Anisotropic Chan–Vese segmentation
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Abstract. In this paper we study a variant to Chan–Vese (CV) segmentation model with rectilinear anisotropy. We show existence of minimizers in the 2-phases case and how they are related to the (anisotropic) Rudin-Osher-Fatemi (ROF) denoising model. Our analysis shows that in the natural case of a piecewise constant on rectangles image (PCR function in short), there exists a minimizer of the CV functional which is also piecewise constant on rectangles over the same grid that the one defined by the original image. In the multiphase case, we show that minimizers of the CV multiphase functional also share this property in the case that the initial image is a PCR function. We also investigate a multiphase and anisotropic version of the Truncated ROF algorithm, and we compare the solutions given by this algorithm with minimizers of the multiphase anisotropic CV functional.

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1 Introduction
Image segmentation consists in partitioning a given image into multiple segments in which pixels share some characteristics. One of the most relevant models in the field of image segmentation is the Mumford-Shah (MS) model, introduced by the authors in \cite{25}. This model was the seed of a very successful approach to the problem: variational techniques with level set formulations. A particular case of the Mumford-Shah model is the case in which the objective function is piecewise constant inside some domains with finite perimeters. This model was introduced by Chan and Vese in the 2-phases and multiphases framework in \cite{14} and \cite{29}, respectively. They are known as Chan-Vese (CV) models and play a cornerstone role in diverse recent applications of image processing (e.g. see \cite{16, 26, 30, 31}).

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The starting point of this paper is the recent study [11] about the relationship between the CV model in image segmentation and Rudin-Osher-Fatemi’s (ROF) model in image denoising (see [28]). In [11], the authors show that a thresholding in the ROF model’s solution provides a partial minimizer of the two phases CV functional, which can be written as

\[ \text{CV}_2(\Lambda, c_1, c_2) = \text{Per}(\Lambda; \Omega) + \mu \left( \int_{\Lambda} (c_1 - f)^2 \, dx + \int_{\Omega \setminus \Lambda} (c_2 - f)^2 \, dx \right). \]

A partial minimizer in the sense that one can obtain a minimizer in each of its three variables; i.e. letting \( \Lambda_u := \{ x : u(x) \geq \frac{c_1 + c_2}{2} \} \), with \( u \) being the minimizer of the ROF functional (see [12, Proposition 2.6]), one has

\[
\begin{cases}
\Lambda_u \in \arg \min_{\Lambda : \text{Per}(\Lambda) < +\infty} \text{CV}_2(\Lambda, c_1, c_2) \\
\left( \frac{1}{|\Lambda|} \int_{\Lambda} f \, dx, \frac{1}{|\Omega \setminus \Lambda|} \int_{\Omega \setminus \Lambda} f \, dx \right) \in \arg \min_{(c_1, c_2) \in [0, 1]^2} \text{CV}_2(\Lambda, c_1, c_2).
\end{cases}
\]

Despite this, whether \( (\Lambda_u, \frac{1}{|\Lambda_u|} \int_{\Lambda_u} f \, dx, \frac{1}{|\Omega \setminus \Lambda_u|} \int_{\Omega \setminus \Lambda_u} f \, dx) \) is a true minimizer of \( \text{CV}_2 \) remains as an open problem. On the other hand, as stated in [11], it is of interest to understand if this relationship is still valid in some variants of both CV and ROF models.

Our work focuses in particular anisotropic variants of these models. Our motivation comes from some well known features of the anisotropic \( \ell_1 \) version of the ROF model, such as sharp recovery of edges (see [15, 17]), exact computability (e.g. see [10, 19, 22]) and an observed reduction of the staircasing effect detected in the isotropic version (see [15, 27]). Because of that, we establish the anisotropic case as the \( \ell_1 \) one, i.e. we replace in the CV models the usual total variation by the total variation with respect to \(| \cdot |_1\), defined by \(|v|_1 := |v_1| + |v_2|\) for \( v = (v_1, v_2) \in \mathbb{R}^2 \).

Our main objectives are the next ones: First, we show that there is a global minimizer of the two phases anisotropic CV functional

\[ \text{ACV}_\mu(\Lambda, c_1, c_2) := \text{Per}(\Lambda; \Omega) + \mu \left( \int_{\Lambda} (c_1 - f)^2 \, dx + \int_{\Omega \setminus \Lambda} (c_2 - f)^2 \, dx \right), \]

whose first component is an upper level set of a minimizer to the anisotropic ROF functional in \( L^2(\Omega) \), which is defined as follows:

\[ \text{AROF}_\lambda(u) := |Du|_1(\Omega) + \frac{\lambda}{2} \int_\Omega (u - f)^2 \, dx. \]
In order to show this result, we need to assume that $\Omega$ is a rectangle and that the data considered belong to a suitable space. Namely, we will assume that $f$ is piecewise constant on rectangles (denoted by PCR and defined in Def. [4]). We point out that this restriction is harmless from the point of view of applications. Our strategy is as follows: First of all, we generalize the ACV functional, which is defined only for sets of finite perimeter, to an energy functional defined on $L^2(\Omega) \times [0,1]^2$ as follows

$$G_\mu(u, c_1, c_2) := |Du|_1(\Omega) + \mu \int_\Omega (u(\cdot - f))^2 + (1 - u)(\cdot - f))^2 dx + \int_\Omega I_{[0,1]}(u) dx,$$

where $I_{[0,1]}$ is the indicator function on $[0,1]$. In relation to the above, we note that $G_\mu(\cdot, c_1, c_2)$ and AROF$\lambda(\cdot)$ are solely finite on the space of bounded variation functions, $BV(\Omega)$. On the other hand, the indicator function restricts the range of $G_\mu(\cdot, c_1, c_2)$ to $[0,1]$. Additionally, we remark that if $E$ is a set with finite perimeter and $u = \chi_E$, $G_\mu(u, c_1, c_2) = ACV_\mu(E, c_1, c_2)$.

On these functionals, we prove existence of a global minimizer of $G_\mu$ through a direct variational method. Then, we show that a truncation of the solution to the AROF functional (i.e. an upper level set of the solution) yields the first component of a minimizer to $G_\mu$. To show that this is indeed the case, we rely on the description of explicit solutions of AROF obtained in [22] for any PCR datum and on the corresponding Euler-Lagrange equations to both functionals. In doing so, we find a global minimizer both of $G$ and of ACV. All the results concerning the 2-phases case are worked out in Section [3].

**Remark 1.** In the 1-dimensional case ($\Omega = [a,b] \subset \mathbb{R}$), it was shown in [24] that there is a minimizer to the CV problem with first component satisfying the following two properties:

(a) The boundary of the set belongs to a sole level set of the datum $f$ in a multivalued sense.

(b) If $f$ is piecewise constant, then the boundary of the set is contained in the jump set of $f$.

The above properties are shown to be false in the anisotropic case as shown in Example[3] as well as in the two–dimensional case of the standard CV model (see [24], Remark 4.2). However, as a by-product of our result, we obtain that, in the case of a PCR datum, the first component of a minimizer to the ACV$\mu$ functional is also a PCR function; i.e the minimizing set is a rectilinear polygon. Moreover, its essential boundary belongs to a grid generated
by the datum itself. This last property permits to design a trivial algorithm to compute a minimizer of the ACV$_\mu$ functional.

In second place, we deal with the $n$-phases anisotropic CV model. In this case, we decide to slightly modify the original (anisotropic) functional, which reads as

$$ACV^n_\mu(\Omega, c) := \sum_{i=1}^n \left( \text{Per}_1(\Omega_i; \Omega) + \mu \int_{\Omega_i} (c_i - f)^2 \, dx \right),$$

with $\mu > 0$, $\Omega := \{\Omega_i\}_{i=1}^n \in \mathcal{P}_n(\Omega)$, where, by $\mathcal{P}_n(\Omega)$ we denote the set of all nonempty $n$-parts (disjoint) partition of $\Omega$ and $c := \{c_i\}_{i=1}^n \in [0, 1]^n$. The proposed modification consists in not counting more than once the length of the possible overlaps of the boundaries of the partition. To do this, we propose the following energy functional:

$$G^n_\mu(\Sigma, c) = \sum_{i=1}^n \left( \text{Per}_1(\Sigma_i; \text{int}(\Sigma_i)) + \mu \int_{\Sigma_i \setminus \Sigma_{i-1}} (c_i - f)^2 \, dx \right),$$

with $\mu > 0$ and $c := \{c_i\}_{i=1}^n \in [0, 1]^n$ and $\Sigma := \{\Sigma_i\}_{i=0}^n \in \mathcal{P}_n^*(\Omega)$, where $\mathcal{P}_n^*(\Omega) := \{\{\Lambda_i\}_{i=0}^n : \emptyset = \Lambda_0 \subseteq \Lambda_i \subseteq \Lambda_{i+1} \subseteq \Lambda_n = \Omega\}$.

We observe that defining $\Omega_i := \Sigma_i \setminus \Sigma_{i-1}$, the unique difference between both functionals is that in ACV$_\mu$ some edges will count more than twice (in the case that an edge belongs to the boundary of more than two different upper level sets) in the length term while in $G^\mu$ they are counted only once.

As in the 2-phases case, the existence of a minimizer $(\Sigma^*, c^*)$ follows from the direct method in calculus of variations (see Proposition 2). Moreover, one can assume that $0 \leq c_{i+1}^* \leq c_i^* \leq 1$ for any $i = 1, \ldots, n - 1$. Next, we observe that

$$\Sigma^* \in \arg\min_{\Lambda \in \mathcal{P}_n^*(\Omega)} G^n_\mu(\Lambda, c^*) \quad \text{with} \quad c_i^* = \frac{1}{|\Sigma_i^* \setminus \Sigma_{i-1}^*|} \int_{\Sigma_i^* \setminus \Sigma_{i-1}^*} f(x) \, dx.$$

In the case that $f \in \text{PCR}(\Omega)$, by the tools developed in [22], we can show that $\Sigma^*$ is a rectilinear polygon whose boundaries lie on the grid generated by $f$. These results concerning the multiphase case are the core of Section 4.

