WELL POSEDNESS FOR THE MOTION OF A COMPRESSIBLE LIQUID WITH FREE SURFACE BOUNDARY.

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ABSTRACT. We study the motion of a compressible perfect liquid body in vacuum. This can be thought of as a model for the motion of the ocean or a star. The free surface moves with the velocity of the liquid and the pressure vanishes on the free surface. This leads to a free boundary problem for Euler’s equations, where the regularity of the boundary enters to highest order. We prove local existence in Sobolev spaces assuming a "physical condition", related to the fact that the pressure of a fluid has to be positive.

1. Introduction

We consider Euler’s equations
describing the motion of a perfect compressible fluid body in vacuum:

\[(1.2) \quad (\partial_t + V^k \partial_k) \rho + \rho \text{div} V = 0, \quad \text{div} V = \partial_k V^k \quad \text{in} \quad D,\]

where \(V^k = \delta^{ki} v_i = v_k\) and we use the summation convention over repeated upper and lower indices. Here the velocity \(V = (V^1, ..., V^n)\), the density \(\rho\) and the domain \(D = \cup_{0 \leq t \leq T} \{t\} \times D_t, D_t \subset \mathbb{R}^n\) are to be determined. The pressure \(p = p(\rho)\) is assumed to be a given strictly increasing smooth function of the density. The boundary \(\partial D_t\) moves with the velocity of the fluid particles at the boundary. The fluid body moves in vacuum so the pressure vanishes in the exterior and hence on the boundary. We therefore also require the boundary conditions on \(\partial D = \cup_{0 \leq t \leq T} \{t\} \times \partial D_t:\)

\[(1.3) \quad (\partial_t + V^k \partial_k)|_{\partial D} \in T(\partial D),\]
\[(1.4) \quad p = 0, \quad \text{on} \quad \partial D.\]

Constant pressure on the boundary leads to energy conservation and it is needed for the linearized equations to be well posed. Since the pressure is assumed to be a strictly increasing function of the density we can alternatively think of the density as a function of the pressure and for physical reasons this function has to be non negative. Therefore the density has to be a non negative constant \(\rho_0\) on the boundary and we will in fact assume that \(\rho_0 > 0\), which is the case of liquid. We hence assume that

\[(1.5) \quad p(\rho_0) = 0 \quad \text{and} \quad p'(\rho) > 0, \quad \text{for} \quad \rho \geq \rho_0, \quad \text{where} \quad \rho_0 > 0\]

From a physical point of view one can alternatively think of the pressure as a small positive constant on the boundary. By thinking of the density as function of the pressure the incompressible case can be thought of as the special case of constant density function.

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The motion of the surface of the ocean is described by the above model. Free boundary problems for compressible fluids are also of fundamental importance in astrophysics since they describe stars. The model also describes the case of one fluid surrounded by and moving inside another fluid. For large massive bodies like stars gravity helps holding it together and for smaller bodies like water drops surface tension helps holding it together. Here we neglect the influence of gravity which will just contribute with a lower order term and we neglect surface tension which has a regularizing effect.

Given a bounded domain $\mathcal{D}_0 \subset \mathbb{R}^n$, that is homeomorphic to the unit ball, and initial data $V_0$ and $\rho_0$, we want to find a set $\mathcal{D} \subset [0, T] \times \mathbb{R}^n$, a vector field $V$ and a function $\rho$, solving (1.1)-(1.4) and satisfying the initial conditions

\begin{align}
\{ x; (0, x) \in \mathcal{D} \} &= \mathcal{D}_0, \\
V &= V_0, \quad \rho = \rho_0 \quad \text{on} \quad \{0\} \times \mathcal{D}_0.
\end{align}

In order for the initial-boundary value problem (1.1)-(1.7) to be solvable initial data (1.7) has to satisfy certain compatibility conditions at the boundary. By (1.2), (1.4) also implies that $\text{div} V|_{\partial \mathcal{D}} = 0$. We must therefore have $\rho_0|_{\partial \mathcal{D}_0} = \overline{\rho}_0$ and $\text{div} V_0|_{\partial \mathcal{D}_0} = 0$. Furthermore, taking the divergence of (1.1) gives an equation for $(\partial_t + V^k \partial_k) \text{div} V$ in terms of only space derivatives of $V$ and $\rho$, which leads to further compatibility conditions. In general we say that initial data satisfy the compatibility condition of order $m$ if there is a formal power series solution in $t$, of (1.1)-(1.7) $(\tilde{\rho}, \tilde{V})$, satisfying

\begin{align}
(\partial_t + \tilde{V}^k \partial_k)^j (\tilde{\rho} - \overline{\rho}_0) |_{\{0\} \times \partial \mathcal{D}_0} = 0, \quad j = 0, \ldots, m - 1.
\end{align}

Let $N$ be the exterior unit normal to the free surface $\partial \mathcal{D}_t$. Christodoulou\cite{C2} conjectured the initial value problem (1.1)-(1.8), is well posed in Sobolev spaces under the assumption

\begin{align}
\nabla_N p \leq -c_0 < 0, \quad \text{on} \quad \partial \mathcal{D}, \quad \text{where} \quad \nabla_N = N^i \partial_{x^i}.
\end{align}

Condition (1.9) is a natural physical condition. It says that the pressure and hence the density is larger in the interior than at the boundary. Since we have assumed that the pressure vanishes or is close to zero at the boundary this is therefore related to the fact that the pressure of a fluid has to be positive.

In general it is possible to prove local existence for analytic data for the free interface between two fluids. However, this type of problem might be subject to instability in Sobolev norms, in particular Rayleigh-Taylor instability, which occurs when a heavier fluid is on top of a lighter fluid. Condition (1.9) prevents Rayleigh-Taylor instability from occurring. Indeed, if this condition is violated Rayleigh-Taylor instability occurs in a linearized analysis.

In the irrotational incompressible case the physical condition (1.9) always hold, see \cite{W1,2,CL}, and \cite{W1,2} proved local existence in Sobolev spaces in that case. \cite{W1,2} studied the classical water wave problem describing the motion of the surface of the ocean and showed that the water wave is not unstable when it turns over. Ebin\cite{E1} showed that the general incompressible problem is ill posed in Sobolev spaces when the pressure is negative in the interior and the physical condition is not satisfied. Ebin\cite{E2} also announced a local existence result for the incompressible problem with surface tension on the boundary which has a regularizing effect so (1.9) is not needed then.

In \cite{CL}, together with Christodoulou, we proved a priori bounds in Sobolev spaces in the general incompressible case of non vanishing curl, assuming the physical condition (1.9) for the pressure. We also showed that the Sobolev norms remain bounded as long as the physical condition hold and the second fundamental form of the free surface and the first order derivatives of the velocity are bounded.
Usually, existence follows from similar bounds for some iteration scheme, but the bounds in [CL] used all the symmetries of the equation and so only hold for modifications that preserve all the symmetries. In [L1] we showed existence for the linearized equations and in [L3] we proved local existence for the nonlinear incompressible problem with non vanishing curl, assuming that (1.9) holds initially.

For the corresponding compressible free boundary problem with non-vanishing density on the boundary, there are however in general no previous existence or well-posedness results. Relativistic versions of these problems have been studied in [C1,DN,F,FN,R] but solved only in special cases. The methods used for the irrotational incompressible case use that the components of the velocity are harmonic to reduce the equations to equations on the boundary and this does not work in the compressible case since the divergence is non vanishing and the pressure satisfies a wave equation in the interior. To be able to deal with the compressible case one therefore needs to use interior estimates as in [CL,L1]. Let us also point out that in nature one expects fluids to be compressible, e.g. water satisfies (1.5), see [CF]. For the general relativistic equations there is no special case corresponding to the incompressible case.

In [L2] we showed existence for the linearized equations in the compressible case and here we prove local existence for the nonlinear compressible problem:

**Theorem 1.1.** Suppose that \( p = p(\rho) \) is a smooth function satisfying (1.5). Suppose also that initial data \( v_0, \rho_0 \) and \( D_0 \) are smooth satisfying the compatibility conditions (1.8) to all orders \( m \) and \( D_0 \) is diffeomorphic to the unit ball. Then if the physical condition (1.9) hold at \( t = 0 \) there is a \( T > 0 \) such that (1.1)-(1.4) and (1.6)-(1.7) has a smooth solution for \( 0 \leq t \leq T \). Furthermore (1.9) hold for \( 0 \leq t \leq T \), with \( c_0 \) replaced by \( c_0/2 \).

A few remarks are in order. The existence of smooth solutions implies existence of solutions in Sobolev spaces if one has a priori bounds in Sobolev spaces. In the incompressible case we had already proven a priori energy bounds in Sobolev spaces in [CL] as well as a continuation result, that the solution remains smooth as long as the physical condition is satisfies and the solution is in \( C^2 \). Similar bounds in Sobolev spaces hold also in the compressible case. We also remark that there are initial data satisfying the compatibility conditions to all orders, see section 16. If only finitely many compatibility conditions are satisfied then we get existence in \( C^k \) for some \( k \). What then is essential is that the physical condition hold and this and the existence time only depends on a bound of finitely many derivatives of initial data. This makes it possible to construct a sequence of smooth solutions converging to a solution in Sobolev norms since we have a uniform lower bound for the existence times, in terms of the Sobolev norm.

A few remarks about the proof are also in order. As in the incompressible case [L3] we will use the Nash-Moser technique to prove local existence. However, because of the presence of the boundary problem for the wave equation for the enthalpy one has to take as many time derivatives of the equations as space derivatives. Therefore one has to use interpolation in time as well and one might just as well do smoothing also in time in the application of the Nash-Moser technique although there is no loss of regularity in the time direction. Because of this all our constants will depend on a lower bound of the time interval. This will make it a bit more delicate since we will also need to choose a small time, in order that the physical and coordinate conditions should hold. However, at the same time certain estimates are more natural when one includes time derivatives up to the highest order.

The plan of the paper is as follows. We will assume that the reader is somewhat familiar with the notation in [L3]. We will also assume the existence proofs given in [L1,L2,L3] for the inverse of the linearized operator so we will only prove improved estimates here. In section 2 we formulate Euler’s equations in the Lagrangian coordinates and derive the linearized equations in these coordinates. Here we also define a modified linearized operator which is easier to first consider. In section 3 we define the orthogonal projection onto divergence free vector fields, the normal operator and decompose the
equation onto a divergence free part and a wave equation for the divergence. In section 4 we construct
the families of tangential vector fields, define the modified Lie derivatives with respect to these and
calculate its commutators with the normal operator and other operators that occur in the linearized
equation. In section 5 we derive estimates of derivatives of a vector field in terms of the curl the
divergence and tangential derivatives or the normal operator. In section 6 we give tame estimates for
the Dirichlet problem, in section 7 we give tame estimates for the wave equation and in section 8 we give
tame estimates for the divergence free part. Then in section 9 we put these estimates together to get
tame estimates for the inverse of the modified linearized operator. In section 10 we give estimates of the
enthalpy in terms of the coordinate. In section 11 we show that the physical and coordinate conditions
can be satisfied for small times if they hold initially. In section 12 we then get tame estimates for
the inverse of the linearized operator. In section 13 we give tame estimates for the second variational
derivative. In section 14 we construct the smoothing operators needed for the Nash-Moser iteration.
Finally, in section 15 we construct the Nash-Moser iteration that proves local existence of a smooth
solution. In section 16 we show that one can construct a large class of initial data that satisfy the
compatibility conditions to all orders. Most of the steps of the proof above works for the pressure any
smooth strictly increasing functions of the density. However, the estimates for the enthalpy in terms of
the coordinate simplified if one assume that the pressure is a linear function of the density and we first
do the proof in this case and then in section 17 give the additional estimates needed for the general
case.

2. Lagrangian coordinates and the linearized equation.

Let us introduce Lagrangian coordinates in which the boundary becomes fixed. Let \( \Omega \) be a the
unit ball in \( \mathbb{R}^n \) and let \( f_0 : \Omega \to D_0 \) be a diffeomorphism. By a theorem in [DM] the volume form
\( \kappa_0 = \det (\partial f_0/\partial y) \) can be arbitrarily prescribed up to a multiplicative constant and by a scaling of
the equations we can also assume that the volume of \( D_0 \) is that of the unit ball. Assume that \( v(t,x),
p(t,x), (t,x) \in D \) are given satisfying the boundary conditions (1.3)-(1.4). The Lagrangian coordinates
\( x = x(t,y) = f_t(y) \) are given by solving

\[
\frac{dx}{dt} = V(t,x(t,y)), \quad x(0,y) = f_0(y), \quad y \in \Omega
\]

Then \( f_t : \Omega \to D_t \) is a diffeomorphism, and the boundary becomes fixed in the new \( y \) coordinates. Let
us introduce the notation

\[
D_t = \left. \frac{\partial}{\partial t} \right|_{y=constant} = \left. \frac{\partial}{\partial t} \right|_{x=constant} + V^k \frac{\partial}{\partial x^k},
\]

for the material derivative. The partial derivatives \( \partial_t = \partial/\partial x^i \) can then be expressed in terms of
partial derivatives \( \partial_a = \partial/\partial y^a \) in the Lagrangian coordinates. We will use letters \( a,b,c,...,f \) to denote
partial differentiation in the Lagrangian coordinates and \( i,j,k,... \) to denote partial differentiation in
the Eulerian frame.

In these coordinates Euler’s equation (1.1) become

\[
\rho D_t^2 x_i + \partial_t p = 0, \quad (t,y) \in [0,T] \times \Omega
\]

and the continuity equation (1.2) become

\[
D_t \rho + \rho \text{div} V = 0, \quad (t,y) \in [0,T] \times \Omega
\]
Here the pressure $p = p(\rho)$ is assumed to be smooth strictly increasing function of the density $\rho$. With $h$, the enthalpy, i.e. $h'(\rho) = p'(\rho)/\rho$ and $h = 0$ when $p = 0$, (2.3) becomes

$$D_t^2 x_i + \partial_i h = 0, \quad (t, y) \in [0, T] \times \Omega$$

Since $h$ is a strictly increasing function of $\rho$ we can solve for $\rho = \rho(h)$ as a function of $h$ and with $e(h) = \ln \rho(h)$ (2.4) become

$$D_t e(h) + \text{div} V = 0$$

Euler’s equations are now replaced by

$$D_t^2 x_i + \partial_i h = 0, \quad (t, y) \in [0, T] \times \Omega, \quad \text{where} \quad \partial_i = \frac{\partial y^a}{\partial x^i} \frac{\partial}{\partial y^a}$$

and taking the divergence of (2.7) using that $[D_t, \partial_i] = -\partial_i V^k \partial_k$ we obtain a wave equation for the enthalpy

$$D_t^2 e(h) - \Delta h - (\partial_i V^k) \partial_k V^i = 0, \quad h \bigg|_{\partial \Omega} = 0$$

Here $e(h)$ is a given smooth strictly increasing function and

$$\Delta h = \sum_i \partial_i^2 h = \kappa^{-1} \partial_a (\kappa g^{ab} \partial_b h) \quad \text{where} \quad g_{ab} = \delta_{ij} \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b}$$

and $g^{ab}$ is the inverse of the metric $g_{ab}$ and $\kappa = \det (\partial x/\partial y) = \sqrt{\det g}$. The initial conditions are

$$x_i \big|_{t=0} = f_0, \quad D_t x_i \big|_{t=0} = v_0,$$

$$h \big|_{t=0} = e^{-1}(\ln \rho_0), \quad D_t h \big|_{t=0} = -\text{div} V_0 / e'(e^{-1}(\ln \rho_0))$$

where $e^{-1}$ is the inverse function of $e(h)$. If $(x, h)$ satisfies (2.7)-(2.8) with initial data of the form (2.10)-(2.11) then $(x, h)$ also satisfies (2.6). By a theorem in [DM] the volume form can be arbitrarily prescribed so we can in fact choose it so $\kappa_0 = \det (\partial f_0/\partial y) = 1/\rho_0$, in which case $e(h) = -\ln \kappa$, since this is true when $t = 0$ and since $D_t \ln \kappa = \text{div} V$. Hence we are left we two independent initial data, $f_0$ and $v_0$.

In order for (2.8) to be solvable we must have the following condition on $e(h)$ and coordinate condition:

$$c_1^{-1} \leq e'(h) \leq c_1, \quad \sum_{a,b} |g^{ab}| + |g_{ab}| \leq c_1^2, \quad |\partial x/\partial y|^2 + |\partial y/\partial x|^2 \leq c_1^2$$

for some constant $0 < c_1 < \infty$. In order for (2.7) to be solvable we must have the physical condition:

$$\nabla N \cdot h \leq -c_0 < 0, \quad \text{on} \quad \partial \Omega, \quad \text{where} \quad \nabla N = N^i \partial x_i.$$
(2.14) is true initially. By continuity, the conditions on the inverse of the metric and on the enthalpy are then true also for small times, with $c_1$ replaced by $2c_1$ and $c_0$ replaced by $c_0/2$. In the iteration we construct we have to make sure that the iterates are small enough and the time is small enough that these conditions remain small for all the iterates. This will be discussed in section 11. In order to solve the wave equation on a bounded domain one also needs compatibility conditions on initial data. These conditions are so that the initial conditions are compatible with the boundary condition $h|_{\partial \Omega} = 0$. These conditions are:

\begin{equation}
D^k_t h|_{\partial \Omega} = 0, \quad \text{when } t = 0, \quad \text{for } k = 0, ..., m - 1
\end{equation}

For $k = 0, 1$ this is simply conditions on initial data and for $k \geq 2$ one can use (2.8) and (2.7) to express it in terms of lower time derivatives of $h$, so it can be calculated in terms of the initial conditions. (2.15) is called the $m^{th}$ order compatibility condition. We will assume that our initial data are smooth and satisfies the $m^{th}$ order condition for all $m$. This will be used to construct an approximate solution that satisfy the equation to all orders as $t \to 0$, and in fact the initial conditions will be turned into an inhomogeneous term that vanishes to all orders as $t \to 0$. This will also be discussed in following sections. We prove the following theorem:

\textbf{Theorem 2.1.} Suppose that initial data $(v_0, f_0, \rho_0)$ in (2.10)-(2.11) are smooth and the compatibility conditions (2.15) hold for all orders $m$. Suppose also that (2.13) and (2.14) hold when $t = 0$. Then there is $T > 0$ such that (2.7)-(2.8) have a smooth solution $x \in C^\infty([0, T] \times \overline{\Omega})$, if $e(h)$ is a smooth strictly increasing function with $e(0) = 0$. Furthermore (2.13) and (2.14) hold for $0 \leq t \leq T$ with $c_0$ replaced by $c_0/2$ and $c_1$ replaced by $2c_1$.

We remark that if $x \in C^\infty([0, T] \times \overline{\Omega})$ and the conditions in the theorem hold then (2.8) has a solution $h \in C^\infty([0, T] \times \overline{\Omega})$. Theorem 1.1 follows from Theorem 2.1 since initially we assumed that $D_0$ is diffeomorphic to the unit ball, so (2.13) holds initially. We will first prove Theorem 2.1 when $e(h) = ch$ is a linear functions of $h$. This corresponds to $p = p(\rho) = c_0(\rho - \rho_0)$, where $c, \rho_0 > 0$ are positive constants. The result is true in general when the pressure is any strictly increasing smooth function of the density. Only the estimates for the enthalpy in terms of the coordinates have to be modified and we show how to do this in section 17. The reason we first pick a linear function is that in this case we get a linear wave equation for the enthalpy and that simplifies the estimates and makes it more similar to the incompressible case where we have a linear elliptic equation for the pressure.

Let us now, define the Euler map, that will be used to find the solution of Euler’s equations. A solution of Euler’s equations is given by $\Theta(x, h) = 0$, where $\Theta = (\Theta_0, ..., \Theta_n)$ is given by

\begin{equation}
\Theta_i(x, h) = D^2_t x_i + \partial_i h, \quad i = 1, ..., n,
\end{equation}

and

\begin{equation}
\Theta_0(x, h) = D^2_t e(h) - \Delta h - (\partial_i V^k)\partial_k V^i, \quad h|_{\partial \Omega} = 0.
\end{equation}

We have assumed that our initial conditions satisfy compatibility to all orders, i.e. there are smooth functions $(x_0, h_0)$ satisfying the initial conditions (2.10)-(2.11) and

\begin{equation}
D^k_t \Theta(x_0, h_0)|_{t=0} = 0, \quad \text{for all } k \geq 0, \quad h_0|_{\partial \Omega} = 0.
\end{equation}

In the process of solving $\Theta(x, h) = 0$ we will only consider functions $(x, h)$ which has the same time derivatives as $(x_0, h_0)$ when $t = 0$ and which satisfy $h|_{\partial \Omega} = 0$. Let us therefore introduce the notation

\begin{equation}
C^\infty_{00}([0, T] \times \overline{\Omega}) = \{ u \in C^\infty([0, T] \times \overline{\Omega}); D^k_t u|_{t=0} = 0, \text{ for all } k \geq 0 \}
\end{equation}

and for short, let $C_{00}^\infty = C_{00}^\infty ([0, T] \times \bar{\Omega})$ and $C^\infty = C^\infty ([0, T] \times \bar{\Omega})$. (2.18) then says that $\Theta(x_0, h_0) \in C_{00}^\infty$ and if $x - x_0 \in C_{00}^\infty$ then it follows that also $\Theta_0(x, h_0) \in C_{00}^\infty$. Then by [H1] we can solve the equation $\Theta_0(x, h) = 0$, with initial conditions (2.10)-(2.11) and boundary conditions $h|_{\partial \Omega} = 0$. The result in [H1] is formulated for vanishing initial conditions and instead an inhomogeneous term that vanishes to all orders as $t \rightarrow 0$. However, we can turn the problem into this by considering $\Theta_0(h) = \Theta_0(x, h + h_0) - \Theta_0(x, h_0) = -\Theta_0(x, h_0)$, with vanishing initial data for $\tilde{h}$. This gives a solution in $h \in C_{00}^\infty$ satisfying the boundary condition $\tilde{h}|_{\partial \Omega} = 0$.

Let $(x_0, h_0)$ be the formal solution given above. For $x$ satisfying $x - x_0 \in C_{00}^\infty$, we now the define the Euler map $\Phi = (\Phi_1, \ldots, \Phi_n)$ to be

$$(2.20) \quad \Phi_i(x) = D_t^2 x_i + \partial_i h, \quad i = 1, \ldots, n$$

where $h = \Psi(x)$ is given implicitly by solving

$$(2.21) \quad D_t^2 c(h) - \Delta h - (\partial_i V^j)(\partial_j V^i) = 0, \quad h|_{\partial \Omega} = 0$$

with initial conditions (2.11). A solution of Euler’s equations is given by

$$(2.22) \quad \Phi(x) = 0$$

By the preceding argument we can, in fact, find a solution $h$ to (2.21) such that $h - h_0 \in C_{00}^\infty$ if $x - x_0 \in C_{00}^\infty$. The reason we choose to consider the map $\Phi(x)$ instead of $\Theta(x, h)$ is that we must make sure that $h|_{\partial \Omega} = 0$ and that the physical condition is satisfied, since the linearized operator is not invertible otherwise. Alternatively, one could also have tried to only consider $h$ satisfying these conditions, but it seems much more difficult to preserve these conditions in the smoothing process used in the Nash-Moser iteration. The main work will now be to prove tame estimates for the inverse of the linearized operator.

In the Nash-Moser iteration we will in fact solve for

$$(2.23) \quad \Phi(u) = \Phi(u + x_0) - \Phi(x_0)$$

Let $F = \Phi(x_0)$, when $t \geq 0$ and $F = 0$ when $t < 0$, and for $\delta \geq 0$ let $F_\delta(t, y) = F(t - \delta, y)$. Then $F_\delta \in C_{00}^\infty$. We will solve for

$$(2.24) \quad \Phi(u) = F_\delta - F_0$$

The Nash-Moser theorem says that if the linearized operator is invertible and we have tame estimates for its inverse and for the second variation of the operator then in fact we have a solution of (2.24) if the right hand side is small in $C_{00}^\infty$. But the right hand side of (2.24) tends to zero in $C^\infty$ when $\delta \rightarrow 0$ so (2.24) has a solution for some $\delta > 0$ and hence

$$(2.25) \quad \Phi(u + x_0) = 0, \quad 0 \leq t \leq \delta$$

As pointed out above at each step of the iteration we will only have functions $u$ that vanish to all orders as $t \rightarrow 0$. This condition can in fact be preserved by smoothing operators.

In order to solve $\Phi(x) = 0$ we must show that the linearized operator is invertible. Let us therefore calculate the linearized equations. If $\delta$ is a variation in the Lagrangian coordinates, i.e. derivative w.r.t. a parameter when $t$ and $y$ are fixed. Since $[\delta, \partial_i \partial y^a] = 0$ it follows that

$$(2.26) \quad [\delta, \partial_i] = \left(\delta \frac{\partial y^a}{\partial x^i}\right) \frac{\partial}{\partial y^a} = - (\partial_i \delta x^j) \partial_j,$$

where we used the formula for the derivative of the inverse of a matrix $\delta A^{-1} = -A^{-1}(\delta A)A^{-1}$. It follows that $[\delta - \delta x^k \partial_k, \partial_i] = 0$ and by (2.20)

$$(2.27) \quad D_t^2 \delta x_i - (\partial_k D^2 x_i) \delta x^k - \partial_i (\delta x^k \partial_k h - \delta h) = \Phi'(x) \delta x_i - \delta x^k \partial_k \Phi_i$$

It follows from this and (2.21) that we have:
Lemma 2.2. Let \( \mathcal{F} = \mathcal{F}(r,t,y) \) be a smooth function of \((r,t,y) \in K = [-\varepsilon, \varepsilon] \times [0,T] \times \overline{\Omega}, \varepsilon > 0, \) such that \( \mathcal{F}_{|r=0} = x. \) Then \( \Phi(\mathcal{F}) \) is a smooth function of \((r,t,y) \in K, \) such that \( \partial \Phi(\mathcal{F})/\partial r|_{r=0} = \Phi'(x)\delta x, \) where \( \delta x = \partial \mathcal{F}/\partial r|_{r=0} \) and the linear map \( L_0 = \Phi'(x) \) is given by

\[
(2.28) \quad \Phi'(x) \delta x_i = D_i^2 \delta x + (\partial_i \partial_k h) \delta x^k - \partial_i (\delta x^k \partial_k h - \delta h)
\]

where

\[
(2.29) \quad D_i^2 (\varepsilon(h)\delta h) - \delta x^k \partial_k D_i^2 \varepsilon(h) - \Delta (\delta h - \delta x^k \partial_k h) - 2(\partial_i V^k) \partial_k (\delta V^i - \delta x^l \partial_l V^i) = 0, \quad \delta h \bigg|_{\partial \Omega} = 0
\]

In order to use the Nash-Moser iteration scheme to obtain a solution of \((2.13)\) we must show that linearized operator is invertible and that the inverse satisfies tame estimates:

Theorem 2.3. Suppose that \( x_0, h_0 \in C^\infty([0,T] \times \overline{\Omega}) \) is a formal solution at \( t = 0, \) i.e. \((2.18)\) hold. Suppose that \((2.13)\) and \((2.14)\) hold when \( t = 0. \) Let

\[
(2.30) \quad |||u|||_{r,\infty} = \sup_{0 \leq t \leq T} \sup_{y \in \Omega} \sum_{|\alpha| \leq r} |\partial_t^\alpha u(t, y)|
\]

Then there is a \( T_0 = T(x_0, h_0) > 0, \) depending only on upper bounds for \( |||x_0|||_{r_0+4,\infty} + |||h_0|||_{r_0+4,\infty}, \) where \( r_0 = \lfloor n/2 \rfloor + 1, c_0^{-1} \) and \( c_1, \) such that the following hold. If \( x - x_0 \in C_{00}^{\infty}, h \) is the solution to \((2.21)\) with \( h - h_0 \in C_{00}^{\infty} \) and

\[
(2.31) \quad T \leq T_0, \quad |||x - x_0|||_{r_0+4,\infty} \leq 1,
\]

then \((2.13)\) and \((2.14)\) hold for \( 0 \leq t \leq T \) with \( c_0 \) replaced by \( c_0/2 \) and \( c_1 \) replaced by \( 2c_1. \) Furthermore, linearized equations

\[
(2.32) \quad \Phi'(x) \delta x = \delta \Phi, \quad \text{in} \ [0,T] \times \overline{\Omega},
\]

where \( \delta \Phi \in C_{00}^{\infty} \) has a solution \( \delta x \in C_{00}^{\infty}. \) The solution satisfies the estimates

\[
(2.33) \quad |||\delta x|||_{a,\infty} \leq C_a (|||\delta \Phi|||_{a+r_0,\infty} + |||\delta \Phi|||_{0,\infty} |||x - x_0|||_{a+r_0+4,\infty}), \quad a \geq 0
\]

where \( C_a = C_a(x_0, h_0, c_0^{-1}, c_1) \) is bounded when \( a \) is bounded.

Furthermore \( \Phi \) is twice differentiable and the second derivative satisfies the estimates

\[
(2.34) \quad |||\Phi''(x)(\delta x, \epsilon x)|||_{a,\infty}
\leq C_a \left( |||\delta x|||_{a+2r_0+4,\infty} |||\epsilon x|||_{0,\infty} + |||\delta x|||_{0,\infty} |||\epsilon x|||_{a+2r_0+4,\infty} + |||x - x_0|||_{a+3r_0+6,\infty} |||\delta x|||_{0,\infty} |||\epsilon x|||_{0,\infty} \right)
\]

Theorem 2.1 follows from Theorem 2.3 (or rather a version with Hölder norms.) and the Nash-Moser theorem; Theorem 15.1. Theorem 2.3 follows from Lemma 11.3 and Theorem 12.2. More precisely, we will first prove an \( L^2 \) estimate

\[
(2.35) \quad |||\delta x(t)|||_r + |||\delta x(t)|||_r \leq C \sum_{s=1}^r |||x|||_{r+s+4-s,\infty} \int_0^t |||\delta \Phi(\tau)|||_s \, d\tau,
\]
where

\[(2.36) \quad \|u(t)\|_r = \sum_{|\alpha| \leq r} \|\partial_t^{\alpha} u(t, \cdot)\|_{L^2(\Omega)}\]

and we will first prove this estimate for a lower order modification, \(L_1\), of the linearized operator to be described below. Furthermore in Theorem 9.1 we will first prove the estimate for \(L_1\) expressed in the Lagrangian coordinates, i.e. when also the vector field is expressed in the Lagrangian frame to be described below.

It will follow from estimating the solution of the wave equation (2.29) that \(\delta h\) will have the same regularity as \(\delta x\). Since

\[(2.37) \quad [\partial_i, D_t^2] \delta x^i = 2(\partial_i V^k) \partial_k (\delta V^i - \delta x^i \partial_t V^i) + (\partial_k D_t V^i) \partial_i \delta x^k + \delta x^i \partial_t ((\partial_l V^k) \partial_k V^i)\]

we get by taking the divergence of (2.28):

\[(2.38) \quad D_t^2 \text{div} \delta x - \delta x^l \partial_l (\text{div} D_t V - (\partial_l V^k) \partial_k \delta x^i) - \Delta (\delta x^k \partial_k h - \delta h) + 2(\partial_i V^k) \partial_k (\delta V^i - \delta x^i \partial_t V^i) = \text{div}(\Phi'(x) \delta x) - (\partial_i \delta x^i) \partial_t \Phi^i - \delta x^i \partial_t \text{div} \Phi\]

Hence if we add (2.29) and (2.38) we get

\[(2.39) \quad \text{div}(\Phi'(x) \delta x) = D_t^2 (\text{div} \delta x + e'(h) \delta h) + (\partial_i \delta x^i) \partial_t \Phi^i\]

There is also a similar identity for the curl, see [L3];

\[(2.40) \quad \text{curl}(\Phi'(x) \delta x) = \mathcal{L}_{D_t} \text{curl} \left(D_t \delta x - \delta x^k \partial_v k \right) + (\partial_i \delta x^k) \partial_j \Phi_k - (\partial_j \delta x^k) \partial_i \Phi_k\]

where \(\mathcal{L}_{D_t}\) is the space time Lie derivative with respect to the vector field \(D_t = (1, V)\):

\[(2.41) \quad \mathcal{L}_{D_t} \sigma_{ij} = D_t \sigma_{ij} + (\partial_i V^j) \sigma_{ij} + (\partial_j V^i) \sigma_{ij}\]

restricted to the space components. It can be integrated along characteristics since it is invariant under changes of coordinates:

\[(2.42) \quad D_t \left(a_a^i a_b^j \sigma_{ij} \right) = a_a^i a_b^j \mathcal{L}_{D_t} \sigma_{ij}, \quad \text{where} \quad a_a^i = \partial x^i / \partial y^a\]

We now want to modify the linearized operator by adding a lower order term so as to remove the last term on the right in (2.39) or else replace it by a lower order term proportional to \(\text{div} \delta x\) and \(\delta h\) so that the equation \(\text{div}(\Phi'(x) \delta x) = 0\) gives an estimate for \(\text{div} \delta x\) in terms of \(\delta h\) which we have better control of than \(\partial \delta x\). In [L3] we used the modification

\[(2.43) \quad L_1 \delta x^i = L_0 \delta x^i - \delta x^l \partial_l \Phi^i + \delta x^i \text{div} \Phi\]

\[= D_t^2 \delta x_i - (\partial_k D_t^2 x_i) \delta x^k - \partial_t (\delta x^k \partial_k h - \delta h) + \delta x^i \text{div} \Phi\]

where

\[(2.44) \quad \text{div} \Phi = D_t \text{div} V + \Delta h + (\partial_i V^j) \partial_j V^i = D_t \text{div} V + D_t^2 e(h)\]
Then we get

\[(2.45) \quad \text{div}(L_1 \delta x) = D_t^2 \left( \text{div} \delta x + e'(h) \delta h \right) + (\text{div} \delta x) \text{div} \Phi \]

$L_1$ is a lower order modification of the linearized operator, that reduces to the linearized operator at a solution of $\Phi(x) = 0$. We will first prove that $L_1$ is invertible using that it has a nice equation for the divergence. Then once we inverted $L_1$ we will obtain estimates for $L_1$ that are so good that they can be used to iterate and deal with lower order modifications like $L_0$. However, as pointed out in [Ha] one do not need to invert the operator exactly but only do so up to a quadratic error. But this operator $L_0$ or some modification of it is likely to show up at other places so it seems important to show that a more general class of operators are invertible and to have good estimates for them.

