Transverse momentum diffusion and collisional jet energy loss in non-Abelian plasmas

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Abstract

We consider momentum broadening and energy loss of high momentum partons in a hot non-Abelian plasma due to collisions. We solve the coupled system of Wong-Yang-Mills equations on a lattice in real time, including binary hard elastic collisions among the partons. The collision kernel is constructed such that the total collisional energy loss and momentum broadening are lattice spacing independent. We find that the transport coefficient $\hat{q}$ corresponding to transverse momentum broadening receives sizeable contributions from a power-law tail in the $p_{\perp}$-distribution of high-momentum partons. We establish the scaling of $\hat{q}$ and of $dE/dx$ with density, temperature and energy in the weak-coupling regime. We also estimate the nuclear modification factor $R_{AA}$ due to elastic energy loss of a jet in a classical Yang-Mills field.

PACS numbers: 11.15.Kc, 12.38.Mh, 24.10.Lx, 24.85.+p, 25.75.Bh
I. INTRODUCTION

The study of high transverse momentum jets produced in heavy-ion collisions can provide information on the properties of the hot QCD plasma produced in the central rapidity region \([1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]\). After the discovery of jet quenching at the Relativistic Heavy Ion Collider (RHIC) \([13, 14]\) a lot of progress has been made towards using jets as a quantitative tomographic probe of the QGP. Jet quenching refers to the suppression of high transverse momentum hadrons, such as \(\pi^0\) and \(\eta\) mesons in central \(A + A\) collisions compared to expectations from measurements in \(p + p\) collisions. Whereas pions and \(\eta\)-mesons exhibit the same suppression at high \(p_\perp\), direct photons were found to be unsuppressed \([15]\). This indicates that the observed suppression is related to the absorption (energy loss) of energetic partons in the medium.

In this paper we study collisional energy loss and momentum broadening of massless high momentum partons traversing a non-Abelian plasma. Soft multi-particle interactions are treated by solving the coupled system of Wong-Yang-Mills equations in real time. In addition, particles can undergo hard elastic collisions.

So far, estimates based on perturbative QCD (pQCD) of the strength of the coupling of jets to a plasma are sensitive to infrared cutoffs. Also, they are often restricted to systems that are (at least locally) in thermal equilibrium. The problem does not arise in the Wong-Yang-Mills simulation \([16, 17, 18, 19, 20, 21, 22, 23, 24, 25]\), since the soft sector is described by classical chromo-fields. It is separated from the hard sector corresponding to hard elastic pQCD processes. The soft sector is non-perturbative but in an essentially classical way because the occupation number of field modes below the saturation momentum (or temperature) are large \([26, 27]\).

It is well known that a cutoff independent collisional energy loss can be obtained by resumming soft interactions \([28, 29]\). In the present paper (see, also, Refs. \([30, 31]\)) we show by explicit implementation that this can also be achieved within the framework of a transport theory by treating the soft interactions via classical Yang-Mills fields defined on a lattice. We find that, in practice, this works even for physical values of the gauge coupling, \(g \sim 2\), so long as a weak-coupling (resp. continuum-limit) condition specified in Eq. \([25]\) below is satisfied. Within this framework, we are able to also consider the interesting problem of elastic energy loss of a jet propagating through a classical non-Abelian field, which might be...
relevant for describing the early stages of a high-energy collision of large nuclei; see below.

The main purpose of this paper is to extract lattice-spacing independent results for the transport coefficient $\hat{q}$ associated with broadening of the momentum distribution of gluon jets, as well as for collisional energy loss $dE/dx$. Previous publications [30, 31] already presented a calculation of $\hat{q}$ within this approach, however lacking the detailed analysis shown here as well as a computation of $dE/dx$. Furthermore, in this paper we extract the entire $p_\perp^2$ distribution of jets passing through a thermal plasma (not only its first moment $\hat{q}$). We also address the scaling of $\hat{q}$ and $dE/dx$ with the particle density, temperature, and jet energy. The scaling laws turn out to agree, qualitatively, with pQCD expectations although the overall magnitude of $\hat{q}$ and $dE/dx$ is found to receive substantial corrections.

We extract a value for $\hat{q}$ of $3.6 \pm 0.3$ GeV$^2$fm$^{-1}$ at $T = 400$ MeV in a thermal SU(3) background for a parton with energy $E = 19.2$ GeV. For the collisional energy loss we obtain $dE/dx = 1.6 \pm 0.4$ GeV fm$^{-1}$.

This paper is organized as follows: We introduce the Boltzmann-Vlasov equations as well as the Wong equations for non-Abelian plasmas in Sec. II and discuss the lattice implementation in Sec. III. We outline how collisions are included into the Wong-Yang-Mills simulation in Sec. IV and describe how the separation between the soft and hard sector is done in Sec. V. After discussing the initialization of the simulation in Sec. VI, we present results for collisional energy loss and for momentum broadening in Sec. VII. Finally, we close with conclusions in Sec. VIII.