Thirdly, we discuss about a possible relationship between the minimizers of a variant of ACV$_\mu^n$ and the truncated AROF$_\lambda$ functional, from a general point of view. For this purpose, we define the following general ACV$_\mu^n$ variant as

$$CV^n_{\varphi, \mu}(\Omega, c) := \sum_{i=1}^n \left( \text{Per}_\varphi(\cup_{j=1}^i \Omega_i; \Omega) + \mu_i \int_{\Omega_i} (c_i - f)^2 \, dx \right)$$

(3)
where \( \mu := \{ \mu_i \}_{i=1}^n \in [0, +\infty)^n \), \( \Omega := \{ \Omega_i \}_{i=1}^n \in \mathcal{P}_n(\Omega) \) and \( c := \{ c_i \}_{i=1}^n \in [0, 1]^n \). Similarly, the truncated ROF functional for \( n \) phases is defined as

\[
\text{TROF}^n_{\varphi, \lambda}(\Sigma, \tau) := \sum_{i=1}^{n-1} \left( \text{Per}_\varphi(\Sigma_i; \Omega) + \lambda \int_{\Sigma_i} (\tau_i - f) \, dx \right)
\]  

(4)

where \( \lambda > 0, \Sigma := \{ \Sigma_i \}_{i=0}^n \in \mathcal{P}_n^*(\Omega) \) and \( \tau := \{ \tau_i \}_{i=1}^{n-1} \in [0, 1]^{n-1} \). Here, these generalizations depend on a positively 1-homogeneous convex function \(| \cdot |\varphi\). Note that \( CV^n_{\varphi, \mu} = G^n_\mu \) if \(| \cdot |\varphi\) is the 1-norm and \( \mu_i = \mu \) for every \( i \).

In Section 5 we prove that, in the 2-phases case, it is possible to obtain a relationship between the minimizers of \( \text{ACV}^2_{\mu} \) and \( \text{TROF}^2_{1, \lambda} \). However, we prove that a similar relationship in the multiphase case cannot be true, in general. Finally, we remark the relationship between \( \text{TROF}^n_{\varphi, \lambda} \) and \( \text{AROF}_\lambda \) in the anisotropic case.

The paper finishes with some applications of our results. In Section 6 we show the strength of them with some examples on 2-phases and multiphase image segmentation, by comparing them with similar isotropic processes.

2 Preliminaries

Before starting, we prescribe some notations and we provide a basic knowledge on bounded variation functions and anisotropies.

2.1 Notations

Throughout the paper, \( \Omega \subset \mathbb{R}^N \) will denote an open bounded set with boundary \( \partial \Omega \) and \( \nu^\Omega \) will denote the outer unit exterior normal at a point on the boundary, when defined. We denote by \( L^p(\Omega) \) (\( 1 \leq p < +\infty \)), \( L^\infty(\Omega) \) and \( \mathcal{M}(\Omega)^M \) the set of Lebesgue \( p \)-integrable functions in \( \Omega \), the set of essentially bounded measurable functions in \( \Omega \) and the set of finite vector Radon measures on \( \Omega \), respectively. In the case that \( M = 1 \), we omit the index. For any measurable set \( E \subset \mathbb{R}^N \) with respect to the Lebesgue measure in \( \mathbb{R}^N \), we denote by \( |E| \) the Lebesgue measure of \( E \). By \( \mathcal{H}^{N-1} \), we denote the \((N-1)\)-dimensional Haussdorff measure. We denote by \( BV(\Omega) \) the Banach space of functions of bounded variation in \( \Omega \) with the norm defined next:

\[
BV(\Omega) := \{ u \in L^1(\Omega) : Du \in \mathcal{M}(\Omega)^N \},
\]

\[
||u||_{BV(\Omega)} := ||u||_{L^1(\Omega)} + ||Du||_{\mathcal{M}(\Omega)^N},
\]

where \( Du \) is the distributional gradient of \( u \). If \( u \in BV(\Omega) \), then the measure \( Du \) can be decomposed into its absolutely continuous part and singular with
respect to the Lebesgue measure $\mathcal{L}^N$:

$$Du = \nabla u \mathcal{L}^N + D^s u,$$

with $\nabla u$ being the Radon-Nikodym derivative of $Du$ with respect to $\mathcal{L}^N$. We use standard notations and results for bounded variation functions such as those in [3]. In particular, $J_u$ will denote the jump set of $u \in BV(\Omega)$ and $\nu^u$ the Radon-Nikodym of $Du$ with respect to its total variation; i.e. $\nu^u = \frac{Du}{|Du|}$. Moreover, $D^s u$ can be split into the jump part of the measure $D^j u$ and its Cantor part $D^c u$ and

$$D^j u = (u^+ - u^-)\nu^u \mathcal{H}^{N-1}|_{J_u},$$

with $u^+$ and $u^-$ being the approximate limits at a jump point. We will assume the criterion that $\nu^u$ is oriented at $\mathcal{H}^{N-1}$-a.e. point on $J_u$ in such a way that $u^+ > u^-$. We consider the space

$$X_\Omega = \{z \in L^\infty(\Omega; \mathbb{R}^N) : \text{div} z \in L^2(\Omega)\}.$$

In [4, Theorem 1.2], the weak trace on the boundary of a bounded Lipschitz domain $\Omega$ of the normal component of $z \in X_\Omega$ is defined. Namely, it is proved that the formula

$$\langle [z, \nu^\Omega], \rho \rangle := \int_\Omega \rho \text{div} z + \int_\Omega z \cdot \nabla \rho$$

defines a linear operator $[\cdot, \nu^\Omega] : X_\Omega \to L^\infty(\partial\Omega)$ such that

$$\| [z, \nu^\Omega] \|_{L^\infty(\partial\Omega)} \leq \| z \|_{L^\infty(\Omega)},$$

for all $z \in X_\Omega$ and $[z, \nu^\Omega]$ coincides with the pointwise trace of the normal component if $z$ is smooth. From now on, the above definitions will be considered in the special case that $N = 2$.

### 2.2 Anisotropies and Chan-Vese model

Next, we refer about isotropic and anisotropic models. This classification depends on the norm in the total variation used in each model. As follows, we introduce its definition according to [2].

**Definition 1.** Let $\Omega \subset \mathbb{R}^2$ be an open set, $|\cdot|_\varphi : \mathbb{R}^2 \to [0, +\infty]$ be a convex, positively 1-homogeneous function such that $|x|_\varphi > 0$ if $x \neq 0$ and let $u$ be
a function in \( L^2(\Omega) \). Then, the total variation of \( u \) with respect to \(| \cdot |_\varphi\), denoted by \(|D u|_\varphi(\Omega)\), is defined as

\[
|D u|_\varphi(\Omega) := \sup \left\{ \int_\Omega u \text{div} \eta \, dx : \eta \in C^1_c(\Omega; \mathbb{R}^2), |\eta|^\ast_{\varphi} \leq 1 \right\},
\]

where \(| \cdot |^\ast_{\varphi}\) denotes the convex conjugate of \(| \cdot |_{\varphi}\). We note that \(|D u|_\varphi(\Omega) < +\infty\) if and only if \( u \in BV(\Omega) \), in the case that \(| \cdot |_{\varphi}\) is a norm. In this context, we write \( B_{\varphi}(x; R) \) the ball with radius \( R > 0 \) centered at \( x \in \mathbb{R}^N \) with respect to the \(| \cdot |_{\varphi}\) distance. In addition, we denote by \( \text{Per}_{\varphi}(E; \Omega) = |D \chi_E|_{\varphi}(\Omega) \) the \( \varphi\)-perimeter of \( E \) in \( \Omega \), where \( E \subseteq \Omega \). When \( \Omega = \mathbb{R}^N \), we omit it and we simply write \( \text{Per}_{\varphi}(E) \). Finally, for a finite perimeter set \( E \), we denote by \( \partial^* E \) its reduced boundary, according to the notation in [7].

Remark 2. If \(| \cdot |_{\varphi}\) is the Euclidean norm, the definition above coincides with the total variation of the measure \( Du \). The models in which the Euclidean norm is used are called isotropic. Otherwise, we will speak about anisotropies, the models where these are used are called anisotropic. Throughout this work, if the function \(| \cdot |_{\varphi}\) is not explicitly stated, it will refer to the anisotropic model when \(| \cdot |_{\varphi}\) is the 1-norm, i.e. \(|(v_1, v_2)|_1 := |v_1| + |v_2| \) for \((v_1, v_2) \in \mathbb{R}^2\); and the \( \ell_1 \) total variation of \( u \in BV(\Omega) \) will be denoted by \(|D u|_1(\Omega)\).

Next we introduce the Chan-Vese model in a general \(| \cdot |_{\varphi}\) setting:

Definition 2. Let \( \Omega \subset \mathbb{R}^2 \) be an open set, \(| \cdot |_{\varphi}\) be as Def. 1 and \( f \) be an \( L^2(\Omega) \) function such that \( f(\Omega) \subseteq [0, 1] \). We consider the following functional:

\[
\text{CV}_{\varphi, \mu}(\Sigma, c_1, c_2) = \text{Per}_{\varphi}(\Sigma; \Omega) + \mu \left( \int_{\Sigma} (c_1 - f)^2 \, dx + \int_{\Omega \setminus \Sigma} (c_2 - f)^2 \, dx \right),
\]

where \( \Sigma \subseteq \Omega \), \( c_1, c_2 \in [0, 1] \) and \( \mu > 0 \). The associated model to the above functional consists in finding a 3-tuple \((\Sigma, c_1, c_2)\) such that

\[
(\Sigma, c_1, c_2) \in \arg \min_{\Lambda \subseteq \Omega \atop s_1, s_2 \in [0, 1]} \text{CV}_{\varphi, \mu}(\Lambda, s_1, s_2) \quad \text{for a fixed } \mu.
\]

According to this definition, the CV and ACV models correspond to \( \text{CV}_{\varphi, \mu} \) model when \(| \cdot |_{\varphi} = | \cdot |_2\) and \(| \cdot |_{\varphi} = | \cdot |_1\), respectively; and their functionals are denoted by \( \text{CV}_\mu \) and \( \text{ACV}_\mu \), respectively.

