Orbital integrals and $K$-theory classes

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Abstract

Let $G$ be a semisimple Lie group with discrete series. We use maps $K_0(C^*_r G) \to \mathbb{C}$ defined by orbital integrals to recover group theoretic information about $G$, including information contained in $K$-theory classes not associated to the discrete series. An important tool is a fixed point formula for equivariant indices obtained by the authors in an earlier paper. Applications include a tool to distinguish classes in $K_0(C^*_r G)$, the (known) injectivity of Dirac induction, versions of Selberg’s principle in $K$-theory and for matrix coefficients of the discrete series, a Tannaka-type duality, and a way to extract characters of representations from $K$-theory. Finally, we obtain a continuity property near the identity element of $G$ of families of maps $K_0(C^*_r G) \to \mathbb{C}$, parametrised by semisimple elements of $G$, defined by stable orbital integrals. This implies a continuity property for $L$-packets of discrete series characters, which in turn can be used to deduce a (well-known) expression for formal degrees of discrete series representations from Harish-Chandra’s character formula.

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1 Introduction

Let $G$ be a real semisimple Lie group. Its reduced $C^*$-algebra $C^*_rG$ is the closure in $\mathcal{B}(L^2(G))$ of the algebra of convolution operators by functions in $L^1(G)$. It represents the tempered dual of $G$ as a ‘noncommutative space’ in the sense of noncommutative geometry, and encodes all tempered representations of $G$. Its $K$-theory $K_*(C^*_rG)$ is a natural invariant to consider. This $K$-theory is described explicitly in terms of equivariant indices of Dirac operators on $G/K$, for a maximal compact subgroup $K < G$, in the Connes–Kasparov conjecture. This was proved in various cases by Pening- ton and Plymen [35], Wassermann [42], Lafforgue [32] and finally in general by Chabert, Echterhoff and Nest [13].

Despite this explicit knowledge about the structure of $K_*(C^*_rG)$, it remains a challenge to extract explicit representation theoretic information from this $K$-theory group. There has been a good amount of success in this direction for classes in $K_*(C^*_rG)$ corresponding to discrete series representations, for groups having such representations. For example, Lafforgue [31] used $K$-theory to recover Harish-Chandra’s criterion $\text{rank}(G) = \text{rank}(K)$ for the existence of discrete series representations.

The von Neumann trace $\tau_e$ on $C^*_rG$, defined by $\tau_e(f) = f(e)$ for $f$ in a dense subalgebra, induces a map on $K_0(C^*_rG)$. On classes corresponding to
the discrete series, this gives the formal degrees of such representations. But this trace maps all other classes to zero (see Proposition 7.3 in \[15\]). It has recently become clear that a natural generalisation of the von Neumann trace involving orbital integrals can be used to extract much more information from \(K_0(C^*_rG)\). For a semisimple element \(g \in G\), the orbital integral \(\tau_g(f)\) of a function \(f\) on \(G\) is the integral of \(f\) over the conjugacy class of \(g\). This integral converges for \(f\) in Harish-Chandra’s Schwartz algebra, which has the same \(K\)-theory as \(C^*_rG\). That leads to maps

\[ \tau_g: K_0(C^*_rG) \to \mathbb{C}. \]  

(1.1)

If \(D\) is an elliptic operator on a \(\mathbb{Z}_2\)-graded vector bundle over a manifold \(M\), \(G\)-equivariant for a proper, cocompact action by \(G\) on \(M\), then one has the equivariant index

\[ \text{index}_G(D) \in K_0(C^*_rG). \]

In \[25\], the authors proved a fixed point formula for the numbers

\[ \tau_g(\text{index}_G(D)). \]  

(1.2)

They showed that Harish-Chandra’s character formula for the discrete series is a special case of this fixed point formula, much as Weyl’s character formula is a special case of the Atiyah–Segal–Singer \[8\] or Atiyah–Bott \[5\] fixed point formulas, as proved in \[6\]. Also, Shelstad’s character identities for \(L\)-packets of representations follows from a \(K\)-theoretic argument involving \(\tau_g\), in the case of discrete series representations \[27\].

For discrete groups, orbital integrals (now sums over conjugacy classes) are also useful tools in \(K\)-theory. The main result in \[40\] is a fixed point theorem for \(1.2\) in the discrete group case, which has consequences to orbifold geometry, positive scalar curvature metrics, and trace formulas. Gong \[17\] and Samurkaš \[36\] used such maps on the \(K\)-theory of maximal group \(C^*\)-algebras to deduce information about rigidity of manifolds. And Xie and Yu have an article in preparation about an APS-type index theorem involving \(\tau_g\).

For semisimple Lie groups \(G\), the results in \[25, 27, 32\] mentioned above show that classes in \(K_0(C^*_rG)\) corresponding to the discrete series contain a great deal of information about those representations. But it was long unclear what (representation theoretic) information can be recovered from other classes. That question was important motivation for this paper. As a concrete example, it was not known what information the generator of \(K_0(C^*_r\text{SL}(2,\mathbb{R}))\) corresponding to the limits of discrete series (or to the non-spherical principal series) contains.
In the present paper, we investigate further properties and applications of the maps \((1.1)\) for semisimple Lie groups, many of them related to the fixed point formula for \((1.2)\). This starts with an explicit expression for \(\tau_g\) applied to \(K\)-theory generators defined via Dirac induction (Theorem 3.1). That result shows that \(\tau_g\) is the zero map on \(K\)-theory if \(\text{rank}(G) \neq \text{rank}(K)\), but it has interesting consequences if \(\text{rank}(G) = \text{rank}(K)\). These include

- a way to use the maps \(\tau_g\) to distinguish elements of \(K_0(C^*_r G)\) (Corollary 4.1);

- an embedding of \(K_0(C^*_r G)\) into the spaces of distributions on \(G^\text{reg}\) or \(G\) (Corollary 4.2);

- an induction formula from \(K\)-equivariant indices to \(G\)-equivariant ones (Corollary 4.6);

- versions of Selberg’s vanishing principle for classes in \(K_0(C^*_r G)\) (Corollary 4.7) and matrix coefficients of the discrete series (Corollary 4.8);

- a Tannaka-type duality result (Corollary 4.9);

- a result relating the value of \(\tau_g\) on \(K\)-theory generators to characters of representations (Corollary 5.3).

Furthermore, Dirac induction is known to be injective (indeed, bijective), but we recover this injectivity independently as well.

In the last bullet point above, Corollary 5.3 explicitly states that \(\tau_g\) maps a \(K\)-theory class to the value at \(g\) of the character of one of the irreducible direct summands of the representation it corresponds to naturally. The values at \(g\) of these characters are equal up to a sign, and they add up to zero if that representation is reducible. So the value at \(g\) of one of these characters is the most relevant information one could have expected to obtain by applying \(\tau_g\). This, to a large extent, answers the question if and what representation theoretic information is contained in classes in \(K_0(C^*_r G)\) if \(\text{rank}(G) = \text{rank}(K)\), even those not corresponding to the discrete series. In particular, the generator of \(K_0(C^*_r \text{SL}(2, \mathbb{R}))\) corresponding to the limits of discrete series determines the characters of these representations on \(K\).

For a fixed element \(x \in K_0(C^*_r G)\), we will see that \(\tau_g(x)\) does not depend continuously on \(g\), for example at the identity element \(e\). Theorem 6.2 states that a modified version of \(\tau_g\), related to \(L\)-packets of representations in the Langlands program, has better continuity properties at \(e\). That implies continuity of certain finite sums of discrete series characters (Corollary 4.4).
And that can be used to take the limit as $g \to e$ in Harish-Chandra’s character formula for the discrete series to obtain expressions for formal degrees of discrete series representations.

We hope that the various applications of orbital integrals to $K$-theory of group $C^*$-algebras in this paper help to demonstrate the relevance of orbital integrals as a tool to study such $K$-theory groups. In future work, we hope to generalise the results and their applications in this paper to more general groups.

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2 Preliminaries

Throughout this paper, let $G$ be a connected semisimple Lie group with finite centre. Let $K < G$ be a maximal compact subgroup. For any Lie group, we will denote its Lie algebra by the corresponding gothic letter.

Fix a $K$-invariant inner product on $\mathfrak{g}$, and let $\mathfrak{p} \subset \mathfrak{g}$ be the orthogonal complement to $\mathfrak{k}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

2.1 Dirac induction

The map $\text{Ad}: K \to \text{SO}(\mathfrak{p})$ lifts to $\tilde{\text{Ad}}: \tilde{K} \to \text{Spin}(\mathfrak{p})$, for a double cover $\tilde{K}$ of $K$. Let $\Delta_\mathfrak{p}$ be the standard representation of $\text{Spin}(\mathfrak{p})$, viewed as a representation of $\tilde{K}$ via $\tilde{\text{Ad}}$. Let $\hat{K}_{\text{Spin}}$ be the set of irreducible representations $V$ of $\tilde{K}$ such that $\Delta_\mathfrak{p} \otimes V$ descends to a representation of $K$. Let $R_{\text{Spin}}(K)$ be the free abelian group generated by $\hat{K}_{\text{Spin}}$.

