Topological defect brane-world models

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Abstract

5-dimensional homogeneous and isotropic models with a bulk cosmological constant and a minimally coupled scalar field are considered. We have found that in special cases the scalar field can mimic a frustrated (i.e. disordered) networks of topological defects: cosmic strings, domain walls and hyperdomain walls. This equivalence enabled us to obtain 5-dimensional instantonic solutions which can be used to construct brane-world models. In some cases, their analytic continuation to a Lorentzian metric signature give rise to either 4-dimensional flat or inflating branes. Models with arbitrary dimensions (D > 5) are also briefly discussed.

1 Introduction

For nearly a century, starting with Kaluza–Klein (KK) theories, which were seeking for a unification between gravity and electromagnetism, higher dimensional cosmological models have been quite often present in scientific research. These theories has received a great attention during the last years due to some seminal papers published on brane-worlds, and extra dimensions models, which are able to provide an explanation for fundamental physical troubles such as the hierarchy problem. The geometry as well as the topology of the extra dimensions has to be compatible with the effective 4-dimensional (4–D) universe where we live in, i.e. the extra dimension should be hidden: they can be compact as in the old KK theories or large and infinite as was firstly pointed out by Rubakov and Shaposhnikov, Akama and others.

The allowance of more than four dimensions, 4+n with n > 0, for the space-time introduces a new Planck scale, $M_{PL,4+n}$, which can be considered as a fundamental one and is related to the usual 4–dimension Planck mass $M_{PL}$. This relation depends on the topology as well as on the geometry of the extra dimensions and allows a solution of the hierarchy problem by considering that $M_{PL,4+n}$ is of the order of the weak scale $M_{EW}$. In the Arkani-Dimopoulou-Dvali (ADD) model, the extra dimensions are compact and $M_{PL} = M_{PL,4+n} R^n$, being $R$ the size of the extra dimensions. The hierarchy problem in this case can be solved whenever n ≥ 3 (see e.g. [3]). However, more recently, Rundall and Sundrum proposed a model (RS1) where turned out to be enough introducing a unique compact extra dimension. One of the main differences between both models lies on the dependence of the warp factor on the extra dimension. The RS1 model, inspired by string theory, places our universe in a flat 3–brane, i.e. a 4–D flat hypersurface, with negative tension, embedded in a 5–D Anti de Sitter (AdS) space-time, while the second brane with positive tension is hidden. Later on, Randall and Sundrum constructed another model (RS2) where the extra dimension is infinite and the universe is lying on a flat 4–D brane with positive tension, being the bulk a 5–D AdS space-time.

The framework of brane-worlds has been used in cosmology to describe the birth of branes. In particular, Garriga and Sasaki constructed an interesting instanton brane-world model able to describe the birth of an inflating brane in the semiclassical approximation. The brane is surrounded by a 5–D AdS
space. However, the general 5-D solution used in this model can present a singularity when the warp factor vanishes.

In the present paper, we have obtained 5-D solutions which can be used to construct singularity-free brane-world instantons which, in some cases, can describe inflating 3-branes. Indeed, we have obtained a brane-world model where the brane is inflating and the warp factor has the same dynamical behaviour as in the RS1 and RS2 models [11, 12] (see Eqs. (4.4) and (4.5) below).

In order to get singularity-free 5-D instantons, we have considered the bulk filled with matter, which we have modeled by a minimally coupled scalar field whose energy density and pressure satisfy a perfect fluid state equation. This scalar field can mimic the behaviour of different networks of frustrated (i.e. chaotically distributed) topological defects (NFTD) such as cosmic strings, domain walls and hyperdomain walls [12–14], depending on the specific state equation we choose.

The NFTD can be formed during phase transitions. They do not satisfy scaling solutions and the number of topological defects do not change in the comoving volume. Up to our knowledge, these defects were first introduced in cosmology by Kibble, and could have occasionally dominated the energy density of the early universe [20], so changing its dynamical behaviour. Vilenkin proposed [13] a scenario to describe a network of frustrated cosmic strings where the strings do not intercommute, assuming the probability for such physical process to be nearly zero. The equivalence between such NFTD and scalar fields has enabled us to obtain 5-dimensional instantonic solutions which can be used to construct brane-world models.

The paper is organized as follows. In the next section, we describe the model and get a constraint equation that must satisfy the scale factor of a 5-D Euclidean homogeneous and isotropic space endowed with a 5-D cosmological constant and filled with a scalar field whose energy density and pressure satisfy a perfect fluid state equation \( P = (\alpha - 1) \rho \). In section 3, we obtain the instantonic solutions for \( \alpha = 3/4 \), i.e. when the bulk is filled with a network of frustrated cosmic strings. Many of these solutions describe asymptotically AdS wormholes. In section 4, we get the 5-D Euclidean solutions when the state equation for the scalar field reads \( P = -1/2p \). This scalar field can describe a network of frustrated domain walls. As an example, one of the solutions (Eq. (1.4)) is used to construct a brane-world instanton which can describe the birth of an inflating brane from nothing in the semiclassical approximation and whose scale factor coincides with the one of RS1 and RS2 models in section 5. In section 5, we analyze the case \( \alpha = 1/4 \), which may correspond to a bulk with a network of frustrated hyperdomain walls (3-D objects). Finally, we summarize and generalize our model for an arbitrary number of dimensions \( D > 5 \) in section 6.

