Liouville type theorems for stable solutions of elliptic system involving the Grushin operator.

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Abstract
We examine the degenerate elliptic system

\[-\Delta_s u = v^p, \quad -\Delta_s v = u^\theta, \quad u, v > 0 \text{ in } \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}, \quad \text{where } s \geq 0 \text{ and } p, \theta > 0.\]

We prove that the system has no smooth stable solution provided \(p, \theta > 0\) and \(N_s < 2 + \alpha + \beta\), where

\[\alpha = \frac{2(p + 1)}{p\theta - 1} \quad \text{and} \quad \beta = \frac{2(\theta + 1)}{p\theta - 1}.\]

This result is an extension of some result in [15]. In particular, we establish a new integral estimate for \(u\) and \(v\) (see Proposition 1.1), which is crucial to deal with the case \(0 < p < 1\).

Keywords: Stable solutions, Liouville-type theorem, Grushin operator, Critical exponents, Elliptic system.

1. Introduction
We start by noting that throughout this article, \(N_s := N_1 + (1 + s)N_2\) is called the homogeneous dimension associated to the Grushin operator:

\[\Delta_s = \Delta_x + |x|^{2s} \Delta_y,\]

where \(s \geq 0\), and

\[\Delta_x := \sum_{i=1}^{N_1} \frac{\partial^2}{\partial x_i^2}, \quad \text{and} \quad \Delta_y := \sum_{j=1}^{N_2} \frac{\partial^2}{\partial y_j^2},\]

are Laplace operators with respect to \(x \in \mathbb{R}^{N_1}, \ y \in \mathbb{R}^{N_2}\) and \(|x|^{2s} = \left(\sum_{i=1}^{N_1} x_i^2\right)^s\).

In this paper, we are interested in the classification of stable solutions to the following degenerate elliptic system

\[-\Delta_s u = v^p, \quad -\Delta_s v = u^\theta, \quad u, v > 0 \text{ in } \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}, \quad \text{where } p, \theta > 0. \tag{1.1}\]

In the case \(s = 0\), the system (1.1) becomes

\[-\Delta u = v^p, \quad -\Delta v = u^\theta, \quad u, v > 0 \text{ in } \mathbb{R}^N, \quad \text{where } p, \theta > 0. \tag{1.2}\]

It is well studied and there are many interesting results on classifying the solutions of this system for various \(p\) and \(\theta\).
In the last decade, increased attention has focused on the system \(1.1\). We note that the Sobolev critical hyperbola
\[
\frac{1}{p+1} + \frac{1}{\theta+1} = \frac{N-2}{N},
\]
which is introduced independently by Mitidieri \[13\] and Van der Vorst \[27\], plays a crucial role in the analysis of \(1.1\). Among the main interests, the Lane–Emden conjecture says that there has no positive classical solution if and only if the pair \((p,\theta)\) lies below the Sobolev critical hyperbola, i.e.,

**Conjecture.** Let \(p,\theta > 0\). If the pair \((p,\theta)\) is subcritical i.e., if \((p,\theta)\) satisfies
\[
\frac{1}{p+1} + \frac{1}{\theta+1} > \frac{N-2}{N}, \tag{1.3}
\]
Then system \(1.2\) has no positive classical solutions.

The full Lane-Emden conjecture is still open. Only partial results for nonexistence are known, and many researchers have made contribution in pushing the progress forward. We shall briefly present some important recent developments of this conjecture.

The case of radial solutions was solved by Mitidieri \[12\] and Serrin-Zou \[20\] constructed positive radial solutions on and above the critical hyperbola, i.e
\[
\frac{1}{p+1} + \frac{1}{\theta+1} \leq \frac{N-2}{N},
\]
which means that the nonexistence theorem is optimal for radial solutions. For non-radial solutions of the Lane-Emden system, certain Liouville theorems are known. Denote the scaling exponents of system \(1.2\) by
\[
\alpha = \frac{2(p+1)}{p\theta-1}, \quad \beta = \frac{2(\theta+1)}{p\theta-1}, \quad p\theta > 1. \tag{1.4}
\]
Then subcritical condition \(1.3\) is equivalent to
\[
N < 2 + \alpha + \beta, \quad \text{for} \quad p\theta > 1. \tag{1.5}
\]
There are no positive classical super-solutions to \(1.2\) if
\[
p\theta \leq 1 \quad \text{or} \quad p\theta > 1 \quad \text{and} \quad \max \{\alpha, \beta\} > N-2,
\]
see \[12, 23, 21\]. This implies the conjecture for \(N = 1, 2\). Also, in \[1\] the conjecture is true for
\[
\min \{\alpha, \beta\} > \frac{N-2}{2}, \quad \text{with} \quad (\alpha, \beta) \neq \left(\frac{N-2}{2}, \frac{N-2}{2}\right).
\]

In dimension \(N = 3\), Serrin-Zou \[21\] gave a proof for the nonexistence of polynomially bounded solutions, an assumption that was relaxed later by Poláčik, Quittner and Souplet \[25\]. More recently, in dimension \(N = 4\), the conjecture was settled completely by Souplet \[24\] who provided, in dimensions \(N \geq 5\), a more restrictive new region for the exponents \((p,\theta)\) that insures that if \((p,\theta)\) satisfies \(1.5\), along with \(\max \{\alpha, \beta\} > N-3\), then system \(1.2\) has no positive classical solutions.

On the other direction, the Liouville type theorem for the class of stable solutions for both system has attracted the attention of many mathematicians. We refer \[2, 10, 9, 13\] for Lane–Emden system. The nonexistence of positive stable classical solutions of \(1.2\) was examined in \[3\]. In particular, Cowan \[3\] proved that if \(N \leq 10\), \(1.2\) has no stable solution for any \(2 \leq p \leq \theta\). This result was then extended in \[10\] to the Lane–Emden system with weights. Among other things, we also mention that Hajlaoui et.al. \[9\] improved the previous works \[3, 10\] and mainly obtained a new comparison property which is key to
deal with the case $1 < p \leq \frac{4}{3}$.