2.3 PCR functions and ROF problems

Definition 3. Let \( \Omega \subset \mathbb{R}^2 \) be an open set, \(| \cdot |_{\varphi}\) be as Def. 1 and \( f \) be an \( L^2(\Omega) \) function. The \( \varphi\)-ROF functional is defined as follows:

\[
\text{ROF}_{\varphi, \lambda}(u) = |D u|_{\varphi}(\Omega) + \frac{\lambda}{2} \int_\Omega (u - f)^2 \, dx,
\]

(5)
where \( u \in L^2(\Omega) \) and \( \lambda > 0 \). The associated model to the above functional consists in finding the minimum of \( \text{ROF}_{\varphi, \lambda} \), i.e., solving the problem

\[
\min_{u \in L^2(\Omega)} \text{ROF}_{\varphi, \lambda}(u)
\]

The minimizer, which we denote by \( w_\lambda \), satisfies the associated Euler-Lagrange equation to \( \text{ROF}_{\varphi, \lambda} \), which is the next one:

\[
-\text{div}(z_{w_\lambda}) = \lambda (w_\lambda - f),
\]

with \( z_{w_\lambda} \in \partial|Dw_\lambda|_\varphi \) (here the symbol \( \partial \) denotes the subdifferential of the anisotropic total variation; see [23] for a characterization). According to these definitions, we denote by \( \text{ROF}_\lambda \) and \( \text{AROF}_\lambda \) models those \( \text{ROF}_{\varphi, \lambda} \) ones where \( |\cdot|_\varphi \) are the Euclidean norm and 1-norm, respectively.

As we work with the AROF model when \( f \) belongs to a particular family of functions, \( \text{PCR}(\Omega) \), we next introduce the definition of this space.

**Definition 4.** Let \( \Omega \) be a rectangle. We say that \( w \) is piecewise constant on rectangles and we write \( w \in \text{PCR}(\Omega) \) if \( w \) has a finite number of level sets of positive \( L^2 \) measure, and each one is a rectilinear polygon up to an \( L^2 \)-null set.

In order to work with the results introduced in [22], we need to present some notation used in that paper. These terms are introduced now.

**Definition 5.** Let \( \Omega \) be a bounded rectangle in \( \mathbb{R}^2 \). We denote by \( G \) any set of horizontal and vertical lines on \( \Omega \), which we will call a grid on \( \Omega \), and any of the rectangles in the partition generated by the grid we will call it cell. Let \( F \) be a rectangular polygon contained in \( \Omega \) and let \( f \in \text{PCR}(\Omega) \). We denote

- by \( \mathcal{F}(G) \) the set of all rectangular polygons which satisfy that each one of their sides is a segment of adjacent vertices of \( G \).

- by \( G(F) \) the minimal grid such that each side of \( F \) is contained in one line of \( G \), by \( \mathcal{Q}_f \) the partition of \( \Omega \) provided by the level-sets of \( f \) and by \( G_f \) the grid \( \bigcup_{\Sigma \in \mathcal{Q}_f} G(\Sigma) \).

It is immediate to observe that \( \text{PCR}(\Omega) \subset BV(\Omega) \). Moreover, \( J_u = \bigcup_{F \in \mathcal{Q}_u} \partial F \) for \( u \in \text{PCR}(\Omega) \). In [22], the subdifferential of the anisotropic total variation (in the case that \( |\cdot|_\varphi = |\cdot|_1 \)) on PCR functions was characterized as follows:
Lemma 1. Let $f \in \text{PCR}(\Omega)$. Then, $v \in \partial |Df|_1$ if and only if $v \in L^2(\Omega)$ and there exists $z \in X_{\Omega}$ such that $v = -\nabla z$, $||z||_\infty \leq 1$ a.e. and

$$[z, \nu^\Omega] = 0, \quad [z, \nu^F](x) = \begin{cases} -1 & \text{if } f|_F < f|_{F'}, \ x \in \partial F \cap \partial F' \\ 1 & \text{if } f|_F > f|_{F'}, \ x \in \partial F \cap \partial F' \end{cases},$$

with $F \neq F' \in Q_f$.

Given $f \in \text{PCR}(\Omega)$, in [22, Theorem 5] a minimizer $w_\lambda$ of AROF$_\lambda$ was obtained via a finite algorithm. In particular, it can be constructed as a PCR(\Omega) function with all level sets $\{F_i\}_{i=1}^l \subseteq \mathcal{F}(G_f)$. Moreover, $w_\lambda|_{F_k} > w_\lambda|_{F_{k+1}},$

$$\per_1(\{w_\lambda > \tau\}; \Omega) + \lambda \int_{\{w_\lambda > \tau\}} (w_\lambda - f) \, dx = 0, \quad \text{for any } 0 < \tau < 1,$$

and, by (6) and Lemma 1 we obtain that there exists $z_{w_\lambda} \in \partial |Dw_\lambda|_1$ such that

$$-\nabla(z_{w_\lambda}) = \lambda(w_\lambda - f)$$

$$[z_{w_\lambda}, \mathcal{L}(w_\lambda > \tau)] = -1 \quad \text{for any } 0 < \tau < 1.$$

Observe that, thanks to Lemma 1, we obtain that

$$\partial |Dw_\lambda|_1 \subseteq \partial |D\chi_{\{w_\lambda > \tau\}}|_1, \quad \text{for any } 0 < \tau < 1.$$

3 2-phases ACV model

In this section we prove the existence of a PCR minimizer of the 2-phases anisotropic Chan-Vese problem and its relationship with solutions to the AROF problem.

Definition 6. Let $\Omega \subset \mathbb{R}^2$ be an open set and let $f \in L^2(\Omega)$. Then, we define the generalized functional of ACV$_\mu$ as follows: $G_\mu : L^2(\Omega) \times [0, 1]^2 \to [0, +\infty[,$

$$G_\mu(u, c_1, c_2) = |Du|_1(\Omega) + \mu \int_{\Omega} \left(u(c_1 - f)^2 + (1 - u)(c_2 - f)^2\right) dx + \int_{\Omega} I_{[0,1]}(u),$$

where $\mu > 0$. Concerning the generalised functional proposed above, we note that the partition $\{\Lambda, \Omega \setminus \Lambda\}$ has been relaxed by $u$ and $1 - u$, providing a convex behaviour in terms of $u$. This function $u$ has a range restriction on $[0,1]$, imposed by the indicator function of the interval $[0,1]$;

$$I_{[0,1]}(x) := \begin{cases} 0 & \text{if } x \in [0,1] \\ +\infty & \text{otherwise.} \end{cases}$$
The associated model to the above functional consists in finding a 3-tuple \((u, c_1, c_2)\) such that

\[
(u, c_1, c_2) \in \arg \min_{(w, a, b) \in [0,1]^2} G_{\mu}(w, a, b).
\]

**Proposition 1.** There exists \((u, c_1, c_2) \in BV(\Omega) \times [0,1]^2\) such that it is minimizer of \(G_{\mu}\) and \(u(\Omega) \subseteq [0,1]\).

**Proof.** The proof easily follows from the direct method in the calculus of variations. We give a sketch of it by the sake of completeness. Since \(G_{\mu} \geq 0\) in \(L^2(\Omega) \times [0,1] \), there exists \(m = \inf \{G_{\mu}(w) : w \in L^2(\Omega) \times [0,1]\}\). We take a minimizing sequence \(\{(u_n, c_{1,n}, c_{2,n})\}_{n \in \mathbb{N}}\). Since \(u_n(\Omega) \subseteq [0,1]\), then there is a subsequence (not relabeled) and \(u \in L^2(\Omega)\) such that \(u_n \rightharpoonup u\) weakly in \(L^2(\Omega)\) and that \(u(x) \in [0,1]\) a.e. in \(\Omega\). Moreover, from the lower semicontinuity of the anisotropic total variation and the coercivity of the \(|\cdot|_1\) norm, we may assume that \(u_n \to u\) in \(L^1(\Omega)\). On the other hand, since \(c_{i,n} \in [0,1]\), for \(i = 1,2\), then there is a subsequence (not relabeled) and \(c_i \in [0,1]\) such that \(c_{i,n} \to c_i\) for \(i = 1,2\). Finally, from the lower semicontinuity of the anisotropic total variation we get that

\[
G_{\mu}(u, c_1, c_2) \leq \lim \inf G_{\mu}(u_{n}, c_{1,n}, c_{2,n}) = m,
\]

which shows that \((u, c_1, c_2)\) is a minimizer. \(\square\)

We now fix \(0 \leq c_2 < c_1 \leq 1\) and consider \(G_{\mu,c_1,c_2} : L^2(\Omega) \to [0, +\infty]\) defined by

\[
G_{\mu,c_1,c_2}(u) := G_{\mu}(u, c_1, c_2).
\]  

Observe that, in the case that \(c_1 = c_2\), the only minimizers are constant functions.

**Theorem 1.** Let \(0 \leq c_2 < c_1 \leq 1\) and let \(w\) be the minimizer of (5) with parameter \(\lambda = \mu(c_1 - c_2)\). Then,

\[
u = \chi_{\Sigma} \quad \text{s.t.} \quad \Sigma := \left\{ x \in \Omega : w(x) > \frac{c_1 + c_2}{2} \right\}
\]

is a minimizer of \(G_{\mu,c_1,c_2}\).

**Proof.** First of all, we observe that the functional \(G_{\mu,c_1,c_2}\) is convex and lower semicontinuous in \(L^2(\Omega)\). Therefore, the existence of a minimizer \(u \in BV(\Omega)\) to (7) is guaranteed. Moreover, applying [21, Theorem 2.1], its Euler-Lagrange equation is given by

\[
\text{div} z_u = \mu(c_1 - c_2)(c_1 + c_2 - 2f) + h,
\]  

(8)
where \( z_u \in \partial |Du|_1 \) and \( h \in \partial \mathbb{I}_{[0,1]}(u) \).