2 The model

As it was stressed before, we are interested in getting 5-D instanton solutions which can be used to construct brane-world instantons [11, 21–23], free from any singularity at the origin of the extra-coordinate. For this purpose, we consider now a 5-D space filled with a minimally coupled scalar field, \( \psi \), and a cosmological constant, \( \Lambda_5 \). The Lorentzian action of this system has the form

\[
S = \frac{1}{2k_5^2} \int d^5X \sqrt{|g^{(5)}|} \left\{ R[g^{(5)}] - 2\Lambda_5 \right\} + \int d^5X \sqrt{|g^{(5)}|} \left\{ -\frac{1}{2} g^{(5)MN} \partial_M \psi \partial_N \psi - V(\psi) \right\} + S_{YGH},
\]

(2.1)

where \( k_5^2 \) is the 5-D gravitational constant and \( S_{YGH} \) is the boundary term [22]. The metric of the Lorentzian 5-D space \( g^{(5)} \) is taken to be homogeneous and isotropic, so it can be written as

\[
g^{(5)} = g^{(5)MN} dX^M \otimes dX^N = -e^{2\gamma(\tau)} d\tau \otimes d\tau + e^{2\beta(\tau)} g^{(4)}_{\mu\nu} dx^\mu \otimes dx^\nu.
\]

(2.2)

In this expression, \( g^{(4)} \) represents the metric of a 4-D Einstein space; i.e. \( R_{\mu\nu}[g^{(4)}] = \lambda g^{(4)}_{\mu\nu} \), with constant scalar curvature; \( R[g^{(4)}] = R_4 = 4\Lambda := 12k \) where \( k = \pm 1, 0 \). Unless otherwise stated, the Einstein spaces are considered as spaces of constant curvature. However, all the solutions we will obtain are valid for arbitrary Einstein spaces with constant scalar curvature \( R_4 = 12k \), \( k = \pm 1, 0 \).

Analogously to the case for Eq. (2.3), we consider the scalar field to be homogeneous. Therefore, its energy density and pressure are given by:

\[
\rho = \frac{1}{2} e^{-2\gamma(\tau)} \dot{\psi}^2 + V(\psi) = -T_0^0, \\

P = \frac{1}{2} e^{-2\gamma(\tau)} \dot{\psi}^2 - V(\psi) = T_\mu^\mu, \quad \mu = 1, \ldots, 4.
\]

(2.3)

Furthermore, we suppose that the scalar field satisfies the state equation

\[
P = (\alpha - 1) \rho,
\]

(2.4)

where \( \alpha \) is a constant. The conservation equation for the scalar field energy momentum tensor implies \( \rho = A a^{\alpha-4}\rho \), where \( A \) is an arbitrary constant. This results in an inverse power-law for the scalar field
potential $V$ in term of the scale factor $a = \exp[\beta(\tau)]$; i.e. $V = (1 - 2\Lambda)Aa^{-4\alpha}$, whenever $\alpha > 0$. This potential can be rewritten in terms of the scalar field $\varphi$ (see e.g. [24]). Therefore, Eq. (2.4) strongly constrains the form of the potential $V(\varphi)$. Depending on the values taken on by the parameter $\alpha$, the scalar field can mimic different kinds of matter as we will discuss shortly.

This model can equivalently be considered as a 5-D space-time filled with a perfect fluid with action

$$S = -\frac{V_4}{\kappa_5^2} \int d\tau \left[ 6e^{-\gamma y_0} \beta^2 + e^{\gamma y_0} U \right],$$

where $\gamma_0 = 4\beta$, $V_4 = \int d^4x \sqrt{|g^{(4)}|}$, the overdot denotes derivative with respect to the Lorentzian time $\tau$, while the potential $U$ is defined as

$$U = e^{2\gamma y_0} \left( -\frac{1}{2} R_5 e^{-2\beta} + \Lambda_5 + \kappa_5^2 \rho \right).$$

From Eqs. (2.5), (2.6) the scale factor $a$, in the proper time gauge ($\gamma = 0$), must satisfy:

$$\left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} - \frac{\Lambda_5}{6} - \frac{1}{6} \kappa_5^2 A a^{-4\alpha} = 0. \tag{2.7}$$

As it has been said before, in this model the matter in the 5-D space can be equivalently represented by either a scalar field or a perfect fluid. In both cases, we have to specify the state equation; i.e. the parameter $\alpha$ given in Eq. (2.4), to find the dynamical behaviour of the warp factor $a$. There are many values of $\alpha$ which represent relevant kinds of matter. For example, if $\alpha = 0, 1, 5/4$, the matter content corresponds to a 5-D cosmological constant, dust and radiation, respectively. In this paper, we will consider scalar fields which could mimic a matter content described by networks of frustrated cosmic strings (1-D objects), domain walls (2-D objects), and also hyperdomain walls (3-D objects); i.e. $\alpha = 1/2$, and also hyperdomain walls (3-D objects); i.e. $\alpha = 1/4$. These topological defects may have been formed during phase transitions and their evolution afterwards does not correspond to scaling solutions [23]. On the other hand, we shall suppose that the number of topological defects does not change in the comoving volume. In this paper, we will not study the underlying field theory that can give rise to such topological defects.

We analytically continue the constraint equation (2.7) into the Euclidean proper “time” $y$, $\tau \rightarrow -iy$, so that we will obtain 5-D instantons. In term of $y$ Eq. (2.7) becomes

$$\left( \frac{da}{dy} \right)^2 - k + \frac{\Lambda_5}{6} a^2 + \frac{1}{6} \kappa_5^2 A a^{-4\alpha+2} = 0. \tag{2.8}$$

From the previous equation, it can be seen that for negative cosmological constant and matter content with $\alpha > 0$, the dominant term for large scale factor ($a \rightarrow +\infty$) is proportional to $\Lambda_5 a^2$. Therefore, in this limit the instanton solutions are asymptotically Anti de Sitter (AdS). Indeed, this behaviour is shown by all instantons for the kind of matter considered in this work when the warp factor is defined at large $|y|$ and $\Lambda_5 < 0$. On the other hand, the instanton scale factor will obviously depend on the matter content, given by the cosmological constant and the energy density $\rho$, as well as the topology of the Euclidean space ($k = \pm 1, 0$), as it is shown by Eq. (2.8). Throughout the paper $r_1, a_1, y_1, \tau_0$ and $\tau_1$ will mean integration constants.