Let $V \in \hat{K}_{\text{Spin}}$. Then we have the $G$-equivariant vector bundle

$$E_V := G \times_K (\Delta_\mathfrak{p} \otimes V) \to G/K.$$ 

Let $\{X_1, \ldots, X_{\dim(G/K)}\}$ be an orthonormal basis of $\mathfrak{p}$. Let $c_\mathfrak{p}: \mathfrak{p} \to \text{End}(\Delta_\mathfrak{p})$ be the Clifford action. Let $L: \mathfrak{g} \to \text{End}(C^\infty(G))$ be the infinitesimal left regular representation. Consider the Dirac operator

$$D_V := \sum_{j=1}^{\dim(G/K)} L(X_j) \otimes c_\mathfrak{p}(X_j) \otimes 1_V.$$
on
\[ \Gamma^\infty(E_V) = (C^\infty(G) \otimes \Delta_p \otimes V)^K. \]

If \( G/K \) has a \( G \)-invariant Spin-structure (which is the case precisely if \( \Delta_p \) descends to \( K \)), then \( D_V \) is the Spin-Dirac operator on \( G/K \) coupled to the bundle \( G \times_K V \to G/K \), see Proposition 1.1 in [34]. In any case, \( D_V \) is a \( G \)-equivariant elliptic differential operator, and has an index

\[ \text{index}_G(D_V) \in K_\dim(G/K)(C^*_rG). \]

Here \( C^*_rG \) is the reduced group \( C^* \)-algebra of \( G \), and \( \text{index}_G \) is the analytic assembly map [10]. If \( \dim(G/K) \) is even, then \( \Delta_p \), and hence \( E_V \), has a natural \( \mathbb{Z}_2 \)-grading with respect to which \( D_V \) is odd. Then \( \text{index}_G(D_V) \in K_0(C^*_rG) \). If \( \dim(G/K) \) is odd, then there is no such grading, and \( \text{index}_G(D_V) \in K_1(C^*_rG) \). So in general, we have

\[ \text{index}_G(D_V) \in K_{\dim(G/K)}(C^*_rG). \]

**Dirac induction** is the map

\[ \text{D-Ind}^G_K : R_{\text{Spin}}(K) \to K_{\dim(G/K)}(C^*_rG) \]

given by

\[ \text{D-Ind}^G_K[V] = \text{index}_G(D_V), \]

with \( V \) as above. By the Connes–Kasparov conjecture, proved in [13, 32, 42], this map is an isomorphism of abelian groups.

From now on, we suppose that \( G/K \) is even-dimensional, since the \( K \)-theory group \( K_0(C^*_rG) \) we study is zero otherwise.

### 2.2 Orbital integrals and a fixed point formula

Let \( g \in G \) be a semisimple element. Let \( Z_G(g) < G \) be its centraliser. Let \( d(hZ_G(g)) \) be the left invariant measure on \( G/Z_G(g) \) determined by a Haar measure \( dg \) on \( G \). The **orbital integral** with respect to \( g \) of a measurable function \( f \) on \( G \) is

\[ \tau_g(f) := \int_{G/Z_G(g)} f(hgh^{-1}) d(hZ_G(g)), \]

if the integral converges. Harish-Chandra proved that the integral converges for \( f \) in the Harish-Chandra Schwartz algebra \( C(G) \), see Theorem 6 in [19].
The subalgebra $C(G) \subset C^*_r G$ is dense and closed under holomorphic functional calculus (see Theorem 2.3 in [25]). Hence we obtain a map

$$\tau_g : K_0(C^*_r G) = K_0(C(G)) \rightarrow \mathbb{C}.$$ 

Note that $\tau_e$ is the usual von Neumann trace.

Let $M$ be a Riemannian manifold with a proper, isometric, cocompact action by $G$. Let $E \rightarrow M$ be a $G$-equivariant, Hermitian, $\mathbb{Z}_2$-graded vector bundle. Let $D$ be an odd, self-adjoint, $G$-equivariant, elliptic differential operator on $E$. Then we have

$$\text{index}_G(D) \in K_0(C^*_r G).$$ 

In [25], the authors proved a fixed-point formula for the number $\tau_g(\text{index}_G(D))$, for almost all $g \in G$. Consequences include Harish-Chandra’s character formula for the discrete series (Theorem 16 in [19]; see Corollary 2.6 in [25]) and Shelstad’s character identities in the case of discrete series representations ([37]; see Theorem 2.5 in [27]). In this paper, we explore further consequences.

To state the fixed-point formula in [25], let $\mathcal{N} \rightarrow M^g$ be the normal bundle to the fixed point set $M^g$ of $g$ in $M$. Let $\sigma_D$ be the principal symbol of $D$. Let $c^g \in C_c(M^g)$ be nonnegative, and such that for all $m \in M^g$,

$$\int_{Z_G(g)} c^g(h_m) dh = 1,$$

for a fixed Haar measure $dh$ on $Z_G(g)$ compatible with $dg$ and $d(hZ_G(g))$. If $G/K$ is odd-dimensional, then $K_0(C^*_r G) = 0$, so $\tau_g(\text{index}_G(D)) = 0$.

**Theorem 2.1.** If $G/K$ is even-dimensional, then for almost all semisimple $g \in G$, we have $\tau_g(\text{index}_G(D)) = 0$ if $g$ is not contained in any compact subgroup of $G$, and

$$\tau_g(\text{index}_G(D)) = \int_{TM^g} c^g \frac{\text{ch}([\sigma_D]_{\text{supp}(c^g)})(g) \cdot \text{Todd}(TM^g \otimes \mathbb{C})}{\text{ch}([\Lambda \mathcal{N} \otimes \mathbb{C}](g))} \ (2.1)$$

if it is.

Here $\text{ch} : K^0(\text{supp}(c^g)) \rightarrow H^*(\text{supp}(c^g))$ and $\text{ch} : K^0(TM^g|_{\text{supp}(c^g)}) \rightarrow H^*(TM^g|_{\text{supp}(c^g)})$ are Chern characters, and Todd denotes the Todd class.
Remark 2.2. Explicitly, Theorem 2.1 holds for the semisimple $g \in G$ with finite Gaussian orbital integral (FGOI), see Definition 7 in [25]. That condition means that the integral

$$\int_{G/Z_G(g)} e^{-d(e,hgh^{-1})^2} d(hZ_G(g))$$

converges, where $d$ is the $G$-invariant Riemannian distance on $G$. It was shown in Proposition 4.2 in [25] that almost every element of $G$ has FGOI.

In this paper, whenever a result is stated for almost all $g$, what is meant is that it holds for semisimple elements with FGOI, and possibly also with dense powers in a maximal torus.

3 A fixed point formula on $G/K$

Let $T < K$ be a maximal torus. Let $\tilde{T} < \tilde{K}$ be its inverse image in $\tilde{K}$. Fix a set $R_+^c$ of positive roots of $(\mathfrak{t}^\mathbb{C}, \mathfrak{t}^\mathbb{C})$. Let $\rho_c$ be half the sum of the elements of $R_+^c$. Let $V \in \hat{\tilde{K}}_{\text{Spin}}$. Let $\lambda \in \mathfrak{t}^\mathbb{R}$ be its highest weight with respect to $R_+^c$.

For any finite-dimensional (actual or virtual) representation $W$ of $K$ or $\tilde{K}$, we denote its character by $\chi_W$. For any function $\varphi$ on $\tilde{K}$ that descends to a function on $K$, we will use the same notation $\varphi$ for both the function on $\tilde{K}$ and $K$. E.g., we have $\chi_{\Delta_p} \chi_V \in C^\infty(K)$.

In the case where $T$ is a Cartan subgroup of $G$, i.e. rank$(G) = \text{rank}(K)$, fix a set of positive noncompact roots $R_+^{nc}$ of $(\mathfrak{g}^\mathbb{C}, \mathfrak{t}^\mathbb{C})$ such that the character $\chi_{\Delta_p}$ of the graded representation $\Delta_p$ of $\tilde{K}$ satisfies

$$\chi_{\Delta_p}|_T = \prod_{\alpha \in R_+^{nc}} (e^{\alpha/2} - e^{-\alpha/2}). \quad (3.1)$$

Such a choice of positive noncompact roots can always be made, see for example pages 17 and 18 of [7], Remark 2.2 in [34] and (5.1) in [9]. In the equal-rank case, we write $R_+^* := R_+^c \cup R_+^{nc}$. We will denote half the sums of the elements of $R_+^c$ and $R_+^{nc}$ by $\rho$ and $\rho_n$, respectively.

Let $W_K := N_K(T)/T$ be the Weyl group of $(K,T)$.

Theorem 3.1. (a) If rank$(G) = \text{rank}(K)$, then for almost all $g \in T$,

$$\tau_g(D-\text{Ind}_K^G[V]) = (-1)^{\dim(G/K)/2} \chi_V \chi_{\Delta_p}(g)$$

$$= (-1)^{\dim(G/K)/2} \sum_{w \in W_K} \varepsilon(w) e^{w(\lambda + \rho_c)} \prod_{\alpha \in R_+^c} (e^{\alpha/2} - e^{-\alpha/2})(g).$$

(In particular, the right hand sides are well-defined.)
(b) If \( \text{rank}(G) \neq \text{rank}(K) \), then for almost all \( g \in T \),

\[
\tau_g(\text{D-Ind}_K^G[V]) = 0.
\]

Let \( a \subseteq p \) be an abelian subspace such that \( Z_g(t) = t \oplus a \). Let \( c \in C_c(a) \) be a function whose integral over \( a \) is 1. Let \( \sigma_{D_V} \) be the principal symbol of \( D_V \).