II. BOLTZMANN-VLASOV EQUATION FOR NON-ABELIAN GAUGE THEORIES

The classical transport theory for non-Abelian plasmas has been established by Heinz and Elze [32, 33, 34, 35]. Here, we solve numerically the classical transport equation for hard gluons with adjoint SU(2) color charge $q = q^a \tau^a$, where the $\tau^a$ are the color generators, including hard binary collisions

$$p^\mu \left( \partial_\mu + g q^a F^a_{\mu\nu} \partial_\nu + g f^{abc} A^b_\mu(x) q^c \partial_\nu \right) f = C.$$  

(1)

$f = f(x, p, q)$ denotes the single-particle phase space distribution, $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$ is the gauge field strength tensor, $g$ the gauge coupling, $A^a_\mu$ the soft gauge field,
and $C$ is the collision term to be defined below. It is coupled self-consistently to the Yang-Mills equation for the soft gluon fields,

$$D_\mu F^{\mu\nu} = j^\nu = g \int \frac{d^3p}{(2\pi)^3} dq \, q^\nu f(x, p, q),$$

with $\nu^\mu = (1, \mathbf{p}/p)$. For $C = 0$ these equations reproduce the “hard thermal loop” effective action near equilibrium [36, 37, 38]. However, the full classical transport theory also includes some higher n-point vertices of the dimensionally reduced effective action for static gluons [39] beyond the hard-loop approximation. The back-reaction of the long-wavelength fields on the hard particles (“bending” of their trajectories) is taken into account. This is essential for achieving cutoff independent results for the transport coefficient $\hat{q}$ and for the energy loss $dE/dx$ of high momentum partons.

When the phase-space density is parametrically small, $f = \mathcal{O}(1)$, which is the case for hard momenta, the collision term is given by

$$C = \frac{1}{4E_1} \int_{p_2} \int_{p'_1} \int_{p'_2} (2\pi)^4 \delta^{(4)}(p'_1 + p'_2 - p_1 - p_2) \left(f'_1 f'_2 |M_{12'1'-2}|^2 - f_1 f_2 |M_{1'-2'-1}|^2\right),$$

with $\int_{p_i} = \int \frac{d^3p_i}{(2\pi)^3 2E_i}$. The matrix element $M$ includes all $gg \rightarrow gg$ tree-level diagrams shown in Fig. 1 and color factors as appropriate for the SU(2) gauge group.

![FIG. 1: Processes contributing to $gg \rightarrow gg$ scattering at leading order.](image)

We employ the test particle method and replace the continuous distribution $f(x, p, q)$ by a large number of test particles [40]:

$$f(x, p, q) = \frac{1}{N_{\text{test}}} \sum_i \delta^3(x - x_i(t))(2\pi)^3 \delta^{(3)}(p - p_i(t))\delta^{(N^2-1)}(q - q_i(t)),$$

which leads to the Wong equations [16] (also see [41, 42])

$$\dot{x}_i(t) = v_i(t),$$

$$\dot{p}_i(t) = gg_i^a(t) (E^a(t) + v_i(t) \times B^a(t)),$$

$$\dot{q}_i(t) = -ig v_i^\mu(t) [A_\mu(t), q_i(t)].$$
Here, \(\mathbf{x}_i(t)\), \(\mathbf{v}_i(t)\), and \(q_i(t)\), are the position, velocity\(^1\) and color charge of the \(i^{th}\) test particle. \(N_{\text{test}}\) denotes the number of test particles per physical particle, \(A_\mu(t) = A_\mu^a(t)\tau^a\) and the commutator commutes color generators \(\tau^a\). The last equation (7) describes the precession of the color charge \(q_i(t)\) due to the color fields.

Writing the current in terms of the individual test particles, the Yang-Mills equation for the soft gluon fields becomes

\[
D_\mu F^{\mu\nu} = J^\nu = \frac{g}{N_{\text{test}}} \sum_i q_i v_i^\nu \delta(\mathbf{x} - \mathbf{x}_i(t)).
\]  

(8)

The theory without collisions as given by equations (5-7) coupled to the lattice Yang-Mills equations (8) was first solved in [43] to study Chern-Simons number diffusion in non-Abelian gauge theories at finite temperature. It was applied later also to the problem of gauge-field instabilities in anisotropic SU(2) plasmas [24, 25]. Our numerical implementation is based on the improved formulation detailed in [25] where the non-Abelian currents, generated by the hard particle modes on the lattice sites, are “smeared”. This technique makes simulations in three dimensions on large lattices possible in practice.

III. REAL-TIME LATTICE SIMULATION

The time evolution of the Yang-Mills field is determined by the standard Hamiltonian method in \(A^0 = 0\) gauge [17, 43, 44]. The temporal gauge is particularly useful because it allows for a simple identification of the canonical momentum as the electric field

\[
\mathbf{E}^a = -\dot{\mathbf{A}}^a.
\]  

(9)

In addition, time-like link variables \(U\), defined below, become simple identity matrices.

The lattice Hamiltonian in this gauge is given by [45]

\[
H_L = \frac{1}{2} \sum_i \mathbf{E}_{L,i}^2 + \frac{1}{2} \sum_\square (N_c - \text{ReTr}U^\square) + \frac{1}{N_{\text{test}},L} \sum_j |\mathbf{p}_{L,j}|, 
\]  

(10)

including the particle contribution \(1/N_{\text{test},L} \sum_j |\mathbf{p}_{L,j}|\). The plaquette is defined by

\[
U^\square = U_x(i)U_y(i + \hat{x})U_y^\dagger(i + \hat{y})U_x^\dagger(i), 
\]  

(11)

\(^1\) We consider only massless particles here so that \(|\mathbf{v}_i| = 1\).
with the link variable

$$U_\mu(i) = e^{i a g A_\mu(i)}.$$  \hfill (12)

Note that the index $\mu$ on $U$ is merely an indicator of its direction and not a Lorentz index. The shifts $\hat{x}$ and $\hat{y}$ are one lattice spacing in length and directed into the $x$- or $y$-direction, respectively. \footnote{We set $\tau_a = \sigma_a$, the Pauli matrices, without the usual factor of 1/2, i.e., the commutation relation reads $[\tau^a, \tau^b] = 2\delta^{ab}$. Another factor of 1/2 is absorbed into the $A$-field, which has to be taken into account when calculating the physical fields $E$ and $B$ from it.}

