq \left( \int_Q (A_{\varphi, S_l(Q)}(v_{\omega} Q, 1 S \sigma_{2,1} Q))^\frac{p_1}{p_1} \sigma_{1} dx \right)^{1/p_1} \cdot \left( \int_{Q \cap \Omega_{i+2}} f_1^{1/p_1} \sigma_1 \right)^{1/p_1} \]

\[ \lesssim [u]_{A_p}^{1/p} \left( \sum_{Q_{l,\eta} \in S_l(Q)} v_{\omega}(Q_{l,\eta}) \right)^{1/p_1} \sigma_{1} \quad \left( \sum_{Q_{l,\eta} \in S_l(Q)} \sigma_{2}(Q_{l,\eta}) \right)^{1/p_1} \]

\[ \times \left( \int_{Q \cap \Omega_{i+2}} f_1^{p_1} \sigma_1 \right)^{1/p_1} \quad \text{(by Lemma 2.4 and Lemma 2.6)} \]

\[ \leq [u]_{A_p}^{1/p} [v_{\omega}]_{A_\infty}^{1/p'} v_{\omega}(Q)^{1/p'} \left( \sum_{Q_{l,\eta} \in S_l(Q)} \sigma_{2}(Q_{l,\eta}) \right)^{1/p_1} \left( \int_{Q \cap \Omega_{i+2}} f_1^{p_1} \sigma_1 \right)^{1/p_1}. \]
where Lemma 2.3 is used in the last step. Recall that we have the convention (5.2). By Hölder’s inequality, we get

\[
J_1 = \sum_{l \in \mathbb{Z}, Q \in Q_l} \left( \int_{E_l(Q)} A_{\mathcal{G}, \mathcal{S}}(f_1 \sigma_1 1_{Q \cap \Omega_{l+2}, 1} \sigma_2 1_Q)(x) v_\bar{w}(E_l(Q)) \right)^{1-p} v_\bar{w}(E_l(Q))^{1-p}
\]

\[
\lesssim \bar{w} \left[ A_{\mathcal{G}, \mathcal{S}}(v_\bar{w}) \right]_{A_\infty}^{p/p'} \left( \sum_{l \in \mathbb{Z}, Q \in Q_l} \sum_{Q_{l, \eta} \in \mathcal{S}_l(Q)} \sigma_2(Q_{l, \eta}) \right)^{p/p_2} \left( \sum_{l \in \mathbb{Z}, Q \in Q_l} \int_{Q \cap \Omega_{l+2}} f_1^{p_1} \sigma_1 \right)^{p/p_1}
\]

\[
\lesssim \bar{w} A_{\mathcal{F}}[v_\bar{w}]_{A_\infty}^{p/p'} \sigma_2_{A_\infty}^{p/p_2} \| f_1 \|_{L^{p_1}(\sigma_1)}^{p/p_1}.
\]

Next we estimate \(J_2\). Since \(E_l(Q) \subset \Omega_{l+2}\) for \(R \in Q_{l+2}\) with \(R \subset Q\), \(A_{\mathcal{G}, \mathcal{S}}(v_\bar{w} 1_{E_l(Q)}, \sigma_2 1_Q)(x)\) is a constant for \(x \in R\). We have

\[
\int_{E_l(Q)} A_{\mathcal{G}, \mathcal{S}}(f_1 \sigma_1 1_{Q \cap \Omega_{l+2}, 1} \sigma_2 1_Q)(x) v_\bar{w} dx = \sum_{R \in Q_{l+2}, R \subset Q} \int_{E_l(Q)} A_{\mathcal{G}, \mathcal{S}}(v_\bar{w} 1_{E_l(Q)}, \sigma_2 1_Q)(x) \sigma_1 dx \cdot E_R^{\sigma_1} f_1
\]

\[
\leq 16 \sum_{R \in Q_{l+2}, R \subset Q} \int_{E_l(Q)} A_{\mathcal{G}, \mathcal{S}}(v_\bar{w} 1_{E_l(Q)}, \sigma_2 1_Q)(x) \sigma_1 dx \cdot E_R^{\sigma_1} f_1
\]

\[
+ \sum_{R \in Q_{l+2}, R \subset Q} \int_{E_l(Q)} A_{\mathcal{G}, \mathcal{S}}(v_\bar{w} 1_{E_l(Q)}, \sigma_2 1_Q)(x) \sigma_1 dx \cdot E_R^{\sigma_1} f_1.
\]

Similarly to (3.7), for \(x \in E_l(Q)\), we have

\[
A_{\mathcal{G}, \mathcal{S}}(f_1 \sigma_1 1_{Q \cap \Omega_{l+2}, \sigma_2 1_Q})(x) \leq 2A_{\mathcal{G}, \mathcal{S}_l(Q)}(f_1 \sigma_1 1_{Q \cap \Omega_{l+2}, \sigma_2 1_Q})(x).
\]

Consequently, by setting \(S(G) = \bigcup_{l \in \mathbb{Z}, Q \in Q_l} S_l(Q)\), we have

\[
\sum_{l \in \mathbb{Z}, Q \in Q_l} \left( \int_{E_l(Q)} A_{\mathcal{G}, \mathcal{S}}(\sigma_1 1_{Q \cap \Omega_{l+2}, \sigma_2 1_Q})(x) v_\bar{w} dx \right)^p \cdot (E_R^{\sigma_1} f_1)^p \cdot v_\bar{w}(E_l(Q))^{1-p}
\]

\[
\lesssim \sum_{l \in \mathbb{Z}, Q \in Q_l} \int_{E_l(Q)} A_{\mathcal{G}, \mathcal{S}_l(Q)}(\sigma_1 1_{Q \cap \Omega_{l+2}, \sigma_2 1_Q})^p v_\bar{w} dx \cdot (E_R^{\sigma_1} f_1)^p
\]

\[
= \sum_{G \in G} \sum_{l \in \mathbb{Z}, Q \in Q_l} \int_{E_l(Q)} A_{\mathcal{G}, \mathcal{S}_l(Q)}(\sigma_1 1_{Q \cap \Omega_{l+2}, \sigma_2 1_Q})^p v_\bar{w} dx \cdot (E_R^{\sigma_1} f_1)^p.
\]
\[
\leq \sum_{G \in \mathcal{G}} \int_G (A_{\mathcal{G}, \mathcal{S}(G)}(\sigma_1 1_G, \sigma_2 1_G))^p v_{\omega G} dx \cdot (E_G^{\sigma_1} f_1)^p
\]

(by the disjointness of \(E_l(Q)\))
\[
\lesssim \sum_{G \in \mathcal{G}} [\tilde{w}]_{A_p} \left( \sum_{\Gamma(Q) = G} \sum_{Q_i, \eta \in \mathcal{S}(Q)} \sigma_1(Q_i, \eta) \right)^{p/p_1}
\times \left( \sum_{\Gamma(Q) = G} \sum_{Q_i, \eta \in \mathcal{S}(Q)} \sigma_2(Q_i, \eta) \right)^{p/p_2}
\cdot (E_G^{\sigma_1} f_1)^p
\quad \text{(by Lemma 2.4)}
\]
\[
\leq [\tilde{w}]_{A_p} \left( \sum_{Q \in \mathcal{G}} \int_G M(\sigma_1 1_G) \cdot (E_G^{\sigma_1} f_1)^{p_1} \right)^{p/p_1}
\times \left( \int_S M(\sigma_2 1_S) \right)^{p/p_2}
\quad \text{(by Lemma 2.3)}
\]
\[
\lesssim [\tilde{w}]_{A_p} \left[ \sigma_1 \right]_{A_{\infty}}^{p_1/p_1} \left[ \sigma_2 \right]_{A_{\infty}}^{p_2/p_2} \sigma_2(S)^{p_2/p_2} \cdot \left( \sum_{G \in \mathcal{G}} \sigma_1(G) \cdot (E_G^{\sigma_1} f_1)^{p_1} \right)^{p/p_1}
\]
\[
\lesssim [\tilde{w}]_{A_p} \left[ \sigma_1 \right]_{A_{\infty}}^{p_1/p_1} \left[ \sigma_2 \right]_{A_{\infty}}^{p_2/p_2} \sigma_2(S)^{p_2/p_2} \cdot \|f_1\|_{L^{p_1}(\sigma_1)}^{p/p_1}
\quad \text{(3.9)}
\]

where (3.5) is used in the last step.

For the \(E_R^{\sigma_1} f_1 > 16E_{1(Q)}\) part, by Hölder’s inequality, we have
\[
\sum_{R \in Q_{l+2}, R \subset Q} \int_R A_{\mathcal{G}, \mathcal{S}}(v_{\omega R} 1_{E_l(Q)}, \sigma_2 1_Q)(x) \sigma_1 dx \cdot E_R^{\sigma_1} f_1
\]
\[
\leq \left( \sum_{R \in Q_{l+2}, R \subset Q} \sigma_1(R)^{-p_1'/p_1} \left( \int_R A_{\mathcal{G}, \mathcal{S}}(v_{\omega R} 1_{E_l(Q)}, \sigma_2 1_Q)(x) \sigma_1 dx \right)^{p_1'/p_1} \right)^{1/p_1'}
\]
\[
\times \left( \sum_{R \in Q_{l+2}, R \subset Q} \sigma_1(R)(E_R^{\sigma_1} f_1)^{p_1} \right)^{1/p_1}
\]
\[
\leq \left( \int_{Q \cap Q_{l+2}} A_{\mathcal{G}, \mathcal{S}}(v_{\omega R} 1_{E_l(Q)}, \sigma_2 1_Q)(x) \sigma_1 dx \right)^{1/p_1'}
\]
\[
\times \left( \sum_{R \in Q_{l+2}, R \subset Q} \sigma_1(R)(E_R^{\sigma_1} f_1)^{p_1} \right)^{1/p_1}
\]
\[ \lesssim \left( \int_{Q \cap \{l \in \mathbb{Z} \}} (A_{\mathcal{G}, S}(Q)(v \bar{w} 1_{E_l(Q)}, \sigma_2 1_Q))^{p_1} \sigma_1 dx \right)^{1/p_1} \]
\[ \times \left( \sum_{R \in Q_{l+2}, R \subset Q \atop \varepsilon_R^1 f_1 > 16 \varepsilon_{R_1}^1 l(Q) / l(Q)} \sigma_1(R) (\mathbb{E}_R^\sigma f_1)^{p_1} \right)^{1/p_1} \]
\[ \lesssim [\bar{w}]_{A_p}^{1/p} [v \bar{w}]_{A_\infty}^{1/p} \left( \sum_{Q \subset Q_{l+2}, R \subset Q} v \bar{w}(Q) \right)^{1/p'} \left( \sum_{Q \subset Q_{l+2}, R \subset Q} \sigma_2(Q, \eta) \right)^{1/p_2} \]
\[ \times \left( \sum_{R \in Q_{l+2}, R \subset Q \atop \varepsilon_R^1 f_1 > 16 \varepsilon_{R_1}^1 l(Q) / l(Q)} \sigma_1(R) (\mathbb{E}_R^\sigma f_1)^{p_1} \right)^{1/p_1}, \]

