Coding Rule for Periodic Orbits in the One-dimensional Map

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A new coding rule for periodic orbits in unimodal one-dimensional maps is derived. The best-known example of a family of unimodal maps is the logistic map. The band merging is observed in the bifurcation diagram of the logistic map. Let $a_m^k$ ($k \geq 1$) be the critical value at which $2^k$-band merges into $2^{k-1}$-band. At $a > a_m^k$, the diverging orbit appears and thus 1-band disappears. The relations $a_m^{k+1} < a_m^k$ for $k > 0$ hold. Let $s_q$ be the code for periodic orbit of period $q$ in the parameter interval $(a_m^k, a_m^{k+1})$. Assume that the code $s_q$ represented by symbols 0 and 1 is known. In the interval $(a_m^{k+1}, a_m^k)$, there exists the periodic orbit of period $2^k \times q$ ($k \geq 1$). Let its code be $s_{2^kq}$. Let $D$ be the doubling operator defined by the substitution rules as $0 \Rightarrow 11$ and $1 \Rightarrow 01$. The following coding rule is derived. Operating $k$ times of $D$ to $s_q$, the code $s_{2^kq}$ is determined.

**Key words:** One-dimensional Map, Bifurcation Diagram, Coding Rule, Periodic Orbits, Doubling Operator

1. Introduction

In this paper, the coding rule for periodic orbits in unimodal one-dimensional maps is discussed. The best-known example of a family of unimodal maps is the logistic map. Let $q$ be the period of periodic orbit. The periodic orbit is represented by a set of $q$ symbols. This set is called code. The method using the code to classify periodic orbits has been introduced by Metropolis-Stein-Stein (Metropolis et al., 1973). The kneading theory by Milnor-Thurston (Milnor and Thurston, 1988) inherits this method and it gives the useful method to calculate the topological entropy (see also Nagashima and Baba, 1999). In this paper, we use two symbols 0 and 1 to represent codes.

Using the bifurcation diagram displayed in Fig. 1 of the logistic map, we explain the problem discussed in this paper. In Fig. 1, for example, a natural number 3 means the periodic window of period-3 orbit. There exist infinitely many windows in the parameter interval $(a_m^4, a_m^0 = 4)$ where $a_m^k$ ($k \geq 1$) is the critical value that $2^k$ ($k \geq 1$) bands merge into $2^{k-1}$ bands and $a_m^0$ is the critical value that one band disappears. For example, a window of period-3 orbit corresponds to the domain of a stability of period-3 orbit. We remark that the interval $(a_m^{k+1}, a_m^k)$ ($k \geq 0$) in the bifurcation diagram is $2^k$-band.

Let us consider the window of period-3 orbit in 1-band. The origin of this window is the appearance of period-3 orbits with codes 001 and 011 which appear through the tangent bifurcation. In the bifurcation diagram, the stable periodic orbit with code 001 is observed. In the following, the periodic orbit with code 001 is abbreviated as the periodic orbit 001.

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There exists the window of period-2 $\times$ 3 orbit (2 $\times$ 3 in Fig. 1) in the interval $(a_m^3, a_m^0)$.

2. Preparations

2.1 Code for periodic orbit

We introduce the logistic map $f$.

$$f : x_{n+1} = ax_n(1-x_n).$$

(1)

Here $0 < a < 4$ and $0 < x_0 < 1$. The fixed point $P$ located at $x = 0$ is stable at $0 < a < 1$. A new fixed point $Q$ appears at $a = 1$ and its position is $x = 1 - 1/a$ ($a > 1$). The fixed point $Q$ occurs the period-doubling bifurcation at $a = 3$.

Here, we show the coding method introduced by Metropolis-Stein-Stein. Take the particular orbit starting from the initial point at $x = 1/2$ and coming back to the initial point. This type orbit is called the superstable periodic orbit. If the orbital point enters into the region satisfying $x > 1/2$, we give a symbol $R$. If the orbital point enters into the region satisfying $x < 1/2$, we give a symbol $L$. Suppose that the following coding is obtained.

$$1/2 \rightarrow R \rightarrow L \rightarrow R \rightarrow R \rightarrow 1/2.$$  

(2)
Here, a fraction 1/2 represents an initial point and the arrow (→) means the orbital order. The code is determined as \( RLRR \). The number of symbols in code does not accord with the length of period. Thus, in the kneading theory, a symbol \( C \) is added in front of this code and new code \( CRLRR \) is defined. Symbol \( C \) means the center of interval.

Next, using \( CRLRR \), we explain the minimum representation for code. The position represented by \( C \) is \( x = 1/2 \) and the mapping function has the maximum point at this point. This fact implies that next position represented by \( R \) (next symbol of \( C \)) is the maximum orbital point. In the unimodal map, the maximum orbital point is mapped to the minimum one. Thus, the position represented by \( L \) (next symbol of \( R \)) is the minimum one. We name the representation \( LRRCR \) the minimum representation for code. In the following, we use the minimum representation for codes.