Eq. (10) is given in lattice units, which are chosen such that all lattice variables are dimensionless:

$$E_L^a = \frac{g a^2}{2} E^a, \quad B_L^a = \frac{g a^2}{2} B^a, \quad p_L = \frac{a}{4} p, \quad Q_L^a = \frac{1}{2} q^a, \quad N_{\text{test}, L} = \frac{1}{g^2} N_{\text{test}},$$ \hfill (13)

with the lattice spacing $a$. $H_L$ is hence related to the physical Hamiltonian by $H = 4/(g^2 a) H_L$. To convert lattice variables to physical units we will fix the lattice length $L$ in fm, which will then determine the physical scale for $a$. All other dimensionful quantities can then be determined from Eqs. (13). The Hamiltonian (10) determines the energy density of the system and enters the equations of motion for the fields, e.g.,

$$\frac{d}{dt} E_L = \{ E_L, H_L \},$$ \hfill (14)

with the Poisson bracket $\{\cdot, \cdot\}$. Our lattice has periodic boundary conditions in all spatial directions.

IV. COLLISIONS

The collision kernel (3) is similar to that used in (parton) cascade simulations \cite{46, 47, 48, 49, 50, 51, 52, 53, 54, 55}. Here, it is restricted to hard binary collisions since soft multi-parton interactions are mediated by interactions with the collective Yang-Mills field. This way, we are able also to study collective phenomena and their contribution to isotropization and thermalization. In particular, we can in principle also study systems away from equilibrium (see \cite{30}) for which the scale corresponding to the Debye-mass squared in an isotropic system becomes negative \cite{56, 57, 58, 59}. In this case it can obviously not damp the propagator to act as a cut off for the momentum exchange in the infrared.
To complete our dual particle/field description, we need to specify the separation scale $k^*$ between the field and particle degrees of freedom. We will discuss this separation scale in detail below. For now it will serve as a lower bound for the exchanged momenta for binary elastic particle collisions. All softer momentum exchanges are mediated by the fields.

The collision term (3) is incorporated using the stochastic method introduced and applied in [54, 60, 61]. We do not interpret the cross section in a geometrical way as done in [46, 48, 49, 50, 51, 52] but determine scattering processes in a stochastic manner by sampling possible transitions in a volume element per time interval. This collision algorithm can be extended to include inelastic processes $gg \leftrightarrow ggg$ as done in [54, 55], which will also be incorporated in the future in our simulations.

The collision rate in a spatial volume element $\Delta^3 x$ per unit phase space for a particle pair with momenta in the range $(p_1, p_1 + \Delta^3 p_1)$ and $(p_2, p_2 + \Delta^3 p_2)$ follows from Eq. (3)

$$\frac{\Delta N_{\text{coll}}}{\Delta t (2\pi)^3 \Delta^3 x \Delta^3 p_1} = \frac{1}{2E_1} \frac{\Delta^3 p_2}{(2\pi)^3 2E_2} \frac{f_1 f_2}{f_1 f_2} \times \frac{1}{2} \int \frac{d^3 p_1'}{(2\pi)^3 2E_1} \frac{d^3 p_2'}{(2\pi)^3 2E_2} |M_{12 \rightarrow 1'2'}|^2 (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_1' - p_2').$$

Expressing the distribution functions as

$$f_i = \frac{\Delta N_i}{(2\pi)^3 \Delta^3 x \Delta^3 p_i}, \quad i = 1, 2,$$

and employing the usual definition of the cross section for massless particles [62]

$$\sigma_{22} = \frac{1}{4s} \int \frac{d^3 p_1'}{(2\pi)^3 2E_1} \frac{d^3 p_2'}{(2\pi)^3 2E_2} |M_{12 \rightarrow 1'2'}|^2 (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_1' - p_2'),$$

one obtains the total collision probability in a volume element $\Delta^3 x$ and time interval $\Delta t$:

$$P_{22} = \frac{\Delta N_{\text{coll}}}{\Delta N_1 \Delta N_2} = \tilde{v}_{\text{rel}} \sigma_{22} \frac{\Delta t}{\Delta^3 x}. \quad (18)$$

$\tilde{v}_{\text{rel}} = s/2E_1 E_2$ denotes the relative velocity, where $s$ is the invariant mass of the particle pair. $P_{22}$ is a number between 0 and 1.\(^3\) Whether or not a collision occurs is sampled stochastically as follows: We compare $P_{22}$ to a uniformly distributed random number between 0 and 1. If the random number is less than $P_{22}$, the collision does occur. Otherwise, there is no collision

\(^3\) In practice one has to choose suitable $\Delta^3 x$ and $\Delta t$ to ensure that $P_{22} < 1$. 

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between the two particles in that time step. Since we represent each physical particle by $N_{\text{test}}$ test particles, we have to rescale the cross section as $\sigma \rightarrow \sigma/N_{\text{test}}$. This leads to

$$P_{22} = \bar{v}_{\text{rel}} \frac{\sigma_{22}}{N_{\text{test}}} \frac{\Delta t}{\Delta x^3}.$$  \hspace{1cm} (19)

To determine this probability, we require the total cross section $\sigma_{22}$. To leading order in $\alpha_s$, it follows from the differential cross section obtained from the diagrams in Fig. \[63, 64, 65\]:

$$\frac{d\sigma}{dt} = 4\pi\alpha_s^2 \frac{N_c^2}{s^2} \left(3 - \frac{tu}{s^2} - \frac{su}{t^2} - \frac{st}{u^2}\right),$$  \hspace{1cm} (20)

with $N_c$ the number of colors. The invariant Mandelstam variables are

$$s = (p_1 + p_2)^2, \quad t = (p_1 - p_1')^2, \quad u = (p_1 - p_2')^2.$$  \hspace{1cm} (21)