**Lemma 3.2.** For almost all \( g \in T \),

\[
\tau_g(\text{D-Ind}_K^G[V]) = \int_{T a} c \frac{\text{ch}(\sigma_{D_V}|_{\text{supp}(c)}(g))}{\text{ch}(a \times \Lambda(p/a \otimes \mathbb{C})(g))}.
\]

**Proof.** Let \( g \in T \) be such that its powers are dense in \( T \), and with FGOI (see Remark 2.2). By Proposition 4.2 in [25], almost all elements of \( T \) have these two properties.

We have \( G/K \cong p \) as \( K \)-spaces, hence in particular as \( T \)-spaces. Hence

\[
(G/K)^g = (G/K)^T = p^{\text{Ad}(T)} = a.
\]

Set \( A := \exp(a) \); this is the centraliser of \( g \) in \( \exp(p) \). We have \( p = a \oplus p/a \) as representations of \( T \). So the normal bundle in \( G/K = p \) to \( (G/K)^g = a \) is \( a \times p/a \to a \). The Todd class of the trivial bundle \( T(G/K)^g \otimes \mathbb{C} \to (G/K)^g \) is 1. Hence the claim follows from Theorem 2.1.

Let us compute \( [\sigma_{D_V}|_{\text{supp}(c)}] \). Let \( \beta_a \in K^0(a) \) be the Bott generator. (Note that \( a \) is even-dimensional since \( G/K \) is.) Let \( \pi : T a \to a \) be the tangent bundle projection, and \( \pi|_{\text{supp}(c)} : \text{supp}(c) \times a \to \text{supp}(c) \) its restriction. Note that

\[
\Delta_p \cong \Delta_a \otimes \Delta_{p/a}
\]

as graded representations of \( \widetilde{T} \). These descend to \( T \) after tensoring with \( V \).

**Lemma 3.3.** Under the isomorphism

\[
K_0^T(\text{supp}(c) \times a) \cong K_0(\text{supp}(c) \times a) \otimes R(T),
\]

we have

\[
[\sigma_{D_V}|_{\text{supp}(c)}] \mapsto \pi|_{\text{supp}(c)}^{*} \beta_a \otimes [\Delta_{p/a} \otimes V].
\]

**Proof.** Let \( c_a : a \to \text{End}(\Delta_a) \) be the Clifford action. The class \( \pi|_{\text{supp}(c)}^{*} \beta_a \in K^0(\text{supp}(c) \times a) \) is defined by the vector bundle homomorphism

\[
A : \text{supp}(c) \times \Delta_a^+ \to \text{supp}(c) \times \Delta_a^-.
\]

\[\text{We absorb a possible sign in the definition of } \beta_a; \text{ see Lemma 4.1 in [15].}\]
given by
\[ A_Y = c_a(Y) \]
for all \( Y \in \text{supp}(c) \).

We have
\[ (G \times_K (\Delta_p^+ \otimes V))|_a \cong a \times \Delta_p^+ \otimes V \]
as \( T \)-vector bundles. So
\[ \pi|_{\text{supp}(c)}^*( (G \times_K (\Delta_p^+ \otimes V))|_{\text{supp}(c)}) = (\text{supp}(c) \times a) \times \Delta_p^+ \otimes V. \]

Let \( X, Y \in a \), so that, using the above identification, we get
\[ \sigma_{D_V}(X,Y) = c_p(Y) \otimes 1_V : \Delta_p^+ \otimes V \to \Delta_p^- \otimes V. \quad (3.2) \]

Since \( Y \in a \), the map (3.2) equals the odd endomorphism
\[ c_a(Y) \otimes 1_{\Delta_p/a \otimes V} \in \text{End}(\Delta_a \otimes \Delta_p/a \otimes V) \]
Together with the above form of the class \( \pi|_{\text{supp}(c)}^* \beta_a \), this implies the claim.

Lemma 3.4. Suppose that \( \text{rank}(G) = \text{rank}(K) \). Then
\[ \bigwedge p \otimes \mathbb{C} = (-1)^{\dim(G/K)/2} \Delta_p \otimes \Delta_p \]
as graded representations of \( T \).

Proof. The set of positive noncompact roots \( R_n^+ \) determines a complex structure on \( p \) such that \( p^{1,0} \) is the sum of the positive noncompact root systems. As graded representations of \( T \), we have
\[ \bigwedge p \otimes \mathbb{C} = \bigwedge p^{1,0} \otimes \bigwedge p^{0,1} = \bigwedge_C p \otimes (\bigwedge_C p)^*. \]
The element \( \rho_n \in i t^* \) is integral for \( \tilde{T} \), and \( \Delta_p \otimes \mathbb{C}_{\rho_n} \) descends to a representation of \( T \). We have
\[ \bigwedge_C p = (-1)^{\dim(G/K)/2} \Delta_p \otimes \mathbb{C}_{\rho_n} \]
as graded representations of \( T \); see for example the proof of Lemma 5.5 in \[25]. Since \( \Delta_p^* \cong (-1)^{\dim(G/K)/2} \Delta_p \), we conclude that
\[ \bigwedge p \otimes \mathbb{C} = \Delta_p \otimes \Delta_p^* = (-1)^{\dim(G/K)/2} \Delta_p \otimes \Delta_p \]
The nontrivial element of the kernel of the covering map \( \tilde{K} \to K \) acts on \( \Delta_p \) as \( \pm 1 \); therefore \( \Delta_p \otimes \Delta_p \) descends to a representation of \( T \). \[\square\]
Lemma 3.5. Let \( c \) be a nonnegative, compactly supported, continuous function on \( \mathbb{R}^{2n} \) with integral 1. Let \( \beta \in K^0(\mathbb{R}^{2n}) \) be the Bott class, and \( \pi|_{\text{supp}(c)} : \text{supp}(c) \times \mathbb{R}^{2n} \rightarrow \text{supp}(c) \) where \( \pi : T\mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n} \) the natural projection. Then
\[
\int_{\mathbb{R}^{2n} \times \mathbb{R}^{2n}} c \, \text{ch}(\pi|_{\text{supp}(c)} \beta) = 0.
\]  

(3.3)

Proof. By Proposition 6.11 in [41], the integral (3.3) equals the \( L^2 \)-index of the Spin-Dirac operator on \( \mathbb{R}^{2n} \). That index is zero because the \( L^2 \)-kernel of this Dirac operator is zero. Indeed, the Spin-Dirac operator on \( \mathbb{R}^{2n} \) only has continuous spectrum, see for example Theorem 7.2.1 in [16].

Proof of Theorem 3.1. Lemma 3.3 implies that
\[
\text{ch}(\sigma_{D_{\mathcal{V}}|_{\text{supp}(c)}}(g)) = \text{ch}(\pi|_{\text{supp}(c)} \beta_a)(\chi_{\Delta_p/a}(\mathcal{V})(g).
\]
Furthermore,
\[
\text{ch}([a \times \wedge \mathfrak{p}/a \otimes \mathbb{C}](g)) = \chi_{\wedge \mathfrak{p}/a \otimes \mathbb{C}}(g)
\]
in the graded sense. So by Lemma 3.2
\[
\tau_g(\text{D-Ind}_{K}^G[\mathcal{V}]) = \frac{\chi_{\Delta_p/a}(\mathcal{V})}{\chi_{\wedge \mathfrak{p}/a \otimes \mathbb{C}}}(g) \int_{T_a} c \, \text{ch}(\pi|_{\text{supp}(c)} \beta_a).
\]

If rank(\( G \)) \neq \text{rank}(\mathcal{K})\), then \( a \) is nonzero, and the claim follows from Lemma 3.5. If rank(\( G \)) = rank(\( \mathcal{K} \)), then Lemma 3.4 implies that
\[
\tau_g(\text{D-Ind}_{K}^G[\mathcal{V}]) = (-1)^{\dim(G/K)/2} \frac{\chi_{\mathcal{V}}}{\chi_{\Delta_p}}(g);
\]
in particular, the right hand side is well-defined. The claim now follows from Weyl’s character formula and (3.1). (Note that (\( \mathcal{K}, T \)) and (\( K, T \)) have the same Weyl group \( W_K \), since they have the same root system.)

Remark 3.6. If \( g = e \), then \( \tau_e(\text{D-Ind}_{K}^G[\mathcal{V}]) \) is the \( L^2 \)-index of \( D_{\mathcal{V}} \) by Proposition 4.4 in [41]. That index is zero if the kernel of \( D_{\mathcal{V}} \) is zero. Theorem 3.1 shows that, in the equal-rank case, the more general trace \( \tau_g \) yields nonzero information even in cases where the kernel of \( D_{\mathcal{V}} \) is zero (see also Section 5.2).

4 Consequences

Suppose from now on that rank(\( G \)) = rank(\( \mathcal{K} \)).
4.1 Distinguishing $K$-theory classes

As a consequence of Theorem 3.1, the traces $\tau_g$ `separate points' on $K_0(C_r^*G)$, or distinguish all elements of $K_0(C_r^*G)$, in the following sense.