We use two symbols 0 and 1 in consideration of correspondence with the binary representation where a symbol 0 (1) means \( L \) (\( R \)). Thus, the code for \( P \) is 0 and that of \( Q \) is 1. Next, we give a meaning of symbol \( C \). Two periodic orbits which appear through the tangent bifurcation constitute (0-1)-pair which means a pair of the stable and the unstable periodic orbits (Hall, 1994). In the two dimensional map, (0-1)-pair means the saddle-node pair. There exist periodic orbit with code where \( C \) is replaced by 0 and that with code where \( C \) is replaced by 1. The code \( LRRCR \) means two periodic orbits 01101 and 01111. The set of these codes is an example of (0-1)-pair.

We comment on the codes for periodic orbits which appear through the period-doubling bifurcation. For example, let us consider the code 0111, which is the code for the daughter periodic orbit which appears through the period-doubling bifurcation of the period-2 orbit. We exchange a symbol 1 at the second-to-last to 0 and have new code 0101 which is the repetition of word 01. Thus, 0101 is meaningless as a code. The code for the periodic orbit which appears through the period-doubling bifurcation does not have a partner code of (0-1)-pair.

The code obtained here is the same as the code determined by the tent map \( T \) defined on \([0, 1]\). In the following, we explain this fact. The logistic map \( f \) at \( a = 4 \) is converted into the tent map \( T \):

\[
T : X_{n+1} = 1 - 2X_n - 1
\]

by the translation formula

\[
x_n = \sin^2((\pi/2)X_n).
\]

Thus, the logistic map at \( a = 4 \) and the tent map \( T \) are conjugate. Here, we take the orbit of logistic map at \( a = 4 \). If the orbit enters the interval \([0, 1/2]\), the symbol is defined as 0. If the orbit enters the interval \([1/2, 1]\), the symbol is defined as 1. For the point \( x = 1/2 \), we can use 0 or 1. This is originated from the fact that there are two representations to an irreducible fraction, for example, \( 1/2 = * \), \( 10_\infty \) and \( 1/2 = 01_\infty \). Here, a symbol \( * \) is a decimal point and the right hand sides are the binary representation.

For example, suppose that the code 011 is obtained. In the tent map, there exists the periodic orbit 011. Conversely, the periodic orbit in the tent map exists in the logistic map. From these facts, we can study the periodic orbit with a given code in the tent map. Translating the orbital points in the tent map by Eq. (4), we have the orbital points in the logistic map at \( a = 4 \). The orbital order of periodic points in the tent map is the same as that in the logistic map.

2.2 Block representation

First, we introduce two block symbols \( E(2) = 01 \) and \( F(2) = 11 \) (Yamaguchi and Tanikawa, 2009, 2016). Block symbol \( E(2) = 01 \) represents the code for the daughter periodic orbit which appears through the period-doubling bifurcation of \( Q \). Block symbol \( F(2) = 11 \) is introduced for convenience sake and there is no periodic orbit represented by \( F(2) \). Suppose that the periodic orbit of period...
\[ q = 2n \ (n \geq 2) \] is written by \( E(2) \) and \( F(2) \). We say the block symbol as block briefly. Since the first symbol of block \( F(2) \) is 1, the first block of the minimum representation is \( E(2) \).

Let us consider the block code that the number of blocks are greater than or equal to 2. The minimum representation begins with \( E(2)F(2) \). In order to prove this fact, we confirm the large/small relation between \( E(2)F(2) = \bullet \ 0111 \) and \( S = 0101 \). Translating them into the binary one, we obtain \( 0101 \) for \( 0111 \) and \( 0110 \) for \( 0101 \). The translation procedure is given in Appendix B. Since the relation \( 0101 < \bullet \ 0110 \) holds, the claim is proved. In the following discussions, we use the abbreviated notations \( E \) and \( F \).

### 2.3 Intervals and symbols

We explain the structure of bifurcation diagram displayed in Fig. 1. In the left side of Fig. 1, the accumulation of the period-doubling bifurcation is observed.

We decrease the parameter value of \( a \) from \( a = a^0_m = 4 \). At the critical point \( a = a^1_m = 3.678573 \), one band splits into two bands. At the critical point \( a = 3.592572 \), two bands split into four bands. Let \( a^\infty_m = 3.569945 \) be the accumulation point of the band splitting. The critical value \( a^\infty_m \) is also the accumulation point of period-doubling bifurcation. Increasing \( a \) from \( a^\infty_m \), we can observe the band merging.

Next, we give the relation of the bifurcation diagram and the shape of mapping function. Figure 2(a) represents the shape of \( f(x) \) at \( a = a^0_m \) where \( f(1/2) = 1 \). The orbit starting from \( x = 1/2 \) reaches the fixed point \( P \) at \( x = 0 \). It is an example of superstable orbit.

The closed interval \( \text{Int}(A_0) \) is defined as \( [1/2, 1] \) and \( \text{Int}(B_0) \) is defined as \( [0, 1/2] \). We remark that \( \text{Int}(A_0) \) includes \( Q \). The mapping function \( f(x) \) is a unimodal function which has two monotonic branches. Let \( M_0 \) be the transition matrix representing the transitions between \( \text{Int}(A_0) \) and \( \text{Int}(B_0) \).

\[
M_0 = \begin{pmatrix}
A_0 & B_0 \\
A_0 & 1 & 1 \\
B_0 & 1 & 1
\end{pmatrix}.
\] (5)

In \( M_0 \), \( A_0 \) (\( B_0 \)) expresses \( \text{Int}(A_0) \) (\( \text{Int}(B_0) \)). For example, the first row means that the image of \( \text{Int}(A_0) \) covers \( \text{Int}(A_0) \) and \( \text{Int}(B_0) \) once. The eigenvalue of \( M_0 \) is 2, and thus the topological entropy at \( a = a^1_m \) is \( \ln 2 \).