Using $t = -q^2$, with $q$ the momentum transfer, and the identity $s + t + u = 0$ for massless particles, we can express the total cross section for processes with $\sqrt{q^2}$ larger than $k^*$ as

$$\sigma_{22} = \int_{k^*}^{s/2} \frac{d\sigma}{dq^2} dq^2.$$  \hspace{1cm} (22)

The momentum transfer is then determined stochastically in the center-of-momentum frame of the two colliding particles from the probability distribution

$$P(q^2) = \frac{1}{\sigma_{22}} \frac{d\sigma}{dq^2}.$$  \hspace{1cm} (23)

In Eq. (22) we have introduced the cutoff $k^*$ for point-like binary collisions. To avoid double-counting, this cutoff should be on the order of the hardest field mode that can be represented on the given lattice, $k^* \simeq \pi/a$.

V. SEPARATION SCALE

The scattering processes in the regime of hard momentum exchange are described by elastic binary collisions, while soft momentum exchanges are mediated by the fields. A scattering in the soft regime corresponds to deflection of a particle in the field of the other(s).

Physically, the separation scale $k^*$ should be sufficiently small so that the soft field modes below $k^*$ are highly occupied \[44\] and hence can be described classically. On the other hand, $k^*$ should be sufficiently large to ensure that hard modes can be represented by particles and that collisions are described by \[3\], which is valid only for low occupation numbers since the
Bose enhancement factor \((1 + f)\) is approximated by 1. In practice \(g \sim 1\), and we choose \(k^*\) to be on the order of the temperature. At the same time, \(k^*\) is related to the hardest available field mode, which on a cubic lattice is given by \(\sqrt{3\pi}/a\). Obviously, any matching between soft and hard regimes can only be done approximately, because the lattice on which the field modes are defined is cubic, while the momentum space cutoff of the hard collision integral is implicitly spherical.

The “soft” scale is given by

\[
m_D^2 = \frac{2g^2N_c}{N_c^2 - 1} \int \frac{d^3 p}{(2\pi)^3} \frac{f(p)}{|p|} \sim \frac{\pi^2}{2} \frac{g^2N_c}{N_c^2 - 1} \frac{n}{p_h},
\]

where \(N_c = 2\) is the number of colors and \(n\) denotes the number density of hard gluons, summed over two helicities and \(N_c^2 - 1\) colors. Also, \(p_h \approx 3T\) is the typical momentum of a hard particle from the medium.

To allow for reliable numerical simulations one should have \(m_D L \gg \pi\) and \(m_D a \ll \pi\). The first condition ensures that the relevant soft modes actually fit on the lattice while the latter ensures that the lattice can resolve the wavelength \(1/m_D\) to good precision.

As we have argued above, we choose the inverse lattice spacing to be on the order of the temperature of the medium. Thus, with (24) the condition \(m_D a \ll \pi\) roughly translates to

\[
\frac{g^2N_c}{N_c^2 - 1} \frac{n}{T^3} \ll 1.
\]

In order to satisfy this relation, which is essentially the weak-coupling condition, at \(g \sim 1\), we perform the numerical simulations below for an extremely hot and undersaturated medium: \(T^3 \gg n\). This ensures that the simulations are carried out near the continuum limit. We verify below that transverse momentum broadening of a high-energy jet passing through a thermal medium is independent of \(T\) if the density and the ratio of jet momentum to temperature is fixed. One may therefore obtain a useful weak-coupling estimate of \(\langle p_\perp^2 \rangle\) (resp. for the related transport coefficient \(\tilde{q}\)) by extrapolating our measurements down to temperatures relevant to present heavy-ion collisions.

**VI. Initialization**

We consider a heat-bath of Boltzmann distributed particles with a density of \(n = \{5, 10, 20\}\) fm\(^{-3}\) and an average particle momentum of \(3T = \{6, 12, 18, 24\}\) GeV. For a given
lattice (resp. \( k^* \)) we take the initial energy density of the thermalized fields to be
\[
\varepsilon_{\text{fields}} = \int \frac{d^3 k}{(2\pi)^3} k \hat{f}_{\text{Bose}}(k) \Theta(k^* - k),
\]
where
\[
\hat{f}_{\text{Bose}}(k) = \frac{n \pi^2}{T^3 \zeta(3)} \frac{1}{e^{k/T} - 1}
\]
is a Bose distribution normalized to the assumed particle density \( n \), and \( \zeta \) is the Riemann zeta function.

The initial field amplitudes are sampled from a Gaussian distribution:
\[
\langle A^a_i(x) A^b_j(y) \rangle = 4 \mu^2 g^2 \delta_{ij} \delta^{ab} \delta(x - y).
\]
To thermalize the initial fields (approximately), we match their Fourier spectrum to the classical limit of the Bose distribution. Hence, the initial spectrum is gauge-fixed to Coulomb gauge and a filter is applied such that
\[
A_i \sim 1/k
\]
(in continuum notation). Setting \( E_i = 0 \) initially\(^4\), Gauss’s law implies that the local charge density at time \( t = 0 \) vanishes. We ensure that any particular initial condition satisfies exact local charge neutrality. The charge smearing algorithm for SU(2) explicitly exploits (covariant) current conservation and hence Gauss’s law is satisfied exactly by construction at all times \(^{25}\).