**Corollary 4.1.** Let $x \in K_0(C_r^*G)$. If $\tau_g(x) = 0$ for all $g$ in a dense subset of $T$, then $x = 0$.

**Proof.** Let $x \in K_0(C_r^*G)$. By surjectivity of Dirac induction, we can write

$$x = \sum_{V \in \hat{K}_{\Spin}} m_V \text{D-Ind}^G_K[V],$$

for $m_V \in \mathbb{Z}$, finitely many nonzero. By Theorem 3.1, we have for almost all $g \in T$,

$$\tau_g(x) = (-1)^{\dim(G/K)/2} \sum_{V \in \hat{K}_{\Spin}} m_V \frac{\chi_V \chi_{\Delta_p}}{(g)}(g).$$

So if $\tau_g(x) = 0$ for all $g$ in a dense subset of $T$, then by continuity and conjugation invariance of the characters $\chi_V$, we find that

$$\sum_{V \in \hat{K}_{\Spin}} m_V \chi_V = 0.$$

So $m_V = 0$ for all $V$, i.e. $x = 0$. \hfill $\Box$

4.2 $K$-theory and distributions

Let $G^{\text{reg}} \subset G$ be the subset of regular elements.

**Corollary 4.2.** The map

$$\tau: K_0(C_r^*G) \to D'(G^{\text{reg}})$$

defined by

$$\langle \tau(x), f \rangle = \int_{G^{\text{reg}}} \tau_g(x) f(g) \, dg$$

for $x \in K_0(C_r^*G)$ and $f \in C^\infty_c(G^{\text{reg}})$, is a well-defined, injective group homomorphism.

**Proof.** Let $x \in K_0(C_r^*G)$. By surjectivity of Dirac induction, we can write $x = \text{D-Ind}^G_K[y]$, for some $y \in R_{\Spin}(K)$. Theorem 3.1 implies that the function $g \mapsto \tau_g(x)$ equals an analytic function almost everywhere on the set
of elliptic elements of $G$. Theorem 2.1 implies that this function equals zero almost everywhere on the set of non-elliptic elements of $G$. So $g \mapsto \tau_g(x)$ equals an analytic function almost everywhere on $G$. Furthermore, that analytic function is bounded on compact subsets of $G^{\text{reg}}$. This implies that $\tau(x)$ is a well-defined distribution on $G^{\text{reg}}$.

If $\tau(x) = 0$, then $\tau_g(x) = 0$ for almost all $g \in G^{\text{reg}}$, in particular for almost all elements of $T$. Hence Corollary 4.1 implies that $x = 0$. □

**Remark 4.3.** As noted in the proof of Corollary 4.2, the first part of Theorem 2.1 implies that $\tau(x)$ is zero outside the set of regular elliptic elements of $G$.

**Remark 4.4.** We will describe the map $\tau$ in Corollary 4.2 explicitly in terms of characters of representations in Section 5. There we will see that $\tau(x)$ equals the character of a tempered representation of $G$ almost everywhere on the set of regular elliptic elements, and zero almost everywhere outside the set of elliptic elements. Therefore, it extends to a distribution on all of $G$ by Harish-Chandra's regularity theorem.

### 4.3 Injectivity of Dirac induction

We have used the surjectivity of Dirac induction in the proof of Corollary 4.1 (which is justified because the Connes–Kasparov conjecture has been proved). Theorem 3.1 implies injectivity of Dirac induction.

**Corollary 4.5.** Dirac induction is injective.

**Proof.** Let $y \in R_{\text{Spin}}(K)$, and suppose that $\text{D-Ind}_K^G(y) = 0$. Then $\tau_g(\text{D-Ind}_K^G(y)) = 0$ for all $g \in T$. Theorem 3.1 implies that for almost all $g \in T$,

$$\frac{X_y}{\chi_{\Delta_y}}(g) = 0.$$ 

So $\chi_y = 0$, i.e. $y = 0$. □

### 4.4 An induction formula

Let $M$ be an even-dimensional Riemannian manifold with a $G$-equivariant Spin$^c$-structure. Let $E \to M$ be a $G$-equivariant, Hermitian vector bundle. Let $D^E_M$ be the Spin$^c$-Dirac operator on $M$ twisted by $E$. By Abels’ theorem [1], there is a $K$-invariant submanifold $N \subset M$ such that $M \cong G \times_N N$ via
the action map $G \times N \to M$. Furthermore, $N$ has a $K$-equivariant Spin$^c$-structure on $N$ compatible with the one on $M$, see Proposition 3.10 in [24]. The Spin$^c$-Dirac operator $D_N^E$ on $N$, twisted by $E|_N$, has the property that

$$D\text{-Ind}_K^G(\text{index}_K(D_N^E)) = \text{index}_G(D_M^G) \in K_0(C^*_r G).$$

(4.1)
See Theorem 5.2 in [25] and Proposition 4.7 in [22].

Theorem 3.1 and surjectivity of Dirac induction imply that the following diagram commutes for all $g$ in the dense subset of $T$ in Theorem 3.1:

$$
\begin{array}{ccc}
K_0(C^*_r G) & \xrightarrow{\tau_g} & \mathbb{C} \\
\text{D-Ind}_K^G \downarrow & & \downarrow \\
R_{\text{Spin}}(K) & \xrightarrow{(-1)^{\dim(G/K)/2} \text{ev}_g / \chi_{\Delta_p}(g)} & \mathbb{C}.
\end{array}
$$

Here $\text{ev}_g$ denotes evaluation of characters of representations at $g$; note that the bottom arrow is well-defined.

The equality (4.1) and commutativity of (4.2) imply the following formula for induction from slices.

**Corollary 4.6.** We have, for almost all $g \in T$,

$$\tau_g(\text{index}_G(D_N^E)) = (-1)^{\dim(G/K)/2} \text{index}_K(D_N^E)(g) / \chi_{\Delta_p}(g).$$

Note that the right hand side can be computed via the Atiyah–Segal–Singer fixed point formula [8].

Induction formulas like Corollary 4.6 we used in various settings to deduce results about $G$-equivariant indices from results about $K$-equivariant indices [18, 22, 23, 24, 25]. The case $g = e$ is not covered by Corollary 4.6; that case is Corollary 53 in [18].

### 4.5 Selberg’s principle

The Selberg principle is a vanishing result for orbital integrals of certain convolution idempotents on $G$. See [12, 28, 29] for approaches to this principle in the spirit of noncommutative geometry. Theorem 2.1 implies a version of this principle.

**Corollary 4.7** ($K$-theoretic Selberg principle). For almost all $g$ not contained in compact subgroups of $G$, the map

$$\tau_g : K_0(C^*_r G) \to \mathbb{C}$$

is zero.
Proof. Theorem 2.1 implies that for almost all \( g \) not contained in compact subgroups of \( G \), and all \( V \in R_{\text{Spin}}(K) \), we have

\[
\tau_g(\text{D-Ind}_K^G[V]) = 0.
\]

So surjectivity of Dirac induction implies the claim.

Corollary 4.7 has a purely representation theoretic consequence.

**Corollary 4.8** (Selberg principle for matrix coefficients of the discrete series). Let \( \pi \) be a discrete series representation of \( G \). Let \( v \) be a \( K \)-finite vector in the representation space of \( \pi \), and \( m_{v,v} \) the corresponding matrix coefficient. For all \( g \) not contained in compact subgroups of \( G \), we have

\[
\tau_g(m_{v,v}) = 0.
\]

**Proof.** Let \( d_\pi \) be the formal degree of \( \pi \). By rescaling, we may assume that \( v \) has norm 1. Then \( d_\pi \tilde{m}_{v,v} \) is an idempotent in \( C_\tau^*G \). Let \( [\pi] \in K_0(C_\tau^*G) \) be its \( K \)-theory class. Since \( v \) is \( K \)-finite, the function \( m_{v,v} \) lies in Harish-Chandra’s Schwartz algebra \( C(G) \). Therefore, for all semisimple \( g \in G \),

\[
\tau_g(m_{v,v}) = \frac{1}{d_\pi} \tau_g([\pi]).
\]

By Corollary 4.7 the number is zero for almost all \( g \) not contained in compact subgroups. The claim therefore follows by continuity of \( m_{v,v} \).

### 4.6 A Tannaka-type duality

We now suppose that the representation \( \Delta_p \) of \( \tilde{K} \) descends to \( K \). This is true of we replace \( G \) by a double cover if necessary. Then Dirac induction is defined on \( R(K) \).

The \( K \)-theory group \( K_0(C_\tau^*G) \) and its elements contain nontrivial information about \( G \) and its representations, see e.g. [25, 27, 32]. But just the isomorphism class of \( K_0(C_\tau^*G) \) as an abelian group contains no information about \( G \) whatsoever: this group is always free, with countably infinitely many generators. It turns out, however, that the combination of the isomorphism class of \( K_0(C_\tau^*G) \), the topological space \( T \) and the maps \( \tau_g: K_0(C_\tau^*G) \to \mathbb{C} \), for \( g \in T \), together determine the Cartan motion group \( K \ltimes \mathfrak{p} \) and vice versa. The tempered representation theory of \( K \ltimes \mathfrak{p} \) is closely related to that of \( G \); this is the Mackey analogy [3, 20, 21, 33, 38, 43]. Also, the analytic assembly map for \( G \) can be defined in terms of a continuous deformation from \( K \ltimes \mathfrak{p} \) to \( G \), see pp. 23–24 of [10] and [20].
This is vaguely analogous to the fact that the irrational rotation algebras 
\(A_\lambda\), for irrational \(\lambda\) in \([0, 1/2]\), have the same 
K-theory \(\mathbb{Z} \oplus \mathbb{Z}\), but are
determined up to isomorphism by the pair \((K_0(A_\lambda), \tau)\) where \(\tau\) is a natural trace. This is because the image of \(\tau\) is \(\mathbb{Z} + \lambda\mathbb{Z}\).