A new approach which is independently obtained in [15], allows the authors to prove the following Liouville theorem for classical stable solutions of (1.2) for any $p, \theta > 0$, satisfying (1.5).

**Theorem A.** If $p, \theta > 0$ satisfies (1.3), then (1.2) has no smooth stable solution.

In the case $0 < p < 1$, the main tools are based on the relationship between the stability for the Lane-Emden system (1.2) and the stability for a fourth order problem, called the $m$-biharmonic equation

$$\Delta_m^2 u := \Delta(|\Delta u|^{m-2} \Delta u) = u^\theta \quad \text{in} \quad \mathbb{R}^N.$$  

The case $p > 1$ was handled by the results of [9]. The main motivation was to provide a proof of the Lane-Emden conjecture dealing with positive stable solutions.

Coming back to the Lane-Emden system (1.1) for the general case $s \geq 0$, the Liouville property is less understood and is more complicated to deal with than $s = 0$, because the operator $\Delta_s$ no longer has symmetry and it degenerates on the manifold $\{0\} \times \mathbb{R}^{N^2}$ which causes further mathematical difficulties.

In [6] the author extended some of Cowan results [3], in order to prove the nonexistence of positive stable solutions of (1.1) with $4/3 < p \leq \theta$ and $N_s \leq 10$, or $1 < p \leq \min\{\frac{4}{3}, \theta\}$ with additional assumption. The main idea used in [6] is a combination of stability inequality, comparison principle and bootstrap argument. After that, this idea was exploited by Mtiri [16], the author has obtained some Liouville theorems for stable solutions of (1.1), see Theorem B below. This results improves the bound given in [6].

Our main objective is to classify the stable solution of (1.1) for any $p, \theta > 0$, and the general case $s \geq 0$. In order to state our results more accurately, we define the notion of stability where we consider a general system given by

$$-\Delta_s u = f(x, v), \quad -\Delta_s v = g(x, u) \quad \text{in} \quad K, \quad \text{a bounded regular domain} \quad \subset \mathbb{R}^N,$$

where $f, g \in C^1(K \times \mathbb{R})$. Following Montenegro [17], a smooth solution $(u, v)$ of (1.6) is said to be stable in $K$ if the following eigenvalue problem

$$-\Delta \xi = f_v(x, v)\xi + \eta \xi, \quad -\Delta \zeta = g_u(x, u)\xi + \eta \zeta \quad \text{in} \quad K$$

has a nonnegative eigenvalue $\eta$, with a positive smooth eigenfunctions pair $(\xi, \zeta)$.

The main result in this paper is the following

**Theorem 1.1.** If $p, \theta > 0$ and $N_s$ satisfying $N_s < 2 + \alpha + \beta$, then (1.1) has no smooth stable solution.

To prove Theorem 1.1, we borrow crucially the idea in [15] who established Theorem A. Without loss of generality, we consider only $\theta > p > 0$ and $p \theta > 1$. As we will see soon, the $\theta > p \geq 1$ case can be handled by the results in [16], so our main concern is the case

$$\theta > \frac{1}{p} > 1 > p > 0.$$  

The main difficulty arises from the fact that there has no works in literature dealing with stable solutions (radial or not) for the $m$-biharmonic–type Grushin equation. Consequently, it is difficult to use the technique developed in [15]. We overcome this difficulty, we shall derive new the important integral estimate for $u$ and $v$ which is crucial to deal with the case $0 < p < 1$. The following Proposition transforms our notion of a stable solution of (1.1) into an inequality which allows the use of arbitrary test functions.

**Proposition 1.1.** Let $(u, v)$ denote a stable solution of (1.1) with $0 < p < 1$. Then

$$\theta p \int_{\mathbb{R}^N} u^{\theta-1} \gamma^2 \, dx \, dy \leq \int_{\mathbb{R}^N} v^{1-p|\Delta_s \gamma|^2} \, dx \, dy,$$

(1.7)

for all $\gamma \in C^\infty_c(\mathbb{R}^N)$.   

Remark 1.1. • If \( s = 0 \), we obtain a similar result in [13] see Theorem A.
• Our approach is more easier than those developed in [13], for \( p, \theta > 0 \). To the best of our knowledge, no general Liouville type result was known for stable solution of (1.1) with \( 0 < p < 1 \).
• We note also that the method used in the present paper can be applied to study the weighted systems, and to more general class of degenerate operator, such as the \( \Delta_s \) operator (see [13, 8]) of the form

\[
\Delta_s := \sum_{j=1}^{N} s_j^2 \Delta_{x(j)} \quad s := (s_1, \ldots, s_N) : \mathbb{R}^N \to \mathbb{R}^N,
\]

where \( s_i : \mathbb{R}^N \to \mathbb{R}, \quad i = 1, \ldots, N \), are nonnegative continuous functions satisfying some properties such that \( \Delta_s \) homogeneous of degree two with respect to a group dilation in \( \mathbb{R}^N \).