The above procedure ensures that there is no large discontinuity of the energy density when going from the field to the particle regime. This way we are able to vary the separation scale \( k^* \) about the temperature \( T \) by varying the lattice spacing. Fig.\(^2\) shows the distribution of field modes and particles and the separation scale \( k^* \sim T \).

VII. MOMENTUM DIFFUSION AND ENERGY LOSS OF HIGH MOMENTUM PARTONS

Having initialized the background particles and fields, we can now add a few high-momentum test particles propagating along a given (“longitudinal”) direction which represent the jets. Their density should be sufficiently low so that they do not influence the thermal background significantly and so their mutual interaction is minimized.

\(^4\) Equipartitioning of electric and magnetic fields is achieved very rapidly within a few time steps.
FIG. 2: (Color online) Bose distribution and its low and high-momentum limits, used for the initial fields and particles, respectively. Physically, the separation \( k^* \) should be on the order of the temperature \( T \). The band between \( T/2 \) and \( 2T \) roughly indicates the region within which we vary \( k^* \).

We always initialize “bunches” of test particles, which represent one physical hard momentum parton (“jet”). A bunch corresponds to \( N_{\text{test}} \) particles in the same lattice cell. The physical color charge is independent of \( N_{\text{test}} \). If, in fact, the color charges of all test particles representing one jet add to zero, no coherent radiation is emitted (colorless jet). Such jets can only suffer collisional energy loss\(^5\). In the particle-in-cell simulation radiative energy loss is not consistently included (see e.g. [66]). Initializing a bunch of test particles with aligned color vectors, leading to a net current on the lattice, will hence not correspond to the correct physical bremsstrahlung process. We postpone the consistent implementation of radiative energy loss to future work.

As detailed above, colorless bunches of test particles permit us to restrict to collisional energy loss and momentum broadening due to elastic collisions only. We first demonstrate that in our approach both

\[
\hat{q} = \frac{1}{\lambda \sigma} \int d^2 p_\perp p_\perp^2 \frac{d\sigma}{dp_\perp^2},
\]

\(^5\) Note that individual test particles from the bunch are of course colored and hence they collide not only with hard thermal particles but also with the modes of the thermal fields.
and the differential energy loss $dE/dx$ are independent of the separation scale $k^*$. Here and in what follows, $p_\perp$ denotes the momentum transverse to the initial jet momentum. $\hat{q}$ can be extracted from the squared transverse momentum of the test-particles accumulated up to a time $t$:

$$\hat{q} = \frac{\langle p_\perp^2 \rangle(t)}{t}.$$  \hspace{1cm} (29)

Fig. 3 depicts the contributions to $\hat{q}$ due to soft and hard collisions, respectively, as well as the total. In these simulations the gluon density of the medium was taken to be $n = 5 \text{ fm}^{-3}$, the temperature $T = 4 \text{ GeV}$, and the jet energy is 16 times the average thermal momentum ($48T$). $\hat{q}$ is shown as a function of the separation scale $k^* \sim \sqrt{3\pi/a}$.

The curves for the total and the soft contributions were averaged over $\sim 80$ runs for each point. The curve corresponding to the hard sector was obtained by subtracting the result without hard collisions (soft sector only) from the total. The error bars indicate one standard deviation about the mean.

We find that the total value is constant to a good approximation although the contribution due to soft scatterings changes considerably. Below $k^* \approx T$, the soft sector contributes less than 10%, while it starts dominating around $k^* \approx 3T$. It is evident, therefore, that transport coefficients obtained in the leading logarithmic (LL) approximation from the pure Boltzmann approach (without soft fields) are rather sensitive to the infrared cutoff $k^*$, unless the energy $\sqrt{s}$ is extremely high. In LL approximation,

$$\hat{q} = n \frac{4\pi \alpha_s^2 N_c^2}{N_c^2 - 1} \ln \left( \frac{C^2 Q^2}{k^*} \right),$$  \hspace{1cm} (30)

where $Q^2 \simeq s$ is the upper bound for the momentum transfer and $C$ is a constant.

Fig. 4 repeats the same analysis for the collisional energy loss per unit path length, $dE/dx$. Again we find a constant total energy loss, and a similar dependence on $k^*$ of the partial contribution due to soft interactions as for $\hat{q}$. We have also verified the $k^*$-independence for different temperatures, densities and jet energies. Thus, the above-mentioned matching of soft and hard processes provides estimates for $\hat{q}$ and $dE/dx$ which are independent of the artificial separation scale $k^*$ (and of the lattice spacing $a$). It cures the infrared divergence of the perturbative hard-scattering cross section and does not rely on infrared cutoffs from equilibrium physics such as the Debye mass; thus, calculations are not restricted to equilibrium. On the other hand, the matching procedure might have to be modified for
other observables which are sensitive to very different scales. This should be analyzed in the future.

Next, we turn to the density and temperature dependence of $\hat{q}$ and $dE/dx$. Figs. 5 and 6 show the linear rise of $\hat{q}$ and $dE/dx$ with the density, which is expected from Eq. (28) and
the perturbative results (30) and (31) below, respectively. We will use this linear dependence below to extrapolate to larger densities (e.g., \( n_{\text{thermal}}(T = 500 \text{ MeV}) \approx 32 \text{ fm}^{-3} \) for pure glue in SU(3)).