In Fig. 2(b), the functions \( f^2(x) \) and \( f^4(x) \) around \( x = 1/2 \) at \( a = a^1_m \) are displayed. The superstable orbit displayed by arrowed line goes to the fixed point \( Q \) where the condition \( f^4(1/2) = 1 - 1/a \) holds. From this condition, the critical value \( a^1_m \) is determined. In Fig. 2(b), there are two intersection points of \( f^2(x) \) and the diagonal line. The right intersection point is \( Q \) and the left one is \( \alpha_0 \) which is the daughter orbital point which appears through the period-doubling bifurcation of \( Q \). The other point \( \alpha_1 \) is not displayed in Fig. 2(b) since it locates in the right side of \( Q \). The period of daughter orbit is 2. We say it period-2 orbit. There are four intersection points of \( f^2(x) \) and the diagonal line. New intersection points \( \beta_0 \) and \( \beta_2 = f^2(\beta_0) \) are the daughter orbital points which appear through the period-doubling bifurcation of \( \alpha_0 \). Note that \( \beta_0 \) is the minimum point in orbital points of period-2 orbit.

Let the interval sandwiched between dashed lines located in the region \( x \leq 1/2 \) (\( x \geq 1/2 \)) be \( \text{Int}(A_1) \) (\( \text{Int}(B_1) \)). The interval including the fixed point \( \alpha_0 \) of \( f^2(x) \) is \( \text{Int}(A_1) \). \( \text{Int}(A_1) \) includes \( \beta_0 \) and \( \text{Int}(B_1) \) includes \( \beta_2 \). We remark that \( \text{Int}(A_1) \) and \( \text{Int}(B_1) \) for \( k \geq 1 \) include both end points (Nagashima and Baba, 1999).

The function \( f^2(x) \) is a monotonic decreasing function in \( \text{Int}(A_1) \) and is a monotonic increasing one in \( \text{Int}(B_1) \). From the unimodal property of \( f^2(x) \), the transition matrix \( M_1 \) representing the transitions between \( \text{Int}(A_1) \) and \( \text{Int}(B_1) \) is obtained as follows.

\[
M_1 = \begin{pmatrix}
A_1 & B_1 \\
A_1 & 1 & 1 \\
B_1 & 1 & 1
\end{pmatrix}.
\] (6)

The eigenvalue of \( M_1 \) is 2, and thus the topological entropy at \( a = a^1_m \) is \( \ln 2 \).

In Fig. 2(c), the left intersection point of \( f^4(x) \) and the diagonal line is \( \beta_0 \) and two intersection points \( \gamma_0 \) and \( \gamma_4 \) of \( f^4(x) \) and the diagonal line are the daughter periodic points which appear through the period-doubling bifurcation of \( \beta_0 \). We say the periodic orbit of \( \gamma_0 \) and \( \beta_4 \) period-2 orbit. Two points \( \beta_0 \) and \( \gamma_0 \) locate in \( \text{Int}(A_2) \), and \( \epsilon_0 \) and \( \gamma_5 \) in \( \text{Int}(B_2) \).

In Fig. 2(d), the left intersection point of \( f^8(x) \) and the diagonal line is \( \gamma_0 \) and two intersection points \( \delta_0 \) and \( \delta_8 \) of \( f^8(x) \) and the diagonal line are the daughter periodic points which appear through the period-doubling bifurcation of \( \gamma_0 \). Two points \( \gamma_0 \) and \( \delta_0 \) locate in \( \text{Int}(A_3) \), and \( \beta_0 \) and \( \delta_8 \) in \( \text{Int}(B_3) \).

The transition matrix \( M_k \) (\( k \geq 1 \)) at \( a = a^m_m \) is determined.

\[
M_k = \begin{pmatrix}
A_k & B_k \\
A_k & 1 & 1 \\
B_k & 1 & 1
\end{pmatrix}.
\] (7)

The critical value \( a^m_m \) (\( k \geq 0 \)) is determined by the relation \( f^{2^{k-1}x+2^k}(1/2) = f^{2^k}(1/2) \) (see Appendix C). The eigenvalue of \( M_k \) is 2, and thus the topological entropy at \( a = a^m_m \) (\( k \geq 0 \)) is \( \ln (2^k) \). At \( a^\infty_m \), the topological entropy is zero.

If the orbital point by \( f^2 \) locates in \( \text{Int}(A_k) \) (\( \text{Int}(B_k) \)), let the symbol of orbital point be \( \text{Symb}(A_k) \) (\( \text{Symb}(B_k) \)). These symbols are constructed by the symbols 0 and 1. For example, we consider the orbit in the parameter range \( [a^{k^{l+1}}_m, a^{k^l}_m] \) \( (k \geq 1) \). Suppose that \( x_0 \) exists in \( \text{Int}(A_k) \) or \( \text{Int}(B_k) \). The orbital point \( x_{2^k} = f^{2^k}(x_0) \) does not escape from these intervals. This fact means that the code for periodic orbit by \( f^{2^k} \) is represented by \( \text{Symb}(A_k) \) and \( \text{Symb}(B_k) \).