2. Liouville Type Theorem.

This section is devoted to the proof of Theorem 1.1. For convenience, we always denote by \( C \) a generic constant whose concrete values may change from line to line or even in the same line. If this constant depends on an arbitrary small number \( \epsilon \), then we may denote it by \( C_\epsilon \). We also use Young inequality in the form

\[
ab \leq c a^q + C_\epsilon b^{q'} \quad \text{for} \quad q, q' > 1 \quad \text{satisfying} \quad \frac{1}{q} + \frac{1}{q'} = 1
\]

As mentioned before, we need only to consider the case \( \theta > p \) and \( p \theta > 1 \). We split the proof into two cases:

\[
\theta > p \geq 1 \quad \text{and} \quad \theta > p^{-1} > 1 > p > 0.
\]

2.1. The case \( \theta > p^{-1} > 1 > p > 0 \).

Here we handle the case \( 0 < p < 1 \). We begin with proving the integral estimate for \( u \) and \( v \)

**proof of Proposition 1.1.** By the definition of stability, there exist smooth positive functions \( \xi, \zeta \) and \( \eta \geq 0 \) such that

\[
-\Delta_s \xi = pv^{p^{-1}} \zeta + \eta \xi, \quad -\Delta_s \zeta = \theta u^{\theta^{-1}} \xi + \eta \zeta \quad \text{in} \quad \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}.
\]

Using \((\xi, \zeta)\) as super-solution, \((\min_{\Omega} \xi, \min_{\Omega} \zeta)\) as sub-solution, and the standard monotone iterations, we can claim that there exist positive smooth functions \( \varphi, \chi \) verifying

\[
-\Delta_s \varphi = pv^{p^{-1}} \chi, \quad -\Delta_s \chi = \theta u^{\theta^{-1}} \varphi \quad \text{in} \quad \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}.
\]

Therefore, we have

\[
\theta u^{\theta^{-1}} \varphi = \Delta_s \left( \frac{1}{p} v^{1-p} \Delta_s \varphi \right) \quad \text{in} \quad \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}.
\]

Let \( \gamma \in C_c^2(\mathbb{R}^N) \). Multiplying the above equation by \( \gamma^2 \varphi^{-1} \) and integrating by parts, there holds

\[
\int_{\mathbb{R}^N} \theta u^{\theta^{-1}} \varphi \gamma^2 dx dy = \frac{1}{p} \int_{\mathbb{R}^N} v^{1-p} \Delta_s \varphi \Delta_s (\gamma^2 \varphi^{-1}) dx dy
\]

\[
= \frac{1}{p} \int_{\mathbb{R}^N} v^{1-p} \Delta_s \varphi \left( -4 \gamma \frac{\nabla \varphi \cdot \nabla s \gamma}{\varphi^2} + \frac{2 |\nabla s \gamma|^2}{\varphi} + \frac{2 \gamma \Delta_s \gamma}{\varphi^3} + \frac{2 \gamma^2 |\nabla \varphi|^2}{\varphi^3} - \frac{\gamma^2 \Delta_s \varphi}{\varphi^2} \right) dx dy. \tag{2.1}
\]
Using Cauchy-Schwarz’s inequality and the fact that \(-\Delta_s \varphi > 0\), we get

\[
-4 \int_{\mathbb{R}^N} \frac{v^{1-p}}{p} \Delta_s \varphi \nabla_s \varphi \cdot \nabla_s \gamma dxdy \\
\leq -2 \int_{\mathbb{R}^N} \frac{v^{1-p}}{p} \Delta_s \varphi \frac{\vert \nabla_s \gamma \vert^2}{\varphi} dxdy - 2 \int_{\mathbb{R}^N} \frac{v^{1-p}}{p} \Delta_s \varphi \frac{\varphi^2}{\varphi^3} dxdy. \tag{2.2}
\]

Combining (2.1) and (2.2), one obtains, using again the Cauchy-Schwarz inequality,

\[
\int_{\mathbb{R}^N} \theta u^{\theta-1} \gamma^2 dxdy \leq \frac{2}{p} \int_{\mathbb{R}^N} v^{1-p} \Delta_s \varphi \frac{\varphi^2}{\varphi^3} dxdy - \frac{1}{p} \int_{\mathbb{R}^N} v^{1-p} \frac{(\Delta_s \varphi)^2}{\varphi^2} \gamma^2 dxdy \\
\leq \frac{1}{p} \int_{\mathbb{R}^N} v^{1-p} \frac{(\Delta_s \varphi)^2}{\varphi^2} \gamma^2 dxdy + \frac{1}{p} \int_{\mathbb{R}^N} v^{1-p} (\Delta_s \gamma)^2 dxdy - \frac{1}{p} \int_{\mathbb{R}^N} v^{1-p} \frac{(\Delta_s \varphi)^2}{\varphi^2} \gamma^2 dxdy \\
= \frac{1}{p} \int_{\mathbb{R}^N} v^{1-p} (\Delta_s \gamma)^2 dxdy.
\]

The proof is completed. \(\square\)

As a consequence of Proposition 1.1, we derive immediately the following integral estimate for stable solution \((u, v)\) of system (1.1), in the case \(p \in (0, 1)\) and \(p \theta > 1\), which is a crucial tool in our approach.

**Lemma 2.1.** Let \((u, v)\) be a stable solution of (1.1), with \(p \in (0, 1)\). Then, for any integer

\[
k \geq \max \left(\frac{1 + p + \beta(\theta + 1)}{2p}, \frac{1}{4}, \frac{\alpha(\theta + 1)}{4}\right),
\]

there exists a positive constant \(C = C(N, \epsilon, p, k)\) such that for any \(\zeta \in C^2(\mathbb{R}^N)\) satisfying \(0 \leq \zeta \leq 1\),

\[
\int_{\mathbb{R}^N} v^{p+1} \zeta^{4k} dxdy + \int_{\mathbb{R}^N} u^{\theta+1} \zeta^{4k} dxdy \leq C \left[ \int_{\mathbb{R}^N} \left( \left(\Delta_s \zeta \right)^\frac{p+1}{p} + \left| \nabla_s \zeta \right|^2 \frac{\varphi^2}{\varphi^3} + \left| \nabla_s \zeta \right| \frac{\varphi^2}{\varphi^3} \right)^\frac{p}{2} dxdy \right]. \tag{2.3}
\]

Here \(\alpha\) and \(\beta\) are defined in (1.4).

