Non-Markovian dynamics with fermions

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Employing the quadratic fermionic Hamiltonians for the collective and internal subsystems with a linear coupling, we studied the role of fermionic statistics on the dynamics of the collective motion. The transport coefficients are discussed as well as the associated fluctuation-dissipation relation. Due to different nature of the particles, the path to equilibrium is slightly affected. However, in the weak coupling regime, the time-scale for approaching equilibrium is found to be globally unchanged. The Pauli-blocking effect can modify the usual picture in open quantum system. In some limits, contrary to boson, this effect can strongly hinder the influence of the bath by blocking the interacting channels.

I. INTRODUCTION

Investigations of the behavior of dissipative quantum non-Markovian collective subsystems beyond the weak coupling or high temperature limits triggered significant interest in exactly solvable models \cite{1–14}. The internal subsystem in these models is represented by a collection of bosonic harmonic oscillators, which interacts with the collective bosonic subsystem, modeled by a harmonic oscillator as well, via a linear coupling between the coordinates and/or momenta. The quantum-mechanical motion of a fermionic oscillator (two-level collective subsystem) coupled linearly to dissipative fermionic environment requires an intensive study \cite{6}. The interest in considering fermionic baths is however growing up due to the possibility of creating and manipulating rather small fermionic systems in condensed matter, atomic and nuclear physics. In particular, the approaching phase to equilibrium of these systems would greatly help to understand how they can reach the thermodynamic limits \cite{15}.

In the present paper, we use the quadratic fermionic Hamiltonians for the collective and internal subsystems with a linear coupling, and present the detailed analysis of the role of the fermionic statistics (in the comparison with the bosonic one) in the dynamics of the collective motion. As an example, we compare the decays of excited collective states of the bosonic and fermionic harmonic oscillators.

We use the Langevin approach \cite{10–14} which is widely applied to find the effects of fluctuations and dissipations in macroscopical systems. The Langevin method in the kinetic theory significantly simplifies the calculation of non-equilibrium quantum and thermal fluctuations and provides a clear picture of the dynamics. Many problems in solid state physics, condensed matter, chemistry, nuclear and atomic physics can be described using the Langevin equations in the space of relevant collective coordinates.

In Sec. II.A we present a fully quantum-mechanical derivation of non-Markovian Langevin equations. These equations fulfill the quantum fluctuation-dissipation theorem as shown in Sec. II.B. In Secs. II.C and II.D the analytical expressions for occupation number are derived. The time-dependent transport coefficients which take the memory effects into consideration are obtained in Sec. II.E. The results of illustrative numerical calculations of diffusion and friction coefficients and level populations are presented in Sec. III. Sec. III deals with the decay of excited state for the collective plus internal bosonic or fermionic subsystems. A summary is given in Sec. IV.

II. FORMALISM

The Hamiltonian of whole system (the heat bath plus collective subsystem) is assumed here to have the following form \cite{6}:

\begin{equation}
H = H_c + H_b + H_{cb} = \hbar \omega a^+ a + \sum_\nu \hbar \omega_\nu a^\nu_+ a^\nu + \sum_\nu g_\nu \left( a^\nu_+ a + a^+ a^\nu \right),
\end{equation}

which explicitly depends on the collective (heat bath) creation \( a^+ \) \((a^\nu_+)\) and annihilation \( a \) \((a^\nu)\) operators. Contrary to the usual assumption, these operators are considered here as the fermionic operators and satisfy the following permutation relations:

\begin{equation}
aa^+ + a^+ a = 1, aa = a^+ a^+ = 0, a_\nu a^\nu_+ + a^\nu_+ a_\nu = \delta_{\nu,\nu'}, a_\nu a_{\nu'} + a_{\nu'} a_\nu = a^\nu_+ a^\nu + a^\nu a^\nu_+ = 0.
\end{equation}

In Eq. (1) the terms \( H_c, H_b, \) and \( H_{cb} \) are the Hamiltonians of the collective subsystem depending on the frequency \( \hbar \omega \) of the fermionic oscillator, of the intrinsic bath subsystem, and of the collective-bath interaction, respectively. The
model of heat bath is an assembly of the fermionic oscillators \( \nu \) with frequencies \( \hbar \omega_\nu \). More precisely, we assumed that the bath is a collection of two-level fermionic systems depicted in Fig. 1. The use of two-level picture is helpful to simplify slightly the derivations given below while keeping the main physical effects associated with the fermionic nature of the bath.

\[ g_\nu \]

\[ \hbar \Omega \]

\[ n_\nu(\omega) \]

\[ \mu \]

\[ \Omega \]

\[ \omega \]

\[ \mu \]

The coupling to the heat bath is linear in the collective and bath operators and corresponds to the energy being transferred to and from the bath or collective subsystem by the creation and absorption of bath or collective subsystem quanta. The real constant \( g_\nu \) determines the coupling strength between the collective oscillator and bath oscillator \( \nu \). The coupling term can have important consequences on the dynamics of the collective subsystem by altering the effective collective potential and by allowing energy to be exchanged with the thermal reservoir, thereby, allowing the collective subsystem to attain the thermal equilibrium with the heat bath.