Things are somewhat easier to see if we express the vector field in the Lagrangian frame:

\[(2.46) \quad W^a = \frac{\partial y^a}{\partial x^i} \delta x^i, \quad \omega_{ab} = \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b} (\partial_i v_j - \partial_j v_i). \]

Then

\[(2.47) \quad D_t W^a = \frac{\partial y^a}{\partial x^k} (\delta V^k - \delta x^l \partial_i V^k), \quad D_t^2 W^a = \frac{\partial y^a}{\partial x^k} (D_t^2 \delta x^i - \delta x^l \partial_i D_t^2 x^k - 2(\delta V^k - \delta x^l \partial_i V^k) \partial_k v_i). \]

Since also

\[(2.49) \quad D_t g_{ab} = \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b} (\partial_i v_j + \partial_j v_i) \]

the modified linearized operator (2.43) in the Lagrangian frame become

\[(2.50) \quad L_1 W^a = g_{ab} D_t^2 W^b - \partial_a ((\partial_c h) W^c - \delta h) + (D_t g_{ac} - \omega_{ac}) D_t W^c + \text{div} \Phi W^a. \]

Also, the divergence is invariant,

\[(2.51) \quad \text{div} \delta x = \text{div} W = \kappa^{-1} \partial_a (\kappa W^a) \]

Now, $D_t$ does not commute with taking the divergence, if $\kappa^{-1} D_t \kappa = \text{div} V \neq 0$, so we will replace it by a modified time derivative that does:

\[(2.52) \quad \dot{D}_t = D_t + \text{div} V, \quad \text{i.e.} \quad \dot{D}_t W^a = D_t W^a + (\text{div} V) W^a = \kappa^{-1} D_t (\kappa W^a) \]

It then follows that

\[(2.53) \quad \dot{D}_t \text{div} W = \text{div} \dot{D}_t W \]

if $\sigma = \ln \kappa$ then $\dot{\sigma} = D_t \sigma = \text{div} V$, see [L1]. With this notation we have $\dot{D}_t^2 = (D_t + \text{div} V)(D_t + \text{div} V) = D_t^2 + 2\dot{\sigma} D_t + \dot{\sigma}^2 + \ddot{\sigma} = D_t^2 + 2\dot{\sigma} D_t + \ddot{\sigma} - \dot{\sigma}^2$ so

\[(2.54) \quad D_t^2 = \dot{D}_t^2 - 2\dot{\sigma} \dot{D}_t + \dot{\sigma}^2 - \ddot{\sigma}, \quad D_t = \dot{D}_t - \dot{\sigma} \]
Hence, with $\dot{W} = \dot{D}_1 W \text{ and } \ddot{W} = \dot{D}_2^2 W$, we can write the equation (2.50) as

\begin{equation}
L_1 W^a = \dot{W}^a - g^{ab} \partial_b ((\partial_c h) W^c - \delta h) - B_1 \dot{W}^a - B_0 W^a \tag{2.55}
\end{equation}

where

\begin{align*}
B_1 \dot{W}^a &= -g^{ab} (D_t g_{bc} - \omega_{bc}) \dot{W}^c + 2\dot{\sigma} W^a, \\
B_0 W^a &= g^{ab} (D_t g_{bc} - \omega_{bc}) \dot{\sigma} W^c - (D_t^2 c(h) + \dot{\sigma}^2) W^a, \tag{2.56, 2.57}
\end{align*}

Taking the divergence of (2.55) gives

\begin{equation}
\text{div} L_1 W = \dot{D}_2^2 \text{div} W - \Delta ( (\partial_c h) W^c - \delta h) - \text{div} B_1 \dot{W} - \text{div} B_0 W \tag{2.58}
\end{equation}

On the other hand by (2.29)

\begin{equation}
\dot{D}_2^2 (e'(h) \delta h) - \Delta ( \delta h - (\partial_c h) W^c ) + \text{div} B_1 \dot{W} - 2\dot{\sigma} \text{div} \dot{W} + \text{div} B_0 W + (D_t^2 c(h) + \dot{\sigma}^2) \text{div} W = 0 \tag{2.59}
\end{equation}

If we add (2.59) to (2.58) we get

\begin{equation}
\text{div} L_1 W = \dot{D}_2^2 (\text{div} W + e'(h) \delta h) + \text{div} \Phi \text{ div} W \tag{2.60}
\end{equation}

as it should be, by (2.45). With $\varphi = \text{div} W + e'(h) \delta h$ we can hence alternatively write (2.59):

\begin{equation}
\dot{D}_2^2 (e'(h) \delta h) - \Delta (\delta h - (\partial_c h) W^c) + \text{div} B_1 \dot{W} + \text{div} B_0 W - 2\dot{\sigma} \varphi + (D_t^2 c(h) + \dot{\sigma}^2) \varphi - \text{div} \Phi e'(h) \delta h = 0 \tag{2.61}
\end{equation}

or

\begin{equation}
\dot{D}_2^2 (e'(h) \delta h) - \Delta \delta h + \partial_t ( (\partial^i \delta x^k) \partial_k h ) + (\partial^i \delta x^k) \partial_t \partial_k h - 2(\partial_i V^k) \partial_k \delta V^i = 0, \quad \delta h \big|_{\partial \Omega} = 0 \tag{2.62}
\end{equation}

We now, also want to express $L_0 = \Phi'(x)$ is these coordinates. In order to do this we must transform the term $\delta x^k \partial_k \Phi^i$ in (2.25) to the Lagrangian frame. If $\Phi^a = \Phi^i \partial y^a / \partial x^i$, then $(\delta x^k \partial_k \Phi^i) \partial y^a / \partial x^i = W^c \nabla_c \Phi^a$, where $\nabla_c$ is covariant differentiation, see e.g. [CL]. Hence by (2.43)

\begin{equation}
L_0 W^a = L_1 W^a - B_3 W^a, \quad \text{where} \quad B_3 W^a = -W^c \nabla_c \Phi^a + W^a \text{div} \Phi \tag{2.63}
\end{equation}

3. The projection and the normal operator. 

The energy and curl estimates.

Let us now also define the projection $P$ onto divergence free vector fields by

\begin{equation}
PU^a = U^a - g^{ab} \partial_b p_U, \quad \Delta p_U = \text{div} U, \quad \frac{\partial p_U}{\partial x^i} = 0 \tag{3.1}
\end{equation}

(Here $\Delta q = \kappa^{-1} \partial_a (\kappa g^{ab} \partial_b q)$.) $P$ is the orthogonal projection in the inner product

\begin{equation}
\langle U, W \rangle = \int_\Omega g_{ab} U^a W^b \kappa dy \tag{3.2}
\end{equation}
and its operator norm is one:

\[(3.3) \quad \|PW\| \leq \|W\|\]

For a function \( f \) that vanishes on the boundary define \( A_f W^a = g^{ab} A_f W_b \), where

\[(3.4) \quad A_f W_a = -\partial_a ((\partial_c f) W^c - q), \quad \triangle ((\partial_c f) W^c - q) = 0, \quad q|_{\partial\Omega} = 0\]

This is defined for general vector fields but it is symmetric in the divergence free class. If \( U \) and \( W \) are divergence free then

\[(3.5) \quad \langle U, A_f W \rangle = \int_{\partial\Omega} n_a U^a (-\partial_c f) W^c dS\]

where \( n \) is the unit conormal. If \( f|_{\partial\Omega} = 0 \) then \( -\partial_c f|_{\partial\Omega} = (\nabla_N f) n_c \). It follows that \( A_f \) is a symmetric operator on divergence free vector fields, and in particular

\[(3.6) \quad A = A_h\]

is positive since we assumed that \(-\nabla_N h \geq c > 0 \) on the boundary. We have

\[(3.7) \quad |\langle U, A_f W \rangle| \leq \|\nabla_N f/\nabla_N p\|_{L^\infty(\partial\Omega)} \langle U, AU \rangle^{1/2} \langle W, AW \rangle^{1/2}\]

Note also that \( A_f \) only depends on \( \nabla_N f \) on the boundary so we can replace \( f \) by something with the same first order derivative that is supported in a neighborhood of the boundary. We now want to estimate the norm of \( A_f \). Now the projection has norm one so we can drop \( q \) in (3.6). If \( S \) is a tangential vector field then \( S^a \partial_a ((\partial_c f) W^c) = (\partial_c S f) W^c + (\partial_c f) L_S W^c \), where the Lie derivative, \( L_S \) is defined in the next section. Furthermore if \( R \) is the normal vector fields then we can replace \( f \) by the distance \( d \) to the boundary times the value of \( \nabla_N f \) at the boundary extended to be constant along the normal. Replacing \( f \) by this function we see that \( \partial_c f \) is then independent of the radial variable so \( R^a \partial_a ((\partial_c f) W^c) = (\partial_c f) RW^c \). In conclusion we get that

\[(3.8) \quad \|A_f W\| \leq C \sum_{S \in S} \|\nabla_N S f\|_{L^\infty(\partial\Omega)} \|W\| + C \|\nabla_N f\|_{L^\infty(\partial\Omega)} \|\partial W\|\]

where \( S \) is a spanning set of tangential vector fields.

Let us now also define the projected multiplication operators \( M_\beta \) with a two form \( \beta \) by

\[(3.9) \quad M_\beta W_a = P(\beta_{ab} W^b)\]

Since the projection has norm one it follows that

\[(3.10) \quad \|M_\beta W\| \leq \|\beta\|_\infty \|W\|\]

Furthermore we define the operator taking vector fields to one forms

\[(3.11) \quad G W_a = M_\beta W_a = P(g_{ab} W^b)\]

Then \( G \) acting on divergence free vector fields is just the identity \( I \).
Let $L_1$ be the modified linearized operator in (2.55). We now want to project the equation

\begin{equation}
(3.12) \quad L_1 W = F
\end{equation}

to the divergence free vector fields: We will decompose $L_1$ into 3 parts. We write

\begin{equation}
(3.13) \quad W = W_0 + W_1, \quad W_0 = PW, \quad W_1^a = g^{ab} \partial_b q_1,
\end{equation}

Then if $\dot{g}_{ab} = \dot{D}_t g_{ab}$, where $\dot{D}_t = D_t - \dot{\sigma}$, we have $\partial_a D_t q_1 = D_t (g_{ab} W^b) = \dot{g}_{ab} W_1^b + g_{ab} \dot{W}_1^b$ and $\partial_a D_t^2 q_1 = \ddot{g}_{ab} W_1^b + 2 \dot{g}_{ab} \dot{W}_1^b + g_{ab} \ddot{W}_1^b$. Hence

\begin{equation}
(3.14) \quad \dddot{W}_1^a = g^{ab} \partial_b D_t^2 q_1 - 2 g^{ab} \ddot{g}_{bc} \dot{W}_1^c - g^{ab} \dddot{g}_{bc} W_1^c
\end{equation}

Since the projection of a gradient of a function that vanishes on the boundary vanishes it follows that

\begin{equation}
(3.15) \quad P \dddot{W}_1^a = B_2(W_1, \dddot{W}_1)^a, \quad \text{where} \quad B_2(W_1, \dddot{W}_1)^a = -P(2 g^{ab} \dot{g}_{bc} \dddot{W}_1^c + g^{ab} \dddot{g}_{bc} W_1^c)
\end{equation}

Let $L_{10} = PL_1$. Since $\text{div} \dddot{W}_1 = 0$ and $\delta h$ vanishes the boundary it follows from projecting (2.55) that,

\begin{equation}
(3.16) \quad L_{10} W_0 = \dddot{W}_0 + AW_0 - B_{10} \dddot{W}_0 - B_{00} W_0
\end{equation}

\begin{equation}
(3.17) \quad L_{10} W_1 = AW_1 - B_{11} \dddot{W}_1 - B_{01} W_1
\end{equation}

where

\begin{equation}
(3.18) \quad B_{00} W = PB_1 W, \quad B_{11} W^a = PB_1 W^a + 2 P(g^{ab} \dot{g}_{bc} W^c) \quad B_{01} W^a = PB_0 W^a + P(g^{ab} \dot{g}_{bc} W^c).
\end{equation}

Hence the projection of (3.12) becomes

\begin{equation}
(3.19) \quad L_{10} W_0 = -L_{10} W_1 + PF,
\end{equation}

Here, by (2.60)

\begin{equation}
(3.20) \quad W_1^a = g^{ab} \partial_b q_1, \quad \Delta q_1 = \varphi - \epsilon'(h) \delta h, \quad q_1 \mid_{\partial \Omega} = 0
\end{equation}

where

\begin{equation}
(3.21) \quad D_t^2 \varphi + \text{div} \Phi \varphi = \text{div} F + \text{div} \Phi \epsilon'(h) \delta h,
\end{equation}

and by (2.59)

\begin{equation}
(3.22) \quad D_t^2 (\epsilon'(h) \delta h) - \Delta \delta h = \Delta \left( (\partial_t h) W^c \right) - \text{div} B_1 \dddot{W} + 2 \dot{\sigma} \text{div} \dddot{W} - \text{div} B_0 W - (D_t^2 \epsilon(h) + \dot{\sigma}^2) \text{div} W
\end{equation}

Hence we have obtained a system of equations for $(W_0, W_1, \varphi, \delta h)$.

We now want to show how to show the main idea of how to obtain estimates for the divergence free equation. Let $\dddot{W} = \dddot{D}_t W = D_t W + (\text{div} V) W = \kappa^{-1} D_t (\kappa W) = D_t W + \kappa W$. Then $\dddot{W}$ is divergence free if $W$ is divergence free. Let us now derive the basic energy estimate for equations of the form

\begin{equation}
(3.23) \quad \dddot{W} + AW = H
\end{equation}
where $A$ is either the normal operator or the smoothed out normal operator. Now, for any symmetric operator $B$ we have

$$
\frac{d}{dt} \langle W, BW \rangle = 2 \langle \dot{W}, BW \rangle + \langle W, \dot{B}W \rangle
$$

where $\dot{B}$ is the time derivative of the operator $B$ considered as an operator from the divergence free vector fields to the one forms, see section 4. $\dot{B}$ is defined by (4.24) with $T = D_t$. For the two operators that we will consider here this is given by (4.39) and (4.40). Note that, the projection in (4.24) comes up here since we take the inner product with a divergence free vector field in (3.24). Let the lowest order energy $E_0 = E(W)$ be defined by

$$
E(W) = \langle \dot{W}, \dot{W} \rangle + \langle W, (A + I)W \rangle.
$$

Since $\langle W, W \rangle = \langle W, GW \rangle$, it follows that

$$
\dot{E}_0 = 2 \langle \dot{W}, \dot{W} + (A + I)W \rangle + \langle \dot{W}, \dot{G}W \rangle + \langle W, (\dot{A} + \dot{G})W \rangle
$$

It follows from (4.26)-(4.36) and (3.7) and (3.10) that

$$
|\langle W, \dot{A}W \rangle| \leq \|h/h\|_\infty \langle W, AW \rangle, \quad |\langle W, \dot{G}W \rangle| \leq \|\dot{g}\|_\infty \langle W, W \rangle
$$

The last two terms are hence bounded by a constant times the energy so it follows that

$$
|\dot{E}_0| \leq \sqrt{E_0} (2\|H\| + c\sqrt{E_0}), \quad c = \|h/h\|_\infty + \|\dot{g}\|_\infty + 2
$$

Now, let $w = \dot{W}$, $\dot{w} = \dot{\dot{W}}$ and $\ddot{w} = \dddot{W}$. i.e. $w_a = g_{ab} W^b$, $\dot{w}_a = g_{ab} \dot{W}^b$ and $\ddot{w}_a = g_{ab} \ddot{W}^b$. Observe that $\ddot{w} \neq D_t w$ etc. since $\dot{W} = \dddot{W}$. In fact

$$
D_t w_a = \dot{g}_{ab} W^b + \ddot{w}_a, \quad D_t \ddot{w}_a = \dot{g}_{ab} \dddot{W}^b + \dddot{w}_a
$$

where $\dot{g}_{ab} = \dddot{g}_{ab}$ and $\dddot{W}^a = \dddot{D}_t W^a$. Now our equation says that

$$
\dddot{w} + A W = H, \quad H = -B_0 W - B_1 \dot{W} + F
$$

Since $\text{curl} Av = 0$ it follows that

$$
|\text{curl} \dddot{w}| \leq C (|\partial W| + |W| + |\partial \dot{W}| + |\dot{W}| + |\text{curl} F|)
$$

and hence

$$
|D_t \text{curl} w| \leq C (|\text{curl} \dddot{w}| + |\partial W| + |W|),
$$

$$
|D_t \text{curl} \dddot{w}| \leq C (|\partial W| + |W| + |\partial \dot{W}| + |\dot{W}| + |\text{curl} F|)
$$
4. The tangential vector fields and Lie derivatives

Following [L1], we now construct the tangential vector fields, that are time independent expressed in the Lagrangian coordinates, i.e. that commute with $D_t$. This means that in the Lagrangian coordinates they are of the form $S^a(y) \partial / \partial y^a$. Furthermore, they will satisfy,

\[(4.1) \quad \partial_a S^a = 0,\]

Since $\Omega$ is the unit ball in $\mathbb{R}^n$ the vector fields can be explicitly given. The vector fields corresponding to rotations, span the tangent space of the boundary and are divergence free in the interior. Furthermore they span the tangent space of the level sets of the distance function from the boundary in the Lagrangian coordinates

\[(4.2) \quad \frac{\partial}{\partial y^b} - \frac{\partial}{\partial y^a} \]

away from the origin $y \neq 0$. We will denote this set of vector fields by $S_0$. We also construct a set of divergence free vector fields that span the full tangent space at distance $d(y) \geq d_0$ and that are compactly supported in the interior at a fixed distance $d_0/2$ from the boundary. The basic one is

\[(4.3) \quad \frac{\partial}{\partial y^1} - \frac{\partial}{\partial y^2} \]

which satisfies (4.1). Furthermore we can choose $f, g, h$ such that it is equal to $\partial / \partial y^1$ when $|y^i| \leq 1/4$, for $i = 1, \ldots, n$ and so that it is 0 when $|y^i| \geq 1/2$ for some $i$. In fact let $f$ and $g$ be smooth functions such that $f(s) = 1$ when $|s| \leq 1/4$ and $f(s) = 0$ when $|s| \geq 1/2$ and $g'(s) = 1$ when $|s| \leq 1/4$ and $g(s) = 0$ when $|s| \geq 1/2$. Finally let $h(y^3, \ldots, y^n) = f(y^3) \cdots f(y^n)$. By scaling, translation and rotation of these vector fields we can obviously construct a finite set of vector fields that span the tangent space when $d \geq d_0$ and are compactly supported in the set where $d \geq d_0/2$. We will denote this set of vector fields by $S_1$. Let $\mathcal{S} = S_0 \cup S_1$ denote the family of tangential space vector fields and let $\mathcal{T} = \mathcal{S} \cup \{D_t\}$ denote the family of space time tangential vector fields.

Let the radial vector field be

\[(4.4) \quad R = \frac{\partial}{\partial y^a} \]

Now,

\[(4.5) \quad \partial_a R^a = n \]

is not 0 but for our purposes it suffices that it is constant. Let $\mathcal{R} = \mathcal{S} \cup \{R\}$. Note that $\mathcal{R}$ span the full tangent space of the space everywhere. Let $\mathcal{U} = \mathcal{S} \cup \{R\} \cup \{D_t\}$ denote the family of all vector fields. Note also that the radial vector field commutes with the rotations;

\[(4.6) \quad [R, S] = 0, \quad S \in S_0 \]

Furthermore, the commutators of two vector fields in $S_0$ is just $\pm$ another vector field in $S_0$. Therefore, for $i = 0, 1$, let $\mathcal{R}_i = \mathcal{S}_i \cup \{R\}$, $\mathcal{T}_i = \mathcal{S}_i \cup \{D_t\}$ and $\mathcal{U}_i = \mathcal{S}_i \cup \{R\} \cup \{D_t\}$.
Let \( \mathcal{U} = \{U_i\}_{i=1}^M \) be some labeling of our family of vector fields. We will also use multindices \( I = (i_1, ..., i_r) \) of length \( |I| = r \). so \( U^I = U_{i_1} \cdots U_{i_r} \) and \( \mathcal{L}_U^I = \mathcal{L}_{U_{i_1}} \cdots \mathcal{L}_{U_{i_r}} \), where \( \mathcal{L}_U \) is the Lie derivative, defined below. Sometimes we will write \( \mathcal{L}_U^I \), \( U \in \mathcal{S}_0 \) or \( I \in \mathcal{S}_0 \), meaning that \( U_{i_k} \in \mathcal{S}_0 \) for all of the indices in \( I \).

Let us now introduce the Lie derivative of the vector field \( W \) with respect to the vector field \( T \);

\[
\mathcal{L}_T W^a = TW^a - (\partial^c T^a)W^c
\]

We will only deal with Lie derivatives with respect to the vector fields \( T \) constructed in the previous section. For those vector fields \( T \) we have

\[
[D_t, T], \quad \text{and} \quad [D_t, \mathcal{L}_T] = 0
\]

The Lie derivative of a one form is defined by

\[
\mathcal{L}_T \alpha_a = T\alpha_a + (\partial^c T^a)\alpha_c,
\]

The Lie derivative also commute with exterior differentiation, \([\mathcal{L}_T, d] = 0\) so

\[
\mathcal{L}_T \partial_a q = \partial_a Tq
\]

if \( q \) is a function. The Lie derivative of a two form is given by

\[
\mathcal{L}_T \beta_{ab} = T\beta_{ab} + (\partial^c T^a)\beta_{cb} + (\partial^c T^b)\beta_{ac}
\]

Furthermore if \( w \) is a one form and \( \text{curl} \, w_{ab} = dw_{ab} = \partial_a w_b - \partial_b w_a \) then since the Lie derivative commutes with exterior differentiation:

\[
\mathcal{L}_T \text{curl} \, w_{ab} = \text{curl} \, \mathcal{L}_T w_{ab}
\]

We will also use that the Lie derivative satisfies Leibniz rule, e.g.

\[
\mathcal{L}_T (\alpha_c W^c) = (\mathcal{L}_T \alpha_c)W^c + \alpha_c \mathcal{L}_T W^c, \quad \mathcal{L}_T (\beta_{ac} W^c) = (\mathcal{L}_T \beta_{ac})W^c + \beta_{ac} \mathcal{L}_T W^c.
\]

Furthermore, we will also treat \( D_t \) as if it was a Lie derivative and set

\[
\mathcal{L}_{D_t} = D_t
\]

Now of course this is not a space Lie derivative. It can however be interpreted as a space time Lie derivative. But the important thing is that it satisfies all the properties of the other Lie derivatives we are considering. The reason we want to call it \( \mathcal{L}_{D_t} \) is simply a matter of that we will apply products of Lie derivatives and \( D_t \) applied to the equation and since they behave in exactly the same way it is more efficient to have one notation for them.

In [L1] we used extensively that the Lie derivatives with respect to the vector fields above preserved the divergence free condition. This is no longer true if \( \kappa \) is not a constant. since \( \text{div} \, U = \kappa^{-1} \partial_a (\kappa U^a) \). This is no longer the case if \( U \) is not divergence free. One could modify the vector fields by multiplying them by \( \kappa^{-1} \). However, instead we will essentially multiply the vector field we apply them to with \( \kappa \). The modified Lie derivative is now for any of our tangential vector fields defined by

\[
\hat{\mathcal{L}}_U W = \mathcal{L}_U W + (\text{div} \, U)W,
\]
They preserves the divergence free condition, in fact

\[ \text{div} \mathcal{L}_U W = \hat{U} \text{div} W, \quad \text{where} \quad \hat{U} f = U f + (\text{div} U) f / \]

if \( f \) is a function. This definition is invariant and (4.17) holds for any vector field \( U \). However, in general, since we are considering Lie derivatives only with respect to the vector fields constructed above and only expressed in the Lagrangian coordinates it is simpler to use the definition

\[ \mathcal{L}_U W = \kappa^{-1} \mathcal{L}_U (\kappa W) = \mathcal{L}_U W + (U \sigma) W, \quad \text{where} \quad \sigma = \ln \kappa \]

Due to (4.1), \( \text{div} S = S \sigma \) if \( S \) is any of the tangential vector fields and \( \text{div} R = R \sigma + n \), if \( R \) is the radial vector field. For any of the tangential vector fields it then follows that

\[ \text{div} \mathcal{L}_U W = \hat{U} \text{div} W, \quad \text{where} \quad \hat{U} f = U f + (U \sigma) f. \]

This has several advantages. The commutators satisfy \([\mathcal{L}_U, \mathcal{L}_T] = \tilde{\mathcal{L}}_{[U, T]}\), since this is true for the usual Lie derivative. Furthermore, this definition is constant with our previous definition of \( \hat{D}_t \).

However, when applied to one forms we want to use the regular definition of the Lie derivative. Also, when applied to two forms most of the time we use the regular definition: However, when applied to two forms it turns out to be sometimes convenient to use the opposite modification:

\[ \tilde{\mathcal{L}}_T \beta_{ab} = \mathcal{L}_T \beta_{ab} - (U \sigma) \beta_{ab}, \quad \hat{U} = U - (U \sigma) \]

We will most of the time apply the Lie derivative to products of the form \( \alpha_a = \beta_{ab} W^b \):

\[ \mathcal{L}_T (\beta_{ab} W^b) = (\tilde{\mathcal{L}}_T \beta_{ab}) W^b + \beta_{ab} \tilde{\mathcal{L}}_TW \]

since the usual Lie derivative satisfies Leibniz rule. Using the modified Lie derivative we indicated in \([L2]\) how to extend the existence theorem in \([L1]\) to the case when \( \kappa \) is no longer constant, i.e. \( D_t \sigma = \text{div} V \neq 0 \). This will be carried out in more detail here.

We will now calculate commutator between Lie derivatives and the operator defined in the previous section, i.e. the normal operator and the multiplication operators. It is easier to calculate the commutator with Lie derivatives of these operators considered as operators with values in the one forms. The one form \( w \) corresponding to the vector fields \( W \) is given by lowering the indices

\[ w_a = W_a = g_{ab} W^b \]

For an operator \( B \) on vector fields we denote the corresponding operator with values in the one forms by \( \overline{B} \). These are related by

\[ \overline{BW_a} = g_{ab} BW^b, \quad \overline{BW}^a = g^{ab} \overline{B}_a \]

Most operators that we consider will map onto the divergence free vector fields so we will project the result afterwards to stay in this class. Furthermore, in order to preserve the divergence free condition we will use the modified Lie derivative. If the modified Lie derivative is applied to a divergence free vector field then the result is divergence free so projecting after commuting does not change the result. As pointed out above, for our operators it is easier to commute Lie derivatives with the corresponding operators from the divergence free vector fields to the one forms. Let \( B_T \) be defined by

\[ B_T W^a = P \left( g^{ab} \left( \mathcal{L}_T \overline{B} W_b - \overline{B}_b \tilde{\mathcal{L}}_T W \right) \right) \]
In particular if \( B \) is a projected multiplication operator \( \mathcal{P}_a W = P(\beta_{ab} W^b) = \beta_{ab} W^b - \partial_a q \), where \( q \) vanishes on the boundary is chosen so that \( \text{div } BW = 0 \) then

\[
(4.25) \quad \mathcal{L}_T \mathcal{P}_a W = \beta_{ab} \mathcal{L}_T W^b + (\mathcal{L}_T \beta_{ab}) W^b + \partial_a Tq
\]

and if we project to the divergence free vector fields then the term \( \partial_a Tq \) vanishes since if \( T \) is a tangential vector field then \( Tq = 0 \) as well. It therefore follows that \( B_T \) is another projected multiplication operator:

\[
(4.26) \quad \mathcal{P}_a W = P(\mathcal{L}_T \beta_{ab}) W^b
\]

In particular, we will denote the time derivative of an operator by \( \dot{B} = B_{Di} \) and for a projected multiplication operator this is

\[
(4.27) \quad \dot{B} W = B_{Di} W = P(\mathcal{D}_t \beta_{ab}) W^b
\]

If \( B \) maps onto the divergence free vector fields

\[
(4.28) \quad \mathcal{L}_T BW^a = \mathcal{L}_T (g^{ab} \mathcal{P}_b W) = (\mathcal{L}_T g^{ab}) \mathcal{P}_b W + g^{ab} \mathcal{L}_T \mathcal{P}_b W
\]

Here \( \mathcal{L}_T g^{ab} = -g^{ac} g^{bd} \mathcal{L}_T g_{cd} \), and if \( B \) maps onto the divergence free vector fields then \( \mathcal{L}_T B \) is also divergence free so the left hand side is unchanged if we do so and we get:

\[
(4.29) \quad \mathcal{L}_T BW^a = -P(g^{ab} (\mathcal{L}_T g_{bc}) BW^c) + P(g^{ab} (\mathcal{L}_T \mathcal{P}_b W - \mathcal{P}_b \mathcal{L}_T W)) + B \mathcal{L}_T W^a
\]

By (4.26) applied the \( G_{ab} = P(g_{ab} W^b) \) we see that \( G_T W = P((g^{ab} \mathcal{L}_T g_{bc}) W^c) \) so the first term in the right of (4.29) is \( G_T BW^a \). The second term is by definition (4.24) \( B_T W \) so we get

\[
(4.30) \quad \mathcal{L}_T BW = B \mathcal{L}_T W + B_T W - G_T BW
\]

The most important property of the projection is that it almost commutes with Lie derivatives with respect to tangential vector fields: i.e. let \( \mathcal{P}_a u_a = u_a - \partial_a p_U \). Then

\[
(4.31) \quad \mathcal{P}_a \mathcal{L}_T \mathcal{P}_a u_a = \mathcal{P}_a \mathcal{L}_T u_a
\]

since \( \mathcal{L}_T \partial_a p_U = \partial_a T p_U \) vanishes when we project again since \( T p_U \) vanishes on the boundary. We have just used this fact above. We have already calculated commutators between Lie derivatives and the projected multiplication operators so let us now also calculate the commutator between the Lie derivative with respect to tangential vector fields and the normal operator. Recall that the normal operator is defined by \( A_f W^a = g^{ab} A_f W_b \), where

\[
(4.32) \quad A_f W_a = -\partial_a ((\partial_c f) W^c - q), \quad \triangle ((\partial_c f) W^c - q) = 0, \quad q_{|\partial \Omega} = 0
\]

and \( f \) is function that vanished on the boundary. Hence since the Lie derivative commutes with exterior differentiation:

\[
(4.33) \quad \mathcal{L}_T A_f W_a = -\partial_a ((\partial_c f) \mathcal{L}_T W^c + (\partial_c \mathcal{L}_f W^c) + (\partial_c T \sigma) f W^c - T q)
\]
However, now since \( q \) vanishes on the boundary it follows that \( Tq \) also vanish on the boundary and so does \( (\partial_c T\sigma)fW^c \). Therefore the last two terms vanish when we project again so we get

\[
P\left(g^{ab}\mathcal{L}_T A_f^j W_b\right) = P\left(g^{ab}\mathcal{L}_T \hat{A}_f W_b\right) + P\left(g^{ab} A_T^j W_b\right)
\]

Let us now change notation so \( A = A_h \), where \( h \) is the enthalpy, see section 3. Then we have just calculated \( A_T \) defined by (4.24) to be \( A_T = A_{F,h} \), i.e.

\[
A_T = A_{T,h}, \quad \text{if} \quad A = A_h
\]

In particular, if \( T = D_t \) is the time derivative we will use the notation \( \hat{A} = A_{D_t} \) which then is

\[
\hat{A}W = A_{D_t}W = A_{D_t,h}W
\]

We can now also calculate higher order commutators:

**Definition 4.1.** If \( T \) is a vector fields let \( B_T \) be defined by (4.24). If \( T \) and \( S \) are two tangential vector fields we define \( B_{T,S} = (B_S)_T \) to be the operator obtained by first using (4.24) to define \( B_S \) and then define \( (B_S)_T \) to be the operator obtained from (4.24) with \( B_S \) in place of \( B \). Similarly if \( S^I = S^{i_2} \cdots S^{i_r} \) is a product of \( r = |I| \) vector fields then we define

\[
B_I = \left( \cdots (B_{S^{i_1}}) \cdots \right)_{S^{i_k}}
\]

If \( B \) is a multiplication operator \( BW^a = P\left(g^{ab} \beta_{bc} W^c\right) \) then

\[
B_I W = P\left(g^{ab} (\mathcal{L}_T^I \beta_{bc}) W^c\right).
\]

In particular if \( G \) is the identity operator \( GW^a = P\left(g^{ab} \beta_{bc} W^c\right) \) then

\[
G_I W = P\left(g^{ab} (\mathcal{L}_T^I \beta_{bc}) W^c\right).
\]

If \( A \) is the normal operator then

\[
A_I W^a = P\left(g^{ab} \partial_b ((\partial_c \hat{T}^I p) W^c) \right)
\]

With \( B_T \) as in (4.4) we have proven that if \( B \) maps onto the divergence free vector fields then

\[
\mathcal{L}_T BW = BW_T + B_TW - G_T BW, \quad W_T = \hat{\mathcal{L}}_T W
\]

Repeating this, gives for a product of modified Lie derivatives:

\[
\mathcal{L}_T^I BW = c^{I_1\ldots I_k} G_{I_3} \cdots G_{I_k} B_{I_1} W_{I_2} \quad W_J = \hat{\mathcal{L}}_T^J W
\]

where the sum is over all combinations of \( I = I_1 + \ldots + I_k \), and \( c^{I_1\ldots I_k} \) are some constants, such that 
\( c^{I_1\ldots I_k} = 1 \) if \( I_1 + I_2 = I \). Let us then also introduce the notation

\[
G_{I_1 I_2} = c^{I_1\ldots I_k} G_{I_3} \cdots G_{I_k},
\]

where the sum is over all combination such that \( I_3 + \ldots I_k = I - I_1 - I_2 \). With this notation we can write (4.42)

\[
\mathcal{L}_T^I BW = G_{I_1 I_2} B_{I_1} W_{I_2}
\]

where again \( G_{I_1 I_2} = 1 \) if \( I_1 + I_2 = I \). Also let

\[
\tilde{G}_{I_1\ldots I_k} = 0, \quad \text{if} \quad I_2 = I, \quad \text{and} \quad \tilde{G}_{I_1\ldots I_k} = G_{I_1\ldots I_k}, \quad \text{otherwise.}
\]

Then we also have

\[
\hat{\mathcal{L}}_T^I BW = BW_I + \tilde{G}_{I_1 I_2} B_{I_1} W_{I_2}
\]
5. Estimates for Derivatives of a Vector Field in Terms of the Curl, the Divergence and Tangential Derivatives or the Normal Operator.