**Proof.** First, for any \(\epsilon \in (0, 1)\) and \(\eta \in C^2(\mathbb{R}^N)\), there holds

\[
\int_{\mathbb{R}^N} v^{1-p} (\Delta_s (\epsilon \eta))^2 dxdy = \int_{\mathbb{R}^N} v^{1-p} (u \Delta_s \epsilon \eta + 2 \nabla_s u \nabla_s \eta + \eta \Delta_s u)^2 dxdy \\
\leq (1 + C \epsilon) \int_{\mathbb{R}^N} v^{p+1} \eta^2 dxdy + \frac{C}{\epsilon} \int_{\mathbb{R}^N} v^{1-p} u^2 |\Delta_s \eta|^2 dxdy + C \int_{\mathbb{R}^N} v^{1-p} |\nabla_s u|^2 |\nabla_s \eta|^2 dxdy. \tag{2.4}
\]

Take \(\eta = \zeta^{2k}\) with \(k \geq \frac{1}{p} + 1 > 2\) and \(0 \leq \zeta \leq 1\). Apply Young’s inequality, we get

\[
\int_{\mathbb{R}^N} v^{1-p} u^2 |\Delta_s (\zeta^{2k})|^2 dxdy \leq C_k \int_{\mathbb{R}^N} v^{1-p} u^2 \left( |\Delta_s \zeta|^2 + |\nabla_s \zeta|^4 \right) \zeta^{4k(1-p)} + \frac{2(4k-2(1+p))}{1+p} dxdy \\
\leq C_{p, k} \int_{\mathbb{R}^N} v^{p+1} \zeta^{4k} dxdy + C_{\epsilon, k, p} \int_{\mathbb{R}^N} u^{\frac{4+p}{2}} \left( |\Delta_s \zeta|^2 + |\nabla_s \zeta|^4 \right) \frac{\varphi^2}{\varphi^3} \zeta^{4k-2(\frac{p}{2}+1)} dxdy,
\]

and

\[
\int_{\mathbb{R}^N} v^{1-p} |\nabla_s u|^2 |\nabla_s (\zeta^{2k})|^2 dxdy = 4k^2 \int_{\mathbb{R}^N} v^{1-p} |\nabla_s u|^2 |\nabla_s \zeta|^2 \zeta^{4k(1-p)} + \frac{2(4k-2(1+p))}{1+p} dxdy \\
\leq C_{p, k} \epsilon^2 \int_{\mathbb{R}^N} v^{p+1} \zeta^{4k} dxdy + C_{p, k} \frac{C}{\epsilon^2} \int_{\mathbb{R}^N} |\nabla_s u|^\frac{p+1}{p} |\nabla_s \zeta|^\frac{p+1}{p} \zeta^{4k-\frac{p+1}{p}} dxdy.
\]

Combining all these inequalities with (2.4), we get the following estimates.
\[
\int_{\mathbb{R}^N} v^{1-p} |\Delta_s(u\zeta^{2k})|^2\,dx\,dy \\
\leq (1 + C_{p,k}) \int_{\mathbb{R}^N} v^{p+1} \zeta^{4k}\,dx\,dy + \frac{C_{p,k}}{\epsilon^2} \int_{\mathbb{R}^N} |\nabla_s u|^{\frac{p+1}{p}} |\nabla_s \zeta|^{\frac{p+1}{p}} \zeta^{4k-\frac{p+1}{p}}\,dx\,dy \\
+ C_{\epsilon,k,p} \int_{\mathbb{R}^N} u^{\frac{4k}{p+1}} (|\Delta_s \zeta|^2 + |\nabla_s \zeta|^4)^{\frac{p+1}{2p}} \zeta^{2k-2(\frac{p+1}{2})}\,dx\,dy. 
\]

(2.5)

**Proof.** Since the simple calculation implies that

\[
\int_{\mathbb{R}^N} |\nabla_s u|^{\frac{p+1}{p}} |\nabla_s \zeta|^{\frac{p+1}{p}} \zeta^{4k-\frac{p+1}{p}}\,dx\,dy \\
\leq 4 \int_{\mathbb{R}^N} v^{p+1} \zeta^{4k}\,dx\,dy + C \int_{\mathbb{R}^N} \frac{4k}{p+1} \left( |\nabla_s \zeta|^{\frac{2(p+1)}{p}} + |\nabla_s \zeta|^{\frac{p+1}{p}} \right) \zeta^{4k-\frac{2(p+1)}{p}}\,dx\,dy.
\]

(2.6)

We will use also the following lemma

**Lemma 2.2.** Let \( k \geq \frac{4-p}{2p} > 1 \), there exists a positive constant \( C = C_{N,c,p,k} > 0 \) such that for any \( \epsilon > 0 \), and \( \zeta \in C_c^\infty(\mathbb{R}^N) \) with \( 0 \leq \zeta \leq 1 \), there holds

\[
\int_{\mathbb{R}^N} |\nabla_s u|^{\frac{p+1}{p}} |\nabla_s \zeta|^{\frac{p+1}{p}} \zeta^{4k-\frac{p+1}{p}}\,dx\,dy \\
\leq C_{N,c,p,k} \int_{\mathbb{R}^N} u^{\frac{p+1}{p}} (|\nabla_s \zeta|^{\frac{2(p+1)}{p}} + |\nabla_s \zeta|^{\frac{p+1}{p}} \zeta^{4k-\frac{2(p+1)}{p}})\,dx\,dy.
\]