\[ a^+_\nu(t) = a^+_\nu(0) + \frac{g_\nu}{\hbar \omega_\nu} a^+_\nu(0) e^{i \omega_\nu t} - \frac{g_\nu}{\hbar \omega_\nu} a^+_\nu(t) + \frac{g_\nu}{\hbar \omega_\nu} \int_0^t e^{i \omega_\nu (t-\tau)} \frac{d a^+_\nu(\tau)}{d \tau} d\tau, \]

\[ a_\nu(t) = a_\nu(0) + \frac{g_\nu}{i \hbar \omega_\nu} a(0) e^{-i \omega_\nu t} - a_\nu(0) + \frac{g_\nu}{\hbar \omega_\nu} \int_0^t e^{-i \omega_\nu (t-\tau)} \frac{d a(\tau)}{d \tau} d\tau. \]
Substituting Eqs. (5) in (3) and eliminating the bath variables from the equations of motion of the collective subsystem, we obtain a set of integro-differential stochastic dissipative equations:

\[
\frac{d}{dt}a^+(t) = i\omega a^+ + \frac{i}{\hbar} \sum_\nu g_\nu \left(a_\nu^+(0) + \frac{g_\nu}{i\omega_\nu}a^+(0)\right)e^{i\omega_\nu t} - i \sum_\nu \frac{g_\nu^2}{\hbar^2 \omega_\nu}a^+(t) + i \sum_\nu \frac{g_\nu^2}{\hbar^2 \omega_\nu} \int_0^t e^{i\omega_\nu(t-\tau)} \frac{d\nu^+(\tau)}{d\tau} d\tau, \\
\frac{d}{dt}a(t) = -i\omega a - \frac{i}{\hbar} \sum_\nu g_\nu \left(a_\nu(0) + \frac{g_\nu}{i\omega_\nu}a(0)\right)e^{-i\omega_\nu t} + i \sum_\nu \frac{g_\nu^2}{\hbar^2 \omega_\nu}a(t) - i \sum_\nu \frac{g_\nu^2}{\hbar^2 \omega_\nu} \int_0^t e^{-i\omega_\nu(t-\tau)} \frac{d\nu(\tau)}{d\tau} d\tau. 
\]  

(6)

Introducing the renormalized frequency and creation operators, respectively,

\[
\Omega = \omega - \sum_\nu \frac{g_\nu^2}{\hbar^2 \omega_\nu}, \\
\tilde{a}_\nu^+ = a_\nu^+ + \frac{g_\nu}{i\omega_\nu}a_\nu^+, 
\]

we obtain from Eqs. (6) the Langevin type equations

\[
\frac{d}{dt}a^+(t) = ia^+(t)\Omega + iF^+(t) + i \int_0^t K^*(t-\tau) \frac{d\nu^+(\tau)}{d\tau} d\tau, \\
\frac{d}{dt}a(t) = -ia(t)\Omega - iF(t) + i \int_0^t K(t-\tau) \frac{d\nu(\tau)}{d\tau} d\tau, 
\]

(8)

with the dissipative kernels

\[
K^*(t) = \sum_\nu \frac{g_\nu^2}{\hbar^2 \omega_\nu} e^{i\omega_\nu t}, \\
K(t) = \sum_\nu \frac{g_\nu^2}{\hbar^2 \omega_\nu} e^{-i\omega_\nu t},
\]

and random forces

\[
F^+(t) = \sum_\nu F^+_\nu(t) = \frac{1}{\hbar} \sum_\nu g_\nu \tilde{a}_\nu^+(0) e^{i\omega_\nu t}, \\
F(t) = \sum_\nu F_\nu(t) = \frac{1}{\hbar} \sum_\nu g_\nu \tilde{a}_\nu(0) e^{-i\omega_\nu t}.
\]

B. Fluctuation-dissipation relations

The fluctuation-dissipation relations are the relations between the dissipation of a collective subsystem and the fluctuations of random forces. Those relations express the non-equilibrium behavior of the system in terms of equilibrium or quasi-equilibrium characteristics. They ensure that the system approaches the equilibrium state.

For the correlation of the random force, one can obtain

\[
<< F^+_\nu(t) >> = << F_\nu(t) >> = << F^+_\nu(t)F^+_\nu(t) >> = << F_\nu(t)F_\nu(t) >> = 0
\]

and

\[
<< F^+_\nu(t)F_\nu(\tau) >> = \frac{g_\nu^2}{\hbar^2} << \tilde{a}_\nu^+(0)\tilde{a}_\nu(0) >> e^{i\omega_\nu(t-\tau)} = \frac{g_\nu^2}{\hbar^2} e^{i\omega_\nu(t-\tau)} n_\nu, \\
<< F_\nu(t)F^+_\nu(\tau) >> = \frac{g_\nu^2}{\hbar^2} << \tilde{a}_\nu(0)\tilde{a}_\nu^+(0) >> e^{-i\omega_\nu(t-\tau)} = \frac{g_\nu^2}{\hbar^2} e^{-i\omega_\nu(t-\tau)} [1 - n_\nu],
\]