## 3 Cosmic string instantons

Networks of frustrated cosmic strings (NFCS) can crop up in a theory where strings neither intercommute nor pass through each other [12]. They have been studied in 4-D cosmology where they could eventually dominate the energy density of the universe [20]. In this section, we consider that the energy density $\rho$ in our model behaves like that of a NFCS in a 5-D space; i.e. $\rho \propto a^{-3}$. The last expression reminds us the energy density of a 4-D space filled with dust. The dynamical behaviour of the scale factor is equivalent in both cases. However, they actually correspond to space-times with different dimensionality, topology and matter content.

In the time gauge $\gamma = \sqrt{a}$ $\Rightarrow$ $dr := dy/\sqrt{a}$ and for $\alpha = 3/4$, the constraint equation (2.8) becomes:

$$\left( \frac{da}{dr} \right)^2 - ka + \text{sign}(\Lambda)|\Lambda|a^3 + \bar{A}^2 = 0, \tag{3.1}$$

where we have introduced the notation: $\Lambda = \Lambda_5/6$ and $\bar{A}^2 = (1/6)\kappa^2 A$. For the particular chosen gauge, $\gamma = \sqrt{a}$, we can obtain an analytical expression for the scale factor when the 5-D instanton is sliced into flat, spherical and hyperbolic sections ($k = \pm 1, 0$) and also for positive, zero and negative cosmological constant.
3.1 Negative cosmological constant: $\Lambda_5 < 0$

The 5-D instantons filled with a scalar field whose energy density behave like a NFCS, and are 5-D asymptotically AdS wormholes, when $\Lambda_5$ is negative and the sections $r = \text{const}$ have constant curvature. The scale factor of these instantons for $p = \frac{k}{3\Lambda_5} + \frac{4\Lambda_5}{m} > 0$ reads

\[
a(r) = a_1 + \lambda^2 \frac{1 - \text{cn} \left( \lambda \sqrt{|\Lambda|} \left( r - r_1 \right) | m \right)}{1 + \text{cn} \left( \lambda \sqrt{|\Lambda|} \left( r - r_1 \right) | m \right)}, \quad 0 \leq \lambda \sqrt{|\Lambda|} |r - r_1| < 2K(m),
\]

where

\[
a_1 = \left[ \frac{A^2}{2|\Lambda|} + \sqrt{p} \right]^{1/3} - \left[ -\frac{A^2}{2|\Lambda|} + \sqrt{p} \right]^{1/3}, \quad \lambda = \left[ 3a_1^2 + \frac{k}{|\Lambda|} \right]^{1/4}, \quad m = \frac{1}{2} - \frac{3a_1}{4A^2}.
\]

The function $\text{cn}(u|m)$ is a Jacobian Elliptic function while $K(m)$ is the complete Jacobian Elliptic integral of the first kind \[26\]. For $r = r_1$, the scale factor is equal to the minimum radius (throat) of the wormhole, $a_1$. Independently of the curvature of the 4-D sections ($k = \pm 1, 0$), the scale factor $a$ tends to infinity when $\lambda \sqrt{|\Lambda|} |r - r_1| \to 2K(m)$ (asymptotic AdS behaviour).

When the parameter $p$ vanishes, i.e. when $\Lambda = -A/(27A^4)$, and the wormhole is sliced into hyperbolic sections, the solution to equation (3.1) becomes

\[
a(r) = a_1 \left\{ 1 + \frac{3}{2} \tan^2 \left( \sqrt{\frac{3a_1 |\Lambda|}{8}} (r - r_1) \right) \right\}, \quad 0 \leq \sqrt{\frac{3a_1 |\Lambda|}{8}} |r - r_1| < \frac{\pi}{2},
\]

where $a_1$ is defined by Eq. (3.3).

Finally, the scale factor of the instanton for negative values of $p$ can be written as

\[
a(r) = \frac{a_1}{\text{cn}^2 \left( \lambda \sqrt{|\Lambda|} (r - r_1) | m \right)} - a_3 \text{sc}^2 \left( \lambda \sqrt{|\Lambda|} (r - r_1) | m \right), \quad 0 \leq \lambda \sqrt{|\Lambda|} |r - r_1| < K(m),
\]

where the function $\text{sc}(u|m)$ again is a Jacobian Elliptic function and

\[
a_1 = \frac{1}{27|\Lambda|^2} \cos(\theta/3), \quad a_2 = -\frac{1}{27}[\cos(\theta/3) + \sqrt{3} \sin(\theta/3)], \quad a_3 = \frac{1}{27}[\cos(\theta/3) + \sqrt{3} \sin(\theta/3)],
\]

\[
s = \left( \frac{1}{27|\Lambda|^2} \right)^{1/2}, \quad \theta = \arctan \left( \frac{2|\Lambda| \sqrt{p}}{A^2} \right), \quad \lambda = \frac{1}{2}(a_1 - a_2)^{1/2}, \quad m = \frac{a_3 - a_2}{a_1 - a_2}.
\]

The parameter $\theta$ takes values on the interval $(0, \pi/2)$. This can be deduced by taking into account that $\cos(\theta) = \frac{A^2}{2|\Lambda|} s$ and $\sin(\theta) = \sqrt{p}/s$. On the one hand, the negativity of $p$ in the case under consideration forbids the value $\theta = 0$. On the other hand, the existence of a non zero matter content in the model ($\Lambda \neq 0$) does not allow the value $\theta = \pi/2$.