(by Lemma 2.3 and Lemma 2.6)

\[ \lesssim [\bar{w}]_{A_p}^{1/p} [v \bar{w}]_{A_\infty}^{1/p} \left( \sum_{l \in \mathbb{Z}, Q \subset Q_{l+2}} \left( \sum_{R \in Q_{l+2}, R \subset Q} v \bar{w}(E_l(Q)) \int_R A_{\mathcal{G}, S}(v \bar{w} 1_{E_l(Q)}, \sigma_2 1_Q)(x) \sigma_1 dx \cdot \mathbb{E}_R^\sigma f_1 \right) \right)^{p} \]
\[ \lesssim [\bar{w}]_{A_p}^{1/p} [v \bar{w}]_{A_\infty}^{1/p} \left( \sum_{l \in \mathbb{Z}, Q \subset Q_{l+2}} \left( \sum_{R \in Q_{l+2}, R \subset Q} \sigma_2(Q, \eta) \right) \right)^{p/p_2} \]
\[ \times \left( \sum_{l \in \mathbb{Z}, Q \subset Q_{l+2}} \left( \sum_{R \in Q_{l+2}, R \subset Q} \sigma_1(R) (\mathbb{E}_R^\sigma f_1)^{p_1} \right) \right)^{p/p_1} \]
\[ \lesssim [\bar{w}]_{A_p}^{1/p} [v \bar{w}]_{A_\infty}^{1/p} \left[ \sigma_2^{p/p_2} \|S\|_{L^p(\sigma_1)} \right]^{p} \]

where Lemma 2.3 is used in the last step. It follows from (3.2) and Hölder’s inequality that

\[ \sum_{l \in \mathbb{Z}, Q \subset Q_{l+2}} \left( \sum_{R \in Q_{l+2}, R \subset Q} \int_R A_{\mathcal{G}, S}(v \bar{w} 1_{E_l(Q)}, \sigma_2 1_Q)(x) \sigma_1 dx \cdot \mathbb{E}_R^\sigma f_1 \right) \]
\[ \lesssim [\bar{w}]_{A_p}^{1/p} [v \bar{w}]_{A_\infty}^{1/p} \left[ \sum_{l \in \mathbb{Z}, Q \subset Q_{l+2}} \left( \sum_{R \in Q_{l+2}, R \subset Q} \sigma_2(Q, \eta) \right) \sigma_1(R) (\mathbb{E}_R^\sigma f_1)^{p_1} \right]^{p/p_1} \]

where Lemma 2.3 and (3.3) are used in the last step.

Putting (3.8), (3.9) and (3.10) together, we get

\[ J_2 \lesssim [\bar{w}]_{A_p}^{1/p} [v \bar{w}]_{A_\infty}^{1/p} \left[ \sigma_2^{p/p_2} \|S\|_{L^p(\sigma_1)} \right]^{p} \]

This completes the proof. \(\square\)

Now we continue to prove Theorem 1.1.
3.1 Estimate of $I_1$

In this subsection, we consider $I_1$. We have

$$
\int_{E(Q)} A_{g,S}(f_1\sigma_11_{Q\setminus\Omega_{i+2}}, f_2\sigma_21_{Q\setminus\Omega_{i+2}})(x)v_{\tilde{w}}dx
= \int_{Q\setminus\Omega_{i+2}} A_{g,S}(f_1\sigma_11_{Q\setminus\Omega_{i+2}}, v_{\tilde{w}}1_{E_i(Q)})(x)f_2\sigma_2dx
\leq \left( \int_{Q\setminus\Omega_{i+2}} (A_{g,S}(f_1\sigma_11_{Q\setminus\Omega_{i+2}}, v_{\tilde{w}}1_{Q}))^{p/2}\sigma_2dx \right)^{1/p_2} \cdot \left( \int_{Q\setminus\Omega_{i+2}} f_2^{p_2}\sigma_2 \right)^{1/p_2}
\lesssim [\tilde{w}]_{A_p}[v_{\tilde{w}}]_{A_\infty}^{|p/p'|}([\sigma_1]_{A_\infty})^{1/|p_1|} + [\sigma_2]_{A_\infty} \cdot v_{\tilde{w}}(Q)^{1/p'}
\times \left( \int_{Q\setminus\Omega_{i+2}} f_1^{p_1}\sigma_1 \right)^{1/p_1} \cdot \left( \int_{Q\setminus\Omega_{i+2}} f_2^{p_2}\sigma_2 \right)^{1/p_2},
$$

where Lemma 2.5 and Lemma 2.6 are used. By Hölder’s inequality, we have

$$
I_1 \lesssim [\tilde{w}]_{A_p}[v_{\tilde{w}}]_{A_\infty}^{|p/p'|}([\sigma_1]_{A_\infty})^{1/|p_1|} + [\sigma_2]_{A_\infty} \cdot v_{\tilde{w}}(Q)^{1/p'}
\times \left( \sum_{l \in \mathbb{Z}, Q \in Q_i} \int_{Q\setminus\Omega_{i+2}} f_1^{p_1}\sigma_1 \right)^{p/p_1} \cdot \left( \sum_{l \in \mathbb{Z}, Q \in Q_i} \int_{Q\setminus\Omega_{i+2}} f_2^{p_2}\sigma_2 \right)^{p/p_2}
\leq [\tilde{w}]_{A_p}[v_{\tilde{w}}]_{A_\infty}^{|p/p'|}([\sigma_1]_{A_\infty})^{1/|p_1|} + [\sigma_2]_{A_\infty} \cdot \|f_1\|_{L^{p_1}([\sigma_1])} \cdot \|f_2\|_{L^{p_2}([\sigma_2])}.
$$

3.2 Estimates of $I_2$ and $I_3$

In this subsection, we estimate $I_2$ and $I_3$. Since they are similar, we only estimate $I_2$ and the other one can be estimated similarly by the symmetry. We have

$$
\int_{E_i(Q)} A_{g,S}(f_1\sigma_11_{Q\setminus\Omega_{i+2}}, f_2\sigma_21_{Q\cap\Omega_{i+2}})(x)v_{\tilde{w}}dx
= \int_{Q\cap\Omega_{i+2}} A_{g,S}(f_1\sigma_11_{Q\setminus\Omega_{i+2}}, v_{\tilde{w}}1_{E_i(Q)})(x)f_2\sigma_2dx
= \sum_{R \in \mathbb{Q}_{i+2}} \int_{\hat{R}} A_{g,S}(f_1\sigma_11_{Q\setminus\Omega_{i+2}}, v_{\tilde{w}}1_{E_i(Q)})(x)f_2\sigma_2dx.
$$

Since $E_i(Q) \subset \Omega_{i+2}$ and $\hat{R} \subset Q_{i+2}$, $A_{g,S}(f_1\sigma_11_{Q\setminus\Omega_{i+2}}, v_{\tilde{w}}1_{E_i(Q)})(x)$ is a constant for $x \in \hat{R}$. Therefore,

$$
\int_{E_i(Q)} A_{g,S}(f_1\sigma_11_{Q\setminus\Omega_{i+2}}, f_2\sigma_21_{Q\cap\Omega_{i+2}})(x)v_{\tilde{w}}dx
= \sum_{R \in \mathbb{Q}_{i+2}} \int_{\hat{R}} A_{g,S}(f_1\sigma_11_{Q\setminus\Omega_{i+2}}, v_{\tilde{w}}1_{E_i(Q)})(x)\sigma_2dx \cdot \mathbb{E}_R^{p_2} f_2
$$
\[ \leq 16 \sum_{R \in Q_{t+2}, R \subseteq Q} \int_{R} A_{\sigma, S}(f \sigma_11_{Q \setminus \Omega_{t+2}}, v \bar{w})_1E_1(Q)E_2(x) \sigma_2 dx - E_{\bar{\Gamma}(Q)}^2 f_2 \\
+ \sum_{R \in Q_{t+2}, R \subseteq Q} \int_{R} A_{\sigma, S}(f \sigma_11_{Q \setminus \Omega_{t+2}}, v \bar{w})_1E_1(Q)E_2(x) \sigma_2 dx - E_{\bar{\Gamma}(Q)}^2 f_2, \]

(3.11)

where \( \bar{\Gamma}(Q) \) are the principal cubes with respect to \( f_2 \) and \( \sigma_2 \).

**3.2.1 The part with \( E^2_R f_2 \leq 16E^2_{\bar{\Gamma}(Q)} f_2 \)**

For this part, we have

\[ \sum_{R \in Q_{t+2}, R \subseteq Q} \int_{R} A_{\sigma, S}(f \sigma_11_{Q \setminus \Omega_{t+2}}, v \bar{w})_1E_1(Q)E_2(x) \sigma_2 dx - E_{\bar{\Gamma}(Q)}^2 f_2 \]

\[ \leq \int_{Q} A_{\sigma, S}(f \sigma_11_{Q \setminus \Omega_{t+2}}, v \bar{w})_1E_1(Q)E_2(x) \sigma_2 dx - E_{\bar{\Gamma}(Q)}^2 f_2. \]

For \( \tilde{G} \in \tilde{G} \), where \( \tilde{G} \) is the set consisting of principal cubes with respect to \( f_2 \) and \( \sigma_2 \), set

\[ g_1(x) = \sum_{l \in Z \cap Q_{l}, \bar{\Gamma}(Q) = \tilde{G}} f_1(x)1_{Q \setminus \Omega_{t+2}}(x). \]