(9)

where \(n_\nu = \frac{1}{1 + \exp \left[ \frac{(\hbar \omega_\nu - \mu)}{kT} \right]} \) is the equilibrium Fermi-Dirac distribution of the occupation numbers for fermions depending on the temperature \(T\). The parameters \(\mu\) is the chemical potential that offers the possibility to fix arbitrarily
the initial occupation of the states to which the system is coupled to. In most applications shown below, we will simply assume \( \mu = 0 \). However, as we will see, the change of \( \mu \) significantly affects the evolution. Here, the symbol \( \langle\langle ... \rangle\rangle \) denotes the average over the bath. Using the expressions \([9]\), one can find the fluctuation dissipation relations

\[
K^*(t - \tau) = \sum_{\nu} \frac{\langle\langle F^\nu(t) F^\nu(\tau) \rangle\rangle}{\omega_{\nu} n_{\nu}},
\]

\[
K(t - \tau) = \sum_{\nu} \frac{\langle\langle F^\nu(t) F^\nu(\tau) \rangle\rangle}{\omega_{\nu} [1 - n_{\nu}]},
\]

**C. Analytical expressions for occupation number**

It is convenient to introduce the spectral density \( \rho(w) \) of the heat bath excitations, which allows us to replace the sum over different two-level systems \( \nu \) by integral over the frequency: \( \sum_{\nu} \rightarrow \int_0^\infty dw \rho(w) \). Let us consider the following spectral function \([2]\)

\[
\frac{g_w^2}{\hbar^2 \omega_{\nu}} \rightarrow \rho(w) g^2(w) = \frac{1}{\pi} g_0 \frac{\gamma^2}{\gamma^2 + w^2},
\]

where the memory time \( \gamma^{-1} \) of the dissipation is inverse to the bandwidth of the heat bath excitations which are coupled to collective subsystem. This is the Ohmic dissipation with the Lorenzian cutoff (Drude dissipation). The relaxation time of the heat bath should be much less than the characteristic collective time, i.e. \( \gamma \gg \omega \).

Employing Eq. \([12]\), we obtain the expressions for the dissipative kernels:

\[
K^*(t) = \frac{1}{\pi} g_0 \gamma^2 \int_0^\infty dw \frac{e^{iw\gamma t}}{\gamma^2 + w^2} = \frac{1}{2} g_0 \gamma^2 e^{-\gamma t} + i \frac{1}{\pi} g_0 \gamma^2 \int_0^\infty dw \frac{\sin[wt]}{\gamma^2 + w^2},
\]

\[
K(t) = \frac{1}{\pi} g_0 \gamma^2 \int_0^\infty dw \frac{e^{-iw\gamma t}}{\gamma^2 + w^2} = \frac{1}{2} g_0 \gamma^2 e^{-\gamma t} - i \frac{1}{\pi} g_0 \gamma^2 \int_0^\infty dw \frac{\sin[wt]}{\gamma^2 + w^2}.
\]

To leading order in \( g_0 \), these dissipative kernels are approximated in the following way \([2]\)

\[
K^*(t) = K(t) = \frac{2}{\pi} g_0 \gamma^2 \int_0^\infty dw \cos(wt) = g_0 \gamma e^{-\gamma t}.
\]

Besides the specific nature of the bath (see Fig. \([1]\), setting to zero the imaginary part of \( K(t) \) is the only approximation that is made in the present work. Comparing with other approach \([16]\), it could be shown that this approximation does not influence the non-equilibrium evolution in the weak coupling regime. In the limit \( \gamma \rightarrow \infty \), the dissipative kernels \([14]\) have a well-known Markovian form

\[
K(t) = 2g_0 \delta(t).
\]

For the sake of simplicity, in further analytical calculations we use the dissipative kernels \([14]\).

For the average occupation number \( n(t) \), one can obtain (see Appendix A):

\[
n(t) = \langle\langle a^+(t) a(t) \rangle\rangle = \bar{A}^*(t) \bar{A}(t) \langle\langle a^+(0) a(0) \rangle\rangle + \langle\langle \bar{F}^+(t) \bar{F}(t) \rangle\rangle
\]

with the explicit expression

\[
\langle\langle \bar{F}^+(t) \bar{F}(t) \rangle\rangle = \frac{1}{\pi} g_0 \gamma^2 \int_0^\infty dw \frac{w}{\gamma^2 + w^2} B_w^*(t) B_w(t) n_\nu(\omega),
\]

where

\[
B_w^*(t) = \frac{e^{itw}(z_1 - z_2)(iw + \gamma) + e^{iz_1}(-iw + z_2)(z_1 + \gamma) + ie^{iz_2}(w + iz_1)(z_2 + \gamma)}{(w + iz_1)(z_1 - z_2)(w + iz_2)},
\]

and

\[
\bar{A}^*(t) = e^{iz_1} \frac{z_1 + \gamma - ig_0 \gamma}{z_1 - z_2} + e^{iz_2} \frac{z_2 + \gamma - ig_0 \gamma}{z_2 - z_1}.
\]
The expressions for $z_1$ and $z_2$ are given in Appendix A. The equilibrium population of the fermionic oscillator is obtained by taking $t \to \infty$ limit in \ref{eq:20}. The first term of $B_w^\ast(t)$ contributes only to the asymptotic value of the occupation number:

$$n(t \to \infty) = \langle \langle \hat{F}^+(t) \hat{F}(t) \rangle \rangle \mid_{t \to \infty} = \frac{1}{\pi} g_0 \gamma^2 \int_0^\infty dw \frac{w}{|w + i z_1|^2 |w + i z_2|^2} n_\nu(\omega). \quad (20)$$