Fig. 7 shows that \( \hat{q} \) is approximately independent of \( T \) as long as the ratio of the jet energy to the temperature \( E/T \) as well as the density \( n \) are fixed. From (30) we expect

\[ \hat{q} \sim c n, \quad c \approx 0.26 \text{ GeV}^2 \text{ fm}^{-2} \]

FIG. 5: (Color online) Linear dependence of \( \hat{q} \) on the density \( n \). \( T = 4 \text{ GeV}, g = 2, E/T = 48, k^* \approx 1.16 T \). The line shows the best linear fit.

\[ \frac{dE}{dx} \sim c' n, \quad c' \approx 0.014 \text{ GeV fm}^{-2} \]

FIG. 6: (Color online) Linear dependence of \( dE/dx \) on the density \( n \). \( T = 4 \text{ GeV}, g = 2, E/T = 48, k^* \approx 1.16 T \). The line shows the best linear fit.
at most a logarithmic dependence on $T$, because $Q^2 \simeq s$ and $\langle s \rangle = 6ET$. The simulation shows that this dependence is very weak.

Fig. 7 shows $dE/dx$ dropping approximately like $\sim 1/T$. This behavior is expected from the perturbative LL result for elastic energy loss (see, for example, ref. [67])

$$\frac{dE}{dx} = n \left( \frac{16\pi \alpha_s^2 N_c^2}{N_c^2 - 1} \right) \frac{E}{s} \ln \left( C' \frac{Q^2}{k^*} \right),$$

where $s$ is the center-of-mass energy for a process involving scattering of the jet from a hard thermal excitation, $Q^2 \simeq s$ is the upper limit for the momentum transfer, and $C'$ is a constant. Because $\langle s \rangle = 6ET$ this leads to $dE/dx \sim 1/T$. Additionally, $T$ also appears in the logarithm, but this dependence turns out to be weak.

We can now extrapolate to temperatures which are accessible in practice, for which direct computations can not be performed due to the numerical reasons explained above. For an ideal gas of thermal gluons at a temperature $T$ the density $n = 16T^3\zeta(3)/\pi^2$ (for $N_c = 3$). Using the linear dependence of $\hat{q}$ on $n$ confirmed above, and its independence on $T$ for fixed $n$ and $E/T$, we find $\hat{q} \approx 7 \pm 0.6 \text{ GeV}^2\text{fm}^{-1}$ at $T = 500\text{ MeV}$. This number has been rescaled to the color factors appropriate to SU(3)\(^6\). Since $E/T$ is fixed, this result

\(^6\) We divide the results by the prefactors given in Eqs. (30) and (31), respectively, which correspond to $N_c = 2$, and multiply by the prefactors appropriate for $N_c = 3$. 

FIG. 7: (Color online) (In-)dependence of $\hat{q}$ on the temperature $T$. $n = 5 \text{ fm}^{-3}$, $g = 2$, $E/T = 48$, $k^* \approx 1.16T$ (squares) and $k^* \approx 1.73T$ (circles). 

\[ T \text{[GeV]} \]
\[ \hat{q} \text{[GeV}^2\text{fm}^{-1}] \]
\[ k^* = 1.16T \quad \bullet \]
\[ k^* = 1.73T \quad \circ \]
corresponds to a jet energy of $E = 48T = 24\text{GeV}$. The quoted error arises from using different possible fits, including a logarithmic dependence on $T$ or not, and from different choices for $k^*$. For a temperature of $T = 400\text{MeV}$ and the corresponding thermal gluon density, we find $\hat{q} \approx 3.6\pm0.3\text{GeV}^2\text{fm}^{-1}$ ($E = 19.2\text{GeV}$). We emphasize that our simulations do not account for quarks and anti-quarks which would provide a sizeable contribution to the thermal density. Nevertheless, such values for $\hat{q}$ are within the range extracted from RHIC data \cite{68, 69, 70}.

In Fig. 8 we present a possible extrapolation of $dE/dx$ to temperatures around $500\text{MeV}$. We find $dE/dx \approx 0.35 - 0.6\text{ GeVfm}^{-1}$ at $T = 500\text{ MeV}$, and for a jet energy of $E = 48T = 24\text{GeV}$. Adjusting the color factors as appropriate for SU(3) and extrapolating to the thermal density of gluons we find $dE/dx \approx 2.5 \pm 0.6\text{ GeV fm}^{-1}$. At $T = 400\text{MeV}$, the result is $dE/dx \approx 1.6 \pm 0.4\text{ GeV fm}^{-1}$ ($E = 19.2\text{GeV}$).

In Figs. 9 and 10 we show how $\hat{q}$ and $dE/dx$ depend on the energy $E$ of the jet. The behavior is logarithmic, in agreement with the perturbative expectation; compare to Eqs. (30) and (31), using $Q^2 \sim ET$ (the explicit factor of $E$ in the numerator cancels since $\langle s \rangle = 6ET$). A fit of the numerical result to (30), using $Q^2 \approx s$, leads to $C \approx 1.45$, but is good only if
an additional prefactor of \( \sim 0.63 \) is allowed. This suggests that the perturbative result does not describe the numerical solution very well, which could perhaps be expected at \( g = 2 \). Repeating the analysis for \( dE/dx \) via Eq. (31), we find \( C' \approx 7.7 \); again, a multiplicative factor needs to be included, this time it is \( \sim 0.14 \). Thus, for the jet energies considered here, there is a smaller "K factor" relative to pQCD at LL for \( dE/dx \) than for \( \hat{q} \); note that the former is sensitive also to longitudinal momentum exchanges while the latter is not.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig9}
\caption{(Color online) Jet-energy dependence of \( \hat{q} \) for \( T = 4 \) GeV and \( n = 5 \) fm\(^{-3} \). \( k^* \approx 1.16T \). The line shows the fit to Eq. (30), with an overall multiplicative factor of 0.63.}
\end{figure}

We have also determined the full \( p_{\perp}^2 \)-distribution of the high-momentum partons traversing the hot medium in order to assess the relative contributions from various processes to its first moment \( \hat{q} \). We find that over time the initial \( \delta \)-function broadens to a Gaussian distribution with a power-law tail. This enhancement of transverse momentum broadening reflects the well known result from QED and is in line with the findings for QCD in Refs. [71, 72]. The enhancement has also been discussed within the higher twist formalism in Ref. [73]. What is perhaps less obvious is the relative magnitude of the Gaussian and power-law parts, which may be expected to be time dependent. However, for time scales typical of heavy-ion collisions we do not observe a large relative shift of these contributions.