(2.7)
and

\[ C_p \int_{\mathbb{R}^N} u|\nabla^2 u||\nabla u|^{\frac{p+1}{p}} |\nabla s \zeta|^{\frac{p+1}{p}} \zeta^{2-\frac{4k-2}{p}} \frac{dxdy}{|\nabla^2 u|^{\frac{p+1}{p}} |\nabla u|^{\frac{p+1}{p}} |\nabla s \zeta|^{\frac{p+1}{p}} \zeta^{4k-\frac{p}{p}} dxdy} \]

\[ \leq C_{e,p} \int_{\mathbb{R}^N} u\frac{|\nabla^2 u|^{\frac{p+1}{p}} |\nabla u|^{\frac{p+1}{p}} |\nabla s \zeta|^{\frac{p+1}{p}} \zeta^{2-\frac{4k+2}{p}} dxdy \quad + C_{e,p} \int_{\mathbb{R}^N} |\nabla u|^{\frac{p+1}{p}} |\nabla s \zeta|^{\frac{p+1}{p}} \zeta^{4k-\frac{p+1}{p}} dxdy \]  

\[ \leq C_{e,p} \int_{\mathbb{R}^N} u\frac{|\nabla^2 u|^{\frac{p+1}{p}} |\nabla u|^{\frac{p+1}{p}} |\nabla s \zeta|^{\frac{p+1}{p}} \zeta^{4k-\frac{p+1}{p}} dxdy + C_{e,p} \int_{\mathbb{R}^N} |\nabla^2 u|^{\frac{p+1}{p}} |\nabla u|^{\frac{p+1}{p}} |\nabla s \zeta|^{\frac{p+1}{p}} \zeta^{4k-\frac{p+1}{p}} dxdy. \]  

(2.8)

Now we shall estimate the integral

\[ \int_{\mathbb{R}^N} |\nabla^2 u|^{\frac{p+1}{p}} \zeta^{4k} dxdy. \]

For the last inequality, we have used the following estimates from [18, Proof of Theorem 2.6]

\[ \|\nabla x_{i,j} u\|_{L^2(\mathbb{R}^{N_1})} \leq C(N_1)\|\Delta u\|_{L^2(\mathbb{R}^{N_1})}, \quad \text{for} \quad 1 \leq i, j \leq N_1, \]

and

\[ \|\nabla y_{h,k} u\|_{L^2(\mathbb{R}^{N_2})} \leq C(N_2)\|\Delta u\|_{L^2(\mathbb{R}^{N_2})}, \quad \text{for} \quad 1 \leq h, k \leq N_2. \]

Let \( \psi^r \in C_0^\infty(\mathbb{R}^N) \), with \( r > 2 \). By direct calculations, we get

\[ |\nabla^2 u| \leq C_r \left[ u \left( |\nabla\psi|^2 \psi^{-2} + |\nabla^2 \psi| \psi^{-1} \right) + |\nabla u||\nabla\psi| \psi^{-1} + |\nabla^2 (u\psi)| \right]. \]

Consider \( \psi = \zeta \), and \( r = \frac{4kp}{p+1} \geq 2 \), so that \( k \geq \frac{p+1}{2p} \). We can claim, for any \( 0 \leq \zeta \leq 1 \), there exists \( C_{p,k} > 0 \) such that

\[ \int_{\mathbb{R}^N} |\nabla^2 u|^{\frac{p+1}{p}} \zeta^{4k} dxdy \]

\[ \leq C_{p,k} \int_{\mathbb{R}^N} |\nabla^2 u|^{\frac{p+1}{p}} |\nabla u|^{\frac{p+1}{p}} |\nabla s \zeta|^{\frac{p+1}{p}} \zeta^{4k-\frac{p+1}{p}} dxdy + C_{p,k} \int_{\mathbb{R}^N} |\nabla u|^{\frac{p+1}{p}} |\nabla s \zeta|^{\frac{p+1}{p}} \zeta^{4k-\frac{p+1}{p}} dxdy \]

(2.9)

\[ + C_{p,k} \int_{\mathbb{R}^N} u^{\frac{p+1}{p}} \left( |\nabla^2 u|^{\frac{p+1}{p}} |\nabla s \zeta|^{\frac{p+1}{p}} \zeta^{4k-\frac{p+1}{p}} dxdy. \]

Clearly \( u\zeta \in H_0^2(\mathbb{R}^N) \), so from a standard approximation and scaling argument imply then

\[ \int_{\mathbb{R}^N} |\nabla^2 (u\zeta^{\frac{4kp}{p+1}})|^{\frac{p+1}{p}} dxdy \leq C_{N,p} \int_{\mathbb{R}^N} |\Delta s (u\zeta^{\frac{4kp}{p+1}})|^{\frac{p+1}{p}} dxdy \]

\[ \leq C_{N,p,k} \int_{\mathbb{R}^N} |\nabla u|^{\frac{p+1}{p}} |\nabla s \zeta|^{\frac{p+1}{p}} \zeta^{4k-\frac{p+1}{p}} dxdy + C_{N,p} \int_{\mathbb{R}^N} u^{p+1} \zeta^{4k} dxdy \]

(2.10)

Combining (2.7)–(2.10), we obtain the estimate for the first left term in (2.6):

\[ Q_1 \leq C_{N,p,k} \int_{\mathbb{R}^N} u^{p+1} \zeta^{4k} dxdy + C_{N,p,k} \epsilon \int_{\mathbb{R}^N} |\nabla u|^{\frac{p+1}{p}} |\nabla s \zeta|^{\frac{p+1}{p}} \zeta^{4k-\frac{p+1}{p}} dxdy \]

\[ + C_{N,p,k} \int_{\mathbb{R}^N} u^{\frac{p+1}{p}} \left( |\nabla s \zeta|^{\frac{p+1}{p}} |\nabla s \zeta|^{\frac{p+1}{p}} \zeta^{4k-\frac{p+1}{p}} dxdy. \]
Furthermore, by Young’s inequality,