Then by the disjointness of \( E_l(Q) \) in \( l \in Z \) and \( Q \in Q_l \),

\[ \sum_{l \in Z \cap Q_{l}} \left( \sum_{R \in Q_{t+2}, R \subseteq Q} \int_{R} A_{\sigma, S}(f \sigma_11_{Q \setminus \Omega_{t+2}}, v \bar{w})_1E_1(Q)E_2(x) \sigma_2 dx - E_{\bar{\Gamma}(Q)}^2 f_2 \right)^p \cdot v \bar{w}(E_l(Q))^{1-p} \]

\[ \leq \sum_{l \in Z \cap Q_{l}} v \bar{w}(E_l(Q))^{1-p} \left( \int_{Q} A_{\sigma, S}(f \sigma_11_{Q \setminus \Omega_{t+2}}, v \bar{w})_1E_1(Q)E_2(x) \sigma_2 dx \right)^p \]

\[ \times \left( E_{\bar{\Gamma}(Q)}^2 f_2 \right)^p \]

\[ \leq \sum_{G \in \tilde{G}} \sum_{l \in Z \cap Q_{l}, \bar{\Gamma}(Q) = \tilde{G}} \int_{E_l(Q)} (A_{\sigma, S}(f \sigma_11_{Q \setminus \Omega_{t+2}}, \sigma_2 l_{\tilde{G}}))^{p} v \bar{w} dx \cdot (E_{\bar{\Gamma}(Q)}^2 f_2)^p \]

\[ \leq \sum_{G \in \tilde{G}} \int_{E_l(Q)} (A_{\sigma, S}(g_1 \sigma_1, \sigma_2 l_{\tilde{G}}))^{p} v \bar{w} dx \cdot (E_{\bar{\Gamma}(Q)}^2 f_2)^p \]

\[ \leq [\bar{w}]_{A, p}^{p/p_1}, [\sigma_2]_{A, p_1}^{p/p_1} + [v \bar{w}]_{A, \infty}^{1/p} \sum_{G \in \tilde{G}} \left( \int_{G} \sigma_2^{p/p_1} \right)^{p/p_1} (E_{\bar{\Gamma}(Q)}^2 f_2)^p \]

\[ \leq [\bar{w}]_{A, p}^{p/p_1}, [\sigma_2]_{A, p_1}^{p/p_1} + [v \bar{w}]_{A, \infty}^{1/p} \sum_{G \in \tilde{G}} \left( \int_{G} \sigma_2^{p/p_1} \right)^{p/p_1} (E_{\bar{\Gamma}(Q)}^2 f_2)^p \]

17
where, again, Lemma 2.5 and Lemma 2.6 are used in the last step. Therefore, by Hölder's inequality, we get

\[
\lesssim [\bar{\omega}]_{\mathcal{A}_{p}} [\sigma_{2}]^{p/p_{2}} [\sigma_{1}]^{1/p_{1}} + [v_{\bar{\omega}}]_{\mathcal{A}_{p}} \left( \sum_{G \in \mathcal{G}} \| f_{1} G \|^2_{\mathcal{A}_{1}} \right)^{p/p_{1}} \times \left( \sum_{G \in \mathcal{G}} \sigma_{2}(G) \| \mathbb{E}_{G}^{\sigma_{2}} f_{2} \|^{p}_{p_{2}} \right)^{p/p_{2}} 
\]

\[
\lesssim [\bar{\omega}]_{\mathcal{A}_{p}} [\sigma_{2}]^{p/p_{2}} [\sigma_{1}]^{1/p_{1}} + [v_{\bar{\omega}}]_{\mathcal{A}_{p}} \left( \sum_{G \in \mathcal{G}} \| f_{1} G \|^2_{\mathcal{A}_{1}} \right)^{p/p_{1}} \cdot \| f_{2} \|^{p}_{L^{p_{2}}(\sigma_{2})}, \quad (3.12)
\]

where (3.6) is used in the last step.

3.2.2 The part with \( \mathbb{E}_{R}^{\sigma_{2}} f_{2} > 16 \mathbb{E}_{\Gamma(Q)}^{\sigma_{2}} f_{2} \)

By Hölder's inequality, we have

\[
\sum_{R \in Q_{l_{2}} \cap \Omega_{l_{2}}} \left( \int_{R} A_{\mathcal{G},\mathcal{S}} (f_{1} \sigma_{1} 1_{Q_{l_{2}}} \cap \Omega_{l_{2}}, v_{\bar{\omega}} 1_{E_{1}(Q_{l_{2}}} (x) \sigma_{2} dA_{R} \mathbb{E}_{R}^{\sigma_{2}} f_{2} \right)
\]

\[
\leq \left( \sum_{R \in Q_{l_{2}} \cap \Omega_{l_{2}}} \sigma_{2}(R) \| \mathbb{E}_{R}^{\sigma_{2}} f_{2} \|^{p}_{p_{2}} \right)^{1/p_{2}} \left( \int_{R} A_{\mathcal{G},\mathcal{S}} (f_{1} \sigma_{1} 1_{Q_{l_{2}}} \cap \Omega_{l_{2}}, v_{\bar{\omega}} 1_{E_{1}(Q_{l_{2}}} (x) \sigma_{2} dA_{R} \mathbb{E}_{R}^{\sigma_{2}} f_{2} \right)^{p/p_{2}}
\]

where, again, Lemma 2.5 and Lemma 2.6 are used in the last step. Therefore, by Hölder’s inequality, we get

\[
\sum_{l \in \mathbb{Z}, Q \in Q_{l}} \left( \sum_{R \in Q_{l_{2}} \cap \Omega_{l_{2}}} \left( \int_{R} A_{\mathcal{G},\mathcal{S}} (f_{1} \sigma_{1} 1_{Q_{l_{2}}} \cap \Omega_{l_{2}}, v_{\bar{\omega}} 1_{E_{1}(Q_{l_{2}}} (x) \sigma_{2} dA_{R} \mathbb{E}_{R}^{\sigma_{2}} f_{2} \right) \right).
\]
\[
\frac{1}{L^2} \left( \int_{E_l(Q)} A_{g_1} \left( f_1 \sigma_1 1_{Q \setminus \Omega_{i+2}}, f_2 2_{Q \setminus \Omega_{i+2}} \right)(x) \, v_\Omega \, dx \right)^p v_\Omega(E_l(Q))^{1-p} 
\]

\[
\lesssim \| \tilde{A}_R \|_{p/p'} \left( \| \sigma_1 \|_{A_\infty}^{1/p_1} + \| \sigma_2 \|_{A_\infty}^{1/p_2} \right) \left( \sum_{l \in \mathbb{Z} \cap Q} \int_{Q \setminus \Omega_{i+2}} f_1^{p} \sigma_1 \right)^{p/p_1} 
\]

\[
\cdot \left( \sum_{l \in \mathbb{Z} \cap Q} \sum_{R \in Q_{i+2}} \sigma_2(\hat{R}) \left( \frac{E^R}{R} f_2 \right)^{p_2} \right)^{p/p_2} 
\]

\[
\lesssim \| \tilde{A}_R \|_{p/p'} \left( \| \sigma_1 \|_{A_\infty}^{1/p_1} + \| \sigma_2 \|_{A_\infty}^{1/p_2} \right) \| f_1 \|_{L^{p_1}(\sigma_1)} \cdot \| f_2 \|_{L^{p_2}(\sigma_2)}. 
\]

where (3.4) is used in the last step.

Combining (3.11), (3.12) and (3.13), we get

\[
I_2 \lesssim \| \tilde{A}_R \|_{p/p'} \left( \| \sigma_1 \|_{A_\infty}^{1/p_1} + \| \sigma_2 \|_{A_\infty}^{1/p_2} \right) \| f_1 \|_{L^{p_1}(\sigma_1)} \cdot \| f_2 \|_{L^{p_2}(\sigma_2)}. 
\]

By symmetry, we also have

\[
I_3 \lesssim \| \tilde{A}_R \|_{p/p'} \left( \| \sigma_1 \|_{A_\infty}^{1/p_1} + \| \sigma_2 \|_{A_\infty}^{1/p_2} \right) \| f_1 \|_{L^{p_1}(\sigma_1)} \cdot \| f_2 \|_{L^{p_2}(\sigma_2)}. 
\]

### 3.3 Estimate of \( I_4 \)

Similarly to the previous arguments, we have

\[
\int_{E_l(Q)} A_{g_1, S}(f_1 \sigma_1 1_{Q \setminus \Omega_{i+2}}, f_2 2_{Q \setminus \Omega_{i+2}})(x) v_\Omega \, dx 
\]

\[
= \sum_{R \in Q_{i+2}} \sum_{R \in Q_2} \int_{\hat{R}} A_{g_1, S}(v_\Omega 1_{E_l(Q)}, \sigma_1 1_R)(x) \sigma_2 \, dx \cdot \frac{E^R}{R} f_1 \cdot \frac{E^R}{R} f_2 
\]

\[
\leq \sum_{R \in Q_{i+2}, R \subseteq Q_2} \sum_{R \in Q_2} \int_{\hat{R}} A_{g_1, S}(v_\Omega 1_{E_l(Q)}, \sigma_1 1_R)(x) \sigma_2 \, dx 
\]

\[
\times \frac{E^R}{R} f_1 \cdot \frac{E^R}{R} f_2 
\]

\[
+ \sum_{R \in Q_{i+2}, R \subseteq Q_2} \sum_{R \in Q_2} \int_{\hat{R}} A_{g_1, S}(v_\Omega 1_{E_l(Q)}, \sigma_1 1_R)(x) \sigma_2 \, dx 
\]

\[
\times \frac{E^R}{R} f_1 \cdot \frac{E^R}{R} f_2 
\]

\[
+ \sum_{R \in Q_{i+2}, R \subseteq Q_2} \sum_{R \in Q_2} \int_{\hat{R}} A_{g_1, S}(v_\Omega 1_{E_l(Q)}, \sigma_1 1_R)(x) \sigma_2 \, dx 
\]

\[
\times \frac{E^R}{R} f_1 \cdot \frac{E^R}{R} f_2. 
\]
3.3.1 Estimate of $\tilde{w}$

$$\times \mathbb{E}_R^{\sigma_1} f_1 \cdot \mathbb{E}_R^{\sigma_2} f_2$$

$$+ \sum_{E_R^{\sigma_1} \subset \mathbb{R}^n} \sum_{E_R^{\sigma_2} \subset \mathbb{R}^n} \int_R A_{\varphi, S}(v_{\tilde{w}}1_{E_1(Q)}, \sigma_11_R)(x)\sigma_2dx$$

$$\times \mathbb{E}_R^{\sigma_1} f_1 \cdot \mathbb{E}_R^{\sigma_2} f_2$$

$$:= I_{41}(l, Q) + I_{42}(l, Q) + I_{43}(l, Q) + I_{44}(l, Q).$$

3.3.1 Estimate of $I_{41}$

We have

$$\sum_{E_R^{\sigma_1} \subset \mathbb{R}^n} \sum_{E_R^{\sigma_2} \subset \mathbb{R}^n} \int_R A_{\varphi, S}(v_{\tilde{w}}1_{E_1(Q)}, \sigma_11_R)(x)\sigma_2dx$$

$$\times \mathbb{E}_R^{\sigma_1} f_1 \cdot \mathbb{E}_R^{\sigma_2} f_2$$

$$\leq 16^2 \int_{E_1(Q)} A_{\varphi, S}(\sigma_11_Q, \sigma_21_Q)(x)\sigma_2dx \cdot \mathbb{E}_F^\sigma f_1 \cdot \mathbb{E}_G^\sigma f_2.$$