On the other hand, as \( w \) is minimizer of (5), the Euler-Lagrange equation with respect to this functional is fulfilled, i.e., we have

\[
\text{div} \ z_w = \mu(c_1 - c_2)(w - f),
\]

where \( z_w \in \partial |Dw|_1 \subset \partial |Du|_1 \) by (2.3). Thus, we let \( z_u := z_w \in \partial |Du|_1 \) and we will prove that (8) is satisfied by \( z_w \). Proving this, we show that \( u \) is a minimizer of \( G_{\mu,c_1,c_2} \).

From (9), we get that (8) is satisfied if, and only if

\[
\mu(c_1 - c_2) \left( w - \frac{c_1 + c_2}{2} \right) = h.
\]

Therefore, we only need to show that \( h \in \partial \mathbb{I}_{[0,1]}(u) \). We distinguish two cases:

- If \( w < \frac{c_1 + c_2}{2} \), we have that \( u = 0 \) and \( h \in ]-\infty,0] = \partial \mathbb{I}_{[0,1]}(0) \).
- If \( w \geq \frac{c_1 + c_2}{2} \), we have that \( u = 1 \) and \( h \in [0,\infty[ = \partial \mathbb{I}_{[0,1]}(1) \).

Therefore, \( u \) satisfies (8) and the proof is finished.

\[\square\]

**Remark 3.** Let \( w \) be a minimizer of (5). Since each coordinate of a minimizer of \( G_{\mu} \) fulfills its Euler-Lagrange equation, if

\[
c_1 = \frac{\int_{\Sigma} f \, dx}{|\Sigma|}, \quad c_2 = \frac{\int_{\Omega \setminus \Sigma} \chi_{R \setminus \Sigma}}{|\Omega \setminus \Sigma|} \quad \text{where} \quad \Sigma := \left\{ x \in \Omega : w(x) \geq \frac{c_1 + c_2}{2} \right\},
\]

then, \((\chi_{\Sigma}, c_1, c_2)\) is a minimizer of \( G_{\mu} \) if \( \mu = \lambda/(c_1 - c_2) \). The existence of such a triplet \((\Sigma, c_1, c_2)\) satisfying (10) is guaranteed by Proposition 1 and Theorem 1.

Now we conclude with the existence and characterization of some minimizers of \( ACV_{\mu} \):

**Theorem 2.** Let \( w \) be the minimizer of \( \text{AROF}_\lambda \). Then, there is \( 0 \leq c_2^* < c_1^* \leq 1 \) verifying (10), \((\chi_{\Sigma}, c_1^*, c_2^*)\) is a minimizer of \( G_{\mu} \) and \((\Sigma, c_1^*, c_2^*)\) is a minimizer of \( ACV_{\mu} \) if \( \mu = \lambda/(c_1^* - c_2^*) \).

**Proof.** First, we note that it is obvious that, given \( E \subset \Omega \) of finite perimeter,

\[
G_{\mu}(\chi_E, c_1, c_2) = ACV_{\mu}(E, c_1, c_2).
\]
Then,
\[
\min_{u \in L^2(\Omega) \atop c_1, c_2 \in [0, 1]} G_\mu(u, c_1, c_2) \leq \min_{\Lambda \subset \Omega \atop c_1, c_2 \in [0, 1]} \text{ACV}_\mu(\Lambda, c_1, c_2) \tag{11}
\]
On the other hand, we know that there exist minimizers of $G_\mu$ by Proposition 1 and each coordinate of each minimizer satisfies its respective Euler-Lagrange equation. Let us denote by $(u^*, c_1^*, c_2^*)$ one minimizer. By Theorem 1, $\Sigma$ satisfies
\[
G_\mu(u^*, c_1^*, c_2^*) = G_\mu(\chi_{\Sigma}, c_1^*, c_2^*) \geq \min_{c_1, c_2 \in [0, 1]} \text{ACV}_\mu(\chi_{\Sigma}, c_1, c_2).
\]
Therefore, $(c_1^*, c_2^*)$ fulfill the equalities in Remark 3. Finally, (11) and (3) imply that
\[
\min_{\Lambda \subset \Omega \atop c_1, c_2 \in [0, 1]} \text{ACV}_\mu(\Lambda, c_1, c_2) \leq G_\mu(\chi_{\Sigma}, c_1^*, c_2^*) \leq \min_{\Lambda \subset \Omega \atop c_1, c_2 \in [0, 1]} \text{ACV}_\mu(\Lambda, c_1, c_2),
\]
thus showing that $(\Sigma, c_1^*, c_2^*)$ is a minimizer of $\text{ACV}_\mu$.

**Example 1.** Next, we show that the properties described in Remark 1 are not satisfied in the anisotropic 2-dimensional case either, using the previous result and the characterization of AROF$\lambda$’s solutions in [22]. For that, we provide a counterexample.

Let us consider $\Omega = [-1, 1]^2$ and $f = \chi_{A_1 \cup A_2}$ with

\[
A_1 = [-1/2, 1/2]^2, \quad A_2 = [1/4, 1/2] \times [1/2, 3/4].
\]

According to [22, Theorem 5], the AROF$_{16}$ minimizer in this case is

\[
w = \frac{3}{4} \chi_{A_1} + \frac{1}{2} \chi_{A_2} + \frac{9}{94} \chi_{\Omega \setminus (A_1 \cup A_2)}.
\]

By Theorem 3, we know that $(A_1, 1, 1/48)$ is a minimizer of ACV$_{768/47}$, whose jump set is not contained in the jump set of $f$ (and, therefore, in a sole level set of $f$).

Finally, in algorithm 1 we present the classical alternate algorithm that will allow us to approximate a solution of the ACV model, depending on a tolerance term $\varepsilon_{\text{tol}}$ and an iteration maximum $n_{\text{max}}$. 

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Algorithm 1: ACV approximate minimizer

Initiation: \( k = 0, \mu > 0, (c_1, c_2) \in [0,1] \) s.t. \( c_1 > c_2 \).

\( w_0 \leftarrow \text{AROF}_{\mu(c_1-c_2)} \) minimizer.

\( \Lambda \leftarrow \text{ACV}_\mu \) minimizer via Theorem 1.

\[ \text{while } |\Lambda_k \triangle \Lambda_{k-1}| > \varepsilon_{\text{tol}} \land k < n_{\text{max}} \text{ do} \]

\[ c_1 \leftarrow \frac{1}{|\Lambda_k|} \int_{\Lambda_k} f \, dx, \quad c_2 \leftarrow \frac{1}{|\Omega \setminus \Lambda_k|} \int_{\Omega \setminus \Lambda_k} f \, dx, \quad k \leftarrow k + 1 \]

\( w_k \leftarrow \text{AROF}_{\mu(c_1-c_2)} \) minimizer

\( \Lambda_k \leftarrow \text{ACV}_\mu \) minimizer via Theorem 1

end while

return \((\Lambda_k, c_1, c_2)\)

Observe that we are imposing the condition \( c_1 > c_2 \) at each step. This condition is guaranteed if initially it is so. Since the proof of this fact is a special case of the multiphase case, we refer to Proposition 3.

4 Multiphase ACV model

This section is devoted to the study of the anisotropic multiphase Chan-Vese model. First of all, we prove that there exists a minimizer to (2).

Proposition 2. Let \( \Omega \subset \mathbb{R}^2 \) be an open set and \( f \in L^2(\Omega) \). Then, there exists \( \Sigma \in \mathcal{P}_n^*(\Omega) \) and \( c \in [0,1]^n \) such that

\[ (\Sigma, c) \in \arg \min_{\Lambda \in \mathcal{P}_n^*(\Omega)} \min_{a \in [0,1]^n} G^n_\mu(\Lambda, a). \]

Proof. The proof is very similar to the proof of Proposition 1. In particular, for a minimizing sequence \((\Sigma^k, c^k) = ((\Sigma^k_0, \ldots, \Sigma^k_n); (c^k_1, \ldots, c^k_n))\), we can take \( u^k_i := \chi_{\Sigma^k_i} \) and work exactly as in Proposition 1. Finally, from the convergence a.e. obtained from the lower semicontinuity of the anisotropic total variation, we conclude that the weak limits in \( L^2(\Omega) \), \( u_i \) are of the form \( u_i = \chi_{\Sigma_i} \) and that \( \Sigma := (\Sigma_0, \ldots, \Sigma_n) \in \mathcal{P}_n^*(\Omega) \). The rest of the proof is totally analogous and we omit it.

Theorem 3. If \( c = \{c_i\}_{i=1}^n \) satisfies \( 1 \geq c_i \geq c_{i+1} \geq 0 \), then there exists a minimizer \( \Sigma \) to \( G^n_\mu(\cdot, c) \) for any initial datum \( f \in \text{PCR}(\Omega) \) and it satisfies that each one of its components belongs to \( F(G_f) \); i.e: \( \Sigma \) is a rectangular partition of \( \Omega \) whose components have their boundary on \( G_f \).

Proof. First of all, we rewrite \( G^n_\mu(\cdot, c) \) as follows:

\[ G^n(\Sigma) := G^n_\mu(\Sigma, c) = \sum_{i=1}^{n-1} \left( \text{Per}_1(\Sigma_i; \text{int}(\Sigma_{i+1})) + \mu \int_{\Sigma_i} g_{c_i, c_{i+1}} \, dx \right). \]

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where \( g_{c_i,c_{i+1}}(x) = (c_i - c_{i+1})(c_i + c_{i+1} - 2f(x)) \).