If at $t = 0$ the collective subsystem is in the ground state, then $\langle a^+(0) a(0) \rangle = 0$ and

$$n(t) = \frac{1}{\pi} g_0 \gamma^2 \int_0^\infty dw \frac{w}{\gamma^2 + w^2} B_w^\ast(t) B_w(t) n_\nu(\omega). \quad (21)$$

This compact expression is rather interesting to identify the effect of the Fermi statistics. Indeed, three contributions to the evolution can be clearly separated. First, we easily identify the influence of the specific spectral function. Here, we will assume large $\gamma$ value to insure that the system is coupled to a large set of states in the bath. The second contribution stems from $B_w^\ast(t) B_w(t)$. This term only depends on the original equation of motion [see Eqs. (3) and (4)]. These two factors contain the specific treatment of the bath considered here. Due to the two-level nature of the bath, their contributions turn out to be identical to those of bosonic bath. The only direct effect of the fermionic nature of the bath is the Fermi-Dirac statistics that determines the initial occupancies $n_\nu(\omega)$.

Note that we will also compare the results with the reference case of a system coupled to a bosonic bath. This case is simply obtained by employing the following replacement

$$\frac{1}{1 + \exp \left[ \frac{h\nu}{kT} \right]} \rightarrow \frac{1}{\exp \left[ \frac{h\nu}{kT} \right] - 1},$$

in Eq. \ref{eq:20}. One can obtain the expression for the occupation number of bosonic system (collective bosonic oscillator plus internal bosonic oscillators). This expression for bosonic system was firstly derived in Ref. \cite{2}.

**D. Analytical expressions for level population at $g_0 \ll 1$ and at high and low temperatures**

Retaining terms to leading order in $g_0$, we obtain at $g_0 \ll 1$ that

$$z_1 = -\gamma + i g_0 \gamma,$$
$$z_2 = i \Omega - g_0 \gamma,$$

$$\tilde{A}^\ast(t) = e^{z_1 t} \left( \frac{z_1 + \gamma - i g_0 \gamma}{z_1 - z_2} + e^{z_2 t} \left( \frac{z_2 + \gamma - i g_0 \gamma}{z_2 - z_1} \right) \right) = e^{z_2 t},$$

$$\tilde{A}^\ast(t) \tilde{A}(t) = e^{(z_2 + z_2^t) t} = e^{-2 g_0 \Omega t},$$

and

$$n(t) = \langle a^+(0) a(0) \rangle \gg e^{-2 g_0 \Omega t} + \frac{1}{\pi} g_0 \gamma^2 \int_0^\infty dw \frac{1}{\gamma^2 + w^2} \frac{w}{1 + \exp \left[ \frac{h\nu}{kT} \right]} \times$$

$$\left\{ f_1 + f_2 e^{-2 \gamma t} + f_3 e^{-2 g_0 \Omega t} + 2 \text{Re}[f_4 e^{(z_1 - i \omega) t}] + 2 \text{Re}[f_5 e^{(z_2 - i \omega) t}] + 2 \text{Re}[f_6 e^{(z_1 + z_2^t) t}] \right\}. \quad (23)$$

The analytical expressions for the functions $f_i$ are presented in Appendix B. Another good presentation is as follows

$$n(t) = \langle a^+(0) a(0) \rangle \gg e^{-2 g_0 \Omega t} + n(t \to \infty) [1 + e^{-2 g_0 \Omega t}], \quad (24)$$

$$n(t \to \infty) = \frac{1}{\pi} g_0 \gamma^2 \int_0^\infty dw \frac{w}{1 + \exp \left[ \frac{h\nu}{kT} \right]} \frac{1}{[w^2 - g_0 w \gamma + (1 + g_0^2) \gamma^2] [w^2 - 2 \omega \Omega + (1 + g_0^2) \Omega^2]^2]. \quad (25)$$

At high temperatures ($kT \gg h\nu$) and $h\gamma \gg kT$, the main contribution to the integral \ref{eq:25} arises in the neighborhood of $|w| \approx \Omega$ and yields the usual Fermi-Dirac population

$$n_F = \frac{1}{\exp \left[ \frac{h\nu}{kT} \right] + 1}, \quad (26)$$
showing that the bath imposes its temperature to the system. Similarly, for bosonic systems, in this limit, we obtain

$$n_B = \frac{1}{\exp \left[ \frac{\hbar \gamma}{kT} \right] - 1}.$$  (27)

At $\gamma \to \infty$ and $g_0 \ll 1,$

$$n(t \to \infty) = \frac{g_0}{\pi} \int_0^{\infty} dw \frac{w}{\exp \left[ \frac{\hbar w}{kT} \right] \pm 1} \left( w^2 - 2w\Omega + \Omega^2 \right).$$  (28)

At the low-temperature limit ($kT \ll \hbar \omega$), in (28) one can employ the following expansion:

$$\frac{1}{[w^2 - 2w\Omega + \Omega^2]} \approx \frac{1}{\Omega^2} \left[ 1 + 2\frac{w}{\Omega} + \left( \frac{w}{\Omega} \right)^2 \right].$$  (29)

