Rapidity Divergences and Deep Inelastic Scattering in the Endpoint Region

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Abstract

The deep inelastic scattering cross section in the endpoint region, \( x \sim 1 \), has been subjected to extensive analysis. We revisit this process, and show that in the endpoint individual factors in the factorized hadronic tensor have rapidity divergences. We regulate these divergences using a recently introduced rapidity regulator, and find that each factor requires a different scale to minimize large rapidity logarithms. Interestingly, in this case running in rapidity is non-perturbative. In addition, we give an operator definition of the parton distribution function in the endpoint region, and remark on the issues that should be considered in constructing this function.

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I. INTRODUCTION

Deep inelastic scattering (DIS) has been crucial in developing our understanding of QCD since the first high energy experiments at the Stanford linear accelerator in 1967 [1]. These early experiments gave rise to Feynman’s parton model, and subsequent DIS experiments have allowed us to further refine our understanding of the structure of nucleons. In this paper we explore DIS in a corner of phase space, where the light-cone momentum fraction, $x$, of the struck quark nears its maximal value, $x \sim 1$. Ours is not the first analysis that has scrutinized this endpoint regime. Factorization and resummation of the DIS cross section for $x \sim 1$ was first investigated in Refs. [2–5] using QCD factorization methods. Later, with the development of soft collinear effective theory (SCET) [6–8], DIS in the endpoint region was revisited in the context of effective field theory [9–15].

In this work we use SCET to study the $x \sim 1$ region of DIS, and focus on the definition of each term in the factorized form of the hadronic tensor. We repeat the derivation of the factorization of the DIS hadronic tensor into a hard coefficient, a jet function, a collinear factor, and a soft function. Each of these pieces is well defined in SCET. The hard coefficient comes from the matching of SCET onto QCD, while the jet function, collinear factor, and soft function are matrix elements of SCET operators. The jet function consists of all radiation that is collinear to the final state, while the collinear factor consists of all radiation collinear to the initial state. The soft function includes soft radiation from both the initial and final state. Though the properties of the hard coefficient and jet function are well known, the collinear factor and soft function have not been explored as thoroughly, and it is these latter two objects that we focus our attention on.

We show through an explicit one-loop calculation that rapidity divergences in the collinear factor and the soft function cancel against each other. In order to rigorously determine the collinear matrix element and the soft matrix element, we must admit a rapidity regulator into SCET, for which in this paper we adopt the choice that has already been introduced in Refs. [16, 17]. Our calculations have a number of interesting aspects. First, we find that label momentum conservation prohibits the emission of real collinear radiation from the initial state that is not also soft. Then, combining the fact that SCET is defined with a collinear zero-bin [18] in which any overlap of collinear degrees of freedom with soft ones must be subtracted from the collinear sector, the collinear factor only has virtual contributions as
was first pointed out in Ref. [15]. Second, since the soft and collinear functions are both described in SCET II, there exists another soft zero-bin [18] in which any overlap of the soft degrees of freedom with collinear ones also has to be subtracted from soft contributions. Third, we find that the choice of scale which minimizes large rapidity logarithms is different in the collinear factor and soft function so that a resummation of rapidity logarithms is needed. Remarkably, we notice this running in rapidity is non-perturbative, and therefore it can be absorbed into the non-perturbative soft function. Finally, we construct a definition for the parton distribution function (PDF) in the endpoint region that is consistent with two principles: that the PDF does not depend on the final state, and that it has DGLAP type running. The soft function could not be included in the PDF because it encodes information from both initial and final states, a fact we account for in our definition by introducing a vacuum matrix element of soft Wilson lines only sensitive to the initial state. This results in a final formula for the DIS hadronic tensor that is a convolution of the PDF with a new soft function containing only final state radiation.

II. FACTORIZATION

In this section, we repeat the derivation of the factorization with the DIS hadronic tensor using SCET. We work in Breit frame where the incoming proton moves along the $-\hat{z}$ direction with energy much larger than the proton mass $m_p$, so that the proton momentum is

$$p^\mu = \frac{\sqrt{s}}{2} n^\mu + \frac{m_p^2}{2\sqrt{s}} \bar{n}^\mu,$$

(1)

where $n^\mu = (1, 0, 0, -1)$, $s = (p + k)^2$, is the center-of-mass energy squared, and $m_p$ is the proton mass. Particles collinear to the proton have momentum

$$p^\mu_n = \frac{1}{2} \bar{n} \cdot p_n n^\mu + \frac{1}{2} n \cdot p_n \bar{n}^\mu + p^\mu_{n,\perp},$$

(2)

where components differ parametrically in their sizes: $\bar{n} \cdot p_n \sim \sqrt{s}$, $p^\mu_{n,\perp} \sim \Lambda_{\text{had}} / \sqrt{s}$, and $n \cdot p_n \sim (\Lambda_{\text{had}} / \sqrt{s})^2$, with $\Lambda_{\text{had}} \sim m_p$ a typical hadronic scale. The incoming proton is struck by a virtual gluon of momentum $q^\mu$ with large invariant mass squared: $-q^2 \equiv Q^2$. The final state momentum is restricted by momentum conservation to be $p_X = p + q$ with invariant mass squared

$$M_X^2 = (p + q)^2 = \frac{Q^2}{x} (1 - x) + m_p^2 \approx \frac{Q^2}{x} (1 - x), \quad x = \frac{Q^2}{2p_p \cdot q}.$$
In the endpoint region $x \sim 1$ the invariant mass is small compared to $Q$. At this point we take $Q\sqrt{1-x} \gg \Lambda_{\text{had}}$, however, we do not fix the scale $Q(1-x)$ relative to $\Lambda_{\text{had}}$ since, as we will see, the rapidity regulator separates these scales. Thus the total final state momentum in the endpoint region is collinear, and any final state collinear particle will have momentum
\[
p_{\bar{n}} = \frac{1}{2} n \cdot p_{\bar{n}} \bar{n}^\mu + \frac{1}{2} \bar{n} \cdot p_{\bar{n}} n^\mu + p_{\bar{n},\perp}^\mu ,
\] (4)

