The Nehari Problem for the Paley–Wiener Space of a Disc

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Abstract
There is a bounded Hankel operator on the Paley–Wiener space of a disc in \( \mathbb{R}^2 \) which does not arise from a bounded symbol.

Keywords
Hankel operator · Paley–Wiener space · Several variables

Mathematics Subject Classification
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1 Introduction

Let \( \mathbb{D} \) be the unit disc in \( \mathbb{R}^2 \). The Paley–Wiener space \( \text{PW}(\mathbb{D}) \) is the subspace of \( L^2(\mathbb{R}^2) \) comprised of functions \( f \) whose Fourier transforms \( \hat{f} \) are supported in \( \overline{\mathbb{D}} \). For a tempered distribution \( \varphi \), we consider the Hankel operator \( H_\varphi \) defined by the equation

\[
\hat{H}_\varphi f(\eta) = \int_{\mathbb{D}} \hat{f}(\xi) \hat{\varphi}(\xi + \eta) \, d\xi, \quad \eta \in \mathbb{D},
\]  

on the dense subset of \( \text{PW}(\mathbb{D}) \) comprised of functions \( f \) such that \( \hat{f} \) is smooth and compactly supported in \( \overline{\mathbb{D}} \).

We are interested in the characterization of the symbols \( \varphi \) such that \( H_\varphi \) extends by continuity to a bounded operator on \( \text{PW}(\mathbb{D}) \). If \( \varphi \) is in \( L^\infty(\mathbb{R}^2) \), then clearly

\[
\|H_\varphi f\|_2 \leq \|f\|_2 \|\varphi\|_{\infty}.
\]
Since \( \xi + \eta \) is in \( 2D \) whenever \( \xi \) and \( \eta \) are in \( D \), \( H_\varphi = H_\psi \) for any \( \psi \) such that the restrictions of \( \hat{\psi} \) and \( \hat{\varphi} \) to \( 2D \) coincide (as distributions in \( 2D \)). We thus find that

\[
\|H_\varphi\| \leq \inf \{ \|\psi\|_\infty : |\hat{\psi}|_{2D} = |\hat{\varphi}|_{2D} \}. \tag{3}
\]

We say that the Hankel operator \( H_\varphi \) has a bounded symbol if the quantity on the right hand side of (3) is finite. We have just demonstrated that if \( H_\varphi \) has a bounded symbol, then \( H_\varphi \) is bounded. We wish to explore the converse.

**Question** Does every bounded Hankel operator on \( PW(D) \) have a bounded symbol?

In the classical one-dimensional setting, where the role of \( D \) is played by the half-line \( \mathbb{R}_+ = [0, \infty) \), Nehari [6] gave a positive answer to this question. We therefore refer to affirmative answers to analogous questions as Nehari theorems. Our question for \( PW(D) \) was first raised implicitly by Rochberg [9, Sec. 7], after he had proved that Nehari’s theorem holds for the Paley–Wiener space \( PW(I) \) of a finite interval \( I \subset \mathbb{R} \).

It was conditionally\(^1\) shown in [1] that the Nehari theorem holds for the Paley–Wiener space \( PW(P) \) of any convex polygon \( P \). However, in view of C. Fefferman’s negative resolution [3] of the disc conjecture for the Fourier multiplier of a disc, it would not be surprising to see differing results for \( PW(P) \) and \( PW(D) \).

The main purpose of the present note is to establish the following.

**Theorem 1** There is a bounded Hankel operator on \( PW(D) \) which does not have a bounded symbol.

Minor modifications of our proof show that if \( P_n \) is an \( n \)-sided regular polygon, then the optimal constant in the inequality

\[
\inf \{ \|\psi\|_\infty : |\hat{\psi}|_{2P_n} = |\hat{\varphi}|_{2P_n} \} \leq C_n\|H_\varphi\|_{PW(P_n)}
\]

satisfies \( C_n \geq c_\varepsilon n^{1/2-\varepsilon} \) for any fixed \( \varepsilon > 0 \). Here, \( c_\varepsilon > 0 \) denotes a constant which depends only on \( \varepsilon \). Conversely, the conditional argument of [1] yields that \( C_n \leq cn \) for some absolute constant \( c > 0 \). Analogous estimates for Fourier multipliers associated with polygons were considered in [2].

Finally, let us remark that Ortega-Cerdà and Seip [7] have shown that Nehari’s theorem also fails for (small) Hankel operators on the infinite-dimensional torus. However, Helson [4] proved that if the Hankel operator is in the Hilbert–Schmidt class \( S_2 \), then it is induced by a bounded symbol. We are led to the following.