Then, we obtain

$$n(t \to \infty) = \frac{g_0}{\pi} \frac{1}{\Omega^2} \int_0^{\infty} dw \frac{w}{\exp \left[ \frac{\hbar w}{kT} \right] \pm 1} \left[ 1 + 2\frac{w}{\Omega} + \left( \frac{w}{\Omega} \right)^2 \right].$$ (30)

from which one can derive the following analytical expressions

$$n_B(t \to \infty) = \frac{g_0}{\pi} \left( \frac{kT}{\hbar \Omega} \right)^2 \left[ \zeta(2) + 4\frac{kT}{\hbar \Omega} \zeta(3) + 18 \left( \frac{kT}{\hbar \Omega} \right)^2 \zeta(4) \right]$$  (31)

and

$$n_F(t \to \infty) = \frac{g_0}{\pi} \left( \frac{kT}{\hbar \Omega} \right)^2 \left[ \frac{1}{2} \zeta(2) + \frac{3}{4} \frac{kT}{\hbar \Omega} \zeta(3) + \frac{63}{4} \left( \frac{kT}{\hbar \Omega} \right)^2 \zeta(4) \right],$$  (32)

for bosonic and fermionic systems, respectively. Here, $\zeta(n)$ is the Raman zeta-function. We see that the $n_{F,B}(t \to \infty)$ is nearly linear in $g_0$ in this regime. At $T \to 0,$

$$n_F(t \to \infty)/n_B(t \to \infty) \to \frac{1}{2}.$$  (33)

This limit will be explicitly studied in Sec. III.B.

**E. Friction and diffusion coefficients**

In order to calculate the friction and diffusion coefficients, we employ Eq. (48) and obtain the following equations:

$$\frac{d}{dt} \approx \frac{d\tilde{A}^+(t)/dt}{\tilde{A}^+(t)} \approx \frac{d\tilde{A}(t)/dt}{\tilde{A}(t)} \approx \frac{d\tilde{F}(t)}{\tilde{F}(t)},$$

and

$$\frac{d}{dt} \approx \frac{d[\tilde{A}^+(t)\tilde{A}(t)]/dt}{\tilde{A}^+(t)\tilde{A}(t)} \approx \frac{d[\tilde{A}(t)\tilde{A}^+(t)]/dt}{\tilde{A}(t)\tilde{A}^+(t)} \approx \frac{d[\tilde{F}(t)\tilde{F}^+(t)]/dt}{\tilde{F}(t)\tilde{F}^+(t)}.$$  (34)
As seen from the structure of these equations, the dynamics of collective subsystem is determined by the time-dependent friction

\[
\lambda(t) = \frac{1}{2} \frac{d \ln \tilde{A}(t) \tilde{A}(t)}{dt} = -\frac{1}{2} \frac{d[\tilde{A}^*(t) \tilde{A}(t)]}{dt} = -\frac{1}{2} \frac{d \tilde{A}^*(t)/dt}{\tilde{A}^*(t)} + \frac{d \tilde{A}(t)/dt}{\tilde{A}(t)}
\]  

(35)

and diffusion

\[
D_{a+a}(t) = -\frac{1}{2} \frac{d[\tilde{A}^*(t) \tilde{A}(t)]}{dt} \ll \tilde{F}^+(t) \tilde{F}(t) \gg + \frac{1}{2} \frac{d \ll \tilde{F}^+(t) \tilde{F}(t) \gg}{\tilde{A}^*(t) \tilde{A}(t)}
\]

\[
D_{a+a}(t) = \lambda(t) \ll \tilde{F}^+(t) \tilde{F}(t) \gg + \frac{1}{2} \frac{d \ll \tilde{F}^+(t) \tilde{F}(t) \gg}{\tilde{A}^*(t) \tilde{A}(t)}
\]

\[
D_{aa}(t) = \lambda(t) \ll \tilde{F}^+(t) \tilde{F}(t) \gg + \frac{1}{2} \frac{d \ll \tilde{F}^+(t) \tilde{F}(t) \gg}{\tilde{A}^*(t) \tilde{A}(t)}
\]

(36)

(37)

coefficients. Therefore, we have obtained the local in time Markovian-type equations for the first and second moments, but with a general form of transport coefficients which explicitly depend on time. The non-Markovian effects are taken into consideration through this time dependence. It can be shown that the appropriate equilibrium distribution is achieved in the course of time evolution. At \( t \to \infty \) the system reaches the equilibrium state \( \ll a^+ a \gg = \frac{d}{a} \ll a a^+ \gg = \frac{d}{a} \ll a^+ a^+ \gg = \frac{d}{a} \ll a a^+ \gg = 0 \) and the asymptotic diffusion coefficients can be derived from the above expressions:

\[
D_{a+a}(t \to \infty) = \lambda(t \to \infty) n(t \to \infty),
\]

\[
D_{aa^+}(t \to \infty) = \lambda(t \to \infty) [1 - n(t \to \infty)],
\]

(38)

(39)

where the asymptotic friction coefficient

\[
\lambda(t \to \infty) = -\frac{1}{2} [z_2 + z_2^*].
\]

(40)

It is interesting to mention that the time evolution of the friction coefficient and consequently its asymptotic limit do not depend on the fact that the bath is composed of fermions or bosons. However, the specific quantum natures of the bath enter into the diffusion coefficients through the appearance of occupation probabilities in Eqs. (38) and (39).