Similarly to (3.7), for $x \in E_1(Q)$, we have

$$A_{\varphi, S}(\sigma_11_Q, \sigma_21_Q)(x) \leq 2A_{\varphi, S_i}(\sigma_11_Q, \sigma_21_Q)(x).$$

By Lemma 2.21, we have

$$\sum_{l \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_l} v_{\tilde{w}}(E_1(Q))^{1-p}I_{41}(l, Q)^p$$

$$\leq \sum_{l \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_l} v_{\tilde{w}}(E_1(Q))^{1-p} \left( \int_{E_1(Q)} A_{\varphi, S}(\sigma_11_Q, \sigma_21_Q)(x)v_{\tilde{w}}dx \right)^p$$

$$\leq \sum_{l \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_l} \left( \int_{E_1(Q)} A_{\varphi, S_i}(\sigma_11_Q, \sigma_21_Q)(x)v_{\tilde{w}}dx \right)^p$$

$$\leq \sum_{l \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_l} \int_{E_1(Q)} (A_{\varphi, S_i}(\sigma_11_Q, \sigma_21_Q))^p v_{\tilde{w}}dx \cdot (\mathbb{E}_F^\sigma f_1 \cdot \mathbb{E}_G^\sigma f_2)^p$$

$$\leq \sum_{l \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_l} \left[ \bar{w} \right]_{A_F} \left( \sum_{G \in \mathcal{G}} \sum_{l \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_l} \sigma_1(Q_{l, Q}, \eta) (\mathbb{E}_G^{\sigma_1} f_1)^{p_1} \right)^{p/p_1}$$

$$\times \left( \sum_{G \in \mathcal{G}} \sum_{l \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_l} \sigma_2(Q_{l, Q}, \eta) (\mathbb{E}_G^{\sigma_2} f_2)^{p_2} \right)^{p/p_2}.$$
(by Lemma 2.4 and Hölder’s inequality)
\[
\leq [\tilde{w}]_{A_F} (\sigma_1)_{A_\infty} (\sigma_2)_{A_\infty} \left( \sum_{G \in \mathcal{G}} \sigma_1(G) \left( \mathbb{E}_G^\sigma f_1 \right)^{p_1} \right)^{p/p_1} \\
\times \left( \sum_{G \in \mathcal{G}} \sigma_2(G) \left( \mathbb{E}_G^\sigma f_2 \right)^{p_2} \right)^{p/p_2} \\
\lesssim [\tilde{w}]_{A_F} (\sigma_1)_{A_\infty} (\sigma_2)_{A_\infty} \| f_1 \|_{L^p(\sigma)} \| f_2 \|_{L^p(\sigma_2)},
\]
where (3.5) and (3.6) are used in the last step.

3.3.2 Estimates of I_{42} and I_{43}

We have
\[
\sum_{R \in \mathcal{Q}_{1+2}, \bar{R} \in \mathcal{Q} \atop R \leq 16 \mathbb{E}_R^{\sigma_1} f_1} \sum_{R \in \mathcal{Q}_{1+2}, \bar{R} \in \mathcal{Q} \atop \bar{R} > 16 \mathbb{E}_R^{\sigma_2} f_2} \int_R A_{\varphi, S}(v_{\tilde{\varphi}} 1_{E_i(Q)}, \sigma_1, \sigma_2)(x) dx \cdot \mathbb{E}_R^\sigma f_1 \cdot \mathbb{E}_R^{\sigma_2} f_2 \\
\leq \sum_{R \in \mathcal{Q}_{1+2}, \bar{R} \in \mathcal{Q} \atop \bar{R} > 16 \mathbb{E}_R^{\sigma_2} f_2} \int_R A_{\varphi, S}(v_{\tilde{\varphi}} 1_{E_i(Q)}, \sigma_1, \sigma_2)(x) dx \cdot \mathbb{E}_R^\sigma f_1 \cdot \mathbb{E}_R^{\sigma_2} f_2.
\]

For simplicity, set
\[
g_Q := \sum_{R \in \mathcal{Q}_{1+2}, \bar{R} \in \mathcal{Q} \atop \bar{R} > 16 \mathbb{E}_R^{\sigma_2} f_2} \mathbb{E}_R^{\sigma_2} f_2.
\]

We have
\[
\sum_{i \in \mathbb{Z}, Q \in \mathcal{Q}_i} v_{\tilde{\varphi}}(E_i(Q))^{1-p} \left( \int_{E_i(Q)} A_{\varphi, S}(1_{Q}, g_Q, \sigma_2)(x) v_{\tilde{\varphi}} dx \cdot \mathbb{E}_G^\sigma f_1 \right)^p \\
= \sum_{G \in \mathcal{G}} \sum_{i \in \mathbb{Z}, Q \in \mathcal{Q}_i \atop \Gamma(Q) = G} v_{\tilde{\varphi}}(E_i(Q))^{1-p} \left( \int_{E_i(Q)} A_{\varphi, S}(1_{Q}, g_Q, \sigma_2)(x) v_{\tilde{\varphi}} dx \cdot \mathbb{E}_G^\sigma f_1 \right)^p.
\]

Set
\[
h_G := \sup_{Q \in \mathcal{Q}_i \atop \Gamma(Q) = G} g_Q.
\]

We have
\[
\sum_{G \in \mathcal{G}} \| h_G \|_{L^p(\sigma_2)}^{p_2} = \sum_{G \in \mathcal{G}} \int h_G^{p_2} \sigma_2 \\
\leq \sum_{G \in \mathcal{G}} \sum_{Q \in \mathcal{Q}_{1+2}, \bar{R} \in \mathcal{Q} \atop \Gamma(Q) = G} \int g_Q^{p_2} \sigma_2 \\
= \sum_{G \in \mathcal{G}} \sum_{Q \in \mathcal{Q}_{1+2}, \bar{R} \in \mathcal{Q} \atop \Gamma(Q) = G} \sum_{R \in \mathcal{Q}_{1+2}, \bar{R} \in \mathcal{Q} \atop \bar{R} > 16 \mathbb{E}_R^{\sigma_2} f_2} \left( \mathbb{E}_{\bar{R}}^{\sigma_2} f_2 \right)^{p_2} \sigma_2(\bar{R}).
\]
Since \( E_l(Q) \) are disjoint, we have

\[
\sum_{l \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_l} v_w(E_l(Q))^{1-p} I_{42}(l, Q)^p
\leq \sum_{G \in \mathcal{G}} \sum_{l \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_l} v_w(E_l(Q))^{1-p} \left( \int_{E_l(Q)} A_{\gamma, S}(\sigma_1 1_Q, g_Q 1_Q)(x) v_w dx \cdot \mathbb{E}_G f_1 \right)^p
\leq \sum_{G \in \mathcal{G}} \sum_{l \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_l} \int_{E_l(Q)} (A_{\gamma, S}(\sigma_1 1_{G}, h_Q 1_Q))^p v_w dx \cdot (\mathbb{E}_G f_1)^p
\leq \sum_{G \in \mathcal{G}} \int_G (A_{\gamma, S}(\sigma_1 1_{G}, h_Q 1_Q))^p v_w dx \cdot (\mathbb{E}_G f_1)^p
\leq [\tilde{w}] A_{p, \mathcal{P}}^{p/p_1} ([v_w]_{A_\infty}^{1/p'} + [\sigma_2]_{A_\infty}^{1/p_2}) \sum_{G \in \mathcal{G}} \sigma_1(G)^{p/p_1} (\mathbb{E}_G f_1)^p \| h_G \|_{L^p_{Q_2}(\sigma_2)}^{p/p_2}
\quad \text{(by Lemma 2.5)}
\leq [\tilde{w}] A_{p, \mathcal{P}}^{p/p_1} ([v_w]_{A_\infty}^{1/p'} + [\sigma_2]_{A_\infty}^{1/p_2}) \| f_1 \|_{L^p_{Q_1}(\sigma_1)}^{p/p_1} (\sum_{G \in \mathcal{G}} \| h_G \|_{L^p_{Q_2}(\sigma_2)}^{p/p_2})^{p/p_2}
\quad \text{(by Hölder’s inequality and 3.5)}
\leq [\tilde{w}] A_{p, \mathcal{P}}^{p/p_1} ([v_w]_{A_\infty}^{1/p'} + [\sigma_2]_{A_\infty}^{1/p_2}) \| f_1 \|_{L^p_{Q_1}(\sigma_1)}^{p/p_1} \| f_2 \|_{L^p_{Q_2}(\sigma_2)}^{p/p_2}.
\]

By symmetry, we get

\[
\sum_{l \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_l} v_w(E_l(Q))^{1-p} I_{43}(l, Q)^p \leq [\tilde{w}] A_{p, \mathcal{P}}^{p/p_1} ([v_w]_{A_\infty}^{1/p'} + [\sigma_1]_{A_\infty}^{1/p_2})^p
\times \| f_1 \|_{L^p_{Q_1}(\sigma_1)}^{p/p_1} \| f_2 \|_{L^p_{Q_2}(\sigma_2)}^{p/p_2}.
\]

### 3.3.3 Estimate of \( I_{44} \)

We have

\[
\sum_{R \in \mathcal{Q}_{l+2, R \subset Q}^{\mathcal{G}}} \sum_{E_{E_l(Q)}^{16\mathcal{P}_{\mathcal{G}}}} \int_R A_{\gamma, S}(v_w 1_{E_l(Q)}, \sigma_1 1_{R})(x) \sigma_2 dx \cdot \mathbb{E}_R f_1 \cdot \mathbb{E}_R f_2
\leq \left( \sum_{R \in \mathcal{Q}_{l+2, R \subset Q}^{\mathcal{G}}} \sigma_1(R)^{-p_1/p_2} \left( \int_R A_{\gamma, S}(v_w 1_{E_l(Q)}, g_Q \sigma_2)(x) \sigma_1 dx \right)^{p_1} \right)^{1/p'_1}
\quad \cdot \left( \sum_{R \in \mathcal{Q}_{l+2, R \subset Q}^{\mathcal{G}}} \sigma_1(R)(\mathbb{E}_R f_1), \mathcal{P}_{\mathcal{G}} \right)^{1/p_1}.
\]