**Question** Does every Hankel operator on \( PW(D) \) in \( S_2 \) have a bounded symbol?

In this context, we mention that Peng [8] has characterized when \( H_\varphi \) is in the Schatten class \( S_p \), for \( 1 \leq p \leq 2 \), in terms of the membership of \( \varphi \) in certain Besov spaces adapted to \( 2D \). In particular, \( H_\varphi \) is in \( S_2 \) if and only if

\[
\int_{2D} |\hat{\varphi}(\xi)|^2 (2 - |\xi|)^{3/2} d\xi < \infty.
\]

\(^1\) The arguments in [1] rely on Nehari’s theorem for \( \mathbb{R}_+ \times \mathbb{R}_+ \) as a black box. It was long believed that the Nehari theorem had been proven in this setting, but a significant flaw was recently observed in the available reasoning. We refer to [5, Sect. 10] for a detailed discussion.
2 Proof of Theorem 1

If the Nehari theorem were to hold for \( \text{PW}(D) \), there would by the closed graph theorem exist an absolute constant \( C < \infty \) such that

\[
\inf \left\{ \| \psi \|_{\infty} : \hat{\psi} \big|_{2D} = \hat{\varphi} \big|_{2D} \right\} \leq C \| H_{\varphi} \| \tag{4}
\]

for every bounded Hankel operator on \( \text{PW}(D) \). To prove Theorem 1, we will construct a sequence of symbols which demonstrates that no such \( C < \infty \) can exist.

We begin with an upper bound for \( \| H_{\varphi} \| \). Guided by the following lemma, our plan is to construct \( \varphi \) such that \( H_{\varphi} \) admits an orthogonal decomposition. For a symbol \( \varphi \), define

\[
D_{\varphi} = \{ \eta \in D : \xi + \eta \in \text{supp} \hat{\varphi} \text{ for some } \xi \in D \}.
\]

Lemma 2 Suppose that \( \varphi = \varphi_1 + \varphi_2 \) and that \( D_{\varphi_1} \cap D_{\varphi_2} = \emptyset \). Then,

\[
H_{\varphi} = H_{\varphi_1} \oplus H_{\varphi_2}.
\]

Proof Let \( f \) be any function in \( \text{PW}(D) \) such that \( \hat{f} \) is smooth and compactly supported in \( D \). Since \( H_{\varphi} f = H_{\varphi_1} f + H_{\varphi_2} f \) by linearity of the integral (1), it is sufficient to demonstrate that \( H_{\varphi_1} f \perp H_{\varphi_2} f \). It follows directly from the definition of the Hankel operator (1) that

\[
\text{supp} \hat{H}_{\varphi_1} f \subseteq D_{\varphi_1} \quad \text{and} \quad \text{supp} \hat{H}_{\varphi_2} f \subseteq D_{\varphi_2}.
\]

By the assumption that \( D_{\varphi_1} \cap D_{\varphi_2} = \emptyset \), we therefore conclude that

\[
\langle H_{\varphi_1} f, H_{\varphi_2} f \rangle = \langle \hat{H}_{\varphi_1} f, \hat{H}_{\varphi_2} f \rangle = 0.
\]

In particular, if \( D_{\varphi_1} \cap D_{\varphi_2} = \emptyset \), then

\[
\| H_{\varphi} \| = \max(\| H_{\varphi_1} \|, \| H_{\varphi_2} \|).
\]

Let us next explain the construction of \( \varphi \). Consider a radial smooth bump function \( \hat{b} \) which is bounded by 1, equal to 1 on \( \frac{1}{2}D \) and compactly supported in \( D \). For a real number \( 0 < r < 1/2 \), set \( \hat{b}_r(\xi) = \hat{b}(\xi/r) \). Note that

\[
\| \hat{b}_r \|_1 \leq \pi r^2.
\]

For \( j = 1, 2, \ldots, n \), we let \( \hat{\varphi}_j \) be the function obtained by translating \( \hat{b}_r \) by \( 2 - r \) units in the direction \( \theta_j = 2\pi (j - 1)/n \), as measured with respect to the positive \( \xi_1 \)-axis in the \( \xi_1 \xi_2 \)-plane. We set
Fig. 1 Plots of $D(w)$ and the corresponding disc sector from the proof of Lemma 3, for $w = 1.1, w = 1.5,$ and $w = 1.8$

\[ \varphi = \varphi_1 + \varphi_2 + \cdots + \varphi_n. \quad (6) \]

Since $0 < r < 1/2$, it is clear that $\text{supp} \hat{\varphi} \subseteq 2\mathbb{D} \setminus \mathbb{D}$. Let $r_0 = 1 - \frac{1}{\sqrt{2}} = 0.29 \ldots$

**Lemma 3** If $n \geq 2$ and $r = \min(r_0, (2/n)^2)$, then

\[ D_{\varphi_j} \cap D_{\varphi_k} = \emptyset \]

for every $1 \leq j \neq k \leq n$.