We give the remark about coding. If \( \text{Int}(A_0) \) is defined, \( \text{Symb}(A_0) \) is also determined. \( \text{Int}(B_0) \) and \( \text{Symb}(B_0) \) are also defined. At \( a = 4 \), their explicit representations by 0 and 1 are determined. We remark that \( \text{Symb}(A_0) = 1 \) and \( \text{Symb}(B_0) = 0 \). For the other symbols \( \text{Symb}(A_k) \) and \( \text{Symb}(B_k) \) \( (k \geq 1) \), the same facts hold. The length of \( \text{Symb}(A_k) \) and \( \text{Symb}(B_k) \) represented by 0 and 1 is \( 2^k \).

The structure of windows in the interval \( [a^{k^{l+1}}_m, a^{k^l}_m] \) is similar to that in the interval \( [a^{k^{l+1}}_m, a^{k^l}_m] \). Thus, if the periodic
orbit of period $q$ with code $s_q$ exists in the interval $(a^1_m, a^0_m)$, the corresponding periodic orbit of period $2^k \times q$ with code $s_{2^k \times q}$ exists in the interval $(a^{k+1}_m, a^k_m)$. It is noted that $s_q$ is represented by Symb($A_0$) and Symb($B_0$) and $s_{2^k \times q}$ is represented by Symb($A_k$) and Symb($B_k$). Thus, our problem is renewed as Problem 2.1.

**Problem 2.1.** Derive the rule to determine Symb($A_k$) and Symb($B_k$) from Symb($A_0$) = 1 and Symb($B_0$) = 0.

### 3. Coding Rule for the Period-doubling Bifurcation

Only the coding rule to determine codes for periodic orbits which appear through the period-doubling bifurcation of the fixed point $Q$ has been known. In this section, we make clear the period-doubling bifurcation of $Q$ and derive Coding rule 3.1. In order to answer Problem 2.1, we need Coding rule 3.1.

Using Fig. 3, we explain the period-doubling bifurcation. Let two daughter periodic points appeared from $Q$ be $\xi_0$ and $\xi_1$. We remark that these notations for period-2 orbit are different from those in Subsec. 2.3.

Orbital point $\xi_0$ moves to the region $x < 1/2$ across $x = 1/2$ with the increase in parameter $a$. If $\xi_0$ does not move to the region $x < 1/2$, the symbols of $\xi_0$ and $\xi_1$ are 1. The code 11 of period-2 orbit is obtained but it is repetition of the code $P_0 = 1$ of $Q$. This is a contradiction. As a result, one orbital point of daughter periodic points which is near to $x = 1/2$ moves to the region $x < 1/2$. Its symbol becomes 0. The code $P_1 = 01$ for period-2 orbit is determined.

Next, after the period-doubling bifurcation of period-2 orbit, new two daughter periodic points $\zeta_0$ and $\zeta_2$ are born from $\xi_0$. Let the point which is near to $x = 1/2$ be $\zeta_0$. We increase the parameter $a$ furthermore. The point $\zeta_0$ moves to the region $x > 1/2$ across $x = 1/2$. Schematic orbit starting from $\zeta_0$ is displayed in Fig. 4(a). We obtain the code 1101 for period-4 orbit. Its minimum representation is $P_2 = \{01, 1\}$ and it is rewritten as $P_2 = P_1 P_0 P_0$. The notation $P_1 P_0 P_0$ means that we write the code for $P_1 P_0$ in this order.

New daughter periodic points $\eta_0$ and $\eta_4$ appear from $\zeta_0$ of period-4 orbit. Let the point which is near to $x = 1/2$ be $\eta_0$. 
Fig. 3. The parameter $a$ increases from the upper figure to the lower one. The orbital points $\xi_0$ and $\xi_1$ are the daughter periodic points appeared from $Q$. The arrows represent the direction of movement of $\xi_0$ and $\xi_1$ when $a$ increases. The orbital points $\zeta_0$ and $\zeta_2$ are the daughter periodic points appeared from $\xi_0$, and $\eta_0$ and $\eta_4$ are those appeared from $\zeta_0$.

Fig. 4. (a) Period-4 orbit. (b) Period-8 orbit.

Schematic orbit starting from $\eta_0$ is displayed in Fig. 4(b). The point $\eta_0$ moves to the region $x < 1/2$ across $x = 1/2$. The code for period-8 orbit starting from $\eta_0$ is 01011101. Its minimum representation is $P_3 = 0111 \cdot 01 \cdot 01 = P_2 P_1 P_1$. Thus, Coding rule 3.1 is obtained.

**Coding rule 3.1.** Let $P_0$ be the code for $Q$ and $P_k$ ($k \geq 1$) be the code for daughter periodic orbit which appears through the period-doubling bifurcation of $Q$. The code $P_{k+1}$ ($k \geq 1$) is determined by the recursive rule as

$$P_{k+1} = P_k P_{k-1} P_{k-1}$$

where $P_0 = 1$ and $P_1 = 01$.

Here, we define the doubling operator $D$.