The first part of the lemma below says that one can get a point wise estimate of any first order derivative of a vector field by the curl, the divergence and derivatives that are tangential at the boundary. The second part says that one can also get estimates in $L^2$ with a normal derivative instead of tangential derivatives. The last part says that we can get the estimate for the normal derivative from the normal operator. The lemma is formulated in terms in the Eulerian frame, i.e. in terms the original Euclidean coordinates. Later we will reformulate it in the Lagrangian frame and then also get estimates for higher derivatives in similar fashion.

**Lemma 5.1.** Let $\tilde{N}$ be a vector field that is equal to the normal $N$ at the boundary $\partial D_t$ and satisfies $|\tilde{N}| \leq 1$ and $|\partial \tilde{N}| \leq K$. Let $q_{ij} = \delta_{ij} - \tilde{N}^k \tilde{N}^j$. Then

\begin{align}
(5.1) & \quad |\partial \beta|^2 dx \leq C (q^{kl} \delta_{ij} \partial_k \beta_i \partial_j \beta_j + |\text{curl} \beta|^2) \\
(5.2) & \quad \int_{D_t} |\partial \beta|^2 dx \leq C \int_{D_t} (\delta_{ij} \tilde{N}^k \tilde{N}^l \partial_k \beta_i \partial_l \beta_j + |\text{curl} \beta|^2 + |\text{div} \beta|^2 + K^2 |\beta|^2) \, dx
\end{align}

Suppose that $\delta_{ij} \alpha_j$ is another vector field that is normal at the boundary and let $A \beta_i = \partial_i (\alpha_k \beta^k - q)$ and $q$ is chosen so that $\text{div} A \beta = 0$ and $q|_{\partial\Omega} = 0$. Then

\begin{align}
(5.3) & \quad \int_{D_t} \delta_{ij} \alpha_k \alpha_l \partial_k \beta_i \partial_l \beta_j \leq C \int_{D_t} (\delta_{ij} A \beta_i A \beta_j + |\alpha|^2 (|\text{curl} \beta|^2 + |\text{div} \beta|^2) + |\partial \alpha|^2 |\beta|^2) \, dx
\end{align}

This lemma was proven in [L3]. We will now state some results that were proven in [L3] and give some further definitions.

**Definition 5.1.** For $\mathcal{V}$ and $\mathcal{V}'$ any of the family of vector fields introduced in section 4, and for $\beta$ a two form, a one form, a function or a vector field we define

\begin{align}
(5.4) & \quad |\beta|^r_{\mathcal{V}} = \sum_{|I| \leq r, I \in \mathcal{V}} |L^I_{\mathcal{U}} \beta|, \quad |\beta|^V_{r, s} = \sum_{|I| \leq r, I \in \mathcal{V}, j \leq s} |L^I_{\mathcal{U}} D^j_{\mathcal{F}} \beta|,
\end{align}

where $L^I_{\mathcal{U}}$ is a product of Lie derivatives. Furthermore let

\begin{align}
(5.5) & \quad |\beta|^r_{\mathcal{V}} = \sum_{|\alpha|+k \leq r} |\partial^a_{y} D^k_{t} \beta|, \quad |\beta|^V_{r, s} = \sum_{|\alpha| \leq r, k \leq s} |\partial^a_{y} D^k_{t} \beta|
\end{align}

where $|\beta|$ is the point wise norm. Furthermore let $[\beta]^V_0 = 1$, $[\beta]_0 = 1$ and

\begin{align}
(5.6) & \quad [\beta]^V_{s} = \sum_{s_1 + \cdots + s_k \leq s, s_i \geq 1} |\beta|^{V}_{s_1} \cdots |\beta|^{V}_{s_k}, \quad [\beta]_{s} = \sum_{s_1 + \cdots + s_k \leq s, s_i \geq 1} |\beta|_{s_1} \cdots |\beta|_{s_k},
\end{align}

We note that if $\beta$ is a function then $L^I_{\mathcal{U}} \beta = U \beta$ and in general it is equal to this plus terms proportional to $\beta$. Hence (5.4) is equivalent to $\sum_{|I| \leq r, I \in \mathcal{V}} |U^I \beta|$. In particular if $\mathcal{R}$ denotes the family of space vector fields then $|\beta|^\mathcal{R}$ is equivalent to $|\beta|$, with a constant of equivalence independent of the metric. Note also that if $\beta$ is the one form $\beta_{\alpha} = \partial_{\alpha} q$ then $L^I_{\mathcal{U}} \beta = \partial U^I q$ so $|\partial q|^V_{s} = \sum_{|I| \leq r, I \in \mathcal{V}} |\partial U^I q|$. 20
Let \( c_1 \) be a constant such that

\[
\sum_{a,b} (|g_{ab}| + |g^{ab}|) \leq c_1^2, \quad |\partial x/\partial y|^2 + |\partial y/\partial x|^2 \leq c_1^2
\]

Let \( K_{i0} \) respectively \( K_i \), for \( i \geq 1 \), denote continuous increasing functions of

\[
c_1 + ||x||_{i,0,\infty}, \quad \text{respectively} \quad ||x||_{i,\infty} + c_1 + T^{-1},
\]

which also depends on the order \( r \) of differentiation. Here, the norms are as in Definition 5.3.

We note that the bounds for the metric \( g \) and its inverse follows from the bounds for the the Jacobian of the coordinate and its inverse since

\[
g_{ab} = \delta_{ij}(\partial x^i/\partial y^a)(\partial x^j/\partial y^b) \quad \text{and} \quad g^{ab} = \delta^{ij}(\partial y^a/\partial x^i)(\partial y^b/\partial x^j).
\]

It also follows that we have a bound for \( \kappa = \det(\partial x/\partial y) = \sqrt{\det g} \) and its inverse. Furthermore, if we have a bound for \( \kappa^{-1} \) the bound for the inverse of the metric follows from the bound for the metric, since an upper bound for the eigenvalues and a lower bound for their product gives a lower bound for the eigenvalues, since they are all positive by assumption. Moreover we note that also a lower bound \( |\partial x/\partial y|^2 \geq c_1^{-2} \) follows.

In what follows it will be convenient to consider the norms of \( \hat{\mathcal{L}}_t^I W = \kappa^{-1}\mathcal{L}_t^I(\kappa W) \) if \( W \) is a vector field and of \( \hat{\mathcal{L}}_t^I g = \kappa\mathcal{L}_t^I(\kappa^{-1} g) \), if \( g \) is the metric. The reason for this is simply that \( \text{div}(\hat{\mathcal{L}}_t^I W) = \hat{U}^I \text{div} W \) and \( \mathcal{L}_t^I \text{curl} w = \text{curl}(\mathcal{L}_t^I W) \) and when we lower indices \( w_a = g_{ab} W^b = (\kappa^{-1} g_{ab})(\kappa W^b) \) and apply the Lie derivative to the product we get \( \mathcal{L}_t^I w_a = (\mathcal{L}_t^I g_{ab}) W^b + g_{ab} \mathcal{L}_t^I W^b \).

The following Lemma was proven in [L3] for the cases without time derivatives. The cases with time derivatives follows from these by applying them to \( W \) replaced by \( \hat{D}_t^I W \), noting that \( \text{div} \hat{D}_t^I W = \hat{D}_t \text{div} W \) and \( \mathcal{L}_t^I \text{curl} (\hat{D}_t^I W)_b = \kappa \mathcal{L}_t^I \text{curl} W + \text{lower order terms} \) is equal to \( \hat{D}_t^I \text{curl} W + \text{lower order terms} \).

This is similar to the calculation in [L1,L3] that \( \text{curl}(\mathcal{L}_S^I W)_b = (\kappa^{-1} \mathcal{L}_S^I W)_b, \) where \( (\mathcal{L}_S^I W)_b = g_{ab} \mathcal{L}_S^I W^b \), is equal to \( \mathcal{L}_S^I \text{curl} W + \text{lower order terms} \).

We have:

**Lemma 5.2.** Let \( W \) be a vector field and let \( w_a = g_{ab} W^b \) be the corresponding one form. Let \( \kappa = \det(\partial x/\partial y) = \sqrt{\det g} \). Then

\[
|\kappa| + |\kappa^{-1}| \leq K_{10}, \quad |U^I \kappa| + |U^I \kappa^{-1}| \leq K_{10} c_{I_1 \ldots I_k} |U^{I_1}g| \cdots |U^{I_k}g|
\]

where the sum is over all \( I_1 + \ldots + I_k = I \) and \( K_{10} \) is as in Definition 5.2.

With notation as in Definition 5.1 and section 4 we have

\[
|\kappa W|^R_r \leq K_{10} (|\text{curl} w|^R_{r-1} + |\kappa \text{div} W|^R_{r-1} + |\kappa W|^S_{r-1} + \sum_{s=0}^{r-1} |g/\kappa|^R_{r-s}|\kappa W|^R_s)
\]

We also have

\[
|\kappa W|^R_r \leq K_{10} \sum_{s=0}^{r} |g/\kappa|^R_{r-s} (|\text{curl} w|^R_{r-s-1} + |\kappa \text{div} W|^R_{r-s-1} + |\kappa W|^S_{r-s}),
\]

where the first two terms in the sum should be interpreted as 0 if \( s = 0 \).

The inequalities (5.10)-(5.11) also hold with \( (R,S) \) replaced by \( (U,T) \) and moreover,

\[
|\kappa W|^R_{r,s} \leq K_{10} (|\text{curl} w|^R_{r-1,s} + |\kappa \text{div} W|^R_{r-1,s} + |\kappa W|^S_{r,s} + \sum_{i+j+r+s-1,i \leq r,j \leq s} |g/\kappa|^U_{r+s-i-j}|\kappa W|^R_{i,j})
\]
and

\[ |\kappa W|^{R}_{r,s} \leq K_1 \sum_{i \leq j} \left[ g/|\kappa| \right]^{U}_{r+s-i-j} \left( |\text{curl} W|^{R}_{i-1,j} + |\kappa \text{div} W|^{R}_{i-1,j} + |\kappa W|^{S}_{i,j} \right), \]

where the sums are only over positive indices and if \( i = 0 \) then the first two terms should be interpreted as 0.

**Definition 5.3.** For \( V \) any of the family of vector fields in section 4 let

\[ \|W\|_{r}^{V} = \sum_{|I| \leq r, I \in V} \|\mathcal{L}^{I}_{U} W\|, \quad \|W\|_{r,\infty}^{V} = \sum_{|I| \leq r, I \in V} \|\mathcal{L}^{I}_{U} W\|_{\infty} \]

and let

\[ \|W\|_{r} = \sum_{|\alpha| + k \leq r} \|D_{x}^{\alpha} \partial_{y}^{k} W\|, \quad \|W\|_{r,\infty} = \sum_{|\alpha| + k \leq r} \|D_{x}^{\alpha} \partial_{y}^{k} W\|_{\infty} \]

where \( \|W\| = \|W(t)\| = \|W(t, \cdot)\|_{L^{2}(\Omega)} \), \( \|W\|_{\infty} = \|W(t)\|_{\infty} = \|W(t, \cdot)\|_{L^{\infty}(\Omega)} \). Let the mixed norms be defined by

\[ \|W\|_{r,s}^{V} = \sum_{j=0}^{s} \|D^{j}_{x} W\|_{r}^{V}, \quad \|W\|_{r,s}^{V} = \sum_{j=0}^{s} \sum_{|\alpha| \leq r} \|D^{j}_{x} \partial_{y}^{\alpha} W\| \]

and

\[ \|W\|_{r,s,\infty}^{V} = \sum_{j=0}^{s} \|D^{j}_{x} W\|_{r,s,\infty}^{V}, \quad \|W\|_{r,s,\infty} = \sum_{j=0}^{s} \sum_{|\alpha| \leq r} \|D^{j}_{x} \partial_{y}^{\alpha} W\|_{\infty} \]

Furthermore, let

\[ \|\alpha\|_{r,\infty} = \sup_{0 \leq t \leq T} \|\alpha(t, \cdot)\|_{r,\infty}, \quad \|\alpha\|_{r,s,\infty} = \sup_{0 \leq t \leq T} \|\alpha(t, \cdot)\|_{r,s,\infty}, \]

and

\[ \left[ \left[ \left[ \beta \right] \right] \right]_{r,\infty} = \sum_{r_{1} + \cdots + r_{k} \leq r, r_{i} \geq 1} \|\|\beta\|\|_{r_{1},\infty} \cdots \|\beta\|\|_{r_{k},\infty} \]

It follows from the discussion after Definition 5.1 and the beginning of Lemma 5.3 that \( \|W\|_{r} \) is equivalent to \( \|W\|^{R}_{r} \) with a constant of equivalence depending only on the dimension. As with the point wise estimates it will sometimes be convenient to instead use \( \|\mathcal{L}^{I}_{U} W\| = \|\kappa^{-1} \mathcal{L}^{I}_{U}(\kappa W)\|. \) This in particular true for the family of space tangential vector fields \( S \). However instead of introducing a special notation we then write \( \|\kappa W\|^{S}_{r} \), since \( \kappa \) is bounded from above and below by a constant \( K_{1} \) this is equivalent with a constant of equivalence \( K_{1} \). Furthermore, by interpolation \( \|\kappa W\|^{S}_{r} \leq K_{1}(\|g\|_{r}, \|W\|^{S}_{r}) \) and \( \|W\|^{S}_{r} \leq K_{1}(\|g\|_{r}, \|W\|^{S}_{r}) \), and in our inequalities we will have lower order terms of this form anyway. Note also that it follows from the interpolation inequalities in Lemma 5.7 that

\[ \|\|g\|\|_{0,\infty} \leq \left[ \left[ \left[ g \right] \right] \right]_{r,\infty} \leq K_{1} \|g\|_{r,\infty} \]

Here the constant \( K_{1} \) depends on a lower bound for \( T > 0 \) and in fact tends to infinity as \( T \to 0 \). Most of the time this does not matter, but when estimating lower norms of the enthalpy we can not use this estimate.

The following lemmas were proven in [L3] for the cases without time derivatives. Their generalizations to including time derivatives is immediate.
Lemma 5.3. We have with a constant $K_{10}$ as in Definition 5.2, we have

\begin{equation}
\|\kappa W\|_{r,0} \leq K_{10} (\|\text{curl } w\|_{r-1,0} + \|\kappa \text{div } W\|_{r-1,0} + \|\kappa W\|_{r,0}^S + K_{10} \sum_{s=0}^{r-1} \|g\|_{r-s,0,\infty} \|\kappa W\|_{s,0})
\end{equation}

and

\begin{equation}
\|\kappa W\|_{r,0} \leq K_{10} \sum_{s=0}^{r} \|g\|_{r-s,0,\infty} (\|\text{curl } w\|_{s-1,0} + \|\kappa \text{div } W\|_{s-1,0} + \|\kappa W\|_{s,0}^S)
\end{equation}

where the two terms with $s-1$ should be interpreted as 0 when $s = 0$. We also have with a constant $K_1$ as in Definition 5.2, we have

\begin{equation}
\|\kappa W\|_{r} \leq K_{10} (\|\text{curl } w\|_{r-1} + \|\kappa \text{div } W\|_{r-1} + \|\kappa W\|^T + K_{10} \sum_{s=0}^{r-1} \|[\![g]\!]\|_{r-s,\infty} \|\kappa W\|_s)
\end{equation}

and

\begin{equation}
\|\kappa W\|_{r} \leq K_{10} \sum_{s=0}^{r} \|[\![g]\!]\|_{r-s,\infty} (\|\text{curl } w\|_{s-1} + \|\kappa \text{div } W\|_{s-1} + \|\kappa W\|^T)
\end{equation}

where the two terms with $s-1$ should be interpreted as 0 when $s = 0$. Furthermore

\begin{equation}
\|\kappa W\|_{r,s} \leq K_{10} (\|\text{curl } w\|_{r-1,s} + \|\kappa \text{div } W\|_{r-1,s} + \|\kappa W\|_{r,s}^S + \sum_{i \leq s, j \leq s, i+j \leq r-s} \|[\![g]\!]\|_{r+s-i-j,\infty} \|\kappa W\|_{i,j})
\end{equation}

and

\begin{equation}
\|\kappa W\|_{r,s} \leq K_{10} \sum_{i \leq s, j \leq s} \|[\![g]\!]\|_{r+s-i-j,\infty} (\|\text{curl } w\|_{i-1,j} + \|\kappa \text{div } W\|_{i-1,j} + \|\kappa W\|_{i,j}^S)
\end{equation}

where the two terms with $i-1$ should be interpreted as 0 if $i = 0$.

The following lemma, proven in [L3], gives a bounds of derivatives of a vector field by the curl, the divergence and the normal operator. We have:

**Lemma 5.4.** Let $c_0 > 0$ be a constant such that $|\nabla_N h| \geq c_0 > 0$, let $K'_1$ and $K'_2$ be constants such that $\|\nabla_N h\|_{L^\infty(\partial\Omega)} + c_1 \leq K'_1$ and $\sum_{S \in S} \|\nabla_N S h\|_{L^\infty(\partial\Omega)} + |\partial g| + c_1 + K'_1 \leq K'_2$ Then

\begin{equation}
c_0 \|\partial W\| \leq (\|AW\| + K'_1 (\|\text{curl } w\| + \|\text{div } W\|) + K'_2 \|W\|)
\end{equation}

By Lemma 5.4 we also have

\begin{equation}
c_0 \|\partial \hat{\mathcal{L}}^j_T W\| \leq K'_2 (\|\text{curl } (\hat{\mathcal{L}}^j_T W)_b\| + \|\text{div } \hat{\mathcal{L}}^j_T W\| + \|A \hat{\mathcal{L}}^j_T W\| + \|\hat{\mathcal{L}}^j_T W\|)
\end{equation}

Here $(\hat{\mathcal{L}}^j_T W)_{bb} = g_{ab} \hat{\mathcal{L}}^j_T W^b$, and as in the proof of Lemma 5.3 in [L3] we see that $\text{curl } (\hat{\mathcal{L}}^j_T W)_b$ is equal to $\text{curl } \mathcal{L}^j_T w = \mathcal{L}^j_T \text{curl } w$, plus terms of lower order, where $\mathcal{L}^j_T w = g_{ab} W^b$. In particular we see that we can get any space tangential derivative in this way so we also get:
Lemma 5.5. With $K'_2$ as in Lemma 5.4 we have

\[
(5.29) \quad c_0 \|W\|_{r,0} \leq K'_2\left(\|\text{curl}\,w\|_{r-1,0} + \|\text{div}\,W\|_{r-1,0} + \|W\|^S_{r-1,A} + \sum_{s=0}^{r-1} \|g\|_{\infty,r-s}\|W\|_{s,0}\right)
\]

where

\[
(5.30) \quad \|W\|^S_{s,A} = \sum_{|I|=s, I \in S} \|A\mathcal{L}_S^I W\|
\]

We also have

\[
(5.31) \quad c_0 \|W\|_r \leq K'_2\left(\|\text{curl}\,w\|_{r-1} + \|\text{div}\,W\|_{r-1} + \|W\|^T_{r-1,A} + \|\dot{W}\|^T_{r-1} + \sum_{s=0}^{r-1} \|g\|_{\infty,r-s}\|W\|_s\right)
\]

where

\[
(5.32) \quad \|W\|^T_{s,A} = \sum_{|I|=s, I \in T} \|A\mathcal{L}_T^I W\|
\]

Let us now state the interpolation inequalities that we will use. For a proof of Lemma 5.6 see e.g. [H1,H2] for the $L^\infty$ estimate and [CL] for the $L^2$ estimate.

Lemma 5.6. There are constants $C_r$, respectively $C_{r,T}$ depending on $r$ respectively on $r$ and $T$, such that if $\beta, W \in C^\infty([0,T] \times \overline{\Omega})$ is a two form, a function or a vector field, then

\[
(5.33) \quad \|\beta\|_{s,\infty} \leq C_{r,T}\|\beta\|_{0,\infty}^{1-s/r}\|\beta\|_{r,\infty}^{s/r}
\]

\[
(5.34) \quad \|W\|_s \leq C_r\|W\|_{0}^{1-s/r}\|W\|_r^{s/r}
\]

Here $\|W\|_s = \|W(t)\|_s$. Furthermore, for $\beta \in C^\infty_0([0,T] \times \overline{\Omega})$, see (2.19), (5.33) holds with constants independent of $T$.

The consequence we will use is that

Lemma 5.7. There are constants $C$ depending on $T$ and $r$, such that if $f, f_i, \alpha, \beta, W \in C^\infty([0,T] \times \overline{\Omega})$ are functions, two forms or vector fields, then

\[
(5.35) \quad \|f\|_{s_1,\infty} \cdots \|f\|_{s_k,\infty} \leq C\|f\|_{0,\infty}^{k-1}\|f\|_{r,\infty}
\]

\[
(5.36) \quad \|\beta\|_{r-s,\infty} \leq C\|\alpha\|_{s,\infty} + \|\beta\|_{0,\infty}\|\alpha\|_{r,\infty} + \|\beta\|_{0,\infty}\|\alpha\|_{r,\infty}
\]

\[
(5.37) \quad \|\beta\|_{r-s,\infty} \|W\|_s \leq C\|\beta\|_{s,\infty}\|W\|_r + \|\beta\|_{r,\infty}\|W\|_0
\]

\[
(5.38) \quad \|f_1\|_{s_1,\infty} \cdots \|f_k\|_{s_k,\infty} \leq C\sum_{i=1}^k \|f_1\|_{0,\infty} \cdots \|f_{i-1}\|_{0,\infty} \|f_i\|_{r,\infty} \|f_{i+1}\|_{0,\infty} \cdots \|f_k\|_{0,\infty}
\]

where $r = s_1 + \ldots + s_k$. Here $\|W\|_s = \|W(t)\|_s$. Furthermore, for $\alpha, \beta, f, f_i \in C^\infty_0([0,T] \times \overline{\Omega})$, see (2.19), (5.35)-(5.38) holds with constants independent of $T$.

Proof. This follows from using Lemma 5.6 on each factor and then using the inequality $A^{s/r}B^{1-s/r} \leq A + B$, see [L3]. □
6. Tame estimates for the Dirichlet problem.

In this section we will give tame estimates for the Dirichlet problem

\[ \triangle q = f, \quad \text{in} \quad [0, T] \times \Omega, \quad q|_{\partial \Omega} = 0 \]

Let us recall the definition of the mixed norms

\[ \|q\|_{r,s} = \sum_{|\alpha| \leq r, k \leq s} \|D^k \partial^\alpha_y q\| \]

In the proofs that follow we will use the interpolations

\[ \|g\|_{r,\infty} \|g\|_{r,\infty} \leq K_1 \|g\|_{s+r,\infty}, \quad \|g\|_{s+1,\infty} \|g\|_{r+1,\infty} \leq K_2 \|g\|_{s+r+1,\infty} \]

where \( K_1 \) and \( K_2 \) are as in Definition 5.2.

**Theorem 6.1.** Suppose that \( q \) is a solution of the Dirichlet problem, \( q|_{\partial \Omega} = 0 \) and \( W^a = g^{ab} \partial_b q \) then

\[ \|W\|_{r+1,s} + \|\partial q\|_{r+1,s} \leq K_1 \sum_{i \leq r, j \leq s} \|[\partial g]_{r+s-i-j,\infty}\| \|\triangle q\|_{i,j} + K_2 \sum_{j=0}^{s} \|[\partial g, D_t g]_{r+s+1-j,\infty}\| W_{0,j} \]

if \( r, s \geq 0 \), and

\[ \|W\|_{r+1,s} + \|q\|_{r+2,s} \leq K_2 \sum_{i \leq r, j \leq s} \|[\partial g, D_t g]_{r+s-i-j,\infty}\| \|\triangle q\|_{i,j} \]

Moreover if \( P \) is the orthogonal projection onto divergence free vector fields and \( W \) is any vector field then,

\[ \|PW\|_{r,s} \leq K_1 \sum_{i \leq r, j \leq s} \|[\partial g, D_t g]_{r+s-i-j,\infty}\| \|W\|_{i,j} \]

**Remark.** Note that the constant in (6.4)-(6.6) are independent of \( T \) whereas those in (6.7)-(6.8) depends on a lower bound for \( T > 0 \).

First two useful lemmas:
Lemma 6.2. Suppose that $S \in \mathcal{S}$ and $q|_{\partial \Omega} = 0$, and

\[(6.9) \quad \hat{L}_S W^a = g^{ab} \partial_b q + F^a.\]

Then

\[(6.10) \quad \|\hat{L}_S W\| \leq K_{10}(\|\text{div} W\| + \|F\|)\]

Proof. Let $W_S = \hat{L}_S W$.

\[(6.11) \quad \int_{\Omega} g^{ab} W_S^a W_S^b \kappa dy = \int_{\Omega} W_S^a \partial_a q \kappa dy + \int_{\Omega} W_S^a g^{ab} F^b \kappa dy\]

If we integrate by parts in the first integral on the right, using that $q$ vanishes on the boundary we get

\[(6.12) \quad \int_{\Omega} W_S^a \partial_a q \kappa dy = - \int_{\Omega} \text{div}(W_S) q \kappa dy\]

However $\text{div} W_S = \hat{S} \text{div} W$. Then we can integrate by parts in the angular direction. $S = S^a \partial_a$, $\hat{S} = S + \text{div} S$ so we get $\int_{\Omega} (\hat{S} f) \kappa dy = \int_{\Omega} \partial_a (S^a f \kappa) dy = 0$, where $\partial_a S^a = 0$. Hence we get

\[(6.13) \quad \int_{\Omega} W_S^a \partial_a q \kappa dy = \int_{\Omega} \text{div}(W) (Sq) \kappa dy\]

Here $|Sq| \leq |\partial q| \leq K_{10}(|W_S| + |F|)$ so it follows that

\[(6.14) \quad \|W_S\|^2 \leq K_{10} \|\text{div} W\| (\|W_S\| + \|F\|) + K_{10} \|W_S\| \|F\|\]

and using that the inequality $A^2 \leq K(A + B)B$ implies that $A \leq (2K + 1)B$ we get for some other constant $K_{10}$:

\[(6.15) \quad \|W_S\| \leq K_{10}(\|\text{div} W\| + \|F\|) \quad \square\]

Lemma 6.3. Let $W^a = g^{ab} w_b$ Then

\[(6.16) \quad \hat{D}_l^k W^a = g^{ab} D_l^k w_b - \sum_{i=0}^{k-1} \binom{k}{i} g^{ab}(\hat{D}_l^{k-i} g_{bc}) \hat{D}_l^i W^c\]

Proof. We have $D_l^k w_b = D_l^k (\kappa^{-1} g_{bc} \kappa W^c) = \sum_{i=0}^{k} \binom{k}{i} (D_l^{k-i} (\kappa^{-1} g_{bc})) \hat{D}_l^i (\kappa W^c)$, which proves the lemma. \quad \square

Proof of Theorem 6.1. To simplify notation let us denote $W^a = W^a_1 = g^{ab} \partial_b q$. If we apply $L^I_T$ to $w_a = g_{ab} W^b$ we get

\[(6.17) \quad \partial_a T^I q = g_{ab} W^b_I + \hat{c}^I l^2 g_{l_1 a} W^b_{l_2}, \quad W_I = \hat{L}_T^I W, \quad g_{l a b} = \hat{L}_T^I g_{a b}\]
and the sum is over all combinations \(I = I_1 + I_2, \tilde{c}^{I_1 I_2}\) are constants such that \(\tilde{c}^{I_1 I_2} = 0\) if \(I_2 = I\). We assume that \(\hat{L}_I^t = \hat{L}_S^t \hat{L}_D^t\), where \(J \in S\) and \(|J| = r + 1\). If we write \(\hat{T}^I = \hat{S}^T^K\), \(W_I = \hat{L}_S W_K\), and use Lemma 6.2 we get since \(\text{div} W_K = \hat{T}^K \text{div} W = \hat{T}^K \triangle q = \kappa^{-1} T^K (\kappa \triangle q)\)

(6.18) \[\|W_I\| \leq K_{10}\|\hat{T}^K \triangle q\| + K_{10} \tilde{c}^{I_1 I_2} \|g_I\|_\infty \|W_I\|\]

or if we sum over all of them

(6.19) \[\|\kappa W\|_{r+1,s} \leq K_{10}\|\kappa \Delta q\|_{r,s} + K_{10} \sum_{i \leq r+1, j \leq s, i+j \leq r+s} \|\{[g]\}_{r+s+i-j,\infty}\| \kappa W\|_{i,j}^S\]

We now want to apply Lemma 5.3 to \(W^a = g^{ab} \partial_b q\). Then \(\text{curl} w = 0\) and \(\text{div} W = \triangle q\).

(6.20) \[\|\kappa W\|_{r+1,s} \leq K_{10}\|\kappa \Delta q\|_{r,s} + K_{10} \|W\|_{r+1,s}^S + K_{10} \sum_{i \leq r+1, j \leq s, i+j \leq r+s} \|\{[g]\}_{r+s+i-j,\infty}\| \kappa W\|_{i,j},\]

(6.19) and (6.20) therefore gives for \(r, s \geq 0\),

(6.21) \[\|\kappa W\|_{r+1,s} \leq K_{10}\|\kappa \Delta q\|_{r,s} + K_{10} \sum_{i \leq r+1, j \leq s, i+j \leq r+s} \|\{[g]\}_{r+s+i-j,\infty}\| \kappa W\|_{i,j},\]

Using induction it follows that

(6.22) \[\|\kappa W\|_{r+1,s} \leq K_{10} \sum_{i \leq r, j \leq s} \|\{[g]\}_{r+s-i-j,\infty}\| \kappa \Delta q\|_{i,j} + K_{10} \sum_{j=0}^{s} \|\{[g]\}_{s+1-j,\infty}\| \kappa W\|_{0,j}\]

(6.4) follows from this and \(\partial_a q = g^{ab} \partial_b q\). (Replacing \(\kappa\) by 1 just causes more terms of the same form.)