**Corollary 4.9.** The

- abelian group \(K_0(C^*_r G)\) up to isomorphism;
- pointed topological space \((T, \{e\})\) up to homeomorphism; and
- family of group homomorphisms \(\tau_g: K_0(C^*_r G) \to \mathbb{C}\), for \(g \in T\)
together determine the Cartan motion group \(K \ltimes p\), and vice versa.

**Proof.** Write

\[ K_0(C^*_r G) = \bigoplus_{j \in \mathbb{Z}} \mathbb{Z} \]

and let \(e_j\) be a generator of the \(j\)th copy of \(\mathbb{Z}\).

Consider the function \(\chi_j: T \to \mathbb{R}\) given by \(\chi_j(g) = \tau_g(e_j)\). By Theorem 3.1 there is a function \(\psi \in C^\infty(T)\), not unique but independent of \(j\), and there are uniquely determined integers \(d_j\) such that for all \(j\),

\[ \lim_{g \to e} \psi(g) \chi_j(g) = d_j, \]

where at least one of the integers \(d_j\) equals 1. (Indeed, take \(\psi = \chi_{\Delta p}|_T\) and \(d_j\) plus or minus the dimensions of the irreducible representations of \(K\).) By replacing \(e_j\) by \(-e_j\) where necessary, we can make sure that all integers \(d_j\) are positive.

Fix \(j_0 \in \mathbb{Z}\) such that \(d_{j_0} = 1\). Then, again by Theorem 3.1

\[ |\chi_{\Delta p}|_T| = |\chi_{j_0}|^{-1}. \]

And \(\overline{\chi_{\Delta p}} = -\chi_{\Delta p}\), so \(\chi_{\Delta p}\) is imaginary-valued. Hence

\[ \chi_{\Delta p}|_T = \pm i|\chi_{j_0}|^{-1}. \]

We cannot resolve the sign ambiguity with the data we have, but we will not need to.

The characters of irreducible representations \(V_j\) of \(K\) are now determined by

\[ \chi_{V_j}|_T = \chi_{\Delta p}|_T \chi_j = \pm i|\chi_{j_0}|^{-1} \chi_j, \]
with the sign chosen such that \( \pm i|\chi_{j_0}^{-1} \chi_j| > 0 \) near the identity element. This determines the representations \( V_j \) of \( K \), and their tensor products and the underlying vector spaces. By Tannaka duality \[39\], this determines \( K \).

To recover \( p \) as a \( K \)-representation, set \( \psi := i|\chi_{j_0}^{-1} \). Then

\[
\psi = \pm \chi_{\Delta_p}|_T = \pm \prod_{\alpha \in R^+_n} \left( e^{\alpha/2} - e^{-\alpha/2} \right).
\]

This implies that for all \( X,Y \in t \),

\[
\frac{d}{dt} \bigg|_{t=0} \psi(X + tY) = \psi(\exp(X)) \sum_{\alpha \in R^+_n} \frac{\langle \alpha, Y \rangle}{2} \coth(\langle \alpha, X \rangle/2).
\]

The term on the right hand side corresponding to \( \alpha \) equals the same term with \( \alpha \) replaced by \( -\alpha \). But otherwise this expression determines the weights \( \alpha \) up to signs. In this way, we recover the set \( R_n \) of \( t \)-weights of \( p \otimes \mathbb{C} \) as a complex representation of \( T \). And hence \( p \) as a real representation of \( T \), and therefore as a representation of \( K \). This determines \( K \rtimes p \).

Conversely, the Cartan motion group \( K \rtimes p \) determines its maximal compact subgroup \( K \) and the quotient \( p = (K \rtimes p)/K \) as a representation of \( K \). And \( K \) determines the pair \( (T, \{ e \}) \) up to conjugacy. The \( K \)-theory group \( K_0(C^*_r G) \) is isomorphic to \( R(K) \) via Dirac induction. Furthermore, \( K \) and \( p \) determine the characters \( \chi_V \), for \( V \in \hat{K} \) and \( \chi_{\Delta_p} \), and the dimension \( \dim(G/K) = \dim(p) \). Hence, by Theorem \[3.1\] this determines the maps \( \tau_g : K_0(C^*_r G) \cong R(K) \to \mathbb{C} \), for \( g \in T \).

**Remark 4.10.** In Corollary \[4.9\] \( T \) may be replaced by a dense subset. Also, one only needs the neighbourhoods of the identity element, not all of its topology. And as stated in the corollary, one does not need the group structure of \( T \).

**Remark 4.11.** If \( G = K \) is compact, then the triple \( (K_0(C^*_r G), (T, \{ e \}), (\tau_g)_{g \in T}) \) determines the ring \( R(G) \) of characters of \( G \). That in turn determines the tensor products of representations of \( G \), and forgetful maps to finite-dimensional complex vector spaces. So in this case, Corollary \[4.9\] reduces to Tannaka duality for compact groups \[39\] (which was used in the proof of Corollary \[4.9\]).

**Remark 4.12.** If the representation \( \Delta_p \) of \( \tilde{K} \) does not descend to \( K \), then we only recover the ring \( R_{\text{Spin}}(K) \) in the proof of Corollary \[4.9\] and cannot directly apply Tannaka duality.
5 Characters

Again, we suppose that the representation $\Delta_p$ of $\tilde{K}$ descends to $K$. We may need to replace $G$ by a double cover for this assumption to hold. This assumption is now not essential; see Remark 5.1.

5.1 Characters and $\tau_g$

The structure of the $C^*$-algebra $C^*_r G$ and its $K$-theory was described by Wassermann [42] and Clare, Crisp and Higson [14]. We can use this to relate values of $\tau_g$ on $K$-theory classes to values of characters of representations.

Let $P = MAN < G$ be a cuspidal parabolic and $\sigma$ in the set $\hat{M}_{ds}$ of discrete series representations of $M$. Consider the bundle of Hilbert spaces $\mathcal{E}_{P,\sigma} \to \hat{A}$ whose fibre at $\nu \in \hat{A}$ is $\text{Ind}_{G}^{P}(\sigma \otimes \nu \otimes 1_N)$. (This can be topologised by viewing it as a trivial bundle in the compact picture of induced representations.) Let $\text{Ind}_{P}^{G}(\sigma)$ be the Hilbert $C_0(\hat{A})$-module of continuous sections of $\mathcal{E}_{P,\sigma}$ vanishing at infinity. The group

$$W_\sigma := \{w \in N_K(a)/Z_K(a); w \sigma = \sigma\}$$

acts on $\mathcal{K}(\text{Ind}_{P}^{G}(\sigma))$ via Knapp–Stein intertwiners; see Theorem 6.1 in [14]. Let $\mathcal{K}(\text{Ind}_{P}^{G}(\sigma))^{W_\sigma}$ be the fixed point algebra of this action. Then

$$C^*_r G \cong \bigoplus_{P,\sigma} \mathcal{K}(\text{Ind}_{P}^{G}(\sigma))^{W_\sigma}$$

where the sum runs over a set of cuspidal parabolics $P = MAN$ and $\sigma \in \hat{M}_{ds}$. This is Theorem 6.8 in [14]. See also Theorem 8 in [42].

Now let $P$ and $\sigma$ be such that

$$\mathcal{K}(\text{Ind}_{P}^{G}(\sigma))^{W_\sigma} \neq 0$$

is nonzero, hence infinite cyclic. (This is equivalent to the condition that $W_\sigma$ equals the $R$-group $R_\sigma$, see Lemma 10 in [42].) Let $b(P, \sigma) \in K_0(C^*_r G)$ be the generator of this summand of $K_0(C^*_r G)$ in the image under Dirac induction of the $\mathbb{Z}_{\geq 1}$-span of $\tilde{K}$ inside $\tilde{R}(K)$.

Let $\eta \in i t^*_M$ be the Harish-Chandra parameter of $\sigma$, and $\tilde{\eta} \in i t^*$ its extension by zero on the orthogonal complement of $t_M$ in $t$. For any positive root system $\tilde{R}^+$ of $(g^C, t^C)$ for which $\tilde{\eta}$ is dominant, let $\pi_{G}(\tilde{\eta}, \tilde{R}^+)$ be the corresponding (limit of) discrete series representation of $G$. We need the following version of Schmid’s character identities. This is Lemma 12 in [42] in the equal rank case, but with information included about the infinitesimal characters of the limits of discrete series representations that occur.
Proposition 5.1. There are $2^\dim(A)$ choices of positive roots $R_1^+, \ldots, R_{2^\dim(A)}^+ \subset R$, obtained from $R^+$ by the application of all combinations of $\dim(A)$ commuting reflections in simple noncompact roots, such that

$$\text{Ind}^G_P(\sigma \otimes 1_A \otimes 1_N) = \bigoplus_{j=1}^{2^\dim(A)} \pi^G(\tilde{\eta}, R_j^+).$$

Proof. This is a special case of Theorem 13.3 in [30] for the maximal parabolic $G$ in the equal-rank group $G$.