We now proceed as in \[22, Lemma 2\]. Given \( \Sigma \in P^*_n(\Omega) \), and \( \varepsilon > 0 \), we will construct \( \Sigma^* \in P^*_n(\Omega) \) such that \( \Sigma^*_i \in \mathcal{F}(G_f) \) and

\[ G^n(\Sigma^*) < G^n(\Sigma) + \varepsilon. \]

We divide the proof into three steps:

1. **Smoothing**: We will construct a variant of \( \Sigma \), denoted by \( \tilde{\Sigma} \) such that \( \tilde{\Sigma} \in P^*_n(\Omega) \) and their components have smooth boundaries. For that, we define \( \tilde{\Sigma}_{i,\delta} \) as

\[ \tilde{\Sigma}_{i,\delta} = \{ x \in \Omega : \psi_{\delta} \ast \chi_{\Sigma_i} \geq t \}, \]

where \( \psi_{\delta} \) is a standard smooth approximation of unity, \( \delta \) is a positive parameter close to 0 and \( t \in (0, \frac{1}{2}) \). As a consequence of this definition and following the idea of \[22, Lemma 2\], we know that \( \tilde{\Sigma}_{i,\delta} \) is smooth and it is possible to set values \( t \) and \( \delta \) such that

\[ G^n(\tilde{\Sigma}) < G^n(\Sigma) + \varepsilon, \]

with \( \tilde{\Sigma} = \{ \tilde{\Sigma}_i \}_{i=1}^{n-1} \in P^*_n(\Omega) \).

2. **Squaring**: Next, we will construct a variant of \( \tilde{\Sigma} \), denoted by \( \hat{\Sigma} \), such that \( \hat{\Sigma} \in P^*_n(\Omega) \) and their components have rectangular boundaries. For this end, we will apply a squaring process on the components of \( \tilde{\Sigma} \) in each cell of \( G_f \), avoiding an increase of the energy at \( G^n \). So, we consider a closed cell \( C \) of \( G_f \) and we suppose that \( C^o \cap (\cup_{i=1}^{n-1} \partial \tilde{\Sigma}_i) = C^o \cap (\cup_{i=m_1}^{m_2} \partial \hat{\Sigma}_i) \). We split \( \{ \Sigma_i \}_{i=m_1}^{m_2} \) into those sets whose index satisfies \( g_{c_i,c_{i+1}} > 0 \) in the cell \( C \) and the rest, denoting them by \( \{ \hat{\Sigma}_i \}_{i=m_1}^{b-1} \) and \( \{ \hat{\Sigma}_i \}_{i=m_2}^{b_2} \) respectively, providing a well defined division thanks to the condition \( c_i \geq c_{i+1} \), and the fact that \( f \) is constant in \( C^o \).

First, we work with the sets \( \hat{\Sigma}_i \in \{ \hat{\Sigma}_i \}_{i=m_1}^{m_2} \). Following the idea of \[22, Lemma 2, Step 2\], we construct a covering \( W^i \) of \( \partial(\hat{\Sigma}_i \cap C^o) \), made up of squares, such that

\[ \text{Per}_1(\hat{\Sigma}_i \cup W^i; \text{int}(\hat{\Sigma}_{i+1})) \leq \text{Per}_1(\hat{\Sigma}_i; \text{int}(\hat{\Sigma}_{i+1})) \text{ and } L^2(\hat{\Sigma}_i \cup W^i) \geq L^2(\hat{\Sigma}_i) \]

(12)

Now, we apply a similar argument on \( \{ \hat{\Sigma}_i \}_{i=m_1}^{b-1} \). In this case, we repeat the previous approach on \( \Sigma_{i+1} \setminus \hat{\Sigma}_i \in \{ \Sigma_{i+1} \setminus \hat{\Sigma}_i \}_{i=m_1}^{b-1} \), defining a cover
$W^i$ such that (12) is fulfilled replacing $\tilde{\Sigma}_i$ by $\tilde{\Sigma}_{i+1} \setminus \tilde{\Sigma}_i$. As a result, we have that
\[
\text{Per}_1(\tilde{\Sigma}_i \setminus W^i; \text{int}(\tilde{\Sigma}_{i+1})) \leq \text{Per}_1(\tilde{\Sigma}_i; \text{int}(\tilde{\Sigma}_{i+1})) \quad \text{and} \quad L^2(\tilde{\Sigma}_i \setminus W^i) \leq L^2(\tilde{\Sigma}_i)
\]
(13)

Then, we define
\[
\hat{\Sigma}_c = \{\hat{\Sigma}_c^i\}_{i=1}^n := \begin{cases} 
\tilde{\Sigma}_i \setminus W^i & \text{if } i \in \{m_1, \ldots, i_0 - 1\} \\
\tilde{\Sigma}_i \cup W^i & \text{if } i \in \{i_0, \ldots, m_2\} \\
\Sigma_i & \text{otherwise}
\end{cases}
\]
which satisfies that $G^n(\hat{\Sigma}_c) \leq G^n(\Sigma)$ given the behaviour of $g_{c,\epsilon_i+1}$ at each index $i$ and the above inequalities (12) and (13). Moreover, their components satisfy that their boundaries are rectangular at cell $C$. Adjusting the covers $W^i$, we assure the inclusion condition $\hat{\Sigma}_c^i \subset \hat{\Sigma}_c^{i+1}$.

If we repeat this process on each cell of $G_f$ adding to the changes on $\tilde{\Sigma}$, we define a collection $\hat{\Sigma} \in \mathcal{P}^*_n(\Omega)$ such that their components have rectangular boundaries and
\[
G^n(\hat{\Sigma}) \leq G^n(\Sigma).
\]

3. **Aligning**: All in a row, we will define a collection, denoted by $\Sigma^* \in \mathcal{P}^*_n(\Omega)$, whose components are rectilinear polygons on $F(G_f)$. For that, we modify the boundaries of the components of $\hat{\Sigma}$ averting an increase of the energy at $G^n$.

First, for each $i = 1, \ldots, n - 1$, we consider the minimal grid, denoted by $G^i$, which contains $G_f$ and $\partial \hat{\Sigma}_{i+1}$. Following a similar strategy to [22, Lemma 2, Step 3], we modify $\hat{\Sigma}_1$ transporting segments of $\partial \hat{\Sigma}_1$ into $G^1$ while the variation of area and $\ell$--perimeter is controlled. This process, carried out segment by segment, provides a new $\hat{\Sigma}_{(1),1} \subset \Sigma_2$ such that $\partial \hat{\Sigma}_{(1),1} \subset G^1$ and $G^n(\hat{\Sigma}_{(1)}) \leq G^n(\hat{\Sigma})$, where $\hat{\Sigma}_{(1)}$ is equal to $\hat{\Sigma}$ with the exception of first element, which is replaced by $\hat{\Sigma}_{(1),1}$.

Now, we take $\Sigma_2$ and we repeat the previous method but with a slight tweak. In this case, if we move a segment $s$ of $\partial \Sigma_2$ such that $s \subset \partial \hat{\Sigma}_{(1),1}$, we also modify the respective part of the boundary of $\partial \hat{\Sigma}_{(1),1}$ in consequence. Then, this procedure gives us $\Sigma_{(2),1}$ and $\Sigma_{(2),2}$, variants of $\Sigma_{(1),1}$ and $\Sigma_2$, such that $\partial \hat{\Sigma}_{(2),1}, \partial \hat{\Sigma}_{(2),2} \subset G^2$ and $G(\hat{\Sigma}_{(2)}) \leq G(\hat{\Sigma})$, where $\hat{\Sigma}_{(2)}$ is equal to $\hat{\Sigma}$ with the exception of first two elements, which are replaced by $\hat{\Sigma}_{(2),1}$ and $\hat{\Sigma}_{(2),2}$. In addition, we note that the previous inclusion
is fulfilled by the condition $c_i > c_{i+1}$. Rehashing this scheme for each $i$ up to $n - 1$, we have defined a collection $\hat{\Sigma}_{(n-1)}$ whose components have their boundaries in $G^{n-1}$. Moreover, since $\partial \hat{\Sigma}_n = \partial \Omega \subset G_f$, we get $G^{n-1} = G_f$. As a result, $\Sigma^* := \{\Sigma^*_i\}_{i=1}^n := \hat{\Sigma}_{(n-1)} \in \mathcal{P}^*_n(\Omega)$ is a collection which satisfies that

$$G^n(\Sigma^*) \leq G^n(\Sigma) + \varepsilon \quad \text{and} \quad \Sigma^*_i \subset \mathcal{F}(G_f) \quad \text{for each } i.$$ 

The proof finishes as in [22, Theorem 3] and thus, we omit it. □

**Remark 4.** Observe that given $(\tilde{\Sigma}, c)$ a minimizer to $G^n_\mu$, then $\tilde{\Sigma}$ is a minimizer to $G^n_\mu(\cdot, c)$ and that $c$ can be reordered in such a way that $1 \geq c_i \geq c_{i+1} \geq 0$ for all $i = 1, \ldots, n - 1$. Therefore, by Theorem 3, we obtain that there exists $\Sigma \in \mathcal{P}^*_n(\Omega)$ such that $\Sigma$ is a rectangular partition of $\Omega$ whose components have their boundary on $G_f$ and $(\Sigma, c)$ is also a minimizer to $G^n_\mu$.

We next show that the classical two step algorithm also leads to a PCR function whose components have boundaries in $G_f$, at any iteration. The algorithm reads as follows:

**Algorithm 2:** $G^n_\mu$ approximate minimizer.

| Initiation: | $k = 0$, $\mu > 0$, $c = \{c_i\}_{i=1}^{n-1} \in [0,1]^{n-1}$ s.t. $c_i > c_{i+1}$.
| $\Sigma^0 \leftarrow G^n_\mu(\cdot, c)$ PCR minimizer via Theorem 3 |
| while $\sum_{i=0}^n |\Sigma_i - \Sigma_{i-1}|^2 > \varepsilon_{\text{tol}} \land k < n_{\text{max}}$ do |
| $c_i \leftarrow \frac{1}{|\Sigma_i \setminus \Sigma_{i-1}|} \int_{\Sigma_i \setminus \Sigma_{i-1}} f \, dx$, $k \leftarrow k + 1$ |
| $\Sigma^k \leftarrow G^n_\mu(\cdot, c)$ PCR minimizer via Theorem 3 |
| end while |
| return $(\Sigma^k, c)$ |

In order to be able to apply Theorem 3 we only need to show next result.