\[ Q_2 = -\frac{p+1}{p} \int_{\mathbb{R}^N} u \left( |\nabla_s u| |\nabla_s \zeta| \right)^{\frac{p+1}{p}} - \nabla_s^2 \zeta \left( \nabla_s \zeta, \nabla_s u \right) \zeta^{4k_{\frac{p}{p+1}}} \, dx \, dy \\
- (4k - \frac{p+1}{p}) \int_{\mathbb{R}^N} u |\nabla_s u|^{\frac{p+1}{p}} \nabla_s \zeta \cdot (\nabla_s u - \nabla_s \zeta) \zeta^{4k_{\frac{p}{p+1}}} - \frac{1}{p} \, dx \, dy \\
\leq C_{p,k} \int_{\mathbb{R}^N} u \left( |\nabla_s u| |\nabla_s \zeta| \right)^{\frac{p+1}{p}} \left( |\nabla_s \zeta|^2 + |\nabla_s^2 \zeta| \right) \zeta^{4k_{\frac{p}{p+1}} - \frac{p+1}{p}} \, dx \, dy \\
\leq C_{p,k} \int_{\mathbb{R}^N} u \left( |\nabla_s \zeta|^{2(p+1)} + |\nabla_s^2 \zeta| \right) \zeta^{4k_{\frac{p}{p+1}} - \frac{2(p+1)}{p}} \, dx \, dy.
\]

Combining the last two inequalities with \((2.12)\), we get readily

\[ (1 - C_{N,p,k}) \int_{\mathbb{R}^N} |\nabla_s u| \left| \nabla_s \zeta \right| \zeta^{4k_{\frac{p}{p+1}}} \, dx \, dy \]
\[ \leq C_{p,N,k} \int_{\mathbb{R}^N} u = \frac{p+1}{p} \left( |\nabla_s \zeta|^{\frac{p+1}{p}} + |\nabla_s^2 \zeta| \right) \zeta^{4k_{\frac{p}{p+1}} - \frac{2(p+1)}{p}} \, dx \, dy \\
+ C_{N,p,k} \epsilon \int_{\mathbb{R}^N} v^{p+1} \zeta^{4k} \, dx \, dy.
\]

Take \(\epsilon\) small enough, the lemma follows. \(\Box\)

Now, using Lemma \((2.2)\) and \((2.5)\), we obtain also

\[ \int_{\mathbb{R}^N} v^{1-p} |\Delta_s (u\zeta^{2k})|^2 \, dx \, dy \leq C \int_{\mathbb{R}^N} u^{\frac{p+1}{p}} \left( |\Delta_s \zeta|^{\frac{p+1}{p}} + |\nabla_s \zeta|^{\frac{2(p+1)}{p}} + |\nabla_s^2 \zeta| \right) \zeta^{4k_{\frac{p}{p+1}} - \frac{2(p+1)}{p}} \, dx \, dy \\
+ (1 + C) \int_{\mathbb{R}^N} v^{p+1} \zeta^{4k} \, dx \, dy. \quad (2.11)
\]

Thanks to the approximation argument, the stability property \((1.7)\), with \(\gamma = u\zeta^{2k}\) where \(\zeta \in C_0^\infty(\mathbb{R}^N)\), we get

\[ \theta_p \int_{\mathbb{R}^N} u^{\theta+1} \zeta^{4k} \, dx \, dy - (1 + C) \int_{\mathbb{R}^N} v^{p+1} \zeta^{4k} \, dx \, dy \]
\[ \leq C \int_{\mathbb{R}^N} u^{\frac{p+1}{p}} \left( \Delta_s \zeta \right)^{\frac{p+1}{p}} \zeta^{4k_{\frac{p}{p+1}} - \frac{2(p+1)}{p}} \, dx \, dy. \quad (2.12)
\]

Multiplying the equation \(-\Delta_s v = u\zeta^{4k}\) by \(u\zeta^{4k}\) and integrating by parts, there holds

\[ \int_{\mathbb{R}^N} v^{p+1} \zeta^{4k} \, dx \, dy - \int_{\mathbb{R}^N} u^{p+1} \zeta^{4k} \, dx \, dy \leq -\int_{\mathbb{R}^N} \Delta_s (u\zeta^{4k}) \, dx \, dy + \int_{\mathbb{R}^N} v |\nabla_s u| |\nabla_s (u\zeta^{4k})| \, dx \, dy.
\]

Using Young’s inequality and applying again Lemma \((2.2)\) we can conclude that for any \(\epsilon > 0\), there exists \(C = C_{N,p,k} > 0\) such that