To prove (6.5) we need estimates for the last terms on the right in (6.4). By (6.4) with \(r = 0\) we have

(6.23) \[\|\partial^2 q\|_{0,s} \leq K_{10} \sum_{j=0}^{s} \|\{[g]\}_{s-j,\infty}\| \Delta q\|_{0,j} + K_{10} \sum_{j=0}^{s} \|\{[g]\}_{s+1-j,\infty}\| \partial q\|_{0,j}\]

Since \(\hat{D}_i^a \Delta q = \kappa^{-1} \partial_a D_i^a (\kappa g^{ab} \partial_b q)\) we have

(6.24) \[\|\Delta D_i^a q\| \leq \|\hat{D}_i^a \Delta q\| + K_{10} \sum_{j=0}^{s-1} \|\{[g]\}_{s-j,\infty}\| \partial^2 D_i^a q\| + K_{10} \sum_{j=0}^{s-1} \|\{[g]\}_{s+1-j,\infty}\| \partial D_i^a q\|\]

and using (6.23) it follows that

(6.25) \[\|\Delta D_i^a q\| \leq K_{10} \sum_{j=0}^{s} \|\{[g]\}_{s-j,\infty}\| \Delta q\|_{0,j} + K_{10} \sum_{j=0}^{s-1} \|\{[g]\}_{s+1-j,\infty}\| \partial q\|_{0,j}\]

We have \(\int_{\Omega} g^{ab} (\partial_a q)(\partial_b q) \kappa dy = -\int_{\Omega} (\Delta q) q \kappa dy\) and there is a constant \(K_{10}\), depending just on the volume of \(\Omega\), i.e. \(\int_{\Omega} \kappa dy\), such that \(\|q\| \leq K_{10}\|\Delta q\|\), see [SY]. Hence in addition we have

(6.26) \[\|q\| + \|\partial q\| \leq K_{10}\|\Delta q\|, \quad \text{if} \quad q|_{\partial \Omega} = 0\]
(6.26) applied to $D_t^s q$ in place of $q$ together with (6.25) gives

$$(6.27) \quad \|\partial q\|_{0,s} \leq K_{10} \sum_{j=0}^{s} \left[ [[g]] \right]_{s-j,\infty} \|\triangle q\|_{0,j} + K_{10} \sum_{j=0}^{s-1} \left[ [[g]] \right]_{s+1-j,\infty} \|\partial q\|_{0,j},$$

where the last term should be interpreted as 0 if $s = 0$, and inductively it follows that

$$(6.28) \quad \|\partial q\|_{0,s} \leq K_{20} \sum_{j=0}^{s} [[[(\partial g, D_t g)]]]_{s-j,\infty} \|\triangle q\|_{0,j}.$$ 

It remains to prove the estimates for the projection (6.6). We have $W = W_0 + W_1$, where $W_0 = PW$, and $W_1 = g^{ab} \partial_b g$ where $\triangle q = \text{div} W$ and $q|_{\partial\Omega} = 0$. Proving (6.6) for $r \geq 1$ reduces to proving it for $r = 0$ by using (6.4), since $\hat{R}^l \triangle q = \text{div} (\hat{L}_B^l W)$ and replacing $\kappa$ by 1 just produces more terms of the same form. (6.6), for $r = 0$ and $s = 0$ follows since the projection has norm 1, $\|PW\| \leq \|W\|$. Since the projection of $g^{ab} D_t^k w_{1b} = g^{ab} \partial_b D_t^k q$ vanishes we obtain from Lemma 6.3:

$$(6.29) \quad \|P \hat{D}_t^k W_1\| \leq K_1 \sum_{i=0}^{k-1} \left[ [[g]] \right]_{k-i} \|\hat{D}_t^i W_1\|.$$ 

Since also $P \hat{D}_t^k W_0 = \hat{D}_t^k W_0$ we have

$$(6.30) \quad (I - P) \hat{D}_t^k W_1 = (I - P) D_t^k W$$

and hence since the projection has norm one

$$(6.31) \quad \|\hat{D}_t^k W_1\| + \|\hat{D}_t^k W_0\| \leq K_1 \|\hat{D}_t^k W\| + K_1 \sum_{i=0}^{k-1} \left[ [[g]] \right]_{k-i} \|\hat{D}_t^i W_1\|.$$ 

Using induction we therefore obtain

$$(6.33) \quad \|\hat{D}_t^k W_0\| + \|\hat{D}_t^k W_1\| \leq K_1 \sum_{j=0}^{k} \left[ [[g]] \right]_{k-j,\infty} \|\hat{D}_t^j W\|.$$ 

Since as before replacing $\kappa$ by 1 just produces more terms of the same form this proves (6.6) also for $r = 0$. (6.7)-(6.8) follows from interpolation. □

7. Tame estimates for the wave equation.

Existence of solutions for a wave equation with Dirichlet boundary conditions and initial conditions satisfying some compatibility conditions is well known, see e.g. [H3]. The result in [H3] is stated with vanishing initial condition but we will reduce to that case by subtracting from a function that solves the equation to all orders as $t \to 0$. We consider the Cauchy problem for the wave equation on a bounded domain with Dirichlet boundary conditions:

$$(7.1) \quad D_t^2 (e^t \psi) - \triangle \psi = f, \quad \text{in} \quad [0, T] \times \Omega, \quad \psi|_{\partial\Omega} = 0,$$

$$(7.2) \quad \psi|_{t=0} = \psi_0, \quad D_t \psi|_{t=0} = \psi_1.$$
Here
\begin{equation}
\triangle \psi = \frac{1}{\sqrt{\det g}} \partial_a \left( \sqrt{\det g} g^{ab} \partial_b \psi \right),
\end{equation}
where \( g^{ab} \) is the inverse of the metric \( g_{ab} \) and \( \det g = \det \{ g_{ab} \} = \kappa^2 \), in our earlier notation. We assumed that \( e' \) is positive, \( g^{ab} \) is symmetric and positive definite (since the metric is), and that \( g^{ab} \) and \( e' \) are smooth satisfying:
\begin{equation}
e' + 1/e' \leq c'_1, \quad \sum_{a,b} (|g^{ab}| + |g_{ab}|) \leq c^2_1,
\end{equation}
for some constants \( 0 < c_1 < c'_1 < \infty \).

We will apply this theorem to
\begin{equation}
D^2_t (e'(h) \delta h) - \triangle \delta h = -\partial_i \left( (\partial^i \delta x^k) \partial_k h \right) - (\partial^i \delta x^k) \partial_i \partial_k h + 2(\partial_i V^k) \partial_k \delta V^i, \quad \delta h \big|_{\partial \Omega} = 0
\end{equation}
with vanishing initial conditions and the right hand side vanishing to all orders as \( t \to 0 \). We will also need some estimates for the equation for the enthalpy itself, which is best dealt with by writing \( h = \tilde{h} + h_0 \) where \( h_0 \) is the smooth approximate solution. In particular, in case \( e(h) = c^{-2} h \) we get the equation
\begin{equation}
c^{-2} D^2_t \tilde{h} - \triangle \tilde{h} = - (c^{-2} D^2_t h_0 - \triangle h_0 - (\partial_i V^j)(\partial_j V^i) )
\end{equation}
with vanishing initial conditions but where the right hand side vanishes to all orders as \( t \to 0 \) and is smooth for \( t \geq 0 \). In both cases we have vanishing initial conditions.

In the first case, (7.5), we then also want to use the special form of \( f \) in (7.1).
\begin{equation}
f = f_1 + \kappa^{-1} \partial_a (\kappa F^{a}_1)
\end{equation}
For the lowest order energy estimate it will be useful to take advantage of the special form (7.7) because it will allow us to estimate \( F \) with one less space derivative and one more time derivative instead and in the estimate for the divergence equation we have one more time derivative than space derivatives. Let
\begin{equation}
g_s = \||g||_{s+1, \infty}, \quad h_s = \||h||_{s+2, \infty},
\end{equation}
and let \( K'_2 \) denote a continuous function of
\begin{equation}
h_0 + g_0 + c'_1 + T^{-1} + r
\end{equation}
which in what follows also depends on the order \( r \) of differentiation. We will use that by interpolation;
\begin{equation}
(g_s + h_s)(g_r + h_r) \leq K'_2 (g_{r+s} + h_{r+s}).
\end{equation}
Theorem 7.1. Suppose that initial data in (7.2) vanishes and \( f \) in (7.1) or \( f_1 \) and \( F_1 \) in (7.7) are smooth and vanish to all orders as \( t \to 0 \). Suppose also that \( e' = e'(h) \) in (7.1) is a smooth function of \( h \) and that \( h \) and \( g \) are smooth. Then for \( r \geq 1 \) the solution of (7.1) satisfies the estimates

\[
\|\psi\|_{0,r} + \|\psi\|_{1,r-1} \leq K_2^r \sum_{s=1}^{r} (h_{r-s} + g_{r-s}) \int_{0}^{t} (\|f_1\|_{0,s-1} + \|F_1\|_{0,s}) \, d\tau
\]

Furthermore

\[
\|\psi\|_{r} \leq K'_2 \sum_{s=1}^{r} (h_{r-s} + g_{r-s}) \int_{0}^{t} (\|f_1\|_{0,s-1} + \|F_1\|_{0,s}) \, d\tau + \|f\|_{s-2}
\]

and

\[
\|\psi\|_{r} \leq K'_2 \sum_{s=1}^{r} (h_{r-s} + g_{r-s}) \int_{0}^{t} (\|\dot{f}\|_{s-2} + \|f\|_{0}) \, d\tau
\]

where \( \|f\|_{s-2} \) should be interpreted as \( 0 \) if \( s = 1 \).

Proof. Let us first prove (7.11). Let \( E_r \) be as in Lemma 7.2 and let

\[
\tilde{E}_r^2 = E_r^2 + \int_{\Omega} \psi^2 \kappa dy + \sum_{s=0}^{r-1} \int_{\Omega} g_{ab}(\dot{D}_t^s F_1^a)(\dot{D}_t^s F_1^b) \kappa dy
\]

Then it follows that \( \tilde{E}_r \) satisfy the same inequality as \( E_r \) in Lemma 7.2 does, even without the last term in (7.19) since the norm of \( \psi \) is included in \( \tilde{E}_1 \):

\[
\frac{d\tilde{E}_r}{dt} \leq K'_2 \sum_{s=1}^{r} (h_{r-s} + g_{r-s}) (\tilde{E}_s + \|f_1\|_{0,s} + \|f_1\|_{0,s-1})
\]

The highest order energy in the right hand side \( \tilde{E}_s \) with \( s = r \) can be removed by multiplying with the integrating factor \( e^{Kt} \) and integrating. We get

\[
\tilde{E}_r \leq K'_2 \sum_{s=1}^{r} (h_{r-s} + g_{r-s}) \int_{0}^{t} (\|f_1\|_{0,s-1} + \|F_1\|_{0,s}) \, d\tau + K'_2 \sum_{s=1}^{r-1} (h_{r-s} + g_{r-s}) \int_{0}^{t} \tilde{E}_s \, d\tau
\]

We claim that if \( r \geq 1 \)

\[
\tilde{E}_r \leq K'_2 \sum_{s=1}^{r} (h_{r-s} + g_{r-s}) \int_{0}^{t} (\|f_1\|_{0,s-1} + \|F_1\|_{0,s}) \, d\tau
\]

If \( r = 1 \), we have just proven it in (7.16) and in general it follows from (7.16) using induction an interpolation.

We claim that (7.11) follows from (7.17). In fact \( \|\dot{D}_t^{r-1}(g^{ab}\partial_b\psi)\| \leq C \tilde{E}_r \) and by Lemma 6.3 \( \dot{D}_t^{r-1}(g^{ab}\partial_b\psi) = g^{ab}\partial_b D_t^{r-1}\psi + \sum_{s=0}^{r-2} (r-1) g^{ab}(\dot{D}_t^{s-1}g_{ab})(\dot{D}_t^{r-1}g_{ab}) \). If follows that \( \|\partial D_t^{r-1}\psi\| \leq \sum_{i=1}^{r-1} g_{r-s} \tilde{E}_s \) which together with (7.16) and interpolation prove (7.11).

Finally, (7.12) follows from (7.11) and Lemma 7.3. and (7.13) follows from (7.12) using that \( \|f\|_{s-2} \leq \int_{0}^{t} \|\dot{f}\|_{s-2} \, d\tau \).
Lemma 7.2. Suppose that $g^{ab}$ and $e' = e'(h)$ are smooth and satisfy (7.4) for $r \geq 1$,

\begin{equation}
E_r(t) = \left( \sum_{s=0}^{r-1} \frac{1}{2} \int_{\Omega} e'(D_t^{s+1} \psi)^2 + g_{ab}(\dot{D}_t^s \Psi^a)(\dot{D}_t^s \Psi^b) \kappa dy \right)^{1/2}, \quad \Psi^a = g^{ab} \partial_b \psi + F_1^a
\end{equation}

Then for $r \geq 1$

\begin{equation}
\frac{dE_r}{dt} \leq K_2^r \sum_{s=1}^{r} \left( |h_{r-s} + g_{r,s}|(E_s + \|F_1\|_{0,s} + \|f_1\|_{0,s-1}) + K(g_{r-1} + h_{r-1})\|\psi\|_0 \right)
\end{equation}

Proof. We will prove that $dE_r^2/dt$ is bounded by $E_r$ times the right hand side of (7.19), and (7.19) follows from this since $dE_r/dt = (dE_r^2/dt)/(2E_r)$. With the notation $\dot{D}_t = D_t + \text{div} V$ and $\ddot{D}_t = D_t - \text{div} V$ we have $\dot{D}_t(e'g^2) = (\dot{D}_t e')g^2 + 2e'g\dot{D}_t g$, for any functions $e'$ and $g$, and since also $D_t\kappa/\kappa = \text{div} V$ it follows that

\begin{equation}
\frac{dE_r^2}{dt} = \sum_{s=0}^{r-1} \int_{\Omega} \left( e'(D_t^{s+1} \psi)(D_t^{s+2} \psi) + g_{ab}(\dot{D}_t^s \Psi^a)(\dot{D}_t^s \Psi^b) \right) \kappa dy
\end{equation}

+ \frac{1}{2} \sum_{s=0}^{r-1} \int_{\Omega} \left( (\dot{D}_t e')(D_t^{s+1} \psi)^2 + (\dot{D}_t g_{ab})(\dot{D}_t^s \Psi^a)(\dot{D}_t^s \Psi^b) \right) \kappa dy.

Here the terms on the second row are bounded by $K_2^r E_r^2$. Therefore it remains to look on the terms on the first row. Since $\dot{D}_t(e') = \kappa^{-1} D_t(\kappa e')$ it follows that

\begin{equation}
e' D_t^{s+2} \psi = \dot{D}_t^s D_t^2(e' \psi) + \sum_{i=0}^{s+1} B_i^s D_t^i \psi,
\end{equation}

\begin{equation}
B_i^s = \sum_{j=\max(0,i-2)}^{s} \kappa^{-1} c_{ij}^s (D_t^{s-j} \kappa)(D_t^{j+2-i} e')
\end{equation}

Furthermore, using Lemma 6.3 we get, since $\Psi_a = \partial_a \psi + F_a,$

\begin{equation}
\dot{D}_t^{s+1} \Psi^a = \dot{D}_t^a \partial_a D_t^{s+1} \psi + g_{ab} D_t^{s+1} E_b - \sum_{i=0}^{s+1} \left( \begin{array}{c} s+1 \\ i \end{array} \right) g^{ab}(\dot{D}_t^{s+1-i} g_{bc})\dot{D}_t^i \Psi^c
\end{equation}

Since $e' = e'(h)$ it follows that the $L^2$ norm of the sums in (7.21) and (7.22) are bounded by $K_2^r \sum_{s=0}^{r} (g_{r-s} + h_{r-s}) E_s$, which is included in the right hand side of (7.19) and so is $\|g^{ab} D_t^{s+1} E_a\|$. Therefore it remains to consider

\begin{equation}
\sum_{s=0}^{r-1} \int_{\Omega} \left( (D_t^{s+1} \psi)(\dot{D}_t^s D_t^2(e' \psi) + (\dot{D}_t^s \Psi^a)(\partial_a D_t^{s+1} \psi)) \right) \kappa dy
\end{equation}

\begin{equation}
= \sum_{s=0}^{r-1} \int_{\Omega} \left( (D_t^{s+1} \psi)(\dot{D}_t^s D_t^2(e' \psi) - \kappa^{-1} \partial_a (\kappa \dot{D}_t^s \Psi^a)) \right) \kappa dy = \sum_{s=0}^{r-1} \int_{\Omega} \left( (D_t^{s+1} \psi)(\dot{D}_t^s f_1) \kappa dy
\right)
\end{equation}

Here we have integrated by parts using that $D_t^{s+1} \psi |_{\partial \Omega} = 0$, that $\dot{D}_t^2(e' \psi) - \kappa^{-1} \partial_a (\kappa \Psi^a) = f_1$ and that $\dot{D}_t^s(\kappa^{-1} \partial_a (\kappa \Psi^a)) = \kappa^{-1} \partial_a (\kappa \dot{D}_t^s \Psi^a)$. \hfill \Box

One can get additional space regularity from taking time derivatives of the equation (7.1) and solving the Dirichlet problem for the Laplacian.
Lemma 7.3. If \( r \geq 1 \) then
\[
\|\psi\|_r \leq K_2^r \sum_{i=1}^{r} (g_{r-i} + h_{r-i})(\|\psi\|_0, i + \|\psi\|_1, i-1 + \|f\|_{i-2})
\]
where \( \|f\|_{i-2} \) is to be interpreted as 0 if \( i = 1 \).

Proof. We can use the estimates for the Dirichlet problem:
\[
\Delta \psi = D_t^2 (e' \psi) - f, \quad \psi|_{\partial \Omega} = 0
\]
from Theorem 6.1, since also \( e' = e'(h) \).
\[
\|\psi\|_{r-s, s} \leq K_2^r \sum_{i=r-s-2, j \leq s+2} (g_{r-i-j} + h_{r-i-j})(\|\psi\|_{i, j} + \|f\|_{i, j-2})
\]
where \( \|f\|_{i, j-2} \) is to be interpreted as 0 if \( j - 2 < 0 \). Hence using induction and interpolation we get
\[
\|\psi\|_r \leq K_2^r \sum_{i=1}^{r} (g_{r-i} + h_{r-i})(\|\psi\|_0, i + \|\psi\|_1, i-1 + \|f\|_{i-2})
\]
where \( \|\psi\|_1, i-1 \) is to be interpreted as 0 if \( i = 0 \) and \( \|f\|_{i-2} \) is to be interpreted as 0 if \( i - 2 < 0 \).

Theorem 7.4. Suppose that \( \phi \) is a solution of
\[
D_t^i \phi - \Delta(p' \phi) = f, \quad \phi|_{t=0} = \dot{\phi}|_{t=0} = 0.
\]
where \( p' = p'(h) = 1/e'(h) \) and \( f \) vanishes to all orders as \( t \to 0 \); \( D_t^k f|_{t=0} = 0, \) for \( k \geq 0 \). Let
\[
W_1 = \nabla q, \quad \Delta q = \phi, \quad q|_{\partial \Omega} = 0
\]
and
\[
F_1 = \nabla q', \quad \Delta q' = f, \quad q'|_{\partial \Omega} = 0
\]
Then for \( r \geq 1 \)
\[
\|W\|_{r+1} \leq K_2^r \sum_{s=1}^{r} (h_{r-s} + g_{r-s}) \int_0^t (\|F_1\|_{s-1} + \|F_1\|_0) \, d\tau
\]
Proof. By Theorem 7.1
\[
\|\phi\|_r \leq K_2^r \sum_{s=1}^{r} (h_{r-s} + g_{r-s}) \int_0^t (\|F_1\|_{s-1} + \|F_1\|_0) \, d\tau
\]
which by inverting the Laplacian in (7.29) proves that \( \|\partial W_1\|_r \) is bounded by the right hand side of (7.31) and it only remains to prove the estimate for \( \|D_t^{r+1} W_1\| \). Using (2.54) we can write (7.28) as
\[
\dot{D}_t^2 \phi - 2\sigma \dot{D}_t \phi + k'' \phi - \Delta(p' \phi) = f, \quad k'' = \sigma^2 - \ddot{\sigma}
\]
Proof. By Sobolev’s lemma, we estimate the right-hand side of (7.1) as follows:

\[
\text{div} (\tilde{W}_1 - \nabla (p' \text{ div } W_1) - 2\tilde{\sigma}\tilde{W}_1 + \kappa''W_1 - F_1) = -2(\partial_a\tilde{\sigma})\tilde{W}_1^a + (\partial_a\kappa'')W_1^a
\]

and hence

\[
\text{div} (\tilde{D}_t^j\tilde{W}_1 - \tilde{D}_t^j\nabla (p' \text{ div } W_1) - \tilde{D}_t^j(2\tilde{\sigma}\tilde{W}_1 - k'W_1 + F_1)) = -\tilde{D}_t^j(2(\partial_a\tilde{\sigma})\tilde{W}_1^a - (\partial_a\kappa'')W_1^a)
\]

We claim that

\[
P\tilde{D}_t^jW_1 = PB_j(W_1, ..., \tilde{D}_t^{j-1}W_1), \quad B_j(W_1, ..., \tilde{D}_t^{j-1}W_1) = -\sum_{i=0}^{j-1} \left(\begin{array}{c} j \\ i \end{array}\right) (\tilde{D}_t^{j-i}g_{ab})\tilde{D}_t^iW_1
\]

In fact, \(0 = PD_t^j\partial_aq = PD_t^j(g_{ab}W_1^b) = P\sum_i \left(\begin{array}{c} j \\ i \end{array}\right) (\tilde{D}_t^{j-i}g_{ab})\tilde{D}_t^iW_1\). Furthermore, let

\[
\triangle q^j = -\tilde{D}_t^j(2(\partial_a\tilde{\sigma})\tilde{W}_1^a - (\partial_a\kappa'')W_1^a), \quad q^j|_{\partial\Omega} = 0
\]

To say that \(\text{div } H = 0\) is equivalent to saying that \((I - P)H = 0\) so it follows from (7.35)-(7.37) that

\[
\tilde{D}_t^jW_1 = PB_j(W_1, ..., \tilde{D}_t^{j-1}W_1) + (I - P)\tilde{D}_t^{j-2}(-\nabla (p' \text{ div } W_1) + 2\tilde{\sigma}\tilde{W}_1 - k''W_1 + F_1) + \nabla q^{j-2}
\]

Here

\[
\tilde{D}_t^{j-2}(\nabla^a (p' \text{ div } W_1)) = \sum_{i=0}^{j-2} \sum_{m=0}^{i} \left(\begin{array}{c} j-2 \\ i-m \end{array}\right) \tilde{D}_t^{j-2-i}g^{ab}\partial_b((D_t^{j-m} p'(h))(D_t^m \text{ div } W_1))
\]

and by interpolation

\[
\|\tilde{D}_t^{j-2}(\nabla^a (p' \text{ div } W_1))\| \leq K'_2 \sum_{s=0}^{j-2+1} (g_{j-2-s} + h_{j-3-s})\|\text{div } W_1\|_s
\]

Since the projection \(P\) maps \(L^2\) to \(L^2\) and since inverting (7.37) maps \(L^2\) to \(H^1\) (in fact to \(H^2\)) it therefore follows that

\[
\|D_t^{r+1}W_1\| \leq K'_2 \sum_{s=0}^{r} (g_{r-s}||D_t^sW_1|| + (g_{r-1-s} + h_{r-2-s})||\text{div } W_1||_s) + K'_2 \sum_{s=0}^{r-1} g_{r-1-s}||D_t^sF_1||
\]

(7.31) follows from this if we use (7.32) to estimate \(\phi = \text{div } W_1\) and use that \(||D_t^rF_1|| \leq \int_0^t ||D_t^r\tilde{F}_1|| d\tau\). \(\square\)

Corollary 7.5. With assumptions as in Theorem 7.1 we have

\[
\|\psi\|_{r,\infty} \leq K'_2((g_{r+r_0-1} + h_{r+r_0-1})\|f\|_{0,\infty} + \|f\|_{r+r_0-1,\infty})
\]

where \(r_0 = [n/2] + 1\).

Proof. By Sobolev’s lemma \(\|\psi\|_{r,\infty} \leq C\|\psi\|_{r+r_0}\), which in turn can be estimated by Theorem 7.1. Then we estimate the \(L^2\) norms of \(f\) by \(L^\infty\) norms and use interpolation. \(\square\)

Let us now prove that the solution of (7.1) depends smoothly on parameters if the metric \(g\) and the inhomogeneous term \(f\) do. We have:
Lemma 7.6. Suppose that $f \in C^m\left(\mathbb{B}^k, C_0^\infty([0,T] \times \overline{\Omega})\right)$, $g^{ab} \in C^m\left(\mathbb{B}^k, C^\infty([0,T] \times \overline{\Omega})\right)$, where $\mathbb{B} = \{r \in \mathbb{R}^k, |r| \leq \varepsilon\}$. Suppose also that $g$ satisfies the coordinate condition uniformly in $\mathbb{B}^k$. Let $\psi$ be the solution of

$$(7.43) \quad \square \psi = c^2 D_t^2 \psi - \triangle \psi = f, \quad \psi|_{\partial \Omega} = 0 \quad \psi|_{t=0} = D_t \psi|_{t=0} = 0$$

where $\triangle$ is given by (7.3). Then $\psi \in C^m\left(\mathbb{B}^k, C_0^\infty([0,T] \times \overline{\Omega})\right)$.

Proof. We note that it suffices to prove the statement in the theorem for $m = 1$, since the general case follows from this by induction. In fact, assuming that $\psi \in C^l\left(\mathbb{B}^k, C_0^\infty([0,T] \times \overline{\Omega})\right)$, $l < m$ then for $|\alpha| \leq l$:

$$(7.44) \quad \square D^\alpha_r \psi = D^\alpha_r f - \sum_{\beta + \gamma = \alpha, |\beta| < |\alpha|} c^\alpha_{\beta \gamma} \square \gamma D^\beta_r \psi,$$

where $\gamma = (\gamma_j, ... \gamma_1)$ are ordered multi-indices with $\gamma_i \in \{1, ..., k\}$. Here $D^\gamma_r = D_{r\gamma} \cdots D_{r\gamma_1}$, where $D_{r\gamma} = \partial / \partial r^\gamma$, and $\square \gamma$ are the repeated commutators defined inductively by $\square^{(\alpha, \gamma)} = [D_r, \square \gamma]$, where $\square^{(\alpha)} = \square \alpha$. The right hand side of (7.44) is then in $C^1\left(\mathbb{B}^k, C_0^\infty([0,T] \times \overline{\Omega})\right)$ so it follows from the statement in the theorem for $m = 1$ that $\psi \in C^{l+1}\left(\mathbb{B}^k, C_0^\infty([0,T] \times \overline{\Omega})\right)$.

Let us write $\psi_r$, $g_r$, $f_r$, and $\square_r$, to indicate the dependence of $r$. First we will prove that $f_r \in C(\mathbb{B}^k, C_0^\infty)$ and $g_r \in C(\mathbb{B}^k, C^\infty)$ implies that $\psi_r \in C(\mathbb{B}^k, C_0^\infty)$. We will only prove this for $r = 0$ since the proof in general is just a notational difference from the proof for $r = 0$. We have

$$(7.45) \quad \square_r (\psi_r - \psi_0) = f_r - f_0 - (\square_r - \square_0) \psi_0$$

The right hand side is in $C(\mathbb{B}^k, C_0^\infty)$ and tends to 0 in $C(\mathbb{B}^k, C^N)$, for any $N$, as $r \to 0$. Since in Corollary 7.5 we have uniform bounds for $\square^{-1}_r$, it follows that $\psi_r - \psi_0$ tends to 0 in $C(\mathbb{B}^k, C^N)$, for any $N$, as $r \to 0$, i.e. $\psi_r \in C(\mathbb{B}^k, C_0^\infty)$.

Let us now assume that $f_r \in C^1(\mathbb{B}^k, C_0^\infty)$ and $g_r \in C^1(\mathbb{B}^k, C^\infty)$. Let $\dot{\psi}_r = D_r \psi_r$ be defined by

$$(7.46) \quad \square_r \dot{\psi}_r = \dot{f}_r - \square_r \psi_r, \quad \dot{\psi}_r|_{\partial \Omega} = 0, \quad \dot{\psi}_r|_{t=0} = D_t \psi_r|_{t=0} = 0,$$

where $\square_r = [D_r, \square]$, $\dot{f}_r = D_r f_r$ and $D_r = (D_{r1}, ..., D_{rk})$. Since the right hand side of (7.46) is in $C(\mathbb{B}^k, C_0^\infty)$ it follows as above that $\dot{\psi}_r \in C(\mathbb{B}^k, C_0^\infty)$. It remains to prove that $\dot{\psi}_r$ is differentiable. We have

$$(7.47) \quad \square_r (\psi_r - \psi_0 - \dot{\psi}_0) = f_r - f_0 - r \dot{f}_0 + (\square_r - \square_0 - r \triangle_r) \psi_0 + r (\square_r - \square_0) \dot{\psi}_0$$

Then the right hand side divided by $r$ tends to 0 in $C^N$, for any $N$, as $r \to 0$. In view of the uniform bounds for $\square^{-1}_r$ in Corollary 7.5 it follows that $(\psi_r - \psi_0 - \dot{\psi}_0)/r$, tends to 0 in $C^N$ for any $N$, as $r \to 0$. Hence $\dot{\psi} \in C^1(\mathbb{B}^k, C_0^\infty)$. □

8. Tame estimates for the divergence free equation.

Let

$$(8.1) \quad \dot{W} + AW + B_0 W + B_1 \dot{W} = H, \quad \text{div} H = 0, \quad W|_{t=0} = \dot{W}|_{t=0} = 0$$
where \( H \) is smooth and vanishes to order \( r \) as \( t \to 0 \), i.e. \( D_t^k H|_{t=0} = 0 \), for \( k \leq r \). Here \( B_i W^a = P(g^{ab} \beta_{bc} W^c) \), \( i = 0, 1 \), are projected multiplication operators. Let

\[
\begin{align*}
(8.2) & \quad g_s = \|g\|_{s+1, \infty}, \quad h_s = \|h\|_{s+2, \infty}, \quad \beta_s = \|\beta^0\|_{s, \infty} + \|\beta^1\|_{s, \infty}, \quad k_s = g_s + h_s + \beta_s
\end{align*}
\]

and let \( K_2'' \) denote a continuous function of

\[
\begin{align*}
(8.3) & \quad k_0 + c_0^{-1} + c_1 + T^{-1} + r,
\end{align*}
\]

which in what follows also depends on the order \( r \) of differentiation. We will use that by interpolation;

\[
(k_s k_r) \leq K_2'' k_{r+s}, \quad \text{for } r, s \geq 0.
\]

We will prove the following estimates

**Theorem 8.1.** Suppose that \((2.4)-(2.5)\) hold for \(0 \leq t \leq T\) and suppose also that \( x \) is smooth for \(0 \leq t \leq T\), \( T \leq 1\). Then \((8.1)\) has a smooth solution for \(0 \leq t \leq T\). It satisfies the estimates

\[
(8.4) \quad \|\dot{W}\|_r + \|W\|_r \leq K_2'' \sum_{s=0}^r k_{r-s} \int_0^t \|H\|_s \, d\tau,
\]

for \( r \geq 0 \) and for \( m = 0, 1, 2 \).

\[
(8.5) \quad \sum_{j=0}^{m+1} \|\dot{D}_t^j W\|_r \leq K_2'' \sum_{s=0}^r \sum_{j=0}^m k_{r+m-j-s} \int_0^t \|\dot{D}_t^j H\|_s \, d\tau,
\]

Furthermore, for \( r \geq 1 \),

\[
(8.6) \quad \|\ddot{W}\|_{r-1} + \|\dot{W}\|_{r-1} + \|\dot{W}\|_r + \|W\|_r
\]

\[
\leq K_2'' \sum_{s=0}^r (k_{r-1-s} \|\ddot{H}\|_s + k_{r-s} \|\dot{H}\|_s + k_{r+1-s} \|H\|_s + \|\text{curl}H\|_s) \, d\tau.
\]

As pointed out before, existence of smooth solutions for \((8.1)\) was proven in [L3] so we only need to prove the estimates. The theorem with the norms replaced by norms with just space derivatives was proven in [L3]. The theorem here is actually simpler to prove than the one there.

First we note that we can reduce to the case \( B_0 = B_1 = 0 \) since the general case follows from this case. Let us show this for \((8.4)\). Assume that \((8.4)\) holds for the case when \( B_0 = B_1 = 0 \). Then using

\[
(8.4) \text{ applied to the equation } \dot{W} + AW = H_1 = H = B_0 W + B_1 \dot{W},
\]

using that, by Theorem 6.1, 

\[
\|B_1 W\|_s \leq K_2'' \sum_{k=0}^s (\beta_{s-k} + g_{s-k}) \|W\|_k
\]

and \((8.4)\),

\[
(8.7) \quad \|\dot{W}\|_r + \|W\|_r \leq K_2'' \sum_{s=0}^r (g_{r-s} + h_{r-s} + \beta_{r-s}) \int_0^t (\|H\|_s + \|W\|_s + \|\dot{W}\|_s) \, d\tau
\]

We can now first remove the highest order terms from the right hand side by a Grönwall type of argument since they occur in the left. In fact let \( f(t) = \int_0^t \|W\|_r + \|\dot{W}\|_r \, d\tau \). Then by \((8.7)\) \( f' \leq K_2'' f + K_2'' \|H\|_r + \Sigma_{r-1} \), where \( \Sigma_{r-1} \) is the sum of the first \( r - 1 \) terms in the right of \((8.7)\). Hence
multiplying by the integrating factor $e^{K_2'' t}$ and integrating gives $f(t) \leq K_2'' \int_0^t \|H\|_r + \sum_{r-1} d\tau$. It follows that

\[(8.8) \quad \|\dot{W}\|_r + \|W\|_r \leq K_2'' \sum_{s=0}^{r} (g_{r-s} + h_{r-s} + \beta_{r-s}) \int_0^t \|H\|_s d\tau + K_2'' \sum_{s=0}^{r-1} (g_{r-s} + h_{r-s} + \beta_{r-s}) \int_0^t (\|W\|_s + \|\dot{W}\|_s), d\tau\]

This proves (8.4) for $r = 0$. (8.4) for $r \geq 1$ now follows by induction, using (8.4) for $s \leq r - 1$ in the terms on the second row of (8.8) together with the interpolation (8.4). The proof of that we can reduce to the case $B_0 = B_1 = 0$ also for (8.5) follows in a the same way, using that we have already proven the estimate for smaller $m$. The prove that (8.6) can be reduced to the case $B_0 = B_1 = 0$, we also estimate the lower order terms using (8.5).