The cases described by Eqs. (3.4) and (3.5) represent also Euclidean asymptotically AdS wormholes. In fact, the scale factor varies from $a_1$ ($a_1 = a(r_1)$), corresponding to the radius of the wormhole throat, up to infinity as $\sqrt{3a_1 |\Lambda|/8} |r - r_1| \to \pi/2$ or $\lambda \sqrt{|\Lambda|} |r - r_1|$ approaches $K(m)$ for (3.4) and (3.5), respectively.

All the above solutions can be continued back into the Lorentzian time ($\tau \to \tau - \tau_1 \sqrt{\alpha}$). In this case, it can be seen that the Lorentzian solutions represent 5-D collapsing FRW universes as the scale factors then vary between 0 and $a_1$ where the explicit expression of $a_1$ depends on the type of 4-d geometry ($k = \pm 1, 0$) and the parameter $p$.

3.2 Positive cosmological constant: $\Lambda_5 > 0$

Unlike for negative cosmological constant, there are not Euclidean solutions for $r = \text{const}$ with hyperbolic ($k = -1$) or flat ($k = 0$) geometry, when the 5-D space is filled with NFCS and positive cosmological constant. This is because in these cases the differential equation (3.1) would imply $(\frac{dn}{dr})^2 < 0$.

It will however be seen that there are instantonic solutions with spherical $r = \text{const}$ sections. The explicit form of such solutions (for $k = 1$) depends on the parameter $p = \frac{k}{3\Lambda_5} + \frac{4\Lambda_5}{m}$. If $p \geq 0$, then there are no Euclidean solution because Eq. (3.3) would then imply $(\frac{dn}{dr})^2 < 0$. The situation is rather different for $p < 0$, as in this case there is an Euclidean 5-D space whose scale factor $a$ is

\[
a(r) = \frac{a_3}{\text{dn}^2 \left( \lambda \sqrt{|\Lambda|} (r - r_1) | m \right)} - a_2 \text{sm}^2 \left( \lambda \sqrt{|\Lambda|} (r - r_1) | m \right), \quad 0 \leq \lambda \sqrt{|\Lambda|} |r - r_1| \leq K(m).
\]

(3.7)
The functions \( \text{dn}(u|m) \) and \( \text{sd}(u|m) \) again are Jacobian Elliptic functions, the parameters \( \lambda, a_1, a_2, a_3 \) \( s, \theta \) are defined in Eq. (3.6) with the obvious substitution \( |\Lambda| \) by \( \Lambda \), and

\[
m = \frac{a_1 - a_3}{a_1 - a_2}. \tag{3.8}
\]

The warp factor of this instanton takes on values between \( a_3 \) for \( r = r_1 \) and \( a = a_1 \) for \( \lambda \sqrt{|\Lambda|} (r - r_1) = K(m) \). The parameter \( \theta \) takes on values in the interval \( \left( \frac{\pi}{4}, \pi \right) \) as \( \cos(\theta) = -\frac{\Lambda^2}{2\Lambda s} \) and \( \sin(\theta) = \frac{\sqrt{p}}{s} \).

The presence of matter in the model \( \left( \bar{\Lambda}, \Lambda \right) \) space-time with a minimum scale factor \( K \) – cosmological constant, the solution to Eq. (4.1) reads

\[
\text{We are now going to describe 5–D Euclidean spaces filled with a matter content with state equation}
\]

\[
a \text{to a “static trivial solution” with constant scale factor,}
\]

\[
a = a_1 \text{ and an asymptotically de Sitter (dS) space-time with a minimum scale factor } a = a_1.
\]

### 3.3 Zero cosmological constant: \( \Lambda_5 = 0 \)

If \( \Lambda_5 \) vanishes, there is a unique non-trivial instantonic solution with the topology \( \mathbb{R} \times S^4 \) and the following scale factor:

\[
a(r) = \bar{A}^2 + \frac{1}{4} (r - r_1)^2, \quad r \in \mathbb{R}. \tag{3.9}
\]

This instanton is a 5–D asymptotically flat wormhole whose throat is located at \( r = r_1 \). It is worth mentioning that a fine tuning between the matter content and the spatial curvature: \( ka = \bar{A}^2 \implies a = a_1 := \bar{A}^2 \), leads to a “static trivial solution” with constant scale factor, \( a_1 \), and topology \( \mathbb{R} \times S^4 \). This static instanton is unstable.

There is a baby universe branching off from the wormhole throat. Bearing in mind that the energy density of a NFCS in a 5–D space scales respect to the warp factor \( a \) similarly to as dust did in a 4–D space, the behaviour of this baby universe is similar to the dynamics of a 4–D closed FRW universe filled with dust.

### 4 Domain wall instantons

Domain walls may have been formed during phase transition when the vacuum manifold has disconnected components. Particularly interesting are the networks of frustrated domain walls (NFDW) \([16]\) whose evolution does not approach a scaling solution. They have been sketched recently in the literature to give account of the dark matter in the universe \([17]\) as well as the dark energy \([19]\). While in a 4–D universe the energy density of a NFDW is proportional to \( a \), in a 5–D space it behaves as \( a^2 \). Therefore, the presence of a scalar field in a 5–D space with an energy density similar to the one of a NFDW implies the appearance of an effective curvature term. This has interesting consequences as we will see shortly.