As before, let $\rho_c$ be half the sum of the compact positive roots. By Lemma 15(i) in [42], the element $\tilde{\eta} - \rho_c$ is dominant for $K$. It is integral because $\Delta_p$ descends to $K$; this implies that $\rho_n$ and hence $\tilde{\eta} - \rho + \rho_n$ is integral.

Proposition 5.2 (Wassermann). Let $V_{\tilde{\eta} - \rho_c} \in \hat{K}$ have highest weight $\tilde{\eta} - \rho_c$. Then

$$\text{D-Ind}^G_K[V_{\tilde{\eta} - \rho_c}] = b(P, \sigma).$$

Proof. See the last page of [42]. This uses Proposition 5.1.

Proposition 5.1 and Harish-Chandra’s character formula for (limits of) discrete series representations imply that the character of the representation $\text{Ind}^G_P(\sigma \otimes 1_A \otimes 1_N)$ naturally associated to the $K$-theory generator $b(P, \sigma)$ is zero on $T$, if this representation is reducible. (See Subsection 5.2 for an example.) Therefore, it is a useful property of the map $\tau_g$ that it maps $b(P, \sigma)$ to the possibly nonzero value of an irreducible summand of that representation.

Corollary 5.3. For almost all $g \in T$, $\tau_g(b(P, \sigma))$ equals the value at $g$ of the character of one of the irreducible summands of $\text{Ind}^G_P(\sigma \otimes 1_A \otimes 1_N)$. The values at $g$ of the characters of these summands at $g$ are all equal up to a sign.

Proof. Proposition 5.2 and Theorem 3.1 imply that

$$\tau_g(b(P, \sigma)) = \tau_g(\text{D-Ind}^G_K[V_{\tilde{\eta} - \rho_c}]) = (-1)^{\dim(G/K)/2} \sum_{w \in W_K} \varepsilon(w) e^{w\tilde{\eta}} \prod_{\alpha \in R^+} (e^{\alpha/2} - e^{-\alpha/2})(g).$$

By Harish-Chandra’s character formula (extended coherently to the limits of discrete series), the right hand side is the value at $g$ of the character of $\pi^G(\tilde{\eta}, R^+)$. That formula also shows that on $T$, the character of $\pi^G(\tilde{\eta}, R^+_j)$ equals the character of $\pi^G(\tilde{\eta}, R_j^+)$ modulo a sign, for $j = 1, \ldots, 2^\dim(A)$. Hence the claim follows from Proposition 5.1.

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Remark 5.4. If the representation $\Delta_p$ does not descend to $K$, then the analogue of Corollary 5.3 relates $\tau_g(b(P,\sigma))$ to characters of the corresponding representations of a double cover of $G$.

5.2 Example: non-spherical principal series and limits of discrete series of $\text{SL}(2,\mathbb{R})$

Consider the case where $G = \text{SL}(2,\mathbb{R})$, $K = T = \text{SO}(2)$, and $P = MAN < \text{SL}(2,\mathbb{R})$ is the minimal parabolic of upper triangular matrices, where $M = \{\pm I\}$. Then $\hat{M}_{ds} = \{\sigma_+,\sigma_-,\}$, where $\sigma_+$ is the trivial representation of $M$ in $\mathbb{C}$ and $\sigma_-$ is the nontrivial one. Now we have Morita equivalences

\[
\mathcal{K}(\text{Ind}_P^G(\sigma_+))^W_{\sigma_+} \sim C_0([0,\infty));
\]

\[
\mathcal{K}(\text{Ind}_P^G(\sigma_-))^W_{\sigma_-} \sim C_0(\mathbb{R}) \rtimes \mathbb{Z}_2.
\]

See Example 6.11 in [14]. So the pair $(P,\sigma_+)$ does not contribute to $K_0(C^*_r(\text{SL}(2,\mathbb{R})))$, whereas $(P,\sigma_-)$ contributes a summand $\mathbb{Z}$, generated by

\[b(P,\sigma_-) = \text{D-Ind}_K^G[C_0].\]

Let $\alpha \in i\mathfrak{t}^*$ be the root mapping $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ to $2i$. Set $R^+ := \{\alpha\}$. Let

\[g = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} \in T,
\]

where $\varphi \in \mathbb{R} \setminus 2\pi \mathbb{Q}$. Theorem 3.1 now yields

\[\tau_g(b(P,\sigma_-)) = \frac{1}{2i\sin \varphi}.
\]

This is the value at $g$ of the character of the limit of discrete series representation $\pi^G(0,R^+)$, and minus the value at $g$ of the character of the limit of discrete series representation $\pi^G(0,-R^+)$. The direct sum of these two representations is the non-spherical principal series representation $\text{Ind}_P^G(\sigma_- \otimes 1_A \otimes 1_N)$. The character of that representation is zero at $g$.

Some authors, including the authors of this paper, have wondered if the $K$-theory generator $b(P,\sigma_-)$ can be detected by suitable maps out of $K_0(C^*_r(\text{SL}(2,\mathbb{R})))$, and if representation theoretic information can be recovered from it. This example shows that the answer to both questions is yes.
6 Stable orbital integrals and continuity at the group identity

This section is independent of the rest of this paper. In particular, it does not depend on Theorem 3.1.

It follows from Theorem 3.1 that, for a fixed \( x \in K_0(C^*_r G) \), the function

\[
g \mapsto \tau_g(x)
\]
on the set of semisimple elements \( g \) of \( G \), is not continuous if \( G \) is noncompact. In particular, it is not continuous at the identity element. Theorem 3.1 does imply that this function is continuous almost everywhere. Already in the compact case, it is a nontrivial question if the right hand side of the fixed point formula (2.1) depends continuously on \( g \), for example as \( g \to e \) (as pointed out in Section 8.1 in [11]). It turns out that a version of \( \tau_g \) involving stable orbital integrals has better continuity properties near the identity element. (This comes at the cost of mapping more elements to zero, however. See Section 5.2 where the stable orbital integral of the class in \( K_0(C^*_r SL(2 R)) \) associated to the limits of discrete series is shown to be zero.)

6.1 Continuity at \( e \)

Let \( G_\mathbb{C} \) be a complex semisimple Lie group, and \( G < G_\mathbb{C} \) a real form of \( G_\mathbb{C} \). Let \( g \) be a semisimple element of \( G \).

**Definition 6.1.** The stable conjugacy class of \( g \) in \( G \) is

\[ (g)_s := \{ hgh^{-1} \in G : h \in G_\mathbb{C} \}; \]

the intersection of the conjugacy class \((g)_{G_\mathbb{C}} \) of \( g \) in \( G_\mathbb{C} \) with \( G \).

For every \( f \) in the Harish-Chandra Schwartz algebra \( \mathcal{C}(G) \), the stable orbital integral of \( f \) with respect to \( g \) is

\[
\tau_g^s(f) := \sum_{g'} \tau_{g'}(f) = \sum_{g'} \int_{G/Z_G(g')} f(hg'h^{-1})dh(Z_G(g')),
\]

where the sum is over representatives \( g' \) of \( G \)-conjugacy classes in \((g)_s \), i.e.,

\[ (g)_s = \sqcup g'(g'). \]

Stable conjugacy classes are relevant to the notion of an \( L \)-packet of representations and Shelstad’s character identities. See [37].
The map $\tau_s^g: K_0(C^*_r G) = K_0(C(G)) \to \mathbb{C}$ induced by $\tau_g^s$, has better continuity properties in $g$ than $\tau_g$. Let $S \subset G$ be the set of elements $g$ for which Theorem 2.1 holds, see Remark 2.2. Then $G \setminus S$ has measure zero, so in particular $S$ is dense.

**Theorem 6.2.** For all $x \in K_0(C^*_r G)$,

$$\lim_{g \to e; g \in S} \tau_g^s(x) = \tau_e(x).$$

(Note that $\tau_e = \tau_e^s$.)

Let $K < G$ be maximal compact. If $\text{rank}(G) \neq \text{rank}(K)$, then Theorem 6.2 follows from Theorem 3.1(b) and the fact that $\tau_e$ is identically zero on $K_0(C^*_r G)$. So assume from now on that $\text{rank}(G) = \text{rank}(K)$.

Theorem 6.2 implies a continuity property of characters of $L$-packets of discrete series representations.

As before, let $T < K$ be a maximal torus, and set $W_K := N_K(T)/T$. Let $W_G$ be the Weyl group of the root system of $(g^\mathbb{C}, t^\mathbb{C})$. Fix representatives $w \in W_G$ of all classes $[w] \in W_G/W_K$. For any discrete series representation with Harish-Chandra parameter $\lambda$, we denote its global character by $\Theta_{\lambda}$.

**Corollary 6.3.** Let $\pi$ be a discrete series representation of $G$ with Harish-Chandra parameter $\lambda \in i\mathfrak{t}^\ast$. Then

$$\lim_{g \to e; g \in T^{\text{reg}}} \sum_{[w] \in W_G/W_K} \Theta_{w\lambda}(g) = d_{\pi}$$

where $d_{\pi}$ is the formal degree of $\pi$.

This corollary will be proved after we prove Theorem 6.2. As a consequence, one can take the limit as $g \to e$ in Harish-Chandra’s character formula to obtain an expression for $d_{\pi}$; see e.g. page 25 of [7]. See also Proposition 50 in [18].