**Definition 3.2.** The doubling operator $D$ is defined.

$$D : 0 \Rightarrow 11 \equiv F, \ 1 \Rightarrow 01 \equiv E.$$  \hfill (9a)

$$D : E \Rightarrow EF, \ F \Rightarrow EE.$$  \hfill (9b)

Here, the notation $0 \Rightarrow 11$ means the replacing 0 with 11.

Using $D$, we rewrite Coding rule 3.1 in Substitution rule 3.3.

**Substitution rule 3.3.** Operating $k$ times of $D$ to $P_0$, the code $P_k$ is determined. After rewriting $P_k$ in the minimum representation, $P_k$ is obtained.

4. Coding Rule as an Answer to Problem 2.1

4.1 Coding rule

In this subsection, we derive Coding rule 4.1 which is an answer to Problem 2.1. First, we take out the periodic orbits related to period-3 orbit from the Sharkovskii ordering.

$$3 \triangleright 2 \times 3 \triangleright 2^2 \times 3 \triangleright 2^3 \times 3 \triangleright \cdots.$$  \hfill (10)

Here, $3 \triangleright 2 \times 3$ means the fact that period-3 orbit implies the existence of period-2 $\times 3$.

The window of period-3 orbit is in 1-band, that of period-2 $\times 3$ in 2-band, and so on. The two codes for period-3
orbits are 011 and 001. Here, we use the code 011. Note that we obtain the same results mentioned below even if the code 001 is used. Using the tent map, we confirm the order relations of orbital points of period-3 orbit 011 and display schematic orbit in Fig. 5(a) where we place the orbital points to the equal distance. Two intervals $I_1$ and $I_2$ are defined in Fig. 5(a) where $I_1$ and $I_2$ include both end points. We name the graph representing the transitions between $I_1$ and $I_2$ Stefan diagram (Stefan, 1977, Devaney, 2003). Here, we call the diagram displayed in Fig. 5(b) $C_3$.

From $C_3$, it is easy to see the existence of period-6 orbit starting from $I_1$ and coming back to $I_1$.

$$I_1 \rightarrow I_2 \rightarrow I_2 \rightarrow I_2 \rightarrow I_1 \rightarrow I_2 \rightarrow I_1. \quad (11)$$

Here, the relation $I_1 \rightarrow I_2$ means that the image of $I_1$ covers $I_2$.

Since the interval $I_1$ includes $x = 1/2$, we can use 0 and 1 for the orbit in $I_1$. On the other hand, the interval $I_2$ locates in the region satisfying $x < 1/2$. The symbol of the orbital point in $I_2$ is 1. We obtain the code 01301 of period-6 orbit. The other period-6 orbit 01311 also exists. These constitute (0-1)-pair. Using blocks $E$ and $F$, 01301 is represented as $EFE$ and 015 as $EFF$. Just after the tangent bifurcation, period-6 orbit $EFE$ is unstable and that with $EFF$ is stable just after the appearance (see Appendix D). Thus, all periodic orbits which are appeared through the period-doubling bifurcation or the tangent bifurcation in the interval $(a_{m-2}^2, a_{m}^1)$ are coded by two symbols Symb($A_1$) and Symb($B_1$).

Using the tent map, we confirm the order of period-6 (2×3) orbital points with block code $EFE$ and display them in Fig. 6. Five intervals $I_k$ ($k = 1, 2, \cdots, 5$) are defined and the transition matrix $M_6$ among them are obtained.

$$M_6 = \begin{pmatrix}
I_1 & I_2 & I_3 & I_4 & I_5 \\
I_1 & 0 & 0 & 0 & 1 & 1 \\
I_2 & 0 & 0 & 0 & 0 & 1 \\
I_3 & 0 & 0 & 1 & 1 & 0 \\
I_4 & 0 & 1 & 0 & 0 & 0 \\
I_5 & 1 & 0 & 0 & 0 & 0 
\end{pmatrix}. \quad (12)$$

For example, the first row means that transitions from $I_1$ to $I_4$ and to $I_5$ are permitted.

We can construct Stefan diagram $C_6$ displayed in Fig. 7. The orbit starting from $I_3$, passing through $I_4$ and coming back to $I_3$ does not exist. Thus, $I_3$ is deleted in Fig. 7. We name Fig. 7 the simplified Stefan diagram. We define the long cycle and the short one in $C_6$. Let the long cycle

$$I_1 \rightarrow I_4 \rightarrow I_2 \rightarrow I_5 \rightarrow I_1 \quad (13)$$

be $R_o$ and the short one

$$I_1 \rightarrow I_4 \rightarrow I_1 \quad (14)$$

be $R_s$. Here, we define $R_o = R_sR_s$. The length of $R_o$ is the same as that of $R_i$.