#### 4.1 Negative cosmological constant: \( \Lambda_5 < 0 \)

We are now going to describe 5–D Euclidean spaces filled with a matter content with state equation \( P = -1/2p \). The matter content can either be modeled by a scalar field or a perfect fluid. Both cases can describe effectively a NFDW. For convenience we rewrite the constraint equation (2.8) (for \( \alpha = 1/2 \)) in the proper Euclidean time gauge as

\[
\left( \frac{da}{dy} \right)^2 + \text{sign}(\Lambda)|\Lambda|a^2 - k_{eff} = 0. \tag{4.1}
\]

In this equation, \( k_{eff} = k - \bar{A}^2 \) is the effective spatial curvature parameter. This parameter encodes the geometry of the \( y = \text{const} \) sections by means of \( k \) and the matter content through \( \bar{A}^2 \). For a negative cosmological constant, the solution to Eq. (4.1) reads

\[
a(y) = \sqrt{\frac{k_{eff}}{\Lambda}} \cosh \left( \sqrt{|\Lambda|}(y - y_1) \right), \quad y \in \mathbb{R}, \tag{4.2}
\]

when \( k_{eff} \) is negative (\( k_{eff} < 0 \)). This instanton describes a 5–D asymptotically AdS wormhole whose throat is located at the hypersurface \( y = y_1 \). Even if the behaviour of the scale factor of the wormhole is similar for all negative values of the parameter \( k_{eff} \), its topology can be different as it can be sliced into flat, hyperbolic or spherical \( y = \text{const} \) sections. Indeed, for a given 5–D cosmological constant, the radius of the wormhole throat get narrower, when the instanton is sliced into spherical 4–D hypersurfaces, than in hyperbolic 4–D ones.

The behaviour of the warp factor \( a \) is different for strictly positive values of \( k_{eff} \). In this case

\[
a(y) = \sqrt{\frac{k_{eff}}{\Lambda}} \sinh \left( \sqrt{|\Lambda|}(y - y_1) \right), \quad y \in \mathbb{R}, \tag{4.3}
\]
while the topology of the instanton is \( \mathbb{R} \times S^3 \). In contrast with the solution described by Eq. (4.3), the scale factor vanishes at \( y = y_1 \) and could induce a singularity at this hypersurface (\( y = y_1 \)). It happens when \( k_{eff} < 1 \), because in this case the scalar curvature blows up when \( y \) approaches \( y_1 \). However, for \( k_{eff} = 1 \) (absence of matter), the solution (4.3) describes an Euclidean AdS space when \( y = \text{const} \) sections are spherical and the scalar curvature is well-defined at the origin of the extra-coordinate \( y = y_1 \). The situation can be rather different for instantons whose scale factor equals solution (4.1) with \( k_{eff} = k = 1 \), sliced into more general 4-D Einstein spaces with \( R[g^{(4)}] = 12 \). In the later case, this is so because the invariant \( C^2 \) related to the 5-D Weyl tensor of the 5-D instanton by \( C^2 = C_{\nu\rho\sigma\tau} C^{\nu\rho\sigma\tau} \) can be divergent at the origin of the extra-coordinate. For \( k_{eff} = 1 \), the instanton solution (4.3) has been used to construct a brane-world model [1].

Finally, for vanishing \( k_{eff} \), i.e. when the 5-D instanton has spherical sections at \( y = \text{const} \) and a matter content with \( \tilde{A}^2 = 1 \), the warp factor is:

\[
a(y) = a_1 \exp(\pm \sqrt{\Lambda} y), \quad a_1 > 0, \quad y \in \mathbb{R}. \quad (4.4)
\]

This scale factor coincides with the one of an Euclidean 5-D AdS space sliced into 4-D flat spatial sections. However, an instanton, with the scale factor (4.4), sliced into 4-D spherical sections is not an Euclidean 5-D AdS. In fact, its scalar curvature depends on the Euclidean extra coordinate, \( y \),

\[
R[g^{(5)}] = \frac{12}{a_1^2} \exp(\mp 2 \sqrt{\Lambda} y) - 20|\Lambda|,
\]

and get asymptotically AdS for large values of \( a(y) \). It is clear that solution (4.4) with the fine tuning \( \tilde{A}^2 = 1 \) is unstable: any matter fluctuations \( \tilde{A}^2 + \delta \tilde{A}^2 \) will result in transition of (4.4) into either (4.2) or (4.3), depending on the sign of \( \delta \tilde{A}^2 \), with \( k_{eff} = \delta \tilde{A}^2 \) and the same topology \( \mathbb{R} \times S^3 \).

Instanton (4.4) is nonsingular at any finite value of \( |y| \). It can be used to construct a brane-world instanton by a similar procedure of cutting and gluing of the original instanton [11, 21]. Some of this gravitational instantons with branes can describe the creation of brane-worlds.

**4.2 Positive cosmological constant: \( \Lambda_5 > 0 \)**

For a space filled with a scalar field, with \( P = -1/2\rho \), and a positive cosmological constant, there is an unique Euclidean solution when \( y = \text{const} \) sections have constant scalar curvature. The solution to Eq. (4.1) reads

\[
a(y) = \sqrt{\frac{k_{eff}}{\Lambda}} \sin \left( \sqrt{\Lambda} |y - y_1| \right), \quad 0 \leq \sqrt{\Lambda} |y - y_1| \leq \frac{\pi}{2}.
\]

and the effective spatial curvature parameter must be \( 0 < k_{eff} \leq 1 \); i.e. \( k = 1 \) and \( \tilde{A}^2 < 1 \). In the limiting case \( k_{eff} = 1 \) (absence of matter), the instanton describes a 5-D sphere, that equivalently is an Euclidean dS space.