22
where Lemma 2.5 and Lemma 2.6 are used in the last step. It follows that

\[
\begin{align*}
&\lesssim [\tilde{w}]_{A_p} [v\tilde{w}]_{A_\infty}^p ([\sigma_1]_{A_\infty}^1 + [\sigma_2]_{A_\infty}^2)^p \left( \sum\sum_{l \in \mathcal{Z}, Q \in \mathcal{Q}_l} \| g\tilde{w} \|_{L^2_p(\sigma_2)}^{p_2} \right)^{p/p_2} \\
&\quad \cdot \left( \sum\sum_{l \in \mathcal{Z}, Q \in \mathcal{Q}_l} \sum_{R \in \mathcal{Q}_{l+2, \mathcal{R}_C}} E_R^{s_1} \| f_1 \|_{E_R^{s_1}}^{p_1} \right)^{p/p_1},
\end{align*}
\]

where Lemma 2.5 and Lemma 2.6 are used in the last step. It follows that

\[
\begin{align*}
&\sum_{l \in \mathcal{Z}} \sum_{Q \in \mathcal{Q}_l} v\tilde{w}(E_l(Q))^{1-p} I_{44}(l, Q)^p \\
&= \sum_{l \in \mathcal{Z}} \sum_{Q \in \mathcal{Q}_l} v\tilde{w}(E_l(Q))^{1-p} \left( \sum\sum_{R \in \mathcal{Q}_{l+2, \mathcal{R}_C}} E_R^{s_1} \| f_1 \|_{E_R^{s_1}} \right)^{p/p_1} \left( \sum\sum_{l \in \mathcal{Z}, Q \in \mathcal{Q}_l} \sum_{R \in \mathcal{Q}_{l+2, \mathcal{R}_C}} E_R^{s_2} \| f_2 \|_{E_R^{s_2}} \right)^{p/p_2} \\
&\quad \cdot \left( \sum\sum_{l \in \mathcal{Z}, Q \in \mathcal{Q}_l} \sum_{R \in \mathcal{Q}_{l+2, \mathcal{R}_C}} E_R^{s_1} \| f_1 \|_{E_R^{s_1}} \right)^{p/p_1} \left( \sum\sum_{l \in \mathcal{Z}, Q \in \mathcal{Q}_l} \sum_{R \in \mathcal{Q}_{l+2, \mathcal{R}_C}} E_R^{s_2} \| f_2 \|_{E_R^{s_2}} \right)^{p/p_2} \\
&= [\tilde{w}]_{A_p} [v\tilde{w}]_{A_\infty}^p ([\sigma_1]_{A_\infty}^1 + [\sigma_2]_{A_\infty}^2)^p \times \left( \sum\sum_{l \in \mathcal{Z}, Q \in \mathcal{Q}_l} \sum_{R \in \mathcal{Q}_{l+2, \mathcal{R}_C}} E_R^{s_1} \| f_1 \|_{E_R^{s_1}}^{p_1} \right)^{p/p_1} \\
&\quad \times \left( \sum\sum_{l \in \mathcal{Z}, Q \in \mathcal{Q}_l} \sum_{R \in \mathcal{Q}_{l+2, \mathcal{R}_C}} E_R^{s_2} \| f_2 \|_{E_R^{s_2}}^{p_2} \right)^{p/p_2} \\
&\lesssim [\tilde{w}]_{A_p} [v\tilde{w}]_{A_\infty}^p ([\sigma_1]_{A_\infty}^1 + [\sigma_2]_{A_\infty}^2)^p \| f_1 \|_{L^{p_1}(\sigma_1)} \cdot \| f_2 \|_{L^{p_2}(\sigma_2)},
\end{align*}
\]

where (3.3) and (3.4) are used in the last step.

Summing up the above arguments, we get

\[
I_4 \lesssim [\tilde{w}]_{A_p} \left( [\sigma_1]_{A_\infty}^1 [\sigma_2]_{A_\infty}^2 + [v\tilde{w}]_{A_\infty}^p ([\sigma_1]_{A_\infty}^1 + [\sigma_2]_{A_\infty}^2)^p \right) \times \| f_1 \|_{L^{p_1}(\sigma_1)}^{p_1} \cdot \| f_2 \|_{L^{p_2}(\sigma_2)}^{p_2}.
\]

This completes the proof of (1.1).
3.4 Sharpness of the strong type estimates

Finally, we prove the sharpness. We use the example in [19]. That is,

\[ R_1(\vec{f})(x) = p.v. \int_{(\mathbb{R}^n)^m} \frac{\sum_{j=1}^{m}(x_1 - (y_j)_1)}{(\sum_{j=1}^{m} |x - y_j|^2)^{(nm+1)/2}} f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m. \]

Assume that 0 < \varepsilon < 1. Let

\[ f_i(x) = |x|^{(n-\varepsilon)} \chi_{(0,1]^n}(x) \quad \text{and} \quad w_i(x) = |x|^{(n-\varepsilon)(p_i-1)}, \quad i = 1, \ldots, m. \]

Then we have \( v_{\vec{w}} = |x|^{(n-\varepsilon)(mp-1)} \), \( [\vec{w}]_{A_{mp}} = [v_{\vec{w}}]_{A_{mp}} \approx (1/\varepsilon)^{mp-1} \) and \( [\sigma_i]_{A_{\infty}} \lesssim 1/\varepsilon \quad i = 1, \ldots, m. \)

Moreover,

\[ \|R_1(\vec{f})\|_{L^p(v_{\vec{w}})} \geq (1/\varepsilon)^{m+1/p} \quad \text{and} \quad \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(w_i)} \approx (1/\varepsilon)^{1/p}. \]

It follows that our result is sharp whenever \( \max_i \{p_i\} \leq p'/ (mp - 1) \).

This completes the proof of Theorem 1.1.

4 Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2. We begin with the Whitney decomposition.

Proposition 4.1 [6, Proposition 7.3.4] Let \( \Omega \) be an open nonempty proper subset of \( \mathbb{R}^n \). Then there exists a family of closed cubes \{Q_j\}_j such that

(i) \( \bigcup_j Q_j = \Omega \) and the \( Q_j \)'s have disjoint interiors;

(ii) \( \sqrt{n}l(Q_j) \leq \text{dist} (Q_j, \Omega^c) \leq 4\sqrt{n}l(Q_j) \);

(iii) if the boundaries of two cubes \( Q_j \) and \( Q_k \) touch, then

\[ \frac{1}{4} \leq \frac{l(Q_j)}{l(Q_k)} \leq 4; \]

(iv) there exists some constant \( 1 < \gamma < 5/4 \) such that \( \sum_j \chi_{\gamma Q_j}(x) \leq C_n. \)

Next we give a weak type estimate for the multilinear maximal function. Recall that the multilinear maximal function is defined by

\[ M(\vec{f}) = \sup_{Q \ni \vec{x}} \prod_{i=1}^{m} \frac{1}{|Q|} \int_Q |f_i(y_i)| dy_i \]

and the dyadic maximal function is defined by

\[ M^\delta(\vec{f})(x) = \sup_{Q \ni \vec{x}, Q \in \mathcal{D}} \prod_{i=1}^{m} \frac{1}{|Q|} \int_Q |f_i(y_i)| dy_i. \]
Lemma 4.2 Let \( \tilde{P} = (p_1, \ldots, p_m) \) with \( 1/p = 1/p_1 + \cdots + 1/p_m \) and \( 1 < p_1, \ldots, p_m < \infty \). Suppose that \( \tilde{w} = (w_1, \ldots, w_m) \) with \( \tilde{w} \in A_{\tilde{P}} \). Then

\[
\| M(\tilde{f}) \|_{L^{p,\infty}(\tilde{w})} \leq C_{m, \tilde{P}} [\tilde{w}]^{1/p} \prod_{i=1}^{m} \| f_i \|_{L^{p_i}(w_i)}.
\]

Proof. In [3], the authors proved that there exists \( 2^n \) family of dyadic grids \( \mathcal{D}_\beta \) such that

\[
M(\tilde{f})(x) \leq 6^m \sum_{\beta=1}^{2^n} M_{\mathcal{D}_\beta}(\tilde{f})(x),
\]

where

\[
M_{\mathcal{D}_\beta}(\tilde{f})(x) = \sup_{Q \ni x, Q \in \mathcal{D}_\beta} \prod_{i=1}^{m} \frac{1}{|Q|} \int_Q |f_i(y_i)|dy_i.
\]

For some fixed dyadic grid \( \mathcal{D} \),

\[
\{ x \in \mathbb{R}^n : M_{\mathcal{D}}(\tilde{f}) > \alpha \} = \bigcup_k Q_k,
\]

where \( \{Q_k\}_k \) are disjoint dyadic cubes in \( \mathcal{D} \) and

\[
\prod_{i=1}^{m} \frac{1}{|Q_k|} \int_{Q_k} |f_i(y_i)|dy_i > \alpha.
\]

It follows that

\[
\alpha^p \left( \sum_k v_{\tilde{w}}(Q_k) \right) \leq \sum_k \left( \prod_{i=1}^{m} \frac{1}{|Q_k|} \int_{Q_k} |f_i|dy_i \right)^p v_{\tilde{w}}(Q_k)
\]

\[
\leq \sum_k \left( \prod_{i=1}^{m} \int_{Q_k} |f_i|^{p_i}w_i dy_i \right)^{p/p_i} v_{\tilde{w}}(Q_k) \prod_{i=1}^{m} \sigma_i(Q_k)^{p/p_i} |Q_k|^{-mp/p_i}
\]

\[
\leq [\tilde{w}]_{A_{\tilde{P}}} \sum_k \left( \prod_{i=1}^{m} \int_{Q_k} |f_i|^{p_i}w_i dy_i \right)^{p/p_i}
\]

\[
\leq [\tilde{w}]_{A_{\tilde{P}}} \prod_{i=1}^{m} \| f_i \|_{L^{p_i}(w_i)}^{p/p_i}
\]

Hence

\[
\| M_{\mathcal{D}}(\tilde{f}) \|_{L^{p,\infty}(\tilde{w})} = \sup_{\alpha > 0} \alpha v_{\tilde{w}}(\{ x \in \mathbb{R}^n : M_{\mathcal{D}}(\tilde{f}) > \alpha \})^{1/p}
\]

\[
= \sup_{\alpha > 0} \alpha \left( \sum_k v_{\tilde{w}}(Q_k) \right)^{1/p}
\]

\[
\leq [\tilde{w}]_{A_{\tilde{P}}}^{1/p} \prod_{i=1}^{m} \| f_i \|_{L^{p_i}(w_i)}.\]
This completes the proof. □

The following result can be proved similarly to [IS p. 1240] and we omit the details.