Let $s_2$ be the symbol of orbital point in $I_2$. Remember the fact that $I_2$ extends over the regions satisfying $x < 1/2$ and $x > 1/2$. Thus, there exist the periodic orbit satisfying the condition $s_2 = 0$ and that satisfying the condition $s_2 = 1$. For our purpose, we choose the periodic orbit satisfying the condition $s_2 = 1$. Thus, we obtain $R_o = 0111$ and $R_i = 0101$. 
Using blocks $E$ and $F$, $R_o$ is represented as $EFE \equiv \text{Symb}(A_2)$ and $R_i$ as $EE \equiv \text{Symb}(B_2)$. We also have the following relations.

$$\text{Symb}(A_2) = \text{Symb}(A_1)\text{Symb}(B_1),$$  \hspace{1cm} (15a)
$$\text{Symb}(B_2) = \text{Symb}(A_1)\text{Symb}(A_1).$$  \hspace{1cm} (15b)

All periodic orbits which are appeared through the period-doubling bifurcation or the tangent bifurcation in the interval $(a_{m^3}, a_{m^2}^1]$ are coded by two symbols $\text{Symb}(A_2)$ and $\text{Symb}(B_2)$.

From the simplified Štefan diagram $C_6$, we obtain that the periodic orbit $R_o, R_i, R_o$ exists. Its code is represented as $EFE \cdot EE \cdot EF$. This implies the existence of period-12 $(2^2 \times 3)$ orbit. The partner of (0-1)-pair is $EE \cdot EE \cdot EF$.

Using the tent map, we confirm the order of periodic orbit $EFE \cdot EE \cdot EF$ and display them in Fig. 8. Eleven intervals are defined and the transition matrix $M_{11}$ among them are obtained.

$$M_{12} = \begin{pmatrix}
I_1 & I_2 & I_3 & I_4 & I_5 & I_6 & I_7 & I_8 & I_9 & I_{10} & I_{11} \\
I_1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
I_2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
I_3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
I_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
I_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
I_7 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
I_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
I_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
I_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
I_{11} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.  \hspace{1cm} (16)$$

The interval $I_2$ extends over the regions satisfying $x < 1/2$ and $x > 1/2$. Thus, we choose the periodic orbit satisfying the condition that the symbol of orbital point in $I_2$ is 0. Thus, we obtain $EFE \equiv \text{Symb}(A_2)$ for the block code for $R_o$ and $EFE \equiv \text{Symb}(B_2)$ for that of $R_i$. $\text{Symb}(A_3)$ and $\text{Symb}(B_3)$ are represented by using $\text{Symb}(A_2)$ and $\text{Symb}(B_2)$.

$$\text{Symb}(A_3) = \text{Symb}(A_2)\text{Symb}(B_2),$$  \hspace{1cm} (19a)
$$\text{Symb}(B_3) = \text{Symb}(A_2)\text{Symb}(A_2).$$  \hspace{1cm} (19b)

All periodic orbits which are appeared through the period-doubling bifurcation or the tangent bifurcation in the interval $(a_{m^3}, a_{m^2}^1]$ are coded by two symbols $\text{Symb}(A_3)$ and $\text{Symb}(B_3)$. Summarizing the results obtained above, we obtain Coding rule 4.1. Coding rule 4.1 is proved in Subsec. 4.2.

**Coding rule 4.1.** Let $\text{Symb}(A_1)$ and $\text{Symb}(B_1)$ be the symbols in the interval $(a_{m^3}, a_{m^2}^1]$ where $k \geq 0$. $\text{Symb}(A_3)$ and $\text{Symb}(B_3)$ are determined by the following coding rules.

$$\text{Symb}(A_3) = 1 \equiv P_0, \hspace{1cm} (20a)$$
$$\text{Symb}(B_3) = 0, \hspace{1cm} (20b)$$
$$\text{Symb}(A_1) = E \equiv P_1, \hspace{1cm} (20c)$$
$$\text{Symb}(B_1) = F, \hspace{1cm} (20d)$$
$$\text{Symb}(A_k) = \text{Symb}(A_{k-1})\text{Symb}(B_{k-1}) \equiv P_k \ (k \geq 2), \hspace{1cm} (20e)$$
$$\text{Symb}(B_k) = \text{Symb}(A_{k-1})\text{Symb}(A_{k-1}) \equiv P_{k-1}P_{k-1} \ (k \geq 2).$$  \hspace{1cm} (20f)

Using the doubling operator $D$, Coding rule 4.1 is renewed as Substitution rule 4.2.

**Substitution rule 4.2.** Suppose that the code $s_q$ for the periodic orbit of period $q$ in the interval $(a_{m^3}, a_{m^2}^1]$ is known. Let $s_{2^k \cdot q}$ be the code for period-$2^k \times q$ orbit in the interval $(a_{m^3}, a_{m^2}^1]$. Applying $k$ times of the doubling operator $D$ to $s_q$, the code $s_{2^k \cdot q}$ is determined.

Using Coding rule 4.1, we have Proposition 4.3.

**Proposition 4.3.** Let $s$ be the code of periodic orbit which is appeared through the period-doubling bifurcation or the tangent bifurcation in $(a_{m^3}, a_{m^2}^1]$ $(k \geq 1)$. The code $s$ is represented by $\text{Symb}(A_m)$ and $\text{Symb}(B_m)$ $(m = k, k - 1, \ldots, 0)$.

We give two remarks. For example, the period-2 orbit exists in $(a_{m^3}, a_{m^2}^1]$. This is not the period orbit which is appeared through the period-doubling bifurcation in $(a_{m^3}, a_{m^2}^1]$. Thus, its code is not represented by $\text{Symb}(A_2)$ or $\text{Symb}(B_2)$.