In the semiclassical approximation, solution (4.6) describes a quantum path for the creation of a 5-D asymptotically dS space-time from nothing. However, similarly to solution (4.3), this instanton is singular at the hypersurface \( y = y_1 \) when \( k_{eff} < 1 \), because the scalar curvature reaches infinite value when \( y \) approaches \( y_1 \).
4.3 Zero cosmological constant: $\Lambda_5 = 0$

If the 5-D cosmological constant vanishes, the scale factor solution of Eq. (5.2) is

$$a(y) = a_1 \pm \sqrt{k_{eff}} (y - y_1), \quad -a_1 \leq \pm \sqrt{k_{eff}} (y - y_1),$$

while the instanton topology corresponds to $\mathbb{R} \times S^4$ and $0 \leq k_{eff} \leq 1$. When the effective curvature parameter equals unity (absence of matter), the previous solution describes a flat 5-D Euclidean space sliced into 4-D spherical sections. On the other hand, if $k_{eff} = 0$, the instanton has an arbitrary constant warp factor which can have a runaway behaviour similar to Einstein universe. Therefore, in this case, the solution is unstable. For the remaining cases, $0 < k_{eff} < 1$, the instanton is asymptotically flat and has a singularity at the origin of the extra dimension $y = y_1 \mp a_1/\sqrt{k_{eff}}$.

5 Hyperdomain wall instantons

Since we are working in 5-D spaces, there can still be another kind of topological defects which might have been formed during early phase transition. We have called them hyperdomain walls and they are 3-D objects. Analogously, to strings and domains walls, these topological defects could give rise to networks of frustrated hyperdomain walls (NFW) depending on the underlying broken symmetry. The energy density of a NFW scales as $a^{-1}$ and can also be effectively modeled either by a scalar field or by a perfect fluid, with the state equation $P = -3/4 \rho$ (this corresponds to $\alpha = 1/4$ in Eq. (2.4)). A 5-D universe filled with this matter content has a dynamical behaviour similar to a 4-D universe filled with NFW, as the energy density in both cases has the same behaviour respect to the scale factor.

5.1 Negative cosmological constant: $\Lambda_5 < 0$

We can rewrite the constraint equation (2.4) in the proper Euclidean time gauge for $\alpha = 1/4$ as follows:

$$\left( \frac{da}{dy} \right)^2 + \text{sign}(\Lambda)|\Lambda|a^2 + \bar{A}^2 a - k = 0.$$  

(5.1)

If $\Lambda_5 < 0$ and the 5-D space is filled with a scalar field with $P = -3/4 \rho$, there is a solution which describes Euclidean asymptotically AdS wormholes whose scale factor reads

$$a(y) = \frac{1}{2|\Lambda|} \left\{ \bar{A}^2 - \sqrt{\bar{q}} \cosh \left[ \sqrt{|\Lambda|} (y - y_1) \right] \right\}, \quad y \in \mathbb{R}.$$  

(5.2)

Here, the parameter $q$, defined as $q := \bar{A}^4 - 4k|\Lambda|$ and $k = \pm 1, 0$; is strictly positive. This wormhole can be sliced into flat, hyperbolic or spherical 4-D sections and its throat radius, $a_1 = a(y_1)$, located at $y = y_1$ is smaller when $k = 1$. There is a collapsing baby universe branching off from the throat of the wormhole. When the 4-D geometry is flat or hyperbolic, the scale factor of the baby universe takes values on $[0, a_1]$, while for $k = 1$, the baby universe has a minimum radius different from zero $a_0 = (\bar{A}^2 - \sqrt{\bar{q}})/(2|\Lambda|)$. In this case, $q > 0$ and $k = 1$, the Lorentzian metric of the universe interpolates between the wormhole solution (5.3), with $k = 1$, and the instanton

$$a(y) = \frac{1}{2|\Lambda|} \left\{ \bar{A}^2 - \sqrt{\bar{q}} \cosh \left[ \sqrt{|\Lambda|} (y - y_0) \right] \right\}, \quad 0 \leq \sqrt{|\Lambda|} |y - y_0| \leq \arccosh \left( \frac{\bar{A}^2}{\sqrt{\bar{q}}} \right).$$  

(5.3)

The scale factor (5.3) vanishes at $y = \arccosh(\bar{A}^2/\sqrt{\bar{q}})$ and reaches its maximum value, $a_0$, at $y = y_0$, where $y_0$ is an integration constant.

When the parameter $q = 0$, i.e. $\bar{A}^2 = 4k|\Lambda|$, there are 5-D instanton solutions only for $k = 1$, and the scale factor reads

$$a(y) = \left( a_1 - \frac{1}{\sqrt{|\Lambda|}} \right) \exp \left[ \sqrt{|\Lambda|} (y - y_1) \right] + \frac{1}{\sqrt{|\Lambda|}}.$$  

(5.4)

On the one hand, if $a_1 > 1/\sqrt{|\Lambda|}$, $\sqrt{|\Lambda|} (y - y_1) \in \mathbb{R}$, the scale factor is larger than $1/\sqrt{|\Lambda|}$ and approaches $1/\sqrt{|\Lambda|}$ when $\sqrt{|\Lambda|} (y - y_1) \to -\infty$. This solution is asymptotically AdS (it can be easily checked, that scalar curvature $R \to -20|\Lambda|$ as $a(y) \to +\infty$). On the other hand, if $a_1 < 1/\sqrt{|\Lambda|}$, $\sqrt{|\Lambda|} (y - y_1) \in (-\infty, -\ln(1 - a_1 \sqrt{|\Lambda|})]$ and the scale factor can take on values in the interval $(0, 1/\sqrt{|\Lambda|})$. When

\footnote{A small perturbation on the effective spatial curvature parameter leads to a linear growth of the instanton scale factor.}