We now in what follows assume that $B_0 = B_1 = 0$. Of course we could have included these operators in the calculations that follows but the argument becomes more clear without them. Lowering the indices in (8.1):

\[(8.9) \quad G\dot{W} + AW = GH\]

Let $L_T^I$, $I \in T$, stand for a product of Lie derivatives of $|I|$ vector fields in $T$ and let $W_I = \hat{L}_T^I W$. If we apply repeatedly apply Lie derivatives $L_T$ and the projection in between, see section 4, we obtain

\[(8.10) \quad e^{I_1 I_2} (G_{I_1} \dot{W}_{I_2} + A_{I_1} W_{I_2} - G_{I_1} H_{I_2}) = 0\]

where the sum is over all combination of $I_1 + I_2 = I$ and $e^{I_1 I_2} = 1$. Here $G_I$ and $A_I$ are the operators given by (4.39) and (4.40). If we raise the indices again we get

\[(8.11) \quad \dot{W}_I + AW_I = -\tilde{c}^{I_1 I_2} A_{I_1} W_{I_2} - \tilde{c}^{I_1 I_2} G_{I_1} \dot{W}_{I_2} + c^{I_1 I_2} G_{I_1} H_{I_2}\]

where $\tilde{c}^{I_1 I_2} = 1$, if $|I_2| < |I|$, and $\tilde{c}^{I_1 I_2} = 0$ if $|I_2| = |I|$.

Let us define energies

\[(8.12) \quad E_I = \langle \dot{W}_I, \dot{W}_I \rangle + \langle W_I, (A + I) W_I \rangle, \quad E_I^T = \sum_{|I| \leq s, I \in T} \sqrt{E_I}\]

Note that in the sum we also included all time derivatives $\hat{L}_D$. The reason for this is that when calculating commutators second order time derivatives show up in the first term on the right in (8.10). We get by differentiating (8.11), see the end of section 3,

\[(8.13) \quad \dot{E}_I = 2\langle \dot{W}_I, \dot{W}_I \rangle + (A + I) W_I + \langle \dot{W}_I, G W_I \rangle + \langle W_I, (\dot{A} + \dot{G}) W_I \rangle\]

We now, want to estimate the right hand side by $E_r^T$, where $r = |I|$. Here, by (3.27), the last two terms can be bounded by $(h_0 c_0^{-1} + \theta_0) E_I$. Therefore, (8.12) can be estimated by the $L^2$ norm of the right hand side of (8.10). In the sums with $c^{I_1 I_2}$ in (8.9), we have $|I_2| < |I|$. Since we included time derivatives up to highest orders in $E_r^T$ it follows that $\|\dot{W}_{I_2}\| \leq E_r^T$. Here $G_{I_1}$ is a bounded operator so the term with $G_{I_1}$ can be controlled. $A_{I_1}$ is an operator of order one so the term with $A_{I_1}$ can be controlled by a constant times $\|\partial W_{I_2}\| + \|W_{I_2}\|$. However, at this point we only have control of tangential derivatives.
of $W$. One could combine the estimate here with the curl estimate given later to get control over all
derivatives up to order $r = |I|$. Instead we will add a lower order term to the energy such that the
its time derivative cancels the terms with $A_I$ and replaces them with lower order terms that may be
controlled. Let

$$D_I = 2\varepsilon^{I_1 I_2} \langle W_I, A_I W_{I_2} \rangle$$

where the sum is over all $I_1 + I_2 = I$, with $|I_2| < |I|$. Then

$$\dot{D}_I = 2\varepsilon^{I_1 I_2} (\langle \dot{W}_I, A_I W_{I_2} \rangle + \langle W_I, \dot{A}_I W_{I_2} \rangle + \langle W_I, A_I \dot{W}_{I_2} \rangle).$$

Hence

$$\dot{E}_I + \dot{D}_I = 2\varepsilon^{I_1 I_2} (\langle W_I, \dot{A}_I W_{I_2} \rangle + \langle W_I, A_I \dot{W}_{I_2} \rangle) + \langle W_I, \dot{A} W_I \rangle + 2\langle \dot{W}_I, -\varepsilon^{I_1 I_2} G_I \dot{W}_{I_2} + \varepsilon^{I_1 I_2} G_I H_{I_2} + W_I \rangle + \langle \dot{W}_I, \dot{G} W_I \rangle + \langle W_I, \dot{G} W_I \rangle$$

Here, the terms on the first row can be controlled using (3.7) and the terms on the terms on the second
row can be controlled using (3.10). We get

$$|\langle U, A_J W \rangle| \leq ||h||_{s+1,\infty} c_0^{-1} ||U||_{s+1}^{1/2} ||W||_{s+1}^{1/2}, \quad |\langle U, G_J W \rangle| \leq ||g||_{s,\infty} ||U|| ||W||, \quad |J| = s$$

Hence, we have proven that

$$|\dot{E}_I + \dot{D}_I| \leq CE_r \varepsilon^{I_1 I_2} \sum_{s=0}^{r} (h_{r-s} c_0^{-1} + g_{r-s}) (E_s + ||H||_s^2), \quad |D_I| \leq CE_r \varepsilon^{I_1 I_2} \sum_{s=0}^{r-1} h_{r-s} c_0^{-1} E_s^T$$

so it follows that

$$E_r^T \leq C(h_0 c_0^{-1} + g_0) \int_0^t (E_r^T + ||H||_r) dt + C \sum_{s=0}^{r-1} (h_{r-s} c_0^{-1} + g_{r-s}) \left( \int_0^t (E_s^T + ||H||_s^2) dt + E_s^T \right)$$

Using induction and interpolation (8.4) we get the same inequality without $E_s^T$, for $s \leq r - 1$, but
instead multiplied by a constant $K'_2$ depending on (8.3) and $r$. Then, by a Grönwall type of argument,
see the beginning of the proof, we can also remove $E_r^T$ from the right hand side of (8.17), i.e., with
$f(t) = \int_0^t E_r^T dt$, we get and inequality $f' \leq K'_2 f + K'_2 \sum_{s=0}^r (h_{r-s} + g_{r-s}) ||H||_s^2$ and multiplying
by the integrating factor $e^{K'_2 t}$ gives:

**Lemma 8.2.** For $r \geq 0$ we have

$$E_r^T \leq K'_2 \sum_{s=0}^r (h_{r-s} + g_{r-s}) \int_0^t ||H||_s^2 dt$$

This proves the estimates for tangential derivatives. We now also want to have estimate for the curl
and then by the results in section 5 the estimates for all derivatives will follow from this.

Let $w_a = g_{ab} W^b$ and $\dot{w}_a = \dot{g}_{ab} W^b$ and let $\text{curl} w_{ab} = \partial_a w_b - \partial_b w_a$. Then $D_t w_a = \dot{w}_a + \dot{g}_{ab} W^b$,
where $\dot{g}_{ab} = \dot{D}_t g_{ab}$ and $\dot{W}^b = \dot{D}_t W^b$. It follows that $D_t \text{curl} w_{ab} = \text{curl} \dot{w}_{ab} + \partial_a (\dot{g}_{bc} W^c) - \partial_b (g_{ac} W^c)$.
Since the curl of \( A \) vanishes it follows that \( \text{curl} \, \dot{w} = \text{curl} \, H \), if \( \dot{w} = g_{ab} \dddot{W} \). Since the curl commutes with the Lie derivative it therefore follows that

\[
|D_t \text{curl} \, \dot{w}|_{r-1}^{l_t} \leq C \sum_{s=0}^{r} g_{r-s} |\dot{W}|_{s}^{l_t} + |\text{curl} \, H|_{r-1}^{l_t}
\]

(8.21)

\[
|D_t \text{curl} \, w|_{r-1}^{l_t} \leq C \sum_{s=0}^{r} g_{r-s} |W|_{s}^{l_t}
\]

(8.22)

and by Lemma 5.3:

\[
|W|_{r}^{l_t} \leq K_1 \sum_{s=1}^{r} g_{r-s} (|\text{curl} \, w|_{s}^{l_t} + |\text{div} \, W|_{s}^{l_t} + |W|_{s}^{T}) + K_1 g_{r-1} |W|_{0}
\]

(8.23)

\[
|\dot{W}|_{r}^{l_t} \leq K_1 \sum_{s=1}^{r} g_{r-s} (|\text{curl} \, \dot{w}|_{s-1}^{l_t} + |\text{div} \, \dot{W}|_{s-1}^{l_t} + |\dot{W}|_{s}^{T}) + K_1 g_{r-1} |\dot{W}|_{0}
\]

(8.24)

Let

\[
C_r^{l_t} = \| \text{curl} \, \dot{w} \|_{r-1} + \| \text{curl} \, w \|_{r-1}.
\]

(8.25)

Since \( \text{div} \, W = \text{div} \, \dot{W} = 0 \) it now follows that

\[
|\dot{C}_r^{l_t}| \leq K_1 \sum_{s=0}^{r} g_{r-s} (|\dot{W}|_{s} + |W|_{s}) + C \| \text{curl} \, H \|_{r-1} \leq K_1 \sum_{s=0}^{r} g_{r-s} (C_s^{l_t} + E_s^{T}) + C \| \text{curl} \, H \|_{r-1}.
\]

(8.26)

This together with Lemma 8.2 and the argument for its proof gives

**Lemma 8.3.** For \( r \geq 0 \) we have

\[
|\dot{W}|_{r} + |W|_{r} + E_r^{T} \leq K_2^n \sum_{s=0}^{r} (g_{r-s} + h_{r-s}) \int_{0}^{t} \|H\|_{s} \, d\tau.
\]

(8.27)

This proves the first part of Theorem 8.1.

In order to prove the second part of Theorem 8.1 we replace the energy in (8.10) by

\[
E_{r,m}^{T} = \sum_{I_1+I_2 \in T, |I_1| \leq r, |I_2| = m} \sqrt{E_I}
\]

(8.28)

i.e. we take two additional time derivatives. Noting that, the argument leading up to Lemma 8.2 only requires that we have at least as many time derivatives as tangential space derivatives so it follows from its proof that

**Lemma 8.4.** For \( r \geq 0 \) we have

\[
E_{r,m}^{T} \leq K_2^n \sum_{s=0}^{r} \sum_{j=0}^{m} k_{r+m-j-s} \int_{0}^{t} \|\hat{D}_i H\|_{s}^{T} \, d\tau
\]

(8.29)
We have $\tilde{w}_a = -A_a W + H_a$ and $\tilde{w}_b = g_{ab} D_t \tilde{W}^b = D_t \tilde{w}_a - (D_t g_{ab}) W^b$ so, since $\text{curl} A W = 0$, $\text{curl} \tilde{w} = \text{curl} H$ and $\text{curl} \tilde{w}_{ab} = \text{curl}(D_t H)_{ab} - \partial_a (D_t g_{bc}) W^b + \partial_b ((D_t g_{ac}) W^c)$. Hence $|\text{curl} \tilde{w}|_{s-1} = |\text{curl} H|_{s-1}$ and $|\text{curl} \tilde{w}|_{s-1} \leq K_1 \sum_{j=0}^{s} g_{s-j} |W|_j + |D_t \text{curl} H|_{s-1}$ so using the estimates (8.23)-(8.23) with $(W, \tilde{W})$ replaced by $(\tilde{W}, \tilde{W})$ gives:

\begin{equation}
|\tilde{W}|^T_r \leq K_1 \sum_{s=0}^{r} g_{r-1-s} |\tilde{W}|^T_s + K_1 \sum_{s=1}^{r} g_{r-1-s} |\text{curl} H|_{s-1}
\end{equation}

\begin{equation}
|\tilde{W}|^T_r \leq K_1 \sum_{s=0}^{r} g_{r-1-s} (|W|^T_s + |\tilde{W}|^T_s) + K_1 \sum_{s=1}^{r} g_{r-1-s} |D_t \text{curl} H|_{s-1}
\end{equation}

Using the estimate in Lemma 8.3, with $r$ replaced by $r-1$, the estimate in Lemma 8.4, the estimate $\|D_t^k \text{curl} H\|_{s-1} \leq \int_0^t \|D_t^{k+1} \text{curl} H\|_{s-1} d\tau$, and interpolation gives:

**Lemma 8.5.** For $r \geq 0$ and $m = 0, 1, 2$, we have

\begin{equation}
\sum_{j=0}^{m+1} \|D_t^j W\|_r + E_{r,m} \leq K_2' \sum_{s=0}^{r} k_{r+s-m-j} \int_0^t \|D_t^j H\|_s d\tau.
\end{equation}

We now want to use the estimate for the additional time derivatives to get estimates for $\|W\|_r$ and $\|\tilde{W}\|_r$ in (8.6) since by Lemma 5.5:

\begin{equation}
c_0 \|W\|_r \leq K_2'' (\|\text{curl} w\|_{r-1} + \|\text{div} W\|_{r-1} + \|W\|_{r-1}^T + \|\tilde{W}\|_{r-1}^T + \sum_{s=0}^{r-1} g_{r-1-s} \|W\|_s)
\end{equation}

\begin{equation}
c_0 \|\tilde{W}\|_s \leq K_2'' (\|\text{curl} \tilde{w}\|_{r-1} + \|\text{div} \tilde{W}\|_{r-1} + \|\tilde{W}\|_{r-1}^T + \|\tilde{W}\|_{r-1}^T + \sum_{s=0}^{r-1} g_{r-1-s} \|\tilde{W}\|_s)
\end{equation}

where

\begin{equation}
\|W\|_{r,A}^T = \sum_{|I|=s, I \in \mathcal{T}} \|A \hat{L}_I^T W\|
\end{equation}

The terms of order $r-1$ or less in (8.33)-(8.34) can be estimated by Lemma 8.3 and Lemma 8.5 with $r$ replaced by $r-1$. Therefore it only remains to estimate the terms involving the curl and the operator $A$. The estimate for $\|A W_J\|$ with $|J| \leq r-1$ we get from (8.11);

\begin{equation}
A W_J = -c^{J_1 J_2} A_{J_1} W_{J_2} - c^{J_1 J_2} (G_{J_1} \tilde{W}_{J_2} - G_{J_1} H_{J_2})
\end{equation}

where $c^{J_1 J_2} = 0$ if $|J_2| = r-1$. Here the terms in the parenthesis can be estimated by Lemma 8.5, with $r$ replaced by $r-1$. Note that $\|H_{J_2}\| \leq \int_0^t \|\tilde{H}_{J_2}\| d\tau$ since we assume that $H$ vanishes to all orders as $t \to 0$. Since by (3.8) $A_{J_1}$ is order one and $|J_2| \leq r-2$ in the first term on the right it follows that also this term can be estimated by Lemma 8.3 with $r$ replaced by $r-1$. We hence get

\begin{equation}
\|W\|_{r-1,A}^T \leq K_2'' \sum_{s=0}^{r-1} \int_0^t (k_{r-1-s} \|\tilde{H}\|_s + k_{r-s} \|\tilde{H}\|_s + k_{r+1-s} \|H\|_s) d\tau
\end{equation}

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From (8.9) we also get the estimate for $\|A\dot{W}_J\|$ with $|J| \leq r - 1$ by letting one of the derivatives in (8.9) be a time derivative so $I = J + \{D_t\}$. If we write this out we get

$$A\dot{W}_J = -\tilde{c}^{J_1J_2}A_{J_1}\dot{W}_{J_2} - c^{J_1J_2}\dot{A}_{J_1}W_{J_2} - c^{J_1J_2}(G_{J_1}\dot{W}_{J_2} + \dot{G}_{J_1}\dot{W}_{J_2} - G_{J_1}H_{J_2} - \dot{G}_{J_1}H_{J_2})$$

where $\tilde{c}^{J_1J_2} = 0$ if $|J_2| = r - 1$. Here the terms in the parenthesis can be estimated by Lemma 8.3, with $r$ replaced by $r - 1$. Since $A_{J_1}$ and $\dot{A}_{J_1}$ is order 1, see (3.8), we can also estimate all the other by Lemma 8.3 with $r$ replaced by $r - 1$, apart from the term with $A\dot{W}_J$. This term is estimated by $\|A\dot{W}_J\| \leq C_0(\|W\|_r + \|W\|_{r-1})$, where the last term can again be estimated by Lemma 8.3 with $r$ replaced by $r - 1$. We hence get

$$\|A\dot{W}_J\| \leq K_0'' \sum_{s=0}^{r-1} \int_0^t (k_{r-1-s}\|\tilde{H}\|_{s} + k_{r-s}\|H\|_{s} + k_{r+1-s}\|H\|_{s}) d\tau$$

(8.39) and (8.4) together with (8.37) and (8.39) therefore gives

$$c_0\|W\|_r \leq K_0'' \|\text{curl } w\|_{r-1} + K_0'' \sum_{s=0}^{r-1} \int_0^t (k_{r-1-s}\|\tilde{H}\|_{s} + k_{r-s}\|H\|_{s} + k_{r+1-s}\|H\|_{s}) d\tau$$

(8.40) and

$$c_0\|\dot{W}\|_r \leq h_0c_0^{-1}\|\text{curl } w\|_{r-1} + K_0'' \|\text{curl } w\|_{r-1} + K_0'' \sum_{s=0}^{r-1} \int_0^t (k_{r-1-s}\|\tilde{H}\|_{s} + k_{r-s}\|H\|_{s} + k_{r+1-s}\|H\|_{s}) d\tau$$

(8.41)

It therefore only remains to control the curl. With $C_r^{\text{curl}} = \|\text{curl } w\|_{r-1} + \|\text{curl } w\|_{r-1}$ it hence follows from (8.26) and (8.40)-(8.41) that

$$|\dot{C}_r^{\text{curl}}| \leq K_0'' C_r^{\text{curl}} + K_0'' \sum_{s=0}^{r-1} \int_0^t (k_{r-1-s}\|\tilde{H}\|_{s} + k_{r-s}\|H\|_{s} + k_{r+1-s}\|H\|_{s}) d\tau + \|\text{curl } H\|_{r-1}$$

(8.42)

where we also used that, by Lemma 8.3, $\|\dot{W}\|_0 + \|W\|_0 \leq K_0'' \int_0^t \|H\|_0 d\tau$. Integrating this equation gives a bound for $C_r^{\text{curl}}$ in terms of the integral in (8.38). Using this bound in (8.40)-(8.41) then gives

Lemma 8.6. For $r \geq 1$ we have

$$\|\dot{W}\|_r + \|W\|_r \leq K_0'' \sum_{s=0}^{r-1} \int_0^t (k_{r-1-s}\|\tilde{H}\|_{s} + k_{r-s}\|H\|_{s} + k_{r+1-s}\|H\|_{s}) d\tau$$

(8.43)

This concludes the proof of Theorem 8.1.

9. Existence and tame estimates for the inverse of the modified linearized operator.

We want to show existence and tame estimates for the inverse of the linearized operator. However, first we will show existence and estimates for the modified linearized operator $L_1$ given by (2.55):

$$L_1W = F, \quad W|_{t=0} = \dot{W}|_{t=0} = 0$$
where $F$ is smooth and vanishes to all orders as $t \to 0$.

Let
\begin{equation}
(9.2) \quad n_s = \|g\|_{s+2, \infty} + \|\omega\|_{s+1, \infty} + \|h\|_{s+3, \infty},
\end{equation}
and let $K_3''$ be denote a continuous function of
\begin{equation}
(9.3) \quad n_0 + c_0^{-1} + T^{-1} + c_1 + r
\end{equation}
which in what follows also depends on the order of differentiat ion $r$. In the proof that follows we will use the interpolation
\begin{equation}
(9.4) \quad n_sn_r \leq K_3''n_{s+r}
\end{equation}

**Theorem 9.1.** Suppose that (2.4)-(2.5) hold for $0 \leq t \leq T$ and suppose also that $x$ is smooth for $0 \leq t \leq T$ an that $T \leq 1$. Then (9.1), where $F$ is smooth and vanishing to all orders as $t \to 0$, has a smooth solution for $0 \leq t \leq T$. It satisfies the estimates
\begin{equation}
(9.5) \quad \|\dot{W}\|_r + \|W\|_r \leq K_3'' \sum_{s=1}^{r} n_{r-s} \int_0^t \|F\|_s \, d\tau, \quad r \geq 1
\end{equation}

**Proof of Theorem 9.1.** The proof of existence and the estimate (9.5) for the modified linearized operator uses the orthogonal decomposition of the vector field into its divergence free part and a gradient of a function vanishing on the boundary. The solution will be further divided into four parts and for each of them we use either Theorem 6.1, Theorem 7.1 or Theorem 8.1. This gives us the estimates in Theorem 9.2. These estimates imply the estimate (9.5). Furthermore, the estimates holds for iterates, i.e given an iterate for $W$ we define $\delta h = \Psi'(x)W$ by (9.10) and then define $W_{ij}$ by (9.8)-(9.20) and (9.12)-(9.13). This gives us a new iterate for $W$. Theorem 9.2 then gives us uniform bounds for the iterates and applied to the equations for differences of iterates gives us convergence, see [L2,L3].

Now, the solution $W$ of
\begin{equation}
(9.6) \quad L_1W = F
\end{equation}
can be obtained as the sum of four terms, see section 3,
\begin{equation}
(9.7) \quad W = W_0 + W_1, \quad W_1 = W_{10} + W_{11}, \quad W_0 = W_{00} + W_{01}
\end{equation}
where $W_0$ is divergence free and
\begin{equation}
(9.8) \quad W_{ai} = g^{ab}\partial_bq_{1i}, \quad i = 0, 1, \quad q_{1i}|_{\partial\Omega} = 0, \quad \triangle q_{11} = \varphi, \quad \triangle q_{10} = -e'(h)\delta h
\end{equation}
and $\varphi$ satisfies an ordinary differential equation:
\begin{equation}
(9.9) \quad D_t^2\varphi + \text{div} \Phi \varphi = \text{div} F + \text{div} \Phi e'(h)\delta h, \quad \varphi|_{t=0} = \dot{\varphi}|_{t=0} = 0
\end{equation}
where $\text{div} \Phi = D_t^2e(h) + D_t \text{div} V$ and $\delta h$ satisfies the wave equation
\begin{equation}
(9.10) \quad D_t^2(e'(h)\delta h) - \triangle \delta h = \triangle ((\delta h)W^c) - \text{div} B_1\dot{W} + 2\dot{\sigma} \text{div} \dot{W} - \text{div} B_0W - (D_t^2e(h) + \dot{\sigma}^2) \text{div} W,
\end{equation}
\begin{equation}
(9.11) \quad \delta h|_{\partial\Omega} = 0, \quad \delta h|_{t=0} = \dot{\delta h}|_{t=0} = 0.
\end{equation}
Here, the divergence free parts satisfy the evolution equation for the normal operator, (3.16)

(9.12) \( \dot{W}_{00} + AW_{00} - B_{10} W_{00} - B_{00} W_{00} = -AW_{10} + B_{11} \dot{W}_1 + B_{01} W_1 + PF, \quad W_{00}|_{t=0} = \dot{W}_{00}|_{t=0} = 0 \)

and

(9.13) \( \dot{W}_{01} + AW_{01} - B_{10} W_{01} - B_{00} W_{01} = -AW_{11}, \quad W_{01}|_{t=0} = \dot{W}_{01}|_{t=0} = 0 \)

If

(9.14) \( E_r = \sum_{i,j=0}^{1} \| \dot{W}_{ij} \|_r + \| W_{ij} \|_r + \| \delta h \|_r \)

then we get from Theorem 9.2 that for \( r \geq 1 \)

(9.15) \( E_r \leq K_3'^r \sum_{s=1}^{r} n_{r-s} \int_0^t (\| F \|_s + E_s) \, d\tau \)

Using a Grönwall type of argument and induction as in section 8 it follows that for \( r \geq 1 \) we have

(9.16) \( E_r \leq K_3'^r \sum_{s=1}^{r} n_{r-s} \int_0^t \| F \|_s \, d\tau \)

which proves Theorem 9.1. □

**Theorem 9.2.** If \( r \geq 1 \) we have

(9.17) \( \| \dot{W}_{11} \|_r + \| W_{11} \|_r \leq K_3'^r \sum_{s=1}^{r} n_{r-s} \int_0^r (\| F \|_s + \| W_{10} \|_s) \, d\tau \)

(9.18) \( \| \dot{W}_{11} \|_r \leq K_3'^r \sum_{s=1}^{r} n_{r-s} (\| F \|_s + \| W_{10} \|_s + \int_0^r \| F \|_s + \| W_{10} \|_s \, d\tau) \)

(9.19) \( \| \dot{W}_{01} \|_r + \| W_{01} \|_r \leq K_3'^r \sum_{s=1}^{r} n_{r-s} \int_0^r (\| F \|_s + \| W_{10} \|_s) \, d\tau \)

(9.20) \( \| \dot{W}_{00} \|_r + \| W_{00} \|_r \leq K_3'^r \sum_{s=1}^{r} n_{r-s} \int_0^r (\| F \|_s + \| \partial W_{10} \|_s) \, d\tau \)

(9.21) \( \| \dot{W}_{10} \|_r + \| W_{10} \|_{r+1} \leq K_3'^r \sum_{s=1}^{r} n_{r-s} \int_0^t (\| \dot{W} \|_s + \| W \|_s) \, d\tau \)

where \( W = W_0 + W_1, \quad W_1 = W_{10} + W_{11} \) and \( W_0 = W_{00} + W_{01} \).

**Proof of Theorem 9.2.** (9.17) follows directly from applying Lemma 9.3 below to (9.9). Here we write \( \text{div} \ W_1 = \text{div} \ F + \text{div} \Phi e'(h) \delta h = \text{div} \ (F - \text{div} \Phi W_{10}) + (\partial_a \text{div} \Phi) W_{10}^a \) and use Theorem 6.1. This also gives the additional estimate (9.18).

To prove (9.19) we use the second part of Theorem 8.1 applied to (9.13). Since \( \text{curl} \, AW_{11} = 0 \) and we have an estimate for and additional time derivative in (9.18), (9.19) follows.
By the first part of Theorem 8.1:

\begin{equation}
\|\dot{W}_{00}\|_r + \|W_{00}\|_r \leq K_3'' \sum_{s=0}^{r} n_{r-1-s} \int_0^r (\|F\|_s + \|\partial W_{10}\|_s + \|\dot{W}_1\|_s + \|W_1\|_s) \, d\tau
\end{equation}

Using (9.21) and (9.17) gives (9.20). Note that the sum above starts at \( s = 0 \) but since the lower norms are included in the higher norms and since we replaced \( n_{r-s-1} \) by \( n_{r-s} \) we can start the sum in (9.20) from \( s = 1 \).

The estimate (9.21) follows from Theorem 7.1 applied to (9.10). If \( f \) denotes the right hand side of (9.10) then

\begin{equation}
\|f\|_{s-2} \leq K''_3 \sum_{k=0}^{s} n_{s-1-k} (\|W\|_k + \|W\|_k)
\end{equation}

Since \( \|f\|_0 \leq K'_3 (\|W\|_2 + \|\dot{W}\|_1) \) we obtain from (7.13) for \( r \geq 2 \)

\begin{equation}
\|\delta h\|_r \leq K''_3 \sum_{s=0}^{r} n_{r-1-s} \int_0^t (\|\dot{W}\|_s + \|W\|_s) \, d\tau
\end{equation}

For \( r = 1 \) we use (7.12), which gives (9.24) also for \( r = 1 \) if we write \( f = f_1 + \triangle (h_c W^c) \). This proves that \( \|\delta h\|_r \) is bounded by the right hand side of (9.21). To get the estimate also for \( W_{10} \) we use Theorem 6.1 to estimate the solution of the Dirichlet problem in (9.8) using that \( \|e'(h)\delta h\|_r \leq K''_3 \sum_{s=0}^{r} h_{r-s,\infty} \|\delta h\|_s \). This gives that \( \|\partial W_{10}\|_r \) and \( \|W_{10}\|_r \) are bounded by the right hand side of (9.21) but it remains to show that \( \|D_r^{c+1} W_{10}\| \) is bounded by the right hand side of (9.21) which follows from Theorem 7.4.

**Lemma 9.3.** Let \( \varphi \) be the solution of

\begin{equation}
D_t^2 \varphi + k \varphi = f, \quad \varphi|_{t=0} = \varphi|_{\tau=0} = 0,
\end{equation}

where \( D_j^t f|_{t=0} = 0 \) for \( j \geq 0 \) and \( k = \text{div} \Phi = D_t^2 e(h) + D_t \text{div} V \). Then for \( r \geq 1 \) we have

\begin{equation}
\|\varphi\|_{r-1} + \|\varphi\|_{r-1} \leq K''_3 \sum_{s=0}^{r-1} n_{r-1-s} \int_0^t \|f\|_s \, d\tau,
\end{equation}

\begin{equation}
\|\varphi\|_{r-1} \leq K''_3 \sum_{s=0}^{r-1} n_{r-1-s} \left( \|f\|_s + \int_0^t \|f\|_s \, d\tau \right)
\end{equation}

Furthermore let \( W_1 \) and \( F_1 \) be defined by

\begin{align}
W_1 &= \nabla q, \quad \Delta q = \varphi, \quad q|_{\partial \Omega} = 0 \\
F_1 &= \nabla q', \quad \Delta q' = f, \quad q'|_{\partial \Omega} = 0
\end{align}

Then for \( r \geq 1 \) we have

\begin{equation}
\|\dot{W}_1\|_r + \|W_1\|_r \leq K''_3 \sum_{s=0}^{r} n_{r-s} \int_0^t \|F_1\|_s \, d\tau,
\end{equation}

\begin{equation}
\|\dot{W}_1\|_r \leq K''_3 \sum_{s=0}^{r} n_{r+1-s} \left( \|F_1\|_s + \int_0^t \|F_1\|_s \, d\tau \right)
\end{equation}
Proof of Lemma 9.3. (9.25) is an ordinary differential equation and (9.26)-(9.27) follows as in the proof of Proposition 10.1 in [L3]. (9.25) can be written

\[ \dot{D}t^2\varphi - 2\dot{\sigma}\dot{D}t\varphi + k'\varphi = f, \quad k' = \dot{\sigma}^2 + D_t^2e(h) \]

and so

\[ \text{div} (\dot{W}_1 - 2\dot{\sigma}\dot{W}_1 + k'W_1 - F_1) = -2(\partial_\alpha\dot{\sigma})\dot{W}_1^\alpha + (\partial_\alpha k')W_1^\alpha \]

and hence

\[ \text{div} (\dot{D}_t^i\dot{W}_1 - \dot{D}_t^i(2\dot{\sigma}\dot{W}_1 - k'W_1 + F_1)) = -\dot{D}_t^i(2(\partial_\alpha\dot{\sigma})\dot{W}_1^\alpha - (\partial_\alpha k')W_1^\alpha) \]

We claim that

\[ PD_t^iW_1 = PB_j(W_1, ..., \dot{D}_t^{j-1}W_1), \quad B_j(W_1, ..., \dot{D}_t^{j-1}W_1) = \sum_{i=0}^{j-1} \binom{j}{i}(\dot{D}_t^{j-i}g_{ab})\dot{D}_t^iW_1 \]

In fact \(0 = PD_t^i\partial_q = PD_t^i(g_{ab}W_1^a) = P\sum_{i=0}^{j-1} \binom{j}{i}(\dot{D}_t^{j-i}g_{ab})\dot{D}_t^iW_1\). Furthermore, let

\[ \triangle q^j = -\dot{D}_t^i(2(\partial_\alpha\dot{\sigma})\dot{W}_1^\alpha - (\partial_\alpha k')W_1^\alpha), \quad q^j|_{\partial \Omega} = 0 \]

To say that \(\text{div} H = 0\) is equivalent to saying that \((I - P)H = 0\) so it follows from (9.34)-(9.36) that

\[ \dot{D}_t^iW_1 = PB_j(W_1, ..., \dot{D}_t^{j-1}W_1) + (I - P)(2\dot{\sigma}\dot{W}_1 - k'W_1 + F_1) + \nabla q^{j-2} \]

Since the projection \(P\) maps \(L^2\) to \(L^2\) and since inverting (9.36) maps \(L^2\) to \(H^1\) (in fact to \(H^2\)) it therefore follows that:

\[ \|D_t^{r+1}W_1\| \leq K_3^n \sum_{s=0}^{r} n_{r-s}\|D_t^sW_1\| + \sum_{s=0}^{r-1} n_{r-2-s}\|D_t^sF_1\| \]

Sine it also follows from (9.26) that

\[ \|\partial \dot{W}_1\|_{r-1} + \|\partial W_1\|_{r-1} \leq K_3'' \sum_{s=0}^{r} n_{r-s}\int_{0}^{t}\|F_1\|_s d\tau, \]

\[ \|\partial \dot{W}_1\|_{r-1} \leq K_3'' \sum_{s=0}^{r} n_{r-s}\left(\|F_1\|_s + \int_{0}^{t}\|F_1\|_s d\tau\right) \]

the lemma follows from also estimating \(\|D_t^sF_1(t, \cdot)\| \leq \int_{0}^{t} \|D_t^{s+1}F_1(\tau, \cdot)\| d\tau. \) □

10. Estimates for the enthalpy in terms of the coordinate.

We have now proved that the linearized operator is invertible. However, since we think of \(h = \Phi(x)\) as a functional of \(x\) we must also estimate the \(L^\infty\) norms of \(h\) in terms of the \(L^\infty\) norms of \(x\). For the corresponding problem for the incompressible case in [L3] we could take advantage of the Schauder estimates. However for the wave equation there are no \(L^\infty\) estimates that do not lose regularity.
For wave equations it is best to get the $L^\infty$ norms from the $L^2$ norms using Sobolev’s lemma. These estimates where obtained in Corollary 7.5. However, in the estimates there the $L^\infty$ norm also occurred in the right hand side, due to that we assumed that $e' = e'(h)$. This estimate can easily be improved by estimating the $L^2$ norm of the solution of the nonlinear wave equation instead. We will however for simplicity assume that $e(h) = ch$, so that $e'(h) = c$, where $0 < c < \infty$ is a constant. In that case there are no $h$ terms in the results in section 7. We have

(10.1) \[ D_t^2 ch - \Delta h = (\partial_i V^j)(\partial_j V)^i, \quad h|_{\partial \Omega} = 0 \]

The estimates in Corollary 7.5 were however formulated for vanishing initial data. Therefore let $\tilde{h} = h - h_0$ and where $h_0$ satisfies the equation (10.1) to all orders as $t \to 0$, if $x - x_0$ vanishes to all orders as $t \to 0$, see section 2. It follows that

(10.2) \[ D_t^2 (c\tilde{h}) - \Delta \tilde{h} = -D_t^2 (ch_0) + \Delta h_0 - (\partial_i V^j)(\partial_j V)^i \]

vanishes to all orders as $t \to 0$. We can therefore apply Corollary 7.5, which gives:

**Lemma 10.1.** We have

(10.3) \[ \|g\|_{s+1, \infty} + \|\omega\|_{s, \infty} \leq K_1 \|x\|_{s+2, \infty} \]

Suppose that $e(h) = ch$, where $0 < c < \infty$. Then for $r \geq 2$

(10.4) \[ \|h\|_{r, \infty} \leq K_2 \|x\|_{r+r_0+1, \infty} \]

Here $K_i$ are as in Definition 5.2 and $K_2$ also depends on a bound for $\|h_0\|_{r+r_0+1, \infty}$.