where $n \cdot p_{\bar{n}} \sim Q$, $p_{\bar{n},\perp} \sim Q \lambda$ and $\bar{n} \cdot p_{\bar{n}} \sim Q^2 \lambda^2$ with $\lambda \sim \sqrt{1-x}$. Finally, we have
\[
q^\mu = \frac{Q}{2} (\bar{n}^\mu - n^\mu) ,
\] (5)

and
\[
s = Q^2 + M_X^2 + m_p^2 = Q^2 + \mathcal{O}(\lambda^2) .
\] (6)

Thus in the Breit frame, up to correction of order $\lambda^2$ the large lightcone momentum component of the proton is
\[
\bar{n} \cdot p \approx Q .
\] (7)

In our analysis we follow a two-step procedure: in the first step we match from QCD onto SCET$_I$ where the offshellness of collinear momentum scales as $p_c^2 \sim Q^2 \lambda^2$. In the next step we integrate out the final state collinear fields and match onto SCET$_{II}$ where the offshellness of collinear fields scale as $p_c^2 \sim \Lambda_{\text{had}}^2$. The first step is straightforward and has been covered in detail in Ref. [9]. In this work we are only concerned with the second one. For our purposes the best way to proceed is to factor the DIS differential cross section in SCET$_I$ and then to match onto SCET$_{II}$. Following Ref. [19], the DIS cross section is
\[
\sigma = \frac{d^3k'}{2|k'|(2\pi)^3} \frac{\pi e^4}{sQ^4} L_{\mu\nu}(k,k') W^{\mu\nu}(p,q) ,
\] (8)

where $k$ and $k'$ are the incoming and outgoing lepton momenta, $q = k - k'$, and
\[
L_{\mu\nu} = 2(k_\mu k'_\nu + k_\nu k'_\mu - k \cdot k' g_{\mu\nu}) .
\] (9)

The DIS hadronic tensor is
\[
W^{\mu\nu}(p,q) = \frac{1}{2} \sum_\sigma \int d^4x e^{iq \cdot x} \langle h(p,\sigma)|J^\mu(x)J^\nu(0)|h(p,\sigma)\rangle , \
J^\mu(x) = \bar{\psi}(x)\gamma^\mu \psi(x) ,
\] (10)
with external proton state \( h(p, \sigma) \) with momentum \( p \) and spin \( \sigma \). The QCD current in Eq. (10) matches onto an SCET current of the form

\[
J^\mu_{\text{eff}}(x) = \bar{\chi}_{n,\omega_2} \gamma_\perp^\mu \chi_{n,\omega_1}(x) + \text{h.c.},
\]

where \( \text{h.c.} \) stands for the hermitian conjugate. The matching is

\[
J^\mu(x) \to \sum_{\omega_1,\omega_2} C(\omega_1,\omega_2; \mu_q, \mu) \left( e^{-\frac{i}{2} \omega_1 n \cdot x} e^{\frac{i}{2} \omega_2 n \cdot x} \chi_{n,\omega_2} \gamma_\perp^\mu \chi_{n,\omega_1}(x) + \text{h.c.} \right),
\]

where the matching coefficient \( C(\omega_1,\omega_2; \mu_q, \mu) \) depends on a factorization scale \( \mu_q \) at which the matching onto QCD is carried out, and a running scale \( \mu \). Here

\[
\gamma_\perp^\mu \equiv \gamma^\mu - \frac{1}{2} \gamma^\parallel\gamma^\mu - \frac{1}{2} \gamma^\parallel\gamma^\mu.
\]

From Eq. (10) we determine the hadronic tensor in SCET",

\[
W_{\text{eff}}^{\mu\nu} = \sum_{\omega_1,\omega_2,\omega_1',\omega_2'} C^*(\omega_1,\omega_2; \mu_q, \mu) C(\omega_1',\omega_2'; \mu_q, \mu) \int \frac{d^4x}{4\pi} e^{-\frac{i}{2} (Q-\omega_1)n \cdot x} e^{\frac{i}{2} (Q-\omega_2)n \cdot x}
\]

\[
\times \frac{1}{2} \sum_{\sigma} \delta_{\bar{n},\bar{p},Q} \langle h_n(p, \sigma)|T\left[ \bar{\chi}_{n,\omega_1} \gamma_\perp^\mu \chi_{n,\omega_1}(x) \right] T\left[ \bar{\chi}_{n,\omega_2'} \gamma_\perp^\nu \chi_{n,\omega_2'}(0) \right]|h_n(p, \sigma) \rangle
\]

\[
= \sum_{\omega_1,\omega_2,\omega_1',\omega_2'} \delta_{Q,\omega_1} \delta_{Q,\omega_2} C^*(\omega_1,\omega_2; \mu_q, \mu) C(\omega_1',\omega_2'; \mu_q, \mu) \int \frac{d^4x}{4\pi}
\]

\[
\times \frac{1}{2} \sum_{\sigma} \delta_{\bar{n},\bar{p},Q} \langle h_n(p, \sigma)|T\left[ \bar{\chi}_{n,\omega_1} \gamma_\perp^\mu \chi_{n,\omega_1}(x) \right] T\left[ \bar{\chi}_{n,\omega_2'} \gamma_\perp^\nu \chi_{n,\omega_2'}(0) \right]|h_n(p, \sigma) \rangle,
\]