\[ (1 - C) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} v^{p+1} \zeta^{4k} \, dx \, dy - \int_{\mathbb{R}^N} u^{\theta+1} \zeta^{4k} \, dx \, dy \]
\[ \leq C \int_{\mathbb{R}^N} u^{\frac{p+1}{p}} \left( \Delta_s \zeta \right)^{\frac{p+1}{p}} \zeta^{4k_{\frac{p}{p+1}} - \frac{2(p+1)}{p}} \, dx \, dy. \quad (2.13)
\]
Now, multiplying (2.13) by $\frac{1 + 2C\epsilon}{1 - C\epsilon}$, adding it with (2.12), we get
\[
C\epsilon \int_{\mathbb{R}^N} v^{p+1}\zeta^k dx dy + \left(p\theta - \frac{1 + 2C\epsilon}{1 - C\epsilon}\right) \int_{\mathbb{R}^N} u^{\theta+1}\zeta^k dx dy \\
\leq C \int_{\mathbb{R}^N} u^{\frac{p+1}{p}} \left(\Delta_s \zeta^{\frac{p+1}{p}} + |\nabla_s \zeta|^{\frac{2(p+1)}{p}} + |\nabla^2_s \zeta|^{\frac{p+1}{p}}\right) \zeta^{4k - \frac{2(p+1)}{p}} dx dy.
\]
As $\theta > p^{-1} > 1$. Fix $0 < \epsilon < \frac{p\theta - 1}{C(p\theta + 2)}$, there holds
\[
\int_{\mathbb{R}^N} v^{p+1}\zeta^k dx dy + \int_{\mathbb{R}^N} u^{\theta+1}\zeta^k dx dy \\
\leq C \int_{\mathbb{R}^N} u^{\frac{p+1}{p}} \left(\Delta_s \zeta^{\frac{p+1}{p}} + |\nabla_s \zeta|^{\frac{2(p+1)}{p}} + |\nabla^2_s \zeta|^{\frac{p+1}{p}}\right) \zeta^{4k - \frac{2(p+1)}{p}} dx dy.
\]
Apply Young’s inequality, we deduce then
\[
\int_{\mathbb{R}^N} v^{p+1}\zeta^k dx dy + \int_{\mathbb{R}^N} u^{\theta+1}\zeta^k dx dy \\
\leq C\epsilon \int_{\mathbb{R}^N} \left(|\Delta_s \zeta|^{\frac{p+1}{p}} + |\nabla_s \zeta|^{\frac{2(p+1)}{p}} + |\nabla^2_s \zeta|^{\frac{p+1}{p}}\right)^{\frac{p\beta}{p\beta + 1}} dx dy + \epsilon' \int_{\mathbb{R}^N} |u|^{\theta+1}\zeta^{4k(p\theta - 1) - 2(\theta + 1)} dx dy \\
\leq C\epsilon \int_{\mathbb{R}^N} \left(|\Delta_s \zeta|^{\frac{p+1}{p}} + |\nabla_s \zeta|^{\frac{2(p+1)}{p}} + |\nabla^2_s \zeta|^{\frac{p+1}{p}}\right)^{\frac{p\beta}{p\beta + 1}} dx dy + \epsilon' \int_{\mathbb{R}^N} |u|^{\theta+1}\zeta^{4k} dx dy.
\]
Choose $k \geq \frac{\beta(p+1)}{4} = \frac{\alpha(\theta + 1)}{4}$ so that $4k \leq \frac{4kp\theta + 1}{p+1} - 2(\theta + 1)$, for the last line. Take $\epsilon'$ small enough, the estimate (2.20) is proved. \(\square\)

We are now in position to conclude. Choose $\chi$ a cut-off function in $C_c^\infty(\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}, [0, 1])$, such that
\[
\chi = 1 \text{ on } B_1 \times B_2, \quad \text{and} \quad \chi = 0 \text{ outside } B_2 \times B_2^{1+\epsilon}.
\]
For $R > 0$, put $\eta_R(x, y) = \chi(\frac{x}{R}, \frac{y}{R^{1+\epsilon}})$, it is easy to verify that there exists $C > 0$ independent of $R$ such that
\[
|\nabla_x \eta_R| \leq \frac{C}{R} \quad \text{and} \quad |\nabla_y \eta_R| \leq \frac{C}{R^{1+\epsilon}},
\]
\[
|\nabla^2_x \eta_R| + |\Delta_x \eta_R| \leq \frac{C}{R^2} \quad \text{and} \quad |\nabla^2_y \eta_R| + |\Delta_y \eta_R| \leq \frac{C}{R^{2(1+\epsilon)}}.
\]
Applying (2.20) with $\zeta = \eta_R(x, y)$, we get
\[
\int_{B_R \times B_R^{1+\epsilon}} u^{\theta+1} dx dy \leq \int_{\mathbb{R}^N} u^{\theta+1}\eta_R^{4k} dx dy \leq CR^{2N_2 - \frac{2(p+1)}{p}}.
\]
Since under our assumptions $N_2 < \frac{2(p+1)(\theta + 1)}{p\theta - 1} := 2 + \alpha + \beta$, the desired claim follows by letting $R \to \infty$. \(\square\)

2.2. The case $\theta > p \geq 1$.

Let us recall a consequence of Theorem 1.1 (with $\alpha = 0$ there) in [10].

**Theorem B.** Let $x_0$ be the largest root of the polynomial
\[
H(x) = x^4 - p\theta\alpha\beta \left[4x^2 - 2(\alpha + \beta)x + 1\right].
\]

1. If $\frac{4}{3} < p \leq \theta$ then (1.1) has no stable solution if $N_2 < 2 + 2x_0$. 

2. If $1 < p \leq \min\left(\frac{4}{3}, \theta\right)$, then (1.1) has no bounded stable solution, if

$$N_s < 2 + 2x_0 \left[\frac{p}{2} + \frac{(2 - p)(p\theta - 1)}{(\theta + p - 2)(\theta + 1)}\right].$$

For $\theta > p \geq 1$, we can proceed similarly as the proof of Proposition in [13], we sketch a proof here for the reader’s convenience. Performing the change of variables $x = \frac{\beta}{2} z$ in (2.1), a direct computation shows that $H(x) = \left(\frac{\beta}{2}\right)^4 L(z)$ where

$$L(z) := z^4 - \frac{16p\theta(p + 1)}{\theta + 1} z^2 + \frac{16p\theta(p + 1)(p + \theta + 2)}{(\theta + 1)^2} - \frac{16p\theta(p + 1)^2}{(\theta + 1)^2}.$$

Denote by $z_0$ the largest root of $L$, hence $x_0 = \frac{\beta}{2} z_0$ and $H(x) < 0$ if and only if $L(z) < 0$. For $\theta > p \geq 1$, there holds

$$L(p + 1) = (p + 1)^4 - \frac{16p\theta(p + 1)^3}{\theta + 1} + \frac{16p\theta(p + 1)^2(p + \theta + 2)}{(\theta + 1)^2} - \frac{16p\theta(p + 1)^2}{(\theta + 1)^2}$$

$$= (p + 1)^4 - \frac{16p\theta(p + 1)^3}{\theta + 1} + \frac{16p\theta(p + 1)^2}{(\theta + 1)^2} + \frac{16p\theta(p + 1)^2}{(\theta + 1)^2} - \frac{16p\theta(p + 1)^2}{(\theta + 1)^2}$$

$$= (p + 1)^2 \left[(p + 1)^2 - \frac{16p^2\theta}{(\theta + 1)} + \frac{16p^2\theta}{(\theta + 1)^2}\right]$$

$$= \left(\frac{p + 1}{\theta + 1}\right)^2 [(p + 1)^2(\theta + 1)^2 - 16p^2\theta^2] < 0.$$
for all $\theta > p \geq 1$, and $N_s < 2 + \alpha + \beta$, (1.1) has no smooth stable solution.