**Proof.** The first inequality follows directly from the definitions and interpolation. If $f$ denotes the right hand side of (10.2) then

(10.5) \[ \|f\|_{s, \infty} \leq K_2 (\|h_0\|_{s+2, \infty} + \|x\|_{s+2, \infty}) \]

As pointed out above if $e(h) = ch$ then $h_r$ in section 7 vanishes and by Corollary 7.5 and interpolation we have

(10.6) \[ \|\tilde{h}\|_{r, \infty} \leq K_2 (\|x\|_{r+r_0+1, \infty} + \|h_0\|_{r+r_0+1, \infty}) \]

However since we are just looking on fixed initial data we can also include the norms of $h_0$ in the constants and since in fact we also have a lower bound for $\|x\|_{1, \infty} \geq Cc_1 > 0$, by the coordinate condition, the lemma follows. \[ \square \]

**Remark.** Note that $K_2$ in (10.4) depends on $h_0$. However, $h_0$ is a function which is fixed once we fixed initial data so this just leads to an $r$ dependence of the constant.

It now follows that with $K_i$ as in Definition 5.2 and $K'_i$ as in sections 7, 8, and 9 we have

(10.7) \[ K'_i \leq K_{i+r_0+1} \]
11. Estimates for the physical condition and coordinate condition.

We assume that the physical condition and the coordinate condition initially at time 0 for some constants \( c_0 > 0 \) and \( c_1 < \infty \) and we need to show that this implies that they will hold with \( c_0 \) replaced by \( c_0/2 \) and \( c_1 \) replaced by \( 2c_1 \), for \( 0 \leq t \leq T \), if \( T \) is sufficiently small.

Now, for the coordinate condition this is easy since we can just estimate the physical condition by the time derivative of \( g \), which can be estimated by \( \|x\|_{2,\infty}^2 \).

**Lemma 11.1.** Let \( M(t) = \sup_{y \in \Omega} \sqrt{\|\partial x/\partial y\|^2 + \|\partial y/\partial x\|^2} \). Then

\[
M(t) \leq 2M(0), \quad \text{for} \quad t \leq T, \quad \text{if} \quad T\|x\|_{2,\infty} M(0) \leq 1/8
\]

Let \( N(t) = \sup_{y \in \partial \Omega} |\nabla_N h|^{-1} \). Then assuming that \( T \) is so small that (11.1) hold we have

\[
N(t) \leq 2N(0), \quad \text{for} \quad t \leq T, \quad \text{if} \quad T\|h\|_{2,\infty} M(0) N(0) \leq 1/8
\]

**Proof.** We have \( |D_t \partial x/\partial y| \leq \|x\|_{2,\infty} \) and \( |D_t \partial y/\partial x| \leq \|\partial y/\partial x\|^2 |D_t \partial x/\partial y| \) so \( M'(t) \leq (1 + M^2)\|x\|_{2,\infty} \leq 2M^2\|x\|_{2,\infty} \), since also \( M(t) \geq 1 \). Hence

\[
M(t) \leq M(0)(1 - 2\|x\|_{2,\infty} M(0)t)^{-1}, \quad \text{when} \quad 2\|x\|_{2,\infty} M(0)t < 1.
\]

Now, \( \nabla_N h = N^a\partial_a h \), where \( N \) is the unit normal, so \( D_t \nabla_N h = \nabla_N D_t h + (D_t N^a)\partial_a h = \nabla_N D_t h + (D_t N^a)g_{ab} N^b \nabla_N h \), since \( h|_{\partial \Omega} = 0 \). Furthermore \( 0 = D_t(g_{ab} N^a N^b) = 2g_{ab}(D_t N^a)N^b + (D_t g_{ab})N^a N^b \) and \( N^a = (\partial y^a/\partial x^i)N^i \), where \( \delta ij N^i N^j = 1 \). Hence \( |D_t \nabla_N h| \leq M (|\partial D_t h| + |\partial D_t x| |\nabla_N h|) \)

Therefore if \( N(t) = \sup_{y \in \partial \Omega} |\nabla_N h|^{-1} \), we have \( N' \leq M\|h\|_{2,\infty} N^2 + M\|x\|_{2,\infty} N/2 \) and if we use (11.1) and multiply with the integrating factor, \( \tilde{N}(t) = N(t)e^{-tM(0)}\|x\|_{2,\infty} \) we get \( \tilde{N}' \leq 2e^{1/8} M(0)\|h\|_{2,\infty} \tilde{N}^2 \).

Hence

\[
N(t) \leq N(0)e^{1/8} (1 - N(0)2e^{1/8} M(0)\|h\|_{2,\infty} t)^{-1}, \quad \text{when} \quad N(0)2e^{1/8} M(0)\|h\|_{2,\infty} t < 1
\]

This proves the lemma. \( \square \)

To satisfy the condition in (11.1) we just need to choose \( T \) so small that \( T\|x\|_{2,\infty} c_1 \leq 1/8 \).

Remarking that \( x = u + x_0 \), where \( x_0 \) is a fixed and that in the Nash-Moser iteration we will only apply our estimates to functions satisfying \( \|u\|_{r_0 + 4,\infty} \leq 1 \).

However for the physical condition this is a bit more difficult. One has to control the \( \|h\|_{2,\infty} \) and the estimate for \( h \) in terms of \( x \) used interpolation so they are in terms of a constant \( K \) that is a continuous function of \( T^{-1} \) and tends to infinity as \( T \to 0 \). In the compressible case we never used interpolation in time so the corresponding estimate there was easier. We therefore have to redo the estimates for the wave equation for the lowest norms without using interpolation in time. This is however, follows from standard estimates for the wave equation. Those we have here also work if we do not use interpolation. If we do not use interpolation, then the estimates in sections 5,6 and 7 still hold, with constants independent of \( T \leq 1 \), but instead of depending linearly on the highest norms of \( x \) they are polynomials in the highest norms of \( x \) occurring in the estimates.

**Lemma 11.2.** There are continuous increasing functions \( C_r \) such that for \( T \leq 1 \) we have

\[
\|h\|_{r,\infty} \leq C_r \left( c_1, \|x\|_{r + r_0 + 1,\infty}, \|h_0\|_{r + r_0 + 1,\infty} \right)
\]

**Proof.** The proof is the same as the proof of Lemma 10.1 using that the estimates in sections 5,6 and hold with constant of the form above. \( \square \)

Summing up, we have hence proven that
Lemma 11.3. Let $C_2$ be as in Lemma 11.2 and let $c_1$ and $c_0$ be constants such that the coordinate condition (2.13) and the physical condition (2.14) hold when $t = 0$. Suppose $0 < T \leq 1$ is fixed such that

$$
(11.6) \quad Tc_1(1 + \|x_0\|_2,\infty) \leq 1/8,
$$

$$
(11.7) \quad ||u||_{r_0 + 4, \infty} \leq 1
$$

where $C_2$ is as in Lemma 11.2. Then for $0 \leq t \leq T$, the coordinate condition hold with $c_1$ replaced by $2c_1$ and $c_0$ replaced by $c_0/2$ if

$$
(11.8) \quad \|h\|_{2,\infty} \leq C_2(c_1, ||u + x_0||_{r_0 + 3,\infty}, ||h_0||_{r_0 + 3,\infty})
$$

In view of (11.2), the physical condition with $c_0$ replaced by $c_0/2$ hold if $T$ is so small that (11.5) hold and

$$
(11.9) \quad Tc_1C_2(2c_1, ||u + x_0||_{r_0 + 3,\infty}, ||h_0||_{r_0 + 3,\infty}) \leq c_0/8
$$

Recalling again that in the Nash-Moser iteration we will only consider $u$ for which (11.7) hold. From now on we will therefore assume that $0 < T \leq 1$ is fixed and so small that (11.6) hold.

12. Tame estimates for the inverse of the linearized operator in terms of the coordinate.

We have

Theorem 12.1. Suppose that $T > 0$ is so small that the conditions in Lemma 11.3 hold. Suppose also that $x = u + x_0$ and $\delta \Phi$ are smooth in $[0, T] \times \overline{\Omega}$ and that $\delta \Phi$ and $u$ vanish to infinite order as $t \to 0$. Then there are constants $K$, depending on the approximate solution $(x_0, h_0)$, on $(c_0, c_1)$ and on $r$, such that there is a smooth solution $\delta x$ of

$$
(12.1) \quad \Phi'(x)\delta x = \delta \Phi, \quad \text{in} \quad [0, T] \times \overline{\Omega} \quad \delta x|_{t=0} = \delta x|_{t=0} = 0
$$

satisfying

$$
(12.2) \quad \|\delta \dot{x}\|_r + \|\delta x\|_r \leq K \sum_{s=1}^r \|x\|_{r + r_0 + 4 - s, \infty} \int_0^t \|\delta \Phi\|_s \, d\tau
$$

for $r \geq 1$ if

$$
(12.3) \quad ||u||_{r_0 + 4, \infty} \leq 1
$$

Moreover

$$
(12.4) \quad \|x\|_{r + r_0 + 4, \infty} \leq K + ||u||_{r + r_0 + 4, \infty}
$$
Proof. First show existence the equation where the vector field is expressed in the Lagrangian frame, $W^a = (\partial y^a / \partial x^1) \delta x^1$ and $F^a = (\partial y^a / \partial x^1) \delta \Phi^a$:  

\begin{equation}
(12.5) \quad L_0 W = F, \quad \text{in} \quad [0, T] \times \Omega \quad W|_{t=0} = W|_{t=0} = 0
\end{equation}

and that it satisfies the estimate

\begin{equation}
(12.6) \quad \|\dot{W}\|_r + \|W\|_r \leq K \sum_{s=1}^{r} \|x\|_{r+r_0+4-s, \infty} \int_0^t \|F\|_s d\tau
\end{equation}

for $r \geq 1$ if We have already proved this for $L_1 W = F$ in Theorem 9.1, using Lemma 10.1 and Lemma 11.3. Therefore it remains to prove the result for $L_0 W = L_1 W - B_3 W = F$, where $B_3$ is given by (2.63). We have $\|B_3 W\|_s \leq K \sum_{s=0}^{k} \|x\|_{s+r_0+3-k, \infty} \|W\|_k \leq K \sum_{k=1}^{s} \|x\|_{s+r_0+4-k, \infty} \|W\|_k$, if $s \geq 1$. Applying the theorem to the equation $L_1 W = F + B_3 W$ and using interpolation we get that for $r \geq 1$

\begin{equation}
(12.7) \quad \|\dot{W}\|_r + \|W\|_r \leq K \sum_{s=1}^{r} \|x\|_{r+r_0+4-s, \infty} \int_0^t (\|F\|_s + \|W\|_s) d\tau
\end{equation}

If we put up an iteration $L_1 W^{k+1} = F - B_3 W^k$, for $k \geq 0$ and $W^0 = 0$ then $(12.6)$ is going to be true with $W$ in the right hand side replaced by $W^{(k)}$ and in the left by $W^{k+1}$. It is easiest to first show convergence to a smooth solution and then afterwards prove the estimate $(12.2)$. To show convergence we can just consider the estimate $(12.6)$ where we include the norms of $x$ in the constants and estimate the lower order norms by higher order norms. Let $\dot{W}^{k+1} = W^{k+1} - W^k$. Then $L_1 \dot{W}^{k+1} = -B_3 W^k$, for $k \geq 1$. Hence if $E^k = \sum_{j=1}^{k} \|D_1 \dot{W}^j\|_r + \|\dot{W}\|_r$ we have $E^{k+1} \leq C \int \|F\|_r + E^k d\tau$. Using a Grönwall type of argument one therefore get uniform bounds $E^k \leq C^r \int_0^t \|F\|_r d\tau$, for $0 \leq t \leq T \leq 1$. This proves convergence of to a smooth solution. W. Once we have a smooth solution it will satisfy the estimate $(12.7)$. By a Grönwall type of argument and induction as in section 8 it follows that the solution also satisfies the estimate $(12.6)$, for some other constant $K$.

Finally, we want to deduce the estimate for $\delta x$ and $\delta \Phi$. The estimate $(12.6)$ is in terms of $W^a = \delta x \partial y^a / \partial x_i$ and $F^a = \delta \Phi \partial y^a / \partial x^i$, turning them into an estimate for $\delta x$ and $\delta \Phi$ just produces lower order terms of the same form:  

\begin{equation}
(12.8) \quad \|\dot{\delta x}\|_r + \|\delta x\|_r \leq K (\|\dot{W}\|_r + \|W\|_r + \|x\|_{r+2, \infty} \|W\|_0), \quad \|F\|_r \leq K (\|\delta \Phi\|_r + \|x\|_{r+1, \infty} \|\delta \Phi\|_0),
\end{equation}

where we used the interpolation inequality in Lemma 5.7. By $(12.6)$ $\|W\|_0 \leq \|W\|_1 \leq K \int_0^t \|F\|_1 d\tau \leq K \int_0^t \|\delta \Phi\|_1 d\tau$ and by interpolation

\begin{equation}
(12.9) \quad \|x\|_{r+r_0+4-s, \infty} \|x\|_{s+1, \infty} \leq K \|x\|_{r+r_0+3, \infty}, \quad 1 \leq s \leq r
\end{equation}

Hence $(12.3)$ follows. \( \square \)

If we also use Sobolev’s lemma, $r_0 = [n/2] + 1$, and estimate the integrals by the $L^\infty$ norms and use interpolation and $(12.2)$ we get

\begin{equation}
(12.10) \quad \|\dot{\delta x}\|_r \leq K (\|\delta \Phi\|_{r+r_0, \infty} + \|x\|_{r+2r_0+4, \infty} \|\delta \Phi\|_{0, \infty}), \quad r \geq 0
\end{equation}

Furthermore we want to turn it into estimates for

\begin{equation}
(12.11) \quad \Phi(u) = \Phi(u + x_0) - \Phi(x_0)
\end{equation}

see (2.23). Then $\dot{\Phi}'(u) = \Phi'(u + x_0)$. Let $\psi(u)$ denote the right inverse of $\dot{\Phi}'(u)$. Then since $\|x\|_{r, \infty} \leq \|u\|_{r, \infty} + \|x_0\|_{r, \infty}$ we can again include $\|x_0\|_{r, \infty}$ in the constants. Hence we have proven that

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Theorem 12.2. Suppose that $T > 0$ is so small that the conditions in Lemma 11.3 hold. Suppose also that $x = u + x_0$ and $\tilde{g}$ are smooth in $[0,T] \times \tilde{\Omega}$ and that $\tilde{g}$ and $u$ vanish to infinite order as $t \to 0$. Then there are constants $K$, depending on the approximate solution $(x_0,h_0)$, on $(c_0,c_1)$ and on $r$, such that the linearized operator $\Phi'(u)$ has a right inverse $\psi(u)$ satisfying

\begin{equation}
\|\psi(u)\|_{r,\infty} \leq K (\|\tilde{g}\|_{r+r_0,\infty} + \|u\|_{r+2r_0+4,\infty} \|\tilde{g}\|_{0,\infty}), \quad r \geq 0
\end{equation}

if

\begin{equation}
\|u\|_{r_0+4,\infty} \leq 1
\end{equation}

13. Tame estimate for the second variational derivative.

We now first want to show that the Euler map $\Phi(x)$ given by (2.20)-(2.21) is $C^k$, i.e. that $\Phi(x)$ depends smoothly on $k$ parameters if $x$ does. To be more precise, with $B^k = \{ r \in \mathbb{R}^k ; |r| \leq 1 \}$, we want to show that $\Phi(x) - \Phi(x_0) \in C^\infty(B^k, C^{\infty}_0)$ if $x - x_0 \in C^\infty(B^k, C^{\infty}_0)$, where $x_0$ is the approximate solution satisfying (2.18) and $C^{\infty}_0$ is given by (2.19). That $D^2_r$ and $\partial_i$ in (2.20) depends smoothly on parameters is obvious so we only need to prove that $h = \Psi(x)$, given by (2.21) does. Subtracting off the approximate solution $h_0$ of (2.21) we get (10.2), where the right hand side is in $C^k(B^k, C^{\infty}_0)$. Hence it follows from Lemma 7.6 that $h - h_0 \in C^k(B^k, C^{\infty}_0)$.

We must now also obtain tame estimates for the second variational derivative. If $x$ depends smoothly on the parameter $r$ then the variational derivative of $x$ is $\delta x(t,y) = \partial x(r,t,y)/\partial r |_{r=0}$, we can e.g. take $x = x(t,y) + r \delta x(t,y)$. The first variational derivative $\Phi'(x)$ of the Euler map is given by

\begin{equation}
\Phi'(x)\delta x_i = \delta \Phi(x)_i = \partial \Phi(x)/\partial r |_{r=0} = D^2_r \delta x_i - \partial_i h \partial_i \delta x^k + \partial_i h'(\delta x),
\end{equation}

where $\delta h = h'(\delta x) = \Psi'(x)\delta x$ satisfies

\begin{equation}
D^2_r (c \delta h) - \Delta \delta h = -\delta \Delta p - \partial_k p \Delta \delta x^k - 2(\partial_i \partial_k p) \partial_i \partial_k \delta x^k, \quad \delta \Delta h = 2 \partial_k V^i \partial_i \delta x^k \partial_i V^j - 2 \partial_k V^i \partial_i \delta v^k
\end{equation}

and $\delta v |_{\partial \Omega} = 0$, where $\delta v = D_r \delta x$.

Now, let $x$ depend smoothly on two parameters $r$ and $s$, such that $\partial^2 x/\partial r \partial s = 0$, and also set $\epsilon x = \partial x/\partial s |_{s=0}$, e.g. $x = x(t,y) + r \delta x(t,y) + s \epsilon x(t,y)$. Then the second variational derivative is given by

\begin{equation}
\Phi''(x)(\delta x, \epsilon x)_i = \epsilon \delta \Phi(x)_i = \partial \left( \partial \Phi(x)/\partial r |_{r=0} \right)/\partial s |_{s=0}.
\end{equation}

We have

Lemma 13.1.

\begin{equation}
\Phi''(\epsilon x, \delta x)_i = \partial_k h \left( \partial_i \partial_j \delta x + \partial_i \partial_j \epsilon x^k \right) - \partial_k h'(\delta x) \partial_i \partial_j \delta x^k - \partial_k h'(\delta x) \partial_i \epsilon x^k + \partial_i h''(\epsilon x, \delta x)
\end{equation}

where $h' = \Psi'(x)$ and $h'' = \Psi''(x)$. The estimates for $h' = \Psi'(x)$ and $h'' = \Psi''(x)$ must also be obtained.
Lemma 13.2. Let $h = \Psi(x)$ and let $\delta h = h'(\delta x) = \Psi'(x)\delta x$ be the variational derivative. We have

$$\|\delta h\|_{r,\infty} \leq K\left(\|\delta x\|_{r+r_0+1,\infty} + \|x\|_{r+2r_0+3,\infty}\|\delta x\|_{0,\infty}\right)$$

and with $h''(\delta x, \varepsilon x) = \Psi''(x)(\delta x, \varepsilon x)$ the second variational derivative, we have

$$\|h''(\delta x, \varepsilon x)\|_{r,\infty} \leq K\left(\|\delta x\|_{r+3r_0+6,\infty}\|\varepsilon x\|_{0,\infty} + \|\varepsilon x\|_{r+3r_0+6,\infty}\|\delta x\|_{0,\infty} + \|x\|_{r+3r_0+6,\infty}\|\delta x\|_{0,\infty}\|\varepsilon x\|_{0,\infty}\right)$$

Proof of Lemma 13.2. The proof is similar to the proof of Lemma 10.1. We have

$$D^2_t(ch) - \triangle h = (\partial_i V^j)(\partial_j V^i), \quad \triangle h = \kappa^{-1}\partial_a(\kappa g^{ab}\partial_b h)$$

It follows that

$$D^2_t(\epsilon\delta h) - \triangle \delta h = \delta((\partial_i V^j)(\partial_j V^i)) + \kappa^{-1}\partial_a(\delta(\kappa g^{ab}\partial_b h)) - \text{div}\, \delta x \kappa^{-1}\partial_a(\kappa g^{ab}\partial_b h),$$

since $\delta\kappa = \kappa\text{div}\, \delta x$, see [L1]. Using the estimate for $h$ in terms of $x$ in Lemma 10.1 and Corollary 7.5 gives if $f$ denotes the right hand side of (13.8)

$$\|f\|_{r,\infty} \leq K\left(\|\delta x\|_{r+2,\infty} + \|x\|_{r+3r_0+3,\infty}\|\delta x\|_{0,\infty}\right)$$

and hence by Corollary 7.5

$$\|\delta h\|_{r,\infty} \leq K\left(\|\delta x\|_{r+r_0+1,\infty} + \|x\|_{r+2r_0+3,\infty}\|\delta x\|_{0,\infty}\right)$$

In the proof we use the interpolation inequalities in Lemma 5.7 and the fact that $\|x\|_{r_0+4,\infty} \leq K$.

To calculate the second variation we apply $\epsilon$ to this, where we assumed that $\epsilon\delta x = 0$. Note that $\epsilon\text{div}\, \delta x = -(\partial_i\epsilon x^k)(\partial_k \delta x^i)$.

$$D^2_t(\epsilon\delta h) - \triangle \epsilon\delta h = \epsilon\delta((\partial_i V^j)(\partial_j V^i)) - \text{div}\, \epsilon x \kappa^{-1}\partial_a(\delta(\kappa g^{ab}\partial_b h)) - \text{div}\, \delta x \kappa^{-1}\partial_a(\epsilon(\kappa g^{ab}\partial_b h)) + \delta(\epsilon x \text{div}\, \delta x + (\partial_i\epsilon x^k)(\partial_k \delta x^i))\kappa^{-1}\partial_a(\kappa g^{ab}\partial_b h) - \text{div}\, \delta x \kappa^{-1}\partial_a(\epsilon(\kappa g^{ab}\partial_b h)) + \kappa^{-1}\partial_a(\delta(\kappa g^{ab}\partial_b h)) - \text{div}\, \delta x \kappa^{-1}\partial_a(\kappa g^{ab}\partial_b h) + \kappa^{-1}\partial_a(\epsilon(\kappa g^{ab}\partial_b h)) - \text{div}\, \delta x \kappa^{-1}\partial_a(\kappa g^{ab}\partial_b h)$$

Here $\delta g^{ab} = -g^{ac}g^{bd}\delta g_{cd}$, $\delta g_{ab} = \delta_{ij}(\partial_a \delta x^i)(\partial_b \delta x^j) + \delta_{ij}(\partial_a x^i)(\partial_b \delta x^j)$ so since $\epsilon\delta x = 0$, we have $\epsilon\delta g_{ab} = 2\delta_{ij}(\partial_a \delta x^i)(\partial_b \delta x^j)$. The first terms on the right of (13.11) gives raise to term of the form

$$(\partial\epsilon x)(\partial \delta v), \quad (\partial \delta v)(\partial \delta x)(\partial v), \quad (\partial \delta v)(\partial \epsilon x)(\partial \delta v), \quad (\partial \epsilon x)(\partial \delta x)(\partial \delta v)(\partial \delta v)$$

multiplied by powers of $\partial y/\partial x$ or $\partial x/\partial y$. If $f_1$ denotes any of these terms then

$$\|f_1\|_{r,\infty} \leq K\left(\|\delta x\|_{r+4,\infty}||\epsilon x||_{0,\infty} + ||\epsilon x||_{r+4,\infty}||\delta x||_{0,\infty} + ||x||_{r+5,\infty}||\delta x||_{0,\infty}||\epsilon x||_{0,\infty}\right)$$

The terms on the second and third row in (13.11) gives raise to terms of the form

$$(\partial^2 \delta x)(\partial \epsilon x)(\partial h), \quad (\partial \delta x)(\partial^2 \epsilon x)(\partial h), \quad (\partial \delta x)(\partial \epsilon x)(\partial^2 h), \quad (\partial \delta x)(\partial \epsilon x)(\partial h)(\partial^2 x)$$
multiplied by powers of \( \partial y / \partial x \) or \( \partial x / \partial y \). These can be estimated by (13.13) plus

\[
(13.15) \quad K |h||h||r+4, \infty | \| \delta x || r+3, \infty || \epsilon x || r+3, \infty \leq K |x||x||r+4, \infty || \delta x || r+3, \infty || \epsilon x || r+3, \infty
\]

by Lemma 10.1. The terms on the last row in (13.11) gives raise to terms of the form

\[
(13.14) \quad (\partial^2 \delta x)(\partial \epsilon h), \quad (\partial^2 \epsilon x)(\partial \delta h), \quad (\delta x)(\partial \epsilon h)(\partial^2 \delta x), \quad (\partial \delta x)(\partial \epsilon h)(\partial^2 \delta x)
\]

multiplied by powers of \( \partial y / \partial x \) or \( \partial x / \partial y \). These can be estimated by

\[
(13.15) \quad K \left( |\delta x||r+3, \infty | ||\epsilon h||r+3, \infty + ||\epsilon x||r+3, \infty + ||\delta h||r+3, \infty + ||\epsilon h||r+3, \infty \right)
\]

If we also use (13.5) we see that this can be estimated by

\[
(13.16) \quad K \left( |\delta x||r+3, \infty | ||\epsilon x||r+3, \infty + ||\epsilon x||r+3, \infty + ||\delta x||r+3, \infty + ||\epsilon x||r+3, \infty + ||\delta x||r+3, \infty + ||\epsilon x||r+3, \infty \right)
\]

Using interpolation again this is bounded by

\[
(13.17) \quad K \left( |\delta x||r+4, \infty | ||\epsilon x||r+4, \infty + ||\epsilon x||r+4, \infty + ||\delta x||r+4, \infty + ||\epsilon x||r+4, \infty + ||\delta x||r+4, \infty + ||\epsilon x||r+4, \infty \right)
\]

It follows that

\[
(13.18) \quad |D^2_x (\epsilon h) - \Delta \delta h| r, \infty \leq K \left( |\delta x||r+4, \infty | ||\epsilon x||r+4, \infty + ||\epsilon x||r+4, \infty + ||\delta x||r+4, \infty + ||\epsilon x||r+4, \infty + ||\delta x||r+4, \infty + ||\epsilon x||r+4, \infty \right)
\]

Hence by Corollary 7.5:

\[
(13.19) \quad |\epsilon h||r, \infty \leq K \left( |\delta x||r+2, \infty | ||\epsilon x||r+2, \infty + ||\epsilon x||r+2, \infty + ||\delta x||r+2, \infty + ||\epsilon x||r+2, \infty + ||\delta x||r+2, \infty + ||\epsilon x||r+2, \infty \right)
\]

Theorem 13.3. Suppose that \( T > 0 \) is so small that the assumptions in Lemma 11.3 hold. Then there are constants \( K \) depending on the approximate solution \( (x_0, h_0) \), on \( (c_0, c_1) \) and on \( r \) such that

\[
(13.20) \quad |\Phi''(u + x_0)(\epsilon x, \delta x)| r, \infty \leq K \left( |\delta x||r+2, \infty | ||\epsilon x||r+2, \infty + ||\epsilon x||r+2, \infty + ||\delta x||r+2, \infty + ||u||r+2, \infty + ||\delta x||r+2, \infty + ||\epsilon x||r+2, \infty \right)
\]

if

\[
(13.21) \quad |u||r, \infty \leq 1.
\]
14. The smoothing operators.

We will work in Hölder spaces since the standard proofs of the Nash-Moser theorem uses Hölder spaces. The Hölder norms for functions defined on a compact convex set $B \in \mathbb{R}^{1+n}$ are given by, if $k < a \leq k + 1$, where $k \geq 0$ is an integer,

$$
\|u\|_{a,\infty} = \|u\|_{H^a} = \sup_{(t,y),(s,z) \in B} \frac{\|\partial^a u(t,y) - \partial^a u(s,z)\|}{\|t-y - (s,z)\|^{a-k}} + \sup_{(t,y) \in B} |u(t,y)|
$$

and $\|u\|_{H^0} = \sup_{(t,y) \in B} |u(t,y)|$. Since we use the same notation for the $C^k$ norms, $\|u\|_{k,\infty}$ we will indicate the difference by simply using letters $a, b, c, d, e, f$ etc for the Hölder norms and $i, j, k, \ldots$ for the $C^k$ norms. Since a Lipschitz continuous function is differentiable almost everywhere and the norm of the derivative at these points is bounded by the Lipschitz constant, we conclude that for integer values this is the same if the $L^\infty$ norm of $\partial^a u$ for $|a| \leq k$, and furthermore, since all our functions are smooth it is the same as the supremum norm. Our tame estimates for the inverse of the linearized operator and the second variational derivative are only for $C^k$ norms with integer exponents. However, since $\|u\|_{k,\infty} \leq C\|u\|_{a,\infty} \leq C\|u\|_{k+1,\infty}$, if $k \leq a \leq k + 1$, see (14.2), they also hold for non integer values with a loss of of one more derivative.

In [L3] we used smoothing only in the space directions but here we will use smoothing also in the time direction. Therefore we define the Hölder space time norms as above.

They satisfy

$$
\|u\|_{a,\infty} \leq C\|u\|_{b,\infty}, \quad a \leq b
$$

and they also satisfy the interpolation inequality

$$
\|u\|_{c,\infty} \leq C\|u\|_{a,\infty}^{\lambda} \|u\|_{b,\infty}^{1-\lambda}
$$

where $0 \leq a \leq c \leq b$, $0 \leq \lambda \leq 1$ and $c = \lambda a + (1 - \lambda)b$. Furthermore, the Hölder spaces are rings:

$$
\|uv\|_{a,\infty} \leq C(\|u\|_{a,\infty}\|v\|_{0,\infty} + \|u\|_{0,\infty}\|v\|_{a,\infty})
$$

For the Nash-Moser technique, apart from tame estimates one also needs smoothing operator $S\theta$ that satisfy the following properties with respect to the Hölder norms:

**Lemma 14.1.** Let $\|u\|_a$ denote the Hölder norms in (14.1) with $B = [0,T] \times \Omega$, where $T \leq 1$. Let $C_0^\infty = C_0^\infty([0,T] \times \Omega)$ be as in (14.11). Then there is a family of smoothing operators $S\theta : C_0^\infty \to C_0^\infty$, $1 \leq \theta < \infty$ such that

$$
\|S\theta u\|_{a,\infty} \leq C\|u\|_{b,\infty}, \quad a \leq b
$$

$$
\|S\theta u\|_{a,\infty} \leq C\theta^{a-b}\|u\|_{b,\infty}, \quad a \geq b
$$

$$
\|(I - S\theta)u\|_{a,\infty} \leq C\theta^{a-b}\|u\|_{b,\infty}, \quad a \leq b
$$

$$
\|((S\theta - S\theta)u\|_{a,\infty} \leq C\theta^{a-b}\|u\|_{b,\infty}, \quad a \geq 0
$$

where the constants $C$ only depend on the dimension and an upper bound for $a$ and $b$.