Let $s_{2 \times 3}$ be the code of periodic orbit which is appeared through the tangent bifurcation in $(a_{m^3}, a_{m^2}^1]$. The code $s_{2 \times 3}$ is represented by $\text{Symb}(A_3)$ and $\text{Symb}(B_1)$. Using Coding rule 4.1, we obtain that $s_{2 \times 3}$ is represented by $\text{Symb}(A_0)$ and $\text{Symb}(B_0)$.

Finally, we show two examples of how to use of Substitution rule 4.2. We pay attention to the period-5 orbit locating between the windows of period-3 orbit and period-4 orbit. This is not included in the Sharkovskii ordering. The codes
Fig. 8. Periodic orbit 011101010111 = EF · EE · EF. Interval $I_4$ includes $x = 1/2$.

Fig. 9. The simplified Štefan diagram $C_{12}$ where $I_1$, $I_6$, and $I_9$ are deleted.

4.2 Proof of Coding rule 4.1

First, we give the proof of Eqs. (20c) and (20d). Next, we give the proof of Eqs. (20e) and (20f).

Proof of Eqs. (20c) and (20d). We consider the situation after the accumulation of period-doubling bifurcations. Therefore, there exist the orbital points $\beta_0$ and $\beta_2$ of period-4 orbit appeared from $\alpha_0$ of period-2 orbit. Suppose that the point $\beta_0$ locates in the left side of $\alpha_0$. The point $\beta_0 \in \text{Int}(A_1)$ is the minimum point of period-4 orbit. On the other hand, the point $\beta_2$ locates in the right side of $\alpha_0$ and in $\text{Int}(B_1)$. The point $\beta_2$ move to $\text{Int}(B_1)$ (see the proof of Coding rule 3.1). From Coding rule 3.1, the code for orbit starting from $\beta_0$ is $P_2 = P_1P_0P_0 = 01 \cdot 11$. The former part 01 represents Symb($A_1$) = $E$ and thus the latter one 11 represents Symb($B_1$) = $F$. This means that the word for orbit from $\beta_0$ to $\beta_0$ is 11. Thus, we have Symb($A_1$) = $E$ and Symb($B_1$) = $F$. (Q.E.D.)

Proof of Eqs. (20e) and (20f). Assume that the daughter periodic points $\gamma_0$ and $\gamma_4$ appear from $\beta_0$ of period-4 orbit (see Fig. 2(c)). Suppose that $\gamma_0$ ($\gamma_4$) locates in the left (right) side of $\beta_0$ and $\gamma_4$ moves to $\text{Int}(B_2)$. From Coding rule 3.1, the code for the periodic orbit starting from $\gamma_0$ is $P_3 = P_2P_1 = EF \cdot EE$. The point $f^4(\gamma_0)$ enters into $\text{Int}(B_2)$, and $f^4(\gamma_4)$ comes back to $\text{Int}(A_2)$. Thus, the former part $EF$ represents Symb($A_2$) and the latter one $EE$ represents Symb($B_2$). Two symbols are the super-blocks constructed by $E$ and $F$.

The daughter periodic orbit with code $P_{k+1} = A_kB_k$ has its orbital point in $\text{Int}(A_k)$ and $\text{Int}(B_k)$. Coding rule 3.1 guarantees that the super-blocks Symb($A_k$) and Symb($B_k$) are determined by Eqs. (20e) and (20f). (Q.E.D.)

5. Concluding Remarks

The parameter interval $(a_m^{k+1}, a_m^k)$ in the bifurcation diagram is defined. The code of periodic orbit in $(a_m^{k+1}, a_m^k)$ are 00111 and 00101. First, we apply $D$ to the stable periodic orbit 00111. The code for period $2 \times 5$ is $EEFEF$, and that of period $2^2 \times 5$ is $EF \cdot EE \cdot EE \cdot EF \cdot EF$. Next, we apply $D$ to the unstable periodic orbit 00101. The code for period $2 \times 5$ orbit is $EFFEF$, and that of period $2^2 \times 5$ orbit is $EF \cdot EE \cdot EE \cdot EF \cdot EE$. By numerical calculation, the correctness of codes obtained here is confirmed.
is coded by Symb($A_2$) and Symb($B_2$). Coding rule 4.1 to determine Symb($A_2$) and Symb($B_2$) from Symb($A_0$) = 1 and Symb($B_0$) = 0 is derived and its correctness is proved. We can apply Coding rule 4.1 to the periodic orbits in the unimodal maps.

For example, in the window of period-3 orbit, the bifurcation processes similar to the original bifurcation diagram in $a \in (0, 4]$ are observed. For periodic orbits in the window of period-3 orbit, we can consider the same problem discussed in this paper.

Interval $[a_{m+1}^k, a_m^k]$ is regarded as a small world where the basic words are Symb($A_2$) and Symb($B_2$). This concept came out of the symbol dynamics naturally. It is needed to reconsider the relation of periodic orbit and code. As a result, we may provide new concept or interpretation of the bifurcation diagram.

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Appendix A. Sharkovskiı̆ ordering

Theorem A was proved by Sharkovskiı̆ (Sharkovskiı̆, 1964).