The asymptotic diffusion and friction coefficients are related by the well-known fluctuation-dissipation relations connecting diffusion and damping constants. Note that

\[
\left. \frac{d[\tilde{A}^*(t) \tilde{A}(t)]}{dt} \right|_{t \to \infty} = 2 \left. \frac{d \tilde{A}^*(t)/dt}{\tilde{A}^*(t)} \right|_{t \to \infty} = 2z_2,
\]

\[
\left. \frac{d[\tilde{A}(t) \tilde{A}(t)]}{dt} \right|_{t \to \infty} = 2 \left. \frac{d \tilde{A}(t)/dt}{\tilde{A}(t)} \right|_{t \to \infty} = 2z_2^*
\]

and at \( g_0 \ll 1 \)

\[
\lambda(t \to \infty) = g_0 \Omega.
\]

### III. CALCULATED RESULTS

The population, diffusion and friction coefficients depend on the parameters \( \omega, g_0, \) and \( \gamma \). The value of \( \gamma \) should be taken to hold the condition \( \gamma \gg \omega \) or \( \gamma \gg \Omega \). We set \( \gamma/\Omega = 12 \) while the renormalized frequency is taken \( \hbar \Omega = 1 \) MeV. To calculate the level population, friction and diffusion coefficients, we use formulas (19), (20), (40), and (38).
FIG. 2: (Color online) The calculated dependencies of the average occupation numbers \( n(t) \) on time \( t \) for the fermionic and bosonic systems, respectively, labeled by "(f)" and "(b)". The results for different coupling constants \( g_0 \) and temperatures \( T \) are indicated in the plots. Results (a) and (b) [(c) and (d)] correspond to an initially unoccupied, i.e. \( n(t=0)=0 \) [occupied, i.e. \( n(t=0)=1 \)] system state.

A. Non-equilibrium properties

For bosonic and fermionic systems, the time dependence of occupation numbers \( n(t) \) are shown in Fig. 2. Note that here a zero chemical potential is assumed. In Fig. 2 the collective subsystem is initially unoccupied with \( n(0)=0 \) (left hand side), or occupied with \( n(0)=1 \) (right hand side). If the state is initially unoccupied and at low temperature, we see that, before reaching an equilibrium, the non-equilibrium evolution is affected by the fermionic nature of the bath. More precise analyze shows that some oscillations of the occupation numbers exist in the bosonic case. In the fermionic case, these oscillations are smaller. When temperature increases the difference between the Fermi and Bose systems seems to be washed out by the thermal fluctuations. After some transient time, which depends on the coupling strength, the temperature, spectral function, and the level populations reach their equilibrium values. For the bosonic and fermionic systems, we observe that the transient times are almost the same and independent of the initial collective subsystem occupancy. In particular, we see in Fig. 2 that the transient time increases with decreasing coupling strength \( g_0 \). Thus, the transition occurs faster in the case of larger coupling strength.

B. Asymptotic properties

We see in Fig. 2 that after some relaxation time, the collective subsystem reaches an equilibrium. Before equilibrium, at low temperature \( [kT/(\hbar\Omega)]=0.1 \), the occupation numbers increase with increasing \( g_0 \) for both fermionic and bosonic systems. This means that at low temperature the friction strongly influences the dynamics of occupation numbers. At intermediate temperature \( [kT/(\hbar\Omega)]=1 \), the ratio between the asymptotic occupation numbers for bosonic and fermionic systems in this figure is about of factor 2 for different values of \( g_0 \). As discussed in Sec. II.D, it is anticipated that at high temperature and weak coupling the asymptotic equilibrium essentially reflects the fermionic or bosonic nature of the bath that is imposed to the collective subsystem. Deeper insight in the equilibrium properties can be obtained using Eq. (30) for the equilibrium occupation number. For two coupling strengths \( g_0=0.1 \) and \( g_0=0.001 \), the dependencies of \( n(t \rightarrow \infty) \) on temperature \( T \) are systematically investigated in Fig. 3. The Fermi-Dirac and Bose-Einstein distributions are also shown as the references. At high temperature, the occupation numbers for the fermionic and bosonic systems correspond to the Fermi-Dirac and Bose-Einstein populations when the temperature is above a certain threshold and the equilibrium properties of the system is imposed by the heat-bath. This is an indirect argument of the correctness of our method. At low temperatures, there are noticeable deviations from the
usual Fermi-Dirac and Bose-Einstein populations behavior. For the bosonic system, this effect was firstly found in Ref. [2].