where \( T \) denotes time ordering, \( \bar{T} \) anti-time ordering, and \( h_n(p, \sigma) \) denotes the SCET proton state. Here we have inserted an explicit Kronecker delta that ensures the proton label momentum is equal to \( Q \) as fixed by momentum conservation in Eq. (7). Usoft gluons in SCET1 can be decoupled from collinear modes via the BPS phase redefinition \[8\], and the hadronic tensor above can be factored into the matrix elements of operators in each of the two collinear sectors and the usoft sector:

\[
W_{\text{eff}}^{\mu\nu} = \frac{-g_{\mu\nu}}{2} N_c \sum_{\omega_1,\omega_2} C^*(Q, Q; \mu_q, \mu) C(\omega_1',\omega_2'; \mu_q, \mu) \int \frac{d^4x}{4\pi}
\]

\[
\times \frac{1}{2} \sum_{\sigma} \delta_{\bar{n},\bar{p},Q} \langle h_n(p, \sigma)|\bar{\chi}_{n,Q}(x) \frac{i}{2} \chi_{n,\omega_1}(0)|h_n(p, \sigma) \rangle
\]

\[
\times \langle 0| \frac{1}{2} \bar{\chi}_{n,Q}(x) \chi_{n,\omega_2}(0) |0 \rangle
\]

\[
\times \frac{1}{N_c} \langle 0| \text{Tr} \left( T \left[ Y_n^\dagger(x) \bar{Y}_n(x) \right] T \left[ \bar{Y}_n^\dagger(0) Y_n(0) \right] \right) |0 \rangle.
\]
The Wilson lines $Y_n$ and $\bar{Y}_n$ are associated with soft radiation from the initial and final state respectively and are defined as

$$Y_n(x) = P \exp \left( ig \int_{-\infty}^x ds \, n \cdot A_{us}(sn) \right)$$

$$\bar{Y}_n(x) = P \exp \left( ig \int_x^\infty ds \, \bar{n} \cdot A_{us}(s\bar{n}) \right).$$

Next we define a jet function

$$\langle 0 | \frac{i}{2} \chi_{n,\omega_\perp}(x) \bar{Y}_n(\omega_\perp) | 0 \rangle \equiv Q \delta(\bar{n} \cdot x) \delta^{(2)}(x) \int dr \, e^{-\frac{i}{2} r \cdot x} J_n(r; \mu),$$

and a soft function

$$\frac{1}{N_c} \langle 0 | \text{Tr} \left( \bar{T} \left[ Y_n(\omega_\perp) \bar{Y}_n(\omega_\perp) \right] \bar{T} \left[ \bar{Y}_n(0) Y_n(0) \right] \right) | 0 \rangle \equiv \int d\ell \, e^{-\frac{i}{2} \ell \cdot x} S(\ell; \mu).$$

Furthermore, using label momentum conservation we simplify the collinear matrix element in the $n$ sector:

$$\langle h_n(p, \sigma) | \bar{\chi}_{n,Q}(x) \frac{i}{2} \chi_{n,\omega'_\perp}(0) | h_n(p, \sigma) \rangle = \delta_{Q,\omega'_\perp} \langle h_n(p, \sigma) | \bar{\chi}_{n}(x) \frac{i}{2} \delta_{P,2Q} \chi_{n}(0) | h_n(p, \sigma) \rangle,$$

where $P = \bar{n} (P + P')$, and $\bar{n} \cdot \bar{P}$ is the large component of the proton momentum. Combining with the above relations, Eq. (15) becomes

$$W^{\mu\nu}_{\text{eff}} = -g^{\mu\nu}_{\perp} H(Q; \mu, \mu) \int dr d\ell \, J_n(r; \mu) S(\ell; \mu) \int \frac{d n \cdot x}{4\pi} e^{-\frac{i}{2} (r + \ell) \cdot n \cdot x}$$

$$\times \frac{1}{2} \sum_{\sigma} \delta_{n,\bar{P},Q} \langle h_n(p, \sigma) | \bar{\chi}_{n}(n \cdot x) \frac{i}{2} \delta_{P,2Q} \chi_{n}(0) | h_n(p, \sigma) \rangle,$$

where

$$H(Q) = Q |C(Q, Q; \mu, \mu)|^2.$$  

Finally, we introduce a $n$-collinear function

$$C_n(Q - k; \mu) = \int \frac{d n \cdot x}{4\pi} e^{ik \cdot n \cdot x} \frac{1}{2} \sum_{\sigma} \delta_{n,\bar{P},Q} \langle h_n(p, \sigma) | \bar{\chi}_{n}(n \cdot x) \frac{i}{2} \delta_{P,2Q} \chi_{n}(0) | h_n(p, \sigma) \rangle$$

$$= \frac{1}{2} \sum_{\sigma} \delta_{n,\bar{P},Q} \langle h_n(p, \sigma) | \bar{\chi}_{n}(0) \frac{i}{2} \delta_{P,2Q} \delta(i\bar{n} \cdot \partial - k) \chi_{n}(0) | h_n(p, \sigma) \rangle.$$

We do not identify the above function with the PDF for reasons we will discuss later. Using the above definition in Eq. (20) we arrive at our final expression for the factored form of the
DIS hadronic tensor in SCET\(_I\) :

\[
W_{\text{eff}}^{\mu\nu} = -g_\perp^{\mu\nu} H(Q; \mu_q, \mu) \int dr d\ell J_n(r; \mu) S(\ell; \mu) C_n(Q + r + \ell; \mu).
\] (23)

The \(\mu\) dependence of the hard coefficient \(H\) is such that it exactly cancels the \(\mu\) dependence of the product of the collinear and soft functions.