The proof is finished. □

References

[1] J. Busca and R. Manasevich, A Liouville-type theorem for Lane-Emden systems, Indiana University Mathematics Journal 51 (2002), no. 1, 37–52.

[2] C. Cowan, Liouville theorems for stable lane emden systems and biharmonic problems. *Nonlinearity*, **26** (2013), 2357–2371.

[3] C. Cowan, Liouville theorems for stable lane emden systems and biharmonic problems. *Nonlinearity*, **26** (2013), 2357–2371.

[4] C. Cowan and N. Ghoussoub, Regularity of semi-stable solutions to fourth order nonlinear eigenvalue problems on general domains, *Calc. Var. PDE*. **49** (2014), 291-305.

[5] AT. Duong and QH. Phan, Liouville type theorem for nonlinear elliptic system involving Grushin operator. J Math Anal Appl. **454**, 2 (2017),785–801.

[6] Duong AT, Phan QH. Liouville type theorem for nonlinear elliptic system involving Grushin operator. J Math Anal Appl. 2017, 454 2:785–801.

[7] L. Dupaigne, A. Farina and B. Sirakov, Regularity of the extremal solutions for the Liouville system, in: Geometric Partial Differential Equations, in: Publications of the Scuola Normale Superiore/CRM Series, vol. **15** (2013), pp. 139-144.

[8] Franchi B, Lanconelli E. Une métrique associée à une classe d’opérateurs elliptiques dégénérés. Rend. Sem. Mat. Univ. Politec. Torino. 1983:105–114, conference on linear partial and pseudodifferential operators (Torino, 1982).

[9] H. Hajlaoui, A. Harrabi and F. Mtiri, Liouville theorems for stable solutions of the weighted Lane-Emden system, *Discrete Contin. Dyn. Syst.*, **37** (2017), 265-279.

[10] L. Hu, Liouville type results for semi-stable solutions of the weighted Lane-Emden system, *J. Math. Anal. Appl*. **432** (2015), 429-440.

[11] Kogoj AE, Lanconelli E. On semilinear $\Delta_{\lambda}$-Laplace equation. Nonlinear Anal. 2012;75(12): 4637–4649.

[12] E. Mitidieri; Nonexistence of positive solutions of semilinear elliptic systems in RN, Differential Integral Equations 9 (1996) 465-479.

[13] Enzo Mitidieri, A Rellich type identity and applications, Comm. Partial Differential Equations 18 (1993), no. 1-2, 125–151, DOI 10.1080/03605309308820923. MR1211727 (94c:26016).

[14] E. Mitidieri and S. Pohozaev, A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities, *Tr. Mat. Inst. Steklova* **234** (2001), 1-384.

[15] F. Mtiri and D. Ye, *Liouville theorems for stable at infinity solutions of Lane-Emden system*, Nonlinearity **32** (2019), 910-926.

[16] Foued Mtiri. On the classification of solutions to a weighted elliptic system involving the Grushin operator. arXiv:2007.03009v1 [math.AP] 6 Jul 2020

[17] M. Montenegro, Minimal solutions for a class of elliptic systems. *Bull. London Math. Soc.*, **37** (2005), 405–416.
[18] L. Negro and G. Metafune, C.Spina $L^p$ estimates for Baouendi-Grushin operators
arXiv:1907.10439v1 [math.AP] 24 Jul 2019

[19] P. Poláčik, P. Quittner and P. Souplet, Singularity and decay estimates in superlinear problems via Liouville-type theorems. I. Elliptic equations and systems, Duke Math. J., 139 (2007), 555–579.

[20] J. Serrin, H. Zou; Existence of positive solutions of the Lane-Emden system, Atti Semin. Mat. Fis. Univ.Modena 46 (1998) 369-380.

[21] J. Serrin and H. Zou, Non-existence of positive solutions of Lane-Emden systems, Diff. Int. Eq. 9 (1996), 635-653

[22] J. Serrin and H. Zou, Non-existence of positive solutions of Lane-Emden systems, Diff. Inte. Equations 9 (1996), 635-653.

[23] M.A.S. Souto, A priori estimates and existence of positive solutions of non-linear cooperative elliptic systems, Diff. Int. Eq. 8 (1995) 1245-1258.

[24] Philippe Souplet, The proof of the Lane–Emden conjecture in four space dimensions. Advances in Mathematics 221 (2009) 1409–1427.

[25] P. Poláčik, P. Quittner and P. Souplet, Singularity and decay estimates in superlinear problems via Liouville-type theorems. I. Elliptic equations and systems, Duke Math. J., 139 (2007), 555–579.

[26] X. Yu, Liouville type theorem for nonlinear elliptic equation involving Grushin operators, Commun. Contemp. Math. 17(5) (2015) 1450050.

[27] R. C. A. M. Van der Vorst, Variational identities and applications to differential systems, Arch. Rational Mech. Anal. 116 (1992), no. 4, 375–398, DOI 10.1007/BF00375674. MR1132768 (93d:35043).