The last property, (14.8) follows from (14.6) for $a \geq b$ and from (14.7) for $a \leq b$. Alternatively, it follows from the following stronger property

$$
\|\frac{d}{dt} S\theta u\|_{a,\infty} \leq C\theta^{a-b-1}\|u\|_{b,\infty}, \quad a \geq 0.
$$
For functions supported in the interior of a compact set $K$ there there are smoothing operators, see [H1], that satisfy the above properties (14.5)-(14.9), with respect to the Hölder norms. These are constructed as follows. Let the Fourier transform $\hat{\phi} \in C_0^\infty$ be 1 in a neighborhood of the origin and set $\phi_\theta(z) = \theta^{1+n}\phi(\theta z)$, and $S_\theta u = \chi \phi_\theta \ast u$, where $\chi \in C_0^\infty$ is 1 on a neighborhood of $K$. However we have functions defined on the compact set $[0,T] \times \overline{\Omega}$ that do not have compact support in $\Omega$. Therefore we need to extend these functions to have compact support in some larger set, without increasing the Hölder norms with more than with a multiplicative constant. There is a standard extension operator in [S] that turns to have these properties, see Lemma 14.2 below.

First however, we note that we will only apply the smoothing operators to functions that vanish to all orders as $t \to 0$. Hence we can extend these functions to be 0 for $t \leq 0$ without changing the Hölder norm. Then we extend the functions defined for $t \leq T$ to functions supported in $[0,2]$, using the extension in Lemma 14.2, for $y \in \overline{\Omega}$ fixed, noting that Hölder continuity in $(t,y)$ follows from differentiability and Hölder continuity in each direction using the triangle inequality and the linearity of the extension operator in Lemma 14.2. Then we want to extend the functions defined in $\overline{\Omega} = \{y; |y| \leq 1\}$ to functions supported in $\{y; |y| \leq 2\}$. In order to do this we first remove a region around the origin and introduce polar coordinates $r \geq 0$ and $\omega \in S^{n-1}$, which is a nonsingular change of variables away from the origin. Then we use the extension operator in Lemma 14.2 to, for fixed $t$ and $\omega$ extend in the radial direction from function defined for $r \leq 1$ to functions supported in $r \leq 2$. By the remark above, Hölder continuity in $(t,r,\omega)$ follows from Hölder continuity in each direction.

Doing the extensions above we hence obtain an extension $\tilde{u}$ of $u$ defined in $[0,T] \times \overline{\Omega}$ such that

$$
(14.10) \quad ||\tilde{u}||_{a,\infty} \leq C||u||_{a,\infty}, \quad a \geq 0 \quad \text{supp}(\tilde{u}) \subseteq \{(t,y); 0 \leq t \leq 2, |y| \leq 2\}
$$

for $u$ in

$$
(14.11) \quad C_0^\infty = C_0^\infty([0,T] \times \overline{\Omega}) = \{u \in C^\infty([0,T] \times \overline{\Omega}), D_k^\infty u|_{t=0} = 0, k \geq 0\}
$$

We note that, in fact the constant in (14.10) is independent of $T$.

Once we have the extension operator we can use the smoothing operators in [H1,H2], defined for compactly supported functions, applied to the extension of our function. Let us call the smoothing operators defined in [H1,H2] $\hat{S}_\theta$. These satisfy the properties (14.5)-(14.9). By (14.10) the smoothing operators $\hat{S}_\theta u$ given by the restriction of $\hat{S}_\theta \tilde{u}$ to $[0,T] \times \overline{\Omega}$ then also satisfy the properties (14.5)-(14.9) if $u$ is in (14.11). However, $\hat{S}_\theta u$ is not in (14.11) anymore. In our estimates we will only apply the smoothing operators to functions that vanish to all orders as $t \to 0$ and in our estimates we need also $S_\theta u$ to vanish to all orders as $t \to 0$. We therefore have to modify our smoothing operators so that this is true. Let $\chi(t) \in C^\infty$ be a function such that $\chi(t) = 0$, when $t \leq 0$ and $\chi(t) = 1$, when $t \geq 1$, and let $\chi_\theta(t) = \chi(\theta t)$. Then

$$
(14.12) \quad S_\theta u = \chi \hat{S}_\theta \tilde{u})|_{[0,T] \times \overline{\Omega}}
$$

is in (14.11) and we claim that for functions $u$ in (14.11), (14.5)-(14.9) hold. This follows from Lemma 14.3 below, since the smoothing operators defined in [H1,H2] are convolution operators of the form in Lemma 14.3.

**Lemma 14.2.** There is a linear extension operator $\mathcal{E}xt : H^a((-\infty,0]) \to H^a((-\infty,\infty))$, where $H^a$ are the Hölder spaces, such that $\mathcal{E}xt(f) = f$ when $r \leq 0$, and

$$
(14.13) \quad ||\mathcal{E}xt(f)||_{a,\infty} \leq C||f||_{a,\infty}
$$
Here $C$ is bounded when $a$ is bounded. Furthermore, if $r \geq -c$ in the support of $f$, where $c > 0$, then $r \leq c$ in the support of $\mathcal{E}t(x)(f)$.

**Proof.** Let $\mathcal{E}t(x)(f)(r) = \tilde{f}(r)$, where $\tilde{f}(r) = f(r)$, when $r \leq 0$, and

$$
(14.14) \quad \tilde{f}(r) = \int_1^\infty f(r - 2\lambda r) \psi_1(\lambda) d\lambda, \quad r > 0
$$

where $\psi_1$ is a continuous function on $[1, \infty)$, such that

$$
(14.15) \quad \int_1^\infty \psi_1(\lambda) d\lambda = 1, \quad \int_1^\infty \lambda^k \psi_1(\lambda) d\lambda = 0, \quad k > 0, \quad |\psi_1(\lambda)| \leq C_N(1 + \lambda)^{-N}, \quad N \geq 0
$$

The existence of such a function was proved in [S] where the extension operator was also introduced. In [S] it was proven that this operator is continuous on the Sobolev spaces but it was not proven there that it is continuous on the Hölder spaces so we must prove this.

First we note that if $f \in C^k$ then the extension is in $C^k$. In fact

$$
(14.16) \quad \tilde{f}^{(j)}(r) = \int_1^\infty f^{(j)}(r - 2\lambda r)(1 - 2\lambda)^j \psi_1(\lambda) d\lambda, \quad r > 0
$$

From the continuity of $\partial^j r f$ and (14.14)-(14.15) it follows that $\lim_{r \to +0} \tilde{f}^{(j)}(r) = f^{(j)}(0)$, that $\tilde{f}$ is in $C^k$, and that $\|\tilde{f}\|_{k, \infty} \leq C_k\|f\|_{k, \infty}$, if $k$ is an integer.

Suppose now that $k < a \leq k + 1$ where $k$ is an integer. We must now estimate

$$
(14.17) \quad \sup_{r, \rho} \frac{|\tilde{f}^{(k)}(r) - \tilde{f}^{(k)}(\rho)|}{|r - \rho|^{a-k}}
$$

by $C_k\|\partial^k r f\|_{a-k, \infty}$. If $r = 0$ and $\rho = 0$ there is nothing to prove. Also if $r < 0 < \rho$ or $\rho < 0 < r$, then $|r - \rho| \geq |\rho|$ and $|r - \rho| \geq |r|$ so in this case, we can reduce it to two estimates with either $r = 0$ or $\rho = 0$. Also it is symmetric in $r$ and $\rho$ so it only remains to prove the assertion when $r > \rho \geq 0$. It follows from the Hölder continuity of $f^{(k)}$ and the last estimate in (14.15) that for $r, \rho \geq 0$,

$$
(14.18) \quad \left| \int_1^\infty (f^{(k)}(r - 2\lambda r) - f^{(k)}(\rho - 2\lambda \rho))(1 - 2\lambda)^k \psi_1(\lambda) d\lambda \right| \leq C_k\|f^{(k)}\|_{a-k, \infty} |r - \rho|^{a-k}
$$

which proves the lemma. □

**Lemma 14.3.** Let $\chi(t) \in C^\infty$ be a function such that $\chi(t) = 0$, when $t \leq 0$ and $\chi(t) = 1$, when $t \geq 1$, and let $\chi_0(t) = \chi(\theta t)$. Let the Fourier transform $\hat{\phi} \in C_0^\infty$ be $1$ in a neighborhood of the origin and let $\chi_1 \in C^\infty$ be $1$ on a neighborhood of $\{(t, y); 0 \leq t \leq 2, |y| \leq 2\}$. Set $S_\theta = \chi_1 \phi_\theta * u$, where $\phi_\theta((t, y)) = \phi(\theta(t, y)) / \theta^{1+n}$. Then

$$
(14.19) \quad \|(1 - \chi_\theta)S_\theta u\|_{a, \infty} \leq C|\theta|^{a-b}\|u\|_{b, \infty},
$$

if $u$ is smooth and vanishes for $t \leq 0$ and for $t \geq 2$.

**Proof.** First we note that by interpolation it suffices to prove the estimate for $a = k$ an integer. Since $u$ vanishes to infinite order as $t \to 0$ we have if $k < b \leq k + 1$

$$
(14.20) \quad |u(t, y)| = \left| \int_0^t (\partial^k_x u)(s, y) \frac{(t - s)^{k-1}}{(k - 1)!} ds \right| \leq \int_0^t s^{b-k}\|u\|_{b, \infty} \frac{(t - s)^{k-1}}{(k - 1)!} ds \leq C_b t^b\|u\|_{b, \infty}.
$$

Since $\phi$ is fast decaying we have if $|\alpha| = k$,

$$
(14.21) \quad |D^\alpha ((1 - \chi_\theta)\chi_1 \phi_\theta * u)| \leq C_N t^{k-b}\|u\|_{b, \infty} \int_0^\infty \int_0^\infty \frac{|s|^{b} ds dy}{(1 + |\theta t - s| + |\theta x - y|)^N}
$$

for any $N$. Here the integral is uniformly bounded when $\theta t \leq C$ so the lemma follows. □
15. The Nash Moser Iteration.

At this point, given the results stated in sections 11-14, the problem is now reduced to a completely standard application of the Nash-Moser technique. One can just follow the steps of the proof of \[AG, H1, H2, K1\] replacing their norms with our norms. The main difference is that we have a boundary, but we have constructed smoothing operators that satisfy the required properties for the case with a boundary. The proof of the Nash-Moser that we outline below is similar to the one in \[L3\]. The only difference is that now we also smooth in time. We will follow the formulation from \[AG\] which however is similar to \[H1, H2\]. The theorem in \[AG\] is stated in terms of Hölder norms, with a slightly different definition of the Hölder norms for integer values. However, the only properties that are used of the norms are the smoothing properties and the interpolation property in section 14, which we proved with the usual definition, i.e. the one used in \[H1\].

Let us also change notation and call \(\tilde{\Phi}(u)\) from last section \(\Phi(u)\). For \(k < a \leq k + 1\), where \(k \geq 0\) is an integer, let

\[
||u||_{a,\infty} = ||u||_{H^a} = \sup_{(t,y),(s,z) \in [0,T] \times \Omega} \left| \sum_{|a| = k} \frac{\partial^a u(t,y) - \partial^a u(s,z)}{|(t,y) - (s,z)|^{a-k}} \right| + \sup_{(t,y) \in [0,T] \times \Omega} |u(t,y)|
\]

and \(\|u\|_{H^\alpha} = \sup_{(t,y) \in [0,T] \times \Omega} |u(t,y)|\). The estimates we proved for the inverse of the linearized operator and the second derivative of the operator where in terms of \(L^\infty\) norms, i.e. Hölder norms for integer values. However, since \(\|u\|_{k,\infty} \leq \|u\|_{a,\infty} \leq \|u\|_{k+1}\), if \(k \leq a \leq k + 1\), it follows that they also hold for non integer values with loss of an additional derivative.

\((H_1)\): \(\Phi\), is twice differentiable and satisfies

\[
\|\Phi''(u)(v_1, v_2)\|_{a,\infty} \leq C_a \left( \|v_1\|_{a+\mu,\infty} \|v_2\|_{a+\mu',\infty} + \|v_1\|_{\mu',\infty} \|v_2\|_{a+\mu,\infty} + \|u\|_{a+\mu,\infty} \|v_1\|_{\mu',\infty} \|v_2\|_{\mu',\infty} \right),
\]

where \(\mu = 3r_0 + 8\), for \(u, v_1, v_2 \in C^\infty_0\), if

\[
\|u\|_{\mu',\infty} \leq 1, \quad \mu' = r_0 + 4
\]

where \(K\) is the constant in Lemma 12.1

\((H_2)\): If \(u \in C^\infty_0\) satisfies (15.3) then there is a linear map \(\psi(u)\) from \(C^\infty_0\) to \(C^\infty_0\) such that \(\Phi'(u)\psi(u) = I_d\). It satisfies

\[
\|\psi(u)g\|_{a,\infty} \leq C_a \left( \|g\|_{a+\lambda,\infty} + \|g\|_{0,\infty} \|u\|_{a+\lambda,\infty} \right),
\]

where \(\lambda = 2r_0 + 6\).

**Theorem 15.1.** Suppose that \(\Phi\) satisfies \((H_1), (H_2)\) and \(\Phi(0) = 0\). Suppose that \(\mu \geq \mu'\) and let \(\alpha > \lambda + \mu + \mu', \alpha \notin \mathbb{N}\). Then

i) There is neighborhood \(W_\delta = \{f \in C^\infty_0; \|f\|_{a+\lambda,\infty} \leq \delta^2\}, \delta > 0\), such that, for \(f \in W_\delta\), the equation

\[
\Phi(u) = f
\]

has a solution \(u = u(f) \in C^\infty_0\). Furthermore,

\[
\|u(f)\|_{a,\infty} \leq C \|f\|_{a+\lambda,\infty}, \quad a < \alpha
\]

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In the proof, we construct a sequence \( u_j \in C_0^\infty \) converging to \( u \), that satisfy \( \|u_j\|_{\mu'} \leq 1 \) and \( \|S_i u_j\|_{\mu'} \leq 1 \), for all \( j \), where \( S_i \) is the smoothing operator in (15.7). The estimates (15.2) and (15.4) will only be used for convex combinations of these and hence within the domain (15.3) for which these estimates hold.

Following [H1,H2,AG,K1,K2] we set

\[
(15.7) \quad u_{i+1} = u_i + \delta u_i, \quad \delta u_i = \psi(S_i u_i)g_i, \quad u_0 = 0, \quad S_i = S_{\theta_i}, \quad \theta_i = \theta_0 2^i, \quad \theta_0 \geq 1
\]

and \( g_i \) are to be defined so that \( u_i \) formally converges to a solution. We have

\[
(15.8) \quad \Phi(u_{i+1}) - \Phi(u_i) = \Phi'(u_i)(u_{i+1} - u_i) + e_i'' = \Phi'(u_i)\psi(S_i u_i)g_i + e_i'' \\
= (\Phi'(u_i) - \Phi'(S_i u_i))\psi(S_i u_i)g_i + g_i + e_i'' = e_i' + e_i'' + g_i
\]

where

\[
(15.9) \quad e_i' = \Phi'(u_i) - \Phi'(S_i u_i)\delta u_i \\
(15.10) \quad e_i'' = \Phi(u_{i+1}) - \Phi(u_i) - \Phi'(u_i)\delta u_i \\
(15.11) \quad e_i = e_i' + e_i''
\]

Therefore

\[
(15.12) \quad \Phi(u_{i+1}) - \Phi(u_i) = e_i + g_i
\]

and adding, we get

\[
(15.13) \quad \Phi(u_i) = \sum_{j=0}^{i} g_j + S_i E_i + e_i + (I - S_i) E_i, \quad E_i = \sum_{j=1}^{i-1} e_j
\]

To ensure that \( \Phi(u_i) \to f \) we must have

\[
(15.14) \quad \sum_{j=0}^{i} g_j + S_i E_i = S_i f
\]

Thus

\[
(15.15) \quad g_0 = S_0 f, \quad g_i = (S_i - S_{i-1})(f - E_{i-1}) - S_i e_{i-1}
\]

and

\[
(15.16) \quad \Phi(u_i) = S_i f + e_i + (I - S_i) E_i
\]

Given \( u_0, u_1, ..., u_i \) these determine \( \delta u_0, \delta u_1, ..., \delta u_i \) which by (15.9)-(15.10) determine \( e_1, ..., e_{i-1} \), which by (15.15) determine \( g_i \). The new term \( u_{i+1} \) is the determined by (15.7).
Lemma 15.2. Assume that \( \|u_i\|_{\mu',\infty} \leq 1 \), \( \|u_{i+1}\|_{\mu',\infty} \leq 1 \), and \( \|S_i u_i\|_{\mu',\infty} \leq 1 \). Then
\[
\|e_i^\prime\|_{r,\infty} \leq C_r \left( \|\delta u_i\|_{\mu',\infty} + \|\delta u_i\|_{r,\mu} \right) + C_r \|S_i u_i\|_{r,\mu,\infty} \|\delta u_i\|_{r,\mu,\infty}
\]
and
\[
\|e_i^\prime\|_{r,\infty} \leq C_r \left( \|\delta u_i\|_{r,\mu,\infty}^2 + \|\delta u_i\|_{r,\mu,\infty} \right)
\]

Proof. The proof of (15.17) makes use of
\[
(\Phi'(u_i) - \Phi'(S_i u_i)) \delta u_i = \int_0^1 \Phi''(S_i u_i + s(I - S_i) u_i)(u_i - S_i u_i, \delta u_i) ds
\]
together with (15.2). The proof of (15.18) makes use of
\[
\Phi(u_{i+1}) - \Phi(u_i) - \Phi'(u_i) \delta u_i = \int_0^1 (1 - s)\Phi''(u_i + s \delta u_i)(\delta u_i, \delta u_i) ds
\]
together with (15.2). □

Let \( \bar{\alpha} > \alpha \) and \( \bar{\alpha} - \mu > 2\alpha - \mu - \mu' \). Throughout the proof \( C_\alpha \) will stand for constants that depend on \( a \) but are independent of \( i \) and \( n \) in (15.21).

Our inductive assumption \( (H_n) \) is,
\[
\|\delta u_i\|_{a,\infty} \leq \delta \theta_i^{\bar{\alpha} - \alpha}, \quad 0 \leq a \leq \bar{\alpha}, \quad i \leq n
\]
If \( n = 0 \) then \( \delta u_0 = \psi(0)S_0 f \), and if \( a \leq \bar{\alpha} \), we have \( \|\delta u_0\|_{a,\infty} \leq C_\bar{\alpha}\|f\|_{\alpha + \lambda,\infty} \leq C_\bar{\alpha} \delta^2 \), so it follows that (15.21) hold for \( n = 0 \) if we choose \( \delta \) so small that \( C_\bar{\alpha} \delta \leq \theta_0^{\bar{\alpha} - \alpha} \). We must now prove that \( (H_n) \) implies \( (H_{n+1}) \) if \( C_\alpha \delta \leq 1 \), where \( C_\alpha' \) is some constant that only depends on \( \bar{\alpha} \) but is independent of \( n \).

Lemma 15.3. If (15.21) hold then with a constant \( C_\alpha \) independent of \( i \leq n \)
\[
\sum_{j=0}^i \|\delta u_j\|_{a,\infty} \leq C_\alpha \delta \left( \min(i, 1/|\alpha - a|) + 1 \right) (\theta_i^{\alpha - \alpha} + 1), \quad 0 \leq a \leq \bar{\alpha}
\]

Proof. Using (15.21) we get \( \sum_{j=0}^i \|\delta u_j\|_{a,\infty} \leq C_\alpha \delta \sum_{j=0}^i 2^{j(a - \alpha)} \) and noting that \( \sum_{j=0}^i 2^{-sj} \leq C(\min(1 + 1/s, i) + 1) \), if \( s > 0 \), (15.22) follows. □

It follows from (15.22):

Lemma 15.4. If \( (H_n) \), i.e. (15.21), hold, and \( \bar{\alpha} > \alpha \), then for \( i \leq n + 1 \) we have
\[
\|u_i\|_{a,\infty} \leq C_\alpha \delta \left( \min(i, 1/|\alpha - a|) + 1 \right) (\theta_i^{\alpha - \alpha} + 1), \quad 0 \leq a \leq \bar{\alpha}
\]
(15.23)
\[
\|S_i u_i\|_{a,\infty} \leq C_\alpha \delta \left( \min(i, 1/|\alpha - a|) + 1 \right) (\theta_i^{\alpha - \alpha} + 1), \quad a \geq 0
\]
(15.24)
\[
\|(I - S_i) u_i\|_{a,\infty} \leq C_\alpha \delta \theta_i^{\alpha - \alpha}, \quad 0 \leq a \leq \bar{\alpha}
\]
(15.25)
where the constants are independent of \( n \).

Proof. The proof of (15.23) is just summing up the series \( u_{i+1} = \sum_{j=0}^i \delta u_j \), using Lemma 15.3. (15.24) follows from (15.23) using (14.5) for \( a \leq \bar{\alpha} \) and (14.6), with \( b = \bar{\alpha} \) for \( a \geq \bar{\alpha} \). (15.25) follows from (14.7) with \( b = \bar{\alpha} \) and (15.23) with \( a = \bar{\alpha} \). □

Since we have assumed that \( \alpha > \mu' \), we note that in particular, it follows that
\[
\|u_i\|_{\mu',\infty} \leq 1 \quad \text{and} \quad \|S_i u_i\|_{\mu',\infty} \leq 1 \quad \text{for} \quad i \leq n + 1 \quad \text{if} \quad C_\mu \delta \leq 1.
\]

As a consequence of Lemma 15.4 and Lemma 15.2 we get
Lemma 15.5. If \((H_n)\) is satisfied and \(\alpha > \mu'\), then for \(i \leq n\),

\[
\begin{align*}
\|e'_i\|_{a,\infty} & \leq C_a \theta_i^{a-(2\alpha-\mu-\mu')} , \\
\|e''_i\|_{a,\infty} & \leq C_a \theta_i^{a-(2\alpha-\mu-\mu')}
\end{align*}
\]

where the constants are independent of \(n\).

As a consequence of Lemma 15.5 and (14.8) we get

Lemma 15.6. If \((H_n)\) is satisfied, then for \(i \leq n + 1\),

\[
\begin{align*}
\|S_i e_{i-1}\|_{a,\infty} & \leq C_a \theta_i^{a-(2\alpha-\mu-\mu')} , \\
\|(S_i - S_{i-1})f\|_{a,\infty} & \leq C_a \theta_i^{a-\beta}\|f\|_{\beta,\infty} , \\
\|(I - S_i)f\|_{a,\infty} & \leq C_a \theta_i^{a-\beta}\|f\|_{\beta,\infty}
\end{align*}
\]

Furthermore, if \(\alpha - \mu > (2\alpha - \mu - \mu')\):

\[
\begin{align*}
\|S_i S_{i-1} E_{i-1}\|_{a,\infty} & \leq C_a \theta_i^{a-(2\alpha-\mu-\mu')} , \\
\|(I - S_i)E_i\|_{a,\infty} & \leq C_a \theta_i^{a-(2\alpha-\mu-\mu')}
\end{align*}
\]

Here the constants \(C_a\) are independent of \(n\).

Proof. (15.29) follows from (15.27). For \(a \leq \alpha - \mu\) we use (14.5) with \(b = a\) and for \(a \geq \alpha - \mu\), we use (14.6) with \(b = \alpha - \mu\). (15.30) follows from (14.8) and (15.31) follows from (14.7). Now, \(E_i = \sum_{j=0}^{i-1} e_j\) so by Lemma 15.5 \(\|E_i\|_{a,\infty} \leq C_a \theta_i^{a-(2\alpha-\mu-\mu')} \leq C_a \delta^2 \theta_i^{a-(2\alpha-\mu-\mu')}\), since we assumed that the exponent is positive. (15.32) follows from this and (14.8) with \(b = \alpha - \mu\) and similarly (15.33) follows from (14.7) with \(b = \alpha - \mu\). \(\Box\)

It follows that:

Lemma 15.7. If \((H_n)\) is satisfied, \(\alpha - \mu > (2\alpha - \mu - \mu')\), and \(\alpha > \mu'\) then for \(i \leq n + 1\),

\[
\|g_i\|_{a,\infty} \leq C_a \theta_i^{a-(2\alpha-\mu-\mu')} + C_a \theta_i^{a-\beta}\|f\|_{\beta,\infty} , \quad a \geq 0.
\]

Using this lemma and (15.4) we get

Lemma 15.8. If \((H_n)\) holds, \(\alpha - \mu > (2\alpha - \mu - \mu')\), \(\alpha > \mu'\), \(\alpha > \lambda\) then, for \(i \leq n + 1\), we have

\[
\|\delta u_i\|_{a,\infty} \leq C_a \theta_i^{a+\lambda-(2\alpha-\mu-\mu')} + C_a \|f\|_{\beta,\infty} \theta_i^{a+\lambda-\beta} , \quad a \geq 0.
\]

Proof. Using Lemma 15.7, (15.24), and (15.4) we get

\[
\|\delta u_i\|_{a,\infty} \leq C_a \left(\|g_i\|_{a+\lambda,\infty} + \|f\|_{\beta,\infty} \|S_i u_i\|_{a+\lambda,\infty}\right) \\
\leq C_a \left(\theta_i^{a+\lambda-(2\alpha-\mu-\mu')} + \|f\|_{\beta,\infty} \theta_i^{a+\lambda-\beta}\right) \\
+ C_a \left(\delta^2 \theta_i^{\lambda-(2\alpha-\mu-\mu')} + \|f\|_{\beta,\infty} \theta_i^{\lambda-\beta}\right) \delta(i,1/|\alpha - a - \lambda|) + 1)(\theta_i^{a+\lambda-\alpha} + 1)
\]

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Using that $\alpha > d$ we get (15.35). □

If we now pick $\beta = \alpha + \lambda$, and use the assumptions that $\lambda + \alpha < 2\alpha - \mu - \mu'$, and $\|f\|_{\alpha + \lambda, \infty} \leq \delta^2$, we get that for $i \leq n + 1$,

\[(15.37) \quad \|\delta u_i\|_{a, \infty} \leq C_a \delta^2 \theta_i^{a-\alpha}, \quad a \geq 0,\]

If we pick $\delta > 0$ so small that

\[(15.38) \quad C_a \delta \leq 1,\]

the assumption $(H_{n+1})$ is proven.

The convergence of the $u_i$ is an immediate consequence of Lemma 15.3:

\[(15.39) \quad \sum_{i=0}^{\infty} \|u_{i+1} - u_i\|_{a, \infty} \leq C_a \delta, \quad a < \alpha\]

It follows from Lemma 15.6 that

\[(15.40) \quad \|\Phi(u_i) - f\|_{a, \infty} \leq C_a \delta^2 \theta_i^{a-\alpha-\lambda}\]

which tends to 0, as $i \to \infty$, if $a < \alpha + \lambda$. In particular it follows from (15.39) that $\|u_j\|_{\mu', \infty} \leq 1$, if $\delta > 0$ is chosen small enough.

It remains to prove $u \in C_{00}^\infty$. Note that in Lemma 15.8 we proved a better estimate than $(H_n)$. In fact if we let $\gamma = 2(\alpha - \mu) - (\alpha + \lambda) > 0$ and $\alpha' = \alpha + \gamma$, then $\|f\|_{\alpha', \infty} \leq C$ implies that

\[(15.41) \quad \|\delta u_i\|_{a, \infty} \leq C_a \theta_i^{a-\alpha'}, \quad a \geq 0\]

Using this new estimate, in place of $(H_n)$, we can go back to Lemma 15.4-Lemma 15.8 and replace $\alpha$ by $\alpha'$. Then it follows from Lemma 15.8 that

\[(15.42) \quad \|\delta u_i\|_{a, \infty} \leq C_a \theta_i^{a+\gamma-2(\alpha'-\mu) + C_a \theta_i^{a+\lambda-\beta}} \|f\|_{\beta, \infty}\]

and if we now pick $\gamma' = 2(\alpha' - \mu) - (\lambda - \alpha') = 2\gamma$ and $\alpha'' = \alpha' + \gamma' = \alpha + 2\gamma$, and use that $\|f\|_{\alpha', \gamma', \infty} \leq C$ we see that

\[(15.43) \quad \|\delta u_i\|_{a, \infty} \leq C_a \theta_i^{a-\alpha''}, \quad a \geq 0\]

Since the gain $\gamma > 0$ is constant, repeating this process yields that (15.41) holds for any $\alpha'$ and hence that (15.39)-(15.40) hold for any $a \geq 0$, (without $\delta$). It follows that $u_j$ is a Cauchy sequence in $C^k([0, T] \times \Omega)$, for any $k$, and since $u_j \in C_{00}^\infty$ it follows that $u_j \to u \in C_{00}^\infty$ and $\Phi(u_j) \to f \in C_{00}^\infty$, and since $\Phi$ is continuous it follows that $\Phi(u) = f$. (15.6) follows from (15.37) with $\delta^2 = \|f\|_{\alpha + \lambda, \infty}$. This concludes the proof of Theorem 15.1.

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16. Existence of initial data satisfying the compatibility conditions.

In this section we show that there are initial data satisfying the compatibility conditions. We do not attempt to find the most general class of initial data that do so, the purpose is simply to show that our local existence theorem is not about the empty set. The set of initial data we construct can then easily be extended to a much larger set using essentially the same proof.

Let us therefore start by making some simplifying assumptions. First we assume that \( e(h) = h \). We now want to find a formal power series solution in \( t \) of the system

\[
(16.1) \quad D_t^2 x_i = -\partial_i h, \quad D_t^2 h - \triangle h = (\partial_i V^j)(\partial_j V^i), \quad v_i = D_t x_i
\]

with initial data

\[
(16.2) \quad x|_{t=0} = f_0, \quad D_t x|_{t=0} = v_0, \quad h|_{t=0} = h_0, \quad D_t h|_{t=0} = h_1 = -\text{div} V_0
\]

**Lemma 16.1.** Let \( h_l = D^l_t h|_{t=0} \), for \( l \geq 0 \) and let \( V_0 = v|_{t=0} \). If (16.1) and (16.2) holds then

\[
(16.3) \quad h_{l+2} = \triangle h_l + F_{l+2}(h_{-1}, \ldots, h_{l-1}), \quad l \geq 0
\]

where \( F_2 = (\partial_i V_0^j)(\partial_j V_0^i) \) and in general \( F_{l+2} = F_{l+2}(h_{-1}, \ldots, h_{l-1}) \) is a sum of the form

\[
(16.4) \quad F_{l+2} = C_{\alpha_1 \ldots \alpha_n}^{l_1 \ldots l_n} h_{i_1}^{\alpha_1} \ldots h_{i_n}^{\alpha_n}, \quad h_{i}^{\alpha} = \partial_x^\alpha h_i, \quad l \geq 0, \quad h_{-1}^{\alpha} = \partial_x^\alpha V_0^{i\alpha}, \quad \alpha = (\alpha', i_k)
\]

with

\[
(16.5) \quad |\alpha_1| + l_1 + \ldots + |\alpha_n| + l_n = l + 2, \quad n \geq 2, \quad -1 \leq l_i \leq l - 1, \quad 1 \leq |\alpha_i| + l_i \leq l + 1
\]

**Proof.** If we us that \( [D_t, \partial_i] = -\partial_i V^k \partial_k \) we obtain from differentiating the first equation in (16.1)

\[
(16.6) \quad \partial_i D_t^l h = -D_t^{l+1} v_i + a^{k_1 \ldots k_n} (\partial V_{k_1}) \cdots (\partial V_{k_n}) V_{k_n}
\]

where \( V_k = D_k^l V \). Here the sum is over \( k_1 + \ldots + k_n = l + 2 - n, \quad n \geq 2, \quad k_n \geq 1, \) and terms in the sum consists of contractions over \( n - 1 \) pairs of indices. From differentiating \( \text{div} V = \partial_i V^i \)

\[
(16.7) \quad D_t^{l+1} \text{div} V = \text{div} D_t^{l+1} V + a^{k_1 \ldots k_n} (\partial V_{k_1}) \cdots (\partial V_{k_n})
\]

where the sums are over \( k_1 + \ldots + k_n = l + 2 - n, \quad n \geq 2 \) and terms in the sum consists of contractions over \( n \) pairs of indices.