It is now straightforward to match Eq. (23) onto SCET\(_II\). The jet function \(J_n(r; \mu)\) characterizes the final state with typical offshellness \(p_x^2 \sim Q^2(1 - x)\), and thus it must be integrated out at the scale \(\mu_c \sim Q\sqrt{1 - x}\). This is the equivalent of carrying out an operator product expansion (OPE) on \(W_{\text{eff}}^{\mu\nu}\). In practice the jet function becomes a coefficient which is calculated at the collinear scale \(\mu_c\). The usoft gluons of SCET\(_I\) become soft gluons in SCET\(_II\), so \(S(\ell; \mu)\) remains unchanged, and because the off-shellness of the \(n\) collinear degrees of freedom changes from \(p_c^2 \sim Q^2(1 - x)\) in SCET\(_I\) to \(p_c^2 \sim \Lambda_{\text{had}}^2\) in SCET\(_II\), \(C_n\) remains unchanged. Furthermore, as was pointed out in Refs. [16, 17] the factorization of soft and collinear modes in SCET\(_II\) requires an additional regulator which separates rapidity regions. Finally, as we will show below there can be no collinear radiation into the final state in the \(x \sim 1\) region so the collinear function can be expressed as

\[
C_n(Q - k; \mu, \nu) = Z_n(\mu, \nu) \delta_{\bar{n}, \tilde{p}, Q} \delta(k),
\] (24)

where \(\nu\) plays the role of a dimensionless rapidity scale separating soft and collinear rapidity regions. Thus, in SCET\(_II\) the hadronic tensor is

\[
W_{\text{eff}}^{\mu\nu} = -g_\perp^{\mu\nu} H(Q; \mu_q, \mu_c) \int d\ell J_n(\ell; \mu_c, \mu) \phi_\perp^{n_s}(Q(1 - x) + \ell; \mu),
\] (25)

with

\[
\phi_\perp^{n_s}(Q(1 - x) + \ell; \mu) = \delta_{\bar{n}, \tilde{p}, Q} Z_n(\mu, \nu) S(\ell; \mu, \nu),
\] (26)

being a non-perturbative function. The scale \(\nu\) that minimizes rapidity logarithms in the collinear factor is different from the \(\nu\) that minimizes rapidity logarithms in the soft function. Moreover, while the collinear factor depends only on initial state collinear radiation, the soft function depends on both initial and final state soft radiation. As we will discuss in more detail in Sect. [VI] this difference makes an identification of \(\phi_\perp^{n_s}\) with the PDF problematic.

Our expression agrees with the expression in Ref. [15] up to the appearance of the rapidity.
regulator which however was not considered in their work. The hard and jet functions and their evolution have been studied extensively in the literature to which we refer the reader for details.

III. THE COLLINEAR FUNCTION

In this section we study the collinear function $C_n(Q - k; \mu, \nu)$. We argue that label momentum conservation and the zero bin subtraction allows no real radiation of $n$-collinear particle into the final state and correspondingly this function involves only virtual corrections, and then show this explicitly at one loop in perturbation theory. In addition, up to the same order we show the need for a rapidity regulator and determine the value of the rapidity scale $\nu$ which minimizes rapidity logarithms.

The label momentum conserving Kronecker delta $\delta_{\bar{n}, \tilde{p}, Q}$ in Eq. (22) forces the external proton label momentum to be equal to $Q$, while the Kronecker delta $\delta_{\bar{P}, 2Q}$ requires that each $\chi_n$ field has total label moment $Q$ as well. Thus, any momentum that flows from the $\bar{\chi}_n$ field on the left side to the $\chi_n$ field on the right must have zero label momentum. Any field that causes momentum to flow in this way corresponds to real radiation (as it must cross the cut). Since SCET is formulated with an explicit zero-bin subtraction, collinear fields with zero label momentum vanish, which means that there can be no real radiation of $n$-collinear particles. This is just a manifestation of momentum conservation: only soft radiation from the initial state into the final state is allowed otherwise we are no longer in the $x \sim 1$ region.

Let us consider an explicit calculation of $C_n(Q - k; \mu, \nu)$ to order $\alpha_s$ using external parton states. The $O(\alpha_s^{(0)})$ Feynman diagram is shown in Fig. [I] and gives the tree level result

\[
C_n(Q - k)^{(0)} = \delta_{\bar{n}, \tilde{p}, Q} \delta_{\bar{n}, \tilde{p}, Q} \delta(n \cdot p_r - k) m_0
\]

(27)

\[
= \delta_{\bar{n}, \tilde{p}, Q} \delta(k) m_0
\]

FIG. 1: The $O(\alpha_s^{(0)})$ Feynman diagram for the $n$ collinear function. The dashed lines are collinear quarks, the grey circles are vertices where momentum is injected, and the gap indicates a cut.
where $\vec{n} \cdot \vec{p}$ is the $\mathcal{O}(1)$ quark label momentum, and $p_r$ is the $\mathcal{O}(\lambda^2)$ quark residual momentum, which is zero for an on-shell quark. The two Kronecker deltas in the first line come directly from the definition of the operator in Eq. (22); in the second line one of them has been set to one. Here

$$m_0 = \frac{1}{2} \sum_\sigma \xi^\sigma \bar{\xi}_1 \xi^\sigma,$$

(28)

where $\xi^\sigma$ are SCET quark spinors with spin $\sigma$.