**Proposition 3.** Let $c = \{c_i\}_{i=1}^n$ satisfy $1 \geq c_i \geq c_{i+1} \geq 0$, and let $\Sigma$ be the PCR($f$) minimizer of $G^n_\mu(\cdot, c)$ obtained in Theorem 3. Then, $\tilde{c}_i > \tilde{c}_{i+1}$, for $i = 1, \ldots, n - 1$ with $\tilde{c}_i$ defined as

$$\tilde{c}_i := \frac{1}{|\Sigma_i \setminus \Sigma_{i-1}|} \int_{\Sigma_i \setminus \Sigma_{i-1}} f(x) \, dx.$$ 

**Proof.** First of all, note that, given $\Sigma_{i-1}$ and $\Sigma_{i+1}$, $\Sigma_i$ is a minimizer of

$$\text{Per}_1(\Sigma_{i-1}, \text{int}(E)) + \text{Per}_1(E, \text{int}(\Sigma_{i+1})) + \mu(c_i - c_{i+1}) \int_E (c_i + c_{i+1} - 2f) \, dx,$$
for all sets $\Sigma_{i-1} \subset E \subset \Sigma_{i+1}$. Note that the minimizer does not depend on
the behaviour of $f$ on $\Sigma_{i-1}$ and on $\Sigma_{i+1}$. Therefore, $\Sigma_i \setminus \Sigma_{i-1}$ minimizes in
$\Sigma_{i+1} \setminus \Sigma_{i-1}$, the following functional:

$$\text{Per}_1(E, \text{int}(\Sigma_{i+1} \setminus \Sigma_{i-1})) + \mu(c_i - c_{i+1}) \int_E (c_i + c_{i+1} - 2f) \, dx. \quad (14)$$

Since $\emptyset$ is admissible, and $\Sigma_i \setminus \Sigma_{i-1}$ is a minimizer, it follows that

$$\int_{\Sigma_i \setminus \Sigma_{i-1}} (c_i + c_{i+1} - 2f) \, dx \leq 0.$$  

In case of equality, the perimeter is 0 and then, $\Sigma_i$ coincides either with $\Sigma_{i-1}$ or $\Sigma_{i+1}$ and that phase will be removed passing from $n$– to $(n-1)$–phases. Therefore, we can suppose that

$$\tilde{c}_i > \frac{c_i + c_{i+1}}{2}$$

Therefore, if $\tilde{c}_i \leq \tilde{c}_{i+1}$, it follows that

$$\int_{\Sigma_{i+1} \setminus \Sigma_i} (c_i + c_{i+1} - 2f) \, dx < 0.$$  

This implies that $\Sigma_{i+1} \setminus \Sigma_{i-1}$ has strictly less energy in (14) than $\Sigma_i \setminus \Sigma_{i-1}$, a contradiction. \qed

5 Relationship between ACV and Truncated AROF models

This section is aimed at showing the relation between $\text{CV}^n_{\varphi, \mu}$ and $\text{TROF}^n_{\varphi, \lambda}$
minimizers with respect to $|\cdot|_{\varphi}$, functionals defined in (3) and (4), respectively. Throughout this section, $\text{TROF}^n_{\lambda}$ will denote the anisotropic version
of $\text{TROF}^n_{\varphi, \lambda}$.

In the anisotropic 2-phases case, if $\mu$ is a constant vector, we have that
the minimizer of $\text{TROF}^2_{\lambda}((\emptyset, \cdots, \Omega), \tau)$ is provided by the upper level set
$\Sigma_{\tau} := \{x \in \Omega : w_{\lambda}(x) > \tau\}$ where $w_{\lambda}$ is the solution of $\text{AROF}_{\lambda}$, by analogy
on the result [12, Proposition 2.6]. In addition, by Theorem 2, we have that

**Corollary 1.** Let $w$ be the minimizer of $\text{AROF}_{\lambda}$ and $\Sigma_{\tau}$ the $w$’s level set solution of
$\text{TROF}^2_{\lambda}((\emptyset, \cdots, \Omega), \tau)$ such that $\tau$ is defined as

$$\tau = \frac{c_1 + c_2}{2}, \text{ with } c_1 := \frac{\int_{\Sigma_{\tau}} f \, dx}{|\Sigma_{\tau}|}, \quad c_2 := \frac{\int_{\Omega \setminus \Sigma_{\tau}} f \, dx}{|\Omega \setminus \Sigma_{\tau}|}. \quad (15)$$

Then, $(\Sigma_{\tau}, c_1, c_2)$ is a minimizer of $\text{ACV}_{\mu}$ if $\lambda = \mu(c_1^* - c_2^*)$ and $(\chi_{\Sigma_{\tau}}, c_1, c_2)$ is a minimizer of $\mathcal{G}_\mu$. 

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In view of this result, which connects $AROF_\lambda$, $ACV_\mu$ and the anisotropic $TROF_\lambda$, it is relevant to propose similar relationships on the multiphase case. However, as we will show, the relationship between $CV_\lambda(\Omega, \mu)$ and $TROF_\lambda$, presented in [11] in the isotropic case, does not hold, in general. The connection is presented with respect to a generic anisotropy $| \cdot |_\phi$ next:

**Relationship $CV^n_{\phi, \mu}$ - $TROF^n_{\phi, \lambda}$**: Let $(\Sigma, \tau)$ be a pair in $P_n^s(\Omega) \times [0, 1]^n$ such that

$$TROF^n_{\phi, \lambda}(\Sigma, \tau) \leq TROF^n_{\phi, \lambda}(\Lambda, \tau)$$

for any feasible $\Lambda \in P_n^s(\Omega)$; and

$$\tau_i = \frac{c_i + 1 + c_i}{2} \text{ with } c_i := \int_{\Omega_i} f \, dx, \quad \Omega_i := \Sigma \setminus \Sigma_{i-1},$$

for each $i \in \{1, \ldots, n-1\}$. Provided that $c_i > c_{i+1}$, we define $\mu := \{\mu_i\}_{i=1}^n$ as follows

$$\mu_1 = \frac{\lambda}{2(c_1 - c_2)}, \quad \mu_n = \frac{\lambda}{2(c_n - c_{n-1})},$$

$$\mu_i = \frac{\lambda(c_{i-1} - c_{i+1})}{2(c_{i-1} - c_i)(c_i - c_{i+1})} \text{ for } 1 < i < n,$$ (16)

The possible relationship is this one: Letting $\Omega := \{\Omega_i\}_{i=1}^n$ and $c := \{c_i\}_{i=1}^n$, is it true that

$$CV^n_{\phi, \mu}(\Omega, c) \in \arg \min_{\Lambda \in P_n^s(\Omega)} CV^n_{\phi, \mu}(\Lambda, c)?$$ (17)

To show that this is not the case, we need some preliminary results. These results are an adaptation of the results in [5, Section 9]. Since our aim is to show some very particular examples, we only consider some specific settings.

**Definition 7.** Let $A \subset \mathbb{R}^2$ be a bounded open set with Lipschitz boundary. We say that $A$ is calibrable with respect to the anisotropy $\phi$ if there exists $z \in X_A$ such that $|z|_\phi \leq 1$ a.e. in $A$, div $z$ is constant in $A$ and $[z, \nu^A] = -|\nu^A|_\phi$ at $\partial A$.

**Lemma 2.** Let $A \subset \mathbb{R}^2$ be a bounded open set with Lipschitz boundary, and let $\Omega := B_{\phi^*}(0; R)$ be such that $\overline{A} \subset \Omega$. Then, there exists $z \in X_{\Omega \setminus \overline{A}}$ such that $|z|_\phi \leq 1$ a.e. in $\Omega \setminus \overline{A}$, div $z$ is constant in $\Omega \setminus \overline{A}$ and $[z, \nu^\Omega] = 0$ at $\partial \Omega$ and $[z, \nu^A] = -|\nu^A|_\phi$ at $\partial A$ if and only if

$$\Omega \setminus \overline{A} \in \arg \min_{E \in \mathcal{E}} \left\{ \frac{\operatorname{Per}(E; \Omega \setminus \overline{A}) - H^1(\partial^* E \cap \partial \overline{A})}{|E|} \right\},$$ (18)

with $\mathcal{E} := \{E \subset \Omega \setminus \overline{A} : \operatorname{Per}(E) < +\infty, |E| > 0\}$.

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Proof. The proof follows the same ideas as the proof of [5, Theorem 5].

Suppose first that there is a vector field \( z \) satisfying the hypothesis of the Lemma. Then, by Grenn-Gauss theorem,

\[
\int_{\Omega \setminus \overline{A}} \text{div} \, z \, dx = - \int_{\partial(\Omega \setminus \overline{A})} [z, \nu] \, d\mathcal{H}^1 = \text{Per}_\varphi(A).
\]

Therefore, \( \text{div} \, z = \frac{\text{Per}_\varphi(A)}{|\Omega \setminus \overline{A}|} \). Applying once again Green-Gauss theorem, for any \( E \in \mathcal{E} \), we obtain

\[
\frac{|E| \text{Per}_\varphi(A)}{|\Omega \setminus \overline{A}|} = \int_E \text{div} \, z \, dx = - \int_{\partial E} [z, \nu] \, d\mathcal{H}^1 \geq \mathcal{H}^1(\partial^* E \cap \partial A) - \text{Per}_\varphi(E; \Omega \setminus \overline{A}),
\]

which shows the first implication.