\footnote{This Lorentzian metric has the form of equation (2.4), with $\gamma = 0$, where the scale factor $a$ is obtained either from (5.2) or from (5.3) with the help of the Wick rotation: $y - y_1 \to i(\tau - \tau_0)$ or $y - y_0 \to i(\tau - \tau_0)$, i.e., $a(\tau) = 1/(2|\Lambda|) \left\{ \bar{A}^2 - \sqrt{\bar{q}} \cos \left[ \sqrt{|\Lambda|} (\tau - \tau_0) \right] \right\}$, $0 \leq |\tau - \tau_0| \leq \pi/\sqrt{|\Lambda|}$.}
\[ \sqrt{\Lambda}(y - y_1) = -\ln(1 - a_1 \sqrt{\Lambda}), \] a vanishes and the scalar curvature diverges, while for \( \sqrt{\Lambda}(y - y_1) \to -\infty, \) a approaches \( 1/\sqrt{\Lambda}. \) In both cases, it is impossible to perform an analytical continuation to a Lorentzian metric whose proper time would be \( \tau. \) This can be easily seen by checking that the extrinsic 5-D curvature does not vanish for any finite value of the Euclidean proper time \( y. \) It must be noted that there is a "static solution", where the scale factor takes a very particular value \( a = 1/\sqrt{\Lambda}. \) Similar to the 4-D static Einstein universe, this instanton is unstable: a small variation of the cosmological constant or the matter content (through \( \Lambda \)), leads to an exponential increasing or decreasing of the scale factor (with \( y \)).

Finally, when the parameter \( q \) is negative, there is a 5-D Euclidean solution whose scale factor is

\[ a(y) = \frac{1}{2\sqrt{\Lambda}} \left\{ \sqrt{\Lambda^2 + \frac{1}{4} \ln(1 - a_2 \sqrt{\Lambda})} \right\}, \quad \sqrt{\Lambda}(y - y_1) \geq -\arcsinh \left[ \frac{\Lambda^2}{\sqrt{-q}} \right]. \]  

The topology of the previous instanton is \( \mathbb{R} \times S^4 \) and it is asymptotically AdS. The scale factor can take any positive value. Moreover, it vanishes when \( \sqrt{\Lambda}(y - y_1) = -\arcsinh \left[ \frac{\Lambda^2}{\sqrt{-q}} \right], \) where the scalar curvature diverges and consequently the instanton is singular at that 4-D hypersurface.

### 5.2 Positive cosmological constant: \( \Lambda_5 > 0 \)

It can be easily seen that for positive bulk cosmological constant, the Euclidean equation (5.1) has a unique instanton solution which is sliced into 4-D spherical \( y = \text{const} \) sections and whose scale factor reads

\[ a(y) = \frac{1}{2\Lambda} \left\{ -\Lambda^2 + \sqrt{\Lambda^4 + 4\Lambda \cos \left[ \sqrt{\Lambda}(y - y_1) \right]} \right\}, \quad 0 \leq \sqrt{\Lambda}|y - y_1| \leq \arccos \left( \frac{\Lambda^2}{\sqrt{\Lambda^4 + 4\Lambda}} \right). \]  

The scale factor of this instanton solution vanishes when \( \sqrt{\Lambda}(y - y_1) = \arccos \left( \frac{\Lambda^2}{\sqrt{\Lambda^4 + 4\Lambda}} \right). \) Moreover, the vanishing of the scale factor induces a singularity as the scalar curvature diverges in this hypersurface. It must be also noted that there is a 5-D Lorentzian universe branching off from this instantonic solution at \( y = y_1. \) In fact, if we perform the analytical continuation \( \sqrt{\Lambda}(y - y_1) \to i\sqrt{\Lambda}(\tau - \tau_1), \) we obtain a 5-D asymptotically dS universe.

### 5.3 Zero cosmological constant: \( \Lambda_5 = 0 \)

Analogously to the positive cosmological constant case, the constraint equation (5.1) has a unique non-trivial solution when \( \Lambda_5 = 0. \) \( \) The scale factor of the instanton is

\[ a(y) = \frac{1}{\Lambda^2} \left[ 1 - \Lambda^4 (y - y_1)^2 \right], \quad 0 \leq |y - y_1| \leq \frac{2}{\Lambda^2}, \]  

and has the topology \( \mathbb{R} \times S^4. \) This Euclidean solution has a singularity at \( |y - y_1| = 2/\Lambda^2, \) where the scale factor vanishes. There is a 5-D universe branching off from the hypersurface corresponding to the maximum radius of the scale factor, i.e. \( y = y_1. \)

### 6 Conclusions

In the present paper we have described 5-D gravitational instantons whose geometry is homogeneous and isotropic in a space which is filled with a cosmological constant, \( \Lambda_5, \) and a minimally coupled scalar field, \( \varphi, \) that satisfies a perfect fluid state equation. In particular, we have chosen the 5-D space to be sliced into 4-D constant curvature spaces. However, all our solutions are valid when the \( y = \text{const} \) sections are 4-D Einstein spaces with constant scalar curvature. The scalar field potential, \( V(\varphi), \) is strongly constrained by the perfect fluid state equation \( P = (\alpha - 1)\rho. \) Indeed, we have seen that in terms of the scale factor \( a, \) \( V(\varphi) \) satisfies an inverse power law, whenever \( \alpha > 0. \)

The matter content in the model can equivalently be represented either by a minimally coupled scalar field or by a perfect fluid with a given state equation. In both cases, it can be noticed that in the Euclidean sector, the energy density \( \rho \) is frozen. On the other hand, we have seen that \( \varphi \) can mimic different types of networks of frustrated topological defects, including cosmic strings (\( \alpha = 3/4), \) domain walls (\( \alpha = 1/2) \) and hyperdomain walls (\( \alpha = 1/4). \) For all of these equations of state and for the different possibilities of the cosmological constant (positive, negative and zero) we have got the behaviour of the scale factor \( a. \) These instantons are asymptotically AdS when \( \Lambda_5 \) is negative and the scale factor takes on large values \( (a \to +\infty). \) Moreover, we have checked that some of the Euclidean solutions are wormholes.