Three of the five $\mathcal{O}(\alpha_s)$ Feynman diagram for $C_n(Q-k; \mu, \nu)$ are shown in Fig. 2. The remaining two diagrams are obtained by the reflection of diagrams (a) and (b) about a vertical axis through the middle of the diagram. The amplitude corresponding to diagram

(a) is

$$im_{(a)} = im_0 (2g_s^2 C_F) \delta_{\vec{n}, \vec{\tilde{q}}, Q} \delta(\vec{n} \cdot p_r - k) \mu^2 \epsilon \sum_{\vec{n} \cdot \vec{q} \neq 0} \int \frac{d^D q_r}{(2\pi)^D} \frac{1}{\vec{n} \cdot q} \frac{1}{(p-q)^2 + i\epsilon q^2 + i\epsilon},$$

(29)

where we work in $D = 4 - 2\epsilon$ dimensions, and the external quark states have momentum $p^\mu$. This diagram gives a virtual correction since the gluon does not cross the cut. The Kronecker delta in front of the square brackets ensures that the external state label momentum is set equal to $\tilde{Q}$. The sum over the gluon label momentum is restricted to those values where $\vec{n}_1 \cdot \vec{q} \neq 0$ because collinear fields in SCET collinear fields are defined with a zero-bin subtraction so that they are nonzero only for nonzero label momentum. This prevents a double counting of degrees of freedom [18]. The amplitude obtained from diagram (b) is

$$im_{(b)} = -im_0 (2g_s^2 C_F) \delta_{\vec{n}, \vec{\tilde{q}}, Q} \mu^2 \epsilon \times \sum_{\vec{n} = \vec{q} \neq 0} \delta_{\vec{n}, \vec{q}, 0} \int \frac{d^D q_r}{(2\pi)^{D-1}} \frac{1}{\vec{n} \cdot q} \frac{1}{(p-q)^2 + i\epsilon q^2 + i\epsilon} \delta(q^2) \delta(\vec{n} \cdot p_r - k),$$

(30)
and corresponds to real radiation as the gluon crosses the cut. Note, because of label momentum conservation the real gluon must have zero label momentum, as enforced by the second Kronecker delta in the expression above. However since all collinear fields are defined with a zero-bin subtraction the sum over gluon momentum label does not include $\bar{n} \cdot \bar{q} = 0$ as explicitly indicated, which leads $im_{(b)} = 0$ and there is no real collinear radiation in the amplitude. Finally, the amplitude for diagram (c) is zero for the same reason.

Including the contribution from the reflected diagrams which are not shown in Fig. 1 the total collinear contribution will be twice that in Eq. (29).

$C_{n}(Q - k)^{(1)} = m_{0}\delta_{\bar{n} \cdot \bar{p}, Q} \delta(k) (4g_{s}^{2}C_{F}) \mu^{2\epsilon} \frac{d^{D}q}{(2\pi)^{D}} \frac{1}{\bar{n} \cdot (p - k)} \frac{1}{\bar{n} \cdot q (p - q)^{2} + i\epsilon} q^{2} + i\epsilon, \quad (31)$

where the $[0]$ subscript indicates that the integral requires a zero-bin subtraction. As was thoroughly discussed in Ref. [17] the integral in Eq. (31) contains a rapidity divergence that must be regulated properly. Here we will adopt the procedure of Ref. [17], and introduce a gluon mass to regulate infrared (IR) divergences. Then, in agreement with Ref. [17] we find

$C_{n}(Q - k)^{(1)} = m_{0}\delta_{\bar{n} \cdot \bar{p}, Q} \delta(k) \frac{\alpha_{s}C_{F}}{\pi} w^{2} \left\{ e^{\gamma \epsilon} \Gamma(\epsilon) \left( \frac{\mu^{2}}{m_{g}^{2}} \right)^{\epsilon} + \frac{1}{\epsilon} \left[ 1 + \ln \frac{\nu}{\bar{n} \cdot p} \right] \left[ 1 + \ln \frac{\mu^{2}}{m_{g}^{2}} + 1 - \frac{\pi^{2}}{6} \right] \right\}. \quad (32)$

Here $\eta$ is the rapidity regulator and $\nu$ the running rapidity scale. Clearly the logarithms in the expressions are minimized for a choice $\nu \sim \bar{n} \cdot p = Q$ and $\mu \sim \Lambda_{\text{had}}$. The divergences in $\eta$ and $\epsilon$ must be absorbed into appropriate counter-terms.

IV. THE SOFT FUNCTION

Next we turn our attention to the soft function defined in Eq. (18). Our aim is to calculate the soft function to one loop so that we can isolate the poles in $\eta$, and determine the scale which minimizes rapidity logarithms. At tree level we have the trivial result

$S(\ell)^{(0)} = \delta(\ell). \quad (33)$

The one loop result is given by the sum of the diagrams in Fig. 3 and their reflections about a vertical axis through the middle of the diagram. The gap between the vertices indicates a cut in the diagram, so diagram (a) corresponds to a virtual contribution, while diagram (b) corresponds to a real contribution. Again, in agreement with Ref. [17] we obtain
FIG. 3: Feynman diagrams for the one loop evaluation of the soft function. There are two additional diagrams which are obtained by reflecting about a vertical axis through the middle of the diagram. The double lines indicate Wilson lines which produce the gluons, and $n$ & $\bar{n}$ label the direction of the Wilson lines. The gap between vertices indicates a cut.