**Proof** Throughout this proof, we identify $\mathbb{R}^2$ with $\mathbb{C}$. We consider first a simpler situation. For a point $w$ in $2\mathbb{D} \setminus \sqrt{2}\mathbb{D}$, let

\[ D(w) = \{ \eta \in \mathbb{D} : \xi + \eta = w \text{ for some } \xi \in \mathbb{D} \}. \]

In other words, $D(w)$ is the intersection of the discs defined by $|\xi| < 1$ and $|w - \xi| < 1$. To find the intersection of the corresponding circles, we set $\xi = e^{i\theta}$ and let $\theta^\pm$ denote the solutions of the equation

\[ 1 = |w - e^{i\theta}| \quad \iff \quad \theta^\pm = \arg w \pm \arccos \left( \frac{|w|^2}{2} \right). \]

Let $P_0$ denote the origin, $P_\pm$ the points $e^{i\theta^\pm}$, and $P_w$ the point $w$. The law of cosines implies that the angle $\angle P_0 P_\pm P_w$ is greater than or equal to $\pi/2$ if and only if $|w| \geq \sqrt{2}$. If this holds, then the intersection of the two discs is contained in the disc sector defined by the origin and the two points $P_\pm$. See Fig. 1.

Suppose therefore that $|w| \geq \sqrt{2}$ and set $I(w) = (\theta^-, \theta^+)$. If $\xi$ is in $D(w)$, we have just seen that $\arg \xi$ is in $I(w)$. It follows that if $w_1$ and $w_2$ are points in $2\mathbb{D} \setminus \sqrt{2}\mathbb{D}$, then

\[ I(w_1) \cap I(w_2) = \emptyset \quad \implies \quad D(w_1) \cap D(w_2) = \emptyset. \quad (7) \]

Our goal is now to estimate

\[ I_{\varphi_j} = \bigcup_{w \in \text{supp} \hat{\varphi}_j} I(w). \]
Since supp $\hat{\psi}_j$ is contained in a disc with center $(2 - r)e^{i\theta_j}$ and radius $r$, straightforward geometric arguments show that if $w$ is in supp $\hat{\psi}_j$, then
\[ |w| \geq 2(1 - r) \quad \text{and} \quad |\arg w - \theta_j| \leq \arctan\left(\frac{r}{2 - r}\right). \]

To ensure that $|w| \geq \sqrt{2}$ we require that $r \leq r_0 = 1 - \frac{1}{\sqrt{2}}$. Moreover, if $\theta^\pm$ correspond to the point $w$ as above, then
\[ |\theta^\pm - \theta_j| \leq \arccos(1 - r) + \arctan\left(\frac{r}{2 - r}\right) \leq 2\sqrt{r} + r \leq 3\sqrt{r}. \]

Here, we used that $2 - r \geq 1$ and that $\arctan(r) \leq r$ for $0 \leq r \leq 1$. This shows that $I_{\psi_j} \subseteq \left(\theta_j - 3\sqrt{r}, \theta_j + 3\sqrt{r}\right)$.

Let $\phi$ be as in (6), with $n \geq 2$ and $r = \min(r_0, (2/n)^2)$. It then follows from Lemmas 2, 3, (2), and (5) that
\[ \|H_\psi\| = \|H_{\psi_j}\| \leq \|\psi_j\|_\infty \leq \|\hat{\psi}_j\|_1 = \|\hat{\psi}_r\|_1 \leq \pi r^2. \quad (8) \]

A lower bound for the left hand side in (4) will be established through duality.