Figs. 11 and 12 show the distribution of the high-momentum “jet” test particles after \( t \approx 2.8 \) fm and \( t \approx 5.2 \) fm, respectively, in a double-logarithmic plot versus \( p_{\perp}^2/t \). We scale \( p_{\perp}^2 \) by the inverse time so that the basic features of the distribution are nearly time independent.
FIG. 10: (Color online) Jet-energy dependence of $dE/dx$ for $T = 4$ GeV and $n = 5$ fm$^{-3}$. $k^* \approx 1.16T$. The line shows the fit to Eq. (31), with an overall multiplicative factor of 0.143.

FIG. 11: (Color online) $p_{\perp}^2$-distribution of the high-momentum ($E/T = 48$) “jet”-partons after $t \approx 2.8$ fm for $T = 4$ GeV and $n = 20$ fm$^{-3}$ ($\hat{q} \approx 5.16$ GeV$^2$fm$^{-1}$).

The low-$p_{\perp}$ part follows a Gaussian distribution in $p_{\perp}$. The power-law tail at large $p_{\perp}$ behaves approximately as $p_{\perp}^{-4}$. This is expected for particles experiencing only few scatterings since in the high-energy limit the differential cross section $d\sigma/dp_{\perp}^2 \sim p_{\perp}^{-4}$, c.f. Eq. (20). This is the probability distribution for the momentum transfer in a single hard collision.
FIG. 12: (Color online) $p_{\perp}^2$-distribution of the high-momentum ($E/T = 48$) “jet”-partons after $t \approx 5.2 \text{ fm}$ for $T = 4 \text{ GeV}$ and $n = 20 \text{ fm}^{-3}$ ($\hat{q} \approx 5.16 \text{ GeV}^2 \text{fm}^{-1}$).

In both figures we also indicate the value of $\hat{q}$ to show that the power-law tail contributes significantly to this transport coefficient. For the densities, temperatures and jet energies considered here it is clearly not a very good approximation to determine the transport coefficient $\hat{q}$ from the Gaussian part of the distribution alone as this would underestimate $\hat{q}$ substantially: discarding the power-law tail from Figs. 11, 12 gives $\hat{q}_{\text{Gauss}} \approx 0.6 \text{ GeV}^2 \text{fm}^{-1}$.

Note also that transverse momenta on the order of the temperature ($T = 4 \text{ GeV}$), such as the separation scale $k^* \approx 1.2T$, correspond to $p_{\perp}^2/t \approx 5.2 \text{ GeV}^2 \text{fm}^{-1}$ in Fig. 11 and to $p_{\perp}^2/t \approx 3 \text{ GeV}^2 \text{fm}^{-1}$ in Fig. 12. Above this value for $p_{\perp}^2/t$ the distribution is due almost entirely to hard collisions (we have checked that multiple soft collisions do not contribute much in that region).

Finally, we also provide an estimate for the nuclear modification factor $R_{AA}$ of the jet spectrum due to elastic energy loss in a classical Yang-Mills field\(^7\). This is of relevance for collisions of heavy nuclei at high energies: the large number of gluons produced in the central rapidity region can be described as a classical field for a short time [18, 19, 20, 21, 22, 23] until the field modes decohere and thermalize [25]. These classical fields produced in the early stage of the collision also exhibit long-range correlations in rapidity [75], which we

\(^7\) Classical radiative energy loss has recently been considered in ref. [74] but is not taken into account here.
presently neglect.

We proceed as follows. From our simulations presented above, the elastic energy loss at a density $n$ and for a separation scale $k^* (\equiv \sqrt{3\frac{2}{n}}) = \sqrt{3} \ 2T$ can be parameterized as

$$\frac{dE}{dx} = K \ n \frac{16\pi \alpha_s^2 N_c^2}{N_c^2 - 1} \ \frac{1}{6T} \ \ln \left( \frac{C'^2 6ET}{k^{*2}} \right),$$

(32)

with $K = 0.143$ and $C' = 22.975$. This form for $dE/dx$ has been established numerically in the weak-coupling regime specified by Eq. (25) for $N_c = 2$; in what follows, we extrapolate it to $N_c = 3$ and to physical density and temperature. Note also that for such large $k^*$ (on the order of the so-called “saturation momentum” $Q_s$) most of the energy density is due to the classical field. We evaluate this expression as a function of the jet energy for $N_c = 3$, $T = 400$ MeV and the corresponding thermal density of gluons. This corresponds to an energy density of about 17 GeV/fm$^3$, which is an appropriate average over the first 1 fm/c of a central Au+Au collision at RHIC energy [18, 19, 20, 21, 22, 23].