In Fig. 3, the calculated ratio $n_F(t \to \infty)/n_B(t \to \infty)$ is compared to the completely thermalized limit as a function of temperature. At $[kT/(\hbar\Omega) = 1]$, both fermionic and bosonic systems are above the threshold mentioned and the factor $1/2$ can be simply explained using the approximate formula:

$$\frac{n_F(t \to \infty)}{n_B(t \to \infty)} \approx \tanh \left( \frac{\hbar\Omega}{2kT} \right)$$

leading to a value of $0.46$. We see that above of certain temperature, which depends on the coupling strength, the occupation numbers follow the statistics imposed by the bath for all coupling strengths considered. The low temperature regime is less straightforward to understand. In this case, the quantum fluctuations compete with the statistical fluctuations and a clear deviation from the thermal equilibrium prescription is observed. In particular, we see that the transition temperature between quantal and statistical effects strongly depends on the interaction strength. The low temperature regime is significantly affected by the nature of the bath, leading to a factor of $1/2$ between the $n_F(t \to \infty)$ and $n_B(t \to \infty)$ at $T \to 0$. This aspect has been analytically studied with Eqs. (31) and (32). The result of these equations are shown in Fig. 3 and perfectly match the numerical result in the low temperature regime (below $kT/\hbar\Omega = 0.1$). The influence of quantum statistics is further illustrated in the ratio shown in Fig. 4.
As seen, this will have a direct influence the diffusion process. Note that because the calculations are not sensitive to the de-excitation constant $\gamma$, there is no influence of the spectral function.

FIG. 5: (Color online) The calculated dependence of the average occupation number $n(t)$ on time $t$ for the fermionic system. The results for different chemical potentials $\mu$ and initial average occupation number $n(0)$ are indicated.

C. Non-zero chemical potential and Pauli principle effect

At given temperature, it is expected that a change of the chemical potential significantly affects the asymptotic behavior and the overall dynamics in the fermionic case. When the chemical potential becomes comparable to or bigger than the state energy $\hbar \Omega$ and the collective subsystem is already occupied at initial time, we anticipate that it will not decay towards the bath due to the fact that the bath states are already occupied at the relevant energy. This situation is depicted in Fig. 1. Examples of time-evolution with different initial chemical potentials for the bath are shown in Fig. 5 for the collective subsystem that is either fully occupied or fully empty. Since we are above of the threshold, we observe that the asymptotic behavior is independent of the initial conditions for the collective subsystem. The Pauli principle effect is perfectly seen in the case of $\mu/(\hbar \Omega) = 5$ and a fully occupied collective subsystem at initial time. The collective subsystem occupation remains almost constant and equal to its initial value.

D. Transport properties: fluctuation and dissipation

The asymptotic diffusion coefficients as the functions of $g_0$ are shown in Fig. 6. As already mentioned, the friction is the same for the Fermi or Bose systems. Therefore, we do not expect that the dissipative aspects strongly depend on the nature of the particles. An indirect proof of this was the fact that the time-scale before reaching an equilibrium was weakly dependent on the statistical properties. Both friction and diffusion exhibit a monotonic increase with increasing coupling strength.

At low temperature, the difference between the asymptotic diffusion coefficients for fermionic and bosonic systems is small but increases slightly with increasing $g_0$ [upper part of Fig. 6]. At high temperature, the deviation between fermionic and bosonic systems is larger and almost the same for all coupling constants [lower part of Fig. 6]. The difference between two cases considered here directly stems from the bounded values of the occupation numbers in the Fermi systems. Because $0 \leq n(t=\infty) \leq 1$, from Eqs. (38) and (39), we see that the diffusion coefficients will verify

$$D_{a^+a}(t \to \infty) \leq \lambda(t \to \infty), \quad D_{aa^+}(t \to \infty) \leq \lambda(t \to \infty).$$

In bosonic systems, similar equations can be derived except that $D_{aa^+}(t \to \infty) = \lambda(t \to \infty)[1 + n(t \to \infty)]$, and there is no restriction on the value of $n(t \to \infty)$. This quenching of the diffusion coefficient reflects the difference between the Fermi and Bose heat baths. At any temperature, the reduction factor can be directly estimated as

$$\frac{D_{a^+a}^{\text{Fermi}}(t \to \infty)}{D_{a^+a}^{\text{Bose}}(t \to \infty)} = \frac{n_F(t \to \infty)}{n_B(t \to \infty)}.$$

(43)
that is seen in Fig. 4. As shown in this figure, the fluctuations are reduced in the fermionic case compared to the bosonic case both at high and low temperature. This leads to a rather subtle effect of the nature of the bath. At high temperature, the quenching directly reflects the thermal fluctuations that are smaller in the Bose systems compared to the Fermi systems. At lower temperature, the main effect is a difference in the quantal zero-point motions.

IV. SUMMARY

The non-Markovian quantum Langevin equations and fluctuation-dissipation relations are derived for the fermionic system (the collective fermionic oscillator plus fermionic heat-bath). The explicit expressions for the time-dependent level population, friction and diffusion coefficients are derived for the case of linear coupling between the collective and internal subsystems. The results of numerical calculations of diffusion and friction coefficients and level population are also shown and compared to the bosonic case. At high temperature, regardless of the strength of coupling between the bath and collective subsystem, the collective subsystem relaxes towards occupation numbers imposed by the bath nature (bosonic or fermionic). At sufficiently low temperatures, the effect of the coupling is macroscopically observable as the deviation of level population from the Fermi-Dirac population. In this case the quantum fluctuations plays a significant role. In particular, a quenching of the asymptotic population compared to bosons systems is anticipated. This affects the diffusion process while leaving the friction unchanged. With the recent progress in manipulating Fermi gas, it might be interesting to explore this effect experimentally.