**Lemma 4** Suppose that $\hat{f}$ is smooth and compactly supported in $2\mathbb{D}$. Then,
\[ \frac{|\langle \hat{f}, \hat{\phi}\rangle|}{\|f\|_1} \leq \inf \left\{ \|\psi\|_\infty : \hat{\psi}\big|_{2\mathbb{D}} = \hat{\phi}\big|_{2\mathbb{D}} \right\}. \]

**Proof** Obviously,
\[ \frac{|\langle f, \psi\rangle|}{\|f\|_1} \leq \|\psi\|_\infty, \]

and when $\hat{f}$ is supported in $2\mathbb{D}$ and $\hat{\psi}|_{2\mathbb{D}} = \hat{\phi}|_{2\mathbb{D}}$, we have that
\[ \langle f, \psi\rangle = \langle \hat{f}, \hat{\psi}\rangle = \langle \hat{f}, \hat{\phi}\rangle. \]

We now need to choose a test function $f$ adapted to the symbol $\phi$ of (6). It turns out that $f = f_1 + f_2 + \cdots + f_n$, where $f_j = \phi_j$ for $j = 1, 2, \ldots, n$, will do. By our choice of $n \geq 2$ and $r = \min(r_0, (2/n)^2)$, it is clear that supp $\hat{f}_j \cap$ supp $\hat{f}_k = \emptyset$ for every $1 \leq j \neq k \leq n$, since the converse statement would contradict Lemma 3.
Exploiting this, we find that
\[ |\langle f, \varphi \rangle| = \|f\|_2 = \|\hat{f}\|_2 = n\|\hat{b}_r\|_2^2 \geq \frac{\pi}{4}nr^2. \] (9)

To get an upper bound for \( \|f\|_1 \), we split the integral at some \( R > 0 \),
\[ \|f\|_1 = \int_{|x| \leq R} |f(x)| \, dx + \int_{|x| > R} |f(x)| \, dx = I_1 + I_2. \]

For the first integral, we use the Cauchy–Schwarz inequality,
\[ I_1 \leq \sqrt{\pi R} \left( \int_{|x| \leq R} |f(x)|^2 \, dx \right)^{1/2} \leq \sqrt{\pi R} \|f\|_2 = \sqrt{\pi R} \|\hat{f}\|_2 \leq \pi R \sqrt{n}r, \]
where we again exploited that \( \text{supp} \, \hat{f}_j \cap \text{supp} \, \hat{f}_k = \emptyset \) for \( 1 \leq j \neq k \leq n \). For the second integral, we note that \( b \) is rapidly decaying, since \( \hat{b} \) is smooth and compactly supported. In particular, for every \( \kappa \geq 1 \), there is a constant \( A_\kappa \) such that
\[ \int_{|x| > \varrho} |b(x)| \, dx \leq \frac{A_\kappa}{\varrho^{\kappa-1}}, \] (10)
holds for every \( \varrho > 0 \). We constructed \( \hat{f}_j \) by translating \( \hat{b}_r \) by \( 2^{-r} \) units in direction \( \theta_j \), so there is a unimodular function \( g_j \) such that
\[ f_j(x) = g_j(x)b_r(x) = g_j(x)r^2b(rx). \]
Thus \( |f(x)| \leq nr^2b(rx) \) and (10), with \( \varrho = Rr \), yields
\[ I_2 \leq n \int_{|x| > R} r^2|b(rx)| \, dx = n \int_{|x| > R} |b(x)| \, dx \leq \frac{A_\kappa}{(Rr)^{\kappa-1}} \left( \frac{n}{(Rr)^{\kappa-1}} \right). \]

Combining our estimates for \( I_1 \) and \( I_2 \) and choosing \( R = n^{1/(2\kappa)}/r \), we find that
\[ \|f\|_1 = I_1 + I_2 \leq (\pi + A_\kappa)n^{1/2+1/(2\kappa)}. \] (11)

Inserting the estimates (9) and (11) into Lemma 4, we obtain
\[ \frac{\pi r^2n^{1/2-1/(2\kappa)}}{4(\pi + A_\kappa)} \leq \inf \{ \|\psi\|_\infty : \|\hat{\psi}\|_{2\mathbb{D}} = \|\hat{\varphi}\|_{2\mathbb{D}} \}. \] (12)

**Final part of the proof of Theorem 1** To finish the proof of Theorem 1, we combine (8) and (12) to conclude that the constant \( C \) in (4) must satisfy
\[ \frac{n^{1/2-1/(2\kappa)}}{4(\pi + A_\kappa)} \leq C \]
for any fixed $\kappa \geq 1$ and every integer $n \geq 2$. Choosing some $\kappa > 1$ and letting $n \to \infty$, we obtain a contradiction. \hfill \Box

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