**Lemma 4.3** Let $T$ be an $m$-linear Calderón-Zygmund operator and $Q$ be a cube. Set $Q^* = 10\sqrt{n}Q$ and $Q^{**} = 10\sqrt{n}Q^*$. Suppose that $x, z \in Q$ and $y \in Q^*$. Then

$$|T(f_1, \cdots, f_{i-1}, f_i\chi_{Q'}, f_{i+1}, \cdots, f_m)(x) - T(f_1, \cdots, f_{i-1}, f_i\chi_{Q''}, f_{i+1}, \cdots, f_m)(y)| \leq CM(f_1, \cdots, f_m).$$

Next we give a characterization of the weak boundedness of multilinear Calderón-Zygmund operators.

**Lemma 4.4** Let $1 < p, p_1, p_2 < \infty$ and $\vec{w} \in A_p$, where $\vec{w} := (w_1, w_2)$ and $\vec{p} := (p_1, p_2)$ with $1/p = 1/p_1 + 1/p_2$. Suppose that $T$ is a multilinear Calderón-Zygmund operator. Then the following assertions are equivalent.

(i) $\|T(f_1\sigma_1, f_2\sigma_2)\|_{L^{p, \infty}(v_{\vec{w}})} \leq C \prod_{i=1}^2 \|f_i\|_{L^{p_i}(\sigma_i)}$;

(ii) $\int_Q |T(f_1\sigma_1\chi_Q, f_2\sigma_2\chi_Q)(x)|v_{\vec{w}}(x)dx \leq C' \prod_{i=1}^2 \|f_i\|_{L^{p_i}(\sigma_i)}v_{\vec{w}}(Q)^{1/p'}$ for all cubes $Q \subset \mathbb{R}^n$ and all functions $f_i \in L^{p_i}(\sigma_i), i = 1, 2$.

**Proof.** (i) $\Rightarrow$ (ii): By the weak type boundness of $T$, we have

$$\int_Q |T(f_1\sigma_1\chi_Q, f_2\sigma_2\chi_Q)(x)|v_{\vec{w}}(x)dx$$

$$= \int_0^\infty v_{\vec{w}}\{x \in Q : |T(f_1\sigma_1\chi_Q, f_2\sigma_2\chi_Q)(x)| > \lambda\}d\lambda$$

$$\leq \int_0^\infty \min\{v_{\vec{w}}(Q), \lambda^{-p}\|T(f_1\sigma_1\chi_Q, f_2\sigma_2\chi_Q)\|_{L^{p, \infty}(v_{\vec{w}})}\}d\lambda$$

$$= p'\|T(f_1\sigma_1\chi_Q, f_2\sigma_2\chi_Q)\|_{L^{p, \infty}(v_{\vec{w}})}v_{\vec{w}}(Q)^{1/p'}$$

$$\leq p'C \prod_{i=1}^2 \|f_i\|_{L^{p_i}(\sigma_i)}v_{\vec{w}}(Q)^{1/p'}.$$

(ii) $\Rightarrow$ (i): Let $\Omega$ be an open set containing $\{x : |T(f_1\sigma_1, f_2\sigma_2)(x)| > \lambda\}$. Form the Whitney decomposition to $\Omega$, we get Whitney cubes $Q_j$. Set $Q_j^* = 10\sqrt{n}Q_j$ and $Q_{j}^{**} = 10\sqrt{n}Q_{j}^*$. Let $\gamma$ be defined as that in Proposition 4.1 In the following, we prove that

$$v_{\vec{w}}\{x \in \mathbb{R}^n : |T(f_1\sigma_1, f_2\sigma_2)(x)| > 2\lambda, M(f_1\sigma_1, f_2\sigma_2)(x) \leq \beta\lambda\}$$

$$\lesssim \beta v_{\vec{w}}(\Omega) + \mathcal{T}_s \beta^{-p} \lambda^{-p} \prod_{i=1}^m \|f_i\|_{L^{p_i}(\sigma_i)}^p, \quad (4.1)$$

where

$$\mathcal{T}_s = \sup_{\|f_i\|_{L^{p_i}(\sigma_i)} \leq 1} \sup_{i=1, 2} v_{\vec{w}}(Q)^{-1/p'} \int_Q |T(f_1\sigma_1\chi_Q, f_2\sigma_2\chi_Q)(x)|v_{\vec{w}}(x)dx.$$

By Whitney’s decomposition, we only need to estimate

$$v_{\vec{w}}\{x \in Q_j : |T(f_1\sigma_1, f_2\sigma_2)(x)| > 2\lambda, M(f_1\sigma_1, f_2\sigma_2)(x) \leq \beta\lambda\}.$$
Assume that there exists some $z_j \in Q_j$ such that $M(f_1 \sigma_1, f_2 \sigma_2)(z_j) \leq \beta \lambda$. Otherwise, is zero. By the property of Whitney decomposition, we can also choose some $y_j \in Q_j$ such that $y_j \in \Omega$. Since $\{x : T(f_1 \sigma_1, f_2 \sigma_2)(x) > \lambda\} \subset \Omega$, we have $|T(f_1 \sigma_1, f_2 \sigma_2)(y_j)| \leq \lambda$.

For any $f_i$, $i = 1, 2$, denote $f^0_i = f_i \chi_{\gamma Q_j}$ and $f^\infty_i = f_i \chi_{(\gamma Q_j)^c}$. We consider every $f^\alpha_i$ separately, where $\alpha_i = 0$ or $\infty$.

Similarly as that in [18], we consider first the case $\alpha_1 = \alpha_2 = \infty$. For $x \in Q_j$, we have
\[
\begin{align*}
&T(f_1^\infty \sigma_1, f_2^\infty \sigma_2)(x) - T(f_1^\infty \sigma_1, f_2^\infty \sigma_2)(y_j) \\
\leq & T(f_1^\infty \sigma_1 \chi_{(Q_j)^c}, f_2^\infty \sigma_2)(x) - T(f_1^\infty \sigma_1 \chi_{(Q_j)^c}, f_2^\infty \sigma_2)(y_j) \\
+ & |T(f_1 \sigma_1 \chi_{Q_j^\ast \gamma Q_j}, f_2^\infty \sigma_2)(x) - T(f_1 \sigma_1 \chi_{Q_j^\ast \gamma Q_j}, f_2^\infty \sigma_2)(y_j)| \\
+ & |T(f_1 \sigma_1 \chi_{Q_j^\ast \gamma Q_j}, f_2 \sigma_2 \chi_{(Q_j)^c})(x) - T(f_1 \sigma_1 \chi_{Q_j^\ast \gamma Q_j}, f_2 \sigma_2 \chi_{(Q_j)^c})(y_j)| \\
+ & |T(f_1 \sigma_1 \chi_{Q_j^\ast \gamma Q_j}, f_2 \sigma_2 \chi_{(Q_j)^c})(y_j)| \\
\leq & C_1 M(f_1 \sigma_1, f_2 \sigma_2)(z_j) + |T(f_1 \sigma_1 \chi_{Q_j^\ast \gamma Q_j}, f_2 \sigma_2 \chi_{(Q_j)^c})(y_j)|.
\end{align*}
\]

Next, suppose that $\alpha_i = \infty$ for $i = 1$ or $2$. Without loss of generality, assume that $i = 1$. We have
\[
\begin{align*}
&T(f_1^\infty \sigma_1, f_2^0 \sigma_2)(x) - T(f_1^\infty \sigma_1, f_2^0 \sigma_2)(y_j) \\
\leq & T(f_1^\infty \sigma_1 \chi_{Q_j^c}, f_2^0 \sigma_2)(x) - T(f_1^\infty \sigma_1 \chi_{Q_j^c}, f_2^0 \sigma_2)(y_j) \\
+ & |T(f_1 \sigma_1 \chi_{Q_j^\ast \gamma Q_j}, f_2^0 \sigma_2)(x) - T(f_1 \sigma_1 \chi_{Q_j^\ast \gamma Q_j}, f_2^0 \sigma_2)(y_j)| \\
\leq & C_2 M(f_1 \sigma_1, f_2 \sigma_2)(z_j) + |T(f_1 \sigma_1 \chi_{Q_j^\ast \gamma Q_j}, f_2^0 \sigma_2)(y_j)|.
\end{align*}
\]

Hence
\[
\begin{align*}
&T(f_1 \sigma_1, f_2 \sigma_2)(x) - T(f_1^0 \sigma_1, f_2^0 \sigma_2)(x) - T(f_1 \sigma_1, f_2 \sigma_2)(y_j) \\
\leq & C_3 M(f_1 \sigma_1, f_2 \sigma_2)(z_j) + |T(f_1^0 \sigma_1, f_2^0 \sigma_2)(y_j)| \\
+ & |T(f_1 \sigma_1 \chi_{Q_j^\ast \gamma Q_j}, f_2 \sigma_2 \chi_{(Q_j)^c})(y_j)| \\
+ & |T(f_1 \sigma_1 \chi_{Q_j^\ast \gamma Q_j}, f_2 \sigma_2 \chi_{(Q_j)^c})(y_j)|.
\end{align*}
\]

Consequently, for any $0 < \delta < 1/2$,
\[
\begin{align*}
&T(f_1 \sigma_1, f_2 \sigma_2)(x) - T(f_1^0 \sigma_1, f_2^0 \sigma_2)(x) - T(f_1 \sigma_1, f_2 \sigma_2)(y_j) \delta \\
\leq (C_3 \beta)^\delta \lambda^\delta + \Sigma^\delta,
\end{align*}
\]

where
\[
\begin{align*}
\Sigma' &= |T(f_1^0 \sigma_1, f_2^0 \sigma_2)(y_j)| + |T(f_1 \sigma_1 \chi_{Q_j^\ast \gamma Q_j}, f_2 \sigma_2 \chi_{(Q_j)^c})(y_j)| \\
&+ |T(f_1 \sigma_1 \chi_{Q_j^\ast \gamma Q_j}, f_2^0 \sigma_2)(y_j)| + |T(f_1^0 \sigma_1, f_2 \sigma_2 \chi_{(Q_j)^c})(y_j)|.
\end{align*}
\]