\[ m_\nu = \delta(\ell) \frac{2\alpha_s C_F}{\pi} w^2 \left[ -\frac{e^{\gamma_E}}{\eta} \left( \frac{\mu}{m_g} \right)^{2\epsilon} + \frac{1}{2\epsilon^2} + \frac{1}{\epsilon} \ln \frac{\mu}{\nu} + \ln^2 \frac{\mu}{m_g} - \ln \frac{\mu^2}{m_g^2} \ln \frac{\nu}{m_g} - \frac{\pi^2}{24} \right] \quad (34) \]

for the sum of diagram (a) in Fig. 3 and its mirror image. The real contribution from diagram (b) and its mirror image is

\[ \tilde{m}_r = -4C_F g_s \mu^2 w^2 \nu^n \int \frac{d^D k}{(2\pi)^{D-1}} \delta(k^2 - m_g^2) \delta(\ell - k^+) |2k^3|^{-\eta} \frac{1}{k^+ k^-} \quad (35) \]

\[ = -\frac{\alpha_s C_F}{\pi} \left( \frac{e^{\gamma_E}}{\eta} \frac{\mu^2}{m_g^2} \right)^\epsilon w^2 \nu^n \theta(\ell) \frac{1}{\ell^{1+\eta}} \Gamma(\epsilon). \]

As pointed out in Ref. [18] in SCET$_\Pi$ there are also zero bin subtractions for the soft modes: any overlap with the $n$-collinear or $\bar{n}$-collinear region must be removed. In the virtual contribution all these subtractions vanish which is why we have not yet considered them. In the real contribution, however, these subtractions are not zero. The overlap of the integral in Eq. (35) with the $n$-collinear region is given by taking the limit $k^+ \gg k^-$ with $k^+ k^- \sim k_\perp^2$

\[ s_n = -4C_F g_s \mu^2 w^2 \nu^n \int \frac{d^D k}{(2\pi)^{D-1}} \delta(k^2 - m_g^2) \delta(\ell - k^+) |k^+|^{-\eta} \frac{1}{k^+ k^-} \quad (36) \]

\[ = -\frac{\alpha_s C_F}{\pi} \left( \frac{e^{\gamma_E}}{\eta} \frac{\mu^2}{m_g^2} \right)^\epsilon w^2 \nu^n \theta(\ell) \frac{1}{\ell^{1+\eta}} \Gamma(\epsilon), \]
which is the same as the result in Eq. (35). The $\bar{n}$-collinear subtraction is given by taking the limit $k^- \gg k^+$ with $k^+k^- \sim k_\perp^2$ in the first line of Eq. (35):

$$s_{\bar{n}} = -4C_F g_s^2 \mu^2 w^2 \nu^n \int \frac{d^D k}{(2\pi)^{D-1}} \delta(k^2 - m_g^2) \delta(\ell - k^+) |k^-|^{-\eta} 1\frac{1}{k^+} \frac{1}{k^-}$$  (37)

Thus the zero bin subtracted real contribution is

$$m_r = \bar{m}_r - s_n - s_{\bar{n}} = -s_{\bar{n}}$$  (38)

where the plus-function of the dimensionfull variable $\ell$ is given in terms of the definition of a dimensionless variable $x = \ell/\kappa$

$$\frac{1}{(\ell)_+} = \frac{1}{\kappa(x)_+} + \ln \kappa \delta(\kappa x),$$  (39)

with

$$\frac{1}{(x)} \equiv \lim_{\beta \to 0} \left[ \frac{\theta(x - \beta)}{x} + \ln \beta \delta(x) \right].$$  (40)

As was pointed out in Ref. [20, 21] the net effect of the zero bin subtraction without a rapidity regulator is to divide by the square of the soft matrix element, which is equivalent in perturbation theory to subtracting the soft contribution. Here we obtain a similar result, however with a rapidity regulator care must be taken since each of the zero-bin subtractions have the absolute value of different momentum components raised to the power $-\eta$. Adding the virtual and real contributions gives the one loop expression for the soft function

$$S(\ell)(1) = \frac{\alpha_s C_F}{\pi} w^2 \left\{ - \frac{e^{\epsilon \gamma_E} \Gamma(\epsilon)}{\eta} \left( \frac{\mu}{m_g} \right)^{2\epsilon} \delta(\ell) + \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{m_g^2} \right) \left[ \frac{1}{(\ell)_+} - \ln \nu \delta(\ell) \right] \right\}.$$  (41)

When this expression is written in terms of the dimensionless variable $z = \ell/\kappa$ we find

$$S(\ell)(1) = \frac{\alpha_s C_F}{\mu \pi} w^2 \left\{ - \frac{e^{\epsilon \gamma_E} \Gamma(\epsilon)}{\eta} \left( \frac{\mu}{m_g} \right)^{2\epsilon} \delta(z) + \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{m_g^2} \right) \left[ \frac{1}{z_+} - \ln \frac{\nu}{\kappa} \delta(\ell) \right] \right\}.$$  (42)

The sum of the collinear one-loop expression, Eq. (32), and the expression above is independent of the rapidity regulator, as it must be. However, a different choice $\nu \sim \ell \sim Q(1 - x)$ from that for the collinear one-loop expression in Eq. (32) minimizes the rapidity logarithm in Eq. (42) above. Thus, running in $\nu$ is necessary.
V. RENORMALIZATION & RUNNING

The divergences in \( \epsilon \) and \( \eta \) in Eq. (32) and Eq. (42) can be subtracted by suitable counter terms, which we define by

\[
C_n(Q - k)^R = Z_n^{-1}C_n(Q - k)^B
\]
\[
S(\ell)^R = \int d\ell' Z_s(\ell - \ell')^{-1} S(\ell')^B,
\]
where the superscripts \( R \) and \( B \) indicate renormalized and bare. To extract \( Z_n \) we need the wave function renormalization factor at one loop

\[
Z_\psi = 1 - \frac{\alpha_s C_F}{4\pi \epsilon}. \tag{43}
\]