**Theorem A.** Consider the following ordering on the set of natural numbers (Sharkovskiı̆ ordering):

$$3 > 5 > 7 > 9 > \cdots$$

$$2 \times 3 > 2 \times 5 > 2 \times 7 > 2 \times 9 > \cdots$$

$$2^2 \times 3 > 2^2 \times 5 > 2^2 \times 7 > 2^2 \times 9 > \cdots$$

$$2^3 \times 3 > 2^3 \times 5 > 2^3 \times 7 > 2^3 \times 9 > \cdots$$

$$\cdots > 2^4 \times 3 > 2^4 \times 5 > 2^4 \times 2 > 2 > 1.$$  

Let $f$ be a one-dimensional continuous map from interval to itself. If $f$ has a period-$n$ orbit and the relation $n > m$ in the Sharkovskiı̆ ordering holds, then $f$ has a period-$m$ orbit.

Appendix B. Translation procedure

We introduce the translation procedure from code for the tent map ($X_{n+1} = 1 - |2X_n - 1|$) to that of the binary map ($X_{n+1} = 2X_n$ (mod 1)) (Yamaguchi and Tanikawa, 2016).

**Procedure B.** Let $w = s_1s_2s_3 \cdots s_q$ be a given code. If the parity of $w$ is even, we prepare $s = s_1s_2 \cdots s_q$ and $t = t_1t_2 \cdots t_q$ where $t_1 = s_1$. If the parity of $w$ is odd, we prepare $s = w$. We rewrite the suffixes as $s = s_1s_2 \cdots s_{2q}$ and prepare $t = t_1t_2 \cdots t_{2q}$ where $t_1 = s_1$.

If the parity of $w$ is even, for $2 \leq k \leq q$, determine $t_k$ by the following rules (a) or (b). If the parity of $w$ is odd, for $2 \leq k \leq 2q$, determine $t_k$ by the following rules (a) or (b).

After $t$ is determined, output $t$.

(a) If $\Sigma_{j=1}^{k-1} t_j$ is odd, we let $t_k = 1 - s_k$.

(b) If $\Sigma_{j=1}^{k-1} t_j$ is even, we let $t_k = s_k$.

Appendix C. How to determine the critical value

All orbital points exist in the region $[x_{\text{min}}, x_{\text{max}}]$ where $x_{\text{max}} = f(1/2)$ ($\leq 1$) and $x_{\text{min}} = f^2(1/2)$ ($\geq 0$). Here, we use the fact that the maximum point $x_{\text{max}}$ is mapped to the minimum point $x_{\text{min}}$ in the unimodal map. In the logistic map, $f(1/2)$ gives the maximum point.

First, we derive the equation to determine $a_m^k$ ($k = 1$). In Fig. 2(b), $x_{\text{min}}$ is the left edge of Int($A_1$). The relations $f^3(x_{\text{min}}) = x_Q$, $x_{\text{min}} = f^2(1/2)$ and $f(x_Q) = x_Q$ hold. In these equations, $x_Q$ represents the position of $Q$. Using these relations, we obtain the equation to determine $a_m^1$.

$$f^3(1/2) = f^4(1/2).$$  

The critical value $a_m^1$ determined by Eq. (C.1) is equal to that by $f^4(1/2) = x_Q = 1 - 1/a$. In fact, the left hand side of Eq. (C.1) is rewritten as $f^5(1/2) = f(f^4(1/2)) = f(x_Q) = x_Q$.

Next, we derive the equation to determine $a_m^2$ ($k = 2$). In Fig. 2(c), $x_{\text{min}}$ is the left edge of Int($A_2$). Let $x_{\text{min}}$ be the position of $a_0$. From the relations $f^4(x_{\text{min}}) = x_{\text{min}}$, $x_{\text{min}} = f^2(1/2)$ and $f^2(x_{\text{min}}) = x_{\text{min}}$, we have the equation to determine $a_m^2$.

$$f^8(1/2) = f^6(1/2).$$  

We derive the equation to determine $a_m^3$ ($k = 3$). In Fig. 2(d), $x_{\text{min}}$ is the left edge of Int($A_3$). Let $x_{\text{min}}$ be the position of $a_0$. We have the relations $f^8(x_{\text{min}}) = x_{\text{min}}$, $x_{\text{min}} = f^2(1/2)$ and $f^2(x_{\text{min}}) = x_{\text{min}}$, and obtain the equation to determine $a_m^3$.

$$f^{14}(1/2) = f^{10}(1/2).$$  

Repeating this procedure, the following equation to determine $a_m^k$ ($k \geq 1$) is derived.

$$f^{2^{k-1} \times 3^2+2}(1/2) = f^{2^{k+2}}(1/2).$$  

Appendix D. Parity of code and the stability of periodic orbit

We define the parity of code and give the stability of periodic orbit (Hall, 1994).

**Definition D.1.** If the number of 1 included in the code is even (odd), the parity of code is even (odd).

**Property D.2.**

(i) Suppose that the parity of code $s$ is even. The periodic orbit with code $s$ is unstable.

(ii) Suppose that the parity of code $s$ is odd. The periodic orbit with code $s$ is stable just after the appearance and occurs the period-doubling bifurcation.

Property D.2 is applicable to all periodic orbits which are appeared through the period-doubling bifurcation or the tangent bifurcation.

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