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\(^3\text{A fine tuning between the matter content and the spatial curvature: } k = \Lambda^2a, \text{ leads to a "static trivial solution" with constant scale factor, } a = \Lambda^{-2}, \text{ and 4-D spherical } y = \text{const sections. This solution is unstable. For example, small fluctuations around its Lorentzian counterpart will grow with time quadratically.}\)
The obtained instantons are useful to construct nonsingular brane-world models, such as we have explicitly done in section 4 for solution [1,4]. There, we have seen that an Euclidean space with a negative cosmological constant and a NFDW matter content when \(k_{eff} = 0\), i.e. \(A = 1\) and \(k = 1\), has a scale factor similar to those of RS1 and RS2 models [4]. This space can be used to build a brane-world instanton by a cutting and gluing procedure. Some of the advantages of this model are that it is not singular and can describe the birth of an infatiguing brane in the semiclassical approximation. Here, infatation is due only to the geometrical construction. Since the sections of constant Euclidean "time" \((y = \text{const})\) are spherical, the analytical continuation along the azimuthal coordinate, \(\chi\), of the 4-D sphere, \(\chi \rightarrow iHt + \pi/2\), results in a brane whose geometry is a 4-D dS space-time. Here, the bulk energy density \(\rho\) remains frozen after the analytical continuation.

The cutting and gluing procedure can also be used to construct models with an arbitrary number of parallel branes [3]. These branes can either be inflating or static, after their birth, whenever the topology of the \(y = \text{const}\) are \(S^4\) or \(\mathbb{R}^4\), respectively. It must be noted that we can also use instantons with a singularity at some hypersurface \(y = \text{const}\), by introducing an additional brane and removing from the brane-world instanton the singular region. This could lead to the appearance of branes with negative tension.

It is worth remarking that model \((2.2)\) can be generalized for an arbitrary number of dimensions: \(g^{(d)} \rightarrow g^{(d)}\) and \(\mathbb{R} \times M^d \rightarrow \mathbb{R} \times M^d\), where the d-dimensional manifold \(M^d\) undergoes a topological splitting into \(n\) Einstein spaces: \(M^d = \prod_{i=1}^{n} M^{d_i}\), \(g^{(d)} = \sum_{i=1}^{n} g^{(d_i)}\), \(R_{\mu\nu}[g^{(d_i)}] = \lambda g^{(d_i)}\), \(d = \sum_{i=1}^{n} d_i\), with \(\lambda_i > 0\) and \(\lambda = \prod_{i=1}^{n} \lambda_i\). Now, using the conformal transformation \(g^{(d_i)} (\lambda_i/\lambda) g^{(d_i)}\), it can be easily shown [3] that the constituent manifold \(M^d\) is also the Einstein space: \(R_{\mu\nu}[g^{(d)}] = \lambda g^{(d)}\), and the scale factor \(a(\tau)\) satisfies the \((D = d + 1)\)-dimensional analog of Eq. \((2.7)\):

\[
(\frac{\dot{a}}{a})^2 + \frac{R_d}{d(d-1) a^2} - 2 \frac{\Lambda_D}{d(d-1)} A a^{-\alpha_d} = 0, \tag{6.1}
\]

where \(R[g^{(d)}] = R_d = d\lambda := d(d-1)k\). Obviously, cosmic strings correspond to \(\alpha = (d-1)/d\), domain walls have \(\alpha = (d-2)/d\), etc. For example, the warp factor of a model filled with a network of \((d-2)\)-dimensional topological defects is described in Euclidean region by Eq. \((6.1)\) (in the case of fine tuning \(2\kappa^2 D_A d(d-1) \equiv A^2 = 1\) and negative cosmological constant \(\Lambda \equiv 2\Lambda_D d(d-1)\)).

Thus, the brane-world instantons, as well as the birth of the brane-worlds from them, can be constructed for this \((d + 1)\)-dimensional model following a completely parallel procedure to that was applied to solution \([2,3]\). By instance, in the particular case \(M^d = S^4 \times \prod_{i=1}^{n} S^{d_i}\), the analytic continuation \(\chi \rightarrow iHt + \pi/2\) with respect to the coordinate \(\chi\) of the 4-sphere \(S^4\) results in the following Lorentzian metric:

\[
ds^2_L = dg^2 + H^2 a^2(y)(-dt^2 + \frac{1}{H^2} \cosh^2 Ht d\Omega^2_4) + \frac{r^2_i}{H^2} d\Omega^2(d_i), \tag{6.2}
\]

where \(r_i, i = 1, \ldots, n\) are the radii of the \(d_i\)-spheres and \(n\) the number of spheres. We thus arrive at a brane-world model with inflating 4-D part of the brane, plus frozen, compactified and unobservable (for \(r_i \lesssim 10^{-17}\) cm) \((d - 4)\) dimensions on it. This scenario, with a brane of codimension one, is of interest because:

1. It points out to the very interesting possibility for the unification of new brane-world scenarios with the standard Kaluza–Klein approach.
2. It gives a possibility for the localization of the gauge fields on the brane [3,28, 29].

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4 As it follows from equation (2) of reference [29], only for \(n > 0\), where \(n\) is the number of additional compactified dimensions, such localized gauge fields are normalizable.
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