27
Note that \(|Q^*_j \cap \Omega^c|\) is comparable with \(|Q^*|\) due to the property of the Whitney decomposition. Integrating over \(y_j \in Q^*_j \cap \Omega^c\), we have

\[
\frac{1}{|Q^*_j\cap \Omega^c|} \int_{Q^*_j\cap \Omega^c} \left| T(f_1\sigma_1, f_2\sigma_2)(x) - T(f^0_1\sigma_1, f^0_2\sigma_2)(x) \right| dy_j \\
- T(f_1\sigma_1, f_2\sigma_2)(y_j) \right|^\delta dy_j
\leq \frac{1}{|Q^*_j\cap \Omega^c|} \int_{Q^*_j\cap \Omega^c} \Sigma^\delta dy_j
\leq \frac{1}{|Q^*_j\cap \Omega^c|} \int_{Q^*_j\cap \Omega^c} \ively dy_j
\leq (C_3\beta\lambda)\delta + C_4\mathcal{M}(f_1\sigma_1, f_2\sigma_2)(z_j)^\delta,
\]

where Kolmogorov’s inequality and the \(L^1 \times L^1 \to L^{1/2,\infty}\) boundedness of \(T\) are used, see [38 p. 1239]. Since

\[
|T(f_1\sigma_1, f_2\sigma_2)(x) - T(f^0_1\sigma_1, f^0_2\sigma_2)(x) - T(f_1\sigma_1, f_2\sigma_2)(y_j)|^\delta
\geq |T(f_1\sigma_1, f_2\sigma_2)(x) - T(f^0_1\sigma_1, f^0_2\sigma_2)(x)|^\delta - |T(f_1\sigma_1, f_2\sigma_2)(y_j)|^\delta
\geq |T(f_1\sigma_1, f_2\sigma_2)(x) - T(f^0_1\sigma_1, f^0_2\sigma_2)(x)|^\delta - \lambda^\delta,
\]

we have

\[
|T(f_1\sigma_1, f_2\sigma_2)(x) - T(f^0_1\sigma_1, f^0_2\sigma_2)(x)| \leq (1 + C_5\beta)\lambda.
\]

It follows that for \(\beta \leq (2C_5)^{-1}\),

\[
|T(f_1\sigma_1, f_2\sigma_2)(x) - T(f^0_1\sigma_1, f^0_2\sigma_2)(x)| \leq (1 + C_5\beta)\lambda \leq 3\lambda/2.
\]

Denote

\[
E_j = \{x \in Q_j : |T(f_1\sigma_1, f_2\sigma_2)(x)| > 2\lambda; \mathcal{M}(f_1\sigma_1, f_2\sigma_2)(x) \leq \beta\lambda\}.
\]

Then we have

\[
E_j \subset \{x \in Q_j : |T(f^0_1\sigma_1, f^0_2\sigma_2)(x)| > \lambda/2\}.
\]

Therefore,

\[
\sum_j v_{\sigma}(E_j) \leq \beta^{-p} \sum_{j:v_{\sigma}(E_j) > \beta v_{\sigma}(\gamma Q_j)} v_{\sigma}(E_j) \left( \frac{2}{\lambda v_{\sigma}(\gamma Q_j)} \int_{E_j} |T(f^0_1\sigma_1, f^0_2\sigma_2)| v_{\sigma} \right)^p
+ \beta \sum_{j:v_{\sigma}(E_j) \leq \beta v_{\sigma}(\gamma Q_j)} v_{\sigma}(\gamma Q_j)
\]

\[
:= I + II.
\]

By Hölder’s inequality, we have

\[
I \leq \left( \frac{2}{\beta\lambda} \right)^p T^p \sum_j \prod_{i=1}^2 \left( \int_{\gamma Q_j} |f_i(y_i)|^{p_i}\sigma_i dy_i \right)^{p/p_i}
\leq \left( \frac{2}{\beta\lambda} \right)^p T^p \prod_{i=1}^2 \left( \int_{\mathbb{R}^n} \left( \sum_j \chi_{\gamma Q_j} |f_i(y_i)|^{p_i}\sigma_i dy_i \right) \right)^{p/p_i}
\leq \beta^{-p} \lambda^{-p} T^p \prod_{i=1}^2 \|f_i\|_{L^{p_i}(\sigma_i)}^p.
\]

28
On the other hand, by the property of Whitney’s decomposition, 
\[ H \leq \beta \sum_j v_{\bar{w}}(\gamma Q_j) \leq C_n \beta v_{\bar{w}}(\Omega). \]

This proves (11). Taking the infimum over \( \Omega \), we have
\[ v_{\bar{w}}\{x \in \mathbb{R}^n : |T(f_1 \sigma_1, f_2 \sigma_2)(x)| > 2\lambda; \mathcal{M}(f_1 \sigma_1, f_2 \sigma_2)(x) \leq \beta \lambda \} \]
\[ \leq \beta v_{\bar{w}}\{x \in \mathbb{R}^n : |T(f_1 \sigma_1, f_2 \sigma_2)(x)| > \lambda \} + T^p_* \beta^{-p} \lambda^{-p} \prod_{i=1}^2 \|f_i\|_{L^p(\sigma_i)}^p. \]

It follows that
\[ \|T(f_1 \sigma_1, f_2 \sigma_2)\|_{L^p,\infty(v_{\bar{w}})}^p = \sup_{\lambda > 0} (2\lambda)^p v_{\bar{w}}\{|T(f_1 \sigma_1, f_2 \sigma_2)\| > 2\lambda \} \]
\[ \leq \sup_{\lambda > 0} (2\lambda)^p v_{\bar{w}}\{|T(f_1 \sigma_1, f_2 \sigma_2)\| > 2\lambda; \mathcal{M}(f_1 \sigma_1, f_2 \sigma_2)(x) \leq \beta \lambda \} \]
\[ + \sup_{\lambda > 0} (2\lambda)^p v_{\bar{w}}\{\mathcal{M}(f_1 \sigma_1, f_2 \sigma_2)(x) > \beta \lambda \} \]
\[ \leq \sup_{\lambda > 0} 2^p \beta C_6 \lambda^p v_{\bar{w}}\{x \in \mathbb{R}^n : |T(f_1 \sigma_1, f_2 \sigma_2)(x)| > \lambda \} \]
\[ + (2^p \|\mathcal{M}\|_{L^p(w_1) \times L^p(w_2) \rightarrow L^{p,\infty}(v_{\bar{w}})} + C_6 T^p_*) \beta^{-p} \prod_{i=1}^2 \|f_i\|_{L^p(\sigma_i)}^p. \]
\[ = 2^p \beta C_6 \|T(f_1 \sigma_1, f_2 \sigma_2)\|_{L^p,\infty(v_{\bar{w}})}^p + (2^p \|\mathcal{M}\|_{L^p(w_1) \times L^p(w_2) \rightarrow L^{p,\infty}(v_{\bar{w}})} + C_6 T^p_*) \beta^{-p} \prod_{i=1}^2 \|f_i\|_{L^p(\sigma_i)}^p. \]

Let \( \beta = \min\{2C_5^{-1}, (2^{p+1}C_6)^{-1}\} \). By Lemma 4.2, we get
\[ \|T(f_1 \sigma_1, f_2 \sigma_2)\|_{L^p,\infty(v_{\bar{w}})} \leq (T_* + [\bar{w}]_{A_{\bar{P}}}) \prod_{i=1}^2 \|f_i\|_{L^p(\sigma_i)}^p. \tag{4.3} \]

The following is a characterization of the weak boundedness of \( A_{\mathcal{R},\mathcal{S}} \).

**Lemma 4.5** Let \( 1 < p, p_1, p_2 < \infty \) and \( \bar{w} \in A_{\bar{P}}, \) where \( \bar{w} := (w_1, w_2) \) and \( \bar{P} := (p_1, p_2) \) with \( 1/p = 1/p_1 + 1/p_2 \). Suppose that \( \mathcal{R} \) is a dyadic grid and \( \mathcal{S} \) is a sparse family in \( \mathcal{R} \). Then the following assertions are equivalent.

(i). \( \|A_{\mathcal{R},\mathcal{S}}(f \sigma_1, f \sigma_2)\|_{L^p(\bar{w})} \leq C \prod_{i=1}^2 \|f_i\|_{L^p(\sigma_i)} \); 

(ii). \( \int_Q A_{\mathcal{R},\mathcal{S}}(f \sigma_1 \chi_Q, f \sigma_2 \chi_Q)(x) v_{\bar{w}}(x) \, dx \leq C \prod_{i=1}^2 \|f_i\|_{L^p(\sigma_i)} v_{\bar{w}}(Q)^{1/p'} \) for all cubes \( Q \subset \mathbb{R}^n \) and all functions \( f_i \in L^p(\sigma_i), \) \( i = 1, 2; \)