It follows from (16.1) that \( D_t (D_t h + \text{div} V) = 0 \). Hence

\[
(16.8) \quad \triangle D_t^l h - D_t^{l+2} h = e^{k_1 \ldots k_n}(\partial V_{k_1}) \cdots (\partial V_{k_n}) + a^{k_1 \ldots k_n} (\partial V_{k_1}) \cdots (\partial^2 V_{k_n-1}) V_{k_n}
\]

where \( k_n \geq 1 \) in the last sum. Finally we note that we can turn \( V_{k+1} \), for \( k \geq 1 \) into \( h_k \):

\[
(16.9) \quad V_{k+1} = -\partial h_k + a^{k_1 \ldots k_n} (\partial V_{k_1}) \cdots (\partial V_{k_n})
\]

where \( k_n \geq 1 \) and \( V_1 = -\partial h \). This proves the general form (16.4) and it remains to prove the range of the indices in (16.5). To prove this we note that the terms in (16.8) are contractions over \( n \) pairs of indices and this is still true for the terms we obtain by using (16.9) to replace the factors of \( V \) by factors of \( h \). This proves that \( |\alpha_1| + \ldots + |\alpha_n| = 2n \). On the other hand when we replace factors of \( V_{k+1} \) by \( h_k \) the number of time derivatives go down by one for each factor, so we conclude that \( l_1 + \ldots + l_n = l + 2 - 2n \). This proves (16.5) apart from the last statement. That \( |\alpha_i| \geq 1 \) is clear and if \( l_i = -1 \) then \( |\alpha_i| \geq 2 \), in view of (16.4), so in general \( |\alpha_i| + l_i \geq 1 \), and since \( n \geq 2 \), (16.5) follows.

We will now obtain a formal power series solution in the distance to the boundary of the system for \( h_1 \) in Lemma 16.1. In order to do this, we will first choose simpler initial data for (16.2),

\[
(16.10) \quad f_0 = y, \quad v_0 = \partial \phi, \quad k_0 = -\triangle \phi
\]

where \( \phi \) is to be determined. Let \( h_{-1} = -\phi \), then (16.3) hold also for \( l = -1 \) with \( F_1 = 0 \).
Lemma 16.2. Suppose that $g_{ab} = \delta_{ab}$. Suppose also that $h_{0,1}$ and $h_{1,1}$ are smooth for $l \geq -1$, and let $F_l$ be as in Lemma 16.1, and $F_1 = 0$. Then the system

$$\Delta h_l = h_{l+2} + F_{l+2}(h_{-1}, ..., h_{l-1}), \quad l \geq -1$$

with boundary conditions

$$h_l|_{\partial \Omega} = h_{0,l}, \quad \nabla_N h_l|_{\partial \Omega} = h_{1,l} \quad l \geq -1$$

has a formal power series solution in the distance to the boundary:

$$\tilde{F}_l(r, \omega) \sim \sum h_{n,l}(\omega)(1 - r)^n/n!$$

Let $\chi$ be smooth such that $\chi(d) = 1$, when $|d| \leq 1$, $\chi(d) = 0$, when $|d| \geq 2$ and $\chi \geq 0$. Then there are $\varepsilon_{ln} > 0$ such that

$$\tilde{F}_l(r, \omega) = \sum \chi((1 - r)/\varepsilon_{ln})h_{n,l}(\omega)(1 - r)^n/n!$$

are smooth functions and such that (16.11) hold to infinite order at the boundary.

Proof. We have

$$\Delta = \nabla_N^2 + \text{tr} \theta \nabla_N + \overline{\Delta}$$

where $\theta$ is the second fundamental form of the boundary and $\overline{\Delta}$ is the tangential Laplacian on the boundary. In the case $g_{ab} = \delta_{ab}$, $\nabla_N = \partial_r$ and $\text{tr} \theta = (n - 1)/r$, where $r$ is the radial derivative. Furthermore $\overline{\Delta} = r^{-2}\Delta_\omega$, where $\Delta_\omega$ is the angular Laplacian on $S^{n-1}$. Hence we have the system

$$\partial_r^2 h_l = -\frac{1}{r^2} \Delta_\omega h_l + h_{l+2} - \frac{n+1}{r} \partial_r h_l + F_{l+2}(h_{-1}, ..., h_{l-1}), \quad l \geq -1$$

We want this system to be satisfied to all orders at the boundary, so if

$$h_{k,l} = (\partial_r^k h_l)|_{\partial \Omega}, \quad F_{m,l} = (\partial_r^m F_l)|_{\partial \Omega}$$

we want

$$h_{k,l} = \sum_{2i+2j+m \leq k, m \leq k+1} c_{ijkm} \Delta_\omega^i h_{m,l+2j} + \sum_{2i+2j+m \leq k} d_{ijkm} \Delta_\omega^i F_{m,l+2j}$$

to hold for all $k \geq 2$ and $l \geq -1$, where $h_{0,l}$ and $h_{1,l}$ are the given boundary conditions.

We now want to use induction. Note that the first term in the right of (16.18) contains $h_{k',l'}$ for $k' + l' \leq k + l$, and $k' < k$, and the second term contains $h_{k',l'}$ for $k' + l' < k + l$, by the last inequality in (16.5). Assume that we found

$$h_{k,l}, \quad \text{for} \quad k + l \leq N, \quad l \geq -1, \quad k \geq 0$$

such that (16.18) hold for $k + l \leq N$ and $k \geq 2$. Note that if $N = 0$ then $k \leq 1$ so there is nothing to prove. We know want to find $h_{k,l}$ for $k + l = N + 1$ such that (16.18) hold also for $k + l \leq N + 1$. This is again proven by induction. Assume that in addition to (16.19) we found

$$h_{k,l}, \quad \text{for} \quad k + l = N + 1, \quad 0 \leq k \leq M, \quad l \geq -1$$

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such that (16.18) hold for \( k \geq 2 \). Note that for \( M \leq 1 \) there is nothing to prove. Since \( m \) in the first sum on the right of (16.18) is less than \( k \) in the left it follows that we can find \( h_{k+1,l-1} \) such that (16.18) hold.

In order to prove (16.14) we note that \( h_{m,l} \) are smooth functions on \( S^{n-1} \). Hence we can use the usual trick of choosing \( \varepsilon_{ml} \) so small that \( (\|h_{m,l}\|_{m+1} + 1)\varepsilon_{ml} \leq 1/2 \), in which case the sum converges in \( H^m \) for any \( m \) and \( r \). \( \Box \)

Now, we want to find a formal power series solution in \( t \) of the system (16.1) with initial data in (16.2) of the form

\[
(16.21) \quad f_0(y) = \tilde{f}_0(y) + y, \quad v_0 = \tilde{v}_0 - \partial_t \tilde{h}_{-1}, \quad h_0 = \tilde{h}_0 + \tilde{h}_0 \quad h_1 = -\text{div} \tilde{V}_0 + \Delta \tilde{h}_{-1}
\]

where \( \tilde{h}_0, \tilde{h}_{-1} \) are given by Lemma 16.2 and \( \tilde{f}_0, \tilde{v}_0 \) and \( \tilde{h}_0 \) vanish to infinite order at the boundary. Let \( h_l \), for \( l \geq 2 \) be defined by (16.3). Then it inductively follows that

\[
(16.22) \quad h_l = \tilde{h}_l + \tilde{h}_l,
\]

where \( \tilde{h}_l \) are as in Lemma 16.2 and \( \tilde{h}_l \) vanish to infinite order at the boundary. Therefore if we choose boundary data in (16.12) such that \( h_0, l = 0 \) for \( l \geq 0 \), \( h_{1,0} \leq \epsilon_0 < 0 \). Then it follows that we can choose \( \tilde{h}_0 \) so that \( h_0 > 0 \) in \( \Omega \) and \( \nabla_N h_0|_{\partial \Omega} \leq -c_0 < 0 \). Moreover it follows that

\[
(16.23) \quad h_l|_{\partial \Omega} = 0, \quad l \geq 0
\]

and hence the compatibility conditions are satisfied to all orders.

We can now construct smooth functions in \([0,T] \times \overline{\Omega} \), satisfying the initial conditions (16.2) and the equations (16.2) to infinite order as \( t \to 0 \):

**Lemma 16.3.** Suppose that initial data \( f_0, v_0 \) and \( h_0 \) and \( h_1 = -\text{div} V_0 \) satisfy the compatibility conditions for all orders, i.e. if \( h_l \) for \( l \geq 2 \) are defined by (16.3) then

\[
(16.24) \quad h_l|_{\partial \Omega} = 0, \quad \text{for} \quad l \geq 0.
\]

Then there are smooth functions \( (x,h) \) in \([0,T] \times \overline{\Omega} \), such that (16.2) hold, (16.1) is satisfied to infinite order as \( t \to 0 \) and \( h|_{\partial \Omega} = 0 \).

**Proof.** Let \( \chi \) be as in Lemma 16.2 and set

\[
(16.25) \quad h(t,y) = \sum_{l=0}^{\infty} \chi(t/\varepsilon_l)h_l(y)t^l/l!
\]

where \( \varepsilon_l > 0 \) are chosen so that \( (\|h_l\|_m + 1)\varepsilon_m \leq 1/2 \). Then it follows that the sum converges in \( H^m \) for any \( m \) so \( h \) is smooth and satisfies \( h|_{\partial \Omega} = 0 \) and \( D^l_t h|_{t=0} = h_l \). Furthermore let \( x(t,y) \) be defined by

\[
(16.26) \quad D^2_t x_i = -\partial_i h, \quad x_i|_{t=0} = f_0, \quad D_t x|_{t=0} = v_0. \quad \Box
\]
17. The general case, when the enthalpy is a strictly increasing function of the density.

We will now outline how to generalize the existence result obtained for \( e(h) = ch \) to the case when \( e(h) \) is a smooth strictly increasing function satisfying \( 1/c_1' \leq e'(h) \leq c_1' \). First we will show that the functional \( h = \Psi(x) \), i.e. the solution of (2.21), exist for \( x \) in a bounded set, \( \|u\|_{r_0+r_1, \infty} \leq 1 \), and for \( T \) sufficiently small. Since in the Nash-Moser iteration we need a bound for \( T \) that we can obtain such a bound as well as a bound for \( \|a\|_{r_0+r_1, \infty} \), we will first show that we can obtain such a bound as well as a bound for \( \|a\|_{r_0+3, \infty} \) independent of \( T \). Local existence for the nonlinear wave equation follows from a standard argument using essentially the same estimates, so it is just a question of showing that we have \( a \) priori bounds up to some time \( T > 0 \) that only depends on the approximate solution \( (x_0, h_0) \) and is independent of \( x = u + x_0 \) as long as \( \|u\|_{r_0+r_1, \infty} \leq 1 \). Once we have a bound for \( \|h\|_{4, \infty} \), the bound for higher derivatives follows from this.

The equation we study in this section is

\[
(17.1) \quad D_t(e'(h)D_t h) - \triangle h = f, \quad h \big|_{t=0} = 0, \quad f = (\partial_i V^j)(\partial_j V^i)
\]

where

\[
(17.2) \quad e' + 1/e' \leq c_1', \quad \sum_{a,b} |g^{ab}| + |g_{ab}| \leq c_1', \quad |\partial x/\partial y|^2 + |\partial y/\partial x|^2 \leq c_1'
\]

for some constant \( 0 < c_1' < \infty \). Let \( K'_{c0} \) denote a continuous function of \( c_1' \) that also depends on the order of differentiation \( r \) but is independent of a lower bound for \( T \).

We prove the following Theorem:

**Theorem 17.1.** Let \( r_0 = [n/2] + 1 \) be the Sobolev exponent and let \( k \geq 1 \). There are continuous function \( C_k \) and \( D_k \) such that if \( T > 0 \) is so small that

\[
(17.3) \quad TC_1(\|x_0\|_{r_0+2+1, \infty}, \|h_0\|_{r_0+2+1, \infty}) \leq 1,
\]

and \( u = x - x_0 \) is small that

\[
(17.4) \quad \|u\|_{r_0+2+k, \infty} \leq 1
\]

then (17.1) has a smooth solution for \( 0 \leq t \leq T \) satisfying

\[
(17.5) \quad \|h\|_{k, \infty} \leq D_k(\|x_0\|_{r_0+2+k, \infty}, \|h_0\|_{r_0+2+k, \infty}).
\]

Furthermore, there is a continuous function \( K_2' \) of

\[
(17.6) \quad \|x\|_{2, \infty} + \|h\|_{2, \infty} + \|h_0\|_{2, \infty} + 1/T + c_1'
\]

depending also on \( r \) such that

\[
(17.7) \quad \|h\|_{r, \infty} \leq K_2'(\|x\|_{r+r_0+2, \infty} + \|h_0\|_{r+r_0+2, \infty})
\]

**Proof.** Since by Sobolev’s Lemma

\[
(17.8) \quad \|h\|_{1, \infty} \leq C(\|h\|_{1, r_0} + \|h\|_{0, r_0+1})
\]
where $C$ is independent of $T$. It follows from Lemma 17.2 that

$$(17.9) \quad \frac{d\tilde{E}_{r}}{dt} \leq M_{r}(\tilde{E}_{r} + 1)^{r+2}, \quad r \geq r_{0}$$

where $M_{r} = K_{10}^{r+1}\|x\|_{r+3,\infty}^{(r+2)^{2}}$. Integrating this inequality gives $-d(\tilde{E}_{r} + 1)^{-(r+1)}/dt \leq M_{r}(r + 1)$ and hence $(\tilde{E}_{r}(t) + 1)^{-(r+1)} \geq (\tilde{E}_{r}(0) + 1)^{-(r+1)} - M_{r}(r + 1)t$. If $M_{r}(r + 1)t \leq (\tilde{E}_{r}(0) + 1)^{-(r+1)}/2$ it follows that $\tilde{E}_{r}(t) + 1 \leq 2(\tilde{E}_{r}(0) + 1)$. Since $\tilde{E}_{r}(0) \leq K_{20}^{r+1}(\|h_{0}\|_{1,s} + \|h_{0}\|_{0,s+1})$ this proves the first part of the theorem for $k = 1$ and for $k \geq 2$ it follows from also using Theorem 17.3.

It follows from Lemma 17.2 and interpolation in space time that

$$(17.10) \quad \frac{d\hat{E}_{r+1}}{dt} \leq K_{2}'(\hat{E}_{r+1} + \|x\|_{r+3,\infty}), \quad \text{if} \quad \hat{E}_{r+1} = \sum_{s=0}^{r} \|x\|_{r+2-s} \hat{E}_{s+1}$$

Multiplying by the integrating factor $e^{K_{2}t}$ and integrating, we get

$$(17.11) \quad \hat{E}_{r+1}(t) \leq K_{2}'(\hat{E}_{r+1}(0) + \|x\|_{r+3,\infty})$$

By Theorem 7.3 $\|h\|_{r+1} \leq K_{2}' \hat{E}_{r+1}$ and hence

$$(17.12) \quad \|h(t, \cdot)\|_{r+1} \leq K_{2}' \left( \sum_{s=0}^{r} \|x\|_{r+2-s} \|h_{0}(0, \cdot)\|_{s+1} + \|x\|_{r+3} \right)$$

Using Sobolev’s lemma and interpolation the estimate for $\|h\|_{r,\infty}$ follows. □

**Lemma 17.2.** Suppose that $g^{ab}$ and $e' = e'(h)$ are smooth and satisfy (7.4). For $s \geq 0$ let

$$(17.13) \quad E_{s+1}(t) = \left( \frac{1}{2} \int_{\Omega} e'(D_{t}^{s+1}h)^{2} + g_{ab}(\tilde{D}_{t}^{s}H^{a})(\tilde{D}_{t}^{s}H^{b})\kappa dy \right)^{1/2}, \quad H^{a} = g^{ab}\partial_{b}h,$$

$E_{0}(t) = \int_{\Omega} h^{2}\kappa dy$ and $\tilde{E}_{r+1} = \sum_{s=0}^{r} E_{s}$. Then for $r \geq 0$

$$(17.14) \quad \|h\|_{1,r} + \|h\|_{0,r+1} \leq C \sum_{s=0}^{r} \left( \left[\left[\partial x\right]\right]_{r-s,\infty} \tilde{E}_{1+s} \right)$$

$$(17.15) \quad \tilde{E}_{1+r} \leq C \sum_{s=0}^{r} \left( \left[\left[\partial x\right]\right]_{r-s,\infty} (\|h\|_{1,s} + \|h\|_{0,s+1}) \right)$$

and

$$(17.16) \quad \frac{d\tilde{E}_{r+1}}{dt} \leq K_{10}' \sum_{k=0}^{r+1} \|h\|_{k,\infty}^{\min(r+1-k,r)} \left( \sum_{s=0}^{r} \left[\left[\partial x\right]\right]_{r+1-k-s,\infty} \tilde{E}_{s+1} + \left[\left[\partial x\right]\right]_{r+2-k,\infty} \right)$$

**Proof.** The proof is similar to that of Lemma 7.2. We have

$$(17.17) \quad \frac{dE_{s+1}^{2}}{dt} = \int_{\Omega} \left( e'(D_{t}^{s+1}h)(D_{t}^{s+2}h) + g_{ab}(\tilde{D}_{t}^{s}H^{a})(\tilde{D}_{t}^{s+1}H^{b}) \right) \kappa dy$$

$$+ \frac{1}{2} \int_{\Omega} \left( (\tilde{D}_{t}e')(D_{t}^{s+1}h)^{2} + (\tilde{D}_{t}g_{ab})(\tilde{D}_{t}^{s}H^{a})(\tilde{D}_{t}^{s}H^{b}) \right) \kappa dy.$$
Here the terms on the second row are bounded by $K_{10}^1(\|h\|_{1,\infty} + \|g\|_{1,\infty})E_{s+1}^2$. The first row of the right hand side is up to lower order

\[
(17.18) \quad \int_\Omega \left( (D_t^{s+1}h)(\hat{D}_t^s D_t(e'D_t h)) + (\hat{D}_t^s H^a)(\partial_a D_t^{s+1}h) \right) \kappa dy
= \int_\Omega (D_t^{s+1}h) \left( \hat{D}_t^s D_t(e'D_t h) - \kappa^{-1}\partial_a (\kappa \hat{D}_t^s H^a) \right) \kappa dy = \int_\Omega (D_t^{s+1}h)(\hat{D}_t^s f) \kappa dy,
\]

where we have integrated by parts using that $D_t^{s+1}h|_{\partial \Omega} = 0$, that $D_t(e'D_t h) - \kappa^{-1}\partial_a (\kappa H^a) = f$ and that $\hat{D}_t^s (\kappa^{-1}\partial_a (\kappa H^a)) = \kappa^{-1}\partial_a (\kappa \hat{D}_t^s H^a)$. In fact, using Lemma 6.3 we get, since $H_a = \partial_a h$,

\[
(17.19) \quad \hat{D}_t^{s+1}H^a - g^{ab}\partial_a D_t^{s+1}h = -\sum_{i=0}^{s+1} \binom{s+1}{i} g^{ab}(\hat{D}_t^{s+1-i}g_{bc})\hat{D}_t^i H^c
\]

and, since $\hat{D}_t D_t^2 e(h) = \kappa^{-1}D_t^2 (\kappa D_t^2 e(h))$,

\[
(17.20) \quad \hat{D}_t^2 D_t^2 e(h) - e'(h)D_t^{s+2}h = \sum_{i=0}^{s+1} \binom{s}{i} \kappa^{-1}(D_t^{s-i}h)(D_t^{2+i}e(h)) + D_t^{s+2}e(h) - e'(h)D_t^{s+2}h.
\]

Here the right hand side is bounded by a constant times

\[
(17.21) \quad \sum_{i=0}^{s+1} \kappa^{-1}|D_t^{s-i}h| \sum_{s_1+\ldots+s_k=s+i, s_i \geq 1, k \geq 1} |e^{(k)}(h)||D_t^{i}h| \ldots |D_t^{k}h| + \sum_{s_1+\ldots+s_k=s+2, s_i \geq 1, k \geq 2} \kappa^{-1}(D_t^{s-i}h)(D_t^{2+i}e(h)) + D_t^{s+2}e(h) - e'(h)D_t^{s+2}h.
\]

The $L^2$ of this can be estimated by Theorem 7.3:

\[
(17.22) \quad K_{1,0}^1 \sum_{k=0}^{s+1} \|h\|_{1,\infty} \left( \sum_{i=1}^{s+1} \left( \left[ [\partial x] \right]_{s+2-k-i, \infty} \right)_{s+2-k-i, \infty} \right)
\]

By Lemma 6.3 $\hat{D}_t^s (g^{ab}\partial_a h) = g^{ab}\partial_a D_t^s h + \sum_{i=0}^{s-1} \binom{s}{i} g^{ab}(\hat{D}_t^{s-i}g_{bc})\hat{D}_t^i (g^{cb}\partial_b h)$ so it follows that $\|\partial D_t^s h\| \leq \sum_{i=0}^{s-1} \left( \left[ [\partial x] \right] \right)_{s+2-i, \infty} \hat{E}_{s+1,i}$ Since also $\|\hat{D_t} f\| \leq K_{10} \left[ [\partial x] \right]_{s+2}$ the lemma follows. □

**Theorem 17.3.** Suppose that $D_t^2 e(h) - \Delta h = f$, where $f = (\partial_i V_j)(\partial_j V^i)$. Suppose also that $|e^{(k)}(h)| \leq C_k$. Then we have

\[
(17.23) \quad \|h\|_{r} \leq K_{1,0}^1 \sum_{k=1}^{r} \|h\|_{1,\infty} \left( \sum_{i=1}^{k} \left[ [\partial x] \right]_{r-k-i, \infty} \right)_{r-k-i, \infty} \|h\|_{i,1,\infty} + \|h\|_{1,1,1} + \left[ [\partial x] \right]_{r-k, \infty}
\]

and

\[
(17.24) \quad \|h\|_{r} \leq K_{1,0}^1 \sum_{k=1}^{r} \|h\|_{1,\infty} \left( \sum_{i=1}^{k} \left[ [\partial x] \right]_{r-k-i, \infty} \right)_{r-k-i, \infty} \|h\|_{0,1,\infty} + \|h\|_{1,1,1} + \left[ [\partial x] \right]_{r-k, \infty}
\]

where

\[
(17.25) \quad [h]^j = \sum_{r_1+\ldots+r_k=r, r_i \geq 1, k \geq j} |h|_{r_1} \ldots |h|_{r_k}, \quad \left[ [\partial x] \right]_{r, \infty} = \sum_{r_1+\ldots+r_k=r, r_i \geq 1} \|\partial x\|_{r_1, \infty} \ldots \|\partial x\|_{r_k, \infty}
\]

**Proof.** The first inequality follows from Lemma 17.4 below and interpolation and the second follows from the first and Lemma 17.6 below. □
Lemma 17.4. Suppose that \( D_t^2 e(h) - \Delta h = f \), where \( f = (\partial_i V^j)(\partial_j V^i) \). Suppose also that \( |e^{(k)}(h)| \leq C_k \). With notation as in Definition 5.1 we have

\[
[h]_{r,s} \leq K_{10}' \sum_{i+j \leq r+s} [\partial x]_{r+s-i-j} [h]_{i,j,1}
\]

where \([\partial x]_l\) is as in Definition 5.1,

\[
[h]_{r,s} = \sum_{r_1 + \ldots + r_k = r, \ s_1 + \ldots + s_k = s, \ s_i, r_i \geq 1} |h|_{r_1, s_1} \ldots |h|_{r_k, s_k},
\]

\[
[h]_{r,s,m} = \sum_{r_1 + \ldots + r_k = r, \ s_1 + \ldots + s_k = s, \ s_i, r_i \geq 1, s_i \leq m,}
\]

and \([h]_{0,0} = 1, [h]_{0,0,1} = 1\).

Proof. If \( h = \eta(e) \) is the inverse of \( e(h) \), then

\[
|h|_{r,s} \leq C \sum_{r_1 + \ldots + r_k = r, \ s_1 + \ldots + s_k = s, \ r_i + s_i \geq 1} |\eta^{(k)}(e)| |e|_{r_1, s_1} \ldots |e|_{r_k, s_k}.
\]

Since,

\[
|e|_{r,s} \leq |\Delta h|_{r,s-2} + |f|_{r,s-2}, \quad s \geq 2
\]

where

\[
|\Delta h|_{r,s-2} \leq K_{10} \sum_{1 \leq i \leq r+2, j \leq s-2} [\partial x]_{r+s-i-j} |h|_{i,j}, \quad s \geq 2
\]

and

\[
|f|_{r,s-2} \leq K_{10} [\partial x]_{r+s}, \quad s \geq 2
\]

we obtain

\[
|e|_{r,s} \leq K_{10} \sum_{i \leq r+2, j \leq s-2} [\partial x]_{r+s-i-j} |h|_{i,j}, \quad s \geq 2
\]

It follows that

\[
[h]_{r,s,m} \leq K_{10}' \sum_{i+j \leq r+s} [\partial x]_{r+s-i-j} [h]_{i,j,m-1}, \quad m \geq 2
\]

and the lemma follows by induction.

\[
\square
\]

Lemma 17.5. For \( r \geq 1 \) we have with notation as in Definition 5.3,

\[
[h]_r \leq K_{10}' \sum_{j=0}^{r} [h]^j \left( \sum_{i=1}^{r-j} \left( [\partial x] \right)_{r-i-j,i} \|h\|_{0,i} + \|h\|_{1,i-1} + \left( [\partial x] \right)_{r-j,1} \right)
\]

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Proof. We have
\[ |\nabla h|_{r,s} \leq |e(h)|_{r,s+2} + |f|_{r,s} \]
where
\[ |e(h)|_{r,s} \leq e'(h)|h|_{r,s} + C \sum_{k=2}^{r+s} [e(k)(h)] [h]_{r,s}^k, \]
where \[ [h]_{r,s}^k = \sum_{r_1 + \ldots + r_k = r, s_1 + \ldots + s_k = s} |h|_{r_1,s_1} \cdots |h|_{r_k,s_k} \]
and by the previous lemma and interpolation in space only
\[ \|[h]_{r,s}^k\| \leq K'_1 \sum_{i+j \leq r+s, j \geq k-1, i \geq 1} \|[\partial x]\|_{r+s-i-j, \infty} \|h\|_{1,\infty}^i (\|h\|_{i,0} + \|h\|_{i-1,1}) \]
Using Theorem 6.1 we get for \( r \geq 2 \)
\[ \|h\|_{r+2, s} \leq K_1 \sum_{i \leq r, j \leq s} \|[\partial x]\|_{r+s-i-j, \infty} \|h\|_{i,j+2} + K_1 \sum_{j=0}^{s} \|[\partial x]\|_{r+s+1-j, \infty} \|h\|_{1,j,\infty} \]
\[ + K'_1 \|[\partial x]\|_{r+s+2, \infty} + K'_1 \sum_{i+j \geq r+s+2, i, j \geq 1} \|[\partial x]\|_{r+s+2-i-j, \infty} \|h\|_{1,\infty}^i (\|h\|_{i,0} + \|h\|_{i-1,1}) \]
Hence using induction, we get
\[ \|h\|_r \leq K'_1 \sum_{1 \leq i \leq r} \|[\partial x]\|_{r-i, \infty} (\|h\|_{0,i} + \|h\|_{1,i-1}) \]
\[ + K'_1 \|[\partial x]\|_{r-i, \infty} \|h\|_{1,\infty}^i \|h\|_i \]
Since \( \|h\|_1 \leq \|h\|_{0,1} + \|h\|_{1,0} \) the lemma now follows from induction. \( \square \)

Alternatively, one can use interpolation interpolation in space-time.

Lemma 17.6. We have
\[ \|D_t^1 h\|_{L^2/(m-1)} \leq C_T \|h\|_{L^\infty}^{l/m} \|D_t^m h\|_{L^2}^{1-l/m} \]
\[ \|D_t^1 h \cdots D_t^k h\|_{L^2} \leq C_T \|h\|_{L^\infty}^{k-1} \|D_t^m h\|_{L^2} \]
where \( m = l_1 + \ldots + l_k \). Furthermore, if \( h = \tilde{h} + h_0 \) where \( \tilde{h} \) vanish to infinity order as \( t \to 0 \) then, with a constant independent of \( T \) but depending on \( h_0 \) we have
\[ \|D_t^1 h \cdots D_t^k h\|_{L^2} \leq C_m (\|h\|_{L^\infty} \|h_0\|_{L^\infty,m}) (\sum_{j=0}^{m} \|D_t^j h\|_{L^2} + 1) \]

Proof. The interpolation inequalities are standard and it is also standard that the constant is independent of \( T \) if \( h \) vanish to infinite order as \( t \to 0 \). Hence in the product we can estimate
\[ \|D_t^1 h\|_{L^2/(m-1)} \leq \|D_t^1 \tilde{h}\|_{L^2/(m-1)} + \|D_t^1 h_0\|_{L^\infty}. \]

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References

[AG]. S. Alinhac and P. Gerard, *Opérateurs pseudo-differentiels et theorem de Nash-Moser*, Inter Editions and CNRS, 1991.

[BG]. M.S. Baouendi and C. Gouaouic, *Remarks on the abstract form of nonlinear Cauchy-Kovalevsky theorems*, Comm. Part. Diff. Eq. 2 (1977), 1151-1162.

[C1]. D. Christodoulou, *Self-Gravitating Relativistic Fluids: A Two-Phase Model*, Arch. Rational Mech. Anal. 130 (1995), 343-400.

[C2]. D. Christodoulou, *Oral Communication* (August 95).

[CK]. D. Christodoulou and S. Klainerman, *The Nonlinear Stability of the Minkowski space-time*, Princeton Univ. Press, 1993.

[CL]. D. Christodoulou and H. Lindblad, *On the motion of the free surface of a liquid.*, Comm. Pure Appl. Math. 53 (2000), 1536-1602.

[CF]. R. Courant and K. O. Friedrichs, *Supersonic flow and shock waves*, Springer-Verlag, 1977.

[Cr]. W. Craig, *An existence theory for water waves and the Boussinesq and Korteweg-de Vries scaling limits*, Comm. in P. D. E. 10 (1985), 787-1003.

[DM]. B. Dacorogna and J. Moser, *On a partial differential equation involving the Jacobian determinant.*, Ann. Inst. H. Poincare Anal. Non. Lineaire 7 (1990), 1-26.

[DN]. S. Dain and G. Nagy, *Initial data for fluid bodies in general relativity*, Phys. Rev. D 65 (2002).

[E1]. D. Ebin, *The equations of motion of a perfect fluid with free boundary are not well posed.*, Comm. Part. Diff. Eq. 10 (1987), 1175–1201.

[E2]. D. Ebin, *Oral communication* (November 1997).

[F]. H. Friedrich, *Evolution equations for gravitating ideal fluid bodies in general relativity*, Phys. Rev. D 57 (1998).

[FN]. H. Friedrich and G. Nagy, *The initial boundary value problem for Einstein’s vacuum field equation*, Comm. Math. Phys. 201 (1999), 619-655.

[Ha]. R. Hamilton, *Nash-Moser Inverse Function Theorem*, Bull. Amer. Math. Soc. (N.S.) 7 (1982), 65–222.

[H1]. Hörmander, *The boundary problem of Physical geodesy*, Arch. Rational Mech. Anal. 62 (1976), 1-52.

[H2]. Hörmander, *Implicit function theorems* (1977), Lecture Notes (Stanford ).

[H3]. Hörmander, *The analysis of Linear Partial Differential Operators III*, Springer Verlag.

[K1]. S. Klainerman, *On the Nash-Moser-Hörmander scheme*, unpublished lecture notes.

[K2]. S. Klainerman, *Global solutions of nonlinear wave equations*, Comm. Pure Appl. Math. 33 (1980), 43–101.

[L1]. H. Lindblad, *Well posedness for the linearized motion of an incompressible liquid with free surface boundary.*, Comm. Pure Appl. Math., (2003).

[L2]. Well posedness for the linearized motion of a compressible liquid with free surface boundary., Comm. Math. Phys. (2003).

[L3]. Well posedness for the motion of an incompressible liquid with free surface boundary., preprint: [http://xxx.lanl.gov/abs/math.AP/0402327](http://xxx.lanl.gov/abs/math.AP/0402327), to appear in the Annals of Math..

[Na]. V.I. Nalimov, *The Cauchy-Poisson Problem* (in Russian),, Dynamika Splosh. Sredy 18 (1974,), 104-210.

[Ni]. T. Nishida, *A note on a theorem of Nirenberg*, J. Diff. Geometry 12 (1977), 629-633.

[R]. A. D. Rendall, *The initial value problem for a class of general relativistic fluid bodies*, J. Math. Phys. (1992), 1047-1053.

[SY]. Schoen and Yau, *Lectures on Differential Geometry*, International Press, 1994.

[S]. E. Stein, *Singular integrals and differentiability properties of functions* (1970), Princeton University Press.

[W1]. S. Wu, *Well-posedness in Sobolev spaces of the full water wave problem in 2-D*, Invent. Math. 130 (1997), 39-72.

[W2]. S. Wu, *Well-posedness in Sobolev spaces of the full water wave problem in 3-D*, J. Amer. Math. Soc. 12 (1999), 445-495.

[Y]. H. Yoshinara, *Gravity Waves on the Free Surface of an Incompressible Perfect Fluid* 18 (1982), Publ. RIMS Kyoto Univ., 49-96.