![FIG. 13: (Color online) Nuclear modification factor $R_{AA}(p_{t})$ of jets due to elastic energy loss in a classical Yang-Mills field produced in the early stage of a relativistic heavy-ion collision at RHIC. The band indicates the uncertainty originating from the extrapolation of $dE/dx$ to physical temperatures (compare to Fig. 8).](image)

The initial transverse momentum distribution of jets at RHIC can be parameterized
approximately as

\[
\frac{dN_i}{d^2p_{\perp}dy} \sim \frac{1}{p_{\perp}^{n+2}}
\]

with \( n \approx 4 \). Here, \( p_{\perp} \) denotes the momentum of a jet transverse to the colliding ion beams. In the central region (\( y \sim 0 \)) it is equal to the jet energy. The final distribution due to interactions with the background is then given by

\[
\frac{dN_f}{d^2p_{\perp}dy} = \int d^2p'_{\perp} \delta^{(2)}(p_{\perp} - (1 - \epsilon)p'_{\perp}) \frac{dN_i}{d^2p'_{\perp}dy} = \frac{1}{(1 - \epsilon)^2} \frac{dN_i}{d^2p'_{\perp}dy} = \frac{1}{p_{\perp}^{n+2}} (1 - \epsilon)^n.
\]

(34)

Here, \( \epsilon \) denotes the fractional energy loss up to a time \( \tau \), which we take to be 1 fm/c:

\[
\epsilon(p_{\perp}) = \frac{\tau}{p_{\perp}} \frac{dE}{dx}(p_{\perp}).
\]

(35)

Thus, the nuclear modification factor \( R_{AA} \) at the parton level (neglecting hadronization) can be written as

\[
R_{AA}(p_{\perp}) = \frac{dN_f/d^2p_{\perp}dy}{dN_i/d^2p_{\perp}dy} = (1 - \epsilon(p_{\perp}))^n.
\]

(36)

We find that \( \epsilon(p_{\perp}) \) is on the order of 10% and that it decreases with increasing jet energy. However, due to the relatively steep initial spectrum of produced particles at RHIC, this can lead to \( \sim 30\% - 50\% \) suppression in the \( p_{\perp} \)-range between 5 GeV and 20 GeV; see Fig. 13. Clearly, the experimentally observed flat \( R_{AA} \approx 0.2 \) can not be accounted for fully by early-stage elastic energy loss in the classical field background. Nevertheless, our result shows that this contribution is significant and that it can not be neglected.

**VIII. SUMMARY AND CONCLUSIONS**

We have studied collisional energy loss as well as momentum broadening of high-momentum gluon jets in a hot and dense non-Abelian SU(2) plasma by solving the coupled system of Wong-Yang-Mills equations in real time on a lattice. We separate the soft from the hard momentum exchange interactions by introducing a separation scale \( k^* \). This separation scale is given by the inverse lattice spacing \( \sim 1/a \), which determines the magnitude of the highest momentum field modes that can be represented on the lattice. We fix its physical value to be on the order of the temperature. Momentum exchanges below that scale are
mediated by the classical fields, those above the separation scale by direct elastic collisions between the particles. The latter were implemented through the pQCD collision kernel and the stochastic method for determining scattering probabilities.

We restricted to collisional energy loss by simulating effectively colorless jets (at the scale set by the lattice spacing). We were able to obtain lattice-spacing and hence separation-scale independent results for \( \hat{q} \) and \( dE/dx \) in a static and weakly coupled plasma. The dependence on temperature and density, as well as on the jet energy was found to follow, qualitatively, expectations from pQCD. We then extrapolated our simulation results to thermal densities and to more realistic temperatures which could not be simulated directly. For a thermal gluon plasma (no quarks and anti-quarks) at \( T = 400 \) MeV, and with color factors adjusted to the SU(3) gauge group, we estimate \( \hat{q} \approx 3.6 \pm 0.3 \) GeV\(^2\)fm\(^{-1}\) and \( dE/dx \approx 1.6 \pm 0.4 \) GeVfm\(^{-1}\), for a jet energy of 19.2 GeV. The errors are mainly due to the required extrapolation. At finite time (on the order of the transverse dimension of the collision zone in a heavy-ion collision), the \( p_T^2 \)-distribution of the high-momentum partons is found to be well approximated by a Gaussian distribution at low \( p_T \) and to exhibit a power-law tail \( \sim p_T^{-4} \) at high \( p_T \). The first moment of the distribution, i.e. the transport coefficient \( \hat{q} \), receives a large contribution from the power-law tail.

We have also provided a first estimate of the (elastic) energy loss of a jet traversing a classical Yang-Mills field, which might emerge in the early stage of a collision of large nuclei at high energy. For a field energy density of about 15 GeV/fm\(^3\), the fractional energy loss over a time interval of \( \tau \simeq 1 \) fm/c amounts to about 10% – 20%. Once convoluted with the steep \( \sim 1/p_T^6 \) initial spectrum of produced hard particles, this results in a nuclear modification factor \( R_{AA} \simeq 0.5 – 0.8 \), indicating that energy loss in the early stage (before the onset of hydrodynamic behavior of the hot medium) may give a significant contribution to the observed \( R_{AA} \) at RHIC.

**Acknowledgments**

We thank Oliver Fochler, Charles Gale, Sangyong Jeon, Berndt Müller, and Zhe Xu for helpful discussions and comments. A.D. thanks J. Jalilian-Marian and D. Kharzeev for emphasizing the importance of energy loss in a classical Yang-Mills field.

The numerical simulations were performed at the Center for Scientific Computing (CSC) of
Goethe University, Frankfurt am Main. M.S. and B.S. were in part supported by DFG Grant GR 1536/6-1. B.S. gratefully acknowledges a Richard H. Tomlinson Fellowship awarded by McGill University as well as support from the Natural Sciences and Engineering Research Council of Canada. Y.N. is supported by Japan MEXT grant No. 20540276. M.S. and Y.N. acknowledge support from the Yukawa Institute for Theoretical Physics during the “Entropy Production Before QGP” workshop.

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