Let us suppose now that \( \Omega \setminus \overline{A} \) is a minimizer of the functional

\[
E \mapsto \frac{\text{Per}_\varphi(E; \Omega \setminus \overline{A}) - \mathcal{H}^1(\partial^* E \cap \partial A)}{|E|},
\]

among all sets in \( \mathcal{E} \). We now define the functional

\[
F(\xi) := \int_{\Omega \setminus \overline{A}} (\text{div} \xi)^2 \, dx, \quad \text{with} \quad \xi \in X_{\Omega \setminus \overline{A}}.
\]

Therefore, arguing as in [6, Proposition 6.1 and Theorem 6.7] and [7, Proposition 3.5 and Theorem 5.3], one can prove that the following variational problem has a solution with unique divergence:

\[
\min \left\{ F(\xi) : \begin{array}{l}
\xi \in X_{\Omega \setminus \overline{A}}, \\
|\xi|^*_\varphi \leq 1 \text{ a.e. in } \Omega \setminus \overline{A} \\
[\xi, \nu^\Omega] = 0 \text{ at } \partial \Omega \quad \text{and} \quad [\xi, \nu^A] = -|\nu^A|^*_\varphi \text{ at } \partial A
\end{array} \right\}.
\]

Moreover, given any minimizer \( \xi_{\min} \), \( \text{div} \xi_{\min} \in L^\infty(\Omega \setminus \overline{A}) \cap BV(\Omega \setminus \overline{A}) \) and, letting \( Q_\mu \) be the \( \mu \) upper level set of \( \text{div} \xi_{\min} \) in \( \Omega \setminus \overline{A} \), \( Q_\mu \) has finite perimeter, and

\[
\int_{Q_\mu} \text{div} \xi_{\min} \, dx = \mathcal{H}^1(\partial^* Q_\mu \cap \partial A) - \text{Per}_\varphi(Q_\mu; \Omega \setminus \overline{A}). \tag{19}
\]

Were \( \text{div} \xi_{\min} \) not constant in \( \Omega \setminus \overline{A} \) and equal to \( \frac{\text{Per}_\varphi(A)}{|\Omega \setminus \overline{A}|} \), there will be \( \mu_0 > \frac{\text{Per}_\varphi(A)}{|\Omega \setminus \overline{A}|} \) such that \( Q_{\mu_0} \) is nonempty. Therefore, by (19), we obtain that \( \Omega \setminus \overline{A} \) cannot be a minimizer of the above functional. \( \square \)
**Theorem 4.** Let \( A \subset \mathbb{R}^2 \) be a bounded open set with Lipschitz boundary such that \( A = \bigcup_{i=1}^m C_i \) with \( C_i \) disjoint, convex and calibrable, and let \( \Omega := B_{r^*}(0; R) \) be such that \( A \subset \subset \Omega \). Let \( 1 \leq k \leq m \), \( \{i_1, \ldots, i_k\} \subset \{1, \ldots, m\} \) and consider the following variational problem

\[
(P)_{i_1, \ldots, i_k} := \min_{E \in \mathcal{E}_{i_1, \ldots, i_k}} \left\{ \text{Per}_{\varphi}(E; \Omega) + \frac{\text{Per}_{\varphi}(A)}{|\Omega \setminus A|} |E \setminus A| \right\},
\]

with \( \mathcal{E}_{i_1, \ldots, i_k} := \left\{ \bigcup_{j=1}^k C_{i_j} \subseteq E \subseteq \Omega \setminus \bigcup_{j=k+1}^m C_{i_j} : \text{Per}_{\varphi}(E) < +\infty \right\} \).

Then, there exists \( z \in X_{\Omega \setminus \overline{A}} \) such that \(|z|^*_{\varphi} \leq 1 \) a.e. in \( \Omega \setminus \overline{A} \), \( \text{div} z \) is constant in \( \Omega \setminus \overline{A} \) and \([z, \nu^A] = 0 \) at \( \partial \Omega \) and \([z, \nu^A] = -|\nu^A|^*_{\varphi} \) at \( \partial A \) if and only if \( \bigcup_{j=1}^k C_{i_j} \) is a solution to \( (P)_{i_1, \ldots, i_k} \) for any \( \{i_1, \ldots, i_k\} \subset \{1, \ldots, m\} \).

**Proof.** Suppose first that there exist a vector field \( z \) as in the hypothesis and let us consider \( E \in \mathcal{E}_{i_1, \ldots, i_k} \) and \( D := E \setminus \overline{A} \). Then, by Green-Gauss theorem,

\[
\frac{\text{Per}_{\varphi}(A)}{|\Omega \setminus A|} |D| = \int_D \text{div} z \, dx \geq -\text{Per}_{\varphi}(E; \Omega \setminus A) + \mathcal{H}^1(\partial^* D \cap \bigcup_{j=1}^k \partial C_{i_j}).
\]

Therefore,

\[
\sum_{j=1}^k \text{Per}_{\varphi}(C_{i_j}) \leq \sum_{j=1}^k \text{Per}_{\varphi}(C_{i_j}) + \frac{\text{Per}_{\varphi}(A)}{|\Omega \setminus A|} |D| + \text{Per}_{\varphi}(E; \Omega \setminus A)
\]

\[
-\mathcal{H}^1(\partial^* D \cap \bigcup_{j=1}^k \partial C_{i_j}) \leq \frac{\text{Per}_{\varphi}(A)}{|\Omega \setminus A|} |D| + \text{Per}_{\varphi}(E; \Omega),
\]

thus showing that \( \bigcup_{j=1}^k C_{i_j} \) is a solution to \( (P)_{i_1, \ldots, i_k} \).

Now, let us suppose that \( \bigcup_{j=1}^k C_{i_j} \) is a solution to \( (P)_{i_1, \ldots, i_k} \) for any \( \{i_1, \ldots, i_k\} \subset \{1, \ldots, m\} \). We only need to show that, in this specific case of \( A \), \([18]\) holds. Let \( E \) be a set of finite perimeter such that \( E \subseteq \Omega \setminus \overline{A} \) and let \( C_{i_j} \) for \( j = 1, \ldots, k \) be the ones such that \( \partial E \cap C_{i_j} \neq \emptyset \). Then, by minimality,

\[
\sum_{j=1}^k \text{Per}_{\varphi}(C_{i_j}) \leq \text{Per}_{\varphi} \left( E \cup \bigcup_{j=1}^k C_{i_j}; \Omega \right) + \frac{\text{Per}_{\varphi}(A)}{|\Omega \setminus A|} |E|
\]

\[
\leq \text{Per}_{\varphi}(E; \Omega) + \sum_{j=1}^k \text{Per}_{\varphi}(C_{i_j}) - \mathcal{H}^1(\partial^* E \cap \partial A) + \frac{\text{Per}_{\varphi}(A)}{|\Omega \setminus A|} |E|.
\]
Therefore,
\[ \text{Per}_\varphi(E; \Omega) - \mathcal{H}^1(\partial^* E \cap \partial A) + \frac{\text{Per}_\varphi(A)}{|\Omega \setminus A|} |E| \geq 0. \]

This implies \((18)\).

**Corollary 2.** Let \( A \subset \mathbb{R}^2 \) be a bounded open set with Lipschitz boundary such that \( A = \bigcup_{i=1}^m C_i \) with \( C_i \) disjoint, convex and calibrable, and let \( \Omega := B_{\varphi^*}(0; R) \) be such that \( \overline{A} \subset \Omega \). Suppose that
\[ \text{dist}_\varphi(C_i, \bigcup_{j \neq i} C_j \cup (\mathbb{R}^2 \setminus \overline{\Omega})) > \text{Per}_\varphi(C_i) \quad \text{for any } i = 1, \ldots, m. \] (20)

Let \( f := \sum_{i=1}^m \alpha_i \mathbb{1}_{C_i} \) with \( \alpha_i > 0 \). Then, for
\[ \lambda \geq \max_{i=1, \ldots, m} \left\{ \frac{1}{\alpha_i} \left( \frac{\text{Per}_\varphi(A)}{|\Omega \setminus A|} + \frac{\text{Per}_\varphi(C_i)}{|C_i|} \right) \right\}, \] (21)
the solution to ROF\(_{\varphi, \lambda}\) model is given by
\[ u = \sum_{i=1}^m \left( \alpha_i - \frac{\text{Per}_\varphi(C_i)}{\lambda |C_i|} \right) \mathbb{1}_{C_i} + \frac{\text{Per}_\varphi(A)}{\lambda |\Omega \setminus A|} \mathbb{1}_{\Omega \setminus A}. \]

**Proof.** Since each \( C_i \) is calibrable, we can construct a vector field \( z_i \in X_{C_i} \) such that \( \text{div} \ z_i = \frac{\text{Per}_\varphi(C_i)}{|C_i|} \) and \( [z_i, \nu_{C_i}] = -|\nu_{C_i}|. \) Moreover, it is easy to show that the hypothesis in Theorem 4 are satisfied. Therefore, we can construct a vector field \( z_{\text{out}} \in X_{\Omega \setminus A} \) such that \( \text{div} \ z_{\text{out}} = \frac{\text{Per}_\varphi(A)}{|\Omega \setminus A|} \), \([z_{\text{out}}, \nu_{C_i}] = -|\nu_{C_i}|, \) and \([z_{\text{out}}, \nu_{\Omega \setminus A}] = 0. \) Furthermore, observe that, in this case \( u|_{C_i} \geq u|_{\Omega \setminus A}. \) Therefore, it is easy to show that, considering
\[ z := \sum_{i=1}^m z_i \mathbb{1}_{C_i} + z_{\text{out}} \mathbb{1}_{\Omega \setminus A}, \]
then \( -\text{div} \ z \in \partial|Du|_\varphi \) and therefore, \( u \) is the solution of the ROF\(_{\varphi, \lambda}\) model by \((6)\). \( \square \)

**Example 2.** In this example, using a case of 3-phases segmentation, we show that the relationship \((17)\) does not hold, in general. Let \( \Omega := B_{\varphi^*}((0, 0), R) \) and \( f = \mathbb{1}_{C_1} + \mathbb{1}_{C_2} + \frac{1}{2} \mathbb{1}_{C_3} \) with \( C_1 := B_{\varphi^*}((0, -L), 1), C_2 := B_{\varphi^*}((0, 0), (1/2)) \) and \( C_3 := B_{\varphi^*}((0, L), 2). \) We take \( R \) and \( L \) large enough so that \( \lambda = 10 \) is feasible with respect to \((21)\), \( A := \bigcup_{i=1}^3 C_i \) satisfies that \( \overline{A} \subset \Omega \) and \( C_i \) fulfils
such that \( \tau \). Let Corollary 3. Thus, Corollary 1 implies the following result.