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Appendix A

To find the solution of Eq. (8), we apply the Laplace transform:

\[ A^+(z) = a^+(0) \frac{1 - iK(z)}{z - i\Omega - ziK(z)} + F^+(z) \frac{i}{z - i\Omega - ziK(z)}, \]  

\[ (44) \]
where $A^+(z) = L\{a^+(t)\}$ and $F^+(z) = L\{F^+(t)\}$.

Substituting the expression (14) for the kernel into Eq. (14) and employing the roots

$$z_1 = \frac{1}{2} \left(-\gamma + i\Omega + i\gamma g_0 - \sqrt{4i\gamma \Omega + (\gamma - i\Omega - i\gamma g_0)^2}\right)$$

and

$$z_2 = \frac{1}{2} \left(-\gamma + i\Omega + i\gamma g_0 + \sqrt{4i\gamma \Omega + (\gamma - i\Omega - i\gamma g_0)^2}\right)$$

of the equation

$$(z + \gamma)(z - i\Omega) - ig_0\gamma z = 0,$$  \tag{45}

one can rewrite the equation for the creation operator in the following form

$$A^+(z) = a^+(0) \frac{z + \gamma - ig_0\gamma}{(z + \gamma)(z - i\Omega) - ig_0\gamma z} + F^+(z) \frac{i(z + \gamma)}{(z + \gamma)(z - i\Omega) - ig_0\gamma z},$$ \tag{46}

$$= a^+(0) \frac{z + \gamma - ig_0\gamma}{(z - z_1)(z - z_2)} + F^+(z) \frac{i(z + \gamma)}{(z - z_1)(z - z_2)}.$$  \tag{47}

The exact solution $a^+(t)$ in terms of roots $z_i$ can be given by the residue theorem. Then, the explicit solution for the original is

$$a^+(t) = a^+(0) \hat{A}^+(t) + i\hat{F}^+(t),$$ \tag{48}

where

$$\hat{A}^+(t) = L^{-1} \left\{ \frac{z + \gamma - ig_0\gamma}{(z - z_1)(z - z_2)} \right\}$$ \tag{49}

$$= e^{tz_1} \frac{z_1 + \gamma - ig_0\gamma}{z_1 - z_2} + e^{tz_2} \frac{z_2 + \gamma - ig_0\gamma}{z_2 - z_1}$$

and

$$\hat{F}^+(t) = L^{-1} \left\{ F^+(z) \frac{i(z + \gamma)}{(z - z_1)(z - z_2)} \right\}$$ \tag{50}

$$= \sum_{\nu} \frac{g_{\nu}}{\hbar} a^+_{\nu}(0) \left[ e^{iz_1} \frac{z_1 + \gamma}{(z_1 - z_2)(z_1 - i\omega_{\nu})} + e^{iz_2} \frac{z_2 - \gamma}{(z_2 - z_1)(z_2 - i\omega_{\nu})} + e^{i\omega_{\nu}} \frac{i\gamma - \omega_{\nu}}{(z_2 - i\omega_{\nu})(iz_1 + \omega_{\nu})} \right].$$

Here, $L^{-1}$ is the inverse Laplace transform.

Appendix B

The constants in the Eq. (23) are the following:

$$f_1 = \frac{w^2 + \gamma^2}{(w^2 - 2g_0\gamma + (1 + g_0^2)\gamma^2)(w^2 - 2w\Omega + (1 + g_0^2)\Omega^2)};$$

$$f_2 = \frac{g_0^2\gamma^2}{(w^2 - 2g_0\gamma + (1 + g_0^2)\gamma^2)((1 + g_0^2)\gamma^2 - 4g_0\gamma\Omega + (1 + g_0^2)\Omega^2)};$$

$$f_3 = \frac{\gamma^2 - 2g_0\gamma\Omega + (1 + g_0^2)\Omega^2}{(w^2 - 2w\Omega + (1 + g_0^2)\Omega^2)((1 + g_0^2)\gamma^2 - 4g_0\gamma\Omega + (1 + g_0^2)\Omega^2)};$$

$$f_4 = \frac{ig_0(w + i\gamma)}{(w^2 - 2g_0\gamma + (1 + g_0^2)\gamma^2)(\gamma - ig_0\gamma - (-i + g_0)\Omega)(w + i(i + g_0)\Omega)}. $$
\[ f_5 = \frac{(-iw + \gamma)(\gamma - (-i + g_0)\Omega)}{(w - (-i + g_0)\gamma)((i + g_0)\gamma + (-1 - ig_0)\Omega)(w^2 - 2w\Omega + (1 + g_0^2)\Omega^2)}, \]

\[ f_6 = \frac{g_0\gamma(-1 + g_0)\Omega)}{(iw + \gamma - ig_0\gamma)(w + i(i + g_0)\Omega)((1 + g_0^2)\gamma^2 - 4g_0\gamma\Omega + (1 + g_0^2)\Omega^2)}. \]

One can show that at \( t = 0 \)

\[ f_1 + f_2 + f_3 + f_4 + f_4^* + f_5 + f_5^* + f_6 + f_6^* = 0. \]

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