Then the one loop collinear counter term is

\[
Z_n = 1 + \frac{\alpha_s C_F}{\pi w^2} \left\{ \frac{e^{\epsilon \gamma_E} \Gamma(\epsilon)}{\eta} \left( \frac{\mu}{m_g} \right)^{2\epsilon} - \frac{1}{\epsilon} \left( \frac{3}{4} + \ln \frac{\nu}{\bar{n} \cdot p} \right) \right\}. \tag{44}
\]

The one-loop soft counter term is

\[
Z_s(\ell) = \delta(\ell) + \frac{\alpha_s C_F}{\pi w^2} \left\{ \frac{e^{\epsilon \gamma_E} \Gamma(\epsilon)}{\eta} \left( \frac{\mu}{m_g} \right)^{2\epsilon} \delta(\ell) + \frac{1}{\epsilon} \left[ \frac{1}{1 + \ln(\bar{n} \cdot p)} - \ln \nu \delta(\ell) \right] \right\}. \tag{45}
\]

A non-trivial check on this result is to verify that these counter terms obey the consistency condition

\[
Z_H Z_J^\bar{n}(\ell) = Z_n^{-1} Z_s^{-1}(\ell), \tag{46}
\]
where \( Z_H \) is the square of the counter term of the SCET DIS current, and \( Z_J^\bar{n}(\ell) \) is the jet-function counter term. The one loop expression for \( Z_H \) was first given in the appendix of Ref. [22]. Converting their expression from \( 4 - \epsilon \) dimensions to \( 4 - 2\epsilon \) dimensions and squaring gives

\[
Z_H = 1 - \frac{\alpha_s C_F}{2\pi} \left( \frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{2}{\epsilon} \ln \frac{\mu^2}{Q^2} \right), \tag{47}
\]
where \( Q^2 = \bar{n} \cdot p \bar{n} \cdot p_X \), with \( p_X^\mu \) the final state momentum. The one-loop expression for \( Z_J^\bar{n}(\ell) \) can be obtained from Ref. [9]

\[
Z_J^\bar{n}(\ell) = \delta(\ell) + \frac{\alpha_s C_F}{4\pi} \left[ \left( \frac{4}{\epsilon^2} + \frac{3}{\epsilon} - \ln \frac{\bar{n} \cdot p_X}{\mu^2} \right) \delta(\ell) - \frac{4}{\epsilon} \frac{1}{\ell_+} \right]. \tag{48}
\]

Thus at one loop

\[
Z_H Z_J^\bar{n}(\ell) = \delta(\ell) + \frac{\alpha_s C_F}{4\pi} \left\{ \left[ -\frac{3}{\epsilon} + \frac{4}{\epsilon} \ln(\bar{n} \cdot p) \right] \delta(\ell) - \frac{4}{\epsilon} \frac{1}{\ell_+} \right\}. \tag{49}
\]
Adding the inverse of Eq. (44) and the inverse of Eq. (45) we find that at one loop the expression for \(Z_n^{-1}Z_s^{-1}(\ell)\) agrees with the above expression satisfying the consistency condition.

We can extract the one-loop anomalous dimensions from the counter terms above. The \(\mu\) anomalous dimensions are

\[
\gamma_n^{\mu}(\mu, \nu) = \frac{2\alpha_s C_F}{\pi} \left( \frac{3}{4} + \ln \frac{\nu}{\bar{n} \cdot p_\nu} \right) 
\]

\[
\gamma_s^{\mu}(\ell; \mu, \nu) = \frac{2\alpha_s C_F}{\pi} \left[ \frac{1}{\ell^+} - \ln \nu \delta(\ell) \right],
\]

and the \(\nu\) anomalous dimensions are

\[
\gamma_n^{\nu}(\mu, \nu) = \frac{\alpha_s C_F}{\pi} \ln \frac{\mu^2}{m_g^2}
\]

\[
\gamma_s^{\nu}(\mu, \nu) = -\frac{\alpha_s C_F}{\pi} \ln \frac{\mu^2}{m_g^2}.
\]

As is immediately obvious from the presence of \(m_g\) in these expressions, the running in \(\nu\) is not perturbative. The running in \(\mu\) and \(\nu\) are independent and can be carried out in any order. The one-loop \(\mu\)-running factor for the collinear function is

\[
C_n(Q - k; \mu, \nu_c) = U(\mu, \mu_0, \nu_c)C_n(Q - k; \mu_0, \nu_c)
\]

\[
U(\mu, \mu_0, \nu_c) = e^{\omega(\mu, \mu_0)} \left[ \frac{\nu_c}{\bar{n}_1 \cdot \vec{p}} \right]^\omega(\mu, \mu_0),
\]

where \(\nu_c\) is the collinear rapidity scale and

\[
\omega(\mu, \mu_0) = \frac{4C_F}{\beta_0} \ln \left[ \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)} \right].
\]

The one-loop \(\mu\)-running factor for the soft function is

\[
S(\ell; \mu, \nu_s) = \int dr U(\ell - r; \mu, \mu_0, \nu_s)S(\ell; \mu_0, \nu_s)
\]

\[
U(\ell - r; \mu, \mu_0, \nu_s) = \left( e^{2\gamma_E \nu_s} \right)^{-\omega(\mu, \mu_0)} \frac{1}{\Gamma(\omega(\mu, \mu_0)) \left[ (\ell - r)^{1 - \omega(\mu, \mu_0)} \right]_+}.
\]

Despite the fact that the \(\nu\) running is non-perturbative we give the expression for \(\nu\) running factor

\[
S(\ell; \mu_s, \nu) = V(\mu_s, \nu, \nu_0)S(\ell; \mu_s, \nu_0)
\]

\[
V(\mu_s, \nu, \nu_0) = \left[ \nu \right]^{\omega(\mu_s, m_g)}.
\]
If we first run in $\mu$ then $\nu_c \sim Q$ and $\nu_s \sim \Lambda_{\text{had}}$ in the expressions above. After running in $\mu$ we would run the soft function in $\nu$ with $\mu \sim \mu_c$ in Eq. (55). Conversely, we can first run the soft function in $\nu$ to the scale $\nu_s \sim Q$, where $\mu \sim \Lambda_{\text{had}}$ in Eq. (55). Then $\nu_c = \nu_s$ in the expressions above. This second approach is equivalent to running both the collinear factor and soft function at once by adding their anomalous dimensions as was done in Ref. [15]. Since the running in $\nu$ is non-perturbative it should be absorbed into a non-perturbative function.