This expression for each \( i \). Then, by Corollary 2 the solution of ROF_{\varphi, \lambda} problem is exactly given by

\[
w = \frac{8}{10} \chi_{C_1} + \frac{6}{10} \chi_{C_2} + \frac{4}{10} \chi_{C_3} + \frac{\text{Per}_{\varphi}(A)}{10 |\Omega \setminus \overline{A}|} \chi_{\Omega \setminus \overline{A}} \tag{22}
\]

Now, we define \( c_1, c_2 \) and \( c_3 \) as follows:

\[
c_1 = \int_{C_1} \frac{f}{|C_1|} \, dx, \quad c_2 = \int_{C_2 \cup C_3} \frac{f}{|C_2 \cup C_3|} \, dx, \quad c_3 = \int_{\Omega \setminus \overline{A}} \frac{f}{|\Omega \setminus \overline{A}|} \, dx.
\]

where \( c_1 = 1, c_2 = 9/17 \) and \( c_3 = 0 \). By enlarging \( R \) if necessary, we assume that the latter term of (22) is bounded on top by \( c_2/2 \). Thus, we may assure that

\[
\Sigma_1 := \left\{ x \in \Omega : w > \frac{c_1 + c_2}{2} \right\} = \Omega, \quad \Sigma_2 := \left\{ x \in \Omega : w > \frac{c_2 + c_3}{2} \right\} = A.
\tag{23}
\]

If we suppose that (17) is true, then \( \{ \Omega_i \}_{i=1}^3 := \{ \Sigma_1, \Sigma_2 \setminus \Sigma_1, \Omega \setminus \Sigma_2 \} = \{ C_1, C_2 \cup C_3, \Omega \setminus \overline{A} \} \) satisfies that

\[
\text{CV}^3_{\varphi, \mu}(\{ \Omega_i \}_{i=1}^3, \{ c_i \}_{i=1}^3) \leq \text{CV}^3_{\varphi, \mu}(\{ \Lambda_i \}_{i=1}^3, \{ c_i \}_{i=1}^3), \quad \forall \{ \Lambda_i \}_{i=1}^3 \in \mathcal{P}_3(\Omega), \tag{24}
\]

where \( \mu = \{85/8, 1445/72, 85/9\} \), as in (10). However, if we define the following disjoint partition \( \{ \Omega_i^* \}_{i=1}^3 := \{ C_1 \cup C_2, C_3, \Omega \setminus \overline{A} \} \), we have that

\[
\text{CV}^3_{\varphi, \mu}(\{ \Omega_i^* \}_{i=1}^3, \{ c_i^* \}_{i=1}^3) = \frac{725}{72} \text{Vol}(C_1) < \text{CV}^3_{\varphi, \mu}(\{ \Omega_i \}_{i=1}^3, \{ c_i \}_{i=1}^3) = \frac{733}{72} \text{Vol}(C_1),
\]

which leads us to a contradiction with respect to (24). It should be noted that this reasoning is valid for any anisotropy \( | \cdot |_{\varphi} \).

Finally, we would like to point out a relationship between AROF_{\lambda} and TROF_{\lambda}. First we note that, for a fixed \( \tau \in [0, 1]^{n-1} \), any \( \Sigma \in \mathcal{P}_n^*(\Omega) \) satisfies this expression

\[
\text{TROF}_{\varphi, \lambda}(\Sigma, \tau) = \sum_{i=1}^{n-1} \text{TROF}^2_{\varphi, \lambda}(\{ \emptyset, \Sigma_i, \Omega \}, \tau_i). \tag{25}
\]

Thus, Corollary 1 implies the following result.

**Corollary 3.** Let \( w \) be the AROF_{\lambda} minimizer. Then, for a \( \tau \in [0, 1]^{n-1} \) such that \( \tau_i > \tau_{i+1} \), the minimizer \( \Sigma \in \mathcal{P}_n^*(\Omega) \) of TROF_{\lambda}^n(\cdot, \tau) is provided by

\[
\Sigma_i = \left\{ x \in \Omega : w(x) > \tau_i \right\} \quad \text{for } i = 1, \ldots, n-1.
\tag{26}
\]
This fact allows us to define an algorithm that, by calculating the minimizer of $\text{AROF}_\lambda$, we can obtain a minimizer of the functional $\text{TROF}_n^{\lambda}$. In Section 6 we show that it can be an advantageous segmentation tool in certain situations. The algorithm, defined as the previous ones, is the following one:

**Algorithm 3:** TROF$_n^{\lambda}$ approximate minimizer

| Initiation: | $k = 0$, $\{\tau_i\}_{i=1}^{n-1} \in [0, 1]^{n-1}$ s.t. $\tau_i > \tau_{i+1}$. |
|-------------|---------------------------------------------------------------|
| $w_0 \leftarrow \text{AROF}_\lambda$ minimizer. |
| $\Sigma^0 \leftarrow \text{TROF}_n^{\lambda}$ minimizer via Corollary 3 |

while $\sum_{i=0}^{n-1} |\Sigma^k_i \setminus \Sigma^{k-1}_i|^2 > \varepsilon_{\text{tol}} \land k < n_{\text{max}}$ do

$\tau_i \leftarrow \frac{1}{|\Sigma^k_{i+1} \setminus \Sigma^k_i|} \int_{\Sigma^k_{i+1} \setminus \Sigma^k_i} f \, dx$, $k \leftarrow k + 1$. |

$\Sigma^k \leftarrow \text{TROF}_n^{\lambda}$ minimizer via Corollary 3 |

end while

return $\Sigma^k$

**Remark 5.** We should note that, although the relationship between $\text{CV}_n^{\varphi, \mu}$ and $\text{TROF}_n^{\varphi, \lambda}$ (17) does not hold, in general, the segmentation provided by the $\text{TROF}_n^{\varphi, \lambda}$ functional provides satisfactory results both in the isotropic case and in our case: the anisotropic one (e.g. see figure 2). Moreover, in [11] it was shown that algorithm 3 in the isotropic formulation, converges. Their proof can be easily adapted in the anisotropic framework and thus we can prove that the above algorithm (directly) and Algorithm 1 (applying Corollary 2) converge. Therefore, in the next section we will make use of algorithm 3 when studying some multiphase segmentations.

6 Applications

In the previous sections, we have looked into an analytical approach to AROF$_\lambda$, ACV$_\mu$, CV$_n^{\varphi, \mu}$ and TROF$_n^{\varphi, \lambda}$, which gives us a robust tool to find solutions to these variational models. In the literature, CV and ROF models have been studied from an approximative perspective, where successful methods of resolution have been proposed which lead to an approximation of the theoretical minimizer (see e.g. [1, 9, 13, 20, 21]). In this section we will show that the theoretical study of these models provides efficient segmentation algorithms and characterizations of minimizers that are of great interest. Therefore, we show the performance of algorithms 1 and 3 comparing them with other ones based on level sets; and we present a possible application of theorems 2 and 3.
To exemplify the performance of algorithms 1 and 3, we compare these algorithms with the similar isotropic ones proposed in [11]. Specifically, in figures 1 and 2 we compare algorithms 1 and 3 with their isotropic counterparts. For this, we set $n_{\text{max}} = 200$, $\varepsilon_{\text{tol}} = 10^{-3}$, we initialise $c_i$ and $\tau_i$ using FCM processes (see [8]) and we approximate the AROF$\lambda$ and ROF$\lambda$ minimizers via standard Split-Bregman algorithms. Moreover, the selected examples are those segmentations that differ the least, in quadratic terms, from the original images.

In figures 1 and 2, we observe that our anisotropic model is less influenced by the applied noise than the isotropic one. Thus, it provides a better reconstruction of word Pass (compare figures 1c and 1d) or of the edges of a shaded cube (compare figures 2c and 2d). This fact may be correlated with a worse performance on the isotropic TV on these noises. As the compared algorithms are, to some extent, analogous to ours, these examples show how the models studied in this work provide some advantages in certain situations where isotropic versions of these do not give adequate segmentations.

![Figure 1: Comparison of 2-phases segmentation.](image)
Finally, we discuss the application of Theorem 2 and 3 as a reliability analysis tool. According to the previous results, we could consider that a segmentation using ACV$_\mu$ or $G^n_\mu$ functionals is a good segmentation if it fits properly with the grid of the cells as much as possible, which is the same as resembling the shape of an exact minimizer. In addition, although this does not imply that this fitting behaviour indicates an accurate approximation, it does make explicit the performance of poor segmentations. To illustrate this application, we consider figure 3. Here, we segment $3a$ via ACV model using two different methods: the anisotropic version of Getreur’s implementation (see isotropic one in [18]) and our algorithm 1. Comparing them, we note the difference between segmentation $3c$ and $3d$. We observe that $3c$ is defective due to the lack of adjustment with the grid of image $3a$, a mismatch that may be rectified by defining other initial parameters. Moreover, this inaccuracy is identified without comparing the segmentation with the exact minimizer $3b$. 

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Figure 2: Comparison of 4-phases segmentations.
In consequence, we consider this application as a setting up test which detects incorrect parameter initializations without comparing the approach with the exact minimizer directly. In the same spirit, although computing the exact minimizer of $ACV_\mu$ by brute force (i.e. minimizing the functional within the sets in $\mathcal{F}(G_f)$) is a lengthy computation, it can be used as a test for comparing the performance of any segmentation algorithm in some simple images, in the sense that the grid associated to the image is sufficiently small.

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