### 6.2 A $K$-theoretic character identity

Let $G_c$ be a compact inner form of $G$, which exists because $\text{rank}(G) = \text{rank}(K)$. Inner forms are defined for example in Chapter 2 of [2], but the only properties we need are that $G_c$ is a real form of $G$, and $T$ identifies with a Cartan subgroup of $G_c$. So pairs $(G, T)$ and $(G_c, T)$ have the same root system. The positive root system $R^+$ determines a $G$-invariant complex
structure on $G_c/T$. For any integral $\nu \in i\mathfrak{t}^*$, consider the holomorphic line bundles

$$L^G_\nu := G \times_T \mathbb{C}_\nu \to G/T;$$
$$L^G_{c\nu} := G_c \times_T \mathbb{C}_\nu \to G_c/T.$$ 

Let $\bar{\partial}_{L^G_\nu}$ and $\bar{\partial}_{L^G_{c\nu}}$ be the Dolbeault operators on $G/T$ and $G_c/T$, respectively, coupled to these line bundles.

In [27], the authors prove a $K$-theoretic analogue of Shelstad’s character identities [37], and deduce Shelstad’s character identity in the case of the discrete series.

**Theorem 6.4.** For all integral $\nu \in i\mathfrak{t}^*$ and all $g \in S$,

$$\tau_g(\text{index}_{G_c}(\bar{\partial}_{L^G_{c\nu}} + \bar{\partial}^*_{L^G_{c\nu}})) = \sum_{[w] \in W_G/W_K} \tau_g(\text{index}_{G_c}(\bar{\partial}_{L^G_w}^{-1}_\nu + \bar{\partial}^*_{L^G_w}^{-1}_\nu)).$$

**Proof.** This is (3.6) in [27]. There, $\nu$ is regular but that property is not used in the proof of the above equality. \hfill \Box

### 6.3 Dolbeault operators

We will use some properties of the Dolbeault–Dirac operators in Theorem 6.4 to deduce Theorem 6.2.

First of all, every element of $K_0(C^*_r G)$ is the index of a Dolbeault–Dirac operator on $G/T$. Indeed, let $V \in \bar{K}_{\text{Spin}}$, and let $\lambda \in i\mathfrak{t}^*$ be its highest weight with respect to the positive compact roots chosen earlier. Then $\lambda - \rho_n$ is a weight of $\Delta_p \otimes V$, so it is integral for $T$. Consider the holomorphic, $G$-equivariant line bundle

$$L^G_{\lambda - \rho_n} := G \times_T \mathbb{C}_{\lambda - \rho_n} \to G/T.$$ 

Let $\bar{\partial}_{L^G_{\lambda - \rho_n}}$ be the Dolbeault operator on $G/T$ coupled to $L^G_{\lambda - \rho_n}$.

**Proposition 6.5.** We have

$$\text{D-Ind}_K^G[V_\lambda] = (-1)^{\dim(G/K)} \text{index}_G(\bar{\partial}_{L^G_{\lambda - \rho_n}} + \bar{\partial}^*_{L^G_{\lambda - \rho_n}}).$$

**Proof.** This is proved in Section 5 of [26] in the case where $\lambda + \rho_c$ is regular for $G$, but that assumption is not necessary for the arguments. \hfill \Box

**Lemma 6.6.** We have for all $w \in W_G$ and all $g \in S$,

$$\tau_{wg^{-1}}(\text{index}_G(\bar{\partial}_{L^G_{\lambda - \rho}} + \bar{\partial}^*_{L^G_{\lambda - \rho}})) = \tau_g(\text{index}_G(\bar{\partial}_{L^G_{w^{-1}(\lambda - \rho)}} + \bar{\partial}^*_{L^G_{w^{-1}(\lambda - \rho)}})).$$

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Proof. In the case of Dolbeault operators twisted by holomorphic vector bundles, and finite fixed point sets, the fixed point formula in Theorem 2.1 simplifies considerably, see Corollary 6.3 in [26]. For any $h \in T$ with dense powers, and any integral $\nu \in i(t^*)$, this yields

$$\tau_h(\text{index}_G(\tilde{\partial}_G + \tilde{\partial}_G^*)) = \sum_{xT \in (G/T)^h} \frac{\text{tr}(g|_{(L_G^\nu)_{xT}})}{\det_C(1 - g^{-1}|_{T_{xT}G/T})}. \tag{6.1}$$

Now, for $w \in W_G$, we have $(G/T)^{wgw^{-1}} = (G/T)^T = N_K(T)/T$. And for $x \in N_K(T)$,

$$(L_G^\nu)_{xT} = \mathbb{C}\text{Ad}^*(x)_\nu;$$

$$T_{xT}G/T = \bigoplus_{\alpha \in R^+} \mathbb{C}\text{Ad}^*(x)_{\alpha}.$$ 

as complex representations of $T$, where we use the complex structure on $G/T$ defined by $R^+$. So

$$\tau_{wgw^{-1}}(\text{index}_G(\tilde{\partial}_L^\nu + \tilde{\partial}_L^\nu^*)) = \sum_{xT \in N_K(T)/T} \frac{\text{tr}(wgw^{-1}|_{\mathbb{C}\text{Ad}^*(x)(\lambda - \rho)})}{\det_C(1 - wg^{-1}w^{-1}|_{\bigoplus_{\alpha \in R^+} \mathbb{C}\text{Ad}^*(x)_{\alpha}})}$$

$$= \sum_{yT \in w^{-1}N_K(T)w/T} \frac{\text{tr}(g|_{\mathbb{C}\text{Ad}^*(y^{-1}w^{-1})(\lambda - \rho)})}{\det_C(1 - g^{-1}|_{\bigoplus_{\alpha \in R^+} \mathbb{C}\text{Ad}^*(y^{-1}w^{-1})_{\alpha}})}$$

$$= \sum_{yT \in w^{-1}N_K(T)w/T} \frac{\text{tr}(g|_{\mathbb{C}\text{Ad}^*(y^{-1})(\lambda - \rho)})}{\det_C(1 - g^{-1}|_{\bigoplus_{\alpha \in R^+} \mathbb{C}\text{Ad}^*(y^{-1})_{\alpha}})}. \tag{6.2}$$

(In the last step, we substituted $y = w^{-1}xw$.)

Finally, $w^{-1}N_K(T)w = N_K(T)$, and

$$\bigoplus_{\alpha \in R^+} \mathbb{C}\text{Ad}^*(y^{-1})_{\alpha} = T_{yT}G/T$$

as complex representations of $T$, with respect to the complex structure defined by the positive root system $w^{-1}R^+$ with respect to which $w^{-1}(\lambda - \rho)$ is dominant. So by (6.1), the expression (6.2) equals

$$\tau_g(\text{index}_G(\tilde{\partial}_L^\nu_{w^{-1}(\lambda - \rho)} + \tilde{\partial}_L^\nu_{w^{-1}(\lambda - \rho)})).$$

\end{proof}
Lemma 6.7. We have for all integral $\nu \in i^*$,
\[
\tau(e(\text{index}_{G_c}(\bar{\partial}_{L^c} + \bar{\partial}^*_{L^c}))) = \tau(e(\text{index}_{G}(\bar{\partial}_{L^c} + \bar{\partial}^*_{L^c}))).
\]

Proof. By Connes–Moscovici’s $L^2$-index formula, Theorem 5.2 in [15], we have
\[
\tau(e(\text{index}_{G}(\bar{\partial}_{L^c} + \bar{\partial}^*_{L^c}))) = \varepsilon(\text{ch}(\bigwedge C_{g^c} \otimes C_{\nu}) \hat{A}(g_c, T))[g_c/t],
\]
for the same sign $\varepsilon \in \{\pm 1\}$. Here $\text{ch}: R(T) \to H^*(g, T, \mathbb{R})$ is the relative Chern character, and the characteristic classes $\hat{A}$ in $H^*(g, T, \mathbb{R})$ are defined in Section 4 of [15]. The right hand side of the first line only depends on the representations $\bigwedge C_{g^c} \otimes C_{\nu}$ and $g/t$ of $T$, and similarly for the right hand side of the second line. Since $g/t$ and $g_c/t$ both equal the sum of the positive root spaces as complex representations of $T$, we find that the two expressions are equal.

6.4 Proofs of Theorem 6.2 and Corollary 6.3

To finish the proof of Theorem 6.2, we need a final lemma.

Lemma 6.8 (Arthur). We have for all $g \in T^\reg$,
\[
\tau^s_g = \sum_{[w] \in W_G/W_K} \tau_{wgw^{-1}}.
\]

Proof. In Section 27 (p.194) of [4], it is pointed out that two elements $g, g' \in T^\reg$ are conjugate if and only if $g = w_K g' w_K^{-1}$ for some $w_K \in W_K$, and stably conjugate if and only if $g = w_G g' w_G^{-1}$ for some $w_G \in W_G$.

Proof of Theorem 6.2. By surjectivity of Dirac induction and Proposition 6.5 every $x \in K_0(C^*_r G)$ is represented by the equivariant index
\[
x = \text{index}_G(\bar{\partial}_{L^c} + \bar{\partial}^*_{L^c})
\]
for an integral element $\nu \in i^*$.