(iii). \( \int_Q A_{\mathcal{R},\mathcal{S}}(f \sigma_1 \chi_Q, f \sigma_2 \chi_Q)(x) v_{\bar{w}}(x) \, dx \leq C \prod_{i=1}^2 \|f_i\|_{L^p(\sigma_i)} v_{\bar{w}}(Q)^{1/p'} \) for all dyadic cubes \( Q \in \mathcal{S} \) and all functions \( f_i \in L^p(\sigma_i), \) \( i = 1, 2. \)
Proof. (i)⇒(ii) is similar to the proof of Lemma 4.4. (ii)⇒(iii) is obvious. We only need to prove (iii)⇒(i).
For any \( t > 0 \), denote \( \Omega_t := \{ x \in \mathbb{R}^n : A_{\varphi, \mathcal{S}}(|f_1| \sigma_1, |f_2| \sigma_2)(x) > t \} := \bigcup_\zeta P_\zeta \), where \( P_\zeta \) are pairwise disjoint maximal cubes in \( \Omega_t \). We have
\[
\sum_{Q_{j,k} \ni P_\zeta} \prod_{i=1}^2 \frac{1}{|Q_{j,k}|} \int_{Q_{j,k}} |f_i(y_i)| \sigma_i dy_i > t,
\]
and
\[
\sum_{Q_{j,k} \not\ni P_\zeta} \prod_{i=1}^2 \frac{1}{|Q_{j,k}|} \int_{Q_{j,k}} |f_i(y_i)| \sigma_i dy_i \leq t.
\]
Therefore, for \( x \in E_\zeta = P_\zeta \cap \Omega_{2t} \), we have
\[
2t < A_{\varphi, \mathcal{S}}(|f_1| \sigma_1, |f_2| \sigma_2)(x) \leq A_{\varphi, \mathcal{S}}(|f_1| \sigma_1 P_\zeta, |f_2| \sigma_2 P_\zeta)(x) + t.
\]
That is,
\[
A_{\varphi, \mathcal{S}}(|f_1| \sigma_1 P_\zeta, |f_2| \sigma_2 P_\zeta)(x) > t, \quad x \in E_\zeta.
\]
It follows that
\[
(2t)^p v_\varpi(\Omega_{2t}) = (2t)^p \sum_\zeta v_\varpi(E_\zeta)
\]
\[
\leq 2^p \sum_{v_\varpi(E_\zeta) > \beta v_\varpi(P_\zeta)} v_\varpi(E_\zeta) \left( \frac{1}{v_\varpi(E_\zeta)} \int_{E_\zeta} A_{\varphi, \mathcal{S}}(|f_1| \sigma_1 P_\zeta, |f_2| \sigma_2 P_\zeta)(x) dx \right)^p
\]
\[
+ (2t)^p \sum_{v_\varpi(E_\zeta) \leq \beta v_\varpi(P_\zeta)} \beta v_\varpi(P_\zeta)
\]
\[
\leq 2^p \beta^{1-p} \sum_\zeta v_\varpi(P_\zeta)^{1-p} \left( \int_{P_\zeta} A_{\varphi, \mathcal{S}}(|f_1| \sigma_1 P_\zeta, |f_2| \sigma_2 P_\zeta)(x) dx \right)^p
\]
\[
+ 2^p \beta \| A_{\varphi, \mathcal{S}}(|f_1| \sigma_1, |f_2| \sigma_2) \|^p_{L^{p,\infty}(v_\varpi)}
\]
\[
\leq C^p 2^p \beta^{1-p} \sum_\zeta \left( \int_{P_\zeta} |f_1|^{p_1} \sigma_1 \right)^{p/p_1} \cdot \left( \int_{P_\zeta} |f_2|^{p_2} \sigma_2 \right)^{p/p_2}
\]
\[
+ 2^p \beta \| A_{\varphi, \mathcal{S}}(|f_1| \sigma_1, |f_2| \sigma_2) \|^p_{L^{p,\infty}(v_\varpi)}
\]
\[
\leq C^p 2^p \beta^{1-p} \| f_1 \|^p_{L^{p_1}(\sigma_1)} \| f_2 \|^p_{L^{p_2}(\sigma_2)} + 2^p \beta \| A_{\varphi, \mathcal{S}}(|f_1| \sigma_1, |f_2| \sigma_2) \|^p_{L^{p,\infty}(v_\varpi)}.
\]
By setting \( \beta = 2^{-p-1} \) and taking the supremum of \( t \), we get the conclusion desired. \( \square \)

Now we are ready to prove Theorem 1.2

Proof of Theorem 1.2. By setting the Banach space \( \mathcal{X} \) to be \( L^1_{v_\varpi}(Q) \), we see from (2.1) that
\[
\int_Q |T(f_1 \sigma_1 \chi_Q, f_2 \sigma_2 \chi_Q)(x)| v_\varpi(x) dx \leq \sup_{\varphi, \mathcal{S}} \int_Q A_{\varphi, \mathcal{S}}(|f_1| \sigma_1 \chi_Q, |f_2| \sigma_2 \chi_Q)(x) v_\varpi(x) dx.
\]
Hence
\[
v_{\vec{w}}(Q)^{-1/p'} \int_Q |T(f_1 \sigma_1 \chi_\mathcal{Q}, f_2 \sigma_2 \chi_\mathcal{Q})(x)|v_{\vec{w}}(x)dx
\]
\[
\lesssim \sup_{\mathcal{D}, \mathcal{S}} v_{\vec{w}}(Q)^{-1/p'} \int_Q A_{\mathcal{D}, \mathcal{S}}(|f_1| \sigma_1 \chi_\mathcal{Q}, |f_2| \sigma_2 \chi_\mathcal{Q})(x)v_{\vec{w}}(x)dx.
\]

For fixed $\mathcal{D}, \mathcal{S}$, by Lemma 4.5 it suffices to estimate
\[
v_{\vec{w}}(Q)^{-1/p'} \int_Q A_{\mathcal{D}, \mathcal{S}}(|f_1| \sigma_1 \chi_\mathcal{Q}, |f_2| \sigma_2 \chi_\mathcal{Q})(x)v_{\vec{w}}(x)dx
\]
for dyadic cube $Q \in \mathcal{S}$. By Lemma 2.5, we have
\[
v_{\vec{w}}(Q)^{-1/p'} \int_Q A_{\mathcal{D}, \mathcal{S}}(|f_1| \sigma_1 \chi_\mathcal{Q}, |f_2| \sigma_2 \chi_\mathcal{Q})(x)v_{\vec{w}}(x)dx
\]
\[
\leq v_{\vec{w}}(Q)^{-1/p'} \int_Q A_{\mathcal{D}, \mathcal{S}}(v_{\vec{w}} \chi_\mathcal{Q}, |f_2| \sigma_2 \chi_\mathcal{Q})|f_1| \sigma_1(x)dx
\]
\[
\leq v_{\vec{w}}(Q)^{-1/p'} \left( \int_Q (A_{\mathcal{D}, \mathcal{S}}(v_{\vec{w}} \chi_\mathcal{Q}, |f_2| \sigma_2 \chi_\mathcal{Q}))^{p'_1} \sigma_1 dx \right)^{1/p_1}
\]
\[
\cdot \left( \int_Q |f_1|^{p_1} \sigma_1(x)dx \right)^{1/p_1}
\]
\[
\lesssim [\vec{w}]_{A_{p}}^{1/p} [v_{\vec{w}}]_{A_{\infty}}^{1/p'} \left( [\sigma_1]_{A_{\infty}}^{1/p_1} + [\sigma_2]_{A_{\infty}}^{1/p_2} \right) \|f_1\|_{L^{p_1}(\sigma_1)} \|f_2\|_{L^{p_2}(\sigma_2)}.
\]

By (4.3), we get the conclusion desired. \qed

References

[1] S. Buckley, Estimates for operator norms on weighted spaces and reverse Jensen inequalities, Trans. Amer. Math. Soc., 340 (1993), 253–272.

[2] D. Cruz-Uribe, J.M. Martell, C. Pérez, Sharp weighted estimates for classical operators, Adv. Math., 229 (2012), 408–441.

[3] W. Damián, A.K. Lerner and C. Pérez, Sharp weighted bounds for multilinear maximal functions and Calderón-Zygmund operators, [http://arxiv.org/abs/1211.5115](http://arxiv.org/abs/1211.5115)

[4] X.T. Duong, R. Gong, L. Grafakos, J. Li, L. Yan, Maximal operator for multilinear singular integrals with non-smooth kernels, Indiana Univ. Math. J., 58 (2009), 2517–2541.

[5] X.T. Duong, L. Grafakos, and L. Yan, Multilinear operators with non-smooth kernels and commutators of singular integrals, Trans. Amer. Math. Soc. 362 (2010) 2089–2113.

[6] L. Grafakos, Modern Fourier Analysis, Second Edition, Springer-Verlag, 2008.
[7] L. Grafakos, R.H. Torres, Multilinear Calderón-Zygmund theory, Adv. Math. 165 (2002) 124–164.

[8] L. Grafakos, R.H. Torres, Maximal operator and weighted norm inequalities for multilinear singular integrals, Indiana Univ. Math. J. 51 (2002) 1261–1276.

[9] G. Hu and D. Yang. Maximal commutators of BMO functions and singular integral operators with non-smooth kernels on spaces of homogeneous type, J. Math. Anal. Appl., 354 (2009), 249–262.

[10] T.P. Hytönen, The sharp weighted bound for general Calderón-Zygmund operators, Ann. of Math., 175 (2012), 1473–1506.

[11] T. Hytönen and M. Lacey, The $A_p-A_\infty$ inequality for general Calderón-Zygmund operators, Indiana Univ. Journal of Math. (to appear).

[12] T. Hytönen, M. Lacey, H. Martikainen, T. Orponen, M. Reguera, E. Sawyer, I. Uriarte-Tuero, Weak and strong type estimates for maximal truncations of Calderón-Zygmund operators on $A_p$ weighted spaces (2011), available at http://arxiv.org/abs/1103.5229

[13] T. Hytönen and C. Pérez, Sharp weighted bounds involving $A_\infty$, J. Anal. & P.D.E. In Press.

[14] M. Lacey, S. Petermichl, M. Reguera, Sharp $A_2$ inequality for Haar shift operators, Math. Ann., 348 (2010), 127–141.

[15] M. Lacey, E. Sawyer and I. Uriarte-Tuero, A characterization of two weight norm inequalities for maximal singular integrals with one doubling measure, J. Anal. & P.D.E. 5(2012), 1–60.

[16] A.K. Lerner, On an estimate of Calderón-Zygmund operators by dyadic positive operators, J. Anal. Math., (to appear).

[17] A.K. Lerner, A simple proof of the $A_2$ conjecture, Int. Math. Res. Not. 2012; doi: 10.1093/imrn/rns145.

[18] A.K. Lerner, S. Ombrosi, C. Pérez, R.H. Torres, R. Trujillo-González, New maximal functions and multiple weights for the multilinear Calderón-Zygmund theory, Adv. Math., 220 (2009), 1222–1264.

[19] K. Li, K. Moen and W. Sun, Weighted estimates for multilinear maximal functions and Calderón-Zygmund operators, preprint.

[20] H. Lin, Y. Meng, and D. Yang. Weighted estimates for commutators of multilinear Calderón-Zygmund operators with non-doubling measures, Acta Math. Sci. Ser. B, 30 (2010), 1–18.

[21] C. Pérez, S. Treil and A. Volberg, On $A_2$ conjecture and corona decomposition of weights, Available at http://arxiv.org/abs/1006.2630
[22] S. Petermichl, The sharp bound for the Hilbert transform on weighted Lebesgue spaces in terms of the classical $A_p$ characteristic, Amer. J. Math., 129 (2007), 1355–1375.

[23] S. Petermichl, The sharp weighted bound for the Riesz transforms, Proc. Amer. Math. Soc., 136 (2008), 1237–1249.

[24] S. Shi, Z. Fu, and S. Lu. Weighted estimates for commutators of one-sided oscillatory integral operators, Front. Math. China, 6 (2011), 507–516.

[25] Q. Xue and Y. Ding. Weighted estimates for the multilinear commutators of the Littlewood-Paley operators, Sci. China Ser. A, 52 (2009), 1849–1868.