VI. DEFINITION OF THE PARTON DISTRIBUTION FUNCTION

Finally we consider the definition of the parton distribution function. One may be tempted to define the PDF as in Eq. (26) above, since both the collinear and soft function need to be run to the same common $\mu$ scale. This is problematic because, while the collinear function is only sensitive to initial state interactions, the soft function depends on both initial and final state interactions as evidenced by the soft Wilson lines running to infinity. To keep the PDF universal, we want to require that the PDF only depend on properties of the initial state. While this appears to suggest that we define the PDF as consisting solely of the collinear factor, this is not a satisfactory choice, since we loose the property that the PDF has DGLAP type running.

One solution is to introduce an additional operator matrix element and define the PDF as

$$f_{q}^{ns}(z; \mu) \equiv \frac{1}{2} \sum_{\sigma} \langle h_{n}(p, \sigma)|\bar{\chi}_{n}(0)\frac{i}{2} \gamma_{5} \chi_{n}(0)|h_{n}(p, \sigma)\rangle \times \int \frac{\text{d}n \cdot x}{4\pi} e^{i\frac{Q_{z_{n}}}{\mu} \cdot n} \frac{1}{N_{c}}\langle 0|\text{Tr} \left[ T \left[ Y_{n}^{\dagger}(n \cdot x)Y_{n}(n \cdot x) \right] T \left[ Y_{n}^{\dagger}(0)Y_{n}(0) \right] \right] |0\rangle.$$  

In this definition only the initial state interactions appear, which ensures the universality of the PDF. In addition, this combination of matrix elements has DGLAP type running. The downside of introducing the soft vacuum matrix element above, is that we need to divide by it somewhere else. As a result we must define a new soft function

$$\bar{S}(\ell; \mu) \equiv \int \frac{\text{d}n \cdot x}{4\pi} e^{i\frac{\ell_{n}}{\mu} \cdot x} \frac{1}{N_{c}}\langle 0|\text{Tr} \left[ T \left[ Y_{n}^{\dagger}(n \cdot x)\bar{Y}_{n}(\bar{n} \cdot x) \right] T \left[ \bar{Y}_{n}^{\dagger}(0)Y_{n}(0) \right] \right] |0\rangle \times \left[ \int \frac{\text{d}n \cdot x}{4\pi} e^{i\frac{\ell_{n}}{\mu} \cdot x} \frac{1}{N_{c}}\langle 0|\text{Tr} \left[ T \left[ Y_{n}^{\dagger}(n \cdot x)Y_{n}(n \cdot x) \right] T \left[ Y_{n}^{\dagger}(0)Y_{n}(0) \right] \right] |0\rangle \right]^{-1}.$$
This function encodes soft final state interactions. In terms of these functions the factored form of the hadronic DIS tensor in SCET is

\[ W_{\text{eff}}^{\mu\nu} = -g_{\perp}^{\mu\nu} H(Q; \mu_q, \mu_c) \int dz \int dr \, J_{\bar{n}}(Qz - r; \mu_c, \mu) f_{q_s}^a(z; \mu) S(r; \mu). \]

VII. CONCLUSIONS

In this paper we have revisited DIS in the endpoint region \( x \sim 1 \) with the goal of a clearer understanding of the individual factors in the factorized hadronic tensor. We use a two-step process where we first match QCD onto SCET\(_I\) at a scale \( \sim Q \) and then match onto SCET\(_{II}\) at a scale \( \sim Q \sqrt{1 - x} \). In agreement with previous results we find that the hadronic tensor factors into a hard coefficient, a jet function, a collinear factor, and a soft function.

By considering the collinear factor and soft function in perturbation theory we find that these functions need a rapidity regulator to be well defined. The scale which minimizes rapidity logarithms in the collinear factor is \( \sim Q \), while the scale which minimizes rapidity logarithms in the soft function is \( \sim Q(1 - x) \). We find that running in rapidity is non-perturbative, and should be absorbed into the non-perturbative soft function.

In addition, we find some interesting subtleties in the one loop calculations. First being that in the collinear factor real radiation is prohibited by label momentum conservation this function only includes virtual contributions. Second, that in the one loop computation of the soft function the overlap of the soft degrees of freedom with collinear degrees of freedom needs to be subtracted.

Finally, we consider the proper definition of the PDF. Our derivation of the factored form of the DIS hadronic tensor makes explicit that while the collinear factor only depends on the initial state interactions, the soft function depends both on initial and final state interactions. This means that if we do not want to include information from the final state in the PDF we must not include the soft function in the definition of the PDF. We do, however, want to preserve the DGLAP running of the PDF, and the collinear factor does not have such a running. Thus we are forced to introduce a vacuum matrix element of soft Wilson lines that depend only on the initial state direction. This results in a PDF that is only sensitive to the initial state direction, but also has DGLAP running. This gives a hadronic tensor that is a convolution of the PDF with a soft function. In a future publication we will examine the
implications of rapidity running and our definition of the PDF on Drell-Yan in the endpoint region \[23\].

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