Let $g \in S$. By Theorem 6.4 and Lemmas 6.6 and 6.8 we have
\[
\tau^s_g(x) = \tau^s_g(\text{index}_G(\bar{\partial}_{L^c} + \bar{\partial}^*_{L^c})) = \tau_g(\text{index}_{G_c}(\bar{\partial}_{L^c} + \bar{\partial}^*_{L^c})).
\]
Since $G_c$ is compact, this expression is continuous in $g$. And by Lemma 6.7,
\[
\tau(e(\text{index}_{G_c}(\bar{\partial}_{L^c} + \bar{\partial}^*_{L^c}))) = \tau(e(\text{index}_G(\bar{\partial}_{L^c} + \bar{\partial}^*_{L^c}))) = \tau_e(x).
\]
Proof of Corollary 6.3. For $w \in W_G$, let $[\pi_{w\lambda}] \in K_0(C^*_r G)$ be the class defined by the discrete series representation with Harish-Chandra parameter $w\lambda$. By Propositions 5.1 and 5.2 in [25], we have for all $g \in T^{reg}$,

$$\sum_{[w] \in W_G/W_K} \Theta_{w\lambda}(g) = \sum_{[w] \in W_G/W_K} \tau_g([\pi_{w\lambda}])$$

$$= (-1)^{\dim(G/K)/2} \sum_{[w] \in W_G/W_K} \tau_g(\text{index}_G(\bar{\partial}_{L^G_{w(\lambda-\rho)}} + \bar{\partial}^{*}_{L^G_{w(\lambda-\rho)}})).$$

Lemmas 6.6 and 6.8 imply that the right hand side equals

$$(-1)^{\dim(G/K)/2} \tau_g(\text{index}_G(\bar{\partial}_{L^G_{\lambda-\rho}} + \bar{\partial}^{*}_{L^G_{\lambda-\rho}})).$$

As $g \to e$ through the set $S$ in Theorem 6.2 that result implies that the limit of the above expression is

$$(-1)^{\dim(G/K)/2} \tau_e(\text{index}_G(\bar{\partial}_{L^G_{\lambda-\rho}} + \bar{\partial}^{*}_{L^G_{\lambda-\rho}})) = \tau_e([\pi_\lambda]) = d_{\pi}.$$

The claim now follows from continuity of characters on the regular set. □

References

[1] Herbert Abels. Parallelizability of proper actions, global $K$-slices and maximal compact subgroups. Math. Ann., 212:1–19, 1974/75.

[2] Jeffrey Adams, Dan Barbasch, and David A. Vogan, Jr. The Langlands classification and irreducible characters for real reductive groups, volume 104 of Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 1992.

[3] Alexandre Afgoustidis. How tempered representations of a semisimple Lie group contract to its Cartan motion group. ArXiv:1707.00240.

[4] James Arthur. An introduction to the trace formula. Harmonic analysis, the trace formula, and Shimura varieties, 1–263, Clay Math. Proc., 4, Amer. Math. Soc., 2005.

[5] Michael Atiyah and Raoul Bott. A Lefschetz fixed point formula for elliptic complexes. I. Ann. of Math. (2), 86:374–407, 1967.

[6] Michael Atiyah and Raoul Bott. A Lefschetz fixed point formula for elliptic complexes. II. Applications. Ann. of Math. (2), 88:451–491, 1968.
[7] Michael Atiyah and Wilfried Schmid. A geometric construction of the discrete series for semisimple Lie groups. *Invent. Math.*, 42:1–62, 1977.

[8] Michael Atiyah and Graeme Segal. The index of elliptic operators. II. *Ann. of Math. (2)*, 87:531–545, 1968.

[9] Michael Atiyah and Isadore Singer. The index of elliptic operators. III. *Ann. of Math. (2)*, 87:546–604, 1968.

[10] Paul Baum, Alain Connes, and Nigel Higson. Classifying space for proper actions and K-theory of group C*-algebras. In *C*-algebras: 1943–1993 (San Antonio, TX, 1993), volume 167 of Contemp. Math., pages 240–291. American Mathematical Society, Providence, RI, 1994.

[11] Nicole Berline, Ezra Getzler, and Michèle Vergne. *Heat kernels and Dirac operators*. Grundlehren Text Editions. Springer-Verlag, Berlin, 2004. Corrected reprint of the 1992 original.

[12] Philippe Blanc and Jean-Luc Brylinski. Cyclic homology and the Selberg principle. *J. Funct. Anal.*, 109(2):289–330, 1992.

[13] Jérôme Chabert, Siegfried Echterhoff, and Ryszard Nest. The Connes-Kasparov conjecture for almost connected groups and for linear p-adic groups. *Publ. Math. Inst. Hautes Études Sci.*, 97:239–278, 2003.

[14] Pierre Clare, Tyrone Crisp, and Nigel Higson. Parabolic induction and restriction via C*-algebras and Hilbert C*-modules. *Compos. Math.*, 152(6):1286–1318, 2016.

[15] Alain Connes and Henri Moscovici. The $L^2$-index theorem for homogeneous spaces of Lie groups. *Ann. of Math. (2)*, 115(2):291–330, 1982.

[16] Nicolas Ginoux. *The Dirac spectrum*, volume 1976 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009.

[17] Sherry Gong. Finite part of operator K-theory for groups with rapid decay. *J. Noncommut. Geom.*, 9(3):697–706, 2015.

[18] Hao Guo, Mathai Varghese, and Hang Wang. Positive scalar curvature and poincare duality for proper actions. ArXiv:1609.01404.

[19] Harish-Chandra. Discrete series for semisimple Lie groups. II. Explicit determination of the characters. *Acta Math.*, 116:1–111, 1966.
[20] Nigel Higson. The Mackey analogy and $K$-theory. In Group representations, ergodic theory, and mathematical physics: a tribute to George W. Mackey, volume 449 of Contemp. Math., pages 149–172. Amer. Math. Soc., Providence, RI, 2008.

[21] Nigel Higson. On the analogy between complex semisimple groups and their Cartan motion groups. In Noncommutative geometry and global analysis, volume 546 of Contemp. Math., pages 137–170. Amer. Math. Soc., Providence, RI, 2011.

[22] Peter Hochs. Quantisation commutes with reduction at discrete series representations of semisimple groups. Adv. Math., 222(3):862–919, 2009.

[23] Peter Hochs and Varghese Mathai. Spin-structures and proper group actions. Adv. Math., 292:1–10, 2016.

[24] Peter Hochs and Varghese Mathai. Quantising proper actions on Spin$^c$-manifolds. Asian J. Math., 21(4):631–685, 2017.

[25] Peter Hochs and Hang Wang. A fixed point formula and Harish-Chandra’s character formula. Proc. London Math. Soc., 00(3):1–32, 2017.

[26] Peter Hochs and Hang Wang. A fixed point theorem on noncompact manifolds. Ann. K-theory, 2017. to appear, ArXiv:1512.07812.

[27] Peter Hochs and Hang Wang. Shelstad’s character identity from the point of view of index theory. arXiv:1711.00992, 2017.

[28] Pierre Julg and Alain Valette. Twisted coboundary operator on a tree and the Selberg principle. J. Operator Theory, 16(2):285–304, 1986.

[29] Pierre Julg and Alain Valette. L’opérade de co-bord tordu sur un arbre, et le principe de Selberg. II. J. Operator Theory, 17(2):347–355, 1987.

[30] Anthony W. Knapp and Gregg J. Zuckerman. Classification of irreducible tempered representations of semisimple groups. II. Ann. of Math. (2), 116(3):457–501, 1982.

[31] Vincent Lafforgue. Banach KK-theory and the Baum-Connes conjecture. In Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), pages 795–812. Higher Ed. Press, Beijing, 2002.
[32] Vincent Lafforgue. $K$-théorie bivariante pour les algèbres de Banach et conjecture de Baum-Connes. *Invent. Math.*, 149(1):1–95, 2002.

[33] George W. Mackey. On the analogy between semisimple Lie groups and certain related semi-direct product groups. pages 339–363, 1975.

[34] Rajagopalan Parthasarathy. Dirac operator and the discrete series. *Ann. of Math. (2)*, 96:1–30, 1972.

[35] M. G. Penington and Roger J. Plymen. The Dirac operator and the principal series for complex semisimple Lie groups. *J. Funct. Anal.*, 53(3):269–286, 1983.

[36] Süleyman Kağan Samurkaş. Bounds for the rank of the finite part of operator $K$-theory. ArXiv:1705.07378.

[37] Diana Shelstad. Characters and inner forms of a quasi-split group over $\mathbb{R}$. *Compositio Math.*, 39(1):11–45, 1979.

[38] Qijun Tan, Yi-Jun Yao, and Shilin Yu. Mackey analogy via $D$-modules for $\text{SL}(2, \mathbb{R})$. *Internat. J. Math.*, 28(7):1750055, 20, 2017.

[39] Tadao Tannaka. Über den Dualitätssatz der nichtkommutativen topologischen Gruppen. *Tohoku Math. J.*, 45:1–12, 1939.

[40] Bai-Ling Wang and Hang Wang. Localized index and $L^2$-Lefschetz fixed-point formula for orbifolds. *J. Differential Geom.*, 102(2):285–349, 2016.

[41] Hang Wang. $L^2$-index formula for proper cocompact group actions. *J. Noncommut. Geom.*, 8(2):393–432, 2014.

[42] Antony Wassermann. Une démonstration de la conjecture de Connes-Kasparov pour les groupes de Lie linéaires connexes réductifs. *C. R. Acad. Sci. Paris Sér. I Math.*, 304(18):559–562, 1987.

[43] Shilin Yu. Mackey analogy as deformation of $D$-modules. ArXiv:1510.02650.